Localizability-Constrained Deployment of Mobile Robotic Networks with Noisy Range Measurements

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Abstract—When nodes in a mobile network use relative noisy measurements with respect to their neighbors to estimate their positions, the overall connectivity and geometry of the measurement network has a critical influence on the achievable localization accuracy. This paper considers the problem of deploying a mobile robotic network implementing a cooperative localization scheme based on range measurements only, while attempting to maintain a network geometry that is favorable to estimating the robots’ positions with high accuracy. The quality of the network geometry is measured by a “localizability” function serving as potential field for robot motion planning. This function is built from the Cramér-Rao bound, which provides for a given geometry a lower bound on the covariance matrix achievable by any unbiased position estimator that the robots might implement using their relative measurements. We describe gradient descent-based motion planners for the robots that attempt to optimize or constrain different variations of the network’s localizability function, and discuss ways of implementing these controllers in a distributed manner. Finally, the paper also establishes formal connections between our statistical considerations such as symmetric rearrangements of subsets of nodes. This property is known to be tightly linked to the possibility of uniquely reconstructing the nodes’ positions from range measurements [10], and hence is useful to enforce for a rigid network, e.g., if each robot measures its distance with respect to all others, yet it will be essentially impossible for them to estimate their positions accurately as soon as small errors and their statistical characteristics, and in particular it is insufficient by itself to predict the achievable accuracy of the position estimates under noisy range measurements. For example, a set of robots that are almost aligned can form a rigid network, e.g., if each robot measures its distance with respect to all others, yet it will be essentially impossible for them to estimate their positions accurately as soon as small errors contaminate the range measurements, these errors being greatly amplified by the poor network geometry.

The position estimation performance achievable for a given geometry under noisy relative measurements is more accurately captured by statistical notions such as the Cramér-Rao Bound (CRB) [11], which provides a lower bound on the covariance matrix of any possible unbiased position estimate. Several references compute the CRB for sensor networks...
performing cooperative localization with a variety of sensing modalities [6], [12]–[14], but with a focus on static nodes. The CRB has also been used extensively to guide the motion of mobile sensors performing signal processing tasks such as tracking a target [15] or estimating the parameters of an spatial process [16]. In contrast, we use the CRB as the basis for a multi-robot motion planning strategy that supports more accurate position estimates for the robots themselves.

We formulate our problem more precisely in Section II where we also show that there is in fact a close connection between the CRB and a certain notion of weighted rigidity introduced in [7], [8], for a specific choice of weights. In Section III we propose a potential-field based motion planning method to guide the robots along trajectories that maintain the CRB of the network low, as a measure of the localizability of the robots. Descending the gradient of a potential field is a standard tool to deploy groups of robots performing various tasks, from source seeking to formation control and coverage control [17]–[21]. Potential fields can encode constraints on the robots’ paths such as obstacle avoidance [22], communication constraints [23] and, as we discuss here, localizability constraints. It is generally desirable with such methods to obtain motion planning algorithms that can be implemented in a distributed fashion, with the robots communicating only with a restricted number of neighbors, but possibly through multiple iterations. This issue is addressed in Section IV. Finally, Section V summarizes our approach and briefly illustrates it via simulations.

II. PROBLEM STATEMENT AND LOCALIZABILITY DEFINITION

Consider a network of \( n + m \) mobile robots evolving in a 2D space, i.e., with positions \( \mathbf{p}_i = [x_i, y_i]^T \in \mathbb{R}^2 \), \( i = 1, \ldots, n + m \) expressed in a global common reference frame. We assume that \( m \) robots know their position perfectly, e.g., they could in fact be static nodes whose position is known to be fixed. Call these \( m \) robots anchors and without loss of generality we choose their indices to be \( 1, \ldots, n + m \). Let \( \mathbf{p} = [\mathbf{p}_1^T, \ldots, \mathbf{p}_n^T] \in \mathbb{R}^{2n} \) be a column vector containing the (unknown) positions of the remaining robots, and let \( \mathbf{p}_n^T \in \mathbb{R}^{2(n+m)} \).

