A General Method for Constructing Ramanujan Formulas for $1/\pi^\nu$

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Abstract

In this article we give the theoretical background for generating Ramanujan type $1/\pi^\nu$ formulas. As applications of our method we give a general construction of $1/\pi^4$ series and examples of $1/\pi^6$ series. We also study the elliptic alpha function whose values are useful for such evaluations.

keywords $\pi$-formulas; elliptic alpha function; Ramanujan; elliptic functions; singular modulus; approximations; numerical methods

1 Introduction

The standard definitions of the Elliptic Integrals of the first and second kind respectively (see [9],[4],[18]) are:

$$K(x) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - x^2 \sin^2(t)}} \quad \text{and} \quad E(x) = \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2(t)} dt.$$  \hspace{1cm} (1)

$$K(x) = \frac{\pi}{2} x F_1 \left( \frac{1}{2}, \frac{1}{2}, 1; x^2 \right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\left( \frac{1}{2} \right)_n \left( \frac{1}{2} \right)_n}{(1)_n} \frac{x^{2n}}{n!}$$  \hspace{1cm} (2)

In the notation of Mathematica we have

$$K(x) = \text{EllipticK}[x^2] \quad \text{and} \quad E(x) = \text{EllipticE}[x^2].$$  \hspace{1cm} (3)

Also we have (see [7],[9]):

$$K(k) = \frac{dK(k)}{dk} = \frac{E(k)}{k(1 - k^2)} - \frac{K(k)}{k}.$$  \hspace{1cm} (4)

The elliptic singular moduli $k_r$ is defined to be the solution of the equation:

$$\frac{K(\sqrt{1 - w^2})}{K(w)} = \sqrt{r}.$$  \hspace{1cm} (5)

In Mathematica notation

$$w = k = k_r = k|\tau| = \text{InverseEllipticNomeQ}[e^{-\pi \sqrt{\tau}}]^{1/2}.$$  \hspace{1cm} (6)
The complementary modulus is given by \( k_r' = 1 - k_r^2 \). (For evaluations of \( k_r \) see [7],[17],[27])

Also we will need the following relation satisfied by the elliptic alpha function (see [7]):

\[
a(r) = \frac{\pi}{4K(k_r'^2)} - \sqrt{r} \left( E(k_r) \frac{K(k_r)}{K(k_r') - 1} \right) .
\]  

(7)

Our method requires finding derivatives of powers of the elliptic integral \( K \) which always can be expressed in terms of \( K \) and \( E \). Hence from (7) and (4) we need to know \( k_r \) and \( a(r) \). (Here we carry out these evaluations using Mathematica programs.

In the literature (see [7] and Wolfram pages) the function \( a(r) \) is not widely known.

As for the singular moduli, the elliptic alpha function can be evaluated from modular equations. The case of \( a(4r) \) is given in [7] chapter 5:

\[
a(4r) = (1 + k_r)^2 a(r) - 2\sqrt{r} k_r
\]  

(8)

The authors of [22] have proven the formula for \( a(9r) \):

\[
\frac{a(9r)}{\sqrt{r}} - k_r^2 = 1 - k_r k_r' - 3M_{3r}^2 - \frac{1}{3M_{3r}} + \frac{1}{3M_{3r}^2} + \frac{1}{M_{3r}^2} \left( \frac{a(r)}{\sqrt{r}} - \frac{k_r^2}{3} \right)
\]  

(9)

where \( M_{3r} \) is a root of the polynomial equation

\[
27M_{3r}^4 - 18M_{3r}^2 - 8(1 - 2k_r^2)M_{3r} - 1 = 0
\]  

(10)

In Section 2 we review and extend the method for constructing \( \pi^{-n} \) series based on \( a(r) \). In the next section, following Ramanujan (see [4]), we give and prove a formula for the evaluation of \( a(25r) \), when \( a(r) \) is known. We conclude this note with some examples.

