Thermodynamics and fluctuations in finite-time quantum heat engines under reservoir squeezing

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We investigate the thermodynamics and fluctuations of a finite-time quantum Otto engine alternatively driven by a hot squeezed and a cold thermal reservoir. We show that reservoir squeezing significantly enhances the performance by increasing the thermodynamic efficiency and the power, and enables higher stability by decreasing the relative power fluctuations and speeding up the convergence of quantum efficiency to its most probable value. We also demonstrate the counternuitive result that the efficiency can be larger than the Otto limit in the finite-time operation. Experimental demonstration of this quantum heat engine can be available, based on a single-electron spin pertaining to a trapped 40Ca+ ion [1]. We provide a general framework for reliably studying the finite-time nanoengine in finite time operation which accounts for quantum friction and coherence, deriving important insights into the novel thermodynamic behaviors beyond the classical thermal machines.

Quantum heat engines have become a laboratory reality, notably the recent experiments realizing quantum Otto heat engines on nuclear magnetic resonance [2, 3] and nitrogen-vacancy centers in diamond [4]. These thermal machines wherein, apart from the working substance, the reservoirs may be finite-dimensional and thus non-thermal [5, 6], have access to nano-scale open systems in which quantum effects manifest themselves, such as coherence [7–19], entanglement [20–24], correlations [25–28], quantum measurements [29–33], and squeezing [34–40]. The quantum engines in the presence of these additional freedoms may outperform their classical counterparts [41–52]. This constitutes one of the central issues in quantum thermodynamics.

For microscopic systems, heat and work are no longer deterministic [53–57] as is the case for macroscopic systems. As a result, the efficiency and power for quantum heat engines are stochastic, and both of them are fluctuating. The power fluctuations, together with the efficiency fluctuations [58, 59], as a limiting factor for the practical usefulness in heat engines, measure the machine stability [55]. Ideally, the quantum heat engine should have high efficiency (small entropy production), large power, and small fluctuations for these thermodynamic variables measuring performance. Strong emphasis has been put on the finite-time thermodynamics of the quantum heat engines, and in particular on fluctuations of power and efficiency [59–66].

Unlike the previous studies considering nanoengines [2, 3, 34, 35, 51, 52] where quasistatic and local-equilibrium approximations were required and thus some quantum effects were tied to ignoring, we develop a formalism for analyzing the performance and stability for quantum heat engines by overcoming these limitations. We show that both efficiency and power are enhanced by reservoir squeezing with the advantage of decreasing fluctuations of efficiency and power, which is the generic case for finite-time cyclic heat engines driven by non-thermal reservoirs. The result that the efficiency can be enhanced by speeding up the machine even in the absence of squeezing is in stark contrast to previous reports [18, 67–69]. In particular, we find the counter-intuitive result that the efficiency can beat the Otto limit when and only when the unitary driving proceeds in finite time. The result relies only on purely quantum origin and it would not hold when either unitary driven stroke or thermal-contact process satisfies the quasi-static limit.

We consider a quantum Otto engine cycle working between a hot squeezed and a cold thermal bath [see Fig. 1(a)]. This engine cycle consists of two unitary and two isochoric strokes. Firstly, unitary compression from state \( \rho_{t_0} \) to \( \rho_{t_1} \) with \( t_0 = 0 \): the energy gap is enlarged by a spin–1/2 system with Hamiltonian \( H_{ch}(t) = \frac{\omega(t)}{\tau} (\cos\frac{\omega(t)}{2\tau_{ch}})\sigma_x + \sin\frac{\omega(t)}{2\tau_{ch}}\sigma_z \), where \( \omega(t) = \omega_c (1 - t/\tau_{ch}) + \omega_h (t/\tau_{ch}) \) with \( \tau_{ch} = t_1 \) and \( 0 \leq t \leq \tau_{ch}, \) and \( \sigma_{x,y,z} \) are the Pauli matrices. The driven Hamiltonian does not commute at different times, generating quantum coherence in the energy basis of the system. Secondly, isochoric heating from state \( \rho_{t_1} \) to \( \rho_{t_2} \): the system is weakly coupled to a hot squeezed reservoir at inverse temperature \( \beta_h \) during time duration \( t_h \) with \( \tau_h = t_2 - t_1 \), while its Hamiltonian keeps a constant as \( H_{ch}(t) = H_{ch}(t_1) = \frac{\omega_h}{\tau} \sigma_z /2 \). Thirdly, unitary expansion from state \( \rho_{t_2} \) to \( \rho_{t_3} \): the driven Hamiltonian \( H_{ch}(t_3 - t) = H_{ch}(t) \) is realized by reversing the protocol used in the unitary compression, such that the expansion Hamiltonian takes the same time as the compression Hamiltonian, namely, \( \tau_{dri} = \tau_{hc} = \tau_{ch} \). Lastly, isochoric cooling from state \( \rho_{t_3} \) to \( \rho_{t_4} \): the system is weakly coupled with a cold thermal reservoir at inverse
(b) Transition probability as a function of driving time $\tau$.

