Relativistic mass and spin of a Kerr black hole in terms of observations

Alfredo Herrera-Aguilar,1,2 R. A. Lizardo-Castro,3 and Ulises Nucamendi2,c

1Instituto de Física, Benemérita Universidad Autónoma de Puebla, Apartado Postal J-48, 72570, Puebla, Puebla, México.
2Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, Ciudad Universitaria, CP 58040, Morelia, Michoacán, México.
3Facultad de Ciencias Físico Matemáticas, Universidad Autónoma de Puebla CP 72570, Puebla, Pue., México.

(Dated: May 29, 2018)

In this paper, we derive closed formulas for the mass $M$ and the spin parameter $a = J/M$ (here $J$ is the black hole angular momentum in natural units) of a Kerr black hole in terms of a minimal quantity of observational data, namely, the red-/blueshifts of photons emitted by massive particles (stars or gas) moving on geodesics around the black hole and their respective orbital radius. It turns out that given a set of two (three) stars revolving around the black hole, the aforementioned formulas involve just eight (twelve) observational data.

Keywords: Kerr black hole, mass and spin, red/blueshifts, black hole rotation curves.

PACS numbers: 11.27.+d, 04.40.-b, 98.62.Gq

I. INTRODUCTION

During last decades we have witnessed a growing interest in the search for astrophysical black holes, whose existence has become a fundamental scientific issue, as well as for different methods to characterize their parameters (see for instance [1-3]). On the one hand there is a vast dynamic evidence indicating that in each galaxy there are millions of black holes with stellar masses; moreover, at the center of almost all galaxies there exists a supermassive black hole (with masses that range from millions to billions of solar masses), including a black hole hosted at the center of the Milky Way itself, called SgrA*. On the other hand, recent discoveries of gravitational waves [4-10] have also given rise to a new golden era in the area of black hole astrophysics since in the near future more sensitive detectors will be built, starting the physics of sensitive detectors will be built, starting the physics of high precision gravitational waves that will allow probing regions increasingly closer to the event horizons of these black hole objects (for an interesting review see [11]).

Regarding the center of our Galaxy, astronomers have estimated the mass of SgrA* and the distance from the Earth to its center employing the Keplerian central potential, which handles the stellar orbits like Keplerian orbits, and using high-resolution near-infrared techniques [12]. The authors estimated that the distance from Earth to the center of the Milky Way is $R_0 = 8.3 \pm 0.35$ kpc and its mass is $M = (4.31 \pm 0.06 |_{stat} \pm 0.36 |_{nu}) \times 10^6 M_\odot$. Here we should mention that in their reported results for the mass $M$, the statistical error is independent of $R_0$ and the main uncertainty for the parameter $M$ is due to the uncertainty in the distance from Earth to the SgrA* center.

However, within this Newtonian approach the authors cannot compute the black hole angular momentum. Nevertheless, there are various techniques which can bound or estimate the rotation parameter of the black hole, for instance, by using flare emissions with a certain period, the value has been bounded by the following estimation $0.70 \pm 0.11 M \leq a \leq M$ [13], while an analysis that implements high-frequency quasi-period oscillation renders the following value $a \sim 0.996 M$ [14-16]. These estimations show us that SgrA* has a close to extremal value of spin, making static black hole configurations inappropriate to describe the dynamics of this astrophysical object as well as the physics in the vicinity of its event horizon.

In the near future, the BlackHoleCam project will join three different experiments: the Event Horizon Telescope [17], which will focus on the black hole event horizon looking for emissions from the relativistic plasma accreting onto SgrA*; GRAVITY [18], which will track the stars orbiting SgrA* with a near-infrared interferometer at the Very Large Telescope (VLT), and a set of radio telescopes (including ALMA) for studying and detecting a radio pulsar in tight orbit about SgrA*. The main objective of the BlackHoleCam project is measuring with high accuracy the SgrA* parameters, assuming that a black hole is hosted in the center of our Galaxy. On the other side, the Strong Gravity EU project [19] will analyze multi-wavelength spectral and fast timing observations of systems containing different kinds of black holes with the aim of estimating the mass and rotation parameters through measurements of X-ray radiation [20-21], a region where precision is considerable better in comparison with the infrared and radio regions. Finally, the MICADO project at the Extremely Large Telescope [22] will analyze velocity profiles and proper motions of surrounding the SgrA* stars and/or gas by spectroscopy.
with accuracies that improve the VLT/NACO ones about a factor of five, specifically the stellar proper motions of order of 10 μas/yr (400 m/s) can be detected within a few years of observations with this facility; moreover, MICADO will also possess an array of 4x4 near-infrared detectors sensitive to the wavelength range 0.8 – 2.5 μm.

