Limitations of Bayesian Evidence applied to cosmology

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ABSTRACT

There has been increasing interest by cosmologists in applying Bayesian techniques, such as Bayesian Evidence, for model selection. A typical example is in assessing whether observational data favour a cosmological constant over evolving dark energy. In this paper, the example of dark energy is used to illustrate limitations in the application of Bayesian Evidence associated with subjective judgements concerning the choice of model and priors. An analysis of recent cosmological data shows a statistically insignificant preference for a cosmological constant over simple dynamical models of dark energy. It is argued that for nested problems, as considered here, Bayesian parameter estimation can be more informative than computing Bayesian Evidence for poorly motivated physical models.

Key words: cosmology: methods: data analysis – cosmological parameters – cosmology observations – cosmology: theory.

1 INTRODUCTION

Bayesian techniques for estimating the posterior distributions of cosmological parameters are now well established in astronomy (see Lahav & Liddle 2006, and references therein). In the last few years, cosmologists have become increasingly interested in statistical techniques for model selection [for an early application see Jaffe 1996 and for recent summaries see Liddle et al. (2006) and Trotta (2008)]. This subject has a long history and is discussed at length in Jeffreys’ classic monograph (Jeffreys 1961). The aim of model selection is to provide a measure by which to rank competing models. A model that is highly predictive should clearly be favoured over a model that is not. Model selection, in effect, quantifies Occam’s Razor by penalizing complicated models with many parameters that need to be finely tuned to match the data. A topical example of model selection applied to cosmology is in assessing whether observational data favour dynamical dark energy over a cosmological constant (for recent discussions see Liddle et al. 2006; Szydlowski, Kurek & Krawiec 2006; Sahlén, Liddle & Parkinson 2007; Serra, Heavens & Melchiorri 2007). This is the example that we will use in this paper.

Models can be ranked by computing the Bayesian Evidence, \( E(M) \), defined as the probability of the data \( D \) given the model \( M \),

\[
E(M) = \int d\theta P(D|\theta M)\pi(\theta | M),
\]

where \( \pi(\theta | M) \) is the prior distribution of model parameters \( \theta \) and \( P(D|\theta M) \) is the likelihood of the parameters under model \( M \). The ratio of the Evidence for two models,

\[
B_{12} = E(M_1)/E(M_2),
\]

also known as the Bayes factor, provides a measure with which to discriminate between the models. If each model is assigned equal prior probability, the Bayes factor gives the posterior odds of the two models. A value for \( B_{12} \) of order unity indicates that there is little to choose between the two models but a value of, say, \( B_{12} \lesssim 0.01 \) suggests that the data strongly favour model 2 over model 1.\(^1\) The computation of Evidence can be challenging since it requires the evaluation of an integral [equation (1)] over the entire likelihood function. This can take many hours of supercomputer time if the cost of evaluating likelihood function at a single point within a multi-dimensional parameter space is large. Rather than compute the Evidence, some authors have used proxies such as the Bayesian Information Criterion (BIC), which can be computed from the maximum of the likelihood function. The BIC and various information theoretic criteria for model selection are discussed by Liddle (2004, 2007) and by Trotta (2008); these will not be discussed in any detail in this paper, which will focus on the Evidence defined by equation (1).

The application of Bayesian Evidence to cosmology has not met with uniform approval. The methodology has been attacked vigorously recently by Linder & Miquel (2008), and defended even more vigorously by Liddle et al. (2007). This author agrees with the statistical analysis of Liddle et al. (2007). Nevertheless, our conclusions are more in sympathy with those of Linder and Miquel, namely that Bayesian Evidence is of limited use for many applications to cosmology.

The main reason for reaching this conclusion is that the very concept of a model is subjective in many cosmological applications. Ideally, one would like to test a physically well-motivated model.

\(^1\) Many authors have used qualitative guidelines suggested by Jeffreys (1961) to interpret Bayes factors (see Section 3).
rather than adopting a phenomenological parameterization, but this is rarely possible in cosmology because the underlying physics is poorly understood. As is evident from equation (1) computation of Bayesian Evidence requires assumptions concerning the prior distributions of any parameters specifying a model. Again, in cosmology we rarely have strong guiding principles to help us choose priors. Bayesian Evidence is useful in situations where hypotheses are well motivated and when there are symmetries, or other information, to guide the choice of priors. (Specific examples are given in the textbook by Mackay 2003.) Perceived difficulties with ‘subjective’ choices of priors are, of course, at the heart of the long-standing debate between ‘Frequentists’ and ‘Bayesians’ (see for example, Kendall & Stuart 1979, Section 21; Jaynes 2003) and this paper has nothing new to add to this well-worn discussion. But even if one approaches statistics from a Bayesian point of view, as this author does, the difficulties involved in defining models and priors must be appreciated when interpreting Bayesian Evidence.

