SPECTRAL RADIUS OF A STAR WITH ONE LONG ARM

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Abstract. A tree is said to be starlike if exactly one vertex has degree greater than two. In this paper, we will study the spectral properties of $S(n, k \cdot 1)$, that is, the starlike tree with $k$ branches of length 1 and one branch of length $n$. The largest eigenvalue $\lambda_1$ of $S(n, k \cdot 1)$ satisfies $\sqrt{k+1} \leq \lambda_1 < k/\sqrt{k-1}$. Moreover, the largest eigenvalue of $S(n, k \cdot 1)$ is equal to the largest eigenvalue of $S(k \cdot (n+1))$, which is the starlike tree that has $k$ branches of length $n-1$. Using the spectral radii of $S(n, k \cdot 1)$ we can show that there is a sequence of Salem numbers that converges to each integer $>1$.

1. Introduction

A tree which has exactly one vertex of degree greater than two is said to be starlike. Spectral properties of starlike trees are recently studied in [LG01, LG02, BS98].

Let $P_n$ be the path with $n$ vertices. We denote $S(n_1, n_2, \cdots, n_k)$ a starlike tree in which removing the central vertex $v_1$ leaves disjoint paths such that $S(n_1, n_2, \cdots, n_k) - v_1 = P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_k}$.

We say that the starlike tree $S(n_1, n_2, \cdots, n_k)$ has $k$ branches and the lengths of branches are $n_1, n_2, \cdots, n_k$. It will be assumed that $n_1 \geq n_2 \geq \cdots \geq n_k$.

For a simple graph $G$ of order $n$, the spectrum of $G$ is the set of eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its adjacency matrix $A$. The characteristic polynomial $\det(\lambda I - A)$ of $A$ is called the characteristic polynomial of $G$, denoted $\phi(G, \lambda)$ or simply $\phi(G)$. It is known that if $G$ is a graph and $v$ is any vertex, then

$$\phi(G) = \lambda \phi(G - v) - \sum_u \phi(G - v - u) - 2 \sum_C \phi(G - C),$$

where the first summation is over vertices $u$ adjacent to the vertex $v$ and the second summation is over all cycles $C$ embracing the vertex $v$. Applying to the starlike trees we obtain

$$\phi(S(n_1, n_2, \cdots, n_k)) = \lambda \prod_{i=1}^k \phi(P_{n_i}) - \sum_{i=1}^k \phi(P_{n_i-1}) \prod_{j \in I_i} \phi(P_{n_j})$$

(1)

where $I_i = \{1, 2, \cdots, k\} \setminus \{i\}$.

Using Equation (1), Lepović and Gutman [LG01] determine the bounds for the largest eigenvalues of starlike trees.

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Theorem 1.1. [LG01, Theorem 2] If $\lambda_1$ is the largest eigenvalue of the starlike tree $S(n_1, n_2, \cdots, n_k)$, then
\[ \sqrt{k} \leq \lambda_1 < \frac{k}{\sqrt{k-1}} \]
for any positive integers $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$.

The lower bound $\sqrt{k}$ for $\lambda_1$ is realized by the star on $k + 1$ vertices. The upper bound can be achieved asymptotically by the starlike trees with $n_1 = n_2 = \cdots = n_k = n$. In such case, we will denote this starlike tree by $S(k \cdot n)$ instead of $S(n_1, n_2, \cdots, n_k)$.

In this paper, we will discuss the spectral properties of a star with one long arm; let $S(n, k \cdot 1)$ be the starlike tree with $k$ branches of length 1 and one branch with length $n$. Note that $S(n, k \cdot 1)$ is a tree on $n + k + 1$ vertices.

Two nonisomorphic graphs with the same spectrum are called cospectral. It is known that no two starlike trees are cospectral [LG02]. However, the spectral radius does not distinguish starlike trees. We will show that there are infinitely many pairs of nonisomorphic starlike trees that have the same spectral radius.

Theorem A. For any positive integer $k \geq 3$, starlike trees $S(n, k \cdot 1)$ and $S(k \cdot (n + 1))$ have the same largest eigenvalue.

