NUMERICAL SIMULATIONS FOR THE ENERGY-SUPERCRITICAL NONLINEAR WAVE EQUATION

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ABSTRACT. We carry out numerical simulations of the defocusing energy-
supercritical nonlinear wave equation for a range of spherically-symmetric ini-
tial conditions. We demonstrate numerically that the critical Sobolev norm of
solutions remains bounded in time. This lends support to conditional scatter-
ing results that have been recently established for nonlinear wave equations.

1. INTRODUCTION

In recent years, there has been a great deal of progress in understanding the
long-time behavior of solutions to nonlinear dispersive partial differential equations.
One line of research has focused on the scattering problem for large solutions under
optimal regularity assumptions on the initial conditions, particularly in the setting
doing nonlinear Schrödinger equations (NLS) and wave equations (NLW).
Progress in this direction was precipitated especially by the development of new

texts (e.g. the concentration compactness approach to induction on energy)
that were developed in order to establish global well-posedness and scattering (i.e.
asymptotically linear behavior) for certain special cases, e.g. the mass-
and energy-critical NLS. Outside of these special cases, however, current techniques are often
limited to proving conditional results, in which one shows that scatte-
ing occurs
under the assumption of a priori bounds for a critically-scaling Sobolev norm. In
this paper, we will present numerical simulations for the energy-supercritical NLW
that lend support to the veracity of these assumed critical bounds. A similar study
was carried out in [10] in the setting of the energy-supercritical NLS.

To describe the problem and our results more precisely, we introduce the equa-
tions

\[ i\partial_t u + \Delta u = \mu |u|^p u, \quad u : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C} \] (NLS)

and

\[ -\partial^2_t u + \Delta u = \mu |u|^p u, \quad u : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}. \] (NLW)

In each case, the parameter \( \mu \) yields either the defocusing (\( \mu > 0 \)) or focusing
(\( \mu < 0 \)) case, and \( p > 0 \) is the power of the nonlinearity. These are Hamiltonian
PDE, with the conserved energy given by

\[ E(u) = \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u|^2 + \frac{\mu}{p+2} |u|^{p+2} \, dx \quad \text{for NLS} \]

and

\[ E(u, \partial_t u) = \int_{\mathbb{R}^d} \frac{1}{2} |\partial_t u|^2 + \frac{1}{2} |\nabla u|^2 + \frac{\mu}{p+2} |u|^{p+2} \, dx \quad \text{for NLW}. \] (1.1)
Both equations also enjoy a scaling symmetry, namely
\[
\lambda^\frac{2}{p} u(\lambda^2 t, \lambda x) \quad \text{for NLS}
\]
\[
\lambda^\frac{2}{p} u(\lambda t, \lambda x) \quad \text{for NLW},
\]
which defines a notion of critical regularity for these equations. In particular, if we define
\[
s_c = \frac{d}{2} - \frac{2}{p},
\]
then one finds that the \(\dot{H}^{s_c}\)-norm of \(u|_{t=0}\) for NLS and the \(\dot{H}^{s_c} \times \dot{H}^{s_c - 1}\) norm of \((u, \partial_t u)|_{t=0}\) for NLW are invariant under the rescaling (1.2). Generally speaking, these are the optimal spaces for initial data in terms of the well-posedness theory of (NLS) and (NLW); see e.g. [7, 8, 43].

The main topic of this paper is the question of scattering. We say that a forward-global solution \(u\) to (NLW) scatters (in \(\dot{H}^{s_c} \times \dot{H}^{s_c - 1}\)) if there exists a solution \(v(t)\) to the linear wave equation such that
\[
\lim_{t \to \infty} \|(u(t), \partial_t u(t)) - (v(t), \partial_t v(t))\|_{\dot{H}^{s_c} \times \dot{H}^{s_c - 1}} = 0.
\]
An analogous definition holds for solutions to (NLS).

A special case of (NLS) and (NLW), called the energy-critical case, occurs when the scaling symmetry (1.2) leaves the energy of the solution invariant as well. This corresponds to choosing \(p = \frac{4d}{d-2}\) in dimensions \(d \geq 3\), or equivalently \(s_c = 1\). For the case of NLS, there is also the mass-critical case corresponding to \(p = \frac{4d}{d}\), in which case the scaling symmetry preserves the mass (i.e. the \(L^2\) norm), which is a conserved quantity for NLS (but not for NLW). For these special cases, conservation of energy/mass yields a priori control over the critical Sobolev norm (in the defocusing case, at least). Ultimately, this provides enough control over solutions to establish global well-posedness and scattering, although proving this is a very challenging problem that required the work of many mathematicians over many years to settle definitively (see [1, 3, 9, 11–15, 25–27, 29–31, 34, 35, 39, 41, 42, 43, 52, 53, 58, 60–64]):

**Theorem 1.1** (Global well-posedness and scattering). For the defocusing case of the mass- and energy-critical NLS or energy-critical NLW, arbitrary initial data in the critical Sobolev space lead to global solutions that scatter. Similar results hold in the focusing case, provided one imposes suitable size restrictions on the mass/energy.

