Abstract

We prove Csorba’s conjecture that the Lovász complex $\text{Hom}(C_5, K_n)$ of graph multimorphisms from the 5–cycle $C_5$ to the complete graph $K_n$ is $\mathbb{Z}/2\mathbb{Z}$–equivariantly homeomorphic to the Stiefel manifold, $V_{n-1,2}$, the space of (ordered) orthonormal 2–frames in $\mathbb{R}^{n-1}$. The equivariant piecewise-linear topology that we need is developed along the way.

Keywords: Stiefel manifolds, PL topology, group action, graph morphism complexes

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1. Introduction

In his remarkable proof of the Kneser conjecture [14], Lovász gave a lower bound for the chromatic number of a graph using equivariant algebraic topology. This was essentially done via a functor (the edge complex, defined below) from the category of graphs and graph morphisms to the category $G$–$\text{TOP}$ (topological spaces equipped with an action of the group $G$ and $G$–maps, i.e., continuous functions commuting with the given actions). In the case of the edge complex functor, the group $G$ is of order two, and the actions are free. The edge complex of the complete graph $K_n$ is (equivariantly homeomorphic to) the sphere $S^{n-2}$ with the antipodal action, and the punchline is provided by the Borsuk-Ulam theorem.

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In [6], Csorba observed that, for \( n = 2, n = 3, \) and \( n = 4 \), the 5–cycle complex (defined below) of the complete graph \( K_n \), denoted \( \text{Hom}(C_5, K_n) \), is equivariantly homeomorphic to the Stiefel manifold \( V_{n-1,2} \) of (ordered) orthonormal 2-frames in the Euclidean space \( \mathbb{R}^{n-1} \), and he conjectured that this was true for all \( n \). The non-equivariant version of Csorba's conjecture was proven by C. Schultz in [18], who also proved that \( \text{Hom}(C_5, K_n) \) is equivariantly homotopy equivalent to \( V_{n-1,2} \). In this note we give a proof of the equivariant Csorba conjecture: \( \text{Hom}(C_5, K_n) \) is equivariantly homeomorphic to \( V_{n-1,2} \) with respect to the actions described below. This provides a rather mysterious combinatorial model for the Stiefel manifolds \( V_{n-1,2} \) which are fundamental topological spaces.

The edge complex and the (5–)cycle complex are special cases of the Lovász graph multimorphism complex, which we proceed to define precisely now. A graph morphism \( f \) is a function from \( V_\Gamma \) to \( V_\Lambda \) (where \( V_\Gamma \) denotes the vertex set of the graph \( \Gamma \)) so that the image of an edge of \( \Gamma \) is an edge of \( \Lambda \). (In particular, a graph morphism from \( \Gamma \) to the complete graph \( K_n \) is just an admissible coloring of the vertices of \( \Gamma \).) A multimorphism from \( \Gamma \) to \( \Lambda \) is a relation \( \phi \subseteq V_\Gamma \times V_\Lambda \) such that:

(i) \( \phi(v) := \{ w \in V_\Lambda | (v, w) \in \phi \} \) is non-empty for all \( v \in V_\Gamma \), and

(ii) any function \( f \subseteq \phi \) is a graph morphism.

In other words, \( \phi \) is a multimorphism when there is a function \( f \subseteq \phi \) and any function \( f \subseteq \phi \) is a graph morphism.

The Lovász multimorphism complex \( \text{Hom}(\Gamma, \Lambda) \) [3] is a bifunctor (contravariant in the first variable, covariant in the second) assigning a regular cell complex to the (simple) graphs \( \Gamma, \Lambda \). Any regular cell complex is determined by its face poset ([5] p200, [15] Ch 3 §1), and the face poset of \( \text{Hom}(\Gamma, \Lambda) \) is the set of multimorphisms from \( \Gamma \) to \( \Lambda \) ordered by inclusion. Geometrically, the cells of the complex \( \text{Hom}(\Gamma, \Lambda) \) are products of simplices; the cell corresponding to the multimorphism \( \phi \):

\[
\prod_{v \in V_\Gamma} \Delta \phi(v)
\]

where \( \Delta A \) is the full simplex on the vertex set \( A \). \( \text{Hom}(\Gamma, \Lambda) \) is the union of all these cells indexed by the multimorphisms. We will identify each cell with the multimorphism \( \phi \) indexing it and refer to it as such. The vertices (0–cells) of \( \text{Hom}(\Gamma, \Lambda) \) are the graph morphisms from \( \Gamma \) to \( \Lambda \), the vertices of a given cell \( \phi \) are obtained by choosing any \( w \) from \( \phi(v) \) for each \( v \in V_\Gamma \).
The composition of two multimorphisms is also a multimorphism, hence \( \text{Hom}(\Gamma, \Lambda) \) is (bi-)functorial for multimorphisms (not just for morphisms, this extended functoriality was put to very good use in \([20]\)). That is, a multimorphism \( \alpha: \Gamma' \to \Gamma \) induces a (cellular) map \( \text{Hom}(\Gamma, \Lambda) \to \text{Hom}(\Gamma', \Lambda) \), and a multimorphism \( \beta: \Lambda \to \Lambda' \) induces a (cellular) map \( \text{Hom}(\Gamma, \Lambda) \to \text{Hom}(\Gamma, \Lambda') \).

If \( G \) and \( H \) are groups acting on the graphs \( \Gamma \) and \( \Lambda \) respectively, then there is an action of \( G \times H \) on \( V_\Gamma \times V_\Lambda \), which restricts to an action on \( \text{Hom}(\Gamma, \Lambda) \). In particular, a graph \( \Gamma \) equipped with an action of a group \( G \) defines the functor \( \text{Hom}(\Gamma, -) \) from the category of graphs and graph (multi-)morphisms to \( G-\text{TOP} \), the equivariant category of \( G \)-spaces and \( G \)-maps. Lovász’s proof of the Kneser conjecture \([14]\) essentially employs the \textbf{edge complex} functor \( \text{Hom}(K_2, -) \) with \( G \) being the automorphism group of (the edge) \( K_2 \), the complete graph on 2 vertices.

If we denote the vertices of \( K_2 \) with + and -, then the nontrivial element of \( G \) is the involution on \( \text{Hom}(K_2, \Lambda) \) switching \( \phi(+) \) and \( \phi(-) \), the subsets (of \( V_\Lambda \)) assigned to the two vertices of \( K_2 \). This action is free because \( \phi(+) \) and \( \phi(-) \) are distinct since they are nonempty and disjoint. In particular, \( \text{Hom}(K_2, K_n) \) is a poset consisting of pairs \((A, B)\) of nonempty disjoint subsets of \( \{1, \ldots, n\} \), ordered by component-wise inclusion, where \( A = \phi(+) \) and \( B = \phi(-) \). \( \text{Hom}(K_2, K_n) \) can be geometrically realized as the \((n-2)\)-sphere of radius two with respect to the \(L^1\) norm (inducing the taxicab metric) in the hyperplane orthogonal to the diagonal vector \((1, 1, \ldots, 1)\) in the Euclidean space \( \mathbb{R}^n \), where \( \phi(+) \) and \( \phi(-) \) are the sets of coordinates with positive and negative values respectively. Switching \( \phi(+) \) and \( \phi(-) \) yields the antipodal action.

Interestingly, long before the definition of \( \text{Hom}(\Gamma, \Lambda) \), the underlying spaces of \( \text{Hom}(K_m, K_n) \) figured prominently in two unrelated applications of equivariant algebraic topology (neither one involving chromatic numbers or graphs): in Alon’s elegant Necklace Splitting Theorem (with \( m \) prime) \([1]\) and the solution of the prime power case of the Topological Tverberg Problem (\([16]\) and \([22]\)), which was conjectured by Bárány, Shlossman, and Szűcs in \([2]\).

The Lovász conjecture (proven by Babson and Kozlov \([4]\), see also \([20]\)) is about the (odd) \textbf{cycle complexes} \( \text{Hom}(C_m, \Lambda) \) where \( C_m \) is the \( m \)-gon. \( \text{Hom}(C_m, \Lambda) \) also has an involution induced by a reflection (of the \( m \)-gon). When \( m \) is odd, any such reflection flips an edge and hence induces a free
action on $\text{Hom}(C_m, \Lambda)$ as above. The Lovász conjecture reduces to a computation involving the equivariant cohomology of $\text{Hom}(C_m, K_n)$.

A graph multimorphism $\phi$ in $\text{Hom}(\Gamma, \Lambda)$ is also determined by specifying $\phi^{-1}(w) := \{ v \in V_{\Gamma} \mid (v, w) \in \phi \}$ for each $w \in V_{\Lambda}$. Clearly, for any $\phi$, each $\phi^{-1}(w)$ is independent (i.e., no two elements are adjacent in $\Gamma$). The **independence complex** $\text{ind}(\Gamma)$ of a graph $\Gamma$ is the simplicial complex with vertex set $V_{\Gamma}$ and simplices being independent subsets of $V_{\Gamma}$. Equivalently, $\text{ind}(\Gamma)$ is the flag complex (the largest simplicial complex whose 1–skeleton is the given graph) of the edge complement of $\Gamma$ (the graph with the same vertices as $\Gamma$ but with the complementary edge set). Any flag complex is the independence complex of the edge complement of its 1–skeleton.

When $\Lambda = K_n$, the only condition on $\phi^{-1}(w)$ is being in $\text{ind}(\Gamma)$, so $\text{Hom}(\Gamma, K_n)$ consists of $\phi$ such that:

(i) $\phi(v)$ is non-empty for all $v \in V_{\Gamma}$, and

(ii) $\phi^{-1}(j)$ is independent for $j = 1, \ldots, n$.

We can identify $\text{Hom}(\Gamma, K_n)_{>\phi}$ with the face poset of the join (over $j = 1, \ldots, n$) of the links of the $\phi^{-1}(j)$ in $\text{ind}(\Gamma)$. Since $\text{Hom}(\Gamma, K_n)_{<\phi}$ is the face poset of the boundary of a product (over the vertices $v \in V_{\Gamma}$) of the simplices on $\phi(v)$, $\text{Hom}(\Gamma, K_n)$ is a (closed) manifold if $\text{ind}(\Gamma)$ is a PL (piecewise-linear) sphere. In fact, the converse is also true \[6\]. The complex $\text{Hom}(C_m, K_n)$ is a manifold only when $m = 5$ since this is the only time when $\text{ind}(C_m)$ is a sphere: $\text{ind}(C_3)$ is 3 points, $\text{ind}(C_4)$ is the disjoint union of two edges, $\text{ind}(C_5)$ is a pentagon, and $\text{ind}(C_m)$ with $m > 5$ has maximal simplices of different dimensions.

On the topological side, the orthogonal group $O_2$ acts on the Stiefel manifold $V_{n-1,2} := \{ (x, y) \in S^{n-2} \times S^{n-2} \mid x \cdot y = 0 \}$ (with the Grassmannian $Gr_{n-1,2}$ of 2–planes in $(n - 1)$–space as the quotient). The group $O_2$ is the semi-direct product of rotations $SO_2$ with a subgroup generated by an arbitrary reflection. Two natural involutions on $V_{n-1,2}$ are (i) $(x, y) \mapsto (x, -y)$ and (ii) $(x, y) \mapsto (y, x)$. Since any two reflections are conjugate via a rotation, these give equivalent actions. An explicit map $V_{n-1,2} \to V_{n-1,2}$ interchanging the actions (i) and (ii) is $(x, y) \mapsto \frac{1}{\sqrt{2}}(x + y, x - y)$. Schultz \[15\] used the action (i) on the Stiefel manifold. Below, we will use the (equivalent) action (ii). The corresponding involution on $\text{Hom}(C_5, K_n)$ is induced by any reflection of the pentagon $C_5$ (they all give equivalent actions).