(c) Coherence and Kullback-Leibler divergence at $t = 100\gamma$. The parameters are $h = 1$, $\omega_c/2\pi = 1000$, $\omega_b/2\pi = 2250$, $\beta_c = 2/(\hbar \omega_c)$, $\beta_h = 1/(\hbar \omega_h)$, $\tau_c = 3$, and $\gamma_c = \gamma_h = 3$.

FIG. 1: (a) Illustration of a spin-1/2 system operating with a quantum Otto cycle alternatively driven by a hot squeezed and a cold thermal bath. The system states at times $t = t_i$ with $i = 0, 1, 2, 3$ are denoted by $\rho_{t_i}$, while $\rho_{t}$ are the initial states of the four strokes in a cycle, respectively. In each cycle the machine produces the total work $-\langle w_{tot} \rangle$ by absorbing average heats from the hot and cold baths, $q_h$ and $q_c$, where $q_h = -\langle w_{tot} \rangle - q_c$ due to the energy conservation.

The dynamics of the system during a unitary stroke where no heat is exchanged is given by $\Delta = \{H, \rho\}$. In the limit cycle $|\tau|$, where a periodic steady state is achieved with all the periodic variables.

The dynamics of a quantum system weakly coupled to a heat reservoir of inverse temperature $\beta$ can be described by quantum master equation in Lindblad form [71, 72]

$$\frac{d\rho}{dt} = -i[H, \rho] + \mathcal{L}(\rho),$$

where $\mathcal{L}$ is the Lindblad super operator describing heat dissipation responsible for driving the system to the steady state where $\rho = \rho_\text{eq} = \exp(-\beta H) / \text{Tr}(\exp(-\beta H))$. The dynamics of the system during a unitary stroke where no heat is exchanged is given by $\frac{d\rho}{dt} = -i[H, \rho]$ [3, 70].

The system states at the respective ends of the two driven strokes are then $\rho_{t_1} = U_{hc}\rho_0U_{hc}^\dagger$ and $\rho_{t_3} = U_{hc}\rho_{t_2}U_{hc}^\dagger$, where $U_{hc} = \mathcal{T}_\omega \exp(-\frac{i}{\hbar} \int_{t_2}^{t_3} dt H_{hc}(t))$ and $U_{hc} = \mathcal{T}_\omega \exp(-\frac{i}{\hbar} \int_{t_1}^{t_2} dt H_{hc}(t))$, with the time-ordering operator $\mathcal{T}_\omega$. Also, it is noted that in an iso-choric where the system Hamiltonian is fixed, the dynamics (1) becomes $d\rho_{t}/dt = \mathcal{L}(\rho_{t})$, where $\rho_{t}$ is replaced by $\rho_{eq} = \mathcal{S}(\rho_{t})\rho_{eq}\mathcal{S}(\rho_{t})$ with $\mathcal{S}(r) = \exp(r^* \sigma_+ - r\sigma_-)$ being dependent on both squeezing parameter $r$ and $\sigma_\pm = (\sigma_x \pm i \sigma_y)/2$ in the presence of reservoir squeezing. Hence, the quantum Lindblad equation determines density matrices $\rho_i$ ($i = 0, 1, 2, 3$) for the spin system [See Supplementary Material (SM), Sec. I [73]].

