BOLTZMANN’S ENTROPY AND KÄHLER-RICCI SOLITONS

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ABSTRACT. We study a Boltzmann’s type entropy functional (which appeared in existing literature) defined on Kähler metrics of a fixed Kähler class. The critical points of this functional are gradient Kähler-Ricci solitons, and the functional was known to be monotonically increasing along the Kähler-Ricci flow in the canonical class.

In this article, we derive and analyze the second variation formula for this entropy functional, and show that all gradient Kähler-Ricci solitons are stable with respect to this entropy functional. Furthermore, using this result, we give a new proof that gradient shrinking Kähler-Ricci solitons are stable with respect to the Perelman’s entropy in a fixed Kähler class.

1. INTRODUCTION

In this article, we examine an entropy functional defined on the space of Kähler potentials of a compact Kähler manifold whose first Chern class has a definite sign. This functional, which will be denoted by $\mathcal{H}$ in this article, appeared in some existing literature related to the Kähler-Ricci flow and solitons.

Let $X^n$ be a compact Kähler manifold whose first Chern class $c_1(X)$ has a definite sign denoted by $\lambda \in \mathbb{R}$. Let $\omega_0$ be a Kähler metric such that $\lambda \omega_0 \in c_1(X)$. Denote the space of Kähler potentials by:

$$\mathcal{K} = \{ \varphi \in C^\infty(X, \mathbb{R}) : \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \}.$$  

Given any $\varphi \in \mathcal{K}$, we denote $\omega_\varphi := \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi$. Since $\lambda[\omega_\varphi] = \lambda[\omega_0] = c_1(X) = [\text{Ric}(\omega_\varphi)]$, the $\partial \bar{\partial}$-lemma asserts that there exists a unique smooth function $f_\varphi$ on $X$ such that:

$$\lambda \omega_\varphi - \text{Ric}(\omega_\varphi) = \sqrt{-1} \partial \bar{\partial} f_\varphi;$$ and

$$\int_X e^{-f_\varphi} \omega_\varphi^n = \int_X \omega_0^n.$$  

Note that $\omega_\varphi$ is in the same Kähler class as $\omega_0$, and therefore the total volume

$$[\omega_\varphi]^n := \int_X \omega_\varphi^n$$

is independent of $\varphi \in \mathcal{K}$. We define the entropy functional $\mathcal{H} : \mathcal{K} \to \mathbb{R}$ by:

$$\mathcal{H}(\varphi) := \frac{1}{[\omega_0]^n} \int_X f_\varphi e^{-f_\varphi} \omega_\varphi^n.$$
This \( \mathcal{H} \)-functional can be expressed as a special form of Boltzmann’s entropy in statistical thermodynamics. We first rewrite \( f_\varphi \) as:

\[
f_\varphi = -\log \left( e^{-f_\varphi \omega_\varphi^n} / \omega_\varphi^n \right) = -\log \left( e^{-f_\varphi \omega_\varphi^n} / \left[ \omega_\varphi^n \right]^n \right).
\]

Denote \( d\nu_\varphi := e^{-f_\varphi \omega_\varphi^n} / \left[ \omega_\varphi^n \right]^n \) and \( d\mu_\varphi := \omega_\varphi^n / \left[ \omega_\varphi^n \right]^n \). Note that both \( d\nu_\varphi \) and \( d\mu_\varphi \) are probability measures on \( X \). Under these notations, the \( \mathcal{H} \)-functional can be written as:

\[
\mathcal{H}(\varphi) = -\int_X d\nu_\varphi \log \left( \frac{d\nu_\varphi}{d\mu_\varphi} \right) d\mu_\varphi = -\int_X \log \left( \frac{d\nu_\varphi}{d\mu_\varphi} \right) d\nu_\varphi
\]

which is the Boltzmann’s entropy with respect to the two measures \( d\nu_\varphi \) and \( d\mu_\varphi \).

On a compact Riemannian manifold \((M, g)\), Lott and Villani studied in [12] the Boltzmann’s entropy (different from \( \mathcal{H} \) in this article) of the form:

\[
H_{d\mu_0}(d\nu_t) := \int_X d\nu_t \log \left( \frac{d\nu_t}{d\mu_0} \right) d\mu_0,
\]

where \( d\mu_0 \) is a fixed measure and \( d\nu_t \) is a geodesic path of measures (absolutely continuous with respect to \( d\mu_0 \)) in the Wasserstein space \( \mathcal{P}_2(M) \). They showed that this entropy is convex (i.e. \( \frac{\partial^2}{\partial t^2} H_{d\mu_0}(d\nu_t) \geq 0 \)) for any geodesic paths \( d\nu_t \) if and only if \((M, g)\) has non-negative Ricci curvature.

Concerning our \( \mathcal{H} \)-functional defined in (1.1), there are several interesting results about it in the subject of the Kähler-Ricci flow. In [15], Section 6) (see also [11]), it was proved that the \( \mathcal{H} \)-functional \[^1\] is monotonically increasing along the normalized Kähler-Ricci flow \( \frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) + \lambda \omega \) for the case \( c_1(X) > 0 \) assuming the initial metric \( \omega_0 \in \{ 1 \}_{c_1(X)} \). In fact it is also true in the cases of \( c_1(X) = 0 \) and \( c_1(X) < 0 \) (see the discussion in Section 6 in this article). Furthermore, the critical points of this functional are Kähler potentials \( \varphi \) such that \( \omega_\varphi \) is a gradient Kähler-Ricci soliton, meaning that:

\[
\text{Ric}(\omega_\varphi) + \nabla^2 f_\varphi = \lambda \omega_\varphi
\]

and so \( \nabla f_\varphi \) is a real holomorphic vector field. In particular, if \( f_\varphi \) is a constant function, then \( \omega_\varphi \) is a Kähler-Einstein metric.

This \( \mathcal{H} \)-functional also plays a role in the unpublished result due to Perelman (see [17][2]) in which he proved that the diameter and scalar curvature are uniformly bounded along the normalized Kähler-Ricci flow \( \frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) + \lambda \omega \) starting with \( \omega_0 \in \{ 1 \}_{c_1(X)} \) > 0. In Section 2 of [17] and Section 6 of [2], monotonicity of Perelman’s \( W \)-functional was used to show that the Ricci potential \( f_\varphi \) is uniformly bounded from below along the normalized Kähler-Ricci flow. In fact, a uniform bound for \( \mathcal{H} \) is sufficient to prove a uniform lower bound for \( f_\varphi \) along the flow. The lower bound of \( \mathcal{H} \) follows from the monotonicity along the Kähler-Ricci flow, and the upper bound can be deduced using Jensen’s inequality. Both are easier to obtain than the analogous results of the \( W \)-functional.

\[^1\] In [15], the letter b was used to denote the \( \mathcal{H} \)-functional. In [10], the author adopted a different sign convention for this functional.
In \cite{10}, Proposition 2.2, the author derived an upper bound for \( H \) in relation to the maximal compact subgroup of \( \text{Aut}(M, \omega_0) \). Using this result, it was proved in \cite{10} Corollary 2.6 that if a Kähler-Ricci soliton exists, then it maximizes Perelman’s \( \mu \)-functional.

In the recent article \cite{6}, Donaldson also pointed out that the \( H \)-functional and the Ding’s functional \( F \) introduced in \cite{5} are related by \( \frac{dF}{dt} \leq H \) along the normalized Kähler-Ricci flow starting from \( \omega_0 \in \frac{1}{c_1(X)} \).

