Universal distribution of threshold forces at the depinning transition

Andrei A. Fedorenko, Pierre Le Doussal and Kay Jörg Wiese
CNRS-Laboratoire de Physique Théorique de l’École Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex, France

(Dated: July 10, 2006)

We study the distribution of threshold forces at the depinning transition for an elastic system of finite size, driven by an external force in a disordered medium at zero temperature. Using the functional renormalization group (FRG) technique, we compute the distribution of pinning forces in the quasi-static limit. This distribution is universal up to two parameters, the average critical force, and its width. We discuss possible definitions for threshold forces in finite-size samples. We show how our results compare to the distribution of the latter computed recently within a numerical simulation of the so-called critical configuration.

I. INTRODUCTION

The dynamics of elastic objects driven by an external force in disordered media has attracted considerable theoretical and experimental interest during the last years [1,2,3]. The reason is twofold. On one hand, elastic objects in disordered media exhibit the rich behavior of glassy systems and thus their study can help us to understand the physics of more complex systems such as spin glasses [4] or random field systems [5]. On the other hand, the motion of elastic objects in disordered media is an adequate description of many experimental systems. One can divide these systems into two classes. The first class comprises periodic systems, the most prominent examples being charge density waves (CDW) in solids. These start sliding when the applied electric field becomes large enough [6]. Vortex lines in disordered superconductors form a quasi-ordered periodic Bragg glass phase [7,8]. The second class includes propagating interfaces such as domain walls in magnetically or structurally ordered systems [9], interfaces between immiscible fluids in porous media [10] or dislocation lines in metals [11]. To unify the mathematical description of these different systems one uses the notion of “manifolds”. In all these systems the interplay between quenched disorder and elasticity leads to a complicated response of the system to an applied external force. At zero temperature, a driving force \( f > f_c \) exceeding a certain threshold value \( f_c \) is required to set the elastic manifold into motion. This depinning transition shares many features with critical phenomena [12]: characteristic lengths diverge close to the transition as \( \xi \sim (f - f_c)^{-\nu} \) and the system becomes extremely sensitive to small perturbations. Following the description of standard critical phenomena, one can identify the ordered phase with the moving phase with force \( f > f_c \), and the order parameter with the velocity \( v \) which vanishes as \( v \sim (f - f_c)^{\nu/\beta} \) at the transition. One also introduces the dynamic exponent \( z \) which relates time and space by \( t \sim x^z \). The critical force \( f_c \), which must be tuned to reach the scale-invariant regime, plays a role similar to the critical temperature in thermal phase transitions. There are many subtleties however within this analogy, since depinning is a non-equilibrium transition at zero temperature, where quenched disorder dominates. As the corresponding static problem of elastic manifolds in disordered media [13], the depinning problem suffers from two peculiarities, when compared to standard critical phenomena: First an infinite set of operators becomes relevant simultaneously be-
temperature) where violation of FSS was found \[^{26}\]. In the
depinning problem, one difficulty is to define properly the crit-
ical force and its fluctuations in the limit of large interface (in-
ternal) size \(L\). A recent and efficient algorithm \[^{27}\] allows to
obtain exactly the critical force \(f_c\) of an interface in a periodic
medium of period \(M\) (i.e. a cylinder), as well as the so-called
critical configuration. The latter is defined as the last block-
ing configuration as \(f\) is increased up to \(f_c\), which also defines
\(f_c = f_c(L, M)\). One can refer to this definition as an extremal
configuration in a given sample. The finite-size sample-to-
sample distribution of \(f_c(L, M)\) was computed numerically
\[^{28}\] and found to depend on the aspect ratio \(k = M/L^d\) of
the cylinder. This should be expected since for large \(k\) one
recovers a zero-dimensional problem and the interface will be
blocked by rare disorder configurations, hence dominated by
extremal statistics. However this results seems to depend on
the precise definition and one may ask whether a more fun-
damental definition exists, with no need to specify a value
for \(k\). An alternative approach is to define the observables
at the depinning transition as the time average in the mov-
ing state at fixed velocity \(v\), in the limit \(v \to 0^+\). This def-
nition, to which we refer as the quasi-static depinning limit,
is usually associated to the FRG approach of the depinning
transition. Observables calculated in this approach must, a
priori, be distinguished from those computed in the critical
configuration. Since the time average is usually performed
in a steady state, to avoid dominance by history dependence,
it also requires specifying boundary conditions. It is widely
believed that both approaches give the same result, at least
for \((N = 1)\) component interfaces, since in the limit of in-
finitesimal systems \((L \to \infty)\) all quasi-static configurations should
have the same statistical properties and the critical force \(f_c\)
should be self-averaging. However it is less clear how these
approaches compare when applied to finite-size fluctuations
where the dispersion in local pinning forces becomes impor-
tant.

In the present paper we study the distribution of the thresh-
old forces by means of the functional renormalization group.
Within the field theory we propose two definitions of the criti-
cal force \(f_c(L)\) in finite size \(L\) and show that they are identical
to one loop in the renormalized theory. We compute the cu-
mulants of \(f_c(L)\) and extract the distribution which is found
to be universal, up to a shift in \(f\) (the critical force \(f_c\)) and one
scale-parameter (the width of the distribution). All results are
valid within the \(\epsilon = 4 - d)\) expansion and extrapolations to
low dimension are discussed. The critical force studied here is
defined from a fixed center of mass ensemble. As we point out
it can be, in principle, obtained in numerics. Since it does not
refer to any transverse size \(M\), it is more fundamental than the
one used in the numerical studies on a cylinder. We discuss
how the latter one can in principle be recovered.

The paper is organized as follows. Section II introduces
the model and the FRG treatment of the depinning transition.
In Section III we compute the bare distribution of threshold
forces to one-loop order using an improved perturbation the-
yory, and renormalize it for the case of an elastic interface. In
Section IV we discuss the renormalization for periodic sys-
tems. In Section V we discuss the relation between the dis-
tribution of critical forces, in the quasi-static limit, and in the
critical configuration.

II. MODEL AND FRG DESCRIPTION

Let us consider the motion of a one-component elastic man-
ifold with short-range elasticity. The configuration of the
manifold can be described by a scalar displacement field \(u_{xt}\),
where \(x\) denotes the \(d\) dimensional internal coordinate of the
manifold. We study the over-damped dynamics of a manifold
in the disordered medium which obeys the following equation of
motion

\[
\eta \partial_t u_{xt} = c \nabla^2 u_{xt} + F(x, u_{xt}) + f,
\]

where \(\eta\) is the bare friction and \(c\) the elasticity. The quenched
random force \(F(x, u)\) can be chosen Gaussian with zero mean
and variance

\[
\overline{F(x, u) F(x', u')} = \Delta(u - u') \delta^d(x - x').
\]

For periodic systems the function \(\Delta(u)\) is periodic, while for
interfaces it decays exponentially for large \(u\). In the latter
case, in contrast to the statics, at depinning both random bond
(RB) and random field (RF) microscopic disorder renormalize
to the same fixed point, which has RF characteristics, so that
we can restrict ourselves to the latter case. \(a\) is the width of the
function \(\Delta(u)\). To make the problem well-defined we imply
an UV cutoff at scale \(\Lambda^{-1}\). We consider a finite system of size
\(L\) with periodic boundary conditions. The size \(L\) serves as an
IR cutoff i.e. it plays the role of the mass in the corresponding
field theory. One can easily see that due to the tilt symmetry
the elastic constant remains uncorrected to all orders so that
we are free to fix \(c = 1\).

