Turbo Coded Single User Massive MIMO

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Summary: This work deals with turbo coded single user massive multiple input multiple output (SU-MMIMO) systems, with and without precoding. SU-MMIMO has a much higher spectral efficiency compared to multi-user massive MIMO (MU-MMIMO) since independent signals are transmitted from each of the antenna elements (spatial multiplexing). MU-MMIMO that uses beamforming has a much lower spectral efficiency, since the same signal (with a delay) is transmitted from each of the antenna elements. In this work, expressions for the upper bound on the average signal-to-noise ratio (SNR) per bit and spectral efficiency are derived for SU-MMIMO with and without precoding. We propose a performance index \( f(N_t) \), which is a function of the number of transmit antennas \( N_t \). Here \( f(N_t) \) is the sum of the upper bound on the average SNR per bit and the spectral efficiency. We demonstrate that when the total number of antennas \( (N_{tot}) \) in the transmitter and receiver is fixed, there exists a minimum value of \( f(N_t) \), which has to be avoided. Computer simulations show that the bit-error-rate (BER) is nearly insensitive to a wide range of the number of transmit antennas and re-transmissions, when \( N_{tot} \) is large and kept constant. Thus, the spectral efficiency can be made as large as possible, for a given BER and \( N_{tot} \).

Keywords: Flat fading, precoding, re-transmissions, single user massive MIMO, spectral efficiency, turbo codes.

1. Introduction

Wireless telecommunication standards for 6G and beyond, aim for peak data rates per user of the order of 100 gigabits per second (Gbps). This can only be achieved using antenna arrays having a large number of antenna elements at both the transmitter and receiver (single user massive multiple input multiple output (SU-MMIMO)) and carrier frequencies of the order of terahertz (mmwave frequencies). Much of the existing literature on massive MIMO deals with multi user case (MU-MMIMO) [1–18], where the base station is equipped with a large number of antennas and each user has only one antenna. SU-MMIMO has not yet been studied, excepting for a few works with equal number of transmit and receive antennas and ideal receiver [19, 20], orthogonal frequency division multiplexing (OFDM)-based practical receiver which estimates timing, carrier frequency offset and channel impulse response [21, 22], analysis of probability of erasure (probability of not detecting an OFDM frame when it is present) [23, 24], ideal receiver with unequal number of transmit and receive antennas with precoding [25, 26]. SU-MMIMO is also presented in [27, 28]. SU- and MU-MMIMO for distributed antenna systems (antennas that are spatially far apart) is studied in [29], which is quite different from what is presented in this work (for example see Fig. 2 of [21]).

Let us look at the differences between SU- and MU-MMIMO [27, 28]:

1. In MU-MMIMO, beamforming is possible only in the downlink, whereas in SU-MMIMO, beamforming is possible in both uplink and downlink.

2. Spatial multiplexing is not possible in MU-MMIMO since the user (mobile handset) has only one antenna, whereas in SU-MMIMO spatial multiplexing is possible in both uplink and downlink.

The difference between beamforming and spatial multiplexing is enumerated below [27, 28]:

1. Beamforming has a lower spectral efficiency, since the same signal (with a delay) is transmitted from a large number of antenna elements. Spatial multiplexing has a higher spectral efficiency, since independent signals are transmitted from a large number of antenna elements.

2. Beamforming yields a highly directive pencil beam. Spatial multiplexing requires a rich scattering channel for effective operation and has no directivity.

This work describes two methods of implementing SU-MMIMO with unequal number of transmit and receive antennas, namely:

1. With precoding [25, 26].

2. Without precoding.

SU-MMIMO without precoding and equal number of transmit and receive antennas has been described earlier in [19, 20]. We now briefly discuss the topic of precoding. Precoding at the transmitter is a technique that dates back to the era of voiceband modems or wired communications [30–36]. The term “precoding” is quite generic and refers to one or more of the many different functionalities, as given below:

1. It compensates for the distortion introduced by the channel. Note that channel compensation at the receiver is referred to as equalization [37–43]. Here, channel compensation implies removal or minimization of intersymbol interference (ISI).
2. It performs error control coding, besides channel compensation.

3. It shapes the spectrum of the transmitted signal, and renders it suitable for propagation over the physical channel. Note that most channels do not propagate a dc signal and precoding is used to remove the dc component in the message signal. At this point, it is important to distinguish between a message signal and the transmitted signal.

