Precision measurements for the Higgsploding Standard Model

Valentin V. Khoze,¹ Joey Reiness,¹ Michael Spannowsky,¹ and Philip Waite¹

¹Institute for Particle Physics Phenomenology, Department of Physics, Durham University, Durham, DH1 3LE, UK
E-mail: valya.khoze@durham.ac.uk, joey.y.reiness@durham.ac.uk, michael.spannowsky@durham.ac.uk, p.a.waite@durham.ac.uk

Abstract: Higgspllosion is the mechanism that leads to exponentially growing decay rates of highly energetic particles into states with very high numbers of relatively soft Higgs bosons. In this paper we study quantum effects in the presence of Higgspllosion. First, we provide a non-perturbative definition of Higgspllosion as a resolved short-distance singularity of quantum propagators at distances shorter than the inverse Higgspllosion energy scale, $E^*$. We then consider quantum effects arising from loops in perturbation theory with these propagators on internal lines. When the loop momenta exceed the Higgspllosion scale $E^*$, the theory dynamics deviates from what is expected in the standard QFT settings without Higgspllosion. The UV divergences are automatically regulated by the Higgspllosion scale, leading to the change of slopes for the running couplings at the RG scales $\mu > E^*$. Thus, the theory becomes asymptotically safe. Further, we find that the finite parts are also modified and receive power-suppressed corrections in $1/E^*$. We use these results to compute a set of precision observables for the Higgsploding Standard Model. These and other precision observables could provide experimental evidence and tests for the existence of Higgspllosion in particle physics.
1 Introduction

A conventional wisdom is that in the description of nature based on a local quantum field theory, one should always be able to probe shorter and shorter distances with higher and higher energies. Specifically, the characteristic length scales probed are $\Delta x \sim 1/E$ where $E$ is the energy scale, a momentum transfer, or the virtuality that is achieved in the experiment. In the asymptotic regime where $E \to \infty$, one expects to probe $\Delta x \to 0$.

Higgsplosion [1] is a dynamical mechanism, or a new phase of the theory, which presents an obstacle to this length-scale/energy principle at energies above a certain value $E_*$, referred to as the Higgsplosion energy scale. Beyond this energy scale the dynamics of the system are changed drastically [2]: distance scales below $|x| \lesssim 1/E_*$ cannot be resolved in interactions; UV divergences are regulated; the theory becomes asymptotically free and the Hierarchy problem of the Standard Model (SM) is therefore absent. This effect can also be depicted in the short-distance scaling behaviour of the propagator of a scalar particle,
\[ \Delta(x) := \langle 0 | T(\phi(x) \phi(0)) | 0 \rangle \sim \begin{cases} m^2 e^{-m|x|} & : \text{for } |x| \gg 1/m \\ 1/|x|^2 & : \text{for } 1/E_* \ll |x| \ll 1/m \\ E_*^2 & : \text{for } |x| \lesssim 1/E_*\end{cases} \]  

where for \(|x| \lesssim 1/E_*\) one enters the Higgsplasion regime.

In the simplest settings described by a quantum field theory of a massive scalar field \(\phi\) with mass \(m\) and coupling \(\lambda\), we show in Section 2 how Eq. 1.1 is linked to the growing multi-particle decay rates. Furthermore, we show that the Higgsplasion energy scale is set by \(E_* = C \frac{m}{\lambda}\), where \(C\) is a model-dependent constant of \(O(100)\). This expression holds in the weak-coupling limit \(\lambda \to 0\). In this respect, it resembles the \(SU(2)\) sphaleron, which has a mass scale of \(M_{\text{sph}} = \text{const} \frac{mW}{\alpha_w} [3, 4]\). However, while the sphaleron is a phenomenon of the non-Abelian gauge-Higgs sector of the Standard Model, Higgsplasion arises due to its scalar sector only.

The fundamental ingredient for the theory is the value of the Higgsplasion scale \(E_*\). It is the scale where the rate for the process \(1^* \to n \times h\) grows exponentially for large enough \(n\). The factorial growth of the rate has been calculated before at leading order [5–8], one-loop resummed [8–11], or using a semiclassical approach [12–14]. However, Higgsplasion itself has not been taken into account in those calculations. Thus, in Section 3, we extend their approach by including Higgsplasion, and, for the first time, calculate the loop-corrected rates in a self-consistent way.

After the Higgsplasion scale \(E_*\) is established we can evaluate its phenomenological impact on precision observables, such as \(gg \to h^{(*)}, h \to \gamma\gamma, h \to Z\gamma, B \to Xs\gamma\) or \(g - 2\). We calculate these precision observables explicitly in Section 4, and conclude with a discussion of our findings in Section 5.

2 The propagator and Higgsplasion basics

2.1 The Dyson propagator

In the introduction we pointed out that the central object in a theory with Higgsplasion is the propagator (1.1), and that Higgsplasion manifests itself in resolving the short-distance singularity at \(x^2 \leq 1/E_*^2\), where \(E_*\) is the characteristic (high-)energy scale of Higgsplasion. To explain what we mean by this and how the effect of Higgsplasion modifies the familiar structure of the propagator, it is worthwhile first to summarise the basic elements and the interplay between the propagator for a massive scalar field \(\phi\), its self-energy \(\Sigma(p^2)\), and the partial width \(\Gamma_n(p^2)\). This is the aim of this section.

Our technical discussion in this and the following section will be for a quantum field theory of a single massive scalar degree of freedom. The specific models we consider are the \(\phi^4\) theory with the unbroken \(Z_2\) symmetry,

\[ \mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4} \phi^4, \]  

where \(E_* \approx V_0\).
and a similar model with spontaneous symmetry breaking (SSB), where the scalar field has a non-zero VEV \( \langle \phi \rangle = v \),

\[
\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{\lambda}{4} (\phi^2 - v^2)^2 , \tag{2.2}
\]

\[
\phi(x) = v + h(x) , \quad m_h = \sqrt{2\lambda} v . \tag{2.3}
\]

In the SSB case (2.2)-(2.3), \( m_h \) is the mass of the physical scalar field \( h(x) \). As in our earlier work [1, 2, 14] we will view the SSB theory (2.2) as a simplified model for the Standard Model Higgs sector in the unitary gauge.

In a generic QFT model with a massive scalar, we can define the following quantities:

1. The Feynman propagator of \( \phi \) is the Fourier transform of the 2-point Green function,

\[
\Delta(p) = \int d^4 x e^{i p \cdot x} \langle 0 | T(\phi(x) \phi(0)) | 0 \rangle = i \frac{p^2 - m_0^2 - \Sigma(p^2) + i\epsilon}{p^2 - m_0^2} , \tag{2.4}
\]

where \( m_0 \) is the bare (unrenormalised) mass of the scalar field \( \phi \).

2. The self-energy \( \Sigma(p^2) \) is the sum of all one-particle-irreducible (1PI) diagrams contributing to the 2-point function,

\[
- i \Sigma(p^2) = \sum -(1\text{PI}) - . \tag{2.5}
\]

The right-hand side of Eq. (2.4) can be interpreted in perturbation theory as the sum over the infinite series of the bare propagators and the \( \Sigma(p^2) \) insertions,

\[
\frac{i}{p^2 - m_0^2 - \Sigma(p^2)} = \frac{i}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2} \sum_{n=1}^{\infty} \left( -i \Sigma(p^2) \frac{i}{p^2 - m_0^2} \right)^n . \tag{2.6}
\]

Hence Eq. (2.4) gives the full quantum propagator, also known as the Dyson propagator [15–17], valid in perturbative and non-perturbative quantum field theories.

3. The physical (or pole) mass \( m \) is defined as the pole of the quantum propagator (2.4),

\[
m^2 - m_0^2 - \Sigma(m^2) = 0 , \quad \text{or} \quad m^2 = m_0^2 + \Sigma(m^2) . \tag{2.7}
\]

4. The field renormalisation constant \( Z_\phi \) is determined from the slope of \( \Sigma(p^2) \) at \( m^2 \),

\[
Z_\phi = \left( 1 - \frac{d\Sigma}{dp^2} \right)_{p^2=m^2}^{-1} . \tag{2.8}
\]

Using the definition of the pole mass (2.7) and the renormalisation constant \( Z_\phi \) in (2.8), the full propagator (2.4) can be written as,

\[
\Delta(p) = \frac{i Z_\phi}{p^2 - m^2 - Z_\phi [\Sigma(p^2) - \Sigma(m^2) - \Sigma'(m^2)(p^2 - m^2)]} . \tag{2.9}
\]
5. The $Z\phi$ constant is used to define the renormalised quantities $\Delta_R(p)$ and $\Sigma_R(p^2)$,

$$\Delta_R(p) = Z\phi^{-1} \Delta(p),$$  \hspace{1cm} (2.10)
$$\Sigma_R(p) = Z\phi \left( \Sigma(p^2) - \Sigma(m^2) - \Sigma'(m^2)(p^2 - m^2) \right).$$  \hspace{1cm} (2.11)

Hence, the result for the renormalised propagator in terms of all finite quantities is,

$$\Delta_R(p) = \frac{i}{p^2 - m^2 - \Sigma_R(p^2) + i\epsilon}. \hspace{1cm} (2.12)$$

6. The optical theorem provides the physical interpretation of the imaginary part of the self-energy in terms of the momentum-scale dependent decay width $\Gamma(p^2)$,

$$- \text{Im} \Sigma_R(p^2) = m \Gamma(p^2), \hspace{1cm} (2.13)$$

with the decay width being determined by the partial widths of $n$-particle decays at energies $s \geq (nm)^2$,

$$\Gamma(s) = \sum_{n=2}^{\infty} \Gamma_n(s), \quad \Gamma_n(s) = \frac{1}{2m} \int \frac{d\Phi_n}{n!} |\mathcal{M}(1 \to n)|^2. \hspace{1cm} (2.14)$$

Here $\mathcal{M}$ is the amplitude for the $1^* \to n$ process, the integral is over the $n$-particle Lorentz-invariant phase space, and $1/n!$ is the Bose-Einstein symmetry factor for $n$ spin-zero particles produced in the final state.

7. The origin of Higgsplasion [1] is that the scattering amplitudes $\mathcal{M}(1 \to n)$, and consequently the decay rates into the $n$-particle final states, grow factorially with $n$ in the large-$n$ limit, $\frac{1}{m} |\mathcal{M}_n|^2 \sim n! \lambda^n \sim e^{n \log(\lambda n)}$. When $n$ scales linearly with the available energy, $n \sim \sqrt{s}/m$, this translates into the exponential dependence of the decay rate $\Gamma(s)$ on $\sqrt{s}$. It was further argued in [1, 14] that there is a sharp transition between the exponential suppression, $\Gamma_n(s < E^2_*)/m \ll 1$, and the exponential growth, $\Gamma_n(s > E^2_*)/m \gg 1$, for the $n$-particle rate at a certain characteristic energy scale $E_*$ (and in a large-$n$ limit that is still allowed by kinematics, $n \lesssim \sqrt{s}/m$). Hence in a Higgsploding theory, the propagator,

$$\Delta_R(p) = \frac{i}{p^2 - m^2 - \text{Re} \Sigma_R(p^2) + im\Gamma(p^2) + i\epsilon}, \hspace{1cm} (2.15)$$

is effectively cut off at $p^2 \geq E^2_*$ by the exploding width $\Gamma(p^2)$ of the propagating state into the high-multiplicity final states. The incoming highly energetic state decays rapidly into the multi-particle state made out of soft quanta with momenta $k_i^2 \sim m^2 \ll E^2_*$. The width of the propagating degree of freedom becomes much greater than its mass: it is no longer a simple particle state. In this sense, it has become a composite state made out of the $n$ soft particle quanta of the same field $\phi$.

The main purpose of the summary above is to demonstrate that there are no apparent subtleties that arise when accounting for the UV-renormalisation effects in the expression
for the renormalised propagator (2.15). This expression is general and its validity is not restricted to, for example, the narrow width approximation.