In this paper, I will use the problem of dark energy to illustrate the points outlined in the previous paragraph. In the next section, I begin with a discussion of ‘skater’ models (Linder 2005; Sahlén et al. 2007) to demonstrate the subjectivity involved in defining a model. The skater parameterization is clearly unphysical and should be thought of as an approximation to a more complex model requiring more free parameters and uncertain priors. This is typical of many parameterizations of evolving dark energy. I will then discuss the constraints on simple dynamical models of dark energy, including a ‘thawing’ field evolving in a linear potential and a ‘freezing’ tracker model, to illustrate problems associated with priors. Some general comments on model selection are presented in Section 3 and the conclusions are summarized in Section 4.

2 TESTING DARK ENERGY

The discovery that the Universe is accelerating has stimulated an enormous amount of interest in dynamical models of dark energy (see for example, the comprehensive review by Copeland, Sami & Tsujikawa 2006). A particularly simple class of models is based on a scalar field $\phi$ evolving in a potential $V(\phi)$. The equation of motion of the field is

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi),$$

where dots denote time derivatives, $H = \dot{a}/a$ where $a$ is the scale factor, and the prime denotes a derivative with respect to the field value $\phi$. In this paper, we focus on comparing this class of dynamical models with the hypothesis that the dark energy is a cosmological constant, that is $V = V_0 = constant$.

2.1 The difficulty of defining a physically well-motivated model.

In a ‘skating’ model (Linder 2005), the potential is assumed to be flat, $V = V_0$, but the scalar field has some kinetic energy, $\dot{\phi} \neq 0$. This leads to a non-trivial equation of state that evolves as

$$\frac{d\omega}{d\ln a} = -3(1 - w^2).$$

What prior should we choose for $\phi$? The equation of motion gives $\dot{\phi} \propto a^{-1}$, that is the field velocity decays adiabatically as the Universe expands. So a physically well motivated prior for $\phi$ would be a delta function centred around the value $\phi = 0$, making the model indistinguishable from a cosmological constant.

Sahlén, Liddle & Parkinson (2005, 2007) have derived constraints on skater models using distant supernovae and other cosmological data. The data do not constrain strongly the field velocity $\dot{\phi}_0$ at the present day and so the limits on $\dot{\phi}_0$ simply reflect the maximum value allowed by their choice of prior on the kinetic contribution to the cosmic density at high redshift. It can be argued that if the likelihood function is flat over the full domain of a parameter, then the Evidence is independent of the prior (Liddle et al. 2007). But how does one specify the domain of a parameter? As discussed in the next section, if the likelihood does vary over the parameter range, perhaps because the data have been used to suggest the range, then the posterior distributions of other parameters, such as $V_0$, and the Evidence (equation 1) will depend on the choice of prior.

In fact, the problem is more serious than outlined above. We have described the skater model here because it is easy to see that it is a proxy for a more complicated model involving more parameters (and priors). This is true of many simple parameterizations of dynamical dark energy. How could skating behaviour be realized in practice? In the following example,

$$V(\phi) = \frac{M^{4+w}}{\phi^\alpha} + V_0,$$

the first term in equation (5) is a ‘tracker’ potential with an attractor solution (Steinhardt, Wang & Zlatev 1999). It is therefore easy to arrange for the field to follow the attractor solution at high redshift and then glide on to the constant part of the potential with some finite $\dot{\phi}$ at low redshift. As a specific example, assume $V_0 = 2H_0^2$, $\alpha = 4$, $M^{4+w} = 0.05H_0^2$, then at the present day $\Omega_{\phi_0} = 0.71$, $w_{\phi_0} = -0.97$ and $\dot{\phi}_0/H_0 = 0.246$ (actually outside the range on $\dot{\phi}_0/H_0$ permitted by the priors assumed by Sahlén et al. 2007). We would argue that equation (5) with attractor initial conditions is a physically better motivated model than a simple flat potential with some arbitrary choice of initial $\phi$. Of course, this is a more complicated ‘skater-like’ model and requires the specification of priors on three parameters $M$, $\alpha$ and $V_0$. Furthermore, as the above example shows, for reasonable values of $\alpha$ it is difficult to get a substantial deviation from $w_{\phi_0} = -1$. The additional parameters therefore allow models that show deviations from the dynamics of a cosmological constant at low redshift, but the differences are small.

