A WALL CROSSING FORMULA FOR DEGREES OF REAL CENTRAL PROJECTIONS

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Abstract. The main result is a wall crossing formula for central projections defined on submanifolds of a real projective space. Our formula gives the jump of the degree of such a projection when the center of the projection varies. The fact that the degree depends on the projection is a new phenomenon, specific to real algebraic geometry. We illustrate this phenomenon in many interesting situations. The crucial assumption on the class of maps we consider is relative orientability, a condition which allows us to define a $\mathbb{Z}$-valued degree map in a coherent way. We end the article with several examples, e.g. the pole placement map associated with a quotient, the Wronski map, and a new version of the real subspace problem.

0. Introduction

One of the fundamental problems in real algebraic geometry concerns the solutions of systems of real algebraic equations. Since, in general, even the existence of real solutions is not guaranteed, it is important to find a priori lower bounds for the number of these solutions.

In the last years several important developments have taken place in this direction, which are related to problems in enumerative geometry [DeK], [IKS], [FK], [W1], [W2], [OT2], or to the study of certain polynomial systems which often have interesting applications [EG1], [EG2], [SS]. In many cases these lower bounds are provided by the degrees of certain maps, for instance central projections of projective submanifolds [EG1], [EG2], [Sa], [SS]. For a projective $m$-dimensional submanifold $X \subset \mathbb{P}^{N-1}_\mathbb{C}$, and a central projection $\pi : \mathbb{P}^{N-1}_\mathbb{C} \dashrightarrow \mathbb{P}^m_\mathbb{C}$ whose center does not intersect $X$, one gets a finite map $\pi|_X : X \to \mathbb{P}^m_\mathbb{C}$ whose degree with respect to complex orientations coincides with the number of points of a fibre $(\pi|_X)^{-1}(p)$ if multiplicities are taken into account. Note that this fibre can be regarded as the set of solutions of a system of homogeneous algebraic equations. This degree can also be identified with the number of points of the intersection $X \cap H$ for a general codimension $m$ projective subspace $H \subset \mathbb{P}^{N-1}_\mathbb{C}$; hence it is cohomologically determined and independent of the choice of the central projection $\pi$ (as long as its center does not intersect $X$). This is an important example of the fundamental principle of conservation of numbers. It is well known that this principle does not hold in real algebraic geometry. One of the goals of this article is to show that in real geometry, this principle should be replaced by a wall crossing formula. Such wall crossing formulae play an important role in gauge theory [DoK], [OT1], but apparently the wall crossing phenomenon (jump of an invariant in a well controlled way) has not been studied until now in real algebraic geometry. The general principle of wall crossing is very simple: one has a parameter space $\mathcal{P}$ which parameterizes a certain class of maps, and a wall $\mathcal{W} \subset \mathcal{P}$ of bad maps. If certain conditions are fulfilled,
one can define a degree map
\[ \text{deg} : \pi_0(P \setminus W) \to \mathbb{Z} \]
which associates to any chamber \( C \in \pi_0(P \setminus W) \) an integer. A wall crossing formula computes the difference \( \text{deg}(C_+) - \text{deg}(C_-) \) between the integers associated to two adjacent chambers. In this article we will prove such a wall crossing formula for the space of central projections restricted to a smooth, \textit{not necessarily algebraic} submanifold \( X \) of a real projective space. We are not aware of any systematic results in this generality. Note in particular that standard algebro-geometric techniques like e.g. elimination theory, do not apply.

The crucial assumption on the class of maps we consider is \textit{relative orientability}, a condition which allows us to define a \( \mathbb{Z} \)-valued degree map in a coherent way.

The wall crossing phenomenon for real central projections \( P(V) \supset X \to P(W) \) pointed out by our result is in striking contrast to the invariance of the degree of a section in a relatively oriented vector bundle (see [OT2]). We describe now briefly the results of this article:

In the first section we introduce the important notion of relative orientation of a map between topological manifolds, and we define the degree of a relatively oriented map between closed manifolds of the same dimension. For relatively \textit{orientable} maps \( f \) (so for maps which can be relatively oriented, but have not been endowed with a relative orientation) one can define the absolute degree \( |\text{deg}(f)| \), which is similar to Kronecker’s concept of \textit{characteristic} and Hopf’s \textit{absolute degree}. In the differentiable case the degree of a relatively oriented map can be computed using the local degrees at the points of a finite fibre. The special case of a Real finite holomorphic map \( f : X \to Y \) between Real complex manifolds of the same dimension is particularly important. If the restriction
\[ f(\mathbb{R}) : X(\mathbb{R}) \to Y(\mathbb{R}) \]
is relatively orientable, then there are fundamental estimates and comparison formulae for the sum of the multiplicities of the points of a real fibre \( f(\mathbb{R})^{-1}(y) \), \( y \in Y(\mathbb{R}) \):
\[
|\text{deg}(f(\mathbb{R}))| \leq \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_xf(\mathbb{R}) \leq \text{deg}(f) ,
\]
\[
|\text{deg}(f(\mathbb{R}))| \equiv \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_xf(\mathbb{R}) \equiv \text{deg}(f) \pmod{2} .
\]

In the second section we study central projections. Let \( V, W \) be real vector spaces, \( f \in \text{Hom}(V, W) \) and
\[ [f] : P(V) \setminus P(\ker(f)) \to P(W) \]
the induced morphism. Let \( X \subset P(V) \) be a compact submanifold with \( \dim(X) = \dim(P(W)) \) and \( X \cap P(\ker(f)) = \emptyset \). The induced map
\[ [f]_X : X \to P(W) \]
is relatively orientable if and only if
\[ w_1(X) = (\dim(X) + 1)w_1(\lambda_{V;X}) , \]
where $\lambda_{V,X} := \lambda_V|_X$ denotes the restriction to $X$ of the tautological line bundle $\lambda_V$ of $\mathbb{P}(V)$. It follows in particular that the relative orientability of such central projections $[f]_X$ is independent of $f$.

The tensor product $T_X \otimes \lambda_{V,X}$ can be written as $Y_X/\lambda_{V,X}$, where $Y_X$ is a subbundle of the trivial bundle $\mathcal{V}_X := X \times V$. With this definition we obtain a canonical identification

$$T_X = \text{Hom}(\lambda_{V,X}, Y_X/\lambda_{V,X}).$$

The data of a relative orientation of a map $f \in \text{Hom}(V,W)$ with $X \cap \mathbb{P}(\ker(f)) = \emptyset$ is equivalent to the data of a bundle isomorphism

$$\mu : X \times \det(W) \to \det(Y_X).$$

In our situation the space $\mathcal{P}$ parameterizing the relevant maps is the vector space $\text{Hom}(V,W)$, and the wall associated with the submanifold $X \subset \mathbb{P}(V)$ is:

$$\mathcal{W}_X := \{ f \in \text{Hom}(V,W) \mid X \cap \mathbb{P}(\ker(f)) \neq \emptyset \}.$$

A point $f_0 \in \mathcal{W}_X$ on the wall is called regular when $f_0$ is surjective, the intersection $X \cap \mathbb{P}(\ker(f_0))$ consists only of one point $\xi_0$, and $\ker(f_0) \cap Y_{\xi_0} = \xi_0$. Let $\mathcal{W}_X^0$ be the subspace of regular points in $\mathcal{W}_X$. Our first result in section 2 shows that the wall $\mathcal{W}_X \subset \text{Hom}(V,W)$ is a smooth hypersurface in all regular points $f_0 \in \mathcal{W}_X^0$ and identifies the normal line $N_{\mathcal{W}_X^0,f_0}$ with $\text{Hom}(\det(Y_{\xi_0}),\det(W))$ via a canonical isomorphism

$$\psi_{f_0} : N_{\mathcal{W}_X^0,f_0} \to \text{Hom}(\det(Y_{\xi_0}),\det(W)).$$

This is not a standard result, since the manifold $X$ is not supposed to be algebraic. It also shows that the choice of an orientation parameter $\mu : X \times \det(W) \to \det(Y_X)$ serves a second, completely different purpose: $\mu^{-1} : \det(Y_{\xi_0}) \to \det(W)$ defines a generator of the normal line $N_{\mathcal{W}_X^0,f_0}$ for every $f_0 \in \mathcal{W}_X^0$ with $\ker(f_0) \cap Y_{\xi_0} = \xi_0$.

The main result in this section is the wall crossing formula (see Theorem 2.14):

**Theorem** (Wall-crossing formula) Let $f_0 \in \mathcal{W}_X^0$ be a regular point on the wall with $\ker(f_0) \cap Y_{\xi_0} = \xi_0$. Consider a smooth map

$$f : N_{\mathcal{W}_X^0,f_0} \to \text{Hom}(V,W)$$

whose differential $f_* : 0$ is a right splitting of the exact sequence

$$0 \to T_{\mathcal{W}_X^0,f_0} \to T_{\text{Hom}(V,W),f_0} \to N_{\mathcal{W}_X^0,f_0} \to 0.$$ 

Then for every sufficiently small $\tau \in N_{\mathcal{W}_X^0,f_0} \setminus \{0\}$ we have:

1. $f(\tau) \in \text{Hom}(V,W) \setminus \mathcal{W}_X$ and $[f(\tau)]_X$ is a local diffeomorphism at $\xi_0$.
2. The local degree of $[f(\tau)]_X$ at $\xi_0$ is

$$\text{deg}_*(f(\tau),\mu,\xi_0)([f(\tau)]_X) = \text{sign}(\mu_{\xi_0} \circ (\psi_{f_0}(\tau))).$$

3. $\text{deg}_*[f(\tau)]_X - \text{deg}_*[-f(\tau)]_X = 2\text{sign}(\mu_{\xi_0} \circ (\psi_{f_0}(\tau))).$

As a consequence we obtain a general formula which computes the degree difference $\text{deg}_*[g_1,\mu,\xi_0]([g_1]_X) - \text{deg}_*[g_0,\mu,\xi_0]([g_0]_X)$ for a smooth path $(g_t)_{t \in [0,1]}$ in $\text{Hom}(V,W)$ with $g_0, g_1 \in \text{Hom}(V,W) \setminus \mathcal{W}_X$, which intersects the wall $\mathcal{W}_X$ only in regular points with transversal intersection.

The third result in section 2 describes the irregular locus $\mathcal{W}_X \setminus \mathcal{W}_X^0$ of the wall. We show that $\mathcal{W}_X \setminus \mathcal{W}_X^0$ is closed and its complement in $\text{Hom}(V,W)$ is connected. This implies that any two points $g_0, g_1 \in \text{Hom}(V,W) \setminus \mathcal{W}_X$ can be connected
by a smooth path \((g_t)_{t \in [0, 1]}\) which intersects \(W_X\) only along \(W^0_X\) with transversal intersection. In other words, the difference \(\deg_{\nu(g_1, \mu)}([g_1]_X) - \deg_{\nu(g_0, \mu)}([g_0]_X)\) can always be computed by our difference formula.

Important applications of our difference formula are the following general properties of the degree map

\[
\deg^X : \pi_0(\Hom(V, W) \setminus W_X) \to \mathbb{Z}.
\]

1. All values of the degree map are congruent modulo 2.
2. If \(a \in \text{im}(\deg^X)\), then any integer \(c\) with \(-|a| \leq c \leq |a|\) which has the same parity as \(a\) also belongs to \(\text{im}(\deg^X)\).

The second “no gaps” property can be considered as a strong existence result; it shows that the phenomenon exhibited by the theorem of Brockett and Segal ([Sc] p. 41, see also section 2.4.2 below) is completely general.

In the third section we discuss examples. First we describe a large class of important Real complex manifolds, the so called conjugation manifolds [HHP], for which the orientability of \(f(\mathbb{R})\) can be easily checked. This class contains Grassmann manifolds and toric manifolds with their standard Real structures. Then we discuss Wronski projections, the universal pole placement map, and a real subspace problem. Note that in many interesting situations one has a canonical relative orientation of the considered projections. In these situations one obtains canonical signs corresponding to the points of a regular fibre.

1. Degrees of real maps

By topological manifold we always mean a topological manifold which is Hausdorff and paracompact.

Let \(M\) be an \(n\)-dimensional topological manifold, and \(\mathcal{O}_M\) its orientation sheaf; for every point \(m \in M\) we have

\[
\mathcal{O}_{M, m} := H_n(M, M \setminus \{m\}, \mathbb{Z}).
\]

Note that for every locally constant sheaf \(\xi\) on \(M\) and for every open neighborhood \(U\) of \(m\) one has canonical identifications

\[
H_n(M, M \setminus \{m\}, \xi) = H_n(U, U \setminus \{m\}, \xi|_U) = H_n(U, U \setminus \{m\}, \xi_m) = H_n(M, M \setminus \{m\}, \xi_m) = \mathcal{O}_{M, m} \otimes \xi_m,
\]

and

\[
H^n(M, M \setminus \{m\}, \xi) = H^n(U, U \setminus \{m\}, \xi|_U) = H^n(U, U \setminus \{m\}, \xi_m) = H^n(M, M \setminus \{m\}, \xi_m) = \text{Hom}(\mathcal{O}_{M, m}, \xi_m).
\]

Suppose now that \(M\) is connected and closed. The canonical fundamental class of \(M\) in cohomology is the canonical generator \(\{M\}\) of \(H^n(M, \mathcal{O}_M)\); for every \(m \in M\) this class can be written as

\[
\{M\} = j_m(c_m)
\]

where \(j_m : H^*(M, M \setminus \{m\}, \mathcal{O}_M) \to H^*(M, \mathcal{O}_M)\) is the natural morphism, and \(c_m = \text{id}_{\mathcal{O}_{M, m}}\) is the canonical generator of \(H^n(M, M \setminus \{m\}, \mathcal{O}_M)\).