By Theorem 1.1 the largest eigenvalue $\lambda_1$ of $S(n, k \cdot 1)$ satisfies
\[ \sqrt{k + 1} \leq \lambda_1 < \frac{k + 1}{\sqrt{k}}. \]
As a consequence of Theorem A, we have a sharper upper bound for the starlike tree $S(n, k \cdot 1)$.

Theorem B. If $\lambda_1$ is the largest eigenvalue of $S(n, k \cdot 1)$, then
\[ \sqrt{k + 1} \leq \lambda_1 < \frac{k}{\sqrt{k - 1}} \]
for any positive integers $n \geq 1$ and $k \geq 3$.

There are two special algebraic integers related to the largest eigenvalue of starlike trees. A Salem number is an algebraic integer $\alpha > 1$, all of whose other conjugates have modulus $\leq 1$, with at least one conjugate of modulus 1. A Pisot number is an algebraic integer $\beta > 1$, all of whose other conjugates have modulus $< 1$. With Theorem B and the work of McKee–Rowlinson–Smyth [MRS99], we have the following corollary.

Corollary 3.2. For $n \geq 2$ and $k \geq 3$ let $\lambda_1$ be the largest eigenvalue of the starlike tree $S(n, k \cdot 1)$. Then the number $t > 1$ defined by
\[ \sqrt{t} + \frac{1}{\sqrt{t}} = \lambda_1, \]
(2)
is a Salem number.

Using the spectral properties of the starlike tree $S(n, k \cdot 1)$, the author studied the stretch factors of pseudo-Anosov mapping classes of closed orientable surfaces. In particular, the number $t$ defined by (2) is the stretch factor of a pseudo-Anosov mapping class from Thurston’s construction whose configuration graph is $S(n, k \cdot 1)$. For more about this topic, see [Shi16].

2. Bounds for the largest eigenvalue

In this section we will prove main theorems of this paper. Lepović and Gutman [LG01] show that the number $t > 1$, defined by $t = \frac{\sqrt{t} + 1}{\sqrt{t}} = \lambda_1$, where $\lambda_1$ is the largest eigenvalue of $S(k \cdot (n+1))$, is the root of the polynomial equation

$$t^n + 3 - (k - 1)t^{n+2} + (k - 1)t - 1 = 0. \tag{3}$$

To prove Theorem A we will show that when $\lambda_1$ is the largest eigenvalue of $S(n, k \cdot 1)$, the number $t$ given by $t = \frac{\sqrt{t} + 1}{\sqrt{t}} = \lambda_1$, is again the root of the polynomial (3).

Proof of Theorem A. Equation (1) reduces to

$$\phi(S(n, k \cdot 1)) = \lambda \phi(P_n) \phi(P_1)^k - \left( k \phi(P_n) \phi(P_1)^{k-1} + \phi(P_{n-1}) \phi(P_1)^k \right)$$

$$= \lambda^{k+1} \phi(P_n) - k \lambda^{k-1} \phi(P_n) - \lambda^k \phi(P_{n-1})$$

$$= \lambda^{k-1} \left( \lambda^2 \phi(P_n) - k \phi(P_n) - \lambda \phi(P_{n-1}) \right).$$

Therefore the largest eigenvalue of $S(n, k \cdot 1)$ is the root of

$$\lambda^2 \phi(P_n) - k \phi(P_n) - \lambda \phi(P_{n-1}) = 0. \tag{4}$$

By substituting $\lambda = 2 \cos \theta$, we get $\phi(P_n) = \sin(n + 1) \theta / \sin \theta$ (see [CDS95, p.73]) and Equation (4) becomes

$$\left( 4 \cos^2 \theta - k \right) \frac{\sin(n + 1) \theta}{\sin \theta} - 2 \cos \theta \frac{\sin n \theta}{\sin \theta} = 0. \tag{5}$$

By setting $t^{1/2} = e^{i \theta}$, we have

$$\lambda = 2 \cos \theta = t^{1/2} + t^{-1/2}$$

and

$$\sin n \theta = \frac{t^{n/2} - t^{-n/2}}{2i}.$$ 

By substituting and simplifying, Equation (5) becomes

$$t^{n+3} - (k - 1)t^{n+2} + (k - 1)t - 1 = 0. \tag{6}$$

If $t^*$ is a root of Equation (6), then the number $\lambda^*$, defined by $\lambda^* = \sqrt{t^*} + 1 / \sqrt{t^*}$, is a root of Equation (4). Since Equation (6) is identical with Equation (3) we can conclude that the largest eigenvalue of $S(n, k \cdot 1)$ is equal to the largest eigenvalue of $S(k \cdot (n+1))$. \[\square\]
Theorem B follows directly from Theorem A and the work of Lepović and Gutman.

