GEOMETRIC QUANTIZATION OF DIRAC MANIFOLDS

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Abstract

We study geometric quantization of Dirac manifolds. We introduce the notion of Dirac-Chern class for a complex line bundle over a Dirac manifold and show Poincaré’s lemma for Dirac manifolds to establish their prequantization condition. Additional to this, we introduce a polarization for a Dirac manifold $M$ and discuss procedures for quantization in two cases where $M$ is compact and where $M$ is not compact.

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1 Introduction

Physics provides us with a lot of interesting subjects to study in mathematics. Quantization is the one of them, which gives the relationship between observables in a classical system and a quantum system. There are several kinds of quantization, such as canonical quantization, Feynman path integral quantization, geometric quantization, Moyal quantization, Weyl-Wigner quantization and so on. In the paper, we focus on geometric quantization. Geometric quantization consists of two procedures: prequantization and polarization. Prequantization assigns to a symplectic manifold $S$ a Hermitian line bundle $L \to S$ with a connection whose curvature 2-form is the symplectic structure. Then, the Poisson algebra of smooth functions on $S$ acts faithfully on the space $\Gamma^\infty(S, L)$ of smooth sections of $L$. A polarization is the procedure which reduces $\Gamma^\infty(S, L)$ to a subspace $\mathcal{A} \subset \Gamma^\infty(S, L)$ appropriate for physics so that a subalgebra of $C^\infty(S)$ may still act on $\mathcal{A}$.

The study of geometric quantization in symplectic geometry goes back to the theory of B. Kostant [16] and J. Souriau [20]. Later, it was extended to presymplectic manifolds by C. Günther [13] and developed by M. Gotay and J. Śniatycki [12] and I. Vaisman [25]. Also, geometric quantization of Poisson manifolds was algebraically investigated by J. Huebschmann [15] and studied in terms of Hermitian line bundles by I. Vaisman [26] and then in terms of $S^1$-bundles by D. Chinea, J. Marrero and M. de Leon [6]. In the case where Poisson manifolds are twisted by closed 3-forms, their geometric quantization is studied by F. Petalidou [18]. In [31], prequantization was extended to Dirac manifolds in terms of Dirac-Jacobi structures appeared in A. Wade [28] by A. Weinstein and M. Zambon.

The purpose of the paper is to extend the geometric quantization problem to Dirac manifolds and to investigate it. Dirac manifolds are introduced by T. Courant [7] to unify approaches to the geometry of Hamiltonian vector fields and their Poisson algebras, which are thought of generalizations of both...
presymplectic manifolds and Poisson manifolds. Our approach to prequantization is different from the one suggested in [31].

The paper is organized as follows: In Section 2, we review the fundamentals of Dirac manifolds. In Section 3, after reviewing the Lie algebroid cohomology and the connection theory of Lie algebroids, we introduce the first Dirac-Chern classes of line bundles over Dirac manifolds and show that it does not depend on a choice from their connections. Section 4 is devoted to the formulation of prequantization for Dirac manifolds. We define Poisson structure on the space of admissible functions with a certain characteristic distribution and construct the representation of the Poisson algebra. We show Poincaré’s lemma for Dirac manifolds and formulate the condition for prequantization of them to be realized. In Section 5, we introduce polarizations for Dirac manifolds and develop the quantization process of them, basing on the discussion in Section 4.

Throughout the paper, every smooth manifold is assumed to be paracompact. We denote by \( \Gamma_\infty(M,E) \) the space of smooth sections of a smooth vector bundle \( E \rightarrow M \). Especially, if \( E = T M \), we often write \( X(M) \) for \( \Gamma_\infty(M,T M) \). We use \( \Omega^k(M) \) and \( X^k(M) \) for \( \Gamma_\infty(M,\wedge^k T^* M) \) and \( \Gamma_\infty(M,\wedge^k T M) \), respectively.

The symbol \( i \) denotes the imaginary unit.

2 Dirac manifolds

2.1 Definition

Let \( M \) be a finite dimensional smooth manifold. We define symmetric and skew-symmetric operations on a vector bundle \( \mathbb{T} M := T M \oplus T^* M \) over \( M \) as

\[
\langle (X, \xi), (Y, \eta) \rangle_+ := \frac{1}{2} \{ \xi(Y) + \eta(X) \} \in C^\infty(M)
\]

and

\[
\llangle (X, \xi), (Y, \eta) \rrangle := ([X, Y], L_X \eta - i_Y d\xi) \in \Gamma^\infty(M, \mathbb{T} M)
\]

for all \( (X, \xi), (Y, \eta) \in \Gamma^\infty(M, \mathbb{T} M) \). Here \( L_X \eta \) stands for the Lie derivative on \( \eta \) by \( X \) and \( i_Y d\xi \) for the interior product of \( \xi \) with \( V \). A subbundle \( D \subset \mathbb{T} M \) is called a Dirac structure on \( M \) if the following conditions are satisfied:

(D1) \( \langle \cdot, \cdot \rangle_+ \mid_D = 0 \);
(D2) \( D \) has rank equal to \( \text{dim}(M) \);
(D3) \( \llangle \Gamma^\infty(M,D), \Gamma^\infty(M,D) \rrangle \subset \Gamma^\infty(M,D) \).

A smooth manifold \( M \) together with Dirac structure \( D \subset \mathbb{T} M \) is called a Dirac manifold, denoted by \( (M,D) \). In addition to the natural pairing \( \langle \cdot, \cdot \rangle_+ \), one defines a skew-symmetric pairing \( \langle \cdot, \cdot \rangle_- \) as

\[
\langle (X, \xi), (Y, \eta) \rangle_- := \frac{1}{2} \{ \xi(Y) - \eta(X) \} \in C^\infty(M).
\]

Example 2.1 Suppose that \( M \) be a presymplectic manifold with a presymplectic form \( \omega \). The 2-form \( \omega \) induces the bundle map

\[ \omega^\flat : \mathfrak{X}(M) \rightarrow \Omega^1(M), \quad X \mapsto i_X \omega. \]

One can obtain the subbundle \( \text{graph}(\omega^\flat) \) in \( \mathbb{T} M \) as

\[
\text{graph}(\omega^\flat)_m := \{ (X_m, i_{X_m} \omega_m) \in T_m M \oplus T^*_m M | X_m \in T_m M \} \quad (m \in M)
\]
and can verify that graph(ω) satisfies the three conditions (D1) – (D3) in the above. Therefore, (M, graph(ω)) is a Dirac manifold. Similarly, any symplectic manifold M defines a Dirac structure on M.

Example 2.2 Similarly to Example 2.1, any Poisson manifold (P, π) defines a Dirac structure. Indeed, the 2-vector field π induces the bundle map

$$\pi^\sharp: \Omega^1(P) \rightarrow \mathfrak{X}(P), \quad \alpha \mapsto \{ \beta \mapsto \pi(\beta, \alpha) \}.$$ 

and the subbundle graph(π^\sharp) given by

$$\text{graph}(\pi^\sharp)_m := \{ (\pi^\sharp(\xi_m), \xi_m) \in T_m M \oplus T_m^* M \mid \xi_m \in T_m^* M \} \quad (m \in M).$$

It can be easily verified that (P, graph(π^\sharp)) is a Dirac manifold.

Example 2.3 We let F ⊂ TM be a regular distribution and denote by F^o the annihilator of F in T^* M. Then, a vector bundle F ⊗ F^o → M is Dirac structure on M.

Lemma 2.1 If Condition (D3) holds, then it holds that

$$(\mathcal{L}_{X_1}\xi_2)(X_3) + (\mathcal{L}_{X_2}\xi_3)(X_1) + (\mathcal{L}_{X_3}\xi_1)(X_2) = 0$$

for any (X_1, ξ_1), (X_2, ξ_2), (X_3, ξ_3) ∈ Γ^∞(M, D).

Proof. By noting that

$$\xi_i(X_j) + \xi_j(X_i) = 0 \quad (i, j = 1, 2, 3),$$

the left-hand side in (2.1) is calculated to be

$$\begin{align*}
(\mathcal{L}_{X_1}\xi_2)(X_3) + (\mathcal{L}_{X_2}\xi_3)(X_1) + (\mathcal{L}_{X_3}\xi_1)(X_2) \\
= X_1(\xi_2(X_3)) - \xi_2([X_1, X_3]) + X_2(\xi_3(X_1)) - \xi_3([X_2, X_1]) + X_3(\xi_1(X_2)) - \xi_1([X_3, X_2]) \\
= \frac{1}{2} \left\{ X_1(\xi_2(X_3)) - X_1(\xi_3(X_2)) - \xi_2([X_1, X_3]) + \xi_2([X_3, X_1]) \\
+ X_2(\xi_3(X_1)) - X_2(\xi_1(X_3)) - \xi_3([X_2, X_1]) + \xi_3([X_1, X_2]) \\
+ X_3(\xi_1(X_2)) - X_3(\xi_2(X_1)) - \xi_1([X_3, X_2]) + \xi_1([X_2, X_3]) \right\} \\
= \frac{1}{2} \left\{ -(i_{X_1}d\xi_2)(X_1) + \xi_1([X_2, X_3]) - (i_{X_2}d\xi_3)(X_2) + \xi_2([X_3, X_1]) - (i_{X_3}d\xi_1)(X_3) + \xi_3([X_1, X_2]) \right\} \\
= \frac{1}{2} \left\{ \langle [X_2, \xi_3], (X_3, \xi_1) \rangle, (X_1, \xi_1) \rangle - (\mathcal{L}_{X_1}\xi_2)(X_1) \\
+ \langle [X_3, \xi_3], (X_1, \xi_1) \rangle, (X_2, \xi_2) \rangle - (\mathcal{L}_{X_2}\xi_3)(X_2) \\
+ \langle [X_1, \xi_1], (X_2, \xi_2) \rangle, (X_3, \xi_3) \rangle - (\mathcal{L}_{X_3}\xi_1)(X_3) \right\}. \\
\end{align*}$$

So, we have

$$\begin{align*}
(\mathcal{L}_{X_1}\xi_2)(X_3) + (\mathcal{L}_{X_2}\xi_3)(X_1) + (\mathcal{L}_{X_3}\xi_1)(X_2) \\
= \frac{1}{3} \left\{ \langle [X_1, \xi_1], (X_2, \xi_2) \rangle, (X_3, \xi_3) \rangle + \langle [X_2, \xi_2], (X_3, \xi_3) \rangle, (X_1, \xi_1) \rangle \\
+ \langle [X_3, \xi_3], (X_1, \xi_1) \rangle, (X_2, \xi_2) \rangle \right\}. \\
\end{align*}$$
Lemma 2.1 that
at each 
(2.2), one obtains that
by the condition (D1). This indicates that
where
Proposition 2.2
Proof
where the symbol
(2.2)
follows from the condition (D1) that
element
It is easy to check that
From Condition (D3) it follows that
It is easy to check that

\[
\ker \rho = D \cap T^* M \quad \text{and} \quad \ker \rho^* = D \cap TM,
\]
where

\[
D \cap T^* M := D \cap ((0) \oplus T^* M), \quad D \cap TM := D \cap (TM \oplus (0)).
\]
Here, we remark that
are thought of as a subbundle of either
(resp.
(resp. 
).
Proposition 2.2 Given a Dirac manifold \((M, D)\), one has the characteristic equations
\[
\rho(D) = (D \cap T^* M)^\circ \quad \text{and} \quad \rho^*(D) = (D \cap TM)^\circ,
\]
where the symbol \(^\circ\) stands for the annihilator.
Proof. Suppose that a vector field \(X \in \mathfrak{X}(M)\) belongs to \(\rho(D)\), that is, there exists a 1-form \(\xi\) such that \((X, \xi) \in D\). Letting \(\eta\) be a 1-form such that \((0, \eta) \in D \cap T^* M\), we have
\[
\eta(X) = \langle (X, \xi), (0, \eta) \rangle_+ = 0
\]
by the condition (D1). This indicates that \(\rho(D) \subset (D \cap T^* M)^\circ\). Furthermore, from the left equation in (2.2), one obtains that
\[
\dim \rho_m(D_m) = \dim T_m M - \dim \ker \rho_m = \dim T_m M - \dim (D_m \cap T_m^* M)
\]
at each \(m \in M\). Therefore, it holds that \(\rho(D) = (D \cap T^* M)^\circ\). Similarly, it can be verified that \(\rho^*(D) = (D \cap TM)^\circ\).

For each \(m \in M\), we define a bilinear map \(\Omega_m\) on the subspace \(\rho_m(D_m) \subset T_m M\) as
\[
\Omega_m(X_m, Y_m) := \xi_m(Y_m) \quad (Y_m \in \rho_m(D_m)),
\]
where \(\xi_m\) is an element in \(T_m^* M\) such that \((X_m, \xi_m) \in D_m\). To see the well-definedness of \(\Omega_m\), we take any element \((X_m, \xi), (X_m, \xi_m^*') \in D_m\). It follows from Proposition 2.2 that \(\xi_m - \xi_m^* \in \rho_m(D_m)^\circ\). This implies that \(\xi_m(Y_m) = \xi_m^*(Y_m)\) for any \(Y_m \in \rho_m(D_m)\). Therefore, the map \(\Omega_m\) is well-defined. In addition, it follows from the condition (D1) that \(\Omega\) is skew-symmetric. Next, let \(X, Y, Z\) be any element in \(\Gamma^\infty(M, \rho(D))\), that is, there exist \(\xi, \eta, \zeta\) such that \((X, \xi), (Y, \eta), (Z, \zeta) \in \Gamma^\infty(M, D)\). Then, it follows from Lemma 2.1 that
\[
X(\Omega(Y, Z)) - Y(\Omega(X, Z)) + Z(\Omega(X, Y))
\]
Therefore, the 2-form $\Omega$ satisfies the 2-cocycle condition over $\rho(D)$. As a result, one can obtain a presymplectic form $\Omega$ on $\rho(D)$ by (2.3). The symbol $\Omega^\flat$ denotes the bundle map induced from $\Omega$. That is, $\Omega^\flat$ is the map $\Omega^\flat : \rho(D) \to \rho(D)^*$ which assigns $\Omega^\flat(X) = \Omega(X, \cdot)$ to $X \in \rho(D)$. One easily finds that $\ker \Omega^\flat = D \cap TM$.