In the context of wireless multiple input, multiple output (MIMO) systems, the main task of the precoder is to remove interchannel interference (ICI), either for single-user or multi-user case [44–50]. It should be observed that precoding requires knowledge of the channel state information (CSI) at the transmitter, which is usually fed back by the receiver to the transmitter. The receiver estimates CSI from a known training signal that is sent by the transmitter. CSI usually refers to the channel impulse response (CIR) or its statistics (mean and covariance), depending on the type of precoder used. Thus, precoding requires the channel to be time invariant or wide sense stationary (WSS) over at least one transmit and receive duration. Moreover, precoding can only be performed on systems employing time division duplex (TDD), which is a method of half duplex telecommunication. In other words, the channel needs to be reciprocal, that is, the CIR from the transmitter to receiver must be identical to that from receiver to transmitter.

In this work, we describe an elegant precoding method which reduces ICI in single user massive MIMO systems and compare it with the case without precoding [19,21]. Rayleigh flat fading channel is assumed. If the channel is frequency selective, orthogonal frequency division multiplexing (OFDM) can be used e.g. single input single output (SISO) OFDM [43,51,52], single input multiple output OFDM [53] or MIMO OFDM [21,22,54,55].

This work is organized as follows. Section 2 describes the signal model with and without precoding. In Section 3 precoding for SU-MMIMO is discussed. The case without precoding is presented in Section 4. Section 5 presents the simulation results and Section 6 gives the conclusions.

2. Signal Model

Consider a precoded MIMO system with $N_t$ transmit and $N_r$ receive antennas, as shown in Fig. 1 [25]. The precoded received signal in the $k^{th}$ ($0 \leq k \leq N_{rt} - 1, k$ is an integer), re-transmission is given by

$$\tilde{\mathbf{R}}_k = \mathbf{H}_k \mathbf{H}_k^H \mathbf{S}^p + \mathbf{W}_k$$

(1)

where $\tilde{\mathbf{R}}_k \in \mathbb{C}^{N_r \times 1}$ is the received vector, $\mathbf{H}_k \in \mathbb{C}^{N_r \times N_t}$ is the channel matrix and $\mathbf{W}_k \in \mathbb{C}^{N_r \times 1}$ is the additive white Gaussian noise (AWGN) vector. The transmitted symbol vector is $\mathbf{S}^p \in \mathbb{C}^{N_r \times 1}$, whose elements are drawn from an $M$-ary constellation. Boldface letters denote vectors or matrices. Complex quantities are denoted by a tilde. However tilde is not used for complex symbols $\mathbf{S}^p$. The elements of $\mathbf{H}_k$ are statistically independent, zero mean, circularly symmetric complex Gaussian with variance per dimension equal to $\sigma_{H_k}^2$, as given by (2) of [19]. Similarly, the elements of $\mathbf{W}_k$ are statistically independent, zero mean, circularly symmetric complex Gaussian with variance per dimension equal to $\sigma_{W_k}^2$, as given by (3) of [19].

The system model without precoding is shown in Figure 2 which is similar to Figure 1 of [19], excepting that here the number of transmit antennas is not equal to the number of receive antennas. The received signal without precoding, in the $k^{th}$ re-transmission is given by (see also (1) of [19])

$$\hat{\mathbf{R}}_k = \mathbf{H}_k \mathbf{S} + \mathbf{W}_k$$

(2)

where $\mathbf{S} \in \mathbb{C}^{N_r \times 1}$ whose elements are drawn from an $M$-ary constellation. In this work, the elements of $\mathbf{S}^p$ and $\mathbf{S}$ are turbo coded and mapped to a QPSK constellation with coordinates $\pm 1 \pm j$, as depicted in Fig. 1. Moreover, here $\mathbf{H}_k$ is an $N_r \times N_t$ matrix, whereas in [19] $\mathbf{H}_k$ is an $N \times N$ matrix. We assume that $\mathbf{H}_k$ and $\mathbf{W}_k$ are independent across re-transmissions, hence (4) in [19] is valid with $N$ replaced by $N_r$. We now proceed to analyze the signal models in (1) and (2).

