Double integral inequalities of Hermite-Hadamard-type for \( \phi_h \)-convex functions on linear spaces

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Abstract

The concept of \( \phi_h \)-convexity is extended for functions defined on closed \( \phi_h \)-convex subsets of linear spaces. Consequently, some double integral inequalities of Hermite-Hadamard type defined on time-scaled linear spaces are established for \( \phi_h \)-convex functions.

1 Introduction

A well celebrated, fundamental inequality for a convex function \( f \) is the classical Hermite-Hadamard’s inequality:

\[
\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2},
\]

where \( a, b \in \mathbb{R} \) with \( a < b \).

It was first suggested by Hermite in 1881. But this result was nowhere mentioned in literature and was not widely known as Hermite’s result. A leading expert on the history and theory of complex functions, Beckenbach [2], wrote that the inequality (1.1) was proven by Hadamard in 1893. In general, (1.1) is now known as the Hermite-Hadamard inequality. It has several extensions and generalizations for univariate and multivariate convex functions and its classes on classical intervals.

The concept of the theory of time scales was initiated by Stefan Hilger (see [10]) in order to unify and extend the theory of difference and differential calculus in a consistent way. In this theory, the delta and nabla calculus are introduced. A linear combination of these delta and nabla dynamics, the diamond-\( \alpha \) calculus on time scales was developed by Sheng et al. [12]. Since the advent of this notion, several authors have extended the classical Hermite-Hadamard inequality (1.1) to time scales via the diamond-alpha dynamic calculus for univariate convex functions (see Dinu [5]) and the references therein.

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Recently, Fagbemigun and Mogbademu [6] introduced the time-scaled version of some classes of convex functions, including a more generalized class of \( \phi_h \)-convex function on time scales as given below:

**Definition 1.1.** [6] Let \( \mathbb{J}_T \subset \mathbb{T} \rightarrow \mathbb{R} \) be a nonzero non negative function with the property that \( h(t) > 0 \) for all \( t \geq 0 \). A mapping \( f : \mathbb{I}_T \rightarrow \mathbb{R} \) is said to be \( \phi_h \)-convex on time scales if

\[
f(\lambda x + (1-\lambda)y) \leq \left( \frac{\lambda}{h(\lambda)} \right)^s f(x) + \left( \frac{1-\lambda}{h(1-\lambda)} \right)^s f(y),
\]

for \( s \in [0,1], 0 \leq \lambda \leq 1 \) and \( x,y \in \mathbb{I}_T \).

**Remark 1.1.**

(i) If \( s = 1 \) and \( h(\lambda) = 1 \), then \( f \in SX(\mathbb{I}_T) \), i.e., \( f \) is convex on time scales (see [5]).

(ii) If \( s = 1, h(\lambda) = 1 \), where \( \lambda = \frac{1}{2} \), then \( f \in J(\mathbb{I}_T) \) is mid-point convex on time scales (see [6]).

(iii) If \( s = 0 \), then \( f \in P(\mathbb{I}_T) \) is \( P \)-convex on time scales (see [6]).

(iv) If \( h(\lambda) = \lambda \frac{1}{2} \), then \( f \in SX(h, \mathbb{I}_T) \) is \( h \)-convex on time scales (see [6]).

(v) If \( s = 1, h(\lambda) = 2\sqrt{\lambda(1-\lambda)} \), then \( f \in MT(\mathbb{I}_T) \) is \( MT \)-convex on time scales (see [6]).

In a more recent paper, the authors [7] introduced a more general calculus of \( \diamond \)-\( \phi_h \) dynamics for a single variable function on time scales as follows;

**Definition 1.2.** [7] Let \( h : \mathbb{J}_T \subset \mathbb{T} \rightarrow \mathbb{R} \) be a real valued function, with the property that \( h(t) > 0 \) for all \( t \geq 0 \). The \( \diamond \)-\( \phi_h \) dynamic derivative of a function \( f : \mathbb{T} \rightarrow \mathbb{R} \) in \( t \in \mathbb{T} \) is defined to be the number denoted by \( f^{\diamond\phi_h}(t) \) (when it exists), with the property that for any \( \epsilon > 0 \), there is a neighbourhood \( U \) of \( m \) such that, for all \( n \in U \), \( 0 \leq s \leq 1 \) and \( 0 \leq \lambda \leq 1 \), with \( \mu_{mn} = \sigma(m) - n \) and \( \nu_{mn} = \rho(m) - n \), where \( m,n \in \mathbb{T}_k^* \), then,

