A system $M$ of equivalence relations on a set $E$ is semirigid if only the identity and constant functions preserve all members of $M$. We construct semirigid systems of three equivalence relations. Our construction leads to the examples given by Zádori in 1983 and to many others and also extends to some infinite cardinalities. As a consequence, we show that on every set of at most continuum cardinality distinct from 2 and 4 there exists a semirigid system of three equivalence relations.

1. Introduction

A binary relation on a set $E$ is a subset $\rho$ of the cartesian product $E \times E$. We write $x \rho y$ instead of $(x,y) \in \rho$. A map $f : E \to E$ preserves $\rho$ if

$$x \rho y \Rightarrow f(x) \rho f(y)$$

for all $x,y \in E$. A binary system on the set $E$ is a pair $M := (E, (\rho_i)_{i \in I})$ where each $\rho_i$ is a binary relation on $E$. An endomorphism of $M$ is any map $f : E \to E$ preserving each $\rho_i$. The identity map on $E$ is an endomorphism of $M$. If there is no other endomorphism, $M$ is rigid. Provided that the relations $\rho_i$ are reflexive, the constant maps are endomorphisms, too. We say $M$ is semirigid if the identity map and the constant maps are the only endomorphisms.

Rigidity and semirigidity have attracted some attention (eg see [11], [23], [25]). Systems of equivalence relations lead to prototypes of semirigid systems. Indeed, as mentioned by R. S. Pierce [17] (Problem 2, p. 38), if a set $E$ has at least three elements, only the constant functions and the identity map preserve all equivalence relations on $E$. From this, it follows that:

**Lemma 1.1.** If a set $\{\rho_i : i \in I\}$ of equivalence relations generates by means of joins and meets (possibly infinite) the lattice $\text{Eqv}(E)$ of equivalences relations on $E$, then $M := (E, (\rho_i)_{i \in I})$ is semirigid.

The converse does not hold. Indeed, according to Strietz [18], if $E$ is finite with at least four elements, four equivalences are needed to generate $\text{Eqv}(E)$.
by joins and meets whereas, as shown by Zádori (1983) [26], for every set \( E \), whose size \( |E| \) is finite and distinct from 2 and 4, there is a semirigid system made of three equivalence relations. Zádori’s result reads as follows.

**Theorem 1.2.** Let \( A := \{0, \ldots, n-1\} \). The following system of three equivalence relations \( \rho, \sigma, \tau \) on \( A \) is semirigid.

*Case \( n = 2k + 1, \ k \geq 1:*

\[
\rho = \{(0,1,\ldots,k-1), \{k,\ldots,2k\}\}, \\
\sigma = \{(0,k), \{1,k+1\}, \ldots, \{k-1,2k-1\}\}, \\
\tau = \{(0,k+1), \{1,k+2\}, \ldots, \{k-1,2k\}\}.
\]

*Case \( n = 2k + 2, \ k \geq 2:*

\[
\rho = \{(0), \{1,2,\ldots,k\}, \{k+1,\ldots,2k+1\}\}, \\
\sigma = \{(0,1,k+1), \{2,k+2\}, \ldots, \{k,2k\}\}, \\
\tau = \{(1,k+2), \{2,k+3\}, \ldots, \{k-1,2k\}, \{0,k,2k+1\}\}.
\]

The case \( n \) even is represented by the colored graph of Figure 1 in [13]. In this graph, connected components formed by single colored edges represent the blocks of a partition into equivalence classes. For \( n \) odd, simply delete the node 0 which is located on top of the graph and relabel conveniently the vertices.

In [13] we investigated semirigid systems. In this paper we describe a general construction of semirigid systems of three equivalence relations which includes Zádori’s.

### 1.1. Results

Let \( \mathbb{R} \) denote the set of real numbers and \( \mathbb{R} \times \mathbb{R} \) its cartesian square. We define three equivalence relations \( \simeq_0, \simeq_1, \simeq_2 \) on \( \mathbb{R} \times \mathbb{R} \). We denote by \( p_1 \) and \( p_2 \) the first and second projections from \( \mathbb{R} \times \mathbb{R} \) onto \( \mathbb{R} \) and by \( p_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) the map \( p_0 := p_1 + p_2 \). Next, for \( i = 0, 1, 2 \), we define \( \simeq_i \) as the kernel of \( p_i \), i.e. \( u \simeq_i v \) for all \( u, v \in \mathbb{R} \times \mathbb{R} \) such that \( p_i(u) = p_i(v) \). Finally, we set \( \mathcal{R} := (\mathbb{R} \times \mathbb{R}, (\simeq_0, \simeq_1, \simeq_2)) \).

The system \( \mathcal{R} \) is, by far, not semirigid. But, there are many subsets \( C \) of the plane for which the system \( \mathcal{R} \upharpoonright C \) induced by \( \mathcal{R} \) on \( C \) is semirigid. In Section 3 we introduce the notion of *monogenic subset* (see Definitions 3.2 given in Subsection 3.2) and with the simple notion of symmetry in the plane we prove:

**Theorem 1.3.** If a finite subset \( C \) of \( \mathbb{R} \times \mathbb{R} \) is monogenic and has no center of symmetry in \( \mathbb{R} \times \mathbb{R} \) then \( \mathcal{R} \upharpoonright C \) is semirigid.

A simple example, which is at the origin of this result, is the following. Let \( \mathbb{N} \) denotes the set of non-negative integers. For \( n \in \mathbb{N} \) set \( T_n := \{(i,j) \in \mathbb{N} \times \mathbb{N} : i + j \leq n\} \). Then \( T_n \) satisfies the hypotheses of Theorem 1.3. Hence \( \mathcal{R} \upharpoonright T_n \) is semirigid. This yields an example of a semirigid system of three pairwise
isomorphic equivalence relations on a set having \( \frac{(n+1)(n+2)}{2} \) elements. Now, set 
\[ T_{n,2} := \{(i,j) \in T_n : i + j \in \{n - 1, n\}\} \quad \text{and} \quad T_{n,2}' := T_{n,2} \cup \{(0,0)\}. \]
Both sets satisfy the hypotheses of Theorem [1.3], hence the induced systems \( \mathcal{R} \upharpoonright T_{n,2} \) and
$\mathcal{R} \upharpoonright T_{n,2}'$ are semirigid. As it is easy to see, these two examples are isomorphic to those of Zádori.

In the vein of Theorem 1.3 we prove the following result (see Proposition 3.15 for a more precise statement).
Theorem 1.4. For each cardinal $\kappa = 3$ or $4 < \kappa \leq 2^{\aleph_0}$, there exists a semirigid system of three equivalences on a set of cardinality $\kappa$.

Our study leads to several questions that we mention in the paper. Among these are the following.

Problems 1.5. (1) Does the conclusion of Theorem 1.4 extend to every cardinal $\kappa > 2^{\aleph_0}$?

(2) A complete description or a useful characterization of semirigid systems is not known. Find algorithms deciding in reasonable time whether a system of $\kappa$ equivalence relations on a set of size $n$ is semirigid or not. The algorithmic complexity of the problem does not seem to be known.

(3) Systems of equivalence relations leading to Theorem 1.3 are subsystems of systems of equivalence relations associated with three directions in the plane. Describe finite semirigid systems which are embeddable into the system associated with three directions in the plane.

(4) Describe a test to decide whether a systems of three equivalence relations is embeddable into a system of equivalences associated to three directions in the plane.

Concerning item (3) and (4) of Problems 1.5 let us mention that the class of systems embeddable into a system of equivalences given by three directions in the plane is closed under embeddability. This fact leads to the following problems

Problems 1.6. Which are the finite minimal non representable systems? How many non isomorphic are they? Finitely many?

1.2. Links with universal algebra and combinatorics. To conclude, we indicate first how our study fits in the context of universal algebra.

The set of congruences of a universal algebra on a set $E$ is a basic example of a set of equivalence relations. This set ordered by inclusion forms a lattice, in fact a sublattice of the lattice $\text{Eqv}(E)$. One of the oldest unsolved problem in universal algebra is the finite lattice representation problem:

*Is every finite lattice isomorphic to the congruence lattice of a finite algebra?* (see [14, 15]).