Below, starting in Section III, we will find it convenient to
work in the comoving frame. To that end we shift \(u_{xt} \to v t + u_{xt}\),
\(\{u_{xt}\} = 0\) and \(f \to f - \eta v\), where \(v = L^{-d}(\int_x \partial_t u_{xt})\)
is the velocity of the center of mass. Here the angular brack-
ets stand for the average over different initial configurations
(since we are studying zero temperature dynamics) and the
overline denotes the average over disorder distribution. We
will assume that a steady state attractor has been reached,
hence that averages depend only on time differences and not
on a specific choice of initial conditions.

To study the dynamics of an elastic manifold efficiently, we
use the formalism of generating functionals. Introducing the
response field \(\tilde{u}_{xt}\) one can compute the average of the observ-
able \(A[u_{xt}]\) over dynamic trajectories with different initial
conditions for a particular disorder configuration as follows

\[
\langle A[u_{xt}] \rangle = \int \mathcal{D}[u] \mathcal{D}[\tilde{u}] A[u_{xt}] e^{-S_F[u, \tilde{u}]}.
\]

\(S_F\) is the action for a particular realization of the disorder (a
particular sample). To compute the average of the observables
which explicitly depends on the random force at the position
of the manifold we introduce the source \( J_{xt} \) for the random force \( F \) so that the corresponding action reads

\[
S_F[u, \tilde{u}] = \int_{xt} i\tilde{u}_{xt}(\eta\partial_t - \nabla^2)u_{xt} - \int_{xt} i\tilde{u}_{xt}\{F(x, u_{xt}) + f_{xt}\} - \int_{xt} J_{xt}F(x, u_{xt}).
\]

(4)

After averaging over the disorder distribution, any observable which depends on the displacement field and the random force at the position of the manifold can be computed as follows

\[
\left\langle A[u_{xt}]F(x_1, u_{xt}, t_1)\ldots F(x_n, u_{xt}, t_n) \right\rangle = \frac{\delta}{\delta J_{xt, 1} \ldots \delta J_{xt, t_n}} \left[ \int D[u] D[\tilde{u}] A[u_{xt}] e^{-S[u, \tilde{u}]} \right]_{J=0},
\]

\( S[u, \tilde{u}] \) is the effective action, which can be split into two parts: the free part \( S_0 \) being quadratic in fields and the interaction part \( S_{int} \) containing all non-linear terms

\[
S_0 = \int_{xt} i\tilde{u}_{xt}(\eta\partial_t - \nabla^2)u_{xt} - \int_{xt} i\tilde{u}_{xt}f_{xt},
\]

\[
S_{int} = \frac{1}{2} \int_{xtt'} (i\tilde{u}_{xt} + J_{xt})(\Delta(u_{xt} - u_{xt'})) (i\tilde{u}_{xt'} + J_{xt'}).
\]

Setting \( J_{xt} = 0 \) we recover the action used in Refs. [16,17]. The quadratic part \( S_0 \) gives the free response

\[
\left\langle u_{xt}, i\tilde{u}_{xt=0} \right\rangle = R_{q,t} = \frac{\Theta(t)}{\eta} e^{-q^2\eta t/\eta},
\]

(5)

while the free correlation function is \( C_{q,t} = \left\langle u_{q,t} u_{q,0} \right\rangle = 0 \) at zero temperature. The splitted diagrammatics for the perturbation theory in disorder \( \Delta \) was developed in Refs. [16,17]. It is known that naive perturbation theory, obtained by taking for \( \Delta \) an analytic function exhibits the property of dimensional reduction and fails to describe the physics, giving for example an incorrect roughness exponent. The physical reason for this is the existence of a large number of metastable states.

Let us briefly sketch the FRG analysis of the system under consideration. Power counting shows that the whole function \( \Delta(u) \) becomes relevant below \( d_{ac} = 4 \) and thus one has to renormalize the whole function. To extract the scaling behavior one has to study the flow of the renormalized function \( \Delta \) under changing the IR cutoff towards infinity. Various choices for the IR cutoff were discussed in Refs. [16,17]. A convenient choice is to add a small mass \( m \), so that the scaling behavior can be extracted from the effective action \( I[u, \tilde{u}] \) of the theory as \( m \) decreases to zero. To study the finite-size distribution of threshold forces, we use, as in [23], the system size \( L \) as the natural IR cutoff. Then any integral over momentum \( q \) has to be replaced by the sum according to the rule \( \int_q \to \sum_{q^2 \leq \Lambda^2} \), where the sum runs over all \( q = 2\pi k/L \), \( k \in \mathbb{Z}^d \), \( k \neq 0 \). Exclusion of the zero mode means that we are working in an ensemble of fixed center of mass, a point further discussed in Section \( \mathbb{V} \).

Let us define the rescaled disorder as

\[
\Delta(u) = \frac{1}{\varepsilon I_1} L^{2\zeta - \epsilon} \Delta(u L^{-\zeta}),
\]

(6)

where \( I_1 = L^\nu I_1 = \int |q|^{-2} \) is the one-loop integral. It was shown in Refs. [14,15] that the FRG equation, i.e. the flow equation for the running disorder correlator can be written to one-loop order as

\[
L\partial_L \tilde{\Delta}(u) = (\varepsilon - 2\zeta) \tilde{\Delta}(u) + \zeta u \tilde{\Delta}'(u) - \frac{1}{2} \left( \tilde{\Delta}(u) - \tilde{\Delta}(0) \right)^2,
\]

(7)

the two loop flow equation being obtained in Refs. [16,17]. “0” means a derivative at fixed bare quantities. The flow of the correlator is such that \( \Delta(u) \) acquires a cusp at the origin \( u = 0 \) at the Larkin scale \( L_c \simeq (c^2 a^2/\Delta(0))^{1/\epsilon} \). Beyond the Larkin scale \( (L > L_c) \) the renormalized correlator is singular and perturbation theory breaks down. Nevertheless, the flow tends to a non-trivial fixed-point (FP) solution \( \tilde{\Delta}'(u) \) with a new value for the roughness exponent which controls the large-scale behavior. There are two FPs which describe interfaces and periodic systems correspondingly. The former FP has \( \zeta = \varepsilon/3 + O(\epsilon^2) \), while the latter one has \( \zeta = 0 \) due to periodicity. The renormalization of the mobility gives the value of the dynamic exponent

\[
z = 2 + L \frac{d}{dL} \eta_R \bigg|_{L=0} = 2 - \tilde{\Delta}''(0) + O(\tilde{\Delta}^2),
\]

(8)

where \( \eta_R \) is the renormalized mobility. Other critical exponents can be computed using the scaling relations

\[
\nu = \frac{1}{2 - \zeta} = \frac{\beta}{z - \zeta},
\]

(9)

