A $t$-MOTIVIC INTERPRETATION OF SHUFFLE RELATIONS FOR MULTIZETA VALUES

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Abstract. Thakur [Tha10] showed that, for $r, s \in \mathbb{N}$, a product of two Carlitz zeta values $\zeta_A(r)$ and $\zeta_A(s)$ can be expressed as an $\mathbb{F}_p$-linear combination of $\zeta_A(r + s)$ and double zeta values of weight $r + s$. Such an expression is called shuffle relation by Thakur. Fixing $r, s \in \mathbb{N}$, we construct a $t$-module $E'$. To determine whether an $(r + s)$-tuple $c$ in $\mathbb{F}_q(\theta)^{r+s}$ gives a shuffle relation, we relate it to the $\mathbb{F}_q[t]$-torsion property of the point $v_c \in E'(\mathbb{F}_q[\theta])$ constructed with respect to the given $(r + s)$-tuple $c$. We also provide an effective criterion for deciding the $\mathbb{F}_q[t]$-torsion property of the point $v_c$.

1. Introduction

Let $A := \mathbb{F}_q[\theta]$ be the polynomial ring in the variable $\theta$ over the finite field of $q$ elements $\mathbb{F}_q$ where $q$ is a power of a prime $p$. We denote by $A_+$ the set of monic polynomials in $A$ and let $k := \mathbb{F}_q(\theta)$ be the field of fractions of $A$. For $(s_1, \ldots, s_r) \in \mathbb{N}^r$, Thakur [Tha04] introduced the multizeta value $\zeta_A(s_1, \ldots, s_r)$ defined by

$$\zeta_A(s_1, \ldots, s_r) := \sum_{(a_1, \ldots, a_r) \in A_r} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in \mathbb{F}_q((1/\theta)).$$

Here $\sum_{i=1}^r s_i$ is called the weight and $r$ is called the depth of the presentation $\zeta_A(s_1, \ldots, s_r)$. In particular, depth one multizeta values are called Carlitz zeta values initiated by Carlitz [Car35] and depth two multizeta values are called double zeta values. In [Tha09], Thakur showed that each multizeta value is non-vanishing.

We fix positive integers $r, s \in \mathbb{N}$ and let $n := r + s$. By [Tha10], we know that the product of two zeta values $\zeta_A(r)$ and $\zeta_A(s)$ can be expressed as an $\mathbb{F}_p$-linear combination of $\zeta_A(n)$ and double zeta values of weight $n$. Such an expression is called shuffle relation by Thakur. Chen [Che15] derived an explicit formula of a shuffle relation, which is given by

$$\zeta_A(r)\zeta_A(s) - \zeta_A(r, s) - \zeta_A(s, r) =$$

$$\zeta_A(n) + \sum_{i+j=n \atop (i-1)j} \left( (-1)^{s-1} \binom{j-1}{s-1} + (-1)^{r-1} \binom{j-1}{r-1} \right) \zeta_A(i, j).$$

We want to study shuffle relations with coefficients in $k$. By a shuffle relation for multizeta values over $k$, we mean that the following identity holds:

$$\zeta_A(r)\zeta_A(s) - \zeta_A(r, s) - \zeta_A(s, r) = b_0 \zeta_A(n) + \sum_{i=1}^{n-1} a_i \zeta_A(i, n - i)$$

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for some \( a_i, b_0 \in k \). We are interested in \( n \)-tuples of coefficients \( \mathcal{C} = (b_0, a_1, \ldots, a_{n-1}) \in k^n \) satisfying the equation (1.2).

In this paper we provide a \( t \)-motivic interpretation of shuffle relations for multizeta values over \( k \). Let \( r, s, n \) be given as above. We construct Frobenius modules (see \( \S2.2 \)) \( M' \) and \( M_\mathcal{C} \) associated with the given \( n \)-tuple of coefficients \( \mathcal{C} = (b_0, a_1, \ldots, a_{n-1}) \in k^n \), which fits into the short exact sequence of Frobenius modules

\[
0 \to M' \to M_\mathcal{C} \to 1 \to 0
\]

so \( M_\mathcal{C} \) represents a class in \( \text{Ext}^1_{\mathcal{C}}(1, M') \). For more details, we refer readers to [2.3] and [2.4]. To determine whether the given \( n \)-tuple of coefficients \( \mathcal{C} = (b_0, a_1, \ldots, a_{n-1}) \in k^n \) satisfy a shuffle relation over \( k \) as in the equation (1.2), we relate it to whether \( M_\mathcal{C} \) is an \( F_q[t] \)-torsion class in \( \text{Ext}^1_{\mathcal{C}}(1, M') \). More precisely, if the \( n \)-tuple of coefficients \( \mathcal{C} \in k^n \) satisfy the equation (1.2), then we show that the Frobenius module \( M_\mathcal{C} \) represents an \( F_q[t] \)-torsion class in \( \text{Ext}^1_{\mathcal{C}}(1, M') \). Conversely, if the Frobenius module \( M_\mathcal{C} \) represents an \( F_q[t] \)-torsion class in \( \text{Ext}^1_{\mathcal{C}}(1, M') \), then the \( n \)-tuple of coefficients \( \mathcal{C} \in k^n \) satisfy the equation (1.2) in the case \( (q - 1) \nmid n \). In the case \( (q - 1) \mid n \), the \( n \)-tuple of coefficients \( \mathcal{C} \in k^n \) satisfy the equation (1.2) modulo \( \bar{\pi}^n \). We state the above results in the Theorem 2.6.

Following the strategy in [CPY19], we give an effective criterion for the \( F_q[t] \)-torsion property of \( M_\mathcal{C} \). With the \( t \)-motivic interpretation of shuffle relations for multizeta values over \( k \) given in the Theorem 2.6, we can effectively determine whether the given \( n \)-tuple of coefficients \( \mathcal{C} = (b_0, a_1, \ldots, a_{n-1}) \in k^n \) satisfy a shuffle relation over \( k \) (resp. shuffle relation over \( k \) modulo \( \bar{\pi}^n \)) in the case \( (q - 1) \nmid n \) (resp. \( (q - 1) \mid n \)). We also provide some examples in [3.6].

After we worked out this project, we found that there is another approach to determine a given \( n \)-tuple \( \mathcal{C} = (b_0, a_1, \ldots, a_{n-1}) \in k^n \) satisfying a shuffle relation over \( k \) by using the results provided by Chang [Cha16]. We fix \( r, s \in \mathbb{N} \) and put \( n := r + s \). By combining a shuffle relation over \( k \) and the relation (1.1), we have a relation of the form

\[
(1.3) \quad \bar{b}_0 \zeta_A(n) + \sum_{i=1}^{n-1} \bar{a}_i \zeta_A(i, n - i) = 0
\]

where \( \bar{a}_i, \bar{b}_0 \in k \). So we have a one-to-one correspondence between shuffle relations over \( k \) and relations over \( k \) of the form (1.3). By [Cha16] Thm. 5.1.1, Thm. 6.1.1, we have an effective process to check whether a given sequence \( \bar{a}_i, \bar{b}_0 \in k \) satisfies (1.3) in the case \( (q - 1) \nmid n \). In the case \( (q - 1) \mid n \), this effective process can check whether a given sequence \( \bar{a}_i, \bar{b}_0 \in k \) satisfies (1.3) modulo \( \bar{\pi}^n \). Hence we achieve the same result from this approach. For more details, we refer readers to [4.7].

Comparing with the results provided in this paper, let us consider the analogue question in the classical case. For fixed positive integer \( d \) and \( d \)-tuple of positive integer variable \( (s_1, \ldots, s_d) \) with \( s_1 > 1 \), the classical multiple zeta value is defined by

\[
\zeta(s_1, \ldots, s_d) := \sum_{k_1 > \cdots > k_d > 0} k_1^{-s_1} \cdots k_d^{-s_d} \quad (\text{see } [Zha16]).
\]

We fix positive integers \( r, s > 1 \) and let \( n := r + s \). There are two well-known formulas. One is shuffle product, also known as Euler’s decomposition formula:
\[ \zeta(r)\zeta(s) = \sum_{i \geq 2, j \geq 1 \atop i+j=r+s} \left[ \binom{i-1}{r-1} + \binom{j-1}{s-1} \right] \zeta(i,j), \]

and the other is shuffle product:

\[ \zeta(r)\zeta(s) = \left( \sum_{n_1=n_2} + \sum_{n_1>n_2} + \sum_{n_1<n_2} \right) \frac{1}{n_1^n n_2^n} = \zeta(r+s) + \zeta(r,s) + \zeta(s,r). \]

Note that we used to think the field of rational functions \( k \) as an analogue of the field of relational numbers \( \mathbb{Q} \). Studying \( n \)-tuples of coefficients \( \mathbf{C} = (b_0, a_1, \ldots, a_{n-1}) \in \mathbb{Q}^n \) satisfying the following equation:

\[ \zeta(r)\zeta(s) - \zeta(r,s) - \zeta(s,r) = b_0 \zeta(n) + \sum_{i=2}^{n-1} a_i \zeta(i,n-i) \]

appeals to us, and we wonder if there is any criterion for a given \( n \)-tuple of rational numbers satisfying the equation (1.4). But in this case it is still an unknown problem.

The paper is organized as follows. In \( \S \) we set up some essential preliminaries first, and then we state our main theorem, Theorem 2.6 which gives a \( t \)-motivic interpretation of shuffle relations. We prove our main theorem in \( \S \) and provide a necessary condition for a shuffle relation in \( \S \). We state an effective criterion whether \( M_\mathcal{E} \) is \( \mathbb{F}_q[t] \)-torsion in \( \text{Ext}_t^1(1,M') \) and write down an algorithm in \( \S \), \( \S \) respectively. We also provide some examples in \( \S \). In \( \S \) we give another approach to our result. The crucial property which makes our criterion effective is the identification of \( \text{Ext}_t^1(1,M') \) as a \( t \)-module defined over \( A \) in which \( M_\mathcal{E} \) corresponds to an integral point. However, its proof is essentially the same as \( [CPY19] \) and so we leave the detailed proof in the appendix.

