$L_2$- and $S_{p,q}^r B$-discrepancy of (order 2) digital nets

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L₂- and $S^r_{p,q}B$-discrepancy of (order 2) digital nets

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Dick proved that all order 2 digital nets satisfy optimal upper bounds of the $L_2$-discrepancy. We give an alternative proof for this fact using Haar bases. Furthermore, we prove that all digital nets satisfy optimal upper bounds of the $S^r_{p,q}B$-discrepancy for a certain parameter range and enlarge that range for order 2 digital nets. $L_p$, $S^r_{p,q}F$- and $S^r_{p}H$-discrepancy is considered as well.

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1 Introduction and results

Let $N$ be some positive integer and let $\mathcal{P}$ be a point set in the unit cube $[0,1)^d$ with $N$ points. Then the discrepancy function $D_P$ is defined as

$$D_P(x) = \frac{1}{N} \sum_{z \in \mathcal{P}} \chi_{[0,x)}(z) - x_1 \cdot \ldots \cdot x_d,$$

for any $x = (x_1, \ldots, x_d) \in [0,1)^d$. By $\chi_{[0,x)}$ we mean the characteristic function of the interval $[0,x) = [0, x_1) \times \ldots \times [0, x_d)$, so the term $\sum_z \chi_{[0,x)}(z)$ is equal to $\#(\mathcal{P} \cap [0,x))$. 
This means that $D_P$ measures the deviation of the number of points of $P$ in $[0,x)$ from the fair number of points $N[0,x) = N x_1 \cdots x_d$ which would be achieved by a (practically impossible) perfectly uniform distribution of the points of $P$, normalized by the total number of points.

Usually one is interested in calculating the norm of the discrepancy function in some normed space of functions on $[0,1)^d$ to which the discrepancy function belongs. A well known result concerns $L_p([0,1)^d)$-spaces for $1 < p < \infty$. There exists a constant $c_{p,d} > 0$ such that for every positive integer $N$ and all point sets $P$ in $[0,1)^d$ with $N$ points, we have

$$\|D_P\|_{L_p([0,1)^d)} \geq c_{p,d} \frac{(\log N)^{(d-1)/2}}{N}. \quad (2)$$

It was proved by Roth [R54] for $p = 2$ and by Schmidt [S77] for arbitrary $1 < p < \infty$. The best value for $c_{2,d}$ can be found in [HM11]. Furthermore, there exists a constant $C_{p,d} > 0$ such that for every positive integer $N$, there exists a point set $P$ in $[0,1)^d$ with $N$ points such that

$$\|D_P\|_{L_p(\mathbb{Q}^d)} \leq C_{p,d} \frac{(\log N)^{(d-1)/2}}{N}. \quad (3)$$

It was proved by Davenport [D56] for $p = 2, d = 2$, by Roth [R80] for $p = 2$ and arbitrary $d$ and finally by Chen [C80] in the general case. The best value for $C_{2,d}$ can be found in [DP10] and [FPPS10].

There are results for $L_1([0,1)^d)$- and star ($L_\infty([0,1)^d)$-) discrepancy though there are still gaps between lower and upper bounds, see [HS1], [S72], [BLV08]. As general references for studies of the discrepancy function we refer to the monographs [DP10], [NW10], [M99], [KN74] and surveys [B11], [Hi14], [M13c].

Roth’s and Chen’s original proofs of (3) were probabilistic. Explicit constructions of point sets with good $L_p$-discrepancy in arbitrary dimension have not been known for a long time. Chen and Skriganov [CS02] (see also [CS08] and [DP10]) gave constructions with optimal bound of the $L_2$-discrepancy and Skriganov [S06] later proved the $L_p$ bound. The constructions of Chen and Skriganov were order 1 digital nets with large Hamming weight. Dick and Pillichshammer [DP14a] (see also [DP14b]) gave alternative constructions. Their constructions are order 3 digital nets. Dick [D14] proved then the following result.

**Theorem 1.1.** There exists a constant $C_{d,b,v} > 0$ such that for every positive integer $n$
and every order 2 digital \((v, n, d)\)-net \(P_n^b\) in base \(b\) we have
\[
\left\|D_{P_n^b}|L_2([0, 1]^d)\right\| \leq C_{d, b, v} \frac{n^{(d-1)/2}}{b^{n}}.
\]

In this work we give an alternative proof for this fact.

Furthermore, there are results for the discrepancy in other function spaces, like Hardy spaces, logarithmic and exponential Orlicz spaces, weighted \(L_p\)-spaces, BMO (see [B11] for results and further literature).

Here, we are interested in Besov \((S^r_{p, q}, B([0, 1]^d))\), Triebel-Lizorkin \((S^r_{p, q}, F([0, 1]^d))\) and Sobolev \((S^r_p H([0, 1]^d))\) spaces with dominating mixed smoothness. Triebel [T10] proved that for all \(1 \leq p, q \leq \infty\) with \(q < \infty\) if \(p = \infty\) and all \(r \in \mathbb{R}\) satisfying \(1/p - 1 < r < 1/p\), there exists a constant \(c_{p, q, r, d} > 0\) such that for every integer \(N \geq 2\) and all point sets \(P\) in \([0, 1]^d\) with \(N\) points, we have
\[
\left\|D_P|S^r_{p, q} B([0, 1]^d)\right\| \geq c_{p, q, r, d} N^{r-1} (\log N)^{(d-1)/q} \tag{4}
\]

and with the additional condition that \(q > 1\) if \(p = \infty\) there exists a constant \(C_{p, q, r, d} > 0\) such that for every positive integer \(N\), there exists a point set \(P\) in \([0, 1]^d\) with \(N\) points and we have
\[
\left\|D_P|S^r_{p, q} B([0, 1]^d)\right\| \leq C_{p, q, r, d} N^{r-1} (\log N)^{(d-1)(1/q + 1 - r)}.
\]

Hinrichs [Hi10] proved for \(d = 2\) that for all \(1 \leq p, q \leq \infty\) and all \(0 \leq r < 1/p\) there exists a constant \(C_{p, q, r} > 0\) such that for every integer \(N \geq 2\) there exists a point set \(P\) in \([0, 1]^2\) with \(N\) points such that
\[
\left\|D_P|S^r_{p, q} B([0, 1]^2)\right\| \leq C_{p, q, r} N^{r-1} (\log N)^{1/q}.
\]

Markhasin [M13b] proved that for all \(1 \leq p, q \leq \infty\) and all \(0 < r < 1/p\) there exists a constant \(C_{p, q, r, d} > 0\) such that for every integer \(N \geq 2\) there exists a point set \(P\) in \([0, 1]^d\) with \(N\) points such that
\[
\left\|D_P|S^r_{p, q} B([0, 1]^d)\right\| \leq C_{p, q, r, d} N^{r-1} (\log N)^{(d-1)/q}. \tag{5}
\]

Explicit point sets with optimal bounds of \(S^r_{p, q} B\)-discrepancy used in [M13b] are the already mentioned point sets by Chen and Skriganov. In \(d = 2\) also (generalized) Hammersley point sets can be used (see [Hi10], [M13a]). Our goal is to prove that there are way more point sets with optimal bounds of the \(S^r_{p, q} B\)-discrepancy. Furthermore
there are results for spaces $S^r_{p,q} F([0,1)^d)$ and $S^r_p H([0,1)^d)$ in [M13c].

