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Recent Results in Continuous-Time Network Information Theory

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Abstract—In this paper, we propose to use Brownian motions to formulate continuous-time multiuser Gaussian networks and derive the capacity regions of a continuous-time white Gaussian multiple access channel with/without feedback, a continuous-time white Gaussian interference channel without feedback and a continuous-time white Gaussian broadcast channel without feedback. These “complete” results stand in stark contrast to the status quo of network information theory in discrete-time, where the capacity regions of the all the above-mentioned channels are known only for a handful of special scenarios. For certain cases, our results echo, from a different perspective, the folklore that “a continuous-time channel is the limit of bandwidth limited discrete-time ones as the bandwidth tends to infinity”.

I. INTRODUCTION

Extending Shannon’s fundamental theorems on point-to-point communications to general networks with multiple sources and destinations, network information theory aims to establish the fundamental limits on information flows in networks and the optimal coding schemes that achieve these limits. Despite great efforts from within the information theory community and numerous elegant results and techniques developed during the last 60 years, the theory remains far from complete and there is still a severe lack of understanding on even the most basic one-hop channels, such as multiple access channels (MAC), interference channels (IC), broadcast channels (BC), and so on. Complete and explicit characterizations of the capacity regions of such channels, as innocent as they may look, are widely believed to be intractable, if not impossible.

Continuous-time channels were considered as early as Shannon in his original treatment of information theory. Since then, continuous-time channels, especially point-to-point continuous-time Gaussian channels, have been studied as one of the major subjects in information theory. On the other hand, to the best of our knowledge, the frontier of the research on network information theory has rather been focusing on the discrete-time setting only. In a way, this phenomenon can find its source from Shannon’s original treatment, where continuous-time channels has been converted to associated discrete-time ones before being analyzed. Shannon’s arguments, as elaborated below, convincingly articulate that “a continuous-time channel is the limit of bandwidth limited discrete-time ones as the bandwidth tends to infinity”, which has become a folklore in information theory.

In this paper, refraining from shifting the workspace, we will directly work within the continuous-time setting. More specifically, given average power constraints, we use Brownian motions to formulate the problems and we will give complete characterizations of the capacity regions of a number of continuous-time one-hop channels possibly with feedback. It is our formulation of the problems that equip us with established tools and techniques from stochastic analysis, which enable us to translate some classical ideas and techniques from the discrete-time setting to the continuous-time one. So, in this sense, our results in the continuous-time setting are rather natural offsprings from a marriage of the classical information theory and stochastic analysis, but not corollaries from an analysis on the associated discrete-time channels, as done in Shannon’s treatment.

As emphasized before, the crux of our approach is an appropriate mathematical formulation of white Gaussian channels, which, depending on the subjects, disciplines or the level of rigorousness required, may be defined very differently, many of which are rather vague. One conventional formulation is to use the so-called white Gaussian noise as follows:

\[ Y(t) = X(t) + Z(t), \quad t \geq 0 \quad (1) \]

where \( X(t) \) and \( Y(t) \) are the continuous-time channel input, output, respectively, and \( Z(t) \) is the white Gaussian noise. Unfortunately, for very subtle reasons, white Gaussian noises do not exist as “regular” stochastic processes, which, immediately implies that even defining the information capacity of (1) is a tricky mathematical problem. To circumvent this issue, an approach proposed by Shannon is to assume the channel has bandwidth limit \( W \). Then, assuming \( X \) has average power limit \( P \) and \( Z \) has flat power spectral density 1, one can represent the continuous-time channel by sampling the input and output every \( 1/(2W) \) seconds:

\[ Y_n^{(W)} = X_n^{(W)} + Z_n^{(W)}, \quad n \geq 1, \quad (2) \]

where the noise process \( \{Z_n^{(W)}\} \) is i.i.d. with variance 1. It is well-known from the theory for discrete-time Gaussian channels that the capacity (per second) of the channel in (2) can be computed as

\[ C^{(W)} = W \log \left(1 + \frac{P}{2W}\right). \]
The infinite bandwidth capacity $C$ of the channel in (1) is defined as

$$C = \lim_{W \to \infty} C^{(W)},$$

(3)