Although we do not have much to say at present about the real part of the self-energy at $p^2 \gg m^2$ (in the regime of interest where $n \gg 1$), it is sufficient for our purposes to consider only the imaginary part, which is determined by the multi-particle decay rate $\Gamma_n(p^2)$. As soon as $\text{Im } \Sigma(p^2) = -m \Gamma_n(p^2)$ becomes exponentially large, which occurs at $E_*$, the propagators develop sharp exponential form-factors and vanish, thus providing a cut-off above $E_*$ for the integrals over loop momenta. Potential cancellations between the imaginary and real parts are impossible. Essentially, it is sufficient to have the Higgsplosion of the absolute value $|\Sigma(p^2)|^2 = (\text{Re } \Sigma(p^2))^2 + (\text{Im } \Sigma(p^2))^2$.

For a more detailed introduction to Higgsplosion and some of its applications we refer the reader to Refs. [1, 2, 14] and [11, 18] and references therein. In particular, we mention here that Higgsplosion solves the fine-tuning problem of the Higgs mass by removing quantum contributions to $m_0^2$ from all states with masses or momenta greater than the Higgsplosion scale $E_*$ as explained in [1]. We also note the observation of [2] that no UV divergences remain in the theory above the Higgsplosion scale $E_*$. The running couplings become flat above $E_*$ and thus flow to UV fixed points, rendering the theory with Higgsplosion asymptotically safe.

### 2.2 Continuation to the Euclidean space

In practice, all perturbative calculations in a theory, independently of whether or not it is in the Higgsplosion regime, are carried out in Euclidean space. In the absence of Higgsplosion, the use of the Euclidean signature $p^2 = p_0^2 + \vec{p}^2$ facilitates the UV regularisation of divergent integrals over loop momenta. The analytic continuation of the momentum variable is achieved with the standard Wick rotation, $p_0^E = ip_0$ with $\vec{p}_E = \vec{p}$, so that the propagator (2.12) becomes,

$$\Delta_R(p) = \frac{-i}{p^2 + m^2 + \Sigma_R(p^2)}. \quad (2.16)$$

In the coordinate representation, using the imaginary time,

$$\tau = it, \quad (2.17)$$

the Dyson propagator reads,

$$\Delta_R(x_1, x_2) = \langle 0|\phi(x_1)\phi(x_2)|0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2 + \Sigma_R(p^2)} e^{i p_0 \Delta \tau + i \vec{p} \Delta \vec{x}}. \quad (2.18)$$

When the theory enters the Higgsplosion regime [1], the self-energy undergoes a sharp exponential growth. This behaviour,

$$\Sigma_R(p^2) \sim \begin{cases} 0 & : \text{for } p^2 < E_*^2 \\ \infty & : \text{for } p^2 \geq E_*^2 \end{cases} \quad (2.19)$$

---

1Dimensional regularisation, cut-off regularisation and even Pauli-Villars regularisation all are defined and normally carried out in the Euclidean signature.
is captured by restricting the integration domain over the loop momenta. As a result, the integral over the 4-momentum in (2.16) becomes cut off by Σ outside the ball of radius $E_*$, 

$$
\Delta_R(x_1, x_2) = \int_{p^2 \leq E_*^2} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} e^{i\mu_0 \Delta \tau + i\vec{p} \Delta \vec{x}}.
$$

For non-vanishing $|\Delta x| > 0$ this integral is regular and can be straightforwardly evaluated. This leads to short-distance behaviour of the propagator of the form,

$$
\Delta_R(x, 0) \sim \begin{cases} 
1/|\Delta x|^2 & : \text{for } 1/E_* \ll |\Delta x| \ll 1/m \\
E_*^2 & : \text{for } |\Delta x| \lesssim 1/E_*
\end{cases}. \quad (2.21)
$$

This result is in agreement with Eq. (1.1). It is also clear that the absence of the singularity at $|x|^2 \to 0$ is a consequence of the dynamic integration cut-off for momenta $p^2 > E_*^2$ (i.e. above the Higgsplosion scale of the self-energy).

At the same time, in the opposite limit where the point splitting $\Delta x$ goes to zero, we recover from Eq. (2.20) the loop integral corresponding to the tadpole diagram,

$$
\Delta_R(x, x) = \int_{p^2 \leq E_*^2} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2}.
$$

It follows that the appearance of the Higgsplosion scale $E_*$ renders this loop integral finite. In the absence of Higgsplosion, the closed loop integral is quadratically divergent in the UV, as expected, and requires UV regularisation either by imposing a UV cut-off $\Lambda_{UV}$, or via dimensional or other type of UV regularisation. This is an example of the general result that Higgsploding theories are UV-finite [1, 2].

It should be noted that even in the limit of infinite $E_*$ or $\Lambda_{UV}$, the integral in Eq. (2.20) is finite for a non-vanishing end point separation. It is is regulated instead by the inverse separation $1/(\Delta t)^2$ or $1/(\Delta \vec{x})^2$. Of course, to obtain a closed loop, we will ultimately have to send this separation to zero.

### 2.3 Multi-particle decay width

In the preceding section we explained that the theory enters the Higgsplosion phase if the decay rate of a highly virtual / highly energetic single particle state into multi-particle final states becomes exponentially large. To compute this decay rate $\Gamma_n(s)$, we need to find the amplitudes for the $1^+ \to n$ processes at energies $\sqrt{s}$ and integrate them over the $n$-particle phase space, as in Eq. (2.14).

We are interested in keeping the number of particles $n$ in the final state as large as possible, that is, near the maximum number allowed by the phase space, $n \lesssim n_{\text{max}} = \sqrt{s}/M_h$. We can therefore take the bosons in the final state to be non-relativistic. Such
$n$-point amplitudes were studied in detail in scalar QFT in Refs. [5–8, 19], with the result,

\[
\text{Model (2.1)}: \quad \mathcal{A}_{1 \rightarrow n}(p_1 \ldots p_n) = n! \left( \frac{\lambda}{8m^2} \right)^{\frac{n-1}{2}} \exp \left[ -\frac{5}{6} n \varepsilon \right], \quad (2.23)
\]

\[
\text{Model (2.2)}: \quad \mathcal{A}_{1 \rightarrow n}(p_1 \ldots p_n) = n! \left( \frac{1}{2v} \right)^{n-1} \exp \left[ -\frac{7}{6} n \varepsilon \right], \quad (2.24)
\]

\[n \rightarrow \infty, \quad \varepsilon \rightarrow 0, \quad n\varepsilon = \text{fixed}.
\]

As indicated, these tree-level amplitudes are computed in the double-scaling limit with large multiplicities $n \gg 1$ and small non-relativistic energies of each individual particle, $\varepsilon \ll 1$, where,

\[
\varepsilon = \sqrt{\Delta - nm} \quad \frac{1}{nm} E_n^{\text{kin}} \simeq \frac{1}{n} \frac{1}{2m^2} \sum_{i=1}^{n} \vec{p}_i^2, \quad (2.25)
\]

so that the total kinetic energy per particle mass $n\varepsilon$ in the final state is fixed.

The pre-exponential factors on the right-hand side of Eqs. (2.23)-(2.24) correspond to the tree-level amplitudes (or more precisely, currents with one incoming off-shell leg) computed on the $n$-particle thresholds. In the elegant formalism of Brown [5], these amplitudes for all $n$ arise from the generating functional $\phi_0$, which is given by a spatially-uniform solution to the classical equations of motion\(^2\). These solutions are easily found by solving the classical equations of motion corresponding to models (2.1)-(2.2), and read,

\[
\text{Model (2.1)}: \quad \phi_0(t) = \frac{z(t)}{1 - \lambda z(t)^2/(8m^2)}, \quad \text{where} \quad z = z_0 e^{int}, \quad (2.26)
\]

\[
\text{Model (2.2)}: \quad \phi_0(t) = v \left( \frac{1 + z(t)/(2v)}{1 + z(t)/(2v)} \right), \quad \text{where} \quad z = z_0 e^{imn t}, \quad (2.27)
\]

Then the amplitudes on $n$-particle mass thresholds are given by differentiating these classical generating functions $n$ times with respect to the source variable $z$,

\[
\mathcal{A}_{\text{tree} \ 1 \rightarrow n}^{\text{thr}} := \langle n | \phi(0) \rangle_{\text{tree}} = \left( \frac{\partial}{\partial z} \right)^n \phi_0(z) |_{z=0} = \left\{ \begin{array}{ll}
\left( \frac{\lambda}{8m^2} \right)^{\frac{n-1}{2}} n! \left( \frac{1}{2m^2} \right)^{n-1} & \varepsilon = \text{fixed}
\end{array} \right. \quad (2.28)
\]

These are exact expressions for tree-level amplitudes valid for any value of $n$ [5].

The amplitudes in Eqs. (2.23)-(2.24) go beyond the threshold results (2.28) by accounting for the kinematic dependence in the non-relativistic limit of the external momenta. These were derived as a non-relativistic deformation of the Brown’s generating functions method in Refs. [8] and [19] for the unbroken and the broken theories respectively. The amplitudes now contain an exponential form-factor that depends on the kinetic energy of the final state $n\varepsilon$. But, importantly, the factorial growth $\sim \lambda^{n/2} n!$ characteristic to the multi-particle amplitude on mass threshold remains. Its occurrence can be traced back to

\[2\text{Tree level is equivalent to the leading-order expansion in } \hbar, \text{ thus reducing the quantum problem to classical solutions. Furthermore, the vanishing external 3-momenta of the on-the-threshold amplitudes selects spatially-uniform time-dependent solutions [5].}\]
the factorially growing number of Feynman diagrams at large $n$ and the lack of destructive interference between the diagrams in the scalar theory. In Section 3 we will compute the leading-order quantum corrections to these tree-level amplitudes, in the case where a non-trivial finite Higgsplison scale $E_*$ is present.

To obtain the self-energy, $\text{Im} \Sigma_R(p^2) = -m \Gamma(p^2)$, we integrate the amplitudes\(^3\) \((2.23)-(2.24)\) over the $n$-particle phase-space for large $n$,

$$\Gamma_n(s) = \frac{1}{2m} \int \frac{d\Phi_n}{n!} |M_{1 \to n}|^2.$$ \hfill (2.29)$$

For the unbroken theory \((2.1)\), one finds \([12, 13]\),

$$\Gamma_n(s) \propto R(\lambda; n, \varepsilon) = \exp \left[ n \left( \log \frac{\lambda n}{16} + \frac{3}{2} \log \frac{\varepsilon}{3\pi} + \frac{1}{2} - \frac{17}{12} \varepsilon + Q(\lambda n, \varepsilon) \right) \right],$$ \hfill (2.30)$$

and for the model with SSB \((2.2)\) \([19]\),

$$\Gamma_n(s) \propto R(\lambda; n, \varepsilon) = \exp \left[ n \left( \log \frac{\lambda n}{4} + \frac{3}{2} \log \frac{\varepsilon}{3\pi} + \frac{1}{2} - \frac{25}{12} \varepsilon + Q(\lambda n, \varepsilon) \right) \right].$$ \hfill (2.31)$$

In particular, note that the ubiquitous factorial growth of the large-$n$ amplitudes translates into the $\frac{1}{n!}|M_n|^2 \sim n! \lambda^n \sim e^{n \log(\lambda n)}$ factor in the rates $R$ above. The term $Q(\lambda n, \varepsilon)$ on the right-hand side of both equations above indicates quantum corrections: at tree level $Q(\lambda n, \varepsilon) \equiv 0$.

These rates can also be computed using an alternative semi-classical method formulated by Son in Ref. \([12]\). This is an intrinsically non-perturbative approach, with no reference in its outset made to perturbation theory. The path integrals contributing to the amplitudes and the rates are computed in the steepest descent method, controlled by two large parameters, $1/\lambda \to \infty$ and $n \to \infty$. More precisely, the limit is,

$$\lambda \to 0, \quad n \to \infty, \quad \text{with \ \ \lambda n = \text{fixed}, \ \ \varepsilon = \text{fixed}.}$$ \hfill (2.32)$$

The semi-classical computation carried out in \([12]\), in the regime where,

$$\lambda n = \text{fixed} \ll 1, \quad \varepsilon = \text{fixed} \ll 1,$$ \hfill (2.33)$$

provided an alternative derivation of the tree-level perturbative results for non-relativistic final states \((2.30)-(2.31)\), with the expressions being correctly reproduced. Remarkably, this semi-classical calculation also reproduces the leading-order quantum corrections arising from resumming one-loop effects. These were previously computed perturbatively in the $\varepsilon \to 0$ limit in \([8–10]\),

$$Q_{1\text{-loop}}(\lambda n) = \begin{cases} -\lambda n \text{ Re } F/8 & : \text{Model (2.1)} \\ + \lambda n \text{ } 2B & : \text{Model (2.2)} \end{cases}.$$ \hfill (2.34)$$

\(^3\)Note that the conventionally-normalised amplitudes $M_{1 \to n}$ are obtained from the currents $A_{1 \to n}$ in \((2.23)-(2.24)\) by the LSZ amputation of the single off-shell incoming line, $M_{1 \to n} := (s - M_0^2) \cdot A_{1 \to n}(p_1 \ldots p_n)$. 

