Semi-inner products and the concept of semi-polarity

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Abstract

The lack of an inner product structure in general Banach spaces yields the motivation to introduce a semi-inner product with a more general axiom system than that determining a Hilbert space (it misses the requirement for symmetry). We use the semi-inner product on a finite dimensional real Banach space \((X, \| \cdot \|)\) to generalize the concept of polarity which depends on the Euclidean structure of the underlying vector space. For this purpose we construct a map on \((X, \| \cdot \|)\), called normality map. This normality map is also interesting for itself, e.g., for studying isoperimetricas in higher-dimensional normed spaces.

**Keywords:** anti-norm, gauge function, isoperimetrix, Minkowski space, normality, normed space, polarity, semi-inner product, simplectic bilinear form, support function

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1 Introduction

Motivated by the lack of the inner product structure in general Banach spaces, Lumer defined in \([10]\) semi-inner product spaces. In this way he carried over Hilbert space arguments to the theory of Banach spaces. Semi-inner product spaces are subject of many investigations, but mainly from the viewpoint of functional analysis; see, e.g., the book \([3]\) and the references there.

We apply semi-inner products for purely geometric purposes. More precisely, we use the semi-inner product structure of finite dimensional real Banach spaces to generalize the concept of polarity which depends on the Euclidean structure of the underlying vector space.

Let \(X\) be a real vector space. A **semi-inner product** on \(X\) is a real function \([\cdot, \cdot]\) on \(X \times X\) with the following properties:

(i) \([x + y, z] = [x, z] + [y, z], [\lambda x, y] = \lambda [x, y]\) for all real \(\lambda\),

(ii) \([x, x] > 0\), when \(x \neq 0\),

(iii) \(|[x, y]|^2 \leq [x, x][y, y]\).

If on \(X\) a semi-inner product is defined, then \(X\) is said to be a **semi-inner product space**. A semi-inner product \([\cdot, \cdot]\) on a vector space induces a norm \(\| \cdot \|\) by setting \(\|x\| = \sqrt{[x, x]}\). Conversely, every Banach space \((X, \| \cdot \|)\) can be transformed into a semi-inner product space (see \([11]\) Theorem 1) in the following way. Let \(S := \{x \in X : \|x\| = 1\}\) be the unit sphere of \((X, \| \cdot \|)\), and \(X^*\) be the dual space of \(X\). On \(X^*\) one can define a norm \(\| \cdot \|^*\), called the **dual norm**, in the usual way, i.e.,

\[
\|f\|^* := \sup\{f(x) : \|x\| = 1\} \quad \text{for} \quad f \in X^*.
\]
If $S^*$ is the unit sphere of $(\mathbb{X}^*, \| \cdot \|^*)$, then for any $x \in S$ there exists, by the Hahn-Banach Theorem, at least one functional (exactly one functional if the norm is smooth) $f_x \in S^*$ with $f_x(x) = 1$. For any $\lambda x \in \mathbb{X}$, where $x \in S$, we choose $f_{\lambda x} \in \mathbb{X}^*$ such that $f_{\lambda x} = \lambda f_x$. Then a semi-inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{X}$ is defined by

$$\langle x, y \rangle := f_y(x). \quad (2)$$

Let now $(\mathbb{X}, \| \cdot \|)$ be a normed space (i.e., a finite dimensional real Banach space) with origin $o$ and unit ball $B = \{x \in \mathbb{X} : \|x\| \leq 1\}$, which is a compact, convex subset of $\mathbb{X}$ with boundary $S$ centered at its interior point $o$. Let $S_c$ be the unit sphere with respect to the Euclidean norm, i.e., the norm induced by an inner product on $\mathbb{X}$. A vector $x \neq 0$ is normal to a vector $y \neq 0$, denoted by $x \perp y$ if for any real $\lambda$ the inequality $\|x\| \leq \|x + \lambda y\|$ holds; see, e.g., [14] § 6.