2.2 Dependence on priors

To make contact with previous work, we first analyse the simple phenomenological model with a constant equation of state parameter $w_0$. Evidence computations for this model have been presented by Liddle et al. (2006) and Serra et al. (2007), using broadly similar data to those used here.

We use a compilation from the following web address http://braeburn.pha.jhu.edu/~ariess/R06/Davis07_R07_WV07 for listing redshifts, distance moduli and their errors for Type 1a supernovae. These data were used in Davis et al. (2007) and consist of combined data from Wood-Vasey et al. (2007) and Riess et al. (2007). Supernovae with redshifts less than 0.02 were discarded to limit systematic errors associated with local peculiar velocities, leaving 181 supernovae with a maximum redshift of 1.755 (sn1997ff). Following Sahlén et al. (2007), in addition to the constraints on luminosity distances from Type 1a supernovae, we add constraints on the cosmic microwave background (CMB) peak shift parameter $\mathcal{R}$ at the redshift of decoupling and on the baryon...
acoustic scale parameter \( \Lambda \) at the characteristic depth of the Sloan Digital Sky Survey (Eisenstein et al. 2005), assuming Gaussian distributions with

\[
\mathcal{R}(z_{\text{dec}} = 1089) = 1.70 \pm 0.03, \quad (6a) \\
A(z = 0.35) = 0.474 \pm 0.017. \quad (6b)
\]

For definitions of the parameters \( \mathcal{R} \) and \( A \) and references to the numerical values listed in (6a, 6b), we refer the reader to Wang & Mukherjee (2006) and Sahbén et al. (2007).

Spatial curvature is assumed to be zero, thus a model is specified by the Hubble parameter \( h \) (in units of, 100 km s\(^{-1}\) Mpc\(^{-1}\)) and the equation of state parameter \( w_0 \). To facilitate comparison with Serra et al. (2007), we adopt the identical flat priors for \( \Omega_m \) and \( h \) with ranges \( 0.1 \leq \Omega_m \leq 0.5 \) and \( 0.56 \leq h \leq 0.72 \) (and we do not attempt to justify these choices). For \( w_0 \) we will adopt a flat prior over the range \( -1 \leq w_0 \leq -1/3 \), that is excluding the ‘phantom’ regime \( w_0 < -1 \), and a flat prior over the range \( -2 \leq w_0 \leq -1/3 \).

Fig. 1 shows likelihood contours in the \( w_0 - \Omega_m \) plane marginalized over the Hubble parameter \( h \). The results are broadly compatible with the analysis presented by Serra et al., though the supernova sample used here is larger and so the contours in Fig. 1 are somewhat tighter than theirs. Note that the peak of the likelihood function is close to the cosmological constant value \( w_0 = -1 \). There is no evidence of a shift of the contours below the phantom divide line seen in some earlier analyses (e.g. Riess et al. 2004). There is evidence that the older High-z Supernovae Search Team (HZZST) data pull the solutions to \( w_0 < -1 \) (see Nessers & Perivolaropoulos 2007) indicative of (unknown) systematic errors in the earlier data. The HZZST data are not included in the sample used here.

The Evidence ratios for a cosmological constant and the constant \( w_0 \) model are listed in Table 1 for various choices of prior on \( w_0 \). The results in Table 1 agree well with those of Lidelle et al. (2006) who used the distant supernovae data of Astier et al. (2006). The Evidence ratios in Table 1 are about a factor of two larger than those computed by Serra et al. (2007). Much of this difference is caused because the latter authors include the old HZZST data which pull the likelihood function further into the phantom regime hence penalising the \( \Lambda \) model.

