A dynamical model for hierarchy and modular organization: The trajectories en route to the attractor at the transition to chaos

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Abstract. We show that the full features of the dynamics towards the Feigenbaum attractor, present in all low-dimensional maps with a unimodal leading component, form a hierarchical construction with modular organization that leads to a clear-cut emergent property. This well-known nonlinear model system combines a simple and precise definition, an intricate nested hierarchical dynamical structure, and emergence of a power-law dynamical property absent in the exponential-law that governs the dynamics within the modules. This classic nonlinear system is put forward as a working example for complex collective behavior.

1. Introduction
Prototypical complex systems are inherently hierarchical [1]-[3]. Hierarchical systems consist of interrelated subsystems or modules, each of which has also a hierarchical constitution. These modular structures form levels down to a lowest one made of elementary units. Examples of complex systems made of smaller complex systems are widespread: astrophysical objects, social organizations, biological organisms, molecules and atoms themselves, etc. In these kinds of systems there can be many subsystems. For example, tissues put individual living beings together and tissues consist of cells, a cell has nuclei and organelles and these are made of macromolecules such as proteins. The proteins in turn are made of many atoms, and electrons and nucleons, which are comprised of hadrons, quarks, etc, form the atoms. In describing a hierarchy it is necessary to define how the subsystems interact and how these interactions associate with the dynamical processes within and between the structures. This applies to physical, biological as well as to social complex systems.

The dynamical properties of hierarchies, within subsystems and between those that interconnect them, are essential for the understanding of complex systems. A purely static description is insufficient for the comprehension of the modular organization and functioning of the entire system, and, most importantly, for the appreciation of the properties of the system called emergent properties, those that result in capabilities and behaviors of the whole that are not attributable to any of the separate subsystems or modules. Emergence ‘arises’ from the dynamic interactions of parts within the whole. Experience accumulated in studying complex systems have established that the theoretical generation of dynamical hierarchies from a formal
system is not trivial and that the quest for a possible general explanation for hierarchical structures is at least elusive. However, it is believed that genuine emergence is strongly associated with nonlinearity, and that a hierarchy of building blocks operating at the “Edge of Chaos” is a characteristic of complex systems.

Here we take the view that it is possible to generate hierarchical systems with dynamical organization from a formal starting point in the form of a simple closed-form nonlinear dynamical system. And also that nonlinear dynamics tuned at the transition to chaos can lead to an emergent property. We illustrate this standpoint by resorting to the properties of the simplest well-known nonlinear iterated map, the iconic one-dimensional unimodal map, as represented by the quadratic logistic map [4], [5].

Figure 1. Left panel: Absolute value of attractor positions for the logistic map \( f_\mu(x) \) in logarithmic scale as a function of \(- \ln (\mu_\infty - \mu)\). Right panel: Absolute value of trajectory positions for \( f_\mu(x) \) at \( \mu_\infty \) with initial condition \( x_0 = 0 \) in logarithmic scale as a function of the logarithm of time \( t \), also shown by the numbers close to the circles. The arrows indicate the equivalence between the diameters \( d_{n,0} \) in the left panel, and position differences \( D_n \) with respect to \( x_0 = 0 \) in the right panel.

Elements in our analysis are the following: i) The preimages of the periodic attractors along the bifurcation cascade that leads to the transition to chaos appear entrenched in a fractal hierarchical structure of increasing complexity as period doubling develops. ii) The limiting form of this rank structure results in an infinite number of families of well-defined phase-space gaps in the positions of the attractor at the transition to chaos, the so-called Feigenbaum attractor. iii) The gaps in each of these families can be ordered with decreasing width in accordance to power laws and are seen to appear sequentially in the dynamics generated by uniform distributions of initial conditions. iv) The emergent power law with log-periodic modulation associated with
the rate of approach of trajectories towards the Feigenbaum attractor is explained in terms of the progression of gap formation. v) The relationship between the law of rate of convergence to the Feigenbaum attractor and the inexhaustible hierarchy feature of the preimage structure is elucidated. A detailed account of these properties is given in Refs. [6]-[8]

2. Brief recollection of the dynamics towards the Feigenbaum attractor

We start by recalling that unimodal maps are dissipative due to their one-dimensionality and consequently their dynamical properties are associated with sets of positions called attractors as they capture at sufficiently long times most trajectories. Other trajectories evolve towards related sets of positions called repellors. The trajectories associated to the period-doubling route to chaos in unimodal maps exhibit intricate dynamical properties that follow concerted patterns. At the period-doubling accumulation points, the transitions to chaos, periodic attractors become multifractal before turning chaotic. We call the attractor at the transition to chaos the Feigenbaum attractor when the Lyapunov exponent vanishes as it changes sign [4], [5]. There are two sets of properties associated with the attractors involved in these dissipative systems: those of the dynamics inside the attractors and those of the dynamics towards the attractors. These properties have been characterized in detail, the organization of trajectories and of the sensitivity to initial conditions at the attractor are described in Ref. [6], while the features of the rate of approach of an ensemble of trajectories to the attractor has been explained in Ref. [7].

