ON THE HOLLENBECK-VERBITSKY CONJECTURE AND M.
RIESZ THEOREM FOR VARIOUS FUNCTION SPACES

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Abstract. Let $P_+$ be the Riesz’s projection operator and let $P_− = I − P_+$.
We consider two-sided estimates of \[ \| (|P_+ f|^s + |P_− f|^s)^{\frac{1}{s}} \|_{L^p(T)} \] in terms of
Lebesgue $p$-norm of the function $f \in L^p(T)$. For some values of parameters $0 < s < \infty$ and $p$, our inequalities are sharp, while in other cases we find the family
of test functions which we conjecture to be sharp. Also, we obtain the right
asymptotic of the constants for large $s$ as well as the appropriate vector-valued
inequalities. This proves the conjecture of Hollenbeck and Verbitsky on the
Riesz projection operator in some cases. As a consequence of inequalities we
have in the paper we get Riesz-type theorems on conjugate harmonic functions
for various function spaces. In particular, slightly general version of Stout’s
theorem for Lumer Hardy spaces were obtained by a new approach.

1. Introduction and the results

1.1. Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and let $T = \{z \in \mathbb{C} : |z| = 1\}$
be the unit circle. For $0 < p \leq \infty$ we denote by $L^p(T)$ the Lebesgue space on the
unit circle. The space $H^p(T)$ contains all $\varphi \in L^p(T)$ for which all negative Fourier
coefficients are equal to zero, i.e.,
\[ \hat{\varphi}(n) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) e^{-int} dt = 0 \]
for all integers $n < 0$.

The Riesz projection operator $P_+ : L^p(T) \to H^p(T)$ and the co-analytic projection
operator $P_− = I − P_+$, where $I$ be the identity operator on $L^p(T)$, operate in
the following way:
\[ P_+ f(\zeta) = \sum_{n=0}^{\infty} \hat{f}(n) \zeta^n, \quad P_− f(\zeta) = \sum_{n=-\infty}^{-1} \hat{f}(n) \zeta^n \]
for $f(\zeta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \zeta^n$.

1.2. For a function $f$ on $U$ and $r \in (0, 1)$ we denote by $f_r$ the function $f_r(\zeta) = f(r \zeta)$, $\zeta \in U$. The harmonic Hardy space $h^p$, for $p \in (0, \infty)$, consists of all harmonic
complex-valued functions $U$ on $U$ for which the integral mean
\[ M_p(U, r) = \left\{ \int_{T} |U_r(\zeta)|^p |d\zeta| \right\}^{1/p} \]
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remains bounded as $r$ approaches 1. Since $|U|^p$ for $1 \leq p < \infty$ is subharmonic on $U$, the integral mean $M_p(U, r)$ is increasing in $r$. The norm on $H^p$ in this case is given by

$$\|U\|_p = \lim_{r \to 1} M_p(U, r).$$

The analytic Hardy space $H^p$ is the subspace of $h^p$ that contains all analytic functions. For the theory of Hardy spaces in the unit disk we refer to [1, 2, 3, 15] as well as to the Pavlović book for more on the Hardy space theory in the unit disc. Since $|f|^p$ is subharmonic for every $0 < p < \infty$, the norm on the Hardy space $H^p$ may be introduced as

$$\|f\|_p = \lim_{r \to 1} M(r, f).$$

For $p = \infty$ the space $H^\infty$ contains all bounded analytic functions in the unit disc.

It is well known that for $f \in H^p$ the radial boundary value $f^*(\zeta) = \lim_{r \to 1-} f(r\zeta)$ exists for almost every $\zeta \in T$ and $f^*$ belongs to the space $H^p(T)$. Moreover, we have the isometry relation have $\|f^*\|_{L^p(T)} = \|f\|_{H^p}$. Therefore, we may identify the spaces $H^p$ and $H^p(T)$.

Moreover, we may identify the spaces $H^p(T)$ and $H^p$ via the isometry $f \to f^*.$

If we identify the Hardy space $H^p$ and the space $H^p(T)$, then $P_+$ may be represented as the Cauchy integral (with density $f^*$):

$$P_+ f(z) = \frac{1}{2\pi i} \int_T \frac{f^*(\zeta)}{\zeta - z} d\zeta, \quad z \in U.$$  

1.3. From the Parseval equality it follows that the operator $P_+$ is an orthogonal projection of $L^2(T)$ onto $H^2(T)$ and therefore we have $\|P_+: L^2(T) \to H^2(T)\| = 1$. The question for $p \neq 2$ is more complex. It is a classical result that Riesz projection $P_+$ is a bounded operator as an operator on $L^p(T)$ space for every $1 < p < \infty$. This is the classical result obtained by M. Riesz. Nowadays there exist many different proofs of this fact. There exists an elementary proof of Stein base on Hardy–Stein equality. But it needed a long time to find the exact norm of the Riesz projection. In 2000, Hollenbeck and Verbitsky used the method which involves a plurisubharmonic minorant in a certain inequality and finally proved that the norm is equal to $\frac{1}{\sin \frac{\pi}{p}}$. The idea in [5] is to find a plurisubharmonic minorant $F(z, w)$ in $\mathbb{C}^2$ with $F(0, 0) = 0$ of the function

$$\Phi(z, w) = \frac{1}{\sin^p \frac{\pi}{p}} |w + \overline{z}|^p - \max\{|z|^p, |w|^p\}.$$  

The minorant may be found in the following form $F(w, z) = c\Re(wz)^\frac{p}{2}$ if $1 < p < 2$. In other words, we have

$$\max\{|z|^p, |w|^p\} \leq \frac{1}{\sin^p \frac{\pi}{p}} |z + \overline{w}|^p - c\Re(wz)^\frac{p}{2}$$

for a positive constant $c$. If we integrate the inequality for $z = P_+ f(\zeta)$ and $w = P_- f(\zeta)$, with respect to the circle $rT$, and if we have on mind that the function $\Re(P_+ f(\zeta) P_- f(\zeta))^\frac{p}{2}$ is subharmonic in the unit disc equal zero for $\zeta = 0$, we obtain
the inequality
\[
\left\{ \int_T \max\{|P_+ f(\zeta)|^p, |P_- f(\zeta)|^p\} |d\zeta| \right\}^{\frac{1}{p}} \leq \frac{1}{\sin \frac{\pi}{2p}} \left\{ \int_T |f(\zeta)|^p |d\zeta| \right\}^{\frac{1}{p}}.
\]

Some partial results on the norm of this operator were known much earlier by Gohberg and Krupnik. This method has been recently used by Kalaj [7] in order to prove the Hollenbeck and Verbitsky conjecture in a special case. We are able here to simplify Kalaj’s proof. This method will be also used in this paper. What Hollenbeck and Verbitsky proved was earlier known as Gohberg and Krupnik conjecture (they proved some special cases of the conjecture, i.e., for the numbers which may be represented in the form \( p = 2^n \), where \( n \in \mathbb{N} \)). Let us say that the Riesz projection operator is not bounded on \( L^1(T) \), and moreover it does not exist a bounded projection of \( L^1(T) \) onto \( H^1(T) \). This was shown by Newman, and generalize by Rudin.

1.4. In 2010, Hollenbeck and Verbitsky posed the problem [6] of finding the optimal constant \( C_{s,p} \) for the inequality
\[
\| |P_+ f|^s + |P_- f|^s \|_{L^p(T)} \leq C_{s,p} \| f \|_{L^p(T)}
\]
for \( p \in (1, \infty) \) and a positive number \( s \).

Hollenbeck and Verbitsky conjectured that
\[
C_{s,p} = \frac{2^{\frac{1}{2}}}{2 \cos \frac{\pi}{2p}} \text{ if } 1 < p < 2 \text{ and } 0 < s < \sec \frac{\pi}{2p},
\]
and
\[
C_{s,p} = \frac{2^{\frac{1}{2}}}{2 \sin \frac{\pi}{2p}} \text{ if } 2 < p < \infty \text{ and } 0 < s < \csc \frac{\pi}{2p}.
\]

We prove that the conjecture is correct for \( 0 < s \leq 2 \). It seems that the case \( s > 2 \) is more complex.

This conjecture was motivated by considering some family functions.

As we have said, the case \( p = \infty \) is solved by Hollenbeck and Verbitsky [5] in 2000 by method of plurisubharmonic minorant. Since the lower estimates were already known, they obtained that \( C_{\infty,p} = \frac{1}{\sin \frac{\pi}{p}} \), i.e. they proved that
\[
\| \max\{|P_+(f)|, |P_-(f)|\} \|_{L^p(T)} \leq C_{\infty,p} \| f \|_{L^p(T)} \text{ (} f \in L^p(T)\).\]

This method has been recently used by Kalaj for \( s = 2 \) in order to prove the Hollenbeck and Verbitsky conjecture in this special case. We are able here to simplify Kalaj’s proof.

We would like to cover the vector-valued case, i.e., we may suppose that \( f \) is be vector-valued, i.e., it may maps the unit ball into \( \mathbb{C}^n \). It is maybe surprising that the constants \( C_{s,p} \) remains the same.

1.5. We will also consider the reverse inequalities, i.e., inequalities of the form
\[
\| f \|_{L^p(T)} \leq \tilde{C}_{s,p} \| |P_+ f|^s + |P_- f|^s \|_{L^p(T)}
\]
with the best constant \( \tilde{C}_{s,p} \). The existence of the constants \( \tilde{C}_{s,p} \) in above inequality is a consequence of the Banach open mapping theorem.

The case \( s = 2 \) was considered by Kalaj [7]. He used the reverse inequalities for \( p > 2 \) in order to derive the direct inequality for \( 1 < p < 2 \).
1.6. We have several aims in the paper.
(1) To confirm the Hollenbeck and Verbitsky conjecture in some cases.
(2) To obtain the best possible constants for the reverse inequality.
(3) To discuss the vector-valued case.
(4) To show that the constant we have in some inequalities and which are optimal
for functions in Hardy space are useful in some other function spaces such as Lumer’s
Hardy spaces, weighted Bergman spaces and Fock spaces.

The paper is separated in several sections. We expose the results on (1) in the
next section. Here we establish inequalities for complex numbers with best possible
constants and with reminder which is plurisubharmonic function of two complex
variables. These inequalities are be used later to obtain the upper estimates for the
$L^p$ norm of $(|P_+|^s + |P_-|^s)^{\frac{1}{s}}$. Then we prove optimality of the estimates. At the
end of the paper we considered the vector-valued analytic mappings, where we can
also use the method of plurisubharmonic minorants.

