New estimations for the Berezin number inequality

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Abstract
In this paper, by the definition of Berezin number, we present some inequalities involving the operator geometric mean. For instance, it is shown that if \( X, Y, Z \in \mathcal{L}(\mathcal{H}) \) such that \( X \) and \( Y \) are positive operators, then

\[
\text{ber}'((X \# Y)Z) \leq \text{ber'}\left(\frac{(Z^*YZ)_{1q}}{q} + \frac{X_{1p}^2}{p}\right) - \frac{1}{p} \inf_{\lambda \in \Omega} \left(\left\langle \hat{X}(\lambda)\right\rangle - \left\langle (Z^*YZ)(\lambda)\right\rangle\right)^2,
\]

in which \( X \# Y = X^{1/2}(X^{-1/2}YZX^{-1/2})X^{1/2}, \ p \geq q > 1 \) such that \( r \geq \frac{3}{q} \) and \( \frac{1}{p} + \frac{1}{q} = 1. \)

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1 Introduction and preliminaries
We denote the \( C^* \)-algebra of all bounded linear operators on a separable complex Hilbert space \( \mathcal{H} \) with \( \mathcal{L}(\mathcal{H}) \). An operator \( X \in \mathcal{L}(\mathcal{H}) \) is called positive if \( \langle Xx, x \rangle \geq 0 \) for every \( x \in \mathcal{H} \), and in this case we write \( X \geq 0. \) The numerical range and numerical radius of \( X \in \mathcal{L}(\mathcal{H}) \) are respectively defined by \( W(X) := \{\langle Xf, f \rangle : f \in \mathcal{H}, \|f\| = 1\} \) and \( w(X) := \sup\{|f| : f \in W(X)\}. \) We denote by \( \mathcal{F}(\Omega) \) the set of all complex-valued functions on a nonempty set \( \Omega. \) Let \( \mathcal{H} = \mathcal{H}(\Omega) \subset \mathcal{F}(\Omega) \) be a Hilbert space. The Riesz representation theorem makes certain that a functional Hilbert space has a reproducing kernel, which is a function \( k_\lambda : \Omega \times \Omega \to \mathcal{H}, \) that is called the reproducing kernel enjoying the reproducing property \( k_\lambda := \langle k_\lambda(z), \lambda \rangle \in \mathcal{H}, (\lambda \in \Omega) \) such that \( f(\lambda) = \langle f, k_\lambda \rangle, \) in which \( \lambda \in \Omega \) and \( f \in \mathcal{H}, \) (see [18]).

For \( \{\xi_n(z)\}_{n \geq 0}, \) an orthonormal basis of the space \( \mathcal{H}(\Omega), \) the reproducing kernel can be presented as follows:

\[
k_\lambda(z) = \sum_{n=0}^{\infty} \xi_n(\lambda)\xi_n(z)
\]

(see [2, 18] and the references therein). Throughout the paper, \( \mathcal{H} = \mathcal{H}(\Omega) \) for some nonempty set \( \Omega. \) If \( X \in \mathcal{L}(\mathcal{H}), \) then the Berezin symbol of \( X \) is the function \( \hat{X} \) with

\[
\hat{X}(\mu) := \langle X\hat{k}_\mu, \hat{k}_\mu \rangle_{\mathcal{H}}, \quad (\mu \in \Omega),
\]

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where $\tilde{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ is the normalized reproducing kernel of $H$ (see [7]). Karaev in [13–15] defined the Berezin set and the Berezin number for operator $X$ as follows:

$$\text{Ber}(X) := \{\tilde{X}(\lambda) : \lambda \in \Omega\}$$

and

$$\text{ber}(X) := \sup\{|\tilde{X}(\lambda)| : \lambda \in \Omega\},$$

respectively. Moreover, the Berezin number of two operators $X$, $Y$ satisfies the following properties:

(i) $\text{ber}(\nu X) = |\nu| \text{ber}(X)$ for all $\nu \in \mathbb{C}$;

(ii) $\text{ber}(X + Y) \leq \text{ber}(X) + \text{ber}(Y)$.

