INVARIANT SUBMANIFOLDS OF \((LCS)_{\nu}\)-MANIFOLDS WITH RESPECT TO QUARTER SYMMETRIC METRIC CONNECTION

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Abstract. The object of the present paper is to study invariant submanifolds of \((LCS)_{\nu}\)-manifolds with respect to quarter symmetric metric connection. It is shown that the mean curvature of an invariant submanifold of \((LCS)_{\nu}\)-manifold with respect to quarter symmetric metric connection and Levi-Civita connection are equal. An example is constructed to illustrate the results of the paper. We also obtain some equivalent conditions of such notion.

1. Introduction

It is known that a connection \(\nabla\) on a Riemannian manifold \(M\) is called a metric connection if there is a Riemannian metric \(g\) on \(M\) such that \(\nabla g = 0\), otherwise it is non-metric. In 1924, Friedman and Schouten \[8\] introduced the notion of semi-symmetric linear connection on a differentiable manifold. In 1932, Hayden \[10\] introduced the idea of metric connection with torsion on a Riemannian manifold. In 1970, Yano \[33\] studied some curvature tensors and conditions for semi-symmetric connections in Riemannian manifolds. In 1975, Golab \[9\] defined and studied quarter symmetric linear connection on a differentiable manifold. A linear connection \(\nabla\) in an \(n\)-dimensional Riemannian manifold is said to be a quarter symmetric connection \[9\] if torsion tensor \(T\) is of the form

\[
T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = A(Y)K(X) - A(X)K(Y)
\]

where \(A\) is a 1-form and \(K\) is a tensor of type (1,1). If a quarter symmetric linear connection \(\nabla\) satisfies the condition

\[
(\nabla_X g)(Y,Z) = 0
\]

for all \(X, Y, Z \in \chi(M)\), where \(\chi(M)\) is a Lie algebra of vector fields on the manifold \(M\), then \(\nabla\) is said to be a quarter symmetric metric connection. For a contact metric manifold admitting quarter symmetric connection, we can take \(A = \eta\) and \(K = \phi\) and hence \((1.1)\) takes in the form

\[
T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y.
\]
The relation between Levi-Civita connection $\nabla$ and quarter symmetric metric connection $\nabla$ of a contact metric manifold is given by

\begin{equation}
\nabla_XY = \nabla_XY - \eta(X)\phi Y.
\end{equation}

Quarter symmetric connection also studied by Ali and Nivas [1], Anitha and Bagewadi ([2] - [4]), De and Uddin [6], Hui [12], Mishra and Pandey [16], Mukhopadhya et al. [17], Prakash [20], Rastogi [21], Siddesha and Bagewadi [32], Yano and Imai [35] and many others.

In particular if $\phi X = X$ and $\phi Y = Y$, then quarter symmetric reduces to a semi-symmetric connection [8]. The semi-symmetric connection is the generalized case of quarter symmetric metric connection and it is important in the geometry of Riemannian manifolds.

In 2003, Shaikh [23] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds), with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [14] and also by Mihai and Rosca [15]. Then Shaikh and Baishya ([25], [26]) investigated the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology. The $(LCS)_n$-manifolds is also studied by Hui [11], Hui and Atceken [13], Prakash [19], Shaikh and his co-authors ([24] - [31]) and many others.

The present paper deals with the study of invariant submanifolds of $(LCS)_n$-manifolds with respect to quarter symmetric metric connection. Section 2 is concerned with some preliminaries which will be used in the sequel. $(LCS)_n$-manifolds with respect to quarter symmetric metric connection is studied in section 3. In section 4, we study invariant submanifolds of $(LCS)_n$-manifolds with respect to quarter symmetric metric connection. It is proved that the mean curvature of an invariant submanifold with respect to quarter symmetric metric connection and Levi-Civita connections are equal. In this section, we construct an example of such notion to illustrate the result. Section 5 consists with the study of recurrent invariant submanifolds of $(LCS)_n$-manifold with respect to quarter symmetric metric connection. We obtain a necessary and sufficient condition of the second fundamental form of an invariant submanifold of a $(LCS)_n$-manifold with respect to quarter symmetric metric connection to be recurrent, 2-recurrent and generalized 2-recurrent. Some equivalent conditions of invariant submanifold of $(LCS)_n$-manifolds with respect to quarter symmetric metric connection are obtained in this section.

2. Preliminaries

The covariant differential of the $p^{th}$ order, $p \geq 1$, of a $(0,k)$-tensor field $T$, $k \geq 1$, defined on a Riemannian manifold $(M, g)$ with the Levi-Civita connection $\nabla$ is denoted by $\nabla^p T$. According to [22] the tensor $T$ is said to be recurrent, respectively 2-recurrent, if the following condition holds on $M$

\begin{equation}
(\nabla T)(X_1, X_2, \cdots, X_k; X)T(Y_1, Y_2, \cdots, Y_k) = (\nabla T)(Y_1, Y_2, \cdots, Y_k; X)T(X_1, X_2, \cdots, X_k),
\end{equation}
respectively

\[(2.2) \quad (\nabla^2 T)(X_1, X_2, \cdots, X_k; X, Y)T(Y_1, Y_2, \cdots, Y_k) =
(\nabla^2 T)(Y_1, Y_2, \cdots, Y_k; X, Y)T(X_1, X_2, \cdots, X_k),\]

where \(X, Y, X_1, Y_1, \cdots, X_k, Y_k \in T\tilde{M} \). From (2.1) it follows that at a point \(x \in \tilde{M}\)
if the tensor \(T\) is non-zero then there exists a unique 1-form \(\pi\) respectively, a \((0,2)\)
tensor \(\psi\), defined on a neighbourhood \(U\) of \(x\), such that

\[(2.3) \quad \nabla T = T \otimes \pi, \quad \pi = d(\log \|T\|)\]

respectively,

\[(2.4) \quad \nabla^2 T = T \otimes \psi,\]

holds on \(U\), where \(|T|\) denotes the norm of \(T\) and \(|T|^2 = g(T, T)\).

