Hahn-Banach theorems for $\kappa$-normed spaces

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Abstract

For a new class of topological vector spaces, namely $\kappa$-normed spaces, an associated quasisemilinear topological preordered space is defined and investigated. This structure arise naturally from the consideration of a $\kappa$-norm, that is a distance function between a point and a $G_\delta$-subset. For it, analogs of the Hahn-Banach theorem are proved.

1 Introduction.

Normable topological vector spaces play important role in functional analysis, but their class does not encompass all locally convex spaces. A new class of topological spaces, namely the class of $\kappa$-metric topological spaces, was introduced earlier in works [8, 9], where a regular $\kappa$-metric is defined as a non-negative function $\rho : X \times 2^X \to \mathbb{R}$ satisfying axioms $(N1 - 4, 5(b))$ below, $2^X_0$ being the family of all canonical closed subsets of $X$. It is worth to mention that $\kappa$-metric spaces may be non-metrizable. One of the most important examples of $\kappa$-metric spaces are locally compact groups and generalized loop groups [3]. Different distance functions for subsets of linear normed spaces were studied in [1]. Topological vector spaces with $\kappa$-metrics satisfying additional conditions related with the linearity of these spaces were defined and studied in [4, 5]. For such $\kappa$-normed spaces (see Definition 1.1 below) analogs of theorems about fixed point, closed graph and open mapping have been proved. Free $\kappa$-normed spaces generated by $\kappa$-metric uniform spaces with uniformly continuous $\kappa$-metrics have also been studied, as well as categorial properties of $\kappa$-normed spaces relative to products, projective
and inductive limits. Duality of $\kappa$-normed spaces and their applications has also been investigated.

On the other hand, Hahn-Banach theorems play very important role for locally convex spaces \[6, 7, 10\]. In particular these theorems are used for investigations concerning barrelledness properties of locally convex spaces \[2\]. In this work a structure of quasisemilinear preordered topological space on $X \times S$, where $S$ is a family of subsets of a topological vector space $X$, is naturally defined and investigated. This structure in general does not reduce to a linear or even a semilinear one. For it, analogs of the Hahn-Banach theorems are proved. These theorems can be used for further investigations of analogs of barrelledness for such quasisemilinear spaces and their duality. Analogs of separation theorems for quasisemilinear preordered spaces can be deduced using the Hahn-Banach theorems. In turn, this can serve for a continuation of investigations of applications of $\kappa$-normed spaces to differential equations and economic problems (see also \[5\]).

1.1. Definition. A topological vector space $X$ over the field $K = \mathbb{R}$ or $\mathbb{C}$ or non-Archimedean supplied with the family $S_X := 2^X$ of all canonical closed subsets or the family $S_X := 2^X_\delta$ of all closed $G_\delta$-subsets is called $\kappa$-normed if there exists on $X \times S_X$ a non-negative function $\rho(x, C)$, called a $\kappa$-norm, satisfying the following conditions:

- \((N1)\) $\rho(x, C) = 0$ if and only if $x \in C$;
- \((N2)\) if $C \subset C'$, then $\rho(x, C) \geq \rho(x, C')$;
- \((N3)\) the map $x \mapsto \rho(x, C)$ is uniformly continuous for each fixed $C \in S_X$;
- \((N4)\) for each increasing transfinite sequence $\{C_\alpha\}$ with $C := \text{cl}(\bigcup \alpha C_\alpha) \in S_X$ then $\rho(x, C) = \inf \alpha \rho(x, C_\alpha)$, where $\text{cl}_X(A) = \text{cl}(A)$ denotes the closure of a subset $A$ in $X$;
- \((N5)(a)\) $\rho(x + y, \text{cl}(C_1 + C_2)) \leq \rho(x, C_1) + \rho(y, C_2)$ and
- \((N5)(b)\) $\rho(x, C_1) \leq \rho(x, C_2) + \bar{\rho}(C_2, C_1)$ (with the maximum instead of the sum on the right sides of the inequalities in the non-Archimedean case),
- \((N6)\) $\rho(\lambda x, \lambda C) = |\lambda|\rho(x, C)$ for each $K \ni \lambda \neq 0$;
- \((N7)\) $\rho(x + y, y + C) = \rho(x, C)$.

If we consider the empty set $\emptyset$ as an element of $S_X$, then

- \((N8)\) $\rho(x, \emptyset) = \infty$ and $\rho(x, C) < \infty$ for each $x \in X$ and $\emptyset \neq C \in S_X$.

The space $X \times S_X$ is called $\kappa$-normed if there is given a fixed $\kappa$-norm $\rho$. We denote the $\kappa$-normed space by $(X, S_X, \rho)$. For two closed subsets $A$ and $B$ in $X$ we denote by $A \hat{+} B$ the set $\text{cl}_X(A + B)$. If we put $k \circ A := A \hat{+} \ldots \hat{+} A$, \[2\]
in general from \( k \circ A = k \circ B \) does not follow \( A = B \), and \( kA \) may be not equal to \( k \circ A \) as may be easily seen. The main results of this paper are Theorems 2.8, 2.10, 2.11 and 2.14.

2 Hahn-Banach theorems for \( \kappa \)-normed spaces.

2.1.1. Notes and Definitions. Consider a family \( S_X \) of subsets of a topological vector space \( X \) over the field \( K \), either \( \mathbb{Q} \) or \( \mathbb{R} \) or \( \mathbb{C} \), such that

1. \( \emptyset \in S_X \);
2. there is defined an addition ‘\( \hat{+} \)’ in \( S_X \) making it a commutative semigroup with zero element, that is, \( S_X^2 \ni (A, B) \mapsto A \hat{+} B \in S_X \) such that it is commutative and associative with zero element \( \emptyset \);
3. \( S_X \) is preordered by the inclusion relation, that is, \( A \leq B \) if and only if \( A \subset B \) or \( A = B \), which obviously is (3i) reflexive and (3iii) transitive. Additionally we shall require that (3iii) \( A_1 \leq B_1 \) and \( A_2 \leq B_2 \) implies that \( A_1 \hat{+} B_1 \leq A_2 \hat{+} B_2 \) and (3iv) \( A \hat{+} B \leq C \hat{+} B \) and \( B \leq G \) implies that \( A \hat{+} G \leq C \hat{+} G \);
4. it is invariant relative to the multiplicative group \( K^* := K \setminus \{0\} \), that is, \( aA \in S_X \) for each \( a \in K^* \) and \( A \in S_X \); moreover, it is semilinear relative to \( K^* \), that is, \( a(A \hat{+} B) = aA \hat{+} aB \), \( (ab)A = a(bA) \), \( 1A = A \) for each \( A \) and \( B \) in \( S_X \) and each \( a \) and \( b \) in \( K^* \);
5. \( S_X \) is invariant relative to shifts on all vectors from \( X \), that is, \( v + A \in S_X \) for each \( v \in X \) and \( A \in S_X \).

Due to Condition (4), \( S_X \) is idempotent free, that is,
6. from \( aA = aB \) for some \( a \in K^* \), it follows \( A = B \).

Conditions (3, 4) imply that
7. \( aA \leq aB \) for each \( A \leq B \) in \( S_X \) and \( a \in K^* \).

Conditions (3, 5) imply that
8. \( A \leq A \hat{+} B \) for each \( A \) and \( B \) in \( S_X \) with \( 0 \in B \).

If a family \( S_X \) of subsets of \( X \) satisfies Conditions (1 – 5) we shall call it a \( K^* \)-quasisemilinear preordered space. Without (3) we shall call it a \( K^* \)-quasisemilinear space. If \( A \hat{+} B \leq A' \hat{+} B \) implies that \( A \leq A' \) we say that \( S_X \) is monotonically cancellative.

