Methods from Multiscale Theory and Wavelets
Applied to Nonlinear Dynamics

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Abstract. We show how fundamental ideas from signal processing, multiscale theory and wavelets may be applied to nonlinear dynamics.

The problems from dynamics include iterated function systems (IFS), dynamical systems based on substitution such as the discrete systems built on rational functions of one complex variable and the corresponding Julia sets, and state spaces of subshifts in symbolic dynamics. Our paper serves to motivate and survey our recent results in this general area. Hence we leave out some proofs, but instead add a number of intuitive ideas which we hope will make the subject more accessible to researchers in operator theory and systems theory.

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1. Introduction

In the past twenty years there has been a great amount of interest in the theory of wavelets, motivated by applications to various fields such as signal processing, data compression, tomography, and subdivision algorithms for graphics (Our latest check on the word “wavelet” in Google turned up over one million and a half results, 1,590,000 to be exact). It is enough here to mention two outstanding successes of the theory: JPEG 2000, the new standard in image compression, and the WSQ (wavelet scalar quantization) method which is used now by the FBI to store its fingerprint database. As a mathematical subject, wavelet theory has found points of interaction with functional and harmonic analysis, operator theory, ergodic theory and probability, numerical analysis and differential equations. With the explosion of information due to the expansion of the internet, there came a

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need for faster algorithms and better compression rates. These have motivated the research for new examples of wavelets and new wavelet theories.

Recent developments in wavelet analysis have brought together ideas from engineering, from computational mathematics, as well as fundamentals from representation theory. This paper has two aims: One to stress the interconnections, as opposed to one aspect of this in isolation; and secondly to show that the fundamental Hilbert space ideas from the linear theory in fact adapt to a quite wide class of nonlinear problems. This class includes random-walk models based on martingales. As we show, the theory is flexible enough to allow the adaptation of pyramid algorithms to computations in the dynamics of substitution theory as it is used in discrete dynamics; e.g., the complex dynamics of Julia sets, and of substitution dynamics.

Our subject draws on ideas from a variety of directions. Of these directions, we single out quadrature-mirror filters from signal/image processing. High-pass/low-pass signal processing algorithms have now been adopted by pure mathematicians, although they historically first were intended for speech signals, see [55]. Perhaps unexpectedly, essentially the same quadrature relations were rediscovered in operator-algebra theory, and they are now used in relatively painless constructions of varieties of wavelet bases. The connection to signal processing is rarely stressed in the math literature. Yet, the flow of ideas between signal processing and wavelet math is a success story that deserves to be told. Thus, mathematicians have borrowed from engineers; and the engineers may be happy to know that what they do is used in mathematics.

Our presentation serves simultaneously to motivate and to survey a number of recent results in this general area. Hence we leave out some proofs, but instead we add a number of intuitive ideas which we hope will make the subject more accessible to researchers in operator theory and systems theory. Our theorems with full proofs will appear elsewhere. An independent aim of our paper is to point out several new and open problems in nonlinear dynamics which could likely be attacked with the general multiscale methods that are the focus of our paper.

In Section 2 below, we present background material from signal processing and from wavelets in a form which we hope will be accessible to operator theorists, and to researchers in systems theory. This is followed in Section 3 by a presentation of some motivating examples from nonlinear dynamics. They are presented such as to allow the construction of appropriate Hilbert spaces which encode the multiscale structure. Starting with a state space $X$ from dynamics, our main tool in the construction of a multiscale Hilbert space $H(X)$ is the theory of random walk and martingales. The main part of our paper, Sections 4 and 5 serves to present our recent new and joint results.
2. Connection to signal processing and wavelets

We will use the term “filter” in the sense of signal processing. In the simplest case, time is discrete, and a signal is just a numerical sequence. We may be acting on a space of signals (sequences) using the operation of Cauchy product; and the operations of down-sampling and up-sampling. This viewpoint is standard in the engineering literature, and is reviewed in [19] (see also [10] and [63]) for the benefit of mathematicians.

A numerical sequence \((a_k)\) represents a filter, but it is convenient, at the same time, also to work with the frequency-response function. By this we mean simply the Fourier series (see (3.3) below) corresponding to the sequence \((a_k)\). This Fourier series is called the filter function, and in one variable we view it as a function on the one-torus. (This is using the usual identification of periodic functions on the line with functions on the torus.) The advantage of this dual approach is that Cauchy product of sequences then becomes pointwise product of functions.

We will have occasion to also work with several dimensions \(d\), and then our filter function represents a function on the \(d\)-torus. While signal processing algorithms are old, their use in wavelets is of more recent vintage. From the theory of wavelets, we learn that filters are used in the first step of a wavelet construction, the so-called multiresolution approach. In this approach, the problem is to construct a function on \(\mathbb{R}\) or on \(\mathbb{R}^d\) which satisfies a certain renormalization identity, called the scaling equation, and we recall this equation in two versions (2.2) and (3.2) below. The numbers \((a_k)\) entering into the scaling equation turn out to be the very same numbers the signal processing engineers discovered as quadrature-mirror filters.

A class of wavelet bases are determined by filter functions. We establish a duality structure for the category of filter functions and classes of representations of a certain \(C^*\)-algebra. In the process, we find new representations, and we prove a correspondence which serves as a computational device. Computations are done in the sequence space \(\ell^2\), while wavelet functions are in the Hilbert space \(L^2(\mathbb{R})\), or some other space of functions on a continuum. The trick is that we choose a subspace of \(L^2(\mathbb{R})\) in which computations are done. The choice of subspace is dictated by practical issues such as resolution in an image, or refinement of frequency bands.

We consider non-abelian algebras containing canonical maximal abelian sub-algebras, i.e., \(C(X)\) where \(X\) is the Gelfand space, and the representations define measures \(\mu\) on \(X\). Moreover, in the examples we study, it turns out that \(X\) is an affine iterated function system (IFS), of course depending on the representation. In the standard wavelet case, \(X\) may be taken to be the unit interval. Following Wickerhauser et al. [30], these measures \(\mu\) are used in the characterization and analysis of wavelet packets.

Orthogonal wavelets, or wavelet frames, for \(L^2(\mathbb{R}^d)\) are associated with quadrature-mirror filters (QMF), a set of complex numbers which relate the dyadic...
scaling of functions on $\mathbb{R}^d$ to the $\mathbb{Z}^d$-translates. In the paper [53], we show that generically, the data in the QMF-systems of wavelets are minimal, in the sense that it cannot be nontrivially reduced. The minimality property is given a geometric formulation in $\ell^2(\mathbb{Z}^d)$; and it is then shown that minimality corresponds to irreducibility of a wavelet representation of the Cuntz algebra $\mathcal{O}_N$. Our result is that this family of representations of $\mathcal{O}_N$ on $\ell^2(\mathbb{Z}^d)$ is irreducible for a generic set of values of the parameters which label the wavelet representations. Since MRA-wavelets correspond to representations of the Cuntz algebras $\mathcal{O}_N$, we then get, as a bonus, results about these representations.

**Definition 2.1.** (Ruelle’s operators.) Let $(X, \mu)$ be a finite Borel measure space, and let $r : X \to X$ be a finite-to-one mapping of $X$ onto itself. (The measure $\mu$ will be introduced in Section 4 below, and it will be designed so as to have a certain strong invariance property which generalizes the familiar property of Haar measure in the context of compact groups.) Let $V : X \to [0, \infty)$ be a given measurable function. We then define an associated operator $R = R_V$, depending on both $V$ and the endomorphism $r$, by the following formula

$$R_V f(x) = \frac{1}{\# r^{-1}(x)} \sum_{r(y) = x} V(y) f(y), \quad f \in L^1(X, \mu).$$  \hspace{1cm} (2.1)

Each of the operators $R_V$ will be referred to as a Ruelle operator, or a transition operator; and each $R_V$ clearly maps positive functions to positive functions. (When we say “positive” we do not mean “strictly positive”, but merely “non-negative”.)

We refer to our published papers/monographs [54, 40, 45] about the spectral picture of a positive transition operator $R$ (also called the Ruelle operator, or the Perron–Frobenius–Ruelle operator). Continuing [18], it is shown in [54] that a general family of Hilbert-space constructions, which includes the context of wavelets, may be understood from the so-called Perron–Frobenius eigenspace of $R$. This eigenspace is studied further in the book [19] by Jorgensen and O. Bratteli; see also [55]. It is shown in [54] that a scaling identity (alias refinement equation)

$$\varphi(x) = \sqrt{N} \sum_{k \in \mathbb{Z}} a_k \varphi(Nx - k)$$  \hspace{1cm} (2.2)

may be viewed generally and abstractly. Variations of (2.2) open up for a variety of new applications which we outline briefly below.

**3. Motivating examples, nonlinearity**

In this paper we outline how basic operator-theoretic notions from wavelet theory (such as the successful multiresolution construction) may be adapted to certain state spaces in nonlinear dynamics. We will survey and extend our recent papers on multiresolution analysis of state spaces in symbolic shift dynamics $X(A)$, and on a class of substitution systems $X(r)$ which includes the Julia sets; see, e.g., our papers [42, 43, 44, 45]. Our analysis of these systems $X$ starts with consideration
of Hilbert spaces of functions on $X$. But already this point demands new tools. So the first step in our work amounts to building the right Hilbert spaces. That part relies on tools from both operator algebras, and from probability theory (e.g., path-space measures and martingales).

First we must identify the appropriate measures on $X$, and secondly, we must build Hilbert spaces on associated extension systems $X_\infty$, called generalized solenoids.

The appropriate measures $\mu$ on $X$ will be constructed using David Ruelle’s thermodynamical formalism \cite{ruelle}: We will select our distinguished measures $\mu$ on $X$ to have minimal free energy relative to a certain function $W$ on $X$. The relationship between the measures $\mu$ on $X$ and a class of induced measures on the extensions $X_\infty$ is based on a random-walk model which we developed in \cite{random}, such that the relevant measures on $X_\infty$ are constructed as path-space measures on paths which originate on $X$. The transition on paths is governed in turn by a given and prescribed function $W$ on $X$.

In the special case of traditional wavelets, $W$ is the absolute square of a so-called low-pass wavelet filter. In these special cases, the wavelet-filter functions represent a certain system of functions on the circle $T = \mathbb{R}/\mathbb{Z}$ which we outline below; see also our paper \cite{traditional}. Even our analysis in \cite{traditional} includes wavelet bases on affine Cantor sets in $\mathbb{R}^d$. Specializing our Julia-set cases $X = X(r)$ to the standard wavelet systems, we get $X = T$, and the familiar approach to wavelet systems becomes a special case.

According to its original definition, a \textit{wavelet} is a function $\psi \in L^2(\mathbb{R})$ that generates an orthonormal basis under the action of two unitary operators: the dilation and the translation. In particular,

$$\{ U^j T^k \psi \mid j, k \in \mathbb{Z} \}$$

must be an orthonormal basis for $L^2(\mathbb{R})$, where

$$Uf(x) = \frac{1}{\sqrt{2}} f \left( \frac{x}{2} \right), \quad Tf(x) = f(x-1), \quad f \in L^2(\mathbb{R}), x \in \mathbb{R}. \quad (3.1)$$

One of the effective ways of getting concrete wavelets is to first look for a \textit{multiresolution analysis} (MRA), that is to first identify a suitable telescoping nest of subspaces $(V_n)_{n \in \mathbb{Z}}$ of $L^2(\mathbb{R})$ that have trivial intersection and dense union, $V_n = U V_{n-1}$, and $V_0$ contains a \textit{scaling function} $\varphi$ such that the translates of $\varphi$ form an orthonormal basis for $V_0$.

The scaling function will necessarily satisfy an equation of the form

$$U \varphi = \sum_{k \in \mathbb{Z}} a_k T^k \varphi, \quad (3.2)$$

called the scaling equation (see also \cite{scaling}).

To construct wavelets, one has to choose a \textit{low-pass filter}

$$m_0(z) = \sum_{k \in \mathbb{Z}} a_k z^k \quad (3.3)$$
and from this obtain the scaling function \( \phi \) as the fixed point of a certain cascade operator. The wavelet \( \psi \) (mother function) is constructed from \( \phi \) (a father function, or scaling function) with the aid of two closed subspaces \( V_0 \), and \( W_0 \). The scaling function \( \phi \) is obtained as the solution to a scaling equation (3.2), and its integral translates generate \( V_0 \) (the initial resolution), and \( \psi \) with its integral translates generating \( W_0 := V_1 \ominus V_0 \), where \( V_1 \) is the next finer resolution subspace containing the initial resolution space \( V_0 \), and obtained from a one-step refinement of \( V_0 \).

Thus the main part of the construction involves a clever choice for the low-pass filter \( m_0 \) that gives rise to nice orthogonal wavelets.

### 3.1. MRAs in geometry and operator theory

Let \( X \) be a compact Hausdorff space, and let \( r : X \to X \) be a finite-to-one continuous endomorphism, mapping \( X \) onto itself. As an example, \( r = r(z) \) could be a rational mapping of the Riemann sphere, and \( X \) could be the corresponding Julia set; or \( X \) could be the state space of a subshift associated to a 0–1 matrix.

