A Study of Dynamics of the Tricomplex Polynomial $\eta^p + c$

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Abstract

In this article, we give the exact interval of the cross section of the so called Mandelbric set generated by the polynomial $z^3 + c$ where $z$ and $c$ are complex numbers. Following that result, we show that the Mandelbric define on the hyperbolic numbers $\mathbb{D}$ is a square with its center at the origin. Moreover, we define the Multibrot sets generated by a polynomial of the form $Q_{p,c}(\eta) = \eta^p + c$ ($p \in \mathbb{N}$ and $p \geq 2$) for tricomplex numbers. More precisely, we prove that the tricomplex Mandelbric has four principal slices instead of eight principal 3D slices that arise for the case of the tricomplex Mandelbrot set. Finally, we prove that one of these four slices is an octahedron.

1 Introduction

In 1982, A. Douady and J. H. Hubbard [2] studied dynamical systems generated by iterations of the quadratic polynomial $z^2 + c$. One of the main result of their work was the proof that the well-known Mandelbrot set for complex numbers is a connected set. It is also well known that the Mandelbrot set is bounded by a discus of radius 2 and cross the real axis on the interval $[-2,\frac{1}{4}]$. Considering functions of the form $z^p + c$ where $z, c$ are complex numbers and $p$ may be an integer, a rational or a real number greater than 2, T. V. Papathomas, B. Julesz, U. G. Gujar and V. G. Bhavsar [12, 6] explored the sets generated by these functions called Multibrot sets. The last two authors remark that these polynomials generate rich fractal structures. This was the starting point for other researchers like D. Schleicher ([17, 7]), E. Lau [7], X. Sheng

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and M. J. Spurr [20], X.-D. Liu, W.-Y. Zhu [8] and many others to study symmetry groups in Multibrot sets, their connectivity and the discus that bound these sets.

In 1990, P. Senn [18] suggested to define the Mandelbrot set for another set of numbers: the hyperbolic numbers (also called duplex numbers). He remarked that the Mandelbrot set for this number structure seemed to be a square. Four years later, a proof of this statement was giving by W. Metzler [9].

Unless these works were done in the complex plane, so in 2D, several mathematicians questioned themselves on a generalization of the Mandelbrot set in three dimensions. In 1982, A. Norton [11] succeeded to bring a first method to visualize fractals in 3D using the quaternions. In 2000, D. Rochon [14] used the bicomplex numbers set \( M(2) \) to give a 4D definition of the so-called Mandelbrot set and made 3D slices to get the Tetrabrot. Later, X.-y. Wang and W.-j. Song [23] follow D. Rochon’s work to establish the Multibrots sets for bicomplex numbers. Several years before, D. Rochon and V. Garant-Pelletier ([5], [4]) gave a definition of the Mandelbrot set for multicomplex numbers denoted \( M_n \) and restrict their explorations to the tricomplex case. Particularly, they found eight principal slices of the tricomplex Mandelbrot set and proved that one of these 3D slices, typically named the Perplexbrot, is an octahedron of edges \( \frac{2\sqrt{3}}{3} \).

In this article, we investigate the Multibrot sets for complex, hyperbolic and tricomplex numbers respectively denoted \( M_p, H_p \) and \( M_3^p \) when \( p \) is an integer greater than one. We emphasize on the sets \( M^3, H^3 \) and \( M_3^3 \) respectively called the Mandelbric, Hyperbric and tricomplex Mandelbric. Precisely, in the second section, we recall some definitions and properties of tricomplex numbers denoted \( M(3) \). We remark that complex and hyperbolic numbers are embedded in \( M(3) \) and also that there are other interesting subsets of \( M(3) \). In the third section, we review the definition and properties of Multibrots sets. Particularly, we prove that the set \( M^3 \) crosses the real axis on \( \left[ -\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}} \right] \). In section four, we define Multibrot sets for hyperbolic numbers. Particularly, based on Metzler’s article (see [9]), we prove that \( H^3 \) is a square where its center is the origin. Finally, in the fifth section, we define the tricomplex Multibrot sets corresponding to the polynomial \( \eta^p + c \) where \( \eta, c \) are tricomplex numbers and \( p \geq 2 \) is an integer. We obtain, for the case where \( p = 3 \), that there is four principal 3D slices of \( M_3^3 \) instead of eight like it is showed in [5] for \( M_3^2 \). Moreover, we prove that one of these four slices, typically named the Perplexbric, is an octahedron of edges \( \frac{2\sqrt{3}}{3\sqrt{3}} \).

2 Tricomplex numbers

In this section, we begin by a short introduction of the tricomplex numbers space \( M(3) \). One may be refer to [1], [4] and [22] for more details on the next properties.

A tricomplex number \( \eta \) is composed of two coupled bicomplex numbers \( \zeta_1, \zeta_2 \) and an imaginary unit \( i_3 \) such that

\[
\eta = \zeta_1 + \zeta_2 i_3
\]

where \( i_3^2 = -1 \). The set of such tricomplex numbers is noted \( M(3) \). Since \( \zeta_1, \zeta_2 \in M(2) \), we can write them as \( \zeta_1 = z_1 + z_2 i_2 \) and \( \zeta_2 = z_3 + z_4 i_2 \) where \( z_1, z_2, z_3, z_4 \in M(1) \simeq \mathbb{C} \). In that way, (2.1) can be rewritten as

\[
\eta = z_1 + z_2 i_2 + z_3 i_3 + z_4 i_3
\]
shows the results after multiplying each tricomplex imaginary unity two by two. The zero divisors. The representation \((\gamma_1, \gamma_2)\) of tricomplex numbers with addition \(+\) and multiplication \(\cdot\) because it allows to do these operations term-by-term. In fact, we have the following theorem (see [1]):

### Table 1: Products of tricomplex imaginary units

|        | 1   | i₁ | i₂ | i₃ | i₄ | j₁ | j₂ | j₃ |
|--------|-----|----|----|----|----|----|----|----|
| 1      | 1   | i₁| i₂ | i₃ | i₄ | j₁ | j₂ | j₃ |
| i₁     | i₁  | -1|  j₁|  j₂| -j₃| -i₂| -i₃| i₄ |
| i₂     | i₂  | j₁| i₁ | -j₂| -i₁| i₄ | -i₂| j₃ |
| i₃     | i₃  | j₂| j₃ | -1 | -j₁| i₄ | -i₁| -i₂|
| i₄     | i₄  |-j₃| -j₂| -j₁| -1 | i₃ | i₂ | i₁ |
| j₁     | j₁  |-i₂| -i₁| i₄ | i₃ | 1  | -j₃| -j₂|
| j₂     | j₂  |-i₃| i₄ | -i₁| i₂ | -j₃| 1  | -j₁|
| j₃     | j₃  | i₄ | -i₃| -i₂| i₁ | -j₂| -j₁| 1  |

Table 1 shows the results after multiplying each tricomplex imaginary unity two by two. The set of tricomplex numbers with addition \(+\) and multiplication \(\cdot\) forms a commutative ring with zero divisors.

