GROUPS WITH A POLYNOMIAL DIMENSION GROWTH

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Abstract. We show that finitely generated groups with a polynomial dimension growth have Yu’s property A and give an example of such groups.

§1 Introduction

In the asymptotic geometry of discrete groups the growth of functions associated to the group is of great importance. Probably the most known concept in the area is the volume growth of a group, i.e., the growth of the capacity of $r$-balls $B_r(e)$ when the radius $r$ tends to infinity. We consider finitely generated discrete groups supplied with the word metric and look at them as geometric objects. Following Gromov [Gr1] one can define different geometric characteristics of a discrete group. In particular one can speak about dimension of $r$-balls and its growth.

First we give an informal description of dimension. A finite metric space $X$ of diameter $r$ can be assign a dimension on the scale $\lambda < r$ by means of $\lambda$-approximations of $X$ by a finite polyhedra. Under $\lambda$-approximation we assume roughly speaking an isometric imbedding of $X$ into a regular neighborhood $N(K)$ (in some normed space) of a finite polyhedron $K$ with simplexes of size $\lambda$. The minimal dimension of such $K$ is the dimension of $X$ on the scale $\lambda$. If $\lambda(t)$ is a sufficiently slowly tending to infinity function, then the growth of the dimension of the $t$-ball $B_t(e)$ on the scale $\lambda(t)$ when $t$ goes to infinity is the dimension growth of the group $\Gamma$. We will give precise definitions in the next section in the language of covers.

In contrast with the volume growth of a group all classical groups do not exhibit any dimension growth at all. The dimension of the balls in a typical classical group

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Γ is constant and equals the Gromov asymptotic dimension asdim Γ. Groups with finite asymptotic dimension behave nicely in the following sense: Many famous conjecture were proved for them such as the coarse Baum-Connes and the classical Novikov conjectures [Yu1], Gromov’s hypersphericity conjecture [Dr2], the K-theoretic integral Novikov conjecture [CG], the mod p Higson conjecture and the integral Novikov conjecture [DFW]. The asymptotic finite dimensionality was checked for a large class of groups [Gr1],[DJ],[BD1],[BD2],[CG],[Ji],[Ro3].

In [Yu2] Guoliang Yu extended his result on the Novikov conjecture to the groups having so called property A. In this paper we show that all finitely generated groups with the polynomial dimension growth have property A. Thus for groups with the polynomial dimension growth the Novikov conjecture holds true.

Additionally, in this paper we give examples of finitely generated asymptotically infinite dimensional groups with the polynomial dimension growth.

The dimension growth like the volume growth in the case of finitely generated groups can be at most exponential. Celebrated Gromov’s group [Gr2] containing an expander is an example of a group with a nonpolynomial dimension growth. In fact one can show that the growth there is exponential. We conclude the introduction with an open question.

**Question.** Are there finitely generated groups of the intermediate dimension growth?

### §2 Dimension growth of a metric space

Let \((X, \rho)\) be a metric space and let \(U\) be a cover of \(X\). A number \(\lambda > 0\) is called a Lebesgue number for \(U\) if for every set \(A \subset X\) of diameter \(\leq \lambda\) there is an element \(U \in U\) such that \(A \subset U\). The Lebesgue number \(L(U)\) of a cover \(U\) is the infimum of all Lebesgue numbers. The multiplicity \(m(U)\) of a cover \(U\) is the maximal number of elements of \(U\) with a nonempty intersection. We define a function

\[
ad_X(\lambda) = \min \{m(U) \mid L(U) \geq \lambda\} - 1.
\]

Clearly, \(ad_X(\lambda)\) is monotone. We call this function the asymptotic dimension function of a metric space \(X\).

**Proposition 2.1.** For a finitely generated group \(\Gamma\) with the word metric there is \(a > 0\) such that \(ad_\Gamma(\lambda) \leq e^{a\lambda}\)

**Proof.** There is \(a > 0\) such that \(|B_\lambda(x)| \leq e^{a\lambda}\) for all \(x \in \Gamma\). We consider the cover \(U\) of \(\Gamma\) by the \(\lambda\)-balls \(\{B_\lambda(x) \mid x \in \Gamma\}\). Clearly, \(L(U) \geq \lambda\). We estimate the multiplicity \(m(U)\) of \(U\). If for different \(x_i\) we have

\[
y \in B_\lambda(x_1) \cap \cdots \cap B_\lambda(x_k)
\]
then \( \rho(y, x_i) \leq \lambda \) for all \( i \). Hence \( k \leq |B_\lambda(x)| \leq e^{a\lambda} \). Thus, \( m(U) \leq e^{a\lambda} \). □

The asymptotic dimension function has a direct relation to Gromov’s asymptotic dimension.

**Definition [Gr1].** Let \( X \) be a metric space. Then \( \text{asdim} \ X \leq n \) if for every \( r < \infty \) there are uniformly bounded \( r \)-disjoint families \( U^0, \ldots, U^n \) of subsets of \( X \) such that \( \bigcup_i U^i \) is a cover of \( X \).

A family \( F \) of a subsets of a metric space \((X, \rho)\) is called \( r \)-disjoint if \( \rho(F, F') = \inf \{ \rho(x, x') \mid x \in F, x' \in F' \} \geq r \) for all \( F, F' \in F, \ F \neq F' \). We denote by

\[
\text{diam}(F) = \sup \{ \text{diam}(F) \mid F \in F \}.
\]

A family \( F \) is called uniformly bounded if \( \text{diam}(F) < \infty \).

We use the notation \( l_2 \) for the Hilbert space of square summable sequences. Let \( \Delta \) denote the standard infinite dimensional simplex in \( l_2 \)

\[
\Delta = \{(x_1, x_2, \ldots) \in l_2 \mid \sum x_i = 1, x_i \geq 0 \}.
\]

A subcomplex \( K \subset \Delta \) taken with the restricted metric is called uniform. A map \( f : X \to K \) from a metric space to a uniform complex is called uniformly cobounded if there is \( D \) such that \( \text{diam} f^{-1}(\text{St}(v, K)) \leq D \) for all \( v \in K \) where \( \text{St}(v, K) \) is the star of a vertex \( v \) in the complex \( K \).

