On the moving plane method for boundary blow-up solutions to semilinear elliptic equations

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Abstract: We consider weak solutions to $-\Delta u = f(u)$ on $\Omega_1 \setminus \Omega_0$, with $u = c \geq 0$ in $\partial \Omega_1$ and $u = +\infty$ on $\partial \Omega_0$, and we prove monotonicity properties of the solutions via the moving plane method. We also prove the radial symmetry of the solutions in the case of annular domains.

Keywords: Elliptic equations, boundary blow-up, symmetry of solutions, radial symmetry, moving plane method

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1 Introduction

Let $\Omega_0$ and $\Omega_1$ be two bounded smooth domains of $\mathbb{R}^N$, $N \geq 2$, such that $\Omega_0 \subset \Omega_1$. Moreover, let $f \in C^1([0, +\infty))$ and $c \geq 0$. We consider weak solutions to the problem

$$
\begin{align*}
-\Delta u &= f(u) \quad \text{in } \Omega_1 \setminus \Omega_0, \\
u > c & \quad \text{in } \Omega_1 \setminus \Omega_0, \\
u &= c \quad \text{on } \partial \Omega_1, \\
u &= +\infty \quad \text{on } \partial \Omega_0,
\end{align*}
$$

i.e., we consider $u \in C^1(\overline{\Omega_1 \setminus \Omega_0})$ such that

$$
\int_{\Omega_1 \setminus \Omega_0} \nabla u \nabla \varphi = \int_{\Omega_1 \setminus \Omega_0} f(u) \varphi \quad \text{for all } \varphi \in C_c^\infty(\Omega_1 \setminus \Omega_0)
$$

and

$$
\lim_{x \to x_0} u(x) = +\infty \quad \text{for all } x_0 \in \partial \Omega_0.
$$

When $c = 0$, actually we deal with the case of positive solutions. Necessary and sufficient conditions for the existence of solutions to (1.1) are provided by the classical results of Keller [18] and Osserman [19], under suitable assumptions on the nonlinearity. The literature regarding boundary blow-up solutions is really wide (see, for example, [1–7, 12–17, 20, 21]). Here we exploit an adaptation of the celebrated moving plane technique (see [8]) in order to obtain monotonicity properties of the solutions to (1.1). The domain that we consider is not convex and the solutions are not in $H^1_0(\Omega_1 \setminus \Omega_0)$ as in the classical case. This is the same difficulty...
that occurs when dealing with the study of the uniqueness, symmetry and monotonicity properties of solutions to singular semilinear elliptic equations, see [9–11]. These problems exhibit in fact some similarities with problem (1.1) although the proofs cannot be adapted to our case.

In the second part of the paper, we prove the radial symmetry of the solutions on annular domains, under suitable assumptions. In our setting, this cannot be done just using the moving plane method, since the domain is not convex, and we prove that the solution is radially symmetric showing directly that the angular derivative is zero. The technique is based on a refined maximum principle for the linearized equation.

Let us introduce some notations. Let \( \nu \) be a direction in \( \mathbb{R}^N \), with \( |\nu| = 1 \). Given a real number \( \lambda \), we set

\[
T_\lambda^\nu = \{ x \in \mathbb{R}^N : x \cdot \nu = \lambda \}
\]

and

\[
x_\lambda^\nu = R_\lambda^\nu(x) = x + 2(\lambda - x \cdot \nu)\nu,
\]

that is, the reflection of \( x \) trough the hyperplane \( T_\lambda^\nu \). We will make the following assumption throughout the paper:

(A) \( \Omega_0 \) and \( \Omega_1 \) are strictly convex with respect to the \( \nu \)-direction and symmetric with respect to \( T_0^\nu \).

Moreover, we set

\[
\Omega_\lambda^\nu = \{ x \in \Omega_1 : x \cdot \nu < \lambda \} \setminus R_\lambda^\nu(\Omega_0) \quad \text{and} \quad (\Omega_\lambda^\nu)' = R_\lambda^\nu(\Omega_\lambda^\nu).
\]

Also we set

\[
a(\nu) = \inf_{x \in \Omega_1} x \cdot \nu \quad \text{and} \quad b(\nu) = \inf_{x \in \Omega_0} x \cdot \nu.
\]

Observe that, by assumption (A), it follows that \( \Omega_\lambda^\nu \) is nonempty and \( (\Omega_\lambda^\nu)' \subset \Omega_1 \setminus \Omega_0 \) for any \( a(\nu) < \lambda \leq 0 \).

We define

\[
u_\lambda^\nu(x) = u(x^\nu_\lambda) \quad \text{for all} \ x \in \Omega_\lambda^\nu
\]

and for any \( a(\nu) < \lambda \leq 0 \).

We are now ready to state our main results.

