ON DIFF(M)-PSEUDO-DIFFERENTIAL OPERATORS AND THE GEOMETRY OF NON LINEAR GRASSMANNIANS

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ABSTRACT. We consider two principal bundles of embeddings with total space $Emb(M, N)$, with structure groups $Diff(M)$ and $Diff_+(M)$, where $Diff_+(M)$ is the group of orientation preserving diffeomorphisms. The aim of this paper is to describe the structure group of the tangent bundle of the two base manifolds:

$B(M, N) = Emb(M, N)/Diff(M)$ and $B_+(M, N) = Emb(M, N)/Diff_+(M)$.

From the various properties described, an adequate group seems to be a group of Fourier integral operators, which is carefully studied. A surprising fact is that the group of diffeomorphisms does not embed canonically to this group of operators, and that for the tangent space $TB_+(S^1, N)$, the group obtained is a subgroup of some group $GL_{res}$. Since the constructions of the paper disqualifies the invariant polynomials of $Vect(M)$ for the polynomials on another Lie algebra, we suggest some constructions that could lead to knot invariants through a theory of Chern-Weil forms in the spirit of [27].

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INTRODUCTION

Given $M$ and $N$ two Riemannian manifolds without boundary, with $M$ compact, the space of smooth embeddings $Emb(M, N)$ is currently known as a principal bundle with structure group $Diff(M)$, where $Diff(M)$ naturally acts by composition of maps. The base

$$B(M, N) = Emb(M, N)/Diff(M)$$

is known as a Fréchet manifold, and there exists some local trivializations of this bundle. We focus here on the base manifold, which seems to carry a richer structure than $Emb(M, N)$ itself.

This paper gives the detailed description of the structure group of the tangent bundle of connected components of $T B(M, N)$. This structure group can be slightly different when changing of connected component of $B(M, N)$. It is viewed as an extension of the group of automorphisms $Aut(E)$ of a vector bundle $E$ by some group of pseudo-differential operators. We show that this group is a regular Lie group (in the sense that it carries an exponential map), and that it is also a group of Fourier integral operators, which explains the notations $FIO_{Diff}$ and $FCl_{Diff}$ ("Cl" for "classical"). All these groups are constructed along a short exact sequence of the type

$$0 \to PDO \to FIO \to Diff \to 0',$$

where $PDO$ is a group of pseudo-differential operators, $FIO$ is a group of Fourier integral operators, and $Diff$ is a group of diffeomorphisms; this sequence plays a central role in the proofs. The theorems described are general enough to be applied to many groups of diffeomorphisms: volume preserving diffeomorphisms, symplectic diffeomorphisms, hamiltonian diffeomorphisms, and to groups of pseudo-differential operators: classical or non-classical, bounded or unbounded, compact and so on, but we concentrate our efforts on $Diff(M)$ and $Diff_+(M)$, the group of orientation preserving diffeomorphisms. The constructions are made for operators acting on smooth sections of trivial or non trivial bundles. For a non trivial bundle $E$, the group of automorphisms of the bundle plays a central role in the description, because easy arguments suggest that there is no adequate embedding of the group of diffeomorphisms of the base manifold into the group of automorphisms of the bundle. Specializing to $M = S^1$, given a vector bundle $E$ over $S^1$, the groups $FIO_{Diff}(S^1, E)$ and in particular $FCl^{0,*}_{Diff_+}(S^1, E)$ is of particular interest, where $FCl^{0,*}_{Diff_+}(S^1, E)$ is defined through the short exact sequence:

$$0 \to Cl^{0,*}(S^1, E) \to FCl^{0,*}_{Diff_+}(S^1, E) \to Diff_+(S^1) \to 0,$$
where \( C_{\text{Diff}}^{0,*}(S^1, E) \) is the group of bounded classical pseudo-differential operators and \( \text{Diff}_+(S^1) \) is the group of orientation-preserving diffeomorphisms. Given any Riemannian connection on the bundle \( E \), if \( \epsilon \) is the sign of this connection, it appears that \( [F C_{\text{Diff}}^{0,*}(S^1, E), \epsilon] \) is a set of smoothing operators. Thud, it is a subgroup of the group

\[
\text{Gl}_{\text{res}} = \{ u \in \text{Gl}(L^2(S^1, E)) | [\epsilon, u] \text{ is Hilbert-Schmidt} \}
\]

Even if the inclusion is not a bounded inclusion, this result extends the results given in [45] on the group \( \text{Diff}_+(S^1) \) (which inclusion map into \( \text{Gl}_{\text{res}} \) is not bounded too) and in [26] for the group \( C_{\text{Diff}}^{0,*}(S^1, E) \). We get a non-trivial cocycle on the Lie algebra of \( F C_{\text{Diff}}^{0,*}(S^1, E) \) by the Schwinger cocycle, extending results obtained in [26, 28] for a trivial bundle \( E \).

