A CLASSIFICATION ALGORITHM FOR COMPLEX SINGULARITIES OF CORANK AND MODALITY UP TO TWO

JANKO BÖHM, MAGDALEEN S. MARAIS, AND GERHARD PFISTER

Abstract. In [Arnold et al. 1985], Arnold has obtained normal forms and has developed a classifier for, in particular, all isolated hypersurface singularities over the complex numbers up to modality 2. Building on a series of 105 theorems, this classifier determines the type of the given singularity. However, for positive modality, this does not fix the right equivalence class of the singularity, since the values of the moduli parameters are not specified. In this paper, we present a simple classification algorithm for isolated hypersurface singularities of corank \( \leq 2 \) and modality \( \leq 2 \). For a singularity given by a polynomial over the rationals, the algorithm determines its right equivalence class by specifying a polynomial representative in Arnold’s list of normal forms.

1. Introduction

In his classical paper on singularities [Arnold 1974], Arnold has classified all isolated hypersurface singularities over the complex numbers with modality \( \leq 2 \). He has given normal forms in the sense of polynomial families with moduli parameters such that every stable equivalence class of function germs contains at least one (but only finitely many) elements of these families. We refer to such elements as normal form equations. Two germs are stably equivalent if they are right equivalent after the direct addition of a non-degenerate quadratic form. Two function germs \( f, g \in m^2 \subset \mathbb{C}[x_1, \ldots, x_n] \), where \( m = \langle x_1, \ldots, x_n \rangle \), are right equivalent, written \( f \sim g \), if there is a \( \mathbb{C} \)-algebra automorphism \( \phi \) of \( \mathbb{C}[x_1, \ldots, x_n] \) such that \( \phi(f) = g \). Using the Splitting Lemma, any germ with an isolated singularity at the origin can be written, after choosing a suitable coordinate system, as the sum of two functions on disjoint sets of variables. One function that is called the non-degenerate part, is a non-degenerate quadratic form, and the other part, called the residual part is in \( m^3 \). The Splitting Lemma is implemented in Singular as part of the library classify.lib [Krüger 1997].

In [Arnold et al. 1985], Arnold has made this classification explicit by describing an algorithmic classifier, which is based on a series of 105 theorems. This approach determines the type of the singularity in the sense of its normal form. However, the values of the moduli parameters are not determined, that is, no normal form equation is given. Arnold’s classifier is implemented in classify.lib.

Classification of complex singularities has a multitude of practical and theoretical applications. The classification of real singularities in [Marais and Steenpaß 2015a, 2016; Böhm, Marais and Steenpaß 2015b] is based on determining the complex type of the singularity.

In this paper, we develop a determinant for complex singularities of modality \( \leq 2 \) and corank \( \leq 2 \), which computes, for a given rational input polynomial, a normal form equation in its equivalence class. For singularities with non-degenerate Newton boundary, our determinant is based on a simple and uniform approach, which does not require a case-by-case analysis (except for some trivial final steps to read off the values of the moduli parameters according to Arnold’s choice of the normal form). Two series of cases with degenerate Newton boundary are handled with more specific methods. Here, we use results of [Luengo and Pfister 1990] to compute a normal form. In this way, we obtain an approach which does not only determine the moduli
parameters, but also allows for an elegant implementation. We have implemented our algorithm in the SINGULAR-library classify2.lib (Böhm, Marais and Pfister 2016).

It is important to note that two different normal form equations do not necessarily represent two different right equivalence classes. In (Marais and Steenpaß 2016) the complete structure of the equivalence classes for, in particular, complex singularities of modality 1 and corank 2 is determined, in the sense that all equivalences between normal form equations are described. All normal form equations in the right equivalence class of a given unimodal corank 2 singularity can, hence, be determined by combining our classifier with the results in (Marais and Steenpaß 2016). There is not yet a similar complete description of the structure of the equivalence classes of bimodal singularities.

This paper is structured as follows: In Section 2 we give the fundamental definitions and provide the prerequisites on singularities and their classification. In Section 3 we develop a general algorithm for the classification of complex singularities of modality \( \leq 2 \) and corank \( \leq 2 \). Essentially, the algorithm is structured into a subalgorithm for elimination below the Newton polygon, and a subalgorithm for elimination on and above the Newton polygon, which also determines the values of the moduli parameters. The algorithm for the two series of germs of modality 2 with degenerate Newton boundary is discussed in Section 4.

2. Definitions and Preliminary Results

In this section, we give some basic definitions and results, as well as some notation that will be used throughout the paper.

**Definition 1.** Let \( K \subset \mathbb{C}[x_1, \ldots, x_n] \) be a union of equivalence classes with respect to the relation \( \sim \). A normal form for \( K \) is given by a smooth map

\[
\Phi : B \rightarrow \mathbb{C}[x_1, \ldots, x_n] \subset \mathbb{C}[x_1, \ldots, x_n]
\]

of a finite-dimensional \( \mathbb{C} \)-linear space \( B \) into the space of polynomials for which the following three conditions hold:

1. \( \Phi(B) \) intersects all equivalence classes of \( K \),
2. the inverse image in \( B \) of each equivalence class is finite,
3. \( \Phi^{-1}(\Phi(B) \setminus K) \) is contained in a proper hypersurface in \( B \).

The elements of the image of \( \Phi \) are called normal form equations.

**Remark 2.** Arnold has chosen a normal form for each of the corank 2 singularities of modality \( \leq 2 \). He has also associated a type to each normal form, see Table 1. We denote the normal form corresponding to the type \( T \) by \( NF(T) \). For \( b \in \text{par}(NF(T)) := \Phi^{-1}(K) \) with \( K \) as in Definition 1 we write \( NF(T)(b) := \Phi(b) \) for the corresponding normal form equation.

In the following, we give a short account on weighted jets, filtrations, and Newton polygons. See (Arnold 1974) and (de Jong and Pfister 2000) for more details.

**Definition 3.** Let \( w = (c_1, \ldots, c_n) \in \mathbb{N}^n \) be a weight on the variables \( (x_1, \ldots, x_n) \). The \( w \)-weighted degree on \( \text{Mon}(x_1, \ldots, x_n) \) is given by \( w \text{-deg}(\prod_{i=1}^{n} x_i^{s_i}) := \sum_{i=1}^{n} c_i s_i \). If the weight of all variables is equal to 1, we refer to the weighted degree of a monomial as the standard degree of \( m \) and write \( \text{deg}(m) \) for \( w \text{-deg}(m) \). We use the same notation for terms of polynomials.

We call a polynomial \( f \in \mathbb{C}[x_1, \ldots, x_n] \) quasihomogeneous or weighted homogeneous of degree \( d \) with respect to the weight \( w \) if \( w \text{-deg}(t) = d \) for any term \( t \) of \( f \).

**Definition 4.** Let \( w = (w_1, \ldots, w_s) \in (\mathbb{N}^n)^s \) be a finite family of weights on the variables \( (x_1, \ldots, x_n) \). For any monomial (or term) \( m \in \mathbb{C}[x_1, \ldots, x_n] \), we define the piecewise weight with respect to \( w \) as

\[
w \text{-deg}(m) := \min_{i=1, \ldots, s} w_i \text{-deg}(m).
\]

