EQUIVARIANT POLYHARMONIC MAPS

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Abstract. We study $O(d)$-equivariant polyharmonic maps and their associated heat flows. We are mainly interested in blowup phenomena for the higher order flows. Such results have been hard to prove, due to the inapplicability of the maximum principle to these higher order flows. We believe that the ideas developed herein could be useful in the study of other higher order parabolic equations. We prove that blowup occurs in the biharmonic map heat flow from $B(0,1;4)$ into $S^4$. To our knowledge, this was the first example of blowup in the higher order polyharmonic map heat flows. We provide Mathematica code that computes our symmetry reduction for the polyharmonic map heat flow of any order. This code is then used to explicitly compute our symmetry reductions for the harmonic, as a check, and biharmonic cases. Next, the possible $O(d)$-equivariant biharmonic maps from $\mathbb{R}^4$ into $S^4$ are classified. Finally, we show that there exists, in contrast to the harmonic map analogue, equivariant biharmonic maps from $B(0,1;4)$ into $S^4$ that wind around $S^4$ as many times as we wish.

1. Introduction

In this work we are interested in (extrinsic) polyharmonic maps and their associated heat flows. Let $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ be open and bounded, and $N$ a smooth compact Riemannian manifold without boundary which is isometrically embedded in some Euclidean space. The poly-energies that we study are, for $u \in H^m(\Omega; N)$,

$$E_m(u; \Omega) = \int_{\Omega} |G_m u|^2 \, dx,$$

where

$$G_m u := \begin{cases} D\Delta^{\frac{m-1}{2}} u & \text{if } m \text{ odd}, \\ \Delta^{\frac{m}{2}} u & \text{if } m \text{ even}. \end{cases}$$

Critical points of these poly-energies are called polyharmonic maps. The Euler-Lagrange equations associated to (1.1) are

$$\begin{cases} (-\Delta)^m u(x) \perp T_{u(x)}N \text{ on } \Omega, \\ D^m u = D^m g \text{ on } \partial \Omega \text{ for } |\alpha| \leq m - 1, \end{cases}$$

where $g$ is boundary data, and the above is interpreted in the distributional sense for $u \in H^m(\Omega; N)$.

We are also interested in the $L^2$-gradient flows of (1.1):

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^m u(t,x) \perp T_{u(t,x)}N \text{ on } \mathbb{R}^+ \times \Omega, \\ D^m u = D^m g \text{ on } \Gamma(\mathbb{R}^+ \times \Omega) \text{ for } |\alpha| \leq m - 1, \end{cases}$$

where $g$ is initial/boundary data, and $\Gamma(U)$ denotes the parabolic boundary of $U \subset \mathbb{R}^{1+d}$. We may replace $\Omega$ with a Riemannian manifold $M$. However, for concreteness, and since our study does not need this generality, our presentation will only consider flat domains.
The energies (1.1) for \( m \geq 2 \) are higher order analogues of the Dirichlet energy, the \( m = 1 \) case. Different higher order energies have been proposed. For example, the intrinsic bi-energy

\[
H_2(u; \Omega) = \int_{\Omega} |(\Delta u(x))^T|^2 \, dx,
\]

where \((\Delta u(x))^T\) is the orthogonal projection of \( \Delta u(x) \) onto \( T_{u(x)}N \). The intrinsic energies do not depend on the embedding of \( N \) into Euclidean space, whereas the extrinsic energies do. This makes the intrinsic energies more natural from a geometric perspective. However, the intrinsic energies lack coercivity, in contrast to the extrinsic energies, which makes them difficult to work with from an analytic perspective. In this work we focus solely on the extrinsic energies.

There is a large literature concerning harmonic maps. The study of higher order polyharmonic maps is more recent. In [He90, He91, He91a], Hélein proved that weakly harmonic maps from surfaces are smooth. Chang, Wang, and Yang in [CWY99] prove the analogue of this result for extrinsic biharmonic maps, in the critical dimension \( d = 4 \), from flat domains into spheres. They also prove partial regularity for biharmonic maps in the supercritical case in analogy with the results of Evans in [Eva91]. For a different but related approach, see [Str03] whose methods extend to \( p \)-harmonic maps and biharmonic maps on the Heisenberg group. Chang, Wang, and Yang’s study of biharmonic maps is connected to applications in four dimensional conformal geometry, for example see [CGY99] and [XY02]. These results were generalized by Wang in [Wan04b, Wan04c, Wan04a]. The case of intrinsic biharmonic maps was studied by Ku in [Ku08]. Interior regularity for intrinsic and extrinsic polyharmonic maps in the critical dimension, that is \( d = 2m \), was proven in [GS09], see also [GSZG09] for the special case of extrinsic polyharmonic maps into spheres. Boundary regularity was settled in [LW09] by Lamm and Wang.

The corresponding heat flow has has also received attention. In particular, Lamm in [Lam04] studies the extrinsic biharmonic map heat flow in the subcritical setting, and in the critical setting with a small initial energy assumption. He is able to show global existence and sub-convergence to a smooth biharmonic map. Then in [Lam05], he proves the analogue of Eells and Sampson’s result, see [ES64], for the intrinsic biharmonic map heat flow in dimensions less than or equal to four. In [Gas06, Wan07] the analogue of Struwe’s work in [Str85] is undertaken for the extrinsic polyharmonic map heat flows.

The work in [Lam04, Gas06, Wan07] is heavily inspired by the ideas in [Str85]. This is because Struwe avoids the maximal principle and relies instead on \( L^2 \)-estimates and interpolation inequalities. These ideas generalize to the higher order cases where the maximal principle does not hold. One may also draw parallels to the work of Kuwert and Schätzle on the Willmore flow, see [KS01], where they also make use of interpolation inequalities.

In [CG89], Coron and Ghidaglia showed that finite time blowup could occur in the supercritical harmonic map heat flow. Then Chang, Ding, and Ye in [CDY92] showed that finite time blowup also occurred in the critical dimension. Chang, Ding, and Ye worked with an equivariant ansatz. More specifically, they considered solutions, denoted by \( u : B(0,1;d) \to S^d \), that satisfy

\[
(1.5) \quad u(x) = T(\psi)(x) := \begin{cases} \left( \frac{x}{|x|} \sin \psi(|x|), \cos \psi(|x|) \right) & \text{for } x \in B(0,1;d) - \{0\}, \\ \hat{e}_{d+1} & \text{if } x = 0, \end{cases}
\]
where \( \psi : [0,1] \to \mathbb{R} \) and \( \psi(0) = 0 \). The condition \( \psi(0) = 0 \) ensures continuity of \( u \) at the origin, if \( \psi \) is itself continuous at zero. For reasons that will become clear later, we call maps that satisfy \[ (1.5) \] equivariant.

More precisely, Chang, Ding, and Ye’s result says that if \( |\psi(0,1)| > \pi \) then the corresponding solution to the harmonic map heat flow blows up in finite time. Prior to this, Chang and Ding, in [CD91], show that if \( \sup_{r \in [0,1]} |\psi(0,r)| \leq \pi \) then the corresponding local in time classical solution to the harmonic map heat flow is in fact global in time. Only when \( |\psi(0,1)| < \pi \) do we have sub-convergence to a harmonic map, see [KW84] for further details. These proofs rely heavily on the comparison/maximum principle.

Our main motivation for this study is to try to extend the work in [CD91, CDY92] to the higher order cases. The lack of a maximum principle makes this task difficult. Looking to the future the following ideas may be useful. In [GP02] Galaktionov and Pohozaev use the technique of majorizing operators in order to obtain a comparison principle for the bi-heat equation. These ideas may allow us to construct barriers in order to prove global existence in a similar way as in [CD91]. However, this will not allow a proof of finite time blowup using barriers. In [RS13], Raphaël and Schweyer look at finite time blowup of the 1-corotational energy critical harmonic map heat flow while avoiding the maximum principle. Instead they rely on energy methods and modulation theory.

Symmetric and equivariant biharmonic maps have already been studied in [WOY14, MR13, GZ12]. In [WOY14], Wang, Ou, and Yang study rotationally symmetric intrinsic biharmonic maps from \( S^2 \) into \( S^2 \). Similar to this work, they compute the corresponding symmetry reduction, and classify their class of symmetric intrinsic biharmonic maps. In [MR13], Montaldo and Ratto examine a more general class of equivariant intrinsic biharmonic maps. They consider maps that are equivariant with respect to Riemannian submersions. They setup machinery to compute the corresponding symmetry reductions, and use this to explicitly compute the symmetry reduction in some concrete cases. As applications they prove the stability of specific proper, that is non-harmonic, intrinsic biharmonic maps from \( T^2 \) into \( S^2 \) among a certain class of equivariant maps. Moreover, they construct a counterexample to a generalization to intrinsic biharmonic maps of Sampson’s maximum principle for harmonic maps, see [Sam78].

In [GZ12], Gastel and Zorn studied a fourth-order ODE very similar to our symmetry reduction of the equivariant biharmonic map equation \[ (2.6) \] (with \( \partial_t \psi = 0 \)). Their ODE arises when trying to construct biharmonic maps of cohomogeneity one between spheres using joins of two harmonic eigenmaps. In contrast to this work, they use purely variational methods. Their stated reason for this choice being that purely ODE methods would cause difficulties due to their ODE being fourth-order and having ill-posed boundary conditions. Although the questions studied in [GZ12] are quite different than the ones studied here, we have found success in using elementary ODE methods to study our similar fourth-order ODE. To us, this demonstrates that ODE methods can be useful in exploring such questions. It may be of interest in future work to see if a synthesis of the ideas in [GZ12] and here can yield deeper insights. Next, we outline the structure of the rest of this paper.

In Section 2 we first prove that the smooth flow of \[ (1.3) \] preserves the ansatz given by \[ (1.5) \]. We refer the reader to [CD91] Lemma 2.2 for a proof of this using the maximum principle in the harmonic case, see also [Gro91] Lemma 4.2 where a similar argument using the maximum principle is used for the axially symmetric harmonic map heat flow. Unfortunately, the maximum principle is not available in
the higher order polyharmonic heat flows. In our approach we show the equivalence of maps satisfying (1.5) and of maps which are, in a natural sense, $O(d)$-equivariant. Then we show that the differential operator in (1.3) is symmetric with respect to this $O(d)$-equivariance. Next, we present Mathematica code that computes the symmetry reduction of the polyharmonic map heat flow of any order. This code is then used to explicitly compute the symmetry reductions of the harmonic, as a check, and biharmonic cases.