Coming back to \( \text{Emb}(M, N) \), one could suggest that \( \text{Aut}(E) \) is sufficient as a structure group, but we refer the reader to earlier works such as [11, 8, 27] to see how pseudo-differential operators can arise from Levi-Civita connections of Sobolev metrics when the adequate structure group for the \( L^2 \) metric is a group of multiplication operators. Moreover, especially for \( M = S^1 \), taking the quotient

\[
B_+(S^1, N) = \text{Emb}(S^1, N)/\text{Diff}_+(S^1),
\]

we show that there is a sign operator \( \epsilon(D) \), which is a pseudo-differential operator of order 0, and coming intrinsically from the geometry of \( \text{Emb}(S^1, N) \), such that the recognized structure group of \( TB_+(S^1, N) \) is \( F C_{\text{Diff}}^{0,*}(S^1, E) \subset \text{Gl}_{\text{res}} \). We finish with the starting point of this work, which was a suggestion of Claude Roger, saying the any well-defined Chern-Weil form of a connected component of \( TB(S^1, N) \) can be understood as an invariant of a knot, whose homotopy class is exactly a connected component of \( B(S^1, N) \). If one considers oriented knots, we get connected components of \( B_+(S^1, N) \). The work begun has not been completely successful yet, but it is a pleasure to suggest at the end of this paper some Chern-Weil froms that may lead to knot invariants, extending the approach of [27].

1. Preliminaries on algebras and groups of operators

Now, we fix \( M \) the source manifold, which is assumed to be Riemannian, compact, connected and without boundary, and the target manifold which is only assumed Riemannian. We note by \( \text{Vect}(M) \) the space of vector fields on \( TM \). Recall that the Lie algebra of the group of diffeomorphisms is \( \text{Vect}(M) \), which is a Lie-subalgebra of the (Lie-)algebra of differential operators, which is itself a subalgebra of the algebra of classical pseudo-differential operators.

1.1. Differential and pseudodifferential operators on a manifold \( M \).

**Definition 1.1.** Let \( \text{DO}(M) \) be the graded algebra of operators, acting on \( C^\infty(M, \mathbb{R}) \), generated by:

- the multiplication operators: for \( f \in C^\infty(M, \mathbb{R}) \), we define the multiplication operator

\[
M_f : g \in C^\infty(M, \mathbb{R}) \mapsto f g \text{ (by pointwise multiplication)}
\]

- the vector fields on \( M \): for a vector field \( X \in \text{Vect}(M) \), we define the differentiation operator

\[
D_X : g \in C^\infty(M, \mathbb{R}) \mapsto D_X g \text{ (by differentiation, pointwise)}
\]
Multiplication operators are operators of order 0, vector fields are operators of order 1. For $k \geq 0$, we note by $DO^k(M)$ the differential operators of order $\leq k$.

Differential operators are local, which means that

$$\forall A \in DO(M), \forall f \in C_\infty^\infty(M, \mathbb{R}), supp(A(f)) \subset supp(f).$$

The inclusion $Vect(M) \subset DO(M)$ is an inclusion of Lie algebras. The algebra $DO(M)$, graded by the order, is a subalgebra of the algebra of classical pseudo-differential operators $Cl(M)$, which is an algebra that contains the square root of the Laplacian, and its inverse. This algebra contains trace-class operators on $L^2(M, \mathbb{R})$. An exposition of basic facts on pseudo-differential operators defined on a vector bundle $E \to M$ can be found in [13] for definition of pseudo-differential operators and of their order, (local) definition of symbols and spectral properties.

We assume known the definition of the algebra of pseudo-differential operators $PDO(M, E)$, classical pseudo-differential operators $Cl(M, E)$. When the vector bundle $E$ is assumed trivial, i.e. $E = M \times \mathbb{R}^p$ with $\mathbb{R}^p = \mathbb{R}$ or $\mathbb{C}$, we use the notation $Cl(M, V)$ or $Cl(M, \mathbb{K}^p)$ instead of $Cl(M, E)$. These operators are pseudolocal, which means that

$$\forall A \in PDO(M, E), \forall f \in L^2(M, E), \text{ if } f \text{ is smooth on } K, \text{ then } A(f) \text{ is smooth on } K.$$

**Definition 1.2.** A pseudo-differential operator $A$ is log-polyhomogeneous if and only if its formal symbol reads (locally) as

$$\sigma(A)(x, \xi) \sim_{|\xi| \rightarrow +\infty} \sum_{j=0}^\alpha \sum_{k=-\infty}^{\alpha'} \sigma_{j,k}(x, \xi)(\log(|\xi|))^j,$$

where $\sigma_{j,k}$ is a positively $k$–homogeneous symbol.

The set of log-polyhomogenous pseudo-differential operators is an algebra.

A global symbolic calculus has been defined independently by two authors in [7], [33], where we can see how the geometry of the base manifold $M$ furnishes an obstruction to generalize local formulas of composition ans inversion of symbols. We do not recall these formulas here because they are not involved in our computations. More interesting for this article is to precise when the local formulas of composition of formal symbols extend globally on the base manifold.

