Abstract. We construct spectral triples on a class of particular inductive limits of matrix-valued function algebras. In the special case of the Jiang-Su algebra we employ a particular $AF$-embedding.

1. Introduction

According to the noncommutative differential geometry program [3, 4] both the topological and the metric information on a noncommutative space can be fully encoded as a spectral triple on the noncommutative algebra of coordinates on that space. Nowadays several noncommutative spectral triples have been constructed, with only a partial unifying scheme emerging behind some families of examples, e.g. quantum groups and their homogeneous spaces, like quantum spheres and quantum projective spaces. (see e.g., [5, 6, 9]) Also some preservation properties with respect to the product, inductive limits or extensions of algebras have been investigated.

Most of these constructions are still awaiting however a proper analysis of such properties as smoothness, dimension (summability) and other conditions selected by Connes. As a testing ground for these and related matters as large as possible class of examples should be investigated, including some important new algebras.

In [12] a general way to construct a spectral triple on arbitrary quasi diagonal $C^*$-algebras was exhibited. However, in that case one cannot expect summability. Instead, summability was obtained in [11] for certain inductive family of coverings, and $p$-summability with arbitrary $p$ for any AF-algebra through the construction in [2].

In the present paper we elaborate a construction that extends the latter mentioned approach to a wider class of particular inductive limits of matrix-valued function algebras whose connecting morphisms have a certain peculiar form. In particular this construction applies to the Jiang-Su algebra $\mathcal{Z}$ (cf. [8]), which was originally constructed in terms of an explicit particular inductive limit of dimension drop algebras. The aim therein was to obtain an example of an infinite-dimensional stably finite nuclear simple unital $C^*$-algebras with exactly one tracial state and with the same $K$-theory of the complex numbers. The importance of the Jiang-Su algebra $\mathcal{Z}$ stems from the fact that under some other hypothesis $\mathcal{Z}$-stability entails classification in terms of the Elliott invariant as proved in [15].

The organization of the paper is the following: In the first section we recall the definition of the Jiang-Su algebra and construct a particular $AF$-embedding for it. In the second section we compute the image of elements belonging to a dense subalgebra of the Jiang-Su algebra under the representation obtained by composing the forementioned $AF$-embedding with the representation appearing in [2]. In the last section we use the above
results to check that some of the Dirac operators considered in [2] give rise to a spectral triple for the Jiang-Su algebra.

2. Spectral triple on the Jiang-Su algebra

Let $B$ be an inductive limit of $C^*$-algebras $B = \lim(B_i, \phi_i)$, with $B_0 = \mathbb{C}$ and where every $B_i$ is a unital sub-$C^*$-algebra of the algebra of continuous-valued functions on the interval with values in $M_{n_i}$ for some natural numbers $n_i$ containing a dense $*$-subalgebra of Lipschitz functions and for $l > i$ natural numbers. The connecting morphisms $\phi_{i,i+l}$ take the form

\[
\phi_{i,i+l}(f) = u_{i,i+l} \begin{pmatrix} f \circ \xi_{i, i+k}^{i+1} \otimes 1_{N_i}^{i+1} & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f \circ \xi_{i, l}^{i+l} \otimes 1_{N_i}^{l+i} \end{pmatrix} \begin{pmatrix} u_{i,i+l}^* \\

\end{pmatrix}
\]

for some natural numbers $k_i^{i+1}$, $n_i^{i+1}$, ..., $n_i^{l+i}$, a unitary $u_{i,i+l} \in C([0,1], M_{n_i^{i+1}})$ and some paths $\xi_{i,1}^{i+1}, ..., \xi_{i,l}^{i+1}$ satisfying

\[
|\xi_{i,r}^{i+l}(x) - \xi_{i,r}^{i+l}(y)| \leq \frac{1}{2^l}, \quad \text{for } 1 \leq r \leq k_i^{i+1}, \ x, y \in [0,1].
\]

