Research Article

Metric Dimension Threshold of Graphs

Meysam Korivand, Kazem Khashyarmanesh, and Mostafa Tavakoli

1Department of Pure Mathematics, Faculty of Mathematical Sciences and Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran
2Department of Applied Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran

Correspondence should be addressed to Kazem Khashyarmanesh; khashyar@ipm.ir

Received 1 June 2022; Revised 16 August 2022; Accepted 22 August 2022; Published 14 September 2022

Academic Editor: Faranak Farshadifar

Copyright © 2022 Meysam Korivand et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let $G$ be a connected graph. A subset $S$ of vertices of $G$ is said to be a resolving set of $G$, if for any two vertices $u$ and $v$ of $G$ there is at least a member $w$ of $S$ such that $d(u, w) \neq d(v, w)$. The minimum number $t$ that any subset $S$ of vertices $G$ with $|S| = t$ is a resolving set for $G$, is called the metric dimension threshold, and is denoted by $\dim th(G)$. In this paper, the concept of metric dimension threshold is introduced according to its application in some real-world problems. Also, the metric dimension threshold of some families of graphs and a characterization of graphs $G$ of order $n$ for which the metric dimension threshold equals $2$, $n - 2$, and $n - 1$ are given. Moreover, some graphs with equal the metric dimension threshold and the standard metric dimension of graphs are presented.

1. Introduction

Throughout this paper, all graphs considered are assumed to be finite, simple, undirected, and connected. For a graph $G$, the vertex set and the edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. We remind that all notations and terminologies are standard here and taken mainly from the standard books of graph theory. For instance as usual, we denote the path, the cycle, the star, and the complete graph on $n$ vertices by $P_n$, $C_n$, $S_n$, and $K_n$, respectively. The distance between two vertices $u$ and $v$, denoted by $d(u, v)$, is the number of edges in a shortest path from $u$ and $v$. Let $G$ be a graph with $v \in V(G)$. The eccentricity $\epsilon(v)$ is the distance between $v$ and a vertex farthest from $v$ in $G$. The diameter $\text{diam}(G)$ of $G$ is the greatest eccentricity among the vertices of $G$. For an ordered subset $S = \{v_1, v_2, \ldots, v_t\}$ of vertices and a vertex $v$ in a connected graph $G$, the metric $S$-representation of $v$ is the vector $r(v|S) = (d(v, v_1), \ldots, d(v, v_t))$. The set $S$ is a resolving set for $G$ if every two vertices of $G$ have distinct $S$-representations. Particularity, a vertex $w \in V(G)$ resolves a pair of vertices $u, v \in V(G)$ if $d(v, w) \neq d(u, w)$. The minimum cardinality of a resolving set of $G$ is called metric dimension of $G$ and denoted by $\dim(G)$. This concept was first introduced by Slater in 1975 [22]. To see more details about metric dimension of graphs, we recommend references [2, 5, 9, 26].

Also, several versions of the standard metric dimension have been introduced and studied. In the following, we will discuss some of these versions.

In 2004, Sebő and Tannier introduced and studied a stronger version of the standard metric dimension which is called the strong metric dimension [21]. In the standard metric dimension, a resolving set uniquely specifies the position of vertices; however, this set cannot distinguish distances in the graph. Sebő and Tannier’s goal was to introduce a set called a strong resolving set so that in addition to vertices, they also distinguish distances. For the sake of it, they said, one vertex $a$ strongly resolves vertices $b$ and $c$, if there exist the smallest path between $b$ (respectively $c$) and $a$ such that $c$ (respectively $b$) belongs to the path. As well, in [11], Okamoto et al. introduced a special version of the standard metric dimension. In this way, instead of all vertices in the standard metric dimension, only adjacent vertices have a different representation from the metric generator. The cardinality of the smallest set which resolves every two adjacent vertices is called the local metric dimension. Researchers have had significant results on this concept. Among the interesting correlations obtained between the local metric dimension and
the independent set, it is shown that \( n - \beta(G) \) is an upper bound for the local metric dimension of a graph \( G \) with order \( n \). The classification of graphs that apply to the above boundary equation is still open. For more results, the reader can refer to papers [4, 17, 24].

In addition, other versions of the metric dimension had been introduced, including \( k \)-metric dimension, adjacency dimension, resolving partitions, edge metric dimension, mixed metric dimension, and fractional metric dimension [1, 7, 8, 13–15].