A polynomial \( f \) is called piecewise homogeneous of degree \( d \) with respect to \( w \) if \( w \text{-deg}(t) = d \) for any term \( t \) of \( f \).
Table 1. Normal forms of singularities of modality $\leq 2$ and corank $\leq 2$ as given in Arnold et al. [1985]

|        | Complex normal form | Restrictions | Complex normal form | Restrictions |
|--------|--------------------|--------------|--------------------|--------------|
| Simple |                    |              |                    |              |
| $A_k$  | $x^{k+1}$          | $k \geq 1$   | $J_{3,0}$          | $x^3 + bx^3y^3 + y^3 + cxy^3$ $4b^3 + 27 \neq 0$ |
| $D_k$  | $x^2y + y^{k-1}$   | $k \geq 4$   | $J_{3,p}$          | $x^3 + x^3y^3 + ay^3 + p$ $p > 0$, $a_0 \neq 0$ |
| $E_6$  | $x^3 + y^4$       | -            | $Z_{1,0}$          | $x^3y + dx^3y^3 + cxy^3 + y^3$ $4d^3 + 27 \neq 0$ |
| $E_7$  | $x^3 + xy^3$      | -            | $Z_{1,p}$          | $x^3y + x^3y^3 + ay^3 + p$ $p > 0$, $a_0 \neq 0$ |
| $E_8$  | $x^3 + y^5$       | -            |                    |              |
| $X_9$  | $x^3 + ax^2y^2 + y^4$ | $a^2 \neq 1$ | $W_{1,0}$          | $x^4 + ax^2y^3 + y^6$ $a_0^2 \neq 4$ |
| $J_{10}$ | $x^3 + ax^2y^2 + y^6$ | $4d^3 + 27 \neq 0$ |                    |              |
| $J_{10+k}$ | $x^3 + ax^2y^2 + ay^{k+1}$ | $a \neq 0$, $k > 0$ | $W_{1,2q-1}$ | $(x^2 + y^3)^2 + axy^{k+q}$ $q > 0$, $a_0 \neq 0$ |
| $X_{9+k}$ | $x^3 + ax^2y^2 + ay^{k+1}$ | $a \neq 0$, $k > 0$ |                    |              |
| $Y_{1+s}$ | $x^r + ax^2y^2 + y^5 + x^{s+1}$ | $a_0 \neq 0$, $r, s > 4$ | $E_{19}$       | $x^3 + xy^7 + ay^{11}$ - |
| $E_{12}$ | $x^3 + y^4 + ay^5$ | -            | $E_{20}$           | $x^3 + y^{11} + axy^4$ - |
| $E_{13}$ | $x^3 + xy^5 + ay^6$ | -            | $E_{17}$           | $x^3y + y^6 + axy^4$ - |
| $E_{14}$ | $x^3 + y^6 + axy^7$ | -            | $E_{18}$           | $x^3 + x^4y^7 + axy^4$ - |
| $Z_{11}$ | $x^3 + xy^6 + axy^7$ | -            |                    |              |
| $Z_{12}$ | $x^3 + x^4y^6 + ax^4y^7$ | -            |                    |              |
| $Z_{13}$ | $x^3y^5 + y^6 + axy^7$ | -            | $W_{17}$           | $x^4 + xy^5 + ay^7$ - |
| $W_{12}$ | $x^3 + y^7 + 2ax^2y^3$ | -            | $W_{18}$           | $x^3 + y^7 + ax^2y^3$ - |
| $W_{13}$ | $x^3 + xy^7 + ay^8$ | -            |                    |              |
|        |                    |              |                    |              |

Unimodal

|        | Complex normal form | Restrictions | Complex normal form | Restrictions |
|--------|--------------------|--------------|--------------------|--------------|
|        |                    |              |                    |              |
| $J_{10+k}$ | $x^3 + ax^2y^2 + ay^{k+1}$ | $a \neq 0$, $k > 0$ | $W_{1,2q}$ | $(x^2 + y^3)^2 + axy^{k+q}$ $q > 0$, $a_0 \neq 0$ |

**Definition 5.** Let $w$ be a (piecewise) weight on $\mathbb{M}(x_1, \ldots, x_n)$.

(1) Let $f = \sum_{i=0}^{\infty} f_i$ be the decomposition of $f \in \mathbb{C}[x_1, \ldots, x_n]$ into weighted homogeneous monomials $f_i$ of $w$-degree $i$. The **weighted $j$-jet** of $f$ with respect to $w$ is

$$w-\text{jet}(f, j) := \sum_{i=0}^{j} f_i.$$  

The sum of terms of $f$ of lowest $w$-degree is the **principal part** of $f$ with respect to $w$.

(2) A power series in $\mathbb{C}[x_1, \ldots, x_n]$ has **filtration** $d \in \mathbb{N}$ with respect to $w$ if all its monomials are of $w$-weight degree $d$ or higher. The power series of filtration $d$ form a sub-vector space

$$E_d^w \subset \mathbb{C}[x_1, \ldots, x_n].$$

(3) A power series $f \in \mathbb{C}[x_1, \ldots, x_n]$ is **weighted $k$-determined** with respect to the weight $w$ if

$$f \sim w-\text{jet}(f, k) + g \quad \text{for all } g \in E_{k+1}^w.$$  

We define the **weighted determinacy** of $f$ as the minimum number $k$ such that $f$ is $k$-determined.

**Definition 6.** Let $w \in \mathbb{N}^n$ be a single weight. A power series $f \in \mathbb{C}[x_1, \ldots, x_n]$ is called **semi-quasihomogeneous** with respect to $w$ if its principal part with respect to $w$ is non-degenerate, that is, has finite Milnor number. The principal part is then called the **quasihomogeneous part** of $f$.

**Notation 7.** (1) If the weight of each variable is 1, we write $E_d$ and $\text{jet}(f, j)$ instead of $E_d^w$ and $w-\text{jet}(f, j)$, respectively.

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1We say that $f$ is (semi-)quasihomogeneous if there exists a weight $w$ such that $f$ is (semi-)quasihomogeneous with respect to $w$.
(2) If for a given type $T$, $w \cdot \text{jet}(\text{NF}(T)(b), j)$ is independent of $b \in \text{par}(\text{NF}(T))$, we denote it by $w \cdot \text{jet}(T, j)$.

There are similar concepts of jets and filtrations for coordinate transformations:

**Definition 8.** Let $\phi$ be a $C$-algebra automorphism of $\mathbb{C}[[x_1, \ldots, x_n]]$ and let $w$ be a weight on $	ext{Mon}(x_1, \ldots, x_n)$.

1. For $j > 0$ we define $w \cdot \text{jet}(\phi, j) := \phi^w_j$ as the automorphism given by
   \[
   \phi^w_j(x_i) := w \cdot \text{jet}(\phi(x_i), w \cdot \deg(x_i) + j) \quad \text{for all } i = 1, \ldots, n.
   \]
   If the weight of each variable is equal to 1, that is, $w = (1, \ldots, 1)$, we write $\phi_j$ for $\phi^w_j$.

2. $\phi$ has filtration $d$ if, for all $\lambda \in \mathbb{N}$,
   \[
   (\phi - \text{id})E^w_{\lambda} \subset E^w_{\lambda + d}.
   \]

**Remark 9.** Note that $\phi_0(x_i) = \text{jet}(\phi(x_i), 1)$ for all $i = 1, \ldots, n$. Furthermore note that $\phi^w_0$ has filtration $\leq 0$, and that, for $j > 0$, $\phi^w_j$ has filtration $j$ if $\phi^w_{j-1} = \text{id}$.

The following definition gives an infinitesimal analogue of the above definition.

**Definition 10.** A formal vector field $v = \sum_i v_i \partial_{x_i}$ has filtration $d$ with respect to a weight $w$, if the directional derivative of $v$ raises the filtration by not less than $d$, that is,
   \[
   \text{for all } g \in E^w_{\delta}, \quad L_v(g) := \sum_i v_i \partial g \partial x_i \in E^w_{\delta + d}.
   \]

In a similar way as (Marais and Steenpaß 2015a, Proposition 8), one can prove:

**Proposition 11.** Let $f, g \in \mathbb{C}[[x_1, \ldots, x_n]]$ be two power series with $f \sim g$. Let $w \in \mathbb{N}^n$ and suppose that the maximal weighted filtration of $f$ with respect to $w$ is $k$. Furthermore, let $\phi$ be a $C$-algebra automorphism of $\mathbb{C}[[x_1, \ldots, x_n]]$ such that $\phi(f) = g$. If $\text{jet}(f, k)$ factorizes as
   \[
   w \cdot \text{jet}(f, k) = f_1^{s_1} \cdots f_i^{s_i}
   \]
   in $\mathbb{C}[x_1, \ldots, x_n]$, then $w \cdot \text{jet}(g, k)$ factorizes as
   \[
   w \cdot \text{jet}(g, k) = \phi_0^w(f_1)^{s_1} \cdots \phi_0^w(f_i)^{s_i}.
   \]

**Definition 12.** Let $f = \sum_{i,j} a_{i,j} x^i y^j \in \mathbb{C}[[x,y]]$, let $T$ be a corank 2 singularity type. We call
   \[
   \text{supp}(f) := \{x^i y^j \mid a_{i,j} \neq 0\}
   \]
   \[
   \text{supp}(T) := \text{supp}(\text{NF}(T)(b))
   \]
where $b \in \text{par}(\text{NF}(T))$ is generic, the support of $f$ and of $T$, respectively. Let
   \[
   \Gamma_+(f) := \bigcup_{x^i y^j \in \text{supp}(f)} ((i, j) + \mathbb{R}^2_+)
   \]
   \[
   \Gamma_+(T) := \bigcup_{x^i y^j \in \text{supp}(T)} ((i, j) + \mathbb{R}^2_+)
   \]
and let $\Gamma(f)$ and $\Gamma(T)$ be the boundaries in $\mathbb{R}^2$ of the convex hulls of $\Gamma_+(f)$ and $\Gamma_+(T)$, respectively. Then:

1. $\Gamma(f)$ and $\Gamma(T)$ are called the Newton polygons of $f$ and $T$, respectively.