3. Precoding

The $i^{th}$ element of $\tilde{\mathbf{R}}_k$ in (1) is

$$\tilde{R}_{k,i} = \tilde{F}_{k,i,i} S_i + \tilde{I}_{k,i} + \tilde{W}_{k,i} \quad \text{for } 1 \leq i \leq N_r$$

(3)

where

$$\tilde{F}_{k,i,i} = \sum_{j=1}^{N_r} |\tilde{H}_{k,i,j}|^2$$

$$\tilde{I}_{k,i} = \sum_{j=1}^{N_r} \tilde{F}_{k,i,j} S_j$$

$$\tilde{F}_{k,i,j} = \sum_{l=1}^{N_r} \tilde{H}_{k,i,l} \tilde{H}_{k,j,l}^H \quad \text{for } i \neq j.$$ 

(4)

The desired signal in (3) is $F_{k,i,i} S_i$, the interference term is $\tilde{I}_{k,i}$ and the noise term is $\tilde{W}_{k,i}$. Now

$$E\left[\tilde{F}_{k,i,i}^2\right] = E \left[\sum_{j=1}^{N_r} |\tilde{H}_{k,i,j}|^2 \sum_{l=1}^{N_r} |\tilde{H}_{k,i,l}|^2 \right]$$

$$= E \left[\sum_{j=1}^{N_r} \tilde{F}_{k,i,j}^2 + \tilde{R}_{k,i,j}^2\right]$$

$$= E \left[\sum_{j=1}^{N_r} \tilde{R}_{k,i,j}^2\right] + \tilde{R}_{k,i,i}^2.$$

$$E\left[\tilde{I}_{k,i}^2\right] = E \left[\sum_{j=1}^{N_r} \tilde{I}_{k,i,j}^2 \right]$$

$$= E \left[\sum_{j=1}^{N_r} \tilde{I}_{k,i,j}^2\right].$$

$$E\left[\tilde{W}_{k,i}^2\right] = E \left[\tilde{W}_{k,i}^2\right].$$
Input bits length $L_{d1}$

Interleaver

Rate-$1/2$ RSC encoder 1

Rate-$1/2$ RSC encoder 2

Map to QPSK

MUX DEMUX and precode

Tx 1

Rx $N_t$

Encoder 1 $L_{d1}$ QPSK symbols

Encoder 2 $L_{d1}$ QPSK symbols

Total length $L_d$

Output bit stream

Turbo decoder

Average over $k$ and concatenate

Rx $N_r$

Fig. 1. System model with precoding.

Input bits length $L_{d1}$

Interleaver

Rate-$1/2$ RSC encoder 1

Rate-$1/2$ RSC encoder 2

Map to QPSK

MUX and DEMUX

Tx 1

Rx $N_t$

Encoder 1 $L_{d1}$ QPSK symbols

Encoder 2 $L_{d1}$ QPSK symbols

Total length $L_d$

Output bit stream

Turbo decoder

Average over $k$ and concatenate

Rx $N_r$

Fig. 2. System model without precoding.
\[
\sum_{i=1}^{N_t} \tilde{H}_{k,i,i,l}^2 + \tilde{H}_{k,i,i,Q}^2 = 4\sigma_H^4 N_t(N_t + 1)
\]

where the subscript “\( l \)” denotes the in-phase part and the subscript “\( Q \)” denotes the quadrature part of a complex quantity and the following relation has been used [56, 57]

\[
E[X^4] = 3\sigma_X^4
\]

where \( X \) is a zero-mean, real-valued Gaussian random variable with variance \( \sigma_X^2 \). Moreover from (4) and (2) in [19]

\[
E[F_{k,i,i}] = 2\sigma_H^2 N_t.
\]

We also have

\[
E\left[\tilde{F}_{k,i,i}^2\right] = E\left[\sum_{j \neq i}^{N_r} \tilde{F}_{k,i,j} S_j \right]
\]

\[
\times \sum_{j=1}^{N_r} P_{av} E\left[\tilde{F}_{k,i,j}^2 \tilde{F}_{k,i,l}^2\right] \delta_K(j - l)
\]

\[
= \sum_{j=1}^{N_r} P_{av} E\left[\tilde{F}_{k,i,j}^2\right]\left[\tilde{F}_{k,i,l}^2\right] \delta_K(j - l)
\]

(8)

where \( \delta_K(\cdot) \) is the Kronecker delta function [19, 43], we have assumed independence between \( \tilde{F}_{k,i,j} \) and \( S_j \) and [19]

\[
E[S_j S_l^*] = P_{av} \delta_K(j - l) = 2\delta_K(j - l).
\]