Due to work of Brolin [25] and Ruelle [72], it is known that for these examples, \( X \) carries a unique maximal entropy, or minimal free-energy measure \( \mu \) also called strongly \( r \)-invariant; see Lemma 4.1.3 (ii), and equations (4.20) and (4.27)–(4.28) below. For each point \( x \in X \), the measure \( \mu \) distributes the “mass” equally on the finite number of solutions \( y \) to \( r(y) = x \).

We show that this structure in fact admits a rich family of wavelet bases. Now this will be on a Hilbert space which corresponds to \( L^2(\mathbb{R}) \) in the familiar case of multiresolution wavelets. These are the wavelets corresponding to scaling \( x \) to \( N \cdot x \), by a fixed integer \( N \), \( N \geq 2 \). In that case, \( X = T \), the circle in \( \mathbb{C} \), and \( r(z) = z^N \). So even in the “classical” case, there is a “unitary dilation” from \( L^2(X) \) to \( L^2(\mathbb{R}) \) in which the Haar measure on \( T \) “turns into” the Lebesgue measure on \( \mathbb{R} \).

Our work so far, on examples, shows that this viewpoint holds promise for understanding the harmonic analysis of Julia sets, and of other iteration systems. In these cases, the analogue of \( L^2(\mathbb{R}) \) involves quasi-invariant measures \( \mu_\infty \) on a space \( X_\infty \), built from \( X \) in a way that follows closely the analogue of passing from \( T \) to \( \mathbb{R} \) in wavelet theory. But \( \mu_\infty \) is now a path-space measure. The translations by the integers \( \mathbb{Z} \) on \( L^2(\mathbb{R}) \) are multiplications in the Fourier dual. For the Julia examples, this corresponds to the coordinate multiplication, i.e., multiplication by \( z \) on \( L^2(X, \mu) \), a normal operator. We get a corresponding covariant system on \( X_\infty \), where multiplication by \( f(z) \) is unitarily equivalent to multiplication by \( f(r(z)) \). But this is now on the Hilbert space \( L^2(X_\infty) \), and defined from the path-space measure \( \mu_\infty \). Hence all the issues that are addressed since the mid-1980’s for \( L^2(\mathbb{R}) \)-wavelets have analogues in the wider context, and they appear to reveal interesting spectral theory for Julia sets.
3.2. Spectrum and geometry: wavelets, tight frames, and Hilbert spaces on Julia sets

**Background.** Problems in dynamics are in general irregular and chaotic, lacking the internal structure and homogeneity which is typically related to group actions. Lacking are the structures that lie at the root of harmonic analysis, i.e., the torus or \( \mathbb{R}^d \).

Chaotic attractors can be viewed simply as sets, and they are often lacking the structure of manifolds, or groups, or even homogeneous manifolds. Therefore, the “natural” generalization, and question to ask, then depends on the point of view, or on the application at hand.

Hence, many applied problems are typically not confined to groups; examples are turbulence, iterative dynamical systems, and irregular sampling. But there are many more. Yet harmonic analysis and more traditional MRA theory begins with \( \mathbb{R}^d \), or with some such group context.

The geometries arising in wider contexts of applied mathematics might be attractors in discrete dynamics, in substitution tiling problems, or in complex substitution schemes, of which the Julia sets are the simplest. These geometries are not tied to groups at all. And yet they are very algorithmic, and they invite spectral-theoretic computations.

Julia sets are prototypical examples of chaotic attractors for iterative discrete dynamical systems; i.e., for iterated substitution of a fixed rational function \( r(z) \), for \( z \) in the Riemann sphere. So the Julia sets serve beautifully as a testing ground for discrete algorithms, and for analysis on attractors. In our papers [44], [42] we show that multiscale/MRA analysis adapts itself well to discrete iterative systems, such as Julia sets, and state space for subshifts in symbolic dynamics. But so far there is very little computational harmonic analysis outside the more traditional context of \( \mathbb{R}^d \).

**Definition 3.2.1.** Let \( S \) be the Riemann sphere, and let \( r: S \to S \) be a rational map of degree greater than one. Let \( r^n \) be the \( n \)th iteration of \( r \), i.e., the \( n \)th iterated substitution of \( r \) into itself. The Fatou set \( F(r) \) of \( r \) is the largest open set \( U \) in \( S \) such that the sequence \( r^n \), restricted to \( U \), is a normal family (in the sense of Montel). The **Julia set** \( X(r) \) is the complement of \( F(r) \).

Moreover, \( X(r) \) is known to behave (roughly) like an attractor for the discrete dynamical system \( r^n \). But it is more complicated: the Julia set \( X(r) \) is the locus of expanding and chaotic behavior; e.g., \( X(r) \) is equal to the closure of the set of repelling periodic points for the dynamical system \( r^n \) in \( S \).

In addition, \( X = X(r) \) is the minimal closed set \( X \), such that \( |X| > 2 \), and \( X \) is invariant under the branches of \( r^{-1} \), the inverses of \( r \).

While our prior research on Julia sets is only in the initial stages, there is already some progress. We are especially pleased that Hilbert spaces of martingales have offered the essential framework for analysis on Julia sets. The reason is that martingale tools are more adapted to irregular structures than are classical Fourier methods.
To initiate a harmonic analysis outside the classical context of the tori and of $\mathbb{R}^d$, it is thus natural to begin with the Julia sets. Furthermore, the Julia sets are already of active current research interest in geometry and dynamical systems.

But so far, there are only sporadic attempts at a harmonic analysis for Julia sets, let alone the more general geometries outside the classical context of groups. Our prior work should prepare us well for this new project. Now, for Julia-set geometries, we must begin with the Hilbert space. And even that demands new ideas and new tools.

To accommodate wavelet solutions in the wider context, we built new Hilbert spaces with the use of tools from martingale theory. This is tailored to applications we have in mind in geometry and dynamical systems involving Julia sets, fractals, subshifts, or even more general discrete dynamical systems.

The construction of solutions requires a combination of tools that are somewhat non-traditional in the subject.

A number of wavelet constructions in pure and applied mathematics have a common operator-theoretic underpinning. It may be illustrated with the following operator system: $\mathcal{H}$ some Hilbert space; $U: \mathcal{H} \to \mathcal{H}$ a unitary operator; $V: \mathbb{T} \to \mathcal{U}(\mathcal{H})$ a unitary representation. Here $\mathbb{T}$ is the 1-torus, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and $\mathcal{U}(\mathcal{H})$ is the group of unitary operators in $\mathcal{H}$.

The operator system satisfies the identity

$$V(z)^{-1} UV(z) = zU, \quad z \in \mathbb{T}. \quad (3.4)$$

**Definition 3.2.2.** We say that $U$ is homogeneous of degree one with respect to the scaling automorphisms defined by $\{V(z) \mid z \in \mathbb{T}\}$ if (3.4) holds. In addition, $U$ must have the spectral type defined by Haar measure. We say that $U$ is homogeneous of degree one if it satisfies (3.4) for some representation $V(z)$ of $\mathbb{T}$.

In the case of the standard wavelets with scale $N$, $\mathcal{H} = L^2(\mathbb{R})$, where in this case the two operators $U$ and $T$ are the usual $N$-adic scaling and integral translation operators; see (3.1) above giving $U$ for the case $N = 2$.

If $V_0 \subset \mathcal{H}$ is a resolution subspace, then $V_0 \subset U^{-1}V_0$. Set $W_0 := U^{-1}V_0 \ominus V_0$, $Q_0 :=$ the projection onto $W_0$, $Q_k := U^{-k}Q_0U^k$, $k \in \mathbb{Z}$, and

$$V(z) = \sum_{k=-\infty}^{\infty} z^k Q_k, \quad z \in \mathbb{T}. \quad (3.5)$$

Then it is easy to check that the pair $(U,V(z))$ satisfies the commutation identity (3.4).

Let $(\psi_i)_{i \in I}$ be a Parseval frame in $W_0$. Then

$$\{U^k T_n \psi_i \mid i \in I, k, n \in \mathbb{Z}\} \quad (3.6)$$

is a Parseval frame for $\mathcal{H}$. (Recall, a Parseval frame is also called a normalized tight frame.)

Turning the picture around, we may start with a system $(U,V(z))$ which satisfies (3.4), and then reconstruct wavelet (or frame) bases in the form (3.6).
To do this in the abstract, we must first understand the multiplicity function calculated for \( \{ T_n \mid n \in \mathbb{Z} \} \) when restricted to the two subspaces \( V_0 \) and \( W_0 \). But such a multiplicity function may be be defined for any representation of an abelian \( C^* \)-algebra acting on \( \mathcal{H} \) which commutes with some abstract scaling automorphism \( V(z) \).

This technique can be used for the Hilbert spaces associated to Julia sets, to construct wavelet (and frame) bases in this context.

**Wavelet filters in nonlinear models.** These systems are studied both in the theory of symbolic dynamics, and in \( C^* \)-algebra theory, i.e., for the Cuntz–Krieger algebras.

It is known [25, 71] that, for these examples, \( X \) carries a unique maximal entropy, or minimal free-energy measure \( \mu \). The most general case when \((X, r)\) admits a strongly \( r \)-invariant measure is not well understood.

The intuitive property of such measures \( \mu \) is this: For each point \( x \) in \( X \), \( \mu \) distributes the “mass” equally on the finite number of solutions \( y \) to \( r(y) = x \). Then \( \mu \) is obtained by an iteration of this procedure, i.e., by considering successively the finite sets of solutions \( y \) to \( r^n(y) = x \), for all \( n = 1, 2, \ldots \); taking an average, and then limit. While the procedure seems natural enough, the structure of the convex set \( K(X, r) \) of all the \( r \)-invariant measures is worthy of further study. For the case when \( X = \mathbb{T} \) the circle, \( \mu \) is unique, and it is the Haar measure on \( \mathbb{T} \).

The invariant measures are of interest in several areas: in representation theory, in dynamics, in operator theory, and in \( C^* \)-algebra theory. Recent work of Dutkay–Jorgensen [43, 44, 42] focuses on special convex subsets of \( K(X, r) \) and their extreme points. This work in turn is motivated by our desire to construct wavelet bases on Julia sets.

Our work on \( K(X, r) \) extends what is known in traditional wavelet analysis. While linear wavelet analysis is based on Lebesgue measure on \( \mathbb{R}^d \), and Haar measure on the \( d \)-torus \( \mathbb{T}^d \), the nonlinear theory is different, and it begins with results from geometric measure theory: We rely on results from [25], [60], [51], and [61]. And our adaptation of spectral theory in turn is motivated by work by Baggett et al. in [51].

Related work reported in [61] further ties in with exciting advances on graph \( C^* \)-algebras pioneered by Paul Muhly, Iain Raeburn and many more in the \( C^* \)-algebra community; see [3, 10], [13], [20], [21], [31], [35], [65], [66], [68].

We now prepare the tools in outline, function systems and self-similar operators which will be used in Subsection 4.1 in the construction of specific Hilbert bases in the context of Julia sets.

Specifically, we outline a construction of a multiresolution/wavelet analysis for the Julia set \( X \) of a given rational function \( r(z) \) of one complex variable, i.e., \( X = \text{Julia}(r) \). Such an analysis could likely be accomplished with some kind of wavelet representation induced by a normal operator \( T \) which is unitarily equivalent to a function of itself. Specifically, \( T \) is unitarily equivalent to \( r(T) \), i.e.,
\[ UT = r(T)U, \quad \text{for some unitary operator } U \text{ in a Hilbert space } \mathcal{H}. \] (3.7)

Even the existence of these representations would seem to be new and significant in operator theory.

There are consequences and applications of such a construction: First we will get the existence of a finite set of some special generating functions \( m_i \) on the Julia set \( X \) that may be useful for other purposes, and which are rather difficult to construct by alternative means. The relations we require for a system \( m_0, \ldots, m_{N-1} \) of functions on \( X(r) \) are as follows:

\[
\frac{1}{N} \sum_{y \in X(r)} m_i(y) m_j(y) h(y) = \delta_{i,j} h(x) \quad \text{for a.e. } x \in X(r),
\] (3.8)

where \( h \) is a Perron–Frobenius eigenfunction for a Ruelle operator \( R \) defined on \( L^\infty(X(r)) \). Secondly, it follows that for each Julia set \( X \), there is an infinite-dimensional group which acts transitively on these systems of functions. The generating functions on \( X \) we are constructing are analogous to the more familiar functions on the circle \( T \) which define systems of filters in wavelet analysis. In fact, the familiar construction of a wavelet basis in \( \mathcal{H} = L^2(\mathbb{R}) \) is a special case of our analysis.

In the standard wavelet case, the rational function \( r \) is just a monomial, i.e., \( r(z) = z^N \) where \( N \) is the number of frequency bands, and \( X \) is the circle \( T \) in the complex plane. The simplest choice of the functions \( m_0, \ldots, m_{N-1} \) and \( h \) in this case is \( m_k(z) = z^k, \ z \in T, \ 0 \leq k \leq N-1, \) and \( h(z) \equiv 1, \ z \in T. \) This represents previous work by many researchers, and also joint work between Jorgensen and Bratteli [19].

