QUANTUM E(2) GROUPS FOR COMPLEX DEFORMATION PARAMETERS

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Abstract. We construct a family of $q$-deformations of $E(2)$ group for nonzero complex parameters $|q| < 1$ as locally compact braided quantum groups over the circle group $T$ viewed as a quasitriangular quantum group with respect to the unitary $R$-matrix $R(m, n) := (q/\bar{q})^{mn}$ for all $m, n \in \mathbb{Z}$. For real $0 < |q| < 1$, the deformation coincides with Woronowicz’s $E_q(2)$ groups. As an application, we study the braided analogue of the contraction procedure between $SU_q(2)$ and $E_q(2)$ groups in the spirit of Woronowicz’s quantum analogue of the classic Inönü-Wigner group contraction. Consequently, we obtain the bosonisation of braided $E_q(2)$ groups by contracting $U_q(2)$ groups.

1. Introduction

Quantum $E(2)$ groups were constructed as $q$-deformations of the double cover of the group of motions of the Euclidean plane $E(2)$ for real deformation parameters $0 < q < 1$ [22, 23]. They were used as a prototype for constructing a large class of examples of locally compact quantum groups [28, 29]. Here we construct a family of $q$-deformations of the same group for complex deformation parameters $0 < |q| < 1$. Consider the pair of operators $(v, n)$ satisfying the following conditions

\[ v \text{ is unitary}, \quad n \text{ is normal}, \quad vnv^* = qn, \quad \text{Sp}(n) = \mathbb{C}|q|, \] \[ \text{(1.1)} \]

where $\mathbb{C}|q| := \{ \lambda \in \mathbb{C} : |\lambda| \in |q|\mathbb{Z} \} \cup \{0\}$. Let $q = |q|\Phi_q$ and $n = |n|\Phi_n$ be the polar decompositions of $q$ and $n$, respectively. The explicit meaning of the commutation relation $v nv^* = qn$ is $v\Phi_n v^* = \Phi_q \Phi_n$ and $v|n|v^* = |q||n|$.

The spectral condition of $n$ implies that $C_0(\mathbb{C}|q|)$ is generated by the unbounded normal operator $n$ in the sense of [26, Definition 3.1]. The commutation relation $v nv^* = qn$ induces the following action $\alpha : \mathbb{Z} \to \text{Aut}(C_0(\mathbb{C}|q|))$:

\[ \alpha_k(f)(\lambda) = f(q^k\lambda), \quad \text{for all} \ k \in \mathbb{Z}, \ f \in C_0(\mathbb{C}|q|) \text{ and } \lambda \in \mathbb{C}|q|. \] \[ \text{(1.2)} \]

Let $B = C_0(\mathbb{C}|q|) \rtimes_{\alpha} \mathbb{Z}$ be the associated crossed product $C^*$-algebra. By virtue of [23, Theorem 1.6] and the universal property of the $C^*$-algebra crossed products, the pair of operators $(v, n)$ in (1.1) generate the $C^*$-algebra $B$. In particular, $v, n \notin B$ but these are affiliated with $B$, see [23, Definition 1.1].

For real $0 < q < 1$, one has a well-defined nondegenerate $^*$-homomorphism $\Delta_B : B \to \mathcal{M}(B \otimes B)$ satisfying

\[ \Delta_B(v) = v \otimes v, \quad \Delta_B(n) = v \otimes n + n \otimes v^*. \] \[ \text{(1.3)} \]
Here $\otimes$, $\mathcal{M}(B)$ and $+$ denote the minimal tensor product of $C^*$-algebras, multiplier algebra of $B$ and the closure of the sum of the operators, respectively. Also, $\Delta_B(n)$ is affiliated with $B \otimes B$. The pair $(B, \Delta_B)$ is Woronowicz’s $E_q(2)$ group \[22\][24].

However, the passage from the real to the complex deformation parameter $q$ reveals two interesting features. Firstly, the $C^*$-algebra $B$ for the deformation parameter $|q|$ is isomorphic to the one for the complex parameter $q$, see Proposition \[5.4\]. Secondly, $\Delta_B(n)$ and $\Delta_B(n^*)$ do not commute (even formally). In other words, $(B, \Delta_B)$ fails to be a $C^*$-quantum group unless $q = \overline{q}$.

We address this issue by replacing the ordinary tensor product $\otimes$ by the twisted tensor product $\otimes_R$ from \[14\] associated to the bicharacter $R : \mathbb{Z} \times \mathbb{Z} \to T$ defined by

$$R(m, n) := \zeta^{mn}, \quad \text{for all } m, n \in \mathbb{Z} \text{ with } \zeta := \frac{q}{\overline{q}} = (\Phi_q)^2. \quad (1.4)$$

For that matter, we view the circle group $T$ as a quasitriangular quantum group with respect to $R$ as explained in Example \[2.11\].

Now consider the category $\mathcal{C}^*\mathfrak{alg}(T)$ whose objects are $C^*$-algebras equipped with an action of $T$ and morphisms are $T$-equivariant nondegenerate $^*$-homomorphisms. Clearly, $C(T)$ is an object of $\mathcal{C}^*\mathfrak{alg}(T)$ with respect to the translation action of $T$. Also, the universal property of $B$ ensures that $(\nu, \lambda) \to \nu$ and $(\eta, \lambda) \to \eta \lambda$ for all $\lambda \in T$ extends uniquely to an action of $T$ on $B$ (see also \[5.3\]). The monoidal structure $\Box_B$ on $\mathcal{C}^*\mathfrak{alg}(T)$ is defined as follows: $C \Box_B D$ is a $C^*$-algebra for all objects $(C, \delta_C)$, $(D, \delta_D)$ of $\mathcal{C}^*\mathfrak{alg}(T)$ with injective nondegenerate $^*$-homomorphisms $j_1 : C \to \mathcal{M}(C \Box_B D)$ and $j_2 : D \to \mathcal{M}(C \Box_B D)$ such that the $^*$-algebra generated by $\{j_1(c)j_2(d) \mid c \in C, d \in D\}$ is dense in $C \Box_B D$. On the homogeneous elements $c_k \in C$ and $d_l \in D$ of degree $k$ and $l$, that is $\delta_C(c_k) = c_k \otimes z^k$ and $\delta_D(d_l) = d_l \otimes z^l$, the canonical embeddings $j_1, j_2$ commute up to $\zeta^{kl}$:

$$j_1(c_k)j_2(d_l) = \zeta^{kl}j_2(d_l)j_1(c_k). \quad (1.5)$$

In particular, we consider the twofold twisted tensor product $B \Box_B B$ and define an analogue of \[1.3\] in $\mathcal{C}^*\mathfrak{alg}(T)$ by

$$\Delta_B(v) := j_1(v)j_2(v), \quad \Delta_B(n) := j_1(v)j_2(n) + j_1(n)j_2(v^*). \quad (1.5)$$

The primary goal of this article is to prove that $E_q(2) = (B, \Delta_B)$ is a braided $C^*$-quantum group over the quasitriangular quantum group $T$ with respect to $R$. This is essentially contained in Theorem \[5.6\]. Subsequently, we construct an ordinary $C^*$-quantum group $\mathbb{H} = (C, \Delta_C)$ with an idempotent quantum group homomorphism with image $T$. In the Hopf-algebraic context this process is known as bosonisation which was discovered by Radford \[17\] and extensively studied by Majid \[23\][24]. Therefore, $\mathbb{H} = (C, \Delta_C)$ is the analytic counterpart of the bosonisation of $E_q(2)$, which is a new example of (non-compact) $C^*$-quantum group.

The double cover of $E(2)$ may be constructed by contracting $SU(2)$. This is one of the well studied examples of the classic Inönü-Wigner group contraction \[7\]. In the purely algebraic setting, the quantum analogue of the contraction procedure was investigated by Celeghini, Giachetti, Sorace and Tarlini \[3\][5] and applied to construct new examples of quantum groups. Their work motivated Woronowicz \[25\] to study the contraction procedure for $C^*$-quantum groups. In this article, it was shown that for real $0 < q < 1$, the contraction of the compact quantum groups $SU_q(2)$ respect to the (closed quantum) subgroup $T$ coincides with $E_q(2)$ groups.

Deformations of $SU(2)$ group for complex $q$ satisfying $0 < |q| < 1$ was considered in \[8\]. They are braided compact quantum groups over the quasitriangular quantum group $T$ with respect to $R$ and $U_q(2)$ groups are the associated bosonisation. Now $T$ is a closed quantum subgroup of braided $SU_q(2)$ groups \[20\] Proposition 4.1] in the sense of Woronowicz \[6\] Definition 3.2]. Similarly, we observe that $T$ is also a closed quantum subgroup of braided $E_q(2)$ in Remark \[5.10\]. Then we prove a braided
analogue of the contraction procedure between SU_q(2) and E_q(2) in Theorem 5.6. 
More precisely, the T-equivariant contraction of braided SU_q(2) groups with respect 
to T are isomorphic to braided E_q(2) groups. Consequently, we show that H can 
be constructed by contracting U_q(2) with respect to the same subgroup T.

Let us briefly outline the structure of this article. We have gathered all the 
necessary preliminaries in Section 2. In general, the category of Yetter-Drinfeld 
representations of a C^*-quantum group G, denoted by YDRep(G), is a braided 
monoidal category, and additionally the braiding isomorphisms are unitary operators. 
In short, we call such a category as a unitarily braided monoidal category. 
Motivated by [27], a general theory of braided C^*-quantum groups [19] is developed 
using manageable braided multiplicative unitaries [15, 18] in YDRep(G), where G 
is a regular C^*-quantum group.

Suppose G = (A, ∆_A) is a regular quasitriangular C^*-quantum group with 
respect to a unitary R-matrix R ∈ U(Ā ⊗ Ā), where Ĥ = (Ĥ, ∆_Ĥ) is the dual of G. 
Then the representation category Rep(Ĥ) of Ĥ is a unitarily braided monoidal category 
[14 Section 3]. Therefore, it is natural to work with manageable braided 
multiplicative unitaries in Rep(Ĥ) instead of YDRep(Ĥ) whenever Ĥ is a quasitriangular 
quantum group and construct braided C^*-quantum groups out of them. 
This is done in Section 3 using the results presented in the Appendix A dealing 
with the Yetter-Drinfeld representations of Ĥ. In particular, the construction of 
brained C^*-quantum groups over Ĥ is presented in Theorem 5.6.

From Section 2 onwards we fix Ĥ to be the circle group T (viewed as a quantum 
and the R-matrix R defined in (1.4). Let L be a Hilbert space equipped 
with an orthonormal basis {e_{ij}}_{i,j∈Z}. A concrete realisation of (1.1) on L is given by
\[ ve_{i,j} := e_{i-1,j}, \quad ne_{i,j} := q^t e_{i,j+1}. \] (1.6)
In fact, any Hilbert space realisation of (1.4) is either one dimensional or infinite 
dimensional, and the direct integral of all infinite dimensional irreducible representations 
of (1.1) is unitarily equivalent to (1.6), see [23 Section 3 (B)]. Therefore, 
(1.6) defines a faithful nondegenerate \( t \)-representation of B on L.

We identify L ∼= ℓ^2(Z) ⊗ ℓ^2(Z) and use the canonical representation of T on 
the second tensor factor. This makes L an object of Rep(T) and R induces the unitary 
braiding X ∈ U(L ⊗ L). On standard basis elements \( \{e_{i,j} \otimes e_{k,l}\}_{i,j,k,l \in \mathbb{Z}} \) of L ⊗ L 
the action of X is given by \( e_{i,j} \otimes e_{k,l} \rightarrow \zeta^{-jl} e_{k,l} \otimes e_{i,j} \). Starting with these data, 
we construct a manageable braided multiplicative unitary \( F ∈ U(L ⊗ L) \) in the 
category Rep(Ĥ), see Theorem 4.4.

The main results of this article are presented in Section 5. We apply Theorem 5.6 
to \( \mathbb{H} \) and construct E_q(2) = (B, ∆_Ĥ) as a braided group over T from it in Theorem 6.6. 
The associated bosonisation \( \mathbb{H} = (C, ∆_C) \) is described in Theorem 6.19. 
In the final section, we present the contraction procedure between \( \mathbb{H} \) and U_q(2) 
group.

2. Preliminaries

All Hilbert spaces and C^*-algebras (which are not explicitly multiplier algebras) 
are assumed to be separable. For a C^*-algebra A, let \( \mathcal{M}(A) \) be its multiplier algebra 
and let \( \mathcal{U}(A) \) be the group of unitary multipliers of A. The unit of \( \mathcal{M}(A) \) is denoted 
by \( 1_A \). For two norm closed subsets X and Y of a C^*-algebra A and \( T ∈ \mathcal{M}(A) \), set
\[ XTY := \{ xTy : x ∈ X, y ∈ Y \} \] –closed linear span.

Let \( \mathcal{C}^∗\text{alg} \) be the category of C^*-algebras with nondegenerate \( \text{t} \)-homomorphisms 
\( \varphi : A → \mathcal{M}(B) \) as morphisms \( A → B \); let \( \text{Mor}(A, B) \) denote the set of morphisms.
We use the same symbol for an element of \( \text{Mor}(A, B) \) and its unique unital extension from \( \mathcal{M}(A) \) to \( \mathcal{M}(B) \).

Let \( \mathcal{H} \) be a Hilbert space. A representation of a C*-algebra \( A \) is a nondegenerate *-homomorphism \( \pi: A \to \mathcal{B}(\mathcal{H}) \). Since \( \mathcal{B}(\mathcal{H}) = \mathcal{M}(\mathbb{C}(\mathcal{H})) \) and the nondegeneracy conditions \( \pi(A) \mathbb{K}(\mathcal{H}) = \mathbb{K}(\mathcal{H}) \) and \( \pi(A)\mathcal{H} = \mathcal{H} \) are equivalent; hence \( \pi \in \text{Mor}(A, \mathbb{K}(\mathcal{H})) \). The identity operator on \( \mathcal{H} \) is denoted by \( \text{id}_{\mathcal{H}} \). Whereas the the unit element of \( \mathcal{M}(\mathbb{K}(\mathcal{H})) = \mathcal{B}(\mathcal{H}) \) is denoted by \( 1_{\mathcal{H}} \).

We write \( \Sigma \) for the tensor flip \( \mathcal{H} \otimes K \to K \otimes \mathcal{H} \), \( x \otimes y \mapsto y \otimes x \), for two Hilbert spaces \( \mathcal{H} \) and \( K \). We write \( \sigma \) for the tensor flip isomorphism \( A \otimes B \to B \otimes A \) for two C*-algebras \( A \) and \( B \).

