Extensions of trivial inertial blocks

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Abstract

In this paper, we generalize the work of Külshammer and Puig on extensions of nilpotent blocks to get the graded algebra structure of extensions of trivial inertial blocks with an abelian defect group condition. This structure has a uniqueness property, which shows a way to compare different extensions of trivial inertial blocks. By such a way, we prove Rouquier’s conjecture for trivial inertial blocks.

1. Introduction

1.1. Throughout this paper, $p$ is a prime number and $O$ is a complete discrete valuation ring with an algebraically closed residue field $k$ of characteristic $p$. Let $G$ be a finite group and let $H$ be a normal subgroup of $G$. Let $b$ be a $G$-stable block of $H$ over $O$, a central primitive idempotent of the group algebra $OH$. The ideal $OHb$ of $OH$ generated by $b$ is an algebra with the identity element $b$, which is the block algebra of $b$. Let $Q$ be a defect group of the block $b$, which is a maximal $p$-subgroup of $H$ such that $Br_Q^{OH}(b) \neq 0$ (see Paragraph 2.5 for $Br_Q^{OH}$). Set $G = N_G(Q)$, $H = N_H(Q)$ and $K = (H \times H)\Delta(G)$, where $\Delta(G)$ is the diagonal subgroup of $G \times G$. Let $b$ be a block of $H$ such that $Br_Q^{OH}(b) = Br_Q^{GH}(b)$; such a block $b$ is the Brauer correspondent of $b$ in $H$. Clearly $b$ is $G$-stable. Let $P$ be a maximal $p$-subgroup of $G$ such that $Br_P^{OH}(b) \neq 0$. By [10, Proposition 5.3], the intersection $P \cap H$ is a defect group of $b$ in $H$. So we assume that $Q = P \cap H$.

1.2. Clearly $OHb$ is a ring extension of $OHb$ and has an obvious $G/H$-graded algebra structure; we call it an extension of the block $b$. Block extensions are typically Clifford-theoretic settings and naturally arise in the local block theory. A $(b, H)$-Brauer pair $(R, b_R)$ consists of a $p$-subgroup $R$ and a block $b_R$ of $C_H(R)$ over $k$. It is well known that $b_R$ is also a block of $N_H(R, b_R)$, where $N_H(R, b_R)$ denotes the stabilizer of $(R, b_R)$ under the $H$-conjugation action. An important idea of the local block theory is to glue the information of such block algebras $kN_H(R, b_R)b_R$ to get the information of the block $b$. The block algebra $kN_H(R, b_R)b_R$ is an extension of the block $b_R$ of $C_H(R)$.

1.3. Külshammer and Puig characterized the block algebra $kN_H(R, b_R)b_R$ when the block $b_R$ of $C_H(R)$ has the center of $R$ as a defect group. More generally, assuming that the block $b$ of $H$ is nilpotent (see [3]), in [10] they determined the algebraic structure of the extension $OHb$. The main results in [10] are [10, Theorems 1.8 and 1.12], which generalize the main result in [6]. Later we strengthened [10, Theorem 1.12] to get [21, Corollary 3.15], by which, it is not difficult to prove that when the block $b$ is nilpotent, there is an $OK$-module such that its restriction to $H \times H$ induces a basic Morita equivalence between $OHb$ and $O\mathbb{H}b$ (see the next paragraph). On the other hand, assuming that $Q$ is abelian and that the index of $H$ in $G$ is coprime to $p$, Rouquier conjectures that there is a complex $C$ of $OK$-modules whose restriction to $H \times H$ induces a splendid Rickard equivalence between $OHb$ and $O\mathbb{H}b$ (see [23, 5.2.3]). Rouquier’s conjecture implies the well known Broué’s conjecture.

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1.4. Any $O(H \times \mathbb{H})$-module $U$ can be identified with an $(OH, \mathbb{H})$-bimodule and vice versa, by the equalities $x \cdot u \cdot y^{-1} = (x, y)u$ and $(x, y) \cdot u = xuy^{-1}$ for any $x \in H$, any $y \in \mathbb{H}$ and any $u \in U$. In this paper we will freely use these identifications. An indecomposable $O(H \times \mathbb{H})$-module $M$ induces a basic Morita equivalence (see [16]) between $OHb$ and $OH\mathbb{H}b$ if $M$ induces a Morita equivalence between $OHb$ and $OH\mathbb{H}b$ and its source modules have their $O$-rank prime to $p$; in this case, the block $b$ of $H$ over $O$ is inertial (see [20]). As the title of this paper shows, we are mainly interested in the trivial inertial blocks, which are special cases of the inertial blocks. The block $b$ of $H$ is a trivial inertial block if there is a $p$-permutation $O(H \times \mathbb{H})$-module inducing a Morita equivalence between $OHb$ and $OH\mathbb{H}b$.

1.5. In this paper, we characterize the $G/H$-graded algebra structure of the extension $OGb$ when $P$ is abelian and the block $b$ is a trivial inertial block. We find a relationship between this $G/H$-graded algebra structure and Rouquier’s conjecture, and prove Rouquier’s conjecture when blocks are trivial inertial blocks. In order to state our results precisely, we need the theory of pointed groups in [12], which is introduced in 2.6 below for convenience. Set $\alpha = \{b\}$. Then $G_\alpha$ and $H_\alpha$ are pointed groups on $OH$. Since $Br_P^{OH}(b) \neq 0$, we choose a primitive idempotent $i$ in $(OH)^P$ such that $bi = i$ and such that $Br_P^{OH}(i) \neq 0$, where $(OH)^P$ is the centralizer of $P$ in $OH$. We denote by $\gamma$ the conjugacy class of $i$ in $(OH)^P$. Since $P$ is maximal such that $Br_P^{OH}(b) \neq 0$, $P_\gamma$ is a defect pointed group of $G_\alpha$. Similarly we take a defect pointed group $Q_\delta$ of $H_\alpha$. Moreover $Q_\delta$ can be chosen so that $Q_\delta$ is contained in $P_\gamma$. We denote by $N_G(Q_\delta)$ the stabilizer of $Q_\delta$ under the $G$-conjugation action. Set $E = N_G(Q_\delta)/C_H(Q)$.

1.6. Let $R_\varepsilon$ and $T_\delta$ be local pointed groups on $OH$. For any $x \in G$ such that $(R_\varepsilon)^x \leq T_\delta$, we denote by $\varphi^T_{R_\varepsilon, x}$ the group homomorphism $\varphi^T_{R_\varepsilon, x} : R \to T$, $u \mapsto uxu^{-1}$. We call the pair $(\varphi^T_{R_\varepsilon, x}, \dot{x})$ a $(G, \dot{G})$-homomorphism from $R_\varepsilon$ to $T_\delta$, where we set $\dot{G} = G/H$ and $\dot{x}$ is the image of $x$ in $\dot{G}$. By [10, Theorem 1.8] or [21, Theorem 3.5], we can see that local pointed groups $R_\varepsilon$ on $OH$ such that $Q_\delta \leq R_\varepsilon \leq P_\gamma$ play an important role in determining the algebra structure of block extensions. So we define a category $E_{(P_\gamma, H, G)}$, where the objects are local pointed groups $R_\varepsilon$ such that $Q_\delta \leq R_\varepsilon \leq P_\gamma$ and, for any pair of objects $R_\varepsilon$ and $T_\delta$, the morphisms from $R_\varepsilon$ to $T_\delta$ are $(G, \dot{G})$-homomorphisms from $R_\varepsilon$ to $T_\delta$; the composition in $E_{(P_\gamma, H, G)}$ is determined by the composition of group homomorphisms and the product in $\dot{G}$. The category $E_{(P_\gamma, H, G)}$ is the local structure of the block extension $OGb$. Note that (1) is the unique local point of $R$ on $OQ$ for any subgroup $R$ of $P$. In the proposition below, we show that the local structure of the block extension $OGb$ is controlled by a finite group when $P$ is abelian.

Proposition 1.7. Keep the notation as above and assume that $P$ is abelian. There are a finite group $L$ containing $P$ and a surjective group homomorphism $\pi : L \to E$ with the kernel $Q$ such that $\pi(u)$ is the image of $u$ in $E$ for any $u \in P$ and such that with the identification of $L/Q$ and $E$ through the isomorphism $L/Q \cong E$ induced by $\pi$, the functor $\tau : E_{(P_\gamma, H, G)} \to E_{(P_\gamma, H, G)}(Q, L)$ mapping an object $R_\varepsilon$ onto $R_{\{1\}}$ and a morphism $(\varphi^T_{R_\varepsilon, x}, \dot{x})$ onto $(\varphi^T_{R_\varepsilon, x}, \pi(x))$ is an isomorphism of categories. Moreover for another such a pair of $L'$ and $\pi'$, there is a group isomorphism $\varphi : L \to L'$, unique up to conjugation, such that $\pi' \circ \varphi = \pi$ and such that $\varphi(u) = u$ for any $u \in P$.

1.8. We denote by $U$ the simple factor of $(OH)^Q$ determined by the point $\delta$ and by $s_\delta$ the canonical surjective homomorphism $(OH)^Q \to U$. Explicitly $U$ is the simple factor of $(OH)^Q$ such that $s_\delta(\delta) \neq \{0\}$. Clearly the $N_G(Q_\delta)$-conjugation stabilizes $\delta$ and thus $U$. For a ring $R$, we denote by $R^*$ the multiplicative group of $R$. We construct the subgroup $N_G(Q_\delta)$ of the direct product $N_G(Q_\delta) \times U^*$ consisting of all elements $(x, s_\delta(a))$ such that the $x$- and $s_\delta(a)$-conjugations have the
it is easily checked that $k^*$ and $C_H(Q)$ as subgroups of $N_G(Q_δ)$. Then it is easily checked that $k^*$ is central in $\hat{N}_G(Q_δ)$, that $C_H(Q)$ is normal in $\hat{N}_G(Q_δ)$ and that the intersection of $k^*$ and $C_H(Q)$ is trivial. Since any $k$-algebra automorphism on $U$ is inner, the quotient of $\hat{N}_G(Q_δ)$ by $k^*$ is isomorphic to $N_G(Q_δ)$. That is to say, $\hat{N}_G(Q_δ)$ is a $k^*$-group with $k^*$-quotient $N_G(Q_δ)$ (see 2.4 below).

1.9. For an object $R_ε$ of $E_{(P_δ, H, G)}$, we denote by $E_{G, \hat{G}}(R_ε)$ its automorphism group in $E_{(P_δ, H, G)}$. The map $N_G(Q_δ) → E_{G, \hat{G}}(Q_δ), x → (\varphi^Q_δ, x, \hat{x})$ induces a group isomorphism $N_G(Q_δ)/C_H(Q) → E_{G, \hat{G}}(Q_δ)$, through which we identify the two groups. We set $\hat{E}_{G, \hat{G}}(Q_δ) = \hat{N}_G(Q_δ)/C_H(Q)$. Then the inclusion $k^* \subset \hat{N}_G(Q_δ)$ induces an injective group homomorphism $k^* → \hat{E}_{G, \hat{G}}(Q_δ)$ so that we can identify $k^*$ as a subgroup of $\hat{E}_{G, \hat{G}}(Q_δ)$. Moreover $k^*$ is central in $\hat{E}_{G, \hat{G}}(Q_δ)$ and the quotient of $\hat{E}_{G, \hat{G}}(Q_δ)$ by $k^*$ is isomorphic to $E_{G, \hat{G}}(Q_δ)$. That is to say, $\hat{E}_{G, \hat{G}}(Q_δ)$ is a $k^*$-group with $k^*$-quotient $\hat{E}_{G, \hat{G}}(Q_δ)$ (see 2.4 below). We denote by $\hat{L}$ the pull-back through the homomorphism $π$ and the canonical homomorphism $\hat{E}_{G, \hat{G}}(Q_δ) → E_{G, \hat{G}}(Q_δ)$. Then $\hat{L}$ is a $k^*$-group with $k^*$-quotient $L$ and so is the opposite group $\hat{L}^\circ$ (see 2.4).

1.10. We denote by $K$ the inverse image of $N_H(Q_δ)/C_H(Q)$ in $L$. Clearly $K$ is normal in $L$ and thus the twisted group algebra $O_L\hat{L}^\circ$ (see 2.4) has an obvious $L/K$-graded algebra structure. The homomorphism $π$ induces a group isomorphism $L/K ≅ N_G(Q_δ)/N_H(Q_δ)$. On the other hand by [12, Theorem 1.2], $H$ acts on the set of all defect pointed groups of $H_δ$. Clearly the $G$-conjugation action stabilizes this set and thus by Frattini argument, we have $G = HN_G(Q_δ)$. Then the inclusion $N_G(Q_δ) \subset G$ induces a group isomorphism $\hat{G} ≅ N_G(Q_δ)/N_H(Q_δ)$. We identify $\hat{G}$, $N_G(Q_δ)/N_H(Q_δ)$ and $L/K$ through the above two group isomorphisms. Then the twisted group algebra $O_L\hat{L}^\circ$ (see 2.2) becomes a $\hat{G}$-graded algebra. Since $k$ is perfect, the inverse image of $P$ in $\hat{L}$ splits uniquely and thus we can identify $P$ with a subgroup of $\hat{L}$. The inclusion $P \subset \hat{L}$ induces a $P$-interior algebra structure on $O_L\hat{L}^\circ$. Set $(OG)_γ = i(OG)i$. Clearly there is a group homomorphism $P → (OG)_γ^*$ mapping $u$ onto $u\hat{I}$ for any $u \in P$ and thus $(OG)_γ$ is a $P$-interior algebra. Since $H$ is normal in $G$, the algebra $O\hat{G}b$ has an obvious $\hat{G}$-graded algebra structure and so does $(OG)_γ$. By [10, 2.14.1], $O\hat{G}b$ and $(OG)_γ$ are Morita equivalent. A full matrix algebra $S$ over $O$ is a determinant one $P$-interior algebra if $S$ is a $P$-interior algebra and the image in $S$ of any element of $P$ has determinant one.

Theorem 1.11. Keep the notation as above and assume that $P$ is abelian and that the block $b$ of $H$ is a trivial inertial block. Then there is a determinant one $P$-interior full matrix algebra $S$ over $O$ such that we have a $\hat{G}$-graded $P$-interior algebra isomorphism (see 2.2)

$$\gamma_\gamma \equiv S \otimes O_L\hat{L}^\circ.$$ Moreover $S$ has a $P$-stable $O$-basis containing the unity of $S$ and $S$ is unique up to $P$-interior algebra isomorphisms.

Theorem 1.11 is a partial generalization of the work of Külshammer and Puig on extensions of nilpotent blocks. It also partially generalizes [18, Theorem 4.6] on the structure of blocks for $p$-solvable groups and [25, Theorem 45.11] on the structure of blocks with normal defect groups. By Proposition 1.7, the group $\hat{L}$ in Theorem 1.11 has a uniqueness property, which shows us a way to compare different extensions of trivial inertial blocks. By this way, we prove
Theorem 1.12. Assume that $P$ is abelian and that the block $b$ of $H$ is a trivial inertial block. Then there is a $p$-permutation $\mathcal{O}K$-module $M$ such that its restriction $\text{Res}^K_H(\mathcal{O}b)$ induces a Morita equivalence between $\mathcal{O}Hb$ and $\mathcal{O}Hb$.

This paper is divided into six sections. In Section 2 we recollect notation and terminology. In Section 3, we introduce Watanabe isomorphism and show that when $P$ is abelian, any automorphism on $kHb$ induced by the $u$-conjugation for any $u$ of $P$ is inner (see Lemma 3.8), where $b$ is the image of $b$ through the reduction homomorphism $\mathcal{O}H \to kH$. In Section 4, assuming that $P$ is abelian, that the block $b$ of $H$ is inertial, that the index of $H$ in $G$ is a power of $p$ and that $\mathcal{O} = k$, we extend a suitable bimodule inducing a basic Morita equivalence between $kHb$ and $k\mathbb{H}b$ to a $k((H \times \mathbb{H})\Delta(P))$-module (see Proposition 4.10); moreover we prove that with a suitable choice of the bimodule, the extended $k((H \times \mathbb{H})\Delta(P))$-module is a $p$-permutation module when the block $b$ of $H$ is a trivial inertial block (see Corollary 4.11). Lemma 3.8 is the main tool to do such an extension. In Section 5, we consider the situation where $P$ is abelian, the block $b$ of $H$ is a trivial inertial block and the index of $H$ in $G$ is a power of $p$. In this situation, by Proposition 4.11 we prove that the block $b$ of $G$ is inertial (see Proposition 5.2); then we characterize the algebra structure of the extension $\mathcal{O}Gb$ (see Theorem 5.12), by which, we find a Dade $p$-subalgebra $S$ in the 1-component of $(\mathcal{O}G)_\gamma$. In Section 6 we prove Proposition 1.7 and Theorems 1.11 and 1.12.

2. Notation and terminology

2.1. Throughout this paper, all $\mathcal{O}$-modules are $\mathcal{O}$-free finitely generated; all $\mathcal{O}$-algebras have identity elements, but their subalgebras need not be unity ones. Let $A$ be an $\mathcal{O}$-algebra; we denote by $A^\circ$, $Z(A)$, $J(A)$, $1_A$ and $\text{Aut}(A)$ the opposite $\mathcal{O}$-algebra of $A$, the center of $A$, the radical of $A$, the identity element of $A$ and the group of all $\mathcal{O}$- automorphisms of $A$, respectively. Sometimes we write $1$ instead of $1_A$ without confusion. Let $B$ be an $\mathcal{O}$-algebra; a homomorphism $\mathcal{F} : A \to B$ of $\mathcal{O}$-algebras is said to be an embedding if $\mathcal{F}$ is injective and $\mathcal{F}(A) = \mathcal{F}(1_A)B\mathcal{F}(1_A)$. Let $X$ be a finite group, let $\mathcal{C}$ be a $\mathcal{X}$-graded algebra and $C_x$ be the $x$-component of $\mathcal{C}$ for any $x \in X$. The tensor product $A \otimes_\mathcal{O} C$ is an $\mathcal{X}$-graded algebra with the $x$-component $A \otimes_\mathcal{O} C_x$ for any $x \in X$.

2.2. Let $X$ and $Y$ be groups with $Y$ normal in $X$. An $\mathcal{O}$-algebra $B$ is a $Y$-interior $X$-algebra (see $\mathcal{O}Y$-interior $X$-algebra in [7]) if there are group homomorphisms $\varphi : Y \to B^*$ and $\psi : X \to \text{Aut}(B)$ such that $\varphi$ preserves the action of $X$ on $Y$ and that of $X$ on $B^*$ and such that $\psi$ lifts the conjugation action of $Y$ on $B$. For any $x, y \in Y$, any $z \in Z$ and any $a \in B$, we write $\varphi(x)a\varphi(y)$ as $xay$ and $\psi(z^{-1})(a)$ as $a^z$. We call $B$ as $X$-interior algebras and $X$-algebras, respectively, when $X = Y$ and $X = 1$. Note that $X$-interior algebras here are the same as interior $X$-algebras in [12] and as $\mathcal{O}X$-interior algebras in [16]. Let $B'$ be another $Y$-interior $X$-algebra. The tensor product $B \otimes_\mathcal{O} B'$ is a $Y$-interior $X$-algebra with the group homomorphism $Y \to (B \otimes_\mathcal{O} B')^*$, $y \mapsto y_1B \otimes y_2B'$ and with the action of $X$ on $B \otimes_\mathcal{O} B'$ defined by the equality $(a \otimes a')^x = a^x \otimes a'^x$ for any $a \in B$, any $a' \in B'$ and any $x \in X$. An $\mathcal{O}$-algebra homomorphism $\mathcal{F} : B \to B'$ is said to be a $Y$-interior $X$-algebra homomorphism (see [16, 2.4]) if for any $x, y \in Y$, any $z \in Z$ and any $a \in B$, we have $\mathcal{F}(xay) = x\mathcal{F}(a)y$ and $\mathcal{F}(a^z) = \mathcal{F}(a)^z$. We call $\mathcal{F}$ an $X$-interior algebra homomorphism and an $X$-algebra homomorphism, respectively, when $X = Y$ and $X = 1$.

2.3. Let $i$ be an $X$-fixed idempotent in $B$. It is easy to check that $iBi$ is a $Y$-interior $X$-algebra with the group homomorphism $Y \to (iBi)^*$, $y \mapsto yi$ and with the group homomorphism $X \to \text{Aut}(iBi)$ mapping $x$ onto the restriction of $\psi(x)$ to $iBi$ for any $x \in X$. Let $W$ be a subgroup of $X$. Clearly $B$ is a $(W \cap Y)$-interior $X$-algebra with the restrictions of $\varphi$ to $W \cap Y$ and of $\psi$ to $W$, denoted...
by $\text{Res}^X_\vee(B)$. Let $\mathcal{C}$ be an $X$-interior algebra with the structure homomorphism $\rho : X \to \mathcal{C}^*$. Let $\varrho : Z \to X$ be a group homomorphism. The algebra $\mathcal{C}$ with the group homomorphism $\rho \circ \varrho : Z \to \mathcal{C}^*$ is a $Z$-interior algebra, denoted by $\text{Res}_\varrho(\mathcal{C})$.