Many applied problems are typically not confined to groups; examples are turbulence, iterative dynamical systems, and irregular sampling. But there are many more.

Even though attractors such as Julia sets are highly nonlinear, there are adaptations of the geometric tools from the analysis on \( \mathbb{R} \) to the nonlinear setting.

The adaptation of traditional wavelet tools to nonlinear settings begins with an essential step: the construction of an appropriate Hilbert space; hence the martingales.

Jorgensen, Bratteli, and Dutkay have developed a representation-theoretic duality for wavelet filters and the associated wavelets, or tight wavelet frames. It is based on representations of the Cuntz algebra \( O_N \) (where \( N \) is the scale number); see [19, 39, 44, 53]. As a by-product of this approach they get infinite families of inequivalent representations of \( O_N \) which are of independent interest in operator-algebra theory.

The mathematical community has long known of operator-theoretic power in Fourier-analytic issues, e.g., Rieffel’s incompleteness theorem for Gabor systems violating the Nyquist condition, cf. [34]. We now feel that we can address a
number of significant problems combining operator-theoretic and harmonic analysis methods, as well as broadening and deepening the mathematical underpinnings of our subject.

3.3. Multiresolution analysis (MRA)

One reason various geometric notions of multiresolution in Hilbert space (starting with S. Mallat) have proved useful in computational mathematics is that these resolutions are modeled on the fundamental concept of dyadic representation of the real numbers; or more generally on some variant of the classical positional representation of real numbers. In a dyadic representation of real numbers, the shift corresponds to multiplication by $2^{-1}$.

The analogue of that in Hilbert space is a unitary operator $U$ which scales functions by $2^{-1}$. If the Hilbert space $\mathcal{H}$ is $L^2(\mathbb{R})$, then a resolution is a closed subspace $V_0$ in $\mathcal{H}$ which is invariant under $U$ with the further property that the restriction of $U$ to $V_0$ is a shift operator. Positive and negative powers of $U$ then scale $\mathcal{H}$ into a nested resolution system indexed by the integers; and this lets us solve algorithmic problems by simply following the standard rules from number theory.

However, there is no reason at all that use of this philosophy should be restricted to function spaces on the line, or on Euclidean space. Other geometric structures from discrete dynamics admit resolutions as well.

Most MRA, Fourier multiresolution analysis (FMRA) and generalized multiresolution analysis (GMRA) wavelet constructions in pure and applied mathematics have a common operator-theoretic underpinning. Consider the operator system $$(\mathcal{H}, U, V(z))$$ from (3.4) above.

**Proposition 3.3.1.** Once the system $(\mathcal{H}, U, V)$ is given as in (3.4), a scale of closed subspaces $V_n$ (called resolution subspaces) may be derived from the spectral subspaces of the representation $\mathcal{T} \ni z \mapsto V(z)$, i.e., subspaces $(V_n)_{n \in \mathbb{Z}}$ such that $V_n \subset V_{n+1}$,

$$\bigcap V_n = \{0\}, \quad \bigcup V_n \text{ is dense in } \mathcal{H},$$

and $UV_n \subset V_{n-1}$.

Conversely, if these spaces $(V_n)$ are given, then $V(z)$ defined by equation (3.5) can be shown to satisfy (3.4) if

$$Q_0 = \text{the orthogonal projection onto } U^{-1}V_0 \ominus V_0,$$

and $Q_n = U^{-n}Q_0U^n, \quad n \in \mathbb{Z}$.

As a result we note the following criterion.

**Proposition 3.3.2.** A given unitary operator $U$ in a Hilbert space $\mathcal{H}$ is part of some multiresolution system $(V_n)$ for $\mathcal{H}$ if and only if $U$ is homogeneous of degree one with respect to some representation $V(z)$ of the one-torus $\mathcal{T}$. 
In application the spaces $V_n$ represent a grading of the entire Hilbert space $\mathcal{H}$, and we say that $U$ scales between the different levels. In many cases, this operator-theoretic framework serves to represent structures which are similar up to scale, structures that arise for example in the study of fractional Brownian motion (FBM). See, e.g., [59], which offers a Hilbert-space formulation of FBM based on a white-noise stochastic integration.

Moreover, this can be done for the Hilbert space $\mathcal{H} = L^2(X_\infty, \mu_\infty)$ associated with a Julia set $X$. As a result, we get wavelet bases and Parseval wavelets in this context.

This is relevant for the two mentioned classes of examples, Julia sets $X(r)$, and state spaces $X(A)$ of substitutions or of subshifts. Recall that the translations by the integers $\mathbb{Z}$ on $L^2(\mathbb{R})$ are multiplication in the Fourier dual. For the Julia sets, this corresponds to coordinate multiplication, i.e., multiplication by $z$ on the Hilbert space $L^2(X, \mu)$, consequently a normal operator. The construction will involve Markov processes and martingales.

Thus, many of the concepts related to multiresolutions in $L^2(\mathbb{R})$ have analogues in a more general context; moreover, they seem to exhibit interesting connections to the geometry of Julia sets.

**Pyramid algorithms and geometry.** Several of the co-authors’ recent projects involve some variant or other of the Ruelle transfer operator $R$, also called the Ruelle–Perron–Frobenius operator. In each application, it arises in a wavelet-like setting. But the settings are different from the familiar $L^2(\mathbb{R})$-wavelet setup: one is for affine fractals, and the other for Julia sets generated from iteration dynamics of rational functions $r(z)$.

Thus, there are two general themes in this work. In rough outline, they are like this: (1) In the paper [45], Dutkay and Jorgensen construct a new class of wavelets on extended fractals based on Cantor-like iterated function systems, e.g., the middle-third Cantor set. Our Hilbert space in each of our constructions is separable. For the fractals, it is built on Hausdorff measure of the same (fractal) dimension as the IFS-Cantor set in question. In the paper [15], we further introduce an associated Ruelle operator $R$, and it plays a central role. For our wavelet construction, there is, as is to be expected, a natural (equilibrium) measure $\nu$ which satisfies $\nu R = \nu$, i.e., a left Perron–Frobenius eigenvector. It corresponds to the Dirac point-measure on the frequency variable $\omega = 0$ (i.e., low-pass) in the standard $L^2(\mathbb{R})$-wavelet setting. It turns out that our measures $\nu$ for the IFS-Cantor sets are given by infinite Riesz products, and they are typically singular and have support = the whole circle $\mathbb{T}$. This part of our research is related to recent, completely independent, work by Benedetto et al. on Riesz products.

(2) A second related research direction is a joint project in progress with Ola Bratteli, where we build wavelets on Hilbert spaces induced by Julia sets of rational functions $r(z)$ of one complex variable. Let $r$ be a rational function of degree at least 2, and let $X(r)$ be the corresponding Julia set. Then there is a family of Ruelle operators indexed by certain potentials, and a corresponding
family of measures \( \nu \) on \( X(r) \) which satisfy \( \nu R = \nu \), again with the measures \( \nu \) playing a role somewhat analogous to the Dirac point-mass on the frequency variable \( \omega = 0 \), for the familiar \( L^2(\mathbb{R}) \)-MRA wavelet construction.

Independently of Dutkay–Jorgensen \[45\], John Benedetto and his co-authors Erica Bernstein and Ioannis Konstantinidis \[12\] have developed a new Fourier/infinite-product approach to the very same singular measures that arise in the study in \[45\]. The motivations and applications are different. In \[45\] the issue is wavelets on fractals, and in \[12\], it is time-frequency duality.

In this second class of problems, the measures \( \nu \) are far less well understood; and yet, if known, they would yield valuable insight into the spectral theory of \( r \)-iteration systems and the corresponding Julia sets \( X(r) \).

### 3.4. Julia sets from complex dynamics

Wavelet-like constructions are just under the surface in the following brief sketch of problems. The projects all have to do with the kind of dynamical systems \( X \) already mentioned; and \( \Omega \) is one of the standard probability spaces. The space \( X \) could be the circle, or a torus, or a solenoid, or a fractal, or it could be a Julia set defined from a rational map in one complex variable. More generally, consider maps \( r \) of \( X \) onto \( X \) which generate some kind of dynamics. The discussion is restricted to the case when for each \( x \) in \( X \), the pre-image \( r^{-1}(\{x\}) \) is assumed finite. Using then a construction of Kolmogorov, it can be seen that these systems \( X \) admit useful measures which are obtained by a limit construction and by averaging over the finite sets \( r^{-1}(\{x\}) \); see, e.g., Jorgensen’s new monograph \[52\] and Proposition \[13\] for more details. We list three related problems.

1. **Operator algebras**: A more systematic approach, along these lines, to crossed products with endomorphisms. For example, trying to capture the framework of endomorphism-crossed products introduced earlier by Bost and Connes \[16\].
2. **Dynamics**: Generalization to the study of onto-maps, when the number and nature of branches of the inverse is more complicated; i.e., where it varies over \( X \), and where there might be overlaps. (3) **Geometry**: Use of more detailed geometry and topology of Julia sets, in the study of the representations that come from multiplicity considerations. The Dutkay–Jorgensen paper \[45\] on fractals is actually a special case of Julia-set analysis. (One of the Julia sets \( X \) from the above is a Cantor set in the usual sense, e.g., the Julia set of \( r(z) = 2z - 1/z \); or \( r(z) = z^2 - 3 \).)

A main theme will be Hilbert spaces built on Julia sets \( X(r) \) built in turn on rational functions \( r(z) \) in one complex variable. In the case when \( r \) is not a monomial, say of degree \( N \geq 2 \), we construct \( N \) functions \( m_i, i = 0, 1, \ldots, N - 1 \), on \( X(r) \) which satisfy a set of quadratic conditions \[6,8\] analogous to the axioms that define wavelet filters. We have done this, so far, using representation theory, in special cases. But our results are not yet in a form where they can be used in computations. By using results on the Hausdorff dimension of Julia sets, it would also be interesting to identify inequivalent representations, both in the case when the Julia set \( X \) is fixed but the functions vary, and in the case when \( X \) varies.
Moreover, we get an infinite-dimensional “loop group” \( G = G(X, N) \) acting transitively on these sets of functions \( m_i \). The group \( G \) consists of measurable functions from \( X \) into the group \( U_N \) of all unitary \( N \) by \( N \) complex matrices. In particular, our method yields information about this group. Since \( X \) can be geometrically complicated, it is not at all clear how such matrix functions \( X \to U_N(\mathbb{C}) \) might be constructed directly.

The group \( G \) consists of measurable functions \( A: X(r) \to U_N(\mathbb{C}) \), and the action of \( A \) on an \((m_i)\)-system is as follows: \((m_i) \mapsto (m_i^{(A)})\), where

\[
m_i^{(A)}(x) = \sum_j A_{i,j}(r(x))m_j(x), \quad x \in X(r).
\]  

Even when \( N = 2 \), the simplest non-monomial examples include the Julia sets of \( r(z) = z^2 + c \), and the two functions \( m_0 \) and \( m_1 \) on \( X(c) = \text{Julia}(z^2 + c) \) are not readily available by elementary means. The use of operator algebras, representation theory seems natural for this circle of problems.

4. Main results

4.1. Multiresolution/wavelet analysis on Julia sets

We attempt to use equilibrium measures (Brolin \[25\], Ruelle \[71\], Mauldin–Urbanski \[61\]) from discrete dynamical systems to specify generalized low-pass conditions in the Julia-set theory. This would seem to be natural, but there are essential difficulties to overcome. While Ruelle's thermodynamical formalism was developed for transfer operators where the weight function \( W \) is strictly positive, the applications we have in mind dictate careful attention to the case when \( W \) is not assumed strictly positive. The reason is that for our generalized multiresolution analysis on Julia sets, \( W \) will be the absolute square of the low-pass filter.

We begin with one example and a lemma. The example realizes a certain class of state spaces from symbolic dynamics; and the lemma spells out a class of invariant measures which will be needed in our Hilbert-space constructions further into the section. The measures from part (ii) in Lemma 4.1.3 below are often called strongly invariant measures, or equilibrium measures from Ruelle’s thermodynamical formalism, see \[71\].

**Example 4.1.1.** Let \( N \in \mathbb{Z}_+ \), \( N \geq 2 \) and let \( A = (a_{ij})_{i,j=1}^N \) be an \( N \) by \( N \) matrix with all \( a_{ij} \in \{0, 1\} \). Set

\[
X(A) := \{ (x_i) \in \prod_{N} \{1, \ldots, N\} \mid A(x_i, x_{i+1}) = 1 \}
\]

and let \( r = r_A \) be the restriction of the shift to \( X(A) \), i.e.,

\[
r_A(x_1, x_2, \ldots) = (x_2, x_3, \ldots), \quad x = (x_1, x_2, \ldots) \in X(A).
\]

**Lemma 4.1.2.** Let \( A \) be as above. Then

\[
\# r_A^{-1}(x) = \# \{ y \in \{1, \ldots, N\} \mid A(y, x_1) = 1 \}.
\]
It follows that \( r_A : X(A) \to X(A) \) is onto iff \( A \) is irreducible, i.e., iff for all \( j \in \mathbb{Z}_N \), there exists an \( i \in \mathbb{Z}_N \) such that \( A(i, j) = 1 \).