1.7. Now, we will state the results of our paper. We first prove the next theorem:

**Theorem 1.1.** For $f \in L^p(T)$ we have

$$\| (|P_+ f|^s + |P_- f|^s)^{\frac{1}{s}} \|_{L^p(T)} \leq C_{s,p} \| f \|_{L^p(T)}$$

for $1 < p \leq 2$ and $0 < s \leq 2$, with the best possible constant

$$C_{s,p} = \frac{2^{\frac{1}{s}}}{2 \cos \frac{\pi}{2p}}.$$ 

For $s > 2$ we have the following estimates of the best possible constants:

$$C_{s,p} \geq \frac{2^{\frac{1}{s}}}{2 \cos \frac{\pi}{2p}},$$

if $2 \leq s \leq \frac{1}{\cos \frac{\pi}{2p}}$, and

$$C_{s,p} \geq \frac{(1 + \tilde{t}^s)^{\frac{1}{s}}}{\sqrt{1 + \tilde{t}^2 + 2t \cos \frac{\pi}{p}}}$$

if $s \geq \frac{1}{\cos \frac{\pi}{2p}}$ and $\tilde{t}$ is the unique zero of the equation $t^{s-1} - t + \cos \frac{\pi}{p}(t^s - 1) = 0$
in $(0, 1)$.

Similar inequalities hold for $p > 2$.

For $2 < p < \infty$, $0 < s \leq 2$ and $f \in L^p(T)$ we have

$$\left( \int_T (|P_+ f(\zeta)|^s + |P_- f(\zeta)|^s)^{\frac{1}{s}} |d\zeta| \right)^{\frac{1}{p}} \leq \frac{2^{\frac{1}{s}}}{2 \sin \frac{\pi}{2p}} \left( \int_T |f(\zeta)|^p |d\zeta| \right)^{\frac{1}{p}}.$$ 

Also, considering estimates for large $s$, we obtain the asymptotics, as conjectured
by Hollenbeck and Verbitsky:

**Theorem 1.2.** For the constant $C_{s,p}$ as defined above, there holds the following:

$$\lim_{s \to \infty} C_{s,p} = \frac{1}{\sin \frac{\pi}{p}}.$$ 

According the Banach open mapping theorem, we easily conclude the existence
of constants $C_{s,p}$ in the appropriate reverse inequalities. The next theorem contains
our results on these estimates.
Lemma 2.1. In the paper of Hollenbeck and Verbitsky [5], a function such that \( u \parallel \) result od Pichorides [14] gives \( r, t \) points of this function in the interior of \((a function \( u \parallel \)).

1.8. The Riesz projection is closely related to the Hilbert operator \( H \) which maps a function \( u \) in harmonic Hardy space \( h^p(U) \) to the harmonic conjugate \( \tilde{u} \), i.e., a function such that \( u + i\tilde{u} \) is an analytic function on the unit disc. The classical result od Pichorides [14] gives \( ||H|| = tan \frac{\pi}{2p} \), where \( \bar{p} = min\{p, \frac{p}{p-1}\} \). It is not hard to derive this result from the result of Hollenbeck and Verbitsky.

2. The Hollenbeck and Verbitsky conjecture for \( 0 < s \leq 2 \)

2.1. Here we prove the result given in [11]. The theorem contains the Kalaj’s result (for \( s = 2 \)).

The proof of this upper bound for the \( s \)-norm of the Riesz projection \( P_+ \) and co-analytic projection \( P_- \) is given at the end of this subsection. We first prove the needed lemmas. The upper estimate is a consequence of the following result for complex numbers. The proof of this inequality in the Kalaj work in the case \( s = 2 \) at some places uses some nontrivial estimates. The proof given here is considerably simple. In fact, by some transformations we trivially see the estimate for possible stationary points, thus reducing the proof to checking boundary points. For \( s = 2 \) a proof of this inequality may be found in [7]; for \( s = \infty \) the inequality is considered in the paper of Hollenbeck and Verbitsky [5].

Lemma 2.1. For all complex numbers \( z \) and \( w \) and \( 1 < p \leq 2 \), and \( 0 < s \leq 2 \) there holds

\[
- \left( \frac{|z|^s + |w|^s}{2} \right)^\frac{s}{p} + \left( \frac{|z + w|^p}{2p \cos^p \frac{\pi}{2p}} \right) - \tan \frac{\pi}{2p} \Re(zw) \geq 0.
\]

Proof. Since of homogeneity, if we divide it by \( \max\{|z|^p, |w|^p\} \) and if we write the number \( \frac{z}{|z|^2} \) in the polar form, we obtain that we have to prove only the following: for \( 0 < s \leq 2 \), \( 1 < p \leq 2 \), \( 0 \leq r \leq 1 \) and \( -\pi \leq t \leq \pi \) there holds

\[
- \left( \frac{1}{2} + r^s \right)^\frac{s}{p} + \left( \frac{(1 + r^2 + 2r \cos t)^p}{2p \cos^p \frac{\pi}{2p}} \right) - r^t \tan \frac{\pi}{2p} \cos \frac{tp}{2} \geq 0.
\]

Let us denote the left side by \( \Phi(r, t) \). For fixed \( 0 < s \leq 2 \), we will found critical points of this function in the interior of \((r, t) \in [0, 1] \times [0, \pi] \) (since \( \Phi \) is even in \( t \))
as zeroes of the partial derivatives with respect to \( r \) and \( t \). We have the following equations:

\[
\frac{\partial \Phi}{\partial r}(r, t) = \frac{-p}{2} r^{s-1} \left( \frac{1 + r^s}{2} \right)^{\frac{s}{2} - 1} + \frac{p(r + \cos t)}{2p \cos^p \frac{\pi}{2p}} (1 + r^2 + 2r \cos t)^{\frac{s}{2} - 1} - \frac{p}{2} r^\frac{s}{2} \tan \frac{\pi}{2p} \cos \frac{tp}{2} = 0
\]

and

\[
\frac{\partial \Phi}{\partial t}(r, t) = \frac{-pr \sin t}{2p \cos^p \frac{\pi}{2p}} (1 + r^2 + 2r \cos t)^{\frac{s}{2} - 1} + \frac{p}{2} r^\frac{s}{2} \tan \frac{\pi}{2p} \sin \frac{tp}{2} = 0.
\]

For these two equations we obtain

\[-r^{s-1} \left( \frac{1 + r^s}{2} \right)^{\frac{s}{2} - 1} + 2(r + \cos t) \frac{r^\frac{s}{2} \tan \frac{\pi}{2p} \sin \frac{tp}{2}}{2p \cos^p \frac{\pi}{2p}} (1 + r^2 + 2r \cos t)^{\frac{s}{2} - 1} - r^\frac{s}{2} \tan \frac{\pi}{2p} \cos \frac{tp}{2} = 0\]

and

\[
\frac{(1 + r^2 + 2r \cos t)^{\frac{s}{2} - 1}}{2p r^\frac{s}{2} \cos^p \frac{\pi}{2p} \tan \frac{\pi}{2p}} = \frac{\sin \frac{tp}{2}}{2r \sin t}.
\]

Now, using the second equality in the first equation, we obtain

\[-r^s \left( \frac{1 + r^s}{2} \right)^{\frac{s}{2} - 1} + \frac{(r + \cos t) r^\frac{s}{2} \tan \frac{\pi}{2p} \sin \frac{tp}{2}}{\sin t} - r^\frac{s}{2} \tan \frac{\pi}{2p} \cos \frac{tp}{2} \sin t = 0\]

\[-r^s \left( \frac{1 + r^s}{2} \right)^{\frac{s}{2} - 1} + \frac{r^\frac{s}{2} \tan \frac{\pi}{2p} \sin \frac{tp}{2}}{\sin t} (r + \cos t) \frac{\sin \frac{tp}{2}}{\sin t} = 0\]

\[-r^s \left( \frac{1 + r^s}{2} \right)^{\frac{s}{2} - 1} + \frac{r^\frac{s}{2} \tan \frac{\pi}{2p} \sin \frac{tp}{2}}{\sin t} (r \frac{\sin \frac{tp}{2}}{\sin t} - \sin \left( t - \frac{tp}{2} \right)) = 0.\]

In the critical points the function \( \Phi \) has the value

\[
\Phi(r, t) = \frac{-r^{h/2} \tan \frac{\pi}{2p} \sin \frac{tp}{2}}{\sin t} \left( r \frac{\sin \frac{tp}{2}}{\sin t} - \sin \left( t - \frac{tp}{2} \right) \right) + \frac{1 + r^s}{2r^s} \frac{1}{2p} \cos \frac{tp}{2} + \frac{r^\frac{s}{2} \tan \frac{\pi}{2p} (1 + r^2 + 2r \cos t)}{2r \sin t} \frac{\sin \frac{tp}{2}}{\sin t} - r^\frac{s}{2} \tan \frac{\pi}{2p} \cos \frac{tp}{2} \sin t \]

\[
= \frac{r^\frac{s}{2} \tan \frac{\pi}{2p} (1 + r^2 + 2r \cos t)}{2r \sin t} \left( -1 + r^s \frac{\sin \frac{tp}{2}}{\sin t} - \sin \left( t - \frac{tp}{2} \right) \right) + \frac{r^\frac{s}{2} \tan \frac{\pi}{2p} (1 + r^2 + 2r \cos t)}{2r \sin t} \left( -1 + r^s \frac{\sin \frac{tp}{2}}{\sin t} - \sin \left( t - \frac{tp}{2} \right) \right) + \frac{r^\frac{s}{2} \tan \frac{\pi}{2p} (1 + r^2 + 2r \cos t)}{2r \sin t} \left( -1 + r^s \frac{\sin \frac{tp}{2}}{\sin t} - \sin \left( t - \frac{tp}{2} \right) \right) + \frac{r^\frac{s}{2} \tan \frac{\pi}{2p} (1 + r^2 + 2r \cos t)}{2r \sin t} \left( -1 + r^s \frac{\sin \frac{tp}{2}}{\sin t} - \sin \left( t - \frac{tp}{2} \right) \right).
\]

In the critical points we have \( \Phi(r, t) \geq 0 \) and only if there holds the inequality

\[\sin \left( t - \frac{tp}{2} \right) \left( \frac{1 + r^s}{r^s - 1} - 2r \right) + \sin \frac{tp}{2} \left( 1 + r^2 - \frac{1 + r^s}{r^s - 1} \right) \geq 0.\]
If we multiply this inequality by \( r^s \) we obtain an equivalent one
\[
\sin \left( t - \frac{tp}{2} \right) (r - r^{s+1}) + \sin \frac{tp}{2} (r^s - r^2) \geq 0.
\]
But this inequality is obviously true for \( 0 < s < 2 \), since we have
\[
r - r^{s+1} \geq 0, \quad \sin \left( t - \frac{tp}{2} \right) \geq 0, \quad \sin \frac{tp}{2} \geq 0, \quad r^s - r^2 \geq 0
\]
if the numbers \( p, r \) and \( t \) belong to the given intervals.