For a convex body $K$, i.e., a compact, convex subset of $\mathbb{X}$ with nonempty interior and $u \neq o$, let $h(K, u)$ be the support function in direction $u$. The support function of $K$ with respect to the norm $\| \cdot \|$ is defined by $h_B(K, u) = \frac{h(K, u)}{h(B, u)}$. Alternatively, for every $u \neq o$ this normed support function $h_B(K, u)$ can be viewed as the signed distance with respect to $\| \cdot \|$ from the origin $o$ to a supporting hyperplane $H$ of $K$ such that the outer normal of $H$ with respect to $K$ yields a positive inner product with $u$; see, e.g., [2] or [12] § 2. This means that the normed support function $h_B(K, u)$ of $K$ can be expressed as $\sup \{ \langle x, u \rangle : x \in K \}$.

We denote the set of all convex bodies containing the origin $o$ as interior point by $\mathcal{X}_o$. For $K \in \mathcal{X}_o$, let $g(K, \cdot)$ be the gauge function of $K$, i.e.,

$$g(K, x) := \min \{ \lambda \geq 0 : x \in \lambda K \} \quad \text{for} \quad x \in \mathbb{X}.$$ 

Note that $g(B, x) = \|x\|$ for every $x \in \mathbb{X}$.

## 2 Semi-inner product spaces

From now on, let $(\mathbb{X}, \| \cdot \|)$ be a finite dimensional real Banach space which is smooth and strictly convex. We denote by $\langle \cdot, \cdot \rangle$ the semi-inner product induced by the norm $\| \cdot \|$. If $(\mathbb{X}, \| \cdot \|)$ is an inner product space, i.e., the corresponding semi-inner product is, in addition, symmetric, then we denote it by $\langle \cdot, \cdot \rangle$. The following properties are proved in [4] (see also [3], [7] § 2.4, and [9]).

(iv) The homogeneity property: $[x, \lambda y] = \lambda [x, y]$ for all $x, y \in \mathbb{X}$ and all real $\lambda$.

(v) $[y, x] = 0 \iff \|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{R}$.

(vi) The generalized Riesz-Fischer Representation Theorem: To every linear functional $f \in \mathbb{X}^*$ there exists a unique vector $y \in \mathbb{X}$ such that $f(x) = [x, y]$ for all $x \in \mathbb{X}$. Then $[x, y] = [x, z]$ for all $x \in \mathbb{X}$ if and only if $y = z$.

(vii) The dual vector space $\mathbb{X}^*$ is a semi-inner product space by $[f_x, f_y]^* = [y, x]$.

**Remark 2.1.** Property (v) can be written in the form

(v’) $x \neq 0, y \neq 0$ and $[y, x] = 0 \iff x \perp y$. 

Remark 2.2. By Property (vi) we have a one-to-one map \( F : \mathbb{X} \to \mathbb{X}^* \) with \( F : x \mapsto f_x \), where \( f_x \) is determined by \( \mathbb{X} \). Property (vii) implies that \( F \) is norm-preserving.

Remark 2.3. The norm \( \| \cdot \|^\ast \) on \( \mathbb{X}^* \) induced by the semi-inner product \([\cdot,\cdot]^\ast\) is given by
\[
\|f_x\|^\ast = \sqrt{\langle f_x, f_x \rangle} = \sqrt{|x|} = \|x\|. \tag{3}
\]

Proposition 2.1. The norm induced by the semi-inner product \([\cdot,\cdot]^\ast\) on \( \mathbb{X}^* \) coincides with the norm defined by \( \mathbb{X} \).

Proof. Let \( f_x \in \mathbb{X}^* \). Then \( \text{sup}\{f_x(y) : \|y\| = 1\} = \text{sup}\{[y, x] : \|y\| = 1\} \). Since \( |[y, x]|^2 \leq [y, y] \cdot [x, x] = \|x\|^2 \) for all \( y \) with \( \|y\| = 1 \), we get
\[
\text{sup}\{f_x(y) : \|y\| = 1\} \leq \|x\|. \tag{4}
\]

On the other hand, \( \text{sup}\{[y, x] : \|y\| = 1\} \geq \frac{1}{\|x\|} [x, x] = \frac{1}{\|x\|} [x, x] = \|x\| \), which together with (3) yields \( \text{sup}\{f_x(y) : \|y\| = 1\} = \|x\| \).

\[
\]

Proposition 2.2. For the map \( F \) and any \( x, y \in \mathbb{X} \), \( \lambda, \mu \in \mathbb{R} \) we have
\[
\|F(\lambda x + \mu y)\|^\ast \leq |\lambda|\|F(x)\|^\ast + |\mu|\|F(y)\|^\ast.
\]