A tricomplex number has a useful representation using the idempotent elements \(\gamma_2 = \frac{1+i3}{2}\) and \(\gamma_2 = \frac{1-j3}{2}\). Recalling that \(\eta = \zeta_1 + \zeta_2 i₃\) with \(\zeta_1, \zeta_2 \in M(2)\), the idempotent representation of \(\eta\) is

\[
\eta = (\zeta_1 - \zeta_2 i₃) \gamma_2 + (\zeta_1 + \zeta_2 i₃) \gamma_2.
\]
Theorem 1 Let \( \eta_1 = \zeta_1 + \zeta_2 i_3 \) and \( \eta_2 = \zeta_3 + \zeta_4 i_3 \) be two tricomplex numbers. Set \( \eta_1 = u_1 \gamma_2 + u_2 \overline{\gamma}_2 \) and \( \eta_2 = u_3 \gamma_2 + u_4 \overline{\gamma}_2 \) be the idempotent representation (2.9) of \( \eta_1 \) and \( \eta_2 \). Then,

1. \( \eta_1 + \eta_2 = (u_1 + u_3) \gamma_2 + (u_2 + u_4) \overline{\gamma}_2 \);
2. \( \eta_1 \cdot \eta_2 = (u_1 \cdot u_3) \gamma_2 + (u_2 \cdot u_4) \overline{\gamma}_2 \);
3. \( \eta_1^m = u_1^m \gamma_2 + u_2^m \overline{\gamma}_2 \), \( \forall m \in \mathbb{N} \).

Moreover, we define a \( M(3) \)-cartesian set \( X \) of two subsets \( X_1, X_2 \subseteq M(2) \) as follows:

\[
X = X_1 \times_{\gamma_2} X_2 := \{ \eta = \zeta_1 + \zeta_2 i_3 \in M(3) | \eta = u_1 \gamma_2 + u_2 \overline{\gamma}_2, u_1 \in X_1 \) and \( u_2 \in X_2 \} \quad (2.10)
\]

Let define the norm \( \| \cdot \|_3 : M(3) \to \mathbb{R} \) of a tricomplex number \( \eta = \zeta_1 + \zeta_2 i_3 \) as

\[
\| \eta \|_3 := \sqrt{\| \zeta_1 \|^2_2 + \| \zeta_2 \|^2_2} = \sqrt{\sum_{i=1}^{2} |z_i|^2 + \sum_{i=3}^{4} |z_i|^2} = \sqrt{\sum_{i=0}^{7} x_i^2}.
\]

According to the norm (2.11), we say that a sequence \( \{ s_m \}_{m=1}^{\infty} \) of tricomplex numbers is bounded iff there exist a real number \( M \) such that \( \| s_m \|_3 \leq M \) for all \( m \in \mathbb{N} \). Now, according to (2.10), we define two kinds of tricomplex discs:

Definition 1 Let \( \alpha = \alpha_1 + \alpha_2 i_3 \in M(3) \) and set \( r_2 \geq r_1 > 0 \).

1. The open discus is the set

\[
D_3(\alpha; r_1, r_2) := \{ \eta \in M(3) | \eta = \zeta_1 \gamma_2 + \zeta_2 \overline{\gamma}_2, \| \zeta_1 - (\alpha_1 + \alpha_2 i_2) \|_2 < r_1 \) and
\| \zeta_2 - (\alpha_1 + \alpha_2 i_2) \|_2 < r_2 \}.
\]

2. The closed discus is the set

\[
\overline{D}_3(\alpha; r_1, r_2) := \{ \eta \in M(3) | \eta = \zeta_1 \gamma_2 + \zeta_2 \overline{\gamma}_2, \| \zeta_1 - (\alpha_1 + \alpha_2 i_2) \|_2 \leq r_1 \) and
\| \zeta_2 - (\alpha_1 + \alpha_2 i_2) \|_2 \leq r_2 \}.
\]

We end this section by several remarks about subsets of \( M(3) \). Let the set \( C(i_k) := \{ \eta = x_0 + x_1 i_k | x_0, x_1 \in \mathbb{R}, i_k \in \{ i_1, i_2, i_3, i_4 \} \} \). So, \( C(i_k) \) is a subset of \( M(3) \) for \( k = 1, 2, 3, 4 \) and we also remark that they are all isomorphic to \( C \). Furthermore, the set

\[
\mathbb{D}(i_k) := \{ x_0 + x_1 i_k | x_0, x_1 \in \mathbb{R} \}
\]

where \( i_k \in \{ j_1, j_2, j_3 \} \) is a subset of \( M(3) \) and is isomorphic to the set of hyperbolic numbers \( \mathbb{D} \) for \( k \in \{ 1, 2, 3, 4 \} \) (see [16, 19] and [21] for further details about the set \( \mathbb{D} \) of hyperbolic numbers). Moreover, we define three particular subsets of \( M(3) \) (see [5] and [13]).

Definition 2 Let \( i_k, i_1 \in \{ 1, i_1, i_2, i_3, i_4, j_1, j_2, j_3 \} \) where \( i_k \neq i_1 \). The first set is

\[
\mathbb{M}(i_k, i_1) := \{ x_1 + x_2 i_k + x_3 i_1 + x_4 i_k i_1 | x_i \in \mathbb{R}, i = 1, \ldots, 4 \}.
\]
It is easy to see that $M(i_k, i_1)$ is closed under addition and multiplication of tricomplex numbers and that $M(i_k, i_1) \simeq M(2)$ except for the biduplex sets $M(j_1, j_2)$, $M(j_1, j_3)$ and $M(j_2, j_3)$ (see [5]).

**Definition 3** Let $i_k, i_1, i_m \in \{i_1, i_2, i_3, i_4, j_1, j_2, j_3\}$ with $i_k \neq i_1$, $i_k \neq i_m$ and $i_1 \neq i_m$. The second subset is

$$M(i_k, i_1, i_m) := \{ x_1i_k + x_2i_1 + x_3i_m + x_4i_ki_1i_m | x_i \in \mathbb{R}, i = 1, \ldots, 4 \}. \quad (2.15)$$

Using Table 1, we can easily verify that for any tricomplex number $\zeta \in M(i_k, i_1, i_m)$, $\zeta^3 \in M(i_k, i_1, i_m)$. In section 5, this fact will be useful to characterize some principal 3D slices of the tricomplex Mandelbric.

**Definition 4** Let $i_k, i_1, i_m \in \{i_1, i_2, i_3, i_4, j_1, j_2, j_3\}$ with $i_k \neq i_1$, $i_k \neq i_m$ and $i_1 \neq i_m$. The third subset is

$$T(i_m, i_k, i_1) := \{ x_1i_k + x_2i_1 + x_3i_m | x_1, x_2, x_3 \in \mathbb{R} \}. \quad (2.16)$$

This last set is not closed under multiplication depending on the number of times $k$ you multiply a tricomplex number in this set. For $k$ even, two cases may occur depending on the choice of the tricomplex imaginary units: the case that $T(i_m, i_k, i_1) \subseteq M(i_k, i_1)$ or the case that the result of multiplying tricomplex numbers in $T(i_m, i_k, i_1)$ is not closed in the set $M(i_k, i_1)$. The first case arises if one of the imaginary unit $i_k, i_1, i_m$ is 1 or if the product $i_ki_1 = \pm i_m$. Whenever one of these conditions are not fulfilled, the result is not closed in the set $M(i_k, i_1)$. On the other hand, if $k$ is odd, the first case stay the same but the second is always closed in the set $M(i_k, i_1, i_m)$. These facts are a direct consequence of the definition of the tricomplex imaginary units.

