Research Article

On the Convergence and Stability Results for a New General Iterative Process

Kadri Doğan and Vatan Karakaya

Department of Mathematical Engineering, Yıldız Technical University, Davutpaşa Campus, Esenler, 34220 Istanbul, Turkey

Correspondence should be addressed to Kadri Doğan; dogankadri@hotmail.com

Received 14 June 2014; Accepted 11 August 2014; Published 2 September 2014

Academic Editor: Syed Abdul Mohiuddine

We put forward a new general iterative process. We prove a convergence result as well as a stability result regarding this new iterative process for weak contraction operators.

1. Introduction and Preliminaries

Throughout this paper, by \( \mathbb{N} \), we denote the set of all positive integers. In this paper, we obtain results on the stability and strong convergence for a new iteration process (3) in an arbitrary Banach space by using weak contraction operator in the sense of Berinde [1]. Also, we obtain that the iteration procedure (3) can be useful method for solution of delay differential equations. To obtain solution of delay differential equation by using fixed point theory, some authors have done different studies. One can find these works in [2, 3]. Many results of stability have been established by some authors using different contractive mappings. The first study on the stability of the Picard iteration under Banach contraction condition was done by Ostrowski [4]. Some other remarkable results on the concept of stability can be found in works of the following authors involving Harder and Hicks [5, 6], Rhoades [7, 8], Osilike [9], Osilike and Udomene [10], and Singh and Prasad [11]. In 1988, Harder and Hicks [5] established applications of stability results to first order differential equations. Osilike and Udomene [10] developed a short proof of stability results for various fixed point iterations processes. Afterward, in following studies, same technique given in [10] has been used, by Berinde [12], Olatinwo [13], Imoru and Olatinwo [14], Karakaya et al. [15], and some authors.

Let \((E, d)\) be complete metric space and \(T : E \rightarrow E\) a self-map on \(E\); and the set of fixed points of \(T\) in \(E\) is defined by \(F_T = \{p \in E : Tp = p\}\). Let \(\{x_n\}_{n \in \mathbb{N}} \subset E\) be the sequence generated by an iteration involving \(T\) which is defined by

\[
x_{n+1} = f(T,x_n) \quad n = 0, 1, \ldots,
\]

where \(x_0 \in E\) is the initial point and \(f\) is a proper function. Suppose that sequence \(\{x_n\}_{n \in \mathbb{N}}\) converges to a fixed point \(p\) of \(T\). Let \(\{y_n\}_{n \in \mathbb{N}} \subset E\) and set

\[
e_n = d(y_{n+1}, f(T,y_n)) \quad n = 0, 1, \ldots
\]

Then, the iteration procedure (1) is said to be \(T\) stable or stable with respect to \(T\) if and only if \(\lim_{n \to \infty} e_n = 0\) implies \(\lim_{n \to \infty} y_n = p\).

Now, let \(C\) be a convex subset of a normed space \(E\) and \(T : C \rightarrow C\) a self-map on \(E\). We introduce a new two-step iteration process which is a generalization of Ishikawa iteration process as follows:

\[
x_0 = x \in C,
\]

\[
f(T,x_n) = (1 - \varphi_n)x_n + \xi_nT x_n + (\varphi_n - \xi_n)Ty_n, \quad n = 0, 1, \ldots
\]

\[
y_n = (1 - \zeta_n)x_n + \zeta_nT x_n,
\]

\[F_T = \{p \in E : Tp = p\}.\]
for $n \geq 0$, where $\{\xi_n\}$, $\{\varphi_n\}$, and $\{\zeta_n\}$ satisfy the following conditions

\[(C_1) \, \varphi_n \geq \xi_n,\]

\[(C_2) \, \{\varphi_n - \xi_n\}_{n=0}^{\infty}, \{\varphi_n - \zeta_n\}_{n=0}^{\infty}, \{\xi_n\}_{n=0}^{\infty}, \{\zeta_n\}_{n=0}^{\infty} \in [0, 1],\]

\[(C_3) \, \sum_{n=0}^{\infty} \varphi_n = \infty.\]

In the following remark, we show that the new iteration process is more general than the Ishikawa and Mann iteration processes.

**Remark 1.**

1. If $\xi_n = 0$, then (3) reduces to the Ishikawa iteration process in [16].
2. If $\zeta_n = 0$, then (3) reduces to the Mann iteration process in [17].

**Lemma 2 (see [18]).** If $\delta$ is a real number such that $0 \leq \delta < 1$ and $\{\epsilon_n\}_{n \in \mathbb{N}}$ is a sequence of positive real numbers such that $\lim_{n \to \infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n \in \mathbb{N}}$ satisfying

$$u_{n+1} \leq \delta u_n + \epsilon_n \quad n = 0, 1, \ldots$$

one has

$$\lim_{n \to \infty} u_n = 0.\quad (5)$$

**Lemma 3 (see [2]).** Let $\{s_n\}_{n \in \mathbb{N}}$ be a sequence of positive real numbers including zero satisfying

$$s_{n+1} \leq (1 - \mu_n) s_n.\quad (6)$$

If $\mu_n \in (0, 1)$ and $\sum_{n=0}^{\infty} \mu_n = \infty$, then $\lim_{n \to \infty} s_n = 0$.