We recall now some of the basic features of the bifurcation forks that form the period-doubling cascade sequence in unimodal maps, often illustrated by the logistic map \( f_{\mu}(x) = 1 - \mu x^2 \), \(-1 \leq \mu \leq 1\), \( 0 \leq \mu \leq 2 \) [4], [5]. The dynamics towards a particular family of periodic attractors, the superstable attractors [2], makes possible the understanding of the rate of approach to the Feigenbaum attractor, located at \( \mu = \mu_\infty = 1.401155189... \), and highlights the source of its discrete scale invariant property [8]. The family of trajectories associated with these attractors - called supercycles - of periods \( 2^N \), \( N = 1, 2, 3, ... \), are located along the bifurcation forks. The positions (or phases) of the \( 2^N \) attractor are given by \( x_m = f_{\mu(\mu)}(0), m = 1, 2, ..., 2^N \). Notice that infinitely many other sequences of superstable attractors appear at the period-doubling cascades within the windows of periodic attractors for values of \( \mu > \mu_\infty \). Associated to the \( 2^N \)-attractor at \( \mu = \mu_N \) there is a \((2^N - 1)\)-repellor consisting of \( 2^{N-1} \) positions \( y_m, m = 1, 2, ..., 2^{N-1} \). These positions are the unstable solutions, \( |df_{\mu_N}^{(2n-1)}(y)/dy| < 1 \), of \( y = f_{\mu_N}^{(2n-1)}(y), n = 1, 2, ..., N \). The first, \( n = 1 \), originates at the initial period-doubling bifurcation, the next two, \( n = 2 \), start at the second bifurcation, and so on, with the last group of \( 2^{N-1}, n = N \), setting out from the \( N - 1 \) bifurcation. See Fig. 1.

Basic to the understanding of the dynamical properties of unimodal maps is the following far-reaching property: Time evolution at \( \mu_\infty \) from \( t = 0 \) up to \( t \to \infty \) traces the period-doubling cascade progression from \( \mu = 0 \) up to \( \mu_\infty \). There is an underlying quantitative relationship between the two developments. Specifically, the trajectory inside the Feigenbaum attractor with initial condition \( x_0 = 0 \), the \( 2^N \)-supercycle orbit, takes positions \( x_t \) such that the distances between appropriate pairs of them reproduce the diameters \( d_{N,m} \) defined from the supercycle orbits with \( \mu_N < \mu_\infty \). See Fig. 1, where the absolute value of positions and logarithmic scales are used to illustrate the equivalence. This property has been central in obtaining rigorous results for the sensitivity to initial conditions for the Feigenbaum attractor [6]. Other families of periodic attractors share most of the properties of supercycles. We consider explicitly the case of a map with quadratic maximum but the results are easily extended to general nonlinearity \( z > 1 \).

The organization of the total set of trajectories as generated by all possible initial conditions as they flow towards a period \( 2^N \) attractor has been determined in detail [8]. It was found
Figure 2. Top panel: Time of flight $t_f(x)$ for $N = 2$, the black lines correspond to initial conditions that terminate at the attractor positions $x = 0$ and $x \approx -0.310703$, while the grey lines to trajectories ending at $x = 1$ and $x \approx 0.8734$. Right (left) bottom panel: Same as top panel, but plotted against the logarithm of $x - y_1$ ($y_1 - x$). It is evident that the peaks are arranged exponentially around the old repellor position $y_1$, i.e., they appear equidistant in a logarithmic scale.

