Lorentz Covariance of Dirac Electrons in Solids: Dielectric and Diamagnetic Properties

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We study the electrodynamics of Dirac electrons in solids (e.g., bismuth) by comparing it with quantum electrodynamics (QED). It is shown that Lorentz covariance associated with the Dirac electrons in solids results in a remarkable correlation between the dielectric and diamagnetic properties, leading to a significant enhancement in the permittivity directly linked to the well-known phenomenon of large diamagnetism.

The Dirac equation is the cornerstone of relativistic quantum mechanics, and it was originally derived by Dirac requiring the electron wave equation linear in time-derivative to be Lorentz-covariant.\textsuperscript{11} As pointed out by Wolff,\textsuperscript{22} an essentially equivalent equation describes the motion of nonrelativistic electrons in narrow-gap systems with strong spin–orbit coupling such as bismuth.\textsuperscript{3} The Dirac equation in narrow-gap systems is invariant under a Lorentz transformation when the speed of light \(c\) is replaced by an effective speed of light \(c^* = \sqrt{E_G/2m^*}\) with \(E_G\) and \(m^*\) as the band gap and effective electron mass, respectively. As a result, not in the original sense but another type of Lorentz covariance specified by \(c^*\) emerges for the Dirac electrons in solids. The Dirac electron system is not just interesting in itself but also provides a platform to study topological insulators\textsuperscript{4,5} and exotic magnetoelectric effects.\textsuperscript{6}

One of the most interesting phenomena in the Dirac electrons of solids is large diamagnetism which has been experimentally known for many years, e.g. in Bi, and the magnitude of diamagnetism is at a maximum when the chemical potential is located in the band gap.\textsuperscript{7,8} This has been theoretically explained by an interband effect of the magnetic field.\textsuperscript{9} Thus, it is distinct from Landau diamagnetism which results from the Landau quantization of electron orbital motion in metals.\textsuperscript{10} Based on the Luttinger–Kohn representation,\textsuperscript{11} which is equivalent to the standard Bloch representation when linked by a unitary transformation, a general formula for the uniform and static orbital susceptibility has previously been established.\textsuperscript{12} With this formula, the diamagnetic properties of Dirac electron systems and related materials have been intensively studied.\textsuperscript{13–21} However, the dielectric properties and electrodynamics of Dirac electrons in solids have not yet attracted much attention.\textsuperscript{22,23} Apparently, the electrodynamics of Dirac electrons in narrow-gap systems can be taken as a counterpart of quantum electrodynamics\textsuperscript{24–27} (QED) in solids. In Table I, we present a correspondence table for Dirac electrons in bismuth and QED. In particular, the zero-temperature insulator with the greatest diamagnetism corresponds to the vacuum in QED.

In this letter, we report our theoretical results on electric susceptibility \(\chi_e(q, \omega) = e_\omega(q, \omega) - 1\) and magnetic susceptibility \(\chi_m(q, \omega) = 1 - \mu_\omega(q, \omega)\) with a magnitude \(q\) of a wave vector and frequency \(\omega\) of Dirac electrons in solids, where \(e_\omega(q, \omega)\) and \(\mu_\omega(q, \omega)\) are the relative permittivity and permeability, respectively.\textsuperscript{28} We find the relationship between the susceptibilities to be \(\chi_e(q, \omega) = -\left(c^*/c^*\right)^2\chi_m(q, \omega)\) for the zero-temperature insulator, which originates from Lorentz covariance of the Dirac equation and can be considered as the nature of the vacuum in QED that is realized in solids. With this relationship and the explicit evaluation of the charge renormalization factor \(Z_3 \equiv 1/e_\omega(0, 0)\) in solids, we show a significant enhancement of the permittivity directly linked to the large diamagnetism.

First, we note the nature of the vacuum in QED.\textsuperscript{24} The permittivity \(e_\omega\) and permeability \(\mu_\omega\) of the classical vacuum are constants and related to each other by \(e_\omega\mu_\omega = c^{-2}\) due to the Lorentz covariance of Maxwell’s equations.\textsuperscript{29} However, in QED, the vacuum permittivity and permeability are not constants but depend on \(q\) and \(\omega\) as \(e_\omega e_\omega(q, \omega)\) and \(\mu_\omega \mu_\omega(q, \omega)\), respectively. In particular, \(e_\omega(q, \omega)\) describes the vacuum polar-
zation caused by the dynamics of virtually excited particle–antiparticle pairs.\textsuperscript{30} Even in the case of the polarized vacuum, the Lorentz covariance of the Dirac equation makes a desired correlation between the electric and magnetic properties of the vacuum as $\varepsilon_{\nu}(q, \omega)\mu_{\nu}(q, \omega) = 1$.\textsuperscript{31} In the uniform and static limit of $q = \omega = 0$, the previous reduction to $Z_2 \equiv 1/\varepsilon(0, 0) = \mu(0, 0)$. We can then renormalize $\varepsilon_{\nu}(q, \omega)$ and $\mu_{\nu}(q, \omega)$ as $\varepsilon_{\nu}'(q, \omega) = Z_3 \varepsilon_{\nu}(q, \omega)$ and $\mu_{\nu}'(q, \omega) = Z_3^{-1} \mu_{\nu}(q, \omega)$, respectively, such that $\varepsilon_{\nu}'(q, \omega) = \mu_{\nu}'(q, \omega)$ are equal to 1 in the $q = \omega = 0$ limit. Correspondingly, the bare electric charge $e_0$ is assumed to be renormalized as $e_0' = \sqrt{Z_3}e_0$ in QED,\textsuperscript{32} and the physically observable elementary charge, permittivity, and permeability are identified as $e_0$, $\varepsilon_0 e_0\varepsilon_{\nu}'(q, \omega)$, and $\mu_0 \mu'_0\varepsilon'(q, \omega)$, respectively, in QED.\textsuperscript{34,31} This renormalization procedure is also summarized in Table I. In QED, the value of $Z_2$ cannot be determined because it is renormalized into the elementary charge $e$.\textsuperscript{30} It is important to note that this is not the case in solids.

