Entanglement percolation with bipartite mixed states

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Abstract – We develop a concept of entanglement percolation for long-distance singlet generation in quantum networks with neighboring nodes connected by partially entangled bipartite mixed states. We give a necessary and sufficient condition on the class of mixed network states for the generation of singlets. States beyond this class are insufficient for entanglement percolation. We find that neighboring nodes are required to be connected by multiple partially entangled states and devise a rich variety of distillation protocols for the conversion of these states into singlets. These distillation protocols are suitable for a variety of network geometries and have a sufficiently high success probability even for significantly impure states. In addition to this, we discuss possible further improvements achievable by using quantum strategies including generalized forms of entanglement swapping.

The distribution of entanglement through quantum networks is of essential importance to quantum cryptography and distributed quantum computing. However, the generation of entanglement between remote nodes in a network faces a severe obstacle. Due to noise, e.g. in transmission lines, desired maximally entangled states will degrade into mixtures and the re-establishment of high-fidelity entanglement requires sophisticated purification schemes, e.g. involving quantum repeaters [1,2]. Although quantum repeaters are a promising tool for quantum communication they still require a considerable overhead of physical resources or are relatively slow [3]. An alternative scheme was recently proposed by Acín et al. [4] in which ideas from classical bond percolation have been applied to lattice-shaped quantum networks. It was shown that maximally entangled singlet states can be created between arbitrary points of the network, with a probability that is independent of the distance between them, if the network nodes were initially connected by partially entangled pure states with sufficiently high entanglement. The scheme relies on the fact that states of this type can be converted by local operations and classical communication (LOCC) into singlet states with finite probability [5]. If this singlet conversion probability (SCP) exceeds a certain, lattice-geometry–dependent threshold, arbitrarily large clusters of network nodes connected by singlets are created. By successively applying entanglement swapping it is then possible to establish singlets between arbitrarily distant nodes in a cluster. Although being a very promising concept, this classical entanglement percolation (CEP) is not optimal. Indeed it was shown in [4,6–9] that quantum strategies can be used to improve the SCP leading to even more powerful protocols.

CEP, particularly when extended by quantum strategies, offers the possibility to establish entanglement in quantum networks over long distances. However, in previous work this has only been considered for pure states. In this paper we develop a concept of entanglement percolation for mixed states. We start by analyzing the minimum requirements for singlet generation in quantum networks where the nodes are initially connected by a finite number of mixed 2-qubit states. We show that the generation of a singlet between two arbitrary nodes is possible if and only if they are connected by at least two “paths” consisting of a particular class of mixed states (see fig. 1(a)). If this is not the case then a singlet in such a network can never be formed between the nodes using LOCC. Fortunately, it turns out that these mixed states are not only of theoretical interest but arise naturally in systems undergoing amplitude damping which is relevant for many practical setups, e.g. where photon loss or spontaneous photon emission is dominant. Note that, as in previous work on entanglement percolation [4], the sole requirement we impose is that the network initially contains only bipartite...
A rep represents a partially entangled bipartite mixed state, i.e.,

\[
\rho(\alpha, \gamma, \lambda) = \lambda |\psi\rangle \langle \psi| + (1 - \lambda) |01\rangle \langle 01|,
\]

where \( |\psi\rangle = \sqrt{\alpha} |00\rangle + \sqrt{1 - \alpha - \gamma} |11\rangle + \sqrt{\gamma} |01\rangle \)

and \( 0 \leq \lambda \leq 1 \). States like this occur naturally if a state \( |\psi\rangle \) is subject to amplitude damping. We distill a singlet from two states of this type, \( \rho(\alpha, \gamma, \lambda) \) and \( \rho(\beta, \delta, \nu) \), with a finite probability, by performing a C-NOT gate between the qubits in each group and measuring the two target qubits, i.e., the two qubits originally forming the state \( \rho(\beta, \delta, \nu) \), in the computational basis. If in both cases the qubit is found in the state \( |1\rangle \) a pure, entangled state is created. We call this first stage of the protocol pure-state conversion measurement (PCM). In the case of identical states, i.e., \( \alpha = \beta, \gamma = \delta, \lambda = \nu \), PCM already yields a singlet. Otherwise the state can be transformed into a singlet via the “Procrustean method” [11]. The total success probability of this protocol then yields the SCP

\[
p = 2\lambda \nu \min[\alpha(1 - \beta - \delta), \beta(1 - \alpha - \gamma)].
\]

Note that the SCP (2) corresponding to our protocol coincides with the highest possible probability for the conversion of two mixed states into a singlet [10].

