On Gaps Between Primitive Roots in the Hamming Metric

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Abstract

We consider a modification of the classical number theoretic question about the gaps between consecutive primitive roots modulo a prime $p$, which by the well-known result of Burgess are known to be at most $p^{1/4+o(1)}$. Here we measure the distance in the Hamming metric and show that if $p$ is a sufficiently large $r$-bit prime, then for any integer $n \in [1,p]$ one can obtain a primitive root modulo $p$ by changing at most $0.11002786\ldots r$ binary digits of $n$. This is stronger than what can be deduced from the Burgess result. Experimentally, the number of necessary bit changes is very small. We also show that each Hilbert cube contained in the complement of the primitive roots modulo $p$ has dimension at most $O(p^{1/5+\epsilon})$, improving on previous results of this kind.
1 Introduction

Let \( p \) be a fixed prime number. Studying the gaps between consecutive quadratic non-residues and primitive roots modulo \( p \) is a classical number theoretic question, where however not much progress has been made since the work of Burgess \([5, 6]\) that implies that these gaps are at most \( p^{1/4+o(1)} \).

Here we consider a modification of this question where the distances are measured in the Hamming metric (see for example \([18, \text{Section 1.1}]\)). More specifically, we denote by \( \Delta_p \) the smallest number \( s \) such that for any integer \( n \in [1, p] \) one can change at most \( s \) binary digits of \( n \) in order to get a primitive root modulo \( p \).

We use the ideas of \([22]\), which in turn expand on those of \([1]\), to estimate character sums over integers that are close in the Hamming metric to a given integer \( n \) and then derive an estimate on \( \Delta_p \). As a primitive root is also quadratic non-residue, \( \Delta_p \) also gives a bound on the gaps between quadratic non-residues in the Hamming metric.

Let \( \rho_0 = 0.11002786\ldots \) be the unique root of the equation

\[
H(\rho) = 1/2, \quad 0 \leq \rho \leq 1/2,
\]

where

\[
H(\gamma) = -\gamma \log \gamma - (1-\gamma) \log(1-\gamma) \quad 0 < \gamma < 1,
\]

denotes the binary entropy function \((\text{see } [18, \text{Section 10.11}])\).

**Theorem 1.** We have \( \Delta_p \leq (\rho_0 + o(1)) r \) as \( p \to \infty \), where \( r \) is the number of binary digits of \( p \).

Note that an immediate application of the Burgess result \([5]\) would only give 0.25 in place of \( \rho_0 \).

It is also interesting to study the sparsest primitive root and quadratic non-residue. More precisely, let \( W_p \) be the smallest Hamming weight (see \([18, \text{Section 1.1}]\)) of the binary expansion of the primitive roots \( g \in \{1, \ldots, p-1\} \) modulo \( p \). For \( p \neq 2 \), we define \( w_p \) analogously with respect to quadratic non-residues modulo \( p \). Since for \( p > 2 \) a primitive root modulo \( p \) is necessarily a quadratic non-residue modulo \( p \), we have

\[
w_p \leq W_p \leq \Delta_p.
\]

We are not able to improve the above bound for \( W_p \), however, using a recent result of \([2]\) we obtain a more precise estimate on \( w_p \).

Let

\[
\vartheta_0 = \frac{1}{8 \sqrt{\pi}} = 0.07581633\ldots
\]

(2)
Theorem 2. We have \( w_p \leq (\vartheta_0 + o(1)) \frac{r}{\log^2 \log p} \), as \( p \to \infty \), where \( r \) is the number of binary digits of \( p \).

