Narrow resonances and short-range interactions

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Narrow resonances in systems with short-range interactions are discussed in an effective field theory (EFT) framework. An effective Lagrangian is formulated in the form of a combined expansion in powers of a momentum $Q \ll \Lambda$—a short-distance scale—and an energy difference $\delta \epsilon = |E - \epsilon_0| \ll \epsilon_0$—a resonance peak energy. At leading order in the combined expansion, a two-body scattering amplitude is the sum of a smooth background term of order $Q^0$ and a Breit-Wigner term of order $Q^2(\delta \epsilon)^{-1}$ which becomes dominant for $\delta \epsilon \lesssim Q^3$. Such an EFT is applicable to systems in which short-distance dynamics generates a low-lying quasistationary state. The EFT is generalized to describe a narrow low-lying resonance in a system of charged particles. It is shown that in the case of Coulomb repulsion, a two-body scattering amplitude at leading order in a combined expansion is the sum of a Coulomb-modified background term and a Breit-Wigner amplitude with parameters renormalized by Coulomb interactions.

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I. INTRODUCTION

Low-energy dynamics of few-body systems with short-range interactions is most conveniently described by an effective field theory (EFT) [1]. An EFT Lagrangian contains only long-distance degrees of freedom and includes an infinite number of local couplings satisfying symmetries of a given system. Physical observables such as scattering amplitudes can be systematically expanded in powers of a typical momentum $Q$ which is much smaller than a typical scale of the short-distance physics $\Lambda$ [2]. A key role is played by power counting rules allowing to a priori determine which finite set of operators contributes at a given order in a momentum expansion. Systems in which observables have natural values—i.e., set by constants of order unity times an appropriate power of $\Lambda$—can be described by a power counting based on a mass dimension of effective operators. Some systems, however, contain observables whose magnitudes are much different than those expected from a dimensional analysis. This is often the case when short-distance dynamics generates shallow bound or quasibound states. In nuclear physics, a prototypical example is low-energy nucleon-nucleon scattering which is dominated by a shallow bound deuteron [3]. EFT description of observables with unnatural values requires alternative power counting rules. Nucleon-nucleon interactions characterized by a large scattering length can be described by an EFT in which a leading contact four-nucleon coupling scales as $Q^{-1}$. As a result, a loop expansion of a scattering amplitude has to be summed to all orders reproducing a large cross section [4]. A large cross section at low energies can also result from short-range interactions which are not strong enough to bind but can cause a virtual state. Such is the nucleon-nucleon interaction in a singlet channel. A power counting applicable in the case of a shallow bound state also describes systems with a virtual bound state. Because of large cross sections, systems with shallow bound or virtual states are said to display broad two-body resonances.

The focus here is on systems with short-range interactions that display a narrow low-lying resonance. In the vicinity of such a resonance, a two-body scattering amplitude has a sharp peak. An EFT developed here describes a low-lying resonance at energy $\epsilon_0$ of order $Q^2$. Such a resonance is associated with a quasibound state with a lifetime given by the inverse of the resonance width $\Gamma \ll \epsilon_0$.

For a shallow bound and virtual state, a two-body scattering amplitude has a universal form expressed in terms of an effective range expansion (ERE) [5]. Similarly, a low-lying resonance in a system with short-range interactions can be described by an amplitude that has a universal form—a smooth, i.e., background part and a Breit-Wigner term. Both contributions are generated by the short-distance physics. The Breit-Wigner amplitude dominates within a narrow width $\Gamma$ around a peak energy $\epsilon_0$. An EFT framework is particularly useful in capturing the universal character of the short-distance physics. As shown in the following sections, a background and Breit-Wigner term appear as leading contributions in combined expansion in powers of a low-energy momentum $Q$ and energy difference $\delta \epsilon = |E - \epsilon_0| \sim \Gamma \sim Q^3$.

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Before developing such a combined expansion in the context of an EFT, it is useful to discuss a simple potential model that contains a narrow low-lying resonance. This is done in Sec. III followed by the development of an EFT in Sec. IV. In Sec. V the effective theory is generalized to include systems with repulsive Coulomb interactions.

II. A TOY MODEL

A simple model displaying narrow resonances consists of two spin-zero nonrelativistic particles each with mass $M$ with short-range interactions given by a potential containing an attractive square well with range $R$ and a repulsive $\delta$ shell at $r = R$:

$$V(r) = -V_0 \Theta(R - r) + W_0 \delta(r - R), \quad (1)$$

where $\Theta(R - r)$ and $\delta(r - R)$ are step and spherical $\delta$ functions, respectively; $V_0$ and $W_0$ are two positive constants. Narrow resonances in a potential containing only a repulsive $\delta$ function were discussed in Ref. [2].