2. The compact segments of $\Gamma(f)$ or $\Gamma(T)$ are called faces. If $\Delta$ is a face, then the set of monomials of $f$ lying on $\Delta$ is denoted by $\text{supp}(f, \Delta)$ and the sum of the terms lying on $\Delta$ by $\text{jet}(f, \Delta)$. Moreover, we write $\text{supp}(\Delta)$ for the set of monomials corresponding to the lattice points of $\Delta$, and set $\text{supp}(T, \Delta) := \text{supp}(T) \cap \text{supp}(\Delta)$. We use the same notation for a set of faces, considering the monomials lying on the union of the faces.

3. Any face $\Delta$ induces a weight $w(\Delta)$ on $\text{Mon}(x, y)$ in the following way: If $\Delta$ has slope $-\frac{w_x}{w_y}$, in lowest terms, and $w_x, w_y > 0$, we set $w(\Delta) \cdot \deg(x) = w_x$ and $w(\Delta) \cdot \deg(y) = w_y$. 
(4) If \( w_1, \ldots, w_s \) are the weights associated to the faces of \( \Gamma(f) \), respectively \( \Gamma(T) \), ordered by increasing slope, there are unique minimal integers \( \lambda_1, \ldots, \lambda_s \geq 1 \) such that the piecewise weight associated to \( (\lambda_1 w_1, \ldots, \lambda_s w_s) \) by Definition 4 is constant on \( \Gamma(f) \), respectively \( \Gamma(T) \). We denote this piecewise weight by \( w(f) \), respectively \( w(T) \), and the corresponding constant by \( d(f) \), respectively \( d(T) \).

(5) Let \( \Delta_1 \) and \( \Delta_2 \) be faces with weights \( w_1 \) and \( w_2 \), respectively, and let \( w \) be the piecewise weight defined by \( w_1 \) and \( w_2 \). Let \( d \) be the \( w \)-degree of the monomials on \( \Delta_1 \) and \( \Delta_2 \). Then \( \text{span}(\Delta_1, \Delta_2) \) is the Newton polygon associated to the sum of all monomials of \( w \)-degree \( d \).

(6) A monomial \( m \) lies strictly underneath, on or above \( \Gamma(f) \), if the \( w(f) \)-degree of \( m \) is less than, equal to or greater than \( d(f) \), respectively. We use this notation also with respect to \( \Gamma(T) \), \( w(T) \), and \( d(T) \).

**Notation 13.** Given \( f \in \mathbb{C}[[x_1, \ldots, x_n]] \) and \( m \in \text{Mon}(x_1, \ldots, x_n) \), we write \( \text{coeff}(f, m) \) for the coefficient of \( m \) in \( f \).

**Definition 14.** The **Jacobian ideal** \( \text{Jac}(f) \subset \mathbb{C}[[x_1, \ldots, x_n]] \) of \( f \) is generated by the partial derivatives of \( f \in \mathbb{C}[[x_1, \ldots, x_n]] \). The **local algebra** of \( f \) is the residue class ring of the Jacobian ideal of \( f \).

**Definition 15.** Suppose \( f \) is a non-degenerate germ, \( e_1, \ldots, e_\mu \) are monomials representing a basis of the local algebra of \( f \), and \( e_1, \ldots, e_s \) are the monomials in this basis above or on \( \Gamma(f) \). We then call \( e_1, \ldots, e_\mu \) a **system** of the local algebra of \( f \).

**Lemma 16** [Arnold (1974), Corollary 3.3]. Let \( f \) be a semi-quasihomogeneous function with quasi-homogeneous part \( f_0 \), and let \( e_1, \ldots, e_\mu \) be monomials representing a basis of the local algebra of \( f_0 \). Then \( e_1, \ldots, e_\mu \) also represent a basis of the local algebra of \( f \).

**Theorem 17** [Arnold (1974), Theorem 7.2]. Let \( f \) be a semi-quasihomogeneous function with quasi-homogeneous part \( f_0 \) and let \( e_1, \ldots, e_s \) be a system of the local algebra of \( f_0 \). Then \( f_0 \) is equivalent to a function of the form \( f_0 + \sum_{k=1}^{s} c_k e_k \) with constants \( c_k \).

In [Arnold (1974)], the following approach is used to extend the above results to a larger class of singularities of corank 2.

**Definition 18.** A piecewise homogeneous function \( f_0 \) of degree \( d \) satisfies **Condition A**, if for every function \( g \) of filtration \( d + \delta > d \) in the ideal spanned by the derivatives of \( f_0 \), there is a decomposition

\[
g = \sum_i \frac{\partial f_0}{\partial x_i} v_i + g',
\]

where the vector field \( v \) has filtration \( \delta \) and \( g' \) has filtration bigger than \( d + \delta \).

Note that quasihomogeneous functions satisfy Condition A.

**Theorem 19.** Suppose that the principal part \( f_0 \) of the piecewise homogeneous function \( f \) has finite Milnor number and satisfies Condition A. Let \( e_1, \ldots, e_s \) be a system of the local algebra of \( f_0 \). Then \( f_0 \) is equivalent to a function of the form \( f_0 + \sum_k c_k e_k \) with constants \( c_k \).

Following Arnold’s proof of Theorem 17, Theorem 19 can be proven by iteratively applying the following lemma.

**Lemma 20.** Let \( f_0 \in \mathbb{C}[[x_1, \ldots, x_n]] \) be a piecewise homogeneous function of weighted \( w \)-degree \( d_w \) that satisfies Condition A, and let \( e_1, \ldots, e_r \) be the monomials of a given \( w \)-degree \( d' \) in a system of the local algebra of \( f_0 \). Then, for every series of the form \( f_0 + f_1 \), where the filtration of \( f_1 \) is greater than \( d_w \), we have

\[
f_0 + f_1 \sim f_0 + f'_1,
\]

where the terms in \( f'_1 \) of degree less than \( d' \) are the same as in \( f_1 \), and the part of degree \( d' \) can be written as \( c_1 e_1 + \cdots + c_r e_r \) with \( c_i \in \mathbb{C} \).
Proof. Let \( g(x) \) denote the sum of the terms of degree \( d' \) in \( f_1 \). There exists a decomposition of \( g \) of the form

\[
g(x) = \sum \frac{\partial f_0}{\partial x_i} v_i(x) + c_1 e_1 + \cdots + c_r e_r, \quad v_i \in \mathbb{C}[x_1, \ldots, x_n],
\]

since \( e_1, \ldots, e_r \) represent a monomial vector space basis of the local algebra of \( f_0 \) in degree \( d' \).

Let \( d(x_i) \) be the \( w \)-degree of \( x_i \), and let \( v'_i := \text{w-jet}(v_i, d(x_i)) \). Then

\[
g(x) = \sum \frac{\partial f_0}{\partial x_i} v'_i(x) + c_1 e_1 + \cdots + c_r e_r - g'(x),
\]

where \( g'(x) \) has filtration greater than \( d' \). Applying the transformation defined by

\[
x_i \mapsto x_i - v'_i(x)
\]

to \( f = f_0 + f_1 \), we transform \( f \) to

\[
f_0(x) + (f_1(x) + (c_1 e_1(x) + \cdots + c_r e_r(x) - g(x)) + R(x),
\]

where the filtration of \( R \) is greater than \( d' \). \(\square\)

Remark 21. A system of the local algebra is in general not unique. For his lists of normal forms of hypersurface singularities, Arnold has chosen in each case (in particular) a specific system of the local algebra. In the rest of the paper, we call these systems the Arnold systems.