It is convenient to work with a smaller model: $\text{Hom}_{\Gamma}(\Gamma, \Lambda)$, in which two multimorphisms are considered the same if their values differ only on
the independent set \( I \) of vertices. Thus, \( \text{Hom}_I(\Gamma, \Lambda) \) is the subcomplex of \( \text{Hom}(\Gamma \setminus I, \Lambda) \) consisting of the cells \( \phi \) that can be extended to \( \Gamma \). The projection from \( \text{Hom}(\Gamma, \Lambda) \) to \( \text{Hom}_I(\Gamma, \Lambda) \) is a homotopy equivalence \([6]\) since the fibers are contractible. It turns out that \( \text{Hom}(\Gamma, K_n) \) is homeomorphic to \( \text{Hom}_I(\Gamma, K_n) \) when \( \text{ind}(\Gamma) \) is a PL sphere \([18]\). For \( \text{Hom}_I(\Gamma, \Lambda) \) to inherit the \( G \)-action (induced from an action on \( \Gamma \), the set \( I \) needs to be (setwise) \( G \)-invariant. When \( G \) is the group of order two generated by the reflection on \( C_5 \) switching the vertex \( i \) with the vertex \( 6 - i \) for \( i \in \{1, 2, 3, 4, 5\} \), there are two \( G \)-invariant independent subsets: \( \{3\} \) and \( \{2, 4\} \). In \([18]\) Schultz uses \( I = \{2, 4\} \); we use \( I = \{3\} \). We need \( \text{Hom}_I(C_5, K_n) \) and \( \text{Hom}(C_5, K_n) \) to be equivariantly homeomorphic which we get from an equivariant version of a lemma of Schultz \([18]\).

While \( \text{Hom}(C_5, K_n) \) (or \( \text{Hom}_{\{3\}}(C_5, K_n) \), which is equivariantly homeomorphic to it) is the star of this story, the hero is \( \text{Hom}_{\{3\}}(P_4, K_n) \), where \( P_4 \) is the path with four edges from 1 to 5 (the subgraph of \( C_5 \) missing the edge \( \{1, 5\} \)). Our story also features \( P_4 \setminus \{3\} \), which is the disjoint union of two copies of \( K_2 \), namely the edges \( \{1, 2\} \) and \( \{4, 5\} \). The graph morphism \( P_4 \setminus \{3\} \to K_2 \) sending 1 and 5 to + and 2 and 4 to − induces the diagonal embedding of (the sphere) \( \text{Hom}(K_2, K_n) \) in \( \text{Hom}(P_4 \setminus \{3\}, K_n) = \text{Hom}(K_2, K_n) \times \text{Hom}(K_2, K_n) \). We also have the restriction map \( \text{Hom}_{\{3\}}(C_5, K_n) \to \text{Hom}_{\{3\}}(P_4, K_n) \) induced from the inclusion of \( P_4 \) in \( C_5 \) and the inclusion of \( \text{Hom}_{\{3\}}(P_4, K_n) \) into \( \text{Hom}(P_4 \setminus \{3\}, K_n) \). All these maps are equivariant with respect to the involutions induced from the aforementioned \( i \mapsto 6 - i \). (The action on \( \text{Hom}(K_2, K_n) \) is trivial since it maps homeomorphically to the diagonal in \( \text{Hom}(P_4 \setminus \{3\}, K_n) \), the fixed point set of the involution.)

Note that \( \text{Hom}_{\{3\}}(P_4, K_n) \) is not homeomorphic to \( \text{Hom}(P_4, K_n) \): We will show that \( \text{Hom}_{\{3\}}(P_4, K_n) \) is a \((2n - 4)\)-dimensional manifold with boundary while \( \text{Hom}(P_4, K_n) \) is not a manifold, having maximal cells of every dimension from \( 2n - 4 \) to \( 3n - 6 \). (Recall that \( n \geq 3 \).)

The face posets of these Lovász complexes above are actually very easy to describe concretely; all are made up of (ordered) quadruples of nonempty subsets \( A, B, C, D \) of \( \{1, \ldots, n\} \) satisfying further conditions. For any cell \( \phi \), \( A, B, C, D \) are \( \phi(1), \phi(2), \phi(5), \) and \( \phi(4) \) respectively. In \( \text{Hom}(P_4 \setminus \{3\}, K_n) \) we have \( A \cap B = \emptyset = C \cap D \). In \( \text{Hom}_{\{3\}}(P_4, K_n) \) we also have that \( B \cup D \neq \{1, \ldots, n\} \), and \( \text{Hom}_{\{3\}}(C_5, K_n) \) has the further restriction that \( A \cap C = \emptyset \). The diagonal \( \text{Hom}(K_2, K_n) \) has \( A = C \) and \( B = D \) (in addition to \( A \cap B = \emptyset \)).
We will denote these cells as

$$\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

to remind us of their locations in terms of the vertices of the pentagon $C_5$. Such an array should not be thought of as a $2 \times 2$ matrix.

On the topological side we have the involution on $S^{n-2} \times S^{n-2}$ switching the two coordinates, with the diagonal $S^{n-2}$ as the fixed set. An equivariant regular neighborhood of the diagonal is $N := \{(x, y) \in S^{n-2} \times S^{n-2} \mid x \cdot y \geq 0\}$, which is a manifold with boundary $V_{n-1,2} = \{(x, y) \in S^{n-2} \times S^{n-2} \mid x \cdot y = 0\}$. We prove in Section 5 that $\text{Hom}_{\{3\}}(P_4, K_n)$ is a manifold with boundary $\text{Hom}_{\{3\}}(C_5, K_n)$ and also an equivariant regular neighborhood of the diagonal in $\text{Hom}(P_4 \\setminus \{3\}, K_n) \cong \text{Hom}(K_2, K_n) \times \text{Hom}(K_2, K_n)$. From the equivariant PL theory of collapsing, shelling, and regular neighborhoods developed in Section 4, our main result follows:

**Main Theorem.** The regular cell complex $\text{Hom}_{\{3\}}(P_4, K_n)$ is a PL manifold with boundary $\text{Hom}_{\{3\}}(C_5, K_n)$ and is equivariantly homeomorphic (with respect to the involution described above) to $N := \{(x, y) \in S^{n-2} \times S^{n-2} \mid x \cdot y \geq 0\}$, where the involution on $N$ interchanges $(x, y)$ with $(y, x)$. The Stiefel manifold $V_{n-1,2} = \partial N$ is therefore equivariantly homeomorphic to $\text{Hom}_{\{3\}}(C_5, K_n)$, which is equivariantly homeomorphic to $\text{Hom}(C_5, K_n)$.

The structure of the rest of the paper is as follows: We establish notation and give basic definitions and prove that $\text{Hom}(C_5, K_n)$ and $\text{Hom}_{\{3\}}(C_5, K_n)$ are equivariantly homeomorphic in Section 2. The small amount of discrete Morse theory that we use is in Section 3. Section 4 is devoted to the stating the equivariant versions of facts we need from piecewise-linear topology given in [17]. Finally, in Section 5 we specialize to our setup and prove the equivariant Csorba conjecture.

2. Notation and Basics

All simplicial complexes and posets we consider are finite. An (abstract) simplicial complex $K$ on a vertex set $V$ is a collection of (finite) subsets of $V$ such that if $\sigma \subseteq \tau \in K$, then $\sigma \in K$. The dimension of the simplex $\sigma$ is one less than its cardinality, and the $n$–skeleton of the simplicial complex $K$ (denoted $K^n$) is the sub-simplicial complex made up of all simplices of $K$.
with dimension at most $n$. However, we will abuse the notation and use $K^0$ to also mean $\bigcup\{\sigma \mid \sigma \in K\}$. The nonempty simplices $\sigma \in K$ are also called the faces of $K$. The **geometric realization** of $K$ is the topological space

$$|K| := \left\{ \sum_{v \in \sigma} t_v \delta_v \in \mathbb{R}^V \mid \sum_{v \in \sigma} t_v = 1, t_v > 0, \sigma \in K \setminus \{\emptyset\} \right\}$$

where $\delta_v$ is the standard basis vector of $\mathbb{R}^V$ corresponding to $v \in V$. The underlying topological space of a face $\sigma$ of $K$ is

$$|\sigma| := \left\{ \sum_{v \in \sigma} t_v \delta_v \in \mathbb{R}^V \mid \sum_{v \in \sigma} t_v = 1, t_v \geq 0 \right\}.$$

Note that we are using vertical bars to denote both the cardinality of a finite set and the geometric realization of a simplex or a simplicial complex; which one is meant should be clear from the context.

If $\sigma$ is a simplex in a simplicial complex $K$, its **link** is defined $\text{lnk}_K(\sigma) := \{\tau \in K \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in K\}$. If $K$ and $L$ are two simplicial complexes, we define their **join** $K \ast L := \{\sigma \cup \tau \mid \sigma \in K, \tau \in L\}$.

A subcomplex $L$ of a simplicial complex $K$ is called **full** if it satisfies the property that if $\sigma \in K$ and $\sigma \subseteq L^0$, then $\sigma \in L$.

A **regular cell structure** on a (compact Hausdorff) topological space $X$ is a (finite) collection $\{c\}$ of subspaces (called cells or faces), each homeomorphic to a (closed) disk of dimension $d$ for some $d$, such that (1) $X$ is the disjoint union of the relative interiors of its cells (called the open cells and denoted by $\text{int } c$), and (2) the boundary of each cell is a union of (lower dimensional) cells. A topological space with a regular cell structure is a **regular cell complex**. For any simplicial complex $K$, the collection $\{|\sigma|\}_{\sigma \in K \setminus \{\emptyset\}}$ gives a regular cell structure on $|K|$.

For any poset $P$, its **order complex** $\Delta P$ is the simplicial complex whose simplices are chains $a_0 < a_1 < \ldots < a_d$, with $a_i \in P$. The faces of a regular cell complex under inclusion form a poset whose order complex is its barycentric subdivision ([5] p200, [15] Ch 3 §1). Note that this notation is consistent with $\Delta A$ being the full simplex on a vertex set $A$ if the set $A$ is endowed with any total order. Also note that an abstract simplicial complex $K$ is itself a poset and includes the empty simplex (unlike its face poset), so the order complex $\Delta K$ is the cone of the barycentric subdivision of $K$.

A **$G$–simplicial complex**, where $G$ is a (finite) group, is a simplicial complex $K$ equipped with a permutation action of $G$ on the vertex set $V$.
so that the induced action on the subsets of $V$ sends simplices to simplices. Similarly, in a $G$–cell complex, the group action permutes the cells. For any cell (or simplex) $c$, $G_c := \{g \in G \mid gc = c\}$ is called its **stabilizer**. It will be convenient for us to define a **$G$–regular cell complex** to be a topological space with a $G$–action and a regular cell structure with the group $G$ permuting the cells so that every closed cell is $G_c$–homeomorphic to a cone on its boundary with the apex fixed by $G_c$. The geometric realization of a $G$–simplicial complex $K$ is a $G$–regular cell complex with cells $\{|\sigma|\}_{\sigma \in K \setminus \emptyset}$ since the stabilizer of a simplex always fixes its centroid. Similarly, if the cells of a regular cell complex are convex with $G$ acting affinely on each one, the complex is $G$–regular. If the stabilizer of each cell (or simplex) fixes the cell pointwise, the complex is called **admissible**. If $G$ acts on a poset $P$ (preserving the order), then $\Delta P$ is an admissible $G$–simplicial complex.

Just as in the non-equivariant case ([5] p200, [15] Ch 3 §1), the face poset of a $G$–regular cell complex determines its $G$–homeomorphism type:

**Lemma 2.1.** If $X$ is a $G$-regular cell complex with face poset $F$, then $X$ is $G$–homeomorphic to $|\Delta F|$.

The proof below is very similar to the non-equivariant case and is mainly given for completeness.

**Proof.** The proof is by induction on the number of cells in $X$. There is nothing to prove if $X$ consists of a single 0–cell. Now, choose a maximal cell $c$ and define $Y := X \setminus \bigcup_{g \in G} \text{int} \ gc$ with the induced $G$–cell structure. By the induction hypothesis, $Y \approx G|\Delta(F \setminus Gc)|$. Also, $\partial c \approx G_c|\Delta F_{<c}|$. Then we take the cone of $\partial c$ with its apex being a point $x \in \text{int} \ c$ fixed by $G_c$. Extending this coning equivariantly to $G\partial c$ gives the homeomorphism from $|\Delta F|$ to $X$. \hfill \Box

We are specifically interested in the $G$–equivariant homeomorphism type of some Lovász multimorphism complexes $\text{Hom}(\Gamma, K_n)$, for a graph $\Gamma$ with a group of symmetries $G$. The following equivariant version of Lemma 3.5 in [15] enables us, in some cases, to work with the smaller and more convenient $G$–regular cell complexes $\text{Hom}_I(\Gamma, K_n)$, where $I$ is a $G$–invariant independent subset of vertices of $\Gamma$.

