Kinetically dominated curved universes:
Logolinear series expansions

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We develop a method for computing series expansions out of a singularity for solutions to ordinary differential equations when the asymptotic form contains both linear and logarithmic terms. Such situations are common in primordial cosmology when considering expanding out of a singularity in a pre-inflationary phase of the universe. We develop mathematical techniques for generating these expansions, and apply them to polynomial and Starobinsky inflationary potentials. This paper is the first in a programme of work on kinetically dominated curved universes, for which such power series are essential. Code for analytic and numerical computation of logolinear series is provided on GitHub.

I. INTRODUCTION

Even in the latest cosmic microwave background data [1–4], tensions still exist at low multipoles with unexpected features and a suppression of power on large spatial scales both visible by eye in the $C_l$ spectra [5–8].

Beyond the possibility that these effects are due just to chance, there are numerous possible physical explanations for these small anomalies [9], including inflationary models with singularities and discontinuities [10–12], multi-field phase-transitions [13–17], M-theory [18, 19], supergravity [20], just-enough inflation models [21] or kinetic dominance [22–24]. In this paper, we focus on the last of these.

It has been shown in general [22] that classical solutions to the cosmic evolution equations begin in a big-bang singularity with the kinetic energy of the inflaton dominating over its potential energy. For this phase of kinetic dominance the solutions to the background equations take a generic and asymptotically simple form. After this phase of early kinetic dominance, evolution settles into its traditional slow-roll form. There is some evidence for kinetic dominance in current data [24, 25], although issues on how to set quantum initial conditions in such a phase, and the nature of their imprint on the cosmic microwave background are still under theoretical investigation [26, 27]. More importantly, there is more than enough wiggle-room in the currently observed primordial power spectrum of curvature perturbations to allow for these extensions, as well as the capacity for future experiments to constrain them [28].

The asymptotic solutions examined in [22] are merely the first terms in a series expansion. To compute higher order terms in the kinetic dominance approximation, we must consider a more general type of power series, which we term logilinear expansions. We generalise results in [29] from the closed case to open, closed and flat cases, and from a quadratic potential to a field with general $V(\phi)$. We then apply this methodology to some example potentials. These higher order terms may prove useful for improving the stability of numerical codes, for example those computing mode functions of primordial power spectra [30, 31]. It should be noted that whilst it remains true that flat universes are most preferred by current observational data [32], late-time curvature is only weakly coupled to primordial curvature, and thus does not preclude effects at inflation which could be of interest.

The structure of this paper is as follows: In Sec. II we review the critical equations of inflationary cosmology to which we will apply our series expansion methodology. Sec. III establishes notation and identities for logolinear expansions. In Sec. IV we apply logolinear expansions to the equations of the early universe. Secs. V and VI provide concrete applications to specific potentials and we conclude in Sec. VII.

II. BACKGROUND

The background equations for a homogeneous Friedmann-Robertson-Walker spacetime with material content defined by a scalar field are:

$$H + H^2 = -\frac{1}{3m_p^2}(\ddOT - V(\phi)), \quad (1)$$

$$0 = \ddOT + 3H\DDOT + \frac{\d}{\d\phi} V(\phi), \quad (2)$$

where $H = \frac{\d}{\d t} \log a$ is the Hubble parameter, $\phi$ is the homogeneous value of the scalar field, $V(\phi)$ is the scalar potential, $a$ is the scale factor and dots indicate derivatives with respect to cosmic time $t$. One may supplement Eqs. (1) and (2) with a third non-independent equation:

$$H^2 + \frac{K}{a^2} = \frac{1}{3m_p^2}\left(\frac{1}{2}\phi^2 + V(\phi)\right), \quad (3)$$
where \( K \in \{+1, 0, -1\} \) is the sign of the curvature of the universe. For the remainder of this paper we will set the Planck mass to unity \( m_p = 1 \), but note that one may reintroduce \( m_p \) at any time by replacing \( \phi \rightarrow \phi/m_p \), \( V \rightarrow V/m_p^2 \).

Under general conditions [22] the solutions to Eqs. (1) and (2) begin in a singularity at \( t = 0 \) with \( \dot{\phi}^2 \gg V(\phi) \). In this kinetically dominated regime, solutions take the asymptotic forms:

\[
H = \frac{1}{3t}, \quad \phi = \phi_p \pm \sqrt{\frac{2}{3}} \log t, \quad (4)
\]

where \( \phi_p \) is a constant of integration. The approximate solutions in Eq. (4) typically serve to set initial conditions for the numerical solution of differential Eqs. (1) and (2). Applying this prescription naively leads to difficulties, since Eq. (3) along with initial conditions in Eq. (4) effectively set the curvature to be slightly positive. This additional curvature can lead to problems with numerical stability where the (supposedly negligible) curvature terms can come to dominate before inflation begins, leading to collapsing solutions. More importantly, one expects Eqs. (1) and (2) to encompass flat \((K = 0)\), open \((K = -1)\) and closed \((K = +1)\) cases. To avoid confusion, note that the sign of \( K \) is opposite in sense to the usually defined curvature density parameter \( \Omega_K \), for example in a closed universe, \( K = +1 \Rightarrow \Omega_K < 0 \).

