Knot Theory from a Chern-Simons
Gauge Theory Point of View

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Abstract

A brief summary of the development of perturbative Chern-Simons
gauge theory related to the theory of knots and links is presented. Em-
phasis is made on the progress achieved towards the determination of a
general combinatorial expression for Vassiliev invariants. Its form for all
the invariants up to order four is reviewed, and a table of their values for
all prime knots with ten crossings is presented.

1 Introduction

Chern-Simons gauge theory \cite{1,2} provides an excellent framework to study
knot and link invariants. This framework has shown to be very useful in
both, the study of polynomial invariants as the Jones polynomial \cite{3} and its
generalizations, and the study of Vassiliev invariants or numerical invariants
of finite type. Non-perturbative aspects of the theory play a fundamental role
in the first context while perturbative ones are the main tool in the second. In
this paper I will present a brief summary of the results on Vassiliev \cite{4,5,6,7,8}
invariants achieved from perturbative Chern-Simons gauge theory in the last
few years.

\footnote{Invited lecture delivered at the Workshop on Geometry and Physics, held at Medina del
Campo, Spain, September 20 – 22, 1999.}
An important line of investigation in the context of Vassiliev invariants is the search for a universal combinatorial expression. Different approaches \[9, 10, 11, 12\] have been carried out. In the framework of perturbative Chern-Simons gauge theory, explicit combinatorial formulae for all the primitive invariants up to order four have been obtained \[12\]. The context based on this approach seems rather promising to obtain a general combinatorial expression.

As any other gauge theory, Chern-Simons gauge theory can be analyzed for different gauge fixings. Covariant gauges lead to complicated integral expressions for Vassiliev invariants \[13, 14, 15, 16, 17, 18, 19\]. Simpler integral expressions are obtained in non-covariant gauges of the light-cone type \[20, 21, 22, 23, 24\]. However, for a non-covariant gauge fixing of the temporal type one obtains combinatorial expressions \[12, 25\]. It is in the last situation in the one that all intermediate integrals can be done so one ends with combinatorial expressions where only the information contained at the crossings is relevant.

Combinatorial expressions are much preferred to compute invariants and to study their properties. It turns out that the resulting combinatorial expressions can be rewritten in terms of Gauss diagrams. These facts notably simplify their explicit formulae. In this brief presentation I will collect all the expressions based on Gauss diagrams for all the primitive Vassiliev invariants up to order four.

An extended version of this presentation can be found in ref. \[26\]. In the present paper, however, I include a table that was not presented in ref. \[26\] because the corresponding computations were not carried out then. The appendix contains a table with all the primitive Vassiliev invariants up to order four for all prime knots with ten crossings, as computed from the combinatorial expressions provided by Chern-Simons gauge theory.

The paper is organized as follows. In sect. 2, I briefly describe the quantization procedure in the temporal gauge and comment on the essential ingredients which lead to the combinatorial expressions. In sect. 3, the resulting combinatorial expressions are presented in terms of Gauss diagrams. Finally, in sect. 4 aspects of future research directions are described.

2 Perturbative Chern-Simons gauge theory in the temporal gauge

I will begin reviewing the basic elements of Chern-Simons gauge theory. This theory is a quantum field theory whose action is based on the Chern-Simons form associated to a non-abelian gauge group. The fundamental data in Chern-Simons gauge theory are the following: a smooth three-manifold \( M \) which will be taken to be compact, a gauge group \( G \), which will be taken
semi-simple and compact, and an integer parameter $k$. The action of the theory is the integral of the Chern-Simons form associated to a gauge connection $A$ corresponding to a gauge group $G$:

$$S_{CS}(A) = \frac{k}{4\pi} \int_{M} \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$  \tag{1}$$

The exponential of $i$ times this action is invariant under gauge transformations for integer $k$.