Definition 22 (Kouchmireneko [1976]). We say that \( f \in \mathbb{C}[[x, y]] \) has non-degenerate Newton boundary if for every face \( \Delta \) of \( \Gamma(f) \) the saturation of \( \text{jet}(f, \Delta) \) has finite Milnor number. \(\text{2}\)

Remark 23. (1) Note that if \( f \) has non-degenerate Newton boundary and finite Milnor number, then the principal part of \( f \) with respect \( w(f) \) has finite Milnor number.

(2) Also note that for Arnold’s normal forms \( \text{NF}(T) \) of corank and modality \( \leq 2 \) the principal part with respect to \( w(T) \) satisfies Condition A.

(3) Suppose that \( f \) is a function of corank 2 with non-degenerate Newton boundary such that, for one of Arnold’s normal forms \( \text{NF}(T) \) of modality \( \leq 2 \), the support of the principal part \( f_0 \) of \( f \) with respect to \( w(T) \) coincides with that of the principal part of \( \text{NF}(T) \). Then a system of the local algebra of \( f_0 \) is also a system of the local algebra of \( f \).

Remark 24. It follows from Lemma 20 that all hypersurface singularities of corank \( \leq 2 \) and modality \( \leq 2 \) with non-degenerate Newton boundary are finitely weighted determined. Moreover, we explicitly obtain the weighted determinacy for each such singularity.

3. A Classification Algorithm for Corank 2 Complex Simple, Unimodal and Bimodal Singularities

We now describe an algorithm to determine an Arnold normal form equation for a given input polynomial \( f \in \mathbb{C}^1, f \in \mathbb{Q}[x, y] \) of modality \( \leq 2 \). In this section, we limit our discussion on functions with a normal form with non-degenerate Newton boundary. In the case of normal forms with degenerate Newton boundary, our algorithm will resort to special algorithms described in Section 4. Figures 1 to 4 illustrate the modality 2 types with non-degenerate Newton boundary. The figures show in the gray shaded area all monomials which can possibly occur in a polynomial \( f \) of the given type \( T \). The faces of the Newton polygon \( \Gamma(T) \) are shown in blue. The dots with a thick black circle indicate the moduli monomials in the Arnold system. Red dots indicate monomials which are not in \( \text{Jac}(f) \). Monomials occurring in any normal form equation with non-zero coefficients are shown as blue dots.

The structure of our algorithm consists out of two basic steps, see Algorithm 1. We first determine the complex type of \( f \) by removing all the monomials underneath \( \Gamma(T) \), in the semi-quasihomogeneous cases, and all the monomials underneath and on \( \Gamma(T) \) which are not in \( \text{NF}(T) \),

\(\text{2}\)We say that the singularity defined by \( f \) has non-degenerate Newton boundary if there exists a germ \( \tilde{f} \in \mathbb{C}[[x, y]] \) with \( f \sim \tilde{f} \) which has non-degenerate Newton boundary. We use the analogous terminology also for semi-quasihomogeneous.
Suppose no monomials in \( \text{Algorithm 2} \). After that, we determine a normal form equation of \( f \) (using Algorithm 5 in the non-simple cases). More generally, we will formulate the algorithm in a way, that it is applicable to any \( f \in \mathfrak{m}^2 \), and will recognize if \( f \) is of modality \( > 2 \), returning an error in this case.

### Algorithm 1

**Algorithm to classify singularities of modality \( \leq 2 \) corank \( \leq 2 \)**

**Input:** A polynomial germ \( f \in \mathfrak{m}^2 \) over the rationals.

**Output:** \( \text{NF}(f) \) as well as the values of all moduli parameters occurring in a normal form equations that is equivalent to \( f \), if \( f \) is of modality \( \leq 2 \), corank \( \leq 2 \); \textbf{false} otherwise.

1. Apply Algorithm 2 to \( f \).
2. if \( T \) as returned by Algorithm 2 is a simple type then
   return \( (\text{NF}(T), ()) \)
3. Apply Algorithm 5 to the output of Algorithm 2 and return the result.

We first discuss Algorithm 2. If \( f \) is of corank \( \leq 1 \), then \( f \) is of type \( X_{k} \), where \( k = \mu(f) \). Suppose now that \( f \) is of corank 2. Determining \( T \) in the process, we remove all monomials below \( \Gamma(T) \) if \( \Gamma(T) \) has only one face, and all monomials on or below \( \Gamma(T) \) which are not in \( \text{NF}(T) \), if \( \Gamma(T) \) has two faces. Let \( d \) be the maximal filtration of \( f \). If \( f \) is of type \( X_{9} \), nothing has to be done. Note that \( f \) is of type \( X_{9} \) if and only if the \( d \)-jet of \( f \) has 4 different roots over the complex numbers. If \( f \) is not of type \( X_{9} \), then Algorithm 3 will transform \( f \) such that \( \text{supp}(T, d) = \text{supp}(\text{jet}(f, d)) \). Using (Marais and Steenpaß, 2015a, Proposition 8), we find the corresponding linear transformation by factorizing \( \text{jet}(f, d) \).

At this stage we know that \( \text{supp}(\text{jet}(f, d)) \subset \text{supp}(\text{NF}(T)) \). We store the monomials of the \( d \)-jet of \( f \) in \( S_0 = \text{supp}(\text{jet}(f, d)) \). The remainder of Algorithm 2 will proceed in an iterative way, changing \( f \) and \( S_0 \) in the process: In each step of the iteration, we can have one of the following two possibilities for \( \Gamma(f) \):

1. Note that monomials of the form \( x^{m_1} y \) or \( xy^{m_2} \) cannot be intersection points of (finite) faces of \( \Gamma(T) \). If any of the monomials \( m_0 \in S_0 \) which is not of this form lies on two faces of \( \Gamma(f) \), it is clear that \( \Gamma(T) \) has at least two faces with corner point \( m_0 \). The algorithm will then stay in this case. Let \( \Delta_i \) and \( \Delta_j \) be the two different faces of \( \Gamma(f) \) on which \( m_0 \) lies. The corner point in all modality 1 and 2 cases with a Newton polygon with two faces is either \( x^2 y^2 \) or \( x^2 y^3 \). It follows that if \( m_0 \neq x^t y^r \), \( t = 2 \) or \( t = 3 \), then \( f \) is not of modality \( \leq 2 \). Otherwise, using the shape of \( \Gamma_0 := \text{span}(\Delta_i, \Delta_j) \) and the fact that \( m_0 = x^t y^r \) is a corner point of \( \Gamma_0 \), all monomials in \( f \) on \( \Gamma_0 \) of the form \( x^{m_1} y \) or \( xy^{m_2} \) can be removed iteratively, by increasing degree, each time replacing the corresponding terms of the given degree by higher \( w(f) \)-degree terms using Algorithm 4. After each iteration, \( f, \Delta_i, \Delta_j \) and \( \Gamma_0 \) are recalculated. In each iteration, there will either be no terms of the considered form on \( \Gamma_0 \), in which case the process stops, or the number of equivalence classes in the local algebra of \( f \) represented by powers of \( x \) or \( y \) underneath \( \Gamma_0 \) strictly increases, except possibly in the last two steps of the process (where monomials on the final Newton polygon may be removed). Note that, if \( x^{m_1} y \) and \( y^{m_2} \) are largest powers of \( x \) and \( y \) underneath \( \Gamma_0 \), then \( 1, x, \ldots, x^{m_1-1}, y, \ldots, y^{m_2-1} \) represent different equivalence classes. Since \( \mu(f) \) is finite, the process must stop after finitely many iterations. No further monomials on \( \Gamma_0 \) can be removed without creating terms underneath \( \Gamma_0 \). Hence, in all cases in consideration, this algorithm will produce the Newton polygon of the normal form. In fact, if \( \text{supp}(f, \Gamma_0) \) does not coincide with \( \text{supp}(\Gamma_0, \Gamma_0) \) for some type \( \Gamma \) of modality \( \leq 2 \), then the modality of \( f \) is bigger than 2.

