BOUNDARY AT INFINITY OF SYMMETRIC RANK ONE SPACES

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Abstract. It is shown that the canonical Carnot–Carathéodory spherical and horospherical metrics, which are defined on the boundary at infinity of every rank one symmetric space of noncompact type, are visual; i.e., they are bi-Lipschitz equivalent with universal bi-Lipschitz constants to the inverse exponent of Gromov products based in the space and on the boundary at infinity respectively.

§1. Introduction

Let \( M \) be a rank one symmetric space of noncompact type. There are two classes of natural metrics on the boundary at infinity \( \partial_\infty M \). First, these are spherical and horospherical metrics. Given \( o \in M \), for every \( t > 0 \) we consider the Riemannian metric \( ds_t^2 \) induced from \( M \) on the sphere \( S_t \subset M \) of radius \( t \) centered at \( o \). Identifying \( S_t, \partial_\infty M \) with the unit sphere \( U_o M \subset T_o M \) via the radial projection from \( o \), we view \( ds_t^2 \) as the Riemannian metric on \( U_o M \) for all \( t > 0 \). Then there exists a limit

\[
ds_\infty = \lim_{t \to \infty} e^{-t} ds_t,
\]

which is a Carnot-Carathéodory metric. The respective (bounded) interior distance is the spherical metric \( d_\infty \) on \( \partial_\infty M \) based at \( o \).

Similarly, given a Busemann function \( b : M \to \mathbb{R} \) centered at \( \omega \in \partial_\infty M \), for every \( t > 0 \) we can consider the Riemannian metric \( ds_{b,t}^2 \) induced from \( M \) on the horosphere \( H_{b,t} = b^{-1}(t) \). Identifying \( H_{b,t}, \partial_\infty M \setminus \{\omega\} \) with the fixed horosphere \( H = H_{b,0} \) via the radial projection from \( \omega \), we view \( ds_{b,t}^2 \) as the Riemannian metric on \( H \) for all \( t \in \mathbb{R} \). Then there exists a limit

\[
ds_b = \lim_{t \to \infty} e^{-t} ds_{b,t},
\]

which is a Carnot–Carathéodory metric. The respective (unbounded) interior distance is the horospherical metric \( d_b \) on \( \partial_\infty M \setminus \{\omega\} \) associated with \( b \).

Second, since \( M \) is a CAT(-1)-space, in particular, \( M \) is Gromov hyperbolic, there are visibility metrics on the Gromov boundary at infinity of \( M \). This boundary coincides with the geodesic boundary \( \partial_\infty M \), and for \( \xi, \eta \in \partial_\infty M \) we have the Gromov product \( (\xi|\eta)_o \) based at \( o \), and similarly, for \( \xi, \eta \in \partial_\infty M \setminus \{\omega\} \) we have the Gromov product \( (\xi|\eta)_b \) associated with a Busemann function \( b \). Any (bounded) metric on \( \partial_\infty M \) that is bi-Lipschitz equivalent to the function \( (\xi, \eta) \mapsto a^{-(\xi|\eta)_o} \), and any (unbounded) metric on \( \partial_\infty M \setminus \{\omega\} \) that is bi-Lipschitz equivalent to the function \( (\xi, \eta) \mapsto a^{-(\xi|\eta)_b} \) for some \( a > 1 \), are called visibility metrics.

These two classes of metrics are certainly well known, but surprisingly, we did not find in the literature an answer to the natural question as to how they are related (except for...
the case where $M = \mathbb{H}^n$ is real hyperbolic). Our aim in this paper is to fill this gap. We obtain the following.

**Theorem 1.1.** There are constants $c_1, c_2 > 0$ such that for every symmetric rank one space $M$ of noncompact type, the horospherical metric $d_\omega$ on $\partial_\infty M \setminus \{\omega\}$ associated with an arbitrary Busemann function $b : M \to \mathbb{R}$ centered at any point $\omega \in \partial_\infty M$ satisfies

$$c_1 e^{-(\xi,\eta)b} \leq d_\omega(\xi,\eta) \leq c_2 e^{-(\xi,\eta)b}$$

for each $\xi, \eta \in \partial_\infty M \setminus \{\omega\}$.

**Theorem 1.2.** There are constants $c_1, c_2 > 0$ such that for every symmetric rank one space $M$ of noncompact type, the spherical metric $d_\infty$ on $\partial_\infty M$ based at any $\omega \in M$ satisfies

$$c_1 e^{-(\xi,\eta)c} \leq d_\infty(\xi,\eta) \leq c_2 e^{-(\xi,\eta)c}$$

for each $\xi, \eta \in \partial_\infty M$.

**Remark 1.3.** In the case where $M = \mathbb{H}^n$, $n \geq 2$, is real hyperbolic, the metric $2d_\infty$ is isometric to the standard metric of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ for every base point $o \in \mathbb{H}^n$. On the other hand, an easy argument shows (see [BS, sect. 2.4.3]) that the function $(\xi, \eta) \mapsto e^{-(\xi,\eta)c}$ coincides with half of the chordal metric,

$$e^{-(\xi,\eta)c} = \frac{1}{2} |\xi - \eta|$$

for every $\xi, \eta \in S^{n-1}$.

Similarly, for every Busemann function $b : \mathbb{H}^n \to \mathbb{R}$ centered at $\omega \in \partial_\infty \mathbb{H}^n$, the horospherical metric $d_\omega$ is isometric to the standard $\mathbb{R}^{n-1}$, and an easy calculation in the upper half-space model shows that $d_\omega(\xi,\eta) = e^{-(\xi,\eta)b}$ for every $\xi, \eta \in \partial_\infty \mathbb{H}^n \setminus \{\omega\}$.

**Remark 1.4.** For the case where $M = \mathbb{C} \mathbb{H}^2$ is the complex hyperbolic plane, a weaker version of Theorem 1.2 was obtained in [Ku].

**Remark 1.5.** In [Bo] it was proved that the function $(u, u') \mapsto |uu'| := e^{-(u,u')o}$ is a metric on the boundary at infinity $\partial_\infty X$ of any CAT(-1)-space $X$ for every $o \in X$. Moreover, in [FS] it was proved that $\partial_\infty X$ endowed with this metric is a *Ptolemy metric space*; i.e., the Ptolemy inequality $|xy| \cdot |uv| \leq |xu| \cdot |vy| + |xv| \cdot |uy|$ holds true for all $x, y, u, v \in \partial_\infty X$.

§2. Preliminaries

2.1. Symmetric rank one spaces. Every rank one symmetric space $M$ of noncompact type is a hyperbolic space $\mathbb{K} \mathbb{H}^n$ over the real numbers $\mathbb{K} = \mathbb{R}$, or the complex numbers $\mathbb{K} = \mathbb{C}$, or the quaternions $\mathbb{K} = \mathbb{H}$, or the octonions $\mathbb{K} = \mathbb{C} a$, and dim $M = n \cdot \text{dim} \mathbb{K}$, where $n \geq 2$ and $n = 2$ in the case where $\mathbb{K} = \mathbb{C} a$ (see, e.g., [Wo]). We use the standard notation $TM$ for the tangent bundle of $M$ and $UM$ for the subbundle of unit vectors.