Any robot \( i \), including the anchors, can measure some of its Euclidean distances \( d_{ij} \) with respect to other robots \( j \) in a set \( \mathcal{N}_i \subset \{1, \ldots, n + m\} \), which we call its neighbors, and moreover it can also communicate with these neighbors (in order to implement a distributed motion planning algorithm). For simplicity, we assume in this paper that range measurement capabilities are symmetric, i.e., \( j \in \mathcal{N}_i \) if and only if \( i \in \mathcal{N}_j \). The sets \( \mathcal{N}_i \) could change over time, as the geometry of the network evolves, or be dependent on the global configuration \( \mathbf{p} \). The agents with their sets of neighbors then form an undirected ranging graph \( \mathcal{G} = (\mathcal{V}, E) \), with \( |\mathcal{V}| = n + m \) vertices so that an edge \( \{i, j\} \in E \) if \( i \) and \( j \) are neighbors. We also define the indicator function \( 1_{\mathcal{N}_i} \) for a set \( \mathcal{N}_i \) by \( 1_{\mathcal{N}_i}(j) = 1 \) if \( j \in \mathcal{N}_i \) and \( 1_{\mathcal{N}_i}(j) = 0 \) otherwise.

A. Range Measurements and Cooperative Localization

For two agents \( i, j \) capable of measuring their distance \( d_{ij} \), we denote by \( \theta_{ij} \) a measurement by \( i \) of its distance to \( j \) (which could be different from \( \theta_{ji} \), because of measurement errors). We consider here only simple ranging measurement models, namely, the case of distance measurements with random additive Gaussian errors for all nodes, i.e., \( \theta_{ij} = d_{ij} + \epsilon_{ij} \), or with random multiplicative log-normal errors for all nodes, i.e., \( \theta_{ij} = e^{\epsilon_{ij}} d_{ij} \) or equivalently \( \log(\theta_{ij}) = \log(d_{ij}) + \epsilon_{ij} \), with in both cases \( \epsilon_{ij} \sim \mathcal{N}(0, \sigma^2) \), and the same parameter \( \sigma \) for all \( i, j \). We make the standard simplifying assumption that all measurement errors \( \epsilon_{ij} \) are independent. Additive errors are characteristic of ToF-based distance measurements, and multiplicative errors of measurements based on signal strength for example [6].

Starting from the relative distance measurements \( \theta_{ij} \), the robots must estimate their positions in the global common frame of reference. In other words, they must form an estimate \( \hat{\mathbf{p}} \) of \( \mathbf{p} \). We assume \( m \geq 3 \), so that we have enough anchors in order to be able to remove the intrinsic translational and rotational ambiguity for the whole network associated with relative range measurements. The literature addressing this cooperative localization problem is extensive, with both centralized and decentralized algorithms available [1]–[4]. For example, a basic method could be to solve a least-squares problem minimizing the sum of the squared residuals \( r_{ij}^2 = |\theta_{ij}^2 - \|\hat{\mathbf{p}}_i - \hat{\mathbf{p}}_j\|^2|^2 \), which leads via gradient descent to distributed computations of an estimate \( \hat{\mathbf{p}} \). Our deployment methodology is independent of the choice of cooperative localization algorithm implemented by the robots, but requires a real-time estimate \( \hat{\mathbf{p}} \) provided by such an algorithm, as a input signal to the motion planner.

B. Lower Bound on Localization Accuracy

The CRB [11] provides a lower bound on the covariance matrix of any unbiased position estimate \( \hat{\mathbf{p}} \) constructed from the relative measurements \( \Theta := [\theta_{ij}]_{i,j \in \mathcal{N}_i} \) between the robots. This bound depends on the relative positions of the robots, i.e., on the geometry of the ranging network. While it is not necessarily achieved by a particular localization algorithm, we use it here as an indicator of the ability of a network geometry to support accurate position estimation, i.e., of the localizability of the network. Our objective is to design motion planning strategies for the robot network that maintain a high level of localizability, which we aim to achieve concretely by maintaining ranging network geometries associated with a small CRB.

