**Article**

**Certain new subclasses of m-fold symmetric bi-pseudo-starlike functions using Q-derivative operator**

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**Abstract:** In this current study, we introduced and investigated two new subclasses of the bi-univalent functions associated with $q$-derivative operator; both $f$ and $f^{-1}$ are $m$-fold symmetric holomorphic functions in the open unit disk. Among other results, upper bounds for the coefficients $|\rho_{m+1}|$ and $|\rho_{2m+1}|$ are found in this study. Also certain special cases are indicated.

**Keywords:** $m$-fold symmetric bi-univalent functions, analytic functions, univalent function.

**MSC:** 30C45.

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1. **Introduction**

Let $\mathcal{A}$ be the family of holomorphic functions, normalized by the conditions $f(0) = f'(0) - 1 = 0$ which is of the form

$$f(z) = z + \rho_2 z^2 + \rho_3 z^3 + \cdots$$  

(1)

in the open unit disk $\Omega = \{z; z \in \mathbb{C} \text{ and } |z| < 1\}$. We denote by $\mathcal{G}$ the subclass of functions in $\mathcal{A}$ which are univalent in $\Omega$ (for more details see [1]).

The Keobe-One Quarter Theorem [1] state that the image of $\Omega$ under all univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Hence all univalent function $f \in \mathcal{A}$ has an inverse $f^{-1}$ satisfy $f^{-1}(f(z))$ and $f(f^{-1}(v)) = v$ ($|v| < r_0(f)$, $r_0(f) \geq \frac{1}{4}$), where

$$g(v) = f^{-1}(v) = v - \rho_2 v^2 + (2\rho_2^2 - \rho_3)v^3 - (5\rho_2^3 - 5\rho_2\rho_3 + \rho_4)v^4 + \cdots$$  

(2)

A function $f \in \mathcal{A}$ denoted by $\Sigma$ is said to be bi-univalent in $\Omega$ if both $f^{-1}(z)$ ans $f(z)$ are univalent in $\Omega$ (see for details [2–11]).

A domain $\Psi$ is said to be $m$-fold symmetric if a rotation of $\Psi$ about the origin through an angle $2\pi/m$ carries $\Psi$ on itself. Therefore, a function $f(z)$ holomorphic in $\Omega$ is said to be $m$-fold symmetric if

$$f(e^{2\pi i/m}z) = e^{2\pi i}f(z).$$

A function is said to be $m$-fold symmetric if it has the following normalized form

$$f(z) = z + \sum_{\phi=1}^{\infty} \rho_{m\phi+1} z^{m\phi+1} \quad (z \in \Omega, \ m \in \mathbb{N} = \{1, 2, 3, \cdots \}).$$  

(3)

Let $\mathcal{S}_m$ the class of $m$-fold symmetric univalent functions in $\Omega$, that are normalized by (3), in which, the functions in the class $\mathcal{S}$ are one-fold symmetric. Similar to the concept of $m$-fold symmetric univalent functions, we introduced the concept of $m$-fold symmetric bi-univalent functions which is denoted by $\Sigma_m$. Each of the function $f \in \Sigma$ produces $m$-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. 

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The normalized form of \( f(z) \) is given as in (3) and the series expansion for \( f^{-1}(z) \), which has been investigated by Srivastava et al., [12], is given below:

\[
g(v) = f^{-1}(v) = v - \rho_{m+1}v^{m+1} + \left[ (m+1)\rho_m^2 - \rho_{2m+1} \right] v^{2m+1} - \left[ \frac{1}{2}(m+1)(3m+2)\rho_{m+1}^3 - (3m+2)\rho_{m+1}\rho_{2m+1} + \rho_{3m+1} \right].
\]

(4)

Some of the examples of \( m \)-fold symmetric bi-univalent functions are

\[
\left\{ \frac{z^m}{1 - z^m} \right\}^k,
\]

\[
[-\log(1 - z^m)]^k,
\]

and

\[
\left\{ \frac{1}{2} \log \left( \frac{1 + z^m}{1 - z^m} \right)^k \right\}.
\]

For more details on \( m \)-fold symmetric analytic bi-univalent functions (see [5,12–17]).

