A K–theory approach to the tangent invariants of 
blow–ups

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Abstract
We extend the formula for the Chern classes of blow-ups of algebraic varieties due to Porteous and Lascu–Scott, and of symplectic and complex manifolds due to Geiges and Pasquotto, to the blow–ups of almost complex manifolds.

Our approach is based on a concrete partition for the tangent bundle of a blow–up. The use of topological K–theory of vector bundles simplifies the classical approaches.

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1 Introduction

In this paper all manifolds under consideration are in the real and smooth category, which are connected but not necessarily compact and orientable.

Let \( X \subset M \) be a smooth submanifold whose normal bundle has a complex structure, and let \( \tilde{M} \) be the blow–up of \( M \) along \( X \). We present a partition for the tangent bundle \( \tau_{\tilde{M}} \) of \( \tilde{M} \) which implies that, if \( X \subset M \) is an embedding in the category of almost complex manifolds, then the blow-up \( \tilde{M} \) has a canonical almost complex structure, see Theorem 2.3 and Corollary 2.4.

The partition on \( \tau_{\tilde{M}} \) is ready to apply to deduce a formula for the tangent bundle \( \tau_{\tilde{M}} \) of the blow–up \( \tilde{M} \) in the K–theory \( K(\tilde{M}) \) of complex bundles over \( \tilde{M} \), see Theorem 3.2 and Remark 3.3, which in turn yields a formula for the total Chern class \( C(\tilde{M}) \) of \( \tilde{M} \), see Theorem 4.6.

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Historically, the formula for the Chern class of a blow–up of a nonsingular variety was conjectured by J. A. Todd [14] and B. Segre [12], confirmed by I. R. Porteous [11] and by Lascu–Scott [7, 8], generalized to the blow ups of possibly singular varieties along regularly embedded centers by Aluffi [1]. It has also been extended to the blow–ups in the categories of symplectic and complex manifolds by H. Geiges and F. Pasquotto [5]. While establishing the formula in its natural generality we demonstrate also an approach with deserved simplicity.

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2 Geometry of a blow–up

For an Euclidean vector bundle \(\xi\) over a topological space \(Y\) write \(D(\xi)\) and \(S(\xi)\) for the unit disk bundle and sphere bundle of \(\xi\), respectively. If \(A \subset Y\) is a subspace (resp. if \(f : W \to Y\) is a map) write \(\xi \mid A\) (resp. \(f^*\xi\)) for the restriction of \(\xi\) to \(A\) (resp. the induced bundle over \(W\)).

Assume throughout that \(i : X \to M\) is a smooth embedding that embeds \(X\) as a closed subset of \(M\), and whose normal bundle \(\gamma_X\) has a fixed complex structure \(J\). Furnish \(M\) with an Riemannian metric so that the induced metric on \(\gamma_X\) is Hermitian in the sense of Milnor [10, p.156].

Let \(\pi : E = \mathbb{P}(\gamma_X) \to X\) be the complex projective bundle associated with \(\gamma_X\). Since the tautological line bundle \(\lambda_E\) on \(E\) is a subbundle of the induced bundle \(\pi^*\gamma_X\) we can formulate the composition

\[
G : D(\lambda_E) \subset D(\pi^*\gamma_X) \overset{\hat{\pi}}{\to} D(\gamma_X),
\]

where \(\hat{\pi}\) is the obvious bundle map over \(\pi\). Regard \(E \subset D(\lambda_E)\) and \(X \subset D(\gamma_X)\) as the zero sections of the corresponding disk bundles, respectively.

**Lemma 2.1.** The map \(G\) agrees with the projection \(\pi\) on \(E\), and restricts to a diffeomorphism \(D(\lambda_E) \setminus E \to D(\gamma_X) \setminus X\).

**Proof.** Explicitly we have

\[
D(\lambda_E) = \{(l, v) \in E \times \pi^*\gamma_X \mid v \in l \in E, \|v\|^2 \leq 1\},
\]

\[
D(\gamma_X) = \{(x, v) \in X \times \gamma_X \mid v \in \gamma_X \mid x, \|v\|^2 \leq 1\}.
\]

The inverse of \(G\) on \(D(\gamma_X) \setminus X\) is \((x, v) \to (\langle v \rangle, v)\), where \(v \in \gamma_X \mid x\) with \(v \neq 0\) and where \(\langle v \rangle \in E\) denotes the complex line spanned by a non–zero vector \(v\).

It follows from Lemma 2.1 that the map \(G\) restricts to a diffeomorphism

\[
g = G \mid S(\lambda_E) : S(\lambda_E) = \partial D(\lambda_E) \to S(\gamma_X) = \partial D(\gamma_X).
\]
With respect to the metric on $M$ identifying $D(\gamma_X)$ with a tubular neighborhood of $X$ in $M$ we can formulate the adjoint manifold

\[(2.1) \quad \widetilde{M} = (M \setminus D(\gamma_X)) \cup_g D(\lambda_E)\]

by gluing $D(\lambda_E)$ to $(M \setminus D(\gamma_X))$ along $S(\lambda_E)$ using $g$. Moreover, piecing together the identity on $M \setminus D(\gamma_X)$ and the map $G$ yields the smooth map

\[(2.2) \quad f : \widetilde{M} = (M \setminus D(\gamma_X)) \cup_g D(\lambda_E) \to M = (M \setminus D(\gamma_X)) \cup_{id} D(\gamma_X),\]

which is known as the blow-up of $M$ along $X$ with exceptional divisor $E$ [9]. Obvious but useful properties of the map $f$ are listed below.

