ON THE PARTIAL TRANSPOSE OF A HAAR UNITARY MATRIX

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ABSTRACT. We consider the effect of a partial transpose on the limit *-distribution of a Haar distributed random unitary matrix. If we fix the number of blocks, $b$, we show that the partial transpose can be decomposed along diagonals into a sum of $b$ matrices which are asymptotically free and identically distributed. We then consider the joint effect of different block decompositions and show that under some mild assumptions we also get asymptotic freeness.

1. Introduction

Consider a sequence of random matrices $(U_N)_{N \geq 1}$, where the matrix $U_N$ is an $N \times N$ Haar unitary matrix. In our previous paper [10], we studied matrices obtained from Haar unitary matrices via a permutation of the entries. More precisely for $\sigma$ being a permutation of the set $\{(i,j) : 1 \leq i, j \leq N\}$, we considered a matrix $U_N$ defined as $[U_N^\sigma]_{i,j} = [U_N]_{\sigma(i,j)}$. For a sequence of matrices of growing sizes one considers a sequence of permutations $(\sigma_N)_{N \geq 1}$, we identified conditions which give that sequence $U_N^{\sigma_N}$ is asymptotically circular and we also found a conditions on pairs of sequences of permutations $(\sigma_N)_{N \geq 1}$ and $(\sigma'_N)_{N \geq 1}$ such that sequences $U_N^{\sigma_N}$ and $U_N^{\sigma'_N}$ are asymptotically free. Moreover we showed that our conditions are satisfied with probability one by sequences of random uniform permutations. Despite the fact that conditions from [10] cover many examples of permutations, there are important examples of entry permutations not covered by this result. One such example is given by partial transposes, a significant class of entry permutations of matrix entries which plays an important role in quantum information theory. This paper is devoted to a detailed study of asymptotic distribution of partial transposes of Haar unitary matrices.

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Notation 1. In this paper we shall consider the left and right partial transpose of a block matrix. Let $A^T$ be the usual transpose of a matrix $A$. If $N = b \times d$ we can write $M_N(C) = M_b(C) \otimes M_d(C)$ and visualize this as $M_b(M_d(C))$, i.e. $b \times b$ matrices with each entry a $d \times d$ matrix. As operators on the tensor product we let $\Gamma = id \otimes T$ and $\Gamma^{(-1)} = T \otimes id$. We shall write $\Gamma^{(1)}$ for $\Gamma$ and usually also include the size of the blocks as this determines the map. We call $\Gamma^{(-1)}_{b,d}$ the left partial transpose and $\Gamma^{(1)}_{b,d}$ the right partial transpose.

If $N$ is a positive integer we denote by $[N]$ a linearly ordered set with $N$ elements. For simplicity, we will identify $[N]$ to the set $\{1, 2, \ldots, N\}$. Suppose that $N = b \cdot d$ and $\vartheta \in \{-1, 1\}$. To bring our notation with entry permutations we shall define the map $\Gamma^{(\vartheta)}_{b,d} : [N]^2 \to [N]^2$ as follows. First, let $\varphi_{b,d} : [N]^2 \to ([b] \times [d])^2$ given by

$$\varphi_{b,d}(i, j) = (a_1, a_2, a_{-1}, a_{-2})$$

whenever $i = (a_1 - 1)d + a_2$ and $j = (a_{-1} - 1)d + a_{-2}$. The idea here is to have $(a_1, a_2, a_{-1}, a_{-2})$ locate the $(a_2, a_{-2})$ entry of the $(a_1, a_{-1})$ block. Then let $\gamma^{(\vartheta)}_{b,d} : ([b] \times [d])^2 \to ([b] \times [d])^2$ be given by

$$\gamma^{(\vartheta)}_{b,d}(a_1, a_2, a_{-1}, a_{-2}) = (a_\vartheta, a_{-2\vartheta}, a_{-\vartheta}, a_{2\vartheta}).$$

Thus $\gamma^{(1)}_{b,d}$ switches $a_2$ and $a_{-2}$, whereas $\gamma^{(-1)}_{b,d}$ switches $a_1$ and $a_{-1}$. Finally, put

$$\Gamma^{(\vartheta)}_{b,d} = \varphi_{b,d}^{-1} \circ \gamma^{(\vartheta)}_{b,d} \circ \varphi_{b,d}.$$

In the case that $N = bd$ we can decompose the matrix $U_N$ as $b \times b$ block matrix with blocks of size $d \times d$ each. Hence we write $U_N = [U^{(d)}_{i,j}]_{1 \leq i,j \leq b}$. It is natural then to consider the asymptotic join $*$-distribution of $\{U^{(d)}_{i,j}\}_{1 \leq i,j \leq b}$ as $d \to \infty$, this question was settled in [3], see discussion around equations (1), (2) for details. Denote the resulting tuple of non-commutative random variables by $\{v_{i,j}\}_{1 \leq i,j \leq b}$. In section 2 we present a parallel construction of a decomposition of a Haar unitary element $v$ and we explain that for $v_{i,j}$ as above the matrix $v = [v_{i,j}]_{1 \leq i,j \leq b}$ where is indeed a Haar unitar and $v_{i,j}$ generate so called Brown algebra. Next we look at the transpose of $v$ that is $v^t = [v_{j,i}]_{1 \leq i,j \leq b}$ and we present a decomposition of $v^t$ into a sum of $n$ operators which are free and $R$–diagonal, this allows us to find the distribution of $v^t$. Moreover we show that if we have $B$ which is free from $\{v_{j,i}\}_{i,j}$ then $v$ is free from $M_n(B)$. In Section 3 we study asymptotic joint limiting distribution of different partial transposes of the same Haar unitary matrix. We find sufficient conditions under which partial transposes of the same Haar unitary matrix are asymptotically free.
2. Limit distributions and freeness results in the Brown algebra

Definition 2. The non-commutative unitary group $U_{b}^{nc}$, also called the Brown algebra, is the universal $C^{*}$-algebra generated by a unit and $b^2$ operators $\{v_{ij}\}_{i,j=1}^{b}$ such that $\sum_{k=1}^{b}v_{ik}v_{jk}^* = \sum_{k=1}^{b}v_{ki}^*v_{kj} = \delta_{ij}$, i.e. that the matrix $v = (v_{ij})_{i,j=1}^{b} \in M_{b}(U_{b}^{nc})$ is unitary.

The algebra $U_{b}^{nc}$ was constructed by Brown in [2] as the (1, 1) corner algebra of the free product $C^{*}$-algebra $M_{b}(\mathbb{C}) * C(\mathbb{T})$ where $C(\mathbb{T})$ is the $C^{*}$-algebra of continuous functions on the unit circle. We can put a tracial state, $\varphi_{b}$, on $M_{b}(\mathbb{C}) * C(\mathbb{T})$ which is the free product of the normalized trace on $M_{b}(\mathbb{C})$ and the state on $C(\mathbb{T})$ obtained from integration with the normalized Haar measure on $\mathbb{T}$ (see McClanahan [6, §3]). By virtue of the free product construction we have that $M_{b}(\mathbb{C})$ and $C(\mathbb{T})$ are free with respect to $\varphi_{b}$ in $M_{b}(\mathbb{C}) * C(\mathbb{T})$, (see [12 Prop. 1.5.5]). Thus in $M_{b}(\mathbb{C}) * C(\mathbb{T})$ we have a Haar unitary $v \in C(\mathbb{T})$ (given by $v(z) = z$ for $z \in \mathbb{T}$), which is $*$-free from the matrix units $\{e_{ij}\}_{i,j=1}^{b}$. We let $U_{b}^{nc} = e_{11}(M_{b}(\mathbb{C}) * C(\mathbb{T}))e_{11}$ and $v_{ij} = e_{11}ve_{1} \in U_{b}^{nc}$. Then $\{v_{ij}\}_{i,j=1}^{b}$ generate $U_{b}^{nc}$ and $M_{b}(U_{b}^{nc})$ can be identified with $M_{b}(\mathbb{C}) * C(\mathbb{T})$ (see McClanahan [6 Prop. 2.2]). We let $\varphi : U_{b}^{nc} \to \mathbb{C}$ be given by $\varphi(e_{11}fe_{11}) = b\varphi_{b}(e_{11}fe_{11})$ for $f \in C(\mathbb{T})$. Thus $\varphi$ is a trace on $U_{b}^{nc}$.

The free $*$-cumulants of the Haar unitary $v$ are described by saying that $v$ is $R$-diagonal and the non-vanishing ones are given by the signed Catalan numbers $\beta_{r} = (-1)^{r-1}Cat_{r-1}$ where $Cat_{b}$ is the $b^{th}$ Catalan number, see [11 Cor. 15.1]. Using the fact that the matrix units $\{e_{ij}\}_{i,j=1}^{b}$ are $*$-free we may use the free compression result of Nica and Speicher, [11 Thm. 14.18], to conclude that the free $*$-cumulants (relative to $\varphi$) of $\{v_{ij}\}_{i,j=1}^{b}$ are given by

\begin{align*}
(1) \quad & \kappa_{2r}(v_{11,j1}, v_{12,j1}, v_{13,j2}, \ldots, v_{ir,jr}, v_{11,jr}) = b^{1-2r}\beta_{r} \\
(2) \quad & \kappa_{2r}(v_{11,j1}, v_{11,j2}, v_{12,j2}, \ldots, v_{ir,jr}, v_{1r,j1}) = b^{1-2r}\beta_{r}
\end{align*}

and all other free cumulants are zero. Since $\varphi$ is tracial each of (1) and (2) implies the other (see [11 Def. 15.3]).

Let $U_{N}$ be a $N \times N$ Haar distributed random unitary matrix. It was shown by Voiculescu that $U_{N}$ is asymptotically $*$-free from constant matrices (see [11 Thm. 23.14] for a proof using the Weingarten calculus). Suppose that $N = bd$, then we can see $U_{N}$ as a $N \times N$ block matrix with block entries $U_{i,j}$ for $1 \leq i,j \leq n$. Then, for fixed $b$, the joint distribution of the block matrices $\{U_{ij}\}_{i,j=1}^{n}$ converges (as $d \to \infty$) to the joint distribution of $\{v_{ij}\}_{i,j=1}^{b}$ which is given by (1) and (2). This
was already observed by Cébron and Ulrich in [3, Cor. 2.8 and Thm. 3.3].