We begin by introducing the Dirac Hamiltonian in solids, which is effectively identical to the Wolf Hamiltonian that describes low-energy electron excitations in narrow-gap systems.\textsuperscript{2} The Dirac Hamiltonian is given in its second quantized form as

$$H = \sum_k \bar{\psi}_k \left[ \hbar c \gamma^0 k_0 + m^* c^2 \right] \psi_k$$

with $\bar{\psi}_k = \psi_k^\dagger \gamma^0$ and where $k = (k_1, k_2, k_3)$ is a wave vector, $\gamma^0, \gamma^1, \gamma^2$, and $\gamma^3$ are the gamma matrices, and the repeated Roman indexes $i = 1, 2, 3$ are to be summed. Under a canonical transformation, the four components of $\psi_k$ correspond to the conduction and valence band electrons with a spin degeneracy in the Luttinger–Kohn representation.\textsuperscript{2,11,33} In the Dirac Hamiltonian, Eq. (1), anisotropy of the effective mass, which has been considered in the Wolf Hamiltonian, is neglected. In a forthcoming paper, we plan to investigate the effects of anisotropy in comparison between theory and experiment for the permittivity. The coupling of Dirac electrons with an electromagnetic field is obtained by the gauge principle with the electromagnetic scalar and vector potentials as $\phi_\mu(q, \omega)$ and $a_\mu(q, \omega)$, respectively,\textsuperscript{11} resulting in an additional time-dependent Hamiltonian $H'(t) = -e_0 \sum_q J^\mu(q) A_\mu(-q, \omega) e^{-i\omega t}$, where the repeated Greek indexes $\mu = 0, 1, 2, 3$ are to be summed. With the use of $c^\dagger$ instead of the conventional use of $c$, we define a four-current and an electromagnetic four-potential as $J^\mu(q) = (c^\dagger \tilde{p}_\mu(q) \tilde{\psi}_k + c^\dagger \sum_k \tilde{\psi}_{k+q} \gamma^\mu \tilde{p}_k$ and $A_\mu(q, \omega) = (\phi_\mu(q, \omega)/c^\dagger - a_\mu(q, \omega))$, respectively. As shown in Table I, the coupling constant $e_0 (> 0)$ is equal to the elementary charge $e$ in the present case.

The Hamiltonian $H + H'(t)$ is, in fact, similar to that of QED. We are, however, treating an electromagnetic field with classical theory while employing the quantum theory of electrons. With this treatment, the effects of a mutual Coulomb interaction $e_0^2/4\pi\varepsilon_0 q^2$ are included through Maxwell’s equations in matter as follows:

The electric field $E_\mu(q, \omega) = -i q \tilde{p}_\mu(q, \omega) + i\omega a_\mu(q, \omega)$ and the magnetic induction $B_\mu(q, \omega) = i q \times a_\mu(q, \omega)$ induce electric charge density modulations $-e_0 \delta \phi_\mu(q, \omega) = -i q \cdot P_\mu(q, \omega)$ and an electric current $-e_0 \delta J_\mu(q, \omega) = i q \times M_\mu(q, \omega)$ with a polarization $P_\mu(q, \omega) = e_0 \chi_\mu(q, \omega) E_\mu(q, \omega)$ and a magnetization $M_\mu(q, \omega) = e_0 \varepsilon(q, \omega) B_\mu(q, \omega)$.\textsuperscript{28} The induced electric charge and current can be then written in terms of the electromagnetic potentials as

$$-e_0 \delta \phi_\mu(q, \omega) = -e_0 \chi_\mu(q, \omega) \left[ q^2 \tilde{\phi}_\mu(q, \omega) - \omega \cdot q \cdot a_\mu(q, \omega) \right],$$

$$-e_0 \delta j_\mu(q, \omega) = -e_0 \chi_\mu(q, \omega) \left[ q \omega \tilde{\phi}_\mu(q, \omega) - \omega^2 a_\mu(q, \omega) \right]$$

$$+ e_0 c^2 \varepsilon_\mu(q, \omega) \left[ q^2 a_\mu(q, \omega) - q \cdot q \cdot a_\mu(q, \omega) \right],$$

where $q^2 = \tilde{q}^2 = \delta_{ij}q_i q_j$.

Equations (2) and (3) enable us to relate the electric and magnetic susceptibilities $\chi_\mu(q, \omega)$ to the polarization tensor $\Pi_\mu\nu(q, \omega)$, which gives the dynamical four-current $\delta J^\nu(q, \omega) = (c^\dagger \tilde{J}_\nu(q, \omega))$ as $-e_0 \delta J^\nu(q, \omega) = -e_0 \varepsilon_\mu\nu \Pi_\mu\nu(q, \omega) A_\nu(q, \omega)$. When comparing the previous equation with Eqs. (2) and (3), we can write $\Pi_\mu\nu(q, \omega)$ as

$$\Pi_\mu\nu(q, \omega) = \left( \tilde{Q}^2 q^\mu q^\nu - \tilde{Q}^\mu q^\nu \right) \chi_\nu(q, \omega) + \frac{c^2}{c^2} \tilde{Q} \chi_\nu(q, \omega)$$

$$- \left( \tilde{Q}^\mu q^\nu - \tilde{Q}^\nu q^\mu \right) \chi_\nu(q, \omega),$$

where $\tilde{Q} = diag(0, -1, -1, -1)$, $\tilde{Q}^\nu = (0, q)$, $\tilde{Q}^2 = -q^2$, $q^\mu = diag(1, -1, -1, -1)$, $Q^\nu = (\omega/c^*, q)$, and $Q^2 = \omega^2/c^* - q^2$. Equation (4) has a general form that satisfies the charge conservation $Q_\mu \Pi_\mu\nu(q, \omega) = 0$ and gauge invariance $Q_\mu \Pi_\mu\nu(q, \omega) = 0$, where $Q_\mu = (\omega/c^*, -q)$.

With $\mu = \nu$ in Eq. (4), we obtain a standard relationship between the electric susceptibility and the polarization function as $\chi_\mu(q, \omega) = \Pi_\mu\mu(q, \omega)/Q^2$. Multiplying both sides of Eq. (4) by $g_{\mu\nu} = g_{\nu\mu}$ and taking the summation with respect to the repeated Greek indexes, we obtain a useful formula for the magnetic susceptibility as

$$\chi_\mu(q, \omega) = -\frac{c^2}{c^2} \chi_\mu(q, \omega) + \Delta \chi(q, \omega)],$$

$$\Delta \chi(q, \omega) = \frac{3 Q^2 \chi_\mu(q, \omega) + g_{\mu\nu} \Pi_\mu\nu(q, \omega)}{2Q^2}.$$ 

Because the polarization tensor can be expressed by the Kubo formula,\textsuperscript{34} we can now make microscopic calculations based on the Dirac Hamiltonian, Eq. (1), not only for $\chi_\mu(q, \omega)$ but also for $\chi_\mu(q, \omega)$ with the use of Eqs. (5) and (6). The detailed calculations are presented in the Supplemental Material,\textsuperscript{35} where the standard thermal Green function technique for nonrelativistic electron gas\textsuperscript{36} is extended to our “covariant” electron–hole gas. The presented method of calculations is marginally different from that used in QED at finite temperatures and densities\textsuperscript{37} in that we use an integral representation.
We first show our results of the imaginary parts of $\chi^e(q, |\omega|)$ and $\Delta\chi(q, |\omega|)$ for the Dirac electron system at zero temperature with an arbitrary value of the chemical potential $\mu$ as follows:

$$\text{Im}\chi^e(q, |\omega|) = \frac{e_0^2}{16\pi\epsilon_0 \hbar c^2} \int_{-\infty}^{\infty} dx \left(2x^2 - 1\right) \left(\theta(-Q^2) - \theta\left(Q^2 - \frac{4m^2c^2}{\hbar^2}\right)\theta\left(a^2 - x^2\right)\right) \theta\left((x-b_\pm)(b_\pm - x)\right),$$  \hspace{1cm} (7)