### 2.1. C*-quantum groups, actions and representations

Let \( \mathcal{H} \) be a Hilbert space. An element \( \mathcal{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}) \) is called a multiplicative unitary [2 Definition 1.1] if it satisfies the pentagon equation

\[
\mathcal{W}_{23}\mathcal{W}_{12} = \mathcal{W}_{12}\mathcal{W}_{13}\mathcal{W}_{23} \quad \text{in} \ \mathcal{U}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}).
\]  

Furthermore, \( \mathcal{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}) \) is said to be manageable if there is a strictly positive operator \( Q \) on \( \mathcal{H} \) and a unitary \( \mathcal{W} \in \mathcal{U}(\overline{\mathcal{H}} \otimes \mathcal{H}) \) satisfying

\[
\mathcal{W}(Q \otimes Q)\mathcal{W}^* = Q \otimes Q, \quad \langle x \otimes u \ | \ \mathcal{W} \ | \ z \otimes y \rangle = \langle \overline{\mathcal{W}} \ | \ \mathcal{W}^* \otimes Q^{-1}y \ | \ \overline{\mathcal{W}} \rangle
\]

for all \( x, z \in \mathcal{H} \), \( u \in D(Q) \) and \( y \in D(Q^{-1}) \) (see [27 Definition 1.2]). Here \( \overline{\mathcal{H}} \) is the conjugate Hilbert space, and an operator is strictly positive if it is positive and self-adjoint with trivial kernel. The condition \( \mathcal{W}(Q \otimes Q)\mathcal{W}^* = Q \otimes Q \) means that the unitary \( \mathcal{W} \) commutes with the unbounded operator \( Q \otimes Q \).

A C*-quantum group \( \mathcal{G} = (A, \Delta_A) \) consists of a C*-algebra \( A \) and an element \( \Delta_A \in \text{Mor}(A, A \otimes A) \) constructed from a manageable multiplicative unitary \( \mathcal{W} \) [27 Theorem 1.5]. Let \( \tilde{\mathcal{G}} = (\tilde{A}, \tilde{\Delta}_A) \) be the dual C*-quantum group and let \( \mathcal{W} \in \mathcal{U}(\tilde{A} \otimes A) \) be the reduced bicharacter. In particular,

\[
A := \{ (\omega \otimes \text{id}_A)W \ | \ \omega \in \tilde{A}' \}^- \text{closed linear span} \subset \mathcal{B}(\mathcal{H}),
\]

\[
\tilde{A} := \{ (\text{id}_A \otimes \omega)W \ | \ \omega \in A' \}^- \text{closed linear span} \subset \mathcal{B}(\mathcal{H}).
\]

Moreover, \( \Delta_A \) and \( \tilde{\Delta}_A \) are characterised by

\[
(\text{id}_{\tilde{A}} \otimes \Delta_A)W = W_{12}W_{13} \quad \text{in} \ \mathcal{U}(\tilde{A} \otimes A \otimes A),
\]

\[
(\tilde{\Delta}_A \otimes \text{id}_A)W = W_{23}W_{13} \quad \text{in} \ \mathcal{U}(\tilde{A} \otimes \tilde{A} \otimes A).
\]

We reserve the phrase “quantum groups” for C*-quantum groups. An important concept in the theory of quantum groups is the concept of regularity introduced by Baaj and Skandalis in [2]. A quantum group \( \mathcal{G} = (A, \Delta_A) \) is said to be regular if \( (\tilde{A} \otimes 1_A)W(1_{\tilde{A}} \otimes A) = \tilde{A} \otimes A \). Dual of a regular quantum group is again regular. A (right) action of \( \mathcal{G} = (A, \Delta_A) \) on a C*-algebra \( C \) is an injective morphism \( \delta: C \to C \otimes A \) with the following properties:

1. \( \delta \) is a comodule structure, that is,

\[
(\text{id}_C \otimes \Delta_A) \circ \delta = (\delta \otimes \text{id}_A) \circ \delta;
\]

2. \( \delta \) satisfies the Podleś condition:

\[
\delta(C)(1_C \otimes A) = C \otimes A.
\]

We call \((C, \delta)\) a \( \mathcal{G} \)-C*-algebra. We shall drop \( \delta \) from our notation whenever it is clear from the context. A morphism \( f: C \to D \) between two \( \mathcal{G} \)-C*-algebras \((C, \delta_C)\) and \((D, \delta_D)\) is \( \mathcal{G} \)-equivariant if \( \delta_D \circ f = (f \otimes \text{id}_A) \circ \delta_C \). Let \( \text{Mor}^\mathcal{G}(C, D) \) be the set of \( \mathcal{G} \)-equivariant morphisms from \( C \) to \( D \). Let \( \mathcal{G} \text{-alg}(\mathcal{G}) \) be the category with \( \mathcal{G} \)-C*-algebras as objects and \( \mathcal{G} \)-equivariant morphisms as arrows.
A (right) representation of \( G \) on a \( C^* \)-algebra \( D \) is an element \( U \in \mathcal{U}(D \otimes A) \) with
\[
(id_D \otimes \Delta_A)U = U_{12}U_{13} \quad \text{in } \mathcal{U}(D \otimes A \otimes A).
\]
In particular, if \( D = \mathbb{K}(\mathcal{L}) \) for some Hilbert space \( \mathcal{L} \), then \( U \) is said to be a (right) representation of \( G \) on \( \mathcal{L} \).

The tensor product of representations \( U^i \in \mathcal{U}(\mathbb{K}(\mathcal{L}_i) \otimes A) \) for \( i = 1, 2 \) is defined by \( U_{13}^1 U_{23}^2 \in \mathcal{U}(\mathbb{K}(\mathcal{L}_1 \otimes \mathcal{L}_2) \otimes A) \). It is denoted by \( U^1 \otimes U^2 \). An element \( t \in \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2) \) is called an intertwiner if \( (t \otimes 1_A)U^1 = U^2(t \otimes 1_A) \). The set of all intertwiners between \( U^1 \) and \( U^2 \) is denoted by \( \text{Hom}^G(U^1, U^2) \). A routine computation shows that \( \tilde{\mathfrak{p}} \) is associative; and the trivial 1-dimensional representation is a tensor unit.

This gives representations a structure of \( W^* \)-category, which we denote by \( \mathcal{R}\text{ep}(G) \); see [21, Section 3.1–2] for more details. Objects of \( \mathcal{R}\text{ep}(G) \) are pairs \( (\mathcal{L}, U) \) consisting of a Hilbert space \( \mathcal{L} \) and a representation \( U \) of \( G \) on \( \mathcal{L} \).

A covariant representation of \((C, \delta, G)\) on a Hilbert space \( \mathcal{L} \) is a pair \((U, \varphi)\) consisting of a representation \( U \in \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes A) \) of \( G \) and a representation \( \varphi : G \to \mathcal{B}(\mathcal{L}) \) of \( C \) that satisfy the covariance condition
\[
(\varphi \otimes id_A)(\delta(c)) = U(\varphi(c) \otimes 1_A)U^* \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes A)
\]
for all \( c \in C \). A covariant representation is called faithful if \( \varphi \) is faithful. Faithful covariant representations always exists, see for instance [13, Example 4.4].

Example 2.9. Let \( G \) be a locally compact group. Let \( \mathcal{H} := L^2(G) \) with respect to the right Haar measure on \( G \). A routine computation shows that the operator \( (\mathbb{W}\xi)(g_1, g_2) := \xi(g_1g_2, g_2) \) for all \( \xi \in L^2(G \times G) \) is a manageable multiplicative unitary with \( Q = id_\mathcal{H} \) and \( \mathbb{W} = \mathbb{W}^* \) and generates the quantum group \( \hat{G} := (C_0(G), \Delta_C(G)) \) where \( (\Delta_C(G))f(g_1, g_2) := f(g_1g_2) \) for all \( f \in C_0(G) \). In this way, \( G \) is viewed as a quantum group. Moreover, the category \( \mathcal{C}^*\text{alg}(G) \) is equivalent to the category \( \mathcal{C}^*\text{alg}(\hat{G}) \) with \( G \)-\( C^* \)-algebras as objects and \( G \)-equivariant morphisms as arrows. Similarly, \( \mathcal{R}\text{ep}(G) \) is also equivalent to the representation category \( \mathcal{R}\text{ep}(\hat{G}) \) of \( \hat{G} \). Let \( \mu \) be the right regular representation of \( G \) on \( \mathcal{H} \). The dual of \( G \) is \( \hat{G} = (C^*_r(G), \Delta_{C^*_r(G)}) \), where \( \Delta_{C^*_r(G)}(\mu_g) := \mu_g \otimes \mu_g \). Also, \( G \) and \( \hat{G} \) are regular quantum groups, and in addition, \( \hat{G} \) coincides with \( \hat{G} \) as quantum group whenever \( G \) is Abelian.

For simplicity, whenever a locally compact group \( G \) is viewed as a quantum group, we use the same notation \( G \) to denote corresponding quantum group \( \hat{G} = (C_0(G), \Delta_C(G)) \).

2.2. Quasitriangular quantum groups. Let \( G = (A, \Delta_A) \) be a quantum group and let \( \hat{G} = (\hat{A}, \Delta_A) \) be the dual. An element \( R \in \mathcal{U}(\hat{A} \otimes \hat{A}) \) is said to be an \( R \)-matrix on \( \hat{G} \) if it is a bicharacter:
\[
(id_{\hat{A}} \otimes \Delta_A)R = R_{12}R_{13}, \quad (\Delta_A \otimes id_{\hat{A}})R = R_{23}R_{13} \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{A} \otimes \hat{A});
\]
and satisfies the \( R \)-matrix condition:
\[
R(\sigma\Delta_A(\hat{a}))R^* = \Delta_A(\hat{a}) \quad \text{for all } \hat{a} \in \hat{A}.
\]
A quasitriangular quantum group is a quantum group \( \hat{G} = (A, \Delta_A) \) with an \( R \)-matrix \( R \in \mathcal{U}(\hat{A} \otimes \hat{A}) \), see [13, Definition 3.1].

Example 2.11. Every continuous bicharacter \( Z \times Z \to \mathbb{T} \) satisfies the \( R \)-matrix condition (2.10) because \( \hat{A} = C_0(Z) \) is a commutative \( C^* \)-algebra. Thus \( \mathbb{T} \) is quasitriangular with respect to any bicharacter on \( Z \times Z \). In particular, for \( q \in \mathbb{C} \setminus \{0\} \), the bicharacter \( \mathbb{R} : Z \times Z \to \mathbb{T} \) in [13] is an \( R \)-matrix on \( \mathbb{T} \).
Throughout this subsection we assume \( \mathcal{G} = (A, \Delta_A) \) is a quasitriangular quantum group with an \( R \)-matrix \( R \in \mathcal{U}(\hat{A} \otimes \hat{A}) \). Then the categories \( \mathcal{R} \text{Rep}(\mathcal{G}) \) and \( \mathcal{C}^* \text{alg}(\mathcal{G}) \) are of particular interest. More precisely, [13] Proposition 3.2 & Theorem 4.3 show that \( \mathcal{R} \text{Rep}(\mathcal{G}) \) is a unitarily braided monoidal category and \( \mathcal{C}^* \text{alg}(\mathcal{G}) \) is a monoidal category, whenever \( \mathcal{G} \) is a quasitriangular quantum group. We recall the explicit construction of the unitary braiding on \( \mathcal{R} \text{Rep}(\mathcal{G}) \) and the monoidal product \( \boxtimes_R \) on \( \mathcal{C}^* \text{alg}(\mathcal{G}) \). The latter construction was motivated by [16].

Let \( (\alpha, \beta) \) be an \( R \)-Heisenberg pair acting on a Hilbert space \( \mathcal{L} \). More explicitly, \( \alpha, \beta \in \text{Mor}(A, \mathbb{K}(\mathcal{L})) \) and satisfy the commutation relation \( W_{1\alpha}W_{2\beta} = W_{2\beta}W_{1\alpha} R_{13} \in \mathcal{U}(\hat{A} \otimes \hat{A} \otimes \mathbb{K}(\mathcal{L})) \), where \( W_{1\alpha} := ((\text{id}_A \otimes \alpha)W_{13} \) and \( W_{1\beta} := ((\text{id}_A \otimes \beta)W_{23} \in \mathcal{U}(\hat{A} \otimes \hat{A} \otimes \mathbb{K}(\mathcal{L})) \), see [13] Definition 3.1. Existence of \( R \)-Heisenberg pairs is guaranteed by [13] Lemma 3.8.

Suppose \( (\mathcal{L}_i, U^i) \) are objects in \( \mathcal{R} \text{Rep}(\mathcal{G}) \) for \( i = 1, 2 \). The proof of [13] Theorem 4.1 shows that there is a unique solution \( Z \in \mathcal{U}(\mathcal{L}_1 \otimes \mathcal{L}_2) \), independent of the choice of the \( R \)-Heisenberg pair \( (\alpha, \beta) \), of the following equation

\[
U_{1\alpha}^1 U_{2\alpha}^2 Z_{12} = U_{2\beta}^2 U_{1\beta}^1 \quad \text{in} \quad \mathcal{U}(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}).
\]

The braiding unitary is defined by \( \mathcal{L}_1 \otimes \mathcal{L}_2 := Z \circ \Sigma \in \text{Hom}^\mathbb{G}(U^2 \boxtimes U^1, U^1 \boxtimes U^2) \), see [13] Equation 3.2.

Suppose \( (C_i, \delta_i) \) are objects in \( \mathcal{C}^* \text{alg}(\mathcal{G}) \) and \( (U^i, \varphi_i) \) are faithful covariant representations of \( (C_i, \delta_i, \mathcal{G}) \) on \( \mathcal{L}_i \) for \( i = 1, 2 \). Clearly, \( (\mathcal{L}_i, U^i) \) are objects in \( \mathcal{R} \text{Rep}(\mathcal{G}) \) for \( i = 1, 2 \) and let \( \mathcal{L}_1 \otimes \mathcal{L}_2 \) be the unitary braiding. Define \( j_i \in \text{Mor}(C_i, \mathbb{K}(\mathcal{L}_1 \otimes \mathcal{L}_2)) \) by

\[
j_1(c_1) := \varphi_1(c_1) \otimes 1_{\mathcal{L}_2}, \quad j_2(c_2) := \varphi_2(c_2) \otimes 1_{\mathcal{L}_1}, \quad (\mathcal{L}_1 \otimes \mathcal{L}_2) \in \text{Hom}^\mathbb{G}(U^2 \boxtimes U^1, U^1 \boxtimes U^2) \),
\]

for all \( c_i \in C_i \) and \( i = 1, 2 \). Then \( C_1 \boxtimes_R C_2 := \mathcal{O} \in \mathfrak{B}(\mathcal{L}_1 \otimes \mathcal{L}_2) \) is a \( \mathcal{C}^* \)-algebra [13] Theorem 4.6 and \( \delta_1 \otimes \delta_2(j_1(c_1)j_2(c_2)) := (j_1 \otimes \text{id}_A)\delta_1(c_1)(j_2 \otimes \text{id}_A)\delta_2(c_2) \) defines an action of \( \mathcal{G} \) on \( C_1 \boxtimes_R C_2 \) [13] Proposition 4.1. Let \( (C_i', \delta_i') \) be objects in \( \mathcal{C}^* \text{alg}(\mathcal{G}) \) and \( f_i \in \text{Mor}^\mathbb{G}(C_i, C_i') \) for \( i = 1, 2 \). The monoidal product \( f_1 \boxtimes_R f_2 \in \text{Mor}(C_1 \boxtimes_R C_2, C_1' \boxtimes_R C_2') \) in \( \mathcal{C}^* \text{alg}(\mathcal{G}) \) is defined by

\[
(f_1 \boxtimes_R f_2) \circ j_1 = j_1' \circ f_1, \quad (f_1 \boxtimes_R f_2) \circ j_2 = j_2' \circ f_2,
\]

where \( j_i' \in \text{Mor}(C_i', C_1' \boxtimes_R C_2') \) are the canonical morphisms for \( i = 1, 2 \).