2 The general method and the construction of \( 1/\pi^4 \) and \( 1/\pi^6 \) formulas

In Mathematica notation (see [28]):

\[
\phi(z) = \phi_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1; z \right) = \frac{4K^2 (1 - \sqrt{1 - z})}{\pi^2}
\]  

(11)

Let

\[
C_{n,r} = \frac{n!}{(n - r)! r!}
\]

and

\[
F_p(x) := \phi^p(x)
\]  

(12)
and define \(c_p(n)\), with \(p := 2\nu, \nu \in \mathbb{N}\), by

\[
F_p(x) = \left( \sum_{n=0}^{\infty} \frac{C_n^{2\nu}}{64^n} x^n \right)^p = \sum_{n=0}^{\infty} c_p(n)x^n
\]

Also consider the following equation satisfied by the function \(F_{2\nu}(x)\):

\[
A_p x^p F_p(z) + A_{p-1} z^{p-1} F^{(p-1)}(z) + \ldots + A_1 z F^{(1)}(z) + A_0 F^{(0)}(z) = \sum_{n=0}^{\infty} c_{2\nu}(n)z^n \left( C_{2\nu} n^{2\nu} + C_{2\nu-1} n^{2\nu-1} + \ldots + C_1 n + C_0 \right) = \frac{g}{\pi^{2\nu}}
\]

where \(p = 2\nu - 1\).

Set \(z = 1 - (1 - 2w)^2\). Then if \(1 - 2w = k_r^2\) the quantity \(g\) is a function of \(k_r\) and \(a(r)\) and hence \(g\) is algebraic number when \(r \in \mathbb{N}\) and \(A_{2\nu}, A_{2\nu-1}, \ldots, A_1, A_0, C_{2\nu}, C_{2\nu-1}, \ldots C_1, C_0\) can determined from equation (13).

**A series representing \(1/\pi^4\).**

For \(\nu = 2\) we have

\[
A_4 = r^2(1 - 2w)^4(105a(r)^4 - 420a(r)^3\sqrt{r}w + 90a(r)^2rw(-1 + 8w) - 20a(r)r^{3/2}w(1 - 12w + 32 + w^2) + r^2w(-2 + 43w - 192w^2 + 256w^3))^{-1}
\]

\[
A_3 = [2r^{3/2}(1 - 2w)^2(a(r)(5 - 10w) + \sqrt{r}(3 - 23w + 28w^2))]\times
\]

\[
[105a(r)^4 - 420a(r)^3\sqrt{r}w + 90a(r)^2rw(-1 + 8w) - 20a(r)r^{3/2}w(1 - 12w + 32 + w^2) + r^2w(-2 + 43w - 192w^2 + 256w^3)]^{-1}
\]

\[
A_2 = [r(45a(r)^2(1 - 2w)^2 - 30a(r)\sqrt{r}(-1 + 11w - 30w^2 + 24w^3) + r(7 - 140w + 735w^2 - 1400w^3 + 880w^4))]\times
\]

\[
[105a(r)^4 - 420a(r)^3\sqrt{r}w + 90a(r)^2rw(-1 + 8w) - 20a(r)r^{3/2}w(1 - 12w + 32 + w^2) + r^2w(-2 + 43w - 192w^2 + 256w^3)]^{-1}
\]

\[
A_1 = [\sqrt{r}(-210a(r)^3(-1 + 2w) + 90a(r)^2\sqrt{r}(1 - 13w + 20w^2) - 10a(r)r(-2 + 57w - 276w^2 + 304w^3) + r^{3/2}(2 - 115w + 995w^2 - 2640w^3 + 2080w^4))][2(105a(r)^4 - 420a(r)^3\sqrt{r}w + 90a(r)^2rw(-1 + 8w) - 20a(r)r^{3/2}w(1 - 12w + 32 + w^2) + r^2w(-2 + 43w - 192w^2 + 256w^3))]^{-1}
\]
and

$$A_0 = 1$$

The above values of $A_j$ were evaluated with Mathematica on the basis of the first and third term of (13). Setting all the Taylor expansion coefficients of