We need now to check the boundary points. If we consider the function \( \Phi(r, t) \) for \( t \in (-\pi - \epsilon, \pi + \epsilon) \) for some \( \epsilon > 0 \), we see that \( t = \pi \) satisfies \( \frac{\partial \Phi}{\partial t} = 0 \), only for \( r = 0 \) (which will be consider later) or for \( p = 2 \), when we obtain \( \Phi(r, \pi) = -\left(1 + \frac{r^s}{2}\right) + 2 - r \geq 0 \).

For \( t = 0 \), the inequality we want to prove is
\[
-\left(\frac{1 + r^s}{2}\right) + \frac{(1 + r)^p}{2p \cos^p \frac{\pi}{2p}} - r^\frac{\pi}{2p} \tan \frac{\pi}{2p} \geq 0.
\]
Because of
\[
-\left(\frac{1 + r^s}{2}\right) + \frac{(1 + r)^p}{2p \cos^p \frac{\pi}{2p}} - r^\frac{\pi}{2p} \tan \frac{\pi}{2p} \geq 0
\]
\[
-\left(\frac{1 + r^s}{2}\right) + \frac{(1 + r)^p}{2p \cos^p \frac{\pi}{2p}} - r^\frac{\pi}{2p} \tan \frac{\pi}{2p} = \hat{F}(r),
\]
it is enough to show that \( \hat{F}(r) \geq 0 \), which is equivalent to
\[
F(r) = \frac{\hat{F}(r)}{(1 + r)^p} = -\left(\frac{1 + r^2}{2(1 + r)^2}\right)^{p/2} + \frac{1}{2p \cos^p \frac{\pi}{2p}} - \left(\frac{r}{(1 + r)^2}\right)^{p/2} \tan \frac{\pi}{2p} \geq 0.
\]
We will prove that \( F \) is decreasing. From \( F(r) \geq F(1) \), i.e., from \( F(1) \geq 0 \) (this may be considered as \( r = 1 \)) follows the desired conclusion. Indeed,
\[
F'(r) = \frac{p(r - 1)}{2(1 + r)^3} \left( -\left(\frac{1 + r^2}{2(1 + r)^2}\right)^{\frac{p}{2}} + \left(\frac{r}{(1 + r)^2}\right)^{\frac{p}{2}} \tan \frac{\pi}{2p} \right) \leq 0,
\]
since
\[
-\left(\frac{1 + r^2}{2(1 + r)^2}\right)^{\frac{p}{2}} + \left(\frac{r}{(1 + r)^2}\right)^{\frac{p}{2}} \tan \frac{\pi}{2p} \geq 0
\]
is equivalent to
\[
\left(\frac{1 + r^2}{2r}\right)^{\frac{p}{2}} \leq \tan \frac{\pi}{2p},
\]
which is correct, since for considered values of \( r \) and \( p \) holds \( \tan \frac{\pi}{2p} \geq 1 \) and \( \frac{1 + r^2}{2r} \geq 1 \), while \( \frac{p}{2} - 1 \leq 0 \).

On the other hand, for \( r = 0 \) we have
\[
-2^{-\frac{s}{2}} + \frac{1}{2p \cos^p \frac{\pi}{2p}} \geq 0,
\]
which follows from \( \frac{1}{2 \cos \frac{\pi}{2p}} \geq \frac{1}{\sqrt{2}} \geq 2^{-\frac{s}{2}} \) for \( 0 < s \leq 2 \).
For $r = 1$ we have the inequality

$$-1 + \left( \frac{\cos \frac{t}{2}}{\cos \frac{\pi}{2p}} \right)^p - \tan \frac{\pi}{2p} \cos \frac{pt}{2} \geq 0,$$

whose proof can be found in [7] or [18]. □

**Proof of Theorem 1.1.** If we now apply the above lemma for the $z = P_+ f(\zeta)$ and $w = P_- f(\zeta)$, we obtain

$$- \left( \frac{|P_+ f(\zeta)|^s + |P_- f(\zeta)|^s}{2} \right)^\frac{p}{s} + \frac{|P_+ f(\zeta) + P_- f(\zeta)|^p}{2^p \cos \frac{\pi}{2p}}$$

$$- \tan \frac{\pi}{2p} \Re(P_+ f(\zeta) P_- f(\zeta))^\frac{p}{s} \geq 0,$$

i.e.,

$$\left( |P_+ f(\zeta)|^s + |P_- f(\zeta)|^s \right)^\frac{p}{s} \leq \frac{2^p}{2^p \cos \frac{\pi}{2p}} |f(\zeta)|^p$$

$$- 2^p \tan \frac{\pi}{2p} \Re(P_+ f(\zeta) P_- f(\zeta))^\frac{p}{s}.$$

If we integrate the above inequality over $T$, and if we have in mind the inequality

$$\int_T \Re(P_+ f(\zeta) P_- f(\zeta))^\frac{p}{s} \geq 0$$

(because of subharmonicity) we obtain

$$\int_T \left( |P_+ f(\zeta)|^s + |P_- f(\zeta)|^s \right)^\frac{p}{s} \leq \frac{2^p}{2^p \cos \frac{\pi}{2p}} \int_T |f(\zeta)|^p,$$

which is the statement of the theorem. □

**Remark 2.2.** The inequality in Theorem [14] may be derived from Kalaj’s result [7]. The proof above is independent of the case $s = 2$ and its includes the simplification and generalisation of the approach from the Kalaj’s paper.

If we use inequalities between means of order $s$ and of order 2, for $0 < s \leq 2$ we obtain:

$$\left( \frac{|P_+ f(\zeta)|^s + |P_- f(\zeta)|^s}{2} \right)^\frac{p}{s} \leq \left( \frac{|P_+ f(\zeta)|^2 + |P_- f(\zeta)|^2}{2} \right)^\frac{p}{2},$$

i.e.,

$$\left( |P_+ f(\zeta)|^s + |P_- f(\zeta)|^s \right)^\frac{p}{s} \leq 2^p \left( |P_+ f(\zeta)|^2 + |P_- f(\zeta)|^2 \right)^\frac{p}{2}.$$

If we integrate the last inequality, we obtain

$$\int_T \left( |P_+ f(\zeta)|^s + |P_- f(\zeta)|^s \right)^\frac{p}{s} |d\zeta| \leq 2^p \left( \int_T |P_+ f(\zeta)|^2 + |P_- f(\zeta)|^2 \right)^\frac{p}{2} |d\zeta|.$$

Using the result for $s = 2$, i.e., the inequality

$$\int_T \left( |P_+ f(\zeta)|^2 + |P_- f(\zeta)|^2 \right)^\frac{p}{2} |d\zeta| \leq \frac{2^p}{2^p \cos \frac{\pi}{2p}} \int_T |f(\zeta)|^p |d\zeta|,$$

we obtain

$$\left( \int_T \left( |P_+ f(\zeta)|^s + |P_- f(\zeta)|^s \right)^\frac{p}{s} |d\zeta| \right)^\frac{1}{p} \leq \frac{2^p}{2 \cos \frac{\pi}{2p}} \left( \int_T |f(\zeta)|^p |d\zeta| \right)^\frac{1}{p}.$$
Similarly we act in the case $2 < p < \infty$. As in the case $1 < p \leq 2$, we will apply Theorem 2.1 from [7]:

$$
\left( \int_\mathcal{T} \left( |P_+ f(\zeta)|^2 + |P_- f(\zeta)|^2 \right)^\frac{p}{2} \, |d\zeta| \right)^\frac{1}{p} \leq \frac{2^{\frac{1}{p}}}{2 \sin \frac{\pi}{2p}} \left( \int_\mathcal{T} |f(\zeta)|^p |d\zeta| \right)^\frac{1}{p}.
$$

If we use again the inequality between means of order $s$ and $2$ we obtain

$$
\int_\mathcal{T} \left( |P_+ f(\zeta)|^s + |P_- f(\zeta)|^s \right)^\frac{p}{2s} |d\zeta| \leq 2^{\frac{s}{p} - \frac{p}{2s}} \int_\mathcal{T} \left( |P_+ f(\zeta)|^2 + |P_- f(\zeta)|^2 \right)^\frac{p}{2} |d\zeta|
$$

which is the statement of the theorem.

3. LOWER ESTIMATES OF THE CONSTANTS – THE FAMILY OF TEST FUNCTIONS

In this section we give the estimates from below for constants in the Riesz projection inequalities for all $0 < s < \infty$ and $1 < p < \infty$. This will prove the optimality of constants in the previous section.

In this section we will estimate the constants $C_{s,p}$ and $\tilde{C}_{s,p}$ in the inequalities

$$
\| (|P_+ f|^s + |P_- f|^s)^\frac{1}{s} \|_{L^p(\mathcal{T})} \leq C_{s,p} \| f \|_{L^p(\mathcal{T})}
$$

and

$$
\| f \|_{L^p(\mathcal{T})} \leq \tilde{C}_{s,p} \| (|P_+ f|^s + |P_- f|^s)^{1/s} \|_{L^p(\mathcal{T})}
$$

from below.