**Theorem 1.1.** Let \( u \in C^1(\overline{\Omega_1} \setminus \overline{\Omega_0}) \) be a weak solution to (1.1). Then

\[
u_\lambda^\nu \quad \text{in} \ \Omega_\lambda^\nu \quad \text{for any} \ a(\nu) < \lambda < b(\nu).
\]

Consequently, it follows that \( u \) is strictly increasing with respect to the \( \nu \)-direction in the set

\[
\{ x \in \Omega_1 : a(\nu) < x \cdot \nu < b(\nu) \}, \quad \text{with} \ \frac{\partial u}{\partial \nu} > 0.
\]

In order to get symmetry results for the solution to (1.1), we restrict our attention to annular domains. We denote by \( B_R \) the open ball of center \( 0 \) and radius \( R > 0 \) in \( \mathbb{R}^N \). By Theorem 1.1, we immediately deduce the following.

**Corollary 1.2.** Let \( 0 < R_0 < R_1 \) and \( u \in C^1(\overline{B_{R_1}} \setminus \overline{B_{R_0}}) \) be a weak solution to (1.1) in \( B_{R_1} \setminus B_{R_0} \). Then \( \frac{\partial u}{\partial r} < 0 \) in \( B_{R_1} \setminus B_{R_0} \).

In the following we state sufficient conditions in order to deduce the radial symmetry of the solution, once we prove the monotonicity. We set

\[
\nu = -x \cdot \nabla u,
\]

and we denote by \( u_\theta \) the angular derivative of \( u \).

**Theorem 1.3.** Let \( 0 < R_0 < R_1 \) and let \( u \in C^1(\overline{B_{R_1}} \setminus \overline{B_{R_0}}) \) be a weak solution to (1.1) in \( B_{R_1} \setminus B_{R_0} \) that satisfies

\[
u_\lambda = a(\nu) \quad \text{as} \ |x| \to R_0.
\]

If \( f \leq 0 \), then \( u \) is radially symmetric and radially decreasing in \( B_{R_1} \setminus B_{R_0} \).

For the reader’s convenience, let us point out that the condition in (1.3) is inspired by the results in [20, 21].

In Section 2 we give the proof of Theorem 1.1. We prove Theorem 1.3 in Section 3.
2 Proof of Theorem 1.1

Let \( a(v) < \lambda < b(v) \). We define
\[
w_\lambda^v = u - u_\lambda^v.
\]
We need to prove that \( w_\lambda^v < 0 \) in \( \Omega_\lambda^v \). We have
\[
\frac{\int \nabla (\nabla u - \nabla u_\lambda^v) \cdot \nabla \varphi}{\Omega_\lambda^v} = \int_{\Omega_\lambda^v} \frac{f(u) - f(u_\lambda^v)}{u - u_\lambda^v} (u - u_\lambda^v) \varphi \quad \text{for all } \varphi \in C_0^\infty(\Omega_\lambda^v),
\]
so that \( w_\lambda^v \) weakly satisfies
\[
-\Delta w_\lambda^v = c_\lambda(x) w_\lambda^v,
\]
where
\[
c_\lambda(x) = \frac{f(u) - f(u_\lambda^v)}{u - u_\lambda^v} \quad \text{if } u \neq u_\lambda^v \quad \text{and} \quad c_\lambda(x) = 0 \quad \text{if } u = u_\lambda^v.
\]
Observe that \( c_\lambda(x) \in L^\infty(\Omega_\lambda^v) \) and \( c_\lambda(x) \in L^\infty(\Omega_\lambda^v) \) for \( \lambda - a(v) \) small. Moreover, since \( w_\lambda^v = 0 \) on \( \partial \Omega_\lambda^v \cap T_\lambda^v \), it follows that \( w_\lambda^v \) weakly satisfies
\[
-\lambda w_\lambda^v = c - u_\lambda^v < 0 \quad \text{on } \partial \Omega_\lambda^v \cap \partial \Omega_1 \quad \text{and} \quad u < u_\lambda^v \quad \text{on a neighborhood of } \partial \Omega_\lambda^v \cap \partial \Omega_1(\Omega_0),
\]
thanks to the fact that
\[
\lim_{x \to x_0} u_\lambda^v(x) = +\infty \quad \text{for all } x_0 \in \partial \Omega_\lambda^v \cap \partial \Omega_1(\Omega_0).
\]
Then
\[
(w_\lambda^v)^+ \in H^1_0(\Omega_\lambda^v).
\]
Let \( \lambda - a(v) \) be sufficiently small so that the weak maximum principle in small domains works for the operator \( \Delta + c_\lambda(x) \) in \( \Omega_\lambda^v \), recalling that \( c_\lambda(x) \in L^\infty(\Omega_\lambda^v) \) for \( \lambda - a(v) \) small. We get \( w_\lambda^v \leq 0 \) in \( \Omega_\lambda^v \) and, by the strong maximum principle, we obtain
\[
w_\lambda^v < 0 \quad \text{in } \Omega_\lambda^v.
\]
Set now
\[
\mu = \sup \{ \lambda > a(v) : w_\lambda^v < 0 \text{ in } \Omega_\lambda^v \} \quad \text{for } a(v) < t \leq \lambda.
\]