The resolution of Theorem 1.1 required the development of a powerful new set of techniques. The initial breakthrough was due to Bourgain, who introduced the method of ‘induction on energy’ [3]. This technique has been significantly developed and refined. Presently, the typical approach to problems as in Theorem 1.1 follows the so-called ‘Kenig–Merle roadmap’ developed in [31]. One proceeds by contradiction: Assuming the theorem to be false, one constructs a minimal energy counterexample, which (due to minimality) enjoys certain compactness properties. One then shows that such compactness properties are at odds with the dispersive/conservative nature of the equation and ultimately lead to a contradiction; this is often achieved through the use of conservation laws together with certain nonlinear estimates known as virial or Morawetz estimates. For an expository introduction to these techniques, we refer the reader to [40, 65].

\[
1\text{Here } \dot{H}^s \text{ denotes the homogeneous } L^2\text{-based Sobolev space; see Section 1.1.}
\]
Beginning with the work of Kenig and Merle \cite{32}, a great deal of recent research has focused on establishing analogous results beyond the mass- and energy-critical cases. In such cases, the ‘Kenig–Merle roadmap’ naturally leads to a proof of scattering under the assumption of a priori bounds in the critical Sobolev space, where the assumed bounds play the role of the ‘missing conservation law’ at critical regularity. Stated roughly, we have the following conjecture:

**Conjecture 1.2.** For the defocusing NLS or NLW, any solution that remains bounded in the critical Sobolev space is global-in-time and scatters.

By now, the range of positive results of this type is extensive. For the case of NLS, see \cite{19,24,32,48,66,67}; for the case of NLW, see \cite{1,6,18,20,23,33,37,51,54,55}. While some recent remarkable work of Dodson \cite{16,17} has actually established unconditional scattering results at critical regularity for the energy-subcritical NLW with radial initial data, the majority of the scattering results for large data at ‘non-conserved’ critical regularity are conditional in nature.

In this paper, we carry out numerical simulations for the energy-supercritical NLW with radial (i.e. spherically symmetric) initial conditions, where energy-supercritical refers to the condition $p > \frac{4}{d-2}$, or equivalently $s_c > 1$. In the radial setting, (NLW) takes the form

$$-\partial_t^2 u + \partial_r^2 u + \frac{d-1}{r} \partial_r u = \mu |u|^p u, \quad u : \mathbb{R}^d \times (0, \infty) \to \mathbb{R},$$

where we write $u = u(t,r)$, with $r > 0$ and impose the Neumann boundary condition $\partial_r u|_{r=0} \equiv 0$. As in \cite{10}, the restriction to radial solutions provides a significant simplification in the numerical analysis of (NLW). Our main result is to demonstrate (numerically) boundedness of the critical Sobolev norms for large time, thus lending support to the conditional scattering results discussed above. We expect that similar results will hold in the non-radial setting and plan to address this case in future work.

For the sake of concreteness, we focus on two representative cases, namely,

$$(d, p, s_c) = (3, 6, \frac{7}{6}) \quad \text{and} \quad (d, p, s_c) = (5, 2, \frac{3}{2}).$$

These particular cases were considered in the works \cite{5,30,33,37,38}, which established scattering under the assumption of a priori bounds for $(u, \partial_t u)$ in $H^s \times H^{s-1}$. We study a range of choices for $u_0 = u(0, r)$ and $u_1 = \partial_t u(0, r)$ (see Section 3), and in all cases we observe (numerically) that the critical Sobolev norm converges after a short time and, in particular, remains bounded for large times. Additionally, we compute numerically the potential energy (i.e. the $L^{p+2}$ norm), the $L^\infty$ norm, and certain scale-invariant Besov norms. We observe that the Besov norms become relatively small (compared to the Sobolev norms), and that the higher Lebesgue norms decay at a rate that matches solutions to the linear wave equation. All of this behavior is consistent with scattering. We describe our results in detail in Section 4.

At present, existing analytic techniques are generally insufficient to rigorously establish the boundedness in time of the critical Sobolev norm of solutions, unless such norms can be controlled by conserved quantities (although we should mention again the remarkable work of \cite{16,17} for the case of the radial energy-subcritical

\[2\text{Note that radiality is preserved in time. This is a consequence of the fact that the Laplacian commutes with rotations, together with the uniqueness of solutions.}\]
NLW). In particular, this problem seems to be especially challenging in the energy-supercritical regime, as there is no known coercive conserved quantity above the regularity of the energy. Nonetheless, such boundedness is generally expected to hold true in the defocusing setting. Indeed, both the dispersion of the underlying linear equation and the defocusing nature of the nonlinearity tend to cause solutions to spread out and decay. Our results provide additional numerical evidence in support of the belief of boundedness and lend support to the wide range of conditional scattering results for NLW that have been established in recent years. It remains an important open problem in the analysis of nonlinear dispersive PDE to prove rigorously that energy-supercritical Sobolev norms do indeed remain bounded in time.

The rest of this paper is organized as follows: In Section 1.1, we collect some basic notation and preliminaries. In Section 2, we describe the numerical methods we use in this work. In Section 3, we describe the sets of initial conditions used in the numerical simulations. In Section 4, we describe our numerical findings. In Appendix A, we prove a simple scattering result (namely, scattering holds if the critical Besov norm is sufficiently small compared to the critical Sobolev norm), which is relevant to the discussion in Section 4. Finally, in Appendix B we discuss the notion of incoming/outgoing waves, which play a role in our choice of initial conditions.

1.1. Notation and preliminaries. We use the standard notation for Lebesgue norms, e.g.

\[ \|u\|_{L^r_x(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} |u(x)|^r \, dx \right)^{\frac{1}{r}} \]

for \(1 \leq r < \infty\). We denote space-time norms by \(L^q_t L^r_x\), i.e.

\[ \|u\|_{L^q_t L^r_x(I \times \mathbb{R}^d)} = \|u(t)\|_{L^r_x(\mathbb{R}^d)} \|u\|_{L^q_t(I)}. \]

We define Sobolev and Besov norms by utilizing the Fourier transform \(F\). We define the homogeneous \(L^2\)-based Sobolev spaces are then defined by

\[ \|u\|_{\dot{H}^s_{x}(\mathbb{R}^d)} = \|\nabla^s u\|_{L^2(\mathbb{R}^d)} = \|\xi^s \hat{u}\|_{L^2(\mathbb{R}^d)}. \]