Also, we know that

$$\text{ber}(X) \leq w(X) \leq \|X\|$$

for all $X \in \mathcal{L}(H)$. In some recent papers, several Berezin number inequalities have been investigated by authors [3–6, 9, 10, 12, 21, 22].

Assume that $X_1, \ldots, X_n \in \mathcal{L}(H)$ and $p \geq 1$. In [3], the generalized Euclidean Berezin number of $X_1, \ldots, X_n$ is defined as follows:

$$\text{ber}_p(X_1, \ldots, X_n) := \sup_{\lambda \in \Omega} \left(\sum_{i=1}^n |\langle X_i \tilde{k}_\lambda, \tilde{k}_\lambda \rangle|^p\right)^{\frac{1}{p}}.$$

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the Young inequality is the inequality

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q},$$

where $x$ and $y$ are positive real numbers (see [11]). A refinement of (1) was obtained by Kittaneh and Manasrah [17]

$$xy + r_0(x^q - y^q)^2 \leq \frac{x^p}{p} + \frac{y^q}{q},$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$ or equivalently

$$x^\nu y^{1-\nu} + r_0(x^\frac{1}{2} - y^\frac{1}{2})^2 \leq \nu x + (1 - \nu)y,$$

in which $\nu \in [0, 1]$ and $r_0 = \min\{\nu, 1 - \nu\}$.

For positive operators $X, Y \in \mathcal{L}(H)$, the operator geometric mean is the positive operator $X \ngeq Y = X^{\frac{1}{2}}(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^{\frac{1}{2}}X^{\frac{1}{2}}$, where it has the property $X \ngeq Y = Y \ngeq X$. A matrix mean inequality was established by Bhatia and Kittaneh in [8], and later this inequality was generalized in [18]. A matrix Young inequality was obtained by Ando in [1]. The matrix mean inequality and the matrix Young inequality were considered with the numerical radius norm by Salemi and Sheikhhosseini in [19, 20].

In this paper, we get some upper bounds for the Berezin number of the $(X \ngeq Y)Z$ on reproducing kernel Hilbert spaces (RKHS), where $Z \in \mathcal{L}(H)$ is arbitrary, and give some Berezin number inequalities. We also present some inequalities for the generalized Euclidean Berezin number.
2 Main results

We need the following lemma to prove our results (see [16]).

**Lemma 1** Let $X \in L(H)$ be a positive operator, and let $x \in H$ be any vector. If $r \geq 1$, then

$$
\langle Xx, x \rangle^r \leq \langle x^r, x \rangle
$$

and if $0 \leq r \leq 1$, then

$$
\langle x^r, x \rangle \leq \langle Xx, x \rangle^r.
$$

Before giving our next result, we set $\|X\|_{ber} := \sup\{\|\langle X, \lambda \rangle\| : \lambda, \mu \in \Omega\}$ and $m(X) := \inf_{\lambda \in \Omega} |\overline{X}(\lambda)|^2$.

**Theorem 2** Let $X, Y, Z \in L(H)$ be operators such that $X, Y$ are positive. If $p \geq q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
\text{ber}^r((X \preceq Y)Z) \leq \text{ber} \left( \frac{X^{2q}}{p} + \frac{(Z^*YZ)^{2q}}{q} \right) - \frac{1}{p} \inf_{\lambda \in \Omega} \left( \|\overline{X}(\lambda)\|^{\frac{2q}{p}} - \|\overline{Z^*YZ}(\lambda)\|^{\frac{2q}{p}} \right)^2
$$

for all $r \geq \frac{2}{q}$.