Again, a direct application of the Burgess result \( [3] \) gives a weaker bound, namely
\[
w_p \leq \left( \frac{1}{4 \sqrt{e}} + o(1) \right) r.
\]

Finally, we also consider the distribution of primitive roots in so-called Hilbert cubes: For \( a_0, a_1, \ldots, a_d \in \mathbb{F}_p \) write
\[
\mathcal{H}(a_0; a_1, \ldots, a_d) = \left\{ a_0 + \sum_{i=1}^{d} \vartheta_i a_i : \vartheta_i \in \{0, 1\} \right\}.
\]

As in \( [12] \), we define \( f(p) \) as the largest \( d \) such that there are \( a_0, a_1, \ldots, a_d \in \mathbb{F}_p \) with pairwise distinct \( a_1, \ldots, a_d \) such that \( \mathcal{H}(a_0; a_1, \ldots, a_d) \) does not contain a quadratic non-residue modulo \( p \). Furthermore, we define \( F(p) \) as the largest \( d \) such that there are \( a_0, a_1, \ldots, a_d \in \mathbb{F}_p \) with pairwise distinct \( a_1, \ldots, a_d \) such that \( \mathcal{H}(a_0; a_1, \ldots, a_d) \) does not contain a primitive root modulo \( p \). Also, for the complementary sets we define \( \overline{f}(p) \) and \( \overline{F}(p) \) as the largest \( d \) such that \( \mathcal{H}(a_0; a_1, \ldots, a_d) \) is entirely in the set of quadratic non-residues respectively primitive roots modulo \( p \).

As a primitive root is a non-residue, and as a set of residues becomes a set of non-residues by multiplication with one fixed non-residue (recall that 0 is neither residue, nor non-residue) we have:
\[
\overline{F}(p) \leq \overline{f}(p) = f(p) \leq F(p).
\]

Hegyvári and Sárközy \( [12, \text{Theorem 2}] \) give the bound \( f(p) < 12 p^{1/4} \). Here we improve the exponent and also extend the result to \( F(p) \).

Theorem 3. We have \( F(p) \leq p^{1/5+o(1)} \) as \( p \to \infty \).

As for primes with \( (p - 1)/2 \) also a prime the set of non-residues is the same as the set of primitive roots one should not expect that upper bounds on \( \overline{F}(p) \) are generally better than those for \( F(p) \).

From their result, Hegyvári and Sárközy \( [12] \) give an application to the maximal dimension \( d \) of Hilbert cubes in the set of integer squares. We do not follow this path here, but remark that the first two authors have recently improved the bound on \( d \) using a different method, see \( [9] \).

It is likely that the bound \( p^{1/5+o(1)} \) is far from the truth. One may conjecture a bound of \( p^{o(1)} \) or even \( (\log p)^C \) for some positive constant \( C \). Indeed, for the easier problem of subset sums where \( a_0 = 0 \), for \( p \equiv \pm 3 \pmod{8} \) and subset sums avoiding quadratic non-residues modulo \( p \), Csikvári \( [7, \text{Corollary 2.2}] \) has obtained an upper bound \( \log p / \log 2 \). However, it may be difficult to prove a
bound of this type for the general case. It has been observed in [12, 7] that improving the bound on \( f(p) \) to \( p^\alpha \) with \( \alpha < \vartheta_0 \), where \( \vartheta_0 \) is given by (2), is impossible without improving the Burgess bound [5] on the smallest quadratic non-residue. To see this, one simply constructs a Hilbert cube consisting of many small elements \( a_i = i, i = 0, \ldots, d \) where \( d = f(p) \). Here the elements of the Hilbert cube are at most

\[ 1 + 2 + \ldots + d < d^2 = f(p)^2 < p^{2\alpha}. \]

Hence \( 2\alpha \geq 1/(4e^{-1/2}) + o(1) \) unless one improves on the Burgess bound. In fact, the same argument shows that in the bound on \( F(p) \) one cannot go beyond \( 1/8 = 0.125 \) in the exponent without improving the Burgess bound \( g(p) \leq p^{1/4 + o(1)} \) on the least primitive root \( g(p) \) modulo \( p \). In this context, let us recall that, assuming the Generalised Riemann Hypothesis (GRH), Shoup [25] has proved a bound of

\[ g(p) = O((\log p)^6). \quad (4) \]

Finally, let us remark that Theorem 3 immediately gives \( \Delta_p \leq (0.2 + o(1))r \), which is stronger than the bound \( \Delta_p \leq (0.25 + o(1))r \) resulting from applying the Burgess bound, but weaker than our result obtained in Theorem 1 making use of a direct application of exponential sums.

