Gravity is believed to have deep and inherent relation to thermodynamics. We study phase transition and critical behavior in the extended phase spaces of asymptotic anti de Sitter (AdS) black holes in Einstein-Horndeski gravity. We demonstrate that the black holes in Einstein-Horndeski gravity undergo first order phase transitions mimicking the van der Waals-Maxwell gas-liquid system. The key approach in our study is to introduce a proper pressure, which is a natural extension of the case of the Reissner-Nordstrom AdS. We present the criticality exponents for the phase transition processes.

I. INTRODUCTION

The relation between gravity theory and thermodynamics is an interesting and profound issue. The black hole thermodynamics (in fact, spacetime thermodynamics, because the physical quantities in black hole thermodynamics should be treated as the quantities of the globally asymptotic flat manifold) is set up in [1] and confirmed by Hawking radiation [2]. An ordinary thermodynamical process may seem depressing. Rich and varied physical phenomena appear around the critical point. Phase transition and critical phenomena best present the inherent properties of thermodynamic (statistical) system. More and more analogies are found between ordinary thermodynamics and black hole thermodynamics, including several phase transition and critical phenomena, through more and deep studies. These studies reveal significant properties of gravity systems. Further more, it may be helpful to solve some extremely difficult problems beyond gravity through investigations of phase transitions in gravity system. In an asymptotic AdS space, a first order phase transition occurs in a charged black hole (Hawking-Page phase transition) [3]. The Hawking-Page phase transition maps to quark confinement/definement transition in the scenario of AdS/CFT [4], such that gains much more physical significance. The second landmark of first order phase transition in asymptotic AdS space is the phase transition similar to Van der Waals-Maxwell gas/liquid transitions in Reissner-Nordstrom AdS (RN-AdS) black holes [5]. This phase transition is also called P-V criticality in literatures because of the similarity of the P-V diagrams of the AdS gravity and the Van der Waals-Maxwell gas/liquid system.

Recently the behavior of P-V criticality is found in several different spaces [6–9]. A critical step to realize the P-V criticality is to introduce a thermodynamic pressure, which is proportional to the cosmological constant. Under this situation, the mass of a black hole becomes enthalpy rather than energy, which is required by thermodynamic laws. In these laws, the term \( dP \) appears, which implies that the cosmological “constant” may be a variable. There are a large amount of discussions of a variable vacuum energy in contexts of cosmology since the discovery of cosmic acceleration.

Theoretically, we have several motivations to invoke the assumption of a variable cosmological “constant”. If the present theory is not a “fundamental theory”, the constants in the theory may be variables. For example, the cosmological constant becomes a variable in gauged supergravity. And further more, the cosmological constant, Newton constant, and other coupling constants may become variables in quantum theory. The observed values are their vacuum expectations [10]. Thus it may be reasonable to identify the cosmological constant as the pressure in P-V criticality in gravity. Such an identity leads to several sensible results [6]. However, it still needs more examinations since the P-V criticality of gravity system compared to gas/liquid system is fundamentally an analogy. We have no sound statistical mechanics behind the gravity thermodynamics. And thus we are unable to prove that the pressure should be identified as the cosmological constant at microscopic level. We shall show that a cosmological constant is...
not a proper candidate for pressure in thermodynamics in the Einstein-Horndeski theory, and then resent the proper pressure in this theory.

The scalar-tensor gravity has a fairly long history and several different extensions. The Horndeski scalar tensor theory is the most generic scalar tensor theory in which the equations of motion are consisted of terms which have at most second-order derivatives acting on each fields, though it permits higher-order derivatives in the action \[11\]. This property is very similar to the Lovelock gravity. From different considerations, the Horndeski theory is rediscovered in a different form in the studies of cosmology\[12\]. Black holes solutions in a special Einstein-Horndeski gravity are derived in \[13–17\]. Thermodynamics for black holes in Einstein-Horndeski is investigated in \[17\]. It is shown that there is no P-V criticality in black holes in Einstein-Horndeski, if one just treat the cosmological constant as pressure, as usual. We shall demonstrate that the proper pressure is different from the cosmological constant. With this new pressure, the P-V criticality appears.

