Unitarity of Neutral Kaon System

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Abstract

In neutral kaon system, we always use non-hermitian Hamiltonian for convenience of treating decay process, unitarity seems to be lost. If we take decay channels ($\pi\pi$, $\pi\pi\pi$, $\pi\ell\nu$ etc.) into account, however, Hamiltonian of the whole system must be hermitian. We attempt to derive an effective Hamiltonian with respect to only $K^0$, $\bar{K}^0$ states, starting from the hermitian Hamiltonian. For brevity, we take only a $\pi\pi$ state into account as the decay channel in this paper.

We can not avoid an oscillation between $K^0$, $\bar{K}^0$ and $\pi\pi$ states if we start from a hermitian Hamiltonian whose states all have discrete energy levels. We therefore treat the $\pi\pi$ state more appropriately to have a continuous energy spectrum to achieve the decay of $K^0$, $\bar{K}^0$ into $\pi\pi$.

As the consequence, we find a different time evolution from what we expect in the conventional method immediately after the decay starts, though it recovers Fermi’s golden rule for long enough time scale.

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1 Introduction

The neutral kaon system has long served as a probe of fundamental physics. Time evolution equation of neutral kaon system is written by non-hermitian Hamiltonian customarily, unitarity of the system seems to be lost. This is because we don’t take decay channels (ππ, πππ, πℓν ... etc.) of K mesons into account. Hamiltonian of the neutral kaon system must be hermitian with the decay channels being included into the base of this Hamiltonian. We attempt to derive the decay behavior of neutral kaon system effectively, starting from a hermitian Hamiltonian \[1\].

Recently, there are many discussions about CPT and quantum mechanics violation\[2\]. If these violations exist, they must be extremely small. In order to treat these extremely small quantities, we need to describe the time evolution of neutral kaon system as exactly as possible. Therefore, it seems to be important to clarify whether the conventional treatment by use of non-hermitian Hamiltonian gives exact result, relying on the exact treatment. In future, such clarification may affect analysis of CPT and quantum mechanics violation.

In Sec.2, for convenience, we derive time evolution equation of neutral kaon system with \( K^0 \), \( \bar{K}^0 \) and \( \pi\pi \) bases. Here we treat \( \pi\pi \) state as a state with continuous energy spectrum. Then we can avoid an oscillation between \( \pi\pi \) and K meson, and explain a decay phenomenon well.

In Sec.3, assuming CP invariance, we solve the time evolution equation concerning \( K_1 \) and \( \pi\pi \) states perturbatively, and consider the decay process of \( K_1 \).

In Sec.4, we examine the validity of making use of perturbation to solve time evolution equation. We introduce a simplified model where we can solve the equation non-perturbatively, and compare the result with the one obtained by our perturbative method.

Sec.5 is devoted to summary and discussion.

2 Derivation of Time Evolution Equation

For brevity, we consider a system with \( K^0 \), \( \bar{K}^0 \) and \( \pi\pi \) states. In a frame where the \( K^0 \), \( \bar{K}^0 \) are at rest, their 4-momenta are fixed, while \( \pi\pi \) state has an ambiguity of relative momentum \( \vec{k}, \pi(\vec{k})\pi(-\vec{k}) \). If we start from \( \pi\pi \) state as a discrete state, we
can not avoid an oscillation $\pi \pi \leftrightarrow K$. Thus $K^0$, $\bar{K}^0$ are treated as discrete states, while $\pi \pi$ state is a continuous state with a continuous parameter $\vec{k}$.

At arbitrary time $t$, a state $|\psi(t)\rangle_I$ is given as

$$|\psi(t)\rangle_I = C_{K^0}(t)|K^0\rangle + e^{iH_0 t} C_{\bar{K}^0}(t)|\bar{K}^0\rangle + \sqrt{\pi^2} \int d^3k |\pi(\vec{k})\rangle |\pi(-\vec{k})\rangle,$$

where index $I$ implies interaction representation.

We note that the momentum $\vec{P}$ should be preserved in the $K^0$, $\bar{K}^0 \to \pi \pi$ transition, since the interaction Hamiltonian of the form

$$H_{\text{int}}^I = \int d^3x H_{\text{int}}^I,$$

implies that the total momentum should vanish due to the factor $\delta(\vec{P}_K + \vec{P}_\pi + \vec{P}_\pi)$.

We thus have assigned $\vec{k}, -\vec{k}$ for the momentum of two $\pi$’s ($\vec{P}_K = 0$).