which can be easily computed as $P/2$. The above-mentioned conversion to discrete-time channels provides great insights for understanding continuous-time channels by establishing them as the limits of associated discrete-time ones, which invariably prompts a model shift from the continuous-time setting to the discrete-time one in relevant research; see, e.g., [16]. Some careful thinking, however, reveals that the capacity defined as in (3) is not intrinsically based on the continuous-time channels in (1), rather it is taken as the supremum of the capacity of associated discrete-time channels in (2). As a matter of fact, it takes a highly nontrivial proof in [1] to show that it is indeed the operational capacity of (1).

A parallel translation of the above sampling procedure will similarly define infinite bandwidth capacity region for continuous-time multiuser networks, which, however, remains to be proven/disproven to be equal to the operational capacity region of the continuous-time networks, where the feedback may be present. While the equivalence between these two notions of capacity region is certainly well worthy of serious consideration, it is however not the main concern in the present work. Under our approach, we would rather take a “shortcut” to bypass all the obstacles posed by the “irregularity” of white Gaussian noises. More specifically, to formulate Gaussian channels, we would rather follow [11] and consider the following integral version of (2):

$$Y(t) = \int_0^t X(s)ds + B(t),$$

(4)

where, slightly abusing the notation, we still use $Y(t)$ to denote the output corresponding to the input $X(s)$, and $B(t)$ denotes the standard Brownian motion ($Z(t)$ can be viewed as a generalized derivative of $B(t)$). Brownian motions are well-defined stochastic processes and have been extensively studied in probability theory. An immediate and convenient consequence of such a formulation is that many notions from the discrete-time setting, including information capacity, carry over the continuous-time setting. As evidenced by numerous results collected in [11] on point-to-point Gaussian channels, the use of Brownian motions elevate the level of rigorouousness of our treatment, in which we are able to command a wide range of established techniques and tools from stochastic analysis readily available at our disposal.

It has been proven [12] that under the same power constraint, the capacity of (4) is also $P/2$, the same as the operational capacity and infinite bandwidth capacity of (1). The subtle differences and connections between the two above-mentioned formulations for multiuser networks in terms of the corresponding three capacity regions, however, remain to be investigated. In this regard, we observe (see the full version of this manuscript) that that the capacity regions of some channels under our formulation coincide with their infinite bandwidth capacity regions, which naturally leads to our conjecture that the three capacity regions always coincide for any networks. On the other hand, it seems to us that to prove/disprove this conjecture is far from straightforward.

To put our results into a relevant context, we list some key results of network information theory in discrete-time first.

**Gaussian MACs.** When there is no feedback, the capacity region of a Gaussian MAC has been explicitly derived in Wyner [17] and Cover [5]. On the other hand, the capacity region of MACs with feedback is far from understood: Cover and Leung [6] derived an achievable region for a memoryless MAC with feedback, which has recently been improved by Bross and Lapidoth [3]. An interesting result has been obtained by Ozarow [13], who derived the capacity region of a memoryless Gaussian MAC with two users, and showed that in general the capacity region for a discrete memoryless MAC is increased by feedback. The capacity region of more general MACs has also been considered. Unfortunately, none of the above-mentioned work gives an explicit characterization of the capacity region of a generic multiple access channel with feedback, which is widely believed to be highly intractable.

In Section II, we derive the capacity region of a continuous-time white Gaussian MAC with $m$ senders and with/without feedback. It turns out that for such a channel, the feedback does not increase the capacity region.

**Gaussian ICs.** The capacity regions of discrete-time Gaussian ICs are largely unknown except for certain special scenarios, such as ICs with strong interference Sato [15], Han and Kobayashi [10]. Almost all the work on capacity regions of ICs so far only deal with two pairs of senders and receivers. For more than two user pairs, special classes of Gaussian ICs have been examined using the scheme of interference alignment; see an extensive list of references in [9].

In Section III, we derive the capacity region of a continuous-time white Gaussian IC with $m$ pairs of senders and receivers and without feedback.

