An intrinsic Cramér-Rao bound on SO(3) for (dynamic) attitude filtering

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Abstract—In this note an intrinsic version of the Cramér-Rao bound on estimation accuracy is established on the Special Orthogonal group $SO(3)$. It is intrinsic in the sense that it does not rely on a specific choice of coordinates on $SO(3)$: the result is derived using rotation matrices, but remains valid when using other parameterizations, such as quaternions. For any estimator $\hat R$ of $R \in SO(3)$ we give indeed a lower bound on the quantity $E(\log(\hat R R^T))$, that is, the estimation error expressed in terms of group multiplication, whereas the usual estimation error $E(\hat R - R)$ is meaningless on $SO(3)$. The result is first applied to Whaba’s problem. Then, we consider the problem of a continuous-time nonlinear deterministic system on $SO(3)$ with discrete measurements subject to additive isotropic Gaussian noise, and we derive a lower bound to the estimation error covariance matrix. We prove the intrinsic Cramér-Rao bound coincides with the covariance matrix returned by the Invariant EKF, and thus can be computed online. This is in sharp contrast with the general case, where the bound can only be computed if the true trajectory of the system is known.

I. INTRODUCTION

We consider the problem of attitude estimation from vector measurements, where the attitude parameter is static in the first place, and then the dynamic case where it evolves on the special orthogonal group $SO(3)$. Attitude estimation, both in the static and dynamic cases, has been the subject of numerous works due to its potential applications to e.g. aerial vehicles (or satellites) control. The references are too numerous to be exhaustively listed, and the reader is referred to e.g. [17] for an overview of estimation problems on $SO(3)$, or the survey [14], and e.g. the paper [16] for a very recent work on the subject.

In the present paper an intrinsic version of the Cramér-Rao bound on estimation accuracy is established in Section III on the Special Orthogonal group $SO(3)$. It is intrinsic in the sense that it does not rely on a specific choice of coordinates on $SO(3)$. For any estimator $\hat R$ of $R \in SO(3)$ we give a lower bound on the quantity $E(\log(\hat R R^T))$, that is, the average estimation error expressed in terms of group multiplication, and then projected onto a three dimensional vector space using the logarithmic map of $SO(3)$. This error indeed makes sense as $RR^T$ is the rotation that maps the estimated orientation to the true orientation, whereas the usual estimation error $E(\hat R - R)$ is meaningless on $SO(3)$, as $\hat R - R$ is not a rotation matrix, and has no intrinsic counterpart. Taking advantage of the Lie group structure of the space the calculations are rather simple and direct.

Viewing $SO(3)$ as a manifold and choosing an invariant metric, we recover in a simple way the result derived by S. Smith in [21] (see also the recent work of N. Boumal [11], [12]).

As a straightforward application, the result is first applied to the static attitude estimation problem, also known as Whaba’s problem [24], for which we derive a lower bound. Note a (classical) Cramér-Rao lower bound has already been proposed in [13] for the linearized problem. Then, we consider in Section IV the problem of attitude estimation/filtering from vector measurements and angular velocity measurements from a gyroscope (see e.g. [16], [18] and [15] for an implementation) in the degenerate case where the gyroscope is of much higher quality than the other sensors. For systems possessing deterministic dynamics and stochastic output measurements, J. H. Taylor proved in [22] the Cramér-Rao bound is provided by the Extended Kalman Filter (EKF) covariance, linearized around the true unknown trajectory of the system, and thus cannot be computed online. Thanks to the invariance properties of the system, we prove the Cramér-Rao bound does not depend on the true system’s trajectory, and can be computed online.

The Invariant Extended Kalman Filter (IEKF) is a recent methodology to modify the EKF in order to account for the invariance properties of the ambient space when devising EKFs on Lie groups, see [9], [8], and more recently [3] where an IEKF is derived on $SO(3)$ with discrete time observations. A remarkable property of the IEKF, akin to the properties of symmetry-preserving observers [6], [7] from which the IEKF is derived, is that the estimation error system depends on the system’s trajectory in a reduced manner, and sometimes does not depend on it at all, a property shared by the intrinsic Cramér-Rao bound derived in this paper. In fact, the links between both theories go beyond: in the case considered here on $SO(3)$, we prove in Section IV the intrinsic Cramér-Rao bound coincides with the covariance matrix returned by the Invariant EKF (and thus can be computed online).