2.1. Diagonal Decomposition

Given a unital $*$-algebra $\mathcal{A}$ and a matrix $a = (a_{ij})_{i,j=1}^{b} \in M_b(\mathcal{A})$ we let

$$a^t = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{b1} \\ a_{12} & a_{22} & \cdots & a_{b2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1b} & a_{2b} & \cdots & a_{bb} \end{bmatrix}.$$ 

We call $a^t$ the transpose of $a$. Since the entries come from a non-commutative algebra we no longer expect to have $(ab)^t = b^t a^t$. In this sub-section we shall show how to decompose $v^t$ into $b$ pieces each $*$-free from each other. Let

$$s = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & 0 & \vdots & \cdots & 0 \end{bmatrix}.$$ 

Then $s = (s_{ij})_{ij}$ where $s_{ij} = \delta_{i+j-1,j}$ (all indices mod $n$), then and $s^i = 1$ for $i \equiv 0 \pmod b$. If $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear map with $\varphi(1) = 1$, we let $\Phi : M_b(\mathcal{A}) \to \mathbb{C}$ given by $\Phi(a) = \varphi(a_{11} + \cdots + a_{nn})/n$. If $I_b$ denotes the identity matrix of $M_b(\mathcal{A})$ then $\Phi(I_b) = 1$. If $\varphi$ is a trace on $\mathcal{A}$ then $\Phi$ is a trace on $M_b(\mathcal{A})$ and $\varphi(s^i) = 0$ for $i \neq 0$.

2.2. The $*$-distribution of $v^t$

Now let $\mathcal{A} = U_b^{ac}$ and let us return to our Haar unitary $v \in M_b(U_b^{ac})$.

**Definition 3.** Let $w_{1,0} = \text{diag}(v_{11}, \ldots, v_{bb})$ be the $b \times b$ diagonal matrix with diagonal entries $v_{11}, \ldots, v_{bb}$. For $1 \leq k \leq b-1$ let $w_{1,k}$ be the diagonal matrix $\text{diag}(v_{k+1,1}, v_{k+2,2}, \ldots, v_{k,b})$. Then

$$v^t = w_{1,0}s^0 + w_{1,1}s^1 + \cdots + w_{1,b-1}s^{b-1}$$

is the diagonal decomposition of $v^t$. For $0 \leq k \leq b-1$, let $v_k = w_{1,k}s^k$.

Then $v^t = v_0 + \cdots + v_{b-1}$ and we shall show in Theorem 15 that the family $\{v_0, v_1, \ldots, v_{b-1}\}$ is $*$-free. When $b = 3$ our decomposition looks like

$$v_0 = \begin{bmatrix} v_{11} & 0 & 0 \\ 0 & v_{22} & 0 \\ 0 & 0 & v_{33} \end{bmatrix}, v_1 = \begin{bmatrix} 0 & v_{21} & 0 \\ 0 & 0 & v_{32} \\ v_{13} & 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 0 & v_{31} \\ v_{12} & 0 & 0 \\ 0 & v_{23} & 0 \end{bmatrix}.$$
To demonstrate \(\ast\)-freeness we shall restate the results in Eq’s (1) and (2) using the symmetric group.

For a positive integer \(m\) let \([m] = \{1, 2, \ldots, m\}\) and \(S_m\) denote the permutation group of \([m]\). Moreover we let \([\pm m] = \{1, -1, 2, -2, \ldots, m, -m\}\) and \(S_{\pm m}\) denote the permutation group of \([\pm m]\). We shall regard \(S_m\) as the subgroup of \(S_{\pm m}\) of permutations acting trivially on \(\{-1, -2, \ldots, -m\}\). Thus for \(\pi \in S_m\) and \(k \in [m]\) we have \(\pi(-k) = -k\).

Let \(\delta \in S_{\pm m}\) be the permutation with cycle decomposition \((1, -1)(2, -2)\cdots(m, -m)\). Let \(\gamma\) be the permutation in \(S_m\) with cycle decomposition \((1, 2, 3, \ldots, m)\). Following our convention, \(\gamma\delta\gamma^{-1}\) has the cycle decomposition \((1, -1, 2, -2)(-2, 3)\cdots(-m, -m)\). Given \(\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \mathbb{Z}_2^m\) with \(\mathbb{Z}_2 = \{-1, 1\}\), we consider \(\epsilon\) to also be a permutation in \(S_{\pm m}\) by setting \(\epsilon(k) = \epsilon(k)\) for \(k \in [\pm m]\). See Remark 5 for some illustrations of our notation. For a sequence \(i_{\pm 1}, i_{\pm 2}, \ldots, i_{\pm m} \in [n]\) we let \(\ker(i)\) be the partition of \([\pm m]\) such that \(i_r = i_s\) if and only if \(r\) and \(s\) are in the same block of \(\ker(i)\). Since we must deal with \(\ast\)-moments we need a way to record all possible mixed \(\ast\)-moments of the \(u_{ij}\). To this end we will let \(a^{(1)} = a\) and \(a^{(-1)} = a^\ast\) for any element, \(a\), of a \(\ast\)-algebra. The restatement of Eq. (1) and Eq. (2) now becomes Eq. (3) below.

**Lemma 4.** Let \(\epsilon_1, \ldots, \epsilon_m \in \mathbb{Z} = \{-1, 1\}\) and \(i_{\pm 1}, \ldots, i_{\pm m} \in [b]\). Then

\[
(3) \quad \kappa_m(v_{i_1i_{-1}}^{(\epsilon_1)}, \ldots, v_{i_mi_{-m}}^{(\epsilon_m)}) = 0
\]

unless: (i) \(m\) is even; (ii) \(\epsilon_k + \epsilon_{k+1} = 0\) for \(1 \leq k \leq m - 1\); (iii) \(\ker(i) \geq \epsilon\gamma\delta\gamma^{-1}\epsilon\). When these conditions are satisfied the cumulant is \(b!^{-m}\beta_{m/2}\).

**Remark 5.** As there are only two possible \(\epsilon\)’s which produce a non-zero cumulant there are only two possible values for \(\epsilon\gamma\delta\gamma^{-1}\epsilon\). When \(\epsilon = (1, -1, \ldots, 1, -1)\) we have \(\epsilon\gamma\delta\gamma^{-1}\epsilon = (1, m)(-1, -2)(2, 3)(-3, -4)\cdots(-m, -m)\). When \(\epsilon = (-1, 1, \ldots, -1, 1)\) we have \(\epsilon\gamma\delta\gamma^{-1}\epsilon = (-1, -m)(1, 2)(-2, -3)(3, 4)\cdots(-(m-2), -(m-1))(m-1, m)\). See Figures 1 and 2 for illustrations. The condition in Lemma 4 becomes either,

\[
i_1 = i_m, i_{-1} = i_{-2}, i_2 = i_3, i_{-3} = i_{-4}, \ldots, i_{-m} = i_{-(m-1)}
\]
or

\[ i_1 = i_2, i_{-1} = i_{-m}, i_{-2} = i_{-3}, i_3 = i_4, \ldots, i_m = i_{m-1}. \]

**Notation 6.** For a matrix \( a = (a_{ij})_{i,j=1}^b \in M_b(\mathcal{A}), \) let \( a_{ij}^{[1]} = a_{ij} \) and \( a_{ij}^{[-1]} = a_{ji}. \) With this notation \( a_{ij}^{[-1]} \) is the \((i, j)\)-entry of \( a^* \). Note that with our other notation we have \( a_{ij}^{[-1]} = a_{ij}^*. \)

**Remark 7.** With the notation above we have that equation (3) becomes

\[ \kappa_m(v_{i_1i_{-1}}, \ldots, v_{i_mi_{-m}}) = 0 \]

unless: \((i) m \) is even; \((ii) \) \( \epsilon_k = -\epsilon_{k+1} \) for \( 1 \leq k < m; \) \((iii) \) \( \ker(i) \geq \gamma \delta \gamma^{-1}, \) i.e.

\[ i_1 = i_{-m}, i_2 = i_{-1}, i_3 = i_{-2}, \ldots, i_m = i_{-(m-1)}. \]

Notice that a matrix \( a = (a_{ij})_{ij} \) is \( R \)-cyclic exactly when all cumulants \( \kappa_b(a_{i_1i_{-1}}, a_{i_2i_{-2}}, \ldots, a_{i_{m}i_{-b}}) \) vanish except possibly when \((i) b \) is even and \((ii) \) \( \ker(i) \geq \gamma \delta \gamma^{-1}. \) So condition (4) is a combination of \( R \)-cyclicity and \( R \)-diagonality; which might be called \( R^* \)-cyclicity.

**Notation 8.** Recall that \( w_{1,k} = \text{diag}(v_{k+1,1}, v_{k+2,2}, \ldots, v_{k,n}). \) We shall interpret the indices of \( v \) modulo \( n; \) so that when \( b = 5, v_{-2,4} = v_{3,4}. \) With this convention we let

\[ w_{1,k;j} = s^j w_{1,k}s^{-j} = \text{diag}(v_{k+j+1,1}, v_{k+j+2,2}, \ldots, v_{k+j,j}). \]

Then we have \( w_{1,k;j}^* = \text{diag}(v_{k+j+1,1}^*, v_{k+j+2,2}^*, \ldots, v_{k+j,j}^*). \) For \( k \geq 0, \) let

\[ w_{-1,k} = \text{diag}(v_{b-k+1,1}^{[-1]}, v_{b-k+2,2}^{[-1]}, \ldots, v_{b-k,b}^{[-1]}) \]

\[ = \text{diag}(v_{1,b-k+1}^*, v_{2,b-k+2}^*, \ldots, v_{b,b-k}^*) = s^{-k} w_{1,k}^* s^k = w_{1,k,-k}. \]