**Lemma 2.2.** Let $i_E : E \to \widetilde{M}$ (resp. $i_X : X \to M$) be the zero section of $D(\lambda_E)$ (resp. of $D(\gamma_X)$) in view of the decomposition (2.1). Then

i) the normal bundle of $E$ in $\widetilde{M}$ is $\lambda_E$;

ii) $f^{-1}(X) = E$ with $f \circ i_E = i_X \circ \pi$;

iii) $f$ restricts to a diffeomorphism: $\widetilde{M} \setminus E \to M \setminus X$. □

For a manifold $N$ write $\tau_N$ for its tangent bundle. One has then the obvious bundle decompositions

$$
\tau_{D(\lambda_E)} \mid S(\lambda_E) = \tau_{S(\lambda_E)} \oplus \mathbb{R}(\alpha_1); \quad \tau_{M \setminus D(\gamma_X)} \mid S(\gamma_X) = \tau_{S(\gamma_X)} \oplus \mathbb{R}(\alpha_2),
$$

where $\alpha_1$ (resp. $\alpha_2$) is the outward (resp. inward) unit normal field along the boundary $S(\lambda_E) = \partial D(\lambda_E)$ (resp. $S(\gamma_X) = \partial(M \setminus D(\gamma_X))$) with $\mathbb{R}(\alpha_i)$ the trivial real line bundle spanned by the field $\alpha_i$. Moreover, if we let $\tau_g$ be the tangent map of the diffeomorphism $g$ then the decomposition (2.1) of $\widetilde{M}$ indicates the partition

\[(2.3) \quad \tau_{\widetilde{M}} = \tau_{M \setminus D(\gamma_X)} \bigcup_h \tau_{D(\lambda_E)},\]

where the gluing diffeomorphism

$$
h : \tau_{D(\lambda_E)} \mid S(\lambda_E) \to \tau_{M \setminus D(\gamma_X)} \mid S(\gamma_X)
$$

is the bundle map over $g$ with

$$
h(t \alpha_1, u) = (t \alpha_2, \tau_g(u)), \quad u \in \tau_{S(\lambda_E)}, \quad t \in \mathbb{R}.
$$
Indeed, the two bundles $\tau_{D(\lambda_E)} | S(\lambda_E)$ and $\tau_{M \setminus D(\gamma_X)} | S(\gamma_X)$ admit more subtle decompositions with respect to them the gluing map $h$ in (2.3) admits a useful presentation.

Let $p_E : \lambda_E \to E$ and $p_X : \gamma_X \to X$ be the obvious projections. The same notions will be reserved for their restrictions to the subspaces $S(\lambda_E) \subset D(\lambda_E) \subset \lambda_E$ and $S(\gamma_X) \subset D(\gamma_X) \subset \gamma_X$, respectively.

For a topological space $Y$ write $1_C$ (resp. $1_R$) for the trivial complex line bundle $Y \times \mathbb{C}$ (resp. the trivial real line bundle $Y \times \mathbb{R}$) over $Y$. For a complex vector bundle $\xi$ write $\xi^r$ for its real reduction. As example the trivialization over $S(\lambda_E)$

\begin{equation}
(2.4) \quad (p_E^* \lambda_E | S(\lambda_E))^r = \mathbb{R}(\alpha_1) \oplus \mathbb{R}(J(\alpha_1))
\end{equation}

indicates that $p_E^* \lambda_E | S(\lambda_E) = 1_C$, where $J(\alpha_1)$ is a unit tangent vector field on $S(\lambda_E)$.

Let

$$\tilde{g} : g^*(\tau_{M \setminus D(\gamma_X)} | S(\gamma_X)) \to \tau_{M \setminus D(\gamma_X)} | S(\gamma_X)$$

be the induced bundle of $g$ over $S(\lambda_E)$, and let

$$\kappa : \tau_{D(\lambda_E)} | S(\lambda_E) \to g^*(\tau_{M \setminus D(\gamma_X)} | S(\gamma_X))$$

be the bundle isomorphism over the identity of $S(\lambda_E)$ so that $h = \tilde{g} \circ \kappa$ \cite{10} Lemma 3.1. With respect to the Hermitian metric induced from $\gamma_X$ one has the orthogonal decomposition $\pi^* \gamma_X = \lambda_E \oplus \lambda_E^\perp$ in which $\lambda_E^\perp$ denotes the orthogonal complement of $\lambda_E$ in $\pi^* \gamma_X$.

**Theorem 2.3.** The tangent bundle of the blow up $\tilde{M}$ has the partition

$$\tau_{\tilde{M}} = \tau_{M \setminus D(\gamma_X)} \bigcup_{\text{geom}} \tau_{D(\lambda_E)},$$

in which

\begin{align*}
\text{i)} & \quad \tau_{D(\lambda_E)} | S(\lambda_E) = (\pi \circ p_E)^* \tau_X \oplus (p_E^* \lambda_E)^r \oplus p_E^* \text{Hom}(\lambda_E, \lambda_E^\perp)^r; \\
\text{ii)} & \quad g^*(\tau_{M \setminus D(\gamma_X)} | S(\gamma_X)) = (\pi \circ p_E)^* \tau_X \oplus (p_E^* \lambda_E)^r \oplus p_E^* (\lambda_E^\perp)^r.
\end{align*}