Jackson [18,19] introduced the \( q \)-derivative operator \( D_q \) of a function as follows;

\[
D_q f(z) = \frac{f(qz) - f(z)}{(q - 1)z}.
\]

(5)

and \( D_q f(0) = f'(0) \). In case of \( g(z) = z^k \) for \( k \) is a positive integer, the \( q \)-derivative of \( f(z) \) is given by

\[
D_q z^k = \frac{z^k - (zq)^k}{(q - 1)z} = [k]_q z^{k-1}.
\]

As \( q \rightarrow 1^- \) and \( k \in \mathbb{N} \), we get

\[
[k]_q = \frac{1 - q^k}{1 - q} = 1 + q + \cdots + q^k \rightarrow k,
\]

(6)

where \( (z \neq 0, q \neq 0) \). For more details on the concepts of \( q \)-derivative (see [5,20–27]).

**Definition 1.** [28] Let \( f(z) \in \mathcal{A} \), \( 0 \leq \chi < 1 \) and \( \sigma \geq 1 \) is real. Then \( f(z) \in L_{\nu}(\chi) \) of \( \sigma \)-pseudostarlike function of order \( \chi \) in \( \Omega \) if and only if

\[
\Re \left( \frac{z[f'(z)]^\nu}{f(z)} \right) > \chi.
\]

(7)

Babalola [28] verified that, all pseudostarlike function are Bazilevic of type \( \left( 1 - \frac{1}{\sigma} \right) \), order \( \chi^2 \) and univalent in \( \Omega \).

**Lemma 1.** [1] Let the function \( \omega \in \mathcal{P} \) be given by the following series \( \omega(z) = 1 + \omega_1 z + \omega_2 z^2 + \cdots \) \( (z \in \Omega) \). The sharp estimate given by \( |\omega_n| \leq 2 \) \( (n \in \mathbb{N}) \) holds true.

In [29] Girgaonkar et al., introduced a new subclasses of holomorphic and bi-univalent functions as follows:

**Definition 2.** A function \( f(z) \) given by (1) is said to be in the class \( \mathcal{M}_L(\chi) \) \( (0 < \chi \leq 1, (z, v) \in \Omega) \) if \( f \in \mathcal{E} \), \( |\arg(f'(z))| < \frac{\chi \pi}{2} \) and \( |\arg(g'(v))| < \frac{\chi \pi}{2} \), where \( g(v) \) is given by (2).
Theorem 1. A function $f(z)$ given by (1) is said to be in the class $\mathcal{M}_{\Sigma}(\psi)$ ($0 \leq \psi < 1, (z, v) \in \Omega$) if $\theta \in \Sigma$, $\Re[(f'(z))^\psi] > \psi$ and $\Re[(g'(v))^\psi] > \psi$, where $g(v)$ is given by (2).

In this current research, we introduced two new subclasses denoted by $\mathcal{M}_{\Sigma}^{q}(\chi)$ and $\mathcal{M}_{\Sigma}^{q}(\psi)$ of the function class $\Sigma$ and obtain estimates coefficient $|\rho_{m+1}|$ and $|\rho_{2m+1}|$ for functions in these two new subclasses.

2. Main Results

Definition 3. A function $f(z)$ given by (1) is said to be in the class $\mathcal{M}_{\Sigma}^{q}(\chi)$ ($m \in \mathcal{N}, 0 < q < 1, \sigma \geq 1, 0 < \chi \leq 1, (z, v) \in \Omega$) if

$$f \in \Sigma \quad \text{and} \quad |\arg(D_qf(z))^\chi| < \frac{\chi \pi}{2},$$

and

$$|\arg(D_qg(v))^\chi| < \frac{\chi \pi}{2},$$

where $g(v)$ is given by (2).

