Sharp estimate on the supremum of a class of partial sums of small i.i.d. random variables.

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Summary. We take an $L_1$-dense class of functions $\mathcal{F}$ on a measurable space $(X, \mathcal{X})$ together with a sequence of independent, identically distributed $X$-space valued random variables $\xi_1, \ldots, \xi_n$ and give a good estimate on the tail distribution of $\sup_{f \in \mathcal{F}} \sum_{j=1}^n f(\xi_j)$ if the expected values $E|f(\xi_1)|$ are very small for all $f \in \mathcal{F}$. In a subsequent paper [2] we shall give a sharp bound for the supremum of normalized sums of i.i.d. random variables in a more general case. But that estimate is a consequence of the results in this work.

1. Introduction.

This work is part of a more general investigation about the supremum of (normalized) partial sums of bounded, independent and identically distributed random variables if the class of random variables whose partial sums we investigate have some nice properties. It turned out that it is useful to investigate first the case when the expectations of the absolute value of these random variables are very small, and this is the subject of the present paper. In paper [2] we shall get good estimates in the general case when the expectations of the absolute value of the summands may be relatively large with the help of the main result in this paper.

First I recall the notion of $L_1$-dense classes of functions which plays an important role in our investigation, and then I formulate the main result of this paper. After its formulation I make some comments that may help in understanding its content and the motivation behind this investigation.

Definition of $L_1$-dense classes of functions. Let a measurable space $(X, \mathcal{X})$ be given together with a class of $\mathcal{X}$ measurable, real valued functions $\mathcal{F}$ on this space. The class of functions $\mathcal{F}$ is called an $L_1$-dense class of functions with parameter $D \geq 1$ and exponent $L \geq 1$ on the space $(X, \mathcal{X})$ if there exists a finite $\varepsilon$-dense subset $\mathcal{F}_{\varepsilon, \nu} = \{f_1, \ldots, f_m\} \subset \mathcal{F}$ in the space $L_1(X, \mathcal{X}, \nu)$ with $m \leq D\varepsilon^{-L}$ elements, i.e. there exists such a set $\mathcal{F}_{\varepsilon, \nu} \subset \mathcal{F}$ with $m \leq D\varepsilon^{-L}$ elements for which

$$\inf_{f_j \in \mathcal{F}_{\varepsilon, \nu}} \int |f - f_j| \, d\nu < \varepsilon$$

for all functions $f \in \mathcal{F}$. (Here the set $\mathcal{F}_{\varepsilon, \nu}$ may depend on the measure $\nu$, but its cardinality is bounded by a number depending only on $\varepsilon$.)

The main result of this work is the following Theorem 1.

Theorem 1. Let $\mathcal{F}$ be a finite or countable $L_1$-dense class of functions with some parameter $D \geq 1$ and exponent $L \geq 1$ on a measurable space $(X, \mathcal{X})$ such that $\sup_{x \in X} |f(x)| \leq 1$ for all $f \in \mathcal{F}$. Let $\xi_1, \ldots, \xi_n$, $n \geq 2$, be a sequence of independent and identically distributed random variables with values in the space $(X, \mathcal{X})$ with such a distribution $\mu$ for
which the inequality $\int |f(x)|\mu(dx) \leq \rho$ holds for all $f \in \mathcal{F}$ with a number $0 < \rho \leq n^{-200}$. Put $S_n(f) = S_n(f)(\xi_1, \ldots, \xi_n) = \sum_{j=1}^{n} f(\xi_j)$ for all $f \in \mathcal{F}$. The inequality

$$P\left( \sup_{f \in \mathcal{F}} |S_n(f)| \geq u \right) \leq D\rho^{Cu} \quad \text{for all } u > 41L \quad (1.1)$$

holds with some universal constant $1 > C > 0$. We can choose e.g. $C = \frac{1}{50}$.

I introduce an example that may help in understanding better the content of Theorem 1. In particular, it gives some hints why a condition of the type $u > C L$ was imposed in formula (1.1). (We applied this condition with $C = 41$.)

Let us take a set $X = \{x_1, \ldots, x_N\}$ with a large number $N$ together with the uniform distribution $\mu$ on it, i.e. let $\mu(x_j) = \frac{1}{N}$ for all $1 \leq j \leq N$, and define the following class of function $\mathcal{F}$ on $X$. Fix a positive integer $L$, and let the class of functions $\mathcal{F}$ consist of the indicator functions of all subsets of $X$ containing no more than $L$ points. Let us fix a number $n$, and choose for all numbers $j = 1, \ldots, n$ a point of the set $X$ choosing each point with the same probability $\frac{1}{N}$ independently of each other. Let $\xi_j$ denote the element of $X$ we chose at the $j$-th time. In such a way we defined a sequence of independent random variables $\xi_1, \ldots, \xi_n$ on $X$ with distribution $\mu$, and a class of functions $\mathcal{F}$ consisting of non-negative functions bounded by 1 such that

$$\int f(x)\mu(dx) = \frac{L}{N} \quad \text{for all } f \in \mathcal{F}.$$  

Let us introduce the random sums $S_n(f) = \sum_{j=1}^{n} f(\xi_j)$ for all $f \in \mathcal{F}$. We shall estimate first the probability $P_n = P\left( \sup_{f \in \mathcal{F}} S_n(f) \geq n \right)$ and then the probability $P_{u,n} = P\left( \sup_{f \in \mathcal{F}} S_n(f) \geq u \right)$ for $u \leq n$.

It is not difficult to see that $P_n = 1$ if $n \leq L$, and $P_n \leq \binom{N}{L} \left(\frac{L}{N}\right)^n \leq C^L \rho^{-L}$ with $\rho = \frac{L}{N}$, where $C$ is a universal constant. The number $C$ can be chosen as such a constant for which the inequality $p^p \leq C^p p!$ holds for all positive integers $p$. We can choose for instance $C = 4$. In the proof of the above estimate we have exploited that $X$ has $\binom{N}{L}$ subsets containing exactly $L$ points, and the event $\sup_{f \in \mathcal{F}} S_n(f) \geq n$ may occur only if there is a subset of $X$ with $L$ points such that all $\xi_j, 1 \leq j \leq n$, are contained in this subset. Also the estimate $P_{u,n} \leq \binom{n}{u} P_u \leq C^L n^u \rho^{u-L}$ holds, because the event $\sup_{f \in \mathcal{F}} S_n(f) \geq u$ can only happen if there are some indices $1 \leq j_1 < j_2 < \cdots < j_u \leq n$ such that all points $\xi_{j_k}, 1 \leq s \leq u$, are contained in a subset of $X$ of cardinality $L$. The probability of such an event is $P_u$ for all sequences $1 \leq j_1 < j_2 < \cdots < j_u \leq n$, and there are $\binom{n}{u}$ such sequences.

We show that if $N \geq n^{201}$ and $n \geq 41L$, then the above model satisfies the conditions of Theorem 1, and compare the bound we got for $P_{u,n}$ in our previous calculation with the estimate Theorem 1 supplies in this example. To show that the conditions of
Theorem 1 hold in this case we have to prove that the class of functions \( F \) consisting of the indicator functions of all subsets containing \( L \) points of a set \( X \) is an \( L_1 \)-dense class, and to estimate the probability \( P_{u,n} \) with the help of Theorem 1 we have to give a possible value for the parameter and exponent for this \( L_1 \)-dense class. To do this I recall the definition of Vapnik–Červonenkis classes together with a classical result about their properties.

**Definition of Vapnik–Červonenkis classes.** Let a set \( X \) be given, and let us select a class \( D \) of subsets of this set \( X \). We call \( D \) a Vapnik–Červonenkis class if there exist two real numbers \( B \) and \( K \) such that for all positive integers \( n \) and subsets \( S(n) = \{x_1, \ldots, x_n\} \subset X \) of cardinality \( n \) of the set \( X \) the collection of sets of the form \( S(n) \cap D, D \in D \), contains no more than \( Bn^K \) subsets of \( S(n) \). We call \( B \) the parameter and \( K \) the exponent of this Vapnik–Červonenkis class.