This article is organized as follows. In the next section, we briefly review an exact solution in the Einstein-Horndeski theory. In section III, we demonstrate that P-V phase transition really occurs with a new pressure different from the cosmological constant. In section IV, we conclude this paper.

II. THE STATIC BLACK HOLE SOLUTION IN HORNDESKI GRAVITY

The most general action of Horndeski gravity can be seen in the reference \[14\]. In our paper, we just investigate a special case in Horndeski gravity with a non-minimal kinetic coupling, and the corresponding action is written as \[15–17\]

\[
S = \int \sqrt{-g} d^4x \left[ (R - 2\Lambda) - \frac{1}{2} (\alpha g_{\mu\nu} - \eta G_{\mu\nu}) \nabla^\mu \phi \nabla^\nu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right],
\]

where \(\alpha\) is a coupling constant, and \(\eta\) stands for the coupling strength with the dimension of length square. Besides the electromagnetic field \(F_{\mu\nu}\), a real scalar field \(\phi\) also exists, which has a non-minimal kinetic coupling with metric and Einstein tensor. From this action, the dynamical equations are

\[
G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2} (\alpha T_{\mu\nu} + \eta \Xi_{\mu\nu} + E_{\mu\nu}),
\]

\[
\nabla_\mu \left[ (\alpha g_{\mu\nu} - \eta G_{\mu\nu}) \nabla_\nu \phi \right] = 0,
\]

\[
\nabla_\mu F^{\mu\nu} = 0,
\]

where \(T_{\mu\nu}, \Xi_{\mu\nu},\) and \(E_{\mu\nu}\) are defined as

\[
T_{\mu\nu} \equiv \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} \nabla_\mu \nabla_\nu \phi \nabla_\rho \phi,
\]

\[
\Xi_{\mu\nu} \equiv \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi R - 2 \nabla_\rho \phi \nabla_{(\mu} \phi R_{\nu)} - \nabla^\rho \phi \nabla^\lambda \phi R_{\mu\nu\rho\lambda}
- (\nabla_\mu \nabla^\rho \phi)(\nabla_\nu \nabla_\rho \phi) + (\nabla_\mu \nabla_\nu \phi) \Box \phi + \frac{1}{2} G_{\mu\nu}(\nabla_\phi)^2
- g_{\mu\nu} \left[ - \frac{1}{2} (\nabla^\rho \nabla^\lambda \phi)(\nabla_\rho \nabla_\lambda \phi) + \frac{1}{2} (\Box \phi)^2 - \nabla_\rho \phi \nabla_\lambda \phi R^{\rho\lambda} \right],
\]

\[
E_{\mu\nu} \equiv F_{\mu}^\rho F_{\nu}\rho - \frac{1}{2} g_{\mu\nu} F^2.
\]

In our paper, we focus on the static black hole solutions with spherical symmetry to eqs. \[2.2\], \[2.4\], while the metric, scalar field and Maxwell field are simplified

\[
ds^2 = -f(r)dt^2 + g(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

\[
\phi = \phi(r), \quad A = \Psi(r)dt,
\]
Through this simplification, an analytical solution has been solved to be

\[
f(r) = \frac{\alpha^2}{3\eta} - \frac{2M}{r} + \frac{3\alpha + \Lambda\eta}{\alpha - \Lambda\eta} + \left(\frac{\alpha + \Lambda\eta + \frac{\alpha^2q^2}{4\eta}}{\alpha - \Lambda\eta}\right)^2 \tan^{-1}\left(\frac{\sqrt{\alpha}r}{\sqrt{\alpha\eta}}\right) + \frac{\alpha^2q^2}{4\eta^2} - \frac{\alpha^2q^4}{48(\alpha - \Lambda\eta)^2} + \frac{\alpha^4q^4}{16\eta(\alpha - \Lambda\eta)^2},
\]

