Noncommutative coherent states and related aspects of Berezin-Toeplitz quantization

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Dedicated by the first and the third authors to the memory of the second author, with gratitude for his friendship and for all they learnt from him

Abstract

In this paper, we construct noncommutative coherent states using various families of unitary irreducible representations (UIRs) of $G_{nc}$, a connected, simply connected nilpotent Lie group, that was identified as the kinematical symmetry group of noncommutative quantum mechanics for a system of 2-degrees of freedom in an earlier paper. Likewise described are the degenerate noncommutative coherent states arising from the degenerate UIRs of $G_{nc}$. We then compute the reproducing kernels associated with both these families of coherent states and study Berezin-Toeplitz quantization of the observables on the underlying 4-dimensional phase space, analyzing in particular the semi-classical asymptotics for both these cases.

I Introduction

Noncommutative quantum mechanics (NCQM) is an active field of research these days. The starting point here is to alter the canonical commutation relations (CCR) among the
respective positions and momenta coordinates and hence introduce a new noncommuta-
tive Lie algebra structure. Consult [13, 8] for a detailed account on this approach.

There is another approach available where one starts with noncommutative field the-
ory (NCFT) studying quantum field theory on noncommutative space-time. An excellent
pedagogical treatment to noncommutative quantum field theory can be found in [15].
Refer to the excellent review [9] to delve further into the studies of NCFT. A noncom-
mutative quantum field theory is the one where the fields are functions of space-time
coordinates with spatial coordinates failing to commute with each other. Among others,
Snyder and Yang were the leading proponents to introduce the concept of noncommu-
tative structure of space-time (see [14, 16]). Introduction of such assumption of spatial
noncommutativity eliminates ultraviolet (UV) divergences of quantum field theory and
runs parallel to the technique of renormalization as a cure to such annoying divergence
issues in quantum field theory. NCQM can then be seen as nonrelativistic approximation
of NCFT (see [10, 3]).

In yet another approach (see [4], [5]), the authors start from a certain nilpotent Lie
group $G_{nc}$ as the defining group of NCQM for a system of 2 degrees of freedom and
compute its unitary dual using the method of orbits introduced by Kirillov (see [11]).
Although $G_{nc}$ does not contain the Weyl-Heisenberg group $G_{WH}$ as its subgroup, the
unitary dual of $G_{WH}$ is found to be sitting inside that of $G_{nc}$. Various gauges arising in
NCQM are also found to be related to a certain family of unitary irreducible representa-
tions (UIRs) of $G_{nc}$. The Lie group $G_{nc}$ was later identified as the kinematical symmetry
group of NCQM in [6] where various Wigner functions for such model of NCQM were
computed that were found to be supported on relevant families of coadjoint orbits associ-
ated with $G_{nc}$. The purpose of this paper is to construct coherent states arising from
$G_{nc}$, study their properties and discuss the associated Berezin-Toeplitz quantization of
the observables on the 4-dimensional phase space, including the relevant “semi-classical”
asymptotics.

In Section IV we review the relevant facts about the UIRs of the 7-dimensional real
Lie group $G_{nc}$ from [5] and proceed to construct the associated coherent states and
reproducing kernel. The UIRs in question are parameterized by triples $(\hbar, \theta, B) \in \mathbb{R}^3$
satisfying $\hbar^2 - \theta B \neq 0$; the “degenerate” case $\hbar^2 = \theta B$ is then discussed separately in
Section IV. With the coherents states in hand, one constructs the Toeplitz operators
in the standard fashion, and we examine the corresponding “semiclassical limit” of the
resulting Berezin-Toeplitz quantization in Section IV. The main novelty here, of course,
is the presence of the three deformation parameters $\hbar, \theta, B$ instead of the sole Planck
constant $\hbar$, thus making it already somewhat unclear in what manner these three should
be allowed to approach zero in any analogue of the ordinary semi-classical limit where
one just has $\hbar \searrow 0$; note that $(\hbar, \theta, B)$ cannot approach $(0, 0, 0)$ completely unrestrictedly
due to the non-degeneracy condition $\hbar^2 - \theta B \neq 0$. It turns out that the right objects
from this point of view are the renormalized quantities $B := \mathcal{B}/\hbar$ and $T := \vartheta/\hbar$, and nice semiclassical asymptotics are established for the situation when $\hbar, B, T$ all tend to zero, without any restrictions. (Note that $B$ has the physical interpretation of the applied magnetic field, see eqn. (3.6) in [5]. Note also that in the $B,T$ notation the non-degeneracy condition becomes simply $BT \neq 1$, so there is no longer any problem with $B, T, \hbar$ simultaneously approaching zero.)

One can in principle consider the Berezin-Toeplitz quantization and the corresponding semi-classical behaviour also in the degenerate case $\hbar^2 - \vartheta B = 0$. In this case, the underlying Hilbert space carrying the coherent states undergoes a dimension reduction from 4 to 2, and while we still get a unique associated reproducing kernel, it turns out that the measure defining the inner product in the reproducing kernel Hilbert space at hand is no longer uniquely determined. Furthermore, the semi-classical asymptotics of the associated Berezin-Toeplitz operators turn out not to depend at all on the choice of this measure, nor in fact on the deformation parameter $\vartheta$ (or on $B = \hbar^2/\vartheta$), and reduce just to the plain Berezin-Toeplitz deformation quantization (star-product) on the complex plane. Details are supplied in Section V.

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The middle named author, S. Twareque Ali, passed away unexpectedly on January 25 2016, before this manuscript was finished. The other two authors dedicate this paper to him, in memory of his scientific enthusiasm and great and lasting friendship.

II Noncommutative coherent states associated to $G_{nc}$

The geometry of the coadjoint orbits associated with the 7-dimensional real Lie group $G_{nc}$ is studied in detail in [5]. There the orbits were classified based on the triple $(\rho, \sigma, \tau) \in \mathbb{R}^3$. In this paper, we will focus on the family of UIRs corresponding to $(\rho, \sigma, \tau) \in \mathbb{R}^3$ such that $\rho \neq 0$, $\sigma \neq 0$ and $\tau \neq 0$ with $\rho^2\alpha^2 - \gamma\beta\sigma\tau \neq 0$. The UIRs are given by

$$
(U^\rho_{\sigma,\tau}(\theta, \phi, \psi, q, p)f)(r_1, s_2)
= e^{i\rho(\theta + \alpha q_2 s_2 + \alpha p_1 r_1 + \frac{\alpha}{2} q_1 p_1 - \frac{\alpha}{2} q_2 p_2)} e^{i\sigma(\phi + \beta p_1 s_2 - \frac{\beta}{2} p_1 p_2)}
\times e^{i\tau(\psi + \gamma q_2 r_1 + \frac{\gamma}{2} p_2 q_1)} f(r_1 + q_1, s_2 - p_2). \tag{2.1}
$$

Note that in (2.1), $r_1$ has the dimension of length and $s_2$ has that of momentum. By taking the inverse Fourier transform of (2.1) in the second coordinate $s_2$ yields the following
representation on $L^2(\mathbb{R}^2, dr)$:

\[
(U^\rho_{\sigma,\tau}(\theta, \phi, \psi, q, p)f)(r_1, r_2) = e^{i\rho(\theta + \alpha p_{1}r_1 + \beta p_{2}r_2 + \frac{\gamma}{2}q_{1}p_{1} + \frac{\delta}{2}q_{2}p_{2})}e^{i\sigma(\phi + \frac{\alpha}{p_{1}}r_1)}
\times e^{i\tau(\psi + \gamma q_{2}r_1 + \frac{\delta}{2}q_{1}q_{2})}f \left( r_1 + q_1, r_2 + q_2 + \frac{\sigma}{p_{1}}p_{1} \right),
\]

(2.2)

where $f \in L^2(\mathbb{R}^2, dr)$.

**Definition II.1.** For a given fixed vector $\chi \in L^2(\mathbb{R}^2, dr)$ and a fixed point $(q, p) \in \mathbb{R}^4$, the underlying phase space for the 2-dimensional system under study, let us define the following vectors in $L^2(\mathbb{R}^2, dr)$ as

\[
\chi^{nc}_{q,p} = U^\rho_{\sigma,\tau}(0, 0, 0, -q, p)\chi.
\]

(2.3)

Introducing the following change of variables:

\[
\hbar = \frac{1}{\rho \alpha}, \quad \alpha = -\frac{\sigma \beta}{p^2 \alpha^2} \quad \text{and} \quad \beta = -\frac{\tau \gamma}{p^2 \alpha^2},
\]

the vectors $\chi^{nc}_{q,p}$ read

\[
\chi^{nc}_{q,p}(r) = e^{i\frac{\hbar}{\sqrt{2\hbar}}(r - \frac{1}{2}q) \cdot p - \frac{i\alpha \hbar}{2\sqrt{2\hbar}}p_{1}p_{2} + \frac{i\beta \hbar}{\sqrt{2\hbar}}(q_{2}r_{1} - \frac{1}{2}q_{1}q_{2})} \chi \left( r_1 - q_1, r_2 - q_2 - \frac{\theta}{\hbar}p_{1} \right).
\]

Let us now define the noncommutative coherent states using the vectors $\chi^{nc}_{q,p}$ as

\[
\eta^{nc}_{q,p}(r) = e^{i\frac{\hbar}{\sqrt{2\hbar}}(r - \frac{1}{2}q) \cdot p - \frac{i\alpha \hbar}{2\sqrt{2\hbar}}p_{1}p_{2} + \frac{i\beta \hbar}{\sqrt{2\hbar}}(q_{2}r_{1} - \frac{1}{2}q_{1}q_{2})} \eta \left( r_1 - q_1, r_2 - q_2 - \frac{\theta}{\hbar}p_{1} \right),
\]

(2.4)

where $\eta$ is a vector given by $\eta = \frac{\chi}{||\chi||}$.

