Lowest Landau level broadened by a Gaussian random potential with an arbitrary correlation length: An efficient continued-fraction approach

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For an electron in the plane subjected to a perpendicular constant magnetic field and a homogeneous Gaussian random potential with a Gaussian covariance function we approximate the averaged density of states restricted to the lowest Landau level. To this end, we extrapolate the first 9 coefficients of the underlying continued fraction consistently with the coefficients’ high-order asymptotics. We thus achieve the first reliable extension of Wegner’s exact result [Z. Phys. B 51, 279 (1983)] for the delta-correlated case to the physically more relevant case of a non-zero correlation length.

Nearly ideal two-dimensional electronic structures have attracted great attention for more than a decade not only because of their varied and important applications, but also because of the discovery of the quantum Hall effect [1]. For a microscopic understanding of the occurring phenomena it is essential to know the spectral properties of electrons confined to two dimensions under the influence of a perpendicular constant magnetic field taking into account the presence of disorder.

A commonly studied minimal model is that of non-interacting electrons which is characterized by the one-electron Hamiltonian given by the Schrödinger operator

$$\hat{H} := \hat{K} + \hat{V}$$

(1)

$$\hat{K} := \frac{1}{2m} \left( \frac{\hbar}{i} \frac{\partial}{\partial x_1} - \frac{eB}{2} x_2 \right)^2 + \frac{1}{2m} \left( \frac{\hbar}{i} \frac{\partial}{\partial x_2} + \frac{eB}{2} x_1 \right)^2.$$  

(2)

Here $x := (x_1, x_2)$ are Cartesian coordinates of the Euclidean plane $\mathbb{R}^2$, $\hbar$ is Planck’s constant, $e$ is the elementary charge, $m$ is the (effective) mass of the (spinless) electron, and $B > 0$ the strength of a perpendicular magnetic field. The static random potential $V$ is added to mimic the interaction with quenched disorder. Its probability distribution is assumed to be Gaussian with zero mean and Gaussian covariance, that is

$$\overline{V(x)} = 0, \quad \overline{V(x)V(x')} = \sigma^2 \exp \left[ -\frac{(x-x')^2}{2\lambda^2} \right].$$  

(3)

The overbar denotes averaging with respect to the probability distribution. Finally, $\sigma > 0$ is the strength and $\lambda > 0$ the correlation length of the fluctuations of the potential.

The spectral resolution of the unperturbed Hamiltonian $\hat{K}$ reads $\hat{K} = \sum_{n=0}^{\infty} \varepsilon_n \hat{E}_n$, where the eigenvalue $\varepsilon_n := (2n+1)\frac{\hbar^2}{2m}$ is called the $n$-th Landau level and $\hat{E}_n$ denotes the corresponding eigenprojector. With increasing $B$ the magnetic length $l := \sqrt{\hbar/(eB)}$ decreases, while
the degeneracy \( \langle x|\hat{E}_n|x\rangle = (2\pi l^2)^{-1} \) (per area) of each Landau level and the distance between successive levels increase. Hence, for a fixed concentration of electrons, sufficiently high fields, and low temperatures it is reasonable to simplify the model by restricting its Hamiltonian \( \hat{H} \) to the (still infinite dimensional) eigenspace \( \hat{E}_0 L^2(\mathbb{R}^2) \) of the lowest Landau level. This has been done, for example, in the important work of Wegner \( \text{[5]} \), where the averaged restricted density of states (per area)\( \rho(\varepsilon) := \langle y|\hat{E}_0\delta(\varepsilon - \hat{E}_0)\hat{H}\hat{E}_0|y\rangle \) (4)

has been calculated exactly in the delta-correlated limit \( \lambda \downarrow 0, \sigma \to \infty, \lambda^2\sigma^2 = \text{const.} \)

The purpose of the present Letter is twofold. One goal is to present what we think is an accurate extension of Wegner’s result to arbitrary values of \( \sigma \) and \( \lambda \). Clearly, this is of physical interest because in the high-field limit the actual correlation length is no longer small in comparison with the magnetic length. The other goal is to demonstrate thereby that the power of the well-known continued-fraction approach to spectral densities of non-trivial quantum problems \( \text{[2,3]} \) can considerably be enhanced in cases, where one has an \textit{a-priori} information about their asymptotic high-frequency decay.