Similarly, the canonical class of \(M\) in homology is the class \(\{M\} \in H_n(M, \mathcal{O}_M)\) whose image in \(H_n(M, M \setminus \{m\}, \mathcal{O}_M)\) is the canonical generator of this group for every \(m \in M\).
Definition 1.1. Let $M, N$ be differentiable manifolds, and $g : M \to N$ a continuous map. A relative orientation of $g$ is an isomorphism $\nu : g^*(\mathcal{O}_N) \to \mathcal{O}_M$. A relatively oriented map is a pair $(g, \nu)$, where $\nu$ is a relative orientation of $g$. A continuous map $g : M \to N$ is called relatively orientable if it admits a relative orientation.

Since $w_1$ classifies real line bundles on paracompact spaces, we obtain:

Remark 1.2. A continuous map $g : M \to N$ between differentiable manifolds is relatively orientable if and only if $g^*(w_1(T_N)) = w_1(T_M)$.

Remark 1.3. If $H^1(M, \mathbb{Z}_2) = 0$, then any continuous map $g : M \to N$ is relatively orientable. In particular any map $g : S^n \to N$ ($n \geq 2$) is relatively orientable.

Note that if $\dim(M) = \dim(N) =: n$ and $(g, \nu)$ is a relatively oriented differentiable map $M \to N$ between closed connected manifolds, one can define the degree $\deg_{\nu}(g)$ by the formula

$$\deg_{\nu}(g)(M) = \nu_*(g^*(\{N\})),$$

where the right hand term is computed with respect to the standard orientation of $\mathbb{R}^n$. This degree can be defined in an intrinsic way using the map

$$g_x : (U, U \setminus \{x\}) \to (N, N \setminus \{y\})$$

(which is well defined when $U$ is sufficiently small) and writing

$$\nu_x(e_xg_x^*(c_y)) = \deg_{\nu,x}(g)c_x,$$

where $e_x : H^n(U, U \setminus \{x\}, \mathcal{O}_U) \to H^n(M, M \setminus \{x\}, \mathcal{O}_y)$ is the isomorphism defined by excision.

Proposition 1.4. Let $M, N$ be closed connected $n$-manifolds, $g : M \to N$ a smooth map, $\nu : g^*(\mathcal{O}_N) \to \mathcal{O}_M$ a relative orientation, and $y \in N$ a point with $g^{-1}(y)$ finite. Then

$$\deg_{\nu}(g) = \sum_{x \in g^{-1}(y)} \deg_{\nu,x}(g).$$

Proof. Write $\{N\} = j_y(c_y)$, where $c_y \in H^n(N, N \setminus \{y\}, \mathcal{O}_N)$ is the canonical generator. One gets a pull-back class

$$g^*(c_y) \in H^n(M, M \setminus g^{-1}(y), g^*(\mathcal{O}_N)),$$

and $g^*(\{N\})$ can be written as

$$g^*(\{N\}) = J_{M,y}(g^*(c_y)),$$

where $J_{M,y} : H^n(M, M \setminus g^{-1}(y), g^*(\mathcal{O}_N)) \to H^n(M, g^*(\mathcal{O}_N))$ is the canonical map.
For every $x \in g^{-1}(y)$ let $U_x$ be an open neighborhood of $x$ such that $U_{x_1} \cap U_{x_2} = \emptyset$ for $x_1 \neq x_2$. Put $U := \bigcup_{x \in g^{-1}(y)} U_x$, denote by $g_x : (U_x, U_x \setminus \{x\}) \to (N, N \setminus \{y\})$ the restriction of $g$, and note that the inclusion $U \to M$ induces an isomorphism

$$H^n(U, U \setminus g^{-1}(y), (g_U)^* (\sigma_N)) \to H^n(M, M \setminus g^{-1}(y), g^* (\sigma_N)),$$

by excision. We denote by

$$J_{U,y} : H^n(U, U \setminus g^{-1}(y), (g_U)^* (\sigma_N)) \to H^n(M, g^* (\sigma_N))$$

the composition of $J_{M,y}$ with this isomorphism. One has an obvious isomorphism

$$H^n(U, U \setminus g^{-1}(y), (g_U)^* (\sigma_N)) = \bigoplus_{x \in g^{-1}(y)} H^n(U_x, U_x \setminus \{x\}, \sigma_y)$$

and the restriction of $J_{U,y}$ to a summand $H^n(U_x, U_x \setminus \{x\}, \sigma_y)$ is the composition $j_x \circ e_x$ of the isomorphism $e_x : H^n(U_x, U_x \setminus \{x\}, \sigma_y) \to H^n(M, M \setminus \{x\}, \sigma_y)$ with the canonical map $J_x : H^*(M, M \setminus \{x\}, \sigma_y) \to H^*(M, g^* (\sigma_N))$. Therefore

$$\nu g^* \{\{N\}\} = \nu J_{U,y}((g_U)^* (\sigma_y)) = \nu J_{U,y} \left( \sum_{x \in g^{-1}(y)} g_x^*(c_x) \right) = \nu \left( \sum_{x \in g^{-1}(y)} J_x \circ e_x(g_x^*(c_x)) \right)$$

$$= \sum_{x \in g^{-1}(y)} \nu \circ J_x \circ e_x(g_x^*(c_x)) = \sum_{x \in g^{-1}(y)} j_x \circ e_x(g_x^*(c_x)) = \sum_{x \in g^{-1}(y)} \deg_{\nu, x}(g) j_x(c_x)$$

$$= \left\{ \sum_{x \in g^{-1}(y)} \deg_{\nu, x}(g) \right\} \{M\} .$$

by (2).

Proposition 1.5. Let $X$, $Y$ be compact complex manifolds endowed with Real structures, let $f : X \to Y$ be a finite Real holomorphic map such that the induced map $f(\mathbb{R}) : X(\mathbb{R}) \to Y(\mathbb{R})$ is relatively oriented. Then

1. For any $y \in Y(\mathbb{R})$ one has

$$|\deg(f(\mathbb{R}))| \leq \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f(\mathbb{R}) \leq \deg(f) .$$

2. $\deg(f(\mathbb{R})) \equiv \deg(f) \mod 2$.

3. For any $y \in Y(\mathbb{R})$ one has

$$\deg(f(\mathbb{R}))) \equiv \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f(\mathbb{R}) \pmod{2} .$$

Proof. Choose a relative orientation $\nu : f(\mathbb{R})^* (\sigma_Y(\mathbb{R})) \to \sigma_X(\mathbb{R})$.

1. For the first inequality note that

$$|\deg(f(\mathbb{R}))) = \sum_{x \in f(\mathbb{R})^{-1}(y)} \deg_{\nu, x}(f(\mathbb{R}))| \leq \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f(\mathbb{R}) =$$

$$\sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f .$$

For the second inequality, note that for any point $y \in Y(\mathbb{R})$, we have

$$\deg(f) = \sum_{x \in f(\mathbb{R})^{-1}(y)} \text{mult}_x f + \sum_{x \in f(\mathbb{R})^{-1}(y) \setminus X(\mathbb{R})} \text{mult}_x f =$$
(4) \[ \sum_{x \in f^{-1}(y)} \operatorname{mult}_x f(R) + \sum_{x \in f^{-1}(y) \setminus X(R)} \operatorname{mult}_x f \]

and all the terms on the right are positive.

(2) Choose a regular value \( y \in Y(R) \) of \( f(R) \). Then one has \( \deg_{\nu,x}(f(R)) \in \{\pm 1\} \) for every \( x \in f^{-1}(y) \) and

\[ \deg(f(R)) = \sum_{x \in f^{-1}(y)} \deg_{\nu,x}(f(R)) \equiv \sum_{x \in f^{-1}(y)} \operatorname{mult}_x f(R) \pmod{2}. \]

It suffices to note that

\[ \operatorname{mult}_x f(R) = \operatorname{mult}_x f, \quad \text{and} \quad \sum_{x \in f^{-1}(y) \setminus X(R)} \operatorname{mult}_x f \equiv 0 \pmod{2}. \]

(3) We use (2) and note that, by (1) one obviously has

\[ \deg(f) \equiv \sum_{x \in f^{-1}(y) - 1} \operatorname{mult}_x f(R) \pmod{2}. \]

Note that the estimate (1) holds without any transversality assumption on \( f \).

A statement similar to Proposition [1.5] holds for Real sections in Real holomorphic vector bundles over compact complex manifolds endowed with Real structures [OT2].

**Proposition 1.6.** Let \( X \) be compact complex manifold endowed with Real structure, and let \( E \to X \) be a Real holomorphic vector bundle over \( X \) with \( \text{rk}(E) = \dim(X) = n \). Suppose that the real vector bundle \( E(R) \to X(R) \) is relatively orientable, and let \( s \) be a Real holomorphic section of \( E \) with finite zero locus \( Z(s) \).

Then

\[ \begin{align*}
|\deg(E(R))| & \leq \sum_{z \in Z(s) \setminus X(R)} \operatorname{mult}_z (s) \leq \langle e_n(E), [X] \rangle \\
|\deg(E(R))| & \equiv \langle e_n(E), [X] \rangle \pmod{2} \\
|\deg(E(R))| & \equiv \sum_{z \in Z(s) \setminus X(R)} \operatorname{mult}_z (s)
\end{align*} \]

## 2. Wall crossing for degrees of real projections

### 2.1. Projections. The wall associated with a submanifold of \( \mathbb{P}(V) \). Let \( V \) be an \( N \)-dimensional real vector space and let \( \lambda_V \) be the tautological line bundle on the projective space \( \mathbb{P}(V) \). By definition, \( \lambda_V \) is a line subbundle of the trivial bundle \( \mathbb{P}(V) \times V \) and the tangent bundle \( T_{\mathbb{P}(V)} \) can be canonically identified with \( \text{Hom}(\lambda_V, \mathbb{P}(V) \times V) = \lambda_V^* \otimes V/\lambda_V \).

Let \( X \subset \mathbb{P}(V) \) be a compact submanifold of dimension \( m < N - 1 \). We denote by \( T_X \) the tangent bundle of \( X \) regarded as a subbundle of the restriction \( T_{\mathbb{P}(V)}|_X \), and by \( V_X, \lambda_V|_X \) the restrictions of the bundles \( V, \lambda_V \) to \( X \). The tensor product \( T_X \otimes \lambda_V|_X \) is a subbundle of the quotient bundle \( V_X/\lambda_V|_X \), so it can be written...
as $Y_X/\lambda_{V,X}$, where $Y_X \subset V_X$ is the subbundle of $V_X$ defined as the preimage of $T_X \otimes \lambda_{V,X}$ under the epimorphism

$$V_X \twoheadrightarrow V_X/\lambda_{V,X}.$$  

With this definition we obtain a canonical identification

$$(5) \quad T_X = \text{Hom}(\lambda_{V,X}, Y_X/\lambda_{V,X}),$$

which will play an important role in the following constructions. Note that the bundle $Y_X$ can be identified with the dual jet bundle $[J^1(\lambda_{V,X})]^\vee$ of $\lambda_{V,X}$.

Let now $W$ be a real vector space of dimension $m + 1$. A morphism $f \in \text{Hom}(V, W)$ defines the central projection

$$\begin{aligned}
\mathcal{I}_f : & \mathbb{P}(V) \setminus \mathbb{P}(\ker(f)) \rightarrow \mathbb{P}(W), \\
& f^{-1}(r_{f,l}) \cap Y_{l,l} = l,
\end{aligned}$$

and an induced linear map

$$q_{f,l} : l \rightarrow f(l),$$

and an induced linear map

$$q_{f,l} : V/l \rightarrow W/f(l).$$

Let $H'$ be a linear complement of the line $f(l)$ in $W$, and note that $H := f^{-1}(H')$ is a linear complement of $l$ in $V$. For $\varphi \in \text{Hom}(l, H)$, the $[f]$-image of the line $l_\varphi := \{ v + \varphi(v) | v \in l \}$ in $\mathbb{P}(W)$ is $\{ f(v) + f(\varphi(v)) | v \in l \}$, which is the graph of $f \circ \varphi \circ r_{f,l}^{-1}$. This shows that via the identifications

$$T_l(\mathbb{P}(V)) = \text{Hom}(l, V/l), \quad T_{f(l)}(\mathbb{P}(W)) = \text{Hom}(f(l), W/f(l))$$

the differential $[f]_{\ast,l}$ at a point $l \in \mathbb{P}(V) \setminus \mathbb{P}(\ker(f))$ is given by

$$(6) \quad ([f]_{\ast,l}(\varphi) = q_{f,l} \circ \varphi \circ r_{f,l}^{-1}.$$}

The differential of the restriction $[f]_X$ is given by the same formula applied to $\varphi \in T_l(X) = \text{Hom}(l, Y_{l,l}).$ More precisely:

**Remark 2.1.** The differential of the restriction $[f]_X$ at $l \in X \setminus \mathbb{P}(\ker(f))$ is given by the formula:

$$(7) \quad ([f]_X)_{\ast,l}(\varphi) = p_{f,l} \circ \varphi \circ r_{f,l}^{-1}.$$}

where $p_{f,l} : Y_{l,l} \rightarrow W/f(l)$ is the morphism induced by $f$.

**Remark 2.2.** Let $l \in X \setminus \mathbb{P}(\ker(f))$. The following conditions are equivalent:

1. $[f]_X$ is a local diffeomorphism at $l$,
2. $f^{-1}(f(l)) \cap Y_l = l$,
3. $\ker(f) \cap Y_l = \{0\}$.

