Recently, we have demonstrated that there exists a possible relationship between $q$-deformed algebras in two different contexts of Statistical Mechanics, namely, the Tsallis’ framework and the Kaniadakis’ scenario, with a local form of fractional-derivative operators for fractal media, the so-called Hausdorff derivatives, mapped into a continuous medium with a fractal measure. Here, in this paper, we present an extension of the traditional calculus of variations for systems containing deformed-derivatives embedded into the Lagrangian and the Lagrangian densities for classical and field systems. The results extend the classical Euler-Lagrange equations and the Hamiltonian formalism. The resulting dynamical equations seem to be compatible with those found in the literature, specially with mass-dependent and with nonlinear equations for systems in classical and quantum mechanics. Examples are presented to illustrate applications of the formulation. Also, the conserved Nether current, are worked out.

I. INTRODUCTION

The minimum action principle implies that, by minimizing some action variables or functionals, we can obtain the dynamical equations that describe physical phenomena. The formalism related is known as the calculus of variations. But, the classical variational calculus has a major difficult in dealing with nonconservative systems. Here, we claim that the calculus of variations with deformed-derivatives embedded into a Lagrangian or a Lagrangian density is adequate to study both, conservative and nonconservative classical and field systems in order to obtain a version of the respective Euler-Lagrange equations ($E$–$L$), combining both cases. Also, systems with position-dependent mass and nonlinearities can be studied and the $E$–$L$ obtained.

The applications of the formalism presented here may include classical and quantum mechanics, field theory, complex systems and so on. We are not talking about including classical definitions of fractional derivatives nor including operators of integer order acting on a d-dimensional space but we consider a mapping from a fractal coarse-grained (fractal porous) space, which is essentially discontinuous in the embedding Euclidean space to a continuous one. Also, in the construction of the actions to obtain the Euler Lagrange equations (ELE), we have used different definitions for the actions and for the deformed-derivatives and integrals.

In Ref. [5], we adopt the viewpoint to suitably treat nonconservative systems is through Fractional Calculus (FC), since it can be shown that, for example, a friction force has its form stemming from a Lagrangian that contains a term proportional to the fractional derivative, which may be a derivative of any non-integer order. Parallel to the standard FC, there is some kind of the local fractional calculus with certain definitions called local fractional derivatives, for example, the works of Refs.[4, 7–12]. Here we are interested in the related approaches with Hausdorff derivative and the conformable derivative. We think that the most appropriate name of those formalism are deformed-derivatives or even metric or topological derivatives. All of the mentioned approaches seem to be applicable to power-law phenomena. Also, for description of complex systems, the $q$-calculus, in a non-extensive statistic context, has its formal development based on the definition of deformed expressions for the logarithm and exponential, namely, the $q$–logarithm and the $q$–exponential. In this context, an interesting algebra emerges and the formalism of a deformed derivative opened new possibilities for, besides the thermodynamical, other treatment of complex systems, specially those with fractal or multifractal metrics and presenting long-range dynamical interactions.

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The deformation parameter or entropic index, $q$, occupying an important place in the description of those complex systems, describes deviations from standard Lie symmetries and the formalism aimed to accommodate scale invariance in a system with multifractal properties to the thermodynamic formalism. For $q \to 1$, the formalism reverts to the standard one.

Here, we consider the relevant space-time/phase space as fractal or multifractal.[10]

The use of deformed-operators is also justified here based on our proposition that there exists an intimate relationship between dissipation, coarse-grained media and the some limit scale of energy for the interactions. Since we are dealing with open systems, as commented in Ref. [11], the particles are indeed dressed particles or quasi-particles that exchange energy with other particles and the environment. Depending on the energy scale an interaction may change the geometry of space–time, disturbing it at the level of its topology. A system composed by particles and the surrounding environment may be considered nonconservative due to the possible energy exchange. This energy exchange may be the responsible for the resulting non-integer dimension of space–time, giving rise then to a coarse-grained medium. This is quite reasonable since, even standard field theory, comes across a granularity in the limit of Planck scale. So, some effective limit may also exist in such a way that it should be necessary to consider a coarse-grained space–time for the description of the dynamics for the system, in this scale. Also, another perspective that may be proposed is the previous existence of a nonstandard geometry, e.g., near a cosmological black hole or even in the space nearby a pair creation, that imposes a coarse-grained view to the dynamics of the open system. Here, we argue that deformed-derivatives, similarly to the FC, allows us to describe and emulate this kind of dynamics without explicit many-body, dissipation or geometrical terms in the dynamical governing equations. In some way, the formalism proposed here may yield an effective theory, with some statistical average without imposing any specific nonstandard statistics. So, deformed derivatives and/or FC may be the tools that could describe, in a softer way, connections between coarse-grained medium and dissipation at a certain energy scale.

One relevant applicability of our formalism may concern position-dependent systems (see Ref. [17] and references therein) that seems to be more adequate to describe the dynamics of many real complex systems where there could exist long-range interactions, long-time memories, anisotropy, certain symmetry breakdown, nonlinear media, etc.[18].

In the case of quantum mechanics, certain minimum length scale yields a modification in the position momentum commutation relationship or some modification in the underlying space-time that may results in a Schrödinger equation with a position-dependent mass [19].