We are going to explore this \( H \)-functional further in this article by deriving and analyzing the second variation formula for \( H \). For the Perelman’s \( \nu \)-functional introduced in \cite{13}, the Euler-Lagrange’s equation gives gradient shrinking Ricci solitons as its critical points. The second variation of \( \nu \) was discussed and computed in \cite{3, 4} and a notion of stability of shrinking Ricci solitons was developed using the second variation formula. Various works concerning about stability of shrinking Ricci solitons can be found in e.g. \cite{8, 9, 11}. Analogously, from the second variation formula for \( H \), we introduce a stability operator \( S_f \) and a notion of \( H \)-stability for gradient Kähler-Ricci solitons. Our main result is that any critical point (i.e. gradient Kähler-Ricci solitons) is \( H \)-stable:

**Main Theorem.** Let \( \varphi \) be a critical point of the functional \( H \) (so that \( \omega_\varphi \) is a gradient Kähler-Ricci soliton), then for any \( \psi \in T_\varphi K \), we have:

\[
\left. \frac{d^2}{dt^2} \right|_{t=0} H(\varphi + t\psi) \leq 0
\]

and equality holds if and only if \( \nabla \psi \) is a real holomorphic vector field. □

From dynamical system viewpoint, the main theorem and the monotonicity of \( H \) along the Kähler-Ricci flow assert that gradient Kähler-Ricci solitons are “attractors”, and \( -H \) acts as a Lyapunov function for the flow. It is well-known in \cite{1} that the normalized Kähler-Ricci flow in the canonical class converges to Kähler-Einstein metrics when \( c_1(X) = 0 \) or \( c_1(X) < 0 \). In case of \( c_1(X) > 0 \), assuming the existence of a shrinking Kähler-Ricci soliton, the normalized Kähler-Ricci flow in the canonical class was shown in \cite{20} to converge to the soliton under some invariant condition on the initial metric (see also \cite{16, 14, 23, 22, 19}). It is hoped that the main result of this article could bring more insight about the stability of the Kähler-Ricci flow when it approaches to the soliton limit.

In the case of \( c_1(X) > 0 \), the \( H \)-functional is also related to the Perelman’s \( \mu \)-functional, in a sense that \( H \) is an upper barrier of \( \mu \) (up to addition of a constant) and that they coincide at gradient Kähler-Ricci solitons (again up to addition of a constant). Therefore, the main theorem of this article implies (see Proposition 7.1) an earlier result in \cite{21} that Kähler-Ricci solitons are \( \mu \)-stable in a fixed Kähler class, i.e. \( \mu \)-stable in the direction of complex Hessian of potential functions. This is an important result used in many works about the stability of Kähler-Ricci solitons and the convergence of the Kähler-Ricci flow (see e.g. \cite{21, 18, 24, 25}).

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2. Ricci Potential and its Variation

In this section, we study the function \( f_\varphi \) (commonly called the Ricci potential of \( \omega_\varphi \)) and derive the evolution equation for \( f_\varphi \) which will be used often later on. To begin, we express the Ricci potential in a more explicit way in terms of the Ricci potential of \( \omega_0 \):

\[
\lambda \omega_\varphi - \text{Ric}(\omega_\varphi) = \sqrt{-1} \partial \bar{\partial} f_\varphi \quad \implies \quad \lambda \omega_0 + \lambda \sqrt{-1} \partial \bar{\partial} \varphi + \sqrt{-1} \partial \bar{\partial} \log \omega^n_\varphi = \sqrt{-1} \partial \bar{\partial} f_0
\]

\[
\lambda \omega_0 - \text{Ric}(\omega_0) = \sqrt{-1} \partial \bar{\partial} f_0 \quad \implies \quad \lambda \omega_0 + \sqrt{-1} \partial \bar{\partial} \log \omega^n_0 = \sqrt{-1} \partial \bar{\partial} f_0
\]

and so by subtraction, we have:

\[
\sqrt{-1} \partial \bar{\partial} \left( \lambda \varphi + \log \frac{\omega^n_\varphi}{\omega^n_0} \right) = \sqrt{-1} \partial \bar{\partial} (f_\varphi - f_0).
\]

Therefore, \( \lambda \varphi + \log \frac{\omega^n_\varphi}{\omega^n_0} - (f_\varphi - f_0) \) is a constant on \( X \). Using the normalization conditions:

\[
\int_X e^{-f_\varphi} \omega^n_\varphi = \int_X e^{-f_0} \omega^n_0 = [\omega_0]^n,
\]

one can determine this constant and show that the Ricci potential \( f_\varphi \) is given by:

\[
f_\varphi = f_0 + \log \frac{\omega^n_\varphi}{\omega^n_0} + \lambda \varphi + \log \left( \frac{1}{[\omega_0]^n} \int_X e^{-f_0 - \lambda \varphi} \omega^n_0 \right).
\]

**Lemma 2.1** (Evolution Equation of \( f_\varphi \)). Let \( \varphi(t) \) be a 1-parameter smooth family of Kähler potentials in \( K \), where \( t \in (-\varepsilon, \varepsilon) \). Denote \( \psi := \frac{\partial f_\varphi}{\partial t} \), then the Ricci potential \( f_\varphi(t) \) evolves by:

\[
\frac{\partial f_\varphi}{\partial t} = \Delta \psi + \lambda \left( \psi - \frac{1}{[\omega_0]^n} \int_X \psi e^{-f_\varphi} \omega^n_\varphi \right),
\]

where \( \Delta := \Delta_\varphi \) is the Laplacian with respect to \( \omega_\varphi(t) \).

**Proof.** Recall that \( f_\varphi \) is defined by \( \lambda \omega_\varphi - \text{Ric}(\omega_\varphi) = \sqrt{-1} \partial \bar{\partial} f_\varphi \). Therefore, we have:

\[
\lambda (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi) + \sqrt{-1} \partial \bar{\partial} \log \det(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi) = \sqrt{-1} \partial \bar{\partial} f_\varphi.
\]

Differentiating both sides with respect to \( t \), we get:

\[
\sqrt{-1} \partial \bar{\partial} (\lambda \psi) + \sqrt{-1} \partial \bar{\partial} (\text{tr}_\omega \sqrt{-1} \partial \bar{\partial} \psi) = \sqrt{-1} \partial \bar{\partial} \left( \frac{\partial f_\varphi}{\partial t} \right).
\]

Since \( X \) is compact, we have:

\[
\lambda \psi + \Delta \psi + c(t) = \frac{\partial f_\varphi}{\partial t}
\]

where \( c(t) \) is a function of \( t \) only to be determined. By the normalization condition on \( f_\varphi \), we know:

\[
0 = \frac{d}{dt} \int_X e^{-f_\varphi} \omega^n_\varphi
\]

\[
= \int_X \left( -\frac{\partial f_\varphi}{\partial t} + \Delta \psi \right) e^{-f_\varphi} \omega^n_\varphi
\]

where we have used the fact that \( \frac{d}{dt} \log \omega^n_\varphi = \text{tr}_\omega \frac{\partial}{\partial t} \omega_\varphi = \text{tr}_\omega \sqrt{-1} \partial \bar{\partial} \psi = \Delta \psi \). Combining with (2.3), we have:

\[
\int_X (\lambda \psi + c) e^{-f_\varphi} \omega^n_\varphi = 0
\]
and so using the normalization condition on $f_\varphi$, we can complete the proof of the lemma by observing that:

$$c(t) = -\frac{\lambda}{[\omega_0]^n} \int_X \psi e^{-f_\varphi} \omega^n_\varphi.$$ 

\[ \square \]

3. Critical Points of $\mathcal{H}$

The first variation of $\mathcal{H}$ and the Euler-Lagrange’s equation have been studied in [15][10] when $c_1(X) > 0$, in which the critical points of $\mathcal{H}$ were known to be Kähler-Einstein metrics and more generally (shrinking) gradient Kähler-Ricci solitons. The cases of $c_1(X) = 0$ and $c_1(X) < 0$ can be proved in similar ways. We include the detail below for easy reference.