3. Braided \( \mathcal{C}^* \)-Quantum Groups over Quasitriangular Quantum Groups

Let \( \mathcal{G} = (A, \Delta_A) \) be a quasitriangular quantum group with an \( R \)-matrix \( R \in \mathcal{U}(\hat{A} \otimes \hat{A}) \).

**Definition 3.1.** Let \( (\mathcal{L}, U) \) be an object of \( \mathcal{R} \text{Rep}(\mathcal{G}) \). Let \( \mathcal{L} \otimes \mathcal{L} := Z \circ \Sigma \in \text{Hom}^\mathbb{G}(U \boxtimes U, U \boxtimes U) \) be the braiding unitary. A unitary \( F \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L}) \) is said to be a braided multiplicative unitary over \( \mathcal{G} \) relative to \( (U, R) \) if

\[
(1) \quad F \in \text{Hom}^\mathbb{G}(U \boxtimes U, U \boxtimes U);
\]

\[
F_{12}U_{13}U_{23} = U_{13}U_{23}F_{12} \quad \text{in} \quad \mathcal{U}(\mathbb{K}(\mathcal{L} \otimes \mathcal{L}) \otimes A);
\]

(2) \( F \) satisfies the braided pentagon equation:

\[
F_{23}F_{12} = F_{12}(v^\mathcal{L})_{12}F_{12}(v^\mathcal{L})_{23}F_{23} \quad \text{in} \quad \mathcal{U}(\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}).
\]

Recall the bounded co-inverse \( R_A \) of \( \mathcal{G} \), which is an involutive normal antiautomorphism of \( A \), as in [27] Theorem 1.5 (4)]. Then \( U^0 := U^T \otimes R_A \in \mathcal{U}(\mathbb{K}(\mathcal{T} \otimes \mathbb{H}(\mathcal{T})) \otimes A) \) is the contragradient representation of \( U^0 \), where \( T \) is the transposition defined by \( T' := T' \) for all \( T \in \mathbb{B}(\mathcal{T}) \) (see [21] Definition 11]). There is a unique element \( \hat{Z} \in \mathcal{U}(\mathcal{T} \otimes \mathcal{L}) \) satisfying [21] for the representations \( (U^0, U) \).
Suppose \( G \) is constructed from a manageable multiplicative unitary \( \mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}) \) and \( Q \) be the strictly positive operator acting on \( \mathcal{H} \) appearing in (2.2). Let \( \pi \in \text{Mor}(A, \mathbb{K}(\mathcal{H})) \) be the embedding in (A.3) and let \( U := (\text{id}_{\mathbb{K}(\mathcal{L})} \otimes \pi)U \in \mathcal{U}(\mathbb{L} \otimes \mathcal{H}) \).

**Definition 3.4.** A braided multiplicative unitary \( F \in \mathcal{U}(\mathbb{L} \otimes \mathbb{L}) \) over \( G \) relative to \( (U, R) \) is said to be manageable if there is a strictly positive operator \( Q_{\mathcal{L}} \) on \( \mathcal{L} \) and a unitary \( \tilde{F} \in \mathcal{U}(\mathbb{L} \otimes \mathbb{L}) \) such that
\[
\tilde{F}(Q_{\mathcal{L}} \otimes Q_{\mathcal{L}})F^* = Q_{\mathcal{L}} \otimes Q_{\mathcal{L}}, \quad U(Q_{\mathcal{L}} \otimes Q)^* = Q_{\mathcal{L}} \otimes Q,
\]
and
\[
\langle x \otimes u \mid Z^*F \mid y \otimes v \rangle = (\mathfrak{T} \otimes Q_{\mathcal{L}})u \mid \mathfrak{T}Z^* \mid \mathfrak{T} \otimes Q_{\mathcal{L}}^{-1}v \tag{3.5}
\]
for all \( x, y \in \mathcal{L}, u \in \mathcal{D}(Q_{\mathcal{L}}), v \in \mathcal{D}(Q_{\mathcal{L}}^{-1}) \).

The main result of this section is the construction of braided C\(^*\)-quantum groups from manageable braided multiplicative unitaries in \( \mathcal{R}\text{ep}(\mathbb{G}) \).

**Theorem 3.6.** Let \( F \in \mathcal{U}(\mathbb{L} \otimes \mathbb{L}) \) be a manageable braided multiplicative unitary over a regular quantum group \( G = (A, \mathbb{K}_{\mathcal{A}}) \) relative to \( (U, R) \). Let
\[
B := \{(\omega \otimes \text{id}_{\mathbb{K}(\mathcal{L})})F \mid \omega \in \mathbb{B}(\mathcal{L})^+\} \text{ - closed linear span} \subset \mathbb{B}(\mathcal{L}). \tag{3.7}
\]
Then
1. \( B \) is a nondegenerate, separable C\(^*\) subalgebra of \( \mathbb{B}(\mathcal{L}) \);
2. Define \( \beta(b) := U(b \otimes 1_{\mathcal{L}})U^* \) for all \( b \in B \). Then \( \beta \in \text{Mor}(B, B \otimes A) \) and \( (B, \beta) \) is an object of \( \mathcal{C}^{alg}(G) \);
3. \( F \in \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes B) \);
4. There exists a unique \( \Delta_B \in \mathcal{M}(B, B \otimes B) \) characterised by
\[
(i\text{id}_{\mathbb{K}(\mathcal{L})} \otimes \Delta_B)F = \{(i\text{id}_{\mathbb{K}(\mathcal{L})} \otimes j_1)F \} \{(i\text{id}_{\mathbb{K}(\mathcal{L})} \otimes j_2)F\}. \tag{3.8}
\]
Moreover, \( \Delta_B \) is coassociative: \( (\Delta_B \otimes id_B) \circ \Delta_B = (id_B \otimes \Delta_B) \circ \Delta_B \) and satisfies the cancellation conditions: \( \Delta_B(j_1(B)) = B \otimes \Delta_B(B) \).

The pair \((B, \Delta_B)\) is said to be the braided C\(^*\)-quantum group (over \( \mathbb{G} \)) generated by the braided multiplicative unitary \( F \).

**Proof.** Let \( (\mathcal{L}, V) \) be the object in \( \mathcal{R}\text{ep}(\mathbb{G}) \) induced by the \( R \)-matrix \( R \) from \((\mathcal{L}, U)\) in (A.3). The functor \( F : \mathcal{R}\text{ep}(\mathbb{G}) \to \mathcal{YD}\text{Rep}(\mathbb{G}) \) in the Proposition A.2 does not change the underlying Hilbert spaces and the morphisms. Since \( F \in \text{Hom}^G_{\mathbb{G}}(U, \mathbb{U}, U, \mathbb{U}) \) in \( \mathcal{R}\text{ep}(\mathbb{G}) \) it is also an endomorphism of the object \((\mathcal{L}, V)\) in \( \mathcal{YD}\text{Rep}(\mathbb{G}) \); hence \( F \in \text{Hom}^G_{\mathbb{G}}(V, \mathbb{V}, V, \mathbb{V}) \) in \( \mathcal{R}\text{ep}(\mathbb{G}) \). This implies that \((3.3)\) remains same in \( \mathcal{YD}\text{Rep}(\mathbb{G}) \). Hence, \( F \) is also a braided multiplicative unitary in the sense of \( [15] \) Definition 3.2. Let \( \pi \in \text{Mor}(A, \mathbb{K}(\mathcal{H})) \) and \( \hat{\pi} \in \text{Mor}(\hat{A}, \mathbb{K}(\mathcal{H})) \) be the canonical embeddings in (2.3). Define \( R' := (\hat{\pi} \otimes \hat{\pi})R \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}) \) and \( \mathcal{V} := (\text{id}_{\mathbb{K}(\mathcal{L})} \otimes \hat{\pi})V \in \mathcal{U}(\mathcal{L} \otimes \mathcal{H}) \). Using \([12]\) Equation 33 we obtain a concrete realisation of (A.3) as follows
\[
R_{23}U_{12}R'_{23} = U_{12}V_{13} \text{ in } \mathcal{U}(\mathcal{L} \otimes \mathcal{H} \otimes \mathcal{H}).
\]
Manageability of \( F \) implies \( U \) commutes with \( Q_{\mathcal{L}} \otimes Q \). Since \( R' \) is a bicharacter it commutes with \( Q \otimes Q \) by \([12]\) Proposition 3.10. Therefore, \( V_{13} \) commutes with \( Q_{\mathcal{L}} \otimes Q \otimes Q \) which implies that \( \mathcal{V} \) commutes with \( Q_{\mathcal{L}} \otimes Q \). Thus \( F \) is also manageable in the sense of \([15]\) Definition 3.5. Hence \([19]\) Theorem 5.1] completes the proof. In fact, the conclusions (1) \(-\) (3) and, assuming the existence of \( \Delta_B \) characterised by \((3.3)\), the cancellation conditions in (4) follow immediately. Define \( \Delta_B(b) := F(b \otimes 1_{\mathcal{L}})F^* \) for all \( b \in B \). Observe that \((B \mapsto \mathbb{B}(\mathcal{L}), U) \) is a faithful covariant representation of \((B, \beta, \mathbb{G})\) on \( \mathcal{L} \). Hence, we use the braiding operator \( \times \) in
Definition 4.1 to define the canonical embeddings $j_{1,2} \in \text{Mor}^G(B, B \mathbb{R})$ in (2.13).

Then braided pentagon equation (4.3.3) and (1) ensures that $\Delta_B : B \to \mathcal{M}(B \mathbb{R})$ is a $G$-equivariant $^\ast$-homomorphism, satisfies (4.3.4), and is coassociative. Finally, $\Delta_B(B) (B \mathbb{R}) = \Delta_B(B_{j_{1}(B)}j_{2}(B)) = (B \mathbb{R}B)j_{2}(B) = B \mathbb{R}B$ shows that $\Delta_B$ is nondegenerate. □

4. BRAIDED MULTIPlicative UNITARY OF $E_q(2)$ GROUPS

Let $\{e_p\}_{p \in \mathbb{Z}}$ be an orthonormal basis of $H = \ell^2(\mathbb{Z})$. The operator $\tilde{N}e_p := pe_p$ is an unbounded densely defined self adjoint operator with $\text{Sp}(\tilde{N}) = \mathbb{Z}$ and generates the $C^*$-algebra $C_0(\mathbb{Z})$. The operator $ze_p := e_{p+1}$ is unitary and generates the $C^*$-algebra $C(\mathbb{T})$. A simple computation shows that $z$ and $\tilde{N}$ satisfy the following commutation relation

$$z^* \tilde{N} z = \tilde{N} + 1_H. \quad (4.1)$$

Define $W \in \mathcal{U}(H \otimes H)$ by

$$W := (1_H \otimes z)^{\otimes 1_H} = \int_{\mathbb{T}} x^* dE_{\tilde{N}}(s) \otimes dE_z(x),$$

where $dE_{\tilde{N}}$ and $dE_z$ are the spectral measures of $\tilde{N}$ and $z$, respectively. The commutation relation (4.1) gives the action of $W$ on the orthonormal basis $\{e_k \otimes e_l\}_{k,l \in \mathbb{Z}}$ by $e_k \otimes e_l \mapsto e_k \otimes e_{l+k}$. It is easy to verify that $W$ is a manageable multiplicative unitary with $Q = \text{id}_H$ and $\overline{W}(e_k \otimes e_l) := e_k \otimes e_{l+k}$. $W$ generates $\mathbb{T}$:

$$C(\mathbb{T}) = \{(\omega \otimes \text{id}_H)W | \omega \in B(H)_+\}^{\ast}\text{-closed span},$$

$$\Delta_C(\mathbb{T})(z) = W(z \otimes 1_H)W^* = z \otimes z.$$

The dual of $\mathbb{T}$ is $Z$ with the comultiplication map $\Delta_{C_0(\mathbb{Z})}(\tilde{N}) = \tilde{N} \otimes 1_{C_0(\mathbb{Z})} + 1_{C_0(\mathbb{Z})} \otimes \tilde{N}$.

Recall the $R$-matrix $R : \mathbb{Z} \times \mathbb{Z} \to \mathbb{T}$ defined in (1.4). Then $\mathbb{T}$ is a quasitriangular with respect to $R$ and $\mathcal{R}ep(\mathbb{T})$ a unitarily braided monoidal category.

Let $L := H \otimes H$ and fix an orthonormal basis $\{e_{i,j} := e_i \otimes e_j\}_{i,j \in \mathbb{Z}}$ of $L$. Define $U := 1_H \otimes W \in \mathcal{U}(H^{\otimes 2}) \cong \mathcal{U}(L \otimes H)$. Clearly, $U(e_{i,j} \otimes e_p) = e_{i,j} \otimes e_{p+j}$ and denote $U$ by $W$ while viewed as an element of $\mathcal{U}(L \otimes C(\mathbb{T}))$. Also, the first equation in (2.3) shows that $U$ is a representation of $\mathbb{T}$ on $L$. Hence $(L, U)$ is an object of $\mathcal{R}ep(\mathbb{T})$.