Based on the aforementioned dynamical description, we derive the average work $\langle w_{tot} \rangle$, average injection $\langle q_h \rangle$, and work fluctuations, $\delta w_{tot}^2 = \langle w_{tot}^2 \rangle - \langle w_{tot} \rangle^2$, can be given (see SM Sec. II [73]) by

$$-\langle w_{tot} \rangle = h(\omega_h - \omega_c)(\langle n_{t_2} \rangle - \langle n_{t_0} \rangle) + 2\hbar\xi(\langle n_{t_2} \rangle + \omega_h \langle n_{t_0} \rangle) - 2\hbar \omega_h \zeta_h - 2\hbar \omega_c \zeta_c,$$

(2)

$$\langle q_h \rangle = \hbar \omega_h \left[ \langle n_{t_2} \rangle + \langle n_{t_0} \rangle (2\xi - 1) - 2 \zeta_c \right],$$

(3)

$$\delta w_{tot}^2 = h^2 \omega_h^2 \left( \frac{1}{2} - \langle n_{t_2} \rangle^2 - \langle n_{t_0} \rangle^2 \right) + 2 \hbar \omega_c \omega_h \left( \langle n_{t_2} \rangle (1 - 2\xi) + 2 \zeta_c \right) + 2 \langle n_{t_2} \rangle (1 - 2\xi) + 2 \zeta_c + 1 - \xi. \right.$$
states are removed, and \( \langle w_{\text{trls}} \rangle \) represents the additional work that overcomes the inner friction causing unwanted diabatic transitions in instantaneous energy eigenstates. The squeezing results in an increase in the transitionless work and the coherent work, but a decrease in the amount of frictional work which is always negative, as shown in Fig. 2(a). The transitionless work \( \langle w_{\text{trls}} \rangle \) increases with increasing thermal-contact time \( \tau_h \) to reach the maximum value at which the system approaches to the thermal state, but the effects of \( \tau_h \) on both frictional work \( \langle w_{\text{frl}} \rangle \) and coherent work \( \langle w_{\text{coh}} \rangle \) are particularly small, as shown in Fig. 2(b). The coherent work displays the oscillations in quick isochoric stroke, since the coherence \( C(\rho_{\text{t2}}) \), which interferes with the coherence generated during the unitary expansion, is only partially erased.

With consideration of Eqs. (2) and (3), the thermodynamic efficiency, \( \eta_{\text{th}} = -\langle w_{\text{tot}} \rangle/\langle q_t \rangle \), is then obtained as

\[
\eta_{\text{th}} = \eta_{\text{Otto}} + 2(\omega_c/\omega_h)[\xi(\langle n_{\text{t0}} \rangle + \langle n_{\text{t2}} \rangle) - \zeta_{\text{hc}} - \zeta_{\text{ch}}],
\]

where \( \eta_{\text{Otto}} = 1 - \omega_c/\omega_h \) is the so-called Otto efficiency. Because the times taken for two isochoric and two unitary strokes are finite, the quantum coherence and inner friction are created, resulting in that the efficiency depends on both these kinds of quantum effects. Quite interestingly, the efficiency \( \eta_{\text{th}} \) for the heat engine \( \langle q_h \rangle > 0 \) may surpass the Otto efficiency \( \eta_{\text{Otto}} \) if \( \xi(\langle n_{\text{t0}} \rangle + \langle n_{\text{t2}} \rangle) > \zeta_{\text{hc}} + \zeta_{\text{ch}} \). We prove in SM Sec. III [73] that the thermodynamic efficiency, irrelevant to the Carnot bound [8], must be bounded by the generalized Carnot value \( \eta_{\text{gen}}^{\text{C}} = 1 - \beta_{\text{eff}}^{\text{C}}/\beta_c \) with \( \beta_{\text{eff}}^{\text{C}} = \ln(2 \cosh 2r + (e^{\beta_{\text{eff}}^{\text{C}}/\beta_c} - 1)(\cosh 2r - 1))/2 \cosh 2r + (e^{\beta_{\text{eff}}^{\text{C}}/\beta_c} - 1)(\cosh 2r - 1))/\hbar\omega_h \).

