THE FOURTH MOMENT OF CENTRAL VALUES OF QUADRATIC HECKE $L$-FUNCTIONS IN THE GAUSSIAN FIELD

PENG GAO

ABSTRACT. We obtain an asymptotic formula for the fourth moment of central values of a family of quadratic Hecke $L$-functions in the Gaussian field under the generalized Riemann hypothesis (GRH). We also establish lower bounds unconditionally and upper bounds under GRH for higher moments of the same family.

Mathematics Subject Classification (2010): 11M06, 11M41, 11M50

Keywords: central values, Hecke $L$-functions, quadratic Hecke characters, moments

1. INTRODUCTION

Moments of central values of families of $L$-functions have been intensively studied in the literature in order to understand important arithmetic information they carry. Although much progress has been made towards establishing asymptotic formulas for the first few moments for various families of $L$-functions, little is known for the higher moments. In connection with random matrix theory, conjectures on the order of magnitude for the moments were made by J. P. Keating and N. C. Snaith in [18]. A simple and powerful method towards establishing lower bounds of these conjectured results was developed by Z. Rudnick and K. Soundararajan in [23] and [24].

For the family of quadratic Dirichlet $L$-functions, the above method of Rudnick and Soundararajan allows them to show in [24] that for every even natural number $k$,

\[ \sum_{d \in \mathcal{D}} |d|^{\frac{k}{2}} X(\log X)^{\frac{k(k+1)}{2}}, \]

(1.1)

where $\chi_d = \left( \frac{d}{\cdot} \right)$ is the Kronecker symbol and $\mathcal{D}$ denotes the set of fundamental discriminants.

In the other direction, Soundararajan [29] proved that, assuming the generalized Riemann hypothesis (GRH), for all real $k \geq 0$,

\[ \sum_{d \in \mathcal{D}} |d|^{\frac{k}{2}} X(\log X)^{\frac{k(k+1)}{2}+\varepsilon}. \]

(1.2)

The above result was further sharpened by A. J. Harper in [11] to remove the $\varepsilon$ power.

Besides the lower and upper bounds for the moments given in (1.1) and (1.2), asymptotic formulas are known for integers $1 \leq k \leq 4$ in the form

\[ \sum_{0 < d \leq X \ (d, 2) = 1 \ d \ square-free} L\left( \frac{1}{2}, \chi_{8d} \right)^k = XP_{\frac{k+1}{2}}(\log X) + E_k(X), \]

(1.3)

where $P_n(x)$ is an explicit linear polynomials of degree $n$ and $E_k(X)$ is the error term. Here we note that for odd, square-free $d > 0$, the character $\chi_{8d}$ is primitive modulo $8d$ satisfying $\chi_{8d}(-1) = 1$. Note also that we choose to present the asymptotic formulas for a family which is preferred by most recent studies due to technical reasons instead of the family appearing in (1.1) and (1.2).

Evaluation of the first two moments ($k = 1, 2$) in (1.3) was initiated by M. Jutila in [17]. The error terms in Jutila’s results were subsequently improved in [9, 31, 32] for the first moment and in [27, 28] for the second moment. For smoothed first moment, a result of D. Goldfeld and J. Hoffstein in [9] implies that one can take $E_1(x) = O(x^{1/2+\varepsilon})$. An error term of the same size was later obtained by M. P. Young in [32] using a recursive approach. The same approach was then adapted by K. Sono [27] to show that one can take $E_2(x) = O(x^{1/2+\varepsilon})$ for smoothed second moment. The sizes of these error terms then make the expressions given in (1.3) in agreement with a conjecture made by J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith in [3] on asymptotic behaviours of these moments of the family of quadratic Dirichlet $L$-functions.
Theorem 1.1. For every even natural number \(k\), we have
\[
\sum_{\substack{(d,2)=1 \\ N(d) \leq X}}^* L\left(\frac{1}{2}, \chi_{(1+i)^s d}\right)^k \geq k X (\log X)^{k(k+1)/2}.
\]
Here the "\(\ast\)" on the sum over \(d\) means that the sum is restricted to square-free elements \(d\) in \(\mathcal{O}_K\).

The proof of Theorem 1.1 is given in Section 3 and it follows the arguments in the proof of [24] Theorem 2 for the case of Dirichlet \(L\)-functions. Also similar to the remarks given below [24] Theorem 2, the proof of Theorem 1.1 can be applied to give lower bounds for the moments for all rational numbers \(k\), provided that we replace \(L\left(\frac{1}{2}, \chi_{d}\right)^k\) by \(\left|L\left(\frac{1}{2}, \chi_{d}\right)^k\right|\).

Before we state a corresponding result on the upper bounds, we need to introduce some notations. Let \(x \in \mathbb{R}\) such that \(x \geq 10\) and \(z \in \mathbb{C}\), we define
\[
\mathcal{L}(z, x) = \begin{cases} 
\log \log x & |z| \leq (\log x)^{-1}, \\
- \log |z| & (\log x)^{-1} < |z| \leq 1, \\
0 & |z| > 1.
\end{cases}
\]
We further define for \(z_1, z_2 \in \mathbb{C}\),
\[
\mathcal{M}(z_1, z_2, x) = \frac{1}{2} \left(\mathcal{L}(z_1, x) + \mathcal{L}(z_2, x)\right),
\]
\[
\mathcal{V}(z_1, z_2, x) = \frac{1}{2} \left(\mathcal{L}(z_1, x) + \mathcal{L}(z_2, x) + \mathcal{L}(2 \Re(z_1), x) + \mathcal{L}(2 \Re(z_2), x) + 2 \mathcal{L}(z_1 + z_2, x) + 2 \mathcal{L}(z_1 + z_2^*, x)\right).
\]

Then we have the following result on upper bounds for shifted moments of the family \(\mathcal{F}\) given in (1.4) under GRH.

Theorem 1.2. Assume GRH for \(\zeta_K(s)\) and \(L(s, \chi_{(1+i)^s d})\) for all odd, square-free \(d\). Let \(X\) be large and let \(z_1, z_2 \in \mathbb{C}\) with \(0 \leq \Re(z_1), \Re(z_2) \leq \frac{1}{\log X}\), and \(|\Im(z_1)|, |\Im(z_2)| \leq X\). Then for any positive real number \(k\) and any \(\varepsilon > 0\), we have
\[
\sum_{\substack{(d,2)=1 \\ N(d) \leq X}}^* \left|L\left(\frac{1}{2} + z_1, \chi_{(1+i)^s d}\right) L\left(\frac{1}{2} + z_2, \chi_{(1+i)^s d}\right)^k\right| \ll_{k, \varepsilon} X (\log X)^\varepsilon \exp \left(k \mathcal{M}(z_1, z_2, X) + \frac{k^2}{2} \mathcal{V}(z_1, z_2, X)\right).
\]
The proof of Theorem 1.2 is given in Section 2 and our proof follows closely the approaches in [26, 29, 30]. We note that similar results to Theorem 1.2 were obtained for the moments of the Riemann zeta function by V. Chandee [2] and for the moments of all Dirichlet L-functions modulo q by M. Munsch [21].

We now give two consequences of Theorem 1.2. First, by setting \( z_1 = z_2 = 0 \) in Theorem 1.2, we deduce immediately the following upper bounds for moments of the central values of the family \( F \) given in (1.4).

**Corollary 1.3.** Assume GRH for \( \zeta_K(s) \) and \( L(s, \chi_{(1+i)\omega}) \) for all odd, square-free \( d \). For any positive real number \( k \) and any \( \varepsilon > 0 \), we have for large \( X \),

\[
\sum_{(d,2)=1 \atop N(d) \leq X} |L\left(\frac{1}{2}, \chi_{(1+i)\omega}^d\right)|^k \ll_k, \varepsilon X (\log X)^{k(1)/2+\varepsilon}.
\]

The second consequence concerns upper bounds for shifted fourth moment of the family \( F \) given in (1.4), which is what we need in a refined study in Section 5 on the fourth moment under GRH.

**Corollary 1.4.** Assume GRH for \( \zeta_K(s) \) and \( L(s, \chi_{(1+i)\omega}) \) for all odd, square-free \( d \). Let \( X \) be large and let \( z_1, z_2 \in \mathbb{C} \) with \( 0 \leq \Re(z_1), \Re(z_2) \leq \frac{1}{\log X} \) and \( |\Im(z_1)|, |\Im(z_2)| \leq X \). Then

\[
\sum_{(d,2)=1 \atop N(d) \leq X} |L\left(\frac{1}{2} + z_1, \chi_{(1+i)\omega}^d\right)|^2 |L(\frac{1}{2} + z_2, \chi_{(1+i)\omega}^d)|^2 \ll X (\log X)^{4+\varepsilon} \left(1 + \min \left\{ (\log X)^6, \frac{1}{(|\Im(z_1)| - |\Im(z_2)|)^5}\right\}\right).
\]

We shall omit the proof of the above corollary as it is analogous to the proof [26, Theorem 2.4]. With the aide of Corollary 1.4, we are able to obtain a more precise expression for the fourth moment. To present our result, we define constants \( a_k \) for any real number \( k > 0 \) by

\[
(1.9) \quad a_k = 2^{-k(1+k)/2} \prod_{\omega \equiv 1 \mod 1+i^3} \frac{(1 - \frac{1}{N(\omega)})^{k(1+k)/2}}{1 + \frac{1}{N(\omega)}} \left(1 + \frac{1}{\sqrt{N(\omega)}}\right)^{-k} + \left(1 - \frac{1}{\sqrt{N(\omega)}}\right)^{-k} + \frac{1}{N(\omega)}
\]

Here and in what follows, we denote \( \omega \) for a prime number in \( \mathcal{O}_K \), by which we mean that the ideal \( (\omega) \) generated by \( \omega \) is a prime ideal. We also note that the expression \( \omega \equiv 1 \mod (1+i^3) \) indicates that \( \omega \) is a primary element in \( \mathcal{O}_K \) defined in Section 2.4.

Now, we state our result on a conditional asymptotic evaluation on the fourth moment.

**Theorem 1.5.** Assume GRH for \( \zeta_K(s) \) and \( L(s, \chi_{(1+i)\omega}) \) for all odd, square-free \( d \). Then for any \( \varepsilon > 0 \), we have

\[
\sum_{(d,2)=1 \atop N(d) \leq X} |L\left(\frac{1}{2}, \chi_{(1+i)\omega}^d\right)|^4 = \frac{\pi a_4}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot \zeta_K(2)} \left(\frac{\pi}{4}\right)^{10} X (\log X)^{10} + O \left( X (\log X)^{9.5+\varepsilon}\right).
\]

Here the “\( \ast \)” on the sum over \( d \) means that the sum is restricted to square-free elements \( d \) in \( \mathcal{O}_K \) and \( a_4 \) is defined as in (1.9).

Without assuming GRH, we also have the following lower bound for the fourth moment.

**Theorem 1.6.** Unconditionally, we have

\[
\sum_{(d,2)=1} |L\left(\frac{1}{2}, \chi_{(1+i)\omega}^d\right)|^4 \Phi \left(\frac{N(d)}{X}\right) \geq \left(\frac{\pi a_4}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot \zeta_K(2)} \left(\frac{\pi}{4}\right)^{10} + o(1)\right) \left(\frac{N(d)}{X}\right)^{10}.
\]

Here the “\( \ast \)” on the sum over \( d \) means that the sum is restricted to square-free elements \( d \) in \( \mathcal{O}_K \) and \( a_4 \) is defined as in (1.9).

Theorems 1.5 and 1.6 are similar to those of Shen given in [26, Theorems 1.1-1.2] on the fourth moment of the family of quadratic Dirichlet L-functions as well as the result of Soundararajan and Young given in [30, Theorem 1.1] on the second moment of the family of quadratic twists of modular L-functions. The proof for Theorems 1.5 and 1.6 given in Section 5 also proceed along the same lines of the proofs of [26, Theorems 1.1-1.2] and [30, Theorem 1.1].

2. Preliminaries

As a preparation, we first include some auxiliary results needed in the proofs of our theorems.
2.1. Quadratic residue symbol and quadratic Gauss sum. Recall that \( K = \mathbb{Q}(i) \) and it is well-known that \( K \) have class number one. We denote \( U_K = \{ \pm 1, \pm i \} \) and \( D_K = -4 \) for the group of units in \( \mathcal{O}_K \) and the discriminant of \( K \), respectively.

Every ideal in \( \mathcal{O}_K \) co-prime to 2 has a unique generator congruent to 1 modulo \((1+i)^3\) which is called primary. It follows from Lemma 6 on [13, p. 121] that an element \( n = a + bi \in \mathcal{O}_K \) with \( a, b \in \mathbb{Z} \) is primary if and only if \( a \equiv 1 \pmod{4} \), \( b \equiv 0 \pmod{4} \) or \( a \equiv 3 \pmod{4} \), \( b \equiv 2 \pmod{4} \).

For \( n \in \mathcal{O}_K, (n, 2) = 1 \), we denote the symbol \( \left( \frac{a}{n} \right) \) for the quadratic residue symbol modulo \( n \) in \( K \). For a prime \( \varpi \in \mathbb{Z}[i] \) with \( N(\varpi) \neq 2 \), the quadratic symbol is defined for \( a \in \mathcal{O}_K, (a, \varpi) = 1 \) by \( \left( \frac{a}{\varpi} \right) \equiv a^{(N(\varpi)-1)/2} \pmod{\varpi} \), with \( \left( \frac{a}{\varpi} \right) \in \{ \pm 1 \} \). When \( \varpi | a \), we define \( \left( \frac{a}{\varpi} \right) = 0 \). Then the quadratic symbol is extended to any composite \( n \) with \( N(n), 2 = 1 \) multiplicatively. We further define \( \left( \frac{c}{K} \right) = 1 \) for \( c \in U_K \).

The following quadratic reciprocity law (see [8, (2.1)]) holds for two co-prime primary elements \( m, n \in \mathcal{O}_K \):

\[
\left( \frac{m}{n} \right) = \left( \frac{n}{m} \right).
\]

Moreover, we deduce from Lemma 8.2.1 and Theorem 8.2.4 in [1] that the following supplementary laws hold for primary \( n = a + bi \) with \( a, b \in \mathbb{Z} \):

\[
\left( \frac{i}{n} \right) = (-1)^{(1-a)/2} \quad \text{and} \quad \left( \frac{1+i}{n} \right) = (-1)^{(a-b-1-5^2)/4}.
\]

For any complex number \( z \), we define

\[
\overline{e}(z) = \exp \left( 2\pi i \left( \frac{z}{2i} - \frac{z}{2i} \right) \right).
\]

For any \( r \in \mathcal{O}_K \), we define the quadratic Gauss sum \( g(r, \chi) \) associated to any quadratic Hecke character \( \chi \) modulo \( q \) of trivial infinite type and the quadratic Gauss sum \( g(r, n) \) associated to the quadratic residue symbol \( \left( \frac{r}{n} \right) \) for any \((n, 2) = 1 \) by

\[
g(r, \chi) = \sum_{x \mod q} \chi(x) \overline{e}(\frac{rx}{q}), \quad g(r, n) = \sum_{x \mod n} \left( \frac{x}{n} \right) \overline{e}(\frac{rx}{n}).
\]

When \( r = 1 \), we shall denote \( g(\chi) \) for \( g(1, \chi) \) and \( g(n) \) for \( g(1, n) \). Recall from [7, (2.2)] that for primary \( n \), we have

\[
g(n) = \left( \frac{i}{n} \right) N(n)^{1/2}.
\]

A Hecke character \( \chi \) is said to be primitive modulo \( q \) if it does not factor through \( (\mathcal{O}_K/(q'))^\times \) for any divisor \( q' \) of \( q \) such that \( N(q') < N(q) \). Recall from Section [1] that we denote \( \chi_c \) for the quadratic residue symbol \( \left( \frac{\cdot}{n} \right) \) and we define \( \chi_c(n) \) to be 0 when \( 1+i | n \). In Section 2.1 of [8], it is shown that the symbol \( \chi_{(i(1+i))^d} \) defines a primitive quadratic Hecke character modulo \((1+i)^5d\) of trivial infinite type for any odd and square-free \( d \in \mathcal{O}_K \). When replacing \( d \) by \( i^3d \), we see that the symbol \( \chi_{(i+1)^d} \) also defines a primitive quadratic Hecke character modulo \((1+i)^5d\) of trivial infinite type for any odd and square-free \( d \in \mathcal{O}_K \). Our next lemma evaluates \( g(\chi_{(i+1)^d}) \) exactly.

**Lemma 2.2.** For any odd, square-free \( d \in \mathcal{O}_K \), we have

\[
g(\chi_{(i+1)^d}) = N((1+i)^5d)^{1/2}.
\]

**Proof.** It suffices to prove [2.3] with \( d \) replaced by \( jd \), where \( j = 1 \) or \( i \) and \( d \) is primary and square-free. It follows from the Chinese remainder theorem that \( x = j(1+i)^5y + dz \) varies over the residue class modulo \((1+i)^5d\) when \( y \) and \( z \) vary over the residue class modulo \( d \) and \((1+i)^5\), respectively. We then deduce that

\[
g(X_{j(1+i)^5d}) = \sum_{z \mod (1+i)^5} \sum_{y \mod d} \left( \frac{d}{z j(1+i)^5y + dz} \right) \overline{e}(\frac{jjy}{d}) \overline{e}(\frac{z}{j(1+i)^5}).
\]

As \( \chi_{(1+i)^5} \) is a Hecke character of trivial infinite type modulo \((1+i)^5\), we deduce that

\[
\left( \frac{j(1+i)^5}{j(1+i)^5y + dz} \right) = \chi_{j(1+i)^5}((1+i)^5y + dz) = \chi_{j(1+i)^5}(dz).
\]

On the other hand, we denote \( s(z) \) to be the unique element in \( U_K \) such that \( s(z)z \) is primary for any \((z, 2) = 1 \). It follows from the quadratic reciprocity law [2.1] that

\[
\left( \frac{d}{j(1+i)^5y + dz} \right) = \left( \frac{s(z)(1+i)^5y}{d} \right).
\]
We then conclude from (2.10) and (2.11) that
\[ g(\chi(1+i)^{5}d) = \sum_{z \mod (1+i)^{5}} \sum_{y \mod d} \left( \frac{j(1+i)^{5}}{z} \right) \left( \frac{s(z)(1+i)y}{d} \right) \tilde{\varepsilon} \left( \frac{z}{1+i} \right) \]
(2.8)
\[ = N(d)^{1/2} \sum_{z \mod (1+i)^{5}} \left( \frac{j(1+i)^{5}}{z} \right) \left( \frac{s(z)y}{d} \right) \tilde{\varepsilon} \left( \frac{z}{1+i} \right) , \]
where the last equality above follows from (2.24).

