Comparison of quadratic investment portfolio on five stocks of mining companies with risk free assets and without risk free assets

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Abstract. All types of investment have different risks. One way to avoid risk is to diversify the capital fund into several investment assets. The problem is determining the optimal differential weight for each asset. The purpose of this study is to determine the weight of the investment portfolio with risk free assets and without risk free assets. The method used in this study is a quadratic investment portfolio model. The optimization is done by using the Lagrangian Multiplier method. The assets analyzed used several mining stocks and deposits with an interest rate according to the Indonesian Bank rate. The expected results from this study are to obtain the optimal weight allocated to each asset in ordering an investment portfolio. Based on the results of the optimization, it can be used as a consideration for investors in making investments.

1. Introduction
One of the asset development efforts that investors in general can undertake is by developing the asset to some investment. According to Gambrah and Pirvu [1] and Sukono et al. [2], investment is a commitment made at this time on several funds or other resources. The objective of an investor who invests is to get the greatest possible profit with the least possible risk. The risk is generally proportional to the return, that is, an investment asset with a large return is expected to be followed by a large risk, or the same. To minimize the risk, investors can diversify their investments and form portfolios [3-6].

According to Algarvio et al. [7] initiated the modern theory for portfolios, Markowitz used a set approach of portfolio efficiency which is represented as a combination of two separate stocks. The design of a stock portfolio using the Markowitz model was carried out by Tayali and Tolun [8] and Soeryana et al. [9-10] to evaluate the work of several portfolios. Then this model is again used by Dai and Wang [11] and to evaluate portfolio performance based on the Capital Asset Pricing Model (CAPM). Research on investment planning in stocks was also conducted by Sukono et al. [12] using graphs in the process of forming the optimal portfolio from the Optimal Portfolio with Risky Assets (OPRA) Model, which was later followed by Turkmen and Yigit [13] in analyzing stock portfolios with risky assets in the banking sector with a case study of banks in the country. Turkey uses the Markowitz model.

In investing, investors can choose to invest their funds either only in risk free assets or only in risk free assets, or a combination of the two assets. The choice of investors on these assets, depending on the extent of investors' preferences for risk [14-15]. Risk free assets are assets whose rate of return in the future can be ascertained at this time and is indicated by the return variance value equal to zero. One example of risk free assets is short-term deposits or bonds issued by the government. Risky assets are assets whose actual rate of return in the future still contains uncertainty. One example of risk-free assets is stocks [12,16].

From the description of the above statement, the purpose of this study is to perform
optimization to determine the weight when verifying assets, with the aim of the investment results investors get the maximum possible return with little risk. Optimization is carried out using the quadratic investment portfolio method and considering risk-free assets. The evaluation of the portfolio will be carried out using the Sharp Ratio. The implications of this research can provide new insights for investors, and as a basis for developments carried out by further researchers.

2. Material and Methods

2.1. Material

The data used in this study are some mining stock traded at the capital market in Indonesia. The stock data was obtained from https://finance.yahoo.com for the period from 9 September 2017 to 9 September 2020. The data used consisted of 5 stocks in the form of ADRO, ANTM, ENRG, BIPI, and PRBA. Stock selection is done by comparing the Padar Capitalization Value with the ERD value of the stock which is not more than 3, and the ratio is calculated by dividing the average and variance of these stocks.

| No | Stock Code | Distribution Estimates | Company          | Market Capitalization Value | DER | Ratio   |
|----|------------|------------------------|------------------|----------------------------|-----|---------|
| 1  | ADRO       | Normal                 | Adaro Energy     | 44,460,487,180,000         | 0.66| 0.7844  |
| 2  | ANTM       | Normal                 | Aneka Tambang    | 23,189,687,969,625         | 0.69| 0.1868  |
| 3  | ENRG       | Normal                 | Ratu Prabu Energi| 817,032,162,488            | 0.49| 1.0278  |
| 4  | FIRE       | Normal                 | Alfa Energi Investama| 10,407,132,998,000        | 1.76| 1.3467  |
| 5  | DOID       | Normal                 | Delta Dunia Makmur| 5,554,537,748,640         | 1.49| 1.7537  |

Source: idx.co.id

2.2 Methods

The model used in stock portfolio optimization is the Mean-Variance, with and without considerate of risk free asset.