2 Preparations

Throughout the paper the implied constants in the symbols “\( O \)”, “\( \ll \)” and “\( \gg \)” may depend on an integer parameter \( \nu \geq 1 \). We recall that the expressions \( A \ll B, B \gg A \) and \( A = O(B) \) are each equivalent to the statement that \( |A| \leq cB \) for some constant \( c \). As usual, \( \log z \) denotes the natural logarithm of \( z \).

The letter \( p \) (possibly subscripted) always denotes a prime.

We also use \( \mathbb{F}_p \) to denote the finite field of \( p \) elements.

We need the following well-known statement (see, for example, [18, Section 10.11, Lemma 7]):

**Lemma 1.** For any integers \( k \geq \ell \geq 0 \),

\[ \binom{k}{\ell} = 2^{kH(\ell/k) + o(k)}. \]

We note that \( \chi(z) = \chi(z^{p-2}) \) for \( z \in \mathbb{F}_p^\star \) and a multiplicative character \( \chi \) of \( \mathbb{F}_p^\star \).

We need the following statement which follows immediately from the Weil bound, see [14, Chapter 11], and which is essentially [19, Theorem 2].
Lemma 2. For any multiplicative character $\chi$ of $\mathbb{F}_p^*$ of order $m \geq 2$, any integers $M$ and $K$ with $1 \leq K < p$, and any polynomial $F(U) \in \mathbb{F}_p[U]$ with $d$ distinct roots (of arbitrary multiplicity) such that $F(U)$ is not the $m$-th power of a rational function, we have

$$\sum_{u=M+1}^{M+K} \chi(F(u)) \ll dp^{1/2} \log p.$$ 

The following result is a combination of the bounds of Pólya-Vinogradov (for $\nu = 1$) and Burgess (for $\nu \geq 2$), see [14, Theorems 12.5 and 12.6].

Lemma 3. For arbitrary integers $W$ and $Z$ with $1 \leq Z \leq p$, for an arbitrary non-principal multiplicative character $\chi$ of $\mathbb{F}_p^*$, and for an arbitrary positive integer $\nu$, we have

$$\left| \sum_{z=W+1}^{W+Z} \chi(z) \right| \leq Z^{1-1/\nu} p^{(\nu+1)/4\nu^2+o(1)}.$$  

As usual, we use $\mu(d)$ and $\varphi(d)$ to denote the Mőbius and the Euler functions of an integer $d \geq 1$, respectively. We now mention the following well-known characterisation of primitive roots modulo $p$ which follows from the inclusion-exclusion principle and the orthogonality property of characters (see, for example, [17, Exercise 5.14]).

Lemma 4. For any integer $a$, we have

$$\frac{\varphi(p-1)}{p-1} \sum_{d \mid p-1} \mu(d) \sum_{\text{ord} \chi = d} \chi(a) = \begin{cases} 1, & \text{if } a \text{ is a primitive root modulo } p, \\ 0, & \text{otherwise}, \end{cases}$$

where the inner sum is taken over all $\varphi(d)$ multiplicative characters $\chi$ modulo $p$ of order $d$.

We also recall the following bound [2, Theorem 2.1] on short sums of the Legendre symbol $(n/p)$ modulo $p$.

Lemma 5. For every $\varepsilon > 0$ there exists $\delta > 0$ such that, for all sufficiently large primes $p$, the bound

$$\left| \sum_{n \leq N} (n/p) \right| \leq (1-\delta)N$$

holds for all integers $N$ in the range $p^{1/(4\sqrt{e})+\varepsilon} \leq N \leq p$.