Besov spaces are defined using the standard Littlewood–Paley multipliers. In particular, for \(N \in 2\mathbb{Z}\) we let \(\varphi_N\) be denote a smooth bump function supported where \(|\xi| \sim N\), with \(\sum \varphi_N = 1\). We then define the Littlewood–Paley projections \(P_N u\) through the Fourier transform, i.e.

\[ P_N u = \mathcal{F}^{-1} \varphi_N \hat{u}. \]

The Besov norm \(\dot{B}^{s}_{q,c}\) is defined via

\[ \|u\|_{\dot{B}^{s}_{q,c}(\mathbb{R}^d)} = \|N^s P_N u\|_{L^q(\mathbb{R}^d)} \|_{L^r_c(2^c)}. \]

We will frequently consider the Sobolev norm

\[ \| (u, \partial_t u) \|_{H^{s_c}_{x} \times H^{s_c - 1}_x}^2 := \|u\|_{H^{s_c}_x}^2 + \|\partial_t u\|_{H^{s_c - 1}_x}^2, \quad \text{where} \quad s_c > 1, \]

for \(u \in H^{s_c}_x \times H^{s_c - 1}_x\).
as well as the Besov norms
\[ \|u\|_{B^{s_c}_{p,r}} = \sup_{N \in 2^{\mathbb{Z}}} N^{s_c} \|P_N u\|_{L^2} \quad \text{and} \quad \|\partial_t u\|_{B^{s_c-1}_{p,r}} = \sup_{N \in 2^{\mathbb{Z}}} N^{s_c-1} \|P_N \partial_t u\|_{L^2}, \]

Note that for any \( s \), we have
\[ N^s \|\varphi_N u\|_{L^2} \lesssim \|u\|_{\dot{H}^s}, \]
uniformly in \( N \), which implies
\[ \|u\|_{\dot{B}^{s_c}_{p,r}} \lesssim \|u\|_{\dot{H}^{s_c}} \quad \text{and} \quad \|\partial_t u\|_{\dot{B}^{s_c-1}_{p,r}} \lesssim \|\partial_t u\|_{\dot{H}^{s_c-1}}. \]

As mentioned above, the restriction to radial (i.e. spherically symmetric) solutions leads to some simplifications. We have already mentioned the simplification of the PDE (and hence the numerical analysis). Additionally, we may change to spherical coordinates and write
\[ \hat{u}(\xi) = (2\pi)^{-\frac{d}{2}} \int_0^{\infty} \left[ \int_{\partial B(0,1)} e^{-i\xi \cdot \omega} dS(\omega) \right] u(r)r^{d-1} dr \]
\[ = |\xi|^{-\frac{d-2}{2}} \int_0^{\infty} J_{d-2}(r|\xi|)u(r)r^\frac{d}{2} dr, \]
where \( J_{d} \) denotes the standard Bessel function (see e.g. [57]). This informs our numerical computation of the Fourier transform, and therefore the relevant Sobolev and Besov norms.

2. NUMERICAL METHODS

In this section, we present a finite difference method to solve the radial wave equation (1.4). First, we truncate the computational domain to \([0, R_{\max}]\) with \( R_{\max} \) sufficiently large that the truncating effect can be neglected. A homogeneous Neumann boundary condition is considered \( r = 0 \), while a Dirichlet condition is taken at \( r = R_{\max} \):
\[ \partial_r u(t, 0) = 0, \quad u(t, R_{\max}) = 0, \quad \text{for} \ t \geq 0. \quad (2.1) \]
Since \( u_r \equiv 0 \) at point \( r = 0 \), we use L'Hôpital's rule for the third term of (1.4) and thus reduce the wave equation (1.4) to
\[ \partial_r u + d \partial_r u - \mu |u|^p u \quad \text{at} \quad r = 0, \quad (2.2) \]
while \( r \in (0, R_{\max}) \) we have reformulated (1.4) as
\[ \partial_r u = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left( r^{d-1} \frac{\partial u}{\partial r} \right) - \mu |u|^p u \quad \text{for} \ r \in (0, R_{\max}). \quad (2.3) \]

Denote the mesh size \( h = R_{\max}/N \) with \( N \) a positive integer, and define the spatial grid points \( r_j = jh \) for \( 0 \leq j \leq N \). Let \( U_j(t) \) denote the numerical approximation of \( u(t, r_j) \). We then apply the second-order finite difference method to discretize the spatial domain of (2.2) and (2.3) and obtain the semi-discretization scheme as follows:
\[ U_0''(t) = \frac{d}{dr} \left( U_1 - 2U_0 + U_{-1} \right) - \mu |U_0|^p U_0, \]
\[ U_j''(t) = \frac{1}{h^2} \left( \left( \frac{U_{j+1/2}}{r_j} \right)^{d-1} \left( U_{j+1} + U_{j} \right) + \left( \frac{r_j-1/2}{r_j} \right)^{d-1} \left( U_{j-1} + U_{j} \right) \right) - \mu |U_j|^p U_j \quad (2.4) \]
for \( j = 1, 2, \ldots, N-1 \), where \( r_j \pm 1/2 = r_j \pm h/2 \), and \( U_{-1} \) represents the solution at the ghost point \( r_{-1} = -h \). The discretization of boundary conditions (2.1) leads to
\[ \frac{U_1 - U_{-1}}{2h} = 0, \quad U_N = 0. \quad (2.5) \]
The first boundary condition in (2.5) implies that $U_{-1} = U_1$ at any time $t$, and thus can be used to simply the scheme (2.4) at $r = r_0$.