Graph theory has many applications in various sciences, including chemistry, to study some related parameters: see, for example, [18, 19]. Also, there are many applications for this invariance including error correcting codes, robot navigation, privacy in social networks, chemistry, pattern recognition, coin weighing, image processing, and locating intruders in networks [3, 5, 6, 12, 16, 20, 22, 25]. In situations where the points in the (submarine, drone, etc.) robot’s movement space are fixed and without damage, a metric generator set \( S \) is used to locate the robot (submarine, drone, etc.). Now, consider the situation in which the points in the robot movement space are vulnerable. In this case, some vertices of \( S \) could damage and unusable. Therefore, there is a need to replace \( S \) with another metric generator set such as \( S’ \). On the other hand, the problem of finding a metric generator set is NP-hard, and as a result, finding \( S’ \) takes a long time. This reason inspired us to use the following concept in such situations.

For any connected graph \( G \), we define metric dimension threshold, denoted by \( \text{dim}_{th}(G) \), to be the minimum number \( t \) such that any subset \( S \) of vertices \( G \) with \( |S| = t \) is a resolving set for \( G \). In this paper, we determine the metric dimension threshold for some well-known families of graphs. We also characterize graphs \( G \) of order \( n \) for which \( \text{dim}_{th}(G) \) is equal to \( 2, n - 2 \), or \( n - 1 \). Finally, we present some graphs \( G \) with the property that \( \text{dim}(G) = \text{dim}_{th}(G) \).

### 2. Basic Results

In this section, we determine the metric dimension threshold of paths, cycles, and the Petersen graph. Also, we characterize all graphs \( G \) with order \( n \) such that \( \text{dim}_{th}(G) = n - 1 \) or \( n - 2 \).

Clearly, for every connected graph \( G \) with order \( n \), we have \( \text{dim}(G) \leq \text{dim}_{th}(G) \leq n - 1 \). It is especially suitable for the robot if the resolving set \( S \) is of size \( \text{dim}(G) \). In fact, finding the graph \( G \) with the property that \( \text{dim}(G) = \text{dim}_{th}(G) \) is of attention.

**Remarks 1.** Let \( G \) be a connected graph.

(i) Let \( r \) be a positive integer. For every subset \( S \subseteq V(G) \) with \( |S| = r \), we have \( v(S) \neq r(u(S)) \) for \( v, u \in V(G) \), and then, for every subset \( S \subseteq V(G) \) with \( |S| \geq r \), we have \( v(S) \neq r(u(S)) \) for \( v, u \in V(G) \).

(ii) Assume that \( \text{dim}_{th}(G) = k \). Then, it follows from (i) that any subset \( S \) of \( G \) with \( |S| \geq k \) is resolving.

For a vertex \( a \) of \( G \), we use \( N(a) \) to denote the set of its neighbours. Also, for a set \( S \) of vertices of graph \( G \), we set \( N(S) = \bigcup_{a \in S} N(a) \).

**Theorem 1.** For any \( n \geq 3 \), we have \( \text{dim}_{th}(P_n) = 2 \).

*Proof.* At first, we claim that \( \text{dim}_{th}(P_n) \neq 1 \). To see this, we consider the set \( S = N(a) \), where \( a \) is a pendant vertex in \( P_n \). It is not hard to see that \( S \) is not a resolving set of \( P_n \).

Let \( S = \{a, b\} \) be an arbitrary subset of vertices of \( P_n \), with two elements. Since the set included a pendant vertex of \( P_n \) is a resolving set, we may assume that \( a \) and \( b \) are not pendant vertices. Consider arbitrary vertices \( v \) and \( u \) of \( P_n \), and assume that \( d(v, a) = d(u, a) = k \). Let \( P \) and \( P' \) denote the paths between \( a \) and the pendant vertices \( P_n \). Clearly, both \( v \) and \( u \) cannot belong to \( P \) or \( P' \). Without loss of generality, we may assume that \( v \in P \), and \( u \in P' \). Suppose that \( b \notin P_n \). If \( \{v, b\} \) is a resolving set, for any position of \( b \), \( r(v(S)) \neq r(u(S)) \), and the proof is complete.

