THICKENING CALABI-YAU MODULI SPACES

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Abstract. We describe a kind of deformation of the anti-DeRham algebra on a Calabi-Yau manifold $X$. These are in 1-1 correspondence with the total cohomology $\oplus H^i(X, \mathbb{C})$.

In his article [W] in this volume’s precursor, Witten proposed, as a possible approach to constructing a mirror map, a certain extended moduli space $\mathcal{N}$, a thickening of the usual moduli space $\mathcal{M}$ of complex structures on a Calabi-Yau manifold $X$. Witten’s proposal for $\mathcal{N}$ is couched in terms of Conformal Field Theory on $X$, and it is not immediately obvious to the author to what extent it is or can be made rigorous. In any case, it appears to be an intriguing problem of pure deformation theory to construct Witten’s extended moduli directly in terms of ordinary complex geometry, making no appeal to Physics.

The purpose of this note is to describe a type of generalized or ‘exotic’ deformation of complex structure on a Calabi-Yau manifold $X$. Our construction shares with Witten’s the property that the set of first-order deformations coincides with the total DeRham cohomology $\oplus H^i(X)$, with the classical deformations embedded as $H^{1,n-1}(X)$; but the author’s incompetence in physical matters precludes any further comment here as to any real connection between the two constructions.

Our approach focuses on the (Holomorphic) anti-DeRham or Mahr Ed algebra

$$\Omega_X^- = \oplus \Lambda^i \Theta^*$$

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This is a sheaf of graded anticommutative associative algebras. Interestingly, when $X$ is endowed with a holomorphic volume form, $\Omega_X^-$ not only becomes (additively) isomorphic with the DeRham complex $\Omega_X$ but also acquires a (graded) bracket operation, known as the Schouten bracket, extending the usual Lie bracket on $\Theta_X$. This Schouten bracket, which is familiar in symplectic geometry and mechanics, turns $\Omega_X^-$ essentially into a graded Lie algebra. Moreover via the adjoint action, $\Omega^-$ can essentially be identified as the differential graded Lie algebra of derivations on the (associative) Mahr Ed algebra $(\Omega^-, \wedge)$. A suitable graded version of the usual connection between derivations and deformations then yields the realization of the De Rham cohomology, i.e., the hypercohomology of $(\Omega^-, d)$, as deformations of the algebra $(\Omega^-, \wedge)$.

It must be stated that this work is not in final form with numerous basic questions as yet unanswered and even unasked. This applies especially to the higher-order theory. We hope to return to this elsewhere.

1. The Schouten Lie algebra.

Fix an $n$-dimensional complex manifold $X$, not necessarily compact (analogous considerations apply in the real case as well). We denote as usual by $\Theta_X$ and $\Omega_X$ its tangent and cotangent sheaves respectively, and by $\Omega_X = \bigoplus \Lambda^i \Omega_X$ the DeRham algebra. Our focus, however, will be on the latter’s dual, the Mahr Ed algebra

$$\Omega_X^- = \bigoplus \Lambda^i \Theta_X.$$

This is a sheaf of negatively graded, anticommutative associative algebras. Now it is an interesting observation going back to Schouten [S] and Nijenhuis [N] (see also Lichnerowicz [L] and Koszul [K]), that the Mahr Ed algebra carries an additional structure which is essentially that of a graded Lie algebra. To be precise, there is a bracket operation $[\ ,\ ]$ on $\Omega^-$, the Schouten bracket, with the following properties (where $a,b,c$ denote homogenous elements):

0. $[\ ,\ ]$ coincides with the usual Lie bracket on $\Omega_X^- = \Theta_X$;
1. $\deg [a, b] = \deg a + \deg b + 1$;
2. $[a, b] = -\varepsilon(a, b)[b, a]$ where $\varepsilon(a, b) = (-1)^{(\deg a+1)(\deg b+1)}$.
3. A Jacobi identity is satisfied:

\[ \varepsilon(a, c)[a, [b, c]] + \varepsilon(c, b)[c, [a, b]] + \varepsilon(b, a)[b, [c, a]] = 0 \]

