The Angular Momentum-Energy Space

Dan Comănescu

Abstract. In this paper we shall define and study the angular momentum-energy space for the classical problem of plane-motions of a particle situated in a potential field of a central force. We shall present the angular momentum-energy space for some important cases.

Mathematics Subject Classification: 37N05, 70K25, 70K42
Keywords: classical mechanics, particle, angular momentum, energy

1. Introduction

The angular momentum-energy states are used in Classical Mechanics to construct a mathematical model for the plane motions of a particle in a potential field of a central force (see [1], [4] or [5]).

In Astrophysics appears an equation in the angular momentum-energy space describing the stellar distribution around a black hole (see [2]).

In General Relativity a mathematical model of the motions of a particle use the concepts of energy and angular momentum (see [3]).

The objectives of this study are:

• to present the classical concepts of angular momentum and energy;
• to study the Angular Momentum-Energy Space and the Angular Momentum-Energy Space which are corresponding to the uniform rotations;
• to present the Angular Momentum-Energy Space and Angular Momentum-Energy Space which are corresponding to the uniform rotations for some particular force fields.

2. The Movements of a Particle in a Potential Field of a Central Force

We consider a particle situated in a potential field of central force. In the Newtonian Mechanics it is known that a trajectory is contained in a
The Angular Momentum-Energy Space

plane which contains the center of the force. We study the case in which the trajectories are contained in a fixed plane passing through center O. We denote by \( \vec{r} \) the radius vector of the particle, \( r \) the modulus of the radius vector and \( U_s(r) \) the force function. In this case the second law of Newton has the form:

\[
m \ddot{\vec{r}} = -U'_s(r) \frac{\vec{r}}{r}
\]

(1)

Projecting this equation on the natural base of the polar coordinates \((r, \varphi)\) we have:

\[
\begin{cases}
    \ddot{r} = 0 \\
    \dot{r} - r \dot{\varphi}^2 + U'(r) = 0
\end{cases}
\]

(2)

where \( U(r) = U_s(r)/m \) is the force function per unit mass.

We denote by \( J_s \) the angular momentum, \( E_s \) the total energy, \( J = J_s/m \) the angular momentum per unit mass and \( E = E_s/m \) the total energy per unit mass.

An other mathematical model of the motions is obtained using the conservation laws of the angular momentum and total energy. We have:

\[
\begin{cases}
    r^2 \dot{\varphi} = J \\
    \frac{\dot{r}^2}{2} + \frac{J^2}{2r^2} + U(r) = E
\end{cases}
\]

(3)

**Theorem 2.1.**  i) If \((r, \varphi)\) is a solution of (2) and it is not an uniform rotation, then it exists \((J, E) \in \mathbb{R}^2 \) such that \((r, \varphi)\) is a solution of (3).

ii) If \((J, E) \in \mathbb{R}^2 \), \((r, \varphi)\) is a solution of (3) and it is not an uniform rotation, then \((r, \varphi)\) is a solution of (2).

iii) If \( r_0 > 0 \), then it exists an uniform rotation \((r_0, \varphi)\) solution of (2) and (3) if and only if:

\[
\begin{cases}
    J^2 = r_0^3 U'(r_0) \\
    E = U(r_0) + \frac{r_0 U''(r_0)}{2}
\end{cases}
\]

(4)

**Proof.** The propositions i) and ii) are classical results.

iii) Let \((J, E) \in \mathbb{R}^2 \) and \( r_0 > 0 \) such that the relation (4) is true. The uniform rotation \((r_0, \frac{J}{r_0^2})\) is a solution of (2) and (3). Let \((J, E) \in \mathbb{R}^2 \), \( r_0 > 0 \) and \((r_0, \varphi)\) an uniform rotation which is a solution of (2) and (3) then we have:

\[
\dot{\varphi} = \frac{J}{r_0^2}, \quad r_0^2 \dot{\varphi}^2 - U'(r_0) = 0, \quad \frac{J^2}{2r_0^4} + U(r_0) = E.
\]

(5)
The Angular Momentum-Energy Space

We introduce (5)₁ in (5)₂ and we obtain (4)₁. The relation (4)₂ is obtained using (4)₁ and (5)₃.

Hypothesis: In this paper we suppose that a force function per unit mass is a function \( U \in C¹((0, \infty), \mathbb{R}) \).