$$\text{Im}\Delta\chi(q, |\omega|) = \frac{e_0^2Q^2}{32\pi\epsilon_0 \hbar c^2} \int_{-\infty}^{\infty} dx \left(3x^2 - a^2\right) \left(\theta(-Q^2) - \theta\left(Q^2 - \frac{4m^2c^2}{\hbar^2}\right)\theta\left(a^2 - x^2\right)\right) \theta\left((x-b_\pm)(b_\pm - x)\right),$$  \hspace{1cm} (8)

where $a = \sqrt{1 - \frac{4m^2c^2}{Q^2}}$, $b_\pm = \frac{2\mu\pm i\omega\pm Q}{\hbar c}$, and $\theta(x)$ is the Heaviside step function (see the derivation in Sects. 3–5 of Ref. 35). The first terms with $\theta(-Q^2)$ correspond to the contributions from intraband electron excitations, while the second terms with $\theta\left(Q^2 - \frac{4m^2c^2}{\hbar^2}\right)$ correspond to the contributions from virtual electron–hole pairs excited across the band gap $E_G = 2m^*c^2$. Hence, they represent interband effects. We note that $a > 0$, $b_\pm > 0$, and $b_\pm > b_\mp$ by their definitions and whether $a > |b_\pm|$ or $a < |b_\pm|$ can be determined from the identity

$$a^2 - b^2 = \frac{4e^2c^2\left(\mu^2 - m^2c^4\right) - \left(h^2c^4Q^2 \pm 2|\mu| |\omega|\right)^2}{h^2c^4Q^2}.$$  \hspace{1cm} (9)

The imaginary part of $\chi^m(q, |\omega|)$ is then obtained immediately from Eq. (5).

The complex susceptibilities $\chi^e,m(q, \omega)$ can be derived from $\text{Im}\chi^e,m(q, \omega)$ using the Kramers–Kronig relation. The Dirac electron system in solids has a natural bandwidth cutoff $E_\Lambda$ that is caused by the upper limit of energy, and the dispersion of electrons in a solid is regarded as a Dirac dispersion when the energy is below this limit. We therefore define $\chi^e,m(q, \omega)$ as contributions from a Dirac dispersion and a part of the total susceptibility of the solid. Then, the Kramers–Kronig relation leads to

$$\chi^e,m(q, \omega) = -\frac{1}{\pi} \int_{-2E_\Lambda}^{2E_\Lambda} d\omega' \frac{\text{Im}\chi^e,m(q, \omega')}{\omega_\pm - \omega'},$$  \hspace{1cm} (10)

where $\text{Im}\chi^e(m,q,\omega') = \text{sgn}(\omega')\text{Im}\chi^e,m(q, |\omega|')$ and $\omega_\pm = \omega + i\eta$ with $\eta$ being a positive infinitesimal value. It is to be noted that, while the imaginary part of the total susceptibility is properly estimated by the present Dirac Hamiltonian, i.e., by $\text{Im}\chi^e,m(q, \omega)$ for low energies, the real part of the total susceptibility can have extra background contributions from higher energy regions, which have a weak dependence on $\omega$. However, the singular $\omega$ dependence of the real part of the total susceptibility for low energies is correctly described by $\text{Re}\chi^e,m(q, \omega)$ defined in Eq. (10).

For a finite temperature $T$, the susceptibility can be expressed as an integral of the zero-temperature susceptibility with respect to the chemical potential. By denoting them as $\chi^e,m(q, \omega; T, \mu)$ to show their $T$ and $\mu$ dependences explicitly, the finite-temperature electric and magnetic susceptibilities are given by (Sect. 6 of Ref. 35)

$$\chi^e(q, \omega; T, \mu) = \int_{-\infty}^{\infty} d\mu' \chi^e(q, \omega; 0, \mu') \frac{e_0^2}{4k_B T \cosh^2 \frac{\mu'c^2T}{2k_B T}},$$  \hspace{1cm} (11)

$$\chi^m(q, \omega; T, \mu) = -\frac{e_0^2}{c^2} \chi^e(q, \omega; T, \mu) - \frac{e_0^2}{c^2} \int_{-\infty}^{\infty} d\mu' \Delta\chi(q, \omega; 0, \mu') \frac{e_0^2}{4k_B T \cosh^2 \frac{\mu'c^2T}{2k_B T}}.$$  \hspace{1cm} (12)

Using Eq. (12), the $T$ dependence of the nuclear spin relaxation time for the Dirac electron system has recently been calculated.

In the following, we concentrate on narrow-gap insulators at $T = 0$ in which the chemical potential is in the band gap, i.e., $|\mu| < m^*c^2$. For $Q^2 > 0$ and $|\mu| < m^*c^2$, Eq. (9) leads to the constraint of $-a < b_\pm < b_\mp < a$. Therefore, the first terms corresponding to the intraband contributions vanish in Eqs. (7) and (8). For $Q^2 > 4m^2c^2/h^2$ and $|\mu| < m^*c^2$, where $b_\pm < 0$, Eq. (9) leads to $b_\pm < -a < b_\mp$. Thus, the second terms corresponding to the interband contributions reduce to the integrals calculated from $-a$ to $a$. However, because $\int_{-a}^{a} dx (3x^2 - a^2) = 0$, we find that $\text{Im}\Delta\chi(q, |\omega|)$ vanishes. By performing the integration $\int_{-a}^{a} dx (x^2 - 1)$ for $\text{Im}\chi^e(q, |\omega|)$ and using Eq. (5), we obtain

$$\text{Im}\chi^e(q, |\omega|) = -\frac{e_0^2}{24\pi\epsilon_0 \hbar c^2} \Theta\left(\frac{Q^2 - 4m^2c^2}{h^2}\right) a \left(3 - a^2\right).$$  \hspace{1cm} (13)

Because $a$ is a function of only $Q^2$, the imaginary parts of...
\( x_{\text{e,m}}(q, |\omega|) \) depend on \( q \) and \( \omega \) only through \( Q^2 = \omega^2/c^2 - q^2 \).

The substitution of Eq. (13) into Eq. (10) for \( \hbar c' q \ll E_A \) yields

\[
\chi_0(q, \omega) = -\frac{e^2}{c^2} \rho(q, \omega) = -\Omega_2'(Q^2),
\]

(14)

where \( \Omega_2'(Q^2) \equiv -\chi_0(q, \omega) \) is a function of \( Q^2 = \omega^2/c^2 - q^2 \). The presence of the factor \( c^2/e^2 \) is caused by the difference in the effective Lorentz covariance of the Dirac equation for electrons in solids and the true Lorentz covariance of Maxwell’s equations. In fact, if \( c^2 \) is replaced by \( c \), Eq. (14) reduces to \( e_i(q, \omega) \mu_i(q, \omega) = [1 + \chi_0(q, \omega)]/[-\chi_0(q, \omega)] = 1 \) in accordance with the full Lorentz covariance. From Eq. (14), \( \chi_0(q, \omega) \) is opposite in sign to \( \chi_0'(q, \omega) \). The magnitude of \( \chi_0(q, \omega) \) is much smaller than that of \( \chi_0'(q, \omega) \) by the factor of \( e^2/c^2 \sim 10^{-6} \) in solids. Although, the consideration of the anisotropy effects is necessary for an improved quantitative evaluation as exemplified elsewhere.

