Vector Models in $\mathcal{PT}$ Quantum Mechanics

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Abstract We present two examples of non-Hermitian Hamiltonians which consist of an unperturbed part plus a perturbation that behaves like a vector, in the framework of $\mathcal{PT}$ quantum mechanics. The first example is a generalization of the recent work by Bender and Kalveks, wherein the E2 algebra was examined; here we consider the E3 algebra representing a particle on a sphere, and identify the critical value of coupling constant which marks the transition from real to imaginary eigenvalues. Next we analyze a model with SO(3) symmetry, and in the process extend the application of the Wigner-Eckart theorem to a non-Hermitian setting.

Keywords Non-Hermitian quantum mechanics · $\mathcal{PT}$ quantum mechanics · Wigner-Eckhart theorem

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1 Introduction

There are many situations in quantum mechanics wherein the Hamiltonian under consideration can be written as

$$H = H_0 + H_1$$  \hspace{1cm} (1)
where $H_0$ is the unperturbed part and commutes with the generators $T_i$ of symmetry group $G$:

$$[H_0, T_i] = 0$$ (2)

and $H_1$ can be treated like a perturbation and behaves like a vector under $G$.

We wish to generalize this situation in the context of $\mathcal{PT}$ quantum mechanics \cite{1,2}, where the assumption that operators such as the Hamiltonian are Hermitian is relaxed, and replaced by other requirements, notably that the Hamiltonian commutes with the parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) operators.

Interest in non-Hermitian quantum mechanics continues to grow \cite{3}, and recently a number of experiments have observed the so-called $\mathcal{PT}$ phase transition, where the eigenvalues of a $\mathcal{PT}$ Hamiltonian make a transition from being complex to real once a critical value of a coupling constant is reached \cite{4}, \cite{5}, \cite{6}. Thus it is relevant to seek new $\mathcal{PT}$-counterparts to conventional Hamiltonians.

In this work we present two simple cases that can be described as non-Hermitian vector perturbation models where the Hamiltonian can be written as in eq (1); first we consider a particle confined to the surface of a sphere, where the Hamiltonian acts within an infinite dimensional Hilbert space, and next we consider a generic vector perturbation within a finite dimensional Hilbert space and determine the spectrum of eigenvalues using the Wigner-Eckart theorem. We find that for a range of parameters each of these models has a pure real spectrum. At critical values of the coupling the model undergoes $\mathcal{PT}$ transitions wherein the eigenvalues become complex.

### 2 E3 Algebra: particle on a sphere

We begin by generalizing the analysis presented in \cite{7}. They considered the E2 algebra which consists of elements $J, u, v$ such that

$$[J, u] = -iv \quad [J, v] = iu \quad [u, v] = 0. \quad (3)$$

The Hamiltonian

$$h = J^2 + igu$$ (4)

where $J = -i\partial/\partial\theta$, $u = \sin\theta$, $v = \cos\theta$ and $g$ is a constant, represents a 2-dimensional quantum particle restricted to radius $r = 1$.

A generalization of this is the E3 algebra and restricting the particle to the surface of a sphere ($r = 1$). This is described by the Hamiltonian

$$h = L^2 + igu_z$$ (5)

where $L$ obeys

$$[L_i, L_j] = i\epsilon_{ijk}L_k \quad (6)$$
$u$ is a vector operator whose components are given by

\begin{align*}
  u_x &= \sin \theta \cos \phi \\
  u_y &= \sin \theta \sin \phi \\
  u_z &= \cos \theta
\end{align*}

and $g$ is a constant. The remaining commutators are straightforward to calculate;

\[ [L_i, u_j] = i\epsilon_{ijk} u_k \quad [u_i, u_j] = 0. \] (10)

