Phase-Shift Design and Channel Modeling for Focused Beams in IRS-Assisted FSO Systems

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Abstract—As a potential solution for the backhaul of next-generation mobile or low Earth orbit (LEO) satellites, free-space optical (FSO) has received much attention recently. Various techniques have been suggested for employing an intelligent reflecting surface (IRS) in FSO systems, such as anomalous reflection, power amplification, and beam splitting. It is also possible to deliver more power to the receiver (Rx) by collimating or focusing the reflected beam at the Rx lens. In this paper, we propose a phase-shift design of an IRS for beam focusing. In addition, we propose a new pointing error model and an outage performance analysis for the case when the beam width is comparable to or less than the aperture size of the Rx. The analytical results are validated by Monte Carlo simulations. This paper provides essential preliminary results for future research that assumes a focused beam in FSO systems.

Index Terms—Intelligent reflecting surface (IRS), free-space optical (FSO) system, beam focusing, pointing error.

I. INTRODUCTION

Optical wireless systems, such as free-space optical (FSO) systems, are a potential technology for high data rate transmission with license-free operation over a wide range of bandwidth in the optical spectrum [1]. In contrast to radio frequency (RF), FSO systems are immune to electromagnetic interference and have been considered a cost-effective solution for terrestrial backhaul/fronthaul wireless applications for 5G and beyond 5G networks. Recently, the use of an intelligent reflecting surface (IRS) in FSO systems has attracted significant research interest [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]. An IRS is an electromagnetic device with electronically controllable characteristics. Various techniques can be used to manipulate the amplitude, phase, and polarization of the incoming signal using IRS. For example, a power-amplifying IRS module was proposed in [2]. The authors in [3] and [4] introduced power splitting to multiple receivers (Rx) using the IRS.

Anomalous reflection is a fundamental technique that freely adjusts the direction of the reflected beam by manipulating the phase of the incoming beam over the surface of the IRS. Through anomalous reflection using the IRS, we can alleviate the line-of-sight (LOS) requirements, which are essential for establishing FSO links. For the anomalous reflection of an IRS-assisted FSO system, [5] proposed a phase-shift design and pointing error investigation. Moreover, analytical expressions for the end-to-end channel model [6] and various performance metrics [7], [8], [9], [10], [11] were introduced in which both atmospheric turbulence and pointing errors were considered.

Furthermore, there is an increasing interest in techniques that not only control the direction of the beam, but also its size [5], [12]. Using the IRS, we can make the reflected beam to be focused on the Rx to deliver greater power and enhance communication security. Thus, the advantages of FSO systems can be further highlighted. However, research using the IRS to create a focused beam is still in its infancy; not just the IRS phase-shift designs, but research into the pointing error for the focused beam remains an open problem [12]. It is essential to analyze the effect of pointing errors because the focused beam can be more sensitive to pointing errors.

In this paper, we first propose a phase-shift design of the IRS that can freely adjust the beam width as well as the direction of the beam. Then, we present a pointing error model applicable to the focused beam. It should be noted that the conventional Gaussian form pointing error model can be applied only if the beam width is more than twice the diameter of the Rx aperture, and not if the beam is focused smaller [13]. However, the newly proposed pointing error model is applicable to the case when the beam width is similar to, or smaller than, the diameter of the Rx lens. Furthermore, using the proposed pointing error model, the outage probability of the system was investigated when the effects of the pointing error and atmospheric loss were simultaneously considered. Then, the proposed phase-shift design and pointing error model were verified via numerical results.

Finally, we discuss how the outage performance is affected by the beam width, atmospheric loss, transmitted power, and pointing error. These factors are intricately intertwined, making it difficult for system designers to determine the system parameters. However, the presented mathematical modeling can facilitate performance analysis in terms of these factors for the focused beam. In addition, it offers insight into the system design. The size of the beam can be determined to achieve the best performance in a combination of several channel influences.