The operators $u_{i,i+l}$ are unitaries in $C([0,1], M_{n_i^{i+1}})$. The Jiang-Su algebra $Z$ is an inductive limit of prime dimension drop algebras $Z_i$ satisfying a certain universal property. We will use the original construction appearing in [3], where it was proven that given $p_i, q_i, n_i = p_i q_i$ defining the prime dimension drop algebra $Z_i$, there are numbers $k_i^{i+1}$, $k_i^{i+2}$ and $k_i^{i+3}$ such that $n_{i+1} = (k_i^{i+1} + k_i^{i+2} + k_i^{i+3})n_i$ is equal to $n_{i+1} = p_{i+1} q_{i+1}$ for some coprime numbers $p_{i+1}$ and $q_{i+1}$ and that there are a unitary $u_{i,i+1} \in C([0,1], M_{n_i^{i+1}})$ and natural numbers $N_i^{i+1}$, $N_i^{i+2}$ and $N_i^{i+3}$ such that

\[
Z_i \rightarrow Z_{i+1}
\]

\[
\phi_i : f \mapsto u_{i,i+1} \begin{pmatrix} f \circ \xi_{i, i}^{i+1} \otimes 1_{N_i}^{i+1} & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f \circ \xi_{i, l}^{i+1} \otimes 1_{N_i}^{l+i} \end{pmatrix} \begin{pmatrix} u_{i,i+1}^* \\

\end{pmatrix}
\]

is a connecting morphism for $\xi_1 = x/2$, $\xi_2 = 1/2$ and $\xi_3 = (x + 1)/2$. As a consequence, given a natural number $l$, the connecting morphism $Z_i \rightarrow Z_{i+l}$ has the form

\[
\phi_{i,i+l}(f) = u_{i,i+l} \begin{pmatrix} f \circ \xi_{i, i+k}^{i+1} \otimes 1_{N_i}^{i+1} & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f \circ \xi_{i, l}^{i+l} \otimes 1_{N_i}^{l+i} \end{pmatrix} \begin{pmatrix} u_{i,i+l}^* \\

\end{pmatrix}
\]
for some natural numbers \( k_i^{+l}, N_{i,1}^{+l}, \ldots, N_{i,k_i^{+l}}^{+l} \), a unitary \( u_{i,i+l} \in C([0,1], M_{n,i+l}) \) and some paths \( \xi_{i,1}^{+l}, \ldots, \xi_{i,k_i^{+l}}^{+l} \) that have the form

\[
\xi_{i,r}^{+l}(x) = \frac{x + r}{2^l} \quad \text{for} \quad 0 \leq r \leq 2^l - 1
\]

or

\[
\xi_{i,s}^{+l}(x) = \frac{s}{2^l} \quad \text{for} \quad 1 \leq s \leq 2^l - 1,
\]

It follows that the paths appearing in the connecting morphism \( \phi_{l,m} \) satisfy equation \( 2 \) and \( Z \) belongs to the class of inductive limit \( C^* \)-algebras we want to consider.

Note that, given \( B \) as above, after reindexing the sequence \( B_i \), for example sending \( i \mapsto 2i \) we can always suppose that the paths appearing in the connecting morphisms satisfy

\[
|\xi_{i,r}^{+l}(x) - \xi_{i,r}^{+l}(y)| \leq \frac{1}{2^l}
\]

for any \( 1 \leq r \leq k_i^{+l+1} \). This relation will be used for the proof of Lemma 2.1.

Fix a sequence of natural numbers \( n_i \) as above and consider the inductive limit \( A = \lim(A_i, \phi_i^o) \), where \( A_i = C([0,1], M_{n_i}) \) and the connecting morphisms \( \phi_i^o \) are constructed in the same way as above, but they are considered as unital \( * \)-homomorphisms between the \( A_i \)'s. For any \( i, l \in \mathbb{N} \) denote by \( \tilde{\phi}_{i,i+l}^o : A_i \to A_{i+l} \) the \( * \)-homomorphism

\[
\tilde{\phi}_{i,i+l}^o(f) = \begin{pmatrix} f \circ \xi_{i,1}^{+l} \otimes 1_{N_{i,1}^{+l}} & 0 \\ 0 & \ddots \\ 0 & f \circ \xi_{i,k_i^{+l}}^{+l} \otimes 1_{N_{i,k_i^{+l}}}^{+l} \end{pmatrix}.
\]

Let \( u_i \) be the unitary corresponding to the connecting morphism \( A_1 \to A_i \) (or \( B_1 \to B_i \)).