**Lemma 2.2.** Let $\Gamma$ be a graph with a $G$–action, $n \geq 1$, and $I$ a $G$–invariant, independent subset of the vertex set $V_{\Gamma}$. For all $v \in V_{\Gamma}$, define
\[ A_v := \{ J \in \text{ind}(\Gamma) \mid v \in J \}, \quad B_v := \{ J \setminus I \mid J \in A_v \} \]

If there is a \( G \)-homeomorphism \( h : |\Delta \text{ind}(\Gamma)| \to |\Delta \text{ind}(\Gamma \setminus I)| \) such that \( h(|\Delta A_v|) = |\Delta B_v| \) for all \( v \in V_\Gamma \), then \( \text{Hom}(\Gamma, K_n) \) is \( G \)-homeomorphic to \( \text{Hom}_I(\Gamma, K_n) \).

**Proof.** Following [18], we consider the equivariant poset embedding

\[ f : \text{Hom}(\Gamma, K_n) \to \prod_{i=1}^n \text{ind}(\Gamma) = \text{ind}(\Gamma)^{\{1, \ldots, n\}} \]

given by \( \phi \mapsto (\phi^{-1}(i))_i \). Then \( \text{Hom}(\Gamma, K_n) \approx_G \Delta \text{im } f \). The poset \( \text{ind}(\Gamma)^{\{1, \ldots, n\}} \) can naturally be identified with those relations \( \phi \subseteq V_\Gamma \times \{1, \ldots, n\} \) that are multimorphisms from the induced subgraph on the vertices with \( \phi(v) \) nonempty to the complete graph \( K_n \). The additional condition that no \( \phi(v) \) can be empty yields the following description:

\[ \text{im } f = \bigcap_{v \in V_\Gamma} \bigcup_{j=1}^n \prod_{i=1}^n \begin{cases} A_v, & i = j \\ \text{ind}(\Gamma), & i \neq j \end{cases} \]

All the \( A_v \) satisfy the condition that if \( x \in A_v \) and \( x \leq y \), then \( y \in A_v \). Therefore, taking the order complex commutes with unions, and we obtain that

\[ \text{Hom}(\Gamma, K_n) \approx_G \bigcap_{v \in V_\Gamma} \bigcup_{j=1}^n \prod_{i=1}^n \begin{cases} |\Delta A_v|, & i = j \\ |\Delta \text{ind}(\Gamma)|, & i \neq j \end{cases} \]

We use a similar argument for \( \text{Hom}_I(\Gamma, K_n) \). The image of its embedding in \( \text{ind}(\Gamma \setminus I)^{\{1, \ldots, n\}} \) has the additional condition that for each vertex in \( I \), there is some element of \( \{1, \ldots, n\} \) that is not related to any of its neighbors in \( \Gamma \). We have that, for \( v \notin I \), \( B_v \) satisfies the same condition as \( A_v \) above. For \( v \in I \), \( B_v \) also satisfies the condition that if \( x \in B_v \) and \( x \leq y \), then \( y \in B_v \). Hence,

\[ \text{Hom}_I(\Gamma, K_n) \approx_G \bigcap_{v \in V_\Gamma} \bigcup_{j=1}^n \prod_{i=1}^n \begin{cases} |\Delta B_v|, & i = j \\ |\Delta \text{ind}(\Gamma \setminus I)|, & i \neq j \end{cases} \]

Thus, using the \( G \)-homeomorphism from the hypothesis on each coordinate in the product, we obtain that \( \text{Hom}(\Gamma, K_n) \approx_G \text{Hom}_I(\Gamma, K_n) \). \( \square \)
3. Discrete Morse Theory

We will use Robin Forman’s Discrete Morse Theory to find collapsings of simplicial complexes. More thorough discussions of the subject can be found in [8] and [12].

Let $K$ be a finite (abstract) simplicial complex. We use the notation $\sigma \preceq \tau$ if $\sigma < \tau$ and $\dim \sigma = \dim \tau - 1$. By a vector, we mean a pair $(\sigma \preceq \tau)$, where $\tau$ is thought of as the head of the vector and $\sigma$ the tail. A discrete vector field on $K$ is defined to be a collection of vectors $V = \{(\sigma_i \preceq \tau_i) | i \in I\}$ such that each simplex $\rho \in K$ belongs to at most one element of $V$, either as a head or a tail of a vector.

Given a discrete vector field $V$ on $K$, we have the notion of a path, which is a sequence of simplices in $K$ of the form:

$$\sigma_0 \preceq \tau_0 \succ \sigma_1 \preceq \tau_1 \succ \ldots \succ \sigma_{s-1} \preceq \tau_{s-1} \succ \sigma_s$$

where $\forall i : 0 \leq i < s$, $(\sigma_i \preceq \tau_i) \in V$ and $\sigma_i \neq \sigma_{i+1}$. We say a path as above has length $s$. By a cycle we mean a path as above with $\sigma_s = \sigma_0$.

A Morse matching (or a discrete gradient field) is a discrete vector field $V$ with no cycles. The simplices which are unpaired in $V$ are called critical.
An equivalent concept to a Morse matching is a height function on $K$. A **height** (or **Morse**) function is a map $h: K \to \mathbb{R}$ satisfying $\forall \sigma \in K,$

$$|\{\rho \lessdot \sigma \mid h(\rho) \geq h(\sigma)\} \cup \{\tau \rhd \sigma \mid h(\tau) \leq h(\sigma)\}| \leq 1$$

Given a height function $h$, the corresponding Morse matching is the collection of pairs $(\sigma \lessdot \tau)$ for which $h(\sigma) \geq h(\tau)$. Conversely, it is not difficult to construct a height function inducing a given Morse matching ([8], [12]). This height function is clearly not unique. In fact, we may adjust it to be one-to-one and to take values in $\mathbb{N}$ (without changing the Morse matching).

The following is the key lemma from discrete Morse theory we will use in this paper. A more general version for cell complexes can be found in [12], Theorem 11.13.

**Lemma 3.1.** Let $K$ be a finite simplicial complex with a Morse matching whose critical simplices form a subcomplex $L$. Then $K$ collapses simplicially to $L$.

**Proof.** Let $h: K \to \mathbb{N}$ be a one-to-one height function corresponding to the given Morse matching. Define a new height function $\tilde{h}: K \to \mathbb{N}$ as follows:

For $\sigma \in L$, set $\tilde{h}(\sigma) = \dim(\sigma)$. For $\sigma \notin L$, set $\tilde{h}(\sigma) = h(\sigma) + \dim(L)$. Under this new height function, all of the simplices in $L$ remain critical, and the relative heights of all the simplices outside of $L$ are unchanged, preserving their pairings. Therefore, $\tilde{h}$ corresponds to the same Morse matching as $h$, and $\tilde{h}$ is one-to-one on $K \setminus L$.

Now, for $m \in \mathbb{N}$ define

$$K(m) := \{\sigma \in K \mid \exists \tau \geq \sigma \text{ such that } \tilde{h}(\tau) \leq m\}$$

Note that $K(\dim(L)) = L$. For $m \geq \dim(L)$, either $K(m + 1) = K(m)$ or $K(m + 1) = K(m) \cup \{\rho, \tau\}$ where $\rho \lessdot \tau$ and $\tilde{h}(\tau) = m + 1 < \tilde{h}(\rho)$. In the latter case, $K(m + 1)$ collapses to $K(m)$ along the free face $\rho$. Since $K = K(\max\{\tilde{h}(\sigma) \mid \sigma \in K\})$, $K$ collapses to $L$ via a sequence of these collapsings. \qed

4. **Equivariant Neighborhoods**

In this section, we discuss the equivariant theory of $G$-regular neighborhoods with suitable hypotheses, similar to Rourke and Sanderson’s nonequivariant version [17]. Throughout, we assume $G$ is a finite group, all
simplicial complexes are finite, and maps between polyhedra are piecewise-linear. Also, when there is a $G$–action on a space $X$, the product $X \times I$ is assumed to have the $G$–action $g(x,t) = (gx,t)$.

A simplicial complex $K$ **polyhedrally triangulates** a space $X \subset \mathbb{R}^n$ if the 0–skeleton $K^0$ is identified with a subset of $X$ so that the canonical piecewise-linear function from $|K|$ to $\mathbb{R}^n$ mapping $|\sigma|$ of each simplex $\sigma$ to the convex hull of its vertices in $X$ is a homeomorphism onto $X$. Then $X$ is called a **polyhedron**, and two polyhedra are equivalent if there is a piecewise-linear homeomorphism between them. If $K$ is a $G$–complex, the ambient vector space $\mathbb{R}^n$ into which the polyhedron $X$ is $G$–embedded is the underlying space of a (real) linear representation of the group $G$, and the piecewise-linear homeomorphism from $|K|$ to $X$ is a $G$–map, then we have a **polyhedral $G$–triangulation** of a $G$–polyhedron $X$. We implicitly identify each simplex $\sigma$ of $K$ with the image of its geometric realization in $X$. Note that one abstract ($G$–)simplicial complex may triangulate $X$ in multiple ways by having different choices for some of the vertices in $X$, in which case we will give the simplicial complex different names. We will only consider polyhedral triangulations.

Let $Y \subset X$ be polyhedra and $L$ be a subcomplex of $K$ such that $(K, L)$ triangulates $(N, Y)$ where $N$ is a neighborhood of $Y$ in $X$. Define the **simplicial neighborhood** of $L$ in $K$, $N_K(L) := \{ \sigma \in K | \exists \tau \in K \text{ s.t. } \sigma < \tau \text{ and } \tau \cap L^0 \neq \emptyset \}$. Also, define $\hat{N}_K(L) := \{ \sigma \in N_K(L) | \sigma \cap L^0 = \emptyset \}$. The **derived subdivision** $K'$ of $K$ near $L$ is (combinatorially) the simplicial complex with vertex set $K^0 \cup \{v_\tau | \tau \in K \setminus L, \tau \cap L^0 \neq \emptyset \}$, and the simplices are of the form $\sigma \cup \{v_{\tau_1}, \ldots, v_{\tau_m}\}$ where $\sigma \in L$ or $\sigma \in K$ with $\sigma \cap L^0 = \emptyset$ and $\sigma < \tau_1 < \ldots < \tau_m$. Geometrically, $K'$ is realized by selecting the new vertex $v_\tau$ in the interior of each simplex $\tau$ of $K \setminus L$ that intersects a simplex of $L$ and then, in ascending order of dimension, replacing each $\tau$ with the cone (with apex $v_\tau$) on its boundary (which has already been subdivided in the previous steps).

Now suppose $L$ is full in $K$ (i.e., if a set of vertices in $L$ forms a simplex in $K$, then they form a simplex in $L$) and $|\hat{N}_K(L)| = \partial |N_K(L)|$ in $X$. Then $N_1 = |N_{K'}(L)|$ is called a **regular neighborhood** of $Y$ in $X$. If $K$ and $L$ are both admissible $G$–complexes and, when defining $K'$, the set of new vertices $\{v_\tau\}$ is chosen to be $G$–invariant, we say $N_1$ is a $G$–**regular neighborhood**.

**Proposition 4.1.** If $K$ is a $G$–simplicial complex, $|K| \times [0,1]$ has an admissible $G$–triangulation with no new vertices in $|K| \times (0,1)$. 

Proof. Let $F$ be the face poset of $K$. Then the order complex $\Delta(F \times \{0, 1\})$ of the product poset $F \times \{0, 1\}$ satisfies $|\Delta(F \times \{0, 1\})| \approx_G |\Delta F| \times |\Delta \{0, 1\}| \approx_G |K| \times I$. Hence $\Delta(F \times \{0, 1\})$ gives the desired $G$–triangulation. \hfill \Box

Let $\mathcal{K} = \{K_1, K_2, \ldots, K_s\}$ be a collection of simplicial complexes, each one triangulating the polyhedron $X$. For $\phi \in K_1 \times \ldots \times K_s$, let $|\phi| := \cap_{1 \leq i \leq s} |\phi_i|$. Define an equivalence relation on $K_1 \times \ldots \times K_s$ by $\phi \sim \psi$ if $|\phi| = |\psi|$. We define a poset $C_\mathcal{K} := \{\phi \in K_1 \times \ldots \times K_s | |\phi| \neq \emptyset\}/\sim$ with the partial order $[\phi] \leq [\psi] \iff |\phi| \subseteq |\psi|$. We will identify an equivalence class $[\phi]$ with the geometric realization $|\phi|$ of its representatives.

**Proposition 4.2.** Given a collection $\mathcal{K} = \{K_1, \ldots, K_s\}$ of simplicial complexes that triangulate a polyhedron $X$, then $C_\mathcal{K}$ is a regular cell structure on $X$, so that $|\Delta C_\mathcal{K}|$ is a triangulation of $X$ and a common subdivision of $K_1, \ldots, K_s$.