2.1.1. Curvature operator. One of the equivalent characterizations of the Riemannian symmetric spaces is that their curvature tensor is parallel, $\nabla R = 0$; see, e.g., [Wo]. It follows that for every unit vector $u \in U_o M$, where $o \in M$, the eigenspaces $E_u(\lambda)$ of the curvature operator $R(\cdot, u)u : u^\perp \to u^\perp$, where $u^\perp \subset T_o M$ is the subspace orthogonal to $u$, are parallel along the geodesic $\gamma(t) = \exp_o(tu)$, $t \in \mathbb{R}$, and the respective eigenvalues $\lambda$ are constant along $\gamma$. Note that $u^\perp$ is identified with the tangent space $T_o(U_o M)$ of the unit sphere $U_o M$ at $u$.

If $M = \mathbb{K} \mathbb{H}^n$, $n \geq 2$ ($n = 2$ in the case of octonions), then the eigenvalues of the curvature operator $R(\cdot, u)u$ are $\lambda = -1, -4$. The dimensions of the respective eigenspaces are $\dim E_u(-1) = (n - 1) \dim \mathbb{K}$, $\dim E_u(-4) = \dim \mathbb{K} - 1$, $u^\perp = E_u(-1) \oplus E_u(-4)$. In
view of Remark 13, we assume that \( \mathbb{K} \neq \mathbb{R} \). Then \( E_u(-4) \neq \{0\} \) and the sectional curvatures of \( M \) are pinched, \(-4 \leq K_\alpha \leq -1 \). We denote by \( E(\lambda) \) the subbundle of the tangent bundle \( T(U_o M) \) with fibers \( E_u(\lambda), u \in U_o M, \lambda = -1, -4 \).

### 2.2. Boundary at infinity and Carnot–Carathéodory metrics.

#### 2.2.1. Busemann functions and horospheres.

The Busemann function \( b = b_\omega : M \to \mathbb{R} \) associated with a geodesic ray \( \gamma \in \omega \subset \partial_\infty M \) is defined by \( b(x) = \lim_{t \to -\infty} (|x\gamma(t)| - t) \).

In this case, we say that \( b \) is centered at \( \omega \). We denote by \( B(\omega) \) the set of all Busemann functions centered at \( \omega \). Two functions \( b, b' \in B(\omega) \) differ by a constant, and moreover, \( B(\omega) \) is parametrized by the values of \( b \in B(\omega) \) at a fixed point \( o \in M \), which can be arbitrary.

For \( b \in B(\omega) \), \( t \in \mathbb{R} \), the horosphere \( H_{b,t} = b^{-1} (t) \) is a smooth hypersurface in \( M \).

Again, we denote by \( E(\lambda) \) the subbundle of the tangent bundle \( TH_{b,t} \), the fiber of which at every point \( x \in H_{b,t} \) is the eigenspace \( E_u(\lambda) \) of the curvature operator \( R(\cdot, u)u \), where \( u = \text{grad} b(x), \lambda = -1, -4 \). We denote by \( r_{b,t} : H_{b,t} \to \partial_\infty M \setminus \{\omega\} \) the radial projection map that takes every \( x \in H_{b,t} \) to the class \( \gamma(\infty) \in \partial_\infty M \), where \( \gamma : \mathbb{R} \to M \) is a (unique) geodesic with \( \gamma(\infty) = \omega \) passing through \( x \). Then \( r_{b,t} \) is a homeomorphism.

We denote by \( \rho_o : M \setminus \{o\} \cup \partial_\infty M \to U_o M \) the radial projection; \( \rho_o(x) \) is the unit vector tangent to the geodesic ox at \( o \).

#### Lemma 2.1.

For any \( o \in M, \omega \in \partial_\infty M, \) and \( b \in B(\omega) \), the composition \( f = \rho_o \circ r_{b,0} : H = H_{b,0} \to U_o M \) is a Lipschitz embedding; its differential \( df \) is well defined on the \( E(-1) \)-subbundle of \( TH \), preserves the \( E(-1) \)-subbundles of the tangent bundles \( TH \) and \( T(U_o M) \), and is conformal on them.

**Proof.** The composition \( f_s : H \to U_o M \) of the radial projection \( H \to H_{b,s} \) from \( o \) with \( \rho_o \) is a smooth embedding for all sufficiently large \( s \in \mathbb{R} \), which approximates \( f \) as \( s \to \infty \). We show that the differential \( df_s f_s \) is uniformly bounded in \( s \) for every \( x \in H \) and almost preserves the \( E(-1) \)-subbundles as \( s \to \infty \).

For \( x \in H \) and \( v \in E_u(\lambda) \subset \mathcal{H}_x H, u = \text{grad} b(x) \), there is a unique Jacobi vector field \( V \) along the geodesic \( \gamma : \mathbb{R} \to M, \gamma(-\infty) = \omega, \gamma(0) = x, \gamma \) such that \( V(0) = v \) and \( V(s) \to 0 \) as \( s \to -\infty \). The direction field \( V'/|V'| \) is parallel along \( \gamma \), and \( |V(s)| = e^{s\sqrt{|\lambda|}}|v| \).

For \( s \in \mathbb{R} \), we put \( \alpha_s = \angle_{\gamma(s)}(o, \omega), \beta_s = \angle_{\omega}(o, \gamma(s)), \tau_s = |\omega(\gamma(s))| \). Then

\[
(1) \quad \alpha_s \leq 4e^{-\tau_s}/\beta_s,
\]

by comparison with \( H^2 \), and the (generalized) formula for the parallelism angle. Note that \( \beta_s - \angle_o(\omega, \gamma(\infty)) \neq 0 \) as \( s \to \infty \) for all \( x \in H \). Furthermore, \( \lim_{s \to -\infty}(\tau_s - s) = b_\gamma(o) \), where \( \gamma_x = x\gamma(\infty) \). Thus, \( \tau_s = s + b_\gamma(o) + o(1) \) as \( s \to \infty \).

For \( w = df_x(v) \in T(U_o M) \), let \( W \) be the Jacobi field along \( \gamma(s) \) with the initial data \( W(o) = 0, \dot{W}(o) = w \). Then, by the definition of \( w \), the vector \( W(\gamma(s)) \) is the orthogonal projection of \( V(s) \) to \( u'\mathbb{R} \), where \( u' \in U_{\gamma(s)} M \) is tangent to the segment \( \gamma(s) \). The angle between \( V(s) \) and \( W(\gamma(s)) \) is at most \( \alpha_s \). Thus, \( (1 - \alpha_s^2) |V(s)| \leq |W(\gamma(s))| \leq |V(s)| \).

