Dynamics of the relativistic Gross-Pitaevskii equation with harmonic potential: Following the variational approach

F. J. Poveda-Cuevas

Instituto de Física, Universidad Nacional Autónoma de México,
Apartado Postal 20-364, 01000 México D. F., México

R. P. Teles

Department of Physics and Astronomy, Rice University,
6100 Main St., Houston, Texas 77251, USA

Abstract

The role of the collective excitations as well as the free expansion dynamics provide a key diagnostic tools for trapped Bose-Einstein condensations. Based on such dynamics we proposed to study the relativistic version of them in the context of a macroscopic occupation of the ground-state for spin-0 particles. Therefore we used the Higgs model where the external trap is introduced by a non-minimal coupling. Along with variational method, we obtained a nonlinear coupling between dipolar and monopolar modes. Furthermore, the free expansion is no longer ballistic reaching a relativistic confinement.

*Electronic address: fjpovedac@gmail.com
†Electronic address: rafael.fisica.unesp@gmail.com

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I. INTRODUCTION

Phenomena such as superfluidity, superconductivity or topological defects were successfully described by theories of nonlinear fields, and the correspondence between field theories for non-relativistic and relativistic systems has appeared throughout history. For example, the two-fluid model of the Ginzburg-Landau theory [1, 2] has an analogy with the scalar electrodynamics [3–5]. There is also a clear analogy between the nonlinear Schrödinger equation [6] with Abelian Higgs model (AHM) for spinless scalar bosons [7]. An interesting case is the nonlinearity in the equation of motion that describes the dynamics of the macroscopic occupation of ground state for many bosons, i.e. the Gross-Pitaevskii equation (GPE) [8–10]. Thus, the GPE approach preserves some physical properties and characteristics such as superfluidity and vorticity, which are present in Bose-Einstein condensation (BEC) of atomic trapped gases [11]. In fact, during the last decades, the GPE is in agree with the experimental demonstrations of BEC in harmonically trapped ultracold gases [12–14]. These ones have stimulated the interest of many fields in physics [15]. The purpose of this paper is to show a relativistic version of the GPE in a harmonic trap.

There are strong motivations to study the dynamics of GPE in different scenarios. For instance, the analogy with BEC in the relativistic regime can be associated to unified cosmological models and observations related with Dark Energy and Dark Matter [16, 17]. It is remarkable that the hydrodynamics equations of GPE are recovered from the Klein-Gordon equation in a simple model for the gravitational potential [18]. In addition, there are perspectives for using experiments of ultracold quantum gases to learn about the dynamics of high energy particles due to the dynamical universality classes of many body systems far from equilibrium [19]. Finally, the understanding of quantum criticality is through studying the finite-density $O(2)$ model [20], which is important for superfluid-insulator boundary in Bose-Hubbard model being analog to the equation of motion proposed in this article.

The space-like quadratic term in GPE corresponds to the external harmonic potential. This is naturally introduced by coupling the internal degrees of freedom of the atoms with an external field. Actually, GPE can be considered as a Schrödinger equation for a harmonic oscillator with an effective interaction term proportional to the density. In the same spirit, a harmonic potential in the Klein-Gordon equation (KGE) for a complex scalar field can be
introduced by following coupling (adopting the units $\hbar = c = 1$) \cite{21,22}:

$$\hat{p}_i \rightarrow \hat{p}_i - im\omega_i \hat{x}_i, \quad \hat{p}_i^\dagger \rightarrow \hat{p}_i + im\omega_i \hat{x}_i.$$  \hfill (1)

where $\omega_i$ is the external trapping frequency in $i$-direction ($i = 1, 2, 3$). This approach is motivated by the prescription of Moshinsky for Dirac oscillator \cite{23–26}, whose coupling preserves the simple formulation and the analytic solution of the Dirac oscillator \cite{27–29}. Now, we simply have a way to coupling an AHM to external harmonic potential, which is basically a nonlinear KGE for a complex scalar field, $\hat{\Phi}(\mathbf{r}, t) \equiv \hat{\Phi}$. Thus, the equation of motion is given by:

