On a two-weight criteria for multidimensional Hardy type operator in $p$-convex Banach function spaces and some application

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ABSTRACT. The main goal of this paper is to prove a two-weight criteria for multidimensional Hardy type operator from weighted Lebesgue spaces into $p$-convex weighted Banach function spaces. Analogously problem for the dual operator is considered. As application we prove a two-weight criteria for boundedness of multidimensional geometric mean operator and sufficient condition on the weights for boundedness of certain sublinear operator from weighted Lebesgue spaces into weighted Musielak-Orlicz spaces.

Keywords and phrases: Banach function spaces, weights, Hardy type operator, geometric mean operator, certain sublinear operator.

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1. Introduction.

The investigation of Hardy operator in weighted Banach function spaces (BFS) have recently history. The goal of this investigations were closely connected with the found of criterion on the geometry and on the weights of BFS for validity of boundedness of Hardy operator in BFS. Characterization of the mapping properties such as boundedness and compactness were considered in the papers [7], [8], [12], [30] and e.t.c. More precisely, in [7] and [8] were considered the boundedness of certain integral operator in ideal Banach spaces. In [12] was proved the boundedness of Hardy operator in Orlicz spaces. Also, in [30] the compactness and measure of non-compactness of Hardy type operator in Banach function spaces was proved. But in this paper we consider the boundedness of Hardy operator in $p$-convex Banach function spaces and find a new type criterion on the weights for validity of Hardy inequality. Note that the notion of BFS was introduced in [32]. In particular, the weighted Lebesgue spaces, weighted Lorentz spaces, weighted variable Lebesgue spaces, variable Lebesgue spaces with mixed norm, Musielak-Orlicz spaces and e.t.c. is BFS.

In this paper, we establish an integral-type necessary and sufficient condition on weights, which provides the boundedness of the multidimensional Hardy type operator from weighted Lebesgue spaces into $p$-convex weighted BFS. We also investigate the corresponding problems for the dual operator. It is well known that the classical two weight inequality for geometric mean operator is closely connected with the one-dimensional Hardy inequality (see [20]). Analogously, the Pólya-Knopp type inequalities with multidimensional geometric mean
operator are connected with the multidimensional Hardy type operator. Therefore, in this paper, as an application of Hardy inequality we prove the boundedness of multidimensional geometric mean operator and boundedness of certain sublinear operator from weighted Lebesgue spaces into weighted Musielak-Orlicz spaces.

2. Preliminaries

Let \((\Omega, \mu)\) be a complete \(\sigma\)-finite measure space. By \(L_0 = L_0(\Omega, \mu)\) we denote the collection of all real-valued \(\mu\)-measurable functions on \(\Omega\).

Definition 1. [32, 29, 6] We say that real normed space \(X\) is a Banach function space (BFS) if:

(P1) the norm \(\| f \|_X\) is defined for every \(\mu\)-measurable function \(f\), and \(f \in X\) if and only if \(\| f \|_X < \infty\); \(\| f \|_X = 0\) if and only if \(f = 0\) a.e.;

(P2) \(\| f \|_X = \| | f | \|_X\) for all \(f \in X\);

(P3) if \(0 \leq f \leq g\) a.e., then \(\| f \|_X \leq \| g \|_X\);

(P4) if \(0 \leq f_n \uparrow f\) a.e., then \(\| f_n \|_X \uparrow \| f \|_X\) (Fatou property);

(P5) if \(E\) is a measurable subset of \(\Omega\) such that \(\mu(E) < \infty\), then \(\| \chi_E \|_X < \infty\), where \(\chi_E\) is the characteristic function of the set \(E\);

(P6) for every measurable set \(E \subset \Omega\) with \(\mu(E) < \infty\), there is a constant \(C_E > 0\) such that \(\int_E f(x) \, dx \leq C_E \| f \|_X\).

Given a BFS \(X\) we can always consider its associate space \(X'\) consisting of those \(g \in L_0\) that \(f \cdot g \in L_1\) for every \(f \in X\) with the usual order and the norm \(\| g \|_{X'} = \sup \{ \| f \cdot g \|_{L_1} : \| g \|_{X'} \leq 1 \}\). Note that \(X'\) is a BFS in \((\Omega, \mu)\) and a closed norming subspaces.

Let \(X\) be a BFS and \(\omega\) be a weight, that is, positive Lebesgue measurable and a.e. finite functions on \(\Omega\). Let \(X_\omega = \{ f \in L_0 : f \omega \in X \}\). This space is a weighted BFS equipped with the norm \(\| f \|_{X_\omega} = \| f \omega \|_X\). (For more detail and proofs of results about BFS we refer the reader to [6] and [29].)

Note that the notion of BFS was introduced in [32].

Let us recall the notion of \(p\)-convexity and \(p\)-concavity of BFS’s.

Definition 2. [43] Let \(X\) be a BFS. Then \(X\) is called \(p\)-convex for \(1 \leq p \leq \infty\) if there exists a constant \(M > 0\) such that for all \(f_1, \ldots, f_n \in X\)

\[
\left\| \left( \sum_{k=1}^{n} |f_k|^p \right)^{\frac{1}{p}} \right\|_X \leq M \left( \sum_{k=1}^{n} \| f_k \|_X^p \right)^{\frac{1}{p}} \text{ if } 1 \leq p < \infty,
\]
Lemma 1. The following Lemma shows that the variable Lebesgue spaces \( L_p(\Omega) \) if \( p = \infty \). Similarly \( X \) is called \( p \)-concave for \( 1 \leq p \leq \infty \) if there exists a constant \( M > 0 \) such that for all \( f_1, \ldots, f_n \in X \)

\[
\left( \sum_{k=1}^{n} ||f_k||_X^p \right)^{\frac{1}{p}} \leq M \left( \sum_{k=1}^{n} ||f_k||_X \right)^{\frac{1}{q}} \text{ if } 1 \leq p < \infty,
\]

or \( \max_{1 \leq k \leq n} ||f_k||_X \leq M \sup_{1 \leq k \leq n} ||f_k||_X \) if \( p = \infty \).

Remark 1. Note that the notions of \( p \)-convexity, respectively \( p \)-concavity are closely related to the notions of upper \( p \)-estimate (strong \( \ell_p \)-composition property), respectively lower \( p \)-estimate (strong \( \ell_p \)-decomposition property) as can be found in [29].