Remark 1. We have the class $\lim_{q \to 1-1} \mathcal{M}_{\Sigma}^{q}(\chi) = \mathcal{M}_{\Sigma}^{q}(\psi)$ which was introduced and studied by Girgaonkar et al., [29].

Remark 2. We have the class $\lim_{q \to 1-1} \mathcal{M}_{\Sigma}^{q}(\chi) = \mathcal{M}_{\Sigma}^{q}(\psi)$ which was introduced and studied by Srivastava et al., [11].

Theorem 1. Let $f(z) \in \mathcal{M}_{\Sigma}^{q}(\chi)$, ($m \in \mathcal{N}, 0 < q < 1, \sigma \geq 1, 0 < \chi \leq 1, (z, v) \in \Omega$) be given (3). Then

$$|\rho_{m+1}| \leq \frac{2\chi}{\sqrt{(m+1)\sigma \chi^[2m+1]_q - (\chi - \sigma)\sigma [m+1]^2_q}},$$

and

$$|\rho_{2m+1}| \leq \frac{2\chi}{\sigma [2m+1]_q} + \frac{2(m+1)\chi^2}{\sigma^2 [m+1]^2_q},$$

Proof. Using inequalities (1) and (9), we get

$$(D_qf(z))^\chi = [\tau(z)]^\chi,$$

and

$$(D_qg(v))^\chi = [\zeta(v)]^\chi,$$

respectively, where $\tau(z)$ and $\zeta(v)$ in $\mathcal{P}$ are given by the following series

$$\tau(z) = 1 + \tau_{m} z^m + \tau_{2m} z^{2m} + \tau_{3m} z^{3m} + \cdots,$$

and

$$\zeta(v) = 1 + \zeta_{m} v^m + \zeta_{2m} v^{2m} + \zeta_{3m} v^{3m} + \cdots.$$

Clearly,

$$[\tau(z)]^\chi = 1 + \chi \tau_{m} z^m + \left(\chi \tau_{2m} + \frac{\chi(\chi - 1)}{2} \tau_{3m}^2 z^m \right) z^{2m} + \cdots,$$

and

$$[\zeta(v)]^\chi = 1 + \chi \zeta_{m} v^m + \left(\chi \zeta_{2m} + \frac{\chi(\chi - 1)}{2} \zeta_{3m}^2 v^m \right) v^{2m} + \cdots.$$

Also

$$(D_qf(z))^\chi = 1 + \sigma [m+1]_q \rho_{m+1} z^m + \left(\sigma [2m+1]_q \rho_{2m+1} + \frac{\sigma(\sigma - 1)}{2} [m+1]_q \rho_{m+1}^2 \right) z^{2m} + \cdots,$$
From (16) and (18), we obtain

\[ (D_q \xi(v))'' = 1 - \sigma[m + 1]q\rho_{m+1}v^m - \sigma[2m + 1]q\rho_{2m+1}v^{2m} \]
\[ + \left( \sigma(m + 1)[2m + 1]q\rho_{m+1}^2 + \frac{\sigma(\sigma - 1)}{2}[m + 1]q\rho_{m+1}^2 \right) v^{2m} + \ldots \]

Comparing the coefficients in (12) and (13), we have

\[ \sigma[m + 1]q\rho_{m+1} = \chi \tau_m, \quad \sigma[2m + 1]q\rho_{2m+1} + \frac{\sigma(\sigma - 1)}{2}[m + 1]q\rho_{m+1}^2 = \chi \tau_{2m} + \frac{\chi(\chi - 1)}{2} \zeta_m, \]
\[ -\sigma[m + 1]q\rho_{m+1} = \chi \xi_m, \quad -\sigma[2m + 1]q\rho_{2m+1} + \left( \sigma(m + 1)[2m + 1]q + \frac{\sigma(\sigma - 1)}{2}[m + 1]q \right) \rho_{m+1}^2 = \chi \xi_{2m} + \frac{\chi(\chi - 1)}{2} \xi_m. \]