Carrying out the integration in Eq. (10) with Eq. (13), we obtain an explicit expression of \( \Omega_2'(Q^2) \) for \( m^*c^2 \ll E_A \) as

\[
\Omega_2'(Q^2) = -\frac{e^2}{12\pi\epsilon_0\hbar c^3} \left[ \frac{1}{\pi} \log \left( \frac{M_A^2}{m^2} + 1 \right) + \frac{\hbar^2 Q^2}{m^*c^2} \right],
\]

(15)

where \( M_A = 2e^{-5/6}E_A/c^2 \) and \( P_2(s) \) is an analytic function of a complex variable \( s \) given by

\[
P_2(s) = -\frac{1}{5\pi} \left[ \frac{1}{s} + \frac{1}{1 + \sqrt{1 - \frac{4}{s} \log \left( \frac{\sqrt{1 - 4}}{\sqrt{1 - 4} - 1} \right)}} \right].
\]

(16)

By a series expansion with respect to \( s \), we can check that \( P_2(s) \) vanishes at \( s = 0 \). From Eqs. (14)–(16), we see that \( \text{Re} \chi_{\text{e,m}}(q, \omega) \) has a cusp singularity at \( \omega^2 = c^2q^2 + 4m^*c^4/\hbar^2 \) associated with the interband excitations across the band gap. In QED (see Table 1), Eq. (15) corresponds to a well-known result for the bare vacuum polarization function, and \( Z_3[\Omega_2'(Q^2) - \Omega_2(0)] \) describes the physically observable vacuum polarization function up to the second order in renormalized coupling.\(^{24}\)

The relationship between the electric and magnetic susceptibilities, Eq. (14), for a zero-temperature insulator is directly linked to the emergence of Lorentz covariance in our electron system. This is understood as follows: substituting Eq. (14) into Eq. (4) yields the well-known Lorentz covariant form of the polarization tensor as \( \Pi^\mu_\nu(q, \omega) = (Q^2 g^\mu_\sigma - Q^\mu Q^\nu) \Omega_2'(Q^2) \); inversely, if the polarization tensor has the above form, then Eq. (6) leads to \( \Delta \chi(q, \omega) = 0 \) and therefore Eq. (14) as a result. However, for nonzero temperatures, Eq. (14) does not exactly hold even for the insulating regime of \( |\mu(T)| < m^*c^2 \) because the second term in Eq. (12) has nonzero contributions from \( \Delta \chi(q, \omega, 0, \mu^\prime) \) for \( |\mu^\prime| > m^*c^2 \). (Explicit expressions for the susceptibilities in the metallic region of \( |\mu| > m^*c^2 \) will be given in a forthcoming paper.) This is similar to the situation in which \( \mu(q, \omega) \) deviates from \( 1/e_i(q, \omega) \) for QED with nonzero temperatures, but it is a Lorentz covariant theory.\(^{31}\)

In the uniform and static limit of \( q = \omega = 0 \), Eq. (14) reduces to

\[
e_i(0, 0) = 1 - \frac{e^2}{c^2} \rho(0, 0) = \frac{1}{Z_3},
\]

(17)

Noting that \( e_0 = e \) in solids and the fine-structure constant is given by \( \frac{e^2}{4\pi\epsilon_0\hbar c^3} \approx \frac{1}{137} \), we can evaluate \( Z_3^{-1} = 1 - \Pi_2(0) \) to be

\[
\frac{1}{Z_3} \approx 1 + 1.55 \times 10^{-3} \frac{c}{c^2} \log \frac{E_A}{m^*c^2},
\]

(18)

where the bandwidth cutoff \( E_A \) is on the order of \( 1 \) eV. The uniform and static magnetic susceptibility is then given by \( \chi_m(0, 0) = -1.55 \times 10^{-3}(c^2/c) \log(E_A/m^*c^2) \), which is equivalent to the previous result for large diamagnetism.\(^{9,12,17}\)

From Eqs. (17) and (18), we find not only the large diamagnetism \( \chi_m(0, 0) \to -\infty \) but also a large enhancement in the permittivity \( [\chi_m(0, 0) \to 0] \) for \( m^*c^2 \ll E_A \). The physical interpretation is as follows.

In a zero-temperature insulator, virtual electron–hole pairs are created and annihilated dynamically by quantum fluctuations forming a charge distribution of size \( \sim h/m^*c^2 \). In the presence of an electromagnetic field, those electron–hole pairs fluctuate on the length scale \( \sim \hbar c/E_A \); in turn, this change reacts to the field. This effect is called the self-energy of an electromagnetic field. Thus, in the limit of \( m^*c^2 \ll E_A \), the charge distribution behaves as a freely deformable distribution that exhibits perfect screening \( [\chi_m(0, 0) \to 0] \) in the presence of an external charge on one hand and perfect diamagnetism \( [\chi_m(0, 0) \to -\infty] \) in the presence of an external magnetic field on the other hand.\(^{28}\)

In summary, we have studied the electrodynamics of Dirac electrons in a narrow-gap system to find a remarkable correlation between its dielectric and interband properties. Our findings are described by Eqs. (14), (17), and (18). These equations show that both the large diamagnetism and a large enhancement of the permittivity result from virtual electron–hole pair creations across the small band gap, i.e., interband effects associated with an electromagnetic field.

Acknowledgments We thank the very fruitful discussions with Y. Fuseya, T. Hiroswa, H. Matsuura, T. Mizoguchi, and N. Okuma. This work was supported by a Grant-in-Aid for Scientific Research on “Multiferroics in Dirac electron materials” (No.15H02108).

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28) In this letter, we define the magnetic susceptibility \( \chi_m(q, \omega) \) as the response of the magnetization \( \mathbf{M}_q(\omega) \) to the magnetic induction \( \mathbf{B}_q(\omega) \) not to the magnetic field \( \mathbf{H}_q(\omega) \), so that the relative permeability is given by \( \mu_r(q, \omega) = [1 - \chi_m(q, \omega)]^{-1} \). Then \( \chi_m(0,0) \to -\infty \) corresponds to perfect diamagnetism.
29) J. D. Jackson, Classical electrodynamics (Wiley, Hoboken, NJ, 1999).
30) I. Levine et al. (TOPAZ Collaboration), Phys. Rev. Lett. 78, 424 (1997).
31) H. A. Weldon, Phys. Rev. D 26, 1394 (1982).
32) The charge renormalization is originally obtained as \( e_0^* = Z_1^{-1} Z_2^{-1/2} e_0 \), where \( Z_1 \) and \( Z_2 \) are the renormalization factors associated with the electron self-energy and the vertex corrections, respectively, but there is the Ward identity \( Z_1 = Z_2 \).
Supplemental Material for “Lorentz Covariance of Dirac Electrons in Solids: Dielectric and Diamagnetic Properties”

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We derive Eqs. (7), (8), (11), (12), and (15) in the text. Throughout this Supplemental Material, we take $\hbar = k_B = 1$ for simplicity.