It is therefore natural to seek higher order terms to the kinetic dominance solutions in Eq. (4). The singular nature of Eq. (4) as \( t \to 0 \) renders a Taylor expansion inappropriate, as one requires negative powers of \( t \) to describe \( H \) terms. A more general Laurent expansion allowing for negative powers of \( t \) will also not be applicable, as these do not include \( \log t \) terms. By examining Eqs. (1) and (2) one can see that expanding with Eq. (4) as leading order terms, one would need higher order terms such as \( t \log t \).

We are led therefore to consider the more general series expansions introduced by Lasenby and Doran [29] which we term logolinear expansions.

### III. LOGOLINEAR EXPANSIONS

We will consider series expansions for a general function \( x(t) \) in the form:

\[
x(t) = \sum_{j,k} [x^k_j] t^n (\log t)^k, \quad (5)
\]

where \( [x^k_j] \) are twice-indexed real constants defining the series, with square brackets used to disambiguate powers from superscripts. We make convenient definitions for upper and lower indexed functions of \( t \) via partial summation:

\[
x_j(t) = \sum_k [x^k_j] (\log t)^k, \quad [x^k](t) = \sum_j [x^k_j] t^j, \quad (6)
\]

\[
x(t) = \sum_j x_j(t) t^j = \sum_k [x^k](t) (\log t)^k. \quad (7)
\]

In this paper we are deliberately lax with bounds on the limits, as in general the bounds of \( k \) depend on \( j \) in a non-trivial way, and \( j \) will be required to range over non-integer values. This approach was heavily influenced by Graham et al. [33].

Care must be taken with logolinear expansions, as in general they are underdetermined. Writing out the first few terms of Eq. (5) for clarity:

\[
x(t) = \begin{array}{c}
[x^0_0] + t [x^0_1] + t^2 [x^2_0] + \ldots \\
[x^0_1] \log t + t [x^1_2] \log t + \ldots \\
[x^2_0] (\log t)^2 + [x^1_2] (t \log t)^2 + \ldots \\
\vdots \\
\end{array}
\]

\[
(8)
\]

and given that \( t = \exp(\log t) \):

\[
t = 1 + \log t + \frac{1}{2!} (\log t)^2 + \frac{1}{3!} (\log t)^3 + \ldots, \quad (9)
\]

then adding and subtracting \( \alpha \) times Eq. (9) from Eq. (8) yields:

\[
x(t) = \begin{array}{c}
([x^0_0] + \alpha) + ([x^0_1] - \alpha) t + \ldots \\
([x^0_1] + \alpha) \log t + [x^1_2] t \log t + \ldots \\
([x^2_0] + \frac{1}{2!} \alpha) (\log t)^2 + [x^1_2] (t \log t)^2 + \ldots \\
([x^3_0] + \frac{1}{3!} \alpha) (\log t)^3 + [x^1_2] (t \log t)^3 + \ldots \\
\vdots \\
\end{array}
\]

\[
(10)
\]

By setting \( \alpha = [x^0_1] \) we can completely remove the \( t \) term via a redefinition \( [x^0_0] + \frac{1}{2!} [x^0_1] \rightarrow [x^0_0] \). Inspection shows that we can use a similar procedure to remove higher-order \( t \) terms. Similarly, setting \( \alpha = -[x^0_0] \), we may remove the \( \log t \) term by redefining \( [x^0_0] + [x^0_1] \rightarrow [x^0_1] \), \([x^0_0] - \frac{1}{2!} [x^0_1] \rightarrow [x^0_0] \), and again this may be performed for any term.

One may think of the underdetermination of logolinear series as a rather non-trivial gauge freedom which must be carefully controlled when using such series. In practice, we can avoid much of this difficulty by requiring the partial sums in Eq. (6) over \( \log t \) to truncate.

We may formally differentiate and integrate Eq. (5):

\[
\dot{x} = \sum_{j,k} ((j+1)[x^k_{j+1}] + (k+1)[x^k_{j+1}]) t^j (\log t)^k, \quad (11)
\]

\[
\int x \, dt = \sum_{j \geq 0, k} \frac{\sum_{p \geq k} \frac{-p! [x^k_p]}{k! (-j)^{p-k+1}}}{t^j (\log t)^k} + c. \quad (12)
\]

Care must be taken in general with the limits of \( j \) in Eq. (11), as negative \( j \) indices have been introduced during the differentiation, and in deriving Eq. (12) we have used the identity:

\[
\int t^j (\log t)^k \, dt = t^{j+1} \sum_{p=0}^{k} \frac{-k! (\log t)^p}{p! (-j)^{p-k+1}} + c. \quad (13)
\]
Exponentials of the series in Eq. (5) may also be taken. We first define complete ordinary Bell polynomials [34] via:

$$ \exp \left( \sum_j x_j t^j \right) = e^{x_0} \sum_j C_j(x_1, \ldots, x_j) t^j. $$  \hspace{1cm} (14)

We may define and compute $C_j$ recursively via:

$$ C_0 = 1, $$

$$ C_j(x_1, \ldots, x_j) = \sum_{k=1}^{j} C_{j-k}(x_1, \ldots, x_{j-k}) x_k. $$ \hspace{1cm} (15)