\[
\left| \left( \frac{\lambda}{h(\lambda)} \right)^s [f(\sigma(m)) - f(n)]\nu_{mn} + \left( \frac{1-\lambda}{h(1-\lambda)} \right)^s [f(\rho(m)) - f(n)]\mu_{mn} - f^{\diamond\phi_h}(t)\mu_{mn}\nu_{mn} \right| < \epsilon|\mu_{mn}\nu_{mn}|. \tag{1.3}
\]

**Definition 1.3.** [7] Let \( h : \mathbb{J}_T \subset \mathbb{T} \rightarrow \mathbb{R} \) be a real valued function, with the property that \( h(t) > 0 \) for all \( t \geq 0 \). The \( \diamond \)-\( \phi_h \) integral of a function \( f : \mathbb{T} \rightarrow \mathbb{R} \) from \( a \) to \( b \), where \( a,b \in \mathbb{T} \) is given by;

\[
\int_a^b f(t) \diamond\phi_h t = \left( \frac{\lambda}{h(\lambda)} \right)^s \int_a^b f(t) \Delta t + \left( \frac{1-\lambda}{h(1-\lambda)} \right)^s \int_a^b f(t) \nabla t, \tag{1.4}
\]
for all \( s \in [0,1] \), \( \lambda \in [0,1] \) and \( h(t) > 0 \forall t \geq 0 \) provided that \( f \) has a delta and nabla integral on \([a,b]_T\) or \( I_T\).

**Remark 1.2.**[7]

(i) The inequality (1.4) reduces to the diamond-\( \alpha \) integral defined by Sheng et al. [12], if \( s = 1 \), \( h(\lambda) = 1 \) and \( \lambda = \alpha \). Thus, every diamond-\( \alpha \) integrable function on \( T \) is diamond-\( \phi_h \) integrable but the converse is not true (see [7]).

(ii) If \( f \) is diamond-\( \phi_h \) integrable for \( 0 \leq s \leq 1 \), and \( 0 \leq \lambda \leq 1 \), then \( f \) is both \( \Delta \) and \( \nabla \) integrable.

The inequality (1.1) was equally extended to time scales by the authors [7], using the new class of univariate \( \phi_h \)-convex function of [6] to obtain several generalizations of the Hermite-Hadamard inequality on time scales. We present one of such results.

**Theorem 1.1.**[7] Suppose that

(i) \( f : I_T \to \mathbb{R} \) is a continuous \( \phi_h \)-convex function on \( I_T \);

(ii) \( p, q \in (0, 1) \), such that \( p + q = 1 \);

\( g \in C(I_T, \mathbb{R}) \) is symmetric with respect to \( pa + qb = \gamma \) on \([a,b]_T\), for all \( a, b \in I_T \), that is,

\[ g(\gamma - qt) = g(\gamma + pt), \quad \text{for all } t \in [0, b-a]. \]

Then

\[ f(px + qy) \leq p \int_a ^\gamma f(t)g(t) \circ \phi_h t \frac{dt}{\int_a ^\gamma g(t) \circ \phi_h t} + q \int_\gamma ^b f(t)g(t) \circ \phi_h t \frac{dt}{\int_\gamma ^b g(t) \circ \phi_h t} \leq pf(x) + qf(y). \quad (1.5) \]

The two-variable time scales delta, nabla and diamond-\( \alpha \) calculi were introduced by Albrandth and Morian[1], Bohner and Guseinov [3,4], Guseinov [9] and Özkan and Kaymakçağan [11] respectively.