Moreover, with respect to the decompositions i) and ii) the bundle isomorphism $\kappa$ is given by

\begin{align*}
a) & \quad \kappa | (\pi \circ p_E)^* \tau_X = \text{id}; \\
b) & \quad \kappa | (p_E^* \lambda_E)^r = \text{id}; \\
c) & \quad \kappa(b) = b(\alpha_1) \text{ for } b \in \text{Hom}(p_E^* \lambda_E, p_E^* (\lambda_E^\perp)^r).\end{align*}
Proof. It follows from the standard decompositions

\[ \tau_E = \pi^*\tau_X \oplus \text{Hom}(\lambda_E, \lambda_E^\perp)^r, \quad \tau_{D(\lambda_E)} = (pE^*\lambda_E)^r \oplus pE^*\tau_E \]

that

\[ (2.5) \quad \tau_{D(\lambda_E)} = (pE^*\lambda_E)^r \oplus (\pi \circ pE)^*\tau_X \oplus pE^*\text{Hom}(\lambda_E, \lambda_E^\perp)^r. \]

Similarly, it comes from

\[ \tau_{D(\gamma_X)} = pX^*\tau_X \oplus pX^*\gamma_X, \quad \pi^*\gamma_X = \lambda_E \oplus \lambda_E^\perp, \]

as well as the definition of \( f \) that

\[ (2.6) \quad f^*\tau_{D(\gamma_X)} = (pE^*\lambda_E)^r \oplus (\pi \circ pE)^*\tau_X \oplus (pE^*\lambda_E^\perp)^r. \]

One obtains the relations i) and ii) by restricting the decomposition (2.5) and (2.6) to the subspace \( S(\lambda_E) \subset D(\lambda_E) \), respectively.

Finally, properties a), b), c) are transparent in view of the relation \( h = \hat{g} \circ \kappa \), together with the description of \( g \) indicated in the proof of Lemma 2.1. □

A manifold \( M \) is called almost complex if its tangent bundle is furnished with a complex structure \( J_M \) [10, p.151]. Given two almost complex manifolds \((X, J_X)\) and \((M, J_M)\) an embedding \( i_X : X \to M \) is called almost complex if \( \tau_X \) is a complex subbundle of the restricted bundle \( \tau_M | X \). In this situation the normal bundle \( \gamma_X \) of \( X \) has the canonical complex structure \( J \) induced from that on \( \tau_M | X \) and that on \( \tau_X \), hence the blow–up \( \tilde{M} \) of \( M \) along \( X \) is defined. Moreover, in view of the decomposition (2.1) we note that

i) \( J_M \) restricts to an almost complex structure on \( M \setminus D(\gamma_X) \);

ii) the tubular neighborhood \( D(\lambda_E) \) of \( E \) in \( \tilde{M} \) has the canonical almost complex structure so that as a complex vector bundle

\[ \tau_{D(\lambda_E)} = (\pi \circ pE)^*\tau_X \oplus pE^*\lambda_E \oplus \text{Hom}(pE^*\lambda_E, pE^*\lambda_E^\perp) \]

(compare this with (2.5)). Since with respect to the induced complex structures on \( \tau_{D(\lambda_E)} | S(\lambda_E) \) and on \( \tau_{M \setminus D(\gamma_X)} | S(\gamma_X) \) the clutching map \( h \) in (2.3) is \( \mathbb{C} \)-linear by Theorem 2.3, one obtains

Corollary 2.4. If \( i_X : X \to M \) is an embedding in the category of almost complex manifolds, then the blow–up \( \tilde{M} \) has a canonical almost complex structure that is compatible with that on \( M \setminus D(\gamma_X) \) and that on \( D(\lambda_E) \). □

Remark 2.5. The analogue of Corollary 2.4 in the symplectic setting is due to McDuff [9, Section 3], which concludes that if \( i_X : X \to M \) is an embedding of symplectic manifolds, then the blow–up \( \tilde{M} \) admits a symplectic form which coincides with the one on \( M \setminus X \) off the exceptional divisor \( E \). □
3 The tangent bundle of a blow–up

For a topological space $Y$ let $K(Y)$ (resp. $\tilde{K}(Y)$) be the $K$–theory (reduced $K$–theory) of complex vector bundles over $Y$. If $i_X : X \to M$ is an embedding in the category of almost complex manifolds, then the blow–up $\tilde{M}$ has a canonical almost complex structure by Corollary 2.4. In particular, the difference $\tau_{\tilde{M}} - f^*\tau_M$ can be regarded as an element of the ring $\tilde{K}(\tilde{M})$. In Theorem 3.2 below we obtain a formula expressing the element $\tau_{\tilde{M}} - f^*\tau_M \in \tilde{K}(\tilde{M})$ in term of the decomposition $p_E^*\gamma_X = p_E^*\lambda_E \oplus p_E^*\lambda_E^\perp$.

For a relative CW–complex $(Y, A)$ the inclusion $j : (Y, \emptyset) \to (Y; A)$ induces a homomorphism

\[(2.1) \quad j^* : K(Y; A) \to \tilde{K}(Y),\]

where $K(Y; A)$ is the relative $K$–group of the pair $(Y; A)$ defined by

\[(3.2) \quad K(Y; A) =: \tilde{K}(Y/A).\]

In addition, the group $K(Y; A)$ admits another description useful in the subsequent calculation.

**Lemma 3.1** ([2, Theorem 2.6.1]). Any element in the group $K(Y; A)$ can be represented by a triple $[\xi, \eta; \alpha]$ in which $\xi$ and $\eta$ are vector bundles over $Y$ and $\alpha : \xi \mid A \to \eta \mid A$ is a bundle isomorphism.