**Proof.** Using Remark 2.1 we see that $([f]_X)_{\ast,l}$ is an isomorphism if and only if the restriction of

$$q_{f,l} : V/l \rightarrow W/f(l)$$

as $Y_X/\lambda_{V,X}$, where $Y_X \subset V_X$ is the subbundle of $V_X$ defined as the preimage of $T_X \otimes \lambda_{V,X}$ under the epimorphism

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With this definition we obtain a canonical identification

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\end{aligned}$$

and an induced linear map

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and an induced linear map

$$q_{f,l} : V/l \rightarrow W/f(l).$$

Let $H'$ be a linear complement of the line $f(l)$ in $W$, and note that $H := f^{-1}(H')$ is a linear complement of $l$ in $V$. For $\varphi \in \text{Hom}(l, H)$, the $[f]$-image of the line $l_\varphi := \{ v + \varphi(v) | v \in l \}$ in $\mathbb{P}(W)$ is $\{ f(v) + f(\varphi(v)) | v \in l \}$, which is the graph of $f \circ \varphi \circ r_{f,l}^{-1}$. This shows that via the identifications

$$T_l(\mathbb{P}(V)) = \text{Hom}(l, V/l), \quad T_{f(l)}(\mathbb{P}(W)) = \text{Hom}(f(l), W/f(l))$$

the differential $[f]_{\ast,l}$ at a point $l \in \mathbb{P}(V) \setminus \mathbb{P}(\ker(f))$ is given by

$$(6) \quad ([f]_{\ast,l}(\varphi) = q_{f,l} \circ \varphi \circ r_{f,l}^{-1}.$$}

The differential of the restriction $[f]_X$ is given by the same formula applied to $\varphi \in T_l(X) = \text{Hom}(l, Y_{l,l}).$ More precisely:

**Remark 2.1.** The differential of the restriction $[f]_X$ at $l \in X \setminus \mathbb{P}(\ker(f))$ is given by the formula:

$$(7) \quad ([f]_X)_{\ast,l}(\varphi) = p_{f,l} \circ \varphi \circ r_{f,l}^{-1}.$$}

where $p_{f,l} : Y_{l,l} \rightarrow W/f(l)$ is the morphism induced by $f$.
to $Y_l/l$ is an isomorphism. Since the kernel of this restriction is $[f^{-1}(f(l)) ∩ Y_l]/l$, we obtain the equivalence $(1) ⇔ (2)$. On the other hand $f^{-1}(f(l)) = l + \ker(f)$, hence $f^{-1}(f(l)) ∩ Y_l = (l + \ker(f)) ∩ Y_l$. The composition
\[
\ker(f) ∩ Y_l \hookrightarrow (l + \ker(f)) ∩ Y_l \to (l + \ker(f)) ∩ Y_l/l
\]
is surjective because $l \subset Y_l$, and it is injective since $l ∩ \ker(f) = \{0\}$. Therefore one has $f^{-1}(f(l)) ∩ Y_l = l$ if and only if $\ker(f) ∩ Y_l = \{0\}$.

In the special case $X ∩ \mathbb{P}(\ker(f)) = \emptyset$ we obtain a well-defined smooth map $[f]_X : X → \mathbb{P}(W)$ between compact manifolds of the same dimension. The main goal of this section is to study the relative orientability of such a map and, in the relatively orientable case, to study the possible degrees of $[f]_X$ for a fixed submanifold $X$.

Since the projection $[f]_X$ is defined on all of $X$ if and only if $f$ belongs to the open subset
\[
\operatorname{Hom}(V, W)_X := \{ f ∈ \operatorname{Hom}(V, W) | X ∩ \mathbb{P}(\ker(f)) = \emptyset \},
\]
it is natural to study the complement of this open set, namely the wall associated with $X$.

**Definition 2.3.** Let $X ⊂ \mathbb{P}(V)$ be a compact $m$-dimensional submanifold and let $W$ be a real vector space of dimension $n := m + 1$. The wall associated with $X$ is defined by
\[
W_X := \{ f ∈ \operatorname{Hom}(V, W) | X ∩ \mathbb{P}(\ker(f)) ≠ \emptyset \}.
\]

A point $f ∈ W_X$ is called regular if the following conditions are satisfied:

1. $\dim(\ker(f)) = N − n$ or, equivalently, $f$ is an epimorphism,
2. The intersection $X ∩ \mathbb{P}(\ker(f))$ has only one point, denoted by $ξ_f$,
3. $\ker(f) ∩ Y_ξ = ξ_f$.

Denote by $W^n_X$ the subspace of regular points of the wall.

The following proposition shows that the wall $W_X ⊂ \operatorname{Hom}(V, W)$ is a smooth hypersurface at any regular point $f_0$ and identifies the normal line of this hypersurface at $f_0$ canonically with a line depending only on the triple $(X, W, ξ_{f_0})$.

**Proposition 2.4.** Let $X ⊂ \mathbb{P}(V)$ be a smooth $m$-dimensional submanifold. Then:

1. The wall $W_X ⊂ \operatorname{Hom}(V, W)$ is a smooth hypersurface at any regular point $f_0 ∈ W_X$.
2. Denoting $X ∩ \mathbb{P}(\ker(f_0)) = \{ξ_0\}$ we have a canonical isomorphism
\[
\psi_{f_0} : N_{W^n_{X,f_0}} → \operatorname{Hom}(\det(ξ_0), \det(W)).
\]

**Proof.** 1. Consider the incidence varieties
\[
\mathcal{J} := \{(f, ξ, K) ∈ \operatorname{Hom}(V, W) × \mathbb{P}(V) × G_{N−n}(V) | ξ ⊂ K ⊂ \ker(f)\}.
\]
\[
\mathcal{J}_X := \{(f, ξ, K) ∈ \operatorname{Hom}(V, W) × X × G_{N−n}(V) | ξ ⊂ K ⊂ \ker(f)\}.
\]
It’s easy to see that $\mathcal{J}$ (respectively $\mathcal{J}_X$) is a vector bundle over a submanifold of $\mathbb{P}(V) × G_{N−n}(V)$ (respectively $X × G_{N−n}(V)$), so it has a natural manifold structure. Let $q : \mathcal{J} → \operatorname{Hom}(V, W)$, $q_X : \mathcal{J}_X → \operatorname{Hom}(V, W)$ be the projections on the first factor, which are obviously proper smooth maps.
Note that

\[(8) \quad W_X = q_X(J_X).\]

One has a canonical identification

\[T_{(f,\xi,K)}(J) = \{(\varphi,\alpha,\beta) \in \text{Hom}(V, W) \times \text{Hom}(\xi, V/\xi) \times \text{Hom}(K, V/K) | \varphi|_K = f \circ \beta, \beta|_K = \bar{\alpha}\},\]

where \(\bar{\alpha}\) denotes the composition of \(\alpha\) with the natural epimorphism \(V/\xi \to V/K\).

Via this identification we have \(q_{*,(f,\xi,K)}(\varphi,\alpha,\beta) = \varphi\), hence

\[\ker(q_{*,(f,\xi,K)}(\varphi,\alpha,\beta)) = \{(\alpha,\beta) \in \text{Hom}(\xi, V/\xi) \times \text{Hom}(K, \ker(f)/K) | \beta|_K = \bar{\alpha}\}.\]

Similarly

\[\ker(q_{*,(f,\xi,K)}(\varphi,\alpha,\beta)) = \{(\alpha,\beta) \in \text{Hom}(\xi, Y/\xi) \times \text{Hom}(K, \ker(f)/K) | \beta|_K = \bar{\alpha}\}.\]

Let now \(f_0 \in W_X^q\) be a regular point on the wall, and put \(K_0 := \ker(f_0), \xi_0 := \xi_{f_0}\). We will show that \(q_X\) is an immersion at \((f_0,\xi_0, K_0)\) as claimed. Since the fibre \(q_X^{-1}(f_0)\) has only one element and \(q_X\) is proper, it follows that \(q_X(U)\) is a neighborhood of \(f_0\) for every neighborhood \(U\) of \((f_0,\xi_0, K_0)\) in \(J_X\). This implies that the image \(W_X = q_X(J_X)\) is a submanifold of \(\text{Hom}(V, W)\) at \(f_0\) whose germ at \(f_0\) can be identified with the germ at \((f_0,\xi_0, K_0)\) of \(J_X\). A simple dimension count shows that \(W_X\) is a hypersurface at \(f_0\).

2. For the second statement of the proposition we will construct a canonical isomorphism

\[\psi_{f_0} : N_{W_X^q,f_0} \to \text{Hom}(\det(Y_{f_0}), \det(W))\]

as the composition \(u_{f_0} \circ a_{f_0}\) of two canonical isomorphisms:

\[a_{f_0} : N_{W_X^q,f_0} \to \text{Hom}(\xi_0, W/\xi_{f_0}(Y_{f_0}))\]

\[u_{f_0} : \text{Hom}(\xi_0, W/\xi_{f_0}(Y_{f_0})) \to \text{Hom}(\det(Y_{f_0}), \det(W)).\]

We define first

\[A_{f_0} : \text{Hom}(V, W) = T_{f_0} \text{Hom}(V, W) \to \text{Hom}(\xi_0, W/\xi_{f_0}(Y_{f_0}))\]

by

\[(9) \quad A_{f_0}(\varphi) := \varphi|_{\xi_0} \mod f_0(Y_{f_0}).\]

It is easy to see that \(A_{f_0}(\varphi) = 0\) if and only if there exists

\[(\alpha,\beta) \in \text{Hom}(\xi_0, Y_{f_0}/\xi_0) \times \text{Hom}(K_0, V/K_0)\]

such that \((\varphi,\alpha,\beta) \in T_{(f_0,\xi_0,K_0)}(J_X)\), i.e., if and only if \(\varphi \in T_{f_0}(W_X^q)\). Hence \(A_{f_0}\) induces a canonical isomorphism \(a_{f_0} : N_{W_X^q,f_0} \to \text{Hom}(\xi_0, W/\xi_{f_0}(Y_{f_0}))\) as claimed.

For the construction of \(u_{f_0}\) we use the exact sequences:

\[0 \to \xi_0 \hookrightarrow Y_{f_0} \to Y_{f_0}/\xi_0 \to 0,\]
which give standard isomorphisms
\begin{align}
\det(Y_{\xi_0}) &= \xi_0 \otimes \det(Y_{\xi_0} / \xi_0), \\
\det(W) &= \det(f_0(Y_{\xi_0})) \otimes W / f_0(Y_{\xi_0})
\end{align}

Note that one has a second canonical (but non-standard!) isomorphism
\begin{equation}
\det(W) = W / f_0(Y_{\xi_0}) \otimes \det(f_0(Y_{\xi_0}))
\end{equation}
defined by \([w] \otimes \delta \mapsto w \wedge \delta\) (see Remark 2.5 below), which is more convenient in our situation, because the maps we consider here relate the factors of the tensor products in (10), (12) respecting the order.

The isomorphism \(u_{f_0} : \Hom(Y_{\xi_0}, W / f_0(Y_{\xi_0})) \to \Hom(\det(Y_{\xi_0}), \det(W))\) is defined via the isomorphisms (10) and (12) by
\begin{equation}
u_{f_0}(\sigma) = \sigma \otimes \det(\bar{f}_0),\end{equation}where \(\bar{f}_0 : Y_{\xi_0} / \xi_0 \to f_0(Y_{\xi_0})\) is the isomorphism induced by \(f_0\).

Note that if one uses the standard isomorphism (11) for \(\det(W)\) the corresponding formula for \(u_{f_0}\) would be
\[
u_{f_0}(\sigma)(v \otimes \delta) := (-1)^m(\sigma(v) \otimes [\det(\bar{f}_0)(\delta)]).
\]

**Remark 2.5.** Let \(C\) be a finite dimensional real vector space. For every subspace \(A \subset C\) we define the isomorphisms
\[
u_A : \det(A) \otimes \det(C/A) \cong \det(C), \quad \nu_B : \det(C/A) \otimes \det(A) \cong \det(C)
\]
by
\[
u_A(\delta \otimes [c]) = \delta \wedge c, \quad \nu_B([c] \otimes \delta) = c \wedge \delta.
\]

(1) Let 
\[
p : \det(A) \otimes \det(C/A) \to \det(C/A) \otimes \det(A)
\]
be the obvious isomorphism defined by permutation of the factors, and put \(a := \dim(A), c := \dim(C)\). Then one has
\[
u_A^{-1} \circ u_A = (-1)^a(v^{-1} \circ u_B)\cdot\]

(2) Suppose \(C\) has an internal direct sum decomposition \(C = A \oplus B\), and let \(\alpha : A \cong C/B, \beta : B \cong C/A\) be the obvious isomorphisms. Then one has
\[
\det(\alpha) \otimes \det(\beta)^{-1} = \nu_B^{-1} \circ u_A.
\]

In other words, via the isomorphisms \(u_A, v_B\) the tensor product \(\det(\alpha) \otimes \det(\beta)^{-1}\) induces \(\id_{\det(C)}\).
2.2. **Relative orientations of central projections.** Our next goal is to describe explicitly the set of relative orientations of a map \([f]_X\) associated with a morphism \(f \in \text{Hom}(V,W)_X\). We will obtain an important (and surprising) result: relative orientability of a map \([f]_X\) depends only on the embedding \(X \subset \mathbb{P}(V)\), and the set of relative orientations of \([f]_X\) can be canonically identified with a set which is intrinsically associated with this embedding, and is independent of the choice of \(f \in \text{Hom}(V,W)\). First we give a simple description of the line bundle \([f]^*(\det(T_{\mathbb{P}(W)}))\) on \(\mathbb{P}(V) \setminus \mathbb{P}(\ker(f))\):