(2.10)

\[
g(r) = \frac{\alpha^2[4(\alpha - \Lambda\eta)r^4 + 8\eta^2 - \eta q^2]^2}{16\eta^4(\alpha - \Lambda\eta)^2(\alpha r^2 + \eta)^2 f(r)},
\]

(2.11)

\[
\psi^2(r) = -\frac{\alpha^2[4(\alpha + \Lambda\eta)r^4 + \eta q^2][4(\alpha - \Lambda\eta)r^4 + 8\eta^2 - \eta q^2]^2}{32\eta^6(\alpha - \Lambda\eta)^2(\alpha r^2 + \eta)^2 f(r)},
\]

(2.12)

\[
\Psi(r) = \Psi_0 + \frac{1}{4\pi} \frac{\sqrt{\alpha} Q(r)}{\psi(r)} f'(r|) = \frac{1}{4\pi r_h} \left(\frac{2\alpha}{\alpha - \Lambda\eta} + \frac{\alpha q^2}{\eta} - \frac{\alpha q^2}{4r_h^2(\alpha - \Lambda\eta)}\right),
\]

(2.14)

where \(\psi(r) \equiv \phi'(r)\), and \(\Psi_0\) is an integration constant. This solution requires \(\alpha\) and \(\eta\) to have the same sign and \(\alpha \neq \Lambda\eta\). Once \(\alpha = -\Lambda\eta\) and \(q = 0\), the Schwarzschild-AdS solution is recovered and the scalar field becomes trivial [17]. For simplicity but without the loss of generality, we set \(\alpha > 0\) and \(\eta > 0\) in the following. In addition, the temperature of this black hole in Horndeski gravity is

\[
T = \frac{1}{4\pi} \sqrt{g'(r_h) f'(r_h)} = \frac{1}{4\pi r_h} \left(\frac{2\alpha}{\alpha - \Lambda\eta} + \frac{\alpha q^2}{\eta} - \frac{\alpha q^2}{4r_h^2(\alpha - \Lambda\eta)}\right),
\]

(2.14)

where \(r_h\) is the location of outer horizon satisfying \(f(r_h) = 0\). For this formula of temperature, it can be further rewritten as

\[
4r_h^4\Pi^* + (8r_h^2 - 4\Lambda r_h^4 - 16\pi T r_h^3 - q^2) \Pi^* + 16\pi T r_h^3 \Lambda = 0,
\]

(2.15)

where a new parameter \(\Pi^* \equiv \frac{\Pi}{\eta}\) has been defined for convenience of later investigations.

III. P-V CRITICALITY IN THE EXTENDED PHASE SPACE OF BLACK HOLE IN HORNDESKI GRAVITY

There have been a lot of works to investigate P-V criticality in the extended phase space of black hole with asymptotical AdS behaviour. Since the black hole solution in (2.13) also has the asymptotical AdS behaviour, we will investigate the P-V criticality in its extended phase space in Horndeski gravity.

A. No P-V criticality with Pressure defined as \(P = -\frac{1}{8\pi} \Lambda\)

During investigations of P-V criticality in the extended phase space of black hole with asymptotical AdS behaviour, the thermodynamical pressure \(P\) of the system is usually defined as [8]

\[
P = -\frac{1}{8\pi} \Lambda,
\]

(3.1)

while the volume \(V\) of static black hole with spherical symmetry is \(V = \frac{4}{3}\pi r_h^3\). Since the volume \(V\) is a monotonic function of the horizon radius \(r_h\), we can use \(r_h\) to specify the critical behavior instead of \(V\). Note that, in the van der Waals liquid-gas system, one needs first find out the function \(P(V, T)\), therefore, here we should first find out the corresponding function \(P(r_h, T)\).