We assume that $M$ is equipped with charts such that the changes of coordinates are translations and that the vector bundle $E \to M$ is trivial. This is in particular true when $M = S^1 = \mathbb{R} / 2\pi \mathbb{Z}$, or when $M = T^n = \prod_{i=1}^n S^1$. In the case of $S^1$, we use the smooth atlas $ATL$ of $S^1$ defined as follows:

$$ATL = \{\varphi_0, \varphi_1\};$$

$$\varphi_n : x \in [0; 2\pi[ \mapsto e^{i(x+n\pi)} \subset S^1 \text{ for } n \in \{0; 1\}.$$ 

Associated to this atlas, we fix a smooth partition of the unit $\{s_0; s_1\}$. An operator $A : C_\infty(S^1, \mathbb{C}) \to C_\infty(S^1, \mathbb{C})$ can be described in terms of 4 operators

$$A_{m,n} : f \mapsto s_m \circ A \circ s_n \text{ for } (m, n) \in \{0, 1\}.$$

Such a formula is a straightforward application of a localization formula in the case of an atlas $\{\varphi_i\}_{i \in I}$ of a manifold $M$ with associated family of partitions of the unit $\{s_i\}_{i \in I}$, see e.g [13] for details.
Notations. We note by $PDO(M, \mathbb{C})$ (resp. $PDO^o(M, \mathbb{C})$, resp. $Cl(M, \mathbb{C})$) the space of pseudo-differential operators (resp. pseudo-differential operators of order $o$, resp. classical pseudo-differential operators) acting on smooth sections of $E$, and by $Cl^o(M, \mathbb{C}) = PDO^o(S^1, \mathbb{C}) \cap Cl(S^1, \mathbb{C})$ the space of classical pseudo-differential operators of order $o$.

If we set

$$PDO^{-\infty}(M, \mathbb{C}) = \bigcap_{o \in \mathbb{Z}} PDO^o(M, \mathbb{C}),$$

we notice that it is a two-sided ideal of $PDO(M, \mathbb{C})$, and we define the quotient algebra

$$FPDO(M, \mathbb{C}) = PDO(M, \mathbb{C})/PDO^{-\infty}(M, \mathbb{C}),$$

$$FCl(M, \mathbb{C}) = Cl(M, \mathbb{C})/PDO^{-\infty}(M, \mathbb{C}),$$

$$FCl^o(M, \mathbb{C}) = Cl^o(M, \mathbb{C})/PDO^{-\infty}(M, \mathbb{C}),$$

called the algebras of formal pseudo-differential operators. $FPDO(M, \mathbb{C})$ is isomorphic to the set of formal symbols $[7]$, and the identification is a morphism of $\mathbb{C}$-algebras, for the multiplication on formal symbols defined before (see e.g. [13]). At the level of kernels of operators, a smoothing operator has a kernel $K_\infty \in C^\infty(M \times M, \mathbb{C})$, where as the kernel of a pseudo-differential operator is in general smooth only on the off-diagonal region $(M \times M) - \Delta(M)$, where $\Delta(M)$ denotes here, very exceptionnally in this paper, the diagonal set (and not a Laplacian operator). We finish by mentionning that the last property is equivalent to pseudo-locality.

1.2. Fourier integral operators. With the notations that we have set before, a scalar Fourier-integral operator of order $o$ is an operator

$$A : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C})$$

such that, $\forall (i, j) \in I^2$,

$$A_{k,j}(f) = \int_{\text{supp}(s_j)} e^{-i\phi(x, \xi)} \sigma_{k,j}(x, \xi)(s_j^* f)(\xi) d\xi$$

where $\sigma_{k,j} \in C^\infty(\text{supp}(s_j) \times \mathbb{R}, \mathbb{C})$ satisfies

$$\forall (\alpha, \beta) \in \mathbb{N}^2, \quad |D^\alpha_x D^\beta_\xi \sigma_{k,j}(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{\alpha - \beta},$$

and where, on any domain $U$ of a chart on $M$,

$$\phi(x, \xi) : T^* U - U \approx U \times \mathbb{R}^{|\text{dim} M|}_1 \to \mathbb{R}$$

is a smooth map, positively homogeneous of degree 1 fiberwise and such that

$$\det \left( \frac{\partial^2 \phi}{\partial x \partial \xi} \right) \neq 0.$$
Fourier integral operators with same phase function (resp. scalar pseudo-differential operators).