In the same way, one also obtains a skew-symmetric tensor fields $\Pi : \rho^*(D) \times \rho^*(D) \to \mathcal{C}^\infty(M)$ by

$$\Pi_m^\sharp(\xi_m, \eta_m) := \xi_m(Y_m) \quad (\xi_m \in M),$$

where $Y_m$ is a vector in $T_mM$ such that $(Y_m, \eta_m) \in D_m$. The form $\Pi$ defines a map, denoted by $\Pi^\sharp$, from the subspace $\rho^*(D) = (D \cap TM)^*$ to $\rho^*(D)^*$ by

$$\rho_m^*(D_m) \ni \eta \mapsto \{ \xi \mapsto \Pi_m^\sharp(\xi_m) := \xi_m(Y_m) \} \in \rho^*(D)^*$$

for each $m \in M$. Letting $(X, \xi), (Y, \eta)$ be smooth sections of $D$, we have $X = \Pi^\sharp(\xi), Y = \Pi^\sharp(\eta)$ and

$$\llbracket (\Pi^\sharp(\xi), \Pi^\sharp(\eta)) \rrbracket = ([\Pi^\sharp(\xi), \Pi^\sharp(\eta)], \langle \xi, \eta \rangle) \in \Gamma^\infty(M, D),$$

where $\langle \xi, \eta \rangle := \mathcal{L}_{\Pi^\sharp(\xi)}\eta - i_{\Pi^\sharp(\eta)}d\xi$. This implies that

$$\Pi^\sharp(\xi, \eta) = [\Pi^\sharp(\xi), \Pi^\sharp(\eta)] \quad (\forall (X, \xi), (Y, \eta) \in \Gamma^\infty(M, D)).$$

### 2.2 Admissible functions

A smooth function $f$ on a Dirac manifold $(M, D)$ is said to be admissible if there exists a vector field $X_f \in \mathfrak{X}(M)$ such that $(X_f, df)$ is a smooth section of $D$ (see [7]). We note that the vector field $X_f$ is not determined uniquely as exhibited in the next example.

**Example 2.4** Consider the presymplectic structure $\omega = dx_1 \wedge dx_2 + dx_1 \wedge dx_4$ on $\mathbb{R}^4$ and a function $f(x_1, x_2, x_3, x_4) = x_1^2 + k(x_2 + x_4)$ ($k \in \mathbb{R}$). Then, vector fields written in the form

$$X = k \frac{\partial}{\partial x_1} + \varphi_1(x) \frac{\partial}{\partial x_2} + \varphi_2(x) \frac{\partial}{\partial x_3} - (2x_1 + \varphi_1(x)) \frac{\partial}{\partial x_4}, \quad (\varphi_1, \varphi_2 \in \mathcal{C}^\infty(\mathbb{R}^4),$$

turn out to satisfy that $\omega^\flat(X) = df$. Therefore, $f$ is the admissible function on $\mathbb{R}^4, \text{graph}(\omega^\flat))$.

Given a Dirac manifold $(M, D)$, we denote the space of the admissible functions on $(M, D)$ by $\mathcal{C}^\infty_{\text{adm}}(M, D)$. For any admissible function $f, g \in \mathcal{C}^\infty_{\text{adm}}(M, D)$, one defines their bracket $\{ f, g \}$ as

$$\{ f, g \} := X_{g^\prime} f.$$

It can be shown that the bracket (2.5) is both well-defined and skew-symmetric in the same way as the case of $\Omega$. If $f, g$ are admissible, there exist the vector fields $X_f$ and $X_g$ on $M$ such that $(X_f, df), (X_g, dg) \in \Gamma^\infty(D)$. Then, the simple computation yields to that

$$\llbracket (X_g, dg), (X_f, df) \rrbracket = (\llbracket X_f, X_g \rrbracket, d \{ f, g \}) \in \Gamma^\infty(D).$$
This implies that their bracket $[f, g]'$, also, is admissible and satisfies the equation
\[ X_{(f,g)'} + [X_f, X_g] = 0. \] (2.6)

The next proposition can be shown by using (2.6) (see [7]).

**Proposition 2.3** ($C_{\text{adm}}(M, D), \{\cdot, \cdot\}'$) forms a Poisson algebra.

### 3 Lie algebroids

#### 3.1 Basic terminologies

To carry out the procedure of prequantization for Dirac manifolds, the notion of cohomology for Dirac manifold is needed. Before proceeding the discussion, let us recall the definition of Lie algebroid and its cohomology.

**Definition 3.1** A Lie algebroid over $M$ is a smooth vector bundle $A \to M$ with a bundle map $\# : A \to TM$, called the anchor map, and a Lie bracket $\left< \cdot, \cdot \right>$ on the space $\Gamma^\infty(M, A)$ of smooth sections of $A$ such that
\[ \left< \alpha, f\beta \right> = ((\#\alpha f)\beta + f\left< \alpha, \beta \right> \] (3.1)

for any $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma^\infty(M, A)$.

A simple example is a tangent bundle $TM$ over a smooth manifold $M$: the anchor map $\#$ is the identity map, and the bracket $\left< \cdot, \cdot \right>$ is the usual Lie bracket of vector fields. This is called the tangent algebroid of $M$. As is well-known, Poisson manifolds define the structure of Lie algebroid on their cotangent bundles.

**Example 3.1** (Cotangent algebroids) If $(P, \varpi)$ is a Poisson manifold, then a cotangent bundle $T^*P$ is a Lie algebroid: the anchor map is the map $\varpi^\#: \left< \cdot, \cdot \right>$ induced from $\varpi$, $\varpi^\#: T^*P \to TP$, $\alpha \mapsto \left< \beta \mapsto \left< \beta, \varpi^\#(\alpha) \right> = \varpi(\beta, \alpha) \right>$ and the Lie bracket is given by
\[ \left< \alpha, \beta \right> := L_{\varpi^\#(\alpha)}\beta - L_{\varpi^\#(\beta)}\alpha + d(\varpi(\alpha, \beta)) \]
\[ = L_{\varpi^\#(\alpha)}\beta - i_{\varpi^\#(\beta)}d\alpha \]

as in the part immediately before the subsection 2.2. The Lie algebroid $(T^*P \to P, \{\cdot, \cdot\}, \varpi^\#)$ is called a cotangent algebroid.

For other examples and the fundamental properties of Lie algebroid, refer to A. Canna da Silva and A. Weinstein [5] and J.-P. Dufour and N. T. Zung [10].

Let $(A_1 \to M_1, \left[ \cdot, \cdot \right]^1, \#^1)$ and $(A_2 \to M_2, \left[ \cdot, \cdot \right]^2, \#^2)$ be Lie algebroids. A Lie algebroid morphism from $A_1$ to $A_2$ is a vector bundle morphism $\Phi : A_1 \to A_2$ which satisfies
\[ \#^2(\Phi(\alpha)) = \varphi_*(\#^1(\alpha)), \quad (\forall \alpha \in \Gamma^\infty(M_1, A_1)), \]
and, for any smooth sections $\alpha, \beta \in \Gamma^\infty(M_1, A_1)$ written in the forms
\[ \Phi \circ \alpha = \sum_i \xi_i (\alpha'_i \circ \varphi), \quad \Phi \circ \beta = \sum_j \eta_j (\beta'_j \circ \varphi), \]

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where $\xi_i, \eta_j \in C^\infty(M_1)$ and $\alpha'_i, \beta'_j \in \Gamma^\infty(M_2, A_2)$.

$$\Phi \circ \langle \alpha, \beta \rangle^1 = \sum_{i,j} \xi_i \eta_j (\langle \alpha'_i, \beta'_j \rangle^2 \circ \Phi) + \sum_j (\mathcal{L}_{\xi_j} \eta_j)(\beta'_j \circ \Phi) - \sum_i (\mathcal{L}_{\xi_i} \beta_i)(\alpha'_i \circ \Phi)$$

For further discussion of Lie algebroid morphisms, we refer to [10] and K. Mackenzie [17].

Concepts in Lie algebroid theory often appear as generalizations of standard notions in Poisson geometry and differential geometry. The following theorem is an analogue of the splitting theorem by A. Weinstein [29] which states that any Poisson manifold is locally a direct product of symplectic manifold with another Poisson manifold. The splitting theorem for Lie algebroids appears in R. L. Fernandes [11], A. Weinstein [30] and L.-P. Dufor [9]. We refer to [10] for the proof of this theorem.

**Theorem 3.2** (Splitting theorem [9][11][30]) Let $(A \to M, \#, \llbracket, \rrbracket)$ be a Lie algebroid. For each point $m \in M$, there exist a local coordinate chart with coordinates $(x_1, \cdots, x_r, y_1, \cdots, y_s)$ around $m$, where $r = \text{rank}\, \#_m$ and $r + s = \dim M$, and a basis of local sections $\{\alpha_1, \cdots, \alpha_r, \beta_1, \cdots, \beta_s\}$ over an open neighborhood of $m$ such that

$$\langle \alpha_j, \alpha_k \rangle = 0, \quad \langle \alpha_j, \beta_k \rangle = 0, \quad \langle \beta_j, \beta_k \rangle = \sum_\ell f^\ell_{jk}(y) \beta_\ell,$$

$$\#_j = \frac{\partial}{\partial x_j}, \quad dx_j(\#_k) = 0 \quad \mathcal{L}_{\alpha_j} \#_k = 0$$

for all possible indices $j, k, \ell$. Here, $f^\ell_{jk}(y)$ are smooth functions depending only on the variables $y = (y_1, \cdots, y_s)$.

The notion of $A$-connections given below generalizes the usual one of connections on vector bundles (see M. Crainic and R. L. Fernandes [8]).

**Definition 3.3** Let $(A \to M, \#, \llbracket, \rrbracket)$ be a Lie algebroid over $M$ and $E$ a vector bundle over $M$. An $\mathbb{R}$-bilinear map

$$\nabla^A : \Gamma^\infty(M, A) \times \Gamma^\infty(M, E) \to \Gamma^\infty(M, E), \quad (\alpha, s) \mapsto \nabla^A_\alpha s$$

is called an $A$-connection (or the covariant derivative) if it satisfies

1. $\nabla^A_{f \alpha} s = f \nabla^A_\alpha s$;
2. $\nabla^A_\alpha (f s) = f \nabla^A_\alpha s + (\#_\alpha f) s$

for any $f \in C^\infty(M), \alpha \in \Gamma^\infty(M, A)$ and $s \in \Gamma^\infty(M, E)$.

The notion of ordinary connection is the case where $A$ is the tangent algebroid $TM$. We denote by $\nabla^0$ an ordinary connection, that is,

$$\nabla^0 : \mathfrak{X}(M) \times \Gamma^\infty(M, E) \to \Gamma^\infty(M, E), \quad (X, s) \mapsto \nabla^0_X s.$$

When $E \to M$ is a complex vector bundle, an $A$-connection on $E$ is defined as an $A$-connection which is $\mathbb{C}$-linear on $\Gamma^\infty(M, E)$.

Similarly to the case of usual connection theory on vector bundles, one can define the curvature of $A$-connection. The curvature $R^A_\alpha$ of $A$-connection $\nabla^A$ is the map

$$R^A_\alpha : \Gamma^\infty(M, A) \times \Gamma^\infty(M, A) \to \text{End}_\mathbb{C} (\Gamma^\infty(M, E))$$
given by the usual formula
\[ R^A_\alpha(\alpha, \beta) = \nabla^A_\alpha \circ \nabla^A_\beta - \nabla^A_\beta \circ \nabla^A_\alpha - \nabla^A_{[\alpha, \beta]} \]
for any \( \alpha, \beta \in \Gamma^\infty(M, A) \).

For each \( k \in \mathbb{N} \cup \{0\} \), consider the exterior bundle \( \wedge^k A^* \) over \( M \). A smooth section of \( \wedge^k A^* \) is called an \( A \)-differential \( k \)-form. One defines a \( C^\infty(M) \)-multilinear map, called an \( A \)-exterior derivative, \( d_A : \Gamma^\infty(M, \wedge^k A^*) \to \Gamma^\infty(M, \wedge^{k+1} A^*) \) as

\[
(d_A \theta)(\alpha_1, \ldots, \alpha_{k+1}) = \sum_{j=1}^{k+1} (-1)^j \theta_j \left( \theta(\alpha_1, \ldots, \hat{\alpha}_j, \ldots, \alpha_{k+1}) \right) + \sum_{j<k} (-1)^{jk} \theta_j(\alpha_j, \alpha_k, \alpha_{j+1}, \ldots, \alpha_{k+1}),
\]
for any \( \alpha_1, \ldots, \alpha_{k+1} \in \Gamma^\infty(M, A) \).

The following proposition can be verified by a direct computation.

**Proposition 3.4** The differential operator \( d_A \) has the following properties:

1. \( d_A \circ d_A = 0 \);
2. For any \( A \)-differential \( k \)-form \( \theta \) and \( A \)-differential \( \ell \)-form \( \vartheta \),
   \[
d_A(\theta \wedge \vartheta) = (d_A \theta) \wedge \vartheta + (-1)^k \theta \wedge (d_A \vartheta).
\]

From Proposition 3.4, one finds that \( (\Gamma^\infty(M, \wedge^k A^*), d_A) \) forms a chain complex. The cohomology of \( \Gamma^\infty(M, \wedge^k A^*) \) is called the Lie algebroid cohomology, or \( A \)-cohomology (see [3], page 6). By definition, the \( k \)-th cohomology group with coefficients in \( \mathbb{R} \), denoted by \( H^k_L(M, A; \mathbb{R}) \), is given by

\[
H^k_L(M, A; \mathbb{R}) = \frac{\ker \{ d_A : \Gamma^\infty(M, \wedge^k A^*) \to \Gamma^\infty(M, \wedge^{k+1} A^*) \}}{\text{im} \{ d_A : \Gamma^\infty(M, \wedge^{k-1} A^*) \to \Gamma^\infty(M, \wedge^k A^*) \}}.
\]

We denote by \([\sigma]\) the cohomology class of \( \sigma \in \ker \{ d_A : \Gamma^\infty(M, \wedge^k A^*) \to \Gamma^\infty(M, \wedge^{k+1} A^*) \} \).