II. AN INTRINSIC CRAMÉR-RAO BOUND ON SO(3)

A. The classical Cramér-Rao bound

Consider a family of probability densities $p(x, \theta)$ parameterized by a vector $\theta \in \mathbb{R}^2$. Consider an unbiased estimator $\hat \theta(x)$ of the parameter $\theta$ from a sample measurement $x$. The requirement that $\hat \theta$ be unbiased means it must be “good” (for a large sample) whatever $\theta$. Because of this requirement, the estimator can not recover $\theta$ exactly, given a single or a finite number of measurements. This fact is formalized by the well-known existence of a lower bound on the accuracy of the estimator: the so-called Cramér-Rao bound. Mathematically,
it states the average estimation error covariance is lower bounded as follows

\[ P := E(\hat{\theta} - \theta)(\hat{\theta} - \theta)^T \geq J \]

where \( E \) denotes the expectation with respect to the probability law \( p(x, \theta) \), and the matrix \( J \) (the inequality is in the sense of Lower order) is the so-called Fisher information matrix, defined as the Hessian with respect to \( \theta \) of the average log likelihood \( E(\ln(p(x, \theta))) \).

The interesting question raised by S.T. Smith [21], is whether there exists an analogue of this bound for a parameter \( \theta \) that belongs to a Riemannian manifold, and not a vector space anymore. This kind of question can arise in signal processing, where one seeks to estimate for instance a subspace, as in Principal Component Analysis, that is, an element of the Grassman manifold. To answer this question, one must first find a way to compare the estimator \( \hat{\theta} \) and the true parameter \( \theta \), as on a manifold the quantity \( \hat{\theta} - \theta \) has no meaning. This can be done through the Riemann exponential map, and then, [21] proves a Cramér-Rao bound can be produced. Adapting the classical proof to the manifold case, he shows that the (well-defined) error covariance \( P \) is lower bounded by a (well-defined) information matrix \( J \), plus additional terms stemming from the curvature of the ambient space. Unfortunately, the formula is not in closed form. However, for sufficiently small covariance \( P \) it can be expanded up to terms of order \( O(P^{1/2}) \).

The quite inspiring paper [21] draws new links between statistics and geometry. It has been in particular adapted to a tutorial presentation on the intrinsic Cramér-Rao bound, that is, the inverse of the (well-defined) error covariance \( P \), plus additional terms stemming from the curvature of the ambient space. Unfortunately, the formula is not in closed form. However, for sufficiently small covariance \( P \) it can be expanded up to terms of order \( O(P^{1/2}) \).

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Proposition 1: The intrinsic Fisher information matrix also writes

\[ \xi^T R J(\xi) \xi = -E \frac{d^2}{dt^2} \log p(X \mid \exp(t\xi)R) \]

Let \( \hat{R} \) be an unbiased estimator of \( R \) in the sense of the intrinsic (right invariant) error \( R \hat{R}^T \), that is,

\[ \int_X \log[|R\hat{R}^T(X)|]p(X|R)dX = 0 \]

Theorem 1: Let \( P \) be the covariance matrix of the estimation error as defined in (I). We have then

\[ P \geq J(R)^{-1} - \frac{1}{12} (Tr(P)I_3 - P)J(R)^{-1} - \frac{1}{12} J(R)^{-1}(Tr(P)I_3 - P) + O(Tr(P)^2) \]

For small errors, we can neglect the terms in \( P \) on the right hand side (curvature terms) yielding

\[ P = \int_X \log(R\hat{R}(X)^T) \log(R\hat{R}(X)^T)^T p(X|R)dX \geq J(R)^{-1} + C \]

where \( C \) are terms of higher order, linked to the effect of the curvature (or in other words the non-commutativity) of the parameter space \( SO(3) \), hence the letter \( C \), which here stands for “curvature terms”.