In Section 3, we prepare for, and outline our approach to, our deeper study of equivariant biharmonic maps. This study is carried out in Sections 4 and 5. We now discuss the main results of this study.

Firstly, we demonstrate that there are boundary conditions, arising from the restriction of equivariant maps to the boundary, for which there are no equivariant biharmonic extensions.

**Theorem 1.1.** There exists a $K > 0$ such that if

$$u = \Upsilon(\psi) \in C^\infty(B(0, 1; 4); S^4),$$

with $|\psi(1)| \geq K$ and $\partial_r \psi(1) = 0$, then $u$ cannot be a biharmonic map.

This implies that the equivariant biharmonic map heat flow starting from such initial data must blowup in finite time or at infinity. To our knowledge, this was the first blowup result for the higher order polyharmonic map heat flows. Recently in [CL14], an example of finite time blowup for the harmonic map heat flow due to topological reasons was given. Their argument is based on a no neck theorem, and builds upon earlier observations in [QT97]. In [LY13], Liu and Yin prove a no neck theorem for the blowup of a sequence of extrinsic and intrinsic biharmonic maps with bounded energy. Motivated by the arguments in [CL14], Liu and Yin, in [LY14], have used their no neck theorem to show finite time blowup for the biharmonic map heat flow in the critical dimension. More precisely, they prove the following:

**Theorem 1.2 (LY14 Theorem 1.1).** Suppose that $\mathcal{M}'$ is any closed manifold of dimension $m' > 4$ with nontrivial $\pi_4(\mathcal{M}')$ and let $\mathcal{M} = \mathcal{M}' \# T^{m'}$ be the connected sum of $\mathcal{M}'$ with the torus of the same dimension. For any Riemannian metric $g$ on $\mathcal{M}$, we can find (infinitely many) initial maps $u_0 : S^4 \to \mathcal{M}$ such that the biharmonic map heat flow starting from $u_0$ develops a singularity in finite time.

It will be interesting to see if their arguments can be extended to higher order polyharmonic maps. It must be emphasized that the question of finite time blowup for the biharmonic, and other higher order polyharmonic, heat flows into spheres is still open. It is in this latter case that we expect the equivariant ansatz to play an important role.

Secondly, we show that non-constant equivariant biharmonic maps in the critical dimension are unique (modulo dilation).

**Theorem 1.3.** If $u = \Upsilon(\psi) \in C^\infty(\mathbb{R}^4; S^4)$ is a non-constant equivariant biharmonic map, then up to dilation, $\psi(r) = \pm 2 \arctan r$.

This is interesting, because in an upcoming paper the author shows that in the equivariant biharmonic map heat flow blowup coincides with a bubble separating at the origin. Furthermore, this bubble is a non-constant equivariant biharmonic map, hence Theorem 1.3 gives a complete description of these bubbles. By computing the energy of this bubble, we know that if our equivariant initial data has bi-energy
less than or equal to $12|S^3|$ then the resulting flow exists globally in time and sub-converges to a smooth equivariant biharmonic map from $B(0,1;4)$ into $S^4$.

Finally, we show that, in contrast to the harmonic case, there are equivariant biharmonic maps from $B(0,1;4)$ into $S^4$ that wind around $S^4$ as many times as we wish.

**Theorem 1.4.** Let $a \in \mathbb{R}$. Then there exists a biharmonic map

$$u = \Upsilon(\psi) \in C^\infty(B(0,1;4), S^4),$$

such that $\psi(1) = a$.

The paper ends with an appendix which contains a couple of technical proofs which, in the author’s opinion, do not yield much conceptual insight.

**Notation.** Throughout this paper $C$ denotes a positive universal constant. Two different occurrences of $C$ are liable to be different. If our constant depends on some parameter, say $\varepsilon$, then we denote this by writing $C(\varepsilon)$.

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### 2. THE EQUIVARIANT ANSATZ

In this section we show that the ansatz given by (1.5) is preserved by the flow of (1.3). After this, Mathematica code for computing the symmetry reductions is presented. This code can be used to compute the symmetry reduction of the polyharmonic map heat flow of any order. We then use this code to explicitly compute the symmetry reductions for the harmonic, as a check, and biharmonic cases.

**Preservation of symmetry.** We now show that the evolution equation (1.3) preserves the ansatz (1.5). First we need some notation and a definition. We denote the group of $d \times d$ real orthogonal matrices by $O(d)$. For $x \in \mathbb{R}^d$ and $R \in O(d)$, $Rx$ denotes the obvious action of $R$ on $x$, and for $x \in \mathbb{R}^{d+1}$, we set

$$R \cdot x = (R(x_1, \ldots, x_d), x_{d+1}).$$

We let $u : \Omega \rightarrow S^d$, where $\Omega \subset \mathbb{R}^d$ is invariant under the action of $O(d)$, and call $u$ $O(d)$-equivariant, if $R \cdot u(x) = u(Rx)$ for all $x \in \Omega$ and $R \in O(d)$.

Our strategy is simple and is as follows. First we show that $O(d)$-equivariance is preserved by the polyharmonic map heat flow. Then we show that the ansatz given by (1.5) and $O(d)$-equivariance are, essentially, equivalent.

Next, we show that $O(d)$-equivariance is preserved by the evolution of (1.3).

**Lemma 2.1.** Suppose that $d,m \in \mathbb{N}$, $T > 0$, $Q = [0,T] \times B(0,1;4)$, and $u \in C^\infty(Q;S^4)$ is a solution to (1.3) with $g$ as $O(d)$-equivariant initial/boundary data. Then, for each $t \in [0,T]$, $u(t, \cdot)$ is $O(d)$-equivariant and $u(t,0) = \pm \hat{e}_{d+1}$ is constant.
Proof. We let $R \in O(d)$ be arbitrary, and set
\[
v_R(t, x) = R \cdot u(t, R^{-1}x) \text{ for } (t, x) \in Q.
\]
From (1.3), we have
\[
\partial_t u = -(-\Delta)^m u + ((-\Delta)^m u \cdot u) u,
\]
hence
\[
\partial_t v_R(t, x) = -(-\Delta)^m v_R(t, r) + ((-\Delta)^m v_R(t, r) \cdot v_R(t, r)) v_R(t, r),
\]
where we have used properties of orthogonal matrices, and the fact that $u$ is $O(d)$-equivariant. Since $g$ is $O(d)$-equivariant, we have $D_{\alpha}v_R = D_{\alpha}g$ on $\Gamma Q$ for $|\alpha| \leq m - 1$. Therefore, $v_R$ solves (1.3) with the same initial/boundary data. Since we are working in the smooth category, uniqueness of solutions is standard, hence $v_R \equiv u$ and $u(t, x) = R \cdot u(t, R^{-1}x)$ for all $(t, x) \in Q$ and $R \in O(d)$.

Observe that $u(t, 0) = R \cdot u(t, 0)$ for all $R \in O(d)$, hence $u(t, 0)$ must be $\pm \hat{e}_{d+1}$. Since $t \mapsto u(t, 0)$ is continuous, this map must be constant. ■

Next, we show that if a function satisfies (1.5) then it is $O(d)$-equivariant.

Lemma 2.2. Suppose that $d \in \mathbb{N}$ and $u \in C(B(0, 1; d); S^d)$ satisfies (1.5). Then $u$ is $O(d)$-equivariant.

Proof. We let $R \in O(d)$ be arbitrary. Using (1.5), we compute that
\[
R \cdot u(R^{-1}x) = u(x) \text{ for } x \neq 0.
\]
For $x = 0$, we have
\[
R \cdot u(R^{-1}0) = R \cdot u(0) = R \cdot \hat{e}_{d+1} = \hat{e}_{d+1} = u(0).
\]

Finally, we show that if a solution to (1.3) is $O(d)$-equivariant at each time then it is also satisfies (1.5) at each time.

Lemma 2.3. Let $d \in \mathbb{N}$ and $T > 0$. Suppose that $u \in C^\infty([0, T] \times \overline{B(0, 1; d)}; S^d)$, $u(t, \cdot)$ is $O(d)$-equivariant and $u(t, 0) = \hat{e}_{d+1}$ for each $t \in [0, T]$. Then there exists a unique $\psi \in C^\infty([0, T] \times [0, 1])$ such that $\psi(t, 0) = 0$ and $u(t, \cdot) = Y(\psi(t, \cdot))$ for each $t \in [0, T]$.

Proof. Let $R_0 \in O(d)$ be the reflection through the $\hat{e}_1$ axis, that is
\[
R_0 \hat{e}_i = \begin{cases}
\hat{e}_1 & \text{if } i = 1, \\
-\hat{e}_i & \text{otherwise.}
\end{cases}
\]
For $r \in [0, 1]$, we have
\[
u(t, r\hat{e}_i) = R_0 \cdot u(t, R_0^{-1}r\hat{e}_i) = R_0 \cdot u(t, r\hat{e}_i).
\]
Therefore, $u^i(t, r\hat{e}_i) = 0$ for $i \in \{2, 3, \ldots, d\}$, and
\[
(u^1(t, r\hat{e}_1), u^{d+1}(t, r\hat{e}_1)) \in S^1.
\]
Since $u \in C^\infty([0, T] \times \overline{B(0, 1; d)}; S^d)$ and $u(t, 0) = \hat{e}_{d+1}$, there exists a unique $\psi \in C^\infty([0, T] \times [0, 1])$ such that $\psi(t, 0) = 0$ and
\[
u^1(t, r\hat{e}_1) = \sin \psi(t, r), \quad u^{d+1}(t, r\hat{e}_1) = \cos \psi(t, r).
\]
Next, we work in spherical coordinates. We fix an \( \hat{x} \in S^{d-1} \), and let \( R_z \in O(d) \) be a map such that \( R_z \hat{e}_1 = \hat{x} \). Then for \( r > 0 \) we calculate, keeping in mind (2.2) and (2.3),
\[
    u(t, r \hat{x}) = R_z \cdot u(t, r R_z^{-1} \hat{x})
    = R_z \cdot u(t, \hat{e}_1)
    = R_z \cdot (\sin \psi(t, r) \hat{e}_1 + \cos \psi(t, r) \hat{e}_{d+1})
    = \Upsilon(\psi(t, \cdot))(r \hat{x}).
\]
Since \( \hat{x} \) was arbitrary, we are done.