To derive the recursion in Eq. (15) from Eq. (14), take logarithms, differentiate with respect to and then multiply by $t$, multiply the consequent denominator and compare $t^j$ coefficients. The first few terms of Eq. (14) are:

$$ e^{x_0} \left( 1 + x_1 t + \left[ \frac{x_1^2}{2} + x_2 \right] t^2 + \left[ \frac{x_1^3}{6} + x_1 x_2 + x_3 \right] t^3 \right). $$

Note that for Eq. (15), indices should range over integer values, but for Eq. (14) $j$ is allowed to be non-integer. For more detail, see Wilf [35]. To exponentiate a logolinear series therefore, one simply applies the Bell polynomial expansion in Eq. (14) to the lower indexed series expansion in Eq. (6). Care must be taken with the leading term, for example, if $x_0 = [x_0^0] + [x_1^0] \log t$ then

$$ e^{x_0} = e^{[x_0^0] + [x_1^0] \log t} = e^{[x_0^0]} \cdot t^{[x_1^0]}, $$ \hspace{1cm} (16)

we can see that exponentiating a logolinear series in this case will add a constant $x_1^0$-dependent shift to the powers of $j$ indices.

\section*{IV. METHODOLOGY}

We now apply logolinear series to the evolution equations of an inflating universe. We first examine the approach developed in [29], before advocating a clearer approach that allows one to control gauge freedoms with more precision.

\subsection*{A. The Lasenby-Doran approach (log-splitting)}

The approach espoused by Lasenby and Doran [29] is to substitute the partially summed expansions in Eq. (6)

$$ H(t) = \sum_k [H^k](t)(\log t)^k \quad \text{and} \quad \phi(t) = \sum_k [\phi^k](t)(\log t)^k $$

into master Eqs. (1) and (2) to generate a set of recursion relations:

$$ [H^{k+1}] = - \frac{t}{k+1} \left( \frac{1}{3} V(\phi)^k - [\dot{H}^k] - \sum_{p+q=k} [H^p][H^q] + \frac{1}{3} [\phi^p][\phi^q] \right), $$

$$ [\phi^{k+2}] = \frac{t^2}{(k+1)(k+2)} \left( [\phi^{k+1}] + \frac{1}{2} \left[ \phi^{k+1} \right] \right) - \frac{3}{t} \sum_{p+q=k} [H^p] \left( [\phi^q] + [\phi^{q+1}] \right). $$ \hspace{1cm} (17)

When $[H^0]$, $[\phi^0]$ and $[\phi^1]$ are specified, all higher order functions $[H^k]$ may be calculated recursively. Note that in Lasenby and Doran [29] only the $k=0$ case of recursion relations in Eq. (17) is explicitly stated.

We term this approach \textit{log-splitting} since it involves substituting in partially summed series from Eq. (6), which are series in $\log t$ modulated by functions $x^k(t)$.

All that remain to be defined are initial power series $[H^0]$, $[\phi^0]$ and $[\phi^1]$, which should be: (a) power series in $t^{1/3}$ “in order to generate curvature” and (b) chosen so that “successive terms in the series get progressively smaller”. The first of these statements can be seen by examining Eq. (3), but the second is highly non-trivial, and intimately connected to the gauge freedoms indicated in Sec. III. Indeed, the method presented in Lasenby and Doran [29] is only manually applied to the first few terms, as a log-splitting approach is extremely challenging to apply systematically.

\subsection*{B. The lin-splitting approach}

Given aforementioned issues with the approach outlined in the previous section, we now pursue an orthogonal methodology, which instead uses the lower-indexed partial sums from Eq. (6). This presents an easier way to systematically compute higher order terms whilst controlling gauge freedoms.

We begin by defining $N = \log a$, such that $\dot{N} = H$, so that Eqs. (1) and (2) become:

$$ \ddot{N} + N^2 + \frac{1}{3} \left( \phi^2 - V(\phi) \right) = 0, $$ \hspace{1cm} (18)

$$ \ddot{\phi} + 3N \phi + \frac{d}{d\phi} V(\phi) = 0. $$ \hspace{1cm} (19)

Using $N$ rather than $H$ puts Eqs. (1) and (2) on a more equal footing, as it renders both evolution equations second order.

Given that we intend to work with power series in $\log t$, we now transform Eqs. (18) and (19) to logarithmic time, with $x' = \frac{d}{d(\log t)} x$:

$$ N'' - N' + N^2 + \frac{1}{3} \left( \phi^2 - t^2 V(\phi) \right) = 0, $$ \hspace{1cm} (20)

$$ \phi'' - \phi' + 3N' \phi' + t^2 \frac{d}{d\phi} V(\phi) = 0. $$ \hspace{1cm} (21)
Converting to a first order system yields the definitions and equations:

\[
N' = h, \quad \phi' = v,
\]
\[
h' = h - h^2 - \frac{1}{3} v^2 + \frac{1}{3} t^2 V(\phi),
\]
\[
v' = v - 3 v h - t^2 \frac{d}{d\phi} V(\phi).
\] (22)

We now substitute in our series definition from Eq. (6), and note that:

\[
x(t) = \sum_j x_j(t) t^j \quad \Rightarrow \quad x'(t) = \sum_j (x'_j + j x_j) t^j.
\] (23)