Suppose in addition that \( A \) is aperiodic, i.e., there exists \( p \in \mathbb{Z}_+ \) such that \( A^p > 0 \) on \( \mathbb{Z}_N \times \mathbb{Z}_N \). We have the following lemma.

**Lemma 4.1.3.** (D. Ruelle [22, 9]) Let \( A \) be irreducible and aperiodic and let \( \phi \in C(X(A)) \) be given. Assume that \( \phi \) is a Lipschitz function.

(i) Set

\[
(R_\phi f)(x) = \sum_{r_A(y) = x} e^{\phi(y)} f(y), \quad \text{for } f \in C(X(A)).
\]

Then there exist \( \lambda_0 > 0 \),

\[
\lambda_0 = \sup \{ |\lambda| \mid \lambda \in \text{spec}(R_\phi) \},
\]

\( h \in C(X(A)) \) strictly positive and \( \nu \) a Borel measure on \( X(A) \) such that

\[
R_\phi h = \lambda_0 h,
\]

\[
\nu R_\phi = \lambda_0 \nu,
\]

and \( \nu(h) = 1 \). The data are unique.

(ii) In particular, setting

\[
(R_0 f)(x) = \frac{1}{\# r_A^{-1}(x)} \sum_{r_A(y) = x} f(y),
\]

we may take \( \lambda_0 = 1 \), \( h = 1 \) and \( \nu =: \mu_A \), where \( \mu_A \) is a probability measure on \( X(A) \) satisfying the strong invariance property

\[
\int_{X(A)} f \, d\mu_A = \int_{X(A)} \frac{1}{\# r_A^{-1}(x)} \sum_{r_A(y) = x} f(y) \, d\mu_A(x), \quad f \in L^\infty(X(A)).
\]

Our next main results, Theorems 4.1.4 and Theorem 4.2.1, are operator-theoretic, and they address the following question: Starting with a state space \( X \) for a dynamical system, what are the conditions needed for building a global Hilbert space \( H(X) \) and an associated multiscale structure? Our multiscales will be formulated in the context of Definitions 3.2.2 (or equivalently 3.3.1) above; that is, we define our multiscale structure in terms of a natural Hilbert space \( H(X) \) and a certain unitary operator \( U \) on \( H(X) \) which implements the multiscales.

After our presentation of the Hilbert-space context, we turn to spectral theory: In our next result, we revisit Baggett’s dimension-consistency equation. The theorem of Baggett et al. [8] concerns a condition on a certain pair of multiplicity functions for the existence of a class of wavelet bases for \( L^2(\mathbb{R}^d) \). In Theorem 4.2.2 below, we formulate the corresponding multiplicity functions in the context of nonlinear dynamics, and we generalize the Baggett et al. theorem to the Hilbert spaces \( H(X) \) of our corresponding state spaces \( X \).

As preparation for Theorem 4.3.6 below, we introduce a family of \( L^2 \)-martingales, and we prove that the Hilbert spaces \( H(X) \) are \( L^2 \)-martingale Hilbert spaces. In this context, we prove the existence of an explicit wavelet transform.
(Theorem 4.3.6) for our nonlinear state spaces. But in the martingale context, we must work with dilated state spaces $X_\infty$; and in Proposition 4.3.9 and Theorems 5.8.1, 5.8.2, 5.8.9, and 5.8.12, we outline the role of our multiscale structure in connection with certain random-walk measures on $X_\infty$. A second issue concerns extreme measures: in Theorem 5.8.8 we characterize the extreme points in the convex set of these random-walk measures.

**Theorem 4.1.4.** Let $A$ be a unital $C^*$-algebra, $\alpha$ an endomorphism on $A$, $\mu$ a state on $A$ and, $m_0 \in A$, such that

$$\mu(m_0^*\alpha(f)m_0) = \mu(f), \quad f \in A.$$  \hfill (4.1)

Then there exists a Hilbert space $H$, a representation $\pi$ of $A$ on $H$, $U$ a unitary on $H$, and a vector $\varphi \in H$, with the following properties:

$$U\pi(f)U^* = \pi(\alpha(f)), \quad f \in A,$$  \hfill (4.2)

$$\langle \varphi | \pi(f)\varphi \rangle = \mu(f), \quad f \in A,$$  \hfill (4.3)

$$U\varphi = \pi(\alpha(1)m_0)\varphi$$  \hfill (4.4)

$$\text{span}\{ U^{-n}\pi(f)\varphi \mid n \geq 0, f \in A \} = H.$$  \hfill (4.5)

Moreover, this is unique up to an intertwining isomorphism.

We call $(H, U, \pi, \varphi)$ the covariant system associated to $\mu$ and $m_0$.

**Corollary 4.1.5.** Let $X$ be a measure space, $r: X \to X$ a measurable, onto map and $\mu$ a probability measure on $X$ such that

$$\int_X f \, d\mu = \int_X \frac{1}{\# r^{-1}(x)} \sum_{r(y) = x} f(y) \, d\mu(x).$$  \hfill (4.6)

Let $h \in L^1(X)$, $h \geq 0$ such that

$$\frac{1}{\# r^{-1}(x)} \sum_{r(y) = x} |m_0(y)|^2 h(y) = h(x), \quad x \in X.$$

Then there exists (uniquely up to isomorphism) a Hilbert space $H$, a unitary $U$, a representation $\pi$ of $L^\infty(X)$ and a vector $\varphi \in H$ such that

$$U\pi(f)U^{-1} = \pi(f \circ r), \quad f \in L^\infty(X),$$

$$\langle \varphi | \pi(f)\varphi \rangle = \int_X fh \, d\mu, \quad f \in L^\infty(X),$$

$$U\varphi = \pi(m_0)\varphi,$$

$$\text{span}\{ U^{-n}\pi(f)\varphi \mid n \geq 0, f \in L^\infty(X) \} = H.$$  \hfill (4.5)

We call $(H, U, \pi, \varphi)$ the covariant system associated to $m_0$ and $h$. 
4.2. The Baumslag–Solitar group

Our next theorem is motivated by the theory of representations of a certain group which is called the Baumslag–Solitar group, and which arises by the following simple semidirect product construction: the Baumslag–Solitar group $BS(1, N)$ is the group with two generators $u$ and $t$ and one relation $utu^{-1} = tN$, where $N$ is a fixed positive integer. Therefore the two operators $U$ of $N$-adic dilation and $T$ of integral translation operators on $\mathbb{R}$ give a representation of this group. But so do all familiar MRA-wavelet constructions.

Representations of the Baumslag–Solitar group that admit wavelet bases can be constructed also on some other spaces, such as $L^2(\mathbb{R}) \oplus \cdots \oplus L^2(\mathbb{R})$ or some fractal spaces \cite{45}. Hence wavelet theory fits naturally into this abstract setting of special representations of the group $BS(1, N)$ realized on some Hilbert space.

It is already known \cite{39} that the representations that admit wavelets are faithful and weakly equivalent to the right regular representation of the group.

**Theorem 4.2.1.** (i) Let $H$ be a Hilbert space, $S$ an isometry on $H$. Then there exist a Hilbert space $\hat{H}$ containing $H$ and a unitary $\hat{S}$ on $\hat{H}$ such that

$$\hat{S}|_H = S,$$

$$\bigcup_{n \geq 0} \hat{S}^{-n}H = \hat{H}. \quad (4.8)$$

Moreover these are unique up to an intertwining isomorphism.

(ii) If $A$ is a $C^*$-algebra, $\alpha$ is an endomorphism on $A$ and $\pi$ is a representation of $A$ on $H$ such that

$$S\pi(g) = \pi(\alpha(g))S, \quad g \in A, \quad (4.9)$$

then there exists a unique representation $\hat{\pi}$ on $\hat{H}$ such that

$$\hat{\pi}(g)|_H = \pi(g), \quad g \in A, \quad (4.10)$$

$$\hat{S}\hat{\pi}(g) = \hat{\pi}(\alpha(g))\hat{S}, \quad g \in A. \quad (4.11)$$

**Corollary 4.2.2.** Let $X, r,$ and $\mu$ be as in Corollary 4.1.5. Let $I$ be a finite or countable set. Suppose $H \colon X \to B(\ell^2(I))$ has the property that $H(x) \geq 0$ for almost every $x \in X$, and $H_{ij} \in L^1(X)$ for all $i, j \in I$. Let $M_0 \colon X \to B(\ell^2(I))$ such that $x \mapsto \|M_0(x)\|$ is essentially bounded. Assume in addition that

$$\frac{1}{\# r^{-1}(x)} \sum_{y \in r^{-1}(x)} M_0^*(y)H(y)M_0(y) = H(x), \quad \text{for a.e. } x \in X. \quad (4.12)$$

Then there exists a Hilbert space $\hat{K}$, a unitary operator $\hat{U}$ on $\hat{K}$, a representation $\hat{\pi}$ of $L^\infty(X)$ on $\hat{K}$, and a family of vectors $(\varphi_i) \in \hat{K}$, such that

$$\hat{U}\hat{\pi}(g)\hat{U}^{-1} = \hat{\pi}(g \circ r), \quad g \in L^\infty(X),$$
\[ \hat{U}_i \varphi_i = \sum_{j \in I} \hat{\pi}((M_0)_j) \varphi_j, \quad i \in I, \]
\[ \langle \varphi_i \mid \hat{\pi}(f) \varphi_j \rangle = \int_X f H_{ij} \, d\mu, \quad i, j \in I, \quad f \in L^\infty(X), \]
\[ \text{span}\{ \hat{\pi}(f) \varphi_i \mid n \geq 0, \ f \in L^\infty(X), \ i \in I \} = \hat{K}. \]

These are unique up to an intertwining unitary isomorphism. (All functions are assumed weakly measurable in the sense that \( x \mapsto \langle \xi \mid F(x) \eta \rangle \) is measurable for all \( \xi, \eta \in \ell^2(I) \).)

Suppose now that \( H \) is a Hilbert space with an isometry \( S \) on it and with a normal representation \( \pi \) of \( L^\infty(X) \) on \( H \) that satisfies the covariance relation
\[ S \pi(g) = \pi(g \circ r) S, \quad g \in L^\infty(X). \] (4.13)

Theorem 4.2.3. (i) \( V_1 = \hat{S}^{-1}(H) \) is invariant for the representation \( \hat{\pi} \). The multiplicity functions of the representation \( \hat{\pi} \) on \( V_1 \), and on \( V_0 = H \), are related by
\[ m_{V_1}(x) = \sum_{r(y) = x} m_{V_0}(y), \quad x \in X. \] (4.14)

(ii) If \( W_0 := V_1 \ominus V_0 = \hat{S}^{-1} H \ominus H \), then
\[ m_{V_0}(x) + m_{W_0}(x) = \sum_{r(y) = x} m_{V_0}(y), \quad x \in X. \] (4.15)

Proof. Note that \( \hat{S} \) maps \( V_1 \) to \( V_0 \), and the covariance relation implies that the representation \( \hat{\pi} \) on \( V_1 \) is isomorphic to the representation \( \pi^r : g \mapsto \pi(g \circ r) \) on \( V_0 \). Therefore we have to compute the multiplicity of the latter, which we denote by \( m_{V_0}^r \).

By the spectral theorem there exists a unitary isomorphism \( J : H(= V_0) \to L^2(X, m_{V_0}, \mu) \), where, for a multiplicity function \( m : X \to \{0, 1, \ldots, \infty\} \), we use the notation
\[ L^2(X, m, \mu) := \{ f : X \to \bigcup_{x \in X} \mathbb{C}^{m(x)} \mid f(x) \in \mathbb{C}^{m(x)}, \int_X \|f(x)\|^2 \, d\mu(x) < \infty \}. \]
In addition \( J \) intertwines \( \pi \) with the representation of \( L^\infty(X) \) by multiplication operators, i.e.,
\[ (J \pi(g) J^{-1}(f))(x) = g(x) f(x), \quad g \in L^\infty(X), f \in L^2(X, m_{V_0}, \mu), x \in X. \]
Remark 4.2.4. Here we are identifying $H$ with $L^2(X, m_{\ell^2}), \mu)$ via the spectral representation. We recall the details of this representation $H \ni f \mapsto \hat{f} \in L^2(X, m_{\ell^2}, \mu)$.

Recall that any normal representation $\pi \in \text{Rep}(L^\infty(X), H)$ is the orthogonal sum

$$H = \bigoplus_{k \in C} [\pi(L^\infty(X))k],$$

where the set $C$ of vectors $k \in H$ is chosen such that

- $\|k\| = 1$,
- $\langle k | \pi(g)k \rangle = \int_X g(x)v_k(x)^2 \, d\mu(x)$, for all $k \in C$;
- $\langle k' | \pi(g)k \rangle = 0$, $g \in L^\infty(X), k, k' \in C, k \neq k'$; orthogonality.

