The Critical Core Mass of Rotating Planets

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Abstract

The gravitational harmonics measured from the Juno and Cassini spacecraft help us specify the internal structure and chemical elements of Jupiter and Saturn, respectively. However, we still do not know much about the impact of rotation on the planetary internal structure as well as on their formation. The centrifugal force induced by the rotation deforms the planetary shape and partially counteracts the gravitational force. Thus, rotation will affect the critical core mass of the exoplanet. Once the atmospheric mass becomes comparable to the critical core mass, the planet will enter the runaway accretion phase and become a gas giant. We have confirmed that the critical core masses of rotating planets depend on the stiffness of the polytrope, the outer boundary conditions, and the thickness of the isothermal layer. The critical core mass with the Bondi boundary condition is determined by the surface properties. The critical core mass of a rotating planet will increase with the core gravity (i.e., the innermost density). For the Hill boundary condition, the soft polytrope shares the same properties as planets with the Bondi boundary condition. Because the total mass for planets with the Hill boundary condition increases with the decrease of the polytropic index, a higher core gravity is required for rotating planets. As a result, the critical core mass in the Hill model sharply increases. The rotational effects become more important when the radiative and convective regions coexist. Further, the critical core mass of planets with the Hill (Bondi) boundary increases noticeably as the radiative layer becomes thinner (thicker).

Unified Astronomy Thesaurus concepts: Stellar rotation (1629); Exoplanet formation (492); Exoplanet structure (495); Exoplanet evolution (491)

1. Introduction

Core-nucleated instability is one of the mechanisms of planet formation, which involves three different phases (Bodenheimer & Pollack 1986; Pollack et al. 1996; Armitage & Valencia 2010; Piso & Youdin 2014). In the first phase, the rocky core accretes the planetesimal rapidly. In the second phase, the dust will be depleted within the neighboring region and the atmosphere envelope will grow gradually, which is regulated by the Kelvin–Helmholtz (KH) contraction. In the third phase, once the atmospheric envelope mass reaches the critical core mass, the planet will enter the runaway gas accretion phase and be inflated to a gas giant. Note that when runaway accretion occurs, the hydrostatic equilibrium of the envelope is difficult to preserve and the self-gravity of the gas envelope cannot be ignored (Rafikov 2006).

The existence of a critical core mass suggests that when the envelope becomes massive enough and the self-gravity of the envelope dominates, the envelope will collapse (Béthune 2019) and subsequent accretion will take place rapidly on a dynamical timescale. Before runaway accretion, the accretion rate increases continuously with the core mass and the cooling of the envelope (Kanagawa & Fujimoto 2013). Once the planet enters runaway accretion, the accretion rate increases sharply. If the envelope mass approximately equals the core (Bodenheimer & Pollack 1986) or the structure of the gaseous envelope is under thermal and hydrostatic equilibrium (Kanagawa & Fujimoto 2013), the maximum core mass would be attained.

Many factors affect the critical core mass. Lee & Chiang (2015) hold that the metallic elements within the planet can increase the opacity of the envelope. As a result, the critical core mass will increase because the onset timescale of runaway accretion is proportional to the opacity. Kanagawa & Fujimoto (2013) verified that the stiffness of the polytrope—the disk condition around the accreting planets—will change the distribution of mass and density and then determine the critical core mass. Entropy advection (Ali-Dib et al. 2020) may be a candidate mechanism to increase the critical core mass because it can shift the radiative–convective boundary (i.e., RCB) inward. In addition, pebble isolation (Chen et al. 2020) and tidally forced turbulence (Yu 2017) can also change the planet’s RCB and the condition of the critical core mass.

The planet will evolve with strong rotation support (Béthune & Rafikov 2019a) when the core mass exceeds the thermal mass. Further, the planets are rotating as they revolve around the central star due to star–planet tidal interaction, which will influence the interior structure (Béthune & Rafikov 2019b). Thus, rotation is an indispensable factor when considering the formation of planets. We focus here on the rotation of planets. Previous studies assumed that planets are spherically symmetric. However, rotation will actually deform their shape (i.e., an oblate ellipsoid) and weaken the effect of gravity (Maeder 2009). Based on the conclusion of Kanagawa & Fujimoto (2013), we believe that rotation will affect the planet’s critical core mass.

Currently, the internal structure of a rotating planet is investigated by matching the results of simulations of various mechanisms to the gravitational spherical harmonics obtained by the Juno (Kaspi et al. 2010) and Cassini (Iess et al. 2019) spacecraft. In general, there is a gap between the typical rigid-body model of rotating planets (Zharkov & Trubitsyn 1978;
Guillot & Morel 1995; Guillot 2005; Ni 2020) and observation data. The gravitational harmonics derived from the wind velocity in the deep flow (Galanti et al. 2019; Less et al. 2019) and metallic dynamo (Kong et al. 2019) in the rotating planet can match the data. In addition, the spin velocity will increase with the planetary evolution until it gets to the maximum breakup velocity at which the planet dissociates (Ginzburg & Chiang 2020).

We adopt a rotating model proposed by Zeng (2002) to explore the effect of rotation on planetary evolution by obtaining the conditions of the critical core mass. Following Kanagawa & Fujimoto (2013), we assume rotating planets evolve with different boundaries, i.e., Bondi or Hill boundaries. As mentioned above, rotation can change the planet’s shape and partially counteract the gravitational force; our numerical calculations show that the planetary density and mass will condense toward the center and then slow the accretion processes (inhibit runaway accretion). As a result, the planet’s evolutionary timescale will be prolonged. The critical core mass will increase finally.

This paper is structured as follows. In Section 2, we describe the structure of the rotating polytropic planet. In Section 3, we examine the effect of rotation on the critical core mass and characteristic variables. Section 3.1 shows the effect of rotation on a single-layer polytrope. In Section 3.2, we study the effect of rotation in a composite polytropic planet, i.e., a planet with an inner convective layer and an outer radiative layer smoothly connected at the RCB. In Section 4, we summarize the role of rotation in planetary formation.

2. The Model and Assumption

In this section, we take rotation into account. Rotation alters the hydrostatic equilibrium with the coefficient of centrifugal force and then changes the critical core mass. This section is divided into two parts. We exhibit the impact of rotation on the hydrostatic structure in Section 2.1. Section 2.2 shows the polytropic model.

2.1. The Hydrostatic Structure with Rotation

Generally, rotation is a two-dimensional problem. However, Meynet & Maeder (1997) and Maeder (2009) proposed a one-dimensional shellrotational model with constant angular velocity in the equipotential surfaces. The structure parameters of a rotating planet are evaluated by the mean values on equipotential surfaces. However, Zeng (2002) provided an isobaric (equivalent) sphere that can eliminate the mean values of the equipotential surface.