For any \( f \in A_i \) (or \( B_i \)) there is a unique \( \tilde{f} \in A_i \) such that \( f = u_i \tilde{f} u_i^* \). In this way the connecting morphisms take the form

\[
\phi_{i,i+l}^o(f) = u_{i,i+l} \tilde{\phi}_{i,i+l}^o(f) u_{i,i+l}^* = u_{i,i+l} \tilde{\phi}_{i,i+l}^o(\tilde{f}) u_{i,i+l}^*.
\]

Let now \( M = \lim(M_{n_i}, \psi_i) \), where \( \psi_i(a) = a \otimes 1_{n_{i+1}/n_i} \).

**Lemma 2.1.** There is a \( * \)-isomorphism

\[
\alpha : A \to M.
\]

Let \( \gamma \in (1,2) \). A Lipschitz function \( f \in A_i \) with Lipschitz constant \( L_f < \gamma^l \) is sent to

\[
\alpha(f) = \lim_{m \to \infty} \psi_m^\infty(\tilde{\phi}_{i,m}^o(\tilde{f})(0)).
\]
Proof. Define *-homomorphisms

\[(13) \quad \alpha_i : A_i \to M_{n_{i+1}} \]

\[(14) \quad f \mapsto \tilde{\phi}_i(f)(0) \]

and

\[(15) \quad \beta_i : M_{n_i} \to A_i \]

\[(16) \quad a \mapsto u_{i+1} \bar{a} u_{i+1}^* , \]

where \( \bar{a} \in A_i \) is the constant matrix-valued function taking value \( a \in M_{n_i} \). Let now \( \gamma \in (1, 2) \) and take finite sets \( F_i \subset A_i \) consisting of Lipschitz matrix-valued functions with Lipschitz constant less than \( \gamma^i \) and such that their union \( \bigcup_i F_i \) is dense in \( A \). For any \( f \in F_i \) and \( a \in M_{n_i} \) we have

\[(17) \quad \alpha_i \circ \beta_i(a) = \psi_i(a), \]

\[(18) \quad \| \beta_{i+1} \circ \alpha_i(f) - \phi^i_{i,i+1}(f) \| < \frac{\gamma_i}{2}. \]

Hence the result follows by [11] Proposition 2.3.2. \( \square \)

3. The orthogonal decomposition

Let \( \mathcal{H} \) be the Hilbert space considered by Christensen and Antonescu in [2] corresponding to the GNS-representation induced by the unique trace \( \tau \) on \( M \). This trace is given on the finite-dimensional approximants relative to the inductive limit construction by the normalized trace on matrices. Following [2] we want to write \( \mathcal{H} \) as an infinite direct sum of the finite dimensional Hilbert spaces on which the \( M_{n_i} \)'s are represented.

Let \( \mathcal{H}_i = \overline{M_{n_i}} \) and let \( v \in \mathcal{H}_i \). We can consider \( v \) as a matrix of dimension \( n_i \) and for any \( j < i \), we can write \( v \) as a matrix-valued matrix of the form

\[(19) \quad v = \begin{pmatrix} v_{1,1}^{j,i} & \ldots & v_{1,l_j}^{j,i} \\
\vdots & \ddots & \vdots \\
v_{l_j,j}^{j,i} & \ldots & v_{l_j,l_j}^{j,i} \end{pmatrix}, \]

where \( l_j = n_i / n_j \) is the multiplicity of \( M_{n_j} \) in \( M_{n_i} \) and the \( v_{k,l}^{j,i} \) are matrices in \( M_{n_j} \); in particular we can apply the same procedure to these matrices by iteration. With this notation, the projection \( P_{i,j} \) from \( \mathcal{H}_i \) to \( \mathcal{H}_j \) reads
If \( i > 1 \), the projection \( R_j \) from \( \mathcal{H}_j \) to the orthogonal complement of \( \mathcal{H}_{j-1} \) in \( \mathcal{H}_j \) reads for \( w \in \mathcal{H}_j \):

\[
P_{i,j}(v) = \frac{1}{\ell_j} \sum_{k=1}^{\ell_j} v_{k,k}^{j,i} \in M_{n_j}.
\]