In order to evaluate the last sum in (2.8), we note that it suffices to take \( z \) to vary over the reduced residue class modulo \((1+i)^{5}\). One representation of such class consists of the following 16 elements (note that \( \pm 1, \pm i \) consists of the reduced residue class modulo \((1+i)^{3}\) and \( 0, 1 \) consists of the residue class modulo \(1+i)\):
\[ \{ \pm 1, \pm i \} + l(1+i)^{3} + k(1+i)^{4}, \quad l \in \{0, 1\}, k \in \{0, -1\}. \]
We further write \( d = a + bi \) with \( a, b \in \mathbb{Z} \) (recall that \( a \equiv 1 \pmod{4}, b \equiv 0 \pmod{4} \) or \( a \equiv 3 \pmod{4}, b \equiv 2 \pmod{4} \)) and check by direct calculations using (2.22) to see that (2.5) is valid with \( d \) replaced by \( jd \), where \( j = 1 \) or \( i \) and \( d \) is primary and square-free. This completes the proof of the lemma. \( \square \)

Let \( \varphi_{(i)}(n) \) denote the number of elements in the reduced residue class of \( \mathcal{O}_{K}/(n) \), we recall from [8] Lemma 2.2 the following explicitly evaluations of \( g(r, n) \) for primary \( n \).

**Lemma 2.3.**
(i) We have
\[ g(rs, n) = \frac{8}{\pi} g(r, n), \quad (s, n) = 1, \]
g(k, mn) = \( g(k, m)g(k, n) \), \quad \( m, n \) primary and \( (m, n) = 1 \).
(ii) Let \( \varpi \) be a primary prime in \( \mathcal{O}_{K} \). Suppose \( \varpi^{h} \) is the largest power of \( \varpi \) dividing \( k \). (If \( k = 0 \) then set \( h = \infty \).) Then for \( l \geq 1 \),
\[ g(k, \varpi^{l}) = \begin{cases} 
0 & \text{if } l \geq h + 2, \\
\varphi_{(i)}(\varpi) & \text{if } l \leq h \text{ is odd}, \\
-N(\varpi)^{l-1} & \text{if } l \leq h \text{ is even}, \\
\left( \frac{ik \varpi^{h}}{\varpi} \right) N(\varpi)^{l-1/2} & \text{if } l = h + 1 \text{ is odd}, \\
0 & \text{if } l = h + 1 \text{ is even}. 
\end{cases} \]

2.4. **The approximate functional equation.** Let \( \chi \) be a primitive quadratic Hecke character modulo \( m \) of trivial infinite type of \( K \). Let
\[ \Lambda(s, \chi) = (|D_{K}|N(m))^{s/2}(2\pi)^{-s}\Gamma(s)L(s, \chi). \]
A well-known result of E. Hecke shows that \( L(s, \chi) \) has an analytic continuation to the whole complex plane and satisfies the functional equation (see [15] Theorem 3.8)
\[ \Lambda(s, \chi) = W(\chi)(N(m))^{-1/2}\Lambda(1-s, \chi), \]
where \( |W(\chi)| = (N(m))^{1/2} \).

When we take \( \chi = \chi_{(1+i)^{5}d} \) for any odd, square-free \( d \in \mathcal{O}_{K} \), it follows from [15] Theorem 3.8 that we have \( W(\chi_{(1+i)^{5}d}) = g(\chi_{(1+i)^{5}d}) \), so that Lemma 2.2 implies that in this case the functional equation becomes
\[ \Lambda(s, \chi_{(1+i)^{5}d}) = \Lambda(1-s, \chi_{(1+i)^{5}d}). \]

For \( n \in \mathcal{O}_{K} \) and rational integer \( k \geq 1 \), we let \( d_{(i), k}(n) \) denote the analogue on \( \mathcal{O}_{K} \) of the usual function \( d_{k} \) on \( \mathbb{Z} \). Thus \( d_{(i), k}(n) \) equals the coefficient of \( N(n)^{-s} \) in the Dirichlet series expansion of the \( k \)-th power of \( \zeta_{K}(s) \). We shall also write \( d_{(i), 2}(n) \) for \( d_{(i), 2}(n) \). In particular, when \( n \) is primary, we have \( d_{(i), 1}(n) = 1 \) and
\[ d_{(i), 2}(n) = \sum_{d \equiv 1 \mod (1+i)^{3}} 1. \]
We denote also for rational integer \( j \geq 1 \) and any real number \( t > 0 \),
\[
(2.11) \quad V_j(t) = \frac{1}{2\pi i} \int_{(2)} w_j(s)t^{-s} \frac{ds}{s}, \quad w_j(s) = \left( \frac{2^{5/2}}{\pi} \right)^{j} \left( \frac{\Gamma(\frac{1}{2} + s)}{\Gamma(\frac{1}{2})} \right)^{j}.
\]

We shall also write \( V(t), w(t) \) for \( V_2(t), w_2(t) \) respectively in the rest of the paper.

By setting \( j = 1, 2, \alpha_1 = \alpha_2 = 0, G_j(s) = 1 \) in [5, Lemma 2.6], we obtain the following approximate functional equation for \( L(\frac{1}{2}, \chi_{(1+i)^3}) \). Note here that the derivation of [5, Lemma 2.6] assumes a rapid decay of \( G_j(s) \) but the proof also carries over to the case \( G_j(s) = 1 \) due to the rapid decay of \( w_j(s) \).

**Lemma 2.5** (Approximate functional equation). For any odd, square-free \( d \in \mathcal{O}_K \), we have for \( j = 1, 2 \),
\[
(2.12) \quad L(\frac{1}{2}, \chi_{(1+i)^3}) = 2 \sum_{n \equiv 1 \pmod{(1+i)^3}} \frac{\chi_{(1+i)^3}(n)d_{[1, j]}(n)}{N(n)^2} V_j \left( \frac{N(n)}{N(d)^{1/2}} \right).
\]

### 2.6. Poisson summation.

In this section we gather some Poisson summation formulas over \( K \). We first recall that the Mellin transform \( \hat{f} \) for any function \( f \) is defined to be
\[
\hat{f}(s) = \int_0^\infty f(t)t^{s-1} \frac{dt}{t}.
\]

We now state a formula for smoothed character sums over all elements in \( \mathcal{O}_K \).

**Lemma 2.7.** Let \( n \in \mathcal{O}_K \) and let \( \chi \) be a Hecke character \( \pmod{n} \) of trivial infinite type. For any smooth function \( W: \mathbb{R}^+ \to \mathbb{R} \) of compact support, we have for \( X > 0 \),
\[
(2.13) \quad \sum_{m \in \mathcal{O}_K} \chi(m)W \left( \frac{N(m)}{X} \right) = \frac{X}{N(n)} \sum_{k \in \mathcal{O}_K} g(k, \chi)\hat{W} \left( \sqrt{\frac{N(k)X}{N_K(n)}} \right).
\]

The above is also valid when we replace \( \chi \) by \( \frac{1}{\pi} \) and \( g(k, \chi) \) by \( g(k, n) \). Here \( g(k, \chi), g(k, n) \) are defined in (2.3) and
\[
(2.14) \quad \hat{W}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(N(x + yi))e(-t(x + yi)) \, dx \, dy, \quad t \geq 0.
\]

Moreover, the function \( \hat{W}(t) \) is real-valued for all \( t \geq 0 \) and when \( t > 0 \), we have for \( c_s = \varepsilon > 0 \),
\[
(2.15) \quad \hat{W}(t) = \frac{\pi}{2\pi i} \int_{(c_s)} \hat{W}(1 - s)(\pi t)^{-2s} \frac{\Gamma(s)}{\Gamma(1 - s)} \, ds.
\]

**Proof.** This lemma is essentially [8, Lemma 2.7] except for the last assertion. To establish it, we evaluate (2.14) using polar coordinates to see that
\[
(2.16) \quad \hat{W}(t) = \int_{\mathbb{R}^2} \cos(2\pi ty)W(x^2 + y^2) \, dx \, dy = 4 \int_0^{\pi/2} \int_0^{\pi/2} \cos(2\pi tr \sin \theta)W(r^2) \, r \, dr \, d\theta = 2 \int_0^{\pi/2} \int_0^{\pi/2} \cos(2\pi tr^{1/2} \sin \theta)W(r) \, r \, dr \, d\theta.
\]

The first equality above shows that \( \hat{W}(t) \in \mathbb{R} \) for all \( t \geq 0 \).

We now apply inverse Mellin transform to write \( W(t) \) as
\[
W(t) = \frac{1}{2\pi i} \int_{(c_s)} \hat{W}(u)t^{-u} \, du,
\]
where \( c_u = \varepsilon > 0 \). It follows from this and (2.16) that
\[
\hat{W}(t) = \frac{\pi}{2\pi i} \int_0^{\pi/2} \int_0^{\pi/2} \cos(2\pi tr^{1/2} \sin \theta) \frac{1}{2\pi i} \int_{(c_u)} \hat{W}(1 + u)r^{-u} \, dr \, d\theta.
\]
We make some changes of variables (first \( r^{1/2} \rightarrow r \), then \( 2\pi t r \sin \theta \rightarrow r \)) to see that

\[
\tilde{W}(t) = \frac{4}{2\pi i} \int \frac{W(1 + u)(2\pi t)^2 u^{\pi/2}}{(1 + u)^{\Gamma(-u)}} \frac{du}{\Gamma(1 + u)}
\]

(2.17)

where the last line above follows from the relation (see [5] Section 2.4) that

\[
\int_0^\pi \frac{\sin \theta}{\cos \theta} d\theta = \int_0^\infty \frac{\cos \theta}{\sin \theta} \frac{d\theta}{\theta} \Rightarrow \frac{\pi}{2} \frac{\Gamma(\frac{u}{2})}{\Gamma(\frac{u}{2} - 1)}
\]

By a further change of variable \( u \rightarrow -s \) in the last integral of (2.17), we can recast \( \tilde{W}(t) \) as

\[
\tilde{W}(t) = \frac{\pi}{2\pi i} \int \frac{\cos \theta}{\sin \theta} \frac{d\theta}{\theta} \frac{1}{\Gamma(1 - s)} \frac{\Gamma(s)}{\Gamma(1 - s)} ds,
\]

where we can retake \( c_s = \varepsilon \) as well and this completes the proof. \( \square \)

We remark that when \( \chi \) is a primitive Hecke character, we have

\[ g(r, \chi) = \overline{\chi}(r)g(\chi). \]

It follows from this and the expression given in (2.15) that the formula given (2.13) is equivalent to a version of the Poisson summation formula over number fields by L. Goldmakher and B. Louvel in [10, Lemma 3.2] for the case of the Gaussian field.

In the proof of Theorems 1.5 and 1.6, we need to consider a smoothed character sum over odd algebraic integers in \( \mathcal{O}_K \). For this, we quote the following Poisson summation formula from [8, Corollary 2.8], which is a consequence of Lemma 2.7 above.

**Lemma 2.8.** Let \( n \in \mathcal{O}_K \) be primary and \((\frac{m}{n})\) be the quadratic residue symbol \((\mod n)\). For any smooth function \( W : \mathbb{R}^+ \rightarrow \mathbb{R} \) of compact support, we have for \( X > 0 \),

\[
\sum_{\substack{m \in \mathcal{O}_K \\ (m, 1 + i)^3 = 1}} \frac{W\left(\frac{N(m)}{X}\right)}{\sqrt{2N(m)}} = \frac{X}{2N(n)} \left(1 + i\right)^N \sum_{k \in \mathcal{O}_K} (-1)^{N(k)} g(k, n) W\left(\frac{\sqrt{N(k)}}{2N(n)}\right).
\]

**2.9. Analytical behaviors of certain Dirichlet series.** In this section, we discuss the analytical behaviors of several Dirichlet series that are needed in the proof of Theorems 1.5 and 1.6. We first define for \( \Re(\alpha), \Re(\beta) > \frac{1}{2} \),

\[
Z(\alpha, \beta) = \sum_{\substack{n_1, n_2 \equiv 1 \mod (1 + i)^3}} \frac{d_0(n_1)d_0(n_2)}{N(n_1)^\alpha N(n_2)^\beta} \mathcal{P}(n_1n_2),
\]

where we define \( \mathcal{P}(n) \) for primary \( n \) by

\[
\mathcal{P}(n) = \prod_{\substack{\omega \equiv 1 \mod (1 + i)^3 \\ \omega \mid n}} \left(\frac{N(\omega)}{N(\omega) + 1}\right).
\]

Our first result concerns the analytical behaviors of \( Z(\alpha, \beta) \). The proof is similar to [20, Lemma 4.1], so we omit it here.

**Lemma 2.10.** For \( \Re(\alpha), \Re(\beta) > \frac{1}{2} \), we have

\[
Z(\alpha, \beta) = \zeta_K^3(2\alpha)\zeta_K^3(2\beta)\zeta_K^3(\alpha + \beta)Z_1(\alpha, \beta),
\]
where

\[
Z_1(\alpha, \beta) = \left(1 - \frac{1}{4^\alpha}\right)^3 \left(1 - \frac{1}{4^\beta}\right)^3 \left(1 - \frac{1}{2^{\alpha+\beta}}\right)^4 \prod_{\varpi \equiv 1 \mod (1+i)^3} Z_{1, \varpi}(\alpha, \beta),
\]

and where for primary \(\varpi\),

\[
Z_{1, \varpi}(\alpha, \beta) = \left(1 - \frac{1}{N(\varpi)^{2\alpha}}\right) \left(1 - \frac{1}{N(\varpi)^{2\beta}}\right) \left(1 - \frac{1}{N(\varpi)^{\alpha+\beta}}\right)^4 \left(1 + \frac{4}{N(\varpi)^{\alpha+\beta}} + \frac{1}{N(\varpi)^{2\alpha}} + \frac{1}{N(\varpi)^{2\beta}} + \frac{1}{N(\varpi)^{2\alpha+2\beta}}\right)
\]

\[- \frac{1}{N(\varpi)} + \frac{3}{N(\varpi)^{2\alpha+2\beta}} + \frac{2}{N(\varpi)^{2\alpha+4\beta}} + \frac{2}{N(\varpi)^{4\alpha+2\beta}} - \frac{1}{N(\varpi)^{4\alpha+4\beta}}\).

Moreover, \(Z_1(\alpha, \beta)\) is analytic and uniformly bounded in the region \(\Re(\alpha), \Re(\beta) \geq \frac{1}{4} + \varepsilon\).

Let \(g(k, n)\) be defined as in (2.3). We now fix a generator for every prime ideal \((\varpi) \in O_K\) by taking \(\varpi\) to be primary if \((\varpi, 1+i) = 1\) and \(1+i\) for the ideal \((1+i)\) (noting that \((1+i)\) is the only prime ideal in \(O_K\) that lies above the integral ideal \((2) \in \mathbb{Z}\). We also fix 1 as the generator for the ring \(\mathbb{Z}[i]\) itself and extend the choice of the generator for any ideal of \(O_K\) multiplicatively. We denote the set of such generators by \(G\). For any \(k \in O_K\), we shall hence denote \(k_1, k_2\) to be the unique pair of elements in \(O_K\) such that \(k = k_1k_2^2\) with \(k_1\) being square-free and \(k_2 \in G\). In this way, we define

\[
Z(\alpha, \beta, a, k) = \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{d_{[i]}(n_1)d_{[i]}(n_2)}{N(n_1)^a N(n_2)^b} \frac{g(k_1k_2^2, n_1n_2)}{N(n_1n_2)}
\]

Our next lemma gives the analytic properties of \(Z(\alpha, \beta, a, k)\), we omit the proof here since it is similar to [26, Lemma 5.2].