$$A_p z^p F_p(z) + A_{p-1} z^{p-1} F^{(p-1)}(z) + \ldots + A_1 z F^{(1)}(z) + A_0 F^{(0)}(z),$$

with respect to $K(w)$ to be 0, we arrived at the desired values (we used the relations (4) and (7)). The quantity $C_j$ was evaluated from the first equation of relation (13). The general formula produced by this method for $\frac{1}{\pi^4}$ is

$$\sum_{n=0}^{\infty} c_4(n) (k_r k_r')^{2n} [A_4 n^4 + (A_3 - 6A_4) n^3 + (A_2 - 3A_3 + 11A_4) n^2 +$$

$$+(A_1 - A_2 + 2A_3 - 6A_4) n + A_0] = \frac{g}{\pi^4}$$

$$g = 105\pi^4 (105a(r)^4 - 420a(r)^3 \sqrt{w} + 90a(r)^2 rw(-1 + 8w) -$$

$$-20a(r) r^{3/2} w(1 - 12w + 32w^2) + r^2 w(-2 + 43w - 192w^2 + 256w^3))^{-1}$$

with the above values for $A_4, A_3, A_2, A_1, A_0$ (page 3), where $1 - 2w = k_r^2$. It is worth pointing out that Ramanujan-type formulas of order $\nu \geq 4$ are presented here for the first time. The formulas known previously are of order 1,2,3 (see the references of [28] and Guillera’s Pages on the Web).

**Application**

From Wolfram’s Mathworld pages (see Elliptic Lambda Function) and [7] for $r = 2$ we have $k_2 = \sqrt{2} - 1$ and $a(2) = \sqrt{2} - 1$. Hence we get the formula

$$\sum_{n=0}^{\infty} c_4(n) \left( -56 + 40\sqrt{2} \right)^n [462719 + 5 (292072 + 56267\sqrt{2}) n + 6 (268641 + 81580\sqrt{2}) n^2 +$$

$$+4 (134444 + 32155\sqrt{2}) n^3 - 4 (36209 + 34800\sqrt{2}) n^4] = -\frac{4885495}{(-229441 + 162240\sqrt{2}) \pi^4}$$

where

$$c_4(n) = \sum_{s=0}^{n} c_2(s) c_2(n-s)$$

and

$$c_2(n) = \frac{1}{2^{6n}} \sum_{s=0}^{n} \left( \begin{array}{c} 2s \\ s \end{array} \right)^3 \left( \begin{array}{c} 2n-2s \\ n-s \end{array} \right)^3$$

**Remark.**

One can proceed to $k_r$ with higher values of $r \in \mathbb{N}$ to obtain more rapidly convergent $1/\pi^4$ series.

A series representing $1/\pi^6$

For $\nu = 3$ we get
\[
\sum_{n=0}^{\infty} c_6(n) (k_r k'_r)^2 n [A_6 n^6 + (A_5 - 15 A_6) n^5 + (A_4 - 10 A_5 + 85 A_6) n^4 + \\
(A_3 - 6 A_4 + 35 A_5 - 225 A_6) n^3 + (A_2 - 3 A_3 + 11 A_4 - 50 A_5 + 274 A_6) n^2 + \\
(A_1 - A_2 + 2 A_3 - 6 A_4 + 24 A_5 - 120 A_6) n + A_0] = \frac{9}{\pi^6} \tag{15}
\]