We find that after equating coefficients of \(t^j\), Eq. (22) becomes:

\[
N'_j + j N_j = h_j, \quad \phi'_j + j \phi_j = v_j,
\]
\[
h'_j + j h_j = h_j + \frac{1}{3} V(\phi)|_{j-2} - \sum_{p+q=j} h_p h_q + \frac{1}{3} v_p v_q,
\]
\[
v'_j + j v_j = v_j - \frac{dV(\phi)}{d\phi}|_{j-2} - 3 \sum_{p+q=j} v_p h_q.
\] (24)

It is also useful to consider the equivalent of Eq. (3):

\[
\frac{1}{3} V(\phi)|_{j-2} + \sum_{p+q=j} \frac{1}{6} v_p v_q - h_p h_q = \frac{K}{e^{2N_0} e^{\sum_{s=0}^N N_s(t) v^s}} |_{j-\frac{4}{3},}
\] (25)

where exponentiation of logarithmic series was discussed in Sec. III.

For \(j = 0\), Eq. (24) is a non-linear differential equation in zero-indexed functions. Further inspection shows it to be equivalent to the starting equations Eq. (22) with \(V = 0\). Hence we may solve using the kinetically dominated solutions:

\[
N_0 = N_p + \frac{1}{3} \log t, \quad h_0 = \frac{1}{3},
\]
\[
\phi_0 = \phi_p \pm \sqrt{\frac{2}{3}} \log t, \quad v_0 = \pm \sqrt{\frac{2}{3}},
\] (26)

where \(N_p\) and \(\phi_p\) are constants of integration. Whilst we expect there to be four constants of integration a-priori, one of them is fixed by defining the singularity to be at \(t = 0\). As there are only two constants, it is clear that Eq. (26) does not span the full set of solutions to Eq. (24) with \(j = 0\). In fact, Eq. (26) represents a complete solution to \(j = 0\) for only the flat case, as \(K = 0\) effectively sets another integration constant. Nevertheless, we will discover that we may still use Eq. (26) as the base term for the loglinear series, and that the final constant of integration effectively emerges from a consideration of higher order terms.

For \(j \neq 0\), we transfer terms involving \(j\) in the summations from the right hand side to the left, giving a first order linear inhomogeneous vector differential equation:

\[
x'_j + A_j x_j = F_j,
\] (27)

where \(x = (N, \phi, h, v), A_j\) is a (constant) matrix:

\[
A_j = \begin{pmatrix}
0 & 0 & \cdots & -1 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & j - 1 + 2 h_0 \\
0 & 0 & \cdots & 3 v_0 \\
\end{pmatrix}
\] (28)

and \(F_j\) is a vector polynomial in \(\log t\) depending only on earlier series \(x_{p<j}\):

\[
F_j = \begin{pmatrix}
\frac{1}{3} V(\phi)|_{j-2} - \sum_{p+q=j} h_p h_q + \frac{1}{3} v_p v_q \\
- \frac{dV(\phi)}{d\phi}|_{j-2} - 3 \sum_{p+q=j} v_p h_q
\end{pmatrix}
\] (29)

Note that the limits on summations in Eq. (29) now have strict bounds, in contrast with Eq. (24).

At each \(j\), the linear differential Eq. (27) may be solved in terms of a complementary function with four free parameters and a particular integral. These free parameters correspond to the degrees of gauge freedom mentioned in Sec. III.

### C. Complementary function

We may solve the homogeneous version of Eq. (27) exactly, since \(A_j\) is a constant matrix:

\[
\frac{dx_j^{ef}}{d\log t} + A_j x_j^{ef} = 0 \quad \Rightarrow \quad x_j^{ef} = e^{-A_j \log t} [x_j^0],
\] (30)

where \([x_j^0]\) is a constant vector parametrisation initial conditions. To compute the matrix exponential, we first compute eigenvectors and eigenvalues of \(A_j\):

\[
e_\beta = \left( \begin{array}{c} 1 \pm \sqrt{6} \\ -1 \mp \sqrt{6} \end{array} \right), \quad A_j e_\beta = (j + 1) \cdot e_\beta,
\]
\[
e_b = \left( \begin{array}{c} 1 \mp 2 \sqrt{3} \\ 4 \mp \sqrt{6} \end{array} \right), \quad A_j e_b = (j - 3) \cdot e_b,
\]
\[
e_n = \left( \begin{array}{c} 0 \quad 0 \quad 0 \end{array} \right), \quad A_j e_n = j \cdot e_n,
\]
\[
e_\phi = \left( \begin{array}{c} 0 \quad 1 \quad 0 \quad 0 \end{array} \right), \quad A_j e_\phi = j \cdot e_\phi.
\] (31)

Parametrising initial conditions \([x_j^0]\) using the eigenbasis in Eq. (31) with parameters \(\hat{N}, \hat{\phi}, b, \beta, \) one finds:

\[
x_j^{ef} = e^{-A_j \log t} (\hat{N} e_n + \hat{\phi} e_\phi - \frac{9}{14} b e_b + \beta e_\beta)
\]
\[
= \left( \hat{N} e_n + \hat{\phi} e_\phi - \frac{9}{14} b e_b t^{1/3} + \beta e_\beta t^{-1} \right) t^{-j}.
\] (32)
Equation (32) is very interesting. First, all complementary functions (i.e. degrees of gauge freedom) are identical up to a $t^{-j}$ term, which cancels with $t^j$ in the power series in Eq. (6). We may set $\beta = 0$ without loss of generality, as it grows faster than our leading term as $t \to 0$. Choosing $\beta = 0$ therefore amounts to setting the singularity to be at $t = 0$ as an initial condition. We may absorb all $N$ and $\phi$ into our definitions of $N_p$ and $\phi_p$. The only remaining undetermined integration constant is $b$, which amounts to the third integration constant that was missing from Eq. (26). The constant $b$ is controlled by the curvature of the universe via Eq. (25):

$$K = be^{2N_p},$$

where the somewhat cryptic $-\frac{9}{14}$ coefficient in Eq. (32) was chosen so that the curvature relation in Eq. (33) takes a simple form.

