FRACTIONAL CALCULUS AND TIME-FRACTIONAL DIFFERENTIAL EQUATIONS: REVISIT AND CONSTRUCTION OF A THEORY

1,2,3 M. YAMAMOTO

Abstract. For fractional derivatives and time-fractional differential equations, we construct a framework on the basis of the operator theory in fractional Sobolev spaces. Our framework provides a feasible extension of the classical Caputo and the Riemann-Liouville derivatives within Sobolev spaces of fractional orders including negative ones. Our approach enables a unified treatment for fractional calculus and time-fractional differential equations. We formulate initial value problems for fractional ordinary differential equations and initial boundary value problems for fractional partial differential equations to prove the well-posedness and other properties.

Key words. fractional calculus, time-fractional differential equations, fractional Sobolev spaces, operator theory

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dedicated to the memory of Professor Dr. Rudolf Gorenflo who invited the author to studies of fractional calculus

1 Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Meguro, Tokyo 153-8914, Japan
2 Honorary Member of Academy of Romanian Scientists, Ilfov, nr. 3, Bucuresti, Romania
3 Correspondence Member of Accademia Peloritana dei Pericolanti, Palazzo Università, Piazza S. Pugliatti 1 98122 Messina, Italy
4 Peoples’ Friendship University of Russia (RUDN University) 6 Mikhukho-Maklaya St, Moscow, 117198, Russian Federation

e-mail: myama@ms.u-tokyo.ac.jp.
1. Motivations

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial \Omega$, and let $\nu = \nu(x)$ be the unit outward normal vector to $\partial \Omega$ at $x \in \partial \Omega$. We set

$$-Au(x) = \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j v(x)) + \sum_{j=1}^{d} b_j(x)\partial_j v(x) + c(x)v(x), \quad x \in \Omega, \; 0 < t < T,$$

(1.1)

where $a_{ij} = a_{ji}, b_j, c \in C^1(\Omega)$, $1 \leq i, j \leq d$ and we assume that there exists a constant $\kappa > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \kappa \sum_{j=1}^{d} |\xi_j|^2, \quad x \in \Omega, \; \xi_1, ..., \xi_d \in \mathbb{R}.$$

Henceforth $\Gamma(\gamma)$ denotes the gamma function for $\gamma > 0$: $\Gamma(\gamma) := \int_{0}^{\infty} e^{-t} t^{\gamma-1} dt$.

Our eventual purpose is to construct a theoretical framework for treating initial boundary value problems for time fractional diffusion equations with source term $F(x, t)$, which can be described for the case of $0 < \alpha < 1$:

$$\begin{cases}
\partial_t^\alpha u(x, t) = -Au(x, t) + F(x, t), & x \in \Omega, \; 0 < t < T, \\
u(x, 0) = 0, & x \in \Omega, \\
u(x, t) = 0, & x \in \partial \Omega, \; 0 < t < T.
\end{cases}$$

(1.2)

Here, for $0 < \alpha < 1$, we can formally define the pointwise Caputo derivative:

$$\partial_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} (t-s)^{-\alpha} \frac{dv}{ds}(s)ds,$$

(1.3)

as long as the right-hand side exists.

As for classical treatments on fractional calculus and equations, we can refer to monographs Gorenflo, Kilbas, Mainardi and Rogosin [6], Kilbas, Srivastava and Trujillo [16], Podlubny [30] for example.

Henceforth let $L^p(0, T) := \{v; \int_{0}^{T} |v(t)|^p dt < \infty\}$ with $p \geq 1$ and $W^{1,1}(0, T) := \{v \in L^1(0, T); \frac{dv}{dt} \in L^1(0, T)\}$. We define the norm by

$$\|v\|_{L^p(0, T)} := \left( \int_{0}^{T} |v(t)|^p dt \right)^{\frac{1}{p}}, \quad \|v\|_{W^{1,1}(0, T)} := \|v\|_{L^1(0, T)} + \left\| \frac{dv}{dt} \right\|_{L^1(0, T)}.$$

In view of the Young inequality on the convolution, we can directly verify that the classical Caputo derivative (1.3) can be well-defined for $v \in W^{1,1}(0, T)$ and $\partial_t^\alpha v \in L^1(0, T)$. However,
everything is not clear for \( v \notin W^{1,1}(0,T) \), and it is not a feasible assumption that any solutions to (1.2) have the \( W^{1,1}(0,T) \)-regularity in \( t \). The Young inequality is well-known and we can refer to e.g., Lemma A.1 in Appendix in Kubica, Ryszewska and Yamamoto [18].

In (1.3), the pointwise Caputo derivative \( d_t^\alpha \) requires the first-order derivative \( \frac{dv}{ds} \) in any sense. Therefore, in order to discuss \( d_t^\alpha v \) for a function \( v \) which keeps apparently reasonable regularity such as "\( \alpha \)-times” differentiability, we should formulate \( d_t^\alpha v \) in a suitable distribution space. Moreover, such a formulation is not automatically unique. There have been several works for example, Kubica, Ryszewska and Yamamoto [18], Kubica and Yamamoto [19], Zacher [37]. Here we restrict ourselves to an extremely limited number of references. In this article, we extend the approach in Kubica, Ryszewska and Yamamoto [18] to discuss fractional derivatives in fractional Sobolev spaces of arbitrary real number orders and construct a convenient theory for initial value problems and initial boundary value problems for time-fractional differential equations. In [18], the orders \( \alpha \) is restricted to \( 0 < \alpha < 1 \), but here we study fractional orders \( \alpha \in \mathbb{R} \).

The main purpose of this article is to define such fractional derivatives and establish the framework which enables us for example, to uniformly consider weaker and stronger solutions with exact specification of classes of solutions in terms of fractional Sobolev spaces. Thus we intend to construct a comprehensive theory for time-fractional differential equations within Sobolev spaces.

In (1.3), we take the first-order derivative and then \( (1-\alpha) \)-times integral operator : 
\[
\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \cdots ds
\]
finally reach \( \alpha \)-times derivative. As is described in Section 2, our strategy for the definition of the fractional derivative in \( t \), is to consider \( \alpha \)-times derivative of \( v \) as the inverse to the \( \alpha \)-times integral, not via \( \frac{dv}{dt} \).

Moreover, for initial value problems for time-fractional differential equations, we meet other complexity for how to pose initial condition, as the following examples show. **Example 1.1.**

We consider an initial value problem for a simple time-fractional ordinary differential equation:

\[
d_t^\alpha v(t) = f(t), \quad 0 < t < T, \quad v(0) = a.
\]

(1.4)

We remark that (1.4) is not necessarily well-posed for all \( \alpha \in (0,1) \), \( f \in L^2(0,T) \) and \( a \in \mathbb{R} \), because of the initial condition. Especially the order \( \alpha \in \left(0, \frac{1}{2}\right) \) causes a difficulty: Choosing
\[ \gamma > -1 \text{ and} \]
\[ f(t) = t^\gamma \in L^1(0,T), \]
in (1.4), we consider
\[ d_1^\alpha v(t) = t^\gamma, \quad v(0) = a. \quad (1.5) \]
A solution formula is known:
\[ v(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^\gamma ds \quad (1.6) \]
(e.g., (3.1.34) (p.141) in Kilbas, Srivastava and Trujillo [16]). For \( 0 < \alpha < 1 \) and \( \gamma > -1 \), the right-hand side of (1.6) makes sense and we can obtain
\[ v(t) = a + \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)} t^{\alpha+\gamma}, \quad t > 0. \quad (1.7) \]
However, the initial condition \( v(0) = a \) is delicate:
(i) Let \( \alpha + \gamma > 0 \). Then we can readily see that \( v \) given by (1.7) satisfies (1.5).
(ii) Let \( \alpha + \gamma = 0 \). Then (1.7) provides that \( v(t) = a + \Gamma(1-\alpha) \). However this \( v \) does not satisfy \( d_1^\alpha v = t^{-\alpha} \).
(iii) Let \( \alpha + \gamma < 0 \). Then for \( v(t) \) defined by (1.7), we see that \( \lim_{t \downarrow 0} v(t) = \infty \). Therefore (1.7) cannot give a solution to (1.5), and moreover we are not sure whether there exists a solution to (1.5).

We have another classical fractional derivative called the Riemann-Liouville derivative:
\[ D_1^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} v(s) ds, \quad 0 < t < T \quad (1.8) \]
for \( 0 < \alpha < 1 \), provided that the right-hand side exists.
Let \( f \in L^2(0,T) \) be arbitrarily given. In terms of \( D_1^\alpha \), we can formulate an initial value problem:
\[ D_1^\alpha u(t) = f(t), \quad 0 < t < T, \quad (J^{1-\alpha} u)(0) = a. \quad (1.9) \]
The solution formula is
\[ u(t) = \frac{a}{\Gamma(\alpha)} t^{\alpha-1} + J^\alpha f(t), \quad 0 < t < T \quad (1.10) \]
(e.g., Kilbas, Srivastava and Trujillo [16], p.138). Indeed, we can directly verify that $u$ given by (1.10) satisfies (1.9):

$$D_t^{\alpha}J^\alpha f(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} \left( \int_0^s (s-\xi)^{\alpha-1} f(\xi) d\xi \right) ds,$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t f(\xi) \left( \int_0^\xi (\xi-s)^{-\alpha}(s-\xi)^{\alpha-1} ds \right) d\xi,$$

$$= \frac{d}{dt} \int_0^t f(\xi) d\xi = f(t) \quad \text{for } f \in L^1(0,T).$$

and

$$D_t^{\alpha} \left( \frac{a}{\Gamma(\alpha)} t^{\alpha-1} \right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} s^{\alpha-1} ds = 0, \quad 0 < t < T,$$

which means that $D_t^{\alpha}u(t) = f(t)$ for $0 < t < T$. We can similarly verify

$$J^{1-\alpha} \left( \frac{a}{\Gamma(\alpha)} t^{\alpha-1} + J^\alpha f(t) \right) (0) = a.$$

By $0 < \alpha < \frac{1}{2}$, the formula (1.10) makes sense in $L^1(0,T)$, but $u \not\in L^2(0,T)$ for each $f \in L^2(0,T)$ if $a \neq 0$. This suggests that the initial value problem (1.9) may not be well-posed in $L^2(0,T)$.

Furthermore we can consider other formulation:

$$D_t^{\alpha}u(t) = f(t), \quad 0 < t < T, \quad u(0) = a. \quad (1.11)$$

We see that $u(t)$ given by (1.10) satisfies $D_t^{\alpha}u = f$ in $(0,T)$, but it follows from $\alpha - 1 < 0$ that $\lim_{t \downarrow 0} |u(t)| = \infty$ by $\lim_{t \downarrow 0} t^{\alpha-1} = \infty$ if $a \neq 0$, while if $a = 0$, then (1.10) can satisfy $u(0) = 0$, that is, (1.11) by $f$ in some class such as $f \in L^\infty(0,T)$.

The delicacy in the above examples are more or less known, and motivates us to construct a uniform framework for time-fractional derivatives. Moreover, we are concerned with the range space of the corresponding solutions for a prescribed function space of $f$, e.g., $L^2(0,T)$, not only calculations of the derivatives of individually given function $u$. Naturally, for the well-posedness of an initial value problem, we are required to characterize the function space of solutions corresponding to a space of $f$.

In other words, one of our main interests is to define a fractional derivative, denoted by $\partial_t^{\alpha}$, and characterize the space of $u$ satisfying $\partial_t^{\alpha}u \in L^2(0,T)$. Moreover, such $\partial_t^{\alpha}$ should be
an extension of \( d_t^\alpha \) and \( D_t^\alpha \) in a minimum sense in order that important properties of these classical fractional derivatives should be inherited to \( \partial_t^\alpha \).

Throughout this article, we treat fractional integrations and fractional derivatives as operators from specified function spaces to others, that is, we always attach them with their domains and ranges.

Thus we will define a time fractional derivative denoted by \( \partial_t^\alpha \) as suitable extension of \( d_t^\alpha \) satisfying the requirements:

- Such extended derivative \( \partial_t^\alpha \) admits usual rules of differentiation as much as possible. For example, \( \partial_t^\alpha \partial_t^\beta = \partial_t^{\alpha+\beta} \) for all \( \alpha, \beta \geq 0 \).
- It admits a relevant formulation of initial condition even for \( \alpha \in \mathbb{R} \).
- In \( L^2 \)-based Sobolev spaces, there exists a unique solution to an initial value problem for a time-fractional ordinary differential equation and an initial boundary value problem for a time-fractional partial differential equation for \( \alpha > 0 \) and even \( \alpha \in \mathbb{R} \).

In this article, we intend to outline foundations for a comprehensive theory for time-fractional differential equations. Some arguments are based on Gorenflo, Luchko and Yamamoto [7], Kubica, Ryszewska and Yamamoto [18].

This article is composed of eight sections and one appendix:

- Section 2: Definition of the extended derivative \( \partial_t^\alpha \):
  We extend \( d_t^\alpha \) as operator in order that it is well-defined as an isomorphism in relevant Sobolev spaces. We emphasize that our fractional derivative coincides with the classical Riemann-Liouville derivative and the Caputo derivative in suitable spaces, and we never aim at creating novel notions of fractional derivatives but we are concerned with a formulation of a fractional derivative allowing us convenient applications to time-fractional differential equations, as Sections 5 and 6 discuss.
- Section 3: Basic properties in fractional calculus
- Section 4: Fractional derivatives of the Mittag-Leffler functions
- Section 5: Initial value problem for fractional ordinary differential equations
- Section 6: Initial boundary value problem for fractional partial differential equations: selected topics
- Section 7: Application to an inverse source problem:
  For illustrating the feasibility of our approach, we consider one inverse source problem of determining a time-varying function.
- Section 8: Concluding remarks.

2. Definition of the extended derivative $\partial_t^\alpha$

§2.1. Introduction of function spaces and operators

We set

$$
\begin{align*}
(J^\alpha v)(t) := & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds, & 0 < t < T, & \mathcal{D}(J^\alpha) = L^2(0, T), \\
(J_\alpha v)(t) := & \frac{1}{\Gamma(\alpha)} \int_t^T (\xi - t)^{\alpha-1} v(\xi) d\xi, & 0 < t < T, & \mathcal{D}(J_\alpha) = L^2(0, T).
\end{align*}
$$

We can consider $J^\alpha, J_\alpha$ for $v \in L^1(0, T)$, but for the moment, we consider them for $v \in L^2(0, T)$. There are many works on the Riemann-Liouville time-fractional integral operator $J^\alpha$, and we here refer only to Gorenflo and Vessella [11] as monograph, and Gorenflo and Yamamoto [12] for an operator theoretical approach.

We can directly prove

**Lemma 2.1.**

Let $\alpha, \beta \geq 0$. Then

$$
J^\alpha J^\beta v = J^{\alpha+\beta} v, \quad J_\alpha J_\beta v = J_{\alpha+\beta} v \quad \text{for } v \in L^1(0, T).
$$

For $0 < \alpha < 1$, we define an operator $\tau : L^2(0, T) \rightarrow L^2(0, T)$ by

$$
(\tau v)(t) := v(T - t), \quad v \in L^2(0, T).
$$

Then it is readily seen that $\tau$ is an isomorphism between $L^2(0, T)$ and itself.

Throughout this article, we call $K$ an isomorphism between two Banach spaces $X$ and $Y$ if the mapping $K$ is injective, and $KX = Y$ and there exists a constant $C > 0$ such that $C^{-1} ||Kv||_Y \leq ||v||_X \leq C ||Kv||_Y$ for all $v \in X$. Here $|| \cdot ||_X$ and $|| \cdot ||_Y$ denote the norms in $X$ and $Y$ respectively.

We set

$$
0C^1[0,T] := \{ v \in C^1[0,T]; v(0) = 0 \},
$$

$$
^0C^1[0,T] := \{ v \in C^1[0,T]; v(T) = 0 \} = \tau(0C^1[0,T]).
$$
By $H^\alpha(0,T)$ with $0 < \alpha < 1$, we denote the Sobolev-Slobodecki space with the norm $\| \cdot \|_{H^\alpha(0,T)}$ defined by

$$\|v\|_{H^\alpha(0,T)} := \left( \|v\|^2_{L^2(0,T)} + \int_0^T \int_0^T \frac{|v(t) - v(s)|^2}{|t-s|^{1+2\alpha}} \, dt \, ds \right)^{\frac{1}{2}}$$

(e.g., Adams [1]).

By $\overline{X}$ we denote the closure of $X \subset Y$ in a normed space $Y$. We set

$$H^\alpha(0,T) := \overline{C^1[0,T]}^{H^\alpha(0,T)}, \quad H^\alpha(0,T) := \overline{C^1[0,T]}^{H^\alpha(0,T)}.$$  \hspace{2cm} (2.3)

Henceforth we set $H_0(0,T) := L^2(0,T)$ and $H^0(0,T) := L^2(0,T)$.

Then

**Proposition 2.1.**
Let $0 < \alpha < 1$.

(i)

$$H_\alpha(0,T) := \begin{cases} H^\alpha(0,T), & 0 < \alpha < \frac{1}{2}, \\ \{ v \in H^{\frac{1}{2}}(0,T); \int_0^T \frac{|v(t)|^2}{t} \, dt < \infty \}, & \alpha = \frac{1}{2}, \\ \{ v \in H^\alpha(0,T); v(0) = 0 \}, & \frac{1}{2} < \alpha \leq 1. \end{cases}$$

Moreover, the norm in $H_\alpha(0,T)$ is equivalent to

$$\|v\|_{H_\alpha(0,T)} := \begin{cases} \|v\|_{H^\alpha(0,T)}, & \alpha \neq \frac{1}{2}, \\ \left( \|v\|^2_{H^{\frac{1}{2}}(0,T)} + \int_0^T \frac{|v(t)|^2}{t} \, dt \right)^{\frac{1}{2}}, & \alpha = \frac{1}{2}. \end{cases}$$

(ii) Similarly we have

$$H^\alpha(0,T) := \begin{cases} H^\alpha(0,T), & 0 < \alpha < \frac{1}{2}, \\ \{ v \in H^{\frac{1}{2}}(0,T); \int_0^T \frac{|v(t)|^2}{T-t} \, dt < \infty \}, & \alpha = \frac{1}{2}, \\ \{ v \in H^\alpha(0,T); v(T) = 0 \}, & \frac{1}{2} < \alpha \leq 1. \end{cases}$$

Moreover, the norm in $H^\alpha(0,T)$ is equivalent to

$$\|v\|_{H^\alpha(0,T)} := \begin{cases} \|v\|_{H^\alpha(0,T)}, & \alpha \neq \frac{1}{2}, \\ \left( \|v\|^2_{H^{\frac{1}{2}}(0,T)} + \int_0^T \frac{|v(t)|^2}{T-t} \, dt \right)^{\frac{1}{2}}, & \alpha = \frac{1}{2}. \end{cases}$$
We can refer to [18] as for the proof of Proposition 2.1 (i). Since $\tau : H_\alpha(0, T) \rightarrow {}^\alpha H(0, T)$ is an isomorphism, we can verify part (ii) of the proposition.

§2.2. Extension of $d_\alpha t$ to $H_\alpha(0, T)$: intermediate step

We can prove (e.g., [7], [18]):

**Proposition 2.2.**

Let $0 < \alpha < 1$. Then

$$J^\alpha : L^2(0, T) \rightarrow H_\alpha(0, T)$$

is an isomorphism. In particular, $H_\alpha(0, T) = J^\alpha L^2(0, T)$.

By the proposition, we can easily verify that some functions belong to $H_\alpha(0, T)$, although the verification by (2.3) is complicated.

**Example 2.1.**

In view of Proposition 2.2, we will verify that

$$t^\beta \in H_\alpha(0, T) \quad \text{if } \beta > \alpha - \frac{1}{2}.$$ 

Indeed, setting $\gamma := \beta - \alpha$, we see $\gamma > -\frac{1}{2}$, and so $t^\gamma \in L^2(0, T)$. Moreover,

$$J^\alpha t^\gamma = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} s^\gamma ds = \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)} t^{\alpha+\gamma},$$

that is,

$$t^\beta = t^{\alpha+\gamma} = J^\alpha \left( \frac{\Gamma(\alpha + \gamma + 1)}{\Gamma(\gamma + 1)} t^\gamma \right) \in J^\alpha L^2(0, T).$$

Thus we see that $t^\beta \in H_\alpha(0, T)$.

In Proposition 2.4 in [18], the direct proof for $t^\beta \in H_\alpha(0, T)$ is given for more restricted $\beta > 0$ and $0 < \alpha < \frac{1}{2}$, which is more complicated than the proof here.

We set

$$\partial^\alpha_t := (J^\alpha)^{-1} = J^{-\alpha} \quad \text{with} \quad D(\partial^\alpha_t) = H_\alpha(0, T) = J^\alpha L^2(0, T). \quad (2.4)$$

The essence of this extension of $\partial^\alpha_t$ is that we define $\partial^\alpha_t$ as the inverse to the isomorphism $J^\alpha$ on $L^2(0, T)$ onto $H_\alpha(0, T)$. For estimating or treating $\partial^\alpha_t v = g$ later, we will often consider
through $J^\alpha g$, as is already calculated in Example 2.1.

From Proposition 2.2 we can directly derive

**Proposition 2.3.**

There exists a constant $C > 0$ such that

$$C^{-1} \| v \|_{H_\alpha(0, T)} \leq \| \partial_t^\alpha v \|_{L^2(0, T)} \leq C \| v \|_{H_\alpha(0, T)}$$

for all $v \in H_\alpha(0, T)$.

**Example 2.2.**

We return to Example 2.1. Let $\beta > \alpha - \frac{1}{2}$. Then $t^\beta \in H_\alpha(0, T)$ and

$$J^{\alpha t^\beta - \alpha} = \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta + 1)} t^\beta.$$ 

Hence, by the definition of $\partial_t^\alpha$, we obtain

$$t^{\beta - \alpha} = \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta + 1)} \partial_t^\alpha t^\beta,$$

that is,

$$\partial_t^\alpha t^\beta = \frac{(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha} \text{ if } \beta > \alpha - \frac{1}{2} \text{ and } 0 < \alpha < 1.$$  

(2.5)

If $0 < \alpha < \frac{1}{2}$, then $\beta < 0$ is possible and for $\beta < 0$ we have $t^\beta \not\in W^{1,1}(0, T)$. Therefore $d_t^\alpha t^\beta$ cannot be defined directly.

Next we will define $\partial_t^\alpha$ for general $\alpha > 0$. Let $\alpha := m + \sigma$ with $m \in \mathbb{N} \cup \{0\}$ and $0 < \sigma \leq 1$.

Then we define

$$H_m(0, T) := \begin{cases} 
   \{ v \in H^m(0, T); \; v(0) = \cdots = \frac{d^{m-1}v}{dt^{m-1}}(0) = 0 \}, & m \geq 1, \\
   L^2(0, T), & m = 0,
\end{cases}$$

$$H_{m+\sigma}(0, T) := \{ v \in H_m(0, T); \; \frac{d^m v}{dt^m} \in H_\sigma(0, T) \}$$

and

$$\| v \|_{H_{m+\sigma}(0, T)} := \left( \left\| \frac{d^m v}{dt^m} \right\|_{L^2(0, T)}^2 + \left\| \frac{d^m v}{dt^m} \right\|_{H_\sigma(0, T)}^2 \right)^{\frac{1}{2}}.$$ 

We can easily verify that $H_{m+\sigma}(0, T)$ is a Banach space.
We similarly define $\alpha H(0, T)$ for each $\alpha > 0$.

Then, in view of Proposition 2.2, we can prove

**Proposition 2.4.**

Let $m \in \mathbb{N} \cup \{0\}$ and $0 < \sigma \leq 1$. Then $J^{m+\sigma} : L^2(0, T) \rightarrow H_{m+\sigma}(0, T)$ is an isomorphism.

**Proof.**

We can assume that $m \geq 1$. By the definition, $v \in H_{m+\sigma}(0, T)$ if and only if

$$\frac{d^m}{dt^m} v \in H_{\sigma}(0, T), \quad v(0) = \cdots = \frac{d^{m-1}}{dt^{m-1}} v(0) = 0,$$

which implies that $\frac{d^m}{dt^m} v = J^\sigma w$ with some $w \in L^2(0, T)$. By $v(0) = \cdots = \frac{d^{m-1}}{dt^{m-1}} v(0) = 0$, we see that

$$v(t) = \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \frac{d^m}{dt^m} v(s) ds, \quad 0 < t < T.$$ 

Therefore, exchanging the order of the integral, we have

$$v(t) = \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \left( \frac{1}{\Gamma(\sigma)} \int_0^s (s-\xi)^{\sigma-1} w(\xi) d\xi \right) ds$$

$$= \frac{1}{(m-1)!\Gamma(\sigma)} \int_0^t \left( \int_0^t (t-s)^{m-1} (s-\xi)^{\sigma-1} ds \right) w(\xi) d\xi = \frac{1}{\Gamma(m+\sigma)} \int_0^t (t-\xi)^{m+\sigma-1} w(\xi) d\xi$$

$$= J^{m+\sigma} w(t) \quad \text{for } 0 < t < T.$$ 

Hence, $v \in J^{m+\sigma} L^2(0, T)$, which means that $J^{m+\sigma} L^2(0, T) \supset H_{m+\sigma}(0, T)$. The converse inclusion is direct, and we see that $J^{m+\sigma} L^2(0, T) = H_{m+\sigma}(0, T)$. The norm equivalence between $\|v\|_{L^2(0,T)}$ and $\|J^{m+\sigma} v\|_{H_{m+\sigma}(0,T)}$, readily follows from the definition and Proposition 2.2. ■

For $\alpha = m + \sigma$ with $m \in \mathbb{N}$ and $0 < \sigma \leq 1$, we now define $d_t^\alpha$ as extension of the Caputo derivative

$$d_t^\alpha v(t) = \frac{1}{\Gamma(m+1-\alpha)} \int_0^t (t-s)^{m-\alpha} \frac{d^{m+1}}{ds^{m+1}} v(s) ds, \quad 0 < t < T.$$ 

We note that $d_t^\alpha$ requires $(m+1)$-times differentiability of $v$.

