THE TWIN PRIME CONJECTURE

By Yoichi Motohashi

The conjecture
‘there should be infinitely many pairs of primes \( \{p, p + 2\} \)’
has not been conquered yet.

However, a spectacular drama is now unfolding itself in the theory of the distribution of primes. The complete resolution of the conjecture is thus within the range of modern mathematics — perhaps. Luckily enough, I have been witnessing the series of recent great events as a contemporary specialist. The purpose of the present expository talk is to share my excitement with my audience. Any mathematical discovery is an eventual outcome of the rich and long history of our cherished discipline, and the recent amazing discovery by Y. Zhang is a typical instance. I shall describe the essence of the fundamental ideas initiated by GPY (D.A. Goldston, J. Pintz and C.Y. Yildirim) and others which had prepared the way for the discovery, while briefly reviewing the relevant history. You will find all basic ideas are so simple that you will certainly be persuaded that the proverb “small things stir up great” is indeed a truth.

Looking back almost half a century ago, I (then in my 20’s) was eager to learn Yu.V. Linnik’s and A. Selberg’s works in analytic number theory, dreaming the way to the Never-Never Land of prime numbers. They taught me a lot, and I owe them tremendously. I am really happy that their mathematical spirit is still vividly felt in recent developments. Indeed, so many wonders in analytic number theory can be traced back to their ideas. By trekking further and steadily along the way they prepared, you will (I believe) be able to bring us more wonders on primes.

I shall have to be brief in some sections, in order to acquire time for more recent work done by T. Tao and J. Maynard independently, which has made Sections 10 and 11 somewhat less relevant to our main issue of finding infinitely often bounded differences between primes. Nevertheless, you will be better off knowing all the facts that I have put in this text, which I hope will encourage you to delve into the professional literature on primes.

Remark 1: The present text is a substantially improved and augmented version of the one that had been prepared for my talk delivered at the Annual Meeting of the Mathematical Society of Japan (15 March 2014). The expressions that I shall use, whilst being adequate for my present (didactic) purpose, are not always perfectly precise/correct. All facts and details on sieve method and distribution of primes which are needed to understand recent developments are available in my books [10][12].

Remark 2: It is highly recommended to visit T. Tao’s excellent blog:
http://terrytao.wordpress.com/2013/06/03/the-prime-tuples-conjecture-
sieve-theory-and-the-work-of-goldston-pintz-yildirim-motohashi-pintz-and-
zhang/
which has various links to more recent developments.

Remark 3. Because of the digital format specification imposed by arXiv, two diagrams, one of which was kindly put at my disposal by the authors of [15], are

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not included here. To view the diagrams, visit my web-page and download the file EXP2014.pdf.

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1. The conjecture.
Let
\[ \varpi(n) = \begin{cases} 1 & \text{if } n \text{ is a prime}, \\ 0 & \text{if } n \text{ is not a prime}, \end{cases} \]
and put
\[ \pi(x) = \sum_{n<x} \varpi(n), \quad \pi_2(x) = \sum_{n<x} \varpi(n) \varpi(n+2). \]

Anyone who loves mathematics knows
\[ \pi(x) \sim \frac{x}{\log x}. \]

Anyone who ardently loves analytic number theory is bitterly defied by the conjecture
\[ \pi_2(x) \sim C_0 \frac{x}{(\log x)^2}, \quad C_0: \text{an absolute constant,} \quad (1.1) \]
and even by the far more modest statement
\[ \text{The twin prime conjecture: } \lim_{x \to \infty} \pi_2(x) = \infty. \quad (1.2) \]

2. To detect twins.
There are two naive means to detect twin primes:

\begin{align*}
(A) & \quad \varpi(n) \varpi(n+2) > 0, \\
(B) & \quad \varpi(n) + \varpi(n+2) - 1 > 0.
\end{align*}

These are of course equivalent to each other as far as one applies them to individual \( n \)'s, but they are statistically different: always \( \varpi(n) \varpi(n+2) \geq 0 \) but almost always \( \varpi(n) + \varpi(n+2) - 1 = -1 \). It appears that opinions of sieve specialists are now converging upon
\( (A) \) is too strict,
\( (B) \) is more flexible.

But why? It is hard to explain the real situation to people who are not familiar with sieve method. Thus, let me put it bluntly: \( (A) \) is too exact as it gives the definition of \( \pi_2(x) \). A sage (M.J.) in analytic number theory said that exact formulas contain often too much noise. There were a lot of attempts, probably since A.M. Legendre's
time (the late 18th century), to clinch to (1.2) by means of (A); but all eventuated in failure. In fact, GY (Goldston and Yildirim) commenced their investigations in 1999 still brandishing the sharp sword (A). Only in 2004/5, after a few futile (but highly interesting) attempts with (A), did they turn instead to (B). This was a great turning point in their work. Note that GY actually considered primes in tuples: see Section 7. Here I employ an over-simplification in order to make the issue clearer. As far as I know, A. Selberg (1950) was the first who exploited (B), but in a configuration different to GY’s.