A Dirac structure \( D \) over \( M \) can be regarded as a Lie algebroid \( D \to M \) with the bracket \([\cdot, \cdot]\) and the anchor map \( \| \| = \rho = \text{pr}_1|_D \). The distribution \( M \ni m \mapsto \rho_m(D_m) \subset TM \) is called the characteristic distribution. According to Theorem 3.2, the characteristic distribution is integrable in the sense of Stefan [21, 22] and Sussman [23]. The corresponding singular foliation is called the characteristic foliation. The cohomology of \((M, D)\) is defined as the Lie algebroid cohomology \( H^*_L(M, D; \mathbb{R}) \) of the Lie algebroid \((D \to M, \rho, [\cdot, \cdot])\).

The anchor map \( \rho : D \to TM \) has the natural extension to a map \( \wedge^2 \rho : \Gamma^\infty(M, \wedge^2 D) \to \wedge^2(M) \) by

\[
(\wedge^2 \rho(\vartheta))(\alpha_1, \alpha_2) := \rho^\ast(\alpha_1) \wedge \rho^\ast(\alpha_2)(\vartheta)
\]
for any \( \alpha_1, \alpha_2 \in \Omega^1(M) \). The dual \( (\wedge^2 \rho)^* : \Omega^2(M) \to \Gamma^\infty(M, \wedge^2 D^*) \) of \( \wedge^2 \rho \) is explicitly given by

\[
((\wedge^2 \rho)^*)\sigma(\psi_1 \wedge \psi_2) = \sigma(\wedge^2 \rho(\psi_1 \wedge \psi_2)) = \sigma(\rho(\psi_1), \rho(\psi_2))
\]
As noted in the subsection 2.1, one has a skew-symmetric form $(\wedge^2 \rho)^*$ induces a homomorphism

$$(\wedge^2 \rho)^* : H^2_{\text{dR}}(M) \to H^2_{\text{dR}}(M; \mathbb{R}), \quad [\sigma] \mapsto [(\wedge^2 \rho)^* \sigma]. \quad (3.2)$$

from the de Rham cohomology group $H^2_{\text{dR}}(M)$ of $M$ to the Lie algebroid cohomology group $H^2_{\text{Lie}}(M, D; \mathbb{R})$ of $(M, D)$.

Let $(\phi, Q)$ be any $D$-differential $\ell$-form of $(M, D)$, where $\phi \in \Omega^\ell(M)$ and $Q \in \mathfrak{x}'(M)$. Then, the exterior derivative $d_D(\phi, Q)$ for $(\phi, Q)$ is calculated to be

$$(d_D(\phi, Q))((X_1, \xi_1), \cdots, (X_{\ell+1}, \xi_{\ell+1})) = \sum_{j=1}^{\ell+1} (-1)^{j+1} X_j (\phi(X_1, \cdots, \widehat{X_j}, \cdots, X_{\ell+1}) + (\xi_1 \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \xi_{\ell+1})(Q))$$

$$+ \sum_{j<k} (-1)^{j+k} \phi([X_j, X_k], X_1 \cdots, \widehat{X_j}, \cdots, \widehat{X_k}, \cdots, X_{\ell+1})$$

$$+ \sum_{j<k} (-1)^{j+k} ((\mathcal{L}_{X_j} \xi_k - i_{X_j} d\xi_k) \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \widehat{\xi_k} \wedge \cdots \wedge \xi_{\ell+1}) (Q),$$

where $(X_1, \xi_1), \cdots, (X_{\ell+1}, \xi_{\ell+1})$ are smooth sections of $D$. Especially, if $f$ is a smooth function on $M$, $d_D f$ is calculated to be

$$(d_D f)(X, \xi) = X f. \quad (3.3)$$

As noted in the subsection 2.1, one has a skew-symmetric form $\Pi^\ell : \rho^* (D) \to (\rho^* (D))^*$ by $\Pi^\ell(\xi_j) = X_j$ for $(X_j, \xi_j) \in \Gamma^\ell(M, D)$ $(\forall j = 1, \cdots, \ell + 1)$. Noting that

$$\mathcal{L}_{X_j} \xi_k - i_{X_j} d\xi_k = \{\xi_j, \xi_k\},$$

we get

$$(d_D(\phi, Q))((X_1, \xi_1), \cdots, (X_{\ell+1}, \xi_{\ell+1})) = (d\phi)(X_1, \cdots, X_{\ell+1}) + (\partial Q)(\xi_1, \cdots, \xi_{\ell+1}),$$

where $\partial : \mathfrak{x}^*(M) \to \mathfrak{x}^{*+1}(M)$ denotes the contravariant exterior derivative (see I. Vaisman [27]):

$$(\partial Q)(\alpha_1, \cdots, \alpha_{\ell+1}) = \sum_{j=1}^{\ell+1} (-1)^{j+1} \Pi^\ell(\alpha_j) \left( Q(\alpha_1, \cdots, \widehat{\alpha_j}, \cdots, \alpha_{\ell+1}) \right)$$

$$+ \sum_{j<k} (-1)^{j+k} Q([\alpha_j, \alpha_k], \alpha_1 \cdots, \widehat{\alpha_j}, \cdots, \widehat{\alpha_k}, \cdots, \alpha_{\ell+1}),$$

for any $\alpha_1, \cdots, \alpha_{\ell+1} \in \Omega^1(M)$.

**Lemma 3.5** Let $(M, D)$ be a Dirac manifold. The $D$-exterior derivative $d_D : \Gamma^\infty(M, \wedge^* D^*) \to \Gamma^\infty(M, \wedge^{*+1} D^*)$ has the decomposition of exterior differentials $d$ and $\partial$:

$$d_D(\phi, Q) = d\phi + \partial Q.$$

Let $(A \to M, \sharp, \|\cdot\|, \|\cdot\|)$ be a Lie algebroid and $\Phi : M' \to M$ a smooth map. Assume that the differential $d\Phi$ of $\Phi$ is transversal to the anchor map $\sharp : A \to TM$ in the sense that

$$\text{Im} \sharp_{\Phi(x)} + \text{Im} (d\Phi)_x = T_{\Phi(x)} M, \quad (\forall x \in M'). \quad (3.4)$$
Here, $\mathfrak{g}_{\Phi(x)}$ stands for the image of $\mathfrak{g}_{\Phi(x)}$. This assumption leads us to the following condition:

$$\text{im } (id_x \times \sharp_{f(x)}) + T_{(x, \Phi(x))}(\text{graph}(\Phi)) = T_xM' \oplus T_{\Phi(x)}M, \quad (\forall x \in M'), \tag{3.5}$$

where $id_x$ means the identity map on $T_xM'$. The condition ensures that the preimage

$$(id \times \rho)^{-1}T(\text{graph}(\Phi)) = \bigsqcup_{x \in M'} \{(V; \alpha) \mid V \in T_xM', \alpha \in A_{\Phi(x)}, (d\Phi)_x(V) = \sharp(\alpha)\} \tag{3.6}$$

is a smooth subbundle of $(TM' \times A)|_{\text{graph}(\Phi)}$. The vector bundle (3.6) over $\text{graph}(\Phi) \cong M'$ has the structure of Lie algebroid whose anchor map is the natural projection $pr_1$. This Lie algebroid is called the pull-back of Lie algebroid and denoted by $\Phi^*A$ (see P. Higgins and K. Mackenzie [14]). We remark that $f^*A$ has rank $\text{rank}(\Phi^*A) = \text{rank}A - \dim M + \dim M'$.

Let $\Phi : M' \to (M, D)$ be a smooth map from a smooth manifold $M'$ to a Dirac manifold $(M, D)$ which satisfies the condition (3.4). Given a $D$-differential $\ell$-form $\vartheta$, we define a $\Phi^*D$-differential $\ell$ ($\ell > 0$)-form $\Phi^*\vartheta$, called the pull-back of $\vartheta$, as

$$(\Phi^*\vartheta)_x((V_1; (d\Phi)_x(V_1), \xi_1), \ldots, (V_\ell; (d\Phi)_x(V_\ell), \xi_\ell))$$

$$:= \vartheta_{\Phi(x)}(((d\Phi)_x(V_1), \xi_1), \ldots, ((d\Phi)_x(V_\ell), \xi_\ell))$$

for any $V_j \in T_xM'$ and $\xi_j \in T_{\Phi(x)}M$ ($j = 1, 2, \cdots, \ell$). If $\ell = 0$, the pull-back of 0-form $f \in C^\infty(M)$ is defined as $\Phi^*f := f \circ \Phi$. By using Lemma [5.5], it can be easily verified that $\Phi^*$ and $d_D$ commute with each other, that is,

$$\Phi^* \circ d_D = d_D \circ \Phi^*.$$

### 3.2 Dirac-Chern classes of complex line bundles

We let $\varpi : L \to M$ be a complex line bundle over a Dirac manifold $(M, D)$ and $\{(U_j, \varepsilon_j)\}_j$ be a family of pairs which gives local trivializations of $L$. That is, $\{U_j\}_j$ is an open covering of $M$ and $\varepsilon_j$ are nowhere vanishing smooth sections on $U_j$ such that the map

$$U_j \times \mathbb{C} \longrightarrow \varpi^{-1}(U_j), \quad (x, z) \longmapsto z \varepsilon_j(x)$$

for each $j$ is a diffeomorphism.

A $D$-connection

$$\nabla^D : \Gamma^\infty(M, D) \times \Gamma^\infty(M, L) \to \Gamma^\infty(M, L),$$

is also considered as a map from $\Gamma^\infty(M, L)$ to $\Gamma^\infty(M, D^*) \otimes_{C^\infty(M)} \Gamma^\infty(M, L)$ by

$$\Gamma^\infty(M, L) \ni s \longmapsto \{\psi \mapsto \nabla^D_\psi s\} \in \text{Hom}_{C^\infty(M)}(\Gamma^\infty(M, D), \Gamma^\infty(M, L)).$$

On each $U_j$, $\nabla^D e_j$ is written as

$$\nabla^D e_j = 2\pi i \sigma_j \otimes e_j,$$

by using a smooth section $\sigma_j \in \Gamma^\infty(U_j, D^*)$. Since the transition function $g_{jk}$ on $U_j \cap U_k (\neq \emptyset)$ is given by $g_{jk}(x) := \varepsilon_k(x)/\varepsilon_j(x)$ ($x \in U_j \cap U_k$), we have $\varepsilon_k(x) = g_{jk}(x)\varepsilon_j(x).$ From a simple computation, it holds that

$$\nabla^D e_k = (2\pi i g_{jk}\sigma_j + dg_{jk}) \otimes e_j. \tag{3.7}$$
On the other hand,
\[ \nabla^D e_k = 2\pi i g_{jk} \sigma_k \otimes e_j. \]  
(3.8)

It immediately follows from (3.7) and (3.8) that
\[ \sigma_j - \sigma_k = \frac{i}{2\pi} \frac{d\sigma g_{jk}}{g_{jk}}. \]  
(3.9)

As a result, one gets a \( D \)-differential 2-form \( \tau \) defined on the whole of \( M \) by \( \tau = d_D \sigma_j = d_D \sigma_k \) \( (U_j \cap U_k \neq \emptyset) \). It is easy to verify that \( \tau \) satisfies
\[ (R^D_{\psi}(\psi_1, \psi_2))(e_j) = 2\pi i \tau(\psi_1, \psi_2) e_j \quad (\forall \psi_1, \psi_2 \in \Gamma^\infty(M, D)) \]
for each \( j \). That is, \( \tau \) is the curvature 2-section of \( \nabla^D \) (see Remark 3.1 below). Obviously, \( \tau \) defines a second \( D \)-cohomology class \([\tau] \in H^2_D(M, D)\).

**Proposition 3.6** The cohomology class \([\tau] \) determined by the curvature 2-section \( \tau \) does not depend on a choice from \( D \)-connections.

**Proof.** Let \( \nabla' \) be another \( D \)-connection on \( L \to M \) having a curvature \( R' \) of \( \nabla' \) and \( \sigma'_j \) the corresponding local sections in \( \Gamma^\infty(U_j, D') \). Denoting by \( \tau' \) the curvature 2-section corresponding to \( R' \), we have
\[ \tau' - \tau = d_D \sigma'_j - d_D \sigma_j = d_D (\sigma'_j - \sigma_j) \]  
(3.10)
on each \( U_j \). We define an \( \mathbb{C} \)-linear map \( \nabla \) as
\[ \nabla : \Gamma^\infty(M, L) \to \Gamma^\infty(M, D^*) \otimes_{C^\infty(M)} \Gamma^\infty(M, L), \quad s \mapsto (\nabla' - \nabla^D) s. \]

On \( U_j \), it holds that
\[ \nabla_{\psi} e_j = (\nabla'_{\psi} - \nabla^D_{\psi}) e_j = 2\pi i \sigma'_j(\psi) e_j - 2\pi i \sigma_j(\psi) e_j = 2\pi i (\sigma'_j - \sigma_j)(\psi) e_j \]
for any \( \psi \) in \( \Gamma^\infty(U_j, D) \). Putting \( \tilde{\sigma}_j = \sigma'_j - \sigma_j \) for each \( j \), we find that, by (3.9),
\[ \tilde{\sigma}_j - \tilde{\sigma}_k = (\sigma'_j - \sigma'_k) - (\sigma_j - \sigma_k) = \frac{i}{2\pi} \frac{dD g_{jk}}{g_{jk}} - \frac{i}{2\pi} \frac{dD g_{jk}}{g_{jk}} = 0 \]
on \( U_j \cap U_k \neq \emptyset \). Accordingly, there exists a \( D \)-differential 1-form \( \tilde{\sigma} \) over the whole of \( M \) by \( \tilde{\sigma} = \tilde{\sigma}_j = \tilde{\sigma}_k \) on \( U_j \cap U_k (\neq \emptyset) \). Therefore, it follows from (3.10) that
\[ \tau' - \tau = d_D \tilde{\sigma}_j = d_D \tilde{\sigma}. \]

This shows that \([\tau] = [\tau'] \) in \( H^2_D(M, D) \).