The formula (4.16) is obtained by a use of Zorn’s lemma. Here, $v_k^2$ is the Radon-Nikodym derivative of $\langle k | \pi(\cdot)k \rangle$ with respect to $\mu$, and we use that $\pi$ is assumed normal.

For $f \in H$, set

$$f = \sum_{k \in C} \pi(g_k)k, \quad g_k \in L^\infty(X)$$

and

$$\hat{f} = \sum_{k \in C} g_k v_k \in L^2_\mu(X, \ell^2(C)).$$

Then $Wf = \hat{f}$ is the desired spectral transform, i.e.,

$$W \text{ is unitary,}$$

$$W\pi(g) = M(g)W,$$

and

$$\|\hat{f}(x)\|^2 = \sum_{k \in C} |g_k(x)v_k(x)|^2.$$

Indeed, we have

$$\int_X \|\hat{f}(x)\|^2 \, d\mu(x) = \int_X \sum_{k \in C} |g_k(x)|^2 v_k(x)^2 \, d\mu(x) = \sum_{k \in C} \int_X |g_k|^2 v_k^2 \, d\mu = \sum_{k \in C} \langle k | \pi(|g_k|^2)k \rangle = \sum_{k \in C} ||\pi(g_k)k||^2 = \left\| \sum_{k \in C} \pi(g_k)k \right\|_H^2 = \|f\|_H^2.$$

It follows in particular that the multiplicity function $m(x) = m_H(x)$ is

$$m(x) = \# \{ k \in C | v_k(x) \neq 0 \}.$$

Setting

$$X_i := \{ x \in X | m(x) \geq i \}, \quad i \geq 1,$$

we see that

$$H \simeq \bigoplus L^2(X_i, \mu) \simeq L^2(X, m, \mu),$$

and the isomorphism intertwines $\pi(g)$ with multiplication operators.
Returning to the proof of the theorem, we have to find the similar form for the representation \( \pi^r \). Let

\[
\tilde{m}(x) := \sum_{r(y) = x} m_{V_0}(y), \quad x \in X.
\] (4.17)

Define the following unitary isomorphism:

\[
L: L^2(X, m_{V_0}, \mu) \to L^2(X, \tilde{m}, \mu),
\]

\[
(L\xi)(x) = \frac{1}{\sqrt{\# r^{-1}(x)}} (\xi(y))_{r(y) = x}.
\]

(Note that the dimensions of the vectors match because of (4.17).) This operator \( L \) is unitary. For \( \xi \in L^2(X, m_{V_0}, \mu) \), we have

\[
\|L\xi\|^2_{L^2(X, m_{V_0}, \mu)} = \int_X \|L\xi(x)\|^2 d\mu(x)
\]

\[
= \int_X \frac{1}{\# r^{-1}(x)} \sum_{r(y) = x} \|\xi(y)\|^2 d\mu(x)
\]

\[
= \int_X \|\xi(x)\|^2 d\mu(x).
\]

And \( L \) intertwines the representations. Indeed, for \( g \in L^\infty(X) \),

\[
L(g \circ r \xi)(x) = (g(r(y))\xi(y))_{r(y) = x} = g(x)L(\xi)(x).
\]

Therefore, the multiplicity of the representation \( \pi^r: g \mapsto \pi(g \circ r) \) on \( V_0 \) is \( \tilde{m} \), and this proves (i).

(ii) follows from (i).

Conclusions. By definition, if \( k \in C \),

\[
\langle k | \pi(g)k \rangle = \int_X g(x)v_k(x)^2 d\mu(x), \quad \text{and}
\]

\[
\langle k | \pi^r(g)k \rangle = \int_X g(r(x))v_k(x)^2 d\mu(x) = \int_X g(x) \frac{1}{\# r^{-1}(x)} \sum_{r(y) = x} v_k(x)^2 d\mu(x);
\]

and so

\[
m^r(x) = \# \{ k \in C | \sum_{r(y) = x} v_k(y)^2 > 0 \}
\]

\[
= \sum_{r(y) = x} \# \{ k \in C | v_k(y)^2 > 0 \}
\]

\[
= \sum_{r(y) = x} m(y).
\]

Let \( C^m(x) := \{ k \in C | v_k(x) \neq 0 \} \). Then we showed that

\[
C^m(x) = \bigcup_{y \in X, r(y) = x} C^m(y)
\]
and that $C^m(y) \cap C^m(y') = \emptyset$ when $y \neq y'$ and $r(y) = r(y') = x$. Setting $H(x) = \ell^2(C^m(x))$, we have

\[
H(x) = \ell^2(C^m(x)) = \bigoplus_{r(y)=x} \ell^2(C^m(y)) = \bigoplus_{r(y)=x} H(y). \quad \square
\]

4.3. Spectral decomposition of covariant representations: projective limits

We give now a different representation of the construction of the covariant system associated to $m_0$ and $h$.

We work in either the category of measure spaces or topological spaces.

**Definition 4.3.1.** Let $r : X \to X$ be onto, and assume that $\# r^{-1}(x) < \infty$ for all $x \in X$. We define the projective limit of the system:

\[
X \leftarrow X \leftarrow X \cdots \leftarrow X \infty
\]

as

\[
X_\infty := \{ \hat{x} = (x_0, x_1, \ldots) | r(x_{n+1}) = x_n, \text{ for all } n \geq 0 \}.
\]

Let $\theta_n : X_\infty \to X$ be the projection onto the $n$-th component:

\[
\theta_n(x_0, x_1, \ldots) = x_n, \quad (x_0, x_1, \ldots) \in X_\infty.
\]

Taking inverse images of sets in $X$ through these projections, we obtain a sigma-algebra on $X_\infty$, or a topology on $X_\infty$.

We have an induced mapping $\hat{r} : X_\infty \to X_\infty$ defined by

\[
\hat{r}(\hat{x}) = (r(x_0), x_0, x_1, \ldots), \quad \text{and with inverse } \hat{r}^{-1}(\hat{x}) = (x_1, x_2, \ldots).
\]

so $\hat{r}$ is an automorphism, i.e., $\hat{r} \circ \hat{r}^{-1} = \text{id}_{X_\infty}$ and $\hat{r}^{-1} \circ \hat{r} = \text{id}_{X_\infty}$.

Note that

\[
\theta_n \circ \hat{r} = r \circ \theta_n = \theta_{n-1},
\]

Consider a probability measure $\mu$ on $X$ that satisfies

\[
\int_X f \, d\mu = \int_X \frac{1}{\# r^{-1}(x)} \sum_{r(y)=x} f(y) \, d\mu(x).
\]

(4.20)

For $m_0 \in L^\infty(X)$, define

\[
(R \xi)(x) = \frac{1}{\# r^{-1}(x)} \sum_{r(y)=x} |m_0(y)|^2 \xi(y), \quad \xi \in L^1(X).
\]

(4.21)
We now resume the operator-theoretic theme of our paper, that of extending a system of operators in a given Hilbert space to a “dilated” system in an ambient or extended Hilbert space, the general idea being that the dilated system acquires a “better” spectral theory: for example, contractive operators dilate to unitary operators in the extended Hilbert space, and similarly, endomorphisms dilate to automorphisms. This, of course, is a general theme in operator theory; see, e.g., [10] and [38]. But in our present setting, there is much more structure than the mere Hilbert-space geometry. We must adapt the underlying operator theory to the particular function spaces and measures at hand. The next theorem (Theorem 4.3.4) is key to the understanding of the dilation step as it arises in our context of multiscale theory. It takes into consideration how we arrive at our function spaces by making dynamics out of the multiscale setting. Hence, the dilations we construct take place at three separate levels, as follows:

- dynamical systems,

\[(X, r, \mu) \text{ endomorphism } \to (X_\infty, \hat{r}, \hat{\mu}), \text{ automorphism} ;\]

- Hilbert spaces,

\[L_2(X, h \, d\mu) \to (R_{m_0} h = h) \to L_2(X_\infty, \hat{\mu});\]

- operators,

\[S_{m_0} \text{ isometry } \to U \text{ unitary (if } m_0 \text{ is non-singular)},\]

\[M(g) \text{ multiplication operator } \to M_\infty(g).\]

**Definition 4.3.2.** A function \(m_0\) on a measure space is called **singular** if \(m_0 = 0\) on a set of positive measure.

In general, the operators \(S_{m_0}\) on \(H_0 = L^2(X, h \, d\mu)\), and \(U\) on \(L^2(X_\infty, \hat{\mu})\), may be given only by abstract Hilbert-space axioms; but in our **martingale representation**, we get the following two concrete formulas:

\[(S_{m_0} \xi)(x) = m_0(x) \xi(r(x)), \quad x \in X, \xi \in H_0,\]

\[(U f)(\hat{x}) = m_0(x_0) f(\hat{r}(\hat{x})), \quad \hat{x} \in X_\infty, f \in L^2(X_\infty, \hat{\mu}).\]

**Definition 4.3.3.** Let \(X, r, \mu,\) and \(W\) be given as before. Recall \(r: X \to X\) is a finite-to-one map onto \(X\), \(\mu\) is a corresponding strongly invariant measure, and \(W: X \to [0, \infty)\) is a measurable weight function. We define the corresponding Ruelle operator \(R = R_W\), and we let \(h\) be a chosen Perron–Frobenius–Ruelle eigenfunction. From these data, we define the following sequence of measures \(\omega_n\) on \(X\). (Our presentation of a wavelet transform in Theorem 4.3.6 will depend on this sequence.) For each \(n\), the measure \(\omega_n\) is defined by the following formula:

\[\omega_n(f) = \int_X R^n(fh) \, d\mu, \quad f \in L^\infty(X).\]  

(4.22)
To build our wavelet transform, we first prove in Theorem 4.3.4 below that each of the measure families \((\omega_n)\) defines a unique \(W\)-quasi-invariant measure \(\hat{\mu}\) on \(X_\infty\). In the theorem, we take \(W := |m_0|^2\), and we show that \(\hat{\mu}\) is quasi-invariant with respect to \(\hat{r}\) with transformation Radon-Nikodym derivative \(W\).

**Theorem 4.3.4.** If \(h \in L^1(X)\), \(h \geq 0\) and \(Rh = h\), then there exists a unique measure \(\hat{\mu}\) on \(X_\infty\) such that

\[
\hat{\mu} \circ \theta_n^{-1} = \omega_n, \quad n \geq 0.
\]

We can identify functions on \(X\) with functions on \(X_\infty\) by

\[
f(x_0, x_1, \ldots) = f(x_0), \quad f : X \to \mathbb{C}.
\]

Under this identification,

\[
\frac{d(\hat{\mu} \circ \hat{r})}{d\hat{\mu}} = |m_0|^2.
\]

**Theorem 4.3.5.** Suppose \(m_0\) is non-singular, i.e., it does not vanish on a set of positive measure. Define \(U\) on \(L^2(X_\infty, \hat{\mu})\) by

\[
Uf = m_0 f \circ \hat{r}, \quad f \in L^2(X_\infty, \hat{\mu}),
\]

\[
\pi(g)f = gf, \quad g \in L^\infty(X), f \in L^2(X_\infty, \hat{\mu}),
\]

\[
\varphi = 1.
\]

Then \((L^2(X_\infty, \hat{\mu}), U, \pi, \varphi)\) is the covariant system associated to \(m_0\) and \(h\) as in Corollary 4.1.5. Moreover, if \(M_g f = gf\) for \(g \in L^\infty(X_\infty, \hat{\mu})\) and \(f \in L^2(X_\infty, \hat{\mu})\), then

\[
UM_g U^{-1} = M_{g_0}.
\]

The Hilbert space \(L^2(X_\infty, \hat{\mu})\) admits a different representation as an \(L^2\)-martingale Hilbert space. Let

\[
H_n := \{ f \in L^2(X_\infty, \hat{\mu}) \mid f = \xi \circ \theta_n, \xi \in L^2(X, \omega_n) \}.
\]

Then \(H_n\) form an increasing sequence of closed subspaces which have dense union.

We can identify the functions in \(H_n\) with functions in \(L^2(X, \omega_n)\), by

\[
i_n(\xi) = \xi \circ \theta_n, \quad \xi \in L^2(X, \omega_n).
\]

The definition of \(\hat{\mu}\) makes \(i_n\) an isomorphism between \(H_n\) and \(L^2(X, \omega_n)\).