**Proposition 3.1** (First Variation of $\mathcal{H}$). The first variation of $\mathcal{H}$ along a 1-parameter smooth family $\varphi(t)$ of Kähler potentials in $\mathcal{K}$ such that $\frac{\partial \varphi}{\partial t} = \psi$ is given by:

$$\frac{d}{dt}\mathcal{H}(\varphi) = -\frac{1}{[\omega_0]^n} \int_X \psi \left[ \Delta f_\varphi - |\nabla f_\varphi|^2 + \lambda (f_\varphi - \mathcal{H}(\varphi)) \right] e^{-f_\varphi} \omega^n_\varphi.$$ 

**Proof.** Recall that:

$$\mathcal{H}(\varphi) = \frac{1}{[\omega_0]^n} \int_X f_\varphi e^{-f_\varphi} \omega^n_\varphi.$$ 

By direct computations with the use of (2.1), we get:

$$\frac{d}{dt}\mathcal{H}(\varphi) = \frac{1}{[\omega_0]^n} \int_X \left( \frac{\partial f_\varphi}{\partial t} - f_\varphi \frac{\partial f_\varphi}{\partial t} + f_\varphi \Delta \psi \right) e^{-f_\varphi} \omega^n_\varphi$$

$$= \frac{1}{[\omega_0]^n} \int_X \left[ \left( \Delta \psi + \lambda \left( \psi - \frac{\int_X \psi e^{-f_\varphi} \omega^n_\varphi}{[\omega_0]^n} \right) \right) - f_\varphi \left( \Delta \psi + \lambda \left( \psi - \frac{\int_X \psi e^{-f_\varphi} \omega^n_\varphi}{[\omega_0]^n} \right) \right) + f_\varphi \Delta \psi \right] e^{-f_\varphi} \omega^n_\varphi$$

$$= \frac{1}{[\omega_0]^n} \left( \int_X (\Delta \psi - \lambda f_\varphi \psi) e^{-f_\varphi} \omega^n_\varphi + \lambda H(\varphi) \int_X \psi e^{-f_\varphi} \omega^n_\varphi \right).$$

Here we have used the fact that:

$$\int_X \left( \psi - \frac{\int_X \psi e^{-f_\varphi} \omega^n_\varphi}{[\omega_0]^n} \right) e^{-f_\varphi} \omega^n_\varphi = 0.$$ 

Through integration-by-parts, we get:

$$\int_X (\Delta \psi) e^{-f_\varphi} \omega^n_\varphi = -\int_X (\nabla \psi, -\nabla f_\varphi) e^{-f_\varphi} \omega^n_\varphi$$

$$= -\int_X \psi \left( \Delta f_\varphi - |\nabla f_\varphi|^2 \right) e^{-f_\varphi} \omega^n_\varphi.$$ 

Combining with the previous result, we have proved:

$$\frac{d}{dt}\mathcal{H}(\varphi) = -\frac{1}{[\omega_0]^n} \int_X \psi \left[ \Delta f_\varphi - |\nabla f_\varphi|^2 + \lambda (f_\varphi - \mathcal{H}(\varphi)) \right] e^{-f_\varphi} \omega^n_\varphi$$

as desired. \[ \square \]
we define:

\[ \Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{\omega_\partial^n} \int_X \psi \Delta \mathcal{H}(\psi) e^{-\int_\mathcal{H} \omega_\partial^n}. \]

Then along \( \frac{\partial \varphi}{\partial t} = \psi \), we have:

\[ \frac{\partial}{\partial t} \mathcal{H}(\varphi) = \frac{1}{\omega_\partial^n} \int_X \psi \mathcal{H}(\psi) e^{-\int_\mathcal{H} \omega_\partial^n}. \]

For simplicity, we will occasionally denote \( f := f_\varphi \) whenever \( \varphi \) can be understood from the context. Next we introduce three linear operators on \( C^\infty(X, \mathbb{C}) \): given any \( \psi \in C^\infty(X, \mathbb{C}) \), we define:

\[ L f \psi := \Delta \psi - \nabla^i f \nabla_i \psi = \Delta \psi - g^{ij} \nabla_j f \nabla_i \psi, \]

\[ \bar{L} f \psi := \Delta \psi - \bar{\nabla}^i f \bar{\nabla}_i \psi = \Delta \psi - g^{ij} \bar{\nabla}_j f \nabla_i \psi, \]

\[ \Delta f \psi := \Delta \psi - \frac{1}{2} \left( g^{ij} \nabla_j f \nabla_i \psi + g^{ij} \bar{\nabla}_j f \nabla_i \psi \right) \]

where \( \Delta, \nabla \) and the inner product \( \langle \cdot, \cdot \rangle \) are taken with respect to the metric \( \omega_\varphi \). It is clear that \( \Delta f \psi = \frac{1}{2} (L_f + \bar{L}_f) \psi \), and \( \bar{L}_f \psi = \bar{L}_f \psi \), and in particular for real-valued functions \( \psi \), we have \( \bar{L}_f \psi = \bar{L}_f \psi \).

Furthermore, it is helpful to note that \( L f \bar{f} = \bar{L} f \bar{f} = \Delta f \bar{f} = \Delta f - |\nabla f|^2 \), and so the \( L^2 \)-gradient of \( \mathcal{H} \) can be written in three equivalent ways as:

\[ \mathcal{D} \mathcal{H}(\varphi) = - (\Delta f + \lambda \text{Id}) (f - \mathcal{H}) \]

\[ = - (L_f + \lambda \text{Id}) (f - \mathcal{H}) \]

\[ = - (\bar{L}_f + \lambda \text{Id}) (f - \mathcal{H}) . \]

All three of \( L_f, \bar{L}_f \) and \( \Delta f \) are (complex) self-adjoint operators on \( C^\infty(X, \mathbb{C}) \) with respect to the inner product:

\[ (\psi_1, \psi_2)_f := \frac{1}{\omega_\partial^n} \int_X \psi_1 \bar{\psi}_2 e^{-\int_\mathcal{H} \omega_\partial^n} \]

in a sense that \( (L_f (\psi_1), \psi_2)_f = (\psi_1, L_f (\psi_2))_f \) and similarly for \( \bar{L}_f \) and \( \Delta f \). Therefore, their eigenvalues are real.

By a standard argument (see e.g. [2] [15]), it can be shown that when acting on the orthogonal complement of constant functions, both \( -L_f \) and \( -\bar{L}_f \) (and hence for \( -\Delta f \)) have the lowest eigenvalue \( \geq \lambda \). Due to its importance to our upcoming discussions, we state the result below and sketch its proof:

**Lemma 3.3** (c.f. [2] [15]). Given any non-constant function \( \psi \in C^\infty(X, \mathbb{C}) \) of \( -L_f \) such that:

\[ L_f \psi = -\mu \psi, \]

then we have \( \mu \geq \lambda \), and equality holds if and only if \( \nabla^{1,0} \psi := g^{ij} \frac{\partial \psi}{\partial z_j} \frac{\partial}{\partial z_i} \) is a holomorphic vector field.

