Geometric property (T) for non-discrete spaces

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May 27, 2021

Abstract

Geometric property (T) was defined by Willett and Yu, first for sequences of graphs and later for more general discrete spaces. Increasing sequences of graphs with geometric property (T) are expanders, and they are examples of coarse spaces for which the maximal coarse Baum-Connes assembly map fails to be surjective. Here, we give a broader definition of bounded geometry for coarse spaces, which includes non-discrete spaces. We define a generalisation of geometric property (T) for this class of spaces and show that it is a coarse invariant. Additionally, we characterise it in terms of spectral properties of Laplacians. We investigate geometric property (T) for manifolds and warped systems.

1 Introduction

In their paper [8], Willett and Yu introduce geometric property (T) for sequences of graphs, in order to provide examples of coarse spaces for which the maximal coarse Baum-Connes assembly map fails to be surjective. The concept is studied further by the same authors in [9]. There, geometric property (T) is defined for coarse spaces that are discrete with bounded geometry, monogenic, and countable. Such a space has geometric property (T) if each unitary representation of the translation algebra $C_u[X]$ (we call it $C_{cs}[X]$, see Definition 5.3) that has almost invariant vectors, has a non-zero invariant vector. For increasing sequences of finite graphs, this is a strictly stronger property than being an expander. In fact, in this case it is equivalent to the graph Laplacian having spectral gap in the maximal completion of $C_u[X]$ (see [9, Proposition 5.2]). A sequence of finite graphs with geometric property (T) can never have large girth (see [8, Corollary 7.5]).

The aim of this article is to generalise geometric property (T) to spaces satisfying a broader notion of bounded geometry, see Definition 6.7. This broader notion of bounded geometry is explained in Section 3. Spaces with bounded geometry include sequences of graphs with bounded degree, manifolds whose Ricci curvature and injectivity radius are uniformly bounded from below, and warped systems coming from an action of a finitely generated group on a compact manifold. We do not assume monogenicity of the spaces. However, we do assume that the spaces are covered by a countable number of “bounded” sets.

In [9], translation operators are used extensively to study geometric property (T) for discrete spaces. For non-discrete spaces, these operators do not behave as nicely. Therefore we introduce the concept of “operators in blocks”, which are operators with a convenient decomposition (see Definition 5.5).

Any coarse space with bounded geometry can be equipped with a measure $\mu$ that is uniformly bounded and for which a gordo set exists (see Definition 3.6 and Proposition 3.7). We will firstly
define geometric property (T) for a coarse space $X$ equipped with such a measure $\mu$. However, Theorem 8.6 will show that it is in fact independent of the chosen measure $\mu$.

We will consider the Hilbert space $L^2(X, \mu)$. An operator $T$ in $B(L^2(X, \mu))$ has controlled support if the support of a function is not changed much by $T$ (see Definition 5.1). Let $\mathcal{C}_{cs}[X] \subseteq B(L^2(X, \mu))$ be the algebra of operators with controlled support. A representation of this algebra is a unital $\ast$-homomorphism $\rho: \mathcal{C}_{cs}[X] \to B(H)$ for some Hilbert space $H$. For such a representation, the subspace of constant vectors $H_c$ is defined (see Definition 6.6). The coarse space $X$ has geometric property (T) if every unit vector $v \in H^\perp_c$ is “moved enough” by a suitable operator in $\mathcal{C}_{cs}[X]$ (see Definition 6.7).

An important notion in coarse geometry is that of coarse invariance (see Section 2). In [9], it was shown that geometric property (T) is a coarse invariant (see [9, Theorem 4.1]). In this paper we show that our generalisation of geometric property (T) is still a coarse invariant. This gives evidence that we have the “right” definition of property (T).

**Theorem 8.6.** Suppose $(X, \mu)$ and $(X', \mu')$ are coarsely equivalent spaces with bounded geometry, equipped with uniformly bounded measures for which gordo sets exist. Then $(X, \mu)$ has property (T) if and only if $(X', \mu')$ has property (T). In particular, whether $X$ has property (T) is independent of the chosen uniformly bounded measure $\mu$ for which a gordo set exists.

Analogous to [9], we establish a complete characterisation of geometric property (T) in terms of non-amenability for connected coarse spaces of bounded geometry. In fact, for unbounded connected spaces, geometric property (T) is the complete converse of amenability for coarse spaces. This generalizes [9, Corollary 6.1]. The notion of amenability for coarse spaces with bounded geometry is introduced in Lemma 9.1.

**Theorem 9.4.** Let $X$ be a connected coarse space and let $\mu$ be a uniformly bounded measure for which a gordo set exists. If $X$ is bounded, then it is amenable and has geometric property (T). If $X$ is unbounded, it has geometric property (T) if and only if it is not amenable.

Towards the end of this article, we investigate geometric property (T) for Riemannian manifolds. Let $M$ be a Riemannian manifold or a countable disjoint union of Riemannian manifolds with the same dimension. The Riemannian metric defines a coarse structure on $M$. If the Ricci curvature and the injectivity radius of $M$ are uniformly bounded from below, then $M$ has bounded geometry (see Section 10). In this case the manifold Laplacian $\Delta_M$ is an element of the algebra $C^*_{\text{max}}(X)$, and the manifold $M$ has geometric property (T) if and only if $\Delta_M$ has spectral gap in this algebra.

**Theorem 10.4.** Let $(M, g)$ be a Riemannian manifold of bounded geometry equipped with the coarse structure coming from the geodesic metric. Then $M$ has geometric property (T) if and only if the Laplacian $\Delta_M$ has spectral gap in $C^*_{\text{max}}(M)$.

In the last section we discuss warped systems. A group $\Gamma$ acting by homeomorphisms on a Riemannian manifold $M$ gives rise to a coarse spaces called a warped system, see Section 11. In Section 11 we will show that warped systems have bounded geometry if the manifold $M$ is compact. One might expect that the warped system has geometric property (T) if and only if the group has property (T) and the action is ergodic. This remains an open question, though some partial results are given.
2 Preliminaries on coarse spaces

A coarse space is a space with a large-scale geometric structure. They were first described in [4] and further developed in [5]. Let us first recall the definition from [5, Definition 2.3].

**Definition 2.1.** Let $X$ be a set. A coarse structure on $X$ consists of a collection $\mathcal{E}$ of subsets of $X \times X$, whose elements are called the controlled sets, such that:

(i) The diagonal $\{(x,x) \mid x \in X\}$ is an element of $\mathcal{E}$.

(ii) For every $E \in \mathcal{E}$ and $F \subseteq E$, we have $F \in \mathcal{E}$.

(iii) For every $E \in \mathcal{E}$, the inverse $E^{-1} = \{(y,x) \in X \times X \mid (y,x) \in E\}$ is an element of $\mathcal{E}$.

(iv) For every $E,F \in \mathcal{E}$, we have $E \cup F \in \mathcal{E}$.

(v) For every $E,F \in \mathcal{E}$, the composition $E \circ F = \{(x,z) \in X \times X \mid \text{there is } y \in X \text{ such that } (x,y) \in E \text{ and } (y,z) \in F\}$ is an element of $\mathcal{E}$.

A set $X$ equipped with a coarse structure is called a coarse space.

**Example 2.2.** Every metric space $(X,d)$ can be given a coarse structure: let $\mathcal{E} = \{E \subseteq X \times X \mid \sup_{(x,y) \in E} d(x,y) < \infty\}$. For any $R > 0$, we denote $E_R = \{(x,y) \in X \times X \mid d(x,y) \leq R\}$. Then the controlled sets are exactly the subsets of $X \times X$ that are contained in some $E_R$.

**Definition 2.3.** Let $X$ and $Y$ be coarse spaces. A map $f: X \to Y$ is a coarse equivalence if there is a map $g: Y \to X$ such that for any controlled set $E \subseteq X \times X$ the set $(f \times f)(E)$ is controlled, for any controlled set $F \subseteq Y \times Y$ the set $(g \times g)(F)$ is controlled, the set $\{(x,gf(x)) \mid x \in X\}$ is controlled and the set $\{(y,fg(y)) \mid y \in Y\}$ is controlled. The spaces $X$ and $Y$ are called coarsely equivalent if such an $f$ exists.

This definition is easily seen to be equivalent to Definition 2.21 in [5].

An important example of the above is when $Y$ is coarsely dense in $X$.

**Definition 2.4.** Let $X$ be a coarse space and $Y \subseteq X$. The subset $Y$ is called coarsely dense in $X$ if there is a controlled set $E \subseteq X \times X$ such that for all $x \in X$ there is $y \in Y$ with $(x,y) \in E$.

If $Y$ is coarsely dense in $X$, it is easy to show that $X$ is coarsely equivalent to $Y$ with the induced coarse structure.

In this paper we will use the following properties and notation:

**Definition 2.5.** Let $X$ be a coarse space and $E \subseteq X \times X$ a controlled set.

(i) The controlled set $E$ is called symmetric if $E^{-1} = E$.

(ii) For any $x \in X$, we write $E_x = \{y \in X \mid (y,x) \in E\}$.
(iii) For any \( U \subseteq X \), we write \( E_U = \bigcup_{x \in U} E_x \).

(iv) A subset \( U \subseteq X \) is called **bounded** if \( U \times U \) is a controlled set.

(v) A subset \( U \subseteq X \) is called **\( E \)-bounded** if \( U \times U \subseteq E \).

(vi) We write \( E^{\circ n} \) for the \( n \)-fold composition \( E \circ E \circ \cdots \circ E \).

(vii) The set \( E \) generates the coarse structure if for any controlled set \( F \), there is an integer \( n \) such that \( F \subseteq E^{\circ n} \).

For later use we give some basic properties using these definitions.

**Lemma 2.6.** Let \( X \) be a coarse space.

(i) Each controlled set is contained in a symmetric controlled set.

(ii) A subset \( U \subseteq X \) is bounded if and only if it is \( E \)-bounded for some controlled set \( E \).

(iii) If \( E \) is a controlled set and \( x \in X \) then \( E_x \) is \( E \circ E^{-1} \)-bounded (though it is not necessarily \( E \)-bounded).

(iv) If \( U \) is an \( E \)-bounded set, it is contained in \( E_x \) for any \( x \in U \).

**Proof.** These follow directly from the definitions.

Most of the time, we will only consider symmetric controlled sets.

### 3 Bounded geometry

Consider a coarse space \( X \). In [9] it is defined that \( X \) is a discrete space of bounded geometry if for every controlled set \( E \subseteq X \times X \), there is an integer \( N \) such that \( \#E_x \leq N \) for all \( x \in X \). We will define geometric property (T) for a wider class of spaces. We still need a concept of bounded geometry, which we define below.

**Definition 3.1.** Let \( X \) be a coarse space. A controlled set \( F \subseteq X \times X \) is called **covering** if it is symmetric and for every controlled \( E \) there is an \( N \) such that each \( E_x \) can be covered by at most \( N \) sets that are \( F \)-bounded.

**Lemma 3.2.** Let \( X \) be a coarse space and \( F \subseteq X \times X \) a symmetric controlled set. The following are equivalent:

(i) The controlled set \( F \) is covering.

(ii) For each controlled set \( E \) there is a constant \( N \) such that each \( E \)-bounded set \( U \) can be covered by at most \( N \) sets that are \( F \)-bounded.

**Proof.** Suppose \( F \) is covering and let \( E \) be a controlled set. Let \( N \) be such that each \( E_x \) can be covered by at most \( N \) sets that are \( F \)-bounded. Let \( U \) be a non-empty \( E \)-bounded set. For any \( x \in U \) we have \( U \subseteq E_x \), so \( U \) is also covered by at most \( N \) sets that are \( F \)-bounded.

Conversely, suppose that (ii) is true and let \( E \) be a controlled set. There is a constant \( N \) such that each \( E \circ E^{-1} \)-bounded set can be covered by at most \( N \) sets that are \( F \)-bounded. Then each \( E_x \) can also be covered by at most \( N \) sets that are \( F \)-bounded.
Definition 3.3. A space \( X \) has bounded geometry if there is a controlled covering set \( F \subseteq X \times X \).

The definition above is easily seen to be equivalent to the definition given by Roe in [5, Definition 3.8].

Example 3.4. Consider a metric space \( X \) with the corresponding coarse structure. If there exists a controlled covering set, we can enlarge it to \( E_R \) for some \( R > 0 \). Therefore, \( X \) has bounded geometry if and only if \( E_R \) is covering for some \( R > 0 \).

In fact, for many examples of metric spaces, \( E_R \) will be covering for any \( R > 0 \).

Example 3.5. A discrete space of bounded geometry certainly has bounded geometry in this definition, as we can take \( F \) to be the diagonal.

A different way to characterise bounded geometry is by the existence of a measure on \( X \) satisfying certain properties. We will study measures on any sigma algebra on the set \( X \), with as only condition, that singletons must be measurable. A controlled set \( E \subseteq X \times X \) acts on the set of subsets of \( X \), by sending \( U \) to \( E \cap U \). Accordingly, given a measure on \( X \), we say that a controlled set \( E \) is measurable if \( E \cap U \) is measurable for all measurable \( U \subseteq X \). In particular, \( E_x \) is then measurable. Note that \( (E \cap F)_U = E_{FU} \), therefore the composition of measurable controlled sets is again measurable.

Definition 3.6. Let \((X, E)\) be a coarse space, and let \( \mu \) be a measure on \( X \), for any sigma algebra that contains all singletons.

(i) The measure is called \( E \)-uniformly bounded or simply uniformly bounded if for each controlled set \( E \) there is a constant \( C > 0 \) such that each \( E \)-bounded measurable set \( U \) has measure at most \( C \).

(ii) A symmetric controlled set \( E \) is called \( \mu \)-gordo or simply gordo if it is measurable and \( \mu(E_x) \) is bounded away from zero independently of \( x \), for all \( x \in X \).

Proposition 3.7. Let \( X \) be a coarse space. The following are equivalent:

(i) The space \( X \) has bounded geometry.

(ii) There is a uniformly bounded measure \( \mu \) on \( X \) and a gordo set \( E \) for \( \mu \).

(iii) There is a coarsely dense subspace \( Y \subseteq X \) that is a discrete space of bounded geometry.

Proof. Suppose \( X \) has bounded geometry and \( F \subseteq X \times X \) is a controlled covering set. By Zorn’s Lemma there is a maximal subset \( Y \subseteq X \) satisfying \((y, y') \notin F \) for all \( y \neq y' \in Y \). By maximality we know that for each \( x \in X \) there exists \( y \in Y \) such that \((x, y) \in F \), so \( Y \) is coarsely dense in \( X \). To show that \( Y \) is a discrete space of bounded geometry, consider a controlled set \( E \subseteq X \times X \). There exists \( N \) such that every \( E_y \) can be covered by \( F \)-bounded sets \( U_1, \ldots, U_N \). Since each \( F \)-bounded set can contain at most 1 element of \( Y \), we see that \( \#(E_y \cap Y) \leq N \), so \( Y \) has bounded geometry, proving \((i) \implies (iii)\).

Suppose \( Y \subseteq X \) is a coarsely dense discrete space of bounded geometry and let \( \mu \) be the counting measure of \( Y \). Any controlled set \( E \) on \( X \) restricts to a controlled set on \( Y \). Since \( Y \) is a discrete space of bounded geometry any \( E \)-bounded set can contain a bounded number of points of \( Y \). This shows that \( \mu \) is uniformly bounded. Now let \( E \subseteq X \times X \) be a symmetric controlled set such that
for all $x \in X$ there is $y \in Y$ with $(x, y) \in E$. Then each $E_x$ has measure at least 1, so $E$ is gordo, showing (iii) $\implies$ (ii).