The tensor is said to be generalized 2-recurrent if

\[(\nabla^2 T)(X_1, X_2, \cdots, X_k; X, Y) - (\nabla T \otimes \pi)(X_1, X_2, \cdots, X_k; X, Y)T(Y_1, Y_2, \cdots, Y_k) =
(\nabla^2 T)(Y_1, Y_2, \cdots, Y_k; X, Y) - (\nabla T \otimes \pi)(Y_1, Y_2, \cdots, Y_k; X, Y)T(X_1, X_2, \cdots, X_k)\]

holds on \(\tilde{M}\), where \(\phi\) is a 1-form on \(\tilde{M}\). From this it follows that at a point \(x \in \tilde{M}\)
if the tensor \(T\) is non-zero, then there exists a unique \((0,2)\)-tensor \(\psi\), defined on a
neighbourhood \(U\) of \(x\), such that

\[(2.5) \quad \nabla^2 T = \nabla T \otimes \pi + T \otimes \psi\]

holds on \(U\).

An \(n\)-dimensional Lorentzian manifold \(\tilde{M}\) is a smooth connected paracompact
Hausdorff manifold with a Lorentzian metric \(g\), that is, \(\tilde{M}\) admits a smooth symmetric
tensor field \(g\) of type \((0,2)\) such that for each point \(p \in \tilde{M}\), the tensor \(g_p: T_p\tilde{M} \times T_p\tilde{M} \rightarrow \mathbb{R}\) is a non-degenerate inner product of signature \((- , +, \cdots, +)\), where \(T_p\tilde{M}\) denotes
the tangent vector space of \(\tilde{M}\) at \(p\) and \(\mathbb{R}\) is the real number space. A non-zero
vector \(v \in T_p\tilde{M}\) is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies
\(g_p(v, v) < 0\) (resp., \(\leq 0, = 0, > 0\)) [18].

**Definition 2.1.** In a Lorentzian manifold \((\tilde{M}, g)\) a vector field \(P\) defined by

\[g(X, P) = A(X)\]

for any \(X \in \Gamma(T\tilde{M})\), is said to be a concircular vector field [34] if

\[(\nabla_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\},\]

where \(\alpha\) is a non-zero scalar and \(\omega\) is a closed 1-form and \(\nabla\) denotes the operator of
covariant differentiation with respect to the Lorentzian metric \(g\).
Let \( \tilde{M} \) be an \( n \)-dimensional Lorentzian manifold admitting a unit timelike concircular vector field \( \xi \), called the characteristic vector field of the manifold. Then we have

\[
(2.6) \quad g(\xi, \xi) = -1.
\]

Since \( \xi \) is a unit concircular vector field, it follows that there exists a non-zero 1-form \( \eta \) such that for

\[
(2.7) \quad g(X, \xi) = \eta(X),
\]

the equation of the following form holds

\[
(2.8) \quad (\tilde{\nabla}_X \eta)(Y) = \alpha \{g(X, Y) + \eta(X)\eta(Y)\}, \quad (\alpha \neq 0)
\]

\[
(2.9) \quad \tilde{\nabla}_X \xi = \alpha \{X + \eta(X)\xi\}, \quad \alpha \neq 0,
\]

for all vector fields \( X, Y \), where \( \tilde{\nabla} \) denotes the operator of covariant differentiation with respect to the Lorentzian metric \( g \) and \( \alpha \) is a non-zero scalar function satisfies

\[
(2.10) \quad \tilde{\nabla}_X \alpha = (X\alpha) = d\alpha(X) = \rho \eta(X),
\]

\( \rho \) being a certain scalar function given by \( \rho = -(\xi \alpha) \). Let us take

\[
(2.11) \quad \phi X = \frac{1}{\alpha} \tilde{\nabla}_X \xi,
\]

then from (2.9) and (2.11) we have

\[
(2.12) \quad \phi X = X + \eta(X)\xi,
\]

\[
(2.13) \quad g(\phi X, Y) = g(X, \phi Y),
\]

from which it follows that \( \phi \) is a symmetric (1,1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold \( \tilde{M} \) together with the unit timelike concircular vector field \( \xi \), its associated 1-form \( \eta \) and an (1,1) tensor field \( \phi \) is said to be a Lorentzian concircular structure manifold (briefly, \((LCS)_n\)-manifold), \cite{23}. Especially, if we take \( \alpha = 1 \), then we can obtain the LP-Sasakian structure of Matsumoto \cite{14}. In a \((LCS)_n\)-manifold \((n > 2)\), the following relations hold \cite{23}:

\[
(2.14) \quad \eta(\xi) = -1, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),
\]

\[
(2.15) \quad \phi^2 X = X + \eta(X)\xi,
\]

\[
(2.16) \quad \tilde{S}(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X),
\]

\[
(2.17) \quad \tilde{R}(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y],
\]

\[
(2.18) \quad \tilde{R}(\xi,Y)Z = (\alpha^2 - \rho)[g(Y, Z)\xi - \eta(Z)Y],
\]

\[
(2.19) \quad (\tilde{\nabla}_X \phi)Y = \alpha \{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\},
\]
(2.20) \( (X\rho) = d\rho(X) = \beta \eta(X) \),

\[ \tilde{R}(X, Y)Z = \phi \tilde{R}(X, Y)Z + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi \]

for all \( X, Y, Z \in \Gamma(T\tilde{M}) \) and \( \beta = -(\xi\rho) \) is a scalar function, where \( \tilde{R} \) is the curvature tensor and \( \tilde{S} \) is the Ricci tensor of the manifold.