The coefficients \( A_j \) are obtained as in the case \( \nu = 2 \). We present some examples

i) For \( r = 2 \) is

\[
\sum_{n=0}^{\infty} c_6(n) (-56 + 40 \sqrt{2})^n \times \\
\left[ 1 + \frac{(28335058172 - 240070543 \sqrt{2})}{12623771801} n - \frac{(-22911684702 + 3047538900 \sqrt{2})}{12623771801} n^2 - \frac{(1196112280 + 3649618320 \sqrt{2})}{12623771801} n^4 - \frac{(463408744 + 244639040 \sqrt{2})}{3787135403} n^6 \right] = \\
\left( 629823301 - 445352320 \sqrt{2} \right) \pi^6.
\]

ii) For \( r = 7 \) we have \( k_r^2 = \frac{8 - 3 \sqrt{7}}{16} \) and \( a(7) = \frac{\sqrt{7} - 2}{2} \), then

\[
\sum_{n=0}^{\infty} c_6(n) \frac{(-56 + 40 \sqrt{2})^n}{16^n} \times \\
\left[ 1 + \frac{(75313n^2 + 5398980n^3 - 1126755n^4 - 1080450n^5 - 453789n^6)}{34147} \right] = \\
\left( -14417920 \right) \frac{11556387 - 5162500 \sqrt{5}}{34147 \pi^6}.
\]

iii) For \( r = 15, k_r^2 = \frac{(2 - \sqrt{5})^2 (\sqrt{5} - 3) (1 - \sqrt{5})^2}{128} \) and \( a(15) = \frac{\sqrt{15} - \sqrt{5} - 1}{2} \)
then

\[
\sum_{n=0}^{\infty} c_6(n) \left( \frac{47 - 21 \sqrt{5}}{128} \right)^n \times \\
\left[ 1 + \frac{(2871717109830 + 924178552332 \sqrt{5})}{293049243769} n + \frac{(1568959644975 + 6660423786240 \sqrt{5})}{293049243769} n^2 + \frac{(51863088153600 + 23066524139820 \sqrt{5})}{293049243769} n^3 + \right. \\
\left. + \frac{(106483989569175 + 47630637457200 \sqrt{5})}{293049243769} n^4 + \frac{(13026154941675 + 58266415341540 \sqrt{5})}{293049243769} n^5 + \frac{(75619648012725 + 33817435224300 \sqrt{5})}{293049243769} n^6 \right] = \\
\left( 20185088 \right) \frac{11556387 - 5162500 \sqrt{5}}{34147 \pi^6}.
\]
3 A reduction formula for the Elliptic Alpha function \(a(25r)\).

It is clear from the results in section 2 that to obtain rapidly convergent series for \(1/\pi\) and its even powers one requires values of the alpha function \(a(r)\) for large \(r \in \mathbb{N}\), say \(r = 5000\) (see also [29] and [11],[25]). In this section we address this problem and conclude with several examples.

From relations (4),(7) and [4] pages 121-122, chapter 21, if we set 

\[ y = \pi \sqrt{r}, \quad q = e^{-\pi \sqrt{r}}, \quad K(k_r) = K[r], \quad k'_r = \sqrt{1 - k_r^2}, \]

then

\[
1 - 24 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} = \frac{3}{\pi \sqrt{r}} + \left(1 + k_r^2 - \frac{3\alpha(r)}{\sqrt{r}}\right) \frac{4}{\pi^2} K^2[r] \tag{16}
\]

From

\[ k_{r/4} = \frac{2\sqrt{k_r}}{1 + k_r} \quad \text{(17)} \]

and

\[ \alpha(4r) = (1 + k_{4r})^2 \alpha(r) - 2\sqrt{r} k_{4r} \quad \text{(18)} \]

relation (16) becomes

\[
1 - 24 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} = \frac{6}{\pi \sqrt{r}} + \frac{4K^2[r](-6\alpha(r) + \sqrt{r}(1 + k_r^2))}{\pi^2 \sqrt{r}} \tag{19}
\]

If we set

\[ T_{p,r} := \left(1 - 24 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}}\right) - p \left(1 - 24 \sum_{n=1}^{\infty} \frac{nq^{2pn}}{1 - q^{2pn}}\right) \tag{20} \]

then

**Proposition 1.**

\[
\frac{1}{m_p^2 \sqrt{r}} \alpha(p^2r) = \frac{-(1 + k_{p^2r}^2)}{3} + \frac{p(1 + k_{p^2r}^2)}{3m_p^2} + \frac{\pi^2 T_{p,r}}{12K^2[r]} + \frac{\alpha(r)}{\sqrt{r}} \tag{21}
\]