2.2.1. Return Asset

The return of an asset, or it can be called the rate of return of the amount of money invested in a stock within a period can be calculated as follows:

\[
\eta_t = \frac{p_t - p_{t-1}}{p_{t-1}},
\]

with \( \eta_t \) is return to \( i \), \( p_t \) and \( p_{t-1} \) Consecutively is the closing price of shares at time \( t \) and \( t - 1 \) [17].

2.2.2. Portfolio Rate of Return

Referring to Sukono et al. [4], suppose a portfolio has \( N \) assets with each asset are \( \mu_1, \mu_2, ..., \mu_N \) and the return of each asset are \( r_1, r_2, ..., r_N \) with the proportion or weight of each asset is \( w_1, w_2, ..., w_N \). Thus, it can be expressed through vectors as follows:


\[
\mathbf{r} = \left( \begin{array}{c} r_1 \\ r_2 \\ \vdots \\ r_i \\ \vdots \\ r_j \\ \vdots \\ r_N \end{array} \right), \quad \mathbf{\mu} = \left( \begin{array}{c} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_i \\ \vdots \\ \mu_j \\ \vdots \\ \mu_N \end{array} \right), \quad \mathbf{w} = \left( \begin{array}{c} w_1 \\ w_2 \\ \vdots \\ w_i \\ \vdots \\ w_j \\ \vdots \\ w_N \end{array} \right) \text{ and } \mathbf{e} = \left( \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{array} \right),
\]

with \( \mathbf{r} \): return vector, \( \mathbf{\mu} \): mean vector, \( \mathbf{w} \): weight or proportion vector, \( \mathbf{e} \): unit vector, where \( w_i (1, 2, \ldots, N) \) stating the weight or proportion of funds or capital allocated to assets \( i \), and \( \mathbf{e} \) expressing a vector containing the number 1 as much \( N \). The return of an investment portfolio is the sum of the average weight of each return asset in the formation of a portfolio, it can be stated as follows:

\[
\mathbf{r}_p = w_1 r_1 + w_2 r_2 + \ldots + w_N r_N
\]

\[
= \sum_{j=1}^{N} w_j R_j
\]

\[
= \mathbf{w}^T \mathbf{r}.
\]

Total proportion of shares \((w_i, i = 1 \ldots N)\) in a portfolio is equal to one, or if written mathematically:

\[
\sum_{i=1}^{N} w_i = 1.
\]

If the average or expected return on the investment portfolio \( \mu_p = E[\mathbf{r}_p] \), then the average weight of each average return on assets in an investment portfolio, can be expressed in a vector as follows:

\[
E[\mathbf{r}_p] = \sum_{i=1}^{N} w_i \mu_i
\]

\[
= \mathbf{w}^T \mathbf{\mu}.
\]

2.2.3 Portfolio Risk Level

According to Sukono et al. [17], risk is the possibility of deviation or variability of the actual return of an investment with the expected return [8]. Calculation of risk can be done through analysis of variance, covariance, and standard deviation. The variance and standard deviation show the magnitude of the distribution of the random variable among the means, the greater the value of variance and standard deviation, the greater the spread of the random variable.