It is not difficult to see that the subsets of a set \( X \) containing at most \( L \) points constitute a Vapnik–Červonenkis class with exponent \( K = L \) and an appropriate parameter \( B \). (Some calculations show that we can choose \( B = \frac{1 + \varepsilon}{L} \).) I would also recall a classical result (see e.g. [3] Chapter 2, 25 Approximation Lemma) by which the indicator functions of the sets in a Vapnik–Červonenkis class constitute an \( L_1 \)-dense class of functions. (Actually, the work [3] uses a slightly different terminology, and it presents a more general result.) In the book [3] it is proved that if the parameter and exponent of the Vapnik–Červonenkis class are \( B \) and \( K \), then the parameter and exponent of the \( L_1 \)-dense class consisting of the indicator functions of the sets contained in this Vapnik–Červonenkis class can be chosen as \( D = \max(B^2, n_0) \) and \( L = 2K \) with an appropriate constant \( n_0 = n_0(K) \). But it is not difficult to see by slightly modifying the proof that this class of the indicator functions can also be considered as an \( L_1 \)-dense class of functions with exponent \( L = (1 + \varepsilon)K \) and an appropriate parameter \( D = D(K, L, \varepsilon) \) for arbitrary \( \varepsilon > 0 \).

The above considerations show that the class of functions \( F \) considered in the above example is an \( L_1 \)-dense class of functions with exponent \( 2L \) and an appropriate parameter \( D \). It is even an \( L_1 \)-dense class of functions with exponent \( (1 + \varepsilon)L \) and an appropriate parameter \( D(\varepsilon) \) for all \( \varepsilon > 0 \). This means in particular that Theorem 1 can be applied to estimate the probability \( P_{u,n} \) if the numbers \( L, N \) and \( n \) are appropriately chosen. It is not difficult to see that both Theorem 1 and our previous argument provide an estimate of the form \( P_{u,n} \leq \rho^{\alpha u} \) with a universal constant \( 0 < \alpha < 1 \), only the parameter \( \alpha \) is different in these two estimates. (Observe that \( \rho = \frac{L}{2} \geq \int f(x) \mu(dx) \) for all \( f \in \mathcal{F} \) in our example.). To see that we proved such an estimate for \( P_{u,n} \) which implies the inequality \( P_{u,n} \leq \rho^{\alpha u} \) under the conditions of Theorem 1 observe that \( \rho^{\alpha u} - L \leq \rho^{40u/41} \), and \( n^u \leq \rho^{-u/200} \). Moreover, it can be seen that if we are not interested in the value of the universal parameter \( \alpha \), then this estimate is sharp. I also remark that in our example we can give a useful estimate for \( P_{u,n} \) (and not only the trivial bound \( P_{u,n} \leq 1 \)) only in the case \( u > L \).

The main content of Theorem 1 is that a similar picture arises if the supremum of the partial sums defined with the help of an \( L_1 \)-dense class of functions is considered. Namely, Theorem 1 states that if \( \mathcal{F} \) is an \( L_1 \)-dense class of functions that satisfies
some natural conditions, then there are universal constants \(0 < \alpha < 1, C_1 > 1\) and \(C_2 > 0\) such that \(P\left(\sup_{f \in \mathcal{F}} S_n(f) > u\right) \leq D\rho^{-\alpha u}\) if \(n \geq C_1L\) and \(\rho \leq n^{-C_2}\). Here we applied the notations of Theorem 1. We also gave an explicit value for these universal parameters in Theorem 1, but we did not try to find a really good choice. It might be interesting to show on the basis of the calculation of the present paper that we can choose \(C_1 = 1 + \varepsilon\) or \(\alpha = 1 - \varepsilon\) with arbitrary small \(\varepsilon > 0\) if the remaining universal constants are appropriately chosen.

As the above considered example shows the estimate of Theorem 1 holds only if \(u \geq CL\) with a number \(C > 1\). The other condition of Theorem 1 by which \(\rho \leq n^{-C_2}\) with a sufficiently large number \(C_2 > 0\) can be weakened. Actually this is the topic of paper [2] which is a continuation of the present work. In paper [2] I shall consider such \(L_1\)-dense classes of functions \(\mathcal{F}\) for which the parameter \(\rho\) considered in Theorem 1 can be relatively large. On the other hand, in [2] we shall consider only such classes of functions \(\mathcal{F}\) whose elements have the ‘normalizing property’ \(\int f(x)\mu(dx) = 0\) for all \(f \in \mathcal{F}\). In the present work we did not impose such a normalization condition, because in the case \(\rho \leq n^{-\alpha}\) with some \(\alpha > 1\) the lack of normalization has a negligible effect.

Theorem 1 will be proved with the help of Theorem 1A formulated below. After its formulation I shall explain why Theorem 1A can be considered as a very special case of Theorem 1.

**Theorem 1A.** Let \(X = \{x_1, \ldots, x_N\}\) be a finite set of \(N\) elements, and let \(\mathcal{X}\) be the \(\sigma\)-algebra consisting of all subsets of \(X\). Let \(\mu\) denote the uniform distribution on \(X\), i.e. let \(\mu(A) = \frac{|A|}{N}\) for all sets \(A \subset X\), where \(|A|\) denotes the cardinality of a set \(A\). Let \(\mathcal{F}\) be an \(L_1\)-dense class of functions with some parameter \(D \geq 1\) and exponent \(L \geq 1\) on the measurable space \((X, \mathcal{X})\) such that \(0 \leq f(x) \leq 1\) for all \(x \in X\) and \(f \in \mathcal{F}\), and \(\int f(x)\mu(dx) \leq \frac{L}{2}\) for all \(f \in \mathcal{F}\) with some \(\rho > 0\) which satisfies the inequality \(\rho \leq \min\left(\frac{1}{1000}, L^{-20}\right)\). Introduce for all numbers \(p = 1, 2, \ldots\) the \(p\)-fold direct product \(X^p\) of the space \(X\) together with the \(p\)-fold product measure \(\mu_p\) of the uniform distribution \(\mu\) on \(X\), i.e. let each sequence \(x^{(p)} = (x_{s_1}, \ldots, x_{s_p})\), \(x_{s_j} \in X\), \(1 \leq j \leq p\), have the weight \(\mu_p(x^{(p)}) = \frac{1}{N^p}\) with respect to the measure \(\mu\).

For the sake of a simpler argument let us assume that the number \(N\) has the following special form: \(N = 2^kN_0\) with some integer \(k \geq 0\), and a number \(N_0\) that satisfies the inequality \(\frac{1}{16}\rho^{-3/2} < N_0 \leq \frac{1}{8}\rho^{-3/2}\).

Given a function \(f \in \mathcal{F}\) and a positive integer \(p\) let us define the set \(B_p(f) \subset X^p\) for all \(p \geq 2\) by the formula

\[
B_p(f) = \left\{ x^{(p)} = (x_{s_1}, \ldots, x_{s_p}) : x^{(p)} \in X^p, \quad f(x_{s_j}) = 1 \quad \text{for all} \quad 1 \leq j \leq p \right\}, \quad (1.2)
\]

and put

\[
B_p = B_p(\mathcal{F}) = \bigcup_{f \in \mathcal{F}} B_p(f). \quad (1.3)
\]
If \( p \geq 2L \) and \( p \leq \rho^{-1/100} \), then there exist some universal constants \( C_1 > 0 \) and \( 1 > C_2 > 0 \) such that
\[
\mu_{p}(B_{p}) = \mu_{p}(B_{p}(F)) \leq C_{1} D \rho^{C_{2} p}.
\] (1.4)

We can choose for instance \( C_1 = 2 \) and \( C_2 = \frac{1}{4} \).

In Theorem 1A we considered a very special case of the problem discussed in Theorem 1. We took a space of the form \( X = \{x_1, \ldots, x_N\} \) with the uniform distribution \( \mu \) on it, and considered an \( L_1 \)-dense class of functions with some special properties. If we apply it with the choice \( p = n \), then the event \( B_p(F) \) defined in (1.3) agrees with the event \( \sup S_n(f) \geq n \), and formula (1.4) implies the estimate (1.1) with the special choice \( u = n \) for the system \( X, F, \mu \) considered in Theorem 1A.