Remark 2.1. The most important is the case of an attractive force field which is characterized by a force function with the property \( U' > 0 \).

In this paper we use the following notations: the effective force function per unit mass:

\[
V^U_J(r) = \frac{J^2}{2r^2} + U(r)
\]  

(6)

the effective angular momentum per unit mass:

\[
W^U_E(r) = 2r^2(E - U(r))
\]  

(7)

where \( U \) is a force function per unit mass, \( E \) the total energy per unit mass and \( J \) the angular momentum per unit mass. We have \( V^U_J, W^U_E \in C¹((0, \infty), \mathbb{R}) \).

It is easy to see:

Proposition 2.1. If \((r, \varphi)\) is a solution of (3), then for all time-moments we have:

\[
V^U_J(r(t)) \leq E
\]  

(8)

\[
W^U_E(r(t)) \geq J^2
\]  

(9)

The most important notions for the paper are presented in the next considerations.

Definition 2.1. Let \( U \) a force function per unit mass; \((J, E) \in \mathbb{R}²\) is an angular momentum-energy state if it exists a motion of the particle with \( J \) the angular momentum per unit mass and \( E \) the total energy per unit mass. The Angular Momentum-Energy Space \( S_U \) is the set of the angular momentum-energy states.

Remark 2.2. \((J, E) \in \mathbb{R}²\) is an angular momentum-energy state if and only if it exists a solution \((r, \varphi)\) of (2) and (3).

Remark 2.3. An angular momentum-energy state \((J, E)\) is corresponding to an uniform rotation if exists an uniform rotation of the particle with \( J \)
the angular momentum per unit mass and $E$ the total energy per unit mass. We denote by $S_{U}^{u.r}$ the set of the angular momentum-energy states which are corresponding to the uniform rotations.

3. The Angular Momentum-Energy Space

3.1. The set of the angular momentum-energy states which are corresponding to the uniform rotations. This set is characterized by the theorem 1, we have:

$$S_{U}^{u.r} = \{(J, E) \mid \exists s > 0 \ J^2 = s^3U'(s) \text{ and } E = U(s) + \frac{sU'(s)}{2}\}$$  \hfill (10)

It is easy to see that we have the following characterizations of the set of angular momentum-energy states which are corresponding to the uniform rotations:

$$S_{U}^{u.r} = \{(J, E) \mid \exists s > 0 \ (V_{U}^{j})'(s) = 0 \text{ and } E = V_{U}^{j}(s)\}$$  \hfill (11)

$$S_{U}^{u.r} = \{(J, E) \mid \exists s > 0 \ J^2 = W_{E}^{U}(s) \text{ and } (W_{E}^{U})'(s) = 0\}$$  \hfill (12)

We present some interesting properties of the set $S_{U}^{u.r}$.

**Proposition 3.1.** i) If $(J, E) \in S_{U}^{u.r}$, then $(-J, E) \in S_{U}^{u.r}$.

ii) If $r_0 > 0$, then it exists an uniform rotation $r(t) = r_0$ if and only if $U'(r_0) \geq 0$.

iii) If $\min_{r>0} V_{j}^{U}(r) \in \mathbb{R}$ and $E = \min_{r>0} V_{j}^{U}(r)$, then $(J, E) \in S_{U}^{u.r}$.

iv) If $\max_{r>0} W_{E}^{U}(r) \in \mathbb{R}_+$ and $J = \sqrt{\max_{r>0} W_{E}^{U}(r)}$ then $(J, E) \in S_{U}^{u.r}$.

**Proof.** The first and second results are consequences of the characterization (7) of the set $S_{U}^{u.r}$.

iii) In our hypotheses it exists $r_0 > 0$ such that $E = V_{U}^{j}(r_0)$. According to Fermat theorem we have $(V_{U}^{j})'(r_0) = 0$. It is easy to see that the relations (4) are verified and $(J, E) \in S_{U}^{u.r}$.

The proof of iv) is analogue with the demonstration of iii).

3.2. The properties of the Angular Momentum-Energy Space. Firstly we present a characterization of the Angular Momentum-Energy Space using the properties of the force function per unit mass $U$. 