Mathematically, the formula used to calculate variance is:

\[
\sigma_p^2 = \sum_{j=1}^{N} \sum_{i=1}^{N} w_i w_j \sigma_{ij},
\]

\[\text{with } \sigma_p^2 : \text{portfolio return variants}, \sigma_{ij} : \text{covariance between stock returns } i \text{ and } j, r_p : \text{portfolio stock returns}, w_i : \text{proportion invested in shares } i, w_j : \text{the proportion invested in shares } j.\]

Furthermore, the portfolio return variance equation from equation (3) can be expressed in the form of a vector matrix as follows:

\[
\sigma_p^2 = \mathbf{w}^T \Sigma \mathbf{w},
\]

\[\text{with } \Sigma \text{ denotes the variance-covariance matrix.}\]

2.2.4. Efficient Portfolio Selection

Referring to Kalfin et al. [6] who popularized the efficient portfolio selection method. For example, given a portfolio \( p \) with a weight vector \( \mathbf{w} \), investors have two objectives, namely:
1) Maximizing the value of expectations $\mu_p$ from portfolio return,
2) Minimizing portfolio risk as measured by $\sigma_p^2$ or $\sigma_0$.

Based on these two objectives, there are three types of alternative models:

**A. Problem I: Risk Minimization**

For $\mu_0$ is a target average of the returns, then the weighting selection can be obtained using the equation:

$$\min_{w \in \mathbb{R}^N} \frac{1}{2} w^T \Sigma w,$$

Subject to $w^T \mu = \mu_0,$

$$w^T e = 1.$$

The solution of equation (5) is obtained by using the Lagrange multipliers method to become a convex optimization problem with linear constraints.

**B. Problem II: Maximizing Return Expectations**

For $\sigma_0^2$ is the target return of a variance, the weighted selection can be obtained using the equation:

$$\max_{w \in \mathbb{R}^N} w^T \mu,$$

Subject to $w^T \mu = \sigma_0^2,$

$$w^T e = 1.$$

Similar to equation (5), equation (6) can also be solved by the Lagrange multiplier method.

**C. Problem III: Risk Aversion Optimization**

The model in Problem III will be of concern in this study. Based on individual preferences, an investor puts weights with the two objectives mentioned above, which can be expressed algebraically to find the maximum

$$2\tau \mu_p - \sigma_p^2,$$

dengan $\tau \geq 0$.

Parameter $\tau$ called Soeryana’s risk tolerance [10]. Equation [9] is formed as follows. Suppose that the initial capital is given $W_0$, under the portfolio with the weight vector $w$, at the end of the period the capital becomes $W_0(1 + R_p)$ [10]. With $R_p$ is a random variable of portfolio return with weight vector $w$. The utility equation of $W_0(1 + R_p)$ is $u[w_0(1 + R_p)]$. The utility equation will be described using the second-order Taylor series approach, and the utility equation is obtained as follows:

$$u[W_0(1 + R_p)] \approx u(W_0) + W_0 u'(W_0) R_p + \frac{1}{2} W_0^2 u''(W_0) R_p^2.$$  

On the equation (7) obtained the following form of expectations:

$$E \left[ u[W_0(1 + R_p)] \right] \approx u(W_0) + W_0 u'(W_0) \mu_p + \frac{1}{2} W_0^2 u''(W_0) \sigma_p^2 \approx u(W_0) - \frac{W_0^2}{2} u''(W_0) \left[ -\frac{2u'(W_0)}{W_0 u''(W_0)} \mu_p - \sigma_p^2 \right].$$

Based on the above approximation, maximizing the expected utility by approach, is equivalent to maximizing

$$\frac{2}{\tilde{R}_p} \mu_p - \sigma_p^2,$$  

(8)
with \( R_R = -\frac{w_0u''(w_0)}{u'(w_0)} \) represents the relative measure of risk aversion. If taken \( \tau = \frac{1}{R_R} \) then equation (7) is equivalent to (8) [10]. Therefore, Markowitz formed an equation for the optimization problem as follows,

\[
\text{Maximize} \{2\tau \mu_p - \sigma_p^2\},
\]

Subject to: \( \sum_{i=1}^{N} w_i = 1 \),
or it can be explicitly stated as follows,

\[
\text{Maximize} \left\{ 2\tau \sum_{i=1}^{N} w_i \mu_i - \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} \right\},
\]

subject to: \( \sum_{i=1}^{N} w_i = 1 \).