1. Preliminaries

First of all, we introduce our notations. We use capital letters of the alphabet for four-vectors as follows:

$$K^\mu = (k_0, k_1, k_2, k_3), \quad K_\mu = (k_0, -k_1, -k_2, -k_3), \quad K^2 = K_\mu K^\mu = k_0^2 - k^2,$$

$$K'^\mu = (k'_0, k'_1, k'_2, k'_3), \quad K'_\mu = (k'_0, -k'_1, -k'_2, -k'_3), \quad K'^2 = K'_\mu K'^\mu = k'_0^2 - k'^2,$$

$$Q^\mu = (q_0, q_1, q_2, q_3), \quad Q_\mu = (q_0, -q_1, -q_2, -q_3), \quad Q^2 = Q_\mu Q^\mu = q_0^2 - q^2,$$

where $k^2 = k_0^2$, $k'^2 = k'^2$, and $q^2 = q_0^2$ with $k = (k_1, k_2, k_3)$, $k' = (k'_1, k'_2, k'_3)$, and $q = (q_1, q_2, q_3)$, respectively. The Feynman slash notation $\slashed{K}$ and four-vector scalar product $K \cdot K'$ are defined as

$$\slashed{K} = K_\mu \gamma^\mu = k_0 \gamma^0 - k_i \gamma^i, \quad K \cdot K' = K_\mu K'^\mu = k_0 k'_0 - k_0 k'_i = k_0 k'_0 - k \cdot k'.$$

The integrals with respect to four-vectors are given as

$$\int d^4 K = \int_{-\infty}^{\infty} dk_0 \int d^3 k, \quad \int d^4 K' = \int_{-\infty}^{\infty} dk'_0 \int d^3 k'.$$

In addition, we note the following two identities, which will be used for calculations of the polarization tensor in Sect. 3: one is

$$T \sum_{\omega_k} \frac{1}{(i \omega_k + \mu - c^* k_0)(i \omega_k + i \omega_q + \mu - c^* k'_0)} = \frac{f(c^* k_0) - f(c^* k'_0)}{i \omega_q + c^* k_0 - c^* k'_0}, \quad (S.1)$$

where $\omega_k$ ($\omega_q$) is an odd (even) Matsubara frequency and $f(x) = [e^{(x-\mu)/T} + 1]^{-1}$ is the Fermi distribution function;¹¹ the other is

$$\text{Tr} \left[ \gamma^\mu (\slashed{K} + m^* c^+) \gamma^\nu (\slashed{K'} + m^* c^+) \right] = 4 \left[ K'^\mu K'^\nu + K'^\mu K'\nu - (K \cdot K' - m^2 c^2) g^{\mu \nu} \right], \quad (S.2)$$

1.8
which can be easily proved by using the trace identities for the gamma matrices
\[ \text{Tr}(\gamma^\mu) = 0, \quad \text{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}, \quad \text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = 0, \quad \text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \]

2. Integral representation of the thermal Feynman propagator

The thermal Feynman propagator \( S_F(k, i\omega_k) \) is defined as
\[
S_F(k, i\omega_k) \equiv -\frac{1}{2} \int_{-1/T}^{1/T} d\tau \langle T_\tau e^{i(H-\mu N)\tau} \bar{\psi}_k e^{-i(H-\mu N)\tau} \psi_k \rangle e^{i\omega_k \tau},
\]
where the (imaginary) time-ordered product is represented by \( T_\tau \) and \( \langle \cdots \rangle \) denotes the grand canonical average for \( H - \mu N \) with \( N = \sum_k \bar{\psi}_k \gamma^0 \psi_k \) as the electron number operator. Then \( S_F(k, i\omega_k) \) is obtained as
\[
S_F(k, i\omega_k) = \left( (i\omega_k + \mu)\gamma^0 - c^* k_i \gamma^i - m^* c^2 \right)^{-1} = \frac{(i\omega_k - \mu)\gamma^0 - c^* k_i \gamma^i + m^* c^2}{(i\omega_k + \mu)^2 - c^2 k^2 - m^2 c^4}.
\]
To use the standard thermal Green function technique for a covariant field theory, we introduce the following integral representation of the thermal Feynman propagator:
\[
S_F(k, i\omega_k) = \int_{-\infty}^{\infty} dk_0 \frac{K + m^* c^*}{i\omega_k + \mu - c^* k_0} \text{sgn}(k_0) \delta(K^2 - m^2 c^2).
\]
By using
\[
\delta(K^2 - m^2 c^2) = \frac{1}{2 \sqrt{k^2 + m^2 c^2}} \left[ \delta \left( k_0 - \sqrt{k^2 + m^2 c^2} \right) + \delta \left( k_0 + \sqrt{k^2 + m^2 c^2} \right) \right],
\]
we can confirm that the right hand side of Eq. (3.3) equals
\[
\frac{1}{2 \sqrt{k^2 + m^2 c^2}} \left[ \frac{\sqrt{k^2 + m^2 c^2} \gamma^0 - k_i \gamma^i + m^* c^2}{i\omega_k + \mu - c^* \sqrt{k^2 + m^2 c^2}} - \frac{-\sqrt{k^2 + m^2 c^2} \gamma^0 - k_i \gamma^i + m^* c^2}{i\omega_k + \mu + c^* \sqrt{k^2 + m^2 c^2}} \right]
\]
\[
= \frac{(i\omega_k + \mu)\gamma^0 - c^* k_i \gamma^i + m^* c^2}{(i\omega_k + \mu)^2 - c^2 k^2 - m^2 c^4}.
\]