Moreover, with respect to this representation of the group $K(Y; A)$ one has

i) the triple $[\xi, \xi; id]$ represents the zero for any bundle $\xi$ over $Y$;

ii) $[\xi, \eta; \alpha] + [\xi_1, \eta_1; \alpha_1] = [\xi \oplus \xi_1, \eta \oplus \xi_1; \alpha \oplus \alpha_1]$;

iii) $[\xi, \eta; \alpha] \otimes \gamma = [\xi \otimes \gamma, \eta \otimes \gamma; \alpha \otimes id]$;

iv) $j^*[\xi, \eta; \alpha] = \xi - \eta$.

where $\oplus$ means direct sum of vector bundles (homomorphisms), and where $\otimes$ denotes the action $K(X; A) \otimes K(X) \to K(X; A)$ defined by the tensor product of vector bundles.$\Box$

For an embedding $i_X : X \to M$ of almost complex manifolds let $f : \widetilde{M} \to M$ be the blow–up of $M$ along $X$ with exceptional divisor $E$. Consider the composition

\[j_E : K(D(\lambda_E), S(\lambda_E)) \xrightarrow{\cong} K(\widetilde{M}, \widetilde{M} \smallsetminus \overset{\circ}{D(\lambda_E)}) \xrightarrow{j_E} \tilde{K}(\tilde{M})\]
in which the first map is the excision isomorphism. By Lemma 3.1 the trivialization $\varepsilon : 1_{C} | S(\lambda_{E}) \rightarrow 1_{C}$ indicated by (2.4) defines an element $[p_{E}^{*}\lambda_{E}, 1_{C}; \varepsilon] \in K(D(\lambda_{E}), S(\lambda_{E}))$.

**Theorem 3.2.** In the ring $\tilde{K}(\tilde{M})$ one has

\[
\tau_{\tilde{M}} - f^{*}\tau_{M} = j_{E}([p_{E}^{*}\lambda_{E}, 1_{C}; \varepsilon] \otimes p_{E}^{*}\lambda_{E}^{|}).
\]

**Proof.** The partition (2.1) of the blow up $\tilde{M}$ implies the relation

\[
\tau_{\tilde{M}} | M \setminus D(\gamma_{X}) = f^{*}\tau_{M} | M \setminus D(\gamma_{X})
\]

which gives rise to an element $[\tau_{\tilde{M}}, f^{*}\tau_{M}; id] \in K(\tilde{M}; \tilde{M} \setminus D(\lambda_{E}))$ by Lemma 3.1. Moreover, with respect to the excision isomorphism

\[
K(\tilde{M}; \tilde{M} \setminus D(\lambda_{E})) \xrightarrow{\sim} K(D(\lambda_{E}); S(\lambda_{E}))
\]

we have $[\tau_{\tilde{M}}, f^{*}\tau_{M}; id] = [\tau_{D(\lambda_{E})}, f^{*}\tau_{D(\gamma_{X})}; \kappa]$, where

\[
\kappa : \tau_{D(\lambda_{E})} | S(\lambda_{E}) \rightarrow g^{*}(\tau_{M \setminus D(\gamma_{X})}) | S(\gamma_{X}))
\]

is the bundle isomorphism specified in Theorem 2.3. Granted with the decompositions of the bundles $\tau_{D(\lambda_{E})}$ and $f^{*}\tau_{D(\gamma_{X})}$ in (2.4) and (2.5), as well as the decomposition of bundle map $\kappa$ in Theorem 2.3, we calculate

\[
[\tau_{D(\lambda_{E})}, f^{*}\tau_{D(\gamma_{X})}; \kappa]
\]

\[
= [(\pi \circ p_{E})^{*}\tau_{X}, (\pi \circ p_{E})^{*}\tau_{X}; id] + [p_{E}^{*}\lambda_{E}, p_{E}^{*}\lambda_{E}; id]
\]

\[
+ [p_{E}^{*}Hom(\lambda_{E}, \lambda_{E}^{|}), p_{E}^{*}\lambda_{E}^{|}; \kappa'] \quad \text{(by ii) of Lemma 3.1)}
\]

\[
= [p_{E}^{*}Hom(\lambda_{E}, \lambda_{E}^{|}), p_{E}^{*}\lambda_{E}^{|}; \kappa'] \quad \text{(by i) of Lemma 3.1)}
\]

\[
= [p_{E}^{*}(\lambda_{E} \otimes \lambda_{E}^{|}), p_{E}^{*}\lambda_{E}^{|}; \kappa'] \quad \text{(since $Hom(\lambda_{E}, \lambda_{E}^{|}) = \lambda_{E} \otimes \lambda_{E}^{|}$)}
\]

\[
= [p_{E}^{*}\lambda_{E}, 1_{C}; \varepsilon] \otimes p_{E}^{*}\lambda_{E}^{|} \quad \text{(by iii) of Lemma 3.1)}
\]

where $\kappa'$ is the restriction of $\kappa$ to the direct summand $p_{E}^{*}Hom(\lambda_{E}, \lambda_{E}^{|})$ of $\tau_{D(\lambda_{E})}$ (see c) of Theorem 2.3). Summarizing, in the group $K(D(\lambda_{E}), S(\lambda_{E}))$ we have the relation

\[
[\tau_{\tilde{M}}, f^{*}\tau_{M}; id] = [p_{E}^{*}\lambda_{E}, 1_{C}; \varepsilon] \otimes p_{E}^{*}\lambda_{E}^{|}.
\]

Finally, applying the map $j_{E}$ to both sides yields the formula (3.3) by iv) of Lemma 3.1.$\Box$