**Definition 3.7** Let \( L \to M \) be a complex line bundle over a Dirac manifold \((M, D)\) and \( \nabla^D \) any \( D \)-connection on \( L \). The second cohomology class \([\tau] \in H^2_D(M, D)\) by the \( D \)-differential 2-form \( \tau \) which corresponds to the curvature of \( \nabla^D \) is called the first Dirac-Chern class of \( L \to M \). We denote the first Dirac-Chern class of \( L \) by \( c^D_1(L) \). 

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We assume that the line bundle \( L \to M \) has a Hermitian metric \( h \). \( \nabla^D \) is called a Hermitian \( D \)-connection with respect to \( h \) if
\[
\rho(\psi)(h(s_1, s_2)) = h(\nabla^D_{\psi} s_1, s_2) + h(s_1, \nabla^D_{\psi} s_2)
\]
for any smooth section \( s_1, s_2 \) of \( L \) and any smooth section \( \psi \) of \( D \). The following proposition can be shown in a way similar to the case of the ordinary connections on Hermitian line bundles (see [16]).

**Proposition 3.8** The curvature 2-section \( \tau \) of \( \nabla^D \) is a real \( D \)-differential 2-form.

**Remark 3.1** Let \( A \) be any Lie algebroid over \( M \). In general, an \( A \)-connection \( \nabla^A \) on a vector bundle \( \pi : E \to M \) is considered as a \( \mathbb{R} \)-linear map \( \Gamma^\infty(E) \to \Gamma^\infty(A^*) \otimes_{C^\infty(M)} \Gamma^\infty(E) \) satisfying the condition (2) in Definition 3.3. Letting \( \{V_\lambda\}_\lambda \) be an open covering which gives local trivializations of \( E \) and \( s_1, \ldots, s_r \) (\( r = \text{rank} \ E \)) be smooth sections such that \( s_1(p), \ldots, s_r(p) \) is a basis for the fiber \( \pi^{-1}(p) \) for every \( p \in V_\lambda \), one can verify that there exist a matrix \( \Theta_\lambda \) of local sections of \( A^* \) over \( V_\lambda \) such that
\[
\nabla^A s_k = \sum_j \theta^A_{jk} s_j \quad (\theta^A_{jk} \in \Gamma^\infty(V_\lambda, A^*)).
\]
The matrix \( \theta \) is called a connection 1-section (see R. L. Fernandes [17]). In the same manner as the ordinary connection theory, the curvature \( R^A_\lambda \) of \( \nabla^A \) is written as
\[
(R^A_\lambda(\alpha_1, \alpha_2)(s_k)) = \sum_j \kappa_{jk}(\alpha_1, \alpha_2)s_j \quad (\forall \alpha_1, \alpha_2 \in \Gamma^\infty(V_\lambda, A))
\]
on each \( V_\lambda \), where \( \kappa_{jk} \in \Gamma^\infty(V_\lambda, \wedge^2 A^*) \). The matrix \( \kappa = (\kappa_{jk}) \) is called the curvature 2-section of \( \nabla^A \) (see [17]).

### 4 Prequantization of Dirac manifolds

#### 4.1 \( \Omega \)-compatible Poisson structures

Let \((M, D)\) be a Dirac manifold. As is mentioned in Section 2, \((M, D)\) has the presymplectic structure \( \Omega^D : \rho(D) \to \rho(D)^* \) by (2.3). We define a distribution \( V \subset TM \) as
\[
V := \ker \Omega^D = D \cap TM.
\]
Here, we remark again that \( D \cap TM \) is thought of as a subbundle of either \( TM \oplus T^*M \) or \( TM \). We take a subbundle \( \mathcal{H} \subset \rho(D) \) such that
\[
V_m \oplus \mathcal{H}_m = \rho_m(D_m) \quad (\forall m \in M) \tag{4.1}
\]
and fix it. Since \( \ker \Omega^D \cap \mathcal{H} = \{0\} \), it turns out that \( \mathcal{H} \) is isomorphic to \( \mathcal{H}^* := \text{Im} \Omega^D|\mathcal{H} = \text{Im} \Omega^D \) by the restriction map
\[
\Omega^D|\mathcal{H} : \mathcal{H} \cong \mathcal{H}^*.
\]
We denote its inverse map \( (\Omega^D|\mathcal{H})^{-1} : \mathcal{H}^* \to \mathcal{H} \) by and \( \Theta^\sharp \). Then, it can be easily verified that
\[
\Theta^\sharp \circ \Omega^D|\mathcal{H} = \text{id}_{\mathcal{H}} \quad \text{and} \quad \Omega^D|\mathcal{H} \circ \Theta^\sharp = \text{id}_{\mathcal{H}^*}. \tag{4.2}
\]
As is mentioned in Section 2, there exists a bundle map $\Pi^\sharp_m$ defined as

$$\mathcal{V}_m^\circ \ni \eta_m \mapsto \{ \xi \mapsto \xi(\Pi^\sharp_m(\eta_m)) := \xi_m(Y_m) \} \in T_m\mathcal{M}/(D_m \cap T_m\mathcal{M}).$$

We here remark that $\mathcal{V}_m^\circ$ is the annihilator of $\mathcal{V}_m$ in $T_m^*\mathcal{M}$. From the definition of $\Omega$ and Proposition 2.2, the image $\text{im}\, \Omega^\flat$ of $\Omega^\flat$ turns out to be

$$\text{im}\, \Omega^\flat_m = \rho^*(D_m) = (D_m \cap T_m\mathcal{M})^\circ = \mathcal{V}_m^\circ.$$

So, we have $\mathcal{H}^* = \mathcal{V}^\circ$, and find that $\Theta^\flat = \Pi^\sharp$. By Proposition 2.2 any admissible function $f \in C^\infty_{\text{adm}}(M, D)$ satisfies

$$df \in \mathcal{H}^*.$$ 

This allows us to define a vector field as

$$H_f := \Pi^\sharp(df) \in \mathcal{H}.$$

Since $f$ is admissible, there exists a vector field $X_f$ such that $(X_f, df) \in \Gamma^\infty(M, D)$. It is easy to see that $((H_f)_m - (X_f)_m, 0) \in \mathcal{V}_m \subset D_m$ at each $m \in M$. It follows from this that

$$((H_f)_m, (df)_m) = ((H_f)_m - (X_f)_m, 0) + ((X_f)_m, (df)_m) \in D_m.$$ 

So, it turns out that $(H_f, df) \in \Gamma^\infty(M, D)$. For any $f, g \in C^\infty_{\text{adm}}(M, D)$, let us define their bracket $\{ f, g \}$ as

$$\{ f, g \} := H_g f.$$ 

(4.3)

It is easily verified that, for any $f, g \in C^\infty_{\text{adm}}(M, D)$,

$$\{ f, g \} = \Omega(H_g, H_f).$$

Since $(H_f, df), (H_g, dg) \in \Gamma^\infty(M, D)$, we have that

$$([H_g, H_f], df(g, f)) = \{ (H_g, dg), (H_f, df) \} \in \Gamma^\infty(M, D).$$

So, $df(g, f)$, also, is the admissible function. This implies that one can define the operator

$$\{ \cdot, \cdot \} : C^\infty_{\text{adm}}(M, D) \times C^\infty_{\text{adm}}(M, D) \longrightarrow C^\infty_{\text{adm}}(M, D),$$

as (4.3), which is both bilinear and skew-symmetric. The bracket $\{ \cdot, \cdot \}$ satisfies the Leibniz identity. Moreover, from the 2-cocycle condition of $\Omega$ (see Section 2), it follows that

$$0 = H_f(\Omega(H_g, H_h)) - H_g(\Omega(H_f, H_h)) + H_h(\Omega(H_f, H_g))$$

$$+ \Omega(H_h, [H_f, H_g]) + \Omega(H_f, [H_g, H_h]) + \Omega(H_g, [H_f, H_h])$$

$$= - ([L_{H_f}(dh)](H_g) - ([L_{H_g}(df)](H_h) - ([L_{H_h}(dg)](H_f)$$

$$= - \{ [h, f], g \} - \{ [f, g], h \} - \{ [g, h], f \} = 0.$$

That is, the Jacobi identity holds:

$$\{ [f, g], h \} + \{ [g, h], f \} + \{ [h, f], g \} = 0 \quad (\forall f, g, h \in C^\infty_{\text{adm}}(M, D)).$$

(4.4)

Accordingly,

$$([H_f, H_g] + H_{(f,g)}h)h = H_f(H_g h) - H_g(H_f h) + H_{(f,g)}h = \{ [h, g], f \} - \{ [h, f], g \} + \{ [f, g], h \}$$

$$= - \{ [g, h], f \} - \{ [h, f], g \} + \{ [f, g], h \}$$

$$= 0$$

for any admissible function $f, g$ and $h$ on $(M, D)$. Summing up, we can obtain the following result.
Example 4.1 Let us consider a Dirac manifold \( (\mathbb{R}^4, \text{graph}(\omega^b)) \) by the presymplectic form \( \omega = dx_1 \wedge dx_2 + dx_1 \wedge dx_4 \) in Example 2.2. Then, the presymplectic form \( \Omega \) is entirely \( \omega \) and written in the matrix form

\[
\Omega^b = \omega^b = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\]

and consequently, at each \( m \in \mathbb{R}^4 \),

\[
\text{graph} (\omega^b_m) = \left\{ \left( \frac{\partial}{\partial x_1} + \frac{b}{\partial x_2} + c \frac{\partial}{\partial x_3} + d \frac{\partial}{\partial x_4}, -b + d \right) dx_1 + a (dx_2 + dx_4) \right\} | a, b, c, d \in \mathbb{R} \).
\]

The subspace \( V_m = \text{graph} (\omega^b_m) \cap \mathbb{R}^4 \) is given by

\[
V_m = \left\{ \left( \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4}, 0 \right) | a, b \in \mathbb{R} \right\}.
\]

Then, we can take a subspace \( \mathcal{H}_m \) as

\[
\mathcal{H}_m = \left\{ \left( \frac{\partial}{\partial x_1} - \frac{b}{\partial x_4}, b dx_1 + a (dx_2 + dx_4) \right) | a, b \in \mathbb{R} \right\}.
\]

One easily checks that \( \text{graph} (\omega^b_m) = V_m \oplus \mathcal{H}_m \). The vector field \( H_f \) for the admissible function \( f(x) = x_1^2 + k (x_2 + x_4) (k \in \mathbb{R}) \) is given by

\[
H_f = k \frac{\partial}{\partial x_1} - 2x_1 \frac{\partial}{\partial x_4}.
\]

Example 4.2 Let \( (M, F \oplus F^\circ) \) be a Dirac manifold obtained from a regular distribution \( F \subset TM \) (see Example 2.3). Since any vector field \( X \in F \) is embedded in \( F \oplus F^\circ \) with \( X \mapsto (X, 0) \), one finds that \( \mathcal{V} = F \oplus \{0\} \cong F \). It follows from this that \( \mathcal{H} = \{0\} \oplus F^\circ \cong F^\circ \). If \( f \in C^\infty_{\text{adm}}(M, F \oplus F^\circ) \), the vector field \( H_f \) for \( f \) is given by \( H_f = 0 \).

Example 4.3 We consider Dirac manifold \( (\mathbb{R}^2, \text{graph}(\varpi^b)) \) induced from a Poisson bivector \( \varpi = G(x) \partial/\partial x_1 \wedge \partial/\partial x_2 \), where \( G(x) = G(x_1, x_2) \) is a smooth function on \( \mathbb{R}^2 \). We remark that the Dirac structure \( \text{graph}(\varpi^b) \) is written in the form

\[
\text{graph} (\varpi^b) = \left\{ \left( G(x) \left( b \frac{\partial}{\partial x_1} - a \frac{\partial}{\partial x_2} \right), a dx_1 + b dx_2 \right) | a, b \in C^\infty(\mathbb{R}^2) \right\}
\]

and any smooth function on \( \mathbb{R}^2 \), \( \text{graph}(\varpi^b) \) is admissible. The distribution \( \mathcal{V} \) is given by \( \mathcal{V} = \text{graph}(\varpi^b) \cap \mathbb{R}^2 = \{0\} \) and consequently, \( \mathcal{H} = \mathcal{V} = \text{graph}(\varpi^b) \). For a smooth function \( h \), the vector field \( H_h \) is represented as

\[
H_h = G(x) \left( \frac{\partial h}{\partial x_2} \frac{\partial}{\partial x_1} - \frac{\partial h}{\partial x_1} \frac{\partial}{\partial x_2} \right).
\]
4.2 Quantizable Dirac manifolds

Suppose that there exists a Hermitian line bundle $L \xrightarrow{\pi} M$ over $(M, D)$ with a $D$-connection $\nabla^D$ whose curvature is $R^D_C$. For the fixed subbundle $\mathcal{H} \subset D$, we define a map $\hat{\cdot}: \mathcal{C}_\text{adm}^\infty(M, D) \to \text{End}_C(\Gamma^\infty(M, L))$ from the Poisson algebra $(\mathcal{C}_\text{adm}^\infty(M, D), \{\cdot, \cdot\})$ to Lie algebra $(\text{End}_C(\Gamma^\infty(M, L)), [\cdot, \cdot])$ as

$$\hat{f} := -\nabla^D_{(H_f, d\pi)f}s - 2\pi ifs \quad (\forall s \in \Gamma^\infty(M, L))$$

for each $f \in \mathcal{C}_\text{adm}^\infty(M, D)$.

**Proposition 4.2** The map $\hat{\cdot}: \mathcal{C}_\text{adm}^\infty(M, D) \to \text{End}_C(\Gamma^\infty(M, L))$ is a representation, that is, it holds that

$$\{\hat{f}, \hat{g}\} = [\hat{f}, \hat{g}]$$

for all $f, g \in \mathcal{C}_\text{adm}^\infty(M, D)$ if and only if

$$R^D_C((H_f, df), (H_g, dg)) = 2\pi i \Lambda((H_f, df), (H_g, dg)),$$

where $\Lambda$ is the skew-symmetric pairing $\Lambda(\cdot, \cdot) := \langle \cdot, \cdot \rangle_{-}$ in Section 2.