Proof. From the definition of \( F \) we get \( \|F(\lambda x + \mu y)\|^\ast = \|f_{\lambda x + \mu y}\|^\ast = \|\lambda x + \mu y\| \leq \|\lambda x\| + \|\mu y\| = \|f_{\lambda x}\|^\ast + \|f_{\mu y}\|^\ast = |\lambda|\|F(x)\|^\ast + |\mu|\|F(y)\|^\ast. \)

\[
\]

3 The normality map

Let \( \langle \cdot, \cdot \rangle \) be a non-degenerate bilinear simplectic form on \( \mathbb{X} \) (here we assume that the dimension of \( \mathbb{X} \) is even), i.e., \( \langle x, x \rangle = 0 \) and \( \langle x, y \rangle = -\langle y, x \rangle \). The vector space \( \mathbb{X} \) and its dual space \( \mathbb{X}^* \) can be identified via
\[
G : \mathbb{X} \to \mathbb{X}^*, \quad x \mapsto g_x, \quad \text{where } g_x(y) := \langle y, x \rangle; \tag{5}
\]
see [13, § 2.3]. It is easy to see that \( G \) is a one-to-one map. From now on we fix a simplectic form on \( \mathbb{X} \). If \( \mathbb{X} \) is two-dimensional, let \( \langle x, y \rangle \) be the signed area of the parallelogram with vertices \( o, x, x + y, y \). The dual norm \( \| \cdot \|_a \) on \( \mathbb{X}^* \) identified with \( \mathbb{X} \) via the isomorphism \( G \) is called the antinorm on \( \langle \mathbb{X}, \| \cdot \| \rangle \), i.e.
\[
\|x\|_a := \|Gx\|^\ast = \|g_x\|^\ast = \text{sup}\{\langle y, x \rangle : \|y\| = 1\}. \tag{6}
\]

The map defined in Remark 2.2 for the anti-norm is denoted by \( F^a \). The normality relation defined at the beginning is not symmetric, but if \( x \parallel y \), then \( y \) is normal to \( x \) with respect to the antinorm, denoted by \( y \parallel a x \). Since the antinorm of the antinorm is the original norm, i.e. \( \|x\|_{a, a} = \|x\| \), we have
\[
x \parallel y \iff y \parallel a x;
\]
see again [13, § 3]. The product \( J = G^{-1} F : \mathbb{X} \to \mathbb{X} \), which is a one-to-one map, is said to be the normality map of \( \langle \mathbb{X}, \| \cdot \| \rangle \).
Theorem 3.1. For any $x, y \in \mathbb{X}$ and any $\lambda \in \mathbb{R}$ we have
\begin{enumerate}[(i)]
  
  \item $\|x\| = \|Jx\|_a$ and $\|x\|_a = \|J^a x\|$, where $J^a = G^{-1} F^a$;
  
  \item $J(S) = S_a$, where $S_a$ is the unit sphere induced by the antinorm $\| \cdot \|_a$;
  
  \item $[x, y] = \langle x, Jy \rangle$ and $[x, y]_a = \langle x, J^a y \rangle$, where $[\cdot, \cdot]_a$ is the semi-inner product induced by $\| \cdot \|_a$;
  
  \item $x \mapsto Jx$ and $x \mapsto J^a x$;
  
  \item $[Jx, y] = -[Jy, x]$;
  
  \item $[Jx, y]_a = -[J^a y, x]$;
  
  \item $J(\lambda x) = \lambda Jx$;
  
  \item $J(B) = B_a$, where $B_a$ is the unit ball induced by the antinorm $\| \cdot \|_a$;
  
  \item $J^a J = JJ^a = -I$, where $I$ denotes the identity map of $\mathbb{X}$;
  
  \item $x \mapsto -x$ implies $[x, Jx] = 0$ and $[x, J^a x] = 0$.
\end{enumerate}

Proof. Let $x \xrightarrow{F} f_x \xrightarrow{G^{-1}} Jx$ and $x \xrightarrow{F^a} f^a_x \xrightarrow{G^{-1}} J^a x$. Since $G(Jx) = f_x$ and $\|x\|_{a,a} = \|x\|$, we have
\begin{equation}
\|Jx\|_a = \|G(Jx)\|_a^* = \|f_x\|_a^* = \|x\|_a.
\end{equation}
by (3) and (4). By $G(J^a x) = f^a_x$ we get the second equality in (i). The equality (7) yields (ii). According to Definition (5) we have $G(Jx) = \langle \cdot, Jx \rangle$. On the other hand, $G(Jx) = G(G^{-1} F(x)) = F(x) = [\cdot, x]$. The same holds also for $J^a$, and thus we get (iii). Setting $x = Jy$ in the first equality in (iii) and $x = J^a y$ in the second one, we obtain (iv).