In the orthogonal decomposition \( W = W_1 + W_2 \) with respect to the eigenspaces of the curvature operator, the sections \( W_i \) of \( E(-i^2) \), \( i = 1, 2 \), are Jacobi fields along \( \gamma(s) \). Also, we have \( |W_i(\gamma(s))| \leq |W(\gamma(s))| \leq |V(s)| \). The eigenspaces of the curvature
operator $\mathcal{R}(\cdot, u)u$ depend smoothly on the direction $u$, so that for $i = \sqrt{|\lambda|}$ the angle between $W(\gamma(s))$ and $W_i(\gamma(s))$ is $O(\alpha_s)$, whence $|W_i(\gamma(s))| \geq (1 - O(\alpha_s))|V(s)|$.

Since $\tilde{W} = W_i + \tilde{W}_2$, we have $|W_i(\gamma(s))| = \frac{\sinh(\tau_\gamma)}{\tau_\gamma} |w_i|$ for $i = 1, 2$, where $w = w_1 + w_2$ is the orthogonal decomposition with respect to the eigenspaces of the curvature operator. Thus, for $i = \sqrt{|\lambda|}$ we have

$$(1 - O(\alpha_s))|V(s)|/\sinh(i \tau_s) \leq |w_i| \leq |V(s)|/\sinh(i \tau_s),$$

which implies that $|w_i| \sim 2e^{-ib\tau_\gamma(o)}|v|$ as $s \to \infty$. Note that $b\tau_\gamma(o) \geq -\text{dist}(o, H)$ and $b\tau_\gamma(o) \to \infty$ as $x \to \infty$ in $H$.

If $\lambda = -1$, then $|w_2| \leq 2|V(s)|/\sinh 2\tau_\gamma \leq 8e^{-s(2b\tau_\gamma(o))}|v| \to 0$ as $s \to \infty$. This shows that $d_{s_f}$ almost preserves the subbundle $E(-1)$ and is almost conformal on it with the factor $2e^{-b\tau_\gamma(o)}$.

It remains to consider the case where $\lambda = -4$. We already know that $|w_2| \sim 4e^{-2b\tau_\gamma(o)}|v|$ as $s \to \infty$. On the other hand, $|W_1(\gamma(s))| = O(\alpha_s)|V(s)|$, whence $|w_1| = |W_1(\gamma(s))|/\sinh \tau_s \leq O(\alpha_s)e^{2s}|v|/\sinh \tau_s$. Using (1), we obtain $|w_1| \leq ce^{-2b\tau_\gamma(o)}|v|$ for some constant $c > 0$ depending only on $x \in H$, $c = c(x)$. Thus, $|d_{s_f}(v)| \leq |w_1| + |w_2| \leq (c+4)e^{-2b\tau_\gamma(o)}|v|$, and hence, the norm $\|d_{s_f}f_s\|$ is uniformly bounded in $s$ for every $x \in H$. We conclude that $f = \lim_{s \to \infty} f_s$ is Lipschitz.

\textbf{Remark 2.2.} Lemma 2.2 is a refinement of [Pa, Lemme 9.6]. We have added the estimate of $d_{s_f}$ on $E(-4)$ based on the estimate (1), which leads to the conclusion that the embedding $f : H \to U_0 M$ is Lipschitz.

\subsection*{2.2.2. Isometries of $M$ and invariant metrics on a horosphere.}

Given $\omega \in \partial_\infty M$, there is a subgroup $N_\omega$ in the isometry group $G = \text{Isom} M$ (a maximal unipotent group) that leaves invariant every horosphere $H$ in $M$ centered at $\omega$ and is simply transitive on $H$ and on $\partial_\infty M \setminus \{\omega\}$. The group $N = N_\omega$ is a nilpotent Lie group of dimension $\dim N = \dim M - 1$. If $M = \mathbb{C} \mathbb{H}^2$, then $N$ is isomorphic to the classical Heisenberg group.

Fixing a base point $o \in H$, we identify $N$ with the orbit $H = N(o)$, and identify the tangent space $T_o H$ with the Lie algebra $N$ of $N$. Then $N$ leaves invariant the subbundles $E(\lambda)$ of $TH$, and $N = E_1 \oplus E_2$, where $E_i = E_i(-i^2)$, $i = 1, 2$, are the fibers of $E(\lambda)$ at $o$. Moreover, $E_2 = [E_1, E_1]$ in $N$; in particular, $N$ is the minimal subalgebra in $N$ containing $E_1$; see, e.g., [Pa].

The subbundle $E(-4) \subset TH$ is integrable, and the corresponding fibers $F$ are intersections of $K$-lines with $H$, where each $K$-line in $M$ is a totally geodesic subspace isometric to the hyperbolic space $\mathbb{H}^{n-1}_2$ of constant curvature $-4$; see [MO] §20 and [Pa]. Therefore, every fiber $F$ is flat in $H$. The corresponding fibration $\phi : H \to B$ is a Riemannian submersion (with respect to the induced Riemannian metric on $H$) with the base $B$ isometric to $\mathbb{R}^{(n-1)}$, and the kernel of that action is the center $Z$ of $N$, so that $N/Z = \mathbb{R}^{(n-1)}$, and the submersion $\phi$ is equivariant with respect to these actions of $N$ on $H$ and $B$.

If $E_1$ is identified with $\mathbb{K}^{n-1}$ and $E_2$ with $\mathbb{K}^1$, then the Lie algebra structure of $N = E_1 \oplus E_2$ is given by $[v, w] = 2\mathbb{I}nm(v, w)$ for $v, w \in E_1$, where $v = (v_1, \ldots, v_n)$ and $(v, w) = \sum_i v_i w_i$; see [MO] (19.14).

We call the subbundle $E = E(-1) \subset TH$ the polarization on the horosphere $H$. A piecewise smooth curve $\sigma : I \to H$ is said to be $E$-horizontal if $\sigma(t) \in E$ for every $t \in I \subset \mathbb{R}$. Its length is $\ell(\sigma) = \int_I |\dot{\sigma}(t)|dt$. We define the distance $d_E(x, x')$ by taking the infimum of the lengths of $E$-horizontal curves between $x, x' \in H$. This quantity is finite because $E_1$ generates $N$, so that $E$ is completely nonintegrable. The restriction $d_{s_E}^2 = ds_E^2$ of the Riemannian metric $ds^2$ of $M$ will be called the Carnot–Carathéodory
metric, and the corresponding distance $d_E$ is the Carnot–Carathéodory distance on $H$. Furthermore, since $d_H$ is isometric on $E$, $\phi$ preserves the lengths of $E$-horizontal curves.

Let $d_H$ denote the interior distance on $H$ induced from $M$.

**Lemma 2.3.** We have

$$\sup_{x \in H} \frac{d_E^2(o, x)}{d_H(o, x)} \leq 17,$$

where the supremum is taken over all $x \in H$ with $|ox| \leq 1$ in $M$.

