KIRWAN SURJECTIVITY FOR THE EQUIVARIANT DOLBEAULT COHOMOLOGY

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ABSTRACT. Consider the holomorphic action of a compact connected Lie group $K$ on a compact Kähler manifold $M$ which is also Hamiltonian with a moment map $\Phi : M \to \mathfrak{t}^*$. Assume that $0$ is a regular value of the moment map. Weitsman raised the question of what we can say about the cohomology of the Kähler quotient $M_0 := \Phi^{-1}(0)/K$ if all the ordinary cohomology of $M$ is of type $(p,p)$.

In this paper, using the Cartan-Chern-Weil theory we show that in the above context there is a natural surjective Kirwan map from an equivariant version of the Dolbeault cohomology of $M$ onto the Dolbeault cohomology of the Kähler quotient $M_0$. As an immediate consequence, this provides an answer to the question posed by Weitsman.

1. INTRODUCTION

Assume that there is a compact connected Lie group $K$ acting on a compact symplectic manifold $(M, \omega)$ in a Hamiltonian fashion with a moment map $\Phi : M \to \mathfrak{t}^*$, and that $K$ acts freely on the level set $Z := \Phi^{-1}(0)$. Then the quotient space $M_0 := Z/K$ naturally inherits a symplectic structure from that of $M$, and is called a symplectic quotient of the Hamiltonian $K$-manifold $M$. In her fundamental work [Kir84], Kirwan established the important Kirwan surjectivity theorem for compact Hamiltonian $K$-manifolds, which asserts that the Kirwan map $\kappa : H_K(M) \to H(M_0)$ is surjective.

Now assume that the Hamiltonian $K$-manifold $(M, \omega)$ is equivariant Kähler. In other words, assume that the symplectic 2-form $\omega$ is Kähler, and that the action of $K$ is holomorphic. Then the symplectic quotient $M_0$ inherits a Kähler structure from that of $M$, and is called a Kähler quotient of the equivariant Kähler $K$-manifold $M$. It is well known that in this case the action of $K$ naturally extends to an action of $G := K^\mathbb{C}$. Moreover, when $M$ is a non-singular projective variety, and when the action of $G$ is linear, the famous Kempf-Ness theorem [KN79] asserts that the Kähler quotient $M_0$ can be naturally identified with the GIT quotient of $M$ by $G$.

Kirwan studied [Kir84, Sec. 14] the linear action of a reductive algebraic group $G$ on a non-singular projective variety $M$ from the view point of GIT quotient. She showed that there is a Hodge structure on $H_G(M, \mathbb{Q})$, and that the surjective Kirwan map $H_G(M, \mathbb{Q}) \to H(M_0, \mathbb{Q})$ in this case.

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is strictly compatible with the Hodge structures. In particular, this would imply that if the hodge numbers of $M$ satisfy $h^{p,q} = 0$ when $p \neq q$, then the same is true for $M_0$. However, Kirwan’s method is algebro-geometric, and does not apply to the more general case of equivariant Kähler manifolds. Indeed, Weitsman raised the following question in an AIM moment map geometry workshop [AIM04] that took place in August 2004.

**Question**: Suppose $M$ is a compact Kähler manifold and a Hamiltonian $K$-space. Suppose all the ordinary cohomology is of type $(p,p)$. Can we say anything about the cohomology of the quotient?

On a complex manifold, it is well known that if the $\partial\partial$-lemma holds, then there is a pure hodge structure on its ordinary cohomology. On an equivariant Kähler Hamiltonian $K$-manifold $M$, Lillywhite ([Lilly98], [Lilly03]) and Teleman [100] showed that the $\partial_K\partial_K$-lemma holds on the Cartan complex $\Omega_K(M, \mathbb{C})$ of equivariant differential forms, which in particular implies that there is a pure hodge structure on the equivariant De Rham cohomology $H_K(M, \mathbb{C})$. Let $M_0$ be the Kähler quotient of the equivariant Kähler Hamiltonian K-manifold $M$ taken at the zero level set. In this paper we show that the Kirwan map $\kappa : H_K(M, \mathbb{C}) \to H(M_0, \mathbb{C})$ respects the hodge structures on $H_K(M, \mathbb{C})$ and on $H(M_0, \mathbb{C})$ respectively. As an immediate consequence, this implies that if the hodge numbers of $M$ are concentrated on the diagonal, then the same property holds for $M_0$.

Our method relies on the Cartan-Chern-Weil theory in differential geometry, which we briefly explain here. First note that the zero level set $Z := \Phi^{-1}(0)$ is the total space of a principal $K$-bundle $\pi : Z \to M_0$. In this setup, there is a Cartan operator $C : \Omega_K(Z) \to \Omega(M_0)$ in the equivariant De Rham theory, which is a homotopy equivalence from the Cartan complex $\{\Omega_K(Z), d_K\}$ to the De Rham complex $\{\Omega(M_0), d\}$. The definition of the Cartan operator depends on the choice of a connection on $Z$. However, in our situation the restriction of the Kähler metric to $Z$ provides a canonical connection, and thus a canonical Cartan operator $C$. As a result, we have the following commutative diagram.

$$
\begin{array}{ccc}
\Omega_K(M, \mathbb{C}) & \xrightarrow{i^*} & \Omega_K(Z, \mathbb{C}) \\
\downarrow{\kappa} & & \downarrow{C} \\
\Omega(M_0, \mathbb{C}) & & \\
\end{array}
$$

Here the top horizontal map $i^*$ is induced by the inclusion map $i : Z \to M$, and the vertical map is the Cartan operator defined using the canonical connection on $Z$. We define the diagonal map to be the canonical Kirwan map at the level of differential forms.

We observe that the action of $K$ induces a transversely Kähler foliation on $Z$, and that with respect to this transverse Kähler structure, the curvature 2-forms associated to the canonical connection on $Z$ are horizontal forms of type $(1,1)$. As a consequence, the Kirwan map $\kappa : \Omega_K(M) \to \Omega(M_0)$
respects the bi-gradings on $\Omega_K(M)$ and $\Omega(M_0)$ induced by the complex structures on $M$ and and quotient complex structure on $M_0$ respectively. This would enable us to show that the usual Kirwan map at the level of cohomologies is a morphism of Hodge structures, and to show that there is a surjective Kirwan map from the equivariant Dolbeault cohomology of $M$ onto the Dolbeault cohomology of $M_0$.

