The Conformal Four-Point Integrals, Magic Identities and Representations of $U(2,2)$

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Abstract

In [FL1, FL3] we found mathematical interpretations of the one-loop conformal four-point Feynman integral as well as the vacuum polarization Feynman integral in the context of representations of a Lie group $U(2,2)$ and quaternionic analysis. Then we raised a natural question of finding mathematical interpretation of other Feynman diagrams in the same setting. In this article we describe this interpretation for all conformal four-point integrals. Using this interpretation, we give a representation-theoretical proof of “magic identities” that were originally established in [DHSS].

No prior knowledge of physics or Feynman diagrams is assumed from the reader. We provide a summary of all relevant results from quaternionic analysis to make the article self-contained.

1 Introduction

Feynman diagrams are a pictorial way of describing integrals predicting possible outcomes of interactions of subatomic particles in the context of quantum field physics. If at all possible, evaluating these integrals tends to be challenging and usually produces rather cumbersome expressions. Moreover, many Feynman diagrams result in integrals that are divergent in mathematical sense. Physicists have various techniques called “renormalizations” of Feynman integrals which “cancel out the infinities” coming from different parts of the diagrams. (For a survey of renormalization techniques see, for example, [Sm].) However, these renormalization techniques appear very suspicious to mathematicians and attract criticism from physicists as well. For example, if different techniques yield different results, how do you choose the “right” technique? Or, if they yield the same result, what is the underlying reason for that? If one can find an intrinsic mathematical meaning of Feynman diagrams and the corresponding integrals, most of these questions will be resolved.

A number of mathematicians already work on this problem, mostly in the setting of algebraic geometry. See, for example, [M] for a summary of these algebraic-geometric developments as well as a comprehensive list of references. On the other hand, Igor Frenkel has noticed that at least some types of Feynman diagrams can be interpreted in the context of representation theory and quaternionic analysis. In [FL1, FL3, L] we successfully identified the three Feynman diagrams shown in Figure 1 with intertwining operators of certain representations of $U(2,2)$ in the context of quaternionic analysis. Then we raised a natural question of finding mathematical interpretation of other Feynman diagrams in the same setting.

This paper deals with conformal four-point integrals described by the box diagrams. They play an important role in physics, particularly in Yang-Mills conformal field theory. For more details see [DHSS] and references therein. These diagrams have been thoroughly studied by physicists. For example, the integral described by the one-loop Feynman diagram is known to
express the hyperbolic volume of an ideal tetrahedron, and is given by the dilogarithm function \[ DD, W \]; there are explicit expressions for the integrals described by the ladder diagrams in terms of polylogarithms \[ UD \]. Perhaps the most important property of the conformal four-point integrals are the “magic identities” established in \[ DHSS \]. These identities assert that all \( n \)-loop box integrals for four scalar massless particles are equal to each other. We will discuss these “magic identities” in Subsection 4.2.

In this paper we find the representation-theoretic meaning of all conformal four-point integrals. To each such integral, we associate an operator \( L^{(n)} \) on \( \mathcal{H}^+ \otimes \mathcal{H}^+ \), where \( \mathcal{H}^+ \) denotes the space of harmonic functions on the algebra of quaternions \( \mathbb{H} \). We prove that the operator \( L^{(n)} \) is \( u(2, 2) \)-equivariant, sends \( \mathcal{H}^+ \otimes \mathcal{H}^+ \) into itself and, in particular, that the result is a function of two variables that is harmonic with respect to each variable, which is not at all obvious from the construction. We have a decomposition of \( u(2, 2) \)-representations into irreducible components:

\[
(\pi^0 \mathcal{H}^+) \otimes (\pi^0 \mathcal{H}^+) \simeq \bigoplus_{k=1}^{\infty} (\rho_k, \mathcal{H}^+ \otimes \mathbb{C}^k \times \mathbb{C}^k),
\]

Then, by Schur’s Lemma, \( L^{(n)} \) acts on each irreducible component \( (\rho_k, \mathcal{H}^+ \otimes \mathbb{C}^k \times \mathbb{C}^k) \) by multiplication by some scalar \( \mu_k^{(n)} \), and we can find these scalars. This is the essence of the main result (Theorem 14). As an immediate corollary, we obtain the “magic identities” for the operators \( L^{(n)} \): Any two box diagrams with the same number of loops produce the same operator \( L^{(n)} \) on \( \mathcal{H}^+ \otimes \mathcal{H}^+ \). If one can prove that each conformal four-point integral is harmonic with respect to each variable, then one easily obtains the original “magic identities” for the conformal four-point integrals. The proof of Theorem 14 is essentially by evaluating the operators \( L^{(n)} \) on a suitably chosen set of generators of \( \mathcal{H}^+ \otimes \mathcal{H}^+ \). It is pretty elementary, and we think that it is an advantage of this approach.

For example, the integrals described by the ladder diagrams have been evaluated explicitly in \[ UD \]. The two most simple conformal four-point integrals are the one- and two-loop ladder integrals \( l^{(1)}(Z_1, Z_2; W_1, W_2) \) and \( l^{(2)}(Z_1, Z_2; W_1, W_2) \), which can be expressed in terms of the functions

\[
\Phi^{(1)}(x, y) = \frac{1}{\lambda} \left( 2 \text{Li}_2(-\rho x) + 2 \text{Li}_2(-\rho y) + \ln \frac{y}{x} \cdot \ln \frac{1 + \rho y}{1 + \rho x} + \ln(\rho x) \cdot \ln(\rho y) + \frac{1}{3} \pi^2 \right),
\]

and

\[
\Phi^{(2)}(x, y) = \frac{1}{\lambda} \left( 6 \text{Li}_4(-\rho x) + 6 \text{Li}_4(-\rho y) + 3 \ln \frac{y}{x} \cdot (\text{Li}_3(-\rho x) - \text{Li}_3(-\rho y)) \right.
\]

\[
+ \frac{1}{2} \ln^2 \frac{y}{x} \cdot (\text{Li}_2(-\rho x) - \text{Li}_2(-\rho y)) + \frac{1}{4} \ln^2(\rho x) \cdot \ln^2(\rho y)
\]

\[
+ \frac{1}{2} \pi^2 \ln(\rho x) \cdot \ln(\rho y) + \frac{1}{12} \pi^2 \ln \frac{y}{x} + \frac{7}{60} \pi^4 \right).
\]
respectively, where
\[ \lambda(x, y) = \sqrt{(1 - x - y)^2 - 4xy}, \quad \rho(x, y) = \frac{2}{1 - x - y + \lambda}, \]
and \( \text{Li}_N \) denotes the polylogarithm function:
\[ \text{Li}_N(z) = \frac{(-1)^N}{(N-1)!} \int_0^1 \ln^{N-1} \xi \frac{d\xi}{\xi - z}. \]
The expressions for the other ladder integrals are similar.

By contrast, we have very simple expressions for the operators \( L^{(1)} \) and \( L^{(2)} \) on \( \mathcal{H} \otimes \mathcal{H} \). The operator \( L^{(1)} \) is just the projection of \( \mathcal{H} \otimes \mathcal{H} \) onto its first irreducible component \( (\rho_1, \mathcal{K}^+) \) in the decomposition \([1]\). And \( L^{(2)} \) acts on each irreducible component of \( \mathcal{H} \otimes \mathcal{H} \) by multiplication by a scalar, so that if \( x \in \mathcal{H}^+ \otimes \mathcal{H}^+ \) belongs to an irreducible component isomorphic to \( (\rho_k, \mathcal{K}^+ \otimes \mathbb{C}^k \otimes \mathbb{C}^k) \) in the decomposition \([1]\), then
\[ L^{(2)}(x) = \mu_k^{(2)} x, \quad \text{where} \quad \mu_k^{(2)} = \begin{cases} 1 & \text{if } k = 1; \\ \frac{(-1)^{k+1}}{k(k-1)} & \text{if } k \geq 2. \end{cases} \]

Thus we have a representation-theoretical interpretation of an infinite family of Feynman diagrams, and it is reasonable to expect that an even larger class of Feynman diagrams can be interpreted in the same context. Finally, we comment that it is not really necessary to use quaternionic setting to interpret the box diagrams and the corresponding integrals – one could do the same in the setting of analytic functions of four variables instead. However, the vacuum polarization diagram does require quaternionic analysis. Also, this article uses results that have already been stated and proved in quaternionic setting. For these reasons we continue to use quaternions.

The paper is organized as follows. In Section 2 we establish our notations and state relevant results from quaternionic analysis. In Section 3 we state more recent results from \([FL3]\) and \([L]\) that are used in the proofs. In Section 4 we review the box diagrams and the corresponding conformal four-point integrals, state the magic identities and the main result (Theorem 14). In Section 5 we prove Theorem 14, first, in the case of ladder diagrams, and then in general.

## 2 Preliminaries

In this section we establish notations and state relevant results from quaternionic analysis. We mostly follow our previous papers \([FL1]\), \([FL2]\) and \([L]\). A contemporary review of quaternionic analysis can be found in \([Su]\). Quaternionic analysis also has many applications in physics (see, for instance, \([GT]\)).

### 2.1 Complexified Quaternions \( \mathbb{H}_C \) and the Conformal Group \( GL(2, \mathbb{H}_C) \)

We recall some notations from \([FL1]\). Let \( \mathbb{H}_C \) denote the space of complexified quaternions: \( \mathbb{H}_C = \mathbb{H} \otimes \mathbb{C} \), it can be identified with the algebra of \( 2 \times 2 \) complex matrices:
\[ \mathbb{H}_C = \mathbb{H} \otimes \mathbb{C} \simeq \left\{ Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} ; z_{ij} \in \mathbb{C} \right\} = \left\{ Z = \begin{pmatrix} z^0 - iz^3 & -iz^1 - z^2 \\ -iz^1 + z^2 & z^0 + iz^3 \end{pmatrix} ; z^k \in \mathbb{C} \right\}. \]

For \( Z \in \mathbb{H}_C \), we write
\[ N(Z) = \det \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = z_{11}z_{22} - z_{12}z_{21} = (z^0)^2 + (z^1)^2 + (z^2)^2 + (z^3)^2. \]
and think of it as the norm of $Z$. We realize $U(2)$ as

$$U(2) = \{ Z \in \mathbb{H}_C; \ Z^* = Z^{-1} \},$$

where $Z^*$ denotes the complex conjugate transpose of a complex matrix $Z$. For $R > 0$, we set

$$U(2)_R = \{ RZ; \ Z \in U(2) \} \subset \mathbb{H}_C$$

and orient it as in [FL1], so that

$$\int_{U(2)_R} \frac{dV}{N(Z)^2} = -2\pi^3 i,$$

where $dV$ is a holomorphic 4-form

$$dV = dz^0 \wedge dz^1 \wedge dz^2 \wedge dz^3 = \frac{1}{4} dz_{11} \wedge dz_{12} \wedge dz_{21} \wedge dz_{22}.$$

Recall that a group $GL(2, \mathbb{H}_C) \simeq GL(4, \mathbb{C})$ acts on $\mathbb{H}_C$ by fractional linear (or conformal) transformations:

$$h : Z \mapsto (aZ + b)(cZ + d)^{-1} = (a' - Zc')^{-1}(-b' + Zd'), \quad Z \in \mathbb{H}_C,$$

where $h = (a \ b \ c \ d) \in GL(2, \mathbb{H}_C)$ and $\pi^{-1} = (a' \ b' \ c' \ d').$

2.2 Harmonic Functions on $\mathbb{H}_C$

As in Section 2 of [FL2], we consider the space $\tilde{\mathcal{H}}$ consisting of $\mathbb{C}$-valued functions on $\mathbb{H}_C$ (possibly with singularities) that are holomorphic with respect to the complex variables $z_{11}, z_{12}, z_{21}, z_{22}$ and harmonic, i.e. annihilated by

$$\square = 4 \left( \frac{\partial^2}{\partial z_{11} \partial z_{22}} - \frac{\partial^2}{\partial z_{12} \partial z_{21}} \right) = \frac{\partial^2}{(\partial z^0)^2} + \frac{\partial^2}{(\partial z^1)^2} + \frac{\partial^2}{(\partial z^2)^2} + \frac{\partial^2}{(\partial z^3)^2}.$$