The normalization of continuous state is fixed as

$$\langle \pi(\vec{k})\pi(-\vec{k})|\pi(\vec{k}')\pi(-\vec{k}')\rangle = \delta^3(\vec{k} - \vec{k}').$$

Then, $\langle \psi(t)|\psi(t)\rangle_I = 1$ and Eq.(2.1), (2.4) gives

$$|C_{K^0}(t)|^2 + |C_{\bar{K}^0}(t)|^2 + \sqrt{\pi^2} \int d^3k |C_{\vec{k}}(t)|^2 = 1$$

Schrödinger equation is written as

$$i \frac{\partial}{\partial t} |\psi(t)\rangle_S = H |\psi(t)\rangle_S$$

$$H = H_0 + H_{\text{int}}.$$

Here, $H_0$ is free Hamiltonian and $H_{\text{int}}$ is interaction Hamiltonian due to weak interaction. Index $S$ implies Schrödinger representation.

It is related to the interaction picture by

$$|\psi(t)\rangle_I = e^{iH_0 t} |\psi(t)\rangle_S$$

$$i \frac{\partial}{\partial t} |\psi(t)\rangle_I = H_{\text{int}}^I |\psi(t)\rangle_I,$$

with the interaction Hamiltonian being defined by

$$H_{\text{int}}^I = e^{iH_0 t} H_{\text{int}} e^{-iH_0 t}.$$
A "matrix" form of time evolution equation is possible by inserting a complete set

$$1 = |K^0><K^0| + |ar{K}^0><ar{K}^0| + \int d^3k |\pi(k)\pi(-\bar{k})><\pi(k)\pi(-\bar{k})|,$$  \hspace{1cm} (2.10)

to get

$$i\frac{\partial}{\partial t} \begin{pmatrix} <K^0|\psi(t)>_I \\ <\bar{K}^0|\psi(t)>_I \\ <\pi|\psi(t)>_I \end{pmatrix} = \begin{pmatrix} <K^0|H^I_{int}|K^0> & <K^0|H^I_{int}|\bar{K}^0> & <K^0|H^I_{int}|\pi> \\ <\bar{K}^0|H^I_{int}|K^0> & <\bar{K}^0|H^I_{int}|\bar{K}^0> & <\bar{K}^0|H^I_{int}|\pi> \\ <\pi|H^I_{int}|K^0> & <\pi|H^I_{int}|\bar{K}^0> & <\pi|H^I_{int}|\pi> \end{pmatrix} \begin{pmatrix} C_K^0(t) \\ C_{\bar{K}}^0(t) \\ C_{\pi}(t) \end{pmatrix}.$$  \hspace{1cm} (2.11)

Eq.(2.1) and above gives

$$i\frac{\partial}{\partial t} \begin{pmatrix} C_K^0(t) \\ C_{\bar{K}}^0(t) \\ C_{\pi}(t) \end{pmatrix} = \begin{pmatrix} H^I_{K^0 K^0} \\ H^I_{K^0 \bar{K}^0} \\ H^I_{\pi\pi} \end{pmatrix} \begin{pmatrix} C_K^0(t) \\ C_{\bar{K}}^0(t) \\ C_{\pi}(t) \end{pmatrix}.$$  \hspace{1cm} (2.12)

where multiplication of matrices should be understood as

$$H^I_{K^0 K^0} C_{\bar{K}}(t) = \int d^3k H^I_{K^0 \bar{K}} C_{\bar{K}}(t).$$  \hspace{1cm} (2.13)

In Eq(2.12), the Hamiltonian in the interaction representation is related to the time independent Hamiltonian in the Schrödinger representation as

$$\begin{pmatrix} e^{iE_{K^0}t} & 0 & 0 \\ 0 & e^{iE_{\bar{K}^0}t} & 0 \\ 0 & 0 & e^{iE_{\pi}t} \end{pmatrix} \begin{pmatrix} H_{K^0 K^0} & H_{K^0 \bar{K}^0} & H_{K^0 \pi} \\ H_{\bar{K}^0 K^0} & H_{\bar{K}^0 \bar{K}^0} & H_{\bar{K}^0 \pi} \\ H_{\pi K^0} & H_{\pi \bar{K}^0} & H_{\pi \pi} \end{pmatrix} \begin{pmatrix} e^{-iE_{K^0}t} & 0 & 0 \\ 0 & e^{-iE_{\bar{K}^0}t} & 0 \\ 0 & 0 & e^{-iE_{\pi}t} \end{pmatrix} = \begin{pmatrix} H_{K^0 K^0} & H_{K^0 \bar{K}^0} & e^{i\Delta E_{K^0 \bar{K}^0}t} H_{K^0 \pi} \\ H_{\bar{K}^0 K^0} & H_{\bar{K}^0 \bar{K}^0} & e^{i\Delta E_{\bar{K}^0 \bar{K}^0}t} H_{\bar{K}^0 \pi} \\ e^{i\Delta E_{\pi K^0}t} H_{\pi K^0} & e^{i\Delta E_{\pi \bar{K}^0}t} H_{\pi \bar{K}^0} & e^{i\Delta E_{\pi \pi}t} H_{\pi \pi} \end{pmatrix}.$$  \hspace{1cm} (2.14)

where,

$$\Delta E_{\bar{K}^0 \bar{K}^0} = E_{\bar{K}^0} - E_{\bar{K}^0},$$  \hspace{1cm} (2.15)

and so on. CPT invariance implies

$$E_{K^0} = E_{\bar{K}^0} = M_K.$$  \hspace{1cm} (2.16)
3 Solving Time Evolution Equation Perturbatively