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where the constant coefficients $F$ and $B$ will be given in Eqs. (3.30) and (3.49) below. This agreement constitutes an important test of the applicability of the semiclassical method of Son for computations of quantum corrections. In Section 3 we will verify that this agreement persists when the perturbative loop integrals are carried out in the presence of the Higgsplasion scale $E_\ast$.

The semiclassical approach is equally applicable and more relevant to the realisation of the non-perturbative Higgsplasion case where,

$$\lambda n = \text{fixed} \gg 1, \quad \varepsilon = \text{fixed} \ll 1.$$  \hspace{1cm} (2.35)

This calculation was carried out in Ref. [14] for the spontaneously broken theory (2.2), with the result given by,

$$R_n(\lambda; n, \varepsilon) = \exp \left[ \frac{\lambda n}{\lambda} \left( \log \frac{\lambda n}{4} + 0.85 \sqrt{\lambda n} + \frac{1}{2} + \frac{3}{2} \log \frac{\varepsilon}{3\pi} - \frac{25}{12} \varepsilon \right) \right],$$  \hspace{1cm} (2.36)

which is equivalent to Eq. (2.31) with the substitution,

$$Q_{\lambda n \gg 1}(\lambda n) = + 0.85 \sqrt{\lambda n}.$$  \hspace{1cm} (2.37)

Higgsplasion becomes operative when the decay width, or equivalently the rate (2.36), becomes exponentially large. Let us estimate the energy $E_\ast$ where this occurs. The expression (2.36) was derived in the near-threshold limit (2.35), where the parameter $\varepsilon$ is treated as a fixed constant much smaller than one. The initial state energy and the final state multiplicity are related linearly via $E/m_h = (1 + \varepsilon) n$. Hence, for any fixed $\varepsilon$, one can raise the energy to obtain an arbitrarily large $n$, and consequently, a large $\sqrt{\lambda n}$. The negative $\log \varepsilon$ term and the positive $\sqrt{\lambda n}$ term in (2.36) are in competition. Nonetheless, there is always a range of sufficiently high multiplicities, where $\sqrt{\lambda n}$ overtakes the logarithmic term $\log \varepsilon$ for any fixed (however small) value of $\varepsilon$. This leads to the exponentially growing multi-particle rates. It was shown in Ref. [1] that in the model where the multi-particle rates are given by the expression (2.36), and fixing $\lambda = 1/8$, the rates start growing exponentially when $E_\ast \sim 200 m_h$.

In general, the Higgsplasion scale $E_\ast$ is defined as the new dynamically generated scale of the weakly coupled theory, $\lambda \ll 1$, where a sharp transition occurs between the negligibly small multi-particle rates at $E < E_\ast$, and the exponentially large rates at $E > E_\ast$. From general principles, it is clear that this scale would have the parametric dependence of the form,

$$E_\ast = \frac{m_h}{f(\lambda)}, \quad \text{where} \quad f(\lambda)|_{\lambda \to 0} \to 0.$$  \hspace{1cm} (2.38)

The mass $m_h$ is the only dimensionful parameter of the theory and the fact that $E_\ast$ goes to infinity in the free-theory limit $\lambda \to 0$ is also obvious since in this limit no multi-boson production is possible. Hence, the singularity at $\lambda = 0$ implies that the Higgsplasion scale is non-perturbative.

To sharpen this argument for the emergence of $E_\ast$ (as the new dynamical scale in a weakly coupled theory with microscopic massive scalars), we would like to get a better
handle on \( f(\lambda) \) in the context of the model with spontaneously broken symmetry (2.2). The rate, in this case, is given in Eq. (2.36). It will be useful to introduce the more natural rescaled variables for the multi-particle rate \( \mathcal{R} \),

\[
\tilde{n} = \lambda n, \quad \tilde{E} = \frac{\lambda E}{m_h},
\]

so that

\[
\tilde{E} = (1 + \varepsilon) \tilde{n}.
\]

Following closely the discussion in Section 5 of Ref. [14], we now re-write the expression for the rate in the following form,

\[
\mathcal{R}_n(E) \sim \int_0^{\varepsilon_{nr}} d\varepsilon \left( \frac{\varepsilon}{3\pi} \right)^\frac{3}{2} \exp \left[ \frac{\tilde{n}}{\lambda} \left( 0.85 \sqrt{\tilde{n}} + \log \tilde{n} - a \varepsilon \tilde{n}^p - \text{const} \right) \right].
\]

The integral \( \int_0^{\varepsilon_{nr}} d\varepsilon \left( \frac{\varepsilon}{3\pi} \right)^\frac{3}{2} \) appearing on the right-hand side is nothing but the non-relativistic phase-space integral \( \int d\Phi_n \) written in the large-\( n \) limit. The upper limit of the integration, \( \varepsilon_{nr} \), is the non-relativistic kinetic energy per particle per mass, \( \varepsilon_{nr} = E/(nm_h) \ll 1 \) while \( \varepsilon \) is now treated on the right-hand side of (2.41) as the integration variable.

The second point to note is that we introduced an additional term, \(- a \varepsilon \tilde{n}^p\), in the exponent on the right-hand side of (2.41) which represents the sub-leading correction in \( \varepsilon \ll 1 \) to the expression in (2.37). In other words, we have assumed that,

\[
Q_{n\gg 1}(\tilde{n}, \varepsilon) = + 0.85 \sqrt{\tilde{n}} - a \varepsilon \tilde{n}^p + \text{higher orders in } \varepsilon,
\]

where \( a \) and \( p \) are some positive constants. The integral we have to compute is,

\[
\mathcal{R}_n \sim e^{\frac{\tilde{n}}{\lambda} \left( 0.85 \sqrt{\tilde{n}} + \log \tilde{n} - \left( \frac{\lambda}{\tilde{n}^p} \right) \right)} \int d\varepsilon \ e^{\frac{\tilde{n}}{\lambda} \left( \frac{3}{2} \log \varepsilon - a \varepsilon \tilde{n}^p \right)},
\]

where

\[
\tilde{c} = \log 4 + \frac{3}{2} \log 3\pi - \frac{1}{2}.
\]

The saddle-point dominating the integral over \( \varepsilon \) is,

\[
\varepsilon_* = \frac{3}{2a} \frac{1}{\tilde{n}^p},
\]

and the value of the original integral for \( \mathcal{R}_n(\varepsilon) \) at the saddle-point is,

\[
\mathcal{R}_n(\varepsilon_*) \sim \exp \left[ \frac{\tilde{n}}{\lambda} \left( 0.85 \sqrt{\tilde{n}} + \left( 1 - \frac{3p}{2} \right) \log \tilde{n} - \text{const}' \right) \right].
\]

The constant term \( \text{const}' \) is,

\[
\text{const}' = \tilde{c} + \frac{3}{2} \left( 1 - \log \frac{3}{2a} \right) = 1 + \log 4 + \frac{3}{2} \log 2a\pi.
\]

Setting \( a = 1 \), we have \( \text{const}' \simeq 5.14 \).

\[\text{– 10 –}\]
Now, assuming that the coefficient in front of the logarithmic term in \( \varepsilon \) in (2.46) is positive, i.e. \( 0 < p < 2/3 \), the exponent is negative at small \( \tilde{n} \), positive at large \( \tilde{n} \) and crosses zero at some value \( \tilde{n}_* \). For example, for \( p = 1/2 \) and \( a = 1 \), which corresponds to the NLO correction \( -a \varepsilon \tilde{n}^p = -\varepsilon \sqrt{\tilde{n}} \), the value of \( \tilde{n}_* \approx 5.55 \).

In the alternative scenario, where the coefficient in front of the logarithm is negative, for example at \( p = 1 \), the function in the exponent of (2.46) has a more complicated behaviour with a local minimum at intermediate values of \( \tilde{n} \). Nevertheless at larger \( \tilde{n} \), the function is again monotonic and crosses over from negative to positive values at \( \tilde{n}_* \approx 7.2 \).

It then follows that the value of \( \tilde{E}_* = (1 + \varepsilon_*) \tilde{n}_* \approx \tilde{n}_* := C = \text{const} \). As a result, we can write the Higgsplision scale \( E_* \) as,

\[
E_* = C \frac{m_h}{\lambda}.
\]

(2.48)

It is also easy to verify that this conclusion is consistent within the validity of the non-relativistic limit.

The parametric dependence of the Higgsplision energy \( E_* \) on the particle mass and the inverse coupling constant is reminiscent of another famous dynamically induced scale in the electroweak theory – the mass of the sphaleron solution [3, 4], \( M_{\text{sph}} = \text{const} \frac{m_W}{\alpha_w} \).

Both scales are non-perturbative and semi-classical in nature. They do not appear in the Lagrangian of the theory, but rather characterise the energy scale where the transition to novel dynamics involving multi-particle states occurs.\(^4\) The sphaleron, however, does not occur in the pure scalar sector of the theory and requires the \( SU(2) \) gauge theory in the Higgs phase.

3 Systematics of loops with Higgsplision

3.1 Computing a loop with the propagator in the classical background

We will start by considering the scalar field theory (2.1) with unbroken \( Z_2 \) symmetry, and postpone the discussion of the broken theory (2.2) to Section 3.3. As we have already explained, the generating functional for all tree-level amplitudes on \( n \)-particle thresholds is given in this model by the classical solution (2.26).

The aim of this section is to compute the leading-order quantum corrections to these amplitudes in the case where a non-trivial finite Higgsplision scale \( E_* \) is present. The leading order calculation (in the absence of Higgsplision) was performed in [9], extended to the spontaneously broken theory in [10], and generalised in [8] to include all higher-loop effects by exponentiation to the leading order in \( \lambda n \).

We begin by following closely the original leading-loop calculation of Voloshin in [9], and then explain how it should be modified to reflect the appearance of the Higgsplision

\(^4\)In the case of sphalerons, the new dynamics is that of the non-perturbative \((B+L)\)-violating transitions between multi-particle initial and final states.
scale $E_*$. This will allow us to assess the effect of Higgspllosion on the RG running of the parameters of the theory, including their asymptotic safety. We will also see that the so-called finite terms arising from the quantum effect are the same as those computed in [8–10] up to corrections of the order $\mathcal{O}(m^2/E_*^2)$. These considerations will pave the way for computing precision observables in Higgspllosion in Section 4.

The quantum corrections to the tree-level amplitudes (2.28) are obtained by expanding around the classical field, $\phi(x) = \phi_0(x) + \phi_q(x)$, so that the Euclidean Lagrangian (2.1) for the quantum fluctuation $\phi_q$ becomes,

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_q)^2 + \frac{1}{2} \left( m^2 + 3\lambda \phi_0^2 \right) \phi_q^2 + \lambda \phi_0 \phi_q^3 + \frac{\lambda}{4} \phi_q^4 .$$

(3.1)

One then integrates out $\phi_q(x)$ using the background field perturbation theory.

It follows that the generating functional of the amplitudes in the full quantum theory is obtained by promoting the classical solution $\phi_0$ into the quantum expectation value $\langle \phi \rangle = \phi_0 + \langle \phi_q \rangle$. Individual amplitudes are then computed via

$$\langle n | \phi | 0 \rangle = \left( \frac{\partial}{\partial z_0} \right)^n \langle \phi_0 + \langle \phi_q \rangle | z_0 = 0 \rangle .$$

(3.2)

This provides the generalisation to full quantum theory [8, 9] of the tree-level formalism of Brown [5] for computing $1^* \to n$ amplitudes on $n$-particle mass-thresholds.