The above Proposition 1 relates results of Ramanujan in [4] chapter 21 with the evaluation of the alpha function and the evaluations of \(\pi\). Solving (21) for \(T_{p,r}\) we have

\[ T_{p,r} = \frac{4K[r]^2}{\pi^2 \sqrt{r} m_p^2} \left[3\alpha(p^2r) - p\sqrt{r}(1 + k_{p^2r}^2) + (-3\alpha(r) + \sqrt{r}(1 + k_r^2))m_p^2 \right] \tag{22}\]

Hence from (19),(20) and (22) we get another interesting formula

\[
1 - 24 \sum_{n=1}^{\infty} \frac{nq^{2pn}}{1 - q^{2pn}} = \frac{3}{\pi \sqrt{rp}} + \frac{4K[r]^2}{\pi^2 \sqrt{r} pm_p^2} \left[-3\alpha(p^2r) + p\sqrt{r}(1 + k_{p^2r}^2)\right] \tag{23}\]
where
\[ m_p = \frac{K[r]}{K[r^2]} \]  

(24)

From [4] page 463 Entry 4 we have
\[ T_{5,r} = \frac{(x^2 + 22 \cdot q^2 xy + 125 \cdot q^4 y^2)^{1/2}}{(xy)^{1/6}}, \]  

(25)

where \( x = f^6(-q^2) \) and \( y = f^6(-q^{10}). \)

From Ramanujan’s identity
\[ A_r := \frac{1}{R^5(q^2)} - 11 - R^5(q^2) = \frac{x}{q^2 y}, \]  

(26)

where \( R(q) \) is the Rogers-Ramanujan continued fraction
\[ R(q) = \frac{q^{1/5} \cdot q \cdot q^2 \cdot q^3}{1 + 1 + 1 + 1 + \ldots} \]

and
\[ f(-q^2)^6 = \frac{2k_r k'_r K[r]^3}{\pi^3 q^{1/2}} \]  

(27)

we get
\[ T_{5,r} = 4 \left( \frac{(k_r k'_r)^{2/3} \sqrt{125 + 22A_r + A_r^2}}{6 \cdot 2^{1/3} A_r^{5/6}} \right) = 2^{2/3} \left( \frac{(k_r k'_r)^{2/3} [R^5(q^2) + R^5(q^2)]}{3 [R^5(q^2) - 11 - R^5(q^2)]^{5/6}} \right) \]  

(28)

and hence the evaluation
\[ -4 - 24 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} + 120 \sum_{n=1}^{\infty} \frac{nq^{10n}}{1 - q^{10n}} = 2^{2/3} \left( \frac{(k_r k'_r)^{2/3} [R^5(q^2) + R^5(q^2)]}{3 [R^5(q^2) - 11 - R^5(q^2)]^{5/6}} \right) \]  

(29)

But for the evaluation of the Rogers-Ramanujan continued fraction from [26] we have:

**Proposition 2.** (see [30])

If \( q = e^{-\pi \sqrt{r}} \) and \( r \) real positive, then
\[ A_r = a_{4r} = \left( k_r k'_r \right)^{2/3} \left( \frac{w_r}{k_r} + \frac{w'_r}{k'_r} - \frac{w_r w'_r}{k_r k'_r} \right)^3 \]  

(30)

with
\[ m_5 = \frac{w_r}{k_r} + \frac{w'_r}{k'_r} - \frac{w_r w'_r}{k_r k'_r} \] and \( w_r = \sqrt{k_r k_{25r}}, \ w'_r = \sqrt{k'_r k'_{25r}} \)