$$\frac{\partial^2 \hat{\Phi}}{\partial t^2} - \nabla^2 \hat{\Phi} + \sum_{i=1}^{3} \left( m^2\omega_i^2 x_i^2 - m\omega_i \right) \hat{\Phi} + m^2 \hat{\Phi} + \lambda \left( \hat{\Phi}^\dagger \hat{\Phi} \right) \hat{\Phi} = 0, \hfill (2)$$

similarly for $\hat{\Phi}^\dagger$. Note that when $\omega_i \rightarrow 0$ we recover the AHM. This eq.(2) has essentially the same form of GPE except by second order derivative in time, therefore we call it as Relativistic Gross-Pitaevskii equation (RGPE). We approximate the quantum field to classical fields: $\hat{\Phi} = \Phi + \delta \hat{\Phi}$ and $\hat{\Phi}^\dagger = \Phi^* + \delta \hat{\Phi}^\dagger$, such that $\delta \hat{\Phi}$ and $\delta \hat{\Phi}^\dagger$ are negligible, and we consider $\Phi$ and $\Phi^*$ as order parameters. These parameters describe a macroscopic occupation of the ground state for particles with spin-0 in the presence of self-interaction, which is represented by the $\lambda$-parameter.

It is worth mentioning that, although our work has a strongly motivation in possible applications to cosmology and cosmological models, our proposal is only focused on showing some of the dynamic properties of RGPE in the presence of an external potential. We emphasize that a more exhaustive study is required in the field of dark matter and energy physics in order to elucidate the correspondence with the RGPE.

This paper is organized as follows: In section II the variational approach for Klein-Gordon oscillator is briefly introduced. In section III contains the equations of motion and their solution, i.e. the stationary solution, the collective modes, and the free expansion dynamics. The conclusions and outlooks are summarized in section IV.

**II. PRELIMINARIES: VARIATIONAL APPROACH**

In order to evaluate the collective excitations as well as time-of-flight dynamics for RGPE, we proposed to extend the variational method with time-dependent parameters.
This method has already proven to be useful in the studying of trapped Bose-Einstein condensation even in the presence of vortices [30–32]. It consists to write an effective Lagrangian density for a classical complex field \( \Phi \equiv \Phi (\mathbf{r}, t) \), which is given by

\[
L = \left( \frac{\partial \Phi^*}{\partial t} \right) \left( \frac{\partial \Phi}{\partial t} \right) - (\nabla \Phi^*) (\nabla \Phi) - (m^2 \omega_i^2 x_i^2 - m \omega_i + m^2) \Phi^* \Phi - \frac{\lambda}{Q} (\Phi^* \Phi)^2. \tag{3}
\]

In principle the complex field can be given by

\[
\Phi (\mathbf{r}, t) = \phi (\mathbf{r}, t) e^{i \chi (\mathbf{r}, t)}. \tag{4}
\]

Let us elucidate some points about this complex field. Different of the regular Klein-Gordon equation, the wave functions \( \Phi (\mathbf{r}, t) \) and \( \Phi^* (\mathbf{r}, t) \) no longer mean particles with different charges, because now their product represents the charge density of a macroscopic state where particles are coherent and indistinguishable. In other words, since the charge is no longer the difference of particles, they are impossible to count. Here the density determines the charge and the phase dynamics determines currents of the trapped relativistic condensate. Thus, the wave function can be normalized to charge with the amplitude \( \phi (\mathbf{r}, t) \), i.e.

\[
Q = \int \phi (\mathbf{r}, t)^2 d\mathbf{r}, \tag{5}
\]

and the conserved quantities (currents) are given by

\[
j_0 = -\frac{ie}{m} \int \frac{\phi (\mathbf{r}, t)^2 \partial \chi (\mathbf{r}, t)}{\partial t} d\mathbf{r}, \tag{6}
\]

\[
j = -\frac{ie}{m} \int \frac{\phi (\mathbf{r}, t)^2 \nabla \chi (\mathbf{r}, t)}{\partial \mathbf{r}} d\mathbf{r}. \tag{7}
\]