Also, to find more suitable ways to explaining several complex behaviors in nature, Nonlinear (NL) equations have become an important subject subject of study [20], since the applicability of linear equations in physics is usually restricted to idealized systems [21].

An important point to emphasize is that the paradigm we adopt is different from the standard approach in the generalized statistical mechanics context, where the modification of entropy definition leads to the modification of algebra and consequently the derivative concept. We adopt that the mapping to a continuous fractal space leads naturally to the necessity of modifications in the derivatives, that we will call deformed or metric derivatives [22]. The modifications of derivatives brings to a change in the algebra involved, which in turn may conduct to a generalized statistical mechanics with some adequate definition of entropy.

In this paper, we initiate a general variational calculus with metric derivatives embedded into the Lagrangian. The purpose of the present work is to develop the corresponding generalization and for this, we define three options to pursue that will be described in the forthcoming sections.

Our paper is outlined as follows: In Section 2, we cast some mathematical aspects, in Section 3, we develop the variational approach based on embedded deformed-derivatives. In Section 4, we extend the formalism to relativistic independent fields. In Section 5, some applications are presented; in Section 6, the Hamiltonian formalism are addressed and we cast finally our Conclusions and Outlook in Section 7.

II. A GLANCE AT MATHEMATICAL ASPECTS

Hausdorff Derivative

A model that maps hydrodynamics continuum flow in a fractal coarse-grained (fractal porous) space, which is essentially discontinuous in the embedding Euclidean space, into a continuous flow governed by conventional partial differential equations was suggested in Refs. [2, 3]. Using non-conventional partial differential equations based on the model of a fractal continuous flow employing local fractional differential operators, Balankin has suggested essentially that the discontinuous fractal flow in a fractal permeable medium can be mapped into the fractal continuous flow, which is describable within a continuum framework, indicating also that the geometric framework of fractal continuum model is the three-dimensional Euclidean space with a fractal metric.

Employing the local fractional differential operators in connection with the Hausdorff derivative [4], the latter
derivative can be written as \[^{2}\]:

\[
\frac{d^{H}}{dx^{\alpha}}f(x) = \lim_{x \to x'} \frac{f(x') - f(x)}{(x')^{\alpha} - x^{\alpha}} = \left( \frac{x}{l_{0}} + 1 \right)^{1-\zeta} \frac{d}{dx}f = \frac{t_{0}^{-\alpha}}{c_{1}} \frac{d}{dx}f = \frac{d}{dx_{x}}f,
\]

where \(l_{0}\) is the lower cutoff along the Cartesian \(x\)-axis and the scaling exponent, \(\zeta\), characterizes the density of states along the direction of the normal to the intersection of the fractal continuum with the plane, as defined in the work \[^{2}\].

**Conformable Derivative**

Recently, a promising new definition of local deformed derivative, called conformable fractional derivative, has been proposed by the authors in Ref. \[^{14}\] that preserve classical properties and is given by

\[
T_{\alpha}f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}.
\]

If the function is differentiable in a classical sense, the definition above yields

\[
T_{\alpha}f(t) = t^{1-\alpha} \frac{df(t)}{dt}.
\]

Changing the variable \(t \to 1 + \frac{t}{l_{0}}\), we should write \(^{3}\) as \(l_{0} \left( 1 + \frac{t}{l_{0}} \right)^{1-\alpha} \frac{df(t)}{dx_{x}}\), that is nothing but the Hausdorff derivative up to a constant and valid for differentiable functions.

Also, the Riemann Improper Integral \[^{14}\] can be written as:

\[
I_{a}^{\alpha}(f)(t) = \int_{a}^{t} f(x)d^{\alpha}x = \int_{a}^{t} f(x)x^{\alpha-1}dx,
\]

where \(dx^{\alpha} = \frac{dx}{t^{\alpha}}\).

Another similar definition of local deformed derivative with classical properties is that used in Ref. \[^{23}\]:

Let \(f : [0, \infty) \to \mathbb{R}\) and \(t > 0\). Then the local deformed derivative-Katugampola- of \(f\) of order \(\alpha\) is defined by,

\[
\mathcal{D}^{\alpha}(f)(t) = \lim_{\epsilon \to 0} \frac{f(t\epsilon^{\epsilon^{-\alpha}}) - f(t)}{\epsilon},
\]

for \(t > 0\), \(\alpha \in (0, 1)\). If \(f\) is \(\alpha\)-differentiable in some \((0, a)\), \(a > 0\), and \(\lim_{t \to 0^{+}} \mathcal{D}^{\alpha}(f)(t)\) exists, then define

\[
\mathcal{D}^{\alpha}(f)(0) = \lim_{t \to 0^{+}} \mathcal{D}^{\alpha}(f)(t).
\]

The Riemann Improper Integral is also considered in this approach.

**q-derivative in the nonextensive context**

Over the recent decades, diverse formalisms have emerged that are adopted to approach complex systems. Amongst those, we may quote the q-calculus in Tsallis’ version of Non-Extensive Statistics with its undeniable success whenever applied to a wide class of different systems; Kaniadakis’ approach, based on the compatibility between relativity and thermodynamics; Fractional Calculus (FC), that deals with the dynamics of anomalous transport and other natural phenomena, and also some local versions of FC that claim to be able to study fractal and multifractal spaces and to describe dynamics in these spaces by means of fractional differential equations. See the references in \[^{1}\].