Finally, let $\mu$ be a uniformly bounded measure on $X$ and let $F$ be a gordo set for $\mu$. We will show that $F^{\circ 4}$ is a covering set. There is $\varepsilon > 0$ such that $\mu(F_x) \geq \varepsilon$ for all $x$. Let $E$ be a symmetric controlled set. Since $\mu$ is uniformly bounded, there is a constant $C$ such that $\mu(V) \leq C$ for all $F \circ E \circ F$-bounded measurable $V$. Now let $U \subseteq X$ be $E$-bounded. Then $F_U$ is $F \circ E \circ F$-bounded, so all its measurable subsets have measure at most $C$. Let $Y \subseteq U$ be maximal such that the $F_y$ are pairwise disjoint when the $y$ range over $Y$. Since the $F_y$ are all contained in $F_U$ and have measure at least $\varepsilon$ we have $\#Y \leq N$, with $N = \frac{C}{\varepsilon}$. Since $Y$ is maximal we know that for all $x \in U$ there is a $y \in Y$ with $F_x \cap F_y \neq \emptyset$, so $(x, y) \in F \circ F$. This implies that the sets $(F \circ F)_y, y \in Y$ cover $U$. These sets are $F^{\circ 4}$-bounded. By Lemma 4.2 we see that $F^{\circ 4}$ is covering, showing (ii) $\implies$ (i). □

4 Blocking collections

From now on, let $X$ be a coarse space of bounded geometry that can be covered by a countable number of bounded sets. Let $\mu$ be a uniformly bounded measure on $X$ for which a gordo set exists. This measure exists by Proposition 4.1 in applications, there is often already a measure given that satisfies the condition above. For example, if $X$ is a metric space and $\mu$ is a Borel measure, then $\mu$ satisfies the conditions if there is a radius $R > 0$ such that $R$-balls are bounded from below in measure, and for all radii $R > 0$ the $R$-balls are bounded from above in measure.

The definition of geometric property (T) uses the measure $\mu$, but we will show later that it does not depend on it (see Theorem 8.6).

We will start by defining the notion of a blocking collection of subsets of $X$ and proving some elementary results about them.

**Definition 4.1.** Let $E$ be a controlled set and let $\varepsilon > 0$. A collection $(A_i)$ of measurable subsets of $X$ is called $(E, \varepsilon)$-blocking if $\bigcup_i A_i \times A_i \subseteq E$, the $A_i$ are pairwise disjoint and $\mu(A_i) \geq \varepsilon$ for all $i$. A collection $(A_i)$ of measurable subsets of $X$ is called blocking if it is $(E, \varepsilon)$-blocking for some $E, \varepsilon$.

The conditions on the measure $\mu$ ensure that there are sufficiently large blocking collections:

**Lemma 4.2.** There is a blocking collection $(A_i)$ with union $X$.

**Proof.** Let $E \subseteq X \times X$ be a gordo set for $\mu$. By Zorn’s Lemma, there is a maximal subset $I \subseteq X$ such that all the $E_i$ are pairwise disjoint when $i$ ranges over $I$. Then for all $x \in X$ there exists an $i \in I$ such that $E_x \cap E_i \neq \emptyset$, so $(x, i) \in E \circ E$. So the $(E \circ E)_i$ cover $X$, while the $E_i$ are pairwise disjoint. Hence, we can find measurable $A_i$, with $E_i \subseteq A_i \subseteq (E \circ E)_i$ such that $X = \bigsqcup_i A_i$. To construct the $A_i$ explicitly, choose a well-ordering on $I$ and define

$$A_i = E_i \cup \left( (E \circ E)_i \setminus \bigcup_{i' \neq i} E_{i'} \cup \bigcup_{i' < i} (E \circ E)_{i'} \right).$$

Now the measure of the $A_i$ is bounded from below because $E$ is gordo, and the $A_i$ are $E^{\circ 4}$-bounded, showing that the $A_i$ form a blocking collection. □

**Definition 4.3.** Let $E \subseteq X \times X$. We say that $E$ is a controlled set in blocks if $E$ is of the form $\bigsqcup_i A_i \times A_i$, where the $(A_i)$ form a blocking collection.
Note that $\bigsqcup_i A_i \times A_i$ is always a controlled set if the $(A_i)$ form a blocking collection.

**Lemma 4.4.** Let $E \subseteq X \times X$ be a controlled set. Then $E$ is contained in a finite union of controlled sets in blocks.

**Proof.** Assume without loss of generality that $E$ is symmetric. By Lemma 4.2 there is a blocking collection $(A_i)$ with union $X$. Let $F = \bigsqcup_i A_i \times A_i$, which is a controlled set. Define a graph structure on the index set $I$ by connecting $i$ and $j$ if $E_A_i \cap E_A_j \neq \emptyset$.

If $i$ and $j$ are connected by an edge, then $(E \circ E)A_i \cap A_j \neq \emptyset$, and since $A_j$ is $F$-bounded, it follows that $E \circ E (F \circ E)A_i$. Since $\mu$ is uniformly bounded and the $A_i$ are $F$-bounded, we know that the measure of $(F \circ E \circ E)A_i$ is bounded from above, say by $M$, where $M$ does not depend on $i$. Moreover, the measure of $A_j$ is bounded from below uniformly in $j$, say by $\varepsilon > 0$. Since the $A_j$ are also pairwise disjoint, the degree of any vertex $i$ in the graph can be at most $\frac{M}{\varepsilon}$. Then we can colour the graph in $N$ colours, for some integer $N$, such that no two adjacent vertices have the same colour. Now for $1 \leq k \leq N$ define

$$C_k = \{E_{A_i} \mid i \text{ is coloured with the } k\text{-th colour}\}.$$

Then because of the way we coloured the graph, the sets in $C_k$ are disjoint. Moreover they are $E \circ F \circ E$-bounded and bounded from below in measure, so $C_k$ is a blocking collection for each $k$. For $1 \leq k \leq N$ we now define

$$E_k = \bigsqcup \{E_{A_i} \times E_{A_i} \mid i \text{ is coloured with the } k\text{-th colour}\}.$$  

These are controlled sets in blocks. For all $(x, y) \in E$, there is an $i$ with $y \in A_i$. Let $k$ be the colour of $i$. Then $(x, y) \in E_{A_i} \times E_{A_i} \subseteq E_k$. So $E$ is contained in $\bigsqcup_k E_k$. \qed

## 5 The algebra of a coarse space

Analogous to the discrete setting, we now define the algebra of operators with controlled support. We will use the Hilbert space $L^2X = L^2(X, \mu)$.

**Definition 5.1.** Let $E \subseteq X \times X$ be a measurable controlled set and $T \in B(L^2X)$. We say that $T$ is supported on $E$ if for all measurable $U \subseteq X$, all $\xi \in L^2X$ supported on $U$ and all $\eta \in L^2X$ we have

$$\langle \xi, T\eta \rangle = \langle \xi, T\eta |_{E_U} \rangle.$$  

We say that $T$ has controlled support if it is supported on some measurable controlled set.

We can give some equivalent definitions for this:

**Lemma 5.2.** Let $E$ be a measurable symmetric controlled set and $T \in B(L^2X)$. The following are equivalent:

(i) The operator $T$ is supported on $E$.

(ii) For all measurable $U \subseteq X$, all $\xi \in L^2X$ and all $\eta \in L^2X$ supported on $U$ we have

$$\langle \xi, T\eta \rangle = \langle \xi |_{E_U}, T\eta \rangle.$$
Lemma 5.8. The dual of $L^2(X)$ is supported on $E$.

(iii) The operator $T^*$ is supported on $E$.

(iv) For all measurable $U \subseteq X$ and all $\eta \in L^2(X)$ supported on $U$ the function $T\eta$ is supported on $E_U$.

Proof. To prove $(i) \implies (ii)$, we have

$$\langle \xi, T\eta \rangle = \langle \xi|_{E_U}, T\eta \rangle + \langle \xi|_{X \setminus E_U}, T\eta \rangle = \langle \xi|_{E_U}, T\eta \rangle + \langle \xi|_{X \setminus E_U}, T\eta|_{E_{X \setminus E_U}} \rangle = \langle \xi|_{E_U}, T\eta \rangle.$$  

The other direction is similar, and the equivalences $(ii) \iff (iii)$ and $(ii) \iff (iv)$ are clear. 

Clearly linear combinations of operators with controlled support again have controlled support, as does a product of operators with controlled support. So the operators with controlled support form a unital $*$-algebra.

Definition 5.3. Denote by $C_{cs}[X]$ the subalgebra $\{T \in B(L^2(X)) \mid T\text{ has controlled support}\}$ of $B(L^2(X))$. We will denote it $C_{cs}[X, \mu]$ when the measure is not clear from the context.

Remark 5.4. In the case that $X$ is a discrete space of bounded geometry, the algebra $C_{cs}[X]$ is the same as the algebra $C_u[X]$ as defined in [1] Definition 3.1.

Definition 5.5. Let $T \in C_{cs}[X]$, let $E$ be a controlled set and let $\varepsilon > 0$. We say that $T$ is an operator in $(E, \varepsilon)$-blocks if it is supported on $\bigsqcup_i A_i \times A_i$ where $(A_i)$ is some $(E, \varepsilon)$-blocking collection. We say that $T$ is an operator in blocks if it is supported on some controlled set in blocks.

For a measurable subset $U \subseteq X$, denote by $\pi_U: L^2(X) \to L^2(U)$ the restriction map and by $i_U: L^2(U) \to L^2(X)$ extension by 0. An operator $T$ that is supported on $\bigsqcup_i A_i \times A_i$ can be written as $T = \sum_i T_i$, where $T_i = \pi_{A_i} T A_i : L^2 A_i \to L^2 A_i$, using Lemma 5.2. This decomposition makes it very convenient to do computations with operators in blocks. Moreover, every operator in $C_{cs}[X]$ can be written as a finite sum of operators in blocks:

Lemma 5.6. Let $T \in C_{cs}[X]$. We can write $T$ as a finite sum $T_1 + \ldots + T_N$ where each $T_i \in C_{cs}[X]$ is an operator in blocks.

Proof. Let $T$ be supported on the measurable and symmetric controlled set $E$. Let $(A_i)$ be a blocking collection as in the proof of Lemma 5.4 and consider the same colouring on $I$ as in that proof. Let

$$T_k = \sum_{i \text{ with colour } k} \pi_{A_i} T.$$ 

Then $T_k$ is controlled on $\sqcup_i \text{ with colour } k E_{A_i} \times E_{A_i}$, so it is an operator in blocks. Moreover we have $T = T_1 + \ldots + T_N$. 

We define some spaces of functions on $X$:

Definition 5.7. (i) Let $L^2(X)$ denote the $L^2$-functions on $X$ with bounded support.

(ii) Let $LL^2(X)$ denote the measurable functions on $X$ that are locally $L^2$: for each $\varphi \in LL^2(X)$ and each bounded measurable $U$, we have $\int_U |\varphi|^2 < \infty$.

Lemma 5.8. The dual of $L^2(X)$ can be identified with $LL^2(X)$. 

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Definition 6.2. Let \( \rho \) ask the map \( \rho \).

Proof. For any \( \xi \in L^2_X \) and \( \eta \in LL^2_X \), the inner product \( \langle \eta, \xi \rangle \) is finite, so \( \eta \) defines a function in \( (L^2_X)^* \).

Conversely, let \( \varphi \in (L^2_X)^* \). For any bounded measurable \( U \), the function \( \varphi \) can be pulled back along the inclusion \( L^2_U \to L^2_X \) to an element of \( (L^2_U)^* \), which may be viewed as a function \( \eta_U \in L^2_U \). If \( U \subseteq V \) are bounded and measurable, then we find that \( \eta_U = \eta_V|_U \). Therefore, the \( \eta_U \) may be glued together to a function \( \eta \in LL^2_X \).

If \( T \in B(L^2_X) \) has controlled support, it sends \( L^2_X \) to \( L^2_X \), and so does the dual \( T^* \). Taking the dual of \( T^* : L^2_X \to L^2_X \) gives a map \( LL^2_X \to LL^2_X \) which we denote by \( T \) again. Now define the linear map \( \Phi : \mathcal{C}_s[X] \to LL^2_X \) by \( \Phi(T) = T(1_X) \in LL^2_X \). Explicitly, if \( T \) is supported on a controlled set \( E \) then we have
\[
\Phi(T)(x) = (T1_E)(x).
\]

For all \( T, S \in \mathcal{C}_s[X] \) we have \( \Phi(ST) = S\Phi(T) \).

Lemma 5.9. Let \( T \in \mathcal{C}_s[X] \) be such that \( \Phi(T) \) is bounded. Then we can write \( T = T_1 + \ldots + T_N \) where the \( T_k \) are operators in blocks and \( \Phi(T_k) \) is bounded.

Proof. Construct \( T_k \) as in the proof of Lemma 5.8. Then the \( \Phi(T_k) \) are bounded.

6 Geometric property (T)

Definition 6.1. A representation of \( \mathcal{C}_s[X] \) consists of a Hilbert space \( \mathcal{H} \) and a unital \(*\)-homomorphism \( \rho : \mathcal{C}_s[X] \to B(\mathcal{H}) \). If no confusion arises we will omit the map \( \rho \) from the notation.

The standard representation of \( \mathcal{C}_s[X] \) is the inclusion \( \mathcal{C}_s[X] \hookrightarrow B(L^2_X) \). Note that we do not ask the map \( \rho \) to be continuous with respect to the norm on \( \mathcal{C}_s[X] \), inherited from \( B(L^2_X) \).

Definition 6.2. Let \( T, S \in \mathcal{C}_s[X] \).

(i) We write \( T \geq S \) if this inequality holds in \( B(L^2_X) \).

(ii) We write \( T \geq_{\max} S \) if for every representation \( (\rho, \mathcal{H}) \) of \( \mathcal{C}_s[X] \), we have the inequality \( \rho(T) \geq \rho(S) \).

(iii) Let \( \|T\|_{L^2} \) be the norm of \( T \) in \( B(L^2_X) \).

(iv) Define \( \|T\|_{\max} = \sup \|\rho(T)\| \), where the supremum is over all representations \( (\rho, \mathcal{H}) \). We will see in Corollary 6.3 that this is always finite, and hence it also defines a norm on \( \mathcal{C}_s[X] \).

Lemma 6.3. Let \( T \in \mathcal{C}_s[X] \) be an operator in blocks.

(i) If \( T \geq 0 \), then \( T \geq_{\max} 0 \).

(ii) We have \( \|T\|_{L^2} = \|T\|_{\max} \).

Proof. (i) Suppose \( T \) is supported on \( \bigcup_i A_i \times A_i \) where \( (A_i) \) is a blocking collection. Write \( T = \sum_i T_i \) with \( T_i : L^2_A \to L^2_A \). Then \( T^J = \sum_i T_i^{J_i} \) is again an operator in blocks. Now for any representation \( (\rho, \mathcal{H}) \) we have \( \rho(T) = \rho(T^J)^2 \geq 0 \), so we have \( T \geq_{\max} 0 \).
(ii) Clearly we have \(\|T\|_{L^2} \leq \|T\|_{\text{max}}\). Note that \(T^*T\) is again an operator in blocks. We have \(0 \leq T^*T \leq \|T\|_{L^2}^2\), and by part (i), it follows that \(0 \leq \max T^*T \leq \|T\|_{L^2}^2\). So for any representation \((\rho, \mathcal{H})\), we have \(0 \leq \rho(T^*T) \leq \|T\|_{L^2}^2\). So \(\|\rho(T)\| = \|\rho(T)^*\rho(T)\|^{\frac{1}{2}} = \|\rho(T^*T)\|^{\frac{1}{2}} \leq \|T\|_{L^2}\), and hence \(\|T\|_{L^2} = \|T\|_{\text{max}}\).

\[\square\]

If \(T\) is an operator in blocks we will simply write \(\|T\|\) for both the norms \(\|T\|_{L^2}\) and \(\|T\|_{\text{max}}\).