Let $\Delta t > 0$ denote the time step, and define time sequence $t_n = n\Delta t$ for $n = 0, 1, \ldots$. The time domain of the system (2.4) is discretized by using the central difference scheme. The initial condition $\partial_t u(t, r)$ at $t = 0$ is discretized by the backward Euler method to obtain the numerical solution at $t = t_1$. We omit the details for the sake of brevity. In our simulations, we choose the time step $\Delta t$ satisfying

$$\Delta t \leq h \sqrt{\frac{2d-1}{1+3d-1}}$$

(2.6)

to ensure the stability of the scheme. The above condition suggests that the simulations of (1.4) for higher dimensions are generally more time-consuming, since a smaller time step is required to ensure the numerical stability.

### 3. Choice of initial conditions

In this section we describe the specific choices of initial conditions used in this paper. The key assumption on the data is that of spherical symmetry, which reduces the equation to a one-dimensional problem and thereby greatly simplifies the numerical analysis. We begin by considering the cases of Gaussians that are large enough to be safely outside of the small data regime (for otherwise scattering is a known consequence of the small-data well-posedness theory). As in [10], we would also like to consider some initial data for which the underlying linear equation would experience some initial focusing toward the origin; for such data, we would then like to observe that the defocusing nonlinearity counters this effect. The authors of [10] achieved this by multiplying the Gaussian initial data by $e^{\alpha ir^2}$ for some $\alpha > 0$. In the setting of the radial wave equation, it seems natural to consider ‘incoming’ initial data for this purpose (see e.g. [2]), which in our setting refers to the condition

$$u_1 = \partial_r u_0 + \frac{d-2}{r} u_0,$$

(3.1)

where $u_0 = u|_{t=0}$ and $u_1 = \partial_t u|_{t=0}$. We discuss the origin of this condition in more detail in Appendix [B].

Apart from choosing incoming/outgoing initial data (which requires defining $u_1$ precisely in terms of $u_0$), we found that varying the choice of initial velocity $u_1$ plays almost no role in terms of the long-time behavior of the solution. Thus, other than the cases for which we take ‘incoming’ data, we will be content to work with the simplest choice $u_1 = 0$. As in [10], we will then choose $u_0$ to be Gaussian or ‘ring’ initial data of the form $C \exp(-r^2)$ and $Cr^2 \exp(-r^2)$, respectively, where $C \geq 1$ is chosen large enough.

As far as the incoming condition, we note that (3.1) includes the singular term $1/r$ in the expression for $u_1$. While one can verify that $u_1$ still belongs to $L^2 \cap H^{s-1}$ (see Lemma [3.1]), we found that unless $u_0$ vanishes to high enough order at $r = 0$, it is difficult to simulate this condition numerically. Thus we were led only to consider the incoming condition for the ‘ring’ initial data $u_0 = Cr^2 \exp(-r^2)$, in which case the presence of $1/r$ is harmless.
4. Numerical results

We considered three choices of initial conditions. As described in the introduction, for each case we consider the combinations \((d, p) = (3, 6)\) and \((d, p) = (5, 2)\). In the following, we summarize the quantities studied, as well as the corresponding findings. For each case, we study the time evolution of:

(i) The solution \(u(t,r)\). We find that the solution decays over time and travels outward at a constant speed.

(ii) The energy (1.1). Our numerical simulations conserve the energy (1.1), just as the NLW (1.4) does.

(iii) The critical Sobolev norms \(\|u(t)\|_{\dot{H}^{s_c}}\) and \(\|\partial_t u(t)\|_{\dot{H}^{s_c-1}}\), with \(s_c\) as in (1.3). We find that the critical Sobolev norms may initially oscillate, but quickly settle down and converge to a constant. Moreover, our numerical results show that
\[
\lim_{t \to \infty} \|u(t)\|_{\dot{H}^{s_c}} = \lim_{t \to \infty} \|\partial_t u(t)\|_{\dot{H}^{s_c-1}}.
\]

(iv) The potential energy \(\|u(t)\|_{L^{p+2}}\) and the supremum norm \(\|u(t)\|_{L^{\infty}}\). We find that both quantities tend to zero as \(t \to \infty\). More precisely, we observe
\[
\|u(t)\|_{L^{p+2}} \sim (1 + t)^{-\frac{d-1}{2(p+2)}} \quad \text{and} \quad \|u(t)\|_{L^{\infty}} \sim (1 + t)^{-\frac{d-1}{2}},
\]
for sufficiently large \(t\), matching the decay rates for the underlying linear wave equation.

(v) The critical Besov norms \(\|u(t)\|_{\dot{B}^{s_c}_{2,\infty}}\) and \(\|\partial_t u(t)\|_{\dot{B}^{s_c-1}_{2,\infty}}\). We find that the Besov norms remain bounded and become relatively small compared to the critical Sobolev norm as \(t \to \infty\). As discussed in Appendix A, if solutions are sufficiently dispersed in frequency relative to their Sobolev norm, then one can prove scattering by a small-data type argument.

In what follows, we present our numerical results for our three representative cases.

4.1. Case 1. Gaussian data. In this case, we take the initial condition
\[
\begin{align*}
\text{Case 1. Gaussian data.} & \quad \text{In this case, we take the initial condition} \\
& \quad \begin{align*}
u_0 &= 4 \exp(-r^2) , & u_1 &= 0 , & \text{for} & \quad r \geq 0.
\end{align*}
\end{align*}
\]

Figure 1 presents the time evolution of the solution, which shows that the solution decays over time and behaves essentially as a wave packet traveling outward with constant speed.