Hence we get

**Proposition 3.**
\[ \frac{3\alpha(25r)}{m_5 \sqrt{r}} - \frac{3\alpha(r)}{\sqrt{r}} = \]
\[ \frac{5}{m_5^2} (1 + k_{25r}^2) - (1 + k_r^2) - 2^{2/3} A_r^{-5/6} (k_r k_r')^{2/3} [R^5 (q^2) + R^{-5} (q^2)] \]  \hfill (32)

**Proof.**

From (21), (28) is

\[ \frac{3\alpha (25r)}{m_5^2 \sqrt{r}} - \frac{3\alpha (r)}{\sqrt{r}} = -1 + \frac{5}{m_5^2} + \frac{5k_{25r}^2}{m_5^2} - k_r^2 - \frac{2^{2/3} \sqrt{125 + 22 A_r + A_r^7 (k_r k_r')^{2/3}}}{A_r^{5/6}} \]  \hfill (33)

with

\[ A_r/4 = \left( \frac{k_r'}{k_{25r}} \right)^2 \sqrt{\frac{k_r}{k_{25r}}} m_5^{-3} \]  \hfill (34)

The multiplier \( m_5 \) is that of (31) and satisfies also the equation

\[ (5m_5 - 1)^5 (1 - m_5) = 256k_r^2 k_r'^2 m_5. \]  \hfill (35)

The next proposition is a conjecture which is most compactly expressed in terms of the quantity

\[ Y_{\sqrt{-r}} = \frac{1}{6} \left( R \left( e^{-2\pi \sqrt{r}} \right)^{-5} - 11 - R \left( e^{-2\pi \sqrt{r}} \right)^5 \right) = \frac{A_r}{6} : (a) \]

The function \( j_r \) which appears is the \( j \)-invariant (see [6],[17]). For more properties of \( j_r \) and \( A_r \) see [26]:

**Proposition 4.**

As indicated by numerical results, whenever \((r, 5) = \text{GCD}(r, 5) = 1\), then \( \text{deg} \left( Y_{\sqrt{-r/5}} \right) = \text{deg} \left( j_{\sqrt{-r/5}} \right) \).

For a given \( r \in \mathbb{N} \) and \( \text{deg} \left( Y_{\sqrt{-r/5}} \right) = 2, 4, \) or 8, if the smallest nested root of \( j_{\sqrt{-r/5}} \) is \( \sqrt{d} \) then we can evaluate the Rogers-Ramanujan continued fraction with integer parameters.

i) In the case \( \text{deg} \left( Y_{\sqrt{-r/5}} \right) = 2 \) then

\[ Y_{\sqrt{-r/5}} = \frac{l + m \sqrt{d}}{l} \]

with

\[ l^2 - m^2 d = 1 \]

ii) In the case \( \text{deg} \left( Y_{\sqrt{-r/5}} \right) = 4 \) we have

a) If \( U \neq \frac{125}{64} \), then

\[ Y_{\sqrt{-r/5}} = \frac{5}{8} \sqrt{a_0 + \sqrt{-1 + a_0^2}} \left( \sqrt{5 + p - \sqrt{p}} \right) \]  \hfill (36)
where
\[
Y^{-r/5}Y^{*} = \frac{125}{64} \left( a_0 + \sqrt{a_0^2 - 1} \right),
\]
(37)
where \(a_0\) is the positive integer-solution of \(l^2 - m^2d = 1\). Hence \(l = a_0\) and
\(m = d^{-1/2}\sqrt{a_0^2 - 1}\) is a positive integer. The parameter \(p\) is positive rational and can be found directly from the numerical value of \(Y^{-r/5}\).

b) If \(U = \frac{125}{64}\), then
\[
Y^{-r/5} = A + \frac{1}{8} \sqrt{-125 + 64A^2},
\]
(38)
where we set \(A = k + l\sqrt{a}\). Hence a starting point for the evaluation of the integers \(k, l\) is the relation
\[
l^2 = \frac{(A - k)^2}{d} = \text{square of an integer}
\]

iii) If \(\deg \left( Y^{-r_{4-15-1}} \right) = 4\), then we can evaluate \(Y^{-r_{5-1}}\).