**Corollary 6.4.** For any \(T \in \mathcal{C}_{cs}[X]\), the maximal norm \(\|T\|_{\text{max}}\) is finite.

**Proof.** This follows directly from Lemma 6.3 and Lemma 5.6. \(\square\)

Therefore \(\|\cdot\|_{\text{max}}\) defines a norm and we can define the maximal \(C^*\)-algebra:

**Definition 6.5.** The maximal \(C^*\)-algebra \(C^*_{\text{max}}(X)\) is defined as the completion of \(\mathcal{C}_{cs}[X]\) with respect to the maximal norm \(\|\cdot\|_{\text{max}}\).

For a discrete space \(X\) of bounded geometry the \(C^*\)-algebra \(C^*_{\text{max}}(X)\) is the same as the one defined in [9] (there it is called \(C^*_{u,\text{max}}(X)\)). For \(f \in L^\infty X\) denote by \(M_f \in \mathcal{B}(L^2 X)\) the multiplication operator by \(f\). Similar to [9], we define the subspace of the constant vectors.

**Definition 6.6.** Let \(\mathcal{H}\) be a non-degenerate representation of \(\mathcal{C}_{cs}[X]\), and let \(v \in \mathcal{H}\). We call \(v\) a constant vector if \(Tv = M_{\Phi(T)}v\) for all operators \(T \in \mathcal{C}_{cs}[X]\) for which \(\Phi(T)\) is bounded. Here \(\Phi\) is as defined at the end of Section 3. The constant vectors form a closed subspace \(\mathcal{H}_c\) of \(\mathcal{H}\).

Since we can replace \(T\) by \(T - M_{\Phi(T)}\), it is enough to ask that \(Tv = 0\) for all \(T\) with \(\Phi(T) = 0\). Now we are ready to define geometric property (T).

**Definition 6.7.** Let \(X\) be a coarse space with bounded geometry and let \(\mu\) be a uniformly bounded measure for which a gordo set exists. We say that \((X, \mu)\) satisfies geometric property (T) if there are a controlled set \(E\) and constants \(\varepsilon, \gamma > 0\) such that for every non-degenerate representation \((\rho, \mathcal{H})\) and every unit vector \(v \in \mathcal{H}_c^\perp\) there is an operator \(T\) in \((E, \varepsilon)\)-blocks with \(\Phi(T)\) bounded and \(\|\!(T - M_{\Phi(T)})v\!\| \geq \gamma \|T\|\).

In the definition, it is necessary to ask that \(\Phi(T)\) is a bounded function, because otherwise \(M_{\Phi(T)}\) would not be defined. By considering the operator \(T - M_{\Phi(T)}\), we see that we can equivalently ask there to be an operator \(T\) in \((E, \varepsilon)\)-blocks with \(\Phi(T) = 0\) and \(\|Tv\| \geq \gamma \|T\|\).

If \(X\) is a discrete space of bounded geometry with a generating controlled set, and \(\mu\) is the counting measure, we retrieve the definition of [9] (cf. Proposition 3.1 in that paper):

**Lemma 6.8.** Let \(X\) be a discrete space of bounded geometry for which there is a generating controlled set, and let \(\mu\) be the counting measure. The following are equivalent:

(i) The space \((X, \mu)\) satisfies geometric property (T).

(ii) There exists a controlled generating set \(E\) and a constant \(\gamma > 0\) such that for any representation \(\mathcal{H}\) and unit vector \(v \in \mathcal{H}_c^\perp\), there is an operator \(T \in \mathcal{C}_{cs}[X]\) with support in \(E\) such that

\[\|\!(T - M_{\Phi(T)})v\!\| > \gamma \sup_{x,z} \!|T_{xz}|,\]

where \(T_{xz}\) denotes the value at \((x, z)\) of the matrix corresponding to the operator \(T\).
Proof. We certainly have \( \|T\|_{L^2} \geq \sup_{x,z} |T_{xz}| \). With this, the direction (i) \(\implies\) (ii) is trivial, since we may enlarge the controlled set \( E \) to make it generating.

Now suppose (ii) is true, take \( E, \gamma \) satisfying the condition, and let \( \varepsilon = 1 \). Without loss of generality, we may assume \( E \) to be symmetric. There is an integer \( N \) such that \( \#E_x \leq N \) for each \( x \in X \). Let \( \mathcal{H} \) be a representation and \( v \in \mathcal{H}_E^\perp \) a unit vector. We know there is \( T \in \mathcal{C}_{cs}[X] \) supported on \( E \) such that \( \|(T - \Phi(T))v\| > \gamma \sup_{x,z} |T_{xz}| \). For each pair \( (x, z) \in E \) there are fewer than \( 4N \) other pairs \( (x', z') \in E \) with \( \{x, z\} \cap \{x', z'\} \neq \emptyset \). By a graph colouring argument we may write \( T = T_1 + \ldots + T_{4N} \) as a sum of operators controlled on \( E \) with for each \( k \) : if \( (x, z) \in E \) and \( (x', z') \in E \) are different pairs where \( T_k \) is supported, then \( \{x, z\} \cap \{x', z'\} = \emptyset \). Moreover we get \( \sup_{x,z} |T_{xz}| = \sup_k \|T_k\|_{L^2} \). We then have

\[
\left\| \sum_k (T_k - M_{\Phi(T_k)})v \right\| = \| (T - M_{\Phi(T)})v \| \geq \gamma \sup_{x,z} |T_{xz}| = \gamma \sup_k \|T_k\|_{L^2},
\]
	hence there is a \( k \) with \( \| (T_k - M_{\Phi(T_k)})v \| \geq \frac{\gamma}{\sqrt{N}} \|T_k\|_{L^2} \), proving (i).

\[\Box\]

7 Laplacians

In this section we will give a different characterisation of geometric property (T) using spectral properties of Laplacians.

**Definition 7.1.** Let \( E \) be a measurable symmetric controlled set. Define the **Laplacian** \( \Delta^E \in B(L^2 X) \) as

\[
\Delta^E(\xi)(x) = \int_{E_x} (\xi(x) - \xi(y))d\mu(y).
\]

The Laplacian corresponding to \( E \) is obviously supported on \( E \), in particular we have \( \Delta^E \in \mathcal{C}_{cs}[X] \). The Laplacians are positive operators, as is shown by the following lemma.

**Lemma 7.2.** For any measurable symmetric controlled set \( E \) we have \( 0 \leq \max \Delta^E \leq \max 2M \), where \( M = \sup_x \mu(E_x) \).

**Proof.** A straightforward computation shows that for all measurable symmetric controlled sets \( F \), we have

\[
\langle \Delta^F \xi, \xi \rangle = \int_X \overline{\Delta^F \xi(x)} \xi(x)d\mu(x)
\]
\[
= \int_X \int_{F_x} (\overline{\xi(x)} - \overline{\xi(y)})\xi(x)d\mu(y)d\mu(x)
\]
\[
= \frac{1}{2} \int_X \int_{F_x} \left(\overline{(\xi(x) - \xi(y))}\xi(x) + (\overline{\xi(y)} - \overline{\xi(x)})\xi(y)\right) d\mu(y)d\mu(x)
\]
\[
= \frac{1}{2} \int_F |\xi(x) - \xi(y)|^2 d(\mu \times \mu)(x, y)
\]
\[
\geq 0,
\]

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so $\Delta^F \geq 0$. Since $|\xi(x) - \xi(y)|^2 \leq 2|\xi(x)|^2 + 2|\xi(y)|^2$ we also have

$$\langle \Delta^F \xi, \xi \rangle \leq \int_F (|\xi(x)|^2 + |\xi(y)|^2)d(\mu \times \mu)(x, y) = 2 \int_F |\xi(x)|^2d(\mu \times \mu)(x, y)$$

by symmetry. This is equal to $2 \int_F |\xi(x)|^2\mu(F_x)d\mu(x)$, which is at most $2\|\xi\|^2 \sup_x \mu(F_x)$, so we have $\|\Delta^F\|_{L^2} \leq 2 \sup_x \mu(F_x)$. Moreover, if $F_1, F_2$ are disjoint measurable symmetric controlled sets, we see directly that $\Delta^F_1 \cup F_2 = \Delta^F_1 + \Delta^F_2$.

Let $E$ be any measurable symmetric controlled set. By Lemma 6.3 there are symmetric controlled sets in blocks $E_1, \ldots, E_N$ whose union contains $E$. Let $E'_1, \ldots, E'_N$ be pairwise disjoint measurable sets such that $E'_k \subseteq E_k$ for each $k$ and $E = \bigsqcup_k E'_k$. Moreover, we have the following lemma, which shows the converse. 

Lemma 7.3 Let $E$ be a controlled set in blocks and let $T \in \mathcal{C}_{cs}[X]$ be a positive operator supported on $E$ such that $\Phi(T) = 0$. Then

$$T \leq \inf_x \frac{\|T\|}{\mu(E_x)} \Delta^E,$$

where the infimum is taken over all $x$ with $E_x \neq \emptyset$.

Proof. Write $E = \bigsqcup_i A_i \times A_i$ and $T = \sum_i T_i$ with $T_i : L^2 A_i \to L^2 A_i$. It is enough to prove the inequality $T_i \leq \|T_i\| \Delta^{E_i}$ for each $i$, where $E_i = A_i \times A_i$. The space $L^2 A_i$ can be written as the direct sum of the constant functions and the functions with zero integral. For any $\xi \in L^2 A_i$, write $\xi = \xi_c + \xi_d$ where $\xi_c$ is constant and $\int \xi_d = 0$. Then $T_i \xi = T_i \xi_d$ and $\Delta^{E_i} \xi = \mu(A_i) \xi_d$. So we have $\langle T_i \xi, \xi \rangle = \|T_i\| \cdot \|\xi_d\|^2 = \langle \frac{\|T_i\|}{\mu(A_i)} \Delta^{E_i} \xi, \xi \rangle$ showing that $T_i \leq \|T_i\| \Delta^{E_i}$.

Since $\Phi(\Delta^E) = 0$, we have $\Delta^E v = 0$ for all $v \in \mathcal{H}_c$, where $\mathcal{H}$ is a representation of $\mathcal{C}_{cs}[X]$. Moreover, we have the following lemma, which shows the converse.

Lemma 7.4. Let $(\rho, \mathcal{H})$ be a representation of $\mathcal{C}_{cs}[X]$. Then

$$\mathcal{H}_c = \cap \ker(\rho(\Delta^E))$$

where the intersection ranges over all the measurable symmetric controlled sets $E$.

Proof. Let $v \in \cap \ker(\rho(\Delta^E))$ and let $T \in \mathcal{C}_{cs}[X]$ such that $\Phi(T)$ is bounded, we will show that $Tv = M_{\rho(T)}v$, showing that $v \in \mathcal{H}_c$. By Lemma 6.3, we can assume that $T$ is an operator in blocks. We can also assume that $\Phi(T) = 0$ by considering the operator $T - \Phi(T)$. By Lemma 7.3, there is a measurable symmetric controlled set $E$ and a constant $c$ such that $T^*T \leq_{max} c \Delta^E$, since $T^*T$ is a positive operator in blocks. Then $\Delta^E v = 0$ and we get $\|Tv\|^2 = \langle T^*Tv, v \rangle \leq c \langle \Delta^E v, v \rangle = 0$, so $Tv = 0$.

Definition 7.5. Let $T \in C^*_\text{max}(X)$ be a positive operator. We say that $T$ has spectral gap if there is a constant $\gamma > 0$ such that $\sigma_{\text{max}}(T) \subseteq \{0\} \cup [\gamma, \infty)$, where $\sigma_{\text{max}}$ denotes the spectrum of $T$ in the maximal $C^*$-algebra.

Now we can prove the generalisation of [3, Proposition 5.2].
Proposition 7.6. Let $X$ be a coarse space of bounded geometry and let $\mu$ be a uniformly bounded measure for which a gordo set exists. Then $(X, \mu)$ has geometric property (T) if and only if there exists a measurable symmetric controlled set $E$ such that for each representation $(\rho, \mathcal{H})$ we have $\mathcal{H}_c = \ker(\rho(\Delta^E))$ and $\Delta^E$ has spectral gap.

Proof. Suppose $X$ has property (T). Let $E$ be a controlled set and $\varepsilon, \gamma > 0$ constants such that for every representation $\mathcal{H}$ and every unit vector $v \in \mathcal{H}^\perp$, there is an operator $T$ in $(E, \varepsilon)$-blocks such that $\Phi(T) = 0$ and $\|Tv\| \geq \gamma \|T\|$. We will show that for each representation $(\rho, \mathcal{H})$, we have $\mathcal{H}_c = \ker(\rho(\Delta^E))$ and $\Delta^E$ has spectral gap.

Let $v \in \mathcal{H}_c^\perp$ be a unit vector. Let $T$ be an operator in $(E, \varepsilon)$-blocks with $\Phi(T) = 0$ and $\|Tv\| \geq \gamma \|T\|$. Let $T$ be supported on the blocks $(A_i)$ and let $E' = \bigcup_i A_i \times A_i \subseteq E$. By Lemma 7.7 we have $T^* T \leq \frac{\|T\|}{\min_i \mu(A_i)} \Delta^{E'} \leq \frac{\|T\|^2}{\varepsilon} \Delta^E$. Now

$$\gamma^2 \|T\|^2 \leq \|Tv\|^2 \leq \frac{\|T\|^2}{\varepsilon} \langle \Delta^E v, v \rangle \leq \frac{\|T\|^2}{\varepsilon} \|\Delta^E v\|,$$

so $\|\Delta^E v\| \geq \gamma^2 \varepsilon$. This shows that $\ker(\rho(\Delta^E)) \cap \mathcal{H}_c^\perp = \{0\}$, so $\ker(\rho(\Delta^E)) = \mathcal{H}_c$. Moreover, it shows that $\sigma(\rho(\Delta^E)) \subseteq \{0\} \cup [\gamma^2 \varepsilon, \infty)$. Since this holds for every representation $(\rho, \mathcal{H})$, we get $\sigma_{\text{max}}(\Delta^E) \subseteq \{0\} \cup [\gamma^2 \varepsilon, \infty)$, hence $\Delta^E$ has spectral gap.

Now suppose that $E$ is a measurable symmetric controlled set such that for each representation $(\rho, \mathcal{H})$ we have $\mathcal{H}_c = \ker(\rho(\Delta^E))$ and $\Delta^E$ has spectral gap, say $\sigma(\rho(\Delta^E)) \subseteq \{0\} \cup [\gamma, \infty)$. By Lemma 7.3 there are controlled sets in blocks $E_1, \ldots, E_N$ whose union contains $E$. Let $\varepsilon = \min_k \inf_x \mu((E_k)_x) > 0$, where $k$ ranges from 1 to $N$ and $x$ ranges over all elements in $X$ with $(E_k)_x \neq \emptyset$. Now let $(\rho, \mathcal{H})$ be a representation and let $v \in \mathcal{H}_c^\perp$ be a unit vector. Then $v \in \ker(\rho(\Delta^E))$ so $\langle \Delta^E v, v \rangle \geq \gamma$. We also have $\Delta^E \leq \Delta^{E_1} + \ldots + \Delta^{E_N}$ so there is $1 \leq k \leq N$ such that $\langle \Delta^{E_k} v, v \rangle \geq \frac{\gamma}{N}$. Now $\Delta^{E_k}$ is an $(E, \varepsilon)$-operator in blocks and $\|\Delta^{E_k} v\| \geq \frac{\gamma}{N}$, so we see that $X$ has geometric property (T).