Now we reduce some examples of \( p \)-convex and respectively \( p \)-concave BFS. Let \( R^n \) be the \( n \)-dimensional Euclidean space of points \( x = (x_1, \ldots, x_n) \) and let \( \Omega \) be a Lebesgue measurable subset in \( R^n \) and \( |x| = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2} \). The Lebesgue measure of a set \( \Omega \) will be denoted by \( |\Omega| \). It is well known that \( |B(0, 1)| = \frac{\pi^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} + 1 \right)} \), where \( B(0, 1) = \{ x : x \in R^n ; |x| < 1 \} \).

Example 1.1. Let \( 1 \leq q \leq \infty \) and \( X = L_q \). Then the space \( L_q \) is \( p \)-convex (\( p \)-concave) BFS if and only if \( 1 \leq p \leq q \leq \infty \) (\( 1 \leq q \leq p \leq \infty \)).

The proof implies from usual Minkowski inequality in Lebesgue spaces.

Example 1.2. The following Lemma shows that the variable Lebesgue spaces \( L_{q(y)}(\Omega) \) is \( p \)-convex BFS.

Lemma 1. [1] Let \( 1 \leq p \leq q(x) \leq \overline{q} < \infty \) for all \( y \in \Omega_2 \subset R^n \). Then the inequality

\[
||f||_{L_p(\Omega_1)} ||f||_{L_{q(y)}(\Omega_2)} \leq C_{p,q} ||f||_{L_{q(y)}(\Omega_2)} \leq L_p(\Omega_1)
\]

is valid, where \( C_{p,q} = \left( \|\chi_{\Delta_1}\|_\infty^p + \|\chi_{\Delta_2}\|_\infty^p + q \left( \frac{1}{q} - \frac{1}{\overline{q}} \right) \right)^{\frac{1}{p}} \|\chi_{\Delta_1}\|_\infty^p + \|\chi_{\Delta_2}\|_\infty^p \), \( \overline{q} = \text{ess sup}_{\Omega_2} q(x) \), \( \Delta_1 = \{(x, y) \in \Omega_1 \times \Omega_2 : q(y) = p \} \), \( \Delta_2 = \Omega_1 \times \Omega_2 \setminus \Delta_1 \) and \( f : \Omega_1 \times \Omega_2 \rightarrow R \) is any measurable function such that

\[
||f||_{L_p(\Omega_1)} ||f||_{L_{q(y)}(\Omega_2)} = \inf \left\{ \mu > 0 : \int_{\Omega_2} \left( \frac{\|f(\cdot, y)\|_{L_p(\Omega_1)}}{\mu} \right)^{q(y)} \, dy \leq 1 \right\} < \infty
\]

and \( ||f(\cdot, y)||_{L_p(\Omega_1)} = \left( \int_{\Omega_1} |f(x, y)|^p \, dx \right)^{1/p} \).
Analogously, if \(1 \leq q(x) \leq p < \infty\), then \(L_{q(x)}(\Omega)\) is \(p\)-concave BFS.

**Definition 3.** [38, 15]. Let \(\Omega \subset \mathbb{R}^n\) be a Lebesgue measurable set. A real function \(\varphi : \Omega \times [0, \infty) \mapsto [0, \infty)\) is called a generalized \(\varphi\)-function if it satisfies:

a) \(\varphi(x, \cdot)\) is a \(\varphi\)-function for all \(x \in \Omega\), i.e., \(\varphi(x, \cdot) : [0, \infty) \mapsto [0, \infty)\) is convex and satisfies \(\varphi(x, 0) = 0\), \(\lim_{t \to +0} \varphi(x, t) = 0\);

b) \(\psi : x \mapsto \varphi(x, t)\) is measurable for all \(t \geq 0\).

If \(\varphi\) is a generalized \(\varphi\)-function on \(\Omega\), we shortly write \(\varphi \in \Phi\).

**Definition 4.** [38, 14]. Let \(\varphi \in \Phi\) and be \(\rho_\varphi\) defined by the expression

\[
\rho_\varphi(f) := \int_\Omega \varphi(y, |f(y)|) \, dy \quad \text{for all } f \in L_0(\Omega).
\]

We put \(L_\varphi = \{f \in L_0(\Omega) : \rho_\varphi(\lambda_0 f) < \infty \quad \text{for some} \quad \lambda_0 > 0\}\) and

\[
\|f\|_{L_\varphi} = \inf \left\{ \lambda > 0 : \rho_\varphi \left( \frac{f}{\lambda} \right) \leq 1 \right\}.
\]

The space \(L_\varphi\) is called Musielak-Orlicz space.

Let \(\omega\) be a weight function on \(\Omega\), i.e., \(\omega\) is a non-negative, almost everywhere positive function on \(\Omega\). In this work we considered the weighted Musielak-Orlicz spaces. We denote

\[
L_{\varphi, \omega} = \{f \in L_0(\Omega) : f \omega \in L_\varphi\}.
\]

It is obvious that the norm in this spaces is given by

\[
\|f\|_{L_{\varphi, \omega}} = \|f \omega\|_{L_\varphi}.
\]

**Remark 2.** Let \(\varphi(x, t) = t^{q(x)}\) in the Definition 4, where \(1 \leq q(x) < \infty\) and \(x \in \Omega\). Then we have the definition of variable exponent weighted Lebesgue spaces \(L_{q(x)}(\Omega)\) (see [14]).

**Example 1.3.** The following Lemma shows that the Musielak-Orlicz spaces \(L_\varphi\) is \(p\)-convex BFS.

**Lemma 2.** [4] Let \(\Omega_1 \subset \mathbb{R}^n\) and \(\Omega_2 \subset \mathbb{R}^m\). Let \((x, t) \in \Omega_1 \times [0, \infty), \) and \(\varphi(x, t^{1/p}) \in \Phi\) for some \(1 \leq p < \infty\). Suppose \(f : \Omega_1 \times \Omega_2 \mapsto R\). Then the inequality

\[
\left\| \int f(x, \cdot) \right\|_{L_p(\Omega_2)} \leq 2^{1/p} \left\| \int f(\cdot, y) \right\|_{L_p(\Omega_2)}
\]

is valid.
Definition 5. [38] We say that \( \varphi \in \Phi \) satisfies the \( \Delta_2 \)-condition if there exists \( K \geq 2 \) such that
\[
\varphi(y, 2t) \leq K \varphi(y, t)
\] (1.1)
for all \( y \in \Omega \) and all \( t > 0 \). The smallest such \( K \) is called the \( \Delta_2 \)-constant of \( \varphi \).