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2. Preliminaries and The Main Theorems

2.1. Some notations and definitions. Let \( \mathbb{F}_q \) be the finite field with \( q \) elements, where \( q \) is a power of a prime \( p \). Let \( \theta \) be a variable and \( A := \mathbb{F}_q[\theta] \), the polynomial ring in \( \theta \) over \( \mathbb{F}_q \). We denote by \( A_\ast \) the set of monic polynomials in \( A \). Let \( k := \mathbb{F}_q(\theta) \), the field of fractions of \( A \), and define the absolute value \( | \cdot |_\infty \) associated to the infinite place of \( k \) so that \( |\theta|_\infty = q \). Let \( k_\infty \) be the completion of \( k \) with respect to \( | \cdot |_\infty \). Note that \( k_\infty \) is equal to \( \mathbb{F}_q((1/\theta)) \), the field of Laurent series in \( 1/\theta \) over \( \mathbb{F}_q \). Let \( k_\infty \) be a fixed algebraic closure of \( k_\infty \). We denote by \( \overline{k} \) the algebraic closure of \( k \) in \( k_\infty \), and let \( \mathbb{C}_\infty \) be the completion of \( \overline{k} \) with respect to the canonical extension of \( | \cdot |_\infty \).

We recall the characteristic \( p \) multizeta values defined by Thakur.
Definition 2.1 ([Tha04]). For any \( r \)-tuple of positive integers \((s_1, \ldots, s_r) \in \mathbb{N}^r \), we define
\[
\zeta_A(s_1, \ldots, s_r) := \sum_{(a_1, \ldots, a_r) \in A_+^r \atop \deg a_1 > \cdots > \deg a_r} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in \bar{k}_\infty.
\]

Remark 2.2. In [Tha09], Thakur showed that each multizeta value is non-vanishing.

2.2. Anderson-Thakur polynomials. Define \( D_0 := 1 \) and \( D_i := \prod_{j=0}^{i-1} (\theta^{q^j} - \theta^{q^j}) \) for \( i \in \mathbb{N} \). For a non-negative integer \( n \), we express \( n \) as
\[
n = \sum_{i=0}^{\infty} n_i q^i \quad (0 \leq n_i \leq q - 1, \ n_i = 0 \text{ for } i \gg 0),
\]
and we recall the Carlitz factorial
\[
\Gamma_{n+1} := \prod_{i=0}^{\infty} D_i^{n_i} \in A \text{ (see [Tha04]).}
\]

We put \( G_0(y) := 1 \) and define polynomials \( G_n(y) \in \mathbb{F}_q[t, y] \) for \( n \in \mathbb{N} \) by the product
\[
G_n(y) := \prod_{i=1}^{n} (\theta^{q^i} - y^{q^i})
\]
where \( t \) is a new variable independent from \( y \). For \( n = 0, 1, 2, \ldots \), we define the sequence of Anderson-Thakur polynomials \( H_n \in A[t] \) by the generating function identity
\[
\left( 1 - \sum_{i=0}^{\infty} \frac{G_i(\theta)}{D_i^{n+1}} x^{q^i} \right)^{-1} = \sum_{n=0}^{\infty} \frac{H_n}{\Gamma_{n+1}} x^n \text{ (see [AT90, AT09]).}
\]

2.3. Frobenius twisting and Frobenius modules. We consider the Frobenius twisting which is an automorphism on \( \mathbb{C}_\infty((t)) \) defined by
\[
\mathbb{C}_\infty((t)) \to \mathbb{C}_\infty((t)) : f := \sum_i a_i t^i \mapsto f^{(-1)} := \sum_i a_i^{\frac{1}{q^i}} t^i.
\]
Note that the twisting is extended to \( \text{Mat}_n(\mathbb{C}_\infty((t))) \) by acting entry-wisely.

We define \( \bar{k}[t, \sigma] \) to be the non-commutative \( \bar{k}[t] \)-algebra generated by \( \sigma \) with respect to the relation
\[
\sigma f = f^{(-1)} \sigma, \ \forall f \in \bar{k}[t].
\]
A left \( \bar{k}[t, \sigma] \)-module \( M \) is called a Frobenius module if it is free of finite rank over \( \bar{k}[t] \). Morphisms of Frobenius modules are left \( \bar{k}[t, \sigma] \)-module homomorphisms. We let \( \mathcal{F} \) be the category of Frobenius modules.

We denote by \( 1 \) the trivial object in \( \mathcal{F} \). Its underlying space is \( \bar{k}[t] \) subject to the \( \sigma \)-action
\[
\sigma(f) := f^{(-1)}, \ \forall f \in 1.
\]

Let \( \Phi \in \text{Mat}_r(\bar{k}[t]) \) be given. We say that a Frobenius module \( M \) is defined by the matrix \( \Phi \in \text{Mat}_r(\bar{k}[t]) \) if the Frobenius module \( M \) is of rank \( r \) over \( \bar{k}[t] \) with \( \bar{k}[t] \)-basis \( \{f_1, \ldots, f_r\} \subset M \) satisfying
\[
\sigma \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} = \Phi \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} = \begin{pmatrix} \sigma(f_1) \\ \vdots \\ \sigma(f_r) \end{pmatrix}.
\]
2.4. Ext\textsuperscript{1}-modules. Let $M, M'$ be two objects in $\mathcal{F}$ defined by two matrices $\Phi, \Phi'$ respectively for which $\Phi = \left(\begin{array}{c} \Phi' \\ v \\
 \end{array}\right) \in \text{Mat}_r(\bar{k}[t])$ $(r \geq 2)$ for some row vector $v$. Since $M$ fits into the short exact sequence of Frobenius modules

$$0 \to M' \to M \to 1 \to 0,$$

$M$ represents a class in $\text{Ext}_F^1(1, M')$.

The set $\text{Ext}_F^1(1, M')$ forms a group under the Baer sum. Furthermore, it has an $F_q[t]$-module structure given by the following. Given $M_1, M_2$ representing classes in $\text{Ext}_F^1(1, M')$ defined by two matrices $\Phi_1, \Phi_2 \in \text{Mat}_r(\bar{k}[t])$ respectively, we write

$$\Phi_1 = \left(\begin{array}{c} \Phi' \\ v_1 \\
 \end{array}\right), \Phi_2 = \left(\begin{array}{c} \Phi' \\ v_2 \\
 \end{array}\right).$$

Then the Baer sum of the two classes of $M_1, M_2$ is the class of the object $M_1 + M_2 \in \mathcal{F}$ defined by the matrix

$$\left(\begin{array}{c} \Phi' \\ a_{v_1} \\
 \end{array}\right) \in \text{Mat}_r(\bar{k}[t]).$$

Given any $a \in F_q[t]$, the action of $a \in F_q[t]$ on the class of $M_1$ is the class of the object $a * M_1 \in \mathcal{F}$ defined by the matrix

$$\left(\begin{array}{c} \Phi' \\ a_{v_1} \\
 \end{array}\right) \in \text{Mat}_r(\bar{k}[t]).$$

2.5. The Main Theorem.

Definition 2.3. We fix positive integers $r, s \in \mathbb{N}$ and let $n := r + s$ be given. An $n$-tuple $\mathbf{c} = (b_0, a_1, \ldots, a_{n-1}) \in k^n$ is said to have the SR-property if the following equation holds:

(SR) \hfill $\begin{align*}
\zeta_A(r)\zeta_A(s) - \zeta_A(r, s) - \zeta_A(s, r) &= b_0\zeta_A(n) + \sum_{i=1}^{n-1} a_i\zeta_A(i, n-i).
\end{align*}$

Remark 2.4. The equations of the above form, called shuffle relations by Thakur, were first studied by Thakur in [Tha10], and he proved the existence of $n$-tuples in $F_p^n \subset k^n$ having the SR-property in the same paper. Chen [Che15] gave an explicit $n$-tuple in $F_p^n \subset k^n$ having the SR-property.

Definition 2.5. Given $\mathbf{c} = (b_0, a_1, \ldots, a_{n-1}) \in k^n$ and letting $\Gamma_\mathbf{c} \in A$ be the monic least common multiple of the denominators of $\{\frac{a_i}{\Gamma_{n-i}} : i = 1, \ldots, n-1\}$, we put

$$\alpha_i := \frac{a_i\Gamma_\mathbf{c}}{\Gamma_i\Gamma_{n-i}} |_{\theta^t} \in F_q[t], \beta_0 := \frac{b_0\Gamma_\mathbf{c}}{\Gamma_n} |_{\theta^t} \in F_q[t], \gamma_0 := \frac{\Gamma_\mathbf{c}}{\Gamma_i\Gamma_j} |_{\theta^t} \in F_q[t]$$

for each $i$. Then we define the associated Frobenius module $M_\mathbf{c}$ of $\mathbf{c}$ which is defined by the matrix $\Phi_\mathbf{c} \in \text{Mat}_{n+1}(\bar{k}[t])$:

$$\Phi_\mathbf{c} = \left(\begin{array}{cccc}
(t-\theta)^n & (t-\theta)^{n-1}
H_{(n-1)}(t-\theta)^n & \cdots
\vdots
H_{(n-1)-1}(t-\theta)^n & \cdots
\alpha_1 H_{(n-1)-1}(t-\theta)^{n-1} & \cdots & \alpha_{n-1} H_{n-1}(t-\theta)^{n-1}
\Phi_{\mathbf{c}(n+1),1}
\end{array}\right).$$
where
\[ \Phi_{\epsilon(n+1),1} := \beta_0 H_{n-1}^{(-1)}(t - \theta)^n - \gamma_0 H_{r-1}^{(-1)} H_{s-1}^{(-1)}(t - \theta)^n. \]

Let \( \Phi' \in \text{Mat}_n(\overline{k}[t]) \) be the square matrix of size \( n \) in the upper left-hand corner of \( \Phi_{\epsilon} \), i.e.
\begin{equation}
(2.1) \quad \Phi' = \begin{pmatrix}
(t - \theta)^n \\
H_{1-1}^{(-1)}(t - \theta)^n & (t - \theta)^{n-1} \\
\vdots \\
H_{n-1}^{(-1)}(t - \theta)^n & (t - \theta)
\end{pmatrix},
\end{equation}
and let \( M' \) be the Frobenius module defined by \( \Phi' \). Note that the Frobenius module \( M_\epsilon \) represents a class in \( \text{Ext}_{\mathfrak{F}}^1(1, M') \).