In [9] Proposition 5.2 it is actually shown that $\mathcal{H}_c = \ker(\rho(\Delta^E))$ holds for all generating symmetric controlled sets $E$ and that if $X$ has geometric property (T), then $\Delta^E$ has spectral gap for all generating symmetric controlled sets $E$. In our case, we have a similar result, for which we first need another lemma and its corollary.

Lemma 7.7. Let $E$ be a gordo set and $F$ a symmetric measurable controlled set containing $E$ and satisfying $F \circ E = E \circ F$. Let $f \in L^\infty(X)$ be the function $f(x) = \mu((E \circ F)_x)$. Then there is a constant $\delta$ such that

$$\delta \Delta^{F \circ E} \leq_{\text{max}} (2M_f - \Delta^{E \circ F}) M^\perp \Delta^{E \circ F}.$$

Proof. For every $\xi \in L^2 X$ and $x \in X$, we have

$$(2M_f - \Delta^{E \circ F})\xi(x) = f(x)\xi(x) + \int_{(E \circ F)_x} \xi(y)d\mu(y).$$
Then we have

\[
(2M_f - \Delta^{E \circ F})M_f \Delta^{E \circ F} \xi(x) = \Delta^{E \circ F} \xi(x) + \int_{(E \circ F)^c} \frac{1}{f(y)} \Delta^{E \circ F} \xi(y) d\mu(y)
\]

\[
= f(x) \xi(x) - \int_{(E \circ F)^c} \xi(y) d\mu(y) + \int_{(E \circ F)^c} \xi(y) d\mu(y) - \int_{(E \circ F)^c} \frac{1}{f(y)} \int_{(E \circ F)^c} \xi(z) d\mu(z) d\mu(y)
\]

\[
= f(x) \xi(x) - \int_{(E \circ F)^c} \alpha(x, z) \xi(z) d\mu(z),
\]

where

\[
\alpha(x, z) = \int_{(E \circ F)^c \cap (E \circ F)^c} \frac{1}{f(y)} d\mu(y).
\]

Let \( z \in (F \circ F)^c \). There is \( y \in X \) with \((x, y) \in F \) and \((y, z) \in F \). It follows that \( E_y \subseteq (E \circ F)^c \cap (E \circ F)^c \), so \( \alpha(x, z) \geq \int_{E_y} \frac{1}{f(y)} d\mu(y) \). Since \( E \) is gordo and \( f \) is bounded from above, there is a constant \( \delta \) such that \( \alpha(x, z) \geq \delta \) for all \( z \in (F \circ F)^c \). Now define

\[
\beta(x, z) = \begin{cases} 
\alpha(x, z) - \delta & \text{if } (x, z) \in F \circ F, \\
\alpha(x, z) & \text{otherwise}.
\end{cases}
\]

Then \( \beta \) is symmetric, \( \beta(x, z) \geq 0 \) for all \( (x, z) \in X \times X \) and

\[
(2M_f - \Delta^{E \circ F})M_f \Delta^{E \circ F} = \delta \Delta^{F \circ F} + T,
\]

where \( T \in C_{cs}[X] \) is the operator defined by

\[
T \xi(x) = \int_{(E \circ F)^c} \beta(x, y)(\xi(x) - \xi(y)) d\mu(y).
\]

It remains to show that \( T \geq_{\max} 0 \). This is proved similarly to Lemma \ref{lem:mu}. Let \( E_1, \ldots, E_N \) be controlled sets in blocks whose union contains \((E \circ F)^c \). Let \( E_1', \ldots, E_N' \) be measurable symmetric subsets of \( E_1, \ldots, E_N \) respectively, that are disjoint from each other and whose union is \((E \circ F)^c \). For \( 1 \leq k \leq N \) define \( T_k \in C_{cs}[X] \) by

\[
T_k \xi(x) = \int_{(E_k')^c} \beta(x, y)(\xi(x) - \xi(y)) d\mu(y).
\]

An easy computation shows that

\[
\langle T_k \xi, \xi \rangle = \frac{1}{2} \int_{E_k} \beta(x, y)(\xi(x) - \xi(y))^2 d(\mu \times \mu)(x, y),
\]

so \( T_k \geq 0 \). Since \( T_k \) is an operator in blocks, we get \( T_k \geq_{\max} 0 \) by Lemma \ref{lem:max}. So we have \( T = \sum_{k=1}^N T_k \geq_{\max} 0 \).

**Corollary 7.8.** Let \( E \) be a gordo set and \( F \) a symmetric measurable controlled set satisfying \( F \circ E = E \circ F \) and \( E \subseteq F \). Then for every representation \((\rho, H)\), we have \( \ker(\rho(\Delta^{F \circ F})) = \ker(\rho(\Delta^{E \circ F})) \). Moreover, \( \Delta^{F \circ F} \) has spectral gap if and only if \( \Delta^{E \circ F} \) has spectral gap.
Proof. Since \( E \subseteq F \) we have \( E \circ F \subseteq F \circ F \), and therefore \( \Delta^{E \circ F} \subseteq \Delta^{F \circ F} \). It follows that \( \ker(\rho(\Delta^{E \circ F})) \subseteq \ker(\rho(\Delta^{F \circ F})) \) and that if \( \Delta^{E \circ F} \) has spectral gap, then \( \Delta^{F \circ F} \) also has spectral gap. The other direction follows from Lemma 4.7.

**Proposition 7.9.** Let \( X \) be a coarse space of bounded geometry and let \( \mu \) be a uniformly bounded measure. Let \( E \) be a gordo set that generates the coarse structure on \( X \). Let \( E' = E^{\circ 3} \). Then for every representation \( (\rho, H) \) of \( C_{cs}[X] \), we have \( \ker(\rho(\Delta^{E'})) = \mathcal{H}_c \), and \( X \) has geometric property (T) if and only if \( \Delta^{E'} \) has spectral gap.

**Proof.** We apply Corollary 7.8 with \( F = E^{\circ n} \) for \( n \geq 2 \). By induction it follows that for all \( n \geq 3 \) we have \( \ker(\rho(\Delta^{E^{\circ n}})) = \ker(\rho(\Delta^{E^2})) \) and that \( \Delta^{E^{\circ n}} \) has spectral gap if and only if \( \Delta^{E^2} \) has spectral gap.

If \( X \) has geometric property (T), then there is a controlled set \( G \) such that \( \ker(\rho(\Delta^G)) = \mathcal{H}_c \) for every representation \( (\rho, H) \) and \( \Delta^G \) has spectral gap. Since \( E \) is generating, there is some \( n \) such that \( G \subseteq E^{\circ n} \). Then \( \Delta^G \leq \max \Delta^{E^{\circ n}} \). So \( \ker(\rho(\Delta^{E^{\circ n}})) = \ker(\rho(\Delta^G)) \) for every representation \( (\rho, H) \) and \( \Delta^{E^{\circ n}} \) and \( \Delta^G \) have spectral gap.

We will now give a third characterisation of geometric property (T) based more directly on the C*-algebra \( C^*_c(X) \). Note that \( \ker(\Phi) \) is a left ideal in \( C_{cs}[X] \). Let \( I_c(X) = C^*_c(X) \ker(\Phi) \) be the left ideal in \( C^*_c(X) \) generated by \( \ker(\Phi) \).

**Proposition 7.10.** Let \( X \) be a coarse space of bounded geometry and let \( \mu \) be a uniformly bounded measure for which a gordo set exists. Then \( (X, \mu) \) has geometric property (T) if and only if there is a projection \( p \in C^*_c(X) \) generating \( I_c(X) \).

**Proof.** First suppose that \( (X, \mu) \) has geometric property (T). Let \( (\rho, H) \) be a faithful unital representation of \( C^*_c(X) \). By Proposition 7.6 there is a controlled set \( E \) such that \( \ker(\rho(\Delta^E)) = \mathcal{H}_c \) and \( \Delta^E \) has spectral gap. By functional calculaus there is a projection \( p \in I_c(X) \) such that \( \rho(v) = 0 \) for all \( v \in \mathcal{H}_c \) and \( \rho(v) = v \) for all \( v \in \mathcal{H}_c^\perp \). Now for any \( t \in I_c(X) \) we have \( tv = 0 \) for all \( v \in \mathcal{H}_c \), hence \( t = tp \). So \( p \) generates \( I_c(X) \).

Conversely, suppose \( p \in C^*_c(X) \) is a projection that generates \( I_c(X) \). Then we can write \( p = \sum_{j=1}^{K} t_j S_j \) where \( K \) is an integer and \( t_j \in C^*_c(X) \) and \( S_j \in \ker(\Phi) \subseteq C_{cs}[X] \). Choose \( T_j \in C_{cs}[X] \) with \( \| t_j - T_j \| \leq \frac{1}{2K\| S_j \|} \). Let \( T = \sum_{j=1}^{K} T_j S_j \), then \( T \in C_{cs}[X] \), and \( \Phi(T) = 0 \), and \( \| T - p \| \leq \sum_{j=1}^{K} \| t_j - T_j \| \cdot \| S_j \| \leq \frac{1}{2} \). Write \( T = T_1 + \ldots + T_N \) as a sum of operators in blocks and choose a controlled set \( E \) and a constant \( \varepsilon > 0 \) such that they are all in \( (E, \varepsilon) \)-blocks. Now let \( (\rho, H) \) be any representation. For any \( v \in \ker(p) \) we have \( sv = 0 \) for all \( s \in I_c(X) \), hence \( v \in \mathcal{H}_c \). So \( \ker(p) = \mathcal{H}_c \). Now let \( v \in H_c^\perp \) be a unit vector. Then \( pv = v \) so \( \| T v \| \geq \frac{1}{N} \). Then there is \( 1 \leq k \leq N \) such that \( \| T_k v \| \geq \frac{1}{2N} \), showing that \( (X, \mu) \) has geometric property (T).

8 **Coarse invariance**

In this section, we will prove that geometric property (T), as defined in Definition 6.7, does not depend on the measure \( \mu \) chosen, and moreover, that it is a coarse invariant. Our method is inspired by the one employed in [19]. Let \( X \) be a coarse space with bounded geometry, and let \( \mu \) be a uniformly bounded measure for which a gordo set exists. We will start by constructing a discrete space \( Y \) with bounded geometry that is coarsely dense in \( X \), with an appropriate measure \( \nu \). We will show a
correspondence between representations of $\mathcal{C}_{cs}[Y]$ and some of the representations of $\mathcal{C}_{cs}[X]$. This will allow us to see $C^*_\max(Y)$ as a subalgebra of $C^*_\max(X)$. Then we can apply the criterion for geometric property (T) given in Proposition 4.3.10 to conclude that $(X, \mu)$ has geometric property (T) if and only if $(Y, \nu)$ has geometric property (T). After this we will show that geometric property (T) does not depend on the measure as long as the space is discrete. Finally, we will conclude that if $X$ is coarse equivalent to another space $X'$, with a uniformly bounded measure $\mu'$ for which a gordo set exists, then $(X, \mu)$ has geometric property (T) if and only if $(X', \mu')$ has geometric property (T).

By Lemma 4.12 there is a blocking collection with union X. Write $X = \bigsqcup_{y \in Y} U_y$, where $(U_y)$ is a blocking collection and $y$ is a point in $U_y$ (the notation $U_y$ is not to be confused with the notation $E_z$ introduced in Definition 2.35(ii)). Note that $Y$ is coarsely dense in $X$ and that $\mu(U_y)$ is bounded from above and below, uniformly in $y$. Let $\nu$ be the measure on $Y$ defined by $\nu(\{y\}) = \mu(U_y)$. We will first show that $(X, \mu)$ has property (T) if and only if $(Y, \nu)$ has property (T).

**Lemma 8.1.** Let $X$ be a coarse space with bounded geometry and let $\mu$ be a uniformly bounded measure on $X$ for which a gordo set exists. Let $Y$ be a coarsely dense discrete space of bounded geometry with a measure $\nu$. Suppose $X = \bigsqcup_{y \in Y} U_y$ is a disjoint union of measurable sets with $y \in U_y$ and $\nu(y) = \mu(U_y)$, such that $\bigsqcup_{y \in Y} U_y \times U_y$ is a controlled set. Then $(X, \mu)$ has property (T) if and only if $(Y, \nu)$ has property (T).

We need some preparation before we can prove Lemma 8.1. For $x \in X$, let $y(x) \in Y$ be the unique point such that $x \in U_y$. Define the linear maps $\alpha : L^2(Y, \nu) \to L^2(Y, \nu)$ and $\beta : L^2(Y, \nu) \to L^2(X)$ by $\alpha(y) = \frac{1}{\nu(y)} \int_{U_y} \xi$ and $\beta(y) = \eta(y(x))$. Note that these are adjoint to each other and that $ab$ is the identity on $L^2(Y, \nu)$. Let $A = ab$. This is a projection in $\mathcal{C}_{cs}[X]$. The maps $\alpha : A\mathcal{C}_{cs}[X]A \to \mathcal{C}_{cs}[Y]$, $\alpha(T) = aTb$ and $\beta : \mathcal{C}_{cs}[Y] \to A\mathcal{C}_{cs}[X]A$, $\beta(S) = bSa$ are $*$-isomorphisms and inverse to each other. Now if $\mathcal{H}^X$ is a representation of $\mathcal{C}_{cs}[X]$, then $\mathcal{H}^Y = A\mathcal{H}^X$ is a representation of $A\mathcal{C}_{cs}[X]A$ and via $\beta$ also a representation of $\mathcal{C}_{cs}[Y]$. Conversely, every representation of $\mathcal{C}_{cs}[Y]$ gives rise to a representation of $\mathcal{C}_{cs}[X]$.

**Lemma 8.2.** Let $(\rho^X, \mathcal{H}^X)$ be a representation of $\mathcal{C}_{cs}[X]$. Then there is a representation $(\rho^Y, \mathcal{H}^Y)$ such that $A\mathcal{H}^X = \mathcal{H}^Y$, and such that $\rho^Y$ equals the restriction of $\rho^X \circ \beta$ to $A\mathcal{H}^X$. Here $\mathcal{H}^X$ is a completion of a quotient of the algebraic tensor product $\mathcal{C}_{cs}[X] \odot \mathcal{H}^Y$ that we will denote by $\mathcal{C}_{cs}[X] \odot \mathcal{H}^Y$.

**Proof.** First consider the algebraic tensor product $\mathcal{C}_{cs}[X] \odot \mathcal{H}^Y$. We equip this with the sesquilinear form defined on simple tensors by

$$\langle T \odot v, T' \odot v' \rangle = \langle \alpha(A(T')^*TA)v, v' \rangle$$

and extended to $(\mathcal{C}_{cs}[X] \odot \mathcal{H}^Y)^2$ by sesquilinearity. It is easy to see that this is well defined and defines a conjugate symmetric sesquilinear form. We will now show that it is positive semi-definite.

For any operator $T \in \mathcal{C}_{cs}[X]$ there is an integer $M$, such that for every $y \in Y$ there are at most $M$ different $y' \in Y$ such that $\pi_{U_y} T_i U_{y'} \neq 0$ or $\pi_{U_{y'}} T_i U_y \neq 0$. By a graph colouring argument we may write $T = T_1 + \ldots + T_N$, such that for every $1 \leq k \leq N$, and every $y \in Y$, there is at most one $y'$ such that $\pi_{U_y} T_k i U_{y'} \neq 0$ or $\pi_{U_{y'}} T_k i U_y \neq 0$.