2. Suppose no monomials in \( S_0 \), except monomials of the form \( x^{m_1} y \) or \( xy^{m_2} \), lie on two faces of \( \Gamma(f) \). Then \( f \) is not of type \( X_{9+k} \) or \( Y_{r,s} \), since these cases will be recognized to have two faces in the first iteration of the above step. All the monomials in \( S_0 \) lie on only one face of \( \Gamma(f) \). Let \( \Delta \) be this face. If \( f_1 := \text{jet}(f, \Delta) \) is non-degenerate, then \( f \) is a semi-quasihomogeneous germ. Since \( w \cdot \text{jet}(\phi^w_0(f), d(f)) = \phi^w_0(f_1) \) for any automorphism \( \phi \) of
filtration $\geq 0$ with respect to the weight $w$ associated to $\Delta$, $\text{span}(\Delta)$ is an invariant of the type of $f$. The corresponding type $T$ can, hence, be identified. The case $X_9$ will already be recognized as a semi-quasihomogeneous function in the first iteration, and $f$ will be returned by the algorithm without any change. In all other cases, the weight $w$ associated with $\Delta$ will be such that $w \cdot \deg(x) > w \cdot \deg(y)$. If $f_1$ is degenerate, then either $f$ has monomials underneath $\Gamma(T)$, or $\Gamma(T)$ is degenerate. For all semi-quasihomogeneous cases of modality $\leq 2$, except $X_9$, $\text{jet}(T,d)$ is divisible by a power of $x$, and $x$ has the highest multiplicity among all prime factors. Any weighted jet of $\text{NF}(T)$ with respect to a face lying below $\Gamma(T)$ and intersecting $\Gamma(T)$ in $\text{jet}(T,d)$ has the same property. Suppose $\Delta$ is such a face. Then $\text{supp}(T,\Delta) = \{x^n y^m \}$ with $n > m$. Taking into account that the weighted degree of $x$ is greater than the weighted degree of $y$, it follows that $f_1 = g_1^0 y^m$ with $\deg_x(g_1) = 1$. The right equivalence $g_1 \mapsto x$, $y \mapsto y$ transforms $f$ such that $\text{supp}(f,\Delta) = \text{supp}(T,\Delta)$. If the normal form of $f$ has a non-degenerate Newton boundary, but is not semi-quasihomogeneous, we can proceed in the same way: Suppose $\Delta$ lies underneath or on the face of biggest slope of $\Gamma(T)$. If $g_1$ is the factor of highest multiplicity of $f_1$ with $\deg_x(g_1) = 1$, then the right equivalence $g_1 \mapsto x$, $y \mapsto y$ transforms $f$ such that $\text{supp}(f,\Delta) = \text{supp}(T,\Delta)$. We then update $S_0 := \text{supp}(f,\Delta)$ and pass to the next iteration. If $f_1$ does not have any $x$-linear factor, then the normal form of $f$ has a degenerate Newton boundary. In this case, we resort to the algorithms described in Section 4. Since $\mu(f)$ is finite, the same argument as in [1] shows that the iteration terminates after finitely many steps.

We now discuss Algorithm 5, which determines the values of the moduli parameters. Let $w = w(T)$ be the weight associated to $\Gamma(T)$. If $\Gamma(T)$ has only one face $\Delta$, then $\text{supp}(f,\Delta)$ is not necessarily equal to $\text{supp}(\text{NF}(T),\Delta)$. We achieve equality by a weighted linear transformation. In the cases where $\Gamma(T)$ has two faces, equality has already been achieved in Algorithm 2. Above $\Gamma(T)$, we then use the method described in the proof of Lemma 20 to reduce $f$ modulo $\text{Jac}(f_0)$ where $f_0 = \text{jet}(f,\Gamma(T))$: We iteratively apply Algorithm 4 to each term, in the two face case only considering terms in $\text{Jac}(f_0)$, proceeding weighted degree by weighted degree in increasing order (and in each weighted degree according to a total (ordinary) degree ordering). After handling a given weighted degree, if Arnold’s system for type $T$ contains a monomial $m$ of this degree, we write the sum of the remaining terms in the form

$$\frac{\partial f_0}{\partial x} v_1 + \frac{\partial f_0}{\partial y} v_2 + cm,$$

where $v_1, v_2 \in \mathbb{C}[x,y]$ are weighted homogeneous, $c \in \mathbb{C}$, and as

$$\frac{\partial f_0}{\partial x} v_1 + \frac{\partial f_0}{\partial y} v_2,$$

otherwise. By Remark 23 this is always possible. Applying $x \mapsto x - v_1$, $y \mapsto y - v_2$, results in replacing the sum of the remaining terms by a sum of terms which are either in Arnold’s system in the $w$-degree under consideration, or of higher $w$-degree. Since $f$ is weighted $d'$-determined, we stop the iteration when we reach degree $d' + 1$, where $d'$ is the $w$-degree of the highest $w$-degree monomial in Arnold’s system.

**Remark 25.** In the semi-quasihomogeneous cases, line 11 in Algorithm 5 can be omitted, since the reduction modulo $\text{Jac}(f_0)$ is also handled by lines 15 to 18.

**Remark 26.** In Algorithm 5, Arnold’s system can be replaced by any other choice of a system of the local algebra.

**Remark 27.** The algebraic extension of $\mathbb{Q}$ introduced for representing the moduli parameters can arise in two steps of the overall algorithm: Reversal of the linear jet in Algorithm 3 and rescaling of the variables at the end of Algorithm 5. Note that the transformation reversing the linear jet is obtained from the factorization $\text{jet}(f,d) = cg_\alpha g_\beta g_\gamma^\delta g_1^\delta$. Here, a field extension can only occur if $\alpha = \beta = 2$ and $\gamma = \delta = 0$. 
Algorithm 2 Determine the complex type of a corank ≤ 2 singularity of modality ≤ 2 with non-degenerate Newton boundary.

**Input:** A polynomial germ \( f \in \mathbb{m}^2 \) over the rationals.

**Output:** If \( f \) is of modality ≤ 2 and corank ≤ 2, then the complex singularity type \( T \) of \( f \), and a polynomial \( g \) right equivalent to \( f \) such that the set of faces of \( \Gamma(T) \) equals the set of faces of \( \Gamma(w(T) \cdot \text{jet}(g, d(T))) \); \( \text{false} \) otherwise.

1: \( f := \) residual part given by the splitting lemma applied to \( f \), as implemented in classify.lib.
2: if \( \text{corank}(f) \leq 1 \) then
   3: \( \text{return} \ (f, A_{\mu(f)}) \)
4: if \( \text{corank}(f) > 2 \) then
5: \( \text{return} \ false \)
6: if \( f \in E_5 \) then
7: \( \text{return} \ false \) (modality > 2)
8: \( f := \) output of Algorithm [3] applied to \( f \in \mathbb{Q}[x, y] \)
9: \( S_0 := \text{supp} \left( \text{jet}(f, d) \right) \), where \( d := \) maximal filtration of \( f \) w.r.t. the standard grading.
10: while true do
11: \( \text{Let } \Delta_1, \ldots, \Delta_n \text{ be the faces of } \Gamma(f) \text{ ordered by increasing slope.} \)
12: if exist \( i \neq j \) and \( n_1, n_2 > 1 \) with \( m_0 := x^{n_1}y^{n_2} \in \text{supp}(\Delta_i) \cap \text{supp}(\Delta_j) \subseteq S_0 \) then
13: \( \text{return} \ false \) (modality > 2)
14: \( \Gamma_0 := \text{span}(\Delta_1, \Delta_2) \)
15: \( f_1 := \text{jet}(f, \Gamma_0) \)
16: while exist a term of the form \( t = c \cdot x^{n_1 - 1}y^r \) or \( t = c \cdot x^ry^{n_2 - 1} \) in \( f_1 \) do
17: \( f := \) output of Algorithm [4] with input \( f, f_1, t, \) and weights \( w(\Delta_i), w(\Delta_j) \)
18: \( \text{Let } \Delta_1, \ldots, \Delta_n \text{ be the faces of } \Gamma(f). \)
19: \( \Gamma_0 := \text{span}(\Delta_i, \Delta_j), \) where \( i \) and \( j \) are such that \( m_0 \in \text{supp}(\Delta_i) \cap \text{supp}(\Delta_j) \)
20: \( f_1 := \text{jet}(f, \Gamma_0) \)
21: if exists modal 1 or 2 type \( T \) with \( \text{supp}(T, \Gamma_0) = \text{supp}(f_1) \) then
22: \( \text{return} \ (f, T) \)
23: else
24: \( \text{return} \ false \) (modality > 2)
25: else
26: \( \text{Let } \Delta \text{ be the face of } \Gamma(f) \text{ of smallest slope such that } S_0 \subseteq \text{supp}(\Delta). \)
27: \( f_1 := \text{jet}(f, \Delta) \)
28: if \( \mu(f_1) = \infty \) then
29: \( \text{Let } g_1 \text{ be the factor of } f_1 \text{ with highest multiplicity.} \)
30: if \( \text{deg}_x(g_1) = 1 \) then
31: \( \text{Replace } f \text{ by } g_1 \mapsto x, y \mapsto y \text{ applied to } f. \)
32: \( S_0 := \text{supp}(\text{jet}(f, \Delta)) \)
33: else
34: if \( \text{supp}(f, \Delta) = \text{supp}((y^2 - x^3)^2) \) then
35: \( \text{return} \ (f, W^{t}_{1, \mu(f) - 15}) \)
36: else
37: \( \text{return} \ false \) (modality > 2)
38: \( \text{else} \)
39: if exists modal 1 or 2 type \( T \) with \( \Gamma(T) = \Gamma(f_1) \) then
40: \( \text{return} \ (f, T) \)
41: else
42: \( \text{return} \ false \) (modality > 2)
Algorithm 3 Reverse linear jet.