To renormalize the theory, one needs an additional counter-term for the excess force \( f - \eta \nu \), which comes with an UV divergence \( \sim \Lambda^2 \). This term is analogous to the critical temperature shift in the \( \varphi^4 \) theory, and gives the critical threshold force \( f_c^* \). It is zero in the bare theory. We expect that in the limit of an infinite system \( L \to \infty \), the critical force becomes sample independent if there is the same distribution of disorder in each sample and thus \( \lim_{L \to \infty} P_L(f) = \delta(f-f_c^*) \). However the situation is different for finite systems. According to the general theorem for random systems [22] there exists a finite-size scaling correlation length \( \xi_{FS} \) which characterizes the distribution of the observables in an ensemble of samples and which in principle has to be distinguished from the intrinsic correlation length \( \xi \) which enters into correlation functions. Approaching the critical point, the finite-size correlation length diverges similar to the intrinsic correlation length which enters into correlation functions. Therefore it is necessary to consider the full distribution of all sample independent forces.

\[
\xi_{FS} \sim |f - f_c|^\nu_{FS}
\]

In general \( \nu_{FS} \) is different from \( \nu \), and satisfied the inequality

\[
\nu_{FS} \geq 2/(d + \zeta),
\]

(10)

where \( d + \zeta \) is the effective dimension of the disordered system considered. Thus for an ensemble of samples of linear size \( L \) the width of the distribution of critical forces is characterized by a scale \( \xi_{FS} = L \) and reads

\[
\langle (f(L) - f_c^*)^2 \rangle \sim L^{-2/\nu_{FS}},
\]

(11)
For periodic systems $\zeta = 0$ and $\nu = 1/2$ so that $\nu_{PS} \neq \nu$ for $d \leq 4$. For interfaces it was proposed [15] that $\nu_{PS} = \nu$. While $\nu_{PS}$ satisfies (10) with the equal-sign in 1-loop order, at 2-loop order the inequality becomes strict. As discussed in Refs. [15, 17] this difference is closely related to the stability of the FP which controls the scaling behavior.

III. DISTRIBUTION OF THRESHOLD FORCES

A. Perturbation theory

Let us show how the critical force distribution can be computed within “improved” perturbation theory. “Improved” means that we assume the the disorder correlator $\delta(u)$ to be non-analytic, since for analytic disorder perturbation theory gives a zero-threshold force. Then using FRG we renormalize our result to 1-loop order. Effectively, this is a summation of an infinite subset of diagrams. We define the distribution of threshold force as follows

$$P_L(f_c) = \left\langle \delta\left(f_c + L^{-d} \int_x F(x,vt + u_{xt})\right)\right\rangle. \quad (12)$$

From now on we work in the comoving frame and the average is performed with the action $S$ in the quasi-static limit $v \rightarrow 0^+$, as is usually done in the FRG theory of the depinning transition.

Let us introduce the corresponding characteristic function

$$\hat{P}_L(\lambda) = \left\langle e^{-i\lambda f_c(L)} \right\rangle = \int df_c e^{-i\lambda f_c} P_L(f_c) \quad (13)$$

which can be expressed through the cumulants $\overline{(f_c(L))^n}$ as follows

$$\hat{P}_L(\lambda) = \exp\left(\sum_{n=1}^{\infty} \frac{(-i\lambda)^n}{n!} \overline{(f_c(L))^n}\right). \quad (14)$$

The computation of the first cumulant to one loop is trivial. The random force that the interface actually feels in the point $x$ is given by

$$\langle F(x, u_{xt} + vt) \rangle = \left\langle \int_{t_1}^{\infty} \Delta(u_{xt} - u_{xt_1} + v(t - t_1))i\hat{u}_{xt_1} \right\rangle.$$

$$= \int_{t_1}^{\infty} \Delta'(v(t - t_1)) [R_{x=0,t-t_1} - R_{x=0,t=0}] . \quad (15)$$

We will adopt Ito’s prescription in which $R_{x,t=0} = 0$. Note that this corresponds to the definition $\Theta(0) = 0$. Taking the quasi-static limit $v \rightarrow 0^+$, we obtain the well-known expression for the average critical force

$$f_c^* = -\Delta'(0^+) \int_0^\infty dt R_{x=0,t} = -\int_0^\infty \Delta'(0^+) q^2. \quad (16)$$

Note that the critical force diverges at large momentum as $\Lambda^{-d-2}$ and therefore is not universal, i.e. it depends on microscopic parameters. This is analogous to to the shift of the critical temperature in standard critical phenomena, caused by fluctuations. As we know this is shift is also non-universal. However, we expect that the distribution of critical forces for a finite system around the average value are universal, once the distribution is properly normalized. The computation of the the $n$-th cumulant is more tricky. Before considering the general case let us show how this works for the second cumulant. Using the generating function, we can write down the formal expression for the effective force-force correlator

$$\langle F(x_1, u_{x_1,t} + vt) F(x_2, u_{x_2,t} + vt) \rangle = \Delta(0) \delta^d(x_1 - x_2) + \left\langle \int_{t_1}^{t_2} \Delta(u_{x_1,t} - u_{x_1,t_1} + v(t - t_1))i\hat{u}_{x_1,t_1} \times \int_{t_1}^{t_2} \Delta(u_{x_2,t} - u_{x_2,t_2} + v(t - t_2))i\hat{u}_{x_2,t_2} \right\rangle. \quad (17)$$

The first term on the r.h.s. of Eq. (17) is the bare disorder distribution. It is given by Eq. (2) and is a pure Gaussian distribution with zero mean. However the moving manifold explores a different distribution that is an effective distribution which one can observe “sitting” on the moving interface. The second term on the r.h.s. of Eq. (17) as well as the mean value (15) are the deviation of the effective distribution from the bare one. Only connected diagrams contribute to the second cumulant. Integrating the second term in Eq. (17) over fields with the weight $e^{-s}$ we obtain the four connected diagrams shown in Figure I. The corresponding expressions can be rewritten as follows

$$\int_{t_1,t_2} \Delta'(v(t - t_1)) [R_{x_2-x_1, t-t_1} - R_{x_2-x_1, t_1-t_2}]$$

$$\Delta'(v(t - t_2)) [R_{x_1-x_2, t-t_2} - R_{x_1-x_2, t_1-t_2}]. \quad (18)$$

To compute the contribution to the variance of the critical force, we have to integrate over $x_1$ and $x_2$ and then multiply by $L^{-2d}$. This computation is more convenient in Fourier.
Summing all contributions we obtain

$$D_1 + D_2 + D_3 + D_4 = L^{-d}[\Delta'(0^+)]^2$$

\[ \times \int_{q,t,t_2} [R_{q,t-t_1} - R_{q,t_2-t_1}] [R_{q,t-t_2} - R_{q,t_1-t_2}]. \] (19)

Due to causality we have $D_4 = 0$. The other diagrams read

$$D_1 = -D_2 = -D_3 = L^{-d} \int_q \frac{[\Delta'(0^+)]^2}{(q^2)^2}. \quad (20)$$

Summing all contributions we obtain

$$\overline{\langle f_c(L)^n \rangle}_c = L^{-d} \Delta(0) - L^{-d} \int_q \frac{[\Delta'(0^+)]^2}{(q^2)^2}. \quad (21)$$