The matrix element $\langle \phi_q \rangle$ is computed using the Feynman rules following from the action (3.1). It is easy to see that the one-loop contribution to $\langle \phi_q \rangle$ comes from the tadpole diagram, which contains the three-point vertex from (3.1) with two attached propagators – one external, $G(y,x)$, and one forming the loop, $G(x,x)$,

$$\langle \phi_q(y) \rangle_{1\text{-loop}} = (-3\lambda) \int d^4 x G(y,x) \phi_0(x) G(x,x) ,$$

(3.3)

$G(x_1, x_2)$ is the propagator for the scalar field $\phi_q$ in the background of the classical solution,\footnote{To distinguish the propagator in the background of $\phi_0(t)$ from the propagator in the trivial background, we call it $G$ rather than $\Delta$. We also continue working in Euclidean space and thus drop the $T$-ordering in the propagator.}

$$G(x_1, x_2) = \langle 0 | \phi_q(x_1) \phi_q(x_2) | 0 \rangle ,$$

(3.4)

which satisfies the equation,

$$\left( -(\partial^2_{x_1} + m^2 + 3\lambda \phi_0(x_1)^2) \right) G(x_1, x_2) = \delta^{(4)}(x_1 - x_2) .$$

(3.5)

The leading-order quantum correction $\langle \phi_q \rangle_{1\text{-loop}}$ obtained via (3.3) is the solution of the differential equation,

$$\left( -(\partial^2_{x} + m^2 + 3\lambda \phi_0(x)^2) \right) \langle \phi_q(x) \rangle_{1\text{-loop}} = -3\lambda \phi_0(x) G(x,x) .$$

(3.6)
This equation is derived by acting with the differential operator appearing on the left, on both sides of (3.3) and using the definition of the propagator $G$ in (3.5).

It will also be useful for our purposes to write down the equation for the quantum corrected generating function $\phi = \phi_0 + \langle \phi_q \rangle_{\text{1-loop}}$. Using the fact that $\phi_0$ satisfies the classical equation, $-\partial^2 \phi_0 + m^2 \phi_0 + \lambda \phi_3^0 = 0$, and that the quantum correction satisfies Eq. (3.6), it follows from combining the two, that the full generating function $\phi(x)$ is the solution to

$$- \partial^2 \phi(x) + m^2 \phi(x) + \lambda \phi(x)^3 + 3 \lambda \phi_0(x) G(x,x) = 0.$$  \hspace{1cm} (3.7)

We will ultimately need to compute $G(x,x)$ at coincident points, but for now we consider the general case of $x_1 \neq x_2$. It is convenient to use the mixed coordinate-momentum representation $G_\omega(\tau_1,\tau_2)$ for the propagator, i.e. perform the 3D Fourier transform to the 3-momentum, but keep the Euclidean time coordinate $\tau$,

$$G(x_1,x_2) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}(\vec{x}_1-\vec{x}_2)} G_\omega(\tau_1,\tau_2).$$  \hspace{1cm} (3.8)

The partial differential equation (3.5) then becomes an ordinary second order differential equation,

$$\left( -\frac{d^2}{d\tau_1^2} + \omega^2 + 3 \lambda \phi_0(\tau_1)^2 \right) G_\omega(\tau_1,\tau_2) = \delta(\tau_1 - \tau_2),$$  \hspace{1cm} (3.9)

where once again,

$$\omega := \omega_p = \sqrt{p^2 + m^2}.$$  \hspace{1cm} (3.10)

The classical solution (2.26) entering the equation (3.9) should be Wick rotated to the Euclidean time $\tau$. To simplify the resulting expressions, we also introduce a constant shift as in [9],

$$\tau = it + \frac{1}{m} \log \frac{\lambda \zeta_0}{8m^2} - \frac{i \pi}{2m},$$  \hspace{1cm} (3.11)

which gives,

$$\phi_0(\tau) = i \sqrt{\frac{2}{\lambda}} \frac{1}{\cosh(\tau)},$$  \hspace{1cm} (3.12)

Thus, to determine the propagator in the classical background (3.12) one should solve the inhomogeneous second-order linear ODE,

$$\left( -\frac{d^2}{d\tau_1^2} + \omega^2 - \frac{6}{\cosh^2(\tau_1)} \right) G_\omega(\tau_1,\tau_2) = \delta(\tau_1 - \tau_2).$$  \hspace{1cm} (3.13)

This is a textbook problem. The required solution of the inhomogeneous equation is obtained from the two solutions $f_1(\tau)$ and $f_2(\tau)$ to the corresponding homogeneous equation. The result is,

$$G_\omega(\tau_1,\tau_2) = \frac{1}{W}[\Theta(\tau_1 - \tau_2)f_1(\tau_1)f_2(\tau_2) + \Theta(\tau_2 - \tau_1)f_2(\tau_1)f_1(\tau_2)],$$  \hspace{1cm} (3.14)
where \( \Theta \)'s are the step functions and \( W \) is the Wronskian computed for the two homogeneous solutions,
\[
W = \det \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix}.
\]
(3.15)
The solutions of the homogeneous equation
\[
\left( -\left( \frac{d}{d\tau} \right)^2 + \omega^2 - \frac{6}{\cosh^2(\tau)} \right) f_{1,2}(\tau) = 0.
\]
(3.16)
are known [9] thanks to the identification of equation (3.16) as the Scrödinger equation with an exactly solvable potential. The first solution (which is regular at \( \tau \to +\infty \)) can be written in the form,
\[
f_1(\tau) = \left( \omega^2 + 3\omega \frac{(e^{2\tau} - 1)}{e^{2\tau} + 1} + 2 - \frac{12e^{2\tau}}{(e^{2\tau} + 1)^2} \right) e^{-\omega\tau}, \quad \text{where } u = e^\tau,
\]
(3.17)
where here and below we have temporarily set \( m = 1 \) to reduce the clutter. The second solution (which is regular at \( \tau \to -\infty \)) is obtained from \( f_1(\tau) \) by the reflection of its argument, \( f_2(\tau) = f_1(-\tau) \). These expressions can also be checked by a direct substitution into (3.16). Finally, the Wronskian computed for these two solutions is given by [9],
\[
W = 2\omega(\omega^2 - 1)(\omega^2 - 4),
\]
(3.18)
and is independent of the \( \tau \) variable (as indeed should be the case for the second order linear ODE). Once again we remind the reader that the mass in the expressions above was rescaled to \( m = 1 \) (so that e.g. \( \omega = \sqrt{p^2 + 1} \) and is dimensionless), and that it can be easily recovered, when necessary, by dimensional analysis.

To find the expression for the coincident propagator \( G(x,x) \) in (3.3), we must first evaluate the integral in (3.8) for \( G(x + \Delta x, x) \), and then take the limit as \( \Delta x \to 0 \) to close the loop. However, we have to be careful with the order of limits. In fact, the momentum space integration produces a UV-finite result only at non-zero separation between the propagator end points \( x_1 \) and \( x_2 \). When \( \Delta x \) does go to zero, the integral \( \int d^3p \) contains quadratically and logarithmically divergent terms and needs to be regulated. The regulated divergences are then absorbed into the renormalised parameters of the theory – the mass squared term and the coupling constant – after which the UV regulator can be removed. This was the procedure adopted in [9].

Since our main aim is to account for the effects of the Higgsplison scale \( E_* \), which renders the loop integrals over Euclidean 4-momenta finite, as in (2.22), we must now deviate from [9]. Our approach will be to keep a non-vanishing separation between the space-time points, so that our integrals remain finite, before re-writing them as the integrals over \( d^4p \). For these integrals we can implement the Higgsplison cut-off \( p^2 \leq E_*^2 \) directly, as in (2.20), and then safely take limit \( \Delta x \to 0 \) as in (2.22).

To follow the approach outlined above, we expand the integrand in (3.8) in powers of \( 1/\omega \), to isolate the most sensitive terms in the UV, as \( p \sim \omega \to \infty \). With no loss of
generality we can present (3.8) as follows:

\[
G(\tau + \Delta \tau, \tau) = \int \frac{d^3p}{(2\pi)^3} e^{-\omega \Delta \tau} \frac{1}{2\omega} \left( 1 + \frac{1}{\omega} A + \frac{1}{\omega^2} B + O\left(\frac{1}{\omega^3}\right) \right). \tag{3.19}
\]

Here we have set the non-zero separation between the space-time points to be along the
time direction and kept \(\Delta \tau\) positive. The factors \(A\) and \(B\) appearing on the right-hand side of (3.19) are \(\omega\)-independent functions of \(\tau\) and \(\Delta \tau\). Using the formulae for \(f_1(\tau)\), \(f_2(\tau)\) and the Wronskian we find,

\[
A = \frac{12 e^{2\tau}}{(e^{2\tau} + 1)^2} \Delta \tau + O(\Delta \tau^2) = -\frac{3\lambda}{2} \phi_0(\tau)^2 \Delta \tau + O(\Delta \tau^2), \tag{3.20}
\]

\[
B = \frac{12 e^{2\tau}}{(e^{2\tau} + 1)^2} + O(\Delta \tau) = -\frac{3\lambda}{2} \phi_0(\tau)^2 + O(\Delta \tau). \tag{3.21}
\]

To obtain the simple form for \(A\) and \(B\) used above, we anticipate the ultimate \(\Delta \tau \to 0\)
limit and Taylor-expand the expressions in \(A\) and \(B\) to the first non-vanishing order in \(\Delta \tau\).
Note however, that while we can treat \(\Delta \tau\) as a small parameter, the combination \(w \Delta \tau\) is not small at high values of momenta where the exponential factor \(e^{-\omega \Delta \tau}\) regulates the
integrand in (3.19). This factor is therefore left unchanged.

After integrating over \(d^3p\), the first three terms in the brackets on the right-hand side of
(3.19) give rise respectively to the quadratic, linear, and logarithmic dependence on either
the \(1/\Delta \tau\) parameter, or on the UV cut-off momentum scale \(\Lambda_{UV}\), depending on whether
\(\Lambda_{UV} \Delta \tau\) is \(\gg 1\) or \(\ll 1\). The fourth and final term in (3.19) goes as \(\int dw/w^2\) and is regular
in the UV at \(\Delta \tau = 0\). It will be useful to represent (3.19) as follows,

\[
G(\tau + \Delta \tau, \tau) = G_I + G_{II} + G_{III} + G_{finite}. \tag{3.22}
\]

We can now evaluate the integrals \(G_I, G_{II}, G_{III}\) and \(G_{finite}\). For the first integral, we have,

\[
G_I(\tau + \Delta \tau, \tau) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega} e^{-\omega \Delta \tau} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} e^{ip_0 \Delta \tau}, \tag{3.23}
\]

where we have used the familiar relation between the 4D and the 3D integrals for the free
propagator,

\[
\Delta_0(\tau + \Delta \tau, \tau) := \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} e^{ip_0 \Delta \tau} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega} e^{-\omega \Delta \tau}. \tag{3.24}
\]

which is based on the use of the residue theorem.