**Theorem 1.2.** Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $0 < r < 1/p$. There exists a constant $C_{p,q,r,d,b,v} > 0$ such that for every integer $n$ and every order $1$ digital $(v,n,d)$-net $\mathcal{P}_n^b$ in base $b$ we have

$$\| D_{\mathcal{P}_n^b} |S^r_{p,q} F([0,1)^d)]\| \leq C_{p,q,r,d,b,v} b^{\nu(r-1) n^{(d-1)/q}}.$$

**Theorem 1.3.** Let $1 \leq p, q \leq \infty$, $(q > 1$ if $p = \infty$) and $0 \leq r < 1/p$. There exists a constant $C_{p,q,r,d,b,v} > 0$ such that for every positive integer $n$ and every order $2$ digital $(v,n,d)$-net $\mathcal{P}_n^b$ in base $b$ we have

$$\| D_{\mathcal{P}_n^b} |S^r_{p,q} B([0,1)^d)]\| \leq C_{p,q,r,d,b,v} b^{\nu(r-1) n^{(d-1)/q}}.$$

**Corollary 1.4.** Let $1 \leq p, q < \infty$ and $0 < r < 1/\max(p, q)$. There exists a constant $C_{p,q,r,d,b,v} > 0$ such that for every positive integer $n$ and every order $1$ digital $(v,n,d)$-net $\mathcal{P}_n^b$ in base $b$ we have

$$\| D_{\mathcal{P}_n^b} |S^r_{p,q} F([0,1)^d)]\| \leq C_{p,q,r,d,b,v} b^{\nu(r-1) n^{(d-1)/q}}.$$

**Corollary 1.5.** Let $1 \leq p, q < \infty$ and $0 \leq r < 1/\max(p, q)$. There exists a constant $C_{p,q,r,d,b,v} > 0$ such that for every positive integer $n$ and every order $2$ digital $(v,n,d)$-net $\mathcal{P}_n^b$ in base $b$ we have

$$\| D_{\mathcal{P}_n^b} |S^r_{p,q} F([0,1)^d)]\| \leq C_{p,q,r,d,b,v} b^{\nu(r-1) n^{(d-1)/q}}.$$

**Corollary 1.6.** Let $1 \leq p < \infty$ and $0 < r < 1/\max(p,2)$. There exists a constant $C_{p,r,d,b,v} > 0$ such that for every positive integer $n$ and every order $1$ digital $(v,n,d)$-net $\mathcal{P}_n^b$ in base $b$ we have

$$\| D_{\mathcal{P}_n^b} |S^r_p H([0,1)^d)]\| \leq C_{p,r,d,b,v} b^{\nu(r-1) n^{(d-1)/2}}.$$

**Corollary 1.7.** Let $1 \leq p < \infty$ and $0 \leq r < 1/\max(p,2)$. There exists a constant $C_{p,r,d,b,v} > 0$ such that for every positive integer $n$ and every order $2$ digital $(v,n,d)$-net $\mathcal{P}_n^b$ in base $b$ we have

$$\| D_{\mathcal{P}_n^b} |S^r_p H([0,1)^d)]\| \leq C_{p,r,d,b,v} b^{\nu(r-1) n^{(d-1)/2}}.$$

**Theorem 1.8.** Let $1 \leq p < \infty$. There exists a constant $C_{p,d,b,v} > 0$ such that for every
positive integer $n$ and every order 2 digital $(v, n, d)$-net $P^h_n$ in base $b$ we have

$$\left\| D_{P^h_n} L_p([0, 1]^d) \right\| \leq C_{p, d, b, v} \frac{n^{(d-1)/2}}{b^n}. $$

We point out that obviously Theorem 1.1 is a consequence of Theorem 1.8. Nevertheless, we will prove them independently, so that readers without a background in function spaces with dominating mixed smoothness (which is required for the proof of Theorem 1.8) will be able to understand the proof of the $L_2$ bound.

Theorems 1.2 and 1.3 are consistent with older results. Chen-Skriganov point sets are order 1 digital $(v, n, d)$-nets while (generalized) Hammersley point sets are order 2 digital $(0, n, 2)$-nets.

2 Function spaces with dominating mixed smoothness

We define the spaces $S'_{p, q} B([0, 1]^d)$, $S'_{p, q} F([0, 1]^d)$ and $S'_{p, q} H([0, 1]^d)$ according to [10]. Let $S(R^d)$ denote the Schwartz space and $S'(R^d)$ the space of tempered distributions on $R^d$. Let $\varphi_0 \in S(R)$ satisfy $\varphi_0(x) = 1$ for $|x| \leq 1$ and $\varphi_0(x) = 0$ for $|x| > \frac{3}{2}$. Let $\varphi_k(x) = \varphi_0(2^{-k} x) - \varphi_0(2^{-k+1} x)$ where $x \in R$, $k \in N$ and $\varphi_k(x) = \varphi_{k_1}(x_1) \ldots \varphi_{k_d}(x_d)$ where $k = (k_1, \ldots, k_d) \in N_0^d$, $x = (x_1, \ldots, x_d) \in R^d$. The functions $\varphi_k$ are a dyadic resolution of unity since

$$\sum_{k \in N_0^d} \varphi_k(x) = 1$$

for all $x \in R^d$. The functions $F^{-1}(\varphi_k F f)$ are entire analytic functions for every $f \in S'(R^d)$.

Let $0 < p, q \leq \infty$ and $r \in R$. The Besov space with dominating mixed smoothness $S'_{p, q} B(R^d)$ consists of all $f \in S'(R^d)$ with finite quasi-norm

$$\left\| f | S'_{p, q} B(R^d) \right\| = \left( \sum_{k \in N_0^d} 2^{r(k_1+\ldots+k_d)q} \left\| F^{-1}(\varphi_k F f) | L_p(R^d) \right\|^q \right)^{\frac{1}{q}}$$ (6)

with the usual modification if $q = \infty$.

Let $0 < p < \infty$, $0 < q \leq \infty$ and $r \in R$. The Triebel-Lizorkin space with dominating
mixed smoothness $S_{pq}^r F(\mathbb{R}^d)$ consists of all $f \in S'(\mathbb{R}^d)$ with finite quasi-norm

$$\|f|S_{pq}^r F(\mathbb{R}^d)\| = \left\| \left( \sum_{k \in \mathbb{N}_0^d} 2^{r(k_1 + \ldots + k_d)q} \left| \mathcal{F}^{-1}(\varphi_k \mathcal{F} f)(\cdot) \right|^q \right)^{\frac{1}{q}} \|L_p(\mathbb{R}^d) \right\|$$

(7)

with the usual modification if $q = \infty$.