Lemma 3. Let \( \varphi \in \Phi \) and \( 1 < s \leq q(y) \leq q < \infty \). Suppose for all \( C > 0 \) the condition
\[
\varphi(y, Ct) \leq C^{q(y)} \varphi(y, t)
\] (1.2)
holds, where \( y \in \Omega \) and \( t > 0 \).

Then a function \( \varphi \) satisfies the \( \Delta_2 \)-condition, with constant \( K = 2^q \).

Proof. Assume that (1.2) holds. Taking \( C = 2 \) in (1.2), we have
\[
\varphi(y, 2t) \leq 2^{q(y)} \varphi(y, t) \leq 2^q \varphi(y, t).
\]
Thus the inequality (1.1) holds with constant \( K = 2^q \).

Lemma 3 is proved.

It is clear that if \( \varphi(x, t) = t^{q(x)} \), then condition (1.2) satisfies automatically.

The following Lemma characterize bounded, sublinear operators from one Musielak-Orlicz spaces to another.

Definition 6. A mapping \( S \) from one Musielak-Orlicz space \( L_\varphi \) to another Musielak-Orlicz space \( L_\psi \) is said to be sublinear if for all \( f, g \in L_\varphi \) and \( \lambda > 0 \), we have
\begin{enumerate}
  \item \( S(\lambda f) = \lambda S(f) \);
  \item \( S(f + g) \leq S(f) + S(g) \).
\end{enumerate}

Lemma 4. [15] Let \( \varphi, \psi \in \Phi \) and \( \varphi \) and \( \psi \) satisfy the \( \Delta_2 \) condition. Suppose \( S : L_\varphi \mapsto L_\psi \) be sublinear. Then the following conditions are equivalent:
\begin{enumerate}
  \item \( S \) is bounded, i.e. there exists \( C > 0 \) such that \( \|Sf\|_{L_\psi} \leq C \|f\|_{L_\varphi} \);
  \item there exists \( M_1, M_2 > 0 \) such that \( \|f\|_{L_\varphi} \leq C_1 \implies \rho_{L_\psi}(SF) \leq M_2 \).
\end{enumerate}

We note that the Lebesgue spaces with mixed norm, weighted Lorentz spaces and e.t.c. is \( p \)-convex (\( p \)-concave) BFS. Now we reduce more general result connected with Minkowski’s integral inequality.

Let \( X \) and \( Y \) be BFSs on \( (\Omega_1, \mu) \) and \( (\Omega_2, \nu) \) respectively. By \( X[Y] \) and \( Y[X] \) we denote the spaces with mixed norm and consisting of all functions \( g \in L_0(\Omega_1 \times \Omega_2, \mu \times \nu) \) such that \( \|g(x, \cdot)|_Y \in X \) and \( \|g(\cdot, y)|_X \in Y \). The norms in this spaces is defined as
\[
\|g\|_{X[Y]} = \|\|g(x, \cdot)|_Y\|_X, \quad \|g\|_{Y[X]} = \|\|g(\cdot, y)|_X\|_Y.
\]
**Theorem 1.** [43] Let $X$ and $Y$ be BFSs with the Fatou property. Then the generalized Minkowski integral inequality
\[
\|f\|_{X[Y]} \leq M \|f\|_{Y[X]}
\]
holds for all measurable functions $f(x,y)$ if and only if there exists $1 \leq p \leq \infty$ such that $X$ is $p$-convex and $Y$ is $p$-concave.

It is known that $X[Y]$ and $Y[X]$ are BFSs on $\Omega_1 \times \Omega_2$ (see [29].)

### 3. Main results.

We consider the multidimensional Hardy type operator and its dual operator
\[
Hf(x) = \int_{|y|<|x|} f(y) \, dy \quad \text{and} \quad H^*f(x) = \int_{|y|>|x|} f(y) \, dy,
\]
where $f \geq 0$ and $x \in \mathbb{R}^n$.

Now we prove a two-weight criterion for multidimensional Hardy type operator acting from the $p$-concave weighted BFS to weighted Lebesgue spaces.

**Theorem 2.** Let $v(x)$ and $w(x)$ be weights on $\mathbb{R}^n$. Suppose that $X_w$ be a $p$-convex weighted BFSs for $1 \leq p < \infty$ on $\mathbb{R}^n$. Then the inequality
\[
\|Hf\|_{X_w} \leq C \|f\|_{L^p,v}
\]
holds for every $f \geq 0$ and for all $\alpha \in (0,1)$ if and only if
\[
A(\alpha) = \sup_{t>0} \left( \int_{|y|<t} [v(y)]^{-p'} \, dy \right)^{\frac{\alpha}{p'}} \left\| \chi_{\{|z|>t\}} \cdot \left( \int_{|y|<|z|} [v(y)]^{-p'} \, dy \right)^{1-\alpha} \right\|_{X_w} < \infty. \quad (3.2)
\]

Moreover, if $C > 0$ is the best possible constant in (3.1), then
\[
\sup_{0<\alpha<1} \left( \frac{p'}{1-\alpha} \right)^{p'} \left( \frac{p'}{1-\alpha} \right)^{1/p} \frac{A(\alpha)}{(1-\alpha)^{1/p'}} \leq C \leq M \inf_{0<\alpha<1} \frac{A(\alpha)}{(1-\alpha)^{1/p'}}.
\]

**Proof of Theorem 2.** Sufficiency. Passing to the polar coordinates, we have
\[
h(y) = \left( \int_{|z|<|y|} [v(z)]^{-p'} \, dz \right)^{\frac{\alpha}{p'}} = \left( \int_{0}^{\frac{|y|}{s^{n-1}}} \left( \int_{|\xi|=s} [v(s\xi)]^{-p'} \, d\xi \right) \, ds \right)^{\frac{\alpha}{p'}}.
\]
where \(d\xi\) is the surface element on the unit sphere. Obviously, \(h(y) = h(|y|)\), i.e., \(h(y)\) is a radial function.