Partial phase shifts can be found from positive energy solutions of the time-independent Schrödinger equation. In the center-of-mass frame of two particles, an $s$-wave radial wave function in the interior and exterior of the potential in Eq. (1) is given by

$$\chi_0(r) = \begin{cases} N \sin(Kr) & \text{for } r \leq R, \\ N' (\sin(kr)) \cos \delta_0 + \cos(kr) \sin \delta_0 & \text{for } r > R, \end{cases} \quad (2)$$

where $N$ and $N'$ are normalization constants, $\delta_0$ is the $s$-wave phase shift, and $k = \sqrt{ME}$ and $K = \sqrt{MV_0 + k^2}$ are exterior and interior wave numbers, respectively. Because of the radial $\delta$ function in Eq. (1), a logarithmic derivative of the wave function has a discontinuity at $r = R$, namely, $\Delta(\chi_0/\chi_0) = \alpha/R$ with dimensionless constant $\alpha$ defined as

$$\alpha = M R W_0. \quad (3)$$

This yields for the $s$-wave phase shift

$$\delta_0 = -kR + \arctan \left( \frac{kR}{\beta(KR)} \right), \quad (4)$$

with $\beta(x) = x \cot x + \alpha$.

For small values of $\alpha$, the dominant contribution to the phase shift is from an attractive square well with a $\delta$-shell barrier acting as a perturbation. In the limit of infinite $\alpha$, the barrier is impenetrable, and depending on initial conditions the model describes either scattering off a repulsive core at $r = R$ ($\delta_0 = -kR$) or a bound system with a discrete spectrum given by $K_n R = n \pi r$ with positive integers $n$.

Narrow resonances exist for large but finite $\alpha$ when a probability to penetrate the $\delta$-shell barrier virtually vanishes for all but narrow domains around energies $\epsilon_n$ given by zeros of $\beta$. They can be found using an approximate equality $n\pi(1 - n^2/\alpha) \cot[n\pi(1 - n^2/\alpha)] \approx -\alpha$ with corrections of order $\alpha^{-2}$ yielding $K_n R = n\pi[1 - n^2\alpha^{-1} + O(\alpha^{-2})]$. These smeared energy levels correspond to quasistationary states. The form of an $s$-wave phase shift in the vicinity $\epsilon_n$ can be obtained by expanding $\beta$ in powers of $(E - \epsilon_n)$. For the lowest lying resonance, the $s$-wave phase shift is

$$\delta_0 = -kR - \arctan \left( \frac{\gamma \sqrt{E}}{E - \epsilon_0} \right) + \frac{(2\pi - 1)MR^2}{4\pi^2} \frac{\gamma \sqrt{E}}{E - \epsilon_0} + O((E - \epsilon_0)), \quad (5)$$

with $\epsilon_0$ and $\gamma$ given by

$$\epsilon_0 = \frac{\pi^2}{MR^2} \left( 1 - \frac{1}{\alpha} \right) - V_0 \quad \text{and} \quad \gamma = \frac{2\pi^2}{\alpha^2 R \sqrt{M}}. \quad (6)$$

Thus, by changing $V_0$, the energy $\epsilon_0$ can be fine-tuned to have a value much smaller than $1/\alpha R^2$ which sets the high-energy scale.

Using Eq. (5) and exp$(2i \arctan \lambda) = (1 + i\lambda)/(1 - i\lambda)$, an $s$-wave scattering amplitude, $f_0 = (e^{2i\delta_0} - 1)/2ik$, can be written as

$$f_0 = f_0^{(b)} - \frac{1}{\sqrt{M E - \epsilon_0 + i\gamma \sqrt{E}}} e^{2i\delta_0^{(b)}}, \quad (7)$$
where \( f_0^{(b)} = \left(e^{2i\eta_0^{(b)}} - 1\right)/2ik = -R + \ldots \) is the background part of the amplitude corresponding to the first term in Eq. (5) as well as corrections given by the third and higher order terms in Eq. (5). The second term in Eq. (7) is a Breit-Wigner amplitude describing a low-lying resonance with a peak at \( \epsilon_0 \) and a width \( \Gamma = 2\gamma\sqrt{\epsilon_0} \). Note, that the Breit-Wigner amplitude in Eq. (7) saturates the unitarity limit.

