QT-symmetry and weak pseudo-hermiticity

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Abstract
For an invertible (bounded) linear operator Q acting in a Hilbert space \( \mathcal{H} \), we consider the consequences of the QT-symmetry of a non-hermitian Hamiltonian \( H : \mathcal{H} \to \mathcal{H} \) where \( T \) is the time-reversal operator. If \( H \) is symmetric in the sense that \( TH^\dagger T = H \), then QT-symmetry is equivalent to \( Q^{-1} \)-weak-pseudo-hermiticity. But in general this equivalence does not hold. We show this using some specific examples. Among these is a large class of non-\( \mathcal{P}\mathcal{T} \)-symmetric Hamiltonians that share the spectral properties of \( \mathcal{P}\mathcal{T} \)-symmetric Hamiltonians.

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1. Introduction
Among the motivations for the study of the \( \mathcal{P}\mathcal{T} \)-symmetric quantum mechanics is the argument that \( \mathcal{P}\mathcal{T} \)-symmetry is a more physical condition than hermiticity because \( \mathcal{P}\mathcal{T} \)-symmetry refers to ‘spacetime reflection symmetry’ whereas hermiticity is ‘a mathematical condition whose physical basis is somewhat remote and obscure’ [1]. This statement is based on the assumption that the operators \( \mathcal{P} \) and \( T \) continue to keep their standard meanings, as parity (space)-reflection and time-reversal operators, also in \( \mathcal{P}\mathcal{T} \)-symmetric quantum mechanics. But this assumption is not generally true, for unlike \( T \) the parity operator \( \mathcal{P} \) loses its connection to physical space once one endows the Hilbert space with an appropriate inner product to reinstate unitarity. This is because for a general \( \mathcal{P}\mathcal{T} \)-symmetric Hamiltonian, such as \( H = p^2 + x^2 + ix^3 \), the \( x \)-operator is no longer a physical observable, the kets \( |x\rangle \) do not correspond to localized states in space, and \( \mathcal{P} := \int_{-\infty}^{\infty} dx |x\rangle \langle -x| \) does not mean space-reflection [2, 3]. Furthermore, it turns out that one cannot actually avoid using the mathematical operations such as hermitian

1 The space reflection operator is given by \( \int_{-\infty}^{\infty} dx |\xi(x)\rangle \langle \xi^{(-x)}| \), where \( |\xi(x)\rangle \) denote the (localized) eigenkets of the pseudo-hermitian position operator \( X \) [2].
What makes $\mathcal{PT}$-symmetry interesting is not its physical appeal but the fact that $\mathcal{PT}$ is an antilinear operator $^3$. In fact, the spectral properties of $\mathcal{PT}$-symmetric Hamiltonians $^6$ that have made them a focus of recent interest follow from this property. In general, if a linear operator $H$ commutes with an antilinear operator $\Theta$, the spectrum of $H$ may be shown to be pseudo-real, i.e., as a subset of complex plane it is symmetric about the real axis. In particular, non-real eigenvalues of $H$ come in complex-conjugate pairs. If $H$ is a diagonalizable operator with a discrete spectrum the latter condition is necessary and sufficient for the pseudo-hermiticity of $H$ $^7$.

In $^8$, we showed that the spectrum of the Hamiltonian $H = p^2 + z\delta(x)$ is real and that one can apply the methods of pseudo-hermitian quantum mechanics $^2$ to identify $H$ with the Hamiltonian of a unitary quantum system provided that the real part of $z$ does not vanish$^4$.

This Hamiltonian is manifestly non-$\mathcal{PT}$-symmetric. The purpose of this paper is to offer other classes of non-$\mathcal{PT}$-symmetric Hamiltonians that enjoy the same spectral properties.