For example when \( b = 5 \) and \( k = 2 \) we have

\[ w_{1,2} = \text{diag}(v_{31}, v_{42}, v_{53}, v_{14}, u_{25}) \]

and

\[ w_{-1,2} = \text{diag}(v_{41}^{[-1]}, v_{52}^{[-1]}, v_{13}^{[-1]}, v_{24}^{[-1]}, v_{35}^{[-1]}) \]

\[ = \text{diag}(v_{14}^*, v_{25}^*, v_{31}^*, v_{42}^*, v_{53}^*) = s^{-2} w_{1,2}^* s^2. \]
The idea is that putting the minus sign in \( w_{-1,k} \) suggests replacing \( k \) by \(-k\). When \( k = 0 \) we have
\[
w_{1,0} = \text{diag}(v_{11}, \ldots, v_{m}) \quad \text{and} \quad w_{-1,0} = w_{1,0}^*.
\]
Thus for \( \epsilon \in \{-1,1\} \) we can rewrite our definition of \( w \) for \( k \geq 0 \) as
\[
w_{\epsilon,k} = \text{diag}(v_{\epsilon,k+1,1}, \ldots, v_{\epsilon,k,b}).
\]
As before we let \( w_{-1,k,j} = s^j w_{-1,k} s^{-j} \). Then for \( k \geq 0 \)
\[
w_{-1,k,j} = \text{diag}(v_{\epsilon,k+j+1,j+1}, v_{\epsilon,k+j+2,j+2}, \ldots, v_{\epsilon,k,j,j})
\]
\[
= \text{diag}(v_{\epsilon,k+j+1,j+1}^*, v_{\epsilon,k+j+2,j+2}^*, \ldots, v_{\epsilon,k,j,j}^*).
\]
Then for \( \epsilon \in \{-1,1\} \), \( k \geq 0 \), and \( j \in \mathbb{Z} \) we have
\[
w_{\epsilon,k,j} = \text{diag}(v_{\epsilon,k+j+1,j+1}^*, v_{\epsilon,k+j+2,j+2}^*, \ldots, v_{\epsilon,k,j,j}^*).
\]
Recall that \( v_k = w_k s^k \), for \( k > 0 \). Thus
\[
v_k^* = s^{-k} w_{1,k}^* = s^{-k} w_{1,k}^* s^k s^{-k} = w_{-1,k} s^{-k}.
\]
Hence \( v_k(\epsilon) = w_{\epsilon,k} s^k \). The \( m \)th entry of \( w_{\epsilon,k,j} \) is \( v_{\epsilon,k+j+m,j+m} \).

**Notation 9.** Let \( \mathcal{D} \subseteq M_b(U_{nc}^b) \) denote the subalgebra of \( b \times b \) diagonal scalar matrices. We let \( \tilde{\varphi} \) denote the conditional expectation from \( M_b(U_{nc}^b) \) to \( \mathcal{D} \) given by
\[
\tilde{\varphi}((a_{ij})_{ij}) = \text{diag}(\varphi(a_{11}), \ldots, \varphi(a_{bb})).
\]
We let \( \tilde{\kappa}_m \) denote the \( m \)th \( \mathcal{D} \)-valued cumulant. Thus for \( a_1, \ldots, a_m \in M_b(U_{nc}^b) \) we have
\[
\tilde{\kappa}_m(a_1, \ldots, a_m) = \sum_{\pi \in \mathcal{NC}(m)} \mu(\pi, 1_m) \tilde{\varphi}_\pi(a_1, \ldots, a_m).
\]
If \( a_1, \ldots, a_m \) are diagonal matrices in \( M_b(U_{nc}^b) \), as are our matrices \( w_{\epsilon,k,j} \), then the \( \mathcal{D} \)-valued cumulants can just be computed entry wise.

**Proposition 10.** The \( l \)th entry of the diagonal matrix
\[
\tilde{\kappa}_m(w_{\epsilon_1,i_1}, w_{\epsilon_2,i_2,\epsilon_1 i_1}, \ldots, w_{\epsilon_{m,i_{m,\epsilon_{1 i_1 + \cdots + \epsilon_{m-1 i_{m-1}}}}}})
\]
is
\[
\kappa_m(v_{\epsilon_1,j_{1,l} + 1}, v_{\epsilon_2,j_{1+j_{2,l}+1}, \ldots, v_{\epsilon_{m,j_{1+\cdots+j_{m,l}+1}, \ldots, j_{m-1+l}}}})
\]
where \( j_k = \epsilon_k k \).

**Proof.** This follows the previous observation that the \( m \)th entry of \( w_{\epsilon,k,j} \) is \( v_{\epsilon,k+j+m,j+m} \).

Recall that we shall write \( i \equiv j \) to mean equivalence modulo \( b \).
Corollary 11.  
\[ \tilde{\kappa}_m(w_{\epsilon_1,\epsilon_2,\epsilon_3,\ldots,\epsilon_m,i_1,i_2,i_3,\ldots}) \neq 0 \]
only if: (i) \( m \) is even; (ii) \( \epsilon_k i_k + \epsilon_{k+1} i_{k+1} \equiv 0 \) for \( 1 \leq k < m \); (iii) \( i_k \equiv i_{k+1} \) for \( 1 \leq k < m \).

Proof. For a diagonal valued cumulant to be non-zero there has to be a
non-zero entry. By Lemma 4 and Proposition 10 this can only happen
when (i) \( m \) is even; (ii) \( \epsilon_k + \epsilon_{k+1} = 0 \) for \( 1 \leq k < m - 1 \); and (iii) \( \epsilon_k i_k + \epsilon_{k+1} i_{k+1} \equiv 0 \) for \( 1 \leq k < m \). By (ii) this last condition is
equivalent to \( i_k \equiv i_{k+1} \) for \( 1 \leq k < m \). \qed

Lemma 12. Let \( a = (a_{ij})_{i,j} \in M_b(U_n^{nc}) \). We let \( a = c_0 s_0 + \cdots + c_{b-1} s_{b-1} \)
be the diagonal decomposition of a where \( c_0, \ldots, c_{b-1} \) are diagonal
matrices and we let \( c_{i,j} = s^i c_i s^{-j} \).

i) Let \( \pi \) be in \( NC(l) \). If for each block \( V = (j_1, \ldots, j_r) \) of \( \pi \) we
have \( i_1 + \cdots + i_r \equiv 0 \) then
\[ \tilde{\varphi}_\pi(c_{i_1} s^{i_1}, \ldots, c_{i_r} s^{i_r}) = \tilde{\varphi}_\pi(c_{i_1,0}, \ldots, c_{i_1,i_2+\cdots+i_{r-1}}) \]

ii) For any \( \pi \in NC(l) \) we have
\[ \tilde{\varphi}_\pi(c_{i_1} s^{i_1}, \ldots, c_{i_r} s^{i_r}) = \tilde{\varphi}_\pi(c_{i_1,0}, \ldots, c_{i_1,i_2+\cdots+i_{r-1}}) \times \tilde{\varphi}_\pi(s^{i_1}, \ldots, s^{i_r}) \]

Proof. As we have seen
\[ c_{i_1} s^{i_1} \cdots c_{i_r} s^{i_r} = c_{i_1,0} c_{i_2} s^{i_1} \cdots c_{i_r} s^{i_1+\cdots+i_{r-1}} \]
\( c_{i_1,0} c_{i_2} s^{i_1} \cdots c_{i_r} s^{i_1+\cdots+i_{r-1}} \) is a diagonal matrix and \( s^{i_1+\cdots+i_r} \) will be 0 on
the diagonal unless \( i_1 + \cdots + i_r \equiv 0 \). Thus \( \tilde{\varphi}(c_{i_1} s^{i_1} \cdots c_{i_r} s^{i_r}) = 0 \) unless
\( i_1 + \cdots + i_r \equiv 0 \).

To prove (i) note that for \( \tilde{\varphi}_\pi(c_{i_1} s^{i_1}, \ldots, c_{i_r} s^{i_r}) \neq 0 \) we must have that
for each block \( V = (j_1, \ldots, j_r) \) of \( \pi \) we have \( i_1 + \cdots + i_r \equiv 0 \) and thus
\[ \tilde{\varphi}(c_{i_1} s^{i_1} \cdots c_{i_r} s^{i_r}) = \tilde{\varphi}(c_{i_1,j_r} s^{i_1+\cdots+i_{r-1}}) \tilde{\varphi}(s^{i_{r-1}}, \ldots, s^{i_r}) \]
Since this applies for every block we have
\[ \tilde{\varphi}_\pi(c_{i_1} s^{i_1}, \ldots, c_{i_r} s^{i_r}) = \tilde{\varphi}_\pi(c_{i_1,0}, \ldots, c_{i_1,i_2+\cdots+i_{r-1}}) \tilde{\varphi}_\pi(s^{i_{r-1}}, \ldots, s^{i_r}) \]

To prove (ii) note that \( \tilde{\varphi}_\pi(s^{i_1}, \ldots, s^{i_r}) \in \{0,1\} \) with \( \tilde{\varphi}_\pi(s^{i_1}, \ldots, s^{i_r}) = 1 \) only when for each block \( V = (j_1, \ldots, j_r) \) of \( \pi \) we have \( i_1 + \cdots + i_r \equiv 0 \). Thus both sides of the equation in claim (ii) vanish unless
the hypothesis in (i) applies, in which case (ii) follows from (i). \qed

Theorem 13. \( v_0, v_1, \ldots, v_{b-1} \) are *-free over \( D \).
Proof. We shall show that for $d_1, \ldots, d_{m-1} \in \mathcal{D}$ we have $\tilde{\kappa}_m(v_{i_1}^{(1)})d_1, \ldots, v_{i_m}^{(m-1)}d_{m-1}, v_{i_m}^{(m)} = 0$ unless: (i) $m$ is even; (ii) $i_1 \equiv i_2 \equiv \cdots \equiv i_m$; (iii) $\epsilon_k + \epsilon_{k+1} \equiv 0$. We shall let $d_{i,j} = s^i d_i s^{-j}$ and $\tilde{d} = d_{1,i_1} \cdots d_{b-1,i_1 + \cdots + i_{b-1}}$. Recall that $v_{i_k}^{(k)} = w_{i_k,i_k}^{*}$. 