For non-vanishing \(\Delta \tau\) the integral is finite and does not require any UV cut-off. The
expression (3.23) was derived in a theory without Higgsplosion, or in the limit of infinite
Higgsplosion energy scale \(E_s\). But since the expression on the right is the ordinary free
propagator in a trivial background, we know how to modify this expression to account for
the Higgsploding self-energy. Higgsplosion is introduced exactly as in (2.20), and now we
can take the limit \(\Delta \tau \to 0\) and find the contribution of the first term in (3.19) to the closed
loop. It is given by,

\[
G_I(x, x) = \int_{p^2 \leq E_s^2} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} = \frac{1}{16\pi^2} \left( E_s^2 - m^2 \log \frac{E_s^2 + m^2}{m^2} \right). \tag{3.25}
\]
To evaluate \(G_{II}\) and \(G_{III}\) we note the useful identity obtained by differentiating \(G_I(\tau + \Delta \tau, \tau)\) on the first line in (3.23) with respect to \(m^2\),

\[
\frac{\partial}{\partial m^2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega} e^{-\omega \Delta \tau} = -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left( \frac{1}{2\omega^2} + \frac{\Delta \tau}{2\omega^3} \right) e^{-\omega \Delta \tau},
\]

(3.26)

and recognising the expression on the right-hand side as precisely the sum of the integrals appearing in \(G_{II}\) and \(G_{III}\). Since \(B = 12 e^{2\tau}/(e^{2\tau} + 1)^2\) and \(A = B \Delta \tau\), we find that

\[
G_{II} + G_{III} = -\frac{24 e^{2\tau}}{(e^{2\tau} + 1)^2} \frac{\partial}{\partial m^2} G_1.
\]

(3.27)

Differentiating our result for \(G_1\) on the right-hand side of (3.25), we find the closed-form expression for the sum of \(G_{II}\) and \(G_{III}\) at \(\Delta \tau = 0\),

\[
G_{II}(x,x) + G_{III}(x,x) = \frac{1}{2\pi^2} \frac{3 e^{2\tau}}{(e^{2\tau} + 1)^2} \left( \log \frac{E_2^2 + m^2}{m^2} - \frac{E_2^2}{E_2^2 + m^2} \right).
\]

(3.28)

The final term contributing to the propagator in (3.22), \(G_{finite}\), contains no UV-sensitive contributions. It can be computed directly at \(\Delta x = 0\) and one can set \(E_2 \rightarrow \infty\) without the need of introducing any UV cut-off. In this case we find the same expression as in [9],

\[
G_{finite}(x,x) = \frac{1}{2\pi^2} \frac{6 e^{2\tau}}{(e^{2\tau} + 1)^2} - \frac{6 e^{4\tau}}{(e^{2\tau} + 1)^4} F,
\]

(3.29)

where

\[
F = \frac{\sqrt{3}}{2\pi^2} \left( \log \frac{2 + \sqrt{3}}{2 - \sqrt{3}} - i\pi \right).
\]

(3.30)

Collecting the expressions for the individual contributions in equations (3.25), (3.28)-(3.29) we arrive at the final result for the propagator loop in the background field of the unbroken scalar theory with the Higgsplasion energy scale \(E_*\). It can be slightly simplified in the limit \(m^2/E_2^2 \ll 1\), where we get,

\[
G(x,x) = \frac{1}{16\pi^2} \left( E_*^2 - m^2 \log \frac{E_*^2}{m^2} \right) + \frac{1}{2\pi^2} \frac{3 e^{2\tau}}{(e^{2\tau} + 1)^2} \left( \log \frac{E_*^2}{m^2} + 1 \right) - \frac{6 e^{4\tau}}{(e^{2\tau} + 1)^4} F,
\]

(3.31)

or expressing everything in terms of the Brown’s solution \(\phi_0\), we can write an alternative equivalent representation,

\[
G(x,x) = \frac{1}{16\pi^2} \left( E_*^2 - m^2 \log \frac{E_*^2}{m^2} \right) - \frac{1}{2\pi^2} \frac{3 \lambda \phi_0^2}{8} \left( \log \frac{E_*^2}{m^2} + 1 \right) - \frac{3 \lambda^2 \phi_0^4}{32} F.
\]

(3.32)

Recalling the defining equation (3.7) for the quantum-corrected generating functional \(\phi(x)\), makes it obvious that there is a naturally occurring combination,

\[
m^2 \phi_0(x) + \lambda \phi_0(x)^3 + 3\lambda \phi_0(x) G(x,x).
\]

(3.33)
It makes sense to re-absorb the first two terms on the right-hand side of (3.32) into the definitions of the renormalised coupling parameters of the theory in the following way,

\[ \bar{\lambda} = \lambda - \frac{9 \lambda^2}{8\pi^2} \left( \log \frac{E}{m} + \frac{1}{2} \right), \]

(3.34)

\[ \bar{m}^2 = m^2 + \frac{3 \lambda}{16\pi^2} \left( E^2 - m^2 \log \frac{E^2}{m^2} \right), \]

(3.35)

where the subscript \( \ast \) indicates that these renormalised parameters correspond to the theory with the Higgsplosion scale \( E \). Equations (3.34)-(3.35) define a finite renormalisation of the parameters, since \( E \) is a fixed finite non-perturbative scale of the theory; it is neither the RG scale nor the UV cut-off scale. Such finite renormalisation of the model parameters is what we would expect in the theory with Higgsplosion: such a theory does not contain UV divergences and flows to an asymptotically safe theory, where all couplings approach constant values in the UV, as was first explained in [2]. Our new results, Eqs. (3.34)-(3.35), provide a precise definition of these renormalised parameters in the UV fixed point. In Section 3.2 we will interpret these equations in terms of the running parameters and show they have the standard running with the correct slopes at low values of the RG scale, \( \mu < E \), and flatten out and approach fixed points above the Higgsplosion scale \( \mu > E \).

Finally, we are now in a position to compute the leading-order quantum correction \( \langle \phi_q(x) \rangle_{1\text{-loop}} \) to the generating function of \( n \)-point amplitudes on multi-particle mass thresholds. The defining equation for this quantity is Eq. (3.6), and since we have already computed the propagator loop \( G(x, x) \), all the quantities in this equation are known. In terms of the renormalised parameters, (3.34)-(3.35), equation (3.6) (with \( \bar{m} \) set to 1) reads, [9],

\[ \left( -\frac{d^2}{d\tau^2} + 1 - \frac{24 e^{2\tau}}{(e^{2\tau} + 1)^2} \right) \langle \phi_q(\tau) \rangle_{1\text{-loop}} = i 18\lambda \sqrt{\lambda} \frac{e^{5\tau}}{(e^{2\tau} + 1)^5}, \]

(3.36)

The solution is given by the following expression [9],

\[ \langle \phi_q(\tau) \rangle_{1\text{-loop}} = -i \frac{3\lambda}{4} \sqrt{\lambda} \frac{e^{5\tau}}{(e^{2\tau} + 1)^3}, \]

(3.37)

which can be checked by substitution. Adding this quantum correction to the tree-level contribution \( \phi_0(\tau) \), we derive the on-threshold amplitudes, cf. Eq. (2.28),

\[ \langle n|\phi|0 \rangle = \left( \frac{\partial}{\partial z_0} \right)^n \phi|_{z_0=0} = n! \left( \frac{\bar{\lambda}}{8\bar{m}^2} \right)^{(n-1)/2} \left( 1 - \bar{\lambda}(n-1)(n-3) \frac{F}{16} \right). \]

(3.38)

This is the same expression as that derived in [9]. Where did the dependence on the Higgsplosion scale \( E \) go? First, it enters the definition of the renormalised coupling and mass parameters given by Eqs. (3.34)-(3.35). Secondly, there exists an additional effect, which is the correction to the finite part \( F \) itself. Specifically, \( F \) in (3.38) is given by the expression in (3.30) plus the corrections of the order \( \sim m^2/E^2 \) which we have omitted in deriving the expression in (3.30). Section 4 is specifically dedicated to computations of finite corrections from the Higgsplosion scale to precision observables.
3.2 Running couplings

In the absence of Higgsplasion, i.e. in the limit where the Higgsplasion scale is above some regulating UV cut-off, $\Lambda_{UV} < E_s \to \infty$, the divergent terms are absorbed into the renormalised parameters as follows,

$$\bar{\lambda} = \lambda - \frac{9\lambda^2}{8\pi^2} \left( \log \frac{\Lambda_{UV}}{m} + \frac{1}{2} \right),$$

$$\bar{m}^2 = m^2 + \frac{3\lambda}{16\pi^2} \left( \Lambda_{UV}^2 - m^2 \log \frac{\Lambda_{UV}^2}{m^2} \right).$$

These equations are fully analogous to Eqs. (3.34)-(3.35) computed in the presence of Higgsplasion, except the role of $E_s$ is now played by the UV cut-off $\Lambda_{UV}$, which is interpreted as the cut-off on the 4-momentum integration $\int d^4p$.

These expressions for the renormalised couplings have a familiar interpretation in terms of the running couplings at the RG scale $\mu$. In the Wilsonian approach to renormalisation, one begins with the theory defined at some UV cut-off scale $\Lambda_{UV}$, where the bare parameters of the theory are specified. The UV cut-off is then lowered from $\Lambda_{UV}$ to $\mu$ by integrating out all degrees of freedom in the momentum shell $\mu^2 < p^2 < \Lambda_{UV}^2$. The renormalised couplings $\lambda$ and $m^2$ subsequently evolve to new values, specific to this RG scale $\mu$. Hence in order to infer the running couplings $\lambda(\mu)$ and $m^2(\mu)$ from the one-loop computation described above, one has to restrict the integration over the loop momenta to the interval $[\mu, \Lambda_{UV}]$. Adapting the expression for $\bar{\lambda}$ accordingly, ignoring the constant terms and assuming $\mu \gg m$, we find,

$$1/\lambda(\mu) = 1/\lambda(\Lambda_{UV}) + \frac{9}{8\pi^2} \int_\mu^{\Lambda_{UV}} \frac{dp}{p}.$$
This gives the well-established one-loop solution for the running coupling in $(\lambda/4)\phi^4$ theory

$$\lambda(\mu) = \frac{1}{\beta_0 \log \left( \frac{\Lambda_{LP}}{\mu} \right)} , \quad \beta_0 = \frac{9}{8\pi^2} . \quad (3.42)$$

We recognise $\beta_0$ as the one-loop coefficient of the $\beta$ function, $\beta(\lambda) = \frac{9}{8\pi^2} \lambda^2 + \mathcal{O}(\lambda^3)$, and the scale $\Lambda_{LP}$ as the Landau pole of the $\phi^4$ theory without Higgsplasion. Now consider the case where $\Lambda_{UV}$ is greater than the Higgsplasion scale so that $E_* < \Lambda_{UV} \leq \Lambda_{LP}$. When $\mu$ exceeds $E_*$, we enter the Higgsploding regime and the integral contributing to $\lambda(\mu)$ ceases to grow. The $\beta$ function vanishes and the coupling is frozen at $\lambda(\mu = E_*) = \bar{\lambda}_*$, the UV fixed point value given in (3.34). The resulting running of $\lambda(\mu)$ is shown in Fig. 1.

The same logic and assumptions can be applied to the running of the mass. In the absence of Higgsplasion we find,

$$m^2(\mu) = m^2(\Lambda_{UV}) - \frac{3\lambda(\mu)}{16\pi^2} \left( \frac{\Lambda_{UV}^2 - \mu^2}{16\pi^2} \right) \left[ m^2 \log \frac{\Lambda_{UV}^2}{\mu^2} \right] . \quad (3.43)$$

The expression on the right-hand side contains, as expected, quadratic and logarithmic terms depending on the UV cut-off. The quadratic terms in particular, introduce the high degree of fine tuning between the bare mass $m^2(\Lambda_{UV})$ and the radiative corrections. This is the Hierarchy problem intrinsic to QFTs with microscopic scalars. Once again, the effect of Higgsplasion is to freeze the RG evolution at scales above $E_*$. As a result, only the so-called ‘little hierarchy’ problem remains at the scale $E_* \ll M_{Pl}$.

An even more powerful simplification comes from effect of Higgsplasion on the radiative corrections to the light scalar mass, $m^2$, arising from the loop of heavy degrees of freedom $M \gg E_* \gg m$. In this case [1], the radiative correction to the bare $m^2$ parameter is given by (cf. (3.25)),

$$\Delta m^2 = -3\lambda \int_{p^2 \leq E_*^2} \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + M^2} \approx -\frac{3\lambda}{16\pi^2} \frac{E_*^2}{M^2} E_*^2 , \quad \text{where} \quad \frac{E_*^2}{M^2} \ll 1 . \quad (3.44)$$

Returning to the model with a single degree of freedom, we are now in a position to understand the effect of Higgsplasion on the end result of Section 3.1. The on-threshold $1 \to n$ amplitude given in Eq. (3.38), now expressed in terms of the running parameters, reads,

$$\langle n|\phi|0 \rangle = \left( \frac{\partial}{\partial z} \right)^n \phi |_{z=0} = n! \left( \frac{\lambda(\mu)}{8m^2(\mu)} \right)^{n-1} \left( 1 - \lambda(\mu)(n-1)(n-3) \frac{F}{16} \right) , \quad (3.45)$$

with the running parameters $\lambda(\mu)$ and $m^2(\mu)$ approaching constant UV fixed values at values of $\mu \geq E_*$.