The metric-independence of the action (1) implies that the resulting quantum field theory is topological. Appropriate observables lead to vacuum expectation values (vevs) which correspond to topological invariants. A particularly important set of observables is constituted by Wilson loops. They correspond to the holonomy of the gauge connection $A$ along a loop. Given a representation $R$ of the gauge group $G$ and a 1-cycle $\gamma$ on $M$, it is defined as:

$$W_{\gamma}^{R}(A) = \text{Tr}_{R}(\text{Hol}_{\gamma}(A)) = \text{Tr}_{R} \text{P} \exp \int_{\gamma} A.$$  \tag{2}$$

Products of these operators are the natural candidates to obtain topological invariants after computing their vev. These vevs are formally written as:

$$\langle W_{\gamma_1}^{R_1}W_{\gamma_2}^{R_2} \cdots W_{\gamma_n}^{R_n} \rangle = \int [DA]W_{\gamma_1}^{R_1}(A)W_{\gamma_2}^{R_2}(A) \cdots W_{\gamma_n}^{R_n}(A)e^{iS_{CS}(A)},$$  \tag{3}$$

where $\gamma_1, \gamma_2, \ldots, \gamma_n$ are 1-cycles on $M$ and $R_1, R_2$ and $R_n$ are representations of $G$. In (3), the quantity $[DA]$ denotes the functional integral measure. The functional integral in (3) is not well defined. A variety of methods have been proposed to go around this problem and provide some meaning to the right hand side of (3). These methods fall into two categories, perturbative and non-perturbative ones. Witten, in his pioneer work 1988 [1], showed, using non-perturbative methods, that when one considers non-intersecting cycles $\gamma_1, \gamma_2, \ldots, \gamma_n$ without self-intersections, the vevs (3) lead to polynomial invariants as the Jones polynomial and its generalizations.

The construction of the perturbative series expansion of the vev of an operator when dealing with a gauge theory starts with a gauge fixing. The first analysis of Chern-Simons perturbation theory were made in the covariant Landau gauge [13]. Subsequent studies [14, 15, 17] in this gauge led to a framework linked to the theory of Vassiliev invariants, which culminated with the configuration space integral approach [18, 19].

Non-covariant gauges have been also studied in the context of Chern-Simons perturbation theory. The perturbative series which results in the non-covariant light-cone gauge [23] leads to the Kontsevich integral [21]. Vevs
of gauge invariant operators are independent of the gauge chosen and therefore the expressions obtained in the light-cone gauge should be equivalent to the ones obtained in any other gauge. Thus, from a quantum field theory point of view, the configuration space integral which appear in the covariant Landau gauge leads to the same quantities as the Kontsevich integral. The non-covariant temporal gauge leads to alternative expressions for the vevs of gauge-invariant operators which turn out to be combinatorial [12, 25]. Again, gauge invariance implies that the resulting quantities must be the same as the ones in the configuration space approach or in the Kontsevich integral. In the rest of this section I will review the salient features of the analysis of the perturbative series expansion of the vacuum expectation value of a Wilson loop in the temporal gauge.

The gauge-fixing condition in the temporal gauge takes the form

\[ n^\mu A_\mu = 0, \]  

where \( n \) is the unit vector \( n^\mu = (1, 0, 0) \). In this gauge the propagator becomes:

\[
\delta_{ab} \frac{\lambda}{(np)^2} (p_\mu p_\nu - i \frac{\lambda}{\lambda} (np) \epsilon_{\mu\nu\rho} n^\rho) \rightarrow -i \epsilon_{\mu\nu\rho} \frac{n^\rho}{np} \delta_{ab},
\]

where the limit \( \lambda \to 0 \) has been taken. To construct the perturbative series expansion of the vev of a Wilson loop, one needs the Fourier transform of (5).

In the temporal gauge the momentum-space integral that must be carried out has the form:

\[
\Delta(x_0, x_1, x_2) = \int_M \frac{d^3p}{(2\pi)^3} \frac{e^{i(p_0 x_0 + p_1 x_1 + p_2 x_2)}}{p_0^0}.
\]

This integral is ill-defined due to the pole at \( p_0 = 0 \). To make sense of it a prescription has to be given to circumvent the pole. A precise prescription will not be taken. Instead, a rather general form will be used [12],

\[
\Delta(x_0, x_1, x_2) = \frac{i}{2} \text{sign}(x_0) \delta(x_1) \delta(x_2) + f(x_1, x_2),
\]

where \( f(x_1, x_2) \) is a prescription-dependent distribution. The important consequence of the result (6) is that the dependence of \( \Delta(x_0, x_1, x_2) \) on \( x_0 \) has to be of the form \( \text{sign}(x_0) \delta(x_1) \delta(x_2) \). This observation will be crucial in our analysis. The propagator (6) will allow us to introduce the notion of kernel of a Vassiliev invariant and to design a procedure to compute combinatorial expressions for these invariants.