Then the conformal group $GL(2, \mathbb{H}_C)$ acts on $\tilde{\mathcal{H}}$ by two slightly different actions:

$$\pi^0_l(h) : \varphi(Z) \mapsto (\pi^0_l(h)\varphi)(Z) = \frac{1}{N(cZ + d)} \cdot \varphi((aZ + b)(cZ + d)^{-1}),$$

$$\pi^0_r(h) : \varphi(Z) \mapsto (\pi^0_r(h)\varphi)(Z) = \frac{1}{N(a' - Zc')} \cdot \varphi((a' - Zc')^{-1}(-b' + Zd')).$$

where $h = (a' \ b' \ c' \ d') \in GL(2, \mathbb{H}_C)$ and $h^{-1} = (a \ b \ c \ d)$. These two actions coincide on $SL(2, \mathbb{H}_C) \simeq SL(4, \mathbb{C})$ which is defined as the connected Lie subgroup of $GL(2, \mathbb{H}_C)$ with Lie algebra

$$\mathfrak{sl}(2, \mathbb{H}_C) = \{ x \in \mathfrak{gl}(2, \mathbb{H}_C); \ \text{Re}(\text{Tr} \ x) = 0 \} \simeq \mathfrak{sl}(4, \mathbb{C}).$$

We introduce two spaces of harmonic polynomials:

$$\mathcal{H}^+ = \tilde{\mathcal{H}} \cap \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}],$$

$$\mathcal{H} = \tilde{\mathcal{H}} \cap \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}, N(Z)^{-1}]$$

and the space of harmonic polynomials regular at infinity:

$$\mathcal{H}^- = \{ \varphi \in \tilde{\mathcal{H}}; \ N(Z)^{-1} \cdot \varphi(Z^{-1}) \in \mathcal{H}^+ \}. $$
Then
\[ \mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+. \]

Differentiating the actions \( \pi^0_1 \) and \( \pi^0_r \), we obtain actions of \( \mathfrak{gl}(2, \mathbb{H}_C) \simeq \mathfrak{gl}(4, \mathbb{C}) \) which preserve the spaces \( \mathcal{H}, \mathcal{H}^- \) and \( \mathcal{H}^+ \). By abuse of notation, we denote these Lie algebra actions by \( \pi^0_1 \) and \( \pi^0_r \) respectively. They are described in Subsection 3.2 of [FL1].

By Theorem 28 in [FL1], for each \( R > 0 \), we have a bilinear pairing between \((\pi^0_1, \mathcal{H})\) and \((\pi^0_r, \mathcal{H})\):
\[
(\varphi_1, \varphi_2)_R = \frac{1}{2\pi^2} \iint_{Z \in S_R^3} (\tilde{\deg}\varphi_1)(Z) \cdot \varphi_2(Z) \frac{dS}{R}, \quad \varphi_1, \varphi_2 \in \mathcal{H},
\]
where \( S_R^3 \subset \mathbb{H} \) is the three-dimensional sphere of radius \( R \) centered at the origin
\[
S_R^3 = \{ X \in \mathbb{H}; N(X) = R^2 \},
\]
d\( S \) denotes the usual Euclidean volume element on \( S_R^3 \), and \( \tilde{\deg} \) denotes the degree operator plus identity:
\[
\tilde{\deg} f = f + \deg f = f + z_{11} \frac{\partial f}{\partial z_{11}} + z_{12} \frac{\partial f}{\partial z_{12}} + z_{21} \frac{\partial f}{\partial z_{21}} + z_{22} \frac{\partial f}{\partial z_{22}}.
\]

When this pairing is restricted to \( \mathcal{H}^+ \times \mathcal{H}^- \), it is \( \mathfrak{gl}(2, \mathbb{H}_C) \)-invariant, independent of the choice of \( R > 0 \), non-degenerate and antisymmetric
\[
(\varphi_1, \varphi_2)_R = -(\varphi_2, \varphi_1)_R, \quad \varphi_1 \in \mathcal{H}^+, \varphi_2 \in \mathcal{H}^-.
\]

When restricted to \( \mathfrak{u}(2, 2) \), the representations \((\pi^0_1, \mathcal{H}^+)\) and \((\pi^0_r, \mathcal{H}^+)\) become irreducible unitary with respect to the inner product
\[
\langle \varphi_1, \varphi_2 \rangle_{\text{inn. prod.}} = \int_{Z \in S^3_1} (\tilde{\deg}\varphi_1)(Z) \cdot \varphi_2(Z) dS, \quad \varphi_1, \varphi_2 \in \mathcal{H}^+,
\]
(Theorem 28 in [FL1]).

We conclude this subsection with an analogue of the Poisson formula (Theorem 34 in [FL1]). It involves a certain open region \( \mathbb{D}^+_R \) in \( \mathbb{H}_C \) which will be defined in (15).

**Theorem 1.** Let \( R > 0 \) and let \( \varphi \in \tilde{\mathcal{H}} \) be a harmonic function with no singularities on the closure of \( \mathbb{D}^+_R \), then
\[
\varphi(W) = \left( \varphi, \frac{1}{N(Z - W)} \right)_R = \frac{1}{2\pi^2} \iint_{Z \in S^3_R} (\tilde{\deg}\varphi)(Z) \frac{dS}{N(Z - W)} \frac{dS}{R}, \quad \forall W \in \mathbb{D}^+_R.
\]

### 2.3 Representation \((\rho_1, \mathcal{K})\) of \( \mathfrak{gl}(2, \mathbb{H}_C)\)

Let \( \tilde{\mathcal{K}} \) denote the space of \( \mathbb{C} \)-valued functions on \( \mathbb{H}_C \) (possibly with singularities) which are holomorphic with respect to the complex variables \( z_{11}, z_{12}, z_{21}, z_{22} \). We recall the action of \( GL(2, \mathbb{H}_C) \) on \( \tilde{\mathcal{K}} \) given by equation (49) in [FL1]:
\[
\rho_1(h) : f(Z) \mapsto (\rho_1(h)f)(Z) = f((aZ + b)(cZ + d)^{-1}) \frac{N(cZ + d) \cdot N(a' - Zc')}{N(cZ + d) \cdot N(a' - Zc')},
\]
where \( h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{H}_C) \) and \( h^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \). Differentiating the \( \rho_1 \)-action, we obtain an action (still denoted by \( \rho_1 \)) of \( \mathfrak{gl}(2, \mathbb{H}_C) \) which preserves spaces
\[
\mathcal{K}^+ = \{ \text{polynomial functions on } \mathbb{H}_C \} = \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}] \quad \text{and}
\]
\[
\mathcal{K} = \{ \text{polynomial functions on } \{ Z \in \mathbb{H}_C; N(Z) \neq 0 \} \} = \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}, N(Z)^{-1}].
\]

Recall Proposition 69 from [FL1]:
Proposition 2. The representation \((\rho_1, \mathcal{K})\) of \(\mathfrak{gl}(2, \mathbb{H}_C)\) has a non-degenerate symmetric bilinear pairing
\[
\langle f_1, f_2 \rangle = \frac{i}{2\pi^3} \int_{Z \in U(2)_R} f_1(Z) \cdot f_2(Z) \, dV, \quad f_1, f_2 \in \mathcal{K}.
\] (7)
This bilinear pairing is \(\mathfrak{gl}(2, \mathbb{H}_C)\)-invariant and independent of the choice of \(R > 0\).

2.4 The Group \(\mathbb{H}_C^\times\) and Its Matrix Coefficients

We denote by \(\mathbb{H}_C^\times\) the group of invertible complexified quaternions:
\[
\mathbb{H}_C^\times = \{ Z \in \mathbb{H}_C; N(Z) \neq 0 \} \simeq GL(2, \mathbb{C}).
\]

We denote by \((\tau_\frac{1}{2}, S)\) the tautological 2-dimensional representation of \(\mathbb{H}_C^\times\). Then, for \(l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\), we denote by \((\tau_l, V_l)\) the \(2l\)-th symmetric power product of \((\tau_\frac{1}{2}, S)\). (In particular, \((\tau_0, V_0)\) is the trivial one-dimensional representation.) Thus, each \((\tau_l, V_l)\) is an irreducible representation of \(\mathbb{H}_C^\times\) of dimension \(2l + 1\). A concrete realization of \((\tau_l, V_l)\) as well as an isomorphism \(V_l \simeq \mathbb{C}^{2l+1}\) suitable for our purposes are described in Subsection 2.5 of [FL1].

Recall the matrix coefficient functions of \(\tau_l(Z)\) described by equation (27) of [FL1] (cf. [V]):
\[
t_{n, m}^l(Z) = \frac{1}{2\pi i} \int (sz_{12} + z_{22})^{l-m}(sz_{12} + z_{22})^{l+m} s^{-l+n} \frac{ds}{s}, \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots;
\]
\[
m, n \in \mathbb{Z} + l, \quad -l \leq m, n \leq l,
\]
\[
Z = (z_{11}, z_{12}, z_{21}, z_{22}) \in \mathbb{H}_C, \quad \text{the integral is taken over a loop in} \ \mathbb{C} \quad \text{going once around the origin in the counterclockwise direction. We regard these functions as polynomials on} \ \mathbb{H}_C. \quad \text{For example,}
\]
\[
t_{l-1, l}^l(Z) = (z_{11})^{2l}, \quad t_{l-2, l}^l(Z) = (z_{12})^{2l}, \quad t_{l-1, l}^l(Z) = (z_{21})^{2l}, \quad t_{l, l}^l(Z) = (z_{22})^{2l}. \tag{8}
\]

We have the following orthogonality relations with respect to the pairing (3):
\[
\langle t_{n', m'}^l(Z), t_{m, n}^l(Z^{-1}) \cdot N(Z)^{-1} \rangle = -\langle t_{m, n}^l(Z^{-1}) \cdot N(Z)^{-1}, t_{n', m'}^l(Z) \rangle = \delta_{ll'}\delta_{mm'}\delta_{nn'}, \tag{9}
\]
the following orthogonality relations with respect to the inner product (1):
\[
\langle t_{m, n}^l(Z), t_{n', m'}^l(Z) \rangle_{\text{inn. prod.}} = \frac{(l - m)! (l + m)!}{(l - n)! (l + n)!} \delta_{ll'}\delta_{mm'}\delta_{nn'}, \tag{10}
\]
and similar orthogonality relations with respect to the pairing (7):
\[
\langle t_{n', m'}^{l'}(Z) \cdot N(Z)^{k}, t_{m, n}^l(Z^{-1}) \cdot N(Z)^{-k+2} \rangle = \frac{1}{2l + 1} \delta_{kk'}\delta_{ll'}\delta_{mm'}\delta_{nn'}, \tag{11}
\]
where the indices \(k, l, m, n\) are \(l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, m, n \in \mathbb{Z} + l, -l \leq m, n \leq l, k \in \mathbb{Z}\) and similarly for \(l', k', m', n'\) (see, for example, [V]).

One advantage of working with these functions is that they form \(K\)-type bases of various spaces:

**Proposition 3** (Proposition 19 in [FL1], Proposition 5 in [FL3] and Corollary 6 in [FL3]).

The functions
\[
t_{n, m}^l(Z), \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \quad m, n = -l, -l + 1, \ldots, l,
\]
form a vector space basis of \(\mathcal{H}^+ = \{ \varphi \in \mathcal{K}^+; \square \varphi = 0 \};\)
2. The functions
\[ t^l_{mn}(Z) \cdot N(Z)^{-(2l+1)}, \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \quad m, n = -l, -l + 1, \ldots, l, \]
form a vector space basis of \( \mathcal{H}^- \);

3. The functions
\[ t^l_{mn}(Z) \cdot N(Z)^k, \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \quad m, n = -l, -l + 1, \ldots, l, \quad k = 0, 1, 2, \ldots, \]
form a vector space basis of \( \mathcal{H}^+ = \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}] \);

4. The functions
\[ t^l_{mn}(Z) \cdot N(Z)^k, \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \quad m, n = -l, -l + 1, \ldots, l, \quad k \in \mathbb{Z}, \quad (12) \]
form a vector space basis of \( \mathcal{H} = \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}, N(Z)^{-1}] \).