With the generalized nonadditive \(q\)-entropy as motivation, the \(q\)-derivative sets up a deformed algebra and takes into account that the \(q\)-exponential is eigenfunction of \(D_{(q)}\) \[^{15}\]. Borges proposed the operator for \(q\)-derivative as below:

\[
D_{(q)}f(x) = \lim_{y \to x} \frac{f(x) - f(y)}{x \ominus_{q} y} = [1 + (1 - q)x] \frac{df(x)}{dx}.
\]

Here, \(\ominus_{q}\) is the deformed difference operator, \(x \ominus_{q} y = \frac{x - y}{1 + (1 - q)y} \quad (y \neq 1/(q - 1)).\)
The q-Integral has the similar structure of Riemann Improper Integral:

\[ \int_{a}^{t} f(x) dq_x = \int_{a}^{t} \frac{f(x)}{1 + (1 - q)x} dx = \int_{a}^{t} f(x)dq_x; \]  

(8)

where \( dq_x = \lim_{y \to x} x - x(q - y) = \frac{1}{1 + (1 - q)x} dx \).

Recently we have shown that

\[ 1 - q = \frac{(1 - \zeta)}{t_0}. \]  

(9)

So, we concluded that the deformed q-derivative is the first order expansion of the Hausdorff derivative and that there is a strong connection between these formalism by means of a fractal metric.

Now, we highlight some results and relevant properties, which are valid for local forms of derivative addressed here:

**Leibniz rule**

As regards the product rule for derivatives, conformable, Katugampola or Hausdorff follows the rule similar to the usual derivatives:

\[ D^\alpha(fg) = gD^\alpha f + fD^\alpha g, \]

and similarly, for q-derivative:

\[ D^\alpha_q(fg) = gD^\alpha_q f + fD^\alpha_q g. \]

**Integration by parts**

Generalizing, let \((A,B)\) be a subinterval of \((a, b)\). Consider a functional

\[ \int_{a}^{b} f(x)dx^\alpha = \int_{a}^{b} f(x)x^{\alpha-1}dx. \]  

(10)

Then, one can state that the integration by parts holds.

\[ \int_{a}^{b} f(x)D^\alpha g(x)dx^\alpha = f(x)g(x) \big|_{a}^{b} - \int_{a}^{b} g(x)D^\alpha f(x)dx^\alpha. \]  

(11)

The same holds for q-integral.

For details, the reader can consult the refs. [14, 23].

**The chain rule**

For composition of functions holds:

\[ D^\alpha[f \circ g](x) = \frac{df(g(x))}{dg} D^\alpha g(x), \]  

(12)

\[ D_q[f \circ g](x) = \frac{df(g(x))}{dg} D_q g(x). \]  

(13)

### III. VARIATIONAL APPROACH WITH EMBEDDED DERIVATIVES

Our problem here is to search minimizers of a variational problem with deformed derivative embedded into the Lagrangian function \( L \). After the mapping into the fractal continuum, \( L \) will be a \( C^2 \)-function with respect to all its arguments.

Remarks: (i) We consider a fractional variational problem which involves local deformed-derivatives, as Hausdorff, conformable, Katugampola or q-derivative.

(ii) The problem can be easily generalized for Lagrangians which will depend also on higher order deformed-derivatives.

(iii) We assume that \( 0 < \alpha < 1 \).

(iv) To pursue our objectives, we cast three options to address as different approaches:

Option 1: \( \alpha \)-integral or \( q \)-integral, usual, deformed-derivatives embedded
Option 2: $\alpha$–integral or $q$–integral, $\delta$deformed, deformed-derivatives embedded and

Option 3: Usual integral, $\delta$usual, deformed-derivatives embedded, similar to Ref. [25], but here with local deformed or metric derivatives.

We know that deformed-kernels used here can be replaced with other kernels, resulting in a general variational calculus, as in Ref. [27].

Our interpretation of the action above in option 1 and 2 is based on the existence of a succession of specific internal temporal intervals for the system which lead to a anomalous dynamics [26]. The time interval can be thought as distributed with a polynomial distributions of weights or probabilities that are convoluted with the Lagrangian [28]. This leads for the action to be similar to have some similarities with Stieltjes like integral. The action then can be interpreted, in this context as an integration of the Lagrangian, with embedded deformed-derivatives, in a cosmological inhomogeneous time scale. It is similar to time-clock randomization of momenta and coordinates taken from the conventional phase space [29] but without redefining the phase space with some thing like fractional derivative, generalized coordinates or momenta.

Notice that this kind of action is suitable to treat systems with dissipative forces or non-holonomic systems since it include the scale in time letting to consider the effects of internal times of the systems [30]. Thus, systems with memory effects may suitably be analyzed by means of this action. Note yet that any kind of deformed integrals or derivatives needs to be necessarily embedded into the Lagrangian. The only deformed integral could be the own action with its time probability factor convoluted.

Now, we can analyze each case:

**For the option 1:**

To derive the extended version of the Euler–Lagrange equation let us introduce the following $\alpha$–deformed functional

$$ J[y] = \int_\alpha L(x, y, D^\alpha_x y)d^\alpha x, $$

where $L$ is the Lagrangian with embedded deformed derivative $D^\alpha_x y$ and $d^\alpha x = \frac{1}{\Gamma(\alpha)} dx$.