\begin{align}
\tilde{\kappa}_m(w_{i_1,i_1}, & \ldots, w_{i_m,i_m}, s^{\epsilon_1 \epsilon_2} d_1, \ldots, d_{m-1}, w_{i_m,i_m}, s^{\epsilon_m \epsilon_m}) \\
= & \sum_{\pi \in NC(m)} \mu(\pi, 1_m) \tilde{\varphi}_\pi(w_{i_1,i_1}, \ldots, w_{i_m,i_m}, s^{\epsilon_1 \epsilon_2} d_1, \ldots, d_{m-1}, w_{i_m,i_m}, s^{\epsilon_m \epsilon_m}) \\
\equiv & \sum_{\pi \in NC(m)} \mu(\pi, 1_m) \tilde{\varphi}_\pi(s^{\epsilon_1 \epsilon_2}, \ldots, s^{\epsilon_m \epsilon_m}) \tilde{d} \\
= & \sum_{\pi \in NC(m)} \mu(\pi, 1_m) \tilde{\varphi}_\pi(s^{\epsilon_1 \epsilon_2}, \ldots, s^{\epsilon_m \epsilon_m}) \tilde{d} \\
\equiv & \sum_{\pi \in NC(m)} \mu(\pi, 1_m) \sum_{\sigma \in NC(b)} \tilde{\kappa}_\sigma(w_{i_1,i_1}, \ldots, w_{i_m,i_m}, s^{\epsilon_1 \epsilon_2} \cdots + s^{\epsilon_m \epsilon_m-1}) \tilde{d} \\
= & \sum_{\pi \in NC(m)} \mu(\pi, 1_m) \tilde{\varphi}_\pi(w_{i_1,i_1}, \ldots, w_{i_m,i_m}, s^{\epsilon_1 \epsilon_2} \cdots + s^{\epsilon_m \epsilon_m-1}) \tilde{d} \\
(5) & = \tilde{\kappa}_m(w_{i_1,i_1}, \ldots, w_{i_m,i_m}, s^{\epsilon_1 \epsilon_2} \cdots + s^{\epsilon_m \epsilon_m-1}) \tilde{d}.
\end{align}

In the calculation above $(*)_1$ holds by Lemma 12 (ii). By Corollary 14 we know that $\tilde{\kappa}_\sigma(w_{i_1,i_1}, \ldots, w_{i_m,i_m}, s^{\epsilon_1 \epsilon_2} \cdots + s^{\epsilon_m \epsilon_m-1}) = 0$ unless for each block $(i_1, \ldots, i_l)$ of $\sigma$ we have $i_1 + \cdots + i_{l_i} \equiv 0$. If $\sigma \leq \pi$ the same condition holds for any block of $\pi$. Thus for these $\sigma$'s we have $\varphi_\sigma(s^{\epsilon_1 \epsilon_2}, \ldots, s^{\epsilon_m \epsilon_m}) = \varphi_\pi(s^{\epsilon_1 \epsilon_2}, \ldots, s^{\epsilon_m \epsilon_m}) = 1$. This justifies $(*)_2$.

By Corollary 14 the cumulant in $(5)$ vanishes unless: (i) $m$ is even; (ii) $\epsilon_k + \epsilon_{k+1} = 0$ for $1 \leq k < n$; and (iii) $i_1 \equiv \cdots \equiv i_m$. 

\begin{remark}
From [1] Thm. 3.19 we can conclude that $v_0$, $\{v_1, v_{b-1}\}$, ..., $\{v_{b/2-1}, v_{b/2+1}\}$, $v_{b/2}$ are free (assuming $b$ is even). Because of the additional structure of $v$ we get the following stronger conclusion.

**Theorem 15.** $v_0, v_1, \ldots, v_{b-1}$ are $R$-diagonal and $*$-free over $\mathbb{C}$.

Proof. In Theorem 13 we proved freeness over $\mathcal{D}$, the diagonal scalar matrices, so it suffices to show that $\tilde{\kappa}_m(v_{i_1}^{(1)}, \ldots, v_{i_m}^{(m)}) = \kappa_m(v_{i_1}^{(1)}, \ldots, v_{i_m}^{(m)})$. By Lemma 4, $\tilde{\kappa}_m(v_{i_1}^{(1)}, \ldots, v_{i_m}^{(m)})$, the diagonal valued cumulant
in \([3]\), is actually a multiple of the identity matrix. On the other hand 
\(\varphi = \text{tr} \circ \varphi_b\), where \(\varphi_b\) is the state on \(M_b(\mathbb{C}) \otimes U_b^{nc}\) given in Definition
\(2\). Thus \(\kappa_m = \text{tr} \circ \tilde{k}_m\). Hence \(\tilde{k}_m(v_{i_1}, \ldots, v_{i_m}) = \kappa_m(v_{i_1}, \ldots, v_{i_m})\).
Thus we have vanishing of mixed cumulants and hence freeness. \(\square\)

**Theorem 16.** The transpose of a Haar unitary has the same *-distribution as \(b^{-1}(u_1 + \cdots + u_{2^k})\) where \(u_1, \ldots, u_{2^k}\) are \(b^2\) *-free Haar unitaries; i.e., \(v^t \overset{D}{\sim} b^{-1}v^{\oplus 2}\).

*Proof.* From Theorem 15 we only have to show that each \(bv_i\) has same *-distribution as the sum of \(b\) *-free Haar unitaries. We already have shown that \(v_i\) is \(R\)-diagonal, so it remains to show that \(\kappa_{2m}(bv_i, bv_i^*) = \beta_m\). By equation (5) in the proof of Theorem 15 we have
\(\tilde{k}_{2m}(v_i, v_i^*) = \kappa_{2m}(w_{1,1}, w_{-1,1}, \ldots, w_{1,1}, w_{-1,1})\). By Proposition 10, the \(i\)th entry of this diagonal matrix is \(\kappa_{2m}(v_{i+1,t}, v_{i+1,t}^*, \ldots, v_{i+1,t}, v_{i+1,t}^*) = b_1^{-2m}\beta_m\), with the last equality by Equation (1). Thus \(\tilde{k}_{2m}(v_i, v_i^*) = b_1^{-2m}\beta_m\) and hence \(\kappa_{2m}(v_i, v_i^*, \ldots, v_i, v_i^*) = b_1^{-2m}\beta_m\) as claimed. \(\square\)

2.3. Free Independence results in the Brown algebra

The result below is a non-commutative analogue of the asymptotic freeness between Haar unitaries and independent random matrices.

Suppose that \((\mathcal{A}, \varphi)\) is a *-non-commutative probability space such that \(\mathcal{A}\) is a unital *-algebra containing the *-algebra generated by \(\{v_{i,j} : 1 \leq i, j \leq n\}\) and some *-algebra \(\mathcal{B}\) free from the family \(\{v_{i,j} : 1 \leq i, j \leq n\}\).

As before, denote by \(v\) the \(n \times n\) matrix whose \((i, j)\)-entry is \(v_{i,j}\) (in particular, \(v\) is unitary) and denote by \(v^t\) the matrix transpose of \(v\), i.e., the \((i, j)\)-th entry of \(v^t\) is \(v_{j,i}\).

**Theorem 17.** With the notations from above, \(v\) is free from \(M_n(\mathcal{B})\) with respect to \(\Phi = \varphi \circ \text{tr}\).

*Proof.* It suffices to show that

\[
\Phi(v^{(n)}_1A_1v^{(n_2)}A_2 \cdots v^{(n_m)}A_m) = \sum_{\pi \in NC(m)} \nu_{\pi}[v^{(n_1)}v^{(n_2)} \cdots v^{(n_m)}] \cdot \Phi_{\text{Kr}(\pi)}[A_1, \ldots, A_m]
\]

for any positive integer \(m\), any matrices \(A_1, A_2, \ldots, A_m \in M_n(\mathcal{B})\) and any \(n_1, \ldots, n_m \in \{1, *, \}\).

With the notations

\[
\eta_s(i, j) = \begin{cases} (i, j) & \text{if } \eta_s = 1 \\ (j, i) & \text{if } \eta_s = * \end{cases}
\]

and \(v^{(n)} = \begin{cases} v & \text{if } \eta_n = 1 \\ v^* & \text{if } \eta_n = * \end{cases}\),
we have \([v^{(n)}]_{i,j} = v^{(q)}_{η(i,j)}\); thus we get
\[
\Phi(v^{(n)}A_1 v^{(n_2)} A_2 \cdots v^{(n_m)} A_m)
= \sum_{i \rightarrow j} \varphi(v^{(n)}_{η_1(i,j_1)}, \ldots, v^{(n_2)}_{η_2(i,j_2)}, \ldots, a^{(m-1)}_{j_m,i_m} v^{(n_m)}_{η_m(i,j_m)} a^{(m)}_{j_m,i_1})
= \sum_{π \in NC(m)} \sum_{i \rightarrow j} \kappa_π^{ϕ}[v^{(n)}_{η_1(i,j_1)}, \ldots, v^{(n_2)}_{η_2(i,j_2)}, \ldots, v^{(n_m)}_{η_m(i,j_m)}] \cdot \varphi(π)[a^{(1)}_{j_1,i_2}, \ldots, a^{(m)}_{j_m,i_1}],
\]
where \(\rightarrow i = (i_1, \ldots, i_m)\), \(\rightarrow j = (j_1, \ldots, j_m)\) and \(a^{(s)}_{i,j}\) is the \((i,j)\)-th entry of the matrix \(A_s\).