2.4. A $k^*$-group is a group $\hat{X}$ with an injective group homomorphism $\theta : k^* \to Z(\hat{X})$, where $Z(\hat{X})$ is the center of $\hat{X}$, and the quotient $\hat{X}/\theta(k^*) = X$ is called the $k^*$-quotient of $\hat{X}$; usually we omit to mention $\theta$ and write $\lambda x$ instead of $\theta(\lambda)x$ for any $\lambda \in k^*$ and any $x \in \hat{X}$. By [24, Chapter II, Proposition 8] there is a canonical decomposition $O^* \cong k^* \times (1 + J(O))$, through which we regard $k^*$ as a subgroup of $O^*$. We define the twisted group algebra $O \hat{\times} \hat{X}$ to be the $O$-algebra $O \otimes O \hat{\times} \hat{X}$ with the multiplication induced by the product in $\hat{X}$. Let $U$ be another $k^*$-group with $k^*$-quotient $\hat{U}$. A group homomorphism $\phi : \hat{X} \to \hat{U}$ is a $k^*$-group homomorphism if $\phi(\lambda x) = \lambda \phi(x)$ for any $\lambda \in k^*$ and $x \in \hat{X}$. We denote by $\hat{\Theta}$ the inverse image of $\hat{\Theta}$ for any subset $V$ of $X$. If $V$ is a $p$-subgroup of $X$, by [14, 2.10] there is a unique $k^*$-group isomorphism $\hat{\Theta} \cong k^* \times V$. Therefore we can identify $V$ as a subgroup of $\hat{X}$ through this $k^*$-group isomorphism. We denote by $\hat{X}^\circ$ the $k^*$-group with the same underlying group $\hat{X}$ endowed with the group homomorphism $\theta^{-1} : k^* \to Z(\hat{X})$, $\lambda \mapsto \theta(\lambda)^{-1}$. Let $\vartheta : Z \to X$ be a group homomorphism; we denote by $\text{Res}_{\vartheta}(\hat{X})$ the $k^*$-group formed by all pairs $(y, x) \in Z \times \hat{X}$ such that $\vartheta(y)$ is the image of $x$ in $X$, endowed with the group homomorphism mapping $\lambda \in k^*$ on $(1, \lambda)$. The $k^*$-group $\text{Res}_{\vartheta}(\hat{X})$ has $k^*$-quotient isomorphic to $Z$ and thus is a $k^*$-group with $k^*$-quotient $Z$. For more on $k^*$-groups, see [14, §5].

2.5. Let $B$ be an $X$-algebra over $O$. For any subgroup $Z$ of $X$, we denote by $B^Z$ the $O$-subalgebra of all $Z$-fixed elements in $B$. For any subgroup $U$ of $X$, we set $B(U) = k \otimes_O (B^U / \sum V B^V)$, where $V$ runs over the set of proper subgroups of $U$ and $B^V$ is the image of the relative trace map $T_{UV} : B^V \to B^U$. We denote by $\text{Br}_U^B$ the canonical surjective homomorphism $\text{Br}_U^B : B^U \to B(U)$. When $B$ is a $Y$-interior $X$-algebra, the $Y$-interior $X$-algebra on $B$ induces $C_Y(U)$-interior $N_X(U)$-algebra structures on $B(U)$ and on $B^U$ and then $\text{Br}_U^B$ is a $C_Y(U)$-interior $N_X(U)$-algebra homomorphism. When $B$ is equal to the group algebra $O \hat{\times} \hat{X}$, the $k$-algebra homomorphism $kC_X(U) \to B(U)$ sending $x \in C_X(U)$ onto the image of $x$ in $B(U)$ is a $k$-algebra isomorphism (see [25, Proposition 37.5]), through which we identify $B(U)$ with $kC_X(U)$.

2.6. Recall that a pointed group $V_\beta$ on $B$ consists of a subgroup $V$ of $X$ and a $(B^V)^*$-conjugacy class $\beta$ of primitive idempotents of $B^V$. We also say that $\beta$ is a point of $V$ on $B$. Obviously $X$ acts on the set of all pointed groups on $B$ by the equality $(V_\beta)^x = V_{\beta^x}$. We denote by $N_X(V_\beta)$ the stabilizer of $V_\beta$ in $X$. Another pointed group $Z_\gamma$ is contained in $V_\beta$ and we write $Z_\gamma \subseteq V_\beta$ if $Z \subseteq V$ and there exist $i \in \beta$ and $j \in \gamma$ such that $ij = ji = j$. A pointed group $U_\gamma$ on $B$ is local if the image of $\gamma$ in $B(U)$ is not equal to $\{0\}$. A pointed group $U_\gamma$ is a defect pointed group of a pointed group $V_\beta$ on $B$ if $U_\gamma$ is a maximal local pointed group contained in $V_\beta$. Since an $X$-interior algebra structure induces an $X$-algebra structure by the $X$-conjugation, the above terminology on pointed groups applies to $X$-interior algebras. For more on pointed groups, see [12] and [25].

2.7. Set $\hat{X} = X/Y$ and denote by $\hat{x}$ the image of $x$ in $\hat{X}$ for any $x \in X$. Let $Z$ be a subgroup of $X$. Let $\mathcal{A}$ be an $\hat{X}$-graded $Z$-interior algebra such that the image of $z$ in $\mathcal{A}$ is just inside the $\hat{z}$-component of $\mathcal{A}$ for any $z \in Z$. Then the $\hat{X}$-graded $Z$-interior algebra structure on $\mathcal{A}$ induces a $W$-interior $Z$-algebra structure on the 1-component $B$ of the $\hat{X}$-graded algebra $\mathcal{A}$, where we set $W$ to be the intersection $Y \cap Z$. Let $R_e$ be a local pointed group on $B$. Take some $h \in \varepsilon$ and set $\mathcal{A}_e = h \mathcal{A} h$ and $B_e = hBh$. Then it is easy to see that $\mathcal{A}_e$ is an $\hat{X}$-graded $R$-interior algebra with the $\hat{x}$-component $h \mathcal{A}_e h$ for any $\hat{x} \in \hat{X}$ and with the group homomorphism $R \to \mathcal{A}_e^*; u \mapsto uh$ for any $u \in R$, and that $B_e$ is a $(R \cap Y)$-interior $R$-subalgebra of $\mathcal{A}_e$ (see Paragraph 2.3).

2.8. For any $\hat{x} \in \hat{X}$, we denote by $\mathcal{N}_{\mathcal{A}_e}^\hat{x}(R)$ the set of all invertible elements in the $\hat{x}$-component of
\(A_e\) normalizing \(R h\). Notice that \(N_{A_e}^x(R)\) may be empty. We set \(N_{A_e}^x(R) = \bigcup_{z \in X} N_{A_e}^z(R)\). Then it is easily checked that \(N_{A_e}^x(R)\) is a group with respect to the multiplication of \(A_e\) and that \((B_e R)^*\) is a normal subgroup of \(N_{A_e}^x(R)\). We set \(F_A(R_e) = N_{A_e}^x(R)/(B_e R)^*\) and \(\tilde{F}_A(R_e) = N_{A_e}^x(R)/(1 + J(B_e R)^*)\). Since \(B_e R/J(B_e R) \cong k\), the quotient group \(\tilde{F}_A(R_e)\) has a natural \(k^*\)-group structure with \(k^*\)-quotient \(F_A(R_e)\). For any \(a \in N_{A_e}^x(R)\), we denote by \(\hat{a}\) the image of \(a\) in \(\tilde{F}_A(R_e)\). When \(X\) is the trivial group, \(N_{A_e}^x(R)\) is just the normalizer of \(R h\) in \(A_e^*\); in this case, we write \(N_{A_e}^x(R)\) as \(N_{A_e}^x(R)\).

2.9. Now we generalize fusions in [13] to graded algebras. A pair \((\varphi, \hat{x})\) is called an \((A, \hat{X})\)-fusion of \(R_e\) if \(\varphi\) is a group automorphism on \(R\), \(\hat{x}\) is an element of \(\hat{X}\) and there is an element \(a\) in \(N_{A_e}^x(R)\) such that \(a u a^{-1} = (\varphi(u) h)\) for any \(u \in R\). We denote by \(F_{A, \hat{X}}(R_e)\) the set of all \((A, \hat{X})\)-fusions of \(R_e\). \(F_{A, \hat{X}}(R_e)\) is a group with respect to the composition determined by the composition of maps and by the product in \(\hat{X}\). It is easy to prove that the following holds.

2.9.1. The map \(\theta : F_{A, \hat{X}}(R_e) \to F_A(R_e)\) mapping \((\varphi, \hat{x})\) to \(\hat{a}\), where \(a\) is an element in \(N_{A_e}^x(R)\) such that \(a u a^{-1} = \varphi(u) h\) for any \(u \in R\), is a group homomorphism. Moreover the homomorphism \(\theta\) is a group isomorphism if the map \(R \to Rh, u \mapsto uh\) is a group isomorphism.

We set \(\tilde{F}_{A, \hat{X}}(R_e) = \text{Res}_\theta(\tilde{F}_A(R_e))\). That is to say, \(\tilde{F}_{A, \hat{X}}(R_e)\) consists of all triples \((\varphi, \hat{x}, \bar{a}\)\), where \(a\) is an element of \(N_{A_e}^x(R)\) such that \(a u a^{-1} = \varphi(u) h\) for any \(u \in R\). The group \(F_{A, \hat{X}}(R_e)\) has a normal group \(F_{\hat{B}}(R_e)\), which consists of all \((A, \hat{X})\)-fusions \((\varphi, 1)\) of \(R_e\). When \(X\) is the trivial group, we write \(\tilde{F}_{A, \hat{X}}(R_e), F_{A, \hat{X}}(R_e), (\varphi, 1, \bar{a}\) and \((\varphi, 1)\) as \(\tilde{F}_A(R_e), F_A(R_e), (\varphi, \bar{a}\) and \(\varphi\), respectively.

Then \(F_A(R_e)\) is just the usual fusion group in [13].

2.10. Now we fix the setting of the remainder of the paper. Let \(G\) be a finite group and let \(H\) be a normal subgroup of \(G\). Let \(b\) be a \(G\)-stable block of \(H\) over \(O\). Set \(A = O \Gamma g b, B = O \Gamma h b, \hat{G} = G/\hat{H}\). For any \(x \in G\) and any subset \(I\) of \(G\), we denote by \(\hat{x}\) and \(\hat{I}\) the images of \(x\) and \(I\) in \(\hat{G}\), respectively. Clearly \(A\) is a \(\hat{G}\)-graded algebra with the \(\hat{x}\)-component \(O \Gamma h x b\). Obviously, \(\alpha = \{b\}\) is a point of both \(H\) and \(G\) on \(O \Gamma H\). Let \(Q_\delta\) and \(P_\gamma\) be defect pointed groups of \(H_\alpha\) and \(G_\alpha\), respectively, such that \(Q_\delta \leq P_\gamma\). Choosing \(j \in \delta\) and \(i \in \gamma\) such that \(ij = ji\), we set \(A_\gamma = i A\hat{a}, B_\gamma = i B\hat{b}\) and \(B_\delta = j B\hat{b}\). Then \(A_\gamma\) is a \(\hat{G}\)-graded \(P\)-interior algebra, \(B_\gamma\) is a \(Q\)-interior \(P\)-subalgebra of \(A_\gamma\) and the \(Q\)-interior algebra \(B_\delta\) is a source algebra of the block algebra \(B\). Set \(H = N_H(Q)\) and \(G = N_G(Q)\). Let \(b\) be the Brauer correspondent of \(b\) in \(H\) and set \(\alpha' = \{b\}\). Since \(b\) is \(G\)-stable, \(b\) is \(G\)-stable. Thus \(\alpha'\) is a point of both \(G\) and \(H\) on \(O \Gamma H\). We take the local points \(\gamma'\) and \(\delta'\) of \(P\) and \(Q\) on \(O \Gamma H\), respectively, such that \(B_\Gamma^{P H}(\gamma') = B_\Gamma^{Q H}(\gamma')\) and \(B_\Gamma^{Q H}(\delta') = B_\Gamma^{Q H}(\delta')\). Then it is easy to verify that \(P_{\gamma'}\) is a defect pointed group of \(G_{\alpha'\gamma'}\), that \(Q_{\delta'}\) is a defect pointed group of \(H_{\alpha'\delta}\) and that \(Q_{\delta'}\) is contained in \(P_{\gamma'}\). Take \(i \in \gamma'\) and \(j \in \delta'\). Set \(A = O \Gamma g B, B = O \Gamma h B, A_{\gamma'} = i A\hat{a}, B_{\gamma'} = i B\hat{b}\) and \(B_{\delta'} = j B\hat{b}\). For any local pointed group \(R_e\) such that \(Q_\delta \leq R_e \leq P_\gamma\), we set \(C_G(R_e) = N_G(R_e) \cap C_G(R)\) and denote by \(b_e\) the block of \(C_H(R)\) over \(O\) such that \(B_\Gamma^{Q H}(\epsilon) = B_\Gamma^{Q H}(\epsilon)\).

Lemma 2.11. Let \(R_e\) be a local pointed group such that \(R_e \leq P_\gamma\) and \(Q \subset R\). Then we have \(N_G(R_e) = N_{N_G(R_e)}(P_\gamma) C_G(R_e)\).

Proof. Since \(P\) is abelian, by [10, Proposition 5.5.], we have \(P \subset C_G(R_e)\). Clearly there is a pointed group \(C_G(R_e)_\eta\) on \(O H\) such that \(P_\gamma\) is a defect pointed group of \(C_G(R_e)_\eta\). Since \(R_e \leq P_\gamma\), by [10, 2.16.2] the pointed group \(C_G(R_e)_\eta\) is unique. Clearly \(N_G(R_e)\) stabilizes \(C_G(R_e)_\eta\) and thus the set of all defect pointed groups of \(C_G(R_e)_\eta\). Since \(C_G(R_e)\) acts transitively on this set, by Frattini argument we have \(N_G(R_e) = N_{N_G(R_e)}(P_\gamma) C_G(R_e)\).
3. Watanabe isomorphism

Throughout this section, we keep the notation in 2.10; moreover we assume that $P$ is abelian and that the index of $H$ in $G$ is a power of $p$. We denote by $\bar{I}$ the image of any subset $I$ of $\mathcal{O}G$ through the reduction homomorphism $\mathcal{O}G \to kG$.

3.1. Set $E_H(Q_\delta) = E_{H,H/H}(Q_\delta)$, which is a normal subgroup of $E_{G,G}(Q_\delta)$. According to the identification of $E_{G,G}(Q_\delta)$ and $N_G(Q_\delta)/C_H(Q)$ in Paragraph 1.9, we have $E_H(Q_\delta) = N_H(Q_\delta)/C_H(Q)$. We write any element $(\varphi^Q_1)$ in $E_H(Q_\delta)$ as $\varphi^Q_1$. We denote by $S_Q$ the subgroup of all $E_H(Q_\delta)$-fixed elements in $Q$. We set $\mathcal{H}_Q = [Q, N_H(Q_\delta)]$, which is the hyperfocal subgroup (see [17, 1.7]) of $Q_\delta$. Since $E_H(Q_\delta)$ is a $p'$-group (see [25, Theorem 37.9]), by [8, Chapter 5, Theorem 2.3] we have $Q = S_Q \times \mathcal{H}_Q$. Clearly $(Q, \bar{b}_S)$ is a $(b, H)$-Brauer pair and there is a unique $(b, H)$-Brauer pair $(S_Q, \bar{b}_S)$ such that $(S_Q, \bar{b}_S)$ is contained in $(Q, \bar{b}_S)$ (see [4]); we write this inclusion relationship by $(S_Q, \bar{b}_S) \leq (Q, \bar{b}_S)$. By [26, Theorem 2], the map

$$W^Q_{H,\bar{b}} : Z(kH\bar{b}) \to Z(kC_H(S_Q)\bar{b}_S), \ a \mapsto Br^{kH}_{S_Q}(a)\bar{b}_S$$

is a $k$-algebra isomorphism. We call $W^Q_{H,\bar{b}}$ as Watanabe isomorphism. Since $Q_\delta$ is $P$-stable (see [10, Proposition 5.5]), the $(b, H)$-Brauer pair $(Q, \bar{b}_S)$ and then the $(b, H)$-Brauer pair $(S_Q, \bar{b}_S)$ (see [4, Theorem 1.8]) are $P$-stable. Thus the $P$-conjugation induces a $P$-algebra structure on $Z(kC_H(S_Q)\bar{b}_S)$. The $P$-conjugation also induces a $P$-algebra structure on $Z(kH\bar{b})$. Clearly $W^Q_{H,\bar{b}}$ respects the $P$-algebra structures on $Z(kC_H(S_Q)\bar{b}_S)$ and $Z(kH\bar{b})$, so it is a $P$-algebra isomorphism.

3.2. Since the index of $H$ in $G$ is a power of $p$, we have $G = PH$ (see [10, Proposition 5.3]) and then $b$ is a block of $G$. By [10, Proposition 6.2], the obvious $P$-algebra homomorphism $\iota : \mathcal{O}H \to \mathcal{O}G$ determined by the inclusion $\mathcal{O}H \subset \mathcal{O}G$ is a strict semicovering; moreover $P_\gamma$ determines a unique local pointed group $P_\gamma$ on $\mathcal{O}G$ such that $\gamma \subset \bar{\gamma}$. By [10, Corollary 6.3], $P_\gamma$ is a defect pointed group of $G_\alpha$ on $A$; in particular, $A_\gamma$ is a source algebra of the block algebra $A$. Then as in the paragraph above, we have the subgroup $S_P$ of all $E_G(P_\gamma)$-fixed elements in $P$, the hyperfocal subgroup $\mathcal{H}_P$ of $P_\gamma$ and the equality $P = S_P \times \mathcal{H}_P$. Since $A_\gamma = \sum_{u \in P} B_\gamma u$, by [17, Proposition 4.2] we have $\mathcal{H}_P \subset Q$ and thus $P = QS_P$.

Lemma 3.3. Keep the notation and the assumptions as above. Let $R_\varepsilon$ be a local pointed group such that $Q_\delta \leq R_\varepsilon \leq P_\gamma$ and let $R_\bar{\varepsilon}$ be a local pointed group on $\mathcal{O}G$ such that $\varepsilon \subset \bar{\varepsilon}$. Then we have $N_G(R_\varepsilon) = N_G(R_\bar{\varepsilon})$.

Proof. Let $\mathcal{S}$ be the set of all local points $e$ of $R$ on $\mathcal{O}H$ such that $e \subset \bar{e}$. By [10, Proposition 6.2], we have $\mathcal{S} = \{z^e | z \in C_G(R)\}$. Clearly $C_H(R)$ is contained in $N_G(R_\varepsilon)$ and by [10, Proposition 5.5], $P$ is also contained in $N_G(R_\varepsilon)$. Since $C_G(R) = PC_H(R)$, we have $\mathcal{S} = \{e\}$. In particular, $e$ and $\bar{e}$ determine each other. Thus we have $N_G(R_\varepsilon) = N_G(R_\bar{\varepsilon})$.

By [10, Proposition 5.5], we have $N_{N_H(Q_\delta)}(P_\gamma) = N_H(P_\gamma)$.

Lemma 3.4. Keep the notation and the assumptions as above. Then the inclusion $N_H(P_\gamma) \subset N_H(Q_\delta)$ induces a group isomorphism.

3.4.1

$$E_G(P_\gamma) \cong E_H(Q_\delta),$$

which maps an automorphism in $E_G(P_\gamma)$ onto its restriction to $Q$. 

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Proof. By Lemmas 2.11 and 3.3, we have $N_H(Q_δ) = N_{N_H(Q_δ)}(P_γ)C_H(Q) = N_H(P_γ)C_H(Q) = N_H(P_γ)C_H(Q)$. Since $C_H(Q) \cap N_H(P_γ) = C_H(P_γ)$ (see [10, Proposition 5.5]), the inclusion $N_H(P_γ) \subset N_H(Q_δ)$ induces the desired group isomorphism.

Lemma 3.5. Keep the notation and the assumptions as above. Then we have $Q \cap S_P = S_Q$.

Proof. It easily follows from Isomorphism 3.4.1.

3.6. Set $C = C_G(S_P)$ and $D = C_H(S_P)$. Clearly $(P, b_γ)$ is a $(b, G)$-Brauer pair and there is a unique $(b, G)$-Brauer pair $(S_P, b_{SP})$ such that $(S_P, b_{SP}) \leq (P, b_γ)$. The block $b_{SP}$ can be uniquely lifted to a block $b_{SP}$ of $C$ over $O$. Since $G = PH = HS_P$, we have $C = DS_P$ and thus $b_{SP}$ is also a block of $D$ over $O$. Set $\mathcal{E} = OCb_{SP}$ and $\mathcal{F} = ODb_{SP}$. Applying Watanabe isomorphism to $G, P$ and $b$, we get a $k$-algebra isomorphism $W_{G, b}^P : Z(\bar{A}) \to Z(\bar{E})$, $a \mapsto Br_{b_{SP}}(a)b_{SP}$. Actually $Z(\bar{A})$ and $Z(\bar{E})$ admit natural $\bar{G}$-graded algebra structures and $W_{G, b}^P$ is a $\bar{G}$-graded $k$-algebra isomorphism.