**Gaussian BCs.** The capacity regions of discrete-time Gaussian BCs without feedback are well known. And it has been shown by El Gamal [8] that feedback cannot increase the capacity region of a physically degraded Gaussian BC. On the other hand, it was shown by Ozarow and Leung [14] that feedback can increase the capacity of stochastically degraded Gaussian BCs, whose capacity regions are far less understood.

In Section IV, we derive the capacity region of a continuous-time BC with $m$ receivers and without feedback.

Here, we remark that many proofs and technical details have been omitted due to the space limit; for a full version of this manuscript, we refer to http://arxiv.org/abs/1401.3529.

## II. GAUSSIAN MACS

Consider a continuous-time white Gaussian MAC with $m$ users, which can be characterized by

$$Y(t) = \int_0^t X_1(s)ds + \cdots + \int_0^t X_m(s)ds + B(t), \quad t \geq 0,$$

(5)

where $\{B(t) : t \geq 0\}$ is the standard Brownian motion and $\{X_i(s) : s \geq 0\}, i = 1, 2, \ldots, m$, is the input from the $i$-th user, which is a function of $M_i$, the message that user $i$
tries to transmit through the channel. If the channel is with feedback, then \( X_i(s) \) also depends on \( \{ Y(u) : 0 \leq u < s \} \), the channel output up to time \( s \). Note that, with the presence of feedback, the existence and uniqueness of \( Y \) is always a tricky mathematical problem, however, we will not go deep in this respect and simply assume that the input \( X \) is appropriately chosen such that \( Y \) uniquely exists. We will also assume the inputs satisfy the following power constraint: for any \( i = 1, 2, \ldots, m \), there exists \( 0 < P_i < \infty \) such that for all \( t > 0 \)

\[
\frac{1}{t} \int_0^t X_i^2(s)ds \leq P_i. \tag{6}
\]

For \( T, R_1, \ldots, R_m > 0 \), a \( ((e^{TR_1}, \ldots, e^{TR_m}), T) \) code for the MAC as in (5) consists of \( m \) sets of integers \( M_i = \{1, 2, \ldots, e^{TR_i}\} \), the message set for user \( i \), \( i = 1, 2, \ldots, m \), and \( m \) encoding functions, \( X_i : M_i \rightarrow \mathbb{R}^{[0,T]} \), which satisfy the power constraint as in (6), and a decoding function,

\[
g : \mathbb{R}^{[0,T]} \rightarrow M_1 \times \cdots \times M_m.
\]

Assuming that the distribution of messages over the product set \( (M_1, \ldots, M_m) \) is uniform, we define the average probability of error for the \( ((e^{TR_1}, \ldots, e^{TR_m}), T) \) code as

\[
P_e(T) = E[P(g(Y_0^T) \neq (M_1, \ldots, M_m))],
\]

where, for notational simplicity, we have simply written \( \{ Y(u) : 0 \leq u \leq T \} = Y_0^T \). A rate tuple \( (R_1, \ldots, R_m) \) is said to be \textbf{achievable} for the MAC if there exists a sequence of \( ((e^{TR_1}, \ldots, e^{TR_m}), T) \) codes with \( P_e(T) \rightarrow 0 \) as \( T \rightarrow \infty \). The \textbf{capacity region} of the MAC is the closure of the set of all the achievable \((R_1, \ldots, R_m)\) rate tuples.

The following theorem gives an explicit characterization of the capacity region.

**Theorem II.1.** Whether there is feedback or not, the capacity region of the above-mentioned continuous-time white Gaussian MAC is

\[
\{(R_1, \ldots, R_m) \in \mathbb{R}^m_+ : R_i \leq P_i/2, \quad i = 1, 2, \ldots, m\}.
\]

In the following, we will give the proof of Theorem II.1. For notational convenience only, we will assume \( m = 2 \), the case with a generic \( m \) being completely parallel.

We will need the following lemma, whose proof is omitted due to the space limit.