To emphasize a scale separation in the model, it is useful to formulate power counting rules in terms of a small momentum \( Q \) and an energy difference \( \epsilon \equiv |E - \epsilon_0| \). The range \( R \) of the potential in Eq. (1) determines the high-energy scale \( \Lambda \sim 1/R \). If the following scaling is assumed

\[
\alpha \sim Q^{-1}, \quad \epsilon_0 \sim Q^2, \quad \delta \epsilon \lesssim Q^3,
\]

then Eq. (6) yields

\[
\gamma \sim Q^2, \quad \Gamma = 2\gamma\sqrt{\epsilon_0} \sim Q^3.
\]

For a generic momentum of order \( Q \), the second term in Eq. (4) scales as \( kR/\alpha \sim Q^2 \) and is suppressed relative to the first term which is of order unity. However, when energy is such that \( E - \epsilon_0 \sim \delta \epsilon \lesssim Q^3 \), the Breit-Wigner term scales as \( Q^{-1} \) and represents the dominant contribution. The third term in Eq. (5) is of order \( Q^3 \). Thus, an expansion in powers of \( (E - \epsilon_0) \) isolates a term that is subleading everywhere except in a narrow energy domain around \( \epsilon_0 \).

As the center-of-mass energy approaches \( \epsilon_0 \) from below, the phase shift sharply increases and passes \( \pi/2 \) at \( \epsilon_0 \) (modulo \( \pi \)) as can be seen from Eq. (5). As energy goes through a narrow resonance interval region, the phase shift changes by \( \pi \). This behavior of the phase shift should be contrasted with that of resonances due to shallow bound and virtual states. In the latter case, a phase shift increases over a relatively large energy region. Moreover, while for the shallow bound state the phase shift indeed passes \( \pi/2 \) and reaches \( \pi \) at zero energy, it does not necessarily happen for the virtual bound state. Resonances associated with shallow bound and virtual states distinguished mainly by anomalous cross sections are often referred to as broad resonances.

A sharp change in the phase shift leads to a large flux delay given by

\[
\frac{\partial \delta_0(\epsilon_0)}{\partial \epsilon} = -\frac{RM}{2k_0} + \frac{2}{\Gamma} \sim Q^{-3},
\]

where \( k_0 = \sqrt{M\epsilon_0} \). Broad resonances characterized by scattering lengths of order \( Q^{-1} \) cause flux delay of order of \( Q^{-2} \).

It can also be shown that wave functions of quasistationary states given in Eq. (2) which are initially confined to the exterior of the potential in Eq. (1) exponentially decay with a lifetime given by \( \tau = \Gamma^{-1} \).

It is also interesting to point out that once a position of the resonance \( \epsilon_0 \) is fixed, the resonance width \( \Gamma \) is very sensitive to the range of the potential \( R \) and only weakly depends on the variation in \( \alpha \) around \( \alpha \to \infty \) limit. Dependence on \( R \) is due to the great sensitivity of a quasistationary state wave function on boundary conditions at \( r = R \). This greatly contrasts with a situation in the case of a broad resonance associated with a wave function with a size much larger than the range of the potential.

### III. AN EFFECTIVE FIELD THEORY

A key insight from the toy model in the preceding section is that in the vicinity of a narrow low-lying resonance, the scattering amplitude at leading order is a sum of a background term of order \( Q^0 \) and a Breit-Wigner term which scales as \( Q^{-1} \) in a narrow domain of order \( \Gamma \sim Q^3 \) near \( \epsilon_0 \sim Q^2 \). To implement such a scaling in an EFT, two types of couplings at leading order will be used.

As in the preceding section, the focus here is on s-wave resonance. For a system of two spin-zero particles of mass \( M \), an effective field theory Lagrangian at leading order has the form

\[
\mathcal{L}_{\text{LO}} = \Psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M}\right)\Psi + \Phi^\dagger \left(i\partial_0 - \Delta + \frac{\nabla^2}{4M}\right)\Phi - C_0 (\Psi^\dagger \Psi)^2 + g (\Phi^\dagger \Psi \Psi + \Phi \Psi^\dagger \Phi^\dagger),
\]

where \( \Psi^\dagger (x,t) [\Phi(x,t)] \) creates (destroys) the scattering particles and \( \Phi^\dagger (x,t) [\Phi(x,t)] \) creates (destroys) a dimeron field with mass \( 4M + \Delta \) (\( \Delta > 0 \)). The four-point contact interaction with coupling constant \( C_0 \) is the leading term for the background contribution, and a role of the dimeron is to generate the narrow resonance in the vicinity of \( \Delta \).