## 3 Generalized Mandelbrot Sets for Complex Numbers

In this section, we investigate Multibrot sets and recall some of their properties that come from [6, 8, 10, 13, 23]. We also obtain some specific results for the Mandelbric set $M^p$ generate by the complex polynomial $Q_{3,c}(z) = z^3 + c$.

### 3.1 Multibrot sets

We define the Multibrot as follows:

**Definition 5** Let $Q_{p,c}(z) = z^p + c$ a polynomial of degree $p \in \mathbb{N} \setminus \{0, 1\}$. A Multibrot set is the set of complex numbers $c$ which for all $m \in \mathbb{N}$, the sequence $\{Q_{p,c}^m(0)\}_{m=1}^{\infty}$ is bounded, i.e.

$$\mathcal{M}^p = \left\{ c \in \mathbb{C} | \{Q_{p,c}^m(0)\}_{m=1}^{\infty} \text{ is bounded} \forall m \in \mathbb{N} \right\}. \quad (3.1)$$

If we set $p = 2$, we find the well-known Mandelbrot set. The next two theorems provide a method to visualize the $\mathcal{M}^p$ sets (see [8] and [13]).

**Theorem 2** For all complex number $c$ in $\mathcal{M}^p$, we have $|c| \leq 2^{1/(p-1)}$.

**Theorem 3** A complex number $c$ is in $\mathcal{M}^p$ iff $|Q_{p,c}^m(0)| \leq 2^{1/(p-1)} \forall m \in \mathbb{N}$. 

5
Theorem 3 provides a criterion to decide whenever a complex number $c$ belongs to the set $\mathcal{M}^p$. The algorithm used to generate the figures is described in [6]. We use the preset limit $L = 2^{1/(p-1)}$ and the magnitude of maximum iterations $M = 1000$. The images are generated in a square of 1000x1000 pixels. Figures 1(a), 1(b) illustrate respectively the sets $\mathcal{M}^p$ for the values $p = 3$ and $p = 4$.

Now, let $\mathfrak{M}$ denote the family of all generalized Mandelbrot sets $\mathcal{M}^p$, i.e. $\mathfrak{M} := \{\mathcal{M}^p | p \geq 2\}$. The family $\mathfrak{M}$ has the following property (see [23]).

**Theorem 4** All members of the family $\mathfrak{M}$ is a connected set.

We have also some other properties related to the polynomial $Q_{p,c}(z)$ when we iterate it from $z_0 = 0$. The proofs can be found in [13].

**Lemma 1** Set $c > 0$ where $c$ is a real number. Then, the sequence $\{Q_{p,c}^m(0)\}_{m=1}^\infty$ is strictly ascendant. Furthermore, if the sequence $\{Q_{p,c}^m(0)\}_{m=1}^\infty$ is bounded, then it converges to $c_0 > 0$.

**Lemma 2** Set $c < 0$ where $c$ is a real number. Then, the sequence $\{Q_{p,c}^m(0)\}_{m=1}^\infty$ is strictly descendant when $p$ is odd. Furthermore, if the sequence $\{Q_{p,c}^m(0)\}_{m=1}^\infty$ is bounded, then it converges to $c_0 < 0$.

These properties will be useful to prove our next result for the intersection of $\mathcal{M}^3$ with the real axis.
3.2 Roots of a third degree polynomial

Let \( P(x) = x^3 + bx^2 + cx + d \) denote a monic cubic polynomial with real coefficients. Set \( y = x + \frac{b}{3} \) as a Möbius transformation. It reduces the polynomial \( P(x) \) to the polynomial \( Q(y) = y^3 + py + q \) where \( p = c - \frac{b^2}{3} \) and \( q = \frac{2b^3}{27} - \frac{bc^2}{3} + d \). In that case, searching for the roots of \( P(x) \) is equivalent to search the roots of \( Q(y) \).

A complex number \( z \) is a root of \( Q(y) \) iff there exist two complex numbers \( y_1 \) and \( y_2 \) such that

\[
y_1 + y_2 + \frac{p}{3} = 0 \tag{3.2}
\]
\[
y_1y_2 = -\frac{p^3}{27} \tag{3.3}
\]

(see [13]). The last equations can be rewritten as the following equivalent system

\[
y_1^3 + y_2^3 + q = 0 \tag{3.4}
\]
\[
y_1^3y_2^3 = -\frac{p^3}{27}.
\]

In respect to (3.4), we remark that \( y_1, y_2 \) are roots of the polynomial \( T(t) = t^2 - (y_1^3 + y_2^3)t + y_1^3y_2^3 = t^2 + qt - \frac{p^3}{27} \) where its discriminant \( \Delta \) is equal to \( q^2 + \frac{4p^3}{27} \). To settle information on the roots of \( Q(y) \) and so, to get information from the roots of \( P(x) \), we are interesting about the sign of \( D = 27\Delta = 27q^2 + 4p^3 \). In fact, one can prove the following result (see [13] and [3]).

**Theorem 5** Let \( P(x) = x^3 + bx^2 + cx + d \) where \( b, c, d \in \mathbb{R} \) and \( D = 4c^3 + 27d^2 + 4db^3 - b^2c^2 - 18bcd \). Then,

i) if \( D > 0 \), then \( P(x) \) has one real root and two complex roots;

ii) if \( D = 0 \), then \( P(x) \) has three real roots which one is of multiplicity 2;

iii) if \( D < 0 \), then \( P(x) \) has three distinct real roots.

3.3 \( \mathcal{M}^3 \) set: The Mandelbric

In this subsection, we prove that the Mandelbric cross the real axis on the interval \( \left[ -\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}} \right] \) (see Theorem 6). Before we go through the proof of Theorem 6, we first establish some symmetries (see [7] and [20]) in the Mandelbric.

**Lemma 3** Let \( c \in \mathcal{M}^3 \). Then \( \overline{c} \in \mathcal{M}^3 \).

**Proof.** Suppose \( c \in \mathcal{M}^3 \). Then, by Theorem 3, \( |Q_{n,3,c}(0)| \leq 2 \forall m \in \mathbb{N} \). Since \( Q_{n,3,c}(0) = Q_{3,n,c}(0) \), \( |Q_{3,n,c}(0)| = |Q_{n,3,c}(0)| \) for all \( m \in \mathbb{N} \). Thus, \( \overline{c} \in \mathcal{M}^3 \).

**Lemma 3** provides that \( \mathcal{M}^3 \) is symmetrical about the real axis. The next lemma is a step forward to show that \( \mathcal{M}^3 \) is symmetrical about the imaginary axis.

**Lemma 4** Let \( c = x + yi \) where \( x, y \in \mathbb{R} \). If \( c \in \mathcal{M}^3 \), then \( -c \in \mathcal{M}^3 \).
Proof. Let \( c = x + yi \) where \( x, y \in \mathbb{R} \). If \( c \in \mathcal{M}^3 \), then by Theorem 3, \( |Q_{p,c}^m(0)| \leq \sqrt{2} \forall m \in \mathbb{N} \). By induction, we remark that \( \forall m \in \mathbb{N} \), \( Q_{3,-c}^m(0) = -Q_{3,c}^m(0) \), and so \( |Q_{3,-c}^m(0)| = |Q_{3,c}^m(0)| \leq \sqrt{2} \forall m \in \mathbb{N} \). Thus, \(-c \in \mathcal{M}^3 \).

