Particles and Quantum Symmetries

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Abstract

A system of interacting particles equipped with quantum symmetry is described in an abstract algebraic way. The concept of quantum commutativity is used for description of the algebra of quantum states of the system. The graded commutativity and particles in singular magnetic field is considered as an example. The generalized Pauli exclusion principle is also mentioned.

1. INTRODUCTION

In this report particle systems interacting with certain external field and equipped with a quantum symmetry are considered. Our study is based on the assumption that the system is characterized by a given Hopf algebra $H$ which act or coact on an unital and associative algebra $A$. The Hopf algebra $H$ describes quanta of external field and the algebra $A$ describes physical states of the system of particles. The action of $H$ on $A$ represents the process of emission of quanta of the external field. The coaction correspond for the process of absorption. If the algebra $A$ is quantum commutative with respect to the given action or coaction of Hopf algebra $H$, then we say that the system posses a quantum symmetry. Note that quantum commutative algebras have been studied previously by several authors, see [1, 2, 3, 4] for example. In this paper we assume that the Hopf algebra $H$ is a group algebra $kG$, where $G$ is an abelian group and $k \equiv \mathbb{C}$ is the field of complex numbers. In this particular case the coaction of $H$ on $A$ is equivalent to the $G$–gradation of $A$ and the quantum commutativity is equivalent to the well–known graded commutativity [3, 4].

A purely algebraic model of particles in singular magnetic field corresponding to the filling factor $v = \frac{1}{N}$ has been given by the author in [5]. We describe here this model as an example of system with quantum symmetry. This symmetry is described as the gradation of the algebra $A$ by the group $G \equiv Z_2 \oplus \cdots \oplus Z_2$ (N sumands). We generalize our considerations for arbitrary fraction $v = \frac{n}{N}$, where $n$ is the number of charged particles per $N$ fluxes. Note that there is an approach in which the algebra $A$ play the role of noncommutative Fock space. In this attempt the creation and aniliation operators act on the algebra $A$ such that the creation operators act as the multiplication in $A$ and the anihilation ones act as a noncommutative contraction (noncommutative
partial derivatives) \([6, 7, 8]\). One can use such formalism in order to give the algebraic Fock space representation \([4, 11]\).

2. MATHEMATICAL PRELIMINARIES

Let us start with a brief review on the concept of quantum commutativity \([4]\). It is well known that for a quasitriangular Hopf algebra \(H\) there is the so-called \(R\)-matrix \(R = \Sigma R_1 \otimes R_2 \in H \otimes H\) stisfying the quantum Yang–Baxter equation

\[
(R \otimes \text{id})(\text{id} \otimes R)(R \otimes \text{id}) = (\text{id} \otimes R)(R \otimes \text{id})(\text{id} \otimes R).
\] (2.1)

An algebra \(A\) equipped with an action of \(\triangleright : H \otimes A \rightarrow A\) is said to be quantum commutative if the following condition

\[
ab = \Sigma(R_2 \triangleright b)(R_1 \triangleright a)
\] (2.2)

is satisfied for every \(a, b \in A\), see \([4, 2]\) for example. It is interesting that there is a dual notion of quantum commutativity. It is based on the coquasitriangular Hopf algebras (CQHA) \([3, 4]\). Let \(H\) be a Hopf algebra over a field \(k\). We use the following notation for the coproduct in \(H\): if \(h \in H\), then \(\triangle(h) := \Sigma h_1 \otimes h_2 \in H \otimes H\). A coquasitriangular Hopf algebra (a CQTHA) is a Hopf algebra \(H\) equipped with a bilinear form \(\langle -,-\rangle : H \otimes H \rightarrow k\) such that

\[
\Sigma\langle h_1,k_1\rangle k_2 h_2 = \Sigma h_1 k_1 \langle h_2,k_2\rangle, \\
\langle h,kl\rangle = \Sigma \langle h_1,k\rangle \langle h_2,l\rangle, \\
\langle hl,l\rangle = \Sigma \langle h_1,l\rangle \langle k,l_1\rangle
\] (2.3)

for every \(h, k, l \in H\). If such bilinear form \(b\) exists for a given Hopf algebra \(H\), then we say that there is a coquasitriangular structure on \(H\).