4. \([a, \cdot]\) is a derivation of degree \(\deg a + 1\) on the algebra \(\Omega^-\). (In fact, properties 0,1,2,4 already characterize \([\cdot, \cdot]\))

To exploit this structure, define the Schouten Lie algebra of \(X\) by

\[ L_X = \bigoplus_{i=0}^{\dim X - 1} L_X^{-i} := \bigoplus_{i=0}^{\dim X - 1} \Lambda^{i+1} \Theta_X, \]

with the above Schouten bracket \([\cdot, \cdot]\). Then the above properties show that \((L_X, [\cdot, \cdot])\) forms a sheaf of graded Lie algebras on \(X\). Moreover \(L_X\) comes with a faithful graded Lie representation

\[ (1.1) \quad L_X \to \text{Der}_C(\Omega_X^-, \Omega_X^-). \]

Actually, this representation is surjective, i.e., an isomorphism, though we won’t need this fact.

Now suppose given a volume form \(\Phi\) on \(X\), i.e., a nowhere vanishing section of \(\Omega^n_X\). Then interior multiplication by \(\Phi\) induces an additive isomorphism

\[ i_\Phi : \Omega_X^-[n] \to \Omega_X. \]

via \(i_\Phi\), the exterior derivative operator \(d\) on \(\Omega_X\) may be imported over to \(\Omega_X^-\), yielding an operator \(\delta = \delta_\Phi\) of degree +1 with \(\delta^2 = 0\). Note however that as \(\Phi\) is not multiplicative, \(\delta\) loses the all-important derivation property of \(d\). In fact, the failure of \(\delta\) to be a derivation is measured precisely by the Schouten bracket: this is because of the following formula due to Koszul:

\[ [a, b] = (-1)^{\deg b} (\delta(ab) - \delta(a)b - (-1)^{\deg a} \deg b \delta(b)a), \]

\[ (1.2) \quad a, b \in \Omega^- \text{ homogeneous} \]

One consequence of Koszul’s formula (1.2) is that the restriction of \(\delta\) on the Schouten algebra \(L\) may be identified with the commutator with the operator \(\delta\) on \(\Omega^-\):

\[ (1.3) \quad \delta(a) = [a, \delta] := \delta \circ a - (-1)^{\deg a+1} a \circ \delta \]

\[ a \in L; \text{ homogeneous} \]
In particular, we have
\[ \delta |_{L^0} \sim [\cdot, d] |_{\Theta^X} = 0. \]
(of course, this is what makes it sensible to restrict \( \delta \) on \( L \) in the first place).

Note that the RHS of (1.1) a priori carries a differential that is commutator with \( \delta \). Thus the significance of (1.3) is precisely that the representation (1.1) is one of differential graded Lie algebras. In particular, the cohomology of \((L, \delta)\) should be related to properties of \( \Omega^- \) as algebra, such as deformations. This is the basic idea of our approach.

To tie this in later with Dolbeault cohomology, note that the Schouten bracket extends to a pairing
\[ A^{\cdot, q}(L) \times \Omega^- \to A^{0, q}(\Omega^-) =: A^{\cdot, q} \]
where \( A^{\cdot, q}(E) \) denotes real forms of type \((\cdot, \cdot)\) with coefficients in \( E \). Naturally this is compatible with \( \bar{\partial} \).