An approach to this problem is related to our study. Indeed, for $R$ a subset of $\text{Eqv}(E)$, let $\text{Pol}(R)$, resp. $\text{Pol}^1(R)$, be the set finitary operations, resp. unary operations, on $E$ which preserve all members of $R$ and for a set $F$ of finitary operations on $E$, let $\text{Eqv}(F)$ be the set of equivalence relations preserved by all members of $F$. Ordered by inclusion, the set $\text{Eqv}(F)$ is a lattice, the congruence lattice $\text{Cong}(A)$ of the algebra $A := (E,F)$. According to a result of A. Mal’tsev, $\text{Pol}(\text{Eqv}(F))$ is determined by its unary part $\text{Pol}^1(\text{Eqv}(F))$, hence the above problem amounts to prove that every finite lattice $L$ is isomorphic to the congruence lattice of an algebra defined by unary operations.
In general, a sublattice $L$ of $\text{Eqv}(E)$ is not the congruence lattice of an algebra on $E$. Indeed, Pierce’s result implies that relational system $\mathcal{R} := (E, (\rho_i)_{i \in I})$ made of equivalence relations is semirigid if and only if $\text{Cong}(\mathcal{A}_R) = \text{Eqv}(E)$ (where $\mathcal{A}_R := (E, \text{Pol}^1(R))$ and $R := \{\rho_i : i \in I\}$).

We recall that $M_3$ is the lattice on five element made of a 3-element antichain and a top and bottom added. Three equivalences together with the equality relation and the full equivalence relation on $E$ form an sublattice of $\text{Eqv}(E)$ isomorphic to $M_3$ if (a) the meet of any two distinct equivalence relations is the equality relation (this amounts to the fact that the intersection of any pair of equivalence classes belonging to two distinct equivalence relations contains at most one element) and (b) the join of two is the full set $E$ of ordered pairs.

As shown by Zádori [26]:

**Theorem 1.7.** If $\mathcal{R} := (E, (\rho_i)_{i < 3})$ made of three equivalences relations is semirigid then these equivalence relations together with the equality relation and the full equivalence relation on $E$ form a sublattice $L$ of $\text{Eqv}(E)$, isomorphic to the lattice $M_3$.

Provided that $E$ has more than three elements, the lattice $L$, mentionned in Theorem 1.7, is distinct from $\text{Cong}(\mathcal{A}_R)$ (which is equal to $\text{Eqv}(E)$ since $\mathcal{R}$ is semirigid) hence is not the congruence lattice of an algebra on $E$. So our study of semirigidity is somewhat opposite to the representability question mentioned above. Still, the semirigidity notion appears (briefly) in the thesis of W. DeMeo [7] devoted to the representation of finite lattices. The sublattices $L$ of $\text{Eqv}(E)$ such that $\text{Cong}(\mathcal{A}_L) = \text{Eqv}(E)$ (where $\mathcal{A}_L := (E, \text{Pol}^1(L))$) are said to be dense; the fact that, as a lattice, $M_3$ has a dense representation in every $\text{Eqv}(E)$ with $E$ finite on at least five elements, amounting to Zádori’s result, appears in [7] as Proposition 3.3.1 on page 20. According to [3], if $R$ is a set of equivalences on a finite set $E$, the congruence lattice $\text{Cong}(\mathcal{A}_R)$ is made of all equivalence relations definable by primitive positive formulas from $R$ (see [24] for an easy to read presentation). We hope that this result could help in our problems on semirigid relations.

Next, we mention that some systems made of three equivalence relations on the same set play an important role in the study of latin squares and quasigroups. Indeed, behind these objects are 3-nets, objects usually presented in terms of incidence structures rather than equivalence relations. A 3-net of order $n$ is a point-line incidence structure made of $n^2$ points and three classes of $n$ lines such that:

(i) any two lines from different classes are adjacent;
(ii) no two lines from the same class are adjacent;
(iii) any point is incident with exactly one line from each class.

As the reader will easily observe, this amounts to give three equivalence relations $\rho_i, i < 3$, on the set $E$ of points such that each equivalence class of $\rho_i$
intersects in exactly one element each equivalence class of any other \( \rho_j \). Let \( \rho_i, i < 3 \), be three equivalence relations on a set \( E \) which satisfy this condition (we do not demand on \( E \) to be finite); let \( E_i \) be the set of equivalence classes of \( \rho_i \) and \( p_i : E \to E_i \) be the canonical map. Let \( \Phi : E_1 \times E_2 \to E_0 \) be defined by \( \Phi(x, y) := p_0(t) \) where \( t \) is the unique element of the intersection \( x \cap y \). Then the partial maps \( \Phi(x, -) : E_2 \to E_0 \) and \( \Phi(-, y) : E_1 \to E_0 \) are bijective for every \( x \) and \( y \). Conversely, any map \( \Phi \) satisfying these conditions comes from a 3-net. If the elements of the sets \( E_i \) are identified to those of a set \( V \), by means of three bijective maps \( v_i : V \to E_i \) we may then define a quasigroup operation \( \ast \) on \( V \) (see subsection 3.1) by setting \( a \ast b := v_0^{-1}(\Phi(v_1(a), v_2(b))) \). If \( V \) is an \( n \)-element set, e.g. \( V = \{1, \ldots, n\} \), the \( n \) by \( n \) matrix whose coefficient \( a_{i,j} \) is equal to \( i \ast j \) is the table of the quasigroup. Any table of this form is a latin square on the symbols 1, \ldots, \( n \). A given 3-net yields several quasigroups which are said isotopic.

Hundreds of papers and several books, notably \([8, 9]\), have been published on latin squares, quasigroups and 3-nets. We mention few facts connected to the problems we are considering. The graph of a 3-net \( M := (E, \{\rho_i\}_{i < 3}) \) is the graph \( G(M) \) with vertex set \( E \) and edges the pairs of distinct vertices which belong to some equivalence class of some \( \rho_i \). The graph of a latin square is the graph of the corresponding 3-net. Phelps \([16]\) has shown that for each integer \( n \geq 7 \) there is a latin square of order \( n \) whose graph has no proper automorphism. No (reflective) graph with more than a vertex can be semirigid, but it remains to be seen whether the 3-nets of Phelps are semirigid.

**Problem 1.8.** Does there exist a semirigid 3-net of order \( \kappa \) for every cardinal \( \kappa \) (finite or not) larger or equal to 7?

If \( M \) is a 3-net, the equivalence relations \( \{\rho_i\}_{i < 3} \) together with the equality relation and the full equivalence relation on \( E \), form a sublattice of \( \text{Equiv}(E) \), isomorphic to the lattice \( M_3 \) hence satisfy conditions (a) and (b) before Theorem 1.7 above. If \( M \) is induced by a 3-net on some subset then it satisfies condition (a). Conversely a system \( M \) satisfying condition (a) can be extended to a system \( M' \) which forms a 3-net. Furthermore, if \( E \) is finite one can find \( M' \) with order at most \( 2 \times |E| \) (Proposition 3.4). In the finite case, this is a rewording of a result about extension of partial latin squares due to Evans \([10]\). In the infinite case, this follows from Compactness theorem of first order logic. Despite some possible confusion, we call partial 3-net any system induced by some 3-net.

The addition on the set \( \mathbb{R} \) of real numbers is a prototypal example of a quasigroup operation. The 3-net associated with this operation is made of the three equivalence relations \( \simeq_0, \simeq_1 \) and \( \simeq_2 \) on \( \mathbb{R} \times \mathbb{R} \). Our results are about partial 3-nets induced by this 3-net.

This paper is divided into two more sections. In the first section, we recall the definitions that we need. We indicate that systems of equivalence relations
are pre-ultrametric spaces, a fact which may also motivate the study of the semirigidity of these systems. We give a representation theorem for reduced systems of equivalence relations (Theorem 2.8). We end the section with some examples of semirigid systems; Theorem 2.15 contains, with an other proof, the fact that $R \uparrow T_n$ is semirigid. Results announced in the introduction are proved in the last section that is self-contained up to subsection 2.1.

2. SYSTEMS OF EQUIVALENCE RELATIONS AND THEIR REPRESENTATIONS

2.1. Basic definitions. An equivalence relation on a set $E$ is a binary reflexive, symmetric and transitive relation on $E$. A system of equivalence relations on $E$ is a pair $\mathcal{M} := (E, (\rho_i)_{i \in I})$ where each $\rho_i$ is an equivalence relation on $E$. If $F$ is a subset of $E$, the restriction of $\mathcal{M}$ to $F$ is $\mathcal{M} \upharpoonright F := (F, ((F \times F) \cap \rho_i)_{i \in I})$. If $\mathcal{M}' := (E', (\rho'_i)_{i \in I})$ is another system of equivalence relations on $E'$, a homomorphism from $\mathcal{M}$ to $\mathcal{M}'$ is a map $f : E \to E'$ such that for all $x, y \in E, i \in I$:

$$x \rho_i y \text{ implies } f(x) \rho'_i f(y).$$

If the map $f$ is injective and he implication in (2) is an equivalence then $f$ is an embedding; if furthermore $f$ is bijective, $f$ is an isomorphism and the systems $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic. A homomorphism of $\mathcal{M}$ into $\mathcal{M}$ is called an endomorphism of $\mathcal{M}$; the set of endomorphisms of $\mathcal{M}$ is denoted by $\text{End}(\mathcal{M})$.