We will use the family of test functions defined by

$$
f_{\gamma} = \alpha \Re g_{\gamma} + i \beta \Im g_{\gamma},
$$

where $g_{\gamma}(z) = \left( \frac{1 + z}{1 - z} \right)^{\frac{1}{p}}$. Note that $|\Re g_{\gamma}| = \tan |\Im g_{\gamma}|$. Hence, for $\gamma$ tends to $\frac{\pi}{2}$, we have:

$$
\frac{\| (|P_+ f|^s + |P_- f|^s)^\frac{1}{s} \|_{L^p(\mathcal{T})}}{\| f \|_{L^p(\mathcal{T})}} = \frac{(|\alpha + \beta|^s + |\alpha - \beta|^s)^\frac{1}{s}}{2 \left( \alpha^2 \cos^2 \frac{\pi}{2p} + \beta^2 \sin^2 \frac{\pi}{2p} \right)^\frac{1}{2}}.
$$

Therefore, we easily conclude that

$$
C_{s,p} \geq \sup_{\alpha, \beta \in \mathbb{R}} \frac{(|\alpha + \beta|^s + |\alpha - \beta|^s)^\frac{1}{s}}{2 \left( \alpha^2 \cos^2 \frac{\pi}{2p} + \beta^2 \sin^2 \frac{\pi}{2p} \right)^\frac{1}{2}}
$$

while

$$
\tilde{C}_{s,p} \geq \sup_{\alpha, \beta \in \mathbb{R}} \frac{2 \left( \alpha^2 \cos^2 \frac{\pi}{2p} + \beta^2 \sin^2 \frac{\pi}{2p} \right)^\frac{1}{2}}{(|\alpha + \beta|^s + |\alpha - \beta|^s)^\frac{1}{2}} = \left( \inf_{\alpha, \beta \in \mathbb{R}} \frac{(|\alpha + \beta|^s + |\alpha - \beta|^s)^\frac{1}{s}}{2 \left( \alpha^2 \cos^2 \frac{\pi}{2p} + \beta^2 \sin^2 \frac{\pi}{2p} \right)^\frac{1}{2}} \right)^{-1}.
$$

From these inequalities we observe that it is enough to analyze the function

$$
T(\alpha, \beta) = \frac{(|\alpha + \beta|^s + |\alpha - \beta|^s)^\frac{1}{s}}{2 \left( \alpha^2 \cos^2 \frac{\pi}{2p} + \beta^2 \sin^2 \frac{\pi}{2p} \right)^\frac{1}{2}}.
$$
This function is even in both variables, and we can assume, without loss of generality, that $|\alpha + \beta| \geq |\alpha - \beta|$. Also, it is homogeneous, and we can set $\alpha + \beta = 1$ and $\alpha - \beta = t$, with $|t| \leq 1$, by the previous assumption. Therefore, we get:

$$\frac{(|\alpha + \beta|^2 + |\alpha - \beta|^2)^{\frac{1}{2}}}{2\left(\alpha^2 \cos^2 \frac{\pi}{2p} + \beta^2 \sin^2 \frac{\pi}{2p}\right)^{\frac{1}{2}}} = \frac{(1 + |t|^2)^{\frac{1}{2}}}{2\left((\frac{1+1}{2})^2 \cos^2 \frac{\pi}{2p} + (\frac{1+1}{2})^2 \sin^2 \frac{\pi}{2p}\right)^{\frac{1}{2}}} = \frac{(1 + |t|^2)^{\frac{1}{2}}}{\sqrt{1 + t^2 + 2t \cos \frac{\pi}{p}}}$$

Hence, the range of the function $T(\alpha, \beta)$ is the union of the values of the expressions:

$$\frac{(1 + t^*)^{\frac{1}{2}}}{\sqrt{1 + t^2 + 2t \cos \frac{\pi}{p}}}$$
and

$$\frac{(1 + t^*)^{\frac{1}{2}}}{\sqrt{1 + t^2 - 2t \cos \frac{\pi}{p}}}$$
for $0 \leq t \leq 1$.

Since, for $1 < p \leq 2$:

$$\sqrt{1 + t^2 - 2t \cos \frac{\pi}{p}} > \sqrt{1 + t^2 + 2t \cos \frac{\pi}{p}},$$

we conclude:

$$C_{s,p} \geq \sup_{t \in [0,1]} \frac{(1 + t^*)^{\frac{1}{2}}}{\sqrt{1 + t^2 + 2t \cos \frac{\pi}{p}}}$$
and

$$\tilde{C}_{s,p} \geq \left(\inf_{t \in [0,1]} \frac{(1 + t^*)^{\frac{1}{2}}}{\sqrt{1 + t^2 - 2t \cos \frac{\pi}{p}}}\right)^{-1}.$$  

Considering $F(t) = \frac{(1 + t^*)^{2}}{(1 + t^2 + 2t \cos \frac{\pi}{p})}$ we find: $F'(t) = \frac{2s(1 + t^*)}{(1 + t^2 + 2t \cos \frac{\pi}{p})^{2}} \left[t^{s-1} - t + \cos \frac{\pi}{p}(t^s - 1)\right]$, and, therefore: $\text{sgn}F'(t) = \text{sgn} \Phi(t)$, where $\Phi(t) = t^{s-1} - t + \cos \frac{\pi}{p}(t^s - 1)$.

From $\Phi'(t) = (s-1)t^{s-2} - 1 + st^{s-1} \cos \frac{\pi}{p}$ and $\Phi''(t) = (s-1)t^{s-3} \left[ts \cos \frac{\pi}{p} + s - 2\right]$ we see that, in fact, we differ next two cases:

1) If $s \cos \frac{\pi}{p} + s - 2 \geq 0$, then $\Phi''(t) > 0$, $\Phi'$ is increasing and takes the values between $\Phi'(0) = -1$ and $\Phi'(1) = s - 2 + s \cos \frac{\pi}{p} \geq 0$, so there is an $t_0 \in (0,1)$ such that $\Phi'(t_0) = 0$. Thus $\Phi$ decreases on $(0,t_0)$ and increases on $(t_0,1)$, has the minimum in $t_0$ smaller than zero ($\Phi(1) = 0$), and consequently $\Phi$ is positive on $(0,\hat{t})$ and negative on $(\hat{t},1)$. So, $F$ is increasing from $0$ to $\hat{t}$, decreasing from $\hat{t}$ to $1$, therefore $\max_{t \in [0,1]} F(t) = F(\hat{t})$, where $\hat{t}$ is a unique solution of the equation: $t^{s-1} - t + \cos \frac{\pi}{p}(t^s - 1) = 0$, for $s > 2$ and $t \in (0,1)$.

2) If $s \cos \frac{\pi}{p} + s - 2 < 0$, then for $t_1 = \frac{s}{s \cos \frac{\pi}{p}}$, $\Phi'$ has the zero. Consequently $\Phi'$ increases on $[0,t_1]$ and decreases on $[t_1,1]$, so $\Phi' < 0$ and $\Phi$ decreases, which
gives us $\Phi(t) \geq \Phi(1) = 0$. Therefore, in this case, $F$ increases and $\max_{t \in [0,1]} F(t) = F(1) = \frac{4^{1-s}}{\cos^s \frac{\pi}{p}}$.

Analogously to the above considerations, to prove the appropriate statement for $C_{s,p}$, we consider the function $G(t) = \frac{(1+t^p)^2}{(1+t^{2p})^{1+\frac{1}{2t}}}$ for $0 \leq t \leq 1$. We easily calculate $F'(t) = \frac{2s(1+t^p)}{(1+t^{2p})^{1+\frac{1}{2t}}}[t^{s-1}-t+\cos \frac{\pi}{p}(1-t^p)]$ and conclude $\text{sgn} G'(t) = \text{sgn} \Psi(t)$, where $\Psi(t) = ts^{-1}-t+\cos \frac{\pi}{p}(1-t^p)$. Now, from $\Psi'(t) = (s-1)t^{s-2} - st^{s-1}\cos \frac{\pi}{p}$ and $\Psi''(t) = (s-1)t^{s-3}(s-2-ts\cos \frac{\pi}{p})$ we make difference between these three cases:

1) For $\frac{1}{\cos \frac{\pi}{p}} < s < 2$ we have $s-2-ts\cos \frac{\pi}{p} > 0$ and also $s > 1$, so $s-2-ts\cos \frac{\pi}{p}$ increases on $t$ and, since $s-2 < 0$ and $s-2-s\cos \frac{\pi}{p}$ we find that for $t_0 = \frac{s-2}{s-2-s\cos \frac{\pi}{p}}$ we have $\Psi''(t) < 0$ for $t \in (0, t_0)$ and $\Psi''(t) > 0$ for $t \in (t_0, 1)$. Hence, $\Psi'$ decreases on $(0, t_0)$ and increases on $(t_0, 1)$, so $\Psi'(t_0)$ is the minimum of $\Psi'$ on $(0, 1)$. But, $\Psi'(t_0) = (s-1)t_0^{s-2} - 1 - st_0^{s-1}\cos \frac{\pi}{p} = (s-1)t_0^{s-2} - 1 - (s-2)t_0^{s-2} = t_0^{s-2} - 1 > 0$, since $s < 2$ and $t_0 < 1$. This means that $\Psi(t) \geq \Psi(t_0) > 0$, so $\Psi$ increases and $\Psi(t) \leq \Psi(1) = 0$, which implies that $G$ has its minimum for $t = 1$ equal to $G(1) = \frac{4^{1-s}}{\sin^{s-1} \frac{\pi}{p}}$ and $C_{s,p} \geq 2^{1-s} \sin \frac{\pi}{2p}$.

2) For $1 < s < \frac{1}{\cos \frac{\pi}{p}}$ the expression $s-2-s\cos \frac{\pi}{p}$ is negative which gives $\Psi''(t) < 0$ and $\Psi'$ decreases. $\Psi'(1) = s-2-s\cos \frac{\pi}{p} < 0$, while $\lim_{t \to 0^+} \Psi'(t) = +\infty$ so there exists a unique $t_0$ such that $\Psi'(t_0) = 0$. Hence, $\Psi(t)$ increases on $(0, t_0)$ and decreases on $(t_0, 1)$. Since $\Psi(0) = \cos \frac{\pi}{p} < 0$, $\Psi(1) = 0$ we conclude $\Psi(t_0) > 0$ and $\Psi$ has exactly one zero $t^*$ in which $G(t)$ has its minimum. Therefore, for such the $t$ we have $C_{s,p} \geq \frac{1}{\cos \frac{\pi}{p}}$.

3) For $s < 1$ we have $\Psi''(t) > 0$ for every $t \in [0, 1]$, so $\Psi'$ increases, $\Psi'(t) \leq \Psi'(1) = 0$ and $\Psi$ decreases. But, $\Psi(1) = 0$ so $\Psi(t) \geq \Psi(1) = 0$, $G$ is increasing and has minimum in zero. In this case, we conclude $C_{s,p} \geq 1$.

The rest easily follows from the simple observation: If $p > 2$, then $\cos \frac{\pi}{p} = -\cos \frac{\pi}{p}$ for dual exponent $p' = \frac{p}{p-1} < 2$. Consequently, we can change the appropriate signs and functions $\sin$ and $\cos$.

4. **ASYMPTOTICS OF THE CONSTANTS $C_{s,p}$ FOR LARGE $s$**

From our considerations from the previous subsection we conclude that:

$$C_{s,p} \geq \frac{(1+\hat{t})^{\frac{1}{2}}}{\sqrt{1+\hat{t}^2 + 2\hat{t}\cos \frac{\pi}{p}}}$$

where $\hat{t} \in (0, 1)$ is a unique zero of the equation $t^{s-1} - t + \cos \frac{\pi}{p}(t^s - 1) = 0$, for $s$ big enough and $1 < p < 2$.