Finally it may be noted that via \( i_{\phi hi} \), the pairing on \( \Omega^\cdot \) corresponding to exterior product on \( \Omega^- \) is just the so-called Yukawa pairing \( * \). Hence by (1.2) the Schouten bracket on \( \Omega^\cdot \) may be written as
\[ [\alpha, \beta] = \pm (d(\alpha * \beta) - d(\alpha) * \beta \pm d(\beta) * \alpha). \]

2. Exotic deformations.

Now fix a volume form \( \Phi \) on our complex manifold \( X \). Motivated by (1.4), we replace the Schouten algebra \( L_X \) by its subalgebra \( \hat{L}_X \), the restricted Schouten algebra, defined by
\[ \hat{L}^0 = \{ v \in L^0 : L_v \Phi = 0 \}, \quad L_v = \text{Lie derivative}; \]
\[ \hat{L}^{-i} = L^{-i}, \quad i > 0. \]

Note the quasi-isomorphism
\[ \hat{L} \sim \Omega^X[n - 1]. \]
Combining this with the representation of \( \hat{L} \) as the DGL algebra of volume-preserving derivations of the Mahr Ed algebra \( \Omega^- \) should, by general principles,
give rise to a realization of the De Rham cohomology \( H(\Omega^-) \) as a suitable kind of deformations of the ‘algebra with operator’ \( \Omega_X^- \), which may be viewed as a sort of ‘exotic’ or non-classical deformations of the manifold \( X \) itself. We proceed to elaborate this idea.

We begin by defining a generalization of the usual notion of total complex associated to a bicomplex. Let \( K^- \) be a doubly-indexed array (not necessarily a bicomplex) of (abelian) objects with maps \( d', d'' \) of bidegrees \((1,0)\) and \((0,1)\) respectively. To this we may associate as usual the total object \( t(K^-) \) defined by

\[
t^i(K^-) = \bigoplus_{p+q=i} K^{p,q}
\]

By a complexification of \( K^- \) we mean a structure of complex on \( t(K^-) \), i.e. differentials \( d : t^i(K^-) \to t^{i+1}(K^-) \) with \( d \circ d = 0 \), which are compatible with the original \( d', d'' \) in the sense that the map \( K^{p,q} \to K^{p',q} \), induced on subquotients by \( d \), coincides with \( d' \) for \((p', q') = (p+1, q)\) and with \((-1)^qd'' \) for \((p', q') = (p, q+1)\), i.e., representing \( d \) by a matrix, its ‘near diagonals’ should consist of \( d' \) and \( \pm d'' \).

Of course, if \( K^- \) happens to be a bicomplex, its usual associated total complex is a complexification in the above sense, called the standard one. More generally, given a sub-bicomplex \( J^- \subseteq K^- \), a complexification is said to be standard on \( J^- \) if it preserves \( t(J^-) \subseteq t(K^-) \) and agrees with the standard on \( t(J^-) \).

**Theorem.** There is a natural bijection between \( H^i(F^{-j} \hat{L}^-) = H^{i+n-1}(F^{n-1-j} \Omega_X^-) \) and the set of sets of data as follows (up to a natural equivalence):

(i) a surjection \( E^- \to \Omega^- \) of sheaves of graded anticommutative associative algebras, which is an isomorphism in degrees \( < j-n \), and which fits in a diagram with exact rows

\[
\begin{array}{cccccccc}
0 & \to & \Omega^- & \to & A^- & \to & \cdots & \to & A^{-n,i+j-2} & \to & E^{j-n} & \to & \Omega^{j-n} & \to & 0 \\
& & \delta & \downarrow & & & & \downarrow & & \delta & \downarrow & & & \delta & & 0 \\
0 & \to & \Omega^{-j} & \to & A^{-j,0} & \to & \cdots & \to & A^{-j,i+j-2} & \to & E^0 & \to & \mathcal{O} & \to & 0 \\
\end{array}
\]