Let \( M \) be a submanifold of dimension \( m \) of a \((LCS)_n\)-manifold \( \tilde{M} \) \((m < n)\) with induced metric \( g \). Also let \( \nabla \) and \( \nabla^\perp \) be the induced connection on the tangent bundle \( TM \) and the normal bundle \( T^\perp M \) of \( M \) respectively. Then the Gauss and Weingarten formulae are given by

\[ \tilde{\nabla}_XY = \nabla_CY + \sigma(X, Y) \]

and

\[ \tilde{\nabla}_XV = -A_VX + \nabla^\perp_XV \]

for all \( X, Y \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \), where \( \sigma \) and \( A_V \) are second fundamental form and the shape operator (corresponding to the normal vector field \( V \)) respectively for the immersion of \( M \) into \( \tilde{M} \). The second fundamental form \( \sigma \) and the shape operator \( A_V \) are related by

\[ g(\sigma(X, Y), V) = g(A_VX, Y) \]

for any \( X, Y \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \). We note that \( \sigma(X, Y) \) is bilinear and since \( \nabla fX = f\nabla_XY \) for any smooth function \( f \) on a manifold, we have

\[ \sigma(fX, Y) = f\sigma(X, Y) \]

For the second fundamental form \( \sigma \), the first and second covariant derivatives of \( \sigma \) are defined by

\[ (\tilde{\nabla}_X\sigma)(Y, Z) = \nabla_X^\perp(\sigma(Y, Z)) - \sigma(\nabla_XY, Z) - \sigma(Y, \nabla_XZ) \]

and

\[ (\tilde{\nabla}^2\sigma)(Z, W, X, Y) = (\tilde{\nabla}_X\tilde{\nabla}_Y\sigma)(Z, W) \]

\[ = \nabla_X^\perp((\tilde{\nabla}_Y\sigma)(Z, W) - (\tilde{\nabla}_Y\sigma)(\nabla_XZ, W) \]

\[ - (\tilde{\nabla}_X\sigma)(Z, \nabla_YW) - (\tilde{\nabla}_X\nabla_Y\sigma)(Z, W) \]

for any vector fields \( X, Y, Z \) tangent to \( M \). Then \( \tilde{\nabla}\sigma \) is a normal bundle valued tensor of type \((0,3)\) and is called the third fundamental form of \( M \), \( \tilde{\nabla} \) is called the Vander-Waerden-Bortolotti connection of \( \tilde{M} \), i.e. \( \tilde{\nabla} \) is the connection in \( TM \oplus T^\perp M \) built with \( \nabla \) and \( \nabla^\perp \). If \( \tilde{\nabla}\sigma = 0 \), then \( M \) is said to have parallel second fundamental form \([36]\).

The mean curvature vector \( H \) on \( M \) is given by

\[ H = \frac{1}{m} \sum_{i=1}^{m} \sigma(e_i, e_i) \]
where \( \{ e_1, e_2, \cdots, e_n \} \) is a local orthonormal frame of vector fields on \( M \).

A submanifold \( M \) of a \((LCS)_n\)-manifold \( \widetilde{M} \) is said to be totally umbilical if

\[
\sigma(X, Y) = g(X, Y)H,
\]

for any vector fields \( X, Y \in TM \). Moreover if \( \sigma(X, Y) = 0 \) for all \( X, Y \in TM \), then \( M \) is said to be totally geodesic and if \( H = 0 \) then \( M \) is minimal in \( \widetilde{M} \). For a \((0, l)\) tensor field \( T \), \( l \geq 1 \), and a symmetric \((0, 2)\) tensor field \( B \), we have

\[
Q(B, T)(X_1, \cdots, X_l; X, Y) = -T((X \wedge_B Y)X_1, X_2, \cdots, X_l) - \cdots - T(X_1, \cdots, X_{l-1}, (X \wedge_B Y)X_l),
\]

where

\[
(\tilde{X} \wedge B Y)Z = B(Y, Z)X - B(X, Z)Y.
\]

Putting \( T = \sigma \) and \( B = g \) or \( B = \sigma \), we obtain \( Q(g, \sigma) \) and \( Q(\sigma, \sigma) \) respectively.

An immersion is said to be pseudo parallel if

\[
\tilde{R}(X, Y) \cdot \sigma = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \tilde{\nabla}_{[X,Y]})\sigma = L_1 Q(g, \sigma)
\]

for all vector fields \( X, Y \) tangent to \( M \) \( [7] \). In particular, if \( L_1 = 0 \) then the manifold is said to be semiparallel. Again the submanifold \( M \) of a \((LCS)_n\)-manifold \( \widetilde{M} \) is said to be Ricci generalized pseudoparallel \( [7] \) if its second fundamental form \( \sigma \) satisfies

\[
\tilde{R}(X, Y) \cdot \sigma = L_2 Q(S, \sigma).
\]

Also the second fundamental form \( \sigma \) of submanifold \( M \) of a \((LCS)_n\)-manifold \( \widetilde{M} \) is said to be \( \eta \)-parallel \( [36] \) if

\[
(\nabla_X \sigma)(\phi Y, \phi Z) = 0
\]

for all vector fields \( X, Y \) and \( Z \) tangent to \( M \).

**Definition 2.2.** \([5]\) A submanifold \( M \) of a \((LCS)_n\)-manifold \( \widetilde{M} \) is said to be invariant if the structure vector field \( \xi \) is tangent to \( M \) at every point of \( M \) and \( \phi X \) is tangent to \( M \) for any vector field \( X \) tangent to \( M \) at every point of \( M \), that is \( \phi(TM) \subset TM \) at every point of \( M \).

From the Gauss and Weingarten formulae we obtain

\[
\widetilde{R}(X, Y)Z = R(X, Y)Z + A_{\sigma(X, Z)}Y - A_{\sigma(Y, Z)}X,
\]

where \( \widetilde{R}(X, Y)Z \) denotes the tangential part of the curvature tensor of the submanifold.

