We develop two structure theorems for vector valued Siegel modular forms for Igusa’s subgroup $\Gamma_2[2,4]$, the multiplier system induced by the theta constants and the representation $\text{Sym}^2$. In the proof, we identify some of these modular forms with rational tensors with easily handleable poles on $\mathbb{P}^3\mathbb{C}$. It follows that the observed modules of modular forms are generated by the Rankin-Cohen brackets of the four theta series of the second kind.

1 Introduction

A vector valued modular form with respect to a subgroup $\Gamma \subset \text{Sp}(n, \mathbb{Z})$ of finite index is a holomorphic function $f : \mathbb{H}_n \to V$ that transforms under $\Gamma$ as follows:

$$f(M\langle Z \rangle) = j(M, Z) f(Z).$$

Here $j$ denotes a factor of automorphy on a finite dimensional vector space $V$, i.e. a map $j : \Gamma \times \mathbb{H}_n \to \text{GL}(V)$ that is holomorphic in the second variable and satisfies the cocycle relation $j(MN, Z) = j(M, N \langle Z \rangle)j(N, Z)$.

The characters of $\Gamma$ are easy examples of factors of automorphy. Furthermore, a given polynomial representation $\rho : \text{GL}(n, \mathbb{C}) \to \text{GL}(V)$ induces the factor of automorphy $\rho((CZ + D))$. For integral $r$, the functions $\sqrt{\det(CZ + D)}^r$ satisfy the cocycle relation up to $\pm 1$. In order to compensate this error, they are multiplied with multiplier systems $v(M)$ of weight $r/2$.

In [Tsu83], Tsushima calculated the dimension of vector spaces of vector valued cusp forms by means of the Riemann-Roch-Hirzebruch-Theorem. In [Sat86], Satoh combined this result with the decomposition of the vector space of vector valued modular forms into the subspace of cusp forms and the subspace of Eisenstein series, cf. [Ara83]. He obtained
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a structure theorem for certain vector spaces of vector valued modular forms with respect to the full modular group $\text{Sp}(2,\mathbb{Z})$. We should also mention [Ibu12] and [vD13] which used the same strategy. Similarly, [Aok12] and [CvdGG13] estimated or calculated the Hilbert series and found matching generators.

In this article, we use a geometric method to get similar results for Igusa’s group

$$\Gamma_2[2,4] := \{ M \in \Gamma_2[2] : (AB^t)_0 \equiv (CD^t)_0 \equiv 0 \pmod{4} \}$$

instead of the full modular group. We benefit from the fact that the Satake compactification of $\Gamma_2[2,4] \backslash \mathbb{H}_2$ is simply the 3 dimensional projective space $\mathbb{P}^3\mathbb{C}$. This is a consequence of a couple of basic results by Igusa [Igu64a, Going-down process, p.397] that was proven by Runge [Run93].

We shall investigate modular forms being linked to the symmetric square of the standard representation $\rho_e$, i.e.

$$\text{Sym}^2 = \text{Sym}^2(\rho_e) : \text{GL}(2,\mathbb{C}) \longrightarrow \text{Aut}(\text{Sym}^2(\mathbb{C}^2)) .$$

Here $\text{Sym}^2(\mathbb{C}^2)$ denotes the symmetric square of $\mathbb{C}^2$, which can be identified with the space of symmetric $2 \times 2$ matrices. Then, the representation is given by

$$A \mapsto \{ X \mapsto AXA^t \} .$$

In particular, we shall study the spaces of modular forms with respect to representations of the type

$$\text{det}^k \otimes (\text{Sym}^2) , \quad k \in \mathbb{Z}.$$ 

For these vector spaces we shall give generators and compute the dimensions. This can be found either on the following pages or in theorem [10.1 on page 21].

Our approach relies on the fact that in the cases where $k = 3r$ and $k = 3r+1$, these forms can be identified with $\Gamma_2[2,4]$- invariant holomorphic tensors of the form

$$(1) \quad (f_0(Z) \cdot dZ^0 + f_1(Z) \cdot dZ^1 + f_2(Z) \cdot dZ^2) \otimes (dZ^0 \wedge dZ^1 \wedge dZ^2)^{\otimes r}$$

and

$$(2) \quad (g_0(Z) \cdot dZ^1 \wedge dZ^2 + g_1(Z) \cdot dZ^0 \wedge dZ^2 + g_2(Z) \cdot dZ^0 \wedge dZ^1) \otimes (dZ^0 \wedge dZ^1 \wedge dZ^2)^{\otimes r} ,$$
respectively. Here the points in \( \mathbb{H}_2 \) are of the form 
\[
Z = \begin{pmatrix} Z_0 & Z_1 \\ Z_1 & Z_2 \end{pmatrix},
\]

The crucial fact is that the map 
\[ \mathbb{H}_2 \longrightarrow \Gamma_2[2,4]\mathbb{H}_2 \longrightarrow \mathbb{P}^3 \mathbb{C} \]
branches over 10 explicitly given quadrics in \( \mathbb{P}^3 \mathbb{C} \). This implies that \( \Gamma_2[2,4] \)-invariant tensors on \( \mathbb{H}_2 \) correspond to rational tensors on \( \mathbb{P}^3 \mathbb{C} \) which may have poles of certain types along these 10 quadrics.

We shall elaborate on this result in the subsequent lines. The map \( \mathbb{H}_2 \longrightarrow \mathbb{P}^3 \mathbb{C} \) is given by the four theta constants of the second kind \( f_0, \ldots, f_3 \). These are Siegel modular forms of weight 1/2 with respect to a common multiplier system \( v_f \), cf. theorem 4.4 on page 9.

It follows from Runge’s results that the ring of all modular forms of transformation type 
\[ f(M\langle Z \rangle) = v_f^r(M) \cdot \sqrt{\det(CZ + D)}^r \cdot f(Z), \]
which is usually denoted by 
\[ \mathcal{A}(\Gamma_2[2,4]) = \bigoplus_{r \in \mathbb{N}} \left[ \Gamma_2[2,4], r/2, v_f^r \right], \]
is \( \mathbb{C}[f_0, \ldots, f_3] \). This theorem can be found on page 9.

The simplest case are tensors of the form 
\[ f(Z) \cdot (dZ^0 \wedge dZ^1 \wedge dZ^2)^{\otimes r}. \]
They belong to complex valued modular forms transforming as follows 
\[ f(M\langle Z \rangle) = \det(CZ + D)^{3r} \cdot f(Z), \quad M \in \Gamma_2[2,4]. \]

Returning to the vector valued case, we start with fixing some notation. Here and subsequently, \( \mathcal{M}^+ \) stands for the vector space of modular forms of transformation type 
\[ f(M\langle Z \rangle) = v_f^r(M) \cdot \sqrt{\det(CZ + D)}^r \cdot (CZ + D)f(Z)(CZ + D)^t, \quad M \in \Gamma_2[2,4]. \]

It is also possible to twist this vector space by the character \( v_f^2 \). The space \( \mathcal{M}^- \) consists of the modular forms satisfying 
\[ f(M\langle Z \rangle) = v_f^2(M) \cdot v_f^r(M) \cdot \sqrt{\det(CZ + D)}^r \cdot (CZ + D)f(Z)(CZ + D)^t \]
for all \( M \in \Gamma_2 [2, 4] \).