A mapping $T : C \to E$ is said to be contraction if there is a fixed real number $a \in [0, 1)$ such that

$$\|Tx - Ty\| \leq a \|x - y\|\quad (7)$$

for all $x, y \in C$.

This contraction condition has been generalized by many authors. For example, Kannan [19] shows that there exists $b \in [0, 1/2)$ such that, for all $x, y \in C$,

$$\|Tx - Ty\| \leq b (\|x - Tx\| + \|y - Ty\|).\quad (8)$$

Chatterjea [20] shows that there exists $c \in [0, 1/2)$ such that, for all $x, y \in C$,

$$\|Tx - Ty\| \leq c (\|x - Ty\| + \|y - Tx\|).\quad (9)$$

In 1972, Zamfirescu [21] obtained the following theorem.

**Theorem 4 (see [21]).** Let $(X, d)$ be a complete metric space and $T : X \to X$ a mapping for which there exist real numbers $a, b, c$ satisfying $a \in (0, 1)$, $b, c \in (0, 1/2)$ such that, for each pair $x, y \in X$, at least one of the following conditions is performed:

1. $d(Tx, Ty) \leq ad(x, y)$,
2. $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$,
3. $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then $T$ has a unique fixed point $p$ and the Picard iteration $\{x_n\}_{n \in \mathbb{N}}$ defined by

$$x_{n+1} = Tx_n \quad n = 0, 1, 2, \ldots$$

converges to $p$ for any arbitrary but fixed $x_0 \in X$.

In 2004, Berinde introduced the definition which is a generalization of the above operators.

**Definition 5 (see [1]).** A mapping $T$ is said to be a weak contraction operator, if there exist $L \geq 0$ and $\delta \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \delta \|x - y\| + L \|x - Tx\|\quad (11)$$

for all $x, y \in E$.

**Theorem 6 (see [1]).** Let $(E, \|\|)$ be a Banach space. Assume that $C \subseteq E$ is a nonempty closed convex subset and $T : C \to C$ is a mapping satisfying (11). Then $F(T) \neq \emptyset$.

**Definition 7 (see [22]).** Let $\{u_n\}_{n \in \mathbb{N}}$ and $\{v_n\}_{n \in \mathbb{N}}$ be two iteration processes and let both $\{u_n\}_{n \in \mathbb{N}}$ and $\{v_n\}_{n \in \mathbb{N}}$ be converging to the same fixed point $p$ of a self-mapping $T$. Assume that

$$\lim_{n \to \infty} \|u_n - p\| = 0, \quad \lim_{n \to \infty} \|v_n - p\| = 0.\quad (12)$$

Then, it is said that $\{u_n\}_{n \in \mathbb{N}}$ converges faster than $\{v_n\}_{n \in \mathbb{N}}$ to fixed point $p$ of $T$.

The rate of convergence of the Picard and Mann iteration processes in terms of Zamfirescu operators in arbitrary Banach setting was compared by Berinde [22]. Using this class of operator, the Mann iteration method converges faster than the Ishikawa iteration method that was shown by Babu and Vara Prasad [23]. After a short time, Qing and Rhoades [24] showed that the claim of Babu and Vara Prasad [23] is false. There are many studies which have been made on the rate of convergence as given in [15, 25, 26] which are just a few of them.

**2. Main Results**

**Theorem 8.** Let $C$ be a nonempty closed convex subset of an arbitrary Banach space $E$ and let $T : C \to C$ be a mapping satisfying (11). Let $\{x_n\}_{n \in \mathbb{N}}$ be defined through the new iteration (3) and $x_0 \in E$, where $\{\varphi_n - \xi_n\}_{n=0}^{\infty}, \{\varphi_n - \zeta_n\}_{n=0}^{\infty}, \{\xi_n\}_{n=0}^{\infty}, \{\zeta_n\}_{n=0}^{\infty} \in [0, 1]$ with $\varphi_n$ satisfying $\sum_{n=0}^{\infty} \varphi_n = \infty, \varphi_n \geq \xi_n$. Then $\{x_n\}_{n \in \mathbb{N}}$ converges strongly to fixed point of $T$.

**Proof.** From Theorems 4 and 6, it is clear that $T$ has a unique fixed point in $C$ and $F(T) \neq \emptyset$.

From (3), we have

$$\|x_{n+1} - p\|
= \|((1 - \varphi_n) x_n + (\varphi_n - \xi_n) Ty_n + \xi_n Tx_n - p)\|
\leq (1 - \varphi_n) \|x_n - p\| + (\varphi_n - \xi_n) \|Ty_n - p\| + \xi_n \|Tx_n - p\|.$$
\[
\begin{align*}
\|x_{n+1} - p\| &\leq (1 - \varphi_n) \|x_n - p\| + (\varphi_n - \xi_n) \delta \|y_n - p\| \\
&\quad + (\varphi_n - \xi_n) L \|p - Tp\| + \xi_n \|x_n - p\| + \xi_n L \|p - Tp\| \\
&\leq [1 - \varphi_n + \xi_n] \|x_n - p\| + (\varphi_n - \xi_n) \delta \|y_n - p\| \\
&\quad + \varphi_n L \|p - Tp\|. \\
\end{align*}
\]
(13)