A system of equivalence relations $\mathcal{M} := (E, (\rho_i)_{i \in I})$ is reduced if $\cap_{i \in I} \rho_i = \Delta_E$, where $\Delta_E := \{(x, x) : x \in E\}$. Note that if $\mathcal{M}$ is semirigid then it is reduced. Indeed, supposing that $Z := \cap_{i \in I} \rho_i$ is distinct from $\Delta_E$, one choose some $(x, y) \in Z \setminus \Delta_E$. Then the map $f$ defined by $f(x) = y$ and $f(z) = x$ for $z \neq x$ is an endomorphism of $\mathcal{M}$. Hence $\mathcal{M}$ is not semirigid.

2.2. Systems of equivalence relations and ultrametric spaces. Propositions 2.1 and 2.2 of this subsection express the fact that systems of equivalence relations and pre-ultrametric spaces are two faces of the same coin. The study of metric spaces with distance values in a Boolean algebra first appears in Blumenthal [2]. A general study of distances with values in an ordered set is in [19]. Ultrametric spaces with distance values in an ordered set have been studied by Priess-Crampe and Ribenboim in several papers, e.g. [20], [21], [22].

A join-semilattice is an ordered set in which two arbitrary elements $x$ and $y$ have a join, denoted by $x \lor y$, defined as the least element of the set of common upper bounds of $x$ and $y$.

Let $V$ be a join-semilattice with a least element, denoted by $0$. A pre-ultrametric space over $V$ is a pair $\mathcal{D} := (E, d)$ where $d$ is a map from $E \times E$ into $V$ such that for all $x, y, z \in E$:

$$d(x, x) = 0, \ d(x, y) = d(y, x) \text{ and } d(x, y) \leq d(x, z) \lor d(z, y).$$

(3)
The map \( d \) is an ultrametric distance over \( V \) and \( \mathcal{D} \) is an ultrametric space over \( V \) if \( \mathcal{D} \) is a pre-ultrametric space and \( d \) satisfies the separation axiom:

\[
d(x, y) = 0 \text{ implies } x = y.
\]

Given a set \( I \) let \( \mathcal{P}(I) \) be the power set of \( I \). Then \( \mathcal{P}(I) \), ordered by inclusion, is a join-semilattice (in fact a complete Boolean algebra) in which the join is the union, and 0 the empty set.

**Proposition 2.1.** Let \( \mathcal{M} := (E, (\rho_i)_{i \in I}) \) be a system of equivalence relations. For \( x, y \in E \), set \( d_\mathcal{M}(x, y) := \{ i \in I : (x, y) \notin \rho_i \} \). Then the pair \( U_\mathcal{M} := (E, d_\mathcal{M}) \) is a pre-ultrametric space over \( \mathcal{P}(I) \).

Conversely, let \( \mathcal{D} := (E, d) \) a pre-ultrametric space over \( \mathcal{P}(I) \). For every \( i \in I \) set \( \rho_i := \{ (x, y) \in E \times E : i \notin d(x, y) \} \) and let \( \mathcal{M} := (E, (\rho_i)_{i \in I}) \). Then \( \rho_i \) is an equivalence relation on \( E \) and \( d_\mathcal{M} = d \).

Furthermore, \( U_\mathcal{M} \) is an ultrametric space if and only if \( \mathcal{M} \) is reduced.

For a join-semilattice \( V \) with a 0 and for two pre-ultrametric spaces \( \mathcal{D} := (E, d) \) and \( \mathcal{D}' := (E', d') \) over \( V \), a non-expansive mapping from \( \mathcal{D} \) to \( \mathcal{D}' \) is any map \( f : E \to E' \) such that for all \( x, y \in E \):

\[
d'(f(x), f(y)) \leq d(x, y).
\]

**Proposition 2.2.** Let \( \mathcal{M} := (E, (\rho_i)_{i \in I}) \) and \( \mathcal{M}' := (E', (\rho'_i)_{i \in I}) \) be two systems of equivalence relations. A map \( f : E \to E' \) is a homomorphism from \( \mathcal{M} \) into \( \mathcal{M}' \) if and only if \( f \) is a non-expansive mapping from \( U_\mathcal{M} \) into \( U_{\mathcal{M}'} \).

Propositions 2.1 and 2.2 are immediate and the proofs are left to the reader. Still, this suggests to study ultrametric spaces (in the ordinary sense) which are rigid with respect to non-expansive mappings. But notice that a non-trivial ultrametric space could not be semirigid. Indeed:

**Lemma 2.3.** An ultrametric space \( \mathcal{D} := (E, d) \) over a linearly ordered set \( V \) having at least two elements is not semirigid.

**Proof.** If all non-zero distances between elements of \( \mathcal{D} \) are equal then every permutation of \( V \) is an endomorphism of \( \mathcal{D} \), hence \( \mathcal{D} \) is non semirigid. Thus assume that there are \( r, r' \) in \( V \) and \( x, y, x', y' \) in \( E \) such that \( 0 < r = d(x, y) < r' = d(x', y') \). Let \( B(x, r) := \{ z \in E : d(x, z) \leq r \} \). Then \( B(x, r) \notin E \). Indeed, if \( z, z' \in B(x, r) \) then \( d(z, z') \leq d(z, x) \lor d(x, z') \leq r \lor r = r \). Hence, \( \{x', y'\} \notin B(x, r) \). Let \( f \) be the map defined by setting \( f(z) := x \) if \( z \in B(x, r) \) and \( f(z) = z \) otherwise. Then \( f \) is non-expansive. Indeed, let \( z, z' \in E \). Suppose \( z \in B(x, r) \) and \( z' \notin B(x, r) \). Since \( V \) is totally ordered, \( d(x, z') > r \). Since \( d(x, z') \leq d(x, z) \lor d(z, z') \leq r \lor d(z, z') \), we have \( d(z, z') \notin r \). Since \( V \) is totally ordered, we also have \( d(x, z) \leq d(z, z') \). This amounts to \( d(f(z), f(z')) \leq d(z, z') \leq d(z, z') \). The other cases lead trivially to the same inequality. Hence, \( f \) is non-expansive. \( \square \)
Problem 2.4. *Beyond the fact that it is not linearly ordered, what can be said about the order structure of the set of values of the distances of a non-trivial semirigid ultrametric space?*

Contrary to the case of ultrametric spaces, there exist metric spaces (in the ordinary sense) which are rigid with respect to non-expansive maps. In fact there are metrizable topological spaces which are rigid with respect to continuous maps. Indeed, as shown by de Groot [5], there exist connected and locally connected subsets of the plane which are semirigid with respect to continuous maps (see Corollary 3 in [5]).

Problem 2.5. *Describe the semirigid metric spaces with respect to non-expansive mappings.*

2.3. Representation of systems of equivalence relations. We may also represent systems of equivalence relations on finite sets by matrices. Let \( A := (a_{ij})_{j=1,\ldots,n} \) be an \( m \times n \) matrix with non-negative integer coefficients. For each \( j \in \{1,\ldots,n\} \) let \( \rho_j \) be the equivalence relation defined on \( \{1,\ldots,m\} \) by \( i \rho_j j' \) if \( a_{ij} = a_{ij'} \). Then \( \mathcal{M}(A) := (\{1,\ldots,m\},(\rho_j)_{j=1,\ldots,n}) \) is a system of equivalence relations. Conversely:

Lemma 2.6. *Every system of \( n \) equivalence relations on an \( M \)-element set is isomorphic to a system \( \mathcal{M}(A) \) for some \( m \times n \) matrix \( A \) with non-negative integer coefficients.*

*Proof.* Let \( \mathcal{M} \) be a system of \( n \) equivalence relations on an \( m \)-element set. Without loss of generality we may suppose that the \( m \)-element set is \( \{1,\ldots,m\} \) and hence that \( \mathcal{M} = (\{1,\ldots,m\},(\rho_j)_{j=1,\ldots,n}) \). For \( i \in \{1,\ldots,m\} \) and \( j \in \{1,\ldots,n\} \) set \( \rho_j(i) := \{i' \in \{1,\ldots,m\} : i \rho j i'\} \). Let \( \varphi \) be a one-to-one map from the set \( \{\rho_j(i) : i \in \{1,\ldots,m\}, j = 1,\ldots,n\} \) into the set of non-negative integers. Set \( a_{ij} := \varphi(\rho_j(i)) \). \( \square \)

Problem 2.7. *What is the complexity of the problem: decide whether or not the system of equivalence relations associated with an \( m \times n \) matrix is semirigid?*

For reduced systems, Lemma 2.6 has a somewhat simpler form.