If \(\deg \left( Y^{-r_{5-1}} \right) = 8\), the minimal polynomial of \(Y^{-r_{5-1}}/Y^{-r_{4-15-1}}\) is of degree 4 or 8 and is symmetric. Hence it can be reduced to a polynomial of degree at most 4, and hence it is solvable. Thus it remains to evaluate \(Y^{-r_{4-15-1}}\), which can be done with the help of step (ii).

\[
Y^{-r_{5-1}} = \frac{5}{8} \sqrt{a_0 + \sqrt{-1 + a_0^2} \left( \sqrt{p + 5} - \sqrt{p} \right)} 2^{-1} \left( \sqrt{x + 4} - \sqrt{x} \right)
\]
(39)
where \(x = a_1 + b_1\sqrt{d} + c \sqrt{a_2 + b_2\sqrt{d}}\), \(a_1, b_1, a_2, b_2, c\) integers and
\[
Y^{-r_{5-14-1}} = \frac{5}{8} \sqrt{a_0 + \sqrt{-1 + a_0^2} \left( \sqrt{p + 5} - \sqrt{p} \right)}
\]

We give some values of \(Y^{-r/5} = 8^{-1}Ar/5\).

\[
Y^{-1/5} = \frac{5\sqrt{5}}{8}
\]
(40)
\[
Y^{-2/5} = \frac{5}{8} \left( 5 + 2\sqrt{5} \right)
\]
(41)
\[
Y^{-3/5} = \frac{5}{16} \left( 25 + 11\sqrt{5} \right)
\]
(42)
\[
Y^{-4/5} = \frac{5}{16} \left( 25 + 13\sqrt{5} + 5\sqrt{58 + 26\sqrt{5}} \right)
\]
(43)
\[
Y^{-5/5} = \frac{125}{8} \left( 2 + \sqrt{5} \right)
\]
(44)
\[ Y_{\sqrt{-6/5}} = \frac{5}{8} \left( 50 + 35\sqrt{2} + 3\sqrt{5 \left( 99 + 70\sqrt{2} \right)} \right) \] (45)

\[ Y_{\sqrt{-9/5}} = \frac{5}{8} \left( 225 + 104\sqrt{5} + 10\sqrt{1047 + 468\sqrt{5}} \right) \] (46)

\[ Y_{\sqrt{-12/5}} = \frac{5}{16} \left( 1690 + 975\sqrt{3} + 29\sqrt{6755 + 3900\sqrt{3}} \right) \] (47)

\[ Y_{\sqrt{-14/5}} = \frac{5}{8} \left( 1850 + 585\sqrt{10} + 7\sqrt{5 \left( 27379 + 8658\sqrt{10} \right)} \right) \] (48)

\[ Y_{\sqrt{-17/5}} = \frac{5}{8} \left( 5360 + 585\sqrt{85} + 4\sqrt{3613670 + 391950\sqrt{85}} \right) \] (49)

Example.
For \( r = 68 = 4 \cdot 17 \) and from (49) we have \( d = 85 \)

\[ x = a_1 + b_1\sqrt{85} + c\sqrt{a_2 + b_2\sqrt{85}} \]

\[ Y_{\sqrt{-68/5}}/Y_{\sqrt{-17/5}} = 2^{-1} \left( \sqrt{x + 4} - \sqrt{x} \right) \]
\[ a_1 = 2891581250, \; b_1 = 313636050, \; c = 12960 \]
\[ a_2 = 99557521554, \; b_2 = 10798529365 \]

hence

\[ Y_{\sqrt{-68/5}} = Y_{\sqrt{-17/5}} 2^{-1} \left( \sqrt{x + 4} - \sqrt{x} \right) = \]

\[ = \frac{5}{16} \left( 5360 + 585\sqrt{85} + 4\sqrt{3613670 + 391950\sqrt{85}} \right) \left( \sqrt{x + 4} - \sqrt{x} \right) \]

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