We define also the algebra of formal operators, which is the quotient space

\[ \mathcal{F}FIO = FIO/PDO^{-\infty}, \]

which is possible because \( PDO^{-\infty} \) is a closed two-sided ideal. When we consider classical Fourier integral operators, noted \( FCl \), that is operators with classical symbols, we add to this topology the topology on formal symbols \( \mathcal{F} \) which is an ILH topology (see e.g. \( \mathcal{F} \) for state of the art). We want to quote that if the symbols \( \sigma_{m,n} \) are symbols of order 0, then we get Fourier integral operators that are \( L^2 \)-bounded. We note this set \( FIO^0 \). This set is a subset of \( FIO \), and we have

\[ Cl^0 \subset PDO^0 \subset FIO^0 \subset FIO. \]

The techniques used for pseudo-differential operators are also used on Fourier integral operators, especially Kernel analysis. Let us consider a local coordinate operator \( A_{m,n} \) then, using the notation of of the formula \( \mathcal{F} \), the operator \( A_{m,n} \) is described by a kernel

\[ K_{m,n}(x,y) = \int_{\xi} e^{-i(\phi((x,\xi))-y\cdot\xi)}\sigma_{m,n}(x,\xi)d\xi. \]

From this approach one derives the composition and inversion formulas that will not be used in this paper, see e.g. \( \mathcal{F} \), but in the sequel we shall use the slightly restricted class of operators studied in \( \mathcal{F} \) and also in \( \mathcal{F} \) for formal operators.

1.3. **Topological structures and regular Lie groups of operators.** The topological structures can be derived both from symbols and from kernels, as we have quoted before but principally because there is the exact sequence described below with slice. At the level of units of these sets, i.e. of groups of invertible operators, the existence of the slice is also crucial. In the papers \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), the group of invertible Fourier integral operators receives first a structure of topological group, with in addition a differentiable structure, e.g. a Frölicher structure, which recognized as a structure of generalized Lie group, see e.g. \( \mathcal{F} \).

We have to say that, with the actual state of knowledge, using \( \mathcal{F} \), we can give a manifold structure (in the convenient setting described by Kriegl and Michor or in the category of Frölicher spaces following \( \mathcal{F} \)) to the corresponding Lie groups. Let us recall the statement

**Theorem 1.3.** \( \mathcal{F} \) Let \( G, H, K \) be convenient Lie groups or Frölicher Lie groups such that there is a short exact sequence of Lie groups

\[ 0 \to H \to G \to K \to 0 \]

such that there is a local slice \( K \to G \). Then

\[ G \text{ regular } \iff H \text{ and } K \text{ regular.} \]

**Remark 1.4.** In \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), \( \mathcal{F} \), the group \( K \) considered is the group of 1-positively homogeneous symplectomorphisms \( Diff_\omega(T^*M - M) \) where \( \omega \) is the canonical symplectic form on the cotangent bundle. The local section considered enables to build up the phase function of a Fourier integral operator from such a symplectic diffeomorphism inside a neighborhood of \( Id_M \). There is a priori no reason to restrict the constructions to classical pseudo-differential operators of
order 0, and have groups to Fourier integral operators with symbols in wider classes. This remark appears important to us because the authors cited before restricted themselves to classical symbols.

1.4. **Extension of PDO(\(M, E\)) by Aut(\(E\)).** We get now to another group:

**Theorem 1.5.** Let \(H\) be a regular Lie group of pseudo-differential operators acting on smooth sections of a trivial bundle \(E \sim V \times M \to M\). The group \(Diff(M)\) acts smoothly on \(C^\infty(M, V)\), and is assumed to act smoothly on \(H\) by adjoint action. If \(H\) is stable under the \(Diff(M)\)-adjoint action, then there exists a corresponding regular Lie group \(G\) of Fourier integral operators through the exact sequence:

\[
0 \to H \to G \to Diff(M) \to 0.
\]

If \(H\) is a Frölicher Lie group, then \(G\) is a Frölicher Lie group. If \(H\) is a Fréchet Lie group, then \(G\) is a Fréchet Lie group.

**Remark 1.6.** The pseudo-differential operators can be classical, log-polyhomogeneous, or anything else. Applying the formulas of “changes of coordinates” (which can be understood as adjoint actions of diffeomorphisms) of e.g. [13], one easily gets the result.

**Proof of Theorem 1.5.** Let us first notice that the action

\[
(f, g) \in C^\infty(M, V) \times Diff(M) \mapsto f \circ g \in C^\infty(M, V)
\]

can be read as, first a linear operator \(T_g\) with kernel

\[
K(x, y) = \delta(g(x), y) \quad \text{(Dirac \(\delta\) function)}
\]

or equivalently, on an adequate system of trivializations [13],

\[
T_g(f)(x) = \int e^{ig(x) \cdot \xi} \hat{f}(\xi) d\xi.
\]

This operator is not a pseudo-differential operator because it is not pseudolocal (unless \(g = Id_M\), but since

\[
det(\partial_x \partial_\xi(g(x), \xi)) = det(D_x g),
\]

we get that \(T_g\) is a Fourier-integral operator. Notice that another way to see it is the expression of its kernel.