Here the factor $(n-1)!$ results from different contractions of $u_{x,t} - u_{x,t_i}$ and $\hat{u}_{x,t_j}$ ($i, j = 1, \ldots, n$) that form a closed loop. Expanding the integrand in Eq. (22) we find that all terms gives the same contribution up to a factor of $\pm 1$, except for the term composed only from the second response function in each bracket. This term gives a closed loop of response-functions, which is zero by causality. Using the identity

$$\sum_{i=0}^{n-1} (-1)^i C_n^i = (-1)^{n+1}, \quad (23)$$

where $C_n^i$ is a binomial coefficient, we can simplify Eq. (22) to

$$\overline{\langle f_c(L)^n \rangle}_c = -(n-1)!L^{-d(n-1)} \int_q \frac{[\Delta'(0^+)]^n}{q^{2n}}. \quad (24)$$

We are now in a position to construct the characteristic function

$$\ln \hat{P}(\lambda) = -\frac{1}{2} \Delta(0) L^{-d} \lambda^2$$

$$-L^d \int \sum_{q,n=1}^\infty \frac{(-1)^n}{n} \left( \frac{\Delta'(0^+)}{q^2} L^{-d} i\lambda \right)^n. \quad (25)$$

The latter is nothing but the Taylor series of the logarithm, which allows to rewrite Eq. (25) as

$$\hat{P}(\lambda) = \exp \left[ -\frac{1}{2} \Delta(0) L^{-d} \lambda^2 + L^d \int \ln \left( 1 - \frac{\Delta'(0^+)}{q^2} L^{-d} i\lambda \right) \right]. \quad (26)$$

We have also derived Eq. (21) by using direct perturbation theory instead of the generating functional.

The above calculation can be generalized to arbitrary $n$. It can be simplified significantly if one takes into account that all intermediate times $t_i$ must be smaller than the observation time $t$: $t_i < t$ ($i = 1, \ldots, n$). For the $n$-th cumulant ($n > 2$) we have

$$\overline{\langle f_c(L)^n \rangle}_c = \left[ (-1)^n (n-1)! L^{-d(n-1)} \int_q \frac{[\Delta'(0^+)]^n}{q^{2n}} \right] \times \int_{t_1 \ldots t_n} [R_{q,t-t_1} - R_{q,t_2-t_1}] [R_{q,t-t_2} - R_{q,t_3-t_2}] \ldots [R_{q,t-t_n-1} - R_{q,t_n-t_n-1}] [R_{q,t-t_n} - R_{q,t_1-t_n}]. \quad (22)$$

where we have taken into account that $\Delta'(0^+) < 0$.

As follows from the above computation, the distribution of the critical force can be related to the effective action $\Gamma[u, \hat{u}]$ which is a generating functional for one-particle irreducible (1PI) vertex functions $\Gamma^{(E, S)}_{\hat{u}, \ldots, \hat{u}; u, \ldots, u}$ with $S$ external fields $u$ and $E$ external fields $\hat{u}$

$$\Gamma^{(E, S)}_{\hat{u}, \ldots, \hat{u}; u, \ldots, u}(\{\hat{q}_i, \hat{\omega}_i\}, \{q_j, \omega_j\}) = \prod_{i=1}^S \delta u_{q_i, \omega_i} \prod_{j=1}^E \delta \hat{u}_{\hat{q}_j, \hat{\omega}_j} \Gamma[u, \hat{u}] \bigg|_{u=\hat{u}=0}. \quad (27)$$

Indeed as already seen from the bare action the average threshold force can be expressed as vertex function $\Gamma^{(1,0)}_{\hat{u}}(q = 0, \omega = 0)$ in the quasi-static limit $\nu \to 0^+$. Analogously the higher-order cumulants can be identified as the higher-order vertex-functions according to

$$\overline{\langle f_c(L)^n \rangle}_c = L^{-(n-1)d} \Gamma^{(n,0)}_{\hat{u}, \ldots, \hat{u}; u, \ldots, u}(\{q_i = 0, \omega_i = 0\}). \quad (28)$$

The general properties of vertex functions (28) for even $n$ was discussed in Ref. [21]. In particular it was noted that loop diagrams that contribute to the vertex $\Gamma^{(2n,0)}$ precisely cancel each over so that the result is given by minus the missing contribution from acausal loops. It is easy to verify by direct inspection of the Feynman diagrams that definitions (12) and (28) give the the same result in the one-loop approximation, but the question of their equivalence to all orders is open.
functions are distinguished by whether the line entering at
Figure 3. which are defined in Eqs. (35) and (36) and depicted in details in
section some extrapolations to low dimension. We remind the reader that
account: replacing the bare correlator by the running one in a
we will demand a bit more and take an additional effect into
of both. The final result presented here will thus be exact to
momentum (which has to be integrated out), or a combination
integration over momentum) or dependence on the loop mo-
dependence on a mass (as in quantities which do not contain
second case it is interrupted, i.e. there are no restrictions on
outgoing at
FIG. 3: Vertex function $r_{\bar{u}uu}(t, t_1, t_2; q, -q)$ at the tree level. The
functions are distinguished by whether the line entering at $t_2$ and outgoing at $t_1$ is “closed” $r_{\bar{u}uu}$, or “open” $r_{\bar{u}uu}$. In the first case
the time-ordering along the loop is continuous: $t_1 < t_2$, while in the second case it is interrupted, i.e. there are no restrictions on $t_1$ and $t_2$
(see also Figure 2).

B. Renormalization

In this section we focus on the interface problem, periodic systems being considered in the next section. The distribution of the critical forces in (26) has been obtained from the “improved” perturbation theory, and thus, it cannot be reproduced within the Larkin type models in which all observables depend only on $\Delta(0)$. However in the bare theory the disorder correlator is an analytic function so that to make the calculation consistent we have to first renormalize our theory. To that end we replace the bare correlator by the renormalized one. This can be viewed as a partial summation of an infinite series of diagrams. If we want the distribution of $f_c$ strictly to lowest order in $\epsilon = 4 - d$ then the work is essentially done. However, we will demand a bit more and take an additional effect into account: replacing the bare correlator by the running one in a particular diagram we have to be careful because the scale dependence acquired by the correlator may be in form of either dependence on a mass (as in quantities which do not contain integration over momentum) or dependence on the loop momentum (which has to be integrated out), or a combination of both. The final result presented here will thus be exact to lowest order in $\epsilon$ and in addition will contain some effects beyond that order (although a full fledged two loop calculation is not attempted here). This will allow us to discuss in the next section some extrapolations to low dimension.

Let us start from the renormalization of the first cumulant, i.e. the average critical force (16). We remind the reader that the average critical force is a non-universal quantity and afterwards we will subtract it and consider the shifted distribution which is a universal function. According to Eq. (9) the renormalized disorder correlator acquires in the vicinity of the fixed-point a scale dependence. Integration over scales beyond the Larkin scale yields (see Ref. [23, 39] for details):

$$\langle f_c(L) \rangle_c \approx -\Delta^* (0^+) \frac{\Lambda^{2-\zeta}}{2 - \zeta}$$

(29)

where in this formula the UV cutoff $\Lambda$ is meant to be the minimal length of pinned segments of the manifold, i.e., the Larkin length $\Lambda \sim L_c^{-1}$.