Now

\[
E\left[\tilde{F}_{k,i,j}^2\right] = E\left[\sum_{i=1}^{N_t} \tilde{H}_{k,i,i,l} \tilde{H}_{k,i,i,l}^* \right]
\]

\[
\times \sum_{m=1}^{N_t} E\left[\tilde{H}_{k,i,m} \tilde{H}_{k,i,m}^*\right] \delta_K(l - m)
\]

\[
= 4\sigma_H^4 \delta(l - m)
\]

(10)

Substituting (10) in (8) and using (9) we get

\[
E\left[|\tilde{I}_{k,i}|^2\right] = 8\sigma_H^4 N_t(N_t - 1).
\]

(11)

Due to independence between \( \tilde{I}_{k,i} \) and \( \tilde{W}_{k,i} \) in (3), we have from (11) and (3) of [19]

\[
8\sigma_H^4 N_t(N_t - 1) + 2\sigma_W^2 = \sigma_{\text{SNR}}^2.
\]

(12)

Now, each element of \( \mathbf{S}_p \) in (1) carries \( 1/(2N_{rt}) \) bits of information [19]. Therefore, each element of \( \tilde{\mathbf{R}}_k \) also carries \( 1/(2N_{rt}) \) bits of information. Hence, the average signal to interference plus noise ratio per bit of \( \tilde{R}_{k,i} \) in (3) is defined as, using (5), (9) and (12)

\[
\text{SINR}_{av,b} = \frac{E\left[|\tilde{I}_{k,i} + \tilde{W}_{k,i}|^2\right] \times 2N_{rt}}{E\left[|\tilde{I}_{k,i}|^2\right]}
\]

\[
= \frac{8\sigma_H^4 N_t(N_t + 1) \times 2N_{rt}}{8\sigma_H^4 N_t(N_t - 1) + 2\sigma_W^2}.
\]

(13)

When \( \sigma_W^2 = 0 \) in (13), we get the upper bound on \( \text{SINR}_{av,b} \) as given below

\[
\text{SINR}_{av,b,UB} = \frac{8\sigma_H^4 N_t(N_t + 1) \times 2N_{rt}}{2N_{rt}(N_t + 1)}
\]

\[
= \frac{2N_{rt}(N_t + 1)}{N_t - 1}.
\]

(14)

The information contained in \( \mathbf{S}_p \) in (1) is \( N_t/(2N_{rt}) \) bits. Hence the spectral efficiency of the precoded system is

\[
\eta^p = \frac{N_t}{2N_{rt}} \text{ bits per transmission.}
\]

(15)

Note that both (14) and (15) need to be as large as possible to minimize the BER and maximize the spectral efficiency. Let

\[
N_{tot} = N_t + N_r.
\]

(16)

Define

\[
f(N_t) = \text{SINR}_{av,b,UB} + \eta^p
\]

\[
= \frac{2N_{rt}(N_t + 1)}{N_t - 1} + \frac{N_t}{2N_{rt}}
\]

\[
= \frac{2N_{rt}(N_t + 1)}{N_{tot} - N_t - 1} + \frac{N_{tot} - N_t}{2N_{rt}}
\]

(17)

where we have used (16). We need to find \( N_t \) such that \( f(N_t) \) is maximized. The plot of \( \text{SINR}_{av,b,UB} \) (red curve), \( \eta^p \) (blue curve) and \( f(N_t) \) (green curve), as a function of \( N_t \), keeping \( N_{tot} \) fixed, is depicted in Fig. 3 and 4. Note that \( \text{SINR}_{av,b,UB} \) increases monotonically and \( \eta^p \) decreases monotonically, with increasing \( N_t \). We also find that \( f(N_t) \) has a minimum (not maximum) at

\[
N_t = N_{tot} - 2N_{rt}\sqrt{N_{tot} - 1}
\]

(18)
Fig. 3. $\text{SINR}_{av,b,UB}$ and $\eta^p$ as a function of $N_t$ for $N_{tot} = 1024$, with precoding.