3.7. Since $G = HS_P$, we take a complete set $I$ of representatives of $\bar{G}$ in $S_P$. Clearly the $\bar{G}$-graded algebra structure on $\bar{A}$ induces a $\bar{G}$-graded algebra structure on $\bar{A}$, which induces a $\bar{G}$-graded algebra structure on $Z(\bar{A})$. More precisely $Z(\bar{A})$ is a $\bar{G}$-graded algebra with the $v$-component $Z(\bar{A}) \cap (\bar{B}v)$ for any $v \in I$. Clearly the inclusion $C \subset G$ induces a group isomorphism $C/D \cong G/H = \bar{G}$. We identify the quotient $C/D$ with $\bar{G}$ through this isomorphism. The block algebra $\bar{E}$ has an obvious $\bar{G}$-graded algebra structure with the $v$-component $\bar{F}v$ for any $v \in I$. This $\bar{G}$-graded algebra structure on $\bar{E}$ induces a $\bar{G}$-graded algebra structure on $Z(\bar{E})$. More precisely $Z(\bar{E})$ is a $\bar{G}$-graded algebra with the $v$-component $Z(\bar{E}) \cap (\bar{F}v)$ for any $v \in I$ and thus it is a $\bar{G}$-graded $k$-algebra isomorphism. It follows from the uniqueness of the $(b, G)$-Brauer pair $(S_P, b_{SP})$ that the block $b_{SP}$ of $D$ is $P$-stable. Thus the $P$-conjugation induces a $P$-algebra structure on $Z(\bar{F})$. Clearly we have $Z(\bar{A}) \cap \bar{B} = Z(\bar{B})^P$ and $Z(\bar{E}) \cap \bar{F} = Z(\bar{F})^P$. Then by restriction, Watanabe isomorphism $W_{G, b}^P$ induces a $k$-algebra isomorphism $Z(\bar{B})^P \to Z(\bar{F})^P$.

Lemma 3.8. Keep the notation and the assumptions as above. Then there is a unique group homomorphism $\rho : P \to \bar{B}^*$ such that $\rho$ maps $u$ onto $u\bar{b}$ for any $u \in H_P$ and $v$ onto $v(W_{G, b}^P)^{-1}(v^{-1}b_{SP})$ for any $v \in S_P$. In particular for any $u \in P$, the automorphism on $\bar{B}$ induced by the $u$-conjugation is an inner automorphism.

Proof. Take $v \in S_P$. Clearly $v\bar{b}_{SP}$ lies in $Z(\bar{E}) \cap (\bar{F}v)$ and its inverse image $c_v$ through $W_{G, b}^P$ lies in $Z(\bar{A}) \cap (\bar{B}v)$. Set $a_v = c_v^{-1}v$. Then $a_v$ belongs to $\bar{B}^*$ and for any $a \in \bar{B}$, we have $ava^{-1} = a_vaa_v^{-1}$. It is easily checked that the map $\rho : P \to \bar{B}^*$ mapping $u$ onto $u\bar{b}$ for any $u \in H_P$ and mapping $v$ onto $a_v$ for any $v \in S_P$ is a group homomorphism, which induces the $P$-conjugation action on $\bar{B}$.

4. $p$-extensions of basic Morita equivalences

Throughout this section, we keep the notation in 2.10; moreover we assume that $P$ is abelian, that the block $b$ of $H$ is inertial (see Paragraph 1.4), that the index of $H$ in $G$ is a power of $p$, and that $O = k$. So $G = PH, G = PH, b$ is a block of $G$ and $\bar{b}$ is a block of $G$. We denote by $\bar{E}_H(Q_δ)$ the inverse image of $E_H(Q_δ)$ in $E_{G, \bar{G}}(Q_δ)$, where $E_H(Q_δ) = E_{H, H/H}(Q_δ)$.

4.1. Assume that a $(k(H \times H))$-module $W$ induces a Morita equivalence between the block algebras $\bar{B}$ and $\bar{B}$. Then this Morita equivalence induces a $k$-algebra isomorphism $\Psi_{b, b}^W : Z(\bar{B}) \cong Z(\bar{B})$ such
Lemma 4.2. Let isomorphism $Q$ be a module with the algebra structure on the algebra $k$. Then $\Psi$ is a $k$-algebra isomorphism. Similarly the map $\Phi_{b,\delta} : Z(B) \to Z(B_\delta)$, $a \mapsto a_j$ is a $k$-algebra isomorphism. We denote by $\iota$ the group isomorphism $Q \cong \Delta(Q)$, $u \mapsto (u, u)$.

Proposition 4.3. Keep the notation and the assumptions as above. Then there are a $Q$-interior algebra isomorphism $F^N_{\delta,\delta'} : B_\delta \cong \text{Res}_k(\text{End}_k(N)) \otimes_k B_{\delta'}$ and a $k$-algebra isomorphism $\Psi_{\delta,\delta'} : Z(B_\delta) \cong Z(B_{\delta'})$ such that for any $a \in Z(B_\delta)$, we have

$$F^N_{\delta,\delta'}(a) = 1 \otimes F_{\delta,\delta'}(a).$$

Set $S = \text{End}_k(N)$, let $j$ be a primitive idempotent in $S$ and set $M = B_j \otimes_{B_\delta} (S_j \otimes_k B_{\delta'}) \otimes_{B_{\delta'}} jB$. Then $M$ is a $\Delta(P)$-stable indecomposable $k(H \times \bar{H})$-module with a $k\Delta(Q)$-module $N$ as a source and it induces a basic Morita equivalence between $B$ and $B$. In particular, the $k$-algebra isomorphism $\Phi^M_{b,\bar{b}}$ induced by the Morita equivalence is a $P$-algebra isomorphism. Moreover we have

$$\Phi_{b,\delta'} \circ \Psi^M_{b,\bar{b}} = \Psi_{\delta,\delta'} \circ \Phi_{b,\delta}.$$

Proof. Clearly $N_G(Q_\delta) = N_G(Q_{\delta'})$, the simple factors of $(\mathcal{O}H)^Q$ and $(\mathcal{O}\bar{H})^Q$ determined by the points $\delta$ and $\delta'$ are isomorphic as $N_G(Q_\delta)$-algebras and there is a $k^*$-group isomorphism $\hat{N}_G(Q_{\delta'}) \cong \hat{N}_G(Q_{\delta})$ lifting the equality $N_G(Q_\delta) = N_G(Q_{\delta'})$. Set $\hat{G} = G/\bar{H}$. Obviously we have $E_{G,\hat{G}}(Q_\delta) = E_{G,\hat{G}}(Q_{\delta'})$ and the $k^*$-group isomorphism $\hat{N}_G(Q_{\delta}) \cong \hat{N}_G(Q_{\delta'})$ induces a $k^*$-group isomorphism $\hat{E}_{G,\hat{G}}(Q_{\delta}) \cong \hat{E}_{G,\hat{G}}(Q_{\delta'})$ lifting the equality $E_{G,\hat{G}}(Q_\delta) = E_{G,\hat{G}}(Q_{\delta'})$. Set $E_{\bar{H}}(Q_{\delta'}) = E_{\bar{H},\bar{H}/H}(Q_{\delta'})$ and denote by $\hat{E}_{\bar{H}}(Q_{\delta'})$ the inverse image of $E_{\bar{H}}(Q_{\delta'})$ in $E_{\bar{G},\hat{G}}(Q_{\delta'})$. Since $E_{\bar{H}}(Q_{\delta'})$ is equal to $E_{\bar{H}}(Q_{\delta'})$, the isomorphism $\hat{E}_{G,\hat{G}}(Q_{\delta}) \cong \hat{E}_{G,\hat{G}}(Q_{\delta'})$ induces a $k^*$-group isomorphism $\hat{E}_{\bar{H}}(Q_{\delta}) \cong \hat{E}_{\bar{H}}(Q_{\delta'})$ lifting the equality $E_{\bar{H}}(Q_{\delta}) = E_{\bar{H}}(Q_{\delta'})$.

Since the block $b$ of $H$ is inertial, by [20, 2.16.2] there is a $k\Delta(Q)$-module $N$ with vertex $\Delta(Q)$ such that we have a $Q$-interior algebra isomorphism $B_\delta \cong \text{Res}_k(\text{End}_k(N)) \otimes_k k^*(Q \times \hat{E}_H(Q_{\delta}))$. On the other hand, by [14, Proposition 14.6] there is a $Q$-interior algebra isomorphism $B_{\delta'} \cong k^*(Q \times \hat{E}_{\bar{H}}(Q_{\delta'}))$. Since the $k^*$-group isomorphism $\hat{E}_{\bar{H}}(Q_{\delta}) \cong \hat{E}_{\bar{H}}(Q_{\delta'})$ can be extended to a $k^*$-group isomorphism $Q \times \hat{E}_H(Q_{\delta}) \cong Q \times \hat{E}_{\bar{H}}(Q_{\delta'})$, we have a $Q$-interior algebra isomorphism $B_\delta \cong \text{Res}_k(\text{End}_k(N)) \otimes_k B_{\delta'}$, which is denoted by $F^N_{\delta,\delta'}$. Clearly $F^N_{\delta,\delta'}$ induces a $k$-algebra isomorphism $\Psi_{\delta,\delta'} : Z(B_\delta) \cong Z(B_{\delta'})$ such that $F^N_{\delta,\delta'}(a) = 1 \otimes F_{\delta,\delta'}(a)$ for any $a \in Z(B_\delta)$.

It is easy to check that the $(B_\delta, B_{\delta'})$-bimodule $S_j \otimes_k B_{\delta'}$ induces a Morita equivalence between $B_\delta$ and $B_{\delta'}$, that $M$ induces a Morita equivalence between $B$ and $B$, and that $M$ is a $k(H \times \bar{H})$-module with the $k\Delta(Q)$-module $N$ as a source. So $M$ induces a basic Morita equivalence between $B$ and $B$. The $k$-linear map $k\bar{H} \to k\bar{H}$ sending $x$ onto $x^{-1}$ for any $x \in \bar{H}$ is an opposite ring isomorphism and we denote by $\bar{b}^0$ the image of $\bar{b}$ through this opposite ring isomorphism. There
is an obvious $k$-algebra isomorphism $k(H \times \mathbb{H}) \cong kH \otimes_k k\mathbb{H}$ mapping $(y, z)$ onto $y \otimes z$ for any $y \in H$ and any $z \in \mathbb{H}$. We identify $k(H \times \mathbb{H})$ and $kH \otimes_k k\mathbb{H}$ and then $b \otimes b^o$ is a block of $H \times \mathbb{H}$ over $k$. Clearly $b \otimes b^o$ acts on $M$ as the identity map and thus $M$ belongs to the block algebra $k(H \times \mathbb{H})(b \otimes b^o)$. For any $u \in P$, by Lemma 3.8 the automorphism on $k(H \times \mathbb{H})(b \otimes b^o)$ induced by the $(u, u)$-conjugation is an inner automorphism. Thus the $k(H \times \mathbb{H})$-module $M$ is $\Delta(P)$-stable. This implies that the $k$-algebra isomorphism $\Psi_{b, b^o}^M : Z(B) \to Z(B)$ maps $a$ onto $c$ if and only if the equality $am = mc$ holds for any $m \in M$. The equality $am = mc$ holds for any $m \in M$ if and only if the equality $(ja)m = m(cj)$ holds for any $m \in S_f \otimes_k \mathbb{B}_{b^o}$ if and only if $\Psi_{\delta, \delta^o}$ maps $ja$ onto $cj$. Thus the equality 4.3.2 holds.

In the remainder of this section, we always keep the $\Delta(P)$-stable $k(H \times \mathbb{H})$-module $M$ in Proposition 4.3 and will extend it to a $k((H \times \mathbb{H})\Delta(P))$-module in the following Proposition 4.10.

4.4. Clearly $(Q, b_\delta)$ is a $(b, H)$-Brauer pair and there is a unique $(b, H)$-Brauer pair $(S_Q, b_{S_Q})$ such that $(S_Q, b_{S_Q}) \leq (Q, b_\delta)$. Set $E = C_G(S_Q), F = C_H(S_Q), C = kC_G(S_Q)b_{S_Q}$ and $D = kC_H(S_Q)b_{S_Q}$. By Watanabe isomorphism applied to $H, b$ and $Q$, we get a $P$-algebra isomorphism (see 3.1)

$$W^Q_{H, b} : Z(B) \to Z(D), \ a \mapsto Br^k_{S_Q}(a)b_{S_Q}.$$ 

Since $N_H(Q_\delta) = N_H(Q_{b_\delta})$, $S_Q$ is equal to the subgroup of all $N_H(Q_{b_\delta})$-fixed elements in $Q$. We denote by $(S_Q, b_{S_Q})$ the unique $(b, H)$-Brauer pair such that $(S_Q, b_{S_Q}) \leq (Q, b_\delta)$. Set $E = C_G(S_Q), F = C_H(S_Q)$ and $D = kC_H(S_Q)b_{S_Q}$. By Watanabe isomorphism applied to $H, Q$ and $b$, we get a $P$-algebra isomorphism

$$W^Q_{H, b} : Z(B) \to Z(D), \ a \mapsto Br^k_{S_Q}(a)b_{S_Q}.$$ 

Lemma 4.5. With the notation and the assumptions as above, the block $b_{S_Q}$ is the Brauer correspondent of $b_{S_Q}$ in $C_H(S_Q)$.

Proof. Since $(S_Q, b_{S_Q}) \leq (Q, b_\delta)$, we have $Br^k_{Q}(b_{S_Q})b_\delta = b_\delta$. Clearly $Q$ is a defect group of the block $b_{S_Q}$ of $F$ and thus $Br^k_{Q}(b_{S_Q})$ is the sum of all $N_F(Q)$-conjugations of $b_\delta$. Similarly $Br^k_{Q}(b_{S_Q})$ is also the sum of all $N_F(Q)$-conjugations of $b_{\delta^o}$. But since $b_\delta = b_{\delta^o}$ and $N_F(Q) = N_F(Q)$, we have $Br^k_{Q}(b_{S_Q}) = Br^k_{Q}(b_{S_Q})$. The proof is done.

Lemma 4.6. Keep the notation and the assumptions as above and set $\Psi_{b, b}^{S_Q} = W^Q_{H, b} \circ \Psi_{b, b}^M \circ (W^Q_{H, b})^{-1}$. Then the following hold.

4.6.1 $\Psi_{b, b}^{S_Q}$ is equal to $\Psi_{b, b}^{S_Q, b_{S_Q}}$ for some $k(F \times F)$-module $V$ inducing a basic Morita equivalence between $D$ and $D$.

4.6.2 For any $u \in S_Q$, $\Psi_{b, b}^{S_Q, b_{S_Q}}(u b_{S_Q})$ is equal to $u b_{S_Q}$.

Proof. Set $S = \text{End}_k(N)$ and $T = S(\Delta(S_Q))$. Clearly $F_{\delta, \delta^o}^N$ maps $(B_\delta)_R^Q$ onto $(\text{Res}_i(S) \otimes_k \mathbb{B}_{b^o})_R^Q$ for any subgroup $R$ of $Q$ and thus induces a $Q$-interior algebra isomorphism $F_{\delta, \delta^o}^N(S_Q) : (B_\delta)(S_Q) \cong (\text{Res}_i(S) \otimes_k \mathbb{B}_{b^o})(S_Q)$. By [15, Proposition 5.6], there is a $k$-algebra isomorphism

$$(\text{Res}_i(S) \otimes_k \mathbb{B}_{b^o})(S_Q) \cong \text{Res}_i(T) \otimes_k (\mathbb{B}_{b^o})(S_Q)$$
which maps $Br_{\mathcal{S}_Q}^{\text{Res}_S(S) \otimes k \mathcal{B}_{\delta'}}(s \otimes t)$ onto $Br_{\Delta(S_Q)}^S(s) \otimes Br_{\mathcal{S}_Q}^{\mathcal{B}_{\delta'}}(t)$ for any $s \in S^{\Delta(S_Q)}$ and any $t \in \mathcal{B}_{\delta'}^{\mathcal{S}_Q}$; Isomorphism 4.6.3 actually is a $Q$-interior algebra isomorphism. By composing $F_{\delta, \delta'}^N(S_Q)$ and Isomorphism 4.6.3, we get a new $Q$-interior algebra isomorphism

$$F : (\mathcal{B}_\delta)(S_Q) \cong \text{Res}_S(T) \otimes_k (\mathcal{B}_{\delta'})(S_Q).$$

By [25, Corollary 28.7], there is an endo-permutation $k \Delta(Q)$-module $U$ such that $T \cong \text{End}_k(U)$ as $\Delta(Q)$-algebras. Moreover by [4, Proposition 1.5], it is easy to check that $U$ is an indecomposable $k \Delta(Q)$-module with vertex $\Delta(Q)$. We identify $T$ and $\text{End}_k(U)$ through the isomorphism $T \cong \text{End}_k(U)$. We have two $\Delta(Q)$-interior algebra structures lifting the $\Delta(Q)$-algebra structure on $T$: one comes from $T$ itself (see Paragraph 2.5) and the other is determined by the $k \Delta(Q)$-module $U$.

By [25, Proposition 21.5] the two $\Delta(Q)$-interior algebra structures on $T$ coincide.

Set $\eta = Br_{\mathcal{S}_Q}(\delta)$. Then $\eta$ is a local point of $Q$ on $\mathcal{D}$. In particular, $Q_{\eta}$ is contained in the pointed group $F(\mathcal{S}_{b_{\mathcal{S}_Q}})$. Since $Q$ is a defect group of $b_{\mathcal{S}_Q}$, $Q_{\eta}$ is a defect pointed group of the pointed group $F(\mathcal{S}_{b_{\mathcal{S}_Q}})$. Therefore $\mathcal{D}_{\eta} = (\mathcal{B}_\delta)(S_Q)$ is a source algebra of the block algebra $\mathcal{D}$ and then we have a $k$-algebra isomorphism $\Phi_{b_{\mathcal{S}_Q}, \eta} : Z(\mathcal{D}) \cong Z(\mathcal{D}_{\eta})$, $a \mapsto ah$, where we set $h = Br_{\mathcal{S}_Q}(j)$. Similarly $\eta' = Br_{\mathcal{S}_Q}(\delta')$ is a local point of $Q$ on $\mathcal{D}$, $Q_{\eta'}$ is a defect pointed group of the pointed group $F(\mathcal{S}_{b_{\mathcal{S}_Q}})$, $\mathcal{D}_{\eta'} = (\mathcal{B}_{\delta'})(S_Q)$ is a source algebra of the block algebra $\mathcal{D}$ and there is a $k$-algebra isomorphism $\Phi_{b_{\mathcal{S}_Q}, \eta'} : Z(\mathcal{D}) \cong Z(\mathcal{D}_{\eta'})$, $a \mapsto a\mathcal{h}$, where we set $\mathcal{h} = Br_{\mathcal{S}_Q}(j)$.

Let $j$ be a primitive idempotent in $T$. Set $V = \mathcal{D} \otimes_{\mathcal{D}_{\eta}} (T \otimes \mathcal{D}_{\eta'}) \otimes \mathcal{D}_{\eta} \otimes \mathcal{D}_{\eta'}$. Then as in the proof of Proposition 4.3, it is easy to verify that $V$ induces a basic Morita equivalence between $\mathcal{D}$ and $\mathcal{D}$, that $F$ induces a $k$-algebra isomorphism $\Psi_{\eta, \eta'} : Z(\mathcal{D}_{\eta}) \cong Z(\mathcal{D}_{\eta'})$ such that

$$F(d) = 1 \otimes \Psi_{\eta, \eta'}(d)$$

for any $d \in Z(\mathcal{D}_{\eta})$, and that we have the following equality

$$\Phi_{\mathcal{D}_{\eta'}, \eta'} \circ \Psi_{\mathcal{D}_{\eta}, \eta} \circ \Phi_{\mathcal{D}_{\eta}, \eta} = 1.$$

Since the block $b_{\mathcal{S}_Q}$ is the Brauer correspondent of $b_{\mathcal{S}_Q}$ in $C_{\mathcal{H}}(S_Q)$ (see Lemma 4.5), the block $b_{\mathcal{S}_Q}$ of $F$ is inertial.