**Symmetry reduction.** For a function \( f : \mathbb{R}^d \to \mathbb{R} \) such that \( f(x) = f(|x|) = f(r) \), we have
\[
    -\Delta f(x) = -\partial_r^2 f(r) - \frac{d-1}{r} \partial_r f(r) =: (L_1 f)(r) \text{ for } x \neq 0.
\]
We also compute:
\[
    -\Delta \left( \frac{x}{|x|} f(x) \right) = \frac{x}{|x|} \left(-\partial_r^2 f(r) - \frac{d-1}{r} \partial_r f(r) + \frac{d-1}{r^2} f(r) \right) =: \frac{x}{|x|} (L_0 f)(r).
\]
For appropriate \( g_0, g_1 : [0,1] \to \mathbb{R} \), we write
\[
    \{g_0, g_1\} \hat{x}(t, x) = (\hat{x}g_0(t, |x|), g_1(t, |x|)).
\]
We observe that, for \( x \neq 0 \),
\[
    -\Delta \{g_0, g_1\} \hat{x} = \{L_0 g_0, L_1 g_1\} \hat{x},
    \partial_t \{g_0, g_1\} \hat{x} = \{\partial_t g_0, \partial_t g_1\} \hat{x}, \text{ and}
    \{g_0, g_1\} \hat{x} \cdot \{h_0, h_1\} \hat{x} = g_0 h_0 + g_1 h_1.
\]
For \( \mathcal{N} \) a sphere, (1.3) can be written as
\[
    |\partial_t u + (-\Delta)^m u - ((-\Delta)^m u \cdot u) u|^2 = 0.
\]
When \( u \) is \( O(d) \)-equivariant this becomes a PDE solely in terms of \( \psi \).

The following Mathematica code computes this symmetry reduction.

```mathematica
(* functions to calculate the Laplacian and time derivative *)
L1 = Function[expr, -D[expr, {r, 2}] - (d - 1)/r D[expr, r]]
L0 = Function[expr, -D[expr, {r, 2}] - (d - 1)/r D[expr, r] + (d - 1)/r^2 expr]
NegLapl = Function[expr, {L0[expr [[1]]], L1[expr [[2]]]}]

(* calculates the symmetry reduction *)
SymmetryReduce =
Function[{m},
    With[{u = {Sin[[Psi]]t, r]},
        With[{D = Nest[NegLapl, u, m]},
            FullSimplify[D . expr == 0]]]]
```

As a test, we use this to compute the symmetry reduction for the harmonic case:
\[
    -\partial_t \psi = \partial_r^2 \psi + \frac{d-1}{r} \partial_r \psi - \frac{(d-1) \sin(2\psi)}{2r^2},
\]
which is what we expect. The symmetry reduction for the biharmonic case is
\[
\partial_t \psi = -\partial_t^4 \psi - \frac{2(d - 1)}{r} \partial_t^2 \psi + 6(\partial_t \psi)^2 \partial_t^2 \psi - \frac{d - 1}{r^2} \sin(2\psi)(\partial_r \psi)^2 \\
+ \frac{2(d - 1)}{r} (\partial_r \psi)^3 + \frac{1}{r^2} ((d - 1) \cos(2\psi) - d^2 + 5d - 4) \partial_r^2 \psi \\
+ \frac{1}{r^3} ((d^2 - 4d + 3) \cos(2\psi) + 2d^2 - 8d + 6) \partial_t \psi \\
- \frac{3(d^2 - 4d + 3)}{2r^4} \sin(2\psi).
\]
(2.6)

Due to the boundary conditions of (1.3), we have
\[
\partial_t \psi(t, 1) = a_i,
\]
for \( t \in [0, T] \) and \( i \in \{0, \ldots, m - 1\} \), where \( a_i \in \mathbb{R} \).

Due to symmetry, \( \psi \) satisfies conditions at the origin. Let \( R_0 \) be the same as in
(2.1). Arguing similar as in Lemma 2.3, we see that there exists a \( \xi \in C^\infty([0, T] \times [-1, 1]; \mathbb{R}) \) such that \( \xi(0) = 0 \) and
\[
u(t, x_1 e_1) = e_1 \sin \xi(t, x_1) + e_{d+1} \cos \xi(t, x_1),
\]
for \( x_1 \in [-1, 1] \). We also have
\[
u(x_1 e_1) = -\hat{R}_0 \cdot \nu(-x_1 e_1).
\]
This implies that
\[
\sin(\xi(x_1)), \cos(\xi(x_1)) = (\sin(-\xi(-x_1)), \cos(-\xi(-x_1))),
\]
hence \( \xi \) is odd.

Observe that if \( \nu \in C^k(B(0, 1; d); S^d) \) then \( \xi \in C^k([-1, 1]; \mathbb{R}) \). In this case, we have
\[
\partial_{x_1}^2 \xi(0) = 0 \quad \text{whenever} \quad 2i \leq k. \quad \text{Since} \quad \psi = \xi|_{[0, 1]}, \quad \psi \in C^k([0, 1]; \mathbb{R}) \text{ and } \partial_r^2 \psi(0) = 0
\]
whenever \( 2i \leq k \).

3. Critical equivariant biharmonic maps

Now we start to delve deeper into \( O(d) \)-equivariant biharmonic maps in the critical dimension.

After setting \( \partial_t \psi = 0 \) in (2.6) and making the change of variables \( \psi(r) = \varphi(s(r)) \),
where \( s(r) = \log r \), (2.6) becomes the fourth-order autonomous ODE
\[
\partial_s^4 \varphi = -\frac{9}{2} \sin(2\varphi) + (7 + 3 \cos(2\varphi)) \partial_s^2 \varphi + 3(\partial_s \varphi)^2 (2\partial_s^2 \varphi - \sin(2\varphi)).
\]
(3.1)

The boundary condition \( \varphi(0) = 0 \) becomes
\[
\lim_{s \to -\infty} \varphi(s) = 0.
\]
(3.2)

We rewrite this as a first-order system by setting \( \Phi_i = \partial_s^{i-1} \varphi \) for \( i \in \{1, 2, 3, 4\} \):
\[
\partial_s \Phi = \begin{pmatrix}
\Phi_2 \\
\Phi_3 \\
\Phi_4 \\
F_1(\Phi_1, \Phi_3) + \Phi_2^2 F_2(\Phi_1, \Phi_3)
\end{pmatrix},
\]
(3.3)

where
\[
F_1(\Phi_1, \Phi_3) = -\frac{9}{2} \sin(2\Phi_1) + (7 + 3 \cos(2\Phi_1)) \Phi_3, \quad \text{and}
\]
\[
F_2(\Phi_1, \Phi_3) = 3(2\Phi_3 - \sin(2\Phi_1)).
\]
The boundary condition (3.2) becomes

\[ \lim_{s \to -\infty} \Phi_1(s) = 0. \]

Observe that (3.3) and (3.4) are invariant under the transformation \( \Phi \mapsto -\Phi \).

An \( s_0 \in \mathbb{R} \) and initial data \( \Phi_0(s_0) \in \mathbb{R}^4 \) generate a unique solution to (3.3), denoted by \( \Phi_0 : [s_0, s_{\max}) \to \mathbb{R}^4 \), where either \( s_{\max} = \infty \) or \( \lim_{s \to s_{\max}} |\Phi_0(s)| = \infty \). The next lemma shows the equivalence between a solution of (3.3) satisfying (3.4) and it being an orbit in the unstable manifold of the origin of (3.3), denoted from now on by \( W^u(0) \).

**Lemma 3.1.** Suppose that \( \Phi_0 : (-\infty, s_{\max}) \to \mathbb{R}^4 \) solves (3.3). Then the following are equivalent:

1. \( \lim_{s \to -\infty} \Phi_0(s) = 0 \);
2. \( \lim_{s \to -\infty} \Phi_0(s) = 0 \).

For the proof, see the Appendix.

Observe that \( y(s) = 2 \arctan(e^s) \) is a heteroclinic orbit of (3.1), and \( \Upsilon(2 \arctan(\cdot)) \), for \( d \geq 2 \), is the inverse of the stereographic projection of \( S^{d-1} \setminus \{-\hat{e}_d\} \) onto \( \mathbb{R}^{d-1} \).

We set \( Y_i = \partial_i - 1 s y \) for \( i \in \{1, 2, 3, 4\} \).

The first result we will focus on concerning (3.3) states that this heteroclinic orbit gives rise to the only non-constant biharmonic map, up to dilation, from \( \mathbb{R}^4 \) into \( S^4 \).

**Theorem 3.2.** Suppose that \( \Phi_0 : (-\infty, s_{\max}) \to \mathbb{R}^4 \) is a non-trivial orbit in \( W^u(0) \). Then the following dichotomy holds:

1. up to \( s \)-translation \( \Phi_0(s) = Y(s) \) or \( \Phi_0(s) = -Y(s) \) and hence \( s_{\max} = \infty \); or
2. \( \Phi_0 \) blows up in finite time, that is \( s_{\max} < \infty \) and
\[ \lim_{s \to s_{\max}} |\Phi_0(s)| = \infty. \]

Theorem 1.3 is a corollary of this. Next, we outline our strategy for the study of (3.3).

**Strategy.** Our arguments are in part motivated by the harmonic map case. The analogue of (3.1) in the critical harmonic map case is

\[ \partial_s^2 \phi_h = \frac{1}{2} \sin(2\phi_h), \]
\[ \lim_{s \to -\infty} \phi_h(s) = 0. \]

This ODE is a pendulum equation. One can think of it as describing the dynamics of a ball rolling without friction in coordinate space on the potential energy surface \( V(q) = \frac{1}{2} (\cos q)^2 \). After some consideration it is clear that if \( \lim_{s \to -\infty} \phi_h(s) = 0 \) then up to \( s \)-translation there are only two possibilities for \( \phi_h \). These possibilities being the heteroclinic orbits between \( (\phi_h, \partial_s \phi_h) = (0, 0) \) and \( (\phi_h, \partial_s \phi_h) = (\pi, 0) \), and between \( (\phi_h, \partial_s \phi_h) = (0, 0) \) and \( (\phi_h, \partial_s \phi_h) = (-\pi, 0) \). These turn out to be, up to \( s \)-translation, \( \pm y(s) \).