Given a knot \( K \) and one of its regular knot projections, \( K \), on the \( x_1, x_2 \)-plane which is a Morse knot in the \( x_1 \) and \( x_2 \) directions, one has to deal with
the following perturbative series expansion for the vacuum expectation value of the corresponding Wilson loop [12]:

\[ \langle W(K, G) \rangle = \langle W(K, G) \rangle_{\text{temp}} \times \langle W(U, G) \rangle^{b(K)}, \] (8)

being,

\[ \frac{1}{d} \langle W(K, G) \rangle = 1 + \sum_{i=1}^{\infty} v_i(K) x^i, \] (9)

and,

\[ \frac{1}{d} \langle W(K, G) \rangle_{\text{temp}} = 1 + \sum_{i=1}^{\infty} \hat{v}_i(K) x^i. \] (10)

In these expressions \( x \) denotes the inverse of the Chern-Simons coupling constant, \( x = 2\pi i/k \), \( G \) the gauge group, and \( d \) the dimension of the representation carried by the Wilson loop. The function \( b(K) \) is the exponent of the Kontsevich factor, which has been conjectured to be [12],

\[ b(K) = \frac{1}{12}(n_{x_1} + n_{x_2}), \] (11)

where \( n_{x_1} \) and \( n_{x_2} \) are the critical points of the regular projection \( K \) in both, the \( x_1 \) and the \( x_2 \) directions. In (8) \( U \) denotes the unknot and \( \langle W(K, G) \rangle_{\text{temp}} \) is the vacuum expectation of the Wilson loop corresponding to the regular projection \( K \) as computed perturbatively in the temporal gauge with the standard Feynman rules of the theory. Notice that though each of the factors on the right hand side of (8) depends on the regular projection chosen, the left hand side does not. While the coefficients \( v_i(K) \) of the series (9) are Vassiliev invariants the coefficients \( \hat{v}_i(K) \) of (10) are not. The latter depend on the regular projection chosen.

3 Combinatorial expressions in terms of Gauss diagrams

A universal combinatorial formula for Vassiliev invariants could be obtained if the coefficients \( \hat{v}_i(K) \) in (10) could be computed with no integrals left. Unfortunately, this has not been obtained yet to all orders. Only part of the contributions entering \( \hat{v}_i(K) \) have been explicitly written to all orders. These are the kernels introduced in [12]. The kernels are quantities which depend on the knot projection chosen and therefore are not knot invariants. However, at a given order \( i \) a kernel differs from an invariant of type \( i \) by terms that vanish in signed sums of order \( i \). The kernel contains the part of a Vassiliev
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invariant which is the last in becoming zero when performing signed sums, in other words, a kernel vanishes in signed sums of order \( i + 1 \) but does not in signed sums of order \( i \). Kernels plus the structure of the perturbative series expansion seem to contain enough information to reconstruct the full Vassiliev invariants [12].

The expression for the kernels results after considering only the simplest part of the propagator of the gauge field in the temporal gauge. This part involves a double delta function and therefore all the integrals can be performed. The result is a combinatorial expression in terms of crossing signatures after distributing propagators among all the crossings. The general expression can be written in a universal form much in the spirit of the universal form of the Kontsevich integral [21]. Let us consider a knot \( K \) with a regular knot projection \( K \) containing \( n \) crossings. Let us choose a base point on \( K \) and let us label the \( n \) crossings by \( 1, 2, \ldots, n \) as one passes for first time through each of them when traveling along \( K \) starting at the base point. The universal expression for the kernel associated to \( K \) has the form:

\[
N(K) = \sum_{k=0}^{\infty} \left( \sum_{m=1}^{k} \sum_{p_1 \cdots p_m = 1}^{k} \sum_{i_1, \ldots, i_m = 1}^{k} \sum_{i_1 < \cdots < i_m}^{n} \frac{\epsilon_{i_1 \cdots i_m} p_1 \cdots p_m}{(p_1! \cdots p_m!)^2} \right) \sum_{\sigma_1 \in P_1, \ldots, \sigma_m \in P_m} T(i_1, \sigma_1; \ldots; i_m, \sigma_m),
\]