To get rid of physical dimensions we write \( \rho \) in the standardized form

\[
\rho(\varepsilon) =: \frac{\sqrt{1 + (l/\lambda)^2}}{2\pi l^2\sigma} W\left(\frac{\lambda^2}{l^2}, \frac{(\varepsilon - \varepsilon_0)}{\sigma}, \sqrt{1 + (l/\lambda)^2}\right).
\]

In this way we have introduced a one-parameter family of even probability densities on the real line \( \mathbb{R} \) with normalized second moment \( W(a, u) = W(a, -u) \geq 0 \)

\[
\int_{\mathbb{R}} da W(a, u) = 1 = \int_{\mathbb{R}} du W(a, u) u^2.
\]

Here we have used Eq. (7) of \( \text{[4]} \) to normalize the second moment.

In this notation Wegner’s result for the delta-correlated limit reads \( \text{[5,6]} \)

\[
W(0, u) = \frac{2\pi^{-3/2}\exp(u^2)}{1 + [2\pi^{-1/2} \int_0^u d\xi \exp(\xi^2)]^2}.
\]

For the other extreme of the spatial extent of correlations, namely the constantly correlated case \( \lambda = \infty \), one simply has \( \text{[7]} \) the Gaussian

\[
W(\infty, u) = (2\pi)^{-1/2}\exp(-u^2/2).
\]

For intermediate values of \( a \) no exact expression for \( W(a, u) \) is known. What is exactly known, however, is
the fact that $W(a, u)$ falls off for sufficiently large $|u|$ like a Gaussian. More precisely, by Eqs. (17) and (15) of [4] (see also [5])

$$\lim_{u \to \pm \infty} \frac{1}{u^2} \ln(W(a, u)) = -\frac{a + 2}{2a + 2}. \quad (10)$$

In the sequel we will design approximations to the Stieltjes transform

$$R(a, z) := \int_{\mathbb{R}} du \frac{W(a, u)}{z - iu}, \quad \text{Re} \, z > 0, \quad (11)$$

of the standardized density of states $W$ which in turn will yield approximations to $W$ by means of the inversion formula

$$W(a, u) = \frac{1}{\pi} \lim_{v \downarrow 0} \text{Re} \left[ R(a, v + iu) \right]. \quad (12)$$

According to Stieltjes’ classical theory, see for example [9, 10], $R$ can be expanded into a Jacobi-type continued fraction

$$R(a, z) = \sum_{j=1}^{\infty} \frac{r_j(a)}{z}, \quad r_j(a) \geq 0. \quad (13)$$

Here we are using the notation

$$\sum_{j=1}^{\infty} \left( \frac{\Delta_j}{z} \right) = \frac{1}{z} \sum_{j=1}^{\infty} \frac{\Delta_1}{\Delta_2} \cdots \frac{\Delta_j}{z} =: \sum_{j=1}^{\infty} \Delta_j \left( \frac{\Delta_j}{z} \right), \quad (14)$$

for the continued fraction with coefficients $\Delta_1, \Delta_2, \ldots$ and variable $z$.

To derive the continued-fraction coefficients $\{r_j(0)\}$ and $\{r_j(\infty)\}$ corresponding to [5] and [8], respectively, we use the identity

$$\sum_{j=1}^{\infty} \frac{\beta + \gamma j}{z} = \frac{D_{-\beta/\gamma}}{\mu + z}, \quad (15)$$

valid if $\gamma > 0$, $\beta + \gamma > 0$, and $\text{Re} \, z > 0$. Here $D_\nu$ denotes Whittaker’s parabolic cylinder function with index $\nu$, see Section 8.1 of [11]. The identity follows by iteration from the observation that the rhs of (15) obeys

$$T(\beta, \gamma, z) = [z + (\beta + \gamma)T(\beta + \gamma, \gamma, z)]^{-1}. \quad (16)$$

which itself is a consequence of the first of the four recurrence relations for the $D_\nu$’s in Section 8.1.3 of [11]. In fact, Eq. (15) is equivalent to Eq. (14) in §50 of [8]. With the help of (12) one now checks that

$$r_j(0) = \frac{1}{2} + \frac{1}{2} \cdot j, \quad r_j(\infty) = j, \quad (17)$$

for all $j \geq 1$. 