Define $\alpha, \beta \in \text{Mor}(C(\mathbb{T}), \mathcal{R}ep(\mathbb{T}))$ by $\alpha(z) := z$ and $\beta(z) := \tilde{V}$, where $\tilde{V}$ is the unitary operator defined by $\tilde{V}e_p = z^{-p}e_p$, respectively. Then $(\alpha, \beta)$ is an R-Heisenberg pair on $H$ because $z$ and $\tilde{V}$ commute up to $\zeta$:

$$z\tilde{V}e_p = \zeta^{-p}ze_p = \zeta^{-p}e_{p+1} = \zeta \tilde{V}e_{p+1} = \zeta \text{V}e_p.$$ 

In fact, $z$ and $\tilde{V}$ generate noncommutative two torus. Using these we compute the unitary $Z \in \mathcal{U}(L \otimes L)$ in (2.12). Equivalently, $Z$ is uniquely defined by

$$Z_{12} = U_{23}^\ast U_{12}^\ast U_{12} \text{ in } \mathcal{U}(L \otimes L \otimes H), \quad (4.2)$$

where $U_{10} = U$ and $U_{11} = (1_H \otimes 1_H \otimes \tilde{V})^{(1_H \otimes N \otimes 1_H)}$. Clearly, on the basis elements we have $U_{1,2}(e_{k,l} \otimes e_p) = z^{-p} e_{k,l} \otimes e_p$. Therefore,

$$U_{2,2}^\ast U_{1,2}^\ast U_{2,1}^\ast(e_{i,j} \otimes e_{k,l} \otimes e_p) = \zeta^{-(p+j)}U_{2,2}^\ast U_{1,2}^\ast(e_{i,j} \otimes e_{k,l} \otimes e_{p+j})$$

$$= \zeta^{-p} e_{i,j} \otimes e_{k,l} \otimes e_p = z^{-j} e_{i,j} \otimes e_{k,l} \otimes e_p.$$ 

Hence the the unitary braiding operator $\times := Z \circ \Sigma \in \mathcal{U}(L \otimes L)$ is given by

$$Z(e_{i,j} \otimes e_{k,l}) = \zeta^{-(p)} e_{i,j} \otimes e_{k,l}, \quad \times e_{i,j} \otimes e_{k,l} = \zeta^{-j} e_{k,l} \otimes e_{i,j}. \quad (4.3)$$
Define a closed operator \( X = |X| \Phi_X \) on \( \mathcal{L} \otimes \mathcal{L} \) by
\[
|X|e_{i,j} \otimes e_{k,l} = |q|^{k-l+1} e_{i,j} \otimes e_{k,l},
\]
\[
\Phi_X e_{i,j} \otimes e_{k,l} = \zeta^{-j} \Phi_q^{k-1} e_{i-1,j-1} \otimes e_{k-1,l+1},
\]
\[
X e_{i,j} \otimes e_{k,l} = \zeta^{-j} \Phi_q^{k-1} e_{i-1,j-1} \otimes e_{k-1,l+1}.
\]
(4.4)

Then \( X \) is a normal operator because \(|X|\) commutes with its phase \( \Phi_X \) in the polar decomposition and \( \text{Sp}(X) = \mathbb{C}^{[q]} \).

Recall that the operator \( n \), defined by \( ne_{i,j} = q^i e_{i,j+1} \), is closed and injective. Hence it is invertible and on the standard basis elements we have \( n^{-1}e_{i,j} = q^{-i}e_{i,j-1} \).

Define \( P \in \mathcal{U}(\mathcal{L}) \) by
\[
P e_{i,j} = \zeta^j e_{i,j}.
\]
(4.5)

Then the following computation shows \( X = n^{-1}vP \otimes vn \):
\[
(n^{-1}vP \otimes vn)e_{i,j} \otimes e_{k,l} = \zeta^{-j}q^k(n^{-1} \otimes v)e_{i-1,j} \otimes e_{k+l+1} = \zeta^{-j}q^{k-1}e_{i-1,j-1} \otimes e_{k-1,l+1}.
\]
The quantum exponential function \( F_{[q]} : \mathbb{C}^{[q]} \to \mathbb{T} \) is defined in [24 Equation 1.2] by
\[
F_{[q]}(z) = \left\{ \prod_{k=0}^{\infty} \frac{1+|q|^{2k}z}{1+|q|^{2k}z} \right\} \text{if } z \in \mathbb{C}^{[q]} \setminus \{-|q|^{-2k} | k = 0, 1, \ldots \},
\]
otherwise
Since \( F_{[q]} \) is a unitary multiplier of \( C_0(\mathbb{C}^{[q]}) \), we observe \( F_{[q]}(X) \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L}) \).

**Theorem 4.6.** Let \( \mathcal{Y} = \mathcal{W}_{13} \mathcal{W}_{23} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \). Then the operator \( F \) := \( F_{[q]}(\mathcal{X}) \mathcal{Y} \) is a manageable braided multiplicative unitary on \( \mathcal{L} \otimes \mathcal{L} \) over \( \mathbb{T} \) relative to \( (\mathcal{L}, \mathcal{R}) \).

The rest of this section is devoted to the proof of this theorem.

**Lemma 4.7.** Define \( T(\lambda) := F_{[q]}(\lambda X) \) for all \( \lambda \in \mathbb{C}^{[q]} \). Then
\[
F_{[q]}(X)_{23} T(\lambda)_{12} = T(\lambda)_{12} F_{[q]}(\lambda n^{-1}vP \otimes v^2P \otimes vn) F_{[q]}(X)_{23} \text{ in } \mathcal{U}(\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}).
\]

**Proof.** Clearly, the equality holds for \( \lambda = 0 \). Therefore, we fix a nonzero element \( \lambda \in \mathbb{C}^{[q]} \). The operators \( R = \lambda n^{-1}vP \otimes vn \otimes 1_\mathcal{L} = \lambda X_{12} \) and \( S = \lambda n^{-1}vP \otimes v^2P \otimes vn = \lambda (1_\mathcal{L} \otimes v^2P \otimes 1_\mathcal{L}) X_{13} \) are normal, each pair of operators \((|R|, |S|)\) and (\( \Phi_R, \Phi_S \)) strongly commute and
\[
\Phi_R S|\Phi_R = |q|^{-1}|S|, \quad \Phi_S R|\Phi_S = |q||R|, \quad \text{Sp}(R), \text{Sp}(S) = \mathbb{T}^{[q]}.
\]
Also \( R^{-1} S = 1_\mathcal{L} \otimes n^{-1}vP \otimes vn = X_{23} \) is normal with spectrum \( \mathbb{T}^{[q]} \). Then [24 Theorem 2.1-2.2] show that \( R + S \) is normal with spectrum \( \mathbb{T}^{[q]} \) and
\[
F_{[q]}(R^{-1}S)RF_{[q]}(R^{-1}S)^* = R + S.
\]

Since the functional calculus is compatible with conjugation by unitaries, [24 Theorem 3.1] implies
\[
F_{[q]}(R^{-1}S)RF_{[q]}(R^{-1}S)^* = F_{[q]}(F_{[q]}(R^{-1}S)RF_{[q]}(R^{-1}S)^*) = F_{[q]}(R + S) = F_{[q]}(R)F_{[q]}(S).
\]

For \( \lambda \in \mathbb{C}^{[q]} \) define \( F^\lambda := F_{[q]}(\lambda X) \mathcal{Y} \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L}) \).

**Proposition 4.8.** The family of unitaries \( \{F^\lambda\}_{\lambda \in \mathbb{C}^{[q]}} \) commute with \( \mathcal{U} \) and satisfy a variant of the braided pentagon equation (3.3):
\[
F_{23}^{\lambda_1} F_{12}^{\lambda_2} = F_{12}^{\lambda_1} F_{23}^{\lambda_2} \quad \text{in } \mathcal{U}(\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}).
\]
(4.9)
Proof. The unitaries $\mathbb{W}_{25}\mathbb{W}_{15}$ and $\mathbb{W}_{13}\mathbb{W}_{23}$ commute in $U(\mathcal{H}^\otimes 5)$. Identifying $\mathcal{L} = \mathcal{H} \otimes \mathcal{H}$ and $\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{H} = \mathcal{H}^\otimes 5$ we get $U \otimes U$ and $\mathcal{Y}_{12}$ commutes in $U(\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{H})$. Moreover $U \otimes U \mathcal{Y}_{12} \otimes \mathcal{E}_{k,l} \otimes e_{p+j}$ shows the operators $\mathcal{X}_{12}$ and $\mathcal{Y}_{12}$ commute with $U \otimes U$ and $U \otimes U$ commutes with $\mathcal{X}_{12}$ for all $\lambda \in \mathbb{T}^d$. Therefore, $U \otimes U$ commutes with $F_{|q|}(\lambda \mathcal{X})_{12}$ and consequently with $\mathcal{T}_{12}$ in $U(\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{H})$.

The action of $\mathcal{Y}$ on the basis elements of $\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}$ is given by $e_{i,j} \rightarrow q^i e_{i,j+1}$ and $e_{i,j} \otimes e_{k,l} \rightarrow e_{i,j} \otimes e_{k+i+j,l}$, respectively. Therefore,

$$\mathcal{Y}(e_{i,j}) = q^i e_{i,j+1}$$

implies that $\mathcal{Y}$ commutes with $\mathcal{X} \otimes 1_\mathcal{L}$. This implies that $\mathcal{Y}$ is a multiplicative unitary because $\mathcal{Y} \mathcal{X} \mathcal{Y}^* = \mathcal{X} \mathcal{Y} \mathcal{Y}^*$.

Using (4.4) we compute

$$\mathcal{Y} \mathcal{X} \mathcal{Y}^* e_{i,j} = \mathcal{Y}(e_{i,j} \otimes e_{k,l} \otimes e_{s,t})$$

which $(T(\mathcal{L}))_{12} \mathcal{Y}_{12}$ is the family of unitary operators defined in Lemma 4.7.

Now $\mathcal{Y}$ is a multiplicative unitary because $\mathcal{Y}_{12} \mathcal{X}_{12}(e_{i,j} \otimes e_{k,l} \otimes e_{s,t}) = e_{i,j} \otimes e_{k+i+j,l} \otimes e_{s+i+j-t,l}$. This simplifies the last equation to

$$F_{|q|}(\mathcal{X})_{23} = \mathcal{X} \mathcal{Y} \mathcal{X} \mathcal{Y}^*$$

Using (4.4) we compute

$$\mathcal{Y} \mathcal{X} \mathcal{Y}^* \mathcal{X}_{12} \mathcal{Y}_{12} \mathcal{X}_{12}(e_{i,j} \otimes e_{k,l} \otimes e_{s,t})$$

and

$$\mathcal{Y} \mathcal{X} \mathcal{Y}^* \mathcal{X}_{12} \mathcal{Y}_{12} \mathcal{X}_{12}(e_{i,j} \otimes e_{k,l} \otimes e_{s,t})$$

Therefore, $\mathcal{Y}_{12} \mathcal{X}_{12}$ commutes with $F_{|q|}(\mathcal{X})_{23}$ and this implies

$$F_{|q|}(\mathcal{X})_{23} = \mathcal{X} \mathcal{Y} \mathcal{X} \mathcal{Y}^*$$

Next we compute

$$\mathcal{Y}_{12} \mathcal{X}_{12} \mathcal{Y}_{12} \mathcal{X}_{12}(e_{i,j} \otimes e_{k,l} \otimes e_{s,t})$$

and

$$\mathcal{Y}_{12} \mathcal{X}_{12} \mathcal{Y}_{12} \mathcal{X}_{12}(e_{i,j} \otimes e_{k,l} \otimes e_{s,t})$$

This implies

$$F_{|q|}(\mathcal{X})_{23} = \mathcal{X} \mathcal{Y} \mathcal{X} \mathcal{Y}^*$$

Finally, observe that $\mathcal{Y}_{12}$ commutes with $\mathcal{Z}_{23}$ and then the last equation follows from Lemma 4.7. □

In particular, for $\lambda = 1$ the Proposition 1.8 shows that $F$ is a braided multiplicative unitary over $\mathcal{T}$ with respect to $(U, R)$.

It remains to show that $F$ is manageable. We start with the description of the operator $\mathcal{Z}$ appearing in the definition 3.3. The contragradient $U^c = U^\otimes R$ of $U$, where $R(z) := z^*$ is the bounded coinverse of $\mathcal{T}$, acts on the basis elements $\mathcal{E}_{i,j} \otimes e_p$ as $U^c(\mathcal{E}_{i,j} \otimes e_p) = e_{i,j} \otimes e_{p-j}$. A similar calculation for $\mathcal{Z}$ like that of $Z$ yields

$$\mathcal{Z} \mathcal{E}_{i,j} \otimes e_p = \mathcal{Z} \mathcal{E}_{i,j} \otimes e_p = \mathcal{Z} \mathcal{E}_{i,j} \otimes e_p.$$
Next we define the operator $Q_{\mathcal{L}}$ required by definition 3.3

$$Q_{\mathcal{L}}e_{i,j} = |q|^{2}e_{i,j}.$$  

This is a strictly positive operator on $\mathcal{H}$ with spectrum $|q|^2 \cup \{0\}$. A simple calculation shows that $Y$ and $X$ commute with $Q_{\mathcal{L}} \otimes Q_{\mathcal{L}}$ and therefore $F$ commutes with $Q_{\mathcal{L}} \otimes Q_{\mathcal{L}}$. Also the unitary $U$ leaves the first factor of the standard basis vector unchanged and $Q = \text{id}_{\mathcal{H}}$ implies $U$ commutes with $Q_{\mathcal{L}} \otimes Q_{\mathcal{L}}$.

Finally we need a unitary $\tilde{F} \in U(\mathcal{L} \otimes \mathcal{L})$ that satisfies (3.5). It is sufficient to check this if the involved vectors $x, y, u, v$ are standard basis vectors $x = e_{i,j}$, $y = e_{s,t}$, $u = e_{k,l}$ and $v = e_{a,b}$. Using the explicit formulas for $Z$, $\tilde{Z}$, $Y$ and $Q$, we rewrite (3.5) as

$$\zeta^j(e_{i,j} \otimes e_{k,l} | F|q|(X) | e_{s,t} \otimes e_{a+s+t,b}) = |q|^{l-b} \zeta^{-j b}(e_{s,t} \otimes e_{k,l} | \tilde{F} | e_{i,j} \otimes e_{a,b})$$

(4.11)

for all $a, b, i, j, k, l, s, t \in \mathbb{Z}$. To compute the left hand side of (4.11) we shall use the Fourier transform of $F|q|$ restricted on the concentric circles $|z| \in |q|^2$

$$F|q|(z) = \sum_{n \in \mathbb{Z}} F_m(|q|^n) \Phi_{\mathcal{Z}}^m,$$

where $z = \Phi_{\mathcal{Z}}|z|$ and the coefficients $F_m(|q|^n)$ for $m, n \in \mathbb{Z}$ are real and satisfies

$$F_m(|q|^n) = (-|q|)^m F_{-m}(|q|^{n-m}),$$

(4.12)

see [1] or [23 Appendix A].