3. Sieving out noise.
Imitating the definition of $\pi_2(x)$, one might consider

$$\sum_{n<x} (\varpi(n) + \varpi(n+2) - 1).$$

(3.1)

If the sum is positive and large, then the conjecture (1.2) will be resolved. But this argument is awfully absurd, since obviously (3.1) is essentially equal to $2\pi(x) - x$, and one can utter only the nonsense

$$\text{(3.1)} \sim \ -x .$$

(3.2)

Nevertheless! Things should look pretty different if (3.1) is replaced by

$$\sum_{n<x} (\varpi(n) + \varpi(n+2) - 1)W(n).$$

(3.3)

Here $W(n)$’s are non-negative weights. If one succeeds finding a nice sequence $\{W(n)\}$ such that (3.3) tends to positive infinity, then the conjecture (1.2) will be resolved. This must be, however, an extremely difficult task, since such $\{W(n)\}$ should yield a considerable dumping of the terms ‘1’ and simultaneously should not affect much the situation of $n$ being a twin prime. That is, $\{W(n)\}$ is preferably to satisfy

$$W(n) \text{ is } \begin{cases} 
\geq 0 & \text{but very small on average,} \\
1 & \text{when } n \text{ is a large twin prime.}
\end{cases}$$

4. Lovely lambda’s.
In his work mentioned above, Selberg employed the $\Lambda^2$-sieve, his great invention (1947). If translated into our present situation, it becomes:

Consider the quadratic form

$$\sum_{n<x} \left( \sum_{d \mid n(n+2)} \lambda(d) \right)^2,$$

under the side-condition $\lambda(1) = 1$ and $\lambda(d) = 0$ for $d \geq D$,

where $D$ is a parameter to be fixed optimally eventually, but initially satisfying only $D < x^{1/2-\varepsilon}$ with an arbitrary small $\varepsilon > 0$. Expanding the squares out and exchanging the order of summation, we get the main term and the error term. Selberg
diagonalised the main term in a highly original way (in fact an application of M"obius inversion) and found an explicit expression for optimal $\lambda$’s that minimises the main term. It is expedient to know that these optimal $\lambda$’s satisfy

$$\lambda(d) \sim \mu(d) \left( \frac{\log D/d}{\log D} \right)^2,$$  \hspace{1cm} \text{(4.1)}

with $\mu$ being the M"obius function, and to note that

$$\nu(n) > 2 \implies \sum_{d|n} \mu(d)(\log d)^j = 0, \ j \leq 2.$$  \hspace{1cm} \text{(4.2)}

where $\nu(n)$ is the number of prime factors of $n$ which are different to each other. Namely, the choice (4.1) is an \textit{approximation} to (4.2), which explains the fact that Selberg’s $\lambda$’s yield necessary dumping.

We construct, with these \textit{quasi-optimal} $\lambda$’s,

$$W(n) = \left( \sum_{d|n} \lambda(d) \right)^2$$  \hspace{1cm} \text{(4.3)}

to be used in (3.3). We have, with an appropriate $D$,

$$\sum_{n<x} W(n) \sim C_1 \frac{x}{(\log x)^2},$$  \hspace{1cm} \text{(4.4)}

and

$$\sum_{n<x} (\varpi(n) + \varpi(n+2))W(n) \sim C_2 \frac{x}{(\log x)^2}$$  \hspace{1cm} \text{(4.5)}

with certain constants $C_1, C_2 > 0$. Amazing! Compare these with the conjecture (1.1).

It should be noted that Selberg (ca. 1950) examined also the use of the weights

$$\left( \sum_{d_1|n, d_2|n+2 \atop d_1d_2<D} \lambda(d_1, d_2) \right)^2,$$  \hspace{1cm} \text{(4.6)}

but in a configuration different to (4.4)–(4.5) that I briefly mentioned already in Section 2.
5. RH vs. statistics.
The assertion (4.5) is, in fact, a consequence of

The mean prime number theorem

For each $A > 0$ there exists a $\vartheta > 0$ (the level) such that

$$\sum_{q \leq x^\vartheta} \max_{(a,q)=1} \left| \pi(x; a, q) - \frac{\text{li}(x)}{\varphi(q)} \right| \ll x(\log x)^{-A},$$

where $\pi(x; a, q)$ is the number of primes $\leq x$ congruent to $a$ modulo $q$.

6. Powerful modesty.
However, with the best effort one could achieve only $C_2 < C_1$ in (4.4)–(4.5). That is, the asymptotic value $(C_2 - C_1)x/(\log x)^2$ thus attained for (3.3) is negative and large, and so is of no more use to us than the nonsense (3.2). In fact, in order to truly appreciate (4.4)–(4.5) you ought to be well versed in the theory of the distribution of
primes in arithmetic progressions as well as in sieve method. Here, be simply amazed that despite its inability to yield anything about the conjecture (1.2) the assertion comes close to the dreamy asymptotic formula (1.1) at least outwardly, and moreover, there we have \( W(n) = 1 \) whenever \( n \) is a large twin prime. That is, twin primes are probably counted in (4.5) but only in an ineffective way; they must be buried in rubbish. Then, how to make (3.3) more effective and salvage primes proximate to each other?