Of particular importance is the fact that the curvature of the universe depends on a term in $t^{4/3}$. We should therefore expand as power series in $t^{2/3}$ rather than $t$. In our formalism, as espoused in [33], there is nothing preventing $j$ and its summation indices from being non-integer, so we happily do so.

### D. Particular integral

All that remains to be determined is a particular integral of Eq. (27), given that one has the form of $F_j$ at each stage of recursion. The trial solution is $x_j(t) = \sum_{k=0}^{N_j}[F^k_j](\log t)^k$. Defining $F_j = \sum_{k=0}^{N_j}[F^k_j](\log t)^k$ and equating coefficients of $(\log t)^k$ yields

$$(k + 1)[x_j^{k+1}] + A_j [x_j^k] = [F^k_j],$$

(34)

giving a descending recursion relation in $k$:

$$[x_k^{N_j+1}] = 0, \quad [x_j^{k-1}] = A_j^{-1}([F^{k-1}_j] - k[x_j^k]).$$

(35)

The recursion relation in Eq. (35) fails when $A_j$ is non-invertible, which occurs when any of the eigenvalues in Eq. (31) are zero ($j = -1, 0, 4/3$). For these cases, the system is underdetermined, with an infinity of solutions parameterised along the directions of relevant eigenvectors. This infinity of solutions can therefore be carefully absorbed into a corresponding constant of integration.

Similarly, if we were to define an alternative base to the recursion in Eq. (35), then infinite series would be generated. However, all but a finite number of terms would merely contribute to a re-definition of constants $N_p$, $\phi_p$, $b$, or an introduction of nonzero $\beta$, which we disallow due to the consequent shift of the singularity to a non-zero time $t$.

### V. EXAMPLES

We now apply these methods to two examples; first to that of a polynomial potential exemplified by a self-interacting field with a cosmological constant, and second to Starobinsky inflation as an example of a potential containing exponential terms.

The analytical calculations in this section were performed with the aid of the MapleTM (2017) computer algebra package [36, 37]. The numerical calculations were performed using Python 3.6.4, for which the NumPy Polynomial package [38] was particularly useful. All code can be found on GitHub [39].

#### A. Polynomial potentials

Consider a non-trivial polynomial potential:

$$V(\phi) = m_p^2A + \frac{1}{2}m^2\phi^2 + \frac{1}{24}\lambda\phi^4,$$

(36)

which incorporates both a cosmological constant $A$ and a self-interacting $\phi^4$ term. The application of the methods of Sec. IV are straightforward, since we may extract the potential coefficients:

$$V(\phi) |_{j} = m_p^2A\delta_{0j} + \frac{1}{2}m^2\sum_{p+q=j}\phi_p\phi_q + \frac{1}{24}\lambda\sum_{p+q+r+s=j}\phi_p\phi_q\phi_r\phi_s,$$

$$\frac{dV(\phi)}{d\phi} |_{j} = m^2\phi_j + \frac{1}{6}\lambda\sum_{p+q+r+s=j}\phi_p\phi_q\phi_r,$$

(37)

where $\delta_{ij}$ is the Kronecker delta function. As suggested in Sec. IV C, our logolinear series must be in powers of $t^{2/3}$, so that $j = 0, \frac{2}{3}, \frac{1}{3}, 2, \ldots$. Substituting Eq. (37) into the definition of $F_j$ in Eq. (29), we define $j = 0$ solutions via Eq. (26), $j = 4/3$ solutions are computed using the complementary function as:

$$N_4 = -\frac{9}{14}b, \quad \phi_4 = \pm\frac{27\sqrt{6}}{56}b,$$

$$h_4 = -\frac{6}{7}b, \quad v_4 = \pm\frac{9\sqrt{6}}{14}b,$$

(38)