A rotating planet is an oblate ellipsoid, in which the volume on an isobar (Zeng 2002) can be specified as

$$V = \frac{4}{3} \pi r^3 = \frac{4}{3} \pi (2R_e^2 + R_p^2),$$

with the equatorial and polar radii, $R_e$ and $R_p$. $r$ denotes the isobaric radius. Because the planet is equipotential, $\Psi = \text{const}$, the total potential at colatitude $\theta$ can be written as (Maeder 2009)

$$\frac{GM}{r} + \frac{1}{2} \omega^2 r^2 \sin^2 \theta = \frac{GM}{R_p},$$

where $\omega$ is the constant angular velocity. The equatorial and polar radii are given by $R_e = R_p/(1 - \eta)$ and $R_p = r(1 - \eta)^{2/3}$ with $\eta = \omega^2 r^2/2GM$, respectively. Given the angular velocity and total mass, we can obtain the value of the colatitude. Following Maeder (2009), the maximal angular velocity $\omega_{\text{crit}}$ is determined by the critical equatorial radius $R_{e,\text{crit}}$ (Maeder 2009), that is, $\omega_{\text{crit}}^2 = GM/R_{e,\text{crit}}$ with $R_{e,\text{crit}} = 1.5R_p$. When the planet rotates with the breakup velocity, it will disintegrate (Maeder 2009). Thus, the angular velocity should be less than the breakup velocity.

As mentioned above, Maeder (2009) has simplified the two-dimensional rotation problem as a one-dimensional quasi-static model with a constant rotation angle velocity, resulting in the average density of the element volume between two isobars, i.e.,

$$\bar{\rho} = \rho\left(1 - \frac{r^2}{R_e^2}\right)\psi^{-1} = \frac{\rho_{\text{eff}}(1 - r^2 \sin^2 \theta \omega \alpha_p)}{\psi},$$

where $\rho_{\text{eff}}$ is the effective gravitational acceleration and the parameter $\alpha_p = dw/d\psi$ with potential $\Psi$. Following Zeng (2002), the average density in Equation (3) equals $\bar{\rho} = \rho\langle f \rangle$, in which $\langle f \rangle$ is treated as the correction factor. Following Kippenhahn et al. (2012), the mass between two isobars will switch into (Zeng 2002)

$$dM = 4\pi r^2 \bar{\rho} = 4\pi r^2 \rho\langle f \rangle dr.$$  

The equivalent/isobaric sphere structure of a rotating planet is shown in Figure 1 of Zeng (2002). With a constant angular velocity, we can obtain the colatitude of the total potential $\theta$ from Equation (2). Planetary spin imparts a centrifugal acceleration $a_n = \omega^2 r \sin \theta$ that is perpendicular to the axis of rotation. Note that the planet’s gravitational acceleration $g_t = GM/r^2$ ($G$ is the gravitational constant) points to the geometric center of the equivalent sphere. Therefore, there is an angle $\alpha_{\text{eff}}$ between the effective gravitational and gravitational accelerations, and it satisfies

$$\cos \alpha_{\text{eff}} = \frac{g_t - a_n \cos \theta}{g_{\text{eff}}} = \frac{g_t^2 + g_{\text{eff}}^2 - a_n^2}{2g_t g_{\text{eff}}},$$

with

$$g_{\text{eff}} = [(g_t + a_n \sin \theta)^2 + (a_n \cos \theta)^2]^{1/2}.$$  

According to the structural diagram of Zeng (2002), the relationship between the pressure and radius of the rotating planet is satisfied:

$$\frac{dP}{dr} = -\bar{\rho} \frac{d\bar{\rho}}{dr} g_{\text{eff}} \cos \alpha_{\text{eff}},$$

where $g$ is the gravitational acceleration that points to the center. To simplify the relationship between pressure and radius, in Equation (7), Zeng (2002) defined the coefficient of the centrifugal force:

$$f_p = \frac{1}{2} \frac{g_t^2 + g_{\text{eff}}^2 - a_n^2}{g_t^2}.$$  

Thus, the pressure profile will change as follows:

$$\frac{dP}{dr} = -\rho \langle f \rangle g_{\text{eff}} f_p = -\frac{GM}{r^2} \rho\langle f \rangle f_p.$$
A planet spins at a constant angular velocity, the parameter $\alpha_p = d\omega/d\Psi = 1$, and the coefficient of the density ($f_\delta$) is set to 1 (Zeng 2002).

In a rotating planet, the solid core is highly concentrated in the center and immersed in the gaseous envelope (Fujimoto & Tomisaka 1992; Kanagawa & Fujimoto 2013). As the gas component is independent of the solid one, they can be compatible with the different equations of state (Fujimoto & Tomisaka 1992). Their density and pressure satisfy $\rho = \rho_{\text{solid}} + \rho_{\text{gas}}$ and $P = P_{\text{solid}} + P_{\text{gas}}$, in which the subscripts “gas” and “solid” correspond to the gas and solid components, respectively. Following Equations (4) and (9), the solid mass is given by

$$M_{\text{solid}} = \int_0^r 4\pi r^2 \rho_{\text{solid}} (f_\delta) \, dr,$$

and the gaseous mass is

$$M_{\text{gas}} = \int_0^r 4\pi r^2 \rho_{\text{gas}} (f_\delta) \, dr.$$  \hspace{1cm} (10)

The hydrostatic equilibrium can be changed by rotation:

$$\frac{dP_{\text{solid}}}{dr} = -\frac{GM}{r^2} \rho_{\text{solid}} (f_\delta) f_p,$$

$$\frac{dP_{\text{gas}}}{dr} = -\frac{GM}{r^2} \rho_{\text{gas}} (f_\delta) f_p.$$  \hspace{1cm} (11)

The structure should satisfy the boundary conditions, which are described in Sections 2.1.1 and 2.1.2, respectively.

2.1.1. The Internal Boundary

The core radius determines the internal boundary, which can be listed as (Kanagawa & Fujimoto 2013)

$$R_{\text{core}} = \left(\frac{3M_{\text{solid}}}{4\pi\rho_{\text{solid}}}\right)^{1/3},$$  \hspace{1cm} (12)

with a constant solid density. Both the solid and gaseous components contribute to the core mass, thus $M_{\text{core}} = M_{\text{solid}} + M_{\text{gas}} (R_{\text{core}})$ (Kanagawa & Fujimoto 2013). The core density satisfies $\rho_{\text{core}} = \rho_{\text{solid}} + M_{\text{gas}} (R_{\text{core}})/(4\pi R_{\text{core}}^3/3)$, in which $\rho_{\text{solid}} \gg \rho_{\text{gas}}$ (Kanagawa & Fujimoto 2013). The solid mass dominates in the core center.

2.1.2. The Outer Boundary

According to the theory of core accretion, a planet grows in a protoplanetary disk and accretes matter from the disk. The disk parameters would determine the outer boundary condition. The density and pressure at the top of the envelope smoothly connect disk density and pressure (i.e., $\rho_{\text{disk}}$ and $P_{\text{disk}}$; Chiang & Laughlin 2013), which can be given by

$$\rho = \rho_{\text{disk}} = 7.6 \times 10^{-9} \text{ g cm}^{-3}\alpha^{-2.9},$$

$$P = P_{\text{disk}} = 373 \text{ K} \alpha^{-3/7},$$  \hspace{1cm} (13)

where $a$ is the semimajor radius. The outer radius $R_{\text{out}}$ is determined by the Bondi radius,

$$R_B = GM_p/c_s^2,$$  \hspace{1cm} (14)

or Hill radius,

$$R_H = a[M_p/(3(M_p + M_*)]^{1/3}.$$  \hspace{1cm} (15)

The total mass $M_\ast = M_{\text{core}} + M_{\text{gas}}(r)$ with the mass of the center star $M_\ast$ and the sonic speed $c_s = (\gamma_{\text{disk}} P_{\text{disk}}/\rho_{\text{disk}})^{1/2}$ in the protoplanetary disk.