Next, we further study the decay of the solution \(u\). We would like to show decay of both the potential energy (i.e. the \(L^{p+2}\)-norm), as well as pointwise decay (i.e. the \(L^{\infty}\)-norm), both of which are consistent with scattering. In fact, we can show that the decay rate for these quantities matches the decay rate for solutions to the linear equation. To see this, we present in Figure 2 the time evolution of the following quantities and observe boundedness (in fact, convergence) for large \(t\):
\[
(1 + t)^{\frac{d-1}{2(p+2)}} \|u(t)\|_{L^{p+2}} \quad \text{and} \quad t^{\frac{d-1}{2}} \|u(t)\|_{L^{\infty}}.
\]

Figure 3 shows that numerically, the energy (1.1) is conserved over time. Due to a higher power nonlinearity, the energy for the case \((d, p) = (3, 6)\) is much larger than that of \((d, p) = (5, 2)\).

Figure 4 represents the main result of this paper, namely, numerical evidence for the boundedness of the critical Sobolev norms. In fact, after some initial oscillation,
we see that both the $\dot{H}^{s_c}$-norm of $u$ and the $\dot{H}^{s_c-1}$ norm of $\partial_t u$ quickly settle down and converge. Moreover, our numerical results show that
\[
\lim_{t \to \infty} \|u(t)\|_{\dot{H}^{s_c}} = \lim_{t \to \infty} \|\partial_t u(t)\|_{\dot{H}^{s_c-1}}.
\]

Finally, we plot the critical Besov norms (namely, $\dot{B}^{s_c}_{2,\infty}$ for $u$ and $\dot{B}^{s_c-1}_{2,\infty}$ for $\partial_t u$) in Figure 5. While these norms do not converge to zero, we can observe that they
become relatively small compared to the critical Sobolev norms in Figure 4. As discussed in Appendix A, it is possible to prove scattering in such a scenario.
4.2. Case 2. Ring data. In Case 2 we take
\[ u_0 = 10r^2 \exp(-r^2), \quad u_1 = 0, \quad \text{for } r \geq 0. \] (4.3)
As we observed the same phenomena as in Case 1, we will be somewhat brief in our presentation.

Figure 6 presents the time evolution of the solution. It shows that the solution is more oscillating than the case with Gaussian initial conditions; however, similar to Case 1, the solution still decays over time and travels outward.

**Figure 6.** Time evolution of solution of NLW (1.4) with initial condition (4.3).

Figure 7 shows the conservation of energy over time in this case. Compared to the results in Figure 3, the energy in Case 2 is much larger.

**Figure 7.** Conservation of energy of NLW with initial condition (4.3).
Figure 8 presents the critical Sobolev norms, Besov norms, and higher Lebesgue norms over time. It shows that the critical Sobolev norms $\|u\|_{\dot{H}^s}$ and $\|\partial_t u\|_{\dot{H}^{s-1}}$ quickly converge and stay at the same constant after a relatively small time, while the Besov norms are eventually bounded by a relatively small constant compared to the Sobolev norms. The time evolution of the higher Lebesgue norms shows that the decay of $\|u\|_{L^{p+2}}$ and $\|u\|_{L^\infty}$ are qualitatively the same as in Case 1.
4.3. **Case 3. Incoming ring data.** Finally, in Case 3 we take

\[ u_0 = 10r^2 \exp(-r^2), \quad u_1 = \partial_r u_0 + \frac{d-2}{r} u_0, \quad \text{for} \quad r \geq 0. \]  

(4.4)

As described above, such initial data will lead to some initial ‘focusing’ at the level of the linear wave equation. This effect is countered by the defocusing nonlinearity.

Figure 9 illustrates the time evolution of the solution \( u \), where the plot for \( t = 0 \) is the same as that in Figure 6. Due to different initial velocity \( u_1 \), the evolution of solution in Cases 2 and 3 is initially quite different (cf. Figures 6 and 9 for \( t = 1 \)), but after a long time the influence of initial velocity greatly decreases. We also verified that the energy is conserved over time, but we omit the energy plots for the sake of brevity.

![Figure 9. Time evolution of the solution to NLW (1.4) with initial condition (4.4).](image)

The time evolution of the norms in Figure 10 is quite similar to the observations in Figure 8. This suggests that the initial velocity \( u_1 \) ultimately plays an insignificant role in the study of long-time behavior of solution.
Figure 10. Time evolution of the Sobolev norms (first row), Besov norms (second row), and higher norms (last row) for NLW with initial condition (4.3).
Appendix A. A simple scattering result

In this section, we demonstrate that if a solution to \((\text{NLW})\) is sufficiently dispersed in frequency compared to its critical Sobolev norm, then it scatters. For the sake of concreteness, we focus on the particular case \((d, p) = (3, 6)\) considered above, in which case \(s_c = \frac{2}{3}\).

**Proposition A.1.** Suppose \(\|(u_0, u_1)\|_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{3}{2}}} = E\). Then there exists \(\eta_0 = \eta_0(E) > 0\) such that if

\[
\|(u_0, u_1)\|_{\dot{B}^{\frac{3}{2}}_{\infty}, \dot{B}^{\frac{3}{2}}_{\infty}} < \eta < \eta_0,
\]

then the solution to \(-\partial_t^2 u + \Delta u = |u|^6 u\) with data \((u_0, u_1)\) is global in time and obeys the space-time bounds

\[
\|u\|_{L^{12}_t(\mathbb{R} \times \mathbb{R}^3)} + \||\nabla|^\frac{3}{2} u\|_{L^{12}_t(\mathbb{R} \times \mathbb{R}^3)} < \infty.
\]

In particular, \(u\) scatters.