The major contribution of this paper can be summarized as follows. This paper is the first work considering the focused beam case in an IRS-assisted FSO system, thus providing general analytical tools and frameworks for future research on the focused beam of the IRS-assisted FSO system. Moreover, the pointing error model shown in this work is not limited to only IRS-assisted systems but can be extended to other FSO research concerning the narrow beam width smaller than the Rx aperture. Eventually, the analysis results can therefore be applied as a benchmark for determining the optimal tunable parameters of state-of-the-art FSO studies for the focused beam under different channel conditions.

II. SYSTEM MODEL

A. Power Density of Gaussian Beam

This section presents the electric field and power density distributions of the transmitted Gaussian beam on the IRS. Firstly, we consider the two-dimensional (2D) coordinate system of the \( yz \)-plane, where the IRS
exists on the $y$-axis. Note that we can get more straightforward analyses and insights by considering a 2D system. Without loss of generality, the analyses and insights obtained from the 2D system can be expanded to the three-dimensional (3D) system. Assuming that both the centers of the IRS and transmitted beam are at the origin, as illustrated in Fig. 1, the lengths of the IRS and the Rx lens are denoted as $2a_r$ and $2a_t$, respectively. $\theta_r$ and $\theta_t$ are the incident and reflection angles at the IRS, respectively.

Next, we define the electric field and power density of the Gaussian beam in the positive $\hat{z}$ direction from the transmitter using $\hat{y}\hat{z}$-coordinates. The electric field is expressed as follows:

$$E(\hat{y}, \hat{z}; w_0) = E_0 \sqrt{\frac{w_0}{w(\hat{z}, w_0)}} \times \exp \left( -\frac{\hat{y}^2}{w^2(\hat{z}, w_0)} + j\psi(\hat{y}, \hat{z}, w_0) \right)$$

where

$$\psi(\hat{y}, \hat{z}, w_0) = -k\hat{z} - k \frac{\hat{y}^2}{2R(\hat{z}, w_0)} + \zeta(\hat{z}),$$

$\zeta(\hat{z})$ is the axis perpendicular to $\hat{z}$, $E_0$ denotes the electric field at the origin, $w_0$ denotes the beam waist radius, $k = \frac{2\pi}{\lambda}$ is the wave number, $\lambda$ is the optical wavelength, $R(\hat{z}, w_0) = \hat{z}(1 + (\hat{z}/w_0)^2)$ is the curvature radius of the beam’s wavefront at $\hat{z}$, $w_0 = \pi w_0^2/\lambda$ is the Rayleigh range, $\zeta(\hat{z}) = \tan^{-1}(\hat{z}/w_0)$ is the Gouy phase, and $w(\hat{z}, w_0) = w_0 \sqrt{1 + (\hat{z}/w_0)^2}$ is the beam width where the magnitude of the electric field, $|E(\cdot)|$, falls to 1/e of their axial values on the $\hat{z}$ plane. In addition, the power density distribution of the Gaussian beam $I(\cdot) = |E(\cdot)|^2$ is given by

$$I(\hat{y}, \hat{z}; w_0) = \frac{2}{\pi^2 w(\hat{z}, w_0)} \exp \left( -\frac{2\hat{y}^2}{w^2(\hat{z}, w_0)} \right).$$

Then, we express the electric field across the IRS as $E(\gamma \cos \theta_i, \delta_{2z} + y \sin \theta_i; w_0)$, and approximate the power density distribution across the IRS as follows [5]:

$$I_{irs}(y, \delta_{2z}; \theta_i, w_0) \approx \frac{\sqrt{2} \cos \theta_i}{\sqrt{\pi w(\delta_{2z}, w_0)}} \exp \left( -\frac{2 \cos^2 \theta_i y^2}{w^2(\delta_{2z}, w_0)} \right)$$

where the approximation arises when $\delta_{2z} \gg y \sin \theta_i$.