Let again \(\lambda_V\) (respectively \(\lambda_W\)) be the tautological line bundle on \(\mathbb{P}(V)\) (respectively \(\mathbb{P}(W)\)). The family of isomorphisms 

\[
(r_{f,l} : l \rightarrow f(l))_{l \in \mathbb{P}(V) \setminus \mathbb{P}(\ker(f))}
\]
defines a line bundle isomorphism 

\[
r_f : \lambda_V|_{\mathbb{P}(V) \setminus \mathbb{P}(\ker(f))} \xrightarrow{\approx} [f]^*(\lambda_W).
\]

Using the canonical isomorphism 

\[
\det(T_{\mathbb{P}(W)}) = [\wedge^n \lambda^\vee W] \otimes \det(W),
\]
we obtain a canonical isomorphism

\[
(14) [f]^*(\det(T_{\mathbb{P}(W)})) \xrightarrow{(r_f \otimes \text{id}) \otimes \text{id}} [\wedge^n \lambda^\vee V|_{\mathbb{P}(V) \setminus \mathbb{P}(\ker(f))}] \otimes \det(W).
\]

We can prove now

**Lemma 2.6.** Let \(f \in \text{Hom}(V,W)_X\) and let \([f]_X : X \rightarrow \mathbb{P}(W)\) be the projection induced by \(f\). Then:

1. There is a canonical isomorphism
   \[
   [f]_X^*(\det(T_{\mathbb{P}(W)})) = [\wedge^n \lambda^\vee_{V,X}] \otimes \det(W).
   \]
2. The isomorphism class of the line bundle \([f]_X^*(\det(T_{\mathbb{P}(W)}))\) on \(X\) is independent of \(f \in \text{Hom}(V,W)_X\).
3. The restriction \([f]_X : X \rightarrow \mathbb{P}(W)\) is relatively orientable if and only if \(w_1(X) = n\{w_1(\lambda_V)|_X\}\).
4. The data of an isomorphism
   \[
   \nu : [f]_X^*(\det(T_{\mathbb{P}(W)})) \rightarrow \det(T_X)
   \]
   is equivalent to the data of a line bundle isomorphism
   \[
   \mu : X \times \det(W) \xrightarrow{\approx} \det(Y_X),
   \]
   hence to the data of a global trivialization of \(\det(Y_X)\) with fibre \(\det(W)\).

**Proof.** The first statement follows by restricting the isomorphism \([14]\) to \(X\). The statements (2) and (3) are direct consequences of (1) whereas (4) follows from (1) and the canonical isomorphism

\[
\det(T_X) = [\wedge^n \lambda^\vee_{V,X}] \otimes \det(Y_X)
\]
induced by \([8]\).

This proves the following important
Proposition 2.7. Relative orientability of a map \([f]_X : X \to \mathbb{P}(W)\) defined by \(f \in \text{Hom}(V,W)_X\) depends only on the embedding \(X \subset \mathbb{P}(V)\), and is independent of \(f\). The set of relative orientations of such a map \([f]_X : X \to \mathbb{P}(W)\) can be canonically identified with the set of connected components of the space of line bundle isomorphisms
\[
\mu : X \times \text{det}(W) \xrightarrow{\sim} \text{det}(Y)_X,
\]
hence this set is independent of \(f\).

Definition 2.8. A bundle isomorphism \(\mu : X \times \text{det}(W) \xrightarrow{\sim} \text{det}(Y)_X\) will be called orientation parameter for projections \(X \to \mathbb{P}(W)\). Given \(f \in \text{Hom}(V,W)_X\) we denote by
\[
\nu(f,\mu) := [(r^+_f)^{\otimes n} \otimes \mu]
\]
the relative orientation of \([f]_X\) associated with the orientation parameter \(\mu\).

Therefore for every orientation parameter \(\mu : X \times \text{det}(W) \xrightarrow{\sim} \text{det}(Y)_X\) we obtain a well defined map
\[
\deg_X^\mu : \text{Hom}(V,W)_X = \text{Hom}(V,W) \setminus \mathcal{W}_X \to \mathbb{Z},
\]
which obviously descends to a map (denoted by the same symbol):
\[
\deg_X^\mu : \pi_0(\text{Hom}(V,W) \setminus \mathcal{W}_X) \to \mathbb{Z}
\]

Inspired by the terminology used in gauge theory we introduce the following

Definition 2.9. The connected components of \(\text{Hom}(V,W) \setminus \mathcal{W}_X\) will be called chambers.

In the next section we will focus on the following problem: how can one compare the values of the degree map on different chambers. The first step will be a wall-crossing formula which computes the jump of the degree map when one crosses the wall transversally at a regular point. The main ingredient in proving this wall-crossing formula is the following remark which states that, for a given point \(\xi_0 \in X\), the set of isomorphisms \(\text{det}(Y_{\xi_0}) \to \text{det}(W)\) has two (radically different) geometric interpretations:

Remark 2.10. Fix \(\xi_0 \in X\). The orientation parameter \(\mu\) defines an isomorphism \(\mu_{\xi_0}^{-1} : \text{det}(Y_{\xi_0}) \to \text{det}(W)\), which (by Proposition 2.4) also defines a generator of the normal line \(N_{\mathcal{W}_X,f}\) for every \(f \in \mathcal{W}_X^0\) for which \(\xi_f = \xi_0\).

This important remark will play a crucial role in the following section.
By Remark 2.1 the differential
\[ [f]_{*,\xi_0} : T_{\xi_0}(X) = \text{Hom}(\xi_0, Y_{\xi_0}/\xi_0) \to \text{Hom}(f(\xi_0), W/f(\xi_0)) = T_{f(\xi_0)}\mathbb{P}(W) \]
is given by
\[ [f]_{*,\xi_0}(\varphi) = p_{f,\xi_0} \circ \varphi \circ r_{f,\xi_0}^{-1}, \]
where \( r_{f,\xi_0} : \xi_0 \to f(\xi_0), \) \( p_{f,\xi_0} : Y_{\xi_0}/\xi_0 \to W/f(\xi_0) \) are the obvious isomorphisms induced by \( f \). Therefore

**Lemma 2.11.** Let \( \mu \) be an orientation parameter, and suppose that \([f]_X\) is well-defined and a local isomorphism at \( \xi_0 \in X \). Via the isomorphisms (2.3) the local degree \( \deg_{\nu(f,\mu),\xi_0}([f]_X) \) is given by
\[ \deg_{\nu(f,\mu),\xi_0}([f]_X) = \text{sign}(\mu_{\xi_0} \circ (r_{f,\xi_0} \otimes \det(p_{f,\xi_0}))) \].

We will also need the following simple remark concerning the functoriality of the degree map with respect to isomorphisms \( \Phi : W \to W' \).

**Remark 2.12.** Let \( \mu : X \times \text{det}(W) \to \text{det}(Y_X) \) be an orientation parameter, \( f \in \text{Hom}(V,W)_X \) and \( \Phi : W \to W' \) a vector space isomorphism. Then
\[ \deg^X_{\nu(f,\mu),\det(\Phi)}([\Phi \circ f]_X) = \deg^X_{\nu(f,\mu)}([f]_X). \]
In particular, if \( W = W' \) one has:
\[ \deg^X_{\nu(f,\mu)}([\Phi \circ f]_X) = \text{sign}(\det(\Phi))\deg^X_{\nu(f,\mu)}([f]_X). \]

We end this section with an example.

**Example:** (Veronese maps) Let \( W \) be a real vector space of dimension \( n \geq 2 \), and let \( g : \mathbb{P}(W) \to \mathbb{P}(W) \) be a regular real algebraic map. Such a map factorizes as \( g = [f] \circ v_d \), where
\[ v_d : \mathbb{P}(W) \to \mathbb{P}(S^d W) \]
is the Veronese map of degree \( d \), and \( f \in \text{Hom}(S^d(W), W) \) with \( \mathbb{P}(\ker(f)) \cap \text{im}(v_d) = \emptyset \). The positive integer \( d \) is determined by \( g \) and will be called the algebraic degree of \( g \). Applying Lemma 2.6 to the image \( X := v_d(\mathbb{P}(W)) \) and noting that \( v_d(\lambda_{S^d(W)}) = \lambda_{S^d(W)} \), one obtains:
\[ \det(TP(W)) \wedge g^*(\det(TP(W))) = \lambda_{S^d(W)}^{\wedge n(1-d)} \]
This proves the following simple, but interesting, result:

**Proposition 2.13.** A regular real algebraic map \( g : \mathbb{P}(W) \to \mathbb{P}(W) \) of algebraic degree \( d \) is relatively orientable if and only if \( \dim(W)(1-d) \) is even, and in this case it is canonically relatively oriented.

In the case when \( \dim(W)(1-d) \) is even, it is an interesting problem to determine the possible degrees of such regular real algebraic maps with respect to this canonical relative orientation. The case \( \dim(W) = 2 \) is known by the work of Brockett (see [Se] p. 41 or [By] Theorem 2.1) and will be described in detail at the end of section 2.3.
2.3. Wall crossing jump. Suppose now that the condition in the third statement of Lemma 2.6 is satisfied, and fix an isomorphism

\[ \mu : X \times \text{det}(W) \to \text{det}(Y_X). \]

We are interested in the map \( \deg_\mu^X : \pi_0(\text{Hom}(V, W)_X) \to \mathbb{Z} \) defined by

\[ \deg_\mu^X(f) := \deg_{\nu(f, \mu)}([f]_X). \]

Note that \([f] \) can be written as the composition

\[ X \hookrightarrow \mathbb{P}(V) \setminus \mathbb{P}(\ker(f)) \to \mathbb{P}(\text{im}(f)) \to \mathbb{P}(W), \]

where the central map is the projection of \( \mathbb{P}(W) \) onto \( \mathbb{P}(\text{im}(f)) \) with center \( \mathbb{P}(\ker(f)) \).

In other words, we are interested in the variation of the degree of central projections \( \text{deg}_{\nu(f, \mu)}([f]_X) \) as \( f \) varies on a curve which crosses the wall transversally at a regular point.

**Theorem 2.14.** (Wall-crossing formula) Let \( f_0 \in \mathcal{W}_X^0 \) be a regular point on the wall, put \( \xi_0 := \xi_{f_0}, N_0 := N_{\mathcal{W}_X^0, f_0} \), and let

\[ \hat{f} = (f_\tau)_{\tau \in N_0} : N_0 \to \text{Hom}(V, W) \]

be a smooth map such that the differential \( f_{*, 0} \) is a right splitting of the exact sequence

\[ 0 \to T_{\mathcal{W}_X^0, f_0} \to T_{\text{Hom}(V, W), f_0} = \text{Hom}(V, W) \to N_0 \to 0. \]

Then for every sufficiently small \( \tau \in N_0 \setminus \{0\} \) we have:

1. \( f_\tau \in \text{Hom}(V, W)_X \) and \([f_\tau]_X\) is a local diffeomorphism at \( \xi_0 \),
2. \( \deg_{\nu(f_\tau, \mu)}([f_\tau]_X) = \text{sign}(\mu_{\xi_0} \circ (\psi_{f_0}(\tau))) \),
3. \( \deg_{\nu(f_\tau, \mu)}([f_\tau]_X) - \deg_{\nu(f_{\tau - \tau}, \mu)}([f_{\tau - \tau}]_X) = 2\text{sign}(\mu_{\xi_0} \circ (\psi_{f_0}(\tau))). \)

**Proof.** (1) The fact that \( f_\tau \in \text{Hom}(V, W)_X \) for sufficiently small \( \tau \in N_0 \setminus \{0\} \) follows directly from Proposition 2.3, taking into account that the complement \( \mathcal{W}_X = \text{Hom}(V, W)_X \setminus \text{Hom}(V, W)_X \) is a smooth hypersurface at \( f_0 \) and the curve \((f_\tau)_{\tau \in N_0}\) is transversal to this hypersurface at \( f_0 \). In order to prove that \([f_\tau]_X\) is a local diffeomorphism at \( \xi_0 \) we use Remark 2.2. We have to show that \( \ker(f_\tau) \cap Y_{\xi_0} = \{0\} \).

Note that, in general, for two finite dimensional real vector spaces \( A, B \) of the same dimension, the closed subset

\[ \mathcal{W}(A, B) := \{ s \in \text{Hom}(A, B) \mid \ker(s) \neq \{0\} \} \]

of \( \text{Hom}(A, B) \) is a smooth hypersurface at any point \( s_0 \) with \( \dim(\ker(s_0)) = 1 \), and the tangent space at such a point is

\[ T_{s_0}\mathcal{W}(A, B) = \{ \sigma \in \text{Hom}(A, B) \mid \sigma(\ker(s_0)) \subset s_0(A) \}. \]

Therefore a tangent vector \( \sigma \in T_{s_0}(\text{Hom}(A, B)) \) is transversal to \( \mathcal{W}(A, B) \) at \( s_0 \) if and only if the linear map \( \ker(s_0) \to B / s_0(A) \) induced by \( \sigma|_{\ker(s_0)} \) is an isomorphism.