**Proof.** Let $F = \phi^{-1}(\phi(o))$ be the fiber of $\phi$ through $o$. First, we show that the ratio $r(x) = \frac{d_E^2(o, x)}{d_H(o, x)}$ is independent of $x \in F$.

For every $\lambda > 0$ there is a homothety $h_\lambda : N \to N$ that is an automorphism of $N$ and that acts on $F$ as the homothety with coefficient $\lambda^2$ and on $B$ as the homothety with coefficient $\lambda$; see e.g. [Pa]. In particular, $\ell(h_\lambda(\sigma)) = \lambda \ell(\sigma)$ for the length $\ell(\sigma)$ of every $E$-horizontal curve $\sigma$ in $H$. It follows that $r(h_\lambda(x)) = r(x)$ for every $\lambda > 0$. Next, the stabilizer $G_\infty \subset G$ of $o \cup \omega$ acts transitively on the rays $a \subset F$ (see [Mo, §2]). Thus, $r = r(x)$ is independent of $x \in F \setminus \{o\}$.

For any $v, w \in E_1 \subset N = T_o H$, the commutator $[\exp sv, \exp sw] = x_s$ lies in $F = \exp E_2$ for every $s \geq 0$, and $[v, w] = \lim_{s \to 0} \frac{1}{2} ox_s$. Representing the commutator as the end-point of a broken geodesic $\sigma_s$ with four edges, we find that $\sigma_s$ is an $E$-horizontal curve of length $2s(|v| + |w|)$ that connects $o$ and $x_s$. Thus, $d_E(o, x_s) \leq 2s(|v| + |w|)$, while $d_H(o, x_s) = s^2|v| + o(s^2)$. Hence, $r \leq 2(|v| + |w|)^2/|v, w|$. Taking $v, w \in E_1 = \mathbb{K}^{n-1}$ orthonormal, $v = (1, 0, \ldots, 0)$ and $w = (j, 0, \ldots, 0)$, where $j$ is one of $\dim \mathbb{K} - 1$ imaginary units, we find $|v, w| = 2j$. Therefore, $|v| = |w| = 1$, $|v, w| = 2$, and $r \leq 8$.

In the general case, for $x \in N$ with $|ox| \leq 1$ in $M$, there is an $E$-horizontal curve $\sigma$ that connects $o$ and $x'$ so that $\phi \circ \sigma$ is a segment in $B$ and $x' \subset F$, where $F$ is the fiber of $\phi$ through $x$. We have $|x'| \leq |ox| \leq 1$, $d_H(o, x) \geq \ell(\sigma)$, and $d_H(x', x)$, and $\sigma$ lies in a totally real plane in $M$, which is a geodesic subspace isometric to $H^2$. Moreover, $\sigma$ is a horocycle in $H^2$, which yields $\ell(\sigma) \leq 2 \sinh \frac{1}{2}$, and $d_H(x', x) \leq \sinh 1$; see Theorem 2.4 below. Then $d_E(o, x) \leq \ell(\sigma) + d_E(x', x)$, whence

$$\frac{d_E^2(o, x)}{d_H(o, x)} \leq \frac{d_E^2(x', x)}{d_H(x', x)} + \frac{\ell(\sigma)}{2} + 2d_E(x', x) \leq r + 2 \sinh(1/2) + 2\sqrt{r} \sinh 1 \leq 17. \quad \square$$

### 2.3. Negatively pinched Hadamard manifolds.

Let $M$ be a Hadamard manifold with sectional curvatures $-b^2 \leq K_x \leq -a^2$, $a > 0$. Assume that a horosphere $H \subset M$ is fixed. We denote by $|xx'|$ the distance between $x, x' \in H$ in $M$ and by $|xx'|_H$ the induced interior distance in $H$. We shall use the following result from [HI Theorem 4.6].

**Theorem 2.4.** For every $x, x' \in H$ we have

$$\frac{2}{a} \sinh \frac{a}{2} |xx'| \leq |xx'|_H \leq \frac{2}{b} \sinh \frac{b}{2} |xx'|.$$

### 2.4. Gromov hyperbolic spaces.

Most of the facts on hyperbolic spaces needed for what follows, except for information about equiradial points of infinite triangles (see Subsection 2.4.2), can be found, e.g., in [BS]. We discuss these facts for symmetric rank one spaces $M$, though all of them remain true for the general proper $\text{CAT}(−1)$-spaces $X$. In this case, the geodesic boundary at infinity $\partial_{\infty}X$ coincides with the Gromov boundary at infinity. Every $\text{CAT}(−1)$-space is boundary continuous; see [BS, 3.4.2]. This simplifies the definitions of the key notions essentially, compared with the general case of Gromov hyperbolic spaces.
2.4.1. Gromov products and $\delta$-triples. The Gromov product of $x, x' \in M$ with respect to $o \in M$ is $(x|x')_o = \frac{1}{2}(|xo| + |x'o| - |xx'|)$. This quantity is always nonnegative by the triangle inequality.

Being a CAT($-1$)-space, the space $M$ is $\delta$-hyperbolic (in the sense of Gromov); that is, for every triangle $xyz \subset M$, if $y' \equiv xy, z' \equiv xz$, and $|xy'| = |xz'| \leq (y|z)_x$, then $|y'z'| \leq \delta$, with the constant $\delta \leq \delta_{H2}$. It is known that the hyperbolic constant for the real hyperbolic plane $H^2$ equals $\delta_{H2} = 2 \ln \tau = 0.9624 \ldots$, where $\tau$ is the golden ratio, $\tau^2 = \tau + 1$; see [BS].

The condition above implies that for every $a, x, y, z \in M$ the $\delta$-inequality is valid, $(x|y)_a \geq \min\{(x|z)_a, (y|z)_a\} - \delta$ (the converse is also true but with different $\delta$). This inequality can be rewritten in terms of $\delta$-triples. A triple of real numbers $(a, b, c)$ is called a $\delta$-triple if the two smallest of these numbers differ by at most $\delta$. Then the $\delta$-inequality is equivalent to the fact that the numbers $(x|y)_a, (x|z)_a,$ and $(y|z)_a$ form a $\delta$-triple.

The Gromov product of $x, y \in M$ with respect to a Busemann function $b : M \to \mathbb{R}$ is defined by

$$(x|y)_b = \frac{1}{2}(b(x) + b(y) - |xy|).$$

This product, contrary to the standard case $(x|y)_o$, may take arbitrary real values. Busemann functions $b$ and $b'$ centered at one and the same point $\omega \in \partial_\infty M$ differ by a constant, $b - b' = \text{const}$. Therefore, the Gromov products with respect to $b, b'$ differ by the same constant. As above, for every $x, y, z \in M$ the numbers $(x|y)_b, (x|z)_b,$ and $(y|z)_b$ form a $\delta$-triple.