**Corollary 1.** Let \( G \) be a graph. Then, \( \text{dim}_{th}(G) = 1 \) if and only if \( G \in \{P_1, P_2\} \).

**Theorem 2.** For each integer \( n \geq 2 \), we have \( \text{dim}_{th}(C_m) = \begin{cases} 3, & \text{if } m = 2n, \\ 2, & \text{if } m = 2n - 1. \end{cases} \)

*Proof.* Assume first that \( m = 2n \). We claim that \( \text{dim}_{th}(C_m) \neq 2 \). To see this, suppose that \( S = \{a, b, c\} \), for some vertices \( a, b \) and \( c \) of \( G \) with \( d(a, b) = n \). Then, \( N(a) = \{v, u\} \), and so \( r(v(S)) \neq r(u(S)) \).

In the following, we show that \( \text{dim}_{th}(C_m) = 3 \). To achieve this, let \( S = \{a, b, c\} \) be an arbitrary subset of the vertices of \( C_m \). Also, assume that \( v \) and \( u \) are arbitrary vertices of \( C_m \) such that \( d(v, a) = d(u, a) = n \) and \( d(v, b) = d(u, b) \). Thus, \( d(a, b) = d(v, a) + d(v, b) \). Suppose the path between \( a \) and \( b \) includes \( v \) (respectively, \( u \)) are denoted by \( P_v \) (respectively, \( P_u \)). We have \( V(P_v) \cup V(P_u) = V(C_m) \) and \( V(P_v) \cap V(P_u) = \{a, b\} \). Hence, \( v \in V(P_v) \) or \( c \in V(P_u) \). In any cases, it is easy to check that \( d(v, c) \neq d(u, c) \). Therefore, \( r(v(S)) \neq r(u(S)) \).

Now, let \( m = 2n - 1 \), and \( S = \{a, b\} \) be an arbitrary subset of vertices of \( C_m \). Suppose that \( d(v, a) = d(u, a) \) for some vertices \( u \) and \( v \) of \( C_m \). By using a method similar to that we used in above paragraph, one can easily check that \( d(v, b) \neq d(u, b) \). Hence, \( r(v(S)) \neq r(u(S)) \). Since \( \text{dim}(C_m) = 2 \), we have \( \text{dim}_{th}(C_m) = 2 \), which completes the proof.

Recall that, the Kneser graph \( KG(r, s) \) is the graph whose vertices are in a 1–1 correspondence with the \( r \)-sized subsets of \( \{1, 2, 3, \ldots, s\} \), and there is an edge between two vertices if their corresponding \( r \)-subsets are disjoint [10]. The Petersen graph \( KG(2, 5) \) has shown in Figure 1.
Theorem 3. If $P$ is the Petersen graph, then $dim_{th}(P) = 5$.

Proof. Consider the subset $S = \{\{1, 2\}, \{3, 4\}, \{1, 5\}, \{2, 5\}\}$ of vertices of $P$. Then, $d(v(S)) = d(u(S))$, where $v = \{3, 5\}$ and $u = \{4, 5\}$. This implies that $dim_{th}(P) > 4$.

Suppose that $S = \{v_1, v_2, \ldots, v_k\}$ is a subset of $P$, which is not a resolving set for $P$ and seeks a contradiction. Since $diam(P) = 2$, two vertices $u$ and $v$ can be selected in the following cases:

(i) Case 1. $d(v, u) = 2$. Since $P$ is a 3-arc transitive graph [10], without loss of generality, we may assume that $v = \{1, 2\}$ and $u = \{1, 3\}$. Suppose that $w$ is an arbitrary vertex of $P$. If $d(w, u) = 1 = d(w, v)$, then it is easy to see that $w = \{4, 5\}$. Otherwise, we have $d(w, u) = 2 = d(w, v)$. Now, when $1 \in w$, we have $w = \{1, 4\}$ or $w = \{1, 5\}$. Also whenever $1 \notin w$, we have $w = \{2, 3\}$. The only common neighbour of $v$ and $u$ is $\{4, 5\}$. Assume that $v_1 = \{4, 5\}$, and we are looking for elements for which there are other candidate members for $S$. Members of $S$ must be selected with the following conditions:

\[ v_1 \cap v \neq \emptyset \quad \text{and} \quad v_1 \cap u \neq \emptyset \quad \text{for} \quad 2 \leq i \leq 4. \tag{1} \]

(ii) Thus, $1 \in v_i$ or $2, 3 \in v_i$, for $2 \leq i \leq 5$. The vertices obtained in this way are $v_2 = \{1, 4\}$, $v_3 = \{1, 5\}$, and $v_4 = \{2, 3\}$. Now, any other vertex that we consider as $v_5$ resolves vertices $v$ and $u$, which is a contradiction.