(ii) a complexification \( t(A^-, E^-) \) of the middle portion \((A^-, E^-)\) of (2.1), standard on \( A^- \) which fits in an exact sequence of complexes

\[
0 \to \Omega^- / F^{-j-1} \Omega^- \to t(\hat{A}^-, \hat{E}^-) \to \Omega^- \to 0
\]
Let us give a brief indication how the above data are constructed out of an element \( u \in \mathbb{H}^i(F^{-j}\hat{L}^-) \), thought of as a mixed Čech-Dolbeault cohomology class. First, \( u \) gives rise to a 1-cocycle \((a_{\alpha\beta})\) with values in \( \bar{\partial}A^{-j,1+j-1}(L^{-1}) \), the \( \bar{\partial} \)-closed forms and from the action of the latter on \( \Omega^{-} \) we get the \( E^{-} \). Next we have a 0-cochain \((b_{\alpha})\) with values in \( A^{-j+1,1+j-1}(L^{-1}) \) such that \( b_{\alpha} - b_{\beta} = \partial a_{\alpha\beta} \). This yields the differentials \( \delta \) on \( E^{-} \) in (2.1). However, we do not necessarily have a complex yet, i.e. \( \delta^2 \neq 0 \). The additional data of complexification is then obtained by pulling \((b_{\alpha})\) gradually further up the Hodge filtration in the group \( \mathbb{H}^i(F^{-j}\hat{L}^-) \).

Note that aside from ordinary vector fields, the Schouten algebra contains only elements of strictly negative degree, which are nilpotent derivations of index \( \leq n \). On the other hand, corresponding results in the classical case indicate for \( X \) Calabi-Yau that all the deformations involving \( \hat{L}^0 \) are unobstructed, i.e., extend to all orders. Thus our moduli space is an infinitesimal thickening of a smooth space with tangent space essentially \( \oplus H^i(\Theta_X) = \oplus H^{n-1,i}(X) \). In particular for \( n = 3 \) the reduced moduli has tangent space \( H^{2,1} \oplus H^{2,2} \), with the two pieces supposedly interchanged by mirror symmetry.

Note that, as a special case of the above, the deformations corresponding to \( F^{n-2}H^n(X) = \mathbb{H}^1(F^{-1}\hat{L}^{-}) \) correspond to diagrams

\[
0 \to \Omega^{-}[1]_{\leq 0} \to A^{-}[1]_{\leq 0} \to E^{-} \to \Omega^{-} \to 0
\]

with \( E^{-} \) an algebra with operator \( \delta \) satisfying \( \delta^2 = 0 \); i.e. to extensions

\[
[D] \in \text{Ext}_{alg}^{1}(\Omega^{-}, \bar{\partial}A^{1,-}[1]_{\leq 0})
\]

where the Ext is to be interpreted in the above sense.

3. \( H^1 \) and local systems.

To help develop some feeling for the above exotic deformations, we now consider in more detail what is probably the simplest non-classical case, that of \( H^1_{DR} \) (refusing to be bothered by the fact that for most—and by some definition, all—Calabi-Yau manifolds this group actually vanishes!). We do not require \( X \) to be compact or Kählerian.

Now in the case of \( H^1_{DR}(X) = \mathbb{H}^{-n+2}(\hat{L}^{-}) \), the construction of §2 simplifies considerably. It suffices to look at the quotient algebra \( E_{0} \oplus E_{-1} \) of \( E^{-} \); together...
with the differential $\delta : E^{-1} \to E^0$. These fit in an exact diagram
\[
\begin{array}{cccccc}
0 & \to & \mathcal{O} & \to & E^{-1} & \to \Omega^{-1} & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega & \to & E^0 & \to \mathcal{O} & \to 0
\end{array}
\]
(3.1)
where we have identified $\mathcal{O} \simeq \Omega^{-n}, \Omega \simeq \Omega^{1-n}$. Roughly, it is the $H^{0,1}$ part that yields $E^0 \oplus E^{-1}$, while the $H^{1,0}$ part yields the differential $\delta$: in particular, when the $H^{0,1}$ part is zero, i.e. $E^0 \oplus E^{-1}$ is the trivial extension, the essential part of $\delta$ is an operator $\Omega^{-1} \to \Omega$ of the form $u \mapsto \omega \delta(u)$ for some $\omega \in H^0(\Omega)$. Incidentally, a similar simplification holds for any $F^{i-1}H^{1,i}_{DR}(X)$.