It suffices to show that the equality below holds true for any non-crossing partition \(π\):
\[
(6) \quad \kappa_π^{ϕ}[v^{(n)}_1, v^{(n_2)}_2, \ldots, v^{(n_m)}_m] \cdot \Phi(π)[A_1, \ldots, A_m]
= \sum_{i \rightarrow j} \kappa_π^{ϕ}[v^{(n)}_{η_1(i,j_1)}, \ldots, v^{(n_2)}_{η_2(i,j_2)}, \ldots, v^{(n_m)}_{η_m(i,j_m)}] \cdot \varphi(π)[a^{(1)}_{j_1,i_2}, \ldots, a^{(m)}_{j_m,i_1}],
\]

Let \(π \in NC(m)\). One of the blocks of \(π\), say \(B = (t+1, \ldots, s)\), is an interval. Then, denoting by \(\bar{π}\) the non-crossing partition obtained by removing the block \(B\) from \(π\), we have that
\[
k_π^{ϕ}[v^{(n)}_1, v^{(n_2)}_2, \ldots, v^{(n_m)}_m] = K_1 \cdot \kappa_π^{ϕ}[v^{(n)}_1, v^{(n)}_2, \ldots, v^{(n)}_{t-1}, v^{(n_{t+1})}_t, \ldots, v^{(n)}_{s+1}]
\]

\[
\Phi(π)[A_1, \ldots, A_m] = F_1 \cdot \Phi(π)[A_{t+1}, \ldots, A_{s+1}].
\]

Also, we have that
\[
k_π^{ϕ}[v^{(n)}_{η_1(i,j_1)}, \ldots, v^{(n_m)}_{η_m(i,j_m)}]
= K_2(\rightarrow i_α, \rightarrow j_α) \cdot k_π^{ϕ}[v^{(n)}_{η_1(i,j_1)}, \ldots, v^{(n)}_{η_2(i,j_2)}, \ldots, v^{(n_1)}_{η_1(i,j_1)}]
\]

and
\[
\varphi(π)[a^{(1)}_{j_1,i_2}, \ldots, a^{(m)}_{j_m,i_1}]
= F_2(\rightarrow i_α, \rightarrow j_α) \cdot \varphi(π)[a^{(1)}_{j_1,i_2}, \ldots, a^{(t-1)}_{j_{t-1},i_t}, a^{(t)}_{j_t,i_{t+1}} \cdot a^{(s)}_{j_{s+1},i_{s+1}}, a^{(s+1)}_{j_{s+1},i_{s+2}}, \ldots, a^{(m)}_{j_{m-1},i_1}]
\]

where \(\rightarrow i_α = (i_{t+1}, i_{t+2}, \ldots, i_s)\), respectively \(\rightarrow j_α = (j_{t+1}, j_{t+2}, \ldots, j_s)\), and
\[
K_2(\rightarrow i_α, \rightarrow j_α) = \kappa^{ϕ}_{s+1}(u^{(n_1)}_{η_1(i_{t+1},j_{t+1})}, u^{(n_2)}_{η_2(i_{t+2},j_{t+2})}, \ldots, u^{(n_s)}_{η_s(i_s,j_s)})
\]
If \( v \) is R-diagonal, \( K_1 \) cancels unless \( \eta_{l+t} \neq \eta_{l+t+1} \) for all \( l = 1, \ldots, s - t - 1 \); according to (1) and (2), so does \( K_2 \) for any \( \overrightarrow{i}, \overrightarrow{j} \).

Suppose that \( \eta_{l+t} \neq \eta_{l+t+1} \) for all \( l = 1, \ldots, s - t - 1 \). Then

\[
K_1 = (-1)^{s-t-1} \cdot \text{Cat}_{s-t-1}.
\]

On the other hand, utilizing (1) and (2), we have that

\[
K_2(\overrightarrow{i}_\alpha, \overrightarrow{j}_\beta) = n^{1-2r} (-1)^{r-1} \text{Cat}_{r-1} \cdot \delta_{j_t+1}^{i_t+1} \delta_{j_{t+2}}^{i_{t+2}} \delta_{j_{t+3}}^{i_{t+3}} \cdots \delta_{j_s}^{i_s}
\]

therefore

\[
\sum_{\overrightarrow{i}_\alpha, \overrightarrow{j}_\beta} \varphi(a_{j_t,i_t+1}^{(t)} \cdot a_{j_{s+1},i_{s+1}}^{(s)}) K_2(\overrightarrow{i}_\alpha, \overrightarrow{j}_\beta) \cdot F_2(\overrightarrow{i}_\alpha, \overrightarrow{j}_\beta) = \left[ \sum_{\overrightarrow{i}_\alpha, \overrightarrow{j}_\beta} \varphi([A_t A_s]_{j_t,i_{s+1}}) \cdot n^{1-2(s-t)} (-1)^{s-t-1} \text{Cat}_{s-t-1} \right]
\]

\[
\times \prod_{l=t+1}^{s-1} \text{Tr} \circ \varphi(A_l) \cdot | \{ (\overrightarrow{i}_\alpha, \overrightarrow{j}_\beta) : \delta_{j_{t+l}}^{i_{t+l+1}} \delta_{j_{t+l+2}}^{i_{t+l+2}} \cdots \delta_{j_s}^{i_s} = 1 \} |
\]

\[
= \varphi([A_t A_s]_{j_t,i_{s+1}}) \cdot K_1 \cdot F_1
\]

Denoting \( \overrightarrow{i}_\beta = (i_1, i_2, \ldots, i_t, i_{s+1}, i_{s+2}, \ldots, i_m) \) and, respectively \( \overrightarrow{j}_\beta = (j_1, j_2, \ldots, j_t, j_{s+1}, j_{s+2}, \ldots, j_m) \), the relation above gives that

\[
\sum_{\overrightarrow{i}_\alpha, \overrightarrow{j}_\beta} \kappa_{\varphi}^{\pi}[v_{\eta_1(i_1,j_1)}, \ldots, v_{\eta_m(i_m,j_m)}] \cdot \varphi_{\text{Kr}(\pi)}[a_{j_1,i_2}^{(1)}, \ldots, a_{j_m,i_1}^{(m)}] = K_1 \cdot F_1 \cdot \sum_{\overrightarrow{i}_\alpha, \overrightarrow{j}_\beta} \kappa_{\varphi}^{\pi}[v_{\eta_1(i_1,j_1)}, \ldots, v_{\eta_{s+1}(i_{s+1},j_{s+1})}, \ldots, v_{\eta_m(i_m,j_m)}] \cdot \varphi_{\text{Kr}(\pi)}[a_{j_1,i_2}^{(1)}, \ldots, a_{j_{t-1},i_1}^{(t-1)}, [A_t A_s]_{j_t,i_{s+1}}, a_{j_{s+1},i_{s+2}}^{(s+1)}, \ldots, a_{j_m,i_1}^{(m)}]
\]

and (6) follows by induction on \( m \). \( \Box \)

**Remark 18.** \( M_n(\mathcal{B}) \) is not free from \( v^t \). For example, \( v^t \) is not free from the matrix

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}.
\]

If \( v^t \) and \( A \) were free, since \( \Phi(A) = \Phi(v) = 0 \), we would have that

\[
\Phi\left(v^t A(v^t)^* A v^t A(v^t)^* A\right) = 0.
\]
On the other hand, denoting by $a_{i,j}$ the $(i,j)$-th entry of $A$, we have that
\[
\Phi(v^t A(v^t)^* A v^t A(v^t)^*) = \frac{1}{b} \sum_{i_1, \ldots, i_b=1}^{b} \varphi(v_{i_2,i_1} a_{i_2,i_3} v_{i_3,i_4}^* a_{i_4,i_5} v_{i_5,i_6} v_{i_6,i_7} a_{i_7,i_8}^* a_{i_8,i_1}),
\]
where $a_{i,j}$ is the $(i,j)$-th entry of $A$. In particular, $a_{i,j} = 0$ whenever $i > 2$ or $j > 2$, so
\[
\Phi(v^t A(v^t)^* A v^t A(v^t)^*) = \frac{1}{n} \sum_{i_1, \ldots, i_b=1}^{2} \varphi(v_{i_2,i_1} v_{i_3,i_4}^* v_{i_4,i_5} v_{i_5,i_6} a_{i_2,i_3} a_{i_4,i_5} a_{i_6,i_7} a_{i_8,i_1}).
\]
Since $\varphi(v_{i,j}) = 0$ for all $i,j$, there are only three partitions to consider: \{(1,2), (3,4)\}, \{(1,4), (2,3)\}, and \{(1,3,4)\}. If $\pi = \{(1,2), (3,4)\}$, then
\[
\kappa_{\pi} [v_{i_2,i_1}, v_{i_3,i_4}^*, v_{i_4,i_5}, v_{i_5,i_6}] = \kappa_2 (v_{i_2,i_1}, v_{i_3,i_4}^*) \cdot \kappa_2 (v_{i_4,i_5}, v_{i_5,i_6}).
\]
From Equation (11), we have that $\kappa_2 (v_{i_2,i_1}, v_{i_3,i_4}^*) \neq 0$ only if $i_2 = i_3$. Then $a_{i_2,i_3} = 0$ since $a_{i,i} = 0$ for all $i$.

Similarly, if $\pi = \{(1,4), (2,3)\}$, then
\[
\kappa_{\pi} [v_{i_2,i_1}, v_{i_3,i_4}^*, v_{i_4,i_5}, v_{i_5,i_6}] = \kappa_2 (v_{i_2,i_1}, v_{i_3,i_4}^*) \cdot \kappa_2 (v_{i_4,i_5}, v_{i_6,i_7}),
\]
and now $\kappa_2 (v_{i_3,i_4}^*, v_{i_6,i_7}) \neq 0$ only if $i_4 = i_5$, that is $a_{i_4,i_5} = 0$.

Finally, if $\pi = \{(1,2,3,4)\}$, then $\kappa_{\pi} [v_{i_2,i_1}, v_{i_3,i_4}^*, v_{i_4,i_5}, v_{i_5,i_6}] \neq 0$ only if $i_1 = i_4$, $i_2 = i_5$, $i_3 = i_6$, $i_4 = i_7$ and $i_5 = i_8$.

\[
\Phi(v^t A(v^t)^* A v^t A(v^t)^*) = -n^{-4} \sum_{i_1,i_2,i_3,i_5=1}^{2} a_{i_2,i_3} a_{i_3,i_2} a_{i_1,i_5} a_{i_5,i_1} = -4n^{-4} \neq 0.
\]

3. ASYMPTOTIC FREE INDEPENDENCE OF DIFFERENT PARTIAL TRANSPOSES OF A HAAR UNITARY RANDOM MATRIX

3.1. FRAMEWORK AND PREVIOUS RESULTS ON PARTIAL TRANSPOSES

In [9], we gave a necessary and sufficient condition for the asymptotic free independence of different families of partial transposes of Wishart random matrices. More precisely, suppose that $(b_N)_N, (b'_N)_N, (d_N)_N$
and \((d'_N)_N\) are non-decreasing sequences of positive integers such that for each \(N\) we have \(b_N \cdot d_N = b'_N \cdot d'_N = M_N\) and that \(\lim_{N \to \infty} M_N = \infty\). If \(W_N\) denotes a \(M_N \times M_N\) Wishart random matrix, and \(\vartheta, \vartheta' \in \{1, -1\}\), then the asymptotic free independence of the families \((W_N^{\varphi(b'_N,d'_N)})_N\) and 
\((W_N^{\varphi(b_N,d_N)})_N\) is equivalent to the condition

\[
\lim_{N \to \infty} \frac{1}{M_N^2} \left| \left\{ (i,j) \in [M_N]^2 : \Gamma^{(\varphi)}_{b_N,d_N}(i,j) = \Gamma^{(\varphi)}_{b'_N,d'_N}(i,j) \right\} \right| = 0.
\]

The main results of this Section, Theorems 25 and 26, is that condition (7) is equivalent to asymptotic \(*\)-freeness for different partial transposes also in the case of Haar unitary random matrices.