These two quantities are less important if the target is that the system presents a dynamics. Actually when one works with GPE, the phase \( \chi (\mathbf{r}, t) \) plays a fundamental role which must be carefully chosen in order to attain physically consistent results. Nevertheless, in relativistic dynamics we can adopt the gauge which eliminates the small fluctuations of the phase, and keeps physically consistent. Therefore, we adopt \( \chi = \text{const.} \) as the simplest gauge to work. This is basically an uncharge field.

By following the variational principle, the Lagrangian is calculated as

\[
L (t, q_i, \dot{q}_i) = \int \mathcal{L} (t, x_i, q_i, \dot{q}_i) dx_i. \tag{8}
\]
which yields the Euler-Lagrange equations

$$\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0,$$

(9)

where $q_i \equiv q_i(t)$ are the parameters of the amplitude $\phi$, and dot is related with the time derivative.

The variational method needs a suitable Ansatz which describes the ground state of RGPE, such that should be similar to the ground state of the relativistic Klein-Gordon oscillator (KGO) in limit of low interaction strength, i.e. $\lambda \ll 1$.

III. EQUATION OF MOTION FOR A SPHERICAL GROUND STATE

Since the ground state of KGO can be expressed by a Gaussian function, we use this one normalized to the charge $Q$ with the center-of-mass displaced by $\eta_i(t)$ from the center of the harmonic potential, and the Gaussian width is $\sigma_i(t) = \sigma(t)$. This function is given by

$$\phi(t, x_i) = \sqrt{\frac{Q}{\pi^{3/2} \sigma^3}} \prod_{i=1}^{3} e^{-(x_i - \eta_i)^2 / 2 \sigma^2}.$$

(10)

For those who are acquainted with GPE, this Ansatz is so known as ideal gas. We choose it instead Thomas-Fermi Ansatz due to the system has a weak interaction strength when compared with kinetic energy.

By substituting (10) in (8) and changing to a dimensionless scale ($\sigma(t) \rightarrow a_0\sigma(t)$, $\eta_i(t) \rightarrow a_0\eta_i(t)$, and $\tau \rightarrow \omega t$), we obtain the Lagrangian for a spherical field

$$L = \frac{Q}{a_0^2} \left( 3 - \frac{1}{\alpha^2} \right) + \frac{3Q}{2a_0^2} \left\{ \alpha^2 \frac{\dot{\sigma}^2}{\sigma^2} - \frac{1}{\sigma^2} - \sigma^2 - \gamma + \frac{1}{3} \sum_{i=1}^{3} \left( \alpha^2 \frac{\ddot{\eta}_i^2}{\sigma^2} - 2\eta_i^2 \right) \right\},$$

(11)

where the oscillator length is $a_0 = 1/\sqrt{m\omega}$, the dimensionless interaction strength $\gamma = \lambda/3 (2\pi)^{3/2} a_0$, and $\alpha = \sqrt{\omega/m}$ is a dimensionless constant. The Euler-Lagrange equations results in three equations for the center of mass

$$\alpha^2 \left( \ddot{\eta}_i - \frac{2\dot{\eta}_i \dot{\sigma}}{\sigma} \right) + 2\eta_i \sigma^2 = 0,$$

(12)

and a fourth equation for the Gaussian width

$$\alpha^2 \left( \ddot{\sigma} - \frac{\dot{\sigma}^2}{\sigma} + \frac{\sum_{i=1}^{3} \ddot{\eta}_i^2}{3\sigma} \right) + \sigma^3 = \frac{1}{\sigma} + \frac{3\gamma}{2\sigma^2}.$$