If the \( n \)-tuple \( \mathcal{C} \) has the \text{SR}-property, we show that the corresponding Frobenius module \( M_\epsilon \) represents an \( \mathbb{F}_q[t] \)-torsion class in \( \text{Ext}_{\mathfrak{F}}^1(1, M') \). Conversely, if \( M_\epsilon \) represents an \( \mathbb{F}_q[t] \)-torsion class in \( \text{Ext}_{\mathfrak{F}}^1(1, M') \), it is natural to ask if the \( n \)-tuple \( \mathcal{C} \) has the \text{SR}-property.

Our main result is stated as follows.

**Theorem 2.6.** Let \( \mathcal{C} = (b_0, a_1, \ldots, a_{n-1}) \in k^n \) be given.

1. If the \( n \)-tuple \( \mathcal{C} \) has the \text{SR}-property, then the Frobenius module \( M_\epsilon \) represents an \( \mathbb{F}_q[t] \)-torsion class in \( \text{Ext}_{\mathfrak{F}}^1(1, M') \).
2. Suppose that the Frobenius module \( M_\epsilon \) represents an \( \mathbb{F}_q[t] \)-torsion class in \( \text{Ext}_{\mathfrak{F}}^1(1, M') \).
   (a) If \( (q - 1) \nmid n \), then the \( n \)-tuple \( \mathcal{C} \) has the \text{SR}-property.
   (b) If \( (q - 1) \mid n \), then there exists unique \( \tilde{b}_0 \in k \) such that the \( n \)-tuple \( \tilde{\mathcal{C}} = (\tilde{b}_0, a_1, \ldots, a_{n-1}) \) has the \text{SR}-property.

### 3. Proof of the Theorem 2.6

#### 3.1. Some important properties.

**Definition 3.1** ([ABP04]). A formal power series \( \sum_{n=0}^{\infty} a_n t^n \in \overline{k}[t] \) is called *entire* if
\[ \lim_{n \to \infty} \sqrt[n]{|a_n|} = 0, \quad \text{and} \quad [k_\infty(a_0, a_1, \ldots) : k_\infty] < \infty. \]

The set of all entire functions is denoted by \( \mathcal{E} \).

We fix a fundamental period \( \tilde{\pi} \) of the Carlitz module \( \mathcal{C} \) (see [Gos96 Tha04]), and define
\[ \Omega(t) := (-\theta)^{\tilde{\pi}^{-1}} \prod_{i=1}^{\tilde{\pi}} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \mathbb{C}_\infty[t], \]
where \( (-\theta)^{\tilde{\pi}^{-1}} \) is a choice of \( (q - 1) \)th root of \( -\theta \) so that \( \frac{1}{\Omega(\theta)} = \tilde{\pi} \). Note that the power series is entire and we have the functional equation \( \Omega^{(-1)}(t) = (t - \theta)\Omega(t) \) (see [ABP04]).

The following are important properties developed by Anderson and Thakur (see [AT90 AT09]):
\begin{equation}
(3.1) \quad (\Omega^d H_{s-1})^{(d)}(\theta) = \frac{\Gamma_{s-d}(s)}{\tilde{\pi}^s}, \quad \forall s \in \mathbb{N}, \ d \in \mathbb{Z}_{\geq 0},
\end{equation}
where \( S_d(s) \) is the power sum

\[
S_d(s) := \sum_{a \in A, \deg_a a = d} \frac{1}{a^s} \in k.
\]

Furthermore, if we view \( H_n \) as a polynomial in \( F_q[t][\theta] \), then we also have

\[
\deg_a H_n \leq \frac{nq}{q-1}.
\]

Given an \( r \)-tuple of positive integers \( s = (s_1, \ldots, s_r) \) and let \( \Omega \) be the \( r \)-tuple of Anderson-Thakur polynomials \( \Omega := (H_{s_1-1}, \ldots, H_{s_r-1}) \), we define the series

\[
\mathcal{L}_{s, \Omega} := \sum_{i_1 > \cdots > i_r \geq 0} (\Omega^{(i_r)} H_{s_r-1})^{(i_r)} \cdots (\Omega^{(i_1)} H_{s_1-1})^{(i_1)} \in \mathbb{C}_\infty[[t]] \text{ (cf. [AT09]).}
\]

Since \( \Omega \) satisfies the functional equation \( \Omega(t^q) = \frac{\Omega(t)}{t-\theta} \), we have

\[
\mathcal{L}_{s, \Omega} = \Omega^{s_1 + \cdots + s_r} \sum_{i_1 > \cdots > i_r \geq 0} \frac{H_{s_r-1}^{(i_r)}(t) \cdots H_{s_1-1}^{(i_1)}(t)}{[(t - \theta t^q) \cdots (t - \theta t^{q^r})]^s_r \cdots [(t - \theta t^q) \cdots (t - \theta t^{q^1})]^s_1}.
\]

By (3.2), the series \( \mathcal{L}_{s, \Omega} \) is in the Tate algebra \( T \), where

\[
T := \{ f \in \mathbb{C}_\infty[[t]] : f \text{ converges on } |t|_\infty \leq 1 \}.
\]

For \( 1 \leq \ell < j \leq r + 1 \), we define the series

\[
\mathcal{L}_{j, \ell} := \sum_{i_r > \cdots > i_1 \geq 0} (\Omega^{(i_r)} H_{s_r-1})^{(i_r)} \cdots (\Omega^{(i_1)} H_{s_1-1})^{(i_1)} \in \mathbb{C}_\infty[[t]].
\]

Note that we have \( \mathcal{L}_{s, \Omega} = \mathcal{L}_{r+1,1} \), and (3.1) gives

\[
\mathcal{L}_{r+1,1}(\theta) = \pi^{-\sum s_i} \Gamma_{s_1} \cdots \Gamma_{s_r} \zeta_A(s_1, \ldots, s_r).
\]

**Remark 3.2.** We also have the following properties:

1. Chang [Cha14, Lem. 5.3.1] showed that \( \mathcal{L}_{j, \ell} \) is actually an entire function for all \( \ell, j \) with \( 1 \leq \ell < j \leq r + 1 \).
2. By [CPY19, Prop. 2.3.3], we have for \( 1 \leq \ell < j \leq r + 1 \),

\[
\mathcal{L}_{j, \ell}(\theta t^N) = \mathcal{L}_{j, \ell}(\theta) t^N \quad \text{for all } N \in \mathbb{N}.
\]
3. The equation (3.1) and Remark 2.2 give that \( \mathcal{L}_{j, \ell} \) is non-vanishing at \( \theta t^N \) for all \( \ell, j \) with \( 1 \leq \ell < j \leq r + 1 \), \( N \in \mathbb{Z}_{\geq 0} \).

3.2. **A key lemma.**

**Lemma 3.3.** Let \( \{s_i\}_{i=1}^I \subseteq \mathbb{Z}_{\geq 0} \) be a strictly increasing finite sequence, and let \( \{L_i\}_{i=1}^I \subseteq T \) satisfying

\[
L_i(\theta t^N) \neq 0
\]

for all \( N \in \mathbb{N} \cup \{0\}, i \in \{1, \ldots, I\} \) be given. For any \( \{B_i\}_{i=1}^I \subseteq \bar{k}(t) \) satisfying

\[
\sum_{i=1}^I B_i \Omega^{s_i} L_i = 0,
\]

we have \( B_i = 0 \) for all \( i \in \{1, \ldots, I\} \).
Proof. (cf. the proof of [CPY19 Thm. 2.5.2], [Cha16 Thm. 3.1.1]) First, we divide the equation (3.3) by $\Omega^{s_1}$. Then it becomes

\begin{equation}
B_1L_1 + \sum_{i=2}^f B_i\Omega^{\tilde{s}_i}L_i = 0
\end{equation}

where $\tilde{s}_i = s_i - s_1$. Note that each $B_i$ is defined at $\theta^q$ for sufficiently large $N \in \mathbb{N}$ since $B_i$ belongs to $\bar{k}(t)$. Also note that $\Omega$ has a simple zero at $\theta^q$ for each $N \in \mathbb{N}$, and hence (3.4) gives rise to

\[B_1(\theta^q)L_1(\theta^q) = 0\]

for sufficiently large $N \in \mathbb{N}$. By the assumption $L_1(\theta^q) \neq 0$ for all $N \in \mathbb{N}$, which implies that

\[B_1(\theta^q) = 0\]

for all large $N \in \mathbb{N}$, whence $B_1 = 0$. The equation (3.3) becomes

\[\sum_{i=2}^f B_i\Omega^{\tilde{s}_i}L_i = 0.\]

Since $\{\tilde{s}_i\}_{i=2}^f$ is also a strictly increasing finite sequence, we repeat the same process above and then conclude that $B_i = 0$ for all $i \in \{1, \ldots, f\}$.