Define

\[
\mathcal{H} := \{ (\xi_0, \xi_1, \ldots) \mid \xi_n \in L^2(X, \omega_n), R(\xi_{n+1} h) = \xi_n h, \sup_n \int_X R^n(|\xi_n|^2 h) d\mu < \infty \}
\]

with the scalar product

\[
\langle (\xi_0, \xi_1, \ldots) \mid (\eta_0, \eta_1, \ldots) \rangle = \lim_{n \to \infty} \int_X R^n(\xi_n \eta_n h) d\mu.
\]
Theorem 4.3.6. The map \( \Phi : L^2(X_\infty, \hat{\mu}) \rightarrow \mathcal{H} \) defined by
\[
\Phi(f) = (i_n^{-1}(P_n f))_{n \geq 0},
\]
where \( P_n \) is the projection onto \( H_n \), is an isomorphism. The transform \( \Phi \) satisfies
the following three conditions, and it is determined uniquely by them:
\[
\Phi U \Phi^{-1}(\xi_n)_{n \geq 0} = (m_0 \circ r^n \xi_{n+1})_{n \geq 0},
\]
\[
\Phi \pi(g) \Phi^{-1}(\xi_n)_{n \geq 0} = (g \circ r^n \xi_n)_{n \geq 0},
\]
\[
\Phi \varphi = (1, 1, \ldots).
\]

Theorem 4.3.7. There exists a unique isometry \( \Psi : L^2(X_\infty, \hat{\mu}) \rightarrow \hat{H} \) such that
\[
\Psi(\xi \circ \theta_n) = U^{-n} \hat{\pi}(\xi) \hat{U}^n \hat{\varphi}, \quad \xi \in L^\infty(X, \mu).
\]
\( \Psi \) intertwines the two systems, i.e.,
\[
\Psi U = \hat{U} \Psi, \quad \Psi \pi(g) = \hat{\pi}(g) \Psi, \quad \text{for } g \in L^\infty(X, \mu), \quad \Psi \varphi = \hat{\varphi}.
\]

Theorem 4.3.8. Let \((m_0, h)\) be a Perron–Ruelle–Frobenius pair with \( m_0 \) non-singular.

(i) For each operator \( A \) on \( L^2(X_\infty, \hat{\mu}) \) which commutes with \( U \) and \( \pi \), there exists
a cocycle \( f \), i.e., a bounded measurable function \( f : X_\infty \rightarrow \mathbb{C} \) with \( f = f \circ r \),
\( \hat{\mu} \)-a.e., such that
\[
A = M_f,
\]
and, conversely each cocycle defines an operator in the commutant.

(ii) For each measurable harmonic function \( h_0 : X \rightarrow \mathbb{C} \), i.e., \( R_{m_0} h_0 = h_0 \), with
\( |h_0|^2 \leq c h^2 \) for some \( c \geq 0 \), there exists a unique cocycle \( f \) such that
\[
h_0 = E_0(f) h,
\]
where \( E_0 \) denotes the conditional expectation from \( L^\infty(X_\infty, \hat{\mu}) \) to \( L^\infty(X, \mu) \),
and conversely, for each cocycle the function \( h_0 \) defined by (4.25) is harmonic.

(iii) The correspondence \( h_0 \rightarrow f \) in \( \text{iii} \) is given by
\[
f = \lim_{n \rightarrow \infty} h_0 \circ \theta_n
\]
where the limit is pointwise \( \hat{\mu} \)-a.e., and in \( L^p(X_\infty, \hat{\mu}) \) for all \( 1 \leq p < \infty \).

The next result in this section concerns certain path-space measures, indexed
by a base space \( X \). Such a family is also called a process; it is a family of positive
Radon measures \( P_x \), indexed by \( x \) in the base space \( X \). Each \( P_x \) is a measure on the
probability space \( \Omega \) which is constructed from \( X \) by the usual Cartesian product,
i.e., countably infinite Cartesian product of \( X \) with itself. The Borel structure on
\( \Omega \) is generated by the usual cylinder subsets of \( \Omega \). Given a weight function \( W \)
on \( X \), the existence of the measures \( P_x \) comes from an application of a general
principle of Kolmogorov.

Let \( X \) be a metric space and \( r : X \rightarrow X \) an \( N \) to 1 map. Denote by \( \tau_k : X \rightarrow X \),
\( k \in \{1, \ldots, N\} \), the branches of \( r \), i.e., \( r(\tau_k(x)) = x \) for \( x \in X \), the sets \( \tau_k(X) \)
are disjoint and they cover \( X \).
Let $\mu$ be a measure on $X$ with the property
\[
\mu = \frac{1}{N} \sum_{k=1}^{N} \mu \circ \tau_k^{-1}.
\] (4.27)

This can be rewritten as
\[
\int_X f(x) \, d\mu(x) = \frac{1}{N} \sum_{k=1}^{N} \int_X f(\tau_k(x)) \, d\mu(x),
\] (4.28)

which is equivalent also to the strong invariance property.

Let $W, h \geq 0$ be two functions on $X$ such that
\[
\sum_{k=1}^{N} W(\tau_k(x)) h(\tau_k(x)) = h(x), \quad x \in X.
\] (4.29)

Denote by $\Omega$ the multi-index set
\[
\Omega := \Omega_N := \prod_N \{1, \ldots, N\}.
\]

Also, we define
\[
W^{(n)}(x) := W(x)W(r(x)) \cdots W(r^{n-1}(x)), \quad x \in X.
\]

We will return to a more general variant of these path-space measures in Theorem 5.8.12 in the next section. This will be in the context of a system $(X, r, V)$ where $r: X \to X$ is a give finite to one endomorphism, and $W: X \to [0, \infty)$ is a measurable weight function. In this context the measures $(P_x)$ are indexed by a base space $X$, and they are measures on the projective space $X\infty$ built on $(X, r)$, see Definition 4.3.1. The family $(P_x)$ is also called a process: each $P_x$ is a positive Radon measure on $X\infty$, and $X\infty$ may be thought of as a discrete path space.

The Borel structure on $\Omega$, and on $X\infty$, is generated by the usual cylinder subsets. Given a weight function $W$ on $X$, the existence of the measures $P_x := P^W_x$ comes from an application of a general principle of Kolmogorov.

**Proposition 4.3.9.** For every $x \in X$ there exists a positive Radon measure $P_x$ on $\Omega$ such that, if $f$ is a bounded measurable function on $X$ which depends only on the first $n$ coordinates $\omega_1, \ldots, \omega_n$, then
\[
\int_{\Omega} f(\omega) \, dP_x(\omega) = \sum_{\omega_1, \ldots, \omega_n} W^{(n)}(\tau_{\omega_n} \tau_{\omega_{n-1}} \cdots \tau_{\omega_1}(x)) h(\tau_{\omega_n} \tau_{\omega_{n-1}} \cdots \tau_{\omega_1}(x)) f(\omega_1, \ldots, \omega_n).
\] (4.30)
5. Remarks on other applications

5.1. Wavelet sets

Investigate the existence and construction of wavelet sets and elementary wavelets and frames in the Julia-set theory, and the corresponding interpolation theory, and their relationships to generalized multiresolution analysis (GMRA). The unitary system approach to wavelets by Dai and Larson [32] dealt with systems that can be very irregular. And recent work by Ólafsson et al. shows that wavelet sets can be essential to a basic dilation-translation wavelet theory even for a system where the set of dilation unitaries is not a group.

5.2. The renormalization question

When are there renormalizable iterates of \( r \) in arithmetic progression, i.e., when are there iteration periods \( n \) such that the system \( \{ r^{kn}, k \in \mathbb{N} \} \) may be renormalized, or rescaled to yield a new dynamical system of the same general shape as that of the original map \( r(z) \)?

Since the scaling equation (2.2) from wavelet theory is a renormalization, our application of multiresolutions to the Julia sets \( X(r) \) of complex dynamics suggests a useful approach to renormalization in this context. The drawback of this approach is that it relies on a rather unwieldy Hilbert space built on \( X(r) \), or rather on the projective system \( X(r)_{\infty} \) built in turn over \( X(r) \). So we are left with translating our Hilbert-space-theoretic normalization back to the direct and geometric algorithms on the complex plane.

General and rigorous results from complex dynamics state that under additional geometric hypotheses, renormalized dynamical systems range in a compact family.

The use of geometric tools from Hilbert space seems promising for renormalization questions (see, e.g., [64], [15], [23], and [74]) since notions of repetition up to similarity of form at infinitely many scales are common in the Hilbert-space approach to multiresolutions; see, e.g., [55]. And this self-similarity up to scale parallels a basic feature of renormalization questions in physics: for a variety of instances of dynamics in physics, and in telecommunication [64, 15, 23], we encounter scaling laws of self-similarity; i.e., we observe that a phenomenon reproduces itself on different time and/or space scales.

Self-similar processes are stochastic processes that are invariant in distribution under suitable scaling of time and/or space (details below!) Fractional Brownian motion is perhaps the best known of these, and it is used in telecommunication and in stochastic integration. While the underlying idea behind this can be traced back to Kolmogorov, it is only recently, with the advent of wavelet methods, that its computational power has come more into focus, see e.g., [64]. But at the same time, this connection to wavelet analysis is now bringing the operator-theoretic features in the subject to the fore.

In statistics, we observe that fractional Brownian motion (a Gaussian process \( B(t) \) with \( E\{B(t)\} = 0 \), and covariance \( E\{B(t)B(s)\} \) given by a certain
h-fractional law) has the property that there is a number h such that, for all a, the two processes \( B(at) \) and \( \phi^h B(t) \) have the same finite-dimensional distributions. The scaling feature of fractional Brownian motion, and of its corresponding white-noise process, is used in telecommunication, and in stochastic integration; and this connection has been one of the new and more exciting domains of applications of wavelet tools, both pure and applied (Donoho, Daubechies, Meyer, etc. 

This is a new direction of pure mathematics which makes essential contact with problems that are not normally investigated in the context of harmonic analysis.

In our proofs, we take advantage of our Hilbert-space formulation of the notion of similarity up to scale, and the use of scales of closed subspaces \( V_n \) in a Hilbert space \( \mathcal{H} \). The unitary operator \( U \) which scales between the spaces may arise from a substitution in nonlinear dynamics, such as in the context of Julia sets; it may “scale” between the sigma-algebras in a martingale; it may be a scaling of large volumes of data; it may be the scaling operation in a fractional Brownian motion; it may be cell averages in finite elements; or it may be the traditional dyadic scaling operator in \( \mathcal{H} = L^2(\mathbb{R}) \) in wavelet frame analysis.

Once the system \( (\mathcal{H}, (V_n)_n, U) \) is given, then it follows that there is a unitary representation \( V \) of \( \mathbb{T} \) in \( \mathcal{H} \) which defines a grading of operators on \( \mathcal{H} \). Moreover \( U \) then defines a similarity up to scale precisely when \( U \) has grade one, see (3.4) for definition. We also refer to the papers for details and applications of this notion in operator theory.

The general idea of assigning degrees to operators in Hilbert space has served as a powerful tool in other areas of mathematical physics (see, e.g., on phase transition problems); operator theory; and operator algebras; see, e.g., on phase transition problems); operator theory; and operator algebras; see, e.g., on phase transition problems); operator theory; and operator algebras; see, e.g.,

Our research, e.g., in fact already indicates that the nonlinear problems sketched above can be attacked with the use of our operator-theoretic framework.

5.3. Wavelet representations of operator-algebraic crossed products

The construction of wavelets requires a choice of a low-pass filter \( m_0 \in L^\infty(\mathbb{T}) \) and of \( N-1 \) high-pass filters \( m_i \in L^\infty(\mathbb{T}) \) that satisfy certain orthogonality conditions. As shown in, these choices can be put in one-to-one correspondence with a class of covariant representations of the Cuntz algebra \( O_N \). In the case of an arbitrary \( N \)-to-one dynamical system \( r: X \to X \), the wavelet construction will again involve a careful choice of the filters, and a new class of representations is obtain. The question here is to classify these representations and see how they depend on the dynamical system \( r \). A canonical choice of the filters could provide an invariant for the dynamical system.

In a recent paper, the crossed-product of a \( C^* \)-algebra by an endomorphism is constructed using the transfer operator. The required covariant relations are satisfied by the operators that we introduced in.
There are two spaces for these representations: one is the core space of the multiresolution $V_0 = L^2(X, \mu)$. Here we have the abelian algebra of multiplication operators and the isometry $S: f \mapsto m_0 f \circ r$. Together they provide representations of the crossed product by an endomorphism. The second space is associated to a dilated measure $\hat{\mu}$ on the solenoid of $r$, $L^2(X_\infty, \hat{\mu})$ (see [43], [44]). The isometry $S$ is dilated to a unitary $\hat{S}$ which preserves the covariance. In this case, we are dealing with representations of crossed products by automorphisms.

Thus we have four objects interacting with one another: crossed products by endomorphisms and their representations, and crossed products by automorphisms and its representations on $L^2(X_\infty, \hat{\mu})$.

The representations come with a rich structure given by the multiresolution, scaling function and wavelets. Therefore their analysis and classification seems promising.

5.4. KMS-states and equilibrium measures

In [48], the KMS states on the Cuntz–Pimsner algebra for a certain one-parameter group of automorphisms are obtained by composing the unique equilibrium measure on an abelian subalgebra (i.e., the measure which is invariant to the transfer operator) with the conditional expectation. These results were further generalized in [58] for expansive maps and the associated groupoid algebras.