Another advantage is having matrix coefficient expansions such as those described in Propositions 25, 26 and 27 in [FL1]. For convenience we restate Proposition 25 from [FL1]:

**Proposition 4.** We have the following matrix coefficient expansion
\[ \frac{1}{N(Z - W)^{-1}} \cdot \sum_{l,m,n} t^l_{mn}(Z) \cdot t^l_{mn}(W^{-1}), \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \quad m, n = -l, -l + 1, \ldots, l, \quad (13) \]
which converges pointwise absolutely in the region \( \{(Z,W) \in \mathbb{H}_C \times \mathbb{H}_C^*; ZW^{-1} \in \mathbb{D}^+\} \), where \( \mathbb{D}^+ \) is an open region in \( \mathbb{H}_C \) to be defined in (14).

2.5 **Subgroups** \( U(2,2)_R \subset GL(2, \mathbb{H}_C) \) and **Domains** \( \mathbb{D}^+_R, \mathbb{D}^-_R \)

We often regard the group \( U(2,2) \) as a subgroup of \( GL(2, \mathbb{H}_C) \), as described in Subsection 3.5 of [FL1]. That is
\[ U(2,2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{H}_C); \ a, b, c, d \in \mathbb{H}_C, \ a^*d = 1 + b^*b \right\}. \]
The maximal compact subgroup of \( U(2,2) \) is
\[ U(2) \times U(2) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in GL(2, \mathbb{H}_C); \ a, d \in \mathbb{H}_C, \ a^*a = d^*d = 1 \right\}. \]

The group \( U(2,2) \) acts on \( \mathbb{H}_C \) by fractional linear transformations (2) preserving \( U(2) \subset \mathbb{H}_C \) and open domains
\[ \mathbb{D}^+ = \{ Z \in \mathbb{H}_C; \ ZZ^* < 1 \}, \quad \mathbb{D}^- = \{ Z \in \mathbb{H}_C; \ ZZ^* > 1 \}, \quad (14) \]
where the inequalities \( ZZ^* < 1 \) and \( ZZ^* > 1 \) mean that the matrix \( ZZ^* - 1 \) is negative and positive definite respectively. The sets \( \mathbb{D}^+ \) and \( \mathbb{D}^- \) both have \( U(2) \) as the Shilov boundary.

Similarly, for each \( R > 0 \), we can define a conjugate of \( U(2,2) \)
\[ U(2,2)_R = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} U(2,2) \begin{pmatrix} R^{-1} & 0 \\ 0 & 1 \end{pmatrix} \subset GL(2, \mathbb{H}_C). \]

Each group \( U(2,2)_R \) is a real form of \( GL(2, \mathbb{H}_C) \), preserves \( U(2)_R \) and open domains
\[ \mathbb{D}^+_R = \{ Z \in \mathbb{H}_C; \ ZZ^* < R^2 \}, \quad \mathbb{D}^-_R = \{ Z \in \mathbb{H}_C; \ ZZ^* > R^2 \}. \quad (15) \]
These sets \( \mathbb{D}^+_R \) and \( \mathbb{D}^-_R \) both have \( U(2)_R \) as the Shilov boundary.
3 Summary of Results from [FL3] and [L]

3.1 Irreducible Components of \((\rho_1, \mathcal{K})\) and Equivariant Maps

\((\rho_1, \mathcal{K}) \rightarrow (\pi_1^0, \mathcal{H}) \otimes (\pi_0^0, \mathcal{H})\)

First, we state the decomposition theorem:

**Theorem 5** (Theorem 7 in [FL3]). The representation \((\rho_1, \mathcal{K})\) of \(gl(2, \mathbb{H}_C)\) has the following decomposition into irreducible components:

\[
(\rho_1, \mathcal{K}) = (\rho_1, \mathcal{K}^-) \oplus (\rho_1, \mathcal{K}^0) \oplus (\rho_1, \mathcal{K}^+),
\]

where

\[
\mathcal{K}^+ = \mathbb{C} - \text{span of } \{ t_{n,m}(Z) \cdot N(Z)^k ; k \geq 0 \},
\]

\[
\mathcal{K}^- = \mathbb{C} - \text{span of } \{ t_{n,m}(Z) \cdot N(Z)^k ; k \leq -(2l + 2) \},
\]

\[
\mathcal{K}^0 = \mathbb{C} - \text{span of } \{ t_{n,m}(Z) \cdot N(Z)^k ; -(2l + 1) \leq k \leq -1 \}
\]

(see Figure 2).

A tensor product \((\pi_1^0, \mathcal{H}^+) \otimes (\pi_0^0, \mathcal{H}^+)\) of representations of \(gl(2, \mathbb{H}_C)\) decomposes into a direct sum of irreducible subrepresentations, one of which is \((\rho_1, \mathcal{K}^+)\). This decomposition is stated precisely in equation (23). The irreducible component \((\rho_1, \mathcal{K}^+)\) has multiplicity one and is generated by \(1 \otimes 1 \in \mathcal{H}^+ \otimes \mathcal{H}^+\). Thus we have a \(gl(2, \mathbb{H}_C)\)-equivariant map

\[
I : (\rho_1, \mathcal{K}^+) \hookrightarrow (\pi_1^0, \mathcal{H}^+) \otimes (\pi_0^0, \mathcal{H}^+),
\]

which is unique up to multiplication by a scalar. This scalar can be pinned down by a requirement \(I(1) = 1 \otimes 1\).

We consider a map

\[
\mathcal{K} \ni f(Z) \mapsto (I_R f)(W_1, W_2) = \frac{i}{2\pi^3} \int_{Z \in U(2)_R} \frac{f(Z) \, dV}{N(Z - W_1) \cdot N(Z - W_2)} \in \mathcal{H} \otimes \mathcal{H}, \tag{16}
\]
where $H \otimes H$ denotes the Hilbert space obtained by completing $H \otimes H$ with respect to the unitary structure coming from the tensor product of unitary representations $(\pi^0, H)$ and $(\pi^r, H)$. If $W_1, W_2 \in \mathbb{D}_R^-$ or $W_1, W_2 \in \mathbb{D}_R^+$, the integrand has no singularities and the result is a holomorphic function in two variables $W_1, W_2$ which is harmonic in each variable separately.

**Theorem 6** ([Theorem 12 and Corollary 14 in [L]]) When $W_1, W_2 \in \mathbb{D}_R^-$, the map $I_R$ annihilates $\mathcal{H}^- \oplus \mathcal{H}^0$, and its restriction to $\mathcal{H}^+$ coincides with the map $I$.

When $W_1, W_2 \in \mathbb{D}_R^+$, the map $I_R$ annihilates $\mathcal{H}^0 \oplus \mathcal{H}^+$, and its restriction to $\mathcal{H}^-$ produces an equivariant embedding $(\rho_1, \mathcal{H}^-) \hookrightarrow (\pi^0, \mathcal{H}^-) \otimes (\pi^0, \mathcal{H}^-)$.

Next we have a lemma that will be used for evaluating integral operators $L^{(n)}$ on the generators of $(\pi^0, \mathcal{H}^+) \otimes (\pi^0, \mathcal{H}^+)$.

**Lemma 7** (Lemma 18 in [L]). Let $k = 1, 2, 3, \ldots$, and let $z_{ij}$ be $z_{11}$, $z_{12}$, $z_{21}$ or $z_{22}$. Then

\[
(I(z_{ij}))^k(W, W') = \frac{1}{k+1} \sum_{p=0}^{k} (w_{ij}^p) (w_{ij}'^{k-p}).
\]

Finally, we have the following consequence of the proof of this lemma.

**Corollary 8** (Corollary 19 in [L]). Let $k \geq 0$. We have the following orthogonality relations:

\[
\langle N(Z)^{-2-k} : t_{m,n}(Z^{-1}) : t_{m',n'}(Z^{-1}) : t_{p/2-p/2}(Z) \rangle = \begin{cases} 
\frac{1}{p+1} & \text{if } k = 0, l + l' = p/2, \\
m = n = -l \text{ and } m' = n' = -l' & \\
0 & \text{otherwise}.
\end{cases}
\]

### 3.2 Representations $(\omega^l_2, \mathcal{K})$, $(\omega^r_2, \mathcal{K})$ and Their Subrepresentations

Recall that $\mathcal{K}$ denotes the space of $\mathbb{C}$-valued functions on $\mathbb{H}_\mathbb{C}$ (possibly with singularities) which are holomorphic with respect to the complex variables $z_{11}, z_{12}, z_{21}, z_{22}$. We define two very similar actions of $GL(2, \mathbb{H}_\mathbb{C})$ on $\mathcal{K}$:

\[
\omega^l_2(h) : f(Z) \mapsto (\omega^l_2(h)f)(Z) = \frac{f((aZ + b)(cZ + d)^{-1})}{N(cZ + d)^2 \cdot N(a' - Zc')},
\]

\[
\omega^r_2(h) : f(Z) \mapsto (\omega^r_2(h)f)(Z) = \frac{f((aZ + b)(cZ + d)^{-1})}{N(cZ + d) \cdot N(a' - Zc')^2},
\]

where $h = (a \ b \ c \ d) \in GL(2, \mathbb{H}_\mathbb{C})$ and $h^{-1} = (a' \ b' \ c' \ d')$. (These actions coincide on $SL(2, \mathbb{H}_\mathbb{C})$.) Note that $\omega^l_2$ is the action $\omega_2$ in the notations of [L]. Differentiating, we obtain actions of $\mathfrak{gl}(2, \mathbb{H}_\mathbb{C})$ which preserve the spaces $\mathcal{K}$ and $\mathcal{K}^+$ defined by $[\mathfrak{m}]$ and $[\mathfrak{m}]^+$.

**Theorem 9** (Theorem 8 in [L]). The spaces

\[
\mathcal{K}^+ = \mathcal{C} - \text{span of } \{ t_{n,m}^l(Z) \cdot N(Z)^k; \ k \geq 0 \},
\]

\[
\mathcal{K}^- = \mathcal{C} - \text{span of } \{ t_{n,m}^l(Z) \cdot N(Z)^k; \ k \leq -(2l + 3) \},
\]

\[
I_2^- = \mathcal{C} - \text{span of } \{ t_{n,m}^l(Z) \cdot N(Z)^k; \ k \leq -2 \},
\]

\[
I_2^+ = \mathcal{C} - \text{span of } \{ t_{n,m}^l(Z) \cdot N(Z)^k; \ k \geq -(2l + 1) \},
\]

\[
J_2 = \mathcal{C} - \text{span of } \{ t_{n,m}^l(Z) \cdot N(Z)^k; \ -(2l + 1) \leq k \leq -2 \}
\]

and their sums are the only proper subspaces of $\mathcal{K}$ that are invariant under either $\omega^l_2$ or $\omega^r_2$ actions of $\mathfrak{gl}(2, \mathbb{H}_\mathbb{C})$ (see Figure 3).
The irreducible components of \((\varpi_2^l, \mathcal{H})\) and \((\varpi_2^r, \mathcal{H})\) are the subrepresentations

\[(\varpi_2^l, \mathcal{H}^+), \quad (\varpi_2^r, \mathcal{H})^+), \quad (\varpi_2^r, J_2)\]

and the quotients

\[(\varpi_2^l, \mathcal{H}/(I_2^+ \oplus \mathcal{H}^+)) = (\varpi_2^l, I_2^+/(\mathcal{H}^+ + J_2)), \quad (\varpi_2^r, \mathcal{H}/(\mathcal{H}^- + I_2^+)) = (\varpi_2^r, I_2^+/(\mathcal{H}^- + J_2))\]

where \(*\) stands for \(l\) or \(r\).