Any corrections to $F$ due to Higgsplasion are of order $\sim m^2/E_*^2$ and are ignored in the equations above. The leading effect of Higgsplasion enters through the modified running of the mass and coupling. We therefore expect the Higgsplasion result to match the result in [9] at scales below $E_*$. As $\mu$ exceeds $E_*$, the Higgsplasion result begins to deviate from [9] due to the frozen couplings.
3.3 Spontaneously broken theory

The approach in Section 3.1 can also be directly applied to the broken theory, where field $\phi$ acquires a VEV. Voloshin’s calculation [9] for the model shown in Eq. (2.2) was extended to the broken theory by Smith [10]. The new physical scalar boson $h = \phi - \langle \phi \rangle = \phi - v$ has mass $m_h = \sqrt{2\lambda v}$ and a new 3-point coupling. Keeping only leading order shifts and assuming $\mu \gg m_h$, the renormalised parameters are given by,

$$\bar{\lambda} = \lambda - \frac{9\lambda^2}{32\pi^2} \log \frac{\Lambda_{\text{UV}}}{m_h},$$  \hspace{1cm} (3.46)$$
$$m_h^2 = m_h^2 - \frac{3\lambda}{8\pi^2} \Lambda_{\text{UV}}^2.$$ \hspace{1cm} (3.47)

Though some pre-factors and signs differ from the unbroken case, the degrees of divergence remain the same. The $1^* \rightarrow n$ threshold amplitude with one-loop correction can be written as,

$$\langle n|\phi|0 \rangle = n! \left( \frac{\lambda(\mu)}{2m_h^2(\mu)} \right)^{(n-1)/2} (1 + \frac{\lambda(\mu)}{2} n(n-1)B)$$ \hspace{1cm} (3.48)

where

$$B = \frac{\sqrt{3}}{8\pi},$$ \hspace{1cm} (3.49)

is the constant term in the broken theory, analogous to the constant $-F/16$ in the unbroken theory (see Eqs. (3.38), (3.30)). Importantly, and in contrast to the unbroken case, the finite term $B$ is real and the correction is positive, as discussed by Smith [10]. As before, the leading effect of Higgsplosion is through the modified running couplings rather than the minor corrections to $B$.

3.4 Exponentiation of the one-loop corrections

It is worth noting that the exponentiation of the finite terms in the one-loop corrections to the threshold amplitudes, as derived in Ref. [8], will not be modified by the inclusion of Higgsplosion. The argument used in Ref. [8] concerns only the so-called finite part of the quantum correction, and not the UV-sensitive terms where the effect of Higgsplosion is manifest. Hence the entire construction presented in [8] applies equally well to Higgsploding theories. The result for the exponentiated one-loop correction in the broken and unbroken theories respectively is,

$$\langle n|\phi|0 \rangle = \begin{cases} 
\langle n|\phi|0 \rangle_{\text{tree}} \times \exp(n^2 B) & : \text{broken theory} \\
\langle n|\phi|0 \rangle_{\text{tree}} \times \exp(-n^2 F/16) & : \text{unbroken theory}.
\end{cases}$$ \hspace{1cm} (3.50)
4 Precision observables in the presence of Higgsplosion

4.1 Loop calculations for observables in the Standard Model

Many processes in the SM occur at tree level, such as the Drell-Yan production of lepton pairs. These contain no unconstrained momenta, and thus no loop integration is required to calculate them. Only once higher-order perturbative corrections are included, do loop integrals begin to appear. Instead, we will focus on a class of observables which have no tree-level contribution. These loop-induced processes will be modified by Higgsplosion at leading order, so they provide a firm testing ground for investigating Higgsplosion effects.

All the loop-induced processes that we consider arise at the one-loop level. The loop integrals which appear contain various tensor structures and denominators involving external momenta and masses. Following a similar notation to [20], the general one-loop $N$-point tensor integral can be written as

$$ T_{\mu_1...\mu_P}^{P_1}(p_1, ..., p_{N-1}, m_0, ..., m_{N-1}) = \frac{(2\pi)^{4-d}}{i\pi^2} \int d^d k \frac{k^{\mu_1} \cdots k^{\mu_P}}{D_0 \cdots D_{N-1}}, $$

where the denominator contains the propagator factors

$$ D_j = (k + r_j)^2 - m_j^2 + i\epsilon , \quad r_j = \sum_{i=1}^{j} p_i , \quad r_0 = 0. $$

Fig. 2 shows the general $N$-point function with the configuration of momenta that we use. The $p_i$ are the external momenta of the $N$-point integrals, and the $r_i$ are convenient definitions to simplify the notation. The number of spacetime dimensions in these integrals has been written as $d \equiv 4 - \epsilon$, as a precursor to performing dimensional regularisation. The general tensor integrals in Eq. (4.1) can be reduced using the Passarino-Veltman reduction procedure [21] to a set of four independent scalar integrals containing no powers of momenta.
in the numerators. Here, we will focus on observables which reduce to one-point, two-point and three-point scalar integrals. These are:

\[ A_0(m_0^2) = \frac{(2\pi \mu)^\epsilon}{i \pi^2} \int d^d k \frac{1}{k^2 - m_0^2 + i\epsilon}, \quad (4.3) \]

\[ B_0(p_{1i}, m_0^2, m_1^2) = \frac{(2\pi \mu)^\epsilon}{i \pi^2} \int d^d k \prod_{i=0}^1 \frac{1}{(k + r_i)^2 - m_i^2 + i\epsilon}, \quad (4.4) \]

\[ C_0(p_{1i}, p_{2i}, r_{2i}, m_0^2, m_1^2, m_2^2) = \frac{(2\pi \mu)^\epsilon}{i \pi^2} \int d^d k \prod_{i=0}^2 \frac{1}{(k + r_i)^2 - m_i^2 + i\epsilon}. \quad (4.5) \]

These integrals can be evaluated in dimensional regularisation by Feynman parameterisation, wick rotating into Euclidean space, and then carrying out the \(d\)-dimensional spherical integrals. The results are:

\[ A_0(m_0^2) = m_0^2 \left( \frac{2}{\epsilon} - \gamma_E + \ln 4\pi - \ln \frac{m_0^2}{\mu^2} + 1 \right), \quad (4.6) \]

\[ B_0(p_{1i}, m_0^2, m_1^2) = \frac{2}{\epsilon} - \gamma_E + \ln 4\pi - \int_0^1 dx_1 \ln \frac{M_B^2}{\mu^2}, \quad (4.7) \]

\[ C_0(p_{1i}, p_{2i}, r_{2i}, m_0^2, m_1^2, m_2^2) = -\Gamma(3) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{2M_C^2}. \quad (4.8) \]

The dummy variables \(x_1\) and \(x_2\) result from Feynman parameterisation, where we have defined

\[ M_B^2 = (1 - x_1)m_1^2 + x_1 m_2^2 + x_1(x_1 - 1)p_{1i}^2, \quad (4.9) \]

for the \(B_0\) integral, and

\[ M_C^2 = (1 - x_1 - x_2)m_1^2 + x_1 m_2^2 + x_2 m_3^2 + x_1(x_1 + x_2 - 1)p_{1i}^2 - x_1 x_2 p_{2i}^2 + x_2(x_1 + x_2 - 1)r_{2i}^2, \quad (4.10) \]

for the \(C_0\) integral. The \(A_0\) tadpole integral and the \(B_0\) self-energy integral are UV divergent, which is manifest by the \(1/\epsilon\) poles arising from dimensional regularisation. In any physical observable, these UV-divergent contributions must vanish, which is the case for all observables that we consider.

4.2 Procedure for Higgsplash modifications to loop calculations

In the Higgsplash framework, we begin the calculation of the loop integrals for physical processes in the same manner as for the SM. The initial starting point is that we have dimensionally regulated integrals, with the full (infinite) range of momenta allowed. One can then perform Passarino-Veltman decomposition of the tensor integrals to scalar integrals. Higgsplash is subsequently imposed on the scalar integrals, which formally has the effect of separating the integration into two regions. Below the Higgsplash scale, we have the unperturbed propagator of the SM and thus there is no change. Above the Higgsplash scale, the propagator is exponentially suppressed and so the integrand rapidly
goes to zero. In practice, this has the effect of restricting the allowed momenta of the integrand to \( |k^2| \leq E_s^2 \), where \( E_s \) is the Higgsplasion scale. The limit in which \( E_s \to \infty \) corresponds to the original dimensionally regulated integrals, which would arise when performing loop calculations in the SM. The integrals can then be Wick rotated into Euclidean space, where the Higgsplasion scale becomes a Euclidean cut-off such that only momenta satisfying \( k_E^2 \leq E_s^2 \) are allowed. Finally, the integrations can be carried out, with numerical integrations required over the Feynman parameters. Here, we outline the explicit calculations of the UV-divergent integrals \( A_0 \) and \( B_0 \), and the UV-finite integral \( C_0 \).

Starting from Eq. (4.3), we perform the Wick rotation to Euclidean momenta and impose Higgsplasion by restricting the range of allowed momenta to \( k^2 \leq E_s^2 \),

\[
A_0 = -\frac{(2\pi\mu)^\epsilon}{\pi^2} \int_{k^2 \leq E_s^2} d^d k \frac{1}{k^2 + m_0^2}.
\]  

(4.11)

We will suppress the arguments of the scalar integrals from now on, but the arguments from Eqs. (4.3)-(4.5) are implicitly assumed. Carrying out the surface integral over the \( d \)-dimensional sphere, we get

\[
A_0 = -\frac{(2\pi\mu)^\epsilon}{\pi^2} \frac{2\pi^{2-\frac{\epsilon}{2}}}{\Gamma(2-\frac{\epsilon}{2})} \int_{0}^{E_s} dk \frac{k^{3-\epsilon}}{(k^2 + M_B^2)^2},
\]  

(4.12)

The restriction on the momenta from Higgsplasion means that this integral is necessarily finite, so dimensional regularisation is no longer required and the \( \epsilon \to 0 \) limit can be taken. Finally, carrying out the \( k \) integration gives

\[
A_0 = m_0^2 \log \left( \frac{E_s^2}{m_0^2} + 1 \right) - E_s^2.
\]  

(4.13)

This computation of \( A_0 \) is identical to our earlier calculation of the most singular part of the propagator in the background of the Brown solution in Section 3.1. In fact, \( A_0 = -G_I(x,x) \) in Eq. (3.25).

The procedure for \( B_0 \) follows very similar steps. Starting from Eq. (4.4), after doing Feynman parameterisation and imposing Higgsplasion, we get

\[
B_0 = \frac{(2\pi\mu)^\epsilon}{\pi^2} \int_{0}^{1} dx_1 \int_{k^2 \leq E_s^2} d^d k \frac{1}{(k^2 + M_B^2)^2},
\]  

\[
= \frac{(2\pi\mu)^\epsilon}{\pi^2} \frac{2\pi^{2-\frac{\epsilon}{2}}}{\Gamma(2-\frac{\epsilon}{2})} \int_{0}^{1} dx_1 \int_{0}^{E_s} dk \frac{k^{3-\epsilon}}{(k^2 + M_B^2)^2},
\]  

\[
= \int_{0}^{1} dx \left[ \log \left( \frac{E_s^2}{M_B^2} + 1 \right) - \frac{E_s^2}{E_s^2 + M_B^2} \right],
\]  

(4.14)

with \( M_B \) defined in Eq. (4.9). Analogously, \( C_0 \) can be calculated starting from Eq. (4.5),

\[
C_0 = -\frac{(2\pi\mu)^\epsilon}{\pi^2} \frac{1}{\Gamma(3)} \int_{0}^{1} dx_1 \int_{0}^{1-x_1} dx_2 \int_{k^2 \leq E_s^2} d^d k \frac{1}{(k^2 + M_C^2)^3},
\]  

\[
= -\frac{(2\pi\mu)^\epsilon}{\pi^2} \frac{1}{\Gamma(2-\frac{\epsilon}{2})} \int_{0}^{E_s} dk \int_{0}^{1} dx_1 \int_{0}^{1-x_1} dx_2 \frac{k^{3-\epsilon}}{(k^2 + M_C^2)^3},
\]  

\[
= -\frac{1}{\Gamma(3)} \int_{0}^{1} dx_1 \int_{0}^{1-x_1} dx_2 \frac{E_s^4}{2M_C^2(E_s^2 + M_C^2)^2},
\]  

(4.15)
with $M_C$ defined in Eq. (4.10). The expressions in Eqs. (4.13), (4.14) and (4.15) are the final results for the scalar integrals that we use to calculate the effect of Higgsplosion on precision observables.