To prove this result, it is useful to complexify the equation and introduce the variable

\[
v = u - i|\nabla|^{-1} \partial_t u, \quad \text{where} \quad |\nabla|^{-1} = (-\Delta)^{-\frac{1}{4}}.
\]

Then \((u, \partial_t u) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}\) if and only if \(v \in \dot{H}^{s_c}\) (with comparable norms), with a similar statement concerning the Besov norms. The equation \((\text{NLW})\) is then equivalent to

\[
i \partial_t v = -|\nabla|v - |\nabla|^{-1}(\text{Re} \, v|^6 \text{Re} \, v) = 0,
\]

which in turn is equivalent to the Duhamel formulation

\[
v(t) = e^{i|\nabla|}v_0 + i \int_0^t e^{i(t-s)|\nabla|} |\nabla|^{-1}(\text{Re} \, v|^6 \text{Re} \, v) \, ds.
\]

The proof of Proposition A.1 will rely primarily on Strichartz estimates for the operator \(e^{i|\nabla|}\). To begin, we have the following estimates (see, for example, [65]): Let \((q, r)\) and \((\tilde{q}, \tilde{r})\) be wave admissible in 3d, i.e.

\[
\frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}, \quad (q, r) \neq (2, \infty).
\]

Defining

\[
\gamma(q, r) = \frac{3}{2} - \frac{1}{q} - \frac{3}{r},
\]

we have:

\[
\|e^{i|\nabla|} f\|_{L^q_t L^r_x} \lesssim \| |\nabla|^{\gamma(q, r)} f\|_{L^2}, \tag{A.3}
\]

\[
\left\| \int_0^t e^{i(t-s)|\nabla|} F(s) \, ds \right\|_{L^q_t L^r_x} \lesssim \| |\nabla|^{\gamma(q, r) + \gamma(\tilde{q}, \tilde{r})} F\|_{L^q_t L^{\tilde{r}}_x}. \tag{A.4}
\]

The specific space-time norms appearing Proposition A.1 are critical norms for the case \((d, p) = (3, 6)\). That is, they are invariant under the rescaling \((1.2)\).

We define

\[
\|e^{i|\nabla|} f\|_{X(I)} := \|e^{i|\nabla|} f\|_{L_q^\infty(t \times \mathbb{R}^3)} + \||\nabla|^\frac{3}{2} e^{i|\nabla|} f\|_{L^q_t L^r_x(t \times \mathbb{R}^3)}, \tag{A.5}
\]

By the Strichartz estimates above (noting that \(\gamma(4, 4) = \frac{1}{2}\)), we have

\[
\|e^{i|\nabla|} f\|_{X(\mathbb{R})} \lesssim \||\nabla|^\frac{3}{2} f\|_{L^2(\mathbb{R}^3)}.
\]

A less standard ingredient for the proof of Proposition A.1 will be the following refined Strichartz estimate for this norm.

Lemma A.2 (Refined Strichartz estimate). There exists $\theta \in (0, 1)$ such that
\[ \|e^{it|\nabla|}f\|_{X(\mathbb{R})} \lesssim \|f\|_{B^{\theta}_{\infty, \infty}} \|\nabla^q f\|_{L^2_t L^2_x}^{1-q}. \]

Proof. As the Besov norm is controlled by the Sobolev norm, it suffices to prove an estimate of this form for each norm appearing in (A.5). Let us focus on the $L_t^4 L_x^4$ norm, as the proof for the $L_t^{12} L_x^{12}$ norm follows along similar lines.

We employ the Littlewood–Paley frequency decomposition
\[ f = \sum_{N \in 2^\mathbb{Z}} P_N f, \]
as described in Section 1.1. Let us denote
\[ u(t) = e^{it|\nabla|}f, \quad u_N(t) = e^{it|\nabla|}P_N f. \]

By the Littlewood–Paley square function estimate, we may write
\[ \|u\|_{L_t^4 L_x^4}^2 \approx \iint \left( \sum_{N \in 2^\mathbb{Z}} |u_N(t,x)|^2 \right)^2 dx dt \]
\[ \lesssim \iint \sum_{N_1 \leq N_2} |u_{N_1}|^2 |u_{N_2}|^2 dx dt. \]

Now let $\varepsilon > 0$ be a small parameter and define two pairs of sharp wave-admissible exponents $(q_l, r_l)$ and $(q_h, r_h)$ by
\[ \left( \frac{1}{q_l}, \frac{1}{r_l} \right) = \left( \frac{1}{4} + \varepsilon, \frac{1}{4} - \varepsilon \right), \quad \left( \frac{1}{q_h}, \frac{1}{r_h} \right) = \left( \frac{1}{4} - \varepsilon, \frac{1}{4} + \varepsilon \right). \]

Note that
\[ \gamma(q_l, r_l) = \frac{1}{4} + 2\varepsilon, \quad \gamma(q_h, r_h) = \frac{1}{4} - 2\varepsilon. \]

In particular, by (A.3) and Bernstein inequalities, we have
\[ \|u_{N_1}\|_{L_t^{q_l} L_x^{r_l}} \lesssim |||\nabla^{\frac{1}{2} + 2\varepsilon} f_{N_1}\|_{L^2} \lesssim N_{l}^{2\varepsilon} |||f_{N_1}\|_{H^{\frac{1}{2}}}, \]
\[ \|u_{N_2}\|_{L_t^{q_h} L_x^{r_h}} \lesssim |||\nabla^{\frac{1}{2} - 2\varepsilon} f_{N_2}\|_{L^2} \lesssim N_{h}^{-2\varepsilon} |||f_{N_2}\|_{H^{\frac{1}{2}}}. \]

Thus, continuing from above and applying (A.3) and Cauchy–Schwarz, we have
\[ \|u\|_{L_t^{4} L_x^{4}}^2 \lesssim \sum_{N_1 \leq N_2} |||u_{N_1}\|_{L_t^{q_l} L_x^{r_l}}|||u_{N_1}\|_{L_t^{4} L_x^{4}} |||u_{N_2}\|_{L_t^{4} L_x^{4}}|||u_{N_2}\|_{L_t^{q_h} L_x^{r_h}} \]
\[ \lesssim \left( \sup_N \|u_N\|_{L_t^{4} L_x^{4}} \right)^2 \sum_{N_1 \leq N_2} \left( \frac{N_{h}}{N_{l}} \right)^{2\varepsilon} \|f_{N_1}\|_{H^{\frac{1}{2}}} \|f_{N_2}\|_{H^{\frac{1}{2}}} \]
\[ \lesssim \left( \sup_N \|f_N\|_{H^{s}} \right)^2 \|f\|_{H^{s}}^2, \]

which yields the desired estimate.