Özkan and Kaymakçağan [11] gave the following definition of a partial \( \circ_{\alpha_1} \) derivative;

**Definition 1.4.**[11] Let \( f \) be a real-valued function on \( T_1 \times T_2 \). We say that \( f \) has a partial \( \circ_{\alpha_1} \) derivative \( \frac{\partial f(t_1,t_2)}{\partial \circ_{\alpha_1} t_1} \) (with respect to \( t_1 \)) if for each \( \epsilon > 0 \), there exists a neighbourhood \( U_{t_1} \) (open in the relative topology of \( T_1 \)) of \( t_1 \) such that

\[ \left| \alpha_1 [f(\sigma_1(t_1), t_2) - f(s, t_2)] \mu t_1 s + (1 - \alpha_1)[f(\rho_1(t_1), t_2) - f(s, t_2)] \nu t_1 s - f^{\circ_{\alpha_1}}(t_1, t_2) \mu t_1 s \nu t_1 s \right| < \epsilon |\mu t_1 s\nu t_1 s|, \quad (1.5) \]

for all \( s \in U_{t_1} \), where \( U_{t_1} ms = \sigma_1(t_1) - s, \nu t_1 s = \rho_1(t_1) - s. \)
The $\phi_{a_2}$ partial derivative was respectively defined (see [11]).

Motivated by the recent results of these authors; Fagbemigun and Mogbademu [6], Fagbemigun et al. [7] and Özkan and Kaymakçalan [11], we discuss the following new concepts of Fagbemigun and Mogbademu [8].

## 2 Preliminaries

In the sequel, we shall need the following new definitions recently introduced in [8].

Let $T_1$ and $T_2$ be two time scales with $T_1 \times T_2 = \{(x, y) : x \in T_1, y \in T_2\}$ which is a complete metric space with the metric $d$ defined by

$$d((x, y), (x', y')) = ((x - x')^2 + (y - y')^2)^{\frac{1}{2}}, \quad \forall (x, y), (x', y') \in T_1 \times T_2.$$ 

Let $\sigma_i, \rho_i, (i = 1, 2)$ denote respectively the forward jump operator, backward jump operator, and the diamond-$\phi_h$ dynamic differentiation operator on $T_1$.

**Definition 2.1.** Let $f$ be a real-valued function on $T_1 \times T_2$, $h : J_T \subset T \rightarrow \mathbb{R}$ a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. $f$ is said to have a partial $\phi_{(\phi_h)_1}$ derivative $\frac{\partial f(t_1, t_2)}{\partial (\phi_h)_1}(wrt \ t_1)$, at $(t_1, t_2) \in T_1 \times T_2$, if for each $\epsilon > 0$, there exists a neighbourhood $U_{t_1}$ of $t_1$ such that

$$\left| \left( \frac{\lambda}{h(\lambda)} \right)^s [f(\sigma_1(t_1), t_2) - f(m, t_2)] \mu t_1 m \right|$$

$$+ \left( \frac{1 - \lambda}{h(1 - \lambda)} \right)^s |f(\rho_1(t_1), t_2) - f(m, t_2)| \nu t_1 m - f^\phi_{(\phi_h)_1}(t_1, t_2) $\mu t_1 m t_1 m |< \epsilon | \mu t_1 m t_1 m |,$$  \hspace{1cm} \text{(2.1)}

for $s \in [0, 1]$, $0 \leq \lambda \leq 1$ and for all $m \in U_{t_1}$, where $U_{t_1} m = \sigma_1(t_1) - m$, $\nu t_1 m = \rho_1(t_1) - m$.

**Definition 2.2.** Let $f$ be a real-valued function on $T_1 \times T_2$ and $h : J_T \subset T \rightarrow \mathbb{R}$ an increasing function with the property that $h(t) > 0$ for all $t \geq 0$. $f$ is said to have a "partial $\phi_{(\phi_h)_2}$ derivative" $\frac{\partial f(t_1, t_2)}{\partial (\phi_h)_2}(wrt \ t_2)$, at $(t_1, t_2) \in T_1 \times T_2$, if for each $\epsilon > 0$, there exists a neighbourhood $U_{t_2}$ of $t_2$ such that

$$\left| \left( \frac{\lambda}{h(\lambda)} \right)^s [f(t_1, \sigma_2(t_2)) - f(t_1, m)] \mu t_2 m \right|$$

$$+ \left( \frac{1 - \lambda}{h(1 - \lambda)} \right)^s |f(t_1, \rho_2(t_2) - f(t_1, m)| \nu t_2 m - f^\phi_{(\phi_h)_2}(t_1, t_2) \mu t_2 m t_2 m |< \epsilon | \mu t_2 m t_2 m |,$$  \hspace{1cm} \text{(2.2)}

for $s \in [0, 1]$, $0 \leq \lambda \leq 1$ and for all $m \in U_{t_2}$, where $U_{t_2} m = \sigma_2(t_2) - m$, $\nu t_2 m = \rho_2(t_2) - m$.