In the following, we shall use $H$ and $T$ to denote a (separable) Hilbert space and an invertible antilinear operator acting in $H$, respectively. For $H = L^2(\mathbb{R}^d)$, we define $T$ by $^9$

\[
(T\psi)(\vec{x}) := \psi(\vec{x})^*,
\]

for all $\psi \in L^2(\mathbb{R}^d)$ and $\vec{x} \in \mathbb{R}^d$. For $H = \mathbb{C}^N$, we identify it with complex conjugation: for all $\vec{z} \in \mathbb{C}^N$,

\[
T\vec{z} := \vec{z}^*.
\]

2. $\mathcal{QT}$-symmetry

Consider a Hamiltonian operator $H$ acting in $\mathcal{H}$ and commuting with an arbitrary invertible antilinear operator $\Theta$. Because $T$ is also invertible and antilinear, we can express $\Theta$ as $\Theta = QT$, where $Q := \Theta T$ is an invertible linear operator. This suggests the investigation of $\mathcal{QT}$-symmetric Hamiltonians $H$,

\[
[H, QT] = 0,
\]

where $Q$ is an invertible linear operator. Note that $Q$ need not be a hermitian operator or an involution, i.e., in general $Q^\dagger \neq Q$ and $Q^2 \neq 1$.

We can easily rewrite (3) in the form

\[
THT = Q^{-1}HQ.
\]

This is similar to the condition that $H$ is $Q^{-1}$-weakly-pseudo-hermitian $^{12-15}$:

\[
H^\dagger = Q^{-1}HQ.
\]

In fact, (4) and (5) coincide if and only if

\[
TH^T = H.
\]

2 The adjoint $A^\dagger$ of an operator $A : \mathcal{H} \rightarrow \mathcal{H}$ is defined by the condition $\langle \psi | A \phi \rangle = \langle A^\dagger | \phi \rangle$, where $\langle \cdot | \cdot \rangle$ is the defining inner product of the Hilbert space $\mathcal{H}$.

3 This means that $\mathcal{PT}(a_1 \psi_1 + a_2 \psi_2) = a_1^* \mathcal{PT} \psi_1 + a_2^* \mathcal{PT} \psi_2$, where $a_1, a_2$ are complex numbers and $\psi_1, \psi_2$ are state vectors.

4 Otherwise, $H$ has a spectral singularity and it cannot define a unitary time-evolution regardless of the choice of the inner product.
is symmetric\(^5\). For example, let \(\vec{a}\) and \(v\) be respectively complex vector and scalar potentials, \(\vec{x} \in \mathbb{R}^d\) and \(d \in \mathbb{Z}^+\). Then the Hamiltonian\(^6\)

\[
H = \frac{[\vec{p} - \vec{a}(\vec{x})]^2}{2m} + v(\vec{x})
\]

is symmetric if and only if \(\vec{a} = \vec{0}\). Supposing that \(\vec{a}\) and \(v\) are analytic functions, the QT-symmetry of (7), i.e., (4) is equivalent to

\[
\frac{[\vec{p} + \vec{a}(\vec{x})]^2}{2m} + v(\vec{x})^* = \frac{[\vec{p}_Q - \vec{a}(\vec{x}_Q)]^2}{2m} + v(\vec{x}_Q),
\]

where for any linear operator \(L : \mathcal{H} \rightarrow \mathcal{H}\), we have \(L_Q := Q^{-1}LQ\). Similarly, the \(Q^{-1}\)-weak-pseudo-hermiticity of \(H\), i.e., (5) means

\[
\frac{[\vec{p} - \vec{a}(\vec{x})]^2}{2m} + v(\vec{x})^* = \frac{[\vec{p}_Q - \vec{a}(\vec{x}_Q)]^2}{2m} + v(\vec{x}_Q).
\]

As seen from (8) and (9), there is a one-to-one correspondence between QT-symmetric and \(Q^{-1}\)-weak-pseudo-hermitian Hamiltonians of the standard form (7), namely that given such a QT-symmetric Hamiltonian \(H\) with vector and scalar potentials \(v\) and \(a\), there is a \(Q^{-1}\)-weak-pseudo-hermitian Hamiltonian \(H'\) with vector and scalar potentials \(v' = v\) and \(a' = ia\). Note, however, that \(H\) and \(H'\) are not generally isospectral.