That is very difficult. The accumulation of past futile attempts suggests that we ought not to be so daring as to confront (1.2) directly. The strategy GY (2004/5) chose was this: We should be modest. Let us give up trying to directly touch the ‘twin’. Let us consider instead

\[
\sum_{n<x} \left( \sum_{j=1}^{k} \nu(n + h_j) - 1 \right) W(n),
\]

with a new \( \{W(n)\} \). Here \( h_1 < h_2 < \cdots < h_k \) are even integers. They should not be trivial like \( \{2, 4, 6\} \) because one of \( n + 2, n + 4, n + 6 \) is always divisible by 3. A natural condition on the tuple \( \{h_1, h_2, \ldots, h_k\} \) is that

the number of different \( h_j \) mod \( p \) be less than \( p \) for any prime \( p \),

which avoids the redundancy that a member among \( \{n + h_j : j = 1, \ldots, k\} \) is always divisible by a fixed prime. Obviously,

\[
\sum_{j=1}^{k} \nu(n + h_j) - 1 > 0
\]

\( \Rightarrow \) \( \{n + h_1, n + h_2, \ldots, n + h_k\} \) contains at least two primes.

If this holds with infinitely many \( n \), then

\[
\liminf_{t \to \infty} (p_{t+1} - p_t) \leq h_k - h_1,
\]

with \( p_t \) the \( t \)-th prime. Bounded differences between primes should occur infinitely often. The establishment of this will be a tremendous achievement, even though it is perhaps less impressive than the ultimate assertion (1.2).

7. Gem box principle.
We have to choose the weights \( \{W(n)\} \) in (6.1). Here a truly decisive observation was made by GPY (2005): Let \( P(n) = (n + h_1)(n + h_2) \cdots (n + h_k) \). Then,

\[
\nu(P(n)) = k + \ell \text{ with } 0 \leq \ell < k \quad \Rightarrow \quad \text{there are at least } k - \ell \text{ primes among } n + h_1, \ldots, n + h_k.
\]

This is an application of Dirichlet’s pigeon box principle; but I very much prefer gems to pigeons. Here \( n \)’s are actually to be restricted so that (7.1) is valid, which can be realised in a simple way that does not cause any loss of generality.
8. Magical tapering.
The new parameter $\ell \geq 0$ is to be incorporated. In practice, however, it is hard to utilise (7.1) without making any compromise; that would be a return to the stiffness we wished to depart from. I am not very sure if this is what really occurred to them, but GPY seem to have turned to Selberg’s argument which I indicated in the first paragraph of Section 4. The relevant approach is to consider

$$\sum_{n < x} \left( \sum_{d \mid P(n)} \lambda(d) \right)^2, \quad \begin{cases} \lambda(1) = 1, \\ \lambda(d) = 0, \quad d > D. \end{cases}$$

The optimal $\lambda$ satisfies

$$\lambda(d) \sim \mu(d) \left( \frac{\log D/d}{\log D} \right)^k.$$  

Then, GPY practised real magic by introducing

the further tapering factor $\left( \frac{\log D/d}{\log D} \right)^\ell$,

and they constructed the weight

$$W(n) = \left( \sum_{d \mid P(n), d < D} \mu(d) \left( \frac{\log D/d}{\log D} \right)^{k+\ell} \right)^2.$$  

(8.1)

As a matter of fact, this is an approximation to the filtering concerning (7.1), since

$$\nu(P(n)) > k + \ell \implies \sum_{d \mid P(n)} \mu(d)(\log d)^j = 0, \quad j \leq k + \ell.$$  

9. Divine multiplier.
With $W(n)$ as in (8.1), GPY computed asymptotically the sums

$$T^{(1)}(x; k, \ell; D) = \sum_{n < x} W(n),$$

$$T^{(2)}(x; k, \ell; D) = \sum_{n < x} \left( \sum_{j=1}^k \varpi(n + h_j) \right)W(n).$$  

(9.1)

They discovered that, with $D = x^{\vartheta/2}$ ($\vartheta$ as in (5.1)) and a positive $\Delta(x) \approx x(\log x)^{-k}$, one has:

$$\left( T^{(2)}_P - T^{(1)}_P \right)(x; k, \ell; D) \sim \left( \vartheta \cdot \frac{k}{k + 2\ell + 1} \cdot \frac{2\ell + 1}{\ell + 1} - 1 \right) \Delta(x).$$  

(9.2)
This multiplier of $\Delta(x)$ is probably *one of the greatest surprises* in the entire history of number theory. Setting $\ell = \lfloor \sqrt{k} \rfloor$ for instance, we find readily that

$$\text{if } \vartheta > \frac{1}{2} \text{ and } k \text{ large, then } \{n + h_1, n + h_2, \ldots, n + h_k\}$$

contains at least two primes. $\implies$ Bounded differences between primes! \hfill (9.3)

If you had not the extra parameter $\ell$; that is, if you put $\ell = 0$, then (9.2) would be nothing. Without $\vartheta > 1$, which is truly beyond any science fiction, nothing would come out from (9.2) with $\ell = 0$. In fact it is known that (5.1) does not hold for any $\vartheta > 1$.