**Lemma 2.11.** For any \(k \in O_K\), let \(k = k_1k_2^2\) with \(k_1\) square-free and \(k_2 \in G\). Then for \(\Re(\alpha), \Re(\beta) > \frac{1}{2}\), we have

\[
Z(\alpha, \beta, a, k) = L^2(\frac{1}{2} + \alpha, \chi_{ik_1}) L^2(\frac{1}{2} + \beta, \chi_{ik_1}) Z_2(\alpha, \beta, a, k),
\]

where

\[
Z_2(\alpha, \beta, a, k) = Z_{2,1+i}(\alpha, \beta, a, k) \prod_{\varpi \equiv 1 \mod (1+i)^3} Z_{2, \varpi}(\alpha, \beta, a, k).
\]

Here for \(\varpi \mid 2a\),

\[
Z_{2, \varpi}(\alpha, \beta, a, k) = \left(1 - \frac{\chi_{ik_1}(\varpi)}{N(\varpi)^{\frac{1}{2} + \alpha}}\right)^2 \left(1 - \frac{\chi_{ik_1}(\varpi)}{N(\varpi)^{\frac{1}{2} + \beta}}\right)^2,
\]

and for \(\varpi \not\mid 2a\),

\[
Z_{2, \varpi}(\alpha, \beta, a, k) = \left(1 - \frac{\chi_{ik_1}(\varpi)}{N(\varpi)^{\frac{1}{2} + \alpha}}\right)^2 \left(1 - \frac{\chi_{ik_1}(\varpi)}{N(\varpi)^{\frac{1}{2} + \beta}}\right)^2 \sum_{n_1 = 0}^{\infty} \sum_{n_2 = 0}^{\infty} \frac{d_{[i]}(\varpi^{n_1})d_{[i]}(\varpi^{n_2}) g(k, \varpi^{n_1+n_2})}{N(\varpi)^{n_1\alpha+n_2\beta} N(\varpi)^{n_1+n_2}}.
\]

Moreover, \(Z_2(\alpha, \beta, a, k)\) is analytic in the region \(\Re(\alpha), \Re(\beta) > 0\) and for \(\Re(\alpha), \Re(\beta) \geq \frac{1}{\log X}\), we have

\[
(2.21) \quad Z_2(\alpha, \beta, a, k) \ll d_{[i]}^2(\alpha)d_{[i]}^2(\beta)(\log X)^{10},
\]

where the implied constant is absolute.
We define for a prime \( \omega \in \mathcal{O}_K \), \( \alpha, \beta, \gamma \in \mathbb{C} \),
\[
K_1(\alpha, \beta, \gamma; \omega) = \left( 1 - \frac{1}{N(\omega)^{\frac{1}{2}+\alpha}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{\frac{1}{2}+\beta}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{2\alpha+2\gamma}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{2\beta+2\gamma}} \right)^2,
\]
\[
K_2(\alpha, \beta, \gamma; \omega) = \left( 1 - \frac{1}{N(\omega)^{\frac{1}{2}+\alpha}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{\frac{1}{2}+\beta}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{2\alpha+2\gamma}} \right)^2 \left( 1 + \frac{1}{N(\omega)^{2\beta+2\gamma}} \right) \left( 1 + \frac{1}{N(\omega)^{\frac{1}{2}+\alpha+2\gamma}} \right) \left( 1 - \frac{1}{N(\omega)^{\frac{1}{2}+\beta+2\gamma}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{2\alpha+2\gamma}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{2\beta+2\gamma}} \right)^2 + \frac{1}{N(\omega)} \left( 1 - \frac{1}{N(\omega)^{2\alpha+2\gamma}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{2\beta+2\gamma}} \right)^2 + \frac{1}{N(\omega)} \left( 1 - \frac{1}{N(\omega)^{\frac{1}{2}+\alpha+2\gamma}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{\frac{1}{2}+\beta+2\gamma}} \right) \left( 1 - \frac{1}{N(\omega)^{2\alpha+2\gamma}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{2\beta+2\gamma}} \right)^2 + 2 \left( 1 - \frac{1}{N(\omega)^{2\gamma}} \right) \left( 1 - \frac{1}{N(\omega)^{\frac{1}{2}+\alpha}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{\frac{1}{2}+\beta}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{2\alpha+2\gamma}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{2\beta+2\gamma}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{\frac{1}{2}+\alpha+2\gamma}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{\frac{1}{2}+\beta+2\gamma}} \right) \left( 1 - \frac{1}{N(\omega)^{2\alpha+2\gamma}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{2\beta+2\gamma}} \right)^2 \right)\).

With the above notations, we note the following result concerning a Dirichlet series related to \( Z_2(\alpha, \beta, a, k) \).

Lemma 2.12. For \( \Re(\alpha), \Re(\beta) > 0, \Re(\gamma) > \frac{1}{2} \), we have
\[
\sum_{k \in G} \frac{(-1)^{N(k)}}{N(k)^{2\gamma}} Z_2(\alpha, \beta, a, \pm ik^2) = (2^{1-2\gamma} - 1) \zeta_K(2\gamma) \zeta_K^2(2\alpha + 2\gamma) \zeta_K^2(2\beta + 2\gamma) Z_3(\alpha, \beta, \gamma, a),
\]
where
\[
Z_3(\alpha, \beta, \gamma, a) = \prod_{\omega \equiv 1 \mod (1+i)^3} Z_{3,\omega}(\alpha, \beta, \gamma, a),
\]
Here for \( \omega \mid 2a \), \( Z_{3,\omega}(\alpha, \beta, \gamma, a) = K_1(\alpha, \beta, \gamma; \omega) \) and for \( \omega \nmid 2a \), \( Z_{3,\omega}(\alpha, \beta, \gamma, a) = K_2(\alpha, \beta, \gamma; \omega) \).

In addition, we have
(1) \( Z_3(\alpha, \beta, \gamma, a) \) is analytic and uniformly bounded for \( \Re(\alpha), \Re(\beta) \geq \frac{1}{2} + \varepsilon, \Re(\gamma) \geq 2\varepsilon \).
(2) \( Z_3(\alpha, \beta, \gamma, a) \ll (\log X)^{10} \) for \( \Re(\alpha), \Re(\beta) \geq \frac{1}{2} + \frac{1}{2 \log X}, \Re(\gamma) \geq \frac{1}{2} \) with the implied constant being absolute.

Proof. We let \( f(k) = g(k, n)/N(k)^{\gamma} \) and we divide the sum over \( k \) in (2.22) into two sums, according to \( (k, 1+i) = 1 \) or not, to see that
\[
\sum_{k \in G} (-1)^{N(k)} f(\pm ik^2) = \left( 2 \sum_{k \in G} f(\pm 2ik^2) - \sum_{k \in G} f(\pm ik^2) \right),
\]
Note that when \( (n, 1+i) = 1 \), \( g(k, n) = g(2k, n) \) by Lemma 2.3 It follows that we have \( f(\pm 2ik^2) = 4^{-s} f(\pm ik^2) \) so that
\[
\sum_{k \in G} (-1)^{N(k)} f(\pm ik^2) = (2^{1-2s} - 1) \sum_{k \in G} f(\pm ik^2).
\]
We then deduce from this and the definition of \( Z_2 \) given in Lemma 2.11 that
\[
\sum_{k \in G} \frac{(-1)^{N(k)}}{N(k)^{2\gamma}} Z_2(\alpha, \beta, a, \pm ik^2) = (2^{1-2\gamma} - 1) \sum_{k \in G} \frac{1}{N(k)^{2\gamma}} Z_2(\alpha, \beta, a, k^2) = \prod_{\omega \equiv 1 \mod (1+i)^3} \sum_{b \equiv 0 \mod (1+i)^3} \frac{Z_{3,\omega}(\alpha, \beta, a, \omega 2b)}{N(\omega)^{2\gamma} N(\omega)^{2k^2}},
\]
where the last equality above follows from the observation that \( g(\pm ik^2, n) \) is multiplicative with respect to \( k \) and that we have \( g(\pm ik^2, n) = g(k^2, n) \) when \( n \) is primary from Lemma 2.3 The assertions of Lemma 2.12 now follows by arguing similarly to the proof of [20] Lemma 5.3. \( \square \)

Lastly, we note the following result which can be established similar to the proof of [20] Lemma 5.4.

Lemma 2.13. For \( \Re(\alpha), \Re(\beta) > \frac{1}{2}, 0 < \Re(\gamma) < \frac{1}{2} \), we have
\[
\sum_{a \equiv 1 \mod (1+i)^3} \frac{\mu_\mathfrak{a}(a)}{N(a)^{2-2\gamma}} Z_4(\alpha, \beta, \gamma; a) = \frac{\zeta_K(2\alpha + 2\gamma) \zeta_K(2\beta + 2\gamma) \zeta_K^4(\alpha + \beta + 2\gamma)}{\zeta_K^4(\frac{1}{2} + \alpha + 2\gamma) \zeta_K^4(\frac{1}{2} + \beta + 2\gamma)} Z_4(\alpha, \beta, \gamma),
\]
where
\[
Z_4(\alpha, \beta, \gamma) = K_1(\alpha, \beta, \gamma; 1+i) \prod_{\omega \equiv 1 \mod (1+i)^3} \left( K_2(\alpha, \beta, \gamma; \omega) - \frac{1}{N(\omega)^{2-2\gamma}} K_1(\alpha, \beta, \gamma; \omega) \right) \prod_{\omega \equiv 1 \mod (1+i)^3} \left( \frac{1}{1 - \frac{1}{N(\omega)^{1+2\gamma}}} \left( 1 - \frac{1}{N(\omega)^{1+2\gamma}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{1+2\gamma}} \right)^2 \left( 1 - \frac{1}{N(\omega)^{1+2\gamma}} \right)^2 \right).
\]
Moreover, \( Z_4(\alpha, \beta, \gamma) \) is analytic and uniformly bounded for \( \Re(\alpha), \Re(\beta) \geq \frac{3}{8}, -\frac{1}{16} \leq \Re(\gamma) \leq \frac{1}{8} \).
2.14. A mean value estimate for quadratic Hecke \( L \)-functions. In \cite{12} Theorem 1], D. R. Heath-Brown established a powerful quadratic large sieve result for Dirichlet characters. Such result was extended by K. Onodera in \cite{22} to quadratic residue symbols in the Gaussian field. Applying Onodera’s result in a similar fashion as in the derivation of \cite{12} Theorem 2] by Heath-Brown to obtain a mean value estimation for the fourth moment of the family of primitive quadratic Dirichlet \( L \)-functions, we have the following upper bound for the fourth moment of quadratic Hecke \( L \)-functions unconditionally.

Lemma 2.15. Suppose \( \sigma + it \) is a complex number with \( \sigma \geq \frac{1}{2} \). Then

\[
\sum_{(d,2)=1 \atop N(d) \leq X} \ast |L(\sigma + it, \chi(1+i)^d)|^4 \ll X^{1+\varepsilon} (1 + |t|^2)^{1+\varepsilon}.
\]

3. Proof of Theorem 2.1

We recall a result of Landau \cite{19} implies that for an algebraic number field \( F \) of degree \( n = 2 \) and any primitive ideal character \( \chi \) of \( F \) with conductor \( q \), we have for \( X > 1 \),

\[
(3.1) \quad \sum_{N_F(I) \leq X} \chi(I) \ll |N_F(q) \cdot D_F|^{1/3} \log^2(|N_F(q) \cdot D_F|) X^{1/3},
\]

where \( N_F(q), N_F(I) \) denotes the norm of \( q \) and \( I \) respectively, \( D_F \) denotes the discriminant of \( F \) and \( I \) runs over integral ideas of \( F \).

We deduce from (3.1) by partial summation that for odd, square-free \( d \), the series

\[
\sum_{n \equiv 1 \mod (1+i)^3} \frac{\chi(1+i)^d(n)}{\sqrt{N(n)}}
\]

is convergent and equals to \( L(\frac{1}{2}, \chi(1+i)^d) \). In particular, this implies that \( L(\frac{1}{2}, \chi(1+i)^d) \in \mathbb{R} \). It follows from this and the observation that \( k \) is an even natural number that we may further restrict the sum over \( d \) in (1.5) to satisfy \( X/2 < N(d) \leq X \).

We set \( x = X \frac{1}{10} \varepsilon \) and apply Hölder’s inequality to see that

\[
(3.2) \quad \sum_{(d,2)=1 \atop X/2 < N(d) \leq X} \ast L(\frac{1}{2}, \chi(1+i)^d) = \frac{S_1}{S_2-1},
\]

where

\[
S_1 = \sum_{(d,2)=1 \atop X/2 < N(d) \leq X} \ast L(\frac{1}{2}, \chi(1+i)^d) A(d)^k-1, \quad S_2 = \sum_{(d,2)=1 \atop X/2 < N(d) \leq X} \ast A(d)^k.
\]

and

\[
A(d) = \sum_{n \equiv 1 \mod (1+i)^3} \frac{\chi(1+i)^d(n)}{\sqrt{N(n)}}.
\]

In the remaining of the proof, it thus suffices to bound \( S_1 \) and \( S_2 \). We bound \( S_2 \) first by noting that

\[
S_2 \ll \sum_{(d,2)=1} \ast A(d)^k W\left(\frac{N(d)}{X}\right),
\]

where \( W(t) \) is any non-negative smooth function that is supported on \(( \frac{1}{2} - \varepsilon, 1 + \varepsilon )\) for some fixed small \( 0 < \varepsilon < 1/2 \) such that \( W(t) \gg 1 \) for \( t \in (\frac{1}{2}, 1) \).

We now expand \( A(d)^k \) to see that

\[
S_2 = \sum_{n_1, \ldots, n_k \equiv 1 \mod (1+i)^3} \frac{1}{N(n_1 \cdot \ldots \cdot n_k)} \sum_{(d,2)=1} \ast \left( \frac{(1+i)^d}{n_1 \cdot \ldots \cdot n_k} \right) W\left(\frac{N(d)}{X}\right).
\]
We consider the inner sum above by setting \( n = n_1 \cdots n_k \). Using Möbius function to express the condition that \( d \) is square-free, we see that

\[
(3.3) \quad \sum_{(d,2)=1}^{*} \left( \frac{(1+i)^5d}{n} \right) W_n \left( \frac{N(d)}{X} \right) = \sum_{\alpha \equiv 1 \mod (1+i)^3} \sum_{N(\alpha) \leq 2X^{1/2}} \left( \frac{(1+i)^5\alpha^2}{n} \right) \sum_{(d,2)=1}^{*} W_n \left( \frac{N(\alpha^2d)}{X} \right).
\]

Note that for smoothed sums involving any non-principal Hecke character modulo \( \alpha \) of trivial infinite type, we have (see [13] (1.4)) that for \( y \geq 1 \)

\[
(3.4) \quad \sum_{c \equiv 1 \mod (1+i)^3} \chi(c) \Phi \left( \frac{N(c)}{y} \right) \ll \alpha N(\alpha)^{(1+\varepsilon)/2}.
\]

Now, we write \( d = jd' \) with \( j \in U_\alpha \) and \( d' \) being primary and apply the quadratic reciprocity law (2.11) to see that

\[
(3.5) \quad \left( \frac{d}{n} \right) = \left( \frac{j}{n} \right) \left( \frac{n}{d} \right).
\]

Note that if \( n \) is not a square then the symbol \( \left( \frac{d}{n} \right) \) can be regarded as a non-principal Hecke character \( \mod {16n} \) of trivial infinite type. By decomposing the sum over \( d \) in (3.3) into sums over \( j, d' \) and apply (3.4) to the sum over \( d' \), we deduce that the sum over \( d \) in (3.3) is \( \ll N(n)^{1/2+\varepsilon} \). On the other hand, the sum over \( d \) is trivially \( \ll X/N(\alpha)^2 \).

We then conclude that if \( n \) is not a square,

\[
(3.6) \quad \sum_{(d,2)=1}^{*} \left( \frac{(1+i)^5d}{n} \right) W_n \left( \frac{N(\alpha^2d)}{X} \right) \ll \sum_{\alpha \equiv 1 \mod (1+i)^3} \sum_{N(\alpha) \leq 2X^{1/2}} \left( \frac{(1+i)^5\alpha^2}{n} \right) \min \left( N(n)^{1/2+\varepsilon}, \frac{X}{N(\alpha)^2} \right) \ll X^{\frac{1}{4}} N(n)^{\frac{1}{4}+\varepsilon}.
\]

If \( n \) is a perfect square, then by applying the following result for the Gauss circle problem (see [13]) with \( \theta = 131/416 \),

\[
(3.7) \quad \sum_{N(\alpha) \leq x} N(\alpha) \leq \pi x + O(x^\theta)
\]

(together with a routine argument, we see that

\[
(3.8) \quad \sum_{(d,2)=1}^{*} \left( \frac{(1+i)^5d}{n} \right) W_n \left( \frac{N(d)}{X} \right) \ll \sum_{N(d) \leq X} \sum_{(d,2n)=1}^{*} 1 = \frac{2\pi X}{3K(2)} P(n) + O(X^{\frac{1}{4}+\varepsilon} N(n)^{\varepsilon}),
\]

where \( P(n) \) is defined in (2.20).

Using (3.6) and (3.8), we see that

\[
(3.9) \quad S_2 \ll X \sum_{n_1,\ldots,n_k \equiv 1 \mod (1+i)^3} \frac{P(n)}{N(n)} + O \left( X^{\frac{1}{4}+\varepsilon} x^{k(\frac{1}{4}+\varepsilon)} \right).
\]

We note that

\[
(3.10) \quad \sum_{n \equiv 1 \mod (1+i)^3} \frac{d_{i,j,k}(n^2)}{N(n)^2} P(n) \ll \sum_{n_1,\ldots,n_k \equiv 1 \mod (1+i)^3} \frac{P(n)}{N(n)} \ll \sum_{n \equiv 1 \mod (1+i)^3} \frac{d_{i,j,k}(n^2)}{N(n)} P(n).
\]

Similar to [20], Theorem 2, we have that for a positive constant \( C(k) \),

\[
(3.11) \quad \sum_{n \equiv 1 \mod (1+i)^3} \frac{d_{i,j,k}(n^2)}{N(n)^2} P(n) \sim C(k)(\log z)^{k(k+1)/2}.
\]

Applying this in (3.10) and setting \( x = X^{1/4} \) in (3.9), we deduce that

\[
(3.12) \quad S_2 \ll X (\log X)^{k(k+1)/2}.
\]

We evaluate \( S_1 \) next. By applying the approximate functional equation (2.12) for \( L(\frac{1}{2}, \chi(1+i)^{3d}) \), we see that

\[
(3.13) \quad S_1 = 2 \sum_{n \equiv 1 \mod (1+i)^3} \frac{1}{\sqrt{N(n)}} \sum_{n_1,\ldots,n_k \equiv 1 \mod (1+i)^3} \frac{1}{\sqrt{N(n_1 \cdots n_{k-1})}} \sum_{(d,2)=1}^{*} \frac{(1+i)^5d}{n_{i,k-1}} \sum_{X/2 < N(d) \leq X} V_i \left( \frac{N(n)}{\sqrt{N(d)}} \right).
\]
Here we note that (see Lemma 2.1) \( V_1(x) \) is real-valued and smooth on \([0, \infty)\) and the \( j \)-th derivative of \( V_1(x) \) satisfies

\[
V_1(x) = 1 + O(\xi^{1/2-j}) \quad \text{for } 0 < \xi < 1 \quad \text{and} \quad V_1^{(j)}(\xi) = O(e^{-\xi}) \quad \text{for } \xi > 0, \ j \geq 0.
\]

If \( n_{n_1} \cdots n_{k-1} \) is not a square, then using (3.4), (3.14) and partial summation we see that

\[
\sum_{(d,2)=1} \frac{(1+i)^5d}{n_1 \cdots n_{k-1}} V_1 \left( \frac{N(n)}{\sqrt{N(d)}} \right) \ll X^{\frac{1}{4}+\varepsilon} N(n_{n_1} \cdots n_{k-1})^{\frac{1}{2}+\varepsilon} e^{-N(n)/\sqrt{X}}.
\]

If \( n_{n_1} \cdots n_{k-1} \) is an square, then using the right-hand side expression in (3.8) together with (3.14) and partial summation implies that the sum over \( d \) in (3.8) is

\[
\sum_{(d,2)=1} \frac{(1+i)^5d}{n_1 \cdots n_{k-1}} V_1 \left( \frac{N(n)}{\sqrt{N(d)}} \right) = \frac{X}{3\zeta_K(2)} P(n_{n_1} \cdots n_{k-1}) \int_1^X V_1 \left( \frac{\sqrt{2N(n)}}{\sqrt{Xt}} \right) dt + O(X^{\frac{1}{4}+\varepsilon} N(n_{n_1} \cdots n_{k-1})^{\frac{1}{2}+\varepsilon} e^{-N(n)/\sqrt{X}}).
\]

Applying (3.10) and (3.16) in (3.13), we see that the error terms in (3.15) and (3.16) contribute to \( (3.13) \) an amount \( \ll X^{\frac{1}{4}+\varepsilon} x^{(k-1)(\frac{1}{4}+\varepsilon)} X^{\frac{1}{2}+\varepsilon} \ll X^{\frac{3}{4}} \).