Within General Relativity, the so-called Kerr black hole hypothesis states that all isolated rotating astrophysical black holes are described by the Kerr spacetime and are completely defined by just two physical quantities: the mass $M$ and the spin parameter $a = J/M$, where $J$ is the black hole angular momentum in natural units, providing a remarkable prediction of the theory in the strong gravitational field regime. Within the framework of this paper we shall assume that the Kerr black hole solution models real black holes, however, so far there is no exhaustive or direct evidence that all the neutral rotating black holes are described by the Kerr metric and there is still room for considering other spinning black hole configurations, both within General Relativity and modified theories of gravity (see [23, 24] for interesting reviews on this issue, see [28] as well).

Motivated by this fascinating research progress, in Sec. II the authors proposed a novel relativistic method for determining both the mass $M$ and the spin parameter $a$ of a Kerr black hole in terms of directly observed magnitudes with the smallest amount of assumptions and taking advantage of the conserved quantities of a stationary axisymmetric spacetime. Thus, these parameters were expressed in terms of the redshifts $z_{\text{red}}$ and blueshifts $z_{\text{blue}}$ of emitted photons from geodesic particles (stars or gas in certain approximation) circularly orbiting around the Kerr black hole in the equatorial plane, as well as in terms of the orbital radii $r_e$ of these revolving particles. As a result, an eighth order polynomial equation for $M$ in terms of $z_{\text{red}}, z_{\text{blue}}$ and $r_e$ was obtained. This polynomial has no algebraic solution, but a Bayesian fit can help to statistically determine the mass of the black hole. Once an estimation for the mass is obtained, the spin of the black hole can be easily computed through a simple closed expression.

In this work we push forward previous progress and find closed formulas for the Kerr black hole mass $M$ and spin $a$ parameters in terms of directly observed quantities by considering two cases: i) a system of three geodesic particles (stars for instance) which circularly revolve around the black hole in the equatorial plane and ii) a system of two geodesic particles with the same configuration. By taking into account the red- and blueshifts of emitted photons from all the stars at three different points along their circular orbits, we managed to lower the order of the polynomial equation for the mass $M$ to first order for the case i) and to third order for the case ii). The obtained analytical expressions for the relativistic mass and spin parameters of the Kerr black hole will be extremely useful when studying systems like the set of stars orbiting around SgrA* as a first approximation since in this case we have at hand several star orbits that can be combined to lower the corresponding statistical uncertainty.

Our method can also be applied, with relative easiness, to the study of other galactic centers with supermassive black holes, like the core of M31, which hosts a $M_\ast \approx 1.4 \times 10^6 M_\odot$ black hole [30], and to Active galactic Nuclei like Centaurus A, which hosts a $M_\ast \approx 5 \times 10^7 M_\odot$ black hole [31] (see [32] as well). Finally, a suitable modification of this method can be applied to binary systems accretion disks, as well as to the Solar System to relativistically estimate the mass and the rotation parameter of the Sun.

This paper is organized as follows: In Sec. II and III, we briefly review the results presented in [29]. We start in Sec. II by considering the geodesics of massive test particles orbiting around a Kerr back hole and the geodesics of photons emitted by these bodies (by stars or galactic gas, for instance). This analysis makes use of the Killing vector and tensor fields of the Kerr metric to obtain all its conserved quantities (energy, angular momentum and the Carter constant) and allows one to express all the components of the four-velocity/momentum of massive/massless particles in terms of these constants of motion and the parameters of the metric. The final relevant expressions for stable, equatorial and circular geodesic orbits around the Kerr black hole are also quoted. In Sec. III we recall the formulae for computing the kinematical redshifts and blueshifts of emitted photons from stars/gas orbiting around a Kerr black hole. Then, from these black hole rotation curves, we obtain a closed formula for the spin parameter $a$ in terms of the black hole mass, the massive particle orbital radius and the shifts of its emitted photons. This expression allows us to find an eighth order algebraic polynomial equation for the mass $M$ with coefficients depending only on the massive particle orbital radius and the red- and blueshifts of its emitted photons. In Sec IV, by studying a set of three stars orbiting a Kerr black hole and moving along geodesic circular orbits in the equatorial plane, we obtain a linear algebraic equation for the mass parameter $M$ in terms of the red- and blueshifts of photons emitted by these stars and their orbital radii; the corresponding analytical formulae for both the mass $M$ and the spin $a$ of the Kerr black hole are quoted. Similarly, in Sec. V we analyze a system of two geodesic particles with circular equatorial orbits around a Kerr black hole and we obtain a cubic polynomial equation for $M$ with one real root and find other relativistic formulae for $M$ and $a$ in terms of the stars’ radii and the red- and blueshifts of emitted photons by these particles. Finally, in Sec. VI, with these expressions at hand we discuss our results and make some concluding remarks regarding the applicability of this method to different astrophysical systems.
II. GEODESIC PARTICLES IN KERR SPACETIME