The quotient representations can be identified as follows:

**Proposition 10** (Proposition 10 in [1]). As representations of \(\mathfrak{gl}(2, \mathbb{H}_C)\),

\[
(\varpi_2^l, \mathcal{H}/(I_2^+ \oplus \mathcal{H}^+)) \simeq (\pi_1^0, \mathcal{H}^+), \quad (\varpi_2^l, \mathcal{H}/(\mathcal{H}^- + I_2^+)) \simeq (\pi_1^0, \mathcal{H}^-),
\]

\[
(\varpi_2^r, \mathcal{H}/(I_2^+ \oplus \mathcal{H}^+)) \simeq (\pi_0^r, \mathcal{H}^+), \quad (\varpi_2^r, \mathcal{H}/(\mathcal{H}^- + I_2^+)) \simeq (\pi_0^r, \mathcal{H}^-),
\]

in all cases the isomorphism map being

\[
\mathcal{H}^+ \ni \varphi(Z) \mapsto \frac{\deg \varphi(Z)}{N(Z)} \in \frac{\mathcal{K}/(\mathcal{H}^- + I_2^+)}{\mathcal{H}/(I_2^+ \oplus \mathcal{H}^+)}.
\]

The inverse of this isomorphism is given by

\[
\mathcal{K}/(\mathcal{H}^- + I_2^+) \ni f(Z) \mapsto \left\langle f(Z), \frac{1}{N(Z-W)} \right\rangle_Z = \frac{i}{2\pi^3} \int_{Z \in U(2)_R} \frac{f(Z) dV}{N(Z-W)} \in \mathcal{H}. \quad (17)
\]

We extend the \(\pi_1^0\) and \(\pi_0^r\) actions of \(GL(2, \mathbb{H}_C)\) on \(\widetilde{\mathcal{H}}\) to \(\widetilde{\mathcal{K}}\). Differentiating these actions, we obtain actions of \(gl(2, \mathbb{H}_C)\), which preserve \(\mathcal{K}, \mathcal{H}^+\) (and, of course, \(\mathcal{H}^-, \mathcal{H}^+\)). These actions are given by the same formulas as in Subsection 3.2 of [FL2]. Then we have a bilinear pairing between \((\varpi_2^l, \mathcal{H})\) and \((\pi_0^r, \mathcal{H})\) that formally looks the same as (17):

\[
\langle f_1, f_2 \rangle = \frac{i}{2\pi^3} \int_{Z \in U(2)_R} f_1(Z) \cdot f_2(Z) dV, \quad (18)
\]
except now the \( \mathfrak{gl}(2, \mathbb{C}) \)-actions on the first and second components are different: \( f_1 \in (\varpi^1, \mathcal{H}) \) and \( f_2 \in (\varpi^0, \mathcal{H}) \). This bilinear pairing is \( \mathfrak{gl}(2, \mathbb{C}) \)-invariant, non-degenerate and independent of the choice of \( R > 0 \). In other words, the representations \((\varpi^1, \mathcal{H})\) and \((\varpi^0, \mathcal{H})\) are dual to each other. Similarly, we have a bilinear pairing between \((\varpi^0, \mathcal{H})\) and \((\varpi^1, \mathcal{H})\) given by the same formula (18).

Now, let us restrict \( f_2 \) to \((\varpi^0, \mathcal{H}) \subset (\varpi^0, \mathcal{H})\). Then, by (11), this pairing annihilates all \( f_1 \in (\varpi^1, \mathcal{H}^+ \oplus J_2 \oplus \mathcal{H}^+) \). Hence this pairing descends to a pairing between \((\varpi^0, \mathcal{H})\) and \((\varpi^1, \mathcal{H})\). By Proposition 10, the latter representation is isomorphic to \((\varpi^0, \mathcal{H})\).

Thus we obtain the following expression for a \( \mathfrak{gl}(2, \mathbb{C}) \)-invariant bilinear pairing between \((\varpi^0, \mathcal{H})\) and \((\varpi^1, \mathcal{H})\):

\[
(\varphi_1, \varphi_2) = \frac{i}{2\pi^3} \int_{Z \in U(2)_R} (\deg \varphi_1)(Z) \cdot \varphi_2(Z) \frac{dV}{N(Z)}, \quad \varphi_1, \varphi_2 \in \mathcal{H}. \tag{19}
\]

(This pairing is independent of the choice of \( R > 0 \).) Comparing the orthogonality relations (11) and (11), we see that the pairings (11) and (11) coincide when \( \varphi_1 \in \mathcal{H}^+, \varphi_2 \in \mathcal{H}^- \) (but differ for other choices of \( \varphi_1 \) and \( \varphi_2 \)).

### 4 Conformal Four-Point Integrals and Magic Identities

In this section we introduce the conformal four-point integrals \( l^{(n)}(Z_1, Z_2; W_1, W_2) \) represented by the \( n \)-loop box diagrams and explain the “magic identities” due to [DHSS] that assert that integrals represented by diagrams with the same number of loops are, in fact, equal to each other. Then we introduce integral operators \( L^{(n)} \) on \( \mathcal{H}^+ \otimes \mathcal{H}^+ \) and state the main results of this article.

#### 4.1 Conformal Four-Point Integrals

In this subsection we explain how to construct the box diagrams and the corresponding conformal four-point integrals.

As in [DHSS], we use the coordinate space variable notation (as opposed to the momentum notation). With this choice of variables, the one- and two-loop box (or ladder) diagrams are represented as in Figure 4. The simplest conformal four-point integral is the one-loop box integral

\[
l^{(1)}(Z_1, Z_2; W_1, W_2) = \frac{i}{2\pi^3} \int_{T \in U(2)_R} \frac{dV}{N(Z_1 - T) \cdot N(Z_2 - T) \cdot N(W_1 - T) \cdot N(W_2 - T)}.
\]
Here, \( r > 0 \), \( Z_1, Z_2 \in \mathbb{D}_r^- \) and \( W_1, W_2 \in \mathbb{D}_r^+ \). Then we have the two-loop box integral

\[
-4\pi^6 \cdot l^{(2)}(Z_1, Z_2; W_1, W_2) = \int_{T_1 \in U^{(2)}_{r_1}} \int_{T_2 \in U^{(2)}_{r_2}} \frac{|Z_1 - W_1|^2 \cdot |T_1 - T_2|^{-2} dV_{T_1} dV_{T_2}}{|Z_1 - T_1|^2 \cdot |Z_2 - T_2|^2 \cdot |W_1 - T_1|^2 \cdot |W_1 - T_2|^2 \cdot |W_2 - T_2|^2},
\]

where we write \(|Z - W|^2\) for \( N(Z - W) \) in order to fit the formula on page. Here, \( r_1 > r_2 > 0 \), \( Z_1, Z_2 \in \mathbb{D}_{r_1}^-, W_1, W_2 \in \mathbb{D}_{r_2}^+ \). The factor \(|Z_1 - W_1|^2 = N(Z_1 - W_1)\) in the numerator is not involved in integration and gives \( l^{(2)} \) desired conformal properties (Lemma 17).

In general, one obtains the integral from the box diagram by building a rational function by writing a factor

\[
\begin{cases} 
N(Y_i - Y_j)^{-1} & \text{if there is a solid edge joining variables } Y_i \text{ and } Y_j; \\
N(Y_i - Y_j) & \text{if there is a dashed edge joining variables } Y_i \text{ and } Y_j,
\end{cases}
\]

and then integrating over the solid vertices. The issue of contours of integration (and, in particular, their relative position) will be addressed at the end of this subsection.

The box diagrams are obtained by starting with the one-loop box diagram (Figure 4) and adding the so-called “slingshots”, as explained in [DHSS]. Figures 5 and 6 show the two possible results of attaching a slingshot to the one-loop diagram; these are called the two-loop box diagrams. Then Figures 7 and 8 show two different results of attaching a slingshot to the two-loop box diagrams; these are called the three-loop box diagrams. Figure 11 shows a particular example of an \( n \)-loop box diagram called the \( n \)-loop ladder diagram. The reason for the “box”, “ladder” and “loop” terminology becomes apparent when one switches to the momentum variables, see Figure 9 and more figures are given in [DHSS].

In order to specify the cycles of integration, we introduce a partial ordering on the variables in each \( n \)-loop box diagram \( d^{(n)} \). For the one-loop box diagram (Figure 4) the relations are

\[ W_1, W_2 \prec T \prec Z_1, Z_2. \]
Figure 7: Attaching a slingshot to a two-loop box diagram

Figure 8: Another way of attaching a slingshot to a two-loop box diagram

Figure 9: One-, two- and n-loop box or ladder diagrams in momentum variables
Suppose that an \( n \)-loop box diagram \( d^{(n)} \) is obtained from an \( (n-1) \)-loop diagram \( d^{(n-1)} \) by adding a slingshot. Then \( d^{(n)} \) will have one new relation for each solid edge of the slingshot, plus those implied by the transitivity property. Suppose, by induction, that the partial ordering for the variables in \( d^{(n-1)} \) are already specified. We label the solid vertices in \( d^{(n-1)} \) as \( T_1, \ldots, T_{n-1} \). There are exactly four ways of attaching a slingshot to \( d^{(n-1)} \) – so that one of \( Z_1, Z_2, W_1 \) or \( W_2 \) becomes a solid vertex and gets relabeled as \( T_n \).

- If \( d^{(n)} \) is obtained from \( d^{(n-1)} \) by adding the slingshot so that \( Z_1 \) becomes a solid vertex, the relations in \( d^{(n-1)} \) carry over to \( d^{(n)} \) with \( Z_1 \) replaced with \( T_n \). Then we get new relations
  \[
  W_2 \prec T_n \prec Z_1, Z_2
  \]
  (plus those implied by the transitivity property).

- If \( d^{(n)} \) is obtained from \( d^{(n-1)} \) by adding the slingshot so that \( Z_2 \) becomes a solid vertex, the relations in \( d^{(n-1)} \) carry over to \( d^{(n)} \) with \( Z_2 \) replaced with \( T_n \). Then we get new relations
  \[
  W_1 \prec T_n \prec Z_1, Z_2
  \]
  (plus those implied by the transitivity property).

- If \( d^{(n)} \) is obtained from \( d^{(n-1)} \) by adding the slingshot so that \( W_1 \) becomes a solid vertex, the relations in \( d^{(n-1)} \) carry over to \( d^{(n)} \) with \( W_1 \) replaced with \( T_n \). Then we get new relations
  \[
  W_1, W_2 \prec T_n \prec Z_2
  \]
  (plus those implied by the transitivity property).

- If \( d^{(n)} \) is obtained from \( d^{(n-1)} \) by adding the slingshot so that \( W_2 \) becomes a solid vertex, the relations in \( d^{(n-1)} \) carry over to \( d^{(n)} \) with \( W_2 \) replaced with \( T_n \). Then we get new relations
  \[
  W_1, W_2 \prec T_n \prec Z_1
  \]
  (plus those implied by the transitivity property).

This completely defines the partial ordering on the variables in \( d^{(n)} \). We choose real numbers \( r_1, \ldots, r_n > 0 \) such that \( r_i < r_j \) whenever \( T_i \prec T_j \) (it is easy to check that such a choice is always possible). Then each \( T_k \) gets integrated over \( U(2)_{r_k} \). Finally,

\[
Z_i \in \mathbb{D}_{r_{\max,i}}^-, \quad \text{where } r_{\max,i} = \max\{r_k; T_k \prec Z_i\}, \quad i = 1, 2; \quad (20)
\]

\[
W_i \in \mathbb{D}_{r_{\min,i}}^+, \quad \text{where } r_{\min,i} = \min\{r_k; W_i \prec T_k\}, \quad i = 1, 2. \quad (21)
\]

If desired, by Corollary 90 in [2], the integrals over various \( U(2)_{r_k} \)’s can be replaced by integrals over the Minkowski space \( \mathbb{M} \) via an appropriate “Cayley transform”. This means that these integrals are what the physicists call “the off-shell Minkowski integrals”.

### 4.2 Magic Identities

In this subsection we state the so-called “magic identities” due to J. M. Drummond, J. Henn, V. A. Smirnov and E. Sokatchev [DHSS]. Informally, they assert that all conformal four-point box integrals obtained by adding the same number of slingshots to the one-loop integral are equal. In other words, only the number of slingshots matters and not how they are attached.
Using the bilinear pairing (19), we obtain integral operators as shown in Figure 10. Finally, they apply induction on the number of loops or slingshots. These identities can be represented by the box diagrams, such as their kernels:

\[ l^{(n)}(Z_1, Z_2; W_1, W_2) = \tilde{l}^{(n)}(Z_1, Z_2; W_1, W_2). \]

In particular, we can parametrize the conformal four-point integrals by the number of loops in the diagrams and choose a single representative from the set of all \( n \)-loop diagrams, such as the \( n \)-loop ladder diagram (Figures 9 and 11).

