A Note on the Generalized and Universal Associated Legendre Equations*

Keegan L. A. Kirk,† Kyle R. Bryenton,† and Nasser Saad§
School of Mathematical & Computational Sciences, University of Prince Edward Island, 550 University Avenue, Charlottetown, PEI C1A 4P3, Canada

(Received January 17, 2018; revised manuscript received April 9, 2018)

Abstract A class of second-order differential equations commonly arising in physics applications are considered, and their explicit hypergeometric solutions are provided. Further, the relationship with the Generalized and Universal Associated Legendre Equations are examined and established. The hypergeometric solutions, presented in this work, will promote future investigations of their mathematical properties and applications to problems in theoretical physics.

DOI: 10.1088/0253-6102/70/1/19

Key words: universal associated Legendre polynomials, generalized associated Legendre equation, hypergeometric series, exact solutions

1 Introduction

Recently, the Universal Associated Legendre Polynomials

\[ P_{\ell m}^n(r) = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(2\ell - m + 1)}{2^m \nu! (\ell - m - 2\nu)! (\ell - \nu + 1)!} r^{\ell - m - 2\nu} \]

has been the subject of many interesting studies.[1–7] These polynomials are solutions to the differential equation (see Ref. [2] and the references therein):

\[ (1 - r^2) \frac{d^2 P_{\ell m}^n(r)}{dr^2} - 2r \frac{d P_{\ell m}^n(r)}{dr} + \left( \lambda - \frac{m^2}{1 - r^2} - \frac{a + br + cr^2}{1 - r^2} \right) P_{\ell m}^n(r) = 0, \quad (-1 \leq r \leq 1), \quad a, c \in \mathbb{R}, \quad (2) \]

in which \( b = 0, m' = \sqrt{a + c + m^2}, \lambda = \ell'(\ell' + 1) - c, \ell' = m' + n, n = 0, 1, 2, \ldots \) Through partial-fraction decomposition of the rational coefficient of the \( P_{\ell m}^n(r) \) term, Eq. (2) is shown to be a slight modification of the well-known Generalized Associated Legendre Equation[8–19]

\[ (1 - r^2) \frac{d^2 F(r)}{dr^2} - 2r \frac{d F(r)}{dr} + \left( k(k+1) - \frac{n^2}{2(1+r)} - \frac{m^2}{2(1-r)} \right) F(r) = 0. \]

The differential equation (3) was introduced first by Bateman in his analysis of the harmonic equations (Ref. [8], page 389). Following Bateman’s work, this was later intensively studied in a series of research articles by Kuiipers and Meulenbeld for complex-valued parameters \( k, m \), and \( n \).[9–17] The recent book of Virchenko and Fedotov[19] was devoted to the subsequent development of the theory of the Generalized Associated Legendre Functions and their applications.

As we shall prove in the present work, both differential equations (2) and (3) are members of a more general class of differential equations characterized by

\[ (r - \xi_1)(\xi_2 - r) \frac{d^2 F(r)}{dr^2} + (a_1 r + b_1) \frac{d F(r)}{dr} + \left( \lambda + \frac{a_2 r + b_2}{(r - \xi_1)} \frac{a_3 r^2 + b_3 r + c_3}{(r - \xi_1)} \right) F(r) = 0, \]

where \( a_j, b_j, j = 1, 2, 3, c_3 \) are real parameters and

\[ \xi_1 < r < \xi_2. \]

The exact solutions of the differential equation (4) are given, along with their relations to the published solutions of the Generalized and Universal Associated Legendre Differential Equations (2) and (3). New solvable classes of differential equations useful for the analysis of quantum systems are obtained.[20–22]

2 Exact Solutions

The differential equation (4) has three regular singular points, \( r \in \{\xi_1, \xi_2, \infty\} \) with exponents \( \mu_1, \mu_2, \) and \( \mu_\infty \), respectively, determined as the roots of the indicial (quadratic) equations:

\[ \mu_1^2 + \left( \frac{a_1 \xi_1 + b_1}{\xi_2 - \xi_1} - 1 \right) \mu_1 + \left( \frac{a_2 \xi_1 + b_2 \xi_1 + b_2 + c_3}{(\xi_2 - \xi_1)^2} \right) = 0, \]

where \( \xi_1 < r < \xi_2 \).