Now, given \((A, g) \in H \times Diff(M)\), we define

\[
A_g = T_g \circ A.
\]

We get here a set \(G\) of operators which is set-theoretically isomorphic to \(H \times Diff(M)\). Since \(H\) is invariant under the adjoint action of the group \(Diff(M)\), \(G\) is a group, and from the beginning of this proof, we get that \(G\) is a group, and that there is the short exact sequence announced:

\[
0 \to H \to G \to Diff(M) \to 0,
\]

with a global slice

\[
g \in Diff(M) \mapsto T_g \in G.
\]

Since the adjoint action of \(Diff(M)\) is assumed smooth on \(H\), we can endow \(G\) with the product Frölicher structure to get a regular Frölicher Lie group. Since \(Diff(M)\) is a Fréchet Lie group, if \(H\) is a Fréchet Lie group, then \(G\) is a Fréchet Lie group. \(\Box\)
Remark 1.7. Some restricted classed of such operators are already considered in the literature under the name of $G$–pseudo-differential operators, see e.g. [43], but the groups considered are discrete (amenable) groups of diffeomorphisms.

Definition 1.8. Let $M$ be a compact manifold and $E$ be a (finite rank) trivial vector bundle over $M$. We define
\[ FIO_{Diff}(M, E) = \{ A \in FIO(M, E) | \phi_A(x, \xi) = g(x) \cdot \xi; g \in Diff(M) \}. \]

The set of invertible operators $FIO^*_{Diff}(M, E)$ is obviously a group, that decomposes as
\[ 0 \to PDO^*(M, E) \to FIO^*_{Diff}(M, E) \to Diff(M) \to 0 \]
with global smooth section
\[ g \in Diff(M) \mapsto (f \in C^\infty(S^1, E) \mapsto f \circ g). \]
Hence, Theorem 1.5 applies trivially to the following context:

Proposition 1.9. Let $FCl^0_{Diff}(M, E)$ be the set of operators $A \in FIO^*_{Diff}(M, E)$ such that $A$ has a 0-order classical symbol. Then we get the exact sequence:
\[ 0 \to Cl^0(M, E) \to FCl^0_{Diff}(M, E) \to Diff(M) \to 0 \]
and $FCl^0_{Diff}(M, E)$ is a regular Fréchet Lie group, with Lie algebra isomorphic, as a vector space, to $Cl^0(M, E) \oplus Vect(M)$.

Notice that the triviality of the vector bundle $E$ is here essential to make a $Diff(M)$–action on smooth section of $C^\infty(M, E)$. Let us assume now that $E$ is not trivial. At the infinitesimal level, trying to extend straightway, one gets a first condition for the extension.

Lemma 1.10. (see e.g. [5]) Let us fix a 0–curvature connection $\nabla$ on $M$. Then $X \in Vect(M) \mapsto \nabla_X \in DO^1(M, E)$ is a one-to-one Lie algebra morphism.

We remark that the analogy with the setting of trivial bundles $E$ stops here since the group $Diff(M)$ cannot be recovered in this group of operators. For example, when $M = S^1$, if $E$ is non trivial, the (infinitesimally) flat connection ensures that the holonomy group $H$ is discrete, but it cannot be trivial since the vector bundle $E$ is not. On non trivial bundle $E$, let us consider the group of bundle automorphism $Aut(E)$. The gauge group $DO^0(M, E)$ is naturally embedded in $Aut(E)$ and the bundle projection
\[ E \to M \]
induces a group projection
\[ \pi : Aut(E) \to Diff(M). \]
Therefor we get a short exact sequence
\[ 0 \to DO^0(M, E) \to Aut(E) \to Diff(M) \to 0. \]
Following [1] there exists a local slice $U \subset Diff(M) \to Aut(E)$, where $U$ is a $C^0$–open neighborhood on $Id_M$, which shows that $Aut(E)$ is a regular Fréchet Lie group. Therefore, the smallest group spanned by $PDO^*(M, E)$ and $Aut(E)$ is such that:

- the projection $E \to M$ induces a map $Aut(E) \to Diff(M)$ with kernel $DO^0(M, E) = Aut(E) \cap PDO(M, E)$
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• \( Ad_{Aut(E)}(PDO(M,E)) = PDO(M,E) \)
therefore we can consider the space of operators on \( C^\infty(M,E) \)

\[ FIO^*_\text{Diff}(M,E) = Aut(E) \circ PDO^*(M,E). \]

**Lemma 1.11.** The map

\[ (B, A) \in Aut(E) \times PDO^*(M,E) \mapsto \pi(B) \in Diff(M) \]

induces a “phase map”

\[ \tilde{\pi} : FIO^*_\text{Diff}(M,E) \to Diff(M). \]

**Proof.** Let \((B, A, (B', A')) \in Aut(E) \times PDO^*(M,E). \)

\[
\begin{align*}
B \circ A &= B' \circ A' & \iff & \text{Id}_E \circ A = B^{-1} \circ B' \circ A' \\
& \iff & \text{Id}_E \circ A = B^{-1} \circ B' \circ A' & \in PDO^*(M,E) \\
& \Rightarrow & B^{-1} \circ B' \in DO^{0,*}(M,E) \\
& \iff & \pi(B^{-1} \circ B') = \text{Id}_M \\
& \iff & \pi(B) = \pi(B')
\end{align*}
\]

\( \square \)

The next lemma is obvious:

**Lemma 1.12.** \( FIO^*_\text{Diff}(M,E) \) is a group.

**Lemma 1.13.** \( \text{Ker}(\tilde{\pi}) = PDO^*(M,E) \)

**Proof.** Let \( B \circ A \in FIO^*(M,E) \) such that

\[ \tilde{\pi}(B \circ A) = \pi(B) = \text{Id}_M. \]

Then \( B \in DO^{0,*}(M,E) \) and \( B \circ A \in PDO^*(M,E). \)

These results show the following theorem:

**Theorem 1.14.** There is a short exact sequence of groups :

\[ 0 \to PDO^*(M,E) \to FIO^*_\text{Diff}(M,E) \to Diff(M) \to 0 \]

and, if \( H \subset PDO^*(M,E) \) is a regular Fréchet or Frölicher Lie group of operators that contains the gauge group of \( E \), if \( K \) is a regular Fréchet or Frölicher Lie subgroup of Diff(M) such that there exists a local section \( K \to Aut(E) \), the subgroup \( G = K \circ H \) of \( FIO^*_\text{Diff}(M,E) \) is a regular Fréchet Lie group from the short exact sequence:

\[ 0 \to H \to G \to K \to 0. \]

**1.5. Diffeomorphisms and kernel operators.** Let \( g \in Diff(M) \). Then a straightforward computation on local coordinates shows that the kernel of \( T_g \) is

\[ K_g = \delta(g(x), y) \]

where \( \delta \) is the Dirac \( \delta \)-function. These operators also read locally as

\[ T_g(f) = \int_M e^{ig(x) \cdot \xi} \hat{f}(\xi) d\xi \]

on the same system of local trivializations used in [13], p.30-40.
1.6. **Renormalized traces.** $E$ is equipped this an Hermitian products $<.,.>$, which induces the following $L^2$-inner product on sections of $E$:

$$\forall u, v \in C^\infty(S^1, E), \quad (u, v)_{L^2} = \int_{S^1} < u(x), v(x) > \, dx,$$

where $dx$ is the Riemannian volume.

**Definition 1.15.** $Q$ is a **weight** of order $s > 0$ on $E$ if and only if $Q$ is a classical, elliptic, admissible pseudo-differential operator acting on smooth sections of $E$, with an admissible spectrum.

Recall that, under these assumptions, the weight $Q$ has a real discrete spectrum, and that all its eigenspaces are finite dimensional. For such a weight $Q$ of order $q$, one can define the complex powers of $Q$, see e.g. [8] for a fast overview of technicalities. The powers $Q^{-s}$ of the weight $Q$ are defined for $\Re(s) > 0$ using with a contour integral,

$$Q^{-s} = \int_{\Gamma} \lambda^s (Q - \lambda I)^{-1} d\lambda,$$

where $\Gamma$ is an “angular” contour around the spectrum of $Q$.

Let $A$ be a log-polynomial homogeneous pseudo-differential operator. The map $\zeta(A, Q, s) = s \in \mathbb{C} \mapsto \text{tr}(AQ^{-s}) \in \mathbb{C}$, defined for $\Re(s)$ large, extends on $\mathbb{C}$ to a meromorphic function with a pole of order $q + 1$ at 0 ([23]). When $A$ is classical, $\zeta(A, Q,.)$ has a simple pole at 0 with residue $\frac{1}{q} \text{res} A$, where $\text{res}$ is the Wodzicki residue ([54], see also [17]). Notice that the Wodzicki residue extends the Adler trace [4] on formal symbols. Following [23], we define the renormalized trace, see e.g. [8], [43] for the renormalized trace of classical operators.

**Definition 1.16.** $\text{tr}^Q A = \lim_{z \to 0} (\text{tr}(AQ^{-z}) - \frac{1}{q} \text{res} A)$.

On the other hand, the operator $e^{-tQ}$ is a smoothing operator for each $t > 0$, which shows that $\text{tr} A e^{-tQ}$ is well-defined and finite for $t > 0$. From the function $t \mapsto \text{tr} A e^{-tQ}$, we recover the function $z \mapsto \text{tr}(AQ^{-z})$ by the Mellin transform (see e.g. [50], pp. 115-116), which shows the following lemma:

**Lemma 1.17.** Let $A, A'$ be classical pseudo-differential operators, let $Q, Q'$ be weights.

$$\forall t > 0, \text{tr} A e^{-tQ} = \text{tr} A' e^{-tQ'} \iff \begin{cases} \text{tr}^Q(A) = \text{tr}^Q(A') \\ \text{res}(A) = \text{res}(A') \end{cases}$$

If $A$ is trace class, $\text{tr}^Q(A) = \text{tr}(A)$. The functional $\text{tr}^Q$ is of course not a trace on $\text{Cl}(M, E)$. Notice also that, if $A$ and $Q$ are pseudo-differential operators acting on sections on a real vector bundle $E$, they also act on $E \otimes \mathbb{C}$. The Wodzicki residue $\text{res}$ and the renormalized traces $\text{tr}^Q$ have to be understood as functional defined on pseudo-differential operators acting on $E \otimes \mathbb{C}$. In order to compute $\text{tr}^Q(A, B)$ and to differentiate $\text{tr}^Q A$, in the topology of classical pseudo-differential operators, we need the following ([8], see also [30] for the first point):