Let \( \mathcal{M} := (E_i,(\rho_i)_{i \in I}) \) be a system of equivalence relations. For \( x \in E \) and for \( i \in I \) let \( \rho_i(x) := \{y \in E : x \rho y\} \) be the equivalence class of \( x \) and let \( E_i := \{\rho_i(x) : x \in E\} \) be the set of equivalence classes. Let \( \overline{E} := (E_i)_{i \in I} \) be a family of sets and let \( \prod_{i \in I} E_i \) be their cartesian product. For \( x \in \prod_{i \in I} E_i \) denote by \( x_i \) its \( i \)-th projection, hence \( x = (x_i)_{i \in I} \). Let \( \mathcal{M}(\overline{E}) := (\prod_{i \in I} E_i,(\sim_i)_{i \in I}) \) where \( x \sim_i y \) if \( x_i = y_i \). Clearly this system is reduced. If all \( E_i \) are equal to the same set \( W \), we denote the system by \( \mathcal{M}(W,I) \).

Theorem 2.8. *Every reduced system can be embedded into a system of the form \( \mathcal{M}(V) \).*
Proof. Let $\mathcal{M} := (E, (\rho_i)_{i \in I})$. Let $\overline{E} := (E_i)_{i \in I}$ where $E_i := \{\rho_i(x) : x \in E\}$ is the set of equivalence classes of $\rho_i$ and let $\rho : E \to \prod_{i \in I} E_i$ be defined by setting $\rho(x) := (\rho_i(x))_{i \in I}$. Then, the map $\rho$ is an embedding from $\mathcal{M}$ into $\mathcal{M}(\overline{E})$: First, $\rho$ is one-to-one. Indeed, suppose that $\rho(x) = \rho(y)$ for some $x, y \in E$. Then $\rho(x)_i = \rho(y)_i$, that is $\rho_i(x) = \rho_i(y)$, for all $i \in I$. Since $\mathcal{M}$ is reduced, $x = y$. Next, $\rho$ is an homomorphism of $\mathcal{M}$ into $\mathcal{M}(\overline{E})$. Suppose that $x\rho_i y$ for some $x, y$ and $i \in I$. Then $\rho_i(x) = \rho_i(y)$, that is $\rho(x)_i = \rho(y)_i$, which amounts to the fact that $\rho(x) \sim_i \rho(y)$. The proof of the converse is similar.

**Corollary 2.9.** Every reduced system can be embedded into a system of the form $\mathcal{M}(W, I)$, where $\mathcal{M}(W, I) = (W^I, (\simeq_i)_{i \in I})$.

Proof. $\mathcal{M}(\overline{E})$ is embeddable into $\mathcal{M}(W, I)$, where $W := \bigcup_{i \in I} E_i$. Indeed, to $x \in \overline{E} := \prod_{i \in I} E_i$ associate $\varphi(x) \in W^I$ defined by $\varphi(x)_i = x_i$.

Let $\mathcal{M} := (E, (\rho_i)_{i \in I})$ be a system of equivalence relations and $f$ be a selfmap of $E$. It is immediate to see that $f$ is an endomorphism of $\mathcal{M}$ if and only if for each $i \in I$, the map $f$ induces a selfmap $f_i$ on the set $E_i$ of equivalence classes of $\rho_i$. Furthermore, if $\mathcal{M}$ is reduced, the family $(f_i)_{i \in I}$ determines $f$.

**Theorem 2.11** below expresses this fact and characterizes families $(f_i)_{i \in I} \in E_i$ that come from an endomorphism.

**Notation 2.10.** Let $E \subseteq W^I$ and for $i \in I$ denote $E_i$ the image of the $i$-th projection of $E$ in $W$, that is, with our notations, $E_i := \{x_i : x \in E\}$. Note that $E \subseteq \prod_{i \in I} E_i$. For $a \in \prod_{i \in I} W^{E_i}$, denote by $a$ the map from $\prod_{i \in I} E_i$ into $W^I$ defined by setting:

$$a((x_i)_{i \in I}) := (a_i(x_i))_{i \in I} \text{ for all } x \in \prod_{i \in I} E_i. \tag{6}$$

**Theorem 2.11.** Let $E \subseteq W^I$. A map $f : E \to W^I$ is a homomorphism of $\mathcal{M}(W, I) \upharpoonright E$ into $\mathcal{M}(W, I)$ if and only if $f = \hat{a} \upharpoonright E$ for some $a \in \prod_{i \in I} W^{E_i}$. Moreover, if such an element $a$ exists it is unique.

Proof. Suppose that $f$ is a homomorphism. Let $i \in I$ and let $t \in E_i$. Let $x, x' \in E$ such that $x_i = x'_i = t$. Then $x \sim_i x'$. Since $f$ is a homomorphism, $f(x) \sim_i f(x')$, amounting to $f(x)_i = f(x')_i$. Set $a_i(t) := f(x)_i$. Clearly $f(x) = (a_i(x_i))_{i \in I}$. Set $a := (a_i)_{i \in I}$. We have $f = \hat{a} \upharpoonright E$. The converse is immediate.

**Corollary 2.12.** Let $E \subseteq W^I$. Then $\mathcal{M}(W, I) \upharpoonright E$ is semirigid if and only if for every $a \in \prod_{i \in I} E_i$ such that for every $x \in E$,

$$a_i(x_i)_{i \in I} \in E \tag{7}$$

then either each $a_i$ is a constant function or each $a_i$ is the identity function on $E_i$.

**Corollary 2.13.** There is a bijective correspondence between $\text{End}(\mathcal{M}(W, I))$ and $(W^W)^I$.
Proof. Let $a \in (W^W)^I$. Then $\hat{a}$ is an endomorphism of $\mathcal{M}(W, I)$. Theorem 2.11 ensures that every endomorphism has this form. \qed

Example 2.14. Let $W = \{0, 1\}$. Then $\mathcal{M}(W, I)$ has $4^{|I|}$ endomorphisms. Each endomorphism $f$ of $\mathcal{M}(W, I)$ can be expressed as:

$$f(x) = (\alpha_i x_i + \beta_i)_{i \in I}$$

where $\alpha_i \in \{0, 1\}$, $\beta_i \in \{0, 1\}$, and the sum is modulo 2.

2.4. Examples of semirigid systems.

Theorem 2.15. Let $W := \mathbb{N}$, $n \in \mathbb{N}$ and let $I$ be a set (not necessarily finite) with at least three elements. Set:

$$E := \{(x_i)_{i \in I} \in \mathbb{N}^I : \sum_{i \in I} x_i = n\}.$$

Then $\mathcal{M}(\mathbb{N}, I) \upharpoonright E$ is semirigid.

Proof. Observe first that $E_i = \{0, \ldots, n\}$ for each $i \in I$. Apply Corollary 2.12. Let $a \in \prod_{i \in I} E_i$ be such that Condition (7) of Corollary 2.12 holds.

Claim 2.16. If $i, j \in I$ with $i \neq j$ then for all $t < n$:

$$a_i(t + 1) - a_i(t) = a_j(1) - a_j(0).$$

Indeed, let $k$ be distinct from $i$ and $j$. Let $x := (x_i)_{i \in I}$ and $x' := (x'_i)_{i \in I}$ be defined by setting $x_l = x'_l = 0$ if $l \in I \setminus \{i, j, k\}$, $x_i = x'_i + 1 = t + 1$, $x_j = x'_j - 1 = 0$ and $x_k = x'_k = n - t - 1$. Then $x, x' \in E$. Since Condition (7) holds, $(a_l(x_l))_{l \in I}$ and $(a_l(x'_l))_{l \in I}$ belong to $E$, that is $\sum_{l \in I} a_l(x_l) = \sum_{l \in I} a_l(x'_l) = n$. Thus $\sum_{l \in I} (a_l(x_l) - a_l(x'_l)) = a_i(t + 1) - a_i(t) + a_j(0) - a_j(1) = 0$, proving our claim.