If \( S \) is an abstract family of sets of a topological space \( X \) and there is an abstract preorder ‘\( \leq \)’ on \( S \) such that the pair \((X, S)\), with the preorder \( (x, A) \leq (y, B) \) if and only if \( x = y \) and \( A \leq B \), may be equipped
with an addition ‘+’ defined by \((x, A) + (y, B) = (x + y, A \hat{+} B)\), where ‘\(\hat{+}\)’ satisfies Conditions (1−8), then \((X, S)\) will be called an (abstract) \(K^*\)-quasisemilinear preordered space. Providing \(S\) with a topology such that the algebraic operations on \((X, S)\) are continuous; that is, relative to the topologies \(\tau_X\) on \(X\), \(\tau_S\) on \(S\) and \(\tau_K\) on \(K\), the maps \(\hat{+} : (S, \tau_S)^2 \to (S, \tau_S)\); 
\[ + : (X, \tau_X) \times (S, \tau_S) \to (S, \tau_S)\]; 
\(K^* \times S \ni (a, A) \mapsto aA \in S\) are continuous, then \((X, S)\) is called a topological \(K^*\)-quasisemilinear space. If in (1−8) \(K^*\) is substituted by \(K_+ := K \cap (0, \infty)\), then it is called a (topological) \(K_+\)-quasisemilinear space.

Let \(P\) be either \(K^*\) or \(K_+\). For a subset \(J\) of a linear space \(X\) over \(K\) we define, as usually, \(sp_{\mathbf{P}} J := \{x : x = \sum_{j=1}^{n} a_j p_j; a_j \in \mathbf{P}, p_j \in J, n \in \mathbb{N}\}\). For a subset \(J\) in \(S_X\) we shall write \(S_{\mathbf{P}} J := \{A : A = a_1 A_1 + a_2 A_2 + \ldots + a_n A_n; a_j \in \mathbf{P}, A_j \in J, n \in \mathbb{N}\}\).

A subset \(F\) of \(S_X\) is called hereditary if \(A \subseteq B\) whenever \(A \leq B \in F\). If \(F \subseteq S_X\) and \(S_X\) satisfies Conditions (1−8), then there exists a least \(P\)-quasisemilinear preordered space containing \(F\), denoted by \(\hat{F}\), such that \(\hat{F}\) is the intersection of all \(P\)-quasisemilinear preordered subspaces of \(S_X\) containing \(F\). The set \(\{A : A \in S_X, \text{ there are } B \in S_X \text{ and } C \in F \text{ such that } A \hat{+} B \leq C\}\) is called the hereditary face in \(S_X\) generated by the subset \(F\) of \(S_X\).

For a linear space \(X\) its family \(\Omega_X\) of all singletons in \(X\) satisfies Conditions (1−8) with the ordinary sum and with the relation \(x \leq y\) meaning \(x = y\) in \(X\). Hence, we get the notion of the hereditary face of a subset \(J\) in \(X\). For \(X \times S_X\) considered as \(\Omega_X \times S_X\) Conditions (1−8) also are satisfied with the preorder \((x, A) \leq (y, B)\) if and only if \(x = y\) and \(A \leq B\), so we get the notion of the hereditary face in \(X \times S_X\) generated by a subset \(P\) in \(X \times S_X\).

**2.1.2. Definitions and Conventions.** For a topological vector space \(X\) and a family of subsets \(S_X\) as in §2.1.1. a mapping \(f : X \times S_X \to [-\infty, \infty]\) is called sublinear if it satisfies Conditions (1−5):

1. \(f(0, \emptyset) = 0\);
2. \(f(v + x, v + A) = f(x, A)\) for each \(v\) and \(x\) in \(X\) and each \(A \in S_X\);
3. \(f(x + y, \emptyset) \leq f(x, \emptyset) + f(y, \emptyset)\) for each \(x\) and \(y\) in \(X\);
4. \(f(0, A + B) \leq f(0, A) + f(0, B)\) for each \(A\) and \(B\) in \(S_X\) with the convention \(\infty + (-\infty) = \infty\);
5. \(f(0, aA) = af(0, A)\) for each \(a \in K_+\).
It is called superlinear, if it satisfies Conditions (S1, S2, S5) and
(S3)' \( f(x + y, \emptyset) \geq f(x, \emptyset) + f(y, \emptyset) \) for each \( x \) and \( y \) in \( X \);
(S4)' \( f(0, A + B) \geq f(0, A) + f(0, B) \) for each \( A \) and \( B \) in \( S_X \) with the

convention \( \infty + (-\infty) = -\infty \).

A mapping \( f : X \times S_X \to [-\infty, \infty] \) is called semilinear if it satisfies
(S1, S2, S5) and
(S3)'' \( f(x, \emptyset) \) is linear by \( x \in X \) on \( X \times \emptyset \);
(S4)'' \( f(0, A + B) = f(0, A) + f(0, B) \) for each \( A \) and \( B \) in \( S_X \) and takes
no more than one of the values \(-\infty, \infty\).

Conditions (S2, S5) imply that
(S6) \( f(ax, aA) = af(x, A) \) for each \( a \in K_+ \), since \( f(ax, aA) = f(0, a(A - x)) = af(0, A - x) = af(x, A) \). From (S2, S3) it follows that (S7) \( f(x + y, A + B) \leq f(x, A) + f(y, B) \), since \( f(x + y, A + B) = f(0, A - x + B - y) \leq f(0, A - x) + f(0, B - y) = f(x, A) + f(y, B) \).

In view of 2.1.1.(8) and 2.1.2.(S1 – S4) or 2.1.2.(S1, S2, S3', S4') for each nonvoid \( A \) in \( S \) there exists \( x \in A \) such that \( f(x, A) \geq 0 \) for \( f \) sublinear or semilinear, or \( f(x, A) \leq 0 \) for \( f \) superlinear.

For an inequality \( \sum a_i \leq \sum b_j \) if the expression \( \infty - \infty \) occurs on the left side we use the convention \( \infty - \infty = -\infty \), if \( \infty - \infty \) occurs on the right side we use the convention \( \infty - \infty = \infty \).

Let \( F \) be a \( P \)-quasisemilinear preordered subspace of \( S_X \). A mapping
\( f : F \to [-\infty, \infty] \) is called monotonely cancellative if \( f(A) \leq f(B) \) whenever \( A + C \leq B + C \), with \( A \) and \( B \) in \( F \), for some \( C \in S_X \). A mapping \( f : F \to [-\infty, \infty] \) is called monotone if \( f(A) \leq f(B) \) for each \( A \leq B \) in \( F \). These
definitions extend trivially to mappings \( f : H \times F \to [-\infty, \infty] \) where \( H \) is a linear subspace of \( X \) and \( F \) is a set in \( S_X \) satisfying Conditions 2.1.1. (1–8)
with \( H \) instead of \( X \).

2.1.3. Definitions. Let \( p : X \times S_X \to (-\infty, \infty] \) be sublinear, \( H \) be a linear subspace in \( X \) and \( F \) be a subfamily in \( S_X \) satisfying Conditions (1–8)
with \( H \) instead of \( X \) and let \( g : H \times F \to (-\infty, \infty] \) be semilinear, monotone
and dominated by \( p \). An extension \( h \) of \( g \) on \( X \times S_X \), \( h : X \times S_X \to (-\infty, \infty] \),
is called a monotone Hahn-Banach extension if \( h \) is semilinear, monotone and
dominated by \( p \).

If \( p \) is sublinear, then from \( p(x, A) < \infty \) and \( p(y, B) < \infty \) it follows \( p(x + y, A + B) \leq p(x, A) + p(y, B) < \infty \) and \( p(ax, aA) = ap(x, A) < \infty \) for each \( a \in K_+ \), hence the set \( \{ (x, A) \in X \times S_X : p(x, A) < \infty \} \) is a \( P \)-quasisemilinear
preordered subspace of \( X \times S_X \). Analogously, if \( g \) is semilinear, then the set
\{(x, A) \in X \times S_X : g(x, A) < \infty\} is a P-quasisemilinear preordered subspace of \(X \times S_X\).

2.1.4. Remarks and Notations. In §2.1.2, (S5) the restriction \(a > 0\) is necessary since \(f(0, aA) = af(0, A)\) for each \(a \in K^\ast\) would imply \(f(0, A) = 0\) for each \(A\) in \(S_X\) such that \(A = -A\) (for instance, if \(A\) is a balanced subset \(A\) in \(X\)).