Following Bender and Kalveks we consider the case of even time reversal: for a wave function $\psi(\theta, \phi)$ the time reversal operator $T$ is manifested as complex conjugation:

\[ T\psi(\theta, \phi) = \psi^*(\theta, \phi) \] (11)

hence $T^2 = 1$. It is easy to verify the action of $T$ on the elements of the algebra: $T L_i T = -L_i$ and $T u_i T = u_i$. The parity operator $P$ takes $\psi$ into the antipodal point:

\[ P\psi(\theta, \phi) = \psi(\pi - \theta, \phi + \pi) \] (12)

so $P^2 = 1$; elements transform under parity as $P L_i P = L_i$ and $P u_i P = -u_i$. Note that the Hamiltonian $h$ in eq (5) commutes with the combined operation $\mathcal{PT}$ but not with $P$ or $T$ individually. Now let us determine the eigenvalue spectrum of this Hamiltonian. We wish to solve

\[ h\psi(\theta, \phi) = \lambda\psi(\theta, \phi) \] (13)

and we try the general solution:

\[ \psi(\theta, \phi) = f(\theta)e^{im\phi}. \] (14)

For convenience we define $\eta = \cos \theta$; this simplifies the eigenvalue equation for $f$:

\[-\left(1 - \eta^2\right) \frac{\partial^2 f}{\partial \eta^2} + 2\eta \frac{\partial f}{\partial \eta} + \frac{m^2}{1 - \eta^2} f + ig\eta f = \lambda f \] (15)

where $m$ is a fixed integer. If we let

\[ h_0 = -(1 - \eta^2) \frac{\partial^2 f}{\partial \eta^2} + 2\eta \frac{\partial f}{\partial \eta} + \frac{m^2}{1 - \eta^2} f \] (16)

then the Hamiltonian we wish to solve is

\[ h_0 f + ig\eta f = \lambda f. \] (17)

We impose the boundary condition that the solution must be regular at $\eta = \pm 1$.

Let us choose basis elements

\[ |l\rangle \rightarrow N_l P_l |m\rangle(\eta) \] (18)
The first six eigenvalues for $m = 0$ (blue) and $m = 1$ (green) are shown. Intercepts on the $E$ axis are given by $\ell(\ell + 1)$ for $\ell = 0$ to 6. For the case of $m = 0$, we find that the spectrum is entirely real for $0 \leq g < 1.899$ at which point there is a transition to one pair of complex conjugate eigenvalues in the spectrum. At $g = 11.45$ there is a second transition, to two pairs of complex conjugate eigenvalues. Similarly for the case of $m = 1$, we find one complex conjugate eigenvalue pair at $g = 5.41$, and two pairs at $g = 19.04$. In these computations the Hamiltonian is truncated to a $100 \times 100$ matrix; we have verified that the relevant part of the spectrum is insensitive to the truncation.

where $l = |m|,|m| + 1, \ldots$, $P_{l,|m|}$ are the associated Legendre polynomials, with conventional normalization factor

$$N_l = \sqrt{\frac{(2l + 1)}{2}} \sqrt{\frac{(l - |m|)!}{(l + |m|)!}}. \quad (19)$$

The $P_{l,m}$'s satisfy

$$h_0 P_{l,|m|}(\eta) = l(l + 1)P_{l,|m|}(\eta) \quad (20)$$

so the matrix of $h_0$ in this basis is diagonal. The matrix elements of the potential term, $i\eta g$, can easily be determined from the normalization and recursion relations of the $P_{l,m}$'s. By diagonalizing the truncated Hamiltonian matrix we can numerically obtain the eigenvalues of eq (17); see Fig. (1).

3 $\mathcal{PT}$ Vector Model in Finite-Dimensional Hilbert Space

$E^3$ may also be regarded as a realization of the $\mathcal{PT}$ vector model with symmetry group $SO(3)$ and for which the Hilbert space is infinite dimensional. Now we wish to turn out attention to realizations of the $\mathcal{PT}$ vector model with finite dimensional Hilbert spaces. Let us write a simple, generic Hamiltonian $H = H_0 + H_I$ where

$$H_0 = L_x^2 + L_y^2 + L_z^2 \quad (21)$$

and $V_z$ is the $z$ component of a vector operator.

Our task is to obtain a matrix representation of the total Hamiltonian, solve for its eigenvalues and determine what value of the non-Hermitian perturbation cause the eigenvalues to become complex.
Naturally we choose to work with the angular momentum eigenstates $|\ell, m\rangle$; the action of $H_0$ on these states is well known, and we can utilize the Wigner-Eckart theorem to determine the action of $H_I = V_z$.