By Proposition 2.4, the inverse to $J^\alpha$ exists for each $\alpha \geq 0$, and by $J^{-\alpha}$ we denote the inverse:

$$J^{-\alpha} := (J^\alpha)^{-1}.$$
As the extension of such \( d_t^\alpha \) to \( H_{m+\sigma}(0, T) \), we define

\[
\partial_t^{m+\sigma} = J^{-m-\sigma} \quad \text{with} \quad D(\partial_t^{m+\sigma}) = H_{m+\sigma}(0, T).
\] (2.7)

Thus

**Proposition 2.5.**

Let \( \alpha, \beta \geq 0 \).

(i) \[
\partial_t^\alpha : H_{\alpha+\beta}(0, T) \rightarrow H_\beta(0, T)
\]

and

\[
J^\alpha : H_\beta(0, T) \rightarrow H_{\alpha+\beta}(0, T)
\]

are isomorphisms.

(ii) It holds

\[
J^{-\alpha} J^\beta = J^{-\alpha+\beta} \quad \text{on} \quad D(J^{-\alpha+\beta}).
\]

Here we set

\[
D(J^{-\alpha+\beta}) = \begin{cases} 
L^2(0, T) & \text{if } -\alpha + \beta \geq 0, \\
H_{\alpha-\beta}(0, T) & \text{if } -\alpha + \beta < 0.
\end{cases}
\]

**Proof.**

Part (i) is seen by (2.7) and Proposition 2.4. Part (ii) can be proved as follows. Let \( -\alpha + \beta \geq 0 \). If \( \alpha = \beta \), then \( J^{-\alpha} J^\alpha = I \): the identity operator on \( L^2(0, T) \) by (2.7) and the conclusion is trivial. Let \( \beta > \alpha \). Set \( \gamma := \beta - \alpha > 0 \). Let \( v \in L^2(0, T) \) be arbitrary. Then, by Lemma 2.1, we have

\[
J^{-\alpha} J^\beta v = J^{-\alpha} (J^{\alpha+\gamma} v) = J^{-\alpha} (J^\alpha J^\gamma v) = (J^{-\alpha} J^\alpha) J^\gamma v = J^\gamma v.
\]

Since \( J^\gamma v = J^{-\alpha+\beta} v \), we obtain \( J^{-\alpha} J^\beta v = J^{-\alpha+\beta} v \) for each \( v \in L^2(0, T) \).

Next let \( -\alpha + \beta < 0 \). Given \( v \in H_{\alpha-\beta}(0, T) \) arbitrarily, by Proposition 2.4 we can find \( w \in L^2(0, T) \) such that \( v = J^{\alpha-\beta} w \). Then

\[
J^{-\alpha} J^\beta v = J^{-\alpha} J^\beta (J^{\alpha-\beta} w) = J^{-\alpha} (J^\beta (J^{\alpha-\beta} w)) = J^{-\alpha} J^\alpha w = w
\]

by \( \beta, \alpha - \beta > 0 \) and Lemma 2.1. Therefore, \( J^{-\alpha} J^\beta v = w \). Since \( v = J^{\alpha-\beta} w \) implies \( w = (J^{\alpha-\beta})^{-1} v \), we have

\[
w = (J^{\alpha-\beta})^{-1} v = J^{-(\alpha-\beta)} v = J^{-\alpha+\beta} v,
\]
that is, \( J^{-\alpha} J^\beta v = J^{-\alpha+\beta} v \) for each \( v \in H_{\alpha-\beta}(0, T) \). Thus the proof of Proposition 2.5 is complete. ■

Next, for \( 0 < \alpha < 1 \), we characterize \( \partial_t^\alpha \) as extension over \( H_\alpha(0, T) \) of the operator \( d_t^\alpha \) defined on \( _0C^1[0, T] \). Let \( X \) and \( Y \) be Banach spaces with the norms \( \| \cdot \|_X \) and \( \| \cdot \|_Y \) respectively, and let \( K : X \to Y \) be a densely defined linear operator. We call \( K \) a closed operator if \( v_n \in D(K), \lim_{n \to \infty} v_n = v \) and \( K v_n \) converges to some \( w \) in \( Y \), then \( v \in D(K) \) and \( K v = w \). By \( \overline{K} \) we denote the closure of an operator \( K \) from \( X \) to \( Y \), that is, \( \overline{K} \) is the minimum closed extension of \( K \) in the sense that if \( \tilde{K} \) is a closed operator such that \( D(\tilde{K}) \supset D(K) \), then \( D(\tilde{K}) \supset D(\overline{K}) \). It is trivial that \( \overline{K} = K \) for a closed operator \( K \). Moreover we say that \( K \) is closable if there exists \( \overline{K} \).

Here we always consider the operator \( d_t^\alpha \) with the domain \( _0C^1[0, T] \): \( d_t^\alpha : _0C^1[0, T] \subset H_\alpha(0, T) \to L^2(0, T) \). Then we can prove (e.g., [15]):

**Proposition 2.6.**
The operator \( d_t^\alpha \) with the domain \( _0C^1[0, T] \) is closable and \( \overline{d_t^\alpha} = \partial_t^\alpha \).

Proposition 2.6 is not used later but means that our derivative \( \partial_t^\alpha \) is reasonable as the minimum extension of the classical Caputo derivative \( d_t^\alpha \) for \( v \in _0C^1[0, T] \).

**Interpretation of \( D(\partial_t^\alpha) = H_\alpha(0, T) \).**
Let \( \alpha > \frac{1}{2} \) and let \( v \in H_\alpha(0, T) \). By the definition (2.3) of \( H_\alpha(0, T) \), we can choose an approximating sequence \( v_n \in _0C^1[0, T] \), \( n \in \mathbb{N} \) such that \( \lim_{n \to \infty} \| v_n - v \|_{H_\alpha(0, T)} = 0 \). By the Sobolev embedding (e.g., Adams [1]), we see that \( H_\alpha(0, T) \subset H^\alpha(0, T) \subset C[0, T] \) if \( \alpha > \frac{1}{2} \), so that \( v \in C[0, T] \) and \( \lim_{n \to \infty} \| v_n - v \|_{C[0, T]} = 0 \). Therefore,

\[
\lim_{n \to \infty} |v(0) - v_n(0)| \leq \lim_{n \to \infty} \| v_n - v \|_{C[0, T]} = 0,
\]

that is, \( v(0) = \lim_{n \to \infty} v_n(0) \). Therefore, \( v \in H_\alpha(0, T) \) with \( \alpha > \frac{1}{2} \), yields \( v(0) = 0 \). In other words, if \( \alpha > \frac{1}{2} \), then \( v \in D(\partial_t^\alpha) \) means that \( v = v(t) \) satisfies the zero initial condition, which is useful for the formulation of the initial value problem in Sections 5 and 6. We can similarly consider and see that \( v \in D(\partial_t^\alpha) \) with \( \alpha > \frac{3}{2} \) yields \( v(0) = \partial_t v(0) = 0 \).
As is directly proved by the Young inequality on the convolution, we see that
\[ D_t^\alpha v \in L^1(0, T) \quad \text{if} \quad v \in W^{1,1}(0, T). \]

However, we do not necessarily have \( D_t^\alpha v \in L^2(0, T) \). Indeed,
\[ D_t^\alpha t^\beta = \frac{\Gamma(1 + \beta)}{\Gamma(1 - \alpha + \beta)} t^{\beta - \alpha} \quad \text{for} \quad \beta > 0. \]

If \( \beta - \alpha < -\frac{1}{2} \) and \( \beta > 0 \), then \( D_t^\alpha t^\beta \notin L^2(0, T) \), in spite of \( t^\beta \in W^{1,1}(0, T) \). This means that if we want to keep the range of \( D_t^\alpha \) within \( L^2(0, T) \), then even \( W^{1,1}(0, T) \) of a space of differentiable functions is not sufficient, although \( D_t^\alpha \) are concerned with \( \alpha \)-times differentiability.

In general, we can readily prove

**Proposition 2.7.**

We have
\[ \partial_t^\alpha v = D_t^\alpha v = \partial_t^\alpha v \quad \text{for} \quad v \in C^1[0, T] \quad (2.8) \]

and
\[ \partial_t^\alpha v(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t (t - s)^{-\alpha} v(s) \, ds = D_t^\alpha v(t) \quad \text{for} \quad v \in H_\alpha(0, T). \quad (2.9) \]

**Proof.**

Equation (2.8) is easily seen. We will prove (2.9) as follows. For arbitrary \( v \in H_\alpha(0, T) \), Proposition 2.2 yields that \( v = J^\alpha w \) with some \( w \in L^2(0, T) \) and \( w = \partial_t^\alpha v \). On the other hand, Lemma 2.1 implies
\[ D_t^\alpha v = \frac{d}{dt} J^{1-\alpha} v = \frac{d}{dt} J^{1-\alpha} (J^\alpha w) = \frac{d}{dt} Jw = w. \]

Therefore, \( D_t^\alpha v = \partial_t^\alpha v \), and then the proof of Proposition 2.7 is complete. 

Equation (2.9) means that \( \partial_t^\alpha \) coincides with the Riemann-Liouville fractional derivative, provided that we consider \( H_\alpha(0, T) \) as the domain of \( \partial_t^\alpha \) and \( D_t^\alpha \). This extension \( \partial_t^\alpha \) of \( \partial_t^\alpha \big|_{C^1[0, T]} \) is not yet complete, and in the next subsection, we will continue to extend.

**§2.3. Definition of \( \partial_t^\alpha \): completion of the extension of \( d_t^\alpha \)**
By the current extension of \( \partial_t^\alpha \), we understand that \( \partial_t^\alpha 1 = 0 \) for \( 0 < \alpha < \frac{1}{2} \), but \( \partial_t^\alpha 1 \) cannot be defined for \( \frac{1}{2} \leq \alpha < 1 \):

\[
\begin{cases}
1 \in H_\alpha(0, T) = \mathcal{D}(\partial_t^\alpha) & 0 < \alpha < \frac{1}{2}, \\
1 \notin \mathcal{D}(\partial_t^\alpha) & \frac{1}{2} \leq \alpha < 1.
\end{cases}
\]

Moreover, we note that \( d_t^\alpha 1 = 0 \) and \( D_t^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \) for all \( \alpha \in (0, 1) \). Therefore, \( \partial_t^\alpha 1 \) is not consistent with neither the classical fractional derivative \( d_t^\alpha \) nor \( D_t^\alpha \), which suggests that our current extension from \( d_t^\alpha \) to \( \partial_t^\alpha \) is not sufficient as fractional derivative. Moreover, as is seen in Section 7, the current \( \partial_t^\alpha \) is not convenient for treating less regular source terms in fractional differential equations. In order to define \( \partial_t^\alpha v \) in more general spaces such as \( L^2(0, T) \), we should continue to extend \( \partial_t^\alpha \).

We recall (2.1) and (2.3) for \( \alpha > 0 \). We can readily verify that \( \tau : H_\alpha(0, T) \rightarrow H^\alpha(0, T) \) is an isomorphism with \( \tau \) defined by (2.2). Then

**Proposition 2.8.**

Let \( \alpha > 0 \) and \( \beta \geq 0 \). Then \( J_\alpha : H^\beta(0, T) \rightarrow H^{\alpha+\beta}(0, T) \) is an isomorphism.

We recall that \( J_\alpha : H^\beta(0, T) \rightarrow H^{\alpha+\beta}(0, T) \) is an isomorphism by Proposition 2.5 (i). Proposition 2.8 can be proved via the mapping \( \tau : L^2(0, T) \rightarrow L^2(0, T) \) defined by (2.2).

When \( J_\alpha \) remains an operator defined over \( L^2(0, T) \), the operator \( J_\alpha \) is not defined for \( f \in L^1(0, T) \), but the integral \( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \) itself exists as a function in \( L^1(0, T) \) for any \( f \in L^1(0, T) \). This is a substantial inconvenience, and we have to make suitable extension of \( J_\alpha|_{L^2(0, T)} \). Our formulation is based on \( L^2 \)-space and we cannot directly treat \( L^1(0, T) \)-space. Thus we introduce the dual space of \( H_\alpha(0, T) \) which contains \( L^1(0, T) \).

The key idea for the further extension of \( \partial_t^\alpha \), is the family \( \{ H_\alpha(0, T) \}_{\alpha > 0} \) and the isomorphism \( \partial_t^\alpha : H^{\alpha+\beta}(0, T) \rightarrow H^\beta(0, T) \) with \( \alpha, \beta > 0 \), which is called a Hilbert scale. We can refer to Chapter V in Amann [2] as for a general reference.

Let \( X \) be a Hilbert space over \( \mathbb{R} \) and let \( V \subset X \) be a dense subspace of \( X \), and the embedding \( V \rightarrow X \) be continuous. By the dual space \( X' \) of \( X \), we call the space of all the bounded linear functionals defined on \( X \). Then, identifying the dual space \( X' \) with itself, we
can conclude that $X$ is a dense subspace of the dual space $V'$ of $V$:

$$V \subset X \subset V'.$$

By $\langle v', f, v \rangle_V$, we denote the value of $f \in V'$ at $v \in V$. We note that

$$\langle v', f, v \rangle_V = (f,v)_X \text{ if } f \in X,$$

where $(f,v)_X$ is the scalar product in $X$.

We note that $H_\alpha(0,T)$ and $^\alpha H(0,T)$ are both dense in $L^2(0,T)$. Henceforth, identifying the dual space $L^2(0,T)'$ with itself, by the above manner, we can define $(H_\alpha(0,T))'$ and $(^\alpha H(0,T))'$. Then

$$\begin{cases}
H_\alpha(0,T) \subset L^2(0,T) \subset (H_\alpha(0,T))' =: H_-\alpha(0,T), \\
^\alpha H(0,T) \subset L^2(0,T) \subset (^\alpha H(0,T))' =: ^{-\alpha} H(0,T),
\end{cases} \quad (2.10)$$

where the above inclusions mean dense subsets. We refer to e.g., Brezis [4], Yosida [36] as for general treatments for dual spaces.

Let $X,Y$ be Hilbert spaces and let $K : X \rightarrow Y$ be a bounded linear operator with $\mathcal{D}(K) = X$. Then we recall that the dual operator $K'$ is the maximum operator among operators $\tilde{K} : Y' \rightarrow X'$ with $\mathcal{D}(\tilde{K}) \subset Y'$ such that $\langle \tilde{K} y, x \rangle_X = \langle y, K x \rangle_Y$ for each $x \in X$ and $y \in \mathcal{D}(\tilde{K}) \subset Y'$ (e.g., [4]).

Henceforth, we consider the dual operator $(J^\alpha)'$ of $J^\alpha : L^2(0,T) \rightarrow H_\alpha(0,T)$ and the dual $(J_\alpha)'$ of $J_\alpha : L^2(0,T) \rightarrow ^\alpha H(0,T)$ by setting $X = L^2(0,T)$ and $Y = H_\alpha(0,T)$ or $Y = ^\alpha H(0,T)$.

Then we can show

**Proposition 2.9.**

Let $\alpha > 0$ and $\beta \geq 0$. Then:

(i) $(J^\alpha)' : H_{-\alpha-\beta}(0,T) \rightarrow H_{-\beta}(0,T)$ is an isomorphism.

In particular, $(J^\alpha)' : H_{-\alpha}(0,T) \rightarrow L^2(0,T)$ is an isomorphism.

(ii) $(J_\alpha)' : ^{-\alpha-\beta} H(0,T) \rightarrow ^{-\beta} H(0,T)$ is an isomorphism.

In particular, $(J_\alpha)' : ^{-\alpha} H(0,T) \rightarrow L^2(0,T)$ is an isomorphism.

(iii) It holds:

$$J^\alpha v = (J_\alpha)' v \quad \text{for } v \in L^2(0,T).$$
Henceforth we write $J'_\alpha := (J_\alpha)'$. We see

$$J'_\alpha(J'_\alpha)^{-1}w = w \quad \text{for } w \in L^2(0, T)$$

and

$$(J'_\alpha)^{-1}J'_\alpha w = w \quad \text{for } w \in -\alpha H(0, T).$$

**Proof.** From Proposition 2.8, in terms of the closed range theorem (e.g., Section 7 of Chapter 2 in [4]), we can see (i) and (ii).

We now prove (iii). We can directly verify $(J_\alpha \gamma, w)_{L^2(0,T)} = (\gamma, J_\alpha w)_{L^2(0,T)}$ for each $\gamma, w \in L^2(0,T)$. Hence, by the maximality property of the dual operator $J'_\alpha$ of $J_\alpha$, we see that $J'_\alpha \supset J_\alpha$. Thus the proof of Proposition 2.9 is complete. ■

Thanks to Propositions 2.5 and 2.8-2.9, for $\beta \geq 0$, we can regard the operators $J_\alpha$ with $\mathcal{D}(J_\alpha) = \beta H(0, T)$ and $J^\alpha$ with $\mathcal{D}(J^\alpha) = H_\beta(0, T)$, and accordingly also the operators $J'_\alpha$ with $\mathcal{D}(J'_\alpha) = -\alpha - \beta H(0, T)$ and $(J^\alpha)'$ with $\mathcal{D}((J^\alpha)') = H_{-\alpha - \beta}(0, T)$. We do not specify the domains if we need not emphasize them.

We can define $J'_\alpha u$ for $u \in L^1(0,T)$ as follows. We note that $J'_\alpha$ is the dual operator of $J_\alpha : \gamma H(0, T) \rightarrow \alpha + \gamma H(0, T)$, where we choose $\gamma > 0$ such that $\mathcal{D}(J'_\alpha) \supset L^1(0, T)$.

To this end, choosing $\gamma > 0$ such that $\alpha + \gamma > \frac{1}{2}$, we regard $J'_\alpha$ as an operator : $-\alpha - \gamma H(0, T) \rightarrow -\gamma H(0, T)$. Then we can define $J'_\alpha u$ for $u \in L^1(0,T)$. Indeed, the Sobolev embedding yields that

$$\alpha + \gamma H(0,T) \subset H^{\alpha + \gamma}(0,T) \subset C[0, T]$$

by $\alpha + \gamma > \frac{1}{2}$. Therefore, any $u \in L^1(0,T)$ can be considered as an element in $(\alpha + \gamma H(0,T))'$ by

$$-\gamma H(0,T) < u, \phi >_{\alpha + \gamma H(0,T)} = \int_0^T u(t)\phi(t)dt \quad \text{for } \phi \in \alpha + \gamma H(0,T),$$

which means that $L^1(0,T) \subset \mathcal{D}(J'_\alpha) = -\alpha - \gamma H(0,T)$ and $J'_\alpha u$ is well-defined for $u \in L^1(0,T)$.

Henceforth, we always make the above definition of $J'_\alpha u$ for $u \in L^1(0,T)$, if not specified.

Then we can improve Proposition 2.9 (iii) as

**Proposition 2.10.**
We have
\[ J'_\alpha u = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} u(s) ds, \quad 0 < t < T \quad \text{for } u \in L^1(0, T). \] (2.11)

We can write (2.11) as
\[ J'_\alpha u = J^\alpha u \quad \text{for } u \in L^1(0, T), \]
when we do not specify the domain \( D(J^\alpha) \).

**Proof of Proposition 2.10.**

Let \( \gamma > \frac{1}{2} \). We consider \( J'_\alpha \) as an operator \( J'_\alpha : -\alpha - \gamma H(0, T) \rightarrow -\gamma H(0, T) \).

Let \( u \in L^1(0, T) \) be arbitrary. Since \( L^1(0, T) \subset -\gamma H(0, T) \) and \( L^2(0, T) \) is dense in \( L^1(0, T) \), we can choose \( u_n \in L^2(0, T), n \in \mathbb{N} \) such that \( u_n \rightarrow u \) in \( L^1(0, T) \) as \( n \rightarrow \infty \). By Proposition 2.9 (iii), we have \( J'_\alpha u_n = J^\alpha u_n \) for \( n \in \mathbb{N} \). By Proposition 2.9 (ii), we obtain
\[ J'_\alpha u_n \rightarrow J'_\alpha u \quad \text{in } -\gamma H(0, T). \] (2.12)

On the other hand, by \( u \in L^1(0, T) \), the Young inequality on the convolution implies
\[ \| J^\alpha u_n - J^\alpha u \|_{L^1(0, T)} \leq \frac{1}{\Gamma(\alpha)} s^{\alpha - 1} \| u_n - u \|_{L^1(0, T)}, \]
which yields \( J^\alpha u_n = J'_\alpha u_n \rightarrow J^\alpha u \) in \( L^1(0, T) \), that is, \( J^\alpha u_n \rightarrow J^\alpha u \) in \( -\gamma H(0, T) \). Therefore, with (2.12), we obtain \( J'_\alpha u = J^\alpha u \). Thus the proof of Proposition 2.10 is complete. \( \blacksquare \)

Now we complete the extension of \( d^\alpha_t \) with the domain \( _0C^1[0, T] \).

**Definition 2.1.**

For \( \beta \geq 0 \), we define
\[ \partial_t^\alpha := (J'_\alpha)^{-1} \quad \text{with} \quad D(\partial_t^\alpha) = H_\beta(0, T) \quad \text{or} \quad D(\partial_t^\alpha) = -\beta H(0, T). \] (2.13)

Thus

**Theorem 2.1.** Let \( \alpha > 0 \) and \( \beta \geq 0 \). Then \( \partial_t^\alpha : -\beta H(0, T) \rightarrow -\alpha - \beta H(0, T) \) and
\( \partial_t^\alpha : H_{\alpha + \beta}(0, T) \rightarrow H_\beta(0, T) \) are both isomorphisms.
The extension $\partial_t^\alpha : -\beta H(0, T) \rightarrow -\alpha-\beta H(0, T)$ is not only a theoretical interest, but also useful for studies of fractional differential equations even if we consider all the functions within $L^2(0, T)$, as we will do in Sections 6 and 7.

We consider a special case $\beta = 0$. Then $(J_{\alpha}')^{-1} : D((J_{\alpha}^{-1})^2) = L^2(0, T) \rightarrow -\alpha H(0, T)$. Then, since Proposition 2.9 (iii) yields

$$(J_{\alpha}')^{-1} \supset (J_{\alpha})^{-1},$$

we see that this $(J_{\alpha}')^{-1} = \partial_t^\alpha$ defined on $L^2(0, T)$ is an extension defined previously in $H_{\alpha}(0, T)$. In particular, this extended $\partial_t^\alpha$ still satisfies

$$(J_{\alpha}')^{-1} H_{\alpha}(0, T) = L^2(0, T).$$

Our completely extended derivative $\partial_t^\alpha$ of $d_t^\alpha$ with $D(d_t^\alpha) = _0C^1[0, T]$, operates similarly to the Riemann-Liouville fractional derivative $D_t^\alpha$, and is equivalent to $\frac{d}{dt} J^{1-\alpha}$ associated with the domain $-\beta H(0, T)$ and the range $-\alpha-\beta H(0, T)$ with $\alpha > 0$ and $\beta \geq 0$.

Proposition 2.10 enables us to calculate $\partial_t^\alpha u = f$ provided that $u, f \in L^1(0, T)$, and we show

**Example 2.3.**

Let $0 < \alpha < 1$. Choosing $\alpha + \gamma > \frac{1}{2}$ we consider $\partial_t^\alpha$ with $D(\partial_t^\alpha) = -\gamma-\alpha H(0, T) \supset L^1(0, T)$. Then $1 \in D(\partial_t^\alpha)$, and we have

$$\partial_t^\alpha 1 = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}.$$  

Indeed, since $\frac{1}{\Gamma(1-\alpha)} t^{-\alpha} \in L^1(0, T)$, Proposition 2.10 yields

$$J_{\alpha}' \left( \frac{1}{\Gamma(1-\alpha)} t^{-\alpha} \right) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}s^{-\alpha} ds \right) = 1.$$  

Therefore, the definition justifies $\partial_t^\alpha 1 = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}$.

Before proceeding to the next section, we will provide two propositions.

**Proposition 2.11.**

Let $0 < \alpha < 1$. Then

$$\partial_t^\alpha u(t) = \frac{d}{dt} \left( \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s) ds \right) \quad \text{in} \quad (C^\infty_0(0, T))',$$

for $u \in L^1(0, T)$, where $\frac{d}{dt}$ is taken in $(C^\infty_0(0, T))'$. 

In the proposition, we note \( \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s) ds \in L^1(0,T) \subset (C_0^\infty(0,T))^\prime \). We remark that we cannot take the pointwise differentiation of \( \int_0^t (t-s)^{-\alpha} u(s) ds \) in general for \( u \in L^1(0,T) \).

Proposition 2.11 enables us to calculate \( \partial_t^{\alpha} u = f \) for \( u \in L^1(0,T) \) even if \( \partial_t^{\alpha} u \) cannot be defined within \( L^1(0,T) \), as Example 2.4 (a) shows.

Example 2.4.

(a) Let \( \frac{1}{2} < \alpha < 1 \) and let

\[
h_{t_0}(t) = \begin{cases} 
0, & t \leq t_0, \\
1, & t > t_0 
\end{cases}
\]

(2.16)

with arbitrary \( t_0 \in (0,T) \). We can verify that \( h_{t_0} \in H_{1-\alpha}(0,T) \) if \( \alpha > \frac{1}{2} \). Indeed

\[
\int_0^T \int_0^T \frac{|h_{t_0}(t) - h_{t_0}(s)|^2}{|t-s|^{3-2\alpha}} ds dt = 2 \int_0^{t_0} \left( \int_0^T \frac{1}{|t-s|^{3-2\alpha}} ds \right) dt ds
\]

\[
= \frac{1}{1-\alpha} \int_0^{t_0} ((t_0-s)^{-2+2\alpha} - (T-s)^{-2+2\alpha}) ds < \infty
\]

by \( -2 + 2\alpha > -1 \).