The phase space for the 2-dimensional noncommutative system is $\mathbb{R}^4$. The underlying observables are functions defined over the phase space variables $q, p$. These functions are taken to be elements of the Hilbert space $L^2(\mathbb{R}^4, d\nu(q, p))$ equipped with the measure $d\nu(q, p) = \frac{|h^2 - B\theta|}{4\pi^2 \hbar^4} dq \, dp$. The noncommutative coherent states $\eta^{nc}_{q,p}$, given by (2.4), satisfy the resolution of identity as stated by the following lemma:

**Lemma II.2.** The vectors $\eta^{nc}_{q,p}$ defined as the noncommutative coherent states by (2.4) satisfy the following integral relation:

\[
\int_{\mathbb{R}^4} \left| \eta^{nc}_{q,p} \right> \left< \eta^{nc}_{q,p} \right| d\nu(q, p) = I,
\]

(2.5)

where $I$ is the identity operator on $L^2(\mathbb{R}^2, dr)$. 


Proof. Introducing the following changes of variables:

\[ q_{1\text{nc}} = q_1, \]
\[ q_{2\text{nc}} = q_2 + \frac{\vartheta}{\hbar} p_1, \]
\[ p_{1\text{nc}} = p_1 + \frac{B}{\hbar} q_2, \]
\[ p_{2\text{nc}} = p_2, \]

the noncommutative coherent states \( \eta_{q_\varphi}^{\text{nc}} \) appearing in (2.4), can neatly be written as

\[ \eta_{q_\varphi}^{\text{nc}}(r) = e^{\frac{i}{\hbar} (r - \frac{1}{2} q^{\text{nc}})} p^{\text{nc}} \eta (r - q^{\text{nc}}), \]

where \((q_{1\text{nc}}, q_{2\text{nc}})\) and \((p_{1\text{nc}}, p_{2\text{nc}})\) are denoted as \( q^{\text{nc}} \) and \( p^{\text{nc}} \), respectively.

The associated measures transform as

\[ dq^{\text{nc}} dp^{\text{nc}} = \frac{|h^2 - B\vartheta|}{h^2} dq dp. \]

Let us now choose two compactly supported smooth functions \( f \) and \( g \) in \( L^2(\mathbb{R}^2, dr) \). One then obtains,

\[
\int_{\mathbb{R}^4} \langle f | \eta_{q_\varphi}^{\text{nc}} \rangle \langle \eta_{q_\varphi}^{\text{nc}} | g \rangle d\nu(q, p) \\
= \frac{|h^2 - B\vartheta|}{4\pi^2 h^4} \int_{\mathbb{R}^4} \langle f | \eta_{q_\varphi}^{\text{nc}} \rangle \langle \eta_{q_\varphi}^{\text{nc}} | g \rangle dq dp \\
= \frac{1}{4\pi^2 h^2} \int_{\mathbb{R}^4} \langle f | \eta_{q_\varphi}^{\text{nc}} \rangle \langle \eta_{q_\varphi}^{\text{nc}} | g \rangle dq^{\text{nc}} dp^{\text{nc}} \\
= \frac{1}{4\pi^2} \left[ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{i}{\hbar} (r - r') \cdot p^{\text{nc}}} \overline{f(r)} \eta (r - q^{\text{nc}}) \overline{\eta (r' - q^{\text{nc}})} g(r') dr dr' \right] dq^{\text{nc}} d \left( \frac{p^{\text{nc}}}{h} \right) \\
= \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}^2} \delta^{(2)} (r - r') \overline{f(r)} \eta (r - q^{\text{nc}}) \overline{\eta (r' - q^{\text{nc}})} g(r') dr dr' \right] dq^{\text{nc}} \\
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{f(r)} \eta (r - q^{\text{nc}}) \overline{\eta (r - q^{\text{nc}})} g(r) dr dq^{\text{nc}} \\
= ||\eta||^2 \langle f | g \rangle \\
= \langle f | g \rangle.
\]

Using the continuity of the inner product of the underlying Hilbert space and the fact that the compactly supported smooth functions are dense in \( L^2(\mathbb{R}^2, dr) \), one can extend the above equality to any pair of functions \( f, g \) in \( L^2(\mathbb{R}^2, dr) \) and hence proving the lemma. \( \square \)
Let us rewrite (2.7) and observe that
\[
\eta^{nc}_{q,p}(r) = e^{i\hbar (r - \hat{q}^{nc})\cdot p^{nc}} \eta(r - q^{nc}) = e^{i\hbar (r - \frac{1}{2} q^{nc}) + \frac{\hbar}{2} r^{nc}} e^{\left(-q^{nc}_{1} \frac{\partial}{\partial x_{1}} - q^{nc}_{2} \frac{\partial}{\partial x_{2}}\right)} \eta(r) = e^{i\hbar \left[ p^{nc} - q^{nc}_{1} (-i\hbar \frac{\partial}{\partial x_{1}} ) - q^{nc}_{2} (-i\hbar \frac{\partial}{\partial x_{2}})\right]} \eta(r) = e^{i\hbar (\xi^{T} \omega^{nc} X) \eta(r),}
\] (2.10)

where \(\xi\) and \(X\) are 4 \times 1 column vectors given by
\[
\xi = \begin{bmatrix} q^{nc}_{1} \\ q^{nc}_{2} \\ p^{nc}_{1} \\ p^{nc}_{2} \end{bmatrix} = \begin{bmatrix} q_{1} \\ q_{2} + \frac{\theta}{\pi} p_{1} \\ p_{1} + \frac{\theta}{\pi} q_{2} \\ p_{2} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} \hat{Q}^{nc}_{1} \\ \hat{Q}^{nc}_{2} \\ \hat{P}^{nc}_{1} \\ \hat{P}^{nc}_{2} \end{bmatrix} = \begin{bmatrix} r_{1} + i\hbar \frac{\partial}{\partial x_{1}} \\ r_{2} \\ -i\hbar \frac{\partial}{\partial x_{2}} \\ -\frac{\theta}{\pi} r_{1} - i\hbar \frac{\partial}{\partial x_{2}} \end{bmatrix},
\] (2.11)

while \(\omega^{nc}\) in (2.10) is given by the following 4 \times 4 matrix:
\[
\omega^{nc} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ \frac{\hbar}{\pi} & 0 & 0 & -\frac{\hbar^{2}}{\pi^{2}} \\ \hbar^{2} & 0 & 0 & \frac{\hbar}{\pi} \\ 0 & 1 & 0 & 0 \end{bmatrix}.
\] (2.12)

The 4 entries of the column vector \(X\), i.e. \(\hat{Q}^{nc}_{i}\), \(\hat{P}^{nc}_{i}\) for \(i = 1, 2\), in (2.11) represent the non-central generators of \(G_{nc}\) in the Landau gauge representation defined on \(L^{2}(\mathbb{R}^{2}, dr)\) as obtained in (5).

**Definition II.3.** The unitary operator \(D^{nc}(q,p)\) that generates the noncommutative coherent state vectors \(\eta^{nc}_{q,p}\) by acting upon the normalized ground state vector \(\eta\) is defined as the noncommutative displacement operator.
\[
\eta^{nc}_{q,p}(r) = D^{nc}(q,p)\eta(r) = e^{i\hbar (\xi^{T} \omega^{nc} X) \eta(r)},
\] (2.13)

where \(\xi, X\) and \(\omega^{nc}\) are as given in (2.11) and (2.12).