We shall study the graded \( A(\Gamma_2 [2, 4], v_f) \)-modules

\[
M^+ (\Gamma_2 [2, 4]) := \bigoplus_{r \in \mathbb{Z}} M^+_r \quad \text{and} \quad M^- (\Gamma_2 [2, 4]) := \bigoplus_{r \in \mathbb{Z}} M^-_r .
\]

The module \( M^+ \) contains the so called Rankin-Cohen brackets. These are constructed by means of scalar valued modular forms \( f, g \) and derivatives, i.e.

\[
\{ f, g \} = f \cdot Dg - g \cdot Df.
\]

There is a similar construction \( \{ f, g, h \} \) which defines an element in \( M^- \), cf. definition 9.1 on page 19.

The main results of this thesis (theorems 8.7 and 9.3 on pages 18 and 20, respectively) are

\[
M^+ (\Gamma_2 [2, 4]) = \sum_{0 \leq i < j \leq 3} (\mathbb{C}[f_0, \ldots, f_3]) \{ f_i, f_j \}
\]

and

\[
M^- (\Gamma_2 [2, 4]) = \sum_{0 \leq i_1 < i_2 < i_3 \leq 3} (\mathbb{C}[f_0, \ldots, f_3]) \{ f_{i_1}, f_{i_2}, f_{i_3} \} .
\]

For any given degree \( r \), we shall exhibit explicit bases of \( M^+_r \) and \( M^-_r \). Consequently, we obtain the Hilbert functions

\[
\dim M^+_r = 3 \cdot \binom{r + 1}{3} + 2 \cdot \binom{r}{2} + \binom{r - 1}{1}
\]

and

\[
\dim M^-_r = 3 \cdot \binom{r - 2}{3} + \binom{r - 3}{2}.
\]

The case of modular forms with trivial multiplier system follows easily from the above lines as shown in theorem 4.4.
Note that the modules $\mathcal{M}^+(\Gamma_2[2,4])$ and $\mathcal{M}^-(\Gamma_2[2,4])$ contain the $\Gamma_2[2,4]$-invariant holomorphic tensors shown in eqs. (1) and (2). Using them, we shall give a brief overview of the structure theorems’ proofs. For the sake of simplicity, we restrict ourselves to presenting the case $\mathcal{M}^+(\Gamma_2[2,4])$. As shown in theorem 6.4, these tensors correspond to rational tensors on $\mathbb{P}^n\mathbb{C}$ having poles of certain types along 10 quadrics, that are given in lemma 6.3 on page 11. These tensors become holomorphic after pulling them back along 2-coverings that are ramified over the quadrics. We refer to definition 5.1 on page 10 for the exact wording. If the parameter $r \in \mathbb{N}$ of eq. (1) is even, then we can work out this condition explicitly, cf. corollary 7.3 on page 13. This shows the equalities $M^+_6 = M^+_6$ for even $r$, cf. lemma 8.6 on page 18. The case of arbitrary $\mathcal{M}_s^+$ can be reduced to the above ones by multiplying with monomials in the $f_a$, cf. theorem 8.7. This reduction uses the very simple structure of the ring of modular forms, i.e. $A(\Gamma_2[2,4],v_f) = \mathbb{C}[f_0,dots,f_3]$.

This article is based on the author’s PhD thesis [Wie13] under the supervision of E. Freitag who should be thanked for many fruitful discussions. In the thesis the proofs are presented in more detail.

2 Symplectic groups

The symplectic group is the subgroup

$$\text{Sp}(n, R) := \{ M \in \text{SL}(2n, R) : J_n [M] = J_n \}$$

of the linear group of a commutative unital ring $\text{GL}(2n, R)$ where $J_n$ denotes the involution. The group $\text{Sp}(n, \mathbb{Z})$ or $\Gamma_n$ for short acts on the Siegel upper half-space, i.e. the set $\mathbb{H}_n := \{ Z \in \text{Sym}^2(C^n) : \text{Im } Z \text{ pos. def.} \}$, by $(M, Z) \mapsto M(Z) = (AZ + B)(CZ + D)^{-1}$. The kernel of the natural group homomorphism $\text{Sp}(n, \mathbb{Z}) \to \text{Sp}(n, \mathbb{Z}/q\mathbb{Z})$ is the principal congruence subgroup $\Gamma_n[q]$. Let $A$ be a square matrix then $(A)_0$ denotes the diagonal vector $(a_{11},\ldots,a_{nn})$. Then Igusa’s group of level $q$ is defined to be

$$\Gamma_n[q,2q] := \{ M \in \Gamma_n[q] : (AB^t)_0 \equiv (CD^t)_0 \equiv 0 \mod 2q \}.$$

Suppose that $q$ is even then $\Gamma_n[q,2q]$ is normal in the full modular group.

Lemma 2.1

The non-trivial subgroups of $(\Gamma_2[2,4]/\pm I_4)$ of finite order are all of order 2. Their generators
are conjugated in \( \Gamma_2/\pm I_4 \) to the image of the matrix
\[
\begin{pmatrix}
1 & -1 & 0 \\
-1 & 1 & -1 \\
0 & 1 & 0
\end{pmatrix}.
\]

**Proof**

It follows from rather basic algebraic facts that the group \( \Gamma_2[4] \) acts without fixed points. Therefore, [Run93]’s Lemma 5.3 on page 23 completes the proof.

### 3 Modular forms of half integral weight

Let \( \Gamma \) be a congruence subgroup in \( \text{Sp}(n, \mathbb{R}) \), \( v \) a multiplier system of weight \( r/2 \) and \( \rho \) be a rational representation of \( \text{GL}(n, \mathbb{C}) \) on a vector space \( V \). Then,
\[
v(M)\sqrt{\det CZ + D^r} \rho(CZ + D)
\]
is a factor of automorphy.

By a vector valued modular form with respect to this factor of automorphy we mean a holomorphic function \( f : \mathbb{H}_n \to V \) which transforms as
\[
f(M(Z)) = v(M)\sqrt{\det CZ + D^r} \rho(CZ + D)f(Z)
\]
under \( \Gamma \).

In the case where \( n = 1 \), the usual condition at the cusps has to be added. For \( n \geq 2 \), the Koecher principle ensures this, cf. [Fre83, Hilfssatz 4.11, p.175].