The Gromov product of points $\xi, \eta \in \partial_\infty M$ in the boundary at infinity with respect to $o \in M$ is the limit

$$(\xi|\eta)_o = \lim_{s,t \to \infty} (\gamma(s)|\rho(t))_o,$$

where $\gamma = \omega \xi$ and $\rho = \omega \eta$ are geodesic rays (the limit exists by the monotonicity of the Gromov product). Note that $(\xi|\xi)_o = \infty$ for every $\xi \in \partial_\infty M$. Again, for any pairwise distinct points $\xi, \eta, \xi, \zeta \in \partial_\infty M$, the numbers $(\xi|\eta)_o, (\xi|\zeta)_o,$ and $(\eta|\zeta)_o$ form a $\delta$-triple.

Finally, the Gromov product of $\xi, \eta \in \partial_\infty M \setminus \{\omega\}$ with respect to a Busemann function $b : M \to \mathbb{R}$ centered at $\omega \in \partial_\infty M$ is the limit

$$(\xi|\eta)_b = \lim_{s,t \to \infty} (\gamma(s)|\rho(t))_b,$$

where $\gamma = \omega \xi, \rho = \omega \eta$ are geodesics in $M$ with $\gamma(\infty) = \xi, \rho(\infty) = \eta$ (the limit exists by the monotonicity of the Gromov product). For different Busemann functions $b, b' : M \to \mathbb{R}$ with one and the same center $\omega$, the Gromov products $(\xi|\eta)_b$ and $(\xi|\eta)_b'$ differ by a constant, $(\xi|\eta)_b - (\xi|\eta)_b' = \text{const}$ for every $\xi, \eta \in \partial_\infty M \setminus \{\omega\}$. For any pairwise distinct points $\xi, \eta, \zeta \in \partial_\infty M \setminus \{\omega\}$, the numbers $(\xi|\eta)_b, (\xi|\zeta)_b,$ and $(\eta|\zeta)_b$ form a $\delta$-triple.

2.4.2. Equiradial points of a triangle. For every triangle $xyz \subset M$, there are three spheres centered at its vertices that are pairwise tangent to each other from outside. The tangent points lie on the sides of the triangle and are called the equiradial points. The collection of the equiradial points $u \in yz, v \in xz, w \in xy$ is uniquely determined by the conditions $\|uz\| = |vz| = (x|y)_x, \|wv\| = |wy| = (x|z)_y,$ and $\|wx\| = |wx| = (y|z)_x$.

If instead of spheres we take horospheres centered at vertices at infinity, then we obtain the definition of equiradial points of an infinite triangle in $M$. In this case, the existence of equiradial points and their relationship with the corresponding Gromov products is not completely obvious.

**Proposition 2.5.** For any pairwise distinct points $\xi, \eta, \zeta \in \partial_\infty M$, the infinite triangle $\xi \eta \zeta$ in $M$ possesses a unique collection of equiradial points $u \in \eta \zeta, v \in \xi \zeta, w \in \xi \eta$. 
Moreover, we have \((\eta|\zeta)_b = b(v) = b(w)\) for every \(b \in B(\xi)\), \((\xi|\zeta)'_b = b'(u) = b'(w)\) for every \(b' \in B(\eta)\), and \((\eta|\eta)'_w = b''(u) = b''(v)\) for every \(b'' \in B(\zeta)\).

**Proof.** There are uniquely determined points \(v \in \xi \zeta\) and \(w \in \xi \eta\) such that \((\eta|\zeta)_b = b(v) = b(w)\) for every \(b \in B(\xi)\). We take a function \(b\) so that \((\eta|\zeta)_b = 0\). Next, we fix a function \(b' \in B(\zeta)\) such that \(b'(v) = 0\) and let \(u = \eta \zeta \cap b'^{-1}(0)\). After that we fix a function \(b'' \in B(\eta)\) such that \(b''(u) = 0\) and let \(w' = \eta \zeta \cap b''^{-1}(0)\). We show that \(b(w') = 0\), i.e., \(w' = w\).

Denote by \(\eta(t)\zeta(t) \subset \eta \zeta\) the segment of length \(2\delta\) centered at \(u\). Let \(\eta : [0, \infty) \to \xi \eta\) be the natural parametrization of the ray \(w' \eta\), \(\eta(0) = w'\), \(\eta(\infty) = \eta\), and let \(\zeta : [0, \infty) \to \xi \zeta\) be the natural parametrization of the ray \(v\zeta\), \(\zeta(0) = v\), \(\zeta(\infty) = \zeta\). Since \(b'(u) = b'(v) = 0\), \(b''(w') = b''(u) = 0\), we have \(b(\xi \zeta(t)) = t, b(\eta \xi(t)) = b(u) + t, \) and \(|\eta(\zeta(t)) - t| = o(1)\) as \(t \to \infty\). Therefore,

\[
0 = (\eta|\zeta)_b = \lim_{t \to \infty} (\eta\zeta(t))_b = \frac{1}{2} \lim_{t \to \infty} (b(\eta(t)) + b(\zeta(t))) = \frac{1}{2} \lim_{t \to \infty} (b(\eta(t)) - t + b(\zeta(t))) = b(w')/2.
\]

This means that the horospheres \(b^{-1}(0), b^{-1}(0), b''^{-1}(0)\) centered at \(\xi, \eta, \zeta\), respectively, touch each other from outside; i.e., the points \(u, v, w\) are equiradial for the infinite triangle \(\eta \xi \zeta\).

Since \((\eta|\zeta)_b = b(v) = b(w)\) and the equiradial points are determined uniquely, we also have \((\xi|\eta)_w = b'(u) = b'(v), (\xi|\zeta)_w = b''(u) = b''(w)\).

**Remark 2.6.** The equiradial points of infinite triangles in \(M\) with one or two vertices in \(M\) also exist and are determined uniquely, and the corresponding identities for the Gromov products are also valid for them. This can be proved by similar arguments.

**Lemma 2.7.** Let \(u \in \eta \zeta, v \in \xi \zeta, w \in \xi \eta\) be the equiradial points of an infinite triangle in \(M\) with pairwise distinct vertices \(\xi, \eta, \zeta \in \partial_M\). Then \(|uv|, |uw|, |vw| \leq \delta\), where \(\delta > 0\) is the hyperbolicity constant for \(M\).

**Proof.** The equiradial points of every finite triangle in \(M\) are at a distance of at most \(\delta\) from each other by the definition of \(\delta\)-hyperbolicity. Thus, it suffices to find a family of finite triangles in \(M\), the collections of equiradial points of which approximate the points \(u, v, w\).