We see that if \( \phi_h \) is an \( s \)-translation of \( \pm y(s) \) then \( |\phi_h| < \pi \). Therefore, if we have equivariant initial data \( u_0 = \Upsilon(\psi_0) \) in the harmonic map heat flow from \( B(0, 1; 2) \) into \( S^2 \) such that \( |\psi_0(1)| \geq \pi \) then the flow must blowup either in finite time or
at infinity, because it cannot sub-converge to a harmonic map. Theorem 1.1 is the analogue of this observation for the biharmonic map case.

Our situation is more complicated than the one encountered when studying (3.5), because instead of a one-dimensional coordinate space we now have a two-dimensional coordinate space. This adds a lot of flexibility to the possible dynamics. Moreover, unlike (3.5), (3.1) does not have a convenient Hamiltonian formulation. In spite of this, the author found it fruitful to consider (3.1) as the following coupled system of second order ODE:

\[
\begin{align*}
\partial_s^2 \Phi_1 &= \Phi_3 \\
\partial_s^2 \Phi_3 &= F_1(\Phi_1, \Phi_3) + (\partial_s \Phi_1)^2 F_2(\Phi_1, \Phi_3),
\end{align*}
\]

and to think about the ‘forces’ acting on the system in the coordinate space \((\Phi_1, \Phi_3)\), see Figure 1.

Our arguments are also inspired by the ideas in [Gas02]. There the shooting method along with a pendulum equation interpretation was used to show the existence of singularities of the first kind in the harmonic map and Yang-Mills heat flows.

Much of our analysis revolves around finding positive invariant sets on which we approximate (3.3) by simpler ordinary differential inequalities that still give us enough control of the orbits in \(W^u(0)\). We find it convenient to divide the life of each orbit in \(W^u(0)\) into three stages:

**Early life.** This is when the orbit is still close to the origin and its dynamics are well approximated by the linearization of (3.3) at the origin.

**Mid life.** This is the most delicate stage to analyze, because we must deal with the intricacies of the ‘forces’ acting on \(\Phi\) when \(|\Phi_3|\) is not so large, see Figure 1.

**Late life.** This is when \(|\Phi_3| \gg 1\). In such a case (3.3) behaves in a manner similar to

\[
\partial_s Z = \begin{pmatrix} Z_2 \\ Z_3 \\ Z_4 \\ (7 + 6Z_2^2)Z_3 \end{pmatrix}.
\]

This equation is much easier to analyze and one can prove blowup for it. In fact one can use this equation when \(|\Phi_3| \geq 1\).

4. **Finite time blowup or heteroclinic orbit**

The aim of this section is to prove Theorem 3.2 and to collect some facts along the way which will be used in our later arguments. We let

\[
\begin{align*}
W_+ &= \{ x \in \mathbb{R}^4 : (x_1, x_3), (x_2, x_4) \in \Lambda_+ \}, \\
W_- &= \{ x \in \mathbb{R}^4 : (x_1, x_3), (x_2, x_4) \in \Lambda_- \},
\end{align*}
\]

where

\[
\begin{align*}
\Lambda_+ &= \{ x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 2x_1 \}, \\
\Lambda_- &= \{ x \in \mathbb{R}^2 : x_1 \leq 0, x_2 \leq 2x_1 \}.
\end{align*}
\]

The following sets will also be useful:

\[
\begin{align*}
W^*_+ &= W_+ \cap \{ x \in \mathbb{R}^4 : x_3 \neq 0 \}, \\
W^*_- &= W_- \cap \{ x \in \mathbb{R}^4 : x_3 \neq 0 \}.
\end{align*}
\]

Now we consider the different life stages of orbits in \(W^u(0)\).
\[
\Phi_3 = 1
\]

\[
\Phi_3 = -1
\]

\[
\Phi_1
\]

\[
\Phi_3
\]

\[
(\pi/2, 0)
\]

\[
(-\pi/2, 0)
\]

Some orbits in \(W^u(0)\)

\[
F_1(\Phi_1, \Phi_3) = 0
\]

\[
(Y_1, Y_3)
\]

\[
(\Phi_3, F_1(\Phi_1, \Phi_3))
\]

\textbf{Figure 1.} This represents the portion of the coordinate space that is of interest for mid life orbits in \(W^u(0)\). Note that there is the additional force \((\partial_s \Phi_1)^2 F_2(\Phi_1, \Phi_3)\). This force acts to repulse solutions away from the graph of \(\Phi_3 = \frac{1}{2} \sin(2\Phi_1)\) in coordinate space. Observe that \(Y_3 = \frac{1}{2} \sin(2Y_1)\).

\textbf{Early life.} The next lemma gives sufficient control over the orbits in \(W^u(0)\) early in their life when their dynamics are still well approximated by the linearization of (5.3) at the origin.

\textbf{Lemma 4.1.} Suppose that \(\Phi^0 : (-\infty, s_{\text{max}})\) is a non-trivial orbit in \(W^u(0)\) and \(\sigma > 0\). Then there exists an \(s_0 \in (-\infty, s_{\text{max}})\) and an \(s\)-translation of \(Y\), denoted by \(Y^0\), such that either:

1. \[
\Phi_1^0(s_0) = Y_1^0(s_0),
\]

   \[
   \Phi_1^0(s_0) - Y_1^0(s_0) \in W^*_+ \cup W^*_-, \text{ and}
   \]

   \[
   |\Phi_3^0(s_0) - Y_3^0(s_0)| < \sigma;
   \]

2. \[
   \Phi_1^0(s_0) = Y_1^0(s_0),
   \]

   \[
   -\Phi_1^0(s_0) - Y_1^0(s_0) \in W^*_+ \cup W^*_-, \text{ and}
   \]

   \[
   |\Phi_3^0(s_0) + Y_3^0(s_0)| < \sigma;
   \]

3. \[
   \Phi_1^0(s_0) = Y_1^0(s_0); \text{ or}
   \]

4. \[
   \Phi_1^0(s_0) = -Y_1^0(s_0).
   \]

\textbf{Proof.} Via the Stable Manifold theorem, see [Per91, Section 2.7] for a proof, \(W^u(0)\) is a smooth 2-manifold embedded in \(\mathbb{R}^4\).
Since we can choose \( s \) as we like, we can arrange for \( Y \) for \( i \leq 3 \).

Recall that (3.3) is invariant under the transformation \( \Phi \):

\[
\partial_s \Phi_{\text{lin}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & 0 & 10 & 0 \end{pmatrix} \Phi_{\text{lin}} =: A \Phi_{\text{lin}}.
\]

The eigenvalues of \( A \) are \(-3, -1, 1, 3\) with the corresponding eigenvectors

\[
\begin{pmatrix} 1 \\ -3 \\ 9 \\ -27 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ 3 \\ 9 \\ 27 \end{pmatrix}.
\]

Therefore, the tangent plane of \( W^u(0) \) at the origin coincides with the linear subspace spanned by \((1, 1, 1)^T\) and \((1, 3, 9, 27)^T\), hence \( W^u(0) \) may be locally written as a graph over the \( \Phi_1 - \Phi_3 \) plane:

\[
\begin{cases}
\Phi_2(\Phi_1, \Phi_3) = \frac{3}{4} \Phi_1 + \frac{1}{4} \Phi_3 + G_2(\Phi_1, \Phi_3), \\
\Phi_4(\Phi_1, \Phi_3) = -\frac{9}{4} \Phi_1 + \frac{13}{4} \Phi_3 + G_4(\Phi_1, \Phi_3),
\end{cases}
\]

where

\[
\frac{\partial(G_2, G_4)}{\partial(\Phi_1, \Phi_3)}(0, 0) = 0.
\]

Observe that, since \( \Phi^0 \) is not the trivial orbit, the dynamics of (3.3) give, for any \( s_0 \in (-\infty, s_{\text{max}}) \), an \( s \in (-\infty, s_0] \) such that \( \Phi^0_i(s) \neq 0 \). Therefore, we may find an \( s_0 \) sufficiently negative so that \( \Phi^0_i(s_0) \neq 0 \) and \( \Phi^0(s_0) \) is as small as we wish.

Next, we assume that \( \Phi^0_i(s_0) > 0 \) and \( s_0 \) is sufficiently negative. We take \( Y^0 \) to be an \( s \)-translation of the heteroclinic orbit \( Y \) such that \( Y^0_i(s_0) = \Phi^0_i(s_0) \). Note that \( Y^0 \) is also an orbit in \( W^u(0) \) and may be parameterized by \( \Phi_1 \). If \( \Phi^0_i(s_0) = Y^0_i(s_0) \) then \( \Phi^0_i(s_0) = Y^0_i(s_0) \), since locally around the origin \( W^u(0) \) is a graph over the \( \Phi_1 - \Phi_3 \) plane. This is the case of (3).

From (4.2), we have

\[
\begin{cases}
\frac{\partial \Phi_2}{\partial \Phi_1}(\Phi_1, \Phi_3) = \frac{1}{4} + o(\Phi_1, \Phi_3) \to 0(1), \\
\frac{\partial \Phi_4}{\partial \Phi_1}(\Phi_1, \Phi_3) = \frac{13}{4} + o(\Phi_1, \Phi_3) \to 0(1).
\end{cases}
\]

If \( \Phi^0_2(s_0) > Y^0_3(s_0) \) then \( \Phi^0(s_0) - Y^0(s_0) \) is \( W^*_+ \). On the other hand, if \( \Phi^0_2(s_0) < Y^0_3(s_0) \) then \( \Phi^0(s_0) - Y^0(s_0) \) is \( W^*_- \).

Since we can choose \( s_0 \) sufficiently negative so that \( \Phi^0_2(s_0) \) and \( Y^0(s_0) \) are as small as we like, we can arrange for \( |\Phi^0_2(s_0) - Y^0_3(s_0)| < \alpha \). This is the case of (1).

Recall that (3.3) is invariant under the transformation \( \Phi \mapsto -\Phi \). Therefore, if \( \Phi^0_i(s_0) < 0 \) then we may argue the same as above but with \( -\Phi_0 \) instead of \( \Phi_0 \). This leads to the cases of (2) and (4). There are no more cases to consider.

Due to symmetry, it suffices to only consider the cases (1) and (3) of Lemma 4.1.

**Mid life.** Now that we have \( \Phi^0 - Y^0 \in W_+ \cup W_- \), we can approximate (3.3) by a simpler ordinary differential inequality.

