Bethe Ansatz solution for quantum spin-1 chains with boundary terms

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Abstract

The procedure for obtaining integrable open spin chain Hamiltonians via reflection matrices is explicitly carried out for some three-state vertex models. We have considered the 19-vertex models of Zamolodchikov-Fateev and Izergin-Korepin, and the $\mathbb{Z}_2$-graded 19-vertex models with $sl(2|1)$ and $osp(1|2)$ invariances. In each case the eigenspectrum is determined by application of the coordinate Bethe Ansatz.

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1 Introduction

One-dimensional quantum spin chain Hamiltonians and classical statistical systems in two spatial dimensions on a lattice (vertex models), share a common mathematical structure responsible by our understanding of these integrable models [1, 2, 3]. If the Boltzmann weights underlying the vertex models are obtained from solutions of the Yang-Baxter (YB) equation the commutativity of the associated transfer matrices immediately follow, leading to their integrability.

The Bethe Ansatz (BA) is a powerful method in the analysis of integrable quantum models. There are several versions: coordinate BA [4], algebraic BA [5], analytical BA [6], etc. The simplest version is the coordinate BA. In this framework one can obtain the eigenfunctions and the spectrum of the Hamiltonian from its eigenvalue problem. It is really simple and clear for the two-state models like the six-vertex models but becomes tricky for models with a higher number of states.

The algebraic BA, also known as Quantum Inverse Scattering method, is an elegant and important generalization of the coordinate BA. It is based on the idea of constructing eigenfunctions of the Hamiltonian via creation and annihilation operators acting on a reference state. Here one uses the fact that the YB equation can be recast in the form of commutation relations for the matrix elements of the monodromy matrix which play the role of creation and annihilation operators. From this monodromy matrix we get the transfer matrix which commutes with the Hamiltonian.

Imposing appropriate boundary conditions the BA method leads to a system of equations, the Bethe equations, which are useful in the thermodynamic limit. The energy of the ground state and its excitations, velocity of sound, etc., may be calculated in this limit. Moreover, in recent years we witnessed another very fruitful connection between the BA method and conformal field theory. Using the algebraic BA,
Korepin [7] found various representations of correlators in integrable models and more recently Babujian and Flume [8] developed a method from the Algebraic BA which reveals a link to the Gaudin model, rendering solutions of the Knizhnik-Zamolodchikov equations for the SU(2) Wess-Zumino-Novikov-Witten conformal theory in the quasiclassical limit.

Integrable quantum systems containing Fermi fields have been attracting increasing interest due to their potential applications in condensed matter physics. The prototypical examples of such systems are the supersymmetric generalizations of the Hubbard and t-J models [9]. They lead to a generalization of the YB equation associated with the introduction of the a Z_{2} grading [10] which leads to appearance of additional signs in the YB equation.

When considering systems on a finite interval with independent boundary conditions at each end, we have to introduce reflection matrices to describe such boundary conditions. Integrable models with boundaries can be constructed out of a pair of reflection K-matrices K^{±}(u) in addition to the solution of the YB equation. Here K^{-}(u) and K^{+}(u) describe the effects of the presence of boundaries at the left and the right ends, respectively.

Integrability of open chains in the framework of the quantum inverse scattering method was pioneered by Sklyanin relying on previous results of Cherednik [11]. In reference [12], Sklyanin has used his formalism to solve, via algebraic BA, the open spin-1/2 chain with diagonal boundary terms. This model had already been solved via coordinate BA by Alcaraz et al [13]. The Sklyanin original formalism was extended to more general systems by Mezincescu and Nepomechie in [14].

In this paper we consider the coordinate version of the BA for the trigonometric three-state vertex models with a class of boundary terms derived from diagonal reflection K-matrices. These models are well-known in the literature: the Zamolodchikov-Fateev (ZF) model or A_{1}^{(1)} model [15], the Izergin-Korepin (IK) model or A_{2}^{(2)} model [16] and two Z_{2}-graded models, named the sl(2|1) model and the osp(1|2) model [17].

In the context of the coordinate BA, we propose here a new parametrization of wavefunctions. This result is important since it allows us to treat these 19-vertex models in the same way. Moreover the coordinate BA for these three-states models becomes simple as for two-state models terms [13].

The main goal in this paper is to reveal the common structure of these 19-vertex models with boundary terms which permits us to apply the BA method, unifying old and new results.

The paper is organized as follows: We introduce the algebraic tools in Section 2. In section 3, we apply the coordinate BA method for a general open chain Hamiltonian associated with four 19-vertex models. In sections 4,5,6 and 7 the energy eigenspectra and the corresponding Bethe equations are presented for each model. In section 8 we discuss about the graded and non graded solutions for 19-vertex models. Section 9 is reserved for the conclusion.

2 Description of the model

To determine an integrable vertex model on a lattice it is first necessary that the bulk vertex weights be specified by an R-matrix R(u), where u is the spectral parameter. It acts on the tensor product V \otimes V for a given vector space V and satisfy a special system of functional equations, the YB equation

\[ R_{12}(u)R_{13}(u + v)R_{23}(v) = R_{23}(v)R_{13}(u + v)R_{12}(u), \]

in V \otimes V \otimes V, where R_{12} = R \otimes 1, R_{23} = 1 \otimes R, etc.

An R matrix is said to be regular if it satisfies the property R(0) = P, where P is the permutation matrix in V \otimes V: P(\alpha \otimes \beta) = \beta \otimes \alpha) for \alpha, \beta \in V. In addition, we will require [14] that R(u)
satisfies the following properties

\[
\begin{align*}
\text{regularity} : & \quad R_{12}(0) = f(0)^{1/2} P_{12}, \\
\text{unitarity} : & \quad R_{12}(u) R_{12}^{t_2}(u) = f(u), \\
\text{PT - symmetry} : & \quad P_{12} R_{12}(u) P_{12} = R_{12}^{t_{12}}(u), \\
\text{crossing - symmetry} : & \quad R_{12}(u) = U_1 R_{12}^{t_2}(-u - \rho) U_1^{-1},
\end{align*}
\]

(2.2)

where \( f(u) = x_1(u)x_1(-u), \) \( t_i \) denotes transposition in the space \( i, \) \( \rho \) is the crossing parameter and \( U \) determines the crossing matrix

\[ M = U^t U = M^t. \]  

(2.3)

Note that unitarity and crossing-symmetry together imply the useful relation

\[ M_1 R_{12}^{t_2}(-u - \rho) M_1^{-1} R_{12}^{t_2}(u - \rho) = f(u). \]  

(2.4)

The boundary weights then follow from \( K \)-matrices which satisfy boundary versions of the YB equation [12, 14]: the reflection equation

\[ R_{12}(u - v) K_1^-(u) R_{12}^{t_{12}}(u + v) K_2^-(v) = K_2^-(v) R_{12}(u + v) K_1^-(u) R_{12}^{t_{12}}(u - v), \]  

(2.5)

and the dual reflection equation

\[ R_{12}(-u + v)(K_1^+)^{t_1}(u) M_1^{-1} R_{12}^{t_{12}}(-u - v - 2\rho) M_1(K_2^+)^{t_2}(v) \]

\[ = (K_2^+)^{t_2}(v) M_1 R_{12}(-u - v - 2\rho) M_1^{-1}(K_1^+)^{t_1}(u) R_{12}^{t_{12}}(-u + v). \]  

(2.6)

In this case there is an isomorphism between \( K^- \) and \( K^+ \):

\[ K^-(u) \rightarrow K^+(u) = K^-(-u - \rho)^t M. \]  

(2.7)

Therefore, given a solution to the reflection equation (2.5) we can also find a solution to the dual reflection equation (2.6).

In the framework of the quantum inverse scattering method, we define the Lax operator from the \( R \)-matrix as \( L_{aq}(u) = R_{aq}(u), \) where the subscript \( a \) represents auxiliary space, and \( q \) represents quantum space. The row-to-row monodromy matrix \( T(u) \) is defined as a matrix product over the \( N \) operators on all sites of the lattice,

\[ T(u) = L_{aN}(u) L_{aN-1}(u) \cdots L_{a1}(u). \]  

(2.8)

The main result is the following: if the boundary equations are satisfied, then the Sklyanin’s transfer matrix

\[ t(u) = \text{Tr}_a \left( K^+(u)T(u)K^-(u)T^{-1}(-u) \right), \]  

(2.9)

forms a commuting family

\[ [t(u), t(v)] = 0, \quad \forall u, v \]  

(2.10)

The commutativity of \( t(u) \) can be proved by using the unitarity and crossing-unitarity relations, the reflection equation and the dual reflection equation. It implies integrability of an open quantum spin chain whose Hamiltonian (with \( K^-(-0) = 1 \)), can be obtained as

\[ H = \sum_{k=1}^{N-1} H_{k,k+1} + \frac{1}{2} \left. \frac{dK_1^-(u)}{du} \right|_{u=0} + \frac{\text{tr}K_0^+(0)H_{N,0}}{\text{tr}K^+(0)}, \]  

(2.11)
and whose two-site terms are given by

\[ H_{k,k+1} = \left. \frac{d}{du} P_{k,k+1} R_{k,k+1}(u) \right|_{u=0}, \]  

in the standard fashion.

Here we will extend our discussions to include the \( \mathbb{Z}_2 \)-graded vertex models. Therefore, let us describe some useful informations about the graded formulation.

Let \( V = V_0 \oplus V_1 \) be a \( \mathbb{Z}_2 \)-graded vector space where 0 and 1 denote the even and odd parts respectively. Multiplication rules in the graded tensor product space \( V \otimes V \) differ from the ordinary ones by the appearance of additional signs. The components of a linear operator \( A \otimes B \in V \otimes V \) result in matrix elements of the form

\[ (A \otimes B)_{\alpha \beta} = (-)^{p(\beta)(p(\alpha)+p(\gamma))} A_{\alpha \gamma} B_{\beta \delta}. \]  

The action of the graded permutation operator \( \mathcal{P} \) on the vector \( |\alpha \rangle \otimes |\beta \rangle \in V \otimes V \) is defined by

\[ \mathcal{P} |\alpha \rangle \otimes |\beta \rangle = (-)^{p(\alpha)p(\beta)} |\beta \rangle \otimes |\alpha \rangle \implies (\mathcal{P})_{\alpha \beta} = (-)^{p(\alpha)p(\beta)} \delta_{\alpha \beta}. \]  

The graded transposition \( st \) and the graded trace \( str \) are defined by

\[ (A^{st})_{\alpha \beta} = (-)^{p(\alpha)+1} A_{\beta \alpha}, \quad str A = \sum_{\alpha} (-)^{p(\alpha)} A_{\alpha \alpha}. \]  

where \( p(\alpha) = 1 \) (0) if \( |\alpha \rangle \) is an odd (even) element.

For the graded case the YB equation and the reflection equation remain the same as above. We only need to change the usual tensor product to the graded tensor product.