Case 2. $d(v, u) = 1$. In this situation, by using a method similar to that we used in Case 1, one can obtain a contradiction. \qed

Theorem 4. Let $G$ be a connected graph of order $n \geq 3$. Then, $dim_{th}(G) = n - 1$ if and only if there exist vertices $v$ and $u$ of $G$ such that $N(v) - \{u\} = N(u) - \{v\}$.

Proof. ($\Rightarrow$) Assume that $dim_{th}(G) = n - 1$. Then, there exist two vertices $v$ and $u$ of $G$ such that $V(G) - \{v, u\}$ is not a resolving set. Hence, the distance of $v$ and $u$ from vertices of $V(G) - \{v, u\}$ is the same. We claim that there exists a vertex $w$ with $u \neq w$ in $V(G)$ such that $w$ is adjacent to $v$. To achieve this aim, suppose that, for any vertex $w$ with $u \neq w$ in $V(G)$, we have $w = v$ and seek a contradiction. Since $G$ is a connected graph, we have $v \sim u$. Thus, for any vertex $x \in V(G) - \{v, u\}$, we have $d(x, v) = d(x, u) + 1$ which is a contradiction. This means that $N(v) - \{u\} \neq \emptyset$. Since the distance of any vertex of $N(v) - \{u\}$ from $v$ is 1, we have $N(v) - \{u\} = N(u) - \{v\}$.

($\Leftarrow$) Assume that there exists vertices $v$ and $u$ of $G$ such that $N(v) - \{u\} = N(u) - \{v\}$. Clearly, $V(G) - \{v, u\}$ cannot be a resolving set of $G$. Thus, $|V(G) - \{v, u\}| < dim_{th}(G)$. Therefore, $dim_{th}(G) = n - 1$, and the proof is complete. \qed

The following corollaries are immediate from Theorem 4.

Corollary 2. There is an infinite number of graph $G$ with $dim_{th}(G) = n - 1$.

Corollary 3. Let $G$ be a complete multipartite graph with order $n$. Then, $dim_{th}(G) = n - 1$.

Corollary 4. Let $G$ be a graph with order $n$. Then, $dim_{th}(G) = n - 1$ if and only if $G$ satisfies the following conditions:

(a) There exist vertices $v, u, \text{ and w of G such that } w \in N(v)$

(b) $N(v) - \{u, w\} = N(u) - \{v\}$

(a3) The vertex $v_i$ is a pendant vertex of $G$ with $v_i \in N(V_i)$, $N(V_i)$ for $1 \leq i \leq 2$ and $3 \leq j \leq 4$, where $V_i$ is the distance partite set of $G[V(G) - \{v, w\}]$. Reference to

(b) For any vertices $v, u, \text{ and w of G such that } v \in N(u)$, we have $N(v) \cap \{u\} \neq N(u) - \{v\}$.

Proof. If $dim_{th}(G) = n - 2$, then there exist vertices $v, u, \text{ and w of G with property that } A = V(G) - \{v, u, w\}$ is not a resolving set. Thus, there exist two vertices of $\{u, v, w\}$ with the same $A$-representation. Assume that $r(vA) = r(uA)$. Since the set $V(G) - \{v, u\}$ must resolves vertices $v$ and $u$, we have $d(v, u) \neq d(w, u)$. Suppose that $N(u) \cap \{v, u\} = \emptyset$. Then, $N(u) \subseteq A$. For every $w \in N(u)$, since $w \in A$, we have $d(v, w) \neq d(u, w)$. This implies that $d(v, w) = d(u, w)$ which is a contradiction. Similarly, it is impossible to have $\{v, u\} \subseteq N(w)$. Hence, $v \in N(w)$ and $u \neq N(w)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{petersen_graph}
\caption{The Petersen graph $KG(2, 5)$.}
\end{figure}
For (a2), it is enough to show that $N(v) \cap N(u) \neq \emptyset$. Assume that $N(v) \cap N(u) = \emptyset$, and we seek a contradiction. Since $G$ is connected, we may have a vertex $x \in A \cup \{v\}$ such that $x \in N(u)$. If $N(u) = \{v\}$, then, for any member of $A$, as $x$, we have $d(x, u) = d(x, v) + 1$, a contradiction. Otherwise, if $N(u) \neq \{v\}$, then there exists a vertex $y \in A$ such that $y \in N(u)$. Thus, $y \notin N(v)$. Hence, $y$ resolves vertices $v$ and $u$, which is a contradiction.