**Proof** Using the Cauchy–Schwarz inequality, we get

$$
\|X \preceq Y)Z(\lambda)\|^r = \|\left(\left(\frac{X^2}{q} \frac{1}{q} X^2\right)^{\frac{2}{q}} \frac{1}{q} X^2 Y \frac{1}{q} X^2 Z \frac{1}{q} X^2\right)(\lambda)\|^r
$$

$$
= \left\| \left(\frac{X^2}{q} \frac{1}{q} X^2 Y \frac{1}{q} X^2 Z \frac{1}{q} X^2\right) \right\|^r
$$

$$
\leq \left\| \left(\frac{X^2}{q} \frac{1}{q} X^2 Y \frac{1}{q} X^2 Z \frac{1}{q} X^2\right) \right\|^r \cdot \|X \preceq Y)Z(\lambda)\|^r
$$

$$
= \left(\frac{X^2}{q} \frac{1}{q} X^2 Y \frac{1}{q} X^2 Z \frac{1}{q} X^2\right) \cdot \left(\frac{X^2}{q} \frac{1}{q} X^2 Y \frac{1}{q} X^2 Z \frac{1}{q} X^2\right)
$$

$$
= \left(\overline{Z^*YZ}(\lambda)\right)^{\frac{2}{q}} (\overline{X}(\lambda))^{\frac{2}{q}}
$$

for all $\lambda \in \Omega$. By using the Young inequality and (2), we get

$$
(\overline{X}(\lambda))^{\frac{2}{q}} (\overline{Z^*YZ}(\lambda))^{\frac{2}{q}} \leq \frac{1}{p} (\|X \preceq Y)Z(\lambda)\|^{\frac{2q}{p}} + \frac{1}{q} (\|Z^*YZ\|^{\frac{2q}{p}})
$$

$$
- \frac{1}{p} \left( (\|X \preceq Y)Z(\lambda)\|^{\frac{2q}{p}} - (\|Z^*YZ\|^{\frac{2q}{p}}) \right)^2,
$$

and it follows from inequality (4) that

$$
\frac{1}{p} (\|X \preceq Y)Z(\lambda)\|^{\frac{2q}{p}} + \frac{1}{q} (\|Z^*YZ\|^{\frac{2q}{p}}) \leq \frac{1}{p} \left( (\|X \preceq Y)Z(\lambda)\|^{\frac{2q}{p}} - (\|Z^*YZ\|^{\frac{2q}{p}}) \right)^2,
$$

$$
\text{ber}^r((X \preceq Y)Z) \leq \text{ber} \left( \frac{X^{2q}}{p} + \frac{(Z^*YZ)^{2q}}{q} \right) - \frac{1}{p} \inf_{\lambda \in \Omega} \left( \|\overline{X}(\lambda)\|^{\frac{2q}{p}} - \|\overline{Z^*YZ}(\lambda)\|^{\frac{2q}{p}} \right)^2.
$$
Proposition 5 Let $X, Y, Z \in \mathcal{L}(\mathcal{H})$ such that $X, Y$ are positive, and let $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$
\| (X \pm Y) Z \|_{\text{ber}} \leq \left( \frac{X^{\frac{p}{2}}}{p} + \frac{(Z^* Y Z)^{\frac{q}{2}}}{q} \right)_{\text{ber}} - \frac{1}{p} \left( \| X \|_{\text{ber}}^{\frac{p}{2}} - \| (Z^* Y Z) \|_{\text{ber}}^{\frac{q}{2}} \right)^2
$$

for all $\lambda \in \Omega$. Since $(\frac{X^{\frac{p}{2}}}{p} + \frac{(Z^* Y Z)^{\frac{q}{2}}}{q})_{\text{ber}}(\lambda)$ is positive, then we have

$$
\sup_{\lambda \in \Omega} \| (X \pm Y) Z(\lambda) \|_{\text{ber}} \leq \sup_{\lambda \in \Omega} \left( \frac{X^{\frac{p}{2}}}{p} + \frac{(Z^* Y Z)^{\frac{q}{2}}}{q} \right)_{\text{ber}}(\lambda) - \frac{1}{p} \inf_{\lambda \in \Omega} \left( \| X(\lambda) \|_{\text{ber}}^{\frac{p}{2}} - \| (Z^* Y Z)(\lambda) \|_{\text{ber}}^{\frac{q}{2}} \right)^2
$$