In general, the dual reflection equation which depends on the unitarity and cross-unitarity relations of the \( \mathcal{R} \)-matrix takes different forms for different models. For the models considered in this paper, we write the graded dual reflection equation in the following form [18]:

\[ \mathcal{R}^{s_{t1} s_{t2}}_{21}(-u+v)(K^+)^{s_{t1}}(u)M^{-1}_{1} \mathcal{R}^{s_{t1} s_{t2}}_{12}(-u-v-2p)M_{1}^{-1}(K^+)^{s_{t2}}(v) \]

\[ = (K^+)^{s_{t2}}(v)M_{1} \mathcal{R}^{s_{t1} s_{t2}}_{12}(-u-v-2p)M_{1}^{-1}(K^+)^{s_{t1}}(u) \mathcal{R}^{s_{t1} s_{t2}}_{21}(-u+v), \]  

and we will choose a common parity assignment: \( p(1) = p(3) = 0 \) and \( p(2) = 1 \), the BFB grading.

Now, using the relations

\[ \mathcal{R}^{s_{t1} s_{t2}}_{12}(u) = I_{1} R_{21}(u) I_{1}, \quad \mathcal{R}^{s_{t1} s_{t2}}_{21}(u) = I_{1} R_{12}(u) I_{1} \quad \text{and} \quad IK^+(u)I = K^+(u) \]  

with \( I = \text{diag}(1, -1, 1) \) and the property \([M_{1}M_{2}, \mathcal{R}(u)] = 0\) we can see that the isomorphism (2.7) holds with the BFB grading.

The three-state vertex models that we will consider are the Zamolodchikov-Fateev (ZF) model, the Izergin-Korepin (IK) model, the \( sl(2|1) \) model and the \( osp(1|2) \) model. Their \( \mathcal{R} \)-matrices have a common form

\[ \mathcal{R}(u) = \begin{pmatrix}
  x_1 & x_2 & y_5 & x_6 & x_7 \\
  x_2 & x_3 & x_5 & x_6 & x_7 \\
  y_5 & y_6 & x_2 & x_6 & x_5 \\
  y_6 & y_7 & y_5 & x_3 & x_2 \\
  y_7 & y_6 & x_2 & x_3 & x_1
\end{pmatrix}, \]  

(2.18)
satisfying the properties (2.1–2.4) together with their graded version.

In order to derive the bulk Hamiltonian, it is convenient to expand the normalized $R$-matrix ($R = PR$) around the regular point $u = 0$

$$R(u, \eta) = 1 + u(\alpha^{-1}H + \beta I) + o(u^2),$$

with $\alpha$ and $\beta$ being scalar functions. Therefore $H_{k,k+1}$ in (2.11) is the $H$ in (2.19) acting on the quantum spaces at sites $k$ and $k + 1$.

Using a spin language, this is a spin 1 Hamiltonian. In the basis where $S^z_k$ is diagonal with eigenvectors $|+, k\rangle, |0, k\rangle, |-, k\rangle$ and eigenvalues $1, 0, -1$, respectively, the bulk Hamiltonian density acting on two neighboring sites is given by

$$H_{k,k+1} = \begin{pmatrix}
    z_1 & \bar{z}_5 & 1 & \bar{z}_6 & z_3 \\
    1 & \epsilon \bar{z}_5 & z_5 & \epsilon z_4 & \epsilon z_6 \\
    \bar{z}_3 & z_6 & z_7 & \bar{z}_5 & 1 \\
    z_3 & \bar{z}_6 & 1 & \epsilon z_7 & \epsilon z_5 \\
    z_1 & \bar{z}_5 & \epsilon \bar{z}_6 & z_3 & 1
\end{pmatrix},$$

which can be easily written in terms of the usual spin-1 operators:

$$H_{k,k+1} = \epsilon z_4 + \frac{1}{2}(\bar{z}_5 - z_5)[S^z_k - S^z_{k+1}] + \frac{1}{2}(z_5 + \bar{z}_5 - 2\epsilon z_4)[(S^z_k)^2 + (S^z_{k+1})^2]$$

$$+ \frac{1}{4}(2z_1 - 2z_7 + \bar{z}_7)S^z_k S^z_{k+1} + \frac{1}{4}(2z_1 + z_7 + \bar{z}_7 + 4\epsilon z_4 - 4z_5 - 4\bar{z}_5)(S^z_k S^z_{k+1})^2$$

$$+ \frac{1}{4}(2z_1 - 2z_7 - 2z_5 + 2\bar{z}_5)[(S^z_k)^2S^z_{k+1} - S^z_k (S^z_{k+1})^2]$$

$$+ \frac{1}{4}\epsilon z_3[(S^z_k S^+_{k+1})^2 + (S^z_k S^+_{k+1})^2]$$

$$- \frac{1}{4}[z_6 S^z_k S^-_{k+1} + z_6 S^+_{k+1} S^z_k S^-_{k+1} + S^z_k S^+_{k+1}(z_6 S^z_{k+1} S^-_{k+1} + z_6 S^-_{k+1} S^z_{k+1})]$$

$$+ \frac{1}{2}[S^+_k S^z_{k+1} S^-_{k+1} + S^-_{k+1} S^z_k S^+_{k+1} + S^z_k S^+_{k+1} S^-_{k+1} + S^+_k S^-_{k+1} S^z_{k+1} S^-_{k+1} S^z_{k+1} S^-_{k+1} S^z_{k+1}]].$$

Here we have used the sign $\epsilon = \pm 1$ to explicitly include the graded ($\epsilon = -1$) models for which the tensor products in (2.21) are BFB graded and

$$S^z = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & -1
\end{pmatrix}, \quad S^+ = \sqrt{2} \begin{pmatrix}
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    0 & 0 & 0
\end{pmatrix}, \quad S^- = \sqrt{2} \begin{pmatrix}
    0 & 0 & 0 \\
    1 & 0 & 0 \\
    0 & 1 & 0
\end{pmatrix}.$$ (2.22)

Finally, we can consider the boundary terms of $H$ which are derived from the diagonal solutions $K^\pm = \text{diag}(k_{11}^\pm, k_{22}^\pm, k_{33}^\pm)$ of the reflection equations. In general, at the first site (left) it has the form

$$\frac{1}{2} \alpha \frac{dK^-(u)}{du} = \begin{pmatrix}
    l_{11} & 0 \\
    0 & l_{22} \\
    0 & 0
\end{pmatrix}, \quad l_{ii} = \frac{1}{2} \alpha \frac{d\tilde{k}^-_{ii}(u)}{du} \bigg|_{u=0}, \quad i = 1, 2, 3.$$

(2.23)
and for the last site (right) it has the form

\[
\frac{\text{tr}_0 K_0^+(0) H_{N,0}}{\text{tr} K^+(0)} = \begin{pmatrix} r_{11} & r_{22} & r_{33} \\ r_{22} & \end{pmatrix}.
\] (2.24)

To compute the term \(\text{tr}_0 K_0^+(0) H_{N,0}\) we have to use \((2.21)\) in order to get \(H_{N,0}\). In practice, it is equivalent to take the trace of \(K^+(0) H_{21}\) where

\[
H_{21} = P_{12} H_{12} P_{12} = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix},
\] (2.25)

with \(h_{ij}\) being 3 by 3 matrices and \(H_{12}\) is given by \((2.20)\). The result for the right boundary term is

\[
r_{11} = \frac{z_1 k_{11}^+(0) + \epsilon z_5 k_{22}^+(0) + \epsilon z_7 k_{33}^+(0)}{k_{11}^+(0) + c k_{22}^+(0) + k_{33}^+(0)}, \quad r_{22} = \frac{z_3 k_{11}^+(0) + z_4 k_{22}^+(0) + \epsilon z_6 k_{33}^+(0)}{k_{11}^+(0) + c k_{22}^+(0) + k_{33}^+(0)}, \quad r_{33} = \frac{z_7 k_{11}^+(0) + \epsilon z_5 k_{22}^+(0) + z_4 k_{33}^+(0)}{k_{11}^+(0) + c k_{22}^+(0) + k_{33}^+(0)}.
\] (2.26)

The factor \(\alpha\) in \((2.23)\) is due to the normalization of \(H\) in \((2.19)\) and \(\epsilon\)'s appear in \((2.26)\) to take into account the graded traces. Therefore the most general diagonal boundary terms can be written as

\[
\text{b.t.} = \frac{1}{2} (l_{11} - l_{33}) S_1^z + \frac{1}{2} (l_{11}' + l_{33}') (S_1^z)^2 + l_{22} l_{11} + \frac{1}{2} (r_{11} - r_{33}) S_N^z + \frac{1}{2} (r_{11}' + r_{33}') (S_N^z)^2 + r_{22} l_{11},
\] (2.27)

where \(l_{ii}' = l_{ii} - l_{22}\) and \(r_{ii}' = r_{ii} - r_{22}\) for \(i = 1, 2, 3\).

### 3 The coordinate Bethe Ansatz

In this section results are presented for a open quantum spin chain of \(N\) atoms each with spin 1 described by the Hamiltonian \((2.21)\) with the boundaries term \((2.27)\):

\[
H = \sum_{k=1}^{N-1} H_{k,k+1} + \text{b.t.}
\] (3.1)

At each site, the spin variable may be +1, 0, −1, so that the Hilbert space of the spin chain is \(H^{(N)} = \otimes^N V\) where \(V = C^3\) with basis \(\{|+\rangle, |0\rangle, |-\rangle\}\). The dimension of the Hilbert space is \(\dim H^{(N)} = 3^N\). We can see that \(H\) commutes with the third component of the spin

\[
[H, S_T^z] = 0, \quad S_T^z = \sum_{k=1}^{N} S_k^z,
\] (3.2)

This allows us to divide the Hilbert space of states into different sectors, each labelled by the eigenvalue of the number operator \(r = N - S_T^z\). We shall denote by \(H_{n}^{(N)}\) the subspace of \(H^{(N)}\) with \(r = n\). We can see that \(\dim H^{(N)} = \sum_{r=0}^{N} \dim H_{r}^{(N)}\) with

\[
\dim H_{r}^{(N)} = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \left( \begin{array}{c} N \\ r - 2j \end{array} \right) \left( \begin{array}{c} N - r + 2j \\ j \end{array} \right),
\] (3.3)
where \([\frac{k}{z}]\) means the integer part of \(\frac{k}{z}\) and \(\binom{a}{b}\) denotes the binomial number.