For (a3), suppose that $w$ is not a pendant vertex. Hence, $w$ has some neighbours in $A$, and so it is easy to show that $w$ is not a member of $N(v) \cap N(u)$. However, if $w \in N(V_1) \cap N(V_2)$ or $w \in N(V_2) \cap N(V_1)$, for $3 \leq j \leq e(u)$, one can see that the vertices in $A$ keep their distance from $v$ and $u$. Suppose, contrary to our claim, that $w \in N(V_1)$, for an integer $i$ with $3 \leq i \leq e(u)$. Then, there exists vertex $z \in V_i$ such that $w$ is adjacent to $z$. Now, if $i = 3$, then $d(z, u) = 3$. Since $d(z, v) = 2$, we have $d(y, u) \neq d(y, v)$, which is impossible. If $i \geq 5$, then the smallest path between $u$ and $z$ contains $v$. Thus, $z$ resolves $v$ and $u$, a contradiction. If $i = 4$, then we have two cases. First, suppose that $v$ is adjacent to $u$. Then, $d(z, u) = 3$, and so $d(z, v) \neq d(z, v)$. Finally, if $v$ is not adjacent to $u$, one can check that $d(z, u) = 4$. Therefore, in both cases, $A$ is a resolving set, whenever $w \in N(V_1)$, for $3 \leq i \leq e(u)$, which is a contradiction.

For the converse, assume that $G$ satisfies the conditions (a) and (b). In view of Theorem 4, the statement (b) implies that $\dim_{gh}(G) < n - 1$. Let $A = V(G) - \{v, u, w\}$ be the subset of vertices $G$, where $v, u, w$ are the vertices described in (a). One can easily check that $r(v(A)) = r(u(A))$. This implies that $\dim_{gh}(G) > n - 3$. Hence, $\dim_{gh}(G) = n - 2$, and the proof is complete.

Now, we consider the family of graphs $G$ of order $n$ with the property that $\dim_{gh}(G) = n - 2$. Clearly, the smallest one is $P_4$. For a positive integer $k$, let $G_k$ be the graph with $V(G_k) = \{a_1, a_2, \ldots, a_k\} \cup \{v, u, w\}$ and $E(G_k) = \{a_1 v, a_2 v, \ldots, a_k v\} \cup \{a_1 u, a_2 u, \ldots, a_k u\} \cup \{v, w\}$. By matching $G_k$ to $G$ in the previous theorem, one can check that $\dim_{gh}(G_k) = k + 1$. Thus, we have the following corollary.

**Corollary 5.** There are an infinite number of graphs $G$ with $\dim_{gh}(G) = n - 2$.

For vertices $v$ and $u$ of a graph $G$, we shall use $F(v, u)$ to denote the set of those vertices $w \in V(G)$ with the property that $d(w, u) = d(w, v)$. Now, we define $F(G)$ to be the greatest $|F(v, u)|$ for $v, u \in V(G)$. Hence,

$$F(G) = \max_{v \in V(G)}|F(v, u)|. \quad (2)$$

One can easily check that $1 \leq F(G) \leq n - 2$. In the next theorem, we present a lower bound for $\dim_{gh}(G)$ in terms of $F(G)$.

**Theorem 6.** Let $G$ be a graph. Then, $F(G) + 1 \geq \dim_{gh}(G)$.

**Proof.** Suppose that $F(G) = |F(v, u)|$, for vertices $v$ and $u$ of $G$. Since the set $F(v, u)$ does not resolve vertices $v$ and $u$, the result follows.

It is obvious that the above upper bound is sharp. For instance, if $P_n$ is a path of length $n$, then $F(G) + 1 = \dim_{gh}(G) = 2$. Moreover, all graphs $G$ with the property that $\dim_{gh}(G) = n - 1$ satisfy in the equality $F(G) = n - 2$. In fact, Theorem 4 states that $F(G) = n - 2$ if and only if $\dim_{gh}(G) = n - 1$. So, in view of the previous theorem, we can express the following proposition.