We would like to point out that technically we only assume that the $K$ action on the zero level set $Z$ is locally free, so that our result applies to the general case when the Kähler quotient $M_0$ is an orbifold. We note that to show there is a quotient Kähler structure structure on $M_0$, it is equivalent to show that the foliation on $Z$ induced by the locally free $K$-action is transversely Kähler; moreover, the Hodge structure on $H^*_K(M)$ is naturally isomorphic to the Hodge structure on $H^*_B(Z)$, the basic cohomology of $Z$ as a foliated manifold. Thus to establish the main result of this paper it suffices to show that the Kirwan map is a morphism of Hodge structures from $H^*_K(M)$ to $H^*_B(Z)$. It has been long known that the cohomology of a Kähler orbifold exhibits properties very similar to that of a Kähler manifold. However, it has been difficult to find a self-contained reference in the literature until more recently [BBFMT17] appears. Our foliation approach offers an alternative simple treatment of hodge theory on a generic Kähler quotient.

This paper is organized as follows. Section 2 reviews equivariant De Rham cohomology theory, especially the definition of the Cartan operator. Section 3 explains for an abstract double complex how the $dd'$-lemma would lead to a hodge type decomposition. Section 4 reviews the $\overline{\partial}_K\partial_K$-lemma for an equivariant Kähler manifold. Section 5 presents some background materials on transversely Kähler foliations. Section 6 proves that on an equivariant Kähler manifold the Kirwan map is a morphism of Hodge structures.

2. Review of Equivariant Cohomology Theory

We begin with a rapid review of equivariant de Rham theory and refer to [CS99] for a detailed account. Let $K$ be a compact connected Lie group and let $\Omega_K(M) = (\mathfrak{t}^* \otimes \Omega(M))^K$ be the Cartan complex of the K-manifold $M$. By definition an element of $\Omega_K(M)$ is an equivariant polynomial from $\mathfrak{t}$ to $\Omega(M)$ and is called an equivariant differential form on $M$. The bigrading of the Cartan complex is defined by $\Omega^{ij}_K(M) = (S^j(\mathfrak{t}^*) \otimes \Omega^{i-j}(M))^K$. It is equipped with a vertical differential $d \otimes d'$, which is usually abbreviated to $d$, and a horizontal differential $d'$, which is defined by $d'\alpha(\xi) = -i(\xi)\alpha(\xi)$. Here $i(\xi)$ denotes inner product with the vector field on $M$ induced by $\xi \in \mathfrak{t}$. As a total complex, $\Omega_K(M)$ has the grading $\Omega^*_K(M) = \bigoplus_{i+j=r} \Omega^{ij}_K(M)$ and the total differential $d_K = d + d'$. The total cohomology $\ker d_K/\im d_K$ is the equivariant De Rham cohomology $H^*_K(M)$.

Now suppose that $K$ acts freely on $M$. Let $\xi_1, \cdots, \xi_k$ be a basis of $\mathfrak{t}$, and let $c_{ij}^{kl}$’s be the structure constants of the Lie algebra $\mathfrak{t}$ relative to the given
basis. Then there exist 1-forms $\theta^1, \cdots, \theta^k$, called the connection 1-forms, such that
\[ \iota(\xi_i)\theta^j = \delta^j_i, \quad L(\xi_i)\theta^j = -c^j_{il}\theta^l, \]
where $\mathcal{L}(\xi_i)$ denotes the Lie derivative with the vector field on $M$ induced by $\xi_i \in \mathfrak{k}$. In this context, $\forall 1 \leq l \leq k$, the curvature 2-form $\mu^l$ is the 2-form defined by
\[ (2.1) \quad \mu^l = d\theta^l + \frac{1}{2} c^l_{ij}\theta^i\theta^j. \]

We say a form $\gamma \in \Omega(M)$ is \textit{horizontal}, if $\forall \xi \in \mathfrak{k}$, $\iota(\xi)\gamma = 0$. We say a form $\gamma \in \Omega(M)$ is \textit{basic}, if it is horizontal, and if $L(\xi)\alpha = 0$, $\forall \xi \in \mathfrak{k}$. We will denote by $\Omega_{\text{hor}}(M)$ the space of horizontal forms on $M$, and by $\Omega_{\text{bas}}(M)$ the space of basic forms on $M$. By definition, the curvature 2-form $\mu^l$ is horizontal, $\forall 1 \leq l \leq k$. It is also clear that $\forall \gamma \in \Omega_{\text{bas}}(M)$, $d\gamma \in \Omega_{\text{bas}}(M)$. Thus we have a differential complex of basic forms $\{\Omega_{\text{bas}}(M), d\}$.

\textbf{Theorem 2.1.} ([CS99, Thm 3.4.1]) Every form $\alpha \in \Omega^*(M)$ can be written uniquely as
\[ \alpha = \sum_i \theta^i h_i, \]
where $h_i \in \Omega_{\text{hor}}(M)$.

As a direct consequence of Theorem 2.1 we have the projection operator
\[ (2.2) \quad \text{Hor} : \Omega(M) \to \Omega_{\text{hor}}(M). \]
It further gives rise to the following projection operator on the Cartan complex.
\[ (2.3) \quad \text{Hor} : \Omega_k(M) = (S^* \otimes \Omega(M))^K \to (S^* \otimes \Omega_{\text{hor}}(M))^K. \]

\textbf{Definition 2.2. The Cartan operator}
\[ (2.4) \quad C : \Omega_k(M) \to \Omega_{\text{bas}}(M) \]
is the composition of the projection operator $\text{Hor}$ and the map
\[ (S^* \otimes \Omega_{\text{hor}}(M))^K \to \Omega_{\text{bas}}(M) \]
coming from the “evaluation map”
\[ x^l \otimes \alpha \mapsto \mu^l \alpha. \]

\textbf{Remark 2.3.} Set $\theta = \sum_{i=1}^k \theta^i \otimes \xi_i$. Then we get a $\mathfrak{k}$-valued connection one form satisfying
\[ (2.5) \quad \iota(\xi)\theta = \xi, \quad \forall \xi \in \mathfrak{k}, \quad \iota(R_h)^*\theta = \text{ad}(h^{-1})\theta, \quad \forall h \in K. \]
Here $\text{ad}$ denotes the adjoint representation of $K$ on $\mathfrak{k}$. $\forall x \in M$, let $V_x$ be the tangent space to the group orbit $K \cdot x$ at $x$, and let $H_x$ be the kernel of the one form $\theta_x$. Then we get a connection on $M$, i.e., a $K$-invariant splitting
\[ (2.6) \quad T_x(M) = V_x \oplus H_x, \quad x \in M. \]
Conversely, any $K$-invariant splitting as given in (2.6) determines a $\mathfrak{k}$-valued connection one form $\theta$ satisfying (2.5). A close inspection shows that the definition of the Cartan map $C$ does not depend on the choice of a basis $\xi_1, \cdots, \xi_k$ in $\mathfrak{k}$. It depends only on the connection given in (2.6).