In order to see how the unitarity is effectively lost we simplify the situation by assuming CP invariance, and consider only \((K_1, \pi\pi)\) subsystem where \(K_1\) denotes CP even eigenstate, i.e \(K_2\) decouples from the system. We thus consider \(K_1 \rightarrow \pi\pi\) decay and solve time evolution equation by perturbative method. For convenience, we treat \(\pi\) as a massless particle. Refering to Eq.(2.14), we obtain

\[
(C_{K_1}(t) \\ C_{K'}(t)) = T e^{i \int_0^t \hat{H}(t')dt'} (C_{K_1}(0) \\ C_{K'}(0)),
\]

(3.1)

\[
\hat{H}(t) = \begin{pmatrix}
0 & e^{i(E_{K_1} - E_{K'})t} H_{K_1 K'} \\
e^{-i(E_{K_1} - E_{K'})t} H_{K_1 K'} & 0
\end{pmatrix},
\]

(3.2)

where \(T\) stands for time ordered product and we have taken an interaction representation where

\[
<K_1|H_{int}|K_1> = 0, \quad <\pi\pi|H_{int}|\pi\pi> = 0.
\]

(3.3)

Inserting Eq.(3.2) into Eq.(3.1) and expanding in powers of \(\hat{H}(t)\) to second order, we obtain

\[
\begin{pmatrix}
C_{K_1}(t) \\ C_{K'}(t)
\end{pmatrix} = \begin{pmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{pmatrix} \begin{pmatrix}
C_{K_1}(0) \\ C_{K'}(0)
\end{pmatrix},
\]

(3.4)

\[
H_{11} = 1 + \int d^3 k \frac{e^{i \Delta E_{K_1} k t} - 1 - i \Delta E_{K_1} k^t}{(\Delta E_{K_1 k})^2} |H_{K_1 k}|^2
\]

(3.5)

\[
H_{12} = \frac{e^{i \Delta E_{K_1} k t} - 1}{\Delta E_{K_1 k}} H_{K_1 k}
\]

(3.6)

\[
H_{21} = -\frac{e^{-i \Delta E_{K_1} k t} - 1}{\Delta E_{K_1 k}} H_{K_1 k}
\]

(3.7)

\[
H_{22} = 1 + \frac{1}{\Delta E_{K_1 k}} \left(\frac{e^{i \Delta E_{K_1} \bar{k} t} - 1}{\Delta E_{K_1 \bar{k}}} + \frac{e^{-i \Delta E_{K_1} \bar{k} t} - 1}{\Delta E_{K_1 \bar{k}}}\right) H_{\bar{k} K_1} H_{K_1 k}
\]

(3.8)

and take initial conditions into consideration,

\[
C_{K_1}(0) = 1, \quad C_{K'}(0) = 0.
\]

(3.9)

Then

\[
|C_{K_1}(t)|^2 = \{1 + 2 \int d^3 k \frac{\cos(E_{K_1} - E_{K'})t - 1}{(E_{K_1} - E_{K'})^2} |H_{K_1 k}|^2 + O(H^4)\} |C_{K_1}(0)|^2.
\]

(3.10)
Time derivative of Eq.(3.10) is
\[
\frac{\partial}{\partial t} |C_{K_1}(t)|^2 = \{-2 \int d^3k \frac{\sin(E_{K_1} - E_{\bar{k}})t}{E_{K_1} - E_{\bar{k}}} |H_{K_1\bar{k}}|^2 + O(H^4)} |C_{K_1}(t)|^2. \quad (3.11)
\]
For the assumed interaction in Eq.(2.2) and Eq.(2.3), \( |H_{K_1\bar{k}}|^2 \propto \frac{1}{\sqrt{E_{K_1}E_{\bar{k}}}} \). Thus,
\[
\frac{\partial}{\partial t} |C_{K_1}(t)|^2 \propto -\int_{M_{K_1}}^{\infty} dk \frac{\sin kt}{k} |C_{K_1}(t)|^2 \\
= -\{\pi - \text{si}(M_{K_1}t)} |C_{K_1}(t)|^2, \quad (3.12)
\]
where \( \text{si} \) is sine integral function and \( M_{K_1} \) denotes the \( K_1 \) mass. With \( t \to \infty \), Eq.(3.12) reduces to what we expect from Fermi’s golden rule,
\[
\frac{\partial}{\partial t} |C_{K_1}(t)|^2 \propto -\text{const} |C_{K_1}(t)|^2. \quad (3.13)
\]
This equation indicates a decay process, not an oscillation. We should, however, note that when time \( t \) is relatively small, our result shows a clear difference from the conventional result expected from the golden rule.