By (iii) and the anti-symmetry of $\langle \cdot, \cdot \rangle$ it follows that
\begin{equation}
[Jx, y] = \langle Jx, Jy \rangle = -\langle Jy, Jx \rangle = -[Jy, x],
\end{equation}
which is (v). According to (iii) we get $[Jx, y]_a = \langle Jx, J^a y \rangle = -\langle J^a y, Jx \rangle = -[J^a y, x]$. The homogeneity of $[\cdot, \cdot]$ and $\langle \cdot, \cdot \rangle$ yields (vi), which together with (ii) implies (vii). Applying the second implication of (iv) to $Jx$, we have $Jx \mapsto J^a(Jx)$, i.e., $J^a(Jx) \mapsto Jx$. This together with the first implication of (iv) gives $x = \lambda J^a(Jx)$ for some $\lambda \in \mathbb{R}$, because the considered space is strictly convex. By (i), $\|x\| = \|Jx\|_a$. On the other hand, (i) also implies that $\|Jx\|_a = \|J^a(Jx)\|$. We thus get $\lambda = \pm 1$. The fact that $\langle x, Jx \rangle = [x, x] > 0$ for any $x \neq 0$ yields $\lambda = -1$. By (v) it follows that $[J(J^a x), y] = -[Jy, J^a x]$, which together with (ix) yields the first relation of (x) implying $[x, J^a y] = [J(J^a y), J^a x] = -[y, J^a x]$.

Remark 3.1. The last argument in the proof of Theorem 3.1 implies that for every $x \neq 0$ the pair $(x, Jx)$ is positively oriented.

Remark 3.2. If $(\mathbb{X}, \| \cdot \|)$ is the Euclidean plane, then $J : \mathbb{X} \to \mathbb{X}$ is simply the rotation about the origin by the angle of $90^\circ$.

Remark 3.3. If $(\mathbb{X}, \| \cdot \|)$ is two-dimensional, then $J(S) = S_a$ is the isoperimetrix of $(\mathbb{X}, \| \cdot \|)$.

Remark 3.4. The normality map $J$ also appears in [6, p. 308] as $T$. 

4 The concept of semi-polarity

The concept of polarity (polar duality) is a very important tool in several areas of convexity. Let $\mathbb{X}$ be a real vector space. For a set $X \subseteq \mathbb{X}$, the subset of $\mathbb{X}^*$ defined by
\[
\{ f \in \mathbb{X}^* : f(x) \leq 1 \text{ for all } x \in X \}
\]
is called the polar of $X$. Usually the polar of a set $U$ is identified with the subset
\[
\{ y \in \mathbb{X} : \langle x, y \rangle \leq 1 \text{ for all } x \in X \}
\]
of $\mathbb{X}$ via the canonical isomorphism between $\mathbb{X}^*$ and $\mathbb{X}$, induced by an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{X}$. In this way the polar depends on an Euclidean structure on $\mathbb{X}$. The following theorem summarizes some of the most important properties of the polar set of a given set; see, e.g., [15, §1.6 and Remark 1.7.7], [17, §2.8], [1, §3], and [5, §4.1, p. 56].

**Theorem 4.1.** Let $M$ and $N$ be sets in $\mathbb{X}$, let $B$ be the Euclidean unit ball of $\mathbb{X}$, and let $\lambda$ be a non-zero scalar. Then

(i) $M \subseteq N$ implies $N^* \subseteq M^*$;
(ii) $(M \cup N)^* = M^* \cap N^*$;
(iii) $(\lambda M)^* = (1/\lambda)M^*$;
(iv) $B^*_e = B_e$.

If $M \in \mathbb{X}_0$, then

(v) $M^{**} = M$,
(vi) $g(M^*, x) = h(M, x)$ and $h(M^*, x) = g(M, x)$.