In contrast to the average work, the average efficiency of the quantum Otto engine may be ill-defined due to the possible divergence of the stochastic efficiency [59, 60]. Hence, we resort to large deviation theory associated with the exponential decay of probabilities of large fluctuations, assuming that the quantum engine proceeds in the long-time limit. The large deviation function of quantum efficiency can be given by [75]

\[
J(\eta) = -\min_{\varphi_2} \phi(\varphi_2 \eta, \varphi_2),
\]

where \( \phi(\varphi_1, \varphi_2) = \ln(e^{\varphi_1 q_h} + e^{\varphi_2 w_{\text{frl}}}) \).

In Fig. 2(c), the power oscillates as a function of the driving time \( \tau_{\text{dr}} \), and very quick driving speed results in poor power output. In our model, where the driving time \( \tau_{\text{dr}} \) is much smaller than the thermal-contact time \( \tau_h \) and thus the total cycle period \( \tau_{\text{cyc}} \) is dominated by \( \tau_h \), the contribution of the driving time to the power mainly comes from quantum inner friction which is responsible for irreversible work in each cycle. The efficiency increases with increasing driving time, although not monotonically.

We observe from Fig. 2(d) that the power first increases in small \( \tau_h \) and then decreases with further increase in \( \tau_h \). During the fast hot isochoric stroke the decoherence of the system is suppressed, yielding the additional, coherent work \( \langle w_{\text{coh}} \rangle \) which is responsible for the oscillation. Because the transitionless work \( \langle w_{\text{trls}} \rangle \) dominating the total work increases faster than linearly with increasing \( \tau_h \), the power increases with increasing \( \tau_h \) to a certain maximum value and then decreases gradually. The shapes of the efficiency and power curves are similar, except that \( \eta_{\text{th}} \) increases with \( \tau_h \) to reach its maximum value consistent with \( \eta_{\text{gen}}^{\text{C}} \). The oscillations of both the power and the efficiency in Fig. 2(d) come from the effect of the dynamical interference between the residual coherence after the second stroke and the coherence generated in the third stroke. Interestingly, in the large squeezing case (\( r = 1 \)) leads to large coherence [see Fig. 1 (c)] and thus large interference effect, which
accounts for large oscillations of these two performance measures, but these two measures become equivalent to their respective dephased counterparts in the long time $\tau_h$ where coherence is full erased as they should [see Fig. 2(b)]. By suitably controlling over the driving and thermalization times, the quantum engine may run in a favorable regime where both efficiency and power can be enhanced, as shown in Fig. 2(d).

The coherent work $\langle w_{coh} \rangle$ in Fig. 2(b) may contribute to the increase of the extracted net work. If the machine parameters are properly adjusted, the faster the unitary and thermal-contact processes are performed, the greater the contribution of the coherence to the total work extracted, since coherent work increases with speeding up these processes. The increase of the extracted work with shortening of time may lead to increase power [see Fig. 2(c) (up)], and, surprisingly, causes efficiency to surpass the Otto limit that is reached when coherence is fully erased or it is not generated along the unitary stroke. This, the main message from Fig. 2(e) (bottom), confirms our theoretical prediction (5) that quantum coherence, of purely quantum origin, can lead to a marked difference in machine performance.