Analogously, with a $q$–deformed functional we can write:

$$ J[y] = \int_q L(x, y, D_q y)d_q x. $$

We will find the condition that $J[y]$ has a local minimum. To do so, we consider the new fractional functional depending on the parameter $\varepsilon$.

Consider for the variable $y(x)$:

$$ y(x) = y^*(x) + \varepsilon \eta(x); \quad y^*(x) \text{ is the objective function, and } \eta(a) = \eta(b) = 0, \varepsilon \text{ is a the parameter}. $$

So, applying the deformed derivative, we obtain:

$$ D^\alpha_x y(x) = D^\alpha_x y^*(x) + \varepsilon D^\alpha_x \eta(x). $$

Using the chain rule and the well known $\delta$–variational processes relative to the $\varepsilon$ parameter, we can write

$$ \delta_\varepsilon L = \frac{\partial L}{\partial y} \eta + \frac{\partial L}{\partial D^\alpha_x y} D^\alpha_x \eta. $$

Since integration by part holds with deformed integral similarily of integer and, using that the usual transversality condition for an extreme value, one obtain that $\delta_\varepsilon J = 0$ implies that

$$ \frac{\partial L}{\partial y} - D^\alpha_x (\frac{\partial L}{\partial D^\alpha_x y}) = 0. \quad (14) $$

If $L$ depends on different derivative orders, as $L = L(x, y, D^\alpha_x y, D^\beta_x y, D^\gamma_x z)$, then one gets that the $E – L$ equations can be read as

$$ \frac{\partial L}{\partial y} - D^\alpha_x (\frac{\partial L}{\partial D^\alpha_x y}) - D^\beta_x (\frac{\partial L}{\partial D^\beta_x y}) - (\frac{\partial L}{\partial D^\gamma_x z})(\alpha - \beta)x^{-\beta} = 0 $$

for $y$ variable and for $z$–variable as
\[ \frac{\partial L}{\partial z} - \partial^2 \! \! \! \! \! \! \, \frac{\partial L}{\partial D^2 z} - (\frac{\partial L}{\partial D^2 z})((\alpha - \gamma)x^{-\gamma} = 0. \]

We can now go to the next option for analyzes.

**For the option 2:**

Here, for \( \delta \)-deformed approach we mean that the total derivative now takes into account the deformed derivatives. So, we can write for \( \delta \)-deformed process applied to the Lagrangian

\[ \delta^\alpha L = \frac{\partial L}{\partial y} D^\alpha y + \frac{\partial L}{\partial D^2 y} \partial^\alpha D^2 y, \]

where \( D^\alpha y = \eta(x)\varepsilon^{1-\alpha} \). Now, using the integration by parts, one gets

\[ \frac{\partial L}{\partial y} - D^\alpha (\frac{\partial L}{\partial D^2 y}) = 0. \] (15)

For \( q \)-derivative we can similarly write:

\[ \frac{\partial L}{\partial y} - D_{q,x} (\frac{\partial L}{\partial D_{q,x} y}) = 0. \] (16)

We can see that the results are analogous to those obtained by the option 1 approach. We can pursue now results from the next option.

**For the option 3:**

The Lagrangian \( L \) can be \( L = L(x, y, D^\alpha y, D^2 y, D^2 z) \), where embedded deformed derivatives are considered again. In this case we consider the fractional action \( J[y] = \int L(x, y, D^\alpha y, D^2 y, D^2 z) \) and the usual \( \delta \)-variational processes

\[ \delta_L = \frac{\partial L}{\partial y} \eta + \frac{\partial L}{\partial D^2 y} D^\alpha \eta_1 + \frac{\partial L}{\partial D^2 y} D^2 \eta_1 + \frac{\partial L}{\partial z} \eta_2 + \frac{\partial L}{\partial D^2 z} D^7 \eta_2, \]

with \( D^\alpha \eta = x^{1-\alpha} \frac{d \eta}{dx} \) and so on.

The resulting \( E - L \) equations are:

\[ \frac{\partial L}{\partial y} - \frac{d}{dx} (x^{1-\alpha} \frac{\partial L}{\partial D^2 y}) = 0, \] (17)

\[ \frac{\partial L}{\partial z} - \frac{d}{dx} (x^{1-\gamma} \frac{\partial L}{\partial D^2 z}) = 0. \] (18)

For \( q \)-derivative we will have:

\[ \frac{\partial L}{\partial y} - l_{01} \frac{d}{dx} [(1 + (1 - q_1 x)) \frac{\partial L}{\partial D_{q_1,x} y}] - \frac{d}{dx} [(1 + (1 - q_2 x)) \frac{\partial L}{\partial D_{q_2,x} y}] = 0, \] (19)

\[ \frac{\partial L}{\partial z} - l_{02} \frac{d}{dx} [(1 + (1 - q_3 x)) \frac{\partial L}{\partial D_{q_3,x} z}] = 0. \] (20)
IV. RELATIVISTIC, INDEPENDENT FIELDS

Now, we can proceed to pursue equivalent approaches to field theory, based on independent relativistic fields. Here, $\phi = \phi + \epsilon_1^\mu \delta \phi$, $\psi = \psi + \epsilon_2^\mu \delta \psi$,

$$\partial_\mu \phi = \partial_\mu \tilde{\phi} + \epsilon_1^\mu \partial_\mu \delta \phi,$$

$$\partial_\mu \psi = \partial_\mu \tilde{\psi} + \epsilon_2^\mu \partial_\mu \delta \psi.$$

Here, $\delta \phi, \delta \psi$ are arbitrary; $\tilde{\phi}, \tilde{\psi}$ are the objective fields and $\mu = 0, 1, 2, 3$, following the index spatial-temporal derivative, $\partial_\mu$.