**Proposition 2.2.** For a metric space \( X \) the following conditions are equivalent:

1. \( \text{asdim} \ X \leq n \);
2. For every \( \lambda < \infty \) there is a uniformly bounded cover \( U \) of \( X \) with \( L(U) \geq \lambda \) and with \( m(U) \leq n + 1 \);
3. For every \( \epsilon > 0 \) there is a uniformly cobounded \( \epsilon \)-Lipschitz map \( f : X \to K \subset \Delta \subset l_2 \) to an \( n \)-dimensional uniform polyhedron.

**Proof.** (1) \( \Rightarrow \) (2). Let \( \lambda < \infty \) be given. Apply (1) with \( r > 2\lambda \) to obtain the families \( U^0, \ldots, U^n \). Then consider a cover \( U = \{ N_\lambda(U) \mid U \in U^i, i = 0, \ldots, n \} \) where \( N_\lambda(U) = \{ x \in X \mid \rho(x, U) \leq \lambda \} \) denotes the \( \lambda \)-neighborhood of \( U \). Since \( r > 2\lambda \) and each family \( U^i \) is \( r \)-disjoint, we have \( m(U) \leq n + 1 \). Clearly, \( L(U) \geq \lambda \).

(2) \( \Rightarrow \) (3). Let \( U \) be a uniformly bounded cover of \( X \) with \( m(U) \leq n + 1 \) and with \( L(U) \geq \lambda \). Then the partition of unity

\[
\phi_U(x) = \frac{\rho(x, X \setminus U)}{\sum_{V \in U} \rho(x, X \setminus V)}
\]
defines the projection \( p_U : X \to \Delta \subset l_2(U) \). It was shown in [BD2], Proposition 1, that \( p_U \) is \((2n+3)^2/L(U)\)-Lipschitz. The condition \( m(U) \leq n + 1 \) implies that \( p_U(X) \) lies in an \( n \)-dimensional subcomplex of \( \Delta \). Note that \( p_U^{-1}(St(u, K)) = U \) for every vertex \( u \) corresponding the element \( U \in U \). Since \( diam(U) < \infty \), \( p_U \) is uniformly cobounded. Given \( \epsilon > 0 \) we can take \( \lambda \) large enough that the map \( p_U \) will be \( \epsilon \)-Lipschitz.

(3) \( \Rightarrow \) (1). Let \( \mathcal{V}^i = \{ St(b_\sigma, \beta^2 K) \mid \sigma \subset K, \dim \sigma = i \} \), where \( b_\sigma \) denotes the barycenter of a simplex \( \sigma \) and \( \beta^i K \) denotes the \( i \)-th barycentric subdivision of \( K \). There are constants \( c \) and \( d \) depending only on \( n \) such that the family \( \mathcal{V}^i \) is \( c \)-disjoint and \( diam(\mathcal{V}^i) \leq d \) for all \( i \leq n \). We define \( \mathcal{U}^i = f^{-1}(\mathcal{V}^i) \) for a \((d/\lambda)\)-Lipschitz uniformly cobounded map \( f : X \to K \) to a uniform \( n \)-dimensional complex. Then the family \( \mathcal{U}^i \) is \( \lambda \)-disjoint for all \( i \). If \( v \in \sigma \) is a vertex, then \( St(b_\sigma, \beta^2 K) \subset St(v, K) \). Hence \( diam(f^{-1}(St(b_\sigma, \beta^2 K))) \leq diam(f^{-1}(St(v, K)) \leq D \) and hence \( \mathcal{U}^i \) is uniformly bounded for every \( i \). \( \square \)

**Corollary 2.3.** The limit \( \lim_{\lambda \to \infty} ad_X(\lambda) = \text{asdim } X \).

The asymptotic dimension is a coarse invariant, i.e. it is invariant under coarse isomorphisms. The **Coarse category** [Ro1-3] is a quotient category of the category of metric spaces and coarse morphisms. A map \( f : (X, d_X) \to (Y, d_Y) \) between metric spaces is called a coarse morphism if it is

1. **metrically proper**, i.e., the preimage \( f^{-1}(C) \) is bounded for every bounded set \( C \subset Y \);
2. **coarsely uniform**, i.e., \( d_Y(f(x), f(x')) \leq \rho(d_X(x,x')) \) for all \( x, x' \in X \) for some function \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \) (depending on \( f \)).

Two morphisms \( f, g : X \to Y \) are equivalent if they are in finite distance, i.e., there is \( D < \infty \) such that \( d_Y(f(x), g(x)) \leq D \) for all \( x \in X \). The coarse category is defined as the quotient category under this equivalence relation. Thus, a map \( f : X \to Y \) between metric spaces is a coarse isomorphism if there is a coarse morphism \( h : Y \to X \) such that \( hf \) is in bounded distance to \( 1_X \) and \( fh \) is in bounded distance to \( 1_Y \). Clearly, \( \text{asdim } X = \text{asdim } Y \) for coarsely isometric spaces. We note that a quasi-isometry is a coarse isomorphism.

Our main examples of metric spaces are finitely generated groups with the left-invariant word metric. Since for every two finite symmetric generating sets of such group the identity map is a quasi-isometry, we obtain that the asymptotic dimension of a finitely generated group is a well-defined group invariant. Moreover, the growth of the asymptotic dimension function \( ad_\Gamma(\lambda) \) is an invariant of a group \( \Gamma \) though it is not a coarse invariant.

We recall that a morphism \( f : X \to Y \) in an abstract category is called a **monomorphism** if for every two morphisms \( k, l : Z \to X \) with \( f \circ k = f \circ l \) it follows that \( k = l \). A
map of metric spaces \( f : X \to Y \) that represents a monomorphism in the coarse category is called a *coarse imbedding* [Ro3]. In the literature it is often called by an overused name a *uniform imbedding*.