Using this remark we see that the map \( \hat{f} = (\hat{f}_\tau)_{\tau} : N_0 \to \text{Hom}(Y_{\xi_0}, W) \) given by \( \hat{f}_\tau = f_\tau|_{Y_{\xi_0}} \) is transversal to \( \mathcal{W}(Y_{\xi_0}, W) \) at \( f_0 \). Indeed, using the notations introduced in the proof of Proposition 2.3, the map

\[ \ker \hat{f}_0 = \xi_0 \to W / f_0(Y_{\xi_0}) \]
induced by \( \tilde{f}_{\ast,0}(\tau) \) is precisely \( A_{f_\ast}(f_{\ast,0}(\tau)) \) by the definition of \( A_{f_\ast} \). On the other hand, since \( f_{\ast,0} \) is a right inverse of the canonical projection \( \text{Hom}(V, W) \to N_0 \), we see that \( A_{f_\ast}(f_{\ast,0}(\tau)) = a_{f_\ast}(\tau) \), which is nonzero for \( \tau \in N_0 \setminus \{0\} \) because \( a_{f_\ast} \) is an isomorphism by Proposition 2.4.

(2) We suppose first that \( f \) is an affine map, so it has the form

\[
f_\tau = f_0 + \phi(\tau),
\]

where \( \phi : N_0 \to \text{Hom}(V, W) \) is a linear map (which coincides with the differential \( f_{\ast,0} \)). As we have seen in the proof of (1) our assumption about the differential \( f_{\ast,0} \) implies that \( \phi(\tau) \) is a lift of \( \tau \), so that \( a_{f_\ast}(\tau) = A_{f_\ast}(\phi(\tau)) \). Therefore for \( \tau \neq 0 \) the morphism

\[
a_{f_\ast}(\tau) = A_{f_\ast}(\phi(\tau)) = \tilde{\phi}(\tau)|_{\xi_0} : \xi_0 \to W/f_0(Y_{\xi_0})
\]

induced by \( \phi(\tau) \) is an isomorphism. For every \( \tau \in N_0 \setminus \{0\} \) we obtain a direct sum decomposition

\[
W = \phi(\tau)(\xi_0) \oplus f_0(Y_{\xi_0}),
\]

which is independent of \( \tau \), since \( N_0 \) is 1-dimensional. Now fix \( \tau \in N_0 \setminus \{0\} \). We have an obvious commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & \xi_0 & \longrightarrow & Y_{\xi_0} & \longrightarrow & Y_{\xi_0}/\xi_0 & \longrightarrow & 0 \\
& & \downarrow{r_{\phi(\tau),\xi_0}} & & \downarrow{f_\tau|_{Y_{\xi_0}}} & & \downarrow{p_0 + \phi(\tau)_0} & & \\
0 & \longrightarrow & \xi_0 & \longrightarrow & W & \longrightarrow & W/\phi(\tau)(\xi_0) & \longrightarrow & 0,
\end{array}
\]

where

\[
p_0 : Y_{\xi_0}/\xi_0 \to W/\phi(\tau)(\xi_0), \quad \phi(\tau)_0 : Y_{\xi_0}/\xi_0 \to W/\phi(\tau)(\xi_0)
\]

are the linear maps induced by \( f_0 \), and \( \phi(\tau) \) respectively. Note that \( p_0 \) is an isomorphism, because it can be written as a composition of isomorphisms:

\[
Y_{\xi_0}/\xi_0 \to f_0(Y_{\xi_0}) \to W/\phi(\tau)(\xi_0).
\]

The diagram \[19\] shows that

(a) \( r_{f_\tau,\xi_0} = r_{\phi(\tau),\xi_0} \) for every sufficiently small \( \tau \in N_0 \setminus \{0\} \),

(b) \( p_{f_\tau,\xi_0} = p_0 + \phi(\tau)_0 \) for every sufficiently small \( \tau \in N_0 \setminus \{0\} \).

Using formula \[16\] of Lemma 2.11 and taking into account (a), (b) we see that, via the canonical isomorphisms

\[
det(Y_{\xi_0}) = \xi_0 \otimes det(Y_{\xi_0}/\xi_0),
\]

\[
det(W) = \phi(\tau)(\xi_0) \otimes det(W/\phi(\tau)(\xi_0)),
\]

we have for every sufficiently small \( \tau \in N_0 \setminus \{0\} \)

\[
\deg_{\nu(f_\tau,\mu),\xi_0}([f_\tau]_{\nu}) = \text{sign}(\mu_{\xi_0} \circ (r_{\phi(\tau),\xi_0} \otimes det(p_0 + \phi(\tau)_0)))
\]

\[
= \text{sign}(\mu_{\xi_0} \circ (r_{\phi(\tau),\xi_0} \otimes det(p_0))).
\]
For the last equality we used \( \lim_{t \to 0} (p_0 + t \phi(\tau)_0) = p_0 \) and the continuity of the determinant. Now we use the canonical isomorphism

\[
v : W/f_0(Y_{\xi_0}) \otimes \det(f_0(Y_{\xi_0})) \to \det(W)
\]

and we apply Remark 2.5 to the subspaces \( A := \phi(\tau)(\xi_0) \) (for \( \tau \neq 0 \), \( B := f_0(Y_{\xi_0}) \) of \( W \). Therefore let

\[
\alpha : \phi(\tau)(\xi_0) \to W/f_0(Y_{\xi_0}) \quad \beta : f_0(Y_{\xi_0}) \to W/\phi(\tau)(\xi_0)
\]

be the isomorphisms associated with the direct sum decomposition \([W]\). Using the second statement of Remark 2.5 we see that the equality (2.3) remains true if we replace \( r_{\varphi,\xi_0} \) by

\[
\alpha \circ r_{\varphi(\tau),\xi_0} : \xi_0 \to W/f_0(Y_{\xi_0})
\]

and \( p_0 \) by

\[
\beta^{-1} \circ p_0 : Y_{\xi_0}/\xi_0 \to f_0(Y_{\xi_0})
\]

But

\[
\alpha \circ r_{\varphi(\tau),\xi_0} = \tilde{\phi}(\tau) \mid_{\xi_0} : \xi_0 \to Y_{\xi_0}/\xi_0 \quad \beta^{-1} \circ p_0 = \tilde{f}_0 : Y_{\xi_0}/\xi_0 \to f_0(Y_{\xi_0})
\]

Therefore

\[
(21) \quad \deg_{\varphi(f_{\tau},\mu),\xi_0}([f_{\tau}]_{X}) = \text{sign}(\mu_{\xi_0} \circ (\tilde{\phi}(\tau) \mid_{\xi_0} \otimes \det(\tilde{f}_0))) \forall \tau \in N_0 \setminus \{0\}
\]

Now recall that \( \tilde{\phi}(\tau) \mid_{\xi_0} = a_{f_0}(\tau) \) and that

\[
a_{f_0}(\tau) \otimes \det(\tilde{f}_0) = u_{f_0}(a_{f_0}(\tau)) = \psi_{f_0}(\tau)
\]

by the definitions of \( u_{f_0} \) and \( \psi_{f_0} \). This proves the claim in the case of an affine map \( f \).

In order to prove the statement for a general map \( f \) note that the space \( \mathcal{F}_{f_0} \) of maps \( f : N_0 \to \text{Hom}(V,W) \) satisfying the hypotheses of the theorem is a closed affine subspace of the Fréchet space \( \mathcal{C}^{\infty}(N_0,\text{Hom}(V,W)) \). Fix a norm on the line \( N_0 \). For a bounded (with respect to the \( \mathcal{C}^{\infty} \)-topology) subset \( K \subset \mathcal{F}_{f_0} \) we can find \( \varepsilon > 0 \) such that \([f_{\tau}]_{X}\) is defined and is a local diffeomorphism at \( \xi_0 \) for every \( f \in K \) and every \( \tau \in N_0 \setminus \{0\} \) with \( ||\tau|| < \varepsilon \). This shows that the map

\[
f \mapsto \deg_{\varphi(f_{\tau},\mu),\xi_0}([f_{\tau}]_{X}) \quad \text{for small } \tau \in N_0 \setminus \{0\}
\]

is locally constant on \( \mathcal{F}_{f_0} \). But this space is connected and contains affine maps.

(3) Note first that it is sufficient to prove the claimed formula for a special map \( f : N_0 \to \text{Hom}(V,W) \) satisfying the hypothesis of the theorem. This is the case since, for two such maps \( f, g \), the points \( f(\tau), g(\tau) \) belong to the same chamber (see Definition 2.9) for any sufficiently small \( \tau \in N_0 \setminus \{0\} \).

We will construct a special map \( f : N_0 \to \text{Hom}(V,W) \) which satisfies the hypotheses of the theorem, is affine, and has the following remarkable property:

**P.** There exists \( \xi_0 \in \mathbb{P}(W) \) such that the subspace \( L_0 := f^{-1}_\tau(\xi_0) \) is independent of \( \tau \in N_0 \) and \( \mathbb{P}(L_0) \) is transversal to \( X \) at any intersection point.

The existence of such a map solves our problem. Indeed, since \( X \) is compact and \( \mathbb{P}(L_0) \) is transversal to \( X \) at any intersection point, the intersection \( F := \mathbb{P}(L_0) \cap X \)
is finite. By (1) we know that, for every sufficiently small \( \tau \in N_0 \setminus \{0\} \), the map \( [f_\tau]_X \) is defined on all of \( X \). On the other hand our transversality condition implies that \( \zeta_0 \) is a regular value of these maps.

Taking \( \tau = 0 \) in the first condition of \( P \) we see that \( \xi_0 \in F \). Applying Proposition 1.4 to \( [f_\tau]_X \) for sufficiently small \( \tau \in N_0 \setminus \{0\} \), we obtain

\[
\deg_{\nu(f_\tau, \mu)}([f_\tau]_X) = \deg_{\nu(f_\tau, \mu), \xi_0}([f_\tau]_X) + \sum_{\xi \in P \setminus \{\xi_0\}} \deg_{\nu(f_\tau, \mu), \xi}([f_\tau]_X).
\]

Since the second term is obviously independent of \( \tau \neq 0 \) we get

\[
\deg_{\nu(f_\tau, \mu)}([f_\tau]_X) - \deg_{\nu(f_\tau, \mu), \xi_0}([f_\tau]_X) = \deg_{\nu(f_\tau, \mu)}([f_\tau]_X) - \deg_{\nu(f_\tau, \mu), \xi_0}([f_\tau]_X),
\]

so the result follows from (2).

We conclude the proof of the theorem with the construction of an affine map \( f = (f_\tau)_{\tau \in N_0} \) which satisfies the hypothesis of the theorem and has the property \( P \).

Put \( K_0 := \ker(f_0) \). Since \( f_0 \) is an epimorphism, we have \( \dim(K_0) = N - n \) and the space \( L_{K_0} \) of \((N - n + 1)\)-dimensional linear subspaces \( L \subset V \) with \( K_0 \subset L \) can be identified with \( \mathbb{P}(W) \) via \( f_0 \). The subset

\[
L_0 := \{ L \in L_{K_0} \mid L \cap Y_{\xi_0} = \xi_0 \}
\]

is non-empty and Zariski open, hence open and dense in \( L_{K_0} \). Let \( L_0 \in L_0 \) an element which corresponds to a regular value \( \zeta_0 \) of the projection

\[
[f_0]_X : X \setminus \{\xi_0\} \to \mathbb{P}(W).
\]

The existence of such a point follows from Sard’s theorem.

Note that \( \mathbb{P}(L_0) \) is transversal to \( X \) at any intersection point \( \xi \in \mathbb{P}(L_0) \cap X \). Indeed, the condition \( L_0 \cap Y_{\xi_0} = \xi_0 \) implies that \( \mathbb{P}(L_0) \) is transversal to \( X \) at \( \xi_0 \), whereas the condition that \( \zeta_0 = f_0(L_0) \) is a regular value of \( [f_0]_X : X \setminus \{\xi_0\} \to \mathbb{P}(W) \) implies that \( \mathbb{P}(L_0) \) is transversal to \( X \) at any point \( \xi \in \mathbb{P}(L_0) \cap X \setminus \{\xi_0\} \).

Therefore the obtained \((N - n + 1)\)-dimensional subspace \( L_0 \) has the properties:

(a) \( K_0 \subset L_0 \),
(b) \( L_0 \cap Y_{\xi_0} = \xi_0 \),
(c) \( \mathbb{P}(L_0) \) is transversal to \( X \) at any intersection point.

Now we choose a complement for each of the three inclusions in the chain

\[
\xi_0 \subset K_0 \subset L_0 \subset V.
\]

Let \( U_0 \) be an arbitrary complement of \( \xi_0 \) in \( K_0 \), \( l_0 \) an arbitrary complement of \( K_0 \) in \( L_0 \), and \( M_0 \) a complement of \( L_0 \) in \( V \) which is contained in \( Y_{\xi_0} \). The latter complement exists, because by (b) any complement of \( \xi_0 \) in \( Y_{\xi_0} \) is also a complement of \( L_0 \) in \( V \).