Corollary 1 Let \( c = x + yi \) where \( x, y \in \mathbb{R} \). If \( c \in \mathcal{M}^3 \), then \( c = -x + yi \) is in \( \mathcal{M}^3 \).

Proof. Let \( c = x + yi \) and \( c \in \mathcal{M}^3 \). We want to prove that \( c' = -x + yi \in \mathcal{M}^3 \). By hypothesis and Lemma 3, \( c' \in \mathcal{M}^3 \). Therefore, by Lemma 4, \(-c = -x + yi \), we have that \( c' \in \mathcal{M}^3 \).

With this last result, the next proof will be simplifies.

Theorem 6 The Mandelbric cross the real axis on the interval \([2/3\sqrt{3}, 2/3\sqrt{3}]\).

Proof. By the Corollary 1, we can restrict our proof to the interval \([0, 2/3\sqrt{3}]\). Let \( R_{3,c}(x) = x^3 - x + c \) where \( c \in \mathbb{R} \) and \( D = -4 + 27c^2 \). We start by showing that no point \( c > \frac{2}{3\sqrt{3}} \) lies in \( \mathcal{M}^3 \). In this case, \( D > 0 \) and \( R_{3,c} \) has a real root (see Theorem 5) and this root is given by

\[
x_0 = \sqrt[3]{\frac{c}{2} + \sqrt{\frac{c^2 - \frac{4}{27}}{2}}} + \sqrt[3]{\frac{c}{2} - \sqrt{\frac{c^2 - \frac{4}{27}}{2}}}.
\] (3.5)

Suppose that \( c \in \mathcal{M}^3 \), i.e. \( \{Q_{3,c}^m(0)\}_{m=1}^\infty \) is bounded for all \( m \in \mathbb{N} \) (see Definition 5). Then, Lemma 1 implies that \( \{Q_{3,c}^m(0)\}_{m=1}^\infty \) is strictly ascendant and it converges to \( c_0 > 0 \). Since \( Q_{3,c}^m(0) \) is a polynomial function for all \( m \in \mathbb{N} \), we have that

\[
c_0 = \lim_{m \to \infty} Q_{3,c}^{m+1}(0)
= Q_{3,c} \left\{ \lim_{m \to \infty} Q_{3,c}^m(0) \right\}
= Q_{3,c}(c_0).
\] (3.6)

Thus, \( c_0 \) is a real root of \( R_{3,c} \) and \( c_0 = x_0 \). However, since \( \frac{c}{2} > -\frac{c}{2} \), we have that

\[
x_0 = \sqrt[3]{\frac{c}{2} + \sqrt{\frac{c^2 - \frac{4}{27}}{2}}} + \sqrt[3]{\frac{c}{2} - \sqrt{\frac{c^2 - \frac{4}{27}}{2}}}< 0.
\] (3.9)

This is a contradiction with \( x_0 = c_0 > 0 \). Thus, \( c \notin \mathcal{M}^3 \).

Next, we show that for \( 0 \leq c \leq \frac{2}{3\sqrt{3}} \), \( c \) lies in \( \mathcal{M}^3 \). Obviously, \( c = 0 \) is in \( \mathcal{M}^3 \). Suppose that \( 0 < c \leq \frac{2}{3\sqrt{3}} \). In this case, \( D \leq 0 \) and by the Theorem 5, \( R_{3,c}(x) \) has the following three real roots (see [13]) :

\[
\left( \frac{c}{2} + i \sqrt{-\frac{c^2 + \frac{4}{27}}{2}} \right)^{1/3} + \left( -\frac{c}{2} - i \sqrt{-\frac{c^2 + \frac{4}{27}}{2}} \right)^{1/3}.
\] (3.10)
Following De Moivre’s formula, one of these roots can be expressed as follows:

\[ a = \frac{2}{\sqrt{3}} \cos \left( \frac{\theta}{3} \right) \]  
(3.11)

for \( c \in \left( 0, \frac{2}{\sqrt{3}} \right] \) and \( \theta = \arctan \left( \frac{\sqrt{3} - D}{3c} \right) + \pi \) where \( \pi \leq \theta < \frac{\pi}{3} \). We prove by induction that \( |Q_{m,3,c}^m(0)| < a \) \( \forall m \in \mathbb{N} \). For \( m = 1 \), we have that \( |Q_{3,c}(0)| = |c| < a \) because \( |c| < \frac{1}{\sqrt{3}} \leq a \).

Indeed, since \( \pi \leq \theta < \frac{\pi}{2} \), we obtain \( \frac{1}{\sqrt{3}} \leq a < 1 \). Now, suppose that \( |Q_{k,3,c}^k(0)| < a \) for a \( k \in \mathbb{N} \). Then, since \( R_{3,c}(a) = a^3 - a + c = 0 \) and \( c > 0 \),

\[ |Q_{3,c}^{k+1}(0)| = |(Q_{3,c}^k(0))^3 + c| \leq |(Q_{3,c}^k(0))^3| + |c| < a^3 + c = a. \]

Thus, the proposition is true for \( k + 1 \) and \( |Q_{3,c}^m(0)| < a \) \( \forall m \in \mathbb{N} \). Since \( a \leq \sqrt{2} \), then by the Theorem 3 we have \( c \in \mathcal{M}^3 \).

In conclusion, \( \mathcal{M}^3 \cap \mathbb{R}_+ = \left[ 0, \frac{2}{\sqrt{3}} \right] \). \( \square \)

4 Multibrot Sets for Hyperbolic Numbers

Previously, we treated the Multibrot sets for complex numbers. In this section, we propose an extension of the Mandelbrot set for hyperbolic numbers called the Hyperbrots and we prove that \( \mathcal{M}^3 \) for hyperbolic numbers denoted \( H^3 \) is a square of side length \( \frac{2}{\sqrt{3}} \sqrt{2} \).

4.1 Definition of the sets \( H^p \)

Based on the works of Metzler and Senn (see [9] and [18] respectively) on the hyperbolic Mandelbrot set, we define the Hyperbrots as follows:

**Definition 6** Let \( Q_{p,c}(z) = z^p + c \) where \( z, c \in \mathbb{D} \) and \( p \geq 2 \) an integer. The Hyperbrots are defined as the sets

\[ H^p := \left\{ c \in \mathbb{D} \mid \{Q_{p,c}^m(0)\}_{m=1}^\infty \text{ is bounded } \forall m \in \mathbb{N} \right\}. \]  
(4.1)

Metzler proved that \( H^2 \) is a square with diagonal length \( 2\frac{1}{2} \) and of side length \( \frac{2}{3} \sqrt{2} \). We use the same approach to prove that \( H^3 \) is a square with diagonal length \( \frac{2}{3\sqrt{3}} \) and side length \( \frac{2}{3\sqrt{3}} \sqrt{2} \). For the next part, we note a hyperbolic numbers \( z \) as \( (u, v)^\top \) and the fixed number \( c \) as \( (a, b)^\top \) where \( ^\top \) is the transpose of a column vector in \( \mathbb{R}^2 \).

4.2 Special case: \( H^3 \)

First, we recall some of the basic tools introduced by Metzler.

