SHARP LIOUVILLE THEOREMS

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Abstract. Consider the equation \( \text{div}(\varphi^2 \nabla \sigma) = 0 \) in \( \mathbb{R}^N \), where \( \varphi > 0 \). Berestycki, Caffarelli and Nirenberg \cite{4} proved that if there exists \( C > 0 \) such that \( \int_{B_R}(\varphi \sigma)^2 \leq CR^2 \) for every \( R \geq 1 \) then \( \sigma \) is necessarily constant. In this paper we provide necessary and sufficient conditions on \( 0 < \Psi \in C([1, \infty)) \) for which this result remains true if we replace \( R^2 \) with \( \Psi(R) \) in any dimension \( N \). In the case of the convexity of \( \Psi \) for large \( R > 1 \) and \( \Psi' > 0 \), this condition is equivalent to \( \int_1^\infty \frac{1}{\Psi'} = \infty \).

1. Introduction and main results

In 1978 E. De Giorgi \cite{5} made the following conjecture:

**Conjecture.** Let \( u \in C^2(\mathbb{R}^N) \) be a bounded solution of the Allen-Cahn equation \( -\Delta u = u - u^3 \) which is monotone in one direction (for instance \( \partial u/\partial x_N > 0 \) in \( \mathbb{R}^N \)). Then \( u \) is a 1-dimensional function (or equivalently, all level sets \( \{u = s\} \) of \( u \) are hyperplanes), at least if \( N \leq 8 \).

This conjecture was proved in 1997 for \( N = 2 \) by Ghoussoub and Gui \cite{8}, and in 2000 for \( N = 3 \) by Ambrosio and Cabré \cite{2}. In dimensions \( N \geq 9 \), del Pino, Kowalczyk, and Wei \cite{6} established that the conjecture does not hold, as suggested in De Giorgi’s original statement. The conjecture remains still open for dimensions \( 4 \leq N \leq 8 \).

In the proof of the conjecture for \( N \leq 3 \), it is used the following Liouville-type theorem due to H. Berestycki, L. Caffarelli and L. Nirenberg \cite{4}:

**Theorem 1.1.** Let \( \varphi \in L^\infty_{\text{loc}}(\mathbb{R}^N) \) be a positive function. Assume that \( \sigma \in H^1_{\text{loc}}(\mathbb{R}^N) \) satisfies \( \sigma \text{ div}(\varphi^2 \nabla \sigma) \geq 0 \) in \( \mathbb{R}^N \) in the distributional sense. For every \( R > 0 \), let \( B_R = \{|x| < R\} \) and assume that there exists a constant independent of \( R \) such that

\[
\int_{B_R}(\varphi \sigma)^2 dx \leq CR^2
\]

for every \( R \geq 1 \).

Then \( \sigma \) is constant.

To deduce the conjecture for \( N \leq 3 \) from this theorem, the authors made the following reasoning: if \( u \) is a solution in De Giorgi’s conjecture, consider the functions \( \varphi := \partial u/\partial x_N > 0 \) and \( \sigma_i := \partial_{x_i} u/\partial_{x_N} u \), for \( i = 1, \ldots, N - 1 \).

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Since both $\partial_x u$ and $\varphi$ solves the same linear equation $-\Delta v = (1 - 3u^2)v$, an easy computation shows that $\text{div}(\varphi^2 \nabla \sigma_i) = 0$. In dimensions $N \leq 3$ it is proved that there exists $C > 0$ such that $\int_{B_R} |\nabla u|^2 \, dx \leq CR^2$, for every $R \geq 1$. Applying Theorem 1.1 gives $\sigma_i$ is constant for every $i = 1, \ldots, N - 1$.

Motivated by the useful application of Liouville-type theorems to these kind of problems, a natural question is to find functions $0 < \Psi \in C([1, \infty))$, for which Theorem 1.1 remains true if we replace $CR^2$ with $\Psi(R)$. In this way, we make the following definitions:

**Property (P).** We say that a function $0 < \Psi \in C([1, \infty))$ satisfies (P) if it has the following property: if $\varphi \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ is a positive function, $\sigma \in H^1_{\text{loc}}(\mathbb{R}^N)$ satisfies

\begin{equation}
\sigma \text{ div } (\varphi^2 \nabla \sigma) \geq 0 \quad \text{in } \mathbb{R}^N
\end{equation}

in the distributional sense and

$$
\int_{B_R} (\varphi \sigma)^2 \, dx \leq \Psi(R) \quad \text{for every } R \geq 1,
$$

then $\sigma$ is necessarily constant.

**Property (P').** We say that a function $0 < \Psi \in C([1, \infty))$ satisfies (P') if it has the following property: if $\varphi \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ is a positive function, $\sigma \in H^1_{\text{loc}}(\mathbb{R}^N)$ satisfies

\begin{equation}
\text{div } (\varphi^2 \nabla \sigma) = 0 \quad \text{in } \mathbb{R}^N
\end{equation}

in the distributional sense and

$$
\int_{B_R} (\varphi \sigma)^2 \, dx \leq \Psi(R) \quad \text{for every } R \geq 1,
$$

then $\sigma$ is necessarily constant.