10. Divide and conquer.

The assertion (9.3) is indeed wonderful, if only one can leap beyond the barrier $\vartheta = \frac{1}{2}$ in (5.1).

Let me be a little bit personal: I may count myself as one of the earliest people who tried seriously to make this leap, of course without any surmise of recent developments. I was aware at least that not the large sieve but the dispersion method of Linnik is the key. But I could publish only a short report (1976) which relied still on the large sieve; my work relevant to the dispersion method was utterly incomplete, which was inevitable because of my meagre experience with the theory of exponential sums à la A. Weil. Later BFI (Bombieri, J. B. Friedlander and H. Iwaniec (1986)) made a remarkable progress in this direction. Their main result is valid with any $\vartheta < \frac{4}{7}$, but under a restriction on the moduli of the arithmetic progressions which makes it inadequate for the computation of the second sum in (9.1).

Thus a genuinely new insight was needed into the problem (6.1) and the barrier problem. In this situation an idea occurred to MP (2005) (see [11][14] as well); actually we each independently had essentially the same idea, which involved the use of some corner-cutting in order to break the stalemate. On my side: soon after getting the first version of GPY (from G in early April 2005) I realised that a *smoothing* could be applied to the summation variable $d$ in (8.1). That is, we need not sum over all $d < D$ but it suffices to restrict ourselves to those $d$ which have relatively small prime divisors only; I mean that even after applying such a corner-cutting the multiplier of $\Delta(x)$ in (9.2) does not change essentially, although $\Delta(x)$ itself ought to be altered accordingly.

Actually, MP (2005/6) modified the argument of GGPY (GPY and S. Graham (2005)) in order to incorporate this smoothing. Let me nevertheless employ an asymptotic expression for the sake of temporary convenience. Then, what MP did is the same as to replace (8.1) by

$$W(n) = \left( \sum_{d | P(n), d < D}^{(\omega)} \mu(d) \left( \frac{\log D/d}{\log D} \right)^{k+\ell} \right)^2, \hfill (10.1)$$

where $\sum^{(\omega)}$ indicates that all prime divisors of $d$ are less than $D^{\omega}$. Then the multiplier in (9.2), of course under the new setting, is found to be larger than

$$\vartheta_{MP} \cdot \frac{k}{k + 2\ell + 1} \cdot \frac{2\ell + 1}{\ell + 1} - 1 - \exp(-3k\omega/8), \hfill (10.2)$$
provided that one has, for any given \( A > 0 \),
\[
\sum_{q \leq x^{\vartheta_{\text{MP}}}}^{(\omega)} \sum_{\substack{(a,q) = 1 \\ P(a) \equiv 0 \mod q}} \left| \pi(x; a, q) - \frac{\text{li}(x)}{\varphi(q)} \right| \ll x (\log x)^{-A},
\]  
(10.3)

where \( \sum^{(\omega)} \) means that all prime factors of \( q \) are less than \( x^{\omega} \). Here I am not very precise, since MP tacitly assumed for the sake of convenience that \( \ell \approx \sqrt{k} \), \( \omega \approx 1/\sqrt{k} \) with \( k \) large; however, these assumptions are not of critical importance for the application in question, that is, to detect infinitely often bounded differences between primes. I remark also that the hypothetical mean prime number theorem which is required by MP is a consequence of (10.3); that is, MP assumed in fact somewhat less. Anyway we have:

\[
\vartheta_{\text{MP}} > \frac{1}{2} \text{ in (10.3)} 
\implies \text{bounded differences between primes occur infinitely often.} 
\]  
(10.4)

Why is this important? Because, with (10.3), instead of (5.1), the feasibility of a proof by the dispersion method of Linnik becomes much higher. More precisely, the smoothing yields a quasi-infinitely factorable structure in the moduli set \( \{q\} \); namely, we now have instead

\[
\{q_1 q_2 : q_1 \leq Q_1, q_2 \leq Q_2\},
\]

effectively for any multiplicative decomposition \( Q_1 Q_2 \leq x^{\vartheta_{\text{MP}}} \). In practice, we put the summation over \( q_1 \), say, outside and consider the dispersion of the inner sum over \( q_2 \), via the Cauchy inequality. We will be able to detect more cancellation than with the ordinary setting (5.1). Further, we may appeal to R.C. Vaughan’s reduction argument (1980), or the like, in dealing with the sums over primes. This strategy is nothing other than “divide et impera”.

11. From nowhere.

As to the proof of (10.3) for a \( \vartheta_{\text{MP}} > \frac{1}{2} \), I was somehow inclined to be optimistic; and I thought I would have ‘time’. Thus, in the mean time, I was playing with automorphic \( L \)-functions, enjoying some success, but for too long perhaps. Then, in early April last year I felt a jolt. The epicentre was an unknown mathematician named Y. Zhang; I mean that the man had not been known among specialists. Soon I got a copy of his paper (probably a draft). I felt as if I had seen it some 7 years ago, for its overall strategy was the same as that of MP(2005/6).