In the case of the $L_t^{12} L_x^{12}$ norm, the situation is actually simpler because we work away from the ‘strictly admissible’ line $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$. In this case we put one of the lowest frequency pieces $u_{N_1}$ in $L_t^{12} L_x^{12+}$ and one of the highest frequency pieces $u_{N_6}$ in $L_t^{12} L_x^{12-}$. All of the remaining pieces are taken out with a supremum in $N$ of

3 Bernstein inequalities refer to the following general estimates for frequency-localized functions:
\[ \|P_N f\|_{L^2(\mathbb{R}^d)} \lesssim N^{\frac{1}{2} - \frac{1}{r_1}} \|P_N f\|_{L^{r_1}(\mathbb{R}^d)}, \quad 1 \leq r_1 \leq r_2 \leq \infty, \]
\[ \||\nabla|^s P_N f\|_{L^r(\mathbb{R}^d)} \sim N^{s} \|P_N f\|_{L^r(\mathbb{R}^d)}, \quad 1 < r < \infty, \quad s \in \mathbb{R}. \]
the $L_{t,x}^{1/2}$ norm, which yields the Besov norm after an application of Strichartz (as above). Using Bernstein and Strichartz for the low and high frequency pieces, we get a gain of $\left(\frac{N_i}{N_0}\right)^2$, which can be used to defeat the logarithmic loss from the sums in $N_2, \ldots, N_5$ and to sum in $N_1, N_6$ via Cauchy–Schwarz, just as above. □

We turn to the proof of Proposition A.1.

**Proof of Proposition A.1.** The standard well-posedness theory for NLW yields a local-in-time solution to (A.1) satisfying the Duhamel formula (A.2). It will therefore suffice to prove the estimate

$$
\|v\|_{X(I)} \lesssim \|v_0\|_{\dot{B}^{\frac{3}{2},\infty}_2}^{\theta} \|v_0\|_{\dot{H}^{\frac{3}{2}}}^{1-\theta} + \|v\|_{X(I)}^7 
$$

on any interval $I \subset \mathbb{R}$ containing $t = 0$, where $\theta \in (0,1)$ is as in Lemma A.2. Indeed, choosing $\eta_0 = \eta_0(E)$ sufficiently small (where $E, \eta_0$ are as in the statement of Proposition A.1), we may make the first term on the right-hand side of (A.6) as small as we wish. Then, by a standard continuity argument, (A.6) implies

$$
\|v\|_{X(\mathbb{R})} \lesssim \|v_0\|_{\dot{B}^{\frac{3}{2},\infty}_2}^{\theta} \|v_0\|_{\dot{H}^{\frac{3}{2}}}^{1-\theta},
$$

giving the desired global space-time bounds for $v$.

It therefore remains to prove (A.6). Recalling the Duhamel formula (A.2), we apply Lemma A.2 (A.4), and the fractional chain rule\(^4\) to estimate

$$
\|v\|_{X(I)} \lesssim \|v_0\|_{\dot{B}^{\frac{3}{2},\infty}_2}^{\theta} \|v_0\|_{\dot{H}^{\frac{3}{2}}}^{1-\theta} + \|\nabla F(u)\|_{L_t^2}^{\frac{3}{2}} \lesssim \|v_0\|_{\dot{B}^{\frac{3}{2},\infty}_2}^{\theta} \|v_0\|_{\dot{H}^{\frac{3}{2}}}^{1-\theta} + \|v\|_{L_t^2}^6 \|\nabla F(v)\|_{L_t^2},
$$

giving (A.6), as desired. □

**Appendix B. Incoming waves for the radial wave equation**

In this section we describe the notion of incoming/outgoing waves for the radial wave equation. In particular, we wish to explain the origin of the ‘incoming’ condition

$$
u_1 = \partial_r u_0 + \frac{d-2}{r} u_1.
$$

Our discussion will be brief—for more details, see [2].

First, consider the radial wave equation in three space dimensions:

$$
u_{tt} - u_{rr} - \frac{2}{r} u_r = 0.
$$

If $u$ and $v$ are related through

$$
u(r) = \frac{1}{r} \int_0^r \nu(\rho) \, d\rho,
$$

then one finds that $u$ solving (B.2) is equivalent to $v$ solving the 1d wave equation

$$
u_{tt} - v_{rr} = 0;
$$

---

\(^4\)This refers to the estimate

$$
\|\nabla F(u)\|_{L_t^r} \lesssim \|F'(u)\|_{L_t^{r_1}} \|\nabla u\|_{L_t^{r_2}}
$$

for $0 < s < 1$ and $1 < r, r_1, r_2 < \infty$ satisfying $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. 


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however, we should now view \( v \) as a solution on the half-line \( r \in (0, \infty) \) with the Neumann boundary condition \( \partial_r v|_{r=0} = 0 \).