Let \(H\) be a CQTHA with coquasitriangular structure given by a biform \(\langle -,-\rangle : H \otimes H \rightarrow k\) and let \(A\) be a (right) \(H\)-comodule algebra with coaction \(\rho\). Then the algebra \(A\) is said to be quantum commutative with respect to the coaction of \((H,b)\) if an only if we have the relation

\[
a b = \Sigma \langle a_1,b_1\rangle b_0 a_0,
\] (2.4)

where \(\rho(a) = \Sigma a_0 \otimes a_1 \in A \otimes H\), and \(\rho(b) = \Sigma b_0 \otimes b_1 \in A \otimes H\) for every \(a, b \in A\), see \([4]\). The Hopf algebra \(H\) is said to be a quantum symmetry for \(A\). Let \(G\) be an arbitrary group, then the group algebra \(H := kG\) is a Hopf algebra for which the comultiplication, the counit, and the antypode are given by the formulae

\[
\triangle(g) := g \otimes g, \quad \eta(g) := 1, \quad S(g) := g^{-1} \quad \text{for } g \in G.
\]

respectively. If \(H \equiv kG\), where \(G\) is an abelian group, \(k \equiv \mathbb{C}\) is the field of complex numbers, then the coquasitriangular structure on \(H\) is given as a bicharacter on \(G\). \([3]\) Note that for abelian groups we use the additive notation. A mapping \(\epsilon : G \times G \rightarrow \mathbb{C} \setminus \{0\}\) is said to be a bicharacter on \(G\) if and only if we have the following relations

\[
\epsilon(\alpha,\beta + \gamma) = \epsilon(\alpha,\beta)\epsilon(\alpha,\gamma), \quad \epsilon(\alpha + \beta,\gamma) = \epsilon(\alpha,\gamma)\epsilon(\beta,\gamma)
\] (2.5)

for \(\alpha, \beta, \gamma \in G\). If in addition

\[
\epsilon(\alpha,\beta)\epsilon(\beta,\alpha) = 1,
\] (2.6)
for $\alpha, \beta \in G$, then $\epsilon$ is said to be a normalized bicharacter. Note that this mapping are also said to be a commutation factor on $G$ and it has been studied previously by Scheunert [12]) in the context of color Lie algebras. We restrict here our attention to normalized bicharacters only. It is interesting that the coaction of $H$ on certain space $E$, where $H \cong kG$ is equivalent to the $G$-gradation of $E$, see [3]. In this case the quantum commutative algebra $A$ becomes graded commutative

$$ab = \epsilon(\alpha, \beta)ba$$

(2.7)

for homogeneous elements $a$ and $b$ of grade $\alpha$ and $\beta$, respectively.

Let $H$ be a CQTHA with coquasitriangular structure $\langle -, - \rangle$. The family of all $H$-comodules forms a category $\mathcal{C} = \mathcal{M}^H$. The category $\mathcal{C}$ is braided monoidal. The braid symmetry $\Psi \equiv \{\Psi_{U,V} : U \otimes V \longrightarrow V \otimes U; U, V \in \text{Ob}\mathcal{C}\}$ in $\mathcal{C}$ is defined by the equation

$$\Psi_{U,V}(u \otimes v) = \Sigma \langle v_1, u_1 \rangle v_0 \otimes u_0,$$

(2.8)

where $\rho(u) = \Sigma u_0 \otimes u_1 \in U \otimes H$, and $\rho(v) = \Sigma v_0 \otimes v_1 \in V \otimes H$ for every $u \in U, v \in V$.

It is known that in an arbitrary braided monoidal category $\mathcal{C}$ there is an algebra $A$ with braided commutative multiplication $m \circ \Psi = m$. Obviously this algebra is quantum commutative.