After considering (3.1) in (2.14), we obtain function \(P(r_h, T)\)

\[
P(r_h, T) = \frac{4r_h^4\Pi^* + (8r_h^2 - 16\pi T r_h^3 - q^2) \Pi^*}{8\pi (16\pi T r_h^3 - 4r_h^4 \Pi^*)}
\]

(3.2)
For this function $P(r_h, T)$, it is a little complicated to analytically investigate whether there is critical behaviour like the van der Waals liquid-gas system. Therefore, we check the criticality through plotting the $P - r_h$ phase diagram for different temperature $T$ and $q$ values, and for most cases the $P - r_h$ phase diagrams are similar to the following diagram. From these diagrams, we can find out that there is no P-V criticality in this case.

![Diagram](image)

**FIG. 1:** the $P - r_h$ phase diagram, from top to bottom, $T = 0.1$ to $0.3$ with $q = 1, \Pi' = 0.3$

### B. P-V criticality with new Pressure defined as $P = \frac{\alpha}{8\pi \eta}$

In the above subsection, we have found that there is no P-V criticality with pressure $P = -\frac{1}{8\pi} \Lambda$. However, for the solution in (2.13), it is an asymptotical AdS solution with the effective AdS radius $l$ satisfied $l^{-2} = \frac{\alpha}{8\pi \eta}$. Therefore, the corresponding pressure is more reasonable to define as

$$P = \frac{\alpha}{8\pi \eta} = \frac{\Pi'}{8\pi}. \quad (3.3)$$

Note that the thermodynamic volume $V = \frac{4}{3}\pi r_h^3$ is a monotonic function of the horizon radius $r_h$, so we can use $r_h$ to specify the critical behavior instead of $V$. After solving (2.15), we obtain two branches for function $P(r_h, T)$

$$P(r_h, T) = \frac{-b \pm \sqrt{b^2 - 4ac}}{8\pi 2a} \quad (3.4)$$

$$a = 4r_h^4 \quad (3.5)$$

$$b = 8r_h^2 - 44r_h^4 - 16\pi T r_h^3 - q^2 \quad (3.6)$$

$$c = 16\pi T r_h^3 \Lambda \quad (3.7)$$

For $'-'$ branch, it is a little meaningless since the function $P(r_h, T)$ in this branch becomes $P(r_h, T) = 0$ when $\Lambda$ is taken $\Lambda = 0$. Therefore, we just focus on the $'+'$ branch for function $P(r_h, T)$, and its detailed form is

$$P(r_h, T) = \frac{-8r_h^2 - 44r_h^4 - 16\pi T r_h^3 - q^2 + \sqrt{(8r_h^2 - 44r_h^4 - 16\pi T r_h^3 - q^2)^2 - 256\pi 4r_h^7 T}}{64\pi r_h^4} \quad (3.8)$$

In this $'+'$ branch for function $P(r_h, T)$, an interesting and special case is $\Lambda = 0$, while function $P(r_h, T)$ becomes

$$P = \frac{\omega_1}{r_h} + \frac{\omega_2}{r_h^2} + \frac{\omega_4}{r_h^4} \quad (3.9)$$

where

$$\omega_1 = T / 2 \quad \omega_2 = -1 / 4 \pi \quad \omega_4 = \frac{q^2}{32 \pi} \quad (3.10)$$
Comparing with the function $P(r_h, T)$ in the P-V criticality in massive gravity [8], we find that our $P(r_h, T)$ function [3.9] has the same formula as the massive gravity case. Therefore, for this '+' branch with $\Lambda = 0$, we can analytically investigate its P-V criticality like the case in massive gravity, and the critical point is obtained

$$r_{hc} = \sqrt{-\frac{6w_4}{w_2}},$$

(3.11)

$$w_{1c} = -\frac{4}{3}w_2\sqrt{-\frac{w_2}{6w_4}},$$

(3.12)

$$P_c = \frac{w_2^2}{12w_4}. $$

(3.13)

while the corresponding critical exponents around critical point are $\alpha = 0, \beta = \frac{1}{2}, \gamma = 1, \delta = 3$ as same as the massive gravity case.