Also, analyzing the case $s \cos \frac{\pi}{p} + s - 2 \geq 0$, we see that $\Phi(-\cos \frac{\pi}{p}) = (s-1)\cos \frac{\pi}{p}|s^{-2}-1+s|\cos \frac{\pi}{p}|s^{-1}-\cos \frac{\pi}{p}|s^{-2}(s-1-s\cos^2 \frac{\pi}{p}) > 0$, since $s-1-s\cos^2 \frac{\pi}{p} = s-2-s\cos^2 \frac{\pi}{p} + 1-s(\cos^2 \frac{\pi}{p} + \cos \frac{\pi}{p}) \geq 1$, because of $s \cos \frac{\pi}{p} + s-2 \geq 0$ and $\cos \frac{\pi}{p} > 0$. Therefore, $\Phi$ is positive on $(0, \hat{t})$ and negative on $(\hat{t}, -\cos \frac{\pi}{p})$, consequently $\hat{t} < |\cos \frac{\pi}{p}|$. Hence, $\hat{t} = -\cos \frac{\pi}{p} + \hat{t}^{-1}(1+\hat{t}\cos \frac{\pi}{p})$ and $\hat{t} \to -\cos \frac{\pi}{p}$,
as \( s \to \infty \), which gives:

\[
\liminf_{s \to \infty} C_{s,p} \geq \frac{1}{\sin \frac{\pi}{p}}.
\]

To obtain the inequality for the same limit from the above, we evoke the following estimate from [5]:

\[
\max\{|z|^p, |w|^p\} \leq 1 \sin \frac{\pi}{p} \|z + \overline{w}|^p - b_p \Re(zw)^\frac{p}{2}.
\]

We choose an arbitrary small \( \epsilon > 0 \) and multiplies both sides by \((1 + \epsilon)^p\), we get:

\[
(1 + \epsilon)^p \max\{|z|^p, |w|^p\} \leq \frac{(1 + \epsilon)^p}{\sin \frac{\pi}{p}} |z + \overline{w}|^p - b_p (1 + \epsilon)^p \Re(zw)^\frac{p}{2}.
\]

Choosing \( s \) such that \((1 + \epsilon)^s \geq 2\), we have:

\[
(\|z\|^s + |w|^s)^\frac{p}{s} \leq 2^\frac{p}{s} \max\{|z|^p, |w|^p\} \leq (1 + \epsilon)^p \max\{|z|^p, |w|^p\}.
\]

Hence, we obtain:

\[
(\|z\|^s + |w|^s)^\frac{p}{s} \leq \frac{(1 + \epsilon)^p}{\sin \frac{\pi}{p}} |z + \overline{w}|^p - b_p (1 + \epsilon)^p \Re(zw)^\frac{p}{2},
\]

from which, after integrating with \( z = P_+ f, w = P_- f \) we get:

\[
\limsup_{s \to \infty} C_{s,p} \leq \frac{1 + \epsilon}{\sin \frac{\pi}{p}}.
\]

Hence, we obtain

\[
\lim_{s \to \infty} C_{s,p} = \frac{1}{\sin \frac{\pi}{p}},
\]

as conjectured in [6].

Let us note that from the preceding consideration it follows that if we let \( s \to \infty \) we obtain an inequality which is dual to the inequality of Hollenbeck and Verbitsky [5]

\[
\|f\|_{L^p(T)} \leq 2 \max\{\sin \frac{\pi}{2^p}, \cos \frac{\pi}{2^p}\} \|\max\{|P_+ f|^s, |P_- f|^s\}|\|.
\]

Proof in the case \( p > 2 \) is quite similar.

5. Sharp constants in reverse inequalities

We mention earlier that as a consequence of the Banach open mapping theorem we get the existence of constants \( C_{s,p} \) such that

\[
\|f\|_{L^p(T)} \leq C_{s,p} \|\max\{|P_+ f|^s, |P_- f|^s\}|^{\frac{1}{s}}.\]

In this section we prove that their values are equal to those conjectured in the third section for \( s \in \mathbb{R} \setminus (1, 2) \).

First, we discuss the case of \( 0 < s \leq 1 \). We have:

\[
(1 + 2r \cos t + r^2)^\frac{p}{2} \leq (1 + 2r + r^2)^\frac{p}{2} = (1 + r)^p \leq (1 + r^s)^\frac{p}{s},
\]

where the last inequality is a consequence of concavity of the function \( f(x) = x^s \), for this range of \( s \).

This implies \( |z + \overline{r}|^p \leq (|z|^s + |w|^s)^\frac{p}{s} \) and consequently:

\[
\|f\|_{L^p(T)} \leq \|\max\{|P_+ f|^s, |P_- f|^s\}|^{\frac{1}{s}}\|_{L^p(T)},
\]

as conjectured in [6].
for every $1 < p < \infty$.

The constant is sharp as the section 4 shows. It can also be seen by very simple example of function $f(z) = 1$.

Now, we briefly discuss more complex case of $s \geq 2$. We see that using similar approach as in the inequalities in the third section, we can prove the appropriate "elementary" inequality using power means and result from [7], which will turned out to be sharp. But, along with it, we give a self-contained proof of it by inspecting properties of the possible stationary points and estimates for boundary points.

We start from the following lemma:

**Lemma 5.1.** For all complex numbers $z$ and $w$ and $1 < p \leq 2$, $s \geq 2$ there holds

$$|z+w|^p \leq 2^p \sin^p \frac{\pi}{2p} \left( \frac{|z|^s + |w|^s}{2} \right)^{\frac{p}{s}} - 2^{\frac{3p}{2} - \frac{p}{2}} \cos \frac{\pi}{2p} \sin^{p-1} \frac{\pi}{2p} |zw|^\frac{s-1}{s} \cos \frac{(\pi - |t + u|)p}{2},$$

where $z = |z| e^{i\theta}$, $w = |w| e^{i\varphi}$.

**Proof.** Since of homogeneity of the expression, we can assume that $|z| = r < 1 = w$, therefore we have to prove:

$$\Phi(r, t) = (1 + 2 \cos t + r^2)^{\frac{p}{s}} - 2^p \sin^p \frac{\pi}{2p} \left( \frac{1 + r^s}{2} \right)^{\frac{p}{s}} \cos \frac{\pi}{2p} \sin^{p-1} \frac{\pi}{2p} |zw|^\frac{s-1}{s} \cos \frac{(\pi - |t + u|)p}{2},$$

for $0 \leq r \leq 1$, $0 \leq t \leq \pi$.

Denote the expression from the left side by $\Phi(r, t)$. We easily get that the equations $\frac{\partial \Phi(r, t)}{\partial r} = 0$ and $\frac{\partial \Phi(r, t)}{\partial t} = 0$ are equivalent with

$$2(r + \cos t)(1 + 2 \cos t + r^2)^{\frac{p}{s}} - 2^p \sin^p \frac{\pi}{2p} \left( \frac{1 + r^s}{2} \right)^{\frac{p}{s}} r^{s-1} \cos \frac{\pi}{2p} \sin^{p-1} \frac{\pi}{2p} \cos \frac{(\pi - |t + u|)p}{2} = 0,$$

and

$$\frac{(1 + r^2 + 2 \cos t)^{p/2} - 2^p \sin^p \frac{\pi}{2p} \cos \frac{\pi}{2p} \cos \frac{(\pi - |t + u|)p}{2}}{2^p \sin t} = \frac{\sin \frac{(\pi - t)p}{2}}{2^p \sin t}.$$

If we express $(1 + r^2 + 2 \cos t)^{p/2} - 2^p \sin^p \frac{\pi}{2p} \cos \frac{\pi}{2p} \cos \frac{(\pi - |t + u|)p}{2}$ from the second equation and put it in the first, we have:

$$\left( \frac{1 + r^s}{2} \right)^{\frac{p}{s}} - 2^p \sin^p \frac{\pi}{2p} \cos \frac{\pi}{2p} \sin^{p-1} \frac{\pi}{2p} (r + \cos t) \cos \frac{(\pi - |t + u|)p}{2} = 0.$$

Now, using the last two equations, in the similar manner as in the third section, we get:

$$\Phi(r, t) = \frac{2^p \sin^p \frac{\pi}{2p} \cos \frac{\pi}{2p} \sin^{p-1} \frac{\pi}{2p}}{2r \sin t} \times \left[ \left( 1 - r^{2-s} \right) \sin \frac{(\pi - t)p}{2} + r \left( 1 - r^{-s} \right) \sin \frac{(\pi p/2 + t(1 - p/2))}{2} \right],$$

which is evidently non-positive for $r \in (0, 1)$ and $t \in (0, \pi)$.
The rest is devoted to the analysis of $\Phi(r, t)$ on boundary points.

For $t = 0$, we easily see that $\frac{\partial \Phi}{\partial t}(r, 0) = 2^p r^p \cos \frac{\pi}{2p} \sin \frac{\pi^p}{2p} \sin^{p-1} \frac{\pi}{2p} > 0$ for $r > 0$.

For $r = 0$, we have $\Phi(0, t) = 1 - 2^p r^p \sin^{1/p} \frac{\pi}{2p} \leq 0$, easily seen to be true.

For $r = 1$, we see that

$$\Phi(1, t) = 2^p \sin^p \frac{\pi}{2p} \left( \frac{\cos \frac{\pi}{2} y}{\sin^p \frac{\pi}{2p}} - 1 \right) + 2 \frac{\pi}{2p} \cot \frac{\pi}{2p} \left( \frac{(\pi - t)p}{2} \right) \leq 0$$

is, after change of variable $y = \frac{\pi - t}{2}$, equivalent with

$$\tilde{F}'(y) = \frac{\sin^p y}{\sin^p \frac{\pi}{2p}} - 1 + 2 \frac{\pi}{2p} \cot \frac{\pi}{2p} \cos py \leq 0.$$

It is enough to prove that $\frac{\sin^p y}{\sin^p \frac{\pi}{2p}} - 1 + \cot \frac{\pi}{2p} \cos py \leq 0$, since the $s \geq 2$ and $\cos py \leq 0$, for $1 \leq p \leq 2$ and $y \in [0, \frac{\pi}{2}]$. We will prove that the function $F(y) = \frac{1 - \cot \frac{\pi}{2p} \cos py}{\sin^p y}$ has its minimum in $t = \frac{\pi}{2p}$ equal to $F(\frac{\pi}{2p}) = \frac{1}{\sin^p \frac{\pi}{2p}}$. We find that

$$F'(y) = \frac{p}{\sin^p \frac{\pi}{2p}} \left[ \cot \frac{\pi}{2p} \cos p(\pi - t) - \cos y \right].$$

If we consider $G(y) = \cot \frac{\pi}{2p} \cos (p - 1)y - \cos y$, then we see that $G'(y) = -(p - 1) \cot \frac{\pi}{2p} \sin (p - 1)y + \sin y \geq \sin y - \sin (p - 1)y \geq 0$, since $\sin y$ is monotone increasing on $[0, \frac{\pi}{2}]$, $1 < p < 2$ and $(p - 1) \cot \frac{\pi}{2p} \leq 1$.