An effective Lagrangian in Eq. (11) is a leading part in a combined expansion in powers of \( Q/\Lambda \) and \( \delta \epsilon/\epsilon_0 = |E - \epsilon_0|/\epsilon_0 \). In an EFT treatment of systems with shallow bound or virtual states a dimeron or dibaryon field is used to describe a leading and subleading effect in an effective range expansion [7]. In the combined expansion, both
FIG. 1: Leading-order contributions to an s-wave $T$ matrix. A square in (a) represents a $C_0$ coupling; (b)–(d) contain a dimeron propagator and $g$ coupling.

four-point and dimeron Yukawa-like couplings contribute at leading order to a two-body scattering. A power counting in the combined expansion is given by

$$C_0 \sim Q^0, \quad g \sim Q, \quad \Delta \sim Q^2, \quad \delta \epsilon \lesssim Q^3. \quad (12)$$

Note, that the dimeron is weakly coupled. In the toy model in Sec. II, the dimeron coupling is modeled by a $\delta$-shell barrier with penetration probability for a system in the lowest quasistationary state given by $\pi^2/\alpha^2 \sim Q^2$.

Higher order terms include relativistic corrections to the kinetic energy, background terms involving even-order derivative couplings and terms of the form $\Phi^\dagger (i\partial_0 - \Delta)^n \Phi$ ($n = 1, 2, \ldots$) describing corrections to a Breit-Wigner amplitude in the vicinity of the resonance.

A two-body $T$ matrix for each partial wave, $T_\ell = - (M/4\pi) f_\ell$, can be expressed as a loop expansion shown in Fig. 1, where each loop contributes a factor of

$$I_0 = \int \frac{d^3q}{(2\pi)^3} \frac{M}{k^2 - q^2 + i\epsilon} = - \frac{M}{4\pi} (\mu + ik) \sim Q, \quad (13)$$

where $k = \sqrt{ME}$ is the magnitude of a relative momentum in a center-of-mass frame and $M/2$ is the reduced mass of a two-body system. A second equality in Eq. (13) follows when a divergent integral is evaluated using dimensional regularization with power-divergence subtraction (PDS) introduced by Kaplan, Savage, and Wise in the context of an EFT for nucleon-nucleon interactions [8]. As can be seen in Eq. (13), both real and imaginary parts of the loop scale as $Q$ provided one chooses a renormalization $\mu$ of order $Q$. Such scaling also follows if one counts powers of the momentum $q$ in the integral in Eq. (13). The on-shell $T$ matrix does not depend on a regularization scheme, as will become explicit below.

Power counting rules given in Eq. (12) make it possible to separate the $T$ matrix into background and Breit-Wigner parts. The former receives contributions from four-point couplings which scale as $Q^0$. At leading order, this contribution is due to a single contact four-point vertex, shown on Fig. 1(a), yielding

$$T_0^{(b)} = \frac{4\pi}{M} C_0 + \ldots, \quad (14)$$

where an ellipsis denotes the higher order corrections coming from loops and four-point derivative couplings.

The Breit-Wigner amplitude is due to the dimeron coupling. Corresponding diagrams shown in Figs. 1(b)–1(d) yield

$$T_0^{(BW)} = \frac{g^2}{E - \Delta} \left(1 + \frac{g^2}{E - \Delta} I_0 + \ldots\right), \quad (15)$$

The first term in $T_0^{(BW)}$ from a tree-level diagram [Fig. 1(b)] is of order $Q^2(E - \Delta)^{-1}$ which is order $Q^0$ for a typical energy $E \sim Q^2$. Each subsequent term in a loop expansion in Eq. (15) is of relative order $Q^3(E - \Delta)^{-1}$ which is of higher order far from a resonance. However, when $(E - \Delta) \sim \delta \epsilon \lesssim Q^3$, each term is of the same relative order, namely, $Q^0$. As a result, the dimeron contribution is nonperturbative in the vicinity of the resonance. The expansion in Eq. (15) has the form of geometric series and can be summed to yield

$$T_0^{(BW)} = \frac{g^2}{E - \Delta - g^2 I_0} + \ldots, \quad (16)$$

where $I_0$ is given in Eq. (13) and an ellipsis stands for higher order terms in $\delta \epsilon$ expansion. Such kinematic enhancement was discussed by Pascalutsa and Phillips in the case of $\pi N$ scattering near the $\Delta(1232)$ resonance [9] and by Bedaque, Hammer, and van Kolck in Ref. [10].
An infinite loop expansion of a dimeron propagator represents a nonperturbative renormalization of $\Delta$ at leading order in $\delta \epsilon$ expansion. Indeed, the Breit-Wigner amplitude in Eq. (16) is independent of the regularization scale $\mu$ provided the following renormalization conditions are satisfied:

$$g = \hat{g}, \quad \Delta = \Delta - \frac{M}{4\pi} \mu g^2 = \Delta - \mu \hat{g},$$

where in the last equality a dimensionless coupling $g_0 = g\sqrt{M/4\pi}$ is introduced. Note, at leading order in the combined expansion, the dimeron coupling constant $g$ is not renormalized, while the “residual mass” $\Delta$ receives additive renormalization of order $Q^3$ consistent with the power counting rules in Eq. (12). Since the renormalization of $\Delta$ is of subleading order, other regularization schemes such as dimensional regularization with minimal subtraction or cutoff regularization can be used [11].

With resonance parameters defined in terms of renormalized dimeron parameters, $\hat{g}$ and $\hat{\Delta}$ as

$$\gamma = g_0^2 \sqrt{M} \sim Q^2, \quad \epsilon_0 = \hat{\Delta} \sim Q^2, \quad \Gamma = 2\gamma \sqrt{\epsilon_0} \sim Q^4$$

the s-wave $T$ matrix at leading order in the combined expansion has the form

$$T_0^{(LO)} = \frac{4\pi C_0}{Q^0} + \frac{4\pi}{M \sqrt{M} E - \epsilon_0 + i\gamma \sqrt{E}} \frac{\gamma}{Q^2 (\delta \epsilon)^{-1}}.$$

The $T$ matrix in the above equation has a universal form describing a narrow low-energy resonance with a peak at $\epsilon_0$ and width $\Gamma = 2\gamma \sqrt{\epsilon_0} \ll \epsilon_0$. It has the same form as the $T$ matrix corresponding to Eq. (17), since the phase shift due to the background scattering is small and the exponential factor is close to unity. Note, that in the potential model in Sec. II $C_0 = -R \sim Q^0$.

According to power counting rules in Eq. (12), the background term is of order unity. Formally one can consider the case in which $C_0$ scales as $Q^{-1}$. This would require nonperturbative treatment of both the dimeron coupling and the four-point coupling in Eq. (11). A Lagrangian similar to the one in Eq. (11) with nonperturbative four-point and dimeron couplings was discussed in Refs. [12, 13, 14] in the context of scattering of ultracold alkali atoms where an effective two-body interaction has a short range.

### IV. CHARGED PARTICLES

Coulomb interactions become nonperturbative at low energies. As a result, EFT treatment for systems of charged particles with short-range interactions has to be modified. In the context of low-energy nucleon-nucleon interactions, Kong and Ravndal developed an EFT applicable to systems of charged particles with shallow bound or virtual states [15]. In a two-body sector, an effective field theory leads to a Coulomb-modified effective range expansion [15, 16, 17].

It is interesting to consider to what extent the effective field theory developed in Sec. III needs to be modified to describe a system of charged particles. In other words, is it possible to construct a consistent power counting in which a two-body $T$ matrix at leading order can be separated into a background term and a Breit-Wigner amplitude?

In the toy model in Sec. III a narrow resonance is due to a quasistationary state “trapped” by a $\delta$-shell barrier inside the short-range potential. Parameters of the resonance—peak energy and the width—as well as the background scattering are generated by the short-range potential in Eq. (11). An alternative picture can be considered in which a short-range attraction is combined with a long-range repulsion such as Coulomb repulsion which provides a potential barrier. Note, the penetration probability for a $\delta$-shell barrier is $\pi^2/\alpha^2 \sim \gamma$, and consequently the resonance width scales linearly with resonance momentum $k_0 = \sqrt{M \epsilon_0}$ [Eq. (9)]. For a Coulomb barrier, the penetration probability is suppressed by the Gamow factor, which at low energies can be written in terms of the Sommerfeld factor to be used below as $C_\eta^2/2\pi \eta$. The Sommerfeld factor is defined as

$$C_\eta^2 = 2\pi \eta \frac{1}{\exp(2\pi \eta) - 1} \approx 2\pi \eta \exp(-2\pi \eta) \quad \text{with} \quad \eta = \frac{1}{k \alpha_B} = \frac{\alpha_{em} Z^2 M}{2k},$$

where $\alpha_B$ is Bohr radius, $\alpha_{em} = e^2/4\pi$ is the fine-structure constant, and $Z$ is an electric charge. The approximation in Eq. (20) is valid for low energies where a Sommerfeld parameter $\eta > 1$. To reproduce a narrow resonance peak, a careful fine-tuning of the parameters of the short-range attraction and long-range repulsion is required.