We need to prove that \( \mu \geq b(v) \). On the contrary, suppose \( \mu < b(v) \). By continuity, it follows that \( w_\mu^v \leq 0 \) in \( \Omega_\mu^v \). Moreover, it is possible to apply the same argument of the first part of the proof to obtain, by the strong maximum principle, that
\[
w_\mu^v < 0 \quad \text{in } \Omega_\mu^v.
\]
Take now \( \varepsilon > 0 \) sufficiently small such that \( \mu + \varepsilon < b(v) \). Given \( \delta > 0 \), we fix a compact set \( K \subset \Omega^v_\mu \) so that \( \mathcal{L}(\Omega^v_\mu \setminus K) < \frac{\delta}{4} \). Since \( w_\mu^v < 0 \) in \( K \), by the continuity of \( u \), it follows that there exists \( M > 0 \) such that \( w_\mu^v \leq M < 0 \) in \( K \). Therefore, we can find \( \varepsilon_0 > 0 \) such that
\[
w_{\mu + \varepsilon}^v \leq \frac{M}{2} < 0 \quad \text{in } K,
\]
whenever \( 0 < \varepsilon < \varepsilon_0 \). Moreover, choosing \( \varepsilon > 0 \) sufficiently small such that \( \mathcal{L}(\Omega^v_{\mu + \varepsilon} \setminus K) < \delta \) and using \( (w_{\mu + \varepsilon}^v)^+ \) as test function in the weak formulation of (1.1), since \( f \) is locally Lipschitz continuous, an application of the Poincaré inequality gives
\[
\int_{\Omega_{\mu + \varepsilon}^v \setminus K} |\nabla (w_{\mu + \varepsilon}^v)^+|^2 \leq \int_{\Omega_{\mu + \varepsilon}^v \setminus K} \frac{f(u) - f(u_{\mu + \varepsilon}^v)}{u - u_{\mu + \varepsilon}^v} (u - u_{\mu + \varepsilon}^v)^2 \leq C^2_p (\Omega_{\mu + \varepsilon}^v \setminus K) \int_{\Omega_{\mu + \varepsilon}^v \setminus K} |\nabla (w_{\mu + \varepsilon}^v)^+|^2,
\]
where
\[
C = \sup_{s, t \in [0, 1]} \frac{f(t) - f(s)}{t - s}.
\]
Choosing $\delta$ sufficiently small, we have $C^2_\mu(\Omega^\nu_{\rho+\varepsilon}\setminus\mathcal{K})C < 1$ so that $(w^\mu_{\rho+\varepsilon})^\nu = 0$ in $\Omega^\nu_{\rho+\varepsilon}$. Hence, $w^\nu_{\rho+\varepsilon} \leq 0$ in $\Omega^\nu_{\rho+\varepsilon}$. By the strong maximum principle, we get
\[ w^\nu_{\rho+\varepsilon} < 0 \quad \text{in } \Omega^\nu_{\rho+\varepsilon}. \]
This gives a contradiction with the definition of $\mu$ and shows that $\mu \geq b(\nu)$, namely, (1.2) is proved.

Now, let $x$ and $x'$ such that $a(\nu) < x \cdot v < x' \cdot v < b(\nu)$. Setting $\lambda = (x \cdot v + x' \cdot v)/2$, we get $x^\nu_\lambda = x'$ and
\[ u(x) < u^\nu_\lambda(x) = u(x'). \]
Therefore, $u$ is strictly increasing with respect to the $v$-direction. To conclude the proof, fix $x \in \Omega^\nu_{\rho+\varepsilon}$ and let $a(\nu) < \lambda < b(\nu)$ be such that $x \in T^\nu_\lambda \cap \Omega_1$. We have that $-\Delta w^\nu_\lambda = c(x)(x)w^\nu_\lambda$, $w^\nu_\lambda > 0$ in $\Omega^\nu_\lambda$ and $w^\nu_\lambda = 0$ in $T^\nu_\lambda \cap \Omega_1$.

By the Hopf lemma, we obtain
\[ 0 < \frac{\partial w^\nu_\lambda}{\partial v} = 2 \frac{\partial u}{\partial v}(x). \]

### 3 Proof of Theorem 1.3

We start by proving the following proposition.