In addition,
\[
\|y_n - p\| = \|(1 - \zeta_n) x_n + \zeta_n Tx_n - p\| \\
\leq (1 - \zeta_n) \|x_n - p\| + \zeta_n \|Tx_n - p\| \\
\leq (1 - \zeta_n) \|x_n - p\| + \zeta_n \delta \|x_n - p\| + \zeta_n L \|p - Tp\| \\
= (1 - \zeta_n (1 - \delta)) \|x_n - p\| + \zeta_n L \|p - Tp\|. \\
\]
(14)

Substituting (14) in (13), we have the following estimates:
\[
\begin{align*}
\|x_{n+1} - p\| &\leq [1 - \varphi_n + \xi_n \delta] \|x_n - p\| \\
&\quad + (\varphi_n - \xi_n) \delta [(1 - \zeta_n (1 - \delta)) \|x_n - p\| + \zeta_n L \|p - Tp\|] \\
&\quad + \varphi_n L \|p - Tp\| \\
&\leq [1 - \varphi_n + \xi_n \delta + (\varphi_n - \xi_n) \delta (1 - \zeta_n (1 - \delta))] \|x_n - p\| \\
&\quad + [(\varphi_n - \xi_n) \delta \zeta_n + \varphi_n] L \|p - Tp\|. \\
\end{align*}
\]
(15)

Since \(\|p - Tp\| = 0\), we have
\[
\begin{align*}
\|x_{n+1} - p\| &\leq (1 - \varphi_n (1 - \delta)) \|x_n - p\| \\
\|x_{n+1} - p\| &\leq (1 - \varphi_n (1 - \delta)) \|x_n - p\| \\
\|x_{n+1} - p\| &\leq (1 - \varphi_n (1 - \delta)) \|x_{n-1} - p\| \\
\|x_{n+1} - p\| &\leq (1 - \varphi_n (1 - \delta)) \|x_{n-2} - p\| \\
&\quad \vdots \\
\|x_2 - p\| &\leq (1 - \varphi_1 (1 - \delta)) \|x_1 - p\| \\
\|x_1 - p\| &\leq (1 - \varphi_0 (1 - \delta)) \|x_0 - p\| \\
\|x_{n+1} - p\| &\leq \prod_{i=0}^{n} [1 - \varphi_i (1 - \delta)] \|x_0 - p\| \\
&\leq \|x_0 - p\| \cdot e^{\sum_{i=0}^{n} [1 - \varphi_i (1 - \delta)]} \\
&= \|x_0 - p\| \cdot e^{(-1 - \delta) \sum_{i=0}^{\infty} \varphi_i}. \\
\end{align*}
\]
for all \(n \in \mathbb{N}\).

Since \(0 < \delta < 1\), \(\varphi_n \in [0, 1]\), and \(\sum_{n=0}^{\infty} \varphi_n = \infty\), we have
\[
\lim_{n \to \infty} \sup \|x_{n+1} - p\| \\
\leq \lim_{n \to \infty} \sup \|x_0 - p\| \cdot e^{(-1 - \delta) \sum_{i=0}^{\infty} \varphi_i} \leq 0.
\]
(17)

So \(\lim_{n \to \infty} \|x_n - p\| = 0\) yields \(x_n \to p \in F(T)\). This completes the proof of theorem.

**Theorem 9.** Let \((E, \| \cdot \|)\) be Banach space and \(T : E \to E\) a self-mapping with fixed point \(p\) with respect to weak contraction condition in the sense of Berinde (11). Let \(\{x_n\}_{n \in \mathbb{N}}\) be iteration process (3) converging to fixed point of \(T\), where \(\varphi_n \geq \xi_n\) and \(\{\varphi_n\}_{n \in \mathbb{N}}\). Let \(\{\xi_n\}_{n \in \mathbb{N}}\) such that \(0 < \varphi \leq \varphi_n\) for all \(n\). Then two-step iteration process is \(T\) stable.

**Proof.** Let \(\{x_n\}_{n \in \mathbb{N}}\) be iteration process (3) converging to \(p\). Assume that \(\{y_n\}_{n \in \mathbb{N}} \subset E\) is an arbitrary sequence in \(E\). Set
\[
\epsilon_n = \|y_{n+1} - (1 - \varphi_n) y_n + (\varphi_n - \xi_n) T y_n + \xi_n T y_n\| \\
n = 0, 1, \ldots.
\]
(18)

Then, we shall prove that \(\lim_{n \to \infty} \epsilon_n = 0\). Using contraction condition (11), we have
\[
\|y_{n+1} - p\| \leq \|y_{n+1} - (1 - \varphi_n) y_n + (\varphi_n - \xi_n) T y_n + \xi_n T y_n\| \\
\quad + \|((1 - \varphi_n) y_n + (\varphi_n - \xi_n) T y_n + \xi_n T y_n - p\| \\
\leq \epsilon_n + \|((1 - \varphi_n) y_n + (\varphi_n - \xi_n) T y_n + \xi_n T y_n - p\| \\
\leq \epsilon_n + (1 - \varphi_n) \|y_n - p\| + (\varphi_n - \xi_n) \|T y_n - p\| \\
+ \xi_n \|T y_n - p\| \\
\leq \epsilon_n + (1 - \varphi_n) \|y_n - p\| + (\varphi_n - \xi_n) \|T y_n - p\| \\
+ \xi_n \|T y_n - p\|
\]
(19)