Applying Hörmander’s inequality for \(L_p(\mathbb{R}^n)\) spaces and after some standard transformations, we have

\[
\|Hf\|_{X,w} = \left\| w(\cdot) \int_{|y|<|x|} f(y) dy \right\|_X = \left\| w(\cdot) \int_{|y|<|x|} [f(y)h(y)v(y)] [h(y)v(y)]^{-1} dy \right\|_X
\]

\[
\leq \left\| w(\cdot) \|fhv\|_{L_p(|y|<|x|)} \left\| \|h v\|^{-1}_{L_{p'}(|y|<|x|)} \right\|_X
\]

\[
= \left\| w(\cdot) fhv \chi_{\{|y|<|x|\}}(\cdot) \left\| \|h v\|^{-1}_{L_{p'}(|y|<|x|)} \right\|_{L_p[X]}
\]

\[
= M \left\| w(\cdot) fhv \chi_{\{|y|<|x|\}}(\cdot) \left\| \|h v\|^{-1}_{L_{p'}(|y|<|x|)} \right\|_X \right\|_{L_p}
\]

By switching to polar coordinates and after some calculations, we get

\[
\left\| \|h v\|^{-1}_{L_{p'}(|y|<|x|)} \right\|_X^{1/p'} = \left( \int_{|y|<|x|} [h(|y|)v(y)]^{-p'} dy \right)^{1/p'}
\]

\[
= \left( \int_0^{|x|} r^{n-1} [h(r)]^{-p'} \left( \int_0^r [v(r\xi)]^{-p'} d\xi \right) dr \right)^{1/p'}
\]

\[
= \left( \int_0^{|x|} \left( \int_0^r s^{n-1} \left( \int_0^1 [v(s\xi)]^{-p'} d\xi \right) ds \right)^{-\alpha} \left( \int_0^r [v(r\xi)]^{-p'} d\xi \right)^{r^{n-1} dr} \right)^{1/p'}
\]

\[
= \frac{1}{(1-\alpha)^{1/p'}} \left( \int_0^{|x|} \frac{d}{dr} \left( \int_0^r s^{n-1} \left( \int_0^1 [v(s\xi)]^{-p'} d\xi \right) ds \right) dr \right)^{1-\alpha} dr^{1/p'}
\]
\[
\frac{1}{(1 - \alpha)^{1/p'}} \left( \int_0^{|x|^{s-1}} \left( \int_{|\xi|=1} [v(s\xi)]^{-p'} d\xi \right) d\eta \right)^{\frac{1-\alpha}{p'}} = \frac{1}{(1 - \alpha)^{1/p'}} \left( \int_{|z|<|x|} [v(z)]^{-p'} dz \right)^{\frac{1-\alpha}{p'}}.
\]

Therefore from the condition (3.2), we obtain

\[
\left\| fhv \right\|_{L^{p/v}(|y|<|z|)} \left\| [hv]^{-1} \right\|_{L^p(|y|<|z|)} \leq \frac{A(\alpha)}{(1 - \alpha)^{1/p'}} \left\| fhv \right\|_{L^p}.
\]

Thus

\[
\left\| Hf \right\|_{X_w} \leq M \frac{A(\alpha)}{(1 - \alpha)^{1/p'}} \left\| fhv \right\|_{L^p} \text{ for all } \alpha \in (0, 1).
\]

**Necessity.** Let \( f \in L_{p,v}(\mathbb{R}^n), \) \( f \geq 0 \) and the inequality (3.1) is valid. We choose the test function as

\[
f(x) = \frac{p'}{1 - \alpha} [g(t)]^{-\frac{p}{p'}} v^{-p'}(x) \chi_{\{|x|<t\}}(x) + [g(|x|)]^{-\frac{p}{p'}} v^{-p'}(x) \chi_{\{|x|>t\}}(x),
\]

where \( t > 0 \) is a fixed number and

\[
g(t) = \int_{|y|<t} v^{-p'}(y) dy = \int_0^t s^{n-1} \left( \int_{|\eta|=1} v^{-p'}(s\eta) d\eta \right) ds.
\]

It is obvious that \( \frac{dg}{dt} = t^{n-1} \int_{|\eta|=1} v^{-p'}(t\eta) d\eta. \) Again by switching to polar coordinates, from the right hand side of inequality (3.1) we get that

\[
\left\| f \right\|_{L_{p,v}} = \left[ \int_{|x|<t} \left( \frac{p'}{1 - \alpha} \right)^p [g(t)]^{-\alpha(p-1)-1} v^{-p'}(x) dx + \int_{|x|>t} [g(|x|)]^{-\alpha(p-1)-1} v^{-p'}(x) dx \right]^{1/p}
\]

\[
= \left[ \left( \frac{p'}{1 - \alpha} \right)^p [g(t)]^{1-p} + \int_t^\infty [g(r)]^{-\alpha(p-1)-1} \left( \int_{|r\xi|=1} v^{-p'}(r\xi) d\xi \right) dr \right]^{1/p}
\]

\[
= \left[ \left( \frac{p'}{1 - \alpha} \right)^p [g(t)]^{1-p} - \frac{1}{\alpha(p-1)} \int_t^\infty \frac{d}{dr} [g(r)]^{-\alpha(p-1)} dr \right]^{1/p}
\]

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After some calculations, from the left hand side of inequality (3.1), we have

\[ \frac{p'}{1 - \alpha} \left[ g(t) \right]^{\alpha(1-p)} + \frac{1}{\alpha(p-1)} \left\{ \left[ g(t) \right]^{-\alpha(p-1)} - \left[ \int_{R^n} v^{-r'}(y) \, dy \right]^{-\alpha(p-1)} \right\}^{1/p} \leq \left[ \left( \frac{p'}{1 - \alpha} \right)^p + \frac{1}{\alpha(p-1)} \right]^{1/p} [h(t)]^{-1}. \]

After some calculations, from the left hand side of inequality (3.1), we have

\[ \|Hf\|_{L_{p,v}} = \left\| \int f(y) \, dy \right\|_{X_w} \geq \| \chi_{\{|\cdot| > t\}} \int f(y) \, dy \|_{X_w} = \left\| \chi_{\{|\cdot| > t\}} \left( \frac{p'}{1 - \alpha} \int_{|y| < t} [g(t)]^{-\frac{\alpha}{p'}} v^{-r'}(y) \, dy + \int_{|y| > t} [g(|y|)]^{-\frac{\alpha}{p'}} v^{-r'}(y) \, dy \right) \right\|_{X_w} \]
\[ = \left\| \chi_{\{|\cdot| > t\}} \left( \frac{p'}{1 - \alpha} \left[ g(t) \right]^{1 - \frac{\alpha}{p'}} + \int_{r < t} [g(r)]^{-\frac{\alpha}{p'}} \left( \int_{|\eta| = 1} v^{-r'}(r \eta) \, d\eta \right) \, dr \right) \right\|_{X_w} \]
\[ = \left\| \chi_{\{|\cdot| > t\}} \left( \frac{p'}{1 - \alpha} \left[ g(t) \right]^{1 - \frac{\alpha}{p'}} + \frac{p'}{1 - \alpha} \int_{t}^{\infty} \frac{d}{dr} [g(r)]^{1 - \frac{\alpha}{p'}} \, dr \right) \right\|_{X_w} \]
\[ = \frac{p'}{1 - \alpha} \left\| \chi_{\{|\cdot| > t\}} \left[ g(\cdot) \right]^{1 - \frac{\alpha}{p'}} \right\|_{X_w}. \]