It follows that each element of $\mathcal{C}_{cs}[X] \odot \mathcal{H}^Y$ is of the form $\sum_{k=1}^N T_k \odot v_k$ with $T_k \in \mathcal{C}_{cs}[X], v_k \in \mathcal{H}^Y$ and the $T_k$ satisfy that for every $y$ there is at most one $y'$ such that $\pi_{U_y} T_i U_{y'} \neq 0$ or $\pi_{U_{y'}} T_i U_y \neq 0$. 

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Let this $y'$ be $f_k(y)$ if it exists, otherwise put $f_k(y) = y$. Then each $f_k$ is an involutive function (i.e. $f_k \circ f_k = id_Y$) and $T_k$ sends $L^2U_y$ to $L^2U_{f_k(y)}$. Define

$$Q = \begin{pmatrix} 0 & T_1 & \cdots & T_N \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in B(L^2X) \otimes M_{N+1}(\mathbb{C}) = B(L^2(X \times \{0,1,\ldots,N\})).$$

For $y \in Y$ define $V_y = U_y \times \{0\} \cup \left( \bigcup_{k=1}^N U_{f_k(y)} \times \{k\} \right) \subseteq X \times \{0,1,\ldots,N\}$. Then $X \times \{0,1,\ldots,N\} = \bigcup_y V_y$ and $Q$ sends each $L^2V_y$ to $L^2V_y$. Let

$$A \otimes I = \begin{pmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{pmatrix} \in B(L^2(X \times \{0,1,\ldots,N\})).$$

Then also $(A \otimes I)Q^*Q(A \otimes I)$ sends each $L^2V_y$ to $L^2V_y$. It is also a positive element so we can define the square root $R = ((A \otimes I)Q^*Q(A \otimes I))^\frac{1}{2} \in (A \otimes I)B(L^2(X \times \{0,1,\ldots,N\}))(A \otimes I)$. It sends each $L^2V_y$ to $L^2V_y$. Denote the entries of $R$ by $R_{kl} \in AB(L^2X)A$. Then $R_{kl}$ sends $L^2U_y$ to $L^2U_{f_k(y)}$. So $R_{kl}$ has controlled support, and we have $R_{kl} \in AC_{cs}[X]A$. We have $R^2 = (A \otimes I)Q^*Q(A \otimes I)$ and looking at the entry at $(k,l)$ gives $\sum_{s=1}^N R_{ks}R_{sl} = AT_k^*T_lA$ and $R_{kl}^* = R_{lk}$ for all $1 \leq k,l \leq N$.

Now we can compute

$$\left\langle \sum_k T_k \otimes v_k, \sum_k T_k \otimes v_k \right\rangle = \sum_{k,l} \left\langle \alpha(AT_k^*T_lA)v_l, v_k \right\rangle = \sum_{k,l,s} \left\langle \alpha(R_{ks}R_{sl})v_l, v_k \right\rangle = \sum_{k,l,s} \left\langle \alpha(R_{sl})v_l, \alpha(R_{sk})v_k \right\rangle = \sum_{s} \left( \sum_k \alpha(R_{sk})v_k \cdot \sum_k \alpha(R_{sk})v_k \right) \geq 0.$$ 

So the sesquilinear form is semi-positive definite. Define $\mathcal{H}^X$ to be the Hilbert space obtained by dividing $C_{cs}[X] \otimes \mathcal{H}^X$ by the kernel of the semi-norm $\|v\| = \langle v,v \rangle^\frac{1}{2}$ and completing with respect to the resulting norm. Now we will define the representation $\rho^X : C_{cs}[X] \rightarrow B(\mathcal{H}^X)$.

Let $T \in C_{cs}[X]$. This acts on $C_{cs}[X] \otimes \mathcal{H}^X$ by $T \cdot (S \otimes v) = TS \otimes v$ and extending by linearity. To show that this defines a bounded map on $\mathcal{H}^X$ we need to show that there is a constant $C$ depending on $T$ such that $\|T w\| \leq C \cdot \|w\|$ for all $w \in C_{cs}[X] \otimes \mathcal{H}^X$. For this we can assume that $T$ is an operator in blocks, by Lemma 5.6. Then $S = (\|T\|^2 - T^*T)^\frac{1}{2}$ is an element in $C_{cs}[X]$ and we have
for all \( w \in C_{cs}[X] \otimes H^Y \):

\[
||T||^2 : ||w||^2 - ||Tw||^2 = (||T||^2 - T^*T)w, w \rangle = \langle Sw, Sw \rangle \geq 0.
\]

So \( ||Tw|| \leq ||T|| : ||w|| \), showing that the action of \( T \) on \( C_{cs}[X] \otimes H^Y \) defines a bounded linear map \( \rho^X(T) \in B(H^X) \). It is now easy to check that this defines a representation of \( C_{cs}[X] \) on \( H^X \).

Finally, note that we have an injection \( H^Y \hookrightarrow C_{cs}[X] \otimes H^Y \) sending \( v \) to \( 1 \otimes v \). This respects the sesquilinear form, so it induces an injection \( H^Y \rightarrow H^X \). For a simple tensor \( T \otimes v \in C_{cs}[X] \otimes H^Y \) it is easy to check that \( A(T \otimes v) = AT \otimes v = 1 \circ \alpha(ATA)v \), hence \( A(C_{cs}[X] \otimes H^Y) = H^Y \) and \( A\mathcal{H}^X = \mathcal{H}^Y \). Then it is also easy to check that the restriction of \( \rho^X \circ \beta \) to \( A\mathcal{H}^X \) equals \( \rho^Y \). \( \square \)

Lemma 8.3. The map \( \beta : C_{cs}[Y] \xrightarrow{\sim} AC_{cs}[X]A \) extends to an isometry \( \beta : C^*_{max}(Y) \xrightarrow{\sim} AC^*_{max}(X)A \).

Proof. Let \( S \in C_{cs}[Y] \) and let \( (\rho^Y, H^Y) \) be a representation such that \( ||S||_{max} = ||\rho(S)|| \). Now \( H^X = C_{cs}[X] \otimes H^Y \) is a representation of \( C_{cs}[X] \) by Lemma 8.2. Moreover, the restriction of \( \rho^X(\beta(S)) \) to \( H^Y = A\mathcal{H}^X \) equals \( \rho^Y(S) \), so \( ||\beta(S)||_{max} \geq ||\rho^X(\beta(S))|| = ||\rho^Y(S)||_{max} = ||S||_{max} \).

Conversely, let \( (\rho^X, H^X) \) be a representation of \( C_{cs}[X] \) such that \( ||\beta(S)||_{max} = ||\rho^X(\beta(S))|| \). We have \( H^X = A\mathcal{H}^X + (1 - A)\mathcal{H}^X \) where \( H^Y = A\mathcal{H}^X \) is a representation of \( C_{cs}[Y] \). The restriction of \( \rho^X(\beta(S)) \) to \( H^Y \) equals \( \rho^Y(S) \) and the restriction of \( \rho^X(\beta(S)) \) to \( (1 - A)\mathcal{H}^X \) equals zero, because \( \beta(S) \in AC_{cs}[X]A \). So \( ||S||_{max} = ||\rho^Y(S)|| = ||\rho^X(\beta(S))|| = ||\beta(S)||_{max} \).

Hence \( \beta : C_{cs}[Y] \rightarrow AC_{cs}[X]A \) is an isometry, and it extends to a bounded linear map:

\[
\beta : C^*_{max}(Y) \rightarrow C^*_{max}(X)A.
\]

Its image is the closure of \( AC_{cs}[X]A \), this is \( AC^*_{max}(X)A \). \( \square \)

Lemma 8.4. We have \( C^*_{max}(X)A \ker(\Phi^X)A = I_c(X)A \) and \( \beta(I_c(Y)) = I_c(X)A \).

Proof. Recall that \( I_c(X) = C^*_{max}(X) \ker(\Phi^X) \), so clearly \( C^*_{max}(X)A \ker(\Phi^X)A \subseteq I_c(X)A \). To show the converse, it is enough to show that \( \ker(\Phi^X)A \subseteq C_{cs}[X]A \ker(\Phi^X)A \). Let \( T \in \ker(\Phi^X)A \). For every \( y, y' \in Y \), the map \( \pi_{U_y}T_{U_{y'}} : L^2U_y \rightarrow L^2U_{y'} \) is the same when composed with \( i_{U_y}, A_{U_{y'}} : L^2U_y' \rightarrow L^2U_{y'} \), so it has rank at most 1. There is an integer \( N \), such that for every \( y \in Y \) there are at most \( N \) different \( y' \in Y \) such that \( \pi_{U_y}T_{U_{y'}} \neq 0 \), because \( T \) has controlled support. Therefore \( \ker(T) \cap L^2U_y \) has rank at most \( N \). Write \( \ker(T) \cap L^2U_y = f_1\mathbb{C} + \cdots + f_N\mathbb{C} \) where the \( f_{k_y} \) are pair-wise perpendicular elements of \( L^2X \) (some of them may be zero). For all \( 1 \leq k \leq N \) and \( y \in Y \) let \( B_{k_y} : L^2U_y \rightarrow L^2U_y \) be a rank-one isometry sending \( f_{k_y} \mathbb{C} \) to \( \mathbb{C} \) (or let \( B_{k_y} = 0 \) if \( f_{k_y} = 0 \)). Then \( B_{k_y}A_{k_y}B_{k_y} = B_{k_y}B_{k_y} \) is the projection on \( f_{k_y}\mathbb{C} \). Then \( \sum_{k=1}^N B_{k_y}A_{k_y}B_{k_y} = \sum_{k=1}^N B_{k_y}A_{k_y} = \sum_{k=1}^N B_{k_y}A_{k_y} \in C_{cs}[X]A \ker(\Phi^X)A \).

For the second claim note that for any \( S \in C_{cs}[Y] \) we have \( \Phi^Y(S) = 0 \) if and only if \( \Phi^X(\beta(S)) = 0 \). Thus \( \beta(I_c(Y)) = \beta(C^*_{max}(Y) \ker(\Phi^Y)) = AC^*_{max}(X)A \ker(\Phi^X)A = I_c(X)A \).

Now we can prove Lemma 8.7.

Proof of Lemma 8.7. We will use the characterisation of geometric property (T) given in Proposition 7.10.

Suppose that \( (X, \mu) \) has geometric property (T). Let \( p \in C^*_{max}(X) \) be a projection generating the left ideal \( I_c(X) \). Since \( I_c(X) \) is a multiple of \( p \), and from this it follows that \( A p A \) is again a projection. Then we see that \( A p A \in I_c(X)A \) and for all \( t \in I_c(X)A \) we have \( t A p A = t A = t \), so \( A p A \) generates \( I_c(X)A \) as a left \( AC^*_{max}(X)A \)-module. So, by Lemma 8.3 we know that \( I_c(Y) \) is generated by a projection, hence \( (Y, \nu) \) has geometric property (T).
Conversely, suppose that \( (Y, \nu) \) has geometric property (T). Let \( q \in C^*_\text{max}(Y) \) be a projection generating the left ideal \( I_c(Y) \). By Lemma \ref{lemma:projection}, we have \( \beta(q) \in AL_c(X)A \). In particular, \( \beta(q)(1-A) = 0 \), and from this it follows that \( \beta(q) + 1 - A \) is again a projection. We have

\[
C^*_\text{max}(X)(\beta(q) + 1 - A) \\
= C^*_\text{max}(X)\beta(q) + C^*_\text{max}(X)(1-A) \\
= C^*_\text{max}(X)AL_c(X)A + C^*_\text{max}(X)(1-A) \\
= I_c(X)A + C^*_\text{max}(X)(1-A) \\
= I_c(X)
\]

by Lemma \ref{lemma:projection}, so \( (X, \mu) \) has geometric property (T). \hfill \Box

Now we show that geometric property (T) is independent of the measure for discrete spaces of bounded geometry.

**Lemma 8.5.** Let \( Y \) be a discrete space of bounded geometry and let \( \nu \) be a measure on \( Y \) such that all points are measurable and have measure uniformly bounded from above and below. Denote the counting measure on \( Y \) by \( \kappa \). Then \( (Y, \nu) \) has property (T) if and only if \( (Y, \kappa) \) has property (T).

**Proof.** Define \( N \in L^\infty(Y) \) by \( N(y) = \nu(y)\frac{1}{\nu(y)} \). We will consider this as a multiplication operator in \( C^*_\text{cs}[Y] \). Consider the linear map \( a: L^2(Y, \nu) \to L^2(Y, \kappa) \) given by \( a(\eta) = N\eta \). This is a unitary map and it induces an isomorphism \( C^*_\text{cs}[Y, \kappa] \to C^*_\text{cs}[Y, \nu] \) sending \( T \) to \( a^*Ta \). For any \( y \in Y \) and large enough \( V \subseteq Y \), we have

\[
\Phi_\nu(a^*Ta)(y) = a^*Ta(1_V)(y) = N^{-1}TN(1_V)(y) = \Phi_\kappa(N^{-1}TN)(y),
\]

so \( \Phi_\nu(a^*Ta) = \Phi_\kappa(N^{-1}TN) \).

We now identify the algebras \( C^*_\text{cs}[Y, \kappa] \cong C^*_\text{cs}[Y, \nu] \cong C^*_\text{cs}[Y] \), just remembering that \( \Phi_\nu(T) = \Phi_\kappa(N^{-1}TN) \). We get \( C^*_\text{max}(Y, \nu) = C^*_\text{max}(Y, \kappa) \). Moreover we have ker(\( \Phi_\nu \)) = N ker(\( \Phi_\kappa \)) N^{-1} and therefore \( I_c(Y, \nu) = N I_c(Y, \kappa) N^{-1} \). We see that \( I_c(Y, \kappa) \) is generated by a projection if and only if \( I_c(Y, \kappa) \) is generated by a projection. By Proposition 7.10 we are done. \hfill \Box

Finally we prove that property (T) does not depend on the chosen measure and is also a coarse invariant.

**Theorem 8.6.** Suppose \( (X, \mu) \) and \( (X', \mu') \) are coarsely equivalent spaces with bounded geometry, equipped with uniformly bounded measures for which gordo sets exist. Then \( (X, \mu) \) has property (T) if and only if \( (X', \mu') \) has property (T). In particular, whether \( X \) has property (T) is independent of the chosen uniformly bounded measure \( \mu \) for which a gordo set exists.

**Proof.** Construct \( Y \) and \( Y' \) as before. Then \( Y, X, X', Y' \) are all coarsely equivalent. Since \( Y \) and \( Y' \) are coarsely equivalent, by a well-known structural result (see e.g. \cite{9} Lemma 4.1) there are coarse dense \( Z \subseteq Y \) and \( Z' \subseteq Y' \) and a bijective coarse equivalence \( f: Z \to Z' \). Taking the counting measure on \( Z \) and \( Z' \), it is easy to see that \( f \) induces an isomorphism between the algebras \( C^*_\text{cs}[Z] \) and \( C^*_\text{cs}[Z'] \) commuting with \( \Phi^Z \) and \( \Phi^{Z'} \), so that \( Z \) has property (T) if and only if \( Z' \) has property (T).