A coarse imbedding \( f : X \to Y \) defines a coarse isometry \( f : X \to f(X) \). It is not difficult to see that a map between metric spaces \( f : X \to Y \) is a coarse embedding if and only if there exist two monotone tending to infinity functions \( \rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[
(*) \quad \rho_1(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \rho_2(d_X(x, x'))
\]

for all \( x, x' \in X \). If \( X \) is a geodesic metric space or a group with the word metric then the function \( \rho_2 \) can be chosen to be linear.

A finitely generated subgroup \( H \subset G \) in a finitely generated group is a typical example of a coarse imbedding. We say that a subgroup \( H \subset G \) has a *polynomial distortion* [Gr1] in \( G \) if the function \( \rho_1(t) \) can be chosen such that the inverse function \( \rho_1^{-1} \) is a polynomial.

For a metric space with \( \text{asdim} X \leq m \) Gromov suggested to study the asymptotic behavior of the function [Gr1]

\[
\gamma_X(\lambda) = \inf \{ \text{diam}(U) \mid L(U) \geq \lambda, \ m(U) \leq m + 1 \}
\]

as the secondary dimensional invariant. In this paper we will refer to \( \gamma_X \) as to Gromov’s function for the inequality \( \text{asdim} X \leq m \). The asymptotic behavior of the function \( \gamma_X(t) \) is not a coarse invariant. It was shown in [DZ] that every metric space \((X, d)\) with the asymptotic dimension \( \text{asdim} X \leq n \) and with bounded geometry is coarsely isomorphic to a metric space \((X, d')\) with a linear Gromov’s function for \( \text{asdim} X' \leq n \). Nevertheless the growth of \( \gamma_X \) is an invariant of quasi-isometries and hence in the case of a finitely generated group \( G \) the growth of \( \gamma_G \) is an invariant of a group.

§3 Property A

By \( l_p, p \geq 1 \), we denote the Banach space of sequences \( \{x_n\} \) with the norm \( \|x\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \). Also \( l_p(Z) \) will denote the corresponding Banach space with a basis indexed by a countable set \( Z \). Thus, \( l_p = l_p(\mathbb{N}) \).

**Definition.** Let \( p \in \mathbb{R}_+ \cup \{\infty\} \). A discrete metric space \((X, d)\) has property \( A_p \) if there is a sequence of maps \( a^n : X \to l_p(X) \) satisfying the conditions \( \|a^n\|_p = 1 \) and \( a^n(y) \geq 0 \) for all \( y, z \in X \), such that

1. there is a function \( R = R(n) \) such that \( \text{supp}(a^n) \subset B_R(z) \) for all \( z \in X \);
2. for every \( K > 0 \), \( \lim_{n \to \infty} \sup_{d(z,w) < K} \|a^n_z - a^n_w\|_p = 0 \).

We include the case \( p = \infty \) here. Recall that the norm \( \|x\|_{\infty} \) is the sup-norm. The following proposition first was proven in [Yu2] for \( p = 1 \).
**Proposition 3.1.** A discrete metric space $X$ with property $A_p$ admits a coarse embedding in $l_p$.

**Proof.** Let $a^n$ be a sequence of maps from the definition of property $A_p$. By passing to a subsequence we may assume that $\sup_{n<\infty}(\|a^n - a^n\|_p) < 2^n$. Let $z_0 \in X$ be a base point. We define a map $f : X \to l_p(X \times \mathbb{N})$ by the formula $f(z)(x, n) = a^n(z) - a^n(z_0)$. The above inequality insures that $f(z) \in l_p(X \times \mathbb{N})$. We shall show that $f$ is a coarse embedding. We may assume that the function $R$ in the definition of property $A_p$ is strictly monotone. Let $S = R^{-1}$ be the inverse function. We define $\rho_1(t) = (2S(t/2) - 2)^{1/p}$ and $\rho_2(t) = (2t + 1)^{1/p}$ and check the inequalities (*). We have

$$
(\|f(z) - f(w)\|_p)^p = \sum_{n=1}^{\infty} (\|a^n_z - a^n_w\|_p)^p \leq \sum_{n=1}^{[d(z,w)]} (\|a^n_z - a^n_w\|_p)^p + \sum_{[d(z,w)]+1}^{\infty} (\|a^n_z - a^n_w\|_p)^p
$$

$$
\leq \sum_{n=1}^{[d(z,w)]} \|a^n_z - a^n_w\|_p + \sum_{[d(z,w)]+1}^{\infty} 2^{-n} \leq 2d(z, w) + 1 = (\rho_2(d(z, w)))^p,
$$

and

$$
(\|f(z) - f(w)\|_p)^p = \sum_{n=1}^{\infty} (\|a^n_z - a^n_w\|_p)^p \geq \sum_{n=1}^{N} (\|a^n_z - a^n_w\|_p)^p = 2N \geq (\rho_1(d(z, w)))^p.
$$

where $N = [S(d(z, w)/2)]$ is the integral part of $S(d(z, w)/2)$. Here we used the fact that the inequality $n \leq [S(d(z, w)/2)]$ implies the inequality $R(n) \leq d(z, w)/2$ which implies that $\text{supp}(a^n_z) \cap \text{supp}(a^n_w) = \emptyset$. The latter implies that $(\|a^n_z - a^n_w\|_p)^p = (\|a^n_z\|_p)^p + (\|a^n_w\|_p)^p = 2.$ □

**Proposition 3.2.** For every finite $p \geq 1$ property $A_1$ is equivalent to property $A_p$.

**Proof.** First we show that for $m \geq p \geq 1$ property $A_p$ implies $A_m$.

Assume that $X$ has property $A_p$. Let $a^n : X \to l_p(X)$ be a sequence of functions satisfying the conditions (1)-(2) from the definition of $A_p$. Then $\sum_{y \in X} (a^n_y)^p = 1$ and $a^n_z(y) \geq 0$ for all $z, y \in X$. We define $b^n_z(y) = a^n_z(y)^m$. Clearly, $\|b^n_z\|_m = 1$. The condition (1) is satisfied automatically. We check the condition (2). In view of the obvious inequality $t^{m/p} + (1 - t)^{m/p} \leq t + (1 - t) = 1$ for $m/p \geq 1$, $t \in [0, 1]$, we have that $|a - b|^{m/p} \leq |a^{m/p} - b^{m/p}|$ for all $a, b \geq 0$. Hence

$$
(\|b^n_z - b^n_w\|_m)^m = \sum_{x \in X} (|b^n_z(x) - b^n_w(x)|^{m/p})^p \leq \sum_{x \in X} |b^n_z(x)^{m/p} - b^n_w(x)^{m/p}|^p = (\|a^n_z - a^n_w\|_p)^p.
$$
This implies the condition (2).