### B. Channel Model

Next, we define the FSO communication link. In this paper, we assume that the intensity modulation and direct detection are employed where the on-off keying (OOK) is used as the modulation scheme [2], [3], [4], [5], [13]. Laser beams propagate through a turbulence channel with additive white Gaussian noise (AWGN) in the presence of pointing errors. The received optical signal is then converted into an electrical signal at the photodetector. The received signal, $y$, can be expressed as $y = \eta h x + n$, where $x$ is the transmit signal of OOK modulation being either 0 or $2Pt$, where $Pt$ is the average transmitted optical power, $\eta$ is the effective photo-electric conversion ratio, $h$ is the channel gain, and $n$ is the signal-independent AWGN with a variance of $\sigma_n^2$.

The channel gain, $h$, can be modeled as $h = h_1 h_a h_\eta$, where $h_1$ represents the attenuation caused by the scattering and absorption of particles in the air, $h_a$ is the atmospheric fading loss factor, and $h_\eta$ is the pointing error loss factor. Note that $h_a$ and $h_\eta$ are both independent random variables, whereas $h_1$ is a constant given by $h_1 = \exp(-\sigma z)$ where $\sigma$ is the attenuation coefficient and $z$ is the link distance.

Because the coherence time of FSO channels is considerably longer than the typical bit interval, owing to the high data rates, the FSO channel is commonly modeled as a slow-fading channel, where the outage probability becomes a relevant metric in performance evaluation [11], [13]. When the signal-to-noise ratio of OOK is given as $(\sqrt{2}Pt \eta h/\sigma_n)^2$ as in [13], [14], the outage probability for the rate requirement $R_0$ is expressed as $P_{out} = \text{Prob}(h < h_0)$, where $h_0 = \sqrt{(2R_0 - 1)\sigma_n^2/(\sqrt{2}Pt \eta)}$ with the general channel capacity formula.

### III. Phase-Shift Design of IRS

In this section, we present the IRS phase-shift design for creating a reflected beam focused on the Rx. Note that, as discussed in Section II-A, we assume the 2D case to get a clearer insight into phase-shift design. By manipulating the phase of the reflected beam over the IRS surface, we can control the convergence of the beam at the Rx. First, we consider the case when the reflected beam is completely focused at the center of the Rx. A focused beam can be created by allowing the beam reflected from the IRS to have a phase profile of the virtual beam propagating from the center of the Rx in the opposite direction.

Assuming a mirror-assisted system, we consider the virtual beam that comes from the Rx lens, whose location is specified by $\theta_r$ and $d_{2z}$. It is assumed that the virtual beam has the same power density distribution as the original beam from the transmitter (Tx) on the IRS, as follows:

$$I_{irs}(y, d_{2z}; \theta_r, w_0) = I_{irs}(y, d_{2z}; \theta_r, \tilde{w}_0),$$

where $\tilde{w}_0$ is the beam waist of the virtual beam. As the solution of the equation $w(d_{2z}, \tilde{w}_0) = w_{eq, \theta_r}$, we obtain

$$\tilde{w}_0 = \frac{w_{eq, \theta_r}^2}{2} - \frac{1}{2\pi} \sqrt{\pi^2 w_{eq, \theta_r}^4 - 4d_{2z}^2 \lambda^2},$$

where $w_{eq, \theta_r} = w_{irs, \cos \theta_r}$ is the equivalent beam width of the virtual beam in a plane perpendicular to the beam line when the virtual beam with an incident angle of $\theta_r$ forms the same beam width, $w_{irs, \cos \theta_r} = w(d_{2z}, w_0)/\cos \theta_r$, as the beam transmitted from Tx over the IRS.