Hence, this implies that

\[ \frac{p'}{1 - \alpha} \left[ \left( \frac{p'}{1 - \alpha} \right)^p + \frac{1}{\alpha(p-1)} \right]^{-1/p} [g(t)]^{\frac{\alpha}{p'}} \left\| \chi_{\{|\cdot| > t\}} \left[ g(\cdot) \right]^{1 - \frac{\alpha}{p'}} \right\|_{X_w} \leq C, \]

i.e.,

\[ \frac{p'}{1 - \alpha} \left[ \left( \frac{p'}{1 - \alpha} \right)^p + \frac{1}{\alpha(p-1)} \right]^{1/p} \leq C \text{ for all } \alpha \in (0, 1). \]

This completes the proof of Theorem 2.

For the dual operator, the below stated theorem is proved analogously.

**Theorem 3.** Let \( v(x) \) and \( w(x) \) are weights on \( R^n \). Suppose that \( X_w \) be a \( p \)-convex weighted BFSs for \( 1 \leq p < \infty \) on \( R^n \). Then the inequality

\[ \|H^* f\|_{X_w} \leq C \|f\|_{L_{p,v}} \]  \hspace{1cm} (3.3)

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holds for every \( f \geq 0 \) and for all \( \gamma \in (0, 1) \) if and only if

\[
B(\gamma) = \sup_{t > 0} \left( \int_{|y| > t} [v(y)]^{-p'} dy \right)^{\gamma} \left( \int_{|y| < t} [v(y)]^{-p'} dy \right)^{1-\gamma} < \infty.
\]

Moreover, if \( C > 0 \) is the best possible constant in (3.3) then

\[
\sup_{0 < \gamma < 1} \frac{p' B(\gamma)}{(1 - \gamma) \left[ \left( \frac{p'}{1 - \gamma} \right)^p + \frac{1}{\gamma(p-1)} \right]^{1/p}} \leq C \leq \inf_{0 < \gamma < 1} \frac{B(\gamma)}{(1 - \gamma)^{1/p'}}.
\]

**Corollary 1.** Note that Theorem 2 and Theorem 3 in the case \( X_w = L_{\varphi,w}, \varphi(x, t^{1/p}) \in \Phi \) for some \( 1 \leq p < \infty, x \in \mathbb{R}^n \) was proved in [4]. In the case \( X_w = L_{q,w}, 1 < p \leq q < \infty \), for \( x \in (0, \infty), \alpha = \frac{s-1}{p-1} \) and \( s \in (1, p) \) Theorem 2 and Theorem 3 was proved in [46]. For \( x \in \mathbb{R}^n \) in the case \( X_w = L_{q(x),w} \) and \( 1 < p \leq q(x) \leq \text{ess sup}_{x \in \mathbb{R}^n} q(x) < \infty \) Theorem 2 and Theorem 3 was proved in [3] (see also [2]).

**Remark 3.** In the case \( n = 1, X_w = L_{q,w}, 1 < p \leq q \leq \infty, \) at \( x \in (0, \infty) \), for classical Lebesgue spaces the various variants of Theorem 2 and Theorem 3 were proved in [20], [10], [27], [28], [36], [37], [45] and etc. In particular, in the Lebesgue spaces with variable exponent the boundedness of Hardy type operator was proved in [13], [14], [17], [21], [26], [34], [35] and etc. For \( X_w = L_{q(x),w}, 1 < p \leq q(x) \leq \text{ess sup}_{x \in \mathbb{R}^n} q(x) < \infty \) and \( x \in [0,1] \) the two-weighted criterion for one-dimensional Hardy operator was proved in [26]. Also, other type two-weighted criterion for multidimensional Hardy type operator in the case \( X_w = L_{q(x),w}, 1 < p \leq q(x) \leq \text{ess sup}_{x \in \mathbb{R}^n} q(x) < \infty \) and \( x \in [0,1] \) was proved in [34] (see also [35]). In the papers [9] and [42] the inequalities of modular type for more general operators was proved. Also, in [11] the Hardy type inequalities with special power-type weights in Orlicz spaces was proved.

### 4. Applications.

Now we consider the multidimensional geometric mean operator defined as

\[
Gf(x) = \exp \left( \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} \ln f(y) dy \right),
\]
Moreover, if \( C > \) then

we find that for \( w \) with \( w \) holds for every \( w \) and for all \( s \in (1, p) \) if and only if

Let \( \phi, w \) \( (0, ||t||) = \sup_{B(0, ||.||)} \exp \left( \frac{1}{B(0, ||.||)} \int_{B(0, ||.||)} \ln \frac{1}{v(y)} dy \right) \) \( \left\| L_{\phi, w} \right\| < \infty. \) (4.2)

Moreover, if \( C > 0 \) is the best possible constant in (4.1) then

\[
\sup_{s > 1} \frac{e^{s}}{(e^{s} + \frac{1}{s-1})^{1/p}} D(s) \leq C \leq 2^{1/p} \inf_{s > 1} e^{\frac{s-1}{p}} D(s).
\]

**Proof of Theorem 4.** Let \( \alpha = \frac{s-1}{p-1} \), where \( 1 < s < p \). We replace \( f \) with \( f^{\beta} \), \( v \) with \( v^{\beta} \), \( w \) with \( \frac{w^{\beta}(x)}{|B(0, ||x||)|} \), \( 0 < \beta < p \), and \( p \) with \( \frac{p}{\beta} \) and \( \phi(x, t) \) with \( \phi(x, t^{1/\beta}) \) in (3.1), (3.2), we find that for \( 1 < s < \frac{p}{\beta} \)

\[
\left\| \frac{w^{\beta}}{|B(0, ||.||)|} H(f^{\beta}) \right\|_{L_{\phi, w}(B(0, ||.||))} = \left\| \left( \frac{1}{|B(0, ||.||)|} \int_{B(0, ||.||)} f^{\beta}(y) dy \right)^{1/\beta} \right\|_{L_{\phi, w}(R^n)}^{\beta/p} \leq C_{\beta} \left( \int_{R^n} |f(y)v(y)|^p dy \right)^{\beta/p}.
\]