3.3. Proof of the Theorem 2.6(1). (cf. the proof of [CPY19 Thm. 2.5.2], [Cha16 Thm. 3.1.1])

For $i = 1, \ldots, n - 1$, we define two series as follows:

\[
\mathcal{L}^{[i,n-i]} := \sum_{\ell_1 > \ell_2 \geq 0} (\Omega^{n-i}H_{(n-i)-1}(\ell_2))^{(\ell_1)}(\Omega^iH_{i-1})^{(\ell_1)},
\]

\[
\mathcal{L}^{[i]} := \sum_{\ell \geq 0} (\Omega^iH_{i-1})^{(\ell)}.
\]

Let $\psi_\epsilon \in \text{Mat}_{(n+1) \times 1}(\mathcal{E})$ be defined by

\[
\psi_\epsilon = \begin{pmatrix}
\Omega^n \\
\Omega^{n-1}\mathcal{L}^{[1]} \\
\vdots \\
\Omega \mathcal{L}^{[n-1]} \\
\beta_0\mathcal{L}^{[n]} + \sum_{i=1}^{n-1} \alpha_i \mathcal{L}^{[i,n-i]} - \gamma_0(\mathcal{L}^{[r]}\mathcal{L}^{[s]} - \mathcal{L}^{[r,s]} - \mathcal{L}^{[s,r]})
\end{pmatrix}.
\]

Then we have $\psi_\epsilon^{(-1)} = \Phi_\epsilon \psi_\epsilon$ and

\[
\psi_\epsilon(\theta) = \tilde{\pi}^{-n}
\begin{pmatrix}
1 \\
\Gamma_1 \zeta_A(1) \\
\vdots \\
\Gamma_{n-1} \zeta_A(n-1) \\
\Gamma_\epsilon[b_0 \zeta_A(n) + \sum_{i=1}^{n-1} a_i \zeta_A(i,n-i) - (\zeta_A(r)\zeta_A(s) - \zeta_A(r,s) - \zeta_A(s,r))]
\end{pmatrix}
\]

is in $\text{Mat}_{(n+1) \times 1}(\bar{k}_\infty)$. 

\[\square\]
Let $v = (0, \ldots, 0, 1) \in \text{Mat}_{1 \times (n+1)}(\overline{k})$. Then the the SR property gives $v \psi_{\mathcal{E}}(\theta) = 0$. Note that the above satisfies the assumption of \cite[Thm. 3.1.1]{ABP04}. By \cite[Thm. 3.1.1]{ABP04}, there exists

$$f = (f_1, \ldots, f_{n+1}) \in \text{Mat}_{1 \times (n+1)}(\overline{k}[t])$$

such that $f \psi_{\mathcal{E}} = 0$ and $f(\theta) = v$. Now we put $\overline{f} = \frac{1}{f_{n+1}} f \in \text{Mat}_{1 \times (n+1)}(\overline{k}(t))$ and note that $\overline{f}$ is regular at $t = \theta$. We claim that

$$(3.5) \quad \overline{f} - \overline{f}(-1) \Phi_{\mathcal{E}} = (0, \ldots, 0).$$

Let us assume this claim first. Then we have the following equation

$$(3.6) \quad \begin{pmatrix} 1 & \cdots & 1 \\ \frac{f_1}{f_{n+1}} & \cdots & \frac{f_n}{f_{n+1}} & 1 \\ \end{pmatrix}^{(-1)} \Phi_{\mathcal{E}} = \begin{pmatrix} \Phi' \\ 1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \\ \frac{f_1}{f_{n+1}} & \cdots & \frac{f_n}{f_{n+1}} & 1 \\ \end{pmatrix}.$$  

The equation (3.6) gives a left $\overline{k}(t)[\sigma]$-module homomorphism between $\overline{k}(t) \otimes (M' \oplus 1)$ and $\overline{k}(t) \otimes M_{\mathcal{E}}$. By \cite[Prop. 2.2.1]{CPY19}, the common denominator of $\frac{f_1}{f_{n+1}}, \ldots, \frac{f_n}{f_{n+1}}$, say $c$, is in $\mathbb{F}_q[t]$. Write $\Phi_{\mathcal{E}} = \begin{pmatrix} \phi' \\ \mu \end{pmatrix}$ for some $\mu \in \text{Mat}_{1 \times (n+1)}(\overline{k}[t])$, and put $c \ast \Phi_{\mathcal{E}} := \begin{pmatrix} \phi' \\ c \mu \end{pmatrix}$, which defines the Frobenius module $c \ast M_{\mathcal{E}}$ representing a class in $\text{Ext}^1_{\mathcal{E}}(1, M')$. The class $c \ast M_{\mathcal{E}}$ represents the trivial class in $\text{Ext}^1_{\mathcal{E}}(1, M')$ since we have the equation

$$(3.7) \quad \begin{pmatrix} 1 & \cdots & 1 \\ \delta_1 & \cdots & \delta_{n+1} & 1 \\ \end{pmatrix}^{(-1)} c \ast \Phi_{\mathcal{E}} = \begin{pmatrix} \phi' \\ 1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \\ \delta_1 & \cdots & \delta_{n+1} & 1 \\ \end{pmatrix},$$

where $\delta_i = c \frac{f_i}{f_{n+1}} \in \overline{k}[t]$ for $i = 1, \ldots, n + 1$, i.e., the class of $M_{\mathcal{E}}$ is $c$-torsion in the $\mathbb{F}_q[t]$-module $\text{Ext}^1_{\mathcal{E}}(1, M')$.

To complete the proof, we need to verify the equation (3.5). Applying the Frobenius twisting $(\cdot)^{-1}$ on the equation $\overline{f} \psi_{\mathcal{E}} = 0$ and subtracting it from the equation $\overline{f} \psi_{\mathcal{E}} = 0$, we have

$$\left( \overline{f} - \overline{f}(-1) \Phi_{\mathcal{E}} \right) \psi_{\mathcal{E}} = 0.$$ 

Let $(B_1, \ldots, B_{n+1}) := \overline{f} - \overline{f}(-1) \Phi$. Note that $B_{n+1} = 0$ and the above equation becomes

$$(3.8) \quad \sum_{i=1}^{n} B_i \Omega^{n-i+1} \mathcal{L}^{[i-1]} = 0,$$

where we define $\mathcal{L}^{[0]} = 1$ for convenience. By Remark \cite[3.2]{3.3}, the equation (3.8) satisfies the hypothesis of Lemma \cite[3.3]{3.3} It follows by Lemma \cite[3.3]{3.3} that $B_i = 0$ for all $i = 1, \ldots, n + 1$, and so we complete the proof.
3.4. **Proof of the Theorem 2.6(2).** (cf. the proof of [CPY19 Thm. 2.5.2])

Suppose the Frobenius module $M_{c}$ represents an $\mathbb{F}_{q}[t]$-torsion class in $\text{Ext}^{1}_{\mathcal{H}}(1, M')$, i.e., there exists $c \in \mathbb{F}_{q}[t] \setminus \{0\}$ such that $c \ast M_{c}$ represents a trivial class in $\text{Ext}^{1}_{\mathcal{H}}(1, M')$.

Note that $c \ast M_{c}$ is defined by the matrix $X$ given as follows:

\[
X = \begin{pmatrix}
(t - \theta)^{n} & (t - \theta)^{n-1} & \cdots & 1 \\
H_{1,1}^{(-1)}(t - \theta)^{n} & (t - \theta)^{n-1} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
H_{n,n}^{(-1)}(t - \theta)^{n} & \cdots & \cdots & (t - \theta) \\
X_{n+1,1} & c_{1}H_{n,n}^{(-1)}(t - \theta)^{n-1} & \cdots & c_{n-1}H_{1,1}^{(-1)}(t - \theta)^{1}
\end{pmatrix}
\]

where

\[
X_{n+1,1} = c \left[ \beta^{0}H_{n,n}^{(-1)}(t - \theta)^{n} - \gamma^{0}H_{1,1}^{(-1)}(t - \theta)^{n} \right],
\]

and there exists $\delta_{1}, \ldots, \delta_{n} \in \bar{k}[t]$ such that

\[
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}^{(-1)} = \begin{pmatrix}
\Phi' \\
\Psi'
\end{pmatrix}.
\]

Consider

\[
Y = \begin{pmatrix}
\Omega^{n} \\
\Omega^{n-1} \mathcal{L}^{[1]} \\
\Omega^{n-2} \mathcal{L}^{[2]} \\
\vdots \\
\Omega^{1} \mathcal{L}^{[n-1]} \\
Y_{n+1,1} & c \sum_{i=1}^{n-1} \alpha_{i} \mathcal{L}^{[n-i]} & c \sum_{i=2}^{n-1} \alpha_{i} \mathcal{L}^{[n-i]} & \cdots & c \alpha_{n-1} \mathcal{L}^{[1]}
\end{pmatrix}
\]

where

\[
Y_{n+1,1} = c \left[ \beta^{0} \mathcal{L}^{[n]} + \sum_{i=1}^{n-1} \alpha_{i} \mathcal{L}^{[i,n-i]} - \gamma^{0}(\mathcal{L}^{[r]} \mathcal{L}^{[s]} - \mathcal{L}^{[r]} - \mathcal{L}^{[s]}) \right],
\]

then we have the relation $Y^{(-1)} = XY$. Putting $\mathcal{D} = (\delta_{1}, \ldots, \delta_{n})$ and $Y' = \begin{pmatrix} I_{n} \\ \mathcal{D} \end{pmatrix} Y$, we have $Y'^{(-1)} = \begin{pmatrix} \Phi' \\ 1 \end{pmatrix} Y'$. Let $\Psi'$ be the square matrix of size $n$ cut from the upper left square of $Y'$. Since we also have $\begin{pmatrix} \Psi' \\ 1 \end{pmatrix}^{(-1)} = \begin{pmatrix} \Phi' \\ 1 \end{pmatrix} \begin{pmatrix} \Psi' \\ 1 \end{pmatrix}$, by [Pap08 §4.1.6], there exists $\nu = (\nu_{1}, \ldots, \nu_{n}) \in \mathbb{F}_{q}(t)^{n}$ such that

\[
Y' = \begin{pmatrix} \Psi' \\ 1 \end{pmatrix} \begin{pmatrix} I_{n} \\ \nu \end{pmatrix},
\]

i.e.,

\[
\begin{pmatrix} I_{n} \\ \mathcal{D} \end{pmatrix} Y = \begin{pmatrix} \Psi' \\ 1 \end{pmatrix} \begin{pmatrix} I_{n} \\ \nu \end{pmatrix}.
\]
Therefore, we have
\[
\nu_1 = \sum_{i=1}^{n} \delta_i \Omega^{n-i+1} \mathcal{L}^i + c \left[ \beta_0 \mathcal{L}^n + \sum_{i=1}^{n-1} \alpha_i \mathcal{L}^{i,n-i} - \gamma_0 (\mathcal{L}^r \mathcal{L}^s - \mathcal{L}^r - \mathcal{L}^s) \right];
\]
\[
\nu_2 = \sum_{i=2}^{n} \delta_i \Omega^{n-i+1} + c \sum_{i=1}^{n-1} \alpha_i \mathcal{L}^{n-i};
\]
\[
\vdots
\]
\[
\nu_n = \sum_{i=n}^{n} \delta_i \Omega^{n-i+1} + c \sum_{i=n-1}^{n-1} \alpha_i \mathcal{L}^{n-i} (= \delta_n \Omega + c \alpha_{n-1} \mathcal{L}^{[1]});
\]
\[
\nu_{n+1} = \delta_{n+1}.
\]

