Abstract

We define and study a certain relative tensor product of subfactors over a modular tensor category. This gives a relative tensor product of two completely rational heterotic full local conformal nets with trivial superselection structures over a common chiral representation category. In particular, we have a new realization of fusion rules of modular invariants. This also gives a mathematical definition of a composition of two gapped domain walls between topological phases.

1 Introduction

The theory of subfactors due to Jones [21] has been a very powerful tool in conformal field theory. We study some aspects of full conformal field theory from a viewpoint of subfactors and modular tensor categories. (We consider only unitary modular tensor categories in this paper.)

We are interested in a subfactor \( N \subset M \) with finite Jones index \([M : N]\). In conformal field theory, it is often useful to formulate a subfactor \( N \subset M \) in terms of a \( Q \)-system \( \Theta = (\theta, w, x) \) where \( \theta \) is an endomorphism of a type III factor \( N \) with separable predual and \( w \in \text{Hom} (\text{id}, \theta) \), \( x \in \text{Hom} (\theta, \theta^2) \) as in [31]. When \( \theta \) is an object of an abstract modular tensor category \( C \), we say \( \Theta \) is a \( Q \)-system on \( C \). (Note that any modular tensor category is realized as a subcategory of \( \text{End}(N) \) for a type III factor \( N \).) It is also often called a \( C^* \)-Frobenius algebra on \( C \). When we have \( x = \varepsilon (\theta, \theta) x \), where \( \varepsilon \) denotes the braiding, we say that the \( Q \)-system \( \Theta \) is local. It is also often said that it is commutative. We say \( \Theta \) is Lagrangian if we have \((\dim \theta)^2 = \dim C\). (See [11] page 153) for the origin of this

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terminology.) See [22] and references therein for more on subfactors and tensor categories. Our basic reference on modular categories is [2]. See [14] for basics of subfactor theory.

Let \( \{ A(I) \} \) be a completely rational local conformal net in the sense of [26], [24], and let \( \mathcal{C} \) be the Doplicher-Haag-Roberts representation category of \( \{ A(I) \} \). (It is a modular tensor category by [26].) A maximal full conformal field theory in the sense of [25] is given by a local Lagrangian \( Q \)-system on \( \mathcal{C} \boxtimes \mathcal{C}^{opp} \) as in [25], where “opp” means the opposite modular tensor category for which the braiding is reversed. (Also see [4, Proposition 6.7].)

Let \( \theta = \bigoplus_{\lambda \in \text{Irr}(\mathcal{C}), \mu \in \text{Irr}(\mathcal{C}^{opp})} Z_{\lambda \mu} \lambda \boxtimes \bar{\mu} \) be the object of such a \( Q \)-system on \( \mathcal{C} \boxtimes \mathcal{C}^{opp} \), where “Irr” means the set of equivalence classes of simple objects in the modular tensor category. The matrix \( Z = (Z_{\lambda \mu}) \) is then a modular invariant in the sense that it commutes with the \( S \)- and \( T \)-matrices arising from \( \mathcal{C} \) as in [4, Proposition 6.6]. Suppose we have two such modular invariants \( (Z^1_{\lambda \mu}) \) and \( (Z^2_{\mu \nu}) \). Then the matrix product \( Z^1 Z^2 \) clearly satisfies the properties of the modular invariant except for the normalization condition \( Z_{00} = 1 \) where 0 denotes the identity object of the modular tensor category \( \mathcal{C} \). It is sometimes possible to have a decomposition \( Z^1 Z^2 = \sum_i Z^{3,i} \) into modular invariants \( Z^{3,i} \). Such decomposition rules of matrix products have been studied under the name of fusion rules of modular invariants in [13], [15, Section 3.1], [16, Remark 5.4 (iii)]. We have a machinery of \( \alpha \)-induction for subfactors as in [32], [6], [7], [8], [9], and it produces a modular invariant as in [7]. It gives a \( Q \)-system on \( \mathcal{C} \boxtimes \mathcal{C}^{opp} \) as in [35], and this is a general form of a maximal full conformal field theory on \( \mathcal{C} \boxtimes \mathcal{C}^{opp} \) as in [4, Proposition 6.7]. The results in [15, Section 3.1], [16, Remark 5.4 (iii)] say that a braided product of \( Q \)-systems on \( \mathcal{C} \) gives a fusion rule of the corresponding \( Q \)-systems on \( \mathcal{C} \boxtimes \mathcal{C}^{opp} \). In this way, we indirectly have an irreducible decomposition of a certain relative tensor product of two local irreducible Lagrangian \( Q \)-systems on \( \mathcal{C} \boxtimes \mathcal{C}^{opp} \).