### Table 1. Evidence ratios.

| Model                  | Prior | \( E_\Lambda/E_Q \) | \( \ln (E_\Lambda/E_Q) \) |
|------------------------|-------|---------------------|--------------------------|
| Constant \( w \)       | \(-1 \leq w_0 < -0.333\) | 3.42 | 1.23 |
| Constant \( w \)       | \(-2 \leq w_0 < -0.333\) | 6.15 | 1.82 |
| Constant \( w \)       | \(-1.4 \leq w_0 < -0.6\) | 2.95 | 1.08 |
| Linear \( w \)         | \( 0 < V_0 < 3 \) | 1.58 | 0.46 |
| Linear \( w \)         | \( 0 < V_0 < 6 \) | 2.59 | 0.95 |
| Inverse power \( w \)  | \( 0 < a < 6 \) | 8.59 | 2.15 |
| Inverse power \( w \)  | \( 0 < a < 2 \) | 2.78 | 1.02 |
| Inverse power \( w \)  | \( 0 < a < 1 \) | 1.40 | 0.33 |

Note: \( E_\Lambda \) denoted the Evidence for a model with a cosmological constant \( \Lambda \) and \( E_Q \) denotes the Evidence for the dynamical dark-energy models discussed in the text. The first three lines list Evidence ratios for the model with constant \( w_0 \). The next two lines list Evidence ratios for the model with a linear potential as discussed in the text. The last three lines list Evidence ratios for tracker models with an inverse power-law potential.

The Evidence ratios in this Table indicate a marginal preference for the \( \Lambda \) model (see Section 3.1 for remarks on the interpretation of Evidence), but none of the Evidence ratios are high and it is easy to change them by factors of a few by changing the range of the prior on \( w_0 \). This is demonstrated in the third line of the Table, where the Evidence has been recomputed assuming a flat prior over the narrower range \(-1.4 \leq w_0 < -0.6\). The dependence on the prior is not particularly serious in this case, because none of the entries in Table 1 provide strong evidence to favour or disfavour the \( \Lambda \) model.

Following on from the discussion in Section 2.1, we would argue that a model with a flat prior on a constant value of \( w_0 \) is not particularly well motivated. We therefore seek a simple dynamical model, derivable from a potential, \( V(\phi) \) with as few free parameters as possible. One such model is based on the linear potential,

\[
V(\phi) = V_0 + V' \phi, \tag{7}
\]

with a negative gradient \( V' \). For discussions of the linear potential see for example Dimopoulos & Thomas 2003; Kallosh et al. 2003; Avelino 2005. The zero point of the field value has no significance and so can be set to zero at some starting redshift \( z_i \). The field is locked by Hubble friction (equation 3) until the Hubble parameter drops sufficiently that the field begins to roll. At this point, the dark energy will show interesting dynamical behaviour. Eventually, the potential will become negative and the Universe will collapse. This simple model therefore displays ‘thawing’ behaviour in the nomenclature of Caldwell and Linder (2005), followed by ‘cosmic doomsday’ in the nomenclature of Kallosh et al. (2003).

This model has a weak dependence on the starting redshift, which we fix to be the decoupling redshift \( z_i = 1089 \). We ‘absorb’ this weak dependence into the priors on \( V_0 \) and \( V' \). A model is therefore specified by these two parameters. The left-hand panel of Fig. 2 shows contours in the \( V_0 - V' \) plane with constant values of \( \Omega_m \) (the present day dark energy density) and \( w_0 \), (the present day dark energy equation of state parameter). The shaded regions delineate areas where no solution exists. Fig. 1 shows that the data favour models with \( \Omega_m \sim 0.27 \) and \( w_0 \lesssim -0.85 \). The marginalized likelihood function for the linear model is therefore expected to delineate a narrow banana centred around the \( \Omega_m \sim 0.7 \) line. This is indeed what is found, as illustrated in the right-hand panel of Fig. 2.

\[3 \text{ Note that the dimensionless parameters appearing in this equation are related to dimensional parameters as } \phi/M_{\text{pl}} \to \phi, V_0 M_{\text{pl}}^{-2} H_0^{-2} \to V_0 \text{ and } V' M_{\text{pl}}^{-1} H_0^{-2} \to V'.\]
To compute the Evidence, priors need to be specified for the parameters \( V_0 \) and \( V' \). But how do we choose these priors? For spatially flat models with, \( V' = 0 \) the value of \( V_0 \) must lie within the range \( 0 \leq V_0 < 3 \). However, if we allow non-zero values of \( V' \), \( V_0 \) can lie outside this range. One possibility would be to choose a flat prior over the region of the \( V_0 - V' \) plane corresponding to models which are accelerating at the present day. However, this choice of prior is hardly compelling.