The partition of the network into two groups \( A \) and \( B \) is arbitrary as long as one group contains \( A \) and the other contains \( B \). Hence, a singlet between \( A \) and \( B \) (dashed line in fig. 1(a)) can only be established if the groups are connected by at least two states of the form (1), when any operation within each group is allowed. This has to be true for all possible partitions. Therefore a necessary condition is found for any strategy aiming to establish a singlet between two nodes of a mixed-state network with a finite probability: There must be at least two distinct “paths” of edges of the form (1) connecting the corresponding nodes.

In fig. 1(a) this is indicated by two spatially distinct paths of bonds, i.e., in this case it is sufficient that each bond in the path contains one edge. Note that the remaining qubits which are not contained in this path are irrelevant and can therefore be in arbitrary states. Although we allowed global operations within a group to demonstrate the necessity of the condition, it turns out that it is also sufficient for a singlet to be formed using LOCC. This is because entanglement swapping transforms two states of the form (1) into a state of the same form with a finite success probability. We can therefore create a state \( \rho(\alpha, \gamma, \lambda) \) between two nodes of the network given that these nodes are connected by a path consisting of states of the same type. Two such states, originating from two paths, are then converted into a singlet by the
Examples are shown in fig. 2(b). Note that for the \( n=2 \) case this yields the same SCP as the protocol leading to eq. (2). In most situations the SCP is inferior to the alternative methods developed below and therefore we will not describe this method here in detail. The derivation of eq. (3) and the corresponding purification protocol can be found elsewhere [13].

The SCPs \( P(n, \alpha, \lambda) \) can be significantly improved by grouping \( n \) identical PMSs into sets of \( m \) and converting each of these sets into a singlet. For example, for \( m=2 \) we attempt to convert pairs of PMSs connecting two nodes A and B into singlets by using PCM. If this fails for a given pair of PMSs the protocol described above. Unfortunately, this scheme leads in general to an exponential decrease of entanglement fidelity [3], and thus success probability, with the number of swapping operations and is therefore impractical. In the following we therefore specialize on particular networks which allow for efficient protocols.

The condition described above is fulfilled by regular networks as shown in fig. 1(b) in which each node is connected to its nearest neighbors by a bond which consists of multiple edges. We assume that each bond is identical but allow for different edges within a single bond. This setup can be generalized to arbitrary geometries, e.g. square or honeycomb, including lattices in higher dimensions. CEP is achieved if each bond can be converted into a singlet with a probability exceeding the percolation threshold (e.g., \( p_{bh} \approx 0.347 \) for a triangular lattice [12]) which is possible if and only if each bond consists of at least two states of the form (1). For the sake of simplicity we assume in the following that each edge is —up to local unitaries— of the form \( |\alpha, \lambda \rangle \equiv \rho(\alpha, \gamma=0, \lambda) \). Setting \( \gamma=0 \) is not a substantial restriction but keeps the following equations manageable. All protocols in this paper can also be performed if \( \gamma \neq 0 \). Note that purifiable mixed states (PMSs) of the type \( \rho(\alpha, \lambda) \) form the state of two entangled atomic ensembles in particular quantum repeater schemes [2].

If there are exactly two edges between each node the SCP is given by eq. (2). However, allowing for more than two edges the SCP can be greatly increased. Indeed, given \( n \) identical PMSs of the form \( \rho(\alpha, \lambda) \), we developed a protocol which makes use of “distillable subspaces” [10]. These subspaces are constructed such that the projection of \( \rho(\alpha, \lambda)^\otimes n \) into the subspace is pure and entangled. Assuming, without loss of generality, that \( \alpha \geq 1/2 \), the corresponding SCP is given by

\[
P(n, \alpha, \lambda) = \sum_{l=0}^{n} \Lambda_{l} = \left(1 - \lambda\right)^{l} \left(\begin{array}{c} n \\ l \end{array}\right) \times \sum_{k=1}^{n-l} \alpha^{n-l-k} \sum_{k=1}^{l} \left(\begin{array}{c} n-l-k \\ k \end{array}\right) \left(\begin{array}{c} n-l \\ k \end{array}\right) \left(\begin{array}{c} l \\ k \end{array}\right) \left(\begin{array}{c} n-k \\ l \end{array}\right)
\]

(3)

Examples are shown in fig. 2(b).