(12)

In this equation \( P_m \) denotes the permutation group of \( p_m \) elements. The factors in the innermost sum, \( T(i_1, \sigma_1; \ldots; i_m, \sigma_m) \), are group factors which are computed in the following way: given a set of crossings, \( i_1, \ldots, i_m \), and a set of permutations, \( \sigma_1, \ldots, \sigma_m \), with \( \sigma_1 \in P_1, \ldots, \sigma_m \in P_m \), the corresponding group factor \( T(i_1, \sigma_1; \ldots; i_m, \sigma_m) \) is the result of taking a trace over the product of group generators which is obtained after assigning \( p_1, \ldots, p_m \) group generators to the crossings \( i_1, \ldots, i_m \) respectively, and placing each set of group generators in the order which results after traveling along the knot starting from the base point. The first time that one encounters a crossing \( i_j \) a product of \( p_j \) group generators is introduced; the second time the product is similar, but with the indices rearranged according to the permutation \( \sigma_j \in P_j \).

The universal formula (12) for the kernels can be written in a more useful way collecting all the coefficients multiplying a given group factor. The group factors can be labeled by chord diagrams. At order \( k \) one has a term for each of the inequivalent chord diagrams with \( k \) chords. Denoting chord diagrams
by $D$, equation (12) can be written as:

$$\mathcal{N}(\mathcal{K}) = \sum_D N_D(\mathcal{K}) D,$$

where the sum extends to all inequivalent chord diagrams. The concept of kernel can be extended to include singular knots by considering signed sums of (13), or, following [8], introducing vacuum expectation values of the operators for singular knots. If $K_j$ denotes a regular projection of a knot $K^j$ with $j$ simple singular crossings or double points, the corresponding universal form for the kernel possesses an expansion similar to (13):

$$\mathcal{N}(K_j) = \sum_D N_D(K_j) D. \quad (14)$$

The general results about singular knots proved in [8] lead to two important features for (14). On the one hand, finite type implies that $N_D(K_j) = 0$ for chord diagrams $D$ with more than $j$ chords. On the other hand, $N_D(K_j) = 2^j \delta_{D, D(K^j)}$, where $D(K^j)$ is the configuration corresponding to the singular knot projection $K^j$. As observed above, kernels constitute the part of a Vassiliev invariant which survives a maximum number of signed sums.

To compute $N_D(K)$ one needs to introduce first the notion of the set of labeled chord subdiagrams of a given chord diagram. This set will be denoted by $S_D$. This set is made out of a selected set of labeled chord diagrams that will be defined now. A labeled chord diagram of order $p$ is a chord diagram with $p$ chords and a set of positive integers $k_1, k_2, \ldots, k_p$, which will be called labels, such that each chord has one of these integers attached. The set $S_D$ is made out of labeled chord diagrams which satisfy two conditions. These conditions are fixed by the form of the series entering the kernels (12). The elements of $S_D$ will be called labeled chord subdiagrams of the chord diagram $D$. They are defined as follows. A labeled chord subdiagram of a chord diagram $D$ with $k$ chords is a labeled chord diagram of order $p$ with labels $k_1, k_2, \ldots, k_p, p \leq k$, such that the following two conditions are satisfied: a) $k_1 + k_2 + \cdots + k_p = k$; b) there exist elements $\sigma_1 \in P_{k_1}, \sigma_2 \in P_{k_2}, \ldots, \sigma_p \in P_{k_p}$ of the permutation groups $P_{k_1}, P_{k_2}, \ldots, P_{k_p}$ such that, after replacing the $j$-th chord diagram by $k_j$ chords arranged according to the permutation $\sigma_j$, for $j = 1, \ldots, p$, the resulting chord diagram is homeomorphic to $D$. The number of ways that permutations $\sigma_1 \in P_{k_1}, \sigma_2 \in P_{k_2}, \ldots, \sigma_p \in P_{k_p}$ can be chosen is called the multiplicity of the labeled chord subdiagram. The multiplicity of a given labeled chord subdiagram, $s \in S_D$, will be denoted by $m_D(s)$.