Therefore, by Proposition 2.2, we can choose \( w \in L^2(0,T) \) such that \( J^{1-\alpha} w = h_{t_0} \). By Proposition 2.11, we can calculate \( \partial_t^{\alpha} w \) because \( w \in L^2(0,T) \subset L^1(0,T) \):

\[
\partial_t^{\alpha} w = \frac{d}{dt} \left( \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} w(s) ds \right) = \frac{d}{dt} J^{1-\alpha} w = \frac{d}{dt} h_{t_0} = \delta_{t_0},
\]

which is a Dirac delta function satisfying

\[
(C_0^\infty(0,T))^\prime < \delta_{t_0}, \psi >_{C_0^\infty(0,T)} \psi(t_0) \quad \text{for all } \psi \in C_0^\infty(0,T).
\]

(b) We have

\[
\partial_t^{\alpha} t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)} t^{\beta-\alpha} \quad 0 < \alpha < 1, \ \beta > -1
\]

(2.17)

in \( -^{\alpha-\gamma}H(0,T) \) with some \( \alpha + \gamma > \frac{1}{2} \). See Example 2.3 as (2.17) with \( \beta = 0 \). We remark that \( \beta - \alpha < -1 \) is possible, and so \( t^{\beta-\alpha} \notin L^1(0,T) \) may occur.
Indeed, by $\beta > -1$, we can see
\[
\frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} s^\beta ds = \frac{\Gamma(1 + \beta)}{\Gamma(2 - \alpha + \beta)} t^{1-\alpha+\beta}.
\]

Taking the derivative in the sense of $(C^\infty_0(0, T))'$ to see
\[
\frac{d}{dt} \left( \frac{\Gamma(1 + \beta)}{\Gamma(2 - \alpha + \beta)} t^{1-\alpha+\beta} \right)
= \frac{\Gamma(1 + \beta)}{\Gamma(2 - \alpha + \beta)} (1 - \alpha + \beta) t^{\beta-\alpha} = \frac{\Gamma(1 + \beta)}{\Gamma(1 - \alpha + \beta)} t^{\beta-\alpha}.
\]
Thus (2.17) is verified.

We cannot define $d^\alpha t^\beta$ in general for $-1 < \beta \leq 0$, but we can calculate $D^\alpha_t t^\beta$. We remark that $\partial^\alpha_t t^\beta$ in the operator sense coincides with the result calculated by $D^\alpha_t t^\beta$. We here emphasize that our interest is not only computations of fractional derivatives, but also formulate $\partial^\alpha_t$ as an operator defined on $-\alpha-\gamma H(0, T)$ with the isomorphy.

(c) For any constant $t_0 \in (0, T)$, we consider a function of the Heaviside type defined by (2.16). By Proposition 2.10 or 2.11, we can see
\[
\partial^\alpha_t h_{t_0}(t) = \begin{cases} 
0, & t \leq t_0, \\
\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, & t > t_0.
\end{cases}
\]
We compare with the case $\alpha = 1$:
\[
\frac{d}{dt} h_{t_0}(t) = \delta_{t_0}(t) : \text{Dirac delta function at } t_0
\]
in $(C^\infty_0(0, T))'$. On the other hand, $\partial^\alpha_t h_{t_0}$ for $\alpha < 1$ does not generate the singularity of a Dirac delta function.

Moreover,
\[
\partial^\alpha_t h_{t_0} \longrightarrow \delta_{t_0} \quad \text{as } \alpha \uparrow 1 \text{ in } (C^\infty_0(0, T))',
\]
which is taken in the distribution sense. More precisely,
\[
\lim_{\alpha \uparrow 1} (C^\infty_0(0, T))' < \partial^\alpha_t h_{t_0}, \psi >_{C^\infty_0(0, T)} = (C^\infty_0(0, T))' < \delta_{t_0}, \psi >_{C^\infty_0(0, T)} = \psi(t_0)
\]
for each \( \psi \in C_0^\infty(0, T) \).
Indeed, by integration by parts, we obtain
\[
(c_0^\infty(0,T))' < \partial_t^\alpha h(t_0), \psi >_{C_0^\infty(0,T)} = \int_{t_0}^T \frac{(t - t_0)^{-\alpha}}{\Gamma(1 - \alpha)} \psi(t) dt
\]

\[
= \left[ \frac{(t - t_0)^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \psi(t) \right]_{t=t_0}^{t=T} - \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \int_{t_0}^T (t - t_0)^{1-\alpha} \psi'(t) dt
\]

\[
= \frac{1}{\Gamma(2-\alpha)} \int_{T}^{t_0} (t - t_0)^{1-\alpha} \psi'(t) dt.
\]

Letting \( \alpha \uparrow 1 \) and applying the Lebesgue convergence theorem, we know that the right-hand side tends to

\[
\frac{1}{\Gamma(1)} \int_{T}^{t_0} \psi'(t) dt = \psi(t_0).
\]

Proof of Proposition 2.11.
We prove by approximating \( u \in L^1(0,T) \) by \( u_n \in \mathcal{C}^{1}[0,T] \), \( n \in \mathbb{N} \) and using Proposition 2.7. As before, we choose \( \gamma > \frac{1}{2} - \alpha \), so that \( L^1(0,T) \subset -^{\alpha-\gamma}H(0,T) \) by the Sobolev embedding: \( ^{\alpha+\gamma}H(0,T) \subset C[0,T] \).

Since we can choose \( u_n \in \mathcal{C}^{1}[0,T] \), \( n \in \mathbb{N} \) such that \( u_n \rightharpoonup u \) in \( L^1(0,T) \) as \( n \to \infty \), we see that \( u_n \rightharpoonup u \) in \( -^{2\alpha-\gamma}H(0,T) \). Hence, Theorem 2.1 yields that \( \partial_t^{\alpha} u_n \rightharpoonup \partial_t^{\alpha} u \) in \(-^{2\alpha-\gamma}H(0,T) \). Since \( C_0^\infty(0,T) \subset L^2(0,T) \) yields \(-^{2\alpha-\gamma}H(0,T) \subset (C_0^\infty(0,T))' \), we see that \( \partial_t^{\alpha} u_n \rightharpoonup \partial_t^{\alpha} u \) in \( (C_0^\infty(0,T))' \), that is,

\[
\lim_{n \to \infty} (c_0^\infty(0,T))' < \partial_t^{\alpha} u_n, \psi >_{C_0^\infty(0,T)} = (c_0^\infty(0,T))' < \partial_t^{\alpha} u, \psi >_{C_0^\infty(0,T)}
\]

for all \( \psi \in C_0^\infty(0,T) \).

On the other hand, by Proposition 2.7, we have \( \partial_t^{\alpha} u_n = D_t^{\alpha} u_n, n \in \mathbb{N} \). Therefore,

\[
\lim_{n \to \infty} (c_0^\infty(0,T))' < D_t^{\alpha} u_n, \psi >_{C_0^\infty(0,T)} = (c_0^\infty(0,T))' < \partial_t^{\alpha} u, \psi >_{C_0^\infty(0,T)} \tag{2.18}
\]

for all \( \psi \in C_0^\infty(0,T) \).

Then, for any \( \psi \in C_0^\infty(0,T) \), we obtain

\[
(c_0^\infty(0,T))' < D_t^{\alpha} u_n, \psi >_{C_0^\infty(0,T)} = (D_t^{\alpha} u_n, \psi)_{L^2(0,T)} = \frac{1}{\Gamma(1-\alpha)} \left( \frac{d}{dt} \int_0^t (t-s)^{-\alpha} u_n(s) ds, \psi \right)_{L^2(0,T)}
\]
\[ = - \frac{1}{\Gamma(1 - \alpha)} \left( \int_0^t (t - s)^{-\alpha} u_n(s) ds, \frac{d\psi}{dt} \right)_{L^2(0, T)}. \]

Since the Young inequality on the convolution yields
\[ \left\| \int_0^t (t - s)^{-\alpha} u_n(s) ds - \int_0^t (t - s)^{-\alpha} u(s) ds \right\|_{L^1(0, T)} \leq \| s^{-\alpha} \|_{L^1(0, T)} \| u_n - u \|_{L^1(0, T)} \rightarrow 0 \]
as \( n \rightarrow \infty \), we have
\[ \lim_{n \rightarrow \infty} (D^\alpha_t u_n, \psi)_{L^2(0, T)} = \left( -\frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} u(s) ds, \frac{d\psi}{dt} \right)_{L^2(0, T)}. \]

Hence, with (2.18), we obtain
\[ \langle C^\infty_0(0, T)' \rangle < \partial^\alpha_t u, \psi > C^\infty_0(0, T) = \lim_{n \rightarrow \infty} \langle C^\infty_0(0, T)' \rangle < D^\alpha_t u_n, \psi > C^\infty_0(0, T) \]
\[ = \left( -\frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} u(s) ds, \frac{d\psi}{dt} \right)_{L^2(0, T)} \]
for all \( \psi \in C^\infty_0(0, T) \). Consequently, (2.19) means
\[ \partial^\alpha_t u(t) = \frac{d}{dt} \left( \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} u(s) ds \right) \]
in the sense of the derivative of a distribution. Thus the proof of Proposition 2.11 is complete. \( \blacksquare \)

3. Basic properties in fractional calculus

In this section, we present fundamental properties of \( \partial^\alpha_t \) in the case of \( D(\partial^\alpha_t) = H_{\alpha}(0, T) \). We can consider for \( \partial^\alpha_t \) with the domain \( L^2(0, T) \) but we here omit.

We set \( \partial^0_t = I \): the identity operator on \( L^2(0, T) \).

**Theorem 3.1.**

Let \( \alpha, \beta \geq 0 \). Then
\[ \partial^\alpha_t (\partial^\beta_t v) = \partial^{\alpha+\beta}_t v \text{ for all } v \in H_{\alpha+\beta}(0, T). \]

This kind of sequential derivatives are more complicated for \( d^\alpha_t \) and \( D^\alpha_t \) when we do not specify the domains. For \( \partial^\alpha_t \), the domain is already installed in a convenient way.
Proof of Theorem 3.1.

By (2.7) we have \( \partial_t^\alpha = (J^\alpha)^{-1} \) in \( \mathcal{D}(\partial_t^\alpha) = H_\alpha(0, T) \) with \( \alpha \geq 0 \). Hence, it suffices to prove

\[
J^{-\alpha}(J^{-\beta}v) = J^{-(\alpha+\beta)}v, \quad v \in H_{\alpha+\beta}(0, T).
\]

Setting \( w = J^{-\beta}v \), we have \( w \in H_\alpha(0, T) \) by Proposition 2.5 (i). Then \( J^{-\alpha}J^{-\beta}v = J^{-\alpha}w \) and

\[
J^{-(\alpha+\beta)}v = J^{-(\alpha+\beta)}J^\beta w = J^{-(\alpha+\beta)+\beta}w = J^{-\alpha}w.
\]

For the second equality to last, we applied Proposition 2.5 (ii) in terms of \( w \in \mathcal{D}(J^{-\alpha}) = H_\alpha(0, T) \). Hence, \( J^{-\alpha}J^{-\beta}v = J^{-(\alpha+\beta)}v \) for \( v \in H_{\alpha+\beta}(0, T) \). Thus the proof of Theorem 3.1 is complete. ■

We define the Laplace transform by

\[
(Lv)(p) = \hat{v}(p) := \lim_{T \to \infty} \int_0^T v(t)e^{-pt}dt
\]

provided that the limit exists.

**Theorem 3.2 (Laplace transform of \( \partial_t^\alpha v \)).**

Let \( u \in H_\alpha(0, T) \) with arbitrary \( T > 0 \). If \( (|\partial_t^\alpha u|)(p) \) exists for \( p > p_0 \), which is some positive constant, then \( \hat{u}(p) \) exists for \( p > p_0 \) and

\[
\hat{\partial_t^\alpha u}(p) = p^\alpha \hat{u}(p), \quad p > p_0.
\]

For initial value problems for fractional ordinary differential equations and initial boundary value problems for fractional partial differential equations, it is known that the Laplace transform is useful if one can verify the existence of the Laplace transform of solutions to these problems. The existence of the Laplace transform is concerned with the asymptotic behavior of unknown solution \( u \) as \( t \to \infty \), which may not be easy to be verified for solutions to be determined. At least the method by Laplace transform is definitely helpful in finding solutions heuristically.

**Corollary 3.1.**

Let \( u \in H_\alpha(0, T) \) with arbitrary \( T > 0 \). If \( \partial_t^\alpha u \in L^r(0, \infty) \) with some \( r \geq 1 \), then

\[
\hat{\partial_t^\alpha u}(p) = p^\alpha \hat{u}(p), \quad p > p_0.
\]
Proof of Theorem 3.2.

First Step. We first prove

Lemma 3.1.

Let $w \in L^2(0, T)$ with arbitrary $T > 0$. If $|\hat{w}(p)|$ exists for $p > p_0$. Then

$$(\hat{J^\alpha w})(p) = p^{-\alpha} \hat{w}(p), \quad p > p_0.$$ 

Proof of Lemma 3.1.

We recall

$$(J^\alpha w)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(s)\,ds, \quad t > 0 \quad \text{for } w \in L^2(0, T).$$

By $w \in L^2(0, T)$, we see that $J^\alpha w \in L^2(0, T)$ for arbitrarily fixed $T > 0$. By the assumption on the existence of $\hat{w}$, we obtain

$$\lim_{T \to \infty} \int_0^T e^{-pt} w(t)\,dt = \hat{w}(p), \quad p > p_0.$$

Choose $T > 0$ arbitrarily. Then $w \in L^2(0, T)$, and

$$\int_0^T e^{-pt} \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(s)\,ds \right) \,dt = \frac{1}{\Gamma(\alpha)} \int_0^T w(s) \left( \int_s^T e^{-pt} (t-s)^{\alpha-1} \,dt \right)\,ds.$$

Changing the variables: $t \to \xi$ by $\xi := (t-s)p$, we have

$$\int_s^T e^{-pt} (t-s)^{\alpha-1} \,dt = p^{-\alpha} e^{-ps} \int_0^{(T-s)p} \xi^{\alpha-1} e^{-\xi} \,d\xi,$$

and so

$$\int_0^T e^{-pt} \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(s)\,ds \right) \,dt = \frac{1}{\Gamma(\alpha)} p^{-\alpha} \int_0^{(T-s)p} e^{-ps} w(s) \left( \int_0^{(T-s)p} \xi^{\alpha-1} e^{-\xi} \,d\xi \right)\,ds$$

$$= \frac{1}{\Gamma(\alpha)} p^{-\alpha} \int_0^{(T-s)p} e^{-ps} w(s) \left( \int_0^{(T-s)p} \xi^{\alpha-1} e^{-\xi} \,d\xi \right)\,ds$$

$$+ \frac{1}{\Gamma(\alpha)} p^{-\alpha} \int_0^{(T-s)p} e^{-ps} w(s) \left( \int_0^{(T-s)p} \xi^{\alpha-1} e^{-\xi} \,d\xi \right)\,ds =: I_1 + I_2.$$

We remark

$$\int_0^{(T-s)p} \xi^{\alpha-1} e^{-\xi} \,d\xi \leq \int_0^{\infty} \xi^{\alpha-1} e^{-\xi} \,d\xi = \Gamma(\alpha).$$
Then, since \( \int_0^\infty e^{-pt}|w(t)|dt \) exists, we see
\[
|I_1| = \left| \frac{1}{\Gamma(\alpha)} p^{-\alpha} \int_0^T e^{-ps}w(s) \left( \int_0^{(T-s)p} \xi^{\alpha-1}e^{-\xi}d\xi \right) ds \right| \\
\leq C p^{-\alpha} \int_0^T e^{-ps}|w(s)|ds 
\rightarrow 0 \quad \text{as} \quad T \rightarrow \infty.
\]

Since \( 0 < s < \frac{T}{2} \) implies
\[
\frac{T}{2} p < (T - s)p < Tp,
\]
we obtain
\[
\int_0^{Tp} \xi^{\alpha-1}e^{-\xi}d\xi \leq \int_0^{(T-s)p} \xi^{\alpha-1}e^{-\xi}d\xi \leq \int_0^{Tp} \xi^{\alpha-1}e^{-\xi}d\xi \leq \int_0^{\infty} \xi^{\alpha-1}e^{-\xi}d\xi = \Gamma(\alpha)
\]
and so
\[
\lim_{T \to \infty} \int_0^{(T-s)p} \xi^{\alpha-1}e^{-\xi}d\xi = \int_0^{\infty} \xi^{\alpha-1}e^{-\xi}d\xi = \Gamma(\alpha) \quad \text{if} \quad 0 < s < \frac{T}{2}
\]
for any \( p > p_0 > 0 \). Since
\[
\lim_{T \to \infty} \int_0^T w(s)e^{-ps}ds
\]
exists for \( p > p_0 \) in view of (3.2) and (3.3), the Lebesgue convergence theorem yields
\[
\lim_{T \to \infty} \int_0^T e^{-ps}w(s) \left( \int_0^{(T-s)p} \xi^{\alpha-1}e^{-\xi}d\xi \right) ds = \Gamma(\alpha) \int_0^{\infty} e^{-ps}w(s)ds,
\]
and we reach
\[
\lim_{T \to \infty} I_2 = \frac{p^{-\alpha}}{\Gamma(\alpha)} \int_0^{\infty} w(s)e^{-ps}\Gamma(\alpha)ds = p^{-\alpha} \hat{w}(p), \quad p > p_0.
\]
Thus
\[
\lim_{T \to \infty} \int_0^T e^{-pt}(J^\alpha w)(t)dt = \lim_{T \to \infty} (I_1 + I_2) = p^{-\alpha} \hat{w}(p), \quad p > p_0.
\]
The proof of Lemma 3.1 is complete. \( \square \)

**Second Step.** We will complete the proof of the theorem. Since \( u \in H_\alpha(0, T) \) for any \( T > 0 \), we can find \( w_T \in L^2(0, T) \) such that \( J^\alpha w_T = u \) in \( (0, T) \). For any \( t \in (0, T) \), we can define \( w \) satisfying \( w(t) = w_T(t) \) for \( 0 < t < T \). Therefore for all \( t > 0 \), we can define \( w(t) \) and \( J^\alpha w(t) = u(t) \) for any \( t \in (0, T) \), that is, for all \( t > 0 \). Therefore \( \partial_t^\alpha u(t) = w(t) \) for \( t > 0 \) and \( w \in L^2(0, T) \) with arbitrary \( T > 0 \).
Since $\partial_t^\alpha u(t) = w(t)$ for $t > 0$ and $|\hat{\partial_t^\alpha u}(p)|$ exists for $p > p_0$, we know that $|\hat{w}(p)|$ exists for $p > p_0$. Therefore Lemma 3.1 yields
\[
(\hat{J^\alpha u})(p) = p^{-\alpha} \hat{w}(p), \quad p > p_0,
\]
that is,
\[
\hat{u}(p) = p^{-\alpha} \partial_t^\alpha u(p), \quad p > p_0.
\]
Thus the proof of Theorem 3.2 is complete. ■

Moreover $\partial_t^\alpha$ with $D(\partial_t^\alpha) = H_\alpha(0,T)$ is consistent also with the convolution of two functions. We set
\[
(u * g)(t) = \int_0^t u(t-s) g(s) ds, \quad 0 < t < T
\]
for $u \in L^2(0,T)$ and $g \in L^1(0,T)$. Then, the Young inequality on the convolution yields
\[
\|u * g\|_{L^2(0,T)} \leq \|u\|_{L^2(0,T)} \|g\|_{L^1(0,T)}.
\]
(3.5)

We prove

**Theorem 3.3.**

Let $\alpha \geq 0$. Then:

\[
J^\alpha(u * g) = (J^\alpha u) * g \quad \text{for } u \in L^1(0,T) \text{ and } g \in L^1(0,T)
\]
(3.6)

\[
\|u * g\|_{H_\alpha(0,T)} \leq C\|u\|_{H_\alpha(0,T)} \|g\|_{L^1(0,T)} \quad \text{for } u \in H_\alpha(0,T) \text{ and } g \in L^1(0,T).
\]
(3.7)

\[
\partial_t^\alpha(u * g) = (\partial_t^\alpha u) * g \quad \text{for } u \in H_\alpha(0,T) \text{ and } g \in L^1(0,T).
\]
(3.8)

**Proof of Theorem 3.3.**

**Proof of (3.6).** By exchange of the order of the integral and change of the variables $s \to \eta := s - \xi$, we can derive
\[
J^\alpha(u * g)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_0^s u(s-\xi) g(\xi) d\xi \right) ds
\]
\[
= \frac{1}{\Gamma(\alpha)} \int_0^t g(\xi) \left( \int_\xi^t (t-s)^{\alpha-1} u(s-\xi) ds \right) d\xi
\]
\[
= \frac{1}{\Gamma(\alpha)} \int_0^t g(\xi) (J^\alpha u)(t-\xi) d\xi = (g * J^\alpha u)(t), \quad 0 < t < T,
\]
which completes the proof of (3.6).

**Proof of (3.7).** Let $u \in H_\alpha(0,T)$ and $g \in L^1(0,T)$. Then, by Proposition 2.5, there exists $w \in L^2(0,T)$ such that $u = J^\alpha w$ and $\|w\|_{L^2(0,T)} \leq C\|u\|_{H_\alpha(0,T)}$. By (3.6) we obtain $u * g = J^\alpha w * g = J^\alpha(w * g)$. Since $w \in L^2(0,T)$ and $g \in L^1(0,T)$ imply $w * g \in L^2(0,T)$, by $u * g \in J^\alpha L^2(0,T)$ we see that $u * g \in H_\alpha(0,T)$. Moreover, by Proposition 2.5 and $u * g = J^\alpha(w * g)$, we have

$$\|u * g\|_{H_\alpha(0,T)} \leq C\|w * g\|_{L^2(0,T)} \leq C\|w\|_{L^2(0,T)}\|g\|_{L^1(0,T)} \leq C\|u\|_{H_\alpha(0,T)}\|g\|_{L^1(0,T)}.$$ 

which completes the proof of (3.7).

**Proof of (3.8).** For arbitrary $u \in H_\alpha(0,T)$, Proposition 2.5 yields that there exists $w \in L^2(0,T)$ such that $u = J^\alpha w$. Applying (3.6), we have $J^\alpha(w * g) = J^\alpha w * g$. Therefore,

$$\partial_t^\alpha(J^\alpha w * g) = \partial_t^\alpha J^\alpha(w * g) = w * g.$$ 

Since $u = J^\alpha w$ is equivalent to $w = \partial_t^\alpha u$, we reach $\partial_t^\alpha(u * g) = (\partial_t^\alpha u) * g$. Thus the proof of Theorem 3.3 is complete. ■

We will show a useful variant of Theorem 3.3.

**Theorem 3.4.**
Let $\gamma > \frac{1}{2}$ and $\beta + \gamma > \frac{1}{2}$. Then for $\partial_t^\beta : -\gamma H(0,T) \longrightarrow -\beta-\gamma H(0,T)$, we have

$$\partial_t^\beta v * u = \partial_t^\beta (v * u)$$

for all $u \in L^1(0,T)$ and $v \in L^1(0,T)$ satisfying $\partial_t^\beta v \in L^1(0,T)$.

We note that $L^1(0,T) \subset -\gamma H(0,T)$ by the Sobolev embedding and $\gamma > \frac{1}{2}$, and so if $v \in L^1(0,T) \subset \mathcal{D}(\partial_t^\beta) = -\gamma H(0,T)$, then so $\partial_t^\beta v \in -\beta-\gamma H(0,t)$ is well-defined. The assumption of the theorem further requires $\partial_t^\beta v \in L^1(0,T)$. On the other hand, for $v, u \in L^1(0,T)$, the Young inequality implies $v * u \in L^1(0,T)$, and so $v * u \in -\gamma H(0,t)$. Hence in the theorem, $\partial_t^\beta (v * u) \in -\beta-\gamma H(0,t)$ is well-defined.

**Proof of Theorem 3.4.**

The Young inequality yields $\partial_t^\beta v * u \in L^1(0,T)$. Consequently, by Proposition 2.10, we see

$$J'_\beta(\partial_t^\beta v * u) = J^\beta(\partial_t^\beta v * u).$$
By (3.6) in Theorem 3.3, we have

\[ J^\beta (\partial_t^\beta v \ast u) = (J^\beta \partial_t^\beta v) \ast u. \]

Since \( \partial_t^\beta v \in L^1(0, T) \), by using the definition (2.13), again Proposition 2.10 implies

\[ J^\beta \partial_t^\beta v = J'_\beta \partial_t^\beta v = J'_\beta (J'_\beta)^{-1} v = v. \]

Hence, \( J'_\beta (\partial_t^\beta v \ast u) = v \ast u \). Operating \( (J'_\beta)^{-1} \) to both sides and noting \( v \ast u \in L^1(0, T) \subset H(0, T) \), by (2.13), we have

\[ (J'_\beta)^{-1} (J'_\beta (\partial_t^\beta v \ast u)) = (J'_\beta)^{-1} (v \ast u) = \partial_t^\beta (v \ast u), \]

that is, \( \partial_t^\beta v \ast u = \partial_t^\beta (v \ast u) \). Thus the proof of Theorem 3.4 is complete. □

**Theorem 3.5 (coercivity).**

Let \( 0 < \alpha < 1 \). Then

\[ \int_0^T v(t) \partial_t^\alpha v(t) dt \geq \frac{1}{2\Gamma(1-\alpha)} T^{-\alpha} \|v\|_{L^2(0,T)}^2 \quad \text{for } v \in H_\alpha(0,T) \]

and

\[ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) \partial_s^\alpha v(s) ds \geq \frac{1}{2} |v(t)|^2 \quad \text{for } v \in H_\alpha(0,T). \]

The proof of Theorem 3.5 can be found in [18]. Theorem 3.5 is not used in this article, and is useful for proving the well-posedness for initial boundary value problems for fractional partial differential equations (see e.g., [18], [19], [27]).

We can generalize Theorem 3.5 to arbitrary \( v \in L^2(0,T) \), but we omit the details for the conciseness.

4. **Fractional derivatives of the Mittag-Leffler functions**

Related to time fractional differential equations, we introduce the Mittag-Leffler functions:

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, \quad z \in \mathbb{C}. \quad (4.1) \]

It is known (e.g., [30]) that \( E_{\alpha,\beta}(z) \) is an entire function in \( z \in \mathbb{C} \).

Henceforth let \( \lambda \in \mathbb{R} \) be a constant and, \( \alpha > 0 \) and \( \alpha \notin \mathbb{N} \). We fix an arbitrary constant \( \Lambda_0 > 0 \) and we assume that \( \lambda > -\Lambda_0 \).
First we prove Proposition 4.1.