Next, we construct the phase-shift profile $\Delta \psi(y)$ which is added to the phase of the reflected beam across the IRS. As in (13) of [5], it is built by adding the phase of the virtual beam in the reversed direction, that is $\psi(y \cos \theta_r, -(d_{2z} + y \sin \theta_r), w_0)$, and subtracting the phase of the incoming beam, that is $\psi(y \cos \theta_r, d_{2z} + y \sin \theta_r, w_0)$. The resulting phase-shift profile across the IRS becomes

$$\Delta \psi(y | \theta_r, \theta_t, w_0, \tilde{w}_0) = \pi - \psi(y \cos \theta_r, d_{2z} + y \sin \theta_r, w_0) + \psi(y \cos \theta_r, -(d_{2z} + y \sin \theta_r), \tilde{w}_0).$$

This phase-shift profile contains the beam’s focusing as well as the anomalous reflection in [5]. The major difference from (13) of [5] lies in $\tilde{w}_0$ of (6) and the sign of $-(d_{2z} + y \sin \theta_r)$ implying that the virtual beam’s propagation direction is opposite to that of [5]. Eventually, the reflected beam is focused at a distance $d_{2z}$ with the beam width of $\tilde{w}_0$. Fig. 1. Illustration of focused beam shaping using IRS.
Furthermore, we consider the case when the beam width at the Rx is larger than \( w_0 \) and smaller than \( w_{irs} \). In this case, we assume that the reflected beam is focused at a greater distance by \( d_f \) than the Rx, where the virtual beam source is located. Then, we must find the virtual beam whose width is \( w_d \) at the Rx lens of distance \( d_f \), and \( w_{irs} \) at the IRS of distance \( d_{irs} + d_f \). To obtain the exact \( w_0 \) of such a beam, the following equations must be solved:

\[
\frac{w(d_{irs} + d_f, w_0)}{\cos \theta_r} = \frac{w(d_{irs}, w_0)}{\cos \phi_r} \quad \text{and} \quad w(d_f, w_0) = w_1. \tag{8}
\]

However, the analytical solution of (8) is very complicated, because it requires the roots of a general cubic equation to obtain \( w_0 \). Instead of an exact analytical solution, we present an approximated solution. Assuming that the Rx is at a great distance from the IRS, i.e., \( d_{irs} \gg z_R \), \( w_0 \) can be approximated as follows [15]:

\[
w_0 \approx \frac{\lambda d_{irs}}{\pi (w_{irs, \theta_r} - w_l)} \tag{9}
\]

Further, we compare the power density distributions of the reflected beam at the Rx, which are obtained by two different methods: the geometric optics and the Huygens-Fresnel principle. The power density of the geometric optics was obtained by the beam propagated from the virtual beam source to the Rx, at a distance \( d_f \) from the source. The Huygens-Fresnel principle treats every point on the IRS as a secondary source. This is expressed as follows:

\[
E_{0x}(y) = \frac{s}{\sqrt{\pi}} \int_{-a}^{a} \exp(jk|\mathbf{r}_1 - \mathbf{r}_0|) \left( \frac{2}{\sqrt{a^2 - x^2}} \right) \frac{\sqrt{a^2 - x^2}}{w_1^2} \exp(-2((x - u)^2 + y^2)w_1^2) \, dy \, dx,
\]

where \( s = \cos \theta_r / \cos \phi_r, \mathbf{r}_0 = \begin{bmatrix} \cos \theta_r & \sin \theta_r \\ -\sin \theta_r & \cos \theta_r \end{bmatrix} \), \( \mathbf{r}_1 = \begin{bmatrix} y \\ x \end{bmatrix} \), and \( \Delta \psi(y) \) is the phase-shift profile across the IRS, given by (7).

The results of this comparison are presented in Fig. 2. The received power density computed from the Huygens-Fresnel integration matches well with the geometric optics computation. Thus, the proposed phase-shift design for focusing beams is verified.