For \(t > 0\), consider \(x \in v \xi, y \in w \eta, z \in u \zeta\) lying at a distance \(t\) from the vertices of the rays, \(|xv| = |yw| = |zu| = t\). As \(t \to \infty\), we have \(wx \to w \xi, uy \to w \eta, vz \to v \zeta\), and each of the distances \(|wx|, |yw|, |zu|\) differs from \(t\) by \(o(1)\). Thus, each of the distances \(|zx|, |zy|, |xz|\) differs from \(2t\) by \(o(1)\). Hence, the Gromov products \((y|z)x, (x|z)y, (x|y)z\) all are equal to \(t\) up to \(o(1)\). This immediately implies that the equiradial points of \(xyz\) approximate the points \(u, v, w\) as \(t \to \infty\).

**Lemma 2.8.** Let \(u \in \eta \zeta, v \in \xi \zeta, w \in \xi \eta\) be the equiradial points of an infinite triangle \(\eta \xi \zeta \subset M\) with pairwise distinct vertices \(\xi, \eta, \zeta \in \partial_M\). Then for points \(w' \in \eta \eta\) and \(v' \in v \zeta\) with \(|w'w| = 1 + \delta = |v'v|\), where \(\delta\) is the hyperbolicity constant of \(M\), we have \(|v'w'| \geq 2\).

**Proof.** We take \(w'' \in \eta \eta\) and \(v'' \in u \zeta\) so that \(|w''u| = 1 + \delta = |v''v|\). Then Lemma 2.7 shows that \(|v'v''| \leq \delta\) and \(|w'w''| \leq \delta\). Thus, \(|v'w'| \geq |v''w''| - 2\delta = 2\). 


§3. Proof of Theorem 1.1

Let $\delta$ denote the hyperbolicity constant of $M$; we have $\frac{1}{2}\delta_{H^2} \leq \delta \leq \delta_{H^2} < 1$ by comparison with $H^2$ and $\frac{1}{2}H^2$. We fix $o \in M$, $\omega \in \partial_{\infty}M$, and a Busemann function $b \in B(\omega)$ with $b(o) = 0$. For $t \in \mathbb{R}$ we denote by $H_{b,t} = b^{-1}(t)$ the horosphere in $M$; let $H = H_{b,0}$, and let $ds_{b,t}^2$ be the Riemannian metric on $H_{b,t}$ induced from $M$. We identify $H_{b,t}, \partial_{\infty}M \setminus \{\omega\}$ with $H$ via the radial projection from $\omega$ and view $ds_{b,t}^2$ as the Riemannian metric on $H$ for all $t \in \mathbb{R}$. Then the limit

$$ds_b = \lim_{t \to \infty} e^{-t}ds_{b,t}$$

exists and coincides with the Carnot–Carathéodory distance $d_E$.

Indeed, for every $x \in H$ and $\nu \in E_x(\lambda) \subset T_xH$, where $u = \text{grad} b(x)$, a unique Jacobi field $V$ along the geodesic $\exp_x tu$, $t \in \mathbb{R}$, with $V(t) \to 0$ as $t \to \infty$ has the parallel direction field $V/[V]$, and $|V(t)| = e^{\sqrt{|\lambda|}t}|v|$. Thus, $e^{-t}ds_{b,t}(v) = e^{(\sqrt{|\lambda|}-1)t}|v|$, and for $v \in E = E(-1)$ we have $e^{-t}ds_{b,t}(v) = |v| = ds_b(v)$ for all $t \in \mathbb{R}$, while $ds_b(v) = \infty$ for all nonzero $v \in E(-4)$.

We view the corresponding Carnot–Carathéodory distance $d_b$ as a metric on $\partial_{\infty}M \setminus \{\omega\}$; this metric is said to be horospherical.

Lemma 3.1. For each $\xi, \eta \in \partial_{\infty}M \setminus \{\omega\}$, we have

$$d_b(\xi, \eta) \geq c_1 e^{-|\xi|}$$

where $c_1 = 2e^{-(1+\delta)}$.

Proof. We assume that $\xi \neq \eta$, because otherwise there is nothing to prove. For $\epsilon > 0$, we take an $E$-horizontal curve $\sigma_\infty \subset \partial_{\infty}M \setminus \{\omega\}$ between $\xi$ and $\eta$ with length $\ell_b(\sigma_\infty) \leq d_b(\xi, \eta) + \epsilon$. Let $\sigma_t \subset H_{b,t}$ be its radial projection from $\omega$. Since the curve $\sigma_t$ connects the points $u_t \in \omega \xi$ and $v_t \in \omega \eta$ with $b(u_t) = b(v_t) = t$ and also is $E$-horizontal on $H_{b,t}$, its length $\ell_{b,t}(\sigma_t)$ is equal to $e^{\sqrt{|\lambda|}t}|v|$. Let $u \in \omega \eta$ and $v \in \omega \xi$ be the equiradial points of the triangle $\omega \xi \eta$. Then $t_0 := (\xi|\eta) = b(u) = b(v)$. We put $t_0 = t_0 + 1 + \epsilon$. For $u_t \in \omega \eta, v_t \in \omega \xi$ we have $|u_t v_t| \geq 2$ by Lemma 2.3. Then $\ell_{b,t}(\sigma_t) \geq |u_t v_t| \geq 2$ and

$$d_b(\xi, \eta) + \epsilon \geq \ell_b(\sigma_\infty) = e^{-t_0}\ell_{b,t}(\sigma_t) \geq c_1 e^{-|\xi|}.$$

Since $\epsilon$ has been chosen arbitrarily, we obtain the required estimate. \qed

Proof of Theorem 1.1 In view of Lemma 3.1 it remains to estimate the horospherical distance $d_b(\xi, \eta)$ from above. We assume that the points $\xi, \eta \in \partial_{\infty}M \setminus \{\omega\}$ are distinct. Let $u \in \omega \eta, v \in \omega \xi$ be the equiradial points of the triangle $\omega \xi \eta$, $t = (\xi|\eta) = b(u) = b(v)$. Then $|uv| \leq \delta < 1$ by Lemma 2.7. Thus, by Lemma 2.3 we have $d_E^2(u, v) \leq 17d_{b,t}(u, v)$, where $d_{b,t}$ is the interior distance on $H_{b,t}$ induced from $M$. By Theorem 2.1 we have $d_{b,t}(u, v) \leq \sinh |uv| \leq \sinh \delta$. Therefore, $d_{b,t}(u, v) \leq 17 \sinh \delta = c_2$. To avoid a standard $\varepsilon$-argument, we assume for simplicity that there is an $E$-horizontal curve $\sigma_\infty \subset H_{b,t}$ between $u, v$ with $\ell_{b,t}(\sigma_t) = d_E(u, v)$. The curve $\sigma_t$ is the radial projection from $\omega$ of an $E$-horizontal curve $\sigma_\infty \subset \partial_{\infty}M \setminus \{\omega\}$ between $\xi, \eta$, and we have

$$d_b(\xi, \eta) \leq \ell_b(\sigma_\infty) = e^{-t_0}\ell_{b,t}(\sigma_t) = e^{-t}d_E(u, v) \leq c_3 e^{-|\xi|}.$$