The original paper [DHSS] gives a proof for the Euclidean metric case and claims that the result is also true for the Minkowski metric case. In the Euclidean case, the box integrals are produced by making all variables belong to \( \mathbb{H} \) and replacing all cycles of integration by \( \mathbb{H} \). Then \( N(X - Y) \) is just the square of the Euclidean distance between \( X \) and \( Y \). There are no convergence issues whatsoever. On the other hand, the Minkowski case (which is the case we consider) is much more subtle. In order to deal with convergence issues, we must consider the so-called “off-shell Minkowski integrals” or make the cycles of integration to be various \( U(2)_r \)'s. Then the relative position of cycles becomes very important. As can be seen in the course of proof of Theorem 14, choosing the “wrong” cycles typically results in integral being zero.

The proof of Theorem 11 given in [DHSS] can be outlined as follows. First, they prove a symmetry relationship for the two-loop integrals represented by the two-loop diagram in Figure 4:

\[ l^{(2)}(Z_1, Z_2; W_1, W_2) = l^{(2)}(Z_2, Z_1; W_1, W_2); \]

this is done by direct computation. Then they prove the magic identity for the integrals represented by the three-loop diagrams in Figures 7 and 8:

\[ l^{(3)}(Z_1, Z_2; W_1, W_2) = \tilde{l}^{(3)}(Z_1, Z_2; W_1, W_2); \]

this is also done by direct computation. These identities can be represented by the box diagrams, as shown in Figure 10. Finally, they apply induction on the number of loops or slingshots.

4.3 Statement of the Main Result

Using the bilinear pairing (19), we obtain integral operators \( L^{(n)} \) on \( (\pi^0_1, \mathcal{H}^+) \otimes (\pi^0_r, \mathcal{H}^+) \) that have the conformal integrals \( l^{(n)} \) as their kernels:

\[
L^{(n)}(\varphi_1 \otimes \varphi_2)(W_1, W_2) = \left( \frac{i}{2\pi^3} \right)^2 \int_{z_1 \in \mathbb{U}(2) R_1} \int_{z_2 \in \mathbb{U}(2) R_2} l^{(n)}(Z_1, Z_2; W_1, W_2) \cdot (\tilde{\deg}_{Z_1} \varphi_1)(Z_1) \cdot (\tilde{\deg}_{Z_2} \varphi_2)(Z_2) \frac{dV_1}{N(Z_1)} \frac{dV_2}{N(Z_2)},
\]

where \( \varphi_1, \varphi_2 \in \mathcal{H}^+, R_1 > r_{\text{max},1}, R_2 > r_{\text{max},2}, W_1 \in \mathbb{D}^+_r, W_2 \in \mathbb{D}^+_r \) (recall that \( r_{\text{max},i} \) and \( r_{\text{min},i} \) are defined in (20) and (21)). First, we state a preliminary version of the main result.
Proposition 12. For each $\varphi_1, \varphi_2 \in \mathcal{H}^+$, the function $L^{(n)}(\varphi_1 \otimes \varphi_2)(W_1, W_2)$ is polynomial and harmonic in each variable. In other words, $L^{(n)}(\varphi_1 \otimes \varphi_2)(W_1, W_2) \in \mathcal{H}^+ \otimes \mathcal{H}^+$. Moreover, the operator

$$L^{(n)} : (\pi^0_1, \mathcal{H}^+) \otimes (\pi^0_r, \mathcal{H}^+) \to (\pi^0_1, \mathcal{H}^+) \otimes (\pi^0_r, \mathcal{H}^+)$$

is $\mathfrak{gl}(2, \mathbb{H}_\mathbb{C})$-equivariant.

Our goal is to compute the actions of these integral operators $L^{(n)}$ on $(\pi^0_1, \mathcal{H}^+) \otimes (\pi^0_r, \mathcal{H}^+)$ and to prove the magic identities for $L^{(n)}$.

The decomposition of $(\pi^0_1, \mathcal{H}^+) \otimes (\pi^0_r, \mathcal{H}^+)$ into irreducible components is well known. This was done in a greater generality, for example, in [JV]. We provide a summary of this result following [FL1, L]. Let $k = 1, 2, 3, \ldots$, and denote by $\mathbb{C}^{k \times k}$ the space of complex $k \times k$ matrices.

Then $\mathcal{K} \otimes \mathbb{C}^{k \times k}$ can be thought of as the space of holomorphic functions on $\mathbb{H}_\mathbb{C}$ (possibly with singularities) with values in $\mathbb{C}^{k \times k}$. Recall the actions $\rho_k$ of $GL(2, \mathbb{H}_\mathbb{C})$ on $\mathcal{K} \otimes \mathbb{C}^{k \times k}$ described by equation (60) in [FL1]:

$$\rho_k(h) : F(Z) \mapsto (\rho_k(h)F)(Z) = \frac{\tau_{k+1}(cZ + d)^{-1}}{N(cZ + d)} \cdot F((aZ + b)(cZ + d)^{-1}) \cdot \frac{\tau_{k+1}(a' - Zc')^{-1}}{N(a' - Zc')},$$

where $h = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in GL(2, \mathbb{H}_\mathbb{C})$, $h^{-1} = (\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix})$, expressions $cZ + d$ and $a' - Zc'$ are regarded as elements of $\mathbb{H}_\mathbb{C}$, and $\tau : \mathbb{H}_\mathbb{C} \to Aut(\mathbb{C}^{l+1}) \subset \mathbb{C}^{(2l+1) \times (2l+1)}$ is the irreducible $(2l+1)$-dimensional representation of $\mathbb{H}_\mathbb{C}$ described in Subsection 2.2.

Differentiating this action, we obtain an action of $\mathfrak{gl}(2, \mathbb{H}_\mathbb{C})$ which preserves $\mathcal{K} \otimes \mathbb{C}^{k \times k}$ and $\mathcal{K}^+ \otimes \mathbb{C}^{k \times k}$. As a special case of Proposition 4.7 in [JV] (see also the discussion preceding the proposition and references therein), we have:

**Theorem 13.** The representations $(\rho_k, \mathcal{K}^+ \otimes \mathbb{C}^{k \times k})$, $k = 1, 2, 3, \ldots$, of $\mathfrak{sl}(2, \mathbb{H}_\mathbb{C})$ are irreducible, pairwise non-isomorphic. They possess inner products which make them unitary representations of the real form $\mathfrak{su}(2, 2)$ of $\mathfrak{sl}(2, \mathbb{H}_\mathbb{C})$.

According to [JV], we have the following decomposition of the tensor product $(\pi^0_1, \mathcal{H}^+) \otimes (\pi^0_r, \mathcal{H}^+)$ into irreducible subrepresentations of $\mathfrak{gl}(2, \mathbb{H}_\mathbb{C})$:

$$(\pi^0_1, \mathcal{H}^+) \otimes (\pi^0_r, \mathcal{H}^+) \simeq \bigoplus_{k=1}^{\infty} (\rho_k, \mathcal{K}^+ \otimes \mathbb{C}^{k \times k})$$

(see also Subsection 5.1 in [FL1]). We outline the proof of this statement. First of all, by Lemma 10 in [FL1], the tensor product $(\pi^0_1, \mathcal{H}^+) \otimes (\pi^0_r, \mathcal{H}^+)$ contains each $(\rho_k, \mathcal{K}^+ \otimes \mathbb{C}^{k \times k})$ with

$$(\rho_1, \mathcal{K}^+)$$

generated by $1 \otimes 1$

and

$$(\rho_k, \mathcal{K}^+ \otimes \mathbb{C}^{k \times k})$$

generated by $(z_{ij} - z'_{ij})^{k-1}$, $k \geq 2$.

Then one checks that the direct sum $\bigoplus_{k=1}^{\infty} (\rho_k, \mathcal{K}^+ \otimes \mathbb{C}^{k \times k})$ exhausts all of $(\pi^0_1, \mathcal{H}^+) \otimes (\pi^0_r, \mathcal{H}^+)$ by comparing the two sides as representations of $U(2) \times U(2)$ or $\mathfrak{u}(2) \times \mathfrak{u}(2)$.

In order to state the full version of the main result, we introduce coefficients $a^k(n, p)$, where $k = 0, 1, 2, \ldots$ and $0 \leq p \leq k$, that are defined by the following recursive relations:

$$a^k(1, p) = \frac{1}{k + 1}, \quad p = 0, 1, \ldots, k,$$

and

$$a^k(n + 1, p) = \sum_{q=p}^{k} \frac{1}{q + 1} \cdot a^k(n, q).$$
Theorem 14. The operator $L^{(n)}$ associated to any $n$-loop box diagram maps $\mathcal{H}^+ \otimes \mathcal{H}^+$ into $\mathcal{H}^+ \otimes \mathcal{H}^+$, and the map

$$L^{(n)} : (\pi^0_\nu, \mathcal{H}^+) \otimes (\pi^0_\nu, \mathcal{H}^+) \rightarrow (\pi^0_\nu, \mathcal{H}^+) \otimes (\pi^0_\nu, \mathcal{H}^+)$$

(26)
is $\mathfrak{gl}(2, \mathbb{H}_C)$-equivariant. If $x \in (\pi^0_\nu, \mathcal{H}^+) \otimes (\pi^0_\nu, \mathcal{H}^+)$ belongs to an irreducible component isomorphic to $(\rho_k, \mathcal{H}^+ \otimes \mathbb{C}^{k \times k})$ in the decomposition (23), then

$$L^{(n)}(x) = \mu^{(n)}_k x, \quad \text{where} \quad \mu^{(n)}_k = \sum_{p=0}^{k-1} (-1)^{k+p+1} \cdot a^{k-1}(n, p) \cdot \binom{k-1}{p}.$$ 

In particular, we obtain the magic identities for operators $L^{(n)}$:

Corollary 15. Let $L^{(n)}$ and $\bar{L}^{(n)}$ be two integral operators corresponding to any two $n$-loop box diagrams, then $L^{(n)} = \bar{L}^{(n)}$, as operators on $\mathcal{H}^+ \otimes \mathcal{H}^+$.

Remark 16. If one can prove that each $n$-loop box integral $I^{(n)}(Z_1, Z_2; W_1, W_2)$ is harmonic in each variable $Z_1$, $Z_2$, $W_1$ and $W_2$, then it is easy to show that Theorem 14 implies the magic identities, as stated in Theorem 17.

4.4 Example: The Case $n = 2$

In this subsection we compute the coefficients $\mu^{(2)}_k$ and show that Theorem 26 in [L] is a special case of Theorem 14 when $n = 2$. We have:

$$a^{k}(2, p) = \sum_{q=p}^{k} \frac{1}{q+1} \cdot a^{k}(1, q) = \frac{1}{k+1} \sum_{q=p}^{k} \frac{1}{q+1}$$

and

$$\mu^{(2)}_k = \sum_{p=0}^{k-1} (-1)^{k+p+1} \cdot a^{k-1}(2, p) \cdot \binom{k-1}{p} = \frac{1}{k} \sum_{p=0}^{k-1} (-1)^{k+p+1} \cdot \binom{k-1}{p} \sum_{q=p}^{k-1} \frac{1}{q+1} = \frac{(-1)^{k+1}}{k} \sum_{q=0}^{k-1} \frac{1}{q+1} \sum_{p=0}^{q} (-1)^p \binom{k-1}{p}.$$ 

If $k = 1$, we obtain $\mu^{(2)}_1 = 1$. So, assume $k \geq 2$. Using an identity

$$\sum_{p=0}^{q} (-1)^p \binom{k}{p} = (-1)^q \binom{k-1}{q},$$

which can be easily proved by induction (see formula 0.15(4) in [GR]), we obtain

$$\mu^{(2)}_k = \frac{(-1)^{k+1}}{k} \sum_{q=0}^{k-1} \frac{(-1)^q}{q+1} \binom{k-2}{q} = \frac{(-1)^{k+1}}{k(k-1)} \sum_{q=0}^{k-1} (-1)^q \cdot \binom{k-1}{q+1} = \frac{(-1)^{k+1}}{k(k-1)}.$$ 

This shows that

$$\mu^{(2)}_k = \begin{cases} 
1 & \text{if } k = 1; \\
\frac{(-1)^{k+1}}{k(k-1)} & \text{if } k \geq 2;
\end{cases}$$

and that Theorem 26 in [L] is a special case of Theorem 14.
5 Proof of Theorem 14

5.1 Preliminary Lemmas

In this subsection we prove two lemmas that are part of our proof of Theorem 14. The first lemma describes an important conformal property of four-point box integrals.