The vector space of modular forms is denoted by \( [\Gamma, (\frac{r}{2}, \rho), v] \). If \( v \) is trivial and \( r = 2k \) is even, then we shorten this to \( [\Gamma, (k, \rho)] \).

The pairs \( (\frac{r}{2}, \rho) \) and \( (\frac{r-2k}{2}, \det k \cdot \rho) \) define the same factor of automorphy. Hence, for an irreducible representation \( \rho \) we always may assume that \( \rho \) is reduced, i.e. it is polynomial and does not vanish on \( \{\det(A) = 0\} \). After this normalization we call \( r/2 \) the weight of a vector valued modular form.

We recall that vector valued modular forms of negative weight are identically zero.

We can construct from a multiplier system of weight \( 1/2 \) its graded algebra of modular forms of half integral weight \( A(\Gamma, v) := \bigoplus_{r \in \mathbb{Z}} [\Gamma, (\frac{r}{2}, v^r)] \).
We want to identify vector valued modular forms with tensors on $\mathbb{H}_n$; therefore, we fix some notation. We consider the wedge product of all 1-forms $dZ^{ij}$ in lexicographic order by
\[
\bigwedge dZ^{ij} = \bigwedge_{1 \leq i < j \leq n} dZ^{ij}.
\]

**Lemma 3.1**
There is an isomorphism between the vector space $[\Gamma, ((n + 1)k, \text{Sym}^2)]$ of vector valued modular forms transforming as
\[
f(M(Z)) = \det(CZ + D)^{(n+1)k} \cdot (CZ + D)f(Z)(CZ + D)^t
\]
and the subspace of $\Gamma$-invariant tensors in $\left(\bigwedge^k \Lambda^{\frac{n(n+1)}{2}} \Omega\right)(\mathbb{H}_n)$, i.e.
\[
\Phi : \left(\bigwedge^k \Lambda^{\frac{n(n+1)}{2}} \Omega\right)(\mathbb{H}_n) \Gamma \rightarrow [\Gamma, ((n + 1)k, \text{Sym}^2)],
\]
\[
\omega = \sum_{i<j} f_{ij} dZ^{ij} \otimes \left(\bigwedge \omega^{ij}\right)^{\otimes k} \rightarrow (f_{ij})_{1 \leq i, j \leq n}.
\]

**Proof**
We just have to observe the derivative of $Z \mapsto M(Z)$, cf. [Fre83, 1.6 Hilfssatz, p.27].

We shall also consider $\Gamma$-invariant tensors in $\left(\bigwedge^k \Lambda^{\frac{n(n+1)}{2}} \Omega\right)(\mathbb{H}_n)$. They can be identified with vector valued modular forms transforming as
\[
f(M(Z)) = \det(CZ + D)^{(n+1)(k+1)}(CZ + D)^{-t}f(Z)(CZ + D)^{-1}
\]
under $\Gamma$, cf. [Fre83, 4.61 Folgerung, p.172].

In the case where $n = 2$, the reduced representation $\det^2(A)A^{-t}XA^{-1}$ is isomorphic to $\text{Sym}^2$. Therefore, we obtain the subsequent result.

**Lemma 3.2**
The vector space $[\Gamma, (3k + 1, \text{Sym}^2)]$ is isomorphic to the subspace of $\Gamma$-invariant tensors in $\left(\bigwedge^k \Lambda^2 \Omega \otimes \Lambda^3 \Omega\right)(\mathbb{H}_2)$. The tensor
\[
(f_0 \ dZ^1 \wedge dZ^2 + f_1 \ dZ^0 \wedge dZ^2 + f_2 \ dZ^0 \wedge dZ^1) \otimes \left(\bigwedge \omega^{ij}\right)^{\otimes k}
\]
is mapped to the vector valued modular form
\[
\begin{pmatrix}
  f_2 & -f_1 \\
  -f_1 & f_0
\end{pmatrix}.
\]
4 Theta series

We call a vector \( m = (m^1, m^2) \in \mathbb{Z}^{2n} \) even if the Euclidean scalar product of \( m^1 \) and \( m^2 \) 
\( \langle m^1, m^2 \rangle \equiv 0 \mod 2. \)

**Definition 4.1 (Theta series)**

On \( \text{Sym}^2(\mathbb{C}^n) \) we define different kinds of theta series:

- **of the first kind**
  
  \[
  \vartheta \bigg[ \begin{bmatrix} m^1 \\ m^2 \end{bmatrix} \bigg] (Z) := \sum_{g \in \mathbb{Z}^n} \exp \left( \pi i \left( Z \left( g + \frac{m^1}{2} \right) + \left( g + \frac{m^1}{2} \right)^t m^2 \right) \right)
  \]
  
  for the characteristic \( m = (m^1, m^2) \in \{0, 1\}^{2n} \subset \mathbb{Z}^{2n}; \)

- **of the second kind**
  
  \[
  f_a(Z) := \vartheta \bigg[ \begin{bmatrix} a \\ 0 \end{bmatrix} \bigg] (2Z) = \sum_{g \in \mathbb{Z}^n} \exp \left( 2\pi i Z \left( g + \frac{a}{2} \right) \right), \text{ for } a \in \{0, 1\}^n.
  \]

If desired we may observe the above parameter \( a \) as an element of (\( \mathbb{F}_2 \))^n.

We shall multiply the different theta series of the first kind with even characteristic

\[
\theta(Z) := \prod_{m \in \{0, 1\}^{2n}_{\text{even}}} \vartheta \bigg[ \begin{bmatrix} m^1 \\ m^2 \end{bmatrix} \bigg] (Z).
\]

The theta series of the first and second kind are holomorphic functions on \( \mathbb{H}_n \). We state
another of their properties that can be found on page 233 of [Igu64b].

**Lemma 4.2**

The theta series of the first and second kind are related in the following manner

\[
\vartheta^2 \bigg[ \begin{bmatrix} m^1 \\ m^2 \end{bmatrix} \bigg] (Z) = \sum_{a \in (\mathbb{F}_2)^n} (-1)^{(a,m^2)} f_{a+m^1}(Z) \cdot f_a(Z).
\]

Runge deduces on page 59 of [Run93] that there is not a single \( Z \) in \( \mathbb{H}_2 \) which is a
common root of all four thetas, i.e. \( f_a(Z) = 0 \).

A fundamental result on the zero locus of Igusa’s cusp form of weight 5 \( \chi_5 \) is the subsequent one, cf. Satz 2 in [Fre65].
Lemma 4.3
The zero locus of the above $\chi_5$ on $H_2$ is $\bigcup_{M \in \Gamma_2} M (\Delta_2)$, where $\Delta_2$ denotes the set of diagonal matrices.