Expanding this result to three dimensions can be achieved using a rotated astigmatic Gaussian beam. Note that the details of the phase-shift design for anomalous reflection assuming rotated astigmatic Gaussian beam for the 3D case are demonstrated in (28) of [5]. However, we refrain from providing the complete derivation of the phase-shift design for the 3D system due to the limitation of the length and thus present only the basic ideas needed to extend the results of the 2D systems to the 3D systems.

As we discussed earlier in (7), in order to create a focused beam other than anomalous reflection, \( \tilde{w}_0 \) must be set according to (6) or (9), and the sign of the parameter associated with the propagation direction of the virtual beam must be the opposite of a normal reflection. The same method can be applied to the 3D system. When extending to a 3D system, suppose that \( \tilde{w}_0 \) for the 2D system corresponds to \( \tilde{w}_0 = \{ \tilde{w}_0, \tilde{w}_{0z} \} \) for the astigmatic Gaussian beam in the 3D system. Similarly, suppose that \( w_{eq, \theta_r} \) in the 2D system corresponds to \( \Psi = \{ \theta_r, \phi_r \} \) and \( \phi_r \) is the rotated angle of Rx from z-axis in the 3D system.

Then, the i-th component of the diagonal matrix \( S_{w_0} \) presented in Corollary 3 in [5] matches to \( 1/w_{eq, i}^2, \Psi \), for \( i \in \{1, 2\} \), and it is obtained by eigenvalue decomposition of the matrix \( A \) defined in Appendix D of [5]. Then, given \( w_{eq, i}, \Psi \), we can obtain \( \tilde{w}_0 \) using (6) or (9) in this paper. Also, the sign of the second parameter in the phase of the astigmatic Gaussian beam of (28) in [5] should be negative.

So far, we have introduced a method to focus the beam at the Rx using the IRS. With the proposed phase-shift design, we can concentrate the transmitted power on the Rx using the IRS. However, the focused beam is more sensitive to pointing errors. The following section examines the effect of the pointing error when the beam is focused, that is, when the beam width is similar to or smaller than the Rx size.

**IV. POINTING ERROR MODELING**

In this section, we examine the effect of pointing errors on the quality of the FSO channels when the beam is focused on a size similar to or smaller than the Rx aperture. Note that the laser beam reflected from the IRS becomes the astigmatic Gaussian beam at the Rx lens in the 3D space. However, we considered only a general Gaussian beam. This is because the main purpose of pointing error modeling is to analyze the influence of pointing errors according to the size of the beam rather than its shape. Based on the results of the general Gaussian beam, we can expand it to the astigmatic Gaussian beam for future work.

**A. Channel Coefficient Modeling**

First, we introduce the conventional pointing error model. The pointing error loss \( h_p \) is obtained by integrating the power density function over the detector area as follows:

\[
h_p(w; w_1, a_1) = \int_{-a_1}^{a_1} \int_{-a_1}^{a_1} \frac{2}{\sqrt{a^2 - x^2} \pi w_1^2} \exp\left(-2\left((x - u)^2 + y^2\right)w_1^2\right) \, dy \, dx,
\]

where the beam propagates in the positive z direction, the Rx lens is located on the x-y-plane, \( w_1 \) is the beam width at the Rx lens, and \( u \) is the displacement of the beam from the center of the Rx.

By approximating the integration range to a square of equal area to the detector, and applying the Taylor series approximation, (10) can be approximated to the conventional Gaussian form model [13], i.e.,

\[
h_p(w; w_1, a_1) \approx A_0 \exp\left(-2\frac{w_1^2}{w_{eq}^2}\right) \tag{11}
\]

where \( w_{eq}^2 = w_1^2 \sqrt{\pi \text{erf}(\tau)/(2 \tau \exp(-\tau^2))} \), \( A_0 = \text{erf}(\tau^2) \) and \( \tau = \sqrt{\pi a_1/(2w_1)} \). However, the conventional Gaussian form (11) is applicable only if the beam width is more than twice the diameter of the Rx aperture.
Therefore, we propose the new pointing error loss $h_p$ for the cases where the beam width is similar to, or smaller than, the diameter of the Rx lens. We also derive the probability density function (PDF) of $h_p$. Consider the case when the beam width is similar to the radius of Rx lenses first. In this case, (10) can be approximated using the Gaussian error function, i.e., $\text{erf}(\cdot)$ as follows:

$$h_p(u; w_l, a_l) \approx \frac{E}{2} \left( \text{erf} \left( \frac{\sqrt{2}}{w_l} \left( \frac{\sqrt{\pi} a_l - |u|}{w_l} \right) \right) + 1 \right),$$

where $E$ is derived from the error function expressed as $E = \text{erf}(\tau)$. In (12), (a) follows from the power of the beam outside the Rx lens is negligibly small when the beam width is similar to or smaller than the radius of Rx aperture.

Assuming a Gaussian fluctuation of the center of beam on both the horizontal and vertical axes of the Rx, $u$ follows a Rayleigh distribution. The PDF of the pointing error loss, $h_p$, is then given by

$$f_{h_p}(h) = \frac{\sqrt{\pi} \lambda(h_p) \exp \left( v(h_p)^2 \right)}{2}$$

where

$$\lambda(h_p) = \frac{\pi}{2} a_l - \frac{w_l}{\sqrt{\pi} w_l^2} v(h_p), \quad v(h_p) = \text{erf}^{-1} \left( \frac{1}{2} h_p - 1 \right),$$

and $\text{erf}^{-1}(\cdot)$ is the inverse of the erf($\cdot$). For $0 \leq h_p \leq A_0$, $A_0 = \frac{E(\pi+1)}{2}$ is the fraction of the collected power at $r = 0$.

Meanwhile, integrating (13) to obtain a performance metric such as an outage probability may be limited in cases where $w_l \ll a_l$. Note that for $w_l \ll a_l$, $E \approx 1$ and $A_0 \approx 1$. Then, $f_{h_p}(A_0)$ becomes infinite, and (13) cannot be integrated from 0 to $A_0$. Thus, we introduce another pointing error model which can be applicable for the case when $w_l \ll a_l$ as follows:

$$h_p(u; a_l) = I_A(u)$$

where $A = [-a_l, a_l]$, $I_A(u)$ is an indicator function with $I_A(u) = 1$ only if $u \in A$, and $I_A(u) = 0$ otherwise. Then, the PDF can be expressed as:

$$f_{h_p}(h_p) = F_u(a_l) \delta(h_p - 1) + (1 - F_u(a_l)) \delta(h_p),$$

where $\delta(\cdot)$ is the impulse function, and $F_u(\cdot)$ is the cumulative probability distribution (CDF) of displacement which is a Rayleigh random variable.

The exact expression, (10), and the proposed approximation model for $h_p(\tau)$ are plotted in Fig. 3 for the three cases when $w_l/a_l \in \{0.1, 0.5, 1.0\}$. For the approximate model of $h_p$, (12) is used for the two cases when $w_l/a_l \in \{0.5, 1.0\}$, and (14) is used when $w_l/a_l = 0.1$. In addition, we compared the proposed approximation models of $h_p$ with the conventional Gaussian form modeling [13]. Note that the exact expression in (10) and the proposed approximation model correspond for the cases of $w_l/a_l \in \{0.1, 1.0\}$. When $w_l/a_l = 0.5$, the proposed approximation $h_p$, model of (12) has a small gap with the exact expression of (10) for the displacements between 5 cm and 15 cm. However, this gap is insignificant compared to the difference between the conventional method (10) and (11), and is negligible in determining the approximate performance. Furthermore, we can observe that the conventional approximation deviates from the exact expression as $w_l/a_l$ decreases.