Lemma 17. For each \( h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{H}_\mathbb{C}) \) sufficiently close to the identity, we have:

\[
l^{(n)}(\tilde{Z}_1, \tilde{Z}_2; \tilde{W}_1, \tilde{W}_2) = N(a' - Z_1c') \cdot N(cZ_2 + d) \cdot N(cW_1 + d) \cdot N(a' - W_2c') \cdot l^{(n)}(Z_1, Z_2; W_1, W_2),
\]

where \( h^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \tilde{Z}_i = (az_i + b)(cZ_i + d)^{-1} \) and \( \tilde{W}_i = (aW_i + b)(cW_i + d)^{-1}, i = 1, 2. \)

Proof. The proof is by induction on \( n; \) for \( n = 1 \) and \( n = 2 \) this is Lemma 14 in [L]. For concreteness, let us assume that the last slingshot is attached to an \((n-1)\)-loop box diagram \( d^{(n-1)} \) so that \( Z_1 \) becomes a solid vertex and gets relabeled as \( T_n \) (the other cases are similar).

Then

\[
l^{(n)}(Z_1, Z_2; W_1, W_2) = \frac{i}{2\pi^3} \int_{T_n \in U(2)_{T_n}} \frac{N(Z_2 - W_2) \cdot l^{(n-1)}(T_n, Z_2; W_1, W_2)}{N(Z_1 - T_n) \cdot N(Z_2 - T_n) \cdot N(W_2 - T_n)} dV_{T_n},
\]

where \( l^{(n-1)}(Z_1, Z_2; W_1, W_2) \) is the conformal four-point integral corresponding to the \((n-1)\)-loop diagram \( d^{(n-1)} \). By induction, we assume that the result holds for \( l^{(n-1)}(Z_1, Z_2; W_1, W_2) \). Using Lemmas 10, 61 from [PL1] and letting \( \tilde{T}_n = (aT_n + b)(cT_n + d)^{-1} \), we obtain

\[
l^{(n)}(\tilde{Z}_1, \tilde{Z}_2; \tilde{W}_1, \tilde{W}_2) = \frac{i}{2\pi^3} \int_{\tilde{T}_n \in U(2)_{\tilde{T}_n}} \frac{N(\tilde{Z}_2 - \tilde{W}_2) \cdot l^{(n-1)}(\tilde{T}_n, \tilde{Z}_2; \tilde{W}_1, \tilde{W}_2)}{N(\tilde{Z}_1 - \tilde{T}_n) \cdot N(\tilde{Z}_2 - \tilde{T}_n) \cdot N(\tilde{W}_2 - \tilde{T}_n)} dV_{\tilde{T}_n}
\]

\[
= N(a' - Z_1c') \cdot N(cZ_2 + d) \cdot N(cW_1 + d) \cdot N(a' - W_2c') \]

\[
\times \frac{i}{2\pi^3} \int_{\tilde{T}_n \in U(2)_{\tilde{T}_n}} \frac{N(Z_2 - W_2) \cdot N(cT_n + d)^2 \cdot N(a' - T_n c')^2}{N(Z_1 - T_n) \cdot N(Z_2 - T_n) \cdot N(W_2 - T_n)} \cdot l^{(n-1)}(T_n, Z_2; W_1, W_2) dV_{\tilde{T}_n}
\]

\[
= N(a' - Z_1c') \cdot N(cZ_2 + d) \cdot N(cW_1 + d) \cdot N(a' - W_2c') \cdot l^{(n)}(Z_1, Z_2; W_1, W_2),
\]

where we are allowed to replace integration over \( \tilde{T}_n \in U(2)_{\tilde{T}_n} \) with \( T_n \in U(2)_{T_n} \) since the integrand is a closed differential form and \( h \) is sufficiently close to the identity.

The second lemma concerns a set of generators of \( \mathcal{H}^+ \otimes \mathcal{H}^+ \).

Lemma 18. As a representation of \( \mathfrak{gl}(2, \mathbb{H}_\mathbb{C}) \), \( (\pi_0^0 \otimes \pi_0^0, \mathcal{H}^+ \otimes \mathcal{H}^+) \) is generated by \( 1 \otimes (z_{11}^1)^k, k = 0, 1, 2, \ldots \). It can also be generated by \( (z_{11}^1)^k \otimes 1, k = 0, 1, 2, \ldots \).

Proof. Recall that \( (z_{11}^1 - z_{11}^1)^k \) generates the irreducible component \( (\rho_{k+1}, \mathcal{H}^+ \otimes \mathcal{C}^{(k+1) \times (k+1)}) \) of \( \mathcal{H}^+ \otimes \mathcal{H}^+ \). We compute the inner product in \( \mathcal{H}^+ \otimes \mathcal{H}^+ \) induced by (4):

\[
\langle 1 \otimes (z_{11}^1)^k, (z_{11} - z_{11}^1)^k \rangle_{\text{inn. prod.}} = (-1)^k \langle 1 \otimes (z_{11}^1)^k, 1 \otimes (z_{11}^1)^k \rangle_{\text{inn. prod.}} = (-1)^k,
\]

by (8) and (10). Since this product is not zero, it follows that the subrepresentation of \( \mathcal{H}^+ \otimes \mathcal{H}^+ \) generated by \( 1 \otimes (z_{11}^1)^k \) contains \( (\rho_{k+1}, \mathcal{H}^+ \otimes \mathcal{C}^{(k+1) \times (k+1)}) \). Therefore, each irreducible component of \( \mathcal{H}^+ \otimes \mathcal{H}^+ \) is contained in the subrepresentation of \( \mathcal{H}^+ \otimes \mathcal{H}^+ \) generated by \( 1 \otimes (z_{11}^1)^k, k = 0, 1, 2, \ldots \). □
Lemma 19. The operator $L^{(n)}(Z_1, Z_2; W_1, W_2)$ is harmonic in $Z_2$, the pairings (3) and (19) agree, and we can rewrite $L^{(n)}$ as

$$L^{(n)}(\varphi_1 \otimes \varphi_2)(W_1, W_2) = \frac{i}{4\pi^5} \int_{Z_1 \in U(2) R_1} \int_{Z_2 \in s^h_2} l^{(n)}(Z_1, Z_2; W_1, W_2) \cdot (\tilde{\deg}_{Z_1} \varphi_1)(Z_1) \cdot (\tilde{\deg}_{Z_2} \varphi_2)(Z_2) \frac{dV_1}{N(Z_1)} \frac{dS_2}{R_2},$$

where $\varphi_1, \varphi_2 \in \mathcal{H}^+$, $R_1 > r_{\text{max},1}$, $R_2 > r_{\text{max},2}$, $W_1 \in \mathbb{D}_{r_{\text{min},1}}$, $W_2 \in \mathbb{D}_{r_{\text{min},2}}$, as before.

5.2 The Case of Ladder Diagrams

In this subsection we prove Theorem 14 in the special case of ladder diagrams. We label the variables as in Figure 11. Since the function $l^{(n)}(Z_1, Z_2; W_1, W_2)$ is harmonic in $Z_2$, the pairings (3) and (19) agree, and we can rewrite $L^{(n)}$ as

$$L^{(n)}(\varphi_1 \otimes \varphi_2)(W_1, W_2) = \frac{i}{4\pi^5} \int_{Z_1 \in U(2) R_1} \int_{Z_2 \in s^h_2} l^{(n)}(Z_1, Z_2; W_1, W_2) \cdot (\tilde{\deg}_{Z_1} \varphi_1)(Z_1) \cdot (\tilde{\deg}_{Z_2} \varphi_2)(Z_2) \frac{dV_1}{N(Z_1)} \frac{dS_2}{R_2},$$

where $\varphi_1, \varphi_2 \in \mathcal{H}^+$, $R_1 > r_{\text{max},1}$, $R_2 > r_{\text{max},2}$, $W_1 \in \mathbb{D}_{r_{\text{min},1}}$, $W_2 \in \mathbb{D}_{r_{\text{min},2}}$, as before.

Lemma 19. The operator $L^{(n)}$ associated to the $n$-loop ladder diagram sends each $1 \otimes (z^{11}_1)^k$, $k = 0, 1, 2, \ldots$, into

$$\sum_{p=0}^{k} a^k(n, p) \cdot (w^{11}_1)^{k-p} \cdot (w^{11}_1)^p,$$

where the coefficients $a^k(n, p)$, $0 \leq p \leq k$, can be computed from the recursive relations (24) and (25). In particular, $L^{(n)}(1 \otimes (z^{11}_1)^k)$ lies in $\mathcal{H}^+ \otimes \mathcal{H}^+$.

Proof. One part of evaluating $L^{(n)}(1 \otimes (z^{11}_1)^k)$ is integrating over $Z_1 \in U(2) R_1$. First, we determine the effect of doing that. Thus we integrate

$$\frac{i}{2\pi^3} \int_{Z_1 \in U(2) R_1} \frac{N(Z_1 - W_1)^{n-1}}{N(Z_1 - T_1) \cdot N(Z_1 - T_2) \cdots \cdot N(Z_1 - T_n)} \frac{dV_1}{N(Z_1)},$$

and we expand each $N(Z_1 - T_j)^{-1}$ as in (13):

$$\frac{1}{N(Z_1 - T_j)} = N(Z_1)^{-1} \cdot \sum_{l, m, n} \ell^l_{m,n}(T_j) \cdot \ell^l_{m,n}(Z_1^{-1}), \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \quad m, n = -l, -l + 1, \ldots, l.$$

Therefore,

$$\frac{1}{N(Z_1 - T_1) \cdot N(Z_1 - T_2) \cdots \cdot N(Z_1 - T_n)} = \frac{1}{N(Z_1)^n} + \text{lower degree terms in } Z_1.$$

On the other hand,

$$N(Z_1 - W_1)^{n-1} = N(Z_1)^{n-1} + \text{lower degree terms in } Z_1.$$
Comparing this with the orthogonality relations (11), we see that the integral (28) is 1.

Thus we end up integrating \((z'^1_{11})^k\) against something that can be described by a reduced diagram in Figure 12. When we integrate out the operator \(L\), the result of integration over \(T\) is given by (29).

Then we integrate out the \(T_1\) variable:

\[
\frac{i}{2\pi^3} \int_{T_1 \in U(2)_{r_1}} \frac{(t_{11})^k \, dV}{N(W_1 - T_1) \cdot N(T_1 - T_2)}
\]

and, by Lemma 7, we get

\[
\sum_{p=0}^{k} a^n(1, p) \cdot (w_{11})^{k-p} \cdot (t'_{11})^p
\]

with each \(a^k(1, p)\) given by (24). Then we integrate the result against \(\frac{1}{N(W_1 - T_2): N(T_2 - T_3)}\) and so on, until we integrate out the \(T_n\) variable. The recursive relation (25) follows from Lemma 7 since the result of integration over \(T_{n+1}\) has to be a linear combination of \((w_{11})^{k-p} \cdot (w'_{11})^p\)'s and the contribution to each term \((w_{11})^{k-p} \cdot (w'_{11})^p\) comes precisely from the terms \((w_{11})^{k-q} \cdot (t'_{11})^q\), \(p \leq q \leq k\), of the previous integration with weights \((q + 1)^{-1}\).