In this work we are able to introduce a new class of solutions, the hypergeometric solutions, to differential equations commonly used in physics applications. The hypergeometric solutions are presented in this work, and their explicit hypergeometric solutions are provided. Further, the relationship with the Generalized and Universal Associated Legendre Equations are examined and established. The hypergeometric solutions, presented in this work, will promote future investigations of their mathematical properties and applications to problems in theoretical physics.
\[ \mu_2^2 - \left( \frac{a_1 \xi_2 + b_1}{\xi_2 - \xi_1} + 1 \right) \mu_2 \]
\[ + \frac{a_2 \xi_2^2 + (a_1 + b_1) \xi_2 + b_2 + c_3}{(\xi_2 - \xi_1)^2} \]
\[ \mu_\infty^2 + (1 + a_1) \mu_\infty + a_3 - \lambda = 0. \]

Equation (9) is reducible to the hypergeometric equation. To this end, the general solutions of Eq. (4) take the form
\[ F(r) = (r - \xi_1)^{\mu_1} (r - \xi_2)^{\mu_2} f(r), \]
where the exponents \( \mu_1 \) and \( \mu_2 \) are evaluated using Eqs. (5) and (6) respectively. The substitution of Eq. (8) into Eq. (4) yields the following hypergeometric-type equation for the function \( f \equiv f(r) \):
\[ (r - \xi_1)(\xi_2 - r) \frac{d^2 f}{dr^2} + [(a_1 - 2 \mu_1 - 2 \mu_2)r + b_1 + 2 \mu_2 \xi_1 + 2 \mu_1 \xi_2] \frac{df}{dr} + [\lambda - a_3 - (\mu_1 + \mu_2)(\mu_1 + \mu_2 - a_1 - 1)] f = 0. \]

Employing the Möbius transformation, \( z = (\alpha r + \beta)/(\gamma r + \delta) \), for \( \alpha \delta - \beta \gamma \neq 0 \), yields
\[ \frac{d^2 f(z)}{dz^2} + \frac{(\alpha - \gamma z)^2(\beta + \alpha \xi_1 - (\delta + \xi_1 \gamma)z)(\beta + \xi_2 \alpha - (\delta + \xi_2 \gamma)z)}{(\alpha \delta - \beta \gamma)^2} \frac{df(z)}{dz} \]
\[ + \frac{2(\alpha - \gamma z)(\beta - \delta z + (\alpha - \gamma z)(\beta + \alpha \delta) \mu_2 + \gamma(\delta z - \beta + (\gamma z - \alpha \xi_2)))}{(\alpha \delta - \beta \gamma)^2} \frac{df(z)}{dz} \]
\[ + [\lambda - a_3 - (\mu_1 + \mu_2)(\mu_1 + \mu_2 - a_1 - 1)] f(z) = 0. \]

Thus, if \( \gamma = 0 \), the change of variables \( r \to z \equiv z(r) \) transforms Eq. (10) into an equation of the same type as that of Eq. (9). This in turn implies
\[ \left( z - \frac{\beta + \alpha \xi_1}{\delta} \right) \left( z - \frac{\beta + \alpha \xi_2}{\delta} \right) \frac{d^2 f(z)}{dz^2} + \left( z(a_1 - 2 \mu_1 - 2 \mu_2) + \frac{\alpha b_1 - \beta a_1 + 2 \mu_2(\beta + \alpha \xi_1) + 2 \mu_1(\beta + \alpha \xi_2)}{\delta} \right) \frac{df(z)}{dz} \]
\[ + [\lambda - a_3 - (\mu_1 + \mu_2)(\mu_1 + \mu_2 - a_1 - 1)] f(z) = 0. \]