One typical example of such fusion rules is given as follows. Let \( \mathcal{C} \) be the modular tensor category corresponding to the WZW-model \( SU(2)_{17} \). Then by [34, Page 202] (and also by [27, Theorem 2.1] and [4, Proposition 6.7]), we have exactly three irreducible local Lagrangian \( Q \)-systems on \( \mathcal{C} \boxtimes \mathcal{C}^{opp} \) and they are labeled with \( A_{17}, D_{10}, E_7 \) as in [10]. (These labels are for the modular invariant matrices. The label \( A_{17} \) corresponds to the identity matrix.) Their nontrivial fusion rules are as follows by [13, Section 5.1], [16, Remark 5.4 (iii)].

\[
D_{10} \otimes D_{10} = 2D_{10}, \\
D_{10} \otimes E_7 = E_7 \otimes D_{10} = 2E_7, \\
E_7 \otimes E_7 = D_{10} \oplus E_7.
\]

We would like to extend this relative product to the irreducible local Lagrangian \( Q \)-systems on \( \mathcal{C}_1 \boxtimes \mathcal{C}_2^{opp} \) and \( \mathcal{C}_2 \boxtimes \mathcal{C}_3^{opp} \) in this paper where \( \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \) can be different. This setting corresponds to a heterotic full conformal field theory.

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2 A relative tensor product of $Q$-systems

We consider a $Q$-system $\Theta = (\theta, w, x)$ where $\theta$ is an endomorphism of a type III factor $N$ with separable predual and $w \in \text{Hom}(\text{id}, \theta)$, $x \in \text{Hom}(\theta, \theta^2)$. We adapt [35, Definition 3.8], which means that such a $Q$-system corresponds to an inclusion $N \subset M$ where $M$ may not be a factor. We have $N' \cap M = \mathbb{C}$ if and only if the $Q$-system $\Theta$ is irreducible.

We recall the following proposition in [33]. (Also see [12, Proposition 3.7, Corollary 3.8].)

**Proposition 2.1** Let $\Theta = (\theta, w, x)$ be an irreducible local $Q$-system where $\theta$ is of the form $\bigoplus_{\lambda \in \text{Irr}(C_1), \mu \in \text{Irr}(C_2^{\text{opp}})} Z_{\lambda \mu}^0 \lambda \boxtimes \mu$ for some modular tensor categories $C_1, C_2$. Then it is Lagrangian if and only if we have the modular invariance property $S_C^1 Z = Z S_C^2$ and $T_C Z = Z T_C$ for the matrix $Z = (Z_{\lambda \mu})$, where $S_C^1, S_C^2, T_C^1, T_C^2$ are the $S$-matrix for $C_1$, $S$-matrix for $C_2$, $T$-matrix for $C_1$ and $T$-matrix for $C_2$, respectively.

This was first raised as a problem in [36, Section 3] in the context of full conformal field theory, and proved by Müger [33] and an unpublished manuscript of Longo and the author. This is valid in a general context of a modular tensor category. (Also see [4, Proposition 5.2].)

Let $(\theta, w, x)$ be a $Q$-system where $\theta$ is of the form $\bigoplus_{\lambda \in \text{Irr}(C_1), \mu \in \text{Irr}(C_1), \nu \in \text{Irr}(C_2)} Z_{\lambda \mu \nu}^0 \lambda \boxtimes \mu \boxtimes \nu$ for some modular tensor categories $C_1, C_2$. By applying the functor $T$ to the $C_1$ component as in [28, Section 4.1], [11, Section 4.2], we obtain a new $Q$-system $T(\Theta) = (T(\theta), w_T, x_T)$ where $T(\theta) = \bigoplus_{\lambda \in \text{Irr}(C_1), \mu \in \text{Irr}(C_1), \nu \in \text{Irr}(C_2)} Z_{\lambda \mu \nu}^0 \lambda \boxtimes \mu \boxtimes \nu$. (We need the braiding structure of $C_1$ in order to define $x_T$.) Note that even if $\Theta$ is irreducible, $T(\Theta)$ is not irreducible in general.