C. Application to Wahba’s problem

Before proving the results let us give an example application. In this subsection we assume measurements are of the form

\[ X_k = R^T d_k + V_k \]

where \( d_k \)'s are some reference vector in \( \mathbb{R}^3 \), and where \( V_k \)'s are independent isotropic and Gaussian noises with covariance matrices \( \sigma_k^2 I d \). This implies the following density form

\[ p(X_k | R) = \frac{1}{\sqrt{2\pi}\sigma_k}(X - R^T d_k)^T (X - R^T d_k) = \frac{1}{2\sigma_k^2} \|X - R^T d_k\|^2 \]

The Wahba’s problem consists in finding the maximum likelihood estimator of \( R \) and has been solved using a Singular Value Decomposition. We now give a lower bound on the estimation accuracy. Let \( \hat{R} \) be an unbiased estimator of \( R \) in the sense of the intrinsic (right invariant) error \( R \hat{R}^T \). After \( n \) measurements, the information matrix is (using the independence of the noises)

\[ \xi^T J_n(\xi) \xi = -\sum_i^n E \frac{d^2}{dt^2} \log p(X_k \mid \exp(t\xi)R) \]

We first note that

\[ \|X - R^T d_k\|^2 = \|X_k\|^2 - 2d_k^T RX_k + \|d\|^2 \]

We have besides the following Taylor expansion

\[ \exp(t\xi) = \expm((t\xi)_\times) = I + t(\xi)_\times + \frac{1}{2} t^2 ((\xi)_\times)^2 + O(t^3) \]

so the quantity we seek is \( -\frac{d^2}{dt^2} [(\xi \times (\xi \times d)]^T (RX_k) \) as \((\xi)_\times^2 \) is symmetric and thus auto adjoint. On average we have \( E(RX_k) = RR^T d_k = d_k \) and thus the term of interest writes on average

\[ E\left(-[(\xi \times (\xi \times d)]^T (RX_k)\right) = -det(\xi, \xi \times d_k, d_k) \]

\[ = det(\xi, d_k, \xi \times d_k) \]

Let \( H_k = (d_k)_\times \) we have thus

\[ det(\xi, d_k, \xi \times d_k) = (\xi \times d_k)^T (\xi \times d_k) = \xi^T H_k^T H_k \xi \]

We have proved the following expression for the information matrix \( J_n(R) = \sum_k 1/\sigma_k^2 H_k^T H_k \).

Proposition 2: For Wahba’s problem [24], the intrinsic covariance matrix defined by (I) where \( \hat{R} \) is any unbiased estimator for a sample of \( n \) independent measurements, satisfies inequality (9) where \( J(R) = \sum_k 1/\sigma_k^2 H_k^T H_k \).

Note that, the result does not depend on the underlying parameter \( R \). This might be explained using theory of equivariant estimators on Lie groups that can be traced back to [20], [19] (see also [5] for a more recent exposure).

D. Proof of the result (6)

Let \( \hat{R} \) be an unbiased estimator of \( R \) in the sense of the intrinsic (right invariant) error \( R \hat{R}^T \), that is,

\[ \int_X \log|\hat{R}^T(X)|p(X|R)dX = 0 \]

So, if we let \( \xi \) be any vector of the Lie algebra and \( t \in \mathbb{R} \), unbiasedness implies in particular \( E[\log(\exp(t\xi)R)]R \hat{R}^T(X) = 0 \) for any \( t \in \mathbb{R} \). Differentiating this equality written as an integral over \( X \) we get

\[ \frac{d}{dt} \int_X \log(\exp(t\xi)R)\hat{R}^T(X)\mid p(X|R)dX = 0 \]

Formally, this implies

\[ \int_X (D \log(\hat{R}^T(X)))(\xi)_\times \hat{R}^T(X)\mid p(X|R) \] 

\[ + \log(\hat{R}^T(X)) \cdot D_2 p(X \mid R)(\xi)_\times \mid p(X|R)(\xi)_\times \mid p(X|R)|dX = 0 \]

(10)

where \( D \) denotes the differential and \( D_2 \) the partial differential with respect to the second argument. For any vector \( u \in \mathfrak{g} \) we have thus:

\[ -\int_X \langle u, D \log(\hat{R}^T(X))(\xi)_\times \hat{R}^T(X)\rangle p(X|R)dX \]

\[ = \int_X \langle u, \log(\hat{R}^T(X)) \rangle D_2 p(X \mid R)(\xi)_\times \mid p(X|R)|dX \]

\[ \leq \sqrt{\int_X \langle u, \log(\hat{R}^T(X)) \rangle^2 p(X|R)dX} \cdot \sqrt{\int_X (D_2 p(X \mid R)(\xi)_\times \rangle)^2 p(X|R)dX} \]