Theorem 1A can be proved by means an appropriate induction, where we can exploit the \( L_1 \)-dense property of the class of functions \( F \). This will be done in Section 2. In Section 3 we prove Theorem 1 with the help of Theorem 1A and a good approximation.

2. The proof of Theorem 1A.

Theorem 1A will be proved by means of induction with respect to the parameter \( k \) (appearing in the definition of the size \( N \) of the set \( X \)). The first result of this section, Lemma 2.1, formulates a result similar to Theorem 1A in the special case when the set \( X \), where the functions \( f \) are defined contains relatively few points. We need it to start our induction procedure.

**Lemma 2.1.** Let us fix a number \( \rho, 0 < \rho < 1 \), and a set \( X = \{x_1, \ldots, x_{N_0}\} \), with \( N_0 \leq \frac{1}{2} \rho^{-3/2} \) points together with a class of functions \( F \) defined on \( X \) which satisfies the following weakened version of the \( L_1 \)-dense property with parameter \( D \geq 1 \) and exponent \( L \geq 1 \). For all \( 0 \leq u \leq 1 \) there is a set of functions \( f_1, \ldots, f_s \) from the class of functions \( F \) with \( s \leq Du^{-L} \) elements in such a way that \( \inf_{1 \leq j \leq s} \int |f - f_j| d\mu \leq u \), where \( \mu \) denotes the uniform distribution on \( X \). Let us also assume that \( \int f(x) d\mu(x) \leq \rho \) and \( f(x) \geq 0 \) for all \( f \in F \) and \( x \in X \). Let us consider an integer \( p \geq 2L \), the set \( B_p = B_p(F) \subset X^p \) introduced in formula (1.3) together with the uniform measure \( \mu_p \) on the \( p \)-fold product \( X^p \) of the space \( X \). The inequality
\[
\mu_{p}(B_{p}) \leq D \rho^{p/4}
\] (2.1)

holds.

**Proof of Lemma 2.1.** Let us choose such a set of functions \( f_1, \ldots, f_s, f_j \in F \) for all \( 1 \leq j \leq s \), with cardinality \( s \leq D \cdot (2N_0)^L \), which has the property that for all \( f \in F \) there is a function \( f_j, 1 \leq j \leq s \), for which the inequality \( \int |f(x) - f_j(x)| d\mu (dx) \leq \frac{1}{2N_0} \) holds. If \( \int |f(x) - f_j(x)| d\mu (dx) \leq \frac{1}{2N_0} \), then \( |f(x) - f_j(x)| \leq \frac{1}{2} \) for all \( x \in X \). This follows from the inequality \( \frac{1}{N_0} |f(x) - f_j(x)| \leq \int |f(x) - f_j(x)| d\mu (dx) \leq \frac{1}{2N_0} \) for all \( x \in X \). As a
consequence, \( \{ x : f(x) = 1 \} \subset \{ x : f_j(x) \geq \frac{1}{2} \} \) for such a pair of functions \( f \) and \( f_j \), and

\[
B_p = B_p(\mathcal{F}) = \bigcup_{f \in \mathcal{F}} B_p(f) \subset \bigcup_{j=1}^s \left\{ (x_{t_1}, \ldots, x_{t_p}) : f_j(x_{t_k}) \geq \frac{1}{2} \quad \text{for all } 1 \leq k \leq p \right\}.
\]

Besides, we have for each \( j \), \( 1 \leq j \leq s \),

\[
\mu_p \left\{ (x_{t_1}, \ldots, x_{t_p}) : f_j(x_{t_k}) \geq \frac{1}{2} \quad \text{for all } 1 \leq k \leq p \right\} = \left( \mu \left\{ x_t : f_j(x_t) \geq \frac{1}{2} \right\} \right)^p \leq (2\rho)^p.
\]

Hence the relations \( p \geq 2L \) and \( N_0 \leq \frac{1}{8}\rho^{-3/2} \) imply that

\[
\mu_p(B_p) \leq s(2\rho)^p \leq D(2N_0)^{p/2}(2\rho)^p \leq D\rho^{p/4}.
\]

Lemma 2.1 is proved.

In our inductive proof we also need a result presented in Lemma 2.2. It is a version of the following heuristic statement. Let us consider the supremum of the integrals \( \int f(x)\mu(dx) \) for all functions \( f \in \mathcal{F} \) of an \( L_1 \)-dense class \( \mathcal{F} \) of non-negative functions bounded by 1 on a finite set \( X \) with respect to the uniform distribution \( \mu \) on \( X \). Let the cardinality of the set \( X \) be \( 2N \), where the number \( N \) is of the form \( N = A2^k \) with some positive integers \( A \) and \( k \), and let the above supremum of integrals be bounded by a number \( \rho_{k+1} \). Then there is a number \( \rho_k \) slightly larger than \( \rho_{k+1} \) with the following property. For most subsets \( Y \subset X \) with cardinality \( N \) the supremum of the integrals of the restrictions of the functions \( f \in \mathcal{F} \) to the set \( Y \) with respect to the uniform distribution on \( Y \) can be bounded by \( \rho_k \).

**Lemma 2.2.** Let us define two sequences of numbers

\[
N_k = 2^k N_0, \quad \text{and} \quad \rho_k = \rho \prod_{j=0}^{k-1} \left( 1 + \frac{3}{N_j^{1/8}} \right)^{-1}, \quad k = 1, 2, \ldots, \quad \rho_0 = \rho, \quad (2.2)
\]

with the help of some starting numbers \( N_0 \) and \( \rho \) which satisfy the relations \( \rho \leq \min(\frac{1}{1000}, L^{-20}) \) and \( \frac{1}{16}\rho^{-3/2} < N_0 \leq \frac{1}{8}\rho^{-3/2} \). Let us fix an integer \( k \geq 0 \), and consider a set \( X = \{ x_1, \ldots, x_{2N_k} \} \) with \( N_{k+1} = 2N_k = N_02^{k+1} \) elements together with an \( L_1 \)-dense class of functions \( \mathcal{F} \) on \( X \) with parameter \( D \geq 1 \) and exponent \( L \geq 1 \) such that

\[
0 \leq f(x) \leq 1 \quad \text{for all points } x \in X \quad \text{and functions } f \in \mathcal{F} \quad \text{and sets } Y \subset X. \quad \text{The following Statement (a) holds.}
\]

(a) The number of sets \( Y \subset X \) such that \( |Y| = N_k \), and \( \sup_{f \in \mathcal{F}} R_Y(f) \leq N_{k+1}\rho_{k+1} \) for all \( f \in \mathcal{F} \). Let us define the quantity \( R_Y(f) = \sum_{x_j \in Y} f(x_j) \) for all functions \( f \in \mathcal{F} \) and sets \( Y \subset X \). The following Statement (a) holds.

\[
(a) \quad \text{The number of sets } Y \subset X \text{ such that } |Y| = N_k, \text{ and } \sup_{f \in \mathcal{F}} R_Y(f) \geq N_k\rho_k \text{ is less than } \binom{2N_k}{N_k} D \exp \left\{ -\frac{1}{100} 2^{k/20} \rho^{-1/20} \right\}.
\]
Proof of lemma 2.2. Let us fix a partition of \( X = \{x_1, \ldots, x_{2N_k}\} \) to two point subsets \( \{x_{j_1}, x_{j_2}\}, \ldots, \{x_{j_{2N_k}-1}, x_{j_{2N_k}}\} \) together with a sequence of iid. random variables \( \varepsilon_1, \ldots, \varepsilon_{N_k} \) with distribution \( P(\varepsilon_l = 1) = P(\varepsilon_l = -1) = \frac{1}{2} \) for all \( 1 \leq l \leq N_k \). Let us define with their help the ‘randomized sum’

\[
U_k(f) = \sum_{l=1}^{N_k} \varepsilon_l (f(x_{j_{2l-1}}) - f(x_{j_{2l}}))
\]

(2.3)

for all \( f \in \mathcal{F} \).