In this Section we shall review the geodesical motion of massive/massless particles in the Kerr background. The Kerr black hole family in Boyer-Linquist coordinates is given by the following metric:

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2,$$

where the components of the metric tensor $g_{\mu\nu}$ are:

$$g_{tt} = \left(1 - \frac{2Mr}{\Sigma}\right), \quad g_{\varphi\varphi} = \left(\frac{2Mar\sin^2\theta}{\Sigma}\right), \quad g_{\theta\theta} = \Sigma, \quad g_{rr} = \Delta,$$

$$\Delta = r^2 + a^2 - 2Mr, \quad \Sigma = r^2 + a^2 \cos^2\theta,$$

and $M^2 \geq a^2$. In terms of these coordinates we also have

$$g_{tt}^2 - g_{\varphi\varphi}g_{tt} = \Delta \sin^2\theta.$$ (3)

Besides, the four-velocity of a geodesic test particle which moves in the Kerr gravitational field is given by:

$$u^\mu = (u^t, u^r, u^\theta, u^\varphi).$$ (4)

If the four-velocity corresponds to a photon, then $u^\mu = k^\mu$, whereas if the four-velocity is associated to a massive test particle (a photons’ emitter, like a star or gas), then $u^\mu = U^\mu$. These four-velocities are normalized, rendering the following condition:

$$-\delta = g_{tt}(u^t)^2 + g_{rr}(u^r)^2 + g_{\varphi\varphi}(u^\varphi)^2 + g_{\theta\theta}(u^\theta)^2 + 2g_{t\varphi}u^tu^\varphi,$$

where $\delta = 0$ if the particles are photons and $\delta = 1$ if they are massive particles.

Due to the existence of timelike and rotation Killing vectors fields, we have two conserved quantities:

$$E_\delta = -g_{\mu\nu}U^\mu u^\nu,$$

$$L_\delta = g_{\mu\nu}U^\mu u^\nu.$$ (7)

Here we define the energy $E \equiv E_1$ and the angular momentum $L \equiv L_1$ per unit mass of massive particles when $\delta = 1$; and the energy $E_0 \equiv E_0$ and the angular momentum $L_0 \equiv L_0$ of emitted photons when $\delta = 0$.

From the last relations, (5 - 7), one can obtain the effective potential for massive test particles [36, 37]:

$$g_{rr}(U^r)^2 + g_{\theta\theta}(U^\theta)^2 + 1 - \frac{E^2 g_{\varphi\varphi} + 2ELg_{\varphi\varphi} + L^2g_{tt}}{(g_{\varphi\varphi} - g_{tt}g_{\varphi\varphi})} = g_{rr}(U^r)^2 + V_{eff} = 0.$$ (8)

The Kerr metric possesses a Killing tensor field that defines one more constant of motion $C_\delta = K_{\mu\nu}u^\mu u^\nu = 2\Sigma (l_\mu w^\mu)(n_\mu w^\mu) - r^2 = \text{const.}$, which is related to the Carter constant $Q_\delta$ [38] in the following way:

$$C_\delta \equiv (L_\delta - aE_\delta)^2 + Q_\delta \equiv \frac{(r^2 + a^2)E_\delta - aL_\delta}{\Delta}^2 - \Sigma^2 (u^\varphi)^2 - \Delta r^2.$$ (9)

This relation yields an expression for the radial velocity:

$$\Sigma^2 (u^\theta)^2 = \frac{(r^2 + a^2)^2E_\delta - aL_\delta}{\Delta}^2 - \Delta \left[r^2 + (L_\delta - aE_\delta)^2 + Q_\delta\right] \equiv V^2(r).$$ (10)

By substituting (10) into (8) we get for the polar velocity:

$$\Sigma^2 (u^\theta)^2 = \frac{a^2(\delta - E_\delta^2) + \frac{L_\delta^2}{\Sigma^2\sin^2\theta}}{\Sigma} \cos^2\theta \equiv \Theta^2(\theta).$$ (11)

From (11) we can read that the Carter constant measures how much the path of the particles departs from the equatorial plane, vanishing when $\theta = \pi/2$.