We have \( \dim(U_0) = N - n - 1 \), \( \dim(l_0) = 1 \), \( \dim(M_0) = n - 1 \). The sum \( l_0 + M_0 \) is a complement of \( K_0 \) in \( V \), hence \( f_0 \) induces an isomorphism \( l_0 + M_0 \to W \). We obtain an induced internal direct sum decomposition of \( W = \zeta_0 \oplus W_0 \) with

\[
\zeta_0 := f_0(l_0) = f_0(L_0), \quad W_0 := f_0(M_0) = f_0(Y_{\xi_0}).
\]

With respect to the internal direct sum decompositions

\[
V = \xi_0 \oplus U_0 \oplus l_0 \oplus M_0, \quad W = \zeta_0 \oplus W_0,
\]
the map \( f_0 \) is given by a matrix of the form
\[
\begin{pmatrix}
0 & 0 & g_0 & 0 \\
0 & 0 & 0 & h_0
\end{pmatrix},
\]
where \( g_0 : k \to \xi_0 \), \( h_0 : M_0 \to W_0 \) are the isomorphisms induced by \( f_0 \). We denote by \( r_\tau : \xi_0 \to \xi_0 \) the morphism defined as the image of \( \tau \) under the composition
\[ N_0 \xrightarrow{a_\xi_0} \text{Hom}(\xi_0, W/f_0(Y_{\xi_0})) = \text{Hom}(\xi_0, W/W_0) \xrightarrow{\sim} \text{Hom}(\xi_0, \xi_0), \]
and we define
\[ f_\tau := \begin{pmatrix} r_\tau & 0 & g_0 & 0 \\
0 & 0 & 0 & h_0
\end{pmatrix}. \]
The map \( \tilde{f} = (f_\tau)_\tau : N_0 \to \text{Hom}(V, W) \) is affine and satisfies the hypothesis of the theorem (because \( A_{f_0} \circ f_\tau \) coincides with \( a_{f_0} \) by definition of \( r_\tau \)). Moreover, since \( h_0 \) is an isomorphism, for every \( \tau \in N_0 \) the subspace \( f_\tau^{-1}(\xi_0) \) coincides with \( L_0 \). Taking into account \( [b] \) we see that \( \tilde{f} \) satisfies property \( P \), which concludes the proof.

**Corollary 2.15.** Let \( f_0 \in W_X^0 \) be a regular point on the wall and let
\[ \gamma = (g_t)_t : (-\varepsilon, \varepsilon) \to \text{Hom}(V, W) \]
be a smooth path such that \( \gamma(0) = f_0 \) and the image \([\gamma(0)]\) of the velocity vector in \( N_{W_X^0,f_0} \) is non-zero. Then for every sufficiently small \( t \in (-\varepsilon, \varepsilon) \) we have:
\begin{enumerate}
\item \( g_t \in \text{Hom}(V, W)_X \) and \([g_t]_X\) is a local diffeomorphism at \( \xi_{f_0} \),
\item \( \text{deg}_{\nu(g_t, \mu)}(\xi_0 ) = \text{sign}(t) \text{sign}(\mu_{\xi_0} \circ (\psi_{f_0}(\gamma(0)))) \),
\item \( \text{deg}_{\nu(g_t, \mu)}([g_t]_X) - \text{deg}_{\nu(g_{-t}, \mu)}([g_{-t}]_X) = 2\text{sign}(t) \text{sign}(\mu_{\xi_0} \circ (\psi_{f_0}(\gamma(0)))) \).
\end{enumerate}

**Proof.** We may suppose that \( \gamma \) is given by \( g_t = f_{\tau_0} \), where \( \tau_0 \in N_{W_X^0,f_0} \setminus \{0\} \) and \( \tilde{f} : N_{W_X^0,f_0} \to \text{Hom}(V, W) \) is a map satisfying the hypothesis of Theorem 2.14.
This implies the following general difference formula for paths which cross the wall transversally in regular points:

**Theorem 2.16.** (difference formula) Let \( \gamma = (g_t)_t : [0,1] \to \text{Hom}(V,W) \) be a smooth path such that

- (1) \( g_0, g_1 \in \text{Hom}(V,W)_X \),
- (2) \( \text{im}(\gamma) \) intersects the wall \( W_X \) only in regular points,
- (3) \( \gamma \) is transversal to \( W^0_X \).

Let \( \gamma^{-1}(W_X) = \{t_1, \ldots, t_k\} \). Then one has

\[
\deg_{\nu(g_t,\nu)}([g_1]_X) - \deg_{\nu(g_0,\nu)}([g_0]_X) = 2 \sum_{i=1}^k \text{sign}(\mu_{\xi_{\gamma(t_i)}} \circ (\psi_{g_{t_i}}([\hat{\gamma}(t_i)]))) ,
\]

where \( [\hat{\gamma}(t_i)] \) denotes the projection of the velocity vector \( \hat{\gamma}(t_i) \) to the normal line \( N_{W^0_X, g_{t_i}} \).

**Proof.** Suppose \( t_1 < \cdots < t_k \). The map \( t \mapsto \deg_{\nu(g_t,\nu)}([g_t]_X) \) is well defined and constant on each of the intervals

\[
[0,t_1), (t_1,t_2), \ldots, (t_{k-1},t_k), (t_k,1] .
\]

The jumps are given by Corollary 2.15.

We will now prove that any two points \( f_0, f_1 \in \text{Hom}(V,W)_X \) can be connected by a path intersecting the wall transversally in finitely many regular points. This result, which has important consequences, is based on the following

**Theorem 2.17.** The irregular locus \( \mathcal{B}_X = W_X \setminus W^0_X \) is closed and the complement \( \text{Hom}(V,W) \setminus \mathcal{B}_X \) is connected.

**Proof.** We shall identify \( \mathcal{B}_X \) with the union of the images of two smooth proper maps

\[
\hat{q}_X : \hat{\mathcal{J}}_X \to \text{Hom}(V,W) , \quad r_X : D_X \to \text{Hom}(V,W)
\]

of index \(-2\), \( n - N - 1 \) respectively. Then the result follows from Lemma 5.7 in [Te].

Denote by \( \Delta \) the diagonal of the product \( X \times X \) and consider the real blow up \( \widetilde{X} \times \widetilde{X}_\Delta \) of \( X \times X \) along \( \Delta \) (see [Wh] section 3, and [Po] section 4 for a similar construction). Set theoretically one has

\[
\widetilde{X} \times \widetilde{X}_\Delta = \{(X \times X) \setminus \Delta \} \cup \mathbb{P}(N_\Delta) = \{(X \times X) \setminus \Delta \} \cup \mathbb{P}(T_X) = \{(X \times X) \setminus \Delta \} \cup \mathbb{P}(Y_X/(\lambda_\nu|_X)) .
\]

Therefore a point \( \zeta \in \mathbb{P}(N_\Delta) \) above a diagonal point \( (\xi,\xi) \in \Delta \) defines a plane \( \pi_\zeta \subset Y_\xi \) containing \( \xi \). We have a natural smooth map

\[
\psi : \widetilde{X} \times \widetilde{X}_\Delta \to G_2(V)
\]

defined in the following way:

\[
\psi(\zeta) := \begin{cases}
\xi + \eta & \text{when } \zeta = (\xi,\eta) \in (X \times X) \setminus \Delta , \\
\pi_\zeta & \text{when } \zeta \in \mathbb{P}(N_\Delta) .
\end{cases}
\]

We define

\[
\mathcal{J}_X := \{(f,K,\zeta) \in \text{Hom}(V,W) \times G_{N-n}(V) \times \widetilde{X} \times \widetilde{X}_\Delta | \psi(\zeta) \subset K \subset \ker(f)\} .
\]
This space has a natural structure of a vector bundle of rank $n^2$ over the incidence variety
\[ \tilde{\mathcal{I}}_X = \{ (\zeta, K) \in \tilde{X} \times X_\Delta \times G_{N-n}(V) \mid \psi(\zeta) \subset K \} . \]

The variety $\tilde{\mathcal{I}}_X$ is a locally trivial fibre bundle over $\tilde{X} \times X_\Delta$ with $n(N-n-2)$-dimensional fibre, hence smooth of dimension $nN-n^2-2$. This shows that $\dim(\tilde{\mathcal{J}}_X) = nN-2$, so the rank of the projection $\tilde{q}_X : \tilde{\mathcal{J}}_X \to \text{Hom}(V,W)$ is -2.

Put now
\[ D_X := \{ (f, L, \xi) \in \text{Hom}(V,W) \times G_{N-n+1}(V) \times X \mid \xi \subset L \subset \ker(f) \} . \]

$D_X$ has a natural structure of a $(N+1)(n-1)$-dimensional manifold, because it is a rank $(n-1)n$ vector bundle over an $n-1 + (n-1)(N-n)$ dimensional basis. Let $r_X : D_X \to \text{Hom}(V,W)$ be the projection on the first factor. This is a smooth proper map of index $n-N-1 \leq -2$. On the other hand, taking into account Definition 2.3 we see that
\[ B_X = \tilde{q}_X(\tilde{\mathcal{J}}_X) \cup r_X(D_X) , \]

hence
\[ \text{Hom}(V,W) \setminus B_X = (\text{Hom}(V,W) \setminus \tilde{q}_X(\tilde{\mathcal{J}}_X)) \setminus r_X(D_X \setminus r_X^{-1}(\tilde{q}_X(\tilde{\mathcal{J}}_X))) . \]

Applying now Lemma 5.7 in [13] to the proper morphisms
\[ \tilde{q}_X, r_X|_{D_X \setminus r_X^{-1}(\tilde{q}_X(\tilde{\mathcal{J}}_X))} : D_X \setminus r_X^{-1}(\tilde{q}_X(\tilde{\mathcal{J}}_X)) \to (\text{Hom}(V,W) \setminus \tilde{q}_X(\tilde{\mathcal{J}}_X)) \]
we see that the natural maps
\[ \pi_0(\text{Hom}(V,W) \setminus \tilde{q}_X(\tilde{\mathcal{J}}_X), f_0) \to \pi_0(\text{Hom}(V,W), f_0) , \]
\[ \pi_0(\text{Hom}(V,W) \setminus B_X, f_0) \to \pi_0(\text{Hom}(V,W) \setminus \tilde{q}_X(\tilde{\mathcal{J}}_X), f_0) \]
are bijections, so that $\text{Hom}(V,W) \setminus B_X$ is connected, as claimed.

**Corollary 2.18.** Any pair $(f_0, f_1) \in \text{Hom}(V,W)_X \times \text{Hom}(V,W)_X$ can be connected by a smooth path $\gamma : [0,1] \to \text{Hom}(V,W)$ which intersects the wall $W_X$ in finitely many regular points, all intersection points being transversal.

**Proof.** Since $\text{Hom}(V,W) \setminus B_X$ is connected, the pair $(f_0, f_1)$ can be connected by a smooth path $\alpha : [0,1] \to \text{Hom}(V,W)$ which intersects the wall only in regular points. Using a well-known transversality principle (see [DoK] p. 143) we find small perturbations of $\alpha$ which coincide with $\alpha$ on a neighbourhood of $\{0,1\}$ and are transversal to the map $W_X^0 \to \text{Hom}(V,W)$.

Therefore Corollary 2.18 states that the difference
\[ \deg_{\nu(f_1, \mu)}([f_1]_X) - \deg_{\nu(f_0, \mu)}([f_0]_X) \]
can always be computed using such a path from $f_0$ to $f_1$ and the difference formula given by Corollary 2.16. Combining with Remark 2.12 one obtains the following important general property of the degree map $\deg^\nu_X : \pi_0(\text{Hom}(V,W) \setminus W_X) \to \mathbb{Z}$:

**Corollary 2.19.** Let $\mu : X \times \det(W) \to \det(Y_X)$ be an orientation parameter. Then:

1. All values of the degree map $\deg^\mu_X : \pi_0(\text{Hom}(V,W) \setminus W_X) \to \mathbb{Z}$ are congruent modulo 2.
(2) If \( a \in \text{im}(\deg^X_\mu) \), then any integer \( c \) with \( -|a| \leq c \leq |a| \) which has the same parity as \( a \) also belongs to \( \text{im}(\deg^X_\mu) \).

Proof. The first statement follows directly from the difference formula. For the second statement use Corollary 2.18 and take into account that
- the jump when crossing the wall transversally at a regular point is \( \pm 2 \),
- the image of \( \deg^X_\mu \) is invariant under the involution \( -\text{id}_Z \), by Remark 2.12.

Note that the congruence (1) does not follow from Proposition 1.5 since \( X \) is not supposed to be algebraic.

2.4. Examples.

2.4.1. Projecting a hyperquadric. Let \( X \) be the hyperquadric of \( \mathbb{P}^n_{\mathbb{R}} \) \((n \geq 2)\) defined by the equation \( x_0^2 - \sum_{i=1}^n x_i^2 = 0 \). Dehomogenizing with respect to \( x_0 \) one gets an obvious identification \( X = S^{n-1} \). The relative orientability condition is always satisfied (it is obvious for \( n \geq 3 \) by Remark 1.3).

Consider the two projections \( [f_0] : \mathbb{P}^n \setminus \{[1,0,\ldots,0]\} \to \mathbb{P}^{n-1} \), \( [f_1] : \mathbb{P}^n \setminus \{[0,0,\ldots,1]\} \to \mathbb{P}^{n-1} \)
given by \( [f_0](x) := [x_1, \ldots, x_n] \), \( [f_1](x) := [x_0, \ldots, x_{n-1}] \).

Via the obvious identification \( X = S^{n-1} \), the first map is just the canonical projection \( S^{n-1} \to \mathbb{P}^{n-1} \). The degrees of the restrictions \( [f_0]_X : X \to \mathbb{P}^{n-1} \), \( [f_1]_X : X \to \mathbb{P}^{n-1} \) with respect to a suitable choice of the trivialization \( \mu \) are
\[
\deg_{\nu([p_0],\mu)}([f_0]_X) = 2 \quad \text{and} \quad \deg_{\nu([p_1],\mu)}([f_1]_X) = 0.
\]

The picture above shows an interesting phenomenon: the projection \( [f_1]_X \), which has degree 0, has some fibres consisting of two points, which however come with opposite signs. For instance, the intersection of \( X \) with the line connecting \( z_0 \) and \( z_1 \) consists of two points, which come with the same sign when considered as elements in the fibre of \( [f_0]_X \), but with opposite signs when considered as elements in
the fibre of $[f_1]_X$.