**Proposition 1.** Let $G$ be a graph with $F(G) = n - 3$. Then, $\dim_{gh}(G) = n - 2$.

3. **Graphs with $\dim(G)\dim_{gh}(G)$**

In Corollary 4, we show that the only graph $G$ of order $n$ with $\dim(G) = \dim_{gh}(G) = n - 1$ is the complete graph $K_n$. In this section, we want to investigate graphs $G$ of order $n$ with the property that $\dim(G) = \dim_{gh}(G) = k$, for $k \in [2, n - 2]$. Moreover, we characterize all graphs $G$ with $\dim_{gh}(G) = 2$.

Graphs with metric dimension two were studied by Sudhakara and Hemanth Kumar in [23]. Also, Chartrand et al. [6] provided a characterization of graphs of order $n$ and metric dimension $n - 2$. In the following, we will use the results in [6, 23] to obtain a characterization of graphs $G$ of order $n$ with $\dim(G) = \dim_{gh}(G) \in (2, n - 2)$.

**Theorem 7** [6, Theorem 4]. Let $G$ be a connected graph of order $n \geq 4$. Then, $\dim(G) = n - 2$ if and only if $G = K_{s, t}$, $G = K_{s, t} + \overline{K}_l(s \geq 1, t \geq 2)$, or $G = K_s + (K_t \cup K_l)(s \geq 1, t \geq 2)$.

**Theorem 8.** If $G$ is a graph with order $n \geq 4$ and $\dim(G) = n - 2$, then $\dim_{gh}(G) = n - 1$.

**Proof.** Let $G = K_{s, t}$, or $G = K_s + \overline{K}_l(s \geq 1, t \geq 2)$. Then, $G$ is a multipartite graph and $\dim_{gh}(G) = n - 1$.

Now, assume that $G = K_s + (K_t \cup K_l)(s \geq 1, t \geq 2)$. If $s = t = 1$, then $n < 4$, which is a contradiction. Suppose that $s \geq 2$ or $t \geq 2$. Without loss of generality, we may assume that $s \geq 2$. Hence, for two arbitrary vertices $a, b \in V(K_s)$, we have $N(v) = \{u\} = N(u) = \{v\}$. Thus, by Theorem 4, $\dim_{gh}(G) = n - 1$.

**Corollary 6.** There is no graph $G$ of order $n$ with property that $\dim(G) = \dim_{gh}(G) = n - 2$.

**Theorem 9** [23, Theorem 6.1]. Let $G$ be a graph which is not a path with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $\{v_{i_1}v_{i_2}, \ldots, v_{i_k}\}$ be the distance partition of $V(G)$ with reference to the vertex $v$, where $k_i$ is the eccentricity of $v_i, 1 \leq i \leq n$. The metric dimension of $G$ is 2 if and only if there exist vertices $v_i$ and $v_j$ such that $|V_{i-1} \cap V_{i+1}| \leq 1$ for every $i$ with $1 \leq k \leq e(v_i)$ and $1 \leq \ell \leq e(v_j)$.

**Lemma 1.** Let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $\{v_{i_1}v_{i_2}, \ldots, v_{i_k}v_{i_r}\}$ be the distance partition of $V(G)$ with reference to the vertex $v_i$. For every $v_i$ and $v_j (1 \leq i, j \leq n)$, and $k$ and $\ell$ with $1 \leq k \leq e(v_i)$ and $1 \leq \ell \leq e(v_j)$, we have $|V_{i-1} \cap V_{i+1}| \leq 1$ if and only if $G \in \{P_n, C_{2n-1}\}$, for a $n \geq 2$.1
Proof. Only one implication requires proof. Suppose that \( G \) is a graph of order \( n \geq 4 \). We claim that \( \deg(v_i) \leq 2 \), for every \( i \) with \( 1 \leq i \leq n \). Suppose, contrary to our claim, that there exists \( i \) with \( 1 \leq i \leq n \) such that \( \deg(v_i) \geq 2 \). Let \( N(v_i) = \{ v_1, v_2, v_3 \} \). Then, \( N(v_i) \subseteq V_{31} \). We need to consider the following four cases for \( G[N(v_i)] \).

Case 1. \( G[N(v_i)] = K_{3} \). Thus, \( \{ v_1, v_2, v_3 \} \subseteq V_{32} \). This implies that \( |V_{11} \cap V_{32}| > 1 \), which is a contradiction.

Case 2. \( G[N(v_i)] = K_{3} \cup K_{1} \). Suppose that \( v_1 \) is adjacent to \( v_2 \). Hence, \( \{ v_1, v_2 \} \subseteq V_{32} \). So we have \( |V_{11} \cap V_{32}| > 1 \), which is a contradiction.