These derivatives are also denoted by $f^\phi_{(\phi_h)_1}(t_1, t_2)$ and $f^\phi_{(\phi_h)_2}(t_1, t_2)$ respectively.
Before we define the double diamond-$\phi_h$ dynamic integral, we shall employ the following remark of [3].

**Remark 2.1.** Let $f$ be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$. If the delta ($\Delta$) and nabla ($\nabla$) integrals of $f$ exist on $\mathbb{T}_1 \times \mathbb{T}_2$, then the following types of integrals can be defined:

(i) $\Delta\Delta$-integral over $R^0 = [a, b] \times [c, d]$, which is introduced by using partitions consisting of subrectangles of the form $[\alpha, \beta] \times [\gamma, \vartheta]$;

(ii) $\nabla\nabla$-integral over $R^1 = [a, b] \times (c, d]$, which is introduced by using partitions consisting of subrectangles of the form $[\alpha, \beta] \times [\gamma, \vartheta]$;

(iii) $\Delta\nabla$-integral over $R^2 = [a, b] \times (c, d]$, which is introduced by using partitions consisting of subrectangles of the form $[\alpha, \beta] \times (\gamma, \vartheta]$;

(iv) $\nabla\Delta$-integral over $R^3 = (a, b] \times [c, d]$, which is introduced by using partitions consisting of subrectangles of the form $(\alpha, \beta] \times [\gamma, \vartheta]$.

Now let $\bar{U}(f)$ and $\bar{L}(f)$ denote the upper and lower Darboux $\Delta$-integral of $f$ from $a$ to $b$; $\bar{U}(f)$ and $\bar{L}(f)$ denote the upper and lower Darboux $\nabla$-integral of $f$ from $a$ to $b$ respectively. Given the construction of $U(f)$ and $L(f)$, which follows from the properties of supremum and infimum, we give the following definition.

**Definition 2.3.** Let $f$ be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$, $h : \mathbb{T}_1 \subset \mathbb{T} \to \mathbb{R}$ a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. If $f$ is $\Delta$-integrable on $R^0 = [a, b] \times [c, d]$ and $\nabla$-integrable on $R^1 = (a, b] \times (c, d]$, then it is $\phi_h$-integrable on $R = [a, b] \times [c, d]$ and

$$
\int_R f(t, k) \phi_h(t, k) \, dk = \left( \frac{\lambda}{h(\lambda)} \right)^s \int_{R^0} \int_{R^1} f(t, k) \Delta_1 \Delta_2 k
$$

$$
+ \left( \frac{1 - \lambda}{h(1 - \lambda)} \right)^s \int_{R^1} \int_{R^1} f(t, k) \nabla_1 \nabla_2 k,
$$

for all $s \in [0, 1]$, $0 \leq \lambda \leq 1$ and $t, k \in J_T$.

Since $\bar{U}(f) \geq \bar{L}(f)$ and $\bar{U}(f) \geq \bar{L}(f)$, we obtain the following result.

**Theorem 2.1.** Let $f$ be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$, $h : \mathbb{T}_1 \subset \mathbb{T} \to \mathbb{R}$ a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. If $f$ be $\phi_h$-integrable on $R = [a, b] \times [c, d]$, provided its delta ($\Delta$) and nabla ($\nabla$) integrals exist, then

(i) If $\phi_h = 1$, $f$ is $\Delta\Delta$-integrable on $R^0 = [a, b] \times [c, d]$;

(ii) If $\phi_h = 0$, $f$ is $\nabla\nabla$-integrable on $R^1 = (a, b] \times (c, d]$;

(iii) If $\phi_h = \frac{1}{2}$, $f$ is $\Delta\nabla$-integrable and $\nabla\Delta$-integrable on $R^0$ and $R^1$;

(iv) If $\phi_h = \alpha$, $f$ is double $\phi_h$-integrable on $R = [a, b] \times [c, d]$. 

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3 Double integral inequalities of Hermite-Hadamard type for $\phi_h$-convex functions

Here, we obtain double integral inequalities of Hermite-Hadamard type in which upper and lower bounds for the quantity
\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f \left( \frac{px + qy}{p+q} \right) \circ (\phi_h) \circ (\phi_h) \, dp \, dq,
\]
are provided for the generalized class of $\phi_h$-convex functions (1.2), defined on linear spaces of time scales.