With non-deformed standard derivative, we usually consider the fractional action

$$S = \int dt \int d^3x L(\phi, \partial_\mu \phi, \psi, \partial_\mu \psi, x^\mu),$$

with the usual $\delta -$ process

$$\delta \epsilon L = \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial \partial_\mu \phi} \delta \partial_\mu \phi + \frac{\partial L}{\partial \psi} \delta \psi + \frac{\partial L}{\partial \partial_\mu \psi} \delta \partial_\mu \psi.$$

For the deformed derivative embedded into the Lagrangian we can set $\partial_\mu \rightarrow \partial_\mu^{\alpha, \lambda}$, $\lambda = 0, 1, 2, 3$; $L = L(\phi, \partial_\mu^{\alpha, \lambda} \phi, \psi, \partial_\mu^{\beta, \lambda} \psi, x^\mu)$.

For Option 1,2, the deformed Euler-Lagrange Equations can be written as

$$\frac{\partial L}{\partial \phi} = \partial_\mu^{\alpha, \lambda} \left( \frac{\partial L}{\partial \partial_\mu^{\alpha, \lambda} \phi} \right) = 0$$

(21)

and analogously for $\psi$.

For the option 3 approach, we have:

$$\frac{\partial L}{\partial \phi} - \partial_\mu \left[ (x^\mu)^{1-\alpha, \lambda} \frac{\partial L}{\partial \partial_\mu^{1-\alpha, \lambda} \phi} \right] = 0.$$  

(22)

The possible inclusion of higher order derivatives could lead to an interesting study of Podolski-like systems and will be let to a future publication.

Now we can cast some possible applications to the approaches here stated.

V. APPLICATIONS:

A. Deformed Newtonian Mechanics

We consider in this example the “free quasi-particle”, with $V = 0$ and $L = \frac{1}{2} m (D_t^\alpha x)^2$

For the options 1,2, we have similar results as in the Ref. [24]

For option 3, we obtain interesting results that are cast below.

The $E - L$ equation is

$$\frac{\partial L}{\partial x} = \frac{d}{dt} (t^{1-\alpha} \frac{\partial L}{\partial D_t^{1-\alpha} x}) = 0,$$

that leads to the movement equation

$$2(1 - \alpha) m. t^{1-2\alpha} \frac{dx}{dt} + t^{2-2\alpha} m \frac{d^2 x}{dt^2} = 0.$$  

(23)
Note that we can do some variable transformation, so that \( t \equiv 1 + \frac{t'}{l_0} \Rightarrow dt = \frac{dt'}{l_0} \). This means that here we are dealing with the Hausdorff derivative and there is no problems in the point \( t' = 0 \). The dynamical equation can now be written as

\[
2(1 - \alpha) m (1 + \frac{t'}{l_0} )^{1 - 2\alpha} \frac{dx}{dt'} + (1 + \frac{t'}{l_0} )^{2 - 2\alpha} m \frac{d^2 x}{dt'^2} = 0
\]  

(24)

For a low level fractionality, we define \((1 - \alpha) \equiv \epsilon \). With first order in \( \epsilon \), it can be written as

\[
2\epsilon m \frac{dx}{dt} + (1 + 2\epsilon(1 + \frac{t'}{l_0})ln(1 + \frac{t'}{l_0}))m \frac{d^2 x}{dt'^2} = 0,
\]  

(25)

The equation stated above is in some way a time dependent-mass equation with the presence of friction, showing the appearance of dissipation due to the complex context. This is an speculated result, since we are considering an open system.

Here we would like to observe certain resemblances with some results in Ref. [24], but in that paper it seems that the focus was not the variational approach but only some attempt to construct a deformed Newtonian mechanics, the derivative treated there was only the conformable one and the result on eq. (34) (Ref. [24]) seems to has an absent \( t \)-factor in the dissipative term, as the reader can verify.

Analogously, for \( q \)-derivative in the nonextensive statistics context we can obtain the \( E - L \)equations as

\[
\frac{\partial L}{\partial x} - \frac{\partial}{\partial t}[l_0(1 + (1 - q)t)\frac{\partial L}{\partial D_{q,x}}] = 0,
\]

that leads to the dynamical movement equations as

\[
2l_0^2(1 - q)m[1 + (1 - q)t]\frac{dx}{dt} + l_0^2[1 + (1 - q)t]^2m \frac{d^2 x}{dt'^2} = 0.
\]

For low level fractionality, we again define \( 1 - q \equiv \epsilon \), so, for the first order in \( \epsilon \) parameter results that:

\[
2\epsilon m \frac{dx}{dt} + (1 + 2\epsilon t)m \frac{d^2 x}{dt'^2} = 0.
\]

The equations above contains a time-dependent mass and there is again the presence of friction. For \( V \neq 0 \) and \( \epsilon \to 0 \), \( l_0 \to 1 \), the dissipative and the time-dependent term vanishes and we re-obtain the classical Newtonian law for conservative forces:

\[
m \frac{d^2 x}{dt'^2} = -\frac{\partial V}{\partial x}.
\]

So, the results are evidently consistent with the classical Newtonian mechanics.