Now assume that $X$ has property $A_p$, $p \in \mathbb{N} \setminus \{1\}$, and show that $X$ has property $A_1$. Let $b^n : X \to l_p(X)$ be a corresponding sequence of functions. We define $a_n = (b^n)^p$ and check that

$$
\|a^n_z - a^n_w\|_1 = \sum_{x \in X} |(b^n_z(x))^p - (b^n_w(x))^p| = \sum_{x \in X} |b^n_z(x) - b^n_w(x)| \times | \sum_{i=0}^{p-1} (b^n_z(x))^i (b^n_w(x))^{p-1-i} |
$$

By the Hölder inequality we have

$$
\leq \|b^n_z(x) - b^n_w(x)\|_p \| \sum_{i=0}^{p-1} (b^n_z(x))^i (b^n_w(x))^{p-1-i} \|_q,
$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Since $pq = p + q$, we have

$$(b^n_z(x))^i (b^n_w(x))^{p-1-i} q = (b^n_z(x)/b^n_w(x))^i (b^n_z(x))^p.
$$

Therefore,

$$(b^n_z(x))^i (b^n_w(x))^{p-1-i} q \leq (b^n_z(x))^p,
$$

provided $b^n_z(x) \leq b^n_w(x)$. Symmetrically we obtain that

$$(b^n_z(x))^i (b^n_w(x))^{p-1-i} q \leq (b^n_z(x))^p,
$$

provided $b^n_z(x) \geq b^n_w(x)$. Thus, $(b^n_z(x))^i (b^n_w(x))^{p-1-i} q \leq (b^n_z(x))^p + (b^n_w(x))^p$. Hence

$$
\| (b^n_z(x))^i (b^n_w(x))^{p-1-i} \|_q = (\sum_x (b^n_z(x))^i (b^n_w(x))^{p-1-i} q)^{1/q} \leq (\sum_x (b^n_z(x))^p + (b^n_w(x))^p)^{1/q} = 2^{1/q}.
$$

Therefore,

$$
\| a^n_z - a^n_w \|_1 \leq 2^{1/q} p \| b^n_z - b^n_w \|_p,
$$

and the condition (2) holds for $a^n$. \qed

The equality $A_1 = A_2$ was proven in [Tu]. It was shown in [HR] that property $A_1$ coincides with Yu’s property $A$ [Yu2] for metric spaces $X$ of bounded geometry. Also it was proven in [HR] that property $A$ ($= A_1 = A_2$) for finitely generated groups $\Gamma$ with word metrics is equivalent to the topological amenability of the natural action of $\Gamma$ on the Stone-Čech compactification $\beta \Gamma$. 

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Theorem 3.3. Suppose that a discrete metric space $(X, \rho)$ has a polynomial dimension growth. Then $X$ has property $A_p$.

Proof. Since $ad_X(\lambda)$ has a polynomial growth, there is $p > 1$ such that

$$\lim_{\lambda \to \infty} ad_X(\lambda)/\lambda^p = 0.$$ 

In view of Proposition 3.2 it suffices to prove that $X$ has property $A_p$. Given $n < \infty$ we take an open uniformly bounded covering $U = \{U_i\}_{i \in J}$ of $X$ with the Lebesgue number $L(U) \geq n + 1$ and with the multiplicity $m(U) \leq ad_X(n + 1) + 1$. We shrink the cover $U$ by taking sets $\tilde{U} = N_n(U) = U \setminus N_n(X \setminus U)$ for all $U \in U$ where $N_n(A)$ denotes the $n$-neighborhood of $A$. We consider an irreducible subcover $V$ of this new cover, i.e., a subcover with

$$\tilde{V} \setminus \left( \bigcup_{V' \in \tilde{V} \setminus \{V\}} V' \right) \neq \emptyset$$

for all $\tilde{V} \in \tilde{V}$. Then we consider a cover $V = \{U \in U \mid \tilde{U} \in \tilde{V}\}$. Note that $L(V) \geq n$ and $m(V) \leq m(U) \leq d(n + 1) + 1$. Let the cover $V$ be indexed by a (countable) set $J$: $V = \{V_j\}_{j \in J}$. Since the cover $\tilde{V}$ is irreducible we can define an injective map $\xi : J \to X$ such that $\xi(j) = y_j \in V_j$. This gives us an embedding $l_p(J) \subset l_p(X)$. Denote $\phi_i(x) = \rho(x, X \setminus U_i)$. Then for fixed $x \in X$ the family $\{\phi_i(x)\}$ defines a nonzero element $\phi_x \in l_p(J)$. We define a map $a^n : X \to l_p(J) \subset l_p(X)$ as $a^n_z = \phi_z/\|\phi_z\|_p$ for every $z \in \Gamma$. Assume that $\text{diam}(U) \leq R$. Then

$$\text{supp}(a^n_z) = \{y_i \mid \phi_i(z) \neq 0\} = \{y_i \mid z \in U_i \in U\} \subset B_R(z)$$

and the condition (1) from the definition of property $A_p$ is satisfied. To verify the condition (2) we take $w, z \in X$ with $\rho(z, w) \leq K$ and consider the triangle inequality

$$\|a^n_z - a^n_w\|_p \leq \|a^n_z - \phi_w/\|\phi_z\|_p\|_p + \|\phi_w - a^n_w\|_p.$$ 