Now we have from tensor algebra

\[
(\tilde{R}(X, Y) \cdot \sigma)(Z, U) = R^\perp(X, Y)\sigma(Z, U) - \sigma(R(X, Y)Z, U) - \sigma(Z, R(X, Y)U)
\]

for all vector fields \( X, Y, Z \) and \( U \), where

\[
R^\perp(X, Y) = [\nabla^\perp_X, \nabla^\perp_Y] - \nabla^\perp_{[X,Y]}.
\]
In an invariant submanifold $M$ of a $(LCS)_n$-manifold $\tilde{M}$, the following relations hold [31]:

(2.36) \[ \nabla_X \xi = \alpha \phi X, \]

(2.37) \[ \sigma(X, \xi) = 0, \]

(2.38) \[ R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \]

(2.39) \[ S(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X), \quad \text{i.e., } Q\xi = (n-1)(\alpha^2 - \rho)\xi, \]

(2.40) \[ (\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \]

(2.41) \[ \sigma(X, \phi Y) = \phi \sigma(X, Y) = \sigma(\phi X, Y) = \sigma(X, Y) = \sigma(\phi X, \phi Y). \]

3. $(LCS)_n$-MANIFOLD WITH RESPECT TO QUARTER SYMMETRIC METRIC CONNECTION

Let $\widetilde{\nabla}$ be a linear connection and $\nabla$ be the Levi-Civita connection of a $(LCS)_n$-manifold $\tilde{M}$ such that

(3.1) \[ \widetilde{\nabla}_X Y = \nabla_X Y + U(X, Y), \]

where $U$ is a $(1,1)$ type tensor and $X, Y \in \Gamma(T\tilde{M})$. For $\widetilde{\nabla}$ to be a quarter symmetric metric connection on $\tilde{M}$, we have

(3.2) \[ U(X, Y) = \frac{1}{2}[T(X, Y) + T'(X, Y) + T'(Y, X)], \]

where

(3.3) \[ g(T'(X, Y), Z) = g(T(Z, X), Y). \]

From (3.2) and (3.3) we get

(3.4) \[ T'(X, Y) = \eta(X)\phi Y - g(Y, \phi X)\xi. \]

So,

(3.5) \[ U(X, Y) = \eta(Y)\phi X - g(Y, \phi X)\xi. \]

Therefore a quarter symmetric metric connection $\widetilde{\nabla}$ in a $(LCS)_n$-manifold $\tilde{M}$ is given by

(3.6) \[ \widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \]
Let \(\tilde{R}\) and \(\hat{R}\) be the curvature tensors of a \((LCS)_n\)-manifold \(\tilde{M}\) with respect to the quarter symmetric metric connection \(\tilde{\nabla}\) and the Levi-Civita connection \(\tilde{\nabla}\) respectively. Then we have

\[
\tilde{R}(X,Y)Z = \hat{R}(X,Y)Z + (2\alpha - 1) [g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X]
\]

\[
+ \alpha [\eta(Y)X - \eta(X)Y] \eta(Z)
\]

\[
+ \alpha [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \xi,
\]

where \(\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z\) and \(X, Y, Z \in \chi(\tilde{M})\).

By suitable contraction we have from (3.7) that

\[
\tilde{S}(Y,Z) = \hat{S}(Y,Z) + (\alpha - 1)g(Y, Z) + (n\alpha - 1)\eta(Y)\eta(Z)
\]

\[-(2\alpha - 1)a\phi Y,
\]

where \(\tilde{S}\) and \(\hat{S}\) are the Ricci tensors of \(\tilde{M}\) with respect to \(\tilde{\nabla}\) and \(\hat{\nabla}\) respectively and \(a = \text{trace}\phi\). Also we have

\[
\tilde{r} = \hat{r} - (2\alpha - 1)a^2 - (n - 1),
\]

where \(\tilde{r}\) and \(\hat{r}\) are the scalar curvature of \(\tilde{M}\) with respect to \(\tilde{\nabla}\) and \(\hat{\nabla}\) respectively. The relation (3.8) can be written as

\[
\tilde{Q}Y = \hat{Q}Y + (\alpha - 1)Y + (n\alpha - 1)\eta(Y)\xi - (2\alpha - 1)a\phi Y,
\]

where \(\tilde{Q}\) and \(\hat{Q}\) are the Ricci operator of \(\tilde{M}\) with respect to the connections \(\tilde{\nabla}\) and \(\hat{\nabla}\) respectively such that \(g(\tilde{Q}X, Y) = \tilde{S}(X, Y)\) and \(g(\hat{Q}X, Y) = \hat{S}(X, Y)\) for all \(X, Y \in (\chi M)\). Hence we get

\[
\tilde{R}(X,Y)\xi = \hat{R}(X,Y)\xi - \alpha[\eta(Y)X - \eta(X)Y]
\]

\[= (\alpha^2 - \alpha - \rho) [\eta(Y)X - \eta(X)Y],
\]

\[
\tilde{R}(X,\xi)Y = (\alpha^2 - \alpha - \rho) [\eta(Y)X - g(X, Y)\xi]
\]

\[-\hat{R}(\xi, X)Y,
\]

\[
\tilde{S}(Y, \xi) = (n - 1)(\alpha^2 - \alpha - \rho)\eta(Y),
\]

\[
\tilde{Q}\xi = (n - 1)(\alpha^2 - \alpha - \rho)\xi.
\]
Theorem 3.1. In a \((\text{LCS})_n\)-manifold \(\tilde{M}\) with respect to quarter symmetric metric connection \(\tilde{\nabla}\) we have
\[
\begin{align*}
\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y &= 0, \\
\tilde{R}(X,Y,Z,U) + \tilde{R}(Y,X,Z,U) &= 0, \\
\tilde{R}(X,Y,Z,U) + \tilde{R}(X,Y,U,Z) &= 0, \\
\tilde{R}(X,Y,Z,U) - \tilde{R}(Z,U,X,Y) &= 0.
\end{align*}
\]

4. Invariant submanifolds of \((\text{LCS})_n\)-Manifolds with respect to quarter symmetric metric connection

Let \(M\) be an invariant submanifold of a \((\text{LCS})_n\)-manifold \(\tilde{M}\) with the Levi-Civita connection \(\tilde{\nabla}\) and quarter symmetric metric connection \(\tilde{\nabla}\). Let \(\nabla\) be the induced connection on \(M\) from the connection \(\tilde{\nabla}\) and \(\tilde{\nabla}\) be the induced connection on \(M\) from the connection \(\tilde{\nabla}\).