Let $X_T$ be a vector space over the time-scaled field $K$ and let $x, y$ be monotonically increasing functions in $X_T$, $x \neq y$. Let the segment generated by $x, y$ be defined by
\[
[a, b] : \{(1 - \lambda)x + \lambda y, \quad \lambda \in [0, 1]\}.
\]
We consider the function $f : [x, y]_{I_T} \subset T \rightarrow \mathbb{R}$ and the attached function $g(x, y) : [0, 1] \subset T \rightarrow \mathbb{R}$ defined by
\[
g(x, y)(\lambda) := f((1 - \lambda)x + \lambda y), \quad \lambda \in [0, 1].
\]
Note that $f$ is $\phi_h$-convex on $[x, y]$ if and only if $g(x, y)$ is $\phi_h$-convex on $[0, 1]$.

The concept of $\phi_h$-convexity in Definition 1.1 can be extended for functions defined on closed $\phi_h$-convex subsets of the linear spaces in the same way as on classical intervals by replacing the interval $I_T$ by the corresponding closed $\phi_h$-convex subset $E$ of the linear space $X_T$.

It is well-known that if $(X, || \cdot ||)$ is a normed linear space, then the function $f(x) = ||x||^p, p \geq 1$ is convex on $X$.

Using the elementary inequality $(a + b)^s \leq a^s + b^s$ that holds for any $a, b \geq 0$ and $s \in [0, 1]$, we have for the function $g(x) = ||x||$ that
\[
g(\lambda x + (1 - \lambda)y) = \left( \frac{\lambda}{h(\lambda)} ||x|| + \frac{(1 - \lambda)}{h(1 - \lambda)} ||y|| \right)^s \leq \left( \frac{\lambda}{h(\lambda)} ||x|| + \frac{(1 - \lambda)}{h(1 - \lambda)} ||y|| \right)^s \leq \left( \frac{\lambda}{h(\lambda)} \right)^s (||x||)^s + \left( \frac{(1 - \lambda)}{h(1 - \lambda)} \right)^s (||y||)^s \leq \left( \frac{\lambda}{h(\lambda)} \right)^s g(x) + \left( \frac{(1 - \lambda)}{h(1 - \lambda)} \right)^s g(y),
\]
for any $x, y \in X_T, \lambda \in [0, 1]$ and $h$ a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, which shows that $g$ is $\phi_h$-convex on $I_T$.

With this concept, an equivalent definition to definition 1.1 is given as follows.
Definition 3.1. Let $h$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. Then inequality (1.2) can be re-written as

$$f \left( \frac{px + qy}{p + q} \right) \leq \left( \frac{p}{h(p)} \right)^s f(x) + \left( \frac{q}{h(q)} \right)^s f(y),$$

(3.1)

for all $p, q \geq 0$ with $p + q > 0$, $s \in [0, 1]$ and $x, y \in I_T$.

We can now establish double Hermite-Hadamard-type integral inequalities for the class of $\phi_h$-convex functions on time scales linear spaces.

Theorem 3.1. Let $h$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. Let $E \phi_h$-convex set in a linear space of a time scale interval $X_T \subset T$ and $f : E \subseteq X_T \rightarrow \mathbb{R}$ be an integrable $\phi_h$-convex function with respect to the function $\phi_h$ defined on the set $E$ with $\phi_h$ Lebesgue integrable on $[a, b]_T \times [c, d]_T$. Then, for any $a, b, c, d \geq 0$, with $b > a, d > c$ and for any $a, b, c, d \in E$, $s \in [0, 1]$, we have:

$$f \left( \frac{I(a, b; c, d)}{(b - a)(d - c)} x + \frac{I(c, d; a, b)}{(b - a)(d - c)} y \right) \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f \left( \frac{px + qy}{p + q} \right) \circ (\phi_h)_1 q \circ (\phi_h)_2 p$$

$$\leq \frac{I_h(p)(a, b; c, d)}{(b - a)(d - c)} f(x) + \frac{I_h(q)(c, d; a, b)}{(b - a)(d - c)} f(y),$$