3. Imaginary part of the polarization tensor

By applying the Kubo formula,\(^3\) the (retarded) polarization tensor \( \Pi_R^{\mu\nu}(q, \omega) \) is given by
\[
\Pi_R^{\mu\nu}(q, \omega) = \frac{ie_0^2}{\epsilon_0 c^2} \int_0^{\infty} dt \langle \left[ e^{i(H-\mu N)t} J^{\mu}(q) e^{-i(H-\mu N)t}, J^{\nu}(-q) \right] \rangle e^{i\omega t}.
\]
The retarded polarization tensor \( \Pi_R^{\mu\nu}(q, \omega) \) can be obtained from analytic continuation of the thermal polarization tensor \( \Pi^{\mu\nu}(q, i\omega_q) \) defined as
\[
\Pi^{\mu\nu}(q, i\omega_q) \equiv \frac{e_0^2}{\epsilon_0 c^2} \int_0^{1/T} d\tau \langle e^{i(H-\mu N)\tau} J^{\mu}(q) e^{-i(H-\mu N)\tau} J^{\nu}(-q) \rangle e^{i\omega_q \tau}.
\]
Then Wick’s theorem leads to
\[
\Pi^{\mu\nu}(q, i\omega_q) = -\frac{e_0^2}{\epsilon_0} \frac{T}{(2\pi)^3} \sum_\omega \int d^3 k \text{Tr} \left[ \gamma^\mu S_F(k, i\omega_k) \gamma^\nu S_F(k + q, i\omega_k + i\omega_q) \right].
\]
The substitution of Eq. (S.3) into this equation yields

\[
\Pi^{\mu\nu}(q, i\omega_q) = \frac{e^2}{\varepsilon_0} \frac{T}{(2\pi)^3} \sum_{\omega_k} \int d^3k d^3k' \text{Tr} \left[ \gamma^\mu S_1(k, i\omega_k) \gamma^\nu S_1(k', i\omega_k + i\omega_q) \right] \delta^3(k-k'+q)
\]

\[
= \frac{e^2}{\varepsilon_0} \int d^3k d^3k' \text{sgn}(k_0) \delta(K^2-m^2c^2) \int d^3k' \text{sgn}(k'_0) \delta(K'^2-m^2c^2) \times T \sum_{\omega_k} \frac{\text{Tr} [\gamma^\mu(K + m^2c^2) \gamma^\nu(K' + m^2c^2)]}{(i\omega_k + \mu - c^2k_0)(i\omega_k + \mu - c^2k'_0)} \delta^3(k-k'+q).
\]

Using Eqs. (S.1) and (S.2), we obtain \(\Pi^{\mu\nu}(q, i\omega_q)\) as

\[
\Pi^{\mu\nu}(q, i\omega_q) = -\frac{e^2}{2\pi^2\varepsilon_0} \int d^4K \text{sgn}(k_0) \delta(K^2-m^2c^2) \int d^4K' \text{sgn}(k'_0) \delta(K'^2-m^2c^2) \times \frac{K^\mu K'^\nu + K'^\mu K^\nu - (K \cdot K' - m^2c^2)g^{\mu\nu}}{i\omega_q + \mu - c^2k_0 - c^2k'_0} [f(c^*k_0) - f(c^*k'_0)] \delta^4(K-K' + q).
\]

Now we perform the analytic continuation of \(\Pi^{\mu\nu}_R(q, \omega) = \Pi^{\mu\nu}(q, i\omega_q)\) with \(\eta\) being a positive infinitesimal value. The imaginary part of \(\Pi^{\mu\nu}_R(q, \omega)\) is then given by

\[
\text{Im} \Pi^{\mu\nu}_R(q, \omega) = \frac{e^2}{4\pi^2\varepsilon_0 c} \int d^4K \text{sgn}(k_0) \delta(K^2-m^2c^2) \int d^4K' \text{sgn}(k'_0) \delta(K'^2-m^2c^2) \times \frac{2K^\mu K'^\nu + 2K'^\mu K^\nu - 2Q^\mu Q^\nu + Q^2 g^{\mu\nu}}{i\omega_q + \mu - c^2k_0 - c^2k'_0} [f(c^*k_0) - f(c^*k'_0)] \delta^4(K-K' + Q).
\]

(\text{S.4})

4. Imaginary parts of \(\chi_e(q, \omega)\) and \(\Delta\chi(q, \omega)\)

From Eq. (S.4), we obtain the following expressions for the imaginary parts of \(\chi_e(q, \omega) = \Pi^{\mu\nu}_R(q, \omega)/q^2\) and \(\Delta\chi(q, \omega) = \left[ 3Q^2\chi_e(q, \omega) + g_{\mu\nu}\Pi^{\mu\nu}_R(q, \omega) \right]/2q^2:\n
\[
\text{Im} \chi_e(q, \omega) = \frac{e^2}{4\pi^2\varepsilon_0 c q^2} \int d^4K \text{sgn}(k_0) \delta(K^2-m^2c^2) \int d^4K' \text{sgn}(k'_0) \delta(K'^2-m^2c^2) \times \frac{2k^2_0 + 2k'^2_0 - 2q^2_0 + Q^2}{i\omega_q + \mu - c^2k_0 - c^2k'_0} [f(c^*k_0) - f(c^*k'_0)] \delta^4(K-K' + Q),
\]

(\text{S.5})

\[
\text{Im} \Delta\chi(q, \omega) = \frac{e^2}{8\pi^2\varepsilon_0 c q^2} \int d^4K \text{sgn}(k_0) \delta(K^2-m^2c^2) \int d^4K' \text{sgn}(k'_0) \delta(K'^2-m^2c^2) \times \left[ 3Q^2(2k^2_0 + 2k'^2_0 - 2q^2_0 + Q^2) + 2q^2(2m^2c^2 + Q^2) \right]
\]
Noting that the integrands in Eqs. (S.5) and (S.6) do not depend on \( k \) and \( k' \) except in the delta functions, we first consider the integral

\[
\int \int d^3k k' \delta (K^2 - m^* c^2) \delta (K'^2 - m^* c^2) \delta^3 (k - k' + q).
\]

This integral equals

\[
\int d^3k \delta \left( k^2 - k_0^2 - m^* c^2 \right) \delta \left( k_0^2 - (k + q)^2 - m^* c^2 \right)
= \frac{\pi}{2q} \int_0^\infty dk^2 \delta \left( k^2 - k_0^2 + m^* c^2 \right) \theta \left( k^2 - \frac{(k_0^2 - k_0^2 - q^2)^2}{4q^2} \right)
= \frac{\pi}{2q} \theta \left( 4q^2 (k_0^2 - m^* c^2) - (k_0^2 - k_0^2 - q^2)^2 \right).
\]

Then Eq. (S.5) leads to

\[
\text{Im} \chi_s (q, \omega) = \frac{e_0}{4\pi^2 c^* \varepsilon_0 q^2} \int_{-\infty}^{\infty} dk_0 \ \text{sgn}(k_0) \int_{-\infty}^{\infty} dk_0' \ \text{sgn}(k_0') \left( 2k_0^2 + 2k_0'^2 - 2q_0^2 + Q^2 \right)
\times \frac{\pi}{2q} \theta \left( 4q^2 (k_0^2 - m^* c^2) - (k_0^2 - k_0^2 - q^2)^2 \right) \left[ f(c^* k_0) - f(c^* k_0') \right] \delta(k_0 - k_0' + q_0),
\]

\[
= \frac{e_0}{4\pi^2 c^* \varepsilon_0 q^2} \int_{-\infty}^{\infty} dk_0 \ \text{sgn}(k_0, q_0) \left( 2k_0 + q_0 \right)^2 - q_0^2 \right]
\times \frac{\pi}{2q} \theta \left( -Q^2 (2k_0 + q_0)^2 + q^2 (Q^2 - 4m^* c^2) \right) \left[ f(c^* k_0) - f(c^* k_0 + c^* q_0) \right] .
\]