Now $e_{s,t} \otimes e_{a+s+t,b}$ is an eigenvector of $|X|$ with eigenvalue $|q|^{a+t+1}$ and $\Phi_{\mathcal{X}}^m$ acts on it by $e_{s,t} \otimes e_{a+s+t,b} \rightarrow \zeta^{-mt+\frac{m(m-1)}{2}} \Phi_{\mathcal{X}}^m(e_{s,t} \otimes e_{a+s+t,b})$

Thus

$$\zeta^j(e_{i,j} \otimes e_{k,l} | F|q|(X) | e_{s,t} \otimes e_{a+s+t,b})$$

$$= \sum_{m \in \mathbb{Z}} \zeta^j(e_{i,j} \otimes e_{k,l} | \Phi_{\mathcal{X}}^m e_{s,t} \otimes e_{a+s+t,b})$$

$$= \sum_{m \in \mathbb{Z}} \zeta^j(-mt+\frac{m(m-1)}{2} \Phi_{\mathcal{X}}^m e_{s,t} \otimes e_{a+s+t,b})$$

$$= \zeta^j(-l-b)(l-b+1) \Phi_{\mathcal{X}}^m e_{s,t} \otimes e_{a+s+t,b})$$

$$= \zeta^j(-l-b)(l-b+1) \Phi_{\mathcal{X}}^m e_{s,t} \otimes e_{a+s+t,b})$$

(4.11)

The last expression is nonzero if and only if $l-b = s-i = t-j = a+s+t-k$. We also notice that $\zeta = \Phi_{\mathcal{X}}^2$.

Therefore, the left hand side of (4.11) becomes

$$\zeta^j(e_{i,j} \otimes e_{k,l} | F|q|(X) | e_{s,t} \otimes e_{a+s+t,b})$$

$$= (-|q|)^{l-b} \zeta^j \Phi_{\mathcal{X}}^m e_{s,t} \otimes e_{a+s+t,b})$$

(4.11)

Now we define an unbounded normal operator $\tilde{X} = |X| \Phi_{\mathcal{X}}$ with spectrum $\mathcal{Z}^{2}$

$$|\tilde{X}|(e_{i,j} \otimes e_{k,l}) := |q|^{k-i+1} e_{i,j} \otimes e_{k,l}, \quad \Phi_{\mathcal{X}}(e_{i,j} \otimes e_{k,l}) := -\Phi_{\mathcal{X}}^q e_{i+1,j} \otimes e_{k+1,l+1}.$$
and a unitary operator \( \tilde{Y}(e_{ij} \otimes e_{kl}) := e_{ij} \otimes e_{k+i+j,l} \). We show that the unitary \( \tilde{F} := F_{|q|}(\tilde{X})^* \tilde{Z}^2 \tilde{Y} \) satisfies (4.11):

\[
|q|^{l-b} \zeta^{-jb} \overline{e_{s,t}} \otimes e_{k,l} | \tilde{F} | \overline{e_{s,t}} \otimes e_{a,b}
\]

\[
= |q|^{l-b} \zeta^{-jb} \overline{e_{s,t}} \otimes e_{k,l} | \tilde{F}_{|q|}(\tilde{X})^* \overline{e_{s,t}} \otimes e_{a+i+j,b}
\]

\[
= \sum_{m \in \mathbb{Z}} |q|^{l-b} \zeta^{jb} \overline{e_{s,t}} \otimes e_{k,l} | F_{m}(\tilde{X})^* | \overline{e_{s,t}} \otimes e_{a+i+j,b}
\]

\[
= \sum_{m \in \mathbb{Z}} |q|^{l-b} \zeta^{jb} F_{m}(|q|^{k-s+1}) \overline{e_{s,t}} \otimes e_{k,l} | (\Phi_q^*)^{m} | \overline{e_{s,t}} \otimes e_{a+i+j,b}
\]

\[
= (|q|^{l-b} \zeta^{jb} \Phi_q (s-k) F_{b-i}(|q|^{k-s+1}) \delta_{k,a-i+b} \delta_{j,t-l+i} \delta_{k+m,a+i+j} \delta_{t+m,b}
\]

5. Braided \( E_q(2) \)-Groups and Bosonisation

Fix \( q \in \mathbb{C} \) with \( 0 < |q| < 1 \). Now we are going to describe the braided \( C^* \)-quantum group constructed from the manageable braided multiplicative unitary \( \tilde{F} \) in Theorem 4.6. Hence we shall use all the notations introduced in the previous section.

Recall the \( \mathbb{Z} \)-action \( \alpha \) on \( C_0(\mathbb{C}[^q]) \) defined by (1.2): \( (\alpha_m f)(\lambda) := f(q^m \lambda) \) for \( m \in \mathbb{Z} \), \( f \in C_0(\mathbb{C}[^q]) \), \( \lambda \in \mathbb{C}[^q] \). Let \( \beta \) be the \( \mathbb{Z} \)-action on \( C_0(\mathbb{C}[^q]) \) obtained by replacing \( q \) by \( |q| \) in the definition of \( \alpha \) (1.2). Following a similar set of arguments used in [8, Theorem 2.3] we first prove the following result.

**Proposition 5.1.** The \( C^* \)-algebras \( C_0(\mathbb{C}[^q]) \rtimes_{\alpha} \mathbb{Z} \) and \( C_0(\mathbb{C}[^q]) \rtimes_{\beta} \mathbb{Z} \) are isomorphic.

**Proof.** Let \( q = |q| e^{i\theta} \) be the polar decomposition of \( q \). For any element \( \lambda \in \mathbb{C}[^q] \), define

\[
g_{\theta}(\lambda) = \begin{cases} \lambda e^{-i \theta \log |q|} |\lambda| & \lambda \neq 0 \smallskip \\
0 & \lambda = 0.\end{cases}
\]

If \( \lambda = |q|^m e^{i\theta} \neq 0 \), then \( g_{\theta}(\lambda) = \lambda e^{-im\theta} \). Thus \( g_{\theta} \) is continuous at all nonzero \( \lambda \in \mathbb{C}[^q] \). Let \( \{\lambda_k\} \) be a nonzero sequence in \( \mathbb{C}[^q] \) such that \( \lambda_k \to 0 \) as \( k \to \infty \). Then

\[
g_{\theta}(\lambda_k) = \lambda_k e^{-i \theta \log |q|} |\lambda_k| \quad \text{for all } |\lambda_k| \neq 0.
\]

Since \( e^{-i \theta \log |q|} |\lambda_k| \) is bounded for all \( |\lambda_k| \neq 0 \) we have \( g_{\theta}(\lambda_k) \to 0 \) as \( k \to \infty \). Hence, \( g_{\theta} \) is continuous at all \( \lambda \in \mathbb{C}[^q] \). Also, \( g_{\theta} \) separates points of \( \mathbb{C}[^q] \) and \( \lim_{\lambda \to \infty} g(\lambda) = +\infty \). Therefore, \( g_{\theta} \) generates \( C_0(\mathbb{C}[^q]) \), see [26, Section 3, Example 2].

Similarly, \( g_{-\theta} \) is also continuous and it is the inverse of \( g_{\theta} \). So, \( g_{\theta} \) is a homeomorphism of \( \mathbb{C}[^q] \). Therefore, \( g_{\theta}(n) \) also generates \( C_0(\mathbb{C}[^q]) \). A simple computation gives

\[
g_{\theta}(q \lambda) = q \lambda e^{-i \theta \log |q|} |\lambda| = q e^{-i \theta} \lambda e^{-i \theta \log |q|} |\lambda| = |q| g_{\theta}(\lambda).
\]

Then using functional calculus and the commutation relation (1.1) we obtain

\[
vg(n)e^{\ast} = g(qn) = |q| g_{\theta}(n).
\]

This gives a canonical \( \mathbb{Z} \) action, denoted by \( \beta \), on \( C_0(\mathbb{C}[^q]) \) replacing \( q \by |q| \) in (1.2) and the map \( n \mapsto g_{\theta}(n) \) is \( \mathbb{Z} \)-equivariant; hence it extends to an isomorphism of crossed products \( C_0(\mathbb{C}[^q]) \rtimes_{\alpha} \mathbb{Z} \) and \( C_0(\mathbb{C}[^q]) \rtimes_{\beta} \mathbb{Z} \).

\( \square \)
5.1. **The underlying C*-algebra.** For any closed densely defined operator $T$ acting on a Hilbert space $K$ the $z$-transform $z_T \in B(K)$ of $T$ is defined by $z_T := T(1_K + T^*T)^{-\frac{1}{2}}$. Moreover, $T$ is affiliated with a C*-algebra $E$, denoted by $T \gamma_E$, if $z_T \in \mathcal{M}(E)$ and $(1_K - z_Tz_T)^{\frac{1}{2}}E$ is dense in $E$. If $T \in B(K)$ and $T \gamma_E$, then $T \in \mathcal{M}(E)$, see [23, Example 1].

Consider the crossed product C*-algebra $B = C_0(\mathbb{T}^q) \rtimes \mathbb{Z}$. Recall that $B$ is generated by $v$ and $n$ in the sense of [26, Definition 3.1]. In particular, this means $v \in \mathcal{M}(B)$ and $v \gamma B$. Now $z_{vn} = vz_n$ implies $vn \gamma B$. Therefore, the operators $v, vn \gamma B$. It is also easy to verify that the pair of operators $(v, vn)$ satisfy the conditions of [26, Theorem 3.3]; hence $v, vn$ also generate $B$.

Since $vn \gamma B$ and $F_{|q|} \in \mathcal{M}(C_0(\mathbb{T}^q))$ we get $n^{-1}vP \otimes vn = X \gamma K(\mathcal{L}) \otimes B$ and consequently $F_{|q|}(X) \in U(K(\mathcal{L}) \otimes B)$. Also observe that

$$\mathcal{Y} = (1_{\mathcal{H} \otimes \mathcal{H}} \otimes \nu^*)^N(\omega \otimes e_{i,j}) \in U(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{L}) \cong U(\mathcal{L} \otimes \mathcal{L}),$$

where $\mathcal{Y}$ is the self adjoint densely defined operator acting on $\mathcal{H}$ defined in Section 4.

It was already observed in the proof of Proposition 1.3 that $\mathcal{Y}$ is a multiplicative unitary. It is also easy to verify that $\mathcal{Y}$ is manageable with $Q_{\mathcal{L}} = id_{\mathcal{L}}$ and $\mathcal{Y}((\mathcal{T}^q) \otimes e_{i,j}) := \mathcal{T}^q \otimes e_{i,j}$, and also generates $\mathcal{T}$. Thus $\mathcal{Y} \in U(K(\mathcal{L}) \otimes C(\mathcal{T})) \subset U(K(\mathcal{L}) \otimes B)$, and consequently $\mathcal{Y} \in U(K(\mathcal{L}) \otimes B)$.

On the other hand, $B' = \{\omega \otimes id_{\mathcal{L}}|\omega \in \mathbb{B}(\mathcal{L})\}_{CL^S}$ is a C*-algebra. Therefore $B' \subseteq \mathcal{M}(B)$ and

$$B'B = \{\omega \otimes id_{\mathcal{L}}|\omega \in \mathbb{B}(\mathcal{L})|, b \in B\} \text{-closed linear span}
= \{\omega \otimes id_{\mathcal{L}}|\omega \in \mathbb{B}(\mathcal{L})|, b \in B\} \text{-closed linear span}
= \{\omega \otimes id_{\mathcal{L}}(m \otimes b)| m \in \mathbb{K}(\mathcal{L}), \omega \in \mathbb{B}(\mathcal{L})|, b \in B\} \text{-closed linear span} = B.$$

**Proposition 5.2.** The C*-algebra $B'$ coincides with $B$.

*Proof.* It is sufficient to show that the operators $v$ and $vn$ are affiliated with $B'$. Because this will imply the faithful representation $\varphi: B \to B(\mathcal{L})$ in (1.0) is an element of $\text{Mor}(B, B')$ and hence $BB' = B'$.

We consider the following family of unitaries $\{T^\lambda(\lambda)\}_{\lambda \in \mathbb{C} \setminus \{0\}}$ on $U(\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L})$ defined by

$$T^\lambda(\lambda) = F_{|q|}(\lambda n^{-1}vP \otimes P \otimes vn)Y_{13}.\tag{5.3}$$

Using (1.3) and (1.6) we compute

$$\langle (vn \otimes 1_L)X^*e_{i,j} \otimes e_{k,l} \rangle = \zeta^{ij}X(vn \otimes 1_L)e_{k,l} \otimes e_{i,j} = \zeta^{ij}q^kXe_{k-1,l+1} \otimes e_{i,j} = (P \otimes vn)e_{i,j} \otimes e_{k,l}.\tag{5.4}$$

Combining this with Proposition 5.3 we get

$$(F^\lambda)_{12}F_{23}^\lambda (F^\lambda)_{23} = X_{23}F^\lambda X_{23} = F_{|q|}(\lambda n^{-1}vP \otimes P \otimes vn)Y_{13} = T^\lambda(\lambda).$$

Expression at the extreme left of the above chain of equalities belongs to $U(K(\mathcal{L}) \otimes K(\mathcal{L}) \otimes \mathcal{B}')$, so is $F_{|q|}(\lambda n^{-1}vP \otimes P \otimes vn)Y_{13}$ for all $\lambda \in \mathbb{C} \setminus \{0\}$, $\mathbb{Y} \in U(\mathcal{L} \otimes \mathcal{L})$ and $X \gamma K(\mathcal{L}) \otimes K(\mathcal{L})$ with Spec$(X) = \mathcal{L}$. Then, by virtue of [24, Proposition 5.2], the map $\mathcal{C}^q \ni \lambda \rightarrow F_{|q|}(\lambda X)$ is $\mathcal{M}(K(\mathcal{L}) \otimes K(\mathcal{L}))$-continuous and so is the map $\lambda \rightarrow F^\lambda = \mathcal{F}(\lambda X)\mathcal{Y}$. Therefore, $\{T^\lambda(\lambda)\}_{\lambda \in \mathbb{C} \setminus \{0\}}$ is a strictly continuous family of elements of $U(K(\mathcal{L}) \otimes K(\mathcal{L}) \otimes \mathcal{B}')$. In particular, for $\lambda = 0$ we have $T^0(\lambda) = \mathbb{Y}_{13}$; hence $\mathbb{Y} \in U(K(\mathcal{L}) \otimes \mathcal{B}')$. Now the slices $(\omega \otimes id_{K(\mathcal{L})})\mathcal{Y}$ for $\omega \in \mathbb{B}(\mathcal{L})$ are dense in $C(\mathcal{T})$ and $v \in C(\mathcal{T})$ imply $\mathbb{v} \in \mathcal{M}(\mathcal{B}')$. Also the map $\mathcal{C}^q \ni \lambda \rightarrow F_{|q|}(\lambda n^{-1}vP \otimes P \otimes vn) = T^\lambda(\lambda)T^0(\lambda)^* \in U(K(\mathcal{L}) \otimes K(\mathcal{L}) \otimes \mathcal{B}')$ is strictly
continuous. Finally, combining \cite{24} Proposition 5.2 and \cite{19} Proposition 6.11 give \((n^{-1}v \eta P \otimes P \otimes v n) \eta \mathbb{K}(L) \otimes \mathbb{K}(L) \otimes B'\) and consequently \(v n \eta B'\).