**Remark II.1.** A few remarks on the quantum mechanical limits of the above formulations are in order. From (2.11) and (2.12), it can easily be seen that as \(\vartheta \to 0\), \(B \to 0\), the noncommutative displacement operator \(D^{nc}(q,p)\) approaches the canonical displacement operator:
\[
D^{nc}(q,p) \xrightarrow{\vartheta \to 0, B \to 0} \exp \left( i \frac{\hbar}{\pi} \begin{bmatrix} q_{1} & q_{2} & p_{1} & p_{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{Q}_{1} \\ \hat{Q}_{2} \\ \hat{P}_{1} \\ \hat{P}_{2} \end{bmatrix} \right) = \exp \left( i \frac{\hbar}{\pi} (p \hat{Q} - q \hat{P}) \right),
\] (2.14)
where \( \hat{Q} = (\hat{Q}_1, \hat{Q}_2) = (r_1, r_2) \) and \( \hat{P} = (\hat{P}_1, \hat{P}_2) = (-i\hbar \frac{\partial}{\partial r_1}, -i\hbar \frac{\partial}{\partial r_2}) \) are the standard quantum mechanical representation of the position and momentum operators of the underlying 2-dimensional system defined on \( L^2(\mathbb{R}^2, dr) \). The cases \( \mathcal{B} \to 0 \) or \( \vartheta \to 0 \) can similarly be studied. For example,

\[
\begin{align*}
\omega^{nc}_{\mathcal{B} \to 0} \to & \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & \vartheta \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\
\omega^{nc}_{\vartheta \to 0} \to & \begin{bmatrix} 0 & 0 & -1 & 0 \\ -\frac{\vartheta}{\beta} & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

(2.15)

Now in view of (2.10), the noncommutative coherent states \( \eta^{nc}_{q,p} \) can be read off as

\[
\eta^{nc}_{q,p}(r) = e^{\frac{1}{2}(p_1^{nc} \hat{Q}_1 + p_2^{nc} \hat{Q}_2 - q_1^{nc} \hat{P}_1 - q_2^{nc} \hat{P}_2)} \eta(r).
\]

(2.16)

Let us choose an element \( \phi \in L^2(\mathbb{R}^2, dr) \) and consider the map \( \Phi : \mathbb{R}^4 \to \mathbb{C} \) for a fixed vector \( \phi \in L^2(\mathbb{R}^2, dr) \) by

\[
\Phi(q,p) = \langle \eta^{nc}_{q,p} | \phi \rangle.
\]

(2.17)

It is immediate that \( \Phi \in L^2(\mathbb{R}^4, d\nu(q,p)) \). Now consider the isometry map

\[
W : L^2(\mathbb{R}^2, dr) \to L^2(\mathbb{R}^4, d\nu)
\]

(2.18)

defined by \( W \phi = \Phi \). The range \( \mathcal{W} \) of this isometry map, a closed subspace of \( L^2(\mathbb{R}^4, d\nu) \), is a reproducing kernel Hilbert space (RKHS) and the function \( K^{nc} : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{C} \) with

\[
K^{nc}(q,p; q', p') = \langle \eta^{nc}_{q,p} | \eta^{nc}_{q', p'} \rangle
\]

(2.19)

is its reproducing kernel.

Let us now recall a few facts about \( G_{nc} \) from [4]. The group composition rule for \( G_{nc} \) are given by

\[
(\theta, \phi, \psi, q,p)(\theta', \phi', \psi', q', p') = (\theta + \theta' + \alpha \xi((q,p), (q', p'))), \phi + \phi' + \beta \xi'((q,p), (q', p'))
\]

\[
, \psi + \psi' + \gamma \xi''((q,p), (q', p'))), q + q' + p + p',
\]

(2.20)

where the 3-inequivalent local exponents of the abelian group of translations in \( \mathbb{R}^4 \) is given by

\[
\xi((q,p), (q', p')) = \frac{1}{2} [q_1 p_1' + q_2 p_2' - p_1 q_1' - p_2 q_2'],
\]

\[
\xi'((q,p), (q', p')) = \frac{1}{2} [p_1 l_2 - p_2 l_1],
\]

\[
\xi''((q,p), (q', p')) = \frac{1}{2} [q_1 q_1' - q_2 q_1'].
\]

(2.21)
Proposition II.2. Provided one chooses the ground state vector $\eta$ in (2.16) to be the following normalized Gaussian function:

$$\eta(r) = \frac{1}{\sqrt{\pi s}} e^{-\frac{r^2}{2s^2}},$$

(2.22)

then, the reproducing kernel $K^{nc}$ (see 2.14) associated with the Lie group $G_{nc}$ reads

$$K^{nc}((q,p),(q',p')) = e^{\frac{4}{2\pi^2}((q,p),(q',p')) - \frac{1}{4\pi^2}(p-p')^2} e^{\frac{1}{4\pi^2}((q',p')^2)} e^{\frac{im}{2\pi^2}((q,p),(q',p'))}$$

$$\times e^{-\frac{i}{4\pi^2}((q,p),(q',p'))^2} e^{\frac{i\eta}{\pi^2}((q,p),(q',p'))^2} e^{-\frac{i\eta}{\pi^2}((q,p),(q',p'))^2}$$

(2.23)

where $\xi$, $\xi'$ and $\xi''$ are all given by (2.21). Also, $s$ stands for the standard deviation associated with the position vector $r = (r_1, r_2)$ and hence has the dimension of length.

Proof. Using (2.16), one finds that

$$\langle \eta_{q^{nc}}^{nc}|\eta_{q'^{nc}}^{nc}\rangle = \langle \eta|e^{-\frac{i}{2}(p^{nc}\hat{Q}-q^{nc}\hat{P})}e^{\frac{i}{2}(p'^{nc}\hat{Q}-q'^{nc}\hat{P})}\eta\rangle$$

$$= e^{-\frac{i}{2\pi}((p^{nc},q^{nc})+(p'^{nc},q'^{nc}))} \int_{\mathbb{R}^2} \eta(r)e^{-\frac{i}{2}(p^{nc}-p'^{nc})\hat{Q}+(q^{nc}-q'^{nc})\hat{P}} \eta(r) dr$$

$$= e^{-\frac{i}{2\pi}((p^{nc},q^{nc})+(p'^{nc},q'^{nc}))} \int_{\mathbb{R}^2} \eta(r)e^{-\frac{i}{2}(p^{nc}-p'^{nc})\hat{Q}} \eta(r) dr$$

(2.24)

where

$$\tilde{\eta} = e^{\frac{i}{2}(q^{nc}-q'^{nc})}\hat{P}\eta.$$

(2.25)

Therefore, (2.24) now reads

$$\langle \eta_{q^{nc}}^{nc}|\eta_{q'^{nc}}^{nc}\rangle = e^{-\frac{i}{2\pi}((p^{nc},q^{nc})+(p'^{nc},q'^{nc}))} \int_{\mathbb{R}^2} \eta(r)e^{-\frac{i}{2}(p^{nc}-p'^{nc})\hat{Q}+(q^{nc}-q'^{nc})\hat{P}} \eta(r) dr$$

$$= e^{-\frac{i}{2\pi}((p^{nc},q^{nc})+(p'^{nc},q'^{nc}))} \int_{\mathbb{R}^2} \eta(r)e^{-\frac{i}{2}(p^{nc}-p'^{nc})\hat{Q}} \eta(r+q^{nc}-q'^{nc}) dr$$

$$= e^{-\frac{i}{2\pi}((p^{nc},q^{nc})+(p'^{nc},q'^{nc}))} \int_{\mathbb{R}^2} \eta(r)e^{-\frac{i}{2}(p^{nc}-p'^{nc})}\hat{Q} \eta(r+q^{nc}-q'^{nc}) dr$$

$$= e^{-\frac{i}{2\pi}((p^{nc},q^{nc})+(p'^{nc},q'^{nc}))} \int_{\mathbb{R}^2} \eta(r)e^{-\frac{i}{2}(p^{nc}-p'^{nc})}\hat{Q} \eta(r+q^{nc}-q'^{nc}) dr$$

$$= e^{-\frac{i}{2\pi}((p^{nc},q^{nc})+(p'^{nc},q'^{nc}))} \int_{\mathbb{R}^2} \eta(r)e^{-\frac{i}{2}(p^{nc}-p'^{nc})}\hat{Q} \eta(r+q^{nc}-q'^{nc}) dr$$

$$= e^{-\frac{i}{2\pi}((p^{nc},q^{nc})+(p'^{nc},q'^{nc}))} \int_{\mathbb{R}^2} \eta(r)e^{-\frac{i}{2}(p^{nc}-p'^{nc})}\hat{Q} \eta(r+q^{nc}-q'^{nc}) dr$$

$$= e^{-\frac{i}{2\pi}((p^{nc},q^{nc})+(p'^{nc},q'^{nc}))} \int_{\mathbb{R}^2} \eta(r)e^{-\frac{i}{2}(p^{nc}-p'^{nc})}\hat{Q} \eta(r+q^{nc}-q'^{nc}) dr$$