**Definition 7** Let \( (u, v)^\top, (x, y)^\top \in \mathbb{R}^2 \). We define two multiplication operations \( \diamond \) and \( * \) on \( \mathbb{R}^2 \) as

\[ \begin{pmatrix} u \\ v \end{pmatrix} \diamond \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} ux + vy \\ vx + uy \end{pmatrix} \]  
(4.2)

\[
\begin{pmatrix} u \\ v \end{pmatrix} \ast \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} ux \\ vy \end{pmatrix}.
\] (4.3)

Remark 1 The operation \( \ast \) correspond to the multiplication \( \cdot \) of two hyperbolic numbers as we adopted the two dimensional vector notation. We use the symbols \( \ast_{on} \) and \( \ast_{on} \) to denote the \( n \) consecutive multiplications \( \ast \circ \ast \circ \cdots \circ \ast \) and \( \ast \circ \ast \circ \cdots \circ \ast \) respectively.

With the usual addition operation \(+\) on \( \mathbb{R}^2 \), \( (\mathbb{R}^2, +, \ast) \) and \( (\mathbb{R}^2, +, \ast) \) are commutative rings with unity.

Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the following matrix
\[
T := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.
\] (4.4)

Then, \( T \) is an isomorphism between \( (\mathbb{R}^2, +, \ast) \) and \( (\mathbb{R}^2, +, \ast) \). Now, we define
\[
H_{p,c} \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) := \begin{pmatrix} x \\ y \end{pmatrix} \ast_{op} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}
\] (4.5)
and we generalize a result that is included in the proof of Metzler for the case \( p = 2 \).

Lemma 5 For all \( m \in \mathbb{N} \), we have that
\[
T \mathbb{H}_{p,c}^m \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} Q_{a,b}^m(x-y) \\ Q_{a+b}^m(x+y) \end{pmatrix}
\] (4.6)
where \( T \) is the matrix of equation (4.4) and \( Q_{p,c}(z) = z^2 + c \) with \( z, c \in \mathbb{R} \).

The proof can be found in [13]. It is similar to the one of Metzler gave in his article (see [9]). We just replace \( \ast \) by \( \ast_{op} \), \( \ast \) by \( \ast_{op} \) and the degree 2 of \( P_{(a,b)} \) by the integer \( p \geq 2 \). Hence, Lemma 5 allows to separate the dynamics of \( H_{p,c} \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) \) in terms of the dynamics of two real polynomials \( Q_{p,a-b}(x-y) \) and \( Q_{p,a+b}(x+y) \).

Theorem 7 \( \mathcal{H}^3 = \left\{ c = (a, b)^\top \in \mathbb{R}^2 \mid |a| + |b| \leq \frac{2}{3\sqrt{3}} \right\} \).

Proof. By Lemma 5 and the remark right after, \( \{H_{p,c}^m(0)\}_{m=1}^{\infty} \) is bounded \( \forall m \in \mathbb{N} \) iff the real sequences \( \{Q_{3,a-b}^m(0)\}_{m=1}^{\infty} \) and \( \{Q_{3,a+b}^m(0)\}_{m=1}^{\infty} \) are bounded \( \forall m \in \mathbb{N} \). But, according to Theorem 6, these sequences are bounded iff
\[
|a-b| \leq \frac{2}{3\sqrt{3}} \text{ and } |a+b| \leq \frac{2}{3\sqrt{3}}
\] (4.7)
Then, by a simple computation we obtain \( |a| + |b| \leq \frac{2}{3\sqrt{3}} \). Conversely, if \( |a| + |b| \leq \frac{2}{3\sqrt{3}} \) is true, then by the properties of the absolute value, we obtain the inequalities in (4.7). Thus, we obtain the following characterization for \( \mathcal{H}^3 \), \( \mathcal{H}^3 = \left\{ c = (a, b)^\top \in \mathbb{R}^2 \mid |a| + |b| \leq \frac{2}{3\sqrt{3}} \right\} \).

Figure 2 represent an image of the set \( \mathcal{H}^3 \).
5 Generalized Mandelbrot sets for Tricomplex numbers

In this section, we use the set of tricomplex numbers to generalize the Multibrot sets. Particularly, we give some basic properties of the tricomplex Multibrot and we continue the exploration of the Multibrot started in [5]. We will concentrate our exploration on the case $M_3$ corresponding to the polynomial $Q_{3,c} = \eta^3 + c$ with $\eta, c \in M(3)$.

5.1 Definitions and properties of $M_3^p$

The authors of [23] defined the bicomplex Multibrot sets as follows:

**Definition 8** Let $Q_{p,c} = \zeta^p + c$ where $\zeta, c \in M(2)$ and $p \geq 2$ is an integer. The bicomplex Multibrot set is define as the set

$$M_2^p := \left\{ c \in M(2) \mid \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded} \right\}.$$

In [23], they proved that $M_2^p$ can be expressed as a $M(2)$-cartesian set and is connected. In the next theorem, we improve their result concerning the bounded discus of $M_2^p$ in conformity with our Theorem 2.

**Theorem 8** Let $M_3^p$ denote the bicomplex Multibrot sets for integers $p \geq 2$. Then, the following inclusions hold:

$$M_3^p \subset D_2(0, 2^{1/p^2}, 2^{1/p^2}) \subset B_2^1(0, 2^{1/p^2}).$$

(5.1)
Proof. We know from [23] that \( \mathcal{M}_2^p = \mathcal{M}_1^p \times_{\gamma_2} \mathcal{M}_2^p \). Moreover, by Theorem 2, we know that \( \mathcal{M}_1^p \subset \mathcal{B}_1^2(0, 2^{\frac{1}{p-1}}) \). So, combining the both last statements, we proved the left inclusion. For the right inclusion, we use this following inclusion \( \mathcal{D}_2(a, r_1, r_2) \subset \mathcal{B}_2^2(a, \sqrt{r_1^2 + r_2^2}) \) (see [1]) with \( a = 0 \) and \( r_1 = r_2 = 2^{\frac{1}{p-1}} \).

Now, the tricomplex Multibrot sets are defined analogously to the bicomplex ones:

**Definition 9** Let \( Q_{p,c} = \eta^p + c \) where \( \eta, c \in \mathbb{M}(3) \) and \( p \geq 2 \) an integer. The tricomplex Multibrot set is define as the set

\[
\mathcal{M}_3^p := \left\{ c \in \mathbb{M}(3) \mid \{ Q_{m,0}(0) \}_{m=1}^\infty \text{ is bounded} \right\}.
\]

We have the following theorem that characterize \( \mathcal{M}_3^p \) set as a \( \mathbb{M}(3) \)-cartesian product of \( \mathcal{M}_2^p \).