Let \(\sigma\) and \(\tilde{\sigma}\) be second fundamental form with respect to Levi-Civita connection and quarter symmetric metric connection respectively.

Then we have
\[
(4.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + \tilde{\sigma}(X,Y).
\]
From (3.6) and (4.1) we get
\[
(4.2) \quad \nabla_X Y + \sigma(X,Y) = \tilde{\nabla}_X Y + \eta(Y)\phi X - g(\phi X,Y)\xi,
\]
\[
(4.3) \quad \nabla_X Y + \sigma(X,Y) + \eta(Y)\phi X - g(\phi X,Y)\xi.
\]
Since \(M\) is invariant so \(\phi X, \xi \in TM\) and therefore equating tangential and normal parts we get
\[
(4.4) \quad \nabla_X Y = \tilde{\nabla}_X Y + \eta(Y)\phi X - g(\phi X,Y)\xi,
\]
\[
(4.5) \quad \tilde{\sigma}(X,Y) = \sigma(X,Y).
\]
i.e., the second fundamental form with respect to \(\tilde{\nabla}\) and \(\tilde{\nabla}\) are same.

Also from (3.6) and (4.1) we can say that the invariant submanifold \(M\) admits quarter symmetric metric connection.

Hence we get the following:

Theorem 4.1. Let \(M\) be an invariant submanifold of a \((\text{LCS})_n\)-manifold \(\tilde{M}\) with the Levi-Civita connection \(\tilde{\nabla}\) and quarter symmetric metric connection \(\tilde{\nabla}\) and \(\nabla\) be the induced connection on \(M\) from the connection \(\tilde{\nabla}\) and \(\tilde{\nabla}\) be the induced connection on \(M\) from the connection \(\tilde{\nabla}\). If \(\sigma\) and \(\tilde{\sigma}\) are the second fundamental forms with respect to the Levi-Civita connection and quarter symmetric metric connection respectively then
(i) $M$ admits quarter symmetric metric connection.
(ii) The second fundamental forms with respect to $\tilde{\nabla}$ and $\nabla$ are equal.

We now prove the following:

**Theorem 4.2.** The mean curvature of an invariant submanifold $M$ of a $\mathrm{(LCS)}_n$-manifold $\tilde{M}$ remains invariant under quarter symmetric metric connection.

**Proof.** Let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis of $TM$. Therefore, from (3.1) we get

$$\overline{\sigma}(e_i, e_i) = \sigma(e_i, e_i).$$

Summing up for $i = 1, 2, \ldots, m$ and dividing by $m$ we get the desired result. \qed

**Corollary 4.1.** The invariant submanifold $M$ of $\mathrm{(LCS)}_n$-manifold $\tilde{M}$ with respect to quarter symmetric metric connection is minimal with respect to the quarter symmetric metric connection if and only if it is minimal with respect to the Levi-Civita connection.

**Corollary 4.2.** The invariant submanifold $M$ of $\mathrm{(LCS)}_n$-manifold $\tilde{M}$ is totally umbilical with respect to the quarter symmetric metric connection if and only if it is totally umbilical with respect to the Levi-Civita connection.

We now construct an example:

**Example 4.1.** Let us consider the 5-dimensional manifold $\tilde{M} = \{(x, y, z, u, v) \in \mathbb{R}^5 : (x, y, z, u, v) \neq (0, 0, 0, 0, 0)\}$, where $(x, y, z, u, v)$ are the standard coordinates in $\mathbb{R}^5$. The vector fields

$$e_1 = e^{-z} \frac{\partial}{\partial x}, \quad e_2 = e^{-z} \frac{\partial}{\partial y}, \quad e_3 = e^{-2z} \frac{\partial}{\partial z}, \quad e_4 = e^{-z} \frac{\partial}{\partial u}, \quad e_5 = e^{-z} \frac{\partial}{\partial v}$$

are linearly independent at each point of $\tilde{M}$.

Let $\tilde{g}$ be the metric defined by

$$\tilde{g}(e_i, e_j) = \begin{cases} 1, & \text{for } i = j \neq 3, \\ 0, & \text{for } i \neq j, \\ -1, & \text{for } i = j = 3. \end{cases}$$

Here $i$ and $j$ runs over 1 to 5.

Let $\eta$ be the 1-form defined by $\eta(Z) = \tilde{g}(Z, e_3)$, for any vector field $Z \in \chi(\tilde{M})$. Let $\phi$ be the (1,1) tensor field defined by $\phi e_1 = e_1$, $\phi e_2 = e_2$, $\phi e_3 = 0$, $\phi e_4 = e_4$, $\phi e_5 = e_5$. Then using the linearity property of $\phi$ and $\tilde{g}$ we have $\eta(e_3) = -1$, $\phi^3 U = U + \eta(U) \xi$ and $\tilde{g}(\phi U, \phi V) = \tilde{g}(U, V) + \eta(U)\eta(V)$, for every $U, V \in \chi(\tilde{M})$. Thus for $e_3 = \xi$, $(\phi, \xi, \eta, g)$ defines a Lorentzian paracontact structure on $\tilde{M}$. Let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$ with respect to the metric $\tilde{g}$. Then we have $[e_1, e_3] = e^{-2z} e_1$, $[e_2, e_3] = e^{-2z} e_2$, $[e_4, e_3] = e^{-2z} e_4$, $[e_5, e_3] = e^{-2z} e_5$.

Now, using Koszul’s formula for $\tilde{g}$, it can be calculated that $\tilde{\nabla}_e_1 e_1 = e^{-2z} e_3$, $\tilde{\nabla}_e_1 e_3 = e^{-2z} e_1$, $\tilde{\nabla}_e_2 e_2 = e^{-2z} e_3$, $\tilde{\nabla}_e_2 e_3 = e^{-2z} e_2$, $\tilde{\nabla}_e_4 e_3 = e^{-2z} e_4$, $\tilde{\nabla}_e_4 e_4 = e^{-2z} e_3$, $\tilde{\nabla}_e_5 e_3 = e^{-2z} e_5$, and $\tilde{\nabla}_e_5 e_5 = e^{-2z} e_3$. 


and rest of the terms are zero.