Let $\mathcal{D}([0,1)^d)$ consist of all complex-valued infinitely differentiable functions on $\mathbb{R}^d$ with compact support in the interior of $[0,1)^d$ and let $\mathcal{D}'([0,1)^d)$ be its dual space of all distributions in $[0,1)^d$. The Besov space with dominating mixed smoothness $S_{pq}^r B([0,1)^d)$ consists of all $f \in \mathcal{D}'([0,1)^d)$ with finite quasi-norm

$$\|f|S_{pq}^r B([0,1)^d)\| = \inf \left\{ \left\| g|S_{pq}^r B(\mathbb{R}^d) \right\| : g \in S_{pq}^r B(\mathbb{R}^d), g|_{[0,1)^d} = f \right\}.$$  

(8)

The Triebel-Lizorkin space with dominating mixed smoothness $S_{pq}^r F([0,1)^d)$ consists of all $f \in \mathcal{D}'([0,1)^d)$ with finite quasi-norm

$$\|f|S_{pq}^r F([0,1)^d)\| = \inf \left\{ \left\| g|S_{pq}^r F(\mathbb{R}^d) \right\| : g \in S_{pq}^r F(\mathbb{R}^d), g|_{[0,1)^d} = f \right\}.$$  

(9)

The spaces $S_{pq}^r B(\mathbb{R}^d)$, $S_{pq}^r F(\mathbb{R}^d)$, $S_{pq}^r B([0,1)^d)$ and $S_{pq}^r F([0,1)^d)$ are quasi-Banach spaces. We define the Sobolev space with dominating mixed smoothness as

$$S_{p}^r H([0,1)^d) = S_{pq}^r F([0,1)^d).$$

(10)

If $r \in \mathbb{N}_0$ then it is denoted by $S_{pq}^r W([0,1)^d)$ and is called classical Sobolev space with dominating mixed smoothness. An equivalent norm for $S_{pq}^r W([0,1)^d)$ is

$$\sum_{\alpha \in \mathbb{N}_0^d; 0 \leq \alpha \leq r} \left\| D^{\alpha} f|L_p([0,1)^d) \right\|.$$

Of special interest is the case $r = 0$ since

$$S_{pq}^0 H([0,1)^d) = L_p([0,1)^d).$$

The Besov and Triebel-Lizorkin spaces can be embedded in each other (see [T10] or [M13c, Corollary 1.13]). We point out that the following embedding is a combination of well known results and might look odd at the first glance.
Lemma 2.1. Let $0 < p, q < \infty$ and $r \in \mathbb{R}$. Then we have

\[ S_{\text{max}(p,q),q}^r B([0,1]^d) \hookrightarrow S_{pq}^r F([0,1]^d) \hookrightarrow S_{\text{min}(p,q),q}^r B([0,1]^d). \]

3 Haar and Walsh bases

We denote $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$. Let $b \geq 2$ be an integer. We denote $\mathbb{D}_j = \{0,1,\ldots,b^j-1\}$ and $\mathbb{B}_j = \{1,\ldots,b-1\}$ for $j \in \mathbb{N}_0$ and $\mathbb{D}_{-1} = \{0\}$ and $\mathbb{B}_{-1} = \{1\}$. For $j = (j_1,\ldots,j_d) \in \mathbb{N}_{-1}^d$ let $\mathbb{D}_j = \mathbb{D}_{j_1} \times \cdots \times \mathbb{D}_{j_d}$ and $\mathbb{B}_j = \mathbb{B}_{j_1} \times \cdots \times \mathbb{B}_{j_d}$. For a real $a$ we write $a_+ = \max(a,0)$ and for $j \in \mathbb{N}_-$ we write $|j|_+ = j_1 + \ldots + j_d$.

For $j \in \mathbb{N}_0$ and $m \in \mathbb{D}_j$ we call the interval

\[ I_{j,m} = [b^{-j}m, b^{-j}(m+1)) \]

the $m$-th $b$-adic interval in $[0,1)$ on level $j$. We put $I_{-1,0} = [0,1)$ and call it the 0-th $b$-adic interval in $[0,1)$ on level $-1$. For any $k = 0,\ldots,b-1$ let $I_{j,m}^k = I_{j+1,km+k}$. We put $I_{-1,0}^1 = I_{-1,0} = [0,1)$. For $j \in \mathbb{N}_{-1}^d$ and $m = (m_1,\ldots,m_d) \in \mathbb{D}_j$ we call

\[ I_{j,m} = I_{j_1,m_1} \times \cdots \times I_{j_d,m_d} \]

the $m$-th $b$-adic interval in $[0,1]^d$ on level $j$. We call the number $|j|_+$ the order of the $b$-adic interval $I_{j,m}$. Its volume is $b^{-|j|_+}$.

Let $j \in \mathbb{N}_0$, $m \in \mathbb{D}_j$ and $l \in \mathbb{B}_j$. Let $h_{j,m,l}$ be the function on $[0,1)$ with support in $I_{j,m}$ and the constant value $e^{2\pi ik}$ on $I_{j,m}^k$ for any $k = 0,\ldots,b-1$. We put $h_{-1,0,1} = \chi_{I_{-1,0}}$ on $[0,1)$.

Let $j \in \mathbb{N}_{-1}^d$, $m \in \mathbb{D}_j$ and $l = (l_1,\ldots,l_d) \in \mathbb{B}_j$. The function $h_{j,m,l}$ given as the tensor product

\[ h_{j,m,l}(x) = h_{j_1,m_1,l_1}(x_1) \cdots h_{j_d,m_d,l_d}(x_d) \]

for $x = (x_1,\ldots,x_d) \in [0,1)^d$ is called a $b$-adic Haar function on $[0,1)^d$. The functions $h_{j,m,l}$, $j \in \mathbb{N}_{-1}^d$, $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$ are called $b$-adic Haar basis on $[0,1)^d$.

The following result is [ML3c] Theorem 2.1.

Theorem 3.1. The system

\[ \left\{ b^{-|j|_+} h_{j,m,l} : j \in \mathbb{N}_{-1}^d, m \in \mathbb{D}_j, l \in \mathbb{B}_j \right\} \]

is an orthonormal basis of $L_2([0,1)^d)$, an unconditional basis of $L_p([0,1)^d)$ for $1 < p < \infty$.
and a conditional basis of $L_1([0,1)^d)$. For any function $f \in L_2([0,1)^d)$ we have
\[
\|f\|_{L_2([0,1)^d)}^2 = \sum_{j \in \mathbb{N}^d} b^{j|j|} \sum_{m \in \mathbb{B}_j, l \in \mathbb{B}_j} |\langle f, h_{j,m,l} \rangle|^2.
\]

The following result is [M13c, Theorem 2.11].

**Theorem 3.2.** Let $0 < p, q \leq \infty$, $(q > 1$ if $p = \infty$) and $1/p - 1 < r < \min(1/p, 1)$. Let $f \in D'([0,1)^d)$. Then $f \in S_{pq}^\sigma B([0,1)^d)$ if and only if it can be represented as
\[
f = \sum_{j \in \mathbb{N}^d} b^{j|j|} \sum_{m \in \mathbb{B}_j, l \in \mathbb{B}_j} \mu_{j,m,l} h_{j,m,l}
\]
for some sequence $(\mu_{j,m,l})$ satisfying
\[
\left( \sum_{j \in \mathbb{N}^d} b^{j|j| (r-1/p+1)q} \left( \sum_{m \in \mathbb{B}_j, l \in \mathbb{B}_j} |\mu_{j,m,l}|^p \right)^{q/p} \right)^{1/q} < \infty.
\]

The convergence of (11) is unconditional in $D'([0,1)^d)$ and in any $S_{pq}^\sigma B([0,1)^d)$ with $\rho < r$. The representation (11) of $f$ is unique with the $b$-adic Haar coefficients $\mu_{j,m,l} = \langle f, h_{j,m,l} \rangle$. The expression (12) is an equivalent quasi-norm on $S_{pq}^\sigma B([0,1)^d)$.

For $\alpha \in \mathbb{N}$ with the $b$-adic expansion $\alpha = \beta_{a_1-1} b^{a_1-1} + \cdots + \beta_{a_\nu-1} b^{a_\nu-1}$ with $0 < a_1 < a_2 < \cdots < a_\nu$ and digits $\beta_{a_1-1}, \ldots, \beta_{a_\nu-1} \in \{1, \ldots, b-1\}$, the NRT weight of order $\sigma \in \mathbb{N}$ is given by
\[
\varrho_\sigma(\alpha) = a_\nu + a_{\nu-1} + \cdots + a_{\max(\nu-\sigma+1,1)}.
\]
Furthermore, $\varrho_\sigma(0) = 0$.