Later, we shall use the subsequent two theorems.

**Theorem 4.4 (Structure theorem for $A(\Gamma_2 [2, 4], v_f)$)**
The functions $f_0, \ldots, f_3$ are modular forms of weight $1/2$ with respect to Igusa’s group $\Gamma_2 [2, 4]$ and a common multiplier system $v_f$. Nota bene $v_f^4 = 1$.
In particular, the whole ring of modular forms with respect to $\Gamma_2 [2, 4]$ is generated by the theta constants, i.e.

$$A(\Gamma_2 [2, 4], v_f) = \bigoplus_{r \in \mathbb{N}} \left[ \Gamma_2 [2, 4], \frac{r}{2}, v_f^r \right] = \mathbb{C} [f_0, f_1, f_2, f_3].$$

The modular form $\chi_5$ has a trivial multiplier system on $\Gamma_2 [2, 4]$ and hence is not contained in $A(\Gamma_2 [2, 4], v_f)$. The following variant of the previous theorem is also true.

**Theorem 4.5 (Structure theorem for $A^{int}(\Gamma_2 [2, 4])$)**
The graded algebra of scalar modular forms with respect to $\Gamma_2 [2, 4]$ and the trivial multiplier system can be decomposed

$$A^{int}(\Gamma_2 [2, 4]) = \bigoplus_{r \in \mathbb{N}} \left[ \Gamma_2 [2, 4], r \right]$$

$$= \left( \bigoplus_{d \geq 0} \mathbb{C}_{4d} [f_0, f_1, f_2, f_3] \right) \bigoplus \left( \bigoplus_{d \geq 0} \mathbb{C}_{4d} [f_0, f_1, f_2, f_3] \right) \cdot \chi_5$$

This version can be found in [Run93, Remark 3.15, p.74]. And theorem 4.4 is a rather simple consequence using the methods of sections 8 to 11, cf. theorem 11.1 on page 21.

5 Covering holomorphic tensors

We call a holomorphic, surjective and finite covering map $f : M \to N$ between two $n$ dimensional complex manifolds simple covering map if $f (\text{Ram} (f))$ is a smooth hypersurface. Prototypes are the $k$-th $n$-dimensional standard elements

$$p_n^k : \mathbb{E}^n \to \mathbb{E}^n, \quad (z^1, \ldots, z^n) \mapsto (z^1, \ldots, z^{n-1}, (z^n)^k).$$
where \( n, k > 0 \). Indeed, any simple covering is locally isomorphic to a standard element. This follows from the topological classification of unramified covering maps of the space \( \mathbb{E}^{n-1} \times \mathbb{E}^* \).

Let \( R \) be a closed submanifold of codimension 1 of \( N \). Then, for each point \( p \in R \) there is a chart \( V \to \mathbb{E}^n \) that sends \( R \) to \( z^n = 0 \). Hence, there is a simple covering \( U \to V \) that ramifies over \( R \cap V \).

**Definition 5.1 (Covering holomorphic tensors)**

Let \( D \) be an effective divisor on a \( n \)-dimensional complex manifold \( M \), we define \( \Omega^\otimes k(M, D) \) as the space of tensors \( \omega \in \Omega^\otimes k(M \setminus \text{supp} D) \) with the supplementary property:

Let \( Y \) be an irreducible component of \( D \) and \( q \) a point in \( Y \) that is a regular point in \( \text{supp} D \). Then, there exist an open neighbourhood of \( q \in V \) and a simple covering \( p : U \to V \) with the subsequent properties:

1. \( p \) is ramified over \( Y \cap V \);
2. \( p \) is isomorphic to the standard element \( p^k_n \), where \( k = D(Y) + 1 \);
3. \( \omega \)'s pullback \( p^* \omega \) is holomorphically extendable to the whole of \( U \).

Of course, the third condition in the previous definition is independent of the chosen \( p \). It is not hard to show that these tensors are meromorphic on \( M \). Later, we shall use tensors of the following type (the notation should be self explanatory) \( \left( \Lambda^n \Omega \otimes (\Lambda^n \Omega) \otimes^k \right) (M, D) \), where \( n = \text{dim} M \).

### 6 The quotient space \( \Gamma_2[2, 4]\backslash \mathbb{H}_2 \)

The considerations in section 5 were aimed at the group \( \Gamma_2[2, 4] \) acting on \( \mathbb{H}_2 \) and the induced quotient manifold \( \Gamma_2[2, 4]\backslash \mathbb{H}_2 \). Lemma 2.1 on page 5 implies that all elements of finite order in \( \Gamma_2[2, 4] \) are conjugated to the matrix

\[
\begin{pmatrix}
1 & 0 \\
-1 & 0 \\
0 & 1 \\
0 & -1 \\
\end{pmatrix}
\]

The fixed point set of \( \Gamma_2[2, 4] \) is the disjoint union of the \( \Gamma_2 \) conjugates of the diagonal \( \Delta_2 \). Its image \( \mathcal{R} \) in \( \Gamma_2[2, 4]\backslash \mathbb{H}_2 \) is the ramification divisor (its multiplicities are just 1). We deduce from lemma 4.3 on the previous page the succeeding lemma.
Lemma 6.1
The ramification divisor $\mathcal{R}$ is the zero locus of $\chi_5 = \prod_{m \in \{0,1\}^{even}} \vartheta [m]$. Moreover, the image of the zero locus of an individual $\vartheta [m]$ is one of the above mentioned 10 components of the type $M(\Delta_2)$.

The four theta series of the second kind $f_0, \ldots, f_3$ have no common root. Therefore they induce a holomorphic map
\[ F : \mathbb{H}_2 \rightarrow \mathbb{P}^3 \mathbb{C}, \quad Z \mapsto [f_0(Z), \ldots, f_3(Z)]. \]

From Runge’s results ([Run93]) one can deduce the following striking theorem.

Theorem 6.2
The map $\phi : \mathcal{R}_2[2,4]^{\mathbb{H}_2} \rightarrow \mathbb{P}^3 \mathbb{C}$ is an open, holomorphic embedding. The image’s complement is an analytic subset of codimension 2.

Instead of observing divisors, functions, and tensors on $\mathcal{R}_2[2,4]^{\mathbb{H}_2}$ we can study their counterparts on $\mathbb{P}^3 \mathbb{C}$. In particular, we can consider $\mathcal{R}$ as a divisor on $\mathbb{P}^3 \mathbb{C}$.

Lemma 6.3
The ramification divisor $\mathcal{R}$ considered on $\mathbb{P}^3 \mathbb{C}$ is the sum of 10 quadrics given by the 10 polynomials known from lemma 4.2 on page 8. Prototypes are $Q_1(z_0, \ldots, z_3) = (z_0)^2 - (z_1)^2 + (z_2)^2 - (z_3)^2$ and $Q_8(z_0, \ldots, z_3) = 2(z_0 z_2 + z_1 z_3)$.