If \(P\) is either \(K^\ast\) or \(K_\ast\), consider the test relation for \(x \in X\) and \(A \in S_X\) consisting of the equality \(y + nx = l + m\) and the inequality \(B \stackrel{\vdash}{\raisebox{0.5ex}{$\circ$}} n_1 \circ (a_1 A) \vdots \ldots \vdots \vdots \vdots n_k \circ (a_k A) + C \leq J + G + C\) with \(m\) in \(X\), \(y\) and \(l\) in \(H\), \(B\) and \(J\) in \(F\), \(Z \ni n_i > 0\) and \(a_i \in K_\ast\), with \(i = 1, \ldots, k\), \(k \in \mathbb{N}\), \(n := n_1 a_1 + \ldots + n_k a_k\), and \(G\) and \(C\) in \(S_X\). A special test relation is characterized by the condition \(C = \emptyset\). Given as above a sublinear mapping \(p : X \times S_X \to (-\infty, \infty]\) and a semilinear and monotone mapping \(g\) on \(H \times F\) dominated by \(p\), let us define the numbers \(\xi(x, A)\) and \(\xi_0(x, A)\) belonging to \([-\infty, \infty]\) by the following formulas:

\[
\xi(x, A) := \inf\{n^{-1}(p(m, G) + g(l, J) - g(y, B)) : y + nx = l + m \text{ and } B \vdash n_1 \circ (a_1 A) \vdots \ldots \vdots n_k \circ (a_k A) + C \leq J + G + C\} \text{ and }
\]

\[
\xi_0(x, A) := \inf\{n^{-1}(p(m, G) + g(l, J) - g(y, B)) : y + nx = l + m \text{ and } B \vdash n_1 \circ (a_1 A) \vdots \ldots \vdots n_k \circ (a_k A) \leq J + G\},
\]

where the infimum is taken by all test relations and special test relations, respectively. These functions are also denoted by \(\xi_{p,g}(x, A)\) and \(\xi_{0,p,g}(x, A)\), respectively.

Let \(E\) be \(P\)-quasisemilinear subspace of \(X \times S_X\). Denote by \(\bigtriangleup E\) the set of all \(x \in X\) and \(A \in S_X\) for which there exist \(n_i \in \mathbb{N}\), \(a_i \in K_\ast\), \(k \in \mathbb{N}\), \(l\) and \(y\) in \(H := \pi_X(E)\) with

(i) \(y + nx = l\),

and there exist \(B\) and \(C\) in \(F := \pi_S(E)\) and \(Q \in S_X\) such that

(ii) \(B \vdash n_1 \circ (a_1 A) \vdots \ldots \vdots n_k \circ (a_k A) + Q = C + Q\), where variables \(n, n_i, a_i, k\) are the same as above, \(\pi_X : X \times S_X \to X\) and \(\pi_S : X \times S_X \to S_X\) are the natural projections on \(X\) and \(S_X\) respectively.

Also \(\bigtriangleup_0 E\) denotes the set of all \(x \in X\) and all \(A \in S_X\) for which there exist \(n, a_i, k, m\) in \(X\), \(l\) and \(y\) in \(H\) with

(iii) \(y + nx = l\),

and there exist \(B\) and \(C\) in \(E\) such that

(iv) \(B \vdash n_1 \circ (a_1 A) \vdots \ldots \vdots n_k \circ (a_k A) = C\).

2.1.5. Notes and Definitions. Consider a \(\kappa\)-normed space \((X, S, \rho)\), where \(X\) is a topological vector space with a topology \(\tau_X\), \(S\) is a family of
subsets such that each canonical closed subset belongs to $S$ and $\rho : X \times S \to [0, \infty)$ is a $\kappa$-norm. Let us define on $S$ several topologies.

Let $\tau_1$ denote the topology generated by the base $W(A, V) := \{B \in S : A \subset B \subset A + V\}$, where $A \in S$, $V \in \tau_X$, $0 \in V$, $S = 2^X$.

Let $\tau_2$ be the topology generated by the base $W(A, b) := \{B \in S : \bar{\rho}(A, B) < b\}$, where $\bar{\rho}(A, B) := \sup_{a \in A} \rho(a, B)$, $2_o^X \subset S \subset 2_s^X$.

The topology $\tau_3$ induced by the metric $D$ defined by the equation $D(A, B) := \bar{\rho}(A, B) + \bar{\rho}(B, A)$, $2_o^X \subset S \subset 2_s^X$.

For each $b > 0$ and each $A \in S$ there exists a neighborhood $V$ of zero in $X$ such that $|\rho(x, A + V) - \rho(x, A)| < b$ for each $x \in X$ (see Corollary 5 [4]). If $A \subset B$ in $S$, then $\bar{\rho}(A, B) = 0$, hence $D(A, A + V) < b$. Consequently, the topology $\tau_1$ on $S$ is not weaker than $\tau_3$. Then consider on $X \times S$ the topologies $\zeta_j := \tau_X \times \tau_j$, where $j = 1, 2, 3, 4$.

2.1.6. Proposition. Relative to each of the topologies $\zeta_j$, where $j = 1, 2, 3, 4$, $X \times S$ is a topological P-quasisemilinear preordered space.

Proof. Since the space $X \times S$ is a P-quasisemilinear preordered space and $X$ is a topological vector space, it remains to verify the continuity of algebraic operations in $X \times S$.

$\tau_1$. For each $V \in \tau_X$ with $0 \in V$ there are $V_j \in \tau_X$, $0 \in V_j$, with $j = 1$ and $j = 2$ such that $V_1 + V_2 \subset V$. If $A_j \subset B_j \subset A_j + V_j$ for $j = 1$ and $j = 2$ with $A_j$ and $B_j$ in $S$, then $A_1 + A_2 \subset B_1 + B_2 \subset A_1 + A_2 + (V_1 + V_2)$. Hence the addition $(A, B) \mapsto (A + B)$ is continuous from $S^2$ to $S$. Besides, since $x + U + W(A, V) \subset W(x + A, U + V) \subset W(x, A, G)$, where $0 \in U \cap V$, $U + V \subset G$, $G \in \tau_X$, the addition $(x, A) \mapsto x + A$ is continuous from $X \times S$ into $S$.

On the other hand, given $a \in K^*$ and $A \in 2_o^X$, for each $b > 0$ and each neighbourhood $U$ of zero in $X$ there exists a neighbourhood $V_1$ of zero in $X$ such that for each $c$ with $|a - c| < b$ then $cV_1 \subset U$. If $A \subset B \subset A + V_1$, then

(i) $cA \subset cB \subset cA + cV_1 \subset cA + U$. Since $X$ is a topological vector space, then for each $0 \in A \in 2_o^X$ and each $0 \in V_2 \in \tau_X$, $a \in K^*$ and $b > 0$ there exist $A_1 \in 2_o^X$ and $0 < \delta < b$ for which

(ii) $aA \subset aA_1 \subset aA + V_2$ whenever $|a - c| < \delta$ and $c \neq 0$, moreover, $0 \in A_1$. By the definition, if $B \in cW(A_1, V_1)$ then there exists $B_1 \in S$ such that $B = cB_1$ and $A_1 \subset B_1 \subset A_1 + V_1$, hence $cA_1 \subset B \subset cA_1 + V_1$. Choose $V_1$ and $V_2$ such that $V_1 + V_2 \subset U$ and take $V = V_1 \cap V_2$, then from Inclusions
(i, ii) it follows, that

$$aA \subset cA_1 \subset B \subset cA_1 + V_1 \subset aA + V_1 + V_2 \subset aA + U$$

for each $c$ such that $|a - c| < \delta$. Therefore, $\{c : |a - c| < \delta\}W(A_1, V) \subset W(aA, U)$ and hence the multiplication on scalars from $K^* \times S$ into $S$.

$\tau_2$. If $\bar{\rho}(A_j, B_j) < b/2$ for $A_j$ and $B_j$ in $S$, where $j = 1, 2$, then $\bar{\rho}(A_1 + A_2, B_1 + B_2) = \sup_{a_1 \in A_1, a_2 \in A_2} \bar{\rho}(a_1 + a_2, B_1 + B_2) \leq \sup_{a_1 \in A_1} \bar{\rho}(a_1, B_1) + \sup_{a_2 \in A_2} \bar{\rho}(a_2, B_2) = \bar{\rho}(A_1, B_1) + \bar{\rho}(A_2, B_2) < b$, hence $W(A_1, b/2) + W(A_2, b/2) \subset W(A_1 + A_2, b)$.