The action of \(H_{k,k+1}\) on two neighboring sites is read directly of (2.20)

\[
\begin{align*}
H_{k,k+1} |++\rangle &= z_1 |++\rangle, \\
H_{k,k+1} |+0\rangle &= \xi_5 |+0\rangle + |0+\rangle, \\
H_{k,k+1} |0-\rangle &= \xi_5 |0-\rangle + |-0\rangle, \\
H_{k,k+1} |-+\rangle &= \xi_7 |+-\rangle + \xi_6 |00\rangle + z_3 |-+\rangle, \\
H_{k,k+1} |00\rangle &= \xi_4 |00\rangle + \xi_6 |+-\rangle + \xi_6 |-+\rangle, \\
H_{k,k+1} |-+\rangle &= z_7 |-+\rangle + z_6 |00\rangle + z_3 |+-\rangle,
\end{align*}
\]

(3.4)

and the boundary terms (2.27) only see the sites 1 and \(N\). Therefore, we can write

\[
b.t. |i, ..., j\rangle = E_{ij} |i, ..., j\rangle,
\]

(3.5)

where \(E_{ij} = l_{ii} + r_{jj}\), \(i, j = 1, 2, 3\), with the notation \((+, 0, -) = (1, 2, 3)\). \(l_{ii}\) and \(r_{jj}\) are given by (2.23) and (2.25), respectively.

### 3.1 Sector \(r=0\)

The sector \(H^{(N)}_0\) contains only one state, the reference state, with all spin value equal to +1, \(\Psi_0 = \prod_k |+, k\rangle\), satisfying \(H\Psi_0 = E_0\Psi_0\), with \(E_0 = (N - 1)z_1 + \xi_1\). All other energies will be measured relative to this state. It means that we will seek eigenstates of \(H\) satisfying \((H - E_0)\Psi_r = E_r\Psi_r\), in every sector \(r\).

### 3.2 Sector \(r=1\)

In \(H^{(N)}_1\), the subspace of states with all spin value equal to +1 except one with value 0. There are \(N\) states \(|k[0]\rangle = |++0 + + \cdots +\rangle\) which span a basis of \(H^{(N)}_1\). The Ansatz for the eigenstate is thus of the form

\[
\Psi_1 = \sum_{k=1}^N a(k) |k[0]\rangle.
\]

(3.6)

The unknown wavefunction \(a(k)\) determines the probability that the spin variable has the value 0 at the site \(k\).

When \(H\) acts on \(|k[0]\rangle\), it sees the reference configuration, except in the vicinity of \(k\), and using (3.4) we obtain the eigenvalue equations

\[
(\xi_1 + 2z_1 - z_5 - \xi_5)a(k) = a(k - 1) + a(k + 1), \quad (1 < k < N)
\]

(3.7)

At the boundaries, we get slightly different equations

\[
(\xi_1 + \xi_{11} - \xi_{21} + z_1 - z_5)a(1) = a(2),
(\xi_1 + \xi_{11} - \xi_{12} + z_1 - \xi_5)a(N) = a(N - 1).
\]

(3.8)

We now try as a solution

\[
a(k) = a(\theta)\xi^k - a(-\theta)\xi^{-k},
\]

(3.9)
where $\xi = e^{i\theta}$, $\theta$ being some particular momentum fixed by the boundary conditions. Substituting this in equation (3.7) we obtain the eigenvalue

$$E_1 = -2z_1 + z_5 + z_5 + \xi + \xi^{-1}. \quad (3.10)$$

We want equations (3.7) to be valid for $k = 1$ and $k = N$ also, where $a(0)$ and $a(N + 1)$ are defined by (3.9). Combining (3.8) with (3.7) we get the end conditions

$$a(0) = \Delta_1 a(1), \quad \Delta_1 = E_{21} - E_{11} + z_1 - \bar{z}_5 = z_1 - \bar{z}_5 - l_{11},$$

$$a(N + 1) = \Delta_2 a(N), \quad \Delta_2 = E_{12} - E_{11} + z_1 - z_5 = z_1 - z_5 - r_{11}. \quad (3.11)$$

Compatibility between the end conditions (3.11) yields

$$\frac{a(\theta)}{a(-\theta)} = \xi^{-2} \frac{\Delta_1 - \xi}{\Delta_1 - \xi^{-1}} = \xi^{-2N} \frac{\Delta_2 - \xi^{-1}}{\Delta_2 - \xi}, \quad (3.12)$$

or

$$\xi^{2N} = \left( \frac{\Delta_1 \xi - 1}{\Delta_1 - \xi} \right) \left( \frac{\Delta_2 \xi - 1}{\Delta_2 - \xi} \right). \quad (3.13)$$

Therefore, the energy eigenvalue of $H$ in the sector $r = 1$ is given by

$$E_1 = (N - 3)z_1 + l_{11} + r_{11} + z_5 + \bar{z}_5 + \xi + \xi^{-1}, \quad (3.14)$$

with $\xi$ being solution of (3.13).

### 3.3 Sector $r=2$

In the Hilbert space $H_2^{(N)}$ we have $N$ states of the type $|k[-]\rangle = \left(\pm \frac{\mp}{k}\right)$ and $N(N - 1)/2$ states of the type $|k_1[0], k_2[0]\rangle = \left(\pm \frac{\mp}{k_1 k_2}\right)$. We seek these eigenstates in the form

$$\Psi_2 = \sum_{k_1 < k_2}^N a(k_1, k_2) |k_1[0], k_2[0]\rangle + \sum_{k=1}^N b(k) |k[-]\rangle. \quad (3.15)$$

Following Bethe [4], the wavefunction $a(k_1, k_2)$ can be parametrized using the superposition of plane waves (3.9) including the scattering of two pseudoparticles with momenta $\theta_1$ and $\theta_2$, ($\xi_j = e^{i\theta_j}$, $j = 1, 2$):

$$a(k_1, k_2) = \sum_{\epsilon_P} \epsilon_P \left\{ a(\theta_1, \theta_2) \xi_1^{k_1} \xi_2^{k_2} - a(\theta_2, \theta_1) \xi_2^{k_1} \xi_1^{k_2} \right\}, \quad (3.16)$$

where the sum extends over the negations of $\theta_1$ and $\theta_2$, and $\epsilon_P$ is a sign factor ($\pm 1$) that changes sign on negation. The parametrization of $b(k)$ is still undetermined at this stage.

Before we try to parametrize $b(k)$ let us consider the Schrödinger equation $H \Psi_2 = E_2 \Psi_2$. From the explicit form of $H$ acting on two sites (3.4) we derive the following set of eigenvalue equations:

- Equations for $|k_1[0]\rangle$ and $|k_2[0]\rangle$ far in the bulk ($1 < k_1 < k_2 + 1 < N$)

$$(\tilde{E}_2 + 4z_1 - 2z_2 - 2z_5) a(k_1, k_2) = a(k_1 - 1, k_2) + a(k_1 + 1, k_2) + a(k_1, k_2 - 1) + a(k_1, k_2 + 1). \quad (3.17)$$
• Equations for $|k[-]|$ in the bulk ($1 < k < N$)
\[
(\mathcal{E}_2 + 2z_1 - z_7 - \bar{z}_7)b(k) = z_3b(k - 1) + z_3b(k + 1) + \bar{z}_6a(k - 1, k) + z_6a(k, k + 1).
\]
(3.18)

• Equations for two $|k[0]|$ neighbors in the bulk ($1 < k < N - 1$)
\[
(\mathcal{E}_2 + 3z_1 - z_5 - \bar{z}_5 - \epsilon z_4)a(k, k + 1) = a(k - 1, k + 1) + a(k, k + 2) + \epsilon z_6b(k) + \epsilon \bar{z}_6b(k + 1).
\]
(3.19)

In addition we have seven conditions to be satisfied at the free ends of the chain:

• Five equations involving at least one state $|k[0]|$ at one of the ends
\[
(\mathcal{E}_2 + \mathcal{E}_{11} - \mathcal{E}_{21} + 3z_1 - 2z_5 - \bar{z}_5)a(1, k_2) = a(2, k_2) + a(1, k_2 - 1) + a(1, k_2 + 1),
\]
(3.20)
\[
(\mathcal{E}_2 + \mathcal{E}_{12} - 3z_1 - z_5 - 2\bar{z}_5)a(k_1, N) = a(k_1 - 1, N) + a(k_1 + 1, N) + a(k_1, N - 1),
\]
(3.21)
\[
(\mathcal{E}_2 + \mathcal{E}_{11} - \mathcal{E}_{22} + 2z_1 - z_5 - \bar{z}_5)a(1, N) = a(2, N) + a(1, N - 1),
\]
(3.22)
\[
(\mathcal{E}_2 + \mathcal{E}_{11} - \mathcal{E}_{21} + 2z_1 - z_5 - \epsilon z_4)a(1, 2) = a(1, 3) + \epsilon z_6b(1) + \epsilon \bar{z}_6b(2),
\]
(3.23)
\[
(\mathcal{E}_2 + \mathcal{E}_{12} + 2z_1 - z_5 - \epsilon z_4)b(N - 1, N) = a(N - 2, N) + \epsilon z_6b(N - 1) + \epsilon \bar{z}_6b(N).
\]
(3.24)

• Two equations with the state $|k[-]|$ at one of the ends
\[
(\mathcal{E}_2 + \mathcal{E}_{11} - \mathcal{E}_{31} + z_1 - z_7)b(1) = z_3b(2) + z_6a(1, 2),
\]
(3.25)
\[
(\mathcal{E}_2 + \mathcal{E}_{11} - \mathcal{E}_{13} + z_1 - \bar{z}_7)b(N) = z_3b(N - 1) + \bar{z}_6a(N - 1, N).
\]
(3.26)

By simples substitution the Ansatz (3.16) solves the equations (3.17) provided
\[
\mathcal{E}_2 = -4z_1 + 2z_5 + 2\bar{z}_5 + \xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1}.
\]
(3.27)

It immediately follows that the eigenvalues of $H$ are a sum of single pseudoparticle energies.

The parametrization of $b(k)$ can now be determined in the following way: subtracting Eq.(3.19) from Eq.(3.17) for $k_1 = k, k_2 = k + 1$, we get
\[
\epsilon \bar{z}_6b(k + 1) + \epsilon z_6b(k) = a(k, k) + a(k + 1, k + 1) - (z_1 + \epsilon z_4 - z_5 - \bar{z}_5)a(k, k + 1).
\]
(3.28)

for which we can find $b(k)$ in terms of $a(k_1, k_2)$.