We estimate \(\|y_n - p\|\) in (19) as follows:
\[
\|y_n - p\| = \|(1 - \zeta_n) y_n + \zeta_n T y_n - p\| \\
\leq (1 - \zeta_n) \|y_n - p\| + \zeta_n \|T y_n - p\| \\
\leq (1 - \zeta_n) \|y_n - p\| + \zeta_n \delta \|y_n - p\| \\
+ \zeta_n \|T y_n - p\| \\
\]
(20)
Substituting (20) in (19), we have

\[
\|y_{n+1} - p\|
\leq \epsilon_n + (1 - \varphi_n + \xi_n \delta) \|y_n - p\| + \varphi_n L \|p - TP\|
+ (\varphi_n - \xi_n - \epsilon_n \delta) \|y_n - p\|
+ \epsilon_n L \|p - TP\|
\]
\[
= \epsilon_n + [1 - \varphi_n + \xi_n \delta + (\varphi_n - \xi_n) \delta (1 - \zeta_n (1 - \delta))] \|y_n - p\|
\times \|y_n - p\| + [\varphi_n - \xi_n] \|T \| T T y_n - p\|.
\]

Since \( \|p - TP\| = 0 \), we have

\[
\|y_{n+1} - p\|
\leq \epsilon_n + [1 - \varphi_n + \xi_n \delta + (\varphi_n - \xi_n) \delta (1 - \zeta_n (1 - \delta))] \|y_n - p\|
\times \|y_n - p\| + [1 - \varphi_n + \xi_n \delta + (\varphi_n - \xi_n) \delta (1 - \zeta_n (1 - \delta))]
\]
(22)

Since \( 0 < 1 - \varphi_n (1 - \delta) < 1 \) and using Lemma 2, we obtain
\[
\lim_{n \to \infty} y_n = p.
\]

Conversely, letting \( \lim_{n \to \infty} y_n = p \), we show that
\[
\lim_{n \to \infty} \epsilon_n = 0
\]
as follows:

\[
e_n = \|y_{n+1} - p\| - (1 - \varphi_n) \|y_n - p\| - (\varphi_n - \xi_n) T y_n - \xi_n T y_n\|
\leq \|y_{n+1} - p\| + \|p - (1 - \varphi_n) \|y_n - (\varphi_n - \xi_n) T y_n - \xi_n T y_n\|
\leq \|y_{n+1} - p\| + (1 - \varphi_n) \|y_n - p\| + (\varphi_n - \xi_n) \|T y_n - p\|
+ \xi_n \|T y_n - p\|
\leq \|y_{n+1} - p\| + (1 - \varphi_n + \delta \xi_n) \|y_n - p\|
+ (\varphi_n - \xi_n) \|T y_n - p\|
\leq \|y_{n+1} - p\| + (1 - \varphi_n + \delta \xi_n)
\times \|y_n - p\| + (\varphi_n - \xi_n) \|T y_n - p\|
\leq \|y_{n+1} - p\| + [1 - \varphi_n + \delta \xi_n]
\times \|y_n - p\| + (\varphi_n - \xi_n) \|T y_n - p\|
\leq \|y_{n+1} - p\| + (1 - \varphi_n + \xi_n)
\times \|y_n - p\| + (\varphi_n - \xi_n) \|T y_n - p\|
\leq \|y_{n+1} - p\| + [1 - \varphi_n + \xi_n] \|y_n - p\|.
\]
(23)

Since \( \lim_{n \to \infty} \|y_n - p\| = 0 \), it follows that \( \lim_{n \to \infty} \epsilon_n = 0 \).

Therefore the iteration scheme is \( T \) stable.

Example 10 (see [24]). Let \( T : [0, 1] \to [0, 1] \), \( Tx = x/2 \), \( \varphi_n, \xi_n, \delta_n = 0, n = 1, 2, \ldots, 15 \), and \( \xi_n = 1/2, 2/\sqrt{n}, \varphi_n = 1/2, 1/2, 2/\sqrt{n} \), \( \delta_n = 0, 4/\sqrt{n} \), for all \( n \geq 16 \). It is easy to show that \( T \) is a weak contraction operator satisfying (I) with a unique fixed point 0. Furthermore, for all \( n \geq 16 \), \( 4/\sqrt{n}, 1/2 - 2/\sqrt{n}, 1/2 + 2/\sqrt{n} \in [0, 1] \), and \( \sum_{i=0}^{15} (1 + 2/\sqrt{n}) = \infty \). Then the new iterative process is faster than the Ishikawa iterative process. Assume that \( u_0 = w_0 \neq 0 \) is initial point for

the new and Ishikawa iterative processes, respectively. Firstly, we consider the new iterative process, and we have