Input: A polynomial $f \in m^3 \subset \mathbb{Q}[x,y]$ with jet($f, 4$) $\neq 0$.

Output: $g \in m^3 \subset K[x,y]$, where $K$ is an algebraic extension field of $\mathbb{Q}$, such that $g \sim f$, and, in case $f$ is of type $T \neq X_9$ of modality $\leq 2$, then supp(jet($g, d$)) = supp($T, d$) where $d$ is the maximal filtration of $f$ w.r.t. the standard grading.

1. Factorize jet($f, d$) $= cg_1^\alpha g_2^\beta g_3^\gamma g_4^\delta$ over $\mathbb{C}$, where $0 \neq c \in \mathbb{Q}$, $g_1$, $g_2$, $g_3$ and $g_4$ are monic in $x$ and pairwise coprime, and $4 \geq \alpha \geq \beta \geq \gamma \geq \delta \geq 0$.
2. if $\beta, \gamma, \delta = 0$ then
3. if $g_1 \neq c'y, c' \in \mathbb{Q}$ then
4. Replace $f$ with $g_1 \mapsto x$, $y \mapsto y$ applied to $f$.
5. else
6. Replace $f$ with $x \mapsto y$, $y \mapsto x$ applied to $f$.
7. if $\gamma, \delta = 0$ then
8. Replace $f$ with $g_1 \mapsto x$ and $g_2 \mapsto y$ applied to $f$.
9. if $\alpha = 2$ and $\beta, \gamma = 1$ and $\delta = 0$ then
10. if $g_1 \neq c'y, c' \in \mathbb{Q}$ then
11. Replace $f$ with $g_1 \mapsto x$, $y \mapsto y$ applied to $f$.
12. else
13. Replace $f$ with $x \mapsto y$, $y \mapsto x$ applied to $f$.
14. Write $f = a_0x^4 + a_1x^3y + a_2x^2y^2 + R, a_0, a_1 \in \mathbb{Q}, a_2 \in \mathbb{Q}^\times$ and $R \in E_5$.
15. Replace $f$ with $y \mapsto y - \frac{a_0}{a_2}x$, $x \mapsto x$ applied to $f$.
16. return $f$

Algorithm 4 Remove term via partials.

Input: $f$, $t_0 \in K[x,y]$ over a field $K$, with $t$ a term of $f$, and weights $u_1, u_2 \in \mathbb{Z}^2$.

Output: $g \in K[x,y]$ such that $f \sim g$. If called with input as in Algorithms 2 or 5 then $f = g + t$+ terms of higher $(u_1, u_2)$-degree than $t$.

1. $m_x :=$ the sum of the terms of $\frac{\partial f}{\partial x}$ of lowest $u_2$-degree
2. $m_{x,y} :=$ the term of $m_x$ of lowest $u_1$-degree
3. $m_y :=$ the sum of the terms of $\frac{\partial f}{\partial y}$ of lowest $u_1$-degree
4. $m_{y,x} :=$ the term of $m_y$ of lowest $u_2$-degree
5. if $m_{x,y} \nmid t$ then
6. \[
\begin{align*}
\alpha : K[x,y] & \rightarrow K[x,y] \\
x & \mapsto x - t/m_{x,y} \\
y & \mapsto y
\end{align*}
\]
7. return $\alpha(f)$
8. if $m_{y,x} \nmid t$ then
9. \[
\begin{align*}
\alpha : K[x,y] & \rightarrow K[x,y] \\
x & \mapsto x \\
y & \mapsto y - t/m_{y,x}
\end{align*}
\]
10. return $\alpha(f)$
11. return $f$
Algorithm 5 Determine the moduli parameters of a normal form equation of a corank 2 uni- or bimodal singularities.

Input: $f \in \mathbb{R}^3 \subset \mathbb{K}[x, y]$, a germ of modality 1 or 2 and corank 2 of type $T$ over an algebraic extension field $\mathbb{K}$ of $\mathbb{Q}$, as returned by Algorithm 2. In particular, the set of faces of $\Gamma(T)$ equals the set of faces of $\Gamma(w(T) \cdot \text{jet}(f, d(T)))$.

Output: The normal form of $f$, as well as the values of all moduli parameters occurring in a normal form equations that is equivalent to $f$, specified as elements of an algebraic extension field of $\mathbb{K}$.

1. if $T = W_{1, \mu-15}^2$ for some $\mu$ then return result of Algorithm 6 applied to $f$
2. $w := w(T)$ and $d := d(T)$
3. if $\Gamma(T)$ has exactly one face $\Delta$ then Apply a weighted homogeneous transformation to $f$ such that $\text{supp}(f, \Delta) = \text{supp}(T, \Delta)$.
4. $d' := \text{highest } w\text{-degree of a monomial in Arnold’s system of } T$
5. $f_0 := w\cdot \text{jet}(f, d)$
6. for $j = 1, \ldots, d'$ do
7. for all terms $t$ of $f$ of $w$-degree $j$ in increasing order by a total degree ordering do
8. if $\Gamma(T)$ has exactly one face then
9. $f := \text{result of Algorithm 4 with input } f, f_0, t$ and $(1, 1), (1, 1)$
10. else
11. if $t \in \text{Jac}(f_0)$ then
12. $f := \text{result of Algorithm 4 with input } f, f_0, t$ and $w_2, w_1$
13. if exists monomial $m$ of $w$-degree $j$ in Arnold’s system then
14. Write $w\cdot \text{jet}(f, j) - w\cdot \text{jet}(f, j - 1) = \frac{\partial f_0}{\partial x}v_1 + \frac{\partial f_0}{\partial y}v_2 + cm$ with $c \in \mathbb{K}$, $v_1, v_2 \in \mathbb{K}[x, y]$ weighted homogeneous.
15. else
16. Write $w\cdot \text{jet}(f, j) - w\cdot \text{jet}(f, j - 1) = \frac{\partial f_0}{\partial x}v_1 + \frac{\partial f_0}{\partial y}v_2$ with $v_1, v_2 \in \mathbb{K}[x, y]$ weighted homogeneous.
17. Apply $x \mapsto x - v_1, y \mapsto y - v_2$ to $f$.
18. Delete all terms in $f$ of $w$-degree $> d'$.
19. Apply transformation $x \mapsto ax, y \mapsto by$ over an algebraic extension of $\mathbb{K}$ to transform the non-parameter terms to the terms of $\text{NF}(T)$.
20. Read off the parameters $a_i$.
21. return $(\text{NF}(T), (a_i))$