We now consider the renormalization of the second cumulant (variance). The corresponding bare expression can be expressed through the 2-point vertex function as follows

$$\langle f_c(L)^2 \rangle_c = L^{-d} \left[ \Delta(0) - \int q^4 \frac{\Delta'(0^+)^2}{q^4} \right] + O(\Delta^3)$$

(30)

As it was shown in Ref. [17, 21], the 2-point vertex function does not depend on times and scales,

$$\Gamma^{(2),uu}(q) = L^{2\zeta-\epsilon} \frac{1}{I_1 \epsilon} \tilde{\Delta}^*(0) F_2(qL),$$

(31)

with $F_2(z) = B z^{-2\zeta} + O(\ln z/z^2)$ for large $z$ and $F_2(0) = 1$. Note the the constant $B$ depends on the IR cutoff scheme [21]. Combining Eqs. (30) and (31) we obtain

$$\langle f_c(L)^2 \rangle_c = L^{2\zeta-\epsilon} \frac{1}{I_1 \epsilon} \tilde{\Delta}^*(0).$$

(32)

We note that this is consistent with the finite-size scaling prediction:

$$\langle f_c(L)^2 \rangle_c \sim L^{-2/\nu}$$

(33)

using $\nu = 1/(2 - \zeta)$. As we will show below, the full (shifted) distribution is also consistent with this scaling.

To proceed further, let us first consider some typical diagrams contributing to the third cumulant of the critical force, which are shown in Figure 3. To renormalize them at lowest order, we have replaced the disorder lines by the three-point vertices defined as follows

$$\Gamma^{(2,1),uu}(t, t_1, t_2; q_1, q_2) = \Gamma^{(+)uu}(t, t_1, t_2; q_1, q_2) + \Gamma^{(-)uu}(t, t_1, t_2; q_1, q_2).$$

(34)

At tree level, the vertex function $\Gamma^{(2),uu}$ can be expressed by diagrams shown in Figure 3 and the corresponding expressions read

$$\Gamma^{(+)uu}(t, t_1, t_2; q_1, q_2) = \Delta^*(0^+) \text{sign}(t - t_1) \delta(t - t_2),$$

(35)

$$\Gamma^{(-)uu}(t, t_1, t_2; q_1, q_2) = \Delta^*(0^+) \text{sign}(t_1 - t) \delta(t_1 - t_2).$$

(36)

Then the summation of diagrams contributing to the the n-th cumulant with $n > 2$ can be carried out along the lines used for the bare cumulant and gives...
\[
\overline{\langle f_c(L)^n \rangle}_c = (-1)^n (n - 1)! L^{-d(n-1)} \int_q \int_{t_1} \int_{t_2} \int_{\tau_1} \int_{\tau_2} \left[ \Gamma^{(2,1)}_{\bar{u} u} (t, t_2, \tau_1, q, -q) R_{q, \tau_1 - t_2} \right] \times \\
\left[ \Gamma^{(2,1)}_{\bar{u} u} (t, t_3, \tau_2, q, -q) R_{q, \tau_2 - t_3} \right] \cdots \left[ \Gamma^{(2,1)}_{\bar{u} u} (t, t_n, \tau_{n-1}, q, -q) R_{q, \tau_{n-1} - t_n} \right] \left[ \Gamma^{(2,1)}_{\bar{u} u} (t, t_1, \tau_n, q, -q) R_{q, \tau_n - t_1} \right].
\]

Substituting the tree-level expressions (34) to Eq. (37) we recover the bare cumulant (22).

In Appendix A we compute the vertex function \( \Gamma^{(+)}_{\bar{u} u} \) to one-loop order and obtain its large-\( q \) asymptotics which reads

\[
\int_t \Gamma^{(+)}_{\bar{u} u} (t, t_1, t_2; q, -q) = A L^{\zeta - \epsilon} (q L)^\psi,
\]

where times \( t_1 \) and \( t \) are taken infinitely apart (hence it is a quasi-static quantity). The amplitude \( A \) and the exponent \( \psi \) are given by

\[
A = \frac{1}{\varepsilon \bar{I}_1} \tilde{\Delta}^\lambda (0^+) (1 + \mathcal{O}(\epsilon)),
\]

\[
\psi = \frac{4}{9} \varepsilon + \mathcal{O}(\epsilon^2),
\]

and we argue that \( \psi \) is a new exponent (see Appendix A). Taking into account this momentum dependence in Eq. (37) should result in an improved renormalization scheme (compared to simply replacing the bare quantities by \( q \)-independent but scale-dependent renormalized parameters) with a different form for the \( q \) summations appearing below.

After substituting Eq. (38) in Eq. (34), the integration over times can be performed in the same way as for the bare cumulant and gives for \( n > 2 \).

\[
\overline{\langle f_c(L)^n \rangle}_c = -(n - 1)! A^n L^{n(\zeta - 2)} \times L^d \int_q \frac{1}{(q L)^{(2-\psi)}}.
\]

Note that \( A \sim \tilde{\Delta}^\lambda (0^+) < 0 \). To construct the characteristic function let us redefine \( \lambda \rightarrow (2\pi)^{2-\psi} \lambda / |A| L^{\tilde{\zeta}/2} \), that corresponds to measuring \( f_c \) in units of \( |A| L^{\tilde{\zeta}/2} / (2\pi)^{2-\psi} \). This is a non-universal scale as the value of \( \tilde{\Delta}^\lambda (0^+) \) is not universal at the depinning transition. However as we now show, once rescaled the (shifted) distribution is universal.

The characteristic function can be written as

\[
\ln \tilde{P}(\lambda) = -\frac{1}{2} \sigma^2 \lambda^2 - \sum_{\mathbf{k} \in \mathbb{Z}^d, \mathbf{k} \neq 0} \sum_{n=3}^{\infty} \frac{1}{n} \left( \frac{i \lambda}{|\mathbf{k}|^{2-\psi}} \right)^n,
\]

where

\[
\sigma^2 = \frac{(2\pi)^4 \tilde{\Delta}^\lambda (0)}{A^2 \bar{I}_1 \varepsilon} = \frac{\epsilon \bar{I}_1}{(\Delta^\lambda (0))^2} \frac{\tilde{\Delta}^\lambda (0)}{2(2\pi)^4} = \frac{6\pi^2}{\varepsilon} (1 + \mathcal{O}(\varepsilon)).
\]

To compute this universal ratio to lowest order in \( \epsilon \) we have used the 1-loop FRG fixed-point equation evaluated at \( u = 0 \), i.e. \( (\epsilon - 2\zeta) \tilde{\Delta}^\lambda (0) = \tilde{\Delta}^\lambda (0)^{1/2} \) and used \( \zeta = \epsilon/3 + \mathcal{O}(\epsilon^2) \), as well as \( \epsilon \bar{I}_1 = 1/(8\pi^2) + \mathcal{O}(\epsilon) \). Summing over \( n \) we obtain

\[
\ln \tilde{P}(\lambda) = -\frac{1}{2} \sigma^2 \lambda^2 + \sum_{\mathbf{k} \in \mathbb{Z}^d, \mathbf{k} \neq 0} \left[ \frac{i \lambda}{|\mathbf{k}|^{2-\psi}} \frac{1}{|\mathbf{k}|^{2-\psi}} \ln \left( 1 - \frac{i \lambda}{|\mathbf{k}|^{2-\psi}} \right) \right].
\]