Proof of Theorem B. A star with \( k + 2 \) vertices is a subgraph of \( S(n, k \cdot 1) \) and its largest eigenvalue is \( \sqrt{k + 1} \). By the interlacing theorem we have 
\[
\sqrt{k + 1} \leq \lambda_1.
\]

On the other hand, Lepović and Gutman also show that Equation (6) has a zero in the interval \((k - 2, k - 1)\) and it follows that \( \lambda_1 < k/\sqrt{k - 1} \) (See [LG01, proof of Theorem 2]). Therefore we have
\[
\sqrt{k + 1} \leq \lambda_1 < \frac{k}{\sqrt{k - 1}}.
\]

\[\square\]

Remark. In the paper of Lepović and Gutman, they study the properties of the polynomial
\[
t^{n+2} - (k - 1)t^{n+1} + (k - 1)t - 1
\]
and one can easily see that all results are also true for Equation (6).

3. Algebraic integers associated with starlike trees

It is known that a starlike tree has at most one eigenvalue \( \lambda > 2 \). We say that a starlike tree is hyperbolic if it has exactly one eigenvalue greater than 2. It happens that all starlike trees are hyperbolic except \( S(n - 3, 1, 1) \), for \( n \geq 4 \), \( S(5, 2, 1), S(4, 2, 1), S(3, 3, 1), S(3, 2, 1), S(2, 2, 2), S(2, 2, 1), \) and \( S(1, 1, 1, 1) \) [LG01, Theorem 1]. Hence for \( n \geq 2 \) and \( k \geq 3 \), \( S(n, k \cdot 1) \) is hyperbolic.

Let \( \lambda_1 \) be the largest eigenvalue of \( S(n_1, n_2, \ldots, n_k) \). If the starlike tree is hyperbolic, then the number \( t > 1 \), defined by \( \sqrt{t} + 1/\sqrt{t} = \lambda_1 \), is associated with the dynamical complexity of an automorphism of an orientable surface (for more about this topic, see [Lei04] or [Shi16]). In particular, \( t \) is a special algebraic integer, characterized by the following theorem.

Theorem 3.1. [MRS99, Corollary 9] Let \( S \) be a starlike tree whose largest eigenvalue \( \lambda_1 \) is not an integer, and suppose that \( S \) is hyperbolic. Then \( t > 1 \), defined by \( \sqrt{t} + 1/\sqrt{t} = \lambda_1 \), is a Salem number. If \( \lambda_1 \) is an integer then \( t \) is a quadratic Pisot number.

Now we have the following result.

Corollary 3.2. For \( n \geq 2 \) and \( k \geq 3 \) let \( \lambda_1 \) be the largest eigenvalue of the starlike tree \( S(n, k \cdot 1) \). Then the number \( t > 1 \) defined by
\[
\sqrt{t} + \frac{1}{\sqrt{t}} = \lambda_1,
\]
is a Salem number.
Proof. By Theorem B we have
\[ k + 1 < \lambda_1^2 < \frac{k^2}{k - 1} = k + 1 + \frac{1}{k - 1} \]
and hence \( \lambda_1 \) is not an integer. By Theorem 3.1, \( t \) is a Salem number. \( \square \)

Let \( Q_n(t) \) be the polynomial in Equation (6) and let \( \rho(Q_n(t)) \) be the largest real root of \( Q_n(t) \). Let \( m \) be any fixed positive integer. It is shown that for sufficiently large \( n \), \( Q_n(t) \) has a root in the interval \((k-1 - \frac{1}{10^m}, k-1)\) (see the proof of Corollary 2.1. in [LG01]). This implies that
\[ \lim_{n \to \infty} \rho(Q_n(t)) = k - 1. \]

Since the largest root of \( Q_n(t) \) is a Salem number for each \( n \), there is a sequence of Salem numbers that converges to each integer greater than 1.

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