(3.2)

where

$$I(a, b; c, d) = \int_a^b \left( \int_c^d \left( \frac{p}{p + q} \right) \circ (\phi_h)_1 q \circ (\phi_h)_2 p \right),$$

$$I(c, d; a, b) = \int_a^b \left( \int_c^d \left( \frac{q}{p + q} \right) \circ (\phi_h)_1 q \circ (\phi_h)_2 p \right),$$

$$I_h(p)(a, b; c, d) = \int_a^b \int_c^d \left( \frac{p}{h(p)} \right)^s \circ (\phi_h)_1 q \circ (\phi_h)_2 p,$$

and

$$I_h(q)(c, d; a, b) = \int_a^b \int_c^d \left( \frac{q}{h(q)} \right)^s \circ (\phi_h)_1 q \circ (\phi_h)_2 p,$$

for all $p, q \geq 0$ with $p + q > 0$ and $x, y \in X_T$.

Proof. Consider the function $g_{x,y} : [0, 1] \subset T \rightarrow \mathbb{R}$ defined by $g_{x,y}(\lambda) = f(\lambda x + (1 - \lambda)y)$. This function is $\phi_h$-convex on $[0, 1] \subset T$ and by Jensen’s double integral inequality of real functions on time scales, we have

$$g_{x,y} \left( \int_a^b \int_c^d \left( \frac{p}{p + q} \right) \circ (\phi_h)_1 q \circ (\phi_h)_2 p \right)$$

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\[
\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g_{x,y} \left( \frac{p}{p+q} \right) \phi_{(\phi_h)_2} \, p, \quad (3.3)
\]

which is equivalent to
\[
f \left[ \int_a^b \int_c^d \left( \frac{p}{p+q} \right) \phi_{(\phi_h)_1} \, q \phi_{(\phi_h)_2} \, p \int_c^d \left( \frac{p}{p+q} \right) \phi_{(\phi_h)_1} \, q \phi_{(\phi_h)_2} \, p \right] \leq f \left( \int_a^b \int_c^d \left( \frac{p}{p+q} \right) \phi_{(\phi_h)_1} \, q \phi_{(\phi_h)_2} \, p \right) + \left( 1 - \frac{\int_a^b \int_c^d \left( \frac{q}{p+q} \right) \phi_{(\phi_h)_1} \, q \phi_{(\phi_h)_2} \, p}{(b-a)(d-c)} \right) y \right] \]

By definition 3.1 and using a simple calculation, we have
\[
f \left( \int_a^b \int_c^d \left( \frac{p}{p+q} \right) \phi_{(\phi_h)_1} \, q \phi_{(\phi_h)_2} \, p \right) x + \left( \frac{\int_a^b \int_c^d \left( \frac{q}{p+q} \right) \phi_{(\phi_h)_1} \, q \phi_{(\phi_h)_2} \, p}{(b-a)(d-c)} \right) y \right] \]

which proves the first part of Theorem 3.1.

Under the same assumption of definition 3.1, \( f \) is \( \phi_h \)-convex. Thus, integrating inequality (3.1) on the rectangle \([a, b] \times [c, d] \subseteq T\) over \( \phi_{(\phi_h)_1} \, q \phi_{(\phi_h)_2} \, p \) gives
\[
\int_a^b \int_c^d f \left( \frac{p}{p+q} \right) \phi_{(\phi_h)_1} \, q \phi_{(\phi_h)_2} \, p \]
\[
\leq f(x) \int_a^b \int_c^d \left( \frac{p}{p+q} \right) \phi_{(\phi_h)_1} \, q \phi_{(\phi_h)_2} \, p \]

and the second part of inequality (3.2) is satisfied.

**Theorem 3.2.** Let \( h \) be a non zero non negative function with the property that \( h(t) > 0 \) for all \( t \geq 0 \). Let \( E \) be a \( \phi_h \)-convex set in a linear space of a time scale interval \( X_T \subseteq T \) and \( f : E \subseteq X_T \rightarrow \mathbb{R} \) be an integrable \( \phi_h \)-convex function with respect to the function \( \phi_h \) defined on the set \( E \) with the mapping \([0,1] : \lambda \rightarrow f((1-\lambda)x + \lambda y)\) Lebesgue integrable on \([a, b] \times [c, d] \subseteq T\). Then for all \( p, q \geq 0, x, y \in E \) with \( p + q > 0 \), \( s \in [0,1] \) and \( x, y \in X_T \),
Hence, integrating inequality (3.7) on the rectangle $[a, b] \times [c, d]$ gives

$$p \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left( f \left( \frac{px + qy}{p + q} \right) + f \left( \frac{px + qy}{p + q} \right) \right) \circ_{(\phi_h)} 1 \circ_{(\phi_h)} 2 P$$