2.4.2. Degree of rational functions. Let $\mathbb{R}F_n^*$ be the space of pairs of monic polynomials $(p, q)$ with real coefficients of degree $n$ with no common factor. Writing

$$p(t) = t^n + \sum_{i=0}^{n-1} a_i t^i, \quad q(t) = t^n + \sum_{i=0}^{n-1} b_i t^i$$

we see that $\mathbb{R}F_n^*$ can be identified with the Zariski open subset of $\mathbb{R}^{2n}$ defined by the condition $\text{Res}(p, q) \neq 0$.

A pair $(p, q) \in \mathbb{R}F_n^*$ defines a rational function

$$f_{pq}(t) = \frac{p(t)}{q(t)},$$

hence $\mathbb{R}F_n^*$ can be identified with the space of rational functions of degree $n$ sending $\infty$ to $1$. Such a rational function $f_{pq}$ has an extension $F_{pq} : \mathbb{P}_R^1 \to \mathbb{P}_R^1$. Taking the homogenizations $P \in \mathbb{R}[t_0, t_1]_n$, $Q \in \mathbb{R}[t_0, t_1]_n$ we see that $F_{pq}$ is given in homogeneous coordinates by

$$F_{pq}(t_0, t_1) := [P(t_0, t_1), Q(t_0, t_1)].$$

The set of possible degrees $\text{deg}(F_{pq})$ when $(p, q)$ varies in the space $\mathbb{R}F_n^*$ is known. The result is given by Brockett’s Theorem (see Segal [Se] p. 41 for an independent proof, or [By] Theorem 2.1).

**Theorem 2.20.** The space $\mathbb{R}F_n^*$ has $n + 1$ connected components $\mathbb{R}F_n^*$, where

$$u + v = n, \quad u \geq 0, \quad v \geq 0$$

and a pair $(p, q) \in \mathbb{R}F_n^*$ belongs to $\mathbb{R}F_n^*$ if and only if $\text{deg}(F_{pq}) = u - v$.

Therefore the set of possible degrees is $-n, -n + 2, \ldots, n - 2, n$. Note that the degree of the map $F_{pq}^* : \mathbb{P}_C^1 \to \mathbb{P}_C^1$ defined by a pair $(p, q) \in \mathbb{R}F_n^*$ is always $n$. We see that in this case all values of the real degree allowed by the general estimates and congruences in Proposition [15] can occur.

The pair of polynomials corresponding to the rational function

$$g_{uv}(t) = 1 + \sum_{i=1}^{n} \frac{-1}{t + i} + \sum_{j=1}^{n} \frac{1}{t - j}$$

is an element of $\mathbb{R}F_{uv}$.

Note that the map $F_{pq}$ associated with a pair $(p, q) \in \mathbb{R}F_n^*$ can be identified with the composition $[\pi_{pq}] \circ v_n$, where $v_n : \mathbb{P}_R^n \to \mathbb{P}_C^n$ is the Veronese map of degree $n$, and $\pi_{pq} : \mathbb{R}^{n+1} \to \mathbb{R}^2$ is the projection defined by $(P, Q)$. Therefore the theorem of Brockett identifies $n + 1$ chambers in the complement

$$\text{Hom}(\mathbb{R}^{n+1}, \mathbb{R}^2)_{v_n(P_1)} = \text{Hom}(\mathbb{R}^{n+1}, \mathbb{R}^2) \setminus \mathcal{V}_{v_n(P_1)}.$$

3. Examples

3.1. Conjugation manifolds. Let $(X, \tau)$ a topological space endowed with an involution such that $H^\text{odd}(X, \mathbb{Z}_2) = 0$. We recall from [III] that a cohomology frame for $(X, \tau)$ is a pair $(\kappa, \sigma)$, where

1. $\kappa : H^\ast(X, \mathbb{Z}_2) \to H^\ast(X^\tau, \mathbb{Z}_2)$ is a group isomorphism.
(2) \( \sigma : H^2(X, \mathbb{Z}_2) \to H^{2*}_{\mathbb{Z}_2}(X, \mathbb{Z}_2) \) is a group morphism which is a section of the restriction map \( \rho : H^{2*}_{\mathbb{Z}_2}(X, \mathbb{Z}_2) \to H^{2*}(X, \mathbb{Z}_2) \), such that the following conjugation equation holds:

\[
r \circ \sigma(a) = \kappa(a)u^m + q(a) \forall a \in H^{2m}(X, \mathbb{Z}_2).
\]

Here \( r \) denotes the restriction map \( H^{2*}_{\mathbb{Z}_2}(X, \mathbb{Z}_2) \to H^{2*}(X^\tau, \mathbb{Z}_2) = H^*(X, \mathbb{Z}_2)[u] \), and \( q(a) \) is an element in \( H^*(X, \mathbb{Z}_2)[u] \) whose degree with respect to \( u \) is smaller than \( m \).

If a cohomology frame for \((X, \tau)\) exists, then it is unique, natural with respect to equivariant maps, and \( \kappa, \sigma \) are automatically ring isomorphisms (not only group isomorphisms).

A \( \mathbb{Z}_2 \)-space \((X, \tau)\) which admits a cohomology frame is called conjugation space. Examples of conjugation spaces are: all complex Grassmann manifolds (with the standard Real structure), all toric manifolds (with their standard Real structure).

Let now \((X, \tau)\) be a paracompact conjugation space with cohomology frame \((\kappa, \sigma)\), and let \((E, \bar{\tau})\) a Real complex vector bundle on \( X \). Denote by \( \bar{\kappa} \) the image of the total Chern class \( c(E) \) in \( H^*(X, \mathbb{Z}_2) \). Then \( \kappa(\bar{\kappa}(E)) = w(E^\tau) \) (see [HIPI p. 950].

**Proposition 3.1.** Let \((X, \tau), (Y, \iota)\) be Real complex manifolds which are conjugation spaces with respect to their Real structures, and let \( f : X \to Y \) be a Real holomorphic map. Then \( f \) is relatively orientable if and only if \( f^*(c_1(Y)) \equiv c_1(X) \mod 2 \).

**Proof.** One has

\[
f(\mathbb{R})^*w_1(T_Y(\mathbb{R})) = f(\mathbb{R})^*(\kappa_Y(\bar{\kappa}_1(T_Y)))
\]

\[= \kappa_X(f^*(\bar{\kappa}_1(T_Y))) = \kappa_X(c_1(T_X)) = w_1(T_X(\mathbb{R})),
\]

where \( (\kappa_X, \sigma_X), (\kappa_Y, \sigma_Y) \) are the cohomology frames of \((X, \tau)\) and \((Y, \iota)\) respectively, \( T_X, T_Y \) the two tangent bundles regarded as complex vector bundles, \( X(\mathbb{R}), Y(\mathbb{R}) \) the fixed point loci, and \( f(\mathbb{R}) \) the map \( X(\mathbb{R}) \to Y(\mathbb{R}) \) induced by \( f \). It suffices to apply Remark [1.2].

### 3.2. Plücker embeddings of real Grassmann manifolds

Let \( V_0 \) be a real vector space of dimension \( p + q \), \( G_q(V_0) \) the Grassmann manifold of \( q \)-planes in \( V_0 \). Take \( V = \wedge^q V_0 \) and let \( X \) be the image of the Plücker embedding \( Pl : G_q(V_0) \to \mathbb{P}(V) \). Denoting by \( U \) the tautological rank \( q \) bundle on \( G_q(V_0) \) we have a natural identification \( T_{G_q(V_0)} = \text{Hom}(U, V_0/U) \), which shows that

\[
w_1(G_q(V_0)) = (pq + 1)w_1(U).
\]

On the other hand it is well-known that \( Pl^*(\lambda_U) = \det(U) \). In our case we have \( n = m + 1 = pq + 1 \). Since on any Grassmann manifold one has \( w_1(U) \neq 0 \), we have

**Corollary 3.2.** A map \( [f]_X : Pl(G_q(V_0)) \to \mathbb{P}(W) \) associated with a linear map \( f \in \text{Hom}(\wedge^q V_0, W) \) satisfying \( \ker(f) \cap Pl(G_q(V_0)) = 0 \) is relatively orientable if and only if \( pq + 1 \equiv p + q \mod 2 \), i.e., iff \( p \) and \( q \) are not both even.

Consider now the special case when \( V_0 = S^{p+q-1}W_0^\vee \), where \( W_0 = \mathbb{R}^2 \), and take \( W : = S^p W_0^\vee \). This special case is important because we have a standard linear epimorphism

\[
\varphi_{\text{Wronski}} : \wedge^q(S^{p+q-1}W_0^\vee) \to S^p W_0^\vee, \quad \varphi_{\text{Wronski}}(F_1 \wedge \cdots \wedge F_q) := W(F_1, \ldots, F_q),
\]
where
\[ W : [S^{p+q-1}(W_0^\vee)]^q \to S^{pq}W_0^\vee \]
denotes the homogeneous Wronskian. Alternatively, \( \varphi_{\text{Wronski}} \) can be defined as the composition of the \textit{standard isomorphism}
\[ \wedge^q(S^{p+q-1}W_0^\vee) \to S^q(S^{p}W_0^\vee) \]
with the natural projection \( S^q(S^{p}W_0^\vee) \to S^{pq}W_0^\vee \) (see [AC]).

Up to a constant factor the map \( \varphi_{\text{Wronski}} \) can also be obtained using the inhomogeneous Wronskian
\[ w : (\mathbb{R}[s]_{\leq p+q-1})^q \to \mathbb{R}[s]_{\leq pq} \]
via the obvious identifications \( \mathbb{R}[s]_{\leq k} \simeq S^k(W_0^\vee) \) (see [AC] section 2.8). With this remark the results of [EG1] (where the inhomogeneous Wronskian is used) apply. First, it is well known that \( P(\ker(\varphi_{\text{Wronski}})) \cap G_q(S^{p+q-1}W_0^\vee) = \emptyset \), so we have a well-defined projection
\[ [\varphi_{\text{Wronski}}] : G_q(S^{p+q-1}W_0^\vee) \to P(S^{pq}W_0^\vee) \]
for which the degree is known (see [EG1]).

When \( p \) and \( q \) are not both even, then \( [\varphi_{\text{Wronski}}] \) is relatively orientable and:
\[ |\deg([\varphi_{\text{Wronski}}]) = \begin{cases} 0 & \text{if } p, q \text{ are both odd} \\ I(p,q) & \text{if } p + q \text{ is odd}. \end{cases} \]

Here \( I(\cdot, \cdot) \) is symmetric, and for \( 2 \leq p \leq q \) the integer \( I(p,q) \) is given by:
\[ I(p,q) = \frac{1!2! \cdots (p-1)!((q-1)!(q-2)! \cdots (q-(p-1))!(pq)!}{(q-p+2)!(q-p+4)! \cdots (q+p-2)!\left(\frac{q-p+1}{2}\right)!\left(\frac{q-p-1}{2}\right)! \cdots \left(\frac{q+p-1}{2}\right)!}. \]

**Remark 3.3.** Using the first formula in Lemma 2.6 and Lemma 15 in [OT2] one obtains a canonical isomorphism
\[ \det(T^\vee_{P(G_q(S^{p+q-1}W_0^\vee))}) \otimes [f]^*\left(\det(T^\vee_{P(S^{pq}W_0^\vee)}) = \right. \]
\[ [\det(W_0)^\vee]^\otimes q(\nu-1)]^\otimes (p-1)(q-1) \otimes \left[\det(U)^\vee\right]^\otimes (p-1)(q-1), \]
for any \( f \in \text{Hom}(\wedge^q(S^{p+q-1}W_0^\vee), S^{pq}W_0^\vee) \) with \( P(\ker(f)) \cap P(G_q(S^{p+q-1}W_0^\vee)) = \emptyset \). This shows that \( [f] \) is canonically relatively oriented if and only if either \( p, q \in 2\mathbb{N} + 1 \), or \( p \in 2\mathbb{N} + 1 \) and \( q \in 4\mathbb{N} \), or \( p \in 2\mathbb{N} \) and \( q \in 4\mathbb{N} + 1 \).

Therefore, if the pair \((p,q)\) satisfies one of these three conditions then, for any regular value \( [P] \in P(S^{pq}W_0^\vee) \) of the Wronski map \( [\varphi_{\text{Wronski}}] \), one can associate an intrinsic sign to any element in the fibre \( [\varphi_{\text{Wronski}}]^{-1}([P]) \) (without having to orient the plane \( W_0 \)). It would be interesting to have a geometric interpretation of these intrinsic signs.

### 3.3 Universal pole placement map

Let \( W_0 \) be a real plane, \( V_0 \) the trivial bundle \( \mathbb{P}(W_0) \times V_0 \) on the projective space \( \mathbb{P}(W_0) \) with fibre a \( p+q \) dimensional vector space \( V_0 \). For \( \nu \in \mathbb{N} \) denote by \( \text{Quot}^\nu_{\mathbb{P}(W_0)}(V_0) \) the quotient space of equivalence classes of quotients \( s : Y_0 \to Q \) of \( V_0 \) with \( \text{rk}(\ker(s)) = p \) and \( \det(\ker(s)) \simeq H^2(\mathbb{P}(W_0), -\nu) \).