It is of interest to note the following cases (see SM Sec. IV [73]): (i) while the efficiency is independent of squeezing in the case of large difference between two reservoir temperatures, it is sensitively dependent on the squeezing parameter $r$ in the linear response regime where the difference between two reservoir temperatures is small; (ii) the efficiency depends on the degree of squeezing in the low-temperature and high-temperature limits; (iii) In the latter case, by using endoreversible condition [76] we can reproduce the expression for the efficiency at maximum power [34, 35]: $\eta_{mp} = 1 - \sqrt{\text{soch}(2r)/\beta_c}$. The root-mean-square relative fluctuation of power, $\sqrt{\delta P^2}/P$, which is equivalent to the coefficient of variation of the work, $\sqrt{\delta w_{tot}^2}/(w_{tot})$. It measures the dispersion of the probability distribution and thus describe the machine stability. The relative power fluctuation decreases quickly as squeezing parameter $r$ increase, as shown in Figs. 3 (a) and 3 (b), showing that reservoir squeezing leads to an increase both average work and in work fluctuation, but the increase in the fluctuation is much less than in the average value. Fig. 3(b) shows that the oscillation timescale of power fluctuation with respect to the thermal-contact time $\tau_h$ agrees with the corresponding power and efficiency Fig. 2(d). The relative power fluctuation $\sqrt{\delta P^2}/P$ decreases while thermal-contact time $\tau_h$ or driving time $\tau_{dri}$ increases. In physical terms, the larger the thermal-contact time or driving time (quick driving accounting for quantum coherence) is, the closer the system to the stationary state, so the non-equilibrium thermal fluctuation of the power is expected to decrease.

We plot the large deviation function of stochastic efficiency for the quantum Otto engine in Fig. 3(c), where the curve has a maximum when the stochastic efficiency $\eta = \eta_C (\eta_C^\text{gen}$ reduces to $\eta_C$ if $r = 0$) and a minimum at $\eta = \eta_h$. The function $J(\eta)$ is situated between a maximum at the generalized Carnot efficiency $\eta_C^\text{gen}$ and a minimum at the thermodynamic efficiency $\eta_h$, which recovers the special case when squeezing was absent [61, 63]. We find that the standard thermodynamic efficiency is the most likely value, and the generalized Carnot efficiency is the least likely. Furthermore, the rate function $J(\eta)$ is strictly larger in presence of squeezing than the case without squeezing, with the exception of the point $\eta = \eta_h$. Figure 3(c) shows that the convergence of the heat engine towards the thermodynamic efficiency is improved by including the reservoir squeezing. This may be understood by noting that quantum efficiency fluctuations, which can be related to machine stability, are suppressed under reservoir squeezing.

Experimentally, a quantum Otto engine alternatively driven by a thermal and a squeezed bath can be implemented by employing the spin of the valence electron pertaining to a single trapped $^{40}\text{Ca}^+$ ion [1, 78] confined in a Paul trap. As the magnetic field along $z$ direction yields a Zeeman splitting, the Hamiltonian of the spin system can be given by $H = \hbar \omega_z \sigma_z/2$. The coupling between the spin and harmonic motion is mediated via an optical standing wave by a spin-dependent optical dipole force along the oscillation ($x$) direction, which reads $\hbar \Delta_{sw} \sin(k_{sw}\hat{x})\sigma_z/2$, with the amplitude of the standing wave $\Delta_{sw}$ and effective wave number $k_{sw}$. The unharmonic term, $\hbar \Delta_{sw} \sin(k_{sw}\hat{x})\sigma_z/2$, will be re-
sponsible for realization of the squeezed state of the motion [1, 79]. For example, the frequencies of the spin system, determined by magnetic field which may be along x or z direction, varies from $2\pi \times 8$MHz to $2\pi \times 14$MHz in each cycle. The experimental parameters available allow to observe the machine performance, which confirms our theoretical prediction based on the choice of values for $\omega_c$ and $\omega_h$ falling into a relatively large range (see SM Sec. V [73]).

In summary, we have presented a unified thermodynamic theory for a squeezed-bath-driven quantum Otto engine whereby all the variables are periodic and the efficiency is irrelevant to the Carnot value. We have shown that the engine under squeezing can outperform its non-squeezing counterpart by dramatically enhancing efficiency and power output, and even that the efficiency at positive power may beat the quantum Otto limit. We have demonstrated that reservoir squeezing significantly decreases relative power fluctuations and leads to faster convergence of the machine efficiency to its most probable value. Our findings demonstrate the potential of quantum engines fueled by nonthermal reservoirs [80] to realize ideal nano-scale engines with more efficient, larger power, and higher stability.