2.2. The Homology Relation with Rotation

The hydrostatic relation of characteristic variables in a rotating planet is defined as follows:

$$U = \frac{d \log M}{d \log r} = -\frac{\rho}{M/(4\pi r^2)},$$

$$V = -\frac{d \log P}{d \log r} = \frac{GM/r}{P/\rho} f_p,$$  \hspace{1cm} (16)

where Equation (19) is the same as in Chandrasekhar (1939), Hayashi et al. (1962), and Kanagawa & Fujimoto (2013). When planets evolve without rotation, $f_p$ is equal to 1.

We employ a polytropic equation of state to describe the relationship between pressure and density:

$$P = K\rho^{1+1/n},$$  \hspace{1cm} (17)

with the polytropic index of $n = 1$–5. The adiabatic constant $K$ is determined by the disk pressure and density, which can be read as (Kanagawa & Fujimoto 2013)

$$K = P_{\text{disk}}/\rho_{\text{disk}}^{1+1/n}.$$  \hspace{1cm} (18)

Combined with Equations (19)–(20), the polytropic index will connect to the homology invariant and satisfies (Kippenhahn et al. 2012; Kanagawa & Fujimoto 2013)

$$n/n + 1 = \frac{d \log \rho}{d \log r} = \frac{d \log P}{d \log r}.$$  \hspace{1cm} (19)

Simplifying Equation (8), we can obtain $f_p = 3 - 2(1 - \eta)^{-2/3}$. The hydrostatic equilibrium will switch to

$$d \log U = -\frac{U + V n/(n + 1) - 3}{(\alpha + 1) U + V/(n + 1) - (3\alpha + 1)},$$  \hspace{1cm} (20)

with $\alpha = 4n/[9(1 - \eta)^{5/3} - 6(1 - \eta)] > 0$. In the nonrotating planet, $\alpha = 0$.

These parameters can characterize the properties of rotating planets through the simple log $U - \log V$ diagram. In addition, the radial radius satisfies

$$d \log r = \frac{d \log M}{U} = -\frac{d \log P}{V}$$

$$= \frac{d \log (V/U)}{(\alpha + 2) U + V - (3\alpha + 4)}.$$  \hspace{1cm} (21)

The quantity $U/V$ in the rotating state represents the ratio of mass to pressure between the two isobars, i.e., $[d \log M/d \log P]$, which is consistent with the nonrotating state of Kanagawa & Fujimoto (2013). In contrast with the nonrotating state, the centrifugal acceleration coefficient $f_p$ introduced by the spin will change the value of $U/V$, which is approximately $U/V \propto (r^4P)/(M^2f_p^3)$, determined by mass, radius, pressure, and the angular rotation velocity. Following Equation (25), there is a critical line

$$(\alpha + 2) U + V - (3\alpha + 4) = 0.$$  \hspace{1cm} (22)
When $\alpha = 0$, Equation (26) is consistent with Kanagawa & Fujimoto (2013). When $(\alpha + 2)U + V - (3\alpha + 4) > 0$, $U/V$ will increase from the exterior to the interior along the structure lines. Instead, $U/V$ will increase from the inner to the outer shell. There are vertical and horizontal lines, defined from the differentials of $U$ and $V$, shown as follows:

$$U + nV/(n + 1) - 3 = 0,$$

$$U + V/(n + 1) - (3\alpha + 1) = 0.$$  

For $n \geq 3$, these three lines intersect. Hence, $U$ and $V$ become:

$$U = [(3(\alpha + 1) - 2n - 3)/(\alpha + 1) - 1],$$

$$V = 2(n + 1)/(\alpha + 1) - 1.$$  

The outer boundary conditions, connecting to the homology invariants, are given by

$$U_{\text{surf,B}} = \gamma_{\text{disk}}^3(M_p/M_0)^2,$$

$$V_{\text{surf,B}} = \gamma_{\text{disk}} f_{p,\text{surf}},$$

with the coefficient of the centrifugal force at the surface $f_{p,\text{surf}}$ and the characteristic mass $M_0 = [(1/4\pi G)(P_\text{disk}/\rho_\text{disk})^3]$ for the Bondi boundaries. The Hill boundaries can be given as follows:

$$U_{\text{surf,H}} = 4\pi^3\rho_\text{disk}/(3M_\star),$$

$$V_{\text{surf,H}} = (3M_\star/4\pi^3\rho_\text{disk})^{1/3}(M_p/M_0)^{2/3}f_{p,\text{surf}}.$$  

The density between the top of the core and the bottom of the envelope is discontinuous (Kanagawa & Fujimoto 2013), in which core density will jump to a high level. Other parameters are continuous. The jump condition (Kanagawa & Fujimoto 2013) connecting the $U$–$V$ plane shows

$$U_{1e}/U_{1i} = V_{1e}/V_{1i} = \rho_{1e}/\rho_{1i},$$

where the subscripts 1e and 1i correspond to the bottom and base of the envelope, respectively.

### 3. Results

Rotation may change the critical core mass and the homologous relation. In Section 3.1, we discuss the effect of rotation in a single model. Section 3.2 lists the critical mass of a rotating planet in the composite polytrope.

#### 3.1. The Results for a Single Polytrope

In Section 3.1.1, we explore how rotation changes the critical core mass in a single polytrope. In Section 3.1.2, the homology structure of a rotating planet shows the different features.

**3.1.1. The Critical Core Mass of a Single Polytrope**

To find the critical core mass, we first need to ensure the relationship between the core mass and the total mass. The suitable solid mass determines the evolution of this core mass, which can be searched by solving the Equations (19) and (20). The input parameters are listed in Table 1. Figure 1 shows the core mass evolving as a function of the total mass in the Bondi and Hill models, respectively. The characteristic variables of the critical models are listed in Table 2.

Early in the evolution, the total mass is approximately that of the core because the envelope mass is much lower than that of the core ($M_\text{env} = M_p - M_\text{core} \ll M_\text{core}$; Rafikov 2006; Kanagawa & Fujimoto 2013). We can integrate Equation (20) with the polytropic relation from the outside to the inside. Finally, the density distribution satisfies

$$\rho = \rho_\text{disk}\left[\frac{1 + V_{\text{surf}} f_p}{n + 1} \left(\frac{R_{\text{out}}}{r} - 1\right)\right]^{\gamma},$$

with the surface parameter $V_{\text{surf}} = GM_\text{core}p_{\text{disk}}/R_{\text{out}}^2\rho_\text{disk}$. The density profile is gentle relatively when $V_{\text{surf}} f_p/(n + 1) < 1$. As $V_{\text{surf}} f_p/(n + 1) > 1$, the density curve will become steeper. We can assume both the core mass and total mass for the nonrotating and rotating planets are constant. The density will decrease with the coefficient of centrifugal force ($f_p$) induced by rotation. However, in this situation, the core density (core mass) changes in a manner opposite to our assumption. To counteract this, we can obtain the same core mass by reducing the planetary mass. When the total mass is known, the core gravity rises sharply as a result of storing so much mass within the planet. Consequently, the core mass increases significantly.