Such an approach was developed by Higa, Hammer, and van Kolck in Ref. [18] within a framework of a “halo” EFT [19] applicable to halo nuclei [20]. Higa et al. constructed an EFT for low-energy $\alpha$-$\alpha$ scattering which displays
an s-wave resonance at $\epsilon_0 \approx 92$ keV and width $\Gamma \approx 5.6$ eV in the center-of-mass frame. According to a power counting in Ref. [18], coefficients of a Coulomb-modified effective range expansion are such that in the vicinity of $\epsilon_0$ the scattering amplitude has a Breit-Wigner–like shape. Resonance parameters at leading order are given in terms of a Coulomb-modified s-wave scattering length $a_C^0$ and effective range $r_0$ by

$$\epsilon_0 = \frac{2}{a_C^0 \tilde{r}_0 M} \quad \Gamma = \frac{4C_0^2}{\tilde{r}_0 M} \sqrt{\frac{2}{a_C^0 \tilde{r}_0}} \quad \text{with} \quad \tilde{r}_0 = \frac{1}{3a_B} - r_0,$$

where the Sommerfeld factor is evaluated at $k_0 = \sqrt{M \epsilon_0}$.

Here an alternative possibility is discussed based on the EFT developed in Sec. III. In essence, it is assumed that a low-energy resonance is generated by short-range dynamics. Thus, it can be described in the combined expansion used in the case of purely short-range interactions.

In this approach, an s-wave phase shift at leading order in the combined expansion can be written as a sum of a Coulomb-modified background contribution $\delta_C^b$, Breit-Wigner term $\delta_{BW}^C$, and the pure Coulomb phase shift $\sigma_0 = \arg(1 + i\eta)$ always present in the case of charged particles, that is,

$$\delta_0 = \sigma_0 + \delta_C^b + \delta_{BW}^C + \ldots,$$

where an ellipsis represents higher order corrections in a combined expansion. The Breit-Wigner phase shift $\delta_{BW}^C$ has the same form as in the case of a purely short-range interaction [Eq. (4)] with parameters $\epsilon_0^C$ and $\gamma_C^r$ renormalized by Coulomb interactions at short distances, as will be shown below. Accordingly, an s-wave $T$ matrix at leading order can be written as

$$T_0^{(LO)} = T_0^C - \frac{4\pi}{M k \cot \delta_C^b - i\kappa} + \frac{4\pi}{M \sqrt{M}} \frac{1 + \tan \delta_C^b}{1 - \tan \delta_C^b} \frac{\gamma_C^r e^{2i\sigma_0}}{E - \epsilon_0^C + i\gamma_C^r \sqrt{E}},$$

where $T_0^C$ is the pure Coulomb $T$ matrix given by an infinite sum of ladder diagrams with static photons, shown in Fig. 2. At very low energies, a background phase shift $\delta_C^b \ll 1$ is small. Consequently, $\tan \delta_C^b \ll 1$ and can be neglected in the third term with corrections of higher order in the combined expansion.

Electromagnetic interactions are included in an effective Lagrangian in Eq. (11) by replacing ordinary derivatives with covariant derivatives and adding a kinetic term for the electromagnetic field. Feynman diagrams are evaluated in the Coulomb gauge in which leading electromagnetic effects are due to the exchange of static longitudinal photons, while the exchange of transverse photons is suppressed by additional powers of momentum. This results in two types of Coulomb modifications—one from photon exchanges on external particle lines and the other due to the photon-exchange contributions inside the loop (Figs. 3 and 4). As shown in Ref. [15], both of these contributions are nonperturbative at low energies and have to be summed to all orders in the fine structure constant $\alpha_{em}$. Finding
this infinite sum of ladder diagrams (Fig. 2) is equivalent to evaluating the Feynman diagrams on a basis of Coulomb functions instead of plane waves. As a result, external lines develop a factor $C_0^2 e^{2i\sigma_0}$, while a Coulomb-dressed loop evaluated using PDS regularization is given by

$$I_0 \equiv I_0^C = -\frac{M}{4\pi} \left( \mu - \frac{2}{a_B} \ln \frac{\mu a_B \sqrt{\pi}}{2} + \frac{3C_E - 2}{a_B} + \frac{2}{a_B} h(\eta) + ikC_n^2 \right),$$

(24)

where $C_E = 0.5772\ldots$ is the Euler constant, and function $h(\eta) = \eta^2 \sum_{n=1}^{\infty} [n(n^2 + \eta^2)]^{-1} - C_E - \ln \eta$ can be expanded at low energies as $h(\eta) = 1/12\eta^2 + 1/120\eta^4 + \cdots$.

