A NEW APPROACH FOR THE COMPUTATION OF THE TAME DEGREE

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ABSTRACT. In this paper we present a new procedure to compute the tame degree of an affine semigroup that gives rise to a faster algorithm for full affine semigroups.

Let $M$ be an affine monoid, that is, a finitely generated submonoid of $\mathbb{N}^e$ for some positive integer $e$. An element $a$ in $M$ is an atom if it cannot be expressed as the sum of two nonzero elements of $M$. It is well known that $M$ is atomic (every element in $M$ can be expressed as a finite sum of atoms; see for instance [11]), and the atoms of $M$ are precisely $M^* \setminus (M^* + M^*)$, where $M^* = M \setminus \{0\}$. Moreover, this set is finite (see [14]).

Let $A = \{m_1, \ldots, m_k\}$ be the set of atoms of $M$, and let $A$ be the $e \times k$-matrix whose columns are the elements of $A$. Then the map

$$\varphi : \mathbb{N}^k \to M, \quad \varphi(a_1, \ldots, a_k) = a_1m_1 + \cdots + a_km_k = A(a_1, \ldots, a_k)^T$$

is surjective (and known as the factorization homomorphism of $M$). Fixed $m \in M$, its set of factorizations is

$$Z(m) = \varphi^{-1}(m) = \{(a_1, \ldots, a_k) \mid a_1m_1 + \cdots + a_km_k = m\}.$$

By Dickson’s lemma, this set has always finitely many elements. We will also write $Z(X) = \bigcup_{x \in X} Z(x)$ for any subset $X$ of $M$.

For a factorization $z = (z_1, \ldots, z_k)$ of $m$, its length is defined as $|z| = z_1 + \cdots + z_k$, which is the number of atoms involved in this factorization. Given two factorizations $z = (z_1, \ldots, z_k)$ and $z' = (z'_1, \ldots, z'_k)$, define

$$z \wedge z' = (\min(z_1, z'_1), \ldots, \min(z_k, z'_k)),$$

which is the infimum of $z$ and $z'$ with respect to the usual partial ordering in $\mathbb{N}^k$, and it is the translation of greatest common divisor to additive notation.

Let $e_i$ be the $i$th row of the $k \times k$ identity matrix. Given $m \in M$ and $m_i \in A$ such that $m - m_i \in M$ (equivalently, $m \in m_i + M$), the tame degree of $m$ with respect to $m_i$ is defined as the least nonnegative integer $t$ such that for every $z \in Z(m)$, there exists $z' \in Z(m)$ with $z' - e_i \in \mathbb{N}^k$ (or in other words, the $i$th coordinate of $z'$ is nonzero) and $d(z, z') \leq t$. We will denote this invariant by $t(m, m_i)$.

The tame degree of $M$ with respect to $m_i$, $t(M, m_i)$, is then defined as the supremum (maximum in this setting, [6]) of all the tame degrees of the elements of $m_i + M$ with respect to $m_i$.

The tame degree of $M$, $t(M)$, is the maximum of the tame degrees of $S$ with respect to all the atoms. For more properties of this and other nonunique factorization invariants, the interested reader can use [11].

The tame degree of $M$ can be computed by means of the tame degrees of the elements associated to a Graver basis of $A$. We propose next an alternative method based on the computation of minimal factorizations of elements of principal ideals of $M$.

Fixed $m_i \in A$, set $M_i = \text{Minimals} \leq Z(m_i + M)$ and $M_i = \{\varphi(z) \mid z \in M_i\}$, which again, by Dickson’s lemma, has finitely many elements.

**Theorem 1.** Let $M$ be an affine semigroup minimally generated by $A = \{m_1, \ldots, m_k\}$. Then

$$t(M, m_i) = \max\{t(m, m_i) \mid m \in M_i\}.$$
Proof. Since \( M_i \subseteq M \), \( \max\{t(m, m_i) \mid m \in M_i\} \leq t(M, m_i) \).