Fig. 2: (Color online) (a) Probability of generating a singlet using the recycling protocol for \( \alpha = 1/2 \) and \( n = 2, 4, 6, 8, 16 \) (bottom to top). (b) SCP eq. (3) for \( n = 3 \) (red, dashed line), \( n=4 \) (green, dotted line) and \( n=16 \) (blue, dashed-dotted line). As a reference, the \( n=4 \) curve from (a) is replotted. The percolation thresholds for triangular (T), square (S) and honeycomb (H) lattices are given by the horizontal lines.
A B C

Purification to singlet
Swapping

\[ \rho(\alpha, \lambda) \]

\[ \rho(\beta, \nu) \]

Corresponding to Singlet

\[ \rho(\alpha, 1) \]

ρ

\[ \rho(\beta, 1) \]

Convert to Singlet

Fig. 3: (Color online) Possibilities to generate a singlet between two nodes A and C via an intermediate node B. The black, thick lines in the hybrid swapping scheme indicate pure but not maximally entangled states.

Probability never exceeds 1/2 in this case more edges per bond are required in other geometries. Already three edges give a SCP of \( 3\lambda^2\alpha(1 - \alpha) \leq 3/4 \) (see eq. (3)) which is sufficient for square or honeycomb lattices for large enough \( \lambda \) as shown in fig. 2.

Although CEP is a promising tool for entanglement distribution it is known that in a network of pure states the SCP can be improved by certain quantum strategies [4, 6–9]. As we will show in the following, this is also the case in mixed-state networks. A simple example is a protocol involving entanglement swapping consisting of a 1D setup of three nodes A, B and C (see fig. 3). Partially entangled states between A and B and between B and C are joined at B leading to a singlet between A and C. Since the nodes have to be connected by at least two states of the form (1) there are multiple possibilities to create a singlet between nodes A and C as illustrated in fig. 3. We assume here that each bond consists of two edges \( \rho(\alpha, \lambda) \) and \( \rho(\beta, \nu) \). CEP leads to the overall success probability

\[ p_{\text{CEP}} = [2\lambda\nu \min(\alpha(1 - \beta), \beta(1 - \alpha))]^2, \]  

see eq. (2). This can be improved by a hybrid swapping protocol in which we first convert each of the two bonds into a partially entangled pure state using PCM. If \( \alpha \neq \beta \) we proceed by entanglement swapping followed by the Procrustean method (if \( \alpha = \beta \) the method is identical to CEP). The overall success probability for this protocol is

\[ p_h = 2\lambda^2\nu^2[\alpha + \beta - 2\alpha\beta] \min[\alpha(1 - \beta), \beta(1 - \alpha)]. \]  

A third method is given by direct swapping where we first perform entanglement swapping on the states \( \rho(\alpha, \lambda) \) (from bond A–B) and \( \rho(\beta, \nu) \) (from bond B–C) and analogously on the two remaining states (swapping on identical states yields lower SCPs). This leads to two PMSs between A and C. Subsequent purification (using PCM and the Procrustean method) yields the overall success probability

\[ p_d = 2\lambda^2\nu^2\alpha\beta(1 - \alpha)(1 - \beta). \]  

We see that \( p_h \geq p_{\text{CEP}} \) and \( p_h > p_d \). Furthermore, \( p_d > p_{\text{CEP}} \) if \( 2\min[\alpha(1 - \beta), \beta(1 - \alpha)] \geq \max[\alpha(1 - \beta), \beta(1 - \alpha)] \). This means the hybrid protocol has the highest success probability if \( \alpha \neq \beta \). However, if \( \alpha = \beta \) CEP cannot be outperformed by direct or hybrid swapping. The three methods are compared in fig. 4(a): hybrid and
direct swapping can lead to substantially higher SCPs than CEP.