(13)
Then we can keep the term proportional to $\gamma$ since the ground-state will not change much, besides just of a scale factor that can be adjusted. The both above equations of motion can be interpreted as Newton’s equations where each term has a force interpretation such as: quantum pressure ($1/\sigma$), harmonic confinement ($\sigma^3$ and $2\eta_i\sigma^2$), and self-interaction ($3\gamma/2\sigma^2$). These terms are slightly different in the context of GPE, however they have the same meaning. The square velocities and crossed velocities terms are related with either the buoyancy or the drag force. This viscosity behavior can be a part of the quantum pressure due to the physical vacuum. By doing the accelerations and velocities equal to zero ($\ddot{\sigma} = \dot{\eta}_i = \dot{\sigma} = \dot{\eta}_i = 0$), we obtain the equilibrium points being a trivial value for the center of mass
\[ \eta_{0i} = 0, \] (14)
and a polynomial equation of fifth order for the width
\[ \sigma_0^5 - \sigma_0 = \frac{3\gamma}{2\sigma_0}. \] (15)
The solution of eq. (15) is found numerically by using the Newton’s method. This is exactly the same stationary solution for the width equation that comes from GPE.

A. Dipolar and monopolar vibrational modes

The behavior of low-lying collective oscillations are a consequence of small perturbations around the stationary condition in trapped condensates, thus they represent the linear oscillatory modes of the system. Their collective behavior is due to the interaction strength described by the non-linear part from RGPE.

By introducing the deviations from equilibrium points such as
\[ \eta_i (\tau) \approx \eta_{0i} + \delta\eta_i (\tau), \] (16)
\[ \sigma (\tau) \approx \sigma_0 + \delta\sigma (\tau), \] (17)
these ones are expanded in Taylor’s series until first order terms. Thus we obtain two uncoupled equations, where the frequencies of the dipole mode is given by
\[ \alpha_2^2 \omega_d^2 = 2\sigma_0^2, \] (18)
and the frequency of monopole mode is

\[ \alpha^2 \omega_m^2 = 4\sigma_0^2 - \frac{3\gamma}{2\sigma_0^3}. \]  

(19)

Note that the dimensionless frequencies of modes are given by \( \bar{\omega}_i = \omega_i/\omega \).

It is worth mentioning that this monopole mode is degenerate, i.e. there are at least more two quadrupole modes with almost the same frequency. These quadrupole modes can be calculated by using a non-symmetrical trial function. This degeneracy would be a subject to explore as next work.

**B. Nonlinear coupling between dipolar and monopolar modes**

Now that we know how the condensate must respond to small perturbations, we go to solve numerically the nonlinear equations of motion (12) and (13), which can be solved by using the fourth order Runge-Kutta method. Our initial condition is given by \( \eta_{03} = 0 \), and \( \sigma = \sigma_0 + \delta\sigma \); where \( \delta\sigma \) is a deviation from equilibrium configuration. Thus we observe that \( \sigma \) oscillates with \( \omega_m \), and the center of mass does not oscillate. Nevertheless, if we use \( \eta_{03} \neq 0 \) and \( \sigma_0 \) as initial condition, then we observe a beat in both oscillatory motions as shown in fig.1. Therefore the motion of the center of mass is coupled with monopolar mode, where the monopolar mode can be excited due to the motion of the center of mass, however the opposite situation is not true. It is a remarkable result because such motions are not coupled at all for GPE – the motion of the center of mass (dipolar mode) depends only on harmonic potential as a consequence of the generalized Ehrenfest theorem, while other ones such as monopolar and quadrupolar modes are consequence of the nonlinear effects due to the macroscopic occupation.

The difference between classical and relativistic cases is basically a second time derivative in the equation (2). This time derivative is responsible for coupling of both modes. Maybe, we can say that \( \alpha^2 \) is the coupling constant, since this dimensionless parameter delimits how relativistic the system is. This is also responsible for the center of mass oscillates with a distinct frequency than the oscillator.
(a) Solid black line corresponds to the oscillation of the radius, and dashed blue line corresponds to oscillation of the center of mass.