**Theorem 9** \( \mathcal{M}_3^p = \mathcal{M}_2^p \times_{\gamma_2} \mathcal{M}_2^p \).

Proof. Let \( c = c_1 + c_2 i_3 = (c_1 - c_2 i_2)\gamma_2 + (c_1 + c_2 i_2)\overline{\gamma}_2 \) as a tricomplex numbers. So, by Definition 9, \( c \in \mathcal{M}_3^p \) iff \( \{ Q_{m,0}(0) \}_{m=1}^\infty \) is bounded \( \forall m \in \mathbb{N} \). However, from Theorem 1, \( Q_{m,0}^n(0) \) can be expressed with the idempotent representation as follows:

\[
Q_{m,0}^n(0) = Q_{3,c_1-c_2 i_2}^m(0)\gamma_2 + Q_{3,c_1+c_2 i_2}^m(0)\overline{\gamma}_2
\]

for all \( m \in \mathbb{N} \). Moreover, in [1], it is proved for the general case of multicomplex numbers that:

\[
\|\zeta\|_n = \sqrt{\|\zeta_1 - \zeta_2 i_{n-1}\|^2_{n-1} + \|\zeta_1 + \zeta_2 i_{n-1}\|^2_{n-1}}
\]

where \( \zeta = \zeta_1 + \zeta_2 i_n \in \mathbb{M}(n) \). So, \( \{ Q_{m,0}^n(0) \}_{m=1}^\infty \) is bounded iff \( \{ Q_{3,c_1-c_2 i_2}^m(0) \}_{m=1}^\infty \) and \( \{ Q_{3,c_1+c_2 i_2}^m(0) \}_{m=1}^\infty \) are bounded. By Definition 8, we obtain that \( c_1 - c_2 i_2, c_1 + c_2 i_2 \in \mathcal{M}_2^p \). Thus, \( c = (c_1 - c_2 i_2)\gamma_2 + (c_1 + c_2 i_2)\overline{\gamma}_2 \in \mathcal{M}_2^p \times_{\gamma_2} \mathcal{M}_2^p \).

If we combine Theorem 9 with Theorem 2, we get the following statement.

**Theorem 10** Let \( \mathcal{M}_3^p \) be the tricomplex Multibrot set for \( p \in \mathbb{N}\setminus\{0, 1\} \). Then the following inclusion holds:

\[
\mathcal{M}_3^p \subset \mathcal{D}_3(0, 2^{\frac{1}{p-1}}, 2^{\frac{1}{p-1}}).
\]

Finally, in [23], it is proved that the sets \( \mathcal{M}_3^p \) is connected \( \forall p \in \mathbb{N}\setminus\{0, 1\} \). We obtain the same property for \( \mathcal{M}_3^p \).

**Theorem 11** \( \mathcal{M}_3^p \) is a connected set.

Proof. Let define the function \( \Gamma_2 : X_1 \times X_2 \rightarrow X_1 \times_{\gamma_2} X_2 \) with \( X_1, X_2 \subset \mathbb{M}(2) \) and \( X \subset \mathbb{M}(3) \) by \( \Gamma_2(u_1, u_2) = u_1\gamma_2 + u_2\overline{\gamma}_2 \). Obviously, \( \Gamma_2 \) is an homeomorphism. So, if \( X_1, X_2 \) are connected sets, then \( X \) is also a connected set. Thus, by Theorem 9, \( \mathcal{M}_3^p = \mathcal{M}_2^p \times_{\gamma_2} \mathcal{M}_3^p \) and since \( \mathcal{M}_2^p \) is connected (see [23]), \( \mathcal{M}_3^p \) is also a connected set \( \forall p \in \mathbb{N}\setminus\{0, 1\} \).

Theorem 10 is useful to generate the divergence layers of the tricomplex Multibrot sets. We use this information to draw the images of the next part. Moreover, we conjecture that the Fatou-Julia Theorem is true for tricomplex Multibrot sets and use it to give more information about the topology of the sets. For a statement of the generalized Fatou-Julia Theorem, we refer the reader to [5].
5.2 Principal 3D slices of the sets $M_3^3$

We want now to visualize the tricomplex Multibrot sets. Since there are in eight dimensions, we take the same approach from [5] to accomplish this goal. In that way, we may denote the principal 3D slice for a specific tricomplex Multibrot set as $T^p$ and define it as the set

$$ T^p := T^p(i_m, i_k, i_l) = \left\{ c \in \mathbb{R}^5(i_m, i_k, i_l) \mid \{Q_m^c(0)\}_{m=1}^{\infty} \text{ is bounded} \right\} . $$

So the number $c$ has three of its components that are not equal to zero. In total, there is 56 possible combinations of principal 3D slices. To attempt a classification of these slices, we introduce a relation $\sim$ (see [5]).

**Definition 10** Let $T^p_1(i_m, i_k, i_l)$ and $T^p_2(i_m, i_q, i_s)$ two 3D slices of a tricomplex Multibrot set $M^p_3$ that correspond respectively to $Q_{p,c_1}$ and $Q_{p,c_2}$. Then, $T^p_1 \sim T^p_2$ if there exist a bijective function $\varphi : \text{span}_g \{1, i_m, i_k, i_l\} \to \text{span}_g \{1, i_m, i_q, i_s\}$ such that $(\varphi \circ Q_{p,c_1} \circ \varphi^{-1})(\eta) = Q_{p,c_2}(\eta) \forall \eta \in \text{span}_g \{1, i_m, i_q, i_s\}$. In that case, we say that $T^p_1$ and $T^p_2$ have the same dynamics.

If two 3D slices are in relationship in term of $\sim$, then we also say that they are symmetrical. This comes from the fact that their visualizations by a computer give the same images. In [13], it is showed that $\sim$ is also a equivalent relation on the set of 3D slices of $M^p_3$. For the rest of this article, we focus on the principal slices of the $M^p_3$ set, also called the tricomplex Mandelbrot set.

Garant-Pelletier and Rochon [5] showed that $M^3_3$ has eight principal 3D slices. So, according to (5.6) and the eight principal slices defined in [5], we have the next lemma that correspond to the first case discussed in the section 2, i.e. iterates of $Q_{3,c}(0)$ for $m \in \mathbb{N}$ are closed in the set $\mathbb{M}(i_k, i_l)$.

**Lemma 6** (Parisé [13]) We have the following symmetries in term of $\sim$:

1. $T^3(1, i_1, i_2) \sim T^3(1, i_k, i_l) \forall i_k, i_l \in \{i_1, i_2, i_3, i_4\}$;
2. $T^3(1, i_1, i_2) \sim T^3(1, i_j, i_3) \sim T^3(1, i_2, i_j)$;
3. $T^3(i_1, i_2, i_1) \sim T^3(i_k, i_k, i_l)$ for all $i_k, i_l \in \{i_1, i_2, i_3, i_4\}$ and
4. $T^3(1, i_1, i_1) \sim T^3(1, i_k, i_1)$ for $i_k \in \{i_1, i_2, i_3, i_4\}$ and $i_1 \in \{i_1, i_2, i_3, i_4\}$.

Figures 3(a), 3(b), 3(c) and 3(d) illustrates one slices in the four classes of 3D slices of Lemma 6. It seems that figures 3(a) and 3(c) look the same where these correspond to slices $T^3(1, i_1, i_2)$ and $T^3(i_1, i_2, i_1)$. Indeed, we have the next lemma that attest this remark.

**Lemma 7** Slices $T^3(1, i_1, i_2)$ and $T^3(i_1, i_2, i_1)$ have the same dynamics in the sense of the relation $\sim$.