From (16) and (18), we obtain

\[ \tau_m = -\zeta_m, \]

and

\[ 2\sigma[m + 1]q\rho_{m+1}^2 = \chi^2 (\tau_m^2 + \zeta_m^2). \]

Further from (17), (19) and (21), we obtain that

\[ \sigma(\sigma - 1)\chi[m + 1]q\rho_{m+1}^2 + (m + 1)\sigma\chi[2m + 1]q\rho_{m+1}^2 = (\chi - 1)\sigma^2[m + 1]q\rho_{m+1}^2 = \chi^2(\tau_{2m} + \xi_{2m}). \]

Therefore, we have

\[ \rho_{m+1}^2 = \frac{\chi^2(\tau_{2m} + \xi_{2m})}{\sigma[m + 1]q^2(\zeta - \chi) + (m + 1)\sigma\chi[2m + 1]q}. \]

By applying Lemma 1 for the coefficients \( \tau_{2m} \) and \( \xi_{2m} \), then we have

\[ |\rho_{m+1}| \leq \frac{2\chi}{\sqrt{(m + 1)\sigma\chi[2m + 1]q} - (\chi - \sigma)\sigma[m + 1]q}. \]

Also, to find the bound on \( |\rho_{2m+1}| \), using the relation (19) and (17), we obtain

\[ 2\sigma[2m + 1]q\rho_{2m+1} - (m + 1)\sigma[2m + 1]q\rho_{m+1}^2 = \chi(\tau_{2m} - \xi_{2m}) + \frac{\chi(\chi - 1)}{2}(\tau_m^2 - \xi_m). \]

It follows from (20), (21) and (23),

\[ \rho_{2m+1} = \frac{(m + 1)\chi^2\tau_m^2}{2\sigma^2[2m + 1]q} + \frac{\chi(\tau_{2m} - \xi_{2m})}{2\sigma[2m + 1]q}. \]

Applying Lemma 1 for the coefficients \( \tau_m, \tau_{2m}, \xi_m, \xi_{2m} \), then we have

\[ |\rho_{2m+1}| \leq \frac{2\chi}{\sigma[2m + 1]q} + \frac{2(m + 1)\chi^2}{\sigma^2[m + 1]q}. \]

□

Choosing \( q \to 1^{-1} \) in Theorem 1, we get the following result:

**Corollary 1.** Let \( f(z) \in \mathcal{M}_{\Sigma_m}(\chi), (m \in \mathcal{N}, \sigma \geq 1, 0 < \chi \leq 1, (z, v) \in \Omega) \) be given (3). Then

\[ |\rho_{m+1}| \leq \frac{2\chi}{\sqrt{(m + 1)[\sigma\chi m + \sigma^2 m + \sigma^2]}}, \]

where \( \chi \) is defined in (3).
and

\[ |\rho_{m+1}| \leq \frac{2\chi}{\sigma(2m+1)} + \frac{2\chi^2}{\sigma^2(m+1)}. \]  

(26)

Choosing \( m = 1 \) (one-fold case) in Theorem 1, we get the following result:

**Corollary 2.** Let \( f(z) \in \mathcal{M}_\Sigma^q(\chi), (0 < q < 1, \sigma \geq 1, 0 < \chi \leq 1, (z, v) \in \Omega) \) be given (1). Then

\[ |\rho_2| \leq \frac{2\chi}{\sqrt{2\sigma\chi[3]_q - (\chi - \sigma)\sigma[2]_q^2}}, \]  

(27)

and

\[ |\rho_3| \leq \frac{2\chi}{\sigma[3]_q} + \frac{4\chi^2}{\sigma[2]_q^2}. \]  

(28)

Choosing \( q \longrightarrow 1^{-1} \) in Corollary 2, we get the following result:

**Corollary 3.** [29] Let \( f(z) \in \mathcal{M}_\Sigma^q(\chi), (\sigma \geq 1, 0 < \chi \leq 1, (z, v) \in \Omega) \) be given (1). Then

\[ |\rho_2| \leq \frac{2\chi}{\sqrt{2\sigma(2\sigma + \chi)}}, \]  

(29)

and

\[ |\rho_3| \leq \frac{\chi(2\sigma + 3\chi)}{3\sigma^2}. \]  

(30)

**Remark 3.** For one-fold case, we have \( \lim_{q \longrightarrow 1^{-1}} \mathcal{M}_\Sigma^1(\chi) = \mathcal{M}_\Sigma(\chi) \), and we can get the results of Srivastava et al., [11].

**Definition 5.** A function \( f(z) \) given by (3) is said to be in the class \( \mathcal{M}_{\Sigma,m}^{q,\sigma}(\psi) \) \((m \in \mathbb{N}, 0 < q < 1, \sigma \geq 1, 0 \leq \psi < 1, (z, v) \in \Omega) \) if

\[ f \in \Sigma \quad \text{and} \quad \mathbb{R}[(D_q f(z))^\sigma] > \psi, \]  

(31)

and

\[ \mathbb{R}[(D_q g(v))^\sigma] > \psi, \]  

(32)

where \( g(v) \) is given by (2).

**Remark 4.** We have the class \( \lim_{q \longrightarrow 1^{-1}} \mathcal{M}_\Sigma^1(\psi) = \mathcal{M}_\Sigma^1(\chi) \) which was introduced and studied by Girgaonkar et al., [29].

**Remark 5.** We have the class \( \lim_{q \longrightarrow 1^{-1}} \mathcal{M}_\Sigma^1(\psi) = \mathcal{M}_\Sigma(\chi) \) which was introduced and studied by Srivastava et al., [11].

**Theorem 2.** Let \( f(z) \in \mathcal{M}_{\Sigma,m}^{q,\sigma}(\psi), (m \in \mathbb{N}, 0 < q < 1, \sigma \geq 1, 0 \leq \psi < 1, (z, v) \in \Omega) \) be given (3). Then

\[ |\rho_{m+1}| \leq \min \left\{ \frac{2(1 - \psi)}{\sigma[m + 1]_q}, \frac{1 - \psi}{\sigma(\sigma - 1)[m + 1]_q^2 + (m + 1)\sigma[2m + 1]_q} \right\}, \]  

(33)

and

\[ |\rho_{2m+1}| \leq \frac{2(m + 1)(1 - \psi)}{\sigma(\sigma - 1)[m + 1]_q^2 + (m + 1)\sigma[2m + 1]_q} + \frac{2(1 - \psi)}{\sigma[2m + 1]_q}. \]  

(34)

**Proof.** Using inequalities (31) and (32), we get

\[ (D_q f(z))^\sigma = \psi + (1 - \psi)\tau(z), \]  

(35)
and

\[(D_q \psi (v))'' = \psi + (1 - \psi) \zeta (v),\]  \hspace{1cm} (36)

here \(\tau (z)\) and \(\zeta (v)\) in \(\mathcal{P}\) are given by the following series

\[\tau (z) = 1 + \tau_m z^m + \tau_{2m} z^{2m} + \tau_{3m} z^{3m} + \cdots,\]

and

\[\zeta (v) = 1 + \zeta_m v^m + \zeta_{2m} v^{2m} + \zeta_{3m} v^{3m} + \cdots.\]

Clearly,

\[\psi + (1 - \psi) \tau (z) = 1 + (1 - \psi) \tau_m z^m + (1 - \psi) \tau_{2m} z^{2m} + \cdots,\]

and

\[\psi + (1 - \psi) \zeta (v) = 1 + (1 - \psi) \zeta_m v^m + (1 - \psi) \zeta_{2m} v^{2m} + \cdots.\]