3. FUNDAMENTAL ASSUMPTIONS

Let $E$ be a finite dimensional Hilbert space equipped with a basis $\Theta_i, i = 1, ..., N = \text{dim}H$. We assume that $E$ is a space of single particle quantum states for certain system of particles. We also assume that there is a coaction $\rho_E$ of the Hopf algebra $H$ on the space $E$, i.e. a linear mapping $\rho_E : E \longrightarrow E \otimes H$, which define a (right-) $H$-comodule structure on $E$. Then there is a braided monoidal category $\mathcal{C} \equiv \mathcal{C}(E, \rho_H)$ which contains the field $\mathbb{C}$, the space $E$, all tensors product and direct sums of these spaces, and some quotients. An algebra defined as the quotient $\mathcal{A} \equiv \mathcal{A}(E) := TE/I_b$, where $I_b$ is an ideal in $TE$ generated by elements of the form

$$u \otimes v - \Sigma \langle v_1, u_1 \rangle v_0 \otimes u_0$$

(3.1)

and is said to be $\langle -, - \rangle$-symmetric algebra over $E$, in the category $\mathcal{C}$ [3]. The algebra $\mathcal{A}$ describe partitions of particles dressed with quanta of certain physical external field.

Now let us assume that $H = \mathbb{C}G$. In this particular case the category $\mathcal{C} = \mathcal{M}^H$ becomes symmetric. The category $\mathcal{C} = \mathcal{M}^H$, where $H := \mathbb{C}G$ for certain abelian group $G$ and $\langle -, - \rangle$ is a bicharacter like above is denoted by $\mathcal{C} \equiv \mathcal{C}(G, \langle -, - \rangle)$. Observe that the symmetry $\Psi$ in the formula (2.8) can be understood as a cointertwiner for corepresentations $\rho_U$ and $\rho_V$. Note that if $E$ is a $H$-comodule, where $H = \mathbb{C}G$, then $E$ is also a $G$-graded vector space, i.e $E = \bigoplus_{\alpha \in G} E_\alpha$. This means that the above coaction is equivalent to $G$-gradation. The normalized bicharacter on the grading group $G$ has the following form

$$\epsilon(\alpha, \beta) = (-1)^{\langle \alpha | \beta \rangle} q^{<\alpha | \beta>},$$

(3.2)

where $\langle -, - \rangle$ is an integer-valued symmetric bi-form on $G$, and $<-, ->$ is a skew-symmetric integer-valued bi-form on $G$, $q$ is some complex parameter [13]. We use here the so-called standard gradation [14]. This means that the grading group is $G \equiv Z^N := Z \oplus ... \oplus Z$ ($N$-sumands) for arbitrary $q$. If $q = exp(\frac{2\pi i}{n})$, $n \geq 3$, then the grading group
$G \equiv Z^N$ can be reduced to $G = Z_n \oplus \ldots \oplus Z_n$. If $q = \pm 1$, then the grading group $G$ can be reduced to the group $Z_2 \oplus \ldots \oplus Z_2$. In the standard gradation (see [14]) the algebra $\mathcal{A}$ is generated by the relation

$$\Theta_i \Theta_j = \epsilon_{ij} \Theta_j \Theta_i, \quad (3.3)$$

where $\epsilon_{ij} := \epsilon(\sigma_i, \sigma_j)$ for $i, j = 1, \ldots, N$, $\sigma_i$ is the $i$-th generator of the group $G$. In this gradation the $i$-th generator $\Theta_i^\alpha$ is a homogeneous element of grade $\sigma_i$ and describes a particle dressed with a single quantum. Here the quantum commutativity with respect to the algebra $H \equiv \mathbb{C}G$ is the above graded commutativity [12, 13, 14]. Every element $\Theta_\alpha$ of the algebra $\mathcal{A}$ can be given in the form

$$\Theta_\alpha = \Theta_1^{\alpha_1} \ldots \Theta_N^{\alpha_N} \quad (3.4)$$

for $\alpha = \sum_{i=1}^N a_i \sigma_i$, $x_0 \equiv 1$, where 1 is the unit in $\mathcal{A}$.

