ANALYTICAL MODEL OF CYCLIC HEAT EXCHANGE OF THE PLATE OF FINITE SIZES ADJUSTED FOR THE THERMAL RELAXATION

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Abstract. The hyperbolic boundary value problem of heat conduction in a two-dimensional rectangular plate with the third kind boundary conditions was formulated. The model of transient thermal processes in the body takes into account changes in time and along the flow direction of the ambient temperature. An analytical solution was obtained for the temperature field in the plate, adjusted for the phenomena of thermal relaxation and thermal damping.

1. Introduction
Regenerative air pre heaters (RAPH) are intended for utilization of low-potential heat of outgoing gases. Hot and cold flows alternately blow out the same surfaces of the heat transferring body, which is called the nozzle [1, 2]. In the nozzle during the entire heat transfer cycle, non-stationary transient thermal processes occur. Calculating the temperature field of the nozzle at any time is very important problem, since it allows to determine the maximum and minimum temperatures, as well as the areas of their appearance. Knowledge of these temperatures is important in order to avoid the destruction of the nozzle, as well as the condensation of the vapors of certain substances (water, sulfuric acid, etc.) on its surface. The literature presents a sufficiently large number of different models for the calculation of RAPH, the authors of which are: H Hauzen [1], SS Kutateladze [3], VM Datskovski [4], VK Migai et al. [2], In Nusselt [5], Yu.A. Kirsanov [6], etc. A distinctive feature of the developed models is the use for the calculations of the temperature fields in the regenerator nozzle of the classical theory of thermal conductivity based on the Fourier hypothesis:

\[ q = -\lambda \nabla T, \]  

where \( q \) is the heat flux density; \( \lambda \) is the coefficient of thermal conductivity; \( T \) is the temperature. The main disadvantage of equation (1) is the infinite rate of heat propagation, resulting from it [7]. This disadvantage is eliminated by adding to the formula (1) a term, which takes into account the phenomenon of thermal relaxation of the body:

\[ q + \tau_q \frac{\partial q}{\partial \tau} = -\lambda \nabla T, \]  

where \( \tau_q \) is the thermal relaxation time, characterizing the property of thermal inertia or thermal "elasticity" of the body, \( s, \tau \) is the time, s. This hypothesis was proposed by Cattaneo and Vernotte. However, studies have shown that the Cattaneo-Vernotte hypothesis also has drawbacks: along with
the property of thermal inertia, taken into account by thermal relaxation, the thermal conductivity also
has the property of delaying the reaction or damping the temperature to a change in the heat flux.
To take this into account, the Maxwell-Cattaneo-Luikov hypothesis was proposed, which, in addition
to the phenomenon of thermal relaxation, takes into account the phenomenon of thermal damping:

\[ q + \tau_T \frac{\partial q}{\partial \tau} = -\lambda \text{grad}(T + \tau_T \frac{\partial T}{\partial \tau}), \]  

where \( \tau_T \) is the temperature damping time, s.

How experimental and theoretical studies have shown [8], to describe fast transient thermal
processes in a solid (polymethylmethacrylate) it is preferable to use the Maxwell-Cattaneo-Luikov
hypothesis to obtain more accurate results.

A mathematical model of the transient thermal process in the RAPH nozzle in the form of an
infinite cylinder was proposed in [9], adjusted for the Maxwell-Cattaneo-Luikov hypothesis. One of
the assumptions of the model is that the temperatures of the heat carriers are constant throughout the
process. In a real regenerator with a lamellar nozzle, longitudinal flow around the plates with heat
carriers takes place, the temperature of which, due to heat exchange with the nozzle surface, changes
both in time and space.

The aim of the research is the development of an analytical model of thermal processes in the
regenerator nozzle based on the Maxwell-Cattaneo-Luikov hypothesis, taking into account the
thermal dependence of the heat carriers relative to the time and longitudinal coordinate of the
nozzle.

2. The model of cyclic heat transfer of a plate of finite dimensions adjusted for the thermal
relaxation

A nozzle in the form of plates of thickness \( 2h \) and length \( l \) is considered. The heat transfer process is
divided into periods. Cold and hot flows alternately blow the nozzle in the longitudinal direction so
that in the first period it is blown by one flow, and in the next - by another. It is assumed that the heat
transfer coefficients in each period and the thermophysical properties of the nozzle are constant. The
boundary value problem, using equation (3), is presented as follows:

\[ \frac{\partial^2 \theta_j}{\partial t^2} + \frac{\partial \theta_j}{\partial t} = \text{Fo}_{q,j} \left[ \frac{\partial^2 \theta_j}{\partial X^2} \left( \frac{\partial \theta_j}{\partial t} + \kappa \frac{\partial^2 \theta_j}{\partial X^2} \right) + \frac{1}{L^2} \frac{\partial^2 \theta_j}{\partial X^2} \right]; \]  