\section{Construction of the comultiplication map.}
Recall the faithful representation \(\varphi: B \rightarrow B(\mathcal{H})\) in \cite{14}, the unitary operators \(z \in \mathcal{U}(\mathcal{H})\) and \(\U \in \mathcal{U}(L \otimes \mathcal{H})\) from the Section 4. Also recall the actions of \(v \in B'\) on the canonical basis elements \(e_{ij}\), and \(e_{ij} \otimes e_p\) are defined by \(v e_{ij} = e_{ij+1} \otimes e_p\) and \(\U(e_{ij} \otimes e_p) = e_{ij} \otimes e_{p+1}\) respectively. We observe that \(U\) commutes with \(v \otimes 1\).

Theorem 5.9. we obtain the following result.

\[
\delta(v) := v \otimes 1_{C(\mathbb{T})}, \quad \delta(n) = n \otimes z. \tag{5.5}
\]

is a well-defined action of \(\mathbb{T}\) on \(B\). Consequently, \((B, \delta)\) is an object of \(\mathbb{C}^*\text{alg}(\mathbb{T})\) and \((\U, \varphi)\) is a faithful covariant representation of \((B, \delta, \mathbb{T})\) on \(\mathcal{L}\). Next we define the canonical embeddings \(j_1, j_2 \in \text{Mor}(B, B \boxtimes R \mathcal{B})\) in \((5.14)\) using the braiding unitary \(\chi\) defined in \((4.3)\). On the generators \(v\) and \(n\)

\[
\begin{align*}
\Delta_1(v) &:= v \otimes 1\, \mathbb{1}, \quad \Delta_2(v) := \chi(v \otimes 1\, \mathbb{1}) \chi^* = Z(1\, \mathbb{1} \otimes v) Z^* = 1\, \mathbb{1} \otimes v, \\
\Delta_1(n) &:= n \otimes 1\, \mathbb{1}, \quad \Delta_2(n) := \chi(n \otimes 1\, \mathbb{1}) \chi^* = Z(1\, \mathbb{1} \otimes n) Z^* = P \otimes n,
\end{align*}
\tag{5.6}
\]

where \(P \in \mathcal{U}(L)\) is defined in \((4.5)\). Observe that \(\Delta_1(n), \Delta_2(n) \eta B \boxtimes R \mathcal{B}\).

Define \(\Delta_B(b) := \mathbb{F}(b \otimes 1\, \mathbb{1}) \mathbb{F}^*\) for all \(b \in B\). By Theorem 3.5, \((B, \Delta_B)\) is the braided \(C^*\)-quantum group over \(\mathbb{T}\) generated by \(\mathbb{F}\).

Let us compute \(\Delta_B\) on the generators \(v, n\). Since \(\mathbb{Y}\) generates \(\mathbb{T}\) then \(\mathbb{Y}(v \otimes 1\, \mathbb{1}) \mathbb{Y}^* = v \otimes v\). Also \(v \otimes v(e_{ij} \otimes e_k) = e_{i-1,j} \otimes e_{k-1,l}\) shows that it commutes with both the operators \(|X\rangle\) and \(\Phi_X\) in \((4.4)\), so with \(X\). Therefore,

\[
\Delta_B(v) := \mathbb{F}(v \otimes 1\, \mathbb{1}) \mathbb{F}^* = \mathbb{F}_{|q|}(X) \mathbb{Y}(v \otimes 1\, \mathbb{1}) \mathbb{Y}^* \mathbb{F}_{|q|}(X)^* = v \otimes v = j_1(v)j_2(v).
\]

A simple computation shows that \(R := j_1(n)j_2(v^*) = n \otimes v^*,\quad S := j_1(v)j_2(n) = vP \otimes n,\) and

\[
X = R^{-1}S = (j_1(n)j_2(v^*))^{-1}j_1(v)j_2(n) = n^{-1}vP \otimes vn. \tag{5.7}
\]

Then \(R\) and \(S\) are unbounded densely defined normal operators and satisfy the commutation relations \(\cite{24} (0.1)\). Since \(X = R^{-1}S\) is also a normal operator with spectrum \(\mathbb{C} \setminus q\), \(\cite{24} \text{Theorem 2.1-2.2}\) apply and show that \(R + S\) is normal with spectrum \(\mathbb{C} \setminus q\) and

\[
\mathbb{F}_{|q|}(X)(n \otimes v^*) \mathbb{F}_{|q|}(X)^* = \mathbb{F}_{|q|}(R^{-1}S) \mathbb{F}_{|q|}(R^{-1}S)^* = R + S = n \otimes v^* + vP \otimes n.
\]

The following computation

\[
\mathbb{Y}(n \otimes 1\, \mathbb{1}) \mathbb{Y}^*(e_{ij} \otimes e_k) = q^i e_{ij+1} \otimes e_{k+1} = n \otimes v^*(e_{ij} \otimes e_k) \tag{5.8}
\]

gives \(\Delta_B(n) := \mathbb{F}(n \otimes 1\, \mathbb{1}) \mathbb{F}^* = j_1(n)j_2(v^*) + j_1(v)j_2(n).\) Now, \(n \eta B\) and \(\Delta_B \in \text{Mor}(B, B \boxtimes R \mathcal{B})\) imply \(\Delta_B(n) \eta B \boxtimes R \mathcal{B}\). Thus \(\Delta_B\) in \((5.5)\) extends to a coassociative element of \(\text{Mor}^\mathbb{T}(B, B \boxtimes R \mathcal{B})\) and satisfies the cancellation conditions. In summary, we obtain the following result.

Theorem 5.9. The pair \((B, \Delta_B)\) is the braided \(C^*\)-quantum group over \(\mathbb{T}\) generated by the braided multiplicative unitary \(\mathbb{F}\).

Remark 5.10. Define \(p \in \text{Mor}(B, B)\) by \(p(v) = v\) and \(p(n) = 0\). Existence of \(p\) is guaranteed by the universal property of \(B\). Clearly, \(\text{Im}(p) = C(\mathbb{T})\). Furthermore, the restriction of the action \(\delta\) of \(\mathbb{T}\) on \(C(\mathbb{T})\) is trivial and \((5.6)\) implies \(\Delta_B(v) = vP \otimes v\).
v ⊗ v = Δ_C(T)(v). This shows that \( p \in \text{Mor}^+_T(B, C(T)) \) and satisfies \((p \otimes_R p)Δ_B(v) = Δ_C(T)(p(v))\). Hence, \( T \) is a closed quantum subgroup of braided \( E_q(2) \).

If \( q \in \mathbb{R} \), then \( \zeta = 1 \) which implies the braiding operator \( \times = \Sigma \) in (4.3). Consequently, \( F \) is an ordinary manageable multiplicative unitary, \( \mathbb{S}_R \) coincides with \( \otimes \) and (1.3) also coincides with (1.3). In conclusion, we have

**Corollary 5.11.** For real deformation parameters \( 0 < q < 1 \), the deformation \((B, Δ_B)\) coincide with Woronowicz’s \( E_q(2) \) groups.

5.3. **The bosonisation.** The \( R \)-matrix in (4.3) corresponds to the group homomorphism \( Z \ni m \to R(\cdot, m) \in \mathbb{Z} \cong T \) and it induces unique representation \( V \in \mathcal{U}(K(L) \otimes C_0(Z)) \subset \mathcal{U}(L \otimes H) \) of \( Z \) on \( L \) defined by \( V(e_i \otimes \epsilon_p) = \zeta^{-p} \epsilon_{e_i} \otimes \epsilon_p \) satisfying (4.3). Define \( \hat{V} \in \mathcal{U}(C_0(Z) \otimes K(L)) \subset \mathcal{U}(H \otimes L) \) by

\[
\hat{V} = \hat{c}(p) \otimes \epsilon_p = \zeta^{p} \epsilon_{p} \otimes \epsilon_p.
\]

By virtue of Theorem 3.6 over \( T \) relative to \( U, R \) is also a manageable multiplicative unitary over \( T \) in the sense of [15, Definition 3.2].

According to [15, Theorem 3.7 & 3.8]

\[
W := W_{13}W_{23}456F_{14}W_{34}V_{456} \in \mathcal{U}(H \otimes \mathcal{L} \otimes H \otimes L)
\]

is a manageable multiplicative unitary, where \( U \) is the concrete realisation of \( L \otimes H \). Let \( W \) generates the \( C^* \)-quantum group \( \mathbb{H} = (C, Δ_C) \).

The \( C^* \)-algebra \( C := C(T) \otimes \mathbb{S}_R = B = j_{C(T)}(C(T))j_B(B) \subset \mathbb{B}(H \otimes L) \) where, \( j_{C(T)}(z) = z \otimes 1_L \) and \( j_B(b) = \hat{V}^*(1_H \otimes b)\hat{V} \), where \( z \) is the unitary generator of \( C(T) \) and \( b \in B \subset \mathbb{B}(L) \). The concrete realisations of \( v, n, \hat{v} \) give

\[
\begin{align*}
\hat{v}(n) &= 1_H \otimes v, \\
\hat{v}(n) &= 1_H \otimes v, \\
\hat{v}(n) &= 1_H \otimes v, \\
\hat{v}(n) &= 1_H \otimes v,
\end{align*}
\]

where \( \hat{v}(n) \) acts trivially on the fifth leg it commutes with \( W_{23}W_{35} \). Moreover,

\[
\hat{V}^*(1_H \otimes n)\hat{V} = j_B(vn) = P' \otimes vn.
\]

Thus, \( W = W_{14}W_{23}456F_{14}W_{34}V_{456} \) gets simplified as

\[
W = W_{14}W_{23}456F_{14}(n^{-1}vP \otimes vn)\otimes_{23456}W_{25}W_{35}V_{456} \in \mathcal{U}(H^{(6)}).
\]

Since \( V_{456} \) acts trivially on the fifth leg it commutes with \( W_{25}W_{35} \). Moreover,

\[
\hat{V}^*(1_H \otimes vn)\hat{V} = j_B(vn) = P' \otimes vn.
\]

and the comultiplication map \( Δ_C \in \text{Mor}(C, C \otimes C) \) is given by \( Δ_C(c) = W(c \otimes 1_{H^{(6)}}) \) for all \( c \in C \). Clearly,

\[
Δ_C(j_{C(T)}(z)) = j_{C(T)}(z) \otimes j_{C(T)}(z), \quad Δ_C(j_B(v)) = j_B(v) \otimes j_B(v).
\]

Next we compute \( Δ_C(j_B(n)) \). By (5.8) we have

\[
W_{25}W_{35}(j_B(n) \otimes 1_{H^{(6)}})W_{35}^*W_{25}^* = P' \otimes n \otimes 1_H \otimes v^*.
\]

Furthermore, a variant of (5.7) gives

\[
F_{14}(n^{-1}vP \otimes P' \otimes vn)\otimes_{23456}(P' \otimes n \otimes 1_H \otimes v^*)F_{14}(n^{-1}vP \otimes P' \otimes vn)^*_{23456} = P' \otimes n \otimes 1_H \otimes v^* + P' \otimes vP \otimes P' \otimes n.
\]
Now using the concrete realisation of the operators $v, n, P, P', \mathbb{W}$ we compute
\[
\mathbb{W}_{14}\mathbb{W}_{34}(P' \otimes n \otimes 1_{H})e_{p} \otimes e_{i,j} \otimes e_{s} = \zeta^{-p}q^{s}e_{p} \otimes e_{i,j+1} \otimes e_{s+j+1+p} = ((P' \otimes n \otimes z)\mathbb{W}_{14}\mathbb{W}_{34})e_{p} \otimes e_{i,j} \otimes e_{s},
\]
and
\[
\mathbb{W}_{14}\mathbb{W}_{34}(P' \otimes vP \otimes P')e_{p} \otimes e_{i,j} \otimes e_{q} = \zeta^{-p-s}q^{s}e_{p} \otimes e_{i-1} \otimes e_{s+p+j} = ((1_{H} \otimes v \otimes P')\mathbb{W}_{14}\mathbb{W}_{34})e_{p} \otimes e_{i,j} \otimes e_{s}.
\]
Combining the last four calculations we obtain
\[
\Delta_{C}(j_{B}(n)) = j_{B}(n) \otimes j_{C(T)}(z)j_{B}(v^{*}) + j_{B}(v) \otimes j_{B}(n).
\]
Summarising, we have the bosonisation of $E_{q}(2)$.

**Theorem 5.15.** Let $C$ be the universal $C^*$-algebra generated by the the unitaries $\tilde{z}, \tilde{v}$ and the normal operator $\tilde{n}$ with $Sp(\tilde{n}) = \mathbb{C}[q]$ subject to the commutation relations
\[
\tilde{z}n\tilde{z}^{*} = \tilde{n}, \quad \tilde{z}\tilde{n}\tilde{z}^{*} = \zeta\tilde{n}, \quad \tilde{v}\tilde{n}\tilde{v}^{*} = q\tilde{n}.
\]
There exists a unique $\Delta_{C} \in Mor(C, C \otimes C)$ such that
\[
\Delta_{C}(\tilde{z}) = \tilde{z} \otimes \tilde{z}, \quad \Delta_{C}(\tilde{v}) = \tilde{v} \otimes \tilde{v}, \quad \Delta_{C}(\tilde{n}) = \tilde{n} \otimes \tilde{z}\tilde{n}^{*} + \tilde{v} \otimes \tilde{n}
\]
and $H = (\mathbb{C}, \Delta_{C})$ is a $C^{*}$-quantum group. Moreover, there exists an idempotent Hopf $^*$-homomorphism $f \in Mor(C, C)$ with $f(\tilde{z}) = \tilde{z}, f(\tilde{v}) = 1_{C}$ and $f(\tilde{n}) = 0$. Its image is the copy of $C(T)$ generated by $\tilde{z}$ as a closed quantum subgroup of $H$ and its kernel is the copy of $B$ generated by $\tilde{v}, \tilde{n}$ as the braided $E_{q}(2)$ group over $T$.

6. Contraction procedure between braided $SU_{q}(2)$ and $E_{q}(2)$ groups and their respective bosonisations

Throughout this section we fix a complex deformation parameter $0 < |q| < 1$, the unitary $R$-matrix $R: \mathbb{Z} \times \mathbb{Z} \to \mathbb{T}$, defined by \[\Box\] and $T$ is a quasitriangular compact quantum group with respect to $R$. We denote the braided $E_{q}(2)$ constructed in Theorem 5.15 by $(B_{E_{q}}(2), \Delta_{E_{q}}(2))$.