To estimate contribution of the main term in (3.10) to (3.13), we write \( n_{n_1} \cdots n_{k-1} \) as \( rs^2 \) where \( r \) and \( s \) are primary and \( r \) is square-free. Then \( n \) must be of the form \( r\ell^2 \) where \( \ell \) is primary. It follows that the contribution of the main term in (3.10) to (3.13) is

\[
\frac{2X}{3\zeta_K(2)} \sum_{r,s \equiv 1 \mod (1+i)^3} \frac{1}{N(rs)} \sum_{\ell \equiv 1 \mod (1+i)^3} \frac{1}{N(\ell)} \int_1^X P(rs\ell) V_1 \left( \frac{\sqrt{2N(r\ell^2)}}{\sqrt{Xt}} \right) dt.
\]

Note that \( N(r) \leq x^{k-1} < X^{\frac{3}{4}} \), and by a standard argument, we see that the sum over \( \ell \) above is

\[
\frac{2}{3} P(rs) \prod_{\omega \equiv 1 \mod (1+i)^3} \left( 1 - \frac{1}{N(\omega)(N(\omega + 1))} \right) \frac{\pi}{4} \log \frac{\sqrt{X}}{N(r)} + O(1).
\]

It follows that the contribution of the main term in (3.10) to (3.13) is

\[
X \log X \sum_{r,s \equiv 1 \mod (1+i)^3} \frac{1}{N(rs)} P(rs) \gg X \log X \sum_{r \text{ primary and square-free}} \frac{d_{[i,k-1]}(rs^2)}{N(rs)} P(rs) \gg X \log X (\log x)^{k-1+k(k-1)/2}.
\]

We then deduce that

\[
S_1 \gg X (\log X)^{k(k+1)/2}.
\]

The assertion of Theorem 1.2 now follows from (3.2), (3.12) and the above bound.

4. Proof of Theorem 1.2

4.1. A few lemmas. We first include a few lemmas needed in the proof of Theorem 1.2. We denote \( \Lambda_{[i]}(n) \) for the von Mangoldt function on \( K \). Thus \( \Lambda_{[i]}(n) \) equals the coefficient of \( N(n)^{-s} \) in the Dirichlet series expansion of \( \zeta_K(s)/\zeta_K(s) \). Our first lemma provides an upper bound of \( \log |L(s, \chi)| \) in terms of a sum involving prime powers.

**Lemma 4.2.** Let \( \chi \) be a non-principal primitive quadratic Hecke character modulo \( m \) of trivial infinite type. Assume GRH for \( \zeta_K(s) \) and \( L(s, \chi) \). Let \( T \) be a large number and \( x \geq 2 \). Let \( \lambda_0 = 0.56 \ldots \) denote the unique positive real
number satisfying $e^{-\lambda_0} = \lambda_0$. For all $\lambda_0 \leq \lambda \leq 5$ we have uniformly for $|t| \leq T$ and $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \frac{\lambda}{\log x}$ that 

$$(4.1) \quad \log |L(\sigma + it, \chi)| \leq \Re \sum_{(n) \leq x} \frac{\Lambda(n)}{N(n)^{\frac{1}{2}+\sigma+it}} \log N(n) \log x + (\log T + \log N(m) \frac{2}{2})(\frac{1}{2} - \sigma) + \frac{1+\lambda}{\log x} + O\left(\frac{1}{\log x}\right),$$

where the sum $\sum_{(n)}$ means that the sum is over integral ideals of $\mathcal{O}_K$.

Proof. We denote $s$ for $\sigma + it$ and we interpret $\log |L(s, \chi)|$ as $-\infty$ when $L(s, \chi) = 0$. Thus we may suppose $L(\sigma + it, \chi) \neq 0$ in the rest of the proof. Recall the associated function $\Lambda(s, \chi)$ defined as in (2.9). Here $\Lambda(s, \chi)$ is analytical in the entire complex plane since $\chi$ is non-principal. As $\Gamma(s)$ has simple poles at the non-positive rational integers (see [4, §10]), we see from the expression of $\Lambda(s, \chi)$ from (2.9) that $L(s, \chi)$ has simple zeros at $s = 0, -1, -2, \ldots$, these are called the trivial zeros of $L(s, \chi)$. Since we are assuming GRH, we know that the non-trivial zeros of $L(s, \chi)$ are precisely the zeros of $\Lambda(s, \chi)$.

Let $\rho = \frac{1}{2} + i \gamma$ run over the non-trivial zeros of $L(s, \chi)$. We then deduce from [10, Theorem 5.6] and the observation that $L(s, \chi)$ is analytic at $s = 1$ since $\chi$ is non-principal and primitive that

$$(4.2) \quad \Lambda(s, \chi) = e^{A+B}s \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\pi \rho},$$

where $A, B = B(\chi)$ are constants.

Taking the logarithmic derivative on both sides of (4.2) and making use of (2.9), we obtain that

$${- L'(s, \chi) \over L(s, \chi)} = \Gamma'(s) - \log \pi + \frac{1}{2} \log N(m) - \Re B - \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

This implies that

$$(4.3) \quad \Re { L'(s, \chi) \over L(s, \chi)} = \Re \Gamma'(s) - \log \pi + \frac{1}{2} \log N(m) - \Re B - \sum_{\rho} \Re \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

On the other hand, combining the functional equation (2.10) with (4.2), we see that

$$e^{A+B}s \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\pi \rho} = W(\chi)(N(m))^{-1/2} e^{A+B(1-s)} \prod_{\rho} \left(1 - \frac{1-s}{\rho}\right) e^{-\pi \rho^{1-s}}.$$

Taking logarithmic derivative on both sides of the above expression, we obtain that

$$2B = - \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{1-s-\rho} + \frac{1}{\rho} + \frac{1}{\rho}\right) = -2 \sum_{\rho} \frac{1}{\rho}.$$

Here the second equality above follows by noting that the terms containing $1-s-\rho$ and $s-\rho$ cancel as both $\rho, 1-\rho$ are zeros from the functional equation (2.10). We note here that

$$\sum_{\rho} \frac{1}{\rho}$$

is convergent since if $\rho$ is a zero, so is $\overline{\rho}$ and we have

$$\rho^{-1} + \overline{\rho}^{-1} \ll |\rho|^{-2}$$

and we know that $\sum_{\rho} \frac{1}{|\rho|^2}$ is convergent by [15, Lemma 5.5].

We then deduce that

$$\Re(B) = - \sum_{\rho} \Re(\rho^{-1}).$$
Combining the above with (4.4), we see that

\[-\Re \frac{L'}{L}(s, \chi) = \Re \left( \frac{\Gamma'}{\Gamma}(s) - \log \pi + \frac{1}{2} \log N(m) - \Re \sum_{\rho} \left( \frac{1}{s - \rho} \right) \right) \]

\begin{align*}
&= \Re \left( \frac{\Gamma'}{\Gamma}(s) - \log \pi + \frac{1}{2} \log N(m) - F(s) \right) \\
&= \log(|t| + 1) + \frac{1}{2} \log N(m) + O(1) - F(s) \\
&\leq \log T + \frac{1}{2} \log N(m) + O(1) - F(s).
\end{align*}

(4.4)

where the third line above follows from \( \frac{\Gamma'}{\Gamma}(s) = \log s + O(|s|^{-1}) \) by (6) of [4, §10] and where we define

\[F(s) = \Re \sum_{\rho} \frac{1}{s - \rho} = \sum_{\rho} \frac{\sigma - 1/2}{(\sigma - 1/2)^2 + (t - \gamma)^2}.\]

Integrating the last expression given for \(-\Re \frac{L'}{L}(s, \chi)\) in (4.4) from \(\sigma = \Re(s)\) to \(\sigma_0 > \frac{1}{2}\), we obtain by setting \(s_0 = \sigma_0 + it\) that

\begin{align*}
\log |L(s, \chi)| - \log |L(s_0, \chi)| &\leq \left( \log T + \frac{\log N(m)}{2} + O(1) \right)(\sigma_0 - \sigma) - \int_{\sigma}^{\sigma_0} F(u + it)du \\
&\leq (\sigma_0 - \sigma) \left( \log T + \frac{\log N(m)}{2} + O(1) \right),
\end{align*}

(4.5)

where the last inequality above follows from the observation that \(F(s) \geq 0\) for all \(s\) satisfying \(\Re(s) \geq \frac{1}{2}\).

Next, we deduce upon integrating term by term using the Dirichlet series expansion of \(-\frac{L'}{L}(s + w, \chi)\) that

\[\frac{1}{2\pi i} \int_{(c)} -\frac{L'}{L}(s + w, \chi) \frac{x^w}{w^2} dw = \sum_{n \in \mathbb{N}} \frac{\Lambda_i(n) \chi(n)}{N(n)^s} \log \left( \frac{x}{N(n)} \right),\]

where \(c > 2\) is a large number. Now moving the line of integration in the above expression to the left and calculating residues, we see also that

\[\frac{1}{2\pi i} \int_{(c)} -\frac{L'}{L}(s + w, \chi) \frac{x^w}{w^2} dw = -\frac{L'}{L}(s, \chi) \log x - \left( \frac{L'}{L}(s, \chi) \right)' - \sum_{\rho} \frac{x^{\rho - s}}{(\rho - s)^2} - \sum_{k=0}^{\infty} \frac{x^{-k - s}}{(k + s)^2}.
\]

Comparing the above two expressions, we deduce that unconditionally, for any \(x \geq 2\), we have

\[\frac{L'}{L}(s, \chi) = \sum_{n \in \mathbb{N}} \frac{\Lambda_i(n)}{N(n)^s} \log(x/N(n)) + \frac{1}{\log x} \left( \frac{L'}{L}(s, \chi) \right)' + \frac{1}{\log x} \sum_{\rho} \frac{x^{\rho - s}}{(\rho - s)^2} + \frac{1}{\log x} \sum_{k=0}^{\infty} \frac{x^{-k - s}}{(k + s)^2}.\]

(4.6)

We now integrate the real parts on both sides of (4.6) over \(\sigma = \Re(s)\) from \(\sigma_0\) to \(\infty\) to see that for \(x \geq 2\),

\[\log |L(s_0, \chi)| = \Re \left( \sum_{n \in \mathbb{N}} \frac{\Lambda_i(n)}{N(n)^{s_0} \log N(n)} \log(x/N(n)) \right) - \frac{1}{\log x} \frac{L'}{L}(s_0, \chi) + \frac{1}{\log x} \sum_{\rho} \int_{s_0}^{\infty} \frac{x^{\rho - s}}{(\rho - s)^2} d\sigma + O\left( \frac{1}{\log x} \right).
\]

(4.7)

Observe that

\[\sum_{\rho} \left| \int_{\sigma_0}^{\infty} \frac{x^{\rho - s}}{(\rho - s)^2} d\sigma \right| \leq \sum_{\rho} \int_{\sigma_0}^{\infty} \frac{x^{1/2 - \sigma}}{|s_0 - \rho|^2} d\sigma = \sum_{\rho} \frac{x^{1/2 - \sigma_0}}{|s_0 - \rho|^2 \log x} = \frac{x^{1/2 - \sigma_0} F(s_0)}{(\sigma_0 - \frac{1}{2}) \log x}.
\]
Applying this and (4.9) in (4.7), we see that

\[
\log |L(s_0, \chi)| \leq \Re \sum_{n \leq x} \frac{\Lambda(n)}{N(n)^{s_0} \log N(n)} \frac{\log(x/N(n))}{\log x} + \frac{1}{\log x}(\log T + \frac{\log N(n)}{2}) + F(s_0)\left(\frac{x^{\frac{1}{2} - \sigma_0}}{(\sigma_0 - \frac{1}{2}) \log^2 x} - \frac{1}{\log x}\right) + O\left(\frac{1}{\log x}\right).
\]

Adding inequalities (4.3) and (4.8), we deduce that

\[
\log |L(s, \chi)| \leq \Re \sum_{n \leq x} \frac{\Lambda(n)}{N(n)^{s_0} \log N(n)} \frac{\log(x/N(n))}{\log x} + (\log T + \frac{\log N(n)}{2})(\sigma_0 - \sigma + \frac{1}{\log x}) + F(s_0)\left(\frac{x^{\frac{1}{2} - \sigma_0}}{(\sigma_0 - \frac{1}{2}) \log^2 x} - \frac{1}{\log x}\right) + O\left(\frac{1}{\log x}\right) + O(\sigma_0 - \sigma).
\]

The assertion of the proposition now follows by setting \(\sigma_0 = \frac{1}{2} + \lambda / \log x\) with \(\lambda \geq \lambda_0\) and by omitting the term involving \(F(s_0)\) since it is negative. \(\square\)

Our next lemma treats essentially the sum over prime squares in (4.1).

**Lemma 4.3.** Assume GRH for \(\zeta_K(s)\). Let \(\lambda_0\) be given as in Lemma 4.2 and let \(z \in \mathbb{C}\) with \(0 \leq \Re(z) \leq \frac{1}{\log x}\). We have for \(x \geq 10\),

\[
\sum_{\varpi \equiv 1 \mod{(1+i)^3}} \frac{1}{N(\varpi)^{1+\frac{2\lambda_0}{\log x}+2z}} \log \frac{\varpi}{N(\varpi)} = \mathcal{L}(z, x) + O(1),
\]

where \(\mathcal{L}(z, x)\) is given in (4.9).

**Proof.** We first note that under GRH for \(\zeta_K(s)\), the prime ideal theorem has the following form (see [15, Theorem 5.15])

\[
\sum_{\varpi \equiv 1 \mod{(1+i)^3}} \log N(\varpi) = y + O(\sqrt{y} (\log X y)^2).
\]

It follows from this and \(\Re(z) \geq 0\) that we have

\[
\sum_{\varpi \equiv 1 \mod{(1+i)^3}} \frac{1}{N(\varpi)^{1+\frac{2\lambda_0}{\log x}+2z}} \log N(\varpi) \ll \sum_{\varpi \equiv 1 \mod{(1+i)^3}} \frac{1}{N(\varpi)} \log N(\varpi) = O(1).
\]

We then deduce from this that it suffices to establish (4.9) with the left-hand side expression in (4.9) being replaced by \(f(\frac{2\lambda_0}{\log x}+2z, x)\), where

\[
f(z, x) = \sum_{\varpi \equiv 1 \mod{(1+i)^3}} \frac{1}{N(\varpi)^{1+z}} \log N(\varpi) \log x + O(1).
\]

It is easily seen that \(f(\frac{2\lambda_0}{\log x}+2z, x) = O(1)\) when \(|z| \geq 1\) using (4.10) and partial summation. The same procedure also implies that

\[
\sum_{\varpi \equiv 1 \mod{(1+i)^3}} \frac{1}{N(\varpi)} = \log \log x + O(1).
\]
It follows that when $|z| \leq (\log x)^{-1}$, we have

$$f\left(\frac{2\lambda_0}{\log x} + 2z, x\right) = \sum_{w \equiv 1 \mod (1+i)^3, N(w) \leq x^{1/2}} \frac{1}{N(w)} + \sum_{w \equiv 1 \mod (1+i)^3} \frac{1}{2\lambda_0 \log x + 2z} - \frac{1}{N(w)}$$

$$= \log \log x + \sum_{w \equiv 1 \mod (1+i)^3, N(w) \leq x^{1/2}} \frac{\log N(w)}{N(w)} \int_{0}^{2\lambda_0 \log x + 2z} N(w)^u du + O(1)$$

$$= \log \log x + O\left(\frac{1}{\log x}\right) \sum_{w \equiv 1 \mod (1+i)^3, N(w) \leq x^{1/2}} \frac{\log N(w)}{N(w)} + O(1)$$

$$= \log \log x + O(1),$$

where the integral is along the line segment connecting the origin and the point $\frac{2\lambda_0}{\log x} + 2z$ on the complex plane.

In the remaining case when $(\log x)^{-1} \leq |z| \leq 1$, we note that by (4.10) and partial summation,

$$\frac{\partial f}{\partial z} = -\sum_{w \equiv 1 \mod (1+i)^3} \frac{\log N(w)}{N(w)^{1+z}} = -\int_{1}^{\sqrt{x}} \frac{1}{u^{1+z}} \left(d(u + O(u^{1/2}(\log u)^2)) = -\frac{1}{z} + \frac{x^{-z/2}}{z} + O(1).$$

Now we assume that $0 \leq \Re(w)$ and $|w| \geq (\log x)^{-1}$ and we integrate $\partial f/\partial z$ from $1$ to $w$ to see that

$$f(w, x) = -\log w + \int_{1}^{w} \frac{x^{-u/2}}{u \log x} du + O(1) = -\log |w| - \frac{2x^{-w/2}}{u \log x} \bigg|_{1}^{w} - 2 \int_{1}^{w} \frac{x^{-u/2}}{u^2 \log x} du + O(1)$$

$$= -\log |w| - 2 \int_{1}^{w} \frac{x^{-u/2}}{u^2 \log x} du + O(1).$$

We break the integration $\int_{1}^{w} \frac{x^{-u/2}}{u^2 \log x} du$ into two parts, one horizontal integration along the $x$-axis from $1$ to $\Re(w) \gg (\log x)^{-1}$, and the other vertical integration from $\Re(w)$ to $z$. The horizontal integration is

$$\ll \int_{(\log x)^{-1}}^{\Re(z)} \frac{1}{u^2 \log x} du = O(1).$$

If we write $w = \sigma + it$, then the vertical integration can be evaluated by breaking the integral over $t$ for $|t| \leq \Re(z)$ and $|t| > \Re(z)$. We obtain this way that the vertical integration is

$$\ll \int_{-\Re(z)}^{-\Re(z)} \frac{1}{\Re(z)^2 \log x} dt + \int_{\Re(z)}^{\infty} \frac{1}{t^2 \log x} dt \ll \frac{1}{\Re(z) \log x} = O(1).$$

It follows that we have $f(w, x) = -\log |w| + O(1)$ when $0 \leq \Re(w)$ and $|w| \geq (\log x)^{-1}$. In particular, this applies to the case when $w = \frac{2\lambda_0}{\log x} + 2z$, thus completes the proof. \hfill \Box

Lastly, we present a mean value estimation which will be applied to estimate the sum over primes in (4.11) in our proof of Theorem 1.12.