\[
u_{n+1} = (1 - \varphi_n) u_n + (\varphi_n - \xi_n) T ((1 - \zeta_n) u_n + \zeta_n T w_n)
\]

\[
+ \zeta_n T w_n
\]

\[
= (1 - (1/2 + 2/\sqrt{n})) u_n
\]

\[
+ ((1/2 + 2/\sqrt{n}) - (1 - 2/\sqrt{n}))/2)\]
\[
\times T ((1 - 4/\sqrt{n}) u_n + 4/\sqrt{n} T w_n)
\]

\[
+ (1 - 2/\sqrt{n}) T w_n
\]

\[
= (1 - (1/2 + 2/\sqrt{n})) u_n
\]

\[
+ ((1/2 + 2/\sqrt{n}) - (1 - 2/\sqrt{n}))\]
\[
\times (1/2) ((1 - 4/\sqrt{n}) u_n + 4/\sqrt{n} T w_n)
\]

\[
+ (1 - 2/\sqrt{n}) 1/2 u_n
\]

\[
= (1 - 2/\sqrt{n}) u_n
\]

\[
+ 2/\sqrt{n} (1 - 2/\sqrt{n}) u_n + (1 - 1/\sqrt{n}) u_n
\]

\[
= (1 - 2/\sqrt{n} + 2/\sqrt{n} - 4/\sqrt{n} + 4/\sqrt{n} - 1/\sqrt{n}) u_n
\]

\[
= (1 - 1/\sqrt{n} - 4/\sqrt{n} - 1/4) u_n
\]
\[
= \prod_{i=1}^{16} (1 - 1/\sqrt{i} - 4/\sqrt{i} - 1/4) u_0.
\]
(24)

Secondly, we consider the Ishikawa iterative process, and we have

\[
\omega_{n+1} = (1 - \xi_n) \omega_n + \zeta_n T ((1 - \delta_n) \omega_n + \delta_n T w_n)
\]

\[
= (1 - 4/\sqrt{n}) \omega_n + 4/\sqrt{n} T ((1 - 4/\sqrt{n}) \omega_n + 4/\sqrt{n} T w_n)
\]

\[
= (1 - 4/\sqrt{n}) \omega_n + 4/\sqrt{n} 2 ((1 - 4/\sqrt{n}) \omega_n + 4/\sqrt{n} T w_n)
\]

\[
= (1 - 4/\sqrt{n}) \omega_n + 4/\sqrt{n} ((1 - 4/\sqrt{n}) \omega_n + 2/\sqrt{n} w_n)
\]
Figure 1: It shows the value functions found by successive steps of the Ishikawa and new iteration methods.

\[
\left(1 - \frac{4}{\sqrt{n}}\right) w_n + \frac{2}{\sqrt{n}} \left(1 - \frac{2}{\sqrt{n}}\right) w_n
\]

\[
= \left(1 - \frac{4}{\sqrt{n}} + \frac{2}{\sqrt{n}} - \frac{4}{n}\right) w_n
\]

\[
= \frac{n}{\prod_{i=16}^{n} \left(1 - \frac{2}{\sqrt{i}} - \frac{4}{i}\right)} w_0.
\]

(25)

Now, taking the above two equalities, we obtain

\[
\left| \frac{u_{n+1} - 0}{w_{n+1} - 0} \right| = \left| \prod_{i=16}^{n} \left(1 - \frac{1}{\sqrt{i}} - \frac{4}{i} - \frac{1}{4}\right) u_0 \right|
\]

\[
= \frac{n}{\prod_{i=16}^{n} \left(1 - \frac{2}{\sqrt{i}} - \frac{4}{i}\right)}
\]

\[
= \frac{n}{\prod_{i=16}^{n} \left(1 - \frac{1}{\sqrt{i}} + \frac{1}{4}\right) \left(1 - \frac{2}{\sqrt{i}} - \frac{4}{i}\right)}
\]

\[
= \frac{n}{\prod_{i=16}^{n} \left(1 - \frac{2}{\sqrt{i}} - \frac{4}{i}\right)}
\]

(26)

It is clear that

\[
0 \leq \lim_{n \to \infty} \prod_{i=16}^{n} \left(1 - \frac{i - 4\sqrt{i}}{4i - 8\sqrt{i} - 16}\right) = 0.
\]

(27)

Therefore, the proof is completed.

Now, we can give Table 1 and Figures 1 and 2 to support and reinforce our claim in the Example 10.

Finally, we check that this iteration procedure can be applied to find the solution of delay differential equations.

2.1. An Application. Throughout the rest of this paper, the space \( C[a, b]\) equipped with Chebyshev norm \( \|x - y\|_\infty = \max_{t \in [a, b]} |x(t) - y(t)| \) denotes the space of all continuous functions. It is well known that \( C[a, b]\) is a real Banach space with respect to \( \|\cdot\|_\infty \) norm; more details can be found in [2, 27].