In fact, the choice of prior is not particularly critical for the interpretation of Fig. 2 because current data provide relatively poor constraints on \( V' \). This is illustrated by the last two lines in Table 1 which list the Evidence ratios assuming a uniform prior in \( V' \) over the full range shown in Fig. 2 (\( 0 \leq V' < 6 \)) and a uniform prior in \( V_0 \) over the ranges \( 0 \leq V_0 < 3 \) and \( 0 \leq V_0 < 6 \) (excluding the hatched regions). There is no significant evidence to favour \( /\Lambda \) over the dynamical model, and it is clear from inspection of Fig. 2 that the higher Evidence ratio in the last line of Table 1 is largely a consequence of the increased range of the prior on \( V_0 \).

As a final example, we consider a ‘tracker’ model with the Ratra & Peebles (1988) potential

\[
V(\phi) = \frac{M^{4+\alpha}}{\phi^\alpha}.
\]

(8)

An introductory review of this model is given by Martin (2008). As mentioned above, this potential has an attractor solution which drives \( w_0 \) towards the solution

\[
w_0 \rightarrow \frac{\alpha w_B - 2}{\alpha + 2}.
\]

(9)

while the scalar field is subdominant (where \( w_B \) is the equation of state parameter of the background matter). This model is an example of a ‘freezing’ model in the terminology of Caldwell & Linder (2005). We set \( \phi = 0 \) at high redshift (\( z = 10000 \)) and choose the initial value of \( \phi \) so that the field locks on to the attractor solution without overshoot. The low redshift behaviour is therefore fixed by the attractor solution so the model is characterized by the two parameters \( M \) and \( \alpha \) defining the potential (equation 8).

The marginalized likelihood for this model is shown in Fig. 3, (using the same observational data as for Figs 1 and 2). Evidently, the data constrain the power-law index to be \( \alpha \lesssim 1 \). However, from the theoretical point of view, there are no compelling constraints on the spectral index; values as high as \( \alpha = 6 \) or more have been discussed in the literature (Steinhardt, Wang & Zlatev 1999; Martin 2008) and occasionally promoted as a possible solution to the ‘hierarchy’ problem associated with the mass scale \( M \). The Evidence ratios for various assumed prior ranges in \( \alpha \) are listed in the last three lines of Table 1. As expected, these scale almost perfectly with the width of the prior range assumed for \( \alpha \). Nevertheless, none of the Evidence ratios in Table 3 are high and so one cannot reject the potential (equation 8) drawn from a broad uniform prior on \( \alpha \). However, it is obvious from the likelihood function shown in Fig. 3 that we can rule out the potential (equation 8) for specific choices of \( \alpha \gtrsim 1.5 \).

3 COMMENTS ON THE APPLICATION OF MODEL SELECTION

3.1 The Jeffreys scale

The previous section shows that current data, unfortunately, provide relatively weak constraints on simple models of dynamical dark energy. The highest Evidence ratios are about \( \Delta \ln E \sim 2 \) for
certain choices of prior. Plausible variations on the prior of a single parameter can easily change the Evidence ratio by $\Delta \ln E \sim 1$ or more. Many papers on cosmological model selection have adopted the interpretive scale suggested in Appendix B by Jeffreys, which in the view of this author is not sufficiently conservative. Trotta (2008) presents a revised interpretative scale in his review, which accords with one’s intuitive assessment of the relative posterior odds, $B_{12}$ of two models. The two scales are compared in Table 2.

For the dark energy examples summarized in the previous section, if the observational data improve to give, $\Delta \ln E \gg 5$ then it would be reasonable to conclude that there is evidence favouring a cosmological constant. The data are then providing strong enough constraints to overwhelm the changes in the prior volumes illustrated in Table 1 (see Section 3.3). But Evidence ratios of $\Delta \ln E \sim 2$ are clearly too small to achieve this. Many of the problems outlined in the previous section can be overcome by adopting a conservatively high threshold before claiming strong evidence against particular classes of model. The threshold may need to be set higher than $\Delta \ln E = 5$ if one (or both) of the models involves several additional parameters with uncertain priors.