Let us observe that for all \( f \in \mathcal{F} \) the inequality

\[
P(U_k(f) > 2z) \leq \exp \left\{ -\frac{2z^2}{\sum_{l=1}^{N_k} (f(x_{j_{2l-1}}) - f(x_{j_{2l}}))^2} \right\} \leq e^{-z^2/2N_k\rho_{k+1}} \quad \text{for all } z > 0
\]

(2.4)

holds by the Hoeffding inequality (see e.g. [3] Appendix B) and the inequality

\[
\sum_{l=1}^{N_k} (f(x_{j_{2l-1}}) - f(x_{j_{2l}}))^2 \leq 2 \sum_{j=1}^{2N_k} f(x_j)^2 \leq 2R_{k+1}(f) \leq 4N_k\rho_{k+1}.
\]

(2.5)

(In formula (2.5) we exploit the condition \( 0 \leq f(x) \leq 1 \) which implies that \( f(x_j)^2 \leq f(x_j) \).)

Define the (random) set \( V_k = V_k(\varepsilon_1, \ldots, \varepsilon_{N_k}) = \bigcup_{l: \varepsilon_l=1} \{x_{j_{2l-1}}\} \cup \bigcup_{l: \varepsilon_l=-1} \{x_{j_{2l}}\} \). With such a notation we can write

\[
\left\{ \omega: \sum_{s \in V_k(\varepsilon_1(\omega), \ldots, \varepsilon_{N_k}(\omega))} f(x_s) > N_k\rho_{k+1} + z \right\} \subset \left\{ \omega: \sum_{s \in V_k(\varepsilon_1(\omega), \ldots, \varepsilon_{N_k}(\omega))} f(x_s) > \frac{R_{k+1}(f)}{2} + z \right\}
\]

\[
= \{\omega: U_k(f)(\omega) > 2z\}.
\]

Hence

\[
P\left( \left\{ \omega: \sum_{s \in V_k(\varepsilon_1(\omega), \ldots, \varepsilon_{N_k}(\omega))} f(x_s) > N_k\rho_{k+1} + z \right\} \right) \leq e^{-z^2/2N_k\rho_{k+1}} \quad \text{for all } z > 0
\]

(2.6)

by relation (2.4).

I claim that relation (2.6) implies the following Statement (b).

(b) For all \( f \in \mathcal{F} \) and \( z > 0 \) the number of sets \( V \subseteq X \) such that \( |V| = N_k \), and \( \sum_{x \in V} f(x) \geq N_k\rho_{k+1} + z \) is less than or equal to \( e^{-z^2/2N_k\rho_{k+1}} (2N_k)^{2N_k} \).
Indeed, it follows from relation (2.6) that for a fixed partition of the set \( X \) to two point subsets the number of those subsets \( V \subset X \) which contain exactly one point from each element of this partition, (and as a consequence contain exactly \( N_k \) points), and \( \sum_{s \in V} f(x_s) > N_k \rho_{k+1} + z \) is less than or equal to \( 2^{N_k e^{-z^2/2N_k \rho_{k+1}}} \). We get an upper bound for the quantity considered in statement (b) by summing up the number of sets \( V \) with these properties for all partitions of \( X \) to two point subsets, and taking into account how many times we counted each set \( V \) in this procedure. The number of the partitions of \( X \) to two point subsets equals \((2N_k - 1)(2N_k - 3)\cdots 3 \cdot 1 = \frac{(2N_k)!}{2^N_k N_k!}\), and each partition provides at most \( 2^{N_k e^{-z^2/2N_k \rho_{k+1}}} \) sets \( V \) with the desired properties. All sets \( V \) were counted \( N_k! \)-times in this calculation. (A set \( V \), \( |V| = N_k \), was counted in the above calculation as many times as the number of those partitions of \( X \) to two point subsets which have the property that all of their elements contain a fixed element of \( V \).) These considerations imply Statement (b).

Given a number \( 0 \leq u < 1 \) there exist \( s \leq Du^{-L} \) functions \( f_1, \ldots, f_s \) in \( \mathcal{F} \) with the property that for all \( f \in \mathcal{F} \) and sets \( Y \subset X \) one of the functions \( f_j \), \( 1 \leq j \leq s \), satisfies the inequality \( \sum_{x \in Y} |f_j(x) - f(x)| \leq \sum_{x \in X} |f_j(x) - f(x)| \leq uN_{k+1} \). We get this relation by applying the \( L \)-density property of the class \( \mathcal{F} \) (with parameter \( D \) and exponent \( L \)) with the uniform distribution \( \mu \) on \( X \). This has the consequence that if \( \sum_{x \in Y} f(x) \geq N_k \rho_{k+1} + z + 2uN_k \) for some \( Y \subset X \) and \( f \in \mathcal{F} \), then there exists some index \( 1 \leq j \leq s \) such that \( \sum_{x \in Y} f_j(x) \geq N_k \rho_{k+1} + z \) with the same set \( Y \subset X \). Hence Statement (b) implies that the number of sets \( Y \) such that \( |Y| = N_k \) and \( \sum_{x \in Y} f(x) \geq N_k \rho_{k+1} + z + 2uN_k \) with some \( f \in \mathcal{F} \) is less than or equal to \( s \cdot e^{-z^2/2N_k \rho_{k+1}} \left( \frac{2N_k}{N_k} \right) = Du^{-L} e^{-z^2/2N_k \rho_{k+1}} \left( \frac{2N_k}{N_k} \right) \).

Put \( z = N_k \rho_{k+1} \cdot N_k^{-1/8} \) and \( u = \frac{\rho}{N_k} \). With such a choice we get that the number of sets \( Y \subset X \) such that \( |Y| = N_k \) and \( \sup_{f \in \mathcal{F}} R_Y(f) \geq N_k \rho_{k+1} (1 + 3N_k^{-1/8}) = N_k \rho_k \) is less than

\[
D \left( \frac{N_k^{1/8}}{\rho_{k+1}} \right)^L e^{-N_k^{3/4} \rho_{k+1}^{1/2}} \left( \frac{2N_k}{N_k} \right) D \left( \frac{2^{k/8} N_k^{1/8}}{\rho_{k+1}} \right)^L e^{-2^{3k/4} N_k^{3/4} \rho_{k+1}^{1/2}}. \tag{2.7}
\]

It follows from the definition of \( \rho_k \) that \( \frac{1}{8} \rho \leq \rho_{k+1} \leq \rho \), and we also have \( L \leq \rho^{-1/20} \) because of the condition imposed on the number \( \rho \). These relations together with the condition \( \frac{1}{16} \rho^{-3/2} < N_0 \leq \frac{1}{8} \rho^{-3/2} \) of Lemma 2.2 enable us to bound the expression in (2.7) from above by

\[
\left( \frac{2N_k}{N_k} \right) D \left( C_1 2^{k/8} \rho^{-19/16} \right)^{\rho^{-1/20}} e^{-C_2 2^{3k/4} \rho^{-1/8}} \leq \left( \frac{2N_k}{N_k} \right) D \exp \left\{-C_3 2^{k/20} \rho^{-1/20} \right\}
\]

with appropriate constants \( C_1, C_2 \) and \( C_3 \). One can choose e.g. \( C_3 = \frac{1}{100} \), and this implies Statement (a). (In the estimate of the last step we exploited that for a small
number $\rho > 0$ and all positive integers $k$ the term $e^{-C_22^{3k/4}\rho^{-1/8}}$ is much smaller than the reciprocal of $(C_12^{k/8}\rho^{-19/16})^{\rho^{-1/20}}$ which is of order $\exp\left\{-\text{const.}\ \rho^{-1/20}(k + \log\frac{1}{\rho})\right\}$. Lemma 2.2 is proved.