Of course I was truly impressed by the extremely important fact that Zhang cleared away the level barrier in the context of (10.3). The man who came from nowhere struck the target indeed. Therefore, mankind has now

\[
\liminf_{t \to \infty} (p_{t+1} - p_t) < \infty.
\]  
(11.1)

To achieve (10.3), for some \( \vartheta_{\text{MP}} > \frac{1}{2} \), Zhang appealed to P. Deligne’s famous work (1980) on the Weil conjecture; in this respect, he followed, to a large extent,
the work by BFI mentioned above. Thus I am unable to confirm his reasoning on my own but have to rely on the affirmative opinion of experts. I have no courage to exploit any result which I do not fully understand; neither have I any other way than to trust, with considerable caution, competent authors whose claims depend on works which are far beyond my expertise. Nevertheless, here I may try to explain why such heavy machinery comes into play in dealing with (10.3). In essence, it is because of the factoring of various terms and summation intervals, which is described in the previous section. I mean that the strategy there reduces the problem into pieces, all of which are more or less equivalent to counting integers in various arithmetic progressions. To manage this entangled task, presently we have essentially only one means: the Poisson summation formula. Main terms are not troublesome, though often complicated. Real trouble comes naturally from the tail parts, which are expressed in terms of finite or infinite exponential sums. Arguments of the exponentiated terms involve rational numbers with varying numerators and denominators; then Deligne’s work becomes relevant, as it gives strong and uniform bounds for such sums.

12. Phase transition.
Another sensation came more recently from a postdoc: J. Maynard (November 2013), claiming
\[
\liminf_{t \to \infty} (p_{t+1} - p_t) \leq 600. \tag{12.1}
\]
What is really sensational is in his statement that his argument does not incorporate any of the technology used by Zhang; the proof is essentially elementary, relying only on the Bombieri–Vinogradov theorem, i.e., (5.2). This is a true phase transition, and a great gift to all who feel uneasiness when they have to chew works that depend on the highly demanding work of Deligne and A. Weil (1949), even though the efforts of S.A. Stepanov (since 1969) have yielded accessible elementary proofs of some of the consequences of their work.

And more. According to Maynard, Tao (October 2013) got essentially the same idea; and they independently established, only on Rényi’s (5.1),

For each \( m \geq 2 \) there exists a \( k \) such that
\[
\text{with any } \{h_j\} \text{ satisfying (6.2)}
\]
\[
\text{the tuple } \{n + h_1, n + h_2, \ldots, n + h_k\}
\]
contains at least \( m \) primes for infinitely many \( n \). \tag{12.2}

They even got an estimate for \( k \) in terms of \( m \). Fantastic!

Their argument is, to some extent, a realisation as well as an extension of Selberg’s approach (4.6). Hence, in a sense, (12.1) would have been possible to attain in 1965 when (5.2) was established; and (12.2) in 1950! By this I mean that for more than half a century, indeed until a few months ago, no sieve experts had ever tried to seriously look into the ending remark (on p.245) in Selberg’s ‘Lectures on sieves’. I should of course add that the phase transition brought about by Maynard and Tao was an outcome of the sieve movement commenced by Goldston and Yildirim in 1999, without which I suspect that not only Maynard–Tao’s discovery but also the recent wonders concerning bounded differences between primes would have remained under sand, and perhaps would have lain undiscovered for decades to come. Better ideas
always survive; what I described in the last two sections may appear obsolete, at least for now.

The key points of Maynard’s argument are as follows: Basically we are dealing with the quadratic form

$$\sum_{n<x} \left( \sum_{d_j | (n+h_j), \forall j \leq k} \lambda(d_1, d_2, \ldots, d_k) \right)^2, \quad d_1 d_2 \cdots d_k \leq D.$$  \hspace{1cm} (12.3)

We need to be cautious in dealing with the prime factors of $d_j$; but let us ignore this presently: a correct procedure is indicated in Appendix below. Then, in a fashion familiar to those who are experienced in dealing with sums of arithmetical functions in sieve method, an application of Selberg’s change of variables (in fact, an instance of the Möbius inversion) allows one to express $\lambda$’s in terms of any given $F(\xi_1, \xi_2, \ldots, \xi_k)$ as far as $F$ is supported on $\{\xi_1 + \xi_2 + \cdots + \xi_k \leq 1 : \xi_j \geq 0, \forall j \leq k\}$. This is in fact an extension of the argument due to GGPY (2005); their choice corresponds to the specialisation $F(\xi) = f(\xi_1 + \xi_2 + \cdots + \xi_k)$. We let $W(n)$ stand for the squares in (12.3) with such $\lambda$’s, and engage in the evaluation of

$$\sum_{n<x} \left( \sum_{j=1}^k \varpi(n+h_j) - \rho \right) W(n),$$  \hspace{1cm} (12.4)

which is an obvious analogue of (6.1); the parameter $\rho$ is to be fixed later. Actually we need to apply pre-sifting to $n$’s as indicated in (A.3) below, which is not of absolute necessity but for the sake of technical comfort in dealing with $d$’s coming from (12.3). In this way, with $\vartheta$ as in (5.1), we find that the appropriate analogue of the multiplier of $\Delta(x)$ in (9.2) is:

$$\frac{\vartheta}{2} \sum_{j=1}^k J_k^{(j)}(F) - \rho I_k(F),$$  \hspace{1cm} (12.5)

where

$$I_k(F) = \int_0^1 \cdots \int_0^1 F(\xi_1, \xi_2, \ldots, \xi_k)^2 d\xi_1 \cdots d\xi_k,$$

$$J_k^{(j)}(F) = \int_0^1 \cdots \int_0^1 \left( \int_0^1 F(\xi_1, \xi_2, \ldots, \xi_k) d\xi_j \right)^2 d\xi_1 \cdots d\xi_{j-1} d\xi_{j+1} \cdots d\xi_k.$$

If we put $\rho = 1$ and $F(\xi) = (1 - \xi_1 - \cdots - \xi_k)^{\ell}$, then we recover (9.2) due to GPY (2005).

We are naturally interested in the variation problem

$$M_k = \sup_F \frac{\sum_{j=1}^k J_k^{(j)}(F)}{I_k(F)},$$
where the supremum is over functions $F$ that are piece-wise differentiable in the
domain indicated above and such that $I_k(F) \neq 0$, $J_k^{(j)}(F) \neq 0$ for each $j \leq k$. Let

$$\rho = m - 1, \quad m = \inf \{ r \in \mathbb{N} : r \geq \vartheta M_k/2 \}.$$

Then one finds that there are at least $m$ primes in $\{ n + h_1, n + h_2, \ldots, n + h_k \}$ for
infinitely many $n$’s. With a delicate optimisation, Maynard has found

$$M_{105} > 4.002,$$

which together with (5.2) implies (12.1) as there exists $\{ h_1, h_2, \ldots, h_{105} \}$ such that

$$h_{105} - h_1 = 600.$$ More strikingly, he has shown via a simple choice of $F$ that for
sufficiently large $k$

$$M_k > \log k - 2 \log \log k - 2.$$ This implies (12.2).

I repeat: Rényi established his prime number theorem (5.1) in 1948 and the argu-
ment of Manynard and Tao has its root in Selberg’s work of 1950. Thus, more than
60 years ago when I entered elementary school, the notion that bounded differences
between primes occur infinitely often could easily have already belonged to common
knowledge.

**Appendix.** As an induction for students who intend to study Maynard's wo-
 rk, I shall provide details of his arithmetic manipulations in the case $k = 2$, which is
 enough typical so that one may readily infer that the general case is to be settled as is
 shown in (12.5). As to Tao’s argument, the difference is only in the way of computing
 asymptotically the main terms which arise after sieving. He employed Fourier analysis
 in place of the usual method of summing arithmetic functions which Maynard used;
 see Tao’s polymath8 blog, the address of which is given in the references below.

We assume that $N$ tends to infinity, and we put

$$Y = \log \log N, \quad Z = \prod_{p \leq Y} p. \quad (A.1)$$

The rôle of $Y$ or rather that of $Z$ is important, as it makes the co-primality requirement
in various sums easy to attain and also yields crucial truncations after the change of
variables in the mode of Selberg; for the latter, see (A.10), for instance. The prime
number theorem implies $Z \ll (\log N)^2$, which can be regarded to be negligibly small
in our discussion. We choose $c_0 \mod Z$ to satisfy $(Z, (c_0 + h_1)(c_0 + h_2)) = 1$, which
is possible whenever $\{ h_1, h_2 \}$ satisfies the case $k = 2$ of (6.2). We shall work on the
assumption:

$$\lambda(u, v) = 0 \text{ if any of the following holds}$$

$$uv > D, \quad |\mu(uv)| = 0, \quad (uv, Z) > 1. \quad (A.2)$$

With this, we shall consider

$$\sum_{\substack{N \leq n < 2N \mod Z \equiv c_0 \mod Z \quad d_1 \mid (n + h_1), d_2 \mid (n + h_2) \quad \lambda(d_1, d_2) \quad 2 \quad (A.3)$$

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Because of the choice of \( c_0 \) and since \( N \) is large, we have always \( (n + h_1, n + h_2) = 1 \) and thus \( (d_1, d_2) = 1 \) in (A.3), conforming with (A.2). We shall exploit this fact in the sequel without mention.