From $\bar{\rho}(x + A, x + B) = \bar{\rho}(A, B)$ for each $x \in X$ and each $A$ and $B$ in $S$ we have $W(A, b) + x = W(x + A, b)$. In view of Lemma 2 [4] for each $A \in S$ there exists a family $\{U_j : j \in N\}$ such that $A = \cap_{j \in N} U_j$, where $U_j \in \tau_X$ and $U_j \supset \overline{cl}_X(U_{j+1})$ for each $j$. Consider $A \in S$ such that $0 \in A$. Then we can take $0 \in U_j$ for each $j$. Since $(X, \tau_X)$ is the topological vector space, denoting $A + (-B)$ by $\hat{A}^{-B}$ for each $j$ there exists $V_j \in \tau_X$ such that $0 \in V_j$, $V_j - V_j \subset U_j$, $V_j \supset \overline{cl}_X(U_{j+1})$ for each $j$.

Therefore, $\cap_j V_j =: A_1 \in S$ and $A_1 - A_1 \subset A$. In view of §1.1, (N3) for each such $A$ and each $b > 0$ there exists $U$ such that $0 \in U \in \tau_X$ and $|\rho(x + y, A_1^{-1} A_1) - \rho(x, A_1^{-1} A_1)| < b$ for each $y \in U$ and each $x \in X$. In view of Corollary 5 [4] this $U$ can be chosen such that $\bar{\rho}(A_1 - A_1, B) < b$ for each $A_1 \subset U$, $B \subset U$, $A_1 \in S$, $0 \in B \in \tau_X$. By §1.1, (N5) we get the inequality $\bar{\rho}(y + A_1, B) \leq \bar{\rho}(y + A_1, A_1^{-1} A_1) + \bar{\rho}(A_1^{-1} A_1, B)$. Therefore, for each $0 \in A \in S$ and each $b > 0$ there exists $U \in \tau_X$ such that $0 \in U$ and there exists $0 \in A_1 \in S$ such that $x + y + W(A_1, b) \subset x + W(A, 2b)$ for each $x \in X$ and each $y \in U$. For each $C \in S$ there exists $x \in C$ such that $C - x =: A \in S$ and $0 \in S$. Hence the addition $X \times S \ni (x, A) \mapsto x + A \in S$ is continuous. From $\bar{\rho}(aA, aB) = |a|\bar{\rho}(A, B)$ for each $a \in K^*$ and each $x \in X$ and each $B \in S$ we have $\bar{\rho}(aA, aB) = |a|\bar{\rho}(A, B)$, hence $aW(A, b) = W(aA, |a|b)$ for each $a \in K^*$ each $A \in S$ and each $b > 0$. It may be lightly seen that the multiplication is continuous from $K^* \times S$ into $S$.

$\tau_3$. For each $b > 0$ and $A \in S$ there exists $U \in \tau_X$ such that $\bar{\rho}(A, A + U) < b$ (see Corollary 5 in [4]). Setting $W(A, b) = \{b \in S : D(A, B) < b\}$, since $D(x + A, x + B) = D(A, B)$ for each $x \in X$ and $A, B \in S$, then $x + W(A, b) = W(x + A, b)$ for each $x \in X$, each $A \in S$ and each $b > 0$. If $B \in S$ is such that $D(A, B) < b$, then $D(A + x, B) \leq D(A + x, A) + D(A, B) < 2b$ for each $x \in U$, since $\bar{\rho}(A + U, A) = 0$. Hence the addition $(x, A) \mapsto x + A$ is continuous on
$X \times S$. Since $D(A_1 + A_2, B_1 + B_2) \leq D(A_1, B_1) + D(A_2, B_2)$, then the addition $(A, B) \mapsto A + B$ is continuous from $S^2$ into $S$. From $\rho(ax, aA) = |a|\rho(x, A)$ for each $a \in K^*$ and $x \in X$ and $A \in S$, it follows that $D(aa, ab) = |a|D(A, B)$ for each $A$ and $B$ in $S$ and each $a \in K^*$. Therefore, the multiplication on scalars $(a, A) \mapsto aA$ is continuous from $K^* \times S$ into $S$.

2.2. Lemma. (I). $\Delta E$ and $\Delta_0 E$ are $P$-quasisemilinear subspaces such that $\Delta(\Delta E) = \Delta E$, $\Delta_0(\Delta_0 E) = \Delta_0 E$.

(II). If $g$ is a finite semilinear and monotonely cancellative map on $E$, then there exists a unique semilinear and monotonely cancellative extension $h$ on $\Delta E$.

(III). If $g$ is a finite semilinear and monotone map defined on $E$, then $g$ has a unique semilinear and monotone extension to $\Delta_0 E$.

Proof. (I). Equalities 2.1.4. $(i, ii)$ are invariant relative to the multiplication on scalars from $P$ and also relative to shifts on vectors $x$ from $X$, hence $\Delta E$ and $\Delta_0 E$ satisfy Conditions 2.1.1. $(4, 5)$. Addition of two Equalities of type 2.1.4. $(i1, i2)$ for $y_j, x, l_j$ and also for $B_j, A_j, C_j$ and $Q_j$ with $j = 1$ and $j = 2$ shows that $\Delta E$ satisfies Condition 2.1.1. $(2)$; analogously for $\Delta_0 E$ with Equalities 2.1.4. $(ii1, ii2)$. From this Statement $(I)$ follows.

(II). Let $g$ be defined on $E$ and $(x, A)$ be satisfying Conditions 2.1.4. $(i1, i2)$. Then $g(y, B)$ and $g(l, J)$ are in $R$. Therefore, defining $g(x, A)$ by $g(x, A) = n^{-1}(g(l, J) - g(y, B))$ and using semilinearity of $g$ on $P$-quasisemilinear $E$ we get $g(x, A)$ defined on $(sp_{p}x \times (sp_{p}A)$. A construction of $g$ on $\Delta E$ can be done with the help of Conditions 2.1.4. $(i1, i2)$ by considering all $(x_k, A_k)$ in $\Delta E \setminus E$ such that $g$ on $\hat{E}_k$ with $\hat{E}_k := E \cup (x_k, A_k)$ is defined by assigning $g_k = g(x_k, A_k) \in (-\infty, \infty]$ such that $g_k \leq g_l$ for each $(x_k, A_k) \leq (x_l, A_l)$ in $\Delta E$. Considering the family $\mathcal{Y} := \{ (\Sigma, h) : h|_E = g \}$ of $P$-quasisemilinear subspaces $\Sigma$ in $\Delta E$ and semilinear and monotonely cancellative mappings $h$ on $\Sigma$ ordered by $(\Sigma_1, h_1) \leq (\Sigma_2, h_2)$ if and only if $\Sigma_1 \subset \Sigma_2$ and $h_2|_{\Sigma_1} = h_1$, due to the Kuratowski-Zorn lemma we get that there exists a maximal element $(\Sigma, h) \in \mathcal{Y}$. For this maximal element $(\Sigma, h)$ would be $\Sigma \neq \Delta E$, then the construction with $(x_k, A_k) \in \Delta E \setminus \Sigma$ could be continued contradicting $(\Sigma, h)$ maximality, hence there exists $(\Sigma, h) \in \mathcal{Y}$ such that $\Sigma = \Delta E$ and $h|_E = g$.

(III). The proof of the last statement is analogous to that of (II) with the help of Condition 2.1.4. $(ii)$ instead of 2.1.4. $(i)$.

2.3. Lemma. Let $M$ be the hereditary face generated by $P$, where $P := \{(x, A) \in X \times S_X : g(x, A) < \infty \} \cup \{(x, A) \in X \times S_X : p(x, A) < \infty \}$.  

\[ X \times S \]
Then there exists a monotone Hahn-Banach extension of $g$ on $X \times S_X$ if and only if $M \cap \{(x, A) : g(x, A) = \infty\} = \emptyset$, and the restriction of $g$ to $M \cap E$, where $E = H \times F$ (see §2.1.3) has a (necessary finite) semilinear and monotone extension on $M$ dominated by $p|_M$.

**Proof.** If $Q := M \cap \{(x, A) : g(x, A) = \infty\} \neq \emptyset$, then evidently for the extension $g$ on $X \times S_X$ the condition $\text{range}(g) \subset (-\infty, \infty)$ would not be satisfied. On the other hand, if $Q = \emptyset$, such extension exists as in Lemma 2.2.