3. A class of matrix models

Consider two-level matrix models defined on the Hilbert space \(\mathcal{H} = \mathbb{C}^2\) endowed with the Euclidean inner product \(\langle \cdot | \cdot \rangle\). In the following, we explore the QT-symmetry and \(Q^{-1}\)-weak-pseudo-hermiticity of a general Hamiltonian \(H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) for \(Q = \begin{pmatrix} 1 & 0 \\ q & 0 \end{pmatrix}\), where \(a, b, c, d, q \in \mathbb{C}\).

3.1. QT-symmetric two-level systems

Imposing the condition that \(H\) is QT-symmetric (i.e., equation (4) holds) restricts \(q\) to real and imaginary values, and leads to the following forms for the Hamiltonian.

- For real \(q\):
  \[
  H = \begin{pmatrix} a & 0 \\ c & a \end{pmatrix}, \quad a, c \in \mathbb{R}.
  \]

  In this case, \(H\) is a non-diagonalizable operator with a real spectrum consisting of \(a\).

- For imaginary \(q\) (\(q = iq\) with \(q \in \mathbb{R} - \{0\}\)):
  \[
  H = \begin{pmatrix} a - \frac{1}{2}bq & b \\ c + \frac{1}{2}(a - d)q & d + \frac{1}{2}bq \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.
  \]

  In this case the eigenvalues of \(H\) are given by \(E_{\pm} = \frac{1}{2}[a + d \pm \sqrt{(a - d)^2 - b(bq^2 - 4c)}]\).

  Therefore, for \((a - d)^2 \geq b(bq^2 - 4c)\), \(H\) is a diagonalizable operator with a real spectrum;

\(^5\) It is a common practice to identify operators with matrices and define the transpose of an operator \(H\) as the operator whose matrix representation is the transpose of the matrix representation of \(H\). Because one must use a basis to determine the matrix representation, unlike \(H^t := TH^tT\), this definition of transpose is basis-dependent. Note however that \(H^t\) agrees with this definition if one uses the position basis \(\{|\vec{x}\rangle\}\) in \(L^2(\mathbb{R}^d)\) and the standard basis in \(\mathbb{C}^N\).

\(^6\) Non-hermitian Hamiltonians of this form have been used in modeling localization effects in condensed matter physics [16].
and for \((a - d)^2 < b(bq^2 - 4c)\), \(H\) is diagonalizable but its spectrum consists of a pair of (complex-conjugate) non-real eigenvalues. Furthermore, the degeneracy condition \((a - d)^2 = b(bq^2 - 4c)\) marks an exceptional spectral point \([10, 11]\) where \(H\) becomes non-diagonalizable. In fact, for \(a = d\) and \(b = 0\) this condition is satisfied and \(H\) takes the form \((10)\). Therefore, \((11)\) gives the general form of \(QT\)-symmetric Hamiltonians provided that \(q \in \mathbb{R}\).

3.2. \(Q^{-1}\)-weakly-pseudo-hermitian two-level systems

Demanding that \(H\) is \(Q^{-1}\)-weakly-pseudo-hermitian does not pose any restriction on the value of \(q\). It yields the following forms for the Hamiltonian.

- For \(q = 0\):
  \[
  H = \begin{pmatrix}
  a & b_1 + ib_2 \\
  b_1 - ib_2 & d
  \end{pmatrix}, \quad a, b_1, b_2, d \in \mathbb{R}.
  \]  \((12)\)
  In this case \(Q\) is the identity operator and \(H = H^\dagger\). Therefore, \(H\) is a diagonalizable operator with a real spectrum.

- For \(q \neq 0\):
  \[
  H = \begin{pmatrix}
  a_1 + ia_2 & -\frac{2ia_2}{q} \\
  \frac{2ia_2}{q} & a_1 - ia_2
  \end{pmatrix}, \quad a_1, a_2 \in \mathbb{R}, \quad q \in \mathbb{C} - \{0\}.
  \]  \((13)\)
  In this case the eigenvalues of \(H\) are given by \(E_{\pm} = a_1 \pm |a_2||q|^{-1}\sqrt{4 - |q|^2}\). Therefore, for \(|q| < 2\), \(H\) is a diagonalizable operator with a real spectrum; and for \(|q| > 2\), \(H\) is diagonalizable but its spectrum consists of a pair of (complex-conjugate) non-real eigenvalues. Again the degenerate case: \(|q| = 2\) corresponds to an exceptional point where \(H\) becomes non-diagonalizable.