Finally, for our result on Hilbert cubes avoiding primitive roots, we make use of a recent result by Schoen [24, Theorem 3.3] in additive combinatorics. Note that his result is actually only stated for subset sums rather than Hilbert cubes, but this slight generalisation follows immediately.
Lemma 6. For any $a_0 \in \mathbb{F}_p$ and pairwise distinct $a_1, \ldots, a_d \in \mathbb{F}_p$ such that $d \geq 8(p/\log p)^{1/D}$, where $D$ is an integer satisfying

$$0 < D \leq \sqrt{\frac{\log p}{2\log \log p}},$$

the Hilbert cube $(3)$ contains an arithmetic progression of length $L$ where

$$L \geq 2^{-10}(d/\log p)^{1/(D-1)}.$$

3 Double Multiplicative Character Sums

In the following, for a fixed prime $p$ we write $r$ for the non-negative positive integer such that $2^r < p \leq 2^{r+1}$. Given integers $n \in [1, p]$, $k \in [1, r]$ and $l \leq k$, we denote by $U_{k,\ell}(n)$ the set of positive integers $u < 2^k$ whose binary expansions differ from the $k$ most significant binary digits of $n$ in exactly $\ell$ positions (if necessary, we append some leading zeros to the binary expansion of $n$ to guarantee that it is of length $r + 1$). Furthermore, for $m \leq r - k$ we denote by $V_{k,m}(n)$ the set of positive integers $v < 2^{r-k+1}$ whose binary expansions differ from the $r - k + 1$ least binary digits of $n$ in exactly $m$ positions.

Obviously,

$$\#U_{k,\ell}(n) = \binom{k}{\ell} \quad \text{and} \quad \#V_{k,m}(n) = \binom{r-k}{m}. \quad (5)$$

Clearly the binary expansion of any integer of the shape $u2^{r-k+1} + v$, $u \in U_{k,\ell}(n)$, $v \in V_{k,m}(n)$ differs from the binary expansion of $n$ in exactly $\ell + m$ positions. This suggests to consider the following double sum with a multiplicative character $\chi$ of $\mathbb{F}_p^*$:

$$S_n(k, \ell, m; \chi) = \sum_{u \in U_{k,\ell}(n)} \sum_{v \in V_{k,m}(n)} \chi(u2^{r-k+1} + v).$$

We note that the following result is slightly more precise than a bound of Karatsuba [15] (see also [16 Chapter VIII, Problem 9]) that applies to double character sums over arbitrary sets.

Lemma 7. In the notation from above, for any non-trivial multiplicative character $\chi$ of $\mathbb{F}_p^*$ and any positive integer $\nu$, we have

$$|S_n(k, \ell, m; \chi)| \ll \left(\#U_{k,\ell}(n)\right)^{(2\nu-1)/2\nu} \left(\#V_{k,m}(n)\right)^{1/2} 2^{k/2\nu}$$

$$+ \left(\#U_{k,\ell}(n)\right)^{(2\nu-1)/2\nu} \#V_{k,m}(n) 2^{r/4\nu} (\log p)^{1/2\nu}.$$
Proof. Let $K = 2^k$. By the Hölder inequality, we have

$$|S_n(k, \ell, m; \chi)|^{2\nu} \leq \#U_{k,\ell}(n)^{2\nu-1} \sum_{v_1, \ldots, v_{\nu} \in \mathcal{V}_{k,m}(n)} \left| \sum_{\nu \in \mathcal{V}_{k,m}(n)} \chi(u2^{r-k+1} + v) \right|^{2\nu}.$$

Therefore,

$$|S_n(k, \ell, m; \chi)|^{2\nu} \leq \#U_{k,\ell}(n)^{2\nu-1} \sum_{v_1, \ldots, v_{\nu} \in \mathcal{V}_{k,m}(n)} \left| \prod_{i=1}^{K-1} \chi(u2^{r-k+1} + v_i)(u2^{r-k+1} + w_i)^{p-2} \right|.$$