For a measurement model given by the measurements’ probability density \( h(\Theta | \mathbf{p}) \), we define the (symmetric) \( 2n \times 2n \) Fischer Information matrix (FIM) \( F(\mathbf{p}) \) by

\[
F(\mathbf{p}) = -\mathbb{E} \left[ \frac{\partial^2}{\partial \mathbf{p}^2} \ln h(\Theta | \mathbf{p}) \right],
\]

where the matrix inside the expectation operator is the Hessian of \( \ln h(\Theta | \mathbf{p}) \) with respects to the (unknown) position variables \( x_i, y_i, i = 1, \ldots, n \). The CRB states that

\[
\text{Cov}(\mathbf{p}) \geq (F(\mathbf{p}))^{-1},
\]
for any estimate \( \hat{p} \) of \( p \) constructed from the measurements \( \Theta \) that is unbiased, i.e., such that \( \mathbb{E}[\hat{p}] = p \). The notation \( A \succeq B \) means that \( A - B \) is positive semi-definite.

For our measurement models, the FIM can be computed explicitly, see [6] for example. With our variable ordering for \( p \), it can be written in the form \( F(p) = [F_{ij}(p)]_{i,j=1}^N \), where the blocks \( F_{ij} \) for \( i \neq j \) (defined also for indices in \( \{n+1, \ldots, n+m\} \)), are the \( 2 \times 2 \) matrices

\[
F_{ij}(p) := F_{ij}(p, p_j) = -\frac{1}{\sigma_d^2 a^{\alpha}_i} \begin{bmatrix}
(x_i - x_j)^2 & (x_i - x_j)(y_i - y_j) \\
(x_i - x_j)(y_i - y_j) & (y_i - y_j)^2
\end{bmatrix},
\]

and moreover the diagonal \( 2 \times 2 \) blocks are

\[
F_{ii}(p) = -\sum_{k \in N_i} F_{ik}(p_i, p_k) = \sum_{k \in N_i} \frac{1}{\sigma_d^2 a^{\alpha}_i} \begin{bmatrix}
(x_i - x_k)^2 & (x_i - x_k)(y_i - y_k) \\
(x_i - x_k)(y_i - y_k) & (y_i - y_k)^2
\end{bmatrix}.
\]

In the expressions [1] and [2], we set \( \alpha = 1 \) for additive Gaussian noise and \( \alpha = 2 \) for multiplicative log-normal noise. Note that the matrix \( F \) is indeed symmetric since \( F_{ij}(p_i, p_j) = F_{ij}(p_j, p_i) \), with moreover each block \( F_{ij} \) and \( F_{ii} \) also symmetric. In addition, the sparsity pattern of \( F \) is in correspondence with the links in the ranging graph \( G \), i.e., \( F_{ij} = 0 \) if \( j \notin N_i \). Note also that the matrices \( F_{ij} \) for \( j \neq i \) present in \( F \) only depend on the unknown position variables \( p \), however the diagonal matrices \( F_{ii} \) can involve the anchor variables \( p_{n+1}, \ldots, p_{n+m} \), since the anchors are included in some of the sets \( N_i \). This fact is important to be able to obtain a FIM that is invertible.

C. Connections with Rigidity Theory

We conclude this section by making a connection between the FIM above and weighted rigidity theory [8], and show that up to reordering of the variables, the FIM can be viewed as (a submatrix of) a weighted Laplacian matrix [24] of the ranging graph \( G \). First, define \( \tilde{F} \) to be the \( (n+m) \times 2(n+m) \) matrix with blocks \( F_{ij} \) as for \( F \), but including the blocks corresponding to the anchor nodes. Next, reorder the coordinate variables from the order defining \( p \) to \( [x_1, \ldots, x_n, y_1, \ldots, y_n] \) and let \( P \) denote the permutation matrix describing this change of coordinates. The (extended) FIM, built from the entries of [1], [2] with this new ordering, denoted \( \tilde{F}_1 := FP\tilde{P}^{-1} \), is of the form

\[
\tilde{F}_1 = \begin{bmatrix}
F_{xx} & F_{xy} \\ F_{xy}^T & F_{yy}
\end{bmatrix}.
\]