To express the solutions of this equation in terms of hypergeometric functions, one must impose either of the following necessary conditions on \( \alpha \), \( \beta \), and \( \delta \):
\( \beta + \alpha \xi_1 = 0 \), \( \beta + \alpha \xi_2 = 0 \):
\( \beta + \alpha \xi_1 = 0 \), \( \beta + \alpha \xi_2 = 0 \)

\[ \frac{\beta + \alpha \xi_1}{\delta} = 0 \implies \alpha = \frac{\delta}{\xi_2 - \xi_1}, \quad \beta = \frac{\delta \xi_1}{\xi_1 - \xi_2}, \]
\[ \frac{\beta + \alpha \xi_2}{\delta} = 0 \implies \alpha = \frac{\delta}{\xi_1 - \xi_2}, \quad \beta = \frac{\delta \xi_2}{\xi_2 - \xi_1}. \]

Should we impose the conditions as given (i), Eq. (11) reduces to (denoting \( f \to \tilde{f} \))
\[ z(1 - z) \frac{d^2 \tilde{f}(z)}{dz^2} + \left( (a_1 - 2 \mu_1 - 2 \mu_2)z + 2 \mu_1 + \frac{b_1 + \xi_1 a_1}{\xi_2 - \xi_1} \right) \frac{d \tilde{f}(z)}{dz} + [\lambda - a_3 - (\mu_1 + \mu_2)(\mu_1 + \mu_2 - a_1 - 1)] \tilde{f}(z) = 0, \]
with two linearly independent series solutions expressed in terms of the hypergeometric functions as
\[ \tilde{f}_1(z) = _2F_1 \left( \frac{\mu_1 + \mu_2 - a_1 + 1}{2}, \frac{\mu_1 + \mu_2 - a_1 + 1}{2}; \frac{\alpha_1 + 1}{2}; a_3 + \lambda \right), \]
\[ \tilde{f}_2(z) = z^{1 - 2 \mu_1 + \alpha_2}(\xi_1 - \xi_2)_{\xi_1 - \xi_2} \]
\[ \times _2F_1 \left( \frac{1 - a_1}{2} + \mu_2 - \mu_1 + \frac{b_1 + \xi_1 a_1}{\xi_1 - \xi_2}, \frac{\lambda - a_3 + (1 + a_1)^2}{2}; \frac{1 - a_1}{2} + \mu_2 - \mu_1 + \frac{b_1 + \xi_1 a_1}{\xi_1 - \xi_2} + \sqrt{\lambda - a_3 + (1 + a_1)^2} \right). \]

Meanwhile, imposing the condition given in (ii), Eq. (11) reads
\[ z(1 - z) \frac{d^2 \tilde{f}(z)}{dz^2} + \left( (a_1 - 2 \mu_1 - 2 \mu_2)z + 2 \mu_2 + \frac{b_1 + \xi_2 a_1}{\xi_2 - \xi_1} \right) \frac{d \tilde{f}(z)}{dz} + [\lambda - a_3 - (\mu_1 + \mu_2)(\mu_1 + \mu_2 - a_1 - 1)] \tilde{f}(z) = 0. \]
with the two linearly independent solutions
\[ \tilde{f}_1(z) = _2F_1 \left( \frac{\mu_1 + \mu_2 - a_1 + 1}{2}, \frac{\mu_1 + \mu_2 - a_1 + 1}{2}; \frac{\alpha_1 + 1}{2}; a_3, \mu_1 + \mu_2 - a_1 + 1 \right), \]
\[ \tilde{f}_2(z) = z^{1 - 2 \mu_2 + \alpha_2}(\xi_1 - \xi_2)_{\xi_1 - \xi_2} \]
\[ \times _2F_1 \left( \frac{1 - a_1}{2} + \mu_2 - \mu_1 + \frac{b_1 + \xi_1 a_1}{\xi_1 - \xi_2}, \frac{\lambda - a_3 + (1 + a_1)^2}{2}; \frac{1 - a_1}{2} + \mu_2 - \mu_1 + \frac{b_1 + \xi_1 a_1}{\xi_1 - \xi_2} + \sqrt{\lambda - a_3 + (1 + a_1)^2} \right). \]
\[ f_2(z) = z^{1-2\mu_2+(b_1+a_1\xi_2)/(\xi_2-\xi_1)} \times \frac{\Gamma(\mu_1 - \mu_2 - \frac{a_1+1}{2} - \sqrt{\frac{(a_1+1)^2}{2} + \lambda - a_3 + \frac{b_1+a_1\xi_1}{\xi_2-\xi_1}})}{\Gamma(\mu_1 - \mu_2 + \frac{a_1+1}{2} + \sqrt{\frac{(a_1+1)^2}{2} + \lambda - a_3 + \frac{b_1+a_1\xi_1}{\xi_2-\xi_1}}) r - \xi_1} f_1(r) + \frac{\Gamma(\mu_1 + \mu_2 + \frac{a_1+1}{2} + \sqrt{\frac{(a_1+1)^2}{2} + \lambda - a_3 + \frac{b_1+a_1\xi_1}{\xi_2-\xi_1}})}{\Gamma(\mu_1 + \mu_2 - \frac{a_1+1}{2} - \sqrt{\frac{(a_1+1)^2}{2} + \lambda - a_3 + \frac{b_1+a_1\xi_1}{\xi_2-\xi_1}}) r - \xi_1} f_2(r) \]
Communications in Theoretical Physics  
Vol. 70