(2.25)
of representations has been computed in [5]:

\[ \text{the reproducing kernel of } q \in \mathbb{C}^n \text{ for which each of } q = 0, \ldots, n \text{ is satisfied} \]

\[ \int_{\mathbb{R}^2} e^{-\frac{1}{\hbar}[(p, q) - (p', q')]} e^{-\frac{1}{2\pi} |q|^2 - \frac{1}{4\pi} r \cdot (q - q') \cdot \frac{1}{2\pi} |q|^2} \, dr \]

\[ = \frac{1}{\pi \hbar^2} e^{-\frac{1}{2\pi} |(p - p', q - q')|} \]

\[ \times \frac{s^2}{4\pi^2} |p - p'|^2 \cdot \frac{1}{4\pi} |q - q'|^2 + \frac{1}{4\pi} (p - p'), (q - q') \]

\[ = e^{\frac{1}{4\pi} \xi((q, p), (q', p')) - \frac{s^2}{4\pi^2} |p - p'|^2 - \frac{1}{4\pi} |q - q'|^2} \]

\[ \times \frac{1}{\pi \hbar^2} e^{-\frac{1}{2\pi} |(p - p', q - q')|} \]

\[ \times \int_{\mathbb{R}^2} e^{-\frac{1}{2\pi} |(p - p', q - q')|} \left| \int_{\mathbb{R}^2} e^{-\frac{1}{2\pi} |(p - p', q - q')|} \right|^2 \, dr \]

\[ = e^{\frac{1}{4\pi} \xi((q, p), (q', p')) - \frac{s^2}{4\pi^2} |p - p'|^2 - \frac{1}{4\pi} |q - q'|^2} \]

(2.26)

Now writing \( q^{\text{nc}}, p^{\text{nc}} \) in terms of \( q \) and \( p \) with the help of (2.6) and subsequently using (2.21), one finally obtains

\[ \langle \eta_{q, p}^{\text{nc}} | \eta_{q', p'}^{\text{nc}} \rangle \]

\[ = e^{\frac{1}{4\pi} \xi((q, p), (q', p')) - \frac{s^2}{4\pi^2} |p - p'|^2 - \frac{1}{4\pi} |q - q'|^2} \]

\[ \times e^{-\frac{1}{4\pi} \left[ \frac{s^2}{2\pi} (p_1 - p'_1)^2 + 2\pi (q_2 - q'_2) (p_1 - p'_1) \right]} e^{\frac{1}{4\pi} \xi'((q, p), (q', p')) - \frac{s^2}{4\pi^2} (q_2 - q'_2)^2 + 2\pi (p_1 - p'_1) (q_2 - q'_2)} \).

\[ \square \]

Remark II.3. It is worth remarking here that as \( B, \vartheta \to 0 \),

\[ K^{\text{nc}}((q, p), (q', p')) \to e^{\frac{1}{4\pi} \xi((q, p), (q', p')) - \frac{s^2}{4\pi^2} |p - p'|^2 - \frac{1}{4\pi} |q - q'|^2} \]

which is the canonical reproducing kernel that one obtains for the Weyl-Heisenberg group in 2-dimensions.

III Noncommutative coherent states in the degenerate case

In the previous section, we have computed the reproducing kernel \( K^{\text{nc}}((q, p), (q', p')) \) from the generic unitary irreducible representations of \( G_{\text{nc}} \) due to nonzero \( \rho, \sigma \) and \( \tau \) satisfying \( \rho^2 \alpha^2 - \gamma \beta \sigma \tau \neq 0 \). In this section, using similar arguments, we shall compute the reproducing kernel \( K^{\text{nc}}((q, p), (q', p')) \) from the unitary irreducible representations of \( G_{\text{nc}} \) for which each of \( \rho, \sigma \) and \( \tau \) is nonzero and \( \rho^2 \alpha^2 - \gamma \beta \sigma \tau = 0 \) holds. This family of representations has been computed in [5]:

\[ (\hat{U}_{\rho, \delta}^\kappa(\theta, \phi, \psi, q_1, q_2, p_1, p_2) \hat{f}(s)) \]

\[ = e^{iK_{q_1} + i\delta p_2 + i\rho \left( \theta - \alpha q_1 - \frac{\alpha q_2 \beta}{\gamma \sigma} \right)} e^{i\kappa (\phi + \beta p_2 + \frac{\alpha q_2 \beta}{\gamma \sigma} + \delta p_1 p_2)} \]

\[ \times e^{-\frac{\gamma^2}{\sigma^2} \frac{\alpha q_2 \beta}{\gamma \sigma} \hat{f}(s + p_1 + \frac{\alpha q_2}{\beta})} \]

(3.1)

where \( f \in L^2(\mathbb{R}, ds) \). It is to be noted that given \( \rho \neq 0 \), an ordered pair \((\kappa, \delta)\) and \( \zeta \in (-\infty, 0) \cup (0, \infty) \) satisfying \( \rho = \sigma \zeta = \frac{\gamma \beta \tau}{\alpha^2} \), one precisely obtains a unitary irreducible representation of \( G_{\text{nc}} \).
Inverse Fourier transform of (3.1) leads to

\[
(U_{\rho,\zeta}^{\nu,\delta}(\theta, \phi, \psi, q_1, q_2, p_1, p_2)f)(r) = e^{i\rho (\theta + \frac{1}{2} \phi + \frac{\phi^2}{2\zeta} \psi)} + i\kappa q_1 + i\delta q_2 - i\rho \alpha p_1 - i\rho \alpha \eta r q_2 + i\eta (q_1 p_1 - q_2 p_2)
\]

\[
\times e^{i\eta \left( \frac{\phi}{2\zeta} q_1 q_2 - \frac{\eta^2}{4\zeta} p_1 p_2 \right)} f(r - q_1 + \beta \alpha \zeta p_2), \quad (3.2)
\]

where \( f \in L^2(\mathbb{R}, dr) \).

In analogy with section II, given a fixed vector \( \tilde{\chi} \in L^2(\mathbb{R}, dr) \) and a fixed point \((q, p) \in \mathbb{R}^4\), let us first define the vector in \( L^2(\mathbb{R}, dr) \) as:

\[
\tilde{\chi}_{nc,q,p}(r) = U_{\rho,\zeta}^{\nu,\delta}(0,0,0,-q_1,-p_1) \tilde{\chi}. \quad (3.3)
\]

With the following change of variables:

\[
\hbar = \frac{1}{\rho \alpha}, \quad \vartheta = -\frac{\sigma \beta}{\rho^2 \alpha^2},
\]

and recalling that \( \rho = \sigma \zeta = \frac{\gamma \beta \nu}{\rho^2 \alpha^2} \) holds, one can rewrite \( \tilde{\chi}_{nc,q,p} \) suitably as:

\[
\tilde{\chi}_{nc,q,p}(r) = e^{-i\kappa q_1 - i\delta q_2 - \frac{\hbar}{\eta}(p_1 + \frac{\hbar}{\eta} q_2) - \frac{\hbar}{\eta}(q_1 p_1 - q_2 p_2) + \frac{\hbar}{\eta} \left( -\frac{q_1 q_2}{2\hbar} + \frac{\eta^2}{4\hbar} p_1 p_2 \right)} \tilde{\chi}(r + q_1 - \frac{\eta}{\hbar} p_2). \quad (3.4)
\]

**Definition III.1.** Define the degenerate noncommutative coherent states as the following vectors in \( L^2(\mathbb{R}, dr) \):

\[
\tilde{\eta}_{nc,q,p}(r) = e^{-i\kappa q_1 - i\delta q_2 - \frac{\hbar}{\eta}(p_1 + \frac{\hbar}{\eta} q_2) - \frac{\hbar}{\eta}(q_1 p_1 - q_2 p_2) + \frac{\hbar}{\eta} \left( -\frac{q_1 q_2}{2\hbar} + \frac{\eta^2}{4\hbar} p_1 p_2 \right)} \tilde{\eta}(r + q_1 - \frac{\eta}{\hbar} p_2), \quad (3.5)
\]

where \( \tilde{\eta} \) is a vector given by \( \tilde{\eta} = \frac{\tilde{\chi}}{||\tilde{\chi}||} \).

Now the underlying observables are functions in the Hilbert space \( L^2(\mathbb{R}^4, d\tilde{\nu}(q, p)) \) equipped with the measure

\[
d\tilde{\nu}(q, p) = \frac{1}{2\pi \hbar} dq_1 dp_1 d\mu(q_2, p_2), \quad (3.6)
\]

where \( d\mu \) is an arbitrary probability measure on \( \mathbb{R}^2 \).