**Proof.** We show first that the open cells of $C_\mathcal{K}$ are disjoint. In this discussion, let $|\phi| = \cap_{1 \leq i \leq s} |\sigma_i|$ and $|\psi| = \cap_{1 \leq i \leq s} |\tau_i|$. Note that if $|\phi| \cap |\psi| \neq \emptyset$, we have that $|\phi| \cap |\psi| = \cap_{1 \leq i \leq s} (|\sigma_i| \cap |\tau_i|) = \cap_{1 \leq i \leq s} |\sigma_i \cap \tau_i|$. Thus, the intersection of any two closed cells is a closed cell. If $[\phi] < [\psi]$, then, without changing $[\psi]$ or $[\phi]$, we may replace each $\tau_i$ with its minimal face containing $|\psi|$ and each $\sigma_i$ with $\sigma_i \cap \tau_i$. Then $|\psi| \cap \text{int}(|\tau_i|)$ is always nonempty, and for some $i$ we have $|\phi| \subseteq |\partial \tau_i|$. Therefore we have that $|\phi| \subseteq \partial |\psi|$, implying that the intersection of any two distinct closed cells must occur on the boundary of at least one of them, so any two distinct open cells are disjoint.

Each $|\psi|$ is a (nonempty) compact, convex polytope, yielding that $|\psi| \approx D^m$ for some $m \geq 0$. Furthermore, $\partial |\psi| = \bigcup_{[\phi] < [\psi]} |\phi|$: If $x \in \partial |\psi| \subseteq X$, there is a unique $\sigma_i \in K_i$ such that $x \in \text{int}(|\sigma_i|)$, giving some $\phi$ with $x \in |\phi|$ and $[\phi] \leq [\psi]$. However $[\phi] < [\psi]$, $\partial |\psi|$ is not in $\text{int}(|\psi|)$, so there is an $i$ with $x$ not in $\text{int}(|\tau_i|)$, i.e., $\sigma_i \neq \tau_i$. Conversely, if $x \in \bigcup_{[\phi] < [\psi]} |\phi|$, we have $x \in |\phi| \subseteq \partial |\psi|$ as above. This proves that $C_\mathcal{K}$ is a regular cell structure on $X$. Therefore, we have that $|\Delta C_\mathcal{K}| \approx X$. \hfill \Box

**Corollary 4.3.** If $X$ is a $G$–polyhedron and $K$ is a nonequivariant triangulation of $X$, then there is an admissible $G$–triangulation of $X$ which is a subdivision of $K$.

**Proof.** For each $g \in G$, $gK$ is another triangulation of $X$ because $G$ is acting linearly on the ambient representation space. Let $\mathcal{K} = \{gK | g \in G\}$. Using
the notation from the proof of \[\text{4.2}\] each cell \([\phi]\) is given by a map \(\phi : G \to K\). Then \(|\phi| = \bigcap_{g \in G} g|\phi(g)| \subseteq X\). For \(h \in G\), define \((h\phi)(g) := \phi(h^{-1}g)\). This induces an order-preserving \(G\)-action on \(C_K\) because, for any \(\phi\) and any \(h\) in \(G\), \(|h\phi| = \bigcap_{g \in G} g|\phi(h^{-1}g)| = \bigcap_{g \in G} hg|\phi(g)| = h\bigcap_{g \in G} g|\phi(g)| = h|\phi|\). Lastly, \(C_K\) is a \(G\)-regular cell complex because the average of the vertices of any cell is fixed by the cell’s stabilizer. The result follows by Lemma 2.1. □

**Lemma 4.4.** Let \((K, L)\) be an admissible \(G\)-triangulation of \((X, Y)\) with \(L\) full in \(K\) and let \((K_1, L_1)\) be a \(G\)-subdivision of \((K, L)\). Then \(\exists\ G\)-derived subdivisions \(K'\) and \(K'_1\) of \(K\) and \(K_1\) near \(L\) and \(L_1\) such that \(|N_{K'}(L)| = |N_{K'_1}(L_1)|\).

*Proof.* We follow the proof of the non-equivariant version, Lemma 3.7 in [17], and clarify some details with the combinatorial definition of derived subdivisions. We define a map \(f_{L,K} = f : X \to [0, 1]\) by setting \(f(v) = 0\) if \(v \in L^0\) and \(f(v) = 1\) if \(v \in K^0 \setminus L^0\) gives a map on the vertices of \(K\) which can be linearly extended on simplices. Choose \(\epsilon\) small enough so that no vertex of \(K_1 \setminus L_1\) is contained in \(f^{-1}[0, \epsilon]\). Then choose \(G\)-derived subdivisions \(K'\) and \(K'_1\) of \(K\) and \(K_1\) near \(L\) and \(L_1\) respectively with all the new vertices \(v_\tau\) lying in \(f^{-1}(\epsilon)\). We can choose these vertices equivariantly because \(f\) is \(G\)-invariant and \(K\) is admissible. Now we will show that \(|N_{K'}(L)| = f^{-1}[0, \epsilon] = |N_{K'_1}(L_1)|\).

The map \(f\) takes values of 0 or \(\epsilon\) on all of the vertices of \(N_{K'}(L)\) and \(N_{K'_1}(L_1)\), so both of these neighborhoods are contained in \(f^{-1}[0, \epsilon]\). Now let \(x\) be a point in \(f^{-1}[0, \epsilon] \subset |K'| = |K'_1|\); say \(x\) is in the interior of the simplex \(\sigma \cup \{v_{\tau_1}, \ldots, v_{\tau_k}\}\) of \(K'\) (respectively \(K'_1\)) as in the combinatorial definition of a derived subdivision. Then \(x = s_0v_0 + \ldots + s_kv_k + t_1v_{\tau_1} + \ldots + t_mv_{\tau_m}\) where \(\sigma = \{v_0, \ldots, v_k\}\) and \(s_0 + \ldots + s_k + t_1 + \ldots + t_m = 1\). Suppose \(\sigma\) is not in \(L\) (resp. \(L_1\)). Then \(f(v_i) = 1\) (resp. \(f(v_i) > \epsilon\)) for \(i = 0, \ldots, k\). Meanwhile, \(f(v_{\tau_j}) = \epsilon\) for \(j = 1, \ldots, m\). We have then, in both cases, that \(f(x) \geq (s_0 + \ldots + s_k)\epsilon + (t_1 + \ldots + t_m)\epsilon = \epsilon\). This is a contradiction, and \(\sigma\) must be in \(L\) (resp. \(L_1\)), yielding that \(x\) is in \(|N_{K'}(L)|\) (resp. \(|N_{K'_1}(L_1)|\)). □

**Lemma 4.5.** Let \(v\) be a vertex of a \(G\)-simplicial complex \(K\). If \(K'\) is a derived \(G\)-subdivision of \(K\) near \(v\), then \(|\lnk_{K'}(v)|\) is \(G_v\)-homeomorphic to \(|\lnk_K(v)|\).

*Proof.* Assume that the derived vertices are chosen in \(f_{v,K}^{-1}(\epsilon)\) for some \(\epsilon \in (0, 1)\). Then a point in \(|\lnk_{K'}(v)| = f^{-1}(\epsilon)\) is of the form \(\epsilon u + (1 - \epsilon)v\) where
\( u \in |\text{link}_K(v)| \). Mapping this point to \( u \) gives the desired homeomorphism.

**Theorem 4.6.** If \( N_1 \) and \( N_2 \) are \( G \)-regular neighborhoods of \( Y \) in \( X \), then there exists a \( G \)-homeomorphism \( h : X \to X \) that maps \( N_1 \) to \( N_2 \) and is the identity on \( Y \).

**Proof.** The proof mirrors that of the non-equivariant version, Theorem 3.8 in [17]. We have \( N_i = |N_{K_i}(L_i)| \) for \( i = 1, 2 \) as above, where \( K_i \) are \( G \)-triangulations of a neighborhood of \( Y \) in \( X \). As in [17], \( K_1 \) and \( K_2 \) may be subdivided and extended to triangulations of \( |K_1| \cup |K_2| \). Applying [4.2], there is a triangulation \( K_0 \) of \( |K_1| \cup |K_2| \) containing subdivisions of \( K_1 \) and \( K_2 \) and inducing \( L_0 \), a common subdivision of \( L_1 \) and \( L_2 \). By [4.3], \( K_0 \) may be assumed to be a \( G \)-triangulation. By [4.4] for \( i = 1, 2 \), we can find \( G \)-derived subdivisions \( \tilde{K}_i \) and \( \tilde{K}_0 \) of \( K_i \) and \( K_0 \) near \( L_i \) and \( L_0 \) respectively such that \( |N_{\tilde{K}_i}(L_i)| = |N_{\tilde{K}_0}(L_0)| \). Therefore we have that \( N_1 \approx_G |N_{\tilde{K}_1}(L_1)| = |N_{\tilde{K}_0}(L_0)| \approx_G |N_{\tilde{K}_2}(L_2)| = N_2 \) where each \( G \)-homeomorphism is induced by a \( G \)-isomorphism of \( G \)-derived subdivisions. Such an isomorphism is given by changing the placement of each derived vertex \( v \) within the interior of a simplex \( \tau \) touching \( Y \) and fixing the placement of every other vertex in the subdivision, so the corresponding \( G \)-homeomorphism is the identity on \( Y \) itself. Thus, the same is true of the composition of these \( G \)-homeomorphisms. \)

Let \( Y \subset X \) be \( G \)-polyhedra. A **\( G \)-collar** on \( Y \) in \( X \) is a PL \( G \)-embedding \( c : Y \times I \to G X \) such that \( c(y, 0) = y \) and \( c(Y \times [0, 1]) \) is an open neighborhood of \( Y \) in \( X \). Suppose that for every \( a \in Y \) there are (closed, polyhedral) neighborhoods \( U \) and \( V \) of \( a \) in \( X \) and \( Y \) respectively with \( U \cap Y = V \), such that for any \( g \in G \), \( gU \cap U \neq \emptyset \) implies that \( ga = a \) and \( gU = U \). Suppose further that \( U \approx_{G_a} V \times I \) with \( v \mapsto (v, 0) \) on \( V \). Then we say that \( Y \) is **locally \( G \)-collarable** in \( X \) and we have that \( GU \approx_G GV \times I \). Local \( G \)-collarability is equivalent to \( G \)-collarability.

**Theorem 4.7.** Suppose \( Y \subset X \) is locally \( G \)-collarable. Then there is a \( G \)-collar on \( Y \) in \( X \).

**Proof.** The proof is essentially the same as that of the non-equivariant version, Theorem 2.25 in [17], substituting local \( G \)-collars for local collars.

Construct a new \( G \)-polyhedron \( Z := X \cup Y \times [-1, 0] \) by attaching a \( G \)-collar to \( Y \) outside of \( X \), identifying \( Y \subset X \) with \( Y \times \{0\} \). We will construct
a $G$–homeomorphism between $X$ and $Z$ which carries $Y$ to $Y \times \{-1\}$. Then the preimage of $Y \times [-1,0]$ will be a $G$–collar on $Y$ in $X$.

For each $a \in Y$, let $GV_a \times I$ be a local $G$–collar at $a$. Using compactness, cover $Y$ with the interiors of finitely many $GV_{a_1}, \ldots, GV_{a_k}$. Then for each $i$, we will define a $G$–homeomorphism $h_i : Z \rightarrow Z$ which maps the interior of $V_{a_i} \times \{0\}$ into the interior of $V_{a_i} \times [-1,0]$ and is the identity outside of $GV_{a_i} \times [-1,1]$. To do this, we consider a $G$–triangulation $K_{a_i}$ of $GV_{a_i}$. Taking the product of this triangulation with an interval defines a regular $G$–cell structure on $GV_{a_i} \times [-1,1]$. We now equivariantly subdivide this cell complex by its order complex: We must choose a new vertex $v_c$ in the interior of each cell $c$. First, equivariantly select a point $y_\sigma$ in the interior of each simplex $\sigma \in K_{a_i}$. For the cell $|\sigma| \times \{1\}$, choose $v_c = (y_\sigma, 1)$. Likewise, for the cell $|\sigma| \times \{-1\}$, choose $v_c = (y_\sigma, -1)$. Finally, for the cell $|\sigma| \times [-1,1]$, choose $v_c = (y_\sigma, 0)$.

Define a different subdivision simply by moving $v_c$ to $(y_\sigma, -\frac{1}{2})$ when $c$ intersects the relative interior of $GV_{a_i} \times [0,1]$ and making no change in the placement for $v_c$ otherwise. Then the $G$–homeomorphism $h_i$ is given by mapping the first subdivision to the second, since they are both realizations of the same order complex. Note that $h_i$ is the identity except on the relative interior of $GV_{a_i} \times [-1,1]$ where $h_i(y,t) = (y,s)$ with $s < t$. Thus, $h_i$ maps all of the relative interior of $GV_{a_i} \times \{0\}$ into $GV_{a_i} \times (-1,0)$.