**Remark 3.3.** In the group $\tilde{K}(\tilde{M})$ formula (3.3) has the concise expression
\[ \tau_{\tilde{M}} = f^*\tau_M + i_{E!}(\lambda_E^1) \]

where \( i_{E!} \) is the Gysin map in \( K \)-theory

\[ i_{E!} : K(E) \xrightarrow{\psi_E} K(D(\lambda_E), S(\lambda_E)) \xrightarrow{\beta_E} \tilde{K}(\tilde{M}), \]

and where \( \psi_E \) is the Thom isomorphism \( x \to \psi_E(x) = U_E \otimes p_E^*x, x \in K(E) \), with \( U_E \in K(D(\lambda_E), S(\lambda_E)) \) the Thom class of \( \lambda_E \). Indeed, from the general construction \( [2, \text{p.98–99}] \) of Thom classes from the exterior algebras of vector bundles one finds that \( U_E = [p_E^*\lambda_E, 1, \varepsilon] \).

\[ \square \]

4 The Chern class of a blow up

Based on the formula (3.3) we deduce a formula for the total Chern class \( C(\tilde{M}) \) of a blow up \( \tilde{M} \) in the category of almost complex manifolds. In this section the coefficients for cohomologies will be the ring \( \mathbb{Z} \) of integers.

4.1 Preliminaries in Chern classes

Let \( BU \) be the classifying space of stable equivalent classes of complex vector bundles, and let \( c_r \in H^{2r}(BU) \) be the \( r \)-th Chern class of the universal complex vector bundle over \( BU \). Then

\[ H^*(BU) = \mathbb{Z}[c_1, c_2, \cdots]. \]

For a topological space \( Y \) let \([Y, BU]\) be the set of homotopy classes of maps from \( Y \) to \( BU \). In view of the standard identification

\[ \tilde{K}(Y) = [Y, BU] \]

\([13, \text{p.210}]\) we can introduce the total Chern class for elements in \( \tilde{K}(Y) \) as the co–functor \( C : \tilde{K}(Y) \to H^*(Y) \) defined

\[ C(\beta) = 1 + f^*c_1 + f^*c_2 + \cdots, \beta \in \tilde{K}(Y), \]

where \( f : Y \to BU \) is the classifying map of the element \( \beta \). Clearly one has

**Lemma 4.1.** The transformation \( C \) satisfies the next two properties.

i) If \( \xi_i, i = 1,2, \) are two complex vector bundles over \( Y \) with equal dimension and with (the usual) total Chern classes \( C(\xi_i) \), then

\[ C(\xi_1 - \xi_2) = C(\xi_1)C(\xi_2)^{-1}. \]

ii) For a closed subspace \( A \subset Y \) let \( j_A : (Y, \emptyset) \to (Y, A) \) and \( q_A : Y \to Y/A \) be the inclusion and quotient maps, respectively. Then the next diagram commutes:
\[ K(Y; A) = \widetilde{K}(Y/A) \xrightarrow{j_A^*} \widetilde{K}(Y) \]
\[ C \downarrow \quad \downarrow C \quad \square \]
\[ H^*(Y/A) \xrightarrow{q_A^*} H^*(Y) \]

For two complex vector bundles \( \lambda \) and \( \xi \) over a space \( Y \) with \( \dim \lambda = 1 \), \( \dim \xi = m \), and with the total Chern classes \( C(\lambda) = 1 + t \); \( C(\xi) = 1 + c_1(\xi) + c_2(\xi) + \cdots + c_m(\xi) \), respectively, assume that the Chern roots of \( \xi \) is \( s_1, \ldots, s_m \). That is
\[ C(\xi) = \prod_{1 \leq i \leq m} (1 + s_i) \]
with \( c_r(\xi) = e_r(s_1, \ldots, s_m) \) the \( r^{th} \) elementary symmetric function in the roots \( s_1, \ldots, s_m \). The calculation
\[ C(\lambda \otimes \xi) = \prod_{1 \leq i \leq m} (1 + t + s_i) = (1 + t)^m \prod_{1 \leq i \leq m} (1 + \frac{s_i}{1+t}) \]
\[ = (1 + t)^m [1 + \frac{c_1(\xi)}{(1+t)} + \frac{c_2(\xi)}{(1+t)^2} + \cdots + \frac{c_m(\xi)}{(1+t)^m}] \]
shows that
\[ (4.1) \quad C(\lambda \otimes \xi) = \sum_{0 \leq r \leq m} (1 + t)^{m-r} c_r(\xi). \]

For an Euclidean complex line bundle \( \lambda \) over \( Y \) with associated disk bundle \( p_\lambda : D(\lambda) \to Y \) the Thom space \( T(\lambda) \) of \( \lambda \) is the quotient space \( D(\lambda)/S(\lambda) \). In term of Lemma 3.1 the trivialization \( \varepsilon : p_\lambda^*(\overline{\lambda}) \mid S(\lambda) \to 1_C \) indicated by (2.4) defines the element
\[ [p_\lambda^*(\overline{\lambda}), 1_C; \varepsilon] \in K(D(\lambda), S(\lambda)) = \widetilde{K}(T(\lambda)). \]

Given a ring \( A \) and a set \( \{u_1, \ldots, u_k\} \) of \( k \) elements let \( A\{u_1, \ldots, u_k\} \) be the free \( A \) module with basis \( \{u_1, \ldots, u_k\} \).

In addition to Theorem 3.2, the next result will play a key role in establishing the formula for Chern class.