Let $0 < \alpha < 2$. We fix constants $T > 0$ and $\lambda > -\Lambda_0$ arbitrarily. Then

(i) We have $E_{\alpha,1}(-\lambda t^\alpha) - 1 \in H_{\alpha}(0,T)$,

$$\partial_t^\alpha (E_{\alpha,1}(-\lambda t^\alpha) - 1) = -\lambda E_{\alpha,1}(-\lambda t^\alpha), \quad t > 0$$

and there exist constants $C_1 = C_1(\alpha, \Lambda_0, T) > 0$ and $C_2 = C_2(\alpha) > 0$ such that

$$|E_{\alpha,1}(-\lambda t^\alpha)| \leq \begin{cases} C_1 & \text{for } 0 \leq t \leq T \text{ if } -\Lambda_0 \leq \lambda < 0, \\ \frac{C_2}{1+\lambda t^\alpha} & \text{for } t \geq 0 \text{ if } \lambda \geq 0. \end{cases} \tag{4.2}$$

(ii) We have $tE_{\alpha,2}(-\lambda t^\alpha) - t \in H_{\alpha}(0,T)$,

$$\partial_t^\alpha (tE_{\alpha,2}(-\lambda t^\alpha) - t) = -\lambda tE_{\alpha,2}(-\lambda t^\alpha), \quad t > 0.$$ 

Furthermore we can find constants $C_1 = C_1(\alpha, \Lambda_0, T) > 0$ and $C_2 = C_2(\alpha) > 0$ such that

$$|E_{\alpha,2}(-\lambda t^\alpha)| \leq \begin{cases} C_1 & \text{for } 0 \leq t \leq T \text{ if } -\Lambda_0 \leq \lambda < 0, \\ \frac{C_2}{1+\lambda t^\alpha} & \text{for } t \geq 0 \text{ if } \lambda \geq 0. \end{cases} \tag{4.3}$$

A direct proof for the inclusions in $H_{\alpha}(0,T)$ is complicated and through the operator $J^\alpha$, we will provide simpler proofs.

Proof of Proposition 4.1.

(i) Since

$$E_{\alpha,1}(-\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k t^{\alpha k}}{\Gamma(\alpha k + 1)},$$

where the series is uniformly convergent for $0 \leq t \leq T$, the termwise integration yields

$$J^\alpha(E_{\alpha,1}(-\lambda t^\alpha)) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{\Gamma(\alpha k + 1)} \int_0^t (t-s)^{\alpha-1} s^{\alpha k} ds$$

$$= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{\Gamma(\alpha k + \alpha + 1)} t^{\alpha k + \alpha} = -\frac{1}{\lambda} \sum_{j=1}^{\infty} \frac{(-\lambda t^\alpha)^j}{\Gamma(\alpha j + 1)}.$$ 

Here we set $j = k + 1$ to change the indices of the summation. Therefore

$$J^\alpha(E_{\alpha,1}(-\lambda t^\alpha)) = -\frac{1}{\lambda} (E_{\alpha,1}(-\lambda t^\alpha) - 1).$$
Since $\partial_t^\alpha v = (J^\alpha)^{-1}v$ for $v \in H_\alpha(0, T)$ and $E_{\alpha,1}(-\lambda t^\alpha) \in L^2(0, T)$, we obtain

$$-\frac{1}{\lambda}(E_{\alpha,1}(-\lambda t^\alpha) - 1) \in H_\alpha(0, T)$$

and

$$(J^\alpha)^{-1} J^\alpha E_{\alpha,1}(-\lambda t^\alpha) = -\frac{1}{\lambda} \partial_t^\alpha (E_{\alpha,1}(-\lambda t^\alpha) - 1),$$

that is, $\partial_t^\alpha (E_{\alpha,1}(-\lambda t^\alpha) - 1) = -\lambda E_{\alpha,1}(-\lambda t^\alpha)$.

On the other hand, for $0 < \alpha < 2$ and $\beta > 0$, by Theorems 1.5 and 1.6 (p.35) in [30], we can find a constant $C_0 = C_0(\alpha, \beta) > 0$ such that

$$|E_{\alpha,\beta}(-\lambda t^\alpha)| \leq \begin{cases} C_0(1 + |\lambda|t^\alpha)^{-\frac{\beta}{\alpha}} \exp(|\lambda|^\frac{\beta}{\alpha}t) & \text{if } -\Lambda_0 \leq \lambda < 0, \\ \frac{C_0}{1 + \lambda t^\alpha} & \text{if } \lambda \geq 0 \end{cases} \quad (4.4)$$

for all $t \geq 0$. Hence, we can choose constants $C_3 = C_3(\alpha, \beta, \Lambda_0, T) > 0$ and $C_4 = C_4(\alpha, \beta) > 0$ such that

$$|E_{\alpha,\beta}(-\lambda t^\alpha)| \leq \begin{cases} C_3 & \text{for } 0 \leq t \leq T \text{ if } -\Lambda_0 \leq \lambda < 0, \\ \frac{C_4}{1 + \lambda t^\alpha} & \text{for } t \geq 0 \text{ if } \lambda \geq 0. \end{cases} \quad (4.5)$$

In terms of (4.5), we can finish the proof of (4.2). Thus the proof of (i) is complete.

(ii) Since $tE_{\alpha,2}(-\lambda t^\alpha) - t \in L^2(0, T)$, noting $\Gamma(2) = 1 \times \Gamma(1) = 1$, we have

$$J^\alpha(tE_{\alpha,2}(-\lambda t^\alpha)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \sum_{k=0}^\infty \frac{s(-\lambda)^k s^\alpha}{\Gamma(\alpha k + 2)} ds = t \sum_{k=0}^\infty \frac{(-\lambda)^k t^{\alpha k + \alpha}}{\Gamma(\alpha k + \alpha + 2)}$$

$$= -\frac{t}{\lambda} \sum_{j=1}^\infty \frac{(-\lambda)^j t^{\alpha j}}{\Gamma(\alpha j + 2)} \left(1 - \frac{1}{\Gamma(2)}\right),$$

that is,

$$J^\alpha(tE_{\alpha,2}(-\lambda t^\alpha)) = -\frac{1}{\lambda} tE_{\alpha,2}(-\lambda t^\alpha) - t, \quad 0 < t < T.$$ 

Therefore, $tE_{\alpha,2}(-\lambda t^\alpha) - t \in H_\alpha(0, T)$ and

$$-\lambda tE_{\alpha,2}(-\lambda t^\alpha) = \partial_t^\alpha (tE_{\alpha,2}(-\lambda t^\alpha) - t).$$

In terms of (4.5), the proof of the estimate is similar to part (i). Thus we can complete the proof of Proposition 4.1. ■
We set
\[(B_{\lambda}f)(t) := \int_{0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha}) f(s) ds.\] (4.6)

Next we show

**Proposition 4.2.**

Let \(f \in L^2(0, T)\). Then
\[B_{\lambda}f \in H^{\alpha}(0, T) \text{ and } \partial_t^{\alpha}(B_{\lambda}f)(t) = -\lambda(B_{\lambda}f)(t) + f(t), \quad 0 < t < T.\] (4.7)

Moreover there exist constants \(C_5 = C_5(\alpha, \Lambda_0, T) > 0\) and \(C_6 = C_6(\alpha, T) > 0\) such that
\[
\begin{cases}
\|B_{\lambda}f\|_{H^{\alpha}(0, T)} \leq C_5(\alpha, \Lambda_0, T) \|f\|_{L^2(0, T)} & \text{for all } f \in L^2(0, T) \text{ and } \lambda > -\Lambda_0, \\
\|B_{\lambda}f\|_{H^{\alpha}(0, T)} \leq C_6(\alpha, T) \|f\|_{L^2(0, T)} & \text{for all } f \in L^2(0, T) \text{ and } \lambda \geq 0.
\end{cases}
\] (4.8)

The uniformity of estimate (4.8) on \(\lambda \geq 0\) plays an important role in Lemma 6.2 (ii) in Section 6. Such uniformity can be derived from the complete monotonicity of \(E_{\alpha,1}(-\lambda t^\alpha)\) which is characteristic only for \(0 < \alpha < 1\) and see (4.9) below.

Here we prove by using \(\partial_t^{\alpha} = (J^{\alpha})^{-1}\) in \(H^{\alpha}(0, T)\), although other proof by Theorem 3.4 is possible.

**Proof.**

Since in view of (4.5), we can choose constants \(C_7 = C_7(\alpha, \Lambda_0, T) > 0\) and \(C_8 = C_8(\alpha, T) > 0\) such that
\[|E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})| \leq \begin{cases} 
C_7(\alpha, \Lambda_0, T) & \text{for } 0 \leq t \leq T \text{ and } \lambda > -\Lambda_0, \\
C_8(\alpha, T) & \text{for } 0 \leq t \leq T \text{ and } \lambda \geq 0,
\end{cases}\]

we estimate
\[
\left| \int_{0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha}) f(s) ds \right| \leq C \int_{0}^{t} (t-s)^{\alpha-1} |f(s)| ds, \quad 0 < t < T.
\]

Here and henceforth \(C > 0\) denotes generic constants which depend on \(\alpha, \Lambda_0, T\) when we consider \(\lambda > -\Lambda_0\) and does not depend on \(\Lambda_0\) if \(\lambda \geq 0\).

Hence, the Young inequality yields that \(\int_{0}^{t} \!(t-s)^{\alpha-1} |f(s)| ds \in L^2(0, T)\) and so \(B_{\lambda}f \in L^2(0, T)\).
Now
\[ J^\alpha(B_\lambda f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(B_\lambda f)(s)ds \]
\[ = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_0^s (s-\xi)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(s-\xi)^\alpha) f(\xi)d\xi \right) ds \]
\[ = \int_0^t f(\xi) \left( \frac{1}{\Gamma(\alpha)} \int_\xi^t (t-s)^{\alpha-1}(s-\xi)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(s-\xi)^\alpha)ds \right) d\xi. \]

Here
\[ \frac{1}{\Gamma(\alpha)} \int_\xi^t (t-s)^{\alpha-1}(s-\xi)^{\alpha-1} \sum_{k=0}^\infty \frac{(-\lambda)^k(s-\xi)^{\alpha k}}{\Gamma(\alpha k + \alpha)} ds \]
\[ = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{(-\lambda)^k}{\Gamma(\alpha k + \alpha)} \int_\xi^t (t-s)^{\alpha-1}(s-\xi)^{\alpha k+\alpha-1} ds \]
\[ = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{(-\lambda)^k}{\Gamma(\alpha k + \alpha)} \Gamma(\alpha k + \alpha) \Gamma(\alpha k + 2\alpha) (t-\xi)^{\alpha k+2\alpha-1} \]
\[ = -\frac{1}{\lambda} (t-\xi)^{\alpha-1} \sum_{j=1}^\infty \frac{(-\lambda(t-\xi)^\alpha)^j}{\Gamma(\alpha j + \alpha)} = -\frac{1}{\lambda} (t-\xi)^{\alpha-1} \left( E_{\alpha,\alpha}(-\lambda(t-\xi)^\alpha) - \frac{1}{\Gamma(\alpha)} \right). \]

Therefore, we have
\[ J^\alpha(B_\lambda f)(t) = -\frac{1}{\lambda} (B_\lambda f)(t) + \frac{1}{\lambda} \int_0^t (t-\xi)^{\alpha-1} \frac{1}{\Gamma(\alpha)} f(\xi)d\xi \]
\[ = -\frac{1}{\lambda} (B_\lambda f)(t) + \frac{1}{\lambda} (J^\alpha f)(t), \quad 0 < t < T, \]
that is,
\[ (B_\lambda f)(t) = -\lambda J^\alpha(B_\lambda f)(t) + (J^\alpha f)(t) = J^\alpha(-\lambda B_\lambda f + f)(t), \quad 0 < t < T. \]

Hence, \( B_\lambda f \in J^\alpha L^2(0,T) = H_\alpha(0,T) \) by \(-\lambda B_\lambda f + f \in L^2(0,T)\). Therefore, we apply \( \partial^\alpha_t = (J^\alpha)^{-1} \) to reach
\[ \partial^\alpha_t(B_\lambda f) = -\lambda B_\lambda f + f \quad \text{in} \ (0,T). \]
On the other hand, the termwise differentiation of the power series of $E_{\alpha,1}(z)$ yields
\[ \frac{d}{dt}E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha). \]
Furthermore we note the complete monotonicity:
\[ E_{\alpha,1}(-\lambda t^\alpha) > 0, \quad \frac{d}{dt}E_{\alpha,1}(-\lambda t^\alpha) \leq 0, \quad t \geq 0 \tag{4.9} \]
(e.g., Gorenflo, Kilbas, Mainardi and Rogosin [4]). Therefore, for $\lambda \geq 0$, we have
\[ \int_0^T |\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)| dt = \int_0^T \lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) dt \]
\[ = - \int_0^T \frac{d}{dt}E_{\alpha,1}(-\lambda t^\alpha) dt = 1 - E_{\alpha,1}(-\lambda T^\alpha) \leq 1. \tag{4.10} \]
For $-\Lambda_0 < \lambda < 0$, by (4.5) we have
\[ \int_0^T |\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)| dt \leq |\Lambda_0| C \int_0^T t^{\alpha-1} dt, \]
where the constant $C > 0$ depends on $\alpha, \Lambda_0, T$. Consequently,
\[ \lambda \|s^{\alpha-1}E_{\alpha,\alpha}(-\lambda s^\alpha)\|_{L^1(0,T)} \leq C \quad \text{for} \quad \lambda > -\Lambda_0. \]
Applying the Young inequality in (4.6), we obtain
\[ \lambda \|B_{\lambda}f\|_{L^2(0,T)} \leq \lambda \|s^{\alpha-1}E_{\alpha,\alpha}(-\lambda s^\alpha)\|_{L^1(0,T)} \|f\|_{L^2(0,T)} \leq C \|f\|_{L^2(0,T)}. \]
Therefore,
\[ |B_{\lambda}f|_{H_\alpha(0,T)} \leq C \|\partial_t^\beta (B_{\lambda}f)\|_{L^2(0,T)} = C \| - \lambda B_{\lambda}f + f\|_{L^2(0,T)} \]
\[ \leq C(\| - \lambda B_{\lambda}f\|_{L^2(0,T)} + \|f\|_{L^2(0,T)}) \leq C \|f\|_{L^2(0,T)} \quad \text{for all} \quad f \in L^2(0,T). \]
Thus the proof of Proposition 4.2 is complete. ■

We close this section with a lemma which is used in Section 6.

**Lemma 4.1.**
Let $0 < \beta < \alpha$ and $\lambda > 0$. Fixing $\gamma > \frac{1}{2}$, we consider $\partial_t^{\beta} : -\gamma H(0,T) \rightarrow -\beta-\gamma H(0,T)$.
Then
(i) \[ t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) \in L^1(0,T) \subset -\gamma H(0,T). \]
(ii) \[
\partial^\beta_t (t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)) = t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(-\lambda t^\alpha) \in L^1(0, T).
\]

(iii) \[
\partial^\beta_t (t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) \ast v) = (t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(-\lambda t^\alpha) \ast v) \text{ for each } v \in L^2(0, T).
\]

**Proof.**

(i) By (4.4), we see \(|t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)| \leq C t^{\alpha-1}\) for \(t > 0\) and \(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) \in L^1(0, T)\). The embedding \(L^1(0, T) \subset H^{-\gamma}(0, T)\) is derived by the Sobolev embedding.

(ii) By \(\alpha - \beta > 0\), we can similarly verify that \(t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(-\lambda t^\alpha) \in L^1(0, T)\). Therefore, Proposition 2.10 yields

\[
J'_\beta(t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(-\lambda t^\alpha))(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(-\lambda s^\alpha) ds.
\]

By \(\beta > 0\) and \(\alpha - \beta > 0\), we apply the formula (1.100) (p.25) in [30], which can be also directly verified by the expansion of the power series of \(E_{\alpha,\alpha-\beta}(-\lambda s^\alpha)\), so that

\[
\int_0^t (t-s)^{\beta-1} s^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(-\lambda s^\alpha) ds = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha).
\]

Therefore, \(J'_\beta(t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(-\lambda t^\alpha))(t) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\). By \(\partial^\beta_t = (J'_\beta)^{-1}\) in \(L^1(0, T)\), we see part (ii).

Since \(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\), \(\partial^\beta_t (t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\) are both in \(L^1(0, T)\), part (iii) follows from Theorem 3.4. Thus the proof of Lemma 4.1 is complete. ■

5. **Initial value problems for fractional ordinary differential equations**

A relevant formulation of initial value problem for time-fractional ordinary differential equations is our main issue in this section, in order to treat not so smooth data. For keeping the compact descriptions, we are restricted to a simple linear fractional ordinary differential equation. The treatments are similar to Chapter 3 in [18]. We formulate an initial value
problem as follows.

\[
\left\{
\begin{array}{ll}
\partial_t^\alpha (u - a)(t) = -\lambda u(t) + f(t), & 0 < t < T, \\
u - a \in H_\alpha(0,T).
\end{array}
\right.
\]  

(5.1)

As is mentioned in Section 2, if \( \alpha > \frac{1}{2} \), then \( u - a \in H_\alpha(0,T) \subset \mathbb{C}^1[0,T] \) and we know that \( u(t) - a \) is continuous and \( u(0) = a \). Thus in the case of \( \alpha > \frac{1}{2} \), if \( u \) satisfies (5.1), then a usual initial condition \( u(0) = a \) is satisfied.

Our formulation (5.1) coincides with a conventional formulation \( D_t^\alpha (u(t) - a) = f(t) \) of initial value problem, provided that we suitably specify the regularity of \( u \). We emphasize that we always attach fractional derivative operators with the domains such as \( H_\alpha(0,T) \) or \( -^\alpha H(0,T) \) with \( \alpha \geq 0 \), which means that our approach is a typical operator theoretic formulation, for example, similarly to that one is prohibited to consider the Laplacian \( -\Delta \) in \( \Omega \subset \mathbb{R}^d \) not associated with the domain. In other words, the operator \( -\Delta \) with the domain \( \{ u \in H^2(\Omega); u \in H^1_0(\Omega) \} \), is different from \( -\Delta \) with the domain \( \{ u \in H^2_0(\Omega); \nabla u \cdot \nu = 0 \text{ on } \partial \Omega \} \). Here \( \nu = \nu(x) \) is the unit outward normal vector to \( \partial \Omega \).

In particular, if we consider \( \partial_t^\alpha \) with the domain \( H_\alpha(0,T) \) and both sides of (5.1) in \( L^2(0,T) \), we remark that the equality \( \partial_t^\alpha (u - a) = \partial_t^\alpha u - \partial_t^\alpha a \) does not make any sense for \( \alpha > \frac{1}{2} \), because a constant function \( a \) is not in \( H_\alpha(0,T) \). On the other hand, if we consider \( \partial_t^\alpha \) with the domain \( L^2(0,T) \), then we can justify

\[
\partial_t^\alpha (u - a) = \partial_t^\alpha u - \partial_t^\alpha a = \partial_t^\alpha u - \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}
\]

for any \( u \in L^2(0,T) \).

Now we can prove

**Theorem 5.1.**

Let \( f \in L^2(0,T) \). Then there exists a unique solution \( u = u(t) \) to initial value problem (5.1).

Moreover

\[
u(t) = a E_{\alpha,1}(-\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s) ds, \quad 0 < t < T.\]  

(5.2)

Formula (5.2) itself is well-known (e.g., (3.1.34), p.141 in [16]) for \( f \) in some classes. We can
refer also to Gorenflo and Mainardi \[8\], Gorenflo, Mainardi and Srivastva \[9\], Gorenflo and Rutman \[10\], Luchko and Gorenflo \[24\].

On the other hand, we should understand that (5.2) holds in the sense that both sides are in $H_\alpha(0,T)$ for each $f \in L^2(0,T)$.

**Sketch of proof.**

We see that (5.1) is equivalent to

$$J^{-\alpha}(u - a) = -\lambda u + f \quad \text{in } L^2(0,T) \quad \text{and} \quad u - a \in H_\alpha(0,T)$$

and also to

$$u = a - \lambda J^\alpha u + J^\alpha f \quad \text{in } H_\alpha(0,T), \quad (5.3)$$

because $J^\alpha J^{-\alpha}(u - a) = J^\alpha(-\lambda u + f)$.

We conclude that $J^\alpha : L^2(0,T) \longrightarrow L^2(0,T)$ is a compact operator because the embedding $H_\alpha(0,T) \subset H^\alpha(0,T) \longrightarrow L^2(0,T)$ is compact (e.g., \[1\]). On the other hand, we apply the generalized Gronwall inequality (e.g., Lemma A.2 in \[18\]) to $u = -\lambda J^\alpha u$ in $(0,T)$, that is,

$$u(t) = \frac{-\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad 0 < t < T.$$ 

Then we obtain $u(t) = 0$ for $0 < t < T$. Therefore the Fredholm alternative yields the unique existence of $u$ satisfying (5.3).

Finally we define $\tilde{u}(t)$ by

$$\tilde{u}(t) - a := a(E_{\alpha,1}(-\lambda t^\alpha) - 1) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s) ds$$

$$= a(E_{\alpha,1}(-\lambda t^\alpha) - 1) + (B_\lambda f)(t), \quad 0 < t < T.$$ 

Here $B_\lambda$ is defined by (4.6).

By Propositions 4.1 and 4.2, it follows that $\tilde{u}(t)$ satisfies (5.1). Thus the proof of Theorem 5.1 is complete. ■
We can discuss an initial value problem for a multi-term time-fractional ordinary differential equation in the same way:

\[
\begin{align*}
\frac{\partial^\alpha_t}{\partial t^{\alpha}} (u - a) + \sum_{k=1}^{N} c_k \frac{\partial^\alpha_k}{\partial t^{\alpha_k}} (u - a) &= -\lambda u + f(t), \\
u - a &\in H_\alpha(0,T),
\end{align*}
\]  

(5.4)

where \(c_1, ..., c_N \neq 0, 0 < \alpha_1 < \cdots < \alpha_N < \alpha \leq 1\). Theorem 2.1 implies that \(J^\alpha \frac{\partial^\alpha_k}{\partial t^{\alpha_k}} = J^\alpha J^{-\alpha_k} = J^{\alpha - \alpha_k}\). By Proposition 2.2, we can obtain the equivalent equation:

\[
u - a = -\sum_{k=1}^{N} c_k J^{\alpha - \alpha_k} (u - a) - \lambda J^\alpha u + J^\alpha f
\]

and we apply the Fredholm alternative to prove the unique existence of solution, but we omit the details.

We further consider an initial value problem for \(f \in -^\alpha H(0,T)\):

\[
\begin{align*}
\frac{\partial^\alpha_t}{\partial t^{\alpha}} (u - a)(t) &= -\lambda u(t) + f(t), \quad 0 < t < T, \\
u - a &\in L^2(0,T).
\end{align*}
\]  

(5.5)

Similarly to Theorem 5.1, we prove the well-posedness of (5.5) for \(f \in -^\alpha H(0,T)\).

**Theorem 5.2.**

Let \(0 < \alpha < 1\) and \(\alpha \notin \mathbb{N}\). For \(f \in -^\alpha H(0,T)\), there exists a unique solution \(u - a \in L^2(0,T)\) to (5.5). Moreover we can choose a constant \(C > 0\) such that

\[
\|u - a\|_{L^2(0,T)} \leq C(|a| + \|f\|_{-^\alpha H(0,T)})
\]

for all \(a \in \mathbb{R}\) and \(f \in -^\alpha H(0,T)\).

**Example 5.1.**

We consider

\[
\frac{\partial^\alpha_t}{\partial t^{\alpha}} (u - a)(t) = f(t), \quad 0 < t < T
\]

(5.6)

with \(f(t) = \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)} t^{\beta-\alpha}\), where \(\beta > -\frac{1}{2}\). Then, \(u(t) = a + t^\beta\) satisfies (5.5) with \(\lambda = 0\) by (2.17). However, if \(-\frac{1}{2} < \beta < 0\) and \(-\frac{1}{2} < \beta - \alpha\), then \(u, f \in L^2(0,T)\), but \(u(t)\) does not satisfy \(\lim_{t\to 0} u(t) = a\).
Proof of Theorem 5.2.

Setting \( v := u - a \in L^2(0, T) \), we rewrite (5.5) as

\[
\begin{aligned}
(J'_{\alpha})^{-1}v &= -\lambda v - \lambda a + f, \\
v &\in L^2(0, T).
\end{aligned}
\]

By the definition (2.13) of \( \partial_t^\alpha \), equation (5.5) is equivalent to

\[
v = -\lambda J'_{\alpha}v + J'_{\alpha}(-\lambda a + f) \quad \text{in} \quad L^2(0, T).
\]

We set \( g := J'_{\alpha}(f - \lambda a) \in L^2(0, T) \) and \( Pv := -\lambda J'_{\alpha}v \). Then we see that the solution \( v = u - a \) is a fixed point of \( P: v = Pv + g \).

First the operator \( P : L^2(0, T) \to L^2(0, T) \) is a compact operator. Indeed, Proposition 2.9 (iii) yields \( J'_{\alpha}v = J^{\alpha}v \) for \( v \in L^2(0, T) \), and \( J^{\alpha} : L^2(0, T) \to H_\alpha(0, T) \) is an isomorphism by Proposition 2.2. Thus \( J'_{\alpha} \) is a bounded operator from \( L^2(0, T) \) to \( H_\alpha(0, T) \). Since the embedding \( H_\alpha(0, T) \to L^2(0, T) \) is compact, we see that \( J'_{\alpha} : L^2(0, T) \to L^2(0, T) \) is a compact operator (e.g., [1]).

Next we have to prove that \( v = 0 \) in \( L^2(0, T) \) from assumption that \( v = Pv \) in \( (0, T) \). Then, since

\[
J'_{\alpha}v(t) = J^{\alpha}v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1}v(s)ds
\]

by \( v \in L^2(0, T) \), we obtain

\[
v(t) = -\lambda \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{-\alpha}v(s)ds, \quad 0 < t < T.
\]

Hence,

\[
|v(t)| \leq C \int_0^t (t - s)^{\alpha - 1}|v(s)|ds \quad \text{for almost all} \ t \in (0, T).
\]

The generalized Gronwall inequality (e.g., Lemma A.2 in [18]) implies \( v(t) = 0 \) for almost all \( t \in (0, T) \). Therefore, the Fredholm alternative yields the unique existence of a fixed point of \( v = Pv + g \) in \( L^2(0, T) \). The estimate of \( v \) follows from the application of the generalized Gronwall inequality to (5.7) and Proposition 2.9 (ii): \( \|J'_{\alpha}f\|_{L^2(0, T)} \leq C\|f\|_{-\alpha H(0, T)} \). Thus the proof of Theorem 5.2 is complete. □

Remark 5.1.
Now we compare formulation (5.1) with (5.5) for \( f \in L^2(0, T) \).