Proof. By $\phi_h$-convexity of $f$,

$$f(\lambda x + (1 - \lambda)y) \leq \left( \frac{\lambda}{h(\lambda)} \right) \circ f(x) + \left( \frac{1 - \lambda}{h(1 - \lambda)} \right) \circ f(y)$$

(3.4)

and

$$f((1 - \lambda)\phi(x) + \lambda y) \leq \left( \frac{1 - \lambda}{h(1 - \lambda)} \right) \circ f(x) + \left( \frac{\lambda}{h(\lambda)} \right) \circ f(y)$$

(3.5)

for any $0 \leq \lambda \leq 1$, and $s \in [0, 1]$.

Adding (3.4) and (3.5) gives

$$f(\lambda x + (1 - \lambda)y) + f((1 - \lambda)x + \lambda y) \leq \left( \left( \frac{\lambda}{h(\lambda)} \right) \circ f(x) + \left( \frac{1 - \lambda}{h(1 - \lambda)} \right) \circ f(y) \right)$$

(3.6)

By choosing $\lambda = \frac{p}{p+q}$ and $1 - \lambda = \frac{q}{p+q}$ in (3.6), we obtain

$$f \left( \frac{px + qy}{p + q} \right) + f \left( \frac{qx + py}{p + q} \right) \leq \left( \frac{p}{h \left( \frac{p}{p+q} \right)} \right) \circ f(x) + f(y)$$

(3.7)

for any $p, q > 0$ with $p + q > 0$. Then the double integrals

$$\int_a^b \int_c^d f \left( \frac{px + qy}{p + q} \right) \circ_{(\phi_h)} 1 \circ_{(\phi_h)} 2 P$$

and

$$\int_a^b \int_c^d f \left( \frac{qx + py}{p + q} \right) \circ_{(\phi_h)} 1 \circ_{(\phi_h)} 2 P$$

exists since the mapping $[0, 1] : \lambda \to f((1 - \lambda)x + \lambda y)$ is $\phi_h$ Lebesgue integrable on $[a, b] \times [c, d]$.

Hence, integrating inequality (3.7) on the rectangle $[a, b] \times [c, d]$ over $\circ_{(\phi_h)} 1 \circ_{(\phi_h)} 2 P$ gives

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left( f \left( \frac{px + qy}{p + q} \right) + f \left( \frac{px + qy}{p + q} \right) \right) \circ_{(\phi_h)} 1 \circ_{(\phi_h)} 2 P$$

$$\leq \frac{f(x) + f(y)}{(b-a)(d-c)} \int_a^b \int_c^d \left( \frac{p}{h \left( \frac{p}{p+q} \right)} \right) \circ_{(\phi_h)} 1 \circ_{(\phi_h)} 2 P.$$
which is the second inequality of Theorem 3.2.

For the first part, we have, from the $\phi_h$-convexity of $f$ that for any $z_1, z_2 \in E$, $\lambda = \frac{1}{2}$, $s = 1$, $h(\frac{1}{2}) \leq 1$. Thus,

$$f \left( \frac{z_1 + z_2}{2} \right) \leq \frac{1}{2h(\frac{1}{2})} (f(z_1) + f(z_2)).$$

Choosing $z_1 = \frac{px + qy}{p+q}$ and $z_2 = \frac{qx + py}{p+q}$ in (3.8), then for any $p, q \geq 0$, $p+q > 0$, we get

$$f \left( \frac{x + y}{2} \right) \leq \frac{1}{2h(\frac{1}{2})} \left( f \left( \frac{px + qy}{p+q} \right) + f \left( \frac{qx + py}{p+q} \right) \right).$$

(3.9)

Integrating (3.9) on the rectangle $[a, b]_T \times [c, d]_T$ over $\circ(\phi_h), g \circ(\phi_h)_T$ $p$ gives the first part of Theorem 3.2.

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