Although direct and hybrid swapping is used here in a 1D setup, they can easily be embedded into a network of higher dimensions reducing the amount of required resources or leading to higher success probabilities. A simple example is given by a 2D square network as shown in fig. 4(b). Direct swapping can be used to connect the qubits in opposite nodes and subsequent purification leads to a singlet between opposite corners. This method illustrates how two different paths of edges can be used for singlet generation between distant nodes. It does not require two edges per bond. However, if we do have two edges per bond, we can use the hybrid method to convert each bond into a pure state \(|\tilde{\alpha}\rangle = \sqrt{\tilde{\alpha}}|00\rangle + \sqrt{1-\tilde{\alpha}}|11\rangle\) with

\[
\tilde{\alpha} = \max[\alpha(1-\beta), \beta(1-\alpha)]/(\alpha + \beta - 2\alpha\beta),
\]

by using PCM which succeeds with probability \(p_c = \lambda\nu(\alpha + \beta - 2\alpha\beta)\). If this yields only two states \(|\tilde{\alpha}\rangle\) which have a common node (\(B\) or \(C\)), entanglement swapping can be performed followed by the Procrustean scheme. If the conversion succeeds such that all four states have the form \(|\tilde{\alpha}\rangle\), they are connected (e.g. at nodes \(B\) and \(C\)) via “XZ-entanglement swapping” [6] leading to two pure states (between \(A\) and \(D\)) of the form \(|\tilde{\alpha}\rangle\) with

\[
\tilde{\alpha} = (1 + \sqrt{1-16\tilde{\alpha}^2(1-\tilde{\alpha}^2)})/2.
\]

These can be distilled into a singlet with probability \(\min[1, 2(1-\tilde{\alpha}^2)]\) by using a protocol based on majorization theory [14] described in [6,7]. The overall chance of succeeding in generating a singlet is then given by

\[
p_{sq} = 4p_c^2(1-p_c^2)(1-\tilde{\alpha}) + p_c^4 \min(1, 2(1-\tilde{\alpha}^2)).
\]

Connecting two corners via CEP has a success probability of \(p_{CEP} = 1 - (1-p_{CEP})^2\) which can be significantly smaller than eq. (12) as shown in fig. 4(a).

These examples of quantum strategies are only composed of a finite number of nodes and do not exhibit a percolation threshold. However, they can be embedded within larger networks. Two examples are illustrated in fig. 5. We can generate singlets using small networks like hybrid swapping and the square protocol. Doing this results in singlets randomly distributed between neighboring nodes of the larger network. If the probability of generating a singlet exceeds the larger network’s percolation threshold then infinite clusters will form and enable long-distance entanglement distribution. Our quantum strategies have a greater chance of generating a singlet compared to CEP. This means the percolation threshold can be achieved if we use our quantum strategies in cases where this is not possible using CEP.

For example, small swapping arrangements (see fig. 3) can be nested within a 3D face centred cubic (fcc) lattice as shown in fig. 5(a). The percolation threshold for a fcc lattice is given by \(p_h \approx 0.12\) [15]. CEP on this lattice only creates an infinite cluster when \(p_{CEP} > p_h\). However, the SCP of hybrid swapping exceeds \(p_{CEP}\) (see fig. 4(a)). As a consequence, there is a wide range of parameters \(\alpha, \lambda\) such that \(p_h > p_h > p_{CEP}\) and thus an infinite cluster of singlets on the fcc lattice can only be achieved if hybrid swapping is performed. Similarly, the square network arrangement can be nested within a 2D triangular lattice (see fig. 5(b)). In this case, the percolation threshold is satisfied by CEP if \(\tilde{p}_{CEP} > 2\sin(\pi/18)\), i.e. the square SCP exceeds the triangular lattice’s percolation threshold. However, by applying the quantum strategy on the squares we create singlets with probability \(p_{sq} > \tilde{p}_{CEP}\) which can be greater than the threshold in cases when \(\tilde{p}_{CEP}\) is smaller than the threshold (see fig. 4(a)).