2.4. **Lemma.** If $h$ is a monotone Hahn-Banach extension of $g$, then $h \leq \xi_0$. If additionally $h$ is finite or, more generally, if $h$ is monotonely cancellative, then $h \leq \xi$.

**Proof.** If $B+1 \circ (a_1 A) \ldots + n_k \circ (a_k A) \leq E \hat{+} G$ and $y + nx = l + m$ with $n = n_1 a_1 + \ldots + n_k a_k$, then $h(y + nx, B+1 \circ (a_1 A) \ldots + n_k \circ (a_k A)) = h(y, B) + nh(x, A) \leq h(l + m, E \hat{+} G) = h(l, E) + h(m, G)$. Therefore,

(i) $h(x, A) \leq n^{-1}\{h(l, E) + h(m, G) - h(y, B)\} = n^{-1}\{h(m, G) + g(l, E) - g(y, B)\}$ for each special test relation, consequently, $h(x, A) \leq \xi_0(x, A)$ for each $(x, A) \in X \times S_X$. If in addition $h$ is monotonely cancellative and $B+1 \circ (a_1 A) \ldots + n_k \circ (a_k A) + C \leq E \hat{+} G + C$ and $y + nx = l + m$, then $h(y + nx, B+1 \circ (a_1 A) \ldots + n_k \circ (a_k A) + C) \leq h(l + m, E \hat{+} G + C)$ implies $h(y + nx, B+1 \circ (a_1 A) \ldots + n_k \circ (a_k A)) \leq h(l + m, E \hat{+} G)$.

So Inequalities (i) for each test relation give $h(x, A) \leq \xi(x, A)$ for each $(x, A) \in X \times S_X$.

If $h$ is finite, then $h(x, C) \in \mathbb{R}$ for each $C$, consequently, if $A \hat{+} C \leq B \hat{+} C$, then $h(x, A) + h(x, C) \leq h(x, B) + h(x, C)$ and $h(x, A) \leq h(x, B)$, hence $h$ is monotonely cancellative.

2.5. **Lemma.** If $\xi > -\infty$ (see §2.1.4), then

(i) $\xi$ is sublinear and monotonely cancellative;

(ii) $\xi \leq p$;

(iii) $g$ is monotonely cancellative;

(iv) $\xi_{p,h} = \xi_{p,g}$;

(v) $\xi_{s} = \xi$.

In the case of $\xi_0$ and $\Delta_0 E$ instead of $\xi$ and $\Delta E$ the same statements hold, but in (i, iii) only monotonicity is guaranteed.

**Proof.** (i). Since $g$ is semilinear and $p$ is sublinear, then $\xi(ax, aA) = a\xi(x, A)$ for each $a \in K_+$ and $(x, A) \in X \times S$. Evidently, $\xi$ also satisfies Conditions (S1, S2). Since $p$ satisfies (S3, S4) and $g$ satisfies (S3, S4)$^\prime$, then $\xi$ satisfies (S3, S4). To prove monotonely cancellation property of $\xi$ for each
test relation \((y + nx, B + n_1 \circ (a_1 A) + \ldots + n_k \circ (a_k A) + C) \leq (l + m, J + G + C)\) with \(A + D \leq A' + D\) for some \(D \in S\) it is sufficient to find a test relation \((y + nx, B + n_1 \circ (a_1 A) + \ldots + n_k \circ (a_k A') + C') \leq (l + m, J + G + C')\). Let \(C \in S\), put \(C' := C + (l \cdot C)\) as usual and note that \(-1 \cdot (C - C) = C - C\) and \(0 \in C - C\). Since \(S\) is a commutative semigroup, then \(C = A + D\) for \(A\) and \(D\) in \(S\). So, due to Conditions §2.1.1.(2.8) we get \(A' + D \leq (A' + D) + [(A + D) - (A + D)] = (A + D) + (A' - A) + (D - D)\), so it suffices to take \(C' = C + n_1 \circ (a_1 Q) + \ldots + n_k \circ (a_k Q)\) with \(Q := [(A' - A) + (D - D)]\).

(ii). Take a test relation such that \(l = 0\), \(y = 0\) and \(m = nx\), \(E = \emptyset\), \(B = \emptyset\), \(G \geq n_1 \circ (a_1 A) + \ldots + n_k \circ (a_k A)\), then \(x(A) \leq n^{-1}p(m, G) \leq p(x, A)\), consequently, \(\xi \leq p\).

(iii). By the conditions of §2.1.3. \(g\) is monotone and \(g \leq p\), hence, by Lemma 2.4 \(g\) is monotonely cancellative.

(iv). \(\xi_{p,g}\) is defined on \(\Delta E\), since each test relation is a particular case of conditions defining \(\Delta E\). Each test relation with \(g\) and \(E\) is certainly a test relation with \(h\) and \(\Delta E\), consequently, \(\xi_{p,h} \leq \xi_{p,g}\). Consider Conditions 2.1.4.(i1, i2): \(y' + nx = l'\) and \(B' + k \circ A + Q = C' + Q\), then the test relation of §2.1.4 takes the form \(ky + ky' + knx = kl + kl' + km\) and \(k \circ B + k \circ B' + k \circ (n_1 \circ (a_1 A) + \ldots + n_k \circ (a_k A)) + k \circ C \leq k \circ J + k \circ C' + k \circ G + k \circ C\). But this shows that each test relation with \(h\) and \(\Delta E\) is also a test relation with \(g\) and \(E\), since \(g\) is semilinear, hence \(\xi_{p,h} = \xi_{p,g}\).

(v). From (ii, iv) we get \(\xi_{g} \leq \xi_{p,g}\). On the other hand, composition of two subsequent test relations is also a test relation and from the definition of \(\xi\) this statement follows.

For \(\xi_0\) and \(\Delta_0 E\) instead of \(\xi\) and \(\Delta E\) only monotonicity in (i, iii) is guaranteed, since for \(\xi_0\) may be a term \(p(m, G) = \infty\), but for \(\xi\) due to variation of \(C\) this term can be chosen \(p(m, G) < \infty\), otherwise \(p = \infty\) and \(\xi = \infty\) for all arguments that also gives monotone cancellation property of \(\xi\).

2.6. Lemma. Let \(\eta\) be finite sublinear and dominated by \(\xi\). For each \(v \in X\) and each \(D \in S\) define

\[u_{v,D} := \sup \{n^{-1}(\eta(x_1, A_1) + \xi(m_1, G_1) - g(l_1, J_1) + g(y_1, B_1)) : y_1 + x_1 = l_1 + m_1 + nv, B_1 + A_1 + C_1 \leq J_1 + G_1 + n_1 \circ (a_1 D) + \ldots + n_k \circ (a_k D) + C_1\};\]

\[U_{v,D} := \inf \{n^{-1}(\xi(x_2, A_2) + \eta(m_2, G_2) - g(l_2, J_2) - g(y_2, B_2)) : y_2 + x_2 + nv = l_2 + m_2, B_2 + A_2 + n_1 \circ (a_1 D) + \ldots + n_k \circ (a_k D) + C_2 \leq J_2 + G_2 + C_2\};\] where \((y_j, B_j)\) and \((l_j, J_j)\) with \(j = 1\) and \(j = 2\) are in \(E\), \((x_j, A_j), (m_j, G_j)\) and
(z_j, C_j) with j = 1 and j = 2 are in X × S. Then

(i) \( \eta(m + v, D + G) - \xi(m, G) \leq u_{v,D} \leq U_{v,D} \leq \xi(x + v, A + D) - \eta(x, A) \) for each \((x, A)\) and \((m, G)\) in \(X \times S\);

(ii) \(-\infty < u_{v,D} \leq U_{v,D} \leq \infty\) and \(U_{v,D} < \infty\) if \((v, D)\) lies in the hereditary face generated by \:\{\( (z, P) : \xi(z, P) < \infty \)\};

(iii) \(u_{v,D}\) and \(U_{v,D}\) are the same for \(p\) instead of \(\xi\).