Let \( \Phi^0 : (-\infty, s_{\text{max}}) \to \mathbb{R}^4, Y^0 \), and \( s_0 \) be the same as in Lemma 4.1 with \( \Phi^0 - Y^0 \in W_+ \cup W_- \). We set \( X(s) = \Phi^0(s) - Y^0(s) \) for \( s \in [s_0, s_{\text{max}}] \). Note that \( \partial_s X_i = X_{i+1} \) for \( i \in \{1, 2, 3\} \).
Before we prove our next result, we need an estimate.

**Lemma 4.2.** Let 
\[ f(y) = \frac{1}{2} \sin(2y)(3 \cos(2y) - 2) \]
and 
\[ Q(x; f(y)) = \frac{2f(y) + 9 \sin(2x)}{14 + 6 \cos(2x)}, \]
for \( x, y \in \mathbb{R} \). Then there exists a \( c_0 \in (0, 1) \) such that \( \partial_Q Q(x; f(y)) \leq c_0 \) for all \( x, y \in \mathbb{R} \).

For the proof, see the Appendix.

**Lemma 4.3.** If \( X \in W_+ \) then \( \partial_s X_4 \geq 4(X_3 - c_0 X_1) \), and if \( X \in W_- \) then \( \partial_s X_4 \leq 4(X_3 - c_0 X_1) \), where \( c_0 \in (0, 1) \) is from Lemma 4.2.

**Proof.** Note that \( Y_3 = \frac{1}{2} \sin(2Y_1) \). We have 
\[ \partial_s X_4 = F_1(Y_1^0 + X_1, Y_3^0 + X_3) - F_1(Y_1^0, Y_3^0) + (X_2 + Y_2^0)^2 F_2(Y_1^0 + X_1, Y_3^0 + X_3), \]

since \( F_2(Y_1^0, Y_3^0) = 0 \).

If \( X \in W_+ \) then \( F_2(Y_1^0 + X_1, Y_3^0 + X_3) \geq 0 \), hence 
\[ \partial_s X_4 \geq F_1(Y_1^0 + X_1, Y_3^0 + X_3) - F_1(Y_1^0, Y_3^0). \]

On the other hand, if \( X \in W_- \) then \( F_2(Y_1^0 + X_1, Y_3^0 + X_3) \leq 0 \), hence 
\[ \partial_s X_4 \leq F_1(Y_1^0 + X_1, Y_3^0 + X_3) - F_1(Y_1^0, Y_3^0). \]

Next, we study \( F_1(Y_1^0 + X_1, Y_3^0 + X_3) - F_1(Y_1^0, Y_3^0) \). Note that 
\[ F_1(Y_1^0, Y_3^0) = F_1\left(Y_1^0, \frac{1}{2} \sin(2Y_1^0)\right) = F_1(Y_1^0). \]

We are interested in the curve in the \( \Phi_1 - \Phi_3 \) plane such that \( F_1(\Phi_1, \Phi_3) = F_1(Y_1^0) \) for given values of \( Y_1^0 \in (0, \pi) \). This curve can be written as a graph over \( \Phi_1 \):
\[ Q(\Phi_1; F_1(Y_1^0)) = \frac{2F_1(Y_1^0) + 9 \sin(2\Phi_1)}{14 + 6 \cos(2\Phi_1)}. \]

Therefore, 
\[ F_1(Y_1^0 + X_1, Y_3^0 + X_3) - F_1(Y_1^0, Y_3^0) \]
\[ = F_1(Y_1^0 + X_1, Y_3^0 + X_3) - F_1(Y_1^0 + X_1, Y_3^0 + X_3, Q(Y_1^0 + X_1; F_1(Y_1^0))). \]

Lemma 4.2 gives \( \partial_Q Q(\Phi_1; F_1(Y_1^)) \leq c_0 \).

Therefore, if \( X \in W_+ \) then 
\[ Q(Y_1^0 + X_1; F_1(Y_1^0)) \leq Y_3^0 + c_0 X_1, \]
and if \( X \in W_- \) then 
\[ Q(Y_1^0 + X_1; F_1(Y_1^0)) \geq Y_3^0 + c_0 X_1. \]

Therefore, for \( X \in W_+ \):
\[ \partial_s X_4 \geq F_1(Y_1^0 + X_1, Y_3^0 + X_3) - F_1(Y_1^0 + X_1, Y_3^0 + c_0 X_1) \]
\[ \geq \int_{Y_3^0 + c_0 X_1}^{Y_3^0 + X_3} \partial_{\Phi_3} F_1(Y_1^0 + X_1, \Phi_3) d\Phi_3 \]
\[ \geq 4(X_3 - c_0 X_1), \]
Lemma 4.7. Suppose that

\[ 0 < \sigma \] 

Proof. Let \( X \) be arbitrary. First we consider the case where \( \sigma > 0 \) and \( \Phi \) is an \( s \)-translation of \( Y \). Hence, while \( s \rightarrow -\infty \), we can control the growth of \( Y \) at \( s \) is exactly the same.

Remark 4.6. Observe that \( s_1 \leq s_{\text{max}} \), because while \( |\Phi^0| \leq 1 \) we can control the growth of \( |\Phi^0| \).

Proof. We set \( X = \Phi^0 - Y_0 \). First, assume that \( X(s_0) \) is an \( s \)-translation of \( Y \). Lemmas 4.3 and 4.4 give \( \partial^2 s X_3 \geq 2X_3(0) \), hence

\[ \Phi^0(s_0 + s) \geq Y_3^0(s_0 + s) + X_3(0)s^2 \geq -\frac{1}{2} + X_3(0)s^2. \]

Therefore, \( s_1 \in \left( s_0, s_0 + \left( \frac{3}{2X_3(0)} \right)^{\frac{1}{2}} \right) \).

The argument for \( X(s_0) \) is exactly the same.

Late life. The next lemma shows that once an orbit \( \Phi^0 : (-\infty, s_{\text{max}}) \) has exited the region \( |\Phi^0| < 1 \) it blows up in finite time, and in the process \( |\Phi^0| \) diverges to infinity.

Lemma 4.7. Suppose that \( \Phi_0 \in \mathbb{R}^4 \) such that \( \Phi_{0,3} \geq 1 \) and \( \Phi_{0,4} \geq 0 \). Let \( \Phi^0 : [0, s_{\text{max}}) \rightarrow \mathbb{R}^4 \) be the solution to \( \Phi^0 = \Phi_0 \). Then \( \Phi^0 \) blows up in finite time, that is \( s_{\text{max}} < \infty \) and

\[ \lim_{s \rightarrow s_{\text{max}}} |\Phi^0(s)| = \infty. \]

Moreover, \( \Phi^0(s) \rightarrow \infty \) as \( s \rightarrow s_{\text{max}} \).

Proof. Let \( \sigma > 0 \) be arbitrary. First we consider the case where \( \Phi^0(0) \geq \sigma \).

Observe that the set

\[ S_1 = \{ \Phi \in \mathbb{R}^4 : \Phi_2 \geq \sigma, \Phi_3 \geq 1, \Phi_4 \geq 0 \} \]
is positive invariant under the flow described by (3.3). For $\Phi^0 \in S_1$, we have

$$(4.5) \quad \partial_s \Phi^0_i = F(\Phi^0) \left( \frac{\Phi^0}{3} \right)^{\frac{3}{2} \Phi^0_i},$$

where $F(\Phi) \in [c_0, c_1] \subset (0, \infty)$, $c_0 = c_0(\sigma)$, and $c_1 = c_1(\sigma)$.

Observe that in (4.5) $\Phi^0_1$ does not play a significant role. Now we consider rescaled versions of $\Phi^0_2$ and $\Phi^0_3$:

$$z_1 = \frac{\Phi^0_2}{(\Phi^0_4)^{\frac{1}{2}}} \quad \text{and} \quad z_2 = \frac{\Phi^0_3}{(\Phi^0_4)^{\frac{1}{2}}}.$$ 

We differentiate:

$$\begin{align*}
\partial_s z_1 &= \left( \Phi^0_4 \right)^{\frac{1}{2}} z_2 \left( 1 - \frac{F(\Phi^0)}{3} z_1^3 \right), \quad \text{and} \\
\partial_s z_2 &= \left( \Phi^0_4 \right)^{\frac{1}{2}} \left( 1 - \frac{2F(\Phi^0)}{3} z_1^2 z_2^3 \right).
\end{align*}$$

Now (4.5) becomes

$$\partial_s \Phi^0_i = F(\Phi^0) z_1^2 z_2^2 \left( \Phi^0_4 \right)^{\frac{1}{2}}.$$ 

Problems may arise with $z_1(0)$ and $z_2(0)$, if $\Phi^0_1(0) = 0$. In this case we would like to examine $z_1(s)$ and $z_2(s)$ for $0 < s \ll 1$. We have $\Phi^0_i(0) \in S_1$, hence $\Phi^0_i(s) > 0$ for all $s \in [0, s_{\text{max}})$. Therefore, $z_1$ and $z_2$ are well defined for $s \in (0, s_{\text{max}})$, and $z_1(s), z_2(s) \to \infty$ as $s \to 0$. On the other hand, if $\Phi^0_i(0) > 0$ then $z_1(0), z_2(0) > 0$.

Therefore, there exists an $\tilde{s} \in [0, s_{\text{max}})$ such that $\Phi^0(\tilde{s}) \in S_1$, $\Phi^0_i(\tilde{s}) > 0$, and $z_1(\tilde{s}), z_2(\tilde{s}) > 0$. We $s$-translate so that $\tilde{s} = 0$, and set

$$Z = [z_{1, a}, z_{1, b}] \times [z_{2, a}, z_{2, b}] \subset (0, \infty) \times (0, \infty),$$

where

$$z_{1, a} = \min \left\{ z_1(0), 2 \left( \frac{3}{c_1} \right)^{\frac{1}{2}} \right\}, \quad z_{1, b} = \max \left\{ z_1(0), 2 \left( \frac{3}{c_0} \right)^{\frac{1}{2}} \right\},$$

and

$$z_{2, a} = \min \left\{ z_2(0), 2 \left( \frac{3}{2c_0} \right)^{\frac{1}{2}} z_{1, b}^{-1} \right\}, \quad z_{2, b} = \min \left\{ z_2(0), 2 \left( \frac{3}{2c_0} \right)^{\frac{1}{2}} z_{1, a}^{-1} \right\}.$$ 

Observe that $Z$ is a positive invariant set for $(z_1, z_2)$, and $(z_1(0), z_2(0)) \in Z$. Therefore, for $s \in (0, s_{\text{max}})$, we have

$$c_0 z_{1, a}^2 z_{2, a} \left( \Phi^0_4 \right)^{\frac{1}{2}} \leq \partial_s \Phi^0_i \leq c_1 z_{1, b}^2 z_{2, b} \left( \Phi^0_4 \right)^{\frac{1}{2}},$$

$$z_{1, a} \left( \Phi^0_4 \right)^{\frac{1}{2}} \leq \Phi^0_i \leq z_{1, b} \left( \Phi^0_4 \right)^{\frac{1}{2}}, \quad \text{and}$$

$$z_{2, a} \left( \Phi^0_4 \right)^{\frac{1}{2}} \leq \Phi^0_4 \leq z_{2, b} \left( \Phi^0_4 \right)^{\frac{1}{2}}.$$ 

Therefore, $\Phi^0_i$ controls $|\Phi^0|$. We have $\partial_s \Phi^0_i \geq C(\Phi^0_4)^{\frac{1}{2}}$ and $\Phi^0_i(0) > 0$, hence $\Phi^0_i$ diverges to infinity in finite time, that is $\Phi^0$ blows up in finite time.