The chord diagram $D$ itself can be regarded as a labeled chord subdiagram such that its labels, or positive integers attached to its chords, are 1.
It has multiplicity 1. All the elements of $S_D$ except $D$ have a number of chords smaller than the number of chords of $D$. Not all labeled chord diagrams are subdiagrams of $D$. However, given a labeled chord diagram with labels $k_1, k_2, \ldots, k_p$, there can be different sets of permutations leading to $D$. The number of these different sets is the multiplicity introduced above. The elements of the sets $S_D$ for all chord diagrams $D$ up to order four which do not have disconnected subdiagrams are the following:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{を選ぶ} \\
\text{を選ぶ} \\
\end{array}
\end{array}
\end{align*}
\]

The numbers accompanying each labeled chord subdiagram denote their multiplicity. When no number is attached to a chord of a labeled chord diagram it should be understood that the corresponding label is 1.

In order to write the final expression for the kernels the notion of Gauss diagram must be introduced. Given a regular projection $K$ of a knot $K$ we can associate to it its Gauss diagram $G(K)$. The regular projection $K$ can be regarded as a generic immersion of a circle into the plane enhanced by information on the crossings. The Gauss diagram $G(K)$ consists of a circle together with the preimages of each crossing of the immersion connected by a chord. Each chord is equipped with the sign of the signature of the corresponding
crossing. Gauss diagrams are useful because they allow to keep track of the sums involving the crossings which enter in (12) in a very simple form. Let us consider a chord diagram $D$ and one of its labeled chord subdiagrams $s \in S_D$. Let us assume that $s$ has $p$ chords and labels $k_1, k_2, \ldots, k_p$. We define the product,

$$\langle s, G(\mathcal{K}) \rangle,$$  \hspace{1cm} (16)

as the sum over all the embeddings of $s$ into $G(\mathcal{K})$, each one weighted by a factor,

$$\frac{\varepsilon_{k_1} \varepsilon_{k_2} \cdots \varepsilon_{k_p}}{(k_1!k_2! \cdots k_p!)^2},$$  \hspace{1cm} (17)

where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p$ are the signatures of the chords of $G(\mathcal{K})$ involved in the embedding. Using (16) the kernels $N_D(\mathcal{K})$ entering (13) can be written as,

$$N_D(\mathcal{K}) = \sum_{s \in S_D} m_D(s)\langle s, G(\mathcal{K}) \rangle,$$  \hspace{1cm} (18)

where $m_D(s)$ denotes the multiplicity of the labeled subdiagram $s \in S_D$ relative to the chord diagram $D$.

The terms $\langle s, G(\mathcal{K}) \rangle$ entering (18) are related to the quantities $\chi(\mathcal{K})$ de-
It is straightforward to obtain the following relations:

\[
\begin{align*}
\langle \begin{tikzpicture}[baseline=0pt]
\draw (0,0) circle (0.5cm);
\end{tikzpicture} , G(K) \rangle &= \frac{1}{(j!)^2} \chi_1(K), \\
\langle \begin{tikzpicture}[baseline=0pt]
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\end{tikzpicture} , G(K) \rangle &= \chi_2^A(K), \\
\langle \begin{tikzpicture}[baseline=0pt]
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\end{tikzpicture} , G(K) \rangle &= \chi_2^B(K), \\
\langle \begin{tikzpicture}[baseline=0pt]
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\end{tikzpicture} , G(K) \rangle &= \chi_2^C(K), \\
\langle \begin{tikzpicture}[baseline=0pt]
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\end{tikzpicture} , G(K) \rangle &= \chi_2^D(K), \\
\langle \begin{tikzpicture}[baseline=0pt]
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\end{tikzpicture} , G(K) \rangle &= \chi_2^E(K), \\
\langle \begin{tikzpicture}[baseline=0pt]
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\end{tikzpicture} , G(K) \rangle &= \chi_2^F(K),
\end{align*}
\]

where in the first row the relation in the left column applies when \(j\) is odd and the one in the right when \(j\) is even. Notice that in the second relation \(n(K)\) denotes the number of crossings of the regular projection \(K\).