Now, orient the ranging graph arbitrarily and define the following \( |E| \times 2(n+m) \) weighted rigidity matrix \( R(p) \) as the matrix with one row per edge in \( E \), and such that if \( (i, j) \in E \), the corresponding row is

\[
0^T \begin{bmatrix}
\sigma_d^2 a^{\alpha}_i \\
\sigma_d^2 a^{\alpha}_j
\end{bmatrix} 0^T - \begin{bmatrix}
\sigma_d^2 a^{\alpha}_i \\
\sigma_d^2 a^{\alpha}_j
\end{bmatrix} y_i - y_j 0^T - y_i - y_j 0^T - y_i - y_j 0^T,
\]

where \( 0^T \) denotes a zero row vector of appropriate dimensions, and the non-zero entries are in the columns \( i, j, \)

\( n + m + i \) and \( n + m + j \). Straightforward calculations lead to the following result.

Proposition 1: We have \( \tilde{F}_1(p) = R^T(p)R(p) \).

From this remark, the following proposition follows from [8, Proposition 2.15]. Define \( Q_{\mu,\nu}(p) \) for \( \mu \) and \( \nu \) equal to \( x \) or \( y \) as the \( |E| \times |E| \) diagonal matrix with entry \( \frac{\mu - \nu}{\sigma_d^2 a^{\alpha}_i} \) for edge \((i, j) \in E\), with edges ordered as for the rows of \( R \). Let \( B \) be the incidence matrix of the graph \( G \), i.e., the \((n+m) \times |E| \) matrix with entries \( B_{i,(i,j)} = +1 \), \( B_{j,(i,j)} = -1 \), and zero otherwise.

Proposition 2: We have \( \tilde{F}_1(p) = (I_2 \otimes B)Q(p)(I_2 \otimes B^T) \), where \( \otimes \) denotes the Kronecker product and

\[
Q(p) = \begin{bmatrix}
Q_{xx}(p) & Q_{xy}(p) \\ Q_{yx}(p) & Q_{yy}(p)
\end{bmatrix}.
\]

Proposition 2 shows that \( \tilde{F}_1 \) is a (weighted) symmetric rigidity matrix as introduced in [7], [8], for a specific set of weights, namely, edge \((i, j) \in E\) has weight \( 1/(\sigma_d^2 a^{\alpha}_i) \). As a result, the techniques developed in these papers for rigidity maintenance are applicable to keep the CRB low, at least for the E-optimal design approach introduced in the next section. Note also that our weights diverge as the agents get closer, in contrast to the weights introduced in [7], [8] for purposes such as collision avoidance, which remain bounded.

III. POTENTIAL FIELD BASED MOTION PLANNING

A standard technique to design multi-robot deployment algorithms is to let the robots descend the gradient of a potential field (cost function) encoding constraints such as collision avoidance or connectivity and tasks such as coverage-control or source seeking [19]. Here we use this methodology to maintaining good localizability for the group.

Given a real-valued potential function \( f(p) \) measuring the quality of a geometric configuration \( p \), with lower values corresponding to higher quality configurations, potential field based motion planners design trajectories for the robots by obtaining successive configurations \( p^0, p^1, \ldots \) that descend the gradient of \( f \), i.e.,

\[
p^{k+1} = p^k - \gamma_k \nabla f|_{p^k},
\]

where \( \gamma_k \) are some stepsizes, which could be taken constant. For instance, obstacle avoidance controllers can be obtained by designing functions that increase sharply in the neighborhood of an obstacle [22]. The dynamics of the robots are often neglected at this stage, as we do here, and a lower level controller is then necessary to track the resulting trajectories with physical platforms.

For \( f \) sufficiently smooth, the sequence [3] will tend to configurations that remain in a neighborhood of a local minimum of \( f \), and indeed most multi-robot potentials can have many such minima. A further complication comes from the fact that here the current configuration \( p^k \) is not known exactly but is estimated as \( \hat{p}^k \) from a cooperative localization algorithm, in which case one can implement

\[
p^{k+1} = p^k - \gamma_k \nabla f|_{\hat{p}^k}.
\]
Errors in the position estimates can lead to errors in the update directions, but a formal discussion of this issue is outside of the scope of this paper.