Then the inequality

\[
\left\| \left( \frac{1}{|B(0, ||.||)|} \int_{B(0, ||.||)} f^{\beta}(y) dy \right)^{1/\beta} \right\|_{L_{\phi, w}(R^n)} \leq C_{\beta}^{1/\beta} \left( \int_{R^n} |f(y)v(y)|^p dy \right)^{1/p} \) (4.3)
By the L'Hospital rule, we get

\[ A \left( \frac{s - 1}{p - 1} \right) \]

and

\[ B^\beta(s, \beta) < \infty \]

holds if and only if

\[
A \left( \frac{s - 1}{p - 1} \right) = \left[ \sup_{t > 0} \left( \int_{|y| < t} [v(y)]^{-\frac{\beta p}{p - \beta s}} dy \right)^{\frac{s - 1}{p}} \right]^{\frac{\beta s}{\beta}} = B^\beta(s, \beta) < \infty
\]

By the L'Hospital rule, we get

\[
\lim_{\beta \to +0} \left( \frac{1}{|B(0, |x|)|^\frac{p}{p - \beta s}} \int_{|y| < |x|} [v(y)]^{-\frac{\beta p}{p - \beta s}} dy \right)^{\frac{\beta s}{\beta}} = \lim_{\beta \to +0} \exp \left[ \frac{p \ln \frac{1}{|B(0, |x|)|} + (p - \beta s) \ln \left( \int_{|y| < |x|} [v(y)]^{-\frac{\beta p}{p - \beta s}} dy \right)}{p \beta} \right]
\]

\[
= \lim_{\beta \to +0} \exp \left[ -\frac{s}{p} \ln \left( \int_{|y| < |x|} [v(y)]^{-\frac{\beta p}{p - \beta s}} dy \right) + \frac{(p - \beta s) \left( \frac{p}{p - \beta} \right)^2 \int_{|y| < |x|} [v(y)]^{-\frac{\beta p}{p - \beta s}} \ln \frac{1}{v(y)} dy}{p \int_{|y| < |x|} [v(y)]^{-\frac{\beta p}{p - \beta s}} dy} \right]
\]

\[
= \exp \left[ \frac{s}{p} \ln \frac{1}{|B(0, |x|)|} + \frac{\int_{|y| < |x|} \ln \frac{1}{v(y)} dy}{|B(0, |x|)|} \right] = \frac{1}{|B(0, |x|)|^\frac{1}{p}} \exp \left( \int_{B(0, |x|)} \ln \frac{1}{v(y)} dy \right).
\]

Therefore

\[
\lim_{\beta \to +0} B(s, \beta) = \sup_{t > 0} |B(0, t)|^{\frac{s - 1}{p - \beta s}} \int_{|y| > t} \frac{X(|z| > t)(\cdot)}{|B(0, |\cdot|)|^\frac{1}{p}} \exp \left( \int_{B(0, |\cdot|)} \ln \frac{1}{v(y)} dy \right) \bigg|_{L_p, w}
\]

\[ = D(s) < \infty \]
and
\[ \sup_{s > 1} \frac{e^{\frac{s}{p}}}{(e^{s} + \frac{1}{s-1})^{1/p}} D(s) \leq \lim_{\beta \to +0} C_{\beta}^{1/\beta} \leq 2^{1/p} \inf_{s > 1} e^{\frac{s}{p}} D(s). \] (4.5)

Further, we have
\[ \lim_{\beta \to +0} \left( \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} f^\beta(y) \, dy \right)^{1/\beta} = \exp \left( \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} \ln f(y) \, dy \right) = Gf(x). \]

From (4.4) it follows that \( \lim_{\beta \to +0} C_{\beta} = 1 \), and according to (4.2) and (4.5) \( \lim_{\beta \to +0} C_{\beta}^{1/\beta} = C < \infty \). Therefore the inequality (4.1) is valid. Moreover, from (4.3) for \( \beta \to +0 \) we obtain that
\[ \|Gf\|_{L^q_{q - \mu}(\mathbb{R}^n)} \leq C \|f\|_{L^p_{\mu}(\mathbb{R}^n)} \]
and by (4.5)
\[ \sup_{s > 1} \frac{e^{\frac{s}{p}}}{(e^{s} + \frac{1}{s-1})^{1/p}} D(s) \leq C \leq 2^{1/p} \inf_{s > 1} e^{\frac{s}{p}} D(s). \]

This completes the proof of Theorem 4.

**Remark 4.** Let \( \varphi(x, t) = t^q \) and \( n = 1 \). Note that the simplest case of (2.4) with \( v = w = 1 \) and \( p = q = 1 \) was considered in [20] and in [25]. Later this inequality was generalized in various ways by many authors in [12, 22, 23, 24, 31, 39, 40, 41, 46] and etc.

**Corollary 2.** Let \( \varphi(x, t) = t^q \), \( 0 < p \leq q < \infty \) and let \( f \) be a positive function on \( \mathbb{R}^n \). Then
\[ \left( \int_{\mathbb{R}^n} [Gf(x)]^q |x|^{\delta q} \, dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} f^p(x) |x|^{\mu p} \, dx \right)^{1/p} \] (4.6)
holds with a finite constant \( C \) if and only if
\[ \delta + \frac{n}{q} = \frac{\mu}{n} + \frac{n}{p} \]
and the best constant \( C \) has the following condition:
\[ \sqrt{\frac{p}{nq}} e^{\frac{\mu}{n}} |B(0, 1)|^{\frac{1}{q} - \frac{1}{p}} \sup_{s > 1} e^{\frac{s}{p}} (s - 1)^{\frac{1}{q} - \frac{1}{p}} \leq C \leq |B(0, 1)|^{\frac{1}{q} - \frac{1}{p}} e^{\frac{\mu}{n} + \frac{1}{q}}. \]

**Remark 5.** Let \( \varphi(x, t) = t^q \) and \( q = p \). Then the inequality (4.6) is sharp with the constant
\[ C = \frac{e^{\frac{\mu}{n} + \frac{1}{q}}}{\sqrt{n}}. \]
The sufficient conditions for general weights ensuring the validity of the two-weight strong type inequalities for some sublinear operator are given in the following theorem.