According to [12], the asymptotic behavior (10) implies the following asymptotic linear growth for the continued-fraction coefficients

$$\lim_{j \to \infty} \frac{r_j(a)}{j} = \frac{a+1}{a+2}. \quad (18)$$

It is natural to match this linear growth to the first $J < \infty$ coefficients $r_1(a), \ldots, r_J(a)$ to construct the announced approximations $R^{(J)}(a, z)$ to $R(a, z)$ by means of

$$R^{(J)}(a, z) := \sum_{j=1}^{\infty} \frac{r_j^{(J)}(a)}{z^j}. \quad (19)$$

where

$$r_j^{(J)}(a) := \begin{cases} r_j(a) & \text{for } j \leq J \\ r_J(a) + \frac{a+1}{a+2}(j-J) & \text{for } j > J. \end{cases} \quad (20)$$

Not surprisingly, the approximation $R^{(J)}$ to $R$ results in an approximation $W^{(J)}$ to $W$ with the same Gaussian fall-off (10) in the tails. This can be seen as follows. By virtue of (11), the approximation $R^{(J)}$ can be expressed as a terminating continued fraction

$$R^{(J)}(a, z) = \frac{1}{z + \frac{r_1(a)}{z + \cdots + \frac{r_{J-1}(a)}{z + r_J(a)T(r_J(a), \frac{a+1}{a+2}, z)}}}. \quad (21)$$

In view of (12) it is therefore sufficient to show

$$\lim_{u \to \pm\infty} \frac{1}{u^2} \ln(\text{Re}[T(\beta, \gamma, iu)]) = -\frac{1}{2\gamma} \quad (22)$$

and

$$\lim_{u \to \pm\infty} \text{Im}[T(\beta, \gamma, iu)] = 0, \quad (23)$$

because these asymptotic properties are conserved under the (repeated) substitution

$$T(\beta, \gamma, iu) \mapsto [iu + \alpha T(\beta, \gamma, iu)]^{-1} \quad (24)$$

for any $\alpha > 0$. The validity of (22) and (23) can be deduced from an asymptotic evaluation [13] of the following Riccati differential equation

$$\gamma \frac{d}{du}T(\beta, \gamma, iu) = i\beta \left[T(\beta, \gamma, iu)\right]^2 - uT(\beta, \gamma, iu) - i \quad (25)$$

which itself follows from the aforementioned recurrence relations. To summarize, Eq. (15) adds a two-parameter family of terminators for affine-linear extrapolations to the toolkit of [8].
In order to compute the first $J$ continued-fraction coefficients $r_1(a), \ldots, r_J(a)$, we employ the well-known one-to-one correspondence \cite{9,10,2} to the first $J$ even moments $M_2(a), \ldots, M_{2J}(a)$ of the standardized density of states.

According to \cite{4} and \cite{5} one has for the latter

\[
M_{2j}(\lambda^2/l^2) := \int_{\mathbb{R}} du \, W(\lambda^2/l^2, u) u^{2j}
= 2\pi l^2 \left( \frac{1 + l^2/\lambda^2}{\sigma^2} \right)^j \frac{1}{(y|\hat{E}_0 \hat{V} \hat{E}_0)^{2j}|y)} \tag{26}
\]

if $j \geq 1$. We now substitute $\hat{V} = \int_{\mathbb{R}^2} d^2x \, V(x)|x\rangle\langle x|$ into the rhs of \cite{24} and use the standard reduction formula

\[
\prod_{k=1}^{2j} V(x(k)) = \sum_{k=1}^{(2j-1)!!} V(x(P_k(2s-1)))V(x(P_k(2s))) \tag{27}
\]

for the average of a product of $2j$ jointly Gaussian random variables with zero mean. Here $x(1), \ldots, x(2j) \in \mathbb{R}^2$ are $2j$ points of the plane and $P_k$ denotes the $k$-th of the $(2j-1)!!$ permutations of the first $2j$ natural numbers which lead to different sets $\{P_k(1), P_k(2)\}, \ldots, \{P_k(2j-1), P_k(2j)\}$ of $j$ pairs $\{P_k(2s-1), P_k(2s)\}$. Since the covariance function \cite{3} of the random potential $V$ and the position representation $\langle x|\hat{E}_0|x'\rangle = \frac{1}{2\pi l^2} \exp \left[ \frac{i}{2l^2} (x_1 x_2' - x_2 x_1') - \frac{1}{4l^2} (x - x')^2 \right]$ of the eigenprojector corresponding to the lowest Landau level are both Gaussian, one ends up with a sum over $4j$-dimensional Gaussian integrals which can be performed to yield

\[
M_{2j}(a) = \left( 1 + \frac{1}{a} \right)^j \sum_{k=1}^{(2j-1)!!} \frac{1}{\det(A_k(a))}, \quad a > 0. \tag{29}
\]

Here the $2j \times 2j$-matrix $A_k(a)$ is defined through its entries in terms of the Kronecker delta

\[
(A_k(a))_{\mu,\nu} := \left( 1 + \frac{1}{a} \right) \delta_{\mu,\nu} - \delta_{\mu+1,\nu} - \frac{1}{a} \sum_{s=1}^{j} \left( \delta_{\mu,P_k(2s-1)} \delta_{\nu,P_k(2s)} + \delta_{\nu,P_k(2s-1)} \delta_{\mu,P_k(2s)} \right). \tag{30}
\]