**Proof.** Taking \(y = 0\), \(l = 0\), \(J = \emptyset\), \(B = \emptyset\) and omitting indices from the definition of \(U_{v,D}\) we get \(U_{v,D} \leq n^{-1}(-\eta(x, A) + \xi(m, G))\). Then for \(n = 1\), \(m = x + v\) and \(A + D + C = G + C\) due to Lemma 2.5. (i) \(\xi(m, G) = \xi(x + v, A + D)\), hence \(U_{v,D} \leq \xi(x + v, A + D) - \eta(x, A)\).

Choose now \(l_1 = 0, y_1 = 0, n = 1, B = \emptyset, J = \emptyset, x = m + v, A = G + D\), then \(\eta(x, A) = \eta(m + v, D + G)\) and from the definition of \(u_{v,D}\) we get \(\eta(m + v, D + G) - \xi(m, G) \leq u_{v,D}\).

From the equalities \(y_1 + x_1 = l_1 + m_1 + nv\) and \(y_2 + x_2 + nv = l_2 + m_2\) it follows \(y_1 + x_1 - l_1 - m_1 = l_2 + m_2 - y_2 - x_2\). From the inequalities \(B_1 + A_1 + C_1 \leq J_1 + G_1 + n_1 \circ (a_1 D) + \ldots + n_k \circ (a_k D) + C_1\) and \(B_2 + A_2 + n_1 \circ (a_1 D) + \ldots + n_k \circ (a_k D) + C_2 \leq J_2 + G_2 + C_2\) it follows \(B_1 + A_1 + C_1 + B_2 + A_2 + C_2 \leq J_1 + G_1 + n_1 \circ (a_1 D) + \ldots + n_k \circ (a_k D) + C_1 + B_2 + A_2 + C_2 \leq J_2 + G_2 + C_2 + J_1 + G_1 + C_1\). Taking \(C := C_1 + B_2 + A_2 + C_2\) for the first inequality and \(C := C_1 + C_2 + J_1 + G_1\) for the second inequality we get \(u_{v,D} \leq U_{v,D}\).

(ii). The inequality \(u_{v,D} > -\infty\) follows from finiteness of \(\eta\) and \(g\) and taking \(m_1 = 0\), \(G_1 = \emptyset\) for which \(\xi(0, \emptyset) = 0\). Then \(U_{v,D} < \infty\) if there exists \(\xi(m, G) < \infty\), that is the case in the hereditary face.

(iii). Using test relations it is possible to take for each \(b > 0\) a test relation such that \(|\xi(m, G) - g(l, J) + g(y, B) - p(m, G)| < b\). Using semilinearity of \(g\) and substituting arguments \(B' = B + B_j, J' = J + J_j, l' = l + l_j, y' = y + y_j\) we get that \(u_{v,D}\) and \(U_{v,D}\) are the same for \(p\) instead of \(\xi\).

**2.7. Lemma.** Assume that \(\eta\) is finite superlinear mapping on \(X \times S\) and such that \(\xi \geq g \geq \eta\) on \(E\), where \(g\) and \(\xi = \xi_{p,g}\) are defined on \(E\). Let \((v, D) \notin \Delta E\) and \(F := \{(z, P) : z = v_1 + e, P = D_1 + C, (e, C) \in E, e \in X, C \in S, v_1 \in s_p k v, D_1 \in S p D\}\). For each \(\gamma \in R\) denote by \(h_\gamma\) the uniquely determined semilinear extension of \(g\) from \(E\) to \(F\) such that \(h_\gamma(v, D) = \gamma\). Then

(i) \(\xi_{p,h_\gamma} \geq h_\gamma \geq \eta\) on \(F\) if and only if

(ii) \(u_{v,D} \leq \gamma \leq U_{v,D}\).

**Proof.** The existence of \(h_\gamma\) follows from Lemma 2.2. By Lemma 2.5 \(\xi_{p,h_\gamma} \geq \eta\) on \(E\). Suppose (i) is satisfied on \(F\) and \((v, D) \notin \Delta E\). In view of
Lemma 2.5 $h_\gamma$ is monotonically cancellative. Then $h_\gamma(m_2,G_2) + h_\gamma(l_2,J_2) - h_\gamma(y_2,B_2) \geq h_\gamma(x_2 + n v, A_2 + n_1 \circ (a_1 D) + \ldots + n_k \circ (a_k D)) = h_\gamma(x_2, A_2) + n \gamma$, since $h$ is semilinear. On the other hand, $-\eta(x_2, A_2) \geq -\xi(x_2, A_2)$, consequently, $U_{v,D} \geq \gamma$.

Using $\xi \geq \eta$, $h_\gamma \geq \eta$ and monotonic cancellation of $h_\gamma =: h$ we get $\eta(x_1, A_1) - \xi_{p,h}(m_1,G_1) - h(l_1,J_1) + h(y_1,B_1) \leq \eta(x_1, A_1) - h(m_1,G_1) - h(l_1,J_1) + h(y_1,B_1) - h(m_1,G_1) - h(l_1,J_1) = h(x_1 + y_1, A_1 + B_1) - h(m_1 + l_1, G_1 + J_1) \leq nh(v,D) = n \gamma$, since $h(x_1, A_1) + h(y_1, B_1) = h(x_1 + y_1, A_1 + B_1) \leq h(l_1 + m_1 + n v, J_1 + G_1 + n_1 (a_1 D) + \ldots + n_k (a_k D)) = h(l_1, J_1) + h(m_1, G_1) + nh(v,D)$, consequently $u_{v,D} \leq \gamma$.

Suppose now that (ii) is satisfied and $\xi \geq g \geq \eta$ on $E$. In view of Lemma 2.5 we have $\xi_{p,h} \geq h$ on $\Delta E$. Applying Lemma 2.4 to $-g$ and $-\eta$ we have $h \geq \eta$ on $E$. It remains to consider $F \neq \Delta E$. In view of Lemma 2.6.(i) we have $\eta(v,D) \leq u_{v,D}$ (taking $m = 0$ and $G = \emptyset$) also $\xi(v,D) \geq U_{v,D}$ (taking $x = 0$ and $A = \emptyset$). Combining this with (ii) we get $\xi(v,D) \geq \gamma \geq \eta(v,D)$.

2.8. Theorem. Let $p : X \times S_X \to [-\infty, \infty]$ be sublinear and $g : H \times F \to [-\infty, \infty]$ be a finite semilinear, monotone and dominated by $p$ mapping as above.

(i). If $X \times S$ is the hereditary face generated by $F \times F \cup \{(x, C) : p(x, C) < \infty\}$ or more generally by $\{(x, C) : \xi(x, C) < \infty\}$, then a necessary and sufficient condition that $g$ has a finite Hahn-Banach extension $h$ on $X \times S_X$ is that there exists a finite superlinear map $\eta$ on $X \times S_X$ such that $\eta \leq \xi$. If this condition is fulfilled, then $h$ can be chosen such that $\eta \leq h \leq \xi$.

(ii). Then a necessary and sufficient condition that $g$ has a monotone Hahn-Banach extension $h$ on $X \times S_X$ is that, there exists a superlinear map $\eta : X \times S_X \to (-\infty, \infty]$ such that $j \leq \xi_0$. When this condition is fulfilled, then $h$ can be chosen such that $\eta \leq h \leq \xi_0$ on $X \times S_X$.

(iii). If the preordering on $X \times S$ is equality, then $g$ has a finite Hahn-Banach extension on $X \times S$ if and only if there exists a superlinear map $\eta : X \times S \to (-\infty, \infty]$ such that $\eta \leq \xi_0$ and $g$ is cancellative.

Proof. (i) Necessity follows from Lemma 2.4. We prove sufficiency. Consider a maximal pair $(F,h)$ with $F$ a $P$-quasisemilinear subspace of $P$-quasisemilinear preordered space $X \times S$ such that $F \supseteq E$ and $h$ is a finite semilinear extension of $g$ with $\xi_{p,h} \geq \eta$. By Lemma 2.5.(iii) $h$ is monotone on $F$. By Lemma 2.5.(v) $\Delta F = F$. Due to Lemmas 2.6 and 2.7 $\Delta F = X \times S$. Here is employed the assumption that $X \times S$ is the hereditary face generated by $\{\xi < \infty\}$, which gives $[u_{v,D}, U_{v,D}] \neq \emptyset$. By Lemmas 2.4 and 2.7 $\eta \leq h \leq \xi$
Due to Lemma 2.5 for each $z_k$ we get \( B \). Theorem 2.