The map $\mathbb{H}_2 \rightarrow \mathcal{R}_2[2,4]^{\mathbb{H}_2}$ is locally a simple covering ramified over $\mathcal{R}$. Therefore we obtain the following result.

Theorem 6.4
There is a natural isomorphism
\[ \Omega^{\otimes q}(\mathbb{H}_2)^{\Gamma_2[2,4]} \cong \Omega^{\otimes q}(\mathbb{P}^3 \mathbb{C}, \mathcal{R}). \]

The tensors on the left hand side can be considered as certain vector valued Siegel modular forms and the ones on the right hand side can be easily described in algebraic terms.

7 Construction of meromorphic tensors with prescribed poles

For the construction of these tensors, we consider a homogeneous polynomial $Q$ of degree $d$ in $n+1$ variables $X^0, \ldots, X^n$. We always assume that $Q$ is square-free and that $X^0$
does not divide $Q$. We want to describe meromorphic tensors on $\mathbb{P}^n\mathbb{C}$ that are holomorphic outside $Q$’s zero locus $Z(Q)$. Recall that the projective coordinates of $\mathbb{P}^n\mathbb{C}$ are denoted by $z^0, \ldots, z^n$ and the coordinates on the affine space $\mathbb{A}_n$ are

$$\left(\frac{z^1}{z^0}, \ldots, \frac{z^n}{z^0}\right).$$

For the sake of simplicity we take $\omega$ to be of the type $\Lambda^p Q \otimes (\Lambda^n Q)^{\otimes k}$. We introduce a handy notation for the canonical basis elements of $\Lambda^p Q$. Given a subset $I \subset \{1, \ldots, n\}$, say $I = \{i_1, \ldots, i_p\}$ and $i_1 < \cdots < i_p$, then $d_{3}^I$ is short for $d_{3}^{i_1} \wedge \cdots \wedge d_{3}^{i_p}$.

It is not hard to show the following lemma.

**Lemma 7.1**

Let $\omega$ be a meromorphic tensor on $\mathbb{P}^n\mathbb{C}$ which is holomorphic outside $Z(Q)$, then $\omega$ can be written in the form

$$\omega = \sum_{I \subset \{1, \ldots, n\}, |I| = p} \omega_I \, d_{3}^I \otimes (d_{3}^{1} \wedge \cdots \wedge d_{3}^{n})^{\otimes k},$$

where

$$\omega_I(\frac{z^1}{z^0}, \ldots, \frac{z^n}{z^0}) = \frac{A_I(1, \frac{z^1}{z^0}, \ldots, \frac{z^n}{z^0})}{Q^N(1, \frac{z^1}{z^0}, \ldots, \frac{z^n}{z^0})}$$

with the following properties

1. $N$ is a natural number;
2. $A_I$ is a homogeneous polynomial of degree $N \cdot \deg Q$;
3. $(X^0)^{k(n+1)+p}$ divides $A_I$;
4. and it holds for all $J \subset \{2, \ldots, n\}$ with $|J| = p - 1$

\[
(X^0)^{k(n+1)+p+1} \sum_{\substack{1 \leq j \leq n \\text{n. d.}\ \ j \notin J}} (-1)^{\text{pos}(i, J \cup \{j\})} \cdot X^j \cdot A_{J \cup \{j\}},
\]

where $\text{pos}(i, I)$ returns $i$’s position in the ordered set $I$.

For $p = n$ conditions 3 and 4 are merged to

$$3'. (X^0)^{k(n+1)+n+1} \text{ divides } A_{\{1, \ldots, n\}}.$$
We want to describe the space $\Lambda^p \otimes (\Lambda^n \otimes^k) (\mathbb{P}^n, \mathcal{R})$ for the divisor $\mathcal{R} = \sum a_m Z(Q_m)$ where $a_m \geq 0$.

**Theorem 7.2**

We introduce the numbers

$$d := \max_{1 \leq m \leq M} \left\lfloor \frac{a_m + k}{a_m + 1} \right\rfloor$$

and

$$D := \max_{1 \leq m \leq M} \left\lfloor \frac{a_m}{a_m + 1} (k + 1) \right\rfloor.$$

Let $\omega$ be a tensor as in lemma 7.1 where we choose $N$ minimal. We can state necessary and sufficient conditions in terms of $d$ and $D$ for $\omega$ to lie in $\Lambda^p \otimes (\Lambda^n \otimes^k) (\mathbb{P}^n, \mathcal{R})$:

$$\omega \in \Lambda^p \otimes (\Lambda^n \otimes^k) (\mathbb{P}^n, \mathcal{R}) \implies N \leq D,$$

and

$$N \leq d \implies \omega \in \Lambda^p \otimes (\Lambda^n \otimes^k) (\mathbb{P}^n, \mathcal{R}).$$

**Corollary 7.3**

If it holds for all $m$

$$a_m + 1 \mid k,$$

then $d = D$. Hence, the space $\Lambda^p \otimes (\Lambda^n \otimes^k) (\mathbb{P}^n, \mathcal{R})$ is completely determined.

**Proof (Theorem)**

We shall just present the proof in the case where each multiplicity $a_m$ and the exterior power $p$ equal 1.

Let $x$ be a smooth point of $Q$’s zero locus and $Q_m$ its prime factor that vanishes at $x$. We have to consider an open neighbourhood $x \in V \subset \mathbb{P}^n$ and a simple covering $U \to V$ which ramifies over $Q_m = 0$. Without loss of generality, we may assume that $V$ is a subset of $\mathbb{A}^n$ and that the $n$-th partial derivative of $Q_m$ is non-zero in $V$. Now, we can use the considerations from the beginning of section 5 for the submanifold $V \cap Z(Q_m)$.

The covering $p$ is of the form

$$(\zeta^1, \ldots, \zeta^n) \mapsto (\zeta^1, \ldots, \zeta^{n-1}, (\zeta^n)^2 + \varphi(\zeta^1, \ldots, \zeta^{n-1})), $$

where $\varphi$ is implicitly defined by

$$Q_m(1, \zeta^1, \ldots, \zeta^n) = 0 \iff \zeta^n = \varphi(\zeta^1, \ldots, \zeta^{n-1}).$$

\[1\] In the following formula $[x]$ denotes the floor function $\max \{m \in \mathbb{Z} \mid m \leq x\}$. 