§4. Proof of Theorem 1.2

We fix $o \in M$ and, for every $t > 0$, consider the Riemannian metric $ds_t^2$ induced from $M$ on the sphere $S_t \subset M$ of radius $t$ centered at $o$. Identifying $S_t, \partial_{\infty}M$ with the unit
sphere \( U_o M \subset T_o M \) via the radial projection from \( o \), we regard \( ds_t^2 \) as the Riemannian metric on \( U_o M \) for all \( t > 0 \). Then the limit

\[
ds_\infty = \lim_{t \to \infty} e^{-t} ds_t
\]

exists, because, given \( v \in E_o(\lambda) \) with \( u \in U_o M \), there is a unique Jacobi field \( V \) along the geodesic ray \( \gamma(t) = \exp_o tu \), \( t \geq 0 \), with initial data \( V(0) = 0 \), \( V(0) = v \). The direction field \( V/|V| \) is parallel along \( \gamma \) and \( |V(t)| = \sinh(t \sqrt{\lambda}) |v|/\sqrt{\lambda} \) for all \( t \geq 0 \). Thus, \( e^{-t} ds_t(v) = e^{-t}|V(t)| \to |v|/2 \) for \( \lambda = -1 \) and \( \to \infty \) for \( \lambda = -4 \).

We call the subbundle \( E = E(-1) \subset TU_o M \) the polarization on the sphere \( U_o M \). A piecewise smooth curve \( \sigma : I \to U_o M \) is said to be \( E \)-horizontal if \( \sigma'(t) \in E \) for every \( t \in I \subset \mathbb{R} \). Its length is \( \ell(\sigma) = \int_I |\sigma'(t)| dt \). We define the distance \( d_E(u, u') \) by taking the infimum of the lengths of \( E \)-horizontal curves between \( u, u' \in U_o M \). This quantity is finite, because for every horosphere \( H \subset M \) the canonical embedding \( f : H \to U_o M \) is Lipschitz and its differential \( df \) preserves the polarizations on \( H, U_o M \) by Lemma 2.4.

We choose \( \omega \in \partial_\infty M \) so that neither \( u \) nor \( u' \) is tangent to \( \omega \), and take \( b \in B(\omega) \) with \( b(o) = 0 \) and \( H = b^{-1}(0) \). Then \( u, u' \in f(H) \), and there is an \( E \)-horizontal curve \( \sigma \subset H \) between \( x = f^{-1}(u) \) and \( x' = f^{-1}(u') \). Then the curve \( f(\sigma) \subset U_o M \) is \( E \)-horizontal, connects \( u \) and \( u' \), and \( \ell_E(f(\sigma)) \leq 2\ell_E(\sigma) \) by the proof of Lemma 2.4.

We denote by \( d_\infty \) the Carnot–Carathéodory distance on \( \partial_\infty M \) associated with the Carnot–Carathéodory metric \( ds_\infty \). Observe that \( ds_\infty = \frac{1}{2} ds_E \) and \( d_\infty = \frac{1}{2} d_E \). Thus, the argument above shows that for every \( o \in M \), \( \omega \in \partial_\infty M \), and \( b \in B(\omega) \), we have

\[
ds_\infty \leq ds_b \quad \text{and} \quad d_\infty \leq d_b
\]
on \( \partial_\infty M \setminus \{\omega\} \). The following lemma is a minor modification of Lemma 3.1.

**Lemma 4.1.** For each \( \xi, \eta \in \partial_\infty M \), we have

\[
ds_\infty(\xi, \eta) \geq c_1 e^{-\ell(\xi, \eta)}
\]

where \( c_1 = 2e^{-(1+\delta)} \).

**Proof.** We assume that \( \xi \neq \eta \), because otherwise there is nothing to prove. For \( \varepsilon > 0 \) we take an \( E \)-horizontal curve \( \sigma_\infty \subset \partial_\infty M \) between \( \xi \) and \( \eta \) with length \( \ell_\infty(\sigma_\infty) \leq ds_\infty(\xi, \eta) + \varepsilon \). Let \( \sigma_t \subset S_t \) be its radial projection from \( o \). Since the curve \( \sigma_t \) connects the points \( u_t \in o \eta, v_t \in o \xi \) with \( |u_t| = |v_t| = t \) and also is \( E \)-horizontal on \( S_t \), its length \( \ell_t(\sigma_t) \) is \( 2 \sinh(t) \ell_\infty(\sigma_\infty) \).

Let \( u \in o \eta \) and \( v \in o \xi \) be the equiradial points of the triangle \( o \xi \eta \). Then \( t_0 := \ell(\xi, \eta) \) (applied to \( o \xi \eta \). Thus, \( \ell_t(\sigma_t) \geq |u_t v_t| \geq 2 \) and

\[
ds_\infty(\xi, \eta) + \varepsilon \geq \ell_\infty(\sigma_\infty) = \frac{1}{2 \sinh(t_0 + 1 + \delta)} \ell_t(\sigma_t) \geq c_1 e^{-\ell(\xi, \eta)}.
\]

Since \( \varepsilon \) has been chosen arbitrarily, we obtain the required estimate. \( \square \)

**Lemma 4.2.** Assume that points \( \xi, \eta, \omega \in \partial_\infty M \) are pairwise distinct and that points \( v, v' \in \xi \omega \) and \( w, w' \in \xi \eta \) satisfy \( b(v) = b(w), b(v') = b(w') \) for some and, hence, any Busemann function \( b \in B(\xi) \). Then

\[
|vv'| = |ww'| \leq \ln \frac{\sinh A}{a},
\]

where \( A = \max\{|vv'|, |v'w'|\}, a = \min\{|vv'|, |v'w'|\} \).
Proof. We put \( t = b(v) = b(w), \) \( t' = b(v') = b(w') \) and without loss of generality assume that \( t' > t. \) Let \( \sigma_v \in H_{b,v} \) be a shortest curve between \( v' \) and \( w', \) i.e., \( \ell_v(\sigma_v) = \delta_{b,v}(v', w'), \) and let \( \sigma_t \in H_{b,t} \) be the radial projection of \( \sigma_v \) from \( t. \) Then \( e^{t-t'} \delta_{b,t}(v, w) \leq e^{t-t'} \ell_v(\sigma_v) \leq \ell_t(\sigma_t) = \delta_{b,t}(v', w'). \) We have \( \delta_{b,t}(v, w) \geq a, \) and \( \delta_{b,t}(v', w') \leq \sinh A \) by Theorem 2.4. Thus, \( |wv'| = |wv|' = t' - t \leq \ln \frac{\sinh A}{a}. \) □

**Lemma 4.3.** Assume that \( (\xi|\eta)_o \geq 1 + \delta \) for some points \( o \in M \) and \( \xi, \eta \in \partial_\infty M. \) Then for \( \omega \in \partial_\infty M \) opposite to \( \xi \) with respect to \( o, \) i.e., \( o \in \omega \xi, \) and the function \( b \in B(\omega) \) with \( b(o) = 0, \) we have

\[
|(\xi|\eta)_b - (\xi|\eta)_o| \leq c_3 \ln \frac{\sinh(2+3\delta)}{2}.
\]

Proof. We can assume that \( \xi \neq \eta, \) because otherwise \( (\xi|\eta)_b = (\xi|\eta)_o = \infty. \) Furthermore, we have \( (\xi|\omega)_o = o, \) so that \( \eta \neq \omega, \) and \( \omega, \xi, \eta \) are pairwise distinct.