**Lemma II.2.** For any \( \varepsilon > 0 \), there exist two independent Ornstein-Uhlenbeck (OU) processes \( \{ X_i(s) : s \geq 0 \}, i = 1, 2 \), satisfying the following power constraint: for \( i = 1, 2 \), there exists \( P_i > 0 \) such that for all \( t > 0 \),

\[
\frac{1}{t} \int_0^t E[X_i^2(s)]ds = P_i, \tag{7}
\]

such that for all \( T \),

\[
|I_T(X_1, X_2; Y)/T - (P_1 + P_2)/2| \leq \varepsilon, \tag{8}
\]

and

\[
|I_T(X_1; Y|X_2)/T - P_1/2| \leq \varepsilon, |I_T(X_2; Y|X_1)/T - P_2/2| \leq \varepsilon, \tag{9}
\]

moreover,

\[
|I_T(X_1; Y)/T - P_1/2| \leq \varepsilon, |I_T(X_2; Y)/T - P_2/2| \leq \varepsilon, \tag{10}
\]

where

\[
Y(t) = \int_0^t X_1(s)ds + \int_0^t X_2(s)ds + B(t), \quad t \geq 0.
\]

Here (and often in other parts of the paper) the subscript \( T \) means that the conditional mutual information is computed over the time period \([0, T]\).

We are now ready for the proof of Theorem II.1

**Proof of Theorem II.1:**

**The converse part.** In this part, we will show that for any sequence of \( ((e^{TR_1}, e^{TR_2}), T) \) codes with \( P_e(T) \rightarrow 0 \) as \( T \rightarrow \infty \), the rate pair \( (R_1, R_2) \) will have to satisfy

\[
R_1 \leq P_1/2, \quad R_2 \leq P_2/2.
\]

Fix \( T \) and consider the above-mentioned \( ((e^{TR_1}, e^{TR_2}), T) \) code. By the code construction, it is possible to estimate the messages \( (M_1, M_2) \) from the channel output \( Y_0^T \) with a low probability of error. Hence, the conditional entropy of \( (M_1, M_2) \) given \( Y_0^T \) must be small; more precisely, by Fano’s inequality,

\[
H(M_1, M_2|Y_0^T) \leq T(R_1 + R_2)P_e(T) + H(P_e(T)) = Te_T,
\]

where \( e_T \rightarrow 0 \) as \( T \rightarrow \infty \). Now, we can bound the rate \( R_1 \) as follows:

\[
TR_1 = H(M_1) \leq I(M_1; Y_0^T|M_2) + Te_T.
\]

Applying Theorem 6.2.1 in [11], we have

\[
I(M_1; Y_0^T|M_2) = \frac{1}{2} \int_0^T E[(X_1(t)+X_2(t)−\hat{X}_1(t)−\hat{X}_2(t))^2]dt,
\]

where \( \hat{X}_i(t) = E[X_i(t)|Y_0^T, M_2], i = 1, 2 \). Noticing that \( X_2(t) = \hat{X}_2(t) \) for any \( t \), we then have

\[
I(M_1; Y_0^T|M_2) = \frac{1}{2} \int_0^T E[(X_1(t)−\hat{X}_1(t))^2]dt,
\]

which, together with (6), implies that \( R_1 \leq P_1/2 \). A completely parallel argument will yield that \( R_2 \leq P_2/2 \).

**The achievability part.** In this part, we will show that as long as \( (R_1, R_2) \) satisfying

\[
0 \leq R_1 < P_1/2, \quad 0 \leq R_2 < P_2/2, \tag{11}
\]

we can find a sequence of \( ((e^{TR_1}, e^{TR_2}), T) \) codes with \( P_e(T) \rightarrow 0 \) as \( T \rightarrow \infty \). The argument consists of several steps as follows.

**Codebook generation:** For a fixed \( T > 0 \) and \( \varepsilon > 0 \), assume that \( X_1 \) and \( X_2 \) are independent OU processes over \([0, T]\) with respective variances \( P_1 - \varepsilon \) and \( P_2 - \varepsilon \), and that \( (R_1, R_2) \) satisfying (11). Generate \( e^{TR_1} \) independent codewords \( X_{1,i}, i \in \{1, 2, \ldots, e^{TR_1}\}, \) of length \( T \), according to the distribution of \( X_1 \). Similarly, generate \( e^{TR_2} \) independent codewords \( X_{2,j}, j \in \{1, 2, \ldots, e^{TR_2}\}, \) of length \( T \), according to the distribution of \( X_2 \). These codewords (which may not satisfy the power
constraint in (6) form the codebook, which is revealed to the senders and the receiver.