that the paths taken by the full set of trajectories in their way to the supercycle attractors (or to their complementary repellors) are exceptionally structured. We define the preimage $x^{(k)}$ of order $k$ of position $x$ to satisfy $x = h^{(k)}(x^{(k)})$ where $h^{(k)}(x)$ is the $k$-th composition of the map $h(x) \equiv f_{2N}^{(2N-1)}(x)$. The preimages of the attractor of period $2^N$, $N = 1, 2, 3, ...$ are distributed into different basins of attraction, one for each of the $2^N$ phases (positions) that compose the cycle. When $N \geq 2$ these basins are separated by fractal boundaries whose complexity increases with increasing $N$. The boundaries consist of the preimages of the corresponding repellor and their positions cluster around the $2^N - 1$ repellor positions according to an exponential law. As $N$ increases the structure of the basin boundaries becomes more involved. Namely, the boundaries for the $2^N$ cycle develops new features around those of the previous $2^{N-1}$ cycle boundaries, with the outcome that a hierarchical structure arises, leading to embedded clusters of clusters of boundary positions, and so forth. The dynamics associated to families of trajectories always displays a characteristically concerted order in which positions are visited, which in turn reflects the repellor preimage boundary structure of the basins of attraction. That is, each trajectory has an initial position that is identified as a preimage of a given order of an attractor (or repellor) position, and this trajectory necessarily follows the steps of other trajectories with initial conditions of lower preimage order belonging to a given chain or pathway to the attractor.
Figure 3. Same as Fig. 2 for \( N = 3 \). The black lines correspond to initial conditions that terminate at any of the four attractor positions close or equal to \( x = 0 \), while the grey lines to trajectories ending at any of the other four attractor positions close or equal to \( x = 1 \). As the bottom panels show, in logarithmic scale, in this case there are (infinitely) many clusters of peaks (repellor preimages) equidistant from each other.

(or repellor). When the period \( 2^N \) of the cycle increases the dynamics becomes more involved with increasingly more complex stages that reflect the hierarchical structure of preimages. See Figs. 2 and 3, and see Ref. [8] for details. The fractal features of the boundaries between the basins of attraction of the positions of the periodic orbits develop a structure with hierarchy, and this in turn reflects on the properties of the trajectories. The set of trajectories produce an ordered flow towards the attractor or towards the repellor that reflect the ladder structure of the sub-basins that constitute the mentioned boundaries.

Another way by which the preimage structure described above manifests in the dynamics of the supercycles of periods \( 2^N \) is via the successive formation of gaps in phase space (the interval \(-1 \leq x \leq 1\)) that finally give rise to the attractor and repellor multifractal sets. To observe explicitly this process we consider an ensemble of initial conditions \( x_0 \) distributed uniformly across phase space and record their positions at subsequent times. This set of gaps develops in time beginning with the largest one containing the \( k = 0 \) repellor, then followed by a set of two gaps associated with the \( k = 1 \) repellor, next a set of four gaps associated with the \( k = 2 \) repellor, and so forth. This process stops when the order of the gaps \( k \) reaches \( N - 1 \). To facilitate a visual comparison between the process of gap formation at \( \mu = \mu_\infty \) and the dynamics inside the Feigenbaum attractor—as illustrated by the trajectory in Fig. 1 right panel—we plot in Fig. 4 the time evolution of an ensemble composed of \( 10^4 \) trajectories. We use logarithmic scales for both \( |x_t| \) and \( t \) and then superpose on the evolution of the ensemble the positions for the
trajectory starting at $x_0 = 0$. It is clear from this figure that the gaps that form consecutively all have the same width in the logarithmic scale of the plot and therefore their actual widths decrease as a power law, the same power law followed, for instance, by the position sequence $x_t = \alpha^{-N}$, $t = 2^N$, $N = 0, 1, 2, ...$, for the trajectory inside the attractor starting at $x_0 = 0$ (and where $\alpha$ is Feigenbaum’s universal constant). The locations of this specific family of consecutive gaps (the largest gaps for each value of $k$) advance monotonically toward the sparsest region of the multifractal attractor located at $x = 0$. Other gaps cannot be observed in the scale of the figure due to the way the data is plotted. See Refs. [6]-[8] for more details.

Figure 4. Phase-space gap formation for $\mu = \mu_\infty$. The black dots correspond to time evolution of a uniform ensemble of 10000 trajectories as a function of $|x|$ vs $t$, both in logarithmic scales. The open circles are the positions, labeled by the times at which they are reached, for the trajectory inside the Feigenbaum attractor with initial condition $x_0 = 0$, same as right panel in Fig. 1.