**Proof.** Using Proposition 4.1 we have that

$$[\hat{f}, \hat{g}]s = \hat{f}(\hat{g}s) - \hat{g}(\hat{f}s)$$

for any admissible function $f, g$ on $(M, D)$ and any smooth section $s$ of $L \to M$. The bracket $\{f, g\}$ is calculated to be

$$\{f, g\} = \frac{1}{2} ((f, g) - (g, f)) = \frac{1}{2} (df(H_g) - dg(H_f))$$

$$= \langle (H_f, df), (H_g, dg) \rangle_-. $$

From this, we immediately get \ref{4.8} as the necessary and sufficient condition for the map $\hat{\cdot}$ to preserve their brackets. \hfill \Box

**Definition 4.3** A Dirac manifold $(M, D)$ is said to be prequantizable if there exists a Hermitian line bundle $(L, h)$ over $M$ with a Hermitian $D$-connection $\nabla^D$ in the sense that

$$H_f (h(s_1, s_2)) = h(\nabla^D_{(H_f, df)s_1} s_2) + h(s_1, \nabla^D_{(H_f, df)s_2}),$$

which satisfies the condition \ref{4.8}. The line bundle is called the prequantization bundle.
Let us consider the skew-symmetric pairing \( \Lambda \) again. We find that \( \Lambda : \mathcal{F}^{\infty}(M, D) \times \mathcal{F}^{\infty}(M, D) \to \mathcal{C}^{\infty}(M) \) is closed with regard to the differential operator \( d_D \). Indeed, by Lemma \([3.2]\) and the Cartan formula, \( d_D \Lambda \) is calculated to be

\[
(d_D \Lambda)(X, \xi, Y, \eta, Z, \zeta) = X(\eta(Z) - \zeta(Y)) - Y(\xi(Z) - \zeta(X)) + Z(\xi(Z) - \eta(X)) \\
- \Lambda(\{(X, \xi), (Y, \eta), (Z, \zeta)\}) - \Lambda(\{(Y, \eta), (Z, \zeta), (X, \xi)\}) - \Lambda(\{(Z, \zeta), (X, \xi), (Y, \eta)\}) \\
= X(\eta(Z) - \zeta(Y)) - Y(\xi(Z) - \zeta(X)) + Z(\xi(Z) - \eta(X)) - (\mathcal{L}_Y \eta)(Z) - (\mathcal{L}_X \zeta)(X) - (\mathcal{L}_Z \xi)(Y) \\
+ (d\xi)(Y, Z) + (d\eta)(Z, X) + (d\zeta)(X, Y) + \xi(\{[Z, Y]\}) + \eta(\{[Z, X]\}) + \zeta(\{X, Y\}) \\
= 0
\]

for any section \((X, \xi), (Y, \eta)\) and \((Z, \zeta)\) of \( D \). Accordingly, the \( D \)-differential 2-form \( \Lambda \) defines the second cohomology class \([\Lambda]\) in the Lie algebroid cohomology. Additional to this, we have that

\[
\Lambda((X, \xi), (Y, \eta)) = \frac{1}{2}[\xi - \eta] \in \Gamma(\mathcal{F}^{\infty}(M, D)).
\]

That is, it holds that

\[
\Lambda = (\lambda^2 p^\ast \Omega). \tag{4.10}
\]

**Theorem 4.4** A Dirac manifold \((M, D)\) is prequantizable if and only if the \( D \)-cohomology class \([\Lambda]\) of \( \Lambda \) has coefficients in \( \mathbb{Z} \):

\[
[\Lambda] \in H^2_{\mathbb{Z}}(M, D; \mathbb{Z}). \tag{4.11}
\]

Before proceeding with the proof of Theorem \([4.4]\) we show some lemmas needed later. Applying Theorem \([3.2]\) to a Dirac structure \( D \to M \), we find that, for each \( m \in M \), there exist a local coordinates \((x_1, \cdots, x_r, y_1, \cdots, y_s)\) at \( m \) (\( r = \text{rank} \rho_m \) and \( r + s = \text{dim} M \)) and a basis of local sections

\[
\left( \frac{\partial}{\partial x_1}, \lambda_1 \right), \cdots, \left( \frac{\partial}{\partial x_r}, \lambda_r \right), (Y_1, \mu_1), \cdots, (Y_s, \mu_s) \tag{4.12}
\]

over an open neighborhood \( W \) of \( m \) which satisfy

\[
Y_k = \sum_{j=1}^{s} h_{jk}(y) \frac{\partial}{\partial y_j}, \quad \det(h_{jk}(y))_{1 \leq j, k \leq s} \neq 0,
\]

\[
\mathcal{L}_{\frac{\partial}{\partial y_j}} \lambda_k = i_{\frac{\partial}{\partial y_j}} d\lambda_k, \quad \mathcal{L}_{\frac{\partial}{\partial y_j}} \mu_k = i_{\frac{\partial}{\partial y_j}} d\mu_k.
\]

for all possible indices \( j, k \). Let us consider the pull-back of Lie algebroid \( D \to M \) along the projection \( \text{pr}_M : M \times \mathbb{R} \to M \):

\[
D_1 := \text{pr}_M \ast D = \bigcup_{(p, t) \in M \times \mathbb{R}} \{(X_p, f(p, t) (\partial/\partial t)_h) : X_p, \xi_p) \mid \xi_p) \in D_p, f \in \mathcal{C}^{\infty}(M \times \mathbb{R})\}.
\]

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Noting that \( \text{rank}(D_1) = \text{rank} D - \dim M + \dim (M \times \mathbb{R}) = \dim (M \times \mathbb{R}) \), \( D_1 \) has the local basis of the smooth sections on \( W \times \mathbb{R} \)

\[
\left( \frac{\partial}{\partial x_j}, 0; \frac{\partial}{\partial x_j}, \lambda_j \right), \quad (Y_k, 0; Y_k, \mu_k), \quad \left( 0, \frac{\partial}{\partial t}, 0 \right) \quad (1 \leq j \leq r, 1 \leq k \leq s)
\]

induced by (4.12). We denote their dual basis by

\[ \gamma_1, \cdots, \gamma_r, \delta_1, \cdots, \delta_s, \ dt \in \Gamma^\infty(W, D_1^*) , \]

that is, they are the local smooth sections of \( D_1^* \) such that

\[
\gamma_j \left( \frac{\partial}{\partial x_k}, 0; \frac{\partial}{\partial x_k}, \lambda_k \right) = \delta_j (Y_k, 0; Y_k, \mu_k) = \begin{cases} 1 & (j = k) \\ 0 & (j \neq k) \end{cases}, \quad dt \left( 0, \frac{\partial}{\partial t}, 0 \right) = 1
\]

and

\[
\delta_j (Y_k, 0; Y_k, \mu_k) = \gamma_j \left( \frac{\partial}{\partial x_k}, 0; \frac{\partial}{\partial x_k}, \lambda_k \right) = 0, \\
dt (Y_k, 0; Y_k, \mu_k) = dt \left( \frac{\partial}{\partial x_k}, 0; \frac{\partial}{\partial x_k}, \lambda_k \right) = 0
\]

for all possible \( j, k \). A smooth section \( \alpha = (X, f(p, t) \partial/\partial t; X, \xi) \in \Gamma^\infty(W \times \mathbb{R}, D_1) \) is written in the form

\[
\left( \sum_{jk} \left( u_j \frac{\partial}{\partial x_j} + v_k Y_k \right) f(p, t) \frac{\partial}{\partial t} + \sum_{jk} \left( u_j \frac{\partial}{\partial x_j} + v_k Y_k \right) \sum_{k} (u_k \lambda_k + v_k \mu_k) \right).
\]

Then, by a simple computation, one finds that

\[
u_j = \gamma_j (\alpha), \quad v_k = \delta_k (\alpha), \quad f(p, t) = dt (\alpha) \quad (1 \leq j \leq r, 1 \leq k \leq s).
\]

Accordingly, from (3.3), the \( D_1 \)-exterior derivative of a smooth function \( F \) on \( M \times \mathbb{R} \) is represented as

\[
d_{D_1} F = \sum_j \frac{\partial F}{\partial x_j} \gamma_j + \sum_k (Y_k F) \delta_k + \frac{\partial F}{\partial t} dt.
\]

**Proposition 4.5** Let \((M, D)\) be a Dirac manifold and \( D_1 \rightarrow M \) the pull-back of Lie algebroid \( D \) along the natural projection \( \text{pr}_M : M \times \mathbb{R} \rightarrow M \). Then, \( \text{pr}_M^* \) induces the isomorphism

\[
\text{pr}_M^* : H^*_L(M, D; \mathbb{R}) \xrightarrow{\cong} H^*_L(M \times \mathbb{R}, D_1; \mathbb{R}).
\]

The inverse is the homomorphism \( \tau^* : H^*_L(M \times \mathbb{R}, D_1; \mathbb{R}) \rightarrow H^*_L(M, D; \mathbb{R}) \) induced from the inclusion map \( \iota : M \rightarrow M \times \mathbb{R}, \ p \mapsto (p, 0) \).

**Proof.** Since \( \text{pr}_M \circ \iota = id \), it holds that \( \tau^* \circ \text{pr}_M^* = id \). Therefore, it is sufficient to show \( \text{pr}_M^* \circ \tau^* = id \) on \( H^*_L(M \times \mathbb{R}, D_1; \mathbb{R}) \) for the proof. For simplicity, we may assume that \( M \) is an Euclidean space \( \mathbb{R}^{\dim M} \) and \( W \) is a star-shaped open set with respect to the origin \( 0 \in \mathbb{R}^{\dim M} \). We remark that any \( D_1 \)-differential \( \ell \)-form \( \omega \) can be written in the form

\[
\omega = \sum_{I'} f_{I', p} (p, t) \gamma_I \wedge \delta_J + \sum_{J'} g_{J', p} (p, t) dt \wedge \gamma_J \wedge \delta_J.
\]
On the other hand, the

where \( I, I' \) and \( J, J' \) run over all sequences with \( 1 \leq i_1 < i_2 < \cdots < i_r \leq s \), \( 1 \leq i'_1 < i'_2 < \cdots < i'_{r'} \leq s \) (\( c + c' = \ell \)) and \( 1 \leq j_1 < j_2 < \cdots < j_d \leq r \), \( 1 \leq j'_1 < j'_2 < \cdots < j'_{d'} \leq s \) (\( d + d' = \ell - 1 \)), respectively.

For each \( \ell \), we define an operator \( S_\ell \) from \( \Gamma^\infty(W \times \mathbb{R}, \wedge^\ell D_1^*) \) to \( \Gamma^\infty(W \times \mathbb{R}, \wedge^{\ell-1} D_1^*) \) as

\[
S_\ell(\omega) := \sum_{J,J'} \left( \int_0^\ell g_{J,J'}(p,t) \, dt \right) \gamma_J \wedge \delta_{J'}. 
\]

Then, by Proposition 3.4, we have

\[
d_{D_1}(S_\ell(\omega)) = \sum_{J,J'} \left[ \sum_j \frac{\partial f_{J,J'}(p,t)}{\partial x_j}(p,t) \gamma_j + \sum_k \left( f_{I,J'}(p,t) \delta_k + \frac{\partial f_{J,J'}(p,t)}{\partial t}(p,t) \right) \gamma_J \wedge \delta_{J'} \right. \\
\left. - \sum_{J,J'} \left( \int_0^\ell g_{J,J'}(p,t) \, dt \right) \delta_{J'} + \sum_{J,J'} \left( \int_0^\ell g_{J,J'}(p,t) \, dt \right) \gamma_J \wedge (d_{D_1}(\delta_{J'})). \right.
\]

On the other hand, the \( D_1 \)-exterior derivative of \( \omega \) is calculated to be

\[
d_{D_1}(\omega) = \sum_{I,I'} \left[ \sum_j \frac{\partial f_{I,I'}(p,t)}{\partial x_j}(p,t) \gamma_j + \sum_k \left( f_{I,J'}(p,t) \delta_k + \frac{\partial f_{I,I'}(p,t)}{\partial t}(p,t) \right) \gamma_I \wedge \delta_{I'} \right. \\
\left. - \sum_{I,I'} \left( \int_0^\ell g_{I,I'}(p,t) \, dt \right) \delta_{I'} + \sum_{I,I'} \left( \int_0^\ell g_{I,I'}(p,t) \, dt \right) \gamma_I \wedge (d_{D_1}(\delta_{I'})). \right.
\]

Therefore,

\[
S_\ell(d_{D_1}(\omega)) = \sum_{I,I'} \left( \int_0^\ell \frac{\partial f_{I,I'}(p,t)}{\partial t}(p,t) \, dt \right) \gamma_I \wedge \delta_{I'} - \sum_{I,I'} \left( \int_0^\ell \frac{\partial g_{I,I'}(p,t)}{\partial x_j}(p,t) \, dt \right) \gamma_J \wedge \delta_{I'} \\
\left. - \frac{\partial g_{I,I'}(p,t)}{\partial t}(p,t) \, dt \right) \gamma_I \wedge (d_{D_1}(\delta_{I'})). 
\]

As a result, we have that

\[
d_{D_1}(S_\ell(\omega)) + S_\ell(d_{D_1}(\omega)) = \sum_{I,I'} \left( \int_0^\ell \frac{\partial f_{I,I'}(p,t)}{\partial t}(p,t) \, dt \right) \gamma_I \wedge \delta_{I'} + \frac{\partial}{\partial t} \left( \int_0^\ell g_{I,I'}(p,t) \, dt \right) \gamma_I \wedge \delta_{I'} \\
- \sum_{I,I'} \left( \int_0^\ell f_{I,I'}(p,t) \, dt \right) \gamma_I \wedge \delta_{I'} - \sum_{I,I'} \left( \int_0^\ell g_{I,I'}(p,t) \, dt \right) \gamma_I \wedge (d_{D_1}(\delta_{I'})) \\
+ \sum_{I,I'} \left( \int_0^\ell g_{I,I'}(p,t) \, dt \right) \gamma_I \wedge (d_{D_1}(\delta_{I'})). 
\]
\[ \omega - \sum_{I,J} f_{I,J}(p,0) \gamma_I \wedge \delta_J. \] (4.13)