We can write \( Y \) as a bounded disjoint union \( \bigcup_{z \in Z} V_z \) such that \( z \in V_z \). Define the measure \( \lambda \) on \( Z \) by \( \lambda(z) = \nu(V_z) \). Similarly define a measure \( \lambda' \) on \( Y' \). Now by Lemmas \ref{lemma:disjoint} and \ref{lemma:projection} the following are equivalent:
- The space \((X, \mu)\) has property (T).
- The space \((Y, \nu)\) has property (T).
- The space \((Z, \lambda)\) has property (T).
- The space \(Z\) has property (T) with the counting measure.
- The space \(Z'\) has property (T) with the counting measure.
- The space \((Z', \lambda')\) has property (T).
- The space \((Y', \nu')\) has property (T).
- The space \((X', \mu')\) has property (T).

\[\square\]

9 Amenability

In this section, we will consider which connected coarse spaces \(X\) have geometric property (T). Recall that a coarse space \(X\) is connected if \(\{(x, y)\}\) is a controlled set for all \(x, y \in X\). As in [9], it turns out that an unbounded connected coarse space \(X\) has geometric property (T) precisely if it does not satisfy a version of amenability. First, we give two equivalent definitions of amenability for connected coarse spaces. It is a generalisation of amenability for discrete metric spaces of bounded geometry, as defined e.g. in [9].

**Proposition 9.1.** Let \(X\) be a connected coarse space and \(\mu\) a uniformly bounded measure for which a gordo set exists. The following are equivalent:

(i) There is a positive unital linear map \(\varphi : L^\infty X \to \mathbb{C}\) such that if \((A_i)\) is a blocking collection and \(f \in L^\infty X\) satisfies \(\int_{A_i} f = 0\) for all \(i\), and \(f\) is zero outside of the union \(\bigsqcup_i A_i\), then \(\varphi(f) = 0\).

(ii) For each controlled measurable set \(E \subseteq X \times X\) and each \(\varepsilon > 0\) there is a non-empty bounded measurable \(U \subseteq X\) such that \(\mu(E_U) \leq (1 + \varepsilon)\mu(U)\).

In this case, we say that \(X\) is amenable.

**Proof.** First assume (ii). Define the directed set

\[\Lambda = \{(E, \varepsilon) \mid E \subseteq X \times X\ \text{symmetric, measurable, controlled and containing the diagonal,} \ \varepsilon > 0\}\]

with \((E, \varepsilon) \geq (E', \varepsilon')\) if \(E \supseteq E'\) and \(\varepsilon \leq \varepsilon'\). For each \(\lambda = (E, \varepsilon) \in \Lambda\), choose a measurable set \(U_\lambda \subseteq X\) such that \(\mu(E_{U_\lambda}) \leq (1 + \varepsilon)\mu(U_\lambda)\). Define \(\varphi_\lambda \in (L^\infty X)^*\) by \(\varphi_\lambda(f) = \frac{1}{\mu(U_\lambda)} \int_{U_\lambda} f\). Then the \(\varphi_\lambda\) form a net in the unit ball of \((L^\infty X)^*\). By the Banach-Alaoglu theorem there is a limit point \(\varphi\).

We will now show that \(\varphi\) satisfies the condition. So let \((A_i)\) be a blocking collection and \(f \in L^\infty X\) such that \(\int_{A_i} f = 0\) for all \(i\) and \(f\) is zero outside \(\bigsqcup_i A_i\). Let \(E\) be the symmetric measurable controlled set \(\sqcup_i A_i \times A_i \cup \{(x, x) \mid x \in X\}\). Let \(\lambda = (E', \varepsilon)\) with \(E' \supseteq E\). Since \(E_{U_\lambda}\) is a union of some of the \(A_i\) and points outside of \(\bigsqcup_i A_i\), we have \(\int_{E_{U_\lambda}} f = 0\). Then \(\varphi_\lambda(f) = \frac{1}{\mu(U_\lambda)} \int_{U_\lambda} f = \ldots\)
\[-\frac{1}{\mu(U)} \int_{E \setminus U} f \text{ and } |\varphi(f)| \leq \frac{\mu(E \setminus U)}{\mu(U)} \|f\|_\infty \leq \epsilon \|f\|_\infty.\] This shows that $\varphi(f) \to 0$ as $\lambda \to \infty$, hence $\varphi(f) = 0$.

Conversely, assume (i). Let $\varphi: L^\infty X \to \mathbb{C}$ be a positive unital linear map satisfying the condition. Let $E$ be a gordo set and assume for contradiction that there is $\epsilon > 0$ such that for all bounded measurable $U \subseteq X$ we have $\mu(E_U) > (1 + \epsilon) \mu(U)$. Let $(A_i)$ be a blocking collection with union $X$, such that each $A_i$ has measure at least $\epsilon' > 0$. We may assume without loss of generality that $E = \{A_i \times A_j \mid (A_i \times A_j) \cap E \neq \emptyset\}$. Because of bounded geometry, we can use a colouring argument to find involutions $\sigma_1, \ldots, \sigma_N$ on the index set $I$ such that $E_{\sigma_i} = A_{\sigma_i(1)} \cup \cdots \cup A_{\sigma_i(k)}$ for all $i \in I$. For $1 \leq k \leq N$ let $E_k = \bigcup A_i \times A_{\sigma_k(i)}$. Note that $\Delta^{E_k}$ can be viewed as an operator on $L^1$ and also as an operator on $L^\infty X$. For any $f \in L^\infty X$ we have $\int_{A_i \cup A_{\sigma_k(i)}} \Delta^{E_k} f = 0$ for all $i$, so by the condition on $\varphi$ we have $\varphi(\Delta^{E_k} f) = 0$. Equivalently, $(\Delta^{E_k})^* \varphi = 0$.

Let $\mathcal{P}(X) \subset L^1 X$ denote the positive measurable functions on $X$ with integral 1. We can view $L^1 X$ as a subset of $(L^\infty X)^*$ using the standard pairing between $L^1 X$ and $L^\infty X$. By the Goldstine theorem, $\varphi$ is in the weak closure of $\mathcal{P}(X)$. Note that $(\Delta^{E_k})^*$ is a weakly continuous function on $(L^\infty X)^*$ and on $L^1 X$ it is the same as $\Delta^{E_k}$. Therefore, 0 is in the weak closure of the convex set 

$$\{ (\Delta^{E_k} \psi) \mid \psi \in \mathcal{P}(X) \} \subset \bigoplus_{k=1}^N L^1(X).$$

Since norm-closed convex sets are also weakly closed, it follows that 0 is also in the norm closure of this set. So we can choose $\psi \in \mathcal{P}(X)$ such that all $\Delta^{E_k} \psi$ are small in the $L^1$-norm. In particular, we can choose $\psi \in \mathcal{P}(X)$ such that $\sum_i M_i \leq \eta$, where $M_i = \sup_{n} \int_{A_i} |\Delta^{E_k} \psi|$ and we will choose the constant $\eta > 0$ later.

Let $\delta = \frac{1}{1 + \frac{\epsilon}{\epsilon'}} < 1$. For $n \in \mathbb{Z}$ define $U_n = \bigcup \left\{ A_i \mid \int_{A_i} \psi \geq \delta^n \mu(A_i) \right\}$. Since $\int_A \psi = 1$, the $U_n$ have bounded measure, they consist of finitely many of the $A_i$ and they are bounded. Moreover, $U_n = \emptyset$ for $n$ small enough. By assumption, we have $\mu(E_{U_n}) \geq (1 + \epsilon) \mu(U_n)$ for each $n$. For each $i$, define $n_i$ as the minimal number such that $A_i \subseteq E_{U_{n_i}}$ and $N_i$ as the maximal number such that $A_i \not\subseteq U_{N_i}$. Then $\int_{A_i} \psi < \delta^{N_i} \mu(A_i)$, and there is $k$ such that $\int_{A_{\sigma_k(i)}} \psi < \delta^{N_i} \mu(A_{\sigma_k(i)})$.

Now

$$M_i \geq \int_{A_i} |\Delta^{E_k} \psi| = \int_{A_i} \left| \mu(A_{\sigma_k(i)}) \psi - \int_{A_{\sigma_k(i)}} \psi \right| \geq \varepsilon'(\delta^{N_i} - \delta^{N_i}).$$

Define

$$D_n = E_{U_n} \setminus U_{n+1} = \bigcup \left\{ A_i \mid n_i \leq n \leq N_i - 1 \right\}.$$

Then

$$\sum_n \delta^n \mu(D_n) = \sum_i \mu(A_i) \sum_{n=n_i}^{N_i-1} \delta^n = \frac{1}{1-\delta} \sum_i \mu(A_i) (\delta^{n_i} - \delta^{N_i}) \leq \frac{1}{(1-\delta)\varepsilon'} \sum_i M_i \leq \frac{\eta}{(1-\delta)\varepsilon'}.$$

For all $n$ we have

$$(1 + \varepsilon) \mu(U_n) \leq \mu(E_{U_n}) \leq \mu(U_{n+1}) + \mu(D_n).$$

Multiplying this inequality by $\delta^n$ on both sides and adding it for all $n$ we get

$$(1 + \varepsilon) \sum_n \delta^n \mu(U_n) \leq \frac{1}{\delta} \sum_n \delta^n \mu(U_n) + \sum_n \delta^n \mu(D_n),$$

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so
\[
(1 + \varepsilon - \frac{1}{\delta}) \sum_n \delta^n \mu(U_n) \leq \frac{\eta}{(1 - \delta)\varepsilon'}
\]
and
\[
\sum_n \delta^n \mu(U_n) \leq \frac{2\eta}{(1 - \delta)\varepsilon'}.
\]

On the other hand, we have
\[
1 = \int \psi = \sum_n \int_{U_{n+1} \setminus U_n} \psi \leq \sum_n \delta^n (\mu(U_{n+1}) - \mu(U_n)) = \left(\frac{1}{\delta} - 1\right) \sum_n \delta^n \mu(U_n).
\]
Combining these inequalities gives \(1 \leq \frac{2\eta}{\delta \varepsilon'}\). Picking \(\eta = \frac{\delta \varepsilon'}{4}\) gives the desired contradiction. \(\Box\)

The following proposition is a generalisation of [9, Proposition 6.1]. It gives several characterisations of amenability of a coarse space \(X\) in terms of its algebra.

**Proposition 9.2.** Let \(X\) be a connected coarse space and \(\mu\) a uniformly bounded measure for which a gordo set exists. The following are equivalent:

- (i) The space \(X\) is amenable.
- (ii) There is a net \(\xi_\lambda\) of unit vectors in \(L^2 X\) satisfying \(\|T \xi_\lambda - M_{\Phi(T)} \xi_\lambda\| \to 0\) for all \(T \in \mathbb{C}_{cs}[X]\) for which \(\Phi(T)\) is bounded.
- (iii) For all \(T \in \mathbb{C}_{cs}[X]\) with \(\Phi(T) = 0\), we have \(0 \in \sigma(T)\).
- (iv) For all gordo sets \(E\), we have \(0 \in \sigma_{\text{max}}(\Delta^E)\).
- (v) There is a unital representation \(\mathcal{H}\) of \(\mathbb{C}_{cs}[X]\) that has a non-zero constant vector.
- (vi) There is a positive unital linear map \(\varphi : \mathbb{L}^\infty X \to \mathbb{C}\) satisfying \(\varphi(T f) = \varphi(\Phi(T^*) \cdot f)\) for all \(T \in \mathbb{C}_{cs}[X]\) for which \(\varphi(T^*)\) is bounded.

**Proof.** First, assume that \(X\) is amenable. We will use criterion (ii) from Proposition 9.1. As in the proof of Proposition 9.1, define the directed set
\[
\Lambda = \{(E, \varepsilon) \mid E \subseteq X \times X \text{ gordo}, \varepsilon > 0\}
\]
with \((E, \varepsilon) \geq (E', \varepsilon')\) if \(E \supseteq E'\) and \(\varepsilon \leq \varepsilon'\). For each \(\lambda = (E, \varepsilon) \in \Lambda\), choose a bounded measurable subset \(U_\lambda \subseteq X\) such that \(\mu((E \circ E)_{U_\lambda}) \leq (1 + \varepsilon)\mu(U_\lambda)\). Define \(\xi_\lambda = \mu(U_\lambda)^{-\frac{1}{2}} \mathbb{1}_{U_\lambda}\). This is a unit vector in \(L^2 X\).

Let \(T \in \mathbb{C}_{cs}[X]\) such that \(\Phi(T)\) is bounded. We will show that \(\|T \xi_\lambda - M_{\Phi(T)} \xi_\lambda\| \to 0\). We can assume without loss of generality that \(\Phi(T) = 0\). Let \(T\) be supported on \(E\), let \(E' \supseteq E\) and \(\varepsilon > 0\). Then for \(\lambda = (E', \varepsilon)\), the function \(\xi_\lambda\) is constant on \(E_{X \setminus U_\lambda}\), so \(T \xi_\lambda\) is supported on \(E_{U_\lambda}\). Since \(T \mathbb{1}_{(E \circ E)_{U_\lambda}}\) is zero on \(E_{U_\lambda}\), we know that \(T \xi_\lambda\) is the restriction to \(E_{U_\lambda}\) of \(-T \eta_\lambda\), where \(\eta_\lambda = \mu(U_\lambda)^{-\frac{1}{2}} \mathbb{1}_{(E \circ E)_{U_\lambda} \setminus U_\lambda}\). Now
\[
\|T \xi_\lambda\| \leq \|T \eta_\lambda\| \leq \|T\| \cdot \|\eta_\lambda\| = \|T\| \cdot \mu(U_\lambda)^{-\frac{1}{2}} \mu((E \circ E)_{U_\lambda} \setminus U_\lambda)^{\frac{1}{2}} \leq \|T\| \cdot \varepsilon^{\frac{1}{2}},
\]
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We equip this with the partial order $(E, \varepsilon) \geq (E', \varepsilon')$ if $E \supseteq E'$ and $\varepsilon \leq \varepsilon'$. Then $\Lambda$ is a directed set. Let $U$ be an ultrafilter on $P(\Lambda)$ containing $\{\lambda' \geq \lambda\}$ for each $\lambda \in \Lambda$. By assumption, for every $\lambda = (E, \varepsilon) \in \Lambda$ there is a representation $H_\lambda$ of $C_{cs}[X]$ and a unit vector $v_\lambda \in H_\lambda$ satisfying $\|\Delta^E v_\lambda\| \leq \varepsilon$. Note that for $\lambda' \geq \lambda$ we also have $\|\Delta^E v_\lambda\| \leq \varepsilon$. Now let $H = \prod_{\lambda \in \Lambda} H_\lambda/\mathcal{U}$ be the ultraprodut of the $H_\lambda$ and let $v = \lim_\mathcal{U} v_\lambda \in H$. Then $v$ is a unit vector and for each measurable controlled set $E$ we have $\Delta^E v = \lim_\mathcal{U} \Delta^E v_\lambda = 0$. By Lemma 7.4, $v$ is a constant vector, showing $(iv) \implies (v)$.

For $(v) \implies (vi)$, let $v \in H$ be a constant vector of norm 1 and define $\varphi : L^\infty X \to \mathbb{C}$ by $\varphi(f) = \langle Mfv, v \rangle$. Then $\varphi$ is a positive unital linear map and for all $f \in L^\infty X$ and $T \in C_{cs}[X]$ such that $\Phi(T^*)$ is bounded, we have

$$\varphi(Tf) = \langle M_{\Phi(TM)}f, v \rangle = \langle TMfv, v \rangle = \langle M_{\Phi(T^*)}f, v \rangle = \varphi(\Phi(T^*) \cdot f).$$

Finally, for $(vi) \implies (i)$, let $\varphi : L^\infty X \to \mathbb{C}$ be a positive unital linear map satisfying $\varphi(Tf) = \varphi(\Phi(T^*) \cdot f)$ for all $T \in C_{cs}[X]$ such that $\Phi(T^*)$ is bounded. Let $(A_i)$ be a blocking collection and $E = \bigsqcup_i A_i \times A_i$. Let $f \in L^\infty X$ be a function such that $\int_{A_i} f = 0$ for all $i$, and $f$ is zero outside of $\bigsqcup_i A_i$. Then $\Delta^E f = f$ and $\varphi(\Delta^E f) = 0$, so $\varphi(f) = \varphi(\Delta^E f) = 0$. Therefore, $X$ is amenable (using part (i) of Proposition 9.1).

\begin{remark}
If $E'$ is as in proposition 7.9 the properties in Theorem 9.2 are also equivalent to $0 \in \sigma_{max}(\Delta^{E'})$.