(a) For \( f \in L^2(0, T) \), formulations (5.1) and (5.5) are equivalent.

Indeed, we immediately see that (5.1) implies (5.5). Conversely, let \( u \) satisfy (5.5). Then the second condition in (5.5) yields \( u \in L^2(0, T) \), and the first equation in (5.5) concludes that \( \partial_t^\alpha(u - a) \in L^2(0, T) \). By Proposition 2.5, we see that \( u - a \in H_\alpha(0, T) \), which means that \( u \) satisfies (5.1).

(b) In formulation (5.1), as we remarked, we should not decompose \( \partial_t^\alpha(u - a) = \partial_t^\alpha u - \partial_t^\alpha a \), which is wrong for \( \alpha \geq \frac{1}{2} \) because \( a \not\in H_\alpha(0, T) \). We further consider this issue for \( 0 < \alpha < \frac{1}{2} \).

By means of (2.17) with \( \beta = 0 \), for \( 0 < \alpha < \frac{1}{2} \), we see that \( 1 \in H_\alpha(0, T) \) and \( \partial_t^0 1 = \partial_t^\alpha 1 = \frac{1}{\Gamma(1 - \alpha)} t^{-\alpha} \), and also that \( u - a \in H_\alpha(0, T) \) if and only if \( u - a \in H^\alpha(0, T) \). Therefore, we see:

If \( 0 < \alpha < \frac{1}{2} \), then (5.1) is equivalent to

\[
\begin{align*}
\partial_t^\alpha u &= -\lambda u + \frac{a}{\Gamma(1 - \alpha)} t^{-\alpha} + f(t) \quad \text{in } L^2(0, T), \\
u &\in H_\alpha(0, T).
\end{align*}
\]

Assuming that \( f \in L^2(0, T) \) and \( 0 < \alpha < \frac{1}{2} \), we can conclude that (5.1)' is equivalent to (5.5)'. Indeed, \( \frac{a t^{-\alpha}}{\Gamma(1 - \alpha)} \in L^2(0, T) \) and so \( \frac{a t^{-\alpha}}{\Gamma(1 - \alpha)} + f \in L^2(0, T) \). Therefore, by \( u \in L^2(0, T) \), the first equation in (5.5)' yields \( \partial_t^\alpha u \in L^2(0, T) \), which means that \( u \in H_\alpha(0, T) \), and \( \partial_t^\alpha u = \partial_t^\alpha u \) in \( (0, T) \). Thus (5.1)' and (5.5)' are equivalent provided that \( 0 < \alpha < \frac{1}{2} \).

We emphasize that for \( \frac{1}{2} \leq \alpha < 1 \), we cannot make any reformulations of (5.1) similar to (5.1)' by decomposing \( u - a \) into \( u \) and \(-a\). We can discuss similar reformulations also for initial boundary value problems for fractional partial differential equations.

Now we take the widest domain of \( \partial_t^\alpha \) according to classes in time of functions under consideration, and we do not distinguish e.g., \( \partial_t^\alpha \) from \( \partial_t^\alpha \), because there is no fear of confusion.
Example 5.2.
Let $\alpha > \frac{1}{2}$ and let $f(t) := \delta_{t_0}(t)$: the Dirac delta function at $t_0 \in (0, T)$. In particular, we can prove that $|\varphi(t_0)| \leq C\|\varphi\|_{C[0,T]}$ and so $\delta_{t_0} \in (C[0,T])'$:

$$(C[0,T])' < \delta_{t_0}, \psi > C[0,T] = \psi(t_0)$$

for any $\psi \in C[0,T]$. Moreover, by the Sobolev embedding, we can see that $^\alpha H(0, T) \subset H^\alpha(0, T) \subset C[0,T]$ by $\alpha > \frac{1}{2}$. Therefore, $\delta_{t_0} \in -^\alpha H(0, T)$, and

$$-^\alpha H(0,T) < \delta_{t_0} \psi > ^\alpha H(0,T) = \psi(t_0)$$

defines a bounded linear functional on $^\alpha H(0,T)$.

This $\delta_{t_0}$ describes an impulsive source term in fractional diffusion. We will search for the representation of the solution to (5.5) with $f = \delta_{t_0}$ and $a = 0$:

$$\begin{cases}
\partial_t ^\alpha u = -\lambda u + \delta_{t_0} & \text{in } -^\alpha H(0, T), \\
u \in L^2(0, T).
\end{cases} (5.8)$$

Simulating a solution formula for

$$d_t ^\alpha u = -\lambda u + f(t), \quad u(0) = 0 \quad (5.9)$$

(e.g., [16], p.141), we can give a candidate for solution which is formally written by

$$u(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) \delta_{t_0}(s) ds, \quad 0 < t < T. (5.10)$$

Our formal calculation suggests

$$u(t) = \begin{cases}
0, & 0 < t \leq t_0, \\
(t-t_0)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-t_0)^\alpha), & t_0 < t \leq T.
\end{cases} (5.11)$$

Now we will verify that $u(t)$ given by (5.11) is the solution to (5.8). First it is clear that $u \in L^1(0, T)$. Then we will verify $u = -\lambda J'_\alpha u + J'_\alpha \delta_{t_0}$ in $L^2(0, T)$.

Since $u \in L^2(0, T)$, we apply Proposition 2.10 to have

$$J'_\alpha u(t) = J^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds.$$
For $0 < t < t_0$, by (5.11), we see $J_\alpha u(t) = 0$. Next for $t_0 \leq t \leq T$, we have

$$-\lambda J_\alpha u(t) = \frac{-\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(s-t_0)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(s-t_0)^\alpha) \, ds$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^\infty \int_0^t (t-s)^{\alpha-1}(s-t_0)^{\alpha+k-1}ds \left( -\lambda \right)^{k+1} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha+1)}$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^\infty \Gamma(\alpha)(\alpha+k) \int_0^t \frac{1}{\Gamma(\alpha+k+1)}(t-s)^{\alpha+k-1}ds \left( -\lambda \right)^{k+1} \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)}$$

$$= \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{(t-t_0)^{\alpha+k}}{\Gamma(\alpha+k+1)} \lambda^{k+1} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha+1)} = (t-t_0)^{\alpha-1} \sum_{j=1}^\infty \frac{(-\lambda(t-t_0)^\alpha)^j}{\Gamma(\alpha j + \alpha)}. \quad (5.12)$$

For calculations of $J_\alpha u(t)$, we can apply e.g., the formula (1.100) (p.25) in [30], but we here take a direct way.

On the other hand, by the definition of the dual operator $J'_\alpha$, setting $v_0 := J'_\alpha \delta_{t_0}$, we have

$$(v_0, \psi)_{L^2(0,T)} = \delta_{t_0}, \quad J_\alpha \psi \geq \alpha H_0(0,T) \quad \text{for all } \psi \in L^2(0,T).$$

Since $J_\alpha \psi \in \alpha H(0,T) \subset C[0,T]$ by $\alpha > \frac{1}{2}$ and Proposition 2.8, it follows that $v_0$ satisfies

$$-\alpha H_0(0,T) \leq \delta_{t_0}, \quad J_\alpha \psi \geq \alpha H_0(0,T) = (J_\alpha \psi)(t_0) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^T (s-t_0)^{\alpha-1} \psi(s) \, ds.$$

Therefore,

$$\frac{1}{\Gamma(\alpha)} \int_{t_0}^T (s-t_0)^{\alpha-1} \psi(s) \, ds = (v_0, \psi)_{L^2(0,T)} = \int_0^{t_0} v_0(s) \psi(s) \, ds + \int_{t_0}^T v_0(s) \psi(s) \, ds$$

for all $\psi \in L^2(0,T)$. Choosing $\psi \in L^2(0,T)$ satisfying $\psi = 0$ in $(t_0, T)$, we obtain

$$\int_{t_0}^{t_0} v_0(s) \psi(s) \, ds = 0,$$
and so \( v_0(s) = 0 \) for \( 0 < s \leq t_0 \). Hence,

\[
J'_\alpha \delta_{t_0}(s) = v_0(s) = \begin{cases} 
0, & 0 < s \leq t_0, \\
\frac{(s-t_0)^{\alpha-1}}{\Gamma(\alpha)}, & t_0 < s < T.
\end{cases}
\] (5.13)

In other words,

\[
\partial_t^\alpha v_0 = \delta_{t_0},
\] (5.14)

where \( v_0 \in L^2(0, T) \) is defined by (5.13).

Consequently, since

\[
\frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} + (t-t_0)^{\alpha-1} \sum_{k=1}^\infty \frac{(-\lambda(t-t_0)^\alpha)^k}{\Gamma(\alpha k + \alpha)} = (t-t_0)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-t_0)^\alpha),
\]

in terms of (5.12) and (5.13) we reach

\[
-\lambda J'_\alpha u(t) + J'_\alpha \delta_{t_0}(t) = \begin{cases} 
0, & 0 < t \leq t_0, \\
(t-t_0)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-t_0)^\alpha), & t_0 < t < T.
\end{cases}
\]

By (5.11) we verify

\[
u(t) = -\lambda J'_\alpha u(t) + J'_\alpha \delta_{t_0}(t).
\]

Thus we verified that \( u(t) \) given by (5.11) is the unique solution to (5.8).

We recall (4.6):

\[
(B_\lambda f)(t) := \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s) ds, \quad 0 < t < T, \quad \text{for } f \in L^2(0, T).
\]

By Proposition 4.2, we know that \( B_\lambda : L^2(0, T) \rightarrow H_{\alpha}(0, T) \) is a bounded operator.

We close this section with

**Proposition 5.1 (Representation of solution to (5.5) with \( f \in {}^{-\alpha}H(0, T) \)).**

Let \( \lambda > -\Lambda_0 \) be fixed. The operator \( B_\lambda \) can be extended to \( S_\lambda : {}^{-\alpha}H(0, T) \rightarrow L^2(0, T) \) as follows. For \( f \in {}^{-\alpha}H(0, T) \), there exists a sequence \( f_n \in L^2(0, T) \), \( n \in \mathbb{N} \) such that \( f_n \rightarrow f \) in \( {}^{-\alpha}H(0, T) \). Then

\[
\lim_{m, n \to \infty} \|B_\lambda f_n - B_\lambda f_m\|_{L^2(0, T)} = 0
\] (5.15)
and \( \lim_{n \to \infty} B_\lambda f_n \) is unique in \( L^2(0, T) \) independently of choices of sequences \( f_n, n \in \mathbb{N} \) such that \( f_n \to f \) in \( -\alpha H(0, T) \). Hence, setting

\[
S_\lambda f := \lim_{n \to \infty} B_\lambda f_n \quad \text{in} \ L^2(0, T),
\]

we have

\[
\partial_t^\alpha (S_\lambda f) = -\lambda S_\lambda f + f \quad \text{in} \ -\alpha H(0, T). \tag{5.16}
\]

**Proof.**

First, since \( L^2(0, T) \) is dense in \( -\alpha H(0, T) \), we can choose a sequence \( f_n \in L^2(0, T), n \in \mathbb{N} \) such that \( \lim_{n \to \infty} f_n = f \) in \( -\alpha H(0, T) \).

**Verification of (5.15).**

By Proposition 4.2, we see

\[
B_\lambda f = -\lambda J_\alpha' B_\lambda f + J_\alpha' f \tag{5.17}
\]

and

\[
\|B_\lambda f\|_{L^2(0,T)} \leq C\|J_\alpha' f\|_{L^2(0,T)} \quad \text{for} \quad f \in L^2(0,T). \tag{5.18}
\]

In view of (5.18), we obtain

\[
\|B_\lambda f_n - B_\lambda f_m\|_{L^2(0,T)} \leq C\|J_\alpha' f_n - J_\alpha' f_m\|_{L^2(0,T)}.
\]

Proposition 2.9 (ii) yields

\[
\|B_\lambda f_n - B_\lambda f_m\|_{L^2(0,T)} \leq C\|f_n - f_m\|_{-\alpha H(0,T)}.
\]

Since \( \lim_{m,n \to \infty} \|f_n - f_m\|_{-\alpha H(0,T)} = 0 \), we see that \( B_\lambda f_n, n \in \mathbb{N} \) converge in \( L^2(0,T) \).

Similarly we can prove that \( \lim_{n \to \infty} B_\lambda f_n \) is determined independently of choices of sequences \( f_n, n \in \mathbb{N} \) such that \( \lim_{n \to \infty} f_n = f \) in \( -\alpha H(0,T). \) Therefore, \( S_\lambda f \) is well-defined for \( f \in L^2(0,T) \).

**Verification of (5.16).**

For an approximating sequence \( f_n \in L^2(0,T), n \in \mathbb{N} \) such that \( f_n \to f \) in \( -\alpha H(0,T) \), we have

\[
B_\lambda f_n = -\lambda J_\alpha' B_\lambda f_n + J_\alpha' f_n \quad \text{in} \ L^2(0,T) \quad \text{for} \quad n \in \mathbb{N}.
\]

Since \( \lim_{n \to \infty} B_\lambda f_n = S_\lambda f \) in \( L^2(0,T) \), letting \( n \to \infty \), we obtain

\[
S_\lambda f = -\lambda \lim_{n \to \infty} J_\alpha' B_\lambda f_n + \lim_{n \to \infty} J_\alpha' f_n \quad \text{in} \ L^2(0,T).
\]
We apply Proposition 2.9 (ii) to see \( \lim_{n \to \infty} J'_{\alpha} f_n = J'_{\alpha} f \) in \( L^2(0, T) \) by \( \lim_{n \to \infty} f_n = f \) in \( -^\alpha H(0, T) \). Finally Proposition 2.9 (iii) implies that \( J'_{\alpha} |_{L^2(0, T)} : L^2(0, T) \to L^2(0, T) \) is bounded, so that
\[
\lim_{n \to \infty} J'_{\alpha} B_\lambda f_n = \lim_{n \to \infty} J^\alpha B_\lambda f_n = J'_{\alpha} S_\lambda f \quad \text{in} \quad L^2(0, T).
\]
Therefore we reach
\[
S_\lambda f = -\lambda J'_{\alpha} S_\lambda f + J'_{\alpha} f \quad \text{in} \quad L^2(0, T).
\]
By the definition (2.13) of \( \partial_t^\alpha \), this means
\[
\partial_t^\alpha (S_\lambda f) = -\lambda S_\lambda f + f \quad \text{in} \quad -^\alpha H(0, T).
\]
Thus the verification of (5.16) is complete, so that the proof of Proposition 5.1 is finished.

We can discuss more about the representation formula of solution to (5.5) with \( f \in -^\alpha H(0, T) \) in terms of convolution operators, but we will postpone to a future work.

6. Initial boundary value problem for fractional partial differential equations: selected topics

On the basis of \( \partial_t^\alpha \) defined in Section 2, we construct a feasible framework also for initial boundary value problems. We recall that an elliptic operator \(-A\) is defined by (1.1), and we assume all the conditions as described in Section 1 on the coefficients \( a_{ij}, b_j, c \in C^1(\overline{\Omega}) \).

Here we mostly consider the case \( \alpha < 1 \), but cases \( \alpha > 1 \) can be formulated and studied similarly.

By \( \nu = (\nu_1, ..., \nu_d) \) we denote the outward unit normal vector to \( \partial \Omega \) at \( x \) and set
\[
\partial_{\nu x} v = \sum_{i,j=1}^d a_{ij} (\partial_j v) \nu_i \quad \text{on} \quad \partial \Omega.
\]
We define an operator \(-A\) in \( L^2(\Omega) \) by
\[
\begin{cases}
Av(x) = Av(x), & x \in \Omega, \\
\mathcal{D}(A) = \{ v \in H^2(\Omega); \ v|_{\partial \Omega} = 0 \} = H^2(\Omega) \cap H^1_0(\Omega).
\end{cases}
\]
(6.1)

Here \( v|_{\partial \Omega} = 0 \) is understood as the sense of the trace (e.g., [H]).
We can similarly discuss other boundary condition, for example, \( D(A) = \{ v \in H^2(\Omega); \partial_{v^A} v + \sigma(x)v = 0 \text{ on } \partial\Omega \} \) with fixed function \( \sigma(x) \), but we concentrate on the homogeneous Dirichlet boundary condition \( u|_{\partial\Omega} = 0 \).

We formulate the initial boundary value problem by

\[
\partial_t^\alpha (u(x,t) - a(x)) + Au(x,t) = F(x,t) \quad \text{in } L^2(0,T;L^2(\Omega))
\]  

(6.2)

and

\[
u - a \in H_\alpha(0,T;L^2(\Omega)).
\]  

(6.3)

We emphasize that we do not adopt formulation (1.2).

The formulation (6.2) - (6.3) corresponds to (5.1) for an initial value problem for a time-fractional ordinary differential equation. The term \( Au \) in equation (6.2) means that \( u(\cdot,t) \in D(A) = H^2(\Omega) \cap H^1_0(\Omega) \) for almost all \( t \in (0,T) \), that is,

\[
u(\cdot,t)|_{\partial\Omega} = 0 \quad \text{for almost all } t \in (0,T).
\]

In other words, the domain of \( A \) describes the boundary condition which is a conventional way in treating the classical partial differential equations. Like fractional ordinary differential equations in Section 5, we understand that (6.3) means the initial condition.

We first present a basic well-posedness result for (6.2) - (6.3):

**Theorem 6.1.**

Let \( 0 < \alpha < 1 \). Let \( a \in H^1_0(\Omega) \) and \( F \in L^2(0,T;L^2(\Omega)) \). Then there exists a unique solution \( u = u(x,t) \) to (6.2) - (6.3) such that \( u-a \in H_\alpha(0,T;L^2(\Omega)) \) and \( u \in L^2(0,T;H^2(\Omega) \cap H^1_0(\Omega)) \). Moreover there exists a constant \( C > 0 \) such that

\[
\|u - a\|_{H_\alpha(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega) \cap H^1_0(\Omega))} \leq C(\|a\|_{H^1_0(\Omega)} + \|F\|_{L^2(0,T;L^2(\Omega))})
\]

for all \( a \in H^1_0(\Omega) \) and \( F \in L^2(0,T;L^2(\Omega)) \).

Here we remark that

\[
\|v\|_{H_\alpha(0,T;L^2(\Omega))} := \|\partial_t^\alpha v\|_{L^2(0,T;L^2(\Omega))} \quad \text{for } v \in H_\alpha(0,T;L^2(\Omega)).
\]

Unique existence results of solutions are known according to several formulations of initial boundary value problems. In Sakamoto and Yamamoto [32], in the case of symmetric \( A \) where \( b_j = 0 \) for \( 1 \leq j \leq d \) in (1.1), the unique existence is proved by means of the Fourier method, but the class of solutions is not the same as here. In the case where \( b_j \) for
\(j = 1, \ldots, d\) are not necessarily zero, assuming that the initial value \(a\) is zero, Theorem 6.1 is proved in Gorenflo, Luchko and Yamamoto \[7\]. In both \[7\] and \[32\], it is assumed that all the coefficients are independent of \(t\) and and \(c = c(x) \leq 0\) for \(x \in \Omega\). The work Kubica, Ryszewska and Yamamoto \[18\] proved Theorem 6.1 in a general case where \(a_{ij}, b_j, c\) depends both on \(x\) and \(t\) without extra assumption \(c \leq 0\). In other words, Theorem 6.1 is a special case of Theorem 4.2 in \[18\]. Furthermore, there have been other works on the well-posedness for initial boundary value problems and we are restricted to some of them: Bajlekova \[3\], Kubica and Yamamoto \[19\], Luchko \[22, 23\], Luchko and Yamamoto \[26\], Zacher \[37\]. See also Prüss \[31\] for a monograph on related integral equations, and a recent book Jin \[14\] which mainly studies the symmetric \(A\). As for further references up to 2019, the handbooks \[17\] edited by A.Kochubei and Y. Luchko are helpful. Most of the above works discuss the case of \(F \in L^2(0,T; L^2(\Omega))\).

Next we consider less regular \(F\) and \(a\) in \(x\). To this end, we introduce Sobolev spaces of negative orders in \(x\). Similarly to the triple \(^{\alpha}H(0,T) \subset L^2(\Omega) \subset ^{-\alpha}H(0,T)\) as is explained in Section 2, we introduce the dual space \((H^1_0(\Omega))'\) of \(H^1_0(\Omega)\) by identifying the dual space \((L^2(0,T))'\) with \(L^2(0,T)\):

\[ H^1_0(\Omega) \subset L^2(\Omega) \subset (H^1_0(\Omega))' =: H^{-1}(\Omega) \]

(e.g., \[4\]). For less regular \(F\) and \(a\) in the \(x\)-variable, we know

**Theorem 6.2.**

Let \(0 < \alpha < 1\). For the coefficients of \(A\), we assume the same conditions as in Theorem 6.1. Let \(a \in L^2(\Omega)\) and \(F \in L^2(0,T; H^{-1}(\Omega))\). Then there exists a unique solution \(u = u(x,t)\) to (6.2) - (6.3) such that \(u - a \in H_\alpha(0,T; H^{-1}(\Omega))\) and \(u \in L^2(0,T; H^1_0(\Omega))\). Moreover there exists a constant \(C > 0\) such that

\[ \|u - a\|_{H_\alpha(0,T; H^{-1}(\Omega))} + \|u\|_{L^2(0,T; H^1_0(\Omega))} \leq C(\|a\|_{L^2(\Omega)} + \|F\|_{L^2(0,T; H^{-1}(\Omega))}) \]

for all \(a \in L^2(\Omega)\) and \(F \in L^2(0,T; H^{-1}(\Omega))\).

In Theorem 6.2, we consider both sides of (6.2) in \(L^2(0,T; H^{-1}(\Omega))\). The proof of Theorem 6.2 is found in \[18\] and see also \[19\].
Remark 6.1.
As for general $\alpha > 0$, by means of the space defined by (2.6), we can formulate the initial boundary value problem as follows. For $\alpha > 1$, we set $\alpha = m + \sigma$ with $m \in \mathbb{N}$ and $0 < \sigma \leq 1$. Then for $\alpha > 1$ we formulate an initial boundary value problem by

$$
\begin{cases}
\partial_t^\alpha \left( u - \sum_{k=0}^{m} a_k \frac{t^k}{k!} \right) = -Au + F(x, t), & x \in \Omega, \ 0 < t < T, \\
\left( u - \sum_{k=0}^{m} a_k \frac{t^k}{k!} \right) (x, \cdot) \in H_\alpha(0, T) & \text{for almost all } x \in \Omega.
\end{cases}
$$

For $\sigma > \frac{1}{2}$, we can interpret the second condition as usual initial conditions. More precisely,

$$
\left( u - \sum_{k=0}^{m} a_k \frac{t^k}{k!} \right) (x, \cdot) \in H_\alpha(0, T)
$$

if and only if

$$
\begin{cases}
\frac{\partial^k u}{\partial t^k}(\cdot, 0) = a_k, & k = 0, 1, \ldots, m - 1, \\
\frac{\partial^m u}{\partial t^m}(x, \cdot) - a_m \in H_\sigma(0, T).
\end{cases}
$$

In this section, we pick up five topics and apply the results in Sections 2 - 4. We postpone general and complete descriptions to a future work. Moreover, we are limited to the following $A$:

$$
-Av(x) = \sum_{j=1}^{d} \partial_j (a_{ij}(x) \partial_j v(x)) + \sum_{j=1}^{d} b_j(x) \partial_j v + c(x)v, \quad x \in \Omega \quad (6.4)
$$

for $v \in \mathcal{D}(A) := H^2(\Omega) \cap H^1_0(\Omega)$ with

$$
a_{ij} = a_{ji} \in C^1(\overline{\Omega}), \quad b_j \in C^1(\overline{\Omega}) \quad \text{for } 1 \leq i, j \leq d, \quad c \in C^1(\overline{\Omega}), \leq 0 \quad \text{on } \Omega. \quad (6.5)
$$

§6.1. Mild solution and strong solution
We define an operator $L$ as a symmetric part of $A$ by

$$
-Lv(x) = \sum_{j=1}^{d} \partial_j (a_{ij}(x) \partial_j v(x)) + c(x)v, \quad x \in \Omega, \quad \mathcal{D}(L) = H^2(\Omega) \cap H^1_0(\Omega). \quad (6.6)
$$
Then there exist eigenvalues of $L$ and according to the multiplicities, we can arrange all the eigenvalues as

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty.$$ 

Here by $c \leq 0$ in $\Omega$, we can prove that $\lambda_1 > 0$. Moreover we can choose eigenfunctions $\varphi_n$ for $\lambda_n$, $n \in \mathbb{N}$ such that $\{\varphi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$. Henceforth $(\cdot, \cdot)$ and $\| \cdot \|_{L^2(\Omega)}$ denote the scalar product and the norm in $L^2(\Omega)$ respectively and we write $(\cdot, \cdot)_{L^2(\Omega)}$ when we like to specify the space. Thus $L\varphi_n = \lambda_n \varphi_n$, $\|\varphi_n\|_{L^2(\Omega)} = 1$ and $(\varphi_n \varphi_m) = 0$ if $n \neq m$.

For $\gamma \in \mathbb{R}$, we can define a fractional power $L^\gamma$ of $L$ by

$$D(L^\gamma) := \begin{cases} L^2(\Omega) & \text{if } \gamma \leq 0, \\ \{v \in L^2(\Omega); \sum_{n=1}^\infty \lambda_n^{2\gamma} |(v, \varphi_n)|^2 < \infty\} & \text{if } \gamma > 0. \end{cases}$$

We set

$$\|v\|_{D(L^\gamma)} := \left(\sum_{n=1}^\infty \lambda_n^{2\gamma} |(v, \varphi_n)|^2\right)^{\frac{1}{2}} \quad \text{if } \gamma > 0. \quad (6.7)$$

Then it is known that

$$D(L^{\frac{1}{2}}) = H^1_0(\Omega), \quad C^{-1}\|v\|_{H^1_0(\Omega)} \leq \|L^{\frac{1}{2}}v\|_{L^2(\Omega)} \leq C\|v\|_{H^1_0(\Omega)}, \quad v \in H^1_0(\Omega).$$

Here $C > 0$ is independent of choices of $v \in H^1_0(\Omega)$.