where we used the Cauchy Schwarz inequality and the relationships

\[ D_2 p = pD_2 \log p, \] and then \( p = (\sqrt{p})^2 \)
We introduce a basis of $g$ and the matrices $\hat{A}(X) = \log(R\hat{R}(X)^T)$ and $\hat{B}(X)\hat{\xi} = D_2 \log p(X|R)(\xi, R)$. The latter inequality can be re-written as follows:

$$\int_X |\log(R\hat{R}(X)(\xi, R\hat{R}(X))|p(X|R)dX)^2 \leq (\int_X (\hat{A}(X)^T\hat{A}(X)p(X|R)dX)u) \cdot (\int_X (\hat{B}(X)^T\hat{B}(X)p(X|R)dX)\hat{\xi}) \tag{11}$$

Now we compute a second-order development of the left-hand term in the estimation error $Q := R\hat{R}(X)$. To do so, we note $\log(exp(t\xi)Q) = \log(exp(t\xi)exp(log(Q)))$ is equal to (using the Baker-Campbell-Hausdorff formula and keeping only terms up to $t^2$)

$$\xi - \frac{1}{2} ad_{\log(Q)} t \xi + \frac{1}{12} ad_{\log(Q)}^2 t \xi + O(||\log(Q)||^3)|t^2 \xi \tag{12}$$

Differentiating w.r.t. $t$ yields

$$D\log(Q)(\xi, Q) = [I - \frac{1}{2} (\log(Q))_{xx} + \frac{1}{12} (\log(Q))_{xx^2}] \xi \tag{13}$$

Neglecting the third-order terms we get:

$$D\log(Q)(\xi, Q) = [I - \frac{1}{2} (\log(Q))_{xx} + \frac{1}{12} (\log(Q))_{xx^2}] \xi \tag{14}$$

We introduce the latter second-order development in the error in the equation (13):

$$\int_X (I - \frac{1}{2} (\log(R\hat{R}(X)(\xi, R\hat{R}(X))))_{xx} + \frac{1}{12} (\log(R\hat{R}(X)(\xi, R\hat{R}(X)))_{xx^2})_x p(X|R)dX = \frac{1}{12} (\int_X (\hat{A}(X)^T\hat{A}(X)p(X|R)dX)u) \cdot (\int_X (\hat{B}(X)^T\hat{B}(X)p(X|R)dX)\hat{\xi}) \tag{15}$$

This writing uses the fact that $\hat{R}$ is unbiased (which makes the term in front of the factor $\frac{1}{t}$ cancel as $E(\log(Q)) = 0$)

$$\int_X (\hat{A}(X)^T\hat{A}(X) - Tr(\hat{A}(X)\hat{A}(X)^T)I_3)p(X|R)dX\hat{\xi}^2 \leq (\int_X (\hat{A}(X)^T\hat{A}(X)p(X|R)dX)u) \cdot (\int_X (\hat{B}(X)^T\hat{B}(X)p(X|R)dX)\hat{\xi}) \tag{16}$$

We now consider the following system

$$\frac{d}{dt} R_t = R_t(\omega(t)) \times, \quad t_{n-1} \leq t \leq t_n \tag{17}$$

$$Y_t = R_t^T d_n + V_t, \quad t = t_n \tag{18}$$

that is, the motion in space of a solid fixed at a point, having deterministic known angular velocity $\omega(t)$, and noisy measurements at discrete times $t_1 \leq t_2 \leq \cdots$. This fits into the general filtering setting of [22]. We assume that $V_t$ is a Gaussian noise with covariance matrix $\sigma_v^2 I_6$. Our goal is to derive an intrinsic lower Cramér-Rao bound on the estimation error. We will see it follows from the results of Section III indeed, thanks to the facts that $1- R_t$ is a deterministic quantity and 2- due to the invariance of the system, the flow can be explicitly computed.

Such problems arise for attitude estimation in the degenerate case where the gyroscope is conditionally better than the vector sensors. Sensors measuring in the body frame vectors from the fixed frame include magnetometers, that measure the earth magnetic field in the body frame, and accelerometers, that measure the earth gravity vector field in the body frame, under static flight assumptions. For each of these...
sensors, the isotropy assumption of the noise is reasonable technologically, as the measurements are performed using in each case three orthogonal one-axis sensors (accelerometers or magnetometers).