Let $(\theta, w, x)$ be another $Q$-system where $\theta$ is of the form $\bigoplus_{\lambda \in \text{Irr}(C_1), \mu \in \text{Irr}(C_2)} Z_{\lambda \mu}^0 \lambda \boxtimes \mu$ for some modular tensor categories $C_1, C_2$. By applying [20, Corollary 3.10], we have a new $Q$-system $(\theta_1, w_1, s_1)$ with $\theta = \bigoplus_{\lambda \in \text{Irr}(C_1)} Z_{\lambda 0 \lambda}^0 \lambda$ where 0 denotes the identity object of $C_2$. We call it the restriction of $\Theta$ to $C_1$.

Now let $\Theta_1 = (\theta_1, w_1, x_1)$ and $\Theta_2 = (\theta_2, w_2, x_2)$ be $Q$-systems where

$$\theta_1 = \bigoplus_{\lambda \in \text{Irr}(C_1), \mu \in \text{Irr}(C_2^{\text{opp}})} Z_{\lambda \mu}^0 \lambda \boxtimes \mu$$

on $C_1 \boxtimes C_2^{\text{opp}}$ and

$$\theta_2 = \bigoplus_{\mu \in \text{Irr}(C_2), \nu \in \text{Irr}(C_3^{\text{opp}})} Z_{\mu \nu}^0 \mu \boxtimes \nu$$

on $C_2 \boxtimes C_3^{\text{opp}}$ for some modular tensor categories $C_1, C_2, C_3$.

Let $\Theta_1 \boxtimes \Theta_2$ be the tensor product of the two $Q$-systems for which the object is given by

$$\bigoplus_{\lambda \in \text{Irr}(C_1), \mu \in \text{Irr}(C_2^{\text{opp}}), \mu' \in \text{Irr}(C_2), \nu' \in \text{Irr}(C_3^{\text{opp}})} Z_{\lambda \mu \nu}^1 Z_{\mu' \nu'}^2 \lambda \boxtimes \mu \boxtimes \mu' \boxtimes \nu.$$

By applying the $T$ functor to the $C_2$ components, we obtain a new $Q$-system whose object is

$$\bigoplus_{\lambda \in \text{Irr}(C_1), \mu \in \text{Irr}(C_2^{\text{opp}}), \mu' \in \text{Irr}(C_2), \nu \in \text{Irr}(C_3^{\text{opp}})} Z_{\lambda \mu \nu}^1 Z_{\mu' \nu}^2 \lambda \boxtimes \mu \mu' \boxtimes \nu.$$
By restricting this $Q$-system to $\mathcal{C}_1 \boxtimes \mathcal{C}_3^{\text{opp}}$, we obtain a new $Q$-system whose object is

$$\bigoplus_{\lambda \in \text{Irr}(\mathcal{C}_1), \mu \in \text{Irr}(\mathcal{C}_2), \nu \in \text{Irr}(\mathcal{C}_3^{\text{opp}})} Z_{\lambda \mu}^1 Z_{\mu \nu}^2 \lambda \boxtimes \nu.$$ 

**Definition 2.2** We call the above $Q$-system the relative tensor product of $\Theta_1$ and $\Theta_2$ over $\mathcal{C}_2$ and write $\Theta_1 \otimes_{\mathcal{C}_2} \Theta_2$.

From the definition, it is easy to see the following.

**Proposition 2.3** The relative tensor product operation is associative.

To apply this notion to a full conformal field theory, we need the following.

**Proposition 2.4** If two $Q$-systems are both local, then the relative tensor product $\Theta_1 \otimes_{\mathcal{C}_2} \Theta_2$ is also local.

**Proof.** For notational simplicity, we may treat $\mathcal{C}_1 \boxtimes \mathcal{C}_3^{\text{opp}}$ as a single modular tensor category, so we simply write $\mathcal{C}_1$ for $\mathcal{C}_1 \boxtimes \mathcal{C}_3^{\text{opp}}$ as if $\mathcal{C}_3$ were the trivial modular tensor category $\text{Vec}$ of finite dimensional Hilbert spaces.