**B. Deformed Time-Deformed-Schrödinger Equation**

We now proceed with the constructions related to a deformed- field theory, specially that leads to some mass-dependent Schrödinger equations.

To Pursue this objective, let us now consider the Lagrangian density

\[
\mathcal{L} = i\hbar \psi^* D_t \psi + \frac{\hbar^2}{2m} \nabla \psi^* . \nabla \psi - V \psi^* \psi
\]

and take into account the options 1,2.

The \( E - L \) equation leads to the deformed-time-dependent Schrödinger equation (note that \( \nabla \)implies to \( \alpha = 1 \) order in the spatial part of the deformed Euler-Lagrange equations)
\[ i\hbar D_t^\psi = -\frac{\hbar^2}{2m} \nabla^2 \psi - V\psi. \] (26)

In terms of q-derivative, analogously we can show that the resulting \( E - L \) equation is

\[ i\hbar D_{q,t} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi - V\psi = H\psi, \] (27)

Here, let us redefine the \( q \)-derivative as a new scale \( q \)-derivative

\[ D^\lambda_q f(\lambda x) \equiv [1 + (1 - q)\lambda x] \frac{df(x)}{dx}. \] (28)

The solution of eq. (27) with this \( q \)-scale \( q \)-deformed-derivative is

\[ \psi = A e^{\frac{-i\hbar}{\hbar} H t}. \]

We can observe in additions that a solution of the equation of the nonlinear Schrödinger equation

\[ i\hbar \partial_t \psi = H\psi, \]

for \( V = 0 \), is also \( \psi = A e^{\frac{-i\hbar}{\hbar} H t} \) and we can write that

\[ i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} D^\alpha_x D^\alpha_x \psi + V\psi. \] (31)

That is, the nonlinear Schrödinger equation called in Refs. [21] (with \( q = q' = 2 \) compared to the \( q \)-index of the reference) as NRT-like Schrödinger equation and can be thought as resulting from a time \( q \)-scale \( q \)-deformed-derivative applied to the wave function \( \psi \).

**Deformed Spatial Schrödinger equation**

Let us now consider the Lagrangian density expressed by means of the spatial deformed derivatives as

\[ \mathcal{L} = i\hbar \psi^* \psi + \frac{\hbar^2}{2m} D_x^\alpha \psi^* D_x^\alpha \psi - V\psi^* \psi. \] (29)

With option 1,2 the resulting dynamical equation, that is, the spatial deformed-Schrödinger equation is

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} D_x^\alpha D_x^\alpha \psi + V\psi. \] (30)

In terms of \( q \)-derivative, we can show that a similar result is

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} D_{q,x} D_{q,x} \psi + V\psi. \] (31)

Both results are possible expressions of the position-dependent mass, similarly as CAFA-like Schrödinger equation in Ref. [19].

**C. Option 3: Deformed Schrödinger Equation**

Consider again the Lagrangian density as

\[ \mathcal{L} = i\hbar \psi^* \psi + \frac{\hbar^2}{2m} D_x^\alpha \psi^* D_x^\alpha \psi - V\psi^* \psi. \]

By the approach in the option 3, we obtain a form of Cauchy-Euler for Schrödinger equation

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} [2(1-\alpha)x^{-\alpha}D_x^\alpha \psi + x^{2-\alpha}\frac{\partial^2 \psi}{\partial x^2}] + V\psi. \]

For \( q \)-derivative we obtain
\[ i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m}[2(1-q)l_0^2(1+(1-q)x)\frac{\partial \psi}{\partial x} + (1+(1-q)x)l_0^2 \frac{\partial^2 \psi}{\partial x^2}] + V\psi. \]  

(32)

For low fractionality we again define \(1-q = \epsilon\) and considering only first order in \(\epsilon\) we can write a position-dependent mas Schrödinger equations as

\[ \hbar \frac{\partial}{\partial t} \psi = \frac{\hbar^2}{2m}[2(\epsilon l_0^2 \frac{\partial \psi}{\partial x} + (1+2\epsilon x)l_0^2 \frac{\partial^2 \psi}{\partial x^2}] + V\psi. \]  

(33)

VI. HAMILTONIAN FORMALISM

We will now pursue the Hamiltonian formalism that takes into account the embedded deformed derivatives proposed. For the option 1:

Consider \(L = L(t,q,D^\alpha t q)\), where \(q\) is some generalized coordinate and have not to be confused with the entropic parameter. The total derivative of \(L\) is written as

\[ dL = \frac{\partial L}{\partial t} dt + \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial (D^\alpha t q_i)} d(D^\alpha t q_i). \]  

(34)

By the \(E-L\) equation, we can obtain that:

\[ \frac{\partial L}{\partial q_i} = D^\alpha t p_i(D^\alpha t q_i). \]