We further define operator $S(t)$ and $K(t)$ from $L^2(\Omega)$ to $L^2(\Omega)$ by

$$S(t)a := \sum_{n=1}^\infty E_{\alpha,1}(-\lambda_n t^\alpha)(a, \varphi_n) \varphi_n$$

and

$$K(t)a := \sum_{n=1}^\infty t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)(a, \varphi_n) \varphi_n$$

for all $a \in L^2(\Omega)$. Then for $0 \leq \gamma \leq 1$, we can find constants $C_1 > 0$ and $C_2 = C_2(\gamma) > 0$ such that

$$\begin{cases} \|S(t)a\|_{L^2(\Omega)} \leq C_1\|a\|_{L^2(\Omega)}, \\ \|L^\gamma K(t)a\|_{L^2(\Omega)} \leq C_2 t^{\alpha(1-\gamma)-1}\|a\|_{L^2(\Omega)} & \text{for } t > 0 \text{ and } a \in L^2(\Omega). \end{cases} \quad (6.9)$$

The proof of (6.9) is direct by the definition of $S(t)$ and $K(t)$ and can be found e.g., in [7].
Here and henceforth we write $u(t) := u(\cdot, t)$ as a mapping from $(0, T)$ to $L^2(\Omega)$. The proofs of (6.8) - (6.10) can be found e.g., in [7].

We can show

**Proposition 6.1.**

Let $a \in H^1_0(\Omega)$ and $F \in L^2(0, T; L^2(\Omega))$ and let (6.5) hold. The following are equivalent:

(i) $u \in L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega))$ satisfies (6.2) and (6.3).

(ii) $u \in L^2(0, T; L^2(\Omega))$ satisfies

$$u(t) = S(t)a + \int_0^t K(t-s) \sum_{j=1}^d b_j \partial_j u(\cdot, s) ds + \int_0^t K(t-s) F(s) ds, \quad 0 < t < T. \quad (6.10)$$

Equation (6.10) corresponds to formula (5.2) for an initial value problem for fractional ordinary differential equation.

According to the parabolic equation (e.g., Pazy [29]), the solution guaranteed by Theorem 6.1 is called a strong solution of the fractional partial differential equation, while the solution to an integrated equation (6.10) is called a mild solution. Proposition 6.1 asserts the equivalence between these two kinds of solutions under assumption that $a \in H^1_0(\Omega)$ and $F \in L^2(0, T; L^2(\Omega))$.

For the proof of Proposition 6.1, we show two lemmata. Henceforth, for $v \in L^2(\Omega)$, we define $\partial_j v \in H^{-1}(\Omega) := (H^1_0(\Omega))^\prime$, $1 \leq j \leq d$, by

$$H^{-1}(\Omega) < \partial_j v, \varphi >_{H^1_0(\Omega)} := -(v, \partial_j \varphi)_{L^2(\Omega)} \quad \text{for all } \varphi \in H^1_0(\Omega).$$

Then by the definition of the norm of the linear functional, we have

$$\|\partial_j v\|_{H^{-1}(\Omega)} = \sup_{\|\varphi\|_{H^1_0(\Omega)} = 1} |H^{-1}(\Omega) < \partial_j v, \varphi >_{H^1_0(\Omega)}|$$

$$\leq \sup_{\|\varphi\|_{H^1_0(\Omega)} = 1} |(v, \partial_j \varphi)_{L^2(\Omega)}| \leq \|v\|_{L^2(\Omega)}. \quad (6.11)$$

We recall that $L^{-\frac{1}{2}}$ is defined as an operator from $L^2(\Omega)$ to $L^2(\Omega)$ by (6.7), while also $L^\frac{1}{2}: H^1_0(\Omega) \rightarrow L^2(\Omega)$ is defined by (6.7).

Then we can state the first lemma which involves the extension of $L^{-\frac{1}{2}}: L^2(\Omega) \rightarrow L^2(\Omega)$.

**Lemma 6.1.**
(i) The operator $L^{-\frac{1}{2}}$ can be extended as a bounded operator from $H^{-1}(\Omega)$ to $L^2(\Omega)$. Henceforth, by the same notation, we denote the extension. There exists a constant $C > 0$ such that
\[ C^{-1} \| v \|_{H^{-1}(\Omega)} \leq \| L^{-\frac{1}{2}} v \|_{L^2(\Omega)} \leq C \| v \|_{H^{-1}(\Omega)} \] for all $v \in H^{-1}(\Omega)$.

(ii) $K(t)v = L^{-\frac{1}{2}} K(t) L^{-\frac{1}{2}} v$ for $t > 0$ and $v \in H^{-1}(\Omega)$.

(iii) $\| L^{-\frac{1}{2}} (b_j \partial_j v) \|_{L^2(\Omega)} \leq C \| v \|_{L^2(\Omega)}$ for all $v \in L^2(\Omega)$.

The second lemma is

**Lemma 6.2.**

(i) For $a \in L^2(\Omega)$, we have $S(t)a \in D(L)$ for $t > 0$ and
\[ \partial_t^\alpha (S(t)a - a) + LS(t)a = 0, \quad 0 < t < T. \]

Moreover there exists a constant $C_0 > 0$ such that
\[ \| S(t)a - a \|_{H^\alpha(0,T;L^2(\Omega))} \leq C_0 \| a \|_{H^\alpha(\Omega)} \]
for all $a \in H^1_0(\Omega)$. Here $C_0 > 0$ is independent of $a \in H^1_0(\Omega)$ and $T > 0$.

(ii) We have
\[ \partial_t^\alpha \left( \int_0^t K(t-s)F(s)ds \right) + L \left( \int_0^t K(t-s)F(s)ds \right) = F(t), \quad 0 < t < T \]
for $F \in L^2(0,T;L^2(\Omega))$. Moreover, there exists a constant $C_1 = C_1(T) > 0$ such that
\[ \left\| \int_0^t K(t-s)F(s)ds \right\|_{H^\alpha(0,T;L^2(\Omega))} \leq C_1 \| F \|_{L^2(0,T;L^2(\Omega))} \]
for each $F \in L^2(0,T;L^2(\Omega))$.

We note that Lemma 6.2 corresponds to Propositions 4.1 (i) and 4.2.

Let Lemmata 6.1 and 6.2 be proved. Then, the proof of (i) $\rightarrow$ (ii) of Proposition 6.1 can be done similarly to [32]. The proof of (ii) $\rightarrow$ (i) of the proposition can be derived from Lemmata 6.1 and 6.2, and we omit the details. The proofs of the lemmata are provided in Appendix.
§6.2. Continuity at $t = 0$

As is discussed in Sections 1, 2 and 5, the continuity of solutions to fractional differential equations at $t = 0$ is delicate, which requires careful treatments for initial conditions. However, if we assume $F = 0$ in (6.2), then we can prove the following sufficient continuity.

**Theorem 6.3.**

Under (6.4) and (6.5), for $a \in L^2(\Omega)$, the solution $u$ to (6.2) - (6.3) satisfies

$$u \in C([0,T];L^2(\Omega)).$$

The theorem means that $\lim_{t \to 0} \|u(\cdot, t) - a\|_{L^2(\Omega)} = 0$, but the continuity at $t = 0$ breaks if a non-homogeneous term $F \neq 0$, which is already shown in Example 5.1 in Section 5 concerning a fractional ordinary differential equations.

In the case of $b_j = 0$, $1 \leq j \leq d$, the same result is proved in [32] and we can refer also to [14]. However, it seems no proofs for a non-symmetric elliptic operator $A$, although one naturally expect the same continuity. The proof is typical as arguments by the operator theory applied to the classical partial differential equations and we carry out similar arguments for fractional differential equations within our framework.

**Proof.**

**First Step.**

From Lemma 6.1, it follows that the solution $u$ to (6.2) - (6.3) with $F = 0$ is given by

$$u(\cdot, t) = S(t)a + \int_0^t K(t-s) \sum_{j=1}^d b_j \partial_j u(\cdot, s) ds =: S(t)a + M(u)(t), \quad 0 < t < T. \quad (6.12)$$

The proof of Theorem 6.3 is based on an approximating sequence for the solution $u(t)$ constructed by

$$u_1(t) := S(t)a, \quad u_{k+1}(t) := S(t)a + (Mu_k)(t), \quad k \in \mathbb{N}, \ 0 < t < T.$$

First we can prove

$$S(\cdot)a \in C([0,T];L^2(\Omega)). \quad (6.13)$$
Verification of (6.13). For $\ell \le m$, using (4.4) with $\lambda_n > 0$, for $0 \le t \le T$ we estimate
\[
\left\| \sum_{n=\ell}^{m} (a_n \varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n \right\|_{L^2(\Omega)}^2 \le \sum_{n=\ell}^{m} |(a_n \varphi_n)|^2 |E_{\alpha,1}(-\lambda_n t^\alpha)|^2 \le C \sum_{n=\ell}^{m} |(a_n \varphi_n)|^2.
\]
Therefore,
\[
\lim_{\ell,m \to \infty} \left\| \sum_{n=\ell}^{m} (a_n \varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n \right\|_{C([0,T];L^2(\Omega))} = 0,
\]
which means that
\[
\sum_{n=1}^{N} (a_n \varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n \in C([0,T];L^2(\Omega))
\]
converges to $S(t)a$ in $C([0,T];L^2(\Omega))$. Thus the verification of (6.13) is complete. ■

Second Step. We prove
\[
\int_{0}^{t} L^{\frac{\alpha}{2}} K(s) w(t - s) ds \in C([0,T];L^2(\Omega)) \quad \text{for } w \in C([0,T];L^2(\Omega)).
\]
Verification of (6.14).
Let $0 < t < T$ be arbitrarily fixed. For small $h > 0$, we have
\[
\int_{0}^{t+h} L^{\frac{\alpha}{2}} K(s) w(t + h - s) ds - \int_{0}^{t} L^{\frac{\alpha}{2}} K(s) w(t - s) ds
\]
\[
= \int_{t}^{t+h} L^{\frac{\alpha}{2}} K(s) w(t + h - s) ds + \int_{0}^{t} L^{\frac{\alpha}{2}} K(s) (w(t + h - s) - w(t - s)) ds
\]
\[
=: I_1(t,h) + I_2(t,h).
\]
Then by (6.9) we can estimate
\[
\|I_1(t,h)\| \le C \left( \int_{t}^{t+h} s^{\frac{\alpha}{2} - 1} ds \right) \|w\|_{C([0,T];L^2(\Omega))}
\]
\[
= 2C \frac{((t + h) t^{\alpha} - t^{\alpha})}{\alpha} \|w\|_{C([0,T];L^2(\Omega))} \to 0 \quad \text{as } h \to 0.
\]
Next
\[
\|I_2(t,h)\| \le C \int_{0}^{t} s^{\frac{\alpha}{2} - 1} \|w(t + h - s) - w(t - s)\| ds.
\]
We have $\max_{0 \leq s \leq T} \|w(t + h - s) - w(t - s)\| \rightarrow 0$ as $h \to 0$ with fixed $t$ by $w \in C([0, T]; L^2(\Omega))$. Hence, since $s^{\frac{3}{2} - 1} \in L^1(0, T)$, the Lebesgue convergence theorem yields that $\lim_{h \to 0} \|I_2(t, h)\| = 0$. We can argue similarly also for $h < 0$, $t = 0$ and $t = T$, and so the verification of (6.14) is complete. ■

Next we proceed to the proofs of (6.15) and (6.16):

$$u_k \in C([0, T]; L^2(\Omega)), \quad k \in \mathbb{N} \quad (6.15)$$

and

$$u_k \text{ converges in } C([0, T]; L^2(\Omega)) \text{ as } k \to \infty. \quad (6.16)$$

**Third Step: Proof of (6.15).** We will prove by induction. By (6.13), we see that $u_1(t) = S(t)a \in C([0, T]; L^2(\Omega))$. We assume that $u_k \in C([0, T]; L^2(\Omega))$. Then, by Lemma 6.1 (ii), we have

$$M(u_k)(t) = \int_0^t K(t - s) \left( \sum_{j=1}^d b_j \partial_j u_k(s) \right) \, ds = \int_0^t L_{\frac{1}{2}} K(t - s) L_{-\frac{1}{2}} \left( \sum_{j=1}^d b_j \partial_j u_k(s) \right) \, ds.$$

Lemma 6.1 (iii) yields

$$\|L_{-\frac{1}{2}}(b_j \partial_j u_k(t)) - L_{-\frac{1}{2}}(b_j \partial_j u_k(t'))\|_{L^2(\Omega)} \leq C \|u_k(t) - u_k(t')\|_{L^2(\Omega)}, \quad t, t' \in [0, T]$$

for $1 \leq j \leq d$, so that

$$L_{-\frac{1}{2}} b_j \partial_j u_k \in C([0, T]; L^2(\Omega)), \quad 1 \leq j \leq d.$$

Therefore, the application of (6.14) to $M(u_k)$ implies

$$M(u_k) \in C([0, T]; L^2(\Omega)).$$

Consequently,

$$u_{k+1} = S(\cdot)a + M(u_k) \in C([0, T]; L^2(\Omega)).$$

Thus the induction completes the proof of (6.15). ■

**Fourth Step: Proof of (6.16).**

We have

$$(u_{k+2} - u_{k+1})(t) = M(u_{k+1} - u_k)(t)$$
\[
= \int_0^t L^{\frac{1}{2}} K(t - s) \sum_{j=1}^d L^{-\frac{1}{2}} (b_j \partial_j (u_{k+1} - u_k)) ds, \quad k \in \mathbb{N}, \ 0 < t < T.
\]

We set \( v_k := u_{k+1} - u_k \). Then Lemma 6.1 (iii) and (6.9) yield

\[
\|v_{k+1}(t)\| \leq \int_0^t \left\| L^{\frac{1}{2}} K(t - s) \sum_{j=1}^d L^{-\frac{1}{2}} (b_j \partial_j v_k(s)) \right\| ds \leq C \int_0^t (t - s)^{1 - \frac{1}{2}} \|v_k(s)\| ds, \quad k \in \mathbb{N}. \quad (6.17)
\]

Setting \( M_0 := \|v_1\|_{C([0, T]; L^2(\Omega))} \), we apply (6.17) with \( k = 1 \) to obtain

\[
\|v_2(t)\| \leq C \int_0^t (t - s)^{\frac{1}{2} - 1} ds M_0 = \frac{C M_0}{\Gamma \left( \frac{1}{2} \alpha \right)} t^{\frac{1}{2} \alpha}, \quad 0 < t < T.
\]

Therefore, substituting this into (6.17) with \( k = 2 \), we obtain

\[
\|v_3(t)\| \leq \frac{C^2 M_0}{\Gamma \left( \frac{1}{2} \alpha \right)} \int_0^t (t - s)^{\frac{1}{2} - 1} s^{\frac{1}{2} \alpha} ds = C^2 M_0 \frac{\Gamma \left( \frac{1}{2} \alpha + 1 \right)}{\Gamma(\alpha + 1)} t^{\alpha}, \quad 0 < t < T.
\]

Continuing this estimate, we can prove

\[
\|v_{k+1}(t)\| \leq \frac{1}{2} \alpha M_0 C^k \left( \frac{1}{2} \alpha \right)^{k-1} \Gamma \left( \frac{1}{2} \alpha \right) t^{\frac{1}{2} \alpha}, \quad 0 < t < T
\]

for all \( k \in \mathbb{N} \). Consequently,

\[
\|v_{k+1}\|_{C([0, T]; L^2(\Omega))} \leq \frac{C_1}{\Gamma \left( \frac{1}{2} k \alpha + 1 \right)} \left( C \Gamma \left( \frac{1}{2} \alpha \right) \frac{1}{2} \alpha \right)^k, \quad k \in \mathbb{N}.
\]

By the asymptotic behavior of the gamma function, we can verify

\[
\lim_{k \to \infty} \left( C \Gamma \left( \frac{1}{2} \alpha \right) T^{\frac{1}{2} \alpha} \right)^k = 0,
\]

so that \( \sum_{k=0}^\infty \|v_k\|_{C([0, T]; L^2(\Omega))} \) converges. Therefore, \( \lim_{k \to \infty} u_k \) exists in \( C([0, T]; L^2(\Omega)) \). Thus the proof of (6.16) is complete. ■

From the uniqueness of solution to (6.16), we can derive that its limit is \( u \), the solution to (6.2) - (6.3) with \( F = 0 \). Since \( u_k \in C([0, T]; L^2(\Omega)) \) for \( k \in \mathbb{N} \), the limits is also in \( C([0, T]; L^2(\Omega)) \). Thus the proof of Theorem 6.3 is complete. ■

§6.3. Stronger regularity in time of solution
Again we consider (6.2) - (6.3) with $F \neq 0$. Theorem 6.1 provides a basic result on the unique existence of $u$ for $F \in L^2(0, T; L^2(\Omega))$ and $a \in H^1_0(\Omega)$, while Theorem 6.2 is the well-posedness for $a$ and $F$ which is less regular in $x$.

Here we consider stronger time-regular $F$ to improve the regularity of solution $u$. Thanks to the framework of $\partial_t^\alpha$ defined by (2.4), the argument for improving the regularity of solution is automatic.

**Theorem 6.4.**

Let $0 < \alpha < 1$ and $\beta > 0$. We assume $a \in H^2(\Omega) \cap H^1_0(\Omega)$ and

$$F - Aa \in H_\beta(0, T; L^2(\Omega)).$$

(6.18)

Then there exists a unique solution $u$ to (6.2) - (6.3) such that

$$u - a \in H_\beta(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap H_{\alpha+\beta}(0, T; L^2(\Omega)),$$

(6.19)

and we can find a constant $C > 0$ such that

$$\|u - a\|_{H_\beta(0, T; H^2(\Omega))} + \|u - a\|_{H_{\alpha+\beta}(0, T; L^2(\Omega))} \leq C(\|F - Aa\|_{H_\beta(0, T; L^2(\Omega))} + \|a\|_{H^2(\Omega)}).$$

If $0 < \beta < \frac{1}{2}$, then (6.18) is equivalent to that $F \in H_\beta(0, T; L^2(\Omega))$ under condition $a \in H^2(\Omega) \cap H^1_0(\Omega)$. If $\frac{1}{2} < \beta \leq 1$, then (6.18) is equivalent to

$$F \in H_\beta(0, T; L^2(\Omega)), \quad Aa = F(\cdot, 0) \text{ in } \Omega$$

(6.20)

under condition $a \in H^2(\Omega) \cap H^1_0(\Omega)$. Condition (6.18) is a compatibility condition which is necessary for lifting up the regularity of the solution, and is required also for the parabolic equation for instance (see Theorem 6 in Chapter 7, Section 1 in Evans [5]).

From Theorem 6.4, we can easily derive

**Corollary 6.1.**

Let $a = 0$ and $0 < \alpha < 1$. We assume

$$F \in H_\alpha(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)).$$

Then

$$A^2u, \partial_t^{2\alpha}u \in L^2(0, T; L^2(\Omega)).$$
Proof of Theorem 6.4.

We will gain the regularity of the solution $u$ by means of an equation which can be expected to be satisfied by $\partial_t^\alpha u$, although such an equation is not justified for the moment. We consider

$$\partial_t^\alpha v + Av = \partial_\beta(F - Aa), \quad v \in H_\alpha(0, T; L^2(\Omega)). \tag{6.21}$$

Then, by $\partial_t^\beta(F - Aa) \in L^2(0, T; L^2(\Omega))$, Theorem 6.1 yields that $v$ exists and $v \in H_\alpha(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$.

We set $\tilde{u} := J^\beta v + a$. Then Proposition 2.5 (i) implies

$$\tilde{u} - a \in H_{\alpha + \beta}(0, T; L^2(\Omega)) \cap H_\beta(0, T; H^2(\Omega) \cap H_0^1(\Omega)). \tag{6.22}$$

In view of Proposition 2.5 (ii) and $v \in H_\alpha(0, T; L^2(\Omega))$, we have

$$J^{-\alpha}(\tilde{u} - a) = J^{-\alpha} J^\beta v = J^\beta J^{-\alpha} v,$$

and so $J^{-\alpha}(\tilde{u} - a) = J^\beta(J^{-\alpha} v)$. Moreover, by $v \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$, we see

$$A\tilde{u} = A(J^\beta v + a) = J^\beta Av + Aa.$$

Therefore,

$$J^{-\alpha}(\tilde{u} - a) + A\tilde{u} = J^\beta(J^{-\alpha} v + Av) + Aa.$$

In terms of (6.21), we obtain

$$J^{-\alpha}(\tilde{u} - a) + A\tilde{u} = J^\beta(\partial_\beta(F - Aa)) + Aa = F.$$

Therefore, combining (6.22), we can verify that $\tilde{u} - a$ satisfies (6.2) - (6.3), and the uniqueness of solution yields $u = \tilde{u} - a$. Thus the proof of Theorem 6.4 is complete. ■

Proof of Corollary 6.1.

Theorem 6.4 yields

$$u \in H_\alpha(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H_{2\alpha}(0, T; L^2(\Omega)).$$

Consequently, applying also Theorem 3.1, we deduce that $\partial_t^{2\alpha} u \in L^2(0, T; L^2(\Omega))$. Moreover, since $\partial_t^\alpha \partial_t^\alpha = \partial_t^{2\alpha}$ by Theorem 3.1, operating $\partial_t^\alpha$ to (6.2), we obtain

$$\partial_t^\alpha Au = -\partial_t^{2\alpha} u + \partial_t^\alpha F \in L^2(0, T; L^2(\Omega)). \tag{6.23}$$
On the other hand, \( A(\partial_t^\alpha u + Au - F) = 0 \) in \( L^2(0, T; (C_0^\infty(\Omega))') \). Consequently, in terms of (6.23), we reach
\[
A^2 u = -A\partial_t^\alpha + AF \in L^2(0, T; L^2(\Omega)).
\]
Thus the proof of Corollary 6.1 is complete. ■

§6.4. Weaker regularity in time of solution

First we introduce a function space \( -\alpha H(0, T; X) \) for a Banach space \( X \). Indeed, thanks to Proposition 2.9 or Theorem 2.1, the operator \( J'_\alpha : -\alpha H(0, T) \longrightarrow L^2(0, T) \) is surjective and isomorphism for \( \alpha > 0 \), so that we define
\[
\begin{cases}
-\alpha H(0, T; X) := \{ v \in L^2(0, T; X); J'_\alpha v \in L^2(0, T; X) \}, \\
\text{with the norm } ||v||_{-\alpha H(0, T; X)} := ||J'_\alpha v||_{L^2(0, T; X)}.
\end{cases}
\] (6.24)

By \( \partial_t^\alpha := (J'_\alpha)^{-1} : L^2(0, T) \longrightarrow -\alpha H(0, T) \), we see also that
\[
\lim_{n \to \infty} v_n = v \text{ in } L^2(0, T; L^2(\Omega)) \text{ implies } \lim_{n \to \infty} \partial_t^\alpha v_n = \partial_t^\alpha v \text{ in } -\alpha H(0, T). \quad (6.25)
\]

In this subsection, we consider
\[
\partial_t^\alpha (u - a) + Au = F \quad \text{in } -\alpha H(0, T; L^2(\Omega)) \quad (6.26)
\]
and
\[
uu - a \in L^2(0, T; L^2(\Omega)), \quad u \in -\alpha H(0, T; H^2(\Omega) \cap H^1_0(\Omega)). \quad (6.27)
\]

As is argued in Example 5.2, for \( \alpha > \frac{1}{2} \), a source term
\[
F(x, t) = \delta_{t_0}(t)f(x) \in -\alpha H(0, T; L^2(\Omega))
\]
with \( f \in L^2(\Omega) \), describes an impulsive source at \( t = t_0 \), and it is not only mathematically but also physically meaningful to treat singular source \( F \in -\alpha H(0, T; L^2(\Omega)) \) with \( \alpha > 0 \). We here state one result on the well-posedness for (6.26) - (6.27).

**Theorem 6.5.**

Let \( 0 < \alpha < 1 \). We assume
\[
F \in -\alpha H(0, T; L^2(\Omega)), \quad a \in L^2(\Omega).
\]
Then there exists a unique solution \( u \in L^2(0, T; L^2(\Omega)) \cap -^{\alpha}H(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \) to (6.26) - (6.27) and we can find a constant \( C > 0 \) such that

\[
\|u\|_{L^2(0,T;L^2(\Omega))} + \|u\|_{-^{\alpha}H(0,T;H^2(\Omega)\cap H^1_0(\Omega))} \leq C(\|a\|_{L^2(\Omega)} + \|F\|_{-^{\alpha}H(0,T;L^2(\Omega))})
\]  

(6.28)

for each \( a \in L^2(\Omega) \) and \( F \in -^{\alpha}H(0,T;L^2(\Omega)) \).

Proof.

We will prove by creating an equation which should be satisfied by \( J'_\alpha u \), and such an equation can be given by

\[
\partial^\alpha_t J'_\alpha (u - a) + AJ'_\alpha u = J'_\alpha F,
\]

where we have to make justification. Thus we consider the solution \( v \) to

\[
\partial^\alpha_t v + Av = J'_\alpha F + a, \quad v \in H_\alpha(0,T;L^2(\Omega)).
\]

(6.29)

Since \( J'_\alpha F + a \in L^2(0,T;L^2(\Omega)) \), Theorem 6.1 implies the unique existence of solution \( v \in H_\alpha(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega) \cap H^1_0(\Omega)) \). Therefore, Proposition 2.10 yields \( (J'_\alpha)^{-1}v = J^{-\alpha}v = \partial^\alpha_t v \) by \( v \in H_\alpha(0,T;L^2(\Omega)) \). Setting

\[
\tilde{u} := (J'_\alpha)^{-1}v = J^{-\alpha}v,
\]

(6.30)

we readily verify

\[
\tilde{u} \in L^2(0,T;L^2(\Omega)) \cap -^{\alpha}H(0,T;H^2(\Omega) \cap H^1_0(\Omega))
\]

by Propositions 2.2 and 2.9 (ii). Furthermore, since (6.30) implies \( (J'_\alpha)^{-1}v = \partial^\alpha_t v \), and \( v = J'_\alpha \tilde{u} \), we obtain

\[
(J'_\alpha)^{-1}(\tilde{u} - a) + A\tilde{u} - F = (J'_\alpha)^{-1}(\tilde{u} - a + AJ'_\alpha \tilde{u} - J'_\alpha F)
\]

\[
= (J'_\alpha)^{-1}((J'_\alpha)^{-1}v - a + Av - J'_\alpha F) = (J'_\alpha)^{-1}(\partial^\alpha_t v + Av - J'_\alpha F - a) = 0
\]

in \(-^{\alpha}H(0,T;L^2(\Omega))\).