**Theorem 3.1.** The Angular Momentum-Energy Space is characterized by:

\[ S_U = \{ (J, E) \mid E > \inf_{r>0} V_J^U(r) \text{ or } E = \min_{r>0} V_J^U(r) \} \]  

(13)

and

\[ S_U = \{ (J, E) \mid J^2 < \sup_{r>0} W_E^U(r) \text{ or } J^2 = \max_{r>0} W_E^U(r) \} \]  

(14)

**Proof.** We suppose that \((J, E) \in S_U\). It exists \((r, \varphi)\) a solution of (2) and (3). It is easy to see that \(E \geq \inf_{r>0} V_J^U(r)\) and \(J^2 \leq \sup_{r>0} W_E^U(r)\).

If \(E = \inf_{r>0} V_J^U(r)\), then it exists \(r_0 > 0\) such that \(E = V_J^U(r_0) = \min_{r>0} V_J^U(r)\). We deduce that:

\[ S_U \subset \{ (J, E) \mid E > \inf_{r>0} V_J^U(r) \text{ or } E = \min_{r>0} V_J^U(r) \} \]

If \(J^2 = \sup_{r>0} W_E^U(r)\), then it exists \(r_0 > 0\) such that \(J^2 = W_E^U(r_0) = \max_{r>0} W_E^U(r)\). We deduce that:

\[ S_U \subset \{ (J, E) \mid J^2 < \sup_{r>0} W_E^U(r) \text{ or } J^2 = \max_{r>0} W_E^U(r) \} \]

Let \((J, E) \in \{ (J, E) \mid E > \inf_{r>0} V_J^U(r) \text{ or } E = \min_{r>0} V_J^U(r) \}\). If \(E > \inf_{r>0} V_J^U(r)\), then it exists \(r_0 > 0\) such that \(E > V_J^U(r_0)\). We consider the Cauchy problem of differential equations:

\[ \dot{\varphi} = \frac{J}{r^2}, \quad \dot{r} = 2 \sqrt{E - V_J^U(r)}, \quad \varphi(0) = \frac{J}{r_0^2}, \quad r(0) = r_0. \]

According to the Cauchy-Lipschitz Theorem the Cauchy problem has a solution \((r, \varphi)\). This solution is not an uniform rotation and it is a solution of the system (3). Using the Theorem 2.1 we deduce that \((r, \varphi)\) is a solution of (2) and we conclude that:

\[ S_U \supset \{ (J, E) \mid E > \inf_{r>0} V_J^U(r) \text{ or } E = \min_{r>0} V_J^U(r) \} \]

Let \((J, E) \in \{ (J, E) \mid J^2 < \sup_{r>0} W_E^U(r) \text{ or } J^2 = \max_{r>0} W_E^U(r) \}\). If \(J^2 < \sup_{r>0} W_E^U(r)\), then it exists \(r_0 > 0\) such that \(J^2 < W_E^U(r_0)\). We consider the Cauchy problem of differential equations:

\[ \dot{\varphi} = \frac{J}{2u}, \quad \dot{u} = \sqrt{W_E^U(\sqrt{2u}) - J^2}, \quad \varphi(0) = \frac{J}{r_0^2}, \quad u(0) = \frac{r_0^2}{2}. \]
According to the Cauchy-Lipschitz Theorem the Cauchy problem has a solution \((u, \varphi)\). In this situation \((r, \varphi) = (\sqrt{2u}, \varphi)\) is a solution of the system (3). We obtain:

\[
S_U \supset \{(J, E) \mid J^2 < \sup_{r>0} W^U_E(r) \text{ or } J^2 = \max_{r>0} W^U_E(r)\}
\]

We present some properties of the Angular Momentum-Energy Space.

**Theorem 3.2.**

i) If \((J, E) \in S_U\), then \((-J, E) \in S_U\).

ii) If \(k \in \mathbb{R}\), then \(S_{U+k} = S_U + (0, k)\).

iii) Let \(U_1, U_2\) two force functions, if \(U_1 \leq U_2\), then \(S_{U_2} \subset S_{U_1}\).

**Proof.** i) We observe that \(V^U = V^{−J}\) and one obtains the affirmation.

ii) The result is an immediate consequence of the relation \(V^{U+k} = V^U + k\).

iii) We have \(V^U_1(r) \leq V^U_2(r) \forall r > 0\) which implies easily our proposition.

**Remark 3.1.** \(U_1 \leq U_2 \iff \forall r > 0 \text{ we have } U_1(r) \leq U_2(r)\).

Finally we study the conditions of the force function per unit mass \(U\) such that the Angular Momentum-Energy Space is the entire \(\mathbb{R}^2\).