**Lemma 4.4.** Let $X$ and $y$ be real numbers. For fixed $0 < \varepsilon < 1$, let $k$ be a natural number with $y^k \leq X^{1/2-\varepsilon}$. Then for any complex numbers $a(w)$, we have

$$\sum_{\substack{(d,z) = 1 \\ N(d) \leq X}} \sum_{\substack{w \equiv 1 \mod (1+i)^3 \\ N(w) \leq y}} \frac{a(w) X(1+i)^3 d(w)}{N(w)^{2z}} \ll_{\varepsilon} X^{(2k+1)1/2k} \left( \sum_{\substack{w \equiv 1 \mod (1+i)^3 \\ N(w) \leq y}} \frac{|a(w)|^2}{N(w)} \right)^k.$$
Proof. Let \( W(t) \) be any non-negative smooth function that is supported on \((\frac{1}{2} - \varepsilon_1, 1 + \varepsilon_1)\) for some fixed small \(0 < \varepsilon_1 < 1/2\) such that \( W(t) \gg 1 \) for \( t \in (\frac{1}{2}, 1)\). We have that

\[
\left( \sum_{(d,2)=1 \atop N(d) \leq X} \sum_{w \equiv \pi \mod (1+i)^3 \atop N(w) \leq y} \frac{a(w)\chi_{(1+i)^3}d(w)}{N(w)^{2k}} \right)^{2k} \leq \sum_{(d,2)=1 \atop N(d) \leq X} \sum_{w \equiv \pi \mod (1+i)^3 \atop N(w) \leq y} \frac{a(w)\chi_{(1+i)^3}d(w)}{N(w)^{2k}} \left| W\left(\frac{N(d)}{X}\right) \right|^{2k}.
\]

We further expand out the last sum above and treat the sum over \( d \) by applying \((3.3)\) first and then using the smoothed version of Pólya-Vinogradov inequality\((3.4)\) for number fields when the product of the primes involved is not a perfect square to see that we have

\[
\left( \sum_{(d,2)=1 \atop w_1, \ldots, w_k \equiv 1 \mod (1+i)^3 \atop N(w_1), \ldots, N(w_k) \leq y} a(w_1) \cdots a(w_k) \sqrt{N(w_1 \cdots w_k)} \left( \frac{(1+i)^3d}{w_1 \cdots w_k} \right)^2 \left| W\left(\frac{N(d)}{X}\right) \right| \right)^{2k} \leq \sum_{(d,2)=1 \atop w_1, \ldots, w_k \equiv 1 \mod (1+i)^3 \atop N(w_1), \ldots, N(w_k) \leq y \atop w_1 \cdots w_k = \pi} |a(w_1) \cdots a(w_{2k})| \left( \frac{1}{\sqrt{N(w_1 \cdots w_{2k})}} \right)^{2k} + O \left( \sum_{(d,2)=1 \atop w_1, \ldots, w_k \equiv 1 \mod (1+i)^3 \atop N(w_1), \ldots, N(w_k) \leq y \atop w_1 \cdots w_k = \pi} |a(w_1) \cdots a(w_{2k})| y^{2k\varepsilon'} \right).
\]

where we write \( \square \) for a square of an element in \( O_K \).

We take \( \varepsilon' \) small enough so that \( y^{2k\varepsilon'} \ll X \). Then an argument similar to that in the proof of \([30]\) Lemma 6.3 leads to the assertion of the lemma. \( \square \)

4.5. Completion of the proof. With lemmas \([12, 14]\) now available, we proceed to establish an upper bound for the frequency of large values of \( \log |L(\frac{1}{2} + z_1, \chi_{(1+i)^3}d)L(\frac{1}{2} + z_2, \chi_{(1+i)^3}d)| \).

Proposition 4.6. Assume GRH for \( \zeta_K(s) \) and \( L(s, \chi_{(1+i)^3}d) \) for all odd, square-free \( d \in O_K \). Let \( X \) be large and let \( z_1, z_2 \in \mathbb{C} \) with \( 0 \leq \Re(z_1), \Re(z_2) \leq \frac{1}{1000} \), and \( |\Im(z_1)|, |\Im(z_2)| \leq X \). Let \( \mathcal{N}(V; z_1, z_2, X) \) denote the number of odd, square-free \( d \in O_K \) such that \( N(d) \leq X \) and

\[
\log |L(\frac{1}{2} + z_1, \chi_{(1+i)^3}d)L(\frac{1}{2} + z_2, \chi_{(1+i)^3}d)| \geq V + \mathcal{M}(z_1, z_2, X).
\]

Then for \( 10\sqrt{\log \log X} \leq V \leq \mathcal{V}(z_1, z_2, X) \), we have

\[
\mathcal{N}(V; z_1, z_2, X) \ll X \exp \left( -\frac{V^2}{2\mathcal{V}(z_1, z_2, X)} \left( 1 - \frac{25}{\log \log \log X} \right) \right);
\]

for \( \mathcal{V}(z_1, z_2, X) < V \leq \frac{1}{100} \mathcal{V}(z_1, z_2, X) \log \log \log X \), we have

\[
\mathcal{N}(V; z_1, z_2, X) \ll X \exp \left( -\frac{V^2}{2\mathcal{V}(z_1, z_2, X)} \left( 1 - \frac{15V}{\mathcal{V}(z_1, z_2, X) \log \log \log X} \right)^2 \right);
\]

for \( \frac{1}{100} \mathcal{V}(z_1, z_2, X) \log \log \log X < V \), we have

\[
\mathcal{N}(V; z_1, z_2, X) \ll X \exp \left( -\frac{1}{1025} V \log V \right).
\]

Proof. Let \( \lambda_0 \) be the constant defined in Lemma 4.4 and apply this Lemma with \( \lambda = \lambda_0 + \Re(z_i) \log x, i = 1, 2 \) and 

\( T = X \) to \( L(\frac{1}{2} + z_i, \chi_{(1+i)^3}d) \) for \( N(d) \leq X \), we see that for \( 2 \leq x \leq X \),

\[
\log |L(\frac{1}{2} + z_i, \chi_{(1+i)^3}d)| \leq \Re \left( \sum_{n \equiv 1 \mod (1+i)^3 \atop N(n) \leq x} \frac{\Lambda_{(1+i)^3}(n) \chi_{(1+i)^3}d(n)}{N(n)^{\frac{1}{2} + z_i + \frac{1}{2} \log x}} \log \left( \frac{x}{\mathcal{M}(n)} \right) \log x \right) + \frac{3}{2} (1 + \lambda_0) \log x + O \left( \frac{1}{\log x} \right), \quad i = 1, 2.
\]
We then deduce that
\[
\log |L(\frac{1}{2} + z_1, \chi_{(1+i)^5_d})||L(\frac{1}{2} + z_2, \chi_{(1+i)^5_d})|
\]
\begin{equation}
\leq \Re \left( \sum_{\substack{\omega \equiv 1 \mod (1+i)^3 \ \text{and} \ N(\omega) \leq x \ \text{odd and square-free}, \ \omega \neq \pm 1}} \frac{\chi_{(1+i)^5_d}(\omega)}{N(\omega)^{\frac{1}{2} + \frac{M}{\log \log X}}}(N(\omega)^{-i z_1} + N(\omega)^{-i z_2}) \frac{\log \left( \frac{x}{N(\omega)} \right)}{\log x} + 3(1 + \lambda_0) \frac{\log X}{\log x} + O \left( \frac{1}{\log x} \right) \right).
\end{equation}

The terms with \( l \geq 3 \) in the the above sum contribute \( O(1) \). Using the fact \( \sum_{\omega \mid d} \frac{1}{N(\omega)} \ll \log \log \log N(d) \), we deduce from Lemma 4.3 that
\[
\Re \left( \sum_{\substack{\omega \equiv 1 \mod (1+i)^3 \ \text{and} \ N(\omega) \leq x^{1/2} \ \text{odd and square-free}, \ \omega \neq \pm 1}} \frac{\chi_{(1+i)^5_d}(\omega^2)}{2N(\omega)^{1 + \frac{3\lambda_0}{\log X}}}(N(\omega)^{-2i z_1} + N(\omega)^{-2i z_2}) \frac{\log \left( \frac{x}{N(\omega)} \right)}{\log x} \right) = \mathcal{M}(z_1, z_2, x) + O(\log \log \log X)
\]
\[
\leq \mathcal{M}(z_1, z_2, x) + O(\log \log \log X),
\]
where \( \mathcal{M}(z_1, z_2, x) \) is defined as in (1.7).

Applying the above estimation in (1.12), we obtain that
\begin{equation}
\log |L(\frac{1}{2} + z_1, \chi_{(1+i)^5_d})||L(\frac{1}{2} + z_2, \chi_{(1+i)^5_d})|
\end{equation}
\[
\leq \Re \left( \sum_{\substack{\omega \equiv 1 \mod (1+i)^3 \ \text{and} \ N(\omega) \leq x \ \text{odd and square-free}, \ \omega \neq \pm 1}} \frac{\chi_{(1+i)^5_d}(\omega)}{N(\omega)^{\frac{1}{2} + \frac{M}{\log \log X}}}(N(\omega)^{-z_1} + N(\omega)^{-z_2}) \frac{\log \left( \frac{x}{N(\omega)} \right)}{\log x} \right) + \mathcal{M}(z_1, z_2, x) + \frac{5 \log X}{\log x} + O(\log \log \log X).
\]

By taking \( x = \log X \) in (4.13) and bounding the sum over \( \omega \) in (4.13) trivially (with the help of (4.10)), we see that \( N(V; z_1, z_2, X) = 0 \) for \( V > \frac{6 \log X}{\log \log X} \). Thus, we can assume \( V \leq \frac{6 \log X}{\log \log X} \).

In what follows, we shall denote \( V \) for \( V(z_1, z_2, X) \) defined in (1.8) and we note that \( \log X + O(1) \leq V(z_1, z_2, x) \leq 4 \log \log X + O(1) \). We now set \( x = X^{A/V} \) with
\[
A = \left\{ \begin{array}{ll}
\frac{1}{4} \log \log X & 10 \sqrt{\log \log X} \leq V \leq \mathcal{V}, \\
\frac{3}{2} \sqrt{\log \log X} & \mathcal{V} < V \leq \frac{1}{10} \mathcal{V} \log \log \log X, \\
8 & V > \frac{1}{10} \mathcal{V} \log \log \log X.
\end{array} \right.
\]

We further denote \( z = x^{1/\log \log X}, M_1 \) for the real part of the sum in (4.13) truncated to \( N(\omega) \leq z \), \( M_2 \) for the real part of the sum in (4.13) over \( z < N(\omega) \leq x \). We then deduce that
\[
\log |L(\frac{1}{2} + z_1, \chi_{(1+i)^5_d})||L(\frac{1}{2} + z_2, \chi_{(1+i)^5_d})| \leq M_1 + M_2 + \mathcal{M}(z_1, z_2, X) + \frac{5V}{A}.
\]

It follows from this that if \( \log |L(\frac{1}{2} + z_1, \chi_{(1+i)^5_d})||L(\frac{1}{2} + z_2, \chi_{(1+i)^5_d})| \geq V + \mathcal{M}(z_1, z_2, X) \), then we have either
\[
M_2 \geq \frac{V}{A}, \text{ or } M_1 \geq V_1 := V(1 - \frac{6}{A}).
\]

Now, we define
\[
\text{meas}(X; M_1) = \# \{ N(d) \leq X : d \text{ odd and square-free}, M_1 \geq V_1 \}, \quad \text{meas}(X; M_2) = \# \{ N(d) \leq X : d \text{ odd and square-free}, M_2 \geq \frac{V_1}{A} \}.
\]
We then take $m = \lceil 0.9(\frac{V}{2A}) \rceil$ to see that by Lemma 4.4 we have

$$(V/A)^{2m} \text{meas}(X; S_2) \leq \sum_{(d,2)=1}^* |M_2|^{2m} \ll X \frac{(2m)!}{m!2^m} \left( \sum_{w \equiv 1 \mod (1+i)^3, \omega \leq X} \frac{4}{N(\omega)} \right)^m \ll X(3m \log \log \log X)^m,$$

where the last estimation above follows from (4.11) and Stirling’s formula (see [15, (5.112)]), which implies that

$$\frac{(2m)!}{m!2^m} \ll (\frac{2m}{e})^m.$$

We then deduce that

(4.14) \hspace{1cm} \text{meas}(X; M_2) \ll X \exp \left( -\frac{V}{3A} \log V \right).

Next, we estimate $\text{meas}(X; M_1)$. For any $m \leq \frac{(\frac{V}{2A}) \log X}{\log z}$, we obtain using Lemma 4.4 that

(4.15) \hspace{1cm} V_1^{2m} \text{meas}(X; M_1) \leq \sum_{(d,2)=1}^* |M_1|^{2m} \ll X \frac{(2m)!}{m!2^m} \left( \sum_{w \equiv 1 \mod (1+i)^3, \omega \leq z} \frac{|a(\omega)|^2}{N(\omega)} \right)^m,

where

$$a(\omega) = \frac{\Re(N(\omega)^{-z_1} + N(\omega)^{-z_2}) \log \left( \frac{x}{N(\omega)} \right)}{N(\omega)^{-z_2} \log x}.$$

By arguing as in the proof of Lemma 4.3 we see that

$$\sum_{w \equiv 1 \mod (1+i)^3, \omega \leq z} \frac{|a(\omega)|^2}{N(\omega)} \ll \frac{1}{4} \sum_{w \equiv 1 \mod (1+i)^3, \omega \leq z} \frac{1}{N(\omega)} \left( N(\omega)^{-z_1} + N(\omega)^{-z_2} + N(\omega)^{-z_2} + N(\omega)^{-z_2} \right)^2 = \mathcal{V} + O(1).$$

Combining with (4.14), this implies that

$$\text{meas}(X; M_1) \ll X V_1^{-2m} \frac{(2m)!}{m!2^m} (\mathcal{V} + O(1))^m \ll X \left( \frac{2m}{e} \cdot \frac{\mathcal{V} + O(1)}{V_1^2} \right)^m.$$

We now take $m = \lfloor \frac{V^2}{4V} \rfloor$ when $V \leq (\log \log X)^2$ and $m = \lfloor 10V \rfloor$ otherwise to see that in either case, we have for $X$ large,

$$m \leq \frac{1}{2} - 0.1 \frac{\log X}{\log z}.$$

A little calculation then shows that

$$\text{meas}(X; M_1) \ll X \exp \left( -\frac{V^2}{2V} \left( 1 + O \left( \frac{1}{\log \log X} \right) \right) \right) + X \exp (-V \log V).$$

We then deduce from the above and (4.14) that

$$\mathcal{N}(V; z_1, z_2, X) \ll X \exp \left( -\frac{V^2}{2V} \left( 1 + O \left( \frac{1}{\log \log X} \right) \right) \right) + X \exp (-V \log V) + X \exp \left( -\frac{V}{3A} \log V \right).$$

It is then easy to check that this leads to the assertion of the proposition.

□

We now return to the proof of Theorem 1.2 by noting that for $k$ given as in the theorem, Proposition 4.6 implies that for all $V \geq 10\sqrt{\log \log X}$,

(4.16) \hspace{1cm} \mathcal{N}(V; z_1, z_2, X) \ll \begin{cases} X, & V < 10\sqrt{\log \log X}, \\ X(\log X)^{o(1)} \exp \left( -\frac{V^2}{2V(3z_1, z_2, X)} \right), & 10\sqrt{\log \log X} \leq V \leq 4k\mathcal{V}(z_1, z_2, X), \\ X(\log X)^{o(1)} \exp(-4kV), & V > 4k\mathcal{V}(z_1, z_2, X). \end{cases}
Note further that we have
\[
\sum_{(d,2)=1}^{\ast} |L(\frac{1}{2} + z_1, \chi_{(1+i)^5}d)||L(\frac{1}{2} + z_2, \chi_{(1+i)^5}d)|^k = - \int_{-\infty}^{\infty} \exp(kV + kM(z_1, z_2, X))dN(V; z_1, z_2, X)
\]
\[
= k \int_{-\infty}^{\infty} \exp(kV + kM(z_1, z_2, X))dV.
\]
Now the assertion of Theorem 1.2 follows by supplying the bound given in (4.16) to evaluate the integration above.