Now on the other hand, recall the usual interpretation of $H^{1}_{DR}$ in terms of local systems: there is a natural bijection between $H^{1}_{DR}(X)$ and local systems, i.e. locally constant sheaves $\mathcal{L}$ on $X$, which fit in an exact sequence
\[
\begin{array}{cccccc}
0 & \to & \mathbb{C}_X & \to & \mathcal{L} & \to \mathbb{C}_X & \to 0.
\end{array}
\]
(3.2)
Here the $H^{0,1}$ part of the data corresponds to any of the vector bundles $F^i := \mathcal{L} \oplus \Omega^i_X$, viewed as extensions, while the $H^{1,0}$ part corresponds to an integrable connection
$$F^0 \to F^1 = F^0 \otimes \Omega^1_X,$$
or, for that matter, to any of the operators $F^i \to F^{i+1}$ induced by exterior derivative on $\Omega_X$.

The question then is how to go back and forth directly between the data (3.1) and (3.2). The key to this comes from the diagram
\[
\begin{array}{cccccc}
0 & \to & \mathcal{O}_X & \to & E^{-1} & \to \Omega^{-1} & \to 0 \\
\| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_X & \to & F^{-1} & \to \mathcal{O}_X & \to 0 \\
\downarrow & & \downarrow & & \| & & \\
0 & \to & \Omega_X & \to & E^0 & \to \mathcal{O}_X & \to 0 \\
\| & & \downarrow & & \downarrow & & \\
0 & \to & \Omega_X & \to & F^0 & \to \Omega_X & \to 0.
\end{array}
\]
(3.3)
Now given $(E, \delta)$ we may consider the module of differentials $\Omega_E$, which inherits a grading from $E$, and put
$$A' = (\Omega_E)^\cdot \otimes_E \Omega^{\geq -1}.$$We thus have a natural derivation $E \xrightarrow{d_E} \Omega_E \to A'$. By universalit of $d_E$ the operator $\delta : E^{-1} \to E^0$ factors through $A^{-1}$, so we get maps
$$E^{-1} \to A^{-1} \to E^0 \to A^0.$$
and clearly this sequence fits in a diagram like (3.3). We may then simply set

\[ \mathcal{L} = \ker(A^{-1} \to A^0) \]

and check easily that this is the appropriate local system.

To construct \((E^\cdot, \delta)\) from \(\mathcal{L}\) we use a variant of a method used earlier, e.g. in [R]. Consider the sheaves \(B^i = \Omega^i \oplus \mathcal{L} \otimes \Omega^{i+1}\) which we view as filtered:

\[ \Omega^{i+1} \subset \mathcal{L} \otimes \Omega^{i+1} \subset B^i \]

(3.4)

For \(j = 0, -1\) define \(E^j\) as the pairs \((\varphi, u)\) where \(u \in \Omega^j\) and \(\varphi \in \text{Hom}_\mathcal{O}(B^i, B^{i+j})\) is filtration-preserving, induces multiplication by \(u\) on \(\mathcal{L} \otimes \Omega^{i+1}\) and multiplication by \(u + \delta u\) on \(\Omega^{i+1} \oplus \Omega^i = B^i/\Omega^{i+1}\) (this is independent of \(i\)). The operator \(\delta : E^{-1} \to E^0\) is defined by

\[ \delta(\varphi, u) = ([\varphi, d], \delta u) \]

where \(d : B^i \to B^{i+1}\) is the obvious differential. It is not hard to check (compare [R]) that \((E^\cdot, \delta)\) is a deformation as in (3.1) and is the correct one corresponding to \(\mathcal{L}\) as above.

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