**Theorem 5.** Let \( \varphi(x, t^{1/p}) \in \Phi \) for some \( 1 < p < \infty \) and \( x \in \mathbb{R}^n \). Suppose that \( v(x) \) and \( w(x) \) are weight functions on \( \mathbb{R}^n \). Let \( T \) be a sublinear operator acting boundedly from \( L_p(\mathbb{R}^n) \) to \( L_\varphi(\mathbb{R}^n) \) such that, for any \( f \in L_1(\mathbb{R}^n) \) with compact support and \( x \notin \text{supp} \ f \)

\[
|Tf(x)| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^a} \, dy,
\]

where \( C > 0 \) is independent of \( f \) and \( x \). Let there exists \( 1 < p \leq r(x) \leq \text{ess sup} \ r(x) < \infty \) such that, for all \( C > 0 \) \( \varphi(y, Ct) \leq C^{r(y)} \varphi(y, t) \).

Moreover, let \( v(x) \) and \( w(x) \) are weight functions on \( \mathbb{R}^n \) and satisfies the following conditions:

\[
A = \sup_{t > 0} \left( \int_{|y| < t} [v(y)]^{-\frac{1}{p'}} \, dy \right)^{\frac{a}{p'}} \left\| \chi_{\{|x| > t\}} \left( \int_{|y| < |x|} [v(y)]^{-\frac{1}{p'}} \, dy \right)^{\frac{1-a}{p}} \right\|_{L_{\varphi, w}} < \infty,
\]

\[
B = \sup_{t > 0} \left( \int_{|y| > t} [v(y)|y|^a]^{-\frac{1}{p'}} \, dy \right)^{\frac{a}{p'}} \left\| \chi_{\{|x| > t\}} \left( \int_{|y| > |x|} [v(y)|y|^a]^{-\frac{1-a}{p}} \, dy \right)^{\frac{1-a}{p}} \right\|_{L_{\varphi, w}} < \infty.
\]

There exists \( M > 0 \) such that

\[
\sup_{|x|/2 < |y| \leq 4|x|} w(y) \leq M \inf_{|x|/2 < |y| \leq 4|x|} v(x).
\]

Then there exists a positive constant \( C \), independent of \( f \), such that for all \( f \in L_{p,v}(\mathbb{R}^n) \)

\[
\|Tf\|_{L_{p,w}} \leq C \|f\|_{L_{p,v}(\mathbb{R}^n)}.
\]

**Proof of Theorem 5.** Let \( Z = \{0, \pm1, \pm2, \ldots, \} \). For \( k \in Z \) we define \( E_k = \{x \in \mathbb{R}^n : 2^k < |x| \leq 2^{k+1}\} \), \( E_{k,1} = \{x \in \mathbb{R}^n : |x| \leq 2^{k-1}\} \), \( E_{k,2} = \{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^{k+1}\} \), \( E_{k,3} = \{x \in \mathbb{R}^n : |x| > 2^{k-1}\} \). Then \( E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1} \) and the multiplicity of the covering \( \{E_{k,3}\}_{k \in Z} \) is equal to 3.

Given \( f \in L_{p,v}(\mathbb{R}^n) \), we write

\[
|Tf(x)| = \sum_{k \in Z} |Tf(x)| \chi_{E_k}(x) \leq \sum_{k \in Z} |Tf_{k,1}(x)| \chi_{E_k}(x) + \sum_{k \in Z} |Tf_{k,2}(x)| \chi_{E_k}(x) + \sum_{k \in Z} |Tf_{k,3}(x)| \chi_{E_k}(x),
\]

where \( Tf_{k,1} = \sum_{j \geq k} 2^{j(k-1)} T \chi_{E_j} \) and \( Tf_{k,2} = \sum_{j \geq k} 2^{j(k-1)} T \chi_{E_{j,1}} \).
Hence we obtain

\[ |T f_k,3(x)| \chi_{E_k}(x) = T_1 f(x) + T_2 f(x) + T_3 f(x), \]

where \( \chi_{E_k} \) is the characteristic function of the set \( E_k \), \( f_{k,i} = f \chi_{E_{k,i}}, i = 1, 2, 3 \).

First we shall estimate \( \|T_1 f\|_{L_{p,v}} \). Note that for \( x \in E_k, y \in E_{k,1} \) we have \( |y| < 2^{k-1} \leq |x|/2 \). Moreover, \( E_k \cap \text{supp} f_{k,1} = \emptyset \) and \( |x-y| \geq |x| - |y| \geq |x| - |x|/2 = |x|/2 \). Hence by (4.7)

\[
|T_1 f(x)| \leq C \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}^n} \frac{|f_{k,1}(y)|}{|x-y|^n} \, dy \right) \chi_{E_k} \leq C \int_{|y|<|x|/2} \frac{|f(y)|}{|x-y|^n} \, dy \leq C \int_{|y|<|x|} \frac{|f(y)|}{|x-y|^n} \, dy \leq 2^n C |x|^{-n} \int_{|y|<|x|} |f(y)| \, dy
\]

for any \( x \in E_k \). Hence we have

\[
\|T_1 f\|_{L_{p,v}} \leq 2^n C \left( \int_{|y|<|x|} |f(y)| \, dy \right)_{L_{p,v}} \leq 2^n C \left( \int_{|y|<|x|} |f(y)| \, dy \right)_{L_{p,v},|x|^{-n} w}.
\]

By the condition (4.8) and Theorem 2, we obtain

\[
\|T_1 f\|_{L_{p,v}} \leq C_1 \|f\|_{L_{p,v}(R^n)} \tag{4.11}
\]

where \( C_1 > 0 \) is independent of \( f \) and \( x \in R^n \).