Also

\[(D_q f (z))'' = 1 + \sigma [m + 1] q \rho_{m+1} z^m + \left(\sigma [2m + 1] q \rho_{2m+1} + \frac{\sigma (\sigma - 1)}{2} [m + 1] q \rho_{m+1}^2 \right) z^{2m} + \cdots,\]

and

\[(D_q \psi (v))'' = 1 - \sigma [m + 1] q \rho_{m+1} v^m - \sigma [2m + 1] q \rho_{2m+1} v^{2m} + \left(\sigma (m + 1) [2m + 1] q \rho_{m+1}^2 + \frac{\sigma (\sigma - 1)}{2} [m + 1] q \rho_{m+1}^2 \right) v^{2m} + \cdots.\]

Now comparing the coefficients in (35) and (36), we get

\[\sigma [m + 1] q \rho_{m+1} = (1 - \psi) \tau_m,\] \hspace{1cm} (37)

\[\sigma [2m + 1] q \rho_{2m+1} + \frac{\sigma (\sigma - 1)}{2} [m + 1] q \rho_{m+1}^2 = (1 - \psi) \tau_{2m},\] \hspace{1cm} (38)

\[-\sigma [m + 1] q \rho_{m+1} = (1 - \psi) \zeta_m,\] \hspace{1cm} (39)

\[-\sigma [2m + 1] q \rho_{2m+1} + \left(\sigma (m + 1) [2m + 1] q \rho_{m+1}^2 + \frac{\sigma (\sigma - 1)}{2} [m + 1] q \rho_{m+1}^2 \right) \rho_{m+1}^2 = (1 - \psi) \zeta_{2m}.\] \hspace{1cm} (40)

From (37) and (39), we obtain

\[\tau_m = -\zeta_m,\] \hspace{1cm} (41)

and

\[2\sigma [m + 1] q \rho_{m+1}^2 = (1 - \psi)^2 (\tau_m^2 + \zeta_m^2).\] \hspace{1cm} (42)

Also, from (38) and (40), we get

\[\sigma (\sigma - 1) \chi [m + 1] q \rho_{m+1}^2 + (m + 1) \sigma [2m + 1] q \rho_{m+1}^2 = (1 - \psi) (\tau_{2m} + \zeta_{2m}).\] \hspace{1cm} (43)

Applying the Lemma 1 for the coefficients \(\tau_m, \tau_{2m}, \zeta_m, \zeta_{2m}\), we find that

\[|\rho_{m+1}| \leq 2 \sqrt{\frac{(1 - \psi)}{\sigma (\sigma - 1) [m + 1] q + (m + 1) \sigma [2m + 1] q}}.\]

Also, to find the bound on \(|\rho_{2m+1}|\), using the relation (40) and (38), we obtain

\[-(m + 1) \sigma [2m + 1] q \rho_{m+1}^2 + 2\sigma [2m + 1] q \rho_{2m+1}^2 = (1 - \psi) (\tau_{2m} - \zeta_{2m}),\] \hspace{1cm} (44)
Corollary 5. \( \rho_{2m+1} = \frac{(1 - \psi)(\tau_{2m} - \zeta_{2m})}{2\sigma[2m + 1]q} + \frac{(m + 1)}{2}\rho_{m+1}^2. \) (45)

By substituting the value of \( \rho_{m+1}^2 \) from (42), we have

\[
\rho_{2m+1} = \frac{(1 - \psi)(\tau_{2m} - \zeta_{2m})}{2\sigma[2m + 1]q} + \frac{(m + 1)(1 - \psi)^2(\tau_{m}^2 + \zeta_{m}^2)}{4\sigma^2[2m + 1]^2}. \] (46)

Applying the Lemma 1 for the coefficients \( \tau_{m}, \tau_{2m}, \zeta_{m}, \zeta_{2m}, \) we get

\[ |\rho_{2m+1}| \leq \frac{2(1 - \psi)}{\sigma[2m + 1]q} + \frac{2(m + 1)(1 - \psi)^2}{2\sigma^2[2m + 1]^2}. \]