Case 3. \( G[N(v_i)] = P_{3} \). Without loss of generality, we may assume that \( \{ v_1, v_2 \} \subseteq N(v_i) \). This means that \( |V_{11} \cap V_{32}| > 1 \), which is a contradiction.

Case 4. \( G[N(v_i)] = K_{3} \). In this situation, similar to that used in Case 3, one can obtain a contradiction.

So, for any \( i \) with \( 1 \leq i \leq n \), we have \( \deg(v_i) \leq 2 \). This implies that \( G \) is isomorphic to \( P_n \) or \( C_n \), for some positive integer \( n \).

In the following, we show that if \( G \cong C_n \), then \( n \) is an odd integer. On the contrary, suppose that \( G \cong C_{2k} \), for an integer \( k \) with \( k \geq 2 \). Assume that \( E(C_{2k}) = \{ v_1v_2, v_2v_3, \ldots, v_{2k-1}v_{2k}, v_1v_{2k} \} \) and that \( d(v_1, v_{k+1}) = \text{diam}(G) \). Thus, \( N(v_1) = \{ v_2, v_{2k} \} \) and that \( d(v_{k+1}, v_2) = d(v_{k+1}, v_{2k}) = k - 1 \). Therefore, \( |V_{11} \cap V_{k+1|k-1}| > 1 \), which is a contradiction, and the proof is complete. \( \square \)

Theorem 10. Let \( G \) be a graph. Then, \( \dim_{th}(G) = 2 \) if and only if \( G \) is isomorphic to one of the graphs \( P_n \) or \( C_{2n-1} \), for a positive integer \( n \geq 2 \).

Proof. Let \( V(G) = \{ v_1, v_2, \ldots, v_m \} \). First, we suppose that \( G \) is isomorphic to one of the graphs \( P_n \) or \( C_{2n-1} \), for a positive integer \( n \geq 2 \). By Lemma 1, each \( v_i \) and \( v_j \) (\( 1 \leq i, j \leq m \)), \( k \) and \( \ell \) with \( 1 \leq k \leq e(v_i) \) and \( 1 \leq \ell \leq e(v_j) \), implies that \( |V_{ik} \cap V_{j\ell}| \leq 1 \). By the way of contradiction, assume that there exist \( v_i \) and \( v_j \) (\( 1 \leq i, m \)) and \( k, \ell \) which \( 1 \leq k \leq e(v_i) \) and \( 1 \leq \ell \leq e(v_j) \) such that \( |V_{ik} \cap V_{j\ell}| \geq 1 \). Hence, there exist distinct vertices \( u_1 \) and \( u_2 \) in \( V_{ik} \cap V_{j\ell} \). Since \( u_1, u_2 \in V_{ik} \), we have \( d(u_1, v_i) = d(u_2, v_i) = k \) and \( d(u_1, v_j) = d(u_2, v_j) = \ell \). This means that set \( \{ v_i, v_j \} \) can not to be a resolving set of \( G \), which is a contradiction.

The converse implication follows almost immediately from Theorems 1 and 2. \( \square \)

The next corollary is obtained from Theorem 10, [6, Theorem 2], and Theorem 2.

Corollary 7. Let \( G \) be a graph. Then, \( \dim(G) = \dim_{th}(G) = 2 \) if and only if \( G \cong C_{2n-1} \), for a positive integer \( n \geq 2 \).

We end the paper with the following problem.