As a corollary of the proof, we also obtain:

**Corollary 20.** The operators \(L^{(n)}\) associated to the \(n\)-loop ladder diagrams satisfy the following recursive relation:

\[
L^{(n)}(1 \otimes (z'^1_{11})^k) = \frac{1}{k+1} \sum_{p=0}^{k} (w_{11})^{k-p} \cdot L^{(n-1)}(1 \otimes (z'^1_{11})^p).
\]

The proof is as follows. Theorem 14 would have been much easier if we knew in advance that the operator \(L^{(n)}\) is \(\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})\)-equivariant. To deal with this issue, we introduce a closely related integral operator for which \(\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})\)-equivariance is much easier to see. Following notations from [11], we let

\[
\bar{L}^{(n)} : (\mathbb{C}_L, \mathcal{H}) \otimes (\pi^0_r, \mathcal{H}^+) \to (\pi^0_r, \mathcal{H}^+) \otimes (\pi^0_r, \mathcal{H}^+),
\]

\[
\bar{L}^{(n)}(f \otimes \varphi)(W_1, W_2) = \frac{i}{4\pi^3} \int_{Z_1 \in U(2), Z_2 \in R_2} l^{(n)}(Z_1, Z_2; W_1, W_2) \cdot f(Z_1) \cdot (\text{deg}_{Z_2} \varphi)(Z_2) \, dV_1 \frac{dS_2}{R_2},
\]

where \(f \in \mathcal{H}, \varphi \in \mathcal{H}^+, \ R_1 > r_{\text{max}, 1}, \ R_2 > r_{\text{max}, 2}, \ W_1 \in D^{r_{\text{min}, 1}}_+, \ W_2 \in D^{r_{\text{min}, 2}}_+\), as before. The \(\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})\)-equivariance of this operator follows from the \(\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})\)-invariance of the bilinear pairings (7), (18) and Lemma 17. Clearly, we have

\[
L^{(n)} = \bar{L}^{(n)} \circ (N(Z_1)^{-1} \cdot \text{deg}_{Z_1}). \tag{29}
\]
Lemma 21. The operator $\hat{L}^{(n)}$ annihilates $I_2^- \otimes \mathcal{H}^+$. 

Proof. Consider a pure tensor $f \otimes \varphi$ with $f \in I_2^-$ and $\varphi \in \mathcal{H}^+$. Then $f(Z_1)$ is a sum of terms $f_l(Z_1) \cdot N(Z_1)^{-l}$ with $f_l \in \mathcal{H}^+$ and $l \geq 2$. Without loss of generality we can assume that each $f_l$ is homogeneous. As in the proof of Lemma 19 we observe that, as a part of evaluating $\hat{L}^{(n)}(f_l(Z_1) \cdot N(Z_1)^{-l} \cdot \varphi(Z_2))$, one needs to integrate over $Z_1 \in U(2)_{R_l}$:

$$
\frac{i}{2\pi^3} \int_{Z_1 \in U(2)_{R_l}} \frac{f_l(Z_1) \cdot N(Z_1)^{-l} \cdot N(Z_1 - W_1)^{n-1}}{N(Z_1 - T_1) \cdot N(Z_1 - T_2) \cdots \cdot N(Z_1 - T_n)} dV_1.
$$

(30)

Expanding each $N(Z_1 - T_j)^{-1}$ as in \([13]\), we get

$$
\frac{N(Z_1)^{-l}}{N(Z_1 - T_1) \cdot N(Z_1 - T_2) \cdots \cdot N(Z_1 - T_n)} = \frac{1}{N(Z_1)^{n+l}} + \text{lower degree terms in } Z_1.
$$

On the other hand,

$$
f_l(Z_1) \cdot N(Z_1 - W_1)^{n-1} = f_l(Z_1) \cdot N(Z_1)^{n-1} + \text{lower degree terms in } Z_1.
$$

Comparing this with orthogonality relations \([11]\), since $l \geq 2$, we see that the integral (30) and hence $\hat{L}^{(n)}(f_l(Z_1) \cdot N(Z_1)^{-l} \otimes \varphi(Z_2))$ are 0.

Let $\mathfrak{W} \subset \mathcal{K} \otimes \mathcal{H}^+$ denote the subrepresentation of $(\pi^l_2, \mathcal{K}) \otimes (\pi_0^0, \mathcal{H}^+)$ generated by

$$
\{ N(Z)^{-1} \cdot (z_{ij}^1)^k; \ k = 0, 1, 2, 3, \ldots \}.
$$

Thus we have a $\mathfrak{gl}(2, \mathbb{H}_C)$-equivariant map

$$
\hat{L}^{(n)} : (\pi^l_2 \otimes \pi_0^0, \mathfrak{W}) \rightarrow (\pi^0_1, \mathcal{H}^+) \otimes (\pi_0^0, \mathcal{H}^+) =: \mathfrak{W}^0 \subset \mathfrak{W}.
$$

Lemma 22. The operator $\hat{L}^{(n)}$ annihilates $\mathfrak{W} \cap (\mathcal{K}^+ \otimes \mathcal{H}^+)$. 

Proof. We proceed as in the proof of Lemma 28 in \([14]\). Observe that the operator $\hat{L}^{(n)}$ increases the total degree of an element of $\mathcal{K} \otimes \mathcal{H}^+$ by 2. Now, suppose that there exists an element $x \in \mathfrak{W} \cap (\mathcal{K}^+ \otimes \mathcal{H}^+)$ such that $\hat{L}^{(n)}(x) \neq 0$. Since $\hat{L}^{(n)}$ is $\mathfrak{gl}(2, \mathbb{H}_C)$-equivariant, without loss of generality we can assume that $\hat{L}^{(n)}(x)$ belongs to one of the irreducible components of $(\pi^0_1, \mathcal{H}^+) \otimes (\pi_0^0, \mathcal{H}^+)$. Furthermore, we may assume that

$$
\hat{L}^{(n)}(x) = (z_{ij} - z_0'_{ij})^k \quad \text{for some } x \in \mathfrak{W} \cap (\mathcal{K}^+ \otimes \mathcal{H}^+), \quad k = 0, 1, 2, \ldots .
$$

Since $(z_{ij} - z_0'_{ij})^k$ is homogeneous of degree $k$, only the homogeneous component $x'$ of degree $k - 2$ of $x$ contributes anything to $\hat{L}^{(n)}(x)$, and $x' \in \mathcal{K}^+ \otimes \mathcal{H}^+$.

Now, let us regard $\hat{L}^{(n)}$ as a $U(2) \times U(2)$ equivariant map $(\pi^l_2, \mathcal{K}^+) \otimes (\pi_0^0, \mathcal{H}^+) \rightarrow (\pi^0_1, \mathcal{K}^+) \otimes (\pi_0^0, \mathcal{H}^+)$. We have:

$$
\hat{L}^{(n)}(x') = (z_{ij} - z_0'_{ij})^k \in V_2^l \otimes V_2^l.
$$

Since the degree of $x'$ is $k - 2$,

$$
x' \in \bigoplus_{2l + 2l' = k - 2, \ p, l, l' \geq 0} N(Z)^p \cdot (V_l \otimes V_l) \otimes (V_{l'} \otimes V_{l'}).
$$

But $V_l \otimes V_{l'}$ does not contain $V_{\frac{k}{2}}$ unless $l + l' \geq k/2$, which produces a contradiction.
Combining Lemmas 21 and 22 we see that \( \tilde{L}^{(n)} \) descends to a well-defined \( \mathfrak{gl}(2, \mathbb{H}_c) \)-equivariant quotient map

\[
\mathcal{V} / \mathcal{V} \cap ((I_2^l \oplus \mathcal{K}^+ \otimes \mathcal{H}^+) \to \mathcal{H}^+ \otimes \mathcal{H}^+.
\]

(31)

Clearly, this quotient space is a \( \mathfrak{gl}(2, \mathbb{H}_c) \)-invariant subspace of \( (\mathcal{K}^\prime \lbrack (I_2^l \oplus \mathcal{K}^+) \otimes \mathcal{H}^+) \). By Proposition 10 and Lemma 13 we have the following isomorphisms of representations of \( \mathfrak{gl}(2, \mathbb{H}_c) \):

\[
\left( \omega_2^l \otimes \pi_0^r, \mathcal{V} / \mathcal{V} \cap ((I_2^l \oplus \mathcal{K}^+ \otimes \mathcal{H}^+) \right) \simeq \left( \omega_2^l \otimes \pi_0^r, \mathcal{K} / I_2^l \oplus \mathcal{K}^+ \otimes \mathcal{H}^+ \right) \simeq (\pi_0^r, \mathcal{H}^+) \otimes (\pi_0^r, \mathcal{H}^+).
\]

From (29) and Lemma 19 we conclude that the operator \( L^{(n)} \) has image in \( \mathcal{H}^+ \otimes \mathcal{H}^+ \) and the map (26) is \( \mathfrak{gl}(2, \mathbb{H}_c) \)-equivariant.

Since the irreducible components in the decomposition (24) are pairwise distinct, by Schur’s Lemma, \( L^{(n)} \) must act on each irreducible component \( (\rho_k, \mathcal{K}^+ \otimes \mathbb{C}^{k \times k}) \) by multiplication by some scalar \( \mu_k^{(n)} \). Since \( (w_{11} - w'_{11})^{k-1} \) generates \( \mathcal{K}^+ \otimes \mathbb{C}^{k \times k} \), these scalars can be found by computing the ratio of inner products

\[
\mu_k^{(n)} = \frac{\langle L^{(n)}(1 \otimes (z'_{11})^{k-1}), (w_{11} - w'_{11})^{k-1} \rangle_{\text{inn. prod.}}}{\langle 1 \otimes (w'_{11})^{k-1}, (w_{11} - w'_{11})^{k-1} \rangle_{\text{inn. prod.}}} = \sum_{p=0}^{k-1} (-1)^{k+p+1} \cdot \binom{k-1}{p} \cdot \langle L^{(n)}(1 \otimes (z'_{11})^{k-1}), (w_{11})^{k-p-1} \cdot (w'_{11})^{p} \rangle_{\text{inn. prod}}.
\]

This sum can be evaluated using Lemma 19 equation (8) and orthogonality relationships (10). This finishes the proof of Theorem 14 in the special case of ladder diagrams.

5.3 The General Case

Now we prove Theorem 14 in complete generality, where \( L^{(n)} \) is the operator associated to any \( n \)-loop box diagram \( d^{(n)} \). The proof is by induction on the number of loops (or slingshots). The case of the one-loop diagram (Figure 4) was done in [FL1] and [FL3]. So, suppose that the diagram \( d^{(n)} \) is obtained by adding a slingshot to an \( (n-1) \)-loop box diagram \( d^{(n-1)} \). As was mentioned before, there are exactly four ways of doing that — so that one of \( Z_1, Z_2, W_1 \) or \( W_2 \) becomes a solid vertex. For concreteness, let us assume that the slingshot is attached to \( d^{(n-1)} \) so that \( Z_1 \) becomes a solid vertex, the other cases are similar and easier.

Let \( \hat{d}^{(n)} \) be the \( n \)-loop ladder diagram (Figure 11), and let \( \hat{L}^{(n)} \) be the corresponding integral operator. First, we prove the following symmetry property for \( \hat{L}^{(n)} \) (when \( n = 2 \) it is a direct analogue of equation (8) in [DHSS]).

**Lemma 23.** The operator \( \hat{L}^{(n)} : (\pi_0^l, \mathcal{H}^+) \otimes (\pi_0^r, \mathcal{H}^+) \to (\pi_0^l, \mathcal{H}^+) \otimes (\pi_0^r, \mathcal{H}^+) \) has the following symmetry:

\[
\hat{L}^{(n)}(\varphi_1 \otimes \varphi_2)(W_1, W_2) = \hat{L}^{(n)}(\varphi_2 \otimes \varphi_1)(W_2, W_1), \quad \varphi_1, \varphi_2 \in \mathcal{H}^+.
\]

**Proof.** Clearly, this property is true for the generators \( (z_{11} - z'_{11})^k, k \geq 0, \text{of } \mathcal{H}^+ \otimes \mathcal{H}^+ \). Therefore, by the \( \mathfrak{gl}(2, \mathbb{H}_c) \)-equivariance of \( \hat{L}^{(n)} \), it is true for all elements of \( \mathcal{H}^+ \otimes \mathcal{H}^+ \).