\[ \theta_j(X,Y,0) = \theta_{j-1}(X,Y,t_{p,j-1}); \]  

\[ \text{Fo}_{q,j} \int_0^{t_{p,j}} \int_0^l \left[ \theta_{f,j}(Y,t) - \theta_j(Y,t) \right] dY + \frac{\text{Bi}_{y,0,j}}{L^2} \int_0^l \theta_{f,j}(Y,t) dY \]  

\[ + \frac{\text{Bi}_{y,1,j}}{L^2} \left[ \theta_{f,j}(1,t) - \theta_j(1,t) \right] dX = \int_0^1 \int_0^1 \left[ \theta_j(X,Y,t) - \theta_j(X,Y,0) \right] dX \]  

\[ \theta_j(0,Y,t)/cX = 0; \]  

\[ \frac{\partial}{\partial X} \left[ \theta_j(1,Y,t) + \kappa \frac{\partial \theta_j(1,Y,t)}{\partial t} \right] = -\text{Bi}_{y,0,j} \left[ \theta_j(1,Y,t) + \partial \theta_j(1,Y,t)/\partial t \right]; \]  

\[ \frac{\partial}{\partial X} \left[ \theta_j(X,0,t) + \kappa \frac{\partial \theta_j(X,0,t)}{\partial t} \right] = \text{Bi}_{y,0,j} \left[ \theta_j(X,0,t) + \partial \theta_j(X,0,t)/\partial t \right]; \]  

\[ \frac{\partial}{\partial Y} \left[ \theta_j(X,1,t) + \kappa \frac{\partial \theta_j(X,1,t)}{\partial t} \right] = -\text{Bi}_{y,1,j} \left[ \theta_j(X,1,t) + \partial \theta_j(X,1,t)/\partial t \right]. \]

Here \( \theta_j = (T_j - T^*)/(T_{f,max} - T_{f,min}) \) is the relative temperature in the \( j \)-th period;
\( T^* = (T_{f,min} + T_{f,max})/2; j = 0, 1 \) is the period number; \( T_{f,min} \) and \( T_{f,max} \) is the minimum and maximum
of temperatures of heat carriers per cycle; \( t = \frac{\tau}{\tau_q} \) is the relative time from the beginning of the current period; \( X = 2x/\delta_w \); \( Y = y/l \); \( x \) and \( y \) is the coordinates of a point; \( L = 2l/\delta_w \); \( Fo_{q,j} = 4a_{w,j}\frac{\tau_q}{\delta_w^2} \) is the relaxation Fourier number; \( \kappa = \frac{\tau_f}{\tau_q} \); \( Bi_j = \alpha_j\delta_w/(2\lambda_j) \); \( Bi_{y,0,j} = \alpha_{y,0,j}/\lambda_j \); \( Bi_{y,1,j} = \alpha_{y,1,j}/\lambda_j \); \( \alpha_j \), \( \alpha_{x,0,j} \), \( \alpha_{y,1,j} \) is the heat transfer coefficients of the lateral surface, front and back ends of the plate in the \( j \)-period.

Using Fourier transformation [10]

\[
\theta_{L,j}(\mu_{n,j}, y_{m,j}, t) = \int_0^1 K_s(\mu_{n,j}, X) \int_0^1 K_y(y_{m,j}, Y) \theta_j(X, Y, t) dY dX,
\]

with \( \kappa = 1 \), it is obtained the relation [11]:

\[
\theta_{L,w,j}(\mu_{n,j}, y_{m,j}, t) = C_{1,j}f_{1,j}(t) + C_{2,j}f_{2,j}(t) + F_{0,j}F_j(t),
\]

were \( C_{1,j} \) and \( C_{2,j} \) is the constants of integration;

\[
f_{1,j}(t) = \exp(-\zeta_j^2 t) ; \quad f_{2,j}(t) = \exp(-t) ; \quad F_j(t) = \frac{2}{\varrho_j} \int_0^t \sinh \left( \frac{\varrho_j t - \eta}{2} \right) \exp \left( -\beta_j \frac{\eta - l}{2} \right) \theta_{f,j}(Y, \eta) d\eta;
\]

\[
\zeta_j^2 = \frac{\lambda_j Bi_j}{k} \left( \mu_{n,j}^2 + \gamma_{m,j}^2 / L^2 \right) ; \quad \beta_j = 1 + k\zeta_j^2 ; \quad W_j(t) = \sum_{l=0}^{n} d_{l,j} t^l + \sum_{l=1}^{n} d_{l,j} t^{l-1};
\]