Remark. It may be worth remarking that the most important part of Lemma 2.2, relation (2.4) or its consequence (2.6) can be considered as a weakened version of Lemma 3 in [1], and even its proof is based on the ideas worked out in [1]. In formula (2.4) a random sum denoted by $U_k(f)$ was estimated by means of the Hoeffding inequality. To get this estimate we had to bound the variance of the random variable $U_k(f)$, and this was done in formula (2.5). In Lemma 3 of [1] a similar random sum was investigated, but in that case a good asymptotic formula and not only an upper bound was proved for the tail distribution of the random sum. In the proof of that result a sharp version of the central limit theorem was applied instead of the Hoeffding inequality, and we needed a good asymptotic formula and not only a good upper bound for the variance of the random sum we investigated. The proof of the good asymptotic formula for this variance was the hardest part in the proof of Lemma 3 of [1].

Proof of Theorem 1A. Let us fix some numbers $N_0$, $\rho$ and $L$ which satisfy the conditions of Lemma 2.2. Take an integer $k \geq 0$, define the numbers $N_k$ and $\rho_k$ by formula (2.2), consider a space $X = \{x_1, \ldots, x_{N_k}\}$ with $N_k$ elements, and an $L_1$-dense class of functions $\mathcal{F}$ on it with parameter $D \geq 1$ and exponent $L \geq 1$ such that $0 \leq f(x) \leq 1$ for all $x \in X$ and $f \in \mathcal{F}$, and $\int f(x)\mu(dx) \leq \rho_k$ for all $f \in \mathcal{F}$ with the uniform distribution $\mu$ on $X$. Fix an integer $p$ such that $p \geq 2L$, $p \leq \rho^{-1/100}$, and let us also consider the sets $B_p(f)$, $f \in \mathcal{F}$, and $B_p = B_p(\mathcal{F})$ introduced in formulas (1.2) and (1.3). They consist of sequences $x^{(p)} = (x_{s_1}, \ldots, x_{s_p}) \in X^p$ with some nice properties. Let $V(p, \rho, N_0, k) = V_{D,L}(p, \rho, N_0, k)$ denote the supremum of the cardinality of the sets $B_p(\mathcal{F})$ if the supremum is taken for all possible sets $X$ and class of functions $\mathcal{F}$ with the above properties (with parameters $N_k$ and $\rho_k$).

I claim that

$$V(p, \rho, N_0, k) \leq C_k N_k^p Dp^{p/4} \quad \text{for all } k = 0, 1, 2, \ldots$$

with

$$C_k = \prod_{j=0}^{k} (1 + 2^{-j}\rho).$$

Relation (2.8) will be proved by means of induction with respect to $k$. Its validity for $k = 0$ follows from Lemma 2.1. Let us assume that it holds for some $k$, take a set $X$ with cardinality $N_{k+1} = 2N_k$ together with a class of functions $\mathcal{F}$ which satisfies the above conditions with the parameters $D$, $L$, $p$, $\rho_{k+1}$ and $N_{k+1}$, and let us give a good bound on the cardinality of the set $B_p(\mathcal{F})$ defined in (1.2) and (1.3) in this case. To calculate the number of sequences $x^{(p)} = (x_{s_1}, \ldots, x_{s_p}) \in X^p$ which belong to the set $B_p(\mathcal{F})$ let us take all sets $Y \subset X$ with cardinality $|Y| = N_k$, let us bound the number of those sequences $x^{(p)} \in B_p(\mathcal{F})$ for which also the property $x^{(p)} \in Y^p$ holds, and let us
sum up these numbers for all sets $Y \subseteq X$ such that $|Y| = N_k$. Then take into account how many times we counted a sequence $x^{(p)}$ in this summation. I claim that we get the following estimate in such a way:

$$|B_p(\mathcal{F})| \leq N^p_k \left( \frac{\binom{2N_k}{N_k}}{\binom{2N_k-p}{N_k-p}} \right) \left( C_kD\rho^{p/4} + D \exp \left\{ -\frac{1}{100}2^{k/20}\rho^{-1/20} \right\} \right)$$

(2.10)

with the coefficient $C_k$ defined in (2.9).

To prove relation (2.10) let us first observe that if $\mathcal{F}$ is an $L_1$-dense class of functions on the set $X$ with parameter $D$ and exponent $L$, and we restrict the domain where the functions of $\mathcal{F}$ are defined to a smaller set $Y \subseteq X$ then the class of functions we obtain in such a way remains $L_1$-dense with the same parameter $D$ and exponent $L$. Hence if we fix a set $Y$ with cardinality $|Y| = N_k$ for which the property sup$_{f \in \mathcal{F}} R_Y(f) \leq N_k \rho_k$ holds (with the quantity $R_Y(f)$ introduced in the formulation of Lemma 2.2), then the number of those sequences $x^{(p)}$ for which $x^{(p)} \in B_p(\mathcal{F}) \cap Y^p$ can be bounded by our induction hypothesis by $C_kN^p_kD\rho^{p/4}$. We shall bound the number of the sequences $x^{(p)} \in B_p(\mathcal{F}) \cap Y^p$ for the remaining sets $Y$ with cardinality $|Y| = N_k$ by the trivial upper bound $N^p_k$, but the number of such sets $Y$ is less than $\left( \frac{2N_k}{N_k} \right) D \exp \left\{ -\frac{1}{100}2^{k/20}\rho^{-1/20} \right\}$ by Lemma 2.2. This yields the upper bound $C_kN^p_kD\rho^{p/4} \left( \frac{2N_k}{N_k} \right) + N^p_k \left( \frac{2N_k}{N_k} \right) D \exp \left\{ -\frac{1}{100}2^{k/20}\rho^{-1/20} \right\}$ for the sum we get by summing up the number of sequences $x^{(p)} \in Y^p \cap B_p(\mathcal{F})$ for all subsets with $|Y| = N_k$ elements. To prove (2.10) we still have to take into account how many times we counted the sequences $x^{(p)} \in B_p(\mathcal{F})$ in this summation. If all coordinates of a sequence $x^{(p)} \in B_p(\mathcal{F})$ are different, then we counted it $(\frac{2N_k-p}{N_k-p})$-times, because to find a set $Y$, $|Y| = N_k$, containing the elements of this sequence $x^{(p)}$ we have to extend these points with $N_k - p$ new points from the remaining $2N_k - p$ points of $X$. If some coordinates of a sequence $x^{(p)}$ may agree, then we might have counted this sequence with greater multiplicity. The above considerations imply (2.10).

To prove relation (2.8) with the help of (2.10) let us observe that under the conditions of Theorem 1A (In particular, we have $\frac{1}{N_0} \leq 16\rho^{3/2}$, $p^2 \leq \rho^{-1/50} \leq \frac{1}{10}\rho^{-1/6}$, $2N_k - p \geq N_k = 2kN_0$ for all $k = 1, 0, 2, \ldots$, and $\rho > 0$ is sufficiently small.)

$$N^p_k \left( \frac{2N_k}{N_k} \right) = N^p_k \left( \frac{2N_k}{N_k} \right) = N^p_k \frac{2N_k(2N_k-1)\cdots(2N_k-p+1)}{N_k(N_k-1)\cdots(N_k-p+1)}$$

$$= N^p_{k+1} \left( 1 + \frac{1}{2(N_k-1)} \right) \left( 1 + \frac{2}{2(N_k-2)} \right) \cdots \left( 1 + \frac{p-1}{2(N_k-p+1)} \right)$$

$$\leq N^p_{k+1} \exp \left\{ p^2 \frac{2^{k+1}}{N_0} \right\} \leq N^p_{k+1} e^{2^{-(k+1)}\rho^{1/3}} \leq N^p_{k+1} \left( 1 + \frac{1}{3} 2^{-(k+1)} \rho \right),$$

and

$$\exp \left\{ -\frac{1}{100}2^{k/20}\rho^{-1/20} \right\} = \rho^{p/4} \exp \left\{ -\frac{1}{100}2^{k/20}\rho^{-1/20} + \frac{p}{4} \log \frac{1}{\rho} \right\}$$

$$\leq C_k \rho^{p/4} \cdot \frac{1}{3} 2^{-(k+1)} \rho$$
Proof of Lemma 3.1. Let us define for all functions $\mu$ satisfies the inequality

$$\int S_n f \leq \frac{\rho}{2}$$

and $\bar{F}$ simply check that $n \geq 1$ and exponent $L \geq 1$. Theorem 1A is proved.