Note that the dimensionality of the relevant vector space depends on the angular momenta of the multiplets but clearly it is finite. Suppose we consider the two multiplets $|\ell, m\rangle$ and $|\ell + 1, m\rangle$; $m$ takes on values from $-\ell$ to $+\ell$ in the first multiplet and from $-\ell - 1$ to $\ell + 1$ in the second multiplet, so there are $(2\ell + 1) + (2\ell + 3) = 4\ell + 4$ of these states.

The action of $H_0$ on these states is simply
\[ L^2|\ell, m\rangle = \ell(\ell + 1)|\ell, m\rangle \] (23)
\[ L^2|\ell + 1, m\rangle = (\ell + 1)(\ell + 2)|\ell + 1, m\rangle. \] (24)

So all that remains is to determine how $V_z$ acts on these states; here we employ the Wigner-Eckart theorem, which we have extended to the non-Hermitian case as detailed in Appendix A. We find $\langle \ell', m'|V_z|\ell, m\rangle = 0$ unless $m = m'$. Thus we need only to determine
\[ \langle \ell, m|V_z|\ell, m\rangle, \]
\[ \langle \ell + 1, m|V_z|\ell + 1, m\rangle, \]
\[ \langle \ell, m|V_z|\ell + 1, m\rangle, \]
\[ \langle \ell + 1, m|V_z|\ell, m\rangle \]
in order to completely specify $V_z$ in this space. The first two in this list can be expressed in terms of the reduced matrix element $\alpha$ defined in Appendix A; in general we find
\[ \langle \ell, m|V_z|\ell, m\rangle = m\alpha_1 \]
\[ \langle \ell + 1, m|V_z|\ell + 1, m\rangle = m\alpha_2; \]

however we also wish to enforce $\mathcal{P}V_z\mathcal{P} = -V_z$ and $\mathcal{T}V_z\mathcal{T} = -V_z$, which restricts $\alpha_1 = \alpha_2 = 0$. (Determination of $\mathcal{P}$ and $\mathcal{T}$ within this space follows straightforwardly from their action on the spherical harmonics $\mathcal{P}Y_{\ell m}(\theta, \phi) = (-1)^\ell Y_{\ell m}(\theta, \phi)$ and $\mathcal{T}Y_{\ell m}(\theta, \phi) = (-1)^m Y_{\ell,-m}(\theta, \phi)$.)

For the other two types of matrix elements, $\langle \ell, m|V_z|\ell + 1, m\rangle$ and $\langle \ell + 1, m|V_z|\ell, m\rangle$, we find these are proportional to other reduced matrix elements $\beta$ and $\gamma$;
\[ \langle \ell + 1, m|V_z|\ell, m\rangle = f_{\ell m}\beta, \]
\[ \langle \ell, m|V_z|\ell + 1, m\rangle = f_{\ell m}\gamma; \]

where
\[ f_{\ell m} = \left[ \frac{(\ell + 1)^2 - m^2}{(2\ell + 1)(2\ell + 2)} \right]^{1/2}. \] (25)
Note that $f_{\ell m}$ is even in $m$. When we enforce $PV_z P = -V_z$ and $TV_z T = -V_z$, we find this requires $\beta$ and $\gamma$ to be pure imaginary, so we define $\beta = ib$, $\gamma = ic$ for some real numbers $b, c$. Note that in determining these matrix elements we do not assume $V$ is Hermitian; we rely only on the commutators of $V$ with the angular momentum operators. (See Appendix A for details.)

Now let us write down the matrix corresponding to the Hamiltonian $H = H_0 + H_I$. Consider the two-dimensional subspace spanned by the states $|\ell, m\rangle$ and $|\ell + 1, m\rangle$ for a fixed value of $m$ that lies in the range $-\ell, \ldots, \ell$. Within this subspace

$$H_0 = \begin{pmatrix} \ell(\ell + 1) & 0 \\ 0 & (\ell + 1)(\ell + 2) \end{pmatrix}$$

(26)

and

$$V_z = \begin{pmatrix} 0 & -icf_{\ell m} \\ -ibf_{\ell m} & 0 \end{pmatrix}.$$  

(27)