We will use the following technical results (proved in [9, Theorem 3.2]) on partial transposes.

**Lemma 19.**

(i) If \(\vartheta = \vartheta'\), then condition (7) is equivalent to

\[
\lim_{N \to \infty} \frac{\text{l.c.m.}(b_N, b'_N)}{\min(b_N, b'_N)} = \lim_{N \to \infty} \frac{\text{l.c.m.}(d_N, d'_N)}{\min(d_N, d'_N)} = \infty.
\]

Also, for each \(N\), the sets

\[
\{(i,j) \in [M_N]^2 : (\Gamma^{(\varphi)}_{b_N,d_N})^{-1} \circ \Gamma^{(\varphi)}_{b'_N,d'_N}(i,j) = (i,j)\}
\]

and (for \(s = 1, 2\), where \(\pi_s\) is the projection on the \(i\)-th coordinate),

\[
\{(i_1, i_2, j) \in [M_N]^3 : \pi_s \circ \Gamma^{(\varphi)}_{b_N,d_N}(i_1, j) = \pi_s \circ \Gamma^{(\varphi)}_{b'_N,d'_N}(i_2, j)\}
\]

have the same number of elements.

(ii) If \(\vartheta \neq \vartheta'\) then the condition (7) is equivalent to

\[
\lim_{N \to \infty} b_N \cdot d_N = \lim_{N \to \infty} b'_N \cdot d_N = \infty.
\]

Moreover, in this case, condition (7) implies that (for \(s = 1, 2\))

(a) \(|\{(i_1, i_2, j) \in [M_N]^3 : \Gamma^{(\varphi)}_{b_N,d_N}(i_1, j) = \Gamma^{(\varphi)}_{b'_N,d'_N}(i_2, j)\}| = o(M^2)

(b) \(|\{(i_1, i_2, j) \in [M_N]^3 : \pi_s \circ \Gamma^{(\varphi)}_{b_N,d_N}(i_1, j) = \pi_s \circ \Gamma^{(\varphi)}_{b'_N,d'_N}(i_2, j)\}| = o(M^3)

Note that for \(\alpha \neq \beta\) and \(b_N \sim N^\alpha\) and \(b'_N \sim N^\beta\) then

\[
\lim_{N \to \infty} \frac{\text{l.c.m.}(b_N, b'_N)}{\min(b_N, b'_N)} = \infty
\]
Before stating and proving the main theorems of this section, we need to review some results on the unitary Weingarten function.

3.2. ON THE UNITARY WEINGARTEN CALCULUS

**Notation 20.** Recall the unitary Weingarten function \( Wg \). It is a central element of the group ring \( \mathbb{C}[S_n] \) and, by definition, it is the inverse of the function \( \sigma \mapsto N^{\#(\sigma)} \) where \( N \geq n \) and \( \#(\sigma) \) is the number of cycles in the cycle decomposition of \( \sigma \).

Collins [4] showed that for \( U = (u_{ij})_{ij} \) a \( N \times N \) Haar distributed random unitary matrix we have

\[
E(u_{i_1j_1} \ldots u_{i_nj_n} \overline{u_{i_1'j_1'}} \ldots \overline{u_{i_n'j_n'}}) = \sum_{\pi,\sigma \in S_n} Wg_N(\sigma^{-1}\pi)\delta_{i,i\circ\sigma}\delta_{j,j\circ\pi}
\]

where \( \delta_{i,i\circ\sigma} = 1 \) when \( i_1 = i_{\sigma(1)}, \ldots, i_n = i_{\sigma(n)} \) and 0 otherwise.

Collins also showed that for \( \sigma \in S_n \) we have

\[
Wg_N(\sigma) = w_1(\sigma)N^{-2n+\#(\sigma)} + O(N^{-2n+\#(\sigma)-2})
\]

and provided a explicit function for \( w_1(\sigma) \). Indeed \( w_1(\sigma) \) is the product \( \prod_{i=1}^k (-1)^{l_i-1}C_{l_i-1} \) where we decompose \( \sigma \) into a product of cycles \( c_1 \cdots c_k \) and \( l_i \) is the number of elements in the \( i^{th} \) cycle, and \( C_r \) is the \( r^{th} \) Catalan number \( \frac{1}{r+1} \binom{2r}{r} \).

As shown in [8, Prop. 10], we can rewrite equation (8) using pairings. To do so we must recall a lemma about pairings from [7, Lemma 2].

**Lemma 21.** Given two pairings \( p \) and \( q \) of \([n]\) we consider \( p \) and \( q \) to be permutations and then decompose the product \( pq \) into cycles. When we do this we can write \( pq = c_1c'_1c_2c'_2 \cdots c_kc'_k \) with \( c'_i = qc_i^{-1}q \). Moreover the blocks of \( p \vee q \) are \( \{c_1 \cup c'_1, \ldots, c_k \cup c'_k\} \). Thus \( 2\#(p \vee q) = \#(pq) \).

**Notation 22.** Given pairings \( p \) and \( q \) of \([n]\) we denote by \( Wg_N(p,q) \) the value \( Wg_N(\sigma) \) where \( \sigma \in S_{n/2} \) has the same cycle decomposition as \( c_1c_2 \cdots c_k \), where \( pq = c_1c'_1c_2c'_2 \cdots c_kc'_k \). With this notation, Collins’ formula \( (9) \) becomes for \( p, q \in P_2(n) \)

\[
Wg_N(p,q) = w_1(\sigma)N^{-n+\#(pq)/2} + O(N^{-n+\#(pq)/2-2}).
\]

Next, we will remind the main result from [10] concerning the asymptotic behavior (as \( N \to \infty \)) of

\[
E \circ \text{tr}(U^{\sigma_1} \ldots U^{\sigma_n})(U^{\sigma_2} \ldots (U^{\sigma_n})^n)
\]

where \( U \) is a \( N \times N \) Haar unitary random matrix, \( \sigma_1, \ldots, \sigma_n \) are entry permutations of \( N \times N \) matrices, and \( \eta : [n] \to \{1,*\} \).
With the notations
\[ \eta_s(i, j) = \begin{cases} (i, j) & \text{if } \eta_s = 1 \\ (j, i) & \text{if } \eta_s = * \end{cases} \quad \text{and} \quad \varsigma^{(\eta_s)} = \begin{cases} \varsigma & \text{if } \eta_s = 1 \\ \emptyset & \text{if } \eta_s = * \end{cases}, \]
we have that
\[
E(\text{tr}((U_s \sigma_1)^{\eta_1} (U_s \sigma_2)^{\eta_2} \cdots (U_s \sigma_n)^{\eta_n})) = \sum_{i_1, \ldots, i_n=1}^{N} \frac{1}{N} E(u_{\sigma_1 \eta_1(i_1, i_2)} \cdots u_{\sigma_n \eta_n(i_n, i_1)}),
\]
where \((k_s, l_s) = \sigma_s \circ \eta_s(i_s, j_s)\) for \(s = 1, 2, \ldots, n\).

Denote by \(P^s_\sigma(n)\) the set of all pairings \(p\) on \([n]\) such that \(\eta_s \neq \eta_{p(s)}\) and define \((k_s, l_s)_{1 \leq s \leq n}\) via \((k_s, l_s) = \sigma_s \circ \eta_s(i_s, i_{s+1})\), with the convention \(i_{n+1} = i_1\). Applying (10), we then obtain
\[
E \circ \text{tr}((U_s \sigma_1)^{\eta_1} (U_s \sigma_2)^{\eta_2} \cdots (U_s \sigma_n)^{\eta_n}) = \sum_{p, q \in P^s_\sigma(n)} \mathcal{V}_{\sigma, \eta, N}(p, q)
\]
where \(\sigma' = (\sigma_1, \sigma_2, \ldots, \sigma_n)\), and
\[
(11) \quad \mathcal{V}_{\sigma', \eta, N}(p, q) = W_{\mathcal{G}_N}(p, q) \cdot \frac{1}{N} |\mathcal{A}_{\sigma, \eta, N}^{(p,q)}|,
\]
and \(\mathcal{A}_{\sigma, \eta, N}^{(p,q)}\) is the set
\[
\{(i_1, j_1, \ldots, i_n, j_n) \in \mathbb{N}^{2n} : i_1 = j_n, i_{s+1} = j_s \text{ for each } s \in [n-1],
\quad \text{and } k_s = k_{p(s)}, l_s = l_{q(s)} \text{, for each } s \in [n]\}.
\]

For \(S\) a subset of \([m]\), we denote
\[
\mathcal{F}_{\sigma', \eta, N}^{(p,q)}(S) = |\{(i_s, j_s)_{s \in S} : \text{there exists } (i_r, j_r)_{r \notin S} \text{ such that } (i_1, j_1, \ldots, i_m, j_m) \in \mathcal{A}_{\sigma', \eta, N}^{(p,q)}\}|.
\]

We showed in [10] the result below.

**Lemma 23.** If there exists some \(S \subseteq [n]\) interval, i.e. \(S = \{t + 1, t + 2, \ldots, t + r\}\) for some \(r > 0\), such that
\[
\mathcal{F}_{\sigma', \eta, N}^{(p,q)}(S) = o(N^{\left|S\right|+1-\left|B: B \text{ block in } p \lor q \text{ and } B \subseteq S\right|})
\]
then \(\mathcal{V}_{\sigma', \eta, N}(p, q) = o(N^0) = o(1)\).