We find that each \( \nu_i \) is in \( \mathbb{F}_q[t] \) since the right hand side of each equality above is in \( \mathbb{T} \). Now we evaluate \( t = \theta^q \) in each equation above. Note that we work in fields with characteristic \( p \) and \( \Omega \) has a simple zero at \( \theta^q \). So by Remark 3.2\( \text{[2]} \), we get
\[
(3.9)
\]
\[
\nu_1(\theta)^q = (c(\theta) \Gamma c)^q \pi^{-nq} \left[ b_0 \zeta_A(n) + \sum_{i=1}^{n-1} a_i \pi^{-(n-i)} \zeta_A(n - i) - (\zeta_A(r) \zeta_A(s) - \zeta_A(s, r) - \zeta_A(s)) \right]^q;
\]
\[
\nu_2(\theta)^q = \left[ c(\theta) \sum_{i=1}^{n-1} a_i \pi^{-(n-i)} \zeta_A(n - i) \right]^q;
\]
\[
\vdots
\]
\[
\nu_n(\theta)^q = \left[ c(\theta) \sum_{i=n-1}^{n-1} a_i \pi^{-(n-i)} \zeta_A(n - i) \right]^q = \left[ c(\theta) a_{n-1} \pi^{-1} \zeta_A(1) \right]^q).
\]

Taking the \( q \)th root of the both sides for each equation, we have
\[
(3.10)
\]
\[
\nu_1(\theta) = c(\theta) \Gamma c \pi^{-n} \left[ b_0 \zeta_A(n) + \sum_{i=1}^{n-1} a_i \zeta_A(i, n - i) - (\zeta_A(r) \zeta_A(s) - \zeta_A(s, r) - \zeta_A(s)) \right],
\]
and
\[
\nu_2(\theta) = c(\theta) \sum_{i=1}^{n-1} a_i \pi^{-(n-i)} \zeta_A(n - i);
\]
\[
\vdots
\]
\[
\nu_n(\theta) = c(\theta) \sum_{i=n-1}^{n-1} a_i \pi^{-(n-i)} \zeta_A(n - i) (= c(\theta) a_{n-1} \pi^{-1} \zeta_A(1));
\]
\[
\nu_{n+1}(\theta) = \delta_{n+1}(\theta).
\]
From (3.10), we have

\begin{equation}
(3.11) \quad \frac{\nu_1(\theta)}{c(\theta)\Gamma_{\mathcal{E}}} \tilde{\pi}^n = b_0 \zeta_A(n) + \sum_{i=1}^{n-1} a_i \zeta_A(i, n - i) - (\zeta_A(r) \zeta_A(s) - \zeta_A(r, s) - \zeta_A(s, r)).
\end{equation}

Note that if \((q - 1) \nmid n\), then \(\tilde{\pi}^n \notin k_{\infty}\). Since both \(\nu_1(\theta)\Gamma_{\mathcal{E}}\) and the right hand side of (3.11) are in \(k_{\infty}\), we conclude that

\[ b_0 \zeta_A(n) + \sum_{i=1}^{n-1} a_i \zeta_A(i, n - i) - (\zeta_A(r) \zeta_A(s) - \zeta_A(r, s) - \zeta_A(s, r)) = 0. \]

i.e.,

\[ \zeta_A(r) \zeta_A(s) - \zeta_A(r, s) - \zeta_A(s, r) = b_0 \zeta_A(n) + \sum_{i=1}^{n-1} a_i \zeta_A(i, n - i). \]

If \((q - 1) \mid n\), then by [Car35] we know that

\[ \frac{\nu_1(\theta)}{c(\theta)\Gamma_{\mathcal{E}}} \tilde{\pi}^n = b \zeta_A(n) \text{ for some } b \in k. \]

We conclude that

\[ \zeta_A(r) \zeta_A(s) - \zeta_A(r, s) - \zeta_A(s, r) = b \zeta_A(n) + \sum_{i=1}^{n-1} a_i \zeta_A(i, n - i), \]

where \(\tilde{b} = b_0 - b\). Note that the uniqueness of \(\tilde{b}\) is simply a consequence of Remark 2.2.

4. A necessary condition for the \(\text{SR}\)-property

Suppose the \(n\)-tuple \(\mathcal{C} = (b_0, a_1, \ldots, a_{n-1}) \in k^n\) has the \(\text{SR}\)-property. Combining the shuffle relation with the relation (1.1) proved by Chen, we have a relation of the form

\[ b \zeta_A(n) + \sum_{i=1}^{n-1} a_i \zeta_A(i, n - i) = 0. \]

By [Chal16] Thm. 3.1.1, we derive the following necessary condition for shuffle relations.

**Theorem 4.1.** If an \(n\)-tuple \(\mathcal{C} = (b_0, a_1, \ldots, a_{n-1}) \in k^n\) has the \(\text{SR}\)-property, then we have

\[ a_i = 0 \text{ if } (q - 1) \nmid (n - i). \]

5. An effective criterion for the \(\mathbb{F}_q[t]\)-torsion property of \(M_\mathcal{E}\) in \(\operatorname{Ext}^1_{\mathcal{F}}(1, M')\)

In this section, we will provide an effective criterion whether \(M_\mathcal{E}\) is \(\mathbb{F}_q[t]\)-torsion in \(\operatorname{Ext}^1_{\mathcal{F}}(1, M')\). We follow the same idea in [OPY19, §5, §6].
5.1. **Anderson t-modules.** Let \( \tau : \mathbb{C}_\infty \to \mathbb{C}_\infty \) be the \( q \)-th power operator defined by \( x \mapsto x^q \) and let \( \mathbb{C}_\infty[\tau] \) be the twisted polynomial ring in \( \tau \) over \( \mathbb{C}_\infty \) subject to the relation \( \tau \alpha = \alpha^q \tau \) for \( \alpha \in \mathbb{C}_\infty \). We define \( t \)-modules as follows.

**Definition 5.1** ([And86]). Let \( d \in \mathbb{N} \) be given, a \( d \)-dimensional \( t \)-module is a pair \((E, \phi)\), where \( E \) is the \( d \)-dimensional algebraic group \( G_d \) and \( \phi \) is an \( \mathbb{F}_q[t] \)-linear ring homomorphism

\[
\phi : \mathbb{F}_q[t] \to \text{Mat}_d(\mathbb{C}_\infty[\tau])
\]

so that the image of \( t \), denoted by \( \phi_t \), is of the form \( \alpha_0 + \sum_i \alpha_i \tau^i \) with \( \alpha_i \in \text{Mat}_d(\mathbb{C}_\infty) \), and \( \alpha_0 - \theta I_d \) is a nilpotent matrix.

**Remark 5.2.** \( E(\mathbb{C}_\infty) \) is equipped with an \( \mathbb{F}_q[t] \)-module structure via the map \( \phi \).

For a subring \( R \) of \( \mathbb{C}_\infty \) containing \( A \), we say that the \( t \)-module \( E \) is defined over \( R \) if \( \alpha_i \) lies in \( \text{Mat}_d(R) \) for all \( i \geq 0 \).

We take the \( n \)-th tensor power of the Carlitz \( \mathbb{F}_q[t] \)-module as an example. Fixing a positive integer \( n \), the \( n \)-th tensor power of the Carlitz \( \mathbb{F}_q[t] \)-module denoted by \( \mathbb{C}^{\otimes n} \) is an \( n \)-dimensional \( t \)-module defined over \( A \) together with the \( \mathbb{F}_q \)-linear ring homomorphism

\[
[n] : \mathbb{F}_q[t] \to \text{Mat}_n(\mathbb{C}_\infty[\tau])
\]

given by

\[
[t]_n = \theta I_n + N_n + E_n \tau,
\]

where

\[
N_n := \begin{pmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0
\end{pmatrix},
E_n := \begin{pmatrix}
0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 0
\end{pmatrix}.
\]

5.2. **Identification of \( \text{Ext}^1_{\mathbb{F}_q}(1, M') \) and the Anderson \( t \)-module \( E' \).** By an Anderson \( t \)-motive we mean an object \( \mathcal{N} \) in \( \mathcal{F} \) satisfying the following properties.

1. \( N \) is a free left \( \bar{k}[\sigma] \)-module of finite rank.
2. \( (t - \theta)^n N \subseteq \sigma N \) for all sufficiently large integers \( n \).

**Remark 5.3.** We can check directly that \( M' \) is an Anderson \( t \)-motive.

Since \( M' \) is an Anderson \( t \)-motive, we can construct an associated Anderson \( t \)-module \((E', \rho)\) and have the following \( \mathbb{F}_q[t] \)-module isomorphisms established by Anderson

\[
\text{Ext}^1_{\mathbb{F}_q}(1, M') \cong M'/(\sigma - 1)M' \cong E'(\bar{k}).
\]

We state the details in the Theorem 5.4 and Theorem 5.5 which appeared in [CPY19].