Locality of the tensor product $Q$-system $\Theta_1 \boxtimes \Theta_2$ is represented as in Fig. 1. (We follow the graphical convention of [7, Section 3], but compose morphisms from the bottom to the top, which is a converse direction to the one in [7, Section 3].) In this picture, the triple points on the left hand side denote $x_1, x_2, x_2$, respectively. The second braiding on the right hand side is reversed because we have $\mathcal{C}_2^{\text{opp}}$ for this component.

![Figure 1: Locality (1)](image)

From Fig. 1 we connect the wires $\bar{\mu}''$ and $\mu_1''$, the wires $\bar{\mu}$ and $\mu_1$, and the wires $\bar{\mu}'$ and $\mu_1'$ on the both hand sides so that the wires connecting $\bar{\mu}$ and $\mu_1$ go over the ones connecting $\bar{\mu}'$ and $\mu_1'$. Then we obtain Fig. 2. Then the Reidemeister move II on the most right picture of Fig. 2 produces Fig. 3.

Fig. 3 represents the locality of $\Theta_1 \otimes_{\mathcal{C}_2} \Theta_2$. 

\[\square\]
We consider the irreducible decomposition $\Theta_1 \otimes \Theta_2 = \bigoplus_i \Theta^i_3$, which is a finite sum. By [6, Corollary 3.6], this coincides with the corresponding factorial decomposition $M = \bigoplus_i M_i$ where the $Q$-system $\Theta_1 \otimes \Theta_2$ corresponds to an inclusion $N \subset M$ and the one $\Theta^i_3$ corresponds to $N \subset M_i$.

We first list the following lemma. See [11, Definition 5.1] for the definition of Witt equivalence.

**Lemma 2.5** Let $C_1, C_2$ be Witt equivalent modular tensor categories and $\Theta = (\theta, w, x)$ be an irreducible local $Q$-system where $\theta$ is of the form $\bigoplus_{\lambda \in \text{Irr}(C_1), \mu \in \text{Irr}(C_2^{\text{opp}})} Z_{\lambda \mu} \lambda \boxtimes \bar{\mu}$. Then there exists an irreducible local Lagrangian $Q$-system $\tilde{\Theta} = (\tilde{\theta}, \tilde{w}, \tilde{x})$ where $\tilde{\theta}$ is of the form $\bigoplus_{\lambda \in \text{Irr}(C_1), \mu \in \text{Irr}(C_2^{\text{opp}})} \tilde{Z}_{\lambda \mu} \lambda \boxtimes \bar{\mu}$ with $\tilde{Z}_{\lambda \mu} \geq Z_{\lambda \mu}$ for all $\lambda \in \text{Irr}(C_1), \mu \in \text{Irr}(C_2^{\text{opp}})$ and $\tilde{Z}_{00} = Z_{00} = 1$ where $0$ denotes the identity objects of $C_1$ and $C_2$.

**Proof.** Let $\tilde{\mathcal{C}}$ be the modular tensor category arising as the ambichiral category from the $Q$-system $\Theta$ as in [8, Theorem 4.2]. (Note that the ambichiral objects correspond to
Theorem 2.6 If the $Q$-systems $\Theta_1$ and $\Theta_2$ are both Lagrangian, so is each $\Theta_3$.

Proof. Set $Z^3_\mu = \sum_\nu Z^1_\mu Z^2_\nu$ and let $\bigoplus_{\lambda \in \text{Irr}(C_1), \nu \in \text{Irr}(C_2^{\text{opp}})} Z^3_{\lambda \nu} \lambda \otimes \nu$ be the object for $\Theta_3$.

By Proposition 2.1, being Lagrangian for $\Theta_3$ is equivalent to modular invariance property $S_{C_1} Z^3_\mu = Z^3_\mu S_{C_3}$ and $T_{C_1} Z^3_\mu = Z^3_\mu T_{C_3}$ for $Z^3_\mu$, where $S_{C_1}, S_{C_3}, T_{C_1}, T_{C_3}$ are the $S$-matrix for $C_1$, $S$-matrix for $C_3$, $T$-matrix for $C_1$ and $T$-matrix for $C_3$, respectively.