For any $u \in S_Q$, we have $F(uh) = (u, u)1_T \otimes u\mathcal{h}$. Since $(u, u)1_T$ is equal to $1_T$ and since $uh$ and $u\mathcal{h}$ lie in $Z(\mathcal{D}_{\eta})$ and $Z(\mathcal{D}_{\eta'})$ respectively, by 4.6.4 we have

$$\Psi_{\eta, \eta'}(uh) = u\mathcal{h}.$$

Set $B_{\delta, \eta} = \Phi_{b_{\mathcal{S}_Q}, \eta} \circ W_{H, b}^Q \circ \Phi_{b, \delta}$ and $B_{\delta', \eta'} = \Phi_{b_{\mathcal{S}_Q}, \eta'} \circ W_{H, b}^Q \circ \Phi_{b, \delta'}^{-1}$, which are $k$-algebra isomorphisms from $Z(\mathcal{B}_\delta)$ to $Z(\mathcal{D}_{\eta})$ and from $Z(\mathcal{B}_{\delta'})$ to $Z(\mathcal{D}_{\eta'})$, respectively. By Equalities 4.3.1 and 4.6.4, it is easy to check $B_{\delta', \eta'} \circ \Psi_{\delta', \delta} = \Psi_{\delta', \eta} \circ B_{\delta, \eta}$. Then by the equality 4.3.2, we get

$$\Phi_{\mathcal{D}_{\eta'}, \eta'} \circ \Psi_{\mathcal{D}_{\eta}, \eta} \circ W_{H, b}^Q = \Phi_{\mathcal{D}_{\eta'}, \eta} \circ \Psi_{\mathcal{D}_{\eta}, \eta} \circ W_{H, b}^Q = B_{\delta, \eta} \circ \Phi_{\mathcal{D}_{\eta}, \eta} \circ W_{H, b}^Q.$$

Thus we have

$$\Phi_{\mathcal{D}_{\eta'}, \eta'} \circ \Psi_{\mathcal{D}_{\eta}, \eta} \circ W_{H, b}^Q = \Psi_{\eta, \eta'} \circ \Phi_{\mathcal{S}_Q}.$$

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Comparing Equalities 4.6.5 and 4.6.7, we get Statement 4.6.1. Since $\Phi_{bS_Q, \eta}(ub_{S_Q}) = uh$ for any $u \in S_Q$ and $\Phi_{bS_Q, \eta}(ub_{S_Q}) = uh$ for any $u \in S_Q$, Statement 4.6.2 follows from 4.6.6 and 4.6.7.

4.7. We borrow the local point $\tilde{\gamma}$ in Paragraph 3.2. Clearly $(P, b_\tilde{\gamma})$ is a $(b, G)$-Brauer pair. We denote by $(S_P, b_{S_P})$ the unique $(b, G)$-Brauer pair such that $(S_P, b_{S_P}) \leq (P, b_\tilde{\gamma})$. We note that $b_{S_P}$ is also a block of $C_H(S_P)$ (see Paragraph 3.6). We set $C = C_G(S_P), D = C_H(S_P), E = kC_G(S_P)b_{S_P}$ and $F = kC_H(S_P)b_{S_P}$. By Watanabe isomorphism applied to $G$, $P$ and $b$, we get a $k$-algebra isomorphism $W_{G, b}^P : Z(A) \to Z(E)$. The restriction of $W_{G, b}^P$ to $Z(B)^P$ induces a $k$-algebra isomorphism $Z(B)^P \to Z(F)^P$ (see 3.7), denoted still by $W_{G, b}^P$.

4.8. Since $PH = G$, as in Paragraph 3.2 the $P$-algebra homomorphism $B \to A$ is a strict semicovering, $P_\gamma$ determines a local pointed group $P_{\tilde{\gamma}}'$ such that $\gamma' \subset \gamma'$, and $P_{\tilde{\gamma}}'$ is a defect pointed group of the pointed group $G_{\tilde{\gamma}}'$ on $A$. Since $N_G(P_{\tilde{\gamma}}') = N_G(P_{\gamma'}) = N_G(P_{\gamma}) = N_G(P_{\tilde{\gamma}})$ (see Lemma 3.3), $S_P$ is equal to the subgroup of the $E_G(P_{\gamma'})$-fixed elements in $P$. Clearly $(P, b_{\gamma'})$ is also a $(b, G)$-Brauer pair and we denote by $(S_P, b_{S_P})$ the unique $(b, G)$-Brauer pair such that $(S_P, b_{S_P}) \leq (P, b_{\gamma'})$. We note that $b_{S_P}$ is also a block of $C_H(S_P)$ and that $b_{\tilde{\gamma}}$ is equal to $b_{\gamma'}$. We set $C = C_G(S_P), D = C_H(S_P), E = kC_G(S_P)b_{S_P}$ and $F = kC_H(S_P)b_{S_P}$. We apply Watanabe isomorphism to $G$, $P$ and $b$, and get a $k$-algebra isomorphism $W_{G, b}^P : Z(A) \to Z(E)$, which induces a $k$-algebra isomorphism $Z(B)^P \to Z(F)^P$, denoted still by $W_{G, b}^P$. Set $\Psi^P_{S_P} = W_{G, b}^P \circ \Psi^M_{b, b} \circ (W_{G, b}^P)^{-1}$.

Lemma 4.9. With the notation and the assumptions as above, we have $\Psi^P_{S_P}((ub_{S_P})) = \mu b_{S_P}$ for any $u \in S_Q$.

Proof. Set $\mu = Br_{S_Q}^F(\gamma)$. Then $\mu$ is a local point of $P$ on $D$. In particular $P_\mu$ is contained in the obvious pointed group $E_{(ub_{S_P})}$ on the group algebra $kF$. Since $P$ is maximal such that $Br_{P_\mu}^F(b) \neq 0$, $P_\mu$ has to be a defect pointed group of $E_{(ub_{S_P})}$.

Since $E$ is equal to $PF$, as in Paragraph 3.2 the obvious $P$-algebra homomorphism $kF \to kE$ induced by the inclusion $F \subset E$ is a strict semicovering, $P_\mu$ determines a local pointed group $P_{\mu}^\circ$ on $kE$ such that $\mu \subset \mu^\circ$, and $P_{\mu}^\circ$ is a defect pointed group of the pointed group $E_{(ub_{S_P})}$ on the group algebra $kE$. Since the intersection $S_P \cap Q$ is equal to $S_Q$ (see Lemma 3.5) and $S_P$ is the subgroup of all $N_G(P_{\gamma})$-fixed elements in $P$, $N_G(P_{\gamma})$ is contained in $Q$. By Lemma 3.3 we have $N_G(P_{\gamma}) = N_{E}(P_{\mu}) = N_{E}(P_{\mu}) = N_{E}(P_{\mu})$. So $S_P$ is equal to the subgroup of all $N_{E}(P_{\mu})$-fixed elements in $P$.

Without confusion, we denote by $b_{\mu}$ the block of $C_G(P)$ such that $b_{\mu}B_{k}^{E}(\mu) = Br_{k}^{B}(\mu)$. We note that $b_{\mu} = b_{\gamma} = b_{\gamma}$. We claim that $(S_P, b_{S_P}) \leq (P, b_{\mu})$ as $(b_{S_P}, E)$-Brauer pairs. Clearly $(S_Q, b_{S_Q})$ and $(Q, b_\delta)$ are also $(b, G)$-Brauer pairs. Since $(S_Q, b_{S_Q}) \leq (Q, b_\delta)$ as $(b, H)$-Brauer pairs, there is a primitive idempotent $\ell$ in $(kH)^Q$ such that $Br_{k}^{H}(\ell)b_{\delta} \neq 0$ and $Br_{k}^{H}(\ell)b_{S_Q} \neq 0$. Since the index of $H$ in $G$ is a $p$-power, $\ell$ is primitive in $(kG)^Q$ (see [10, Proposition 6.2]). Then by [4, Corollary 1.9] $(S_Q, b_{S_Q}) \leq (Q, b_\delta)$ as $(b, G)$-Brauer pairs. Obviously $(P, b_{\gamma})$ is a $(b, G)$-Brauer pair. Since $Q_{\delta} \subset P_{\gamma}$, the product $Br_{Q}^{H}(i)b_{\delta}$ is not zero. Since $i$ is primitive in $(kG)^P$, by [4, Corollary 1.9] again $(Q, b_{\delta}) \leq (P, b_{\gamma})$ as $(b, G)$-Brauer pairs. Now we have $(S_Q, b_{S_Q}) \leq (Q, b_{\delta}) \leq (P, b_{\gamma})$ as $(b, G)$-Brauer pairs. Then by [4, Theorem 1.7] we have $(S_Q, b_{S_Q}) \leq (S_P, b_{S_P}) \leq (P, b_{\gamma})$ as $(b, G)$-Brauer pairs. This implies $(S_P, b_{S_P}) \leq (P, b_{\gamma})$ as $(b_{S_P}, E)$-Brauer pairs.

We apply Watanabe isomorphism to $E$, $P$ and $b_{S_Q}$, and get a $k$-algebra isomorphism $W_{E, b_{S_Q}}^P : Z(C) \to Z(E), a \mapsto B_{S_P}^{kE}(a)b_{S_P}$, which induces a $k$-algebra isomorphism $Z(D)^P \to Z(F)^P$, denoted
still by $W_{E,b_{QS}}$. Moreover for any $a \in Z(B)^P$, $(W_{E,b_{QS}}^P \circ W_{H,b}^Q)(a)$ is equal to $W_{G,b}^P(a)$. Similarly for any $a \in Z(B)^P$, $(W_{E,b_{QS}}^P \circ W_{H,b}^Q)(a)$ is equal to $W_{G,b}^P(a)$, where $W_{E,b_{QS}}^P$ is the $k$-algebra isomorphism $Z(D)^P \to Z(B)^P$, $d \mapsto Br_{ZS}(d)$.

Let $u \in S_Q$. Clearly $ub_{QS}$ and $ub_{S}$ belong to $Z(D)^P$ and $Z(D)^P$ respectively, and $W_{E,b_{QS}}^P(ub_{QS})$ and $W_{E,b_{S}}^P(ub_{S})$ are equal to $ub_{SP}$ and $ub_{S}$ respectively. Then by 4.6.2 we have

$$u \mathfrak{d}_{SP} = W_{E,b_{QS}}^P (\Psi_{b,b}^Q(ub_{QS})) = (W_{G,b}^P \circ (W_{H,b}^Q)^{-1})(\Psi_{b,b}^Q(ub_{QS})) = (\Psi_{b,b}^P \circ W_{G,b}^P(ub_{QS})) = \Psi_{b,b}^P(ub_{SP}).$$

**Proposition 4.10.** With the notation and the assumptions as above, the $k(H \times H)$-module $M$ in Proposition 4.3 can be extended to a $k((H \times H) \Delta(P))$-module. Set $W = \text{Ind}_{(H \times H) \Delta(P)}^{G \times G}(M)$. Then the $k(G \times G)$-module $W$ induces a Morita equivalence between $A$ and $A$ and the induced $k$-algebra isomorphism $\Psi_{b,b}^{W}: Z(A) \cong Z(A)$ maps $(W_{G,b}^P)^{-1}(vb_{SP})$ onto $(W_{G,b}^P)^{-1}(vb_{SP})$ for any $v \in S_P$.

**Proof.** By Lemma 3.8, there is a group homomorphism $\rho: P \to \mathbb{B}^*$ such that for any $u \in H_P$ and any $v \in S_P$, we have $\rho(u) = ub$ and $\rho(v) = v(W_{G,b}^P)^{-1}(v^{-1}b_{SP})$. Since $N_G(P_\gamma) = N_G(P_\gamma)$ (see Paragraph 4.8), we have $\mathcal{H}_P = [P, N_G(P_\gamma)]$ and $S_P$ is equal to the subgroup of all $N_G(P_\gamma)$-fixed elements in $P$. By Lemma 3.8 applied to $G$, $H$ and $b$, we get a group homomorphism $\varrho: P \to \mathbb{B}^*$ such that for any $u \in H_P$ and any $v \in S_P$, we have $\varrho(u) = ub$ and $\varrho(v) = v(W_{G,b}^P)^{-1}(v^{-1}b_{SP})$.

We claim that the $k(H \times H)$-module $M$ can be extended to $(H \times H) \Delta(P)$ with the $\Delta(P)$-action on $M$ defined by the equality $(u, u) \cdot m = \rho(u)m\varrho(u^{-1})$ for any $(u, u) \in \Delta(P)$ and any $m \in M$. It suffices to show that $(u, u) \cdot m = umu^{-1}$ for any $(u, u) \in \Delta(Q)$ and any $m \in M$. But since $\rho(u) = ub$ and $\varrho(u) = ub$ for any $u \in H_Q$, it suffices to show that $(u, u) \cdot m = umu^{-1}$ for any $(u, u) \in \Delta(S_Q)$ and any $m \in M$. If $S_Q = 1$, then the extension of $M$ to $(H \times H) \Delta(P)$ by the equality above is well-defined. If $S_Q \neq 1$, then by Lemma 4.9 we have

$$(u, u) \cdot m = \rho(u)m\varrho(u^{-1}) = u(W_{G,b}^P)^{-1}(u^{-1}b_{SP})m(W_{G,b}^P)^{-1}(u^{-1}b_{SP})u^{-1} = um(\Psi_{b,b}^M \circ (W_{G,b}^P)^{-1})(ub_{SP})(W_{G,b}^P)^{-1}(u^{-1}b_{SP})u^{-1} = um((W_{G,b}^P)^{-1} \circ \Psi_{b,b}^P)^{-1}(u^{-1}b_{SP})m(W_{G,b}^P)^{-1}(u^{-1}b_{SP})u^{-1} = um(W_{G,b}^P)^{-1}(u^{-1}b_{SP})(W_{G,b}^P)^{-1}(u^{-1}b_{SP})u^{-1} = umu^{-1}.$$ 

Now by [11, Theorem 3.4], the $k(G \times G)$-module $W$ induces a Morita equivalence between $A$ and $A$, which induces a $k$-algebra isomorphism $\Psi_{b,b}^{W}: Z(A) \cong Z(A)$ (see Paragraph 4.1). We denote by $\lambda$ the $\kappa$-linear opposite ring isomorphism $kG \to kG$ sending $x$ onto $x^{-1}$ for any $x \in G$, and identify $k(G \times G)$ and $kG \otimes_k kG$ through the $k$-algebra isomorphism $k(G \times G) \cong kG \otimes_k kG$ mapping $(y, z)$ onto $y \otimes z$ for any $y \in G$ and any $z \in G$. For any $u \in S_P$ and any $m \in M$, in the module $W$ we have

$$1 \otimes m = (1 \otimes (v \otimes v) \cdot ((\rho(v^{-1}) \otimes \lambda(v))m) = ((v \otimes v)(\rho(v^{-1}) \otimes \lambda(v))m) \otimes m$$

$$= (\rho(v^{-1}) \otimes \lambda(v^{-1}g))m = ((W_{G,b}^P)^{-1}(vb_{SP}) \otimes \lambda(((W_{G,b}^P)^{-1}(v^{-1}b_{SP}) \otimes m)$$

So for any $v \in S_P$, $(W_{G,b}^P)^{-1}(vb_{SP})(1 \otimes m)$ is equal to $(1 \otimes m)(W_{G,b}^P)^{-1}(vb_{SP})$ for any $m \in M$. This implies that $\Psi_{b,b}^W$ maps $(W_{G,b}^P)^{-1}(vb_{SP})$ onto $(W_{G,b}^P)^{-1}(vb_{SP})$. 

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We close this section by applying Proposition 4.10 to trivial inertial blocks to get the following

**Corollary 4.11.** Keep the notation by applying the assumptions above and assume that the block $b$ of $H$ is a trivial inertial block. Then there is a $p$-permutation $k((H \times \mathbb{H})\Delta(P))$-module such that its restriction to $H \times \mathbb{H}$ induces a Morita equivalence between $B$ and $\mathbb{B}$.

**Proof.** Clearly $Br^Q_H(\delta')$ is a point on $kC_H(Q)$. Since $Br^Q_H$ identically maps $Br^Q_H(\delta')$ onto $Br^Q_H(\delta)$ and the kernel of $Br^Q_H$ is contained in the radical of $(k\mathbb{H})^Q$, $Br^Q_H(\delta')$ is a local point of $Q$ on $k\mathbb{H}$. Obviously both $\delta'$ and $Br^Q_H(\delta')$ lift the point $Br^Q_H(\delta')$ on $kC_H(Q)$ through the homomorphism $Br^Q_H$. Thus by [25, Lemma 14.5], $\delta'$ and $Br^Q_H(\delta')$ are equal. Since the block $b_1$ of $C_H(Q)$ is also a block of $N_H(Q_\delta)$ with defect group $Q$ and the product $b_1\delta'$ is equal to $\delta'$, $Q_{\delta'}$ is a defect pointed group of $N_H(Q_\delta)$ and then the $Q$-interior algebra $\mathfrak{j}(kN_H(Q_\delta))\mathfrak{j}$ is a source algebra of the block algebra $kN_H(Q_\delta)b_1$. Moreover since $b_1b_1^x = 0$ for any $x \in N_H(Q) - N_H(Q_\delta)$ and $b = Tr_{N_H(Q_\delta)}(b_1)$ (see [1, 2.9]), it is easy to check that the source algebra $\mathfrak{j}(kN_H(Q_\delta))\mathfrak{j}$ is equal to $B_{\delta'}$.

Let $e$ be a point of $N_H(Q_\delta)$ on $B$ such that $Q_\delta \leq N_H(Q_\delta)_e$. Take $e \in e$. By [7, Proposition 4.10], $e_j$ belongs to $\delta$. Without loss of generality, we adjust $j$ so that $j$ and $e_j$ are equal. Then by [7, Proposition 4.10] again, the map $B_{\delta'} \to B_\delta, a \mapsto ae$ is an injective $Q$-interior algebra homomorphism. Since the block $b$ of $H$ is a trivial inertial block, source algebras of the block algebras $B$ and $\mathbb{B}$ are isomorphic and then have the same dimension. So the homomorphism $B_{\delta'} \to B_\delta$ is a $Q$-interior algebra isomorphism.

Consider $j\mathbb{B}$ as a $B_\delta$-module through this isomorphism and set $\mathfrak{m} = B_j \otimes_{B_\delta} \mathbb{B}$. Since there is an obvious $k(H \times \mathbb{H})$-module isomorphism $\mathfrak{m} \cong B_j \otimes_{B_\delta} \mathfrak{B}_{\delta'} \otimes_{\mathbb{B}_{\delta'}} \mathbb{B}$, by Proposition 4.3 the $k(H \times \mathbb{H})$-module $\mathfrak{m}$ is $\Delta(P)$-stable and it induces a Morita equivalence between $B$ and $\mathbb{B}$. By the proof of Proposition 4.10, this $\Delta(P)$-stable $k(H \times \mathbb{H})$-module $\mathfrak{m}$ can be extended to a $k((H \times \mathbb{H})\Delta(P))$-module with the $\Delta(P)$-action on $\mathfrak{m}$ defined by the equality $(u, u) \cdot m = \rho(u)m(g(u^{-1})^{-1})$ for any $(u, u) \in \Delta(P)$ and any $m \in \mathfrak{m}$. We claim that the extended $k((H \times \mathbb{H})\Delta(P))$-module $\mathfrak{m}$ is isomorphic to a direct summand of the $k((H \times \mathbb{H})\Delta(P))$-module $B$ determined by the left and right multiplications of $H$ and $\mathbb{H}$ on $B$ and by the $P$-conjugation. This claim implies that the extended $k((H \times \mathbb{H})\Delta(P))$-module $\mathfrak{m}$ is a $p$-permutation $k((H \times \mathbb{H})\Delta(P))$-module.

Clearly the product $b_1e$ is an $N_H(Q_\delta)$-fixed idempotent in $B$. Since $Q_\delta \leq N_H(Q_\delta)_e$, the product $b_1e$ is not equal to zero. Otherwise, we have $b_1Br^Q_H(e) = 0$ and then $b_1Br^Q_H(j) = 0$; this contradicts with the equality $b_1Br^Q_H(\delta) = Br^Q_H(\delta)$. So the product $b_1e$ has to be equal to $e$. Since $b_1b_1^x = 0$ for any $x \in N_H(Q) - N_H(Q_\delta)$, we have $ee^x = 0$ for any $x \in N_H(Q) - N_H(Q_\delta)$. Set $\ell = Tr_{N_H(Q_\delta)}(e)$. Then $\ell$ is an $\mathbb{H}$-fixed primitive idempotent in $B$ and we have $\mathfrak{m}e = \ell e = \ell e_j = j$. Clearly the product $(B_j)(j\mathbb{B})$ is contained in $B\ell$ and the multiplication determines a $k(H \times \mathbb{H})$-module homomorphism $\mathfrak{m} \to B\ell$. Notice that the homomorphism $\mathfrak{m} \to B\ell$ is also a right $\mathbb{B}$-module homomorphism. By [14, Corollary 3.5], the functor $\mathfrak{m} \mapsto j\mathfrak{m}$ from the category of finitely generated right $\mathbb{B}$-modules to that of finitely generated right $\mathfrak{B}_{\delta'}$-modules is an equivalence of categories. Since this functor maps the right $\mathbb{B}$-module homomorphism $\mathfrak{m} \to B\ell$ onto the right $\mathfrak{B}_{\delta'}$-module isomorphism $(B_j) \otimes j \cong B_j$, the right $\mathbb{B}$-module homomorphism $\mathfrak{m} \to B\ell$ is an isomorphism. Thus the $k(H \times \mathbb{H})$-module homomorphism $\mathfrak{m} \to B\ell$ is an isomorphism. We identify $\mathfrak{m}$ and $B\ell$ through this isomorphism. By Proposition 4.10, $(W^P_{G, b})^{-1}(vBSP)m$ is equal to $m(W^P_{G, b})^{-1}(vBSP)$ for any $v \in SP$ and any $m \in \mathfrak{m}$. So we have $\rho(v)m(g(v^{-1})^{-1}) = \rho(v)(W^P_{G, b})^{-1}(v^{-1}BSP)m(W^P_{G, b})^{-1}(vBSP)v^{-1} = \rho(v)m(v^{-1})$ for any $v \in SP$ and any $m \in \mathfrak{m}$. In particular, the action of $\Delta(P)$ on $\mathfrak{m}$ is the $P$-conjugation action on $\mathfrak{m}$. Thus the extended $k((H \times \mathbb{H})\Delta(P))$-module $\mathfrak{m}$ is a direct summand of the $k((H \times \mathbb{H})\Delta(P))$-module $B$. 14
5. \(p\)-extensions of trivial inertial blocks

Throughout this section, we keep the notation in 2.10; moreover we assume that \(P\) is abelian, that the block \(b\) of \(H\) is a trivial inertial block (see Paragraph 1.4), and that the index of \(H\) in \(G\) is a power of \(p\). So \(G = PH\), \(G = P\mathbb{H}\), \(b\) is a block of \(G\) and \(b\) is a block of \(G\). We denote by \(I\) the image of any subset \(I\) of \(OG\) through the reduction homomorphism \(OG \to kG\). We also borrow the local point \(\gamma\) in Paragraph 3.2.