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We have to observe the pullback $p^*\omega$ of the tensor

$$\omega = \sum_{i=1}^{n} \omega_i \, d_3^i \otimes (d_3^1 \wedge \cdots \wedge d_3^n)^{\otimes k}.$$ 

The coefficient functions are

$$(p^*\omega)_n = (2 \cdot z^n)^{k+1} \, \omega_n \circ p$$
and

$$(p^*\omega)_j = (2 \cdot z^n)^{k} \left( \omega_j \circ p + \omega_n \circ p \cdot \frac{\partial \varphi}{\partial \beta^j} \right) \quad \text{for } j \neq n.$$ 

We have to check whether the pullback $p^*\omega$ is holomorphic on $U$, i.e.

$$\text{ord} \left( (p^*\omega)_j, Z (z^n) \right) \geq 0, \quad 1 \leq j \leq n.$$ 

In the case where $j = n$, this is equivalent to

$$\text{ord} \left( \omega_n \circ p, Z (z^n) \right) \geq -(k + 1).$$ 

The ultrametric inequality yields that it is necessary for $p^*\omega$ to be holomorphic that all coefficients satisfy the above inequality. In the same way we derive sufficient conditions for $p^*\omega$ to be holomorphic. They are

$$\text{ord} \left( \omega_j \circ p, Z (z^n) \right) \geq -\left\lfloor \frac{k+1}{2} \right\rfloor, \quad 1 \leq j \leq n,$$

and the sufficient condition

$$\text{ord} \left( \omega_j, (Q_m) \right) \geq -\left\lfloor \frac{k}{2} \right\rfloor, \quad 1 \leq j \leq n,$$

for $\omega$ to be in the space $\left( \Omega \otimes \mathcal{O} (\Lambda^n \Omega)^{\otimes k} \right)(\mathbb{P}^n \mathbb{C}, \mathcal{R})$. This completes the proof of the theorem. \hfill \Box
8 A structure theorem for vector valued modular forms w.r.t. the multiplier system \( v_f \)

We consider the polynomial \( Q = \prod_m Q_m \) of degree 20 that gives the divisor of \( \chi_5 \) in \( \mathbb{P}^3 \mathbb{C} \), cf. lemma 6.3 on page 11. We recall that

\[
\left[ \Gamma_2 [2, 4], \left( \frac{12k}{2}, \text{Sym}^2 \right), v_f^{12k} \right] \cong \left( \Omega \otimes \mathcal{O} \left( \Lambda^3 \Omega \right)^{\otimes 2k} \right) (\mathbb{P}^3 \mathbb{C}, (Q))
\]

according to lemma 3.1 and theorem 6.4 on pages 7 and 11, respectively. Hence, we can reformulate the results of the previous section in terms of modular forms.

**Lemma 8.1**

If we denote by \( f_a \) the theta constants of the second kind, then the determinant

\[
\begin{vmatrix}
\frac{\partial (f \xi)}{\partial Z^0} & \frac{\partial (f \xi)}{\partial Z^1} & \frac{\partial (f \xi)}{\partial Z^2} \\
\frac{\partial (g \eta)}{\partial Z^0} & \frac{\partial (g \eta)}{\partial Z^1} & \frac{\partial (g \eta)}{\partial Z^2} \\
\frac{\partial (h \zeta)}{\partial Z^0} & \frac{\partial (h \zeta)}{\partial Z^1} & \frac{\partial (h \zeta)}{\partial Z^2}
\end{vmatrix}
\]

equals

\[
c_5 \cdot \chi_5 \left( f_0 \right)^4
\]

with a constant \( c_5 \) in \( \mathbb{C} \).

**Proof**

A proof can be found on pages 15 and 16 of [FSM10].

We shall present an easy and well-known lemma.

**Lemma 8.2**

For \( f \) and \( g \) in \( \left[ \Gamma_2 [2, 4], \frac{1}{2}, v_f \right] \) the Rankin-Cohen bracket

\[
\{ f, g \} := f \cdot Dg - g \cdot Df = \begin{pmatrix}
\frac{\partial g}{\partial Z^0} - g \frac{\partial f}{\partial Z^0} \\
\frac{\partial g}{\partial Z^1} - g \frac{\partial f}{\partial Z^1} \\
\frac{\partial g}{\partial Z^2} - g \frac{\partial f}{\partial Z^2}
\end{pmatrix} \begin{pmatrix}
\frac{\partial f}{\partial Z^0} \\
\frac{\partial f}{\partial Z^1} \\
\frac{\partial f}{\partial Z^2}
\end{pmatrix} = f^2 \cdot D \left( \frac{g}{f} \right)
\]

lies in \( \left[ \Gamma_2 [2, 4], (1, \text{Sym}^2), v_f^2 \right] \).

The next lemma will rely heavily on the just defined Rankin-Cohen brackets.
Lemma 8.3
Every modular form $f \in \Gamma_2[2,4], (6s, \text{Sym}^2)$ is of the form

$$f = \sum_{1 \leq i \leq 3} P_i(f_0, \ldots, f_3) \frac{1}{f_0},$$

where all $P_i$ are homogeneous polynomials of degree $12s - 1$. Conversely, such a sum lies in $\Gamma_2[2,4], (6s, \text{Sym}^2)$ iff it is holomorphic which means

$$f_0 \mid \sum_{1 \leq i \leq 3} f_i \cdot P_i.$$

Proof
A modular form $f \in \Gamma_2[2,4], (6s, \text{Sym}^2)$ can be considered as tensor on $\mathbb{P}^3 \mathbb{C}$ of the form

$$\sum_{i=1}^3 R_i \circ (f_1/f_0) \otimes \cdots \otimes (f_3/f_0)^{\otimes 2s}.$$  

In theorem 7.2 on page 13 we have seen that $R_i$ is of the form

$$R_i = \frac{P_i \cdot f_0^{8s+1}}{\chi_5^s}$$

with $P_i$ a polynomial of degree $12s - 1$ and

$$f_0 \mid \sum_{1 \leq i \leq 3} f_i \cdot P_i.$$

We want to observe this tensor on the upper half plane. The functional determinant of $d(f_1/f_0) \wedge \cdots \wedge d(f_3/f_0)$ is $\chi_5/f_0^4$, cf. lemma 8.1 on the previous page. Hence, the modular form is of the desired type. 

We could have formulated the above theorem replacing $f_0$ by any other $f_a$.