Note that, a priori, the definitions of properties (P) and (P') depend on the dimension $N$. We will show that, in fact, this is not so: if a function $0 < \Psi \in C([1, \infty))$ satisfies (P) (resp. (P')) in some dimension $N_0$, then it satisfies (P) (resp. (P')) in any dimension $N$.

It is obvious that property (P) is stronger than property (P'). In fact, in this paper we will prove that they are equivalent.

With this notation, Theorem 1.1 says that the function $CR^2$ satisfies (P) for every $C > 0$. In [1] the authors formulated the following problem: What is the optimal (maximal) exponent $\gamma_N$ such that $CR^{\gamma_N}$ $(C > 0)$ satisfies (P')?
In [3] it is proved that \( \gamma_N < N \) when \( N \geq 3 \). Also, a sharp choice in the counterexamples of [8] shows that \( \gamma_N < 2 + 2\sqrt{N-1} \) for \( N \geq 7 \). Recently, Moradifam [9] proved that \( \gamma_N < 3 \) when \( N \geq 4 \). Finally, in a recent work [12] the author has proven that \( \gamma_N = 2 \) for every \( N \geq 1 \). In other words, the functions \( CR_k \) do not satisfy (P') for every \( k > 2 \) and \( C > 0 \). On the other hand, the sharpness of the exponent 2 for condition (P) was proved by Gazzola [7].

Moschini [10] proved that \( CR_2(1 + \log R) \) satisfies (P) for every \( C > 0 \). By a classical example [11] it is obtained that \( R^2(1+\log R)^2 \) does not satisfy (P) in dimension \( N = 2 \).

All the results previously exposed are covered by the following theorems:

**Theorem 1.2.** Suppose \( 0 < \Psi \in C([1, \infty)) \). The following conditions are equivalent:

i) \( \Psi \) satisfies (P).

ii) \( \Psi \) satisfies (P').

iii) \( \int_1^{\infty} \frac{1}{h'} = \infty \), for every nondecreasing function \( 0 \leq h \in C([1, \infty)) \) satisfying \( h \leq \Psi \) in \([1, \infty)\).

Note that if \( 0 < \Psi \in C^1([1, \infty)) \) satisfy \( \Psi' > 0 \) in \([1, +\infty)\), the we can take \( h = \Psi \) in iii), obtaining that \( \int_1^{\infty} 1/\Psi' = \infty \) is a necessary condition to have i) and ii), but not sufficient (see Remark 2 below). The next theorem shows that, under convexity conditions on \( \Psi \), this is also a sufficient condition to obtain i) and ii).

**Theorem 1.3.** Suppose \( 0 < \Psi \in C^1([1, \infty)) \) satisfy \( \Psi' > 0 \) in \([1, +\infty)\) and \( \Psi \) is convex in \([R_0, +\infty)\) for some \( R_0 > 1 \). The following conditions are equivalent:

i) \( \Psi \) satisfies (P).

ii) \( \Psi \) satisfies (P').

iii') \( \int_1^{\infty} \frac{1}{\Psi'} = \infty \).

**Remark 1.** For general \( 0 < \Psi \in C([1, \infty)) \), it is possible to prove that if \( \liminf_{x \to \infty} \Psi(x)/x^2 < +\infty \) then \( \Psi \) satisfy (P) and (P'). Therefore, we can restrict our attention to the case \( \lim_{x \to \infty} \Psi(x)/x^2 = +\infty \). Thus, the condition of convexity of \( \Psi \) in Theorem 1.3 seems natural and not too restrictive.

To see that \( \liminf_{x \to \infty} \Psi(x)/x^2 < +\infty \) implies \( \Psi \) satisfy (P) and (P') we will apply Theorem 1.2. Suppose that there exists a divergent sequence \( \{ R_n \} \) and a real number \( C > 0 \) such that \( \Psi(R_n) \leq CR_n^2 \), \( n \geq 1 \) and take a nondecreasing function \( 0 \leq h \in C([1, \infty)) \) satisfying \( h \leq \Psi \) in \([1, \infty)\). Our purpose is to obtain \( \int_1^{\infty} 1/h' = \infty \). To this end, take an arbitrary \( R > 1 \) and consider \( n_0 \in \mathbb{N} \) such that \( R_n > R \) for every \( n \geq n_0 \). Then

\[
R_n - R = \int_R^{R_n} \sqrt{h'} \frac{1}{h'} \leq \left( \int_R^{R_n} h' \right)^{1/2} \left( \int_R^{R_n} \frac{1}{h'} \right)^{1/2}
\]
\[(h(R_n) - h(R))^{1/2} \left( \int_R^{R_n} \frac{1}{h'} \right)^{1/2} \leq (\Psi(R_n))^{1/2} \left( \int_R^{R_n} \frac{1}{h'} \right)^{1/2}\]

\[\leq C^{1/2} R_n \left( \int_R^{\infty} \frac{1}{h'} \right)^{1/2},\]

for every \( n \geq n_0 \). Hence

\[\int_R^{\infty} \frac{1}{h'} \geq \frac{(R_n - R)^2}{C R_n^2},\]

for \( n \geq n_0 \). Taking limit as \( n \) tends to infinity we deduce

\[\int_R^{\infty} \frac{1}{h'} \geq \frac{1}{C}.\]

Since \( R > 1 \) is arbitrary we conclude \( \int_1^{\infty} 1/h' = \infty \), which is the desired conclusion.