**Proposition 3.1.** Let $R_0 < R_1$ and $u \in C^1(\overline{B_{R_1}} \setminus \overline{B_{R_0}})$ be a weak solution to (1.1) in $B_{R_1} \setminus B_{R_0}$. If $f \leq 0$, then
\[ -\Delta v \geq f'(u)v \quad \text{in } B_{R_1} \setminus B_{R_0}. \]

**Proof.** Let $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$. Set $u_i = \frac{\partial u}{\partial x_i}$ for each $i = 1, \ldots, N$. We have
\[ v = -\nabla \left( \sum_{i=1}^N x_i u_i \right) = -\nabla u - \sum_{i=1}^N x_i \nabla u_i. \]

Since $-\Delta u = f(u)$ in $B_{R_1} \setminus B_{R_0}$ and $f \leq 0$, we obtain
\[ -\Delta v = \text{div}(\nabla u + \sum_{i=1}^N x_i \nabla u_i) = 2\Delta u + \sum_{i=1}^N x_i \Delta u_i = -2f(u) - \sum_{i=1}^N x_i f'(u)u_i = -2f(u) + f'(u)v \geq f'(u)v, \]
so that $-\Delta v \geq f'(u)v$. \hfill \square

In the following we will also exploit the fact that
\[ -\Delta u_\theta = f'(u)u_\theta \quad \text{in } B_{R_1} \setminus B_{R_0}, \]
as it follows by direct computation.

**Proof of Theorem 1.3.** We shall actually show that $u_\theta \geq 0$ in $B_{R_1} \setminus B_{R_0}$. By assumption (1.3), choosing $\sigma > 0$ sufficiently small, we have $u_\theta + tv \geq 0$ in $B_{R_1+\sigma} \setminus B_{R_0}$.

Furthermore, by Corollary 1.2, since $v$ is continuous, we have
\[ v = -\frac{x}{|x|} \cdot |x| \nabla u = |x| \frac{\partial u}{\partial v} > 0, \]
where $v = -x/|x|$ with $x \in B_{R_1} \setminus B_{R_0+\sigma}$. Moreover, $v > 0$ on $\partial B_{R_0}$, by the Hopf lemma. Then $v > \theta(\sigma) > 0$ in $B_{R_1} \setminus B_{R_0+\sigma}$. Since $u_\theta$ is bounded in $B_{R_1} \setminus B_{R_0+\sigma}$, for $t$ sufficiently large, we have $u_\theta + tv > 0$ in $B_{R_1} \setminus B_{R_0+\sigma}$.

Hence,
\[ u_\theta + tv \geq 0 \quad \text{in } B_{R_1} \setminus B_{R_0}. \]

Set now
\[ t_0 = \inf\{t > 0 : u_\theta + tv \geq 0 \text{ in } B_{R_1} \setminus B_{R_0}\}. \]
We need to prove that $t_0 = 0$. Conversely, suppose $t_0 > 0$. By the definition of $t_0$ and Proposition 3.1, we obtain

$$\begin{align*}
-\Delta(u_\delta + t_0 \nu) &\geq f'(u)(u_\delta + t_0 \nu) \quad \text{in } B_{R_1} \setminus B_{R_0}, \\
u &\geq 0 \quad \text{in } B_{R_1} \setminus B_{R_0}.
\end{align*}$$

By the strong maximum principle, since $u_\delta = 0$ and $\nu > 0$ on $\partial B_{R_1}$, we get

$$u_\delta + t_0 \nu > 0 \quad \text{in } B_{R_1} \setminus B_{R_0}.$$ 

Since (1.3) is in force, there exists $\delta_0 > 0$ such that

$$u_\delta + (t_0 - \varepsilon) \nu > 0 \quad \text{in } B_{R_0 + \delta_0} \setminus B_{R_0}.$$ 

Moreover, we have

$$u_\delta + t_0 \nu \geq m > 0 \quad \text{in } \overline{B_{R_1} \setminus B_{R_0 + \delta_0}}.$$ 

By continuity, for $\varepsilon > 0$ small, we have that

$$u_\delta + (t_0 - \varepsilon) \nu \geq \frac{m}{2} > 0 \quad \text{in } \overline{B_{R_1} \setminus B_{R_0 + \delta_0}}.$$ 

Resuming, we have that

$$u_\delta + (t_0 - \varepsilon) \nu > 0 \quad \text{in } B_{R_1} \setminus B_{R_0}.$$ 

This contradicts the definition of $t_0$, showing that actually $t_0 = 0$ and $u_\delta \geq 0$ in $B_{R_1} \setminus B_{R_0}$. This is possible only if $u_0 = 0$ in $B_{R_1} \setminus B_{R_0}$, namely, if the solution is radial. We conclude that the solution is also radially decreasing by Theorem 1.1. 

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