**Encoding:** To send message \( i \in \mathcal{M}_1 \), sender 1 sends the codeword \( X_{1,i} \). Similarly, to send \( j \in \mathcal{M}_2 \), sender 2 sends \( X_{2,j} \).

**Decoding:** For any fixed \( \varepsilon > 0 \), let \( T_\varepsilon^{(T)} \) denote the set of jointly typical \((x_1, x_2, y)\) sequences, which is defined as follows:

\[
T_\varepsilon^{(T)} = \{ (x_1, x_2, y) \in \mathbb{R}^{[0,T]} \times \mathbb{R}^{[0,T]} \times \mathbb{R}^{[0,T]} : \\
| \log \frac{d\mu_{X_1,X_2,Y}}{d\mu_{X_1,X_2}}(x_1, x_2, y) - I_T(X_1, X_2; Y) | \leq T\varepsilon, \\
| \log \frac{d\mu_{X_1,X_2,Y}}{d\mu_{X_1}}(x_1, x_2, y) - I_T(X_1; X_2, Y) | \leq T\varepsilon, \\
| \log \frac{d\mu_{X_1,X_2,Y}}{d\mu_{X_2}}(x_1, x_2, y) - I_T(X_2; X_1, Y) | \leq T\varepsilon \}
\]

Based on the received output \( y \in \mathbb{R}^{[0,T]} \), the receiver chooses the pair \((i, j)\) such that

\[(x_{1,i}, x_{2,j}, y) \in T_\varepsilon^{(T)}, \]

if such a pair \((i, j)\) exists and is unique; otherwise, an error is declared. Moreover, an error will be declared if the chosen codeword does not satisfy the power constraint in (6).

**Analysis of the probability of error:** Now, for fixed \( T, \varepsilon > 0 \), define

\[E_{ij} = \{ (X_{1,i}, X_{2,j}, Y) \in T_\varepsilon^{(T)} \}.
\]

By symmetry, we assume, without loss of generality, that (1,1) was sent. Define \( \pi^{(T)} \) to be the event that

\[
\int_0^T (X_{1,1}(t))^2 dt > P_1 T, \quad \int_0^T (X_{2,1}(t))^2 dt > P_2 T.
\]

Then, \( P_\varepsilon^{(T)} \), the error probability for the above coding scheme (where codewords violating the power constraint are allowed), can be upper bounded as follows: for any \( i, j \neq 1 \),

\[
P_\varepsilon^{(T)} = P(\pi^{(T)} \cup E_{11}^{(T)} \cup \cup_{i,j}(i,j) \neq (1,1) E_{ij}) \leq P(\pi^{(T)}) + P(E_{11}^{(T)}) + e^{TR_1} P(E_{11}) + e^{TR_2} P(E_{ij}) + e^{TR_1 + TR_2} P(E_{ij}).
\]

Using the well-known fact that an OU process is ergodic, we deduce that \( P(\pi^{(T)}) \to 0 \) as \( T \to \infty \). And by Theorems 6.6.2 and 6.2.1 in [11], we deduce that

\[
\lim_{T \to \infty} P((X_{1,1}, X_{2,1}, Y) \in T_\varepsilon^{(T)}) = 1 \quad \text{and} \quad \lim_{T \to \infty} P(E_{11}) = 0.
\]

And, for any \( i \neq 1 \),

\[
P(E_{11}) = P((X_{1,i}, X_{2,1}, Y) \in T_\varepsilon^{(T)}) \leq e^{-I_T(X_1; Y|X_2) + \varepsilon T},
\]

where we have used the independence of \( X_1 \) and \( X_2 \), and the consequent fact that

\[I_T(X_1; X_2, Y) = I_T(X_1; X_2) + I_T(X_1; Y|X_2) = I_T(X_1; Y|X_2).
\]