**Proposition 5.1.** With the notation and the assumptions as above, the block \(b\) of \(G\) is inertial.

**Proof.** Let \(P_x\) be the local pointed group on \(OC_H(Q)\) such that \(Br^O_H(\gamma) = Br^{OC_H(Q)}(\mu)\). Since \(Br^O_H(\gamma)\) maps \((OH)^P\) onto \((kC_H(Q))^P\), \(Br^O_Q(\gamma)\) is a local point of \(P\) on \(kC_H(Q)\). Moreover the point \(\mu\) obviously lifts the point \(\tilde{b}\) through the reduction homomorphism \(OC_H(Q) \to kC_H(Q)\).

Since \(Q\) is contained in \(\gamma\), the product \(\tilde{b}\) of \(\gamma\) is not equal to \(\{0\}\). By [10, Proposition 5.5], \(\delta\) is the unique local point of \(P\) on \(OH\) such that \(Q \leq \gamma\). So the \(P\)-conjugation has to stabilize \(b\). Thus \(\tilde{b}\) of \(\gamma\) has to be \(\tilde{b}\) of \(\gamma\). This shows that \(\mu = b\) is contained in the pointed group \(NG(Q)\). Since \(P\) is a maximal \(p\)-subgroup of \(G\) such that \(Br^O_G(b) \neq 0\), it is easily concluded that \(P\) is a defect pointed group of the pointed group \(NG(Q)\).

Take \(\ell \in P\) and set \(B = \ell(OLG(Q))\) and \(E = LG(Q)\). Since the block \(b\) of \(H\) is nilpotent, by [21, Theorem 3.5], there are a finite group \(L\) containing \(P\) and a surjective group homomorphism \(\tau : L \to E\) with the kernel \(Q\) such that \(\tau(u)\) is the image of \(u\) in \(E\) for any \(u \in P\) and such that with the identification of \(L/Q\) and \(E\) through the isomorphism \(L/Q \cong E\) induced by \(\tau\), the functor \(\tau : \mathcal{E}(\mu, H) \to \mathcal{E}(\mu, L)\) mapping an object \(R_u\) onto \(R_u\) and a morphism \((\varphi^T_{R, x}, \tilde{x})\) onto \((\varphi^T_{R, x}, \mu(x))\) is an isomorphism of categories. Clearly the group \(E\) is the product of the subgroups \(N_H(Q)/C_H(Q)\) and \(PC_H(Q)/C_H(Q)\). By Lemma 3.4, \(N_H(Q)/C_H(Q)\) is equal to the image of \(N_H(P)\) in \(E\). Thus \(PC_H(Q)/C_H(Q)\) is normal in \(E\) and then \(P\) is normal in \(L\). Since \(P\) is a Sylow \(p\)-subgroup of \(L\) (see [10, Remark 1.9]), \(P\) has a \(p'\)-complement \(E\) in \(L\).

By [21, Corollary 3.15], there are a \(k\)-group \(\hat{E}\) with \(k\)-quotient \(E\) and a primitive \(P\)-interior full matrix algebra \(T\) such that we have a \(P\)-interior algebra isomorphism \(B \cong T \otimes G OC_P(P < \hat{E})\). By the isomorphism \(\tau\) of categories, it is easy to check that the centralizer \(C_L(P)\) is equal to \(P\) and then that the algebra \((G \otimes L)(P)\) is isomorphic to \(T\), where we set \(\hat{L} = P \times \hat{E}\). Since Ker\((Br^O_L(\gamma)\) is contained in \(J(\gamma)\), \(\gamma\) is the unique local point of \(P\) on \(\gamma\). By [15, Proposition 5.6], \(B(P)\) is isomorphic to \(\mathcal{E}(P) \otimes k(\gamma \gamma)(P)\) and thus to \(kP\), as \(k\)-algebras. Since Ker\((Br^O_L(\gamma)\) is contained in \(J(B)\) (see Lemma 4.2), \(\gamma\) is a local point of \(P\) on \(B\). Clearly \(b\) is also a block of \(N_H(Q)\) and since the index of \(H\) in \(G\) is a \(p\)-power, \(b\) is also a block of \(N_G(Q)\). Since \(P\) is a maximal \(p\)-subgroup of \(G\) such that \(Br^O_G(b) \neq 0\), the \(P\)-interior algebra \(B\) is a source algebra of the block algebra \(\mathcal{O}G(Q)\). By [1, 2.9], \(b\) is equal to 0 for any \(x \in H - N_H(Q)\) and \(B\) is equal to \(\mathcal{T}_{N_H(Q)}/(b)\) and then to \(\mathcal{T}_{N_G(Q)}(b)\). Thus we have \(B = \ell(\mathcal{O}G(Q))\). Since \(B\) is a source algebra of the block algebra \(\mathcal{O}G(b)\), it is easy to check
that there is a $k^*$-group isomorphism $\tilde F_{G, L}(P_{\{1\}}) \cong \tilde E_G(P_{\hat 0})^\circ$, which can be extended to a $k^*$-group isomorphism $P \times \tilde F_{G, L}(P_{\{1\}}) \cong P \times \tilde E_G(P_{\hat 0})^\circ$. Summarizing the above, we have a $P$-interior algebra isomorphism $B \cong T \otimes_{O_*} \tilde O_*(P \times \tilde E_G(P_{\hat 0})^\circ)$. So the block $b$ is inertial (see [20, Paragraph 2.16]).

**Proposition 5.2.** With the notation and the assumptions as above, the block $b$ of $G$ is inertial.

*Proof.* Note that $N_G(P)$ and $N_G(P)$ are equal and that the Brauer correspondents of $b$ and $b$ in $N_G(P)$ coincide. By Corollary 4.11 and [11, Theorem 3.4], we easily conclude that there is a $p$-permutation $k(G \times G)$-module inducing a Morita equivalence between $\mathcal A$ and $\mathcal A$. By [16, Remark 7.8], such a Morita equivalence between $\mathcal A$ and $\mathcal A$ can be lifted to a Morita equivalence between $\mathcal A$ and $\mathcal A$ induced by a $p$-permutation $O(G \times G)$-module. Finally by Proposition 5.1 and [27, Proposition 9], it is easily checked that the block $b$ of $G$ is inertial.

**Remark.** For the later proof, we remark that the block algebras $\mathcal A$ and $\mathcal A$ have isomorphic source algebras (see [16, 7.5]) since there is a $p$-permutation $O(G \times G)$-module inducing a Morita equivalence between $\mathcal A$ and $\mathcal A$.

**5.3.** We begin to show a precise characterization of the $\hat G$-graded $P$-interior algebra structure on $\mathcal A_{\gamma}$. Since $\mathcal A_{\gamma}$ is a source algebra of the block algebra $\mathcal A$, by [20, 2.16.2] and Proposition 5.2 there is a $P$-interior full matrix algebra $S$ over $O$ such that there is a $P$-interior algebra isomorphism

$$\mathcal A_{\gamma} \cong S \otimes_{O_*} \tilde O_* (P \times \tilde E_G(P_{\gamma})^\circ);$$

moreover $S$ is a primitive Dade $P$-algebra (see [16, Theorem 7.2]), unique up to $P$-algebra isomorphisms (see [18, Lemma 4.5]). The $P$-interior algebra $S$ is also $E_G(P_{\gamma})$-stable (see [20, 3.9]). We adjust the $P$-interior algebra structure on $S$ by an $E_G(P_{\gamma})$-stable linear character of $P$ so that $S$ is an $E_G(P_{\gamma})$-stable determinant one $P$-interior full matrix algebra. Then the $P$-interior algebra $S$ is unique up to $P$-interior algebra isomorphisms.

**5.4.** Set $\hat A = S^\circ \otimes_{O_*} \text{Res}_{\hat G}(\mathcal A)$ and $\hat B = S^\circ \otimes_{O_*} \text{Res}_{\hat G}(\mathcal B)$. Then $\hat A$ is a $\hat G$-graded $P$-interior algebra (see 2.1 and 2.2) and $\hat B$ is a $Q$-interior $P$-algebra (see 2.7). By [15, Theorem 5.3], the local pointed group $P_{\gamma}$ determines a local pointed group $P_{\tilde \gamma}$ on $\hat B$ such that for some $i \in \tilde \gamma$, we have

$$i(1 \otimes i) = (1 \otimes i)i = \hat i.$$

Similarly $Q_{\delta}$ also determines a local pointed group $Q_{\tilde \delta}$ on $\hat B$ such that for some $j$ in the unique local point of $Q$ on $S$ and some $j$ in $\delta$, we have $j(j \otimes j) = (j \otimes j)j = j$. We set $\hat A_{\tilde \gamma} = i\hat A_i$, $\hat B_{\tilde \delta} = i\hat B_{\tilde \delta}$, $\hat A_{\delta} = j\hat A_j$ and $\hat B_{\delta} = j\hat B_j$. Then $\hat A_{\tilde \gamma}$ is a $\hat G$-graded $P$-interior algebra and $\hat A_{\delta}$ is a $\hat G$-graded $Q$-interior algebra (see Paragraph 2.7). Since $\hat A = \sum_{u} \hat B_{\tilde \gamma}$ where $u$ run over $P_{\gamma}$, by [17, Proposition 2.8] $P_{\tilde \gamma}$ determines a local pointed group $P_{\tilde \gamma}$ on $\hat A$ such that $\tilde \gamma \subset \tilde \gamma$. We note that the point $\tilde \gamma$ of $P$ on $A$ is determined by the point $\tilde \gamma$ of $P$ on $A$ in the sense of [15, Theorem 5.3].

**Lemma 5.5.** With the notation and the assumptions as above, there is a $P$-interior algebra isomorphism

$$\hat A_{\tilde \gamma} \cong O_* (P \times \tilde E_G(P_{\tilde \gamma})^\circ).$$

*Proof.* By 5.3.1 and 5.4.1, we get a $P$-interior algebra embedding

$$\hat A_{\tilde \gamma} \to S^\circ \otimes_{O_*} S \otimes_{O_*} \tilde O_* (P \times \tilde E_G(P_{\tilde \gamma})^\circ).$$
There is a $P$-interior algebra embedding $d : O \to S^O \otimes S$, which induces another $P$-interior algebra embedding

$$O_*(P \times \hat{E}_G(P_\delta)) \to S^O \otimes S \otimes O_*(P \times \hat{E}_G(P_\delta))$$

mapping $a$ onto $d(1) \otimes a$ for any $a \in O_*(P \times \hat{E}_G(P_\delta))$. Since we have $C_{P \times \hat{E}_G(P_\delta)}(P) = P$, by Lemma 4.2 it is easy to check that $\{1\}$ is the unique local point of $P$ on $O_*(P \times \hat{E}_G(P_\delta))$. Thus by [15, Theorem 5.3], $P$ has a unique local point on $S^O \otimes S \otimes O_*(P \times \hat{E}_G(P_\delta))$. Therefore the local points of $P$ on $S^O \otimes S \otimes O_*(P \times \hat{E}_G(P_\delta))$ determine each other. Thus by [10, Proposition 3.3] we have $\{1\}$ is a local point of $P$ on $\hat{B}_\delta$. This forces that $\{1\}$ is a local point of $P$ on $\hat{B}_\delta$.

So there is an invertible element $a \in \hat{B}_\delta$ such that $\overline{j}^a = \hat{i}$. In particular, the $a$-conjugation induces a $G$-graded $Q$-interior algebra isomorphism $\hat{A}_\delta \cong \operatorname{Res}_Q^P(\hat{A}_\gamma)$, through which we identify $\hat{A}_\gamma$ with $\hat{A}_\delta$. In this case, we have $\overline{j} = \hat{i}$. We denote by $N_{\hat{B}_\delta}(P)$ the normalizer of $P\hat{i}$ in $\hat{B}_\delta$.

**Lemma 5.7.** Keep the notation and the assumptions as above and denote by $\epsilon_{\delta, \gamma}$ the inverse of Isomorphism 3.4.1. Then there is a $k^*$-group isomorphism $\tilde{\epsilon}_{\delta, \gamma} : \hat{E}_H(Q_\delta) \cong \hat{E}_G(P_\gamma)$ lifting the group isomorphism $\epsilon_{\delta, \gamma}$.

**Proof.** Since the index of $H$ in $G$ is a $p$-power, the $P$-algebra homomorphism $OH \to OG$ induced by the inclusion map $H \to G$ is a strict semimorphism (see [10, Example 3.9 and Theorem 3.16]). We denote by $\delta$ the local point of $Q$ on $OG$ such that $\delta \subset \hat{\delta}$. By Lemma 3.3, we have $N_G(Q_\delta) = N_G(Q_\hat{\delta})$. Thus the inclusion $N_H(Q_\delta) \subset N_G(Q_\hat{\delta})$ induces a group isomorphism $E_H(Q_\delta) \cong E_G(Q_\hat{\delta})$. On the other hand, by the proof of Lemma 3.3, $\delta$ and $\hat{\delta}$ determine each other. Thus by [10, Proposition 3.3], the inclusion map $OH \to OG$ induces an $N_G(Q_\delta)$-algebra isomorphism $(OH)(Q_\delta) \cong (OG)(Q_\hat{\delta})$ which induces a $k^*$-group isomorphism $\hat{E}_H(Q_\delta) \cong \hat{E}_G(Q_\hat{\delta})$ lifting the isomorphism $E_H(Q_\delta) \cong E_G(Q_\hat{\delta})$. So it suffices to show that there is a $k^*$-group isomorphism $\tilde{\epsilon}_{\hat{\delta}, \gamma} : \hat{E}_G(Q_\hat{\delta}) \cong \hat{E}_G(P_\gamma)$ lifting the group isomorphism $\epsilon_{\hat{\delta}, \gamma} : E_G(Q_\hat{\delta}) \cong E_G(P_\gamma)$. Since $b_\delta$ and $b_\gamma$ are nilpotent blocks of $C_H(Q)$ and $C_H(P)$ respectively and the index of $H$ in $G$ is a $p$-power, by [10, Proposition 6.5] $Q_\delta$ and $P_\gamma$ are nilcentralized in the sense of [17, 3.1]. Since $Q_\hat{\delta}$ is contained in $P_\gamma$, the isomorphism $\tilde{\epsilon}_{\hat{\delta}, \gamma}$ follows from [19, Theorem 7.16].

**5.8.** We consider the twisted group algebra $O_*(P \times \hat{E}_G(P_\delta))$ and begin to endow it with a $G$-graded $P$-interior algebra structure. Clearly the $k^*$-group isomorphism $\tilde{\epsilon}_{\delta, \gamma}$ can be extended to an injective group homomorphism $Q \times \hat{E}_H(Q_\delta) \to P \times \hat{E}_G(P_\gamma)$, which extends the inclusion $Q \subset P$ and is still denoted by $\tilde{\epsilon}_{\delta, \gamma}$ for convenience. We identify $Q \times \hat{E}_H(Q_\delta)$ with a subgroup of $P \times \hat{E}_G(P_\gamma)$ through $\tilde{\epsilon}_{\delta, \gamma}$, which is normal in $P \times \hat{E}_G(P_\gamma)$.
5.9. Clearly $P \subset P \times \hat{E}_G(P_\gamma)$ and $P \subset G$ induce the following group isomorphisms
\[(P \times \hat{E}_G(P_\gamma))/\langle Q \times \hat{E}_H(Q_\delta) \rangle \cong P/Q \cong G.\]

We identify $\hat{G}$ with $(P \times \hat{E}_G(P_\gamma))/\langle Q \times \hat{E}_H(Q_\delta) \rangle$ through these isomorphisms. As $\mathcal{A}$ is endowed with a $\hat{G}$-graded algebra structure (see Paragraph 2.10), we endow the twisted group algebra $\mathcal{O}_*(P \times \hat{E}_G(P_\gamma)^\circ)$ with an obvious $\hat{G}$-graded algebra structure. Notice that the 1-component of this $G$-graded algebra $\mathcal{O}_*(P \times \hat{E}_G(P_\gamma)^\circ)$ is $\mathcal{O}_*(Q \times \hat{E}_H(Q_\delta)^\circ)$. The inclusion $P \subset P \times \hat{E}_G(P_\gamma)$ induces a $P$-interior algebra structure on $\mathcal{O}_*(P \times \hat{E}_G(P_\gamma)^\circ)$.

**Proposition 5.10.** With the notation and the assumptions as above, there is a $\hat{G}$-graded $P$-interior algebra isomorphism $\hat{A}_{\delta} \cong \mathcal{O}_*(P \times \hat{E}_G(P_\gamma)^\circ)$.

**Proof.** Since $\hat{A}$ is a $\hat{G}$-graded $P$-interior algebra with the 1-component $\hat{B}$, the group $F_{\hat{B}}(P_\gamma)$ (see Paragraph 2.9) makes sense. In this case, $F_{\hat{B}}(P_\gamma)$ actually is equal to the group obtained by applying $F_{\hat{A}}(K_\gamma)$ in [17, 2.3] to $\hat{B}$ and $P_\gamma$. By [17, Proposition 2.8] applied to the inclusion map $\hat{B} \rightarrow \hat{A}$, $F_{\hat{B}}(P_\gamma)$ is a normal subgroup of $F_{\hat{A}}(P_\gamma)$ with a $p'$-power index. On the other hand, by [13, Theorem 3.1], we have $E_{\hat{G}}(P_\gamma) = F_{\hat{A}}(P_\gamma)$ and by [10, Lemma 1.17], we have $F_{\hat{A}}(P_\gamma) = F_{\hat{A}}(P_\gamma)$.

Since $E_{\hat{G}}(P_\gamma)$ is a $p'$-group, so is $F_{\hat{A}}(P_\gamma)$ and thus we have $F_{\hat{B}}(P_\gamma) = F_{\hat{A}}(P_\gamma)$.

For any $\phi \in F_{\hat{B}}(P_\gamma)$, there is an invertible element $a$ in $N_{\hat{B}_p}(P)$ such that $\hat{a} \hat{u} \hat{a}^{-1}$ for any $u \in P$. Since the map $P \rightarrow P_\hat{\theta}$, $u \mapsto u \hat{a}$ is a group isomorphism, it is easy to check that the map $F_{\hat{B}}(P_\gamma) \rightarrow N_{\hat{B}_p}(P)/(\hat{B}_p^\circ)^*$ mapping $\phi$ onto the image of $a$ in $N_{\hat{B}_p}(P)/(\hat{B}_p^\circ)^*$ for any $\phi \in F_{\hat{B}}(P_\gamma)$ is a group isomorphism. Then we have a short exact sequence of group homomorphisms
\[1 \rightarrow (1 + J(\hat{B}_p^\circ)) \cong (\hat{B}_p^\circ)^*/k^* \rightarrow N_{\hat{B}_p}(P)/k^* \rightarrow F_{\hat{B}}(P_\gamma) \rightarrow 1.\]

Since $F_{\hat{B}}(P_\gamma)$ is a $p'$-group, this short exact sequence uniquely splits (see [25, Lemma 45.6]) and thus there is a subgroup $\hat{F}$ of $N_{\hat{B}_p}(P)$ containing $k^*$ such that the restriction to $\hat{F}/k^*$ of the homomorphism $N_{\hat{B}_p}(P)/k^* \rightarrow F_{\hat{B}}(P_\gamma)$ is a group isomorphism.

We consider the conjugation action of $\hat{F}$ on $P_{\hat{\theta}}$, the semidirect product $(P_{\hat{\theta}}) \times \hat{F}$ and the $O$-algebra homomorphism $\theta : \mathcal{O}_*(((P_{\hat{\theta}}) \times \hat{F}) \rightarrow \hat{A}_\gamma$ induced by the inclusions $P_{\hat{\theta}} \subset \hat{A}_\gamma$ and $\hat{F} \subset \hat{A}_\gamma$.