We introduce the $\mathbb{C}[f_0, \ldots, f_3]$-module

$$\mathcal{M}^+(\Gamma_2[2,4]) := \bigoplus_{r \in \mathbb{Z}} \left[ \Gamma_2[2,4], \left( \frac{r}{2}, \text{Sym}^2 \right), v_r^f \right].$$

One of its $\mathbb{C}[f_0, \ldots, f_3]$-submodules is

$$M^+ := \sum_{0 \leq i < j \leq 3} \mathbb{C}[f_0, \ldots, f_3] \{f_i, f_j\}.$$
Theorem 8.4
The Rankin-Cohen brackets are related in the following manner:

\[ R_1 : f_1 \{ f_0, f_2 \} = f_2 \{ f_0, f_1 \} + f_0 \{ f_1, f_2 \} ; \]
\[ R_2 : f_1 \{ f_0, f_3 \} = f_3 \{ f_0, f_1 \} + f_0 \{ f_1, f_3 \} ; \]
\[ R_3 : f_2 \{ f_0, f_3 \} = f_3 \{ f_0, f_2 \} + f_0 \{ f_2, f_3 \} ; \]
\[ R_4 : f_2 \{ f_1, f_3 \} = f_3 \{ f_1, f_2 \} + f_1 \{ f_2, f_3 \} . \]

These are defining relations of the module \( M^+ \). Therefore, the Hilbert function is

\[ \dim M^+_r = 3 \cdot \binom{r+1}{3} + 2 \cdot \binom{r}{2} + \binom{r-1}{1} . \]

Proof
Let \( R \) be an arbitrary relation

\[ \sum_{0 \leq i < j \leq 3} P_{ij} \{ f_i, f_j \} = 0. \]

It is equivalent to the subsequent three relations

\[ f_0 P_{0j} + \sum_{i=1}^{j-1} f_i P_{ij} - \sum_{i=j+1}^{3} f_i P_{ji} = 0, \quad \forall j \in \{1, \ldots, 3\} \]

due to the given relations \( R_1, \ldots, R_4 \) and the linear independence of the brackets \( \{ f_0, f_j \} \).

Applying \( R_1, \ldots, R_4 \), the relation \( R \) can be transformed to a form where \( P_{ij} \) is a polynomial in the variables \( f_0, \ldots, f_j \). In this normal form we see that each \( P_{ij} \) is zero. Indeed, for \( j = 1 \) we obtain

\[ f_0 P_{01} - \sum_{i \geq 1} f_i P_{1i} = 0. \]

Setting the variables \( f_2, f_3 \) zero yields

\[ P_{01} = P_{01}(f_1) = 0. \]

Specialising now \( f_3 \) gives \( P_{12} = P_{12}(f_1, f_2) = 0 \) and hence \( P_{13} = 0 \).

The just proven equality \( P_{12} = 0 \) simplifies the above relation for \( j = 2 \) to

\[ f_0 P_{02} - f_3 P_{23} = 0. \]

A similar line of argument shows

\[ P_{02} = P_{23} = 0. \]

Setting \( j = 3 \) gives the remaining coefficients. \( \square \)
The module $M^+$ can be considered as a submodule of the free module

$$\sum_a \mathbb{C}[f_0, \ldots, f_3] \cdot Df_a.$$ 

In this setting, an element of $M^+$, say $\sum_a P_a \cdot Df_a$, can be characterized by a single $\mathbb{C}[f_0, \ldots, f_3]$-linear equation:

$$\sum_a f_a P_a = 0.$$ 

A short MAGMA [BCP97] code or the above appealing observation, that is due to U. Weselmann, both imply the subsequent theorem.

**Theorem 8.5**

We have

$$\bigcap_{i=0}^3 \frac{f_0 \cdots f_3}{f_i} M^+ = f_0 \cdots f_3 \cdot M^+.$$ 

An immediate consequence is the succeeding lemma.

**Lemma 8.6**

Every modular form $f \in [\Gamma_2[2,4], (6s, \text{Sym}^2)]$ is of the form

$$f = \sum_{0 \leq i < j \leq 3} P_{ij}(f_0, \ldots, f_3) \{f_i, f_j\},$$

where any $P_{ij}$ is a homogeneous polynomials of degree $12s - 1$.

We want to generalize this now to arbitrary weights.

**Theorem 8.7 (Structure theorem)**

We have

$$\mathcal{M}^+(\Gamma_2[2,4]) = \bigoplus_{r \in \mathbb{Z}} \left[ \Gamma_2[2,4], \left( \frac{r}{2}, \text{Sym}^2 \right), v_r^f \right] = \sum_{0 \leq i < j \leq 3} \mathbb{C}[f_0, \ldots, f_3] \{f_i, f_j\}.$$ 

**Proof**

Let $f$ be a modular form of weight $r/2$. If $r$ is a multiply of 12, then we apply lemma 8.6. Therefore, we may assume that $f_i \cdot f$ lies in the right hand side. Now, we can apply theorem 8.5. 

\[ \square \]
We consider the character $v_f^2$ on $\Gamma_2[2,4]$ and the twisted version of $M^+$:

$$M^-(\Gamma_2[2,4]) := \bigoplus_{r \in \mathbb{Z}} \left[ \Gamma_2[2,4], \left( \frac{r}{2}, \text{Sym}^2 \right), v_f^r \cdot v_f^2 \right].$$

It is possible to show that $M^-$ coincides with its $\mathbb{C}[f_0, \ldots, f_3]$-submodule

$$M^- := \sum_{0 \leq i < j < k \leq 3} \mathbb{C}[f_0, \ldots, f_3] \{f_i, f_j, f_k\}.$$

For this purpose, we shall only present the proclaims that differ substantially from their counterparts in section 8.

**Definition 9.1**

For $f, g$ and $h$ in $\mathcal{O}(\mathbb{H}_2)$ and $0 \leq j_1 < j_2 \leq 2$, we define

$$\{f, g, h\}_{(j_1,j_2)} := \begin{vmatrix}
\frac{\partial f}{\partial z_1} & \frac{\partial f}{\partial z_2} \\
\frac{\partial g}{\partial z_1} & \frac{\partial g}{\partial z_2} \\
\frac{\partial h}{\partial z_1} & \frac{\partial h}{\partial z_2}
\end{vmatrix}. $$

Then the Rankin-Cohen 3-bracket of $f, g$ and $h$ is

$$\{f, g, h\} := \begin{pmatrix}
\{f, g, h\}_{(1,2)} & -\{f, g, h\}_{(0,2)} \\
-\{f, g, h\}_{(0,2)} & \{f, g, h\}_{(0,1)}
\end{pmatrix}. $$

A direct computation shows that it holds

$$\{f, g, h\} \in \left[ \Gamma_2[2,4], \left( \frac{5}{2}, \text{Sym}^2 \right), v_5^5 \cdot v_f^2 \right]$$

for all modular forms $f, g, h$ in $[\Gamma_2[2,4], \frac{1}{2}, v_f]$. Similarly as in lemma 8.3 on page 16, we can relate the 2-form $d\left( \frac{f_1}{f_0} \right) \wedge d\left( \frac{f_2}{f_0} \right)$ to the 3-bracket $\{f_0, f_1, f_2\}$ using lemma 3.2 and theorem 6.4 on pages 7 and 11 respectively. In contrast to theorem 8.4 on page 17, the Rankin-Cohen 3-brackets just satisfy a single relation.