Now we can give the generalisation of [9] Corollary 6.1.

\end{remark}

\begin{theorem}
Let $X$ be a connected coarse space and let $\mu$ be a uniformly bounded measure for which a gordo set exists. If $X$ is bounded, then it is amenable and has geometric property (T). If $X$ is unbounded, it has geometric property (T) if and only if it is not amenable.

\end{theorem}

\begin{proof}
If $X$ is bounded, we have already shown that it has geometric property (T), and it is amenable because $\mathbb{1}_X \in L^2 X$ is a constant vector in the standard representation.

Suppose that $X$ is unbounded and not amenable. Then there is a gordo set $E$ such that $0 \not\in \sigma_{max}(\Delta^E)$. Then for all representations $(\rho, \mathcal{H})$ of $C_{cs}[X]$ we have $ker(\rho(\Delta^E)) = \mathcal{H}_c = \emptyset$ and $\Delta^E$ has spectral gap. By Proposition 7.6 the space $X$ has geometric property (T).

Suppose that $X$ is unbounded, has geometric property (T) and is amenable. By Proposition 7.6 there is a measurable controlled set $E$ such that for all representations $(\rho, \mathcal{H})$ of $C_{cs}[X]$, we have $\mathcal{H}_c = ker(\rho(\Delta^E))$, and $\Delta^E$ has spectral gap. Since $X$ is amenable, we have $0 \in \sigma(\Delta^E)$, but then $0$ is an isolated point in the spectrum of $\Delta^E \in B(L^2 X)$. So there is a non-zero $\xi \in ker(\Delta^E)$, which is then a constant vector. Since $\xi \in L^2 X$ and $X$ is unbounded, $\xi$ can not be constant as a function. So there are measurable sets $A, B \subseteq X$ of positive measure such that the convex hulls of $\xi(A)$ and $\xi(B)$ are disjoint. If we intersect $A$ with all of the members of a blocking collection, one of the intersections must have positive measure, since blocking collections are countable. Therefore, we
can assume that $A$ is bounded, and similarly, that $B$ is bounded. Since $X$ is connected, $A \cup B$ is also bounded. Let $T \in C_{cs}(X)$ be the operator defined by

$$T\eta(x) = \begin{cases} \frac{1}{\mu(B)} \int_B \eta - \frac{1}{\mu(A)} \int_A \eta & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Phi(T) = 0$ but $T\xi \neq 0$, hence $\xi$ is not a constant vector and we have a contradiction.

**Corollary 9.5.** Amenability for connected coarse spaces of bounded geometry does not depend on the measure chosen and is a coarse invariant.

**Proof.** This follows from Theorem 8.6 and Theorem 9.4. It is also straightforward to show it directly, using criterion (ii) of Proposition 9.1. □

## 10 Manifolds

Let $(M,g)$ be an $n$-dimensional complete Riemannian manifold, not necessarily connected. Consider the geodesic distance $d$ and volume $\mu$. The metric $d$ defines a coarse structure on $M$, where $E \subseteq M \times M$ is controlled if $\sup_{(x,y) \in E} d(x,y) < \infty$. For any $R > 0$ let $E_R$ be the controlled set $\{(x,y) \in M \times M \mid d(x,y) \leq R\}$. Let $B(x,R) = (E_R)_x$ denote the ball of radius $R$ around a point $x \in M$. In [2], it is defined that $M$ has bounded geometry if the injectivity radius is positive, and the Ricci curvature is bounded below (by a possibly negative constant). In this section, we assume that this is the case.

By Bishop’s inequality (see [2, Theorem 3.101.i]) there are constants $B, B' > 0$ such that $\mu(B(x,R)) \leq B \exp(B'R)$ for every $x \in M$ and $R > 0$. Hence, the measure $\mu$ is uniformly bounded. There are also $R, A > 0$ such that $\mu(B(x,R)) \geq A$ for all $x \in M$, by [1, Proposition 14] (the Proposition is stated for compact manifolds, but the proof works just as well in the non-compact case if the injectivity radius is positive). In particular, the controlled set $E_R$ is gordo, so we can conclude that the manifold $M$ has bounded geometry as a coarse space.

Let $\Delta_M$ be the (positive) Laplacian operator on $M$ and let $\Delta_H = 1 - \exp(-\Delta_M)$. Since the spectrum of $\Delta_M$ is contained in $[0, \infty]$, the spectrum of $\Delta_H$ is contained in $[0, 1]$. In particular, $\Delta_H$ is a bounded operator. We can write

$$\Delta_H f(x) = f(x) - \int_M p(x,y)f(y)d\mu(y)$$

where $p(x,y)$ is the heat kernel on $M$. We need some estimates on this function.

**Lemma 10.1.** (i) There are constants $c, R > 0$ such that $p(x,y) \geq c$ whenever $d(x,y) \leq R$.

(ii) The manifold is stochastically complete: for every $x \in M$ we have $\int_M p(x,y)d\mu(y) = 1$.

(iii) For every $\varepsilon > 0$ there is an $R$ such that $\int_{B(x,R)} p(x,y)d\mu(y) \geq 1 - \varepsilon$ for all $x \in M$.

**Proof.** (i) This follows from [2, Equation 7.43].

(ii) This follows from [3, Theorem 3.4] and the bound on $\mu(B(x,R))$ given above.
(iii) The conditions of [3, Equation 6.40] are met, so there are constants $C, D > 0$ such that

$$p(x, y) \leq C \exp \left( \frac{-d(x, y)^2}{D} \right)$$

for all $x, y \in M$. Now for any $x \in M$ and $R > 0$ we have

$$\int_{M \setminus B(x, R)} p(x, y) d\mu(y) \leq C \int_{M \setminus B(x, R)} \exp \left( \frac{-d(x, y)^2}{D} \right) d\mu(y)$$

$$\leq C \int_R^\infty \frac{2r}{D} \exp \left( \frac{-r^2}{D} \right) V_r(x) dr$$

$$\leq \frac{2BC}{D} \int_R^\infty r^{n+1} \exp \left( \frac{-r^2}{D} \right) dr.$$

As this last integral converges, we know that for every $\varepsilon > 0$ there is an $R > 0$ such that $\int_{M \setminus B(x, R)} p(x, y) d\mu(y) \leq \varepsilon$. Then by part (ii) we get $\int_{B_x, R} p(x, y) d\mu(y) \geq 1 - \varepsilon$ for all $x \in M$.

**Lemma 10.2.** We have $\Delta H \in C^*_\text{max}(M)$.

**Proof.** For any $R > 0$ let $\Delta_{HR} \in C_{cs}[M]$ be the operator defined by

$$\Delta_{HR} f(x) = \int_{B_x, R} p(x, y)(f(x) - f(y)) d\mu(y).$$

Then for $R' > R$ we have

$$(\Delta_{HR'} - \Delta_{HR}) f(x) = \int_{B_x, R'} \setminus B_x, R \ p(x, y)(f(x) - f(y)) d\mu(y).$$

For large enough $R$ we have $\int_{M \setminus B(x, R)} p(x, y) d\mu(y) \leq \varepsilon$. Then we have

$$\langle (\Delta_{HR'} - \Delta_{HR}) \xi, \xi \rangle = \frac{1}{2} \int_{\{(x, y) \in M^2 | R < d(x, y) \leq R' \}} p(x, y) |\xi(x) - \xi(y)|^2 d(\mu \times \mu)(x, y) \leq 2\varepsilon \|\xi\|^2.$$

So

$$0 \leq_{\text{max}} \Delta_{HR'} - \Delta_{HR} \leq_{\text{max}} 2\varepsilon.$$

Therefore the $\Delta_{HR}$ form a Cauchy sequence in $C_{cs}[M]$ with the maximal norm, and they converge to $\Delta_H$. \qed

For any $R > 0$ let $\Delta_R = \Delta_{E_R}$. We will use the estimates of the heat kernel from Lemma 10.1 to prove estimates on $\Delta_H$ in terms of the $\Delta_R$.

**Lemma 10.3.** We have the following estimates on $\Delta_H$:

(i) There are constants $c, R > 0$ such that $\Delta_H \geq_{\text{max}} c \Delta_R$. 

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(ii) For every $\varepsilon > 0$ there are constants $d, R > 0$ such that $\Delta_H \leq \max \varepsilon + d\Delta_R$.

Proof. (i) Take the same constants $c, R$ as in Lemma 10.1 part (i). Then

$$\left(\Delta_H - c\Delta_R\right)f(x) = \int_M \alpha(x,y)(f(x) - f(y))d\mu(y),$$

where $\alpha(x,y) = p(x,y)$ if $d(x,y) > R$ and $\alpha(x,y) = p(x,y) - c$ if $d(x,y) \leq R$. Since $\alpha \geq 0$ everywhere we have $\Delta_H \geq \max c\Delta_R$.

(ii) Let $\Delta_{HR}$ be as in the proof of Lemma 10.2. There is a large enough $R$ such that $\Delta_H \leq \max \varepsilon + \Delta_{HR}$. The function $p(x,y)$ is bounded from above by some constant $d$, and then we get $\Delta_{HR} \leq d\Delta_R$.

For any representation $\mathcal{H}$ of $C^*_\text{max}(M)$, we can consider the unbounded symmetric operator $\Delta_M = -\log(1 - \Delta_H)$ on $\mathcal{H}$. We say that $\Delta_M$ has spectral gap if there is a constant $\gamma > 0$ such that the spectrum of this operator is contained in $\{0\} \cup [\gamma, \infty)$ for each representation $\mathcal{H}$.

**Theorem 10.4.** Let $(M, g)$ be a Riemannian manifold of bounded geometry equipped with the coarse structure coming from the geodesic metric. Then $M$ has geometric property (T) if and only if the Laplacian $\Delta_M$ has spectral gap in $C^*_\text{max}(M)$.

Proof. For any $R > 0$, the controlled set $E_R$ generates the coarse structure on $M$ and is gordo. So we can apply Proposition 7.29 to see that $M$ has geometric property (T) if and only if $\Delta_R$ has spectral gap. Clearly, $\Delta_H$ has spectral gap if and only if $\Delta_M$ has spectral gap. For the remainder of the proof, we use the estimates of Lemma 10.3.

Suppose that $M$ has geometric property (T). By Lemma 10.1 part (i), there are constants $c, R > 0$ such that $\Delta_H \geq c\Delta_R$. Now $\Delta_R$ has spectral gap, and so does $\Delta_H$, and also $\Delta_M$.

Suppose that $\Delta_H$ has spectral gap. There is $\varepsilon > 0$ such that $\sigma_{\max}(\Delta_H) \subseteq \{0\} \cup [2\varepsilon, \infty)$. By Lemma 10.1 part (ii) there are $b, R > 0$ such that $\Delta_H \leq \varepsilon + b\Delta_R$. Then $\Delta_R$ has spectral gap, so $M$ has geometric property (T).

11 Warped systems

Let $\Gamma$ be a finitely generated group with finite generating set $S$. Let $(M, g)$ be a compact Riemannian manifold. Let $d$ be the distance function on $M$ and $\mu$ the measure on $M$ defined by the metric $g$. Let $\alpha: \Gamma \curvearrowright M$ be an action by diffeomorphisms. Let $CM = M \times \{1, 2, \ldots\}$ be the discrete cone. This is equipped with the Riemannian metric $g_C$ that is $t \cdot g$ on $M_t = M \times \{t\}$. Let $d_C$ be the corresponding distance function, given by $d_C(x, y) = t \cdot d(x, y)$ when $x, y \in M_t = M \times \{t\}$ and $d_C(x, y) = \infty$ if $x, y$ are in different $M_t$. Now define the warped system $\text{Warp}(\Gamma \curvearrowright M)$ as the space $CM$ equipped with the largest metric $\delta_T$ for which $\delta_T(x, y) \leq d_C(x, y)$ and $\delta_T(x, s \cdot x) \leq 1$ for all $s \in \Gamma$ and $x, y \in CM$. We will consider $\text{Warp}(\Gamma \curvearrowright M)$ as a coarse space.

In [7], it is shown that that $X = \text{Warp}(\Gamma \curvearrowright M)$ does not depend on the generating set of $\Gamma$ as a coarse space. It is also coarsely equivalent to the coarse disjoint union $\bigsqcup_{t \in \mathbb{Z}} M \times \{t\}$ of subspaces of the warped cone as introduced by Roe in [6] (see [7] Lemma 6.5).

Let $\mu$ be the measure on $X$ defined through the Riemannian metric $g_C$. Since the manifold $M$ is compact, the injectivity radius is positive and the Ricci curvature is bounded below. It follows as
in the previous section that \( F_R = \{ (x, y) \in X \times X \mid d_C(x, y) \leq R \} \) is gordo for any \( R > 0 \), and that its balls are uniformly bounded. By Lemma 11.7 below, it follows that \( \mu \) is uniformly bounded. In particular, \( X \) has bounded geometry.

Let \( \Delta_\Gamma = \sum_{s \in S} \Gamma - s \in \mathbb{C}[\Gamma] \). This can also be viewed as an element in \( C_{cs}[X] \). In [7], it is shown that a coarsely dense subset of \( X \) with discrete bounded geometry, is a family of expanders if and only if there is a constant \( \gamma > 0 \) such that \( \langle \Delta_\Gamma \xi, \xi \rangle \geq \gamma \| \xi \|^2 \) for all \( \xi \in L^2 M \) with \( \int_M \xi d\mu = 0 \). We want to consider when these graphs have geometric property (T), or equivalently by Theorem 8.6 when \( X \) has geometric property (T).

**Definition 11.1.** Let \( \Gamma \) be a group and \((M, \mu)\) a measure space. Let \( \alpha : \Gamma \curvearrowright M \) be a measurable action. The action is called **ergodic** if the only measurable \( \Gamma \)-invariant subsets of \( M \) are the empty set and \( M \) itself.

If \( \Gamma \) has property (T) and the action on \( M \) is ergodic, we might expect the warped system \( X \) to have geometric property (T). At the moment, this is an open question.

**Question 11.2.** Suppose \( \Gamma \) has property (T) and the action \( \alpha : \Gamma \curvearrowright M \) is ergodic. Does \( X \) have geometric property (T)?

The measure-preserving action of \( \Gamma \) on \( M \) gives a map \( \Gamma \to U(L^2 X) \) given by \( g \cdot \xi(x) = \xi(g^{-1}x) \) for \( g \in \Gamma, \xi \in L^2 X \). The image is contained in \( C_{cs}[X] \). Then the map \( \Gamma \to C_{cs}[X] \) gives rise to a map \( \mathbb{C}[\Gamma] \to C_{cs}[X] \). Now every representation of \( C_{cs}[X] \) pulls back to a representation of \( \mathbb{C}[\Gamma] \), therefore the map \( \mathbb{C}[\Gamma] \to C_{cs}[X] \) is contracting for the maximal norms. So it induces a map \( C_{\text{max}}^*(\Gamma) \to C_{\text{max}}^*(X) \). If \( \Gamma \) has property (T), then we know that \( \Delta_\Gamma \) has spectral gap in \( C_{\text{max}}^*(\Gamma) \). So the image of \( \Delta_\Gamma \) also has spectral gap in \( C_{\text{max}}^*(X) \). If \( \ker(\rho(\Delta)) = \mathcal{H}_c \) for any representation \( (\rho, \mathcal{H}) \) of \( C_{cs}[X] \), then we can conclude that \( X \) has property (T) by Proposition 7.6. We can prove the following partial result.