### 4. PARTICLES IN SINGULAR MAGNETIC FIELD

Now let us consider an algebraic model for a system of charged particles moving in two-dimensional space $\mathcal{M}$ with perpendicular magnetic field. Our fundamental assumption here is that the magnetic field is completely concentrated in vertical lines (fluxes). Every charged particle such as electron moving under influence of such singular magnetic field is transformed into a composite system which consists a charge and certain number of magnetic fluxes. We assume that there is in average $n$ electrons per $N$ fluxes. This means that the filling factor is $\nu = \frac{n}{N}$. We also assume that every magnetic flux can be coupled to a charged particle such that the magnetic field of the flux is canceled. In this case we say that the flux is bound to a particle or absorbed by it. The flux which is bound to a particle is said to be a quasiparticle. The particle is also said to be dressed by the flux. The flux not bound to a particle is said to be a quasihole. Our system is characterized by all possible equivalence classes of partitions of quasiparticle states and quasiholes. We are going to construct a quantum commutative algebra $\mathcal{A}$ for description of such partitions. First let us observe that the ”effective” configuration space for the single particle is $\mathcal{M} \equiv \mathbb{R}^2 \setminus \{s_1, \ldots, s_N\}$, where $s_1, \ldots, s_N \in \mathbb{R}^2$ are points of intersection of magnetic lines with the plane. The fundamental group $\pi_1(\mathcal{M})$ of $\mathcal{M}$ is denoted by $G_N$. Let us denote by $\tilde{\sigma}_i$ the homotopy class of paths which arounds the singularity $s_i$ with the winding number equal to 1 and not arounds any other point $s_j, j \neq i$. The fundamental group $G_N$ is a free group generated by $\tilde{\sigma}_i$, $(i = 1, \ldots, N)$ [15].

We introduce an equivalence relation in the loop space $\pi(\mathcal{M})$ as follows: If the phase change of particle moving along a loop $\xi$ in $\mathcal{M}$ is equal to the phase change of particle moving along another loop $\xi'$ in $\mathcal{M}$, then we say that such path are equivalent. The fact that two loops $\xi$ and $\xi'$ are equivalent is denoted by the expression $\xi \equiv_{ph} \xi'$. It is obvious that $\tilde{\sigma}_i \tilde{\sigma}_j \equiv_{ph} \tilde{\sigma}_j \tilde{\sigma}_i$ for $i \neq j$. This means that we have the relation

$$(G_N / \equiv_{ph}) = Z \oplus \ldots \oplus Z \quad (N\text{-sumands}). \quad (4.1)$$

It is known that the corresponding group algebra $H := \mathbb{C}G$ is a coquasitriangular Hopf algebra. The coquasitriangular structure on $H$ is given by a bicharacter $\epsilon : \Gamma \times \Gamma \rightarrow \mathbb{C} - \{0\}$ on $G$. The normalized bicharacter $\epsilon$ for our model is given by the formula

$$\epsilon_{ij} = -(-1)^N (-1)^{\Omega_{ij}}, \quad (4.2)$$
where $\Omega_{ij} = -\Omega_{ji} = 1$, for $i \neq j$, $\Omega_{ii} = 0$, $i, j = 1, 2, \ldots, N$. This means that the grading group is

$$G \equiv Z_2^N := Z_2 \oplus \ldots \oplus Z_2 \quad (N \text{ sumands})$$

(4.3)

Now we describe the the algebra $A$ corresponding to our model. In this case the algebra $A$ is generated by $\Theta^a_i (i = 1, \ldots, N; a = 1, \ldots n)$ and relations

$$\Theta^a_i \Theta^a_j = \epsilon_{ij} \Theta^a_j \Theta^a_i \quad \text{for} \quad a = b,$$

$$\Theta^a_i \Theta^b_j = -\epsilon_{ij} \Theta^b_j \Theta^a_i \quad \text{for} \quad a \neq b,$$

$$(\Theta^a_i)^2 = 0,$$

(4.4)

where $\epsilon$ is given by the formula (4.2). The $G$–gradation of $A$ is given by the relation

$$\text{grade} \Theta^a_i = \sigma_i,$$

(4.5)