### 3.2 Stating and varying priors

It is essential that authors computing Bayesian Evidence state their priors carefully since these are an integral part of the definition of a model. If there are no compelling reasons to guide the choice of priors, then one should demonstrate that the data overwhelm plausible variations in priors before drawing any strong conclusions on particular classes of model. This has not been common practice in the literature. For example, table 4 of Trotta (2008) summarizes Evidence calculations for various cosmological model comparisons. Of the ten entries testing dynamical dark energy against $\Lambda$, only three explore variations in priors. One analysis quoted in this table (Bassett, Corasaniti & Kunz 2004) computes Evidence for simple parameterizations of $w$ (such as $w = w_0 + w_z(z)$) without stating the prior ranges on the parameters. These authors find, $\Delta E_{\Lambda}/E_0 \gtrsim 5 - 6$ suggesting strong evidence favouring $\Lambda$. Such high Evidence ratios are clearly at variance with the conclusions of Section 2.

### 3.3 Model selection compared with parameter estimation

In many cosmological applications of model selection, we are dealing with highly nested problems. In each of the examples discussed in Section 2, the model for dynamical dark energy tends to the $\Lambda$ model as one additional the parameter tends to zero ($w_0 + 1 \rightarrow 0, V \rightarrow 0, \alpha \rightarrow 0$). In each of these cases, the primary question of interest is whether there is any empirical evidence that an additional parameter, $\lambda$ differs from zero. For such highly nested problems$^4$

The Bayes factor for model $M_1$ ($\lambda = 0$) and model $M_2$ ($\lambda$ drawn from a prior distribution $\pi(\lambda)$) is simply

$$B_{12} = \frac{P(D|\lambda = 0, M_2)}{\int P(D|\lambda, M_2) \pi(\lambda|M_2) d\lambda}.\quad (10)$$

(this is related to the Savage–Dickey density ratio, see Trotta 2007) where $P(D|\lambda, M_2) \equiv \mathcal{L}(\lambda)$ is the likelihood function marginalized over all of the common parameters). For a uniform prior in, $\mathcal{L}(\lambda)$ is just the marginalized posterior distribution of $\lambda$ on $M_2$, and from equation (10) we can interpret the likelihood ratio $\mathcal{L}(0)/\mathcal{L}(\lambda)$ as the Bayes factor for two models with delta function priors centred at $\lambda = 0$ and $\lambda$. If we choose $\lambda = \lambda_*$, where $\lambda_*$ corresponds to the peak of the likelihood function, then the Bayes factor is minimized. (For further discussion of the relationship between likelihood ratios and Bayes factors see Gordon & Trotta 2007). Now suppose that the likelihood function is approximated by a Gaussian

$$\mathcal{L}(\lambda) = \mathcal{L}_0 \exp \left[ -\frac{(\lambda - \lambda_*)^2}{2\sigma^2} \right],\quad (11)$$

and that we are interested in whether the parameter $\lambda$ differs from zero. The likelihood ratio $\mathcal{L}(0)/\mathcal{L}(\lambda_*)$ is evidently

$$\frac{\mathcal{L}(0)}{\mathcal{L}(\lambda)} = \exp \left( -\frac{\lambda^2_*}{2\sigma^2} \right).\quad (12)$$

Now assume that under model 2, the parameter $\lambda$ is drawn from a uniform distribution in the range $0 \leq \lambda < \lambda_{\text{max}}$, and assume further that $\lambda_{\text{max}} \gg \lambda_* \gg \sigma$. The data is then ‘informative’ $(\lambda_{\text{max}} \gg \sigma)$ and suggestive that $\lambda$ deviates from zero $(\lambda_* \gg \sigma)$. In this case,

$$\frac{\mathcal{E}(0)}{\mathcal{E}(\lambda)} \approx \exp \left( -\frac{\lambda^2_*}{2\sigma^2} \right) \frac{\lambda_{\text{max}}}{\sqrt{2\pi}\sigma}.\quad (13)$$

If the exponential term dominates in equation (13), the Evidence ratio is exponentially suppressed and the data swamp the dependence on the prior. (In fact, in testing whether a parameter differs from zero, Jeffreys (1961, Section 5.2) suggests using the prior

$$\pi(\lambda) \propto \frac{1}{(1 + \lambda^2/\sigma^2)},\quad (14)$$

since ‘there is nothing in the problem except $\sigma$ to give a scale for $\lambda$’. In other words, the choice of prior is driven by the data and in this case one is reliant on the exponential factor in equation (13) to reject the null hypothesis. Alternatively we might find $\lambda_* \gtrsim \sigma$ suggesting that the parameter $\lambda$ is consistent with zero, with $\lambda_{\text{max}} \gg \sigma$, in which case

$$\frac{\mathcal{E}(0)}{\mathcal{E}(\lambda)} \approx \frac{2}{\sqrt{2\pi}\sigma} \lambda_{\text{max}},$$

which merely tells that under model 2 we need fine tuning of order $\sigma/\lambda_{\text{max}}$ to explain the data.