Now define $h$ to be the composition $h_k \circ \ldots \circ h_1$. Since the interiors of the local $G$–collars cover all of $Y$, $h$ maps all of $Y \times \{0\}$ into $Y \times (-1,0)$. Let $K$ $G$–triangulate $Y$, and thus also $h(Y \times \{0\})$. Then we consider $h(X) \cap [-1,0]$. 

Figure 2: Construction of $h_i$
It has a regular cellular $G$–structure with a face poset isomorphic to that of $|K| \times [-1,0]$. The cells in the former come in three types: simplices of $K$ triangulating $Y \times \{0\}$ (which correspond to the same in the latter complex), simplices of $K$ triangulating $h(Y \times \{0\})$ (which correspond to simplices of $K$ triangulating $Y \times \{-1\}$), and the intersection of $h(X)$ with cells $|\sigma| \times [-1,0]$ for $\sigma \in K$ (which correspond to the cells $|\sigma| \times [-1,0]$). Therefore, $h(X) \cap [-1,0]$ is $G$–homeomorphic to $Y \times [-1,0]$, fixing $Y \times \{0\}$.

We extend this homeomorphism by the identity to the rest of $X$ to get the desired $G$–homeomorphism $\tilde{h}: X \to Z$ carrying $Y$ to $Y \times \{-1\}$. 

**Theorem 4.8.** If $Y \subset X$ is locally $G$–collarable then a $G$–regular neighborhood of $Y$ in $X$ is a $G$–collar.

**Proof.** We have by [4.7] that $Y$ is $G$–collarable. Let $L = L \times 0$ be an admissible $G$–triangulation of $Y$. Let $K$ be the $G$–triangulation of the $G$–collar $Y \times I$ as in [4.1]. Now choose a $G$–derived subdivision $K'$ of $K$ near $L$ such that all of the new vertices have second coordinate $\frac{1}{2}$. Then $|N(L,K')| = Y \times [0,\frac{1}{2}] \approx_G Y \times I$. The result now follows from [4.6].

We will also make use of the notion of bicollarability. We say $Y \subset X$ is $G$–bicollarable in $X$ if there exists a $G$–embedding of $Y \times [-1,1]$ into $X$ with $(y,0) \mapsto y$ for all $y \in Y$ and $Y \times (-1,1)$ mapping to an open neighborhood of $Y$ in $X$.

**Theorem 4.9.** If $N = |N_{K'}(L)|$ is a $G$–regular neighborhood of $Y$ in $X$, then $|\tilde{N}_{K'}(L)|$ is $G$–bicollarable in $X$.

**Proof.** By [4.6] it suffices to consider the case $N = f_{L,K}^{-1}[0,\epsilon]$ for some $\epsilon \in (0,1)$. That is, the derived vertices $\{v_\tau\}$ of $K'$ were chosen in $f^{-1}(\epsilon)$. Let $0 < \epsilon_1 < \epsilon < \epsilon_2 < 1$. Equivariantly, choose alternate derived vertices $\{v^1_\tau\}$ and $\{v^2_\tau\}$ in $f^{-1}(\epsilon_1)$ and $f^{-1}(\epsilon_2)$ respectively, giving derived $G$–subdivisions of $K_1'$ and $K_2'$ of $K$ near $L$. Then there are the natural homeomorphisms $h_i: |\tilde{N}_{K'}(L)| \to G |\tilde{N}_{K'_i}(L)|$ given by sending each $v_\tau$ to $v^i_\tau$. We now define a $G$–bicollar $C: |\tilde{N}_{K'}(L)| \times [-1,1] \to_G cl(|N_{K'_2}(L)| \setminus |N_{K'_1}(L)|)$ by setting

$$C(x,t) = \begin{cases} \left[t|h_1(x) + (1 - |t|)x, \quad -1 \leq t \leq 0 \right. \\ th_2(x) + (1 - t)x, \quad 0 \leq t \leq 1 \end{cases}$$

17
Note that this $G$–bicollarability can alternatively be expressed as $|\check{N}_{K'}(L)|$ being $G$–collarable in both $N$ and in $\text{cl}(X \setminus N)$.

For the remainder of the section, we wish to consider $G$–regular neighborhoods within manifolds. For that purpose, we need to define a $G$–manifold. We will consider $G$–polyhedra and $G$–complexes that are manifolds and that have particularly well-behaved $G$–actions.

First, consider an orthogonal representation $\rho: G \to O_n(\mathbb{R})$. We denote by $S(\rho)$ and $D(\rho)$ the unit sphere and disk respectively in the corresponding representation space. Further, denote by $S_+(\rho)$ the hemisphere with last coordinate nonnegative and similarly for $D_+(\rho)$. These have unique piecewise-linear structures coming from their smooth structures \cite{9}.

We now inductively define a combinatorial $G$–sphere. $S^0$ with a $G$–action is a combinatorial 0–dimensional $G$–sphere. An admissible simplicial $G$–complex $K$ with $|K|$ (PL) $G$–homeomorphic to $S(\rho)$ for some $\rho: G \to O_{n+1}(\mathbb{R})$ is an $n$–dimensional combinatorial $G$–sphere if for every $v \in K^0$, $\text{lnk}_K(v)$ is an $(n-1)$–dimensional combinatorial $G_v$–sphere, itself $G_v$–homeomorphic to $S(\mathbb{R}v^\perp)$, where $\mathbb{R}v^\perp$ is the orthogonal complement in $\rho|_{G_v}$ of the trivial representation $\mathbb{R}v$.

Similarly, we may define a combinatorial $G$–hemisphere by substituting $S_+(\rho)$ and allowing links of vertices to be $n$–dimensional $G$–spheres or $G$–hemispheres in the above definition. Finally, a combinatorial $G$–disk is simply the cone on a $G$–sphere or $G$–hemisphere with a $G$–fixed point.

An admissible simplicial $G$–complex $K$ is an $n$–dimensional combinatorial $G$–manifold if for every $v \in K^0$, $\text{lnk}_K(v)$ is an $(n-1)$–dimensional combinatorial $G_v$–sphere or hemisphere. A $G$–polyhedron $M$ is an $n$–dimensional (PL) $G$–manifold (with boundary) if its admissible $G$–triangulations are $n$–dimensional combinatorial $G$–manifolds. (If it is true for one, it is true for all.) The boundary $\partial M$ is easily shown to be a $G$–submanifold.

**Proposition 4.10.** Let $M$ be an $n$–dimensional $G$–manifold and $M_1$ be an $n$–dimensional $G$–invariant submanifold with $\text{cl}(\partial M_1 \cap \text{int} M)$ $G$–bicollarable in $M$. Then $M_1$ is a $G$–manifold.

**Proof.** Let $K$ be a $G$–triangulation of $M$ with subcomplexes $(K_1, L)$ triangulating $(M_1, \partial M_1)$. We need to show that the link of any vertex $v \in K^0_1$ is a $G_v$–sphere or hemisphere. We consider the case $v \in \text{cl}(\partial M_1 \cap \text{int} M)$. The link of any other vertex of $K_1$ is the same in both $K$ and $K_1$.

Since $\partial M_1$ is $G$–bicollarable, we may consider a closed $G_v$–invariant neighborhood $|\text{st}_L(v)| \times [-1, 1] = U_v$ with $(x, 0)$ identified with $x$ for all $x \in$
| $st_L(v)$ | and | $st_{L_i}(v) \times [0, 1] \subset M_1$. Triangulate $U_v$ in the following way: First triangulate $|\lnk_L(v)| \times [-1, 1]$. Add it to coning $|\lnk_L(v)| \times \{-1\}$ and $|\lnk_L(v)| \times \{1\}$ with $(v, -1)$ and $(v, 1)$ respectively. Let $J$ be this triangulation of $|\lnk_L(v)| \times [-1, 1] \cup |st_L(v)| \times \{-1, 1\}$. Then $J \ast \{v\}$ triangulates $|st_L(v)| \times [-1, 1]$. 

Now subdivide $K$ so that it contains a subdivision of $J \ast \{v\}$ as a subcomplex. Choose a derived $G_v$–subdivision $K'$ near $v$. From the construction in the proof of 4.4 we may assume $|st_K'(v)|$ is an $\epsilon$–neighborhood of $v$ in $|J \ast \{v\}|$. Thus, by 4.5 $|\lnk_L(v)| \times [-1, 1] \cup |st_L(v)| \times \{-1, 1\}$ is $G_v$–homeomorphic to $|\lnk_K'(v)|$, which we know is an $(n-1)$–$G_v$–sphere or hemisphere since $M$ is a $G$–manifold.

Consider the point $w = (v, 1)$. Its stabilizer in $G_v$ is all of $G_v$. Hence, $\lnk_I(w)$ is an $(n-2)$–$G_v$–sphere or hemisphere. But $|\lnk_I(w)|$ is $G_v$–homeomorphic to $|\lnk_L(v)|$, giving us that $|J|$ is the suspension of $|\lnk_L(v)|$ by $w$ and $(v, -1)$, and a $G_v$–sphere or hemisphere. In conclusion, $|\lnk_L(v)| \times [0, 1] \cup |st_L(v)| \times \{1\}$, which is $G_v$–homeomorphic to $|\lnk_K(v)|$, is a $G_v$–hemisphere. (Note that when $\lnk_L(v)$ is a $G_v$–hemisphere, $|\lnk_L(v)| \times [0, 1] \cup |st_L(v)| \times \{1\}$ is actually a quadrant of a $G_v$–sphere where the two nonnegative coordinates give trivial subrepresentations. This is easily seen to be $G_v$–homeomorphic to a $G_v$–hemisphere.) \]

Now assume for the remainder of the section that $Y \subset M$ is a polyhedron and $M$ is an $n$–$G$–manifold. The following is a direct result of 4.10, 4.9 and the non-equivariant Proposition 3.10 from [17].

**Proposition 4.11.** A $G$–regular neighborhood $N$ of $Y$ in $M$ is a $G$–manifold with boundary. If $Y \subset \text{int } M$, $\partial N = |\tilde{N}_{K'}(L)|$ where $L$ and $K'$ are as in the definition of regular neighborhood.

We omit the proofs of the following two results since they are exactly the same as the non-equivariant versions (Theorem 3.11 and Corollary 3.18) in [17], only utilizing $G$–bicollarability in place of collarability of boundaries of manifolds.

**Theorem 4.12** (Simplicial $G$–Neighborhood Theorem). Let $Y$ be a $G$–polyhedron in $\text{int } M^n$ and $N$ be a $G$–neighborhood of $Y$ in $\text{int } M^n$. Then $N$ is $G$–regular if and only if

(i) $N$ is an $n$–manifold with boundary $\partial N$ $G$–bicollarable in $M$.\]
(ii) there are admissible $G$-triangulations $(K, L, J)$ of $(N, Y, \partial N)$ with $L$ full in $K$, $K = N_K(L)$ and $J = N_K(L)$.

**Corollary 4.13.** Suppose $N_1 \subset \text{int } N_2$ are two $G$-regular neighborhoods of $Y$ in $\text{int } M$. Then $\text{cl}(N_2 \setminus N_1)$ is a $G$-collar of $N_1$.

Let $Y \subset X$ be polyhedra such that $X = Y \cup D^m$ and $Y \cap D^m = D^{m-1}$ where $D^m$ and $D^{m-1}$ are disks of dimensions $m \geq 1$ and $m - 1$ respectively and there is a PL homeomorphism $D^m \to D^{m-1} \times I$ with $D^{m-1}$ mapping by the identity to $D^{m-1} \times \{0\}$. Then we say there is an $(m$–dimensional) **elementary collapse** of $X$ onto $Y$. If there is a sequence $X = X_0, X_1, X_2, \ldots, X_k = Y$ of polyhedra such that there is an elementary collapse of $X_{i-1}$ onto $X_i$, we say $X$ **collapses** to $Y$, or $X \searrow Y$. In particular, if $X = |K|$ and $Y = |L|$ with $K$ collapsing simplicially to $L$, then $X$ collapses to $Y$. Note that the dimensions of the individual elementary collapses in a sequence are allowed to vary. We say that a collapse is $m$–dimensional if every elementary step has dimension $\leq m$.