**Theorem 3.3.** The next affirmations are equivalents:

i) \(S_U = \mathbb{R}^2\).

ii) \(\liminf_{r \to 0} r^2 U(r) = -\infty\) or \(\liminf_{r \to \infty} U(r) = -\infty\).

**Proof.** Firstly we suppose that \(S_U = \mathbb{R}^2\), \(\liminf_{r \to 0} r^2 U(r) > -\infty\) and \(\liminf_{r \to \infty} U(r) > -\infty\).

It exists \(0 < r^* < r^{**}\) and \(k^*, k^{**} \in \mathbb{R}^*_+\) such that, if \(r \in (0, r^*)\), then \(U(r) > -\frac{k^*}{r^2}\) and if \(r > r^{**}\), then \(U(r) > -k^{**}\). \(U\) is a continuous function and \([r^*, r^{**}]\) is a compact interval, there exists \(k^{***} > 0\) such that \(U(r) > k^{***}\) for all \(r \in [r^*, r^{**}]\). We introduce \(\bar{k} = \max\{k^*, k^{***} r^*, k^{***} r^{**2}\} > 0\) and we have \(U(r) \geq -\frac{\bar{k}}{r^2}\) for all \(r > 0\). Using the Theorem 3.2. one obtains that \(S_U \subset S_{-\frac{\bar{k}}{r^2}}\). We know that \(S_{-\frac{\bar{k}}{r^2}} \neq \mathbb{R}^2\) (see the §4.3.) and we deduce that \(S_U \neq \mathbb{R}^2\), but this result is a contradiction.

If the affirmation ii) is true, then \(\inf_{r>0} V^U_J(r) = -\infty\) for all \(J \in \mathbb{R}\). According to Theorem 3.1. we obtain that the affirmation i) is true.
4. Particular cases of Angular Momentum-Energy Space

4.1. An Isolated Particle; \( U = 0 \).

\[
S_0 = \{(J, E) \in \mathbb{R}^2 \mid E > 0\} \cup \{(0, 0)\}, \quad S^u_r = \{(0, 0)\}
\]  

(15)

In this case an uniform rotation is an equilibrium point. The angular momentum-energy state \((0, 0)\) is corresponding to all uniform rotations (equilibrium points).

**Remark 4.1.** Let \( k \in \mathbb{R} \). Using the theorem 3 we obtain:

\[
S_k = \{(J, E) \mid E > k\} \cup \{(0, 0)\}
\]

(16)

4.2. Particle in a Gravitational Force Field, \( U = -\frac{k}{r} \). We suppose that the gravitational force is an attraction force (\( k > 0 \)). In this case we have:

\[
S_{-\frac{k}{r}} = \{(J, E) \mid Ej^2 \geq -\frac{k^2}{2}\}, \quad S^u_{-\frac{k}{r}} = \{(J, E) \mid Ej^2 = -\frac{k^2}{2}\}
\]

(17)

4.3. \( U = -\frac{k}{r^2} \) with \( k > 0 \). We have:

\[
S_{-\frac{k}{r^2}} = S_1 \cup S_2 \cup S_3, \quad S^u_{-\frac{k}{r^2}} = \{(-\sqrt{2k}, 0), (\sqrt{2k}, 0)\}
\]

(18)

where:

\[
\begin{align*}
S_1 &= \{(J, E) \mid (J^2 > 2k \text{ and } E > 0)\} \\
S_2 &= \{(J, E) \mid J^2 = 2k \text{ and } E \geq 0\} \\
S_3 &= \{(J, E) \mid J^2 < 2k \text{ and } E \in \mathbb{R}\}
\end{align*}
\]

4.4. Particle in a Hooke Force Field, \( U = \frac{k}{2}r^2 \) with \( k > 0 \). We have:

\[
S_{\frac{k}{2}r^2} = \{(J, E) \mid E \geq \sqrt{k|J|}\} - \{(0, 0)\}, \quad S^u_{\frac{k}{2}r^2} = \{(J, E) \mid E = \sqrt{k|J|}\} - \{(0, 0)\}
\]

(19)