Claim 2.17. $a_i(t) = a_i(0) + t(a_j(1) - a_j(0))$ for all $t \leq n$.

Indeed, this equality holds for $t = 0$; for larger $t$ apply induction and Claim 2.16.

Claim 2.18. The value $\delta := a_i(1) - a_i(0)$ is independent of $i$.

Indeed, from Claim 2.16 we obtain $a_i(1) - a_i(0) = a_j(1) - a_j(0)$.

If $\delta = 0$ then according to Claim 2.17 we have that $a_i(t) = a_i(0)$ for all $i$ and hence all maps $a_i$ are constant, proving that $a$ is constant. Thus, we may suppose $\delta \neq 0$.

Claim 2.19. $\delta = 1$ and $a_i(0) = 0$ for all $i \in I$.

Indeed, from Claim 2.17 we have that $a_i(n) - a_i(0) = n \delta$. Since $a_i(n), a_i(0) \in E_i = \{0, \ldots, n\}$, it follows that $-1 \leq \delta \leq 1$. Suppose that $\delta = -1$. Then $a_i(0) = n$ and $a_i(n) = 0$ for each $i \in I$. Since $|I| \geq 3$, the image $a(x)$ of the sequence $x$
such that \( x_i = n \) and \( x_i = 0 \) for \( l \neq i \) satisfies \( a(x)_i := a_i(x_i) = a_i(n) = 0 \) and \( a(x)_i := a_i(x_i) = a_i(0) = n \). Since \(|I| \geq 3\), this image is not in \( E \). This case is then impossible. The only remaining possibility is \( \delta = 1 \). In this case, \( a_i(t) = t \) for all \( i \) and \( t \). According to Corollary 2.12, \( \mathcal{M}(\mathbb{N}, I) \uparrow E \) is semirigid. \( \square \)

2.4.1. Two special cases. 1) Set \( I := \{0,1,2\} \). Then, the map \( f \) from \( E \) into \( T_n := \{(i,j) \in \mathbb{N} \times \mathbb{N} : i + j \leq n\} \) defined by setting \( f(x_0,x_1,x_2) := (x_1,x_2) \) for all \((x_0,x_1,x_2) \in E\) a is an isomorphism from \( \mathcal{M}(\mathbb{N}, I) \uparrow E \) onto \( \mathcal{R} \uparrow T_n \). Hence, \( \mathcal{R} \uparrow T_n \) is semirigid.

2) Let \( n = 1 \) and \(|I| \geq 3\). In this case, \( E \) is the set of characteristic functions of singletons, that is functions \( \chi_{\{i\}} \) taking value 1 on \( i \) and 0 on elements \( i' \) distinct from \( i \). For each \( i \), clearly \( \sim_i \) is the equivalence relation which puts \( \chi_{\{i\}} \) in one class and all other characteristic functions in an other class. Hence, this system of equivalence relations is isomorphic to \( \mathcal{M} := (I,(\rho_i)_{i \in I}) \), where \( j\rho_i k \) if either \( i = j = k \) or \( j,k \in I \setminus \{i\} \). The fact that the system \( \mathcal{M} \) is semirigid follows directly from Lemma 1.1. Indeed, for all \( i \neq j \), clearly \( \cap \{\rho_k : k \in I \setminus \{i,j\}\} \) is a co-atom in the lattice of equivalence relations, and all co-atoms are obtained this way. Thus the \( \rho_i \)'s generate the lattice of equivalence relations on \( I \).

3. Systems of three equivalences embedded into algebraic structures

The simplest systems of three equivalence relations are those associated to groups. Their endomorphisms are described in Theorem 3.2. These systems are not semirigid, but some induced systems are. In order to obtain some examples of semirigid systems we describe in Lemma 3.1 the endomorphisms defined on induced systems.

Let \( G \) be a group, the operation being denoted by \( \cdot \) and the neutral element 1. Recall that a group homomorphism is any map \( h \) from \( G \) into \( G \) such that \( h(x \cdot y) = h(x) \cdot h(y) \) for all \( x,y \in A \). Set \( E := G \times G \). Denote by \( p_1 \) and \( p_2 \) the first and second projections from \( E \) onto \( G \) and let \( p_0 : E \rightarrow G \) be defined by \( p_0(x,y) := x \cdot y \) (hence \( p_0 = p_1 \cdot p_2 \)). For \( i < 3 \), let \( \zeta_i \) denote the equivalence relation on \( E \) defined for all \( u,v \in E \) by setting \( u \sim_i v \) if \( p_i(u) = p_i(v) \). Finally, set \( \mathcal{M} := (E, (\zeta_i)_{i \leq 3}) \) and let \( C \) be a subset of \( E \).

**Lemma 3.1.** A map \( f : C \rightarrow E \) is a homomorphism from \( \mathcal{M} \uparrow C \) into \( \mathcal{M} \) if and only if for each \( i < 3 \) there is a map \( h_i : p_i(C) \rightarrow G \) such that \( p_i(f(x,y)) = h_i(p_i(x,y)) \) for all \( (x,y) \in C \). In particular, these three maps satisfy:

\[
(8) \quad h_1(x) \cdot h_2(y) = h_0(x \cdot y)
\]

for all \( (x,y) \in C \).

Furthermore, if \( f \) fixes \((1,1)\), then \( h_1, h_2 \) and \( h_0 \) coincide on \( \mathcal{C} := \{x \in A : (1,x) \text{ and } (x,1) \in C\} \) and their common value \( h \) satisfies:

\[
(9) \quad h(x \cdot y) = h(x) \cdot h(y)
\]
for all $x, y$ such that $(x, y) \in C \cap (\check{C} \times \check{C})$ and $x \cdot y \in \check{C}$.

**Proof.** The first part of the lemma holds if $G$ is a magma (rather than a group), that is just a set equipped with a binary operation; the second part holds if in addition this operation has a neutral element, that is an element, denoted by 1, and necessarily unique, such that $e \cdot x = x \cdot e = x$ for all $x \in G$. Since each $p_i$ identifies with the canonical projection onto the quotient of $\mathcal{M}/\sim_i$, a homomorphism $f$ from $\mathcal{M} \mid C$ into $\mathcal{M}$ induces three maps $h_i : p_i(C) \to G$, $i < 3$ (cf. the proof of Theorem 2.11). The fact that Equation (8) holds follows readily from the definitions. Indeed, for $(x, y) \in C$, we have:

$$h_0(x \cdot y) = h_0(p_0(x, y)) = p_1(f(x, y)) \cdot p_2(f(x, y)).$$

For the second part, let $x \in \check{C}$. The relation $(1, 1) \equiv (x, 1)$ yields $(1, 1) = f(1, 1) \equiv f(x, 1)$ hence $f(x, 1) = (h_1(x), 1)$; similarly, $f(1, x) = (1, h_2(x))$. The relation $(x, 1) \equiv_0 (1, x)$ yields $f(x, 1) \equiv_0 f(1, x)$ thus $h_1(x) = h_2(x)$. From $f(1, 1) = (1, 1)$ we obtain $h_2(1) = 1$. Applying Equation (8) to $x$ and $1$ gives $h_1(x) = h_1(x) \cdot h_2(1) = h_0(x \cdot 1) = h_0(x)$. Hence $h_0 = h_1 = h_2$, as claimed. □

**Theorem 3.2.** A map $f : E \to E$ is an endomorphism of $\mathcal{M}$ if and only if $f(x, y) = (a \cdot h(x), h(y) \cdot b)$ for all $x, y \in G$ where $h$ is an endomorphism of $G$ and $(a, b) \in E$.