Let \( v \in o\xi, \) \( u \in o\eta, \) \( w \in \xi \eta \) be the equiradial points of the triangle \( o\xi\eta. \) Then \( |ov| = (\xi|\omega)_o \geq 1 + \delta, \) and thus, there is \( v' \in ov \) such that \( |vv'| = 1 + \delta. \) Also, there is \( w' \in w\eta \) such that \( |ww'| = 1 + \delta. \) Then \( |v'w'| \leq 2 + 3\delta. \) On the other hand, arguing as in Lemma 2.8, we obtain \( |v'w'| \geq 2. \)

Applying a similar argument to the equiradial points \( \tilde{v} \in \xi \omega, \) \( \tilde{w} \in \xi \eta, \) \( \tilde{u} \in \omega \eta \) of the triangle \( \xi\omega\eta, \) we see that

\[
2 \leq |v''w''| \leq 2 + 3\delta
\]

for \( v'' \in \xi \omega, \) \( w'' \in \xi \eta \) with \( b'(v'') = b''(w'') = (\eta|\omega)_o + 1 + \delta \) for any Busemann function \( b' \in B(\xi). \)

Then \( |v''w''| \leq \ln \frac{\sinh(2+3\delta)}{2} \) by Lemma 2.12. It remains to observe that \( |(\xi|\eta)_b - (\xi|\eta)_o| = |\tilde{v}\tilde{w}| = |v'w'|, \) because \( b(o) = 0 \) and \( (\xi|\eta)_b = b(\tilde{v}) = |\tilde{v}||. \) □

Recall that \( d_\infty \) is the spherical metric on \( \partial_\infty M \) centered at \( o \in M. \)

**Lemma 4.4.** If \( \xi, \eta \in \partial_\infty M \) and \( d_\infty(\xi, \eta) \leq 2e^{-(2+\delta)}, \) then \( (\xi|\eta)_o \geq 1 + \delta. \)

Proof. Let \( \xi_t \in o\xi \) and \( \eta_t \in o\eta \) be points with \( |o\xi_t| = |o\eta_t| = t \) for every \( t \geq 0. \) Recall that \( ds_t(v) = 2 \sinh t \cdot d_{S^1}(v) \) for every \( v \in E \subset TU_oM. \) Therefore, \( |\xi_t| \leq d_t(\xi_t, \eta_t) \leq e^td_\infty(\xi, \eta). \) Using the monotonicity of the Gromov product, we obtain

\[
(\xi|\eta)_o \geq (\xi_t|\eta_t)_o = t - \frac{1}{2} |\xi_t \eta_t| \geq t - e^{t-(2+\delta)} = 1 + \delta
\]

for \( t = 2 + \delta. \) □

**Proposition 4.5.** For every \( \xi, \eta \in \partial_\infty M \) with \( d_\infty(\xi, \eta) \leq 2e^{-(2+\delta)}, \) we have

\[
d_\infty(\xi, \eta, o) \leq c'_2 e^{-|\xi|o},
\]

where \( c'_2 = c_2 e^{c_3} \) and \( c_2 \) is the constant from Theorem 1.1.

Proof. By Lemma 1.1 we have \( (\xi|\eta)_o \geq 1 + \delta. \) Take \( \omega \in \partial_\infty M \) opposite to \( \xi \) with respect to \( o, \) and let \( b \in B(\omega), \) \( b(o) = 0. \) Then \( (\xi|\eta)_b \geq (\xi|\eta)_o - c_3 \) by Lemma 1.3 and using Theorem 1.1, we obtain

\[
d_\infty(\xi, \eta, o) \leq d_b(\xi, \eta) \leq c_2 e^{-|\xi|b} \leq c'_2 e^{-|\xi|o}. \]

□

**Proof of Theorem 1.2** Let \( D = \text{diam} \partial_\infty M \) be the diameter of \( \partial_\infty M \) with respect to (any) spherical metric \( d_\infty \) (for \( M = \mathbb{K} \mathbb{H}^n, \) this is clearly independent of \( \mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{Ca} \)
and \( n \geq 2; \) cf. Lemma 2.3). We show that

\[
d_\infty(\xi, \xi') \leq c''_2 e^{-|\xi|o}
\]

for each \( \xi, \xi' \in \partial_\infty M, \) where \( c'_2 = \max\{D e^{1+\delta}, c'_2\}. \)
If \((\xi|\xi')_0 \geq 1 + \delta\), then we argue as in Proposition 4.5, obtaining \(d_\infty(\xi, \xi') \leq c_2 e^{-((\xi|\xi')_0)}\). This allows us to assume that \((\xi|\xi')_0 < 1 + \delta\). Then
\[
d_\infty(\xi, \xi') \leq D \leq De^{1+\delta} e^{-(\xi|\xi')_0}.
\]
The estimate \(d_\infty(\xi, \xi') \geq c_1 e^{-(\xi|\xi')_0}\) was obtained in Lemma 4.1.

References

[Bo] M. Bourdon, Structure conforme au bord et flot géodésique d’un CAT\((-1)\)-espace, Enseign. Math. (2) 41 (1995), 63–102. MR1341941 (96f:58120)

[BS] S. Buyalo and V. Schroeder, Elements of asymptotic geometry, EMS Monogr. in Math., European Math. Soc. (EMS), Zürich, 2007. MR2327160 (2009a:53068)

[FS] T. Foertsch and V. Schroeder, Hyperbolicity, CAT\((-1)\)-spaces and the Ptolemy inequality, arXiv:math/060418v2.

[HI] E. Heintze and H.-C. Im Hof, Geometry of horospheres, J. Differential Geom. 12 (1977), 481–491. MR0512919 (80a:53051)

[Ku] A. Kuznetsov, Visibility metrics on the boundary at infinity for the complex hyperbolic plane, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 353 (2008), 70–92; English transl. in J. Math. Sci. (New York) (to appear).

[Mo] G. D. Mostow, Strong rigidity of locally symmetric spaces, Ann. of Math. Stud., No. 78, Princeton Univ. Press, Princeton, NJ, 1973. MR0385004 (52:5874)

[Pa] P. Pansu, Métries de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un, Ann. of Math. (2) 129 (1989), 1–60. MR0979599 (90e:53058)

[Wo] J. Wolf, Spaces of constant curvature, Univ. of California, Berkeley, CA, 1972. MR0343213 (49:7957)

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