**Lemma III.2.** The vectors \( \tilde{\eta}_{nc,q,p} \) defined as the degenerate noncommutative coherent states by (3.5) satisfy the following integral relation:

\[
\int_{\mathbb{R}^4} |\tilde{\eta}_{nc,q,p}(r)|^2 d\tilde{\nu}(q, p) = \mathbb{I}, \quad (3.7)
\]

where \( \mathbb{I} \) is the identity operator on \( L^2(\mathbb{R}, dr) \).
Proof. Using the following change of variables:

\[ q_1^{nc} = q_1 - \frac{\hbar}{i} p_2 \]
\[ q_2^{nc} = q_2 \]
\[ p_1^{nc} = p_1 + \frac{\hbar}{i} q_2 + 2\kappa \hbar \]
\[ p_2^{nc} = p_2, \]

one can rewrite the degenerate noncommutative coherent states (3.5) as:

\[ \tilde{\eta}_{q,p}^{nc}(r) = e^{-i\delta q^{nc}-(\frac{\hbar}{i}\kappa - 2\kappa \hbar)} \tilde{\eta}(r + q_1^{nc}). \tag{3.9} \]

Observe that \( dq_1 dp_1 = dq_1^{nc} dp_1^{nc} \), by (3.8). Thus for any \( f, g \in L^2(\mathbb{R}, dr) \), we obtain

\[
\int_{\mathbb{R}^4} \langle f | \tilde{\eta}_{q,p}^{nc} | \tilde{\eta}_{q', p'}^{nc} \rangle \, d\nu(q, p) = \frac{1}{2\pi \hbar} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} f(r) e^{-\frac{\hbar}{i}(r-r')(p_1^{nc} + 2\kappa \hbar)} \tilde{\eta}(r + q_1^{nc}) \tilde{\eta}(r' + q_1^{nc}) g(r') \, dr \, dr' \, dq_1^{nc} \, dp_1^{nc} \\
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta(r-r') f(r) \tilde{\eta}(r + q_1^{nc}) \tilde{\eta}(r' + q_1^{nc}) g(r') \, dr \, dr' \, dq_1^{nc} \, dp_1^{nc} \\
= |\tilde{\eta}|^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(r) g(r') \, dr \, dr' \, dq_2^{nc} \, dp_2^{nc} \\
= \langle f | g \rangle. \tag{3.10} \]

The reproducing kernel associated with the degenerate noncommutative coherent states will be given by

\[ \tilde{K}^{nc}((q, p), (q', p')) = \langle \tilde{\eta}_{q,p}^{nc} | \tilde{\eta}_{q', p'}^{nc} \rangle. \tag{3.11} \]

This degenerate reproducing kernel is explicitly computed in the following proposition:

**Proposition III.1.** Provided one chooses the ground state vector \( \tilde{\eta} \in L^2(\mathbb{R}, dr) \) in (3.2) to be the following normalized Gaussian function:

\[ \tilde{\eta}(r) = \frac{1}{\pi^3 s^2} e^{-\frac{r^2}{2s^2}}, \tag{3.12} \]

then, the reproducing kernel \( \tilde{K}^{nc} \) (see (3.11)) associated with the Lie group \( G_{nc} \) reads

\[
\tilde{K}^{nc}((q, p), (q', p')) = e^{\frac{i}{2} \xi((q, p), (q', p')) + \frac{i}{2} \xi''((q, p), (q', p')) + \frac{i}{2} \xi''(\eta((q, p), (q', p')) + \text{in}(q_1 - q_1') + i\delta(q_2 - q_2') - \frac{\hbar}{i\kappa}(q_1 - q_1')^2} \\
\times e^{-\frac{1}{2\kappa^2}(q_1 - q_1')^2 - \frac{2\hbar}{i\kappa}(q_2 - q_2')^2 - \frac{1}{\kappa^2}(p_1 - p_1')^2 + \frac{2\hbar}{2\kappa i}(q_1 p_2 - q_2 p_1') + \frac{2\hbar}{2\kappa^2}(q_1 p_2' + q_2 p_1)} \\
\times e^{-\frac{1}{2\kappa^2}(q_1 p_2 - q_2 p_1)^2 - \frac{2\hbar}{i\kappa}(q_2 - q_2')^2 - \frac{1}{\kappa^2}(p_1 - p_1')^2 + \frac{2\hbar}{2\kappa i}(q_1 p_2 - q_2 p_1') + \frac{2\hbar}{2\kappa^2}(q_1 p_2' + q_2 p_1)} \tag{3.13} \]

where \( \xi, \xi' \) and \( \xi'' \) are all given by (2.21). Also, \( s \) stands for the standard deviation associated with the position coordinate \( r \) and hence has the dimension of length.
Proof. Using (3.9) we get
\[
\langle \eta_{q,p} | \eta_{q',p'} \rangle = e^{-i\delta(q_{nc} - q_{nc}')} - \frac{i}{\pi} (q_{nc} - q_{nc}') - \frac{i}{4\pi} \kappa(p_{nc} - p_{nc}')
\]
\[
\times \int_{\mathbb{R}} e^{-i\delta(p_{nc} - p_{nc}')} \eta(r + q_{nc}) \overline{\eta}(r + q_{nc}') dr
\]
\[
eq e^{-i\delta(q_{nc} - q_{nc}')} - \frac{i}{\pi} (q_{nc} - q_{nc}') - \frac{i}{4\pi} \kappa(p_{nc} - p_{nc}')
\]
\[
\times \frac{1}{\pi^{1/2}} \int_{\mathbb{R}} e^{-i\delta(p_{nc} - p_{nc}')} e^{-\frac{(r+q_{nc})^2}{2\sigma^2}} \overline{e^{-\frac{(r+q_{nc}')(2\sigma^2)}}} dr
\]
\[
eq e^{-i\delta(q_{nc} - q_{nc}')} - \frac{i}{\pi} (q_{nc} - q_{nc}') - \frac{i}{4\pi} \kappa(p_{nc} - p_{nc}')
\]
\[
\times e^{-\frac{h^2}{4\sigma^2}(q_{nc} - q_{nc}')^2 + 2\sigma^2 s^2(p_{nc} - p_{nc}')(q_{nc} - q_{nc}')(q_{nc} - q_{nc}')} + \frac{s^2}{4\sigma^2} h^2 (q_{nc} - q_{nc}')^2},
\]
by the familiar formula for Gaussian integrals
\[
\int_{\mathbb{R}} e^{ar - b - cx^2} dx = \pi^{1/2} e^{-c/2} e^{b^2/c}, \quad c > 0.
\]
A routine manipulation gives (3.13). \hfill \square

IV Toeplitz operators and semiclassical limits

We now proceed to consider a variant of the well-known Berezin-Toeplitz quantization procedure in the context of the coherent states, and resolution of the identity, from the preceding section. Our strategy will be to relate the corresponding Toeplitz operators (defined in (4.4) below) to the analogous operators in the standard setting.

Consider the Fock space
\[
\mathcal{F}_\hbar := \{ f \in L^2(\mathbb{C}^2, e^{-|z|^2} (\pi\hbar)^{-1} dA(z)) : f \text{ is holomorphic on } \mathbb{C}^2 \}
\]
(here \( dA \) stands for the Lebesgue area measure), and let \( V \) be the map
\[
Vf(q,p) := e^{-|z_{nc}|^2/2\hbar} f(z_{nc}), \quad f \in \mathcal{F}_\hbar,
\]
where we have introduced the notation
\[
z_{nc}^j = \sqrt{\frac{\hbar}{2}} q_{nc}^j - \frac{is p_{nc}^j}{\sqrt{2\hbar}}, \quad j = 1, 2.
\]
(4.1)
From the equality
\[
dA(z_{nc}) = \frac{d\mathbf{p}_{nc} d\mathbf{q}_{nc}}{4} = \frac{|h^2 - B\theta|}{4\hbar^2} d\mathbf{p} d\mathbf{q} = \pi^2 h^2 d\nu(q,p),
\]
(4.2)
where as before
\[
d\nu(q,p) = \frac{|h^2 - B\theta|}{4\pi^3 h^4} d\mathbf{q} d\mathbf{p},
\]
(4.3)
one verifies that $V$ is an isometry from $\mathcal{F}_h$ into $L^2(\mathbb{R}^4, d\nu(q, p))$. If $\{e_n\}_n$ is an arbitrary orthonormal basis of $\mathcal{F}_h$, then $\{V e_n\}_n$ will be an orthonormal basis of the image $\text{Ran} V =: W'$ of $V$. Using the standard formula for a reproducing kernel in terms of an orthonormal basis \footnote{2}, and the fact that the reproducing kernel of $\mathcal{F}_h$ is well known to be given by $K_{\mathcal{F}_h}(x, y) = e^{(x, y)/\hbar}$, we see that $W'$ is a reproducing kernel Hilbert space (RKHS) with reproducing kernel

\[
K_{W'}((q, p), (q', p')) = \sum_n V e_n(z^{nc})\overline{V e_n(z'^{nc})} = e^{-|z^{nc}|^2/2\hbar - |z'^{nc}|^2/2\hbar} e_n(z^{nc})\overline{e_n(z'^{nc})} = e^{-|z^{nc}|^2/2\hbar - |z'^{nc}|^2/2\hbar} e(z^{nc} z'^{nc})/\hbar = K^{nc}((q, p), (q', p')),
\]

upon a short computation (cf. \footnote{220}). Since a RKHS is uniquely determined by its reproducing kernel \footnote{2}, it follows that in fact $W' = W$. Thus $V$ is a unitary isomorphism of $\mathcal{F}_h$ onto $W$.