Consider now the case that $G$ is a finite group and $X$ and $Y$ are $G$–polyhedra with $X = Y \cup GD^m$ where $D^m$ (as well as $gD^m$ for each $g \in G$) is a disk as in the definition of an elementary collapse. Suppose we also have that: (1) $gD^m \neq D^m$ implies that $gD^m \cap D^m \subseteq Y$ and (2) there exists a point $y \in D^{m-1}$ fixed by the stabilizer $G_{D^m} = H$ such that $D^{m-1}$ is $H$–homeomorphic to a cone with apex $y$ on some $H$–complex, and (3) $D^m$ is $H$–homeomorphic to $D^{m-1} \times I$, then we say there is an **elementary $G$–collapse** from $X$ to $Y$. (Note then that any point $x \in \{y\} \times (0, 1]$ will have stabilizer $G_x = H$.) A sequence of these is called a $G$–collapse, denoted $X \searrow_G Y$.

A simplicial collapse from admissible $K$ to $L$ gives rise to a $G$–collapse if and only if whenever $(\sigma \leq \tau)$ is in its corresponding Morse matching, so too is $(g\sigma \leq g\tau) \forall g \in G$.

An elementary collapse from an $n$-manifold $M$ to another $n$-manifold $M_1$ where $M = M_1 \cup D^n$ and $D^{n-1} \subset \partial M_1$ and $D^n \setminus D^{n-1} \subset \partial M$ is called an **elementary shelling**. A sequence of such collapses is a **shelling**. Note that every elementary step must be $n$–dimensional.

The equivariant version of shelling requires some additional conditions. Let $M_1 \subset M$ be $n$-manifolds with an elementary $G$-collapse from $M = M_1 \cup GD^n$ to $M_1$ such that: (1) $D^n \cap M_1 = D^{n-1}$ lies in a $G$-collarable subpolyhedron $W \subset \partial M_1$, (2) under the $G_{D^n}$-triangulation $K = \{y\} * L$ of $D^{n-1}$, if $gD^n \neq D^n$, then $gD^n \cap D^n \subseteq |L| \times \{0\}$, and (3) $y \in \partial D^{n-1}$ implies
that, in the $G$–collar on $W$, $(\{y\} \ast \partial|L|) \times I \subset \partial M$. If these three extra conditions are satisfied, this elementary $G$–collapse is called an elementary $G$–shelling. A sequence of these is called a $G$–shelling.

**Lemma 4.14.** If $M$ $G$–shells to $M_1$, then there is a $G$–homeomorphism $h: M \to M_1$ which is the identity outside an arbitrary neighborhood of $M \setminus M_1$.

**Proof.** As in the corresponding proof of Lemma 3.25 in [17], we need only to consider the case of an elementary $G$–shelling. Let $M = M_1 \cup GD^n$ give the elementary $G$–shelling. Denote $G_D^n$ by $H$. Let $K = \{y\} \ast L$ be the $H$–triangulation of $D^{n-1}$ from the definition of elementary $G$–collapse.

Choose a $G$–collar on $GD^{n-1}$ in $M_1$ within the given neighborhood of $M \setminus M_1$. We may consider the disk $D^{n-1} \times [-1,1]$ with $D^n = D^{n-1} \times [-1,0]$, $E^n = D^{n-1} \times [0,1] \subset M_1$, and $D^{n-1} = D^{n-1} \times \{0\}$. Then if $D^n \neq gD^n$, we have that $D^{n-1} \times [-1,1]$ may only intersect $gD^{n-1} \times [-1,1]$ in $|L| \times [0,1]$. We will define an $H$–homeomorphism from $D^{n-1} \times [-1,1]$ to $E^n$ which is the identity on $|L| \times [0,1] \cup |K| \times \{1\}$; such a homeomorphism can then be extended, first equivariantly to all of $G(D^{n-1} \times [-1,1])$ and then by the identity to the rest of $M$. This last extension is possible because either $|L| \times I = \partial D^{n-1} \times I$ or $cl(\partial D^{n-1}) \setminus |L|) \times I = \{y\} \ast \partial|L| \times I \subset \partial M$.

Let $K'$ be a derived $H$–subdivision of $K$ near $y$. We have that $|K| \setminus \{\} \cup |L| \times (-1,0)$ is $H$–homeomorphic to $|K'| = |N_{K'}(y)| \cup |N_{K'}(L)|$ because they
are both $H$–homeomorphic to $|K|$ with an $H$–collar attached outside to $|L|$. Therefore, we have an $H$–homeomorphism from $D^{n-1} \times \{-1, 1\} \cup |L| \times [-1, 1]$ to $D^{n-1} \times \{0, 1\} \cup |L| \times [0, 1]$. Coning the two polyhedra with $(y, 0)$ and $(y, \frac{1}{2})$ gives the desired $H$–homeomorphism from $E^m \cup D^n$ to $E^n$. \hfill \Box

**Theorem 4.15.** Suppose $Y \subseteq X$ are $G$–polyhedra in a $G$–manifold $M$. If $X \setminus G Y$, then a $G$–regular neighborhood of $X$ $G$–shells to a $G$–regular neighborhood of $Y$ in $M$.

**Proof.** We follow the proof of the non-equivariant version, Theorem 3.26 in [17], checking that the conditions of $G$–shelling are satisfied. The proof uses induction on the dimension of the collapse from $X$ to $Y$.

Suppose that the theorem holds when the $G$–collapse is $(m-1)$–dimensional. We now consider the case where there is an $m$–dimensional elementary $G$–collapse from $X$ to $Y$. Let $X = Y \cup GD^m$, with $Y \cap D^m = D^{m-1} \times \{0\}$ where $D^m \approx_{GD^m} D^{m-1} \times I$. For simplicity, we will from now on denote the subgroup $GD^m$ by $H$.

Let $K$ be an admissible $G$–triangulation of $M$ with full subcomplexes $L_2 \leq L_1$ triangulating $Y$ and $X$ respectively. Denote by $J$ the subcomplex triangulating $Z = D^{m-1} \times \{1\} \subset D^m$, and by $GJ$, the resulting $G$–triangulation of its $G$–orbit, $GZ$. Finally, let $y$ be the apex in the $GD^m$–cone structure of $D^{m-1}$. Note then that $\{y\} \times I$ is fixed pointwise by $H$, and any point $(y, t)$ with $t > 0$ has stabilizer exactly $H$. Let $x = (y, \frac{1}{2})$.

As in [17], by breaking up the collapsing into smaller steps, we may assume that there are no vertices of $K^0$ in $D^{m-1} \times (0, 1)$.

Now we choose a derived $G$–subdivision $K'$ of $K$ near $L_2 \cup GJ$. Choose the derived vertices for simplices in $L_1 \setminus (L_2 \cup GJ)$ in $Gp^{-1}(\frac{1}{2})$ ensuring that $x$ is one of them, and denote by $L'$ the new triangulation of $X$. Then $N_{K'}(L')$ gives a $G$–regular neighborhood of $X$, which is the union of $N_{K'}(L_2)$ and $N_{K'}(GJ)$, $G$–regular neighborhoods of $Y$ and $GZ$ respectively. There is an $(m - 1)$–dimensional $H$–collapse from $|J|$ to $(y, 1)$, so the induction hypothesis and [1.14] together imply that $|N_{K'}(J)|$ is an $n$–dimensional $H$–disk. Let $E^n = |N_{K'}(J)|$. As an $H$–disk with $x$ an $H$–fixed point, $E^n$ is seen to be $H$–homeomorphic to $|st_{\tilde{N}_{K'}(J)}(x)| \times I$.

We will show that if $N_{K'}(gJ) \neq N_{K'}(J)$, the two subcomplexes must be disjoint. For such a $g$, suppose there exists a vertex $v = v_t \in N_{K'}(gJ) \cap N_{K'}(J)$. (Note that it must be a derived vertex since $gJ$ and $J$ are themselves disjoint.) Then $\tau \in K$ contains vertices $u$ and $w$ of $gJ$ and $J$ respectively.
Thus, \( \rho = \{u, w\} \in L_1 \) since \( L_1 \) is a full subcomplex of \( K \), but \(|\rho|\) is not contained in \( Y \) and it is not contained in \( GD^m \) since a simplex in \( D^m \) is may only contain vertices from \( L_2 \) and \( J \), not \( gJ \). This contradicts \( X = Y \cup GD^m \), so \( N_{K'}(gJ) \cap N_{K'}(J) \) must be empty. Since we have shown that \( gE_n \neq E^n \) implies \( gE^n \cap E^n = \emptyset \), it must be true that \( G_{E^n} = H \).

We next prove that \(|N_{K'}(J)| \cap |N_{K'}(L_2)|\) is an \((n - 1)\)-disk \( E^{n-1} \) which is \( H\)-homeomorphic to \(|st\tilde{N}_{K'}(J)(x)|\), giving that \( E^n \) is \( H\)-homeomorphic to \( E^{n-1} \times I \) as required. To see this, we show that \( E^{n-1} \) is an \( H\)-regular neighborhood of \( D^{m-1} \times \{1/2\} \) in the \((n - 1)\)-\( H\)-manifold \(|\tilde{N}_{K'}(J)|\), so that we may again invoke the induction hypothesis for \((m - 1)\)-dimensional collapses and ?? (since \( D^{m-1} \times \{1/2\} \) \( H\)-collapses to \( x \) and an \( H\)-regular neighborhood of \( x \) is the desired star of \( x \)).

Let \( P \) be the subcomplex of \( K' \) triangulating \( D^{m-1} \times \{1/2\} \) and let \( Q = \tilde{N}_{K'}(J) \) for brevity. The claim then is that \( N_Q(P) = N_{K'}(L_2) \cap N_{K'}(J) \).

Let \( \sigma \in N_{K'}(L_2) \cap N_{K'}(J) \), we easily see that \( \sigma \) cannot intersect \( L_2^0 \) or \( J^0 \) and must consist only of derived vertices of the form \( v_\rho \). Then there must exist \( u \in L_2^0 \) and \( w \in J^0 \) such that \( \sigma \cup \{u\} \) and \( \sigma \cup \{w\} \) are both simplices of \( K' \). This implies that there exists \( v_\rho \in \sigma \) for some \( \rho \in K \) containing both \( u \) and \( w \). But then \( \{u, w\} \in L_1 \) due to the fullness of \( L_1 \). Thus, \( v_{\{u, w\}} \) is in \( P \) and can be added to \( \sigma \), so \( \sigma \in N_Q(P) \). Hence, we have \( N_{K'}(L_2) \cap N_{K'}(J) \subseteq N_Q(P) \).
For the other inclusion, if \( \sigma \) is in \( N_Q(P) \), it means that there is a \( v_\tau \in P^0 \) such that \( \sigma \cup \{v_\tau\} \) is in \( Q \) for some \( \tau \) which contains vertices from both \( L_2 \) and \( J \). We note again that \( \sigma \) consists only of derived vertices since it is in \( Q = \hat{N}_{K'}(J) \), so let \( \rho \) be the minimal face such that \( v_\rho \in \sigma \cup \{\tau\} \). Then \( \rho \leq \tau \), so we have that \( \rho \in L_1 \). Since \( \rho \) was subdivided, it must contain some vertex \( u \in L_2^0 \). Therefore, \( u \) may be added to \( \sigma \) to get a simplex of \( K' \) intersecting \( L_2^0 \), i.e., \( \sigma \in N_{K'}(L_2) \), and it is already in \( Q \subset \hat{N}_{K'}(J) \). This proves that \( N_{K'}(L_2) \cap N_{K'}(J) = N_Q(P) \). This finishes the proof that \( E^{n-1} \) is an \( H \)-regular neighborhood of \( D^{n-1} \times \{1/2\} \) in \( |Q| \) and therefore \( H \)-homeomorphic to the \((n-1)\)-disk \(|st_Q(x)| \) as explained.

Observe that \( GE^n \cap |N_{K'}(L_2)| \subset |\hat{N}_{K'}(L_2)| \), which is \( G \)-collarable in \(|N_{K'}(L_2)| \) by 4.9.

There is one remaining condition to check for this to be a \( G \)-shelling. Write \( E^{n-1} = |st_Q(x)| \). Then we must verify that \( x \in \partial E^{n-1} \) implies that within the \( G \)-collar on \( |\hat{N}_{K'}(L_2)| \) in \(|N_{K'}(L_2)| \), \(|\{x\} \ast \partial \lnk_Q(x) \times I \subset \partial |N_{K'}(L')|\). It suffices for us to show that every simplex of \( \{x\} \ast \partial \lnk_Q(x) \) lies on \( \partial M \) because the \( G \)-collar is given by moving derived vertices around with simplices of \( K \). Thus, if a simplex \( \sigma \) consisting only of derived vertices lies on \( \partial M \), then \( \sigma \times I \) lies on \( \partial M \), and hence also on \( \partial |N_{K'}(L')| \).