Denote $N^{0} = \mathbb{N} \cup \{0\}$. Consider the Hilbert space $L_{SU_{q}}(2) = \ell^{2}(N^{0} \times \mathbb{Z})$ equipped with an orthonormal basis $\{e_{i,j} : i, j \in N^{0} \times \mathbb{Z}\}$. Denote $L_{E_{q}}(2) = \ell^{2}(\mathbb{Z} \times \mathbb{Z})$ and fix an orthonormal basis $\{e_{i,j} : i, j \in \mathbb{Z}\}$ for it. Recall that $L_{E_{q}}(2)$ is an object of $\mathfrak{Rep}(T)$ with respect to the representation $U(e_{i,j} \otimes e_{p}) = e_{i,j} \otimes e_{p+j}$ defined in Section 4. Identification of the basis vectors of $L_{SU_{q}}(2)$ with the corresponding basis vectors of $L_{E_{q}}(2)$ defines an embedding $L_{SU_{q}}(2) \hookrightarrow L_{E_{q}}(2)$. Furthermore, the restriction of $U$ on $L_{SU_{q}}(2)$ defines a representation of $T$ on it. Consequently, we obtain a $T$-equivariant embedding $\mathbb{B}(L_{SU_{q}}(2)) \hookrightarrow \mathbb{B}(L_{E_{q}}(2))$ and $1_{L_{SU_{q}}(2)}$ is a $T$-equivariant orthogonal projection onto $L_{SU_{q}}(2)$.

Now we recall the braided $SU_{q}(2) = (B_{SU_{q}}(2), \Delta_{SU_{q}}(2))$ group over $T$ with respect to $R$, constructed in \[\Box\]. Define $\alpha, \gamma \in \mathbb{B}(L_{SU_{q}}(2))$ by
\[
\alpha e_{i,j} := \sqrt{1-|q|^{2}}e_{i-1,j}, \quad \gamma e_{i,j} := q^{i}e_{i,j-1}.
\]
Then $B_{SU_{q}}(2)$ is the universal $C^{*}$-algebra generated by $\alpha$ and $\gamma$. Define $f_{\alpha}, f_{\gamma} \in C_{0}(\mathbb{C}[q])$ by $f_{\alpha}(\lambda) = \sqrt{1-|q|^{2}}\chi(\lambda)$ and $f_{\gamma}(\lambda) = \overline{\chi}(\lambda)$, where $\chi$ is the indicator function of the closed unit disc $\{z \in \mathbb{C} | |z| \leq 1\}$. By the property of the continuous functional calculus we have
\[
\alpha = v_{f_{\alpha}}(n), \quad \gamma = f_{\gamma}(n).
\]
Following similar arguments as in \[\Box\] Section 1 and replacing $\mu$ by $|q|$ it is easy to observe
\[
B_{SU_{q}}(2) \subset B_{E_{q}}(2) \quad \text{and} \quad B_{SU_{q}}(2) = 1_{L_{SU_{q}}(2)}B_{E_{q}}(2)1_{L_{SU_{q}}(2)}.
\]
Then the restriction of \( \delta \) in (5.3) \( \delta: B_{SU_q(2)} \rightarrow B_{SU_q(2)} \otimes C(T) \) defines an action of \( C \) on \( B_{SU_q(2)} \). It is easy to verify that \( \delta(\alpha) = \alpha \otimes 1_{C(T)} \) and \( \delta(\gamma) = \gamma \otimes z^* \) is an action of \( T \) on \( B_{SU_q(2)} \), where \( z \) is the unitary generator of \( C(T) \) defined in the beginning of Section 3. In fact, \( \delta(b) = U(b \otimes 1_{C(T)}))U^* \) for all \( b \in B_{SU_q(2)} \).

Consequently \( B_{SU_q(2)} \subset B_{E_q(2)} \) is \( T \)-equivariant, \( B_{SU_q(2)} \otimes_R B_{SU_q(2)} \subset B_{E_q(2)} \otimes_R B_{E_q(2)} \) and the embeddings of \( B_{SU_q(2)} \) into \( B_{SU_q(2)} \otimes_R B_{SU_q(2)} \) are obtained by restricting \( j_1, j_2 \in \text{Mor}(B_{E_q(2)}, B_{E_q(2)} \otimes_R B_{E_q(2)}) \) in (5.6). In fact, a simple computation using (6.2) and (4.5) give \( \gamma \in B_{SU_q(2)} \) for all \( \gamma \in B_{SU_q(2)} \).

The comultiplication map \( \Delta_{SU_q(2)}: B_{SU_q(2)} \rightarrow B_{SU_q(2)} \otimes_B B_{SU_q(2)} \) is defined by
\[
\Delta_{SU_q(2)}(\alpha) = j_1(\alpha)j_2(\alpha) - g_1(\gamma)^*j_2(\gamma),
\]
\[
\Delta_{SU_q(2)}(\gamma) = j_1(\gamma)j_2(\alpha) + j_1(\alpha)^*j_2(\gamma).
\]

Consider the family of inner automorphisms \( \{\tau^k\}_{k \in \mathbb{Z}} \) of \( B_{E_q(2)} \) define by \( \tau^k(\alpha) = \alpha v^k a v^{-k} \) for all \( k \in \mathbb{Z} \). By virtue of (1.1) we have
\[
\tau^k(v) = v, \quad \tau^k(n) = q^k n.
\]
Therefore, \( \tau^k \) is \( T \)-equivariant, that is, \( (\tau^k \otimes \text{id}_{C(T)}) \circ \delta = \delta \circ \tau^k \) for all \( k \in \mathbb{Z} \). Similarly, using (1.5) we observe that \( \tau^k \) is an automorphism of braided \( C^* \)-quantum groups:
\[
(\tau^k \otimes_R \tau^k) \circ \Delta_{E_q(2)} = \Delta_{E_q(2)} \circ \tau^k \quad \text{for all } k \in \mathbb{Z}.
\]

The braided analogue of the contraction procedure [25] is contained in following result.

**Theorem 6.6.** For any \( a \in B_{E_q(2)} \)
\[
\Delta_{E_q(2)}(a) = \lim_{k \to \infty} (\tau^k \otimes_R \tau^k) \Delta_{SU_q(2)}(1_{L_{SU_q(2)}}(\tau^{-k}(a))1_{L_{SU_q(2)}}).
\]

**Proof.** The proof essentially follows from [25] Sections 2 & 3 replacing \( \mu \) by \( |q| \), tensor product \( \otimes \) of \( C^* \)-algebras by \( \otimes_R \) and taking care of certain commutation relations.

Recall the dense \( \ast \)-subalgebra \( B_{E_q(2)} \) of \( B_{E_q(2)} \) defined in [25] Equation 22
\[
B_{E_q(2)} = \cup_{k \in \mathbb{Z}} \tau^k(B_{SU_q(2)}).
\]

Suppose \( a \in B_{E_q(2)} \). For sufficiently large \( k \), \( \tau^{-k}(a) \in B_{SU_q(2)} \), which implies, \( \Delta_{SU_q(2)}(\tau^{-k}(a)) \in B_{SU_q(2)} \otimes_R B_{SU_q(2)} \). Also, \( a \) commutes with \( 1_{L_{SU_q(2)}} \). Therefore, (6.7) coincides with the following expression
\[
\Delta_{E_q(2)}(a) = \lim_{k \to \infty} (\tau^k \otimes_R \tau^k) \Delta_{SU_q(2)}(\tau^{-k}(a)) \quad \text{for all } a \in B_{E_q(2)}.
\]

Define the following elements
\[
t = \prod_{k=1}^{\infty} (1_{L_{SU_q(2)}} - |q|^{2k} \gamma^* \gamma) \in B_{SU_q(2)},
\]
\[
Y = \sum_{r=0}^{\infty} \left( \prod_{l=1}^{r} \frac{1}{1 - |q|^{2l}} \right) \left( -q j_i(\gamma^* j_j(\gamma))^r (j_i(v) j_j(v))^{-r} \right) \in B_{E_q(2)} \otimes_R B_{E_q(2)}.
\]

Note that the first term corresponding to \( r = 0 \) is \( 1_{SU_q(2)} \otimes_R 1_{SU_q(2)} \). Clearly, \( \delta(\gamma^* \gamma) = \gamma^* \gamma \otimes 1 \). Equivalently, \( \gamma^* \gamma \) is a degree zero \( T \)-homogeneous element of \( B_{SU_q(2)} \). Similarly, each term in \( Y \) is also degree zero \( T \)-homogeneous element with respect to the diagonal action \( \delta \otimes \delta \) on \( B_{E_q(2)} \otimes_R B_{E_q(2)} \) mentioned in the Section 2. Therefore, both the elements \( t \) and \( Y \) are degree zero \( T \)-homogeneous elements. We also observe that \( j_i(\alpha) \gamma(\gamma) = j_i(\gamma) j_j(\alpha) \) for all \( k, l = 1, 2, \cdots \)
These give the braided analogue of the formulae \[25\] \[\text{Equation 42-43}]:
\[
\lim_{k \to \infty} k^{\alpha \epsilon} = t^\alpha, \quad \lim_{k \to \infty} \Delta_{SU_q(2)}(\alpha^k)(j_1(v)j_2(v))^{-k} = j_1(t^\alpha)j_2(t^\alpha)Y.
\]
Consequently, for \(a \in B_{SU_q(2)}\) we prove that the right hand side of \([6,8]\) is well defined

\[
\lim_{k \to \infty} \left[ (j_1(v)j_2(v))^k \Delta_{SU_q(2)}(v^{-k}t\alpha^k(t^{-\alpha}at^{-\alpha}))^{-k} \right] = \lim_{k \to \infty} \left[ (j_1(v)j_2(v))^k \Delta_{SU_q(2)}((\alpha^*)^k(t^{-\alpha}at^{-\alpha})\alpha^k)(j_1(v)j_2(v))^{-k} \right]
= \tilde{Y}^*\Delta_{SU_q(2)}(a)\tilde{Y}, \quad \text{where } \tilde{Y} := \Delta_{SU_q(2)}(t^{-\alpha})(j_1(t^\alpha)j_2(t^\alpha))Y \in B_{E_{n(2)}} \boxtimes R B_{E_{n(2)}}.
\]
Let \(\Delta'(a) = \tilde{Y}^*\Delta_{SU_q(2)}(a)\tilde{Y}\) for all \(a \in B_{SU_q(2)}\). For any \(l \in \mathbb{N}\) and \(a \in B_{E_{n(2)}}\), \([6,8]\) implies \((\tau^l \boxtimes R \tau^l)\Delta'(a) = \Delta'((\tau^l)(a))\). Also, \(\tau^l f(n) \to f(0)1_{E_{n(2)}}\) almost uniformly, abbreviated as \(a.u.\) as \(l \to \infty\) for all \(f \in C_0(\mathbb{C}^q_{\infty})\). In particular, we have
\[
a.u. \lim_{l \to \infty} \tau^l(\alpha) = v, \quad a.u. \lim_{l \to \infty} \tau^l(\gamma) = 0, \quad (6.10)
\]
Using the second limit in \([6.10]\) we obtain

\[
a.u. \lim_{l \to \infty} (\tau^l \boxtimes R \tau^l)Y = 1_{E_{n(2)}} \boxtimes L_{E_{n(2)}}, \quad a.u. \lim_{l \to \infty} \tau^l(t^\alpha) = 1_{E_{n(2)}}, \quad (6.11)
\]
Consequently, the bounded sequence \(\{a_l := \tau^l(t)\}_{l \in \mathbb{N}}\) in \(B_{E_{n(2)}}\) satisfies \([25] \text{Equation (39)}\)

\[
a.u. \lim_{l \to \infty} \Delta'(\tau^l(t)) = a.u. \lim_{l \to \infty} \left( (\tau^l R \tau^l)Y^*(t \otimes t)Y \right) = 1_{E_{n(2)}} \boxtimes L_{E_{n(2)}},
\]
Thus, \(\Delta'\) uniquely extends to an element in Mor\(\tilde{\tau}(B_{E_{n(2)}}, B_{E_{n(2)}} \boxtimes R B_{E_{n(2)}})\).

To complete the proof of \([6,8]\), it is sufficient to verify

\[
\lim_{l \to \infty} \Delta'(\tau^l(t^{-\gamma}t^\alpha)) = \Delta_{E_{n(2)}}(v), \quad \lim_{l \to \infty} \Delta'(\tau^l(t^{-\gamma}t^\alpha)) = \Delta_{E_{n(2)}}(n). \quad (6.12)
\]
From the concrete realisation of \(n\) and \(P\) given by \([1.6]\) and \([4.3]\), it is easy to verify that \(nP\) is a normal operator and \(Sp(nP) = \mathbb{C}^q_{\infty}\). Furthermore, \([6.9]\) immediately implies \(j_1(n)j_2(n) = nP \otimes n\). As immediate consequence of \([25] \text{Proposition 1.1}\) we obtain

\[
\lim_{l \to \infty} \|q^{-l\epsilon}(\tau^l(\gamma^*) \boxtimes R \tau^l(\gamma))\psi\| = 0, \quad \text{for all } \psi \in \mathcal{D}(j_1(n)) \cap \mathcal{D}(j_2(n)).
\]
This implies

\[
\lim_{l \to \infty} \|q^{-l\epsilon}(\tau^l \boxtimes R \tau^l)Y \psi - \psi\| = 0, \quad \text{for all } \psi \in \mathcal{D}(j_1(n)) \cap \mathcal{D}(j_2(n)). \quad (6.13)
\]
Then the proof of \([6.12]\) follows similarly from \([25] \text{Section 3 (Step 3)}\) using \([6.3].\)

We end this section with an application of the contraction procedure described above to show that the bosonisations of \(SU_q(2)\) and \(E_q(2)\) are also related via a contraction procedure for the respective braided quantum groups.

Recall the bosonisation \(\mathbb{H} = (C, \Delta_C)\) of \(E_q(2)\) described in Theorem \([5,15]\). For any \(l \in \mathbb{Z}\) set \(\hat{\tau}^l := 1_{C(\mathbb{T})} \boxtimes R \tau^l\). Then \(\{\hat{\tau}^l\}_{l \in \mathbb{Z}}\) is also a one parameter group of automorphisms of \(C = C(\mathbb{T}) \boxtimes R B_{E_{n(2)}}\). Furthermore, \(\hat{\tau}^{-k}(c) \in B_{U_q(2)}\) for \(c \in C\) and sufficiently large \(k\).