Using (3.28) together with (3.17) we can see that (3.23) and (3.24) are readily satisfied. Now we extend the Ansatz (3.16) to $k_1 = k_2 = k$ in order to get a parametrization for the wavefunction $b(k)$:
\[
b(k) = \sum_P \epsilon_P b(\theta_1, \theta_2) \xi_1^P \xi_2^P
\]
\[
= b(\theta_1, \theta_2)\xi_1^k \xi_2^k - b(-\theta_1, \theta_2)\xi_1^{-k} \xi_2^k - b(\theta_1, -\theta_2)\xi_1^k \xi_2^{-k} + b(-\theta_1, -\theta_2)\xi_1^{-k} \xi_2^{-k},
\]
(3.29)
which solves the meeting condition (3.28) provided
\[
\begin{align*}
\Delta b(\theta_1, \theta_2) &= \left( 1 + \xi_1 \xi_2 + \Delta \xi_2 \right) a(\theta_1, \theta_2) - \left( 1 + \xi_1 \xi_2 + \Delta \xi_1 \right) a(\theta_2, \theta_1), \\
\Delta &= z_5 + \xi_5 - z_1 - \varepsilon z_4.
\end{align*}
\] (3.30)

together with \( b(-\theta_1, \theta_2), b(\theta_1, -\theta_2) \) and \( b(-\theta_1, -\theta_2) \) that can be obtained from (3.30) changing the signs of \( \theta_1 \) and \( \theta_2 \).

These relations tell us that the pseudoparticle of the type \(|k[-]\) behaves under the action of \(H\) as the two pseudoparticles \(|k_1[0]\) and \(|k_2[0]\) at the same site \(k\) and its parametrization follows as the plane waves of pseudoparticles \(|k_1[0]\) multiplied by the weight functions \(b(\pm\theta_1, \pm\theta_2)\).

As a consequence of this identification we can see that the equation involving \(b(k)\) (3.18) becomes a **meeting condition** for two states \(|k[0]\). Using the S-matrix language, from (3.18) we get the two-pseudoparticle phase shifts:
\[
\begin{align*}
a(\theta_2, \theta_1) &= \left( \frac{s(\theta_2, \theta_1)}{s(\theta_1, \theta_2)} \right) a(\theta_1, \theta_2), \quad a(\theta_2, -\theta_1) = \left( \frac{s(\theta_2, -\theta_1)}{s(-\theta_1, \theta_2)} \right) a(-\theta_1, \theta_2), \\
a(-\theta_2, \theta_1) &= \left( \frac{s(-\theta_2, \theta_1)}{s(\theta_1, -\theta_2)} \right) a(\theta_1, -\theta_2), \quad a(-\theta_2, -\theta_1) = \left( \frac{s(-\theta_2, -\theta_1)}{s(-\theta_1, -\theta_2)} \right) a(-\theta_1, -\theta_2),
\end{align*}
\] (3.31)

where
\[
\begin{align*}
s(\theta_2, \theta_1) &= (1 + \xi_1 \xi_2 + \Delta \xi_2) [z_3(1 + \xi_1^2 \xi_2^2) - (1 + \xi_1 \xi_2)(\xi_1 + \xi_2) + \Delta \xi_1 \xi_2], \\
\Lambda &= 2(z_1 - z_5 - \bar{z}_5) + z_7 + \bar{z}_7.
\end{align*}
\] (3.32)

The seven remained \(s\)-functions for the phase shift equations (3.31) follow from (3.32) changing the signs of \(\theta_1\) and \(\theta_2\).

At this point we still have to consider the equation (3.22) and four end conditions. We want equation (3.17) to be valid for \(k_1 = 1\) and \(k_2 = N\) also, where \(a(0, k_2)\) and \(a(k_1, N + 1)\) are defined by (3.16). Combining (3.20) and (3.21) with (3.17) we get two end conditions
\[
\begin{align*}
\Delta_1 a(1, k) &= a(0, k), \quad \Delta_1 = z_1 - \bar{z}_5 - \varepsilon_{11} + \varepsilon_{21}, \\
\Delta_2 a(k, N) &= a(k, N + 1), \quad \Delta_2 = z_1 - z_5 - \varepsilon_{11} + \varepsilon_{12},
\end{align*}
\] (3.33)

Substituting (3.16) in (3.33), we obtain the following relations
\[
\begin{align*}
a(-\theta_1, \theta_2) &= \left( 1 - \frac{\Delta_1 \xi_1}{1 - \Delta_1 \xi_1^{-1}} \right) a(\theta_1, \theta_2), \quad a(\theta_1, -\theta_2) = \xi_2^{2N} \left( \frac{\Delta_2 - \xi_2}{\Delta_2 - \xi_2^{-1}} \right) a(\theta_1, \theta_2), \\
a(-\theta_1, -\theta_2) &= \xi_2^{2N} \left( 1 - \frac{\Delta_1 \xi_1}{1 - \Delta_1 \xi_1^{-1}} \right) \left( \frac{\Delta_2 - \xi_2}{\Delta_2 - \xi_2^{-1}} \right) a(\theta_1, \theta_2).
\end{align*}
\] (3.34)

which describe the change of signs of \(\theta_1\) and \(\theta_2\) in \(a(\theta_1, \theta_2)\) and the corresponding pair interchange relations
\[
\begin{align*}
a(\theta_2, -\theta_1) &= \xi_1^{2N} \left( \frac{\Delta_2 - \xi_1}{\Delta_2 - \xi_1^{-1}} \right) a(\theta_2, \theta_1), \quad a(-\theta_2, \theta_1) = \left( 1 - \frac{\Delta_1 \xi_2}{1 - \Delta_1 \xi_2^{-1}} \right) a(\theta_2, \theta_1), \\
a(-\theta_2, -\theta_1) &= \xi_1^{2N} \left( 1 - \frac{\Delta_1 \xi_2}{1 - \Delta_1 \xi_2^{-1}} \right) \left( \frac{\Delta_2 - \xi_1}{\Delta_2 - \xi_1^{-1}} \right) a(\theta_2, \theta_1).
\end{align*}
\] (3.35)
Combining these relations with the phase shift relations (3.31) we get the Bethe equations

\[
\xi_1^{2N} = \left( \frac{1 - \Delta_1 \xi_1}{\Delta_1 - \xi_1} \right) \left( \frac{1 - \Delta_2 \xi_1}{\Delta_2 - \xi_1} \right) \frac{s(\theta_1, \theta_2)}{s(\theta_2, \theta_1)} \frac{s(\theta_2, -\theta_1)}{s(-\theta_1, \theta_2)}, \tag{3.36}
\]

\[
\xi_2^{2N} = \left( \frac{1 - \Delta_1 \xi_2}{\Delta_1 - \xi_2} \right) \left( \frac{1 - \Delta_2 \xi_2}{\Delta_2 - \xi_2} \right) \frac{s(\theta_2, \theta_1)}{s(\theta_1, \theta_2)} \frac{s(\theta_1, -\theta_2)}{s(-\theta_2, \theta_1)}. \tag{3.37}
\]

With these relations we have defined the behavior of the state \( |k[0]\rangle \) at the boundaries. Consequently the equation (3.22), where we have one state \( |k[0]\rangle \) at each end, is also readily satisfied.

Now we recall step by step our procedure described above to see that there is no more function or parameter to be determined, but we still have to solve the equations with the pseudoparticle \(|-\rangle\) at the boundaries.

Similarly, we want equation (3.18) to be valid for \( k = 1 \) and \( k = N \) also, where \( b(0) \) and \( b(N+1) \) are defined by (3.29). Combining (3.25) and (3.26) with (3.18) we obtain two further end conditions

\[
\Delta_3 b(1) = z_3 b(0) + z_6 a(0, 1), \quad \Delta_3 = z_1 - z_7 - \xi_{11} + \xi_{31},
\]

\[
\Delta_4 b(N) = z_3 b(N+1) + z_6 a(N, N+1), \quad \Delta_4 = z_1 - z_7 - \xi_{11} + \xi_{13}. \tag{3.38}
\]

Substituting the already fixed relations for \( a(k_1, k_2) \) and \( b(k) \) and using the Bethe equations we can see that these end conditions are also satisfied. It means that there is no additional end condition due to the presence of the state \( |k[-]\rangle \) at the boundaries. Instead of surprising, this result is in agreement with our parametrization of the wavefunction \( b(k) \), where the dynamics of the pseudoparticle \( |k[-]\rangle \) is understood as the dynamics of two pseudoparticles \( |k[0]\rangle \). Therefore, the energy eigenvalue for the sector \( r = 2 \) is

\[
E_2 = (N - 1)z_1 + l_{11} + r_{11} + \sum_{j=1}^{2} (-2z_1 + z_5 + z_j + \xi_j + \xi_j^{-1}), \tag{3.39}
\]

with

\[
\xi_j^{2N} = \left( \frac{1 - \Delta_1 \xi_j}{\Delta_1 - \xi_j} \right) \left( \frac{1 - \Delta_2 \xi_j}{\Delta_2 - \xi_j} \right) \prod_{k=1, k \neq j}^{2} \frac{s(\theta_j, \theta_k)}{s(\theta_k, \theta_j)} \frac{s(\theta_k, -\theta_j)}{s(-\theta_j, \theta_k)}, \tag{3.40}
\]

where the functions \( s(\theta_j, \theta_k) \) are given by (3.32).

### 3.4 General sector

The above results can now be generalized to arbitrary values of \( r \). In a generic sector \( r \) we build eigenstates of \( H \) as direct products of \( N_0 \) states \( |k[0]\rangle \) and \( N_- \) states \( |k[-]\rangle \), such that \( r = N_0 + 2N_- \). These eigenstates are obtained by superposition of terms of the form

\[
|\phi_r\rangle = |0\rangle \times |\phi_{r-1}\rangle + |\phi_{r-2}\rangle,
\]

with \( |\phi_0\rangle = 1, |\phi_1\rangle = |0\rangle \). For instance, in the sector \( r = 3 \) the eigenstate of \( H \) has the form

\[
\Psi_3 = \sum_{k_1 < k_2 < k_3} a(k_1, k_2, k_3) |k_1[0], k_2[0], k_3[0]\rangle
\]

\[
+ \sum_{k_1 < k_2} \{ b_1(k_1, k_2) |k_1[-], k_2[0]\rangle + b_2(k_1, k_2) |k_1[0], k_2[-]\rangle \}. \tag{3.42}
\]
The Ansatz for the wavefunction of the term with \( N_0 \) states \(|k|0\rangle\) becomes
\[
a(k_1, k_2, \ldots, k_r) = \sum_p \varepsilon_p a(\theta_1, \theta_2, \ldots, \theta_r) \xi_1^{k_1} \xi_2^{k_2} \cdots \xi_r^{k_r},
\]
where the sum extends over all permutations and negations of \( \theta_1, \theta_2, \ldots, \theta_r \) and \( \varepsilon_p \) changes sign at each such mutation.

The Ansatz for the wavefunction of terms with \( N_- \) states \(|k|\rangle\) follows from (3.33) as sum over negations of the terms with \( 2N_- \) states \(|k|0\rangle\) at the same site. For instance, in the sector \( r = 3 \)
\[
b_1(k_1, k_2) = \left( \sum_p \varepsilon_p b_{11}(\theta_1, \theta_2)\xi_1^{k_1} \xi_2^{k_2} \right) \xi_3^{k_3} + \left( \sum_p \varepsilon_p b_{12}(\theta_1, \theta_3)\xi_1^{k_1} \xi_3^{k_3} \right) \xi_2^{k_2}
\]
\[+ \left( \sum_p \varepsilon_p b_{13}(\theta_2, \theta_3)\xi_2^{k_2} \xi_3^{k_3} \right) \xi_1^{k_1},
\]
with similar equation for \( b_2(k_1, k_2) \). In that way we always have a \( \text{far pseudoparticle} \) \(|k|0\rangle\) as a \( \text{viewer} \).