**Lemma 9.2**

The Rankin-Cohen 3-brackets are related in the following manner

$$R_5 : -f_0 \{f_1, f_2, f_3\} + f_1 \{f_0, f_2, f_3\} - f_2 \{f_0, f_1, f_3\} + f_3 \{f_0, f_1, f_2\} = 0.$$
This is a defining relation of the module $M^-$. Hence, the Hilbert function is

$$\dim M^-_r = 3 \cdot \binom{r-2}{3} + \binom{r-3}{2}.$$ 

**Proof**

We just show the relation. Consider the matrix

$$A = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 \\ \frac{\partial f_0}{\partial Z^0} & \frac{\partial f_1}{\partial Z^0} & \frac{\partial f_2}{\partial Z^0} & \frac{\partial f_3}{\partial Z^0} \\ \frac{\partial f_0}{\partial Z^1} & \frac{\partial f_1}{\partial Z^1} & \frac{\partial f_2}{\partial Z^1} & \frac{\partial f_3}{\partial Z^1} \\ \frac{\partial f_0}{\partial Z^2} & \frac{\partial f_1}{\partial Z^2} & \frac{\partial f_2}{\partial Z^2} & \frac{\partial f_3}{\partial Z^2} \end{pmatrix}$$

and its adjoint matrix

$$\text{Adj}(A) = \begin{pmatrix} \frac{\partial f_{(1,2,3)}}{\partial Z^{(0,1,2)}} & -\{f_1, f_2, f_3\}_{(1,2)} + \{f_1, f_2, f_3\}_{(0,2)} - \{f_1, f_2, f_3\}_{(0,1)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_{(0,1,2)}}{\partial Z^{(0,1,2)}} & +\{f_0, f_1, f_2\}_{(1,2)} - \{f_0, f_1, f_2\}_{(0,2)} + \{f_0, f_1, f_2\}_{(0,1)} \end{pmatrix}.$$ 

The product $A \cdot \text{Adj}(A)$ is just $c_5 \cdot \chi_5 \cdot I_4$; this yields the desired relation.

The module $M^-$ can be considered as a submodule of the free module

$$\sum_{0 \leq a < b \leq 3} \mathbb{C} [f_0, \ldots, f_3] \cdot Df_a \wedge Df_b.$$ 

In this setting, an element of $M^-$, say $\sum_{0 \leq a < b \leq 3} P_{ab} \cdot Df_a \wedge Df_b$, can be characterized by four $\mathbb{C} [f_0, \ldots, f_3]$-linear equations:

- $f_1 P_{01} + f_2 P_{02} + f_3 P_{03} = 0$;
- $-f_0 P_{01} + f_2 P_{12} + f_3 P_{13} = 0$;
- $-f_0 P_{02} - f_2 P_{12} + f_3 P_{23} = 0$;
- $-f_0 P_{03} - f_1 P_{13} - f_2 P_{23} = 0$.

This leads eventually to the aforementioned result.

**Theorem 9.3 (Structure theorem)**

We have

$$\mathcal{M}^- (\Gamma_2 [2, 4]) = \bigoplus_{r \in \mathbb{Z}} \left[ \Gamma_2 [2, 4], \left( \frac{r}{2}, \text{Sym}^2 \right), v_f^r \cdot v_f^2 \right] = \sum_{0 \leq i < j < k \leq 3} (\mathbb{C} [f_0, \ldots, f_3]) \{f_i, f_j, f_k\}.$$ 

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10 A structure theorem for vector valued modular forms w.r.t. the trivial mult. system

We can extract from the modules \( \mathcal{M}^+(\Gamma_2[2,4]) \) and \( \mathcal{M}^-(\Gamma_2[2,4]) \) the modular forms with trivial multiplier system.

Theorem 10.1
We have
\[
\bigoplus_{k \in \mathbb{Z}} [\Gamma_2[2,4], (k, \text{Sym}^2)] = \\
\sum_{0 \leq i < j \leq 3} (\mathbb{C}_{2+4\mathbb{Z}}[f_0, \ldots, f_3]) \{f_i, f_j\} \oplus \sum_{0 \leq i_1 < i_2 < i_3 \leq 3} (\mathbb{C}_{1+4\mathbb{Z}}[f_0, \ldots, f_3]) \{f_{i_1}, f_{i_2}, f_{i_3}\}
\]
and
\[
\dim [\Gamma_2[2,4], (k, \text{Sym}^2)] = \begin{cases} 
3 \cdot \left(\frac{2k+1}{3}\right) + 2 \cdot \left(\frac{2k}{3}\right) + \left(\frac{2k-1}{3}\right), & \text{if } k \text{ is even,} \\
3 \cdot \left(\frac{2k-2}{3}\right) + \left(\frac{2k-3}{3}\right), & \text{if } k \text{ is odd.}
\end{cases}
\]

11 A structure theorem for scalar valued modular forms

So far, we have treated tensors of the type
\[
\Lambda^p \Omega \otimes_\mathcal{O} (\Lambda^3 \Omega)^{\otimes k},
\]
where \( p = 1, 2 \). It is worthwhile to mention that our method is also successful for \( p = 3 \). In this case we get the structure theorems for the ring of scalar valued modular forms. Recall that our method relied on the injectivity and ramification behaviour of the map
\[
[f_0, \ldots, f_3] : \Gamma_2[2,4]^{\mathbb{P}^2} \rightarrow \mathbb{P}^3 \mathbb{C}.
\]
More precisely, we obtain the following theorem which is essentially due to Runge.

Theorem 11.1
We have
\[
\bigoplus_{r \in \mathbb{Z}} [\Gamma_2[2,4], \frac{r}{2}, v_r^r] = \mathbb{C}[f_0, \ldots, f_3].
\]
Twisting with $v_f^2$ yields

$$\bigoplus_{r \in \mathbb{Z}} \left[ \Gamma_2 [2, 4], \frac{r}{2}, v_f^r \cdot v_f^2 \right] = \chi_5 \cdot \mathbb{C}[f_0, \ldots, f_3].$$

As a consequence,

$$\bigoplus_{r \in \mathbb{Z}} \Gamma_2 [2, 4], r] = \mathbb{C}_{4\mathbb{Z}}[f_0, \ldots, f_3] \oplus \mathbb{C}_{4\mathbb{Z}}[f_0, \ldots, f_3] \cdot \chi_5.$$

We omit the details of the proof, but stress that we start with tensorial weights, i.e. $k \in 3\mathbb{Z}$, as in the case of vector valued modular forms. Afterwards, we extend the results to all weights again by intersecting appropriate modules. It is quite interesting that Igusa’s modular form $\chi_5$ comes up automatically in our approach. This happens while studying the holomorphicity of tensors by means of the ramification behaviour of $[f_0, \ldots, f_3]$.

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