According to Sukono et al. [4] this approach has several advantages:
i) Only risk tolerance \( \tau \) that must be specified,
ii) Only the first moment \( \mu_i \) and the second moment \( \sigma_{ij} \) required from the return on assets.

2.2.5. Quadratic Investment Portfolio Model Without Risk Free Assets

This section discusses quadratic investment portfolios without risk free assets. Referring to Hult et al. [18] and Sukono et al. [17], that the quadratic investment portfolio model without risk free assets based on Mean-Variance is stated as follows:

\[
\text{Maximize} \{2\tau \mu_p - \sigma_p^2\},
\]

Subject to: \( \sum_{i=1}^{N} w_i = 1 \),
or,

\[
\text{Minimize} \left\{ w^T \Sigma w - 2\tau w^T \mu \right\} \quad \text{subject to:} \quad e^T w = 1.
\]

Lagrange function from equation (9) with \( \lambda \) as a multiplier it can be stated as follows:

\[
L(w, \lambda) = (w^T \Sigma w - 2\tau w^T \mu) + \lambda (e^T w - 1).
\]

Based on Kuhn-Tucker's condition,

The requirements for optimization in equation (9) are \( \frac{\partial L}{\partial w} = 0 \) and \( \frac{\partial L}{\partial \lambda} = 0 \) so it gets:

\[
\frac{\partial L}{\partial w} = 2 \Sigma w - 2\tau \mu + \lambda e = 0,
\]

\[
\frac{\partial L}{\partial \lambda} = w^T e - 1 = 0.
\]

Equation (10) multiply by \( \Sigma^{-1} \)

\[
2 \Sigma^{-1} \Sigma w - 2\tau \Sigma^{-1} \mu + \lambda \Sigma^{-1} e = 0,
\]

\[
w = \tau \Sigma^{-1} \mu - \frac{1}{2} \lambda \Sigma^{-1} e.
\]

Equation (12) multiply \( e^T \):

\[
e^T w = \tau e^T \Sigma^{-1} \mu - \frac{1}{2} \lambda e^T \Sigma^{-1} e \quad \text{because a} \quad e^T w = 1 \quad \text{so as to get}:
\]
substituting equation (12) to equation (13) is obtained:

\[
\begin{aligned}
\mathbf{w} &= \left\{ \frac{1}{e^\Sigma^{-1} e} - \frac{\tau e^\Sigma^{-1} \mu}{e^\Sigma^{-1} e} \right\} \Sigma^{-1} e + \tau \Sigma^{-1} \mu, \\
&= \frac{1}{e^\Sigma^{-1} e} \Sigma^{-1} e + \tau \Sigma^{-1} \mu, \\
&= \frac{1}{e^\Sigma^{-1} e} \Sigma^{-1} e + \tau \Sigma^{-1} \mu, \\
&= \begin{bmatrix}
\tau \Sigma^{-1} \mu \\
\Sigma^{-1} e
\end{bmatrix},
\end{aligned}
\]

(14)

when \( \tau = 0 \) produces a portfolio of minimum variance with weights:

\[
\mathbf{w}^{\text{Min}} = \frac{1}{e^\Sigma^{-1} e} \Sigma^{-1} e,
\]

(15)

whereas for \( \tau > 0 \) obtained:

\[
\mathbf{w}^* = \mathbf{w}^{\text{Min}} + \tau \mathbf{z}^*,
\]

(16)

with \( \mathbf{z}^* = \left\{ \Sigma^{-1} \mu - \frac{\tau e^\Sigma^{-1} \mu}{e^\Sigma^{-1} e} \right\} \Sigma^{-1} e \), \( \mathbf{w}^{\text{Min}} \) is the minimum Mean-Variance portfolio that depends on the covariance matrix \( \Sigma \) but does not depend on vectors \( \mu \), while \( \mathbf{w}^* \) depends on \( \Sigma \) and \( \mu \), and have properties \( \sum_{i=1}^{N} z_i^* = 0 \).