**Sketch of Proof.** Given that \( L_f \psi = -\mu \psi \) for some non-constant \( \psi \in C^\infty(X, \mathbb{C}) \), we have:

\[ \Delta \psi - g^{ij} \nabla_j f \nabla_i \psi = -\mu \psi. \]
holds if and only if
\[ \nabla \nabla \psi = -\lambda \nabla \psi - \nabla \phi \cdot \nabla f = -\mu \nabla \psi \]
Using the commutative formula for covariant derivatives and the fact that \( \lambda g_{i\bar{j}} - R_{i\bar{j}} = \nabla_i \nabla_j f \), one can conclude:
\[ g^{i\bar{j}} \nabla_i \nabla_j \psi - \lambda \nabla_k \psi = g^{i\bar{j}} \nabla_k \nabla_i \psi \cdot \nabla_j f = -\mu \nabla_k \psi \]
\[
\implies g^{i\bar{j}} \nabla_j (e^{-f} \nabla_k \psi) = (\lambda - \mu) \nabla_k \psi.
\]
Finally, by multiplying both sides by \( g^{k\bar{l}} \nabla_k \bar{\psi} \), integrating both sides over \( X \) with respect to the measure \( e^{-f} \omega^n \phi \) and using integration-by-parts, we get:
\[
\int_X |\nabla \nabla \psi|^2 e^{-f} \omega^n = (\mu - \lambda) \int_X |\nabla \psi|^2 e^{-f} \omega^n
\]
where \( |\nabla \nabla \psi|^2 = g^{k\bar{l}} g^{i\bar{j}} (\nabla_k \nabla_i \psi) (\nabla_j \nabla_l \bar{\psi}) \). Therefore, we must have \( \mu \geq \lambda \), and equality holds if and only if \( \nabla_j \nabla_l \bar{\psi} = 0 \) for any \( j \) and \( l \), which is equivalent to saying that \( \nabla^{1,0} \bar{\psi} \) is holomorphic.

If \( \phi \) is a critical point of \( \mathcal{H} \), i.e. \( D\mathcal{H}(\phi) = 0 \), then \( f_\phi \) satisfies the Euler-Lagrange’s equation:
\[
(L_f + \lambda \text{Id}) (f - \mathcal{H}) = 0
\]
or equivalently, \( f_\phi - \mathcal{H}(\phi) \) is an eigenfunction of \( L_f \) with eigenvalue \( \lambda \). By (3.3), \( \nabla f_\phi \) is then real holomorphic. Therefore, the critical potentials \( \phi \) of \( \mathcal{H}(\phi) \) are those which \( \omega_\phi \) is a gradient Kähler-Ricci soliton.

The Kähler-Ricci flow:
\[
\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) + \lambda \omega, \quad \omega(0) = \omega_0
\]
in the canonical class, i.e. \( \lambda \omega_0 \in c_1(X) \), can be regarded as the flow of Kähler potentials:
\[
\frac{\partial \phi}{\partial t} = f_\phi - \mathcal{H}(\phi)
\]
\[
= f_0 + \log \frac{\omega^n}{\omega_0^n} + \lambda \phi - \frac{1}{[\omega_0]^n} \int_X e^{-f_0 - \lambda \phi} \omega^n - \mathcal{H}(\phi)
\]
in a sense that if \( \phi(t) \) satisfies (3.5), then \( \omega(t) := \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi(t) \) satisfies (3.4).

It is interesting to note that using \( -\Delta f \geq \lambda \text{Id} \), we can show that \( \mathcal{H}(\phi) \) is monotonically increasing along the Kähler-Ricci flow \( \frac{\partial \phi}{\partial t} = f_\phi - \mathcal{H}(\phi) \):
\[
\frac{d}{dt} \mathcal{H}(\phi) = -\frac{1}{[\omega_0]^n} \int_X (f_\phi - \mathcal{H}(\phi)) \cdot (\Delta f + \lambda \text{Id}) (f_\phi - \mathcal{H}(\phi)) e^{-f_\phi} \omega^n \geq 0.
\]
Here we have used the fact that \( -\Delta f - \lambda \text{Id} \geq 0 \) acting on non-constant functions and that \( f_\phi - \mathcal{H}(\phi) \) is orthogonal to constant functions (see also [10 Section 2]).

4. Commutator of \( L_f \) and \( \bar{L}_f \)

We will make use of the operators \( L_f \) and \( \bar{L}_f \) in the proof of the main theorem. It is important to note that in general \( L_f L_f \neq L_f \bar{L}_f \). In this section, we will compute the product \( L_f L_f \) and \( L_f \bar{L}_f \) acting on scalar functions, and show that if \( \nabla f \) is real holomorphic, then \( \bar{L}_f \) and \( L_f \) indeed commute with each other.
Lemma 4.1. For any $\psi \in C^\infty(X, \mathbb{R})$, we have:

\begin{align}
(4.1) \quad \bar{L}_f L_f \psi &= \Delta \Delta \psi - 2(\nabla f, \nabla \Delta \psi) - \nabla_j \nabla_i \psi \cdot (\nabla_j \nabla_i f - \nabla_j f \cdot \nabla_i f) \\
&\quad - e^f \nabla_j \left( e^{-f} \nabla_i \nabla_j f \cdot \nabla^j \psi \right) \\
(4.2) \quad L_f \bar{L}_f \psi &= \Delta \Delta \psi - 2(\nabla f, \nabla \Delta \psi) - \nabla_j \nabla_i \psi \cdot (\nabla_j \nabla_i f - \nabla_j f \cdot \nabla_i f) \\
&\quad - e^f \nabla_j \left( e^{-f} \nabla_i \nabla_j f \cdot \nabla^j \psi \right)
\end{align}

If $\varphi$ is a critical point of $\mathcal{H}$ (so that $\nabla f_\varphi$ is real holomorphic), then we have $\bar{L}_f L_f = L_f \bar{L}_f$.

Proof. It suffices to show (4.1) only then (4.2) follows from conjugation.

\begin{align}
(4.3) \quad \bar{L}_f L_f \psi &= \bar{L}_f \left( \Delta \psi - \nabla^i f \cdot \nabla_i \psi \right) \\
&\quad = \underset{L_f(\Delta \psi)}{\Delta \Delta \psi - \nabla^i f \cdot \nabla_i \Delta \psi - \bar{L}_f \left( \nabla^i f \cdot \nabla_i \psi \right)}
\end{align}

For convenience, we use holomorphic normal coordinates (with respect to $\omega_\varphi$) in the rest of computations.