Similarly, we have, for \( j \neq 1 \),

\[P(E_{1j}) \leq e^{-I_T(X_2|Y|X_1) + \varepsilon T},
\]

and for \( i, j \neq 1 \),

\[P(E_{ij}) \leq e^{-I_T(X_1, X_2; Y) + \varepsilon T}.
\]

It then follows that

\[P_\varepsilon^{(T)} \leq P(\pi^{(T)}) + P(E_{11}^{(T)}) + e^{TR_1 + \varepsilon T} I_T(X_1; Y|X_2) + e^{TR_2 + \varepsilon T} I_T(X_2; Y|X_1) + e^{TR_1 + TR_2 + \varepsilon T} - I_T(X_1; X_2; Y).
\]

By Lemma II.2, one can choose independent OU processes \( X_1, X_2 \) such that \( I_T(X_1; Y|X_2)/T \to (P_1 - \varepsilon)/2 \) and \( I_T(X_1, X_2; Y)/T \to (P_1 + P_2 - 2\varepsilon) \) uniformly in \( T \). This implies that with \( \varepsilon \) chosen sufficiently small, we have \( P_\varepsilon^{(T)} \to 0 \), as \( T \to \infty \). In other words, there exists a sequence of good codes (which may not satisfy the power constraint) with low average error probability. Now, a usual argument of “deleting bad codewords” yields a sequence of good codes (which satisfy the power constraint) with \( P_\varepsilon^{(T)} \to 0 \) as \( T \to \infty \), which implies that the rate pair \((R_1, R_2)\) is achievable.

**III. Gaussian ICs**

Consider the following continuous-time white Gaussian interference channel having no feedback and with \( m \) pairs of senders and receivers: for \( i = 1, 2, \ldots, m \),

\[Y_i(t) = a_{1i} \int_0^t X_1(s) ds + \cdots + a_{im} \int_0^t X_m(s) ds + B_i(t), \tag{12}
\]

where \( a_{ij} \in \mathbb{R}, i, j = 1, 2, \ldots, m \), is the channel gain from sender \( j \) to receiver \( i \), all \( B_i(t) \) are (possibly correlated) standard Brownian motions and \( X_i(s) \) is the input from user \( i \), which satisfy the following power constraint: for any \( i = 1, 2, \ldots, m \), there exist \( P_i > 0 \) such that for all \( t > 0 \)

\[
\frac{1}{t} \int_0^t X_i^2(s) ds \leq P_i. \tag{13}
\]

The capacity region of the above continuous-time IC can be routinely defined as in the discrete-time setting; see the full version for more details.

The following theorem explicitly characterizes the capacity region of the above IC:

**Theorem III.1.** The capacity region of the above-mentioned continuous-time white Gaussian IC is

\[
\{(R_1, \ldots, R_m) \in \mathbb{R}^m_+ : R_i \leq a_{ii}^2 P_i/2, \quad i = 1, 2, \ldots, m \}.
\]

**Proof:** For notational convenience only, we only prove the case when \( n = 2 \); the case when \( n \) is generic being similar.

**The converse part.** The proof of this part is parallel to that of the converse part of Theorem II.1 and thus omitted.

**The achievability part.** We only sketch the proof of this part. For arbitrarily small \( \varepsilon > 0 \), by Lemma II.2, one can choose independent OU processes \( X_i \) with respective variances \( P_i - \varepsilon, i = 1, 2 \), such that \( I_T(X_i; Y)/T \) approaches \( a_{ii}^2 (P_i - \varepsilon)/2 \). Then, a parallel random coding argument as in the proof of Theorem II.1 with \( X_j, j \neq i \), being treated
as noise at receiver $i$ shows that the rate pair $(a_{11}^2(P_i - \varepsilon)/2, a_{22}^2(P_2 - \varepsilon)/2)$ can be approached, which yields the achievability part.

IV. GAUSSIAN BCs

In this section, we consider a continuous-time white Gaussian BC with $m$ receivers, which is characterized by: for $i = 1, 2, \ldots, m$,

$$Y_i(t) = \sqrt{\text{snr}_i} \int_0^t X(s)ds + B_i(t), \quad t \geq 0,$$

where $\text{snr}_i$ is the signal-to-noise ratio (SNR) in the channel for user $i$. $B_i(t)$ are (possibly correlated) standard Brownian motions and $X(s)$ is the input, which satisfies the following power constraint: for all $t > 0$, there exists $P > 0$ such that

$$\frac{1}{t} \int_0^t X^2(s)ds \leq P.$$  

The capacity region of the above continuous-time BC can be routinely defined as in the discrete-time setting.