Open Problem. Let \( k \) be a positive integer \( 3 \leq k \leq n - 3 \). Characterize the class of graphs \( G \) with the property that \( \dim(G) = \dim_{th}(G) = k \). Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] S. Arumugam and V. Mathew, "The fractional metric dimension of graphs," Discrete Mathematics, vol. 312, no. 9, pp. 1584–1590, 2012.
[2] R. F. Bailey and P. J. Cameron, "Base size, Metric dimension and other invariants of groups and graphs," Bulletin of the London Mathematical Society, vol. 43, no. 2, pp. 209–242, 2011.
[3] R. F. Bailey and G. I. Yero, "Error-correcting codes from k-resolving sets," Discussiones Mathematicae Graph Theory, vol. 39, no. 2, pp. 341–355, 2019.
[4] G. A. Barragán-Ramírez and J. A. Rodríguez-Velázquez, "The local metric dimension of strong product graphs," Graphs and Combinatorics, vol. 32, no. 4, pp. 1263–1278, 2016.
[5] J. Cáceres, C. Hernando, M. Mora et al., "On the metric dimension of Cartesian products of graphs," SIAM Journal on Discrete Mathematics, vol. 21, pp. 423–441, 2007.
[6] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann, "Resolvability in graphs and the metric dimension of a graph," Discrete Applied Mathematics, vol. 105, no. 1–3, pp. 99–113, 2000.
[7] G. Chartrand, E. Salehi, and P. Zhang, "The partition dimension of a graph," Aequationes Mathematicae, vol. 59, no. 1, pp. 45–54, 2000.
[8] A. Estrada-Moreno, I. G. Yero, and J. A. Rodriguez-Velázquez, "The k-metric dimension of a graph," Applied Mathematics and Information Sciences, vol. 9, no. 6, pp. 2829–2840, 2015.
[9] J. Geneson, "Metric dimension and pattern avoidance in graphs," Discrete Applied Mathematics, vol. 284, pp. 1–7, 2020.
[10] R. L. Graham, M. Grotschel, and L. Lo, Handbook of Combinatorics, The MIT Press, Cambridge MA, 1995.
[11] F. Okamoto, B. Phinezy, and P. Zhang, "The local metric dimension of a graph," Mathematica Bohemica, vol. 135, no. 3, pp. 239–255, 2010.
[12] F. Harary and R. A. Melter, "On the metric dimension of graphs," Ars Combinatoria, vol. 2, pp. 191–195, 1976.
[13] M. Jannesari and B. Omoomi, "The metric dimension of the lexicographic product of graphs," Discrete Mathematics, vol. 312, no. 22, pp. 3349–3356, 2012.
[14] A. Kelenc, D. Kuziak, A. Taranenko, and I. G Yero, "Mixed metric dimension of graphs," Applied Mathematics and Computation, vol. 314, pp. 429–438, 2017.
[15] A. Kelenc, N. Tratnik, and I. G Yero, "Uniquely identifying the edges of a graph: the edge metric dimension," Discrete Applied Mathematics, vol. 251, pp. 204–220, 2018.
[16] S. Khuller, B. Raghavachari, and A. Rosenfeld, "Landmarks in graphs," SIAM Journal on Discrete Mathematics, vol. 7, no. 3, pp. 217–229, 1994.
[17] S. Klavžar and M. Tavakoli, "Local metric dimension of graphs: generalized hierarchical products and some applications," Applied Mathematics and Computation, vol. 364, Article ID 124676, 2020.
[18] J. B Liu, C. Liu, S. Wang, and B. Wei, "Zagreb indices and multiplicative Zagreb indices of Eulerian graphs," Bulletin of
[19] J. B. Liu, J. Zhao, H. He, and Z. Shao, “Valency-based topological descriptors and structural property of the generalized sierpinski networks,” *Journal of Statistical Physics*, vol. 177, pp. 1131–1147, 2019.

[20] R. A. Melter and I. Tomescu, “Metric bases in digital geometry,” *Computer Vision, Graphics, and Image Processing*, vol. 25, no. 1, pp. 113–121, 1984.

[21] A. Sebő and E. Tannier, “On metric generators of graphs,” *Mathematics of Operations Research*, vol. 29, no. 2, pp. 383–393, 2004.

[22] P. J. Slater, “Leaves of trees,” *Congressus Numerantium*, vol. 14, pp. 549–559, 1975.

[23] G. Sudhakara, “Graphs with metric dimension two-a characterization,” *Advances and Applications in Discrete Mathematics*, vol. 60, 2009.

[24] L. Susilowati, M. I. Slamin, M. I. Utoyo, and N. Estuningsih, “The similarity of metric dimension and local metric dimension of rooted product graph,” *Far East Journal of Mathematical Sciences*, vol. 97, no. 7, pp. 841–856, 2015.

[25] R. Trujillo-Rasua and I. G. Yero, “k-metric antidimension: a privacy measure for social graphs,” *Information Science*, vol. 328, pp. 403–417, 2016.

[26] I. G. Yero, A. Estrada-Moreno, and J. A. Rodríguez-Velázquez, “Computing the k-metric dimension of graphs,” *Applied Mathematics and Computation*, vol. 300, pp. 60–69, 2017.