ξ is satisfied with \( \leq k \) in \( x \in Q \) for this \( Z \) put \( p' := p|_Z \). Let \( \xi' \) be defined for such \( p' \) and \( g \) as above. We will show that \( \xi' \) dominates the restriction \( \eta|_Z \). For this it is necessary to prove that when \( B + n_1 \circ (a_1 A) + \ldots + n_k \circ (a_k A) + C \leq E + G + C \) and \( y + n x = l + m \) is satisfied with \( B \) and \( E \) in \( F \) and \( A, C \) and \( G \) in \( Z_2, x \) and \( m \) in \( Z_1, l \) and \( y \) in \( H \) we have

\[
(1) \eta(x, A) \leq n^{-1}(p(m, G) + g(l, E) - g(y, B)).
\]

For \( C \in Z_2 \) there exist \( P \) and \( Q \) in \( S_X \) with \( \xi_0(0, Q) < 0 \) and \( C + P \leq Q \). In view of Condition 3.(iv) we have \( B + n_1 \circ (a_1 A) + \ldots + n_k \circ (a_k A) + Q \leq E + G + Q \). Then substituting \( Q \) on \( B + n_1 \circ (a_1 A) + \ldots + n_k \circ (a_k A) + Q \) we get \( B + n_1 \circ (a_1 A) + \ldots + n_k \circ (a_k A) + B + n_1 \circ (a_1 A) + \ldots + n_k \circ (a_k A) + Q = 2 \circ B + 2 \circ (n_1 \circ (a_1 A) + \ldots + n_k \circ (a_k A)) + Q \leq E + G + B + n_1 \circ (a_1 A) + \ldots + n_k \circ (a_k A) \leq 2 \circ E + 2 \circ G + Q \) and by induction we get \( k \circ B + (k \circ (n_1 \circ (a_1 A) + \ldots + n_k \circ (a_k A)) + Q) \leq k \circ E + (k \circ G + Q) \).

Due to Lemma 2.5 for each \( z \in Z_1 \) there are the following inequalities: \( kn\eta(x, A) + \eta(z, Q) \leq \eta(kn x + z, k \circ (n_1 \circ (a_1 A) + \ldots + n_k \circ (a_k A) + Q) \leq \xi_0(kn x + z, k \circ (n_1 \circ (a_1 A) + \ldots + n_k \circ (a_k A) + Q) \leq \xi_0(km, k \circ G + g(kl, k \circ E) - g(ky, k \circ B) \leq k(p(m, G) + g(l, E) - g(y, B)) + \xi_0(z, Q) \), so dividing by \( kn \) and using that \( k \in \mathbb{N} \) is arbitrary we get Inequality (1).

(iii). Necessity is evident, since \( h \leq \xi_0 \). We prove sufficiency. In view of Theorem 2.8.(i) there exists a monotone Hahn-Banach extension \( h \) of \( g \) such that \( \eta \leq h \leq \xi_0 \), where \( h \) is defined on \( Z \) (see §2.8.(ii)), since \( \xi \leq \xi_0 \) on \( Z \). The rest of the proof follows from Lemma 2.9 given below.

2.9. Lemma. If the preordering on \( X \times S \) is equality and there exists a superlinear map \( \eta : X \times S \to (-\infty, \infty] \) such that \( \eta \leq \xi_0 \). A finite semilinear and cancellative map \( g \) defined on a \( P \)-quasisemilinear subspace \( E \) of \( X \times S \) has a finite semilinear extension \( h \) to all of \( X \times S \).

Proof is analogous to that of Lemma 2.2 with the cancellation property instead of the monotone cancellation, that gives \( h \) on \( \Delta E \). Here is not demanded that \( \eta \leq h \leq \xi_0 \). If \( X \times S \setminus \Delta E =: T \neq \emptyset \), then put \( h(v, D) = \gamma \) for \( (v, D) \in T \) and this gives \( h \) on \( sp_{Kv} \times sp_{PD} \), hence an extension \( \xi_0 \) of \( \xi_0 \) from \( T \) on \( T \cup (sp_{Kv} \times sp_{PD}) \) is defined. Choose \( \gamma \) such that \( \xi_0 = \xi_0 \) on the extended in such way a \( P \)-quasisemilinear space \( T_{v,D} \) generated by \( T \cup \{(v, D)\} \). Then, as above, we get \( h \) on \( \Delta T_{v,D} \). Considering the family of all such extensions and applying to it the Kuratowski-Zorn lemma as in §2.2.
we get $h$ on $X \times S$.

**2.10. Theorem.** Let suppositions of Theorem 2.8 be satisfied and $X \times S$ be monotonely cancellative and $\eta \leq h \leq \xi$. Suppose $v \in X$ and $D \in S$ and $b \in (-\infty, \infty]$. Then $g$ has a monotone Hahn-Banach extension $h$ on $X \times S$ with values in $(-\infty, \infty]$ such that $h(v, D) = b$ if and only if there exists a superlinear map $\eta : X \times S \to (-\infty, \infty]$ such that

(i) $\eta(x, A) \leq \xi(x+nv, A+n_1 \circ (a_1 D)+\ldots+n_k \circ (a_k D)) - nb$ for each $x \in X$, $A \in S$, $k$ and $n_i$ in $N$, $a_i \in K_+$, $i = 1, \ldots, k$, $n = n_1 a_1 + \ldots + n_k a_k$. If Condition (i) is satisfied, then $h$ can be chosen such that $\eta \leq h \leq \xi$.

**Proof.** Necessity is evident. We show sufficiency. Condition (i) is the generalization of the condition $\xi \geq \eta$, since for $v = 0$ and $D = \emptyset$ we get $\eta(x, A) \leq \xi(x, A)$. For the use of Lemma 2.6 in this situation we verify, that when $(v, D) \notin \triangle E$, then $U_{v,D} \geq b$. From $\eta \leq h \leq \xi$ on $E$ and Lemma 2.5 we have $\xi(x_2 + nv, A_2 + n_1 \circ (a_1 D)+\ldots+n_k \circ (a_k D)) \leq \eta(x_2, A_2) + n\xi(v, D)$ on $\triangle(E \cup spK v \times spP D)$, consequently, $-\eta(x_2, A_2) \leq -\xi(x_2, A_2) - n\xi(v, D) + nb$, where $\xi(v, D) \geq b$. Then as in §2.6 we get $U_{v,D} \geq b$.

**2.11. Theorem.** Let $X, S, p, E$ and $g$ be as in §2.8 and assume that $X \times S$ is monotonely cancellative and that $(x, A) \geq (0, \emptyset)$ for each $x \in X$ and $A \in S$. Then $g$ has a monotone Hahn-Banach extension if and only if

(i) $g(x, A) \leq g(y, B) + p(z, C)$ whenever $(x, A) \leq (y + z, B + C)$ with $(x, A)$ and $(y, B)$ in $E$. If $p$ is monotone, then for each $v \in X$ and $D \in S$ there exists a monotone and additive map $h$ on $X \times S$ such that $h \leq p$ and $h(v, D) = p(v, D)$.

**Proof.** Condition 2.10.(i) for $b = \xi(v, D) < \infty$ can be written in the form $\eta(x, A) \leq \lim_{n \to \infty}(\xi(x+nv, A+n_1 \circ (a_1 D)+\ldots+n_k \circ (a_k D)) - n\xi(v, D))$, where $x \in X$ and $A \in S$.

**2.12. Corollary.** Let $X \times S$ be a monotonely cancellative $P$-quasisemilinear preordered space and $p$, $\xi$, $g$ be as usual. Suppose that for a given $(v, D) \in X \times S$.

(i) $\inf\{\xi(x+y+v, A+B+D) - \xi(y+v, B+D) : (y, B) \in X \times S\} > -\infty$ for each $(x, A) \in X \times S$ and

(ii) $-\infty < \xi(y+v, B+D) < \infty$ for some $(y, B) \in X \times S$. Then there exists a monotone Hahn-Banach extension $h$ of $g$ such that $h(v, D) = \xi(v, D)$.