As seen in Figure 1, the critical core mass divides the accretion process into two parts. The core mass first increases with the total mass. The atmospheric mass is much lower than the core, so that the core gravity determines the structure features. Once the core mass becomes comparable to the critical core mass, it will monotonically decrease. Combined with Equations (19) and (36), we can derive the mass ratio between the envelope and core:

$$\frac{M_\text{env}}{M_\text{core}} = \frac{4\pi^3\rho_\text{disk}R_{\text{out}}^3}{M_\text{core}} \times \int_{R_{\text{out}}/R_{\text{core}}}^{1} \xi^{3-n}\left[\frac{f_p V_{\text{surf}}}{n + 1} (1 - \xi)\right]^n d\log \xi,$$

with $\xi = r/R_{\text{out}}$. The atmospheric mass is different in the Bondi and Hill models.

In the Bondi model of $V_{\text{surf}} f_p/(n + 1) < 1$, the density distribution is relatively uniform such that the atmospheric mass is approximately the product of density and volume. As long as $R_{\text{out}} \gg R_{\text{core}}$, the mass ratio will change to $M_\text{env}/M_\text{core} \approx 4\pi^3\rho_\text{disk}R_{\text{out}}^3/3M_\text{core} = (M_\text{core}/M_0)^2/\gamma_{\text{disk}}^3$ (Kanagawa & Fujimoto 2013). Many materials from the protoplanetary disk will be accreted onto the interior because both core gravity and mass will increase slightly due to rotation. Note that the change in the critical core mass is insensitive to the polytropic index because the density for the Bondi model is flattest.

The Hill model covers three scenarios, $V_{\text{surf}} f_p/(n + 1) < 1$, $> 1$, or $\gg 1$. The value of $V_{\text{surf}} f_p/(n + 1)$ affects the mass

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**Table 1**

The Input Parameters

| Parameters | Value |
|-----------|-------|
| $M_\star$ | $1 M_\odot$ |
| $\mu$ | 2.0 |
| $\rho_\text{disk}$ | 5.5 g cm$^{-3}$ |
| $\gamma_{\text{disk}}$ | 4 |
| $\alpha$ | 0.96 |
| $\rho_\text{disk}$ | 0.1 au |
| $\beta_{\text{disk}}$ | $6 \times 10^{-6}$ g cm$^{-3}$ |
| $T_{\text{disk}}$ | 1000 K |
ratio, due to the density distribution. For \( V_{\text{surf}} f_p / (n + 1) < 1 \), it corresponds to the situation of \( n < 3 \). Rotation will flatten the density distribution and then force a mild increase in core gravity to retain the same planetary mass as the nonrotating planet. Thus, the critical core mass will increase slightly. In the case of \( V_{\text{surf}} f_p / (n + 1) > 1 \), the innermost envelope determines the integral. The innermost density is

\[
\rho_{1e} = \rho_{\text{disk}} \left( \frac{f_p}{n + 1} \right)^n \left( \frac{\rho_{\text{core}}}{3 \rho_{\text{disk}}} \right)^{n/3} \left( \frac{M_{\text{core}}}{M_0} \right)^{2n/3}, \quad (38)
\]

where \( R_{\text{out}} / r \gg \min \{ 1, V_{\text{surf}} f_p / (n + 1) \} \) around core. The mass ratio in the Hill model is independent of the outer radius and is determined by the properties of the innermost shell, which can be shown as follows:

\[
\frac{M_{\text{env}}}{M_{\text{core}}} \sim \frac{4\pi R_{\text{core}}^2 \rho_{1e}}{3 M_{\text{core}}} = \left( \frac{1}{3} \right)^{n/3} \left( \frac{f_p}{n + 1} \right)^n \left( \frac{\rho_{\text{disk}}}{\rho_{\text{core}}} \right)^{(3-n)/3} \left( \frac{M_{\text{core}}}{M_0} \right)^{2n/3}. \quad (39)
\]

When \( M_{\text{env}} \sim M_{\text{core}} \), the critical core mass satisfies

\[
M_{\text{crit}, \text{core}} = \sqrt[3]{(n + 1) / f_p} \rho_{\text{disk}} / \rho_{\text{core}}^{n-3/2} \rho_{\text{core}}^{n-3} / M_0. \quad \text{(36)}
\]

As analyzed above, both the core density and the core mass will increase with the reduction in \( f_p \). Thus, the critical core mass may increase moderately. At the same outer boundaries, we note that the atmospheric mass is reduced.

When \( V_{\text{surf}} f_p / (n + 1) \gg 1 \), the density will increase rapidly, resulting in the largest mass concentration being in the middle of the envelope. Therefore, the mass ratio will become

\[
\frac{M_{\text{env}}}{M_{\text{core}}} \sim \left( \frac{f_p}{n + 1} \right)^n \left( \frac{\rho_{\text{disk}}}{\rho_{\text{core}}} \right)^{(3-n)/3} \left( \frac{M_{\text{core}}}{M_0} \right)^{2n/3}, \quad (40)
\]

with the mean density \( \langle \rho_\bullet \rangle = M_\bullet / (4\pi a^3 / 3) \). Core gravity would significantly increase compared to other cases. Further, core mass in a rotating planet will noticeably increase because \( M_{\text{crit}, \text{core}} \propto [(n + 1) / f_p]^{3/2} \).

In summary, the density and mass distributions are relatively uniform in the soft Hill model (\( n > 3 \) when \( V_{\text{surf}} f_p / (n + 1) < 1 \)). The critical core mass will be increased slightly by rotation. For \( n = 3 \), the density distribution in the nonrotating model will increase moderately, which will be weakened by rotation. Hence, the critical core mass for a rotating planet in this state will increase. The changing trend of density in the stiff Hill polytrope \((n < 3) \) when \( V_{\text{surf}} f_p / (n + 1) \gg 1 \) is similar to the case of \( n = 3 \), but core mass increases noticeably. Thus, we hold that the critical core mass increases with the stiffness of the polytrope.

When the atmospheric mass approaches core mass, runaway accretion is triggered so that self-gravity becomes crucial. The planet will expel the envelope mass outward to adapt to the larger mass (Kanagawa & Fujimoto 2013). Rotation can increase the core mass. We hold that the critical core mass of a single rotating polytropic model grows noticeably for the case with a Bondi radius or the softest Hill model (\( n < 3 \)).