**Proof.** Due to the associativity of $\cdot$, if $f$ has the form given above, then it is a homomorphism of $\mathcal{M}$. In particular, for every $u := (a, b) \in E$, the transformation $t_u$ defined on $E$ by setting $t_u(x, y) := (a \cdot x, y \cdot b)$ is an endomorphism of $\mathcal{M}$. Now, let $f$ be an endomorphism of $\mathcal{M}$. Let $(a, b) := f(1, 1)$ and $(a, b)^{-1} := (a^{-1}, b^{-1})$. Since these transformations are endomorphisms of $\mathcal{M}$, the map $g := t_{(a, b)^{-1}} \circ f$ is an endomorphism of $\mathcal{M}$ fixing $(1, 1)$. According to Lemma 3.1, $g$ is of the form $(h, h)$ where $h$ is a homomorphism of $G$. Since $f = t_{(a, b)} \circ g$, $f$ has the form given above. □

### 3.1. Quasigroups, 3-nets and orthogonal systems

A quasigroup is a magma $(G, \cdot)$ of which each translation is bijective (each element $a \in G$ yields the left translation $x \mapsto a \cdot x$ and the right translation $x \mapsto x \cdot a$). Let $G$ be a quasigroup. On $E := G \times G$, the system $\mathcal{M} := (E, (z_i)_{i \leq 3})$ defined as in the case of groups is a 3-net. The first part of Lemma 3.1 applies and, if $G$ has an identity, the second part too. A quasigroup $(G, \cdot)$ with an identity is a loop. It is known that a semigroup $G$ satisfying the identity $(x \cdot y) \cdot (z \cdot x) = x \cdot (y \cdot z) \cdot x$ for all $x, y, z \in G$ is a loop, called a Moufang loop.

**Lemma 3.3.** The 3-net $\mathcal{M}$ on $G \times G$ associated with a Moufang loop $G$ with more than one element has proper automorphisms hence it is not semirigid.

**Proof.** Given an $a \neq 1$, let $f : G \times G \to G \times G$ be defined by setting $f(x, y) := (a \cdot x, y \cdot a)$. This map is an endomorphism of $\mathcal{M}$. It is not constant since $G$ is
a quasigroup. It is not the identity since \( a \neq 1 \). Since it is not the identity, nor a constant map, \( \mathcal{M} \) is not semirigid.

In order to extend Theorem 3.2 to Moufang loops it would suffice that the identity \((a \cdot x) \cdot (y \cdot b) = a \cdot ((x \cdot y) \cdot b)\) holds, but this amounts to associativity (replace \( a \) by 1), hence to the fact that the Moufang loop is a group.

Let us say that two equivalence relations \( \rho \) and \( \tau \) on a set \( E \) are orthogonal if each class of \( \rho \) intersects each class of \( \tau \) in at most one element (that is \( \rho \cap \tau = \Delta_E \)) and strongly orthogonal if each class of \( \rho \) intersects each class of \( \tau \) in exactly one element. According to our introduction, a system of three equivalence relations is a 3-net if and only if these relations are pairwise strongly orthogonal.

The link between the two notions is the following.

**Proposition 3.4.** Let \( \mathcal{M} := (E, (\rho_i)_{i<3}) \) be a system of three equivalence relations. These relations are pairwise orthogonal if and only if \( \mathcal{M} \) is embeddable into a 3-net. Moreover, if \( E \) is finite, then there is a 3-net of order at most \( 2|E| \) in which \( \mathcal{M} \) is embeddable.

**Proof.** We prove first that the orthogonality conditions are necessary. Next, we prove that they suffice by assuming first that \( E \) is finite, in which case we apply a result of Evans. If \( E \) is infinite, we use the diagram method of Robinson, a basic technique of mathematical logic. Namely, we define a theory \( T \) in a first order language; from the finite case of our proposition, we get that it is consistent, hence by the Compactness theorem of first order logic it has some model; it turns out that such a model is a 3-net extending \( \mathcal{M} \).

So suppose that \( \mathcal{M} \) is embeddable into a 3-net \( \mathcal{M}' := (E', (\rho'_i)_{i<3}) \), let \( A \subseteq E' \) be the image of \( E \) by an embedding. Since \( \mathcal{M}' \) is a 3-net, the \( \rho'_i \)'s are pairwise orthogonal, hence their restrictions \( \rho'_i \cap A \times A \) are pairwise orthogonal too. This is equivalent to the that the \( \rho_i \)'s are pairwise orthogonal.

Conversely, suppose that these equivalence relations are pairwise orthogonal. As in the proof of Theorem 2.8, let \( E_i := \{ \rho_i(x) : x \in E \} \) be the set of equivalence classes of \( \rho_i \) and let \( \rho : E \to \prod_{i \in I} E_i \) be defined by setting \( \rho(x) := (\rho_i(x))_{i \in I} \). Let \( e_i := |E_i|, e := \max \{ e_i : i < 3 \} \). Extending each \( E_i \) to a set \( E'_i \) of cardinality \( e \), the map \( \rho \) becomes an embedding of \( \mathcal{M} \) into \( (\prod_{i<3} E'_i, (\sim_i)_{i<3}) \).

Let \( \rho' : E \to E'_1 \times E'_2 \) be defined by setting \( \rho'(x) := (\rho_1(x), \rho_2(x)) \). Since \( \rho_1 \) and \( \rho_2 \) are orthogonal, the system \( (E, (\rho_1, \rho_2)) \) is reduced hence, according to Theorem 2.8, the map \( \rho' \) is an embedding from \( (E, (\rho_1, \rho_2)) \) into \( (E'_1 \times E'_2, (\sim_1, \sim_2)) \).

**Case 1.** Assume that \( e \) is finite. Labelling the elements of \( E'_i \) by the integers from 1 up to \( e \), the range \( E' \) of \( E \) by \( \rho' \) appears as a subset of the grid \( e \times e \) whose elements are labelled by non-negative integers from 1 up to \( e \) (corresponding to \( \rho_0(x) \)) and yields an incomplete latin square. According to Evans [10], this
incomplete latin square extends to a latin square of order 2\(e\). This latin square provides a 3-net of order 2\(e\) in which \(M\) embeds.

**Case 2.** Assume that \(e\) is infinite. The definition of the theory \(T\) goes as follows. The language consists of three binary predicate symbols \(r_i, i < 3\), and infinitely many constant symbols \((c_a)_{a \in E}\). Axioms of \(T\) are such that every model \(\mathcal{N} := (N, (r_i^N)_{i < 3}, (c_a^N)_{a \in E})\) interpreting predicates and constants yields a 3-net \((N, (r_i^N)_{i < 3})\) extending \(M\) through the embedding mapping each \(a \in E\) to \((c_a) \in N\). For that, we define two sets of axioms; the first set about \(r_i, i < 3\), expresses that \((N, (r_i^N)_{i < 3})\) will form a 3-net; the second set that the map \(a \to c_a^N\) is an embedding from \(M\) into \((N, (r_i^N)_{i < 3})\), the axioms needed here are \(\neg(c_a = c_a')\) for every \(a, a'\) distinct in \(E\) and \(r_i(c_a, c_a')\), resp. \(\neg r_i(c_a, c_a')\), whenever \(\rho_i(a, a')\), resp. \(\neg \rho_i(a, a')\), holds in \(M\). Now, if \(F\) is any finite set of axioms, there is a finite subset \(E_F\) of \(E\) such that no constant symbols \(c_a\) with \(a \in E \setminus E_F\) appears among the axioms belonging to \(F\). Applying the finite case of this proposition to \(M \upharpoonright E_F\) we find a model of \(F\) in which all constant symbols \(c_a\) with \(a \in E \setminus E_F\) are interpreted by some arbitrary element \(c \in E_F\). Since every finite set of axioms has a model, Compactness theorem of first order logic ensures that \(T\) has a model. \(\Box\)

3.2. **Triangles and monogenic systems.** Let \(\mathcal{M} := (E, (\rho_i)_{i < 3})\) be a system of three pairwise orthogonal equivalence relations.

**Definitions 3.1.** A triangle \(T\) in \(E\) consists of three elements \(u_0, u_1, u_2\) (not necessarily distinct) of \(E\) such that \(u_0 \sim u_1, u_1 \sim u_2\) and \(u_2 \sim u_0\). For \(X \in C \subseteq E\) we say that \(X\) is \(C\)-closed if any triangle of \(C\) with two elements in \(X\) is included in \(C\). Denote by \(\varphi_C(X)\) the intersection of \(C\)-closed subsets of \(C\) containing \(X\) and denote by \(\delta_C(X)\) the set of \(u \in C\) such that there is a triangle \(\{a, b, u\}\) with \(\{a, b\} \subseteq X\). For \(n \geq 0\) define \(\delta_C^{(n)}(X)\) recursively by \(\delta_C^{(0)}(X) = X\) and \(\delta_C^{(n+1)}(X) := \delta_C(\delta_C^{(n)}(X))\).

Trivially, the collection of \(C\)-closed subsets is stable under intersection and \(\varphi_C(X) = \bigcup \{\delta_C^{(n)}(X) : n \in \mathbb{N}\}\).

**Definitions 3.2.** We say that a subset \(X\) of \(C\) generates \(C\) if \(\varphi_C(X) = C\). We say that \(C\) is monogenic if some subset of \(C\) with at most two elements generates \(C\).