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**Table 2.** Interpretive scales.

| $\ln B_{12}$ | Jeffreys grades | strength of Evidence | $\ln B_{12}$ | Trotta (2008) | strength of Evidence |
|--------------|----------------|----------------------|--------------|----------------|----------------------|
| $<1.15$      | Not worth a mention | -1.0 | Inconclusive | $<1.0$ | |
| 1.15–2.3     | Substantial      | 1.0–2.5 | Weak        | 2.5–5.0 | Moderate to strong |
| 2.3–4.6      | Strong to very strong | 5.0–5.0 | Strong      | > 5 | Strong |

**Table 3.** Likelihood ratio compared to Evidence ratio.

| $\sigma$ | $\ln(\mathcal{L}(1)/\mathcal{L}(w_*))$ | $\ln((E_\lambda/E_\Omega))$ uniform prior |
|----------|-------------------------------------|------------------------------------------|
| 0.1      | 0.03                               | -1.22                                    |
| 0.05     | -0.50                              | 1.77                                     |
| 0.02     | -3.12                              | -0.53                                    |
| 0.015    | -5.55                              | -2.67                                    |
| 0.01     | -12.50                             | -9.22                                    |

4 To simplify the following discussion, we will assume uniform priors on all parameters.
As a specific example\(^5\), imagine that future dark-energy surveys find \(w = -0.95 \pm \sigma \) \([w_0 \equiv (1 - \lambda)\) in the notation used above] and we test model 1 \((w_0 = -1\) the ‘null hypothesis’) against model 2 (uniform prior in \(w_0\)). Table 3 lists the likelihood ratios and Bayes factors for two choices of prior. One can see that for \(\sigma = 0.01\) the data swamp the dependence on the prior and all three entries in Table 3 strongly disfavour \(w_0 = -1\). The likelihood ratio is as informative as the Evidence ratios in the exponentially dominated regime since it matters little whether the posterior odds of two models are \(\sim 10^{-6}\) or \(\sim 10^{-10}\) – the odds are negligible in either case. The case of \(\sigma = 0.015\) is more interesting and is an example of ‘Lindley’s paradox’ (Lindley 1957). The likelihood function indicates a 3.33\(\sigma\) discrepancy with the null hypothesis, yet the Evidence ratio in the third column of Table 3 suggests weak evidence against \(w_0 = -1\). This is simply because the Evidence ratio compares one model that is disfavoured by the data \((w_0 = -1)\) against another model that is disfavoured by the data (uniform distribution of \(\lambda\) over the range \(-1 \leq w < -1/3\), requiring fine tuning at the few per cent level). Lindley’s paradox should not obscure the fact that the likelihood peaks away from \(w_0 = -1\); the likelihood function suggests that \(w_0\) differs from \(-1\), and is therefore informative. If, following future experiments, the contours shown in Figs 1–3 tighten so that zero values for \(1 + w_0, V',\) or \(\alpha\) are exponentially suppressed, then we will have very strong evidence in favour of dynamical dark energy, irrespective of priors. If the Evidence ratios for reasonable choices of priors are still in the ‘ambiguous’ range, \(\ln E_{\lambda}/E_0 \sim 2.5 - 5\) a modest improvement of the data could potentially render the issue decisive.

Finally, let us consider the case relevant to equation (15), namely that the likelihood function peaks at \(\lambda \sim 0\) to within \(\sim \sigma\). The null hypothesis is then favoured if the prior range \(\lambda_{\max} \gg \sigma\). But the Evidence ratio can be reduced to unity by adjusting the prior range to be of order \(\sigma\) (cf. equation 14). An ‘Occam’s Razor’ penalty for a model with additional parameters can be realised only if (i) one has good arguments for choosing the prior ranges of the additional parameters and (ii) the likelihood function is compact with respect to these prior ranges.