Recall that for $f \in L^\infty(\mathbb{C}^2)$, the Toeplitz operator $T_f$ on $\mathcal{F}_h$ is given by

\[
T_f u = P(fu), \quad u \in \mathcal{F}_h,
\]

where $P : L^2(e^{-|z|^2/\hbar} dA(z)) \to \mathcal{F}_h$ is the orthogonal projection. Alternatively, $T_f$ is determined by the property that

\[
\langle T_f u, v \rangle = \int_{\mathbb{C}^2} f(z) u(z)\overline{v(z)} e^{-|z|^2/\hbar} \frac{dz}{(\pi\hbar)^2} \quad \forall u, v \in \mathcal{F}_h.
\]

Similarly, we have Toeplitz operators $T_F$, $F \in L^\infty(\mathbb{R}^4)$, on $W$ defined by

\[
T_F u = \mathcal{P}(Fu), \quad u \in W,
\]

where $\mathcal{P} : L^2(\mathbb{R}^4, d\nu) \to W$ is the orthogonal projection; alternatively, $T_F$ is determined by the property that

\[
\langle T_F u, v \rangle = \int_{\mathbb{R}^4} F(q, p) u(q, p)\overline{v(q, p)} d\nu(q, p) \quad \forall u, v \in W.
\]

Now by a simple change of variable (cf. \footnote{12})

\[
\langle T_F V u, V v \rangle = \int_{\mathbb{R}^4} F(q, p) V u(q, p)\overline{V v(q, p)} d\nu(q, p)
\]

\[
= \int_{\mathbb{R}^4} F(q, p) e^{-|q^{nc}|^2/\hbar} u(z^{nc})\overline{v(z'^{nc})} d\nu(q, p)
\]

\[
= \int_{\mathbb{C}^2} F(i(z^{nc})) e^{-|z^{nc}|^2/\hbar} u(z^{nc})\overline{v(z'^{nc})} dA(z^{nc})/(\pi\hbar)^2,
\]

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for all $u,v \in \mathcal{F}_h$, where $\iota$ is the inverse to the coordinate transformation (4.1):

$$
\iota(\mathbf{z}^\text{nc}) := \left( \frac{\hbar^2 q^\text{nc}_1}{\hbar^2 - B \partial}, \frac{\hbar^2 p^\text{nc}_1}{\hbar^2 - B \partial}, \frac{h \mathbf{B} q^\text{nc}_2}{\hbar^2 - B \partial}, p^\text{nc}_2 \right) \in \mathbb{R}^4,
$$

(4.5)

Consequently,

$$
V^* T_F V = T_{F^\text{fact}}. \tag{4.6}
$$

Remark IV.1. From the last formula one can see what are the commutators of the Toeplitz operators $T_{p_j}, T_{q_k}$, $j = 1, 2$, on $L^2(\mathbb{R}^4, d\nu)$. Indeed, from the formulas for the Toeplitz operators on $\mathcal{F}_h$,

$$
T_{z_j} = z_j, \quad T_{\overline{z}_j} = \hbar \frac{\partial}{\partial z_j},
$$

and the resulting commutator identity

$$
[T_{z_j}, T_{\overline{z}_k}] = -\delta_{jk} \hbar \mathbb{I},
$$

one gets using (4.6)

$$
[T_{P_1}, T_{P_2}] = -iB\hbar^2 \frac{\partial}{\partial^2} \mathbb{I},
$$

$$
[T_{P_1}, T_{Q_1}] = [T_{P_2}, T_{Q_2}] = -i\hbar^3 \frac{\partial}{\partial^2} \mathbb{I},
$$

$$
[T_{Q_1}, T_{Q_2}] = -iB^2 \frac{\partial}{\partial^2} \mathbb{I},
$$

all remaining commutators being zero. Note that these differ from the commutator relations for the corresponding quantum observables (cf. eqn. (3.7) in [3])

$$
[\hat{Q}_j, \hat{P}_k] = i\hbar \delta_{jk} \mathbb{I}, \quad [\hat{Q}_1, \hat{Q}_2] = i \mathbb{I}, \quad [\hat{P}_1, \hat{P}_2] = iB \mathbb{I}
$$

by a factor of $\frac{\hbar^2}{B^2 - \hbar^2}$.

Now from the Berezin-Toeplitz quantization (see e.g. [12], [1]), it is known that for $f, g$, say, smooth with compact support, one has the asymptotic expansion

$$
T_f T_g \approx \sum_{j=0}^{\infty} \hbar^j T_{C_j(f,g)}
$$

as $\hbar \to 0$, in the sense of operator norms, where

$$
C_j(f,g) = (-1)^j \sum_{|\alpha| = j} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial z^\alpha} \frac{\partial^\alpha g}{\partial \overline{z}^\alpha}.
$$
(here the summation is over all multiindices \( \alpha \in \mathbb{N}^2 \) of length \( j \)). From the computation

\[
V^* T_F T_G V = (V^* T_F V) (V^* T_G V) = T_{F \circ T_G},
\]

we thus see that we have an asymptotic expansion, in the sense of operator norms,

\[
T_F T_G \approx \sum_{j=0}^{\infty} \hbar^j \mathcal{C}_j (F \circ \iota, G \circ \iota) \iota^{-1} V,
\]

as \( \hbar \searrow 0 \), with

\[
\mathcal{C}_j (F, G) := \mathcal{C}_j (F \circ \iota, G \circ \iota) \iota^{-1}.
\]

This gives rise also to the associated star-product

\[
F \star G := \sum_{j=0}^{\infty} \hbar^j \mathcal{C}_j (F, G),
\]

so that, heuristically, \( T_F T_G \approx T_{F \star G} \).

In particular, \( \mathcal{C}_0 (F, G) = FG \) (the pointwise product), while

\[
\mathcal{C}_1 (F, G) = \begin{bmatrix} \frac{\partial F}{\partial p_1} & \frac{\partial F}{\partial p_2} & \frac{\partial F}{\partial q_1} & \frac{\partial F}{\partial q_2} \end{bmatrix} \cdot A \cdot \begin{bmatrix} \frac{\partial G}{\partial p_1} \\ \frac{\partial G}{\partial p_2} \\ \frac{\partial G}{\partial q_1} \\ \frac{\partial G}{\partial q_2} \end{bmatrix},
\]

where

\[
A = \begin{bmatrix}
\frac{h(h^4 + B^2 s^4)}{2s^2(h^2 - B \vartheta)^2} & \frac{-iBh}{h} & \frac{-ih^2}{2(h^2 - B \vartheta)} & \frac{h^2(Bs^4 + h^2 \vartheta)}{2s^2(h^2 - B \vartheta)^2} \\
\frac{iBh}{2(h^2 - B \vartheta)} & \frac{-2s^2}{h} & 0 & \frac{-ih^2}{2(h^2 - B \vartheta)} \\
\frac{2h^2}{2(h^2 - B \vartheta)} & \frac{-2h}{h} & \frac{s^2}{h} & \frac{-ih \vartheta}{2(h^2 - B \vartheta)} \\
\frac{h^2(Bs^4 + h^2 \vartheta)}{2s^2(h^2 - B \vartheta)^2} & \frac{ih^2}{2(h^2 - B \vartheta)} & \frac{-ih \vartheta}{h} & \frac{-2h(Bs^4 + \vartheta^2)}{2s^2(h^2 - B \vartheta)^2}
\end{bmatrix}.
\]

The last matrix looks much nicer in terms of the “renormalized” parameters

\[
B = \frac{B}{\hbar}, \quad T := \frac{\vartheta}{\hbar}, \quad S := \frac{s}{\sqrt{\hbar}};
\]
responsible only for re-scaling the Planck constant $\hbar$ namely,

$$A = \begin{bmatrix} \frac{1 + B^2 S^4}{2S^2(1 - BT)^2} & \frac{iB}{2(1 - BT)} & \frac{i}{2(1 - BT)} & \frac{BS^4 + T}{2S^2(1 - BT)^2} \\ \frac{iB}{2(1 - BT)} & \frac{1}{2S^2} & 0 & \frac{iT}{2(1 - BT)} \\ \frac{2(1 - BT)}{BS^4 + T} & 0 & \frac{S^2}{2} & \frac{iT}{S^4 + T^2} \\ \frac{2S^2(1 - BT)^2}{2(1 - BT)} & \frac{i}{2(1 - BT)} & \frac{iT}{2(1 - BT)} & \frac{2S^2(1 - BT)^2}{2(1 - BT)} \end{bmatrix}.$$ 