**Examples 3.5.** If \(G\) is a cyclic group (denoted additively) and \(E := G \times G\), then \(E\) is monogenic (indeed, \(E\) is generated by the pair \(\{(0, 0), (1, 0)\}\) where 1 generates \(G\)). Also for each \(n \in \mathbb{N}\), the set \(T_n = \{(i, j) : i, j \in \mathbb{N} : i + j \leq n\}\) defined in the introductory section is monogenic. Indeed, the set \(X := \{(0, 0), (0, 1)\}\) generates it. Similarly, \(X := \{(0, n), (1, n - 1)\}\) generates each of \(T_n\) and \(T_n\), also defined in the introduction. However, the 8-element set \(U := \{(0, 0), (2, 0), (1, 1), (2, 1), (1, 2), (2, 2), (0, 3), (1, 3)\}\)
is not monogenic (see Figure 3).

Lemma 3.6 (Triangle determinacy property). Let \( i, j, k \) be the three elements of \( \{0, 1, 2\} \) in an arbitrary order and \( u, v, w \in E \) such that \( w \preceq_1 v, w \preceq_j u, u \preceq_k v \). Then any \( w' \in E \) such that \( w' \preceq_i v \) and \( w' \preceq_j u \) is equal to \( w \).

Lemma 3.7. Let \( C \) be a subset of \( E \) and \( f, g \) be two homomorphisms of \( \mathcal{M} \upharpoonright C \) into \( \mathcal{M} \). If \( f \) and \( g \) coincide on a subset \( X \) of \( C \) then \( f \) and \( g \) coincide on \( \varphi_C(X) \).

Proof. It suffices to prove that if \( f \) and \( g \) coincide on a subset \( X \) of \( C \) they coincide on \( \delta_C(X) \). For that, let \( c \in \delta_C(X) \); our aim is to prove that \( f(c) = g(c) \). Let \( T := \{a, b, c\} \) be a triangle with \( \{a, b\} \subseteq X \). The Triangle determinacy applied with \( u := f(a) = g(a) \), \( v := f(b) = g(b) \), \( w := f(c) \) and \( w' := g(c) \) yields \( g(c) = f(c) \).

\(\square\)

Problem 3.8. Are the maps preserving a finite monogenic system either constant or automorphisms?

According to Proposition 3.4 we may suppose that \( \mathcal{M} \) is a 3-net. More can be said if the associated quasigroup is an additive group, particularly if this group is a subgroup of the additive group of a field.

3.3. 3-net of an abelian group. Here, we denote by \( A \) an abelian group, the addition being denoted + and the neutral element 0, and we set \( E := A \times A \).

For each non-empty subset \( X \) of \( E \), let \( D(X) := \{p_i(u) - p_i(v) : u, v \in X, i \in \{1, 2\}\} \). For \( \alpha \in X \), set \( G(X) := \alpha + \langle D(X) \rangle \times \{D(X)\} \) where \( \langle D(X) \rangle \) is the additive subgroup of \( A \) generated by \( D(X) \). Note that the choice of \( \alpha \) is irrelevant.

Lemma 3.9. For every \( X \subseteq C \subseteq E \), \( \varphi_C(X) \) is a subset of \( G(X) \).

Proof. First observe that \( X \subseteq G(X) \), next that for every subgroup \( B \) of \( A \) and any \( \beta \in E \), \( \beta + B \times B \) is \( E \)-closed hence \( \delta_C(X) \subseteq G(X) \) and finally that \( \langle D(X) \rangle = \langle \delta(D(X)) \rangle \).

The next lemma expresses the fact that monogenic systems can be viewed as subsystems of \( \mathbb{Z} \times \mathbb{Z} \) or \( (\mathbb{Z}/n) \times (\mathbb{Z}/n) \) for some integer \( n \).

Lemma 3.10. If \( C \subseteq A \times A \) is monogenic then \( \mathcal{M} \upharpoonright C \) is isomorphic to some system \( M' \upharpoonright C' \) where \( C' \subseteq A' \times A' \) and \( A' \) is a cyclic group. Furthermore, if \( C' \) contains the vertices of a triangle, then we may assume that the pair \( \{(0, 0), (c, 0)\} \) (for some \( c \in A' \)) generates \( C' \).

Proof. We may suppose that \( C \) contains the vertices of a triangle, otherwise \( C \) has at most two elements and the result is trivial. Let \( X := \{a, b\} \) be a subset of \( C \) which generates \( C \). Then there is some triangle \( T := \{a, b, c\} \subseteq C \). Clearly
\{b, c\} and \{a, c\} generate \(C\). Hence, with no loss of generality, we may suppose that \(p_2(a) = p_2(b)\). Thus \(\langle D(X) \rangle = \mathbb{Z}(p_1(b) - p_1(a))\) and
\[
G(X) = a + \mathbb{Z}(p_1(b) - p_1(a)) \times \mathbb{Z}(p_1(b) - p_1(a)).
\]
According to Lemma 3.9, \(C\) is included in \(G(X)\). According to Theorem 3.2 we may translate \(a\) and \(b\) onto \((0, 0)\) and \((p_1(b) - p_1(a), 0)\) respectively. \(\square\)

If \(A\) is the additive group of a field, say \(\mathbb{K}\), multiplication of two elements \(\lambda\) and \(\mu\) of \(A\) will be denoted by \(\lambda \mu\). In this case, a map \(h\) from \(E\) into \(E\) is a \textit{homothety composed with a translation} if for some \(\alpha \in E\) and some \textit{coefficient} \(\lambda \in A\), \(h(u) = \lambda u + \alpha\) for all \(u \in E\).

Note that if \(A'\) is a cyclic subgroup of \(A\) and \(h\) preserves \(E' \coloneqq A' \times A'\) then \(\lambda \in A'\) and \(\alpha \in E'\) (indeed, since \(h\) preserves the additive group \(E'\), \(\alpha = h(0) \in E'\); it follows that \(\lambda u \in E'\) for every \(u \in E'\); hence, for a generator \(a'\) of \(A'\), \(\lambda(a', a') = k(a', a')\) for some \(k \in \mathbb{Z}\); this yields \(\lambda \in A'\)). We will say that \(h\) is a \textit{homothety composed with a translation on} \(E'\).

**Proposition 3.11.** Suppose that \(A\) is the additive group of a field. If a subset \(C \subseteq E \coloneqq A \times A\) is monogenic then every homomorphism of \(\mathcal{M} \upharpoonright C\) into \(\mathcal{M}\) is the restriction to \(C\) of the composition of a homothety with a translation.

**Proof.** Let \(f\) be a homomorphism of \(\mathcal{M} \upharpoonright C\) into \(\mathcal{M}\) and let \(X \coloneqq \{a, b\}\) generates \(C\). Set \(a' \coloneqq f(a)\) and \(b' \coloneqq f(b)\). First suppose that \(a' = b'\). Let \(w\) be the common value and \(c_w\) be the constant map mapping \(a\) and \(b\) to \(w\). Since \(f\) and \(c_w\) coincide on \(X\), Lemma 3.7 ensures that they coincide on \(C = \varphi_C(X)\). Thus \(f\) is constant. Now, assume that \(a' \neq b'\). Let \(\lambda \in A\) be such that \(b' - a' = \lambda(b-a)\), set \(\alpha := a' - \lambda a\) and let \(h\) be the selfmap of \(A\) defined by setting \(h(u) := \lambda u + \alpha\) for every \(u \in A\). This map is an endomorphism of \(\mathcal{M}\). It coincides with \(f\) on \(X\). According to Lemma 3.7, \(f\) coincide with \(h\) on \(C = \varphi_C(X)\). \(\square\)

**Proposition 3.12.** Suppose that \(A\) is \(\mathbb{Z}\) or \(\mathbb{Z}/p\mathbb{Z}\) where \(p\) is prime. If a subset \(C \subseteq E \times E\) is monogenic then every endomorphism of \(\mathcal{M} \upharpoonright C\) is the restriction of the composition of a homothety with a translation on \(E\).

**Proof.** We may extend \(A\) to the additive group of a field. Apply Proposition 3.11. \(\square\)

We recall that an abelian group \(A\) is \textit{torsion-free} if \(nx = 0\) implies \(n = 0\) or \(x = 0\) \((n \in \mathbb{N}, x \in A\)\). An abelian group \(A\) is \textit{divisible} if for every \(a \in A\) and every positive integer \(n\), the equation \(n \cdot x = a\) has a solution. Torsion-free divisible groups are vector spaces over the field \(\mathbb{Q}\) of rational numbers. Also each torsion-free divisible group extends to a field.