\begin{align*}
\bar{L}_f \left( \nabla^i f \cdot \nabla_i \psi \right) \\
= \frac{1}{2} \nabla_j \nabla_j \left( \nabla^i f \cdot \nabla_i \psi \right) + \frac{1}{2} \nabla_j \nabla_j \left( \nabla^i f \cdot \nabla_i \psi \right) - \nabla_j f \cdot \nabla_j \left( \nabla^i f \cdot \nabla_i \psi \right) \\
= \frac{1}{2} \nabla_j \left( \nabla_j \nabla_i f \cdot \nabla_i \psi + \nabla_i f \cdot \nabla_j \nabla_i \psi \right) + \frac{1}{2} \nabla_j \left( \nabla_j \nabla_i f \cdot \nabla_i \psi + \nabla_i f \cdot \nabla_j \nabla_i \psi \right) \\
&\quad - \nabla_j f \cdot \nabla_j \nabla_i f \cdot \nabla_i \psi - \nabla_j f \cdot \nabla_i f \cdot \nabla_j \nabla_i \psi \\
= \frac{1}{2} \nabla_j \nabla_j \nabla_i f \cdot \nabla_i \psi + \frac{1}{2} \nabla_j \nabla_j \nabla_i f \cdot \nabla_i \psi + \frac{1}{2} \nabla_j \nabla_j \nabla_i f \cdot \nabla_i \psi + \frac{1}{2} \nabla_i f \cdot \nabla_j \nabla_j \nabla_i \psi \\
&\quad + \frac{1}{2} \nabla_j \nabla_j \nabla_i f \cdot \nabla_i \psi + \frac{1}{2} \nabla_j \nabla_j \nabla_i f \cdot \nabla_j \nabla_i \psi + \frac{1}{2} \nabla_i f \cdot \nabla_j \nabla_j \nabla_i \psi \\
&\quad - \nabla_j f \cdot \nabla_j \nabla_i f \cdot \nabla_i \psi - \nabla_j f \cdot \nabla_i f \cdot \nabla_j \nabla_i \psi
\end{align*}

Grouping the 5th and 8th terms together, we get:

\begin{align*}
\frac{1}{2} \nabla_j \nabla_j \nabla_i f \cdot \nabla_i \psi + \frac{1}{2} \nabla_i f \cdot \nabla_j \nabla_j \nabla_i \psi \\
&\quad = \frac{1}{2} \nabla_j \nabla_j \nabla_i f \cdot \nabla_i \psi + \frac{1}{2} \nabla_i f \cdot \nabla_j \nabla_j \nabla_i \psi \\
&\quad = \frac{1}{2} \left( \nabla_j \nabla_j \nabla_i f - R_{ji jk} \cdot \nabla_k f \right) \cdot \nabla_i \psi + \frac{1}{2} \nabla_i f \cdot \left( \nabla_i \nabla_j \nabla_j \psi - R_{ji jk} \cdot \nabla_k \psi \right) \\
&\quad = \frac{1}{2} \nabla_j \nabla_j \nabla_i f \cdot \nabla_i \psi + \frac{1}{2} \nabla_i f \cdot \nabla_i \Delta \psi.
\end{align*}

The Riemann curvature terms cancel each other by the Bianchi identity.

Substituting it back in, we get:

\begin{align*}
\bar{L}_f \left( \nabla^i f \cdot \nabla_i \psi \right) \\
&= \nabla_j \nabla_j \nabla_i \psi \cdot \left( \nabla_j \nabla_i f - \nabla_j f \cdot \nabla_i f \right) + \nabla_i f \cdot \nabla_i \Delta \psi \\
&\quad + \nabla_j \nabla_j \nabla_j f \cdot \nabla_i \psi + \nabla_j \nabla_j \nabla_i f \cdot \nabla_j \nabla_i \psi - \nabla_j f \cdot \nabla_j \nabla_j f \cdot \nabla_i \psi.
\end{align*}

It is straightforward to verify that the last three terms sum up to $e^f \nabla_j \left( e^{-f} \nabla_i \nabla_j f \cdot \nabla^i \psi \right)$.

Combining with (4.3), we proved (4.1). \qed
Furthermore, we define the weighted divergence $\text{div}_f$ by:

\[
\text{div}_f X = \frac{1}{2} e^f \left[ \nabla_i (e^{-f} X^i) + \nabla_i (e^{-f} \bar{X}^i) \right] \quad \text{for any vector field } X
\]

\[
\text{div}_f \alpha = \frac{1}{2} e^f \left[ \nabla_i (e^{-f} \alpha_i) + \nabla_i (e^{-f} \bar{\alpha}_i) \right] \quad \text{for any 1-form } \alpha
\]

\[
[\text{div}_f \eta]_B = \frac{1}{2} e^f \sum_A \nabla^A (e^{-f} \eta_{AB}) \quad \text{for any symmetric 2-tensor } \eta
\]

Clearly we have $\text{div}_f \nabla \psi = \Delta_f \psi$ for any $\psi \in C^\infty(X)$. Using the weighted divergence, one can also express $L_fL_f$ as the following:

**Lemma 4.2.** Given any $\psi \in C^\infty(X, \mathbb{R})$, we denote $\nabla \nabla \psi = \psi_{\bar{i}j} \left( dz^i \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^i \right)$ which is a symmetric real 2-tensor. Then we have:

\[
2 \text{div}_f \text{div}_f \nabla \nabla \psi = \Delta \Delta \psi - 2 \langle \nabla f, \nabla \Delta \psi \rangle + \nabla_j \psi \left( \nabla^j f \cdot \nabla_i \nabla \nabla \psi - \nabla^j \nabla^i \psi \right)
\]

and hence from (4.1), we have:

\[
L_fL_f \psi = 2 \text{div}_f \text{div}_f \nabla \nabla \psi - e^f \nabla^i \left( e^{-f} \nabla_{\bar{i}j} f \cdot \nabla^j \psi \right).
\]

**Proof.** It can be proved by straight-forward computations. We first compute:

\[
\left[ \text{div}_f (\nabla \nabla \psi) \right]_j = \frac{1}{2} e^f \nabla^i \left( e^{-f} \nabla_i \nabla_j \psi \right)
\]

\[
= \frac{1}{2} \left( -\nabla^i f \cdot \nabla_i \nabla_j \psi + \nabla^i \nabla_i \nabla_j \psi \right)
\]

\[
= \frac{1}{2} \left( -\nabla^i f \cdot \nabla_i \nabla_j \psi + \nabla_j \Delta \psi \right)
\]

and so we can derive:

\[
\text{div}_f \text{div}_f (\nabla \nabla \psi) = \text{Re} \left\{ e^f \nabla^j \left( e^{-f} \text{div}_f (\nabla \nabla \psi) \right) \right\}
\]

\[
= \text{Re} \left\{ -\nabla^j f \cdot \left[ \text{div}_f (\nabla \nabla \psi) \right]_j + \nabla^j \left[ \text{div}_f (\nabla \nabla \psi) \right]_j \right\}
\]

\[
= \frac{1}{2} \text{Re} \left\{ \nabla_i \nabla_j \psi \cdot \nabla^i f \cdot \nabla^j f - \nabla^j \nabla_j \psi - \nabla^i f \cdot \nabla_i \nabla \Delta \psi - \nabla^i \nabla^j f \cdot \nabla_i \nabla_j \psi + \Delta \Delta \psi \right\}
\]

\[
= \frac{1}{2} \left( \nabla_j \nabla_i \psi \left( \nabla^j f \cdot \nabla^i f - \nabla^j \nabla^i f \right) - 2 \langle \nabla \Delta \psi, \nabla f \rangle + \Delta \Delta \psi \right)
\]

as desired for (4.4). Then, (4.5) follows from (4.4). \qed

**Remark 4.3.** It is also helpful to note that $2 \text{div}_f \text{div}_f = (\nabla \nabla)^*$, which is the adjoint of

\[
\nabla \nabla : C^\infty(X, \mathbb{R}) \to \text{Sym}^2(X)
\]

\[
\psi \mapsto \psi_{\bar{i}j} \left( dz^i \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^i \right)
\]

with respect to the measure $e^{-f} \omega^n$. 