For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, the NRT weight of order $\sigma$ is given by
\[
\varrho_\sigma(\alpha) = \varrho_\sigma(\alpha_1) + \cdots + \varrho_\sigma(\alpha_d).
\]

Let $\alpha \in \mathbb{N}$. The $\alpha$-th $b$-adic Walsh function $\text{wal}_\alpha : [0,1) \to \mathbb{C}$ is given by
\[
\text{wal}_\alpha(x) = e^{2\pi i (\beta_{a_1-1} x_{a_1} + \cdots + \beta_{a_\nu-1} x_{a_\nu})}
\]
for $x \in [0,1)$ with $b$-adic expansion $x = x_1 b^{-1} + x_2 b^{-2} + \cdots$. Furthermore, $\text{wal}_0 = \chi_{[0,1]}$.

Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. Then the $\alpha$-th $b$-adic Walsh function $\text{wal}_\alpha$ on $[0,1)^d$ is
given as the tensor product

\[ \text{wal}_\alpha(x) = \text{wal}_{\alpha_1}(x^1) \ldots \text{wal}_{\alpha_d}(x^d) \]

for \( x = (x^1, \ldots, x^d) \in [0,1)^d \). The functions \( \text{wal}_\alpha, \alpha \in \mathbb{N}^d_0 \) are called \( b \)-adic Walsh basis on \([0,1)^d\).

The \( b \)-adic Walsh function \( \text{wal}_\alpha \) is constant on \( b \)-adic intervals \( I_{\{\varrho_1(\alpha_1), \ldots, \varrho_1(\alpha_d)\}, m} \) for every \( m \in \mathbb{D}_{\{\varrho_1(\alpha_1), \ldots, \varrho_1(\alpha_d)\}} \). The following result is [DP10, Theorem A.11].

**Lemma 3.3.** The system

\[ \{ \text{wal}_\alpha : \alpha \in \mathbb{N}^d_0 \} \]

is an orthonormal basis of \( L_2([0,1)^d) \).

## 4 Digital \((v,n,d)\)-nets

We quote from [DP14a] and [D07] to describe the digital construction method and properties of resulting digital nets. We also refer to [N87] and [NP01].

For an integer \( b \geq 2 \) let \( \mathbb{Z}_b \) denote the commutative ring of integers modulo \( b \). For \( s, n \in \mathbb{N} \) with \( s \geq n \) let \( C_1, \ldots, C_d \) be \( s \times n \) matrices with entries from \( \mathbb{Z}_b \). For \( \nu \in \{0, 1, \ldots, b^n - 1\} \) with the \( b \)-adic expansion \( \nu = \nu_0 + \nu_1 b + \ldots + \nu_{n-1} b^{n-1} \) with digits \( \nu_0, \nu_1, \ldots, \nu_{n-1} \in \{0, 1, \ldots, b-1\} \) the \( b \)-adic digit vector \( \bar{\nu} \) is given as \( \bar{\nu} = (\nu_0, \nu_1, \ldots, \nu_{n-1})^{\top} \in \mathbb{Z}_b^n \). Then we compute \( C_i \bar{\nu} = (x_{i,\nu,1}, x_{i,\nu,2}, \ldots, x_{i,\nu,s})^{\top} \in \mathbb{Z}_b^s \) for \( 1 \leq i \leq d \). Finally we define

\[ x_{i,\nu} = x_{i,\nu,1} b^{-1} + x_{i,\nu,2} b^{-2} + \ldots + x_{i,\nu,s} b^{-s} \in [0,1) \]

and \( x_\nu = (x_{1,\nu}, \ldots, x_{d,\nu}) \). We call the point set \( \mathcal{P}_n^b = \{x_0, x_1, \ldots, x_{b^n-1}\} \) a digital net in base \( b \).

Now let \( \sigma \in \mathbb{N} \) and suppose \( s \geq \sigma n \). Let \( 0 \leq v \leq \sigma n \) be an integer. For every \( 1 \leq i \leq d \) we write \( C_i = (c_{i,1}, \ldots, c_{i,s})^{\top} \) where \( c_{i,1}, \ldots, c_{i,s} \in \mathcal{P}_n^a \) are the row vectors of \( C_i \). If for all \( 1 \leq \lambda_{i,1} < \ldots < \lambda_{i,\eta_i} \leq s, 1 \leq i \leq d \) with

\[ \lambda_{1,1} + \ldots + \lambda_{1,\min(\eta_1,\sigma)} + \ldots + \lambda_{d,1} + \ldots + \lambda_{d,\min(\eta_d,\sigma)} \leq \sigma n - v \]

the vectors \( c_1, c_{1,1}, \ldots, c_{1,\eta_1}, \ldots, c_d, c_{d,1}, \ldots, c_{d,\eta_d} \) are linearly independet over \( \mathbb{Z}_b \), then \( \mathcal{P}_n^b \) is called an order \( \sigma \) digital \((v,n,d)\)-net in base \( b \).

**Lemma 4.1.**
(i) Let $v < \sigma n$. Then every order $\sigma$ digital $(v, n, d)$-net in base $b$ is an order $\sigma$ digital $(v+1, n, d)$-net in base $b$. In particular every point set $P_n^b$ constructed with the digital method is at least an order $\sigma$ digital $(\sigma n, n, d)$-net in base $b$.

(ii) Let $1 \leq \sigma_1 \leq \sigma_2$. Then every order $\sigma_2$ digital $(v, n, d)$-net in base $b$ is an order $\sigma_1$ digital $(\lceil v \sigma_1 / \sigma_2 \rceil, n, d)$-net in base $b$.

Lemma 4.2. Let $P_n^b$ be an order $\sigma$ digital $(v, n, d)$-net in base $b$ then every $b$-adic interval of order $n - v$ contains exactly $b^v$ points of $P_n^b$.

Let $t \in \mathbb{N}_0$ with $b$-adic expansion $t = \tau_0 + \tau_1 b + \tau_2 b^2 + \ldots$. We put $\bar{t} = (\tau_0, \tau_1, \ldots, \tau_{s-1})^\top \in \mathbb{Z}_b^s$ and define

$$D(\mathcal{C}) = \{ t = (t_1, \ldots, t_d) \in \mathbb{N}_0^d \setminus \{(0, \ldots, 0)\} : C_1^\top \bar{t}_1 + \ldots + C_d^\top \bar{t}_d = (0, \ldots, 0) \in \mathbb{Z}_b^s \}.$$

Lemma 4.3. $P_n^b$ is an order $\sigma$ digital $(v, n, d)$-net in base $b$ if and only if $\varrho_\sigma(t) > \sigma n - v$ for all $t \in D(\mathcal{C})$.