A sufficient condition for checking the optimization properties can be made by forming a semidefinite positive Hessian matrix.

2.2.6. Quadratic Investment Portfolio Model with Risk Free Assets

This section discusses quadratic investment portfolios with risk free assets. Referring to Hult et al. [18], that the quadratic investment portfolio model with risk free assets can be illustrated as follows. For example, an investor wants to form a portfolio by setting aside a portion of his capital equal to \( w_0 \) to be invested in risk free assets, where the risk free assets provide a fixed return of \( r_0 \) by time. The proportion of the amount of capital invested in risk free assets is denoted by \( w \) with each asset giving a return of \( r \). Thus, the overall returns from the investment portfolio carried out are stated:

\[
r_p = w_0 + r_0 = \sum_{i=1}^{N} w_i r_i.
\]

(17)

Suppose \( \mathbf{w} = (w_1, \ldots, w_N)^T \) investment portfolio weight vector on risk free assets \( \mathbf{r} = (r_1, \ldots, r_N)^T \) risk free asset return vector, and \( \mathbf{\mu} = (\mu_1, \ldots, \mu_N)^T \) the mean vector of risk free asset returns, so that equation (17) can be expressed as:

\[
r_p = w_0 \mathbf{r}_0 = \mathbf{w}^T \mathbf{r},
\]

therefore, the expectations of portfolio returns \( \mu_p \) is:

\[
\mu_p = E(r_p) = E(w_0 \mathbf{r}_0 + \mathbf{w}^T \mathbf{r}) = w_0 \mathbf{r}_0 + \mathbf{w}^T \mathbf{\mu}.
\]

When given \( \Sigma \) is the covariance matrix, so that the variance or risk of return can be expressed as:

\[
\sigma_p^2 = Var(r_p) = \mathbf{w}^T \Sigma \mathbf{w}.
\]

(18)

The Mean-Variance investment portfolio optimization model with risk free assets is expressed as:

\[
\begin{aligned}
&\text{Maximize} \{ w_0 \mathbf{r}_0 + 2\tau \mathbf{w}^T \mathbf{\mu} - \mathbf{w}^T \Sigma \mathbf{w} \}, \\
&\text{subject to} \ (w_0 + \mathbf{w}^T \mathbf{e}) = 1,
\end{aligned}
\]
or,

\[
\text{Minimize}_{\mathbf{w} \in \mathbb{R}^N} \{ \mathbf{w}^T \Sigma \mathbf{w} - \mathbf{w}_0^T \mathbf{r}_0 - 2\tau \mathbf{w}^T \mathbf{\mu} \}, \tag{19}
\]

subject to \( (\mathbf{w}_0 + \mathbf{w}^T \mathbf{e}) = 1 \),

where \( \mathbf{e} = (1, \ldots, 1) \) is the unit vector. To find the solution of equation (19), the Lagrange multiplier technique can be used. The Lagrang function of equation (19) is:

\[
L(\mathbf{w}, \lambda) = \mathbf{w}^T \Sigma \mathbf{w} - \mathbf{w}_0^T \mathbf{\mu}_0 - 2\tau \mathbf{w}^T \mathbf{\mu} + \lambda (\mathbf{w}_0 + \mathbf{w}^T \mathbf{e} - 1).
\tag{20}
\]

The conditions necessary for optimization in equation (20) are obtained with the first derivative,

\[
\frac{\partial L}{\partial \mathbf{w}} = 2\Sigma \mathbf{w} - 2\tau \mathbf{\mu} + \lambda \mathbf{e} = 0 \tag{21}
\]

\[
\frac{\partial L}{\partial \lambda} = \mathbf{w}_0 + \mathbf{w}^T \mathbf{e} - 1 = 0. \tag{22}
\]

If equations (21) multiplied by the inverse of the covariance matrix \( \Sigma^{-1} \), the following equation is obtained:

\[
2\mathbf{w} - 2\tau \Sigma^{-1} \mathbf{\mu} + \lambda \Sigma^{-1} \mathbf{e} = 0
\tag{23}
\]