### B. Outage Probability

In this section, we obtain the outage probability when considering atmospheric and pointing error losses together with the quality of the FSO channel. Given $f_{h_p}(h_p)$, the PDF of the composite channel coefficient $h$ is expressed as

$$f_h(h) = \int f_{h_p}(h_p) f_{h|h_p}(h|h_p) \, dh_p$$

where (a) is derived from $h = h_i h_p h_u$. Then, we can compute the outage probability as

$$P_{out} = \int_{0}^{A_0} f_h(h) \, dh.$$

By substituting $f_h(h)$ from (17) into (18) and changing the order of integration, the outage probability becomes

$$P_{out} = \int_{0}^{A_0} f_{h_p}(h_p) \int_{0}^{h_p} f_{h_u} \left( \frac{h}{h_p h_i} \right) \, dh \, dh_p$$

where $f_{h_u}(\cdot)$ is the CDF of turbulence-induced channel $h_u$. Note that we consider both log-normal fading for weak turbulence channel and Gamma-Gamma fading for strong turbulence channel.

For the log-normal fading, we have

$$\text{log-normal fading}\quad F_{h_u}(h_u) = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{\log(h_u)}{\sqrt{\sigma^2}} \right),$$

where $\sigma^2 \approx \sigma_R^2/4$ is the variance of the log-normal fading amplitude, that is, $0.5 \log(h_u)$, $\sigma_R^2$ is the Rytov variance, which is given by $\sigma_R^2 = 1.23 C_{10}^2 k^2/8 a_0^{1/6}$ where $C_{10}^2$ is the index of refraction structure parameter. For the Gamma-Gamma fading, we have

$$F_{h_u}(h_u) = \pi \csc(\pi \gamma) \left[ \frac{(\alpha\beta h_u)^\beta}{\Gamma(\alpha,1+\beta)} + \frac{1}{\Gamma(1+\gamma)\Gamma(1+\alpha)} \right],$$

where

$$\alpha = \exp \left( \frac{0.5 \sigma_{10}^2}{(1+1.11 e^{-0.69 a_0^{1/6}})^{1/6}} - 1 \right)^{-1},$$

$$\beta = \exp \left( \frac{0.5 \sigma_{10}^2}{(1+1.06 e^{-0.69 a_0^{1/6}})^{1/6}} - 1 \right)^{-1}.$$
\( \gamma = \alpha - \beta \), and \( pF_q \) is the generalized hypergeometric function for integers \( p \) and \( q \) [16].

Then, the outage probability can be obtained by substituting \( f_{h_p}(h_p) \) from (13) and (15), and \( F_{h_u}(h_u) \) from (20) and (21) into (19), according to the pointing error model and turbulence-induced fading models. Note that for (13), the outage probability cannot be computed in closed form. However, it involves only one finite integral that can be easily computed numerically.

Meanwhile, for (15), the outage probability can be obtained in closed form, and it is given by

\[
\begin{align*}
P_{\text{out}} &= F_u(a_1) \int_0^1 \delta(h_p - 1) F_{h_u} \left( \frac{h_0}{h_p h_1} \right) dh_p \\
&\quad + (1 - F_u(a_1)) \int_0^1 \delta(h_p) F_{h_u} \left( \frac{h_0}{h_p h_1} \right) dh_p \\
&= F_u(a_1) F_{h_u} \left( \frac{h_0}{h_1} \right) - 1 + 1.
\end{align*}
\]

Moreover, from (24), we can also identify that the outage probability reaches a lower limit when \( P_t \) approaches infinity as follows:

\[
P_{\text{out}} \geq \lim_{{P_t \to \infty}} P_{\text{out}} = \lim_{{h_0 \to 0}} F_u(a_1) F_{h_u} \left( \frac{h_0}{h_1} \right) - 1 + 1 = 1 - F_u(a_1).
\]

Note that even if the power increases, the outage probability does not decrease accordingly and gradually converges to the lower limit. In addition, we observe that this lower limit of the outage probability depends only on the probabilistic characteristics of the pointing error, regardless of the atmospheric effect.