4 Comparison with a Non-perturbative Method

In this section, we would like show that our procedure relying on a perturbative method actually reproduces the exact result solved nonperturbatively for some simpified case. This indicates the validity of our method.

4.1 A Simplified Model Solved by Non-perturbative Method

In this subsection, we treat the following system in 1+1 dimension.

\[
H|K_1> = M_{K_1}|K_1> + v \int dk |k>, \quad (4.1)
\]
\[
H|k> = v^*|K_1> + k|k>, \quad (4.2)
\]
where \( k \) is assumed to take \(-\infty \) to \( \infty \), and \( v \) is assumed to be constant. We write eigenvalue and eigenvector of \( H \) as \( H|\omega> = \omega |\omega> \) and define \( |\omega> \) as following
\[
|\omega> = N_{\omega}\{|K_1> + \int dk f_\omega(k)|k>\}. \quad (4.3)
\]
Here, \( N_\omega \) is a normalization factor. From the above we obtain
\[
\omega = M_{K_1} + v^* \int dk f_\omega(k), \quad \omega f_\omega(k) = v + k f_\omega(k), \tag{4.4}
\]
which reads as
\[
f_\omega(k) = \frac{P}{\omega - k} + \frac{\omega - M_{K_1}}{v^*} \delta(\omega - k). \tag{4.5}
\]
Here, \( P \) denotes a principal value\[4\]. The normalization condition for continuous states,
\[
<\omega|\omega'> = \delta(\omega - \omega'), \quad <k|k'> = \delta(k - k'), \tag{4.6}
\]
fixes \( N_\omega \) as,
\[
|N_\omega|^2 = \frac{|v|^2}{\pi^2|v|^4 + (\omega - M_{K_1})^2}. \tag{4.7}
\]
At \( t = 0 \), the state is assumed to be pure \( K_1 \) state,
\[
|\psi(0) >= |K_1 >= \int d\omega |\omega > <\omega|K_1 >= \int d\omega N_\omega^*|\omega >. \tag{4.8}
\]
Time evolution of \( |\psi(t) > \) is uniquely fixed as
\[
|\psi(t) >= \int d\omega e^{-i\omega t} N_\omega^*|\omega >. \tag{4.9}
\]
The ”survival probability” of \( K_1 \) is then given as
\[
|<K_1|\psi(t)>|^2 = |\int d\omega e^{-i\omega t} N_\omega^*|\omega >|^2 = e^{-2\pi|v|^2t}. \tag{4.10}
\]
This expression means that a particle \( K_1 \) decays with \( \Gamma = 2\pi|v|^2 \), and corresponds to the Fermi’s golden rule. Namely, in this simplified model there is no deviation from the conventional result.

### 4.2 Comparison with our Perturbative Model

How about the result obtained by our perturbative method?. From Eq.(3.10), for 1+1 dimensional case we obtain (\(|H_{K_1}\tilde{k}|^2 = |v|^2\))
\[
|C_{K_1}(t)|^2 = \{1 + 2 \int dk \frac{\cos(M_{K_1} - k)t - 1}{(M_{K_1} - k)^2} |v|^2 + O(H^4)}|C_{K_1}(0)|^2. \tag{4.11}
\]
Eq.(4.11) becomes
\[
|C_{K_1}(t)|^2 = \{1 - 2|v|^2\pi t + O(|v|^4)}|C_{K_1}(0)|^2. \tag{4.12}
\]
and time differential is
\[
\frac{\partial}{\partial t}|C_{K_1}(t)|^2 \simeq -2\pi|v|^2|C_{K_1}(t)|^2.
\]
(4.13)

This answer reproduces the exact result Eq.(4.10), and we realize that once we go through differential equation our method actually reproduces the nonperturbative result correctly.

5 Summary and Discussion

In this paper, we have found a clear deviation from the result obtained by conventional Fermi’s golden rule for relatively small \( t \). But as is seen from Eq.(3.12) we can see this difference only for the time duration of order \( t \simeq \frac{1}{M_{K_1}} \). Therefore, it seems to be rather hard to observe such difference. When, however, CPT symmetry is tested, such small deviation might affect the physical quantities, since CPT violation, if any, should be very tiny.

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