and all remaining stages $\frac{2}{3}, \frac{8}{9}, \frac{10}{9}, \ldots$ are computed from the recursion relation in Eq. (35). We find that the first few terms are:
\[
H = \frac{1}{3t} \left( 1 - \frac{6}{t} b t^{1/3} \right) \\
+ \left[ \frac{\lambda}{3} + \left( \frac{4}{2187} + \frac{2\sqrt{6} \phi_p}{729} + \frac{\phi_p^2}{81} + \frac{\sqrt{6} \phi_p^3}{162} + \frac{\phi_p^4}{72} \right) \lambda + \left( \frac{2}{81} + \frac{\sqrt{6} \phi_p}{27} + \frac{\phi_p^3}{18} \right) m^2 \\
+ \left( \frac{4}{729} \right) \left( \frac{2}{243} + \frac{\phi_p^2}{243} + \frac{\sqrt{6} \phi_p^3}{54} \right) \lambda + \left( \frac{2}{27} + \frac{\sqrt{6} \phi_p}{9} \right) m^2 \right] \log t \\
+ \left( \frac{1507b}{383292} - \frac{50b\sqrt{6} \phi_p}{2457} \right) \lambda (log t)^3 + \frac{1507b}{2457} (log t)^4 \right] t^{7/3}, \\
\phi = \phi_p \pm \frac{\sqrt{2}}{3} \log t \pm 27\sqrt{6} b t^{4/3} \\
+ \left[ \frac{50b\Delta}{91} + \left( \frac{43210901b}{1894708179} + \frac{4001293b\sqrt{6} \phi_p}{3321864} + \frac{152143b\phi_p^2}{511056} + \frac{1507\sqrt{6} \phi_p^3}{1902} \right) \lambda \\
+ \left( \frac{152143b}{1660932} + \frac{1507b\sqrt{6} \phi_p}{85176} + \frac{25b\phi_p^2}{19} \right) m^2 \\
+ \left( \frac{4001293b}{97164522} \pm \frac{152143b\sqrt{6} \phi_p}{4982976} \pm \frac{1507b\phi_p^2}{85176} \pm \frac{25\sqrt{6} \phi_p^3}{819} \right) \lambda + \left( \frac{1507b}{42588} \pm \frac{50b\sqrt{6} \phi_p}{273} \right) m^2 \right] \log t \\
+ \left( \frac{152143b}{4982976} + \frac{1507b\sqrt{6} \phi_p}{255528} + \frac{25b\phi_p^2}{273} \right) \lambda (log t)^3 + \frac{1507b}{2457} (log t)^4 \right] t^{7/3}, \\
\phi = \phi_p \pm \frac{\sqrt{2}}{3} \log t \pm 27\sqrt{6} b t^{4/3} \\
+ \left[ \frac{14337\sqrt{6} b^2}{34496} \right] \lambda (log t)^3 + \frac{14337\sqrt{6} b^2}{34496} (log t)^4 \right] t^{10/3}.
\]
FIG. 1. Inflation using $V = \frac{1}{2} m^2 \phi^2$. We see generic features of kinetic initial conditions; the universe initially expands out of a singularity in kinetic dominance with $\phi^2 \gg V, a \propto t^{1/3}$. At later times, the potential comes to dominate $\phi^2 \ll V$, and a period of inflation begins at $t \sim m^{-1}$. Inflation then exits as the field $\phi$ oscillates about the minimum of the potential with the scale factor consequently expanding as some power law. For this example, we choose $m = 10^{-5} m_p$, consistent with current observations. As initial conditions at $t = 1$ we set $\phi_0 = -23 m_p, N_p = 0$, in order to give 50–60 e-folds of inflation, and choose a slightly closed universe $b = 2 \times 10^{-6}$. Using only the first term in the kinetic dominance solution in order to start the numerical integration at $t = m_p^{-1}$ gives the dotted line, whilst using the correction for curvature provided by the next term proves sufficient to match the true solution, which is indistinguishable from the solid curve.

The series in Eqs. (39) and (40) above exhibit some comment-worthy properties. The curvature term $b$ does not begin mixing with potential terms $\lambda, m, \phi_0$ until the sixth term in each series. If one wishes to consider flat models therefore, a reasonable approach is to simply take $t$ and $t^2$ terms from the above as corrections to $H$ and $\phi$ as described in initial conditions in Eq. (4), which effectively sets $b = K = 0$ to first order.

One can see the importance of including higher-order terms in Fig. 1. For numerical codes aiming to provide constraints on the initial curvature of the universe, these corrections will be essential. Fig. 2 shows the accuracy of the log-linear series as the number of terms is increased. In general, they are asymptotic series which provide an excellent approximation to the true solutions before inflation begins.

We may compare Eqs. (39) and (40) with corresponding results from [29] by setting $\lambda = 0$, reintroducing $m_p$ via transformations $\phi \rightarrow \phi/m_p, \phi_0 \rightarrow \phi_0/m_p$, $m \rightarrow \mu/m_p$, setting $m_p^2 = (8\pi)^{-1}$, changing variable definitions via $t = u, \phi_0 = b_0, b = -\frac{28\sqrt{15}}{81} \frac{b_0}{m_p}$ and extracting relevant log $t$ terms:

$$H^0 = \frac{1}{3u} + \frac{32\sqrt{3\pi} b_4}{27} u^{1/3} - \frac{6656\pi b_4^2}{891} u^{5/3} + \left( \frac{\Lambda}{3} + \frac{2\mu^2}{81} + \frac{4\sqrt{3\pi} b_0 \mu^2}{27} + \frac{4\pi b_0^2 \mu^2}{3} \right) u,$$

$$\phi^0 = b_0 \mp b_4 u^{4/3} \pm \frac{118\sqrt{3\pi} b_4^2}{99} u^{8/3} + \left( \mp \frac{3\Lambda}{24\sqrt{3\pi}} \pm \frac{33\mu^2}{1296\sqrt{3\pi}} - \frac{b_0 \mu^2}{36} \mp \frac{\sqrt{3\pi} b_0^2 \mu^2}{6} \right) u^2,$$

$$\phi^3 = \pm \frac{1}{\sqrt{12\pi}} + \left( \pm \frac{3\mu^2}{216\sqrt{3\pi}} - \frac{b_0 \mu^2}{6} \right) u^2. \quad (41)$$

Eq. (41) match precisely with results from [29], up to $\pm$ branches, for which Lasenby and Doran [29] only consider the negative branch.