Changing the integral variable from \( k_0 = (q x - q_0)/2 \) to \( x = q_0/\sqrt{\omega^2 - c^2 q^2} \), we get

\[
\text{Im} \chi_s (q, \omega) = \frac{e_0}{16 \varepsilon_0 c^*} \int_{-\infty}^{\infty} dx \left( x^2 - 1 \right) \frac{\omega^2}{\sqrt{\omega^2 - c^2 q^2}} \theta \left( -Q^2 x^2 + Q^2 - 4m^* c^2 \right)
\times \left[ f \left( \frac{c^* q x - \omega}{2} \right) - f \left( \frac{c^* q x + \omega}{2} \right) \right] .
\]

In the above equation, the Heaviside step function can be separated into two parts as

\[
\theta \left( -Q^2 x^2 + Q^2 - 4m^* c^2 \right) = \theta \left( -Q^2 \right) \theta \left( x^2 - \frac{Q^2 - 4m^* c^2}{Q^2} \right) + \theta \left( Q^2 \right) \theta \left( \frac{Q^2 - 4m^* c^2}{Q^2} \right) - x^2 \right)
= \theta \left( -Q^2 \right) \theta \left( x^2 - a^2 \right) + \theta \left( Q^2 - 4m^* c^2 \right) \theta \left( a^2 - x^2 \right),
\]

where \( a = \sqrt{Q^2 - 4m^* c^2} \) is a positive real function of \( Q^2 \) in the presence of \( \theta \left( -Q^2 \right) \) or \( \theta \left( Q^2 - 4m^* c^2 \right) \). Furthermore, the sign function takes 1 for the constraint imposed by \( \theta \left( -Q^2 \right) \theta \left( x^2 - a^2 \right) \) and \(-1\) for the constraint imposed by \( \theta \left( Q^2 - 4m^* c^2 \right) \theta \left( a^2 - x^2 \right) \). This can be easily understood when the argument of the sign function is written as

\[
x^2 - \frac{\omega^2}{c^2 q^2} = x^2 - \frac{4m^2 c^2}{Q^2} - \frac{Q^2}{q^2}.
\]
Then, we obtain the imaginary part of $\chi_e(q, \omega)$ as
\[
\text{Im} \chi_e(q, \omega) = \frac{e^2_0}{16\pi\varepsilon_0 c^*} \theta(-Q^2) \int_{-\infty}^{\infty} dx \left( x^2 - 1 \right) \theta(\frac{c^* q x - \omega}{2} - \frac{c^* q x + \omega}{2}) \\
- \frac{e^2_0}{16\pi\varepsilon_0 c^*} \theta(Q^2 - 4m^2 c^*) \int_{-\infty}^{\infty} dx \left( x^2 - 1 \right) \theta(\frac{a^2 - x^2}{2}) \left[ f(\frac{c^* q x - \omega}{2}) - f(\frac{c^* q x + \omega}{2}) \right].
\]
(S.7)

In a similar way from Eq. (S.6), we also obtain the imaginary part of $\Delta \chi(q, \omega)$ as
\[
\text{Im} \Delta \chi(q, \omega) = \frac{e^2_0}{32\pi\varepsilon_0 c^*} \frac{Q^2}{q^*} \theta(-Q^2) \int_{-\infty}^{\infty} dx \left( 3x^2 - a^2 \right) \theta(\frac{x^2 - a^2}{2}) \left[ f(\frac{c^* q x - \omega}{2}) - f(\frac{c^* q x + \omega}{2}) \right] \\
- \frac{e^2_0}{32\pi\varepsilon_0 c^*} \frac{Q^2}{q^*} \theta(Q^2 - 4m^2 c^*) \int_{-\infty}^{\infty} dx \left( 3x^2 - a^2 \right) \theta(\frac{a^2 - x^2}{2}) \left[ f(\frac{c^* q x - \omega}{2}) - f(\frac{c^* q x + \omega}{2}) \right].
\]
(S.8)

We note that similar equations as Eqs. (S.7) and (S.8) can be found in the study of quantum electrodynamics at finite temperatures and densities.\(^{4)\)

At zero temperature, Eqs. (S.7) and (S.8) reduce to
\[
\text{Im} \chi_e(q, \omega) = \frac{e^2_0}{16\pi\varepsilon_0 c^*} \theta(-Q^2) \int_{-\infty}^{\infty} dx \left( x^2 - 1 \right) \theta(\frac{x^2 - a^2}{2}) \left[ \theta\left(\mu - \frac{c^* q x - \omega}{2}\right) - \theta\left(\mu - \frac{c^* q x + \omega}{2}\right) \right] \\
- \frac{e^2_0}{16\pi\varepsilon_0 c^*} \theta(Q^2 - 4m^2 c^*) \int_{-\infty}^{\infty} dx \left( x^2 - 1 \right) \theta(\frac{x^2 - a^2}{2}) \left[ \theta\left(\mu - \frac{c^* q x - \omega}{2}\right) - \theta\left(\mu - \frac{c^* q x + \omega}{2}\right) \right] \\
\times \int_{-\infty}^{\infty} dx \left( x^2 - 1 \right) \theta(\frac{a^2 - x^2}{2}) \left[ \theta\left(\mu - \frac{c^* q x - \omega}{2}\right) - \theta\left(\mu - \frac{c^* q x + \omega}{2}\right) \right],
\]
(S.9)

\[
\text{Im} \Delta \chi(q, \omega) = \frac{e^2_0}{32\pi\varepsilon_0 c^*} \frac{Q^2}{q^*} \theta(-Q^2) \int_{-\infty}^{\infty} dx \left( 3x^2 - a^2 \right) \theta(\frac{x^2 - a^2}{2}) \left[ \theta\left(\mu - \frac{c^* q x - \omega}{2}\right) - \theta\left(\mu - \frac{c^* q x + \omega}{2}\right) \right] \\
- \frac{e^2_0}{32\pi\varepsilon_0 c^*} \frac{Q^2}{q^*} \theta(Q^2 - 4m^2 c^*) \int_{-\infty}^{\infty} dx \left( 3x^2 - a^2 \right) \theta(\frac{a^2 - x^2}{2}) \left[ \theta\left(\mu - \frac{c^* q x - \omega}{2}\right) - \theta\left(\mu - \frac{c^* q x + \omega}{2}\right) \right] \\
\times \int_{-\infty}^{\infty} dx \left( 3x^2 - a^2 \right) \theta(\frac{a^2 - x^2}{2}) \left[ \theta\left(\mu - \frac{c^* q x - \omega}{2}\right) - \theta\left(\mu - \frac{c^* q x + \omega}{2}\right) \right].
\]
(S.10)