There is straightforward quantitative way to measure the rate of convergence of an ensemble of trajectories to the attractor and to the repellor that consists in evaluating a single time-dependent quantity. A partition of phase space is made of $N_b$ equally sized boxes or bins and a uniform distribution of $N_c$ initial conditions placed, as above, along the interval $-1 \leq x \leq 1$. The number $r$ of trajectories per box is $r = N_c/N_b$. The quantity of interest is the number of boxes $W_t$ that contain trajectories at time $t$. This is shown in Fig. 5 on logarithmic scales for the first five supercycles of periods $2^1$ to $2^5$ where we can observe the following features: In all cases $W_t$ shows a similar initial nearly-constant plateau, and a final well-defined decay to zero. In between these two features there are $N - 1$ oscillations in the logarithmic scales of the figure. As it can be observed in the left panel of Fig. 5, the duration of the overall decay grows approximately proportionally to the period $2^N$ of the supercycle. We are now in a position to appreciate the dynamical mechanism at work behind the features of the decay rate $W_t$. From our previous discussion we know that, every time the period of a supercycle increases from $2^{N-1}$ to $2^N$ by a shift in the control parameter value from $\mu_{N-1}$ to $\mu_N$, the preimage structure advances one stage of complication in its hierarchy. Along with this, and in relation to the time evolution of the ensemble of trajectories, an additional set of $2^N$ smaller phase-space gaps
develops and also a further oscillation takes place in the corresponding rate $W_t$ for finite period attractors. At $\mu_\infty$ the time evolution tracks the period-doubling cascade progression, and every time $t$ increases from $2^{N-1}$ to $2^N$ the flow of trajectories undergoes equivalent passages across stages in the itinerary through the preimage ladder structure, in the development of phase-space gaps, and in logarithmic oscillations in $W_t$.

**Figure 5.** Left panel: The rate $W(t)$, divided by the number of boxes $N_b$ employed, of approach to the attractor for the supercycles of periods $2^N$, $N = 1, 2, 3, 4$ and 5 in logarithmic scales. The expression shown corresponds to the power-law decay of the developing logarithmic oscillations. Right panel: Superposition of the five curves for $W(t)$ in the left panel via $n$-times repeated rescaling factors shown for the horizontal $x$ and vertical $y$ axis.

In summary, each doubling of the period introduces additional modules or building blocks in the hierarchy of the preimage structure, such that the complexity of these added modules is similar to that of the total period $2^N$ system. As shown in Ref. [7], each period doubling adds also new components in the family of sequentially-formed phase space gaps, and moreover increases in one unit the number of undulations in the transitory log-periodic power-law decay displayed by the fraction $W_t$ of ensemble trajectories still away at a given time $t$ from the attractor (and the repellor). As a consequence of this we have obtained detailed understanding of the mechanism by means of which the discrete scale invariance implied by the log-periodic property [7] in the rate of approach $W_t$ arises.

### 3. The dynamics towards the Feigenbaum attractor as a dynamical hierarchy with modular organization

We proceed now to the identification of the features of the dynamics towards the Feigenbaum attractor as those of a bona fide model of dynamical hierarchy with modular organization. These are: i) Elementary degrees of freedom and the elementary events associated with them.
ii) Building blocks and the dynamics that takes place within them and through adjacent levels of blocks. iii) Self-similarity characterized by coarse graining and renormalization group (RG) operations. iv) Emerging property of the entire hierarchy absent in the embedded building blocks.

The elementary degrees of freedom at the bottom of the hierarchy are the preimages of the attractors of period $2^k$. These are assigned an order $k$, according to the number $k$ of map iterations they require to reach the attractor. The preimages are also distinguished in relation to the position, $x_1, \ldots, x_{2^k}$, of the attractor they reach first. The preimages of each attractor position appear grouped in basins with fractal boundaries. See Fig. 2. The elementary dynamical event is the reduction of the order $k$ of a preimage by one unit. This event is generated by a single iteration of the map for an initial position placed in a given basin. The result is the translation of the position to a neighboring basin of the same attractor position. The (operative) degrees of freedom higher up in the hierarchy are infinite sets of preimages of attractors of period $2^N$, $N > 2$, for which boundary basins are replaced by clusters of boundary basins. See Fig. 3.

The modules or building blocks are clusters of clusters formed by families of boundary basins. The boundary basins of attraction of the $2^N$ positions of the periodic attractors cluster exponentially and have an alternating structure [8]. In turn, these clusters cluster exponentially themselves with their own alternating structure. Furthermore, there are clusters of these clusters of clusters with similar arrangements, and so on. The dynamics associated with the building blocks consists of the flow of preimages through and out of a cluster or block. Sets of preimages ‘evenly’ distributed (say one per boundary basin) across a cluster of order $N$ (generated by an attractor of period $2^N$) flow orderly throughout the structure. If there is one preimage in each boundary basin of the cluster, each map iteration produces a migration from a boundary basin to a neighboring boundary basin in such a way that at all times there is one preimage per boundary basin, except for the inner ones in the cluster that are gradually emptied.