We note here that in physical observables, the dependence on the Higgsplosion scale cancels in the $E_2^* \to \infty$ limit, which recovers the SM result. This is equivalent to the statement that when using cut-off regularisation, observables are finite in the end. The expressions describing the difference in the values of precision observables between the theory with and without Higgsplosion, will have the terms $\sim E_2^2$, $\sim \log E_2^2$ and $E_2^*$-independent terms cancelled. Thus, only the terms suppressed by the inverse powers of $E_2^*$ (also allowing additional logarithms) will survive from the integrals in (4.13)-(4.15) in the results for the deviation of the observables from the Standard Model values. In the language of Section 3 this amounts to the computation of power-suppressed corrections to finite terms $F$ and $B$.

### 4.3 Results for precision observables

Using the expressions from Eqs. (4.13), (4.14) and (4.15), we can now compute the effects on precision observables induced by Higgsplosion. All loop-induced observables will receive corrections that scale like $O(\hat{s}/E_2^2)$. Thus, we focus on observables that can be measured to a high precision. The calculations proceed in accordance with the method outlined in the previous section, where Higgsplosion is imposed after the reduction to scalar integrals. The diagram and amplitude generation is performed by FeynArts [22], and then the Passarino-Veltman reduction and amplitude squaring is performed by FormCalc [23]. The SM expressions for the loop integrals are evaluated using the LoopTools package.

### 4.4 Higgs precision observables

The first class of observables that we consider are the production and decay of an on-shell Higgs boson via loop-induced leading-order processes, i.e. $gg \to h$, $h \to \gamma\gamma$ and $h \to Z\gamma$. Figure 3 shows the leading-order Feynman diagrams for these processes, capturing the dominant Higgs production mechanism at the LHC, along with two of the cleanest Higgs
decay modes. For the production mode, the dominant contribution comes from the top-quark loop, and we neglect the contribution from the lighter fermions. For the decay modes, there are also contributions from $W$-boson loops which interfere destructively with the top-quark loop.

In all three processes the external momentum scale is set by the Higgs mass, $\sqrt{s} \simeq m_h$. This is the only external momentum scale that enters into the loop integrals for these processes, which can be decomposed into $C_0$ triangle integrals. To quantify the relative change due to Higgsplosion, we define the quantities $\hat{\sigma}^*$ and $\hat{\Gamma}^*$ for the partonic cross section and decay width respectively, where the loop integrals have been calculated using the Higgsplosion expressions. These are then compared to their associated SM results, and their relative differences calculated. In Figure 4, we plot the effect of Higgsplosion on the Higgs observables as functions of the Higgsplosion scale, $E^*_\ast$.

Current theoretical uncertainties on $gg \to h$ are $\mathcal{O}(10\%)$, irrespective of the center-of-mass energy of the hard process [24]. Here, a large improvement would be needed to become sensitive to the Higgsplosion scenario. Even if such improvements could be achieved, the predicted experimental sensitivity is $\mathcal{O}(5\%)$ at the LHC with 3000 $\text{fb}^{-1}$ [25]. A higher precision can be achieved for the decay $h \to gg$ at a future electron-positron collider. Ref. [26] gives a predicted precision for BR($h \to gg$) of 1.4\% for the FCCee and 3.3\% for a 250 GeV ILC.

The expected precision for BR($h \to \gamma\gamma$) is 3.0\% at the FCCee. The rate for BR($h \to Z\gamma$) is too small to set tight limits at electron-positron colliders. At the HL-LHC the rate can be limited to $\mathcal{O}(10\%)$ accuracy [27]. Hence, even assuming strong improvements on the theory uncertainties and data from a future circular electron-positron collider it will not be possible to set limits on the Higgsplosion scale in such measurements of a singly-produced on-shell Higgs boson.

A larger effect can be achieved by increasing the interaction scale $\sqrt{s}$. In Fig. 5 we show the impact of Higgsplosion on the process $gg \to h^*$, when varying $\sqrt{s}$ between 10 – 90

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Plot of the effect of Higgsplosion on the partonic cross section for $gg \to h$ (left) and the partial decay width for $h \to \gamma\gamma$ and $h \to Z\gamma$ (right) as functions of the Higgsplosion scale $E^*_\ast$.}
\end{figure}
Figure 5: Plot of the effect of Higgsplosion on the partonic cross section for $gg \to h^*$ as a function of the centre-of-mass energy $\sqrt{s}$. The different curves show different values of the Higgsplosion scale $E^*$. 

TeV. This amounts to producing a Higgs boson far away from its mass-shell or a heavy Higgs boson that could arise in an extension of the Standard Model. The effect becomes $\mathcal{O}(1)$ when $\sqrt{s} \simeq 2E^*$, in close analogy to the $2m_t$ threshold of the $gg \to h$ process in the Standard Model. The three curves in Fig. 5 correspond to different Higgsplosion scales. As corrections from Higgsplosion scale like $\hat{s}/E^2$, the higher the Higgsplosion scale, the large $\sqrt{s}$ has to be to achieve an observable effect. This motivates precision studies at a future high-energy collider to test the realisation of Higgsplosion in nature.

4.5 Flavor observables

As Higgsplosion has a direct effect on all loop-induced processes and virtual corrections, flavor observables that have been measured rather precisely could be used to set a limit on the Higgsplosion scale $E^*$. Relevant observables include rare or semileptonic meson decays and Kaon or $B$-meson mixing parameters [28].

The rate of the rare inclusive decay process $B \to X_s \gamma$ is one of the most important $B$-physics observables as it sets stringent constraints on the parameter space of various extensions of the SM [29]. At lowest order it can be described by the transition $b \to s \gamma$. The effective Hamiltonian for this decay is usually expressed as [30]

$$
\mathcal{H}_{\text{eff}} = \frac{-4G_F}{\sqrt{2}} V_{tb}V_{ts}^* \sum_{i=1}^{8} C_i(\mu) \mathcal{O}_i(\mu),
$$

(4.16)

where $V_{ij}$ are elements of the CKM matrix, $G_F$ is the Fermi constant and $\mu$ is the scale at which the Wilson coefficients $C_i(\mu)$ are evaluated at.

The effect of Higgsplosion is predominantly encoded in the Wilson coefficients. Their relative change from the SM directly modifies the decay rate of $B \to X_s \gamma$ by the same amount. Here, we will focus on the coefficients $C_7$ and $C_8$, which are associated with the
operators

\[ O_7 = \frac{e}{16\pi^2} m_b \langle s_j | \omega_+ \sigma_{\mu\nu} | b_i \rangle \delta_{ij} F^{\mu\nu}, \]  
\[ O_8 = \frac{g_s}{16\pi^2} m_b \langle s_j | \omega_+ \sigma_{\mu\nu} | b_i \rangle T^a_{ij} G^{\mu\nu}_a. \] (4.17) (4.18)

They are calculated from the partonic transitions $b \rightarrow s \gamma$ and $b \rightarrow s g$ respectively. Fig. 6 shows the Wilson coefficients $C_7$ and $C_8$ as functions of the Higgsplison scale, $E_\ast$.

![Figure 6: Plot of the effect of Higgsplison on the Wilson coefficients $C_7$ and $C_8$ for $B \rightarrow X_s \gamma$ as functions of the Higgsplison scale $E_\ast$.](image)

We find that the Higgsplison modifications of $B \rightarrow X_s \gamma$ compared to the SM are small and are, even for a low Higgsplison scale of $E_\ast \simeq 30$ TeV, unobservable, taking theoretical and experimental uncertainties into account. Current measurements of $\text{BR}(B \rightarrow X_s \gamma) \simeq (335 \pm 15) \cdot 10^{-6}$ [31–33] are at the level of 5% accuracy only.

4.6 Anomalous magnetic dipole moment of the electron and muon

The anomalous magnetic dipole moment of the electron is one of the great successes of twentieth century physics, and QED specifically. The precision with which theoretical predictions agree with experimental measurement is about one part in a trillion, which is unprecedented for many areas of physics. However, for a muon there is a discrepancy with the SM prediction [34–37]

\[ \Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{theory}} \simeq 2.90 \cdot 10^{-9}, \] (4.19)

which may be a sign of new physics.

The anomalous magnetic moment of a fermion quantifies how much its magnetic moment $g$ differs from its classical value, which is predicted by the Dirac equation. This is quantified by the expression $a = (g - 2)/2$, where $a$ is the anomalous magnetic moment. In perturbative QED, the tree level result corresponds to the vertex interaction of a charged lepton and photon at zero momentum transfer, and recovers the classical prediction. The
radiative corrections to this vertex can in general be described by the form-factors $F_1$ and $F_2$,

$$\Gamma^\mu = F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m}.$$  \hspace{1cm} (4.20)

The anomalous magnetic moment is then given by $F_2(0)$. We calculate the one-loop contributions to the anomalous magnetic moment of the electron and muon and their deviations due to Higgsplusion. The one-loop result for a charged lepton $\ell$ can be compactly written as

$$a_\ell = \frac{\alpha}{4\pi} \left[ 2B_0(m_\ell^2,0,m_\ell^2) - B_0(0,0,m_\ell^2,m_\ell^2) - B_0(0,0,0,m_\ell^2) - 1 \right].$$  \hspace{1cm} (4.21)

In the SM at one loop, the mass dependence in the $B_0$ integrals cancels so the anomalous magnetic moment is $a_\ell = \alpha/(2\pi)$ for all charged leptons. However, in Higgsplusion the mass dependence remains and changes are induced via the $B_0$ integrals. The effect of Higgsplusion is shown by the plot in Fig. 7. We find that the sensitivity on $a_\mu$ has to be improved by at least two orders of magnitude to be able to set a meaningful limit on the Higgsplusion scale $E_\ast$.

![Figure 7: Plot of the effect of Higgsplusion on the anomalous magnetic moment of the electron and muon as functions of the Higgsplusion scale $E_\ast$.](image)

The anomalous magnetic moment of the electron has been measured to be \cite{38}

$$a_e^{\text{exp}} = 11596521807.3(2.8) \cdot 10^{-13},$$  \hspace{1cm} (4.22)

with an experimental uncertainty of $\delta a_e^{\text{exp}} = 2.8 \cdot 10^{-13}$. Such high precision allows one to set limits on a wide range of new physics models \cite{39}. However, as the relative changes to the anomalous magnetic moments of the electron and muon induced by Higgsplusion are related by

$$1 - a_e^*/a_e^{\text{SM}} \approx \frac{m_e^2}{m_\ell^2},$$  \hspace{1cm} (4.23)

the increased precision for the electron compared with the muon does not translate into a better limit on the Higgsplusion scale.
5 Discussion and Conclusions

The motivation and main task of this paper was to provide the first detailed study of how the characteristic energy scale of Higgsplosion affects the loop integrals in Higgsploding theories.

There are dramatic consequences of Higgsplosion for the theory dynamics at energy scales in the UV. At the same time, we have shown that at currently accessible energies the effects of the Higgsplosion scale remain very small. However, $\mathcal{O}(1)$ effects can be achieved for loop-induced processes or virtual corrections if the interactions are probed close to $\sim 2E_\ast$. This effect can be immediately seen in Fig. 5 for $gg \rightarrow h^\ast$, but it has much broader applicability. It applies to all Standard Model processes, such as Drell-Yan, multi-jet production or the production of electroweak gauge bosons.