**Proposition 1.18.** (i) Given two (classical) pseudo-differential operators $A$ and $B$, given a weight $Q$,

$$\text{tr}^Q[A, B] = -\frac{1}{q} \text{res}(A[B, \log Q]).$$

(1.2)
(ii) Given a differentiable family \( A_t \) of pseudo-differential operators, given a differentiable family \( Q_t \) of weights of constant order \( q \),

\[
(1.3) \quad \frac{d}{dt} \left( tr^{Q_t} A_t \right) = tr^{Q_t} \left( \frac{d}{dt} A_t \right) - \frac{1}{q} \text{res} \left( A_t \left( \frac{d}{dt} \log Q_t \right) \right).
\]

The following "covariance" property of \( tr^Q \) (\cite{8}, \cite{43}) will be useful to define renormalized traces on bundles of operators.

**Proposition 1.19.** Under the previous notations, if \( C \) is a classical elliptic injective operator of order 0, \( tr^C (\left[ A, B \right]) \) is well-defined and equals \( tr^Q A \).

We moreover have specific properties for weighted traces of a more restricted class of pseudo-differential operators (see \cite{18}, \cite{19}, \cite{8}), called odd class pseudo-differential operators following \cite{18}, \cite{19}:

**Definition 1.20.** A classical pseudo-differential operator \( A \) is called odd class if and only if

\[
\forall n \in \mathbb{Z}, \forall (x, \xi) \in T^*M, \sigma_n(A)(x, -\xi) = (-1)^n \sigma_n(A)(x, \xi).
\]

We note this class \( Cl_{\text{odd}} \).

Such a definition is consistent for pseudo-differential operators on smooth sections of vector bundles, and applying the local formula for Wodzicki residue, one can prove \cite{8}:

**Proposition 1.21.** If \( M \) is an odd dimensional manifold, \( A \) and \( Q \) lie in the odd class, then \( f(s) = tr(AQ^{-s}) \) has no pole at \( s = 0 \). Moreover, if \( A \) and \( B \) are odd class pseudo-differential operators, \( tr^Q \left( \left[ A, B \right] \right) = 0 \) and \( tr^Q A \) does not depend on \( Q \).

This trace was first defined in the papers \cite{18} and \cite{19} by Kontsevich and Vishik. We remark that it is in particular a trace on \( DO(M, E) \) when \( M \) is odd-dimensional.

Let us now describe a class of operators which is, in some sense, complementary to odd class:

**Definition 1.22.** A classical pseudo-differential operator \( A \) is called even class if and only if

\[
\forall n \in \mathbb{Z}, \forall (x, \xi) \in T^*M, \sigma_n(A)(x, -\xi) = (-1)^{n+1} \sigma_n(A)(x, \xi).
\]

We note this class \( Cl_{\text{even}} \).

Very easy properties are the following:

**Proposition 1.23.** \( Cl_{\text{even}} \circ Cl_{\text{odd}} = Cl_{\text{odd}} \circ Cl_{\text{even}} = Cl_{\text{even}} \) and

\[
Cl_{\text{even}} \circ Cl_{\text{even}} = Cl_{\text{odd}} \circ Cl_{\text{odd}} = Cl_{\text{odd}}.
\]

Now, following \cite{27}, we explore properties of \( tr^Q \) on Lie brackets.

**Definition 1.24.** Let \( E \) be a vector bundle over \( M \), \( Q \) a weight and \( a \in \mathbb{Z} \). We define :

\[
\mathcal{A}_a^Q = \{ B \in Cl(M, E); \left[ B, \log Q \right] \in Cl^a(M, E) \}.
\]

**Theorem 1.25.** \cite{27}

(i) \( \mathcal{A}_a^Q \cap Cl^0(M, E) \) is an subalgebra of \( Cl(M, E) \) with unit.
(ii) Let $B \in \text{Ell}^r(M, E)$, $B^{-1}A^Q_n B = A^Q_n B$.

(iii) Let $A \in \text{Cl}^0(M, E)$, and $B \in \mathcal{A}^{Q}_{\dim M - b - 1}$, then $\text{tr}^Q[A, B] = 0$.

(iv) For $a < -\frac{\dim M}{2}$, $\mathcal{A}^{Q}_{a} \cap \text{Cl}^{\frac{\dim M}{2} + a}(M, E)$ is an algebra on which the renormalized trace is a trace (i.e. vanishes on the brackets).

We now produce non trivial examples of operators that are in $\mathcal{A}^{Q}_{a}$ when $Q$ is scalar, and secondly we give a formula for some non vanishing renormalized traces of a bracket.