Lemma 4.4. Let $P_n^b$ be an order $\sigma$ digital $(v, n, d)$-net in base $b$ with generating matrices $C_1, \ldots, C_d$. Then

$$\sum_{z \in P_n^b} \text{wal}_t(z) = \begin{cases} b^v & \text{if } t \in D(\mathcal{C}), \\ 0 & \text{otherwise.} \end{cases}$$

We consider the Walsh series expansion of the function $\chi_{[0,x]}$,

$$\chi_{[0,x)}(y) = \sum_{t=0}^{\infty} \hat{\chi}_{[0,x)}(t) \text{wal}_t(y), \quad (13)$$

where for $t \in \mathbb{N}_0$ the $t$-th Walsh coefficient is given by

$$\hat{\chi}_{[0,x)}(t) = \int_0^1 \chi_{[0,x)}(y) \overline{\text{wal}_t(y)} dy = \int_0^x \overline{\text{wal}_t(y)} dy.$$

Lemma 4.5. Let $P_n^b$ be an order $\sigma$ digital $(v, n, d)$-net in base $b$ with generating matrices $C_1, \ldots, C_d$. Then

$$D_{P_n^b}(x) = \sum_{t \in D(\mathcal{C})} \hat{\chi}_{[0,x]}(t).$$

Proof. For $t = (t_1, \ldots, t_d) \in \mathbb{N}_0^d$ and $x = (x_1, \ldots, x_d) \in [0,1]^d$, we have

$$\hat{\chi}_{[0,x)}(t) = \hat{\chi}_{[0,x_1)}(t_1) \cdot \ldots \cdot \hat{\chi}_{[0,x_d)}(t_d).$$
Applying Lemma 4.4 we get
\[ D_P(x) = \frac{1}{b^n} \sum_{z \in P^n} \sum_{t_1, \ldots, t_d=0}^{\infty} \hat{\chi}_{[0,x)}(t) \text{wal}_t(z) - \hat{\chi}_{[0,x)}((0, \ldots, 0)) \]
\[ = \sum_{t \in D(C)} \hat{\chi}_{[0,x)}(t). \]
\[ = \sum_{t \in D(C)} \hat{\chi}_{[0,x)}(t). \]
\[ \square \]

Several constructions of order \( \sigma \) digital \((v,n,d)\)-nets are known. For details, examples and further literature we refer to [DPT14b]. There are especially constructions with a good quality parameter \( v \), e. g. we can construct order 2 digital \((d,n,d)\)-nets in base \( b \) as well as order 1 digital \((0,n,d)\)-nets.

5 Proofs of the results

For two sequences \( a_n \) and \( b_n \) we will write \( a_n \preceq b_n \) if there exists a constant \( c > 0 \) such that \( a_n \leq cb_n \) for all \( n \). For \( t > 0 \) with \( b \)-adic expansion \( t = \tau_0 + \tau_1 b + \ldots + \tau_{\rho_1(t)-1} b^{\rho_1(t)-1} \), we put \( t = t' + \tau_{\rho_1(t)-1} b^{\rho_1(t)-1} \).

The following result is [M13b] Lemma 5.1.

Lemma 5.1. Let \( f(x) = x_1 \cdot \ldots \cdot x_d \) for \( x = (x_1, \ldots, x_d) \in [0,1)^d \). Let \( j \in \mathbb{N}_d, m \in \mathbb{D}_j, l \in \mathbb{B}_j \). Then \( |\langle f, h_j, m, l \rangle| \leq b^{-|j|+} \).

The following result is [M13b] Lemma 5.2.

Lemma 5.2. Let \( z = (z_1, \ldots, z_d) \in [0,1)^d \) and \( g(x) = \chi_{[0,x)}(z) \) for \( x = (x_1, \ldots, x_d) \in [0,1)^d \). Let \( j \in \mathbb{N}_d, m \in \mathbb{D}_j, l \in \mathbb{B}_j \). Then \( \langle g, h_j, m, l \rangle = 0 \) if \( z \) is not contained in the interior of the \( b \)-adic interval \( I_j,m \). If \( z \) is contained in the interior of \( I_j,m \) then \( |\langle g, h_j, m, l \rangle| \leq b^{-|j|+} \).

The following result is [M13b] Lemma 5.9.

Lemma 5.3. Let \( j \in \mathbb{N}_d, m \in \mathbb{D}_j, l \in \mathbb{B}_j \) and \( \alpha \in \mathbb{N}_0^d \). Then
\[ |\langle h_j, m, l, \text{wal}_\alpha \rangle| \leq b^{-|j|+}. \]
If \( q_1(\alpha_i) \neq j_i + 1 \) for some \( 1 \leq i \leq d \) then

\[
\langle h_{j,m,t}, \text{wal}_\alpha \rangle = 0.
\]

The following result is [M13b, Lemma 5.10].

**Lemma 5.4.** Let \( t, \alpha \in \mathbb{N}_0 \). Then

\[
|\langle \hat{\chi}_{[0,\cdot)}(t), \text{wal}_\alpha \rangle| \leq b^{-\max(q_1(t),q_1(\alpha))}.
\]

If \( \alpha \neq t' \) and \( \alpha \neq t \) and \( \alpha' \neq t \) then

\[
\langle \hat{\chi}_{[0,\cdot)}(t), \text{wal}_\alpha \rangle = 0.
\]

**Lemma 5.5.** Let \( C_1, \ldots, C_d \in \mathbb{Z}_b^{\times n} \) generate an order 1 digital \((v, n, d)\)-net in base \( b \).

Let \( \lambda_1, \ldots, \lambda_d, \gamma_1, \ldots, \gamma_d \in \mathbb{N}_0 \). Let \( \omega_{\gamma_1, \ldots, \gamma_d}^{\lambda_1, \ldots, \lambda_d}(\mathcal{C}) \) denote the cardinality of such \( t \in \mathcal{D}(\mathcal{C}) \) with \( q_1(t_i) = \gamma_i \) for all \( 1 \leq i \leq d \) that either \( \gamma_i \leq \lambda_i \) or \( q_1(t'_i) = \lambda_i \). If \( \lambda_1, \ldots, \lambda_d \leq s \) then

\[
\omega_{\gamma_1, \ldots, \gamma_d}^{\lambda_1, \ldots, \lambda_d}(\mathcal{C}) \leq (b - 1)^d b^{(\min(\lambda_1, \gamma_1) - 1) + \cdots + \min(\lambda_d, \gamma_d) - 1 - n + v + s}. \]

**Proof.** Let \( t = (t_1, \ldots, t_d) \in \mathcal{D}(\mathcal{C}) \) with \( q_1(t_i) = \gamma_i \) for all \( 1 \leq i \leq d \) and either \( \gamma_i \leq \lambda_i \) or \( q_1(t'_i) = \lambda_i \). Let \( t_i \) have \( b \)-adic expansion \( t_i = \tau_{i,0} + \tau_{i,1}b + \tau_{i,2}b^2 + \cdots \). Let \( \lambda_i = (c_i,1, \ldots, c_i,s) \), put \( \lambda_i^* = \min(\lambda_i, \gamma_i - 1) \) and \( c_{i,\gamma_i} = (0, \ldots, 0) \) if \( \gamma_i > s, 1 \leq i \leq d \). Then we have

\[
\begin{align*}
&c_{1,\lambda_1^*}^\top \tau_{1,0} + \cdots + c_{1,\lambda_1^*}^\top \tau_{1,\lambda_1^*-1} + c_{1,\gamma_1}^\top \tau_{1,\gamma_1-1} + \\
&\vdots \\
&+ c_{d,\lambda_d^*}^\top \tau_{d,0} + \cdots + c_{d,\lambda_d^*}^\top \tau_{d,\lambda_d^*-1} + c_{d,\gamma_d}^\top \tau_{d,\gamma_d-1} = (0, \ldots, 0)^\top \in \mathbb{Z}_b^n.
\end{align*}
\]

We put

\[
A = (c_{1,1}^\top, \ldots, c_{1,\lambda_1^*}^\top, \ldots, c_{d,1}^\top, \ldots, c_{d,\lambda_d^*}^\top) \in \mathbb{Z}_b^{n \times (\lambda_1^* + \cdots + \lambda_d^*)},
\]

\[
y = (\tau_{1,0}, \ldots, \tau_{1,\lambda_1^*-1}, \ldots, \tau_{d,0}, \ldots, \tau_{d,\lambda_d^*-1})^\top \in \mathbb{Z}_b^{(\lambda_1^* + \cdots + \lambda_d^*) \times 1}
\]

and

\[
w = -c_{1,\gamma_1}^\top \tau_{1,\gamma_1-1} - \cdots - c_{d,\gamma_d}^\top \tau_{d,\gamma_d-1} \in \mathbb{Z}_b^{n \times 1}.
\]