Bound orbits of massive particles occur when $E < 1$, whereas unbounded orbits, like photons’ trajectories, take place when $E_1 \geq 1$.

The remaining components of the four-velocity read:

$$u^t = \frac{1}{\Delta \Sigma} \left\{\left[r^2 + a^2\right]^2 - \Delta a^2 \sin^2\theta\right\} E_\delta - (2Mar)L_\delta\right\};$$ (12)

$$u^\varphi = \frac{1}{\Delta \Sigma \sin^2\theta} \left[(2Mar\sin^2\theta)E_\delta + (\Delta - a^2 \sin^2\theta)L_\delta\right];$$ (13)

being functions of the black hole parameters $M$ and $a$, the orbital radius $r$ and the polar angle $\theta$ of the plane where the geodesic paths lie.

By considering equatorial $U^\theta = 0$ and circular $U^r = 0$ star orbits, then $V_{eff} = 0$ and $V_{eff} = 0$. One also needs the stability condition for circular orbits to hold [38]:

$$V''_{eff}(r) > 0.$$ (14)

For equatorial circular orbits the following condition

$$r > 2M \mp a + 2\sqrt{M \mp a}$$ (15)

must hold, whereas for stable orbits, the following one

$$r > M \left[3 + Z_2 \pm \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)}\right],$$ (16)

$$Z_1 = 1 + \left(1 - \frac{a^2}{M^2}\right)^{1/3}\left[\left(1 + \frac{a}{M}\right)^{1/3} + \left(1 - \frac{a}{M}\right)^{1/3}\right]$$

$$Z_2 = \sqrt{\frac{a^2}{M^2} + Z_1^2},$$

where the $\pm$ signs correspond to co-rotating and counter-rotating photon sources, respectively, with respect to the direction of the black hole’s angular velocity.
Therefore, the expressions for the non-trivial four-velocity components \( U^\rho(r, \pi/2) \) and \( U^t(r, \pi/2) \) of stars become

\[
U^\rho(r, \pi/2) = \frac{\pm M^{1/2}}{r^{3/4}/\sqrt{r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2}}}, \quad (17)
\]

\[
U^t(r, \pi/2) = \frac{\left( r^{3/2} \pm aM^{1/2} \right)}{r^{3/4}/\sqrt{r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2}}} \quad (18)
\]

Similar expressions can be obtained for the components \( k^\mu \) of the photons’ four-momentum (see [29] for details).

**III. REDSHIFTS/BLUESHIFTS OF PHOTONS EMITTED BY GEODESIC PARTICLES**

The frequency of a photon measured by an observer with four-velocity \( U^\mu \) at a point \( C \) reads

\[
\omega_C = -k_\mu U^\mu|_{P_C} \equiv -k_\mu U^\mu_C,
\]

where \( C \) indicates the emission (\( e \)) or detection (\( d \)) points of the measured photons’ frequency. Thus, the general red/blueshift in frequency that light signals emitted by massive particles experience in their path towards an observer, i.e. from the emission point (\( \omega_e \)) till the detection point (\( \omega_d \)), is defined in the following way

\[
1 + z = \frac{\omega_e}{\omega_d}.
\]

For circular equatorial star orbits this quantity becomes

\[
1 + z = \frac{(E_{\gamma} U^t - L_{\gamma} U^\phi)|_C}{(E_{\gamma} U^t - L_{\gamma} U^\phi)|_d} = \frac{U^t_e - b_e U^\phi_e}{U^t_d - b_d U^\phi_d}, \quad (21)
\]

where the apparent impact parameter \( b \equiv \frac{L_{\gamma}}{E_{\gamma}} \) was introduced, and \( E_{\gamma} \) and \( L_{\gamma} \) are defined by (10) and (11), respectively. This quantity is constant along the whole photons’ path \( b_c = b_d \) since the constants of motion \( E_{\gamma} \) and \( L_{\gamma} \) are preserved along the null geodesics followed by the photons from emission till detection.