Here, we recall again that the pull-backs \( \text{pr}_M^*: \Gamma^\infty(M, \wedge^\ell D^*) \rightarrow \Gamma^\infty(M \times \mathbb{R}, \wedge^\ell D_1^*) \) and \( \iota^*: \Gamma^\infty(M \times \mathbb{R}, \wedge^\ell D_1^*) \rightarrow \Gamma^\infty(M, \wedge^\ell D^*) \) are given by

\[ (\text{pr}_M^* \vartheta) \left( \left[ X_1, f_1 \frac{\partial}{\partial t} : X_1, \xi_1 \right], \cdots, \left[ X_\ell, f_\ell \frac{\partial}{\partial t} : X_\ell, \xi_\ell \right] \right) := \vartheta \left( \left[ X_1, \xi_1 \right], \cdots, \left[ X_\ell, \xi_\ell \right] \right). \]

\[ \text{pr}_M^* f = f \circ \text{pr}_M \quad (f \in C^\infty(M)) \]

and

\[ (\iota^* \omega) \left( \left[ X_1, \xi_1 \right], \cdots, \left[ X_\ell, \xi_\ell \right] \right) := \omega \left( \left[ X_1, 0 : X_1, \xi_1 \right], \cdots, \left[ X_\ell, 0 : X_\ell, \xi_\ell \right] \right), \]

\[ \iota^* F = F \circ \iota \quad (F \in C^\infty(M \times \mathbb{R})) \]

respectively. By a simple computation we get

\[ \omega - (\text{pr}_M^* \circ \iota^*) \omega = \omega - \sum_{I,J} f_{I,J}(p,0) \gamma_I \wedge \delta_J. \] (4.14)

From (4.13) and (4.14) it follows that

\[ d_{D_1} \circ S_\ell + S_\ell \circ d_{D_1} = \text{id} - (\text{pr}_M^* \circ \iota^*). \]

Since \( d_{D_1} \circ (\iota \circ \text{pr}_M)^* = (\iota \circ \text{pr}_M)^* \circ d_{D_1} \), it turns out that \( \omega = d_{D_1} (\omega - (\text{pr}_M^* \circ \iota^*) \omega) \) for any \( D_1 \)-differential \( \ell \)-form \( \omega \) such that \( d_{D_1} \omega = 0 \). That is, \( [\omega] = 0 \) in \( H^\ell_1(M \times \mathbb{R}, D_1; \mathbb{R}) \). This completes the proof. \( \square \)

We let \( x_0 \) be any point in \( M \) and suppose that the rank of the anchor map \( \rho_{x_0} : D_{x_0} \rightarrow T_{x_0}M \) at \( x_0 \) is \( r \geq 0 \). For the time being, we assume that \( r \geq 1 \). As mentioned before, there exist a local coordinate \((W; x_1, x_2, \cdots, x_r, y_1, \cdots, y_s) \) \((r + s = \dim M = n) \) around \( x_0 \) and a basis of local sections

\[ \left( \frac{\partial}{\partial x_1}, \lambda_1 \right), \cdots, \left( \frac{\partial}{\partial x_r}, \lambda_r \right), \left( Y_1, \mu_1 \right), \cdots, \left( Y_s, \mu_s \right). \]

over \( W \). We regard \( W \) and \( x_0 \) as \( \mathbb{R}^r \) and the origin \( o \) in \( \mathbb{R}^n \), respectively. The space \( \mathbb{R}^{n-1} \) can be thought of as the subspace \( \{0\} \times \mathbb{R}^{n-1} \) in \( \mathbb{R}^n \). Obviously, \( \mathbb{R}^{n-1} \) is transversal to \( \text{Im} \rho_z \) for any \( z \in \mathbb{R}^{n-1} \) in the sense of (3.4):

\[ \text{Im} \rho_z + \mathbb{R}^{n-1} = \mathbb{R}^n. \]

So, we can define the algebroid restriction \( D^{(1)} \rightarrow \mathbb{R}^{n-1} \) as

\[
D^{(1)}_z = \left\{ \sum_{j=2}^{r} u_j \left( \frac{\partial}{\partial x_j} \right)_z \lambda_j, \left( Y_k \right)_z, \left( \mu_k \right)_z \mid u_j, \alpha_k \in \mathbb{R} \right\} \quad (z \in \mathbb{R}^{n-1}).
\]

For the algebroid restriction, we refer to [10] [17].

**Lemma 4.6** Let \( \text{pr} : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1} \) be the natural projection. The pull-back Lie algebroid \( \text{pr}^*D^{(1)} \) is isomorphic to \( D \) as Lie algebroid.
**Proof.** For each point \( x \in \mathbb{R}^n \), the fiber \( (\text{pr}'D(1))_x \) is given by

\[
(\text{pr}'D(1))_x = \left\{ \sum_{j=2}^r u_j \left( \frac{\partial}{\partial x_j} \right)_x + \sum_{k=1}^s \alpha_k (Y_k)_x ; \right. \\
\left. \sum_{j=2}^r u_j \left( \frac{\partial}{\partial x_j} \right)_x , (\alpha_j)_x \right\} \mid u_j, \alpha_k \in \mathbb{R},
\]

where \( \bar{x} = \text{pr}(x) \in \mathbb{R}^{n-1} \). It is obvious that \( \text{pr}'D(1) \) is isomorphic to \( D \) as a vector bundle over \( \mathbb{R}^n \) by

\[
\left( \sum_{j=2}^r u_j \left( \frac{\partial}{\partial x_j} \right)_x + \sum_{k=1}^s \alpha_k (Y_k)_x ; \sum_{j=2}^r u_j \left( \frac{\partial}{\partial x_j} \right)_x , (\alpha_j)_x \right) \\
\sum_{j=1}^r \sum_{k=1}^s \alpha_k ((Y_k)_x , (\mu_k)_x).
\]

Denoting by \( \# \) the anchor map of \( \text{pr}'D(1) \), one can easily check \( \# = \rho \). Let us denote by \( [\cdot, \cdot] \) the Lie bracket of \( \text{pr}'D(1) \). For any smooth section

\[
\varphi_1 = \left( \sum_{i=1}^r f_i \frac{\partial}{\partial x_i} + \sum_{k=1}^s A_k Y_k ; \sum_{i=1}^r f_i \frac{\partial}{\partial x_i} \right) \circ \pi + \sum_{j=1}^s A_j (Y_j + \mu_j) \circ \pi \\
\varphi_2 = \left( \sum_{k=1}^s g_k \frac{\partial}{\partial x_k} + \sum_{\ell=1}^r B_\ell Y_\ell ; \sum_{k=1}^s g_k \frac{\partial}{\partial x_k} , \mu_k \right) \circ \pi + \sum_{\ell=1}^r B_\ell (Y_\ell + \mu_\ell) \circ \pi
\]

of \( \text{pr}'D(1) \), where \( f_i, g_k, A_j \) and \( B_\ell \) are smooth functions on \( \mathbb{R}^n \), their Lie bracket \( [\varphi_1, \varphi_2]' \) is calculated to be

\[
[\varphi_1, \varphi_2]' = \left( \sum_{i=1}^r f_i \frac{\partial}{\partial x_i} + \sum_{k=1}^s A_k Y_k ; \sum_{i=1}^r f_i \frac{\partial}{\partial x_i} \right) \circ \pi + \sum_{j=1}^s A_j (Y_j + \mu_j) \circ \pi \\
+ \sum_{k=2}^r \left( \sum_{i=1}^r f_i \frac{\partial g_k}{\partial x_i} + \sum_{j} A_j (Y_j g_k) \left( \frac{\partial}{\partial x_k} , A_k \right) \circ \pi + \sum_{\ell=1}^r f_{j\ell} B_{j\ell} \circ \pi \right) \\
- \sum_{k=1}^s \left( \sum_{\ell=1}^r g_k \frac{\partial f_\ell}{\partial x_k} + \sum_{\ell=1}^r B_\ell (Y_\ell f_\ell) \left( \frac{\partial}{\partial x_k} , A_k \right) \circ \pi + \sum_{\ell=1}^r f_{j\ell} B_{j\ell} \circ \pi \right)
\]

(see p157 in [17] for the Lie bracket of the pull-back of Lie algebroid). Consequently, we have

\[
\Phi \circ [\varphi_1, \varphi_2]' = \sum_{j=1}^r A_j B_\ell (Y_j + \mu_j) + \sum_{k=1}^r (\mathcal{L}_{\varphi_1} g_k) \left( \frac{\partial}{\partial x_k} , \lambda_k \right) \\
+ \sum_{i=1}^r (\mathcal{L}_{\varphi_1} B_i) (Y_i + \mu_i) - \sum_{i=1}^r (\mathcal{L}_{\varphi_2} f_i) \left( \frac{\partial}{\partial x_i} , \lambda_i \right) - \sum_{j=1}^r (\mathcal{L}_{\varphi_2} A_j) (Y_j + \mu_j).
\]

These observations completes the proof. \( \square \)

If \( r = 0 \) at the origin \( o \) in \( \mathbb{R}^n \), there locally exists a Poisson vector field \( \Pi \) such that \( \text{rank} \Pi_o^\# = 0 \) and \( D \) is
represented as the graph of $\Pi^j$ near $o$ (see Appendix A.8 in [10]). So, at each $x \in \mathbb{R}^n$, $D_z$ is isomorphic to $T^*_x \mathbb{R}^n \cong T_x \mathbb{R}^n$ by $\rho^*$.

The next result follows immediately Proposition 4.5, similarly to the case of de Rham cohomology.

**Corollary 4.7 (Poincaré’s lemma for Dirac manifolds)** If $W$ is a star-shaped open set in $\mathbb{R}^n$ and $D$ is a Dirac structure over $W$, then $H^2_{\ell}(W,D;\mathbb{R}) = 0$ for $\ell > 0$ and $H^2_0(W,D;\mathbb{R}) = \mathbb{R}$.

**Proof of Theorem 4.4** We assume that

$$[\Lambda] \in H^2_\ell(M,D;\mathbb{Z}).$$

Let $\{W_j\}_j$ be a contractible open covering of $M$. From Corollary 4.7, there exist $D$-differential 1-forms $\sigma_j \in \Gamma^{\infty}(W_j,D^\ast)$ such that

$$\Lambda = d_D \sigma_j$$

on each $W_j$. If $W_j \cap W_k \neq \emptyset$, one immediately finds that $d_D(\sigma_j - \sigma_k) = 0$. So, there exist functions $w_{jk} \in C^{\infty}(W_j \cap W_k)$ such that

$$d_D w_{jk} = \sigma_j - \sigma_k.$$  \hspace{1cm} (4.16)

by using Corollary 4.7 again. We here remark that $W_j \cap W_k$, also, is contractible whenever $W_j$ and $W_k$ are so. From this, it follows that

$$d_D(w_{jk} + w_{kl} - w_{jl}) = 0$$

on $W_j \cap W_k \cap W_l (\neq \emptyset)$. That is, $f_{jk} := w_{jk} + w_{kl} - w_{jl}$ are constant functions which take the values in $\mathbb{Z}$. Let us consider the function $c_{jk} := \exp(-2\pi i w_{jk})$ on $W_j \cap W_k$. Noting (4.16), we have that

$$\sigma_j - \sigma_k = \frac{i}{2\pi} d_D c_{jk}.$$  \hspace{1cm} (4.17)

In addition, those functions $\{c_{jk}\}_{jk}$ satisfy the cocycle condition:

$$c_{jk} c_{kl} = \exp(-2\pi i (w_{jk} + w_{kl})) = \exp(-2\pi i f_{jk}) \exp(-2\pi i w_{jl}) = c_{jl}$$

on $W_j \cap W_k \cap W_l (\neq \emptyset)$. Therefore, one can obtain a line bundle $L \to M$ whose transition functions are $\{c_{jk}\}_{jk}$ and on which $\{\sigma_j\}_j$ determine a connection $\nabla^D$ with curvature $\Lambda$.

We define a Hermitian metric $h$ on $L$ as

$$h_p(s_1,s_2) := \overline{s}_1 z_2,$$

for any section $s_1(p) = (p,z_1)$, $s_2(p) = (p,z_2) \in W_j \times \mathbb{C}$ on each open set $W_j$ of the trivialization. Then, $\nabla^D$ turns out to be a Hermitian connection in the sense of (4.9). Indeed, letting $s_1$, $s_2$ be smooth sections locally written in the form $s_1(p) = g_1(p) e(p)$, $s_2(p) = g_2(p) e(p)$ ($g_1(p),g_2(p) \in \mathbb{C}$), where $e$ is the nowhere vanishing section, and $f \in C^{\infty}_{adm}(M,D)$, we have

$$\left(H_f(h(s_1,s_2))\right)(p) = \left(H_f(h(g_1 e,g_2 e))\right)(p) = \left(H_f(\overline{g}_1 g_2)\right)(p) = (H_f(\overline{g}_1))(p) g_2(p) + \overline{g}_1(p) (H_f g_2)(p).$$

On the other hand,

$$h(\nabla^D_{(H_f df)} s_1, s_2)(p) + h(s_1, \nabla^D_{(H_f df)} s_2)(p)$$
From the assumption, each connection 1-section \( \sigma_j \) is real. Accordingly,
\[
\begin{aligned}
&= h \left( \nabla^D_{(H_f g_1)} s_1 + 2 \pi i \left( g_1 \sigma_f(H_f) e, g_2 e \right) \right) (p) + h \left( g_1 e, (H_f g_2) e_2 + 2 \pi i g_2 \sigma_f(H_f) e \right) (p) \\
&= (H_f g_1)_p g_2 (p) - 2 \pi i g_1 (p) \sigma_f(H_f) e(p) + g_1 (p) (H_f g_2)_p + 2 \pi i g_1 (p) g_2 (p) \sigma_f(H_f).
\end{aligned}
\]

Therefore, we have that
\[
H_f (h(s_1, s_2)) = h (\nabla^D_{(H_f, d_f)} s_1, s_2) + h (s_1, \nabla^D_{(H_f, d_f)} s_2).
\]

This results in that \((M, D)\) is prequantizable.