A. Choice of Potential Function

For illustration purposes, consider potential functions of the form

\[ f(p) = f_{\text{loc}}(p) + \alpha f_{\text{conn}}(p) + \beta f_{\text{task}}(p), \]  

with \( \alpha, \beta \) some parameters weighting particular components of the potential field. The function \( f_{\text{task}} \) aims to deploy the robots to achieve a specific task, for example reach a specific goal in the workspace, or cover an area [19]. Many such potentials have been designed for multi-robot systems. For concreteness, consider the function

\[ f_{\text{task}}(p) = \frac{1}{2} \sum_{i=1}^{n} (x_i - \bar{x}_i)^2. \]  

This function drives the robots toward configurations where robot \( i \) is close to a desired line of \( x \)-coordinate \( \bar{x}_i \). An important aspect in the choice of a potential function is to facilitate distributed gradient computations. Indeed, the updates (4) can be rewritten for agent \( i \) as

\[ p_i^{k+1} = p_i^k - \gamma_k \frac{\partial f_i}{\partial p_i}(p^k), \]  

where \( \frac{\partial f_i}{\partial p_i} \) represents the vector \( [\partial f_i/\partial x_i, \partial f_i/\partial y_i] \). Ideally, computing \( \frac{\partial f_i}{\partial p_i}(p^k) \) for robot \( i \) should be possible by communicating only with a few other robots (its neighbors), to facilitate scaling of the algorithm with the size of the network and avoid communication or computation bottlenecks at certain nodes. Distributed gradient descent updates are trivial for (6), involving in fact no communication between robots, since \( \frac{\partial f_{\text{task}}}{\partial p_i} = [\bar{x}_i - \bar{x}_i], 0] \).

We include in (5) a potential \( f_{\text{conn}} \) to maintain certain pairs of robots sufficiently close. This imposes a priori that we always want to have range measurements during deployment between these pairs. In fact, the localizability potential discussed next could in principle lead to maintaining appropriate links during deployment, but in practice this can lead to numerical difficulties if too many links suddenly disappear, an issue that is left for future work. Denote by \( E_{\text{cons}} \) the set of links to guarantee. Nodes linked by an edge in \( E_{\text{cons}} \) can be kept within a distance \( d_{\text{max}} \) by using a barrier potential such as

\[ f_{\text{conn}}(p) = \sum_{(i,j) \in E_{\text{cons}}} g(\|p_i - p_j\|), \]

where \( g(d) = 0 \) if \( d < d_0 \) and \( g(d) = \left( \frac{1}{d_{\text{max}} - d} - \frac{1}{d_{\text{max}} - d_0} \right)^2 \) if \( d_0 \leq d < d_{\text{max}} \). Here \( d_0 \) is the distance at which the inter-agent distance starts to be penalized. It is straightforward to see that \( \frac{\partial f_{\text{conn}}}{\partial p_i} \) can be computed by agent \( i \) from the position information of its neighbors in \( E_{\text{cons}} \) only.

B. Localizability Potential

The remaining term \( f_{\text{loc}} \) in (3) is the main focus of this paper. We build a potential function from the FIM to attempt to restrict the motion of the group to configurations where the CRB, i.e., \( (F(p))^{-1} \) is sufficiently small. Since the potential field must be scalar, some information provided by the matrix inequality in the CRB will be lost. Various potential functions can be envisioned, as suggested by the literature on optimal design of experiments in statistics [25]. For example, one can try to reach configurations that minimize one of the following functions taken as \( f_{\text{loc}} \)

\[ f_T(p) = -\text{Tr}[F(p)] \quad (\text{T-optimal design}) \]  

\[ f_D(p) = -\ln \det(F(p)) \quad (\text{D-optimal design}) \]  

\[ f_A(p) = \text{Tr}[F(p)^{-1}] \quad (\text{A-optimal design}) \]  

or \( f_E(p) = -\lambda_{\text{min}}(F(p)) \) (E-optimal design).