(b) Spectrum of frequencies for the oscillation of the radius.

(c) Spectrum of frequencies for the oscillation of the center of mass.

Figure 1: (Color online) We used $\alpha = 1$, $\gamma = 1$, $\eta_{01} = \eta_{02} = 0$ and $\eta_{03} = 0.5$.

C. The free expansion dynamics

The time-of-flight pictures constitute the most of common method to measure an usual BEC. This method consists in switching off the harmonic trap and letting the atomic cloud expand freely for some time, typically in order of tens of milliseconds, and then taking
pictures of the expanded cloud [33]. By cutting off the terms responsible for the trap in equations (12) and (13), we are basically switching off the harmonic confinement. Thus the equations of motion become:

$$\ddot{\eta}_i - \frac{2\dot{\eta}_i \dot{\sigma}}{\sigma} = 0,$$

for the center of mass, and

$$\alpha^2 \left( \ddot{\sigma} - \frac{\dot{\sigma}^2}{\sigma} + \frac{3}{3\sigma} \sum_{i=1}^{3} \dot{\eta}_i^2 \right) = \frac{1}{\sigma} + \frac{3\gamma}{2\sigma^2},$$

for the motion of the condensate width. These equations are numerically solved by fourth-order Runge-Kutta method. By using the equilibrium situation as initial condition for the free expansion, we obtain that RGPE results in an hyper-ballistic free expansion when compared with the free expansion from GPE (fig. [2]). The RGPE condensate expands too fast that the width reaches a gigantic size in few milliseconds of free expansion, while GPE results in a ballistic free expansion. This hyper-ballistic expansion is a consequence of the viscosity behavior discussed before. The term $\dot{\sigma}^2/\sigma$ works as a buoyancy force that impulses the expansion even further. Nevertheless, its behavior is completely different if the condensate is oscillating before that the confinement is switched off.

Let us consider that the center of mass is oscillating, consequently the width is also oscillating, and the confinement is switched off at time $\tau_0 = \omega t_0$. Thus we find the time evolution of the velocity of center of mass by integrating equation (20) in time coordinate, which is given by:

$$\dot{\eta}(\tau) = \frac{\dot{\eta}(\tau_0)}{\sigma(\tau_0)} \sigma(\tau)^2. \quad (22)$$

The above velocity can be replaced in equation (21), which yields an artificial harmonic confinement as we can note:

$$\alpha^2 \left( \ddot{\sigma} - \frac{\dot{\sigma}^2}{\sigma} + \frac{3}{3\sigma} \sum_{i=1}^{3} \dot{\eta}_i^2 \right) = \frac{1}{\sigma} + \frac{3\gamma}{2\sigma^2}. \quad (23)$$

This artificial confinement has its own equilibrium point which is calculated by

$$\alpha^2 \left[ \sum_{i=1}^{3} \frac{\dot{\eta}_i^2}{3\sigma(\tau_0)^4} \right] \sigma_{tof}^5 - \sigma_{tof} = \frac{3\gamma}{2}, \quad (24)$$

where the frequency of its linear mode is calculated by introducing a small deviation in condensate width

$$\sigma(\tau) = \sigma_{tof} + \delta\sigma(\tau). \quad (25)$$
Therefore the time-of-flight frequency is given by:

\[ \alpha^2 \omega_{tof}^2 = 4\alpha^2 \left[ \frac{\sum_{i=1}^{3} \dot{\eta}_i(\tau_0)^2}{3\sigma(\tau_0)^4} \right] \sigma_{tof}^2 + \frac{3\gamma}{2\sigma_{tof}^2}. \]  

(26)

The artificial confinement is a directly consequence of the second time derivative on RGPE. Thus we can consider that this is a relativistic effect of the vacuum viscosity, and we name it as relativistic confinement. It can also be interpreted as a competition between the buoyancy force and the drag force.