We also have verified in this sector that the meeting of \(|k|0\rangle\) with \(|k|\rangle\) can be versused as a meeting of three \(|k|0\rangle\) whose phase shift factorizes in a product of two-pseudoparticle phase shifts.

The corresponding energy eigenvalue is a sum of single one-particle energies
\[
E_r = (N - 1) z_1 + l_{11} + r_{11} + \sum_{j=1}^r (-2z_1 + z_5 + z_5 + \xi_j + \xi_j^{-1}),
\]
where \( \xi_j \) are solutions of the Bethe equations
\[
\xi_j^{2N} = \left( 1 - \Delta_1 \xi_j \right) \left( 1 - \Delta_2 \xi_j \right) \prod_{k=1, k \neq j}^r \left( s(\theta_j, \theta_k) \right) \left( s(\theta_k, -\theta_j) \right),
\]
\[j = 1, \ldots, r \]
with
\[
s(\theta_j, \theta_k) = (1 + \xi_j \xi_k + \Delta_j \xi_j)\left[ z_3(1 + \xi_j^3 \xi_k^3) - (1 + \xi_j \xi_k)(\xi_j + \xi_k) + \Lambda \xi_j \xi_k \right]
\]
\[+ \epsilon \xi_j (z_6 + z_6 \xi_j \xi_k) \left( \xi_6 + z_6 \xi_j \xi_k \right),
\]
and
\[
\Lambda = 2(z_1 - z_5 - z_5) + z_7 + \tilde{z}_7,
\]
\[
\Delta_1 = z_1 - z_5 - l_{11} + l_{22}, \quad \Delta_2 = z_1 - z_5 - r_{11} + r_{22}.
\]

Having now built a common ground for all open spin-1 Hamiltonians associated with the 19-vertex models, we may proceed to find explicitly their spectra. We will do that in the next sections.

4 The Zamolodchikov-Fateev model

The simplest three-states vertex model is the ZF 19-vertex \([15]\) or the \( A_1^{(1)} \) model the spin-1 representation \([20]\) and can be constructed from the six-vertex model using the fusion procedure. The \( R \)-matrix which satisfies the YB equation (2.1) has the form (2.18) with
\[
x_1(u) = \sinh(u + \eta) \sinh(u + 2\eta), \quad x_2(u) = \sinh(u \sinh(u + \eta)),
\]
\[
x_3(u) = \sinh(u \sinh(u - \eta)), \quad x_4(u) = \sinh(u \sinh(u + \eta) + \sinh \eta \sinh 2\eta),
\]
\[
y_5(u) = x_5(u) = \sinh(u + \eta) \sinh 2\eta, \quad y_6(u) = x_6(u) = \sinh(u \sinh 2\eta),
\]
\[
y_7(u) = x_7(u) = \sinh \eta \sinh 2\eta.
\]
This $\mathcal{R}$-matrix is regular and unitary, with $f(u) = x_1(u)x_1(-u)$, $P$- and $T$-symmetric and crossing-symmetric with $M = 1$ and $\rho = \eta$. The most general diagonal solution for $K^-(u)$ has been obtained in Ref. [19] and is given by

$$K^-(u, \beta_{11}) = \begin{pmatrix} k^{--}_{11}(u) & 1 \\ k^{--}_{33}(u) & \end{pmatrix},$$

with

$$k^{--}_{11}(u) = -\frac{\beta_{11} \sinh u + 2 \cosh u}{\beta_{11} \sinh u - 2 \cosh u}, \quad k^{--}_{33}(u) = -\frac{\beta_{11} \sinh(u + \eta) - 2 \cosh(u + \eta)}{\beta_{11} \sinh(u - \eta) + 2 \cosh(u - \eta)},$$

where $\beta_{11}$ is the free parameter. By the automorphism (2.7) the solution for $K^+(u)$ follows

$$K^+(u, \alpha_{11}) = K^-(u - \rho, \alpha_{11}) = \begin{pmatrix} k^{++}_{11}(u) & 1 \\ k^{++}_{33}(u) & \end{pmatrix},$$

with

$$k^{++}_{11}(u) = -\frac{\alpha_{11} \sinh(u + \eta) - 2 \cosh(u + \eta)}{\alpha_{11} \sinh(u + \eta) + 2 \cosh(u + \eta)}, \quad k^{++}_{33}(u) = -\frac{\alpha_{11} \sinh u + 2 \cosh u}{\alpha_{11} \sinh(u + 2\eta) - 2 \cosh(u + 2\eta)},$$

where $\alpha_{11}$ is another free parameter.

We recall section 2 to derive the corresponding quantum open spin chain Hamiltonian. It is the quantum open spin chain for the spin-1 $XXZ$ model.

$$H = \sum_{k=1}^{N-1} H_{k,k+1} + \text{b.t.}$$

(4.6)

where the bulk Hamiltonian is given by (2.21) with the weights

$$\epsilon = 1, \quad \alpha = \sin 2\eta, \quad z_1 = 0, \quad z_3 = -1, \quad z_4 = -2 \cosh 2\eta,$$

$$\bar{z}_5 = z_5 = -\cosh 2\eta, \quad \bar{z}_6 = z_6 = 2 \cosh \eta, \quad \bar{z}_7 = z_7 = -1 - 2 \cosh 2\eta.$$ (4.7)

and the boundary terms given by (2.27). For the left boundary (2.23) we get

$$l_{11} = \frac{1}{2} \beta_{11} \sinh 2\eta, \quad l_{22} = 0, \quad l_{33} = \frac{\beta_{11} \cosh \eta - 2 \sinh \eta}{\beta_{11} \sinh \eta - 2 \cosh \eta} \sinh 2\eta,$$

(4.8)

and

$$r_{11} - r_{22} = \frac{\alpha_{11} \cosh \eta - 2 \sinh \eta}{\alpha_{11} \sinh \eta - 2 \cosh \eta} \sinh 2\eta, \quad r_{33} - r_{22} = \frac{1}{2} \alpha_{11} \sinh 2\eta,$$

$$r_{22} = -\frac{1}{4 \sinh 3\eta} \left( \frac{\alpha_{11} \sinh \eta + 2 \cosh \eta}{\alpha_{11} \sinh \eta - 2 \cosh \eta} \right)^2 - 4[1 + 2 \cosh 2\eta].$$ (4.9)

for the right boundary (2.26).

Next, we can use the coordinate BA method as described in section 3 to find the energy eigenvalue (3.45) and the BA equations (3.46). Here we just list the results. The energy spectrum of the Hamiltonian (4.6) for a generic sector $r$ is given by

$$E_r = \sinh 2\eta \left( \frac{1}{2} \beta_{11} + \frac{\alpha_{11} \cosh \eta - 2 \sinh \eta}{\alpha_{11} \sinh \eta - 2 \cosh \eta} \right) + r_{22} + \sum_{j=1}^{r} (-2 \cosh 2\eta + \xi_j + \xi_j^{-1}),$$

(4.10)
with $\xi_j = e^{i\theta_j}$ satisfying the Bethe equations

$$
\xi_{j}^{2N} = \left( \frac{\Delta_1 \xi_j - 1}{\Delta_1 - \xi_j} \right) \left( \frac{\Delta_2 \xi_j - 1}{\Delta_2 - \xi_j} \right) \prod_{k \neq j}^{n} \left( \frac{s(\theta_j, \theta_k)}{s(\theta_k, \theta_j)} \right) \left( \frac{s(\theta_k, -\theta_j)}{-s(-\theta_j, \theta_k)} \right),
$$

where

$$
\Delta_1 = \sinh 2\eta \left( \coth 2\eta - \frac{1}{2} \beta_{11} \right), \quad \Delta_2 = \sinh 2\eta \left( \coth 2\eta - \frac{\alpha_{11} \cosh \eta - 2 \sinh \eta}{\alpha_{11} \sinh \eta - 2 \cosh \eta} \right).
$$

For the ZF model the two-particle phase shift is given by

$$
s(\theta_j, \theta_k) = \frac{1 + \xi_j + \xi_k + \xi_j \xi_k - (\Delta + 2)\xi_j}{1 + \xi_j + \xi_k + \xi_j \xi_k - (\Delta + 2)\xi_k},
$$

with the $s$-functions given by (3.47) and $\Delta = 2 \cosh 2\eta$.

This energy spectrum was already obtained by Mezincescu at al. [19] through a generalization of the quantum inverse scattering method developed by Sklyanin [12], the so-called fusion procedure [20]. This fusion procedure was also used by Yung and Batchelor [21] to solve the ZF vertex-model with inhomogeneities.

For a particular choice of boundary terms, the ZF spin chain has the quantum group symmetry \textit{i.e.}, if we choose $\xi_+ \rightarrow \infty$ ($\beta_{11} = 2 \coth \xi_-$ and $\alpha_{11} = 2 \coth \xi_+$), then the spin chain Hamiltonian (4.1) has $U_q(su(2))$-invariance [19].