**Remark 2.** As said before, if \( 0 < \Psi \in C^1([1, \infty)) \) satisfy \( \Psi' > 0 \) in \( [1, +\infty) \), it is obvious from Theorem 1.2 that the condition \( \int_1^\infty 1/\Psi' = \infty \) is necessary to have i) and ii). To show that this is not sufficient, it suffices to construct functions \( \Psi, h \in C^\infty([1, \infty)) \) satisfying \( 0 < h < \Psi \) and \( 0 < \Psi', h' \) in \( [1, \infty) \) such that \( \int_1^\infty 1/\Psi' = \infty \) and \( \int_1^\infty 1/h' < \infty \).

To do this, for every integer \( n \geq 1 \) define the function \( f_n : [n, n+1/2] \to \mathbb{R} \) by

\[f_n(x) := 7nx + n^3, \quad n \leq x \leq n + 1/2.\]

Clearly

\[f_n(n+1/2) = 7n(n+1/2) + n^3 < f_{n+1}(n+1) = 7(n+1)^2 + (n+1)^3, \quad \text{for every } n \geq 1.\]

Hence, there exists \( 0 < \Psi \in C^\infty([1, +\infty)) \) satisfying \( \Psi' > 0 \) and \( \Psi(x) = f_n(x) \) for every \( n \leq x \leq n + 1/2 \) and \( n \geq 1 \).

It follows that

\[\int_1^\infty \frac{1}{\Psi'} \geq \sum_{n \geq 1} \int_n^{n+1/2} \frac{1}{\Psi'} = \sum_{n \geq 1} \frac{1}{14n} = \infty.\]

Then \( \int_1^\infty 1/\Psi' = \infty \). On the other hand, take an arbitrary \( x \geq 1 \). Then there exists an integer \( n \geq 1 \) such that \( n \leq x < n + 1 \). Thus

\[\Psi(x) \geq \Psi(n) = 7n^2 + n^3 \geq (n + 1)^3 > x^3.\]

Therefore, taking \( h(x) = x^3 \) we have \( \int_1^\infty 1/h' < \infty \), which is our claim.
2. Proof of Theorem 1.2

Proof of Theorem 1.2

It is evident that i) $\Rightarrow$ ii). Therefore we shall have established the theorem if we prove iii) $\Rightarrow$ i) and ii) $\Rightarrow$ iii).

Proof of iii) $\Rightarrow$ i)

Suppose that $0 < \Psi \in C([1, \infty))$ satisfies iii) and $\int_{B_R} (\varphi \sigma)^2 \leq \Psi(R)$ for every $R \geq 1$ where $\varphi \in L^\infty_\text{loc}(\mathbb{R}^N)$ is a positive function and $\sigma \in H^1_\text{loc}(\mathbb{R}^N)$ satisfies (1.1) in the distributional sense. Our purpose is to obtain that $\sigma$ is constant.

If $\inf \Psi = 0$ then there exists a divergent sequence $\{R_n\}$ such that $\Psi(R_n)$ tends to 0 as $n$ tends to $\infty$. Thus $\int_{B_{R_n}} (\varphi \sigma)^2$ also tends to 0, which implies $\sigma = 0$.

Otherwise, let $0 < m := \inf \Psi$ and consider the function

$$h(r) := \frac{1}{2} \int_{B_r} (\varphi \sigma)^2 + \frac{1}{2} m \left(1 - e^{-r}\right), \quad r \geq 1.$$ 

Clearly, $h \leq 1/2 \Psi + 1/2 m \leq \Psi$ in $[1, \infty)$ and $h$ is a positive continuous and nondecreasing function satisfying

$$h'(r) = \frac{1}{2} \left( \int_{|x|=r} (\varphi \sigma)^2 \right) + \frac{1}{2} me^{-r},$$

for almost every $r > 1$. From this $\int_1^\infty 1/h' = \infty$. Taking into account that $1/h'(r) \leq 2e^{r}/m$ for almost every $r > 1$ we have $1/h' \in L^\infty_\text{loc}([1, \infty))$. Thus

$$\int_R^\infty \frac{1}{h'} = \infty, \quad \text{for every } R > 1 \quad (2.1)$$