B. Starobinsky inflation

Starobinsky [40] proposed a modified theory of gravity $(R + R^2)$ as a mechanism for inflation, which remains one of the inflationary theories most consistent with observations [1, 2]. After a conformal transformation to the Einstein frame the theory is equivalent to using a potential:

$$V(\phi) = \Lambda^4 \left( e^{-\phi\sqrt{2/3}} - 1 \right)^2, \quad (42)$$
which is plotted in Fig. 3. In light of discussions in Sec. III of exponentiation of loglinear series, the factor of $\sqrt{2/3}$ in the Starobinsky potential is particularly convenient, as it allows us to keep indices as rational numbers (and is likely not a coincidence). If one were to consider more general exponential inflationary potentials, such as power law inflation [41] ($V \propto e^{-\lambda \phi}$), then we would be forced to introduce additional $t^n$ terms in the power series, where $q$ will not in general be a rational number.

From Eq. (26) we have $\phi_0 = \phi_p \pm 3 \log t$, and Eq. (42) becomes:

$$V(\phi) = \Lambda^4(e^{-\sqrt{2/3} \phi/m_p} - 1)^2,$$

We can see that exponential terms in general will shift the usual $j - 2$ action of the potential terms in Eq. (29) to $j - 2 \pm 2/3$ and $j - 2 \pm 4/3$ for the latter two terms in Eq. (43).

Proceeding as for the polynomial case, $j = 0$ terms are defined as in Eq. (26), $j = 4$ terms follow from the recursion relation in Eq. (35). For $j = 4$ terms, the negative branch is defined as for the polynomial case in Eq. (38), but the positive branch must be handled with care, since the $\frac{2}{3}$ and $\frac{4}{3}$ terms now have contributions from the potential. In this case, we find:

$$N_{\frac{4}{3}} = -\frac{9b}{14} + \frac{27\Lambda^4\Phi_p}{56} - \frac{27\Lambda^4\Phi_p^3}{50},$$

$$\phi_{\frac{4}{3}} \pm \frac{27\sqrt{6}}{56} \left(b + \frac{2\Lambda^4\Phi_p}{9} + \frac{49\Lambda^8\Phi_p^4}{300}\right),$$

$$h_{\frac{4}{3}} = -\frac{6b}{7} - \frac{6\Lambda^4\Phi_p}{7} + \frac{18\Lambda^8\Phi_p^4}{25},$$

$$v_{\frac{4}{3}} = \frac{9\sqrt{6}}{14} \left(b + \frac{2\Lambda^4\Phi_p}{9} + \frac{49\Lambda^8\Phi_p^4}{300}\right),$$

where the definition of $b$ has been judiciously chosen so that the curvature relation in Eq. (33) holds true. All remaining terms are defined via the recursion relation from Eq. (35), and the first few terms for the two branches are:

$$H^+ = \frac{1}{3t} + \frac{3\Phi_p^2\Lambda^4}{5} t^{-1/3} + \left(\frac{6}{7}b - \frac{3\Phi_p\Lambda^4}{25} - 18\Phi_p^2\Lambda^8\right) t^{1/3} + \left(\frac{1}{3} + \frac{11b\Phi_p^2}{2100}\right) \Lambda^4 + \frac{38\Phi_p^2\Lambda^8}{35} + \frac{819\Phi_p^4\Lambda^{12}}{1000} t\right),$$

$$H^- = \frac{1}{3t} - \frac{6b}{7} t^{1/3} + \frac{\Lambda^4}{3} t + \left[\frac{21b^2}{539} - \frac{6\Phi_p\Lambda^4}{11}\right] t^{5/3} + \left[\frac{2b}{91} + \frac{3\Phi_p^2}{13}\right] \Lambda^4 t^{7/3},$$
\[ \phi^+ = \phi_p + \sqrt{\frac{2}{3}} \log t + \frac{3\Lambda^4 \sqrt{\Phi_p^2}}{20} t^{2/3} + \left( \frac{27 \sqrt{6b} + 3 \sqrt{6} \Phi_p \Lambda^4}{56} + \frac{63 \sqrt{6} \Phi_p \Lambda^8}{800} \right) t^{4/3} \]

\[ + \left( \frac{\sqrt{6}}{12} - \frac{33 \sqrt{6} \Phi_p^2}{35} \right) \Lambda^4 - \frac{41 \sqrt{6} \Phi_p \Lambda^8}{140} - \frac{441 \sqrt{6} \Phi_p \Lambda^{12}}{2000} \right) t^2 \]

\[ + \left( \frac{14337 \sqrt{6b^2}}{34496} + \frac{891 \sqrt{6} \Phi_p \Lambda^4}{1568} + \frac{20047 \sqrt{6} \Phi_p \Lambda^8}{86240} + \frac{55263 \sqrt{6} \Phi_p \Lambda^{12}}{35200} \right) t^{10/3} \],

\[ \phi^- = \phi_p - \sqrt{\frac{2}{3}} \log t + \frac{27 \sqrt{6b}}{56} t^{4/3} + \frac{\sqrt{6} \Lambda^4}{12} t^2 + \left[ \frac{6075 \sqrt{6b^2}}{34496} - \frac{15 \sqrt{6} \Phi_p \Lambda^4}{88} \right] t^{8/3} + \left[ \frac{21 \sqrt{6} \Phi_p \Lambda^8}{260} - \frac{9 \sqrt{6b}}{520} \right] \Lambda^4 t^{10/3}. \] 