The function \( f_T \) is the easiest to compute and minimize, unfortunately it typically leads to undesirable paths and configurations, in particular because it does not prevent \( F(p) \) to become singular. The function \( f_E \), where \( \lambda_{\text{min}} \) is the minimum eigenvalue of the FIM, is essentially the potential adopted in previous work on connectivity maintenance [26] (maximizing the first nonzero eigenvalue of the Laplacian matrix of the communication graph) and rigidity maintenance [8] (maximizing the first nonzero eigenvalue of the symmetric rigidity matrix, here \( F_1(p) \)). In view of the connections established in Section II-C the techniques developed in these papers for estimating \( \lambda_{\text{min}} \) and maintaining it above a desired threshold are applicable, but a discussion of the resulting controllers is left for a full version of this paper. In the next section, we focus on the computation of the gradient steps for the functions \( f_D \) and \( f_A \).

IV. GRADIENT COMPUTATIONS FOR THE LOCALIZABILITY POTENTIALS

To simplify the presentation, we assume in this section that the anchor robots are fixed and compute the gradients only for the robots with unknown positions. Computing the gradients of mobile anchors presents no additional difficulty.

A. Partial Derivatives of the FIM

For \( i = 1, \ldots, n \), and \( \nu = x \) or \( y \), the partial derivatives of (8) read

\[ \frac{\partial f_T}{\partial \nu_i}(p) = -\sum_{k=1}^{n} \text{Tr} \left[ \frac{\partial F_{kk}}{\partial \nu_i}(p) \right]. \]  

For (9), we deduce from (27) that

\[ \frac{\partial f_D}{\partial \nu_i}(p) = -\text{Tr} \left[ (F(p))^{-1} \frac{\partial F}{\partial \nu_i}(p) \right]. \]  

Finally, for (10), since \( df_A = \text{Tr}(dF^{-1}) \), \( dF^{-1} = (F^{-1})^T dF (F^{-1}) \) and \( \text{Tr}(A \text{Tr}(B)) = \text{Tr}(BA) \), we have

\[ \frac{\partial f_A}{\partial \nu_i}(p) = -\text{Tr} \left[ (F(p))^{-2} \frac{\partial F}{\partial \nu_i}(p) \right]. \]
In the equations above, we see that the expressions of \( \frac{\partial F}{\partial x_i}(p_i, p_j) \) and \( \frac{\partial F}{\partial y_i}(p_i, p_j) \) are needed. Starting from (11) and (12), we can obtain the following expressions for the partial derivatives of the \( F_{ij} \) blocks. If \( i, j \in \mathcal{N}_i \), then
\[
\frac{\partial F_{ij}}{\partial x_i}(p_i, p_j) = \frac{2}{\sigma^2 \alpha_{ij}} \times \left[ \alpha \left( \frac{\partial (x_i - x_j)^2}{\partial x_i} \right)^2 - (x_i - x_j) \left( y_i - y_j \right) \left( \alpha \left( \frac{\partial (x_i - x_j)^2}{\partial y_i} \right)^2 - \frac{1}{2} \right) \right],
\]
where the symbol * replaces the symmetric term, \( \alpha = 1 \) for additive noise and 2 for multiplicative noise. Similarly,
\[
\frac{\partial F_{ij}}{\partial y_i}(p_i, p_j) = \frac{2}{\sigma^2 \alpha_{ij}} \times \left[ \alpha \left( \frac{\partial (x_i - x_j)^2}{\partial y_i} \right)^2 - (x_i - x_j) \left( y_i - y_j \right) \left( \alpha \left( \frac{\partial (x_i - x_j)^2}{\partial x_i} \right)^2 - \frac{1}{2} \right) \right].
\]