**Theorem 5.4** ([CPY19 Thm. 5.2.1]). Let \( \{x_0, \ldots, x_{n-1}\} \) be a \( \bar{k}[t] \)-basis of \( M' \) on which the \( \sigma \)-action is presented by the matrix \( \Phi' \). Let \( M \in \text{Ext}^1_{\mathbb{F}_q}(1, M') \) be defined by the matrix

\[
\begin{pmatrix}
\Phi' \\
f_0, \ldots, f_{n-1}
\end{pmatrix}.
\]

Then the map

\[
\mu : \text{Ext}^1_{\mathbb{F}_q}(1, M') \to M'/(\sigma - 1)M'
\]

defined by

\[
\mu(M) := f_0x_0 + \cdots + f_{n-1}x_{n-1}
\]

is an isomorphism of \( \mathbb{F}_q[t] \)-modules.
We consider the $n$th tensor power of the Carlitz motive $C^\otimes n \in \mathcal{F}$. The underlying $\bar{k}[t]$-module of $C^\otimes n$ is $\bar{k}[t]$ subject to the $\sigma$-action

$$\sigma(f) := (t - \theta)^{n-1} f^{(-1)}, \ f \in C^\otimes n.$$  

Note that $M'$ fits into the short exact sequence of Frobenious modules

$$0 \longrightarrow C^\otimes n \longrightarrow M' \longrightarrow \bigoplus_{i=1}^{n-1} C^\otimes(n-i) \longrightarrow 0,$$

where the projection map is defined by $\sum_{i=0}^{n-1} f_i x_i \mapsto (f_1, \ldots, f_{n-1})$. As a left $\bar{k}[\sigma]$-module, $C^\otimes j$ is free of rank $j$ with the natural basis $\{(t - \theta)^{j-1}, \ldots, (t - \theta), 1\}$. Hence the set 

$$\{(t - \theta)^{n-1} x_0, \ldots, (t - \theta)x_0, x_0, \ldots, (t - \theta)x_{n-2}, x_{n-2}, x_{n-1}\},$$

denoted by $\{\nu_1, \ldots, \nu_d\}$, is a $\bar{k}[\sigma]$-basis of $M'$.

Define the homomorphism of $\mathbb{F}_q$-vector spaces $\Delta : M' \to \operatorname{Mat}_{d \times 1}(\bar{k})$ by

$$m = \sum_{i=1}^{d} u_i \nu_i \mapsto \Delta(m) := \begin{pmatrix} \delta(u_1) \\ \vdots \\ \delta(u_d) \end{pmatrix},$$

where

$$\delta \left( \sum_i \sigma^i c_i^q \right) = \sum_i c_i^q.$$

We note that the homomorphism $\Delta$ is surjective since

$$\Delta(a_1 \nu_1 + \cdots + a_d \nu_d) = \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix}$$

for $(a_1, \ldots, a_d)^{tr} \in \operatorname{Mat}_{d \times 1}(\bar{k})$. As $t(\sigma - 1)M' \subseteq (\sigma - 1)M'$, the map $\Delta$ induces an $\mathbb{F}_q[t]$-module structure on $\operatorname{Mat}_{d \times 1}(\bar{k})$. We denote by $(E', \rho)$ the $t$-module defined over $\bar{k}$ with $E'(\bar{k})$ identified with $\operatorname{Mat}_{d \times 1}(\bar{k})$ on which the $\mathbb{F}_q[t]$-module structure is given by

$$\rho : \mathbb{F}_q[t] \to \operatorname{Mat}_d(\bar{k}[t])$$

so that

$$\Delta(t(a_1 \nu_1 + \cdots + a_d \nu_d)) = \rho_t \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix}.$$  

**Theorem 5.5 ([CPY19] Thm. 5.2.3).** Let $M'$ be the Frobenius module defined by the matrix $\Phi'$. Let $(E', \rho)$ be the $t$-module whose $\bar{k}$-valued points are $E'(\bar{k})$ identified with $\operatorname{Mat}_{d \times 1}(\bar{k})$, which is equipped with the $\mathbb{F}_q[t]$-module structure via $\rho : \mathbb{F}_q[t] \to \operatorname{Mat}_d(\bar{k}[t])$ through the map $\Delta$ as above. Then we have the following isomorphism of $\mathbb{F}_q[t]$-modules

$$M'/(\sigma - 1)M' \cong E'(\bar{k}).$$

For example, we consider the $n$th tensor power of Carlitz motive $C^\otimes n$. As a left $\bar{k}[\sigma]$-module, $C^\otimes n$ is free of rank $n$ with basis $\{(t - \theta)^{n-1}, \ldots, (t - \theta), 1\}$. We let

$$\Delta_n : C^\otimes n \to \operatorname{Mat}_{n \times 1}(\bar{k})$$

be defined as above with respect to this basis. For \((a_1, \ldots, a_n)^{tr} \in \text{Mat}_{n \times 1}(\bar{k})\), we let
\[
f = a_1(t - \theta)^{n-1} + \cdots + a_{n-1}(t - \theta) + a_n,
\]
so that \(\Delta_n(f) = (a_1, \ldots, a_n)^{tr}\). We can check directly that the multiplication by \(t\) on \(\text{Mat}_{n \times 1}(\bar{k})\) is given by
\[
t \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \Delta_n(tf) = [t]_n \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.
\]
Hence we have the identification
\[
C^{\otimes n}/(\sigma - 1)C^{\otimes n} \cong C^{\otimes n}(\bar{k}).
\]

5.3. A criterion for the \(\mathbb{F}_q[t]\)-torsion property of \(M_\mathfrak{c}\) in \(\text{Ext}^1_{\mathcal{F}}(1, M')\). In this section, we provide a criterion for the \(\mathbb{F}_q[t]\)-torsion property of \(M_\mathfrak{c}\) in \(\text{Ext}^1_{\mathcal{F}}(1, M')\). The strategy of proof is to follow [CPY19] Theorem 6.1.1, and so we put most of the proofs in the Appendix [A].

**Theorem 5.6.** Let \((E', \rho)\) be the associated \(t\)-module given above. Given \(\mathfrak{c} \in k^n\), and let \(v_\mathfrak{c}\) be the integral point in \(E'(A)\) corresponding to \(M_\mathfrak{c}\) via isomorphisms described in the Theorem 5.4 and Theorem 5.3. For \((q - 1) \mid i\), we decompose
\[
i = p^i n_i (q^{h_i} - 1)
\]
such that \(p \nmid n_i\) and \(h_i\) is the greatest integer for which \((q^{h_i} - 1) \mid i\). Put
\[
a = \prod (t^{q^{h_i}} - t)^{p^i},
\]
where the product is taken over integers \(i\) from 1 to \(n\) which are multiples of \((q - 1)\). Then \(M_\mathfrak{c}\) is an \(\mathbb{F}_q[t]\)-torsion class in \(\text{Ext}^1_{\mathcal{F}}(1, M')\) if and only if \(\rho_a(v_\mathfrak{c}) = 0\).

**Proof.** It is clear that \(\rho_a(v_\mathfrak{c}) = 0\) implies that \(M_\mathfrak{c}\) is an \(\mathbb{F}_q[t]\)-torsion class in \(\text{Ext}^1_{\mathcal{F}}(1, M')\). Now we suppose that \(M_\mathfrak{c}\) is an \(\mathbb{F}_q[t]\)-torsion class in \(\text{Ext}^1_{\mathcal{F}}(1, M')\). First, we follow the method in [Cha16, P. 307] to derive the short exact sequence of \(\mathbb{F}_q[t]\)-modules
\[
0 \rightarrow C^{\otimes n}/(\sigma - 1)C^{\otimes n} \rightarrow M'/(\sigma - 1)M' \rightarrow \bigoplus_{i=1}^{n-1} C^{\otimes (n-i)}/(\sigma - 1)C^{\otimes (n-i)} \rightarrow 0.
\]
Note that \(M'\) fits into the short exact sequence of Frobenious modules
\[
0 \longrightarrow C^{\otimes n} \rightarrow M' \rightarrow \bigoplus_{i=1}^{n-1} C^{\otimes (n-i)} \longrightarrow 0,
\]
where the projection map is defined by \(\sum_{i=0}^{n-1} f_i x_i \mapsto (f_1, \ldots, f_{n-1})\). Note also that the \(\mathbb{F}_q[t]\)-linear map \(\sigma - 1\) from \(\bigoplus_{i=1}^{n-1} C^{\otimes (n-i)}\) to itself is injective. By the Snake Lemma, we have our desired short exact sequence of \(\mathbb{F}_q[t]\)-modules.

By the Theorem 5.3 and previous isomorphisms of \(\mathbb{F}_q[t]\)-modules, we have
\[
0 \rightarrow C^{\otimes n}(\bar{k}) \rightarrow E'(\bar{k}) \rightarrow \bigoplus_{i=1}^{n-1} C^{\otimes i}(\bar{k}) \rightarrow 0.
\]
Let $\pi$ denote the surjective map. By the Theorem A.1, $v_e$ is an integral point, and then so is $\pi(v_e)$. In fact, we have $\pi(v_e) \in \bigoplus_{i=1}^{n-1} C^{\otimes i}(k)^{\text{tor}}$ since $M_\epsilon$ is $F_q[t]$-torsion by our assumption. By AT90, Prop. 1.11.2 and CPY19, Lem. 5.1.3, the polynomial \[ b := \prod_{i \in \{1, \ldots, n\} \atop (q-1)i} (p^{hi} - t)^{p^{ei}} \in \mathbb{F}_q[t] \] annihilates $\pi(v_e)$. Hence $\rho_b(v_e) \in \ker \pi \cong C^{\otimes n}(\bar{k})$. By the Theorem A.1 again, $E'$ is defined over $A$. $\rho_b(v_e)$ is also an integral point. Hence $\rho_b(v_e) \in C^{\otimes n}(k)^{\text{tor}}$. By AT90, Prop. 1.11.2 and CPY19, Lem. 5.1.3 again, $\rho_b(v_e)$ is annihilated by \[ (p^{h_n} - t)^{p^{e_n}} \in \mathbb{F}_q[t] \] if $(q-1) \mid n$, otherwise $\rho_b(v_e) = 0$. Therefore $\rho_a(v_e) = 0$. \hfill \Box

6. Algorithm and computational results

6.1. Algorithm. In this section we provide an algorithm to determine, for the given $n$-tuple of coefficients $C = (b_0, a_1, \ldots, a_{n-1}) \in k^n$, whether $M_\epsilon$ is $F_q[t]$-torsion in $\text{Ext}_{\mathbb{F}_q}(1, M')$ or not. (cf. CPY19 §6)

INPUT: $r, s \in \mathbb{N}, n := r + s$, $p$ : a prime, $q$ : a power of $p$, $C = (b_0, a_1, \ldots, a_{n-1}) \in k^n$.