A. Intrinsic Fisher information matrix computation

We suppose that we have at time \( t = 0 \) a prior on the rotation \( R_0 \). This can be modeled as a fictitious measurement \( Y_0 = \exp(\nu) R_0 \), with \( \nu \sim \mathcal{N}(0, P_0) \). The conditional intrinsic information matrix at time \( n \) is (using \([22]\) and the results above)

\[
\xi^T J_n \xi = -E\left( \frac{d^2}{dt^2} \log p(Y_0, Y_1, \ldots, Y_n \mid R_0, R_1, \ldots, \exp(\xi)R_n) \right)
\]

with the log-likelihood that writes

\[
- \log p(Y_0, Y_1, \ldots, Y_n \mid R_0, R_1, \ldots, R_n) = \text{Cst} + \log(Y_1^T R_n^\mathcal{T}) P_0^{-1} \log(Y_n^T R_n^\mathcal{T}) + \frac{1}{2} \sum_{k=1}^{n} ||Y_k - R_k^T d_k||^2
\]

When deriving an intrinsic Cramér-Rao bound for Wahba’s problem in Section II, we have already proved that letting \( \xi \) and the result does not depend on \( \nu \). Now, using the invariance of the dynamics, we see there exists a rotation \( H_k = (d_k)_x \) we have for any \( Q \in SO(3) \) such that \( E(QH_k) = d_k \) that

\[
\frac{d^2}{dt^2} E(||Y_k - Q^T \exp(-t\xi) d_k ||^2) = -2 E(\frac{d^2}{dt^2}(Q^T \exp(-t\xi) Y_k) = 2 \xi^T H_k H_k^T \xi
\]

and the result does not depend on \( Q \). Now, using the invariance of the dynamics, we see there exists a rotation \( A(k, n) \) depending only on \( \omega(s) \), \( t_k \leq s \leq t_n \) such that \( R_n = R_k A(k, n) \). Indeed \( A(k, n) \) is the solution at time \( t_n \) to the differential equation on \( SO(3) \) defined by \( \frac{d}{dt} A_s = A_s \omega(s) \), \( A_t = R_t \) (see eg. [3]). As a result we can write the log likelihood \([19]\) as a function of \( R_n \) only which yields in particular

\[
- \log p(Y_0, Y_1, \ldots, Y_n \mid R_0, R_1, \ldots, \exp(\xi) R_n) = \text{Cst} + \log(Y_1^T R_n^\mathcal{T}) P_0^{-1} \log(Y_n^T R_n^\mathcal{T})
\]

\[
+ \frac{1}{2} \sum_{k=1}^{n} ||Y_k - A(k, n) R_n^\mathcal{T} \exp(-t\xi)d_k||^2
\]

Differeniating the latter inequality twice w.r.t \( t \) and using \([20]\) with \( Q = R_n A(k, n)^T \) (which is valid as \( E(QY_k) = E(R_n A(k, n)^T Y_k) = E(Y_k) = Y_k) \), we get

\[
J_n = - \frac{d^2}{dt^2} \log p(Y_0, Y_1, \ldots, Y_n \mid R_0, R_1, \ldots, \exp(\xi) R_n)
\]

\[
= P_0^{-1} + \sum_{k=1}^{n} \frac{1}{2} H_k^T H_k
\]

B. Intrinsic Cramér-Rao bound

A mere application of Theorem 1 implies

**Proposition 3:** For the considered system \([13]\), at time \( t_n \), the accuracy \( P = E(\log(R_n R_0^\mathcal{T})) \log(R_n R_0^\mathcal{T})^T \) of any unbiased attitude estimator with initial covariance \( P_0 \) is lower bounded according to formula \([9]\), with \( J(R) = P_0^{-1} + \sum_{k=1}^{n} \frac{1}{2} H_k^T H_k \).