4. A Classification Algorithm for Corank 2, Bimodal Singularities with Degenerate Newton Boundary

In this section we give a classification algorithm for the singularities $W_{1, \mu-15}^2$, where $\mu$ is the Milnor number, in Arnold’s list. They have the property that in all coordinate systems the Newton boundary is degenerate, which is the reason that they have to be treated separately. They are of multiplicity 4 and the 4-jet is a 4-th power of a linear homogeneous polynomial. After a suitable automorphism of $\mathbb{C}[[x, y]]$, we may assume that the corresponding polynomial is of the form

$$f = (x^2 + y^3)^2 + \sum_{3i+2j=12+d} w_{ij}x^iy^j, \quad d \geq 1.$$ 

This automorphism was already constructed in the previous section. Singularities of this type have been studied by Luengo and Pfister (1990). It is proved that the Milnor number satisfies $\mu(f) \geq 15 + d$, and equality holds if and only if

$$\sum_{3i+2j=12+d} (-1)^{(i/2)}w_{ij} \neq 0.$$
If the Milnor number $\mu(f) = 15 + d$ is even, then the germ of the curve defined by $f$ is irreducible with semi–group $\langle 4, 6, 12 + d \rangle$. In the odd case, the curve has two branches. Let

$$f = (x^2 + y^3)^2 + \sum_{3i+2j>12} w_{ij}x^iy^j$$

and assume $\mu := \mu(f) < \infty$. Let $> be the weighted degree reverse lexicographical ordering with respect to the weights $(3, 2)$ on $\mathbb{C}[[x, y]]$ with $x > y$.

In [Luengo and Pfister (1990)] it is proved that in case of $\mu$ being even the leading ideal of the Jacobian ideal $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ is generated by $x^3, x^2y^2, xy^{\frac{\mu-3}{2}}$. If $\mu$ is odd, then the leading ideal is generated by $x^3, x^2y^2, xy^{\frac{\mu-3}{2}}, y^{\frac{\mu+1}{2}}$. We obtain a monomial basis of $\mathbb{C}[[x, y]]/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ as $\{x^iy^j\}_{(i, j) \in B}$ with

$$B = \{(i, j) \mid i \leq 2, j \leq 1\} \cup \{(i, j) \mid i \leq 1, 2 \leq j \leq \frac{\mu-4}{2}\}$$

in case that $\mu$ is even and

$$B = \{(i, j) \mid i \leq 2, j \leq 1\} \cup \{(1, j) \mid 2 \leq j \leq \frac{\mu-7}{2}\} \cup \{(0, j) \mid 2 \leq j \leq \frac{\mu-1}{2}\},$$

in case that $\mu$ is odd. Let

$$B_1 := \{(1, \frac{\mu-6}{2}), (1, \frac{\mu-4}{2})\}$$

if $\mu$ is even and

$$B_1 := \{0, \frac{\mu-3}{2}, (0, \frac{\mu-1}{2})\}$$

if $\mu$ is odd.

In [Luengo and Pfister (1990)], the following theorem is proved.

**Theorem 28.** There exists an automorphism $\varphi$ of $\mathbb{C}[[x, y]]$ such that

$$\varphi(f) = (x^2 + y^3)^2 + \sum_{(i, j) \in B_1} w_{ij}x^iy^j.$$

**Note 29.** In particular, it follows that these singularities are bimodal.

**Remark 30.** The normal form given in this way for the case that the Milnor number is odd differs from Arnold’s normal form. Instead of $y^{\frac{\mu-1}{2}}$ and $y^{\frac{\mu+1}{2}}$, he used the monomials $x^3y^{\frac{\mu-3}{2}}$ and $x^2y^{\frac{\mu-1}{2}}$. From a computational point of view, our choice is better. It is easy to convert our normal form to Arnold’s normal form. See Figure 5 for an illustration of the normal forms (using our choice of parameter monomials).

The construction of the automorphism in the theorem is done separately for each weighted degree: Assume we have already

$$f = (x^2 + y^3)^2 + \sum_{3i+2j>12+a} w_{ij}x^iy^j$$

for some $a$ (with Milnor number $\mu = 15 + d$). If $a < d$, then we have

$$\sum_{3i+2j=12+a} (-1)^{[i/2]} w_{ij} = 0.$$

This implies that

$$\sum_{3i+2j=12+a} w_{ij}x^iy^j = l \cdot (x^2 + y^3).$$

We obtain

$$f = \left(x^2 + y^3 + \frac{1}{2}l\right)^2 + \sum_{3i+2j>12+a} \tilde{w}_{ij}x^iy^j$$
for suitable $\tilde{w}_{ij} \in \mathbb{C}$. Now we can choose an automorphism $\varphi$ of $\mathbb{C}[[x,y]]$ such that
\[
\varphi \left( x^2 + y^3 + \frac{1}{2} l \right) = x^2 + y^3 + \text{ terms of weighted degree } \geq \mu
\]
(note that we could even find an automorphism mapping $x^2 + y^3 + \frac{1}{2} l$ to $x^2 + y^3$). We obtain
\[
\varphi(f) = (x^2 + y^3)^2 + \sum_{3i + 2j > 12 + a} \overline{w}_{ij} x^i y^j
\]
for suitable $\overline{w}_{ij} \in \mathbb{C}$.

If $a = d$ then we have
\[
\sum_{3i + 2j = 12 + d} (-1)^{[i/2]} \overline{w}_{ij} \neq 0.
\]
Similarly as before, we can write
\[
\sum_{3i + 2j = 12 + d} w_{ij} x^i y^j = w_{i_0 j_0} x^{i_0} y^{j_0} + l \cdot (x^2 + y^3)
\]
with
\[
(i_0, j_0) = \begin{cases} (0, \frac{\mu-2}{2}) & \text{if } \mu \text{ is odd} \\ (1, \frac{\mu-6}{2}) & \text{if } \mu \text{ is even}. \end{cases}
\]
Since the Milnor number is $15 + d$, we obtain $w_{i_0 j_0} \neq 0$. Using a similar automorphism as in the previous case, we may assume with $a_0 := w_{i_0 j_0}$ (the first modulus), that
\[
f = (x^2 + y^3)^2 + a_0 \cdot x^{i_0} y^{j_0} + \sum_{3i + 2j > 12 + d} w_{ij} x^i y^j.
\]
Note, that $12 + d = \mu - 3$, and we have to compute the normal form of $f$ up to degree $\mu - 1$.

Now we can write
\[
\sum_{3i + 2j = 13 + d} w_{ij} x^i y^j = e \cdot x^{i_1} y^{j_1} + l \cdot (x^2 + y^3)
\]
with
\[
(i_1, j_1) = \begin{cases} (1, \frac{\mu-2}{2}) & \text{if } \mu \text{ is odd} \\ (0, \frac{\mu-6}{2}) & \text{if } \mu \text{ is even}. \end{cases}
\]
Using an automorphism as before, we may assume that $l = 0$.

If $e = 0$, we are done with weighted degree $\mu - 2$.

If $e \neq 0$ we define an automorphism $\varphi$ of $\mathbb{C}[[x,y]]$ by the exponential of the vector field
\[
\delta = c \cdot (3y^2 \frac{\partial}{\partial x} - 2x \frac{\partial}{\partial y})
\]
with
\[
c = (-1)^{\mu-1} \frac{e}{(\mu-3)a_0}.
\]
Since by construction, $\varphi(x^2 + y^3) = x^2 + y^3$, we obtain
\[
\varphi(f) = (x^2 + y^3)^2 + a_0 \cdot x^{i_0} y^{j_0} + \sum_{3i + 2j > 14 + d} \tilde{w}_{ij} x^i y^j
\]
for suitable $\tilde{w}_{ij} \in \mathbb{C}$.

**Remark 31.** Note, that for practical purposes, we have to compute $\varphi$ only up to weighted degree 5, and apply it to $a_0 \cdot x^{i_0} y^{j_0} + \sum_{3i + 2j = 13 + d} w_{ij} x^i y^j$ since we know that $\varphi((x^2 + y^3)^2) = (x^2 + y^3)^2$.