For the case with $\Lambda \neq 0$, the analytical investigation of P-V criticality is a little complicate, because the function $P(r_h, T)$ in (3.8) is complicate. For simplicity, we just first plot the function $P(r_h, T)$ to find out P-V criticality. Indeed, we have found out that there is P-V criticality for some fixed parameters, i.e parameters with $q = 0.3, \Lambda = -0.3$.

![FIG. 2: $q = 0.3, \Lambda = -0.3, T = 0.5 \text{ to } 0.9$](image)

However, it should be pointed out that not all the fixed parameters can obtain the P-V criticality. For example, the following case with parameters $q = 5, \Lambda = -0.3$ has no P-V criticality.
Now we turn to calculate the critical exponents in the Einstein-Horndeski gravity black hole system in this case with parameters $q = 0.3, \Lambda = -0.3$. The critical point is determined as the inflection point in the $P-V$ diagram, i.e.,

$$\frac{\partial P}{\partial r_h} \bigg|_{r_h=r_{hc},T=T_c} = \frac{\partial^2 P}{\partial r_h^2} \bigg|_{r_h=r_{hc},T=T_c} = 0. \quad (3.14)$$

From these two equations, we numerically obtain the critical point as $r_{hc} \approx 0.2598, T_c \approx 0.8002$. On the other hand, in usual thermodynamic system, the critical exponents $\alpha, \beta, \gamma, \text{and} \delta$ are defined as follows

$$C_v \sim |t|^{-\alpha};$$
$$\eta \sim v_l - v_s \sim |t|^{\beta};$$
$$K_T = -\frac{1}{v} \left( \frac{\partial v}{\partial P} \right)_T \sim |t|^{-\gamma};$$
$$P - P_c \sim |v - v_c|^{\delta}; \quad (3.15)$$

where $t = T/T_c - 1$, $C_v$ is isopycnic heat capacity, $v$ stands for specific volume, and $\kappa_T$ is isothermal compressibility. The subscript $c$ stands for critical point, while $l$ and $s$ refer to the two phases of the system; e.g. in the case of normal thermodynamics, $l$ and $s$ may be vapour and liquid phases.

Following the general method proposed in [9], we make the Taylor series expansion near critical point for the equation of state as

$$P = P_c + R t + B t \omega + D \omega^3 + K \omega^2 + ... \quad (3.16)$$

where

$$t = \frac{T}{T_c} - 1 \quad \omega = \frac{r}{r_{hc}} - 1 \quad (3.17)$$

and the expansion coefficients are

$$R = T_c \left[ \left( \frac{\partial P}{\partial T} \right)_{r_h} \right]_c \approx 37.44 \quad \quad \quad B = T_c r_c \left[ \left( \frac{\partial^2 P}{\partial T \partial r_h} \right) \right]_c \approx -35.49 \quad (3.18)$$

$$D = r_c^3 \frac{1}{3!} \left[ \left( \frac{\partial^3 P}{\partial r_h^3} \right) T \right]_c \approx -18.72 \quad \quad \quad K = T_c^2 \frac{1}{2!} \left[ \left( \frac{\partial^2 P}{\partial T^2} \right) r_h \right]_c \approx 3.090 \quad (3.19)$$

The isopycnic heat capacity $C_v$ vanishes since the entropy is also determined by horizon radius, we then have $\alpha = 0$. In order to obtain the critical exponent $\beta$, we also use the Maxwell’s equal area law by keeping the analogy between the
volumes of vapor and liquid phases. For constant $t$, one finds $dP = (Bt + 3D\omega^2)d\omega$, therefore, after using Maxwell’s area law, we obtain the following equation

$$\int_{\omega_l}^{\omega_s} \omega(Bt + 3D\omega^2)d\omega + \int_{\omega_l}^{\omega_s} (Bt + 3D\omega^2)d\omega = 0 \quad (3.20)$$