**Lemma 4.2.** Let \( e \in H^2(Y) \) and \( x \in H^2(T(\lambda)) \) be respectively the Euler class and the Thom class of the oriented bundle \( \lambda \). Then

i) the integral cohomology ring of \( T(\lambda) \) is determined by the additive presentation
\[ (4.2) \quad H^*(T(\lambda)) = \mathbb{Z} \oplus H^*(Y) \{x\}, \]
Moreover, letting \( H \) in view of a) we have the canonical map onto the Thom space \( \lambda(c) \) is

\[
(4.3) \, C([\tilde{\lambda}_{\lambda}(1;\varepsilon) \otimes \tilde{\lambda}]) = \left( \sum_{0 \leq r \leq m} (1 + x)^m r \right) = \mathbb{C}(T(\lambda)).
\]

**Proof.** The presentation (4.2) comes immediately from the Thom isomorphism theorem, which states that product with Thom class \( x \) yields an additive isomorphism

\[
H^r(Y) \cong H^{r+2}(T(\lambda)), \quad y \rightarrow y \cdot x, \quad y \in H^r(X)
\]

for all \( r \geq 0 \). For i) it remains to justify the relation \( x^2 + xe = 0 \).

Let \( p : S(\lambda \oplus 1 \mathbb{R}) \rightarrow Y \) be the sphere bundle of the Euclidean bundle \( \lambda \oplus 1 \mathbb{R} \) and set

\[
D_{+(-)}(\lambda) = \{(u, t) \in S(\lambda \oplus 1 \mathbb{R}) \mid t \geq 0 \}.
\]

It is clear that

a) \( S(\lambda \oplus 1 \mathbb{R}) = D_- \cup D_+ \), \( S(\lambda) = D_- \cap D_+ \),

b) both \( D_\pm(\lambda) \) can be identified with the disk bundle \( D(\lambda) \) of \( \lambda \).

In view of a) we have the canonical map onto the Thom space \( T(\lambda) \)

\[
g : S(\lambda \oplus 1 \mathbb{R}) \rightarrow T(\lambda) = S(\lambda \oplus 1 \mathbb{R})/D_-(\lambda).
\]

Moreover, letting \( u = g^* x \in H^2(S(\lambda \oplus 1 \mathbb{R})) \) the ring \( H^*(S(\lambda \oplus 1 \mathbb{R})) \) has the presentation (see [3 Lemma 4])

\[
H^*(S(\lambda \oplus 1 \mathbb{R})) = H^*(Y)[u]/\langle u^2 + eu \rangle.
\]

The relation \( x^2 + xe = 0 \) on \( H^*(T(\lambda)) \) is verified by the relation \( u^2 + eu = 0 \) on \( H^*(S(\lambda \oplus 1 \mathbb{R})) \), together with the fact that the induced ring map \( q^* \) is monomorphic onto the direct summand \( \mathbb{Z} \oplus H^*(Y)\{u\} \) of \( H^*(S(\lambda \oplus 1 \mathbb{R})) \).

For ii) define over \( S(\lambda \oplus 1 \mathbb{R}) \) the complex line bundle \( \lambda_u \) by

\[
\lambda_u = p^* \bigcap D_+(\lambda) \cup_{\mathbb{C}} 1_{\mathbb{C}} \mid D_-(\lambda) \quad \text{(in view of the partition a)}.
\]

Then \( C(\lambda_u) = 1 + u \in H^*(S(\lambda \oplus 1 \mathbb{R})) \). Moreover, for a vector bundle \( \xi \) over \( Y \) the element

\[
[C(\lambda_u), 1_{\mathbb{C}} \otimes p^* \xi] \in K(S(\lambda \oplus 1 \mathbb{R}), D_-(\lambda))
\]
corresponds to the element \([(p^*_E(\tilde{\lambda}), 1_C; \varepsilon) \otimes p^*_E\xi] \in K(D(\lambda), S(\lambda))\) under the excision isomorphism \(K(S(\lambda \oplus 1_R), D_-(\lambda)) \cong K(D(\lambda), S(\lambda))\) indicated by b), which is also mapped to the element 

\[\lambda_u \otimes p^*\xi - p^*\xi \in \tilde{K}(S(\lambda \oplus 1_R))\]

under the induced homomorphism \(j^*\) of the inclusion \(j : (S(\lambda \oplus 1_R), \emptyset) \to (S(\lambda \oplus 1_R), D_-(\lambda))\) by iv) of Lemma 3.1. It follows from i) of Lemma 4.1 that

\[C(j^*([(\lambda_u, 1_C; \varepsilon) \otimes p^*\xi])] = C(\lambda_u \otimes p^*\xi)C(p^*\xi)^{-1}\]

\[= (\sum_{0 \leq r \leq m} (1 + u)^{m-r}c_r)C(\xi)^{-1}\text{ (by the formula (4.1)).}\]

The commutativity of the diagram in ii) of Lemma 4.1 implies that

\[q^*C([(\lambda_u, 1_C; \varepsilon) \otimes p^*\xi]) = (\sum_{0 \leq r \leq m} (1 + u)^{m-r}c_r)C(\xi)^{-1}\]

One obtains the formula (4.3) from \(q^*(x) = u\) and the injectivity of \(q^*.\)

\[\square\]

### 4.2 The integral cohomology ring of a blow-up

Let \(f : \tilde{M} \to M\) be the blow up of \(M\) along a submanifold \(i_X : X \to M\) whose normal bundle \(\gamma_X\) has a complex structure and with total Chern class \(C(\gamma_X) = 1 + c_1 + \cdots + c_k \in H^*(X), k = \dim_C \gamma_X\). Regard \(D(\lambda_E)\) as a normal disk bundle of the exceptional divisor \(E\) and consider the quotient map onto the Thom space of \(\lambda_E\)

\[q : \tilde{M} \to \tilde{M}/(M \smallsetminus D(\gamma_X) = T(\lambda_E)).\]