\[
d_{i,j} = Bi_j \cos(\mu_{n,j}) \int_0^1 K_j(y_{m,j}, Y) \theta_j(Y, t) dY +
\]

\[
+ \frac{1}{L^2} \left[ Bi_{y,0,j} g_{l,0} + Bi_{y,1,j} K_y(y_{m,j}, Y) \sum_{k=0}^{n} g_{l,k} \right] \int_0^1 \cos(\mu_{n,j}, X) dX =
\]

\[
= Bi_j \cos(\mu_{n,j}) \sum_{k=0}^{n} g_{l,k} S_k(\gamma_{m,j}, Y) + \frac{1}{L^2} \sum_{k=0}^{n} g_{l,k} \mu_{n,j} ;
\]

\[
S_k(\gamma_{m,j}, Y) = \int_0^{\xi} K_j(\gamma_{m,j}, \xi) d\xi =
\]

\[
\begin{cases}
\frac{Bi_{y,0,j} + Sb(\gamma_{m,j})}{\gamma_{m,j}} & \text{for} \ k = 0, \\
\frac{\gamma_{m,j}^2}{k} (\gamma_{m,j}^2 - 1) - \frac{Sb^{(k-1)}(\gamma_{m,j})}{\gamma_{m,j}^k} + \frac{Sb^{(k)}(\gamma_{m,j})}{\gamma_{m,j}^k} & \text{for} \ k > 1;
\end{cases}
\]

\[
Sb(\gamma_{m,j}) = \sin(\gamma_{m,j}) - \frac{Bi_{y,0,j}}{\gamma_{m,j}} \cos(\gamma_{m,j}); \quad K_j(\mu_{n,j}, X) = \cos(\mu_{n,j}, X)
\]

and

\[
K_{y}(\gamma_{m,j}, Y) = \gamma_{m,j} Y + \frac{Bi_{y,0,j}}{\gamma_{m,j}} \sin(\gamma_{m,j}, Y)
\]

is the kernels transformations; \( \mu_{n,j} \) and \( \gamma_{m,j} \) is the roots of the characteristic equations [6]:

\[
\mu_{n,j}(t) = Bi_j, \quad \gamma_{m,j}(\gamma_{m,j} Y + Bi_{y,0,j} / L = \frac{tg(\gamma_{m,j})(\gamma_{m,j}^2 - Bi_{y,0,j} Bi_{y,1,j} / L^2)}{L^2}).
\]

The temperature of the medium is approximated in the coordinate system associated with the plate by the regression equation [6]:

\[
\theta_j(Y, t) = \sum_{k=0}^{n} Y^k \sum_{l=0}^{n} g_{l,k} t^l.
\]
Here $g_{l,k} \ (0 \leq l \leq n_t, \ 0 \leq k \leq n_z)$ is the regression coefficients; $n_t$ is the number of terms of the series in powers of $t$; $n_z$ is the number of terms of the series in powers of $Z$.

In the coordinate system associated with the flow, equation (13) takes the form:

$$\theta_j(Y,t) = \sum_{l=0}^{n_t} (\pm Z)^l b_k \sum_{j=0}^{n_z} g_{l,k} t^j, \quad (14)$$

were $b_k = \sum_{l=0}^{n_t} K_{l,k} Y^{l-k}$; $K_{l,k}$ is the numbers defined by the recurrence formula [6]:

$$K_{l,k} = K_{l,k-1} + K_{l-1,k}; \quad K_{0,0} = 1.$$

It follows from (14) that:

$$F_{m}(t) = b_0 \sum_{l=0}^{n_t} g_{l,0} t^l. \quad (15)$$

After substituting the series (15) into the right-hand side of $F_j(t)$ (12) and integration, we obtain:

$$F_j(t) = \frac{1}{g_j} \left\{ \sum_{l=0}^{n_t} d_{i,j} \left[ I_l(\zeta_j^2 t) - I_{l-1}(\zeta_j^2 t) \right] + \sum_{j=0}^{n_z} d_{j,l} \left[ I_{l-1}(\zeta_j^2 t) - I_{l-2}(\zeta_j^2 t) \right] \right\} =$$

$$= \frac{1}{g_j} \left\{ \sum_{l=0}^{n_t} (-1)^l d_{i,j} ! \left[ \frac{1}{\zeta_j^2 2^{(l+1)}} - 1 + \exp(-t) + \sum_{p=1}^{l} (-1)^p \frac{t^p}{p! \left( \frac{1}{\zeta_j^2 (2^{l-p})} - 1 \right)} \right] + \right. (16)$$

$$+ \sum_{l=0}^{n_t} (-1)^l d_{j,l} ! \left[ \frac{1}{\zeta_j^2 2^{(l+1)}} - 1 + \exp(-t) + \sum_{p=1}^{l} (-1)^p \frac{t^p}{p! \left( \frac{1}{\zeta_j^2 (2^{l-p})} - 1 \right)} \right],$$

were $I_l(\zeta_j^2 t) = \int_0^t \eta^l \exp[\eta(\zeta_j^2 t)] d\eta = \begin{cases} 0 & \text{for } l = 0, \\ [t^l - l I_{l-1}(\zeta_j^2 t)]/a & \text{for } l > 0. \end{cases}$