3.1.2. The Homology Relation of a Single Polytrope

In this section, we mainly discuss the influence of rotation on the homology relation. As shown in Figure 2, rotation will change the relationship between \( U \) and \( V \). Following Equation (36), the
files of the characteristic variables $V$ and $U$ can be expressed as

$$V = (n + 1) \left[ \frac{r}{\text{out}} \left( \frac{n + 1}{V \text{surf} f_p} - 1 \right) + 1 \right], \quad (41)$$

$$U = \frac{4\pi r^3 \rho_{\text{disk}}}{M_{\text{core}}} \left[ 1 + \frac{V \text{surf} f_p}{n + 1} \left( \frac{R \text{out}}{r} - 1 \right) \right]^n. \quad (42)$$

The parameter $V \text{surf} = G M_{\text{core}} \rho_{\text{disk}} / R \text{out} P \text{disk}$ will increase with the core mass. With the constant boundary conditions, the characteristic variables $U$ and $V$ will be affected by the coefficient of the centrifugal force $f_p$.

At the inner edge, the above relation will change into

$$V_{\text{ie}} = \frac{G M_{\text{core}} f_p \rho_{\text{ie}}}{P_{\text{ie}} R_{\text{core}}}$$

$$= f_p \left( \frac{M_{\text{core}}}{M_0} \right)^{2/3} \left( \frac{3 \rho_{\text{disk}}}{\rho_{\text{core}}} \right)^{2/3} \left( \frac{3 \rho_{\text{ie}}}{\rho_{\text{core}}} \right)^{-1/n}, \quad (43)$$

$$U_{\text{ie}} \simeq \frac{3 \rho_{\text{ie}}}{\rho_{\text{core}}}$$

$$\simeq \left( \frac{f_p}{n + 1} \right)^n \left( \frac{\rho_{\text{core}}}{3 \rho_{\text{disk}}} \right)^{n-3/3} \left( \frac{M_{\text{core}}}{M_0} \right)^{2n/3}. \quad (44)$$

Figure 2. The structure line of a single polytropic model evolves with total mass. The left panels denote the Bondi models, and the right panels are the Hill models. From top to bottom, the polytropic index is specified as $N = 1.5$–4. Different colors represent different total masses ($M_\odot$). The values of the total masses are denoted by the corresponding colors. The numbers in parentheses are for the critical model for rotating planets. Dotted lines represent the nonrotating case, and solid lines are for the rotating case. Thick red dashed and solid lines represent the critical models without and with spin, respectively. The solid gray and black lines represent the outer boundaries for the planets without and with rotation, respectively. Long dashed, broken, and dotted lines denote the vertical, critical, and horizontal lines for the nonrotating case, respectively. The coefficients of the centrifugal force derived from rotation are not constant, which increases inward. The critical and horizontal lines are close to the core. Besides, the efficiency of the rotation near the core is small. Thus, the differences of the characteristic lines between planets without/with spin are much less.
with $R_{\text{out}}/r \gg \min[1, V_{\text{surf}} f_p/(n+1)]$. The jump condition satisfies

$$V_{1e} U_{1e}^{1/n} = f_p \left( \frac{M_{\text{core}}}{M_0} \right)^{2/3} \left( \frac{3 \rho_{\text{disk}}}{\rho_{\text{core}}} \right)^{(3-n)/3n}.$$  \hspace{1cm} (45)

The structure line shows the unique characteristics of the Bondi and Hill models in Figure 2. All structure lines for the Bondi model are below the horizontal line. The horizontal line in a rotating planet changes to $V = [(3\alpha + 1) - (\alpha + 1)U/(n+1)]$, where $\alpha > 0$. There are two different structure lines for the nonrotating or rotating case. First, the planet’s mass is below the critical core mass, implying core gravity is effective. The structure line will increase inwardly from the surface boundary until it approaches the horizontal line (Kanagawa & Fujimoto 2013). The centrifugal force, driven by rapid rotation, will weaken part of the gravitational force so that the value of $V_{\text{surf}} \propto \gamma_{\text{disk}} f_p$ at the outer boundary (black dashed line) decreases sharply. While the exterior structural lines are much lower than that of the nonrotating planet, the interior will rise higher with the core mass. Second, the planet’s mass is above the critical core mass so that the gravity of the envelope is effective. The structural line will pass through the outer boundary again and reach the minimum $V_{\text{min}}$, the horizontal line. In a rotating planet, $V_{\text{min}}$ is close to the boundary line. In summary, rotation changes the characteristic variables for the structural line at the same mass, resulting in an increase of critical core mass.

In the Bondi models, $V_{\text{surf}} f_p/(n+1) < 1$, the density distribution is the flattest and is almost determined by the outer boundaries. Rotation will increase the core gravity to keep the same total mass as the nonrotating case because the centrifugal force weakens the gravitational force. Subsequently, core mass increases. The maximum core mass is a transition point between the core gravity and the self-gravity of the envelope (Kanagawa & Fujimoto 2013), which can be derived when the surface structural line approaches the horizontal line:

$$U_{\text{surf}}^{\text{crit}} \simeq \frac{(3\alpha + 1) - \gamma_{\text{disk}} f_p}{(n+1)(\alpha + 1)},$$  \hspace{1cm} (46)

when $n = 3$. The value of $U_{\text{surf}}^{\text{crit}}$ will increase due to $\alpha > 0$ and $f_p < 1$; the corresponding total mass of the critical model will become

$$M_{p}^{\text{crit}} = \gamma_{\text{disk}}^{3/2} \left[ \frac{3\alpha + 1}{\alpha + 1} - \frac{\gamma_{\text{disk}} f_p}{(n+1)(\alpha + 1)} \right]^{1/2} M_0.$$  \hspace{1cm} (47)

combined with Equation (31). The total mass of the nonrotating planet is nearly approximate to the characteristic mass $M_0$ (Kanagawa & Fujimoto 2013). But the critical mass in a rotating planet will increase for $\alpha > 0$ and $f_p < 1$. When $n < 3$, the surface parameter $U_{\text{surf},B}$ is slightly larger than the $U_{\text{surf}}$ so that the final mass of the critical model will increase. If $n > 3$, $U_{\text{surf},B}$ is lower than $U_{\text{surf}}$, the total mass of the critical model will decrease. The core mass can be confirmed by the jump condition. Combining with the results in Table 2, the value of $V_{1e}^{\text{crit}}$ for the critical model at the inner edge approaches $V_{1e}^{\text{crit}} \sim (n + 1)/f_p$. As the core mass can be increased by rotation, the variables at the inner edge may increase.

The structure lines for the rotating planets in the Hill model are shown on the right column of Figure 2. In the soft polytrope ($n > 3$), the inner edges are almost below the horizontal line. Unlike the Bondi model, the structure lines of the Hill model all cross the horizontal, critical, and vertical lines. We analyze the characteristics of the structure line by the value of $V_{\text{surf}} f_p$.