Comparing \((11)\) with \((12)\) and \((13)\) we see that \(QT\)-symmetry and \(Q^{-1}\)-weak-pseudo-hermiticity are totally different conditions on a general non-symmetric Hamiltonian\(^7\). For a symmetric Hamiltonian, we can easily show using \((12)\) and \((13)\) that \(q\) is either real or imaginary and that \(H\) takes the form \((11)\). The converse is also true, i.e., any symmetric Hamiltonian of the form \((11)\) is either real (and hence hermitian) or has the form \((13)\). In summary, \(QT\)-symmetry and \(Q^{-1}\)-weak-pseudo-hermiticity coincide if and only if the Hamiltonian is a symmetric matrix.

4. Unitary \(Q\) and a class of non-\(\mathcal{PT}\)-symmetric Hamiltonians with a pseudo-real spectrum

If \(Q\) is a unitary operator, the \(Q^{-1}\)-weak-pseudo-hermiticity \((5)\) of a Hamiltonian \(H\) implies its \(Q\)-weak-pseudo-hermiticity, i.e., \(H^\dagger = Q^{-1}HQ\). This together with \((5)\) leads in turn to

\[
[H, Q^2] = 0,
\]  \((14)\)
  i.e., \(Q^2\) is a symmetry generator. In the following, we examine some simple unitary choices for \(Q\) and determine the form of the \(Q^{-1}\)-weak-pseudo-hermitian and \(QT\)-symmetric standard Hamiltonians.

\(^7\) Note that this is not in conflict with the fact that in view of the spectral theorems of \([12, 17, 18]\) both of these conditions imply pseudo-hermiticity of the Hamiltonian albeit with respect to a pseudo-metric operator that differs from \(Q^{-1}\) \([15]\).
Consider a standard non-hermitian Hamiltonian (7) in one dimension and let
\[ Q = e^{i\frac{\ell}{2}} \]
for some \( \ell \in \mathbb{R}^+ \). Then introducing
\[ a_1 := \Re(a), \quad a_2 := \Im(a), \quad v_1 := \Re(v), \quad v_2 := \Im(v), \]
where \( \Re \) and \( \Im \) stand for the real and imaginary parts of their argument, and using the identities
\[ Q^{-1}pQ = p, \quad Q^{-1}xQ = x - \ell, \]
we can express the condition of the \( Q^{-1} \)-weak-pseudo-hermiticity of \( H \), namely (9), in the form
\[ a_1(x - \ell) = a_1(x), \quad a_2(x - \ell) = -a_2(x), \quad (17) \]
\[ v_1(x - \ell) = v_1(x), \quad v_2(x - \ell) = -v_2(x). \quad (18) \]
This means that the real parts of the vector and scalar potentials are periodic functions with period \( \ell \) while their imaginary parts are antiperiodic with period \( \ell \). This confirms (14), for \( H \) is invariant under the translation, \( x \rightarrow x + 2\ell \), generated by \( Q^2 \). We can express \( a_1, v_1 \) and \( a_2, v_2 \) in terms of their Fourier series. These have, respectively, the following forms
\[ \ell \text{-periodic real parts:} \sum_{n=0}^{\infty} \left[ c_{kn} \cos \left( \frac{2n\pi x}{\ell} \right) + d_{kn} \sin \left( \frac{2n\pi x}{\ell} \right) \right], \quad (19) \]
\[ \ell \text{-antiperiodic imaginary parts:} \sum_{n=0}^{\infty} \left[ c_{kn} \cos \left( \frac{(2n+1)\pi x}{\ell} \right) + d_{kn} \sin \left( \frac{(2n+1)\pi x}{\ell} \right) \right], \quad (20) \]
where \( c_{kn} \) and \( d_{kn} \) are real constants for all \( k \in \{1, 2\} \) and \( n \in \{0, 1, 2, \ldots\} \).