etc.) will be the focus of future work. Properties of these other classes (such as the weight-function, the recurrence relation, the orthogonality conditions, etc.) will be the focus of future work.

Legendre's Equation (2) as given by Eq. (1), we express Eq. (1) in terms of the hypergeometric function. Using the assumption \( \xi = 1 \), Eq. (2) reduces to the Generalized Associated Legendre Differential Equation. Different choices of other classes \( \xi = 0 \), \( \xi = 1 \). The mathematical properties of these other classes (such as the weight-function, the recurrence relation, the orthogonality conditions, etc.) will be the focus of future work.

3 Connection with the Generalized Associated Legendre Equation

This section serves to demonstrate the relationship between the differential equations (3) and (4). First denote

\[
\xi \in [1, \xi_0, \xi_1, \xi_2]
\]

then, using partial-fraction decomposition, Eq. (4) reads

\[
\begin{align*}
(r - \xi_1)(\xi_2 - r) & \frac{d^2 F(r)}{dr^2} + \left[ (m - n - 2 + 2\mu_1 + 2\mu_2)(1 - m - 2\mu_2)\xi_1 + (1 + n - 2\mu_1)\xi_2 \right] \frac{dF(r)}{dr} \\
+ \left( [k + \frac{n - m}{2} - \mu_1 - \mu_2] [\frac{k + n - m}{2} + \mu_1 + \mu_2] \right) F(r) & = 0 \\
\end{align*}
\]

Using Eq. (28), the indicial equations (5) and (6) may be expressed as

\[
a_3\xi_i^2 + (a_2 + b_3)\xi_i + b_2 + c_3 = \mu_1(\mu_1 - n)(\xi_1 - \xi_2)^2, \quad a_3\xi_i^2 + (a_2 + b_3)\xi_i + b_2 + c_3 = \mu_2(m + \mu_2)(\xi_1 - \xi_2)^2.
\]

Thus, Eq. (29) reduces to

\[
\begin{align*}
(r - \xi_1)(\xi_2 - r) & \frac{d^2 F(r)}{dr^2} + \left[ (m - n - 2 + 2\mu_1 + 2\mu_2)(1 - m - 2\mu_2)\xi_1 + (1 + n - 2\mu_1)\xi_2 \right] \frac{dF(r)}{dr} \\
+ \left( [k + \frac{n - m}{2} - \mu_1 - \mu_2] [\frac{k + n - m}{2} + \mu_1 + \mu_2] \right) F(r) & = 0 \\
\end{align*}
\]

with the exact solutions, determined via Eq. (21), given by:

\[
\begin{align*}
F_1(r) & = (r - \xi_1)^{\mu_1}(\xi_2 - r)^{\mu_2} 2F_1 \left( \begin{array}{c} -k + \frac{n - m}{2} \, \frac{1}{m - 1} \\ \xi_2 - r \end{array} \right) \\
F_2(r) & = (r - \xi_1)^{\mu_1}(\xi_2 - r)^{\mu_2} 2F_1 \left( \begin{array}{c} -k + \frac{n + m}{2} \, \frac{1}{m + 1} \\ \xi_2 - r \end{array} \right)
\end{align*}
\]

as found earlier by Kuipers et al. for \( \mu_1 = n/2 \) and \( \mu_2 = -m/2 \). In this case equation (30) reads

\[
\begin{align*}
(r - \xi_1)(\xi_2 - r) & \frac{d^2 F(r)}{dr^2} + \left[ -2r + \xi_1 + \xi_2 \right] \frac{dF(r)}{dr} + \left( [k + 1] + \frac{n^2(\xi_1 - \xi_2)}{4(r - \xi_1)} - \frac{n^2(\xi_1 - \xi_2)}{4(r - \xi_2)} \right) F(r) = 0.
\end{align*}
\]

For \( \xi_2 = -\xi_1 = 1 \), Eq. (33) reduces to the Generalized Associated Legendre Differential Equation. Different choices of \( \xi_1 \) and \( \xi_2 \) give rise to other interesting classes of differential equations, (e.g. \( \xi_1 = 0, \xi_2 = 1 \)). The mathematical properties of these other classes (such as the weight-function, the recurrence relation, the orthogonality conditions, etc.) will be the focus of future work.

4 Connection with the Universal Associated Legendre Equation

To establish the connection between the solutions of the differential equation (4) and that of the Universal Associated Legendre Equation (2) as given by Eq. (1), we express Eq. (1) in terms of the hypergeometric function. Using the
Legendre Duplication Formula, $\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z + 1/2)/\sqrt{\pi}$, Eq. (1) can be written as

$$F(r) = \sqrt{\frac{2\ell + 1}{2r}} \frac{((2\ell + 1)(\ell' - m')!}{2r(2\ell' + m' + 1)} (1 - r^2)^{m'/2} \sum_{\nu = 0}^{((1/2)(\ell' - m')) (1 - (\ell' - m'))/2\ell' (1 - r^2)^{m'/2}} \nu! (1/2 - \ell' \nu) \end{equation}$$

Further, by means of the Pochhammer identity $\Gamma(z - \nu) = (-1)^\nu \Gamma(z)/(1 - z)_\nu$, we obtain

$$F(r) = \frac{2^m \Gamma((\ell' + 1)/2) \Gamma((\ell' - m' + 1)/2) (2\ell + 1) ((\ell' - m')!}{2r(2\ell' + m' + 1)} (1 - r^2)^{m'/2} \times \sum_{\nu = 0}^{((1/2)(\ell' - m')) (1 - (\ell' - m'))/2\ell' (1 - r^2)^{m'/2}} \nu! (1/2 - \ell' \nu) \end{equation}$$

With the assumption that $-1/2((\ell' - m'))$ or $(1/2)(1 - (\ell' - m')) = 0, -1, -2, \ldots$, this equation may now be written in terms of the hypergeometric equation

$$F(r) = \frac{2^{2\ell'/2}\Gamma((\ell' + 1)/2) \Gamma((\ell' - m' + 1)/2) (2\ell + 1) (2\ell' + m' + 1)}{2r(2\ell' + m' + 1)} (1 - r^2)^{m'/2} \times 2F_1 \left( \frac{1 - (\ell' - m')}{2}, \frac{\ell' - m'}{2} \right) \end{equation}$$

with the understanding that the limit of the right-hand side is well-defined as $r \to 0$. Since

$$\Gamma(\ell'/2) = 2^{\ell'/2} \sqrt{\pi} \Gamma(\ell'/2),$$

it easily follows that

$$F(r) = \frac{2^{2\ell'/2}\Gamma((\ell' + 1)/2) \Gamma((\ell' - m' + 1)/2) (2\ell + 1) (2\ell' + m' + 1)}{2r(2\ell' + m' + 1)} (1 - r^2)^{m'/2} \times 2F_1 \left( \frac{1 - (\ell' - m')}{2}, \frac{\ell' - m'}{2} \right).$$