As we have seen in [45] for the case when $r(z) = z^N$, the covariant representations associated to some low-pass filter $m_0$ are highly dependent on the equilibrium measure $\nu$. As outlined above, we consider here an expansive dynamical system built on a given finite-to-one mapping $r: X \to X$, and generalizing the familiar case of the winding mapping $z \to z^N$ on $T$. In our $r: X \to X$ context, we then study weight functions $V: X \to [0, \infty)$. If the zero-set of $V$ is now assumed finite, we have so far generalized what is known in the $z \to z^N$ case. Our results take the form of a certain dichotomy for the admissible Perron–Frobenius–Ruelle measures, and they fit within the framework of [58]. Hence our construction also yields covariant representations and symmetry in the context of [58]. Our equilibrium measures in turn induce the kind of KMS-states studied in [58].

The results of [58] and [48] restrict the weight function (in our case represented by the absolute square of the low-pass filter, $|m_0|^2$) to being strictly positive and Hölder continuous. These restrictions are required because of the form of the Ruelle–Perron–Frobenius theorem used, which guarantees the existence and uniqueness of the equilibrium measure.

However, the classical wavelet theory shows that many interesting examples require a “low-pass condition”, $m_0(1) = \sqrt{N}$, which brings zeroes for $|m_0|^2$.

Our martingale approach is much less restrictive than the conditions of [58]: it allows zeroes and discontinuities. Coupled with a recent, more general form of Ruelle’s theorem from [49], we hope to extend the results of [58] and be able to find the KMS states in a more general case. Since the existence of zeroes can imply a multiplicity of equilibrium measures (see [41], [19]) a new phenomenon might occur such as spontaneously breaking symmetry.
For each KMS state, one can construct the GNS representation. Interesting results are known \[65\] for the subshifts of finite type and the Cuntz–Krieger algebra, when the construction yields type III\(\lambda\) AFD factors.

The natural question is what type of factors appear in the general case of \(r: X \to X\). Again, a canonical choice for the low-pass filter can provide invariants for the dynamical system.

Historically, von-Neumann-algebra techniques have served as powerful tools for analyzing discrete structures. Our work so far suggests that the iteration systems \((X, r)\) will indeed induce interesting von-Neumann-algebra isomorphism classes.

5.5. Dimension groups

We propose to use our geometric algorithms \[20, 21\] for deciding order isomorphism of dimension groups for the purpose of deciding isomorphism questions which arise in our multiresolution analysis built on Julia sets. The project is to use the results in \[20\] to classify substitution tilings in the sense of \[66, 67\].

5.6. Equilibrium measures, harmonic functions for the transfer operator, and infinite Riesz products

In \[19, 41\], for the dynamical system \(z \mapsto z^N\), we were able to compute the equilibrium measures, the fixed points of the transfer operator and its ergodic decomposition by analyzing the commutant of the covariant representations associated to the filter \(m_0\). In \[43, 44\] we have shown that also in the general case there is a one-to-one correspondence between harmonic functions for the transfer operator and the commutant. Thus an explicit form of the covariant representation can give explicit solutions for the eigenvalue problem \(R_{m_0}h = h\), which are of interest in ergodic theory (see \[9\]).

In some cases, the equilibrium measures are Riesz products (see \[45\]); these are examples of exotic measures which arise in harmonic analysis \[12, 46, 56, 6, 69, 14, 26, 27, 57, 62, 70\]. The wavelet-operator-theoretic approach may provide new insight into their properties.

5.7. Non-uniqueness of the Ruelle–Perron–Frobenius data

A substantial part of the current literature in the Ruelle–Perron–Frobenius operator in its many guises is primarily about conditions for uniqueness of KMS; so that means no phase transition. This is ironic since Ruelle’s pioneering work was motivated by phase-transition questions, i.e., non-uniqueness.

In fact non-uniqueness is much more interesting in a variety of applications: That is when we must study weight functions \(W\) in the Ruelle operator \(R = RW\) when the standard rather restrictive conditions on \(W\) are in force. The much celebrated Ruelle theorem for the transition operator \(R\) gives existence and uniqueness of Perron–Frobenius data under these restrictive conditions. But both demands from physics, and from wavelets, suggest strong and promising research interest in relaxing the stringent restrictions that are typical in the literature, referring to
assumptions on $W$. There is a variety of exciting possibilities. They would give us existence, but not necessarily uniqueness in the conclusion of a new Ruelle-type theorem.

5.8. Induced measures

A basic tool in stochastic processes (from probability theory) involves a construction on a “large” projective space $X_\infty$, based on some marginal measure on some coordinate space $X$. Our projective limit space $X_\infty$ will be constructed from a finite branching process.

Let $A$ be a $k \times k$ matrix with entries in $\{0, 1\}$. Suppose every column in $A$ contains an entry 1.

Set $X(A) := \{(\xi_i)_{i \in \mathbb{N}} \in \prod_{i=0}^{\infty} \{1, \ldots, k\} \mid A(\xi_i, \xi_{i+1}) = 1\}$ and

$$r_A(\xi_1, \xi_2, \ldots) = (\xi_2, \xi_3, \ldots) \quad \text{for} \quad \xi \in X(A).$$

Then $r_A$ is a subshift, and the pair $(X(A), r_A)$ satisfies our conditions.

It is known [72] that, for each $A$, as described, the corresponding system $r_A: X(A) \to X(A)$ has a unique strongly $r_A$-invariant probability measure, $\rho_A$, i.e., a probability measure on $X(A)$ such that $\rho_A \circ R_1 = \rho_A$, where $R_1$ is defined as in (5.1) below.

We now turn to the connection between measures on $X$ and an associated family of induced measures on $X_\infty$, and we characterize those measures $X_\infty$ which are quasi-invariant with respect to the invertible mapping $\hat{r}$, where $X_\infty := \{ \hat{x} = (x_0, x_1, \ldots) \in \prod_{i=0}^{\infty} X \mid r(x_{n+1}) = x_n \}, \hat{r}(\hat{x}) = (r(x_0), x_0, x_1, \ldots)$.

In our extension of measures from $X$ to $X_\infty$, we must keep track of the transfer from one step to the next, and there is an operator which accomplishes this, Ruelle’s transfer operator

$$R_W f(x) = \frac{1}{\# r^{-1}(x)} \sum_{y \in X, r(y) = x} W(y) f(y), \quad f \in L^1(X, \mu). \quad (5.1)$$

In its original form it was introduced in [72], but since that, it has found a variety of applications, see e.g., [49]. For use of the Ruelle operator in wavelet theory, we refer to [54] and [41].

The Hilbert spaces of functions on $X_\infty$ will be realized as a Hilbert spaces of martingales. This is consistent with our treatment of wavelet resolutions as martingales. This was first suggested in [51] in connection with wavelet analysis.

In [43], we studied the following restrictive setup: we assumed that $X$ carries a probability measure $\mu$ which is strongly $r$-invariant. By this we mean that

$$\int_X f d\mu = \int_X \frac{1}{\# r^{-1}(x)} \sum_{y \in X, r(y) = x} f(y) d\mu(x), \quad f \in L^\infty(X). \quad (5.2)$$

If, for example $X = \mathbb{R}/\mathbb{Z}$, and $r(x) = 2x \mod 1$, then the Haar measure on $\mathbb{R}/\mathbb{Z} = \text{Lebesgue measure on } [0, 1)$ is the unique strongly $r$-invariant measure on $X$. 


Suppose the weight function $V$ is bounded and measurable. Then define $R = R_V$, the Ruelle operator, by formula (5.1), with $V = W$.

**Theorem 5.8.1.** (13) Let $r : X → X$ and $X_∞(r)$ be as described above, and suppose that $X$ has a strongly $r$-invariant measure $μ$. Let $V$ be a non-negative, measurable function on $X$, and let $R_V$ be the corresponding Ruelle operator.

(i) There is a unique measure $\hat{μ}$ on $X_∞(r)$ such that
\begin{align*}
(a) \quad & \hat{μ} \circ θ_0^{-1} ≪ μ \quad \text{(set $h = d(μ \circ θ_0^{-1})$)}, \\
(b) \quad & \int_X f \, d\hat{μ} \circ θ_n^{-1} = \int_X R_V^n (fh) \, dμ, \quad n \in \mathbb{N}_0,
\end{align*}

(ii) The measure $\hat{μ}$ on $X_∞(r)$ satisfies
\begin{align*}
& d(\hat{μ} \circ \hat{r}) = V \circ θ_0, \quad (5.3) \\
& R_V h = h. \quad (5.4)
\end{align*}

If the function $V : X → [0, \infty)$ is given, we define $V^n(x) := V(x)V(r(x))⋯V(r^{n-1}(x))$, and set $dμ_0 := V^n dμ_0$. Our result states that the corresponding measure $\hat{μ}$ on $X_∞(r)$ is $V$-quasi-invariant if and only if
\begin{equation}
\frac{d(\hat{μ} \circ θ_0^{-1})}{dμ_0} = (V dμ_0) \circ r^{-1}. \quad (5.5)
\end{equation}

**Theorem 5.8.2.** Let $V : X → [0, \infty)$ be $\mathcal{B}$-measurable, and let $μ_0$ be a measure on $X$ satisfying the following fixed-point property
\begin{equation}
dμ_0 = (V dμ_0) \circ r^{-1}. \quad (5.6)
\end{equation}

Then there exists a unique measure $\hat{μ}$ on $X_∞(r)$ such that
\begin{align*}
& d(\hat{μ} \circ θ_0^{-1}) = V \circ θ_0 \quad \text{and} \\
& \hat{μ} \circ θ_0^{-1} = μ_0. \quad (5.7)
\end{align*}

**Definition 5.8.3.** Let $V : X → [0, \infty)$ be bounded and $\mathcal{B}$-measurable. We use the notation
\begin{equation}
M^V(X) := \{μ ∈ M(X) \mid dμ = (V dμ) \circ r^{-1}\}.
\end{equation}

For measures $\hat{μ}$ on $(X_∞(r), \mathcal{B}_∞)$ we introduce
\begin{equation}
M^V_q(X_∞(r)) := \left\{ \hat{μ} ∈ M(X_∞(r)) \mid \hat{μ} \circ θ_0 ≪ \hat{μ} \text{ and } \frac{d(\hat{μ} \circ \hat{r})}{d\hat{μ}} = V \circ θ_0 \right\}. \quad (5.8)
\end{equation}

As in Definition 5.8.3, let $X$, $r$, and $V$ be as before, i.e., $r$ is a finite-to-one endomorphism of $X$, and $V : X → [0, \infty)$ is a given weight function. In the next theorem, we establish a canonical bijection between the convex set $M^V(X)$ of measures on $X$ with the set of $V$-quasi-invariant measures on $X_∞$, which we call $M^V_q(X_∞(r))$, see (5.8).

The results so far in this section may be summarized as follows:
Theorem 5.8.4. Let $V$ be as in Definition 5.8.3. For measures $\hat{\mu}$ on $X_\infty(r)$ and $n \in \mathbb{N}_0$, define

$$C_n(\hat{\mu}) := \hat{\mu} \circ \theta_n^{-1}.$$ 

Then $C_0$ is a bijective affine isomorphism of $M^V_{\hat{\nu}}(X_\infty(r))$ onto $M^V(X)$ that preserves the total measure, i.e., $C_0(\hat{\mu})(X) = \hat{\mu}(X_\infty(r))$ for all $\hat{\mu} \in M^V_{\hat{\nu}}(X_\infty(r))$.

Theorem 5.8.5. Let $V : X \to [0, \infty)$ be continuous. Assume that there exist some measure $\nu$ on $(X, \mathcal{B})$ and two numbers $0 < a < b$ such that

$$a \leq \nu(X) \leq b, \quad \text{and} \quad a \leq \int_X V^n \, d\nu \leq b \quad \text{for all } n \in \mathbb{N}. \quad (5.9)$$

Then there exists a measure $\mu_0$ on $(X, \mathcal{B})$ that satisfies

$$d\mu_0 = (V \, d\mu_0) \circ r^{-1},$$

and there exists a $V$-quasi-invariant measure $\hat{\mu}$ on $(X_\infty(r), \mathcal{B}_\infty)$.

Theorem 5.8.6. Let $(X, \mathcal{B})$, and $r : X \to X$, be as described above. Suppose $V : X \to [0, \infty)$ is measurable,

$$\frac{1}{\# r^{-1}(x)} \sum_{r(y) = x} V(y) \leq 1,$$

and that some probability measure $\nu_V$ on $X$ satisfies

$$\nu_V \circ R_V = \nu_V. \quad (5.10)$$

Assume also that $(X, \mathcal{B})$ carries a strongly $r$-invariant probability measure $\rho$, such that

$$\rho(\{ x \in X \mid V(x) > 0 \}) > 0. \quad (5.11)$$

Then

(i) $T^n_V (d\rho) = R^n_V(1) \, d\rho$, for $n \in \mathbb{N}$, where 1 denotes the constant function one.

(ii) [Monotonicity] $\cdots \leq R^n_V(1) \leq R^{n+1}_V(1) \leq \cdots \leq 1$, pointwise on $X$.