C. Lowest order in \( \epsilon = 4 - d \)

To obtain the distribution within the \( \epsilon \) expansion it is sufficient to set \( \psi = 0 \) in the formula above, and to compute all sums in \( d = 4 \). Let us first give the skewness and kurtosis to lowest order in the \( \epsilon \) expansion. One uses \( \bar{I}_{1} \):

\[
\sum_{\mathbf{k} \in \mathbb{Z}^d, \mathbf{k} \neq 0} \frac{1}{|\mathbf{k}|^{2p}} = \frac{1}{(p-1)!} \times \int_0^\infty dt \ t^{p-1} \Theta(3, 0, e^{-t}) - 1,
\]

\[
\sum_{\mathbf{k} \in \mathbb{Z}^d, \mathbf{k} \neq 0} \frac{1}{|\mathbf{k}|^6} = 14.8298,
\]

\[
\sum_{\mathbf{k} \in \mathbb{Z}^d, \mathbf{k} \neq 0} \frac{1}{|\mathbf{k}|^8} = 10.2454,
\]

where \( \Theta(3, 0, e^{-t}) = \sum_{\mathbf{k} \in \mathbb{Z}} e^{-t k^2} \) is the elliptic theta function. Hence:

\[
\sigma_3 = \frac{(f - f)^3}{\sigma^3} = \frac{2}{\sigma^3} \sum_{\mathbf{k} \in \mathbb{Z}^d, \mathbf{k} \neq 0} \frac{1}{|\mathbf{k}|^6},
\]

\[
\sigma_4 = \frac{(f - f)^4}{\sigma^4} = -3 \sum_{\mathbf{k} \in \mathbb{Z}^d, \mathbf{k} \neq 0} \frac{1}{|\mathbf{k}|^8},
\]

\[
\sigma_4 = -0.01753 \epsilon^2.
\]

Next one can resum to obtain the characteristic function to lowest order in \( \epsilon \):

\[
\ln \tilde{P}(\lambda) = -\frac{1}{2} \sigma^2 \lambda^2 - F(-i\lambda),
\]

\[
F(-i\lambda) = \int_0^\infty \frac{dt}{t} (e^{i\lambda t} - 1 - i\lambda t + \frac{1}{2} \lambda^2 t^2)
\]

\[
\times [\Theta(3, 0, e^{-t})]^4 - 1].
\]
The sum of two independent random variables \( f = f_0 + \sqrt{f_0^2} f_1 \) where \( f_0 \) is gaussian of variance \( 1 + O(\epsilon) \) and \( f_1 \) is a random variable of order unity with a non trivial distribution, the logarithm of its characteristic function being given (up to a quadratic term) by \( F(-i\lambda) \).

We now analyze the shape of these distributions in physical dimension.

### D. Fourier inversion

In this section we compute the inverse Fourier transform of Eq. (44) in physical dimensions, using our improved scheme.

Let us start by discussing \( d = 1 \). We use a natural extrapolation of our above result, setting \( \epsilon = 3 \) in the above formulae (which are exact to lowest order in \( \epsilon \)). From Eq. (44) we obtain

\[
\hat{P}(\lambda) = \exp \left[ -\pi^2 \lambda^2 \right] \prod_{k=1}^{\infty} \left\{ \left( 1 - \frac{i\lambda}{k^{2/3}} \right)^2 \right\}
\]

\[
\times \exp \left[ i\lambda \left( \frac{2}{k^{2/3}} - \lambda^2 \frac{1}{k^{4/3}} \right) \right], \tag{54}
\]

where we have used \( \psi = 4/3 \) and \( \sigma = \pi \sqrt{2} \). The inverse Fourier transform computed numerically is shown in Figure 4 with \( \psi = 0 \) can formally be considered as the result of improved perturbation theory in non-analytic disorder. The inverse Fourier transform of the latter is also shown in Figure 4 and can not be visually distinguished from the shifted renormalized distribution. The renormalized distribution is more appealing since it guarantees a non-negative defined critical force, which is not the case for the bare one, especially in \( d = 1 \) where the bare averaged critical force \( \langle \phi \rangle \langle \phi \rangle \) is finite. By contrast the averaged renormalized critical force is controlled by the UV cutoff and is of order \( L^{-1/\nu} \) while the typical fluctuation is much smaller, of order \( L^{-1/\nu} \), in the limit of interest, here \( L \gg L_c \). It is interesting that the renormalized distribution is well approximated by the Eq. (54) in which we keep only the first factor with \( k = 1 \):

\[
\hat{P}(\lambda) \approx \exp \left[ -\frac{1}{2} \sigma^2 \lambda^2 \right] \left\{ (1 - i\lambda)^2 \exp \left[ 2i\lambda - \lambda^2 \right] \right\}, \tag{55}
\]

The inverse Fourier transform of Eq. (55) reads

\[
P\langle f \rangle \approx \frac{(\sigma^2 + f^4 + 4 - 2 \sigma^2)}{(2 + \sigma^2)^{5/2}} \sqrt{2\pi} e^{-\frac{(\sigma^2 + f^4 - 2 \sigma^2)}{2(\sigma^2 + f^4 + 4)}}, \tag{56}
\]

and is also shown in Figure 4. The difference in the critical force distributions obtained within improved perturbation theory, renormalized to one-loop theory and approximation (56) is indicated in Figure 5.

For a \( d \)-dimensional system the Fourier transform of the critical force distribution (44) can be written as

\[
\hat{P}(\lambda) = \exp \left[ -\frac{1}{2} \sigma^2 \lambda^2 \right] \prod_{k \in \mathbb{Z}^d} \left\{ \left( 1 - \frac{i\lambda}{|k|^{2-\psi}} \right)^2 \right\}
\]

\[
\times \exp \left[ \frac{i\lambda}{|k|^{2-\psi}} - \frac{1}{2} \frac{\lambda^2}{|k|^{2(2-\psi)}} \right], \tag{57}
\]

Analogously to the case \( d = 1 \) the inverse Fourier transform of Eq. (57) can be well approximated by

\[
\hat{P}(\lambda) = \exp \left[ -\frac{1}{2} \sigma^2 \lambda^2 \right] \left\{ (1 - i\lambda)^2 \exp \left[ 2i\lambda - \lambda^2 \right] \right\}, \tag{58}
\]

which does not depend on \( \psi \).

We now compute the standard deviation, and the kurtosis of the above distributions. The standard deviation reads

\[
(f - \langle f \rangle)^2 \equiv \sigma^2. \tag{59}
\]
The skewness is defined as
\[
\sigma_3 = \frac{(f - \bar{f})^3}{\sigma^3} = \frac{2}{\sigma^3} \sum_{k \in \mathbb{Z}^d, k \neq 0} \frac{1}{|k|^3(2-\psi)} \tag{60}
\]
For \(d = 1\) we obtain \(\sigma_3 = \sqrt{2}/(6\pi) \approx 0.075\) [0.046 for the distribution]. The positive value for the skewness indicates that the right tail of the distribution is heavier than the left tail.