By further considering the kinematic redshift of photons either side of the line of sight that links the Kerr black hole and the observer, one should subtract from Eq. (21) a similar expression for the redshift evaluated at the central value \( z_c \), i.e. at \( b = 0 \) (see Fig. 1), yielding

\[
\zeta = z - z_c \equiv \frac{U^t_c U^\phi_c b_d - U^t_d U^\phi_c b_c}{U^t_d (U^t_d - b_d U^\phi_d)}.
\]

One further needs to consider the light bending generated by the gravitational field of the Kerr black hole, i.e. the mapping \( \beta(r) \) between the apparent impact parameter \( b \) and the location of the photons’ emitter particle (a star/gas) given by its vector position \( r \) in the equatorial plane. In fact, the impact parameter must also be maximized; this happens at the points where \( k^\gamma = 0 \). From Eq. (6) with \( \delta = 0 \) one can obtain its expression for the Kerr metric restricted to the equatorial plane:

\[
\pm a \frac{M^{1/2} \left( 2aM + r_e \sqrt{r_e^2 - 2Mr_e + a^2} \right)}{r_e^{3/2} (r_e - 2M) \sqrt{r_e^2 - 3Mr_e^2 \pm 2aM^2}}.
\]

The quantities \( z \) and \( z' \) are the redshift and blueshift, respectively, when the test particles move in a co-rotating
with the Kerr black hole way, and vice versa when the
probe particles move counter-rotating. In general \( z \neq z' \),
but when \( a = 0 \), the shifts become equal in magnitude
with opposite sign \( z = -z' \).
Thus, in this special case of equatorial and circular orbits,
one can invert the relations \((24)\) and \((25)\) in order to
obtain expressions for the mass and rotation parameters
of the Kerr black hole from the red and blueshifts of
photons emitted by stars.

It is easy to obtain a closed expression for the spin param-
eter depending on observed quantities and the mass:
\[
a^2 = \frac{\alpha r_e^3 (r_e - 2M)}{4 \beta M^2 - \alpha r_e^2},
\]
where \( \alpha \equiv (z + z')^2 \) and \( \beta \equiv (z - z')^2 \). However, when
substituting this relation into \((24)\) and \((25)\) one obtains
an eighth order algebraic equation for the mass
\[
[16 r_e M^3 - (4 \beta M^2 - \alpha r_e^2) (r_e - 2M) (r_e - 3M)]^2
= 4 \alpha r_e^2 M (r_e - 2M)^3 (4 \beta M^2 - \alpha r_e^2)
\]
that cannot be solved in terms of radicals and one must
resort to a statistical fit in order to estimate this quantity
from observational data.

IV. KERR MASS \( M \) AND SPIN \( a \) IN TERMS OF
OBSERVED REDSHIFTS AND BLUESHIFTS

A. A system with three orbiting particles

In this and the following Section we shall turn our at-
tention to express the Kerr black hole parameters \( M \) and
\( a \) in terms of directly measured quantities, namely: the
red- and blueshifts \( z, z' \) and \( z \), as well as the orbital ra-
dius \( r_e \) of the photons’ source that revolve around the
Kerr black hole in circular and equatorial orbits, accom-
plishing the aim stated above.

We shall start by considering a system made-up of
three probe particles that circularly rotate in the equa-
torial plane around a Kerr black hole (for this scheme it
does not matter if a given star/gas co-rotates or counter-
rotates with the black hole) with orbital radii \( r_1, r_2 \) and
\( r_3 \) ordered in the following form \( r_1 < r_2 < r_3 \) (see Fig. 2
for a simple illustration seen from above).

Then, for each of the test particles, labeled by the index
\( i \), the following equation must hold
\[
\gamma_i \equiv \frac{\alpha_i}{\beta_i} = \frac{4 a^2 M^2}{r_i^2 [r_i^2 + a^2 - 2M r_i]}, \quad i = 1, 2, 3;
\]
where \( \alpha_i \equiv (z_i + z'_i)^2 \) and \( \beta_i \equiv (z_i - z'_i)^2 \). Thus, since each
probe particle obeys its own Eq. \((28)\), in our approach we
are considering that each test particle interacts only with
the Kerr black hole, neglecting small corrections coming
from the interactions among the particles themselves.

Then, by dividing equation \((28)\) for the first(second) particle into that of the second(third) probe particle and
solving for \( a^2 \) each of the resulting expressions we obtain
\[
a^2 = \frac{\gamma_2 r_3^3 (r_2 - 2M) - \gamma_1 r_1^3 (r_1 - 2M)}{\gamma_1 r_1^2 - \gamma_2 r_2^2}, \quad (29)
\]
\[
a^2 = \frac{\gamma_3 r_3^3 (r_3 - 2M) - \gamma_2 r_2^3 (r_2 - 2M)}{\gamma_2 r_2^2 - \gamma_3 r_3^2}. \quad (30)
\]
Eqs. \((29)\) and \((30)\) are analytic formulas for the black
hole spin parameter \( a \). By equating these relations we
obtain a linear equation for \( M \), which in turn allows us to
get a relativistic closed formulae for the mass parameter
in terms of the red- and blueshifts and the orbital radii of
all the particles involved in the considered configuration:
\[
M = \frac{1}{2} \sum_{i < j} \frac{(-1)^{i+j+1} \gamma_i \gamma_j r_i^2 r_j^2 (r_j - r_i)}{r_i^2 (r_j - r_i)}. \quad (31)
\]
Once an estimation for the Kerr mass parameter is made
in terms of the quoted observed quantities with the aid of
\((31)\), the resulting value should be inserted in either Eq.
\((29)\) or Eq. \((30)\) in order to get a relativistic estimation
for the value of the spin parameter of the Kerr black hole.