Then \((14)\) corresponds to \( Ay = w \) and we have

\[
\omega_{\gamma_1, \ldots, \gamma_d}^{\lambda_1, \ldots, \lambda_d}(\mathcal{C}) = \#\{ y \in \mathbb{Z}_b^{\lambda_1^* + \cdots + \lambda_d^*} : Ay = w \}.
\]
Since $C_1, \ldots, C_d$ generate an order 1 digital $(v, n, d)$-net, the rank of $A$ is $\lambda_1^* + \ldots + \lambda_d^*$ if $\lambda_1^* + \ldots + \lambda_d^* = n - v$. In this case the solution space of the homogeneous system $Ay = (0, \ldots, 0)$ has dimension 0. If $\lambda_1^* + \ldots + \lambda_d^* > n - v$ then $\text{rank}(A) \geq n - v$ and the dimension of the solution space of the homogeneous system is $\lambda_1^* + \ldots + \lambda_d^* - \text{rank}(A) \leq \lambda_1 + \ldots + \lambda_d - n + v$. This means that for a given $w$ the system $Ay = w$ has at most 1 solution if $\lambda_1^* + \ldots + \lambda_d^* \leq n - v$ and at most $b^{\lambda_1^*+\ldots+\lambda_d^*-n+v}$ otherwise. Finally, there are $(b - 1)^d$ possible choices for $w$ since none of the numbers $\tau_{1, \gamma_1-1}, \ldots, \tau_{d, \gamma_d-1}$ can be 0.

We point out that the condition $\lambda_1, \ldots, \lambda_d \leq s$ is not necessary. It just reduces the technicalities but the results would be the same without it. One would have to define $\lambda_i^* = \min(\lambda_i, s)$ and in the case where $\lambda_i^* > s$ we would get an additional factor $b^{\lambda_i^*-s}$ compensating the restriction.

**Proposition 5.6.** Let $\mathcal{P}_n^b$ be an order 1 digital $(v, n, d)$-net in base $b$. Let $j \in \mathbb{N}_{-1}, m \in \mathbb{D}_j, l \in \mathbb{B}_j$.

1. If $|j|_+ \geq n - v$ then $|\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| \leq b^{-|j|_+-n+v}$ and $|\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| \leq b^{-|j|_+}$ for all but at most $b^n$ values of $m$.
2. If $|j|_+ < n - v$ then $|\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| \leq b^{-|j|_+-n+v} (n - v - |j|_+)^{d-1}$.

**Proof.** For (i), let $|j|_+ \geq n - v$. Since $\mathcal{P}_n^b$ contains exactly $b^n$ points, there are no more than $b^n$ such $m$ for which $I_{j,m}$ contains a point of $\mathcal{P}_n^b$ meaning that at least all but $b^n$ intervals contain no points at all. Thus the second statement follows from Lemma 5.1. The remaining intervals contain at most $b^n$ points of $\mathcal{P}_n^b$ (Lemma 4.2) so the first statement follows from Lemmas 5.1 and 5.2.

We now prove (ii) so let $|j|_+ < n - v$ and $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$. The function $h_{j,m,l}$ can be given (Lemma 3.3) as

\[
h_{j,m,l} = \sum_{\alpha \in \mathbb{N}_0^d} \langle h_{j,m,l}, \text{wal}_\alpha \rangle \text{wal}_\alpha.
\]

We apply Lemmas 4.5, 5.3 and 5.4 and get

\[
|\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| = \left| \sum_{t \in \mathbb{D}(\mathcal{C})} \sum_{\alpha \in \mathbb{N}_0^d} \langle \hat{\chi}_{[0,\cdot)}(t), \langle h_{j,m,l}, \text{wal}_\alpha \rangle \text{wal}_\alpha \rangle \right|
\]

\[
\leq \sum_{t \in \mathbb{D}(\mathcal{C})} \sum_{\alpha \in \mathbb{N}_0^d} \left| \langle \hat{\chi}_{[0,\cdot)}(t), \text{wal}_\alpha \rangle \right| |\langle h_{j,m,l}, \text{wal}_\alpha \rangle|
\]
By Lemma 5.5 we get since

$$\sum_{t \in \mathcal{D}(\mathcal{C})} \sum_{\alpha \in \mathbb{N}_0^d} b_{\max(g_1(\alpha_1), g_1(t_1)) - \ldots - \max(g_1(\alpha_1), g_1(t_d))}$$

$$\leq b^{-|j|+} \sum_{t \in \mathcal{D}(\mathcal{C})} \sum_{\alpha \in \mathbb{N}_0^d} b_{\max(g_1(\alpha_1), g_1(t_1)) - \ldots - \max(g_1(\alpha_1), g_1(t_d))}$$

$$= b^{-|j|+} \sum_{t \in \mathcal{D}(\mathcal{C})} b_{\max(j+1, t_1)} - \ldots - \max(j+1, t_d)}$$

$$= b^{-|j|+} \sum_{\gamma_1, \ldots, \gamma_d = 0}^{\infty} b_{\max(j+1, \gamma_1) - \ldots - \max(j+1, \gamma_d)} \omega_{\gamma_1, \ldots, \gamma_d}^{j+1, \ldots, j+1}(\mathcal{C}). \quad (15)$$

By Lemma 5.5 we get

$$\omega_{\gamma_1, \ldots, \gamma_d}^{j+1, \ldots, j+1}(\mathcal{C}) \leq (b-1)^d b^d$$

since \( j_1 + 1, \ldots, j_d + 1 \leq n - v \leq s \) and \( j_1 + 1 + \ldots + j_d + 1 \leq |j| + d < n - v + d \).

We recall that we have \( g_1(t) > n-v \) for all \( t \in \mathcal{D}(\mathcal{C}) \). This means that \( \omega_{\gamma_1, \ldots, \gamma_d}^{j+1, \ldots, j+1}(\mathcal{C}) = 0 \) whenever \( \gamma_1 + \ldots + \gamma_d \leq n - v \). Therefore \( \omega_{\gamma_1, \ldots, \gamma_d}^{j+1, \ldots, j+1}(\mathcal{C}) = 0 \) if \( \gamma_i \leq j_i \) for all \( 1 \leq i \leq d \).