The main features of Higgsplosion that have been explored in [1, 2] and the present paper are:

1. There are no UV divergences in the theory.

2. The running of the coupling and mass parameters of the theory flattens out when the RG scale $\mu$ exceeds the Higgsplosion scale $E_\ast$ and the parameters reach their UV fixed points. There are no Landau poles and no couplings become negative. Therefore, the Higgsploding theory is asymptotically safe.

3. There are no hierarchically large contributions to the Higgs boson mass arising from potentially very heavy degrees of freedom with masses above $E_\ast$.

4. More generally, all particle states at energies or virtualities above $E_\ast$ rapidly decay into multiple relatively soft Higgs bosons and massive vector bosons. Essentially, at $\sqrt{s} > E_\ast$, all elementary particle states including the Higgs itself become composite states involving high multiplicities $n \sim \sqrt{s/m_h}$ of soft electroweak quanta.

5. The self-consistency of the computational formalism, in particular the applications of the semiclassical formalism of [12] to Higgsplosion [1, 14], requires that the exponentiation of the loop corrections to the tree-level Higgsploding amplitudes, originally proved in [8], remains unaffected when the loop integrations are modified by the presence of the Higgsplosion scale $E_\ast$. In Section 3 we demonstrated that this is indeed the case.

6. In Section 4 we followed this up with the loop formalism in the presence of Higgsplosion and computed a set of Standard Model precision observables. We found that the effects of the Higgsplosion scale on these observables are small and remain unobservable at currently available energies.

7. This of course does not affect the possibility of direct observation of Higgsplosion at energies $\sqrt{s}$ above the $E_\ast$ threshold [19, 40], which would potentially be achievable at future hadron colliders in the 100 TeV range, or in the early Universe (e.g. the decays and interactions of the inflaton during reheating).
It was pointed out in [1] that the direct Higgsplosive production of multiple Higgs bosons in very high energy collisions with $\sqrt{s} > E_*$ does not result in the breakdown of perturbative unitarity even when the rates appear to grow exponentially with energy. The computation of the cross section of physical processes, such as gluon fusion $gg \rightarrow n \times h$ going through an intermediate virtual Higgs boson(s) produced in the $s$-channel, $gg \rightarrow h^* \rightarrow n \times h$, requires the use of the dressed propagators for the intermediate $h^*$. This results in Higgspersion, i.e. a well-behaved cross section for arbitrary $n$ up to very high energies [1]

$$
\sigma_{gg \rightarrow n \times h}^\Delta \sim y_t^2 m_t^2 \log^4 \left( \frac{m_t}{\sqrt{p^2}} \right) \times \frac{1}{p^4 + m_h^4 \mathcal{R}^2} \times \mathcal{R}_n , \tag{5.1}
$$

and thus

$$
\sigma_{gg \rightarrow n \times h} \sim \begin{cases} 
\mathcal{R} & : \text{for } \sqrt{s} \lesssim E_* \text{ where } \mathcal{R} \lesssim 1 \\
1/\mathcal{R} \rightarrow 0 & : \text{for } \sqrt{s} \gtrsim E_* \text{ where } \mathcal{R} \gg 1 
\end{cases} . \tag{5.2}
$$

Hence, by avoiding a breakdown of perturbative unitarity in multi-boson production, the theory can retain consistency and predictivity to much higher, technically even unlimited, energy scales.

It is important to keep in mind that the expression for the cross section in (5.2) does not imply that the physical cross section once again becomes exponentially small when $p^2$ or $s$ is $\gg E_*^2$. This expression was derived under the assumption that all the energy of the collision goes into producing as many soft quanta as kinematically possible. However, at energies exceeding $E_*$, this is no longer the case. Instead it is more advantageous for the initial highly energetic particles to emit one or few hard quanta to lower the energy from $\sqrt{s}$ down to $\sim E_*$. The scattering then proceeds by emitting mostly soft quanta. Thus the correct behaviour for the Higgsplosion cross section is that it saturates at high energies,

$$
\sigma_{gg \rightarrow n \times h} \sim \begin{cases} 
\mathcal{R} & : \text{for } \sqrt{s} \ll E_* \text{ where } \mathcal{R} \ll 1 \\
1 & : \text{for } \sqrt{s} \gtrsim E_* \text{ where } \mathcal{R} \gg 1 
\end{cases} . \tag{5.3}
$$

This leaves the possibility of direct Higgsplosion processes being observable at energies above the Higgsplosion scale.

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References

[1] V. V. Khoze and M. Spannowsky, Higgsplosion: Solving the Hierarchy Problem via rapid decays of heavy states into multiple Higgs bosons, 1704.03447.
[2] V. V. Khoze and M. Spannowsky, The Higgsploing Universe, 1707.01531.
[3] N. S. Manton, Topology in the Weinberg-Salam Theory, Phys. Rev. D28 (1983) 2019.
[4] F. R. Klinkhamer and N. S. Manton, A Saddle Point Solution in the Weinberg-Salam Theory, Phys. Rev. D30 (1984) 2212.

[5] L. S. Brown, Summing tree graphs at threshold, Phys. Rev. D46 (1992) R4125–R4127, [hep-ph/9209203].

[6] E. N. Argyres, R. H. P. Kleiss and C. G. Papadopoulos, Amplitude estimates for multi-Higgs production at high-energies, Nucl. Phys. B391 (1993) 42–56.

[7] M. B. Voloshin, Estimate of the onset of nonperturbative particle production at high-energy in a scalar theory, Phys. Lett. B293 (1992) 389–394.

[8] M. V. Libanov, V. A. Rubakov, D. T. Son and S. V. Troitsky, Exponentiation of multiparticle amplitudes in scalar theories, Phys. Rev. D50 (1994) 7553–7569, [hep-ph/9407381].

[9] M. B. Voloshin, Summing one loop graphs at multiparticle threshold, Phys. Rev. D47 (1993) R357–R361, [hep-ph/9209240].

[10] B. H. Smith, Summing one loop graphs in a theory with broken symmetry, Phys. Rev. D47 (1993) 3518–3520, [hep-ph/9209287].

[11] M. B. Voloshin, Loops with heavy particles in production amplitudes for multiple Higgs bosons, Phys. Rev. D95 (2017) 113003, [1704.07320].

[12] D. T. Son, Semiclassical approach for multiparticle production in scalar theories, Nucl. Phys. B477 (1996) 378–406, [hep-ph/9505338].

[13] M. V. Libanov, V. A. Rubakov and S. V. Troitsky, Multiparticle processes and semiclassical analysis in bosonic field theories, Phys. Part. Nucl. 28 (1997) 217–240.

[14] V. V. Khoze, Multiparticle production in the large $\lambda n$ limit: realising Higgsplosion in a scalar QFT, JHEP 06 (2017) 148, [1705.04365].

[15] F. J. Dyson, The S matrix in quantum electrodynamics, Phys. Rev. 75 (1949) 1736–1755.

[16] J. S. Schwinger, On the Green’s functions of quantized fields. 1., Proc. Nat. Acad. Sci. 37 (1951) 452–455.

[17] J. S. Schwinger, On the Green’s functions of quantized fields. 2., Proc. Nat. Acad. Sci. 37 (1951) 455–459.

[18] J. S. Gainer, Measuring the Higgsplosion Yield: Counting Large Higgs Multiplicities at Colliders, 1705.00737.

[19] V. V. Khoze, Perturbative growth of high-multiplicity W, Z and Higgs production processes at high energies, JHEP 03 (2015) 038, [1411.2925].

[20] A. Denner and S. Dittmaier, Reduction schemes for one-loop tensor integrals, Nucl. Phys. B734 (2006) 62–115, [hep-ph/0509141].

[21] G. Passarino and M. J. G. Veltman, One Loop Corrections for $e^+ e^-$ Annihilation Into $\mu^+\nu^-\mu^-\nu^+$ in the Weinberg Model, Nucl. Phys. B160 (1979) 151–207.

[22] T. Hahn, Generating Feynman diagrams and amplitudes with FeynArts 3, Comput. Phys. Commun. 140 (2001) 418–431, [hep-ph/0012260].

[23] T. Hahn and M. Perez-Victoria, Automatized one loop calculations in four-dimensions and D-dimensions, Comput. Phys. Commun. 118 (1999) 153–165, [hep-ph/9807565].

[24] R. Contino et al., Physics at a 100 TeV pp collider: Higgs and EW symmetry breaking studies, CERN Yellow Report (2017) 255–440, [1606.09408].
[25] CMS collaboration, *Projected Performance of an Upgraded CMS Detector at the LHC and HL-LHC: Contribution to the Snowmass Process*, in *Proceedings, 2013 Community Summer Study on the Future of U.S. Particle Physics: Snowmass on the Mississippi (CSS2013)*: Minneapolis, MN, USA, July 29-August 6, 2013, 2013. 1307.7135.

[26] J. de Blas, M. Ciuchini, E. Franco, S. Mishima, M. Pierini, L. Reina et al., *Electroweak precision observables and Higgs-boson signal strengths in the Standard Model and beyond: present and future*, *JHEP* **12** (2016) 135, [1608.01509].

[27] J. M. No and M. Spannowsky, *A Boost to $h \rightarrow Z\gamma$: from LHC to Future $e^+e^-$ Colliders*, *Phys. Rev.* **D95** (2017) 075027, [1612.06626].

[28] Y. Amhis et al., *Averages of $b$-hadron, $c$-hadron, and $\tau$-lepton properties as of summer 2016*, 1612.07233.

[29] A. J. Buras, M. Misiak, M. Munz and S. Pokorski, *Theoretical uncertainties and phenomenological aspects of $B \rightarrow X(s) \gamma$ decay*, *Nucl. Phys.* **B424** (1994) 374–398, [hep-ph/9311345].

[30] A. Ali and C. Greub, *A Profile of the final states in $B \rightarrow X(s) \gamma$ and an estimate of the branching ratio $BR (B \rightarrow K^* \gamma)$*, *Phys. Lett.* **B259** (1991) 182–190.

[31] CLEO collaboration, S. Chen et al., *Branching fraction and photon energy spectrum for $b \rightarrow s\gamma$*, *Phys. Rev. Lett.* **87** (2001) 251807, [hep-ex/0108032].

[32] Belle collaboration, A. Limosani et al., *Measurement of Inclusive Radiative B-meson Decays with a Photon Energy Threshold of 1.7-GeV*, *Phys. Rev. Lett.* **103** (2009) 241801, [0907.1384].

[33] BABAR collaboration, J. P. Lees et al., *Precision Measurement of the $B \rightarrow X_s \gamma$ Photon Energy Spectrum, Branching Fraction, and Direct CP Asymmetry $A_{CP}(B \rightarrow X_s+d\gamma)$*, *Phys. Rev. Lett.* **109** (2012) 191801, [1207.2690].

[34] Muon g-2 collaboration, G. W. Bennett et al., *Final Report of the Muon E821 Anomalous Magnetic Moment Measurement at BNL*, *Phys. Rev.* **D73** (2006) 072003, [hep-ex/0602035].

[35] F. Jegerlehner and A. Nyffeler, *The Muon g-2*, *Phys. Rept.* **477** (2009) 1–110, [0902.3360].

[36] K. Hagiwara, R. Liao, A. D. Martin, D. Nomura and T. Teubner, *$(g-2)_{\mu}$ and $\alpha(MZ^2)$ re-evaluated using new precise data*, *J. Phys.* **G38** (2011) 085003, [1105.3149].

[37] M. Davier, A. Hoecker, B. Malaescu and Z. Zhang, *Reevaluation of the Hadronic Contributions to the Muon g-2 and to alpha(MZ)*, *Eur. Phys. J.* **C71** (2011) 1515, [1010.4180].

[38] D. Hanneke, S. F. Hoogerheide and G. Gabrielse, *Cavity Control of a Single-Electron Quantum Cyclotron: Measuring the Electron Magnetic Moment*, *Phys. Rev.* **A83** (2011) 052122, [1009.4831].

[39] G. F. Giudice, P. Paradisi and M. Passera, *Testing new physics with the electron g-2*, *JHEP* **11** (2012) 113, [1208.6583].

[40] C. Degrande, V. V. Khoze and O. Mattelaer, *Multi-Higgs production in gluon fusion at 100 TeV*, *Phys. Rev.* **D94** (2016) 085031, [1605.06372].