These expressions suffice to compute all the elements of \( \frac{\partial F}{\partial x_i}(p_i, p_j) \) and \( \frac{\partial F}{\partial y_i}(p_i, p_j) \). For example, we have \( \frac{\partial F_{ij}}{\partial x_i}(p_j, p_i) = \frac{\partial F_{ij}}{\partial x_i}(p_i, p_j) \) by the symmetry of the functions \( F_{ij} \). Then \( \frac{\partial F}{\partial x_i}(p_i, p_k) = -\sum_{k \in \mathcal{N}_i} \frac{\partial F_{ik}}{\partial x_i}(p_i, p_k) \), and for \( k \neq i \), \( \frac{\partial F_{ik}}{\partial y_i}(p_i, p_k) = 0 \) if \( k \) and \( l \) are different from \( i \), or if \( k \) and \( l \) are not neighbors (since then \( F_{ik} = 0 \)). Decomposing the matrices \( \frac{\partial F}{\partial x_i}(p_i, p_j) \) and \( \frac{\partial F}{\partial y_i}(p_i, p_j) \) into 2 × 2 blocks, only the blocks \( (i, i) \), \( (i, j) \) for \( j \in \mathcal{N}_i \), and \( (j, j) \) for \( i \in \mathcal{N}_j \), are non zero.

We can then compute immediately the partial derivatives of \( F \). Namely, starting from (12),
\[
\frac{\partial F}{\partial x_i}(p_i) = \text{Tr} \left[ \sum_{k \in \mathcal{N}_i} \frac{\partial F_{ik}}{\partial x_i}(p_i, p_k) + \sum_{k \in \mathcal{N}_k} \frac{\partial F_{ki}}{\partial x_i}(p_k, p_i) \right].
\]

Using our symmetric graph assumption and \( F_{ik} = F_{ki} \),
\[
\frac{\partial F}{\partial x_i}(p_i) = \frac{4}{\sigma^2} \sum_{k \in \mathcal{N}_i} \frac{1}{d_{ik}} \left( (x_i - x_k) \left( x_i - x_k \right)^2 + (y_i - y_k) \left( y_i - y_k \right)^2 \right),
\]
and similarly
\[
\frac{\partial F}{\partial y_i}(p_i) = \frac{4}{\sigma^2} \sum_{k \in \mathcal{N}_i} \frac{1}{d_{ik}} \left( (y_i - y_k) \left( y_i - y_k \right)^2 + (x_i - x_k) \left( x_i - x_k \right)^2 \right).
\]

We see that these expressions (for \( p_i \)) can be computed by agent \( i \) by communicating only with its neighbors to obtain their relative position estimates \( p_{ik} := p_i - p_k \).

### B. Distributed Computations for D- and A-Optimal Design

For \( f_D \) and \( f_A \), the computations (13) and (14) of agent \( i \) involve the diagonal blocks of \( F^{-1} \frac{\partial F}{\partial y_i} \) and \( F^{-2} \frac{\partial F}{\partial y_i} \), for \( \nu = x \) or \( y \). Computing \( \frac{\partial F}{\partial y_i}(p_i) \) can be done by agent \( i \) from only the knowledge of the relative position estimates \( p_{ij} \) with respect to its neighbors in the ranging graph.
robots to maintain group shapes that enable sufficiently accurate position estimation. Concretely, a potential-field based motion planner aims to maintain network configurations such that the Cramér-Rao Lower Bound on the variance of any unbiased position estimator constructed based on the distance measurements is small. We established connections between this methodology and the problem of rigidity maintenance control, by remarking that the Fischer information matrix (FIM) can be viewed as a type of weighted rigidity matrix. We also discussed distributed implementations of the gradient descent motion planner, for different types of potentials built from the FIM.

VI. CONCLUSIONS

We have considered the problem of deploying a mobile robotic network implementing a cooperative localization scheme to estimate the robots’ positions from noisy relative range measurements, by restricting the trajectories of the robots to maintain group shapes that enable sufficiently accurate position estimation. Concretely, a potential-field based motion planner aims to maintain network configurations such that the Cramér-Rao Lower Bound on the variance of any unbiased position estimator constructed based on the distance measurements is small. We established connections between this methodology and the problem of rigidity maintenance control, by remarking that the Fischer information matrix (FIM) can be viewed as a type of weighted rigidity matrix. We also discussed distributed implementations of the gradient descent motion planner, for different types of potentials built from the FIM.

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