(6)

We note that if the polynomial

$$\prod_{j=1}^{\nu} (2^{r-k+1}U + v_j)(2^{r-k+1}U + w_j)^{p-2} \in \mathbb{F}_p[U]$$

(7)

is a power of another rational function, then every value that occurs in the sequence $v_1, \ldots, v_{\nu}$ and in the sequence $w_1, \ldots, w_{\nu}$ occurs with multiplicity at least 2. Thus, the set of such $v_1, \ldots, v_{\nu}, w_1, \ldots, w_{\nu}$ takes at most $\nu$ distinct values.

We can assume that $r - k > m$ since otherwise the result is trivial. So,

$$\#\mathcal{V}_{k,m}(n) = \binom{r-k}{m} \geq 2.$$

Therefore, there are at most

$$\binom{r-k}{m} + \binom{r-k}{m}^2 + \cdots + \binom{r-k}{m}^\nu \leq 2 \binom{r-k}{m}^\nu$$

subsets of $\mathcal{V}_{k,m}(n)$ with at most $\nu$ elements. When such a subset with $h \leq \nu$ elements is fixed, we can obtain the case described above by placing its elements into $2\nu$ positions. This can be done in no more than $(2\nu)^h \leq (2\nu)^\nu$ ways. So we have at most

$$2(2\nu)^\nu \binom{r-k}{m}^\nu$$

possibilities for vectors $(v_1, \ldots, v_{\nu})$ and $(w_1, \ldots, w_{\nu})$ such that the polynomial (7) is a power of some other rational function. Using
Lemma 2 when the rational function (7) is not a power of another rational function, we can estimate
\[ \sum_{v_1, \ldots, v_{\nu} \in V_{k,m}(n)} \left| \sum_{u=0}^{K-1} \nu \left( (u2^{r-k+1} + v_i)(u2^{r-k+1} + w_i)^{p-2} \right) \right| \]
\[ \ll \left( \frac{r-k}{m} \right)^{\nu} K + \left( \frac{r-k}{m} \right)^{2\nu} p^{1/2} \log p \]
\[ = \#V_{k,m}(n)^{\nu} 2^k + \#V_{k,m}(n)^{2\nu} p^{1/2} \log p. \]
Recalling (6), we obtain the desired result.

4 Proof of Theorem 1

We fix some \( \rho > \rho_0 \) (with \( \rho < \frac{1}{2} \)) where \( \rho_0 \) is the root of the equation (11), and some positive
\[ \varepsilon < 1 - \frac{1}{2H(\rho)}. \]
We define
\[ k = \lfloor (1-\varepsilon)r \rfloor, \quad \ell = \lfloor \rho k \rfloor, \quad m = \lfloor 0.5(r-k) \rfloor. \]
We now show that there exists some \( \delta > 0 \) such that for any nontrivial multiplicative character \( \chi \) of \( \mathbb{F}_p^\ast \), we have
\[ S_n(k, \ell, m; \chi) \ll \#U_{k,\ell}(n) \#V_{k,m}(n)p^{-\delta}. \quad (8) \]
Using Lemma 7 and recalling (5) we see that in order to establish (8), it is enough to show that for an appropriately chosen \( \nu \) and some \( \eta > 0 \) that depends only on \( \rho \) and \( \varepsilon \), we have
\[ \#V_{k,m}(n) \geq 2^{(1+\eta)k/2\nu} \]
and
\[ \#U_{k,\ell}(n) \geq 2^{r/2} p^{2\nu \delta} \log p. \quad (10) \]
Since by our choice of parameters, (5) and Lemma 1 we have
\[ \#V_{k,m}(n) = 2^{r-k+o(r)} = 2^{\varepsilon r+o(r)}, \]
the bound (9) is immediate for all sufficiently large \( \nu \).
Having fixed \( \nu \), we now note that by (5) and Lemma 1 in order to establish (10), for sufficiently small \( \delta > 0 \), it is enough to verify that
\[ H(\rho)(1-\varepsilon) > 1/2, \]
which holds because of our choice of $\varepsilon$.