These modifications occur both for background contributions due to the couplings $C_{2n}$ (Fig. 3) and for dimeron contributions (Fig. 1). In the combined expansion with power counting given in Eq. (12), the leading-order $T$ matrix can be separated into a background term due to four-point particle-particle couplings and a resonance term due to the dimeron coupling. Interference terms involving both interactions are suppressed by additional powers of $Q$ and $\delta\epsilon$. As a result, at leading order in the combined expansion, the background and resonance terms can be evaluated separately. Both of these terms receive Coulomb modifications as discussed above.

In the case of the background scattering, a leading contribution is from $C_0$ vertex which is of order unity according to the power counting in Eq. (12). Nevertheless loop contributions shown in Fig. 3 can have a large magnitude because of the logarithm in Eq. (24). As a a result, a Coulomb-dressed loop expansion should be summed to all orders. Including factors due to the Coulomb interactions on the external lines discussed above and using Eq. (24), a Coulomb-modified background term is

$$T_b^C = \frac{C_0 C_n^2 e^{2i\sigma_0}}{1 - C_0 I_0^C} = -\frac{4\pi}{M} C_n^2 e^{2i\sigma_0} \left( \frac{1}{a_0^C} + \frac{2}{a_B} h(\eta) - iC_n^2 k \right)^{-1},$$

(25)

where a Coulomb-distorted s-wave scattering length $a_0^C$ is defined as

$$\frac{1}{a_0^C} = \frac{4\pi}{MC_0} + \mu - \frac{2}{a_B} \ln \frac{\mu a_B \sqrt{\pi}}{2} + \frac{3C_E - 2}{a_B}.$$

(26)

Equation (25) represents a leading term in a Coulomb-modified effective range expansion. Note, at small energies, $\tan \delta_n^C$ is small due to a Sommerfeld factor.

As in the case of purely short-range interactions, the dimeron coupling contributes at leading order only in a narrow domain around $\epsilon_0$ where it dominates over the background scattering. Summing a Coulomb-dressed loop expansion shown in Fig. 4 to all orders using Eq. (24) and including factors due to Coulomb-dressed external lines, one obtains the following form of the resonance term at leading order:

$$T_{BW}^C = \frac{4\pi}{M} \bar{g}_0^2 C_n^2 e^{2i\sigma_0} \left( E - \Delta - \frac{4\pi}{M} \bar{g}_0^2 I_0^C \right)^{-1} + \cdots,$$

(27)

where corrections include higher order terms in the combined expansion, and $\bar{g}_0$ is defined in Eq. (17).

Since $T_{BW}^C$ is dominant only in a narrow energy domain $(E - \Delta) \sim Q^3$, the Coulomb induced factors in Eq. (27) can be absorbed into regularized constants $\Delta$ and $\bar{g}_0$. Indeed one can define the renormalized coupling $\bar{g}_0$ using a Sommerfeld factor [Eq. (20)] evaluated at $\eta_0 = (k_0 a_B)^{-1} = (a_B \sqrt{MC_0})^{-1}$ via

$$\bar{g}_0^2 = \bar{g}_0^2 C_{\eta_0}^2,$$

(28)

and the renormalized “residual mass” $\tilde{\Delta}$ by

$$\tilde{\Delta} = \Delta + \frac{4\pi}{M} \bar{g}_0^2 C_{\eta_0}^2 \Re \left( I_0^C \right)$$

$$= \Delta - \frac{g_0^2}{C_{\eta_0}^2} \left( \mu - \frac{2}{a_B} \ln \frac{\mu a_B \sqrt{\pi}}{2} + \frac{3C_E - 2}{a_B} \right),$$