**Proof.** Set the numbers $c$ and $c'$ and also the function $\varphi : \mathbb{M}(i_1, i_2) \to \mathbb{M}(i_1, i_2)$ as

$$ c = c_1 + c_2 i_1 + c_3 i_2, \quad c' = c_2 i_1 + c_3 i_2 + c_1 i_1 $$

and

$$ \eta = \varphi(x_1 + x_2 i_1 + x_3 i_2 + x_4 i_1) = x_4 + x_2 i_1 + x_3 i_2 + x_4 i_1. $$
Figure 3: Four 3D slices of the Mandelbric
So,
\[
(\varphi \circ Q_{3,c} \circ \varphi^{-1})(\eta) = \varphi \left( (x_1^3 - 3x_1x_2^2 - 3x_1x_3^2 + 3x_1x_4^2 + 6x_2x_3x_4 + c_1) + (-x_2^3 + 3x_1^2x_2 - 3x_2x_3^2 + 3x_2x_4^2 - 6x_1x_3x_4 + c_2)i_1 \\
+ (-x_3^3 + 3x_1^2x_3 - 3x_2^2x_3 + 3x_3x_4^2 - 6x_1x_2x_4 + c_3)i_2 \\
+ (x_4^3 + 3x_1^2x_4 - 3x_2^2x_4 - 3x_3^2x_4 + 6x_1x_2x_3)j_1 \right) = (x_1^3 + 3x_1^2x_2 - 3x_1x_3^2 + 3x_1x_4^2 - 6x_2x_3x_4 + c_1)i_1 \\
+ (-x_2^3 + 3x_1^2x_2 - 3x_2x_3^2 + 3x_2x_4^2 - 6x_1x_3x_4 + c_2)i_2 \\
+ (x_3^3 + 3x_1^2x_3 - 3x_2^2x_3 + 3x_3x_4^2 - 6x_1x_2x_4 + c_3)i_2 \\
+ (x_4^3 - 3x_1x_2^2 - 3x_1x_3^2 + 3x_1x_4^2 + 6x_2x_3x_4 + c_1)j_1
\]

Thus, by Definition 10, we have the result. \(\square\)

Because \(\sim\) is an equivalence relation, by Lemmas 6 and 7, we have find the first principal slice of \(M_3^4\), we will call this slice the Tetrahedron. Now, for slices that correspond to the second case (where the iterates of \(Q_{3,c}^{4}(0)\) are not closed in \(M(k, i, i)\)) we have a lemma similar to Lemma 6. However, when \(p = 3\), the iterates of \(Q_{3,c}^{4}(0)\) are closed in \(M(k, i, i, i)\) (see section 2).

**Lemma 8** We have the following symmetries:

1. \(T^3(i_1, i_2, i_3) \sim T^3(k, i_1, i_m)\) for \(i_1, i_2, i_m \in \{i_1, i_2, i_3, i_4\}\);

2. Every slices of the form \(T^3(k, i_1, i_m)\) where \(i_ki_1 \neq i_m, i_k, i_1 \in \{i_1, i_2, i_3, i_4\}, i_k \neq i_1\) and \(i_m \in \{i_1, i_2, i_3\}\). Precisely,

\[
\begin{align*}
T^3(i_1, i_2, i_3) & \sim T^3(i_1, i_2, j_3) \sim T^3(i_1, i_3, j_3) \sim T^3(i_1, i_4, j_1) \\
& \sim T^3(i_1, i_4, j_2) \sim T^3(i_2, i_3, j_3) \sim T^3(i_2, i_4, j_1) \\
& \sim T^3(i_2, i_4, j_3) \sim T^3(i_3, i_4, j_3)
\end{align*}
\]

3. Every slices of the form \(T^3(k, j_1, j_m)\) where \(i_k \in \{i_1, i_2, i_3, i_4\}, j_1, j_m \in \{j_1, j_2, j_3\}\) and \(j_1 \neq j_m\). Precisely,

\[
\begin{align*}
T^3(i_1, j_1, j_2) & \sim T^3(i_1, j_1, j_3) \sim T^3(i_1, j_2, j_3) \sim T^3(i_2, j_1, j_3) \\
& \sim T^3(i_2, j_2, j_3) \sim T^3(i_3, j_1, j_3) \sim T^3(i_3, j_2, j_3) \\
& \sim T^3(j_4, j_1, j_3) \sim T^3(j_4, j_2, j_3)
\end{align*}
\]

4. \(T^3(j_1, j_2, j_3)\) with itself.

Proof of Lemma 8 can be found in [13]. The same ideas from the proof of Lemma 6 are used in the proof of Lemma 8 but instead of using the set \(M(k, i)\) we use the set \(M(k, i, i, i, i)\) to define the function \(\varphi\). Figures 4 illustrate one slice in each four classes of 3D slices of Lemma 8. From these figures, we remark that the classes of \(T^3(1, i_1, i_2)\) and \(T^3(i_1, i_2, j_2)\) generate the same images. We notice the same phenomenon for the slices \(T^3(1, j_1, j_1)\) and \(T^3(i_1, j_1, j_2)\) and also \(T^3(j_1, j_2, j_3)\) and \(T^3(j_1, j_2, j_3)\). Indeed, we have this next lemma.

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Figure 4: Four 3D slices of the Mandelbrick
Lemma 9 We have that

1. $T^3(1, i_1, i_2) \sim T^3(i_1, i_2, j_2)$;
2. $T^3(1, i_1, j_1) \sim T^3(i_1, j_1, j_2)$ and;
3. $T^3(j_1, j_2) \sim T^3(j_1, j_2, j_3)$.

Proof. We prove point 1) of this lemma. Let define $c$ and $c'$ as
\[ c = c_1 + c_2 + c_3 + c_4, \quad c' = c_2 + c_3 + c_4. \]

Now, let define $\varphi : M(i_1, i_2) \to M(i_1, i_2, j_2)$ as
\[ \eta = \varphi(x_1 + x_2 + x_3 + x_4) = x_2 + x_3 + x_4. \]

We obtain
\[ Q_{3,c}(\varphi^{-1}(\eta)) = (x_1^3 - 3x_1x_2^2 - 3x_1x_3^2 + 3x_1x_4^2 + 6x_2x_3x_4 + c_1) \]
\[ + (x_1 + x_2 + x_3 + x_4) \]
\[ + (x_1^3 + x_2^3 + x_3^3 + x_4^3 - 6x_1x_2x_3x_4 + c_2) \]
\[ - (x_1^3 + x_2^3 + x_3^3 + x_4^3 - 6x_1x_2x_3x_4 + c_2) \]
and
\[ Q_{3,c'}(\eta) = (x_1^3 - 3x_1x_2^2 - 3x_1x_3^2 + 3x_1x_4^2 + 6x_2x_3x_4 + c_2) \]
\[ + (x_1 + x_2 + x_3 + x_4) \]
\[ + (x_1^3 + x_2^3 + x_3^3 + x_4^3 - 6x_1x_2x_3x_4 + c_2) \]
\[ - (x_1^3 + x_2^3 + x_3^3 + x_4^3 - 6x_1x_2x_3x_4 + c_2) \]
Hence, by applying $\varphi$ on the expression of $Q_{3,c}$, we have that $(\varphi \circ Q_{3,c} \circ \varphi^{-1})(\eta) = Q_{3,c'}(\eta)$ for every $\eta \in M(i_1, i_2, j_2)$. Thus, $T^3(1, i_1, i_2) \sim T^3(i_1, i_2, j_2)$. For the second part, set the numbers $c$ and $c'$, and also the function $\varphi : M(i_1, j_1) \to M(i_1, j_1, j_2)$ as
\[ c = c_1 + c_2 + c_3, \quad c' = c_2 + c_3 + c_4 \]
and
\[ \eta = \varphi(x_1 + x_2 + x_3 + x_4) = x_2 + x_3 + x_4. \]