5. Proof of Theorems 1.5 and 1.6

5.1. Initial treatment. Let \( \Phi \) be a smooth Schwartz class function which is compactly supported on \([\frac{1}{2}, \frac{3}{2}]\) satisfying \( 0 \leq \Phi(t) \leq 1 \) for all \( t \). We apply the approximate functional equation (2.12) to see that
\[
\sum_{(d,2)=1}^{\ast} L(\frac{1}{2}, \chi_{(1+i)^5}d)^a \Phi \left( \frac{N(d)}{X} \right) = \sum_{(d,2)=1}^{\ast} A_{N(d)}(d)^2 \Phi \left( \frac{N(d)}{X} \right),
\]
where
\[
A_{d}(d) = 2 \sum_{n \equiv 1 \mod (1+i)^3} \frac{\chi_{(1+i)^5}d(n)d_{[i]}(n)}{N(n)^{\frac{1}{2}}} V \left( \frac{N(n)}{t} \right).
\]

For two parameters \( U_1, U_2 \) satisfying \( X^{\frac{3}{5}} \leq U_1 \leq U_2 \leq X \), we define
\[
S(U_1, U_2) = \sum_{(d,2)=1}^{\ast} A_{U_1}(d)A_{U_2}(d) \Phi \left( \frac{N(d)}{X} \right).
\]

We let
\[
h(x, y, z) = \Phi \left( \frac{N(x)}{X} \right) V \left( \frac{N(y)}{U_1} \right) V \left( \frac{N(z)}{U_2} \right).
\]

Then applying (5.1) to (5.2) and using the Möbius inversion to remove the square-free condition in (5.2), we obtain that
\[
S(U_1, U_2) = 4 \sum_{(d,2)=1}^{\ast} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{\chi_{(1+i)^5}d(n_1n_2)d_{[i]}(n_1)d_{[i]}(n_2)}{N(n_1n_2)^{\frac{1}{2}}} h(d, n_1, n_2)
\]
\[
= 4 \sum_{(d,2)=1}^{\ast} \sum_{a \equiv 1 \mod (1+i)^3} \mu_{[i]}(a) \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{\chi_{(1+i)^5}d(n_1n_2)d_{[i]}(n_1)d_{[i]}(n_2)}{N(n_1n_2)^{\frac{1}{2}}} h(a^2 d, n_1, n_2)
\]
\[
= 4 \sum_{a \equiv 1 \mod (1+i)^3} \mu_{[i]}(a) \sum_{(d,2)=1}^{\ast} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{\chi_{(1+i)^5}d(n_1n_2)d_{[i]}(n_1)d_{[i]}(n_2)}{N(n_1n_2)^{\frac{1}{2}}} h(a^2 d, n_1, n_2).
\]

Now we separate the terms with \( N(a) \leq Y \) and with \( N(a) > Y \) for some \( Y \leq X \) to be chosen later, writing \( S(U_1, U_2) = S_1 + S_2 \), respectively. We bound \( S_2 \) first in the following result.

Lemma 5.2. Unconditionally, we have \( S_2 \ll X^{1+\epsilon}Y^{-1} \). Under GRH, we have \( S_2 \ll XY^{-1}(\log X)^{46} \).
Proof. We first write $d = lb^2$ with $l$ square-free and $b$ primary. We then let $c = ab$ and apply the definition of $h(x, y, z)$ in (5.3) to see that

$$S_2 = 4 \sum_{c \equiv 1 \mod (1+i)^3} \sum_{a \equiv 1 \mod (1+i)^3} \mu(a) \sum_{(l,2)=1} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{\chi((1+i)\gamma_l(n_1n_2)d_l(n_1)d_l(n_2)}{N(n_1n_2)^{3/2}}h(c^2 l, n_1, n_2)$$

\begin{equation}
(5.4)
\end{equation}

where the last estimation above follows from partial summation and the following estimation for (5.6)

\begin{equation}
\text{Applying the estimation } |\mathcal{F}(\chi(1+i)\gamma_l^2 L_c(1+u, \chi(1+i)\gamma_l)^2 L_c(1+u, \chi(1+i)\gamma_l)^2)\leq |L_c(1+u, \chi(1+i)\gamma_l)^2 | + |L_c(1+u, \chi(1+i)\gamma_l)^2| |
\end{equation}

we can bound $S_2$ by moving the lines of the integrations in (5.4) to $\mathcal{R}(u) = \mathcal{R}(v) = \frac{|x|}{\log X}$ to see that

\begin{equation}
S_2 \ll (\log X)^2 \sum_{c, 2 = 1} d(c) \sum_{(l,2)=1} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{1}{N(l)^{3/2}} \sum_{(l,2)=1} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{1}{N(l)^{3/2}} \sum_{(l,2)=1} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{1}{N(l)^{3/2}} |L_c(1+u, \chi(1+i)\gamma_l)^2 | |d_l|^2 |du||dv|
\end{equation}

By Corollary 1.4, we see that for $|\mathcal{F}(u)| \leq \frac{5X(\log X)^2}{2N(c)^2}$,

\begin{equation}
\sum_{(l,2)=1} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} |L_c(1+u, \chi(1+i)\gamma_l)^2 | |d_l|^2 \ll \frac{X}{N(c)^2}(\log X)^{13}
\end{equation}

Note that Lemma 2.16 implies that

\begin{equation}
\sum_{(l,2)=1} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} |L_c(1+u, \chi(1+i)\gamma_l)^2 | |d_l|^2 \ll \left(\frac{X}{N(c)^2}\right)^{1+\varepsilon} (1 + |\mathcal{F}(u)|^2)^{1+\varepsilon}
\end{equation}

Applying (5.5) in (5.5) when $|\mathcal{F}(u)| \leq \frac{5X(\log X)^2}{2N(c)^2}$ and (5.7) in (5.6) otherwise, together with the observation that $w(u)$ decreases exponentially in $|\mathcal{F}(u)|$, we deduce that

$$S_2 \ll X(\log X)^{15} \sum_{(c,2)=1} \frac{d(c)^2}{N(c)^2} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{1}{N(l)^{3/2}} \sum_{(l,2)=1} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{1}{N(l)^{3/2}} \sum_{(l,2)=1} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{1}{N(l)^{3/2}} \sum_{(l,2)=1} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{1}{N(l)^{3/2}} \sum_{(l,2)=1} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{1}{N(l)^{3/2}} |X^{Y-1}(\log X)^{46}
$$

where the last estimation above follows from partial summation and the following estimation for $x > 2$

\begin{equation}
\sum_{N(c) \leq x} d(c) \ll x(\log x)^{31}
\end{equation}

The unconditionally estimation for $S_2$ is obtained similarly by applying (5.7) in (5.5) for all $u$ and this completes the proof of the lemma. \qed
Next, we treat $S_1$ by applying the Poisson summation formula given in Lemma 2.8 to recast it as

\begin{equation}
S_1 = 4 \sum_{a \equiv 1 \mod (1+i)^3} \frac{\mu_{(a)}}{N(a) \leq Y} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{(-1)^N(k)}{N(n_1 n_2)^2} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{d_{(a)}(n_1) d_{(a)}(n_2)}{N(n_1 n_2)^2} \sum_{(d, 2) = 1} \left( \frac{d}{n_1 n_2} \right) h(a^2 d, n_1, n_2) 
\end{equation}

\begin{equation}
= 2X \sum_{a \equiv 1 \mod (1+i)^3} \frac{\mu_{(a)}}{N(a) \leq Y} \sum_{k \in \mathcal{O}_K} (-1)^N(k) \frac{d_{(a)}(n_1) d_{(a)}(n_2)}{N(n_1 n_2)^2} \left( \frac{N(k) X}{2N(a^2 n_1 n_2)^2}, n_1, n_2 \right),
\end{equation}

where

\[ \tilde{h}(t, y, z) = \Phi(t) V \left( \frac{N(y)}{U_1} \right) V \left( \frac{N(z)}{U_2} \right). \]

Now we write $S_1 = S_1(k = 0) + S_1(k \neq 0)$, where $S_1(k = 0)$ corresponds to the term with $k = 0$. By applying (2.11) and (2.18), we see that when $k \neq 0$,

\begin{equation}
\tilde{h} \left( \frac{N(k) X}{2N(a^2 n_1 n_2)^2}, n_1, n_2 \right) = \frac{\pi}{(2\pi i)^3} \int \int \int \left( \frac{N(a)^2}{N(k)} \right)^s \mathcal{J}(s) w(u) w(v) \frac{1}{N(n_1)^{u-s} N(n_2)^{v-s}} \frac{U_1^u U_2^v X^{-s}}{uv} ds \, du \, dv,
\end{equation}

\begin{equation}
\tilde{h} \left( \frac{N(a)^2}{N(k)} \right)^s \mathcal{J}(s) w(u + s) w(v + s) \frac{1}{N(n_1)^u N(n_2)^v} \frac{U_1^{u+s} U_2^{v+s} X^{-s}}{(u + s)(v + s)} ds \, du \, dv,
\end{equation}

where

\[ \mathcal{J}(s) = \Phi(1 - s) \left( \frac{\pi^2}{2} \right)^{-s} \frac{\Gamma(s)}{\Gamma(1 - s)} \]

and where the last expression in (5.10) follows from moving the lines of the first triple integral in (5.10) to $\Re(s) = \frac{1}{2} + \varepsilon$, $\Re(u) = \Re(v) = \frac{1}{2} + 2\varepsilon$, and a change the variables $u' = u - s, v' = v - s$.

Substituting the last expression in (5.10) to (5.9), we see by using our notation for $k_1, k_2$ given in Section 2.2 that

\begin{equation}
S_1(k \neq 0) = 2X \sum_{a \equiv 1 \mod (1+i)^3} \frac{\mu_{(a)}}{N(a) \leq Y} \sum_{k \in \mathcal{O}_K} (-1)^N(k) \frac{\pi}{(2\pi i)^3} \int \int \int \left( \frac{N(a)^2}{N(k)} \right)^s \mathcal{J}(s) w(u + s) w(v + s)
\end{equation}

\begin{equation}
\times \frac{U_1^{u+s} U_2^{v+s} X^{-s}}{(u + s)(v + s)} Z(\frac{1}{2} + u, \frac{1}{2} + v, a, k) ds \, du \, dv
\end{equation}

\begin{equation}
= \frac{2\pi X}{(2\pi i)^3} \sum_{a \equiv 1 \mod (1+i)^3} \frac{\mu_{(a)}}{N(a) \leq Y} \sum_{k \in \mathcal{O}_K} (-1)^N(k) \frac{\pi}{(2\pi i)^3} \int \int \int \frac{N(a)^2}{N(k)^s} \mathcal{J}(s) w(u + s) w(v + s)
\end{equation}

\begin{equation}
\times \frac{U_1^{u+s} U_2^{v+s} X^{-s}}{(u + s)(v + s)} L^2(1 + u, \chi_{ik_1}) L^2(1 + v, \chi_{ik_1}) Z(\frac{1}{2} + u, \frac{1}{2} + v, a, k) ds \, du \, dv.
\end{equation}

We observe that if we move the lines of integrations over $u, v$ in (5.11) to the left, then we encounter poles at $u = v = 0$ only when $k_1 = \pm i$. For this reason, we further write $S_1(k \neq 0) = S_1(k_1 = \pm i) + S_1(k_1 \neq \pm i)$, where

\begin{equation}
S_1(k_1 = \pm i) = \frac{2\pi X}{(2\pi i)^3} \sum_{a \equiv 1 \mod (1+i)^3} \frac{\mu_{(a)}}{N(a) \leq Y} \sum_{k_2 \in \mathcal{O}_K} (-1)^{N(k_2)} \frac{\pi}{(2\pi i)^3} \int \int \int \frac{N(a)^2}{N(k_2)^s} \mathcal{J}(s) w(u + s) w(v + s)
\end{equation}

\begin{equation}
\times \frac{U_1^{u+s} U_2^{v+s} X^{-s}}{(u + s)(v + s)} \epsilon_k^2 (1 + u) \epsilon_k^2 (1 + v) Z(\frac{1}{2} + u, \frac{1}{2} + v, a, \pm ik_2^2) ds \, du \, dv,
\end{equation}
and

\[ S_1(k_1 \neq \pm i) = \frac{2\pi X}{(2\pi i)^3} \sum_{a \equiv 1 \mod (1+i)^3} \mu_{i\bar{i}}(a) \frac{N(a)^2}{N(a)^2} \sum_{k \in \mathbb{Q}_k \not\equiv Y} \frac{(-1)^{N(k)}}{N(k)^{\frac{3}{2}}} \int_{\frac{1}{2} + \epsilon} \int_{\frac{1}{2} + \epsilon} (c \bar{c}) \int \int \frac{N(a)^2 \mathcal{J}(s)w(u + s)w(v + s)}{c \bar{c}} \times \frac{U^{\psi \epsilon} \tilde{U}^{\psi \epsilon}}{(u + s)(v + s)}L^2(1 + u, \chi_{ik})L^2(1 + v, \chi_{ik})Z_2(1/2 + u, 1/2 + v, a, k) \, ds \, dv. \]

(5.13)

5.3. Computing \( S_1 \): the term \( S_1(k = 0) \)

Note that by Lemma 2.3 we have \( g(0, n) = \varphi_{i\bar{i}}(n) \) if \( n = \square \), and 0 otherwise. Thus we get

\[ S_1(k = 0) = 2X \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{d_{i\bar{i}}(n_1)d_{i\bar{i}}(n_2)}{N(n_1n_2)^{\frac{3}{2}}} \mathcal{P}(n_1n_2) \tilde{h}(0, n_1, n_2) \]

(5.14)

We note that

\[ \sum_{a \equiv 1 \mod (1+i)^3} \frac{\mu_{i\bar{i}}(a)}{N(a)^2} = \frac{4}{3\zeta_K(2)} \prod_{a \equiv 1 \mod (1+i)^3 \atop N(a) \not\equiv Y} \left(1 - \frac{1}{N(a)^2}\right)^{-1} + O(Y^{-1}). \]

Applying this to (5.14), we see that

\[ S_1(k = 0) = \frac{8X}{3\zeta_K(2)} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{d_{i\bar{i}}(n_1)d_{i\bar{i}}(n_2)}{N(n_1n_2)^{\frac{3}{2}}} \mathcal{P}(n_1n_2) \tilde{h}(0, n_1, n_2) \]

\[ + O\left(\frac{X}{Y} \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{d_{i\bar{i}}(n_1)d_{i\bar{i}}(n_2)}{N(n_1n_2)^{\frac{3}{2}}} \tilde{h}(0, n_1, n_2)\right), \]

where we recall that \( \mathcal{P}(n_1n_2) \) is defined in (2.20).

Due to the rapid decay of \( \tilde{h} \), we can estimate the error term above as

\[ \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{d_{i\bar{i}}(n_1)d_{i\bar{i}}(n_2)}{N(n_1n_2)^{\frac{3}{2}}} \tilde{h}(0, n_1, n_2) \ll \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{d_{i\bar{i}}(n_1)d_{i\bar{i}}(n_2)}{N(n_1n_2)^{\frac{3}{2}}} \tilde{h}(0, n_1, n_2), \]

where the last line follows by applying estimations similar to that given in (5.8).

Further note that we have

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi((N(x + yi)) \, dx \, dy = \int_{0}^{2\pi} \int_{0}^{\pi} \Phi(\rho^2) \rho \, d\rho \, d\theta = \pi \tilde{\Phi}(1). \]
We now conclude from the above discussions that
\[ S_1(k = 0) = \frac{8X}{3\zeta_K(2)} \sum_{n_1, n_2 \equiv 1 \mod (1 + i)^3}^{\infty} \frac{d_{[1]}(n_1)d_{[1]}(n_2)}{N(n_1n_2)^{\frac{3}{2}}} \mathcal{P}(0, n_1, n_2) + O \left( \frac{X}{Y} (\log X)^{10} \right) \]
(5.15)
\[
= \frac{8\pi \Phi(1)X}{3\zeta_K(2)} \frac{1}{(2\pi i)^2} \int \int \frac{w(u)w(v)}{uv} U_1^u U_2^v Z(\frac{1}{2} + u, \frac{1}{2} + v) du dv + O \left( \frac{X}{Y} (\log X)^{10} \right),
\]
where \( Z(\frac{1}{2} + u, \frac{1}{2} + v) \) is defined as in (5.16). It follows by moving the line of the integration over \( \mathbb{C} \) from \( \mathbb{R} \) to \( \mathbb{R}(u) = \mathbb{R}(v) = \frac{1}{2} \) by noting that we encounter no poles. We then move the line of the integration over \( u \) to \( \mathbb{R}(u) = -\frac{1}{4} \) to see that we encounter two poles of order 4 at \( v = 0 \) and \( v = -u \) in the process. It follows that
\[
S_1(k = 0) = \frac{8\pi \Phi(1)X}{3\zeta_K(2)} \frac{1}{(2\pi i)^2} \int \int \frac{U_1^u U_2^v}{uv(2u)^3(2v)^3(u + v)^4} \mathcal{E}(u, v) \ du \ dv + O \left( X Y^{-1} (\log X)^{10} \right),
\]
(5.16)
where
\[
\mathcal{E}(u, v) = w(u)w(v)\zeta_K(1 + 2u)(2u)^3\zeta_K(1 + 2v)(2v)^3\zeta_K(1 + u + v)(u + v)^4Z(\frac{1}{2} + u, \frac{1}{2} + v).
\]
Applying Lemma 2.10 again, we see that \( \mathcal{E} \) is analytic for \( \mathbb{R}(u), \mathbb{R}(v) > -\frac{1}{4} + \varepsilon \).

We first move the lines of the integrations in (5.16) to \( \mathbb{R}(u) = \mathbb{R}(v) = \frac{1}{4} \) by noting that we encounter no poles. We then move the line of the integration over \( u \) to \( \mathbb{R}(u) = -\frac{1}{4} \) to see that we encounter two poles of order 4 at \( v = 0 \) and \( v = -u \) in the process. It follows that
\[
\frac{1}{(2\pi i)^2} \int \int \frac{U_1^u U_2^v}{uv(2u)^3(2v)^3(u + v)^4} \mathcal{E}(u, v) \ du \ dv
\]
(5.17)
\[
= \frac{1}{2\pi i} \int \left( \text{Res}_{u=0} + \text{Res}_{v=-u} \right) \left( \frac{U_1^u U_2^v}{uv(2u)^3(2v)^3(u + v)^4} \mathcal{E}(u, v) \right) \ du + O \left( U_1^{\frac{5}{4}} U_2^{-\frac{5}{4}} (\log X)^3 \right).
\]
We treat the contribution from the residue at \( v = 0 \) in (5.17) to see that
\[
I_1(u) = \text{Res}_{u=0} \left( \frac{U_1^u U_2^v}{uv(2u)^3(2v)^3(u + v)^4} \mathcal{E}(u, v) \right)
\]
\[
= \frac{U_1^u}{2^6 \cdot 3! \cdot 10!} \left( \mathcal{E}(u, 0)(u^3(\log U_2)^3 - 12u^2(\log U_2)^2 + 60u \log U_2 - 120) + \mathcal{E}^{(0, 1)}(u, 0)(3u^3(\log U_2)^2 - 24u^2 \log U_2 + 60u) + \mathcal{E}^{(0, 2)}(u, 0)(3u^3 \log U_2 - 12u^2) + \mathcal{E}^{(0, 3)}(u, 0)u^3 \right),
\]
where \( \mathcal{E}^{(i,j)}(u, v) = \frac{\partial^i \partial^j \mathcal{E}(u, v)}{\partial u^i \partial v^j} \).