For the other inequality, take \( n \in m_i + M \) and \( z \in \mathbb{Z}(n) \). If the \( i \)th coordinate of \( z \) is zero, then choose \( z' = z \) in the definition of tame degree. So assume that \( z \cdot e_i = 0 \) (dot product). As \( n \in m_i + M \), \( z \in \mathbb{Z}(m_i + M) \), and consequently, there exists \( x \in M_i \) such that \( x \leq z \) (with respect to the usual partial ordering). Let \( y = z - x \), which is in \( \mathbb{N}^k \). And set \( m = \varphi(x) \). By definition of tame degree, \( m \) follows that \( t \left( x, x' \right) \leq t(m, m_i) \) and \( z' = z' + y \). Then \( d(z, z') = d(x, x') \leq t(m, m_i) \) and also \( z' - e_i = (z' - e_i) + y \in \mathbb{N}^k \). By definition of tame degree, it follows that \( t(n, m_i) \leq t(m, m_i) \). This implies that \( t(n, m_i) \leq \max\{t(m, m_i) \mid m \in M_i\} \), and consequently \( t(M, m_i) \leq \max\{t(m, m_i) \mid m \in M_i\} \).

It is not a coincidence that the same elements needed to compute the \( \omega \)-primality of \( M \) are the same as those needed for the tame degree (compare Theorem 1 with [3, Proposition 3.3]; in that paper the reader can also find the definition of \( \omega \)-primality). We already had evidences of this behavior for numerical semigroups: see [3, Corollary 3] and [3, Remarks 5.9].

In order to compute \( M_i \) one can use \([15\] Algorithm 16\). This is precisely the idea exploited in \([10\) to compute the \( \omega \)-primality. However it turns out that using \texttt{Normaliz} \([4\] to perform this task is faster. In order to compute the minimal elements in \( M_i \), we first calculate the set of minimal nonnegative integer solutions of

\[
\begin{pmatrix}
A & -A
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} = m_i,
\]

and then project on the first \( k \) coordinates, where \((A | -A)\) is the matrix having the columns of \( A \) followed by the columns of \(-A\). From the (finite) resulting set we just take those that are minimal with respect to \( \leq \) in \( \mathbb{N}^k \). Observe that computing a Graver basis of \( A \) is equivalent to finding the set of nonnegative integer minimal solutions of

\[
(A | -A) \begin{pmatrix} x \\ y \end{pmatrix} = 0.
\]

Hence so far, apart from the theoretical result, it seems that we have not gained much with respect to the procedure proposed in \([6\]. A big improvement can be done for full affine semigroups, mainly because in this setting we can use \([3\ Corollary 3.5\]). Let \( G(M) \) be the subgroup of \( \mathbb{Z}^e \) spanned by \( M \). We say that \( M \) is \textit{full} (sometimes known as saturated in the literature) if \( M = \mathbb{N}^e \cap G(M) \). As a particular case of \([3\ Corollary 3.5\], we get the following.

**Proposition 1.** Let \( M \) be a full affine semigroup minimally generated by \( \{m_1, \ldots, m_k\} \), and assume that these elements are the columns of the matrix \( A \). Then

\[
\mathcal{M}_i = \text{Minimals}_{\leq} \left\{ x \in \mathbb{N}^k \mid Ax \geq m_i \right\}.
\]

Let \( G \) be a finitely generated commutative monoid. Let \( H = \{g_1, \ldots, g_n\} \) be a subset of \( G \). The \textit{block monoid} associated to \( H \), denoted \( B(H) \), is the set of sequences \((x_1, \ldots, x_n)\) such that \( x_1 g_1 + \cdots + x_n g_n = 0 \). It is easy to show that \( B(H) \) is a full affine semigroup. Many factorization properties of a monoid can be derived from the factorization properties of the block monoid associated to its class group (see \([11\]). This is why it is worth having dedicated algorithms for this family of affine semigroups.

**Example 1.** We have implemented the procedure presented in this manuscript to compute the tame degree of a full affine semigroup. Let us apply this function to compute the tame degree of \( B(\mathbb{Z}_2^3) \). Let \( B \) be the matrix with columns the nonzero elements of \( \mathbb{Z}_2^3 \). Then \( B(\mathbb{Z}_2^3) \) corresponds with the submonoid of \( \mathbb{N}^7 \) of nonnegative integer solutions of the system of equations

\[ BX \equiv 0 \mod 2. \]

The definition of the affine semigroup in the \texttt{numericalsgps} \([7\] \texttt{GAP} \([9\]) package can be done as follows.

```
gap> c:=[[0, 0, 1, 0, 1, 0, 0], [1, 0, 1, 0, 0, 1, 0], [0, 1, 0, 0, 0, 1, 0], [0, 0, 1, 1], [1, 1, 1, 1, 1, 1, 1]];  
gap> a:=AffineSemigroup("equations",[TransposedMat(m),[2,2,2]]);  
<Affine semigroup>
```
We can compute its atoms by typing.