Therefore, $G$ increases and, for $y \in [0, \frac{\pi}{2p}]$ we have $G \leq 0$, while for $y \in [\frac{\pi}{2p}, \frac{\pi}{2p}]$ it holds $G \geq 0$. This means that $F$ has its minimum in $t = \frac{\pi}{2p}$.

For $t = \pi$, we have $\Phi(r, \pi) = 2^p r^p \sin^p \frac{\pi}{2p} \left[ \frac{1 + r^2}{\sin^p \frac{\pi}{2p}} - \cot \frac{\pi}{2p} \right]$, and after introducing $y = \frac{1 + r^2}{2r} \geq 1$ and $F(y) = \frac{p - 2}{\sin^p \frac{\pi}{2p}} - \frac{\pi}{2p} + y^p$, we find

$$F'(y) = \frac{p(2y - 2)}{\sin^p \frac{\pi}{2p}} - \frac{\pi}{2p} + y^p - 1 = \frac{p}{\sin^p \frac{\pi}{2p}} \left( \frac{2}{\sin^p \frac{\pi}{2p}} \left( 1 + \frac{1}{2(y - 1)} \right) - 1 \right) \geq 0,$$

since $x^{-\frac{1}{2}}$ is monotone increasing, $\left( \frac{1}{2} + \frac{1}{2(y - 1)} \right)^{1 - \frac{1}{2}} \geq \left( \frac{1}{2} \right)^{1 - \frac{1}{2}}$ and $\frac{2}{\sin^p \frac{\pi}{2p}} - 1 - 1 = \left( \frac{2}{\sin^p \frac{\pi}{2p}} \right)^{1 - \frac{1}{2}} - 1$ which is obviously non-negative. Hence, $F(y)$ is increasing and by taking limit as $y \to \infty$, we see that, since $2^p \sin^{-p} \frac{\pi}{2p} - 1 \leq 0$, $\Phi(r, \pi)$ is non-positive for all $0 < r < 1$.

Applying Lemma 5.1 on $z = P_+ f$ and $w = P_- f$ and using subharmonicity of the function $h(z) = |z|^2 \cos \frac{(\pi - i)t}{2}$, for $t \in [-\pi, \pi]$, we get the following sharp inequality:

$$\|f\|_{L^p(T)} \leq 2^{1 - \frac{1}{2}} \sin \frac{\pi}{2p} \left[ |P_+ f|^s + |P_- f|^s \right]^{1/s} \|L^p(T),$$

for $1 \leq p \leq 2$ and $s \geq 2$.

We have managed to find a different proof, but this can be easily proved also by using the result for $s = 2$ and power-mean inequality. Also, using the appropriate result from [7] we conclude:

$$\|f\|_{L^p(T)} \leq 2^{1 - \frac{1}{2}} \cos \frac{\pi}{2p} \left[ |P_+ f|^s + |P_- f|^s \right]^{1/2} \|L^p(T),$$
for $p \geq 2$ and $s \geq 2$.

We get it using
\[ \|f\|_{L^p(T)} \leq 2^{\frac{1}{2}} \cos \frac{\pi}{2p} \| (|P_+ f|^2 + |P_- f|^2)^{\frac{1}{2}} \|_{L^p(T)}, \]
for $p \geq 2$, a result from [7] and the estimate:
\[
\| (|P_+ f|^2 + |P_- f|^2)^{\frac{1}{2}} \|_{L^p(T)} = \int_T (|P_+ f(\zeta)|^2 + |P_- f(\zeta)|^2)^{\frac{1}{2}} \, d\zeta 
\leq 2^{\frac{1}{2}} 2^{\frac{1}{p}} \int_T (|P_+ f(\zeta)|^s + |P_- f(\zeta)|^s)^{\frac{1}{2}} \, d\zeta.
\]

**Remark 5.2.** The reader can see that we have somewhat simpler proofs in both type od inequalities for $1 < p \leq 2$ but we cannot say similar for $p > 2$. In fact, we are able to find different proofs, but using similar methods we need also to estimate values of $r$ and $t$ in the appropriate “elementary” inequality, which can be done, but doesn’t make proof simpler.

**Remark 5.3.** We believe that constants that have appeared in the third section are sharp, but are not able to proved it for all cases by our arguments.

**Remark 5.4.** It can be easily seen that our inequalities have the analogs on the real line, even for the weighted case, as similar proved in [5].

**Remark 5.5.** The conclusion that for $0 < s < 2$ the constants, in both type of inequalities we consider, are sharp is also given in Melentijevi´c’s dissertation([12]).

6. **M. Riesz theorem on conjugate harmonic functions for various function spaces**

6.1. Here we prove the following generalisation of the M. Riesz theorem on harmonic conjugate functions. For example, the next theorem incudes the well known Fock spaces. Fock spaces are obtained for the measure $\omega(z) = e^{-\alpha|z|^2}dA(z)$, where $\alpha > 0$ is a constant. We deduce also the M. Riesz theorem for Besov type spaces.

**Theorem 6.1.** Let $X$ be a set (for example, it may be a space with a norm). Assume that $A : X \to A_p(\omega)$ is a mapping (an operator, not necessarily linear), where $A_{p,\omega}$ is the space of all analytic functions in the domain $D$ which is the unit disc $U$ or the complex plane $\mathbf{C}$. Let
\[ A_p(\omega) = \{ f : \|f\|_{p,\omega} \text{ is finite} \}, \]
where
\[ \|f\|_{p,\omega} = \left\{ \int_D |f(\zeta)|^p \omega(\zeta) d\zeta \right\}^{1/p} ; \]
here $\omega(\zeta)$ is radially symmetric weight function, i.e., we have $\omega(\zeta) = \omega(|\zeta|)$.

Assume that the operator $\Re A$ is bounded; it maps $X$ into $a_p(\omega)$ the space of real parts of functions in $A_p(\omega)$. Then the operator $A$ is also bounded. And there exists a universal constant estimates between norms of the operator.

**Proof.** We have the following inequality
\[ |z|^p \leq \varepsilon_p |(\Re z)|^p - b_p s(z), \]
where \( b_p \) is a positive constant and \( s(z) \) is a subharmonic function, and

\[
(6.1) \quad c_p = \begin{cases} 
\sec \frac{\pi}{2p}, & \text{if } 1 < p \leq 2; \\
\csc \frac{\pi}{2p}, & \text{if } 2 \leq p < \infty.
\end{cases}
\]

For \( x \in X \), let \( Ax \in A_p(\omega) \) be a function denoted by \( f(\zeta) \) for \( \zeta \in D \). Applying the inequality above we obtain

\[
|f(\zeta)|^p \leq c_p^p |\Re f(\zeta)|^p - b_p s(f(\zeta))
\]

Assume first that \( \Re f(0) \geq 0 \) (otherwise we can consider \(-f(\zeta)\)). If we multiply by \( \omega(\zeta) \) and then integrate the above inequality in \( D \) we obtain

\[
\int_D |f(\zeta)|^p \omega(\zeta) d\zeta \leq c_p^p \int_D |\Re f(\zeta)|^p \omega(\zeta) d\zeta - b_p \int_D s(f(\zeta)) \omega(\zeta) d\zeta.
\]

Since \( s(f(\zeta)) \) is a subharmonic function on \( D \), by using polar coordinates and the mean value theorem, we obtain (where the symbol \( a \) stands for 1 of \(+\infty\)):

\[
\int_D s(f(\zeta)) \omega(\zeta) d\zeta = \int_0^a r dr \int_T s(f(\zeta)) \omega(|\zeta|) d\zeta
\geq s(f(0)) \cdot \int_0^a \omega(r) r dr \geq 0.
\]

Therefore

\[
\int_D |f(\zeta)|^p \omega(\zeta) d\zeta \leq c_p^p \int_D |\Re f(\zeta)|^p \omega(\zeta) d\zeta.
\]

Now it follows the inequality we need

\[
\|f\|_{p,\omega} \leq c_p \|\Re f\|_{p,\omega},
\]
i.e.

\[
\|Ax\|_{p,\omega} \leq c_p \|\Re Ax\|_{p,\omega}.
\]

If \( X \) is a space with norm, then it follows

\[
\frac{\|Ax\|_{p,\omega}}{\|x\|_X} \leq c_p \frac{\|\Re Ax\|_{p,\omega}}{\|x\|_X}
\]
for all \( x \neq 0 \). Therefore, if we take \( \sup_{x\neq0} \) we obtain

\[
\|A : X \to A_p(\omega)\| \leq c_p \|\Re A : X \to A_p(\omega)\|,
\]
which we aimed to prove. \( \Box \)

**Remark 6.2.** Of course, one can recover the classical M. Riesz theorem for \( H^p \)-spaces from the above theorem. If we can consider \( A_{p,\alpha} \) and let \( \alpha \to -1 \) for identity operator we obtain M. Riesz theorem the Hardy space \( H^p \). Indeed, we can take the identity operator instead for \( A \) on the space \( A_{p,\alpha}^0 \). Then we obtain (because of the universality of the constant \( c_p \) we can take \( \alpha \to -1 \)).

\[
\|\Id(f)\|_{A_{p,\alpha}} \leq c_p \|\Re \Id(f)\|_{h_{p,\alpha}}.
\]

In other words we have

\[
\|f\|_{A_{p,\alpha}} \leq c_p \|\Re(f)\|_{a_{p,\alpha}}
\]
for all \( f \in A_{p,\alpha} \). If we let above \( \alpha \to -1 \), we obtain the M. Riesz theorem on conjugate harmonic functions in \( H^p \).
Denote
\[ B_{p,\omega} = \{ f : \int_D |f'(\zeta)|^p \omega(\zeta) d\zeta < \infty \} \]
and let \( b_{p,\omega} \) be a harmonic Besov space.