STEP 1. Compute the Anderson-Thakur polynomials $H_0, \ldots, H_{n-1}$ and the polynomial $a$ as in the Theorem 5.6.

STEP 2. Let $M'$ be the Frobenius module defined by $\Phi'$ as in (2.1) with $\bar{k}[t]$-basis $\{x_0, \ldots, x_{n-1}\}$. Let $\{\nu_1, \ldots, \nu_d\}$ be the $\bar{k}[\sigma]$-basis of $M'$ given by \[ \{(t - \theta)^{n-1}x_0, \ldots, (t - \theta)x_0, x_0, \ldots, (t - \theta)x_{n-2}, x_{n-2}, x_{n-1}\}. \]

Identify $M' / (\sigma - 1)M'$ with $\text{Mat}_{d \times 1}(\bar{k})$ via $\{\nu_1, \ldots, \nu_d\}$.

STEP 3. Compute $\beta_0, \alpha_i$ for $i = 1, \ldots, n - 1$ as in Definition 2.5. Consider \[ \left[ \beta_0 H_{n-1}^{(-1)}(t - \theta)^n - c_0 H_{r-1}^{(-1)} H_{n-1}^{(-1)}(t - \theta)^n \right] x_0 + \sum_{i=1}^{n-1} \left( \alpha_i H_{(n-i)-1}^{(-1)}(t - \theta)^{n-i} \right) x_i \]

in $M' / (\sigma - 1)M'$ and multiply it by the polynomial $a$. Write it as the form \[ \sum_{i=1}^d u_i \nu_i \]

which corresponds to the integral point $\rho_a(v_e) = (\delta(u_1), \ldots, \delta(u_d))^{tr} \in E'(A)$ via the $\Delta$ map described in the Theorem 5.5.

OUTPUT: If $\rho_a(v_e)$ is zero, then $M_\epsilon$ is an $F_q[t]$-torsion class in $\text{Ext}_{\mathbb{F}_q}(1, M')$; otherwise, $M_\epsilon$ is not an $F_q[t]$-torsion class in $\text{Ext}_{\mathbb{F}_q}(1, M')$.

6.2. Examples. We use Maple to write a program based on the algorithm. By a “doable” $n$-tuple $C = (b_0, a_1, \ldots, a_{n-1}) \in k^n$, we mean the computation for determining whether $M_\epsilon$ is $F_q[t]$-torsion in $\text{Ext}_{\mathbb{F}_q}(1, M')$ or not can be done within about 10 minutes.

We recheck Chen’s formula (1.1) and search for $C \in k^n - F_p^n$ with the SR-property by the Maple program.
(1) For \( p = q < 30, r, s < 200 \), the computation shows that \( M_\mathcal{C} \) is an \( \mathbb{F}_q[t] \)-torsion class in \( \text{Ext}^1_{\mathcal{D}}(1, M') \) where \( \mathcal{C} \in \mathbb{F}_p^n \) is a “doable” \( n \)-tuple of coefficients coming from the Chen’s formula \((1.1)\). This matches the result of Theorem 2.6(1).

(2) Let \( p = q = 3 \) be given. For \( r, s < 4 \) such that \((q - 1) \nmid n\), “doable” \( n \)-tuples \( \mathcal{C} = (b_0, a_1, \ldots, a_{n-1}) \in k^n - \mathbb{F}_p^n \) with degrees of numerators and denominators of \( b_0, a_i \) less than 4, we list some data below for \( r, s \) and the \( n \)-tuples \( \mathcal{C} \) when \( M_\mathcal{C} \) is \( \mathbb{F}_q[t] \)-torsion in \( \text{Ext}^1_{\mathcal{D}}(1, M') \):

\[
\begin{align*}
(r, s) &= (1, 2), \mathcal{C} = (2, \theta^3 + 2\theta, 0) \in A^3 - \mathbb{F}_3^3, \\
(r, s) &= (1, 2), \mathcal{C} = (0, 2\theta^3 + \theta, 0) \in A^3 - \mathbb{F}_3^3, \\
(r, s) &= (2, 3), \mathcal{C} = (0, 2\theta^3 + \theta, 0, 2, 0) \in A^5 - \mathbb{F}_3^5, \\
(r, s) &= (1, 2), \mathcal{C} = \left(\frac{\theta^3 + 2\theta + 2}{\theta^3 + 2\theta}, 2, 0\right) \in k^3 - A^3, \\
(r, s) &= (1, 2), \mathcal{C} = \left(\frac{2\theta^3 + \theta + 2}{2\theta^3 + \theta}, 1, 0\right) \in k^3 - A^3.
\end{align*}
\]

By Theorem 2.6(2), those \( n \)-tuples \( \mathcal{C} \) have the \textbf{SR}-property, i.e., we have the following shuffle relations:

\[
\begin{align*}
\zeta_A(1)\zeta_A(2) - \zeta_A(1, 2) - \zeta_A(2, 1) &= 2\zeta_A(3) + (\theta^3 + 2\theta)\zeta_A(1, 2), \\
\zeta_A(1)\zeta_A(2) - \zeta_A(1, 2) - \zeta_A(2, 1) &= (2\theta^3 + \theta)\zeta_A(1, 2), \\
\zeta_A(2)\zeta_A(3) - \zeta_A(2, 3) - \zeta_A(3, 2) &= (2\theta^3 + \theta)\zeta_A(1, 4) + 2\zeta_A(3, 2), \\
\zeta_A(1)\zeta_A(2) - \zeta_A(1, 2) - \zeta_A(2, 1) &= \frac{\theta^3 + 2\theta + 2}{\theta^3 + 2\theta}\zeta_A(3) + 2\zeta_A(1, 2), \\
\zeta_A(1)\zeta_A(2) - \zeta_A(1, 2) - \zeta_A(2, 1) &= \frac{2\theta^3 + \theta + 2}{2\theta^3 + \theta}\zeta_A(3) + \zeta_A(1, 2).
\end{align*}
\]

**Remark 6.1.** Fix \( r, s \in \mathbb{N}, p \) a prime and \( q \) a power of the prime \( p \). By a naive analogue of the sum shuffle, we mean the \( n \)-tuples \( \mathcal{C} := (1, 0, \ldots, 0) \in \mathbb{F}_p^n \) has the \textbf{SR}-property. Conjecturally all the \( n \)-tuples \( \mathcal{C} \in \mathbb{F}_p^n \) having the \textbf{SR}-properties come from Chen’s formula \((1.1)\), which implies all the naive analogues of the sum shuffle come from Chen’s formula. From our data, the \( n \)-tuple \( \mathcal{C} \in \mathbb{F}_p^n \) we found with the \textbf{SR}-property comes from Chen’s formula \((1.1)\), so our data do support the conjecture.

### 7. Another Method

**Definition 7.1.** Fixing \( n \), an \( n \)-tuple \( \mathcal{C} = (b_0, a_1, \ldots, a_{n-1}) \in k^n \) is said to have the \textbf{DR}-property if it fits into the following linear relation among double zeta values and \( \zeta_A(n) \):

\[
\text{(DR)} \quad 0 = b_0\zeta_A(n) + \sum_{i=1}^{n-1} a_i\zeta_A(i, n - i).
\]
Fixing \( r, s \in \mathbb{N} \) and letting \( n := r + s \), by combining a shuffle relation and the relation (1.1), we have that the \( n \)-tuple \( \mathbf{C} = (b_0, a_1, \ldots, a_{n-1}) \in k^n \) has the SR-property if and only if the \( n \)-tuple \( \tilde{\mathbf{C}} = (\tilde{b}_0, \tilde{a}_1, \ldots, \tilde{a}_{n-1}) \in k^n \) has the DR-property where
\[
\tilde{b}_0 := b_0 - 1
\]
and
\[
\tilde{a}_i := \begin{cases} 
  a_i - \left[ (-1)^{s-1} \binom{n-i-1}{s-1} + (-1)^{r-1} \binom{n-i-1}{r-1} \right] & \text{if } (q-1) \mid (n-i) \\
  a_i & \text{otherwise}
\end{cases}
\]

Note that finding all possible \( n \)-tuples in \( k^n \) with the DR-property is equivalent to finding all \( n \)-tuples in \( A^n \) having the DR-property. The crucial part is to relate the DR-property to the \( \mathbb{F}_q[t] \)-linear relation among some elements in \( C \otimes (\overline{k}) \). More precisely, put
\[
\mathcal{V} := \{ (s_1, s_2) \in \mathbb{N}^2 : s_1 + s_2 = n \text{ and } (q-1) \mid s_2 \}
\]
For convenience, we label \( \mathcal{V} \) as
\[
\mathcal{V} = \{ s_1, \ldots, s_{|\mathcal{V}|} \}
\]
where \( |\mathcal{V}| \) is the cardinality of the finite set \( \mathcal{V} \). Considering points
\[
\{ \mathbf{v}_n \} \cup \{ \Xi_{s_i} \}_{i=1}^{|\mathcal{V}|} \subset C \otimes (\overline{k})
\]
described in [Cha16, Thm. 2.3.1, Thm. 4.1.1]. We separate them into two cases:

Case I. \((q-1) \nmid n\):
We consider the \( \mathbb{F}_q[t] \)-linear relation:
\[
[\eta]_n(\mathbf{v}_n) + \sum_{i=1}^{|\mathcal{V}|} [\eta_i]_n(\Xi_{s_i}) = 0.
\]
By the proof of [Cha16, Thm. 6.1.1], we can effectively determine if the tuple of polynomials \((\eta, \eta_1, \ldots, \eta_{|\mathcal{V}|}) \in \mathbb{F}_q[t]^{|\mathcal{V}|+1} \) satisfying the above equation. Then by the proof of [Cha16, Thm. 5.1.1], we trace back to an \( n \)-tuple in \( A^n \) having the DR-property.