Now, for arbitrary $1 < R_1 < R_2$ define in the ball $B_{R_2}$ the radial function $\eta$ by

$$\eta(r) := \begin{cases} 
1 & \text{if } 0 \leq r \leq R_1 \\
\int_{R_1}^{R_2} \frac{1}{h'} & \text{if } R_1 < r \leq R_2 \\
\int_{|x|=r}^{R_2} \frac{1}{h'} & \text{if } R_2 \leq |x| \leq R_2 \end{cases}$$

for every $r = |x| \leq R_2$. Multiplying (1.1) by $\eta^2$ and integrating by parts in $B_{R_2}$, we obtain

$$\int_{B_{R_2}} \eta^2 \varphi^2 |\nabla \sigma|^2 \leq -2 \int_{B_{R_2}} \eta \varphi^2 \sigma \nabla \eta \cdot \nabla \sigma$$

$$\leq 2 \left( \int_{B_{R_2}} \eta^2 \varphi^2 |\nabla \sigma|^2 \right)^{1/2} \left( \int_{B_{R_2}} \varphi^2 |\nabla \eta|^2 \right)^{1/2}.$$
Therefore
\[ \int_{B_{R_2}} \eta^2 \varphi^2 |\nabla \sigma|^2 \leq 4 \int_{B_{R_2}} \varphi^2 \sigma^2 |\nabla \eta|^2. \]

Thus
\[ \int_{B_{R_1}} \varphi^2 |\nabla \sigma|^2 \leq \int_{B_{R_2}} \eta^2 \varphi^2 |\nabla \sigma|^2 \leq 4 \int_{B_{R_2}} \varphi^2 \sigma^2 |\nabla \eta|^2 \]
\[ = 4 \int_{R_1}^{R_2} \eta'(r)^2 \left( \int_{|x|=r} (\varphi \sigma)^2 \right) dr \leq 4 \int_{R_1}^{R_2} \eta'(r)^2 2h'(r) dr \]
\[ = \frac{8}{\left( \frac{\int_{R_1}^{R_2} \frac{1}{h'(r)^2} h'(r) dr}{\int_{R_1}^{R_2} \frac{1}{h'} h'} \right)^2} \int_{R_1}^{R_2} \frac{1}{h'(r)^2} h'(r) dr = \frac{8}{\int_{R_1}^{R_2} \frac{1}{h'}}. \]

Fix \( R_1 > 1 \). Applying \((2.1)\) and taking limit in the above inequality as \( R_2 \) tends to \( \infty \) we obtain
\[ \int_{B_{R_1}} \varphi^2 |\nabla \sigma|^2 = 0. \]

Since \( R_1 > 1 \) is arbitrary, \( \sigma \) is constant, which is the desired conclusion.

**Proof of ii) ⇒ iii)**

Suppose that iii) does not hold. That is, there exists a nondecreasing function \( 0 \leq h \in C([1, \infty)) \) satisfying \( h \leq \Psi \) in \([1, \infty)\) and \( \int_{1}^{\infty} \frac{1}{h'} < \infty \). The proof is completed by constructing a positive function \( \varphi \in L^\infty_{\text{loc}}(\mathbb{R}^N) \) and a nonconstant function \( \sigma \in H^1_{\text{loc}}(\mathbb{R}^N) \) satisfying \((1.2)\) in the distributional sense and \( \int_{B_R}(\varphi \sigma)^2 \leq \Psi(R) \) for every \( R \geq 1 \).

First of all, note that \( 0 < \lim_{r \to \infty} h(r) \leq \liminf_{r \to \infty} \Psi(r) \). Since \( \Psi > 0 \) in \([1, \infty)\), we have that \( 0 < m := \inf \Psi \). Consider the odd function \( \mu : \mathbb{R} \to \mathbb{R} \) such that
\[
\mu(r) := \begin{cases} 
\frac{m}{2}(1 - e^{-r}) & \text{if } 0 \leq r \leq 1 \\
\frac{1}{2} \int_{1}^{r} \min \{ h'(s), s^2 \} ds + \frac{m}{2}(1 - e^{-r}) & \text{if } 1 < r
\end{cases}
\]

Clearly \( \mu \) is continuous and increasing in \( \mathbb{R} \) and satisfies, almost everywhere, that
\[ \mu'(r) := \begin{cases} \frac{m}{2} e^{-|r|} & \text{if } 0 \leq |r| \leq 1 \\ \frac{1}{2} \min \{h'(|r|), r^2\} + \frac{m}{2} e^{-|r|} & \text{if } 1 < |r| \end{cases} \]

Therefore

\[ 0 < \frac{1}{\mu'(r)} < \frac{2}{\min \{h'(r), r^2\}} \leq \frac{2}{h'(r)} + \frac{2}{r^2}, \text{ for every } r > 1. \]

Hence \( 1/\mu' \in L^1(1, \infty) \) and it follows immediately \( 1/\mu' \in L^1(\mathbb{R}) \). For this reason, taking any \( 0 < H \in C^\infty(\mathbb{R}^{N-1}) \) satisfying \( \int_{\mathbb{R}^{N-1}} H^2 = 1/2 \), we can define the functions \( \varphi, \sigma : \mathbb{R}^N \to \mathbb{R} \) by

\[ \varphi(x_1, \ldots, x_N) := H(x_1, \ldots, x_{N-1}) \sqrt{\mu'(x_N)} \int_{x_N}^{+\infty} \frac{dr}{\mu'(r)}, \]

\[ \sigma(x_1, \ldots, x_N) := \frac{1}{\int_{x_N}^{+\infty} \frac{dr}{\mu'(r)}}. \]

(If \( N = 1 \), then define \( \varphi(x) = \sqrt{\mu'(x)} \int_{x}^{+\infty} \frac{dr}{\mu'(x)/\sqrt{2}} \) and we apply the same reasoning that in the case \( N > 1 \)).