Next we estimate \( \|T_3 f\|_{L_{p,v}(R^n)} \). It is obviously that, for \( x \in E_k, y \in E_{k,3} \) we have \( |y| > 2|x| \) and \( |x-y| \geq |y| - |x| \geq |y| - |y|/2 = |y|/2 \). Since \( E_k \cap \text{supp} f_{k,3} = \emptyset \), for \( x \in E_k \) by (4.7), we have

\[
|T_3 f(x)| \leq C \int_{|y|>2|x|} \frac{|f(y)|}{|x-y|^n} \, dy \leq 2^n C \int_{|y|>2|x|} \frac{|f(y)|}{|y|^n} \, dy.
\]

Hence we obtain

\[
\|T_3 f\|_{L_{p,v}} \leq 2^n C \left( \int_{|y|>2|x|} |f(y)| |y|^{-n} \, dy \right)_{L_{p,v}} \leq 2^n C \left( \int_{|y|>|x|} |f(y)| |y|^{-n} \, dy \right)_{L_{p,v}}.
\]

By the condition (4.9) and Theorem 3, we obtain

\[
\|T_3 f\|_{L_{p,v}} \leq C_2 \|f\|_{L_{p,v}(R^n)}, \tag{4.12}
\]

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where $C_2 > 0$ is independent of $f$ and $x \in \mathbb{R}^n$.

Let $T : L_p(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n)$. We have

$$
\|Tf_{k,2}\|_{L_p(\mathbb{R}^n)} = \left\| \sum_{k \in \mathbb{Z}} |Tf_{k,2}| \chi_{E_k} \right\|_{L_p(\mathbb{R}^n)}.
$$

By virtue of Lemma 3 it suffices to prove that from $\|f\|_{L_p,\varphi(\mathbb{R}^n)} \leq 1$ implies

$$
\int_{\mathbb{R}^n} \varphi \left( y, w \sum_{k \in \mathbb{Z}} |Tf_{k,2}| \chi_{E_k} \right) dx \leq C, \text{ where } C > 0 \text{ is independent on } k \in \mathbb{Z}.
$$

Finally, we estimate $\|Tf\|_{L_p,\varphi}$. By the $L_p(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n)$ boundedness of $T$ and condition (4.10), we have

$$
\int_{\mathbb{R}^n} \varphi \left( y, w(y) \sum_{k \in \mathbb{Z}} |Tf_{k,2}(y)| \chi_{E_k}(y) \right) dy = \sum_{m \in \mathbb{Z}} \sum_{E_m} \varphi \left( y, w(y) \sum_{k \in \mathbb{Z}} |Tf_{k,2}(y)| \chi_{E_k}(y) \right) dy =
$$

$$
\sum_{k \in \mathbb{Z}} \int_{E_k} \left( C w(y) \|f_{k,2}\|_{L_p(\mathbb{R}^n)} \right)^{r(y)} \varphi \left( y, \frac{|Tf_{k,2}|}{C \|f_{k,2}\|_{L_p(\mathbb{R}^n)}} \right) dy \leq
$$

$$
C_1 \sum_{k \in \mathbb{Z}} \int_{E_k} \left( C w(y) \|f_{k,2}\|_{L_p(\mathbb{R}^n)} \right)^{r(y)} \varphi \left( y, \frac{|Tf_{k,2}|}{C \|f_{k,2}\|_{L_p(\mathbb{R}^n)}} \right) dy \leq
$$

$$
C_2 \sum_{k \in \mathbb{Z}} \sup_{y \in E_k} \left( w(y) \|f\|_{L_p(E_k,2)} \right)^{r(y)} \leq C_2 \sum_{k \in \mathbb{Z}} \sup_{y \in E_k} \left( \|f\|_{L_p(E_k,2)} \right)^{r(y)} \leq
$$

$$
C_3 \sum_{k \in \mathbb{Z}} \sup_{y \in E_k} \left( \|f\|_{L_p,E_k,2} \right)^{r(y)} \leq C_3 \sum_{k \in \mathbb{Z}} \sup_{y \in E_k} \left( \|f\|_{L_p,E_k,2} \right)^{r(y)} \leq
$$

$$
C_3 \left( \sum_{k \in \mathbb{Z}} \left( \int_{E_{k,2}} |f(y)| v(y) \right)^{r/p} \right)^{r/p} \leq C_3 \left( \sum_{k \in \mathbb{Z}} \left( \int_{E_{k,2}} |f(y)| v(y) \right)^{r/p} \right)^{r/p}
$$

$$
C_3 \left( \sum_{k \in \mathbb{Z}} \left[ \int_{E_{k-1}} + \int_{E_k} + \int_{E_{k+1}} \right] |f(y)| v(y)^p dy \right)^{r/p} = C_3 \left( 3 \sum_{k \in \mathbb{Z}} \int_{E_k} |f(y)| v(y)^p dy \right)^{r/p}
$$
\[ 3^{\frac{q}{p}} \left( \|f\|_{L_p,v(R^n)}^p \right)^{\frac{q}{p}} = C_4 \|f\|_{L_p,v(R^n)} \leq C_4. \]

Thus
\[ \|T_2 f\|_{L_p,w} \leq C \|f\|_{L_p,v(R^n)} \tag{4.13} \]

where \( C > 0 \) is independent of \( f \) and \( x \in R^n \).

Combining the inequalities (4.11),(4.12) and (4.13) we obtain the proof of Theorem 5.

In particular, for \( \varphi(x,t) = t^{q(x)} \) by virtue of Remark 2 we have the following Corollary.

**Corollary 3.** [2] Let \( 1 < p \leq q(x) \leq \overline{q} < \infty \) and \( x \in R^n \). Suppose that \( v(x) \) and \( w(x) \) are weight functions defined on \( R^n \) and satisfying conditions (4.8), (4.9) and (4.10) of Theorem 5. Let \( T \) be a sublinear operator satisfies the condition (4.7) of Theorem 5.

Then there exists a positive constant \( C \), independent of \( f \), such that for all \( f \in L_{p,v}(R^n) \)
\[ \|Tf\|_{L_{q(\cdot),w}(R^n)} \leq C \|f\|_{L_{p,v}(R^n)}. \]

**Remark 6.** Note that the condition (4.7) was introduced in [44]. Many interesting operators in harmonic analysis, such as the Calderon-Zigmund singular integral operators, Hardy-Littlewood maximal operators, Fefferman’s singular integrals, Ricci-Stein’s oscillatory singular integrals, Bochner-Riesz means and so on is satisfied the condition (4.7). In the case \( p(x) = p = \text{const} \) for classical Lebesgue spaces the Theorem 5 was proved in [47] (see also [18] and [33]). Also, for classical Lebesgue spaces in [16] and [19] was found new type sufficient conditions on weights for Calderon-Zigmund singular integral operator, whenever the weight functions are radial monotone functions. In particular, the boundedness of certain convolution operator in a weighted Lebesgue space with kernel satisfying the generalized Hörmander’s condition was proved in [5].

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