Similar to the paragraph above, we have another uniquely split short exact sequence
\[1 \rightarrow (1 + J(\hat{A}_\gamma^\circ)) \rightarrow N_{\hat{A}_\gamma}(P)/k^* \rightarrow F_{\hat{A}}(P_\gamma) \rightarrow 1.\]

Clearly the inclusion $\hat{F} \subset N_{\hat{B}_p}(P)$ induces a group isomorphism $F_{\hat{A}}(P_\gamma) \cong \hat{F}/k^*$, which is a section of the homomorphism $N_{\hat{A}_\gamma}(P)/k^* \rightarrow F_{\hat{A}}(P_\gamma)$. Let $\hat{L}$ be the inverse image of $\hat{E}_G(P_\gamma)^\circ$ through Isomorphism 5.5.1. The inclusion $\hat{L} \subset N_{\hat{A}_\gamma}(P)$ induces a group isomorphism $F_{\hat{A}}(P_\gamma) \cong \hat{L}/k^*$, which is also a section of the homomorphism $N_{\hat{A}_\gamma}(P)/k^* \rightarrow F_{\hat{A}}(P_\gamma)$. So $\hat{F}$ and $\hat{L}$ are conjugate in $\hat{A}_\gamma$. Thus the homomorphism $\theta$ is surjective and it is an $O$-algebra isomorphism. Note that $\theta$ maps $\mathcal{O}_*((P_{\hat{\theta}}) \times \hat{F})$ onto $\hat{B}_\gamma$. Clearly the inclusions $\hat{F} \subset N_{\hat{B}_p}(Q)$ and $\hat{F} \subset N_{\hat{B}_p}(P)$ induce $k^*$-group isomorphisms $f_\delta : \hat{F} \cong \hat{F}_B(Q_\delta)$ and $f_\gamma : \hat{F} \cong \hat{F}_{\hat{A}}(P_\gamma)$, respectively. On the other hand, the obvious inclusion $N_{\hat{B}_p}(P) \subset N_{\hat{B}_p}(Q)$ induces a group isomorphism $F_{\hat{A}}(P_\gamma) \cong F_B(Q_\delta)$ which maps any automorphism on $P$. 

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onto its restriction to $Q$. Since it follows from [13, Theorem 3.1] and [10, Lemma 1.17] that $E_H(Q_δ)$ and $E_G(P_γ)$ are equal to $F_δ(Q_δ)$ and $F_δ(P_γ)$ respectively, the group isomorphism $F_δ(P_γ) \cong F_δ(Q_δ)$ coincides with Isomorphism 3.4.1. The inclusion $N_{B_h^γ}(P) \subset N_{B_h^γ}(Q)$ also induces a $k^*$-group isomorphism $f_γ, δ : \hat{F}_δ(P_γ) \cong \hat{F}_δ(Q_δ)$ lifting Isomorphism 3.4.1. Moreover we have $f_γ, δ \circ f_γ = f_δ$.

By [14, Proposition 6.12], [10, 2.12.4] and [15, Proposition 5.11], we have $k^*$-group isomorphisms $g_γ : \hat{E}_G(P_γ)^\circ \cong \hat{F}_δ(P_γ)$ and $g_δ : \hat{E}_H(Q_δ)^\circ \cong \hat{F}_δ(Q_δ)$, which respectively lift the equalities $E_G(P_γ) = F_δ(P_γ)$ and $E_H(Q_δ) = F_δ(Q_δ)$. We set $\hat{ε}_δ, γ = g_δ^{-1} \circ f_γ^{-1} \circ g_δ$. Since $\hat{ε}_δ, γ$ and $\hat{ε}_δ, γ$ both lift the isomorphism $ε_δ, γ$, we can adjust $\hat{ε}_δ, γ$ by a linear character of $E_G(P_γ)$ so that $\hat{ε}_δ, γ$ and $\hat{ε}_δ, γ$ coincide.

Summarizing the above, we have a commutative diagram of $k^*$-group homomorphisms

\[
\begin{array}{ccc}
\hat{E}_G(P_γ)^\circ & \xrightarrow{g_γ} & \hat{F}_δ(P_γ) \\
\hat{E}_H(Q_δ)^\circ & \xrightarrow{g_δ} & \hat{F}_δ(Q_δ)
\end{array}
\]

In particular, this shows that there is an $O$-algebra isomorphism $O_*(P \times \hat{F}) \cong O_*(P \times \hat{E}_G(P_γ)^\circ)$ mapping $ui$ onto $u$ for any $u \in P$ and $O_*(Q \times \hat{F})$ onto $O_*(P \times \hat{E}_G(P_γ)^\circ)$. By composing the inverse of this $O$-algebra isomorphism and Isomorphism 5.10.1, we get the desired $G$-graded $P$-interior algebra isomorphism.

**Lemma 5.11.** With the notation and the assumptions as above, there is a $G$-graded $P$-interior algebra embedding $A_γ \rightarrow S \otimes O_ A_γ$.

**Proof.** Clearly the inclusion map $A_γ \rightarrow S \otimes O_ A_γ$ (see 5.4.1) is a $G$-graded $P$-interior algebra embedding and so is the inclusion map $i : S \otimes O_ A_γ \rightarrow S \otimes O S \otimes O_ A_γ$. By [15, Theorem 5.3], the obvious local point $\{h\}$ of $P$ on $B_γ$ determines a local point $γ'$ of $P$ on $\otimes O B_γ$. Since embeddings preserve localness (see [25, Proposition 15.1]), there is a local point $γ'$ of $P$ on $S \otimes O S \otimes O B_γ$ containing $γ'$.

There is a $P$-interior algebra embedding $e : O \rightarrow S \otimes O S$, which induces a $G$-graded $P$-interior algebra embedding $e \otimes 1 : A_γ \rightarrow S \otimes O S \otimes O_ A_γ$, mapping $a$ onto $a \otimes d(1)$ for any $a \in A_γ$. Obviously $\{h\}$ is a local point of $P$ on $B_γ$. Again since embeddings preserve localness, there is a local point $γ''$ of $P$ on $S \otimes O S \otimes O B_γ$ containing the image of $h$ through Embedding 5.11.2.

Obviously $\{h\}$ is also the unique local point of $P$ on $B_γ$ and then by [15, Theorem 5.3], $P$ has a unique local point on $S \otimes O S \otimes O B_γ$. So $γ'$ is equal to $γ''$. Then by [10, 2.11.2], the embedding $e \otimes 1$ factors through the embedding $i$ to get the embedding in this lemma.

**Theorem 5.12.** With the notation and the assumptions as above, there is a $G$-graded $P$-interior algebra isomorphism $A_γ \cong S \otimes O_ A_γ(P \times \hat{E}_G(P_γ)^\circ)$.

**Proof.** By Lemma 5.11 and Proposition 5.10, there is a $G$-graded $P$-interior algebra embedding $A_γ \rightarrow S \otimes O_ A_γ(P \times \hat{E}_G(P_γ)^\circ)$. By Lemma 4.2, it is easy to check that the identity element of $S \otimes O_ A_γ(P \times \hat{E}_G(P_γ)^\circ)$ is contained in the unique local point of $P$ on $S \otimes O_ A_γ(Q \times \hat{E}_H(Q_δ)^\circ)$. Thus this embedding must be an isomorphism.

### 6. Proofs of Proposition 1.7 and Theorems 1.11 and 1.12

Throughout this section we keep the notation in 2.10; we assume that $P$ is abelian and that the block $b$ of $H$ is a trivial inertial block (see Paragraph 1.4). As the title shows, we will prove the proposition and theorems stated in Section 1, so we also keep the notation there.
6.1. We firstly prove Proposition 1.7 and describe the algebraic structure of $\mathbb{A}_{\gamma'}$. We set $N = N_G(Q_\delta)$, $H = C_H(Q)$ and $\beta = \{b_\delta\}$. Clearly $N_\gamma$ and $H_\delta$ are pointed groups on $\mathcal{O}H$. Let $R_\gamma$ be a local pointed group such that $Q_\delta \leq R_\gamma \leq P$, and let $R_\delta$ be a local pointed group on $\mathcal{O}H$ such that $Br^{\mathcal{O}H}(\delta) = Br^{\mathcal{O}H}(\epsilon)$. Obviously $Br^{\mathcal{O}H}(\epsilon)$ is a local point of $R$ on $kH$. Since $Br^{\mathcal{O}H}$ maps $(\mathcal{O}H)^{R}$ onto $(kH)^{R}$ and $\text{Ker}(Br^{\mathcal{O}H})$ is contained in $J(\mathcal{O}H)$, $R_\epsilon$ has to be a local pointed group on $\mathcal{O}H$. In particular, $R_\epsilon$ is a local pointed group on $\mathcal{O}H$ such that $Br^{\mathcal{O}H}(\epsilon) = Br^{\mathcal{O}H}(\delta)$. Since $Br^{\mathcal{O}H}(\gamma') = Br^{\mathcal{O}H}(\gamma)$ and $Br^{\mathcal{O}H}(\delta') = Br^{\mathcal{O}H}(\delta)$, we take $R$ to be $P$ and $Q$ respectively and then conclude that $P_\gamma$ and $Q_\delta$ are local pointed groups on $\mathcal{O}H$. Moreover $P_\gamma$ and $Q_\delta$ has to be defect pointed groups of $N_\gamma$ and $H_\delta$ respectively since $P_\gamma$ and $Q_\delta$ are defect pointed groups of $G_\alpha$ and $H_\alpha$ respectively. Since the block $b_\delta$ of $H$ is nilpotent (see [3]), by [21, Theorem 3.5] there are a finite group $L$ containing $P$ and a surjective group homomorphisms $\pi : L \to E$ with kernel $Q$ such that $\pi(u)$ is the image of $u$ in $E$ for any $u \in P$ and such that with the identification of $L/Q$ and $\mathcal{E}$ through the isomorphism $L/Q \cong \mathcal{E}$ induced by $\pi$, the functor $\tau : \mathcal{E}(P_\gamma, H, N) \to \mathcal{E}(P_{(1)}, Q, L)$ mapping an object $R_{\gamma'}$ onto $R_{(1)}$ and a morphism $(\varphi_{R, x}, \tilde{x})$ onto $(\varphi_{R, x}, \pi(x))$ is an isomorphism of categories, where $\tilde{x}$ is the image of $x$ in $E$.

6.2. Proof of Proposition 1.7.

Take any two local pointed groups $R_\epsilon$ and $T_\nu$ in $\mathcal{E}(P_\gamma, H, G)$ and let $x$ be an element of $G$ such that $(R_\epsilon)^{-1} \leq T_\nu$. By [10, Proposition 5.3], we have $Q^x = (R^x \cap H) = T \cap H = P \cap H = Q$. Furthermore, since $(Q_\delta)^x = (R_\delta)^x \leq P_\gamma$, by [10, Proposition 5.5] we have $\delta^x = \delta$ and thus $x \in N_G(Q_\delta)$. Let $R^x$ and $T^x$ be local pointed groups on $\mathcal{O}H$ such that $Br^{\mathcal{O}H}(\epsilon) = Br^{\mathcal{O}H}(\delta)$ and $Br^{\mathcal{O}H}(\nu) = Br^{\mathcal{O}H}(\nu')$, respectively. Note that $(R_\epsilon)^{-1} \leq T_\nu$ is equivalent to $(R^x)^{-1} \leq T^x$. We define a functor $\psi : \mathcal{E}(P_\gamma, H, G) \to \mathcal{E}(P_\gamma, H, N)$ mapping an object $R_\epsilon$ onto $R^x_\epsilon$ and a morphism $(\varphi_{R, x}, \tilde{x})$ from $R_\epsilon$ to $T_\nu$ onto a morphism $(\varphi_{R^x, x}, \tilde{x})$ from $R^x_\epsilon$ to $T^x_\nu$. It is trivial to check that this functor is an isomorphism of categories. Set $\tau = \tau \circ \psi$. Then $\tau$ is an isomorphism of categories fulfilling Proposition 1.7. Therefore $L$ and $\pi$ satisfy Proposition 1.7. Suppose that there are another finite group $L'$ and another group homomorphism $\pi' : L' \to E$ satisfying Proposition 1.7. We denote by $\tau'$ the corresponding isomorphism of categories $\mathcal{E}(P_\gamma, H, G) \cong \mathcal{E}(P_{(1)}, Q, L')$. Then $\tau' \circ \psi^{-1}$ is the isomorphism of categories $\mathcal{E}(P_\gamma, H, N) \cong \mathcal{E}(P_{(1)}, Q, L')$ induced by the inclusion $P \subset L'$ and the homomorphism $\pi'$ in the sense of [21, Theorem 3.5]. Then the last statement of Proposition 1.7 follows from the uniqueness part of [21, Theorem 3.5].

6.3. By the proof above, the finite group $L$ and the homomorphism $\pi$ in Proposition 1.7 can be chosen to be the finite group $L$ and the homomorphism $\pi$ respectively. We set $L = \text{Res}_\pi(\hat{E}_N, \epsilon(Q_\delta))$. Since the simple factors of $(\mathcal{O}H)^{Q}$ and $(\mathcal{O}H)^{Q}$ determined by the points $\delta$ and $\delta'$ respectively are isomorphic as $N_G(Q_\delta)$-algebras, it is easily checked that there is a $k^*$-group isomorphism $\hat{E}_{G, \epsilon}(Q_\delta) \cong \hat{E}_N, \epsilon(Q_\delta)$ lifting the equality $E_{G, \epsilon}(Q_\delta) = E_N, \epsilon(Q_\delta)$ and then that there is a $k^*$-group isomorphism $\hat{L} \cong \hat{L}$ lifting the equality $L = L$. We identify $\hat{L}$ and $\hat{L}$ through the $k^*$-group isomorphism $\hat{L} \cong \hat{L}$.

6.4. Since $Q$ is normal in $L$, the twisted group algebra $\mathcal{O}_L \hat{L}^\circ$ has an obvious $L/Q$-graded algebra structure. We identify $L/Q$ and $\mathcal{E}$ through the isomorphism $L/Q \cong \mathcal{E}$ induced by $\pi$. Then $\mathcal{O}_L \hat{L}^\circ$ becomes an $\mathcal{E}$-graded algebra. Since $b_\delta b_\delta x = 0$ for any $x \in G - N$ and $P_\gamma \leq N_\beta$, we have $A_{\gamma'} = iA_i = i(ON)$. Since $P_\gamma$ is a pointed group on $\mathcal{O}H$, $A_{\gamma'}$ has an obvious $\mathcal{E}$-graded algebra structure. By [21, Corollary 3.15], there is a determinant one $P$-interior full matrix algebra $\hat{S}$ over $\mathcal{O}$ such that we have an $\mathcal{E}$-graded $P$-interior algebra isomorphism

6.4.1 $A_{\gamma'} \cong \hat{S} \otimes_{\mathcal{O}} \mathcal{O}_L \hat{L}^\circ$. 
Since \( N_H(Q_δ) \) is normal in \( N_G(Q_δ) \), \( \mathbf{A}_γ \) also has an obvious \( N_G(Q_δ)/N_H(Q_δ) \)-graded algebra structure. We have identified \( G/H \) and \( N_G(Q_δ)/N_H(Q_δ) \) in Paragraph 1.10 and thus \( \mathbf{A}_γ \) becomes a \( \hat{G} \)-graded algebra. Notice that in Paragraph 1.10 we also endowed \( \mathcal{O}_L \mathcal{C} \) with a \( \hat{G} \)-graded algebra structure. Let \( x \) be an element of \( N_G(Q_δ) \), \( \hat{x} \) be the image of \( x \) in \( \mathcal{E} \), \( y_δ \) be an inverse image of \( \hat{x} \) in \( \mathcal{L} \) and \( \hat{y}_δ \) be a lift of \( y_δ \) in \( \hat{\mathcal{L}} \). Then the isomorphism 6.4.1 maps \( i(\mathcal{O}_N H(Q_δ)x)i \) onto \( S \otimes_{\mathcal{O}} (\mathcal{O}_L \mathcal{C}) \hat{y}_δ \) and thus it is also a \( \hat{G} \)-graded \( P \)-interior algebra isomorphism.

6.5. Next we begin to prove Theorem 1.11. We divide its proof into a series of lemmas. Set \( J = PH \), \( J = J/H \) and \( \mathcal{C}_γ = i(\mathcal{O}J)i \). Clearly \( \mathcal{C}_γ \) is a \( J \)-graded \( P \)-interior algebra. By [10, Proposition 6.2], the obvious \( P \)-algebra homomorphism \( \mathcal{O}H \to \mathcal{O}J \) induced by the inclusion \( H \subset J \) is a strict semicovering and there is a unique local point \( \hat{γ} \) of \( P \) on \( \mathcal{O}J \) containing \( γ \). Since \( P_γ \) is a defect pointed group of \( J_α \), by [10, Corollary 6.3] \( P_γ \) is a defect pointed group of \( J_α \) on \( \mathcal{O}J \). In particular, \( \mathcal{C}_γ \) is a source algebra of the block algebra \( \mathcal{O}J \hat{h} \). By Theorem 5.12, we get a determinant one \( P \)-interior full matrix algebra \( S \) over \( \mathcal{O} \) such that there is a \( \hat{J} \)-graded \( P \)-interior algebra isomorphism

\[
\mathcal{C}_γ \cong S \otimes_{\mathcal{O}} \mathcal{O}_s(P \circ \hat{E}_J(P_γ)^{\circ}).
\]

Moreover \( S \) is unique up to \( P \)-interior algebra isomorphisms and it has a \( P \)-stable \( \mathcal{O} \)-basis containing the unity of \( S \).

6.6. Let \( R_ε \) be a local pointed group such that \( Q_δ \leq R_ε \leq P_γ \). Take some \( h \in \varepsilon \) such that \( ih = hi = h \) and \( hj = jh = j \). Set \( A_ε = hAh \) and \( B_ε = hBh \). Clearly \( A_ε = hAh \) is a \( \hat{G} \)-graded \( R \)-interior algebra. Set \( \mathcal{A} = S^0 \otimes_{\mathcal{O}} \text{Res}^\hat{\mathcal{O}}_h(\hat{\mathcal{A}}) \) and \( \mathcal{B} = S^0 \otimes_{\mathcal{O}} \text{Res}^\hat{\mathcal{O}}_h(\hat{\mathcal{B}}) \). Then \( \mathcal{A} \) is a \( \hat{G} \)-graded \( P \)-interior algebra. By [15, Theorem 5.3], \( R_ε \) determines a unique local pointed group \( R_ε \) on \( \hat{\mathcal{B}} \) such that for some \( \hat{h} \in \varepsilon \) and some element \( \ell \) of the unique local point \( \varepsilon_S \) of \( R \) on \( S \), we have \( \hat{h}(\ell \otimes h) = (\ell \otimes h)\hat{h} = \hat{h} \). Set \( \hat{A}_ε = \hat{h}\hat{A}\hat{h} \) and \( \hat{B}_ε = h\hat{B}h \). Clearly \( \hat{A}_ε \) is a \( \hat{G} \)-graded \( R \)-interior algebra.

6.7. Similarly we have the local pointed groups \( P_γ \) and \( Q_δ \) on \( \hat{\mathcal{B}} \) determined by the pointed groups \( P_γ \) and \( Q_δ \) on \( \mathcal{B} \) respectively, the \( \hat{G} \)-graded \( P \)-interior algebra \( \hat{\mathcal{A}}_i = i\hat{\mathcal{A}}i \) for some \( \hat{i} \in \hat{\gamma} \), and the \( \hat{G} \)-graded \( Q \)-interior algebra \( \hat{\mathcal{A}}_j = j\hat{\mathcal{A}}j \) for some \( \hat{j} \in \hat{\delta} \). By Lemma 5.6, we have \( \hat{\gamma} \subset \varepsilon \subset \hat{\delta} \). So as we did in the paragraph above Lemma 5.7, we identify \( \hat{\mathcal{A}}_i \) and \( \hat{\mathcal{A}}_j \) through some suitable \( \hat{G} \)-graded \( \mathcal{R} \)-interior algebra isomorphism, and \( \hat{\mathcal{A}}_i \) and \( \hat{\mathcal{A}}_j \) through some suitable \( \hat{G} \)-graded \( Q \)-interior algebra isomorphism. In this sense, the algebras \( \hat{\mathcal{A}}_i \), \( \hat{\mathcal{A}}_j \) and \( \hat{\mathcal{A}}_j \) are equal. In particular, their identities \( i \), \( h \) and \( j \) are equal and so are their 1-components \( \hat{B}_i \), \( \hat{B}_j \) and \( \hat{B}_j \).

Lemma 6.8. With the above notation, we have \( E_{G,G}(R_ε) \subset F_{A,G}(R_ε) \) and \( E_{G,G}(R_ε) \subset F_{A,G}(R_ε) \).

Proof. Given \( (φ, \hat{x}) \in E_{G,G}(R_ε), \hat{x} \) has a representative \( y \in N_G(R_ε) \) such that \( φ = φ_{R,y}^R \). There is some invertible element \( a_y \) of \( (\mathcal{O}H)^R \) such that \( yhy^{-1} = a_yha_y^{-1} \). Thus \( a_y^{-1}y \) commutes with \( h \). Set \( d_y = (a_y^{-1}y)h \). Clearly \( d_y \) belongs to \( N^R_{\hat{\mathcal{A}}_i}(R) \) and \( d_y \hat{h}d_y^{-1} = \phi(u)h \). So \( (φ, \hat{x}) \in F_{A,G}(R_ε) \) and \( E_{G,G}(R_ε) \subset F_{A,G}(R_ε) \).