For any \( I \subset \{1, \ldots, d\} \) let \( I^c = \{1, \ldots, d\} \setminus I \). We perform an index shift to get

$$|\langle D_{p_n}, h_{j,m,l} \rangle| \leq b^{-|j|+} \sum_{I \subseteq \{1, \ldots, d\}} b_{\max(j_1+1, \gamma_1) - \ldots - \max(j_d+1, \gamma_d)}$$

$$= b^{-|j|+} \sum_{I \subseteq \{1, \ldots, d\}} b_{\sum_{\gamma_1, \ldots, \gamma_d = 0}^{\infty} \sum_{\gamma_1 \leq j_1}^{\gamma_1 \geq j_2+1} \sum_{\gamma_2 \geq j_2+1}^{\gamma_2 \geq j_3+1 \ldots} b_{\sum_{\gamma_2 \in I^c} \gamma_{n_2}}$$

$$= b^{-|j|+} \sum_{I \subseteq \{1, \ldots, d\}} b_{\sum_{\gamma_1, \ldots, \gamma_d = 0}^{\infty} \sum_{\gamma_1 \leq j_1}^{\gamma_1 \geq j_2+1} \sum_{\gamma_2 \geq j_2+1}^{\gamma_2 \geq j_3+1 \ldots} b_{\sum_{\gamma_2 \in I^c} \gamma_{n_2}}$$

$$\leq b^{-|j|+} \sum_{I \subseteq \{1, \ldots, d\}} b_{\sum_{\gamma_1, \ldots, \gamma_d = 0}^{\infty} \sum_{\gamma_1 \leq j_1}^{\gamma_1 \geq j_2+1} \sum_{\gamma_2 \geq j_2+1}^{\gamma_2 \geq j_3+1 \ldots} b_{\sum_{\gamma_2 \in I^c} \gamma_{n_2}}$$

$$\leq b^{-|j|+} \sum_{I \subseteq \{1, \ldots, d\}} b_{\sum_{\gamma_1, \ldots, \gamma_d = 0}^{\infty} \sum_{\gamma_1 \leq j_1}^{\gamma_1 \geq j_2+1} \sum_{\gamma_2 \geq j_2+1}^{\gamma_2 \geq j_3+1 \ldots} b_{\sum_{\gamma_2 \in I^c} \gamma_{n_2}}$$
Proof. According to Lemma 4.1, recall that we have D
\text{Proposition} 5.7.

\[ \begin{align*}
\ldots \sum_{0 \leq j_1 \leq j_1} b^{-r} (r + 1)^{d-1-\#I} \\
\leq b^{-|j|} \sum_{I \subseteq \{1, \ldots, d\}} b^{-|j_1|} - \sum_{\kappa_1 \in I} \frac{(j_{k_1})}{(j_{k_2}+1)} \sum_{\kappa_2 \in I_c} \gamma_{\kappa_1} - \sum_{\kappa_3 \in I_c} (j_{k_2} + 1) \times \\
\leq b^{-|j| - n + v} \sum_{I \subseteq \{1, \ldots, d\}} b^{-|j_1|} - \sum_{\kappa_1 \in I} \frac{(j_{k_1})}{(j_{k_2}+1)} \sum_{\kappa_2 \in I_c} \gamma_{\kappa_1} - \sum_{\kappa_3 \in I_c} (j_{k_2} + 1) \times \\
\leq b^{-|j| - n + v} (n - |j|)^{d-1}.
\end{align*} \]

\[ \square \]

Proposition 5.7. Let \( \mathcal{P}_n^b \) be an order 2 digital \((v, n, d)\)-net in base \( b \). Let \( j \in \mathbb{N}_-^d \), \( m \in \mathbb{D}_j \), \( l \in \mathbb{B}_j \).

(i) If \( |j|_+ \geq n - \lfloor v/2 \rfloor \) then \( \|D_{p_n^b, h_{j,m,l}}\| \leq b^{-|j|+ - n + v/2} \) and \( \|\mu_{j,m,l}(D_{p_n^b})\| \leq b^{-2|j|+} \) for all but \( b^n \) values of \( m \).

(ii) If \( |j|_+ < n - \lfloor v/2 \rfloor \) then \( \|D_{p_n^b, h_{j,m,l}}\| \leq b^{-2n + v} (2n - v - 2|j|)_+^{d-1} \).

Proof. According to Lemma 4.1, \( \mathcal{P}_n^b \) is an order 1 digital \([\lfloor v/2 \rfloor, n, d] \)-net. Hence (i) follows from Proposition 5.6.

We now prove (ii) so let \( |j|_+ < n - \lfloor v/2 \rfloor \) and \( m \in \mathbb{D}_j \), \( l \in \mathbb{B}_j \). We start at (15) and recall that we have \( g_2(t) > 2n - v \) for all \( t \in \mathcal{O}(\mathcal{C}) \). This means that \( \omega_{j_1+1, \ldots, j_d+1}(\mathcal{C}) = 0 \) whenever \( \gamma_1 + \min(\gamma_1, j_1 + 1) + \ldots + \gamma_d + \min(\gamma_d, j_d + 1) \leq 2n - v \). We argue similarly.
to the proof of Proposition 5.6 to get

\[
|\langle D_{\gamma_1}, h_{j_m} \rangle| \leq b^{-|j|} + \sum_{\gamma_1, \ldots, \gamma_d = 0}^\infty b^{-\max(j_1 + 1, \gamma_1) - \ldots - \max(j_d + 1, \gamma_d)} \omega_{j_1, \ldots, j_d + 1}
\]

\[
\leq b^{-|j|} + \sum_{\gamma_1, \ldots, \gamma_d = 0}^\infty b^{-\max(j_1 + 1, \gamma_1) - \ldots - \max(j_d + 1, \gamma_d)}
\]

\[
\leq b^{-|j|} + \sum_{\gamma_1, \ldots, \gamma_d = 0}^\infty b^{-\max(j_1 + 1, \gamma_1) - \ldots - \max(j_d + 1, \gamma_d)}
\]

\[
\leq b^{-|j|} + \sum_{\gamma_1, \ldots, \gamma_d = 0}^\infty b^{-\max(j_1 + 1, \gamma_1) - \ldots - \max(j_d + 1, \gamma_d)}
\]

\[
\leq b^{-|j|} + \sum_{\gamma_1, \ldots, \gamma_d = 0}^\infty b^{-\max(j_1 + 1, \gamma_1) - \ldots - \max(j_d + 1, \gamma_d)}
\]

\[
\leq b^{-|j|} + \sum_{\gamma_1, \ldots, \gamma_d = 0}^\infty b^{-\max(j_1 + 1, \gamma_1) - \ldots - \max(j_d + 1, \gamma_d)}
\]

\[
\leq b^{-|j|} + \sum_{\gamma_1, \ldots, \gamma_d = 0}^\infty b^{-\max(j_1 + 1, \gamma_1) - \ldots - \max(j_d + 1, \gamma_d)}
\]

\[
\leq b^{-|j|} + \sum_{\gamma_1, \ldots, \gamma_d = 0}^\infty b^{-\max(j_1 + 1, \gamma_1) - \ldots - \max(j_d + 1, \gamma_d)}
\]

\[
\leq b^{-|j|} + \sum_{\gamma_1, \ldots, \gamma_d = 0}^\infty b^{-\max(j_1 + 1, \gamma_1) - \ldots - \max(j_d + 1, \gamma_d)}
\]

\[
\leq b^{-|j|} + \sum_{\gamma_1, \ldots, \gamma_d = 0}^\infty b^{-\max(j_1 + 1, \gamma_1) - \ldots - \max(j_d + 1, \gamma_d)}
\]

\[
\leq b^{-|j|} + \sum_{\gamma_1, \ldots, \gamma_d = 0}^\infty b^{-\max(j_1 + 1, \gamma_1) - \ldots - \max(j_d + 1, \gamma_d)}
\]
\[ \times \left( 2n - v - 2 \sum_{\kappa_1 \in I} \gamma_{\kappa_1} - 2 \sum_{\kappa_2 \in I^c} (j_{\kappa_2} + 1) + 1 \right)^{d-1} \]

\[ \leq b^{-|j|+2n+v} \sum_{I\subseteq\{1,\ldots,d\}} b^{\kappa_2+1} \\times \]

\[ \times \left( 2n - v - 2 \sum_{\kappa_1 \in I} (j_{\kappa_1} + 1) - 2 \sum_{\kappa_2 \in I^c} (j_{\kappa_2} + 1) + 1 \right)^{d-1} \]

\[ \leq b^{-2n+v} (2n - v - 2|j|)^{d-1}. \]

\[ \square \]

We are now ready to prove the theorems.