where $\omega_l$ and $\omega_s$ correspond to the large and small volumes of black hole in two different phases. We further consider the condition $P_l = P_s$, i.e. the end point of vapor and the starring point of liquid have same pressure like the usual thermodynamical liquid-gas system, and hence (3.16) implies

$$Bt(\omega_l - \omega_s) + D(\omega_l^3 - \omega_s^3) = 0 \quad (3.21)$$

which means that the second integral of (3.20) vanishes. Therefore, (3.20) is further reduced as

$$Bt(\omega_l^2 - \omega_s^2) + D(\omega_l^4 - \omega_s^4) = 0 \quad (3.22)$$

For these two equations (3.20) and (3.22), the non-trivial solutions are $\omega_l = (-Bt/D)^{1/2}$ and $\omega_s = -(Bt/D)^{1/2}$. It is easy to find that

$$\eta \sim V_l - V_s = (\omega_l - \omega_s)V_c \sim |t|^{1/2} \quad (3.23)$$

which determines the critical exponent $\beta = 1/2$.

The isothermal compressibility $K_T$ can be calculated by $(\partial P/\partial V)_T$ from (3.16), and it is given by

$$(\partial P/\partial V)_T \simeq B/\sqrt{V_c} t \quad (3.24)$$

where $\partial \omega/\partial V = 1/V_c$ has been used. Therefore, near the critical point, the value of $K_T$ is

$$K_T \simeq \frac{1}{Bt} \sim t^{-1} \quad (3.25)$$

which implies $\gamma = 1$. Finally, when $T = T_c$, from (3.16), it yields

$$P - P_c \sim \omega^3 \sim (V - V_c)^3 \quad (3.26)$$

which implies the value of the critical exponent $\delta$ is $\delta = 3$. Obviously, for these derived critical exponents $\alpha = 0, \beta = \frac{1}{2}, \gamma = 1, \delta = 3$, they have the same critical exponents and following scaling laws as the usual thermodynamical liquid-gas system

$$\alpha + 2\beta + \gamma = 2; \quad \gamma = \beta(\delta - 1) \quad (3.27)$$

IV. CONCLUSION AND DISCUSSION

We study the P-V criticality behaviour in the extended phase space in Einstein-Horndeski theory. Einstein-Horndeski theory is the most general covariant scalar-tensor theory [18]. The relation between gravity and thermodynamics is a significant topic in modern theoretical physics. Phase transition and critical phenomenon are the core of thermodynamics and statistical physics. Phase transition of critical phenomenon are also extensively studied in gravity theory.

Surprisingly more or less, it was shown that P-V criticality mimicking the van de Wals-Maxwell gas never occurs in the Einstein-Horndeski theory, since such a criticality occurs in several special cases of Einstein-Horndeski theory, for example the RN-AdS in Einstein theory. Through cautious analysis of the structure of the theory, we demonstrate that the pressure should be $\frac{\alpha}{8\pi\eta}$ rather than the cosmological constant. Physically, $\frac{\alpha}{8\pi\eta}$ uniquely corresponds to the AdS radius, and thus works as an effective cosmological constant. With this new definition of pressure, we find that P-V criticality occurs in the Einstein-Horndeski theory. Furthermore, we obtain the critical exponents in this process and check the compatibility relation between the exponents. We find the exponents follow the ones in mean field theory, which agrees with the previous studies of the P-V criticalities in gravity theory. We deal with an asymptotic anti-de Sitter solution. In the scenario of AdS/CFT, it corresponds a conformal field theory dwelling on the boundary. Our new understanding of the pressure is expected to get more evidences in the CFT sector. There is no uniqueness theorem in Einstein-Horndeski theory. We still need to study critical behaviour of more solutions in spherical symmetry, and further check the critical exponents in the new solutions.
V. ACKNOWLEDGEMENTS

This work is supported by National Natural Science Foundation of China (NSFC) under grant Nos. 11575083, 11565017.

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