(iii) The limit $\lim_{n \to \infty} R^n_V(1)$ exists, $R_V h_V = h_V$, and

$$\rho(\{ x \in X \mid h_V(x) > 0 \}) > 0. \quad (5.12)$$

(iv) The measure $d\mu^{(V)}_n = h_V \, d\rho$ is a solution to the fixed-point problem

$$T^n_V(\mu^{(V)}_n) = \mu^{(V)}_n.$$ 

(v) The sequence $d\mu^{(V)}_n = V^{(n)} h_V \, d\rho$ defines a unique $\hat{\mu}^{(V)}$ as in Theorem 5.8.1, and

(vi) $\mu^{(V)}_n(f) = \int_X R^n_V(f h_V) \, d\rho$ for all bounded measurable functions $f$ on $X$, and all $n \in \mathbb{N}$.

Finally,

(vii) The measure $\hat{\mu}^{(V)}$ on $X_\infty(r)$ satisfying $\hat{\mu}^{(V)} \circ \theta_n^{-1} = \mu^{(V)}_n$ has total mass

$$\hat{\mu}^{(V)}(X_\infty(r)) = \rho(h_V) = \int_X h_V(x) \, d\rho(x).$$
Definition 5.8.7. A measure \( \mu_0 \in M_V^1(X) \) is called relatively ergodic with respect to \((r, V)\) if the only non-negative, bounded \( \mathcal{B} \)-measurable functions \( f \) on \( X \) satisfying
\[
E_{\mu_0}(V f) = E_{\mu_0}(V) f \circ r, \quad \text{pointwise } \mu_0 \circ r^{-1}\text{-a.e.,}
\]
are the functions which are constant \( \mu_0 \)-a.e.

Since we have a canonical bijection between the two compact convex sets of measures \( M_V^1(X) \) and \( M_{qi,1}(X_\infty(r)) \), the natural question arises as to the extreme points. This is answered in our next theorem. We show that there are notions ergodicity for each of the two spaces \( X \) and \( X_\infty(r) \) which are equivalent to extremality in the corresponding compact convex sets of measures.

Theorem 5.8.8. Let \( V : X \to [0, \infty) \) be bounded and measurable. Let
\[
\hat{\mu} \in M_{qi,1}(X_\infty(r)), \quad \text{and } \mu_0 := \hat{\mu} \circ \theta_0^{-1} \in M_V^1(X).
\]
The following assertions are equivalent:

(i) \( \hat{\mu} \) is an extreme point of \( M_{qi,1}(X_\infty(r)) \);
(ii) \( V \circ \theta_0 d\hat{\mu} \) is ergodic with respect to \( \hat{r} \);
(iii) \( \mu_0 \) is an extreme point of \( M_V^1(X) \);
(iv) \( \mu_0 \) is relatively ergodic with respect to \((r, V)\).

We now turn to the next two theorems. These are counterparts of our dimension-counting functions which we outlined above in connection with Theorem 4.2.3; see especially Remark 4.2.4. They concern the correspondence between the two classes of measures, certain on \( X \) (see Theorem 5.8.8), and the associated induced measures on the projective space \( X_\infty \). Our proofs are technical and will not be included. (The reader is referred to [42] for further details.) Rather we only give a few suggestive hints: Below we outline certain combinatorial concepts and functions which are central for the arguments. Since they involve a counting principle, they have an intuitive flavor which we hope will offer some insight into our theorems.

Let \( X \) be a non-empty set, \( \mathcal{B} \) a sigma-algebra of subsets of \( X \), and \( r : X \to X \) an onto, finite-to-one, and \( \mathcal{B} \)-measurable map.

We will assume in addition that we can label measurably the branches of the inverse of \( r \). By this, we mean that the following conditions are satisfied:

The map \( c : X \ni x \mapsto \# r^{-1}(x) < \infty \) is measurable. \hspace{1cm} (5.13)

We denote by \( A_i \) the set
\[
A_i := \{ x \in X \mid c(x) = \# r^{-1}(x) \geq i \}, \quad i \in \mathbb{N}.
\]

Equation (5.13) implies that the sets \( A_i \) are measurable. Also they form a decreasing sequence and, since the map is finite-to-one,
\[
X = \bigcup_{i=1}^{\infty} (A_{i+1} \setminus A_i).
\]
Then, we assume that there exist measurable maps \( \tau_i : A_i \to X \), \( i \in \{1, 2, \ldots\} \) such that

\[
\tau_i(A_i) \in \mathcal{B} \quad \text{for all } i \in \{1, 2, \ldots\}.
\]

Thus \( \tau_i(x), \ldots, \tau_{c(x)}(x) \) is a list without repetitions of the “roots” of \( x, r^{-1}(x) \).

From (5.14) we obtain also that

\[
\tau_i(A_i) \cap \tau_j(A_j) = \emptyset, \text{ if } i \neq j,
\]

and

\[
\bigcup_{i=1}^{\infty} \tau_i(A_i) = X.
\]

In the sequel, we will use the following notation: for a function \( f : X \to \mathbb{C} \), we denote by \( f \circ \tau_i \) the function

\[
f \circ \tau_i(x) := \begin{cases} 
  f(\tau_i(x)) & \text{if } x \in A_i, \\
  0 & \text{if } x \in X \setminus A_i.
\end{cases}
\]

Our Theorem 4.3.4 depends on the existence of some strongly invariant measure \( \mu \) on \( X \), when the system \((X, r)\) is given. However, in the general measurable category, such a strongly invariant measure \( \mu \) on \( X \) may in fact not exist; or if it does, it may not be available by computation. In fact, recent wavelet results (for frequency localized wavelets, see, e.g., [7] and [8]) suggest the need for theorems in the more general class of dynamical systems \((X, r)\).

In the next theorem (Theorem 5.8.9), we provide for each system, \( X \), \( r \), and \( V \) a substitute for the existence of strongly invariant measures. We show that there is a certain fixed-point property for measures on \( X \) which depends on \( V \) but not on the a priori existence of strongly \( r \)-invariant measures, and which instead is determined by a certain modular function \( \Delta \) on \( X \). This modular function in turn allows us to prove a dis-integration theorem for our \( V \)-quasi-invariant measures \( \mu \) on \( X_\infty \). In Theorem 5.8.12 we give a formula for this dis-integration of a \( V \)-quasi-invariant measure \( \mu \) on \( X_\infty \) in the presence of a modular function \( \Delta \). Our dis-integration of \( \mu \) is over a Markov process \( P_x \), for \( x \) in \( X \), which depends on \( \Delta \), but otherwise is analogous to the process \( P_x \) we used in Proposition 4.3.9.

**Theorem 5.8.9.** Let \((X, \mathcal{B})\) and \( r : X \to X \) be as above. Let \( V : X \to [0, \infty) \) be a bounded \( \mathcal{B} \)-measurable map. For a measure \( \mu_0 \) on \((X, \mathcal{B})\), the following assertions are equivalent.

(i) The measure \( \mu_0 \) has the fixed-point property

\[
\int_X V f \circ r \, d\mu_0 = \int_X f \, d\mu_0, \text{ for all } f \in L^\infty(X, \mu_0).
\]

(5.18)
There exists a non-negative, $\mathcal{B}$-measurable map $\Delta$ (depending on $V$ and $\mu_0$) such that
\[
\sum_{r(y)=x} \Delta(y) = 1, \text{ for } \mu_0\text{-a.e. } x \in X, \tag{5.19}
\]
and
\[
\int_X V f \, d\mu_0 = \int_X \sum_{r(y)=x} \Delta(y) f(y) \, d\mu_0(x), \text{ for all } f \in L^\infty(X, \mu_0). \tag{5.20}
\]
Moreover, when the assertions are true, $\Delta$ is unique up to $V \, d\mu_0$-measure zero.

We recall the definitions: if $\mathcal{B}$ is a sigma-algebra on a set $X$ and $r : X \to X$ is a finite-to-one, onto and measurable map, we denote by $X_\infty$ the set
\[
X_\infty(r) := \left\{ (x_0, x_1, \ldots) \in \prod_{n \in \mathbb{N}_0} X \mid r(x_{n+1}) = x_n, \text{ for all } n \in \mathbb{N}_0 \right\}.
\]
We denote the projections by $\theta_n : X_\infty(r) \ni (x_0, x_1, \ldots) \mapsto x_n \in X$. The union of the pull-backs $\theta_n^{-1}(\mathcal{B})$ generates a sigma-algebra $\mathcal{B}_\infty$. The map $r$ extends to a measurable bijection $\hat{r} : X_\infty(r) \to X_\infty(r)$ defined by
\[
\hat{r}(x_0, x_1, \ldots) = (r(x_0), x_0, x_1, \ldots).
\]

Let $V : X \to [0, \infty)$ be a measurable, bounded function. We say that a measure $\mu_0$ on $(X, \mathcal{B})$ has the fixed-point property if
\[
\int_X V f \circ r \, d\mu_0 = \int_X f \, d\mu_0, \quad f \in L^\infty(X, \mu_0).
\]
We say that a measure $\hat{\mu}$ on $(X_\infty(r), \mathcal{B}_\infty)$ is $V$-quasi-invariant if
\[
d(\hat{\mu} \circ \hat{r}) = V \circ \theta_0 \, d\hat{\mu}.
\]
We recall the following result from [44].

**Theorem 5.8.10.** There exists a one-to-one correspondence between measures $\mu_0$ on $X$ with the fixed-point property and $V$-quasi-invariant measures $\hat{\mu}$ on $X_\infty(r)$, given by
\[
\mu_0 = \hat{\mu} \circ \theta_0^{-1}.
\]

**Proposition 5.8.11.** Let $(X, \mathcal{B})$, $r : X \to X$ and be as above, and let $D : X \to [0, \infty)$ be a measurable function with the property that
\[
\sum_{r(y)=x} D(y) = 1. \tag{5.21}
\]
Denote by $D^{(n)}$ the product of compositions
\[
D^{(n)} := D \circ D \circ \cdots \circ D \circ r^{n-1}, \quad n \in \mathbb{N}, \quad D^{(0)} := 1. \tag{5.22}
\]
Then for each \( x_0 \in X \), there exists a Radon probability measure \( P_{x_0} \) on \( \Omega_{x_0} \) such that, if \( f \) is a bounded measurable function on \( \Omega_{x_0} \) which depends only on the first \( n \) coordinates \( x_1, \ldots, x_n \), then
\[
\int_{\Omega_{x_0}} f(\omega) \, dP_{x_0}(\omega) = \sum_{r^n(x_n) = x_0} D^{(n)}(x_n)f(x_1, \ldots, x_n).
\]

(5.23)

**Theorem 5.8.12.** Let \((X, \mathcal{B}), r: X \to X\) and \(V: X \to [0, \infty)\) be as above. Let \( \mu_0 \) be a measure on \((X, \mathcal{B})\) with the fixed-point property \( (5.18) \). Let \( \Delta: X \to [0, 1] \) be the function associated to \( V \) and \( \mu_0 \) as in Theorem 5.8.9, and let \( \hat{\mu} \) be the unique \( V \)-quasi-invariant measure on \( X_\infty(r) \) with
\[
\hat{\mu} \circ \theta_0^{-1} = \mu_0,
\]
as in Theorem 5.8.11. For \( \Delta \) define the measures \( P_{x_0} \) as in Proposition 5.8.11. Then, for all bounded measurable functions \( f \) on \( X_\infty(r) \),
\[
\int_{X_\infty(r)} f \, d\hat{\mu} = \int_{X} \int_{\Omega_{x_0}} f(x_0, \omega) \, dP_{x_0}(\omega) \, d\mu_0(x_0).
\]

(5.24)

### 5.9. Hausdorff measure and wavelet bases

In this problem we propose wavelet bases in the context of Hausdorff measure of fractional dimension between 0 and 1. While our fractal wavelet theory has points of similarity that it shares with the standard case of Lebesgue measure on the line, there are also sharp contrasts.

It is well known that the Hilbert spaces \( L^2(\mathbb{R}) \) has a rich family of orthonormal bases of the following form: \( \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \), \( j, k \in \mathbb{Z} \), where \( \psi \) is a single function \( \in L^2(\mathbb{R}) \). Take for example Haar’s function \( \psi(x) = \chi_I(2x) - \chi_I(2x - 1) \) where \( I = [0, 1] \) is the unit interval. Let \( C \) be the middle-third Cantor set. Then the corresponding indicator function \( \varphi_C := \chi_C \) satisfies the scaling identity (see (2.2)), \( \varphi_C(x) = \frac{1}{2}\varphi_C(x) + \varphi_C(x - \frac{1}{2}) \).

In \([45]\) we use this as the starting point for a multiresolution construction in a separable Hilbert space built with Hausdorff measure, and we identify the two mother functions which define the associated wavelet ONB.

Since both constructions, the first one for the Lebesgue measure, and the second one for the Hausdorff version \((dx)^s\), arise from scaling and subdivision, it seems reasonable to expect multiresolution wavelets also in Hilbert spaces constructed on the scaled Hausdorff measures \( H^s \) which are basic for the kind of iterated function systems which give Cantor constructions built on scaling and translations by lattices.

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