Since $|\rho(z, X \setminus U_j) - \rho(w, X \setminus U_j)| < \rho(z, w), \|\phi_z\|_p \geq n$, and $m(U) \leq 2ad_X(n + 1)$, we obtain the following estimate

$$\|a^n_z - \phi_w/\|\phi_z\|_p\|_p = \frac{1}{\|\phi_z\|_p} \|\phi_z - \phi_w\|_p = \frac{1}{\|\phi_z\|_p} (\Sigma_j |\rho(z, X \setminus U_j) - \rho(w, X \setminus U_j)|^p)^{1/p}$$

$$\leq \frac{1}{\|\phi_z\|_p} (4ad_X(n + 1)\rho(z, w)^p)^{1/p} \leq 4K \frac{ad_X(n + 1)^{1/p}}{n}.$$
We note that
\[
\left\| \frac{\phi_w - a^n_w}{\phi_z} \right\|_p = \left( \frac{1}{\| \phi_z \|_p} - \frac{1}{\| \phi_w \|_p} \right) \| \phi_w \|_p = \frac{\| \phi_z \|_p - \| \phi_w \|_p}{\| \phi_z \|_p} \\
\leq \frac{\| \phi_z - \phi_w \|_p}{\| \phi_z \|_p} \leq 4K \frac{ad_X(n+1)^{1/p}}{n}
\]
by the above estimate. Thus
\[
\| a^n_z - a^n_w \|_p \leq 8K \frac{ad_X(n+1)^{1/p}}{n} \to 0.
\]

\[\Box\]

Theorem 3.3 was proved for bounded \( ad_X(\lambda) \) in [HR] and extended for sublinear \( ad_X(\lambda) \) in [Dr1].

Theorem 3.3 together with Proposition 3.2 and the result of Gouliang Yu [Yu1] imply the following

**Corollary 3.4.** If a group \( \Gamma \) has a polynomial dimension growth then the coarse Baum-Connes conjecture, and hence the Novikov Conjecture holds true for \( \Gamma \).

**Remark 3.5.** Every discrete metric space has property \( A_\infty \).

**Proof.** We define \( a^n_z(x) = \max\{1 - \frac{1}{n}d(x, z), 0\} \). Then \( \text{supp}(a^n_z) \subset B_n(z) \). Let \( d(z, w) \leq K \). There is the inequality
\[
\sup_x \{1 - \frac{1}{n}d(x, z)\}|d(x, w) > n, d(x, z) \leq n\} \leq K/n
\]
since \( K \geq d(z, w) \geq d(x, w) - d(x, z) \geq n - d(x, z) \). Similarly,
\[
\sup_x \{1 - \frac{1}{n}d(x, w)\}|d(x, z) > n, d(x, w) \leq n\} \leq K/n.
\]

Note that \( \sup_x |d(x, z)| - d(x, w)| \leq K/n \). Thus we obtain that \( \| a^n_z - a^n_w \|_\infty \leq K/n \).

Therefore the condition (2) is satisfied. \[\Box\]
§4 Dimension growth of wreath product

Let $G$ and $N$ be a finitely generated groups and let $1 \in G$ and $e \in N$ be their units. The support of a function $f : N \to G$ is the set

$$\text{supp}(f) = \{ x \in N \mid f(x) \neq 1 \}.$$ 

The direct sum $\oplus_N G$ of groups $G$ (or restricted direct product) is the group of functions $C_0(N,G) = \{ f : N \to G \}$ with finite supports. If $N$ is a group, there is a natural action of $N$ on $C_0(N,G)$: $a(f)(x) = f(xa^{-1})$ for all $a \in N$ and $f \in C_0(N,G)$. The semidirect product $C_0(N,G) \rtimes N$ is called a restricted wreath product and it is denoted as $N \wr G$.

REMARK. If $G$ has an element of infinite order, then $\text{asdim} N \wr G = \infty$. Indeed, then $N \wr G$ contains $\mathbb{Z}^m$ for all $m$.

We recall that the product in $N \wr G$ is defined by the formula

$$(f, a)(g, b) = (fa(g), ab).$$

Let $S$ and $T$ be finite generating sets for $G$ and $N$ respectively. Let $1 \in C_0(N,G)$ denote the constant function taking value $1$, and let $\delta^a_v : N \to G$ be the $\delta$-function, i.e., $\delta^a_v(v) = a$ and $\delta^a_v(x) = e_G$ for $x \neq v$.

Then $\mathcal{S} = \{ (\delta^s_e, e), (1, t) \mid s \in S, t \in T \}$ is a generating set for $N \wr G$. Let $\rho$ be the word metric on $N \wr G$ defined by the set $\mathcal{S}$ and let $\| \|_\rho$ be the corresponding norm.

Proposition 4.1. Let $\pi_A : C_0(N,G) = \oplus_{x \in N} G_x \to \oplus_{x \in A} G_x$, $A \subset N$, $G_x = G$, be the projection. Then

1. $\pi_A$ is $1$-Lipschitz with respect to the metric $\rho$;
2. $\| z \| \geq \| a \|$ for all $z \in G_a$, $a \in N$.

Proof. (1) We will use abbreviations $\delta^g_x$ for $(\delta^g_x, e)$ and $t$ for $(1, t)$ for elements of the group $N \wr G$. Every element $w \in N \wr G$ can be presented as a word

$$w = u_1 \delta^g_1 u_2 \delta^g_2 \ldots u_n \delta^g_n u_{n+1}$$

where $u_i \in N \setminus \{ e \}$ and $g_i \in G \setminus \{ 1 \}$ and the multiplication $u \delta$ means the action of $u$ on $\delta$. Moreover, every presentation of $w$ by elements of $\mathcal{S}$ can be reduced to this one by the multiplication in groups $N$ and $G$. Note that for the natural projection $p : N \wr G \to N$ we have $p(w) = u_1 \ldots u_{n+1}$. Then $C_0(N,G)$ consists of elements $w$ which have $u_1 \ldots u_{n+1} = e$ for every presentation of the above type. We note that the norm of $w$ being the minimal number of elements of $\mathcal{S}$ in a presentation of $w$ can be written as

$$\| w \| = \min \{ \sum \| u_i \|_N + \sum \| g_i \|_G \mid w = u_1 \delta^g_1 u_2 \delta^g_2 \ldots u_n \delta^g_n u_{n+1} \}. $$
Since \( x\delta^g = \delta^g x \), for \( w \in C_0(N, G) \) we have
\[
w = u_1\delta^{g_1}_e u_2\delta^{g_2}_e \ldots u_n\delta^{g_n}_e u_{n+1} = \delta^{g_1}_u \delta^{g_2}_u \ldots \delta^{g_n}_u.
\]