Set \( T = \ell \mathcal{S} \). Clearly \( T \) is a Dade \( R \)-algebra and a \( R \)-interior algebra. Since the \( \hat{J} \)-graded \( P \)-interior algebra \( \mathcal{C}_γ \) is \( N_G(P_γ) \)-stable, by the uniqueness of \( S \) the \( P \)-interior algebra \( S \) is \( N_G(P_γ) \)-stable. Since \( N_G(R_ε) = N_{N_G(R_ε)}(P_γ)C_G(R_ε) \) (see Lemma 2.11), by the uniqueness of the local point of \( R \) on \( S \), \( T \) is \( N_G(R_ε) \)-stable. So there is an invertible element \( s_y \) in \( T \) such that \( s_yus_y^{-1} = φ(u)1 \).

Clearly we have \( \hat{h}^{-1}(s_y \hat{h}d_y^{-1})^{-1}(1 \otimes \hat{h}) = (1 \otimes \hat{h})\hat{h}^{-1}(s_y \hat{h}d_y^{-1})^{-1} \) and \( \hat{h}^{-1}(s_y \hat{h}d_y^{-1})^{-1} \) belongs to \( \varepsilon \). In particular, there is some invertible element \( e_y \) of \( \hat{B}^R \) such that \( (s_y \otimes d_y)\hat{h}(s_y \otimes d_y)^{-1} = e_y \). Set \( e_y = e_y^{-1}(s_y \otimes d_y)\hat{h} \).
Then $c_y$ is an invertible element of the $\hat{\cdot}$-component of $\hat{A}$ and we have $c_y u c_y^{-1} = \phi(u)\hat{h}$ for any $u \in R$. Thus we have $(\phi, \hat{x}) \in F_{\hat{\cdot}, \hat{G}}(R_{\hat{x}})$ and $E_{\hat{G}, \hat{G}}(R_{\hat{x}}) \subset F_{\hat{\cdot}, \hat{G}}(R_{\hat{x}})$.

6.9. The isomorphism $\tau$ of categories in Proposition 1.7 induces a group isomorphism $\chi_R : E_{\hat{G}, \hat{G}}(R_{\hat{x}}) \cong E_{\hat{\cdot}, \hat{\cdot}}(R)$, where $E_{\hat{\cdot}, \hat{\cdot}}(R)$ is the automorphism group of $R_{\hat{\cdot}}$ in $\hat{E}_{\hat{\cdot}}(R)$. We also have group isomorphisms $\theta_R : N_L(R) \cong E_{\hat{\cdot}, \hat{\cdot}}(R)$ and $\theta_R : F_{\hat{\cdot}, \hat{\cdot}}(R_{\hat{x}}) \cong F_{\hat{\cdot}, \hat{\cdot}}(R_{\hat{x}})$ (see 1.9 and 2.9.1 respectively). We denote by $\lambda_R$ the composition of $\theta_R$, $\chi_R^{-1}$, the inclusion map $E_{\hat{G}, \hat{G}}(R_{\hat{x}}) \subset F_{\hat{\cdot}, \hat{\cdot}}(R_{\hat{x}})$ and $\theta_R$.

6.10. The inclusion $N_L(R_{\hat{x}}) \subset N_L(Q_{\hat{x}})$ induces a group homomorphism $\theta : E_{\hat{G}, \hat{G}}(R_{\hat{x}}) \to E_{\hat{G}, \hat{G}}(Q_{\hat{x}})$ mapping $(\varphi_{R,x-1}, \hat{x})$ onto $(\varphi_{Q,x-1}, \hat{x})$ for any $x \in N_L(R_{\hat{x}})$. Similarly the inclusion $N_L(R) \subset L$ induces a group homomorphism $\vartheta : E_{L,E}(R) \to E_{L,E}(Q)$. We have $\vartheta \circ \chi_R = \chi_Q \circ \theta$ and then the following commutative diagram of group homomorphisms

\[
\begin{array}{ccc}
N_L(R)/Q & \xrightarrow{\lambda_R} & F_{\hat{\cdot}}(R_e) \\
\downarrow & & \downarrow \\
L/Q & \xrightarrow{\lambda_Q} & F_{\hat{\cdot}}(Q_{\hat{x}}),
\end{array}
\]

where the left downarrow is induced by the inclusion $N_L(R) \subset L$ and the right downarrow is induced by the inclusion $N_{\hat{\cdot}}(R) \subset N_{\hat{\cdot}}(Q_{\hat{x}})$.

6.11. The canonical surjectivity of the homomorphism $\mathcal{N}_{\hat{\cdot}}(Q) \to \mathcal{N}_{\hat{\cdot}}(Q)/k^*$ maps $(\hat{B}_{\hat{\cdot}}^Q)^* \to (\hat{B}_{\hat{\cdot}}^Q)^*/k^*$ and induces a canonical group isomorphism

\[
\left(\mathcal{N}_{\hat{\cdot}}(Q)/k^*\right) \left/ \left(\hat{B}_{\hat{\cdot}}^Q)^*/k^*\right) \cong \mathcal{N}_{\hat{\cdot}}(Q)/(\hat{B}_{\hat{\cdot}}^Q)^* = F_{\hat{\cdot}}(Q_{\hat{x}}).
\]

Set $\mathfrak{M} = \mathcal{N}_{\hat{\cdot}}(Q)/k^*$ and $\mathfrak{R} = (\hat{B}_{\hat{\cdot}}^Q)^*/k^*$. By composing the canonical surjective homomorphism $\mathcal{L} \to \mathcal{L}/Q$, the homomorphism $\lambda_Q$ and the inverse of this canonical group isomorphism, we get a group homomorphism $\mathfrak{P} \to \mathfrak{M}/\mathfrak{R}$. There is an injective group homomorphism $\mathfrak{i} : P \to \mathfrak{M}$ mapping $u$ onto the image of $u\hat{\cdot}$ in $\mathfrak{M}$ for any $u \in P$.

Lemma 6.12. With the notation as above, the homomorphism $\mathfrak{P}$ can be lifted to an injective group homomorphism $\mathcal{L} \to \mathfrak{M}$, which extends the homomorphism $\mathfrak{i}$.

Lemma 6.13. Let $L$ be a finite group, $M$ a group, $Z$ a normal subgroup of $M$ and $\sigma : L \to M = M/Z$ a group homomorphism. Let $\{Z_n\}_{n \in \mathbb{N}}$ be a sequence of subgroups of $Z$, which are normal in $M$. Assume that $Z_0 = Z$, that $Z_n/Z_{n+1}$ is a $p'$-divisible abelian group for any $n \in \mathbb{N}$, and that $Z_n/r = Z$. Assume that, for a Sylow $p'$-subgroup $P$ of $L$, there exists a group homomorphism $\varsigma : P \to M$ lifting the restriction of $\sigma$ to $P$ and fulfilling the following condition

6.13.1 For any subgroup $R$ of $P$ and any $x \in L$ such that $R^x \subset P$, then there is $y_x \in M$ such that $\bar{\sigma}(x) = \bar{y}_x$, where $\bar{y}_x$ denotes the image of $y_x$ in $M$, and such that $\varsigma(u^x) = \varsigma(u)^{y_x}$ for any $u \in R$.

6.13.2 $\varsigma(P)$ acts trivially on $Z$ by the conjugation.

Then there is a group homomorphism $\sigma : L \to M$ lifting $\sigma$ and extending $\varsigma$. Moreover, if $\sigma' : L \to M$ is another group homomorphism lifting $\sigma$ and extending $\varsigma$, there is $z \in Z$ such that $\sigma'(x) = \sigma(x)^z$ for any $x \in L$. 

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Proof. For any $n \in \mathbb{N} \cup \{0\}$, we set $M_n = M/Z_n$; we denote by $\varsigma_n$ the group homomorphism $P \to M_n$ induced by $\varsigma$, and by $y_{x,n}$ the image in $M_n$ of $y_x$ in 6.13.1; there is a canonical group isomorphism $M_n \cong M_{n+1}/(Z_n/Z_{n+1})$; we identify $M_n$ with $M_{n+1}/(Z_n/Z_{n+1})$ through this isomorphism. Clearly $M_0 = M$ and $y_{x,0} = y_x$.

Set $\sigma_0 = \hat{\sigma}$. Clearly $\sigma_0$ extends $\varsigma_0$. Assume that there is a sequence of group homomorphisms $\{\sigma_i : L \to M_{i,j}\}_{0 \leq j \leq k}$ such that for any integer $l$ such that $0 \leq l \leq k - 1$, we have $\sigma_{l+1}$ lifts $\sigma_l$, and such that for any integer $i$ such that $0 \leq l \leq k$, $\sigma_l$ extends $\varsigma_i$. Let $R$ be a subgroup of $P$ and take $x \in L$ such that $R^x \subset P$. Since $\sigma_k$ lifts $\varsigma_0$, $y_{x,k+1}$ and any representative $z_{x,k+1}$ of $\sigma_k(x)$ in $M_{k+1}$ differentiates by some element of $Z/Z_{k+1}$. Moreover since $\varsigma(P)$ acts trivially on $Z$, we have $\varsigma_{k+1}(u^z) = \varsigma_{k+1}(u)\varsigma_{k+1}(z) = \varsigma_{k+1}(u^{z_{x,k+1}})$ for any $u \in P$. Thus $\sigma_k$, $\varsigma_{k+1}$ and $z_{x,k+1}$ satisfy Condition 6.13.1. Since $Z_k/Z_{k+1}$ is a $p'$-divisible abelian group, by [21, Lemma 3.6] there is a group homomorphism $\sigma_{k+1} : L \to M_{k+1}$ lifting $\sigma_k$ and extending $\varsigma_{k+1}$. By induction, we get a sequence of group homomorphisms $\{\sigma_n : L \to M_n\}_{n \in \mathbb{N} \cup \{0\}}$ such that $\sigma_n$ extends $\varsigma_n$ for any $n \in \mathbb{N} \cup \{0\}$ and such that $\sigma_{n+1} \mid \sigma_n$ for any $n \in \mathbb{N} \cup \{0\}$.

Since $\text{lim} Z/Z_n \cong Z$, by [25, Lemma 45.4] canonical homomorphisms $M \to M_n$ induce a group isomorphism $M \cong \text{lim} M_n$. So there is a unique group homomorphism $\sigma : L \to M$ such that $\sigma$ lifts $\varsigma_n$ for any $n \in \mathbb{N} \cup \{0\}$. In particular $\sigma$ lifts $\hat{\sigma}$. Since the restriction of $\sigma$ to $P$ and the homomorphism $\varsigma$ both lift the sequence of group homomorphisms $\{\varsigma_n\}_{n \in \mathbb{N} \cup \{0\}}$, the restriction of $\sigma$ to $P$ has to be equal to $\varsigma$.

We assume that $\sigma' : L \to M$ is another group homomorphism lifting $\hat{\sigma}$ and extending $\varsigma$. Then $\sigma'$ induces a sequence of group homomorphisms $\{\sigma'_n : L \to M_n\}_{n \in \mathbb{N} \cup \{0\}}$ such that $\sigma'_{n+1}$ lifts $\sigma'_n$ for any $n \in \mathbb{N} \cup \{0\}$ and such that $\sigma'_n$ extends $\varsigma_n$ for any $n \in \mathbb{N} \cup \{0\}$, where $\sigma'_0 = \hat{\sigma}$. By the uniqueness part of [21, Lemma 3.6] we find a sequence $\{z_n\}_{n \in \mathbb{N} \cup \{0\}}$ such that $z_n$ belongs to $Z/Z_n$ for any $n \in \mathbb{N} \cup \{0\}$, such that $z_n$ is the image of $z_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$ and such that for any $n \in \mathbb{N} \cup \{0\}$, we have $\sigma'_n(x) = \sigma_n(x)^{z_n}$ for any $x \in L$. Since $\lim Z/Z_n \cong Z$, there is some $z \in Z$ lifting $z_n$ for any $n \in \mathbb{N} \cup \{0\}$. Clearly $\sigma'$ and $\sigma^z$ both lift $\sigma'_n$ for any $n \in \mathbb{N} \cup \{0\}$ and thus $\sigma'(x) = \sigma(x)^z$ for any $x \in L$.

6.14. Proof of Lemma 6.12.

In order to prove this lemma, we apply Lemma 6.13 to the case that $L = \mathbb{L}$, $M = \mathfrak{M}$, $Z = \mathfrak{k}$, $\hat{\sigma} = \mathfrak{p}$ and $\varsigma = i$. By [24, Chapter II, Proposition 8], there is a canonical group isomorphism $\mathfrak{p} \cong 1 + J(\mathfrak{B}^2 \delta)$. We identify $\mathfrak{p}$ with $1 + J(\mathfrak{B}^2 \delta)$. Set $\mathfrak{p}_0 = \mathfrak{p}$ and $\mathfrak{p}_n = 1 + J(\mathfrak{B}^2 \delta)^n$ for any $n \in \mathbb{N}$. Then for any $n \in \mathbb{N} \cup \{0\}$, $\mathfrak{p}_n$ is normal in $\mathfrak{M}$, $\mathfrak{p}_n/\mathfrak{p}_{n+1}$ is a $p'$-divisible abelian group and canonical group homomorphisms $\mathfrak{p} \to \mathfrak{p}/\mathfrak{p}_n$ induces a group isomorphism $\mathfrak{p} \cong \mathfrak{p}_n$ (see [25, Lemma 45.5]). So the group $\mathfrak{p}$ together the sequence $\{\mathfrak{p}_n\}_{n \in \mathbb{N} \cup \{0\}}$ satisfies the assumption in Lemma 6.13.

Let $R$ be a subgroup of $P$ and let $x$ be an element of $\mathbb{L}$ such that $R^x \subset P$. Since $Q$ is normal in $\mathbb{L}$, we have $(Q^x)^P \leq P$. It is easy to see that if the homomorphisms $\mathfrak{p}$ and $\varsigma$ satisfy Condition 6.13.1 for $RQ$, then the homomorphisms $\mathfrak{p}$ and $\varsigma$ satisfy Condition 6.13.1 for $R$. So we assume that $R$ contains $Q$. Since $P$ is an abelian Sylow $p'$-subgroup of $\mathbb{L}$ (see [10, Remark 1.9]), there is $y \in N_\mathbb{L}(P)$ such that $u^y = u^y$ for any $u \in R$ (see [25, Proposition 49.6]). Set $z = xy^{-1}$. Clearly $z$ belongs to $C_\mathbb{L}(R)$. So in order to show that the homomorphisms $\mathfrak{p}$ and $\varsigma$ satisfy Condition 6.13.1, it suffices to show that the homomorphisms $\mathfrak{p}$ and $\varsigma$ satisfy Condition 6.13.1 for any subgroup $R$ of $P$ containing $Q$ such that $R^x = R$. Let $x'$ be the image of $x$ in $\mathbb{L}/Q$ and take a representative $a$ of $\lambda_R(x')$ in $N_{\mathfrak{A}_x}(R)$. Since $\lambda_R$ is the composition of $\vartheta_R$ and $\alpha^{-1}$, the inclusion map $E_{G,R}(R_e) \subset F_{\mathfrak{A}_x}(R)$ and $\vartheta_R$, there is a representative $g$ of $\pi(x)$ in $N_G(R_e)$ such that $\varphi_{R,x-1}^R = \varphi_{R,g^{-1}}^R$, such that $a$ belongs to $N_{\mathfrak{A}_x}(R)$ and such that $\varphi_{R,x-1}^R = \varphi_{R,g^{-1}}^R(u)h$ for any $u \in R$. We denote by $w_x$ the image of $a$ in
$\mathfrak{M}$. Then we have $i(u^x) = i(u)^{w_x}$ for any $u \in R$. Moreover by Diagram 6.10.1, $p$ maps $x$ onto the image of $w_x$ in $\mathfrak{M}/\mathfrak{H}$. So the homomorphisms $p$ and $i$ satisfy Condition 6.13.1.

Set $\tilde{C}_\gamma = \hat{\mathfrak{C}}_\gamma i$. Then $\tilde{C}_\gamma$ is a $\hat{J}$-graded $P$-interior algebra. As in the proof of Lemma 5.5, we prove that $\tilde{C}_\gamma$ is isomorphic to $O_\ast(P \circ \hat{E}_J(p_\gamma)\circ)$ as $\hat{J}$-graded $P$-interior algebras. In particular, $\hat{B}_\gamma$ is isomorphic to $O_\ast(Q \circ \hat{E}_J(p_\gamma)\circ)$ as $Q$-interior $P$-algebras. Since $P = QS_P$ (see 3.3), where $S_P$ is the subgroup of all $E_J(p_\gamma)$-fixed elements of $P$, the $P$-conjugation acts trivially on $(O_\ast(Q \circ \hat{E}_J(p_\gamma)\circ))^Q$. So Condition 6.13.2 is satisfied.

Finally by Lemma 6.13 the homomorphism $p$ can be lifted to a group homomorphism $L \to \mathfrak{M}$ extending the homomorphism $i$. Let $x$ be an element in the kernel of the homomorphism $L \to \mathfrak{M}$. Then the isomorphism $\lambda_Q$ maps the image of $x$ in $L/Q$ onto 1. Thus $x$ has to be inside $Q$. Since the homomorphism $L \to \mathfrak{M}$ extends $i$, $x$ has to be 1. So the homomorphism $L \to \mathfrak{M}$ is injective.

6.15. We denote by $p'$ the homomorphism $L \to \mathfrak{M}$ in Lemma 6.12 and by $\hat{\mathfrak{L}}$ the inverse image of $p'(L)$ in $N_{\hat{A}_\gamma}(Q)$. The group $\hat{\mathfrak{L}}$ with the inclusion map $k^* \to \hat{L}$ becomes a $k^*$-group with the $k^*$-quotient $L$. Since we identify the two groups $\hat{G}$ and $L/K$ in Paragraph 1.10, the twisted group algebra $O_\ast\hat{\mathfrak{L}}$ can be endowed with a $\hat{G}$-graded algebra structure. The inclusion map $P \subset \hat{\mathfrak{L}}$ induces a $P$-interior algebra structure on $O_\ast\hat{\mathfrak{L}}$.

**Lemma 6.16.** With the notation as above, there is a $\hat{G}$-graded $P$-interior algebra isomorphism

$$A_\gamma \cong S \otimes_\mathcal{O} O_\ast\hat{\mathfrak{L}}.$$

**Proof.** The inclusion $\hat{\mathfrak{L}} \subset \hat{A}_\gamma$ induces a $P$-interior algebra homomorphism $O_\ast\hat{\mathfrak{L}} \to \hat{A}_\gamma$. Let $x$ be an element of $\hat{\mathfrak{L}}$ and let $x'$ be the image of $x$ in $L/Q$. We take a representative $a$ of $\lambda_Q(x')$ in $N_{\hat{A}_\gamma}(Q)$ and denote by $w_x$ the image of $a$ in $\mathfrak{M}$. Since $\lambda_Q$ is the composition of $\theta_Q, x^{-1}$, the inclusion map $E_{\hat{G},\hat{C}}(Q_\delta) \subset F_{\hat{A},\hat{C}}(Q_\delta)$ and $\theta_Q, \pi(x)$ has a representative $g$ in $N_C(Q_\delta)$ such that $\varphi_{Q, a^{-1}} = \varphi_{Q, g^{-1}}$, such that $a$ belongs $N_{\hat{A}_\gamma}(Q)$ and such that $a \delta a^{-1} = \varphi_{Q, g^{-1}}(u)\text{id}$ for any $u \in Q$.

Clearly the homomorphism $p$ maps $x$ onto the image of $w_x$ in $\mathfrak{M}/\mathfrak{H}$ and since $a$ belongs to $N_{\hat{A}_\gamma}(Q)$, any inverse image of $x$ in $\hat{\mathfrak{L}}$ lies in the $g$-component of $\hat{A}_\gamma$. Thus the homomorphism $O_\ast\hat{\mathfrak{L}} \to \hat{A}_\gamma$ is a $\hat{G}$-graded algebra homomorphism.