Since \( \{e_1, e_2, e_3, e_4, e_5\} \) is a frame field, then any vector field \( X, Y \in T\widetilde{M} \) can be written as

\[
\begin{align*}
X &= x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5, \\
y &= y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4 + y_5e_5.
\end{align*}
\]

where \( x_i, y_i \in \mathbb{R}, \ i = 1, 2, 3, 4, 5 \) such that

\[
x_1y_1 + x_2y_2 - x_3y_3 + x_4y_4 + x_5y_5 \neq 0
\]

and hence

\[
(4.6) \quad \widetilde{g}(X, Y) = (x_1y_1 + x_2y_2 - x_3y_3 + x_4y_4 + x_5y_5).
\]

Therefore,

\[
(4.7) \quad \nabla_X Y = e^{-2z}[x_1y_2e_1 + x_2y_3e_2 + (x_1y_1 + x_2y_2 + x_4y_4 + x_5y_5)e_3
+ x_4y_3e_4 + x_5y_3e_5]
\]

and

\[
(4.8) \quad \nabla_X Y = e^{-2z}[x_1y_3e_1 + x_2y_3e_2 + (x_1y_1 + x_2y_2 + x_4y_4 + x_5y_5)e_3
+ x_4y_3e_4 + x_5y_3e_5] - y_3(x_1e_1 + x_2e_2 + x_4e_4 + x_5e_5)
- (x_1y_1 + x_2y_2 + x_4y_4 + x_5y_5)e_3
\]

From the above it can be easily seen that \((\phi, \xi, \eta, g)\) is a \((LCS)_5\) structure on \(\widetilde{M}\) with \(\alpha = e^{-2z} \neq 0\) such that \(X(\alpha) = \rho g(X)\) where \(\rho = 2e^{-4z}\). Also \(\nabla_X g = 0\). Thus in an \((LCS)_5\)-manifold quarter symmetric metric connection is given by (4.8).

Let \( f \) be an isometric immersion from \( M \) to \( \widetilde{M} \) defined by \( f(x, y, z) = (x, y, z, 0, 0) \). Let \( M = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\} \), where \((x, y, z)\) are the standard coordinates in \(\mathbb{R}^3\). The vector fields

\[
e_1 = e^{-z}\frac{\partial}{\partial x}, \quad e_2 = e^{-z}\frac{\partial}{\partial y}, \quad e_3 = e^{-2z}\frac{\partial}{\partial z}
\]

are linearly independent at each point of \( M \).

Let \( g \) be the induced metric defined by

\[
g(e_i, e_j) = \begin{cases} 1, & \text{for } i = j \neq 3, \\ 0, & \text{for } i \neq j, \\ -1, & \text{for } i = j = 3. \end{cases}
\]

Here \( i \) and \( j \) runs over 1 to 3.

Let \( \nabla \) be the Levi-Civita connection on \( M \) with respect to the metric \( g \). Then we have \([e_1, e_3] = e^{-2z}e_1, [e_2, e_3] = e^{-2z}e_2\). Now, using Koszul’s formula for \( g \), it can be calculated that

\[
\nabla_{e_1}e_1 = e^{-2z}e_3, \quad \nabla_{e_1}e_3 = e^{-2z}e_1, \quad \nabla_{e_2}e_2 = e^{-2z}e_3, \quad \nabla_{e_2}e_3 = e^{-2z}e_2,
\]

and rest of the terms are zero. Clearly \( \{e_4, e_5\} \) is the frame field for the normal
bundle $T^\perp M$. If we take $Z \in TM$ then $\phi Z \in TM$ and therefore $M$ is an invariant submanifold of $\tilde{M}$. If we take $X, Y \in TM$ then we can express them as

\[
X = x_1 e_1 + x_2 e_2 + x_3 e_3, \\
Y = y_1 e_1 + y_2 e_2 + y_3 e_3.
\]

Therefore

\[
\nabla_X Y = e^{-2z}[x_1 y_3 e_1 + x_2 y_3 e_2 + (x_1 y_1 + x_2 y_2) e_3].
\]

Therefore the second fundamental form

\[
(4.9) \quad \sigma(X, Y) = e^{-2z}(x_4 y_3 e_4 + x_5 y_3 e_5).
\]

Now, for $X, Y \in TM$ we have

\[
\tilde{\nabla}_X Y = e^{-2z}[x_1 y_3 e_1 + x_2 y_3 e_2 + (x_1 y_1 + x_2 y_2) e_3] - y_3 (x_1 e_1 + x_2 e_2)
- (x_1 y_1 + x_2 y_2) e_3.
\]

Tangential part of $\tilde{\nabla}_X Y$ is given by

\[
\nabla_X Y = e^{-2z}[x_1 y_3 e_1 + x_2 y_3 e_2 + (x_1 y_1 + x_2 y_2) e_3] - y_3 (x_1 e_1 + x_2 e_2)
- (x_1 y_1 + x_2 y_2) e_3.
\]

which means $M$ admits quarter symmetric metric connection and the normal part of $\tilde{\nabla}_X Y$ is given by

\[
\tilde{\sigma}(X, Y) = e^{-2z}(x_4 y_3 e_4 + x_5 y_3 e_5)
= \sigma(X, Y),
\]

which implies that the second fundamental forms with respect to $\tilde{\nabla}$ and $\tilde{\nabla}$ are equal. Also $H$ corresponds to Levi-Civita connection as well as quarter symmetric metric connection is zero. Therefore, $M$ is totally umbilical with respect to both the connections.