3. The proof of Theorem 1.

First we prove the following Lemma 3.1 which is a special case of Theorem 1.

**Lemma 3.1.** Let us consider a finite set $X = \{x_1, \ldots, x_{2^k}\}$ with $N = 2^k$ elements together with an $L_1$-dense class of function $F$ on $X$ with parameter $D \geq 1$ and exponent $L \geq 1$ that contains such functions $f \in F$ for which 0 \leq f(x) \leq 1 for all $x \in X$, and $\int f(x) \mu(dx) \leq \rho$ with some $0 < \rho < 1$. Here $\mu$ denotes the uniform distribution on $X$. Let us take the $n$-fold direct product $X^n$ of $X$ with some number $n \geq 2$, and define the function $S_n(f)(x_{s_1}, \ldots, x_{s_n}) = \sum_{j=1}^{n} f(x_{s_j})$ for all $(x_{s_1}, \ldots, x_{s_n}) \in X^n$ and $f \in F$. Let us assume that $\rho \leq n^{-200}$, and $N = 2^k \geq \rho^{-3/2}$. Then the set $B_n(u) \subset X^n$ defined as

$$B_n(u) = \left\{(x_{s_1}, \ldots, x_{s_n}): \sup_{f \in F} S_n(f)(x_{s_1}, \ldots, x_{s_n}) > u \right\}$$

(3.1)

satisfies the inequality

$$\mu_n(B_n(u)) \leq 2D \rho^{u/25} \text{ for all } u \geq 40L,$$  

(3.2)

where $\mu_n$ denotes the uniform distribution on $X^n$.

**Proof of Lemma 3.1.** Let us define for all functions $f \in F$ and integers $j$, $1 \leq j \leq R$, where $R$ is defined by the relation $n < 2^R \leq 2n$, the functions $f_j(x) = \min(2^{-j}, f(x))$ and $\bar{f}_j(x) = 2^j f_j(x)$, $x \in X$. Put $F_j = \{f_j: f \in F\}$ and $\bar{F}_j = \{\bar{f}_j: f \in F\}$. One can simply check that $F_j$ is an $L_1$-dense class with parameter $D$ and exponent $L$, while $\bar{F}_j$ is an $L_1$-dense class with parameter $D2^jL$ and exponent $L$, if $F$ is an $L_1$-dense class with parameter $D$ and exponent $L$. We can also state that $\int f_j(x) \mu(dx) \leq \rho$, and $\int \bar{f}_j(x) \mu(dx) \leq 2^j \rho$ for all $f \in F$.

Let us define for all $f \in F$ and $1 \leq j \leq R$ the following function $H_j(f)$ on $X^n$:

$$H_j(f)(x_{s_1}, \ldots, x_{s_n}) = \text{the number of such indices } l \text{ for which } \bar{f}_j(x_{s_l}) = 1.$$

We can write

$$S_n(f)(x_{s_1}, \ldots, x_{s_n}) \leq \sum_{j=1}^{R} 2^{1-j} H_j(f)(x_{s_1}, \ldots, x_{s_n}) + 1$$
for all $f \in \mathcal{F}$. This formula implies the inequality

$$\sup_{f \in \mathcal{F}} S_n(f)(x_{s_1}, \ldots, x_{s_n}) \leq \sum_{j=1}^{R} 2^{1-j} \sup_{f \in \mathcal{F}} H_j(f)(x_{s_1}, \ldots, x_{s_n}) + 1,$$

and the relation

$$\left\{ (x_{s_1}, \ldots, x_{s_n}): \sup_{f \in \mathcal{F}} S_n(f)(x_{s_1}, \ldots, x_{s_n}) > u \right\} \subset \bigcup_{j=1}^{R} \left\{ (x_{s_1}, \ldots, x_{s_n}): 2^{1-j} \sup_{f \in \mathcal{F}} H_j(f)(x_{s_1}, \ldots, x_{s_n}) > (\sqrt{2} - 1)(u - 1)2^{-j/2} \right\}.$$

Hence

$$\mu_n(B_n(u)) \leq \sum_{j=1}^{R} \mu_n(D_n(u, j)) \quad (3.3)$$

for the set $B_n(u)$ defined in (3.1) by

$$D_n(u, j) = \left\{ (x_{s_1}, \ldots, x_{s_n}): \sup_{f \in \mathcal{F}} H_j(f)(x_{s_1}, \ldots, x_{s_n}) > \frac{\sqrt{2} - 1}{2}(u - 1)2^{j/2} \right\},$$

$$1 \leq j \leq R.$$

We can prove Lemma 3.1 with the help of relation (3.3) if we give a good estimate on the measures $\mu_n(D_n(u))$. This can be done with the help of Theorem 1A.

Indeed, the set $D_n(u, j)$ consists of such sequences $(x_{s_1}, \ldots, x_{s_n}) \in X^n$ which have a subsequence $(x_{s_{p_1}}, \ldots, x_{s_{p_t}})$ with $t = t(j) = \left\lfloor \frac{\sqrt{2} - 1}{2}(u - 1)2^{j/2} \right\rfloor + 1$ elements, where $\lfloor \cdot \rfloor$ denotes integer part, with the property that there is a function $f \in \mathcal{F}$ such that the function $\bar{f}_j(\cdot)$ defined with its help equals 1 in all coordinates of this subsequence. More explicitly,

$$D_n(u, j) = \bigcup_{\{t_1, \ldots, t_t\} \subset \{1, \ldots, n\}} \left( \bigcup_{f \in \mathcal{F}} \left\{ (x_1, \ldots, x_n): \bar{f}_j(x_{s_{t_1}}) = 1, \ldots, \bar{f}_j(x_{s_{t_t}}) = 1 \right\} \right) \quad (3.4)$$

with $t = t(j) = \left\lfloor \frac{\sqrt{2} - 1}{2}(u - 1)2^{j/2} \right\rfloor + 1$.

The outside union in (3.4) consists of $(n) \leq n^{t(j)}$ terms, and the cardinality of the sequences $(x_1, \ldots, x_n)$ in the inner union can be bounded by means of Theorem 1A for each term if it is applied with $p = t(j)$, in the space $X$ consisting of $N = 2^k = N_02^k$ points, for the class of functions $\mathcal{F}_j$ which is an $L_1$-dense class of functions with parameter $D^jL$ and exponent $L$. Moreover, the functions $\bar{f}_j \in \mathcal{F}_j$ satisfy the inequality $\int \bar{f}_j(x) \mu(dx) \leq 2^j \rho$. This means that under the conditions of Lemma 3.1 we can apply Theorem 1A for the class of functions $\mathcal{F}_j$ with parameter $\bar{\rho} = 2^{j+1} \rho$ instead of $\rho$. (We
have to check that all conditions of Theorem 1A hold. In particular, we can state that \( \tilde{\rho} = 2^{j+1} \rho \leq L^{-20} \), since \( \rho \leq n^{-200} \), \( 2^j \leq 2n \), and since we estimate the probability in formula 3.2 only under the condition \( u \geq 40L \), and this probability is zero if \( u > n \), hence we may assume that \( L \leq \frac{n}{40} \). We chose the term \( N_0 \) in the application of Theorem 1A as \( N_0 = 2^{k_0} \) with \( k_0 \) defined by the relation \( \frac{1}{16} \rho^{-3/2} < 2^{k_0} \leq \frac{1}{8} \rho^{-3/2} \), and \( k = k - k_0 \).