5. **Derivation of Eqs. (7) and (8)**

As seen from Eqs. (S.7) and (S.8), the imaginary parts of $\chi_e(q, \omega)$ and $\Delta \chi(q, \omega)$ are odd functions of the frequency $\omega$ while even functions of the chemical potential $\mu$. The former is related to the analyticity while the latter to the particle-hole symmetry. By taking them into account, Eqs. (S.9) and (S.10) can be transformed into
\[
\text{Im} \chi_e(q, \omega) = \frac{e^2_0}{16\pi\varepsilon_0 c^*} \text{sgn}(\omega) \theta(-Q^2) \int_{-\infty}^{\infty} dx \left( x^2 - 1 \right) \theta(\frac{x^2 - a^2}{2}) \left[ \theta(b_+ - x) - \theta(b_- - x) \right]
\]
we can express Eqs. (S.7) and (S.8) as

\[
\text{Im}\Delta\chi(q, \omega) = \frac{e^2}{32\pi\varepsilon_0 c^3} \frac{Q^2}{q^2} \text{sgn}(\omega) \theta(Q^2 - 4m^2 c^2) \int_{-\infty}^{\infty} dx (3x^2 - a^2) \theta(x^2 - a^2) \left[\theta(b_+ - x) - \theta(b_- - x)\right],
\]

where \(b_\pm \equiv \frac{2|q|\pm|d|}{e^2} \). The above equations for \(\text{Im}\chi_c(q, \omega)\) and \(\text{Im}\Delta\chi(q, \omega)\) correspond to Eqs. (7) and (8) in the text, respectively.

6. Derivation of Eqs. (11) and (12)

For the noninteracting electron gas, it is known that the finite-temperature polarizability can be given as an integral of the zero-temperature polarizability with respect to the chemical potential.\(^5\) In the same way as for the electron gas, we can express the finite-temperature susceptibility as an integral of the zero-temperature susceptibility with respect to the chemical potential for our “covariant” electron-hole gas.

Let us write \(\chi_c(q, \omega)\), \(\chi_m(q, \omega)\), and \(\Delta\chi(q, \omega)\) as \(\chi_c(q, \omega; T, \mu)\), \(\chi_m(q, \omega; T, \mu)\), and \(\Delta\chi(q, \omega; T, \mu)\), respectively, to show their dependences on \(T\) and \(\mu\) explicitly. By using Eqs. (S.9) and (S.10) with the identity\(^5\)

\[
f(x) = \frac{1}{e^{(x-\mu)/T} + 1} = \int_{-\infty}^{\infty} d\mu' \frac{\theta(\mu' - x)}{4T \cosh^2 \frac{\mu' - \mu}{2T}},
\]

we can express Eqs. (S.7) and (S.8) as

\[
\text{Im}\chi_c(q, \omega; T, \mu) = \int_{-\infty}^{\infty} d\mu' \frac{\text{Im}\chi_c(q, \omega; 0, \mu')}{4T \cosh^2 \frac{\mu' - \mu}{2T}},
\]

\[
\text{Im}\Delta\chi(q, \omega; T, \mu) = \int_{-\infty}^{\infty} d\mu' \frac{\text{Im}\Delta\chi(q, \omega; 0, \mu')}{4T \cosh^2 \frac{\mu' - \mu}{2T}}.
\]

Because their real parts are related to their imaginary parts by the Kramers–Kronig relation, the finite-temperature electric and magnetic susceptibilities can be given as

\[
\chi_c(q, \omega; T, \mu) = \int_{-\infty}^{\infty} d\mu' \frac{\chi_c(q, \omega; 0, \mu')}{4T \cosh^2 \frac{\mu' - \mu}{2T}},
\]

\[
\chi_m(q, \omega; T, \mu) = -\frac{c^2}{e^2} \left[\chi_c(q, \omega; T, \mu) + \int_{-\infty}^{\infty} d\mu' \frac{\Delta\chi(q, \omega; 0, \mu')}{4T \cosh^2 \frac{\mu' - \mu}{2T}}\right].
\]

The above equations for \(\chi_c(q, \omega; T, \mu)\) and \(\chi_m(q, \omega; T, \mu)\) correspond to Eqs. (11) and (12) in
7. Derivation of Eq. (15)

From Eqs. (10) and (13) in the text, the electric susceptibility $\chi_e(q, \omega)$ is given for $T = 0$ and $|\mu| < m^*c^2$ as

$$\chi_e(q, \omega) = -\frac{1}{\pi} \int_0^{2E_\Lambda} d\omega' \left( \frac{1}{\omega_+ - \omega'} - \frac{1}{\omega_+ + \omega'} \right) \text{Im} \chi_e(q, \omega')$$

$$= \frac{e_0^2}{24\pi^2E_0c^*} \int_0^{2E_\Lambda} d\omega' \frac{2\omega'}{\omega^2 - \omega'_2} a(Q^2) \left[ 3 - a^2(Q^2) \right] \theta(Q^2 - 4m^2c^2)$$

$$= \frac{e_0^2}{24\pi^2E_0c^*} \int_{4m^2c^2}^{4E_\Lambda/c^2} \frac{dQ^2}{Q^2 - Q_+^2} a(Q^2) \left[ 3 - a^2(Q^2) \right],$$

where $a(Q^2) = \sqrt{1 - 4m^2c^2/Q^2}$ with $Q^2 = \omega^2/c^2 - q^2$. For $q \ll E_\Lambda/c^*$, the electric susceptibility $\chi_e(q, \omega)$ becomes a function of only $Q_+^2$. Then we introduce $\Pi_2(Q_+^2) \equiv -\chi_e(q, \omega)$ as

$$\Pi_2(Q_+^2) = -\frac{e_0^2}{24\pi^2E_0c^*} \int_{4m^2c^2}^{4E_\Lambda/c^2} \frac{dQ^2}{Q^2 - Q_+^2} a(Q^2) \left[ 3 - a^2(Q^2) \right].$$

Changing the integral variable from $Q^2$ to $\alpha' = a(Q^2)$, we obtain

$$\Pi_2(Q_+^2) = -\frac{e_0^2}{12\pi^2E_0c^*} \int_0^{\sqrt{1 - m^2c^2/E_\Lambda}} d\alpha' \left[ \frac{2}{1 - \alpha'^2} + \frac{4}{s} \left( 1 + \frac{2}{s} \right) \frac{2(1 - \frac{4}{s})}{1 - \frac{4}{s} - \alpha'^2} \right],$$

where $s = Q_+^2/m^2c^2$. The integral with respect to $\alpha'$ is elementary; for $m^2c^2/E_\Lambda \ll 1$, it leads to

$$\Pi_2(Q_+^2) = -\frac{e_0^2}{12\pi^2E_0c^*} \left[ \log \frac{4E_\Lambda^2}{m^2c^4} + \frac{4}{s} \left( 1 + \frac{2}{s} \right) \sqrt{1 - \frac{4}{s}} \log \frac{\sqrt{1 - \frac{4}{s} + 1}}{\sqrt{1 - \frac{4}{s} - 1}} \right].$$

The above equation corresponds to Eq. (15) in the text.
References

1) For example, see G. F. Giuliani and G. Vignale, Quantum Theory of the Electron Liquid (Cambridge University Press, Cambridge, U.K., 2005).

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