It follows by moving the line of the integration over \( u \) from \( \mathbb{R}(u) = \frac{1}{10} \) to \( \mathbb{R}(u) = -\frac{1}{10} \) that we have
\[
\frac{1}{2\pi i} \int \left( \frac{U_1^u}{uv(2u)^3(2v)^3(u + v)^4} \mathcal{E}(u, v) \right) \ du \ll \frac{U_1^{\frac{5}{4}} U_2^{\frac{5}{4}} (\log X)^3}{2^6 \cdot 3! \cdot 10!}.
\]
Similarly, we have that
\[
\frac{1}{2\pi i} \int \text{Res}_{v=-u} \left( \frac{U_1^u U_2^v}{uv(2u)^3(2v)^3(u + v)^4} \mathcal{E}(u, v) \right) \ du = O \left( U_1^{\frac{5}{4}} U_2^{\frac{5}{4}} (\log X)^3 \right).
\]
Combining (5.16)-(5.17), together with the observation that \( \lim_{s \to 0} \zeta_K(1 + s)s = \pi/4 \) and \( Z_1(\frac{1}{2}, \frac{1}{2}) = 4a_4 \), we obtain that
\[
S_1(k = 0) = \frac{8\pi X}{3\zeta_K(2)} \Phi(1) \left( \frac{\pi}{4} \right)^{10} 4 \cdot 5! a_4 \left( \frac{\pi}{4} \right)^{10} \left( \frac{\pi}{4} \right)^{10} \left( \frac{\pi}{4} \right)^{10} (\log U_1)^{10} (\log U_2)^{10} + O \left( X (\log X)^9 + XY^{-1} (\log X)^{10} \right).
\]
5.4. Computing $S_1$: the term $S_1(k_1 = \pm i)$. In this section, we evaluate $S_1(k_1 = \pm i)$ given in (5.12). Applying Lemma 2.12, we see that

$$S_1(k_1 = \pm i) = \frac{2\pi X}{(2\pi i)^3} \sum_{a \equiv 1 \mod (1+i)^3} \frac{\mu(a \mid a) \mu(a)}{N(a)} \sum_{k_2 \in \mathbb{Z}[i]} (-1)^{N(k_2)} \int \int \int N(a)^{2s} \mathcal{J}(s) w(u + s) w(v + s) \frac{U_1^{u+s} U_2^{v+s} X^s}{(u + s)(v + s)}$$

$$\times \zeta_K^2(1 + u) \zeta_K^2(1 + v)(2^{1-2s} - 1) \zeta_K(2s) \zeta_K^2(1 + 2u + 2v) \zeta_K^2(1 + 2v + 2s) Z_3 \left( \frac{1}{2} + u, \frac{1}{2} + v, s, a \right) ds \, dv.$$

As $Z_3 \left( \frac{1}{2} + u, \frac{1}{2} + v, s \right)$ is analytic in the region $\Re(u), \Re(v) \geq \varepsilon$, $\Re(s) \geq 2\varepsilon$ by (1) of Lemma 2.12 and $(2^{1-2s} - 1) \zeta_K(2s)$ is analytic at $s = \frac{1}{2}$, we move the lines of the integration above to $\Re(u) = \Re(v) = 1, \Re(s) = \frac{1}{10}$ without encountering any poles to see that

$$S_1(k_1 = \pm i) = 2\pi X \sum_{a \equiv 1 \mod (1+i)^3} \frac{\mu(a \mid a) \mu(a)}{N(a)} \int \int \int N(a)^{2s} \mathcal{J}(s) w(u + s) w(v + s) \frac{U_1^{u+s} U_2^{v+s} X^s}{(u + s)(v + s)}$$

$$\times \zeta_K^2(1 + u) \zeta_K^2(1 + v)(2^{1-2s} - 1) \zeta_K(2s) \zeta_K^2(1 + 2u + 2v) \zeta_K^2(1 + 2v + 2s) Z_3 \left( \frac{1}{2} + u, \frac{1}{2} + v, s, a \right) ds \, dv.$$

We extend the sum over $a$ in (5.19) to include all primary elements in $O_K$, introducing an error term

$$2\pi X \sum_{a \equiv 1 \mod (1+i)^3} \frac{\mu(a \mid a) \mu(a)}{N(a)} \int \int \int N(a)^{2s} \mathcal{J}(s) w(u + s) w(v + s) \frac{U_1^{u+s} U_2^{v+s} X^s}{(u + s)(v + s)}$$

$$\times \zeta_K^2(1 + u) \zeta_K^2(1 + v)(2^{1-2s} - 1) \zeta_K(2s) \zeta_K^2(1 + 2u + 2v) \zeta_K^2(1 + 2v + 2s) Z_3 \left( \frac{1}{2} + u, \frac{1}{2} + v, s, a \right) ds \, dv.$$

To facilitate our estimation of the triple integral in the above expression and other similar integrals in what follows, we gather here a few bounds on $\zeta_K(s)$ that hold uniformly in specified regions. On write $s = \sigma + it$, we have

$$\zeta_K(s) \ll (1 + (|t| + 4)^1 - \sigma) \min \left( \frac{1}{|\sigma - 1|}, \log(|t| + 4) \right), \quad \sigma > 1,$$

$$\zeta_K(s) \ll \left( 1 + |s|^2 \right)^{1-\sigma/2 + \varepsilon}, \quad 0 \leq \sigma \leq 1,$$

$$\frac{1}{\zeta_K(s)} \ll \log(|t| + 4), \quad 1 \leq \sigma \leq 2.$$

The first and third estimation above can be established similar to the proofs of [20, Corollary 1.17] and [20, Lemma 6.7], respectively. The second estimation above is the convexity bound for $\zeta_K(s)$ (see [15, Exercise 3, p. 100]).

Also, by applying (3.7), we see that for the $j$-th derivative of $\zeta_K(s)$ with $j \geq 1$, we have for $\Re(s) > 1$,

$$\zeta_K^{(j)}(s) = \sum_{N(n) > 1} \frac{\mu(n)}{N(n)^{1+j}} N(n)^{-s} = \int_1^\infty (-\log u)^j u^{-s} \frac{d}{d u} \sum_{N(n) \leq u}$$

$$= \int_1^\infty (-\log u)^j u^{-s} d\left( \frac{\pi}{4} u + O(u^\theta) \right)$$

$$= \frac{\pi}{4} \frac{j!}{(1-s)^j} + O \left( \int_1^\infty (s + \frac{j}{\log u}) (-\log u)^j u^{-s-1+\theta} du \right).$$

We deduce readily from the above that for $j \geq 1$ and $\Re(s) > 0.4$, we have

$$\zeta_K^{(j)}(s) \ll 1 + |s|.$$

We further note that integrating by parts implies that for $\Re(s) < 1$ and any integer $\nu \geq 1$,

$$\hat{\Phi}(1 - s) \ll \nu \frac{3^{||s||}}{|s||s - 1|^{\nu-1}} \Phi(\nu),$$

where

$$\Phi(\nu) = \max_{0 \leq j \leq \nu} \int_{\frac{1}{2}}^{\frac{1}{2}} |\Phi^{(j)}(t)| dt.$$
We move the lines of the integrations in (5.20) over \( u, v \) to \( \Re(u) = \Re(v) = \frac{1}{\log X} \), \( \Re(s) = \frac{2}{\log X} \) without encountering any poles. Then by (2) of Lemma 2.12 and the estimations given in (5.21) and (5.23), we see that on the new lines of integrations, the expression in (5.20) is

\[
\ll X \log X \sum_{a > Y \atop (a, 2) = 1} \frac{1}{a} \int_{\log X}^{\log X} \int_{\log X}^{\log X} \int (1 + |2s|)^{1+\varepsilon} |\mathcal{J}(s)| \left| \Gamma(u + s + \frac{1}{2}) \right|^2 \left| \Gamma(v + s + \frac{1}{2}) \right|^2 |ds| \left| du \right| \left| dv \right|
\]

\[
\ll X \log X Y^{-1} \int_{\log X}^{\log X} (1 + |2s|)^{1+\varepsilon} \frac{\Gamma(s)}{\Gamma(1 - s)} |\tilde{\Phi}(1 - s)||ds| \ll X Y^{-1} \log X)^{22}\Phi(4),
\]

where the last estimation above follows from (5.22) with \( \nu = 4 \) and the bound \( \frac{\Gamma(s)}{\Gamma(1 - s)} \ll |s|^{2\Re(s) - 1} \).

We conclude from the above discussions and (5.19) that

\[
S_1(k_1 = \pm i) = \frac{2\pi X}{(2\pi i)^3} \sum_{a \equiv 1 \mod (1 + i)^3} N(a)^{1/2} \int_{\frac{1}{2} + \varepsilon}^{1} \int_{\frac{1}{2} + \varepsilon}^{1} N(a)^2 \mathcal{J}(s)w(u + s)w(v + s) \frac{U^{u+s}U^{u+s}X^{-s}}{(u + s)(v + s)}
\]

\[
\times \zeta_K^2(1 + u)\zeta_K^2(1 + v)(2^{-1} - 1)\zeta_K(2s)\zeta_K^2(1 + 2u + 2s)\zeta_K^2(1 + 2v + 2s)Z_4(\frac{1}{2} + u, \frac{1}{2} + v, s) \, ds \, dv
\]

\[
+ O \left( X Y^{-1} \log X)^{22}\Phi(4) \right).
\]

We further apply Lemma 2.13 to deduce from (5.24) that

\[
S_1(k_1 = \pm i) = \frac{2\pi X}{(2\pi i)^3} \left( \int_{\frac{1}{2} + \varepsilon}^{1} \int_{\frac{1}{2} + \varepsilon}^{1} \mathcal{J}(s)(2^{1 - 2s} - 1)\zeta_K(2s)w(u + s)w(v + s) \frac{U^{u+s}U^{u+s}X^{-s}}{(u + s)(v + s)}
\]

\[
\times \zeta_K^2(1 + u)\zeta_K^2(1 + v)\zeta_K^2(1 + 2u + 2s)\zeta_K^2(1 + 2v + 2s)Z_4(\frac{1}{2} + u, \frac{1}{2} + v, s) \, ds \, dv
\]

\[
+ O \left( X Y^{-1} \log X)^{22}\Phi(4) \right).
\]

We now move the lines of the integrations in (5.25) to \( \Re(u) = \Re(v) = \Re(s) = \frac{1}{100} \), without encountering any poles, as \( Z_4(\frac{1}{2} + u, \frac{1}{2} + v, s) \) is analytic and uniformly bounded in the region \( \Re(u), \Re(v) \geq -\frac{1}{2}, -\frac{1}{2} \leq \Re(s) \leq \frac{1}{2} \) by Lemma 2.13.

Next, we move the line of the integration over \( v \) to \( \Re(v) = -\frac{1}{100} + \frac{1}{100} \) to encounter a pole of order 2 at \( v = 0 \) and a pole of order 4 at \( v = -s \) to see that triple integral in (5.25) is

\[
\frac{1}{(2\pi i)^2} \int_{\frac{1}{100}}^{\frac{1}{100}} \int_{\frac{1}{100}}^{\frac{1}{100}} (J_2(u, s) + I_3(u, s)) \, du \, ds + O \left( U_1^{\frac{1}{100}} \frac{1}{U_2^{\frac{1}{100}}} X^{-\frac{1}{100}} \Phi(4) \right),
\]

where the error term above follows from using estimations given in (5.21) and (5.23) and where we denote \( J_2(u, s), I_3(u, s) \) for the residues of the integrand in (5.25) at \( v = 0 \) and \( v = -s \), respectively.

We evaluate the double integral of \( I_2(u, s) \) in (5.26) by writing the integrand in (5.25) as

\[
\frac{U^{u+s}U^{u+s}X^{-s}}{(u + s)(v + s)} \frac{1}{u^2v^2 s(2u + 2s)^3(2v + 2s)^3(u + v + 2s)^2} \mathcal{F}(u, v, s),
\]

where

\[
\mathcal{F}(u, v, s) = \mathcal{J}(s)(2^{1 - 2s} - 1)\zeta_K(2s)w(u + s)w(v + s)
\]

\[
\times \left( \zeta_K(1 + u)u \right)^2 \left( \zeta_K(1 + v)v \right)^2
\]

\[
\times \left( \zeta_K(1 + 2u + 2s)(2u + 2s)(2v + 2s)^3(\zeta_K(1 + u + v + 2s)(u + v + 2s)) \right)
\]

\[
\times Z_4(\frac{1}{2} + u, \frac{1}{2} + v, s).
\]

It is easy to see that \( \mathcal{F}(u, v, s) \) is analytic for \( \Re(u + 2s), \Re(v + 2s) > 0 \) by Lemma 2.12, and that

\[
I_2(u, s) = \frac{U^{u+s}U^{v+s}X^{-s}}{16(u + s)^3(u + s)^4 s^4 u^2} \left( \frac{1}{\mathcal{F}(u, 0, s)}(s(u + 2s) \log U_2 - 10s - 3u) + \mathcal{F}(0, 1, 0)(u, 0, s)(u + 2s) \right).
\]
It follows from this that by moving the line of the double integral in (5.20) involving $I_2(u, s)$ from $\Re(u) = \frac{1}{100}$ to $\Re(u) = -\frac{1}{100} + \frac{1}{\log X}$ and applying (5.21), (5.22), we have

(5.27) \[ \frac{1}{(2\pi i)^2} \int (\frac{}{1}) \int I_2(u, s) \, du \, ds = \frac{1}{2\pi i} \int_{u=0}^{1} \text{Res} \{I_2(u, s)\} \, ds + O \left( U_2 \frac{\pi i}{100} X^{-\frac{1}{100}} (\log X)^5 \right). \]

Note that

$$\text{Res} \{I_2(u, s)\} = \frac{U_1 U_2 X^{-s}}{64 \pi i} \left( F(0, 0, s)(s^2 \log U_1 \log U_2 - 5s \log U_1 - 5s \log U_2 + 26) \right. \]

$$+ \frac{\mathcal{F}(1, 0, 0)}{F(0, 0, s)}(s^2 \log U_2 - 5s) + \frac{\mathcal{F}(0, 1, 0)}{F(0, 0, s)}(s^2 \log U_1 - 5s) + \frac{\mathcal{F}(1, 1, 0)}{F(0, 0, s)}(0, 0, s^2). \]

As one checks that the expression in the parenthesis above is analytic for $-\frac{1}{100} \leq \Re(s) \leq \frac{1}{8}$, we can move the line of the integral in (5.27) involving $\text{Res} \{I_2(u, s)\}$ to $\Re(s) = -\frac{1}{100}$. In this process, we encounter a pole at $s = 0$ so that we have

(5.28) \[ \frac{1}{2\pi i} \int_{u=0}^{1} \text{Res} \{I_2(u, s)\} \, ds = \frac{F(0, 0, 0)}{64} \sum_{j_1 + j_2 + j_3 + j_4 = 10 \atop j_1, j_2, j_3, j_4 \geq 0} \frac{(-1)^{j_2} B(j_4)}{j_1! j_2! j_3! j_4!} (\log^{j_1} U_1)(\log^{j_2} U_2)(\log^{j_3} X)

+ O \left( U_1 \frac{\pi i}{100} X^{-\frac{1}{100}} + (\log X)^9 \right), \]

where

$$B(j) = \begin{cases} 26 & \text{if } j = 0, \\
-5(\log U_1 + \log U_2) & \text{if } j = 1, \\
2 \log U_1 \log U_2 & \text{if } j = 2, \\
0 & \text{if } j \geq 3. \end{cases}$$

To evaluate $F(0, 0, 0)$, we use the fact that $s \Gamma(s) = 1$ when $s = 0$ and the functional equation (2.10) for $\zeta_K(s)$:

$$\pi^{-s} \Gamma(s) \zeta_K(s) = \pi^{-(1-s)} \Gamma(1-s) \zeta_K(1-s)$$

to obtain that $\zeta_K(0) = -\frac{1}{12}$. On the other hand, a direct calculation shows that $Z_1(\frac{1}{2}, \frac{1}{2}, 0) = \frac{16}{3 \zeta_K(2)} a_4$. We then deduce that

$$F(0, 0, 0) = \Phi(1) \zeta_K(0) \left( \frac{\pi}{4} \right)^{10} Z_4(\frac{1}{2}, \frac{1}{2}, 0) = -\frac{4 \Phi(1) a_4}{3 \zeta_K(2)}.$$

Similarly, by moving the line of the integration over $u$ from $\Re(u) = \frac{1}{100}$ to $\Re(u) = \frac{1}{\log X}$ in the double integral of $I_3(u, s)$ in (5.20) and applying estimations given in (5.21), (5.22), we see that

(5.29) \[ \frac{1}{(2\pi i)^2} \int (\frac{}{1}) \int I_3(u, s) \, du \, ds \ll U_1 \frac{\pi i}{100} X^{-\frac{1}{100}} (\log X)^5 \Phi(5). \]

We now summarize our result on $S_1(k_1 = \pm i)$ in the following lemma, by combining the estimations from (5.23)- (5.24).