Next, we turn our attention to showing that $\Phi^0_i \to \infty$ as $s \to s_{\text{max}}$. We let $i_0 \in \mathbb{N}$ be such that $2^{i_0} > \Phi^0_i(0)$. For $i \in \mathbb{N}_0$, we let $s_i$ be defined via $\Phi^0_i(s_i) = 2^{i_0 + i}$. Since $\Phi^0_i$ is monotone increasing and diverges to infinity, these times are well-defined. Because $\partial_s \Phi^0_i \leq C(\Phi^0_i)^{\frac{1}{2}}$, we have that $s_{i+1} - s_i \geq C2^{-\frac{1}{2}(i_0+i)}$. For $s \in [s_i, s_{i+1}]$, we have $\partial_s \Phi^0_i(s) \geq C2^{\frac{1}{2}(i_0+i)}$. Therefore, $\Phi^0_i(s_{i+1}) \geq \Phi^0_i(s_i) + C$ which implies $\Phi^0_i(s) \to \infty$ as $s \to s_{\text{max}}$, since $\Phi^0_i$ is monotone increasing.
Finally, we consider the case in which \( \Phi_2(0) \leq 0 \). Observe that the set

\[
S_2 = \{ \Phi \in \mathbb{R}^1 : \Phi_3 \geq 1, \Phi_4 \geq 0 \}
\]

is positive invariant under the flow described by (3.3). Therefore, while \( \Phi_2^0 \leq 0 \) we have \( |\Phi_2^0| \leq -\Phi_2^0(0) \) and

\[
|\partial_s \Phi_4^i| \leq C \Phi_3^0(1 + |\Phi_2^0(0)|^2).
\]

Therefore, there exists an \( \tilde{s} \in (0, s_{\text{max}}) \) such that \( \Phi_2^0(\tilde{s}) > 0 \). Now by autonomy we may translate, and then apply the previous argument to the new initial data \( \Phi^0(\tilde{s}) \).

**Proof of Theorem 3.2** If \( \Phi_0 = Y^0 \) or \( -\Phi^0 = Y^0 \) where \( Y^0 \) is an \( s \)-translation of \( Y \) then we are done. Otherwise Lemma 1.1 tells us that at some \( s_0 \in (-\infty, s_{\text{max}}) \) we have \( \Phi^0(s_0) - Y_0(s_0) \in W^+_t \cup W^- \) or \( -\Phi^0(s_0) - Y^0(s_0) \in W^+_t \cup W^- \). Since (3.3) is invariant under the transformation \( \Phi \mapsto -\Phi \), it suffices to only consider the case where \( \Phi^0(s_0) - Y^0(s_0) \in W^+_t \cup W^- \). Now Lemma 4.5 shows that there exists an \( s_1 \in (-\infty, s_{\text{max}}) \) such that \( |\Phi_3^0(s_1)| = 1 \). Again due to the invariance of (3.3) under \( \Phi \mapsto -\Phi \), we may assume that \( \Phi_3^0(s_1) = 1 \). Finally, Lemma 1.7 shows that \( \Phi^0 \) must blowup in finite time.

5. Further properties of the unstable manifold

In this section we continue our study of \( W^u(0) \). This leads to the proofs of Theorems 1.1 and 1.3.

Our first result concerns non-trivial orbits \( \Phi^0 \) in \( W^u(0) \) which are not \( s \)-translations of \( \pm Y \). We know that these orbits must exit the region \( |\Phi_3| < 1 \) in finite time. The next lemma tells us that \( (\Phi_1^0, \Phi_2^0) \) stays within a bounded region of \( \mathbb{R}^2 \) up until and including this exit time.

**Lemma 5.1.** Let \( \Phi^0 : (-\infty, s_{\text{max}}) \) be an orbit in \( W^u(0) \) such that \( |\Phi_3^0(\tilde{s})| = 1 \) for some \( \tilde{s} \in (-\infty, s_{\text{max}}) \). Moreover, let \( \tilde{s} \) be the first such time in which \( |\Phi_3^0(\tilde{s})| = 1 \). Then \( |(\Phi_1^0(s), \Phi_2^0(s))| \leq C \) for all \( s \in (-\infty, \tilde{s}) \).

**Proof.** Since \( \Phi^0 \) is non-trivial and not an \( s \)-translation of \( \pm Y \), we are in either Case (1) or Case (2) of Lemma 1.1.

For now we assume that we are in Case (1). This means we have an \( s_0 \in (-\infty, s_{\text{max}}) \), a \( Y^0 \) which is an \( s \)-translation of \( Y \), and an \( X = \Phi^0 - Y^0 \) such that \( X(s_0) \in W^+_t \cup W^- \) and \( X_3(s_0) \neq 0 \) is as small as we like, in particular \( |X_3(s_0)| \leq \frac{1}{4} \). Due to autonomy we may assume that \( s_0 = 0 \). Observe that (4.3) gives

\[
|X_2(0)| \leq C|X_3(0)| \text{ and } |X_4(0)| \leq C|X_3(0)|.
\]

On intervals on which we have uniform control of \( X_3 \) we also have uniform control of \( \Phi_3 \). Therefore, blowup may not happen on such intervals. In what follows we study \( X_3 \) for \( |X_3| \leq 2 \).

We let \( i_0 \) be the largest integer such that \( 2^{-i_0+1} > |X_3(0)| \). We set \( s_0 = 0 \) and \( |X_3(s_i)| = 2^{-i_0+i} \) for \( i \in \mathbb{N} \). For \( s \geq |s_i, s_{i+1}| \), we have \( \partial_s |X_3| \geq 2^{-i_0+1} \), hence \( |X_3(s)| \geq |X_3(s_i)| + 2^{-i_0+1}(s_{i+1} - s_i) \) which implies that \( s_{i+1} - s_i \leq 2 \).

For \( s \in [s_i, s_{i+1}] \), we have \( \partial_s |X_2| \leq 2^{-i_0+i+1} \), hence

\[
|X_2(s)| \leq |X_2(0)| + C \sum_{j=0}^{i-1} 2^{-i_0+j+1} \leq C \left(|X_3(0)| + 2^{-i_0+i+1}\right) \text{ for } s \in [s_0, s_i].
\]
where we have used \((5.1)\). Observe that \(\tilde{s} \in [0, s_{m+1}]\), hence \(|X_2(s)| \leq C\) for all \(s \in (-\infty, \tilde{s})\). This implies that, for all \(s \in (-\infty, \tilde{s})\), \(|\Phi_2^0(s)| \leq C\). Moreover, since \(X(s) \in W_+ \cup W_-\), we have \(|\Phi_0^1(s)| \leq C\).

Finally, we look at what happens if we are in Case (2). Because of the invariance of \((5.3)\) under the transformation \(\Phi \mapsto -\Phi\), we may use the above argument on \(-\Phi^0\) yielding the same conclusion of \(|(\Phi_0^1(s), \Phi_0^2(s))| \leq C\) for all \(s \in (-\infty, \tilde{s})\).

Using this result, we show the existence of boundary conditions arising from the restriction of equivariant maps on \(B(0, 1; 4)\) to \(\partial B(0, 1; 4)\) for which there are no equivariant biharmonic extension.

**Lemma 5.2.** Let \(\Phi \in W^n(0)\). Then there exists a \(K > 0\) such that if \(\Phi_2 = 0\) then \(|\Phi_1| \leq K\).

**Proof.** Let \(\Phi^0 : (-\infty, s_{max}) \to \mathbb{R}^4\) be an orbit in \(W^n(0)\). If \(\Phi^0\) is trivial or an \(s\)-translation of \(\pm Y\) then \(|\Phi_1^0(s)| < \pi\) for all \(s \in (-\infty, s_{max})\), hence if we take \(K > \pi\), then these cases cause us no issues.

Otherwise, we find ourselves in Case (1) or Case (2) of Lemma 4.1. In these cases Lemma 4.5 gives a time \(\tilde{s} \in (-\infty, s_{max})\) such that \(|\Phi_1^0(\tilde{s})| = 1\). We may assume that \(\tilde{s}\) is the first such time. Lemma 5.1 shows that \(|(\Phi_1^0(s), \Phi_2^0(s))| \leq C\) for all \(s \in (-\infty, \tilde{s})\).

Let us assume that \(\Phi_0^0(\tilde{s}) = 1\). If \(\Phi_0^1(\tilde{s}) > 0\) then it will be positive for all \(s \in [\tilde{s}, s_{max}]\), since \(S_1\) from \((4.4)\) is a positive invariant set. Therefore, if \(\Phi_0^2(s) = 0\) then \(s < \tilde{s}\) and \(|\Phi_1^0(s)| \leq C\). On the other hand, if \(\Phi_0^2(s) \leq 0\) then \(\partial_s \Phi_2^0(s) > 1\) for all \(s \in [\tilde{s}, s_{max}]\), since \(S_2\) from \((4.6)\) is a positive invariant set. Therefore, while \(\Phi_2^0(s) \leq 0\) we have \(|\Phi_1^0(s)| \leq |\Phi_2^0(s)|\) for \(s \in [\tilde{s}, s_{max}]\). If \(\Phi_2^0(s) < 0\) for all \(s \in [\tilde{s}, s_{max}]\) then there is nothing more to consider. On the other hand, there is a unique \(s_0 \in [\tilde{s}, s_{max}]\) such that \(\Phi_2^0(s_0) = 0\). Observe that \(s_0 - \tilde{s} \leq |\Phi_2^0(\tilde{s})| \leq C\), hence

\[
(5.2) \quad |\Phi_0^1(s_0)| \leq |\Phi_1^0(\tilde{s})| + (s_0 - \tilde{s})|\Phi_2^0(\tilde{s})| \leq C.
\]

Therefore, if we take \(K > 0\) sufficiently large, these cases also do not cause us any problems.