D. Non-relativistic and ultra-relativistic limits

The regimes present in this system is related with the constant

\[ \alpha^2 = \frac{\hbar \omega}{mc^2}. \]  

(27)

This is a constant that has a ratio of two energy types. The energy as divisor is clearly the vacuum energy given by the rest mass. In the numerator one has \( \hbar \omega \) which may be related
either with photon energy, or with harmonic potential energy. If $\hbar \omega$ comes from harmonic potential, then $\alpha^2$ is interpreted as a constant that determines the degree of confinement. Indeed, it makes sense to explain the limits of $\alpha^2$ ($\alpha \to 0$, and $\alpha \to \infty$), however the equations give us a different evidence. Since $\alpha^2$ appears only in the time-derivatives and the rest mass energy, it means that $\hbar \omega$ is related with momentum (i.e. photon energy). Thus $\alpha^2$ says how relativistic is our system.
The non-relativistic case happens when $\alpha \to 0$, because the equations tend to stationary solution as well as the frequencies of collective modes tends to infinity. In other words, the classical stationary case or the higher confinement situation (if $\alpha^2$ is interpreted as degree of confinement).

In another hand, the ultra-relativistic case happens when $\alpha \to \infty$, because the equations becomes dependent on only accelerations and velocities:

$$\ddot{\eta}_i - \frac{2\dot{\eta}_i \dot{\sigma}}{\sigma} = 0,$$

$$\dot{\sigma} - \frac{\dot{\sigma}^2}{\sigma} + \frac{1}{3} \sum_{i=1}^3 \dot{\eta}^2_i = 0.$$  \hfill (28)

Thus the relativistic confinement is a fraction of ultra-relativistic confinement in addiction of the self-interaction effect, i.e. $\omega^2_{tof} = \frac{4}{3} \omega^2_{ur} + \frac{3\gamma}{2a^2 \sigma_{tof}^4}$, where

$$\lim_{\alpha \to \infty} \omega^2_{tof} = \frac{4}{3} \omega^2_{ur} = \frac{4}{3} \left[ \frac{\sum_{i=1}^3 \dot{\eta}_i (\tau_0)^2}{\sigma (\tau_0)^4} \right] \sigma_{tof}^2.$$  \hfill (29)

This is the lower confinement situation in the another interpretation. The system is relativistic for any other value of $\alpha$.

**IV. CONCLUSIONS**

In this article, we propose the study of a relativistic version for the GPE in the presence of a harmonic external potential. The goal is to point out the differences between RGPE and – already so well known – GPE with respect to the collective excitations as well as the free expansion dynamics. We assume a mean-field as a macroscopic occupation of ground state, and try the variational method usually used with GPE to treat the RGPE.

The spherical harmonic potential is the simplest symmetry to check out such behaviors. The stationary solution of RGPE is the same of GPE. The relativistic system shows itself similar in the category of collective modes. It presents both dipole and monopole modes as non-relativistic case, however they have a nonlinear coupling unlike the non-relativistic case. The nonlinearity of the equations allows that the motion of the center of mass excites the monopole mode, i.e. the dipole mode transfers kinetic energy to condensate width that otherwise is not true.

The free expansion is no longer ballistic becoming faster and faster each time unit. Furthermore, a peculiar expansion behavior appears when this relativistic entity is evolving a
motion of the center of mass before the confinement shutdown. In this conditions, the condensate is released with a initial velocity which yields in an artificial confinement. It means that the condensate oscillates rather than freely expanding.

Our results suggest the existence of a vacuum viscosity as a consequence of the relativistic feature. We based our interpretation on the equations of motion which present forces as either buoyancy-like or drag force-like. These are opened questions to investigate as next works. It is also necessary to extend this subject to numerical simulations as well as a properly study for the dynamic phase. The phase dynamics is able to reduce the hyper-ballistic behavior, since the system presents charge-attraction due to the currents.

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