In addition consider the two dimensional subspace spanned by the states $|\ell + 1, \ell + 1\rangle$ and $|\ell + 1, -\ell - 1\rangle$. These states are not coupled by the perturbation $V_z$ to any other state and hence $V_z = 0$ within this subspace. On the other hand the unperturbed Hamiltonian in this subspace is given by

$$H_0 = \begin{pmatrix} (\ell + 1)(\ell + 2) & 0 \\ 0 & (\ell + 1)(\ell + 2) \end{pmatrix}.$$  

(28)

It is convenient to define

$$h_{\ell + 1} = \begin{pmatrix} (\ell + 1)(\ell + 2) & 0 \\ 0 & (\ell + 1)(\ell + 2) \end{pmatrix}$$  

(29)

and

$$h_m = \begin{pmatrix} \ell(\ell + 1) & -icf_{\ell m} \\ -ibf_{\ell m} & (\ell + 1)(\ell + 2) \end{pmatrix}.$$  

(30)

where $m = -\ell, \ldots, \ell$. The Hamiltonian can now be written as a block-diagonal matrix

$$\begin{pmatrix} h_{\ell + 1} & h_{\ell} & 0 \\ h_{\ell} & h_{\ell - 1} & 0 \\ \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & h_{-\ell} \end{pmatrix}.$$  

(31)

The individual $2 \times 2$ matrices that constitute the Hamiltonian are simple enough that we can obtain analytic expressions for the eigenvalues. The eigenvalues of $h_{\ell + 1}$ are two-fold degenerate and are simply $(\ell + 1)(\ell + 2)$. The eigenvalues of $h_m$ are

$$\lambda_{\ell m}^{\pm} = (\ell + 1)^2 \pm \sqrt{(\ell + 1)^2 - bcf_{\ell m}^2}.$$  

(32)
Fig. 2 Real and imaginary parts of the eigenvalues $\lambda_{\ell,m}$ assuming $b = c$. The blue line at $\lambda = 6$ corresponds to the $2 \times 2$ matrix denoted $h_{\ell+1}$ in the text, with eigenvalues $(\ell+1)(\ell+2)$. The other eigenvalues are $m$-dependent and correspond to the $2 \times 2$ matrices denoted $h_m$ in the text, with eigenvalues given by eq(32). As noted $\lambda_{\ell,m} = \lambda_{\ell,-m}$, so there are only two distinct $m$-dependent curves for $\ell = 1$. In each figure, $|m| = 1$ is plotted in red and $m = 0$ is plotted in green. Note that the transition to complex eigenvalues occurs at $b = 4$ for $\lambda_{\ell,1}^\pm$ and $b = \sqrt{12} \approx 3.47$ for $\lambda_{1,0}^\pm$.

Note that $\lambda_{\ell,m} = \lambda_{\ell,-m}$ so for all $m \neq 0$ the eigenvalues of $h_m$ are also two-fold degenerate. Clearly, the eigenvalues are real provided

$$bc < \frac{(\ell + 1)^2}{f_{\ell m}^2}.$$  \hspace{1cm} (33)

We can make the following observations about the behavior of the eigenvalues. Once $\mathcal{PT}$ symmetry is broken, $\lambda^+$ and $\lambda^-$ form a complex conjugate pair. Since $f_{\ell m}$ has its maximum value for $m = 0$, $\lambda^\pm$ becomes complex for $m = 0$ first. Similarly, $f_{\ell m}$ is minimal for $|m| = \ell$, so $\lambda^\pm$ so these are the last eigenvalues to go complex. For example we consider the case $\ell = 1$ and choose $b = c$ for simplicity. We plot the eigenvalues in Figure 2.

It is worth noting that in the Hermitian case $b = -c$. Hence the condition in eq (33) that ensures the eigenvalues are real is always met.