**Remark 4.2.** For us \( U \) is not defined for \( r = 0 \); this is the reason for which \((0, 0)\) is not a angular momentum-energy state.
4.5. Particle in a Repulsive Elastic force Field, \( U = -\frac{k}{2}r^2 \) with \( k > 0 \).
In this case:
\[
S_{-\frac{k}{2}r^2} = \mathbb{R}^2, \quad S_{-\frac{k}{2}r^2}^{u.r} = \emptyset
\]

\( 4.6. \quad U = -\frac{k}{r} - \frac{q}{r^2} \) with \( k > 0 \) and \( q > 0 \).
\[
S_{-\frac{k}{r} - \frac{q}{r^2}} = \{(J, E) / J^2 \leq 2q \ or \ (J^2 > 2q \ and \ E(J^2 - 2q) \geq -\frac{k^2}{2})\} \quad (21)
\]
\[
S_{-\frac{k}{r} - \frac{q}{r^2}}^{u.r} = \{(J, E) / E < 0 \ and \ E(J^2 - 2q) = -\frac{k^2}{2}\} \quad (22)
\]

4.7. \( U = -\frac{k}{r^{2n}} \) with \( k > 0 \) and \( n > 0 \). If \( n > 1 \), then we have:
\[
S_{-\frac{k}{r^{2n}}} = \mathbb{R}^2, \quad S_{-\frac{k}{r^{2n}}}^{u.r} = \{(J, E) / EJ^\frac{2n}{2n} = (n-1)(2n)\frac{1}{1-n}k^{\frac{1}{1-n}}\} \quad (23)
\]

The case \( n = 1 \) is studied in the §4.3.

If \( n \in (0, 1) \) then:
\[
S_{-\frac{k}{r^{2n}}} = \{(J, E) / EJ^\frac{2n}{2n} \geq -(1-n)(2n)\frac{n}{1-n}k^{\frac{1}{1-n}}\} \quad (24)
\]
\[
S_{-\frac{k}{r^{2n}}}^{u.r} = \{(J, E) / EJ^\frac{2n}{2n} = -(1-n)(2n)\frac{n}{1-n}k^{\frac{1}{1-n}}\} \quad (25)
\]

4.8. \( U = q \sin \frac{1}{r} \) with \( q > 0 \). This case is interesting for theoretical reasons. Using the theorem of characterization of the angular momentum-energy Space we obtain:
\[
S_{q \sin \frac{1}{r}} = \{(J, E) / E > -q\} \cup \{(0, -q)\} \quad (26)
\]

The set \( D_{q \sin \frac{1}{r}} \) of the distances \( r_0 \) for which exists an uniform rotations with \( r(t) = r_0 \) is described by the formula:
\[
D_{q \sin \frac{1}{r}} = \bigcup_{k \in \mathbb{N}}\left[\frac{2}{(4k + 3)\pi}, \frac{2}{(4k + 1)\pi}\right] \quad (27)
\]

If \( r_0 \in \left\{\frac{2}{(2k+1)\pi} / k \in \mathbb{N}\right\} \), then at the distance \( r_0 \) the particle can have an equilibrium state. All equilibrium states have an angular momentum-energy state in the set \( \{(0, q), (0, -q)\} \).
The set of the angular momentum-energy states which are corresponding to the uniform rotations is:

$$S_{q_{s}\sin^{1}r}^{u,r} = \{(J, E) / \exists s > 0 \ J^2 = -qs \cos \frac{1}{s} \text{ and } E = q \sin \frac{1}{s} - \frac{q}{2s} \cos \frac{1}{s}\} \ (28)$$

References

[1] Arnold V.I., Mathematical Methods of Classical Mechanics, Springer Verlag, 1989.

[2] Chon H., Kulsrud R.M., The stellar distribution around a black hole: numerical integration of the Fokker-Plank equation, The Astrophysical Journal, 226: 1087-1108, 1978.

[3] Yi Y.G., General-relativistic equations of motion in terms of energy and angular momentum, European Journal of Physics, 2003, vol. 24, no.4, pp. 413-417.

[4] Comănescu D., Modele și metode în mecanica punctului material, Mirton Publishing House, Timișoara, 2004.

[5] Landau L.D., Lifšit E.M., Mechanics, Ed. Tehnica, București, 1966.

Authors’ affiliation: Department of Mathematics, Faculty of Mathematics and Computer Science, West University of Timișoara, Romania

E-mail: comanescu@math.uvt.ro