Hence, if we denote by \( \mathcal{R}_i = \mathcal{H}_i \oplus \mathcal{H}_{i-1} \), the projection \( Q_j : \mathcal{H_j} \to \mathcal{R}_j \), when applied to an element \( v \in \mathcal{H}_i \) takes the form, for \( 1 \leq s, t \leq \ell_j 

\[
(Q_j(v))_{s,t}^{j-1} = \begin{cases} 
\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} (v_{k,k}^{j,i})_{s,s}^{j-1,j} - \frac{1}{\ell_j-1} \sum_{l=1}^{\ell_j-1} \sum_{k=1}^{\ell_j} (v_{k,k}^{j,i})_{t,t}^{j-1,j} & \text{for } s = t \\
\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} (v_{k,k}^{j,i})_{s,t}^{j-1,j} & \text{for } s \neq t
\end{cases}.
\]

### 4. The commutators

Take \( i < n < m \) and \( v \in \mathcal{H}_m \), \( f \in A_i \). We want to compute the elements \( Q_n(\tilde{\phi}_{i,n}^o(\tilde{f})(0)v) \) and \( \tilde{\phi}_{i,n}^o(\tilde{f})(0)Q_n v \). To this end we want to write \( \tilde{\phi}_{i,n}^o(\tilde{f}) \) as the composition \( \tilde{\phi}_{n,m}^o \circ \tilde{\phi}_{n-1,n}^o \circ \tilde{\phi}_{n-1}^o(\tilde{f}) \).

Let \( k_j^n \) be the amount of different paths appearing in the connecting morphism \( \phi_{j,i} \). If \( 1 \leq j \leq k_{n-1}^n \), we denote by \( \tilde{f} \circ [\xi_{i,j}^{n-1}] \circ \xi_{n-1,j}^n = \tilde{\phi}_{i,n-1}^o(\tilde{f}) \circ \xi_{n-1,j}^n \) the matrix-valued function

\[
(\tilde{f} \circ \xi_{i,1}^{n-1} \circ \xi_{n-1,j}^n \otimes 1_{N_{i,j}^{n-1}})
\]

\[
\vdots
\]

\[
0
\]

\[
\tilde{f} \circ \xi_{i,k_{n-1}^n}^{n-1} \circ \xi_{n-1,j}^n \otimes 1_{N_{i,k_{n-1}^n}^{n-1}}
\]

then we can write

\[
\tilde{\phi}_{i,n}^o(\tilde{f}) = \tilde{\phi}_{n,n-1}^o \circ \tilde{\phi}_{i,n-1}^o(\tilde{f}) =
\]
\[
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
\end{pmatrix}
\]

For \(1 \leq s \leq l_{n-1}^n\), we denote by \(\tilde{\xi}_{n-1,s}^n\) the path
\[
\tilde{\xi}_{n-1,s}^n = \begin{cases} 
\xi_{n-1,1}^n & \text{for } 1 \leq s \leq N_{n-1,1}^n \\
\xi_{n-1,2}^n & \text{for } N_{n-1,1}^n < s \leq N_{n-1,1}^n + N_{n-1,2}^n \\
\vdots & \\
\xi_{n-1,k_{n-1}^n} & \text{for } \sum_{k=1}^{k_{n-1}^n} N_{n-1,k} < s \leq l_{n-1}^n 
\end{cases}
\]

Thus we obtain for \(1 \leq s, t \leq l_{n-1}^n\),
\[
(Q_n \tilde{\phi}^\circ_{1,m} (\tilde{f})(0) v)_{s,t}^{n-1,n} = (Q_n (\tilde{\phi}^\circ_{n,m} \circ \tilde{\phi}^\circ_{n-1,n} \circ \tilde{\phi}^\circ_{1,n-1})(\tilde{f})(0) v)_{s,t}^{n-1,n} = \\
\frac{1}{l_{n-1}^m} \sum_{j=1}^{l_{n-1}^m} (\tilde{f} \circ [\xi_{i}^{n-1}] \circ \tilde{\xi}_{n-1,s}^n)(0) (v_{j,j}^{n,m})_{s,s}^{n-1,n} - \frac{1}{l_{n-1}^m} \sum_{k=1}^{l_{n-1}^m} (v_{j,j}^{n,m})_{k,k}^{n-1,n}
\]

and
\[
(Q_n \tilde{\phi}^\circ_{1,m} (\tilde{f})(0) v)_{s,t}^{n-1,n} = (Q_n (\tilde{\phi}^\circ_{n,m} \circ \tilde{\phi}^\circ_{n-1,n} \circ \tilde{\phi}^\circ_{1,n-1})(\tilde{f})(0) v)_{s,t}^{n-1,n} = \\
\frac{1}{l_{n-1}^m} \sum_{j=1}^{l_{n-1}^m} (\tilde{f} \circ [\xi_{i}^{n-1}] \circ \tilde{\xi}_{n-1,s}^n)(0) (v_{j,j}^{n,m})_{s,t}^{n-1,n} \quad \text{for } s \neq t
\]