Fig. 4. $\text{SINR}_{av,b,UB}$ and $\eta^p$ as a function of $N_t$ for $N_{tot} = 32$, with precoding.
which is obtained by differentiating $f(N_t)$ in (17) with respect to $N_t$ and setting the result to zero. Therefore, the only possible solution is to avoid the minimum. Clearly we require $\text{SINR}_{av,b,UB} > \ln(2)$, since it is the minimum average SNR per bit required for error-free transmission over any type of channel [19]. We also require $\eta_p^b > r_p^b$, where $\eta_p^\text{min}$ is chosen by the system designer. Thus, we arrive at a range of the number of transmit antennas ($N_t, \text{min} \leq N_t \leq N_t, \text{max}$) that can be used, as shown in Fig. 3 and 4. Note that in Fig. 4(b) the minimum of $f(N_t)$ cannot be avoided, since $\eta_p^\text{min}$ would be too small.

Next, similar to (20) in [19], consider

$$
\bar{Y}_i = \frac{1}{N_{rt}} \sum_{k=0}^{N_{rt}-1} \tilde{R}_{k,i} \\
= \frac{1}{N_{rt}} \sum_{k=0}^{N_{rt}-1} \left( \hat{F}_{k,i} + \hat{I}_{k,i} + \tilde{W}_{k,i} \right) \\
= F_i S_i + \hat{U}_i \quad \text{for } 1 \leq i \leq N_r
$$

where $\bar{Y}_i$ is given by (3). $F_i$ is real-valued and

$$
F_i = \frac{1}{N_{rt}} \sum_{k=0}^{N_{rt}-1} \tilde{F}_{k,i} \\
\hat{U}_i = \frac{1}{N_{rt}} \sum_{k=0}^{N_{rt}-1} \tilde{W}_{k,i}
$$

are statistically independent over re-transmissions $(k)$, we have

$$
E \left[ |F_i|^2 \right] = \frac{1}{N_{rt}} E \left[ \sum_{k=0}^{N_{rt}-1} \tilde{F}_{k,i} \sum_{n=0}^{N_{rt}-1} \tilde{F}_{n,i} \right] \\
= \frac{4\sigma_H^2 N_t [N_t + 1 + N_r(N_r - 1)]}{N_{rt}} \\
= \frac{4\sigma_H^2 N_t [N_r(N_r + 1)]}{N_{rt}}
$$

$$
E \left[ |\hat{U}_i|^2 \right] = \frac{\sigma^2_H}{N_{rt}} \\
= \frac{8\sigma_H^2 N_t(N_r - 1) + 2\sigma_W^2}{N_{rt}}
$$

where we have used (5), (7), (12) and the fact that

$$
E \left[ \tilde{U}_{k,i} \right] = 0
$$

where $\tilde{U}_{k,i}$ is defined in (20). Next, we compute the average SINR per bit for $\bar{Y}_i$ in (19). Note that since $\bar{Y}_i$ is a “combination” of $N_{rt}$ re-transmissions, its information content is $N_{rt}/(2N_{rt}) = 1/2$ bit (recall that the information content of $\bar{R}_{k,i}$ in (19) is 1/(2$N_{rt}$) bits). Therefore

$$
\text{SINR}_{av,b,C} = \frac{E \left[ |F_i S_i|^2 \right]}{E \left[ |\hat{U}_i|^2 \right]} \times \frac{2}{8\sigma_H^2 N_t(N_r - 1) + 2\sigma_W^2}
$$

where the subscript “C” denotes “after combining” and we have used (9) and (21). Note that we prefer to use the word “combining” rather than averaging, since it is more appropriate in terms of the “information content” in $\bar{Y}_i$. Once again with $\sigma_W^2 = 0$ and $N_r N_r \geq 1$ we get the approximate upper bound on $\text{SINR}_{av,b,C}$ as

$$
\text{SINR}_{av,b,C, UB} = \frac{8\sigma_H^2 N_t(N_r(N_r + 1) + 1)}{8\sigma_H^2 N_t(N_r - 1)} \approx \frac{2N_r N_t}{N_r - 1} \approx \text{SINR}_{av,b,UB}
$$

when $N_r \gg 1$. Thus, the upper bound on the average SINR per bit before and after combining are nearly identical. Observe that re-transmitting the data increases the upper bound on the average SINR per bit, it does not improve the BER performance, which is seen in Section 5. After concatenation, the signal $\bar{Y}_i$ in (19) for $0 \leq i \leq L_d - 1$ is sent to the turbo decoder. The details of turbo decoding will not be discussed here.