4 Conclusion

We conclude by noting two natural generalizations of our results that deserve further investigation. First the model of a particle on an ordinary 2-sphere considered in section II may be generalized to a particle on a sphere in $n$ dimensions. The $\mathcal{PT}$ transition for this model may be amenable to analytic study in the large $n$ limit and may shed some light on $\mathcal{PT}$ symmetric non-linear sigma models of which it would represent a $0 + 1$ dimensional case [8]. Second the vector model constructed in section III may be easily generalized from the symmetry group SO(3) to any Lie group and therefore represents only one member of a large class of such models.
Appendix A: Wigner-Eckart Theorem

Suppose we have an angular momentum operator $\mathbf{L}$ and a vector operator $\mathbf{V}$ satisfying the commutation relations

$$[L_i, V_j] = i\epsilon_{ijk}V_k.$$  (34)

Let $|\ell, m\rangle$ denote an angular momentum multiplet of total angular momentum $\ell$ and $z$-component $m$. Then according to the Wigner-Eckart theorem the matrix elements of $V_z$ and $V_\pm = V_x \pm iV_y$ between multiplet states are determined by the commutation relations eq (34). In the usual Wigner-Eckart theorem the Cartesian components of the operator $\mathbf{V}$ are assumed to be hermitian. Here we present a non-Hermitian generalization of the theorem.

Following the usual arguments we find the selection rules

$$\langle \ell', m' | V_z | \ell, m \rangle = 0 \quad \text{unless} \quad m' = m \quad (35)$$

$$\langle \ell', m' | V_+ | \ell, m \rangle = 0 \quad \text{unless} \quad m' = m + 1 \quad (36)$$

$$\langle \ell', m' | V_- | \ell, m \rangle = 0 \quad \text{unless} \quad m' = m - 1. \quad (37)$$

Furthermore the matrix elements vanish unless $\ell' = \ell - 1$ or $\ell' = \ell$ or $\ell' = \ell + 1$.

Consider the case $\ell' = \ell$. Generalization of the usual arguments shows that

$$\langle \ell, m + 1 | V_+ | \ell, m \rangle = A (\ell - m)^{1/2} (\ell + m + 1)^{1/2} \quad m = -\ell, \ldots, \ell - 1$$

$$\langle \ell, m | V_z | \ell, m \rangle = Am \quad m = -\ell, \ldots, \ell$$

$$\langle \ell, m - 1 | V_- | \ell, m \rangle = A (\ell - m + 1)^{1/2} (\ell + m)^{1/2} \quad m = -\ell + 1, \ldots, \ell \quad (38)$$

where the proportionality constant $A$ is a complex number called the “reduced matrix element”. Note that for $\mathbf{V}$ hermitian, $A$ would have to be real, but there is no such restriction in the non-hermitian case.

Similarly in the case $\ell' = \ell + 1$ we find

$$\langle \ell + 1, m + 1 | V_+ | \ell, m \rangle = B \left[ \frac{(\ell + m + 2)(\ell + m + 1)}{(2\ell + 2)(2\ell + 1)} \right]^{1/2}$$

$$\langle \ell + 1, m | V_z | \ell, m \rangle = -B \left[ \frac{(\ell - m + 1)(\ell + m + 1)}{(2\ell + 2)(2\ell + 1)} \right]^{1/2}$$

$$\langle \ell + 1, m - 1 | V_- | \ell, m \rangle = -B \left[ \frac{(\ell - m + 1)(\ell - m + 2)}{(2\ell + 2)(2\ell + 1)} \right]^{1/2}, \quad (39)$$

where $m = -\ell, \ldots, \ell$ and $B$ is another complex reduced matrix element.
Finally in the case that $\ell' = \ell - 1$ we find

$$
\langle \ell - 1, m + 1 | V_+ | \ell, m \rangle = -C \left[ \frac{(\ell - m - 1)(\ell - m)}{(2\ell)(2\ell - 1)} \right]^{1/2},
$$

$$
\langle \ell - 1, m | V_z | \ell, m \rangle = -C \left[ \frac{(\ell - m)(\ell + m)}{(2\ell)(2\ell - 1)} \right]^{1/2},
$$

$$
\langle \ell - 1, m - 1 | V_- | \ell, m \rangle = C \left[ \frac{(\ell + m)(\ell + m - 1)}{(2\ell)(2\ell - 1)} \right]^{1/2},
$$

where $C$ is a complex reduced matrix element and $m = -\ell, \ldots, \ell - 2$ in the first line of eq (40), $m = -\ell + 1, \ldots, \ell - 1$ in the second line of eq (40), and $m = -\ell + 2, \ldots, \ell$ in the last line of eq (40).

In the hermitian case the reduced matrix elements satisfy $B = C^*$ but in the non-hermitian case there is no such restriction on the complex elements $B$ and $C$.

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