Defining now the generalized momentum as \(p_i^\alpha = \frac{\partial L}{\partial (D^\alpha t q_i)}\), we obtain

\[ dL = \frac{\partial L}{\partial t} dt + D^\alpha t p_i(D^\alpha t q_i) dq_i + p_i^\alpha d(D^\alpha t q_i). \]  

(35)

Following, defining \(H = H(p_i^\alpha,q_i,t) \equiv p_i^\alpha(D^\alpha t q_i) - L\), we will have

\[ dH = \frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i^\alpha} dp_i^\alpha = dp_i^\alpha D^\alpha t q_i + p_i^\alpha d(D^\alpha t q_i) - dL \]

\[ = dp_i^\alpha D^\alpha t q_i + p_i^\alpha d(D^\alpha t q_i) - \frac{\partial L}{\partial t} dt - D^\alpha t p_i^\alpha dq_i + \frac{\partial L}{\partial (D^\alpha t q_i)} d(D^\alpha t q_i) - p_i^\alpha d(D^\alpha t q_i). \]  

(36)

Thus, we obtain

\[ dH = dp_i^\alpha D^\alpha t q_i - D^\alpha t p_i^\alpha dq_i - \frac{\partial L}{\partial t} dt. \]  

(37)

Comparing the the equations above we can state that

\[ \begin{cases} \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \\ \frac{\partial H}{\partial q_i} = -D^\alpha t p_i^\alpha, \\ \frac{\partial H}{\partial p_i^\alpha} = D^\alpha t q_i. \end{cases} \]  

(38)

For the option 2:

We can rewrite: \(dL \rightarrow d^\alpha L\)

\[ d^\alpha L = \frac{\partial L}{\partial t} d^\alpha t + \frac{\partial L}{\partial q_i} d^\alpha q_i + \frac{\partial L}{\partial (D^\alpha t q_i)} d^\alpha (D^\alpha t q_i) \]  

(39)

or
thus, we can write
\[ D_t^\alpha [p_t^\alpha D_t^\alpha q_i - L] = -\frac{\partial H}{\partial q_i} D_t^\alpha t. \]

Defining again \( H \) as \( H \equiv p_t^\alpha D_t^\alpha q_i - L = H(p_t^\alpha, q_i, t) \) and differentiating results that
\[
\frac{d^\alpha H}{dt^\alpha} = \frac{\partial H}{\partial t} D_t^\alpha t + \frac{\partial H}{\partial q_i}(D_t^\alpha q_i) + \frac{\partial H}{\partial p_i}(D_t^\alpha p_i) = (D_t^\alpha p_i)(D_t^\alpha q_i) + (p_t^\alpha)(D_t^\alpha q_i) - D_t^\alpha L.
\]
Comparing again the equations, results that
\[
\begin{align*}
\frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} D_t^\alpha t - D_t^\alpha \left( \frac{\partial L}{\partial D_t^\alpha q_i} \right) D_t^\alpha q_i, \\
\frac{\partial H}{\partial q_i} &= -\frac{\partial L}{\partial q_i} D_t^\alpha q_i, \\
\frac{\partial H}{\partial p_i} &= D_t^\alpha p_i.
\end{align*}
\]

For the option 3:
We can show that the resulting equations are (remembering that the \( E - L \) in this case is given by \( \frac{\partial L}{\partial q_i} - \frac{1}{t^\alpha} \frac{\partial}{\partial t}(t^{1-\alpha} \frac{\partial L}{\partial D_t^\alpha q_i}) = 0 \))
\[
\begin{align*}
\frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} D_t^\alpha t - D_t^\alpha \left( \frac{\partial L}{\partial D_t^\alpha q_i} \right) D_t^\alpha q_i = (\alpha - 1)t^{-\alpha} p_t^\alpha - D_t^\alpha (p_t^\alpha), \\
\frac{\partial H}{\partial q_i} &= -\frac{\partial L}{\partial q_i} D_t^\alpha q_i = D_t^\alpha q_i.
\end{align*}
\]

To finish this section we want to outline some possible extension for the formalism developed here.
One of such possibilities is to extend to the Pontryagin’s maximum (or minimum) principle used in optimal control theory.
This formalism can be used in the presence of constraints for the state or input controls and will yield a set of deformed-differential equations to study open systems that are interacting with the environment.

A. Noether procedure with deformed derivative

Let us now consider the Lagrangian density as \( L(\phi_i, \partial_\mu \phi_i) \). With the usual \( \delta - \) process, we obtain for the variation of \( L \):
\[
\delta L = \frac{\partial L}{\partial \phi_i} \delta \phi_i + \frac{\partial L}{\partial \partial_\mu \phi_i} \delta \partial_\mu \phi_i.
\]

With \( L(\phi_i, \partial_\mu^\alpha \phi_i) \), we shall have
\[
\delta L = \frac{\partial L}{\partial \phi_i} \delta \phi_i + \frac{\partial L}{\partial (\partial_\mu^\alpha \phi_i)} \delta (\partial_\mu^\alpha \phi_i).
\]

Integrating by parts, considering the \( E - L \) equations and assuming that \( \delta L \) is given by a total deformed-derivative of order \( \alpha_\mu \) of some admissible function \( G^\mu \), \( \delta L = (\partial_\mu^\alpha G^\mu) \), we write
\[
\partial_\mu \mu \frac{\partial L}{\partial (\partial_\mu \phi_i)} (\delta \phi_i) - G^\mu = 0.
\]

The conserved generalized current can be defined as

\[
J^\mu = \frac{\partial L}{\partial (\partial_\mu \phi_i)} (\delta \phi_i) - G^\mu.
\]

Explicit specific cases, like gauge currents, energy-momentum, angular momentum, spin tensors and their corresponding invariances shall be discussed elsewhere in a forthcoming work.