From the equation (23) obtained the weight vector equation \( \mathbf{w} \) as follows:

\[
\mathbf{w} = \tau \Sigma^{-1} \mathbf{\mu} - \frac{\lambda}{2} \Sigma^{-1} \mathbf{e}
\tag{24}
\]

From the equation (24) multiply with \( \mathbf{e}^T \), obtained:

\[
\mathbf{e}^T \mathbf{w} = \tau \mathbf{e}^T \Sigma \mathbf{\mu} - \frac{\lambda}{2} \mathbf{e}^T \Sigma^{-1} \mathbf{e}
\tag{25}
\]

From the equation (25) and \( \mathbf{e}^T \mathbf{w} = \mathbf{w}^T \mathbf{e} \), if substituted in equation (22) is obtained:

\[
1 - \mathbf{w}_0 = \tau \mathbf{e}^T \Sigma^{-1} \mathbf{\mu} - \frac{\lambda}{2} \mathbf{e}^T \Sigma^{-1} \mathbf{e}.
\]

So the amount of value \( -\frac{\lambda}{2} = \frac{1-\mathbf{w}_0 - \tau \mathbf{e}^T \Sigma^{-1} \mathbf{\mu}}{\mathbf{e}^T \Sigma^{-1} \mathbf{e}} \), and if substituted in equation (25), the weight vector equation \( \mathbf{w} \) is obtained as follows:

\[
\mathbf{w} = \tau \Sigma^{-1} \mathbf{\mu} + \frac{1-\mathbf{w}_0 - \tau \mathbf{e}^T \Sigma^{-1} \mathbf{\mu}}{\mathbf{e}^T \Sigma^{-1} \mathbf{e}} \Sigma^{-1} \mathbf{e}.
\tag{26}
\]

A sufficient condition for checking the optimization properties can be done by forming a positive semidefinite Hessian matrix.

### 3. Result and Analysis

In this section, we will discuss the numerical simulation of the mathematical model that has been discussed previously in 2.1. From the calculation results of the selected data, it is obtained the average stock return vector and the unit vector as follows:

\[
\mathbf{w}_0 = (0.0007499 \ 0.0001552 \ 0.0016112 \ 0.0049164 \ 0.0023468 \ 0.000415880 \ 0.000309538 \ 0.000309538 \ 0.000603192
\]

\[
\mathbf{r}_0 = (0.000414580 \ 0.000831530 \ 0.000188906 \ 0.000237002 \ 0.000421488 \ 0.000309538 \ 0.000237002 \ 0.000309538 \ 0.000421488
\]

\[
\Sigma = \begin{bmatrix}
0.000957210 & 0.000415880 & 0.000309538 & 0.000309538 & 0.000603192 \\
0.000414580 & 0.000831530 & 0.000188906 & 0.000237002 & 0.000421488 \\
0.000309538 & 0.000188906 & 0.001570646 & 0.000084131 & 0.000349363 \\
0.000309538 & 0.000237002 & 0.000084131 & 0.000365533 & 0.000314287 \\
0.000603192 & 0.000421488 & 0.000349363 & 0.000314287 & 0.001339935
\end{bmatrix}
\]

and

\[
\mathbf{e} = (1 \ 1 \ 1 \ 1).
\]
Next, we get the inverse of the covariance matrix $\Sigma^{-1}$ which is used for the process of calculating the weight proportions with calculations using Matlab [19]. The results of numerical calculations to find the proportion of weights verified by the Mean-Variance method without using risk free assets are given in Table 2.