**Lemma 5.5.** We have

$$S_1(k_1 = \pm i) = -\frac{8 \pi a_4 \Phi(1) X}{3 \cdot 64 \zeta_K(2)} \left( \frac{\pi}{4} \right)^{10} \sum_{j_1 + j_2 + j_3 + j_4 = 10 \atop j_1, j_2, j_3, j_4 \geq 0} \frac{(-1)^{j_2} B(j_4)}{j_1! j_2! j_3! j_4!} (\log^{j_1} U_1)(\log^{j_2} U_2)(\log^{j_3} X)

+ X \cdot O \left( (\log X)^9 + U_1 \frac{\pi i}{100} U_2 \frac{\pi i}{100} X^{-\frac{1}{100}} + U_2 \frac{\pi i}{100} X^{-\frac{1}{100}} (\log X)^5 \right)

+ X \cdot O \left( U_1 \frac{\pi i}{100} U_2 \frac{\pi i}{100} X^{-\frac{1}{100}} \Phi(4) + Y^{-1} (\log X)^{22} \Phi(4) + U_1 \frac{\pi i}{100} X^{-\frac{1}{100}} (\log X)^5 \Phi(5) \right).$$

**5.6. Computing $S_1$: the term $S_1(k \neq \pm i)$.** In this section, we estimate $S_1(k \neq \pm i)$. We first deduce from (5.13) that

$$S_1(k_1 \neq \pm i) \ll X \sum_{a \equiv 1 \mod (1+i)^3} \frac{1}{N(a)^2} \sum_{k \in O_K} \left| \int \int \int N(a)^2 \mathcal{F}(s) w(u + s) w(v + s) \right|$$

$$\times U_1^{a+s} U_2^{a+s} X^{-s} \left( 1 + u, \chi_{ik_1} \right) L^2(1 + u, \chi_{ik_1}) L^2(1 + v, \chi_{ik_1}) Z_2(\frac{1}{2} + u, a, \frac{1}{2} + v, a, k_1) \, ds \, dv.$$
Let $Z$ be a parameter to be chosen later. We denote $S_{1,1}(k_1 \neq \pm i)$ for the right-hand side expression above truncated to $N(k_1) \leq Z$ and $S_{1,2}(k_1 \neq \pm i)$ for the right-hand side expression above over $N(k_1) > Z$. For $S_{1,1}(k_1 \neq \pm i)$, we shift the the lines of the integrations to $\Re(u) = \Re(v) = -\frac{1}{2} + \frac{1}{\log N}$, $\Re(s) = \frac{3}{4}$. For $S_{1,2}(k_1 \neq \pm i)$, we shift the the lines of the integrations to $\Re(u) = \Re(v) = -\frac{1}{2} + \frac{1}{\log N}$, $\Re(s) = \frac{3}{4}$. Thus we obtain via (2.21) that

$$S_{1,1}(k_1 \neq \pm i) \ll X^{\frac{4}{3}} U_1^{\frac{3}{2}} U_2^{\frac{x}{2}} (\log X)^{10} \sum_{N(a) \leq Y \atop (a, 2) = 1} \frac{d^3_{[i]}(a)}{\sqrt{N(a)}} \int_{-\frac{1}{2} + \frac{1}{\log N}} \int_{-\frac{1}{2} + \frac{1}{\log N}} \int_{-\frac{1}{2} + \frac{1}{\log N}} |J(s)w(u + s)w(v + s)| \, ds \, |du| \, |dv|,$$

(5.31)

$$\times \sum_{k_1 \neq \pm i \atop N(k_1) \leq Z} \frac{d^2_{[i]}(k_1)}{N(k_1)^{\frac{3}{4}}} |L(1 + u, \chi_{ik_1})|^4 \, |ds| \, |du| \, |dv|,$$

where $\sum_{b}$ denotes the sum over all square-free elements in $\mathcal{O}_K$.

By the Cauchy-Schwarz inequality, we have that

$$\sum_{k_1 \neq \pm i \atop N(k_1) \leq Z} \frac{d^2_{[i]}(k_1)}{N(k_1)^{\frac{3}{4}}} |L(1 + u, \chi_{ik_1})|^4 \ll \left( \sum_{k_1 \neq \pm i \atop N(k_1) \leq Z} \frac{d^2_{[i]}(k_1)}{N(k_1)^{\frac{3}{4}}} \right)^2 \left( \sum_{k_1 \neq \pm i \atop N(k_1) \leq Z} \frac{1}{N(k_1)^{\frac{3}{4}}} |L(1 + u, \chi_{ik_1})|^8 \right)^{1/2},$$

(5.32)

$$\ll (\log Z)^{23} \left( \sum_{k_1 \neq \pm i \atop N(k_1) \leq Z} \frac{1}{N(k_1)^{\frac{3}{4}}} |L(1 + u, \chi_{ik_1})|^8 \right)^{1/2}.$$

Similar to our proof of Theorem 1.2 (and hence Corollary 1.3), we can show that under GRH, for $|\Im(u)| \leq T$,

$$\sum_{k_1 \neq \pm i \atop N(k_1) \leq Z} |L(1 + u, \chi_{ik_1})|^8 \ll Z(\log Z)^{37}.$$

Applying the above estimation in (5.32), we deduce that for $|\Im(u)| \leq Z$,

$$\sum_{k_1 \neq \pm i \atop N(k_1) \leq Z} \frac{d^2_{[i]}(k_1)}{N(k_1)^{\frac{3}{4}}} |L(1 + u, \chi_{ik_1})|^4 \ll Z^{1/4}(\log Z)^{224}.$$

(5.33)

Unconditionally, we have similar to Lemma 2.14 that

$$\sum_{k_1 \neq \pm i \atop N(k_1) \leq Z} |L(1 + u, \chi_{ik_1})|^4 \ll Z^{1+\varepsilon}(1 + |\Im(u)|^2)^{1+\varepsilon}.$$

It follows from this and partial summation that we have

$$\sum_{k_1 \neq \pm i \atop N(k_1) \leq Z} \frac{d^2_{[i]}(k_1)}{N(k_1)^{\frac{3}{4}}} |L(1 + u, \chi_{ik_1})|^4 \ll \sum_{k_1 \neq \pm i \atop N(k_1) \leq Z} \frac{1}{N(k_1)^{\frac{3}{4}-\varepsilon}} |L(1 + u, \chi_{ik_1})|^4 \ll Z^{1+\varepsilon}(1 + |\Im(u)|^2)^{1+\varepsilon}.$$

(5.34)

Applying the estimation given (5.33) to (5.31) for $|\Im(u)| \leq Z$ and using (5.34) otherwise, we obtain by noting the exponential decay of $w$ that

$$S_{1,1}(k_1 \neq \pm i) \ll X^{\frac{4}{3}} U_1^{\frac{3}{2}} U_2^{\frac{x}{2}} (\log X)^{10} (\log Y)^{2s} Z^{\frac{1}{2}+\varepsilon}(\log Z)^{224}\Phi(4).$$

(5.35)

Similarly, we have

$$S_{1,2}(k_1 \neq \pm i) \ll X^{-\frac{3}{4}} U_1^{\frac{3}{2}} U_2^{\frac{x}{2}} Y^{3/2} (\log Y)^{2s} Z^{-\frac{3}{4}+\varepsilon}(\log Z)^{224}\Phi(4).$$

(5.36)

We further observe that if instead we apply (5.34) to (5.31) for all $\Im(u)$, then we obtain corresponding estimations for $S_{1,1}(k_1 \neq \pm i)$, $S_{1,2}(k_1 \neq \pm i)$ by replacing $(\log Z)^{224}$ with $Z^{1+\varepsilon}$ in both (5.35) and (5.36). On setting $Z = U_1 U_2 Y X^{-1}$ and keeping in mind that our choices of $Y$ and $Z$ will be at most powers of $X$ (see Section 5.8), we immediately derive from (5.31), (5.35) and (5.36) the following result.

Lemma 5.7. Unconditionally, we have

$$S_1(k \neq \pm i) \ll U_1^{\frac{3}{2}} U_2^{\frac{x}{2}} Y X^{\varepsilon}\Phi(4).$$
5.8. Proof of Theorem 1.5. We now complete the proof for Theorem 1.5 in this section. By setting \( U_1 = U_2 = \frac{X}{(\log X)^{\varepsilon}} \) and \( Y = X^\frac{1}{4} U_1^\frac{1}{2} U_2^{-\frac{1}{2}} \), we deduce from (5.32), (5.18), Lemma 5.2, Lemma 5.4 and Lemma 5.7 that under GRH,

\[
S(U_1, U_2) = \sum_{(d, 2)=1}^\infty |A_U(d)|^2 \Phi \left( \frac{N(d)}{X} \right) = \frac{\pi a_4 \Phi(1)}{2^7 \cdot 3^4 \cdot 7 \cdot \zeta_K(2)} \left( \frac{\pi}{4} \right)^{10} X (\log X)^{10} + O \left( X (\log X)^{9+\varepsilon} + X (\log X)^{10} \right).
\]

Here we note that, similar to [28, Lemma 2.1], we can show that the function \( V \) appearing in the definition of \( A_U(d) \) given in (5.31) is a real-valued function, so that we have \( A_U^2(d) = |A_U(d)|^2 \).

We define \( B_U(d) = L(\frac{1}{2}, \chi_{(1+i)\cdot d})^2 - A_U(d) \) to see that

\[
B_U(d) = \frac{1}{\pi i} \int w(s) L(\frac{1}{2} + s, \chi_{(1+i)\cdot d})^2 \frac{N(d)^s - U^s}{s} ds.
\]

We move the line of the integral in the above expression for \( B_U(d) \) to \( \Re(s) = 0 \) by realizing that \( \frac{N(d)^s - U^s}{s} \) is entire there. Further applying the bound that \( |\frac{N(d)^s - U^s}{s}| \ll \log(\frac{N(d)}{d}) \) for \( t \in \mathbb{R} \), we see that

\[
B_U(d) \ll \log \left( \frac{N(d)}{U} \right) \int_{-\infty}^{\infty} \log(|w(it)|) L(\frac{1}{2} + it, \chi_{(1+i)\cdot d})^2 dt.
\]

It follows that

\[
\sum_{(d, 2)=1}^\infty |B_U(d)|^2 \Phi \left( \frac{N(d)}{X} \right) \ll \left( \log \frac{X}{U} \right)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \log(|w(it_1)|) \log(|w(it_2)|) \sum_{(d, 2)=1}^\infty |L(\frac{1}{2} + it_1, \chi_{sd})|^2 |L(\frac{1}{2} + it_2, \chi_{sd})|^2 \Phi \left( \frac{N(d)}{X} \right) dt_1 dt_2.
\]

We estimation the sum over \( d \) above using Corollary 1.3 when \( |t_1|, |t_2| \leq X \) and Lemma 2.13 otherwise. We then deduce by the exponential decay of \( w(it) \) in \( t \) that

\[
\sum_{(d, 2)=1}^\infty |B_U(d)|^2 \Phi \left( \frac{N(d)}{X} \right) \ll X (\log X)^{9+\varepsilon}.
\]

Combining (5.37) and (5.38), we see that

\[
\sum_{(d, 2)=1}^\infty L(\frac{1}{2}, \chi_{(1+i)\cdot d})^4 \Phi \left( \frac{N(d)}{X} \right) \ll \sum_{(d, 2)=1}^\infty (A_U(d) + B_U(d))^2 \Phi \left( \frac{N(d)}{X} \right)
\]

\[
= \sum_{(d, 2)=1}^\infty A_U(d)^2 \Phi \left( \frac{N(d)}{X} \right) + O \left( \sum_{(d, 2)=1}^\infty |B_U(d)|^2 \Phi \left( \frac{N(d)}{X} \right) + 2 \sum_{(d, 2)=1}^\infty |A_U(d)||B_U(d)||\Phi \left( \frac{N(d)}{X} \right) \right)
\]

\[
= \frac{\pi a_4 \Phi(1)}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot \zeta_K(2)} \left( \frac{\pi}{4} \right)^{10} X (\log X)^{10} + O \left( X (\log X)^{9+\varepsilon} + X (\log X)^{10} \right),
\]

where the last expression above follows from an application of the Cauchy-Schwarz inequality to estimate the sum involving the product of \( |A_U(d)| \) and \( |B_U(d)| \).

We now take \( \Phi(t) \) to be supported on \([1, 2]\) satisfying \( \Phi(t) = 1 \) for \( t \in (1 + \mathcal{Z}^{-1}, 2 - \mathcal{Z}^{-1}) \) and \( \Phi^{(\nu)}(t) \ll \nu \mathcal{Z}^\nu \) for all rational integer \( \nu \geq 0 \). We then deduce that \( \Phi^{(\nu)}(t) \ll \nu \mathcal{Z}^\nu \), and that \( \Phi(1) = 1 + O(\mathcal{Z}^{-1}) \). Thus (5.39) implies that

\[
\sum_{(d, 2)=1}^\infty L(\frac{1}{2}, \chi_{(1+i)\cdot d})^4 \Phi \left( \frac{N(d)}{X} \right) \ll \frac{\pi a_4 \Phi(1)}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot \zeta_K(2)} \left( \frac{\pi}{4} \right)^{10} X (\log X)^{10} + O \left( X (\log X)^{10} \mathcal{Z}^{-1} + X (\log X)^{9.5+\varepsilon} + X (\log X)^{-5} \right).
\]
We then deduce by taking $Z = \log X$ that
\begin{equation}
\sum_{\substack{(d,2)=1 \\ X < N(d) \leq 2X}}^* L\left(\frac{1}{2}, \chi(1+i)^*d\right)^4 \geq \sum_{\substack{(d,2)=1 \\ X < N(d) \leq 2X}}^* L\left(\frac{1}{2}, \chi(1+i)^*d\right)^4 \Phi\left(\frac{N(d)}{X}\right) = \frac{\pi a_4 \hat{\Phi}(1)}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot \zeta_K(2)} \left(\frac{\pi}{4}\right)^{10} X^{(\log X)^{10}} + O\left(X^{(\log X)^{9.5+\epsilon}}\right).
\end{equation}

Similarly, we can choose $\Phi(t)$ in (5.39) such that $\Phi(t) = 1$ for $t \in [1,2]$, $\Phi(t) = 0$ for all $t \notin (1 - Z^{-1}, 2 + Z^{-1})$, and $\Phi^{(i)}(t) \leq \nu Z^\nu$ for all $\nu \geq 0$. Taking $Z = \log X$, we can deduce that
\begin{equation}
\sum_{\substack{(d,2)=1 \\ X < N(d) \leq 2X}}^* L\left(\frac{1}{2}, \chi(1+i)^*d\right)^4 \leq \sum_{\substack{(d,2)=1 \\ X < N(d) \leq 2X}}^* L\left(\frac{1}{2}, \chi(1+i)^*d\right)^4 \Phi\left(\frac{N(d)}{X}\right) = \frac{\pi a_4 \hat{\Phi}(1)}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot \zeta_K(2)} \left(\frac{\pi}{4}\right)^{10} X^{(\log X)^{10}} + O\left(X^{(\log X)^{9.5+\epsilon}}\right).
\end{equation}

Combining (5.40) and (5.41), we obtain that
\begin{equation}
\sum_{\substack{(d,2)=1 \\ X < N(d) \leq 2X}}^* L\left(\frac{1}{2}, \chi(1+i)^*d\right)^4 = \frac{\pi a_4 \hat{\Phi}(1)}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot \zeta_K(2)} \left(\frac{\pi}{4}\right)^{10} X^{(\log X)^{10}} + O\left(X^{(\log X)^{9.5+\epsilon}}\right).
\end{equation}

The assertion of Theorem 1.5 now follows by summing the above over $X = \frac{t}{2}$, $X = \frac{t^2}{4}$, \ldots and then resetting $x$ to be $X$.

5.9. Proof of Theorem 1.6 In this section, we complete the proof for Theorem 1.6. We first apply the Cauchy-Schwartz inequality to see that
\begin{equation}
\sum_{\substack{(d,2)=1}}^* L\left(\frac{1}{2}, \chi(1+i)^*d\right)^4 \Phi\left(d \frac{x}{Y}\right) \geq \frac{A^2}{B},
\end{equation}

where
\begin{align*}
A &= \sum_{\substack{(d,2)=1}}^* A_U(d)L\left(\frac{1}{2}, \chi(1+i)^*d\right)^2 \Phi\left(\frac{N(d)}{X}\right), \\
B &= \sum_{\substack{(d,2)=1}}^* A_U(d)^2 \Phi\left(\frac{N(d)}{X}\right).
\end{align*}

Here we recall that $A_U(d)$ is defined as in (5.1).

We can evaluate $B$ similar to our evaluation of $S(U_1, U_2)$ in Section 5.9 except that now we set $U_1 = U_2 = U = X^{1-4\epsilon}$. By setting $Y = X^{\frac{3}{4}} U_1^{-\epsilon} X^{-\frac{1}{4}}$ again, we deduce from Lemma 5.2, Lemma 5.3 and Lemma 5.7 that unconditionally,
\begin{equation*}
B = \frac{\pi a_4 (1 + O(\epsilon))}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot \zeta_K(2)} \left(\frac{\pi}{4}\right)^{10} \hat{\Phi}(1) X^{(\log X)^{10}} + O\left(X^{(\log X)^{9}} + X \Phi(5)\right),
\end{equation*}

with the implied constant in $O(\epsilon)$ being absolute.

To evaluate $A$, we recast it as
\begin{equation*}
A = 4 \sum_{\substack{(d,2)=1}}^* \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3} \frac{\chi(1+i)^*d(n_1 n_2) d[i](n_1) d[i](n_2)}{N(n_1 n_2)^{\frac{1}{2}}} h_1(d, n_1, n_2),
\end{equation*}

where
\begin{equation*}
h_1(x, y, z) = \Phi\left(\frac{N(x)}{X}\right) V\left(\frac{N(y)}{U}\right) V\left(\frac{N(z)}{N(x)}\right).
\end{equation*}

A similar argument to our evaluation of $B$ above implies that, by taking $Y = X^{\frac{3}{4}} U_1^{-\epsilon} X^{-\frac{1}{4}}$, we have unconditionally,
\begin{equation*}
A = \frac{\pi a_4 (1 + O(\epsilon))}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot \zeta_K(2)} \left(\frac{\pi}{4}\right)^{10} \hat{\Phi}(1) X^{(\log X)^{10}} + O\left(X^{(\log X)^{9}} + X \Phi(5)\right),
\end{equation*}

with the implied constant in $O(\epsilon)$ being absolute.
We now take $Z = \log X$ and take $\Phi$ so that $\Phi(t) = 1$ for $t \in (1 + Z^{-1}, 2 - Z^{-1})$, $\Phi(t) = 0$ for all $t \notin (1, 2)$, and $\Phi^{(\nu)}(t) \ll \epsilon^{\nu}$ for all $\nu \geq 0$. Applying our estimations for $A$ and $B$ in (30), we deduce that

$$\sum_{(d, 2) = 1}^{\infty} \frac{L(1/2, \chi (1 + i \nu) d)}{X < \mathcal{N}(d) \leq 2X} \geq (1 + O(\epsilon)) \frac{\pi a_4}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot \zeta_K(2)} \left(\frac{\pi}{4}\right)^{10} \Phi(1) X (\log X)^{10}.$$

The assertion of Theorem 1.6 now follows by summing the above over $X = \frac{x}{2}$, $X = \frac{x}{3}$, ..., and then resetting $x$ to be $X$.

Acknowledgments. P. G. is supported in part by NSFC grant 11871082.

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