The kurtosis for a Gaussian distribution is three. For this reason, excess kurtosis is defined as
\[
\sigma_4 = \frac{(f - \bar{f})^4}{\sigma^4} - 3 = \frac{3!}{\sigma^4} \sum_{k \in \mathbb{Z}^d, k \neq 0} \frac{1}{|k|^{4(2-\psi)}} \tag{61}
\]
For \(d = 1\) we obtain \(\sigma_4 = -3\zeta(8/3)/\pi^4 \approx -0.04\) [-0.031 for the distribution]. Here \(\zeta(x)\) is the Riemann Zeta function. The small negative excess kurtosis indicates that the distribution is slightly more flat than a Gaussian distribution, while the deviation from the Gaussian distribution is quite small.

IV. PERIODIC SYSTEMS

The renormalization of the critical force distribution for periodic systems requires a separate consideration. Indeed as was shown in Ref. [15], in the periodic case there is an additional relevant operator which is the uniform part of \(\Delta(u)\) so that the RP fixed point is unstable. The flow equation for this operator can be derived by integration of the RG equation over one period [17]
\[
L\partial_L \int_0^1 \tilde{\Delta}(u) du = \varepsilon \int_0^1 \tilde{\Delta}(u) du + O(\Delta^3), \tag{62}
\]
where we have explicitly used \(\zeta = 0\). Thus in the vicinity of the RP FP, the flow of the dimensionless disorder is given by
\[
\tilde{\Delta}(u) = \tilde{\Delta}^*(u) + cL^\varepsilon \tag{63}
\]
where the non-universal constant \(c\) can be estimated as [17]
\[
c = L_c^{-\varepsilon} \int_0^1 \left( \tilde{\Delta}^{(bare)}(u) - \tilde{\Delta}^*(u) \right) du \]
\[
= -L_c^{-\varepsilon} \int_0^1 \tilde{\Delta}^*(u) du = L_c^{-\varepsilon} \left( \frac{\varepsilon^2}{108} + O(\varepsilon) \right) > 0.
\]
This runaway correction to the scaling behavior at the RP FP contributes to all quantities which depend on \(\Delta(0)\) but not to those which depend on \(\Delta'(0)\). Therefore in the case of a periodic system the renormalized second cumulant of the critical force reads
\[
\langle f_c(L)^2 \rangle_c = \frac{1}{I_1 \varepsilon} \tilde{\Delta}^*(0) L^{-d} + \frac{c}{I_1 \varepsilon} L^{-d}. \tag{64}
\]
Higher order cumulants are still given by Eq. (41) with \(\zeta = 0\) and therefore scale with \(L\) as \(\langle f_c(L)^n \rangle_c = L^{-2n}\). We would like to emphasize that the correction to scaling in Eq. (64) describes the sample-to-sample fluctuations and can not be seen in one sample because in each sample there is only one pinned configuration. As a result for \(d < 4\) only the second cumulant of the scaled critical force, which scales as \(L^{-d}\), survives in the limit \(L \to \infty\) resulting in \(\eta_P = 2/d\) and a pure Gaussian distribution for the scaled critical force.

V. DISCUSSION

In the present paper we have computed the renormalized distribution \(P[f - \bar{f}w]\) averaged over all pinned configurations in the limit \(v \to 0^+\), which we identify with the critical force distribution \(P_L(f_c)\). The average critical force is a non-universal quantity which depends on microscopic details of the interactions like the UV cutoff as well as on details of the disorder distribution. After subtraction of the average value, the shifted distribution of the critical force contains only one non-universal scale which can be fixed, e.g. by fixing the second cumulant. The resulting dimensionless distribution is then fully universal, i.e. it does not depend on properties at small scales. We have computed it taking into account only the second cumulant of the bare disorder distribution, since it is the only cumulant relevant in the RG sense. Higher cumulants, which are generated by coarse graining are irrelevant operators and their contribution to the cumulants of the critical force must carry additional dependence on the cutoff. Hence we expect that they result only in a shift of the non-universal expectation values.

Let us now discuss the role of the transverse sample size \(M\) (size of the box). In numerical studies of depinning of elastic interfaces, either via an exact determination of the critical state, or via Langevin dynamics a cylindrical system which is periodic in both directions was studied: longitudinal with period \(L\) and transverse with period \(M\). Hence this is equivalent to a periodic disorder with period \(M\). It is known that a periodic system has a unique pinned configuration for any period \(M\). If \(M \ll L\), this configuration spreads out through the whole box \(L^d \times M\) and there is only one independent pinned configurations. As we have shown for the critical force distribution of critical forces is Gaussian, and thus, for elastic interfaces in the limit \(L \to \infty\), with \(M\) fixed, the distribution of the critical force also becomes Gaussian.

The case where the period \(M\) is taken to infinity at the same time as \(L\) is relevant for elastic interfaces and quite different. The pinned interface has a r.m.s. width \(w = k_w L^\zeta\) so that in a sample of transverse size \(w\) it also has one unique statistically independent pinned configuration. One may then argue that its critical force distribution is \(P_L(f_c)\) whose characteristic function is given by Eq. (43). In the numerical studies one should thus be careful in choosing the size of the periodic box \(M\). If \(M\) scales like \(L^\zeta'\) with \(\zeta' < \zeta\) the system will crossover from the random manifold to the RP FP and the finite-size scaling analysis will result in some mixture of interface and periodic system properties, while the critical force distribution...
will tend to a Gaussian one. On the other hand if $M \gg w$, the sample can be divided into about $M/w$ subsamples, which can be argued to be (almost) statistically independent, with independent pinned configurations. Each configuration has a slightly different critical force which is distributed according to our FRG result. If one defines the total critical force as a maximum of all the critical forces of these subsamples, it becomes $M$-dependent and its shifted distribution tends to the distribution of the extreme value statistics (28). Following Ref. [28] let us introduce for every configuration $\alpha$ of the interface in the sample of width $M = kL$ the depinning force $f_d(\alpha)$ and then associate the threshold force of the whole sample with the following maximal value $f^r_c = \max_{\alpha} \{f_d(\alpha)\}$. In each sample there are only $\approx M/w$ independent pinned configuration, so that the distribution of the maximum of the corresponding critical forces can be written as

$$P_M(f^r_c, M/w) = \frac{d}{df^r_c} \left[ \int_{-\infty}^{f^r_c} df' P_L[f'], \right]^{M/w}.$$  \hspace{1cm} (65)

According to the general theorem of extreme value statistics [33], for large samples, i.e., in the limit $M/w \to \infty$ this distribution approaches the Gumbel distribution. The latter is provided by the tails of the distribution of the critical force for each independent pinning configuration, as given in Eq. (44). According to this distribution the average maximal threshold force of samples of size $M$ behaves as $\ln(k/k_w)$. For large samples with $M \gg w$ it can be extremely large. The above procedure completely washes out all details of the underlying distribution $P_L(f)$ computed here, except for its width, and replaces it by the model-independent function obtained from extreme value statistics. As an illustration, we have plotted the force distribution obtained using (65) for $M/w = 10$ on figure 15.