Thus, with this method we have managed to write rel-
avitistic closed expressions for the mass \( M \) and spin \( a \)
parameters of a Kerr black hole in terms of twelve ob-
servational parameters of three orbiting stars: three dif-
frent redshifts for each particle and their three orbital
radii.

![Figure 2: The line of the Observer to the BH is \( r_d \gg r_e \).
The three orbiting bodies in the configuration are treated like
pointlikes particles and the colored points are the positions of
where the redshifts should be measured. In this analysis, it is
not important the rotating sense of the particles orbits with
respect of the Kerr black hole.](image-url)
whereas for the second test massive particle we have

\[ a^2 = \frac{\alpha_2(r - 2M)}{4\beta_2 M^2 - \alpha_2^2 r^2} \]  

whereas for the second test massive particle we have

\[ a^2 = \frac{\alpha_2(\lambda r^3)(\lambda r - 2M)}{4\beta_2 M^2 - \alpha_2^2 (\lambda r)^2} \]  

By combining Eqs. (32) and (33) we obtain a third order algebraic equation for the ratio \( M/r \):

\[ 8(\alpha_1 \beta_2 - \lambda^3 \alpha_2 \beta_1) \frac{M^3}{r^3} - 4(\alpha_1 \beta_2 - \lambda^4 \alpha_2 \beta_1) \frac{M^2}{r^2} + 2\lambda^2 (\lambda - 1) \alpha_1 \alpha_2 \frac{M}{r} - \lambda^2 (\lambda^2 - 1) \alpha_1 \alpha_2 = 0 \]  

It can be shown that the discriminant \( \Delta_3 \) of Eq. (34) is negative since the cubic coefficient is greater than the quadratic one. Therefore, there is only a real root for the ratio \( M/r \) that can be expressed by the Cardano method in the following way:

\[ \frac{M}{r} = \frac{\alpha_1 \beta_2 - \lambda^4 \alpha_2 \beta_1}{3(\alpha_1 \beta_2 - \lambda^3 \alpha_2 \beta_1)} + \frac{\chi^{2/3} - 2\Psi}{6(4^{1/3})(\alpha_1 \beta_2 - \lambda^3 \alpha_2 \beta_1)^{1/3}} \]  

where we have introduced the following relations

\[ \chi = \Upsilon + \sqrt{4\Psi^2 + \Upsilon^2} \]  

\[ \Upsilon = \beta_2^2 a_1^3 \left[ 2\beta_2 + 9(3\lambda + 2)(\lambda - 1) \lambda^2 \alpha_2 \right] - 3\lambda^2 \alpha_1 \alpha_2 \beta_1 \beta_2 \times \left[ 2\lambda^2 (\alpha_1 \beta_2 + \lambda^2 \alpha_2 \beta_1) + 3(\lambda^2 - 1)(\lambda^3 + 6) \alpha_1 \alpha_2 \right] + \lambda^3 \alpha_2^3 \beta_1 \left[ 9(\lambda - 1)(3\lambda + 2) - \lambda^4 \beta_1 \right] \]  

\[ \Psi = \alpha_1 \beta_2 \left[ 3\lambda^2 (\lambda - 1) \alpha_2 - \beta_2 \right] + 2\lambda^4 \alpha_1 \alpha_2 \beta_1 \beta_2 - \lambda^5 \alpha_2^3 \beta_1 \left[ 3(\lambda - 1) \alpha_1 + \lambda^3 \beta_1 \right] \]  

providing an analytic relativistic formula for the mass of the Kerr black hole in terms of very few observational data.

With this expression at hand, it is straightforward to obtain a closed relativistic formula for the spin parameter of the Kerr black hole:

\[ a^2 = \frac{3\alpha_1 \epsilon \chi^{1/3} \left[ \chi^{2/3} - (\epsilon + \lambda^4 \alpha_2 \beta_1) \chi^{1/3} - 2\Psi \right] r^2}{9\alpha_1 \epsilon^2 \chi^{2/3} - \beta_1 \left[ \chi^{2/3} + 2(\alpha_1 \beta_2 - \lambda^3 \alpha_2 \beta_1) \chi^{1/3} - 2\Psi \right]^2} \]  

where \( \epsilon \equiv \alpha_1 \beta_2 - \lambda^3 \alpha_2 \beta_1 \).