Conversely, suppose that \((M, D)\) is prequantizable, that is, there is the prequantization bundle \((L, \nabla^D)\) over \(M\). Note that the \(s\)-adapted 2-cocycle \(\omega\) is prequantizable, that is, there is the prequantization bundle \((L, \nabla^D)\) over \(M\). The curvature form \(F_{\omega_0}\) corresponding to \(R_0\) satisfies
\[
\omega = (\lambda^2 \rho)^* R_0.
\]

The map \(R_1 : \Gamma^\omega(M, D) \times \Gamma^\omega(M, D) \to \text{End}_C (\Gamma^\omega(M, L))\) defined as
\[
R_1 := R_0 \circ (\rho \times \rho) = (\lambda^2 \rho)^* R_0
\]
is the curvature of a \(D\)-connection \(\nabla_1 := \nabla_0 \circ (\rho \times id)\) on \(L\). Then, the \(D\)-differential 2-form \(\tau_1\) corresponding to \(R_1\) is represented as \(\tau_1 = (\lambda^2 \rho)^* F_{\omega_0}\) by using \(F_{\omega_0}\). Using Proposition \(3.6\) we find that
\[
\lambda = c^D_1 (L) = [\tau_1] = [F_{\omega_0} \circ (\rho \times \rho)] \in H^2_\Lambda (M, D ; \mathbb{Z}).
\]

This completes the proof of Theorem \(4.4\). \(\square\)

We end the section with some examples.

**Example 4.4 (Symplectic manifolds)** We let \((M, \omega)\) be a symplectic manifold and consider the Dirac structure \(D_\omega := \text{graph} (\omega^\flat) \subset TM \oplus T^*M\) induced from the symplectic form \(\omega\) (see also Example \(2.7\)). It is verified that the skew-symmetric 2-cocycle \(\Omega\) by \(\Omega^\flat\) is entirely \(\omega\). Therefore, it follows from the non-degeneracy of \(\omega\) that
\[
\mathcal{V}_m = \ker \Omega^\flat_m = \ker \omega_m = \{0\} \quad (\forall m \in M).
\]

Accordingly, we can take \(D_\omega\) as a subbundle \(\mathcal{H}\) satisfying \(\Omega^\flat\). Then, for any smooth function \(f\) on \(M\), there exists a unique vector field \(H_f\) such that \(df = \omega^\flat (H_f)\). Therefore, \(\Omega\)-compatible Poisson structure coincides with the natural Poisson structure induced from \(\omega\). In this case, the skew-symmetric pairing \(\lambda\) is written as
\[
\lambda ((X, \omega^\flat(X)), (Y, \omega^\flat(Y))) = \omega (X, Y).
\]

The integrability condition \(4.11\) is given by \([\omega] \in H^2_\Lambda (M ; \mathbb{Z}).\)
Example 4.5 (Presymplectic manifolds) As discussed in Example 2.7, given a presymplectic manifold \((M,\omega)\), one obtains a Dirac manifold \((M, D_\omega)\). Similarly to Example 2.4 \(\Omega\) is entirely \(\omega\). The subbundle \(\mathcal{V} = \ker \omega^k\) is given by \(\mathcal{V} = \{(X,0) | \omega^k(X) = 0\} \subset D_\omega\). We let \(\mathcal{H} \subset D_\omega\) be a subbundle which satisfies (4.11) and fix it. For any admissible function \(f\), it holds \(df \in \mathcal{V}^c\). It follows from this that every admissible function is \(d^2\)-closed (see [25] for the definition of “\(d^2\)-closed”). So, their \(\omega\)-compatible Poisson bracket \([f,g]\) for \(f, g \in C^\infty_{\text{adm}}(M, D_\omega)\) can be defined as

\[
[f, g] = \text{sg} \, g(f),
\]

where \(\text{sg} \, g \in \mathfrak{x}(M)\) stands for the horizontal Hamiltonian field [25]. Namely, a Poisson algebra \((C^\infty_{\text{adm}}(M, D_\omega), [\cdot, \cdot])\) consists of \(d^2\)-closed functions on \(M\). The integrability condition (4.11) is given by \([\omega] \in H^2_{\text{dh}}(M; \mathbb{Z})\).

Example 4.6 (Poisson manifolds) Let us consider the case of a Poisson manifold \((P, \sigma)\). As seen in Example 2.2 \((P, \sigma)\) defines a Dirac manifold \((P, D_\sigma)\), where \(D_\sigma = \text{graph}(\sigma^2) \subset TM \oplus T^*M\). One easily finds that \(\rho(D_\sigma) = \text{im} \sigma^2\) and that \(\mathcal{V}\) is given by \(\mathcal{V} = D_\sigma \cap TM = \{0\}\). This permits us to take a subbundle \(D_\sigma = \{(\sigma^2(\alpha), \alpha) | \alpha \in T^*M\}\) as \(\mathcal{H}\). Obviously, every smooth function is admissible function. The skew-symmetric pairing \(\Lambda\) is written as

\[
\Lambda((\sigma^2(\alpha), \alpha), (\sigma^2(\beta), \beta)) = \sigma(\alpha, \beta).
\]

The integrability condition (4.11) is given by \([\sigma] \in H^2_{\text{Lich}}(M; \mathbb{Z})\), where \(H^2_{\text{Lich}}(M; \mathbb{Z})\) denotes Lichnerowicz-Poisson cohomology. Refer to [10] or [27] for the details of the cohomology.

5 Quantization

5.1 \(\alpha\)-density bundles

Let \(V\) be an \(n\)-dimensional vector space over \(\mathbb{C}\) and \(\alpha\) a positive number. A function \(\kappa: V \times \cdots \times V \to \mathbb{C}\) on \(n\)-copies of \(V\) is called a \(\alpha\)-density of \(V\) if it satisfies

\[
\kappa(A v_1, \cdots, A v_n) = |\det A|^\alpha \kappa(v_1, \cdots, v_n) \quad (v_1, \cdots, v_n \in V)
\]

for any invertible linear transformation \(A: V \to V\). Denoting the set of all densities of \(V\) by \(H^{(\alpha)}(V)\), we can check easily that \(H^{(\alpha)}(V)\) is a vector space over \(\mathbb{C}\). Since \(A \in \text{GL}(V)\) acts transitively on bases in \(V\), an \(\alpha\)-density is determined by its value on a single basis. For an alternating covariant \(n\)-tensor \(\omega\), the map \(|\omega|^{(\alpha)}: V \times \cdots \times V \to \mathbb{C}\) defined as

\[
|\omega|^{(\alpha)}(v_1, \cdots, v_n) := |\omega(v_1, \cdots, v_n)|^\alpha \quad (v_1, \cdots, v_n \in V)
\]

is an \(\alpha\)-density over \(V\). If \(\omega\) is nonzero, \(H^{(\alpha)}(V)\) is a 1-dimensional vector space spanned by \(|\omega|^{(\alpha)}\). So, any element \(\kappa \in H^0(V)\) is represented as \(\kappa = c|\omega|^{(\alpha)}\) for some \(c \in \mathbb{C}\).

Let \(M\) be a smooth manifold and \(\alpha\) a positive number. The vector bundle

\[
H^\alpha := \bigoplus_{m \in M} H^{(\alpha)}(T_m M)
\]

over \(M\) is called the \(\alpha\)-density bundle of \(M\). Especially, \(H^{1/2}\) is called the half-density bundle. Let \((U_\alpha; (x^1_\alpha, \cdots, x^\alpha_\alpha))\) be local coordinate chart on \(M\) and \(\omega_\alpha = dx^1_\alpha \wedge \cdots \wedge dx^\alpha_\alpha\). Then, a local trivialization on \(U_\alpha\) is defined to be the map

\[
\Phi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}, \quad \Phi_\alpha(z|\omega_\alpha|_{\rho}) = (p, z).
\]
Letting \((U_{\mu}; (x_1^\mu, \cdots, x_n^\mu))\) be another chart with \(U_\lambda \cap U_\mu \neq \emptyset\) and \(\omega_\mu = dx_1^\mu \wedge \cdots \wedge dx_n^\mu\), we have that
\[
\Phi_\lambda \circ \Phi_\mu^{-1}(p, z) = \Phi_\lambda(z|\omega_\mu)|_p = \Phi_\lambda\left(z\left| \det \left(\frac{\partial x^\lambda_\mu}{\partial x^\nu_j}\right)|_p \right. \right) = \left(p, z\left| \det \left(\frac{\partial x^\lambda_\mu}{\partial x^\nu_j}\right)|_p \right. \right).
\]
That is, \(H^\alpha\) is a complex line bundle whose transition functions are the square roots of the absolute values of the determinants of the matrices
\[
\begin{pmatrix}
\frac{\partial x^\lambda_\mu}{\partial x^\nu_j}(p)
\end{pmatrix}_{1 \leq j, k \leq n} \quad (n = \dim M)
\]
by the coordinate transformations \(x^j_\lambda = x^i_\lambda(x^\mu_1, \cdots, x^\mu_n)\) \((j = 1, \cdots, n)\). A section of \(H^\alpha\) is called an \(\alpha\)-density over \(M\). When \(\alpha = 1/2\), a section of \(H^{1/2}\) is called the half-density. As in the linear case, any \(\alpha\)-density \(\kappa\) on \(U\) can be written in the form \(\kappa = f|\omega|^{(\alpha)}\) for some complex-valued function \(f\). It is easily verified that \(H^\alpha \otimes H^\beta \cong H^{\alpha+\beta}\). Accordingly, for the half densities \(\kappa_1, \kappa_2\) on \(M\), we get a 1-density \(\kappa_1 \otimes \kappa_2\).

Suppose that \((U, \phi)\) is a local coordinate chart on \(M\) and \(\kappa\) is the half density on \(M\) such that the support \(\text{supp} \, \kappa\) of \(\kappa\) is contained in \(U\). The integral of \(\kappa\) over \(M\) is defined as
\[
\int_M \kappa := \int_{\phi^{-1}(U)} (\phi^{-1})^* \kappa.
\]
We here remark that the right-hand side is represented as
\[
\int_{\phi^{-1}(U)} (\phi^{-1})^* \kappa = \int_{\phi^{-1}(U)} f \, |dx_1 \wedge \cdots \wedge dx_n|^{(1/2)}.
\]
If \(\kappa\) is any density on \(M\), the integral of \(\kappa\) over \(M\) is defined as
\[
\int_M \kappa := \sum_j \int_M \phi_j \kappa,
\]
where \(\{\phi_j\}_j\) means a partition of unity subordinate to smooth atlas of \(M\).

### 5.2 Polarization

Let \((M, D)\) be a Dirac manifold and \(\Omega\) the presymplectic form associated with it. The skew-symmetric pairing \(\Lambda\) and the bracket \(\ll\cdot, \cdot\rr\) are naturally extended to operations on the space \(\Gamma^\infty(M, D^\mathbb{C})\) of smooth sections of the complexification \(D^\mathbb{C} := D \otimes_{\mathbb{R}} \mathbb{C}\) of \(D\) by
\[
\Lambda((X + i X', \xi + i \xi'), (Y + i Y', \eta + i \eta')) = \frac{1}{2} \left\{ \xi(Y) - \eta(X) - \xi'(Y') + \eta'(X') + i(\xi(Y') - \eta(X') + \xi'(Y) - \eta'(X)) \right\}
\]
and
\[
\ll(X + i X', \xi + i \xi'), (Y + i Y', \eta + i \eta') \rr.
\]
Given a polarization defines a polarization. for any \(\nabla\) spect to \(q\) prequantization bundle with the Hermitian metric \(\eta\) ing conditions:

\[
\mathcal{L}\eta - i\gamma d\xi - \mathcal{L}\eta' + i\gamma d\xi' + i(\mathcal{L}\eta' - i\gamma d\xi' + \mathcal{L}\eta - i\gamma d\xi),
\]

respectively. As discussed in the previous section, for the subbundle \(D \cap TM \subset D\), we choose a subbundle \(\mathcal{H} \subset D\) which satisfies (4.1) and fix it. We introduce the notion of polarization for Dirac manifold as follows:

**Definition 5.1** A complex subbundle \(\mathcal{P} \subset \mathcal{H}^\mathbb{C}\) is called a (complex) polarization if it satisfies the following conditions:

1. \(\Lambda((\tilde{X}, \hat{\xi}), (\tilde{Y}, \hat{\eta})) = 0\) \((\mathcal{H}(\tilde{X}, \hat{\xi}), (\tilde{Y}, \hat{\eta}) \in \Gamma^\infty(M, \mathcal{P}))\);

2. \(\|\Gamma^\infty(M, \mathcal{P}), \Gamma^\infty(M, \mathcal{P})\| \subset \Gamma^\infty(M, \mathcal{P})\);

The condition (1) can be written in the explicit form

\[
\Omega(X, Y) - \Omega(X', Y') + i(\Omega(X', Y) + \Omega(X, Y')) = 0,
\]

where \((\tilde{X}, \hat{\xi}) = (X + iX', \Omega^1(X) + i\Omega^2(X'))\) and \((\tilde{Y}, \hat{\eta}) = (Y + iY', \Omega^2(Y) + i\Omega^2(Y'))\).

**Example 5.1** Let us consider a Dirac manifold \((\mathbb{R}^{2n}, \text{graph}(\omega^\flat))\) induced from a symplectic manifold \(\mathbb{R}^{2n}\) with the standard symplectic form \(\omega = \sum_j dq_j \wedge dp_j\). Note that the subbundle \(\mathcal{H}\) is the Dirac structure \(\text{graph}(\omega^\flat)\). Then, a subbundle \(\mathcal{P}\) spanned by smooth sections

\[
\left\{ \left( \frac{\partial}{\partial q_j}, dp_j \right) \mid j = 1, \ldots, n \right\}
\]

defines a polarization.