Inverse Fourier transforms give a solution for the boundary-value problem (4)-(10) [10]:

$$\theta_j(X,Y,t) = \sum_{n_j=0}^{n_j} A_{n,j} \cos(\mu_{n,j} X) \sum_{m_j=0}^{n_j} A_{m,j} K_{y,m,j}(Y) \left[ C_{1,j} \exp(-\zeta_j^2 t) + C_{2,j} \exp(-t) + F_{q,j} F_j(t) \right],$$

were $A_{n,j}^{-1} = \int_0^1 \cos^2(\mu_{n,j} X) dX$; $A_{m,j}^{-1} = \int_0^1 K_{y,m,j}(Y) dY$.

In view of the inhomogeneity of the boundary conditions (8) - (10), the non-uniform convergence of the Fourier series in solution (17) arises, which manifests itself in the slow decay of the expansion coefficients $A_n$ and $A_m$ with an unlimited increase in their number [12]. The method of improving convergence proposed by E.M. Kartashov [13] and extended to the multidimensional heat conduction problem in [6], consists in adding to the series (17) the solution of the initial quasistationary boundary value problem (4) - (10) in closed form $\Omega^*(X,Y,t)$ and subtracting the solution of the same problem in the form of a Fourier series $\Omega(X,Y,t)$. By analogy with the solution of the classical Fourier heat equation [6], the solution of problem (4) - (10) in quasistationary formulation has the form:

$$\Omega_j^*(X,Y,t) = \theta_j^*(Y,t) + d\theta_j(Y,t)/dt;$$

$$\Omega_j(X,Y,t) = \sum_{n_j=0}^{n_j} A_{n,j} \cos(\mu_{n,j} X) \sum_{m_j=0}^{n_j} A_{m,j} K_{y,m,j}(Y) F_{q,j} W_j(t) / \theta_j^2.$$

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$$\Omega_j^*(X,Y,t) = \theta_j^*(Y,t) + d\theta_j(Y,t)/dt;$$

$$\Omega_j(X,Y,t) = \sum_{n_j=0}^{n_j} A_{n,j} \cos(\mu_{n,j} X) \sum_{m_j=0}^{n_j} A_{m,j} K_{y,m,j}(Y) F_{q,j} W_j(t) / \theta_j^2.$$
Thus, the solution of the boundary-value problem (4) - (10) with improved convergence of Fourier series is the expression:

\[ \theta_j(X,Y,t) = \sum_{k=0}^{n_k} X^k \left( \sum_{l=0}^{n_l} g_{lk} t^l + \sum_{l=1}^{\infty} g_{lk} t^{l-1} \right) + \sum_{n,j=0}^{\infty} A_{n,j} \cos(\mu_{n,j} x) \sum_{m,j=0}^{\infty} A_{m,j} K \left( v_{m,j} y \right) C_{1,j} \exp(-\xi^2_j t) + C_{2,j} \exp(-r t) + F_{0,j}(t) - W_j(t) / \xi^2_j. \]

Constants of integration \( C_{1,j} \) and \( C_{2,j} \) are found from the initial conditions (5) and (6) after substituting expressions \( W_j(t) = \sum_{l=0}^{n} d_{1,j} t^l + \sum_{l=1}^{\infty} d_{1,j} t^{l-1} \) and the right-hand side of (16) instead of \( F_j(t) \).

3. Conclusion
In quasi-equilibrium and stationary processes the phenomena of thermal relaxation and thermal damping have little effect on the heat transfer process, and the temperature fields are well described by the classical heat conduction equation (1). In thermal transient processes with pronounced nonequilibrium, the effect of these phenomena increases and their negligence is unacceptable. Under these conditions, the Maxwell-Cattaneo-Luikov hypothesis (3) provides an adequate description of thermal transient processes.

The solution of the problem of a transient thermal process in the nozzle of a regenerative air preheater with cyclic boundary conditions of the third kind presented in the article allows one to take into account the influence of the phenomena of thermal relaxation and thermal damping. In addition, the model adjusted for the temperature dependence of the media on time and longitudinal coordinate. Using the proposed solution involves preliminary measurement of the times of thermal relaxation and thermal damping for a specific material of the nozzle.

4. References
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