Conversely, if the real and imaginary parts of both the vector and scalar potentials have respectively the form (19) and (20), the Hamiltonian is \( Q^{-1} \)-weak-pseudo-hermitian. In particular its spectrum is pseudo-real; its complex eigenvalues come in complex-conjugate pairs. These Hamiltonians that are generally non-\( PT \)-symmetric acquire \( QT \)-symmetry provided that they are symmetric, i.e., \( a_1 = a_2 = 0 \). A simple example is
\[ H = \frac{p^2}{2m} + \lambda_1 \sin(2kx) + i\lambda_2 \cos(5kx), \]
where \( \lambda_1, \lambda_2 \in \mathbb{R} \) and \( k := \ell^{-1} \in \mathbb{R}^+ \).

Next, we examine the condition of \( QT \)-symmetry of \( H \), i.e. (8). In view of (16), this condition is equivalent to (18) and
\[ a_1(x - \ell) = -a_1(x), \quad a_2(x - \ell) = a_2(x), \quad (21) \]
which replaces (17). Therefore, \( v \) has the same form as for the case of a \( Q^{-1} \)-weak-pseudo-hermitian Hamiltonian but \( a \) has \( \ell \)-antiperiodic real and \( \ell \)-periodic imaginary parts. In particular, the Fourier series for the real and imaginary parts of \( a \) have respectively the form (20) and (19).

We again see that general \( QT \)-symmetric Hamiltonians of the standard form (7) are invariant under the translation \( x \rightarrow x + 2\ell \). This is indeed to be expected, because in view of \([Q, T] = 0\) we can express (4) in the form
\[ H = Q^{-1}THTQ \]
and use this identity to establish
\[ Q^2H = QTHTQ = QT(Q^{-1}THTQ)TQ = HQ^2. \]
The results obtained in this section admit a direct generalization to higher-dimensional standard Hamiltonians. This involves identifying $Q$ with a translation operator of the form $e^{i\vec{\ell} \cdot \vec{J}}$ for some $\ell \in \mathbb{R}^3 - \{0\}$. It yields $QT$-symmetric and $Q^{-1}$-weak-pseudo-hermitian Hamiltonians with a pseudo-real spectrum that are invariant under the translation $\vec{x} \rightarrow \vec{x} - 2\ell$.

An alternative generalization of the results of this section to (two and) three dimensions is to identify $Q$ with a rotation operator:

$$Q = e^{i\hat{n} \cdot \vec{J}},$$

where $\varphi \in (0,2\pi)$, $\hat{n}$ is a unit vector in $\mathbb{R}^3$ and $\vec{J}$ is the angular momentum operator. Again $[Q,T] = 0$ and we obtain generally non-$\mathcal{PT}$-symmetric, $Q^{-1}$-weak-pseudo-hermitian and $QT$-symmetric Hamiltonians with a pseudo-real spectrum that are invariant under rotations by an angle $2\varphi$ about the axis defined by $\hat{n}$.

Choosing a cylindrical coordinate system whose $z$-axis is along $\hat{n}$, we can obtain the general form of such standard Hamiltonians.

The $Q^{-1}$-weak-pseudo-hermiticity of $H$ implies that the real and imaginary parts of the vector and scalar potentials (that we identify with labels 1 and 2, respectively) satisfy

$$\vec{a}_1(\rho, \theta - \varphi, z) = \vec{a}_1(\rho, \theta, z), \quad \vec{a}_2(\rho, \theta - \varphi, z) = -\vec{a}_2(\rho, \theta, z),$$

$$v_1(\rho, \theta - \varphi, z) = v_1(\rho, \theta, z), \quad v_2(\rho, \theta - \varphi, z) = -v_1(\rho, \theta, z),$$

where $(\rho, \theta, z)$ stand for cylindrical coordinates. Similarly, the $QT$-symmetry yields

$$\vec{a}_1(\rho, \theta - \varphi, z) = -\vec{a}_1(\rho, \theta, z), \quad \vec{a}_2(\rho, \theta - \varphi, z) = \vec{a}_2(\rho, \theta, z).$$