In the same way, for \(1 \leq j \leq l_{n}^m\), we can define paths
\[
\tilde{\xi}_{n,j} = \begin{cases} 
\xi_{m,n,1}^m & \text{for } 1 \leq j \leq N_{n,1}^m \\
\xi_{m,n,2}^m & \text{for } N_{n,1}^m < j \leq N_{n,1}^m + N_{n,2}^m \\
\vdots & \\
\xi_{m,n,k_{n}^m}^m & \text{for } \sum_{k=1}^{k_{n}^m} N_{n,k} < j \leq l_{n}^m 
\end{cases}
\]

and compute for \(1 \leq s, t \leq l_{n-1}^n\),
\[
(Q_n \tilde{\phi}^\circ_{1,m} (\tilde{f})(0) v)_{s,t}^{n-1,n} = (Q_n (\tilde{\phi}^\circ_{n,m} \circ \tilde{\phi}^\circ_{n-1,n} \circ \tilde{\phi}^\circ_{1,n-1})(\tilde{f})(0) v)_{s,t}^{n-1,n} = \\
\frac{1}{l_{n-1}^m} \sum_{j=1}^{l_{n-1}^m} (\tilde{f} \circ [\xi_{i}^{n-1}] \circ \tilde{\xi}_{n-1,s}^n)(0) (v_{j,j}^{n,m})_{s,t}^{n-1,n} \quad \text{for } s \neq t
\]

and
This is a consequence of the fact that
\begin{equation}
1 \sum_{j=1}^{l_n} (\tilde{f} \circ [\xi_{i}^{n-1}] \circ \tilde{\xi}_{n-1,s} \circ \tilde{m}_{n,j})(0)(v_{j,j}^{n,m})_{s,s} \nonumber
\end{equation}
\begin{equation}
- \frac{1}{l_n-1} \sum_{k=1}^{l_n-1} (\tilde{f} \circ [\xi_{i}^{n-1}] \circ \tilde{\xi}_{n-1,k} \circ \tilde{m}_{n,j})(0)(v_{j,j}^{n,m})_{s,s} \nonumber
\end{equation}
for \( s = t \).

Thus we can write the commutators
\begin{equation}
(Q_n(\tilde{\phi}_{i,n}(\tilde{f})(0)v) - \tilde{\phi}_{i,n}(\tilde{f})(0)Q_nv)_{n-1} = \nonumber
\end{equation}
\begin{equation}
\frac{1}{l_n} \sum_{j=1}^{l_n} (\tilde{f} \circ [\xi_{i}^{n-1}] \circ \tilde{\xi}_{n-1,s} \circ \tilde{m}_{n,j} - \tilde{f} \circ [\xi_{i}^{n-1}] \circ \tilde{\xi}_{n-1,s} \circ \tilde{m}_{n,j})(0)(v_{j,j}^{n,m})_{s,t} \nonumber
\end{equation}
and
\begin{equation}
\frac{1}{l_n} \sum_{j=1}^{l_n} ([\tilde{f} \circ [\xi_{i}^{n-1}] \circ \tilde{\xi}_{n-1,s} \circ \tilde{m}_{n,j} - \tilde{f} \circ [\xi_{i}^{n-1}] \circ \tilde{\xi}_{n-1,s} \circ \tilde{m}_{n,j})(0)(v_{j,j}^{n,m})_{s,t} + \nonumber
\end{equation}
\begin{equation}
\frac{1}{l_n-1} \sum_{k=1}^{l_n-1} (\tilde{f} \circ [\xi_{i}^{n-1}] \circ \tilde{\xi}_{n-1,k} \circ \tilde{m}_{n,j} - \tilde{f} \circ [\xi_{i}^{n-1}] \circ \tilde{\xi}_{n-1,k} \circ \tilde{m}_{n,j})(0)(v_{j,j}^{n,m})_{s,t} \nonumber
\end{equation}
for \( s = t \).