4. No Precoding

The block diagram of the system without precoding in similar to Fig. 1 in [19], excepting that now there are $N_t$ transmit and $N_r$ receive antennas. The $i$th element of $H_{t}^r R_k$, where $R_k$ is given by (2), is (similar to (10) of [19])

$$
\bar{Y}_{k,i} = \tilde{F}_{k,i} s_i + \tilde{I}_{k,i} + \tilde{V}_{k,i} \quad \text{for } 1 \leq i \leq N_r
$$

where

$$
\tilde{V}_{k,i} = \sum_{j=1}^{N_r} \tilde{H}_{k,j,i} \tilde{W}_{k,j} \\
\tilde{I}_{k,i} = \sum_{j=1}^{N_r} \tilde{F}_{k,j,i} s_j \\
\tilde{F}_{k,i,j} = \sum_{l=1}^{N_r} \tilde{H}_{k,l,i} \tilde{H}_{k,l,j}
$$

It can be shown that

$$
E \left[ |\tilde{F}_{k,i} |^2 \right] = 4\sigma_H^2 N_r(N_r + 1). \quad \text{(27)}
$$

We also have

$$
E \left[ |\tilde{I}_{k,i} |^2 \right] = 2\sigma_H^2 N_r
$$

and

$$
E \left[ |\tilde{V}_{k,i} |^2 \right] = 8\sigma_H^2 N_r(N_r - 1)
$$
Fig. 5. SINR\textsubscript{av, b, UB} and $\eta$ as a function of $N_t$ for $N_{\text{tot}} = 32$, without precoding.

Fig. 6. SINR\textsubscript{av, b, UB} and $\eta$ as a function of $N_t$ for $N_{\text{tot}} = 1024$, without precoding.
The total power of interference plus noise is

\[
E \left[ \tilde{I}_{k,i} + \tilde{V}_{k,i} \right]^2 = E \left[ \tilde{I}_{k,i} \right]^2 + E \left[ \tilde{V}_{k,i} \right]^2
\]

\[
= 8 \sigma^2_H N_r (N_t - 1) + 4 \sigma^2_W \sigma^2_H N_r
\]

\[
= \sigma^2_{\text{IR}} \text{ (say).} \quad (30)
\]

Observe that the total information emitted by \( N_t \) antennas per transmission is \( N_t/(2N_r) \) bits. Therefore, the information contained in \( \bar{Y}_{k_i} \) in (25) is \( N_t/(2N_rN_t) \) bits. Hence, the average SINR per bit is

\[
\text{SINR}_{\text{av}, b} = \frac{E \left[ \tilde{F}_{k,i,i} S_i \right]^2 \times 2N_r}{E \left[ \tilde{I}_{k,i} + \tilde{W}_{k,i} \right]^2}
\]

\[
= \frac{8 \sigma^2_H (N_r + 1) \times 2N_r}{8 \sigma^2_H (N_r - 1) + 4 \sigma^2_W \sigma^2_H} \quad (31)
\]

and for \( \sigma^2_W = 0 \) we get the upper bound on SINR per bit as

\[
\text{SINR}_{\text{av}, b, UB} = \frac{(N_r + 1) \times 2N_r}{N_t - 1}. \quad (32)
\]

The spectral efficiency without precoding is

\[
\eta = \frac{N_t}{2N_r} \text{ bits per transmission.} \quad (33)
\]

Define

\[
f(N_t) = \text{SINR}_{\text{av}, b, UB} + \eta
\]

\[
= \frac{2N_r(N_r + 1) + N_t}{N_t - 1} \times \frac{2N_r}{N_t - 1} + \frac{N_t}{2N_r} \quad (34)
\]

where \( N_t \) is given by (16). The value of \( N_t \) that minimizes \( f(N_t) \) in (34) is given by

\[
N_t = 2N_r \sqrt{N_t} + 1. \quad (35)
\]

Thus, we can arrive at a range of transmit antennas that can be used, avoiding the minimum of \( f(N_t) \). This is illustrated in Figs. 5 and 6.