Since \( J'_\alpha \) is injective, the application of (6.29) implies \( \partial^\alpha_t (\tilde{u} - a) + A\tilde{u} = F \) and \( \tilde{u} - a \in L^2(0,T;L^2(\Omega)) \). Hence, \( \tilde{u} \) is a solution to (6.26) - (6.27), and (6.28) holds.

We finally prove the uniqueness of solution. Let \( w \in L^2(0,T;L^2(\Omega)) \cap -^{\alpha}H(0,T;H^2(\Omega) \cap H^1_0(\Omega)) \) and \( \partial^\alpha_t w + Aw = 0 \) hold in \(-^{\alpha}H(0,T;L^2(\Omega))\). Then

\[
J'_\alpha \partial^\alpha_t w + J'_\alpha Aw = J'_\alpha \partial^\alpha_t w + A J'_\alpha w = 0 \quad \text{in } L^2(0,T;L^2(\Omega)).
\]

(6.31)
By $w \in L^2(0, T; L^2(\Omega))$ and (2.13), we see
\[ J'_\alpha \partial^\alpha_t w = J'_\alpha (J'_\alpha)^{-1} w = w. \]

On the other hand,
\[ w = (J^\alpha)^{-1} J^\alpha w = \partial^\alpha_t J^\alpha w \]
by (2.4) and $J^\alpha w \in H_\alpha(0, T)$. Hence, $J'_\alpha \partial^\alpha_t w = \partial^\alpha_t J^\alpha w$. Therefore, in terms of (6.31), we have $J^\alpha w \in H_\alpha(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega))$ and $\partial^\alpha_t (J^\alpha w) + A(J^\alpha w) = 0$ in $L^2(0, T; L^2(\Omega))$. Hence, the uniqueness asserted by Theorem 6.1 implies $J^\alpha w = 0$ in $(0, T)$.

Since $J'_\alpha : L^2(0, T) \rightarrow L^2(0, T)$ is injective, we obtain $w = 0$ in $(0, T)$. Thus the proof of Theorem 6.5 is complete.

Finally in this subsection, in order to describe the flexibility of our approach, we show

**Proposition 6.2.**

Let $0 < \alpha < 1$. In (6.26) and (6.27), we assume
\[ a = 0, \quad F \in -^\alpha H(0, T; L^2(\Omega)) \cap -^{2\alpha} H(0, T; H^2(\Omega) \cap H^1_0(\Omega)). \]

Then the solution $u$ satisfies
\[ u \in -^{2\alpha} H(0, T; D(A^2)) \subset -^{2\alpha} H(0, T; H^4(\Omega)). \]

The proposition means that under (6.32), the solution $u$ can hold the $H^4(\Omega)$-regularity in $x$ at the expense of weaker regularity $-^{2\alpha} H(0, T)$ in $t$.

**Proof of Proposition 6.2.**

By Theorem 6.5, the solution $u$ to (6.26) - (6.27) with $a = 0$ exists uniquely. More precisely, we have $(J'_\alpha)^{-1} u + A u = F$ in $-^\alpha H(0, T; L^2(\Omega))$ and
\[ u \in -^\alpha H(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap L^2(0, T; L^2(\Omega)). \]

Therefore $J'_\alpha (J'_\alpha)^{-1} = (J'_\alpha)^{-1} J'_\alpha$ in $L^2(0, T; L^2(\Omega))$ and $J'_\alpha A u = A J'_\alpha u$, and so
\[ (J'_\alpha)^{-1} J'_\alpha u + A J'_\alpha u = J'_\alpha F \in L^2(0, T; L^2(\Omega)). \]

Hence, operating $J'_\alpha$ again, we obtain
\[ (J'_\alpha)(J'_\alpha)^{-1} J'_\alpha u + A J'_\alpha J'_\alpha u = J'_\alpha J'_\alpha F. \]
Consequently, we obtain $J'_\alpha (J'_\alpha)^{-1}J'_\alpha u = (J'_\alpha)^{-1}(J'_\alpha J'_\alpha u)$. Setting $v := J'_\alpha J'_\alpha u$, we have

$$\partial_t^\alpha v + Av = J'_\alpha J'_\alpha F, \quad v \in H_\alpha(0, T; L^2(\Omega)).$$

(6.33)

We note that Proposition 2.9 (iii) yields $J'_\alpha (J'_\alpha u) = J^\alpha (J'_\alpha u)$ by $J'_\alpha u \in L^2(0, T; L^2(\Omega))$ and that $J'_\alpha F \in L^2(0, T; L^2(\Omega))$ by $F \in -^\alpha H(0, T; L^2(\Omega))$ and Proposition 2.9 (ii). Moreover $J'_\alpha (J'_\alpha F) \in H_\alpha(0, T; L^2(\Omega))$ by Propositions 2.9 (iii) and 2.2.

Applying Proposition 2.9 (ii) twice to $F \in -^\alpha H(0, T; H^2(\Omega) \cap H^1_0(\Omega))$, we obtain $J'_\alpha F \in -^\alpha H(0, T; H^2(\Omega) \cap H^1_0(\Omega))$, so that $J'_\alpha (J'_\alpha F) \in L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega))$. Therefore the application of Corollary 6.1 to (6.33), yields

$$A^2 v = A^2 J'_\alpha J'_\alpha u \in L^2(0, T; L^2(\Omega)).$$

By $u \in L^2(0, T; L^2(\Omega))$, we see that $J'_\alpha J'_\alpha u \in L^2(0, T; D(A^2)) \subset L^2(0, T; H^4(\Omega))$. In terms of Proposition 2.9 (ii), it follows $u \in -^\alpha H(0, T; D(A^2))$. Thus the proof of Proposition 6.2 is complete. 

§6.5. Initial boundary value problem for multi-term time-fractional partial differential equations

Let $N \in \mathbb{N}$, $0 < \alpha_1 < \cdots < \alpha_N < \alpha < 1$, $q_k \in C(\overline{\Omega})$ for $1 \leq k \leq N$. We recall that $A$ is defined by (1.1) and (6.1). We discuss an initial boundary value problem for a multi-term time-fractional partial differential equation:

$$\partial_t^\alpha (u - a) + \sum_{k=1}^N q_k(x) \partial_t^{\alpha_k} (u - a) + Au = F$$

(6.34)

and

$$u - a \in H_\alpha(0, T; L^2(\Omega)).$$

(6.35)

Then we can will prove

Theorem 6.6.

For each $a \in H^1_0(\Omega)$ and $F \in L^2(0, T; L^2(\Omega))$, there exists a unique solution $u \in L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega))$ to (6.34) - (6.35). Moreover there exists a constant $C > 0$ such that

$$\|u\|_{L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega))} + \|u - a\|_{H_\alpha(0, T; L^2(\Omega))} \leq C(\|a\|_{H^1_0(\Omega)} + \|F\|_{L^2(0, T; L^2(\Omega))})$$

(6.36)

for each $a \in H^1_0(\Omega)$ and $F \in L^2(0, T; L^2(\Omega))$. 

Here the constant $C > 0$ depends on $\Omega$, $T$, $A$, $\alpha_k$, $q_k$ for $1 \leq k \leq N$.

In the same way as in Subsections 6.3 and 6.4, we can establish stronger and weaker regular solutions, but we omit the details.

The arguments are direct within our framework based on $\partial_t^\alpha : -\beta H(0, T) \rightarrow -\alpha-\beta H(0, T)$ and $\partial_t^\alpha : H_{\alpha+\gamma}(0, T) \rightarrow H_\gamma(0, T)$ with $\gamma \geq 0$ and $\beta \geq 0$. For fractional ordinary differential equations, we mentioned the corresponding initial value problem as (5.4) in Section 5.

The multi-term time-fractional partial differential equations are considered for example in Kian [15], Li, Imanuvilov and Yamamoto [20], Li, Liu and Yamamoto [21] for symmetric $A$ which makes arguments simple. Non-symmetric $A$ contains an advection term $\sum_{j=1}^d b_j \partial_j u$, and so it is physically more feasible model. However, we do not know the existing results on the corresponding well-posedness for non-symmetric $A$. In fact, our method for initial boundary value problems is not restricted to symmetric $A$. Moreover we can treat weaker and stronger solutions in the same way as in Subsections 6.3 and 6.4, and we omit the details.

We can easily extend our method to the case where the coefficients $q_k$, $1 \leq k \leq N$ and $b_j$, $1 \leq j \leq d$ and $c$ depend on time.

Proof.

First Step.

We prove the theorem under the assumption that $b_j = 0$ for $1 \leq j \leq d$ and $c \leq 0$ on $\Omega$.

By Proposition 6.1 and Lemma 6.2, it suffices to prove that there exists a unique $u - a \in H_\alpha(0, T; L^2(\Omega))$ satisfying

$$u(t) - a = S(t)a - a + \int_0^t K(t - s)F(s)ds + \sum_{k=1}^N \int_0^t K(t-s)q_k \partial_t^{\alpha_k}(u(s) - a)ds, \quad 0 < t < T. \quad (6.37)$$

In this step, we prove

Lemma 6.3.

Let $k = 1, \ldots, N$.

(i)

$$\|\partial_t^{\alpha_k} K(t)a\|_{L^2(\Omega)} \leq Ct^{\alpha - \alpha_k - 1}\|a\|_{L^2(\Omega)}, \quad 0 < t < T, \quad a \in L^2(\Omega).$$
(ii) For $G \in L^2(0, T; L^2(\Omega))$, we have $\int_0^t K(t - s)G(s)ds \in H_{\alpha_k}(0, T; L^2(\Omega))$ and

$$
\partial_t^{\alpha_k} \int_0^t K(t - s)G(s)ds = \int_0^t \partial_t^{\alpha_k} K(t - s)G(s)ds, \quad 0 < t < T.
$$

**Proof of Lemma 6.3.**

(i) We set

$$
K_m(t)a := \sum_{n=1}^m t^{\alpha-1} E_{\alpha,a}(-\lambda_n t^\alpha)(a, \varphi_n) \varphi_n \quad \text{for } m \in \mathbb{N} \text{ and } a \in L^2(\Omega).
$$

Then, by Lemma 4.1 (ii), we see that

$$
\partial_t^{\alpha_k} K_m(t)a = \sum_{n=1}^m \partial_t^{\alpha_k} (t^{\alpha-1} E_{\alpha,a}(-\lambda_n t^\alpha))(a, \varphi_n) \varphi_n
$$

$$
= \sum_{n=1}^m t^{\alpha-\alpha_k-1} E_{\alpha,a-\alpha_k}(-\lambda_n t^\alpha)(a, \varphi_n) \varphi_n.
$$

Therefore, applying (4.4), we obtain

$$
\| \partial_t^{\alpha_k} K_m(t)a \|_{L^2(\Omega)}^2 = \sum_{n=1}^m t^{2(\alpha-\alpha_k-1)} \| E_{\alpha,a-\alpha_k}(-\lambda_n t^\alpha) \|^2 (a, \varphi_n)^2
$$

$$
\leq Ct^{2(\alpha-\alpha_k-1)} \sum_{n=1}^m |(a, \varphi_n)|^2. \quad (6.38)
$$

Letting $m \to \infty$, we complete the proof of part (i).

(ii) By Proposition 4.2, we can readily prove $\int_0^t K(t - s)G(s)ds \in H_{\alpha}(0, T; L^2(\Omega)) \subset H_{\alpha_k}(0, T; L^2(\Omega))$. Noting that $G(x, \cdot) \in L^2(0, T)$ for almost all $x \in \Omega$, in terms of Lemma 4.1 (iii), we have

$$
\partial_t^{\alpha_k} \int_0^t K_m(x, t - s)G(x, s)ds = \int_0^t \partial_t^{\alpha_k} K_m(x, t - s)G(x, s)ds, \quad x \in \Omega, \ 0 < t < T
$$

for $m \in \mathbb{N}$.

Similarly to (6.38), we estimate

$$
\left\| \int_0^t (\partial_t^{\alpha_k} K_m - \partial_t^{\alpha_k} K)(t - s)G(\cdot, s)ds \right\|_{L^2(\Omega)} \leq \int_0^t \left\| (\partial_t^{\alpha_k} K_m - \partial_t^{\alpha_k} K)(t - s)G(\cdot, s) \right\|_{L^2(\Omega)} ds
$$
\[
\leq C \int_0^t (t-s)^{\alpha-\alpha_k-1} \left( \sum_{n=m+1}^{\infty} |(G(s), \varphi_n)_{L^2(\Omega)}|^2 \right)^{\frac{1}{2}} ds.
\]

The Young inequality on the convolution yields
\[
\left\| \int_0^t (\partial_\alpha^k K_m - \partial_\alpha^k K)(t-s)G(\cdot, s)ds \right\|_{L^1(0,T;L^2(\Omega))} \\
\leq C \left\| s^{\alpha-\alpha_k-1} \right\|_{L^1(0,T)} \left\| \left( \sum_{n=m+1}^{\infty} |(G(s), \varphi_n)_{L^2(\Omega)}|^2 \right)^{\frac{1}{2}} \right\|_{L^2(0,T)} \\
\leq CT^{\alpha-\alpha_k} \left( \int_0^T \sum_{n=m+1}^{\infty} |(G(s), \varphi_n)_{L^2(\Omega)}|^2 ds \right)^{\frac{1}{2}} \to 0
\]
as \( m \to \infty \) by \( G \in L^2(0,T;L^2(\Omega)) \). Therefore,
\[
\lim_{m \to \infty} \int_0^t \partial_\alpha^k K_m(t-s)G(s)ds = \int_0^t \partial_\alpha^k K(t-s)G(s)ds \quad \text{in} \quad L^1(0,T;L^2(\Omega)). \tag{6.39}
\]

Similarly, using (4.4), we can verify
\[
\lim_{m \to \infty} \int_0^t K_m(t-s)G(s)ds = \int_0^t K(t-s)G(s)ds \quad \text{in} \quad L^1(0,T;L^2(\Omega)). \tag{6.40}
\]
Choosing \( \gamma > \frac{1}{2} \), we see that \( L^1(0,T) \subset \gamma H(0,T) \) and the series in (6.40) converges also in \( \gamma H(0,T;L^2(\Omega)) \). Since \( \partial_\alpha^k : \gamma H(0,T) \rightarrow \gamma^{-\alpha_k} H(0,T) \) is an isomorphism by Theorem 2.1, we obtain
\[
\lim_{m \to \infty} \partial_\alpha^k \int_0^t K_m(t-s)G(s)ds = \partial_\alpha^k \int_0^t K(t-s)G(s)ds \quad \text{in} \quad \gamma^{-\alpha_k} H(0,T;L^2(\Omega)).
\]
Combining this with (6.39), we finish the proof of Lemma 6.3. \( \blacksquare \)

**Second Step.**

We will prove the unique existence of \( u \) to (6.37) by the contraction mapping theorem. We set
\[
Qu(t) := S(t)a + \int_0^t K(t-s)F(s)ds + \sum_{k=1}^{N} \int_0^t K(t-s)q_k \partial_\alpha^k (u(s) - a)ds, \quad 0 < t < T.
\]
By Lemma 6.2, we see
\[ S(t)a - a, \int_0^t K(t - s)F(s)ds \in H_\alpha(0, T; L^2(\Omega)) \quad \text{for } a \in H^1_0(\Omega) \text{ and } F \in L^2(0, T; L^2(\Omega)). \] (6.41)

We estimate \( \sum_{k=1}^N \| Q\alpha - Qv \|_{H_{\alpha_k}(0, T; L^2(\Omega))} \) for \( u - a, v - a \in H_{\alpha_k}(0, T; L^2(\Omega)) \). To this end, we show
\[
\left\| \partial_t^{\alpha_j} \int_0^t K(t - s)w(s)ds \right\|_{L^2(\Omega)} \leq C \int_0^t (t - s)^{\alpha - \alpha_j - 1} \| w(s) \|_{L^2(\Omega)} ds, \quad 1 \leq j \leq N, \ 0 < t < T. 
\] (6.42)

**Verification of (6.42).**

The application of Lemma 6.3 (ii) and (i) yields
\[
\left\| \partial_t^{\alpha_j} \int_0^t K(t - s)w(s)ds \right\|_{L^2(\Omega)} = \left\| \int_0^t \partial_t^{\alpha_j} K(t - s)w(s)ds \right\|_{L^2(\Omega)} 
\leq \int_0^t \left\| \partial_t^{\alpha_j} K(t - s)w(s) \right\|_{L^2(\Omega)} ds \leq C \int_0^t (t - s)^{\alpha - \alpha_j - 1} \| w(s) \|_{L^2(\Omega)} ds.
\]

Thus (6.42) follows directly. \( \blacksquare \)

We note that if \( u - a, v - a \in H_{\alpha_N}(0, T; L^2(\Omega)) \subset H_{\alpha_k}(0, T; L^2(\Omega)) \) with \( 1 \leq k \leq N \), then \( (u - a) - (v - a) = u - v \in H_{\alpha_k}(0, T; L^2(\Omega)) \) for \( 1 \leq k \leq N \). Therefore, using \( q_k \in L^\infty(\Omega) \) for \( 1 \leq k \leq N \), we have
\[
\left\| \partial_t^{\alpha_j} (Q\alpha(t) - Qv(t)) \right\|_{L^2(\Omega)} = \left\| \sum_{k=1}^N \partial_t^{\alpha_j} \int_0^t K(t - s)q_k \partial_{\alpha_k}^s (u(s) - v(s))ds \right\|_{L^2(\Omega)} 
\leq \sum_{k=1}^N \left\| \partial_t^{\alpha_j} \int_0^t K(t - s)q_k \partial_{\alpha_k}^s (u(s) - v(s))ds \right\|_{L^2(\Omega)} 
\leq C \sum_{k=1}^N \int_0^t (t - s)^{\alpha - \alpha_j - 1} \| \partial_{\alpha_k}^s (u(s) - v(s)) \|_{L^2(\Omega)} ds.
\]

Summing up over \( j = 1, \ldots, N \) and applying \( (t - s)^{\alpha - \alpha_j - 1} \leq C(t - s)^{\alpha - \alpha_N - 1} \) for \( 1 \leq j \leq N \), we reach
\[
\sum_{j=1}^N \left\| \partial_t^{\alpha_j} (Q\alpha(t) - Qv(t)) \right\|_{L^2(\Omega)}
\]
Hence, applying (6.42) for estimating \( \sum_{j=1}^{N} \| \partial_{t}^{\alpha_{j}} (Q^{2} u(t) - Q^{2} v(t)) \|_{L^{2}(\Omega)} \), we obtain
\[
\sum_{j=1}^{N} \| \partial_{t}^{\alpha_{j}} (Q^{2} u(t) - Q^{2} v(t)) \|_{L^{2}(\Omega)}
\]
Continuing this estimation, we can reach
\[
\sum_{k=1}^{N} \| \partial_{t}^{\alpha_{k}} (Q^{m} u(t) - Q^{m} v(t)) \|_{L^{2}(\Omega)}
\]
Substituting (6.43) and exchanging the order of the integral, we obtain
\[
\sum_{k=1}^{N} \| \partial_{t}^{\alpha_{k}} (Q^{2} u(t) - Q^{2} v(t)) \|_{L^{2}(\Omega)}
\]
\[
\leq C \int_{0}^{T} \sum_{k=1}^{N} \| \partial_{t}^{\alpha_{k}} (Q^{m} u(t) - Q^{m} v(t)) \|_{L^{2}(\Omega)} d\xi
\]
Continuing this estimation, we can reach
\[
\sum_{k=1}^{N} \| \partial_{t}^{\alpha_{k}} (Q^{m} u(t) - Q^{m} v(t)) \|_{L^{2}(\Omega)}
\]
Consequently, the Young inequality yields
\[
\left( \int_{0}^{T} \left( \sum_{k=1}^{N} \| \partial_{t}^{\alpha_{k}} (Q^{m} u(t) - Q^{m} v(t)) \|_{L^{2}(\Omega)} \right)^{2} d\xi \right)^{\frac{1}{2}}
\]
\[
\leq \left( \frac{CT(\alpha - \alpha_{N})}{\Gamma(m(\alpha - \alpha_{N}))} \right)^{m} \left( \sum_{k=1}^{N} \| \partial_{t}^{\alpha_{k}} (Q^{m} u(t) - Q^{m} v(t)) \|_{L^{2}(\Omega)} \right)
\]
\[
\frac{(CT(\alpha - \alpha_N))^m}{\Gamma(m(\alpha - \alpha_N)) m^{(\alpha - \alpha_N)}} \left( \int_0^T \left( \sum_{k=1}^N \|\partial_t^m (u - v)(s)\|_{L^2(\Omega)} \right)^2 \, ds \right)^{\frac{1}{2}}.
\]

For estimating the right-hand side of the above inequality, we find a constant \( C_N > 0 \) such that
\[
C_N^{-1} \sum_{k=1}^N |\xi_k|^2 \leq \left( \sum_{k=1}^N |\xi_k| \right)^2 \leq C_N \sum_{k=1}^N |\xi_k|^2
\]
for all \( \xi_1, ..., \xi_N \in \mathbb{R} \), we obtain
\[
\sum_{k=1}^N \|\partial_t^{\alpha_k} (Q^m u - Q^m v)\|_{L^2(0,T;L^2(\Omega))} \leq \frac{(CT(\alpha - \alpha_N)T^{\alpha - \alpha_N})^m}{\Gamma(m(\alpha - \alpha_N) + 1)} \sum_{k=1}^N \|\partial_t^{\alpha_k} (u - v)\|_{L^2(0,T;L^2(\Omega))},
\]
that is,
\[
\|Q^m u - Q^m v\|_{H^{\alpha_N}(0,T;L^2(\Omega))} \leq \frac{(CT(\alpha - \alpha_N)T^{\alpha - \alpha_N})^m}{\Gamma(m(\alpha - \alpha_N) + 1)} \|u - v\|_{H^{\alpha_N}(0,T;L^2(\Omega))}
\]
for all \( u - a, v - a \in H^{\alpha_N}(0,T;L^2(\Omega)) \) and \( m \in \mathbb{N} \).

By the asymptotic behavior of the gamma function, we know
\[
\lim_{m \to \infty} \frac{(CT(\alpha - \alpha_N)T^{\alpha - \alpha_N})^m}{\Gamma(m(\alpha - \alpha_N) + 1)} = 0,
\]
and choosing \( m \in N \) sufficiently large, we see that \( Q^m : H^{\alpha_N}(0,T;L^2(\Omega)) \to H^{\alpha_N}(0,T;L^2(\Omega)) \) is a contraction. Thus we have proved the unique existence of a solution \( u \) to (6.37) such that \( u - a \in H^{\alpha_N}(0,T;L^2(\Omega)) \), which completes the proof of Theorem 6.6, provided that \( b_j = 0 \) for \( 1 \leq j \leq d \) and \( c \leq 0 \) in \( \Omega \).

For general \( b_j, c \), setting
\[
-Lv(x) := \sum_{i,j=1}^d \partial_i (a_{ij}(x) \partial_j v)(x)
\]
in place of (6.6), we replace (6.37) by
\[
u(t) - a = S(t)a - a + \int_0^t K(t - s) \sum_{j=1}^d b_j \partial_j u(s) \, ds + \int_0^t K(t - s) cu(s) \, ds
\]
and argue similarly. For the estimation of the second term on the right-hand side of (6.44), in view of Theorem 3.4, we see

$$
\partial_t^{\alpha_k} \int_0^t K(t-s) b_j \partial_j u(s) ds = \partial_t^{\alpha_k} \int_0^t K(s) b_j \partial_j u(t-s) ds
$$

and so we can apply Lemma 6.1 (iii) to argue similarly. Here we omit the details. Thus the proof of Theorem 6.6 is complete. ■

7. Application to an inverse source problem: illustrating example

We construct our framework for fractional derivatives. In this article, within the framework we study initial value problems for fractional ordinary differential equations and initial boundary value problems for fractional partial differential equations which are classified into so called "direct problems". On the other hand, "inverse problems" are important where we are required to determine some of initial values, boundary values, order $\alpha$, and coefficients in the fractional differential equations by observation data of solution $u$.

The inverse problem is indispensable for accurate modelling for analyzing phenomena such as anomalous diffusion. After relevant studies of inverse problems, we can identify coefficients, etc. to determine equations themselves and can proceed to initial value problems and initial boundary value problems. Thus the inverse problem is the premise for the study of the forward problem.

Actually, the main purpose of the current article is to not only establish a theory for direct problems concerning time-fractional differential equations, but also apply the theory to inverse problems. The inverse problems are very various and here we discuss only one inverse problem in order to illustrate how our framework for fractional derivatives works.

Let $0 < \alpha < 1$ and $A$ be the same elliptic operator as the previous sections, that is, defined by (6.1) and (6.4). We consider

$$
\begin{cases}
\partial_t^{\alpha} u + Au = \mu(t) f(x) & \text{in } \mathbb{R}^n H(0,T; L^2(\Omega)), \\
u \in L^2(0, T; L^2(\Omega)),
\end{cases}
$$

(7.1)
where \( \mu \in -\alpha H(0, T), \neq 0 \) and \( f \in L^2(\Omega) \).