Note that both $A$ and the inverse transform $i$ depend only on $B, T$ and $S$, but not on $\hbar$. The parameter $S$, which arises solely from the choice of the vector $\eta$ in (2.22), is in a sense responsible only for re-scaling the Planck constant $\hbar$, and we can choose $S = 1$. It then follows from (1.7) that all $C_j$, $j \geq 0$, will be bidifferential operators with coefficients given by expressions involving only $B, T$, in fact, by rational functions in $B, T$ with powers of $1 - BT$ as the denominators. Replacing the latter by their Taylor expansions around $(B, T) = (0, 0)$, we thus obtain a joint asymptotic expansion for the product $F \ast G$ as $(\hbar, B, T) \to (0, 0, 0)$. Its beginning looks as follows

$$F \ast G \approx FG - \frac{\hbar}{2} \sum_{k=1}^{2} \left( \frac{\partial F}{\partial p_k} - i \frac{\partial F}{\partial q_k} \right) \left( \frac{\partial G}{\partial p_k} + i \frac{\partial G}{\partial q_k} \right)$$

$$+ \left( \frac{Bh}{2} + \frac{Th}{2} \right) \left[ \frac{\partial G}{\partial p_1} \left( \frac{\partial F}{\partial p_2} - i \frac{\partial F}{\partial q_2} \right) - i \frac{\partial F}{\partial p_1} \left( \frac{\partial G}{\partial p_2} + i \frac{\partial G}{\partial q_2} \right) \right]$$

$$+ \frac{\hbar^2}{8} \sum_{k=1}^{2} \left( \frac{\partial}{\partial p_k} + i \frac{\partial}{\partial q_k} \right)^2 G \cdot \left( \frac{\partial}{\partial p_k} - i \frac{\partial}{\partial q_k} \right)^2 F$$

$$+ 2 \left( \frac{\partial}{\partial p_1} + i \frac{\partial}{\partial q_1} \right) \left( \frac{\partial}{\partial p_2} + i \frac{\partial}{\partial q_2} \right) G \cdot \left( \frac{\partial}{\partial p_1} - i \frac{\partial}{\partial q_1} \right) \left( \frac{\partial}{\partial p_2} - i \frac{\partial}{\partial q_2} \right) F$$

$$+ O((\hbar + B + T)^3).$$

(Here the differentiations stop at each $\cdot$, i.e. $\partial F \cdot \partial G$ means $(\partial F)(\partial G)$.)

For the corresponding commutator, we get

$$F \ast G - G \ast F \approx i\hbar \sum_{k=1}^{2} \left( \frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right)$$

$$+ i\hbar B \left( \frac{\partial F}{\partial p_2} \frac{\partial G}{\partial p_1} - \frac{\partial F}{\partial p_1} \frac{\partial G}{\partial p_2} \right)$$

$$+ i\hbar T \left( \frac{\partial F}{\partial q_2} \frac{\partial G}{\partial q_1} - \frac{\partial F}{\partial q_1} \frac{\partial G}{\partial q_2} \right)$$

$$+ \frac{\hbar^2}{8} \left[ 2(\partial_{+1} \partial_{+2} G)(\partial_{-1} \partial_{-2} F) - 2(\partial_{+1} \partial_{+2} F)(\partial_{-1} \partial_{-2} G) \right].$$

\(^1\)Note that as far as length only is concerned, $s$ has the same physical dimension as $\sqrt{\hbar}$.
\[
+ \sum_{k=1}^{2} \left( (\partial_{+k}^2 F)(\partial_{-k}^2 G) - (\partial_{+k}^2 G)(\partial_{-k}^2 F) \right) + O((h + B + T)^3),
\]

where for the sake of brevity, we have denoted \( \partial_{\pm k} = \frac{\partial}{\partial p_k} \pm i \frac{\partial}{\partial q_k}, k = 1, 2 \). The first term is \( i\hbar \{F,G\} \), the Poisson bracket of \( F \) and \( G \), which takes care of the correct semi-classical behavior as \( \hbar \to 0 \); note that it does not contain any \( B \) and \( T \), which come only in the second-order terms.

V  Toeplitz operators and semiclassical limits: the degenerate case

In a similar way as in the preceding section, we can also treat the “degenerate” representation and kernel from Section III. This time, we need the Fock space just on the complex plane,

\[ \tilde{\mathcal{F}}_h := \{ f \in L^2(\mathbb{C}, e^{-|z|^2/(\pi \hbar)} dA(z)) : f \text{ is holomorphic on } \mathbb{C} \}, \]

and let \( \tilde{V} \) be the map

\[ \tilde{V} f(q,p) := e^{i\delta q_2 + \kappa \partial p_2 / \hbar} e^{-|z_{nc}|^2 / 2\hbar} f(z_{nc}), \quad f \in \tilde{\mathcal{F}}_h, \]

where

\[ z_{nc} = \sqrt{\frac{\hbar q_{1c}^2}{2} - \frac{isp_{1c}^2}{\sqrt{2\hbar}}}, \quad (5.1) \]

with \( p_{1c}, q_{1c} \) given by (3.8). One verifies that \( \tilde{V} \) is an isometry from \( \tilde{\mathcal{F}}_h \) into \( L^2(\mathbb{R}^4, d\tilde{\nu}) \):

\[ \int_{\mathbb{R}^4} |\tilde{V} f(q,p)|^2 d\tilde{\nu}(q,p) = \frac{1}{2\pi h} \int_{\mathbb{R}^4} |f(z_{nc})|^2 e^{-|z_{nc}|^2 / \hbar} dq_{1c} dq_{2c} dp_{1c} dp_{2c} \]

\[ = \frac{1}{2\pi^2 h} \int_{\mathbb{R}^2} |f(z_{nc})|^2 e^{-|z_{nc}|^2 / \hbar} dq_{1c} dq_{2c} dp_{1c} dp_{2c} \]

\[ = \frac{1}{\pi h} \int_{\mathbb{C}} |f(z_{nc})|^2 e^{-|z_{nc}|^2 / \hbar} dA(z_{nc}), \]

since \( dA(z_{nc}) = dp_{1c} dq_{1c} / 2 \). Thus if \( \{ e_n \}_n \) is an arbitrary orthonormal basis of \( \tilde{\mathcal{F}}_h \), then \( \{ \tilde{V} e_n \}_n \) will be an orthonormal basis of the image \( \operatorname{Ran} \tilde{V} =: \tilde{W}' \) of \( \tilde{V} \). Using the standard formula for a reproducing kernel in terms of an orthonormal basis, and the fact that the reproducing kernel of \( \tilde{\mathcal{F}}_h \) is well known to be given by \( K_{\tilde{\mathcal{F}}_h}(x,y) = e^{\mp A / \hbar} \), we see that \( \tilde{W}' \)
is a RKHS with reproducing kernel
\[
K_{\tilde{W}'}((q, p), (q', p')) = \sum_n \tilde{V} e_n(z^{nc}) \overline{\tilde{V} e_n(z^{nc})}
\]
\[
e^{i\delta(q_2-q'_2)+i\kappa\theta(p_2-p'_2)/\hbar} e^{-|z^{nc}|^2/2\hbar - |z'^{nc}|^2/2\hbar} \sum_n e_n(z^{nc}) \overline{e_n(z'^{nc})}
\]
\[
e^{i\delta(q_2-q'_2)+i\kappa\theta(p_2-p'_2)/\hbar} e^{-|z^{nc}|^2/2\hbar - |z'^{nc}|^2/2\hbar} e^{z^{nc} \overline{z'^{nc}} / \hbar}
\]
\[
= \tilde{K}_{nc}((q, p), (q', p'))
\]
again upon a short computation. As before, it follows that \( \tilde{W}' = \tilde{W} \), the space for which \( \tilde{K}^{nc} \) is the reproducing kernel. Thus \( \tilde{V} \) is a unitary isomorphism of \( \tilde{F}_h \) onto \( \tilde{W} \).

The Toeplitz operators \( \tilde{T}_F, F \in L^\infty(\mathbb{R}^4, d\tilde{\nu}) \), on \( \tilde{W} \) are now related to the Toeplitz operators \( T_f, f \in L^\infty(\mathbb{C}) \), on \( \tilde{F}_h \) via
\[
\langle \tilde{T}_F \tilde{V} u, \tilde{V} v \rangle = \int_{\mathbb{R}^4} F(q, p) \tilde{V} u(q, p) \overline{\tilde{V} v(q, p)} d\tilde{\nu}(q, p)
\]
\[
= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^4} F(q, p) e^{-|z^{nc}|^2/\hbar} u(z^{nc}) \overline{v(z^{nc})} dq^{nc}_1 dp^{nc}_1 dq^{nc}_2 dp^{nc}_2 d\mu(q_2, p_2)
\]
\[
= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} F(q^{nc}_1, p^{nc}_2, q_2, p^{nc}_1) \frac{\hbar}{2} \frac{dA}{\pi \hbar}
\]
\[
\times u(z^{nc}) \overline{v(z^{nc})} e^{-|z^{nc}|^2/\hbar} dq^{nc}_1 dp^{nc}_1
\]
\[
= \int_{\mathbb{C}} \varrho F(z^{nc}) u(z^{nc}) \overline{v(z^{nc})} e^{-|z^{nc}|^2/\hbar} dA(z^{nc})
\]
\[
= \langle T_{\varrho F} u, v \rangle,
\]
for all \( u, v \in \tilde{F}_h \), where \( \varrho \) is the mapping
\[
\varrho F(z^{nc}) := \int_{\mathbb{R}^2} F(q^{nc}_1, \frac{\hbar}{2} p_2, q_2, p^{nc}_1) \frac{\hbar}{2} \frac{dA}{\pi \hbar} d\mu(q_2, p_2),
\]
\[
q^{nc}_1 = \frac{s}{\sqrt{2\hbar}} (z^{nc} + \overline{z^{nc}}), \quad p^{nc}_1 = \frac{\sqrt{\hbar/2}}{is} (z^{nc} - \overline{z^{nc}}).
\]
Consequently,
\[
\tilde{V}^* \tilde{T}_F \tilde{V} = T_{\varrho F}.
\]  \( \text{(5.2)} \)