**Lemma 11.3.** Suppose that \( \Gamma \) has property (T) and the action \( \alpha : \Gamma \curvearrowright M \) is ergodic. Let \( \mathcal{H} \) be a representation of \( C_{cs}[X] \), and \( v \in \mathcal{H} \) such that \( \Delta_\Gamma v = 0 \). For any \( f \in L^\infty X \) such that \( \int_X f d\mu = 0 \) for all \( t \in \mathbb{N} \), we have \( \langle M_f v, v \rangle = 0 \).

**Proof.** We can assume without loss of generality that \( v \) is a unit vector.

The composition \( \Gamma \overset{\Delta_\Gamma}{\to} C_{cs}[X] \overset{\phi}{\to} \mathcal{B}(\mathcal{H}) \) gives an action of \( \Gamma \) on \( \mathcal{H} \), still denoted by \( \alpha \). For any \( s \in S \), we have \( \| (1 - \alpha_s) v \|^2 = \langle (1 - \alpha_s^*) (1 - \alpha_s) v, v \rangle = \langle (2 - \alpha_s - \alpha_s^*) v, v \rangle \). Since \( 2 - \alpha_s - \alpha_s^* \leq \Delta_\Gamma \) it follows that \( (1 - \alpha_s) v = 0 \). So \( v \) is \( \Gamma \)-invariant.

Now define \( \varphi \in (L^\infty X)^* \) by \( \varphi(f) = \langle M_f v, v \rangle \). Note that \( \varphi \) is a mean on \( L^\infty X \). Moreover, \( \varphi \) is \( \Gamma \)-invariant: for any \( g \in \Gamma, f \in L^\infty X \) we have

\[
\varphi(\alpha_g f) = \langle M_{\alpha_g} f, v \rangle = \langle \alpha_g M_f \alpha_g^* v, v \rangle = \langle M_f v, v \rangle = \varphi(f).
\]

Let \( \mathcal{P}(X) = \{ \psi \in L^1 X \mid \psi \geq 0, \int_X \psi d\mu = 1 \} \). By the Goldstine theorem, \( \varphi \) is in the weak closure of \( \mathcal{P}(X) \). Let \( f \in L^\infty X \) such that \( \int_X f d\mu = 0 \) for all \( t \in \mathbb{N} \). Let \( \varepsilon > 0 \). Then 0 is in the weak closure of the convex set

\[
\left\{ \|(1 - \alpha_s) \psi \| \in \sum_{s \in S} L^1 X \mid \psi \in \mathcal{P}(X), |\psi(f) - \varphi(f)| < \varepsilon \right\}.
\]

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Since norm-closed convex sets are also weakly closed, 0 is also in the norm closure of the set above. Hence there is \( \psi \in \mathcal{P}(X) \) with \( |\psi(f) - \varphi(f)| < \varepsilon \) and \( \|\psi - (1 - \alpha_s)\varphi\|_1 < \varepsilon \) for all \( s \in S \). Now let \( \xi = \psi \mathbf{1} \in L^2X \). This is a unit vector satisfying

\[
\| (1 - \alpha_s)\xi \|_2^2 = \int_X (\xi(s \cdot x) - \xi(x))^2 d\mu(x) \leq \int_X |\xi(s \cdot x) - \xi(x)|^2 = \| (1 - \alpha_s)\psi \|_1 < \varepsilon
\]

for all \( s \in S \). Then \( \| \Delta_\Gamma \xi \|_2 \leq \varepsilon \frac{\mathbf{1}}{|S|} \). Let \( \xi_c \in L^2X \) be the locally constant function defined by \( \xi_c(x) = \frac{1}{\mu(X)} \int_{X_i} \xi \) for \( x \in X_i \). There is \( \gamma > 0 \) such that \( \sigma(\Delta_\Gamma) \subseteq \{0\} \cup [\gamma, \infty) \). Now \( \xi - \xi_c \) is perpendicular to the locally constant functions, and the locally constant functions are the only \( \Gamma \)-invariant functions in \( L^2X \), since the action of \( \Gamma \) is ergodic. From \( \| \Delta_\Gamma (\xi - \xi_c) \|_2 \leq \varepsilon \frac{\mathbf{1}}{|S|} \) it now follows that \( \| (\xi - \xi_c) \|_2 \leq \gamma^{-1} \varepsilon \frac{\mathbf{1}}{|S|} \). Now

\[
|\varphi(f)| \leq |\psi(f)| + \varepsilon
\]

\[
= \left| \int_X (\xi(x))^2 f(x) d\mu(x) \right| + \varepsilon
\]

\[
= \left| \int_X (\xi(x) - \xi_c(x))^2 f(x) d\mu(x) \right| + 2 \left| \int_X (\xi(x) - \xi_c(x)) \xi_c(x) f(x) d\mu(x) \right| + \varepsilon
\]

\[
\leq \gamma^{-2} \varepsilon |S|^2 \cdot \|f\|_\infty + \gamma^{-1} \varepsilon \frac{\mathbf{1}}{|S|} \cdot \|f\|_\infty + \varepsilon.
\]

Since this holds for any \( \varepsilon > 0 \), we conclude that \( \varphi(f) = 0 \).

\[\Box\]

Remark 11.4. In the above, we only used that \( \sigma(\Delta_\Gamma) \subseteq \{0\} \cup [\gamma, \infty) \) instead of the stronger condition \( \sigma_{\text{max}}(\Delta_\Gamma) \subseteq \{0\} \cup [\gamma, \infty) \).

Remark 11.5. To see why the Lemma is progress to the Question, consider that we have to prove that every \( v \in \ker(\rho(\Delta_\Gamma)) \) is constant. For this we can assume that \( v \in H^+ \), and try to show that it is in fact 0. Now from the Lemma it follows that in this case, we have \( v \in (\rho(L^\infty X)H_c)^\perp \), in other words, \( v \) is a vector that is in some way quite far from being constant.

To see this, consider \( f \in L^\infty X \) and \( w \in H_c \). We can write \( f = f_c + g \) with \( f_c \) constant on each component of \( X \) and \( \int_X g d\mu = 0 \) for all \( g \). For all \( t \in \mathbb{R} \) we have \( v + tw \in \ker(\rho(\Delta_\Gamma)) \), so by the Lemma, we have \( \langle M_g (v + tw), v + tw \rangle = 0 \). Using some different values of \( t \) gives \( \langle v, M_g w \rangle = 0 \). Also \( \langle v, M_f w \rangle = 0 \) since \( M_f w \) is a constant vector. So \( \langle v, M_f w \rangle = 0 \).

The converse is also an open question.

Question 11.6. Suppose the action \( \alpha \) is free and \( X \) has geometric property (T). Is it necessary that \( \Gamma \) has property (T) and that the action is ergodic?

We have some partial results on this question. Before we state them, we give some preliminary results.

Lemma 11.7. For \( R > 0 \) and \( g \in \Gamma \) define the controlled sets \( E_R = \{ (x, y) \in X \mid \delta_\Gamma(x, y) \leq R \}, E_g = \{ (g \cdot x, x) \mid x \in X \} \) and \( F_R = \{ (x, y) \in X \mid d_c(x, y) \leq R \} \). For any \( R > 0 \) there is \( R' > 0 \) such that \( E_R \subseteq \bigcup_{g \in S \cap R'} F_{R'} \circ E_g \).

Proof. For any \( s \in S \) the map \( M \to M, x \to s \cdot x \) is Lipschitz. So there is a constant \( L > 0 \) such that \( d_c(s \cdot x, s \cdot y) \leq L \cdot d_c(x, y) \). Let \( R' = L^R R \). For any \( (y, x) \in E_R \), there are \( x_0 = x, x_1, \ldots, x_n \in X \) and \( y_0, y_1, \ldots, y_n = y \in X \) and \( s_1, s_2, \ldots, s_n \in S \) such that \( s_k y_{k-1} = x_k \) for \( k = 1, 2, \ldots, n \) and

\[
n + d_c(x_0, y_0) + \ldots + d_c(x_n, y_n) \leq R.
\]
It follows by induction that
\[ d_c(s_2 \cdots s_2 \cdot s_1 x_0, y_k) \leq L^k d_c(x_0, y_0) + \ldots + L d_c(x_{k-1}, y_{k-1}) + d_c(x_k, y_k). \]
In particular
\[ d_c(g \cdot x, y) \leq L^n d_c(x_0, y_0) + \ldots + L d_c(x_{n-1}, y_{n-1}) + d_c(x_n, y_n) \leq L^n R = R' \]
for \( g = s_n \cdots s_2 \cdot s_1 \). Now \((g \cdot x, x) \in E_g \) and \((y, g \cdot x) \in F_{R'} \) so \((y, x) \in F_{R'} \circ E_g \). Since \( g \in S^{[R]} \) this shows that
\[ E_R \subseteq \bigcup_{g \in S^{[R]}} F_{R'} \circ E_g. \]

Note that \( I = \bigoplus_{t \in \mathbb{N}} B(l^2 X_t) \) is a two-sided ideal in \( C_{\text{cs}}[X] \).

**Lemma 11.8.** Suppose that the action \( \alpha \) is free. Then \( C_{\text{cs}}[X]/I \) is naturally isomorphic to the algebraic cross product \( C_{\text{cs}}[CM]/I \rtimes \Gamma \).

**Proof.** The natural map \( C_{\text{cs}}[CM] \rtimes \Gamma \to C_{\text{cs}}[X] \) descends to \( C_{\text{cs}}[CM]/I \rtimes \Gamma \to C_{\text{cs}}[X]/I \).

To show that it is surjective, let \( T \in C_{\text{cs}}[X] \). Then \( T \) is supported on \( E_R \) for some integer \( R > 0 \). By Lemma 11.7 there is \( R' > 0 \) such that \( E_R \subseteq \bigcup_{g \in S^{[R]}} F_{R'} E_g \). We can use this decomposition to write \( T \) as a finite sum \( T = \sum_g A_g \alpha_g \), with \( A_g \in C_{\text{cs}}[CM] \) showing that the map is indeed surjective.

For injectivity, suppose that \( \sum_{g \in B} A_g \alpha_g \in I \), with \( B \subseteq \Gamma \) finite and \( A_g \in C_{\text{cs}}[CM] \). Then we will show that \( A_g \in I \) for all \( g \in B \). We may as well assume that \( \sum_{g \in B} A_g \alpha_g = 0 \). Since the action is free and by homeomorphisms, for any \( g \in \Gamma \) we have \( \min_{x \in M} d(g \cdot x, x) > 0 \). There is some \( R > 0 \) such that all \( A_g \) are supported on \( F_R \). Now let \( T = \frac{1}{\min_{x \in M} d(h^{-1} x, x)} \). For any \( h \in B \) we know that \( A_g \alpha_g = - \sum_{h \in B \setminus \{g\}} A_h \alpha_h \), so \( A_g \alpha_g \) is supported on \( F_{R'} E_g \cap \bigcup_{h \in B \setminus \{g\}} F_R E_h \). Suppose that \( (x, y) \in F_{R'} E_g \cap F_R E_h \). Then \( d_{c}(x, y) \leq R \) and \( d_{c}(x, h \cdot y) \leq R \), so \( d_{c}(g \cdot y, h \cdot y) \leq 2R \). For some \( t \) we have \( x, y \in X_t \) and then \( d_{c}(g \cdot y, h \cdot y) \geq t \cdot \min_{x \in M} d(h^{-1} x, x) \geq 2R \). So \( t \leq T \). Hence we have \( F_{R} E_g \cap F_{R} E_h \subseteq \bigcup_{t \leq T} X_t \times X_t \), and \( F_{R'} E_g \cap \bigcup_{h \in B \setminus \{g\}} F_R E_h \subseteq \bigcup_{t \leq T} X_t \times X_t \). So \( A_g \alpha_g \) is supported on \( \bigcup_{t \leq T} X_t \times X_t \), and we conclude that \( A_g \in \bigoplus_{t \in \mathbb{N}} B(l^2 X_t) \).

**Lemma 11.9.** Suppose the action \( \alpha \) is free. Then the natural map \( C_{\text{max}}^*(\Gamma) \to C_{\text{max}}^*(X) \) is an injection.

**Proof.** The proof is similar to the proof of Lemma 7.1. Let \( T \) denote the closure of \( I \) in \( C_{\text{max}}^*(X) \). The previous lemma gives \( C_{\text{cs}}[X]/I \cong C_{\text{cs}}[CM]/I \rtimes \Gamma \), hence \( C_{\text{max}}^*(X)/T \cong C_{\text{max}}^*(CM)/T \rtimes_{\text{max}} \Gamma \). Now it suffices to prove that the composition \( C_{\text{max}}^*(\Gamma) \to C_{\text{max}}^*(X) \to C_{\text{max}}^*(X)/T \cong C_{\text{max}}^*(CM)/T \rtimes_{\text{max}} \Gamma \) is injective.

For any \( t \in \mathbb{N} \) define the map \( \varphi_t : C_{\text{cs}}[CM] \to C \) by \( \varphi_t(T) = 1_{\mu(X_t)}(x, T 1_{X_t}) \). This extends to a positive unital map \( \varphi : C_{\text{max}}^*(CM) \to C \). Let \( \varphi \) be a limit point of these maps. Then \( \varphi : C_{\text{max}}^*(CM)/T \to C \) is a ucp map. Since maximal crossed products are functorial for ucp maps, this gives a map \( C_{\text{max}}^*(CM)/T \rtimes_{\text{max}} \Gamma \to C_{\text{max}}^*(\Gamma) \). Now the composition \( C_{\text{max}}^*(\Gamma) \to C_{\text{max}}^*(CM)/T \rtimes_{\text{max}} \Gamma \) is the identity, so the first map is an injection. \( \square \)
As a result, we get that for free actions, $\Delta T$ has spectral gap in $C^*_{\text{max}}(X)$ if and only if $\Gamma$ has property (T). So if $\Gamma$ does not have property (T), for any $\varepsilon > 0$, there is some representation $(\rho, H)$ and a unit vector $v \in H^\perp$ with $\|\Delta T v\| < \varepsilon$. Suppose that $\|\Delta R v\| < 1$ for some $0 < R < 1$. Then one can show that $\|\Delta R v\| \to 0$ as $R \to 0$. Let $u: L^2 X \to L^2 X$ be given by $u(\xi)(x,t) = \xi(x,2t)$. Precomposing the representation $\rho$ by the map $\tau: C_{cs}[X] \to C_{cs}[X], \tau(T) = uT u^*$ multiple times gives vectors $v_0 = v, v_1, \ldots$ such that $\|\Delta v_n\| < \varepsilon$ and $\|\Delta R v_n\| < \varepsilon$. In this case it follows that $X$ does not have geometric property (T). A similar strategy would work for non-ergodic actions. The main obstacle is, that there may be vectors $v$ for which $\|\Delta R v\| = 1$ for all $0 < R < 1$.

Acknowledgement

I would like to thank my advisor Tim de Laat for his support and suggestions. I also thank for Rufus Willett for helpful suggestions, and the reviewer for a very thorough and helpful report. The author is supported by the Deutsche Forschungsgemeinschaft under Germany’s Excellence Strategy - EXC 2044 - 390685587, Mathematics Münster: Dynamics - Geometry - Structure.

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