VII. CONCLUSIONS AND OUTLOOK

In conclusion, we have employed the variational calculus to obtain \( E - L \) equations with deformed derivatives.

The paradigm that governs the standard approach in the context of generalized statistical mechanics was revisited.

To achieve our goals, Lagrangian and Hamiltonian formalisms were studied, some physical reasoning about integration based on the existence of succession of specific internal temporal intervals for the system are furnished and a new class of generalized Lagrangian, Hamiltonian, and action principles are presented.

Position-dependent mass equations and nonlinear equations could be shown to result from the formalism that are in agreement with results found in the scientific literature.

Possible extensions of the subject and the concepts discussed here are also outlined, such as Pontryagin’s maximum (or minimum) principle.

We believe that with this formalism can set up a systematic way to obtain nonstandard equations in several areas of science, without an excessive heuristics. This can avoid to introduce \textit{ad hoc} fields and unnecessary suppositions about strange dynamics.

As an outlook, one of our studies regards the Landau-Lifshiz-Gilbert-Slonczewski equations in the context of metric derivatives and complex systems, to analyse the effects on the damping.

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[1] J. Weberszpil, Matheus Jatkoske Lazo and J.A. Helayël-Neto, Physica A 436, (2015) 399–404.
[2] Alexander S. Balankin and Benjamin Espinoza Elizarraraz, Phys. Rev. E 85, (2012) 056314.
[3] A. S. Balankin and B. Espinoza, Phys. Rev. E 85, (2012) 025302(R).
[4] W. Chen, Chaos, Solitons and Fractals 28 (2006) 923–929.
[5] C. F. L. Godinho, J. Weberszpil, J. A. Helayël-Neto, International Journal of Theoretical Physics, v. 51, p. 1, 2014; arXiv:1208.2266v3 [math-ph].
[6] F. Riewe, Phys. Rev. E 53, (1996) 1890 ; \textit{ibid} 55, (1997)358 , and references therein.
[7] K.M. Kolwankar, A.D. Gangal, In: Fractals: theory and application in engineering, Delft: Springer; 1999.
[8] K.M. Kolwankar, A.D. Gangal, Chaos 6 (1996) 505–23.
[9] K.M. Kolwankar, A.D. Gangal, Phys Rev Lett 80, (1998) 214–217.
[10] A. Babakhani and Varsha Daftardar-Gejji, J. Math. Anal. Appl. 270, (2002) 66–79.
[11] Yan Chen, Ying Yan, Kewei Zhang, J. Math. Anal. Appl. 362 (2010) 17–33.
[12] Alberto Carpinteri and Alberto Sapora, Z. Angew, Math. Mech., 90, (2010) 203–210.
[13] W. Chen, Chaos, Solitons and Fractals 28, (2006) 923–929.
[14] Khalil, R., Al Horani, M., Yousef. A. and Sababheh, M., J. Comput. Appl. Math. 264, (2014) 6570.
[15] Ernesto P. Borges, Physica A 340 (2004) 95 – 101.
[16] Tsallis-Brazilian Journal of Physics, vol. 29, no. 1 (1999).
[17] S. Habib Mazharimousavi, Phys. Rev. A 85, (2012) 034102 .
[18] M. A. Rego-Monteiro and F. D. Nobre, Phys. Rev A 88, (2013) 032105.
[19] R. N. Costa Filho, M. P. Almeida, G. A. Farias, and J. S. Andrade, Jr., Phys. Rev. A 84, (2011) 050102(R).
[20] F. D. Nobre, M. A. Rego-Monteiro and C. Tsallis, EPL, (2012) 97 41001.
[21] F. D. Nobre, M. A. Rego-Monteiro, and C. Tsallis, Phys. Rev. Lett. 106, (2011) 140601.
[22] Alexander Balankin, Juan Bory-Reyes and Michael Shapiro, Phys A, in press, (2015) doi:10.1016/j.physa.2015.10.035.
[23] Douglas R. Anderson and Darin J. Ulness, JOURNAL OF MATHEMATICAL PHYSICS 56, (2015) 063502.
[24] Won Sang Chung, Journal of Computational and Applied Mathematics 290 (2015) 150–158.
[25] T M Atanackovic’´, S Konjik and S Pilipovic’´, J. Phys. A: Math. Theor. 41 (2008) 095201.
[26] Podlubny, I., Fract Calc Appl. Analysis 5(4) (2002) 367-386.
[27] Om Prakash Agrawal, Computers and Mathematics with Applications 59 (2010) 1852–1864.
[28] J. A. Tenreiro Machado , Fractional Calculus and Applied Analysis 6 (1) (2003) 73-80.
[29] A. A. Stanislavsky, Eur. Phys. J. B 49, (2006) 93-101.
[30] El- Nabulsil Ahmad-Rami, Fizika A 14 (2005), no. 4, 289–298.