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Like squeezing, the quantum coherence and correlation are governed by the respective asymptotic forms of $p_K(q, w_{tot}) \propto e^{-K/(q, w_{tot})}$ and $p_K(q) \propto e^{-K/(q, w_{tot})}$. The large deviation functions, $I(q, w_{tot})$ and $J(q)$, describe the exponentially unlikely deviations of $q$ and $\eta$ from their most probable values. The rate function $J(q)$ can be obtained from $I(q, w_{tot})$ by the contraction: $J(q) = \min[I(q, -\eta q)]$. Let us define $q_{h}^{(K)} := \sum_{j=1}^{K} q_{h}^{(j)}/K$, $w^{(K)} := \sum_{j=1}^{K} w_{tot}^{(j)}/K$, $\phi(\varphi_1, \varphi_2) := \lim_{K \to \infty} \frac{1}{K} \ln(e^{K(\varphi_1 q_{h}^{(K)} + \varphi_2 w_{tot}^{(K)})}) = \ln(e^{\varphi_1 q_{h}^{(K)} + \varphi_2 w_{tot}^{(K)}})$, where we have used $e^{\varphi_1 q_{h}^{(K)} + \varphi_2 w_{tot}^{(K)}} = \int dqdw_{tot} e^{-\varphi_1 q_{h}^{(K)} - \varphi_2 w_{tot}^{(K)}} p(q, w_{tot})$. Using the Legendre-Fenchel transform, one then obtains the large deviation function of quantum efficiency (6).

While the vertical dash-double-dot line patterns (black, red, and blue) represent the values of the stochastic efficiency $\eta q_h$ equivalent to the thermodynamic efficiencies $\langle q_{h} \rangle |_{\eta q_h} = 0.461, \langle q_{h} \rangle |_{\eta q_h} = 0.488, \langle q_{h} \rangle |_{\eta q_h} = 0.512$, the vertical dash-dotted lines (black, red, and blue) indicate the values of the stochastic efficiency $\eta$ corresponding to the generalized Carnot efficiencies $\langle q_{h} \rangle |_{\eta q_h} = 0.778, \langle q_{h} \rangle |_{\eta q_h} = 0.817, \langle q_{h} \rangle |_{\eta q_h} = 0.884$. Since the work and heat are fluctuating quantities, the negative values of the stochastic efficiency $\eta$ with $w_{tot}/q_h < 1$ occur when the heat $q_h$ is positive with negative work $w_{tot}$ or vice versa, and the values of the efficiency larger than 1 happen when the heat $q_h$ is positive.

We use large deviation theory to analyze the efficiency statistics. We recall that the large deviation functions of the joint distribution $p(q_h, w_{tot})$ and the efficiency distribution $p_K(q)$ for a large number of cycles ($K \gg 1$) are governed by the respective asymptotic forms of $p_K(q, w_{tot}) \propto e^{-K/(q, w_{tot})}$ and $p_K(q) \propto e^{-K/(q, w_{tot})}$. The large deviation functions, $I(q, w_{tot})$ and $J(q)$, describe the exponentially unlikely deviations of $q_h$ and $\eta$ from their most probable values. The rate function $J(q)$ can be obtained from $I(q, w_{tot})$ by the contraction: $J(q) = \min[I(q, -\eta q_h)]$. Let us define $q_{h}^{(K)} := \sum_{j=1}^{K} q_{h}^{(j)}/K$, $w^{(K)} := \sum_{j=1}^{K} w_{tot}^{(j)}/K$, $\phi(\varphi_1, \varphi_2) := \lim_{K \to \infty} \frac{1}{K} \ln(e^{K(\varphi_1 q_{h}^{(K)} + \varphi_2 w_{tot}^{(K)})}) = \ln(e^{\varphi_1 q_{h}^{(K)} + \varphi_2 w_{tot}^{(K)}})$, where we have used $e^{\varphi_1 q_{h}^{(K)} + \varphi_2 w_{tot}^{(K)}} = \int dqdw_{tot} e^{-\varphi_1 q_{h}^{(K)} - \varphi_2 w_{tot}^{(K)}} p(q, w_{tot})$. Using the Legendre-Fenchel transform, one then obtains the large deviation function of quantum efficiency (6).

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