IV. Links with Invariant Kalman filtering

For the filtering problem of Section III on can derive an Invariant Extended Kalman Filter (IEKF) \([9]\). The IEKF is a novel methodology for devising EKFs on Lie groups, where the EKF is bound to respect the invariances of the problem, and where an intrinsic estimation error is linearized at each step. Moreover, the exponential map allows to map the Kalman correction term to the state space. The IEKF for the problem above is derived on \( SO(3) \) in the recent paper \([3]\), and we briefly recall the principle here. The IEKF equations write

\[
\frac{d}{dt} \hat{R}_n = \hat{R}_n(\omega(t)) \times, \quad t_{n-1} < t < t_n \quad \text{(Propagation)}
\]

\[
(\hat{R}_n)^+ = \exp(K_n(\hat{R}Y_n - d_n))\hat{R}_n, \quad t = t_n \quad \text{(Update)}
\]

where \( K_n \in \mathbb{R}^{3 \times 3} \) is the gain matrix to be tuned as follows. Letting \( \xi_n = \log(\hat{R}R_n^\mathcal{T}) \) be the right invariant estimation error projected in the Lie algebra, the error system has the following remarkable autonomous form

\[
\frac{d}{dt} \xi_n = 0, \quad t_{n-1} < t < t_n \quad \text{(Propagation)}
\]

\[
(\exp(\xi_n))^+= \exp(K_n(\exp(\xi_n)d_n - d_n + V_n)) \exp(\xi_n) \quad \text{(Update)}
\]

During the propagation step, the covariance of the linearized estimation error \( P_t = E(\xi_n^T \xi_n) \) remains fixed, that is,

\[
P_{t_{n+1}} = P_{t_n}
\]

as the linearized dynamics (for the well-chosen estimation error) yields a static system and it was assumed there is no process noise. As concerns the update step, using formula \([7]\), a first order approximation to the error update equation above reads \([3]\)

\[
\xi_n^+ = K_n \xi_n \times d_n + K_n V_n = -K_n H_n \xi_n + K_n V_n
\]

The gain that minimizes the increase in the covariance matrix of the linearized error at the update step is the Kalman gain

\[
K_n = P_n H_n^T (H_n P_n H_n^T + \sigma_n^2 I)^{-1}
\]

leading to the covariance update

\[
(P_n)^+ = P_n^{-1} + \frac{1}{\sigma_n^2} H_n^T H_n
\]

with initial condition the prior covariance matrix \( R_0 \) to be chosen by the user. This proves the following remarkable result

**Proposition 4:** The covariance matrix of the error returned by the IEKF writes

\[
P_n = (P_n^{-1} + \sum_{k=1}^{n} \frac{1}{2} H_k^T H_k)^{-1}
\]

and thus the IEKF returns the Cramér-Rao bound for the associated filtering problem, neglecting the curvature terms.

Note that, it is logical that the curvature terms be ignored by the IEKF as it is based on a first order approximation of
the estimation error. The result is in sharp contrast with the general theory [22] that stipulates that the Cramér-Rao bound is the iekf covariance indeed, but, linearized around the true trajectory, that is unknown to the user. This is why such bounds go by the name of “posterior Cramér-Rao bounds” in the filtering and hidden Markov models (HMM) literature (see e.g. [23]). For invariant systems on SO(3), in the case of deterministic dynamics, we have proved the bound can be computed in real time (that is, the a priori bound coincides with a posteriori bound). This appears as another remarkable feature of dynamical systems defined on Lie groups.

V. Conclusion

In this paper we have derived an intrinsic Cramér-Rao lower bound (ICRLB) on SO(3) in a straightforward way. We have applied it to derive an ICRLB for Wahba’s problem when the noise is isotropic and Gaussian. Then, we have also derived an ICRLB for the problem of filtering on SO(3) a system with deterministic evolution and noisy isotropic Gaussian measurements. We have also proved the intrinsic CRLB is the covariance matrix returned by the invariant EKF on SO(3). This is a remarkable result, as generally the CRLB cannot be computed online as it presupposes to know the true trajectory of the system, which is precisely what one seeks to estimate. It is thus usually reserved for offline simulations to test filters’ efficiency, and we generally speak of “posterior Cramér-Rao bounds” [23].

In the future, we would like to investigate in what ways the intrinsic gradient methods (see [10], [1]) might asymptotically reach the Intrinsic Cramér-Rao bound. We would also like to explore in what ways the present calculations could be extended to a generic Lie group. Besides, we hope it is possible to derive an ICRLB for the filtering problem considered in the present paper, but with a noisy evolution (that is, using noisy gyroscopes). In this case it will certainly not coincide with the covariance returned by the IEKF, but it might be computable online taking advantage of the invariances of the system. Such results could be applied to a wide range of aeronautics estimation problems, like e.g., [2].

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