## 5 The Izergin-Korepin model

The solution of the YB equation corresponding to $A^{(2)}_2$ in the fundamental representation was found by Izergin and Korepin [16]. The $R$-matrix has the form (2.18) with non-zero entries

$$
\begin{align*}
x_1(u) &= \sinh(u - 5\eta) + \sinh \eta, \quad x_2(u) = \sinh(u - 3\eta) + \sinh 3\eta, \\
x_3(u) &= \sinh(u - \eta) + \sinh \eta, \quad x_4(u) = \sinh(u - 3\eta) - \sinh 5\eta + \sinh 3\eta + \sinh \eta, \\
x_5(u) &= -2e^{-u/2} \sinh 2\eta \cosh(u/2 - 3\eta), \quad y_5(u) = -2e^{u/2} \sinh 2\eta \cosh(u/2 - 3\eta), \\
x_6(u) &= 2e^{-u/2+2\eta} \sinh 2\eta \sinh(u/2), \quad y_6(u) = -2e^{u/2-2\eta} \sinh 2\eta \sinh(u/2), \\
x_7(u) &= -2e^{-u+2\eta} \sinh \eta \sinh 2\eta - e^{-2\eta} \sinh 4\eta, \\
y_7(u) &= 2e^{u-2\eta} \sinh \eta \sinh 2\eta - e^{2\eta} \sinh 4\eta.
\end{align*}
$$

This $R$-matrix is regular and unitary, with $f(u) = x_1(u)x_1(-u)$. It is PT-symmetric and crossing-symmetric, with $\rho = -6\eta - i\pi$ and

$$
M = \begin{pmatrix}
  e^{2\eta} & 1 \\
  1 & e^{-2\eta}
\end{pmatrix}.
$$

Diagonal solutions for $K^{-}(u)$ have been obtained in [22]. It turns out that there are three solutions without free parameters, being $K^{-}(u) = 1$, $K^{-}(u) = F^+$ and $K^{-}(u) = F^-$, with

$$
F^\pm(u) = \begin{pmatrix}
  e^{-u} f^{(\pm)}(u) \\
  e^{u} f^{(\pm)}(u)
\end{pmatrix},
$$

where

$$
\begin{align*}
f^{(+)}/(u) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{x - 1} e^{-u(x^2 - 1)} = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \frac{-x}{x^2 - 1} e^{-u(x^2 - 1)} = e^{-u} f^{(-)}(u), \\
f^{(-)}(u) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{x + 1} e^{-u(x^2 + 1)} = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \frac{x}{x^2 + 1} e^{-u(x^2 + 1)} = e^{-u} f^{(+)}(u).
\end{align*}
$$
where we have defined

\[ f(\pm)(u) = \frac{\cosh(u/2 - 3\eta) \pm i \sinh(u/2)}{\cosh(u/2 + 3\eta) \mp i \sinh(u/2)} \]  

(5.4)

By the automorphism (2.7), three solutions \( K^+(u) \) follow as

\[ K^+ = \begin{pmatrix} e^{u-4\eta g(\pm)(u)} & 1 \\ e^{-u+4\eta g(\pm)(u)} & 1 \end{pmatrix}, \]

(5.5)

where we have defined

\[ g(\pm)(u) = \frac{\cosh(u/2 - 3\eta) \pm i \sinh(u/2)}{\cosh(u/2 - 3\eta) \mp i \sinh(u/2 - 6\eta)} \]  

(5.6)

The corresponding quantum open spin chain Hamiltonians is also written as in (4.6), where the bulk term is given by (2.21) with

\[
\begin{align*}
\epsilon &= 1, \quad \alpha = -2 \sinh 2\eta, \quad z_1 = 0, \quad z_3 = \frac{\cosh \eta}{\cosh 3\eta}, \quad z_4 = -\frac{\sinh \eta \sinh 4\eta}{\cosh 3\eta} \\
\zeta_5 &= -e^{-2\eta}, \quad \zeta_6 = -e^{2\eta}, \quad \zeta_7 = \frac{\cosh \eta}{\cosh 3\eta} \left( e^{-4\eta} + 2 \sinh 2\eta \right), \quad \zeta_8 = -\frac{\cosh \eta}{\cosh 3\eta} \left( e^{4\eta} - 2 \sinh 2\eta \right).
\end{align*}
\]

(5.7)

To derive the boundary term (2.27), we will only consider three types of boundary solutions, one for each pair \((K^-(u), K^+(u))\) defined by the automorphism (2.7): \((1, M), (F^+, G^+)\) and \((F^-, G^-)\).

The \((1, M)\) case: For \(K^-(u) = 1\) the left boundary (2.23) vanishes \((l_{11} = l_{22} = l_{33} = 0)\), and the right boundary (2.24) is proportional to the identity. To see this we can substitute (5.7) and \(K^+(0) = \text{diag}(e^{2\eta}, 1, e^{-2\eta})\) in (2.26) to get

\[ r_{11} = r_{22} = r_{33} = -2 \frac{\cosh 4\eta \sinh 2\eta}{\sinh 6\eta}. \]

(5.8)

Therefore, the corresponding open chain Hamiltonian is

\[ H = \sum_{k=1}^{N-1} H_{k,k+1} - 2 \frac{\cosh 4\eta \sinh 2\eta}{\sinh 6\eta} 1_N. \]

(5.9)

The coordinate \(BA\) gives us the corresponding energy spectrum (3.45). For a given sector \(r\) it is given by

\[ E_r = -2 \frac{\cosh 4\eta \sinh 2\eta}{\sinh 6\eta} + \sum_{j=1}^{r} \left( -2 \cosh 2\eta + \xi_j + \xi_j^{-1} \right), \]

(5.10)

where \(\xi_j = e^{i\phi_j}\) are solutions of the Bethe equations

\[ \xi_j^{2N} = \prod_{k=1, k \neq j}^{r} \left( \frac{s(\theta_j, \theta_k)}{s(\theta_k, \theta_j)} \right) \left( \frac{s(\theta_k, -\theta_j)}{s(-\theta_j, \theta_k)} \right), \]

\(j = 1, ..., r\)

(5.11)
The two-particle phase shift for the IK form is given by
\[
s(\theta_j, \theta_k) = \left( \frac{1 + \xi_j \xi_k - \Delta \xi_j}{1 + \xi_j \xi_k - \Delta \xi_k} \right) \left( \frac{1 + \xi_j \xi_k - \xi_j - \xi_k - (\Delta - 2) \xi_k}{1 + \xi_j \xi_k - \xi_j - \xi_k - (\Delta - 2) \xi_j} \right),
\]
where the \(s\)-functions are given by (3.47) and \(\Delta = 2 \cosh 2\eta\).

It was noted by Mezincescu and Nepomechie [24], that this open spin chain Hamiltonian is quantum group invariant. Moreover, the corresponding transfer matrices have been diagonalized by the analytic BA in [23], using the \(U_q(su(2))\) invariance of (5.9). This quantum group invariance was also used by Yung and Batchelor [21] to determine properties of the transfer matrix eigenvalues with inhomogeneities essential to apply the analytical BA.

The \((F^+, G^+)\) and \((F^-, G^-)\) cases: These cases can be treated simultaneously. The matrix elements of the left boundary term are given by
\[
l^{(\pm)}_{11} = \sinh 2\eta \left( \frac{e^{3\eta} \mp i}{\cosh 3\eta} \right), \quad l^{(\pm)}_{22} = 0, \quad l^{(\pm)}_{33} = -\sinh 2\eta \left( \frac{e^{-3\eta} \mp i}{\cosh 3\eta} \right),
\]
and for the right boundary term we have
\[
r^{(\pm)}_{11} - r^{(\pm)}_{22} = -\sinh 2\eta \left( \frac{e^{-3\eta} \mp i}{\cosh 3\eta} \right), \quad r^{(\pm)}_{33} - r^{(\pm)}_{22} = \sinh 2\eta \left( \frac{e^{3\eta} \mp i}{\cosh 3\eta} \right),
\]
\[
r^{(\pm)}_{22} = -\sinh 4\eta \frac{\cosh 7\eta \mp 4i \sinh 3\eta \sinh 2\eta}{\cosh 3\eta \pm i \sinh 2\eta}.
\]

In these cases the corresponding open chain Hamiltonians are not \(U_q(su(2))\)-invariant [23]. Nevertheless, it has recently been argued by Nepomechie [24] that the transfer matrices corresponding to these solutions also have the \(U_q(o(3))\) symmetry, but with a nonstandard coproduct. They can be written in the following form
\[
H^{(\pm)} = \sum_{k=1}^{N-1} H_{k,k+1} + \sinh 2\eta \left( S^+_{1} - S^+_N + \left( \frac{\sinh 3\eta \mp i}{\cosh 3\eta} \right) \left[ (S^+_1)^2 + (S^+_N)^2 \right] \right) + r^{(\pm)}_{22} 1_N.
\]

From the coordinate BA we have find their energy spectra (3.45):
\[
E^{(\pm)}_r = 2 \sinh \eta \left( \frac{\sinh 3\eta \mp i}{\cosh 3\eta} \right) + r^{(\pm)}_{22} + \sum_{j=1}^{r} \left( -2 \cosh 2\eta + \xi_j + \xi_j^{-1} \right),
\]
where \(\xi_j = e^{i\theta_j}\) are solutions of the Bethe equations
\[
\left( \xi_j^{(\pm)} \right)^{2N} = \left( \frac{1 - \Delta_1^{(\pm)} \xi_j}{\Delta_1^{(\pm)} - \xi_j} \right) \left( \frac{1 - \Delta_2^{(\pm)} \xi_j}{\Delta_2^{(\pm)} - \xi_j} \right) \prod_{k=1, k \neq j}^{r} \frac{s(\theta_j, \theta_k)}{s(\theta_k, \xi_j)} \frac{s(\theta_k, -\theta_j)}{s(-\theta_j, \theta_k)},
\]
\[
j = 1, ..., r
\]
where
\[
\Delta_1^{(\pm)} = e^{2\eta} - \sinh 2\eta \left( \frac{e^{3\eta} \mp i}{\cosh 3\eta} \right), \quad \Delta_2^{(\pm)} = e^{-2\eta} + \sinh 2\eta \left( \frac{e^{-3\eta} \mp i}{\cosh 3\eta} \right),
\]
and the two-particle phase shift is still given by (5.12). These cases were also considered in Ref.[21] through the analytical BA with inhomogeneities.
Finally, we note that is interesting to reformulate the Boltzmann weights of the IK model by the following transformation

\[ R(u, \eta) \rightarrow R'(u, \eta) = \frac{1}{2i} R(2u, -\eta - \frac{\pi}{2}). \]  

(5.19)

This \( R' \) matrix differs from the one given in [25] by a gauge transformation. It is regular and unitary, with \( f(u) = x_1(u)x_1(-u) \), \( PT \)-symmetric and crossing-unitarity with \( M' = \text{diag}(e^{-2\eta}, 1, e^{2\eta}) \) and \( \rho' = 3\eta \). After the gauge transformation \( R'_{12}(u) = V^T R_{12}(u) V^{-1} \) with \( V = \text{diag}(e^{-u}, 1, e^u) \), can see that \( M'' = \text{diag}(e^{-4\eta}, 1, e^{4\eta}) \) and \( \rho'' = \rho' \). In this case the solution \( (F'^+, G'^+) \) can be written as

\[ F'^{-} = \text{diag}(1, \frac{\sinh(\eta - \frac{\pi}{2})}{\sinh(\eta + \frac{\pi}{2})}, 1), \quad G'^{+} = -\text{diag}(e^{4\eta} \frac{\sinh(\eta - \frac{\pi}{2})}{\sinh(\eta + \frac{\pi}{2})}, e^{-4\eta}). \]  

(5.20)

This solution was used by Fan in [26] to find the spectrum of the corresponding transfer matrix using the algebraic BA for one and two-particle excited states.