We now see that the bound holds. Using Lemma we now estimate the number $P$ of primitive roots modulo $p$ in the set 

$$Q(k, l, m) = \{u2^{r-k+1} + v : u \in \mathcal{U}_{k,l}(n), v \in \mathcal{V}_{k,m}(n)\}.$$ 

We have

$$P = \sum_{n \in Q(k,l,m)} \frac{\varphi(p-1)}{p-1} \sum_{d | p-1} \frac{\mu(d)}{\varphi(d)} \sum_{\text{ord } \chi = d} \chi(n)$$

$$= \frac{\varphi(p-1)}{p-1} \sum_{d | p-1} \frac{\mu(d)}{\varphi(d)} \sum_{\text{ord } \chi = d} \sum_{n \in Q(k,l,m)} \chi(n)$$

$$= \frac{\varphi(p-1)}{p-1} \#Q(k, l, m) + \frac{\varphi(p-1)}{p-1} \sum_{d > 1: d | p-1} \frac{\mu(d)}{\varphi(d)} \sum_{\text{ord } \chi = d} \sum_{n \in Q(k,l,m)} \chi(n)$$

$$= \frac{\varphi(p-1)}{p-1} \#Q(k, l, m) + O \left( \frac{\varphi(p-1)}{p-1} \sum_{d > 1: d | p-1} \frac{1}{\varphi(d)} \sum_{\text{ord } \chi = d} \frac{\#Q(k, l, m)}{p^\delta} \right)$$

$$= \frac{\varphi(p-1)}{p-1} \#Q(k, l, m) + O \left( \frac{\#Q(k, l, m)}{p^\delta} \sum_{d | p-1} 1 \right).$$

Using the well-known bound

$$\varphi(n) \gg \frac{n}{\log \log(n + 2)}$$

and

$$\sum_{d | p-1} 1 = p^{o(1)}$$

we derive

$$P \gg \frac{\#Q(k, l, m)}{\log \log(p + 1)}.$$

Hence we conclude that for sufficiently large $p$ the set $Q(k, l, m)$ indeed contains primitive roots. Therefore, for a sufficiently large $p$ we have

$$\Delta_p \leq m + \ell \leq (\rho + \varepsilon/2)r.$$

Since $\rho > \rho_0$ and $\varepsilon > 0$ are arbitrary we obtain the desired result.
5 Proof of Theorem 2

We fix some sufficiently small $\varepsilon > 0$ and put

$$N = \left\lceil p^{1/(4\sqrt{e})+\varepsilon} \right\rceil.$$  

By Lemma 4 there exists $\delta > 0$ such that the interval $[1, N]$ contains at least $0.5\delta N$ quadratic non-residues modulo $p$. Let

$$s = \left\lceil \frac{\log N}{\log 2} \right\rceil \quad \text{and} \quad w = \left\lfloor \left( \frac{1}{2} + \varepsilon \right)s \right\rfloor.$$  

Note that

$$\frac{w}{r} = \left( \frac{1}{2} + \varepsilon \right) \left( \frac{1}{4\sqrt{e}} + \varepsilon \right) + o(1), \quad (11)$$

as $p \to \infty$.

Making use of the well-known property that the binary entropy function has maximum $H(1/2) = 1$ and is strictly smaller than 1 outside $1/2$, we conclude that the number of positive integers $n \leq N$ with Hamming weight at least $w$, by Lemma 1, does not exceed

$$\sum_{k=w}^{s} \binom{s}{k} \leq s \frac{2^{sH(w/s)+o(s)}}{w} \leq s \frac{2^{sH(1/2)+o(s)}}{w} \ll 2^s \ll N^\eta,$$

where $\eta < 1$ depends only on $\varepsilon$. Therefore, there is a quadratic non-residue $n \leq N$ of Hamming weight at most $w - 1$. Recalling (11), since $\varepsilon$ is arbitrary, we obtain the desired estimate on $w_p$.