Then \( \pi_A(w) = \delta^{g_1}_{u_1} \ldots \delta^{g_k}_{u_k} \) where \( u_1 \ldots u_m \in A \) for all \( m \leq k \). Thus, a presentation for \( \pi_A(w) \) can be obtained by deletion of \( \delta^{g_r}_e \) for all \( r \) with \( u_1 \ldots u_r \notin A \) from
\[
u_1\delta^{g_1}_e u_2\delta^{g_2}_e \ldots u_n\delta^{g_n}_e u_{n+1}.
\]

If one start from a shortest presentation for \( w \) in the alphabet \( \tilde{S} \), then after the above cancellation he obtains a presentation for \( \pi_A(w) \). It means that \( \|w\| \geq \|\pi_A(w)\| \) for all \( w \in C_0(N, G) \).

(2) For every \( v \in G_a = aG_\epsilon a^{-1} \) and every shortest presentation
\[
v = u_1\delta^{g_1}_e u_2\delta^{g_2}_e \ldots u_n\delta^{g_n}_e u_{n+1}
\]
we should have \( u_1 = a \) and hence \( \|v\| \geq \|a\| \). \( \square \)

**Proposition 4.2.** Let \( \pi : G \to H \) be an epimorphism of finitely generated groups with the kernel \( K = \ker(\pi) \). Let \( U \) be a cover of \( H \) with \( L(U) \geq \lambda \) and \( \text{diam}(U) \leq R \). Let \( V \) be a cover of \( K \) with \( L(V) \geq 6R \) and \( \text{diam}(V) \leq D \) where we \( K \) is taken with the metric restricted from \( G \). Then there is a cover \( W \) of \( G \) with \( L(W) \geq \lambda \), \( \text{diam}(W) \leq D + 2R \), and with \( m(W) \leq m(U)m(V) \). 

**Proof.** Let \( S \) be a finite symmetric generating set for \( G \) and let \( \tilde{S} = \pi(S) \) be a generating set for \( G \). We denote the corresponding metrics on \( G \) and \( H \) by \( d \) and \( \rho \) respectively. Then \( \pi \) is \( 1 \)-Lipschitz for these metrics. Hence for every \( R \) we have \( \pi(N_R(K)) \subset B_R(\epsilon) \) where \( N_R(A) \) denotes the \( R \)-neighborhood of \( A \) and \( B_R(\epsilon) \) denotes the \( R \)-ball centered at \( \epsilon \in H \). Thus, \( N_R(K) \subset \pi^{-1}(B_R(\epsilon)) \). We shall establish the equality \( N_R(K) = \pi^{-1}(B_R(\epsilon)) \).

Indeed, if \( \|\pi(y)\| = k \leq R \) then \( \pi(y) = \pi(s_{i_1}) \ldots \pi(s_{i_k}) \) for some \( s_{i_1}, \ldots, s_{i_k} \in \tilde{S} \). Then \( b = y(s_{i_1} \ldots s_{i_k})^{-1} \in K \) and \( d(y, K) \leq d(y, b) = \|s_{i_1} \ldots s_{i_k}\| \leq k \leq R \), i.e., \( y \in N_R(K) \).

In a metric space \( (X, d) \) we denote by \( N^X_{-R}(A) = A \setminus N_R(X \setminus A) \) the \((-R)\)-neighborhood of \( A \). We define a cover \( \tilde{V} \) of the \( R \)-neighborhood \( N_R(K) \) as follows
\[
\tilde{V} = \{N_R(N^{-K}_{-2R}(V)) \mid V \in V\}.
\]

Since \( L(V) > 5R \), we have
\[
L(\{N^{-K}_{-2R}(V) \mid V \in V\}) \geq 3R.
\]
Then \( L(V) \geq R \geq \lambda \) where \( \bar{V} \) is considered as a cover of \( N_R(K) \). Indeed, for every \( z \in N_R(K) \) we take \( x \in K \) with \( d(x, z) \leq R \). Then we take \( V \in \mathcal{V} \) with \( d(x, K \setminus V) \geq 6R \). Then
\[
d(x, K \setminus (N_{-2R}^K(V))) \geq 4R.
\]
For every \( y \in B_R(z) \cap N_R(K) \) and every \( y' \in K \) with \( d(y, y') \leq R \) we have
\[
d(y, y') \leq d(x, z) + d(z, y) + d(y, y') \leq 3R
\]
and hence \( y' \in N_{-2R}^K(V) \). Therefore
\[
y \in \bar{V} = N_R(N_{-2R}^K(V)).
\]
Thus,
\[
B_R(z) \cap N_R(K) \subset \bar{V}.
\]

Note that
\[
m(\bar{V}) \leq m(V), \quad \text{diam}(\bar{V}) \leq \text{diam}(s\mathcal{V}) + 2R = D + 2R, \quad \text{and} \quad L(\bar{V}) \geq R \geq \lambda.
\]
For every \( U \in \mathcal{U} \) we fix \( z_U \in G \) such that \( \rho(\pi(z_U), H \setminus U) \geq \lambda \). We define
\[
\mathcal{W} = \{ z_U \bar{V} \cap \pi^{-1}(U) \mid U \in \mathcal{U}, \bar{V} \in \mathcal{V} \}.
\]
In view of the equality \( N_R(K) = \pi^{-1}(B_R(e)) \) we have that \( \mathcal{W} \) is a cover of \( G \).

Note that
\[
\text{diam}(z_U \bar{V} \cap \pi^{-1}(U)) \leq \text{diam}(z_U \bar{V}) = \text{diam}(\bar{V}) \leq D + 2R.
\]
We show that \( L(\mathcal{W}) \geq \lambda \). Let \( y \in G \) and let \( U \in \mathcal{U} \) be such that \( \rho((\pi(y), H \setminus U) \geq \lambda \).