**Lemma 4.1.** Let \( i < l < m \leq k \) be natural numbers and let \( \xi_{i}^{l}, \xi_{i}^{m}, \xi_{i}^{k} \) be paths on the interval \([0, 1]\) such that
\begin{equation}
|\xi_{i}^{l}(x) - \xi_{i}^{l}(y)| \leq \frac{1}{2^{l-i}}, \quad \text{for any } x, y \in [0, 1]. \nonumber
\end{equation}

Then, given any \( n > 0 \) and any Lipschitz function in \( C([0, 1], M_{n}) \) with Lipschitz constant \( L_{f} \), we have
\begin{equation}
\|(f \circ \xi_{i}^{l} \circ \xi_{i}^{m})(0) - (f \circ \xi_{i}^{l} \circ \xi_{i}^{k})(0)\| \leq \frac{2^{l}L_{f}}{2^{l}}. \nonumber
\end{equation}

**Proof.** This is a consequence of the fact that \( |\xi_{i}^{l}(x) - \xi_{i}^{l}(y)| \leq \frac{1}{2^{l-i}} \) for every \( x, y \in [0, 1] \). □

5. The spectral triple

Note that if \( D = \sum_{n} \alpha_{n}Q_{n} \) for a certain sequence of real numbers \( \{\alpha_{n}\} \), then the domain of \( D \), \( \text{dom}(D) = \{v \in H : \{\|\alpha_{n}Q_{n}v\| \in l^2(\mathbb{N})\} \) is left invariant under the action of any \( f \in A \); thus in particular, for every \( f \in B \) and it makes sense to consider the (in general unbounded) operator \([D, f]\).
Moreover, it follows from the Hann-Banach extension Theorem, that if \( T \) is an unbounded operator on \( \mathcal{H} \) whose domain contains the algebraic direct sum \( \oplus_{alg} \mathcal{R}_i \) and \( \| TP_n \| \) is uniformly bounded on \( n \), then \( T \) extends (uniquely) to a bounded operator on the whole Hilbert space \( \mathcal{H} \).

Hence, to obtain boundedness of \([D,f]\), we want to compute estimates for \( \| [D,f]P_n \| \) for every \( n \).

For every \( i \in \mathbb{N} \) we will denote by \( LB_i \) the linear subspace of \( B_i \) consisting of Lipschitz functions with Lipschitz constant smaller than \( \gamma^i \) for some \( \gamma \in (1,2) \). Observe that \( \phi_\alpha|_{LB_i} \) is a linear map sending \( LB_i \) into \( LB_{i+1} \) and that the algebraic direct limit \( \bigcup_i LB_i \) is a dense *-subalgebra of \( B \).

**Theorem 5.1.** Let \( D = \sum_n \alpha_n Q_n \), with \( \{ \alpha_n \} \) a diverging sequence of real numbers satisfying \( \alpha_0 = 0 \), \( |\alpha_n| \leq \beta^{2(n-1)} \) with \( \beta < 2 \) and \( n > 0 \). Then \( (\bigcup_i LB_i, \mathcal{H}, D) \) is a spectral triple for \( \mathcal{H} \).

It is \( p \)-summable whenever the sequences of numbers \( \{\alpha_i\} \), \( \{n_i\} \) satisfy

\[
\sum_{i \geq 1} (1 + \alpha_i^2)^{-p/2} (n_i^2 - n_{i-1}^2) < \infty
\]

for some \( p > 0 \).

**Proof.** After reindexing \( i \to 2i \), the *-isomorphism \( \alpha : A \to M \) has the concrete description given in Lemma 2.1. Thus we can compose it with the GNS representation of \( M \) induced by the unique trace \( \tau \).

Let \( l, m \in \mathbb{N} \) and \( v \in \mathcal{H}_l \). Denote by \( \beta^0_{l,m} : \mathcal{H}_l \to \mathcal{H}_m \) and \( \beta^0_{l,\infty} : \mathcal{H}_l \to \mathcal{H} \) the connecting isometries. Note that for \( i < n \in \mathbb{N} \) and \( f \in LB_i \) the action of \( f \) on \( v \) reads

\[
\lim_{m \to \infty} \beta^0_{m,\infty} \tilde{\phi}_{i,m}^o (\tilde{f})(0) \beta^0_{l,m} v,
\]

where we use the convention that \( \tilde{\phi}_{l,m} = \text{id} \) for \( m \leq l \) and \( \beta^0_{l,m} = \text{id} \) for \( m \leq l \). Thus we can write

\[
\|Q_n f v - f Q_n v\| = \|\beta^0_{m,\infty} Q_n \lim_{m \to \infty} \tilde{\phi}_{i,m}^o (\tilde{f})(0) \beta^0_{l,m} v - \lim_{m \to \infty} \beta^0_{m,\infty} \tilde{\phi}_{i,m}^o (\tilde{f})(0) \beta^0_{n,m} Q_n v\|.
\]