One can verify that $(\varphi \circ Q_{3,c'} \circ \varphi^{-1})(\eta) = Q_{3,c'}(\eta)$ for all $\eta \in M(i_1, j_1, j_2)$. Finally, for the last part, set the numbers $c, c'$ and the function $\varphi : M(j_1, j_2) \to M(j_1, j_2, j_3)$ as
\[ c = c_1 + c_2 + c_3, \quad c' = c_1 + c_2 + c_3 \]
and
\[ \eta = \varphi(x_1 + x_2 + x_3 + x_4) = x_2 + x_3 + x_4. \]
Thus, $(\varphi \circ Q_{3,c'} \circ \varphi^{-1})(\eta) = Q_{3,c'}(\eta)$ for every $\eta \in M(j_1, j_2, j_3)$. \qed
Corollary 2  There are four principal 3D slices of the tricomplex Mandelric :
1. \( T^3(1,i_1,i_2) \) called Tetrablic;
2. \( T^3(1,j_1,j_2) \) called Perplexbric;
3. \( T^3(1,i_1,j_1) \) called Hourglassbric;
4. \( T^3(i_1,i_2,j_3) \) called Metabric.

We now treat the second case of Corollary 2 and we show that the Perplexbric is an octahedron of edges \( \frac{2\sqrt{2}}{3\sqrt{3}} \).

5.3 Special case: slice \( T^3(1,j_1,j_2) \)

We had proved in section 4 that the hyperbolic Mandelric (called the Hyperbric) is a square of edges \( \frac{2\sqrt{2}}{3\sqrt{3}} \) (see Theorem 7). Now, our interest is to generalize the Hyperbric in three dimensions. Let adopt the same notation as in [5] for the Perplexbric

\[
P^3 := T^3(1,j_1,j_2) = \left\{ c = c_1 + c_4 j_1 + c_6 j_2 | c_1 \in \mathbb{R} \text{ and } \{Q_{3,c}^n(0)\}_{n=1}^{\infty} \text{ is bounded} \right\}.
\]

Before proving this result, we need this next lemma.

Lemma 10  We have the following characterization of the Perplexbric

\[
P^3 = \bigcup_{y \in \left[\frac{\sqrt{2}}{\sqrt{3}} \pm \frac{\sqrt{6}}{2} \right]} \left\{ T^3 - yj_1 \cap (T^3 + yj_1) \right\} + yj_2
\]

where \( T^3 \) is the Hyperbric (see section 4).

Proof.  By Definition of \( P^3 \) and the idempotent representation, we have that

\[
P^3 = \left\{ c = (d - c_6 j_1) \gamma_2 + (d + c_6 j_1) \gamma_2 | \{Q_{3,c}^n(0)\}_{n=1}^{\infty} \text{ is bounded} \right\}
\]

where \( d = c_1 + c_4 j_1 \in \mathbb{D} \). Furthermore, the sequence \( \{Q_{3,c}^n(0)\}_{n=1}^{\infty} \) is bounded for all \( n \in \mathbb{N} \) iff the two sequences \( \{Q_{3,d-c_6j_1}^n(0)\}_{n=1}^{\infty} \) and \( \{Q_{3,d+c_6j_1}^n(0)\}_{n=1}^{\infty} \) are bounded for all \( n \in \mathbb{N} \). To continue, we make the following remark about hyperbolic dynamics : \( \forall z \in \mathbb{D} \)

\[
\mathcal{H}^3 - z := \left\{ c \in \mathbb{D} | \{Q_{3,c+z}^n(0)\}_{n=1}^{\infty} \text{ is bounded } \forall n \in \mathbb{N} \right\}.
\]

By Definition 6, \( \{Q_{3,d-c_6j_1}^n(0)\}_{n=1}^{\infty} \) and \( \{Q_{3,d+c_6j_1}^n(0)\}_{n=1}^{\infty} \) are bounded for all \( n \in \mathbb{N} \) iff \( d - c_6 j_1, d + c_6 j_1 \in \mathcal{H}^3 \). Therefore, by (5.9), we also have that \( d - c_6 j_1, d + c_6 j_1 \in \mathcal{H}^3 \) iff \( d \in (\mathcal{H}^3 - c_6 j_1) \cap (\mathcal{H}^3 + c_6 j_1) \). Hence,

\[
P^3 = \left\{ c = c_1 + c_4 j_1 + c_6 j_2 | c_1 + c_4 j_1 \in (\mathcal{H}^3 - c_6 j_1) \cap (\mathcal{H}^3 + c_6 j_1) \right\}
\]

\[
= \bigcup_{y \in \mathbb{R}} \left\{ (\mathcal{H}^3 - yj_1) \cap (\mathcal{H}^3 + yj_1) \right\} + yj_2.
\]
In fact, by Theorem 7, 
\[(H^3 - yj_1) \cap (H^3 + yj_1) = \emptyset\] 
whenever \(y \in \left[\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right]^c\). This conduct us to the desire characterization of the Perplexbric :

\[P^3 = \bigcup_{y \in \left[\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right]} \left\{([H^3 - yj_1] \cap (H^3 + yj_1)] + yj_2\right\}.
\]

As a consequence of Lemma 10 and Theorem 7, we have the following result :

**Theorem 12** \(P^3\) is an octahedron of edges \(\frac{2\sqrt{2}}{3 \sqrt{3}}\).

### 6 Conclusion

In this article, we have treated *Multibrot* sets for complex, hyperbolic and tricomplex numbers. Many results presented in this article can be generalized for arbitrary integers of degree \(p \geq 2\).

For the case of complex *Multibrot* sets, it would be grateful if we can grade-up the proof of Theorem 6 for all Multibrots. However, as we can see, the proof is increasing in level of technicality as the degree of the polynomial \(Q_{p,c}\) increase. So, we must find a different approach to prove the following next conjecture.

**Conjecture 1** Let \(M^p\) be the generalized Mandelbrot set for the polynomial \(Q_{p,c}(z) = z^p + c\) where \(z, c \in \mathbb{C}\) and \(p \geq 2\) an integer. Then, we have two cases for the intersection \(M^p \cap \mathbb{R}\) :

1. If \(p\) is even, then \(M^p \cap \mathbb{R} = \left[-2^{\frac{1}{p-1}}, (p - 1)p^{\frac{1}{p-1}}\right];\)

2. If \(p\) is odd, then \(M^p \cap \mathbb{R} = \left[-(p - 1)p^{\frac{1}{p-1}}, (p - 1)p^{\frac{1}{p-1}}\right].\)

This would conduct us to another conjecture about the *Hyperbrots*.

**Conjecture 2** The hyperbrots are squares and the following characterization of Hyperbrots hold:

1. If \(p\) is even, then \(H^p = \left\{c = a + bj \mid 2^{\frac{1}{p-1}} \leq a - b, a + b \leq (p - 1)p^{\frac{1}{p-1}}\right\};\)

2. If \(p\) is odd, then \(H^p = \left\{c = a + bj \mid |a| + |b| \leq (p - 1)p^{\frac{1}{p-1}}\right\}.

Further explorations of 3D slices of the tricomplex *Multibrot* sets are also planned. Particularly, we are interested in the case where the degree of the tricomplex polynomial is an integer \(p > 3\).

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