We will prove with the help of the above relations the inequality

\[
\mu_n(D_n(u, j)) = \frac{|D_n(u, j)|}{N^n} \leq 2nt^{(j)} D^{2jL} (2^{j+1} \rho)^{t(j)/4} \leq 2D(8n^5 \rho)^{t(j)/4} \leq 2D\rho^{t(j)/4} \leq D\rho^{j/25}.
\]  

(3.5)

To get the first estimate in the second line of formula (3.5) observe that under the condition of Lemma 3.1 \( \frac{\sqrt{2}-1}{2}(u - 1) \geq 4L \), hence \( 2^{jL} \leq 2^{j2^{-j/2}t(j)/4} \leq 2^{t(j)/4} \), and by the definition of the number \( R \) we have \( (2^{j+1})^{t(j)/4} \leq (2^{R+1})^{t(j)/4} \leq (4n)^{t(j)/4} \). We imposed the condition \( n \leq \rho^{-1/200} \), and this implies the second inequality. Finally \( t(j) \geq \frac{ju}{4} \). (In the last inequality a \( j = 1 \) parameter is the worst case.) Relation (3.2) follows from (3.3) and (3.5). Lemma 3.1 is proved.

Now we turn to the proof of the main result of this paper.

**Proof of Theorem 1.** We may assume that all functions \( f \in \mathcal{F} \) are non-negative, i.e. \( 0 \leq f(x) \leq 1 \) for all \( f \in \mathcal{F} \) and \( x \in X \), because we can replace the function \( f \) by its absolute value \( |f| \), and apply the result for this new class of functions which also satisfies the conditions of Theorem 1. Next I show that we also may assume that the class of functions \( \mathcal{F} \) contains only finitely many functions, satisfies the same conditions as the original class of function \( \mathcal{F} \) with the only difference that we assume that \( \mathcal{F} \) is an \( L_1 \)-dense class with the same exponent \( L \) but with parameter \( D 2^L \) instead of \( D \).

Indeed, if we have the same upper bound for the probability of \( P \left( \sup_{f \in \mathcal{F}'} S_n(f) > u \right) \)

for all finite subsets \( \mathcal{F}' \subset \mathcal{F} \), then this upper bound remains valid if we take the supremum for all \( f \in \mathcal{F} \). Besides, the conditions of Theorem 1 remain valid if \( \mathcal{F} \) is replaced by an arbirary class of functions \( \mathcal{F}' \subset \mathcal{F} \) with a small modification. Namely, we can state that \( \mathcal{F}' \) is an \( L_1 \)-dense subclass with exponent \( L \) but with a possibly different parameter \( \tilde{D} = D 2^L \). (We had to change the parameter \( D \) of an \( L_1 \)-dense class \( \mathcal{F}' \subset \mathcal{F} \), because if a set of functions \( f_1, \ldots, f_m \) is an \( \varepsilon \)-dense class \( \mathcal{F}_{\varepsilon,\nu} \) appearing in the definition of \( L_1 \)-dense property of the class of functions \( \mathcal{F} \), then these functions \( f_j, 1 \leq j \leq m \), may be not contained in \( \mathcal{F}' \). This problem can be overcome if we choose first an \( \varepsilon/2 \) dense subclass \( \mathcal{F}_{\varepsilon/2,\nu} \) in \( \mathcal{F} \) with at most \( D 2^L \varepsilon^{-L} \) element, and then we replace the functions of this subclass with very close functions from \( \mathcal{F}' \) if this is necessary.)

In the next step I show that we may restrict our attention to the case when the functions of the class of functions \( \mathcal{F} \) (consisting of finitely many functions) take only finitely many values. For this goal first I split up the interval \([0,1]\) to \( n \) subintervals of the following form: \( B_j = (\frac{j-1}{n}, \frac{j}{n}] \), \( 2 \leq j \leq n \), and \( B_1 = [0, \frac{1}{n}] \). (We defined the function \( B_1 \) in a slightly different way in order to guarantee that the point zero is also
contained in some set $B_j$. Then given a class of function $\mathcal{F}$ on a set $X$ that contains finitely many functions $f_1, \ldots, f_R$, we define the following sets $A(s(1), \ldots, s(R)) \subset X$ (depending on $\mathcal{F}$):

$$A(s(1), \ldots, s(R)) = \{x: f_j(x) \in B_{s(j)}, \text{ for all } 1 \leq j \leq R\},$$

where $1 \leq s(j) \leq n$ for all $1 \leq j \leq R$.

In such a way the sets $A(s(1), \ldots, s(R))$ make up a partition of the set $X$. Actually, for the sake of a simpler argument we shall diminish a bit the set $X$, by defining it as the union of those sets $A(s(1), \ldots, s(R))$ for which $\mu(A(s(1), \ldots, s(R))) > 0$ with the measure $\mu$ appearing in Theorem 1. This restriction will cause no problem in our later considerations.

We shall define new functions $\tilde{f}_j(x)$, $1 \leq j \leq R$, by means of the partition of $X$ to the sets $A(s(1), \ldots, s(R))$ by the formula

$$\tilde{f}_j(x) = \frac{\int_{A(s(1), \ldots, s(R))} f_j(x) \mu(dx)}{\mu(A(s(1), \ldots, s(R)))}, \quad 1 \leq j \leq R, \quad \text{if } x \in A(s(1), \ldots, s(R)).$$

We have $|f_j(x) - \tilde{f}_j(x)| \leq \frac{1}{n}$ for all $1 \leq j \leq n$ and $x \in X$. Hence

$$\left| \sup_{1 \leq j \leq R} (S_n(f_j) - S_n(\tilde{f}_j)) \right| \leq 1,$$

for almost all sequences $\xi_1(\omega), \ldots, \xi_n(\omega)$, and as a consequence

$$P \left( \sup_{1 \leq j \leq R} S_n(f_j) > u + 1 \right) \leq P \left( \sup_{1 \leq j \leq R} S_n(\tilde{f}_j) > u \right) \quad (3.6)$$

Let us also observe that the class of functions $\tilde{\mathcal{F}} = \{\tilde{f}_j, 1 \leq j \leq R\}$ also satisfies the conditions of Theorem 1, i.e. $\int \tilde{f}_j(x) \mu(dx) \leq \rho$ for all $1 \leq j \leq R$, and $\tilde{\mathcal{F}}$ is an $L_1$-dense class with parameter $\tilde{D} = D 2^L$ and exponent $L$. (The conditions on the numbers $n$ and $\rho$ clearly remain valid.)

The first relation follows from the identity $\int \tilde{f}_j(x) \mu(dx) = \int f_j(x) \mu(dx)$ which holds because of the identities $\int_{A(s(1), \ldots, s(R))} \tilde{f}_j(x) \mu(dx) = \int_{A(s(1), \ldots, s(R))} f_j(x) \mu(dx)$ for all sets $A(s(1), \ldots, s(R)).$

To prove the $L_1$-dense property of $\tilde{\mathcal{F}}$ let us introduce for all probability measures $\nu$ the probability measure $\tilde{\nu} = \tilde{\nu}(\nu)$ which is defined by the property that for all (measurable) sets $A(s(1), \ldots, s(R))$ and $B \subset A(s(1), \ldots, s(R))$ the identity $\tilde{\nu}(B) = \mu(B) \frac{\nu(A(s(1), \ldots, s(R)))}{\mu(A(s(1), \ldots, s(R)))}$ holds. Because of the special form of the functions $\tilde{f}_j$ if a set of function $\tilde{\mathcal{F}}_{\epsilon, \tilde{\nu}} \subset \tilde{\mathcal{F}}$ is an $\epsilon$-dense subset of $\tilde{\mathcal{F}}$ in the space $(X, \mathcal{X}, \tilde{\nu})$, then it is also $\epsilon$-dense in the space $(X, \mathcal{X}, \nu)$. (In the proof of this statement we exploit that

$$\tilde{\nu}(A(s(1), \ldots, s(R))) = \nu(A(s(1), \ldots, s(R)))$$

14
for all sets $A(s(1), \ldots, s(R))$, and it depends only on the value of a measure $\nu$ on the sets $A(s(1), \ldots, s(R))$ whether a set of functions $\{f_1, \ldots, f_m\} \subset \hat{F}$ is an $\varepsilon$-dense subclass of $\hat{F}$ with respect to the measure $\nu$.