Moreover, as shown in [10] (Proposition 3.1), \(\mathcal{V}_{\sigma', \eta, N}(p, q)\) has the following properties:

\begin{enumerate}
\item [(v1)] \(\mathcal{V}_{\sigma', \eta, N}(p, q) = O(N^0) = O(1)\) for any \(p, q \in P^s_\sigma(n)\).
\item [(v2)] If \(p \lor q\) is crossing, then \(\mathcal{V}_{\sigma', \eta, N}(p, q) = O(N^{-1})\).
\end{enumerate}
(v3) Suppose that $B = \{a_1, a_2, \ldots, a_r\}$ is a block of $p \vee q$, with $a_1 < a_2 < \cdots < a_r$. Then either $V_{\pi, \eta} (p, q) = O(N^{-1})$ or $r$ is even, for each $s \in [r]$ we have that $\eta_{a_s} \neq \eta_{a_{s+1}}$ and $\{p(a_s), q(a_s)\} = \{a_{s-1}, a_{s+1}\}$.

Denoting $\tilde{p}_k$ and $\tilde{q}_k$ are pairings on $[2k]$ given by $\tilde{p}_k (2l) = 2l + 1 (\text{mod } 2k)$ respectively $\tilde{q}_k (2l) = 2l - 1 (\text{mod } 2k)$, the last condition is equivalent to the pair $(p, q)$ of restrictions to $B$ of $p$, respectively $q$, is either $(\tilde{p}_\sigma, \tilde{q}_\tau)$ or $(\tilde{q}_\sigma, \tilde{p}_\tau)$.

(v4) If $\sigma_1 = \sigma_2 = \cdots = \sigma_n = \text{Id}$, then $U$ is $R$-diagonal for each $N$ and the free cumulants are given by

$$
\kappa_2^\sigma(U_N, U^*_N, \ldots, U_N, U^*_N) = (-1)^{r-1} \text{Cat}_{r-1} = \lim_{N \to \infty} V_{\sigma, \eta, \rho} (\tilde{p}_r, \tilde{q}_r)
$$

where $\eta', \eta'' : [2r] \to \{1, *\}$ are given by $\eta'(2s) = \eta''(2s-1) = *$ and $\eta''(2s-1) = \eta''(2s) = 1$ for each $s \in [r]$.

Remark 24. Denote by $NC_{\eta, \text{alt}} (n)$ the (possibly void) set of partitions $\pi$ on $[n]$ such that

- $\pi$ is non-crossing
- if $\{a_1, a_2, \ldots, a_r\}$ is a block of $\pi$ with $a_1 < a_2 < \cdots < a_r$, then $r$ is even and $\eta_1 \neq \eta_2 \neq \ldots \neq \eta_r$.

An immediate consequence of the properties (v1) – (v3) from above is the following.

$$
E \circ \text{tr} \left( (U^{\sigma_1})^{\eta_1} (U^{\sigma_2})^{\eta_2} \cdots (U^{\sigma_n})^{\eta_n} \right) = \sum_{\pi \in NC_{\eta, \text{alt}} (n)} V_{\pi, \eta, \rho} (\pi) + O(N^{-1}),
$$

where $V_{\pi, \eta, \rho} (\pi) = \sum_{p, q \in P_0^\pi (n)} V_{\pi, \eta, \rho} (p, q)$.

3.3. Main Results

Within this section, for $p$ a positive integer, $U_p$ will denote a $p \times p$ Haar unitary random matrix.

Suppose now that $r$ is a positive integer and, for each $i \in [r]$, we have that $\vartheta_i \in \{-1, 1\}$ and $(b_{i,N}), (d_{i,N})$ are two non-decreasing sequences of positive integers such that for every $i, i' \in [r]$,

$$
b_{i,N}d_{i,N} = b_{i',N}d_{i',N} = M_N
$$

for some strictly increasing sequence $(M_N)$.

To simplify the notations, for the rest of the section we will omit the subscript $N$, i.e. we shall write $M, b_s, d_s, \sigma_s$ for $M_N, b_{s,N}, d_{s,N}, \sigma_{s,N}$. 
Proposition 25. If \( s, t \in [r] \) are such that
\[
\liminf_{N \to \infty} \frac{1}{M^2} \left| \{(i, j) \in [M]^2 : (\Gamma_{b_{s}, d_{s}}^{(\theta_{s})})^{-1} \circ \Gamma_{b_{t}, d_{t}}^{(\theta_{t})} (i, j) = (i, j) \} \right| > 0
\]
then \( U_{M}^{\Gamma_{b_{s}, d_{s}}^{(\theta_{s})}} \) and \( U_{M}^{\Gamma_{b_{t}, d_{t}}^{(\theta_{t})}} \) are not asymptotically free.

Proof. Denote \( \tau_{s} = \Gamma_{b_{s}, d_{s}}^{(\theta_{s})} \) and \( \tau_{t} = \Gamma_{b_{t}, d_{t}}^{(\theta_{t})} \).
It suffices to show that \( \lim_{N \to \infty} E \circ \text{tr} \left( U_{M}^{\tau_{s}} (U_{M}^{\tau_{t}})^{*} \right) \neq 0 \).
And indeed
\[
\lim_{N \to \infty} E \circ \text{tr} \left( U_{M}^{\tau_{s}} (U_{M}^{\tau_{t}})^{*} \right) \geq \liminf_{N \to \infty} \sum_{i_{1}, i_{j}=1}^{M} E \left( u_{\tau_{s} (i, j) \tau_{t} (i, j)} \right)
\]
\[
= \liminf_{N \to \infty} \sum_{i_{1}, i_{j}=1}^{M} \frac{1}{M^2} \delta^{\tau_{s} (i, j)} > 0.
\]

Theorem 26. Denote \( \tau_{s} = \Gamma_{b_{s}, d_{s}}^{(\theta_{s})} \) for \( s = 1, 2, \ldots, n \). Then the family
\( U_{M}^{\tau_{1}}, \ldots, U_{M}^{\tau_{r}} \) is asymptotically free if and only if for any \( s \neq t \) we have that \( \Gamma_{b_{s}, d_{s}}^{(\theta_{s})} \) and \( \Gamma_{b_{t}, d_{t}}^{(\theta_{t})} \) satisfy the condition
\[
\lim_{N \to \infty} \frac{1}{M_{N}} \left| \{ (i, j) \in [M_{N}]^2 : \Gamma_{b_{s}, d_{s}}^{(\theta_{s})} (i, j) = \Gamma_{b_{t}, d_{t}}^{(\theta_{t})} (i, j) \} \right| = 0.
\]

Proof. Let \( m \) be a positive integer, and, for each \( k \in [m] \), let \( \eta_{k} \in \{1, *\} \) and let \( (\sigma_{k, \eta})_{N} \in \{ \Gamma_{b_{s}, d_{s}}^{(\theta_{s})} \}_{N} : s \in [r] \} \). With the notations from above, it suffices to show that in the expansion of
\[
E \circ \text{tr} \left( \left( U_{M_{N}}^{\sigma_{1, \eta_{1}}} \right)^{\eta_{1}} \left( U_{M_{N}}^{\sigma_{2, \eta_{2}}} \right)^{\eta_{2}} \cdots \left( U_{M_{N}}^{\sigma_{m, \eta_{m}}} \right)^{\eta_{m}} \right)
\]
all mixed free cumulants cancel asymptotically.

Fix \( \pi \in NC_{\eta, \text{alt}} (m) \). Since \( \pi \) is non-crossing, it has a block which is also a segment, say \((t+1, t+2, \ldots, t+2r)\). Via a circular permutation, we can further suppose that \( t = 0 \), that is \((1, 2, \ldots, 2r)\) is a block in \( \pi \). Denote by \( \eta' \), respectively \( \eta'' \) the restrictions of \( \eta \) to the sets \([2r]\), respectively \([m] \setminus [2r]\). Similarly, denote \( \sigma' = (\sigma_{1}, \ldots, \sigma_{2r}) \) and \( \sigma'' = (\sigma_{2r+1}, \sigma_{2r+2}, \ldots, \sigma_{m}) \). With these notations, \( \pi = [2r] \oplus \pi'' \), for some \( \pi'' \in NC_{\eta'', \text{alt}} (m - 2r) \). Henceforth, for the conclusion to follow from Remark 24 it suffices to show that
\[
V_{\sigma', \eta, M} (\pi) = \mathcal{K} (r, \sigma', \eta') \cdot V_{\sigma'', \eta'', M} (\pi'') + o (M^{0}),
\]
where
\[
\mathcal{K} (r, \sigma', \eta') = \begin{cases} 
\lim_{N \to \infty} \kappa_{2r} \left( \left( U_{M}^{\sigma_{1}} \right)^{\eta_{1}}, \ldots, \left( U_{M}^{\sigma_{2r}} \right)^{\eta_{2r}} \right), & \text{if } \sigma_{1} = \cdots = \sigma_{2r} \\
0, & \text{otherwise}.
\end{cases}
\]
We can also assume that \( \eta_{l+s} \neq \eta_{l+s+1} \) for all \( s \in \{1, \ldots, 2r-1\} \). Otherwise, as \( N \to \infty \), the right-hand side of (12) vanishes according to property (v3) while the left-hand side vanished according to Theorem \([4]\) and property (v4), so the equality holds true.

Let \( p, q \in P_2^0(m) \) be such that \( p \vee q = \pi \). If \( \sigma_s \neq \sigma_{s+1} \) for some \( s \in \{1, 2, \ldots, 2r-1\} \), then property (v3) gives that either \( \lim_{N \to \infty} \mathcal{V}_{\sigma, \eta, M}(\pi) = 0 \) (in particular (12) holds true), or \( s + 1 \in \{p(s), q(s)\} \). If \( r = 1 \), then condition (7) gives that

\[
\mathcal{F}^{(p,q)}_{\sigma, \eta, M}(\{s, s + 1\}) \leq |\{(i_1, i_2, j) \in [M]^3 \mid \tau_1(i_1, j) = \tau_2(i_2, j)\}| = o(M^2).
\]

If \( r > 1 \), then again (7) gives that

\[
\mathcal{F}^{(p,q)}_{\sigma, \eta, M}(\{s, s + 1\}) \leq |\{(i_1, j_1, i_2, j_2) \in [M]^4 \mid i_2 = j_1 \text{ and } \pi_1 \circ \tau_1(i_1, j_1) = \pi_1 \circ \tau_2(j_2, i_2)\}| = o(M^2).
\]

In both situations we have that \( \lim_{N \to \infty} \mathcal{V}_{\sigma, \eta, M}(p, q) = 0 \), according to Lemma \([23]\). The argument for the case \( q(s) = s + 1 \) is similar.