Now, we will consider a delay differential equation such that

\[
\frac{dx}{dt} = g(t, x(t), x(t - \varsigma)), \quad t \in [t_0, b]
\]

(28)

and an assumed solution

\[
x(t) = \phi(t), \quad t \in [t_0 - \varsigma, t_0].
\]

(29)

Assume that the following conditions are satisfied:

\( C_1 \) \( t_0, b \in \mathbb{R}, \varsigma \geq 0, \)

\( C_2 \) \( g \in C(t_0, b) \times \mathbb{R}^2, \mathbb{R}, \)

\( C_3 \) \( \phi \in C(t_0 - \varsigma, t_0], \mathbb{R}, \)

\( C_4 \) there exists the following inequality:

\[
\|g(t, y_1, y_2) - g(t, \lambda_1, \lambda_2)\| \leq K_g \|y_1 - \lambda_1\| + \|y_2 - \lambda_2\| + L\|y_1 - T\|, \]

(30)
for all \( y_i, \lambda_i \in \mathbb{R} \) (\( i = 1, 2 \)) and \( t \in [t_0, b] \) such that \( K_g > 0 \). (C_5) \( 2K_g(b - t_0) < 1 \), and according to a solution of problem (28)-(29) we infer the function \( x \in C([t_0 - \varsigma, b], \mathbb{R}) \) \( \cap C^1([t_0, b], \mathbb{R}) \). The problem can be reconstituted as follows:

\[
(C_6)
\]

\[
x(t) = \begin{cases} 
\varphi(t), & \text{if } t \in [t_0 - \varsigma, t_0], \\
\varphi(t_0) + \int_{t_0}^{t} g(t, x(s), x(s - \varsigma)) ds, & \text{if } t \in [t_0, b].
\end{cases}
\]

(31)

Also, the map \( T : C([t_0 - \varsigma, b], \mathbb{R}) \rightarrow C([t_0 - \varsigma, b], \mathbb{R}) \) is defined by the following form:

\[
T(x)(t) = \begin{cases} 
\varphi(t), & \text{if } t \in [t_0 - \varsigma, t_0], \\
\varphi(t_0) + \int_{t_0}^{t} g(t, x(s), x(s - \varsigma)) ds, & \text{if } t \in [t_0, b].
\end{cases}
\]

(32)

Using weak-contraction mapping, we obtain the following.

**Theorem 11.** We suppose that conditions (C_1)-(C_6) are performed. Then the problem (28)-(29) has a unique solution in \( C([t_0 - \varsigma, b], \mathbb{R}) \) \( \cap C^1([t_0, b], \mathbb{R}) \).

**Proof.** We consider iterative process (3) for the mapping \( T \). The fixed point of \( T \) is shown via \( p \) such that \( Tp = p \).

For the first part, that is, for \( t \in [t_0 - \varsigma, t_0] \), it is clear that \( \lim_{n \to \infty} x_n = p \). Therefore, letting \( t \in [t_0, b] \), we obtain

\[
\|x_{n+1} - p\|_\infty
= \|(1 - \varphi_n) x_n + \xi_n T x_n + (\varphi_n - \xi_n) T y_n - p\|_\infty
\leq (1 - \varphi_n) \|x_n - p\|_\infty + \xi_n \|T x_n - T y_n\|_\infty
+ (\varphi_n - \xi_n) \|T y_n - T p\|_\infty
\leq (1 - \varphi_n) \|x_n - p\|_\infty
+ \xi_n \|T x_n - T p\|_\infty
\]

+ \xi_n \|T y_n - T p\|_\infty
\leq (1 - \varphi_n) \|x_n - p\|_\infty
+ \max_{t \in [t_0 - \varsigma, b]} |T x_n(t) - T p(t)|
+ (\varphi_n - \xi_n) \max_{t \in [t_0 - \varsigma, b]} |T y_n(t) - T p(t)|
\leq (1 - \varphi_n) \|x_n - p\|_\infty
+ \max_{t \in [t_0 - \varsigma, b]} \left| \int_{t_0}^{t} g(t, x_n(s), x_n(s - \varsigma)) ds \right|
- \varphi(t_0) - \int_{t_0}^{t} g(t, p(s), p(s - \varsigma)) ds
+ (\varphi_n - \xi_n) \max_{t \in [t_0 - \varsigma, b]} \left| \int_{t_0}^{t} g(t, y_n(s), y_n(s - \varsigma)) ds \right|
- \varphi(t_0) - \int_{t_0}^{t} g(t, p(s), p(s - \varsigma)) ds
= (1 - \varphi_n) \|x_n - p\|_\infty
+ \max_{t \in [t_0 - \varsigma, b]} \left| \int_{t_0}^{t} g(t, x_n(s), x_n(s - \varsigma)) ds \right|
- \int_{t_0}^{t} g(t, p(s), p(s - \varsigma)) ds
+ (\varphi_n - \xi_n) \max_{t \in [t_0 - \varsigma, b]} \left| \int_{t_0}^{t} g(t, y_n(s), y_n(s - \varsigma)) ds \right|
- \int_{t_0}^{t} g(t, p(s), p(s - \varsigma)) ds
\leq (1 - \varphi_n) \|x_n - p\|_\infty
+ \xi_n \max_{t \in [t_0 - \varsigma, b]} \left| \int_{t_0}^{t} g(t, x_n(s), x_n(s - \varsigma)) ds \right|
- \int_{t_0}^{t} g(t, p(s), p(s - \varsigma)) ds
}\]