for all $\lambda \in \Omega$. This implies that

$$
\text{ber}^r((X \pm Y) Z) \leq \text{ber} \left( \frac{X^{\frac{p}{2}}}{p} + \frac{(Z^* Y Z)^{\frac{q}{2}}}{q} \right)_{\text{ber}} - \frac{1}{p} \inf_{\lambda \in \Omega} \left( \| X(\lambda) \|_{\text{ber}}^{\frac{p}{2}} - \| (Z^* Y Z)(\lambda) \|_{\text{ber}}^{\frac{q}{2}} \right)^2.
$$

(5)

□

Taking the $Z = I$ in inequality (5), we have the following result.

Corollary 3 Let $X, Y \in \mathcal{L}(\mathcal{H})$ be positive operators, and let $p \geq q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$
\text{ber}^r(X \pm Y) \leq \text{ber} \left( \frac{X^{\frac{p}{2}}}{p} + \frac{Y^{\frac{q}{2}}}{q} \right)_{\text{ber}} - \frac{1}{p} \inf_{\lambda \in \Omega} \left( \| X(\lambda) \|_{\text{ber}}^{\frac{p}{2}} - \| Y(\lambda) \|_{\text{ber}}^{\frac{q}{2}} \right)^2
$$

for all $r \geq \frac{2}{q}$.

Corollary 4 Let $X, Y \in \mathcal{L}(\mathcal{H})$ be positive operators. Then

$$
\sqrt{2} \text{ber}(X \pm Y) \leq \text{ber}_2(X, Y) \leq \text{ber}^\frac{1}{2}(X^2 + Y^2).
$$

Proof As in the same arguments in the proof of Theorem 2, if we put $r = p = q = 2$, then we get

$$
\| (X \pm Y) Z \|_{\text{ber}} \leq \left( \frac{X^{\frac{2}{2}}}{2} + \frac{(Z^* Y Z)^{\frac{2}{2}}}{2} \right)_{\text{ber}} - \frac{1}{2} \inf_{\lambda \in \Omega} \left( \| X(\lambda) \|_{\text{ber}}^{2} - \| (Z^* Y Z)(\lambda) \|_{\text{ber}}^{2} \right)^2
$$

Since $\| X(\lambda) \|_{\text{ber}}^{2} \geq 0, \| Y(\lambda) \|_{\text{ber}}^{2} \geq 0, and \ (X^2 + Y^2)(\lambda) \geq 0$, taking the supremum over $\lambda \in \Omega$, we get that

$$
\sqrt{2} \text{ber}(X \pm Y) \leq \text{ber}_2(X, Y) \leq \text{ber}^\frac{1}{2}(X^2 + Y^2).
$$

□
for all \( r \geq \frac{2}{q} \).

**Proof** Indeed, for every \( \lambda, \mu \in \Omega \), we have

\[
\|(X \ast Y)Z\tilde{k}_\lambda, \tilde{k}_\mu\|_p^p = \left\| \left( X^{\frac{1}{2}} \left( \frac{\tilde{X} \ast Y \ast \tilde{X}}{\lambda} \right)^{1} \tilde{X}^{\frac{1}{2}} Z \tilde{k}_\lambda, \tilde{k}_\mu \right) \right\|
\]

\[
= \left\| \left( \tilde{X}^{\frac{1}{2}} \left( X^{\frac{1}{2}} Y \ast \tilde{X} \right)^{1} \tilde{X}^{\frac{1}{2}} Z \tilde{k}_\lambda, \tilde{k}_\mu \right) \right\|
\]

\[
= \left\| \left( \tilde{X}^{\frac{1}{2}} \left( X^{\frac{1}{2}} Y \ast \tilde{X} \right)^{1} \tilde{X}^{\frac{1}{2}} Z \tilde{k}_\lambda, \tilde{k}_\mu \right) \right\|
\]