**Theorem 2.4.** ([CS99, Thm 5.2.1]) The Cartan operator (2.4) is a chain homotopy equivalence from the Cartan complex $\{\Omega_K(M), d_K\}$ to the complex of basic forms $\{\Omega_{bas}(M), d\}$.

We finish this section by recalling the definition of Hamiltonian symplectic manifolds, the Kirwan-Ginzburg equivariant formality theorem, and the Kirwan surjectivity theorem.

**Definition 2.5.** Consider the action of a compact connected Lie group $K$ on a symplectic manifold $(M, \omega)$. Let $\mathfrak{k}^*$ be the dual of the Lie algebra $\mathfrak{k}$ of $K$. We say that the action of $K$ is Hamiltonian, if there exists an equivariant map $\Phi : M \to \mathfrak{k}^*$, called the moment map, satisfying the Hamiltonian equation:

$$-\iota(\xi)\omega = d < \Phi, \xi >, \forall \xi \in \mathfrak{k},$$

where $< \cdot, \cdot >$ denotes the natural pairing between $\mathfrak{k}$ and $\mathfrak{k}^*$.

**Theorem 2.6.** (Kirwan-Ginzburg Equivariant formality theorem) ([Kir84], [Gin87]) Suppose that the action of a compact connected Lie group $K$ on a compact symplectic manifold $(M, \omega)$ is Hamiltonian. Then the Hamiltonian $K$-manifold $M$ is equivariantly formal, i.e., $H_K(M) \cong (\mathfrak{k}^*)^K \otimes H(M)$.

**Theorem 2.7.** (Kirwan surjectivity theorem) ([Kir84]) Consider the Hamiltonian action of a compact connected Lie group $K$ on a compact symplectic manifold $(M, \omega)$ with a moment map $\Phi : M \to \mathfrak{k}^*$. Assume that $0 \in \mathfrak{k}^*$ is a regular value. Then the Kirwan map $\kappa : H_K(M) \to H_K(\Phi^{-1}(0))$ induced by the inclusion map $i : \Phi^{-1}(0) \to M$ is surjective.

3. **The dd’-lemma for an abstract double complex**

Let $(K^{**}, d, d')$ be a double complex which is bounded in the following sense: for each $n$, there are only finitely many non-zero components in the direct sum $K^n = \bigoplus_{i+j=n} K^{i,j}$. Here $d$ is the degree 1 vertical differential, and $d'$ is the degree 1 horizontal differential. Set $D = d + d'$, and $K_{d'} = \ker d' \cap K$. It is easy to see that the inclusion map $K_{d'} \hookrightarrow K$ induces two morphisms of differential complexes as follows.

$$(3.1) \quad (K_{d'}, d) \to (K, d),$$

$$(3.2) \quad (K_{d'}, d) \to (K, D).$$

Clearly, (3.1) and (3.2) further induce two homomorphisms of cohomologies respectively as follows.

$$(3.3) \quad H^{p,q}(K_{d'}, d) \to H^{p,q}(K, d),$$
\begin{equation}
\bigoplus_{p+q=r} \mathbb{H}^{p,q}(K_d^r, d) \to \mathbb{H}^r(K, D).
\end{equation}

**Proposition 3.1.** Assume that the $dd^\prime$-lemma holds for the double complex $(K^{**}, d, d^\prime)$. That is to say that

\begin{equation}
\ker d \cap \text{im } d^\prime = \text{im } d^\prime \cap \ker d = \text{im } dd^\prime
\end{equation}

Then the following properties hold true.

1. The homomorphism (3.3) induced by (3.1) is an isomorphism.
2. The homomorphism (3.4) induced by (3.2) is an isomorphism.

**Proof.** The proof will be left as an exercise for the readers. The injectivity of (3.4) can be easily shown using [DGMS75, Lemma 5.15], though it could also be proved directly using an inductive argument.

**4. EQUIVARIANT DOLBEAULT COHOMOLOGY AND THE $\overline{\delta}_G \delta_G$-LEMMA**

Throughout this section, suppose that there is a holomorphic action of a compact connected Lie group $K$ on a 2n-dimensional complex manifold $M$. On the space of differential forms $\Omega(M)$ the exterior differential $d$ splits as $d = \overline{\delta} + \delta$. Accordingly on the space of equivariant differential forms $\Omega_K(M)$ the operator $1 \otimes d$ splits as $1 \otimes d = 1 \otimes \overline{\delta} + 1 \otimes \delta$. For brevity, we will also abbreviate $1 \otimes \delta$ to $\delta$ and $1 \otimes \overline{\delta}$ to $\overline{\delta}$.

Since the action of $K$ is holomorphic, $\mathbb{H}^{p,q}(M)$ is a $K$-module for any $(p, q)$. Thus the space

\begin{equation}
\Omega_K^{p,q}(M) := \bigoplus_{p' + i = p, q' + i = q} (S^{i} \mathfrak{t}^* \otimes \Omega^{p'q'}(M))^K
\end{equation}

is well defined for any $(p, q)$.