3.4 When is Bayesian Evidence particularly useful?

In the previous subsection, we have argued that the marginalized likelihood function is informative and can provide a good indicator of whether the null hypothesis \((\Lambda)\) is disfavoured compared to dynamical energy models. Let us suppose that \(\Lambda\) is indeed disfavoured by future data. In this case, the likelihood contours in Figs 1–3 would break up into ‘islands’ peaked away from \(\Lambda\). How do we assess between these three parameterizations? Bayesian Evidence is likely to be indispensable for this type of ‘non-nested’ model comparison. Again, the usual caveats should apply:

- we should aim to compare physically well-motivated models (better motivated than the skater model of Section 2.1);
- compare models with as few free parameters as possible (cf. the models of Section 2.2) to limit the sensitivity of the Evidence to prior volumes;
- explore variations in the priors.

\(^5\) We could equally as well have used the example of the spatial curvature \(\Omega_0\) or deviation of the scalar spectral index \(n_s\) from unity, as discussed by Liddle et al. (2007).

4 CONCLUSIONS

Bayesian inference has been applied widely for parameter estimation from cosmological data and is relatively uncontroversial. The Bayesian framework can easily be extended to model selection, but this has proved to be more controversial.

There is nothing wrong with the mathematical framework underlying Bayesian model selection. It is the perceived usefulness of the framework, given difficulties in specifying models that is at the source of the controversy. It is important to recognize that Bayesian model selection differs in a fundamental way from Bayesian parameter estimation. In parameter estimation, the posterior distribution on a parameter is useful because it usually become narrower as the quality of constraining data improves. The sensitivity of the posterior distribution to the prior therefore often diminishes dramatically with better data. This is why Bayesian parameter estimation has proved relatively uncontroversial. (Surprisingly so, since in cases such as estimating the CMB quadrupole there is an irreducible sensitivity to the choice of prior, see Efstathiou 2003).

For Bayesian model selection we need to apply ‘physical intuition’ to select suitable models and the priors on model parameters. Once these are chosen, the data determine the numerical value of the Evidence (i.e. the probability of the data given the model) via equation (1). There is no ‘updating’ involved since the data return a single value of the Evidence given the model. In Section 2, we discussed some of the difficulties associated with defining physically well-motivated models and parameter ranges. Now one can argue, correctly, that the range of a parameter is part of the definition of a model. However, in cosmology, it is often difficult to provide compelling arguments in favour of a particular parameter range. This is certainly the case for the dark-energy tests described in this paper. If we use the data to suggest the parameter ranges, for example, by examining the likelihood function, then a computation of equation (1) will be of limited value since the probability of the data given the model will be high by construction.

Many applications of Bayesian Evidence to cosmology involve highly nested problems in which the primary question of interest is whether a key parameter, \(\lambda\), differs from zero (the ‘null’ hypothesis). For such problems, we have argued that the marginalized likelihood function \(L(\lambda)\) is more informative than Evidence computed for specific, and often poorly motivated, choices of priors. If the likelihood function is exponentially suppressed at \(\lambda = 0\), then we can conclude that there is strong evidence against the null hypothesis for any reasonable choice of priors. Bayesian Evidence is of most use in comparing non-nested models. If we are in the happy situation of having high-quality data that rule out a cosmological constant, then Bayesian Evidence can be used to select between various dynamical models. But the Evidence will only be of interest if the models and priors are physically well motivated.

Finally, we re-iterate that the Evidence calculations presented in Table 1 show no significant evidence in favour of a cosmological constant compared to the dynamical models of dark energy tested here. This conclusion agrees with similar Evidence analyses of Serra et al. (2007) and Liddle et al. (2007), using different models and somewhat different data, but disagrees with the Evidence analysis of Bassett et al. (2004). Several recent papers have used the BIC to claim that a cosmological constant is favoured over dynamical dark energy (Bassett et al. 2004; Davis et al. 2007; Sahlin et al. 2007; Kurek & Szydlowski 2008). However, BIC unfairly penalizes models with many parameters if these parameters are poorly constrained by the data (Liddle 2004, 2007). If this paper is a ‘health warning’ concerning the use of Bayesian Evidence in cosmology, it should be
considered a ‘death certificate’ on the use of approximations such as BIC if the strict criteria for their applicability are not met (see Liddle 2007 for further details). As the models of Section 2 show, current data unfortunately provide relatively weak constraints on simple dynamical models of dark energy.

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