Note that $C_1$ and $C_2$ are Witt equivalent, and so are $C_2$ and $C_3$. Hence $C_1$ and $C_3$ are also Witt equivalent and each $\Theta_3$ has a Lagrangian extension $\Theta'_3$ whose object is $\bigoplus_{\lambda \in \text{Irr}(C_1), \nu \in \text{Irr}(C_2^{\text{opp}})} Z^3_{\lambda \nu} \lambda \otimes \nu$ by Lemma 2.5 and we have $S_{C_1} \hat{Z}^3_\mu = \hat{Z}^3_\mu S_{C_3}$ and $T_{C_1} \hat{Z}^3_\mu = \hat{Z}^3_\mu T_{C_3}$ by Proposition 2.1. By Lemma 2.3 we may write $\hat{Z}^3_\mu = Z^3_\mu + \hat{Z}^3_\mu$, where each $\hat{Z}^3_\mu$ is a non-negative integer.

Since the matrix $\sum_i Z^3_i$ also has the modular invariance property, the matrix $\hat{Z}^3 = \sum_i \hat{Z}^3_i$ also has the modular invariance property. This implies $\sum_{\lambda \nu} S_{C_1,0 \lambda} \hat{Z}^3_{\lambda \nu} S_{C_3,\nu 0} = Z^3_{00}$, but $Z^3_{00} = \sum_i Z^3_{00} = 0$ and $S_{C_1,0 \lambda} > 0$, $S_{C_3,\nu 0} > 0$. We thus have $\hat{Z}^3_{\lambda \nu} = 0$ for all $\lambda \in \text{Irr}(C_1)$ and $\nu \in \text{Irr}(C_3)$. This proves the modular invariance property $S_{C_1} Z^3_\mu = Z^3_\mu S_{C_3}$ and $T_{C_1} Z^3_\mu = Z^3_\mu T_{C_3}$ for $Z^3_\mu$, as desired. □

Note that the use of modular invariance in the last paragraph of the above proof is the same as in [17, p. 726 (5.2)].

This relative tensor product of $Q$-systems looks similar to that of bimodules, but the example of the $A_{17}$-$D_{10}$-$E_7$ modular invariants mentioned in the Introduction shows that their fusion rules do not give a fusion category since the rigidity axiom is not satisfied.

We have interpreted an irreducible local Lagrangian $Q$-system on $C_1 \boxtimes C_2$ as a gapped domain wall between topological phases represented with $C_1$ and $C_2$ in [23, Definition 3.1]. (See [18], [19], [30] for physical treatments of gapped domain walls.) From this viewpoint, the above relative tensor product gives a mathematical definition of the composition of gapped domain walls mentioned in [30, Fig. 1 (d)]. (Note that irreducibility of a $Q$-system is called stability of a gapped domain wall in [30].) A mathematical definition of such a composition has been studied in [29], [1]. It would be interesting to compare the above definition with theirs.

Another construction of fusion product with some formal similarity has been defined in [3]. It would be interesting to find direct relations to their construction.
References

[1] Y. Ai, L. Kong and H. Zheng, Topological orders and factorization homology, arXiv:1607.08422.

[2] B. Bakalov and A. Kirillov, Jr., “Lectures on tensor categories and modular functors”, American Mathematical Society, Providence (2001).

[3] A. Bartels, C. L. Douglas and A. Henriques, Conformal nets III: Fusion of defects, to appear in Mem. Amer. Math. Soc.

[4] M. Bischoff, Y. Kawahigashi and R. Longo, Characterization of 2D rational local conformal nets and its boundary conditions: the maximal case, Doc. Math. 20 (2015), 1137–1184.

[5] M. Bischoff, Y. Kawahigashi, R. Longo and K.-H. Rehren, “Tensor categories and endomorphisms of von Neumann algebras — with applications to quantum field theory”, Springer Briefs in Mathematical Physics, 3, Springer, (2015).

[6] J. Böckenhauer and D. E. Evans, Modular invariants, graphs and α-induction for nets of subfactors I. Comm. Math. Phys. 197 (1998), 361–386.

[7] J. Böckenhauer, D. E. Evans and Y. Kawahigashi, On α-induction, chiral projectors and modular invariants for subfactors, Comm. Math. Phys. 208 (1999), 429–487.

[8] J. Böckenhauer, D. E. Evans and Y. Kawahigashi, Chiral structure of modular invariants for subfactors, Comm. Math. Phys. 210 (2000), 733–784.

[9] J. Böckenhauer, D. E. Evans and Y. Kawahigashi, Longo-Rehren subfactors arising from α-induction, Publ. Res. Inst. Math. Sci. 37 (2001), 1–35.