In this paper we have found a necessary and sufficient condition for the generation of a perfect singlet in a
quantum network where the nodes are initially connected by bipartite mixed qubit states. To form a perfect singlet between two nodes these have to be connected by at least two paths of states of the form (1). If the states forming the bonds in such a network are not of this type a singlet cannot be generated with finite probability. By considering regular networks we have devised efficient protocols for the distillation of singlets and have shown that percolation can be applied to achieve long-distance entanglement distribution. This extends on previous work considering pure states [16].

Furthermore, we have shown that it is principally possible to find quantum strategies for entanglement distribution which outperform CEP. The extension of CEP by such quantum strategies can lead to improved success probabilities for singlet generation, e.g. by nesting small structures like the discussed square setup into large-scale networks [4,6,7]. However, like in the pure-state case, the optimal strategy, i.e. the strategy with the highest possible success probability using the least amount of physical resources, remains unknown. Indeed, an optimal scheme for entanglement distribution might even go beyond the restriction to bipartite entanglement: Networks with multipartite entangled (mixed) states might give further advantages [9,17].

Other schemes for distributing entanglement in quantum networks have been found in the presence of bit-flip noise [18], and Werner states allow the generation of long-range entanglement in a cubic network [17,19]. However, although the resulting states can have a high fidelity they are not perfect singlets. Yet, the ability to purify the states to high, non-unity fidelities make these other strategies very interesting.

Despite this, the simplicity of CEP makes it a very promising framework for the development of long-distance entanglement distribution in quantum networks.

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REFERENCES

[1] Briegel H.-J., Dür W., Cirac J. I. and Zoller P., Phys. Rev. Lett., 81 (1998) 3932.
[2] Duan L.-M., Lukin M. D., Cirac J. I. and Zoller P., Nature, 414 (2001) 413.
[3] Dür W., Briegel H.-J., Cirac J. I. and Zoller P., Phys. Rev. A, 59 (1999) 169; Hartmann L., Kraus B., Briegel H.-J. and Dür W., Phys. Rev. A, 75 (2007) 032310; Dörner U., Klein A. and Jaksh D., Quantum Inf. Comput., 8 (2008) 0468.
[4] Acín A., Cirac J. I. and Lewenstein M., Nat. Phys., 3 (2007) 256.
[5] Vidal G., Phys. Rev. Lett., 83 (1999) 1046.
[6] Perseguers S., Cirac J. I., Acín A., Lewenstein M. and Wehr J., Phys. Rev. A, 77 (2008) 022308.
[7] Lapeyre Jr. G. J., Wehr J. and Lewenstein M., Phys. Rev. A, 79 (2009) 042324.
[8] Cuquet M. and Calsamiglia J., preprint arXiv:0906.2977v1 [quant-ph] (2009).
[9] Perseguers S., Cavalcanti D., Lapeyre Jr G. J., Lewenstein M. and Acín A., preprint arXiv: 0910.2438v1 [quant-ph] (2009).
[10] Jané E., Quant. Inf. Comp., 2 (2002) 348; Chen P. X., Liang L. M., Li C. Z. and Huang M. Q., Phys. Rev. A, 66 (2002) 022309.
[11] Bennett C. H., Bernstein H. J., Popescu S. and Schumacher B., Phys. Rev. A, 53 (1996) 2046.
[12] Bollobás B. and Riordan O., Percolation (Cambridge University Press, Cambridge) 2006.
[13] Broadfoot S., Dorner U. and Jaksh D., in preparation.
[14] Nielsen M. and Vidal G., Quantum Inf. Comput., 1 (2001) 76.
[15] Lorenz C. D. and Ziff R. M., Phys. Rev. E, 57 (1998) 230.
[16] Kieling K. and Eisert J., Quantum and Semi-classical Percolation and Breakdown in Disordered Solids, Lect. Notes Phys., Vol. 762 (Springer, Berlin) 2009, pp. 287-319 (also available at arXiv:0712.1836v1 [quant-ph]).
[17] Perseguers S., preprint arXiv:0910.1459v1 [quant-ph] (2009).
[18] Perseguers S., Jiang L., Schuch N., Verstraete F., Lukin M. D., Cirac J. I. and Vollbrecht K. G. H., Phys. Rev. A, 78 (2008) 062324.
[19] Raussendorf R., Bravyi S. and Harrington J., Phys. Rev. A, 71 (2005) 062313.