For any $\xi \in \mathfrak{k}$, denote by $\xi^{1,0}$ and $\xi^{0,1}$ respectively the $(1, 0)$ and $(0, 1)$ components of the vector field on $M$ induced by $\xi$. Then on the Cartan complex $\Omega_K(M)$ the operator $d^\prime$ splits as $d^\prime = d^{1,0} + d^{0,1}$, where

\begin{align*}
(d^{1,0} \alpha)(\xi) &= i(\xi^{1,0})(\alpha(\xi)), \\
(d^{0,1} \alpha)(\xi) &= i(\xi^{0,1})(\alpha(\xi)), \quad \forall \alpha \in \Omega_K(M).
\end{align*}

Thus the equivariant exterior differential $d_K$ splits as $d_K = \overline{\delta}_K + \delta_K$, where

\begin{align*}
\delta_K &= \delta + d^{1,0}, \\
\overline{\delta}_K &= \overline{\delta} + d^{0,1}.
\end{align*}

It is straightforward to check that

\begin{equation*}
\overline{\delta}_K^2 = 0, \quad \overline{\delta}_K^2 = 0, \quad \delta_K \overline{\delta}_K + \overline{\delta}_K \delta_K = 0.
\end{equation*}

**Definition 4.1.** The equivariant Dolbeault cohomology of $M$, denoted by $\mathbb{H}_K^{p,q}(M, \mathbb{C})$, is defined to be the cohomology of the differential complex $\{\Omega^{p,q}(M), \overline{\delta}_K\}$.

The following result was due to Lillywhite [Lilly98] and Teleman [T00].
Theorem 4.2. ([Lilly03] Thm. 5.1) Consider the holomorphic action of a compact connected Lie group $K$ on a compact Kähler manifold $M$. Assume that the action is equivariantly formal. Then the following properties hold true.

a) \[
H^{p,q}_K(M, \mathbb{C}) \cong \bigoplus_{p'+i=q'+i=q} (S^i(\mathfrak{t}^*))^K \otimes H^{p',q'}_0(M).
\]

b) \[
\ker \delta_K \cap \im \partial_K = \im \delta_K \cap \ker \partial_K = \im \delta_K \partial_K.
\]

Now set $\Omega_{K,\partial_K}(M) = \Omega_K(M) \cap \ker \delta_K$. Since $\delta_K$ anti-commutes with $\partial_K$, we get a differential complex $(\Omega_{K,\partial_K}^r(M), \partial_K)$. The $q$-th cohomology of this differential complex will be denoted by $H(\Omega_{K,\partial_K}^r(M), \delta_K)$.

As an immediate consequence of Theorem 2.6, Theorem 4.2, and Proposition 3.1, we have the following result.

Theorem 4.3. \( a \) The homomorphism
\[
H^{p,q}(\Omega_{K,\partial_K}(M), \delta_K) \rightarrow H^p_K(M).
\]

induced by the inclusion $(\Omega_{K,\partial_K}(M), \delta_K) \hookrightarrow (\Omega_K(M), \delta_K)$ is an isomorphism.

b) The homomorphism
\[
\bigoplus_{p+q=r} H^{p,q}(\Omega_{K,\partial_K}(M), \delta_K) \rightarrow H^r_K(M, \mathbb{C}).
\]

induced by the inclusion $(\Omega_{K,\partial_K}(M), \delta_K) \hookrightarrow (\Omega_K(M), \partial_K)$ is an isomorphism. Thus the data $(H_K(M, \mathbb{R}), H^{p,q}(\Omega_{K,\partial_K}(M)))$ defines a (pure) real Hodge structure of weight $r$.

Definition 4.4. The dimension of the complex vector space $H^{p,q}_K(M, \mathbb{C})$, denoted by $h^{p,q}_K(M)$, is defined to be the equivariant Hodge number of the $K$-manifold $M$.

5. Transversely Kähler foliations

Let $\mathcal{F}$ be a foliation on a smooth manifold $Z$. Throughout this paper we will denote by $\mathfrak{X}(\mathcal{F})$ the space of smooth vector fields which are tangent to the leaves of $\mathcal{F}$, and by $T \mathcal{F}$ the tangent bundle of the foliation. We say that a vector field $X$ on $Z$ is foliated, if $[X, Y] \in \mathfrak{X}(\mathcal{F})$, $\forall Y \in \mathfrak{X}(\mathcal{F})$. We will denote by $\mathfrak{X}(Z, \mathcal{F})$ the space of foliated vector fields on the foliated manifold $(Z, \mathcal{F})$.

Clearly we have that $\mathfrak{X}(\mathcal{F}) \subset \mathfrak{X}(Z, \mathcal{F})$. A transverse vector is an equivalence class in the quotient space $\mathfrak{X}(Z, \mathcal{F})/\mathfrak{X}(\mathcal{F})$. The space of transverse fields, denoted by $\mathfrak{X}(Z/\mathcal{F})$, forms a Lie algebra with a Lie bracket induced from that of $\mathfrak{X}(Z, \mathcal{F})$.

The space of basic forms on $Z$ is defined to be
\[
\Omega_{bas}(Z) = \{ \alpha \in \Omega(Z) | \iota(X)\alpha = \mathcal{L}(X)\alpha = 0, \forall X \in \mathfrak{X}(\mathcal{F}) \}.
\]

Since the exterior differential operator $\partial$ preserves basic forms, we obtain a sub-complex $(\Omega_{bas}^r(Z), d)$ of the de Rham complex, called the basic de Rham
complex. The associated cohomology $H_{B}^{*}(Z)$ is called the basic cohomology.

Let $q$ be the codimension of the foliation $\mathcal{F}$. If $H_{B}^{q}(Z) = \mathbb{R}$, we say that $\mathcal{F}$ is homologically orientable.

Let $Q = T\mathcal{F}/T\mathcal{F}$ be the normal bundle of the foliation. A moment’s consideration shows that for any foliated vector field $X$, and for any $(r, s)$-type tensor

$$\sigma \in C^{\infty}(\bigotimes_{r}^{\ast} Q \otimes \cdots \otimes \bigotimes_{s}^{\ast} Q),$$

the Lie derivative $\mathcal{L}_{X}\sigma$ is well defined.

**Definition 5.1.** A transverse Riemannian metric on a foliation $(Z, \mathcal{F})$ is a Riemannian metric $g$ on the normal bundle $Q$ of the foliation, such that $\mathcal{L}_{X}g = 0$, $\forall X \in \mathfrak{X}(\mathcal{F})$. We say that $\mathcal{F}$ is a Riemannian foliation if there exists a transverse Riemannian metric on $(Z, \mathcal{F})$.