In \cite{12}, Vassiliev invariants up to order four were expressed in terms of these quantities and the crossing signatures. The strategy to obtain them was to start with the kernels \cite{18} and exploit the properties of the perturbative series expansion of Chern-Simons gauge theory. A special role in the construction was played by the factorization theorem proved in \cite{27}. Here, only their final form will be listed.

At second order, the final expression for the invariant is:

\[
\alpha_{21}(K) = \alpha_{21}(U) + \langle \begin{tikzpicture}[baseline=0pt]
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\end{tikzpicture} , G(K) \rangle - \langle \begin{tikzpicture}[baseline=0pt]
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\end{tikzpicture} , G(\alpha(K)) \rangle,
\]

where \(\alpha_{21}(U)\) stands for the value of \(\alpha_{21}\) for the unknot. In this expression \(\alpha(K)\) denotes the ascending diagram \(\alpha(K)\) of a knot projection \(K\). It is defined as the diagram obtained by switching, when traveling along the knot from a base point, all the undercrossings to overcrossings. Ascending diagrams enter often in the combinatorial expressions and it is convenient introduce
the following notation. A bar over a quantity \( L(\mathcal{K}) \) indicates that the same quantity for the ascending diagram has to be subtracted, i.e.:

\[
\tilde{L}(\mathcal{K}) = L(\mathcal{K}) - L(\alpha(\mathcal{K}))
\]  

(21)

where \( \alpha(\mathcal{K}) \) denotes the standard ascending diagram of \( \mathcal{K} \). Using this notation, the final form for the only primitive Vassiliev invariant at order two is:

\[
\alpha_{21}(\mathcal{K}) = \alpha_{21}(U) + \langle \begin{array}{c}
\includegraphics[scale=0.4]{example1}
\end{array} , \tilde{G}(\mathcal{K}) \rangle. 
\]  

(22)

At order three there is only one primitive invariant. It takes the form:

\[
\alpha_{31}(\mathcal{K}) = \langle \begin{array}{c}
\includegraphics[scale=0.4]{example2}
\end{array} + \begin{array}{c}
\includegraphics[scale=0.4]{example3}
\end{array} + 2 \begin{array}{c}
\includegraphics[scale=0.4]{example4}
\end{array} , G(\mathcal{K}) \rangle - \sum_{i=1}^{n} \varepsilon_i(\mathcal{K}) \left[ \langle \begin{array}{c}
\includegraphics[scale=0.4]{example5}
\end{array} , G(\alpha(\mathcal{K})) \rangle \right]_i. 
\]  

(23)

Several comments are in order to explain the quantities entering this expression. The sum is over all crossings \( i, i = 1, \ldots, n \), and \( \varepsilon_i(\mathcal{K}) \) denotes the corresponding signature. The square brackets \( [ \cdot ] \), enclosing a quantity \( L(\mathcal{K}) \) denote:

\[
\left[ L(\mathcal{K}) \right]_i = L(\mathcal{K}) - L(\mathcal{K}_{i+}) - L(\mathcal{K}_{i-}), 
\]  

(24)

where the regular projection diagrams \( \mathcal{K}_{i+} \) and \( \mathcal{K}_{i-} \) are the ones which result after the splitting of \( \mathcal{K} \) at the crossing point \( i \).

Combinatorial expressions for the two primitive invariants at order four have been presented in [12]. Their construction is based on the use of the kernels (18) and the properties of the perturbative series expansion. As in the case of previous orders, these invariants are expressed in terms of the products (16) and the crossing signatures. Their form is more complicated than the ones at lower orders. At order four there are two primitive Vassiliev invariants. The same choice of basis as in [12] is made here. The combinatorial expressions for these two invariants turn out to be:

\[
\alpha_{42}(\mathcal{K}) = \alpha_{42}(U) + \langle 7 \begin{array}{c}
\includegraphics[scale=0.4]{example6}
\end{array} + 5 \begin{array}{c}
\includegraphics[scale=0.4]{example7}
\end{array} + 4 \begin{array}{c}
\includegraphics[scale=0.4]{example8}
\end{array} + 2 \begin{array}{c}
\includegraphics[scale=0.4]{example9}
\end{array} + \begin{array}{c}
\includegraphics[scale=0.4]{example10}
\end{array} + \begin{array}{c}
\includegraphics[scale=0.4]{example11}
\end{array} + \begin{array}{c}
\includegraphics[scale=0.4]{example12}
\end{array} 
\rangle 
\]

\[ + 8 \begin{array}{c}
\includegraphics[scale=0.4]{example13}
\end{array} + 2 \begin{array}{c}
\includegraphics[scale=0.4]{example14}
\end{array} + 8 \begin{array}{c}
\includegraphics[scale=0.4]{example15}
\end{array} + \frac{1}{6} \begin{array}{c}
\includegraphics[scale=0.4]{example16}
\end{array} , \tilde{G}(\mathcal{K}) \rangle 
\]

\[ + \sum_{i,j \in \mathcal{C}_a \atop i > j} \tilde{\varepsilon}_{ij}(\mathcal{K}) \left[ \langle \begin{array}{c}
\includegraphics[scale=0.4]{example17}
\end{array} , G(\alpha(\mathcal{K})) \rangle \right]_{ij}^a 
\]

\[ - 2 \left[ \langle \begin{array}{c}
\includegraphics[scale=0.4]{example18}
\end{array} , G(\alpha(\mathcal{K})) \rangle \right]_i - 2 \left[ \langle \begin{array}{c}
\includegraphics[scale=0.4]{example19}
\end{array} , G(\alpha(\mathcal{K})) \rangle \right]_j 
\]
\[ + \sum_{i > j, i,j \in C} \bar{\varepsilon}_{ij}(K) \left( \left[ \langle \bigoplus, G(\alpha(K)) \rangle \right]_{ij}^b - \left[ \langle \bigoplus, G(\alpha(K)) \rangle \right]_{i} \right. \]
\[ \left. - \left[ \langle \bigoplus, G(\alpha(K)) \rangle \right]_{j} \right), \tag{25} \]

and,
\[ \alpha_{43}(K) = \alpha_{43}(U) + \left( \bigotimes + \bigotimes + \bigoplus + 2 \bigoplus - \frac{1}{6} \bigotimes, \bar{G}(K) \right) \]
\[ + \sum_{i,j \in C_a} \bar{\varepsilon}_{ij}(K) \left( \left[ \langle \bigoplus, G(\alpha(K)) \rangle \right]_{ij}^a - \left[ \langle \bigoplus, G(\alpha(K)) \rangle \right]_{i} \right. \]
\[ \left. - \left[ \langle \bigoplus, G(\alpha(K)) \rangle \right]_{j} \right), \tag{26} \]

In these expressions the explicit dependence on the signatures appears in the quantities \( \bar{\varepsilon}_{ij}(K) \) which are:
\[ \bar{\varepsilon}_{ij}(K) = \varepsilon_{ij}(K) - \varepsilon_{ij}(\alpha(K)) = \varepsilon_i(K)\varepsilon_j(K) - \varepsilon_i(\alpha(K))\varepsilon_j(\alpha(K)). \tag{27} \]

The sums in which these products are involved are over double splittings of the knot projection \( K \) at the crossings \( i \) and \( j \). There are two ways of carrying out these double splittings, depending on the configuration associated to the crossings \( i \) and \( j \). These are described in detail in [12]. In the first one the regular projection \( K \) is split into two while in the second one it is split into three. Splittings of the first type build the set \( C_a \). The ones of the second type build \( C_b \). While only the first one is involved in the invariant \( \alpha_{43} \), both appear in \( \alpha_{42} \). The new quantities entering the sums are:
\[ \left[ L(K) \right]_{ij}^a = L(K) - L(K_{ij}^{a_1}) - L(K_{ij}^{a_2}), \]
\[ \left[ L(K) \right]_{ij}^b = L(K) - L(K_{ij}^{b_1}) - L(K_{ij}^{b_2}) - L(K_{ij}^{b_3}), \tag{28} \]
where \( K_{ij}^{a_1}, K_{ij}^{a_2}, K_{ij}^{b_1}, K_{ij}^{b_2} \) and \( K_{ij}^{b_3} \) are the knot projections which originate after a double splitting of \( K \). As in previous orders, in the expressions (25) and (26), the quantities \( \alpha_{42}(U) \) and \( \alpha_{43}(U) \) correspond to the value of these invariants for the unknot. It has been proved in [12] that the combinatorial expressions
for $\alpha_{42}(K)$ and $\alpha_{43}(K)$ in (25) and (26) are invariant under Reidemeister moves.