Since \( \pi \) is 1-Lipschitz,
\[
d(y, G \setminus \pi^{-1}(U)) \geq \lambda.
\]
Note that
\[
z_U^{-1}y \in z_U^{-1}\pi^{-1}(U) = \pi^{-1}(\pi(z_U)^{-1}U) \subseteq \pi^{-1}(B_R(e)) = N_R(K).
\]
Hence there is \( \bar{V} \in \mathcal{V} \) such that \( d(z_U^{-1}y, N_R(K) \setminus \bar{V}) \geq \lambda \). Hence
\[
d(y, z_U(N_R(K) \setminus \bar{V})) \geq \lambda.
\]
Take \( W = z_U \bar{V} \cap \pi^{-1}(U) \in \mathcal{W} \). Then
\[
G \setminus W = (z_U(N_R(K)) \setminus \bar{V}) \cup (G \setminus \pi^{-1}(U)).
\]
Then
\[
d(y, G \setminus W) \geq \max\{d(y, z_U(N_R(K)) \setminus \bar{V}), d(y, G \setminus \pi^{-1}(U))\} \geq \lambda.
\]
Clearly, \( m(\mathcal{W}) \leq m(\mathcal{U})m(\mathcal{V}) \). \( \square \)
Proposition 4.3. Let $G = G_n \supset G_{n-1} \supset \cdots \supset G_1 \supset G_0 = 1$ be a lower central series for a finitely generated nilpotent group $G$. Then for every $k$ the subgroup $G_{k-1}$ has a polynomial distortion in $G_k$.

Proof. The following more general fact was cited in [Gr1] as well-known: For finitely generated nilpotent groups $H \subset G$ the subgroup $H$ has a polynomial distortion. The proof of this fact was sketched in [Gr1] for nilpotent Lie groups. Using the Mal’cev completion one can derive it for discrete torsion free nilpotent groups. Since every finitely generated nilpotent group is quasi-isometric to a finitely generated torsion free nilpotent group, the result follows in the general case.

For the sake of completeness we give an alternative proof of the proposition. Without loss of generality we may assume that $G$ is torsion free. Let $(x_1, x_2, \ldots, x_m)$ be a Hall’s basis for $G$ [Ha]. Thus $G_k = \langle x_1, \ldots, x_{i_k} \rangle$ for every $k$. Then every element $g \in G$ is uniquely expressible as $g = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, written symbolically as $x^a$, where $a_i \in \mathbb{Z}$, for each $i$, and the group operations on $G$ amount to prescribing polynomials $f = : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ and $i = : \mathbb{Z}^n \to \mathbb{Z}^n$, where $x^a x^b = x^{f(a,b)}$ and $(x^a)^{-1} = x^{i(a)}$. Thus, Hall’s basis defines a bijection $\phi : G \to \mathbb{Z}^m$ by the rule $\phi(g) = a$. We consider the word metric $d_k$ on $G_k$ defined by the set $(x_1, x_2, \ldots, x_{i_k})$. Clearly, $\phi^{-1}$ is 1-Lipschitz. Since the operation laws in $G$ are polynomial there is a polynomial $p(t)$ such that the restriction $\phi|_{B_t(e)} : B_t(e) \to \mathbb{Z}^m$ is $p(t)$-Lipschitz. We denote by $\phi_k$ the restriction of $\phi$ to $G_k$: $\phi_k : G_k \to \mathbb{Z}^{i_k}$. Let $\pi : \mathbb{Z}^{i_k} \to \mathbb{Z}^{i_k-1}$ be the projection. Then the map $\phi_k^{-1} \pi \phi_k : G_k \to G_{k-1}$ is a $G_{k-1}$-equivariant retraction. Then

$$d_{k-1}(x, x') = d_{k-1}(\phi_k^{-1} \pi \phi_k(x), \phi_k^{-1} \pi \phi_k(x')) \leq p(d_k(x, x'))d_k(x, x').$$

□

Proposition 4.4. Let $f : G \to H$ be an epimorphism of finitely generated groups with a finitely generated kernel $K = \ker(f)$ and let $\operatorname{asdim} H \leq n$, $\operatorname{asdim} K \leq k$. Suppose that the Gromov functions $\gamma_H$ and $\gamma_K$ taken for the word metrics on $H$ and $K$ are bounded from above by a polynomial. Also assume that $K$ has a polynomial distortion in $G$. Then Gromov’s function $\gamma_G$ taken for $\operatorname{asdim} G \leq (k+1)(n+1)$ has a polynomial growth.

Proof. Let $p_1(t)$ be a polynomial bound for $\gamma_H$ and let $p_2(t)$ be a polynomial bound for $\gamma_K$. Let $p_3(t)$ be a polynomial such that the inverse function $p_3^{-1}$ is defined and can serve as the lower bound $p_1$ in the inequality (*) for the coarse imbedding $K \subset G$. We define a polynomial $P(\lambda) = p_2(p_3(6p_1(\lambda))) + 2p_1(\lambda)$.

Let $\lambda > 0$ be given. Let $\mathcal{U}$ be a cover of $H$ with $m(\mathcal{U}) \leq n+1$, $L(\mathcal{U}) \geq \lambda$, and with $\operatorname{diam}(\mathcal{U}) \leq p_1(\lambda)$. 
Let $\mathcal{V}$ be a cover of $K$ (taken with the word metric) with $m(\mathcal{V}) \leq k + 1$, $L(\mathcal{V}) \geq p_3(6p_1(\lambda))$, and with $\text{diam}(\mathcal{V}) \leq p_2(p_3(6p_1(\lambda)))$. Then in the subspace metric this cover has the following properties: $L(\mathcal{V}) \geq 6p_1(\lambda)$ and $\text{diam}(\mathcal{V}) \leq p_2(p_3(6p_1(\lambda)))$. We apply Proposition 4.2 to obtain a cover $\mathcal{W}$ of $G$ with $L(\mathcal{W}) \geq \lambda$, $\text{diam}(\mathcal{W}) \leq p_2(p_3(6p_1(\lambda))) + 2p_1(\lambda) = P(\lambda)$ and with $m(\mathcal{W}) \leq (n + 1)(k + 1)$. Thus $\gamma_G \leq P(\lambda)$. □

Clearly, every finitely generated abelian group $G$ has a linear Gromov function $\gamma_G$. In view of Propositions 4.3 and 4.4, by induction we obtain the following.