Expanding the squares and changing the order of summation, we see that the sum equals

\[
(N/Z)S_0 + O(\lambda_{\max}^2 (D \log D)^2),
\]

where \( \lambda_{\max} = \sup |\lambda(u, v)| \) and

\[
S_0 = \sum_{d_1, f_1, d_2, f_2 \atop (d_1, f_1) = (d_2, f_2) = 1} \frac{\lambda(d_1, d_2)\lambda(f_1, f_2)}{[d_1, f_1][d_2, f_2]}.
\]

Because of (A.2), the condition \( (d_1 f_1, d_2 f_2) = 1 \) is equivalent to \( (d_1, f_2)(d_2, f_1) = 1 \). Then we have

\[
S_0 = \sum_{u_1, u_2} \varphi(u_1)\varphi(u_2) \sum_{d_1, f_1, d_2, f_2 \atop (d_1, f_1) = (d_2, f_2) = 1} \frac{\lambda(d_1, d_2)\lambda(f_1, f_2)}{d_1 d_2 f_1 f_2} \sum_{v_1, v_2 \atop (v_1, v_2) = 1} \mu(v_1)\mu(v_2)
\]

\[
= \sum_{u_1, u_2, v_1, v_2} \varphi(u_1)\varphi(u_2) \sum_{d_1, f_1, d_2, f_2 \atop (d_1, f_1) = (d_2, f_2) = 1} \frac{\lambda(d_1, d_2)\lambda(f_1, f_2)}{d_1 d_2 f_1 f_2} \sum_{v_1, v_2 \atop (v_1, v_2) = 1} \mu(v_1)\mu(v_2)
\]

\[
= \sum_{u_1, u_2, v_1, v_2} \varphi(u_1)\varphi(u_2) \mu(v_1)\mu(v_2) \sum_{d_1, f_1, d_2, f_2 \atop (d_1, f_1) = (d_2, f_2) = 1} \frac{\lambda(d_1, d_2)\lambda(f_1, f_2)}{d_1 d_2 f_1 f_2}
\]

\[
Hence, we put
\]

\[
\eta(w_1, w_2) = \mu(w_1)\mu(w_2)\varphi(w_1)\varphi(w_2) \sum_{d_1, d_2 \atop w_1|d_1, w_2|d_2} \frac{\lambda(d_1, d_2)}{d_1 d_2},
\]

and have

\[
S_0 = \sum_{u_1, u_2, v_1, v_2 \atop (u_1 u_2 v_1 v_2, Z) = 1} \mu^2(u_1 u_2 v_1 v_2) \frac{\eta(u_1 v_1, u_2 v_2)\eta(u_1 v_2, u_2 v_1)}{\varphi(u_1)\varphi(u_2)} \cdot \frac{\mu(v_1)\mu(v_2)}{(\varphi(v_1)\varphi(v_2))^2}. \]
Applying the M"obius inversion formula to \((A.7)\), we have
\[
\lambda(d_1, d_2) = \mu(d_1)\mu(d_2)d_1d_2 \sum_{\substack{w_1, w_2 \in \mathbb{Z} \backslash \{0\} \cap [1, D] \atop d_1 | w_1, d_2 | w_2}} \mu^2(w_1w_2) \frac{\eta(w_1, w_2)}{\varphi(w_1)\varphi(w_2)}, \tag{A.9}
\]

The condition \((A.2)\) is readily seen to be well satisfied with any \(\eta(u, v)\) as far as it vanishes for \(uv > D\). Namely, under this specification of \(\eta\) one may regard \((A.9)\) as the definition of \(\lambda\)'s, as we shall do in the sequel. Then, \((A.8)\) implies that
\[
S_0 = \sum_{u_1, u_2} \mu^2(u_1u_2) \frac{\eta^2(u_1, u_2)}{\varphi(u_1)\varphi(u_2)} + O(\eta_{\text{max}}^2 (\log D)^2 / Y), \tag{A.10}
\]
since we have
\[
\sum_{u \leq D} \frac{1}{\varphi(u)} \ll \log D, \quad \sum_{v > 1, (v,Z)=1} \frac{1}{\varphi(v)^2} \ll Y^{-1}. \tag{A.11}
\]

Next, we shall consider
\[
\sum_{N \leq n < 2N, n \equiv c_0 \mod Z} \varpi(n + h_1) \left( \sum_{d_1 | (n+h_1), d_2 | (n+h_2)} \lambda(d_1, d_2) \right)^2. \tag{A.12}
\]

It makes no difference if the condition \(d_1 | (n + h_1)\) is replaced by the apparently stronger condition \(d_1 = 1\), and so we see that \((A.12)\) equals
\[
\frac{1}{\varphi(Z)} (\text{li}(2N) - \text{li}(N)) S_1 + O(\lambda_{\text{max}}^2 E_3(2N, D^2 Z)), \tag{A.13}
\]
where
\[
S_1 = \sum_{d, f} \frac{\lambda(1, d)\lambda(1, f)}{\varphi([d, f])} \tag{A.14}
\]
and
\[
E_l(x, Q) = \sum_{q \leq Q} \tau_l(q) \max_{(a,q)=1} \left| \pi(x; a, q) - \frac{\text{li}(x)}{\varphi(q)} \right|. \tag{A.15}
\]

Here \(\tau_l(q)\) is the number of ways expressing \(q\) as a product of \(l\) factors; in fact, the number of representations of \(q\) as the least common multiple of two integers is bounded by \(\tau_3(q)\). Using the relation
\[
\frac{\varphi(d)\varphi(f)}{\varphi([d, f])} = \sum_{u | (d, f)} \gamma(u), \quad \gamma(u) = \prod_{p | u} (p - 2) \tag{A.16}
\]
we have

\[ S_1 = \sum_u \gamma(u) \left( \sum_{u|d} \frac{\lambda(1,d)}{\varphi(d)} \right)^2. \]  \hfill (A.17)