Since the sequence \( \beta^0_{m,\infty} \tilde{\phi}_{i,m}^o (\tilde{f})(0) \beta^0_{l,m} v \) converges, there is an \( M \) such that

\[
\|\beta^0_{i,k} \tilde{\phi}_{i,k}^o (\tilde{f})(\tilde{f})(0) \beta^0_{l,m} v - \lim_{m \to \infty} \beta^0_{m,\infty} \tilde{\phi}_{i,m}^o (\tilde{f})(0) \beta^0_{l,m} v\| \leq \frac{1}{2^{2(n-1)}}
\]

for any \( k \geq M \). Moreover, by Lemma 4.1 and the discussion preceding it

\[
\|\beta^0_{n,m} \tilde{\phi}_{i,n}^o (\tilde{f})(0) - \tilde{\phi}_{i,m}^o (f)(0) \beta^0_{n,m} Q_n v\| \\
= \|\beta^0_{n,m} (\tilde{\phi}_{i,n}^o (\tilde{f})(0) - \tilde{\phi}_{i,m}^o (f)(0)) \beta^0_{n,m} Q_n v\| \leq \frac{2^{2L} f}{2^{2(n-1)}}
\]
for $m > n$ and
\begin{equation}
Q_n \tilde{\phi}_i^o(\tilde{f})(0) \beta_{l,M}^0 v - \tilde{\phi}_{i,n}(\tilde{f})(0) Q_n \beta_{l,M}^0 v \parallel \leq \frac{2^i L f}{2^{2(n-1)}}.
\end{equation}

We can suppose $M > n$ and obtain
\begin{equation}
Q_n \lim_{m \to \infty} \beta_{n,m}^0 \tilde{\phi}_{i,m}^o(\tilde{f})(0) \beta_{l,m}^0 v - \lim_{m \to \infty} \beta_{m,n}^0 \tilde{\phi}_{i,m}^o(\tilde{f})(0) \beta_{l,m}^0 v \parallel \leq \frac{2^i L f}{2^{2(n-1)}}.
\end{equation}

Thus we obtain
\begin{equation}
\parallel [\alpha_n Q_n, f] P_m \parallel \leq \frac{|\alpha_n|(1 + 2^{2i+1} L f)}{2^{2(n-1)}} \leq (1 + 2^{2i+1} L f)(\beta/2)^{2(n-1)}.
\end{equation}

Hence
\begin{equation}
\parallel [D, f] \parallel \leq \parallel \sum_{n=1}^i \alpha_n Q_n, f \parallel + \parallel \sum_{n > i} \alpha_n Q_n, f \parallel \leq 2 \parallel f \parallel \sum_{n=1}^i |\alpha_n| + (1 + 2^{2i+1} L f) \sum_{n > i} (\beta/2)^{2(n-1)} < \infty
\end{equation}

and $[D, f]$ extends to a bounded operator.
Moreover $D$ has compact resolvent since it has discrete spectrum and its eigenvalues have finite multiplicity. Suppose we have sequences $\{\alpha_i\}, \{n_i\}$ and a real number $p > 0$ as in the statement. Then
\begin{equation}
\text{Tr}((1 + D^2)^{-p/2}) = 1 + \sum_{i \geq 1} (1 + \alpha_i^2)^{-p/2}(n_i^2 - n_{i-1}^2) < \infty.
\end{equation}

As the final comment we observe that by looking at the growth of the dimensions of the matrix algebras appearing in the original construction of the Jiang-Su algebra (cfr. [8]), it is clear that (51) can not be satisfied and the spectral triples exhibited above are not $p$-summable. Also, with the help of Stirling formula it can be seen that $\text{Tr} \exp(-D^2)$ diverges and thus the $\theta$-summability does not hold either.

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