The rest of the proof, that is the case \( \sigma_1 = \cdots = \sigma_2r \), follows the argument for Example 5.3 from \([10]\), as shown below.

Let \( \Gamma_{b,d} \) the common value of \( \sigma_1, \sigma_2, \ldots, \sigma_{2r} \) (here we use again the convention of omitting the index \( N \), that is \( \sigma_j \)'s, \( b \) and \( d \) are depending on \( N \)). Since \( \Gamma_{b,d}^{-1} = (\Gamma_{b,d}^{(1)})^t \), it suffices to show relation (12) for \( \vartheta = 1 \) and the case \( \vartheta = -1 \) follows by taking transposes.

For each \( s \in [2r] \), write

\[
i_s = (\alpha_s - 1)d + \beta_s
\]

with \( \alpha_s \in [b] \) and \( \beta_s \in [d] \). Assuming that \( \vartheta = 1 \), suppose first that \( \eta_1 = 1 \); then, for \( s \in [2r] \),

\[
(k_s, l_s) = \begin{cases} ((\alpha_{s-1} - 1)d + \beta_{s+1}, (\alpha_{s+1} - 1)d + \beta_s) & \text{if } s \text{ is odd} \\ ((\alpha_{s+1} - 1)d + \beta_s, (\alpha_s - 1)d + \beta_{s+1}) & \text{if } s \text{ is even} \end{cases}
\]

From property (v3), we have that \( \lim_{N \to \infty} \mathcal{V}_{\sigma, \eta, M}(p, q) = 0 \) unless the pair \( (p', q') \) is either \( (\tilde{p}_r, \tilde{q}_r) \) or \( (\tilde{q}_r, \tilde{p}_r) \). If \( p' = \tilde{p}_r \) and \( q' = \tilde{q}_r \), then the conditions \( k_s = k_{p(s)} \) and \( l_s = l_{q(s)} \) become

\[
\alpha_1 = \alpha_{2r+1} \\
\beta_s = \beta_{s+2} \text{ for each } s = 1, 2, \ldots, 2r - 1,
\]

\( \vartheta = -1 \) is similar.
while if \( p' = \tilde{q}_r \) and \( q' = \tilde{p}_r \) then the conditions \( k_s = k_{p(s)} \) and \( l_s = l_{q(s)} \) become

\[
\begin{cases}
\beta_1 = \beta_{2r+1} \\
\alpha_s = \alpha_{s+2} \quad \text{for each } s = 1, 2, \ldots, 2r - 1.
\end{cases}
\]

Either way, we obtain that \( \alpha_1 = \alpha_{2r+1} \) and \( \beta_1 = \beta_{2r+1} \), that is \( i_1 = i_{2r+1} \), so property (v1) and the multiplicativity of the leading term in the development of the unitary Weingarten function in (9) gives that

\[
V_{\hat{\sigma}, \eta, M}(\pi) = V_{\hat{\sigma}', \eta', M}(\pi') \cdot V_{\hat{\sigma}'', \eta'', M}(\pi'') + o(M^0),
\]

so it suffices to show that

\[
\lim_{N \to \infty} \kappa_{2r} (U_M^{r,1})^{\eta_1}, (U_M^{r,1})^{\eta_2}, \ldots, (U_M^{r,1})^{\eta_{2r}} = \lim_{N \to \infty} V_{\hat{\sigma}', \eta', M}(\pi') = \sum_{(p', q')} \lim_{N \to \infty} V_{\hat{\sigma}', \eta', M}(p', q')
\]

where the last summation is done over \( (p', q') \in \{(\hat{p}_r, \tilde{q}_r), (\tilde{q}_r, \hat{p}_r)\} \).

For \( r = 1 \), we have that \( \hat{p}_r = \tilde{q}_r = (1, 2) \) so the summation in the right-hand side of (15) has just one term which, according to property (v4) equals 1. On the other hand, Theorem 16 of Section 3 gives that the left-hand side of (15) equals the left-hand side of the first relation of property (v4), that is also 1.

For \( r > 1 \), applying again Theorem 16 of Section 3 and property (v4), we have that the left-hand side of (15) is given by

\[
\lim_{N \to \infty} \kappa_{2r} (U_M^{r,1})^{\eta_1}, (U_M^{r,1})^{\eta_2}, \ldots, (U_M^{r,1})^{\eta_{2r}}
\]

if \( b \to \infty \) and \( d \to \infty \)

\[
\begin{cases}
0 & \text{if } b \to \infty \text{ and } d \to \infty \\
b^{2-2r}(-1)^{r-1} \text{Cat}_{r-1} & \text{if } b \text{ is bounded} \\
2^{2-2r}(-1)^{r-1} \text{Cat}_{r-1} & \text{if } d \text{ is bounded}.
\end{cases}
\]

On the other hand, the summation in the right hand side of (15) has now two terms, one for \( (p', q') = (\hat{p}_r, \tilde{q}_r) \) and one for \( (p', q') = (\tilde{q}_r, \hat{p}_r) \). Let us analyze the case \( (p', q') = (\hat{p}_r, \tilde{q}_r) \). Condition (13) gives that

\[
\mathcal{F}_{\hat{\sigma}, \eta, M}([2r]) \leq b^{2r} d^r = \frac{M^{2r}}{d^r}.
\]

So, if \( d \to \infty \), Lemma 23 gives that \( \lim_{N \to \infty} V_{\hat{\sigma}, \eta, M}(p, q) = 0 \). If \( d \) is bounded, since \( 2r \) is even, (13) gives that \( \alpha_1 = \alpha_{2r+1} \), therefore

\[
i_1 = (\alpha_1 - 1)d + \beta_1 = (\alpha_{2r+1} - 1)d + \beta_{2r+1} = i_{2r+1}.
\]
In particular, each $2(m - 2r)$-tuple $(i_s, j_s)_{2r+1 \leq s \leq m}$ from $\mathcal{A}_{\sigma, q', M}^{(p', q')}$ uniquely determines $\alpha_1$ and $\beta_1$ (via $i_{2r+1}$). Hence we have that

$$|\mathcal{A}_{\sigma, n, M}^{(p, q)}| = \left| \{ (\alpha_s, \beta_s)_{2 \leq s \leq m} : \alpha_s, \beta_s \text{ satisfy conditions } (13) \} \right| \cdot |\mathcal{A}_{\sigma', q', M}^{(p', q')}| = b^{2r-1} \cdot |\mathcal{A}_{\sigma', q', M}^{(p', q')}|.$$

Remember that $\pi = [2r] \oplus \pi''$, so $|\pi| = 1 + |\pi''|$. In particular the first formula from Notation 22 becomes

$$(16) \quad w_1(\pi) = (-1)^{r-1} \text{Cat}_{r-1} \cdot w_1(\pi').$$

So, using (16) and (11), we get

$$\mathcal{V}_{\sigma', n, M}(p, q) = Wg_M(p, q) \cdot \frac{1}{M} |\mathcal{A}_{\sigma, n, M}^{(p, q)}| = (M)^{-m+|\pi|} \cdot w_1(\pi) \cdot \frac{1}{M} |\mathcal{A}_{\sigma, n, M}^{(p, q)}| + O(M^{-2}) = (-1)^{r-1} \text{Cat}_{r-1} \cdot w_1(\pi') \cdot (M)^{-2r+1} \cdot (M)^{-(m-2r)+|\pi|} \frac{1}{M} \times db^{2r-1} |\mathcal{A}_{\sigma', q', M}^{(p', q')}| + O(M^{-2}) = [d^{2-2r} \cdot (-1)^{r-1} \text{Cat}_{r-1}] \cdot [w_1(\pi')(M)^{-(m-2r)+|\pi'|}] \times \frac{1}{M} |\mathcal{A}_{\sigma', q', M}^{(p', q')}| + O(M^{-2}) = [d^{2-2r} \cdot (-1)^{r-1} \text{Cat}_{r-1}] \cdot \mathcal{V}_{\sigma', q', M}(p', q') + O(M^{-2}).$$

In the case $(p', q_{[2r]}) = (\tilde{q}_r, \tilde{p}_r)$, the conditions (14) give that

$$\mathcal{F}_{\sigma, q_{[2r]}}^{(p, q)}([2r]) \leq \frac{M^{2r}}{b^r} \text{ and } |\mathcal{A}_{\sigma, n, M}^{(p, q)}| = b^{2r-1} \cdot |\mathcal{A}_{\sigma', q', M}^{(p', q')}|$$

therefore, using again (11) and (16) we obtain that

$$\mathcal{V}_{\sigma', n, M}(p, q) = \begin{cases} o(M^0) & \text{if } b \to \infty \\ [d^{2-2r} \cdot (-1)^{r-1} \text{Cat}_{r-1}] \cdot \mathcal{V}_{\sigma', q', M}(p', q') + O(M^{-2}) & \text{if } b \text{ is bounded.} \end{cases}$$

The case $\eta_1 = \ast$ is similar, interchanging the symbols $\tilde{p}_r$ and $\tilde{q}_r$. \(\square\)

We conclude this section with some immediate consequences of Theorem 25. First, taking $n = 2$, $\tilde{\vartheta}_1 = 1$, $\tilde{\vartheta}_2 = -1$ and $b_{1,N} = b_{2,N}$, Lemma 19 and Theorem 26 give the following.
Corollary 27. For any $\vartheta, b, d$, we have that $U^{(\vartheta)}_{M,b,d}$ is asymptotically free from its transpose.

In [8] it is proved that unitarily invariant random matrices are asymptotically free from their transposes. An example of random matrices which are not unitarily invariant but are free from their transposes is given by Wigner ensembles (see [3]). Henceforth, Corollary 27 together with Remark 18 gives a new non-trivial class of non-unitarily invariant random matrices which are free from their transposes.

Furthermore, if in Theorem 26 we take $n = 4, b_{1,N} = b_{2,N} = 1, \vartheta_1 = \vartheta_3 = 1, \vartheta_2 = \vartheta_4 = -1$ and $b_{3,N} = b_{4,N}$ such that $b_{3,N}, d_{3,N} \to \infty$, another application of Lemma 19 gives the result below.

Corollary 28. If $b \to \infty$ and $d \to \infty$ then $U_N^{(\vartheta)}$ and their transposes form an asymptotically free family.

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