If instead \( u \) solves the radial equation in five space dimensions, i.e.

\[
   u_{tt} - u_{rr} - \frac{4}{r} u_r = 0,
\]

then (after a short computation) one finds that the same situation arises when \( u \) and \( v \) are related via

\[
   u(r) = \frac{1}{r} \int_0^r (r^2 - \rho^2) v(\rho) \, d\rho.
\]

Now, for the 1d wave equation (with Neumann boundary conditions) one can use the explicit formula for solutions to see that every solution may be decomposed into an incoming and outgoing piece, i.e.

\[
   v(t) = v_{\text{in}}(t) + v_{\text{out}}(t),
\]

where \( v_{\text{in}} \) moves toward \( r = 0 \) and \( v_{\text{out}} \) moves toward \( r = \infty \) (both with speed one). As \( t \to \infty \), the solution becomes increasingly outgoing, while as \( t \to -\infty \) the solution becomes increasingly incoming. Writing \((v_0, v_1)\) for the initial conditions of \( v(t) \), one finds that at time zero the incoming component is

\[
   v_- = \frac{1}{r} (v_0 + \partial_r^{-1} v_1),
\]

while the outgoing component is

\[
   v_+ = \frac{1}{r} (v_0 - \partial_r^{-1} v_1),
\]

where \( \partial_r^{-1} \) denotes the antiderivative. For more detail, see [2].

We are interested in prescribing ‘incoming’ initial data. In this case, at the linear level there is an initial ‘focusing’ of the solution toward the origin, which we can then observe is countered by the defocusing effect of the nonlinearity. In particular, an initial data pair for \( v \) is incoming if \( v_+ = 0 \), that is, if

\[
   (v, \partial_t v)|_{t=0} = (v_0, \partial_r v_0).
\]

Let us now derive equivalent conditions at the level of \((u_0, u_1)\). These are the conditions appearing in our choices of initial data.

For the three-dimensional case, we have the simple relation \( v = \partial_r (ru) \), and the incoming condition becomes

\[
   u_1 = \partial_r u_0 + \frac{1}{r} u_0.
\]

For the five-dimensional case, we instead get \( \partial_r^3 (r^3 u) = 2v + \partial_r v \), and the incoming condition becomes

\[
   u_1 = \partial_r u_0 + \frac{2}{r} u_0.
\]

In general, one derives the condition \((B.1)\).

We close this section with the following lemma, which shows that for sufficiently regular \( u_0 \), the initial velocity \( u_1 \) still belongs to \( L^2 \cap H^s \) for suitable \( s \) (despite the presence of the singular term \( 1/r \)).

**Lemma B.1.** Let \( u \) be a Schwartz function in \( \mathbb{R}^d \), with \( d \geq 3 \). Then \( \frac{1}{|x|^d} u \) belongs to \( H^s \) for any \( 0 \leq s \leq \min\{ \frac{d}{2} - 1, 1 \} \). In fact,

\[
   \| \frac{1}{|x|^d} u \|_{H^s} \lesssim \| u \|_{H^{s+1}}
\]

for any such \( s \).
As (cf. [28, 50]), we may estimate
\[ \| |x|^{-s} u \|_{L^p(\mathbb{R}^d)} \lesssim \| |\nabla|^s u \|_{L^p(\mathbb{R}^d)} \quad \text{for} \quad 1 < p < \frac{d}{s} \]
(see e.g. [59]). In particular, an application of Hardy’s inequality (and the boundedness of Riesz potentials) reduces (B.6) to the commutator estimate
\[ \| \left( \frac{1}{|x|} |\nabla|^s \right) u \|_{L^2} \lesssim \| |\nabla|^{s+1} u \|_{L^2}. \]
For this we will use the Fourier transform. The key fact that we need is
\[ \mathcal{F}[x]^{-\alpha} \sim |\xi|^{\alpha-d} \quad \text{for} \quad 0 < \alpha < d, \]
which can be deduced using scaling and symmetry properties of the Fourier transform, or by a direct computation using the Gamma function (see [59] Lemma 1, p.117). In particular, we are faced with estimating
\[ \| \left( \frac{1}{|x|} |\nabla|^s \right) u \|_{L^2} \sim \left\| \int |\xi - \eta|^{-(d-1)} (|\xi|^s - |\eta|^s) \hat{u}(\eta) \, d\eta \right\|_{L^2_\xi}. \]
As \( 0 < s \leq 1 \), we can estimate this by
\[ \| \int |\xi - \eta|^{-(d-1-s)} |\hat{u}(\eta)| \, d\eta \|_{L^2_\xi} = \| |\xi|^{-(d-1-s)} \ast |\hat{u}(\eta)| \|_{L^2_\xi}. \]
Using the Lorentz-space\(^5\) refinements of Young’s inequality and Hölder’s inequality (cf. [28, 50]), we may estimate
\[ \| |\xi|^{-(d-1-s)} \ast |\hat{u}(\eta)| \|_{L^2_\xi} \lesssim \| |\xi|^{-(d-1-s)} \|_{L^{\frac{s}{d-s}}(\mathbb{R}^d)} \| \hat{u} \|_{L^{\frac{2s}{d-s}}(\mathbb{R}^d)} \lesssim \| |\xi|^{-(s+1)} \|_{L^{\frac{2s}{d-s}}(\mathbb{R}^d)} \| |\xi|^{s+1} \hat{u} \|_{L^2} \lesssim \| |\nabla|^{s+1} u \|_{L^2}, \]
which is acceptable. In the estimates above, we have used \( s \leq \frac{d}{2} - 1 \) to guarantee that the exponents above fall into acceptable intervals. \( \square \)

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\(^5\)Here \( L^{q,\alpha} \) denotes the Lorentz-space defined via the quasi-norm
\[ \| f \|_{L^{q,\alpha}} = \| |\{ f > \lambda \}| \|_{L^q(\mathbb{R}^d)} \| \lambda \|^\alpha_{L^\infty(\mathbb{R}^d)} \| f \|_{L^\infty(\mathbb{R}^d)}. \]
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