Given a polarization \(\mathcal{P}\), we define the subalgebra \(S(\mathcal{P})\) of \((C_{\text{adm}}^\infty(M, D), \{\cdot, \cdot\})\) as

\[
S(\mathcal{P}) := \{ f \in C_{\text{adm}}^\infty(M, D) \mid \forall \psi \in \Gamma^\infty(M, \mathcal{P}) : [(H_f, d\psi), \psi] \in \Gamma^\infty(M, \mathcal{P}) \}.
\]

Suppose that \((M, D)\) is prequantizable and endowed with a polarization \(\mathcal{P}\). Let \(L \to M\) be its prequantization bundle with the Hermitian metric \(h\) on \(L\) and \(\nabla^D\) a Hermitian \(D\)-connection with respect to \(h\). We remark that \(\nabla^D : \Gamma^\infty(M, D) \to \text{End}_\mathbb{C}(\Gamma^\infty(M, L))\) has the natural extension to a map \(\nabla^D : \Gamma^\infty(M, D^\mathbb{C}) \to \text{End}_\mathbb{C}(\Gamma^\infty(M, L))\). Using the extension \(\nabla^D\), we define a map

\[
\delta : \Gamma^\infty(M, D^\mathbb{C}) \times \Gamma^\infty(M, L \otimes H^{1/2}) \to \Gamma^\infty(M, L \otimes H^{1/2})
\]

as

\[
\delta_\psi (s \otimes \kappa) := (\nabla^D \psi) s \otimes \kappa + s \otimes (\mathcal{L}_{\psi} \kappa),
\]

for any \(\psi \in \Gamma^\infty(M, D^\mathbb{C}), s \otimes \kappa \in \Gamma^\infty(M, L \otimes H^{1/2})\). It is easily verified that \(\delta\) is a \(D\)-connection on \(L \otimes H^{1/2}\).

Then, the representation (4.6) of \(C_{\text{adm}}^\infty(M, D)\) can be extended to a map

\[
\hat{\gamma} : C_{\text{adm}}^\infty(M, D) \to \text{End}_\mathbb{C}(\Gamma^\infty(M, L \otimes H^{1/2}))
\]

by setting

\[
\hat{f}(s \otimes \kappa) = -\delta_{i(H_f, df)}(s \otimes \kappa) - 2\pi if(s \otimes \kappa).
\]
The \( \hat{f} \) is also represented as
\[
\hat{f}(s \otimes \kappa) = (\hat{f}s) \otimes \kappa - s \otimes (\mathcal{L}_H \kappa).
\] (5.1)

Since \((M, D)\) is prequantizable, we can check that
\[
[\hat{f}, \hat{g}] (s \otimes \kappa) = [\hat{f}, \hat{g}] (s \otimes \kappa)
\]
for all \(f, g \in C^\infty_{\text{adm}}(M, D)\) in the same manner as the proof for Proposition 4.2.

**Lemma 5.2** It holds that
\[
\delta_\psi (\hat{f} (s \otimes \kappa)) = \hat{f} (\delta_\psi (s \otimes \kappa)) - \delta_{[\psi, (H_f \circ df)]} (s \otimes \kappa)
\]
for any \(\psi \in \Gamma^\infty(M, D^C), s \otimes \kappa \in \Gamma^\infty(M, L \otimes H^{1/2})\) and \(f \in C^\infty_{\text{adm}}(M, D)\).

**Proof.** For any smooth section \(\psi = (X, \xi)\) of \(D^C, s \otimes \kappa\) of \(L \otimes H^{1/2}\) and any admissible function \(f\), we have that
\[
\delta_\psi \circ \hat{f} (s \otimes \kappa) = -\delta_\psi \left( \delta_{(H_f \circ df)} (s \otimes \kappa) + 2\pi i f (s \otimes \kappa) \right)
= -\delta_\psi \left( \delta_{(H_f \circ df)} (s \otimes \kappa) + s \otimes \mathcal{L}_H \kappa \right) - 2\pi i \delta_\psi (f (s \otimes \kappa))
= -\left( \nabla^D_\psi \circ \nabla^D_{(H_f \circ df)} (s \otimes \kappa) - \delta_{(H_f \circ df)} (s \otimes \mathcal{L}_x \kappa) - 2\pi i f (s \otimes \mathcal{L}_x \kappa) \right)
= -\delta_{(H_f \circ df)} (s \otimes \mathcal{L}_x \kappa) - s \otimes \mathcal{L}_H \kappa - 2\pi i f (s \otimes \mathcal{L}_x \kappa).
\]

On the other hand,
\[
\hat{f} \circ \delta_\psi (s \otimes \kappa) = \hat{f} \left( (\nabla^D_\psi (s \otimes \kappa) + s \otimes \mathcal{L}_x \kappa \right)
= \hat{f} \left( (\nabla^D_\psi (s \otimes \kappa) + s \otimes \mathcal{L}_x \kappa \right)
= -\left( \nabla^D_{(H_f \circ df)} (s \otimes \kappa) - 2\pi i f ((\nabla^D_\psi (s \otimes \kappa) - \delta_{(H_f \circ df)} (s \otimes \mathcal{L}_x \kappa) - 2\pi i f (s \otimes \mathcal{L}_x \kappa) \right)
= -\left( \nabla^D_{(H_f \circ df)} (s \otimes \kappa) - \delta_{(H_f \circ df)} (s \otimes \mathcal{L}_x \kappa) - 2\pi i f (s \otimes \mathcal{L}_x \kappa) \right)
= -\left( \nabla^D_{(H_f \circ df)} (s \otimes \mathcal{L}_x \kappa) - s \otimes \mathcal{L}_H \kappa - 2\pi i f (s \otimes \mathcal{L}_x \kappa) \right)
\]

It follows from \(<(X, \xi), (H_f, df)>_+ = 0\) that
\[
\Lambda ((X, \xi), (H_f, df)) = \frac{1}{2} (\xi (H_f) - df (X)) = -Xf.
\]

By using (4.5) and (4.8) these equations yield
\[
(\delta_\psi \circ \hat{f} - \hat{f} \circ \delta_\psi) (s \otimes \kappa) = s \otimes (\mathcal{L}_x \circ \mathcal{L}_H - \mathcal{L}_H \circ \mathcal{L}_x) \kappa - 2\pi i f (X) s \otimes \kappa
= -\left( \left( \mathcal{L}_{(X, H_f)} \right) (s \otimes \kappa) - 2\pi i f (X) s \otimes \kappa \right)
= -\left( \left( \mathcal{L}_{(X, H_f)} \right) (s \otimes \kappa) - 2\pi i f (X) s \otimes \kappa \right)
= -\left( \left( \mathcal{L}_{(X, H_f)} \right) (s \otimes \kappa) - 2\pi i f (X) s \otimes \kappa \right)
= -\left( \left( \mathcal{L}_{(X, H_f)} \right) (s \otimes \kappa) - 2\pi i f (X) s \otimes \kappa \right)
\]
\[ = -\delta_{\psi,(H_f,df)}(s \otimes \kappa) \]

This completes the proof. \(\square\)

We define a subset \(\mathcal{S}_0\) of \(\Gamma^{\infty}(M, L \otimes H^{1/2})\) as

\[ \mathcal{S}_0 := \{ s \otimes \kappa \in \Gamma^{\infty}(M, L \otimes H^{1/2}) | \forall \psi \in \Gamma^{\infty}(M, \mathcal{P}) : \delta_\psi(s \otimes \kappa) = 0 \} \]

and assume that \(\mathcal{S}_0 \neq \{0\}\). Obviously, \(\mathcal{S}_0\) is a linear space over \(\mathbb{C}\). It follows from Lemma 5.2 that \(\hat{f}(\mathcal{S}_0) \subset \mathcal{S}_0\) for every \(f \in S(\mathcal{P})\). Consequently, we obtain the representation of \(S(\mathcal{P})\) by

\[ \hat{\cdot} : S(\mathcal{P}) \rightarrow \text{End}_\mathbb{C} (\mathcal{S}_0), \quad f \mapsto \{ s \otimes \kappa \mapsto -\delta_{(H_f,df)}(s \otimes \kappa) - 2\pi i f (s \otimes \kappa) \}. \]

We proceed with the discussion in the following two cases.

### 5.3 Compact Case

Suppose that \(M\) is compact. The linear space \(\mathcal{S}_0\) has the inner product \(\langle \cdot, \cdot \rangle\) defined as

\[ \langle s_1 \otimes \kappa_1, s_2 \otimes \kappa_2 \rangle := \int_M h(s_1, s_2) \overline{\kappa_1} \kappa_2 \]

for every \(s_1 \otimes \kappa_1, s_2 \otimes \kappa_2 \in \mathcal{S}_0\). By taking the completion \(\overline{\mathcal{S}_0}\) of \(\mathcal{S}_0\), one obtains a Hilbert space \(\mathcal{S}\). The operator \(i\hat{f}\) for \(f \in S(\mathcal{P})\) turns out to be self-adjoint with respect to \(\langle \cdot, \cdot \rangle\). Indeed, we have that

\[
\langle \hat{f}(s_1 \otimes \kappa_1), s_2 \otimes \kappa_2 \rangle + \langle s_1 \otimes \kappa_1, \hat{f}(s_2 \otimes \kappa_2) \rangle \\
= \langle \hat{f}(s_1 \otimes \kappa_1) - s_1 \otimes L_{H_f} \kappa_1, s_2 \otimes \kappa_2 \rangle + \langle s_1 \otimes \kappa_1, \hat{f}(s_2 \otimes \kappa_2) - s_2 \otimes L_{H_f} \kappa_2 \rangle \\
= \int_M \left( h(\hat{f}(s_1), s_2) + h(s_1, \hat{f}(s_2)) \right) \overline{\kappa_1} \kappa_2 - \int_M \left( h(s_1, s_2) (L_{H_f} \overline{\kappa_1}) \kappa_2 - \int_M h(s_1, s_2) \overline{\kappa_1} (L_{H_f} \kappa_2) \right) \\
= -\int_M [L_{H_f}(h(s_1, s_2)) \overline{\kappa_1} \kappa_2] - \int_M h(s_1, s_2) (L_{H_f} \overline{\kappa_1}) \kappa_2 - \int_M h(s_1, s_2) \overline{\kappa_1} (L_{H_f} \kappa_2) \\
= -\int_M L_{H_f} (h(s_1, s_2) \overline{\kappa_1} \kappa_2)
\]

for any \(s_1 \otimes \kappa_1, s_2 \otimes \kappa_2 \in \mathcal{S}\). In the last equality, the symbol \(L_{H_f}(\cdot)\) denotes a Lie derivative of a half-density. We refer to K. Yano [33] for the tensor analysis of Lie derivatives. According to I. Vaisman [24], it holds that

\[ \int_M L_V \kappa = 0 \]

for every vector field \(V\) and every density \(\kappa\) on \(M\). From this it follows that

\[ \langle \hat{f}(s_1 \otimes \kappa_1), s_2 \otimes \kappa_2 \rangle + \langle s_1 \otimes \kappa_1, \hat{f}(s_2 \otimes \kappa_2) \rangle = 0. \]

This directly implies that \(i\hat{f}\) is a self-adjoint operator. Then, the condition (4.7) holds up to the constant factor \(i\).
5.4 Non-compact case

Suppose that $M$ is not compact. We let $\mathcal{Q}$ be the subbundle of $D$ such that

$$Q^2 = \mathcal{P} \cap \overline{\mathcal{P}}$$

and assume that the leaf space $N = M/\mathcal{F}$ is a Hausdorff manifold, where $\mathcal{F}$ is the characteristic foliation which corresponds to the distribution $M \ni m \mapsto \rho_m(\mathcal{Q}_m) \subset T_m M$. We denote by $\pi$ the natural projection from $M$ to $N$. For any $f \in S(\mathcal{P})$ and $(X, \xi) \in \mathcal{Q}$, the vector field $\mathcal{H}(H_f, d_f)$ tangent to each $\pi$-fiber: $\mathcal{H}(H_f, d_f), (X, \xi) = ([H_f, X], \mathcal{L}_H \xi) \in \Gamma^\infty(M, \mathcal{Q})$. Accordingly, it holds that $[H_f, X] \in \rho(\mathcal{Q})$ for any vector field $X \in \rho(\mathcal{Q})$. This means that $H_f$ is a lift of a smooth vector field on $N$. Let $H_N^{1/2}$ be a half density bundle over $N$ and $\kappa_N$ a half density. Then, from (5.1) it holds that

$$\hat{f}(s \otimes \pi^* \kappa_N) = (\hat{f} s) \otimes \pi^* \kappa_N - s \otimes (\mathcal{L}_H \pi^* \kappa_N) = (\hat{f} s) \otimes \pi^* \kappa_N - s \otimes \pi^* (\mathcal{L}_H \pi^* \kappa_N)$$

for any $s \otimes \pi^* \kappa_N \in \Gamma^\infty(M, L \otimes \pi^* H_N^{1/2})$. This enables us to consider the half densities of $M$ which are transversal to $\mathcal{F}$. Since $\mathcal{L}_H(\pi^* \kappa_N) = \pi^* (\mathcal{L}_H \pi^* \kappa_N) = 0$ for $(X, \xi) \in \mathcal{Q}$, we have that

$$\mathcal{L}_H((\pi^* \kappa_N^1) \otimes (\pi^* \kappa_N^2)) = \pi^* (\mathcal{L}_H \pi^* \kappa_N^1) \otimes \pi^* \kappa_N^2 + \pi^* \kappa_N^1 \otimes \pi^* (\mathcal{L}_H \pi^* \kappa_N^2) = 0$$

for every $\pi^* \kappa_N^1, \pi^* \kappa_N^2 \in \pi^* H_N^{1/2}$. In other words, the tensor field $(\pi^* \kappa_N^1) \otimes (\pi^* \kappa_N^2)$ on $M$ is invariant under the flow of $X \in \rho(\mathcal{Q})$. Therefore, there exists a 1-density $\nu_N$ of $N$ onto which $(\pi^* \kappa_N^1) \otimes (\pi^* \kappa_N^2)$ projects. As a result, we let $\mathcal{S}_1$ be the linear subspace of $\mathcal{S}_0$ consisting of the elements in $\Gamma^\infty(M, L \otimes \pi^* H_N^{1/2})$ which have compact support in $N$, and assume that $\mathcal{S}_1 \neq \{0\}$. We define the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{S}_1$ as

$$\langle s_1 \otimes \pi^* \kappa_N^1, s_2 \otimes \pi^* \kappa_N^2 \rangle := \int_N h(s_1, s_2) \nu_N$$

for every $s_1 \otimes \pi^* \kappa_N^1, s_2 \otimes \pi^* \kappa_N^2 \in \mathcal{S}_1$. Replacing $\mathcal{S}_0$ with $\mathcal{S}_1$, we obtain a Hilbert space from $\mathcal{S}_1$ and find that $i\hat{f}$ for $f \in S(\mathcal{P})$ is a self-adjoint operator on $\text{End}_\mathbb{C}(\mathcal{S}_1)$ in a way similar to the compact case.

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