It is easy to check that

\[ 0 < \mu'(r) \leq \frac{1}{2} r^2 + \frac{m}{2} e^{-|r|}, \quad \frac{1}{\mu'(r)} \leq \frac{2}{m} e^{|r|}, \quad r \in \mathbb{R}. \]

From the above it follows that \( 0 < \varphi \in L^\infty_{\text{loc}}(\mathbb{R}^N) \) and \( |\nabla \sigma| \in L^\infty_{\text{loc}}(\mathbb{R}^N) \). Thus \( \sigma \in H^1_{\text{loc}}(\mathbb{R}^N) \). Moreover, an easy computation shows that

\[ \nabla \sigma(x_1, \ldots, x_N) = \left( 0, \ldots, 0, \frac{1}{\mu'(x_N)} \left( \int_{x_N}^{+\infty} \frac{dr}{\mu'(r)} \right)^2 \right), \]

\[ (\varphi^2 \nabla \sigma)(x_1, \ldots, x_N) = (0, \ldots, 0, H^2(x_1, \ldots, x_{N-1})) \],

which implies \( \text{div}(\varphi^2 \nabla \sigma) = 0 \) in \( \mathbb{R}^N \).

Finally taking into account that \( B_R \subset \mathbb{R}^{N-1} \times (-R, R) \), we obtain for every \( R \geq 1 \)

\[ \int_{B_R} (\varphi \sigma)^2 \, dx = \int_{B_R} H^2(x_1, \ldots, x_{N-1}) \mu'(x_N) \, dx \]

\[ \leq \int_{\mathbb{R}^{N-1}} H^2 \, d(x_1, \ldots, x_{N-1}) \int_{-R}^R \mu'(r) \, dr = \frac{1}{2} (\mu(R) - \mu(-R)) = \mu(R) \]
\[ \frac{1}{2} \int_1^R h'(s) \, ds + \frac{m}{2} (1 - e^{-R}) \leq \frac{h(R)}{2} + \frac{m}{2} \leq \frac{\Psi(R)}{2} + \frac{\Psi(R)}{2} = \Psi(R), \]

which completes the proof. \(\Box\)

### 3. Proof of Theorem 1.3

**Proposition 3.1.** Let \( \phi \in C^1([a, b]) \) a convex function satisfying \( \phi' > 0 \) in \([a, b]\). Then

\[ \int_a^b \frac{1}{\phi'} \leq \int_a^b \frac{1}{g'} \]

for every nondecreasing function \( g \in C([a, b]) \) satisfying \( g(a) = \phi(a) \) and \( g \leq \phi \) in \([a, b]\).

Moreover, equality holds if and only if \( g = \phi \).

**Lemma 3.2.** Let \( g \in C([a, b]) \) a nondecreasing function. Let \( p(x) = Ax + B, A > 0, B \in \mathbb{R} \) such that \( g(a) = p(a), g(b) \leq p(b) \). Then

\[ \int_a^b \frac{1}{p'} \leq \int_a^b \frac{1}{g'}. \]

Moreover, equality holds if and only if \( g = p \).

**Proof.**

If \( \int_a^b 1/g' = \infty \) the lemma is trivial. Otherwise, applying Cauchy-Shwartz inequality we obtain

\[ b - a = \int_a^b \sqrt{g'} \frac{1}{\sqrt{g'}} \leq \left( \int_a^b g' \right)^{1/2} \left( \int_a^b \frac{1}{g'} \right)^{1/2} = (g(b) - g(a))^{1/2} \left( \int_a^b \frac{1}{g'} \right)^{1/2}. \]

Hence

\[ \int_a^b \frac{1}{g'} \geq \frac{(b - a)^2}{g(b) - g(a)} \geq \frac{(b - a)^2}{p(b) - p(a)} = \int_a^b \frac{1}{p'}. \]

On the other hand, if equality holds then all the previous inequalities become equalities. This implies that \( g(b) = p(b) \) and that \( \sqrt{g'} \) is a real multiple of \( 1/\sqrt{g'} \). That is, \( g' \) is constant and, since \( g(a) = p(a), g(b) = p(b) \), we obtain \( g = p \). \(\Box\)

**Lemma 3.3.** Let \( g \in C([a, b]) \) a nondecreasing function. For \( 1 \leq i \leq m \) consider \( p_i(x) = A_i x + B_i, A_i > 0, B_i \in \mathbb{R} \) such that \( p_i(a) \leq g(a) \). Define

\[ \overline{g}(x) := \max \{ g(x), p_1(x), p_2(x), ..., p_m(x) \}, \ a \leq x \leq b. \]

Then
Moreover, if $\int_a^b 1/g' < \infty$, then equality holds if and only if $g = \overline{g}$.

**Proof.**

Note that $\overline{g}$ is a nondecreasing continuous function in $[a, b]$. Therefore, the statement of the lemma has sense. If $\int_a^b 1/g' = \infty$ the lemma is trivial.