Every such quotient \( s \) defines an element \( QP(s) \in P(\wedge^pV_0 \otimes \wedge^qW_0^\vee) \) by the formula
\[ QP(s) := [\wedge^pk_s] \]
Here \( k_s \in \text{Hom}(\ker(s), V_0) \) denotes the embedding of \( \ker(s) \) in \( V_0 \), and
\[ \wedge^pk_s \in H^0(\text{Hom}(\wedge^p \ker(s), \wedge^p V_0)) = H^0(\wedge^pV_0 \otimes (\wedge^p \ker(s))^\vee) \]
is the $p$-th exterior power of $k_s$. Since det$(\ker(s)) ^\vee \simeq \mathcal{O}_{P(W_0)}(\nu)$, the section $\wedge^p k_s$ of $\wedge^p V \otimes (\wedge^p \ker(s)) ^\vee$ defines an element of $H^0(\wedge^p V_0(\nu)) = \wedge^p V_0 \otimes S^p W_0 ^\vee$, which is well defined up to multiplication with a non-vanishing scalar. Recall that we have a perfect paring $\wedge^q V_0 \times \wedge^p V_0 \to \det V_0$, which induces an isomorphism

$$\wedge^p V_0 \to (\wedge^q V_0) ^\vee \otimes \det V_0 = \Hom(\wedge^q V_0, \det V_0).$$

Therefore the element $QPl(s) = [\wedge^p k_s]$ can be regarded as an element of

$$\mathbb{P}(\Hom(\wedge^q V_0, S^p W_0 ^\vee \otimes \det(V_0))) = \mathbb{P}(\Hom(\wedge^q V_0, S^p W_0 ^\vee))$$

as claimed. Note that in general the map

$$QPl : \text{Quot}_{P(W_0)}^{p,\nu}(V_0) \to \mathbb{P}(\Hom(\wedge^q V_0, S^p W_0 ^\vee))$$

is not an embedding.

A quotient $[s] \in \text{Quot}_{P(W_0)}^{p,\nu}(V_0)$ defines a central projection

$$\psi[s] : \mathbb{P}((\wedge^q V_0) \setminus \mathbb{P}(\ker(\wedge^p k_s))) \to \mathbb{P}(S^p W_0 ^\vee).$$

Since $\mathbb{P}(\wedge^q V_0)$ contains the image of the Plücker embedding

$$Pl : G_q(V_0) \to \mathbb{P}(\wedge^q V_0),$$

it is interesting to study the composition

$$\phi[s] := \psi[s] \circ Pl : G_q(V_0) \setminus \mathbb{P}(\ker(\wedge^p k_s)) \to \mathbb{P}(S^p W_0 ^\vee)$$

associated with an element $[s] \in \text{Quot}_{P(W_0)}^{p,\nu}(V_0)$. When $\nu = pq$ we can ask if this projection is defined on the all of $G_q(V_0)$, and when $p$ and $q$ are not both even, so that $\phi[s]$ is relatively orientable, one can also ask for the degree.

**Remark 3.4.** Consider the special case where $V_0 = S^{p+q-1}W_0 ^\vee$ and $W_0 = \mathbb{R}^2$. It seems to be well known \cite{EG} that in this case there exists an element

$$s_{\text{Wronski}} \in \text{Quot}_{P(W_0)}^{p,q}(V_0)$$

such that

$$\varphi[s_{\text{Wronski}}] = [\varphi_{\text{Wronski}}]$$

is the Wronski projection introduced in section \S2.2.

Note that the chamber structure of $\Hom(\wedge^q V_0, S^{pq} W_0 ^\vee)_{G_q(V_0)}$ induces a chamber structure on the complement

$$\text{Quot}_{P(W_0)}^{p,q}(V_0) \setminus QPl^{-1}(\mathbb{P}(W_{G_q(V_0)}))$$

of the pull-back of the projectized wall $\mathbb{P}(W_{G_q(V_0)})$ via $QPl$.

The image $\phi[s](U) \in \mathbb{P}(S^{pq} W_0 ^\vee)$ of a $q$-plane $U \in G_q(V_0) \setminus \mathbb{P}(\ker(\wedge^p k_s))$ can be explicitly described as follows: Denote by $j_U : U \to V_0$ the obvious embedding of the trivial rank $q$-bundle $U := \mathbb{P}(W_0) \times U$ in $V_0$, and by $\rho_U : V_0 \to V_0 / U$ the projection onto the quotient bundle. The determinant $\det(\rho_U \circ k_s)$ of the composition $\rho_U \circ k_s : \ker(s) \to V_0 / U$ can be regarded as an element of $\wedge^p (V_0 / U) \otimes S^{pq} W_0 ^\vee$. If this element is non-zero it defines an element

$$PP[s](U) \in \mathbb{P}(\wedge^p (V_0 / U) \otimes S^{pq} W_0 ^\vee) = \mathbb{P}(S^{pq} W_0 ^\vee) ,$$

called the pole placement of $[s] \in \text{Quot}_{P(W_0)}^{p,q}(V_0)$ at $U \in G_q(V_0)$. On the other hand, one can easily prove that $\det(\rho_U \circ k_s)$ is non-zero if and only if $\phi[s]$ is defined at $U$. 
Lemma 3.5. When $\phi_{[s]}$ is defined at $U$, one has

$$\phi_{[s]}(U) = PP[s](U).$$

Proof. Consider the rank $p$ vector bundle $K_s$ on $\mathbb{P}(W_0)$ associated with the sheaf $\ker(s)$, choose $x \in \mathbb{P}(W_0)$ and vectors $l_1, \ldots, l_p \in K_s, x$. Let $(v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q})$ be a basis of $V_0$ such that $(v_{p+1}, \ldots, v_{p+q})$ is a basis of $U$. Let $U' := (v_1, \ldots, v_p)$ and $\hat{k}_s$ the composition of $k_s$ with the projection on $U'$. Regarding $\wedge^p k_s$ as an element in

$$H^0(\mathbb{P}(W_0), \mathcal{H}om(\det(K_s), \mathcal{H}om(\wedge^q V_0, \det(V_0)))) = \mathcal{H}om(\wedge^q V_0, \wedge^{p+q} V_0) \otimes S^p W_0^\vee$$

we have

$$(\wedge^p k_s)(l_1 \wedge \cdots \wedge l_p)(v_{p+1} \wedge \cdots \wedge v_{p+q}) = k_s(l_1) \wedge \cdots \wedge k_s(l_p) \wedge v_{p+1} \wedge \cdots \wedge v_{p+q} = \hat{k}_s(l_1) \wedge \cdots \wedge \hat{k}_s(l_p) \wedge v_{p+1} \wedge \cdots \wedge v_{p+q} = [\det(\rho_U \circ k_s)(l_1 \wedge \cdots \wedge l_p) \otimes (v_{p+1} \wedge \cdots \wedge v_{p+q})],$$

where in the last equality we have used the canonical isomorphism $\det(V_0) = \det(V_0/U) \otimes \det(U)$. Therefore one obtains the following equality in the space $S^p W_0^\vee \otimes \det(V_0)$

$$\wedge^p k_s(v_{p+1} \wedge \cdots \wedge v_{p+q}) = \det(\rho_U \circ k_s) \otimes (v_{p+1} \wedge \cdots \wedge v_{p+q}),$$

which shows that $\wedge^p k_s(v_{p+1} \wedge \cdots \wedge v_{p+q})$ and $\det(\rho_U \circ k_s)$ define the same element in $\mathbb{P}(S^p W_0^\vee)$. 

For a quotient $[s] \in \text{Quot}^{p,q}_{\mathbb{P}(W_0)}(V_0)$ the assignment $U \mapsto PP[s](U)$ defines a rational map $G_q(V_0) \dashrightarrow \mathbb{P}(S^{pq} W_0^\vee)$. We denote by $\text{Rat}(G_q(V_0), \mathbb{P}(S^{pq} W_0^\vee))$ the set of such rational maps. Letting $[s]$ vary in $\text{Quot}^{p,q}_{\mathbb{P}(W_0)}(V_0)$ we obtain a map

$$PP : \text{Quot}^{p,q}_{\mathbb{P}(W_0)}(V_0) \to \text{Rat}(G_q(V_0), \mathbb{P}(S^{pq} W_0^\vee)),$$

which we call the universal pole placement map. Lemma 3.5 shows that the following diagram is commutative:

$$\begin{array}{ccc}
\text{Quot}^{p,q}_{\mathbb{P}(W_0)}(V_0) & \xrightarrow{QPl} & \mathbb{P}(\mathcal{H}om(\wedge^q V_0, S^{pq} W_0^\vee)) \\
PP & & \downarrow Pl^* \\
\text{Rat}(G_q(V_0), \mathbb{P}(S^{pq} W_0^\vee)) & & \\
\end{array}$$

3.4. A real subspace problem. Let $W_0, V_0$ be real vector spaces of dimensions 2 and $p + q$ respectively. We denote by $S^p_0(\mathbb{P}(W_0))$ the subset of elements $s$ in the symmetric power $S^{pq}(\mathbb{P}(W_0))$ consisting of pairwise distinct points.

Subspace problem: Fix a regular algebraic map $\gamma : \mathbb{P}(W_0) \to G_p(V_0)$ of algebraic degree $pq$, and an element $s \in S^p_0(\mathbb{P}(W_0))$. Count the $q$-dimensional linear subspaces $U \subset V_0$ such that

$$\mathbb{P}(U) \cap \mathbb{P}(\gamma(\xi)) \neq \emptyset \quad \forall \xi \in s. \quad (S_{\gamma,s})$$
We will show that this problem has an interesting interpretation which, for general $s \in S_p^q(\mathbb{P}(W_0))$, allows one to associate a sign to every $q$-dimensional subspace $U$ satisfying the condition $(S_{\gamma,s})$ and to compute the total number of solutions of $(S_{\gamma,s})$ when these signs are taken into account. Indeed, for any regular algebraic map $\gamma : \mathbb{P}(W_0) \to G_p(V_0)$ there exists a bundle epimorphism

$$s_\gamma : V_0 \to \mathcal{Q}_\gamma,$$

classifying $\gamma$, i.e., such that

$$\gamma(\xi) = \ker(s_{\gamma,\xi}) \quad \forall \xi \in \mathbb{P}(W_0).$$

The equivalence class $[s_\gamma] \in \text{Quot}^{p,q}_{\mathbb{P}(W_0)}(V_0)$ is well defined, and the assignment $\gamma \mapsto [s_\gamma]$ defines an embedding

$$\text{Mor}^{p,q}(\mathbb{P}(W_0), G_p(V_0)) \to \text{Quot}^{p,q}_{\mathbb{P}(W_0)}(V_0),$$

where $\text{Mor}^{p,q}(\mathbb{P}(W_0), G_p(V_0))$ is the space of regular algebraic morphisms $\mathbb{P}(W_0) \to G_p(V_0)$ of algebraic degree $pq$. Consider the pole placement map $\phi_{[s_\gamma]} : G_q(V_0) \to \mathbb{P}(S^{pq}W_0^\gamma)$ corresponding to the quotient $[s_\gamma]$.

Let $P_s \in \mathbb{R}[W_0]_{pq} = S^{pq}W_0^\gamma$ be a homogeneous polynomial of degree $pq$ on $W_0$ whose set of roots coincides with $s$. This polynomial is determined by $s$ up to multiplication by a non-vanishing constant.

**Lemma 3.6.** The set of solutions of the problem $(S_{\gamma,s})$ can be identified with the fibre $\phi_{[s_\gamma]}$ over $[P_s]$:

$$\phi_{[s_\gamma]}^{-1}([P_s]) = \{ U \in G_q(V_0) \mid \mathbb{P}(U) \cap \mathbb{P}(\gamma(\xi)) \neq \emptyset \quad \forall \xi \in \mathbb{P}(W_0) \}.$$

**Proof.** Indeed, using the definition of $\phi_{[s_\gamma]}$, it follows that for any $U \in G_q(V_0)$ the set of zeros of a polynomial $P_U \in \mathbb{R}[W_0]_{pq}$ representing $\phi_{[s_\gamma]}(U) \in \mathbb{P}(S^{pq}W_0^\gamma)$ coincides with the set of points $\xi \in \mathbb{P}(W_0)$ for which $\det(\rho_U \circ k_{\gamma,\xi}) \neq 0$, i.e., with the set of points $\xi \in \mathbb{P}(W_0)$ for which $\gamma(x) \cap U \neq \emptyset$. If the latter set is $s$ (which has maximal cardinal $pq$) this condition is equivalent to $[P_U] = [P_s]$. Therefore $\phi_{[s_\gamma]}^{-1}([P_s])$ if and only if $U$ satisfies the condition $(S_{\gamma,s})$. 

**Corollary 3.7.** Suppose that $p$ and $q$ are not both even and let $\gamma : \mathbb{P}(W_0) \to G_p(V_0)$ be a regular algebraic map of algebraic degree $pq$ such that

$$[s_\gamma] \in \text{Quot}^{p,q}_{\mathbb{P}(W_0)}(V_0) \setminus \text{QPl}^{-1}(\mathbb{P}(W_{G_q(V_0)})).$$

Then the pole placement map $\phi_{[s_\gamma]} : G_q(V_0) \to \mathbb{P}(S^{pq}W_0^\gamma)$ is well defined. Choose a relative orientation $\nu$ of $\phi_{[s_\gamma]}$. There exists an open dense subset $S_{\gamma} \subset S_p^q(\mathbb{P}(W_0))$ such that for any $s \in S_{\gamma}$ one has

$$\sum_{U \text{ solves } (s_{\gamma,s})} \varepsilon_{U,\nu} = \deg(\phi_{[s_{\gamma,s}]}).$$

**Proof.** The map $s \mapsto [P_s]$ identifies $S_p^q(\mathbb{P}(W_0))$ with an open subset of $\mathbb{R}[W_0]_{pq} = S^{pq}W_0^\gamma$. It suffices to apply Sard theorem to the map $\phi_{[s_\gamma]}$ and to take into account that the set of regular values of a proper smooth map is always open. 

\[\blacksquare\]
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