Finally, we consider the case in which \(\Phi_0^0(\tilde{s}) = -1\). Due to the invariance of \((3.3)\) under the transformation \(\Phi \mapsto -\Phi\), we may apply the above argument to \(-\Phi^0\).

**Theorem 1.1** is a corollary of this.

Next, we show the existence of smooth equivariant biharmonic maps from \(B(0, 1; 4)\) to \(S^4\) that can wind around \(S^4\) as many times as we wish. Before we do this we need some preparatory lemmas. Our arguments are influenced by the ideas in \([GZ12]\). Recall that given an orbit \(\Phi^0 : (-\infty, 0] \to \mathbb{R}^4\) in \(W^n(0)\), \(\psi(r) = \Phi^0_1(\log r)\)

solves \((2.6)\) (with \(\partial_r \psi = 0\) on \([0, 1]\) with \(\psi(0) = 0\). We first wish to verify that given such a \(\psi\), \(u = \Upsilon(\psi)\) is weakly biharmonic.

The next lemma obtains estimates on the derivatives of our solutions \(\psi\) and the corresponding equivariant maps.

**Lemma 5.3.** Let \(\psi \in C([0, 1]; \mathbb{R}) \cap C^\infty((0, 1]; \mathbb{R})\), with \(\psi(0) = 0\), be a solution to \((2.6)\) (with \(\partial_r \psi = 0\)) and

\[ u = \Upsilon(\psi) \in C(B(0, 1; 4); S^4) \cap C^\infty(B(0, 1; 4) \setminus \{0\}; S^4)\]
Then, for \( r > 0 \),

\[
|\psi(r)| \leq Cr, \quad |\partial_r \psi(r)| \leq C, \quad \text{and} \quad |\partial_r^2 \psi(r)| \leq Cr.
\]

Furthermore, \( Du \in L^\infty(B(0, 1; 4)) \) and \( |D^2u(x)| \leq C|x|^{-1} \) for \( x \in \overline{B(0, 1; 4)} - \{0\} \).

In the above inequalities \( C = C(\psi) \).

**Proof.** We set \( \phi(s) = (e^s)^{i} \) and \( \Phi_i^0 = \partial_t^{-1} \phi \) for \( i \in \{1, 2, 3, 4\} \). Recall that \( \Phi^0 \) solves (3.3).

We rewrite (3.3) as

\[
\Phi_i^0(x) = \partial_t \Phi_i^0 + A \Phi_i^0 + \Theta_i^{(0)}(x).
\]

where \( A \) is the same as in (3.1) and \( |\Theta_i^{(0)}(x)| \leq C|\Phi^{(0)}|^3 \) for sufficiently small \( \Phi^{(0)} \). We set

\[
(5.3) \quad V^0(s) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -1 & -3 \\ 1 & 9 & 1 & 9 \\ 1 & 27 & -1 & -27 \end{pmatrix}^{-1} \Phi^0(-s) =: P^{-1} \Phi^0(-s).
\]

We substitute this into (5.3):

\[
(5.4) \quad V^0(s) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} V^0 - P^{-1} G(PV^0) =: DV^0 + \tilde{G}(V^0),
\]

where \( |\tilde{G}(V)| \leq C|V|^3 \) for sufficiently small \( V \). Since we reversed \( s \) in (5.4), we are now interested in the stable manifold at the origin of (5.5). This manifold is tangent to the \( V_1 - V_2 \) plane at the origin, and can be locally written as a graph over this plane with \( V_3 = V_3(V_1, V_2) \) and \( V_4 = V_4(V_1, V_2) \) such that

\[
\frac{\partial(V_3, V_4)}{\partial(V_1, V_2)}(0, 0) = 0.
\]

Our first aim is to show that

\[
|\langle V_3(V_1, V_2), V_4(V_1, V_2) \rangle| \leq C|\langle V_1, V_2 \rangle|^3,
\]

for sufficiently small \( |\langle V_1, V_2 \rangle| \).

We let \( \varepsilon > 0 \), and \( (V_1, V_2) \in \mathbb{R}^2 \) such that \( |\langle V_1, V_2 \rangle| \leq \varepsilon \). We setup the iteration:

\[
\partial_s V^1 = DV^1 \quad \text{with} \quad V^1(0) = (V_1, V_2, 0, 0),
\]

and for \( i \in \mathbb{N} \):

\[
\begin{cases}
\partial_s V^{i+1} = DV^{i+1} + \tilde{G}(V^i), \\
V^{i+1}(0) = (V_1, V_2, V_{3;i+1}, V_{4;i+1}),
\end{cases}
\]

where

\[
(5.7) \quad V_{3;i+1} = -\int_0^\infty e^{-s} \tilde{G}_3(V^i(s)) \, ds, \quad \text{and} \quad V_{4;i+1} = -\int_0^\infty e^{-3s} \tilde{G}_4(V^i(s)) \, ds.
\]

This iteration is used in [Per91, Section 2.7] as part of the proof of the Stable Manifold theorem. In this proof it is shown that, for sufficiently small \( \varepsilon \),

\[
V_3(V_1, V_2) = \lim_{i \to \infty} V_{3;i} \quad \text{and} \quad V_4(V_1, V_2) = \lim_{i \to \infty} V_{4;i}.
\]

Furthermore, it is shown that

\[
(5.8) \quad |V^i(s)| \leq C|\langle V_1, V_2 \rangle| e^{-\alpha s},
\]
Lemma 5.4. Let \( u \) be a solution to (2.6) (with \( \partial \psi = 0 \)) and \( v \) be a solution to (2.8) (with \( \partial \psi = 0 \)). We substitute (5.8) into (5.2), and take the limit \( i \to \infty \) to obtain (5.6).

If \( \Phi^0 \) is an orbit in \( W^u(0) \) then \( V^0 \) is an orbit in the stable manifold of the origin of (5.5). For sufficiently large \( s \), (5.9) and (5.10) give

\[
|V^0_1(s)| \leq Ce^{-s}, \quad |V^0_2(s)| \leq Ce^{-3s}, \quad |V^0_3(s)| \leq Ce^{-3s}, \quad \text{and} \quad |V^0_4(s)| \leq Ce^{-3s}.
\]

For \( r > 0 \), we have

\[
\partial_r \psi(r) = \frac{\Phi^0_2(\log r)}{r} \quad \text{and} \quad \partial_r^2 \psi(r) = \frac{\Phi^0_2(\log r) - \Phi^0_1(\log r)}{r^2}.
\]

Using this, along with (5.2) and (5.9), gives us

\[
|\psi(r)| \leq C(\psi \alpha), \quad |\partial_r \psi(r)| \leq C(\psi), \quad \text{and} \quad |\partial_r^2 \psi(r)| \leq C(\psi)\alpha.
\]

Next, we turn our attention towards the estimates on \( u \). First we focus on \( |Du| \).

Recalling the notation from (2.4), for \( x \neq 0 \), we have

\[
D(\{f_0, f_1\}) : D(\{g_0, g_1\}) = \partial_x f_0 \partial_x g_0 + \partial_x f_1 \partial_x g_1 + \frac{d-1}{r^2} f_0 g_0.
\]

This, (5.10), and \( u = \Upsilon(\psi) \) yield \( Du \in L^\infty(B(0, 1; 4)) \). For \( d = 4 \), we have

\[
|Du|^2 = \frac{1}{\alpha(9\gamma^2 - 18\gamma \alpha \partial_x f_0 + \alpha + 2(\partial^2_f 2\gamma^2 + \alpha \partial_x f_1)\partial_x f_0 + \alpha + 2(\partial^2_f 2\gamma^2 + \alpha \partial_x f_1)\partial_x f_0)\partial_x f_1}
\]

This, (5.10), and \( u = \Upsilon(\psi) \) yield \( |Du| \leq C(\psi)|x|^{-1} \).

Now we wish to show that our solutions \( \psi \) to (2.6) (with \( \partial \psi = 0 \)) give rise to equivariant maps \( u \in H^2 \).

Lemma 5.5. Let \( \psi \in C([0, 1]; R) \cap C^\infty((0, 1]; R) \), with \( \psi(0) = 0 \), be a solution to (2.6) (with \( \partial \psi = 0 \)) and

\[
u = \Upsilon(\psi) \in C(B(0, 1; 4); S^4) \cap C^\infty(B(0, 1; 4) - \{0\}; S^4).
\]

Then \( u \in H^2(B(0, 1; 4); S^4) \).

Proof. Using the growth estimates from Lemma 5.3, it can be shown that \( u \) has weak derivatives in \( L^2 \) up to order two which are equal a.e. to their respective classical derivatives.

Next, we show that our solutions \( \psi \) give rise to weakly biharmonic maps. See [GZ12], for a different approach in a slightly different situation.

Lemma 5.5. Let \( \psi \in C([0, 1]; R) \cap C^\infty((0, 1]; R) \), with \( \psi(0) = 0 \), be a solution to (2.6) (with \( \partial \psi = 0 \)) and

\[
u = \Upsilon(\psi) \in C(B(0, 1; 4); S^4) \cap C^\infty(B(0, 1; 4) - \{0\}; S^4).
\]

Then \( u \) is weakly biharmonic.

Proof. We let \( \eta \in C_c^\infty(B(0, 1; 4); R^5) \) be arbitrary. We wish to show that

\[
\partial_i|_{x=0} E_2(\Pi(u + t\eta)) = 0,
\]

where \( \Pi(x) = \frac{\partial}{\partial x} \) is defined on \( R^5 - \{0\} \).