**Definition 5.2.** A transverse almost complex structure $\mathcal{J}$ on $(Z, \mathcal{F})$ is an almost complex structure $\mathcal{J} : T\mathcal{F}/T\mathcal{F} \to T\mathcal{F}/T\mathcal{F}$ such that $\mathcal{L}_{X}\mathcal{J} = 0$, $\forall X \in \mathfrak{X}(\mathcal{F})$. A transverse almost complex structure $\mathcal{J}$ on $(Z, \mathcal{F})$ is said to be integrable, if $\forall p \in Z$, there exists an open neighborhood $U$ of $p$, such that for any two transverse vector fields $X$ and $Y$ on $U$ with respect to the foliation $\mathcal{F}|_{U}$, the Nijenhaus tensor $N_{\mathcal{J}}(X, Y) = [\mathcal{J}X, \mathcal{J}Y] - \mathcal{J}[\mathcal{J}X, Y] - \mathcal{J}[X, \mathcal{J}Y] - [X, Y]$ vanishes. An integrable transverse almost complex structure is also called a transverse complex structure. The foliation $\mathcal{F}$ is said to be transversely holomorphic if there is a transverse complex structure $\mathcal{J}$ on $(Z, \mathcal{F})$.

Throughout the rest of this section, assume that $\mathcal{F}$ is a foliation on a manifold $Z$ endowed with a transverse almost complex structure $\mathcal{J}$. To simplify notations, we will let $\Omega_{\text{bas}}^{\ast}(Z)$ denote the space of complex basic differential forms, and let $\Omega_{\text{hor}}^{\ast}(Z)$ denote the space of complex horizontal differential forms. Now let $Q_{\mathbb{C}}$ be the complexification of the normal bundle $Q$ of the foliation. For any $\alpha \in \Omega_{\text{hor}}^{r}(Z)$, define a section $\tilde{\alpha} \in C^{\infty}(\wedge^{r} Q_{\mathbb{C}})$ as follows.

$\forall \overline{X}_{1}, \ldots, \overline{X}_{r} \in C^{\infty}(Q_{\mathbb{C}})$,

$$\tilde{\alpha}(\overline{X}_{1}, \ldots, \overline{X}_{r}) = \alpha(X_{1}, \ldots, X_{r}),$$

where $X_{i}$ is a vector field on $Z$ that represents $\overline{X}_{i}$, $1 \leq i \leq r$.

Clearly the definition of $\tilde{\alpha}$ does not depend on the choices of $X_{i}$’s. Indeed this gives us a natural isomorphism

$$\Omega_{\text{hor}}^{r}(Z) \rightarrow C^{\infty}(\wedge^{r} Q_{\mathbb{C}}^{\ast}), \quad \alpha \mapsto \tilde{\alpha}. \quad (5.1)$$

Note that $\mathcal{J}$ determines a decomposition of $\wedge^{r} Q_{\mathbb{C}}^{\ast}$ as follows.

$$\wedge^{r} Q_{\mathbb{C}}^{\ast} = \bigoplus_{p+q=r} (\wedge^{p} Q^{\ast}) \otimes (\wedge^{q} Q^{\ast}), \quad (5.2)$$

where $Q^{\ast}$ are $Q^{\ast}$ are $\sqrt{-1}$ and $-\sqrt{-1}$-eigenbundle of $\mathcal{J}$ respectively. Thus (5.2) together with (5.1) induces a decomposition of $\Omega_{\text{hor}}^{r}(Z)$ as follows.

$$\Omega_{\text{hor}}^{r}(Z) = \bigoplus_{p+q=r} \Omega_{\text{hor}}^{p,q}(Z). \quad (5.3)$$
Definition 5.3. Let $\alpha$ be a complex basic form of type $(p,q)$. In view of the direct sum decomposition (5.3), define $\partial \alpha$ to be the $(p,q+1)$ component of $d\alpha$, and $\bar{\partial} \alpha$ the $(p+1,q)$ component of $d\alpha$.

The proof of the following fact is analogous to the case of complex manifolds, and will be left as an exercise.

Lemma 5.4. The transverse almost complex structure $J$ is integrable if and only if $d = \partial + \bar{\partial}$. In particular, when $J$ is integrable, we have that $\partial^2 = 0$, $\partial \bar{\partial} + \bar{\partial} \partial = 0$, $\partial \bar{\partial} = 0$.

Definition 5.5. Assume that $F$ is a transversely holomorphic foliation on $Z$. The basic Dolbeault cohomology of $Z$, denoted by $H^{p,q}_{\text{bas}}(Z)$, is defined to be the cohomology of the differential complex $\{\Omega^{p,q}_{\text{bas}}(Z), \bar{\partial}\}$. The dimension of the complex vector space $H^{p,q}_{\text{bas}}(Z)$ is defined to be the basic Hodge number $h^{p,q}_{\text{bas}}$ of the transversely holomorphic foliation $(Z,F)$.

Definition 5.6. A transverse Kähler structure on $(Z,F)$ consists of a transverse complex structure $J$ and a transverse Riemannian metric $g$, such that the tensor field $\omega$ defined by $\omega(X,Y) = g(X,JY)$ is anti-symmetric and closed when considered as a 2-form on $Z$ given by the injection $\bigwedge^2 Q^* \to \bigwedge^2 T^*Z$. The 2-form $\omega$ will be called a transverse Kähler form. $F$ is said to be a transversely Kähler foliation if there exists a transverse Kähler structure on $(Z,F)$.

The following result is due to El Kacimi [KA90].

Theorem 5.7. Suppose that $F$ is a homologically orientable transversely Kähler foliation on a compact manifold $Z$. Then on the space of basic forms $\Omega_{\text{bas}}(Z)$ the following $\bar{\partial}d$-lemma holds.

$$\ker \bar{\partial} \cap \text{im} \partial = \text{im} \bar{\partial} \cap \ker \partial = \text{im} \partial \bar{\partial}.$$
We have thus proved that $\mathcal{J}V$ is orthogonal to $T\Sigma$ in $TM|_{\Sigma}$. A simple dimension count shows that the subbundle $\mathcal{J}V$ is the orthogonal complement of $T\Sigma$ in $TM|_{\Sigma}$. Let $E$ be the orthogonal complement of $V$ in $T\Sigma$. Then $E$ is the orthogonal complement of $W := V \oplus \mathcal{J}V$ in $TM|_{\Sigma}$. Since both $W$ and $g$ are invariant under the action of $\mathcal{J}$ and $K$, we obtain a $K$-invariant almost complex structure $\mathcal{J}$ on $E$.