**Inverse source problem.**

Let \( \theta \in L^2(\Omega), \neq 0 \) be arbitrarily chosen, and let \( f \in L^2(\Omega) \) be given. Then determine \( \mu = \mu(t) \in -\alpha H(0, T) \) by data

\[
\int_\Omega u(x,t)\theta(x)dx, \quad 0 < t < T. \tag{7.2}
\]

By Theorem 6.5, we know that the solution \( u \) exists uniquely and \( u \in L^2(0, T; L^2(\Omega)) \cap -\alpha H(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \). Therefore the data (7.2) are well-defined in \( L^2(0, T) \). As other choice of data in the case where the spatial dimensions \( d \leq 3 \), we can choose: \( u(x_0, t) \) for \( 0 < t < T \) with fixed point \( x_0 \in \Omega \). By \( 1 \leq d \leq 3 \), the Sobolev embedding implies \( u \in -\alpha H(0, T; H^2(\Omega)) \subset -\alpha H(0, T; C(\overline{\Omega})) \) and so \( u(x_0, \cdot) \in -\alpha H(0, T) \), which implies that data are well-defined in \( -\alpha H(0, T) \). However we are restricted to data (7.2).

Data (7.2) are spatial average values of \( u(x, t) \) with the weight function \( \theta(x) \). We can choose \( \theta(x) \) in (7.2) such that \( \text{supp } \theta \) is concentrating near one fixed point \( x_0 \in \Omega \), which means that data are mean values of \( u(\cdot, t) \) in a neighborhood of \( x_0 \).

If \( \alpha > \frac{1}{2} \), then \( -\alpha H(0, T) \subset C[0, T] \) by the Sobolev embedding, and so the space \( -\alpha H(0, T) \) can contain a linear combination of Dirac delta functions:

\[
\sum_{k=1}^{N} q_k \delta_{t_k}(t),
\]

where \( q_k \in \mathbb{R}, \neq 0, t_j \in (0, T) \) for \( 1 \leq k \leq N \) and \( H^{-1}(\Omega) < \delta_{t_k}, \varphi > H^1_0(\Omega) = \varphi(t_k) \) for all \( \varphi \in C^\infty_0(0, T) \). Then our inverse source problem is concerned with determination of \( N, q_k, t_k \) for \( 1 \leq k \leq N \).

Now we are ready to state

**Theorem 7.1 (uniqueness).**

We assume that \( f \) satisfies

\[
\int_\Omega \theta(x)f(x)dx \neq 0. \tag{7.3}
\]

If

\[
\int_\Omega \theta(x)u(x,t)dx = 0, \quad 0 < t < T,
\]
then \( \mu = 0 \) in \( -\alpha H(0, T) \).

**Proof.**

The following uniqueness is known: Let \( v \in H_\alpha(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \) satisfy

\[
\partial_t^\alpha v + Av = g(t)f(x), \quad 0 < t < T, \tag{7.4}
\]

where \( g \in L^2(0, T) \) and \( f \in L^2(\Omega) \). Under assumption (7.3), condition

\[
\int_\Omega \theta(x)v(x, t)dx = 0, \quad 0 < t < T, \tag{7.5}
\]

yields \( g = 0 \) in \( L^2(0, T) \).

This uniqueness for \( g \in L^2(0, T) \) can be proved by combining Duhamel’s principle and the uniqueness result for initial boundary value problem for fractional partial differential equation with \( f = 0 \). We can refer also to pp. 411-464 by Li, Liu and Yamamoto in a handbook \[17\], Sakamoto and Yamamoto \[32\]. In particular, Jiang, Li, Liu and Yamamoto \[13\], which establishes the uniqueness for the case of \( f = 0 \) with non-symmetric \( A \). Thus the uniqueness in the inverse source problem of determining \( g \) is classical within the category of \( L^2(0, T) \), and so we omit the proof.

Now we can readily reduce the proof of Theorem 7.1 to the case of \( g \in L^2(0, T) \). We set \( v := J'_\alpha u \). Then, by Propositions 2.5 and 2.9 (iii), we see that \( v \in H_\alpha(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \) satisfy

\[
\partial_t^\alpha v + Av = (J'_\alpha \mu)(t)f(x) \quad \text{in} \quad L^2(0, T; L^2(\Omega)),
\]

where \( J'_\alpha \mu \in L^2(0, T) \).

Noting by Proposition 2.9 (ii) that \( \int_\Omega \theta(x)u(x, \cdot)dx \in L^2(0, T) \), by (7.2) we obtain

\[
0 = J'_\alpha \left( \int_\Omega \theta(x)u(x, \cdot)dx \right) = J^\alpha \left( \int_\Omega \theta(x)u(x, \cdot)dx \right)
= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_\Omega \theta(x)u(x, s)dx \right) ds = \int_\Omega \theta(x)(J^\alpha u)(x, t)dx = 0, \quad 0 < t < T.
\]

That is, we reach (7.5), which yields \( J'_\alpha \mu = 0 \) in \( (0, T) \) by the uniqueness in the inverse problem in the case of \( J'_\alpha \mu \in L^2(0, T) \). Since \( J'_\alpha : -\alpha H(0, T) \rightarrow L^2(0, T) \) is injective by
Proposition 2.9 (ii), we obtain that $\mu = 0$ in $-^{\alpha}H(0,T)$. Thus the proof of Theorem 7.1 is complete. ■

8. Concluding remarks

8.1. Main messages

(A) We establish fractional derivatives as operators in subspaces of the Sobolev-Slobodecki spaces, whose orders are not necessarily integer nor positive, and consider fractional derivatives in spaces of distribution. Accordingly we extend classes of functions of which we take fractional derivatives.

In such distribution spaces, we can justify the fractional calculus, and we can argue as if all the functions under consideration would be in $L^1(0,T)$. Such a typical example is a fractional derivative of a Heaviside function in Example 2.4.

(B) More importantly, with our framework of fractional calculus, we can construct a fundamental theory for linear fractional differential equations, which is easily adjusted to weak solutions and strong solutions, and also classical solutions.

We intend this article to be introductory accounts aiming at operator theoretical treatments of time fractional partial differential equations. Comprehensive descriptions should require more works and so this article can provide the essence of foundations. Next we give some summary and prospects.

8.2. What we have done

(i) In Section 2, we introduced Sobolev spaces $H_\alpha(0,T)$ as subspace of the Sobolev-Slobodecki spaces $H^\alpha(0,T)$, and we defined a fractional derivative $\partial_t^\alpha$ as an isomorphism from $H_\alpha(0,T)$ to $L^2(0,T)$, and finally as isomorphisms from $H_{\alpha+\beta}(0,T) \rightarrow H_\beta(0,T)$ and $-^{\beta}H(0,T) \rightarrow -^{\alpha-\beta}H(0,T)$ for $\alpha > 0$ and $\beta \geq 0$, where $-^{\beta}H(0,T)$ and $-^{\alpha-\beta}H(0,T)$ are defined by the duality and are subspace of the distribution space. The key for the definition of $\partial_t^\alpha$ is an operator theory, and we always attach the fractional derivatives with their domains. The extension procedure is governed by the Riemann-Liouville fractional integral operator $J^\alpha$. 

and $\partial_t^\alpha$ is defined by the inverse to $J^\alpha$ with suitable domain.

(ii) In Section 3, we established several basic properties of $\partial_t^\alpha$, which are naturally expected to hold as formulae of derivatives. We apply some of them to fractional differential equations.

(iii) In Sections 5 and 6, we formulate initial value problems for linear fractional ordinary differential equations and initial boundary value problems for linear fractional partial differential equations to prove the well-posedness and the regularity of solutions within two categories: weak solution and strong solution. Our defined $\partial_t^\alpha$ enables us to treat fractional differential equations in a feasible manner.

(iv) Our researches are strongly motivated by inverse problems for fractional differential equations. The studies of inverse problems can be well based on the formulation proposed in this article. Taking into consideration the great variety of inverse problems, in Section 7, we are obliged to be limited to one simple inverse problem in order to illustrate the effectiveness of our framework.

8.3. What we will do

Naturally we had to skip many important and interesting issues. Moreover many researches are going on and here we give up comprehensive prospects for future research topics. We write some of them, related to our framework.

(I) As fractional partial differential equations, we exclusively consider evolution equations with elliptic operators of second order for spatial part, which is classified into fractional diffusion-wave equation. The mathematical researches should not be limited to such evolution equations. For example, the fractional transport equation is also significant:

$$\partial_t^\alpha u(x,t) + H(x,t) \cdot \nabla u(x,t) = F(x,t).$$

One can refer to Luchko, Suzuki and Yamamoto [25] for a maximum principle, and Namba [28] for viscosity solution.
(II) In this article, we did not apply our framework to nonlinear fractional differential equations, although necessary steps are prepared. We can refer to Lucko and Yamamoto [27] as one recent work.

(III) For fractional partial differential equations, we did not consider non-homogeneous boundary values in spite of the necessity and the importance. We need more delicate treatments and see e.g., Yamamoto [33].

(IV) We should study variants of fractional derivatives according to physical backgrounds. We mention a distributed derivative just as one example:

$$\int_0^1 \partial_t^\alpha u(x,t) d\alpha.$$ 

The work Yamamoto [34] studies similar treatments for a generalized fractional derivative

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} g(t-s) \frac{dv}{ds}(s) ds$$

with $0 < \alpha < 1$ and suitable function $g$, and discusses corresponding initial value problems for time-fractional ordinary differential equations. It is another future issue about how much our framework works for such derivatives.

(V) Our approach is based essentially on the $L^2$-space, and our theory is consistent within $L^2$-based Sobolev spaces of any orders $\alpha \in \mathbb{R}$. Therefore, for example, for treating $L^1$-functions in time as source term $F(x,t)$, we have to embed such functions to an $L^2$-based Sobolev space of negative order. This is not the best possible way for gaining $L^p$-regularity in time with $p \neq 2$, and we should construct the corresponding $L^p$-theory for $\partial_t^\alpha$. We can refer to Yamamoto [35] as for a similar work discussing some fundamental properties in the $L^p$-case with $1 \leq p < \infty$.

(VI) With the aid of our framework for the direct problem, results on inverse problems can be expected to be sharpened in view of required regularity for instance. This should be main future topics after this article.
9. Appendix: Sketches of proofs of Lemmata 6.1 and 6.2

Proof of Lemma 6.1.
(i) By (6.7) and (6.8), noting that
\[ L^{-\frac{1}{2}}v = \sum_{n=1}^{\infty} \lambda_n^{-\frac{1}{2}} (v, \varphi_n)_{L^2(\Omega)} \varphi_n \]
for \( v \in L^2(\Omega) \), we obtain
\[ L^{\frac{1}{2}}L^{-\frac{1}{2}}v = v \quad \text{for} \quad v \in L^2(\Omega) \tag{9.1} \]
and
\[ (L^{\frac{1}{2}}y, z)_{L^2(\Omega)} = (y, L^{\frac{1}{2}}z)_{L^2(\Omega)} \quad \text{for} \quad y, z \in D(L^{\frac{1}{2}}) = H^1_0(\Omega). \tag{9.2} \]
Next we will prove that there exists a constant \( C > 0 \) such that
\[ C^{-1} \|v\|_{H^{-1}(\Omega)} \leq \|L^{-\frac{1}{2}}v\|_{L^2(\Omega)} \leq C \|v\|_{H^{-1}(\Omega)} \quad \text{for} \quad v \in L^2(\Omega). \tag{9.3} \]

Verification of (9.3).
Let \( v \in L^2(\Omega) \). Then we have \( L^{-\frac{1}{2}}v \in H^1_0(\Omega) \) by (6.7) and \( L^{\frac{1}{2}}L^{-\frac{1}{2}}v = v \). Hence, by applying (6.7) again, equalities (9.1) and (9.2) yield
\[ H^{-1}(\Omega) < v, \psi >_{H^1_0(\Omega)} = H^{-1}(\Omega) < L^{\frac{1}{2}}L^{-\frac{1}{2}}v, \psi >_{H^1_0(\Omega)} \]
\[ = (L^{\frac{1}{2}}L^{-\frac{1}{2}}v, \psi)_{L^2(\Omega)} = (L^{-\frac{1}{2}}v, L^{\frac{1}{2}}\psi)_{L^2(\Omega)} \quad \text{for} \quad v \in L^2(\Omega) \text{ and } \psi \in H^1_0(\Omega). \]
Consequently, applying (6.8), we obtain
\[ \|v\|_{H^{-1}(\Omega)} = \sup_{\|\psi\|_{H^1_0(\Omega)}=1} \left| H^{-1}(\Omega) < v, \psi >_{H^1_0(\Omega)} \right| = \sup_{\|\psi\|_{H^1_0(\Omega)}=1} \|L^{-\frac{1}{2}}v, L^{\frac{1}{2}}\psi\|_{L^2(\Omega)} \]
\[ \leq \|L^{-\frac{1}{2}}v\|_{L^2(\Omega)} \sup_{\|\psi\|_{H^1_0(\Omega)}=1} \|L^{\frac{1}{2}}\psi\|_{L^2(\Omega)} \leq C \|L^{-\frac{1}{2}}v\|_{L^2(\Omega)}. \]
Therefore, the first inequality of (9.3) is proved.
Next we have
\[ \|L^{-\frac{1}{2}}v\|_{L^2(\Omega)} = \sup_{\|\psi\|_{H^1_0(\Omega)}=1} \|L^{-\frac{1}{2}}v, \psi\|_{L^2(\Omega)} = \sup_{\|\psi\|_{H^1_0(\Omega)}=1} \|v, L^{-\frac{1}{2}}\psi\|_{L^2(\Omega)}. \]
Since
\[ \|L^{-\frac{1}{2}}\psi\|_{H^1_0(\Omega)} \leq C \|L^{\frac{1}{2}}(L^{-\frac{1}{2}}\psi)\|_{L^2(\Omega)} = C \|\psi\|_{L^2(\Omega)}, \]
by (9.1) we obtain
\[
\sup_{\|\psi\|_{L^2(\Omega)}=1} |(v, L^{-\frac{1}{2}}\tilde{\psi})_{L^2(\Omega)}| \leq \sup_{\|\tilde{\psi}\|_{H_0^1(\Omega)} \leq C} |(v, \tilde{\psi})_{L^2(\Omega)}| \\
\leq C \sup_{\|\mu\|_{H_0^1(\Omega)}=1} |H^{-1}(\Omega) < v, \mu >_{H_0^1(\Omega)}| \leq C \|v\|_{H^{-1}(\Omega)}.
\]
Hence, the second inequality of (9.3) is proved, and the verification of (9.3) is complete. ■

Since \( L^2(\Omega) \) is dense in \( H^{-1}(\Omega) \), in view of (9.3), we can extend \( L^{-\frac{1}{2}} : L^2(\Omega) \rightarrow L^2(\Omega) \) to \( L^{-\frac{1}{2}} : H^{-1}(\Omega) \rightarrow L^2(\Omega) \). More precisely, for any \( v \in H^{-1}(\Omega) \), we choose \( v_n \in L^2(\Omega) \), \( n \in \mathbb{N} \) such that \( \lim_{n \rightarrow \infty} v_n = v \) in \( H^{-1}(\Omega) \). Then (9.3) implies \( \lim_{m,n \rightarrow \infty} \| L^{-\frac{1}{2}}(v_m - v_n) \|_{L^2(\Omega)} = 0 \), that is, a sequence \( \{ L^{-\frac{1}{2}}v_n \}_{n \in \mathbb{N}} \subset L^2(\Omega) \) is a Cauchy sequence and we can define
\[
L^{-\frac{1}{2}}v := \lim_{n \rightarrow \infty} L^{-\frac{1}{2}}v_n \quad \text{in} \quad L^2(\Omega).
\]
Then, by (9.3), the limit is independent of choices of \( \{ v_n \}_{n \in \mathbb{N}} \), and we see that (9.3) holds for \( v \in H^{-1}(\Omega) \). Thus the proof of part (i) of Lemma 6.1 is complete. ■

(ii) By (6.7), we can directly verify
\[
K(t)v = L^\frac{1}{2}K(t)L^{-\frac{1}{2}} \quad \text{for} \quad t > 0 \quad \text{and} \quad v \in L^2(\Omega).
\]
Let \( v \in H^{-1}(\Omega) \) be arbitrarily given. Since \( L^2(\Omega) \) is dense in \( H^{-1}(\Omega) \), there exists a sequence \( v_n \in L^2(\Omega) \), \( n \in \mathbb{N} \) such that \( v_n \rightarrow v \) in \( H^{-1}(\Omega) \) as \( n \rightarrow \infty \). By part (i) and the extended \( L^{-\frac{1}{2}} \), estimate (9.3) holds for \( v \in H^{-1}(\Omega) \). Hence, \( L^{-\frac{1}{2}}v_n \rightarrow L^{-\frac{1}{2}}v \) in \( L^2(\Omega) \).

In terms of (6.9), we see that
\[
L^\frac{1}{2}K(t)L^{-\frac{1}{2}}v_n \rightarrow L^\frac{1}{2}K(t)L^{-\frac{1}{2}}v \quad \text{in} \quad L^2(\Omega)
\]
as \( n \rightarrow \infty \) for \( t > 0 \). Since \( L^\frac{1}{2}K(t)L^{-\frac{1}{2}}v_n = v_n \) by \( v_n \in L^2(\Omega) \), we obtain \( v_n \rightarrow L^\frac{1}{2}K(t)L^{-\frac{1}{2}}v \) in \( L^2(\Omega) \) as \( n \rightarrow \infty \) for \( t > 0 \). Using that \( v_n \rightarrow v \) in \( H^{-1}(\Omega) \), we reach \( L^\frac{1}{2}K(t)L^{-\frac{1}{2}}v = v \) for \( t > 0 \). Thus the proof of part (ii) is complete. ■

(iii) Let \( b_j \in C^1(\Omega) \), \( 1 \leq j \leq d \). Then we will verify
\[
\|b_j \partial_j u\|_{H^{-1}(\Omega)} \leq C \|u\|_{L^2(\Omega)}, \quad 1 \leq j \leq d \quad \text{for all} \quad u \in L^2(\Omega).
\] (9.4)
Here and henceforth constants \( C > 0 \) depend on \( b_j \).

If (9.4) is verified, then we can complete the proof of part (iii) of the lemma as follows: by combining (9.4) with part (i) of the lemma, we obtain

\[
\| L^{-\frac{1}{4}}(b_j \partial_j u) \|_{L^2(\Omega)} \leq C \| b_j \partial_j u \|_{H^{-1}(\Omega)} \leq C \| u \|_{L^2(\Omega)}.
\]

■

Verification of (9.4).

By the definition of \( b_j \partial_j u \) as element in \( H^{-1}(\Omega) \), in terms of \( b_j \in C^1(\Omega) \), we have

\[
\| b_j \partial_j u \|_{H^{-1}(\Omega)} = \sup_{\| \psi \|_{H^1(\Omega)} = 1} |(b_j \partial_j u, \psi)|_{H^0(\Omega)} \leq \sup_{\| \psi \|_{H^1(\Omega)} = 1} \| u \|_{L^2(\Omega)} \| \partial_j \psi \|_{L^2(\Omega)} \leq C \| u \|_{L^2(\Omega)}.
\]

Thus the verification of (9.4), and accordingly the proof of Lemma 6.1 are complete. ■

Proof of Lemma 6.2.

(i) We set

\[
S_m(t)a := \sum_{n=1}^{m} E_{\alpha,1}(\lambda_n t^\alpha)(a, \varphi_n)\varphi_n \quad \text{for } m \in \mathbb{N}.
\]

Proposition 4.1 (i) yields

\[
\partial_t^\alpha (S_m(t)a - a) = \sum_{n=1}^{m} -\lambda_n E_{\alpha,1}(\lambda_n t^\alpha)(a, \varphi_n)\varphi_n = -L \left( \sum_{n=1}^{m} E_{\alpha,1}(\lambda_n t^\alpha)(a, \varphi_n)\varphi_n \right),
\]

that is,

\[
\partial_t^\alpha (S_m(t)a - a) + LS_m(t)a = 0, \quad 0 < t < T, \ m \in \mathbb{N}.
\] (9.5)

Since

\[
LS_m(t)a = t^{-\alpha} \sum_{n=1}^{m} -\lambda_n t^\alpha E_{\alpha,1}(\lambda_n t^\alpha)(a, \varphi_n), \quad 0 < t < T,
\]

by (4.4) we estimate

\[
\| LS_m(t)a \|_{L^2(\Omega)}^2 = t^{-2\alpha} \sum_{n=1}^{m} |(\lambda_n t^\alpha E_{\alpha,1}(\lambda_n t^\alpha)(a, \varphi_n)|^2 \\
\leq Ct^{-2\alpha} \sum_{n=1}^{m} |(a, \varphi_n)|^2, \quad 0 < t < T.
\]
Therefore, \( \lim_{m \to \infty} LS_m(t)a = LS(t)a \) in \( L^2(\Omega) \) for fixed \( t > 0 \). Hence, letting \( m \to \infty \) in (9.5), we obtain \( \partial_t^\alpha (S(t)a - a) + LS(t)a = 0 \) for \( 0 < t < T \).

Next let \( a \in H^1_0(\Omega) \). Since

\[
\lambda_n^\alpha(a, \varphi_n) = (a, L^\frac{1}{2}a) \leq \varphi_n \quad \text{for} \quad a \in H^1_0(\Omega),
\]

we have

\[
LS(t)a = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^\alpha) \lambda_n(a, \varphi_n) \varphi_n = \sum_{n=1}^{\infty} \lambda_n^\alpha E_{\alpha,1}(-\lambda_n t^\alpha) \lambda_n^\alpha(a, \varphi_n) \varphi_n
\]

so that

\[
\|LS(t)a\|_{L^2(\Omega)} \leq C t^{-\frac{\alpha}{2}} \left( \sum_{n=1}^{\infty} |\lambda_n t^\alpha| |E_{\alpha,1}(-\lambda_n t^\alpha)|^2 (L^\frac{1}{2}a, \varphi_n)^2 \right)^{\frac{1}{2}}
\]

\[
\leq C t^{-\frac{\alpha}{2}} \|L^\frac{1}{2}a\|_{L^2(\Omega)} \leq C t^{-\frac{\alpha}{2}} \|a\|_{H^1_0(\Omega)}
\]

by (4.4). Therefore, this means that \( LS(t)a \in L^2(0,T;L^2(\Omega)) \) for \( a \in H^1_0(\Omega) \) and

\[
\|LS(t)a\|_{L^2(0,T;L^2(\Omega))} \leq C \|a\|_{H^1_0(\Omega)}. \tag{9.6}
\]

Since \( \partial_t^\alpha (S(t)a - a) + LS(t)a = 0 \) for \( 0 < t < T \), by (9.6), the proof of part (i) is complete.

\[\blacksquare\]

(ii) We set

\[
\begin{cases}
R(t) := \int_0^t K(t - s)F(s)ds, \\
R_m(t) := \int_0^t K_m(t - s)F(s)ds \\
= \sum_{n=1}^{m} \left( \int_0^t (t - s)^{\alpha - 1} E_{\alpha,1}(-\lambda_n(t - s)^\alpha)F(s) \varphi_n ds \right) \varphi_n(x), \\
F_m := \sum_{n=1}^{m} (F, \varphi_n) \varphi_n
\end{cases}
\]

for \( 0 < t < T \) and \( m \in \mathbb{N} \). Then by Proposition 4.2, we can readily verify

\[
\partial_t^\alpha R_m + LR_m = F_m \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \quad \text{for all} \quad m \in \mathbb{N}. \tag{9.7}
\]
Moreover

\[
\|LR_m(t)\|_{L^2(\Omega)}^2 = \sum_{n=1}^{m} \left| \int_0^t \lambda_n(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)(F(s), \varphi_n) ds \right|^2.
\]

Hence,

\[
\|LR_m\|_{L^2(0,T;L^2(\Omega))^2} = \sum_{n=1}^{m} \int_0^T \left| \int_0^t \lambda_n(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)(F(s), \varphi_n) ds \right|^2 dt
\]

\[
= \sum_{n=1}^{m} \| (\lambda_n s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n s^\alpha) \ast (F(s), \varphi_n)) \|_{L^2(0,T)}^2
\]

\[
\leq \sum_{n=1}^{m} \left( \int_0^T |\lambda_n s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n s^\alpha)| ds \right)^2 \int_0^T |(F(s), \varphi_n)|^2 ds
\]

\[
\leq \int_0^T \sum_{n=1}^{m} |(F(s), \varphi_n)|^2 ds = \|F_m\|_{L^2(0,T;L^2(\Omega))^2}^2.
\]

For the last inequality and the second to last inequality, we applied the Young inequality on the convolution and (4.10) respectively.

Consequently,

\[
\|LR_m\|_{L^2(0,T;L^2(\Omega))^2} \leq C \|F_m\|_{L^2(0,T;L^2(\Omega))^2}, \quad m \in \mathbb{N}.
\]

(9.8)

Since \( F \in L^2(0,T; L^2(\Omega)) \), we see that \( LR_m \to LR \) in \( L^2(0,T; L^2(\Omega)) \) as \( m \to \infty \). By (9.7) we see

\[
\lim_{m \to \infty} \partial_t^\alpha R_m = \lim_{m \to \infty} (F_m - LR_m) = F - LR \quad \text{in } L^2(0,T; L^2(\Omega)).
\]

(9.9)

Since \( \lim_{m \to \infty} R_m = R \) in \( L^2(0,T; L^2(\Omega)) \), it follows from (6.25) that \( \lim_{m \to \infty} \partial_t^\alpha R_m = \partial_t^\alpha R \) in \( -\alpha \) in \( H(0,T; L^2(\Omega)) \). Hence (9.9) yields \( \partial_t^\alpha R = F - LR \) in \( -\alpha \) in \( H(0,T; L^2(\Omega)) \).

In view of (9.7) and (9.8), we reach

\[
\|\partial_t^\alpha R_m\|_{L^2(0,T;L^2(\Omega))} = \| -LR_m + F_m\|_{L^2(0,T;L^2(\Omega))}
\]

\[
\leq \| -LR_m\|_{L^2(0,T;L^2(\Omega))} + \|F_m\|_{L^2(0,T;L^2(\Omega))} \leq C \|F\|_{L^2(0,T;L^2(\Omega))} \quad \text{for each } m \in \mathbb{N}.
\]

Letting \( m \to \infty \), we reach

\[
\|\partial_t^\alpha R\|_{L^2(0,T;L^2(\Omega))} \leq C \|F\|_{L^2(0,T;L^2(\Omega))}.
\]

Thus the proof of Lemma 6.2 is complete. ■
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