(29)

where the real part of the Coulomb-modified loop given in Eq. (13) includes the value of the function $h(\eta_0)$; in the second equality, only the first term in the expansion of $h(\eta)$ is kept. Renormalization conditions in Eqs. (28) and (29) correspond to the those given in Eq. (17) in the case of purely short-range interactions. Note that a bare “residual mass” $\Delta(\mu)$ is very sensitive to the value of the regularization scale $\mu$. Such sensitivity signifies a strong effect of long-range interactions at short distances and is common in effective field theories for systems in which both short- and long-range interactions are present [21, 22].
Expanding the Sommerfeld factor and function \( h(\eta) \) in Eq. (27) around \( k_0 = \sqrt{M_0} \) and using the renormalized constants \( g_0 \) and \( \Delta \) defined in Eqs. (28) and (29), the resonance term \( T_{BW}^C \) can be written as

\[
T_{BW}^C = \frac{4\pi}{M} g_0^2 e^{2i\sigma_0} \left( E - \Delta + ig_0^2 k \right)^{-1} + \cdots ,
\]

where only leading terms in the expansion in powers of \( \delta \epsilon \) are kept.

Resonance parameters \( \epsilon^C_0 \) and \( \gamma^C \) for charged particles can now be defined in the same way as in Eq. (18) in terms of the renormalized constants \( \Delta \) and \( g_0 \)

\[
\gamma^C = g_0^2 \sqrt{M} , \quad \epsilon^C_0 = \Delta .
\]

Finally collecting a leading Coulomb-modified background term [Eq. (25)] and the Breit-Wigner term [Eq. (30)] expressed in terms of \( \epsilon^C_0 \) and \( \gamma^C \) defined in Eq. (31), one obtains an s-wave \( T \) matrix at leading order in the combined expansion

\[
T^{(LO)}_0 = T^C_0 - \frac{4\pi}{M} C^2_{\eta} e^{2i\sigma_0} \left( -\frac{1}{a_0^2} - \frac{2}{a_B} \frac{h(\eta)}{1} - i k C^2_{\eta} \right)^{-1} + \frac{4\pi}{M \sqrt{M} E - \epsilon_0 + i \gamma^C \sqrt{E}} .
\]

The above expression has precisely the form shown in Eq. (23). Similarly, a total s-wave phase shift at leading order in the combined expansion is given by

\[
\delta_0^{(LO)} = \sigma_0 + \arctan \left( k a_0^C C^2_{\eta} \right) - \arctan \left( \frac{\gamma^C \sqrt{E}}{E - \epsilon_0} \right) ,
\]

which is of the form shown in Eq. (22).

V. CONCLUSION

In this paper, effective field theory methods are used to describe a narrow low-lying s-wave resonance in two-body scattering amplitude. In Sec. II a simple potential model with a \( \delta \) shell repulsive barrier and an attractive square-well potential is discussed to illustrate scaling of the resonance parameters with powers of low-energy momentum \( Q \) and \( \delta \epsilon = |E - \epsilon_0| \sim \Gamma \), where \( \epsilon_0 \) is the energy of the resonance peak and \( \Gamma \) is the resonance width. The short-range interaction in Eq. (1) generates a low-lying quasistationary state, which causes a large flux delay and manifests itself as a narrow resonance on top of a smooth repulsive background.

An effective field theory is formulated as a combined expansion in powers of \( Q/\Lambda \) and \( \delta \epsilon/\epsilon_0 \). At leading order, an effective Lagrangian in Eq. (11) contains three bare parameters: a four-point contact coupling constant \( C_0 \sim Q^0 \), a three-point Yukawa-like dimer coupling constant \( g \sim Q \), and a dimer “residual mass” \( \Delta \sim Q^2 \). The four-point coupling generates perturbative background contributions dominant everywhere except within a narrow energy domain around \( \Delta \). For these energies, a dominant contribution is from a dimeron coupling. Loop corrections to the dimeron propagator have to be summed to all orders when \( |E - \Delta| \sim Q^2 \) giving rise to a Breit-Wigner term of order \( Q^{-1} \) [Eq. (19)].

In Sec. III a modification of the EFT in the presence of long-range Coulomb repulsion is discussed. It is shown that a combined expansion can be used to describe a narrow low-lying resonance in systems containing charged particles. As in the case of purely short-range interactions, a Coulomb-modified two-body amplitude contains background and Breit-Wigner terms [Eq. (32)]. The background term has a form of a Coulomb-modified effective range expansion. Strong Coulomb effects at short distances renormalize both dimeron coupling constant \( g \) and “residual mass” \( \Delta \) [Eqs. (28) and (29)]. Systems that can be described by the effective theory developed here include ultracold alkali atoms displaying a narrow Feshbach resonance and low-energy \( \alpha-\alpha \) interactions characterized by a narrow resonance due to the coupling to a long-lived \(^8\)Be isotope.

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