Now let $T_{\Sigma}M$, $V_{\Sigma}$ and $E_{\Sigma}$ be the complexification of $TM$, $V$ and $E$ respectively, and let $E^{\perp,0}_{\Sigma}$ and $E_{\Sigma}^{\perp,1}$ be the $\sqrt{-1}$-eigenbundle and $-\sqrt{-1}$-eigenbundle respectively of the almost complex structure $\mathcal{J} : E_{\Sigma} \to E_{\Sigma}$. We say that a complex tangent vector $A + \sqrt{-1}B$ is tangent to $Z$, where $A, B \in T_{z}M$, $z \in Z$, if both $A$ and $B$ lie in $T_{z}Z$. We first make the following simple observations.

Lemma 6.1.  

a) Suppose that $X \in T^{1,0}_{\Sigma,z}(M)$, where $z \in Z$. Then $X$ is tangent to $Z$ if and only if $X \in E^{\perp,0}_{\Sigma,z}$.

b) If $X_{1}, X_{2} \in C^{\infty}(E^{\perp,0}_{\Sigma})$, then $[X_{1}, X_{2}] \in C^{\infty}(E^{\perp,0}_{\Sigma})$.

Proof.  

a) Let $\nu_{1}, \ldots, \nu_{k}$ be an orthonormal basis of $V_{z}$, and let $\zeta_{1}, \mathcal{J}\zeta_{1}, \ldots, \zeta_{s}, \mathcal{J}\zeta_{s}$ be a basis of $E_{z}$. It is straightforward to check that

$$\nu_{1} - \sqrt{-1}\mathcal{J}\nu_{1}, \ldots, \nu_{k} - \sqrt{-1}\mathcal{J}\nu_{k}, \zeta_{1} - \sqrt{-1}\mathcal{J}\zeta_{1}, \ldots, \zeta_{s} - \sqrt{-1}\mathcal{J}\zeta_{s}$$

form a basis of $T^{1,0}_{\Sigma,z}(M)$. Let $\xi_{1}, \ldots, \xi_{k}$ be a basis of $\xi$, such that $\xi_{i,M}(z) = \nu_{i}$, $1 \leq i \leq k$, and let $\Phi^{i} : M \to \mathbb{R}$ be the $\xi_{i}$-component of the moment map $\Phi$ given by $\Phi^{i}(x) = <\Phi(x), \xi_{i}>, \forall x \in M$. Since $\nu_{i}$ is tangent to $Z = \Phi^{-1}(0)$ for any $j$, $<\nu_{j}, d\Phi^{i}> = 0$. It follows that

$$(6.1) \quad <\nu_{j} - \sqrt{-1}\mathcal{J}\nu_{j}, d\Phi^{i}> = \sqrt{-1}\omega(\nu_{i}, \mathcal{J}\nu_{j}) = \sqrt{-1}g(\nu_{i}, \nu_{j}).$$

Now write $X$ as

$$X = \sum_{i=1}^{k} a_{i}(\nu_{i} - \sqrt{-1}\mathcal{J}\nu_{i}) + \sum_{j=1}^{s} b_{j}(\zeta_{j} - \sqrt{-1}\mathcal{J}\zeta_{j}),$$

where $a_{i}$’s and $b_{j}$’s are complex scalars. Then it follows from (6.1) that $<X, d\Phi^{i}> = \sqrt{-1}a_{i}$. Therefore $X$ is tangent to $Z$ if and only if $a_{i} = 0, \forall 1 \leq i \leq k$. This proves that $X$ is tangent to $Z$ if and only if $X \in E^{\perp,0}_{\Sigma,z}$.

b) Extend $X_{i}$ to a vector field $\tilde{X}_{i}$ on an open neighborhood $U$ of $Z$, $i = 1, 2$. Since $\mathcal{J}$ is an integrable almost complex structure, it follows from the vanishing of the Nijenhuis tensor $N_{\mathcal{J}}(\tilde{X}_{1}, \tilde{X}_{2})$ that $\mathcal{J}[X_{1}, X_{2}] = \sqrt{-1}[X_{1}, X_{2}]$ on $Z$. Thus $[X_{1}, X_{2}]$ is a type $(0, 1)$ vector field. Since $Z$ is a submanifold of $M$, and since both $X_{1}$ and $X_{2}$ are tangent to $Z$, we must have that $[X_{1}, X_{2}]$ is tangent to $Z$. Therefore by Part a) of Lemma 6.1, $[X_{1}, X_{2}]$ must be a section of $E^{\perp,0}_{\Sigma}$. q.e.d.

Proposition 6.2. The foliation $\mathcal{F}$ induced by the action of $K$ on $Z$ is homologically orientable and transversely Kähler.
Proof. Let $Q = \mathcal{T}Z/\mathcal{T}F$ be the normal bundle of the foliation, and let $\psi : E \to Q$ be the restriction to $E$ of the projection map $\text{proj} : \mathcal{T}Z \to \mathcal{T}Z/\mathcal{T}F$. Clearly $\psi$ is an isomorphism from $E$ to $Q$. Define $h = (\psi^{-1})^*g|_E$, and define $\mathcal{J} : Q \to Q$ to be the unique bundle map on $Q$ such that

$$\mathcal{J}\psi(X) = \psi(\mathcal{J}(X)), \ \forall X \in C^\infty(E).$$

Since $g|_E$ and $\mathcal{J} : E \to E$ are $K$-invariant, it is straightforward to check that $h$ is a transverse Riemannian metric, and that $\mathcal{J} : Q \to Q$ is a transverse almost complex structure. The integrability of $\mathcal{J} : Q \to Q$ follows easily from Part b) of Lemma 6.1. Note that $\mathcal{F}$ is a Riemannian foliation with trivial Molino sheaf. It follows that $\mathcal{F}$ is a Killing foliation on a compact manifold $Z$, and is therefore homologically orientable.

Finally, consider the two tensor $\sigma$ on $Q$ given by $\sigma(\cdot, \cdot) = h(\cdot, \mathcal{J}\cdot)$. It is anti-symmetric since $g$ is compatible with $\mathcal{J}$. Moreover, under the injection $\wedge^2 Q^* \to \wedge^2 T^*Z \sigma$ gets mapped to the closed 2-form $\omega|_Z$. This completes the proof of Proposition 6.2. q.e.d.