When $V_{\text{surf}} f_p/(n+1) > 1$, the structure line is above or is below the horizontal line. For the former, the atmospheric mass is less than the core mass so that the core gravity determines the structural signatures. The structure line first increases $U$ and $V$ outward, wherein the value of $V$ for the rotating planet is higher than that of the planet without rotation. Thus, rotation will force the density to drop steeply in the radial coordinate. Subsequently, the structure line decreases $U$ outward. However, rotation in this state will reduce $V$ and then form a flatter descent profile of the density. Thus, core density in a rotating planet is much higher. In the thinner envelope, the structural line near the outer boundary will spiral downwards, which will flatten the corresponding density distribution. For the latter, the structural line first decreases $V$ outward. The value of $V$ for a rotating planet is higher than that of the nonrotating state, implying the density drops more steeply. Once the structure line extends to the critical line, we can get the maximum ($U/V_{\text{max}}$). The structural line turns to increase $V$, the signature of which is similar to that of the former. Most of the atmospheric mass is concentrated in the middle of the planet, i.e., near the intersection with the critical line. Thus, we found that the core mass will increase based on the complete density change trend.

In the case of $V_{\text{surf}} f_p > n + 1$, core mass is determined by the innermost shell. The effect of rotation on the structure line is weaker compared to that of $V_{\text{surf}} f_p > n + 1$. The tails of all structural lines starting at the inner edge will spiral downwards, and the rise in the corresponding mass becomes more gentle. The increase in the critical core mass is relatively slight. The critical models for $V_{\text{surf}} f_p > n + 1$ and $V_{\text{surf}} f_p > n + 1$ show the same signature. The value of $U/V_{\text{max}}$ for a rotating planet is comparable to that of the nonrotating state. The inner edge of the critical model nears the horizontal line, and the slope for the structural line of the critical model before reaching the critical line is almost zero, $d \log V/d \log U = 0$. The values of $V_{1e}^{\text{crit}}$ and $U_{1e}^{\text{crit}}$ for the critical model are listed in Table 2. The values of $V_{1e}^{\text{crit}}$ and $U_{1e}^{\text{crit}}$ also increase, $V_{1e}^{\text{crit}} \sim (n + 1)/f_p$.

In the case of $n = 4$, $V_{\text{surf}} f_p/(n+1)$. The density distribution is relatively modest. When the atmospheric mass is below the core mass, core gravity dominates the evolution processes. Starting at the outer boundary, the slope of the structural line in a rotating planet is negative and is steeper. Compared with the nonrotating planet, the density has a more gentle incremental profile because rotation reduces $V$. In the positive slope, the density distribution becomes steeper because $V$ has increased by rotation. When the atmospheric mass grows to core mass, the structure line for a rotating planet (solid red line) on the right side of the surface boundary is higher than that of the nonrotating state. According to Kanagawa & Fujimoto (2013), the inner edge of the critical model is just above the singular point and approaches

$$U_{1e}^{\text{crit}} = [(3\alpha + 1 - 2n - 3)/(\alpha + 1)n - 1].$$  \hspace{1cm} (48)

Rotation will increase the value of $U_{1e}^{\text{crit}}$ because $\alpha > 0$. The maximum core mass can be derived from Equation (45), which
can be shown a:

\[
M_{\text{crit}} \approx \left[ \frac{3(\alpha + 1) - 2n - 3}{(\alpha + 1)n - 1} \right]^{3/2n} \left( \frac{n + 1}{f_p} \right)^{3/2} \times \frac{3\rho_{\text{disk}}}{\rho_{\text{core}}} \left( \frac{n - 3}{2n} \right) \tag{49}
\]

The critical core mass will increase for \( f_p < 1 \) and \( \alpha > 0 \). When the planet enters the runaway accretion stage, the structure lines will spiral downwards over the inner edge. Rotation increases the \( V \) at the middle part, but the \( V \) at both ends decreases. The inward decrease in mass becomes flatter, steeper, and flatter, respectively. In addition, the decrease in \( V_{\text{te}} \) is more obvious than that of the nonrotating planet. The core mass of a rotating planet may be less than that of the nonrotating state under runaway accretion (see in the right panel of Figure 2).

3.2. The Results for the Composite Polytrope

In the previous section, we discussed the properties of a fully convective planet (a single polytropic model). Instead of a single structure, a two-layer structure with an exterior radiative layer and an interior convective layer is better suited to describe planet structure. (Rafikov 2006). In Section 3.2.1, we study the effect of rotation on the core mass for the composite polytrope. In Section 3.2.2, we research the influence of spin on the structural lines of the composite polytrope.

3.2.1. The Core Mass of the Composite Polytrope

The planetary shape may be a composite polytrope (Kanagawa & Fujimoto 2013), in which the interior convective layer may be covered by a radiative layer (Rafikov 2006). The polytropic index of the radiative layer approaches \( n = \infty \), but \( n_{\text{te}} = 1-5 \) for the convective layer. There is a transition radius, \( R_{\text{t}} = \xi R_{\text{out}} \) (the transition radius ratio \( \xi = 0.1-1 \)), at the RCB. The radiative layer can be treated as an isothermal layer; the temperature in this layer seems to be \( T \approx T_{\text{disk}} \) (Lee et al. 2014; Lee & Chiang 2015). The polytropic equation of state in the radiative layer can be given as (Kippenhahn et al. 2012)

\[
P = K_{\text{disk}} \rho. \tag{50}
\]

The polytropic constant for the radiative envelope is \( K_{\text{disk}} = P_{\text{disk}}/\rho_{\text{disk}} \). Thus, we will derive the density at the transition radius (i.e., the RCB) using Equation (20):

\[
\rho_{t} = \rho_{\text{disk}} \exp \left[ \frac{GM_{\text{core}} \rho_{\text{disk}} f_p}{P_{\text{disk}} R_{\text{out}}} \left( \frac{1}{\xi_{t}} - 1 \right) \right] = \rho_{\text{disk}} \exp \left[ V_{\text{surf}} f_p \left( \frac{1}{\xi_{t}} - 1 \right) \right]. \tag{51}
\]

In the nonrotating case \( (f_p = 1) \), when the value of \( \xi_t \) decreases, \( \rho_t \) becomes larger, and the density in the radiative layer increases exponentially. The variable \( \rho_t \) is a new outer boundary for the convective layer, where the polytropic constant of the convective layer \( K_t = K_{\text{disk}} / \rho_{\text{t}}^{3/4} = P_{\text{disk}} / \rho_{\text{disk}} \rho_{\text{t}}^{3/4} \). The influence of the gravity of the envelope is enhanced because a smaller \( K_t \) requires a larger density to achieve hydrostatic equilibrium. Thus, the density distribution in the convection layer can be modified to be

\[
\rho_{\text{gas}}(r) = \rho_{t} \left( 1 + \frac{GM_{\text{core}} \rho_{\text{t}} f_p}{P_{\text{disk}} (n_{\text{te}} + 1)} \left( \frac{1}{r} - \frac{1}{\xi_{t} R_{\text{out}}} \right) \right)^{n_{\text{te}}} = \rho_{t} \left( 1 + \frac{V_{\text{surf}} f_p}{n_{\text{te}} + 1} \left( R_{\text{out}}^{-1} - \frac{1}{\xi_{t}} \right) \right)^{n_{\text{te}}}. \tag{52}
\]

The values of \( f_p \) and \( \xi_t \) determine the convective density distribution. To obtain the core mass for a rotating case, we can employ Equations (51) and (52) to analyze the features of the composite polytrope. When \( M_{\text{core}}, R_{\text{out}} \), and \( \xi_t \) are constant, the density at the transition shell \( \rho_t \) falls significantly along with \( f_p \), resulting in a significant reduction in \( \rho_{\text{gas}} \), the density at the convective layer. Thus, the value of \( M_{\text{core}} \) in a rotating state is far lower than that in the nonrotating situation, which is the opposite of the initial condition. The same \( M_{\text{core}} \) can be obtained by reducing the total mass or the outer radius. Under these conditions, a much higher core gravity (core mass) is required for the same outer boundary conditions. Based on the analysis above, the core mass becomes larger as \( f_p \) is reduced.