Now let
\[
(i_1, j_1) = \begin{cases} (0, \frac{\mu-1}{2}) & \text{if } \mu \text{ is odd} \\ (1, \frac{\mu-4}{2}) & \text{if } \mu \text{ is even} \end{cases}
\]
and write
\[
\sum_{3i + 2j = 14 + d} \tilde{w}_{ij} x^i y^j = a_1 \cdot x^{i_2} y^{j_2} + l \cdot (x^2 + y^3).
\]
Using an automorphism as in the first case, we may assume \( l = 0 \), and obtain as normal form
\[
(x^2 + y^3)^2 + a_0 x^6 y^3 + a_1 \cdot x^{11} y^{11}.
\]
We summarize the approach in Algorithm 6.

**Algorithm 6** Algorithm to determine parameters for the bimodal singularities of type \( W^1_{1,\mu-15} \):

**Input:** \( f = \gamma \cdot (\alpha x^2 + \beta y^3)^2 \) + terms of weighted \((3,2)\)-degree \( > 12 \in \mathbb{K}[x,y] \) with \( \alpha, \beta, \gamma \in \mathbb{K} \) and \( \mu := \mu(f) < \infty \).

**Output:** A normal form of \( f \) of the form
\[
(x^2 + y^3)^2 + a_0 \cdot x y^{\frac{2-\alpha}{2}} + a_1 \cdot x y^{\frac{2-\alpha}{2}} \quad \text{if} \quad \mu \text{ is even}
\]
\[
(x^2 + y^3)^2 + a_0 \cdot y^{\frac{2-\beta}{2}} + a_1 \cdot y^{\frac{2-\beta}{2}} \quad \text{if} \quad \mu \text{ is odd}
\]

with \( a_0 \neq 0 \), as well as the corresponding moduli parameters of a normal form equation defined over an algebraic extension field of \( \mathbb{K} \).

1. Apply transformation \( x \mapsto ax, y \mapsto by \) over an algebraic extension field of \( \mathbb{K} \) to \( f \) to transform the weighted homogeneous part of \( f \) to \((x^2 + y^3)^2 \).
2. Let \( \succ \) be the local weighted degree reverse lexicographical ordering with weights \((3,2)\) and \( x \succ y \).
3. Compute a standard basis \( G \) of \( \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \) with respect to \( \succ \).
4. Compute \( \mu \) the Milnor number of \( f \), and set \( d := \mu - 15 \).
5. \( a := 13 \)
6. **while** \( a < 12 + d \) **do**
7. \( g := \text{weighted homogeneous part of } f \text{ of degree } a \)
8. Write \( g = l \cdot (x^2 + y^3) \).
9. Construct automorphism \( \varphi \) with \( \varphi(x^2 + y^3 + \frac{1}{2}l) = x^2 + y^3 \) up to degree \( \mu - 1 \).
10. **if** \( l \) is even
11. \( m_0 := x y^{\frac{2-\alpha}{2}}, m_1 := x y^{\frac{2-\alpha}{2}}, m_2 := y^{\frac{2-\alpha}{2}} \)
12. **else**
13. \( m_0 := x y^{\frac{2-\alpha}{2}}, m_1 := y^{\frac{2-\alpha}{2}}, m_2 := y^{\frac{2-\alpha}{2}} \)
14. Write \( g = a_0 \cdot m_0 + l \cdot (x^2 + y^3) \).
15. Construct automorphism \( \varphi \) with \( \varphi(x^2 + y^3 + \frac{1}{2}l) = x^2 + y^3 \) up to degree \( \mu - 1 \).
16. **if** \( l \) is odd
17. \( g := \text{weighted homogeneous part of } \varphi \text{ of degree } 13 + d \)
18. Write \( g = c \cdot m_1 + l \cdot (x^2 + y^3) \).
19. Construct automorphism \( \varphi \) with \( \varphi(x^2 + y^3 + \frac{1}{2}l) = x^2 + y^3 \) up to degree \( \mu - 1 \).
20. **if** \( c \neq 0 \) **then**
21. \( c := (-1)^{\mu-1} \cdot \frac{c}{(\mu-3)a} \)
22. Construct automorphism \( \varphi \) defined by the vector field \( c \cdot (3y^2 \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial y}) \) up to degree 5.
23. \( f := (x^2 + y^3)^2 + \varphi(f - (x^2 + y^3)) \)
24. **if** \( f \) is of type \( W^1_{1,\mu-15} \)
25. \( g := \text{weighted homogeneous part of } f \text{ of degree } 14 + d \)
26. Write \( g = a_1 \cdot m_2 + l \cdot (x^2 + y^3) \).
27. **return** \( \text{NF}(W^1_{1,\mu-15}), (a_0, a_1) \)

**Remark 32.** The approach described in Algorithm 5 in case of a non-degenerate Newton boundary can be adapted to also handle the cases \( W^1_{1,\mu-15} \). However, this strategy requires more iterations than Algorithm 6. To adapt Algorithm 5, we remove lines 1 and 2 and in line 11 we call Algorithm 1 instead of Algorithm 6 if \( f \) is of type \( W^1_{1,\mu-15} \).
Note that in these cases Algorithm 2 does not require a field extension, hence, Algorithm 7 is called with input defined over \( \mathbb{Q} \). Note also that Algorithm 7 is applicable with any choice of a system \( B \) of the local algebra.

**Algorithm 7** Remove terms above the diagonal in cases with degenerate Newton boundary.

**Input:** \( f, f_0 \in \mathbb{Q}[x, y], t \in \mathbb{Q}[x, y] \) a term, and weights \( u_1, u_2 \in \mathbb{Z}^2 \).

**Output:** \( h \in \mathbb{Q}[x, y] \) such that \( f \sim h \).

1. \( w := w(f) \) and \( j := w - \deg(t) \).
2. \( g := \) output of Algorithm 4 with input \( f, f_0, t \) and \( u_1, u_2 \).
3. \( B := \) Arnold’s system of \( \mathbb{Q}[x, y]/\text{Jac}(f) \).
4. if \( t \in \text{Jac}(f_0) \) or \( g \neq f \) or \( (g = f \text{ and } B \text{ contains an element of degree } j) \) then return \( g \)
5. \( m := \) monomial in \( B \) of minimal \( w \)-degree.
6. Factorize \( f_0 = \gamma \cdot g_0^2 \) over \( \mathbb{Q} \) with \( \gamma \in \mathbb{Q} \) and \( g_0 \in \mathbb{Q}[x, y] \) linear.
7. \( \phi := \) automorphism defined by \( (\frac{\partial g_0}{\partial y} \frac{\partial}{\partial x} - \frac{\partial g_0}{\partial x} \frac{\partial}{\partial y}) \) up to \( w \)-degree 5.
8. \( s := \text{coeff}(f, m) \cdot m \).
9. \( t' := w \cdot \text{jet}(\phi(s) - s, j) \).
10. for all \( t \) in \( t' \) in increasing order by standard degree do
11. \( \phi_c \) := automorphism defined by \( c \cdot (\frac{\partial g_0}{\partial y} \frac{\partial}{\partial x} - \frac{\partial g_0}{\partial x} \frac{\partial}{\partial y}) \) up to \( w \)-degree 5.
12. \( h := f_0 + \phi_c(f - f_0) \).
13. for all \( h \) in \( h \) of \( w \)-degree \( j \) in increasing order by standard degree do
14. \( h := \) result of Algorithm 4 with input \( h, f_0, \tilde{l} \) and \( u_1, u_2 \).
15. return \( h \)

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Figure 1. Infinite series of bimodal corank 2 singularities with non-degenerate Newton boundary.

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Figure 2. Exceptional bimodal corank 2 singularities of type E.

Figure 3. Exceptional bimodal corank 2 singularities of type Z.
Figure 4. Exceptional bimodal corank 2 singularities of type W.

Figure 5. Infinite series of bimodal corank 2 singularities with degenerate Newton boundary.

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Janko Böhm, Department of Mathematics, University of Kaiserslautern, Erwin-Schrödinger-Str., 67663 Kaiserslautern, Germany
E-mail address: boehm@mathematik.uni-kl.de

Magdaleen S. Marais, University of Pretoria and African Institute for Mathematical Sciences, Department of Mathematics and Applied Mathematics, Private bag X20, Hatfield 0028, South Africa
E-mail address: magdaleen.marais@up.ac.za

Gerhard Pfister, Department of Mathematics, University of Kaiserslautern, Erwin-Schrödinger-Str., 67663 Kaiserslautern, Germany
E-mail address: pfister@mathematik.uni-kl.de