Thus Theorem 4.1, Theorem 4.2, Corollary 4.1 and Corollary 4.2 are verified. From (2.37) we get

\[
(4.10) \quad \tilde{\sigma}(X, \xi) = 0.
\]

Also from (3.6) we have for $V \in T^\perp M$

\[
\tilde{\nabla}_X V = \tilde{\nabla}_X V + \eta(V) \phi X - g(\phi X, V) \xi
= \tilde{\nabla}_X V
\]

From (2.23) we get

\[
(4.11) \quad \tilde{\nabla}_X V = -A V X + \nabla_X^\perp V
\]
Theorem 4.3. Let \( M \) be an invariant submanifold of a \((LCS)_n\)-manifold \( \widetilde{M} \) with respect to a quarter symmetric metric connection. Then Gauss and Weingarten formulae with respect to quarter symmetric metric connection are given by

\[
\tan(\widetilde{R}(X,Y)Z) = \widetilde{R}(X,Y)Z + \eta(\nabla_Y Z)\phi X - \eta(\nabla_X Z)\phi Y - g(\phi X, \nabla_Y Z)\xi + g(\phi Y, \nabla_X Z)\xi - \eta(\nabla_X Y)\phi Z + A_{\sigma(Y,Z)}X + A_{\sigma(Z,Y)}Y + \nabla_X \eta(Z)\phi Y - \nabla_Y \phi X + \eta(Z)\phi Y - \eta(Z)\phi X + (\alpha - 1)g(\phi Y, Z)\phi Y + (\alpha - 1)g(\phi X, Z)\phi Y - \eta(Z)\phi [X,Y] + g(\phi [X,Y], Z)\xi,
\]

(4.12)

\[
nor(\widetilde{R}(X,Y)Z) = \sigma(X,\nabla_Y Z) - \sigma(Y,\nabla_X Z) + \nabla^\perp_X \sigma(Y, Z) - \nabla^\perp_Y \sigma(X, Z) - \nabla_X \phi X + \eta(Z)\phi(Y) - \eta(Z)\phi(X) - \sigma([X,Y], Z)
\]

(4.13)

Proof. The Riemannian curvature tensor \( \widetilde{R} \) on \( \widetilde{M} \) with respect to quarter symmetric metric connection is given by

\[
\widetilde{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]

(4.14)

By using (2.37), (2.41), (3.6), (4.1), (4.5) and (4.11) in (4.14) we get

\[
\tilde{R}(X,Y)Z = \tilde{R}(X,Y)Z + \sigma(X,\nabla_Y Z) + \eta(\nabla_Y Z)\phi X - g(\phi X, \nabla_Y Z)\xi + A_{\sigma(Y,Z)}X + \nabla^\perp_X \sigma(Y, Z) + \nabla_X \eta(Z)\phi Y - \nabla_Y \phi X + \eta(Z)\phi Y - \eta(Z)\phi X + (\alpha - 1)g(\phi Y, Z)\phi Y + (\alpha - 1)g(\phi X, Z)\phi Y - \eta(\nabla_X Y)\phi Z + A_{\sigma(Z,Y)}Y - \nabla_X \eta(Z)\phi Y + \nabla_Y \phi Z - \eta(Z)\phi Y - \eta(Z)\phi X + \nabla_Y g(\phi X, Z)\xi + (\alpha - 1)g(\phi X, Z)\phi Y - \sigma([X,Y], Z) - \eta(Z)\phi [X,Y] + g(\phi [X,Y], Z)\xi.
\]

(4.15)

Comparing the tangential and normal part of the equation (4.15) we get the Gauss and Weingarten formulae as (4.12) and (4.13).

\( \square \)

5. RECURRENT INVARIANT SUBMANIFOLD OF \((LCS)_n\)-MANIFOLD WITH RESPECT TO QUARTER SYMMETRIC METRIC CONNECTION

We consider invariant submanifold of a \((LCS)_n\)-manifold when \( \sigma \) is recurrent, 2-recurrent, generalized 2-recurrent and \( M \) has parallel third fundamental form with respect to quarter symmetric metric connection. We write the equations (2.26) and
with respect to quarter symmetric metric connection in the form
\begin{equation}
(\nabla X\sigma)(Y, Z) = \nabla^X_\perp(\sigma(Y, Z)) - \sigma(\nabla X Y, Z) - \sigma(Y, \nabla X Z)
\end{equation}
\begin{equation}
(\nabla^2 \sigma)(Z, W, X, Y) = (\nabla X \nabla Y \sigma)(Z, W)
= \nabla^X_\perp((\nabla Y \sigma)(Z, W)) - (\nabla Y \sigma)(\nabla X Z, W)
- (\nabla X \sigma)(Z, \nabla Y W) - (\nabla_{\nabla X Y} \sigma)(Z, W).
\end{equation}

**Theorem 5.1.** Let \( M \) be an invariant submanifold of a \((\text{LCS})_n\)-manifold \( \tilde{M} \) with respect to quarter symmetric metric connection. Then \( \sigma \) is recurrent with respect to the quarter symmetric metric connection if and only if it is totally geodesic with respect to the Levi-Civita connection, provided \( \alpha \neq 1 \).

**Proof.** Let \( \sigma \) be recurrent with respect to quarter symmetric metric connection. Then from (2.3) we get
\begin{equation}
(\nabla X \sigma)(Y, Z) = \pi(X)\sigma(Y, Z),
\end{equation}
where \( \pi \) is a 1-form on \( M \). By using (5.1) and \( Z = \xi \) in the above equation we have
\begin{equation}
\nabla^X_\perp(\sigma(Y, \xi)) - \sigma(\nabla X Y, \xi) - \sigma(Y, \nabla X \xi) = \pi(X)\sigma(Y, \xi),
\end{equation}
which by virtue of (2.37) reduces to
\begin{equation}
- \sigma(\nabla X Y, \xi) - \sigma(Y, \nabla X \xi) = 0.
\end{equation}
Using (2.36), (2.37), (2.41) and (4.4) we obtain \((\alpha - 1)\sigma(X, Y) = 0\). So, we get \( \sigma(X, Y) = 0 \), provided \( \alpha \neq 1 \) i.e., \( M \) is totally geodesic, provided \( \alpha \neq 1 \). The converse statement is trivial. This proves the theorem.