\[
\leq \left\| \left( X^{\frac{1}{2}} Y \ast \tilde{X} \right)^{1} \tilde{X}^{\frac{1}{2}} Z \tilde{k}_\lambda, \tilde{k}_\mu \right\| \cdot \left\| \tilde{k}_\mu \right\|
\]

\[
= \left( X^{\frac{1}{2}} Y \ast \tilde{X} \right)^{1} \tilde{X}^{\frac{1}{2}} Z \tilde{k}_\lambda, \tilde{k}_\mu \left\| \frac{1}{2} \tilde{X}^{\frac{1}{2}} Z \tilde{k}_\lambda, \tilde{k}_\mu \right\|^2
\]

\[
\leq \frac{1}{p} \left( X^{\frac{1}{2}} \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{p}{q}} + \frac{1}{q} \left( Z^* Y \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{q}{q}}
\]

\[
- \frac{1}{p} \left( X^{\frac{1}{2}} \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{p}{q}} - \left( Z^* Y \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{q}{q}}
\]

\[
\leq \frac{1}{p} \left( X^{\frac{1}{2}} \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{p}{q}} + \frac{1}{q} \left( Z^* Y \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{q}{q}}
\]

\[
- \frac{1}{p} \left( X^{\frac{1}{2}} \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{p}{q}} - \left( Z^* Y \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{q}{q}}
\]

\[
\leq \frac{1}{p} \left( X^{\frac{1}{2}} \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{p}{q}} + \frac{1}{q} \left( Z^* Y \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{q}{q}}
\]

so that if we take the supremum over \( \lambda, \mu \in \Omega \) in inequality (6), we get

\[
\|(X \ast Y)Z\|_{\text{ber}} \leq \frac{\frac{1}{p}}{\|X\|_{\text{ber}}} + \frac{\frac{1}{q}}{\|Z^* Y\|_{\text{ber}}}
\]

\[
- \frac{1}{p} \inf_{\mu, \lambda \in \Omega} \left( X^{\frac{1}{2}} \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{p}{q}} - \left( Z^* Y \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{q}{q}}.
\]

**Remark 6** It follows from inequality

\[
\inf_{\mu, \lambda \in \Omega} \left( X^{\frac{1}{2}} \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{p}{q}} - \left( Z^* Y \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{q}{q}}
\]

\[
= \inf_{\mu, \lambda \in \Omega} \left( X^{\frac{1}{2}} \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{p}{q}} + \left( Z^* Y \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{q}{q}} - \frac{1}{p} \left( X^{\frac{1}{2}} \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{p}{q}} \left( Z^* Y \tilde{k}_\lambda, \tilde{k}_\mu \right)^{\frac{q}{q}}
\]

\[
\geq \frac{\|X\|_{\text{ber}}}{\|X\|_{\text{ber}}} + \|Z^* Y\|_{\text{ber}} - 2 \|X\|_{\text{ber}} \|Z^* Y\|_{\text{ber}}
\]

\[
= m(X)^\frac{p}{q} + m(Z^* Y)^\frac{q}{q} - 2 \|Z^* Y\|_{\text{ber}} \|X\|_{\text{ber}}
\]
and inequality (6) that
\[
\| (X \mathcal{Z} Y) Z \|_{\text{ber}} \leq \| X \|_{\text{ber}}^p + \| (Z^* Y Z) \|_{\text{ber}}^q - (m(X))^\frac{p}{2} + (Z^* Y Z) \|_{\text{ber}}^\frac{p}{2} \| X \|_{\text{ber}}^\frac{p}{2}.
\]