Again we can derive the general form of the Fourier series for these potentials. Here, we suffice to give the form of the general symmetric Hamiltonian:

$$H = \frac{\hat{p}^2}{2m} + \sum_{n=0}^{\infty} \left[ e_n(\rho, z) \cos(2n\omega \theta) + f_n(\rho, z) \sin(2n\omega \theta) \right] + i \left[ g_n(\rho, z) \cos((2n+1)\omega \theta) + h_n(\rho, z) \sin((2n+1)\omega \theta) \right],$$

where $e_n$, $f_n$, $g_n$ and $h_n$ are real-valued functions and $\omega := \varphi^{-1} \in \mathbb{R}^+$.  

5. A $QT$-symmetric and non-$\mathcal{PT}$-symmetric Hamiltonian with a real spectrum

In the preceding section we examined $QT$-symmetric Hamiltonians with a unitary $Q$. Because $\mathcal{P}$ is also a unitary operator, $QT$-symmetry with unitary $Q$ may be considered as a less drastic generalization of $\mathcal{PT}$-symmetry. In this section we explore a $QT$-symmetric model with a non-unitary $Q$.

Let $a$ and $a^\dagger$ be the bosonic annihilation and creation operators acting in $L^2(\mathbb{R})$ and satisfying $[a, a^\dagger] = 1$, $q \in \mathbb{C} - \{0\}$, and

$$Q := e^{i\theta q}.$$  

Consider the Hamiltonian operator

$$H = \alpha a^2 + \beta a^\dagger^2 + \gamma [a, a^\dagger] + ma + n a^\dagger,$$

where $\alpha, \beta, \gamma, m, n \in \mathbb{C}$, and demand that $H$ be $QT$-symmetric. Inserting (29) in (4) and using (1) and the identity $Q^{-1} a^\dagger Q = a^\dagger - q$, we obtain

$$a^* = \alpha, \quad \beta^* = \beta, \quad \gamma^* = \gamma, \quad m^* = m - 2\gamma q, \quad n^* = n - 2\beta q, \quad nq = 0.$$

8 The $Q$ considered in section 3 may be viewed as a fermionic analog of (28).
In particular, because \( q \neq 0 \), we have \( n = 0 \) which in turn implies \( \beta = 0 \). Furthermore, assuming that \( \gamma \neq 0 \), we find that \( q \) must be purely imaginary, \( q = iq \) with \( q \in \mathbb{R} - \{0\} \), and \( \Im(\gamma) = q \). In view of these observations, \( H \) takes the following simple form,
\[
H = \alpha a^2 + \gamma [a, a^\dagger] + (\mu + iyq)a,
\]
where \( \mu := \Re(\gamma) \), \( \alpha, \mu \in \mathbb{R} \), and \( \gamma, q \in \mathbb{R} - \{0\} \).

Recalling that \( \mathcal{P}a\mathcal{P} = -a \) and \( \mathcal{T}a\mathcal{T} = a \), we see that for \( \mu \neq 0 \), the \( \mathcal{QT} \)-symmetric Hamiltonian (30) is non-\( \mathcal{PT} \)-symmetric. We also expect that it must have a pseudo-real spectrum. It turns out that actually the spectrum of \( H \) can be computed exactly.

To obtain the spectrum of \( H \) we use its representation in the basis consisting of the standard normalized eigenvectors \( |n\rangle \) of the number operator \( a^\dagger a \). Using the following well-known properties of \( |n\rangle \) [19],
\[
a|n\rangle = \sqrt{n} |n - 1\rangle, \quad a^\dagger |n\rangle = \sqrt{n+1} |n + 1\rangle,
\]
we find for all \( m, n \in \{0, 1, 2, \ldots\} \),
\[
H_{mn} := \langle m|H|n\rangle = \gamma (2n + 1) \delta_{mn} + (\mu + iyq) \sqrt{n} \delta_{m,n-1} + \alpha \sqrt{n(n-1)} \delta_{m,n-2}.
\]
As seen from this relation the matrix \( (H_{mn}) \) is upper-triangular with distinct diagonal entries and up to three nonzero terms in each row. This implies that the eigenvalues \( E_n \) of \( (H_{mn}) \) are identical with its diagonal entries, i.e., \( E_n = \gamma (2n + 1) \). In particular, \( H \) has a discrete, equally spaced, real spectrum that is positive for \( \gamma > 0 \). In the latter case, \( H \) is isospectral to a simple harmonic oscillator Hamiltonian with ground-state energy \( \gamma \).