5. Second Variation of $\mathcal{H}$

Next we derive the second variation formula of $\mathcal{H}$, and show that every Kähler-Ricci soliton is linearly stable with respect to $\mathcal{H}$. We first show:

**Proposition 5.1** (Evolution Equation of $\mathcal{D}\mathcal{H}$). The variation of $\mathcal{D}\mathcal{H}$ along a 1-parameter family $\varphi(t)$ of Kähler potential in $\mathcal{K}$ such that $\frac{\partial \varphi}{\partial t} = \psi$ is given by:

\[
\frac{\partial}{\partial t} \mathcal{D}\mathcal{H} = -2 \operatorname{div}_f \operatorname{div}_f \left( \nabla \nabla \psi \right) - 2 \lambda \Delta_f \psi - \lambda^2 \left( \psi - \frac{1}{[\omega_0]^n} \int_X \psi e^{-f} \omega^n \right) + \frac{\lambda}{[\omega_0]^n} \int_X \psi \mathcal{D}\mathcal{H} e^{-f} \omega^n
\]

**Proof.** First note that $\frac{\partial}{\partial t} \mathcal{D}\mathcal{H} = -(\Delta_f + \lambda \operatorname{Id}) (f - \mathcal{H})$. We compute:

\[
\frac{\partial}{\partial t} (\Delta_f + \lambda \operatorname{Id}) (f - \mathcal{H}) = \frac{\partial}{\partial t} \left( \Delta f - |\nabla f|^2 + \lambda (f - \mathcal{H}) \right)
\]

\[
= -g^{ij} g^{pq} \cdot \nabla_p \nabla_q \left( \frac{\partial \varphi}{\partial t} \right) \cdot \nabla_i \nabla_j f + \Delta \left( \frac{\partial f}{\partial t} \right) + g^{ij} g^{pq} \cdot \nabla_p \nabla_q \left( \frac{\partial \varphi}{\partial t} \right) \cdot \nabla_i f \cdot \nabla_j f
\]

\[
- 2 \left( \nabla \left( \frac{\partial f}{\partial t} \right), \nabla f \right) + \lambda \left( \frac{\partial f}{\partial t} - \frac{\partial \mathcal{H}}{\partial t} \right)
\]

From Lemma 2.1 we have $\frac{\partial f}{\partial t} = \Delta \psi + \lambda \left( \psi - \frac{1}{[\omega_0]^n} \int_X \psi e^{-f} \omega^n \right)$. Therefore:

\[
\frac{\partial}{\partial t} (\Delta f + \lambda \operatorname{Id}) (f - \mathcal{H}) = \nabla_i \nabla_j \psi \left( \nabla^i f \cdot \nabla^j f - \nabla^j \nabla^i f \right) + \Delta \left( \Delta \psi + \lambda \left( \psi - \frac{1}{[\omega_0]^n} \int_X \psi e^{-f} \omega^n \right) \right)
\]

\[
- 2 \left( \nabla \left( \Delta \psi + \lambda \left( \psi - \frac{1}{[\omega_0]^n} \int_X \psi e^{-f} \omega^n \right) \right), \nabla f \right)
\]

\[
+ \lambda \left( \Delta \psi + \lambda \left( \psi - \frac{1}{[\omega_0]^n} \int_X \psi e^{-f} \omega^n \right) - \frac{\partial \mathcal{H}}{\partial t} \right)
\]

\[
= \Delta \Delta \psi - 2 \left( \nabla f, \nabla \Delta \psi \right) + \nabla_i \nabla_j \psi \left( \nabla^i f \cdot \nabla^j f - \nabla^j \nabla^i f \right)
\]

\[
+ 2 \lambda \Delta \psi - 2 \lambda (\nabla \psi, \nabla \psi) + \lambda^2 \left( \psi - \frac{1}{[\omega_0]^n} \int_X \psi e^{-f} \omega^n \right) - \lambda \frac{\partial \mathcal{H}}{\partial t}
\]

\[
= 2 \operatorname{div}_f \operatorname{div}_f (\nabla \nabla \psi) + 2 \lambda \Delta f \psi + \lambda^2 \left( \psi - \frac{1}{[\omega_0]^n} \int_X \psi e^{-f} \omega^n \right) - \lambda \frac{\partial \mathcal{H}}{\partial t}
\]

\[
= 2 \operatorname{div}_f \operatorname{div}_f (\nabla \nabla \psi) + 2 \lambda \Delta f \psi + \lambda^2 \left( \psi - \frac{1}{[\omega_0]^n} \int_X \psi e^{-f} \omega^n \right) - \frac{\lambda}{[\omega_0]^n} \int_X \psi \mathcal{D}\mathcal{H} e^{-f} \omega^n
\]

It completes the proof of 5.1. \(\square\)
Proposition 5.2 (Second Variation of $\mathcal{H}$). Let $\varphi(s,t)$ be a 2-parameter family of potentials in $\mathcal{K}$. Denote $\chi := \frac{\partial \varphi}{\partial s}$ and $\psi := \frac{\partial \varphi}{\partial t}$, then the second variation of $\mathcal{H}(\varphi)$ is given by:

\begin{align}
(5.2) \quad \frac{\partial^2}{\partial s \partial t} \mathcal{H}(\varphi(s,t)) & = \frac{1}{|\omega_0|^n} \int_X \frac{\partial^2 \varphi}{\partial s \partial t} \mathcal{D}(\varphi) e^{-f_{\varphi} \omega^n_\varphi} - \frac{\lambda}{|\omega_0|^n} \int_X (\chi - \chi) (\psi - \psi) \mathcal{D}(\varphi) e^{-f_{\varphi} \omega^n_\varphi} \\
& - \frac{1}{|\omega_0|^n} \int_X \psi \left[ 2 \text{div}_f \text{div}_f \nabla \chi + 2 \lambda \Delta_f \chi + \lambda^2 (\chi - \chi) \right] e^{-f_{\varphi} \omega^n_\varphi}
\end{align}

where $\chi := \frac{1}{|\omega_0|^n} \int_X \chi e^{-f_{\varphi} \omega^n_\varphi}$ and $\psi := \frac{1}{|\omega_0|^n} \int_X \psi e^{-f_{\varphi} \omega^n_\varphi}$ are the averages of $\chi$ and $\psi$ over $X$ with respect to the measure $e^{-f_{\varphi} \omega^n_\varphi}$.