Since \( E^{n-1} \) is an \( H \)-regular neighborhood of \( x \in |Q| \), if \( x \in \partial E^{n-1} \), then \( x \in \partial |Q| = |Q| \cap \partial M \). Thus, \( x \in \partial M \), forcing it to also belong to \( \partial |N_{K'}(L')| \).

Likewise, any simplex of \( \sigma \in \{x\} \ast \partial \lnk_Q(x) \) containing \( x \) lies in \( \partial E^{n-1} \) but not in \( \hat{N}_Q(x) \), forcing \( \sigma \) to be in \( \partial M \) and therefore \( \partial |N_{K'}(L')| \). Hence, we have proven the final condition that this constitutes a \( G \)-shelling of a \( G \)-regular neighborhood of \( X \) to a \( G \)-regular neighborhood of \( Y \).

\textbf{Corollary 4.16.} If \( Y \) \( G \)-collapses to a point, then any \( G \)-regular neighborhood of \( Y \) (in a \( G \)-manifold) is a \( G \)-disk.

\textbf{Corollary 4.17.} A collapsible \( G \)-manifold is a \( G \)-disk.

\textbf{Corollary 4.18.} If \( X \subset \text{int} \ M \) and \( X \searrow Y \) then a \( G \)-regular neighborhood of \( X \) in \( M \) is a \( G \)-regular neighborhood of \( Y \) in \( M \).

\textbf{Proof.} Let \( N_1 \) and \( N_2 \) be \( G \)-regular neighborhoods of \( X \) and \( Y \) respectively in \( M \). By 4.15 \( N_1 \) \( G \)-shells to \( N_2 \), and so by 4.14 they are \( G \)-homeomorphic fixing \( Y \). The homeomorphism carries any \( G \)-triangulation of \( N_2 \) to a \( G \)-triangulation of \( N_1 \). Therefore, by 4.12 \( N_1 \) is a \( G \)-regular neighborhood of \( Y \) in \( M \).
Our final goal for this section is the following result that enables us to recognize $G$–regular neighborhoods from $G$–collapses.

**Theorem 4.19** (Collapsing Criterion for $G$–Regular Neighborhoods). Let $N$ be a $G$–neighborhood of $Y$ in int $M$. Then $N$ is $G$–regular if and only if

(i) $N$ is an $n$–manifold with $\partial N$ $G$–bicollarable in $M$,

(ii) $N \searrow_G Y$.

**Proof.** The proof follows exactly the non-equivariant case (Corollary 3.30 in [17]), but we include it here nonetheless because this theorem is a key tool in proving our main result. For the first implication, suppose $N = |K|$ is regular with $K$ an admissible simplicial neighborhood of $L$; then (i) follows from 4.11 and 4.9. Recall the map $f = f_{L,K}$ from the proof of 4.4. Choose $\epsilon \in (0,1)$ and choose a $G$–derived $K'$ of $K$ near $L$ with all of the new vertices lying in $f^{-1}(\epsilon)$ for a given $\epsilon \in (0,1)$ to obtain $N_1 = |K(L)| = f^{-1}[0,\epsilon]$, a $G$–regular neighborhood of $Y$ in $M$. We have that $N \approx_G N_1$, and we have a $G$–cell structure on $N_1$ whose cells are obtained by intersecting the interior simplices of $K$ with $f^{-1}(0)$, $f^{-1}(\epsilon)$, or $f^{-1}[0,\epsilon]$. We may collapse, along with its orbit, each cell $\sigma \cap f^{-1}[0,\epsilon]$ from its face $\sigma \cap f^{-1}(\epsilon)$ in order of decreasing dimension. That this is a $G$–collapse follows from the admissibility of $K$.

For the other implication, suppose we have $N$ satisfying conditions (i) and (ii). Let $C = \partial N \times [-1,1]$ be a $G$–bicollar with $\partial N = \partial N \times \{0\}$. Then let $N_1 = N \cup (\partial N \times [0,\frac{1}{2}])$, which constitutes a $G$–regular neighborhood of $N$ in $M$ because we can triangulate it to be a simplicial neighborhood. Therefore, by 4.18 since $N \searrow_G Y$, $N_1$ is also a $G$–regular neighborhood of $Y$. But we can define a $G$–homeomorphism on $C$ fixing $\partial N \times \{-1,1\}$ and carrying $\partial N \times \{\frac{1}{2}\}$ to $\partial N \times \{0\}$. We can extend this by the identity to all of $M$, mapping $N_1$ to $N$, showing that the latter is also a $G$–regular neighborhood of $Y$ in $M$. \hfill \Box

5. Main Results

In this section, $G$ is the group $\{\pm 1\}$ unless otherwise noted. Working inside $\text{Hom}(P_4 \setminus \{3\}, K_n)$ ($G$–homeomorphic to the $G$–manifold $S^{n-2} \times S^{n-2}$), we show that $\text{Hom}_{\{3\}}(P_4, K_n)$ is a $G$–regular neighborhood of the diagonal $\text{Hom}(K_2, K_n)$ using the collapsing criterion, i.e., we show that it is a manifold.
of the correct dimension and that it (simplicially) $G$–collapses to the diagonal.

Recall from Section 1 how we represent elements of the posets in question as arrays whose entries $A, B, C, D$ are nonempty subsets of $\{1, \ldots, n\}$.

Define

$$M := \{ \phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A \cap B = \emptyset, \ C \cap D = \emptyset \}$$

$$K := \{ \phi \in M \mid B \cup D \neq \{1, \ldots, n\} \}$$

$$L := \{ \phi \in K \mid A \cap C = \emptyset \}$$

$$S := \{ \phi \in K \mid A = C, \ B = D \}$$

We reiterate that $M$, $K$, and $L$ are the face posets of the $G$–regular cell complexes $\text{Hom}(P_4, K_n)$, $\text{Hom}_3(P_4, K_n)$, and $\text{Hom}_3(C_5, K_n)$ respectively, and $S$ that of the diagonal $\text{Hom}(K_2, K_n)$. By passing to order complexes, we obtain that $\Delta S$ and $\Delta L$ are full $G$–subcomplexes of $\Delta K$, which is a full $G$–subcomplex of $\Delta M$, and they are all admissible. Our goal is to show that $|\Delta K|$ is a $G$–regular neighborhood of $|\Delta S|$ whose boundary is $|\Delta L|$.

**Proposition 5.1.** $\Delta K$ is a $(2n - 4)$–manifold with boundary $\Delta L$.

**Proof.** We show that the link of an element of $K$ is a sphere or a disk of dimension $(2n - 5)$. For any $\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K$, $\text{lnk}_{\Delta K}(\phi) = \Delta K_{<\phi} * \Delta K_{>\phi}$.

For any $\phi \in K$, we obtain an element of its lower link by deleting proper subsets from each of $A$, $B$, $C$, and $D$, at least one of which is nonempty. Therefore, $K_{<\phi}$ is isomorphic to the face poset of $\partial \Delta A * \partial \Delta B * \partial \Delta C * \partial \Delta D$, yielding that $\Delta K_{<\phi}$ is a combinatorial sphere of dimension $|A| + |B| + |C| + |D| - 5$. (Recall that, if $A$ is an unordered set, $\Delta A$ is the full simplex having $A$ as its vertex set, whereas, if $P$ is a poset, $\Delta P$ is its order complex.)

When $\phi \in K \setminus L$, we show that $\Delta K_{>\phi}$ is a sphere of dimension $2n - |A| - |B| - |C| - |D| - 1$, yielding that $\text{lnk}_{\Delta K}(\phi)$ is a sphere of dimension $2n - 5$.

For any $\phi' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in M$ such that $\phi' > \phi$, we have that

$$\emptyset \neq A \cap C \subseteq A' \cap C' \subseteq (B' \cup D')^c$$

so that $\phi' \in K$. Thus, to obtain an element of the upper link of $\phi$, any element of $(A \cup B)^c$ can be added to either $A$ or $B$, but not to both, and
similarly for elements of $(C \cup D)^c$. As a consequence, we have that $K_{>\phi}$ is isomorphic to the face poset of $*_{i=1}^m S^0$ where $m = |(A \cup B)^c| + |(C \cup D)^c|$, and therefore $\Delta K_{>\phi}$ is a sphere of dimension $2n - |A| - |B| - |C| - |D| - 1$ as claimed.

In the case where $\phi \in L$, we claim that $\Delta K_{>\phi}$ is a disk of dimension $2n - |A| - |B| - |C| - |D| - 1$, meaning that $\text{link}_{\Delta K}(\phi)$ is a disk of dimension $2n - 5$. This will finish the proof that $\Delta K$ is a manifold and $\Delta L$ is its boundary. To see that $K_{>\phi}$ is the face poset of a subcomplex of a join of spheres, we consider the various types of elements that we can add to one or more of $A$, $B$, $C$, and $D$ to obtain a larger element of $K$.

1. An element of $A \cap D$ cannot be added anywhere (while remaining in $M$). The same is true for elements of $B \cap D$ and $B \cap C$. Thus, these elements contribute nothing to the upper link.

2. An element of $B \setminus (C \cup D)$ can be added to $C$ or to $D$; doing so will give us something in $K$, since the element was already in $B \cup D$. Thus, each of these elements contributes a copy of $S^0$ to the join of spheres. Similarly, each element of $D \setminus (A \cup B)$ contributes a copy of $S^0$ to the join.

3. An element of $A \setminus D$ can be added to $C$ or to $D$, contributing a copy of $S^0 = \{\pm 1\}$ to the join with $+1$ indicating that the element was added to $C$ and $-1$ indicating $D$. Similarly, an element of $C \setminus B$ can be added to $A (+1)$ or to $B (-1)$. Adding elements of Type 3 to $B$ or $D$ could produce something not in $K$.

4. An element of $(A \cup B \cup C \cup D)^c$ can be added to $A$ or $B$ (but not both) and, at the same time, to $C$ or $D$ (but not both). This contributes a copy of $S^1 = \{\pm 1\} \ast \{\pm 1\}$ (treated as a single coordinate) to the join of spheres with the $+1$’s corresponding to $A$ and $C$ and the $-1$’s corresponding to $B$ and $D$. As with Type 3, adding this type of element to $B$ or $D$ could yield something not in $K$.

To ensure that we remain in $K$, there must be an element of $(B \cup D)^c$ which is not added to $B \cup D$. In terms of coordinates, this means there must be at least one coordinate corresponding to Type 3 or 4 above that has no $-1$’s.

Before proceeding, we define

$$F_{k,l} \subseteq (\ast_{i=1}^k S^1) \ast (\ast_{j=1}^l \{\pm 1\})$$

...to be the subcomplex whose simplices have at least one coordinate from the join with no $-1$’s. (Note that, as before, each copy of $S^1 = \{\pm 1\} \ast \{\pm 1\}$ is
regarded as a single coordinate.) We will prove a lemma (5.2) stating that
\( F_{k,l} \) is a disk of dimension \( 2k + l - 1 \). Assuming that result, since \( K_{>\phi} \) is
isomorphic to the face poset of \((^m\bigcup_{i=1}^n S^0) * F_{k,l} \) where \( k = |(A \cup B \cup C \cup D)^c| \),
\( l = |A \setminus D| + |C \setminus B| \), and \( m = |B \setminus (C \cup D)| + |D \setminus (A \cup B)| \), we have that
\( \Delta K_{>\phi} \) is a disk of dimension
\[
m - 1 + 2k + l = |B \setminus (C \cup D)| + |D \setminus (A \cup B)| \\
+ 2|(A \cup B \cup C \cup D)^c| + |A \setminus D| + |C \setminus B| - 1 \\
= 2n - 2|A| - 2|B| - 2|C| - 2|D| \\
+ 2|A \cap D| + 2|B \cap C| + 2|B \cap D| \\
+ |B \setminus (C \cup D)| + |D \setminus (A \cup B)| \\
+ |A \setminus D| + |C \setminus B| - 1 \\
= 2n - |A| - |B| - |C| - |D| - 1
\]
as we had claimed. \( \square \)

**Lemma 5.2.** For \( k, l \in \mathbb{N} \) such that \( 2k + l - 1 \geq 0 \), \( F_{k,l} \) is a disk of dimension
\( 2k + l - 1 \).