According to i) of Lemma 4.2 one has

**Lemma 4.3.** Let \(t \in H^2(E)\) and \(x \in H^2(T(\lambda_E))\) be the Euler class and Thom class of the oriented bundle \(\lambda_E\), respectively. Then the ring \(H^*(T(\lambda_E))\) is determined by the additive presentation

\[(4.5) H^*(T(\lambda_E)) = \mathbb{Z} \oplus H^*(X)[x, tx, \cdots , t^{k-1}x],\]

together with the relations

\[i) x^2 + tx = 0; \text{ ii) } t^k + c_1 \cdot t^{k-1} + \cdots + c_{k-1} \cdot t + c_k = 0.\]

Consider the Gysin maps of the embeddings \(i_X : X \to M\) and \(i_E : E \to \tilde{M}\) in cohomology
\[
i_{X!} : H^*(X) \xrightarrow{\psi_X} H^*(D(\gamma_X), S(\gamma_X)) \xrightarrow{\approx} H^*(M, M \setminus \tilde{D}(\gamma_X)) \xrightarrow{j^!} H^*(M)
\]
\[
i_{E!} : H^*(E) \xrightarrow{\psi_E} H^*(D(\lambda_E), S(\lambda_E)) \xrightarrow{\approx} H^*(\tilde{M}, \tilde{M} \setminus \tilde{D}(\lambda_E)) \xrightarrow{j^!} H^*(\tilde{M})
\]

where \(\psi_X\) and \(\psi_E\) are the Thom isomorphisms. We shall set
\[
\omega_E = i_{E!}(1) \in H^2(\tilde{M}), \quad \omega_X = i_{X!}(1) \in H^{2k}(M).
\]

Geometrically, if \(M\) is closed and oriented the class \(\omega_X\) (for instance) is the Poincare dual of the cycle class \(i_{X*}[X] \in H_*(M)\).

The next result is shown in [4, Theorem 1].

**Theorem 4.4.** The ring map \(f^* : H^*(M) \to H^*(\tilde{M})\) embeds the ring \(H^*(M)\) as a direct summand of \(H^*(\tilde{M})\), and induces the decomposition

\[
H^*(\tilde{M}) = f^*(H^*(M)) \oplus H^*(X)\{\omega_E, \ldots, \omega_{k-1}^E\}, \quad 2k = \dim \mathbb{R} \gamma_X
\]

that is subject to the two relations

\[
i) \quad f^*(\omega_X) = \sum_{1 \leq r \leq k} (-1)^{r-1} c_{k-r} \cdot \omega_{r}^E; \\
ii) \quad f^*(y) \cdot \omega_E = i_X^*(y) \cdot \omega_E, \quad y \in H^r(M).
\]

Moreover, with respect to the presentations of the rings \(H^*(T(\lambda_E))\) and \(H^*(\tilde{M})\) in (4.5) and (4.6), the induced ring map \(q^*\) is determined by

\[
iii) \quad q^*(t^r x) = (-1)^r \omega_{r+1}^E, \quad r \geq 0. \quad \square
\]

**Remark 4.5.** In [6, p.605] Griffiths and Harris obtained the decomposition (4.6) for blow ups of complex manifolds, while the relations i) and ii) were absent. In the non–algebraic settings partial information on the ring \(H^*(\tilde{M})\) was also obtained by McDuff in [9, Proposition 2.4].

In comparison the structure of \(H^*(\tilde{M})\) as a ring is completely determined by the additive decomposition (4.6), together with the relations i) and ii). Indeed, granted with the fact that \(f^*(H^*(M)) \subset H^*(\tilde{M})\) is a subring, the relations i) and ii) are sufficient to express, respectively, the products of elements in the second summand, and the products between elements in the first and second summands, as elements in the decomposition (4.6). This idea has been applied in [4] to determine the integral cohomology rings of the varieties of complete conics and complete quadrics in the 3-space \(\mathbb{P}^3\), and justify two enumerative problems due to Schubert. \(\square\)
4.3 The Chern class of a blow up

Assume now that \( f : \tilde{M} \to M \) is a blow up in the category of almost complex manifolds, and that the total Chern class of the normal bundle \( \gamma_X \) is

\[
C(\gamma_X) = 1 + c_1 + \cdots + c_k, \quad \dim \gamma_X = 2k.
\]

Applying the transformation \( C : \tilde{K}(M) \to H^*(\tilde{M}) \) to the equality (3.3) and noting the obvious relation

\[
p^*E_{\lambda} = p^*E_X - p^*E_{\lambda E}
\]

in the \( K \)-group \( K(D(\lambda E)) \) one obtains

\[
C(\tilde{M}) \cdot f^*C(M)^{-1} = C(j_E([p^*E, 1_C], \varepsilon \otimes (p^*E_X - p^*E_{\lambda E})))
\]

(by i) of Lemma 4.1)

\[
= q^* \frac{C((p^*E_{\lambda E}, 1_C; \varepsilon) \otimes p^*\gamma_X)}{C(p^*E_{\lambda E}, 1_C; \varepsilon \otimes p^*\gamma_X)} \quad \text{(by ii) of Lemma 4.1)}.
\]

Furthermore, from

\[
C([p^*\lambda_E, 1_C; \varepsilon] \otimes p^*\gamma_X) = (\sum_{0 \leq r \leq k} (1 + x)^{k-r}c_r)C(\gamma_X)^{-1},
\]

\[
C([p^*\lambda_E, 1_C; \varepsilon] \otimes p^*\lambda_E) = (1 + x + t)(1 + t)^{-1}
\]

by (3.3) one gets

\[
C(\tilde{M}) \cdot f^*C(M)^{-1} = q^* (\sum_{0 \leq r \leq k} (1 + x)^{k-r}c_r)(1 + t)(1 + x + t)^{-1}C(\gamma_X)^{-1} - 1
\]

where the second equality comes from iii) of Theorem 4.4. It implies that

\[
C(\tilde{M}) - f^*C(M) = f^*C(M) \cdot g(\omega_E)
\]

with

\[
g(\omega_E) = (\sum_{0 \leq r \leq k} (1 + \omega_E)^{k-r}c_r)(1 - \omega_E)C(\gamma_X)^{-1} - 1 \in H^*(\tilde{M})
\]

a polynomial in \( \omega_E \) with coefficients in \( H^*(X) \).