Tabel 2. Mean-variance investment portfolio without risk free asset

| $\tau$ | ADRO  | ANTM  | ENRG  | FIRE  | DOID  | $\mu$ | $\sigma^2$ | $\frac{\mu}{\sigma^2}$ |
|--------|-------|-------|-------|-------|-------|-------|-----------|------------------------|
| 0      | 0.216024 | 0.396369 | 0.214753 | 0.084916 | 0.087935 | 0.00119 | 0.000516912 | 2.308                   |
| 0.02   | 0.215010 | 0.394647 | 0.215020 | 0.085907 | 0.089414 | 0.00120 | 0.000516919 | 2.323                   |
| 0.04   | 0.213995 | 0.392924 | 0.215287 | 0.086898 | 0.090893 | 0.00120 | 0.00051694 | 2.338                   |
| 0.06   | 0.212981 | 0.391201 | 0.215554 | 0.087888 | 0.092373 | 0.00121 | 0.000516975 | 2.353                   |
| 0.08   | 0.21196  | 0.389478 | 0.215821 | 0.088879 | 0.09385 | 0.00122 | 0.000517025 | 2.368                   |
| 0.1    | 0.210953 | 0.387755 | 0.216088 | 0.089870 | 0.09533 | 0.00123 | 0.000517088 | 2.382                   |
| 0.12   | 0.209938 | 0.386031 | 0.216355 | 0.090861 | 0.096812 | 0.00124 | 0.000517166 | 2.397                   |
| 0.14   | 0.208923 | 0.384307 | 0.216622 | 0.091852 | 0.098293 | 0.00128 | 0.000517257 | 2.411                   |
| 0.16   | 0.20790 | 0.382583 | 0.216889 | 0.092844 | 0.099774 | 0.00125 | 0.000517363 | 2.426                   |
| 0.18   | 0.206892 | 0.380857 | 0.217156 | 0.093836 | 0.10125 | 0.00126 | 0.000517483 | 2.440                   |
| 0.2    | 0.205877 | 0.379132 | 0.217424 | 0.094828 | 0.102737 | 0.00127 | 0.000517617 | 2.455                   |
| 0.22   | 0.204860 | 0.377405 | 0.217691 | 0.095821 | 0.104220 | 0.00127 | 0.000517765 | 2.469                   |
| 0.24   | 0.203844 | 0.375678 | 0.217959 | 0.096814 | 0.105703 | 0.00128 | 0.000517928 | 2.483                   |
| 0.26   | 0.202826 | 0.373950 | 0.218226 | 0.097808 | 0.107187 | 0.00129 | 0.000518105 | 2.497                   |
| 0.28   | 0.201808 | 0.372221 | 0.218494 | 0.098802 | 0.108672 | 0.00130 | 0.000518296 | 2.511                   |
| 0.3    | 0.200790 | 0.370491 | 0.218762 | 0.099797 | 0.11015 | 0.00131 | 0.000518501 | 2.525                   |

by choosing $0 \leq \tau \leq 0.3$ the optimal portfolio is obtained by considering the ratio calculation. The optimal portfolio is achieved when the vector is weighted $w^T = (0.200790, 0.370491, 0.218762, 0.099797, 0.11015)$, successively ADRO, ANTM, ENRG, FIRE, and DOID stocks by generating returns and variances of $0.00131$ and $0.000518501$. The Efficient frontier graph and the ratio between average returns to portfolio variance will be displayed in Figure 1.
To carry out the process of optimizing an investment portfolio with risk free assets, will be allocated a proportion of $w_0 = 10\%$ invested in risk free assets that provide an average return of 7\% per year or equivalent $\mu_0 = 0.0111$ per day. The Mean-Variance portfolio optimization weight with free assets is calculated using the equation (26) and numerical calculations are performed with the results stated in Table 3.