If, in addition, $M$ is centered at $o$, then

(vii) $h(M, x) = \|x\|_M^*$ and $h(M^*, x) = \|x\|_M$ for $x \in X$, where $\| \cdot \|_N$ is the norm induced by the $o$-symmetric convex body $N$.

Now we want to generalize this concept by defining a polarity notion which depends on an arbitrary semi-inner product structure on $\mathbb{X}$.

Let $(\mathbb{X}, \langle \cdot, \cdot \rangle)$ be a normed space with unit ball $B$ and semi-inner product $[\cdot, \cdot]$. For $m \in \mathbb{X}$ we define the set
\[
m^0 := \{ x \in \mathbb{X} : [x, m] \leq 1 \}.
\]
This set is said to be the semi-polar of $m$. Clearly, $o^0 = \mathbb{X}$. Let us suppose that $m \neq o$. Then $f_m(x) = 1$ is the supporting hyperplane of the ball $\frac{1}{\|m\|}B$ at the point $\frac{1}{\|m\|}m$. Thus $m^0$ is the closed half-space bounded by $f_m(x) = 1$ and containing the origin $o$. For a set $M \subseteq \mathbb{X}$ the semi-polar $M^0$ of $M$ is the intersection of all the sets $m^0$ for $m \in M$, i.e.,
\[
M^0 = \bigcap_{m \in M} m^0 = \{ x \in \mathbb{X} : [x, m] \leq 1 \text{ for all } m \in M \}.
\]
Note that for any set $M \subseteq \mathbb{X}$ the semi-polar $M^0$ is a closed convex set containing the origin in its interior. The semi-polar of $\mathbb{X}$ is $\{o\}$. If the underlying space $(\mathbb{X}, \| \cdot \|)$ is Euclidean, then the semi-polar of a set is the polar of this set.
Proposition 4.1. If $\lambda$ is a non-zero scalar and $M, N$ are subset of $\mathbb{X}$, then

(i) $M \subseteq N$ implies $N^\circ \subseteq M^\circ$;
(ii) $(M \cup N)^\circ = M^\circ \cap N^\circ$;
(iii) $(\lambda M)^\circ = (1/\lambda)M^\circ$;
(iv) $B^\circ = B$.

Proof. Taking into account that the semi-inner product is homogeneous with respect to both the arguments, the proof is the same as in the case of polarity; for polarity see, e.g., the proof of Theorem 2.8.1 in [17, p. 100].

The next theorem is an analogue to Theorem 4.1 (v).

Theorem 4.2. If $M \in \mathcal{X}_o$, then $JM = (J^a(M^\circ))^\circ$.

Proof. We have $M^\circ = \{x \in \mathbb{X} : [x, m] \leq 1 \text{ for all } m \in M\}$. Hence, by Theorem 3.1 (x), $[Jm, J^a x] \leq 1$ holds for every $x \in M^\circ$ and every $m \in M$. Therefore $Jm \in (J^a(M^\circ))^\circ$, implying $JM \subseteq (J^a(M^\circ))^\circ$. In order to prove the converse implication, we consider $z \not\in JM$. Then there exists a hyperplane strictly separating $z$ and $JM$. Since $JM \in \mathcal{X}_o$, this hyperplane cannot pass through the origin, and it can be described by $f(x) = 1$, where $f \in \mathbb{X}^*$. Let $u = F^{-1}(f)$. Then

\[ f(z) = [z, u] > 1 \quad \text{and} \]

\[ f(Jm) = [Jm, u] < 1 \quad \text{for any } m \in M. \]

Let us consider $y = (J^a)^{-1} u$. We have $y \in M^\circ$ since by Theorem 3.1 (x), and (9)

\[ [y, m] = [(J^a)^{-1} u, m] = [Jm, J^a(J^a)^{-1} u] < 1. \]

This means that $u \in J^aM^\circ$. Inequality (8) implies that $z \not\in (J^aM^\circ)^\circ$, which completes the proof.

Remark 4.1. Let $(\mathbb{X}, \| \cdot \|)$ be the Euclidean plane. Then Theorem 4.2 implies the following property of polar bodies which also can be checked very easily in a direct way: if $M$ is a convex body and the polar $M^*$ of $M$ is rotated about the origin by the angle of 90°, then the polar of the rotated body is the body obtained from $M$ by the same rotation.