\[
\times \max_{t \in [t_0 - \varsigma, b]} \left| \int_{t_0}^{t} g(t, x_n(s), x_n(s - \varsigma)) ds \right|
- g(t, p(s), p(s - \varsigma)) ds
\]
\[
+ (\varphi_n - \xi_n) \max_{t \in [t_0, b]} \int_{t_0}^{t}  \left| x_n(s) - p(s) \right| \ ds \\
+ |x_n(t) - p(t)| \right| \\
+ L |y_n(s) - T x_n(s)| \ ds \\
+ (\varphi_n - \xi_n) \left( b - t_0 \right) \max_{t \in [t_0, b]} \left| x_n(t) - p(t) \right| \\
+ \xi_n \left( b - t_0 \right) \left| y_n(t) - T x_n(t) \right| \\
+ (\varphi_n - \xi_n) K_g (b - t_0) \\
\times \max_{t \in [t_0, b]} \left| y_n(t) - T y_n(t) \right| \\
= (1 - \varphi_n) \left| x_n - p \right|_{\infty} + 2 \xi_n K_g (b - t_0) \\
\times \max_{t \in [t_0, b]} \left| x_n(t) - p(t) \right| \\
+ 2 (\varphi_n - \xi_n) K_g (b - t_0) \\
\times \max_{t \in [t_0, b]} \left| y_n(t) - p(t) \right| \\
= (1 - \varphi_n) \left| x_n - p \right|_{\infty} + 2 \xi_n K_g (b - t_0) \left| x_n - p \right|_{\infty} \\
+ 2 (\varphi_n - \xi_n) K_g (b - t_0) \left| y_n - p \right|_{\infty} \\
= (1 - \varphi_n) \left| x_n - p \right|_{\infty} + 2 \xi_n K_g (b - t_0) \left| x_n - p \right|_{\infty} \\
+ 2 (\varphi_n - \xi_n) K_g (b - t_0) \left| y_n - p \right|_{\infty}.
\]

Hence, we obtain

\[\| x_{n+1} - p \|_{\infty} \leq (1 - \varphi_n) \| x_n - p \|_{\infty} + 2 \xi_n K_g (b - t_0) \left| x_n - p \right|_{\infty} + 2 (\varphi_n - \xi_n) K_g (b - t_0) \left| y_n - p \right|_{\infty} \}

By continuing this way, we have

\[\| y_n - p \|_{\infty} \]

\[= (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[\leq (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[\leq (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[= (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[= (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[\leq (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[= (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[\leq (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[\leq (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[\leq (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[\leq (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[\leq (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[\leq (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[\leq (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[\leq (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[\leq (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[\leq (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]

\[\leq (1 - \zeta_n) \| x_n - p \|_{\infty} + \zeta_n T x_n - p \|_{\infty} \]
\[ = (1 - \xi_n) \| x_n - p \|_\infty + \xi_n 2K_g (b - t_0) \| x_n - p \|_\infty \]
\[ = (1 - \xi_n (1 - 2K_g (b - t_0))) \| x_n - p \|_\infty. \quad (36) \]

Substituting (36) into (34), we obtain
\[ \| x_{n+1} - p \|_\infty \leq (1 - \phi_n + 2\xi_n K_g (b - t_0)) \| x_n - p \|_\infty \]
\[ + (\phi_n - \xi_n) 2K_g (b - t_0) \times (1 - \xi_n (1 - 2K_g (b - t_0))) \| x_n - p \|_\infty \]
\[ = (1 - \phi_n + 2\xi_n K_g (b - t_0)) \| x_n - p \|_\infty \]
\[ + (\phi_n - \xi_n) 2K_g (b - t_0) \times (1 - \xi_n (1 - 2K_g (b - t_0))) \| x_n - p \|_\infty. \quad (37) \]

Since \( (1 - 2K_g (b - t_0)) < 1 \), we have
\[ \| x_{n+1} - p \|_\infty \leq (1 - \xi_n (1 - 2K_g (b - t_0))) \| x_n - p \|_\infty. \quad (38) \]

We take \( \xi_n (1 - 2K_g (b - t_0)) = \mu_n < 1 \) and \( \| x_n - p \|_\infty = s_n \), and then the conditions of Lemma 3 immediately imply
\[ \lim_{n \to \infty} \| x_n - p \|_\infty = 0. \]

\section*{Conflict of Interests}

The authors declare that there is no conflict of interests regarding the publication of this paper.

\section*{Acknowledgment}

The authors would like to thank Yıldız Technical University Scientific Research Projects Coordination Department under Project no. BAPK 2014-07-03-DOP02 for financial support during the preparation of this paper.

\section*{References}

[1] V. Berinde, "On the convergence of Ishikawa in the class of quasi contractive operators," \textit{Acta Mathematica Universitatis Comenianae}, vol. 73, no. 1, pp. 119–126, 2004.