**Proof.** We proceed by induction on the dimension, \( 2k + l - 1 \). In the initial case, \( F_{0,1} \) has a single \( S^0 \) coordinate which must be \(+1\), so it is a single point, i.e. a disk of dimension 0. To prove \( F_{k,l} \) is a disk, we will show that it is a
\( (2k + l - 1) \)–manifold, show it collapses to a vertex, and then apply Corollary 4.17. There are four types of vertices whose links we need to consider:

1. \(+1\) coming from one of the \( k \) \( S^1 \) coordinates has as its link \( F_{k-1,l+1} \), a
\( (2k + l - 2) \)–disk by induction.
2. \(-1\) coming from one of the \( S^1 \) coordinates has as its link \( S^0 * F_{k-1,l} \), a
\( (2k + l - 2) \)–disk.
3. \(+1\) coming from one of the \( l \) \( \{\pm1\} \) coordinates has as its link \( \bigcup_{i=1}^{2k+l-1} S^0 \),
a \( (2k + l - 2) \)–sphere.
4. \(-1\) coming from one of the \( \{\pm1\} \) coordinates has as its link \( F_{k,l-1} \), a
\( (2k + l - 2) \)–disk.

Now we will define a matching on \( F_{k,l} \). First, we order the coordinates. In each \( S^1 \) coordinate, we also choose one of the two copies of \( \{\pm1\} \) to be distinguished. Associate each simplex in \( F_{k,l} \) with the simplex obtained by inserting or removing \(+1\) to or from the first coordinate lacking a \(-1\) (in the distinguished copy of \( \{\pm1\} \) in the case the first such coordinate is \( S^1 \)).
Doing this does not change which coordinate is the first without a $-1$, so the pairing is well-defined. Every simplex is paired ($\emptyset$ is paired with the vertex with a $+1$ in the first coordinate and nothing in any other coordinate), so if there are no cycles in this matching, $F_{k,l}$.

Suppose there were a cycle. It would have to be of the form:

$$\sigma_0 \prec \tau_0 \succ \sigma_1 \prec \tau_1 \succ \sigma_2 \prec \ldots \prec \tau_{s-1} \succ \sigma_s = \sigma_0$$

where each $\sigma_i$ is paired with $\tau_i$. Also, for $1 \leq i \leq s$, $\sigma_i$ must be $\tau_{i-1}$ minus a vertex $v_i$. Therefore, since this is a cycle, there must be a $j$ such that $\tau_j = \sigma_j \cup \{v_i\}$. For this to be possible, $v_i$ must be a $+1$. Thus, all of the simplices in the cycle must have all the same $-1$ coordinates, but if that is the case, the vertex to be added in any $\sigma_i \prec \tau_i$ pair is always the same, and $v_i$ must be the same for every $i$. This is a contradiction. Therefore, there are no cycles, and we have a Morse matching with a single critical simplex.

In the proof of Proposition 5.4 below, we will use the following lemma, which is essentially an equivariant version of Theorem 3.1 of [11].

**Lemma 5.3.** Let $G$ be any group and $P$ be a finite poset, $h: P \to P$ an order-preserving poset map such that for any $x \in P$, $h(x) \geq x$ (or $h(x) \leq x$). Define $Q$ to be the set of fixed points of $h$. Then $\Delta P$ collapses simplicially to $\Delta Q$. In the case that $h$ is a $G$–map.

**Proof.** We prove it for the case that $h(x) \geq x$, the other case being almost identical. Since $P$ is finite, we may choose $N$ large enough so that for all $x$ in $P$, we have $h^N(x) \in Q$. Now let $\sigma \in \Delta P$ be a chain $x_0 < x_1 < \ldots < x_m$. If $\exists i : 0 \leq i \leq m$ such that $x_i \notin Q$, let $k$ be the largest such $i$. Then we may insert $h^N(x_k)$ into the chain immediately following $x_k$ because $x_k < h^N(x_k) \leq h^N(x_{k+1}) = x_{k+1}$ if $k < m$. Associate to $\sigma$ the chain obtained by inserting $h^N(x_k)$ or by deleting it in the case $h^N(x_k) = x_{k+1}$. Since it is an element of $Q$ being inserted or deleted, the selection of $k$ is not affected, and $x_k$ uniquely determines the other chain in the pair. Therefore, this matching is well-defined. Also, this matching is equivariant if $h$ is a $G$–map.

Suppose there is a cycle

$$\sigma_0 \prec \tau_0 \succ \sigma_1 \prec \tau_1 \succ \sigma_2 \prec \ldots \prec \tau_{s-1} \succ \sigma_s = \sigma_0$$

where $\sigma_i$ is paired with $\tau_i$. Also, for $1 \leq i \leq s$, $\sigma_i$ must be $\tau_{i-1}$ minus a vertex $v_i$. Therefore, since this is a cycle, there must be a $j$ such that $\tau_j = \sigma_j \cup \{v_i\}$. For this to be possible, $v_i$ must be a $+1$. Thus, all of the simplices in the cycle must have all the same $-1$ coordinates, but if that is the case, the vertex to be added in any $\sigma_i \prec \tau_i$ pair is always the same, and $v_i$ must be the same for every $i$. This is a contradiction. Therefore, there are no cycles, and we have a Morse matching with a single critical simplex. $\square$
in the cycle has all the same elements of $P \setminus Q$, so $\exists x \in P \setminus Q$ that is the greatest such element in every simplex. Hence $\tau_j = \sigma_j \cup \{h_\hat{N}(x)\}$ for all $j$, and $y_i = h_\hat{N}(x)$ for all $i$. This is a contradiction because the same element is being added and deleted in consecutive steps. Therefore, we have a Morse matching whose critical simplices are exactly the elements of $\Delta Q$, a subcomplex of $\Delta P$. Thus, $\Delta P G$–collapses to $\Delta Q$.

**Proposition 5.4.** $\Delta K$ simplicially $G$–collapses to $\Delta S$.

**Proof.** The collapsing will occur in three steps. Define

$$K_1 := \{ \phi = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in K \mid A \cap C \neq \emptyset \}$$

$$K_2 := \{ \phi \in K_1 \mid A = C \}$$

First, we collapse $\Delta K$ to $\Delta K_1$. Let $\sigma$ be a chain of the form $\phi_0 \prec \phi_1 \prec \ldots \prec \phi_{m-1} \prec \phi_m$ in $\Delta K$ where $\phi_i = \left( \begin{array}{cc} A_i & B_i \\ C_i & D_i \end{array} \right)$. If $A_0 \cap C_0 = \emptyset$, we want to pair $\sigma$ with another chain for which that is also true. Find the last $k$ such that $A_k \cap C_k = \emptyset$. Then compare $B_k$ and $D_k$ to $B_m$ and $D_m$. If $B_k = B_m$ and $D_k = D_m$, pair $\sigma$ with the chain obtained by adding to (or deleting from) the end of $\sigma$ the element $\left( \begin{array}{cc} A_m \cup (B_m \cup D_m)^c & B_m \\ C_m \cup (B_m \cup D_m)^c & D_m \end{array} \right)$. Otherwise, find the first $l > k$ where $B_l \neq B_k$ or $D_l \neq D_k$. Now pair $\sigma$ with the chain obtained by inserting (or removing if it equals $X_{l-1}$) $\left( \begin{array}{cc} A_l & B_{l-1} \\ C_l & D_{l-1} \end{array} \right)$ before $\phi_l$. Nowhere are we inserting or deleting elements with $A \cap C = \emptyset$, so the selection of $k$ is not affected. In the second case, we are inserting or deleting an element with $B_{l-1} = B_k$ and $D_{l-1} = D_k$, so the selection of $l$ is not affected. Therefore, the matching is well-defined. The critical simplices are exactly those where $A_0 \cap C_0 \neq \emptyset$, forming $\Delta K_1$, a subcomplex. Therefore, if there are no cycles, we have a collapsing from $\Delta K$ to $\Delta K_1$. Also, the pairings are chosen equivariantly, so we will have a $G$–collapse.

Suppose we have a cycle

$$\sigma_0 \prec \tau_0 \succ \sigma_1 \prec \tau_1 \succ \sigma_2 \prec \ldots \prec \tau_{s-1} \succ \sigma_s = \sigma_0$$
Again, for \(1 \leq i \leq s\), \(\sigma_i\) is obtained from \(\tau_{i-1}\) by deleting an element \(\psi_i\), so there must be a pair \(\sigma_j \leq \tau_j = \sigma_j \cup \{\psi_i\}\) coming from our matching. Therefore, \(\psi_i \in K_1\) for all \(i\), which means that all the simplices in our cycle have all of the same elements with \(A \cap C = \emptyset\). Thus, they all have the same \(\phi_k\), so \(B_k\) and \(D_k\) are fixed and we know that every \(\psi_i\) has them as its second column. As a result, the elements after \(\phi_k\) that have \(B \neq B_k\) or \(D \neq D_k\) are not changing as we move through the cycle, implying that \(\psi_i\) is the same for all \(i\). This is a contradiction, so our matching has no cycles. This proves that \(\Delta K G\)-collapse to \(\Delta K_1\).

The next two collapsings are proved by Lemma 5.3. For the first, we define \(h_1 : K_1 \to K_1\) by \(h_1(\phi) = \begin{pmatrix} A \cap C & B \\ A \cap C & D \end{pmatrix}\). This is an order-preserving \(G\)-poset map, and \(h_1(\phi) \leq \phi\). The fixed point set of \(h_1\) is exactly \(K_2\), so Lemma 5.3 implies that \(\Delta K_1 G\)-collapses to \(\Delta K_2\). For the second collapsing, we now define \(h_2 : K_2 \to K_2\) by \(h_2(\phi) = \begin{pmatrix} A & B \cup D \\ A & B \cup D \end{pmatrix}\). This is an order-preserving \(G\)-poset map, \(h_2(\phi) \geq \phi\), and the fixed point set is \(S\). Therefore, the same lemma implies that \(\Delta K_2 G\)-collapses to \(\Delta S\). Hence, \(\Delta K G\)-collapses to \(\Delta S\).

**Theorem 5.5.** \(|\Delta K|\) is a \(G\)-regular neighborhood of \(|\Delta S|\) with boundary \(|\Delta L|\).

**Proof.** \(G\) acts freely outside of \(|\Delta S|\), so \(\partial |\Delta K|\) is \(G\)-bicollarable in \(|\Delta M|\). Now the theorem follows immediately from 4.19 (the collapsing criterion for \(G\)-regular neighborhoods) and Propositions 5.1 and 5.4. \(\Box\)

Now our main result follows easily:

**Main Theorem.** The regular cell complex \(\text{Hom}_{\{3\}}(P_4, K_n)\) is a PL manifold with boundary \(\text{Hom}_{\{3\}}(C_5, K_n)\) and is equivariantly homeomorphic (with respect to the involution described above) to \(N := \{(x, y) \in S^{n-2} \times S^{n-2} : x \cdot y \geq 0\}\), where the involution on \(N\) interchanges \((x, y)\) with \((y, x)\). The Stiefel manifold \(V_{n-2} = \partial N\) is therefore equivariantly homeomorphic to \(\text{Hom}_{\{3\}}(C_5, K_n)\), which is equivariantly homeomorphic to \(\text{Hom}(C_5, K_n)\).

**Proof.** It follows from 2.1 that we have \(\text{Hom}_{\{3\}}(P_4, K^n) \approx_G |\Delta K|\) with the subcomplex \(\text{Hom}_{\{3\}}(C_5, K^n) \approx_G |\Delta L|\). Because \(|\Delta K|\) and \(N\) are both \(G\)-regular neighborhoods of the diagonal, they are equivariantly homeomorphic by 4.6. \(\Box\)
The Stiefel manifold $V_{n-1,2}$ has a natural action of the orthogonal group $O_2$ (with the Grassmannian as the quotient). The equivariant homeomorphism above is with respect to a single reflection of $O_2$. The multimorphism complex $\text{Hom}(C_5, K_n)$ does not have a combinatorial $O_2$ action; however, there is the induced action of the dihedral group $D_5$ (a subgroup of $O_2$) which is the group of symmetries of the cycle $C_5$. It seems natural to ask:

**Question 5.6.** Is $\text{Hom}(C_5, K_n)$ equivariantly homeomorphic to $V_{n-1,2}$ with respect to the action of the dihedral group $D_5$?

Unfortunately, neither of the smaller models $\text{Hom}_{\{3\}}(C_5, K_n)$ or $\text{Hom}_{\{2,4\}}(C_5, K_n)$ is $D_5$-invariant, so it seems that one needs to work with $\text{Hom}(C_5, K_n)$ which does not have an obvious $D_5$-equivariant embedding into $S^{n-2} \times S^{n-2}$. Also, a good (equivariant) combinatorial candidate for $N$ is missing, which is the obstacle to applying the methodology above to answer this question positively.

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