Hence, we will suppose in the rest of the proof that $\int_a^b 1/g' < \infty$. The proof is by induction on $m$.

We first prove the lemma for $m = 1$. To do this, consider the open set $G = \{x \in (a, b) : p_1(x) > g(x)\}$. If $G = \emptyset$, then $\overline{g} = g$ and the lemma follows. Otherwise, $G$ is the countable (possible finite) disjoint union of open intervals. That is, $G = \bigcup_{n \in X} (a_n, b_n)$, where $X \subset \mathbb{N}$ and $p_1(a_n) = g(a_n)$, $p_1(b_n) \geq g(b_n)$ for every $n \in X$. Then

$$
\int_a^b \frac{1}{g'} - \int_a^b \frac{1}{\overline{g}} = \int_G \left( \frac{1}{g'} - \frac{1}{p_1} \right) = \sum_{n \in X} \int_{a_n}^{b_n} \left( \frac{1}{g'} - \frac{1}{p_1} \right).
$$

Applying Lemma 3.2 in each interval $(a_n, b_n)$ we conclude the lemma for the case $m = 1$.

We now proceed by induction. Suppose that the lemma holds for $m - 1 \geq 1$ and we will prove that it holds for $m$. Define

$$h(x) := \max \{g(x), p_1(x), p_2(x), \ldots, p_{m-1}(x)\}, \ a \leq x \leq b.$$

By hypothesis of induction we have

(3.4) \[ \int_a^b \frac{1}{h'} \leq \int_a^b \frac{1}{g'} \, . \]

On the other hand, note that

$$\overline{g}(x) := \max \{g(x), p_1(x), p_2(x), \ldots, p_m(x)\} = \max \{h(x), p_m(x)\}, \ a \leq x \leq b.$$

It is easily seen that $h$ is a continuous nondecreasing function satisfying $p_m(a) \leq g(a) = h(a)$. Therefore applying the case of $m = 1$ (which is yet proved) to functions $h(x)$ and $p_m(x)$, we obtain

(3.5) \[ \int_a^b \frac{1}{\overline{g}} \leq \int_a^b \frac{1}{h'} \, . \]

Combining inequalities (3.4) and (3.5) we obtain the desired inequality (3.3). Finally, if equality holds in (3.3), then equalities also hold in (3.4) and (3.5). This gives $g = h = \overline{g}$ and the proof is completed. \qed
Proof of Proposition 3.1.

We first prove (3.1) in the case \( g(x) < \phi(x) \) for every \( x \in (a, b) \). To do this, for every positive integer \( n \), consider a partition of the interval \((a, b)\) in \(2^n\) subintervals of the same length. That is

\[
(a, b) = \bigcup_{k=1}^{2^n} [x_{k-1,n}, x_{k,n}]; \quad \text{where} \quad x_{k,n} = a + k \frac{b-a}{2^n}; \quad 0 \leq k \leq 2^n.
\]

Consider now the \( 2^n \) lines which are tangent to the graphic of the function \( y = \phi(x) \) at \( x_{k,n} \), \( 1 \leq k \leq 2^n \). That is

\[
p_{k,n}(x) := \phi'(x_{k,n})(x-x_{k,n}) + \phi(x_{k,n}), \quad a \leq x \leq b, \quad 1 \leq k \leq 2^n.
\]

Define

\[
g_n(x) := \max \{ g(x), p_{1,n}(x), p_{2,n}(x), \ldots, p_{2^n,n}(x) \}, \quad a \leq x \leq b.
\]

Note that the convexity of \( \phi \) gives \( g_n(x) \leq \phi(x) \) for every \( a \leq x \leq b \), \( n \geq 1 \).

We claim that \( g_n \rightarrow \phi \) in \( L^\infty(a, b) \) as \( n \rightarrow \infty \). To do this, take an arbitrary \( x \in (a, b) \). Then, for fixed \( n \geq 1 \), there exists \( 1 \leq k \leq 2^n \) such that \( x_{k-1,n} < x \leq x_{k,n} \). Using the convexity and monotonicity of \( \phi \) we deduce

\[
\phi(x) \geq g_n(x) \geq p_{k,n}(x) \geq p_{k,n}(x_{k-1,n}) = \phi'(x_{k,n})(x_{k-1,n} - x_{k,n}) + \phi(x_{k,n}) \geq \phi'(b)(x_{k-1,n} - x_{k,n}) + \phi(x) = -\phi'(b) \frac{b-a}{2^n} + \phi(x).
\]

This gives \( \| \phi - g_n \|_{L^\infty(a, b)} \leq \phi'(b) \frac{b-a}{2^n} \) and the claim is proved.

Now fix \( n_0 > 1 \) and consider \( a_0 = a + (b-a)/2^{n_0} \) and \( b_0 = b - (b-a)/2^{n_0} \).