Also, by using (43) and (45), and applying Lemma 1 we obtain

\[ |\rho_{2m+1}| \leq \frac{2(m + 1)(1 - \psi)}{\sigma(\sigma - 1)[m + 1]^2 + (m + 1)\sigma[2m + 1]} + \frac{2(1 - \psi)}{\sigma[2m + 1]}. \]

This complete the proof. □

Choosing \( q \rightarrow 1^{-1} \) in Theorem 2, we get the following result:

**Corollary 4.** Let \( f(z) \in \mathcal{M}_{\Sigma,m}^c(\psi), \) \( (m \in \mathcal{N}, \sigma \geq 1, 0 \leq \psi < 1, (z, v) \in \Omega) \) be given (3). Then

\[
|\rho_{m+1}| \leq \left\{ \begin{array}{ll}
2 \sqrt{\frac{1 - \psi}{\sigma(\sigma - 1)[m + 1] + (m + 1)\sigma[2m + 1]}} & 0 \leq \psi \leq \frac{m}{1 + 2m}, \\
\frac{2}{\sigma[2m + 1]} & \frac{m}{1 + 2m} \leq \psi < 1,
\end{array} \right.
\]

and

\[
|\rho_{2m+1}| \leq \frac{2(m + 1)(1 - \psi)}{\sigma(\sigma - 1)[m + 1]^2 + (m + 1)\sigma[2m + 1]} + \frac{2(1 - \psi)}{\sigma[2m + 1]}. \]

For one fold case, Corollary 4, yields the following Corollary:

**Corollary 5.** Let \( f(z) \in \mathcal{M}_{\Sigma}^c(\psi), \) \( (\sigma \geq 1, 0 \leq \psi < 1, (z, v) \in \Omega) \) be given (1). Then

\[
|\rho_{2}| \leq \left\{ \begin{array}{ll}
\sqrt{\frac{2(1 - \psi)}{\sigma(2\sigma + 1)}} & 0 \leq \psi \leq \frac{1}{3}, \\
\frac{1 - \psi}{\sigma} & \frac{1}{3} \leq \psi < 1,
\end{array} \right.
\]

and

\[
|\rho_{3}| \leq \frac{(1 - \psi)(2\sigma - 3\psi + 3)}{3\sigma^2}. \]

**Remark 6.** Corollary 5 gives above is the improvement of the estimates for coefficients on \( |\rho_{2}| \) and \( |\rho_{3}| \) investigated by Girgaonkar et al., [29].

**Corollary 6.** [29] Let \( f(z) \in \mathcal{M}_{\Sigma}^c(\psi), \) \( (\sigma \geq 1, 0 \leq \psi < 1, (z, v) \in \Omega) \) be given (1). Then

\[
|\rho_{2}| \leq \sqrt{\frac{2(1 - \psi)}{\sigma(2\sigma + 1)}}.
\]

and

\[
|\rho_{3}| \leq \frac{(1 - \psi)(2\sigma - 3\psi + 3)}{3\sigma^2}.
\]

Taking \( \sigma = 1 \) in Corollary 7, we get the following result:
Corollary 7. \[11\] Let \( f(z) \in \mathcal{M}_{\sigma}^{\psi} \), \((\sigma \geq 1, 0 \leq \psi < 1, (z, \nu) \in \Omega)\) be given (1). Then

\[ |\rho_2| \leq \sqrt{\frac{2(1-\psi)}{3}}, \]

and

\[ |\rho_3| \leq \frac{(1-\psi)(5-3\psi)}{3}. \]

3. Conclusion

In this present paper, two new subclasses indicated by \( \mathcal{M}_{\Sigma,m}^{\chi} \) and \( \mathcal{M}_{\Sigma,m}^{\psi} \) of function class of \( E_m \) was obtained and worked on. Also, the estimates coefficients for \( |p_{m+1}| \) and \( |p_{2m+1}| \) of functions in these classes are determined.

Conflicts of Interest: “The author declares no conflict of interest.”

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