The identity (Ref. [24], Eq. (15.8.6))

$$(-1)^m \frac{(c)_m}{(b)_m} F_1 \left( \frac{m}{c} \right) = \frac{1}{b} \int F_1 \left( \frac{m}{c} \right), \quad m = 0, 1, 2, \ldots,$$

implies

$$F(r) = \frac{(2\ell + 1) (1 + \ell' + m')/2 (2\ell'/2)^{\ell'}/(2\ell + m'/2)}{\sqrt{\pi} ((1 + \ell' + m')/2) (2\ell'/2)^{\ell'}/(2\ell + m'/2)} \times 2F_1 \left( \frac{1 + \ell' + m'}{2}, \frac{\ell' - m'}{2} \right).$$

With $a_1 = 2(\mu_1 + \mu_2 - 1)$, the differential equation (4) reads after an application of partial-fraction decomposition,

$$\xi_2 - r (r - \xi_1) \frac{d^2 F(r)}{dr^2} + (2(\mu_1 + \mu_2 - 1) + b_1) \frac{d F(r)}{dr} + \left( a_3 \xi_2^2 + (a_2 + b_4) \xi_1 + b_2 + c_3 \right) F(r) = 0.$$  

Using the identity (Ref. [24], Eq. (15.8.20)):

$$2F_1 \left( \frac{1 - a}{2}, c \right) = (1 - z)^{c - 1} F_1 \left( \frac{c - a}{2}, \frac{a + c - 1}{2} \right) 4z(1 - z),$$

the solution of Eq. (39), namely,

$$F(r) = (r - \xi_1)^{\mu_1} (\xi_2 - r)^{\mu_2} \left( \frac{r - \xi_1}{\xi_2 - \xi_1} \right)^{1 - (\mu_1 + 2\mu_2 + 2(\mu_1 - 1)\xi_2)/(\xi_2 - \xi_1)} \end{equation}$$

\begin{equation} 
\times 2F_1 \left( \mu_1 + \mu_2 - \frac{a_1 + 1}{2} - \frac{(a_1 + 1)/2}{2(\mu_1 + \mu_2 - 1) + b_1} \frac{d F(r)}{dr} + \left( (a_1 + 1/2)^2 - a_3 + \lambda \right) \mu_1 + \mu_2 - \frac{a_1 + 1}{2} + \sqrt{\left( a_1 + 1/2 \right)^2 - a_3 + \lambda} \right) \left( \frac{r - \xi_1}{\xi_2 - \xi_1} \right) \end{equation}
Using the identity (Ref. [24], Eq. (15.8.7)),
\[
\binom{-m, \gamma}{c-m+1} 2F_1 \left( \frac{1-z}{2} \right) = \binom{c}{m, \gamma} 2F_1 \left( \frac{1-m, \gamma}{c} \frac{1}{z} \right),
\]
and assuming
\[
\frac{2b_1 + (4\mu_2 - 1)\xi_1 + (4\mu_1 - 3)\xi_2}{4(\xi_1 - \xi_2)} + \frac{\sqrt{\lambda - a_3 + (\mu_1 + \mu_2 - 1/2)^2}}{2} = \frac{\ell' - m'}{2},
\]
\[
\frac{2b_1 + (4\mu_2 - 1)\xi_1 + (4\mu_1 - 3)\xi_2}{4(\xi_1 - \xi_2)} - \frac{\sqrt{\lambda - a_3 + (\mu_1 + \mu_2 - 1/2)^2}}{2} = \frac{1 + \ell' + m'}{2},
\]
it follows that \( c = 1/2 \). Finally, the solution (41) now reads
\[
F(r) = \left( \frac{r - \xi_1}{\xi_2 - \xi_1} \right)^{1 - (b_1 + 2a_2 + 2(\mu_1 - \mu_2 - 1)\xi_2)/(\xi_1 - \xi_2)} \left( \frac{1/2}{(\ell' - m')/2} + (\xi_2 - r)\mu_1(\xi_2 - r)^\mu_2 \right)^{(\mu_1 + \mu_2 - 1/2)/2} \left( \frac{1}{2} \left( \ell' + m' \right)^2 \right)^{1/2} \binom{\xi_1 + \xi_2 - 2r}{\xi_1 - \xi_2} 2F_1 \left( \frac{1}{2}, \frac{1}{2} \right),
\]
which reduces, up to a multiplicative constant, to the solution (38) for \( b_1 = 0, \xi_1 = -1 \) and \( \xi_2 = 1 \).