The above arguments suggest that the critical force distribution computed here via the FRG should be compared with the numerics on a cylinder of aspect ratio parameter $k \approx k_w$ defined above from the width. We now propose a more precise statement to identify the critical force computed in this paper. We note that in the calculations performed here within the FRG the position of the center of mass was held fixed (since all momentum integrations excluded the uniform mode $q = 0$). Hence we are working in the fixed center of mass ensemble. This suggests the definition:

$$f_c(u_0, L) = \max_{\alpha(u_0, L)} \{f_d(\alpha(u_0, L))\}$$  \hspace{1cm} (66)

where the maximum is over all configurations $\alpha(u_0, L)$ with center of mass $u_0$ and length $L$ (and periodic boundary conditions along the interface). It can in principle be evaluated numerically by direct enumeration for a discrete interface model. One can then check that it has a well-defined $L \to \infty$ limit with no need for a transverse box, and one can then numerically compute the finite-size distribution for the ensemble of $\alpha(u_0, L)$. This distribution should identify with the one computed here within the FRG (in the massless scheme). It is a more fundamental object than the critical force defined on a cylinder. The latter can then be retrieved in principle as

$$f^r_c = \max_{u_0} f_c(u_0, L)$$  \hspace{1cm} (67)

on the same cylinder, leading to extremal statistics, as discussed above.

The above considerations illustrate that the statistics of the depinning threshold force at finite-size is a rather subtle question. Many questions remain open. It would be interesting to find the proper steady state corresponding to the above definition (66). Also a systematic study of memory effects in the threshold force would be of high interest especially regarding experiments. Indeed, these memory effects may be of importance for aging [36] and hysteresis phenomena [37] controlled in some regimes by the slow dynamics of domain walls. There the observed threshold force may not be the largest one but a threshold force which characterizes a piece of the sample in which the interface got trapped. Hence the data should be interpreted with care to disentangle history effects from finite size effects. It would be very interesting to develop numerical schemes to investigate these questions in particular an efficient algorithm to compute (66).

Acknowledgments

We thank O. Düemmer, W. Krauth, A. Rosso for useful discussions. AAF acknowledges the support from the European Commission through Marie Curie Postdoctoral Fellowship under contract number MIF1-CT-2005-021897. PLD and KJW acknowledge support from ANR program 05-BLAN-0099-01.
and \( \tilde{I}_1 = \tilde{I}_1(0) \), the contributions to \( \Delta_0(0) \) are as follows
\[
\begin{align*}
- \left\{ [I_t(q)] + [I_t(0) + I_t(q)] + [0 + [I_t(0) - I_t(0)]] \right\} \times \Delta''(0) \Delta''(0) \\
+ \left\{ [-I_t(0)] + [0 + [I_t(0) + [0]] \Delta_0(0), \Delta''(0) \right\} (A4)
\end{align*}
\]
where terms in rectangular brackets are in the order of their appearance from diagrams a, b, c and d of figure 7. We remark that contributions proportional to \( \Delta_0(0) \Delta''(0) \) exactly cancel, and we obtain \( \Delta_0(0) \).

To renormalize the vertex function \( \tilde{A}_0(0) \), we have to reexpress the bare disorder correlator by the renormalized dimensionless one:
\[
\Delta_0(u) = m^\varepsilon \left\{ \Delta(u) + \left[ \Delta'(u)^2 + (\Delta(u) - \Delta(0)) \Delta''(u) \right] \tilde{I}_1 \right\}.
\]

Differentiating the latter expression w.r.t. \( u \) we get after rescaling \( \Delta(u) = \frac{1}{\varepsilon I_1} m^{2-\varepsilon} \Delta((um)^\varepsilon) \)
\[
\begin{align*}
\Delta_0(0) &= \frac{m^{2-\varepsilon}}{\varepsilon I_1} \Delta'(0^+) \left[ 1 + \frac{3}{\varepsilon I_1} \Delta'(0) m^\varepsilon \frac{1}{p(p^2 + m^2)^2} \right], \\
\Delta''(0) &= \frac{1}{\varepsilon I_1} m^\varepsilon \Delta''(0^+) + O \left( \Delta'''(0^+), \Delta''(0^+) \Delta''(0^+) \right).
\end{align*}
\]

Omitting the last term in Eq. \( \text{(A1)} \) we obtain the following expression for the renormalized vertex function:
\[
\begin{align*}
\int_{t_2} \Gamma_{\tilde{a}\tilde{u}}^{(+)}(t_1, t_2; q, -q) &= m^{2-\varepsilon} \frac{1}{\varepsilon I_1} \Delta'(0^+) \\
&\times \left[ 1 - 2 \Delta''(0) \frac{1}{\varepsilon I_1} m^\varepsilon [I_t(0) - I_t(0)] \right],
\end{align*}
\]

Note that denominating of the non-periodic systems is described by the fixed point with \( \Delta''(0^+)^* = \frac{2}{9} \varepsilon + O(\varepsilon^2) \). The one-loop integral \( \tilde{I}_1(q) \) reads \( \text{(A7)} \)
\[
\tilde{I}_1(q) = \frac{1}{2} K_d \Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{d}{2} - 1 \right) \int_0^\infty d\alpha \frac{d\alpha}{[1 + \alpha(1 - \alpha) y^2]^{d/2}}.
\]

where \( K_d = 2\pi^{(d-2)/2} (2\pi)^d \Gamma(d/2) \) is area of a d dimensional sphere divided by \( (2\pi)^d \). Taking into account that \( \tilde{I}_1 \equiv \tilde{I}_1(0) = \int_q (q^2 + 1)^{-2} = \frac{1}{2} K_d \Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{d}{2} - 1 \right) \) we obtain
\[
\begin{align*}
\int_{t_2} \Gamma_{\tilde{a}\tilde{u}}^{(+)}(t_1, t_2; q, -q) &= m^{2-\varepsilon} \frac{1}{\varepsilon I_1} \Delta'\left(0^+\right) \\
&\times \left[ 1 + \frac{2}{9} \varepsilon \int_0^\infty d\alpha \ln \left[ 1 + \alpha(1 - \alpha) y^2 \right] + O(\varepsilon^2) \right].
\end{align*}
\]

We are interested in the asymptotic behavior for \( z \to \infty \). In this limit we have
\[
\begin{align*}
\int_0^\infty d\alpha \ln \left[ 1 + \alpha(1 - \alpha) y^2 \right] &= -2 + \frac{\sqrt{y^2 + 2 y}}{y} \left[ \ln 2 \\
&- \ln(2 + y^2 - y \sqrt{4 + y^2}) \right] = -2 + 2 \ln y + O \left( \frac{\ln y}{y^2} \right).
\end{align*}
\]
Matching to a power-law asymptotic behavior we find

\begin{equation}
1 + \frac{2}{9} \varepsilon (-2 + 2 \ln y) + \mathcal{O}(\varepsilon^2) \rightarrow y^{4\varepsilon/9} \left( 1 - \frac{4}{9} \varepsilon \right).
\end{equation}

As a result we obtain for \( q/m \gg 1 \):

\begin{align*}
\int_{t_2} \Gamma_{\hat{u}\hat{u}}(t, t_1, t_2; q, -q) \\
= m^{\varepsilon-\frac{1}{9}} \frac{1}{t_1} \delta^*(0^+) \left( \frac{q}{m} \right)^{4\varepsilon/9} \left( 1 - \frac{4}{9} \varepsilon \right).
\end{align*}

Replacing \( m \) by \( 1/L \) we obtain Eq. \[35\].