Proof. To begin, we recall that:

\[
\frac{\partial}{\partial t} \mathcal{H}(\varphi) = \frac{1}{|\omega_0|^n} \int_X \psi \mathcal{D}(\varphi) e^{-f_{\varphi} \omega^n_\varphi}.
\]

Next we differentiate both sides by $s$:

\[
\frac{\partial^2}{\partial s \partial t} \mathcal{H}(\varphi(s,t)) = \frac{1}{|\omega_0|^n} \int_X \left( \frac{\partial \psi}{\partial s} \mathcal{D}(\varphi) + \psi \frac{\partial}{\partial s} \mathcal{D}(\varphi) \right) e^{-f_{\varphi} \omega^n_\varphi} - \frac{1}{|\omega_0|^n} \int_X \nabla \mathcal{D}(\varphi) \left( \frac{\partial f_{\varphi}}{\partial t} - \Delta \chi \right) e^{-f_{\varphi} \omega^n_\varphi}
\]

Recall from (2.1) that:

\[
\frac{\partial f_{\varphi}}{\partial s} = \Delta \chi + \lambda (\chi - \chi).
\]

From Proposition 5.1 we also have:

\[
\frac{\partial}{\partial s} \mathcal{D}(\varphi) = -2 \text{div}_f (\nabla \chi) - 2 \lambda \Delta_f \chi - \lambda^2 (\chi - \chi) + \frac{\lambda}{|\omega_0|^n} \int_X \chi \mathcal{D}(\varphi) e^{-f_{\varphi} \omega^n_\varphi}.
\]

Substituting these two results back in, we get:

\[
\frac{\partial^2}{\partial s \partial t} \mathcal{H}(\varphi(s,t)) = \frac{1}{|\omega_0|^n} \int_X \frac{\partial^2 \varphi}{\partial s \partial t} \mathcal{D}(\varphi) e^{-f_{\varphi} \omega^n_\varphi} - \frac{1}{|\omega_0|^n} \int_X \psi \left[ 2 \text{div}_f (\nabla \chi) + 2 \lambda \Delta_f \chi + \lambda^2 (\chi - \chi) \right] e^{-f_{\varphi} \omega^n_\varphi} + \frac{\lambda}{|\omega_0|^n} \int_X \psi \chi \mathcal{D}(\varphi) e^{-f_{\varphi} \omega^n_\varphi} - \frac{\lambda}{|\omega_0|^n} \int_X \psi (\chi - \chi) \mathcal{D}(\varphi) e^{-f_{\varphi} \omega^n_\varphi}.
\]

Finally, using the fact that:

\[
(\psi - \psi) (\chi - \chi) \mathcal{D}(\varphi) = \psi (\chi - \chi) \mathcal{D}(\varphi) - \psi \chi \mathcal{D}(\varphi) + \psi \chi \mathcal{D}(\varphi)
\]

and $\int_X \mathcal{D}(\varphi) e^{-f_{\varphi} \omega^n_\varphi} = 0$, we have completed the proof of the proposition.

\[\square\]
Corollary 5.3. In particular, if $\mathcal{D}\mathcal{H}(\varphi(0,0)) = 0$ (i.e. $\omega_0$ is a Kähler-Ricci soliton), then we have:

\begin{align}
(5.3) \quad \frac{\partial^2}{\partial s \partial t} \bigg|_{(s,t)=(0,0)} \mathcal{H}(\varphi(s,t)) &= -\frac{1}{|\omega_0|^n} \int_X \psi \left[ 2 \operatorname{div}_f \operatorname{div}_f \nabla \nabla \chi + 2\lambda \Delta_f \chi + \lambda^2 \left( \chi - \chi^2 \right) \right] e^{-f \omega_0^n} \\
\end{align}

6. $\mathcal{H}$-Stability of Kähler-Ricci Solitons

In the study of functionals in geometric analysis, the second variation formula is often associated with notions of stability. In the previous section we have computed the second variation formula of $\mathcal{H}$. Motivated by the second variation, we introduce:

Definition 6.1 (Stability Operator). In view of Proposition 5.2, we define the stability operator $S_f : T_{\varphi}K \to T_{\varphi}K$ by:

\begin{align}
(6.1) \quad S_f(\psi) := 2 \operatorname{div}_f \operatorname{div}_f \nabla \nabla \psi + 2\lambda \Delta_f \psi + \lambda^2 \left( \psi - \frac{1}{|\omega_0|^n} \int_X \psi e^{-f \omega_0^n} \right)
\end{align}

As such, the second variation of $\mathcal{H}$ at a critical point $\omega_\varphi$ is given by:

\begin{align}
\left. \frac{d^2}{dt^2} \right|_{t=0} \mathcal{H}(\varphi + t\psi) = -\langle \psi, S_f(\psi) \rangle_f
\end{align}

Since the functional $\mathcal{H}$ is monotonically increasing along the Kähler-Ricci flow, we say a Kähler-Ricci soliton $\omega_\varphi$ is stable with respect to $\mathcal{H}$ (or simply $\mathcal{H}$-stable) if and only if $\left. \frac{d^2}{dt^2} \right|_{t=0} \mathcal{H}(\varphi + t\psi) \leq 0$ for any $\psi \in T_{\varphi}K$. We are ready to give the proof of our main theorem that any Kähler-Ricci soliton is stable in this sense.

Theorem 6.2 ($\mathcal{H}$-Stability). Suppose $\varphi$ is a critical point of $\mathcal{H}$, i.e. $\omega_\varphi := \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi$ is a Kähler-Ricci soliton, then we have:

\begin{align}
\left. \frac{d^2}{dt^2} \right|_{t=0} \mathcal{H}(\varphi + t\psi) \leq 0
\end{align}

for any $\psi \in T_{\varphi}K$, and equality holds if and only if $\nabla \psi$ is real holomorphic.

Proof. In view of

\begin{align}
\left. \frac{d^2}{dt^2} \right|_{t=0} \mathcal{H}(\varphi + t\psi) = -\langle \psi, S_f(\psi) \rangle_f
\end{align}

when $\varphi$ is a critical point of $\mathcal{H}$, it suffices to show the stability operator $S_f$ is non-negative definite on $K$. Since $S_f$ is self-adjoint with respect to the $(\cdot, \cdot)_f$ and $S_f(c) = 0$ for any constant $c$, we have $(c, S_f(\psi))_f = 0$ as well and so:

\begin{align}
\langle \psi, S_f(\psi) \rangle_f = \langle \psi - \bar{\psi}, S_f(\psi - \bar{\psi}) \rangle_f.
\end{align}

When $\varphi$ is a Kähler potential such that $\omega_\varphi$ is a Kähler-Ricci soliton, we have $\nabla_i \nabla_j f_\varphi = \nabla_i \nabla_j f_\varphi = 0$ for any $i$ and $j$. Thus, the last term of $\mathcal{L}_f$ in Lemma 4.2 vanishes, and we have:

\begin{align}
\mathcal{L}_f L_f \psi = 2 \operatorname{div}_f \operatorname{div}_f \nabla \nabla \psi
\end{align}
for any $\psi \in T_{\varphi}K$, and so:

\[
S_f(\psi) = L_f L_f \psi + 2\lambda \Delta \psi - 2\lambda (\nabla \psi, \nabla f) + \lambda^2 (\psi - \bar{\psi})
\]

\[
= \bar{L}_f L_f \psi + \lambda (\bar{L}_f + L_f) \psi + \lambda^2 (\psi - \bar{\psi})
\]

\[
= (\bar{L}_f + \lambda Id) (L_f + \lambda Id) (\psi - \bar{\psi})
\]

Note that $\bar{L}_f + \lambda Id \leq 0$ and $L_f + \lambda Id \leq 0$, and that they are self-adjoint and commutative at $t = 0$ (from Lemma 4.1), so they can be simultaneously diagonalized and the product $(\bar{L}_f + \lambda Id)(L_f + \lambda Id)$ is non-negative definite. Since $S_f$ is self-adjoint with respect to the $(\cdot, \cdot)_f$ and $S_f(c) = 0$ for any constant $c$, we have $(c, S_f(\psi)) = 0$ as well and so:

\[
(\psi, S_f(\psi))_f = (\psi - \bar{\psi}, S_f(\psi - \bar{\psi}))_f \geq 0
\]

since $(\bar{L}_f + \lambda Id)(L_f + \lambda Id) \geq 0$. It completes the proof that:

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{H}(\varphi + t\psi) = - (\psi, S_f(\psi)) = - (\psi - \bar{\psi}, S_f(\psi - \bar{\psi})) \leq 0.
\]

Equality holds if and only if

\[
(\bar{L}_f + \lambda Id) (L_f + \lambda Id) (\psi - \bar{\psi}) = 0,
\]

which is equivalent to the fact that $\nabla \psi$ is a real holomorphic vector field. \hfill $\Box$

7. Relation with Perelman’s Entropy

In this section, we focus on the case where $c_1(X) > 0$, and $\omega_0$ is a Kähler metric such that $\lambda \omega_0 \in c_1(X)$ (where $\lambda > 0$). Recall that Perelman’s $W$-functional defined by:

\[
W(g, f, \tau) := \int_X \left[ 2\tau (R + |\nabla f|^2) + f - 2n \right] e^{-f} \omega^n_g.
\]

By taking a suitable $\tau = \tau_0$ such that $[\omega_0]^n = (4\pi \tau_0)^n$, the Perelman’s $\mu$-functional is defined by:

\[
\mu(g) := \inf \left\{ W(g, f, \tau_0) : \int_X e^{-f} \omega^n_g = (4\pi \tau_0)^n \right\}
\]

The first variation of $\mu$ is given by:

\[
\left. \frac{d}{dt} \right|_{t=0} \mu(g + th) = \left. \frac{1}{(4\pi \tau_0)^n} \right|_X \left[ h, \frac{1}{2} g - \tau_0 (\text{Ric} + \nabla^2 f_{\text{min}}) \right] e^{-f_{\text{min}}} dV_g
\]

where $f_{\text{min}}$ is the minimizer such that $\mu(g) = W(g, f_{\text{min}}, \tau_0)$. Therefore, $g$ is a critical metric of $\mu$ if and only if $g$ is a Ricci soliton satisfying:

\[
\text{Ric}(g) + \nabla^2 f_{\text{min}} = \frac{1}{2\tau_0} g.
\]

In our case we have $\lambda \omega_0 \in c_1(X)$, so it is necessary that $\tau_0 = \frac{1}{\lambda}$. Our goal in this section is to show that the Perelman’s $\mu$-functional is concave at Kähler-Ricci solitons along the direction of complex Hessians of potential functions. This result was previously proved by Tian–Zhu in [21] using the second variation of $\mu(g + t\nabla \psi)$. Many dynamical stability results of the Kähler-Ricci flow were established using this results, for instance [21, 18, 24, 25].
We are going to show that the Boltzmann’s type entropy $\mathcal{H}(\varphi + t\psi)$ is an upper barrier of $\mu(g + t\nabla\nabla\psi)$ up to an addition of a constant, and they coincide at $t = 0$ if $\omega_\varphi$ is a Kähler-Ricci soliton. Therefore, if the second variation of $\mathcal{H}(\varphi + t\psi)$ is non-positive at $t = 0$, then so is the second variation of $\mu(g + t\nabla\nabla\psi)$, thus giving a new proof to Tian–Zhu’s result.

**Proposition 7.1** (c.f. [21]). Given a gradient Kähler-Ricci soliton $g$ on $X$ with Kähler form $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$, we have:

\[
\frac{d^2}{dt^2}\bigg|_{t=0} \mu(g + t\nabla\nabla\psi) \leq \frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{H}(\varphi + t\psi) \leq 0.
\]

Furthermore, we have $\frac{d^2}{dt^2}\bigg|_{t=0} \mu(g + t\nabla\nabla\psi) = 0$ if and only if $\nabla\psi$ is a real holomorphic vector field.

**Proof.** For any $t \in (-\varepsilon, \varepsilon)$, by the definition of $\mu$, we have:

\[
\mu(g + t\nabla\nabla\psi) \leq \mathcal{W}(g + t\nabla\nabla\psi, f_{\varphi + t\psi}, \tau_0)
\]

where $f_{\varphi + t\psi}$ is the Ricci potential of $\omega_{\varphi + t\psi}$. By the definition of $\mathcal{W}$, we have:

\[
\mathcal{W}(g + t\nabla\nabla\psi, f_{\varphi + t\psi}, \tau_0) = \frac{1}{|\omega_0|^n} \int_X \left[ 2\tau_0 \left( R + |\nabla f_{\varphi + t\psi}|^2 \right) + f_{\varphi + t\psi} - 2n \right] e^{-f_{\varphi + t\psi}} \omega_\varphi^n + t\psi
\]

\[
= \frac{1}{|\omega_0|^n} \int_X \left[ 2\tau_0 \left( n - \Delta f_{\varphi + t\psi} + |\nabla f_{\varphi + t\psi}|^2 \right) + f_{\varphi + t\psi} - 2n \right] e^{-f_{\varphi + t\psi}} \omega_\varphi^n + t\psi
\]

\[
= 2n(\tau_0 - 1) + \mathcal{H}(\varphi + t\psi).
\]

Here we used the fact that $\int_X \left( -\Delta f + |\nabla f|^2 \right) e^{-f} \omega^n = \int_X \Delta(e^{-f}) \omega^n = 0$. Therefore, for any $t \in (-\varepsilon, \varepsilon)$, we have

\[(7.1) \quad \mu(g_{\varphi} + t\nabla\nabla\psi) \leq \mathcal{W}(g + t\nabla\nabla\psi, f_{\varphi + t\psi}, \tau_0) = \mathcal{H}(\varphi + t\psi) + 2n(\tau_0 - 1).
\]

At $t = 0$, we have $g + t\nabla\nabla\psi = g$ and the Ricci potential $f_\varphi$ coincides with the minimizer $f_{\min}$ such that $\mu(g) = \mathcal{W}(g, f_{\min}, \tau_0)$. Therefore, we have:

\[(7.2) \quad \mu(g) = \mathcal{W}(g, f_\varphi, \tau) = \mathcal{H}(\varphi) + 2n(\tau_0 - 1).
\]

Combining (7.1) and (7.2), we have shown that $\mathcal{H}(\varphi + t\psi) + 2n(\tau_0 - 1)$ is an upper barrier of $\mu(g + t\nabla\nabla\psi)$ and that they are equal at $t = 0$. Therefore, we have:

\[
\frac{d^2}{dt^2}\bigg|_{t=0} \mu(g + t\nabla\nabla\psi) \leq \frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{H}(\varphi + t\psi).
\]

The proposition then follows easily from Theorem 6.2. □

**Remark 7.2.** In [21], the second variation of $\mu(g + t\nabla\nabla\psi)$ computed at a shrinking Kähler-Ricci soliton (using the notations in this article) is given by:

\[
\frac{d^2}{dt^2}\bigg|_{t=0} \mu(g + t\nabla\nabla\psi) = \left( \psi, \left( L_f + \bar{L}_f + \lambda \text{Id} \right)^{-1} L_f L_f(\bar{L}_f + \lambda \text{Id})(L_f + \lambda \text{Id}) \psi \right)_f
\]

which is non-positive since $L_f \leq -\lambda \text{Id}$ and $\bar{L}_f \leq -\lambda \text{Id}$. 

Furthermore, we have $\frac{d^2}{dt^2}\bigg|_{t=0} \mu(g + t\nabla\nabla\psi) = 0$ if and only if $\nabla\psi$ is a real holomorphic vector field.
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