An element \(\alpha \in E\) is a \textit{center of symmetry} for \(C \subseteq E\) if for every \(u \in C\), \(2\alpha - u \in C\).
Proposition 3.13. Let $A$ be torsion-free and $C$ be a monogenic subset of $E$ containing $(0,0)$. If each line going through $(0,0)$ contains only finitely many elements of $C$ and if $C$ has no center of symmetry then $\mathcal{M} \upharpoonright C$ is semirigid.

Proof. Let $f$ be an endomorphism of $\mathcal{M} \upharpoonright C$. According to Lemma 3.10 we may suppose that $A = \mathbb{Z}$. According to Proposition 3.12 the map $f$ is the composition of a homothety with a translation, and there are $\alpha \in E$ and $\lambda \in \mathbb{Z}$ such that $f(u) = \lambda u + \alpha$ for all $u \in E$. We show that $\alpha = 0$ and $\lambda = 1$. Suppose for a contradiction $\alpha \neq 0$. Let us iterate $f$. For each integer $n \geq 1$, the iterated selfmap $f^n$ satisfies $f^n(u) = \lambda^n u + (\sum_{k=0}^{n} \lambda^k)\alpha$ for all $u \in E$. Since $(0,0) \in C$, obviously $(\sum_{k=0}^{n} \lambda^k)\alpha \in C$ for every $n \geq 1$. Hence, since all these elements are multiples of $\alpha$, they all lie on the line going through $(0,0)$ and $\alpha$. Since, by hypothesis this line contains only finitely many elements of $C$, either $\lambda = 0$, in which case $f$ is constant, or $\lambda = -1$, in which case $f$ is a symmetry whose center is $\frac{2}{2}$, a case which is excluded by assumption. Hence $\alpha = 0$ and thus $f(u) = \lambda u$. Since each line going through $(0,0)$ contains only finitely many elements of $C$, proceeding in a similar way as above we get $\lambda \in \{1,-1\}$. Since $f$ is not constant and $(0,0)$ is not a center of symmetry of $C$, similarly we get $\lambda = 1$. In this case, $f$ is the identity on $C$, thus $\mathcal{M}$ is semirigid. □

Theorem 1.3 follows immediately from Proposition 3.13.

Remark 3.14. There are non-monogenic subsets $C$ for which $\mathcal{M} \upharpoonright C$ is semirigid. Indeed, let $A := \mathbb{Z}$ and let $U$ be the ones represented Figure 3. Then $\mathcal{M} \upharpoonright U$ is semirigid but $U$ is not monogenic (the semirigidity was checked by computer).

3.4. Semirigid subsystems of $\mathbb{R} \times \mathbb{R}$. Let $A := \mathbb{Z}$, $E := A \times A$ and

$$B := \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x + y \in \{1,2\}\} \cup \{(0,0)\}.$$ 

Proposition 3.15. The system $\mathcal{M} \upharpoonright B$ induced by the system $\mathcal{M}$ associated with $E$ is semirigid.

Proof. As it is easy to see, $X := \{(0,0),(1,0)\}$ generates $B$. Thus $B$ is monogenic. Furthermore, all other hypothesis of Proposition 3.13 are satisfied. Hence $\mathcal{M} \upharpoonright B$ is semirigid. □

We recall that a subset $X$ of $\mathbb{R}$ is dense if for every $x < y$ in $\mathbb{R}$ there is some $z \in X$ such that $x < z < y$. We also recall that every additive subgroup $D$ of $\mathbb{R}$ is either discrete, in which case $D = \mathbb{Z} \cdot r$ for some $r \in \mathbb{R}$, or dense. Let $D$ be an additive subgroup of $\mathbb{R}$ containing $\mathbb{Z}$. Set $\Delta := \{(x,y) \in D \times D : 0 \leq x, 0 \leq y, x+y \leq 1\}$

Proposition 3.16. If $D$ is a dense subgroup of $\mathbb{R}$ including $\mathbb{Q}$ then the system $\mathcal{M} \upharpoonright (B \cup \Delta)$ induced by the system $\mathcal{M}$ associated with $D \times D$ is semirigid.
Proof. Let $g$ be an endomorphism of $\mathcal{M} \upharpoonright B \cup \Delta$. In particular, $g$ induces a homomorphism of $\mathcal{M} \upharpoonright B$ in $\mathcal{M}$. Since $B$ is monogenic then, according to
Lemma 3.11, the map \( g \upharpoonright B \) is the restriction to \( B \) of the composition of an homothety with a translation, that is \( g(u) = \lambda u + \alpha \) for all \( u \in B \) and some \( \lambda \in \mathbb{R} \) and \( \alpha \in E \). Since \( g \) maps \( B \) into \( B \cup \Delta \), one would easily see that if \( g \) is not constant then \( \lambda = 1 \) and \( \alpha = 0 \) and thus \( g \) is the identity on \( B \). Let \( \alpha := g(0,0) \) and set \( f(u) := g(u) - \alpha \) for every \( u \in B \cup \Delta \). Apply Lemma 3.1 to \( E := D \times D \), \( C := \Delta \) and to the restriction of \( f \) to \( C \). The maps \( h_0, h_1, h_2 \) given by Lemma 3.1 coincide and their common value \( h \) satisfies \( h(x + y) = h(x) + h(y) \) whenever \( x, y, x + y \in [0,1] \cap D \). We claim that \( h(x) = xh(1) \) for every \( x \in [0,1] \cap D \).

We observe first that \( f(0) = 0 \) and \( g(0) = \alpha \), hence \( f \) is 1-Lipschitz, \( g \) is constant or the identity. Indeed, if \( \lambda = 1 \) and \( \alpha = 0 \) and thus \( g \) is the identity on \( B \). Let \( \alpha := g(0,0) \) and set \( f(u) := g(u) - \alpha \) for every \( u \in B \cup \Delta \). Apply Lemma 3.1 to \( E := D \times D \), \( C := \Delta \) and to the restriction of \( f \) to \( C \). The maps \( h_0, h_1, h_2 \) given by Lemma 3.1 coincide and their common value \( h \) satisfies \( h(x + y) = h(x) + h(y) \) whenever \( x, y, x + y \in [0,1] \cap D \). We claim that \( h(x) = xh(1) \) for every \( x \in [0,1] \cap D \).

We observe first that \( h(rx) = rh(x) \) for every rational \( r \) and \( x \) such that \( x, rx \in [0,1] \cap D \). Next we prove that \( h \) is continuous. Our claim follows. The continuity of \( h \) will follows from the fact that \( |h(x)| \leq |x| \) for every \( x \in [0,1] \cap D \) (indeed, from the additivity condition, that means that \( h \) is 1-Lipschitz, hence continuous). So given \( x \in [0,1] \cap D \), let \( X := [0,1] \cap \mathbb{Q} \), \( \alpha := g(0,0) \). In particular, \( \alpha = 0 \), \( h \) is constant on \( B \). Indeed, if \( g \) is constant on \( B \) then, in particular \( \alpha := g(0,0) = g(1,0) \), in which case \( h(1) = p_1(f(1,0)) = p_1(g(1,0) - \alpha) = p_1(0,0) = 0 \). Hence, by our claim, \( h \) is constant on \( [0,1] \cap D \). Thus the image of \( \Delta \) is \( \alpha \) and \( g \) is constant.

If \( g \) is not constant on \( B \) then, as we have seen, \( g \) is the identity on \( B \). In this case \( \alpha = (0,0) \), hence \( f = g \). Thus \( h(1) = p_1(f(1,0)) = p_1(g(1,0)) = p_1(1,0) = 1 \); then, by our claim, \( h \) is the identity on \( [0,1] \cap D \) and \( g \) is the identity on \( \Delta \). Thus, \( g \) is the identity on \( B \cup \Delta \), proving that \( \mathcal{M} \upharpoonright B \cup \Delta \) is semirigid.

\[ \square \]

3.5. Proof of Theorem 1.4. According to Zádorí’s result, there is a semirigid system of three equivalence on a set of size \( \kappa \) for each finite \( \kappa \) distinct from 2 and 4. If \( \kappa = \aleph_0 \), apply Proposition 1.3. If \( \aleph_0 < \kappa < 2^{\aleph_0} \), observe (with the axiom of choice) that there are dense additive subgroups of \( \mathbb{R} \) of cardinality \( \kappa \). Apply Proposition 3.16.

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