By induction, we assume that Theorem 14 — and hence Corollary 15 — hold for \( n - 1 \).
Similarly to the proof of Lemma 19, we first integrate over
the integral is 1. Hence, for the purposes of evaluating
Lemma 24.

\begin{equation}
Z_Z = \sum_{l,p,q} \deg_{Z_1}((-1)^l) \frac{dV_1}{N(Z_1 - T_n)} \cdot N(Z_1) = (t_{11}^{(l)})^k = \frac{k}{k-2-l/2}(T_n).
\end{equation}

Then we integrate out the \( Z_1 \) variable, by Proposition 10 we get

\begin{equation}
\frac{i}{2\pi^3} \frac{dV_1}{N(Z_1 - T_n)} \cdot \frac{t^{(l-1)}(T_n; Z_2; W_1, W_2)}{N(Z_1 - T_n)} dV.
\end{equation}

Using 13, we expand \( N(W_2 - T_n)^{-1} \) and \( t^{(l-1)}(T_n; Z_2; W_1, W_2) \) in terms of basis functions

\begin{equation}
\frac{1}{N(W_2 - T_n)} = N(T_n)^{-1} \cdot \sum_{l,p,q} t_{p,q}^{(l)}(T_n)^{-1} \cdot t_{q,p}^{(l)}(W_2), \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \quad p, q = -l, -l+1, \ldots, l,
\end{equation}

\begin{equation}
t^{(l-1)}(T_n; Z_2; W_1, W_2) = \sum_{l',p',q',m} t_{l',p',q',m}^{l'}(T_n)^{-1} \cdot N(T_n)^{-1-m} \cdot f_{l',p',q',m}(Z_2; W_1, W_2)
\end{equation}

for some functions \( f_{l',p',q',m}(Z_2; W_1, W_2) \). In the diagram \( d^{(1)} \), the number of solid edges at
\( Z_1 \) is one more than the number of dashed edges. This implies that only the terms with \( m \geq 0 \) appear in the expansion of \( t^{(l-1)}(T_n; Z_2; W_1, W_2) \). Thus the integral 33 can be rewritten as

\begin{equation}
\sum_{l',p',q',m} \langle N(T_n)^{-2-m} \cdot t_{p,q}^{l}(T_n)^{-1} \cdot t_{q,p}^{l'}(T_n)^{-1} \cdot t_{l-k/2-k/2}^{l/2}(T_n) \rangle \cdot t_{q,p}^{l'}(W_2) \cdot f_{l',p',q',m}(Z_2; W_1, W_2).
\end{equation}

By Corollary 8, these terms are zero unless \( m = 0, l + l' = k/2, p = q = -l \) and \( p' = q' = -l' \), and 33 becomes

\begin{equation}
\frac{1}{k+1} \sum_{l+l'=k/2} t_{l-l}(W_2) \cdot f_{l-l',-l',0}(Z_2; W_1, W_2)
\end{equation}

\begin{equation}
= \sum_{l',p',q',m} \frac{2l+1}{k+1} \cdot t_{l-l}(W_2) \cdot \langle t_{l-l'}^{l'}(T_n), t_{q,p}^{l'}(T_n)^{-1} \rangle \cdot N(T_n)^{-2-m} \cdot f_{l',p',q',m}(Z_2; W_1, W_2)
\end{equation}

\begin{equation}
= \frac{1}{k+1} \sum_{l+l'=k/2} t_{l-l}(W_2) \cdot \frac{i}{2\pi^3} \frac{dV_{l-l}}{N(T_n)}.
\end{equation}
This proves that
\[ L^{(n)}((z_{11})^k \otimes 1) = \frac{1}{k+1} \sum_{p=0}^{k} (w_{11}^p)^{k-p} \cdot L^{(n-1)}((z_{11})^p \otimes 1), \]

where \( L^{(n-1)} \) denotes the integral operator corresponding to \( d^{(n-1)} \). By induction hypothesis, Corollary 20 and Lemma 23 this implies (32).

From this point on, the proof proceeds as in the ladder case, with trivial modifications. Because of the way the last slingshot is attached, we know that the function \( l^{(n)}(Z_1, Z_2; W_1, W_2) \) is harmonic in \( Z_1 \). Then the pairings (3) and (19) agree, and we can rewrite \( L^{(n)} \) as

\[ L^{(n)}(\varphi_1 \otimes \varphi_2)(W_1, W_2) = \frac{i}{4\pi^5} \int_{Z_1 \in S_{R_1}} \int_{z_2 \in U(2)R_2} l^{(n)}(Z_1, Z_2; W_1, W_2) \cdot (\deg_{Z_1} \varphi_1)(Z_1) \cdot (\deg_{Z_2} \varphi_2)(Z_2) \frac{dS_1}{R_1} dV_2. \]

where \( \varphi_1, \varphi_2 \in \mathcal{H}^+, \ R_1 > r_{\text{max,1}}, \ R_2 > r_{\text{max,2}}, \ W_1 \in \mathbb{D}^+_{r_{\text{min,1}}}, \ W_2 \in \mathbb{D}^+_{r_{\text{min,2}}}, \) as before. We introduce a closely related integral operator

\[ \hat{L}^{(n)} : (\pi_1^0, \mathcal{H}^+) \otimes (\varphi_2, \mathcal{K}) \to (\pi_1^0, \mathcal{K}^+) \otimes (\varphi_2, \mathcal{K}^+), \]

\[ \hat{L}^{(n)}(\varphi \otimes f)(W_1, W_2) = \frac{i}{4\pi^5} \int_{Z_1 \in S_{R_1}} \int_{z_2 \in U(2)R_2} l^{(n)}(Z_1, Z_2; W_1, W_2) \cdot (\deg_{Z_1} \varphi)(Z_1) \cdot f(Z_2) \frac{dS_1}{R_1} dV_2, \]

where \( \varphi \in \mathcal{H}^+, \ f \in \mathcal{K} \). The \( \mathfrak{gl}(2, \mathbb{H}_C) \)-equivariance of this operator follows from the \( \mathfrak{gl}(2, \mathbb{H}_C) \)-invariance of the bilinear pairings (7), (18) and Lemma 17. Clearly, we have

\[ L^{(n)} = \hat{L}^{(n)} \circ (N(Z_2)^{-1} \cdot \deg_{Z_2}). \quad (34) \]

**Lemma 25.** The operator \( \hat{L}^{(n)} \) annihilates \( \mathcal{H}^+ \otimes I_2^- \).

**Proof.** As we have observed earlier, the number of solid edges at \( Z_2 \) is one more than the number of dashed edges. Using this observation, one proceeds exactly as in the proof of Lemma 21 with the roles of the variables \( Z_1 \) and \( Z_2 \) switched. \( \square \)

Let \( \mathfrak{M}' \subset \mathcal{H}^+ \otimes \mathcal{K} \) denote the subrepresentation of \( (\pi_1^0, \mathcal{H}^+) \otimes (\varphi_2, \mathcal{K}) \) generated by

\[ \{ (z_{11})^k \cdot N(Z_2)^{-1} ; \ k = 0, 1, 2, 3, \ldots \}. \]

Thus we have a \( \mathfrak{gl}(2, \mathbb{H}_C) \)-equivariant map

\[ \hat{L}^{(n)} : (\pi_1^0 \otimes \varphi_2, \mathfrak{M}') \to (\pi_1^0, \mathcal{H}^+) \otimes (\pi_1^0, \mathcal{K}^+). \]

The same proof as that of Lemma 22 also shows:

**Lemma 26.** The operator \( \hat{L}^{(n)} \) annihilates \( \mathfrak{M}' \cap (\mathcal{H}^+ \otimes \mathcal{K}^+) \).

Combining Lemmas 25 and 26 we see that \( \hat{L}^{(n)} \) descends to a well-defined \( \mathfrak{gl}(2, \mathbb{H}_C) \)-equivariant quotient map

\[ \mathfrak{M}' \cap (\mathcal{H}^+ \otimes (I_2 \oplus \mathcal{K}^+)) \to \mathcal{H}^+ \otimes \mathcal{H}^+. \quad (35) \]
Clearly, this quotient space is a $\mathfrak{gl}(2, \mathbb{H}_C)$-invariant subspace of $\mathcal{H}^+ \otimes (\mathcal{K}/(I_2 \oplus \mathcal{K}^+))$. By Proposition [10] and Lemma [18] we have the following isomorphisms of representations of $\mathfrak{gl}(2, \mathbb{H}_C)$:

$$
\left( \pi_1^0 \otimes \omega_2^r, \mathcal{Y}' \cap (\mathcal{H}^+ \otimes (I_2^+ \oplus \mathcal{K}^+)) \right) \simeq \left( \pi_1^0 \otimes \omega_2^r, \mathcal{H}^+ \otimes \frac{\mathcal{K}}{I_2^+ \oplus \mathcal{K}^+} \right) \simeq (\pi_1^0, \mathcal{H}^+) \otimes (\pi_1^0, \mathcal{H}^+).
$$

From (34) and Lemma 19 we conclude that the operator $L^{(n)}$ has image in $\mathcal{H}^+ \otimes \mathcal{H}^+$ and the map (26) is $\mathfrak{gl}(2, \mathbb{H}_C)$-equivariant. By (32), the maps $L^{(n)}$ and $\tilde{L}^{(n)}$ coincide on the generators of $\mathcal{H}^+ \otimes \mathcal{H}^+$. Since Theorem 14 is already established for $\tilde{L}^{(n)}$, it follows that the result holds for $L^{(n)}$ as well.

References

[DD] A. I. Davydychev, R. Delbourgo, *A geometrical angle on Feynman integrals*, J. Math. Phys. 39 (1998), no. 9, 4299-4334.

[DHSS] J. M. Drummond, J. Henn, V. A. Smirnov, E. Sokatchev, *Magic identities for conformal four-point integrals*, J. High Energy Phys. 0701 (2007), no. 1, 064, 15 pp.

[FL1] I. Frenkel, M. Libine, *Quaternionic analysis, representation theory and physics*, Advances in Math 218 (2008), 1806-1877; also arXiv:0711.2699.

[FL2] I. Frenkel, M. Libine, *Split quaternionic analysis and the separation of the series for $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})/SL(2, \mathbb{R})$*, Advances in Math 228 (2011), 678-763; also arXiv:1009.2532.

[FL3] I. Frenkel, M. Libine, *Anti De Sitter Deformation of Quaternionic Analysis and the Second-Order Pole*, to appear in IMRN, doi: 10.1093/imrn/rnu083; also arXiv:1404.7098.

[GT] F. Gürsey, C.-H. Tze, *On the role of division, Jordan and related algebras in particle physics*, World Scientific Publishing Co., 1996.

[GR] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th edition, Academic Press, Amsterdam, 2007.

[JV] H. P. Jakobsen, M. Vergne, *Restrictions and expansions of holomorphic representations*, J. Funct. Anal. 34 (1979), no. 1, 29-53.

[L] M. Libine, *The two-loop ladder diagram and representations of $U(2, 2)$*, arXiv:1309.5665, submitted.

[M] M. Marcolli, *Feynman motives*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010.

[Sm] V. A. Smirnov, *Evaluating Feynman integrals*, Springer Tracts in Modern Physics, 211, Springer-Verlag, Berlin, 2004.

[Su] A. Sudbery, *Quaternionic analysis*, Math. Proc. Camb. Math. Soc. 85 (1979), 199-225.

[UD] N. I. Ussyukina, A. I. Davydychev, *Exact results for three- and four-point ladder diagrams with an arbitrary number of rungs*, Phys. Lett. B 305 (1993), no. 1-2, 136-143.
[V] N. Ja. Vilenkin, *Special functions and the theory of group representations*, translated from the Russian by V. N. Singh, Translations of Mathematical Monographs, Vol. 22 American Mathematical Society, Providence, RI 1968.

[W] P. Wagner, *A volume formula for asymptotic hyperbolic tetrahedra with an application to quantum field theory*, Indag. Math. (N.S.) 7 (1996), no. 4, 527-547.

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