**Proof.** For $(x, A) \in X \times S$ put $\eta(x, A) = \inf\{\xi(x+y+v, A+B+D) - \xi(y+v, B+D) : (y, B) \in X \times S\}$. By Condition (i) $\eta > -\infty$. Then as in [10] it can be shown that $\eta \leq \xi$ and $\eta$ is superlinear, since $\xi$ is sublinear and due to Theorem 2.10 we get the statement of this Corollary.
2.13. Definitions and Notes. Let $X \times S$ be a $P$-quasisemilinear pre-ordered space, $p : X \times S \to (-\infty, \infty]$ be sublinear, $E$ be a $P$-quasisemilinear subspace and $g : E \to \mathbb{R}$ be semilinear on $E$. Consider a subset $W$ of $X \times S$. Then $h : X \times S \to (-\infty, \infty]$ is called a $W$-maximal monotone Hahn-Banach extension of $g$ if for each monotone Hahn-Banach extension $h'$ of $g$ from the inequality $h' \geq h$ on $W$ it follows $h' = h$ on $W$.

A sublinear map $\xi : X \times S \to (-\infty, \infty]$ is called of moderate variation if $\inf \{\xi(x+y, A\hat{+}B) - \xi(y, B) : (y, B) \in X \times S\} > -\infty$ for all $(x, A) \in X \times S$. For $\xi$ of moderate variation define a superlinear map $\eta$ by $\eta(x, A) = \inf \{\xi(x+y, A\hat{+}B) - \xi(y, B) : (y, B) \in X \times S\}$, where $(x, A) \in X \times S$.

2.14. Theorem. Let $X, S, p, E$ be as in §2.13 and assume that $X \times S$ is monotonely cancellative. If $\xi = \xi_{p,g}$ is a sublinear map of moderate variation, then for each subset $W$ of $X \times S$ there exists a $W$-maximal monotone Hahn-Banach extension of $g$.

Proof. Consider $\eta(x+y, A\hat{+}B) = \inf \{\xi(x+y+v, A\hat{+}B\hat{+}D) - \xi(v, D) : (v, D) \in X \times S\}$, where $(x, A)$ and $(y, B)$ are in $X \times S$. We have $\eta(x, A) + \eta(y, B) = \inf \{\xi(x + v_1, A\hat{+}D_1) + \xi(y + v_2, B\hat{+}D_2) - \xi(v_1, D_1) - \xi(v_2, D_2) : (v_1, D_1) \text{ and } (v_2, D_2) \in X \times S\} \leq \inf \{\xi(x+y+(v_1 + v_2), A\hat{+}B\hat{+}(D_1 + D_2)) - \xi(v_1, D_1) - \xi(v_2, D_2) : (v_1, D_1) \text{ and } (v_2, D_2) \in X \times S\} \leq \inf \{\xi(x + y + v_1 + 0, A\hat{+}B\hat{+}D_1) - \xi(v_1, D_1) - \xi(0, \emptyset) : (v_1, D_1) \in X \times S\} = \eta(x + y, A\hat{+}B)$, hence $\eta$ satisfies 2.12.(S3)', (S4)'. Since $\xi$ satisfies 2.12.(S6), then $\eta$ also satisfies (S6). Since $\xi$ satisfies (S2), then $\eta$ also satisfies (S2). Evidently, $\eta(0, \emptyset) = 0$, since $\xi(0, \emptyset) = 0$. From $\xi(x+y, A\hat{+}B) \leq \xi(x, A) + \xi(y, B)$ it follows $\eta(x, A) \leq \xi(x+y, A\hat{+}B) - \xi(y, B) \leq \xi(x, A)$. Taking $p = \xi$, $E = \{0, \emptyset\}$ and $b = \xi(v, D)$ we infer from Theorem 2.10 that for each $v \in X$ and each $D \in S$ there exists a monotone semilinear map $h : X \times S \to (-\infty, \infty]$ such that $\eta \leq h \leq \xi$ and $h(v, D) = \xi(v, D)$. The map $\xi$ is of moderate variation, hence $V := \{(x, A) : \xi(x, A) < \infty, (x, A) \in X \times S\} = \{(x, A) : \eta(x, A) < \infty, (x, A) \in X \times S\}$ and $V$ is a hereditary face, since $\xi$ is monotonely cancellative in accordance with Lemma 2.5.(i).

Suppose $\xi < \infty$ and $(v, D) \in X \times S$, consider the set $F := \{(x+y, A\hat{+}B) : (x, A) \in E, x \in spKv, B \in spPD\}$. Define $h$ on $F$ by $h(x+nv, A\hat{+}n_1 \circ (a_1D)+...+n_k \circ (a_kD)) = g(x, A) + n\xi(v, D)$ for each $k$ and $n_i \in \mathbb{N}_0$ and $a_i \in K_+$ and each $(x, A) \in E$, where $n = n_1a_1 + ... + n_ka_k$. Put $\xi' := \xi_{p,h}$, then $\xi'$ is of moderate variation and the associated superlinear map dominates $\eta$. Moreover, if $r : X \times S \to (-\infty, \infty]$ is a monotone Hahn-Banach extension of $g$ and if $r \geq h$ on $F \cap W$, then $r = h$ on $F \cap W$. From the definition of $\xi'$
it follows that
\[ \xi'(x, A) = \min\{\xi(x, A), C_1, C_2\}, \]
where
\[ C_1 := \inf \{ m^{-1}(p(z, Q) + g(l, J) - g(y, B) + n\xi(v, D)) : (y + mx, B + m_1 \circ (a_1A) + \ldots + m_j \circ (a_jA)) \leq (l + nv + z, J + n_1 \circ (d_1D) + \ldots + n_k \circ (d_kD) + Q) \}, \]
\[ C_2 := \inf \{ m^{-1}(p(z, Q) + g(l, J) - g(y, B) - n\xi(v, D)) : (y + nv + mx, B + m_1 \circ (a_1A) + \ldots + m_j \circ (a_jA) + n_1 \circ (d_1D) + \ldots + n_k \circ (d_kD)) \leq (l + z, J + Q) \}, \]
where \( m = m_1a_1 + \ldots + m_ja_j, n = n_1d_1 + \ldots + nkd_k; j, k, m_i, n_i, a_i, d_i \in \mathbb{N} \); (y, B) and (l, J) \in E, m \in \mathbb{N}. In view of Lemma 2.5.(v) \( C_1 \geq \xi(v, D) \), since \( p(z, Q) + n\xi(v, D) \geq \xi(z + nv, Q) + n_1 \circ (d_1D) + \ldots + n_k \circ (d_kD) \). Analogously to Lemma 10 [10] we get:
\[ (i) \quad \xi'(x, A) = \inf \{ m^{-1}(\xi(mx + nv, m_1 \circ (a_1A) + \ldots + m_j \circ (a_jA)) + n_1 \circ (d_1D) + \ldots + n_k \circ (d_kD)) - \xi(nv, n_1 \circ (d_1D) + \ldots + n_k \circ (d_kD)) : j, k, m_i, n_i, a_i, d_i \}. \]

Thus Equation (i) can be rewritten as:
\[ (ii) \quad \xi'(x, A) = \lim_{n/m \to \infty} \{ m^{-1}(\xi(mx + nv, m_1 \circ (a_1A) + \ldots + m_j \circ (a_jA)) + n_1 \circ (d_1D) + \ldots + n_k \circ (d_kD)) - \xi(nv, n_1 \circ (d_1D) + \ldots + n_k \circ (d_kD)) : j, k, m_i, n_i, a_i, d_i \} \]
with natural numbers \( m \) and \( n \), since \( \xi \) satisfies 2.1.2.(56), hence
\[ \xi'(x + z, A + Q) - \xi'(z, Q) = \lim_{n/m \to \infty} \{ m^{-1}(\xi(mx + mz + nv, m_1 \circ (a_1A) + \ldots + m_j \circ (a_jA)) + mQ + n_1 \circ (d_1D) + \ldots + n_k \circ (d_kD)) - \xi(mz + nv, m \circ Q + n_1 \circ (d_1D) + \ldots + n_k \circ (d_kD)) : j, k, m_i, n_i, a_i, d_i \}, \]
and inevitably \( \xi' \) is of moderate variation and the associated superlinear map dominates \( \eta \). The rest of the proof of Theorem 2.14 is analogous to that of Theorem 5 [10] with the help of lemmas given above.

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