The following theorem explicitly characterizes the capacity region of the above BC:

Theorem IV.1. The capacity region of the above-mentioned continuous-time white Gaussian BC is

$$\left\{ (R_1, \ldots, R_m) \in \mathbb{R}^m_+: \frac{R_1}{\text{snr}_1} + \cdots + \frac{R_m}{\text{snr}_m} \leq \frac{P}{2} \right\}.$$

Proof of Theorem IV.1: For notational convenience only, we prove the case when $n = 2$, the case when $n$ is generic being parallel.

The converse part. Without loss of generality, we assume that $\text{snr}_1 \geq \text{snr}_2$. We will show that for any sequence of $((e^{TR_1}, e^{TR_2}), T)$ codes with $P_e(T) \to 0$ as $T \to \infty$, the rate pair $(R_1, R_2)$ will have to satisfy

$$\frac{R_1}{\text{snr}_1} + \frac{R_2}{\text{snr}_2} \leq \frac{P}{2}.$$  

Fix $T$ and consider the above-mentioned $((e^{TR_1}, e^{TR_2}), T)$ code. By the code construction, for $i = 1, 2$, it is possible to estimate the messages $M_i$ from the channel output $Y_{i,0}^T$ with an arbitrarily low probability of error. Hence, by Fano’s inequality, for $i = 1, 2$,

$$H(M_i|Y_{i,0}^T) \leq TR_iP_e(T) + H(P_e(T)) = T\epsilon_i,T,$$

where $\epsilon_i,T \to 0$ as $T \to \infty$. It then follows that

$$TR_1 = H(M_1) \leq I(M_1; Y_{1,0}^T|M_2) + T\epsilon_1,T,$$

$$TR_2 = H(M_2) \leq I(M_2; Y_{2,0}^T|M_2) + T\epsilon_2,T.$$  

It can then be verified that

$$I(M_1, M_2; Y_{2,0}^T) \geq I(M_2; Y_{2,0}^T) = \frac{\text{snr}_2}{\text{snr}_1}I(M_1; Y_{1,0}^T|M_2).$$

Now, applying Theorem 6.2.1 in [11], we have

$$I(M_1, M_2; Y_{2,0}^T) \leq \frac{\text{snr}_2}{2} \int_0^T E[X^2(s)]ds,$

which, together with (17), (18), (19) and (15), immediately implies the converse part.

The achievability part. We only sketch the proof of this part. For an arbitrarily small $\varepsilon > 0$, by Theorem 6.4.1 in [11], one can choose an OU processes $\tilde{X}$ with variance $P - \varepsilon$, such that $I_T(\tilde{X}; Y_1)/T$ approaches $\text{snr}_1(P - \varepsilon)/2$. For any $0 \leq \lambda \leq 1$, let

$$X(t) = \sqrt{\lambda}X_1(t) + \sqrt{1 - \lambda}X_2(t), \quad t \geq 0,$$

where $X_1$ and $X_2$ are independent copies of $\tilde{X}$. Then, by a similar argument as in the proof of Lemm II.2, we deduce that $I_T(X_1; Y_1)/T, I_T(X_2; Y_2)/T$ approach $\text{snr}_1\lambda(P - \varepsilon)/2, \text{snr}_2(1 - \lambda)(P - \varepsilon)/2$, respectively. Then, a parallel random coding argument as in the proof of Theorem II.1 such that

- when encoding, $X_i$ only carries the message meant for receiver $i$;
- when decoding, receiver $i$ treats $X_j$, $j \neq i$, as noise, shows that the rate pair $(\text{snr}_1\lambda(P - \varepsilon)/2, \text{snr}_2(1 - \lambda)(P - \varepsilon)/2)$ can be approached, which immediately establishes the achievability part.

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