From [Str03] (2.1) and (2.2), we have

\[
\partial_i|_{x=0} E_2(\Pi(u + t\eta)) = 2 \int_{B(0, 1; 4)} (\Delta u \cdot \Delta \eta - \sum_{\gamma=1}^5 \Delta u^\gamma \Delta(u^\gamma u \cdot \eta)) \ dx.
\]
We let \( \omega \in C^\infty_c(B(0, 1; 4); [0, 1]) \) be such that \( \omega \equiv 1 \) on \( B(0, \frac{1}{2}; 4) \). For \( R > 0 \), we set \( \omega R(x) = \omega(x/R) \). We have
\[
\partial_t|_{t=0} E_2(\Pi(u + t\eta)) = \partial_t|_{t=0} E_2(\Pi(u + t(\omega R\eta)))
+ \partial_t|_{t=0} E_2(\Pi(u + t((1 - \omega R)\eta))).
\]
Lemma 5.3 gives us
\[
\partial_t|_{t=0} E_2(\Pi(u + t(\omega R\eta))) = o_{R \to 0}(1).
\]
Next, we turn our attention towards \( \partial_t|_{t=0} E_2(\Pi(u + t(1 - \omega R)\eta))) \). Since the support of \((1 - \omega R)\eta\) is bounded away from the origin, and \( u \) is smooth and satisfies the Euler-Lagrange equation (1.2) (with \( m = 2 \)) away from the origin, we have \( \partial_t|_{t=0} E_2(\Pi(u + t((1 - \omega R)\eta))) = 0 \). Therefore, \( \partial_t|_{t=0} E_2(\Pi(u + t\eta)) = o_{R \to 0}(1) \) which gives the desired result after taking the limit \( R \searrow 0 \).

Finally, we prove Theorem 1.4.

**Proof of Theorem 1.4** Due to the invariance of (3.3) under the transformation \( \Phi \mapsto -\Phi \), it suffices to prove the result for \( a \geq 0 \). The \( a = 0 \) case is taken care of by the trivial solution.

Therefore, we let \( a > 0 \) be arbitrary. There exists a non-trivial orbit
\[
\Phi^0 : (-\infty, s_{\text{max}}) \to \mathbb{R}^4
\]
in \( W^u(0) \) which is not an \( s \)-translation of \( \pm Y \). Lemma 4.5 tells us that \( \Phi^0 \) must exit the region \( |\Phi| < 1 \) in finite time. Due to the invariance of (3.3) under the transformation \( \Phi \mapsto -\Phi \), we may assume that there exists some \( s_0 \in (-\infty, s_{\text{max}}) \) such that \( \Phi^0(s_0) = 1 \). Lemma 4.7 tells us that \( \Phi^0(s) \to \infty \) as \( s \nearrow s_{\text{max}} \). Since \( \Phi^0 \) is an orbit in \( W^u(0) \), we also know that \( \Phi^0(s) \to 0 \) as \( s \to -\infty \). Therefore, we may \( s \)-translate \( \Phi^0 \) so that \( \Phi^0(0) = a \). This corresponds to a solution of (3.1) with \( \phi(0) = a \), which after undoing the change of coordinates \( r = e^s \), corresponds to a solution of (2.6) (with \( \partial_0 \psi = 0 \)) such that \( \psi(1) = a \).

Lemmas 5.4 and 5.5 tell us that
\[
u = \tau(\psi) \in C^\infty(B(0, 1; 4) - \{0\}; S^1) \cap H^2(B(0, 1; 4); S^4),
\]
is a weakly biharmonic map. Standard higher interior regularity arguments, see for example [CWY99], yield smoothness of \( u \) on all of \( B(0, 1; 4) \).

**APPENDIX**

**Proof of Lemma 3.1** (2) \( \implies \) (1): This direction is trivial.

(1) \( \implies \) (2): We set \( \tilde{\Phi}_i^0(s) = (-1)^{i+1} \Phi_i^0(-s) \) for \( i \in \{1, 2, 3, 4\} \). Observe that \( \Phi^0 \) solving (3.3) is equivalent to \( \tilde{\Phi}^0 \) solving (3.3). Therefore, after relabeling \( \tilde{\Phi}^0 \) as \( \Phi^0 \) our statement is equivalent to showing that if \( s_0 \in \mathbb{R} \), \( \Phi^0 : [s_0, \infty) \to \mathbb{R}^4 \) solves (3.3), and
\[
\lim_{s \to \infty} \Phi^0_i(s) = 0,
\]
then
\[
\lim_{s \to \infty} \Phi^0(s) = 0.
\]
It is easy to show that if \( x \in C^2([s_0, \infty); \mathbb{R}) \), \( x(s) \to 0 \) as \( s \to \infty \), and \( |\partial_s^2 x(s)| \leq C \) for all \( s \in [s_0, \infty) \) then \( \partial_s x(s) \to 0 \) as \( s \to \infty \). We use this fact, which we call (P1), repeatedly in what follows.
First observe that there cannot exist an \( s_1 \in [s_0, \infty) \) such that \(|\Phi_0(s)| \geq 1\) for all \( s \in [s_1, \infty) \). Indeed, if there were such an \( s_1 \) then eventually \( \Phi_0(s) \) would be the same sign as \( \Phi_0 \) after which we could apply Lemma 4.7 and obtain a contradiction. Therefore, if there is an \( s \in [s_0, \infty) \) such that \( |\Phi_0(s)| \geq 1 \) then there must be an \( s_2 > s_1 \) such that \( |\Phi_0(s)| < 1 \) for all \( s \in [s_2, \infty) \), or else we could apply Lemma 4.7 and obtain a contradiction. Therefore, \( \Phi_0 \) is bounded on \([s_0, \infty)\).

Now we proceed to show, one by one, that \( \lim_{s \to \infty} \Phi_0(s) = 0 \) for \( i \in \{2, 3, 4\} \). First we look at \( \Phi_2 \). The fact that \( \Phi_2 \to 0 \) as \( s \to \infty \), the boundedness of \( \Phi_2 \), and (P1) yield
\[
\lim_{s \to \infty} \Phi_2(s) = 0.
\]

Next, we look at \( \Phi_3 \). Hoping for a contradiction, we assume that \( \Phi_3(s) \neq 0 \) as \( s \to \infty \). From (P2), we know that \( \Phi_3 \) is unbounded, that is there exists a monotone increasing sequence \( \{s_i\}_{i \in \mathbb{N}} \subset [s_0, \infty) \) diverging to infinity such that \( |\Phi_3(s_i)| \to \infty \). From (3.3) and the fact that \(|\Phi_2(s), \Phi_3(s), \Phi_0(s)| \leq C \) on \([s_0, \infty)\), we have that \( |\partial_s \Phi_3(s)| \leq C \) on \([s_0, \infty)\). Therefore, \( |\Phi_3(s)| \geq \frac{1}{2} |\Phi_3(s_i)| \) for \( s \in [s_i, s_i + \frac{1}{2^N} |\Phi_3(s_i)|] \). Observe that over this interval \( \Phi_3 \) is non-vanishing. Therefore, there exists an \( s \in [s_0, \infty) \) such that \( |\Phi_3(s)| \geq 1 \) and \( \Phi_3(s) \) has the same sign as \( \Phi_3(s) \neq 0 \). Lemma 4.7 then yields a contradiction, hence
\[
\lim_{s \to \infty} \Phi_3(s) = 0.
\]

Finally, we look at \( \Phi_4 \). Since
\[
\lim_{s \to \infty} (\Phi_0(s), \Phi_2(s), \Phi_3(s)) = 0,
\]
from (3.3), we have \( \partial_s \Phi_4(s) \to 0 \) as \( s \to \infty \). Now (P1), gives us
\[
\lim_{s \to \infty} \Phi_4(s) = 0.
\]

**Remark A.6.** Observe that in the above proof we make use of Lemma 4.7. Our argument would be circular if Lemma 4.7 depended upon Lemma 3.1. By closely examining the proof of Lemma 4.7 it is clear that this is not the case.

**Proof of Lemma 4.2.** We prove this lemma for \( c_0 = \frac{99}{100} \). It is elementary to compute
\[
\min_{x \in \mathbb{R}} f(x) = -\frac{1}{12} \sqrt{169 + 38\sqrt{19}}.
\]
Since \( f \) is an odd function, we have
\[
|f(y)| \leq \frac{1}{12} \sqrt{169 + 38\sqrt{19}} \leq 2 \text{ for all } y \in \mathbb{R}.
\]
We differentiate:
\[
\partial_x Q(x; f(y)) = \frac{3(9 + 21 \cos(2x) + 2f(y) \sin(2x))}{(7 + 3 \cos(2x))^2}.
\]
By periodicity, what we wish to prove is that \( \partial_x Q(x; f(y)) \leq c_0 \) for all \( x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and \( y \in \mathbb{R} \).

Firstly, \( \partial_x Q \left( \frac{\pi}{2}; f(y) \right) < 0 \) which means we may restrict our attention to \( x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \). We use Weierstrass’ substitution:
\[
\sin(2x) \mapsto \frac{2t}{1 + t^2} \text{ and } \cos(2x) \mapsto \frac{1 - t^2}{1 + t^2} \text{ for } t \in \mathbb{R}.
\]
This transforms the problem into showing that
\[
\frac{3(15 + 2(\tilde{f} - 3t) (1 + t^2))}{2 (5 + 2t^2)^2} \leq c_0,
\]
for all \( t \in \mathbb{R} \) and \( \tilde{f} \in [-2, 2] \). It suffices to show
\[
(A.1) \quad -45 + 50c_0 + (-27 + 40c_0)t^2 + (18 + 8c_0)t^4 - 12(t + t^3) \geq 0,
\]
for all \( t \in \mathbb{R} \). Next we prove this.

We substitute \( c_0 = \frac{99}{100} \) into \( A.1 \) and let \( p \) be the polynomial on the left hand side of the resulting expression, that is
\[
p(t) = \frac{648}{25}t^4 - 12t^3 + \frac{63}{5}t^2 - 12t + \frac{9}{2}.
\]
We calculate:
\[
p'(\frac{2}{5}) < 0, \quad p'(\frac{43}{100}) > 0, \quad \text{and} \quad p''(t) > 0,
\]
for \( t \in \mathbb{R} \). Therefore, \( p \) is convex with its unique global minimum occurring somewhere in \( \left[\frac{2}{5}, \frac{43}{100}\right] \). We use this to estimate:
\[
\min_{t \in \mathbb{R}} p(t) \geq \frac{648}{25} \left(\frac{2}{5}\right)^4 - 12 \left(\frac{43}{100}\right)^3 + \frac{63}{5} \left(\frac{2}{5}\right)^2 - 12 \left(\frac{43}{100}\right) + \frac{9}{2} > 0.
\]
This is what we wished to show. \( \blacksquare \)

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