Vassiliev invariants constitute vector spaces and their normalization can be chosen in such a way that they are integer-valued. Once their value for the unknot has been subtracted off they can be presented in many basis in which they are integers. We will chose here a particular basis in which the numerical values for the invariants up to order four are rather simple:

\begin{align*}
\nu_2(K) &= \frac{1}{4} \tilde{\alpha}_{21}(K), \\
\nu_3(K) &= \frac{1}{8} \tilde{\alpha}_{31}(K), \\
\nu_4^1(K) &= \frac{1}{8} (\tilde{\alpha}_{42}(K) + \tilde{\alpha}_{43}(K)), \\
\nu_4^2(K) &= \frac{1}{4} (\tilde{\alpha}_{42}(K) - 5\tilde{\alpha}_{43}(K)).
\end{align*}

(29)

In these equations the tilde indicates that the value for the unknot has been subtracted, i.e., $\tilde{\alpha}_{ij}(K) = \alpha_{ij}(K) - \alpha_{ij}(U)$. The values of the Vassiliev invariants (29) for all prime knots up to nine crossings have been presented in [26]. The invariants (29) have been computed for torus knots in [28] and [29]. Denoting a generic torus knot by two coprime integers, $p$ and $q$, these invariants take the form:

\begin{align*}
\nu_2(p, q) &= \frac{1}{24} (p^2 - 1)(q^2 - 1), \\
\nu_3(p, q) &= \frac{1}{144} (p^2 - 1)(q^2 - 1)pq, \\
\nu_4^1(p, q) &= \frac{1}{288} (p^2 - 1)(q^2 - 1)p^2q^2, \\
\nu_4^2(p, q) &= \frac{1}{720} (p^2 - 1)(q^2 - 1)(2p^2q^2 - 3p^2 - 3q^2 - 3).
\end{align*}

(30)

The explicit expression of Vassiliev invariants as polynomials in $p$ and $q$ is known up to order six [28]. Of course, up to order four their value agree with the ones computed explicitly from equations (22), (23), (25) and (26). The only torus knots up to ten crossings are $3_1, 5_1, 7_1, 8_19, 9_1$ and $10_{124}$, whose associated coprime integers are $(3,2), (5,2), (7,2), (4,3), (9,2)$ and $(5,3)$, respectively.

In the table collected in the appendix the value of the primitive Vassiliev invariants for all the prime knots with ten crossings are presented. The value of these invariants for prime knots up to ten crossings can be found in [26].
4 Prospects

Though the perspectives are rather promising, the problems inherent to the proper treatment of gauge theories in non-covariant gauges do not permit at the moment to obtain a closed and complete formulation. Much work has to be done to understand the subtle issues related to the use of non-covariant gauges. The kernels plus the properties of the perturbative series expansion are probably enough to compute the explicit form of a given invariant but certainly it does not provide a systematic way of deriving the general universal formula. A proper and complete formulation of the perturbative series in a non-covariant gauge would explain the presence of the Kontsevich factor and will provide a general universal combinatorial formula. It is likely that a lattice formulation of Chern-Simons gauge theory in the temporal gauge could help considerably to make progress in this direction. We expect to report on this and other issues of perturbative Chern-Simons gauge theory in future work.

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Appendix

In this appendix the result of computing the first four primitive Vassiliev invariants in (29), after using the expressions (22), (23), (25) and (26), for all prime knots with 10 crossings, is presented. These quantities have not been computed before using these combinatorial expressions. Their calculation is lengthy but straightforward once the computation of (22), (23), (25) and (26) are programmed and the prime knots are properly labeled.
Table 1: Primitive Vassiliev invariants up to order four for all prime knots with ten crossings.
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