Note that \( a_0 = x_{2^{n-n_0},n} \) and \( b_0 = x_{2^{n-n_0}-n,n} \) for every \( n \geq n_0 \). Since \( [a_0, b_0] \subset (a,b) \) and \( g < \phi \) in \((a,b)\), we deduce that there exists \( \varepsilon_0 > 0 \) (depending on \( n_0 \)) such that \( g(x) < \phi(x) - \varepsilon_0 \) for every \( x \in [a_0, b_0] \). Using \( g_n \rightarrow \phi \) in \( L^\infty(a_0, b_0) \) we can assert that there exists \( n_1 \geq n_0 \) (depending on \( \varepsilon_0 \)) such that \( g(x) < g_n(x) \) for every \( x \in [a_0, b_0] \) and \( n_1 \geq n_0 \). Then

\[
g_n(x) = \max \{ p_{1,n}(x), p_{2,n}(x), \ldots, p_{2^n,n}(x) \}, \quad a_0 \leq x \leq b_0, \quad n \geq n_1.
\]

Consider \( n \geq n_1 \) and \( 2^{n-n_0} < k \leq 2^{n} - 2^{n-n_0} \). Take \( x \in [x_{k-1,n}, x_{k,n}] \).

The convexity of \( \phi \) yields \( g_n(x) = \max \{ p_{k-1,n}(x), p_{k,n}(x) \} \) and consequently \( g_n'(x) \leq \phi'(x_{k,n}) \). This gives

\[
\int_{x_{k-1,n}}^{x_{k,n}} \frac{1}{g_n'} \geq \frac{x_{k,n} - x_{k-1,n}}{\phi'(x_{k,n})}.
\]

Therefore, applying Lemma 3.3 in the interval \([a,b]\) it follows that
\[ \int_{a}^{b} \frac{1}{g'} \geq \int_{a}^{b} \frac{1}{g_{n}} \geq \int_{a_{0}}^{b_{0}} \frac{1}{g_{n}} = \frac{2^{n-2^{n-n_{0}}} \int_{x_{k,n}}^{x_{k-1,n}} \frac{1}{g_{n}} \geq \sum_{k=2^{n-n_{0}+1}}^{2^{n-2^{n-n_{0}}}} \frac{x_{k,n} - x_{k-1,n}}{\phi'(x_{k,n})},} \]

for every \( n \geq n_{1} \). Since \( 1/\phi' \) is continuous in \([a_{0}, b_{0}]\) and \( x_{k,n} - x_{k-1,n} = (b - a)/2^{n} \) we deduce that the right term of the last inequality tends to \( \int_{a_{0}}^{b_{0}} 1/\phi' \) as \( n \) tends to \( \infty \). Thus,

\[ \int_{a}^{b} \frac{1}{g'} \geq \int_{a_{0}}^{b_{0}} \frac{1}{\phi'}. \]

Finally, since \( n_{0} > 1 \) is arbitrary we conclude (3.1) for the case \( g < \phi \) in \((a, b)\).

We now turn out to the general case \( g \leq \phi \) in \((a, b)\) and we proceed to show (3.1). For this purpose, consider the open set \( G = \{ x \in (a, b) : \phi(x) > g(x) \}\). If \( G = \emptyset \), then (3.1) is trivial. Otherwise, \( G \) is the countable (possible finite) disjoint union of open intervals. That is, \( G = \bigcup_{n \in X} (a_{n}, b_{n}) \), where \( X \subset \mathbb{N} \), \( \phi(a_{n}) = g(a_{n}) \), \( \phi(b_{n}) \geq g(b_{n}) \) and \( \phi > g \) in \((a_{n}, b_{n})\) for every \( n \in X \). Applying the previous case in each interval \((a_{n}, b_{n})\) we conclude

\[ \int_{a}^{b} \frac{1}{g'} - \int_{a}^{b} \frac{1}{\phi'} = \int_{G} \left( \frac{1}{g'} - \frac{1}{\phi'} \right) = \sum_{n \in X} \int_{a_{n}}^{b_{n}} \left( \frac{1}{g'} - \frac{1}{\phi'} \right) \geq 0. \]

It remains to prove that equality holds in (3.1) if and only if \( g = \phi \). To this end suppose that we have equality in (3.1) for some \( g \). Take an arbitrary \( x_{0} \in [a, b] \) and consider the function

\[ g_{x_{0}} := \max \{ g(x), \phi'(x_{0}) (x - x_{0}) + \phi(x_{0}) \}, \quad a \leq x \leq b. \]

Clearly \( g_{x_{0}} \) is nondecreasing and satisfies \( g \leq g_{x_{0}} \leq \phi \) in \([a, b]\) and \( g_{x_{0}}(a) = g(a) = \phi(a) \). Hence

\[ \int_{a}^{b} \frac{1}{g_{x_{0}}} \geq \int_{a}^{b} \frac{1}{\phi'} = \int_{a}^{b} \frac{1}{g'}. \]

Applying Lemma 3.3 yields \( g = g_{x_{0}} \) in \([a, b]\). In particular \( g(x_{0}) = g_{x_{0}}(x_{0}) = \max \{ g(x_{0}), \phi(x_{0}) \} = \phi(x_{0}) \). Since \( x_{0} \in [a, b] \) is arbitrary we conclude that \( g = \phi \) in \([a, b]\) and the proposition follows. \( \square \)

**Proof of Theorem 1.3**

Obviously, taking \( h = \Psi \) in Theorem 1.2 it follows immediately i) \( \Rightarrow \) ii) \( \Rightarrow \) iii').