**Proposition 7** Let \( X, Y, Z \in \mathcal{L}(H) \) such that \( X, Y \) are positive, and let \( p \geq q > 1 \), where \( \frac{1}{p} + \frac{1}{q} = 1 \). Then
\[
\left( \| X \|_{\text{ber}} \| Z^* Y Z \|_{\text{ber}} \right)^\frac{r}{2} \leq \| X \|_{\text{ber}}^p + \| (Z^* Y Z) \|_{\text{ber}}^q - \inf_{\lambda, \mu \in \Omega} \left( \langle X \mathcal{K}_\lambda \mathcal{K}_\mu \rangle^\frac{p}{2} - \langle Z^* Y Z \mathcal{K}_\lambda \mathcal{K}_\mu \rangle^\frac{p}{2} \right)^2
\]
for all \( r \geq \frac{2}{q} \).

**Proof** By inequality (2), we have
\[
\langle X \mathcal{K}_\lambda \mathcal{K}_\mu \rangle^\frac{p}{2} \langle Z^* Y Z \mathcal{K}_\lambda \mathcal{K}_\mu \rangle^\frac{p}{2} \leq \frac{1}{p} (\langle X \mathcal{K}_\lambda \mathcal{K}_\mu \rangle^\frac{p}{2} + \langle Z^* Y Z \mathcal{K}_\lambda \mathcal{K}_\mu \rangle^\frac{p}{2})^\frac{p}{2} - \frac{1}{p} (\langle X \mathcal{K}_\lambda \mathcal{K}_\mu \rangle^\frac{p}{2} - \langle Z^* Y Z \mathcal{K}_\lambda \mathcal{K}_\mu \rangle^\frac{p}{2})^2
\]
for all \( \lambda, \mu \in \Omega \) and taking supremum over \( \lambda, \mu \in \Omega \) in the above inequality, we get
\[
\left( \| X \|_{\text{ber}} \| Z^* Y Z \|_{\text{ber}} \right)^\frac{r}{2} \leq \frac{1}{p} (\langle X \mathcal{K}_\lambda \mathcal{K}_\mu \rangle^\frac{p}{2} + \langle Z^* Y Z \mathcal{K}_\lambda \mathcal{K}_\mu \rangle^\frac{p}{2})^\frac{p}{2} - \frac{1}{p} (\langle X \mathcal{K}_\lambda \mathcal{K}_\mu \rangle^\frac{p}{2} - \langle Z^* Y Z \mathcal{K}_\lambda \mathcal{K}_\mu \rangle^\frac{p}{2})^2.
\]

Now, we present the next lemma to obtain our last results.

**Lemma 8** (16) If \( f, g : [0, \infty) \rightarrow \mathbb{R} \) are nonnegative continuous such that \( f(t)g(t) = t \) (\( t \in [0, \infty) \)), then
\[
| \langle Xx, y \rangle | \leq \| f(|X|)x \| g(|X^*|)x \|
\]
where \( X \in \mathcal{L}(H) \) and \( x, y \in H \).

In the next theorem we show an upper bound for the generalized Euclidean Berezin number.
Let $X_i, Y_i, Z_i \in L(\mathcal{H})$ $(1 \leq i \leq n)$. Then

$$\text{ber}^p_n(X^*_i Z_i Y_1, \ldots, X^*_n Z_n Y_n)$$

$$\leq \frac{n^{1+\frac{1}{p}}}{2^r} \text{ber}^p \left( \sum_{i=1}^n \left[ Y_i^* f^2(|Z_i|) Y_i \right]^{rp} + \left[ X_i^* g^2(|Z_i^*|) X_i \right]^{rp} \right)$$

$$- \frac{1}{2} \inf_{\lambda \in \mathbb{R}} \left( \sum_{i=1}^n \left( \sqrt{\left( \left| X_i^* g^2(|Z_i^*|) X_i \right|^p \hat{k}_i, \hat{k}_i \right)} - \sqrt{\left( \left| Y_i^* f^2(|Z_i|) Y_i \right|^p \hat{k}_i, \hat{k}_i \right)} \right)^2 \right),$$

(7)

where $f, g : [0, \infty) \rightarrow \mathbb{R}$ are nonnegative continuous such that $f(t)g(t) = t$ $(t \in [0, \infty))$ and $p, r \geq 1$.