**Theorem 5.2.** Let \( M \) be an invariant submanifold of a \((\text{LCS})_n\)-manifold \( \tilde{M} \) admitting quarter symmetric metric connection. Then \( M \) has parallel third fundamental form with respect to the quarter symmetric metric connection if and only if it is totally geodesic with respect to the Levi-Civita connection, provided \( \alpha \neq 1 \).

**Proof.** Let \( M \) has parallel third fundamental form with respect to quarter symmetric metric connection. Then we have
\begin{equation}
(\nabla X \nabla Y \sigma)(Z, W) = 0.
\end{equation}
Taking \( W = \xi \) and and using (5.2) in the above equation, we get
\begin{equation}
0 = \nabla^X_\perp((\nabla Y \sigma)(Z, \xi)) - (\nabla Y \sigma)(\nabla X Z, \xi)
- (\nabla X \sigma)(Z, \nabla Y \xi) - (\nabla_{\nabla X Y} \sigma)(Z, \xi).
\end{equation}
By using (2.37) and (5.1) in (5.5) we get
\begin{equation}
0 = -\nabla^X_\perp\{\sigma(\nabla Y Z, \xi) + \sigma(Z, \nabla Y \xi)\} - \nabla Y \sigma(\nabla X Z, \xi)
+ \sigma(\nabla Y \nabla X Z, \xi) + 2\sigma(\nabla X Z, \nabla Y \xi) - \nabla X \sigma(Z, \nabla Y \xi)
+ \sigma(Z, \nabla X \nabla Y \xi) + \sigma(\nabla_{\nabla X Y} Z, \xi) + \sigma(Z, \nabla_{\nabla X Y} \xi).
\end{equation}
In view of (2.36), (2.37), (2.41) and (4.4) the equation (5.6) gives

\[
0 = -2(\alpha - 1)\nabla^\perp_X \sigma(Z, Y) + 2(\alpha - 1)\sigma(\nabla_X Z, Y) + 2(\alpha - 1)\eta(Z)\sigma(X, Y) + (\alpha - 1)\sigma(Z, \nabla_X \phi Y) + (\alpha - 1)\sigma(Z, \nabla_X Y) + (\alpha - 1)\eta(Y)\sigma(Z, X).
\]

Putting \(Z = \xi\) in (5.7) and using (2.14), (2.36), (2.37) and (2.41) we obtain

\[2(\alpha - 1)^2 \sigma(X, Y) = 0.\]

Therefore, \(M\) is totally geodesic provided \(\alpha \neq 1\). The converse statement is trivial. This proves the theorem. \(\square\)

**Corollary 5.1.** Let \(M\) be an invariant submanifold of a \((LCS)_n\)-manifold \(\tilde{M}\) admitting quarter symmetric metric connection. Then \(\sigma\) is 2-recurrent with respect to the quarter symmetric metric connection if and only if it is totally geodesic with respect to the Levi-Civita connection, provided \(\alpha \neq 1\).

**Proof.** Let \(\sigma\) be 2-recurrent with respect to quarter symmetric metric connection. Then we have,

\[
(\overline{\nabla}_X \overline{\nabla}_Y \sigma)(Z, W) = \psi(X, Y)\sigma(Z, W),
\]

where \(\psi\) is a 2-form on \(M\). Taking \(W = \xi\) and using (2.37) in the above equation we get

\[
(\overline{\nabla}_X \overline{\nabla}_Y \sigma)(Z, \xi) = 0.
\]

\(\square\)

**Theorem 5.3.** Let \(M\) be an invariant submanifold of a \((LCS)_n\)-manifold \(\tilde{M}\) admitting quarter symmetric metric connection. Then \(\sigma\) is generalized 2-recurrent with respect to the quarter symmetric metric connection if and only if it is totally geodesic with respect to the Levi-Civita connection, provided \(\alpha \neq 1\).

**Proof.** Let \(\sigma\) be generalized 2-recurrent with respect to the quarter symmetric metric connection. Then from (2.5) we have

\[
(\overline{\nabla}_X \overline{\nabla}_Y \sigma)(Z, W) = \psi(X, Y)\sigma(Z, W) + \pi(X)(\overline{\nabla}_Y \sigma)(Z, W),
\]

where \(\psi\) and \(\pi\) are 2-form and 1-form respectively.

Taking \(W = \xi\) in (5.9) and using (2.37) we get

\[
(\overline{\nabla}_X \overline{\nabla}_Y \sigma)(Z, \xi) = \pi(X)(\overline{\nabla}_Y \sigma)(Z, \xi).
\]

In view of (2.36), (2.37), (2.41), (4.4), (5.1) and (5.2) the equation (5.10) gives

\[
-2(\alpha - 1)\overline{\nabla}_X^\perp \sigma(Z, Y) + 2(\alpha - 1)\sigma(\nabla_X Z, Y) + 2(\alpha - 1)\eta(Z)\sigma(X, Y) + (\alpha - 1)\sigma(Z, \nabla_X \phi Y) + (\alpha - 1)\sigma(Z, \nabla_X Y) + (\alpha - 1)\eta(Y)\sigma(Z, X) = -2(\alpha - 1)^2 \pi(X)\sigma(Y, Z).
\]
Putting $Z = \xi$ in (5.11) and using (2.14), (2.36), (2.37) and (2.41) we obtain $2(\alpha - 1)^2 \sigma(X, Y) = 0$. Therefore, $M$ is totally geodesic provided $\alpha \neq 1$. The converse statement is trivial. This proves the theorem. \hfill \Box

From Theorem 5.1, Theorem 5.2, Corollary 5.1 and Theorem 5.3, we can state the following:

**Theorem 5.4.** Let $M$ be an invariant submanifold of a $(LCS)_n$-manifold $\tilde{M}$ admitting quarter symmetric metric connection. Then the following statements are equivalent:

1. $\sigma$ is recurrent
2. $\sigma$ is $2$-recurrent with $\alpha \neq 1$.
3. $\sigma$ is generalized $2$-recurrent with $\alpha \neq 1$.
4. $M$ has parallel third fundamental form, with $\alpha \neq 1$.
5. $M$ is totally geodesic with respect to Levi-Civita connection.

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