**Corollary 4.5.** For every finitely generated nilpotent group $N$ there is $k$ such that $\text{asdim} G \leq k$ and Gromov’s function $\gamma_N$ is bounded from above by a polynomial.

**Theorem 4.6.** Let $N$ be a finitely generated nilpotent group and let $G$ be a finitely generated group with $\text{asdim} G < \infty$. Then the restricted wreath product $N \wr G$ has a polynomial dimension growth.

**Proof.** Since $N$ is nilpotent, by Corollary 4.5 Gromov’s function $\gamma_N$ has a polynomial growth $t^k$, $k \in \mathbb{N}$, for the inequality $\text{asdim} N \leq n$ for some $n$. Let $\lambda$ be given. Then there is an $R$-bounded cover $\mathcal{U}$ of $N$ with $L(\mathcal{U}) > \lambda$, $R \leq \lambda^k$, and with the multiplicity $m(\mathcal{U}) \leq n + 1$. Let $\pi : N \wr G \to N$ be the natural epimorphism. We note that $\pi$ is 1-Lipschitz with respect to the metric $\rho$ on $N \wr G$ and the word metric on $N$ generated by $T$.

Let $r = 6R$. We consider $\bigoplus_{x \in B_r(e)} G_x \subset \ker(\pi) = K$. Let $\text{asdim} G = m$. Then $\text{asdim} G^t \leq lm$ ([DJ]). By Proposition 2.2 there is a uniformly bounded cover $\mathcal{V}$ of $G^{B_r(e)} = \bigoplus_{x \in B_r(e)} G_x$ with $m(\mathcal{V}) \leq |B_r(e)|m + 1$ and with $L(\mathcal{V}) \geq r$. We define a cover $\tilde{\mathcal{V}}$ of

$$K = C_0(N, G) = \oplus_N G = \oplus_{B_r(e)} G \times \bigoplus_{N \setminus B_r(e)} G$$

as follows

$$\tilde{\mathcal{V}} = \{V \times z \mid z \in \bigoplus_{N \setminus B_r(e)} G, \ V \in \mathcal{V}\}.$$ 

We note that for every $z \in \bigoplus_{N \setminus B_r(e)} G$ and every $V \in \mathcal{V}$ the set $V \times z = z(V \times 1)$ is isometric to $V \times 1$. Therefore the cover $\tilde{\mathcal{V}}$ is uniformly bounded. Note that $m(\tilde{\mathcal{V}}) = m(\mathcal{V}) \leq |B_r(e)|(m + 1)$.

We show that $L(\tilde{\mathcal{V}}) \geq r$. Let $f \in K$, $f : N \to G$. We consider two cases.

1. $\text{supp}(f) \subset B_r(e)$. Then there is $V \in \mathcal{V}$ such that $\rho(f, G^{B_r(e)} \setminus V) \geq r$. For every $h$ with $\text{supp}(h) \setminus B_r(e) \neq \emptyset$ in view of Proposition 4.1 we have

$$\rho(f, h) = \rho(f(x), h(x)) \geq \|x\| \geq r$$
for $x \in \text{supp}(h) \setminus B_r(e) \neq \emptyset$. Thus
\[ \rho(f, \oplus_{N \setminus B_r(e)} G) \geq r \]
and hence $\rho(f, K \setminus (V \times 1)) \geq r$.

(2) There is $x \in \text{supp}(f) \setminus B_r(e)$. We decompose $f = f' + \bar{f}$ where $f' = f|_{B_r(e)}$ and $\bar{f} = f|_{N \setminus B_r(e)}$. Since $L(V) \geq r$, there is $V \in \mathcal{V}$ such that
\[ \rho(f', G^{B_r(e)} \setminus V) \geq r. \]
Then $f \in V \times \bar{f} \in \tilde{V}$. Note that
\[ K \setminus (V \times \bar{f}) = ((G^{B_r(e)} \setminus V) \times \bar{f}) \cup (G^{B_r(e)} \times (\oplus_{N \setminus B_r(e)} G \setminus \{\bar{f}\})). \]
By Proposition 4.1
\[ \rho(f, (G^{B_r(e)} \setminus V) \times \bar{f}) \geq \rho(f', G^{B_r(e)} \setminus V) \geq r. \]
If
\[ h \in G^{B_r(e)} \times (\oplus_{N \setminus B_r(e)} G \setminus \{\bar{f}\}), \]
then
\[ f|_{N \setminus B_r(e)} \neq h|_{N \setminus B_r(e)}. \]
Hence there is $y \in N \setminus B_r(e)$ such that $f(y) \neq h(y)$. Then by Proposition 4.1
\[ \rho(f, h) \geq \rho(f(y), h(y)) \geq \|y\| \geq r \]
for all
\[ h \in G^{B_r(e)} \times (\oplus_{N \setminus B_r(e)} G \setminus \{\bar{f}\}). \]
Thus, $\rho(f, K \setminus (V \times \bar{f})) \geq r$.

We apply Proposition 4.2 to the epimorphism $\pi : N \wr_G \to N$ and to the covers $\mathcal{U}$ and $\tilde{\mathcal{V}}$ of $N$ and $K = \ker(\pi)$ to obtain a uniformly bounded cover $\mathcal{W}$ of $N \wr G$ with $L(\mathcal{W}) \geq \lambda$ and with
\[ m(\mathcal{W}) \leq m(\mathcal{U})m(\tilde{\mathcal{V}}) \leq (n + 1)(m + 1)|B_r(e)|. \]
It is known that every finitely generated nilpotent group has a polynomial volume growth. Let $P(t)$ be a monotone polynomial such that $|B_r(e)| \leq P(t)$. Then
\[ m(\mathcal{W}) \leq (n + 1)(m + 1)P(r) = (n + 1)(m + 1)P(6R) \leq (n + 1)(m + 1)P(6\lambda^k). \]
$\square$
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