Proof. For any $\hat{k}_i \in \mathcal{H}(\Omega)$, we have

$$\sum_{i=1}^n \left| X_i^* Z_i Y_i \hat{k}_i, \hat{k}_i \right|^p$$

$$= \sum_{i=1}^n \left| (Z_i Y_i \hat{k}_i, X_i \hat{k}_i) \right|^p$$

$$\leq \sum_{i=1}^n \left\| f(|Z_i|) Y_i \hat{k}_i \right\|^p g(|Z_i^*|) X_i \hat{k}_i \right\|^p \right) \quad \text{(by Lemma 8)}$$

$$= \sum_{i=1}^n \left( f(|Z_i|) Y_i \hat{k}_i, f(|Z_i|) Y_i \hat{k}_i \right)^p g(|Z_i^*|) X_i \hat{k}_i, g(|Z_i^*|) X_i \hat{k}_i \right)^p$$

$$= \sum_{i=1}^n \left( (Y_i^* f^2(|Z_i|) Y_i)^p \hat{k}_i, \hat{k}_i \right)^p \left( X_i^* g^2(|Z_i^*|) X_i \hat{k}_i, \hat{k}_i \right)^p \right)^p$$

$$\leq \sum_{i=1}^n \left( \left( \left( \frac{1}{2} \left( Y_i^* f^2(|Z_i|) Y_i \right)^p \hat{k}_i, \hat{k}_i \right) + \left( \frac{1}{2} \left( X_i^* g^2(|Z_i^*|) X_i \right)^p \hat{k}_i, \hat{k}_i \right) \right) \right)^p \right) \quad \text{(by (2))}$$

$$- \frac{1}{2} \sum_{i=1}^n \left( \sqrt{\left( \left( X_i^* g^2(|Z_i^*|) X_i \right)^p \hat{k}_i, \hat{k}_i \right)} - \sqrt{\left( \left( Y_i^* f^2(|Z_i|) Y_i \right)^p \hat{k}_i, \hat{k}_i \right)} \right)^2$$

$$\leq \frac{n^{1+\frac{1}{p}}}{2^r} \left( \sum_{i=1}^n \left[ Y_i^* f^2(|Z_i|) Y_i \right]^{rp} + \left[ X_i^* g^2(|Z_i^*|) X_i \right]^{rp} \right) \hat{k}_i, \hat{k}_i \right)^p \right)$$

$$- \frac{1}{2} \sum_{i=1}^n \left( \sqrt{\left( \left( X_i^* g^2(|Z_i^*|) X_i \right)^p \hat{k}_i, \hat{k}_i \right)} - \sqrt{\left( \left( Y_i^* f^2(|Z_i|) Y_i \right)^p \hat{k}_i, \hat{k}_i \right)} \right)^2 \right).$$

By taking the supremum on $\hat{k}_i \in \mathcal{H}$ with $\|\hat{k}_i\| = 1$, we reach the desired inequality. □

Selecting $X_i = Y_i = I$ for $i = 1, 2, \ldots, n$ in Theorem 9, we get the next result.
Corollary 10  Let $Z_i \in \mathcal{L}(\mathcal{H})$ (1 $\leq i \leq n$) and $r, p \geq 1$. Then

$$\text{ber}_p(Z_1, \ldots, Z_n) \leq \frac{n^{1-\frac{1}{p}}}{2^p} \text{ber} \left( \sum_{i=1}^{n} (|Z_i|^2)^p + g^2(|Z_i^*|^2)^p \right)$$

$$- \frac{1}{2} \sum_{i=1}^{n} \left( \sqrt{g^2(|Z_i^*|)^2} - \sqrt{f^2(|Z_i|)^2} \right)^2,$$

where $f, g : [0, \infty) \rightarrow \mathcal{R}$ are nonnegative continuous such that $f(t)g(t) = t$ (t $\in [0, \infty)$).