As seen in Figure 3, the evolution of the core mass with the total mass in the Bondi and Hill models is shown in the left and right panels, respectively. Kanagawa & Fujimoto (2013) have verified that the total mass (i.e., \( M_{\text{core}} \)) in the critical model decreases with the transition radius ratio. However, the core mass can be increased by rotation and shows a different signature with different thicknesses of the isothermal layer. Note that the isothermal layer will be removed when \( \xi_t = 1.0 \). For the rotating Bondi models, the breakup speed is proportional to the total mass due to \( \omega_{\text{crit}} = c_s / GM_{\text{p}} \). Thus, the angular velocity increases as \( \xi_t \) decreases. In the smaller envelope, the total mass is much less than that of the core; the parameter \( \eta = \omega^2 r^3 / 2GM_{\text{p}} \approx \omega^2 r^3 / 2GM_{\text{core}} \). If the core mass and radius are the same as those of the nonrotating planet, \( \eta \) will increase as the isothermal layer grows thicker. The coefficient of the centrifugal force, \( f_p = 3 - 2(1 - \eta)^{-2/3} \), under this situation will decrease sharply. The convective density will decrease sharply according to Equations (51) and (52). Following the analysis above, the core mass increases more sharply when the isothermal layer becomes sufficiently thick.

In the Hill model, the breakup speed for a rotating planet is proportional to the sum of the mass of the central star and the planet and can be expressed as \( \omega_{\text{crit}} \propto \sqrt{(M_{\star} + M_{p})} \). Because the total mass of the critical model increases with the decrease in the thickness of the isothermal layer, the angular velocity \( \omega = 0.96 \omega_{\text{crit}} \) will increase. When the isothermal layer is thinner, we assume the same core mass, outer radius, and total mass as the nonrotating planet. The variable \( \eta = \omega^2 r^3 / 2GM_{\text{p}} \approx \omega^2 r^3 / 2GM_{\text{core}} \) at the same radius will increase, resulting in a significant reduction in the coefficient of centrifugal force, \( f_p = 3 - 2(1 - \eta)^{-2/3} \). The convective density and core mass will drop noticeably, which is opposite to the assumption. To obtain the same core mass, we can increase core gravity by decreasing the outer radius or the total mass. When the planet’s mass is given, a higher core gravity is required to keep the same mass as the nonrotating planet. In summary, the increase in the core mass is even sharper as the isothermal layer becomes thinner. In addition, the signature becomes more obvious as the value of \( n_{\text{te}} \) decreases.
3.2.2. The Structure of the Composite Polytrope

The exterior isothermal layer enables the structure line to cross the horizontal and critical lines because of the inward decrease in its thermal energy (Kanagawa & Fujimoto 2013). From Equation (20), we can derive the homologous relation at the transition radius, 

$$\log \left( \frac{V_t}{V_{out}} \right) = \int_{\xi_i}^{\xi} (1 - U) d \log \xi_i + \log \left( \frac{f_{p,t}}{f_{p,out}} \right), \quad (53)$$

with the characteristic variable at the surface $V_{out}$, the coefficient of the centrifugal force at the surface $f_{p,surf}$, and the transition radius $f_{p,t}$. The coefficient of the centrifugal force increases inward and causes a sharp drop of $V_{out} = G M_{disk} / f_{p,out} R_{out} f_{disk}$.

The structural line of the critical model with the Bondi boundary is shown in the right column of Figure 4. If $V_t$ is under the horizontal line, the structure line will retain the same features as a single polytrope. Rotation can decrease the radiative $V$ and steepen the radiative structural line, and then, the density has a more gentle descent profile. However, the slope of the convective structural line is almost similar to that of the nonrotating state. Thus, the change in the density at the transition shell will determine the core gravity. When the radiative layer grows sufficiently thick, $V_t$ is above the horizontal line and the convective structural line will decrease inward.

We can derive the condition of the core mass from the jump condition:

$$V_{t_e} U_{t_e}^{1/n_{t_e}} = \frac{G M_{core}}{R_{core}} f_p \left( \frac{3}{R_{core}} \right)^{1/n_{t_e}} K_i \left( \frac{\rho_{t_e}}{\rho_{disk}} \right)^{(n_{t_e}-3)/3n} \left( \frac{\rho_{t_e}}{\rho_{disk}} \right)^{1/n_{t_e}} f_p. \quad (54)$$

Figure 3. The relationship of the composite polytropes between the core mass and total mass. On the left is the Bondi model, and on the right is the Hill model. Different colors represent different radiation convection boundaries $\xi = 0.2 - 1.0$. 

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Following Kanagawa & Fujimoto (2013), the core mass decreases as the thickness of the isothermal layer increases. When the effect of rotation is included, the condition of core mass can be given as
\[
M_{\text{core}} = \left( \frac{3 \rho_{\text{disk}}}{\rho_{\text{core}}} \right)^{(n_{1e}-3)/2n_{1e}} \left( \frac{1}{\rho_{1e}^{2/n_{1e}} f_p} \right)^{2/3} \times (V_{1e} U_{1e}^{1/n_{1e}})^{2/3} \rho_{\text{disk}}^{2/3n_{1e}} M_{\odot},
\]
with the given values of $U_{1e}$ and $V_{1e}$. When $n_{1e} > 3$, $(n_{1e} - 3)/2n_{1e} < 0$ and then the core mass will increase mildly with the core density. In addition, when $n_{1e} \leq 3$, the effect of the core density is low. Thus, the effect of the core density can be ignored. We note that the core mass, which has dropped at the isothermal layer, will be increased by the decrease in $f_p$.

As seen in the left column of Figure 4, $V_{1e}$ increases slightly compared with that in the nonrotating planet. In addition, the different $U_{1e}$ changes have a mild effect on the core mass. Thus, the core mass in the Bondi model is mainly determined by the polytropic constant $K_t$ according to Equation (55). As mentioned above, the coefficient of the centrifugal force $f_p$ in the Bondi model decreases with $\xi_t$. It will force the density at the transition shell to drop modestly and then increase $K_t$. As a result, the increase in the core mass and the total mass of the critical model is more significant when the isothermal layer grows sufficiently thick. In addition, the total mass of the critical model in the rotating planet will increase because the parameter $\alpha$ increases with the decrease in $f_p$ according to Equation (47).