It remains to prove iii') \( \Rightarrow \) i). Suppose \( \int_{1}^{\infty} \frac{1}{\Psi'} = \infty \). Using again Theorem 1.2 what is left is to show that \( \int_{1}^{\infty} \frac{1}{h'} = \infty \), for every nondecreasing function \( 0 \leq h \in C([1, \infty)) \) satisfying \( h \leq \Psi \) in \([1, \infty)\).
To obtain a contradiction suppose that there exists a nondecreasing function \( 0 \leq h \in C([1, \infty)) \) satisfying \( h \leq \Psi \) in \([1, \infty)\) and \( \int_1^\infty \frac{1}{h'} < \infty \). We first claim that \( \lim_{x \to \infty} \frac{h(x)}{x} = +\infty \). Conversely, suppose that there exist \( M > 0 \) and a divergent sequence \( \{R_n\} \) such that \( h(R_n) \leq MR_n \) for every positive integer \( n \). Applying Cauchy-Shwartz inequality we obtain

\[
R_n - 1 = \int_1^{R_n} \sqrt{R'} \frac{1}{\sqrt{h'}} \leq \left( \int_1^{R_n} h' \right)^{1/2} \left( \int_1^{R_n} \frac{1}{h} \right)^{1/2} \leq (M R_n)^{1/2} \left( \int_1^\infty \frac{1}{h'} \right)^{1/2},
\]

which contradicts that \( \{R_n\} \) diverges.

Consequently there exists \( R_1 := \min \{ R \geq R_0 : h(R) = \Psi'(R_0)(R - R_0) + \Psi(R_0) \} \).

For every \( R > R_1 \) define \( g_R : [R_0, R] \to \mathbb{R} \) by

\[
g_R(x) := \begin{cases}
\Psi'(R_0)(x - R_0) + \Psi(R_0) & \text{if } R_0 \leq x \leq R_1 \\
h(x) & \text{if } R_1 < x \leq R
\end{cases}
\]

It is easily seen that \( g_R \in C([R_0, R]) \) is a nondecreasing function satisfying \( g_R(R_0) = \Psi(R_0) \) and \( g_R \leq \Psi \) in \([R_0, R]\). Then we can apply Proposition 3.1 in the interval \([R_0, R]\) and obtain

\[
\int_{R_0}^R \frac{1}{\Psi'} \leq \int_{R_0}^R \frac{1}{g_R}.
\]

Hence, for arbitrary \( R > R_1 \), we have

\[
\int_{R_1}^R \frac{1}{h'} = \int_{R_0}^R \frac{1}{g'_R} - \int_{R_0}^{R_1} \frac{1}{g'_R} \geq \left( \int_{R_0}^R \frac{1}{\Psi'} \right) - \left( \frac{R_1 - R_0}{\Psi'(R_0)} \right).
\]

Since \( \int_{R_0}^\infty \frac{1}{\Psi'} = \infty \), we can take limit as \( R \) tends to infinity, obtaining

\[
\int_{R_1}^\infty \frac{1}{h'} \geq +\infty. \quad \text{This contradicts our assumption } \int_1^\infty \frac{1}{h'} < \infty.
\]

REFERENCES

[1] G. Alberiti, L. Ambrosio, X. Cabré, On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property, Acta Appl. Math. 65 (2001), 9-33.

[2] L. Ambrosio, X. Cabré, Entire solutions of semilinear elliptic equations in \( \mathbb{R}^3 \) and a conjecture of De Giorgi, J. Amer. Math. Soc. 13 (2000), 725-739.

[3] M. T. Barlow, On the Liouville property for divergence form operators, Canad. J. Math. 50 (1998), 487-496.

[4] H. Berestycki, L. Caffarelli, L. Nirenberg, Further qualitative properties for elliptic equations in unbounded domains, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 25 (1997), 69-94.
[5] E. De Giorgi, *Convergence problems for functionals and operators*, Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), Pitagora, Bologna, 1979, 131-188.

[6] M. del Pino, M. Kowalczyk, J. Wei, *On De Giorgi’s conjecture in dimension $N \geq 9$*, Ann. of Math. 174 (2011), 1485-1569.

[7] F. Gazzola, *The sharp exponent for a Liouville-type theorem for an elliptic inequality*, Rend. Istit. Mat. Univ. Trieste 34 (2002), 99-102.

[8] N. Ghoussoub, C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann. 311 (1998), 481-491.

[9] A. Moradifam, *Sharp counterexamples related to the De Giorgi conjecture in dimensions $4 \leq n \leq 8$*, Proc. Amer. Math. Soc. 142 (2014), 199-203.

[10] L. Moschini, *New Liouville theorems for linear second order degenerate elliptic equations in divergence form*, Ann. Inst. H. Poincar Anal. Non Lineaire 22 (2005), 11-23.

[11] M. H. Protter, H. F. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, N.J. (1967).

[12] S. Villegas, *Optimal power in Liouville theorems*, preprint, arXiv: 2003.04400 (2020).

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