6 Proof of Theorem 3

We fix some $\varepsilon > 0$ and let $d = \left\lceil p^{1/5+\varepsilon} \right\rceil$. Then by Lemma 6 with $D = 5$, for any $a_0, a_1, \ldots, a_d \in \mathbb{F}_p$ with pairwise distinct $a_1, \ldots, a_d$, the set $\mathcal{H}(a_0; a_1, \ldots, a_d)$ contains an arithmetic progression $an + b$, $n = 1, \ldots, N$ of length

$$N \gg \frac{p^{(1/5+\varepsilon)(5/4)}}{(\log p)^{5/4}} \gg p^{1/4+\varepsilon}.$$  

In particular $a \neq 0$, so $a$ has an inverse $\overline{a}$ in $\mathbb{F}_p$.

Thus for any non-principal multiplicative character $\chi$ of $\mathbb{F}_p^*$, by Lemma 3 we have

$$\sum_{n=1}^{N} \chi(an + b) = \chi(a) \sum_{n=1}^{N} \chi(n + \overline{a}b) \ll Np^{-\eta},$$

where $\eta > 0$ depends only on $\varepsilon$. Using Lemma 4 since $\varepsilon > 0$ is arbitrary, we conclude the proof in the same way as in Section 4.
7 Remarks, Experimental Results and Open Problems

It is certainly natural to expect that $\Delta_p = o(r) = o(\log p)$ which follows, for example, from the standard conjectures about gaps between consecutive primitive roots. But possibly it is a little easier to prove.

On the other hand, we do not have any nontrivial lower bounds on $\Delta_p$ apart from the following simple observations:

1. By Dirichlet’s Theorem, asymptotically half of the primes satisfy $p \equiv \pm 1 \mod 8$. For these primes $p$, the residue 2 is a quadratic residue and therefore no primitive root modulo $p$. Thus also no power of 2 is a primitive root, therefore there is no $n$ having Hamming distance 1 to 0 such that $n$ is a primitive root modulo $p$. Hence $\Delta_p \geq 2$ for all primes $p \equiv \pm 1 \mod 8$.

2. On the other hand, a quantitative version of Artin’s conjecture on primitive roots says that the proportion of primes with 2 as a primitive root is given by the Artin constant:

$$A = \prod_{p \text{ prime}} \left(1 - \frac{1}{p(p - 1)}\right) = 0.3739558\ldots. \quad (12)$$

This has been confirmed by Hooley [13] on the assumption of a certain extension of the Riemann Hypothesis. Furthermore, Vinogradov [30] has shown unconditionally that the proportion is at most $A$, see also a very short proof of this by Wiertelak [31]. For a survey on Artin’s original conjecture and its modified version see [21].

The primes with 2 as a primitive root are precisely those primes with $W_p = 1$. Hence the proportion of these primes is expected to be 0.3739... Similarly, the odd primes with $(\frac{2}{p}) = -1$ are precisely those primes with $w_p=1$, and their asymptotic density therefore is 1/2.

3. Computationally, for most primes $p \leq 10^6$ one has $\Delta_p = 2$. We list the number of primes for $p \leq 10^3, 10^4, 10^5, 10^6$ according to their value of $w_p, W_p$ and $\Delta_p$ (for $w_p$, only odd $p$ are considered). As usual, $\pi(x)$ denotes the number of primes up to $x$.

| $\pi(10^j)$ | $w = 1$ | $W = 1$ | $\Delta = 1$ | $w = 2$ | $W = 2$ | $\Delta = 2$ | $w = 3$ | $W = 3$ | $\Delta = 3$ |
|-------------|----------|----------|--------------|----------|----------|--------------|----------|----------|--------------|
| 3           | 168      | 87       | 68           | 12       | 80       | 100          | 133      | 0         | 0            |
| 4           | 1229     | 625      | 471          | 75       | 603      | 756          | 1147     | 0         | 2            |
| 5           | 9592     | 4808     | 3604         | 508      | 4783     | 5985         | 9075     | 0         | 3            |
| 6           | 78498    | 39276    | 29342        | 3915     | 39221    | 49145        | 74565    | 0         | 11           |
Observe that (for example) \(39276 + 39221 + 0 = \pi(10^6) - 1\) (we have \(-1\) as \(p = 2\) is omitted from consideration).