Imitating (A.7), we put

\[ \eta_1(u) = \mu(u) \gamma(u) \sum_{u|d} \frac{\lambda(1,d)}{\varphi(d)}, \]  \hfill (A.18)

so that

\[ S_1 = \sum_u \frac{\eta_1^2(u)}{\gamma(u)}. \]  \hfill (A.19)

Inserting (A.9) into (A.18), we have, after an arrangement,

\[ \eta_1(u) = u \gamma(u) \mu(u) \sum_{(w_1,w_2,Z)=1} \mu^2(w_1 w_2) \frac{\eta(w_1,w_2) \mu(w_2)}{\varphi(w_1) \varphi^2(w_2)} \]

\[ = \frac{u \gamma(u)}{\varphi^2(u)} \sum_{(w_1 u, Z)=1} \mu^2(w_1 u) \frac{\eta(w_1,u)}{\varphi(w_1)} + O(\eta_{\text{max}}(\log D)/Y). \]  \hfill (A.20)

This error term is due to the fact that if \( w_2 \neq u \), then \( w_2/u > Y \). Further, we have

\[ \eta_1(u) = \sum_{(w_1 u, Z)=1} \mu^2(w_1 u) \frac{\eta(w_1,u)}{\varphi(w_1)} + O(\eta_{\text{max}}(\log D)/Y), \]  \hfill (A.21)

since

\[ \frac{u \gamma(u)}{\varphi^2(u)} = \prod_{p|u} \left( 1 - \frac{1}{(p-1)^2} \right) = 1 + O(1/Y), \quad u > 1. \]  \hfill (A.22)

With this, we put

\[ \eta(d_1, d_2) = F \left( \frac{\log d_1}{\log D}, \frac{\log d_2}{\log D} \right), \]  \hfill (A.23)

where \( F \) is as in the last section but with \( k = 2 \). Collecting (A.10), (A.19) and (A.21), we find that we need to evaluate asymptotically the sums

\[ \sum_{w_1, w_2 (w_1 w_2, Z)=1} \frac{\mu^2(w_1 w_2)}{\varphi(w_1) \varphi(w_2)} F \left( \frac{\log w_1}{\log D}, \frac{\log w_2}{\log D} \right)^2, \]  \hfill (A.24)

\[ \sum_u \frac{1}{\gamma(u)} \left( \sum_{w_1 (w_1, Z)=1} \frac{\mu^2(w_1 u)}{\varphi(w_1)} F \left( \frac{\log w_1}{\log D}, \frac{\log u}{\log D} \right) \right)^2. \]
Here one may replace $\mu^2(u_1 u_2)$ by $\mu^2(u_1)\mu^2(u_2)$ and do the same with the factor $\mu^2(w_1 u)$, since $\mu(u_1 u_2) = 0$, for instance, implies that $u_1$ and $u_2$ are divisible by a $u > Y$ and such terms can be discarded in much the same way as is done in (A.10). Thus, the computation can be performed in a fashion quite familiar in the theory of sums of arithmetic functions weighted with smooth functions; in essence it is an application of summation/integration by parts. We may skip the details and show only the end result: The last two sums are asymptotically equal to

$$
(\log D)^2 \left( \frac{\varphi(Z)}{Z} \right)^2 \int_0^1 \int_0^1 F^2(\xi_1, \xi_2) d\xi_1 d\xi_2,
$$

$$
(\log D)^3 \left( \frac{\varphi(Z)}{Z} \right)^3 \int_0^1 \left( \int_0^1 F(\xi_1, \xi_2) d\xi_1 \right)^2 d\xi_2,
$$

respectively, as $D$ tends to infinity.

Now, we choose $D = N^{\vartheta/2 - \varepsilon}$, with $\vartheta$ as in (5.1). Then, the assertions (A.4) and (A.13) yield the multiplier

$$
\frac{\vartheta}{2} \left[ \int_0^1 \left( \int_0^1 F^2(\xi_1, \xi_2) d\xi_1 \right)^2 d\xi_2 + \int_0^1 \left( \int_0^1 F(\xi_1, \xi_2) d\xi_1 \right)^2 d\xi_2 \right]
$$

$$
- \rho \int_0^1 \int_0^1 F^2(\xi_1, \xi_2) d\xi_1 d\xi_2
$$

for the sum

$$
\sum_{\substack{N \leq n < 2N \\ n \equiv c_0 \mod Z}} (\varpi(n + h_1) + \varpi(n + h_2) - \rho) W(n), \quad (A.27)
$$

where $W(n)$'s stand for the squares in (A.3) with $\lambda$'s as in (A.9) along with (A.23). We may skip the estimation of the error terms coming from (A.10) and (A.21) as they should not cause any difficulty. As to the error term in (A.13), we need to eliminate the factor $\tau_3(q)$ in (A.15). This can be achieved via an application of the Cauchy inequality; that is,

$$
E_l^2(x, Q) \ll x (\log Q)^l E_1(x, Q). \quad (A.28)
$$

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