[2] S. M. Soltuz and D. Otrocol, "Classical results via Mann-Ishikawa iteration," \textit{Revue d'Analyse Numerique et de l'Approximation}, vol. 36, no. 2, 2007.

[3] F. Gürsoy and V. Karakaya, "A Picard-S hybrid type iteration method for solving a differential equation with retarded argument," In press, http://arxiv-web3.library.cornell.edu/abs/1403.2546v1.

[4] A. M. Ostrowski, "The round-off stability of iterations," \textit{Journal of Applied Mathematics and Mechanics}, vol. 47, pp. 77–81, 1967.

[5] A. M. Harder and T. L. Hicks, "Stability results for fixed point iteration procedures," \textit{Mathematica Japonica}, vol. 33, no. 5, pp. 693–706, 1988.

[6] A. M. Harder and T. L. Hicks, "A stable iteration procedure for nonexpansive mappings," \textit{Mathematica Japonica}, vol. 33, no. 5, pp. 687–692, 1988.

[7] B. E. Rhoades, "Fixed point theorems and stability results for fixed point iteration procedures," \textit{Indian Journal of Pure and Applied Mathematics}, vol. 21, no. 1, pp. 1–9, 1990.

[8] B. E. Rhoades, "Fixed point theorems and stability results for fixed point iteration procedures. II," \textit{Indian Journal of Pure and Applied Mathematics}, vol. 24, no. 11, pp. 691–703, 1993.

[9] M. O. Osilike, "Stability results for fixed point iteration procedures," \textit{Journal of the Nigerian Mathematics Society}, vol. 26, no. 10, pp. 937–945, 1995.

[10] M. O. Osilike and A. Udomene, "Short proofs of stability results for fixed point iteration procedures for a class of contractive-type mappings," \textit{Indian Journal of Pure and Applied Mathematics}, vol. 30, no. 12, pp. 1229–1234, 1999.

[11] S. L. Singh and B. Prasad, "Some coincidence theorems and stability of iterative procedures," \textit{Computers & Mathematics with Applications}, vol. 55, no. 11, pp. 2512–2520, 2008.

[12] V. Berinde, \textit{Iterative Approximation of Fixed Points}, Editura Efemeride, 2002.

[13] M. O. Olatinwo, "Some stability and strong convergence results for the Jungck-Ishikawa iteration process," \textit{Creative Mathematics and Informatics}, vol. 17, pp. 33–42, 2008.

[14] C. O. Imoru and M. O. Olatinwo, "On the stability of Picard and Mann iteration processes," \textit{Carpathian Journal of Mathematics}, vol. 19, no. 2, pp. 155–160, 2003.

[15] V. Karakaya, K. Doğan, F. Gürsoy, and M. Ertürk, "Fixed point of a new three-step iteration algorithm under contractive-like operators over normed spaces," \textit{Abstract and Applied Analysis}, vol. 2013, Article ID 560258, 9 pages, 2013.

[16] S. Ishikawa, "Fixed points by a new iteration method;" \textit{Proceedings of the American Mathematical Society}, vol. 44, pp. 147–150, 1974.

[17] W. R. Mann, "Mean value methods in iteration;" \textit{Proceedings of the American Mathematical Society}, vol. 4, pp. 506–510, 1953.

[18] V. Berinde, \textit{Iterative Approximation of Fixed Points}, vol. 1912, Springer, Berlin, Germany, 2007.

[19] R. Kannan, "Some results on fixed points," \textit{Bulletin of the Calcutta Mathematical Society}, vol. 10, pp. 71–76, 1968.

[20] S. K. Chatterjea, "Fixed-point theorems;" \textit{Comptes rendus de l’Academie Bulgar de Sciences}, vol. 25, pp. 727–730, 1972.

[21] T. Zamfirescu, "Fixed point theorems in metric spaces," \textit{Archiv der Mathematik}, vol. 23, pp. 292–298, 1972.

[22] V. Berinde, "Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators," \textit{Fixed Point Theory and Applications}, vol. 2004, Article ID 716359, 2004.

[23] G. V. R. Babu and K. N. V. Vara Prasad, "Mann iteration converges faster than Ishikawa iteration for the class of Zamfirescu operators," \textit{Fixed Point Theory and Applications}, vol. 2006, Article ID 49615, 6 pages, 2006.

[24] Y. Qiu and B. E. Rhoades, "Comments on the rate of convergence between Mann and Ishikawa iterations applied to Zamfirescu operators," \textit{Fixed Point Theory and Applications}, vol. 2008, Article ID 387504, 8 pages, 2008.

[25] Z. Xue, "The comparison of the convergence speed between Picard, Mann, Krasnoselskij and Ishikawa iterations in Banach spaces," \textit{Fixed Point Theory and Applications}, vol. 2008, Article ID 387056, 5 pages, 2008.

[26] B. E. Rhoades and Z. Xue, "Comparison of the rate of convergence among Picard, Mann, Ishikawa, and Noor iterations applied to quasicontractive maps," \textit{Fixed Point Theory and Applications}, vol. 2010, Article ID 169062, 12 pages, 2010.

[27] G. Hammerlin and K. H. Hoffmann, \textit{Numerical Mathematics}, Springer, New York, NY, USA, 1991.