where $\sigma_i = (0 \ldots 1 \ldots 0)$ (1 on the $i$–th place), is the $i$–th generator of the grading group $G = Z_2^N$. In our physical interpretation the generator $\Theta^a_i$ describes the partition containing one particle dressing with the $i$–th flux (a quasiparticle) and $N-1$ quasiholes. The statistics of quasiparticles is described by normalized bicharacter. We also have here the generalized Pauli exclusion principle. According to this principle partitions containing two or more anticommuting quasiparticles must be excluded. If $N$ is even, then we obtain the following relations

$$\Theta^a_i \Theta^a_j = -(1)^{\Omega_{ij}} \Theta^a_j \Theta^a_i \quad \text{for} \quad a = b,$$

$$\Theta^a_i \Theta^b_j = -(1)^{\Omega_{ij}} \Theta^b_j \Theta^a_i \quad \text{for} \quad a \neq b,$$

$$(\Theta^a_i)^2 = 0.$$

(4.6)

For odd $N$ we obtain

$$\Theta^a_i \Theta^a_j = (-1)^{\Omega_{ij}} \Theta^a_j \Theta^a_i \quad \text{for} \quad a = b,$$

$$\Theta^a_i \Theta^b_j = (-1)^{\Omega_{ij}} \Theta^b_j \Theta^a_i \quad \text{for} \quad a \neq b,$$

$$(\Theta^a_i)^2 = 0.$$

(4.7)

Let us consider a few examples.

**Example 1.** Let us assume that $N = 2$ and $n = 1$. In this case the algebra $A$ is generating by two generators $\Theta_1$ and $\Theta_2$ satisfying the following relations

$$\Theta_1 \Theta_2 = \Theta_2 \Theta_1,$$

$$\Theta_1^2 = \Theta_2^2 = 0,$$

(4.8)

and describes the systems containing in average one electron per two magnetic fluxes, i.e. the filling factor is $v = \frac{1}{2}$. Observe that there are two partitions with one quasiparticle and one quasiholes, namely $\Theta_1$ and $\Theta_2$. It is interesting that there is one partition which does not contain quasiholes. This is represent by the symmetric monomial $\Theta_1 \Theta_2$.

**Example 2.** If $N = 3$ and $n = 1$, then the filling factor is $v = \frac{1}{3}$ and the algebra $A$ is generated by three generators $\Theta_1, \Theta_2, \Theta_3$ and relations

$$\Theta_1 \Theta_2 = -\Theta_2 \Theta_1,$$

$$\Theta_1^2 = \Theta_2^2 = \Theta_3^2 = 0.$$

(4.9)

We have here three partitions $\Theta_1, \Theta_2$, and $\Theta_3$ which contain one quasiparticle and two quasiholes. Observe that the following partitions $\Theta_1 \Theta_2, \Theta_1 \Theta_3, \Theta_2 \Theta_3$ which contain
two quasiparticles and one quasihole and the partition \( \Theta_1 \Theta_2 \Theta_3 \) which contains three quasiparticles are not physically admissible by the generalized Pauli exclusion principle. Hence in this case the single quasiparticle partitions with two quasiholes can be physically realized!

**Example 3.** For \( N = 3 \) and \( n = 2 \) we have the filling factor \( v = \frac{2}{3} \) and the algebra \( \mathcal{A} \) is generated by \( \Theta^a_i \), where \( a = 1, 2 \) and \( i = 1, 2, 3 \) such that

\[
\Theta^a_i \Theta^a_j = -\Theta^a_j \Theta^a_i, \quad \text{for} \quad a = b, \\
\Theta^a_i \Theta^b_j = \Theta^b_j \Theta^a_i, \quad \text{for} \quad a \neq b,
\]

for \( i \neq j \), \( (\Theta^a_i)^2 = 0 \) and

\[
\Theta^a_i \Theta^a_j = \Theta^a_j \Theta^a_i, \quad \text{for} \quad a = b, \\
\Theta^a_i \Theta^b_j = -\Theta^b_j \Theta^a_i, \quad \text{for} \quad a \neq b,
\]

for \( i = j \). Observe that quasiparticles \( \Theta^a_i \) and \( \Theta^b_j \) for \( i \neq j \) and \( a \neq b \) commutes and partitions with two quasiholes are admissible.

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