We have computed the sum \cite{24} for $j = 1, 2, \ldots, 6$ using the Axiom symbolic system \cite{14} for general $a$. The results for $M_{2j}(a), j = 1, 2, 3, 4,$ are given by the following rational functions
\[ M_2(b-1) = 1 \]
\[ M_4(b-1) = (3b^2 + 2)/(b^2 + 1) \]
\[ M_6(b-1) = (15b^6 + 75b^4 + 102b^2 + 30)/(6b^4 + 6b^2 + 6) \]
\[ M_8(b-1) = (105b^{12} + 2385b^{10} + 32830b^8 + 213697b^6 + 861130b^4 + 2231807b^2 + 1680)/(b^{12} + 296b^{10} + 365b^8 + 2620b^6 + 11854b^4 + 35276b^2 + 69974b + 91906)/b^{12} + 32792b^2 + 1680 + 78025b^8 + 41015b^6 + 12461b^4 + 1954b^2 + 120). \]

(31)

Since the expressions for \( M_{10}(a) \) and \( M_{12}(a) \) are extremely lengthy, we only give, as an example, their values for \( a = 1 \):
\[ M_{10}(1) = 158,659,605,940,126,452,841 \]
\[ M_{12}(1) = 388,336,271,072,847,928,549,926,597,113,071,401,088,677,997,478,405,727,555,223,031 \]
\[ 82,963,680,790,286,536,914,041,227,299,941,666,364,919,341,122,215,602,750 \]

(32)

The moments \( M_{14}(a) \) and \( M_{16}(a) \) have been computed exactly as a reduced fraction for \( a = 1, 2, 4 \), only, by employing a C-program. For the same set of values for \( a \) we also have computed \( M_{18}(a) \), but only have been able to gather up the \( 17!! \approx 3.4 \cdot 10^7 \) terms occurring in (29) to a sum of approximately \( 10^4 \) reduced fractions, the value of which has been computed accurately by floating-point arithmetics. For lack of space we only show the first 35 digits for the example \( a = 1 \):
\[ M_{14}(1) = 47657.946072630475536554559639613945 \ldots \]
\[ M_{16}(1) = 545841.4359250174224295351679205436 \ldots \]
\[ M_{18}(1) = 6980770.3795705620571551127542699920 \ldots \]

From Fig. 1 it can be seen that the resulting continued-fraction coefficients \( r_1(a), \ldots, r_9(a) \) approach quite rapidly their asymptotic behavior given in (18). For example, the relative extrapolation error \( |r_9(a) - r_9(a)|/r_9(a) \) varies between 0.2% for \( a = 4 \) and 0.02% for \( a = 1/4 \).

In Fig. 2 we demonstrate the convergence of the resulting approximations \( W^{(J)}(1, u) \) to \( W(1, u) \) for increasing \( J = 1, 2, \ldots, 9 \). The differences of two successive approximations, \( W^{(J)}(a, u) - W^{(J-1)}(a, u) \), seem to form a nearly alternating sequence with geometric decrease for fixed \( u \). Both observations, taken together, suggest rapid pointwise convergence of the sequence \( \{W^{(J)}\} \). Therefore, we may safely conclude that \( W^{(9)} \) constitutes a reliable approximation to \( W \). Figure 3 shows a plot of the corresponding approximation \( \varrho^{(9)} \) to the averaged density of states \( \varrho \) for different \( \lambda \). Note that, by construction, \( \varrho^{(9)} \) is exact in the limiting cases of a delta-correlated and a constantly correlated random potential; more importantly, \( \varepsilon \to 2\pi F^2 \varrho^{(9)}(\varepsilon + \varepsilon_0) \) is an even probability density with the same first 18 moments and the same Gaussian fall-off in the tails as the true density for general values of the correlation length \( \lambda \).
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FIG. 1. Symbols represent the first 9 continued-fraction coefficients $r_1(a), \ldots, r_9(a)$ for different values of $a$. Note that $r_1(a) = 1$ for all $a$ by (7). The straight lines correspond to the asymptotics of $\{r_j^{(a)}(a)\}$ as defined in (20).
FIG. 2. Difference of two successive approximations to $W$ for $a = 1$ as a function of $u$ given on a linear (a) and a logarithmic (b) scale.

FIG. 3. The approximation $\rho^{(9)}$ to the averaged density of states $\rho$ for different values of the correlation length $\lambda$ as a function of the energy $\epsilon$. 