Bosonisation of \(SU_q(2)\) is the compact quantum group \(U_q(2) = (C_{U_q(2)}, \Delta_{U_q(2)})\) with \(\mathbb{T}\) as a closed quantum subgroup \([3] \text{Section 6}\). The underlying \(C^*\)-algebra \(C_{U_q(2)}\) is the twisted tensor product \(C(\mathbb{T}) \boxtimes R B_{SU_q(2)} \subset B(H \otimes \mathcal{L}_{SU_q(2)}) \subset B(H \otimes \mathcal{L}_{E_{n(2)}})\). The canonical embeddings \(JC(\mathbb{T}), JB_{SU_q(2)}\) are given by \(j_C(\mathbb{T})c = z \otimes 1_{E_{n(2)}}\) and
\( j_{SU_q(2)}(a) = \hat{\Psi}^*(1_H \otimes a)\hat{\Psi} \) for \( a \in SU_q(2) \subset \mathbb{B}(L_{SU_q(2)}) \), where \( \hat{\Psi} \) is the unitary in (A.12). Consequently, (6.3) implies

\[ C_{U_q(2)} \subset C \quad \text{and} \quad C_{U_q(2)} = 1_H \otimes L_{SU_q(2)} C 1_H \otimes L_{SU_q(2)}. \]

From [14, Theorem 6.4], the comultiplication \( \Delta_{U_q(2)} \) is given by \( \Delta_{U_q(2)} := \Psi \circ (id_{C(T)} \otimes \hat{\Delta}_{SU_q(2)}) \), where

\[ \Psi(x) := W_{13} U_{23} \hat{\Psi}^* x_{124} \hat{\Psi}^* U_{23} W_{13} \quad \text{for} \quad x \in \mathbb{B}(H \otimes L_{E_q(2)} \otimes L_{E_q(2)}). \]

We also observe that \( \Delta_C = \Psi \circ (id_{C(T)} \otimes \hat{\Delta}_{SU_q(2)}) \). In particular, we get \( \Delta_C|_{C(T)} = \Delta_{U_q(2)}|_{C(T)} = \Delta_{C(T)} \) as expected.

**Corollary 6.14.** Contraction of \( U_q(2) \) group is isomorphic to the bosonisation of braided \( E_q(2) \) group. Equivalently,

\[ \Delta_C(c) = \lim_{k \to \infty} (\tilde{\tau}^k \otimes \tilde{\tau}^k) \Delta_{U_q(2)}(1_H \otimes \hat{L}_{E_q(2)}(\tilde{\tau}^{-k}(c)) 1_H \otimes \hat{L}_{E_q(2)}) \quad \text{for all} \quad c \in C. \]

(6.15)

**Proof.** Since \( B_{E_q(2)} \) is dense in \( B_{E_q(2)} \otimes C := j_{C(T)}(C(T)) \), \( B_{E_q(2)} \otimes C \) is also dense in \( C \) and the equation (6.15) coincides with the following:

\[ \Delta_C(c) = \lim_{k \to \infty} (\tilde{\tau}^k \otimes \tilde{\tau}^k) \Delta_{U_q(2)}(\tilde{\tau}^{-k}(c)) \quad \text{for} \quad c \in C. \]

(6.16)

For every \( k \in \mathbb{Z} \), the automorphisms \( \tilde{\tau}^k \) act trivially on the first factor of \( C = C(T) \otimes R B_{E_q(2)} \). Subsequently, (6.16) becomes the equality of the restrictions of \( \Delta_C \) and \( \Delta_{U_q(2)} \) on the common closed quantum subgroup \( \mathbb{T} \) for \( c = j_{C(T)}(T) \).

Restriction of the faithful representation \( j_{B_{E_q(2)}} \in Mor(B_{E_q(2)}, \mathbb{K}(H \otimes L_{E_q(2)})) \) on \( C(T) \) is given by \( j_{B_{E_q(2)}}(v) = 1_H \otimes v \). Then \( \tilde{\tau}^k(c) = (1_H \otimes v^k)(1_H \otimes v^{-k}) \) for \( c \in C \subset \mathbb{B}(H \otimes L_{E_q(2)}) \) and \( k \in \mathbb{Z} \). It is also easy to verify that \( v \otimes 1_H \) commutes with \( U \), \( 1_H \otimes v \) commutes with \( \hat{\Psi} \). These imply

\[ (\tilde{\tau}^k \otimes \tilde{\tau}^k)\hat{\Psi}(x) = \Psi((id_{C(T)} \otimes \hat{\Delta}_{SU_q(2)} \tilde{\tau}^k \otimes \hat{\Delta}_{SU_q(2)})(x)) \quad \text{for all} \quad x \in C(T) \otimes R B_{E_q(2)} \otimes R B_{E_q(2)}. \]

Finally, using (6.3) we verify (6.16) for \( c = j_{B_{E_q(2)}}(b) \) with \( b \in B_{E_q(2)} \)

\[ \Delta_C(j_{B_{E_q(2)}}(b)) = \Psi(1_{C(T)} \otimes \hat{\Delta}_{SU_q(2)} \tilde{\tau}^k \otimes \hat{\Delta}_{SU_q(2)}(j_{B_{SU_q(2)}}(b))) \]

\[ = \Psi(1_{C(T)} \otimes \hat{\Delta}_{SU_q(2)}(j_{B_{SU_q(2)}}(b))) \]

\[ = \lim_{k \to \infty} \Psi(1_{C(T)} \otimes \hat{\Delta}_{SU_q(2)}(j_{B_{SU_q(2)}}(b))) \]

\[ = \lim_{k \to \infty} \tilde{\tau}^k \otimes \tilde{\tau}^k \Delta_{U_q(2)}(j_{B_{SU_q(2)}}(b)) \]

\[ = \lim_{k \to \infty} \tilde{\tau}^k \otimes \tilde{\tau}^k \Delta_{U_q(2)}(j_{B_{SU_q(2)}}(b)). \]

\[ \square \]

**Appendix A. Yetter-Drinfeld representation category over quasitriangular quantum groups**

Let \( G = (A, \Delta_A) \) be a quantum group, \( \hat{G} = (\hat{A}, \hat{\Delta}_A) \) be its dual, and \( W \in \mathcal{U}(\hat{A} \otimes A) \) be the reduced bicharacter.

A \( G \)-Yetter-Drinfeld representation is a triple \( (L, U, V) \) consisting of a Hilbert space \( L \), representations \( U \) and \( V \) of \( G \) and \( \hat{G} \) on \( L \) subject to the commutation relation:

\[ V_{12} U_{13} W_{23} = W_{23} U_{13} V_{12} \quad \text{in} \quad \mathcal{U}(\hat{K}(\mathcal{L}) \otimes \hat{A} \otimes A). \]

(A.1)
A morphism between \(\mathbb{G}\)-Yetter-Drinfeld representations \((L_1, U^1, V^1)\) and \((L_2, U^2, V^2)\) is an element \(t \in B(L_1, L_2)\) such that \(t \in \text{Hom}^\mathbb{G}(U^1, U^2)\) in \(\text{Rep}(\mathbb{G})\) and \(t \in \text{Hom}^\mathbb{G}(V^1, V^2)\) in \(\text{Rep}(\hat{\mathbb{G}})\), respectively. Let \(\text{YDRep}(\mathbb{G})\) denote the category of \(\mathbb{G}\)-Yetter-Drinfeld representations. It is easy to verify that the \(\varphi\) operation on \(\text{YDRep}(\mathbb{G})\) defined by \((L_1, U^1, V^1) \overset{\varphi}{\rightarrow} (L_2, U^2, V^2):=(L_1 \otimes L_2, U^1 \varphi U^2, V^1 \varphi V^2)\) is a tensor product with tensor unit \(\mathbb{C}\). This category is already a unitarily braided monoidal category, see [14, Proposition 3.2].

**Proposition A.2.** Let \(\mathbb{G} = (A, \Delta_A)\) be a quasitriangular quantum group with an \(R\)-matrix \(R \in \mathcal{U}(\hat{A} \otimes \hat{A})\). For any object \((L, U)\) in \(\text{Rep}(\mathbb{G})\) the \(R\)-matrix \(R\) induces a canonical object \((L, V)\) in \(\text{Rep}(\hat{\mathbb{G}})\) such that \((L, U, V)\) becomes an object in \(\text{YDRep}(\mathbb{G})\). Moreover, the construction gives an injective braided monoidal functor \(\mathcal{F}: \text{Rep}(\mathbb{G}) \rightarrow \text{YDRep}(\mathbb{G})\) that maps \((L, U)\) to \((L, U, V)\) and leaves the morphisms unchanged.

**Proof.** Since \(R \in \mathcal{U}(\hat{A} \otimes \hat{A})\) is a bicharacter, it is also a quantum group homomorphism from \(\mathbb{G}\) to \(\hat{\mathbb{G}}\). Let \(\Delta_R \in \text{Mor}(A, A \otimes A)\) be the right quantum group homomorphism associated to it [12 Theorem 5.3]. Let \((L, U)\) be an object in \(\text{Rep}(\mathbb{G})\). Consider the unitary \(\hat{V} := U_{12}((\text{id}_{\mathbb{K}(L)} \otimes \Delta_R)U) \in \mathcal{U}(\mathbb{K}(L) \otimes A \otimes A)\). In particular, \(\Delta_R\) satisfies \((\Delta_A \otimes \text{id}_A) \circ \Delta_R = (\text{id}_A \otimes \Delta_R) \circ \Delta_A\). Using the last equation and the fact the \(U\) is a representation, we compute

\[
(id_{\mathbb{K}(L)} \otimes \Delta_A \otimes \text{id}_A)\hat{V} = (id_{\mathbb{K}(L)} \otimes \Delta_A)U_{123} (id_{\mathbb{K}(L)} \otimes ((\Delta_A \otimes \text{id}_A)\Delta_R))U
\]

\[
= U_{13}U_{12} (id_{\mathbb{K}(L)} \otimes ((\Delta_A \otimes \Delta_R)\Delta_A))U
\]

\[
= U_{13}U_{12} (id_{\mathbb{K}(L)} \otimes \Delta_A \Delta_R)U_{12}U_{13}
\]

\[
= U_{13} (id_{\mathbb{K}(L)} \otimes \Delta_R)U_{134} = \hat{V}_{134}.
\]

By [12 Corollary 2.2] the second leg of \(\hat{V}\) is trivial; hence there is a unique element \(V \in \mathcal{U}(\mathbb{K}(L) \otimes A)\) such that

\[
(id_{\mathbb{K}(L)} \otimes \Delta_R)U = U_{12}V_{13} \quad \text{in } \mathcal{U}(\mathbb{K}(L) \otimes A \otimes A).
\]

(A.3)

Also, the first part of the proof of [12 Theorem 5.3] shows that \((L, V)\) is an object in \(\text{Rep}(\hat{\mathbb{G}})\).

Now we show that \((L, U, V)\) is an object in \(\text{YDRep}(\mathbb{G})\). The second condition in (2.4) and the \(R\)-matrix condition (2.10) together imply

\[
W_{23}W_{13}R_{12} = ((\hat{\Delta}_A \otimes \text{id}_A)W)R_{12} = R_{12}((\sigma \circ \hat{\Delta}_A \otimes \text{id}_A)W) = R_{12}(\sigma_2(W_{23}W_{13}))
\]

\[
= R_{12}W_{13}W_{23}.
\]

The bijective correspondence between \(R\) and \(\Delta_R\) in [12 Equation 32], the first condition in (2.4) and the last identity give

\[
W_{34} (\sigma_{34}(\text{id}_A \otimes (\Delta_A \otimes \text{id}_A)\Delta_R)W)W_{34}^* = W_{34} (\sigma_{34}(W_{13}W_{13}R_{14}))W_{34}^*
\]

\[
= W_{34}W_{12}W_{14}R_{13}W_{34}^*
\]

\[
= W_{12}W_{34}W_{14}R_{13}W_{34}^*
\]

\[
= W_{12}R_{13}W_{14}
\]

\[
= (\text{id}_A \otimes (\Delta_R \otimes \text{id}_A)\Delta_A)W.
\]

Then slicing the first leg of the last computation by \(\omega \in \hat{A}^I\) we obtain

\[
W_{23} (\sigma_{23}((\Delta_A \otimes \text{id}_A)\Delta_R(a)))W_{23}^* = (\Delta_R \otimes \text{id}_A)\Delta_A(a) \quad \text{for all } a \in A.
\]
Using this we now compute
\[ U_{12}V_{13}U_{14}W_{34} = ((\text{id}_{\mathcal{L}_2} \otimes (\Delta_R \otimes \text{id}_A)\Delta_A)U)W_{34} \]
\[ = W_{34}(\sigma_{34}((\text{id}_{\mathcal{L}_2} \otimes (\Delta_A \otimes \text{id}_A)\Delta_R)U)) \]
\[ = W_{34}(\sigma_{34}(U_{12}U_{13}V_{14})) = W_{34}U_{12}U_{14}V_{13} = U_{12}W_{34}U_{14}V_{13}. \]
Cancelling the unitary \( U_{12} \) on both sides of the last equation we obtain \( (A.1) \); hence \( (\mathcal{L}, U, V) \) is an object in \( YD\text{Rep}(G) \).

Suppose \( (\mathcal{L}_i, U^i) \) and \( (\mathcal{L}_i, V^i) \) are objects in \( \text{Rep}(G) \) and \( \text{Rep}(\hat{G}) \) respectively, induced by the \( R \)-matrix of \( G \) for \( i = 1, 2 \). Let \( t \in \text{Hom}^\hat{G}(U^1, U^2) \) in \( \text{Rep}(\hat{G}) \). Then
\[ U_{12}^2(t \otimes 1_{A \otimes A})V_{13}^1 = (t \otimes 1_{A \otimes A})U_{12}^1V_{13}^1 = (t \otimes \Delta_R)U \]
\[ = ((\text{id}_{\mathcal{L}_2} \otimes \Delta_R)(U^2)(t \otimes 1_{A \otimes A})) \]
\[ = U_{12}^2V_{13}^1(t \otimes 1_{A \otimes A}). \]
Cancelling the unitary \( U_{12}^2 \) on both sides of the last equation we get \( t \in \text{Hom}^\hat{G}(V^1, V^2) \) in \( \text{Rep}(\hat{G}) \). Thus we have an injective functor \( F: \text{Rep}(G) \rightarrow YD\text{Rep}(\hat{G}) \) that maps objects \( (\mathcal{L}, U) \rightarrow (\mathcal{L}, U, V) \) and leaves the morphisms unchanged. Furthermore, \( F \) preserves the tensor product of representations:
\[ (\text{id}_{\mathcal{L}_1 \otimes \mathcal{L}_2} \otimes \Delta_R)U^1 \hat{\otimes} U^2 = (\text{id}_{\mathcal{L}_1 \otimes \mathcal{L}_2} \otimes \Delta_R)(U_{13}^1U_{23}^2) \]
\[ = U_{13}^1V_{14}^1U_{24}^1V_{24}^2 \]
\[ = U_{24}^1U_{13}^2V_{14}^2V_{24}^1 = (U^1 \hat{\otimes} U^2)(V^1 \hat{\otimes} V^2)(U^1 \hat{\otimes} V^1)V^2(124). \]
In particular, the proof of [14] Theorem 5.3 shows that \( F \) preserves the braiding. Therefore, \( F \) is an injective braided monoidal structure preserving functor. □

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