Here we would like to stress one more time that it is not important at all if a given star/gas is co-rotating or counter-rotating with the Kerr black hole due to the quadratic dependence of the spin parameter \( a \) in Eqs. (32) and (33).

Thus, we managed to write down relativistic closed formulae for the mass \( M \) and spin \( a \) parameters of a Kerr black hole in terms of eight observational measurements of two orbiting stars: three redshifts for each particle and their orbital radii.

It is quite remarkable that we reduced an eighth order polynomial equation for \( M \) with four observational data required in [29], to a cubic algebraic equation for \( M \) with eight needed observational measurements.

\section{V. CONCLUDING REMARKS}

In this paper we managed to write relativistic closed expressions for the mass \( M \) and spin \( a \) parameters of a Kerr black hole in terms of twelve (eight) observational measurements of three (two) orbiting stars: three different redshifts for each particle and their (two) orbital radii.
Here we would like to emphasize that this approach analyzes the black hole rotation curves on the basis of directly measured relativistic invariant quantities: the gravitational red- and blueshifts, in contrast to the tangential velocities, which are coordinate dependent observables. Moreover, astronomers can make use of the two proposed configurations with different orbiting stars in order to reduce the statistical errors involved in just one configuration.

The proposed method can also be used as a null test of the General Relativistic Kerr black hole hypothesis since the expressions for the red-/blueshifts \( z \) and \( z' \) are bounded. Hence, if the red-/blueshifts observational data of a given set of stars orbiting around a black hole do not fall within the range predicted by General Relativity, then the black hole will not be of Kerr type, opening the possibility for black hole solutions of modified theories of gravity to describe the dynamics of the above mentioned revolving stars (see [28]).

We finally would like to quote that our method can be applied to the S0 set of stars orbiting the hypothetical black hole SgrA* as well as to other galaxies as the M31, where two red giants (known as P1 and P2) whose kinematics are consistent with circular stellar disks, revolve around a supermassive black hole known as P3 [31], and Active Galactic Nuclei like Centaurus A (NGC5128) where both molecular gas [31] and red stars [32] orbit around a supermassive black hole, for instance. The method can also be applied to the analysis of intermediate mass black holes, with masses of few thousand solar masses, like those detected in the Arches cluster [33], the star association GCIRS13 in the Galactic center [34], the \( \omega \) Cen [35], among others. Moreover, an appropriate modification of the method can also be applied to binary systems, accretion disks, and the Solar System in order to relativistically estimate the mass and rotation parameter of the central source around which the stars/gas revolve.

### Acknowledgements

A.H.-A. thanks the Raman Research Institute for hospitality and is grateful to Shiv Sethi, Sridhar S., Biswajit Paul, Banibrata Mukhopadhyay, Joseph Samuel, and Adriana González Juárez for fruitful and insightful discussions on this topic. All authors thank SNI for support. A.H.-A. and U.N. also acknowledge support from PRODEP, VIEP-BUAP and CIC-UMSNH. U.N. thanks the CONACYT thematic network project 280908 ‘Agujeros Negros y Ondas Gravitatorias’ for financial support.

---

[1] Z. Q. Shen, K. Y. Lo, M.-C. Liang, P. T. P. Ho and J.-H. Zhao, A size of \( \approx 1 \) au for the radio source Sgr A* at the centre of the Milky Way, Nature (London) **438**, 62 (2005).

[2] A. M. Ghez, S. Salim, N. N. Weinberg, J. R. Lu, T. Do, J. K. Dunn, K. Matthews, M. R. Morris, S. Yelda, E. E. Becklin, T. Kremenek, M. Milosavljevic, and J. Naiman, Measuring Distance and Properties of the Milky Way’s Central Supermassive Black Hole with Stellar Orbits, Astrophys. J. **689**, 1044 (2008).

[3] M. R. Morris, L. Meyer, A. M. Ghez, Galactic center research: manifestations of the central black hole, Research in Astron. Phys. **12**, 995 (2012).

[4] A. Eckart, and R. Genzel, Observations of stellar proper motions near the Galactic Centre, Nature (London) **383**, 415 (1996).

[5] M. B. Begelman, Evidence for Black Holes, Science **300**, 1898 (2003).

[6] B. P. Abbott et al. (LIGO Scientific and Virgo Collaboration), Phys. Rev. Lett. **116**, 061102 (2016).