Now choose a basis $\xi_1, \cdots, \xi_k$ of $\mathfrak{t}$, and let $0^1, \cdots, 0^k$ be connection 1-forms on $Z$ determined by the equations

$$\iota(X)0^i = 0, \ \forall X \in C^\infty(E), \ \forall 1 \leq i \leq k,$$

(6.2) $\iota(X)\xi_j = 0, \forall X \in C^\infty(E), \forall 1 \leq i, j \leq k.$

Then $\forall 1 \leq i \leq k$, we have a curvature two form $\mu^i \in \Omega_{\text{hor}}(Z)$ as given in (2.1). Moreover, we also have a well defined projection operator $\text{Hor} : \Omega(Z) \to \Omega_{\text{hor}}(Z)$ as given in (2.2). The following lemma is a crucial step towards establishing the main result of this paper.

**Lemma 6.3.**

a) Suppose that $\alpha \in \Omega^{p,q}(M)$. Then $\text{Hor}(\iota^*\alpha) \in \Omega^{p,q}_{\text{hor}}(Z)$ is a horizontal form of type $(p, q)$ relative to the direct sum decomposition introduced in (5.3). Here $\iota^*$ is the pullback on differential forms induced by the inclusion map $i : Z \hookrightarrow M$.

b) $\forall 1 \leq i \leq k$, $\mu^i$ is a horizontal form of type $(1, 1)$ relative to the bi-grading introduced in (5.3).

**Proof.**

a) To show that $\text{Hor}(\iota^*\alpha)$ is of type $(p, q)$, it suffices to show that for any vectors $X_1, \cdots, X_r \in \mathcal{E}_{C, r}$, and any vectors $Y_1, \cdots, Y_s \in \mathcal{E}_{C, s}$, where $r + s = p + q$ and $z \in Z$, we have that

$$\text{Hor}(\iota^*\alpha)(X_1, \cdots, X_r, Y_1, \cdots, Y_s) = 0,$$

provided $r \neq p$. By Theorem 2.1, $\iota^*\alpha$ admits a unique expression

$$\iota^*\alpha = \sum_i \theta^i h_i + \text{Hor}(\iota^*\alpha), \ h_i \in \Omega_{\text{hor}}(Z).$$

(6.3)

Here in the above summation $\theta^i = (\theta^i)_1 \cdots (\theta^i)_k$ for some non-negative integers $i_1, \cdots, i_k$ satisfying $i_1 + \cdots + i_k \geq 1$. Therefore (6.3), together with the first half of (6.2), implies that

$$\text{Hor}(\iota^*\alpha)(X_1, \cdots, X_r, Y_1, \cdots, Y_s) = \iota^*\alpha(X_1, \cdots, X_r, Y_1, \cdots, Y_s).$$
However, it follows easily from the definition of the almost complex structure $\mathcal{J}$ on $E$ that $i_*(X_i) \in T^{1,0}_{C^\infty}(M)$, and that $i_*(Y_j) \in T^{0,1}_{C^\infty}(M), \forall 1 \leq i \leq r, \forall 1 \leq j \leq s$. Since $\alpha \in \Omega^{p,q}(M)$, we must have that $\alpha(i_*(X_1), \ldots, i_*(X_r), i_*(Y_1), \ldots, i_*(Y_s)) = 0$ provided $r \neq p$.

b) Let $X_1, X_2 \in C^\infty(E^{1,0}_{C^\infty})$. Since $\mu$ is a real two form, to show Part b) of Lemma 6.3 it suffices to show that $\mu(X_1, X_2) = 0$. By definition,

$$\mu(X_1, X_2) = (d\theta l + c^i_j \theta^i \wedge \theta^j)(X_1, X_2)$$

$$= \theta l(X_1) - \theta l(X_2)$$

$$= 0$$

Here we have used Part b) of Lemma 6.1 to show that the last equality holds.

q.e.d.

Now let $\mathcal{C} : \Omega_K(Z) \to \Omega_{bas}(Z)$ be the Cartan operator as introduced in Definition 2.2, and let $i : Z \to M$ be the inclusion map.

**Definition 6.4.** We define

$$\kappa := \mathcal{C} \circ i^* : \Omega_K(M) \to \Omega_{bas}(Z)$$

to be the Kirwan map at the level of differential forms.

**Remark 6.5.** It is clear from Remark 2.3 that the Kirwan map introduced in Definition 6.4 depends only on the Kähler metric $g$.

**Proposition 6.6.**

a) $\forall \alpha \in \Omega^{p,q}_K(M), \kappa \alpha \in \Omega^{p,q}_{bas}(Z)$.

b) $\forall \alpha \in \Omega^{p,q}_K(M)$,

$$\kappa d_K \alpha = d \kappa \alpha.$$

**Proof.** Part a) of Proposition 6.6 is an immediate consequence of Lemma 6.3. To show that Part b) holds, first note that by Theorem 2.4 $kd_K \alpha = d \kappa \alpha$, $\forall \alpha \in \Omega^{p,q}_K(M)$. Since $d_K = \partial_K + \bar{\partial}_K$ and $d = \partial + \bar{\partial}$, we have that

$$\kappa \partial_K \alpha + \kappa \bar{\partial}_K \alpha = \bar{\partial} \kappa \alpha + d \kappa \alpha.$$

However, by Part a) the Kirwan map $\kappa$ respects the bi-gradings. By comparing the types of the forms we see that

$$\kappa \partial_K \alpha = \bar{\partial} \kappa \alpha.$$

q.e.d.

Now let $\Omega_{bas,\partial} = \Omega_{bas}(Z) \cap \ker \partial$, and let $H^{p,q}(\Omega_{bas,\partial}, \overline{\partial})$ be the cohomologies associated to the differential complex $(\Omega^{p,q}_{bas,\partial}, \overline{\partial})$. From Proposition 6.6 we obtain two chain maps

$$\kappa : (\Omega^{p,q}_K(M), \overline{\partial}_K) \to (\Omega^{p,q}_{bas}(Z), \overline{\partial}),$$

$$\kappa : (\Omega^{p,q}_{K,\partial_K}(M), \overline{\partial}_K) \to (\Omega^{p,q}_{bas,\partial}(Z), \overline{\partial}).$$
We will denote by $\kappa : H^p,q_K(M) \to H^p,q_\partial(Z)$ and $\kappa : H^p,q(\Omega_{K,\partial_k}(M), \partial) \to H^p,q(\Omega_{\text{bas},\partial}(Z), \partial)$ the homomorphisms induced by the chain maps (6.4) and (6.5) respectively, and will also call them Kirwan maps.