[10] A. Cappelli, C. Itzykson and J.-B. Zuber, The A-D-E classification of minimal and \(\mathfrak{A}_{1}^{(1)}\) conformal invariant theories, Comm. Math. Phys. 113 (1987), 1–26.

[11] A. Davydov, M. Müger, D. Nikshych and V. Ostrik, The Witt group of non-degenerate braided fusion categories, J. Reine Angew. Math. 677 (2013), 135–177.

[12] A. Davydov, D. Nikshych and V. Ostrik, On the structure of the Witt group of braided fusion categories, Selecta Math. 19 (2013), 237–269.

[13] E. D. Evans, Fusion rules of modular invariants, Rev. Math. Phys. 14 (2002), 709–731.

[14] D. E. Evans and Y. Kawahigashi, “Quantum Symmetries on Operator Algebras”, Oxford University Press, Oxford (1998).

[15] D. E. Evans and P. Pinto, Subfactor realisation of modular invariants, Comm. Math. Phys. 237 (2003), 309–363.

[16] J. Fuchs, I. Runkel and C. Schweigert, TFT construction of RCFT correlators. I. Partition functions, Nuclear Phys. B 646 (2002), 353–497.
[17] T. Gannon, WZW commutants, lattices and level-one partition functions, *Nuclear Phys. B* 396 (1993), 708–736.

[18] L.-Y. Hung and Y. Wan, Ground state degeneracy of topological phases on open surfaces, *Phys. Rev. Lett.* 114 (2015), 076401.

[19] L.-Y. Hung and Y. Wan, Generalized ADE classification of gapped domain walls, *J. High Energy Phys.* 2015 (2015), 120.

[20] M. Izumi, R. Longo and S. Popa, A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras, *J. Funct. Anal.* 155 (1998), 25–63.

[21] V. F. R. Jones, Index for subfactors, *Invent. Math.* 72 (1983), 1–25.

[22] Y. Kawahigashi, Conformal field theory, tensor categories and operator algebras, *J. Phys. A* 48 (2015), 303001, 57 pp.

[23] Y. Kawahigashi, A remark on gapped domain walls between topological phases, *Lett. Math. Phys.* 105 (2015), 893–899.

[24] Y. Kawahigashi and R. Longo, Classification of local conformal nets. Case $c < 1$, *Ann. of Math.* 160 (2004), 493–522.

[25] Y. Kawahigashi and R. Longo, Classification of two-dimensional local conformal nets with $c < 1$ and 2-cohomology vanishing for tensor categories, *Comm. Math. Phys.* 244 (2004), 63–97.

[26] Y. Kawahigashi, R. Longo and M. Müger, Multi-interval subfactors and modularity of representations in conformal field theory, *Comm. Math. Phys.* 219 (2001), 631–669.

[27] Y. Kawahigashi, R. Longo, U. Pennig and K.-H. Rehren, Classification of non-local chiral CFT with $c < 1$, *Comm. Math. Phys.* 271 (2007), 375–385.

[28] L. Kong and I. Runkel, Morita classes of algebras in modular tensor categories, *Adv. Math.* 219 (2008), 1548–1576.

[29] L. Kong and H. Zheng, The center functor is fully faithful, arXiv: 1507.00503.

[30] T. Lan, J. Wang and X.-G. Wen, Gapped domain walls, gapped boundaries and topological degeneracy, *Phys. Rev. Lett.* 114 (2015), 076402.

[31] R. Longo, A duality for Hopf algebras and for subfactors, *Comm. Math. Phys.* 159 (1994), 133–150.

[32] R. Longo and K.-H. Rehren, Nets of Subfactors, *Rev. Math. Phys.* 7 (1995), 567–597.

[33] M. Müger, On superselection theory of quantum fields in low dimensions, in *XVIth International Congress on Mathematical Physics*, (2010), 496–503, World Sci. Publ.
[34] V. Ostrik, Module categories, weak Hopf algebras and modular invariants, *Transform. Groups* **8** (2003), 177–206.

[35] K.-H. Rehren, Canonical tensor product subfactors, *Commun. Math. Phys.* **211** (2000) 395–406.

[36] K.-H. Rehren, Locality and modular invariance in 2D conformal QFT, in *Mathematical physics in mathematics and physics (Siena, 2000)*, (2001), 341–354.