Hence it is enough to prove the existence of an $L_1$-dense set $\hat{F}_{\varepsilon, \nu'}$ with cardinality bounded by $D\varepsilon^{-L}$ only with respect to such measures $\nu'$ which can be written in the form $\nu' = \tilde{\nu}(\nu)$ with some probability measure $\tilde{\nu}$. In this case the relation we want to check follows from the $L_1$-dense property of the original class of functions $F$ and the inequality $\int |\hat{f}_j - \hat{f}_{j'}|d\tilde{\nu} \leq \int |f_j - f_{j'}|d\tilde{\nu}$ for all pairs $f_j, f_{j'} \in F_j$ and probability measure $d\tilde{\nu}$. The last inequality holds, since

$$\int_{A(s(1), \ldots, s(R))} |\hat{f}_j(x) - \hat{f}_{j'}(x)|d\nu(x) \leq \int_{A(s(1), \ldots, s(R))} |f_j(x) - f_{j'}(x)|d\tilde{\nu}(x)$$

for all sets $A(s(1), \ldots, s(R))$.

Let us observe that for all $k \geq 1$ we can define such a ‘discretized’ probability measure $\mu_k$ on the $\sigma$-algebra $\mathcal{X}$ with atoms $A(s(1), \ldots, s(R))$ in the space $X$ for which

$$|\mu_k(A(s(1), \ldots, s(R)) - \mu(A(s(1), \ldots, s(R))) \leq 2^{-k},$$

and

$$\mu_k(A(s(1), \ldots, s(R)) = \alpha(A(s(1), \ldots, s(R)))2^{-k}$$

(3.7)

with a non-negative integer $\alpha(A(s(1), \ldots, s(R)))$ for all sets $A(s(1), \ldots, s(R))$. (To find such a probability measure $\mu_k$ let us list the sets $A(s(1), \ldots, s(R))$ as $B_1, \ldots, B_Q$, and define the measure $\mu_k$ by the relation $\sum_{1=1}^{\hat{s}} \mu_k(B_l) = \beta_s 2^{-k}$ if $(\beta_s - 1)2^{-k} < \sum_{l=1}^{\hat{s}} \mu_k(B_l) \leq \beta_s 2^{-k}$ with a positive integer $\beta_s$. We assume this relation for all $1 \leq s \leq Q$.)

Clearly,

$$P \left( \sup_{1 \leq j \leq R} S_n(\hat{f}_j) > u \right) = \lim_{k \to \infty} P_{\mu_k} \left( \sup_{1 \leq j \leq R} S_n(\hat{f}_j) > u \right)$$

(3.8)

for all $u > 0$, where $P_{\mu_k}$ means that we consider the probability of the same event as at the left-hand side of the identity, but this time we take iid. random variables $\xi_1, \ldots, \xi_n$ with distribution $\mu_k$ (on the $\sigma$-algebra generated by the atoms $A(s(1), \ldots, s(R))$) in the definition of the random variables $S_n(\hat{f}_j)$.

We shall bound the probabilities at the right-hand side in formula (3.8) for all large indices $k$ by means of Lemma 3.1. This will be done with the help of the following construction. Take a space $\hat{X} = \hat{X}_k = \{x_1, x_2, \ldots, x_{2^k}\}$ with $2^k$ elements and with the uniform distribution $\mu = \mu^{(k)}$ on its points. Let us fix a partition of $\hat{X}$ consisting of some sets $\hat{A}(s(1), \ldots, s(R))$ with $\alpha(A(s(1), \ldots, s(R)))$ elements, where the number $\alpha(\cdot)$ was introduced in (3.7). Let us define the functions $\hat{f}_j(x)$, $1 \leq j \leq R$, $x \in \hat{X}$, by the formula $\hat{f}_j(x) = \frac{x(j)}{n}$, $1 \leq j \leq R$, if $x \in \hat{A}(s(1), \ldots, s(R))$. Take the $n$-fold direct product $\hat{X}^n$ of $\hat{X}$ together with the uniform distribution $\mu_n = \mu^{(k)}$ on it and the
functions $S_n(\hat{f}_j)(x_{t_1}, \ldots, x_{t_n}) = \sum_{l=1}^{n} \hat{f}_j(x_{t_l})$, $1 \leq j \leq R$, if $(x_{t_1}, \ldots, x_{t_n}) \in \hat{X}^n$ on the space $\hat{X}^n$. I claim that

\begin{equation}
P_{\bar{\mu}_k}\left( \sup_{1 \leq j \leq R} S_n(\hat{f}_j) > u \right) = \mu_n^{(k)}\left( \left\{ (x_{t_1}, \ldots, x_{t_n}) : \sup_{1 \leq j \leq R} S_n(\hat{f}_j)(x_{t_1}, \ldots, x_{t_n}) > u \right\} \right) \leq 2D\rho^{u/25}
\end{equation}

if $u > 8L$.

The identity in formula (3.9) holds, since the joint distribution of the random vectors $S_n(\hat{f}_j)(\xi_1, \ldots, \xi_n)$, $1 \leq j \leq R$, where $\xi_1, \ldots, \xi_n$ are independent random variables with distribution $\bar{\mu}_k$ and of the random vectors $S_n(\hat{f}_j)(x_{t_1}, \ldots, x_{t_n})$, $1 \leq j \leq R$, where the distribution of $(x_{t_1}, \ldots, x_{t_n}) \in \hat{X}^n$ is $\mu_n^{(k)}$, agree. To prove the last inequality of (3.9) it is enough to check that for all sufficiently large numbers $k$ the class of functions $\hat{F} = \{\hat{f}_1, \ldots, \hat{f}_R\}$ on the space $\hat{X} = \hat{X}_k$ satisfies the conditions of Lemma 3.1. Namely, the $L_1$-dense property holds with parameter $D = 2D^L$ and exponent $L$, and $\int \hat{f}_j(x)\mu(dx) \leq \rho$ with a number $\rho \leq n^{-200}$ for all $\hat{f}_j \in \hat{F}$.

It is the $L_1$-dense property of the system $\hat{X}, \hat{F}$ that may demand some explanation. Let us observe that it is enough to check this property only for such probability measures $\hat{\nu}$ which have a constant density (with respect to the uniform distribution $\mu^{(k)}$) on all sets $A(s(1), \ldots, s(R))$. This reduction of the probability measures can be justified similarly to the argument we applied to prove the $L_1$-dense property of $\hat{F}$ with the help of the functions $\hat{\nu}(\nu)$. Given a measure $\hat{\nu}$ on $\hat{X}$ with the above property let us correspond to it the measure $\nu$ on $X$ defined by $\hat{\nu}(A(s(1), \ldots, s(R))) = \nu(A(s(1), \ldots, s(R)))$ for all sets $A(s(1), \ldots, s(R))$. Then we get that if a class of functions $\hat{F}_{\varepsilon, \hat{\nu}} = \{\hat{f}_1, \ldots, \hat{f}_n\}$ is an is an $\varepsilon$-dense class of $\hat{F}$ with respect to the measure $\hat{\nu}$, then the class of function $\hat{F}_{\varepsilon, \nu} = \{\hat{f}_1, \ldots, \hat{f}_n\}$ is an is an $\varepsilon$-dense class with respect to the measure $\nu$. The $L_1$-density property of $\hat{F}$ follows from this fact.

Then we get the inequality part of formula (3.9) from Lemma 3.1. Relation (1.1) follows from (3.9), (3.8) and (3.6). We still have to understand that in our estimation the coefficient $2D = 2D^L$ in (3.9) can be replaced by $D$ if we estimate the probability (1.1) only for $u \geq \frac{1}{4}(L + 1)$, and the term $\rho^{u/25}$ in (3.9) is replaced by $\rho^{u/50}$ when turning from (3.9) to formula (1.1). To see this observe that $\rho^{u/25} \leq \rho^{\frac{1}{4}(L+1)/50}. \rho^{u/50} \leq n^{-(L+1)} \rho^{u/50} \leq \frac{1}{2}2^{-L} \rho^{u/50}$ if $u \geq \frac{1}{4}L$, $\rho \leq n^{-200}$, and $n \geq 2$. Theorem 1 is proved.
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