The structural line of the critical model with the Hill boundary is shown in the left column of Figure 4. The structure line of a composite polytrope is different from that of a single

**Figure 4.** Behaviors on the characteristic plane for the critical model with different thicknesses of the isothermal layer. Different colors correspond to different thicknesses. The left panels represent the Bondi models, and the right panels denote the Hill models. The solid and dotted lines correspond to the rotating and the nonrotating planets, respectively.
polytrope. The outer boundary ($V_{\text{out}} = V_{\text{surf,1H}}$) is reduced sharply by the coefficient of the centrifugal force. In addition, the ratio of the characteristic variable $V_{\text{out}}/V_t$ reduces with $\xi_1$ (Kanagawa & Fujimoto 2013), which will force the isothermal layer to spiral downward. The slope of the isothermal layer satisfies

$$\frac{d \log V}{d \log U} \approx \frac{1 - d \log f_p/d \log r}{V - 3}, \quad (56)$$

with $d \log f_p/d \log r < 0$ and $U \ll 1$. When the isothermal layer grows sufficiently thick, the structure will spiral downward, and $V_{\text{out}}$ is below the horizontal line. As $U$ decreases inward, the structure line becomes negative and steepens due to rotation, forcing the mass distribution to decrease gently. However, the changes in the structure line with a positive slope are opposite that of the former. Because $f_p$ increases inward, the slope will gradually become similar to that of the nonrotating planet. $V_t$ is above the critical line. Once $V$ reaches $V_t$, the structure line has turned and then $\log U$ of the adiabatic lines in the stiff polytrope becomes negative due to

$$\frac{d \log V}{d \log U} \approx -1 + (d \log f_p/d \log r)(n_{1e} + 1)/V, \quad (57)$$

with $V > n_{1e} + 1$. The value of $(d \log f_p/d \log r)(n_{1e} + 1)/V$ is tiny as $V > n_{1e} + 1$, resulting in a flatter slope and a mild change in the thinner convective layer. The mass distribution drops steeply as the structural line becomes flatter. Thus, we can get the complete trends of the mass and density distribution.

The condition in the core mass for the Hill model is mainly determined by $V_{1e}$, $U_{1e}$, and $K_t$ following Equation (55). As mentioned above, the $f_p$ in the Hill model will decrease as the isothermal layer is thinner. Compared with the nonrotating planet, rotation under this situation will decrease $\rho_t$ more significantly, forcing a sharper increase in $K_t$. For $n_{1e} \lesssim 3$, the increase in $V_{1e}$ is mild. As the isothermal layer is thinner, $U_{1e}$ in a rotating planet becomes greater. However, rotation decreases $U_{1e}$ when the isothermal layer grows thicker. Thus, rotation will noticeably increase the core mass when the isothermal layer becomes thinner. For $n_{1e} > 3$ and the isothermal layer is thinner, and rotation mildly increases $U_{1e}$. However, there is a sharper increase in $U_{1e}$ and a slight decrease in $V_{1e}$ for a rotating planet when the isothermal layer is thicker. Combined with the polytropic constant $K_t$, the increase in the core mass for the Hill model is significant as the isothermal layer is thinner, especially for the stiff Hill model. In other words, the core mass increases with the decrease in the thickness of the isothermal layer and the polytropic index.

In short, the isothermal layer is a cooling shell, which can reduce the thermal energy or the polytropic index (Kanagawa & Fujimoto 2013). Thus, the core mass will be decreased. However, rotation in the composite polytrope will noticeably increase the core mass compared with that in the nonrotating state. It will change the surface parameters and the signature of the structure line. Although the change in the Bondi model is small, a significant change will be induced in the structure line of the Hill model by the rotation of the isothermal layer.

4. Conclusion

Rotation can deform the shape of a planet. Centrifugal forces, driven by spin, would change the hydrostatic equilibrium. Thus, rotation will alter the critical core mass and determine whether the planet triggers runaway accretion. Because rotation can particularly weaken the gravitational force, the core gravity will increase to retain the same mass as the nonrotating planet, forcing an increase in core mass. The critical core mass is determined by the polytropic index, outer radius, or isothermal layer. The core mass in the Bondi and the soft Hill models will slightly increase owing to rotation. However, the increase in core mass for the stiffer Hill model is more significant with higher core gravity. Rotation can also enhance the critical core mass conditions according to the homology relationship. In addition, the existence of an isothermal layer will increase the planetary density and then decrease the critical core mass. However, planetary rotation can reduce this density by centrifugal force, and new conditions are established for convective density.

The critical core mass then increases considerably with reductions in $\rho_t$ and $f_p$. Because the angular velocity in the Hill model is proportional to the planetary mass, the critical core mass will increase sharply when the radiative layer becomes thinner. However, the case with the Bondi model is the opposite. In short, rotation slightly increases the critical core mass of a single polytrope. It has a tremendous effect on the growth of the critical core mass in the stiff Hill model and a composite polytrope.

The structure and formation of the exoplanet may be deeply affected by rotation. However, in recent years, there have been few studies on how spin affects the formation of planets. This study considers a barotropic state, where the planetary shape is deformed. We have verified the effect of the isothermal layer of a rotating planet on the critical core mass. Rotation will increase the critical core mass and then prolong the evolutionary timescale (i.e., KH contraction timescale), which will inhibit runaway accretion or the formation of gas-giant planets. Subsequently, mini-Neptunes/super-Earths may develop in the postformation stages.

There are some limitations to this work. First, we assumed that an isothermal layer exists with an infinite polytropic index. The isothermal layer will add a new temperature gradient to the planet. However, the trend of this temperature gradient is compatible with cases without rotation. We did not consider the coefficient $f_R > 1$ (Zeng 2002) in the radiative temperature gradient, which will lead to changes in the polytropic relationship between pressure and density. Second, the transition radius (i.e., RCB) is fixed in rotating planets. When the transition radius of the polytropic model evolves with time, the critical core mass and the planetary formation may show different signatures.

In general, rotation is a two-dimensional (Béthune & Rafikov 2019a) or three-dimensional problem (Béthune & Rafikov 2019b). When the core mass is greater than the thermal mass, the envelope gets considerable rotation support (Béthune & Rafikov 2019a). Meanwhile, the one-dimensional rotation model is no longer applicable. However, we can introduce the vortex factor to a one-dimensional model to study the effect of the vortex with rotation support on the structure and evolution of the planet. Spin in this state may have a significant influence on planet formation. This work is based on a simple core accretion model. However, Kurokawa & Tanigawa (2018) proposed that atmospheric recycling is a complex two-dimensional accretion model, which can slow or stall envelope accretion. Thus, atmospheric recycling can also change the accretion rate and the critical core.
mass. In addition, rotating planets may also be affected by the magnetic dynamo (Kong et al. 2019), in which gravitational harmonics would be compatible with the actual data from the Juno or Cassini spacecraft. Rotation works differently under different boundary conditions, so if we consider the evolution of the disk, it may lead to different results.

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