The next theorem shows how the gauge function of the semi-polar of a convex body relates to the normed support function of this body.

Theorem 4.3. If $M \in \mathcal{X}_o$, then

\[ h_B(M^\circ, x) = g(M, x) \quad \text{and} \quad h_B(JM, x) = g(J^aM^\circ, x) \]

for every $x \in \mathbb{X} \setminus \{0\}$. 
Proof. Firstly we prove that the second equation in (10) implies the first one. Applying the second equation for $J^o M^o$ and Theorem 4.2, we get

$$h_B(J(J^o M^o), x) = g(J^o(JM), x) \iff h_B(-M^o, x) = g(-M, x) \iff h_B(-M^o, x) = g(-M, x).$$

The body $J^o M^o$ contains the origin $o$ in its interior, and for any $x \neq o$ we denote by $x_0$ the intersection point of $bd J^o M^o$ and the ray emanating from $o$ and passing through $x$. Let $n \in M^o$ be such that $x_0 = J^o n$. Then for every $m \in M$ we have

$$1 \geq [n, m] = [(J^o)^{-1} x_0, m] \implies 1 \geq [Jm, J^o((J^o)^{-1} x_0)] = [Jm, x_0],$$

by Theorem 3.1 (x). Thus we get $h_B(JM, x_0) = \sup\{[Jm, x_0] : Jm \in JM\} \leq 1$, which yields

$$h_B(JM, x) = h_B(JM, \|x\| / \|x_0\| x_0) \leq \|x\| / \|x_0\| = g(J^o M^o, x).$$

(11)

For every $0 < \lambda < g(J^o M^o, x)$ we have that

$$x \notin \lambda(J^o M^o) = J^o(\lambda M^o) = J^o(\frac{1}{\lambda} M)^o,$$

(12)

by Theorem 3.1 (vii) and Proposition 4.1 (iii). If $y = (J^o)^{-1} x$, implication (12) means that $y \notin (\frac{1}{\lambda} M)^o$, yielding $[y, \frac{1}{\lambda} m_0] > 1$ for some $m_0 \in M$. Hence

$$\lambda \leq [J^o)^{-1} x, m_0] = [Jm_0, J^o((J^o)^{-1} x)] = [Jm_0, x],$$

by Theorem 3.1 (x), and therefore $h_B(JM, x) = \sup\{[Jm, x] : Jm \in JM\} > \lambda$. The last inequality holds for every $0 < \lambda < g(J^o M^o, x)$. Therefore $h_B(JM, x) \geq g(J^o M^o, x)$, which together with (11) implies the second equation of (10).

As a consequence from the above theorem we get relations between the polar of a convex body and the semi-polar of the same body.

**Corollary 4.1.** If $M \in \mathfrak{X}_0$, then $h(M^*, x) = h_B(M^o, x)$ and $h(M^*, x) h(B, x) = h(M^o, x)$.

**Proof.** By Theorem 4.1 (vi), and (10) we have $h(M^*, x) = g(M, x) = h_B(M^o, x)$. 

The next corollary is an analogue of Theorem 4.1 (vii). In this connection we note that if $M$ is centered at the origin, then $J^o M$, $J^o M$, and $M^o$ are also centered at $o$.

**Corollary 4.2.** If $M \in \mathfrak{X}_0$ and, in addition, $M$ is centrally symmetric, then

$$h_B(M^o, x) = \|x\|_M \quad \text{and} \quad h_B(JM, x) = \|x\|_{J^o M^o}.$$  

5 Concluding remarks

The requirements that the underlying normed space $(\mathfrak{X}, \| \cdot \|)$ is smooth, strictly convex and of even dimension are not necessary for the definition of semi-polarity. These requirements are only needed for the purpose that the normality map $J$ is well defined. The normality map $J$ depends on the choice of the simplectic form $\langle \cdot, \cdot \rangle$ on $\mathfrak{X}$. Thus, for dimensions $> 2$ the natural question arises whether $\langle \cdot, \cdot \rangle$ can be chosen in such a manner that $J(S)$ is the isoperimetrix in the sense of Busemann or the isoperimetrix in the sense of Holmes-Thompson; for both these concepts of isoperimetries see, e.g., Chapter 5 of [16], or [11].
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