**Remark V.1.** From the last formula one can again get the commutator relations for the Toeplitz operators \( \tilde{T}_{p_j}, \tilde{T}_{q_k}, j = 1, 2 \), on \( L^2(\mathbb{R}^4, d\tilde{\nu}) \). This time we get \( \tilde{T}_{p_2} = (\int p_2 d\mu(q_2, p_2)) \mathbb{I} \), \( \tilde{T}_{q_2} = (\int q_2 d\mu(q_2, p_2)) \mathbb{I} \) and
\[
[\tilde{T}_{q_1}, \tilde{T}_{p_1}] = -[\tilde{T}_{p_1}, \tilde{T}_{q_1}] = i\hbar \mathbb{I},
\]
all remaining commutators being zero.

Using again the asymptotic expansion known from the Berezin-Toeplitz quantization:
\[
T_f T_g \approx \sum_{j=0}^{\infty} \hbar^j T_0 \mathcal{C}_j(f, g)
\]
as $\hbar \searrow 0$, in the sense of operator norms, where

$$\tilde{C}_j(f, g) = \frac{(-1)^j}{j!} \frac{\partial^j f}{\partial z^j} \frac{\partial^j g}{\partial z^j},$$

one sees from

$$\tilde{V}^* \tilde{T}_F \tilde{T}_G \tilde{V} = T_{\tilde{\varphi}_F} T_{\tilde{\varphi}_G} \approx \sum_{j=0}^{\infty} \hbar^j \tilde{T}_{\tilde{C}_j(\tilde{\varphi}_F, \tilde{\varphi}_G)} = \sum_{j=0}^{\infty} \hbar^j \tilde{V}^* \tilde{C}_j(\tilde{\varphi}_F, \tilde{\varphi}_G) \tilde{V},$$

that there is an asymptotic expansion, in the sense of operator norms,

$$\tilde{T}_F \tilde{T}_G \approx \sum_{j=0}^{\infty} \hbar^j \tilde{T}_{\tilde{C}_j(F, G)},$$

as $\hbar \searrow 0$, with

$$\tilde{C}_j(F, G) := \tilde{\varphi}^* \tilde{C}_j(\tilde{\varphi}_F, \tilde{\varphi}_G). \quad (5.3)$$

Here $\tilde{\varphi}^*$ is in principle any right inverse for $\tilde{\varphi}$, for instance,

$$\tilde{\varphi}^* f(q, p) := f(z^{nc})$$

with the notations (3.8) and (5.1).

**Remark V.2.** In some sense, the freedom of choice for $\tilde{\varphi}^*$, as well as for the probability measure $d\mu$ in (3.6), reflects the “degeneracy” of the representation, as does the reduction of the number of variables of $\eta$. Note that the above choice for $\tilde{\varphi}^*$ has the virtue that it works for all probability measures $d\mu$ and values of $\hbar, \vartheta$.

Not all choices of $d\mu$ and $\tilde{\varphi}^*$, however, are physically relevant. For the associated star-product

$$F \ast G := \sum_{j=0}^{\infty} \hbar^j \tilde{C}_j(F, G), \quad \text{i.e.} \quad \tilde{T}_F \tilde{T}_G \approx \tilde{T}_{F \ast G},$$

we would like to have the usual requirement that $\tilde{C}_0(F, G) = FG$, the pointwise product. Applying $\tilde{\varphi}$ to (5.3), this implies

$$\tilde{\varphi}(FG) = (\tilde{\varphi}F)(\tilde{\varphi}G) \quad \forall F, G.$$

It is easily seen that this is only possible when $d\mu$ is a Dirac mass:

$$d\mu(q_2, p_2) = \delta(q_2 - q_2^*) \delta(p_2 - p_2^*),$$

for some fixed $(q_2^*, p_2^*) \in \mathbb{R}^2$. The functions $F, G$, being elements of $L^\infty(\mathbb{R}^4, d\tilde{\vartheta})$, are then effectively defined only on the plane $(q_2, p_2) = (q_2^*, p_2^*)$ (the complement of this plane has zero measure); and the right inverse $\tilde{\varphi}^*$ becomes simply the ordinary inverse. Viewing
\[ F(q, p) = F(q_1, q_2, p_1, p_2) \] as a function of \( q_1, p_1 \) only, and similarly for \( G, \) we then get as desired \( \tilde{C}_0(F, G) = FG \) (the pointwise product), while

\[
\tilde{C}_1(F, G) = -\frac{1}{2\hbar s^2} \left( \hbar \frac{\partial F}{\partial p_1} - i s^2 \frac{\partial F}{\partial q_1} \right) \left( \hbar \frac{\partial G}{\partial p_1} + i s^2 \frac{\partial G}{\partial q_1} \right).
\]

Note that this expression does not depend at all on the parameter \( \vartheta, \) nor on the choice of the base-point \((q_2^*, p_2^*)\); this can be shown to prevail also for \( \tilde{C}_j, j = 2, 3, \ldots. \) In fact, setting again \( s = \sqrt{\hbar}, \) we arrive via (5.3) just at the formulas for the ordinary Berezin-Toeplitz quantization on \( \mathbb{C} \) in the “free” variables \( q_1, p_1. \)

### VI Conclusion and Outlook

In this paper, we have constructed noncommutative coherent states associated with a system of 2 degrees of freedom by means of the continuous families of UIRs of the kinematical symmetry group \( G_{nc} \) of the underlying system. Subsequently, we computed the pertaining reproducing kernels. Since the generic families of UIRs are indexed by 3 nonzero continuous parameters \( \hbar, B \) and \( \vartheta \) subject to a quadratic constraint \( \hbar^2 - B\vartheta \neq 0, \) it was natural to quest for the Berezin-Toeplitz quantization of the observables on the underlying 4-dimensional Phase space using all 3 deformation parameters instead of the single Planck’s constant \( \hbar. \) In fact, when \( B \) (or \( B = \frac{2B}{\hbar} \)) and \( \vartheta \) (or \( T = \frac{\vartheta}{\hbar} \)) are both zero, the UIRs of \( G_{nc} \) indexed by this single nonzero Planck’s constant \( \hbar \) are nothing but the UIRs of the 2-dimensional Weyl-Heisenberg group \( G_{WH}. \) But the asymptotic analysis of Berezin-Toeplitz quantization pertaining to the generic sector (where \( \hbar^2 - B\vartheta \neq 0 \)) of the unitary dual of \( G_{nc} \) reveals the fact that \( \hbar, B \) and \( \vartheta \) cannot all approach 0 independently due to the imposed quadratic constraint. But upon “renormalizing” the deformation parameters \( B \) and \( \vartheta \) to \( B \) and \( T, \) respectively, one achieves the desired semiclassical asymptotics with \( B, T \) and \( \hbar \) simultaneously approaching 0.

We have subsequently handled the degenerate case \( \hbar^2 - B\vartheta = 0 \) or \( BT = 1 \) by defining the associated family of noncommutative coherent states on a dimensionally reduced Hilbert space and observed the crucial fact that the deformation parameter \( \vartheta = \frac{\hbar^2}{BT} \) disappears from the picture completely yielding the standard setting of Berezin-Toeplitz quantization on the complex plane \( \mathbb{C}. \) Our analysis of the degenerate noncommutative coherent states is propelled by the family of UIRs (see (3.2)) of \( G_{nc}. \) In other words, the group representation theoretic analysis of NCQM conducted in [5] has enabled us to study such degenerate setting in the context of NCQM. As has been insinuated towards the end of Section II that \( B \) here stands for the constant magnetic field applied perpendicular to a charged particle constrained to move on a two dimensional plane. The degenerate case that we have studied here is closely tied with the massless limit of a charged particle moving on such a plane subject to a vertical constant magnetic field (p. 213, [15]).
a model has zero Hamiltonian in the limiting case turning the theory into a topological one. The string theoretic analog of this degenerate case is also discussed there in [15].

In an earlier paper [7] by one of the present authors, noncommutative 4 tori have been constructed explicitly using the unitary dual of $G_{nc}$. We propose to undertake an in-depth study of the pertinent aspects from the point of view of noncommutative geometry, in particular, classifying projective modules over this NC-4 tori, computing connections with constant curvature and the Chern character on the relevant projective modules in near future.

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