**Theorem 6.3.** For \( f \in B_{p,\omega} \) we have
\[ \|f\|_{B_{p,\omega}} \leq c_p \|\Re f\|_{b_{p,\omega}}. \]

**Proof.** The proof is similar as the proof of the above theorem, and it uses the relation
\[ \Re f'(z) = \nabla \Re f(z). \]
Indeed, if in the inequality \(|z|^p \leq c_p^p |\Re z|^p - b_p s(z)\), we in inequality take \( f'(\zeta) \)
\[ |f'(\zeta)|^p \leq c_p^p |\Re f'(\zeta)|^p - b_p s(f'(\zeta)) \]
i.e. if we use the above relation we obtain
\[ |f'(\zeta)|^p \leq c_p^p |\nabla \Re f(\zeta)|^p - b_p s(f'(\zeta)) \]
Now, if we integrate with respect to the \( D \) as in the preceding theorem we obtain
\[ \|f\|_{B_{p,\omega}} \leq c_p \|\Re f\|_{b_{p,\omega}}, \]
which is the statement of our theorem. \qed

The M. Riesz theorem for Bergman spaces was proves by Forelli and Rudin in IUMJ in 1974 and for Bloch spaces By Kalaj and Marković in [8]. In particular if we take the identity operator in the above theorem we obtain the M. Riesz theorem for these type space.

6.2. We prove M. Riesz theorem on conjugate real-valued harmonic function in a domain of \( \mathbb{C} \) that belong to Lumer’s Hardy spaces. We will say something on Lumer’s Hardy spaces firstly.

There are generalizations of Hardy spaces for arbitrary domains in \( \mathbb{C} \). The generalizations we consider in this subsection are known as Lumer’s Hardy spaces [1, 2, 9, 10, 16]. We mention below some facts regarding these spaces that we will need in this section.

The harmonic Lumer’s Hardy space \((Lh)^p(\Omega)\) contains all harmonic complex-valued functions \( U \) on a domain \( \Omega \subseteq \mathbb{C} \) such that the subharmonic function \(|U|^p\) has a harmonic majorant on \( \Omega \). In that case, the function \(|U|^p\) has the least harmonic majorant on \( \Omega \). Let it be denoted by \( H_U \). For \( \zeta_0 \in \Omega \) one introduces a norm on \((Lh)^p(\Omega)\) in the following way:
\[
\|U\|_{p,\zeta_0} = H_U^{1/p}(\zeta_0).
\]

The different norms on \((Lh)^p(\Omega)\) that arise by selecting different elements of the domain \( \Omega \) are mutually equivalent. The analytic Lumer’s Hardy space \((LH)^p(\Omega)\) is the subspace of \((Lh)^p(\Omega)\) that consists of all analytic functions. The two spaces \((Lh)^p(U)\) and \( h^p \) coincide (as do \((LH)^p(U)\) and \( H^p \)). The norms on these spaces are equal, if we select \( \zeta_0 = 0 \) for the Lumer case. Moreover, if \( \Omega \) is a simply connected domain such that \( \partial \Omega \) is a Jordan curve which is sufficiently smooth, the Lumer’s Hardy space \((LH)^p(\Omega)\) is the same as the Smirnov’s Hardy space \( E^p(\Omega) \) which is defined by requirement that the integral means of an analytic function over certain family of curves in the domain \( \Omega \) remains bounded – we refer to the tenth chapter in [1]. Therefore, we have \((LH)^p(U) = E^p(U) = H^p \). This follows from Theorem
which gives a sufficient criterion for coincidence of analytic Lumer’s and Smirnov’s Hardy spaces. This criterion may be easily adapted in order to conclude that the two types of harmonic spaces in a sufficiently smooth domain coincides.

Note that Lumer’s Hardy spaces are conformally invariant. In other words, if \( \Phi \) is a conformal mapping of a domain \( \Omega \) onto \( \Omega \), then \( U \in (Lh)^p(\Omega) \) if and only if \( \tilde{U} = U \circ \Phi \in Lh)^p(\Omega) \). The mapping \( \Phi \) induces an isometric isomorphism \( U \rightarrow \tilde{U} \) of the space \( (Lh)^p(\Omega) \) onto \( (Lh)^p(\tilde{\Omega}) \), since the equality for the least harmonic majorants \( H_U \circ \Phi = H_{\tilde{U}} \) implies that \( \|U\|_{p,\tilde{\zeta}_0} = \|\tilde{U}\|_{p,\zeta_0} \), where \( \zeta_0 \in \Omega \) satisfies \( \zeta_0 = \Phi(\tilde{\zeta}_0) \).

The classical Riesz theorem on conjugate harmonic functions says that for every \( p \in (1, \infty) \) there exists a constant \( c_p \) such that

\[
\|U + iV\|_p \leq c_p \|U\|_p,
\]

where \( U \) is a real-valued function in \( h^p \), \( V \) is a harmonic conjugate to \( U \) on \( \mathbb{U} \), normalized such that \( V(0) = 0 \). See, for instance, [15, Theorem 17.26]. Verbitsky proved [18] that the best possible constant in the Riesz inequality is

\[
(6.3) \quad c_p = \begin{cases} \sec \frac{\pi}{2p}, & \text{if } 1 < p \leq 2; \\ \csc \frac{\pi}{2p}, & \text{if } 2 < p < \infty . \end{cases}
\]

We have conjectured in [11] that the M. Riesz theorem with the Verbitsky constant \( c_p \) is valid in the case of Lumer’s Hardy spaces \((LH)^p(\Omega)\) for every \( p \in (1, \infty) \) and \( \Omega \subseteq \mathbb{C} \). However, we prove this conjecture for analytic functions with the positive real part. Our aim in this section is to prove the Riesz theorem for real-valued harmonic functions in the Lumer’s Hardy space \((Lh)^p(\Omega)\) for which there exists a conjugate with the best possible constant it the case of positive harmonic functions on \( \Omega \). Because of duality we consider the case \( 1 < p < 2 \). This is the content of the following theorem.

**Theorem 6.4.** Let \( \Omega \subseteq \mathbb{C} \) be a domain and \( \zeta_0 \in \Omega \). Assume that for a positive \( U \in (Lh)^p(\Omega) \) there exists a harmonic conjugate of \( U \) on the domain \( \Omega \), denoted by \( V \), and let it be normalized such that \( V(\zeta_0) = 0 \). Then we have the Riesz inequality

\[
(6.4) \quad \|U + iV\|_{p,\zeta_0} \leq c_p \|U\|_{p,\zeta_0},
\]

where the best possible constant is the Verbitsky constant if \( U \) is positive.

**Proof.** In [11] the case \( p = 2 \) is considered. Now, we can adapt the proof for \( p = 2 \) given there [11]. Recall that in [11] we have used the following elementary equality, which is easy to check:

\[
(6.5) \quad |z|^2 = 2(\Re z)^2 - \Re z^2, \quad z \in \mathbb{C}.
\]

For \( p \neq 2 \) we should use an inequality in order to replace the equality given above. If \( 1 < p < 2 \) for \( c_p = (\cos \frac{\pi}{2p})^{-p} \) we have the inequality

\[
(6.6) \quad |z|^p \leq c_p (\Re z)^p - h(z),
\]

where \( h(z) \) is a harmonic function in the domain \( \{z : \Re z > 0\} \), which satisfies \( h(x) > 0 \) for every \( x > 0 \); actually, the function \( h \) is given by \( h(z) = \Re(z^p) \). For the proof see the Kalaj’s paper [7], or the Verbitsky work.

Assume first that \( U > 0 \). Let the analytic function \( U + iV \) be denoted by \( F \), and let \( H_U \) be the least harmonic majorant of the subharmonic function \( |U|^p \) on \( \Omega \). By
applying inequality (6.6) for \( z = F(\zeta), \zeta \in \Omega \), we obtain
\[
|F(\zeta)|^p \leq c_p^p(\Re F(\zeta))^2 - h(F(\zeta)) = c_p^p|U(\zeta)|^2 - h(F(\zeta)) \\
\leq c_p^pH_U(\zeta) - h(F(\zeta)),
\]
which proves that \( c_p^pH_U(\zeta) - h(F(\zeta)) \) is a harmonic majorant of \( |F|^p \) on \( \Omega \). It follows that \( F \in (LH)^p(\Omega) \). Moreover, if \( H_F \) is the least harmonic majorant of \( |F|^p \) on \( \Omega \), we have
\[
H_F(\zeta) \leq c_p^pH_U(\zeta) - h(F(\zeta)).
\]
Since \( F(\zeta_0) = U(\zeta_0) \) is a positive real number, we obtain
\[
\|F\|_{p,\zeta_0}^p = H_F(\zeta_0) \leq c_p^pH_U(\zeta_0) - h(F(\zeta_0)) \leq c_p^pH_U(\zeta_0) = c_p^p\|U\|_{p,\zeta_0}^2.
\]
Finally, we conclude that
\[
\|F\|_{p,\zeta_0} \leq c_p\|U\|_{p,\zeta_0},
\]
which is what we wanted to prove.

The sharpness of \( c_p \) in this case follows from the Verbitsky result. \( \square \)

We give now here a proof of the Riesz inequality but with a constant different from the conjectured one. We use the similar approach as in the Pavlović paper [13]. Assume that the boundary of the domain \( \partial \Omega \) is smooth. Then \( U \) at the boundary may be decomposed as \( U^* = U_1^* - U_2^* \), where \( U_1^* \) and \( U_2^* \) are positive decomposition. Let \( U_1 \) and \( U_2 \) be positive harmonic such that their boundary functions on \( \partial \Omega \) are \( U_1^* \) and \( U_2^* \). We have \( \|U^*\|_p^p = \|U_1^*\|_p^p + \|U_2^*\|_p^p \). Since \( \|F\|_p^p = \|F_1 - F_2\|_p^p \leq 2\|F_1\|_p^p + \|F_2\|_p^p \), it follows that \( \|F\|_p \leq 2c_p\|U\|_p \).

Note that the constant \( c_p \) in the Riesz inequality (6.3) does not depend on \( \zeta_0 \in \Omega \), although the norm of a function in the Lumer’s Hardy space \((Lh)^p(\Omega)\) does. If \( \Omega \) is a simply connected domain with at least two boundary points, this is expected, since the group of all conformal automorphisms of the domain \( \Omega \) acts transitively on \( \Omega \), i.e., for any \( \zeta_0 \in \Omega \) there exists a conformal automorphism \( \Phi \) of \( \Omega \) such that \( \Phi(\zeta_0) = \zeta_0 \). As we have already said, the mapping \( \Phi \) induces an isometric isomorphism of \((Lh)^p(\Omega)\) onto itself. However, for multi-connected domains it is not true, in general, that the group of all conformal automorphisms acts transitively on a domain.

In seventies, Stout [17, Theorem IV.1] proved Riesz’s theorem for Lumer’s Hardy spaces \((LH)^p(\Omega)\) on \( C^2 \)-smooth domains \( \Omega \subseteq C^n \) (without a precise constant in the Riesz inequality). In this case there exists an integral representation of the Lumer’s norm of an analytic function that is used in order to obtain the result.