[7] B. P. Abbott et al. (LIGO Scientific and Virgo Collaboration), Phys. Rev. Lett. **116**, 241103 (2016).

[8] B. P. Abbott et al. (LIGO Scientific and Virgo Collaboration), Phys. Rev. Lett. **118**, 221101 (2017).

[9] B. P. Abbott et al. (LIGO Scientific and Virgo Collaboration), Astrophys. J. Lett. **851**, L35 (2017).

[10] B. P. Abbott et al. (LIGO Scientific and Virgo Collaboration), Phys. Rev. Lett. **119**, 141101 (2017).

[11] V. Cardoso and P. Pani, The observational evidence for horizons: from echoes to precision gravitational-wave physics, arXiv:1707.03021 [gr-qc].
[21] T.K. Das, S. Nag, S. Hegde, S. Bhattacharya, I. Maity, B. Czerny, P. Barai, P.J. Wiita, V. Karas and T. Naskar, Black hole spin dependence of general relativistic multi-transonic accretion close to the horizon, New Astron. 37, 81 (2015).

[22] S. Trippe, R. Davies, F. Eisenhauer, N. M. F. Schreiber, T. K. Fritz and R. Genzel, High Precision Astrometry with MICADO at the European Extremely Large Telescope, Mon. Not. Roy. Astron. Soc. 402, 1126 (2010).

[23] D. Psaltis, Probes and Tests of Strong-Field Gravity with Observations in the Electromagnetic Spectrum, Living Rev. Rel. 11, 9 (2008).

[24] C. Bambi, Testing the Kerr black hole hypothesis, Mod. Phys. Lett. A 26, 2453 (2011).

[25] E. Berti et al., Testing General Relativity with Present and Future Astrophysical Observations, Class. Quant. Grav. 32, 243001 (2015).

[26] T. Johannsen Sgr A* and General Relativity, Class. Quant. Grav. 33, 113001 (2016).

[27] K. Yagi, L.C. Stein, Black Hole Based Tests of General Relativity, Class. Quant. Grav. 33, 054001 (2016).

[28] P. Sheoran, A. Herrera-Aguilar and U. Nucamendi, Mass and spin of a Kerr-MOG black hole and a test of the Kerr black hole hypothesis, arXiv:1712.03344 [gr-qc].

[29] A. Herrera-Aguilar and U. Nucamendi, Kerr black hole parameters in terms of the redshift/blueshift of photons emitted by geodesic particles, Phys. Rev 92, 045024 (2015).

[30] R. Bender et al., Hst stis spectroscopy of the triple nucleus of m31: two nested disks in keplerian rotation around a supermassive black hole, Astrophys. J. 631, 280 (2005).

[31] N. Neumayer, M. Cappellari, J. Reunanen, H. W. Rix, P. P. van der Werf, P. T. de Zeeuw and R. I. Davies, The central parsecs of Centaurus A: High Excitation Gas, a Molecular Disk, and the Mass of the Black Hole, Astrophys. J. 671, 1329 (2007).

[32] J. D. Silge, K. Gebhardt, M. Bergmann and D. Richstone, Gemini/GNIRS observations of the central supermassive black hole in Centaurus A, Astron. J. 130, 406 (2005).

[33] S. F. Portegies Zwart, H. Baumgardt, S. L. W. McMillan, J. Makino, P. Hut and T. Ebisuzaki, The ecology of star clusters and intermediate mass black holes in the Galactic bulge, Astrophys. J. 641, 319 (2006).

[34] J. P. Maillard, T. Paumard, S. R. Stolovy and F. Rigaut, The nature of the Galactic Center source IRS 13 revealed by high spatial resolution in the infrared, A. and A., 423 155-167 (2004).

[35] E. Noyola, K. Gebhardt, M. Kissler-Patig, N. Lutzgendorf, B. Jalali, P. T. de Zeeuw and H. Baumgardt, VLT Kinematics for omega Centauri: Further Support for a Central Black Hole, Astrophys. J. 719, L60 (2010).

[36] J. M. Bardeen, W. H. Press and S. A. Teukolsky, Rotating black holes: locally nonrotating frames, energy extraction, and scalar synchrotron radiation, Astrophys. J. 178, 347 (1972).

[37] D. C. Wilkins, Bound geodesics in the Kerr metric, Phys. Rev. D 5, 814 (1972).

[38] B. Carter, Global structure of the Kerr family of gravitational fields, Phys. Rev. 174, 1559 (1968).

[39] Here we correct a typo made in [24] where the effective potential was mistakenly considered with opposite sign when quoting the results of [30] [37].