It is not difficult to see that for each \( n \in \{0, 1, 2, \ldots\} \) the span of \( \{|0\rangle, |1\rangle, \ldots, |n\rangle\} \), which we denote by \( \mathcal{H}_n \), is an invariant subspace of \( H \). This observation allows for the construction of a complete set of eigenvectors of \( H \) and establishes the fact that \( H \) is a diagonalizable operator with a discrete real spectrum. Therefore, in view of a theorem proven in [20], it is related to a hermitian operator via a similarity transformation, i.e., it is quasi-hermitian [21].

The existence of the invariant subspace \( \mathcal{H}_n \) also implies that the eigenvectors \( |\psi_n\rangle \) of \( H \) corresponding to the eigenvalue \( E_n \) belong to \( \mathcal{H}_n \), i.e., \( |\psi_n\rangle \) is a linear combination of \( |0\rangle, |1\rangle, \ldots, |n - 1\rangle \) and \( |n\rangle \). This in turn allows for a calculation of \( |\psi_n\rangle \). For example,
\[
|\psi_0\rangle = c_0 |0\rangle, \quad |\psi_1\rangle = c_1 \left[ |0\rangle + \frac{2\gamma m}{m^2 + \alpha \gamma} |1\rangle \right],
\]
\[
|\psi_2\rangle = c_2 \left[ |0\rangle + \frac{4\gamma m}{m^2 + \alpha \gamma} |1\rangle + \frac{2\sqrt{2\gamma^2}}{m^2 + \alpha \gamma} |2\rangle \right],
\]
where \( c_0, c_1, c_2 \) are arbitrary nonzero normalization constants and \( m = \mu + iyq \).

### 6. Concluding remarks

It is often stated that \( \mathcal{PT} \)-symmetry is a special case of pseudo-hermiticity because \( \mathcal{PT} \)-symmetric Hamiltonians are manifestly \( \mathcal{P} \)-pseudo-hermitian. This reasoning is only valid for symmetric Hamiltonians \( H \) that satisfy \( H^\dagger = \mathcal{T} H \mathcal{T} \). In general to establish the claim that \( \mathcal{PT} \)-symmetry is a special case of pseudo-hermiticity one needs to make use of the spectral theorems of [12, 17, 18]. Indeed what makes \( \mathcal{PT} \)-symmetric Hamiltonians interesting is the pseudo-reality of their spectrum. This is a general property of all Hamiltonians that are weakly pseudo-hermitian or possess a symmetry that is generated by an invertible antilinear operator. We call the latter \( \mathcal{QT} \)-symmetric.

In this paper, we have examined in some detail the similarities and differences between \( \mathcal{QT} \)-symmetry and \( Q^{-1} \)-weak-pseudo-hermiticity and obtained large classes of symmetric
as well as asymmetric non-$\mathcal{PT}$-symmetric Hamiltonians that share the spectral properties of the $\mathcal{PT}$-symmetric Hamiltonians. In particular, we considered the case that $Q$ is a unitary operator and showed that in this case $Q\mathcal{T}$-symmetry and $Q^{-1}$-weak-pseudo-hermiticity imply $Q^2$-symmetry of the Hamiltonian. We also explored a concrete example of a $Q\mathcal{T}$-symmetric Hamiltonian with a non-unitary $Q$ that is not $\mathcal{PT}$-symmetric. We determined the spectrum of this Hamiltonian, established its diagonalizability and showed that it is indeed quasi-hermitian.

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