Again similar to (20) in [19], we compute the average of \( \bar{Y}_{k,i} \) over all re-transmissions, as given by

\[
\bar{Y}_i = \frac{1}{N_r} \sum_{k=0}^{N_r-1} \tilde{Y}_{k,i}
\]

\[
= \frac{1}{N_r} \sum_{k=0}^{N_r-1} \left( \tilde{F}_{k,i,i} S_i + \tilde{I}_{k,i} + \tilde{V}_{k,i} \right)
\]

\[
= F_i S_i + \bar{U}_i \quad \text{for } 1 \leq i \leq N_t \quad (36)
\]

where \( \bar{Y}_{k,i} \) is given by (25) and

\[
F_i = \frac{1}{N_r} \sum_{k=0}^{N_r-1} \tilde{F}_{k,i,i}
\]

\[
\bar{U}_i = \frac{1}{N_r} \sum_{k=0}^{N_r-1} \left( \tilde{I}_{k,i} + \tilde{V}_{k,i} \right)
\]

\[
= \frac{1}{N_r} \sum_{k=0}^{N_r-1} \tilde{U}_{k,i} \quad (say). \quad (37)
\]

Since \( \tilde{F}_{k,i,i} \) and \( \tilde{U}_{k,i} \) are statistically independent over re-transmissions \( (k) \), we have

\[
E \left[ F_i^2 \right] = \frac{1}{N_r} E \left[ \sum_{k=0}^{N_r-1} \tilde{F}_{k,i,i} \sum_{n=0}^{N_r-1} \tilde{F}_{n,i,i} \right]
\]

\[
= 4 \sigma^2_H N_r \left( N_r + 1 + N_r(N_r - 1) \right)
\]

\[
= \frac{4 \sigma^2_H N_r(N_r - 1) + 4 \sigma^2_W \sigma^2_H}{N_r} \quad (38)
\]

where we have used (27), (28) and (30) and the fact that

\[
E \left[ \tilde{U}_{k,i} \right] = 0 \quad (39)
\]

where \( \tilde{U}_{k,i} \) is defined in (37).

Noting that the average information content of \( \bar{Y}_i \) in (36) is 1/2 bit, the average SINR per bit of \( \bar{Y}_i \) is

\[
\text{SINR}_{\text{av}, b, C} = \frac{E \left[ F_i S_i \right] \times 2}{E \left[ \tilde{U}_i \right]}
\]

\[
= \frac{8 \sigma^2_H (N_r - 1) + 4 \sigma^2_W \sigma^2_H}{N_r} \quad (40)
\]

where the subscript “C” denotes “after combining” and we have used (9) and (38). When \( \sigma^2_W = 0 \) and \( N_r N_t \gg 1 \), we get the upper bound as

\[
\text{SINR}_{\text{av}, b, UB} = \frac{8 \sigma^2_H (N_r N_t + 1) \times 2}{8 \sigma^2_H (N_r - 1) + 4 \sigma^2_W}
\]

\[
\approx 2N_r N_t \frac{N_t}{N_t - 1}
\]

\[
\approx \text{SINR}_{\text{av}, b, UB} \quad (41)
\]

for \( N_r \gg 1 \). Note that re-transmissions increases the upper bound on the average SINR per bit, it does not improve the BER.

5. Simulation Results

In this section, we discuss the results from computer simulations. For the precoded case, the length
Fig. 7. Simulation results with precoding.

Fig. 8. Simulation results without precoding.
of the data bits per “frame” \( (L_d) \) is taken to be the smallest integer greater than 1000, which is an integer multiple of \( N_r \). For the case without precoding, the length of the data bits per “frame” \( (L_d) \) is taken to be the smallest integer greater than 1000, which is an integer multiple of \( N_t \). Note that (see Figs. 1 and 2)

\[
L_d = 2L_{d1}. \tag{42}
\]

The simulations were carried out over \( 10^4 \) frames. The turbo encoder is given by (38) of [19]. Figure 7 gives the bit-error-rate (BER) results with precoding, whereas Figure 8 gives the BER performance without precoding.

- **Fig. 7(a)** gives the bit-error-rate (BER) results for a \( 1 \times 1 \) single input single output (SISO) system \( (N_{tot} = 2) \) with precoding. We get a BER of \( 2 \times 10^{-2} \) at an average SNR per bit of 3.5 dB, with \( N_{rt} = 4 \). The corresponding spectral efficiency is \( \eta^p = 1/8 \) bits per transmission. The BER also does not vary significantly with the number of re-transmissions \( (N_{rt}) \).