As an example for the comments above we note that
\[
\frac{39276}{78498} \approx 0.500344 \quad \text{and} \quad \frac{29342}{78498} \approx 0.373792
\]
are very close to \(1/2\) and the Artin constant \(A = 0.3739558\ldots\) given by (12), respectively.

4. A computer search for \(p \leq 3,000,000\) has produced 24 primes \(p\) with \(\Delta_p = 3\), but none with \(\Delta_p \geq 4\). We list a table of these 24 primes and all classes \(a\) such that the Hamming distance between \(a\) and the closest primitive root is at distance 3.

| \(p\)     | residue classes \(a\) |
|-----------|------------------------|
| 17        | 0, 16                  |
| 67        | 0, 1, 65               |
| 257       | 0, 256                 |
| 1753      | 0                      |
| 2089      | 0                      |
| 8209      | 0, 8196                |
| 8233      | 0, 8226                |
| 65537     | 0, 65536               |
| 77351     | 0                      |
| 111439    | 0                      |
| 114001    | 0                      |
| 164449    | 0                      |
| 239713    | 0                      |
| 262153    | 0, 262144              |
| 514711    | 0                      |
| 924841    | 0                      |
| 929671    | 0                      |
| 947911    | 0                      |
| 1316041   | 0                      |
| 1894369   | 0                      |
| 2097169   | 0, 2097152             |
| 2236879   | 0                      |
| 2493721   | 0                      |
| 2743711   | 0                      |

We note that all the corresponding residue classes are close to either end of the set of residues. Some of these cases can be explained by observing that
often $p$ or the corresponding class are close to a power of 2. Then a small shift may lead to an extra carry of the leading bit.

5. It is well-known that, assuming the GRH, for some constant $C > 0$ and $L = C(\log p)^2$, there are at least $0.4L / \log L$ primes $\ell < L$ that are quadratic non-residues modulo $p$, see [20, Chapter 13]. The same counting argument as in the proof of Theorem 2 implies that in this case $w_p \leq (1 + o(1)) \log r$. With more work, one can also improve the bound $W_p \leq (6 + o(1)) \log r$ that follows under the GRH from the Shoup [25] bound [4]. This in turn may lead to constructions of small sets $\mathcal{G}_p$ that are guaranteed to have a primitive root modulo $p$. Note that finding this primitive root requires full factorisation of $p - 1$, however in several applications (for example, in cryptography or combinatorics), one can simply consecutively use all elements of such sets; see [25, 26] for some related results.

We leave the following questions as open problems for further study. Numerous similar questions could be analogously stated.

**Question 1.** Investigate whether one can improve the bound in Theorem 1, assuming the GRH.

**Question 2.** What can one say about

$$f_i(x) = \frac{1}{\pi(x)} \# \{p \leq x : \Delta_p = i\}?$$

Does the limit $\lim_{x \to \infty} f_i(x)$ exist?

**Question 3.** Examine whether $\Delta_p$ is bounded or not.

**Question 4.** Examine whether $w_p \leq 2$ for most primes, that is, whether

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} w_p = \frac{3}{2}$$

holds.

There are several unrelated results estimating the distance in Hamming metric between some other number theoretic objects, such as reduced residues modulo a composite number [3] and primes, smooth and other special integers [4, 10, 11, 27, 28, 29], or other additive properties of the set of quadratic residues or primitive roots [8, 23]. These new and exciting directions definitely deserve more attention.
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