It is crucial for us to observe from the obvious relation \( g(0) = 0 \) that the polynomial \( g(\omega_E) \) is divisible by \( \omega_E \). Therefore, the relation ii) in Theorem 4.4 is applicable to yield
\[
C(\widetilde{M}) - f^*C(M) = f^*C(M) \cdot g(\omega_E)
\]
\[
= i_X C(M) g(\omega_E)
\]
\[
= C(X) C(\tau_X) g(\omega_E) \quad \text{(since } \tau_M | X = \tau_X \oplus \gamma_X)\).
\]

Summarizing we get

**Theorem 4.6.** With respect to decomposition of the ring \(H^*(\widetilde{M})\) in (4.6), the total Chern class of the blow up \(\widetilde{M}\) is

\[
C(\widetilde{M}) = f^*C(M) + C(X) \left( \sum_{0 \leq r \leq k} (1 + \omega_E)^{k-r} c_r (1 - \omega_E) - \sum_{0 \leq r \leq k} c_r \right). \quad \Box
\]

**Examples 4.7.** The simplest example assuring that Theorem 4.6 is non–trivial in the category of almost complex manifolds is the following one. Recall that the 6–dimensional sphere \(S^6\) has a canonical almost complex structure. The blow–up of \(S^6\) at a point \(X \in S^6\) is diffeomorphic to the complex projective 3–space \(\mathbb{P}^3\), together with an induced almost complex structure \(J\) (see Corollary 2.4). By Theorem 4.6 we have

\[
C(\mathbb{P}^3, J) = 1 - 2x - 4x^3,
\]

where \(x \in H^2(\mathbb{P}^3)\) is the exceptional divisor. This computation shows that \(J\) is different with the canonical complex structure on \(\mathbb{P}^3\).

Similarly, in the recent work [15] H. Yang classified those \((2n-1)\)–connected \(4n\)–manifolds that admit almost complex structures. In term of the Wall’s invariant \(\alpha : H_{2n}(M) \rightarrow \pi_{2n-1}(SO(2n))\) for such a manifold \(M\) [16] the set of isotopy classes of smooth embeddings \(S^{2n} \rightarrow M\) whose normal bundles have complex structures can be identified with an explicit subset \(J(M)\) of the homology group \(H_{2n}(M)\), where \(S^{2n}\) is the \(2n\)–dimensional sphere. Blow–ups of \(M\) along elements in the set \(J(M)\) can provide us with a rich family of almost complex manifolds of dimension \(4n\), to which Therom 4.6 is directly applicable to compute their total Chern classes.

**References**

[1] P. Aluffi, Chern classes of blow-ups, Math. Proc. Cambridge Philos. Soc. 148 (2010), no. 2, 227–242.

[2] M. Atiyah, K-theory, New York-Amsterdam: W.A. Benjamin, Inc. (1967).

[3] H. Duan, The degree of a Schubert variety, Adv. Math., 180(2003), 112-133.

[4] H. Duan and B. Li, Topology of Blow-ups and Enumerative Geometry, arXiv: math.AT/0906.4152.
[5] H. Geiges and F. Pasquotto. A formula for the Chern classes of symplectic blow-ups. J. Lond. Math. Soc. (2) 76(2), (2007), 313–330.

[6] P. Griffith and J. Harris, Principles of algebraic geometry, Wiley, New York 1978.

[7] A. T. Lascu and D. B. Scott. An algebraic correspondence with applications to projective bundles and blowing up Chern classes, Ann. Mat. Pura Appl. (4), 102 (1975), 1–36.

[8] A. T. Lascu and D. B. Scott, A simple proof of the formula for the blowing up of Chern classes, Amer. J. Math. 100(2) (1978), 293–301.

[9] D. McDuff, Examples of simply connected symplectic non-kaehler manifolds, Journal of Differential Geometry, 20(1984), 267–277.

[10] J. Milnor and J. Stasheff, Characteristic classes, Ann. of Math. Studies 76, Princeton Univ. Press, 1975.

[11] I. R. Porteous, Blowing up Chern classes. Proc. Camb. Phil. Soc. 56 (1960), 118–124.

[12] B. Segre, Dilatazioni e varietà canoniche sulle varietà algebriche. Ann. Mat. Pura Appl. (4), 37 (1954), 139–155.

[13] R. M. Switzer, Algebraic topology—homotopy and homology, Classics in Mathematics. Springer-Verlag, Berlin, 2002. xiv+526 pp.

[14] J. A. Todd, Birational transformations with a fundamental surface. Proc. London Math. Soc. (2), 47 (1941), 81–100.

[15] H. Yang, Almost complex structures on (n-1) -connected 2n -manifolds. Topology Appl. 159 (2012), no. 5, 1361–1368.

[16] C.T.C. Wall, Classification of (n-1) -connected 2n -manifolds. Annals of Math. 75 (1962), 163–198.