- **Fig. 7(b)** gives the results for \( N_{tot} = 32 \) and different combinations of transmit \( (N_t) \) and receive \( (N_r) \) antennas. We find that the BER is quite insensitive to variations in \( N_t \), \( N_r \) and \( N_{rt} \). Moreover, the BER at an SNR per bit of 3.5 dB is about \( 2 \times 10^{-6} \), which is a significant improvement over the SISO system. Of all the curves, \( N_t = 25, N_r = 2 \) gives the lowest spectral efficiency of \( \eta^p = 1.75 \) bits/sec/Hz and highest SNR\(_{av,b,UB} = 12.39 \) dB. Of all the curves, \( N_t = 12, N_r = 1 \) gives the highest spectral efficiency \( \eta^p = 10 \) bits/sec/Hz and lowest SNR\(_{av,b,UB} = 1.36 \) dB.

- **Fig. 7(c)** gives the results for \( N_{tot} = 1024 \) for various combinations of \( N_t, N_r \) and \( N_{rt} \). The BER is similar to that of \( N_{tot} = 32 \). Of all the curves, \( N_t = 400, N_r = 1 \) gives the highest spectral efficiency of \( \eta^p = 312 \) bits/sec/Hz and lowest SNR\(_{av,b,UB} = 1.09 \) dB. Of all the curves, \( N_t = 1023, N_r = 2 \) gives the lowest spectral efficiency of \( \eta^p = 0.25 \) and highest SNR\(_{av,b,UB} \rightarrow \infty \).

- **Fig. 8(a)** gives the bit-error-rate (BER) results for a \( 1 \times 1 \) single input single output (SISO) system \( (N_{tot} = 2) \) without precoding. We get a BER of \( 10^{-1} \) at an average SNR per bit of 3.5 dB, with \( N_{rt} = 4 \). The corresponding spectral efficiency is \( \eta = 1/8 \) bits per transmission. Compared to Fig. 3(a) of [19], we find that the BER varies significantly with the number of re-transmissions \( (N_{rt}) \) in Fig. 8(a) of this work. However, it must be noted that the definition of SNR\(_{av,b} \) in (23) of [19] and SNR\(_{av,b} \) (converted to decibels) in (31) of this work, are different. Comparing (23) of [19] and (31) (converted to decibels) in this work for \( N_t = N_r = 1 \), we find that (31) is 3 dB higher. This explains the 3 dB difference between Fig. 3(a) of [19] and Fig. 8(a) of this work.

- **Fig. 8(b)** gives the results for \( N_{tot} = 32 \) and different combinations of transmit \( (N_t) \) and receive \( (N_r) \) antennas. We find that the BER is quite insensitive to variations in \( N_t \), \( N_r \) and \( N_{rt} \). Moreover, the BER at an SNR per bit of 3.5 dB is about \( 10^{-6} \), which is a significant improvement over the SISO system and similar to the precoded system of Fig. 7(b). Of all the curves, \( N_t = 7, N_r = 2 \) gives the lowest spectral efficiency of \( \eta = 1.75 \) bits/sec/Hz and highest SNR\(_{av,b,UB} = 12.39 \) dB. Of all the curves, \( N_t = 16, N_r = 1 \) gives the highest spectral efficiency equal to \( \eta = 8 \) bits/sec/Hz and lowest SNR\(_{av,b,UB} = 5.4 \) dB.

- **Fig. 8(c)** gives the results for \( N_{tot} = 1024 \) for various combinations of \( N_t, N_r \) and \( N_{rt} \). The BER is similar to that of \( N_{tot} = 32 \). Of all the curves, \( N_t = 512, N_r = 1 \) gives the highest spectral efficiency of \( \eta = 256 \) bits/sec/Hz and lowest SNR\(_{av,b,UB} = 3.03 \) dB. Of all the curves, \( N_t = 1, N_r = 2 \) gives the lowest spectral efficiency of \( \eta = 0.25 \) bits/sec/Hz and highest SNR\(_{av,b,UB} \rightarrow \infty \).

### 6. Conclusions

This work presents an elegant method for data detection in turbo-coded massive MIMO with and without precoding. An ideal receiver is assumed. Simulation results show that the BER is quite insensitive to a wide range in the number of transmit antennas and re-transmissions, when the total number of antennas in the transmitter and receiver \( (N_{tot}) \) is large and kept constant. Thus, the spectral efficiency can be made as large as possible for a given BER and \( N_{tot} \). Future work could be to simulate a realistic massive MIMO system with carrier and timing synchronization and channel estimation.

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