Throughout the rest of this section, we will assume that the moment map $\Phi$ is proper. Therefore $Z = \Phi^{-1}(0)$ is compact. The following result is an easy consequence of Proposition 6.2, Theorem 5.7 and Proposition 3.1.

**Corollary 6.7.** Assume that the moment map $\Phi$ is proper. Then we have that

\[(6.6)\quad H^p,q_B(Z) = H^p,q(\Omega_{\text{bas},\partial}(Z), \partial)\]

\[(6.7)\quad H^r_B(Z) = \bigoplus_{p+q=r} H^p,q(\Omega_{\text{bas},\partial}(Z), \partial).\]

In particular, the data $(H_B(Z, \mathbb{R}), H^p,q(\Omega_{\text{bas},\partial}(Z), \partial))$ defines a (pure) real hodge structure of weight $r$. Recall that by Theorem 2.4, $\kappa : (\Omega_K(M), d_K) \to (\Omega_{\text{bas}}(Z), d)$ is also a chain map. It induces a homomorphism $\kappa : H^r_K(M) \to H^r_B(Z)$, which is the usual Kirwan map at the level of cohomologies. Proposition 6.6 implies the following result as an easy consequence.

**Theorem 6.8.** Assume that the equivariant Kähler $K$-manifold $M$ is compact. Then the Kirwan map $\kappa : H^r_K(M, \mathbb{R}) \to H^r_B(Z, \mathbb{R})$ is a morphism of real hodge structures of bi-degree $(0,0)$ with respect to the pure hodge structures described in Theorem 4.3 and Corollary 6.7 respectively.

We are ready to prove the following theorem.

**Theorem 6.9.** Assume that the equivariant Kähler $K$-manifold $M$ is compact. Then the Kirwan map $\kappa : H^p,q_K(M) \to H^p,q_B(Z)$ is surjective. As a consequence, we must have that

\[(6.8)\quad h^p,q_B(M) \leq h^p,q_K(M)\]

**Proof.** We have the following commutative diagram.

\[
\begin{array}{ccc}
\bigoplus_{p+q=k} H^p,q(\Omega_{K,\partial_k}(M), \partial) & \xrightarrow{\kappa} & H^k_K(M) \\
\downarrow \kappa & & \downarrow \kappa \\
\bigoplus_{p+q=k} H^p,q(\Omega_{\text{bas},\partial}(Z), \partial) & \xrightarrow{\kappa} & H^k_B(Z)
\end{array}
\]

By Theorem 4.3 and Corollary 6.7, both the top and the bottom horizontal maps in the above diagram are isomorphisms. By Theorem 2.7 the right
vertical map is surjective. It follows that the left vertical map must be surjective as well. However, since the left vertical map maps each component $H^{p,q}(\Omega_{K,\partial K}(M), \overline\partial K)$ into $H^{p,q}(\Omega_{bas,\partial}(Z), \overline\partial)$, the map

$$\kappa : H^{p,q}(\Omega_{K,\partial K}(M), \overline\partial K) \to H^{p,q}(\Omega_{bas,\partial}(Z), \overline\partial)$$

must be surjective as well for each pair of $(p, q)$. Now consider the commutative diagram

$$
\begin{array}{ccc}
H^{p,q}(\Omega_{K,\partial K}(M), \overline\partial K) & \xrightarrow{\kappa} & H^{p,q}(M) \\
\downarrow & & \downarrow \\
H^{p,q}(\Omega_{bas,\partial}(Z), \overline\partial) & \xrightarrow{\kappa} & H^{p,q}(\partial M_0)
\end{array}
$$

Since the K-manifold $M$ is equivariantly formal, it follows from Theorem 4.3 and Corollary 6.7 that both the top and the bottom horizontal maps are isomorphisms. By our work above, the left vertical map is surjective. It follows that the right vertical map must be surjective as well. q.e.d.

By our assumption, the quotient space $M_0 = Z/K$ is an orbifold. Clearly, the transverse Kähler structure on $Z$ naturally descends to a Kähler structure on $M_0$. We note that there is a natural isomorphism from the complex of basic forms $(\Omega_{bas}(Z), d)$ to the complex of differential forms $(\Omega(M_0), d)$, and that this natural isomorphism respects the type of differential forms. Thus we must have that $H^{p,q}_{bas}(Z) \cong H^{p,q}(M_0)$, which implies that $h^{p,q}_{bas}(Z) = h^{p,q}(M_0)$. The following result is an immediate consequence of Theorem 6.9, which also answers the open question raised by Weitsman.

**Theorem 6.10.** Consider the Hamiltonian action of a compact connected Lie group $K$ on a compact Kähler manifold $(M, \omega)$ with a moment map $\Phi : M \to \mathfrak{t}^*$. Assume that $K$ acts locally freely on the level set $Z = \Phi^{-1}(0)$, and that $M_0 = Z/K$ is the Kähler quotient. Then there is a natural surjective Kirwan map

$$\kappa : H^{p,q}_K(M) \to H^{p,q}_\Sigma(M_0).$$

In particular, if $h^{p,q}(M) = 0$ when $p \neq q$, then $h^{p,q}(M_0) = 0$ when $p \neq q$.

**Proof.** The first half of Theorem 6.10 is obvious. By Part a) of Theorem 4.2 we have that $h^{p,q}_K(M) = 0$ when $p \neq q$. It follows from Theorem 6.9 that $h^{p,q}(M_0) = 0$ when $p \neq q$.

q.e.d.

Finally, we would like to emphasize that the definition of the Kirwan map at the level of differential forms does not depend on the compactness of $M$. As a result, the method developed in this paper may be applied to situations where $M$ is not compact as well. For example, according to the work [L T97] of Lerman and Tolman, any complete simplicial toric variety can be obtained as a symplectic quotient of a complex vector space $C^N$ by the linear action of a compact abelian group $K$. Since $H_K(C^N) \cong$
$\mathbb{C}[x_1, \ldots, x_k]$, and since the Kirwan surjectivity theorem is known to hold in this case, one can easily derive from Proposition 6.6 a symplectic proof of the following well-known result on the hodge numbers of a complete simplicial toric variety.

**Theorem 6.11.** ([CLS11] Thm. 9.3.2) For a complete simplicial toric variety $X$, its hodge number $h^{p,q}(X) = 0$ provided $p \neq q$.

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