| $\tau$ | ADRO    | ANTMT   | ENRG    | FIRE    | DOID    | $\mu$  | $\sigma^2$ | Rasio $\frac{\mu}{\sigma^2}$ |
|--------|---------|---------|---------|---------|---------|--------|------------|-------------------------------|
| 0      | 0.179226| 0.342399| 0.18780 | 0.085307| 0.105263| 0.00226| 0.0004126 | 5.4929                        |
| 0.02   | 0.152443| 0.300252| 0.19207 | 0.109893| 0.145330| 0.00246| 0.0004165 | 5.9100                        |
| 0.04   | 0.125660| 0.258106| 0.19635 | 0.134478| 0.185398| 0.00264| 0.0004282 | 6.2041                        |
| 0.06   | 0.098877| 0.215960| 0.20063 | 0.159064| 0.225465| 0.00284| 0.0004477 | 6.3695                        |
| 0.08   | 0.072094| 0.173814| 0.20490 | 0.183649| 0.265532| 0.00304| 0.0004750 | 6.4140                        |
| 0.1    | 0.045312| 0.131668| 0.20918 | 0.208235| 0.305600| 0.00324| 0.0005102 | 6.3549                        |
| 0.12   | 0.018529| 0.089522| 0.21346 | 0.232820| 0.345667| 0.00343| 0.0005531 | 6.2144                        |

By choosing $0 \leq \tau \leq 0.3$ only $0 \leq \tau \leq 0.12$ which gives a positive weight result. The optimal portfolio is achieved when $\tau = 0.08$ with the weight vector $w^T = (0.072094 \ 0.173814 \ 0.20490 \ 0.183649 \ 0.265532)$, successively ADRO, ANTMT, ENRG, FIRE, and DOID stocks by generating returns and variances of $0.00304$ and $0.0004750$. The picture of Efficient frontier and the ratio between the average return to the variance of the portfolio and risk free assets is shown in Figure 2.
Figure 2. Efficient frontier and ratio portfolio mean-variance with risk free asset

Comparative analysis of the two models is carried out by observing the numerical calculation results in Table 2 and Table 1 are described as follows:

- Risk tolerance for the Mean-Variance model without risk free assets in the mining stock data is $0 \leq \tau \leq 0.3$. For risk tolerance, the Mean-Variance model with moderate risk free assets ranges from mine stocks is $0 \leq \tau \leq 0.12$.

- The minimum portfolio for the Mean-Variance model without risk free assets on mining stocks gets an average return and variance of 0.00119 and 0.000516912, whereas for the method with risk free assets, the average return and variance is obtained 0.00226 and 0.0004126.

- The optimum portfolio for the Mean-Variance model with risk free assets on mining stocks gets an average return and variance of 0.00131 and 0.000518501, whereas for the method with risk free assets, the average return and variance is obtained 0.00304 and 0.0004750.

- The optimum portfolio weight for the Mean-Variance model without risk free assets on mining stocks, obtained from the proportion of shares expressed in vectors $w^T = (0.200790 \ 0.370491 \ 0.218762 \ 0.099797 \ 0.11015)$, respectively ADRO, ANTM, ENRG, FIRE, and DOID shares. Meanwhile, with risk free assets the proportion of mining shares is expressed in vectors $w^T = (0.072094 \ 0.173814 \ 0.20490 \ 0.183649 \ 0.265532)$, which respectively shares ADRO, ANTM, ENRG, FIRE, and DOID.

Comparison of efficient surface curves and the ratio of the Mean-Variance portfolio without and with risk free assets for 5 mining stocks, as in Figure 3.
A clear difference is seen between the efficient surface curve and the ratio between the mean divided by the variance for the Mean-Variance model portfolio with risk free assets, higher than the Mean-Variance model portfolio without risk free assets. From the comparison, it can also be concluded that the Mean-Variance model with risk free assets produces an optimal portfolio return that is more profitable and less risky than the Mean-Variance model without risk free assets.

4. Conclusion
This research has carried out the optimization of the Mean-Variance investment portfolio by entering risk free assets in several mining stocks. From this expansion, a formula can be derived to determine the optimum portfolio weight allocation. From the analysis, it turns out that the efficient surface graph of the combined portfolio without and with risk free assets is above the efficient surface graph of the portfolio without risk free assets. This shows that investment portfolios that combine without and with risk free assets are more profitable than risk free investment portfolios as evidenced by the simulation of five selected mining stock data.

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