6 The \( sl(2|1) \)-model

The solution of the graded YB equation corresponding to \( sl(2|1) \) in the fundamental representation has the form (2.18) with non-zero entries [27, 17]:

\[ \begin{align*}
  x_1(u) &= \cosh(u + \eta) \sinh(2\eta), & x_2(u) &= \sinh u \cosh(u + \eta), \\
  x_3(u) &= \sinh u \cosh(u - \eta), & x_4(u) &= \sinh u \cosh(u + \eta) - \sinh 2\eta \cosh \eta, \\
  y_5(u) &= x_5(u) = \sinh 2\eta \cosh(u + \eta), & y_6(u) &= x_6(u) = \sinh 2\eta \sinh u, \\
  y_7(u) &= x_7(u) = \sinh 2\eta \cosh \eta.
\end{align*} \]  

(6.1)

This \( R \)-matrix is regular and unitary, with \( f(u) = x_1(u)x_1(-u) \). \( P \)- and \( T \)-symmetric and crossing-symmetric with \( M = 1 \) and \( \rho = \eta \). The graded version of the crossing-unitarity relation (2.4) is satisfied with \( f'(u) = x_1(u + i\frac{\pi}{2})x_1(-u - i\frac{\pi}{2}) \).

The most general diagonal solution for \( K^{-}(u) \) has been presented in Ref. [28] and it is given by

\[ K^{-}(u, \beta_{11}) = \begin{pmatrix} k_{11}^{-}(u) & 1 \\ k_{33}^{-}(u) \end{pmatrix}, \]  

(6.2)

with

\[ \begin{align*}
  k_{11}^{-}(u) &= \frac{-\beta_{11} \sinh u + 2 \cosh u}{\beta_{11} \sinh u - 2 \cosh u}, \\
  k_{33}^{-}(u) &= \frac{\beta_{11} \cosh(u + \eta) - 2 \sinh(u + \eta)}{\beta_{11} \cosh(u - \eta) + 2 \sinh(u - \eta)}.
\end{align*} \]  

(6.3)

where \( \beta_{11} \) is the free parameter. Due to the automorphism (2.7) the solution for \( K^{+}(u) \) is given by \( K^{-}(-u - \rho, \frac{1}{4} \alpha_{11}) \) i.e.

\[ K^{+}(u, \beta_{11}) = \begin{pmatrix} k_{11}^{+}(u) & 1 \\ k_{33}^{+}(u) \end{pmatrix}, \]  

(6.4)

where

\[ \begin{align*}
  k_{11}^{+}(u) &= \frac{\alpha_{11} \cosh(u + \eta) - 2 \sinh(u + \eta)}{\alpha_{11} \cosh(u + \eta) + 2 \sinh(u + \eta)}, \\
  k_{33}^{+}(u) &= -\frac{\alpha_{11} \sinh u + 2 \cosh u}{\alpha_{11} \sinh(u + 2\eta) + 2 \cosh(u + 2\eta)}.
\end{align*} \]  

(6.5)
and $\alpha_{11}$ is another free parameter.

The weights for the corresponding bulk Hamiltonian (2.21) are given by

$$
\epsilon = -1, \quad \alpha = \sinh 2\eta, \quad z_1 = 0, \quad z_3 = 1, \quad z_4 = 2 \cosh 2\eta; \\
\xi_5 = z_5 = -\cosh 2\eta, \quad \xi_6 = z_6 = 2 \sinh \eta, \quad \xi_7 = z_7 = 1 - 2 \cosh 2\eta
$$

(6.6)

The left boundary terms of b.t. (2.27) are given by

$$
l_{11} = \frac{1}{2} \beta_{11} \sinh 2\eta, \quad l_{22} = 0, \quad l_{33} = \frac{\beta_{11} \sinh \eta - 2 \cosh \eta}{\beta_{11} \cosh \eta - 2 \sinh \eta} \sinh \eta
$$

(6.7)

and for the right boundary we have

$$
r_{11} - r_{22} = \frac{\alpha_{11} \sinh \eta - 2 \cosh \eta}{\alpha_{11} \cosh \eta - 2 \sinh \eta} \sinh \eta, \quad r_{33} - r_{22} = \frac{1}{2} \alpha_{11} \sinh 2\eta, \\
r_{22} = -\frac{1}{4} \frac{\sinh 4\eta}{\cosh 3\eta} \left( \frac{\alpha_{11} \cosh \eta + 2 \sinh \eta}{\alpha_{11} \cosh \eta - 2 \sinh \eta} \right).
$$

(6.8)

Now, using the coordinate BA we find the energy spectrum (3.45) for the $sl(2|1)$ open chain Hamiltonian:

$$
E_r = \sinh 2\eta \left( \frac{1}{2} \beta_{11} + \frac{\alpha_{11} \sinh \eta - 2 \cosh \eta}{\alpha_{11} \cosh \eta - 2 \sinh \eta} \right) + r_{22} + \sum_{j=1}^{r} (-2 \cosh 2\eta + \xi_j + \xi_j^{-1})
$$

(6.9)

with $\xi_j = e^{i\theta_j}$ satisfying the Bethe equations

$$
\xi_j^{2N} = \left( \frac{\Delta_1 \xi_j - 1}{\Delta_1 - \xi_j} \right) \left( \frac{\Delta_2 \xi_j - 1}{\Delta_2 - \xi_j} \right) \prod_{k \neq j} \left( \frac{s(\theta_j, \theta_k)}{s(\theta_j, \theta_k)} \right) \left( \frac{s(\theta_k, -\theta_j)}{s(-\theta_j, \theta_k)} \right)
$$

(6.10)

where

$$
\Delta_1 = \sinh 2\eta \left( \coth 2\eta - \frac{1}{2} \beta_{11} \right), \quad \Delta_2 = \sinh 2\eta \left( \coth 2\eta - \frac{\alpha_{11} \sinh \eta - 2 \cosh \eta}{\alpha_{11} \cosh \eta - 2 \sinh \eta} \right)
$$

(6.11)

and the two-body phase shift for the $sl(2|1)$ model is given by

$$
s(\theta_j, \theta_k) = \frac{1 - \xi_j - \xi_k + \xi_j \xi_k - (\Delta - 2) \xi_j}{1 - \xi_j - \xi_k + \xi_j \xi_k - (\Delta - 2) \xi_k}
$$

(6.12)

where $\Delta = 2 \cosh 2\eta$.

## 7 The $osp(1|2)$-model

The trigonometric solution of the graded YB equation corresponding to $osp(1|2)$ in the fundamental representation has the form (2.18) with non-zero entries [17]:

$$
x_1(u) = \sinh (u + 2\eta) \sinh (u + 3\eta), \quad x_2(u) = \sinh u \sinh (u + 3\eta) \\
x_3(u) = \sinh u \sinh (u + \eta), \quad x_4(u) = \sinh u \sinh (u + 3\eta) - \sinh 2\eta \sinh 3\eta \\
x_5(u) = e^{-u} \sinh 2\eta \sinh (u + 3\eta), \quad y_5(u) = e^u \sinh 2\eta \sinh (u + 3\eta) \\
x_6(u) = -e^{-u - 2\eta} \sinh 2\eta \sinh u, \quad y_6(u) = e^{u + 2\eta} \sinh 2\eta \sinh u \\
x_7(u) = e^{-u} \sinh 2\eta \left( \sinh (u + 3\eta) + e^{-\eta} \sinh u \right) \\
y_7(u) = e^u \sinh 2\eta \left( \sinh (u + 3\eta) + e^{\eta} \sinh u \right)
$$

(7.1)
This $R$-matrix is regular and unitary, with $f(u) = x_1(u) x_1(-u)$. It is $PT$-symmetric and crossing-symmetric, with $\rho = 3\eta$ and

$$M = \begin{pmatrix} e^{-2\eta} & 1 \\ 1 & e^{2\eta} \end{pmatrix}. \quad (7.2)$$

Diagonal solutions for $K^-(u)$ have been obtained in [29]. It turns out that there are three solutions without free parameters, being $K^-(u) = 1$, $K^-(u) = F^+$ and $K^-(u) = F^-$, with

$$F^\pm = \begin{pmatrix} \mp e^{-2u} f(\pm)(u) & 1 \\ 1 & \mp e^{2u} f(\pm)(u) \end{pmatrix}, \quad (7.3)$$

where we have defined

$$f^+(u) = \frac{\sinh(u+3\eta/2)}{\sinh(u-3\eta/2)}, \quad f^-(u) = \frac{\cosh(u+3\eta/2)}{\cosh(u-3\eta/2)}. \quad (7.4)$$

By the automorphism (2.7), three solutions $K^+(u)$ follow as $K^+(u) = M$, $K^+(u) = G^+$ and $K^+(u) = G^-$, with

$$G^\pm = \begin{pmatrix} \mp e^{2u+4\eta} g(\pm)(u) & 1 \\ 1 & \mp e^{-2u-4\eta} g(\pm)(u) \end{pmatrix}, \quad (7.5)$$

where we have defined

$$g^+(u) = \frac{\sinh(u+3\eta/2)}{\sinh(u+9\eta/2)}, \quad g^-(u) = \frac{\cosh(u+3\eta/2)}{\cosh(u+9\eta/2)}. \quad (7.6)$$

The corresponding quantum open spin chain Hamiltonians is also written as in (4.6), where the bulk term is given by (2.21) with

$$\epsilon = -1, \quad \alpha = \sinh 2\eta, \quad z_1 = 0, \quad z_3 = \frac{\sinh \eta}{\sinh 3\eta}, \quad z_4 = 2\frac{\cosh \eta \sinh 4\eta}{\sinh 3\eta}, \quad z_5 = -e^{2\eta}, \quad z_6 = -e^{-2\eta}, \quad z_7 = -e^{2\eta} + e^{-\eta} \frac{\sinh 2\eta}{\sinh 3\eta}, \quad z_8 = e^{-2\eta} \frac{\sinh 2\eta}{\sinh 3\eta}. \quad (7.7)$$

To derive the boundary term (2.27), we will only consider three types of boundary solutions, one for each pair $(K^-(u), K^+(u))$ defined by the automorphism (2.7): $(1, M)$, $(F^+, G^+)$ and $(F^-, G^-)$.

The $(1, M)$ case: For $K^-(u) = 1$ the left boundary (2.23) vanishes ( $l_{11} = l_{22} = l_{33} = 0$ ), and the right boundary (2.24) is proportional to the identity, for which quantum-algebra invariance is achieved [23]. To see this we can substitute (7.7) and $K^+(0) = \text{diag}(e^{-2\eta}, 1, e^{2\eta})$ in (2.26) to get

$$r_{11} = r_{22} = r_{33} = 2\frac{\cosh 4\eta \sinh 2\eta}{\sinh 6\eta}. \quad (7.8)$$

Therefore, the corresponding open chain Hamiltonian is

$$H = \sum_{k=1}^{N-1} H_{k,k+1} + 2\frac{\cosh 4\eta \sinh 2\eta}{\sinh 6\eta} 1_N \quad (7.9)$$
The coordinate \( \mathcal{B} \) gives us the corresponding energy spectrum (3.45). For a given sector \( r \) it is given by

\[
E_r = 2 \frac{\cosh 4\eta \sinh 2\eta}{\sinh 6\eta} + \sum_{j=1}^{r} (-2 \cosh 2\eta + \xi_j + \xi_j^{-1})
\]

(7.10)

where \( \xi_j = e^{\theta_j} \) are solutions of the Bethe equations

\[
\xi_{2N}^j = \prod_{k=1, k\neq j}^{r} \left( \frac{s(\theta_j, \theta_k)}{s(\theta_k, \theta_j)} \right) \left( \frac{s(\theta_k, -\theta_j)}{s(-\theta_j, \theta_k)} \right),
\]

\( j = 1, ..., r \)

(7.11)

The two-particle phase shift for the \( \text{osp}(1|2) \) model is given by

\[
s(\theta_j, \theta_k) = \frac{1 + \xi_j \xi_k - \Delta \xi_j}{1 + \xi_j \xi_k - \Delta \xi_k} \left( \frac{1 + \xi_j \xi_k + \xi_k + \xi_k - (\Delta + 2) \xi_j}{1 + \xi_j \xi_k + \xi_k + \xi_k - (\Delta + 2) \xi_k} \right),
\]

(7.12)

where the \( s \)-functions are given by (3.47) and \( \Delta = 2 \cosh 2\eta \).