V. NUMERICAL RESULTS

In this section, we provide Monte Carlo simulation results to verify the proposed pointing error model and the outage probability. The simulation parameters are determined as follows. For all simulation cases, we assumed that \( \lambda = 1550 \) nm, and the total length of the propagation path between the Tx to Rx via IRS, \( d_{22} \), was 1 km. Note that the distance only matters in \( h_1 \), and \( h_1 \) depends on the length of the path between Tx and Rx no matter how each distance from Tx and IRS, and distance from IRS and Rx varies.

The weather conditions on the FSO link are characterized by two cases: 1) clear weather with weak turbulence and 2) light fog with strong turbulence, as in [13]. For the clear weather, we assumed that the visibility was 10 km; thus \( h_1 = 0.9 \) for 1 km. In the light foggy weather, the visibility was assumed to be 1 km, and \( h_1 = 0.1158 \) for 1 km. The strength of turbulence was characterized by \( C_{\delta}^2 \) and \( \sigma^2_{\beta} \). We assumed that \( C_{\delta}^2 = 5 \times 10^{-14} \) and \( \sigma^2_{\beta} = 1 \) for the weak turbulence, and \( C_{\delta}^2 = 0.5 \times 10^{-14} \) and \( \sigma^2_{\beta} = 0.1 \) for the strong turbulence.

Fig. 4 compares the outage probability of the analysis and simulation according to the standard deviation of the beam’s displacement, \( \sigma_u \), and transmitted power \( P_t \) for each case of \( w_1/a_1 \in \{0.1, 0.5, 1.0\} \). The simulation and analytical results match closely for both the proposed pointing error models, (12) and (14) for most cases of \( w_1/a_1 \in \{0.1, 1.0\} \). However, when \( w_1/a_1 = 0.5 \) and \( \sigma_u = 0.4 \), there is approximately 2 dB difference between simulation and analysis, which results from the approximate error of the pointing error model. Despite the approximation error, the analysis results are accurate enough to grasp the approximate performance and characteristics of the beam size and displacement. In addition, we observe that the outage probability reaches the lower limit, which increases with \( \sigma_u \). Moreover, the lower limit of the outage probability was equivalent for both atmospheric models. Thus, (25) was verified.

Fig. 5 shows that how the outage probability varies according to the beam width and pointing error. We can obtain a better outage performance (lower outage probability) with a more focused beam with a small \( w_1/a_1 \) when the pointing error is marginal, with \( \sigma_u = 0.1 \). However, when the pointing error is significant, the more focused beam
gets the worse outage performance (high outage probability); when \( w_1/a_1 = 0.1 \) and \( \sigma_u = 0.4 \), the outage probability reaches the lower bound early about at a power of \(-15\) dB and is larger than other cases of \( w_1/a_1 \in \{1.0, 2.0\} \) for the most of transmitted power ranges.

Furthermore, we can observe that the performance gain of \( w_1/a = 0.1 \) over \( w_1/a = 1 \) is less than \(-1\) dB when \( \sigma_u = 0.1 \). Based on this observation, it can be deduced that a focused beam smaller than the size of the Rx has no substantial benefit. We can transmit sufficient power with a beam whose beam width at the Rx is similar to the radius of the Rx aperture. If the beam width is further reduced, it becomes more sensitive to pointing error and only increases the impact of the pointing error.

VI. CONCLUSION

In this paper, we proposed a phase-shift design of an IRS for focusing beams. In addition, we introduced a new pointing error model and an outage probability expression, applicable when the beam width is less than the Rx aperture size, for cases where the conventional model cannot be applied. The simulation validated the proposed phase-shift design, and the outage probability expression was verified. The mathematical modeling presented in this work can facilitate performance analysis according to various factors, such as beam width, power, and pointing error for the focused beam. Thus, this study provides essential preliminary research results for future studies that assume a focused beam in FSO systems.

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