The self-similar feature in the hierarchy is demonstrated by the coarse-graining property amongst building blocks of the hierarchy that leaves the hierarchy invariant when $\mu = \mu_\infty$. That is, clusters of order $N$ can be simplified into clusters of order $N-1$. There is a self-similar structure for the clusters of any order and a coarse graining can be performed on clusters of order $N$ such that these can be reduced to clusters of lower order, the basic coarse graining is to transform order $N$ into order $N-1$. Also coarse graining can be carried out effectively via the RG functional composition and rescaling, as this transformation reduces the order $N$ of the periodic attractor. Automatically the building-block structure simplifies into that of the next lower order and the clusters of boundary basins of preimages are reduced in one unit of involvement.

Dynamically, the coarse-graining property appears as flow within a cluster of order $N$ simplified into flow within a cluster of order $N-1$. As coarse graining is performed in a given cluster structure of order $N$, the flow of trajectories through it is correspondingly coarse-grained. e.g., flow out of a cluster of clusters is simplified as flow out of a single cluster. The RG transformation via functional composition and rescaling of the cluster flow is displayed dynamically since by definition functional composition establishes the dynamics of iterates, and the RG transformation $Rf(x) = -\alpha f(f(-x/\alpha))$ leads for $N$ finite to a trivial fixed point that represents the simplest dynamical behavior, that of the period one attractor. For $N \to \infty$ the RG transformation leads to the self-similar dynamics of the non-trivial fixed point, the period-doubling accumulation point.

There is a modular structure of embedded clusters of all orders. The building blocks, clusters of order $N$, form well-defined sets that are embedded into larger building blocks, sets of clusters of order $N+1$. As the period $2^N$ of the attractor that generates this structure increases the hierarchy extends and as $N$ diverges a fully self-similar structure develops. The RG transformation for $N \to \infty$ no longer reduces the order of the clusters and a nontrivial fixed point arises. The trajectories consist of embedded flows within clusters of all orders.
The entire flow towards the $2^N$-period attractor generated by an ensemble of initial conditions (distributed uniformly across the phase space interval of the map) follows methodically a pattern predetermined by the hierarchical structure of embedded clusters of the preimages.

Each module exhibits a basic kind of flow property. This is the exponential emptying of trajectories within a cluster of order $N$. Trajectories initiated in the boundary basins that form a cluster of order $N$ flow out of it with an iteration time exponential law. The flow is transferred into a cluster of order $N + 1$. This flow is a dynamical module from which a structure of flows is composed. The emerging property that appears when $N \to \infty$ is that there is a power-law emptying of trajectories for the entire hierarchy. The flow of trajectories towards an attractor of period $2^N$ proceeds via a sequence of stage or step flows each within a cluster of a given order. Thus the first is through a cluster of order 1, then through a cluster of order 2, etc., until the last stage is through a cluster of order $N$. The sequence evolves in time via a power law decay that is modulated by logarithmic oscillations. This is the emerging property of the model.

4. Summary and discussion

The remarkable properties of unimodal maps already known for a long time [4], [5] have contributed significantly to the historical development of nonlinear dynamics. The two sets of properties that correspond to the dynamics within and towards the Feigenbaum attractor have been studied in detail a few years ago, in the former case the intricate structure of the sensibility to initial conditions was determined [6], while in the latter case a surprisingly rich dynamical organization was discovered [7]. Additionally, a relationship of statistical-mechanical nature between these two types of dynamical properties was established [7].

Here we have made use of the results from these studies to identify the features of the dynamics of an ensemble of trajectories that approach the Feigenbaum attractor as a genuine model hierarchy where all the elements are clearly defined and evaluated, including the observation of an emergent property, the power-law decay with logarithmic oscillations that accompanies the formation of the band structure of the multifractal attractor, and that is in fact an integral part of a basic statistical-mechanical expression that relates an entropy to a partition function [7]. This statistical mechanics is of a one-parameter deformed type [7].

We have demonstrated that a dynamically-organized hierarchy with well-defined modules arises directly from a simple formal expression with an iteration rule without the need of any other kind of ingredients. We have chosen a classical one-dimensional nonlinear system, a unimodal map as exemplified by the logistic map, to demonstrate that the intricate dynamical behavior at the transition to chaos fulfills the stipulations of such hierarchical system. This may serve as a basic idea to build or design different types of workable hierarchical models appropriate to different purposes.

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5. References

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