Lumer’s Hardy spaces \((Lh)^p(\Omega)\) and \((Lh)^p(\Omega)\) on domains \( \Omega \) in \( C^n \) are defined in a similar way as in the one-dimensional case [9]. However, instead of the harmonic majorant we have to use a pluriharmonic majorant, i.e., a function that is locally the real part of an analytic function on \( \Omega \). Therefore, the Lumer’s Hardy space \((Lh)^p(\Omega)\) contains all pluriharmonic functions \( U \) on \( \Omega \) such that \( |U|^p \) has a pluriharmonic majorant on \( \Omega \). The analytic Lumer’s Hardy space \((Lh)^p(\Omega)\) is the subspace of \((Lh)^p(\Omega)\) that consists of all analytic functions. The norm on \((Lh)^p(\Omega)\) may be introduced with respect to any \( \zeta_0 \in \Omega \) using the least pluriharmonic majorant as in the ordinary case (6.2).

The proof given above works for any dimension. Therefore, we have the dimension-free constant in the Riesz inequality which is, moreover, valid for all domains in \( C^n \).
7. On vector-valued inequalities

7.1. Let \( \hat{D} \) be a domain in \( \mathbb{R}^n \) which contains 0 and has the property that any intersection with the plane containing 0 is a same domain \( D \). For example \( \hat{D} \) may be the unit ball, the closed unit ball, or the whole space. We will use the following general principle for transferring inequalities for functions defined in the domain \( D \) to inequalities for functions in the domain \( \hat{D} \).

**Lemma 7.1.** Let \( \mathcal{L}(x, y, s, t) \) and \( \mathcal{R}(x, y, s, t) \) be real-valued functions defined for \( x \geq 0, y \geq 0 \) and \( s \geq 0, \) and real \( t \). Let \( \mathcal{C} \) be a class of mappings in \( D \) such that

\[
\mathcal{L}(|f(\zeta)|, |f(\eta)|, |f(\zeta) - f(\eta)|, \langle f(\zeta), f(\eta) \rangle) \leq \mathcal{R}(|\zeta|, |\eta|, |\zeta - \eta|, \langle \zeta, \eta \rangle)
\]

for every \( f \in \mathcal{C} \) and \( u \in D \) and \( v \in D \). Here \( \langle \cdot, \cdot \rangle \) stands for the scalar product in \( \mathbb{R}^n \).

Assume that \( \hat{C} \) is a class of mappings in the domain \( \hat{D} \) which has the property that any restriction of a function in \( \hat{C} \) composed by a projections is a mapping in the class \( \mathcal{C} \). Then the above inequality \( (\mathcal{L} \leq \mathcal{R}) \) remains to hold for mappings in the class \( \hat{C} \).

**Proof.** Choose any mapping \( f \in \hat{C} \). Let \( \zeta \) and \( \eta \) be fixed in the domain \( \hat{D} \), and denote \( \Phi = f(\zeta) \) and \( \Psi = f(\eta) \). Our aim is to show that

\[
\mathcal{L}(|\Phi|, |\Psi|, |\Phi - \Psi|, \langle \Phi, \Psi \rangle) \leq \mathcal{R}(|\zeta|, |\eta|, |\zeta - \eta|, \langle \zeta, \eta \rangle).
\]

Let \( D_{\zeta, \eta} \) be the intersection of the domain \( \hat{D} \) and the plane \( P_{\zeta, \eta} \), which contains \( \zeta \), \( 0 \), and \( \eta \), and let \( P_{\Phi, \Psi} \) be the plane which contains \( \Phi \), \( 0 \), and \( \Psi \). Denote by \( \Pi_{\Phi, \Psi} \) the orthogonal projection mapping of the space \( \mathbb{R}^n \) onto the plane \( P_{\Phi, \Psi} \). We will consider the mapping which is composition of the restriction of our mapping \( f \) on the domain \( D_{\zeta, \eta} \), denoted by \( f|_{D_{\zeta, \eta}} \), and the above described projection \( \Pi_{\Phi, \Psi} \).

Therefore, let \( \phi = \Pi_{\Phi, \Psi} \circ f|_{D_{\zeta, \eta}} \). We have \( \phi(\zeta) = \Phi \) and \( \phi(\eta) = \Psi \). Let \( A_1 \) and \( A_2 \) be orthogonal which map the domain in the plane \( D \) onto \( D_{\zeta, \eta} \) and \( P_{\Phi, \Psi} \), respectively.

Since \( \phi : D_{\zeta, \eta} \to D_{\Phi, \Psi} \), then the mapping \( \phi^{-1}_A \circ \phi \circ A_1 : D \to \mathbb{R}^n \) for points \( \zeta = A_1^{-1}(\zeta) \) and \( \eta = A_1^{-1}(\eta) \), we obtain

\[
\mathcal{L}(\tilde{\phi}(\zeta), |\tilde{\phi}(\eta)|, |\tilde{\phi}(\zeta) - \tilde{\phi}(\eta)|, \langle \tilde{\phi}(\zeta), \tilde{\phi}(\eta) \rangle) \leq \mathcal{R}(\tilde{\zeta}, |\tilde{\eta}|, |\tilde{\zeta} - \tilde{\eta}|, \langle \tilde{\zeta}, \tilde{\eta} \rangle).
\]

Since \( A_1^{-1} \) is distance and angle preserving, we obtain

\[
\mathcal{R}(\tilde{\zeta}, |\tilde{\eta}|, |\tilde{\zeta} - \tilde{\eta}|, \langle \tilde{\zeta}, \tilde{\eta} \rangle) = \mathcal{R}(\zeta, |\eta|, |\zeta - \eta|, \langle \zeta, \eta \rangle).
\]

Since \( A_2 \) is also distance and angle preserving, we also have

\[
\mathcal{L}(\tilde{\phi}(\zeta), |\tilde{\phi}(\eta)|, |\tilde{\phi}(\zeta) - \tilde{\phi}(\eta)|, \langle \tilde{\phi}(\zeta), \tilde{\phi}(\eta) \rangle) = \mathcal{L}(\phi(\zeta), |\phi(\eta)|, |\phi(\zeta) - \phi(\eta)|, \langle \phi(\zeta), \phi(\eta) \rangle)
\]

\[
= \mathcal{L}(\Phi, |\Psi|, |\Phi - \Psi|, \langle \Phi, \Psi \rangle).
\]

Our inequality follows. \( \square \)

**Corollary 7.2.** Let \( \mathcal{L}(x, y, s, t) \) and \( \mathcal{R}(x, y, s, t) \) be real-valued functions defined for \( x \geq 0, y \geq 0 \) and \( s \geq 0, \) and real \( t \). Assume that we have inequality

\[
\mathcal{L}(|\zeta|, |\eta|, |\zeta - \eta|, \langle \zeta, \eta \rangle) \leq \mathcal{R}(|\zeta|, |\eta|, |\zeta - \eta|, \langle \zeta, \eta \rangle)
\]

for every \( \zeta \in D \subseteq \mathbb{R}^2 \) and \( \eta \in D \subseteq \mathbb{R}^2 \).
Then this inequality remains valid for every \( \zeta \in \tilde{D} \subseteq \mathbb{R}^n \) and \( \eta \in \tilde{D} \subseteq \mathbb{R}^n \).

Proof. We have just to apply the above lemma for the class which consists of only one function which is identity.

This corollary may be reformulated as follows

**Corollary 7.3.** Let \( L(x, y, s, t) \) and \( R(x, y, s, t) \) be real-valued functions defined for \( x \geq 0, y \geq 0 \) and \( s \geq 0, t \geq 0 \), and real \( t \). Assume that we have inequality

\[
L(|\zeta|, |\eta|, |\zeta \pm \eta|, \angle(\zeta, \eta)) \leq R(|\zeta|, |\eta|, |\zeta \pm \eta|, \angle(\zeta, \eta))
\]

for every \( \zeta \in D \subseteq \mathbb{R}^2 \) and \( \eta \in D \subseteq \mathbb{R}^2 \).

Then this inequality remains valid for every \( \zeta \in \tilde{D} \subseteq \mathbb{R}^n \) and \( \eta \in \tilde{D} \subseteq \mathbb{R}^n \).

This may be reformulates as: if a certain inequality holds for plane vectors and depends only on norm of vectors and angle between them, then the same inequality holds for vectors in the space. In other words if \( F(|z|, |w|, |z \pm w|, \angle(z, w)) \geq 0 \), then we have \( F(|\zeta|, |\eta|, |\zeta \pm \eta|, \angle(\zeta, \eta)) \geq 0 \) for \( \zeta \in \mathbb{R}^n \) and \( \eta \in \mathbb{R}^n \).

### 7.2.

The corollary given above will be applied here for the vector-valued inequalities we need. We prove the following inequality for \( z \) and \( w \) in \( \mathbb{C}^n \):

\[
(||z||^2 + ||w||^2)^{1/2} \leq C_{2,s} ||z + \overline{w}||^p - s(z, w)
\]

where \( s(z, w) \) is plurisubharmonic in \( \mathbb{C}^n \times \mathbb{C}^n \). Indeed, the Kalaj’s inequality states that

\[
(|z|^2 + |w|^2)^{1/2} \leq C_{p,2} |z + \overline{w}|^p - b_{p,2} R(z, \overline{w})^{p/2}
\]

for \( z \in \mathbb{C} \) and \( w \in \mathbb{C} \) and for \( 1 < p < 2 \), where \( b_{p,2} \) is a positive constant. From this inequality we deduce the vector-valued one:

\[
(||z||^2 + ||w||^2)^{1/2} \leq C_{p,2} ||z + \overline{w}||^p - b_{p,2} ||z||^p ||w||^p \cos \frac{p}{2} \angle(z, w).
\]

Therefore, we can take \( s(z, w) = b_{p} ||z||^p ||w||^p \cos \frac{p}{2} \angle(z, w) \).

Hollenbeck and Verbitsky proved the following inequality

\[
\max\{|z|^p, |w|^p\} \leq C_{p,\infty} |z + \overline{w}^p| - b_{p,\infty} R(zw)^{p/2}
\]

for \( 1 < p < 2 \). Since this inequality depends only on \( |z| \) and \( |w| \) and the angle between them, we may conclude that it holds for \( z \in \mathbb{C}^n \) and \( w \in \mathbb{C}^n \). The vector–valued Verbitsky inequality may be written as

\[
\max\{|z|^p, |w|^p\} \leq C_{p,\infty} ||z + \overline{w}||^p - b_{p,\infty} R(z, \overline{w})^{p/2}
\]

here \( \langle \cdot, \cdot \rangle \) is the inner product in \( \mathbb{C}^n \). Inequalities proved in this paper for any \( 0 < s < 2 \) may be transferred for \( z \in \mathbb{C}^n \) and \( w \in \mathbb{C}^n \).

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