Let $\tilde{K}$ be the inverse image of $K$ in $\tilde{\mathfrak{L}}$. Then the homomorphism $O_\ast\hat{\mathfrak{L}} \to \tilde{A}_\gamma$ induces an $O_\ast$-algebra homomorphism $O_\ast\tilde{\mathfrak{L}} \to \tilde{A}_\gamma$. Since $E_{\hat{H}}(Q_\delta)$ is a $\hat{J}$-group and $\tilde{K}$ is the inverse image of $E_{\hat{H}}(Q_\delta)$ in $\tilde{\mathfrak{L}}, \tilde{K}$ has a subgroup $E$ such that $\tilde{K} = Q \circ E$. Thus we have $\tilde{K} = Q \circ \hat{E}$, where $\hat{E}$ is the inverse image of $E$ in $\hat{K}$. We claim that $\hat{B}_\gamma$ is generated by $\hat{E}$ and $\hat{Q}$. Notice that $\hat{B}_\gamma$ is equal to $\hat{B}_\delta$. For any $\phi \in F_{\hat{B}}(Q_\delta)$, there is an invertible element $a$ in $N_{\hat{B}_\gamma}(Q)$ such that $\phi(a)i = a \delta a^{-1}$ for any $u \in Q$. By 2.9.1, the map $F_{\hat{B}}(Q_\delta) \to N_{\hat{B}_\gamma}(Q)/(\hat{B}_\gamma)^*$ mapping $\phi$ onto the image of $a$ in $N_{\hat{B}_\gamma}(Q)/(\hat{B}_\gamma)^*$ for any $\phi \in F_{\hat{B}}(Q_\delta)$ is a $\hat{G}$-graded algebra isomorphism. We consider the following short exact sequence of group homomorphisms

$$\hat{i} \to (\hat{i} + J(\hat{B}_\gamma^Q)) \cong (\hat{B}_\gamma^Q)^*/k^* \to N_{\hat{B}_\gamma}(Q)/k^* \to F_{\hat{B}}(Q_\delta) \to \hat{i}.$$

Clearly $\hat{E}$ is contained in $N_{\hat{B}_\gamma}(Q)$ and trivially intersects $\hat{i} + J(\hat{B}_\gamma^Q)$. So the inclusion $\hat{E} \subset N_{\hat{B}_\gamma}(Q)$ induces an injective group homomorphism $\hat{E}/k^* \to F_{\hat{B}}(Q_\delta)$. On the other hand, by [13, Theorem
3.1] and [10, Lemma 1.17] we have $E_H(Q_δ) = F_B(Q_δ) = F_B(Q_δ)$. Since the groups $E$ and $E_H(Q_δ)$ have the same order, the homomorphism $E/k^* 	o F_B(Q_δ)$ must be a group isomorphism. Moreover it is trivially seen that the inverse of this isomorphism $E/k^* 	o F_B(Q_δ)$ is a section of the group homomorphism $N_{B_1^{25}}(Q)/k^* 	o F_B(Q_δ)$. By Proposition 5.10, there is a $Q$-interior algebra isomorphism $\tilde{B}_{\gamma} \cong O_s(Q \circ \tilde{E}_H(Q_δ)^{o})$. We denote by $\tilde{E}$ the inverse image of $\tilde{E}_H(Q_δ)^{o}$ through this $Q$-interior algebra isomorphism. Then it is easily checked that $\tilde{E}$ is contained in $N_{B_1^{25}}(Q)$ and that the inclusion $\tilde{E} \subset N_{B_1^{25}}(Q)$ induces another section of the group homomorphism $N_{B_1^{25}}(Q)/k^* 	o F_B(Q_δ)$.

Since $E_H(Q_δ)$ is a $p'$-group, by [25, Lemma 45.6] the sequence $6.16.2$ uniquely splits. Thus $\tilde{E}$ and $\tilde{E}'$ are conjugate in $(\tilde{B}_{\gamma}^{25})^{*}$. This implies that $\tilde{B}_{\gamma}$ is generated by $\tilde{E}$ and $Q\tilde{E}$ and then that the homomorphism $O_s\tilde{K} \to \tilde{B}_{\gamma}$ is surjective. Since $\tilde{B}_{\gamma}$ and $O_s\tilde{K}$ have the same $O$-rank $|Q||E_H(Q_δ)|$, the homomorphism $O_s\tilde{K} \to \tilde{B}_{\gamma}$ is an $O$-algebra isomorphism.

We notice that $\tilde{G}$-graded algebras $O_s\tilde{L}$ and $\tilde{A}_\gamma$ are crossed products of $\tilde{G}$. So the homomorphism $O_s\tilde{L} \to \tilde{A}_\gamma$ is a $P$-interior algebra homomorphism. On the other hand, as in Lemma 5.11, we prove that there is a $\tilde{G}$-graded $P$-interior algebra embedding $A_\gamma \to S \otimes _O \tilde{A}_\gamma$. So we get a $\tilde{G}$-graded $P$-interior algebra embedding $A_\gamma \to S \otimes _O O_s\tilde{L}$. Since the identity element of $S \otimes _O O_s\tilde{L}$ is contained in the unique local point of $P$ on $S \otimes _O O_s\tilde{K}$ (see Lemma 4.2 above), this embedding must be a $P$-interior algebra isomorphism.

6.17. Let $j$ be an element of the local point of $Q$ on $S$. By Lemma 4.2, it is easy to check that $j \otimes 1$ is contained in the unique local point of $Q$ on $S \otimes _O \tilde{K}$ and that the inverse image $j'$ of $j \otimes 1$ in $A_\gamma$ through Isomorphism $6.16.1$ is contained in $\delta$. We adjust $j$ so that $j' = j$. Set $A_\delta = jA_j$ and $V = jS_j$. Then Isomorphism $6.16.1$ induces a $\tilde{G}$-graded $Q$-interior algebra isomorphism $A_\delta \cong V \otimes _O \tilde{O_s\tilde{L}}$, through which, we identify $A_\delta$ with $V \otimes _O \tilde{O_s\tilde{L}}$. On the other hand, the $Q$-interior algebra $V$ is $N_G(Q_δ)$-stable (see the second paragraph of the proof of Lemma 6.8). Thus for any $x \in N_G(Q_δ)$ there is an invertible element $s_x$ in $V$ such that $s_xu_x^{-1} = \varphi_{Q,x}^S(u_1)v_1$ for any $u \in Q$. Then $\varphi_{Q,x}^S$ belongs to $F_V(Q_{1V})$ and the map $E_{G,G}(Q_δ) \to F_V(Q_{1V})$ sending $(\varphi_{Q,x}^S, \hat{x})$ onto $\varphi_{Q,x}^S$ for any $x \in N_G(Q_δ)$ is a group homomorphism. Moreover by [10, 2.12.4], this group homomorphism can be lifted to a group homomorphism $\theta : E_{G,G}(Q_δ) \to \tilde{F}_V(Q_{1V})$. We set $\theta(\varphi_{Q,x}^S, \hat{x}) = (\varphi_{Q,x}^S, \hat{u}_x)$. For any $\tilde{y} \in \tilde{L}$, we take a representative $z$ of $\pi(y)$ in $N_G(Q_δ)$. Then it is easy to check that the correspondence

$$L \to \tilde{F}_{A,G}(Q_δ), \tilde{y} \mapsto (\varphi_{Q,z}^S, \hat{z}, u_z \otimes y)$$

is a $k^*$-group homomorphism, where $u_z \otimes y$ is the image of $u_z \otimes \tilde{y}$ in $\tilde{F}_A(Q_δ)$.

**Lemma 6.18.** With the above notation, the inclusion $E_{G,G}(Q_δ) \subset F_{A,G}(Q_δ)$ can be lifted to an injective $k^*$-group homomorphism $\tilde{E}_{G,G}(Q_δ)^{o} \to F_{A,G}(Q_δ)$.

**Proof.** For an element $(x, s_δ(a))$ in $\tilde{E}_{G,G}(Q_δ)^{o}$, we have $s_δ(jxa^{-1}) = s_δ(j)$ and $s_δ(jxa^{-1}) = s_δ(j)$ for any other point $\epsilon$ of $Q$ on $B$. Then by [14, Lemma 6.3] there is a suitable element $c$ of $1 + J(B^Q)$ such that $xa^{-1}c^{-1}$ and $j$ commute each other and such that $j(xa^{-1})j = (xa^{-1}c^{-1})j$. In particular, $j(xa^{-1})j$ belongs to $N_\delta(8)$. For any $a \in N_\delta(8)$, we denote by $\bar{a}$ the image of $a$ in $\tilde{F}_A(Q_δ)$. We define a correspondence $\theta_\delta : \tilde{E}_{G,G}(Q_δ)^{o} \to F_{A,G}(Q_δ)$ sending $(x, s_δ(a))$ to $(\varphi_{Q,x}^S, \hat{x}, j(xa^{-1})j)$.

Let $a'$ be another invertible element of $B^Q$ such that $s_δ(a') = s_δ(a)$. By [14, Lemma 6.3] again, there is a suitable element $c'$ of $1 + J(B^Q)$ such that $xa^{-1}c'$ and $j$ commute and such
that $j(xa^{-1})j = (xa^{-1}c^{-1})j$. Clearly we have $((xa^{-1}c^{-1})j)^{-1}(xa^{-1}c^{-1})j = c'a^{-1}c^{-1}j$ and $c'a^{-1}c^{-1}j$ belongs to $1 + J(B^Q)$. Thus we have $j(xa^{-1})j = xa^{-1}c^{-1}j = xa^{-1}c^{-1}j = j(xa^{-1})j$ and $(\varphi^Q_{Q,x}, \tilde{x}, j(xa^{-1})) = (\varphi^Q_{Q,x}, \tilde{x}, j(xa^{-1}))$. If $(y, s_b(d)) \in \tilde{N}_G(Q_b)$ has the same image as $(x, s_b(a))$ in $\tilde{E}_G,G(Q_b)$, there is some $z \in C_H(Q)$ such that $s_b(d) = s_b(az)$ and $y = xz$. As above we choose $d$ to be $az$. Then we have $(\varphi^Q_{Q,x}, \tilde{x}, j(xa^{-1})) = (\varphi^Q_{Q,xy}, y, j(yd^{-1}))$ and so $\theta_\delta$ is a well defined map.

We take two elements $(x, s_b(a))$ and $(y, s_b(d))$ in $\tilde{N}_G(Q_b)$ such that $xa^{-1}$ and $yd^{-1}$ commute with $j$. We take an element $c''$ of $1 + J(B^Q)$ such that $xy(ad)^{-1}c''$ and $j$ commute and such that $jxy(ad)^{-1}j = hxy(ad)^{-1}c''$. Since $s_b(a^{yd^{-1}}) = s_b(c''a)$, we have

$$\hat{\theta}_\delta((x, s_b(a))(y, s_b(d))) = \hat{\theta}_\delta((xy, s_b(ad))) = (\varphi^Q_{Q,xy}, \tilde{x}y, xy(ad)^{-1}c''_1j)$$
$$= (\varphi^Q_{Q,xy}, \tilde{x}y, xa^{-1}j, yd^{-1}a^{yd^{-1}}c''_1j)$$
$$= (\varphi^Q_{Q,x}, \tilde{x}, xa^{-1}j)((\varphi^Q_{Q,xy}, \tilde{y}, yd^{-1}j) = \hat{\theta}_\delta((x, s_b(a))(y, s_b(d)))$$

and thus $\hat{\theta}_\delta$ is a group homomorphism. Clearly $\hat{\theta}_\delta$ maps $(x, s_b(\lambda a))$ onto $(\varphi^Q_{Q,x}, \tilde{x}, j(x\lambda a^{-1}))$ for any $\lambda \in \mathcal{O}^*$ and lifts the inclusion $E_{G,G}(Q_b) \subset F_{A,G}(Q_b)$. So $\hat{\theta}_\delta$ is an injective $k^*$-group homomorphism.

**Lemma 6.19.** With the notation as above, there is a $k^*$-group isomorphism $\tilde{\mathcal{L}} \cong \mathcal{L}$, which lifts the identity map on $\mathcal{L}$.

**Proof.** By Lemma 6.18, we get an injective $k^*$-group homomorphism $\tilde{E}_{G,G}(Q_b) \to \hat{F}_{A,G}(Q_b)$ lifting the inclusion $E_{G,G}(Q_b) \subset F_{A,G}(Q_b)$. Moreover by the proof of Lemma 6.18, the image of this $k^*$-group homomorphism coincides with the image of Homomorphism 6.17.1. Then by factoring it through Homomorphism 6.17.1, we get a $k^*$-group homomorphism $\tilde{\mathcal{L}} \to \tilde{E}_{G,G}(Q_b)$, which clearly lifts $\pi$. Then this lemma follows from the uniqueness of pull-backs.

**6.20.** Proof of Theorem 1.11.

By Lemmas 6.16 and 6.19, we get a $\hat{G}$-graded $P$-interior algebra $A_\gamma \cong S \otimes_\mathcal{O} O_* \mathcal{L}$ and thus prove the isomorphism 1.11.1. Notice that we identify $\hat{G}$ and $L/K$ in 1.10. We denote by $J$ the inverse image of $\tilde{J}$ in $\mathcal{L}$ and by $\tilde{J}$ the inverse image of $J$ in $\tilde{\mathcal{L}}$. Clearly $J$ is equal to $PK$ and the restriction to $C_\gamma$ of the isomorphism 1.11.1 induces a $P$-interior algebra isomorphism $C_\gamma \cong S \otimes_\mathcal{O} O_* \hat{\mathcal{J}}$. Suppose that there is another determinant one $P$-interior full matrix algebra $S'$ such that there is a $\hat{G}$-graded $P$-interior algebra isomorphism $A_\gamma \cong S' \otimes_\mathcal{O} O_* \mathcal{L}$. Similarly we get another $P$-interior algebra isomorphism $C_\gamma \cong S' \otimes_\mathcal{O} O_* \hat{\mathcal{J}}$. By [16, Theorem 7.2], $S'$ is a Dade $P$-algebra. On the other hand, since $K$ is isomorphic to the semidirect product $Q \circ E_H(Q_b)$ and $P$ is an abelian Sylow $p$-subgroup of $J$ (see [10, Remark 1.9]), $P$ is normal in $\tilde{J}$ and $C_\tilde{J}(P)$ is equal to $P$. So by [18, Lemma 4.5], $S$ and $S'$ are isomorphic as $P$-interior algebras.

**6.21.** Proof of Theorem 1.12.

Set $\mathcal{J} = P \mathcal{H}$ and $C_\mathcal{J} = i(\mathcal{OJ})i$. By [10, Proposition 6.2], the obvious $P$-algebra homomorphism $\mathcal{O} \mathcal{H} \to \mathcal{OJ}$ induced by the inclusion $\mathcal{H} \subset \mathcal{J}$ is a strict semicovering and there is a unique local point $\tilde{g}$ of $P$ on $\mathcal{OJ}$ containing $\tilde{g}$. Since $P_\mathcal{J}'$ is a defect pointed group of $\mathcal{J}_\alpha$, by [10, Corollary 6.3] $P_\mathcal{J}'$ is a defect pointed group of $\mathcal{J}_\alpha$ on $\mathcal{OJ}$. As a particular, $C_\mathcal{J}'$ is a source algebra of the block algebra $\mathcal{OJ}\mathcal{B}$. Since the block $b$ of $H$ is assumed to be a trivial inertial block, by the remark below Proposition 5.2 the source algebras $C_\gamma$ and $C_\mathcal{J}'$ of the block algebras $OJb$ and $\mathcal{OJ}\mathcal{B}$ are isomorphic as $P$-interior algebras. As in the proof of theorem 1.11, the isomorphism 1.11.1 induces a $P$-interior algebra isomorphism $C_\gamma \cong S \otimes_\mathcal{O} O_* \hat{\mathcal{J}}$, where $J$ is the inverse image of $\hat{J}$ in $\mathcal{L}$ and $\tilde{J}$ is the inverse
image of $\mathcal{J}$ in $\hat{L}$. So we have a $P$-interior algebra isomorphism $\mathbb{C}_{\gamma'} \cong S \otimes_O \hat{O}_{\gamma'}$. Similarly the isomorphism 6.4.1 also induces a $P$-interior algebra isomorphism $\mathbb{C}_{\gamma'} \cong S \otimes_O \hat{\mathcal{O}}_{\gamma'}$. Since $P$ is a normal Sylow $p$-subgroup of $\mathcal{J}$ and $C_{\mathcal{J}}(P)$ is equal to $P$, by [18, Lemma 4.5] there is a $P$-interior algebra isomorphism $S \cong S$. Consequently there is a $\hat{G}$-graded $P$-interior algebra isomorphism $\mathbb{A}_i \cong \mathbb{A}_{\gamma'}$. We set $M = \mathcal{A} \otimes_{\mathcal{A}_i} \mathbb{A}$ and $N = \mathcal{B} \otimes_{\mathcal{B}_i} \mathbb{B}$. Then the $O(G \times \hat{G})$-module $M$ induces a Morita equivalence between $\mathcal{A}$ and $\mathcal{A}$ and the $O(H \times \hat{H})$-module $N$ induces a Morita equivalence between $\mathcal{B}$ and $\mathcal{B}$.

Clearly the inclusions $\mathcal{B}_i \subset \mathcal{A}_i$, $\mathbb{A}_i \subset \mathbb{A}$ and $\mathcal{B}_i \subset \mathcal{B}_i$, $\mathbb{A}_i \subset \mathcal{A}_i$ induce a split $O(H \times \hat{H})$-module monomorphism $N \to \text{Res}_{H \times \hat{H}}^{G \times \hat{G}}(M)$. We claim that $K$ stabilizes the image of $N$ in $\text{Res}_{H \times \hat{H}}^{G \times \hat{G}}(M)$. Since $G = H N_G(Q) = H N_G(Q)$, we have $K = (H \times \hat{H}) \Delta(N_G(Q))$. So it suffices to prove that the image of $N$ in $\text{Res}_{H \times \hat{H}}^{G \times \hat{G}}(M)$ is stabilized by $\Delta(N_G(Q))$. Notice that the inclusions $\mathcal{B}_j \subset \mathcal{B}_i$, $\mathbb{A}_j \subset \mathbb{A}_i$ and $\mathcal{B}_j \subset \mathcal{B}_i$, $\mathbb{A}_j \subset \mathbb{A}_i$ induce an $O(H \times \hat{H})$-module isomorphism $\mathcal{B}_j \otimes_{\mathcal{B}_i} \mathbb{B} \cong N$. Given $x \in N_G(Q)$, there is some invertible element $a_x \in (OH)^Q$ such that $x j x^{-1} = a_x j a_x^{-1}$. Therefore $a_x^{-1} x$ centralizes $j$. We adjust the choices of $j$ and $x$ so that $j$ and $x$ correspond to each other through the $\hat{G}$-graded $P$-interior algebra isomorphism $\mathbb{A}_i \cong \mathbb{A}_{\gamma'}$. Thus $a_x^{-1} j x$ corresponds to $j d$ through the isomorphism $\mathbb{A}_i \cong \mathbb{A}_{\gamma'}$, where $d$ is an element in the $x$-component of $\mathbb{A}_{\gamma'}$ such that $j d = d_j$. Then in the module $M$, we have

$x \mathcal{B}_j \otimes j \mathbb{B} x^{-1} = B a_{x^{-1}} j x^{-1} \mathbb{B} = B \otimes j d x^{-1} \mathbb{B} \subset B \otimes j \mathbb{B}$.

Finally we prove that the $\hat{O}K$-module $N$ is a $p$-permutation $\hat{O}K$-module. Since the images of $P$ in $G / H$ and $G / H$ are respectively Sylow $p$-subgroups of $G / H$ and $G / H$ (see [10, Proposition 5.3]), $(H \times \hat{H}) \Delta(P)$ contains a Sylow $p$-subgroup of $K$. So in order to show that the $\hat{O}K$-module $N$ is a $p$-permutation $\hat{O}K$-module, it suffices to show that $\text{Res}_{(H \times \hat{H}) \Delta(P)}^{K}(N)$ is a $p$-permutation $O((H \times \hat{H}) \Delta(P))$-module. Since the isomorphism $\mathbb{A}_i \cong \mathbb{A}_{\gamma'}$ induces a $P$-interior algebra isomorphism $\mathbb{C}_i$ and $\mathbb{C}_{\gamma'}$, it is easy to check that the module $W = O \mathcal{J} \otimes_{\mathbb{C}_i} i \mathcal{J} O$ is a $p$-permutation $O(J \times \mathcal{J})$-module inducing a Morita equivalence between $O \mathcal{J} b$ and $O \mathcal{J} b$. On the other hand, the inclusions $\mathcal{B}_j \subset \mathcal{A}_j$, $i \mathcal{B} \subset i \mathcal{B}$ and $\mathcal{B}_j \subset \mathcal{B}_j$, $i \mathcal{B} \subset i \mathcal{B}$ induce a split $O(H \times \hat{H})$-module monomorphism $M \to \text{Res}_{H \times \hat{H}}^{G \times \hat{G}}(W)$. Then as in the last paragraph, we prove that $\Delta(P)$ stabilizes the image of $N$ in $\text{Res}_{H \times \hat{H}}^{G \times \hat{G}}(W)$. In particular, $N$ can be extended to a $p$-permutation $O((H \times \hat{H}) \Delta(P))$-module. Moreover this extended $O((H \times \hat{H}) \Delta(P))$-module $N$ is the same as $\text{Res}_{(H \times \hat{H}) \Delta(P)}^{K}(N)$ since the inclusions $O \mathcal{J} f \subset A_i$, $i \mathcal{J} \mathcal{O} \subset i \mathcal{J} O$ and $\mathbb{C}_i \subset A_i$ induce an injective $O(J \times \mathcal{J})$-module homomorphism $W \to \text{Res}_{G \times \hat{G}}^{G \times \hat{G}}(M)$. So $\text{Res}_{(H \times \hat{H}) \Delta(P)}^{K}(N)$ is a $p$-permutation $O((H \times \hat{H}) \Delta(P))$-module.

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