Case II. \((q-1) \mid n\):
In this case, we consider the \( \mathbb{F}_q[t] \)-linear relation:
\[
\sum_{i=1}^{|\mathcal{V}|} [\eta_i]_n(\Xi_{s_i}) = 0.
\]
By the proof of [Cha16, Thm. 6.1.1] again, we can effectively determine if the tuple of polynomials \((\eta_1, \ldots, \eta_{|\mathcal{V}|}) \in \mathbb{F}_q[t]^{|\mathcal{V}|} \) satisfying the above equation. We want to use the proof of [Cha16, Thm. 5.1.1] to trace back to an \( n \)-tuple in \( A^n \) with the DR-property. Unfortunately, we can not determine the first coordinate of the \( n \)-tuple in \( A^n \) with the DR-property derived by this process although we know the other coordinates of the \( n \)-tuple in \( A^n \).

In conclusion, we can achieve the same result by the arguments provided in [Cha16]. i.e., we also have an effective criterion to determine whether an \( n \)-tuple in \( k^n \) has the SR-property if \((q-1) \nmid n\). For the case \((q-1) \mid n\), we can not explicit determine the first coordinate of an \( n \)-tuple in \( k^n \).
A.1. Two crucial properties. To derive the Theorem 5.6 we follow the strategy in [CPY19, §5.3]. We need two important properties stated below.

Via the isomorphisms
\[ \operatorname{Ext}_M^1(1, M') \cong M'/(\sigma - 1)M' \cong E'(\overline{k}), \]
we denote the image of the class \( M_\epsilon \in \operatorname{Ext}_M^1(1, M') \) in \( E'(\overline{k}) \) by \( v_\epsilon \).

Theorem A.1. We have that
(1) The associated \( t \)-module \( E' \) given above is defined over \( A \).
(2) \( v_\epsilon \) is an integral point in \( E'(A) \).

In [CPY19, Thm. 5.3.2], they constructed a special set \( \Xi \subset M' \) such that the image of \( \Xi \) via \( \Delta \) is obviously in \( E'(A) \). Furthermore, they proved that the special point is an image of some element in \( \Xi \) and then completed the proof. Here, we follow the same approach.

Proposition A.2. Let \( M' \) be the Frobenius module defined by the matrix \( \Phi' \) in (2.7) with a \( \overline{k}[t] \)-basis \( x_0, \ldots, x_{n-1} \). Let \( \{\nu_1, \ldots, \nu_d\} \) be the \( \overline{k}[\sigma] \)-basis of \( M' \) given by
\[ \{(t - \theta)^{n-1}x_0, \ldots, (t - \theta)x_0, x_0, \ldots, (t - \theta)x_{n-2}, x_{n-2}, x_{n-1}\}. \]

Let \( \Xi \) be the set consisting of all elements in \( M' \) of the form \( \sum_{i=1}^{d} e_i \nu_i \), where \( e_j = \sum_n \sigma^n u_{nj} \) with each \( u_{nj} \in A \). Then for any nonzero \( f \in A[t] \) and any \( 1 \leq \ell \leq n-1 \), we have \( fx_\ell \in \Xi \).

Proof. (cf. [CPY19, Thm. 5.3.2]) We first prove the case when \( \ell = 0 \). We divide \( f \) by \( (t - \theta)^n \) and write
\[ f = g_1(t - \theta)^n + \gamma_1, \]
where \( g_1, \gamma_1 \in A[t] \) with \( \deg \gamma_1 < n \). So
\[ fx_0 = g_1 \sigma x_0 + \gamma x_0 = \sigma g_1^{(1)} x_0 + \gamma_1 x_0. \]

Note that by expanding \( \gamma_1 \) in terms of powers of \( (t - \theta) \) we see that \( \gamma_1 x_0 \) is an \( A \)-linear combination of \( \{\nu_1, \ldots, \nu_n\} \).

Next we divide \( g_1^{(1)} \in A[t] \) by \( (t - \theta)^n \) and write
\[ g_1^{(1)} = g_2(t - \theta)^n + \gamma_2, \]
where \( g_2, \gamma_2 \in A[t] \) with \( \deg \gamma_2 < n \). So
\[ \sigma g_1^{(1)} x_0 = \sigma(g_2(t - \theta)^n + \gamma_2)x_0 = \sigma^2 g_1^{(1)} x_0 + \sigma \gamma_2 x_0. \]

By expanding \( \gamma_2 \) in terms of \( (t - \theta) \) we see that \( \sigma \gamma_2 x_0 \in \Xi \). By dividing \( g_2^{(1)} \) by \( (t - \theta)^n \) and continuing the procedure as above inductively we eventually obtain that \( fx_0 \in \Xi \).

Now for \( \ell \geq 2 \) we suppose that multiplication by any element of \( A[t] \) on \( x_i \) belongs to \( \Xi \) for \( 1 \leq i \leq \ell - 1 \). We prove that \( fx_\ell \in \Xi \) by the induction on the degree of \( f \) in \( t \), and note that the result is valid when \( \deg f \leq n - 1 - \ell \) by expanding \( f \) in terms of powers of \( (t - \theta) \). So we suppose that \( \deg f \geq n - 1 - \ell + 1 \).

We divide \( f \) by \( (t - \theta)^{n-\ell} \) and write
\[ f = g_1(t - \theta)^{n-\ell} + \gamma_1, \]
where \( g_1, \gamma_1 \in A[t] \) with \( \text{deg}_t \gamma_1 < n - \ell \). It follows that

\[
f x_\ell = g_1(t - \theta)^{n - \ell} x_\ell + \gamma_1 x_\ell
\]

\[
= g_1 \left[ \sigma x_\ell - H_{(\ell - 1) - 1}(t - \theta)^n x_0 \right] + \gamma_1 x_\ell
\]

\[
= \sigma g_1^{(1)} (x_\ell - H_{(\ell - 1) - 1} x_0) + \gamma_1 x_\ell
\]

\[
= \sigma g_1^{(1)} x_\ell - \sigma g_1^{(1)} H_{(\ell - 1) - 1} x_0 + \gamma_1 x_\ell.
\]

However, by expanding \( \gamma_1 \) in terms of powers of \( (t - \theta) \) we see that \( \gamma_1 x_\ell \in \Xi \), and by hypothesis \( \sigma g_1^{(1)} H_{(\ell - 1) - 1} x_0 \in A[t] \). Thus to prove the desired result we reduce to prove that \( g_1^{(1)} x_\ell \in A[t] \), which is valid by the induction hypothesis since \( \text{deg}_t g_1^{(1)} = \text{deg}_t g < \text{deg}_t f \). □

**Remark A.3.** By the definition of \( \Delta \) map, \( \Delta(\Xi) \subseteq E'(A) \).

Now, we can prove the Theorem \( A.1 \)

**Proof of the Theorem A.1** (cf. [CPY19 Thm. 5.3.4])

(1) Given any point \((a_1, \ldots, a_d)^t \in E'(k)\), its corresponding element in \( M'/(\sigma - 1)M' \) has a representative of the form \( a_1 \nu_1 + \cdots + a_d \nu_d \). We claim that the element

\[
t \left( \sum_{i=1}^{d} a_i \nu_i \right)
\]

can be expressed as \( \sum_{i=1}^{d} b_i \nu_i \in \Xi \) for which each \( b_i \) is of the form \( b_i = \sum_j \sigma^j c_j \) so that \( c_j \) is an \( A \)-linear combination of \( g^{(1)} \)th powers of the \( a_n \)'s. Then via the map \( \Delta \), the claim implies that the \( t \)-module \( E' \) is defined over \( A \).

We observe that if some

\[
\nu_i \notin \mathcal{S} := \left\{ (t - \theta)^{n-1} x_0, \ldots, (t - \theta) x_{n-2}, x_{n-1} \right\},
\]

then

\[
t a_i\nu_i = a_i(t - \theta)\nu_i + \theta a_i\nu_i = a_i \nu_{i-1} + \theta a_i\nu_i.
\]

Therefore we reduce the claim to the case \( \nu_i \in \mathcal{S} \). To simplify the notation, we denote

\[
\nu_{i_1} := x_{n-1}, \ldots, \nu_{i_n} := (t - \theta)^{n-1} x_0.
\]

Now given any \( 1 \leq \ell \leq n \) we consider \( t a_i \nu_{i_\ell} = a_i t(t - \theta)^{\ell-1} x_{n-\ell} \). Applying Proposition \( A.2 \) to \( t(t - \theta)^{\ell-1} x_{n-\ell} \) we see that \( t a_i \nu_{i_\ell} \) can be written as the form

\[
a_i \sum_{j=1}^{d} \left( \sum_{e_j} \sigma^{e_j} b_{e_j} \right) \nu_j = \sum_{j=1}^{d} \left( \sum_{e_j} \sigma^{e_j} a_i^{j} b_{e_j} \right) \nu_j
\]

for some \( b_{e_j} \in A \), whence the desired result follows.

(2) Note that

\[
\left[ \beta_0 H_{n-1}^{(-1)}(t - \theta)^n - \gamma_0 H_{r-1}^{(-1)} H_{n-1}^{(-1)}(t - \theta)^n \right] x_0 + \sum_{i=1}^{n-1} \left( \alpha_i H_{(n-i)-1}^{(-1)}(t - \theta)^{n-i} \right) x_i
\]

\[
= \sigma \left( \beta_0 H_{n-1} - \gamma_0 H_{r-1} H_{s-1} \right) x_0 + \sum_{i=1}^{n-1} \left( \sigma \alpha_i H_{(n-i)-1} x_i - \sigma \alpha_i H_{(n-i)-1} H_{i-1} x_0 \right).
\]
Applying Proposition [A.2] to the right hand side of the equation above we see that

\[ \left[ \beta_0 H_{n-1}^{-1}(t - \theta)^n - \gamma_0 H_{r-1}^{(-1)} H_{s-1}^{(-1)}(t - \theta)^n \right] x_0 + \sum_{i=1}^{n-1} \left( \alpha_i H_{(n-i)-1}^{-1}(t - \theta)^{n-i} \right) x_i \in \Xi. \]

Since \( v_{\xi} \) is its image via \( \Delta \), the result follows from Remark [A.3].

\[ \square \]

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