**Proof of Theorem 1.** Let \( D_{P_n^k} \) be an order 2 digital \((v,n,d)\)-net in base \( b \). We apply Theorem 3.1 hence we need to prove

\[ \sum_{j \in \mathbb{N}_0^d} b^{|j|+n} \sum_{m \in \mathbb{B}_j, l \in \mathbb{B}_j} |\langle D_{P_n^k}, h_{j,m,l} \rangle|^2 \leq b^{-2n+v} n^{d-1} v. \]

We recall that \( \#\mathbb{D}_j = b^{|j|+} \), \( \#\mathbb{B}_j = b - 1 \). We split the sum and apply Proposition 5.7 to get

\[ \sum_{j \in \mathbb{N}_0^d} b^{|j|+} \sum_{|j|+ < n - \lfloor v/2 \rfloor} |\langle D_{P_n^k}, h_{j,m,l} \rangle|^2 \]

\[ \leq b^{|j|+} b^{|j|+} b^{-4n+2v} (2n - v - 2|j|)^{2(d-1)} \]

\[ \leq b^{2n-v} (2n - v - 2|j|)^{2(d-1)} \]

\[ \leq b^{2n-v} (2n - v - 2n + v)^{2(d-1)} \]

\[ \leq b^{2n-v} n^{d-1} \]

for big intervals,

\[ \sum_{j \in \mathbb{N}_0^d} b^{|j|+} \sum_{|j|+ \geq n - \lfloor v/2 \rfloor} \sum_{m \in \mathbb{B}_j, l \in \mathbb{B}_j} |\langle D_{P_n^k}, h_{j,m,l} \rangle|^2 \]
for middle intervals and
\[
\sum_{j \in \mathbb{N}_d^{-1}} b^{\|j\|} \sum_{m \in D_j, l \in B_j} |\langle D_{P_n}^o, h_j, m, l \rangle| \leq \sum_{j \in \mathbb{N}_d^{-1}} b^{\|j\|} b^{n} b^{-2\|j\|+2n+v} + \sum_{j \in \mathbb{N}_d^{-1}} b^{\|j\|} (b^{\|j\|} - b^n) b^{-4\|j\|}
\]
\[
\leq b^{n-v} \sum_{\kappa=n}^{\infty} b^{-\kappa} (\kappa + 1)^{d-1} + \sum_{\kappa=n}^{\infty} b^{-2\kappa} (\kappa + 1)^{d-1}
\]
\[
\leq b^{-2n+v} n^{d-1} v
\]
for small intervals. □

**Proof of Theorem 1.2.** Let $D_{P_n}$ be an order 1 digital $(v, n, d)$-net in base $b$. We apply Theorem 3.2 hence we need to prove
\[
\sum_{j \in \mathbb{N}_d^{-1}} b^{\|j\|+ (r-1/p+1)q} \left( \sum_{m \in D_j, l \in B_j} |\langle D_{P_n}^o, h_j, m, l \rangle|^{p} \right)^{q/p} \leq b^{n(r-1)q} n^{(d-1)q} b^{vq}.
\]

We recall that $|D_j| = b^{\|j\|}$, $|B_j| = b - 1$. We split the sum and apply Minkowski’s inequality and Proposition 5.6 to get
\[
\sum_{j \in \mathbb{N}_d^{-1}} b^{\|j\|+ (r-1/p+1)q} \left( \sum_{m \in D_j, l \in B_j} |\langle D_{P_n}^o, h_j, m, l \rangle|^{p} \right)^{q/p} \leq b^{n(r-1)q} n^{(d-1)q} b^{vq}.
\]
\[ \leq b^{-(n+v)} q b^{(n-v) r q} (n - v + 1)^{d-1} \]
\[ \leq b^{n(r-1)} q n^{d-1} b^{v(1-r) q} \]

for big intervals,

\[ \sum_{j \in \mathbb{N}} \sum_{n > |j| + |n - v|} b^{|j| + (r-1/p+1)q} \left( \sum_{m \in D_j, l \in B_j} |\langle D_{p_n}, h_{j,m,l} \rangle|^p \right)^{q/p} \]
\[ \leq \sum_{j \in \mathbb{N}} b^{|j| + (r-1/p+1)q} b^{|j| + q/p} b^{(-|j| - n + v)q} \]
\[ \leq b^{-(n+v)q} \sum_{\kappa = n-v}^{n-1} b^{\kappa r q} (\kappa + 1)^{d-1} \]
\[ \leq b^{-(n+v)q} b^{n d-1} \]
\[ \leq b^{n(r-1) q n^{d-1}} b^v \]

for middle intervals and considering the range of \( r \)

\[ \sum_{j \in \mathbb{N}} b^{|j| + (r-1/p+1)q} \left( \sum_{m \in D_j, l \in B_j} |\langle D_{p_n}, h_{j,m,l} \rangle|^p \right)^{q/p} \]
\[ \leq \sum_{j \in \mathbb{N}} b^{|j| + (r-1/p+1)q} b^{n q/p} b^{(-|j| - n + v)q} + \sum_{j \in \mathbb{N}} b^{|j| + (r-1/p+1)q} b^{|j| - b^n q/p} b^{-2|j| - q} \]
\[ \leq b^{n q/p} b^{-(n+v)q} \sum_{\kappa = n}^{\infty} b^{c(r-1/p)q} (\kappa + 1)^{d-1} + \sum_{\kappa = n}^{\infty} b^{c(r-1)q} (\kappa + 1)^{d-1} \]
\[ \leq b^{n q/p} b^{-(n+v)q} b^{n(r-1)/q} n^{d-1} + b^{n(r-1) q n^{d-1}} \]
\[ \leq b^{n(r-1) q n^{d-1}} b^v \]

for small intervals. \( \square \)

**Proof of Theorem 1.3.** Let \( D_{p_n} \) be an order 2 digital \((v, n, d)\)-net in base \( b \). The proof is similar to the proof of Theorem 1.2. We apply Proposition 5.7 instead of 5.6 to get

\[ \sum_{j \in \mathbb{N}} b^{|j| + (r-1/p+1)q} \left( \sum_{m \in D_j, l \in B_j} |\langle D_{p_n}, h_{j,m,l} \rangle|^p \right)^{q/p} \]
\[ \leq \sum_{j \in \mathbb{N}^d_1} b^{\|j\|_{(r-1/p+1)q} \|j\|_{q/p} b^{-2n+v}q} (2n - v - 2\|j\| + (d-1)q)
\]

\[ \leq b^{(-2n+v)q} \sum_{\kappa=0}^{n-v/2-1} b^{\kappa(r+1)q} (2n - v - 2\kappa)(d-1)q (\kappa + 1)^{d-1}
\]

\[ \leq b^{(-2n+v)q} b^{(n-v/2)(r+1)q} (n - v/2 + 1)^{d-1}
\]

\[ \leq b^{n(r-1)q} n^{d-1} b^{v/2(1-r)q}
\]

and analogous results for the other subsums.

\[ \square \]

**Proof of Corollaries 1.4, 1.5, 1.6 and 1.7.** The results for the Triebel-Lizorkin spaces follow from Theorem 2.1 and Theorems 1.2 and 1.3, respectively. The results for the Sobolev spaces then follow in the case \( q = 2 \).

\[ \square \]

**Proof of Theorem 1.8.** The result follows from Corollary 1.7 in the case \( r = 0 \).

\[ \square \]

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