The most striking feature of Eqs. (46) to (49) is that there are no higher-order \(\log t\) terms. This is somewhat to be expected, as under a variable transformation such as \(\phi = e^\varphi\), the evolution Eqs. (1) and (2) can be shown to no longer have any exponential terms. Since \(\varphi\) is also power-law at early times, this means that one would not expect series expansions to require higher-order \(\log t\) terms. It is reassuring that our methodology for loglinear series expansions is robust enough to recover this result without modification.

For the positive branch of the Starobinsky solutions in Eqs. (46) and (48), unlike the polynomial case in Eqs. (39) and (40), we find that there are \(t^{-1/3}\) and \(t^{2/3}\) terms, and potential terms \(\Phi_p\), \(\Lambda\) mix with curvature \(b\) at lower order. At negative \(\phi\) the Starobinsky potential grows faster than curvature as \(t \to 0\), in contrast with the polynomial case. For the negative branch of the Starobinsky solutions in Eqs. (47) and (49), in comparison with the polynomial case much higher order is required before potential terms become included. At positive \(\phi\), the effect of the potential is weak at early times due to its plateau-like nature. Numerical solutions are plotted in Fig. 4. For the positive branch in particular, higher order terms are essential for numerical stability.

**VI. NUMERICAL CONSIDERATIONS**

We now demonstrate how such series may be used to verify the numerical accuracy of kinetically dominated approximations. Scacco and Albrecht [25] consider a \(V = \frac{1}{2} m^2 \phi^2\) potential with kinetic initial conditions. In their work, they parameterise the initial conditions in terms of a dominance factor \(r = 100\), so that at some early time:

\[ \phi = \frac{\dot{\phi}}{r m} = \phi_0, \quad h = \frac{H}{a} = \sqrt{\frac{r^2 + 1}{6} - m \phi_0}, \quad a = a_0. \]

where \(h\) now denotes the conformal Hubble factor. Substituting Eq. (50) into Eq. (3) shows that \(h\) has been chosen to be consistent with a flat universe. Setting initial conditions in this way is common, as it does not require one to specify a time at which to set them. Nevertheless when Scacco and Albrecht [25] go on to solve the Mukhanov-Sazaki equation, they implicitly assume that the initial conditions are set at a conformal time \(\eta = \frac{\eta_0}{a}\) after the start of the universe, where \(\eta_0 = \int_0^t \frac{dt}{a}\). To generate series consistent with Scacco and Albrecht [25] we take the negative branch and \(\Lambda = \lambda = b = 0\) in Eqs. (39) and (40), yielding:
We can therefore see that \( \eta \) are effectively set at \( \eta \), as detailed in Sec. III. The series in Eq. (54) for \( \eta \) is computed by performing a negative exponential on \( N \) and then integrating the consequent series via Eq. (12). We may then use these to test the validity of the assumption that the conditions in Eq. (50) and \((\phi_p, N_p, t)\). Taking \( m = 6 \times 10^{-6} \) as in Eq. (50), a typical set of parameters transforms as:

\[
(\phi_0, a_0, r) = (20.7, 1, 100), \quad \frac{a}{2h} = 98.606
\]

\[
\Rightarrow (\phi_p, N_p, t) = (23.4176, -1.3952, 65.7395)
\]

\[
\Rightarrow \eta = 98.610. \quad (55)
\]

We can therefore see that \( \eta \approx \frac{a}{2h} \), accurate to within a fractional error of \(10^{-4} \). Scacco and Albrecht [25] have indeed set their initial conditions with sufficient accuracy, and having access to these power series makes for simple cross-checking of numerical stability.

VII. CONCLUSION

We developed techniques required for applying log-linear power series to the background differential equations of a Friedmann-Robertson-Walker universe, paying particular attention to details of how to control the gauge freedoms inherent in these expansions. We then applied our methodology to specific cases of polynomial and Starobinsky inflationary potentials, showing that our approach can be successfully applied to finite polynomial and exponential potentials. Future work will involve applying these series to a programme investigating conformally constrained closed universes.

Loglinear expansions could prove useful for improvement of stability of numerical integration codes requiring accurate background solutions such as those codes that solve for mode functions in the Mukhanov-Sasaki equation. We may then use these to test the validity of the assumption that the conditions are effectively set at \( \eta = \frac{a}{2h} \).

We therefore see that \( \eta \approx \frac{a}{2h} \), accurate to within a fractional error of \(10^{-4} \). Scacco and Albrecht [25] have indeed set their initial conditions with sufficient accuracy, and having access to these power series makes for simple cross-checking of numerical stability.

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