5 Conclusion

The classical Generalized and the recent Universal Associated Legendre Equations are members of the more broad class of differential equations given by Eq. (4). We established the hypergeometric solutions of this class of equations and demonstrated that they lead to the Generalized and Universal Associated Legendre hypergeometric solutions. These new solutions open the door for further compelling studies, including the examination of their mathematical properties and the investigation of their applicability to problems in mathematical physics.

References

[1] Chen Chang-Yuan, You Yuan, Lu Fa-Lin, et al., Commun. Theor. Phys. 62 (2014) 331.
[2] Chang-Yuan Chen, Fa-Line Lu, and Dong-Sheng Sun, Appl. Math. Lett. 40 (2015) 90.
[3] Chen Chang-Yuan, You Yuan, Lu Fa-Lin, et al., Commun. Theor. Phys. 66 (2016) 158.
[4] Chen Chang-Yuan, Lu Fa-Lin, Sun Dong-Sheng, et al., Ann. Physics: Elsevier 371 (2016) 183.
[5] G. Yañez-Navarro, Guo-Hua Sun, et al., Commun. Theor. Phys. 68 (2017) 177.
[6] G. Yañez-Navarro, Guo-Hua Sun, Dong-Sheng Sun, et al., J. Math. Phys. 58 (2017) 052105.
[7] Wei Li, Chang-Yuan Chen, and Shi-Hai Dong, Adv. High Energy Phys. 2017 (2017) 1, doi:10.1155/2017/7374256.
[8] H. Bateman, Partial Differential Equations of Mathematical Physics, Cambridge University Press, Cambridge (1932).
[9] L. Kuipers and B. Meulenbeld, Proc. Kon. Ned. Ak. V. Wet. Amsterdam 80 (1957) 436.
[10] L. Kuipers and B. Meulenbeld, Proc. Kon. Ned. Ak. V. Wet. Amsterdam 80 (1957) 444.
[11] L. Kuipers, Math. Scand. 6 (1958) 200.
[12] L. Kuipers and B. Meulenbeld, Arch. Math. 66 (1958) 394.
[13] L. Kuipers and B. Meulenbeld, Proc. Konkl. Nederl. Akad. Wet. Ser. A 61 (1958) 557.
[14] L. Kuipers and B. Meulenbeld, Proc. Kon. Ned. Akad. Wetensch.; Ser. A 61 (1958) 330.
[15] L. Kuipers and B. Meulenbeld, Proc. Kon. Ned. Akad. Wetensch. Ser. A 61 (1958) 186.
[16] L. Kuipers and B. Meulenbeld, Arch. Math. F 66 (1958) 394.
[17] L. Kuipers, Monatsch. Math. 63 (1958) 24.
[18] N. A. Virchenko and I. A. Fedotova, J. Math. Sci. 69 (1994) 1395.
[19] N. A. Virchenko and I. A. Fedotova, Generalized Associated Legendre Functions and Their Applications, World Scientific, Singapore (2001).
[20] L. K. Sharma, Proc. Indian Nat. Sci. Acad. A 36 (1970) 239.
[21] I. A. Khan, Indian J. Pure Appl. Math. 4 (1973) 90.
[22] H. Ciftci, R. L. Hall, N. Saad, and E. Dogu, J. Phys. A: Math. Theor. 43 (2010) 415206.
[23] Earl A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York (1955).
[24] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, et al., NIST Digital Library of Mathematical Functions, http://dlmf.nist.gov/15.8, Release 1.0.17 of 2017-12-22.