In particular, if $X, Y \in \mathcal{L}(\mathcal{H})$, then for all $p \geq 1$ and $0 \leq v \leq 1$

$$\text{ber}_p(X, Y) \leq \frac{1}{2} \text{ber} (|X|^{2vp} + |X^*|^{2(1-v)p} + |Y|^{2vp} + |Y^*|^2) + \inf_{\lambda \in \Omega} \delta(\hat{k}_\lambda),$$

where

$$\delta(\hat{k}_\lambda) = \frac{1}{2} \left( (|X|^{2vp}\hat{k}_\lambda, \hat{k}_\lambda) - (|X^*|^{2(1-v)p}\hat{k}_\lambda, \hat{k}_\lambda) \right)^2$$

$$+ (|Y|^{2vp}\hat{k}_\lambda, \hat{k}_\lambda) - (|Y^*|^2\hat{k}_\lambda, \hat{k}_\lambda)^2.$$ 

In the last theorem, we show another upper bound for $\text{ber}_p(T_1, \ldots, T_n)$.

Theorem 11  Let $Z_i \in \mathcal{L}(\mathcal{H})$ (1 $\leq i \leq n$). Then

$$\text{ber}_p(Z_1, \ldots, Z_n) \leq \frac{1}{2} \left( \sum_{i=1}^{n} \left( \text{ber} (|Z_i|^{2v} + |Z_i^*|^{2(1-v)}) - 2 \inf_{|\lambda| \leq 1} \delta(\hat{k}_\lambda) \right)^p \right)^{\frac{1}{p}}, \quad (8)$$

where $p \geq 1$, $0 \leq v \leq 1$, and $\delta(\hat{k}_\lambda) = (\sqrt{|Z_i^*|^{2(1-v)}\hat{k}_\lambda, \hat{k}_\lambda} - \sqrt{|Z_i|^{2v}\hat{k}_\lambda, \hat{k}_\lambda})^2$. 

Proof  Let $\hat{k}_\lambda \in \mathcal{H}(\Omega)$. Then, by using Lemma 8 and inequality (3), we have

$$\sum_{i=1}^{n} (|Z_i\hat{k}_\lambda, \hat{k}_\lambda|)^p$$

$$\leq \frac{1}{2p} \sum_{i=1}^{n} (|Z_i|^{2v}\hat{k}_\lambda, \hat{k}_\lambda)^p + (|Z_i|^{2(1-v)}\hat{k}_\lambda, \hat{k}_\lambda)^p \quad \text{(by Lemma 8)}$$

$$\leq \frac{1}{2p} \sum_{i=1}^{n} (|Z_i|^{2v}\hat{k}_\lambda, \hat{k}_\lambda) + (|Z_i|^{2(1-v)}\hat{k}_\lambda, \hat{k}_\lambda)$$

$$- (\sqrt{|Z_i|^{2(1-v)}\hat{k}_\lambda, \hat{k}_\lambda})^2$$

$$= \left( \sqrt{|Z_i|^{2(1-v)}\hat{k}_\lambda, \hat{k}_\lambda} - \sqrt{|Z_i|^{2v}\hat{k}_\lambda, \hat{k}_\lambda} \right)^2 \quad \text{(by (3))}$$

Thus

$$\left( \sum_{i=1}^{n} (|Z_i\hat{k}_\lambda, \hat{k}_\lambda|)^p \right)^{\frac{1}{p}} \leq \frac{1}{2} \left( \sum_{i=1}^{n} (|Z_i|^{2v} + |Z_i|^{2(1-v)}\hat{k}_\lambda, \hat{k}_\lambda) \right)^{\frac{1}{p}}.$$
If we get the supremum over all $\hat{k}_\lambda \in \mathcal{H}(\Omega)$ with $\|\hat{k}_\lambda\| = 1$, then we reach the desired result.

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