**The \((F^+, G^+)\) case:** In this case the boundary terms are

\[
l_{11} = e^{-3\eta/2} \sinh 2\eta, \quad l_{22} = 0, \quad l_{33} = e^{3\eta/2} \sinh 2\eta
\]

(7.13)

and

\[
r_{11} - r_{22} = \frac{e^{3\eta/2}}{\sinh(3\eta/2)} \sinh 2\eta, \quad r_{33} - r_{22} = \frac{e^{-3\eta/2}}{\sinh(3\eta/2)} \sinh 2\eta
\]

\[
r_{22} = -\frac{\sinh 4\eta}{\sinh 6\eta} \left( 4 \cosh \left( \frac{3}{2} \eta \right) \cosh \left( \frac{5}{2} \eta \right) - 1 \right)
\]

(7.14)

The energy eigenvalues are

\[
E_r = 2 \sin 2\eta \coth(3\eta/2) + r_{22} + \sum_{j=1}^{r} (-2 \cosh 2\eta + \xi_j + \xi_j^{-1})
\]

(7.15)

with the Bethe equations

\[
\xi_{2N}^j = \left( \frac{1 - \Delta_1 \xi_j}{\Delta_1 - \xi_j} \right) \left( \frac{1 - \Delta_2 \xi_j}{\Delta_2 - \xi_j} \right) \prod_{k=1, k\neq j}^{r} \left( \frac{s(\theta_j, \theta_k)}{s(\theta_k, \theta_j)} \right) \left( \frac{s(\theta_k, -\theta_j)}{s(-\theta_j, \theta_k)} \right),
\]

\( j = 1, ..., r \)

(7.16)

where the phase shift is given by (7.12) and \( \Delta_1 = e^{-2\eta} - \frac{e^{-3\eta/2}}{\sinh(3\eta/2)} \sinh 2\eta, \quad \Delta_2 = e^{2\eta} - \frac{e^{3\eta/2}}{\sinh(3\eta/2)} \sinh 2\eta \)

**The \((F^-, G^-)\) case:** In this case the boundary terms are

\[
l_{11} = -\frac{e^{-3\eta/2}}{\cosh(3\eta/2)} \sinh 2\eta, \quad l_{22} = 0, \quad l_{33} = \frac{e^{3\eta/2}}{\cosh(3\eta/2)} \sinh \eta
\]

\[
r_{11} - r_{22} = \frac{e^{3\eta/2}}{\cosh(3\eta/2)} \sinh 2\eta, \quad r_{33} - r_{22} = -\frac{e^{-3\eta/2}}{\cosh(3\eta/2)} \sinh 2\eta
\]

\[
r_{22} = -\frac{\sinh 4\eta}{\sinh 6\eta} \left( 4 \sinh \left( \frac{3}{2} \eta \right) \sinh \left( \frac{5}{2} \eta \right) - 1 \right)
\]

(7.17)
The corresponding energy eigenvalues are given by

\[ E_r = 2 \sin 2\eta \tanh(3\eta/2) + r_{22} + \sum_{j=1}^{r} (-2 \cosh 2\eta + \xi_j + \xi_j^{-1}) \]  
(7.18)

The Bethe equations are

\[ \xi_j^{2N} = \left( \frac{1 - \Delta_1 \xi_j}{\Delta_1 - \xi_j} \right) \left( \frac{1 - \Delta_2 \xi_j}{\Delta_2 - \xi_j} \right) \prod_{k=1, k \neq j}^{r} \left( \frac{s(\theta_j, \theta_k)}{s(\theta_k, \theta_j)} \right) \left( \frac{s(\theta_j, -\theta_j)}{s(-\theta_j, \theta_k)} \right), \]

\[ j = 1, ..., r \]  
(7.19)

with the phase shift (7.12) and

\[ \Delta_1 = e^{-2\eta} + \frac{e^{-3\eta/2}}{\cosh(3\eta/2)}, \quad \Delta_2 = e^{2\eta} - \frac{e^{3\eta/2}}{\cosh(3\eta/2)} \sinh \eta \]  
(7.20)

8 From non graded to graded solutions

Beside the \( R \)-matrix we also have considered the \( R \)-matrix, which satisfies

\[ R_{12}(u)R_{23}(u+v)R_{12}(v) = R_{23}(v)R_{12}(u+v)R_{23}(u). \]  
(8.1)

Because only \( R_{12} \) and \( R_{23} \) are involved, this equation written in components looks the same as in the non graded case. Moreover, the matrix \( R = PR \) satisfies the usual YB equation (2.1) where \( P \) is the non graded permutation matrix. When the graded permutation matrix \( P \) is used, then \( R = PR \) satisfies the graded version of the YB equation.

Multiplying the \( R \)-matrix for 19-vertex models (2.18) by the diagonal matrix \( \Pi = PP = P \Pi \) we will get graded \( R \)-matrices starting from non graded \( R \)-matrices and vice-versa. The new \( R \)-matrix \( R' = \Pi R \), still has the form (2.18) but with the change of sign of the fifth row due to the grading BFB. The bulk Hamiltonian has the form (2.20) but interchanging the role of the sign \( \epsilon \). Now \( \epsilon = -1 \) for non-graded models and \( \epsilon = 1 \) for graded models.

Let us use this interchange property with the YB solution of the IK model. First we recall the transformation (5.19)

\[ R'(u, \eta) = \frac{1}{2i} R(2u, -\eta - i\pi/2) \Rightarrow H'_{k,k+1}(\eta) = H_{k,k+1}(-\eta - i\pi/2). \]  
(8.2)

The matrix \( R_{IK}(u, \eta) = \Pi R' \) is a solution of the graded version of the YB equation (2.1) and the corresponding vertex model can be named as the graded version of the IK model.

Using the symmetries of the YB solutions for 19-vertex models: \( x_2(u) \rightarrow \pm x_2(u) \) and \( x_6(u) \rightarrow \pm x_6(u) \) with \( y_6(u) \rightarrow \mp y_6(u) \), we can see that this model has the same Boltzmann weights of the \( osp(1|2) \)-model, except for the presence of the factor \( \pm i \) in \( x_6(u) \) and \( \mp i \) in \( y_6(u) \). However, this identification is not so trivial due to the change the signs of the fifth row of \( R \) (BFB grading). Nevertheless, by direct computation we have verified that both models have the same reflection \( K \)-matrices. It means that \( R_{IK}(u, \eta) \) and the \( R(u, \eta) \) of the \( osp(1|2) \) share the same symmetries. Consequently, both open chain Hamiltonians have the same boundary terms. Moreover, from the definition (3.32) we can see that phase shift equations (3.31) are invariant under the replacement \( z_6 \rightarrow \pm i z_6 \) with \( \xi_6 \rightarrow \mp i \xi_6 \). Thus, the coordinate BA previously described, yields the same spectrum for both models. In words, the open spin chain Hamiltonians associated with the graded IK model have the \( osp(1|2) \) - invariance.
This situation is also present in the graded version of the ZF model. In order to see this, we have to reformulate conveniently the Boltzmann weights of the ZF model by the following transformation

$$\mathcal{R}(u, \eta) \rightarrow \mathcal{R}'(u, \eta) = \frac{1}{i} \mathcal{R}(u, \eta - i \frac{\pi}{2}).$$  \hfill (8.3)

The graded version of the ZF model is defined by the following $\mathcal{R}$-matrix

$$\mathcal{R}_{ZF}(u, \eta) = \Pi \mathcal{R}'(u, \eta)$$  \hfill (8.4)

Using again the symmetries of the 19-vertex model we can see, up to a possible canonical transformation: $x_6 \rightarrow x_6'(u) = \pm ix_6(u)$, the non-zero entries of $\mathcal{R}_{ZF}(u, \eta)$ are identified with the Boltzmann weights of the $sl(2|1)$ model (6.1). We also find that both models have the same $K$-matrices and their coordinate Bethe ansätze yield a common spectrum.

We have verified that the inverse situation is also true. The non-graded versions of the graded 19-vertex models are in correspondence with the 19-vertex models of Izergin-Korepin and Zamolodchikov-Fateev.

During the preparation of this paper we learned that the connection between Izergin-Korepin and $osp(1|2)$ models has recently been discussed in Saleur and Wehefritz-Kaufmann [30], where also earlier references are given.

9 Conclusion

In the first part of this paper we have applied the coordinate BA to find the spectra of open spin-1 chain Hamiltonians associated with four 19-vertex models, including two graded models. This procedure was carried out for boundaries derived from diagonal solutions of the reflection equations.

We believe that the method here presented could also be applied for Hamiltonians associated with higher states vertex-models. For instance, in the quantum spin chain $s = 3/2$ XXZ model we have four states: $|k[3/2]\rangle$, $|k[1/2]\rangle$, $|k[-1/2]\rangle$ and $|k[-3/2]\rangle$. It means that the state $|k[1/2]\rangle$ can be parametrized by plane wave and the states $|k[-1/2]\rangle$ and $|k[-3/2]\rangle$ as two and three states $|k[1/2]\rangle$ at the same site, respectively, multiplied by some weight functions.

These weight functions are responsible by the factorized form of the two-body phase shifts of the IK model (5.12) and the $osp(1|2)$ model (7.12). In the ZF model, as well as in the $sl(2|1)$ model, we do not have a factored form for the two-pseudoparticle phase shift because their weight functions (3.30) are constant. It means that the state $|k[-\rangle$ behaves exactly as two states $|k[0]\rangle$ at the same site. This is in agreement with the fact that the ZF model can be constructed by a fusion procedure of two six-vertex models.

There are several issues left for future works. A natural extension of this work is to consider the algebraic version for the BA [26]. Independently, it is interesting to analyse the Bethe Ansatz equations to derive ground state properties, low-lying excitations and the thermodynamic limit.

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