\begin{verbatim}
gap> at:=GeneratorsOfAffineSemigroup(a);
[ [ 0, 0, 0, 0, 0, 2, 0 ], [ 0, 0, 0, 0, 2, 0, 0 ],
  [ 0, 0, 0, 2, 0, 0, 0 ], [ 0, 2, 0, 0, 0, 0, 0 ],
  [ 2, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 1, 0, 1, 0, 1 ],
  [ 0, 1, 0, 0, 0, 1, 0 ], [ 0, 1, 0, 0, 1, 0, 0 ],
  [ 1, 0, 0, 0, 0, 1, 0 ], [ 1, 0, 0, 1, 0, 0, 1 ] ]

Now we can compute its tame degree with the following instruction.

\begin{verbatim}
gap> TameDegreeOfAffineSemigroup(a);
4
\end{verbatim}

We use the package NormalizInterface ([12]) that uses the library of Normaliz to compute the set of minimal nonzero nonnegative solutions of systems of Diophantine equations (Hilbert bases). We can also use 4ti2gap ([2]) that utilizes the 4ti2 ([1]) library for Graver and Hilbert bases computations, but in this example it is slower than Normaliz.

The computation of \( t(\mathcal{B}(\mathbb{Z}_2^3)) \) took 1048757 milliseconds on an i7 laptop with 16GB of memory (Normaliz was compiled without OpenMP, [13], and thus it is running in a single thread). The function TameDegreeOfAffineSemigroup is an implementation of the procedure described in [6]. If we combine Theorem [1] and Proposition [1] then the same output is achieved in 4955 milliseconds. So the speed up is considerable.

There is still another improvement. According to [8], in order to compute \( t(M, u) \) we only have to compute \( |u| = r + 1 \).

\begin{verbatim}
gap> u:=First(at,x->Sum(x)=4);
[ 0, 0, 0, 1, 1, 1, 1 ]

gap> Mu:=minimalElementsPrincipalIdealOfFullAffineSemigroup(u,a);

gap> facts:=List(Mu, x->FactorizationsVectorWRTList(x,at));;

gap> Set(facts,TameDegreeOfSetOfFactorizations);
[ 0, 2, 3, 4 ]
\end{verbatim}

The function minimalElementsPrincipalIdealOfFullAffineSemigroup is just an implementation of Proposition [1].

It takes 2 milliseconds to compute at, the set of atoms of \( \mathcal{B}(\mathbb{Z}_2^3) \). Then 12 to compute Mu. Another 420 milliseconds are required for the calculation of all the factorizations of the elements in Mu. Finally the tame degrees of these sets of factorizations are calculated in 425 milliseconds. This means that with this approach, in less than a second we get the tame degree of \( \mathcal{B}(\mathbb{Z}_2^3) \).

The cardinality of the set of atoms of \( \mathcal{B}(\mathbb{Z}_2^3) \) is 323, and the number of atoms of \( \mathcal{B}(\mathbb{Z}_2^5) \) is 20367. Though there is a considerable speed up compared with the general-purpose algorithm introduced in [6], it is not yet enough to afford the problem of computing \( t(\mathcal{B}(\mathbb{Z}_2^5)) \) with an i7 laptop in a reasonable amount of time; even we would eventually run out of memory.

Some of the numericalsgps functions used in this example are not available yet in the stable version, but can be obtained via \url{https://bitbucket.org/gap-system/numericalsgps}.

**Example 2.** The atoms of \( M = \mathcal{B}(\mathbb{Z}_2 \times \mathbb{Z}_3) \) are

\[
\{(0,0,1,1),(0,0,2,0),(1,1,0,0),(1,0,0,2),(0,0,0,0),(1,0,1,0),(0,1,0,0,2),
(0,1,1,1,0),(0,3,0,0,0),(2,0,0,0),(0,0,1,3),(2,0,1,1,0),(0,0,1,0,3),(2,0,2,0,0),
(2,0,1,0,1),(1,0,0,0,4),(0,1,0,4,0),(0,0,0,6,0),(0,0,0,0,6)\}.
\]

And \( t(M) = 8 \) (19214 milliseconds; process killed after several hours with the procedure presented in [6]).

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