**Lemma 1.26.** Let $Q$ be a weight on $C^{\infty}_0(M, V)$ and let $B$ be a classical pseudo-differential operator of order $b$. If $B$ or $Q$ is scalar, then $[B, \log Q]$ is a classical pseudo-differential operator of order $b - 1$.

**Proposition 1.27.** Let $Q$ be a scalar weight on $C^{\infty}_0(M, V)$. Then

$$\text{Cl}^{a+1}(M, V) \subset \mathcal{A}^{Q}_{a}.$$ 

Consequently,

(i) if $\text{ord}(A) + \text{ord}(B) = -\dim M$, $\text{tr}^Q[A, B] = 0$.

(ii) when $M = S^1$, if $A$ and $B$ are classical pseudo-differential operators, if $A$ is compact and $B$ is of order 0, $\text{tr}^Q[A, B] = 0$.

**Lemma 1.28.** Let $Q$ be a scalar weight on $C^{\infty}_0(M, V)$, and $A, B$ two pseudo-differential operators of orders $a$ and $b$ on $C^{\infty}_0(M, V)$, such that $a + b = -m + 1$ ($m = \dim M$). Then

$$\text{tr}^Q[A, B] = -\frac{1}{q} \text{res}(A[B, \log Q]) = -\frac{1}{q(2\pi)^n} \int_M \int_{|\xi| = 1} \text{tr}(\sigma_a(A)\sigma_b([B, \log Q])).$$

1.7. **Coadjoint action of $Diff(M)$ on renormalized traces.** Let us now explore the action of $Diff(M)$ and of $Aut(E)$ on $\text{tr}^Q(A)$. For this, we get:

**Lemma 1.29.** Let $a \in \mathbb{Z}$. Let $A \in \text{Cl}^a(M, E)$ and let $Q$ be a weight on $E$. Let $B$ be an operator on $C^{\infty}(M, E)$ such that

1. $\text{Ad}_B(\text{Cl}^a(M, E)) \subset \text{Cl}^a(M, E)$

2. $\text{Ad}_B Q$ is a weight of the same order as $Q$

Then

- $\text{Ad}_B A \in \text{Cl}^a(M, E)$
- $\text{Ad}_B Q$ is a weight
- $\text{res}(\text{Ad}_B A) = \text{res}(A)$
- $\text{tr}^{\text{Ad}_B Q}(\text{Ad}_B A) = \text{tr}^Q(A)$.

The properties (1) are true in particular for operators $B \in Aut(E)$.

**Proof.**

Let $Q$ be a weight on $C^{\infty}(M, E)$ and let $A \in \text{Cl}(M, E)$. Let $B \in Aut(E)$. Let $s \in \mathbb{R}_+$, then $Ae^{-sQ}$ is trace class. By [13], we know that $\text{Ad}_B A$ (resp. $\text{Ad}_B Q$) is a classical pseudo-differential operator of the same order (resp. a weight of the same order). Then, since $e^{-sQ}$ is smoothing, $\text{Ad}_B(Ae^{-sQ}, BAE^{-sQ}$ and $e^{-sQ}B^{-1}$ are smoothing, and the following computations are fully justified:
\[ tr \left( Ad_B(Ae^{-sQ}) \right) = tr \left( (BAe^{-\frac{s}{2}Q}) (e^{-\frac{s}{2}Q}B^{-1}) \right) \]
\[ = tr \left( (e^{-\frac{s}{2}Q}B^{-1}) (BAe^{-\frac{s}{2}Q}) \right) \]
\[ = tr \left( (e^{-\frac{s}{2}Q}Ae^{-\frac{s}{2}Q}) \right) \]
\[ = tr \left( (Ae^{-sQ}) \right) \]

So that, by Lemma 1.17, we get the announced property. \( \square \)

2. Splittings on the set of \( S^1 - \)Fourier integral operators

2.1. **The group \( O(2) \) and the diffeomorphism group \( Diff(S^1) \).** Let us consider the \( SO(2) = U(1) \)-action on \( S^1 = \mathbb{R}/\mathbb{Z} \) given by \( (e^{2\pi i \theta}, x) \mapsto x + \theta \). This group acts on \( C^\infty \) by \( (e^{2\pi i \theta}, f) \mapsto f(x + \theta) \) and we have

\[ f(x + \theta) = \int e^{-i(x+\theta) \xi} \hat{f}(\xi) d\xi \]
\[ = \int e^{-i(x,\xi) + \theta \xi} \hat{f}(\xi) d\xi \]

The term \( e^{-i\theta \xi} \) is oscillating in \( \xi \) and does not satisfies the estimates on the derivatives of symbols. So that, this operator is not a pseudo-differential operator but has obviously the form of a Fourier integral operator. The same is for the reflection \( x \mapsto 1 - x \) which corresponds to the conjugate transformation \( z \mapsto \bar{z} \) when representing \( S^1 \) as the set of complex numbers \( z \) such that \( |z| = 1 \). This is a special case of the properties already stated for a general manifold \( M \) given \( g \in Diff(S^1) \), \( g \) acts on \( C^\infty \) by right composition of the inverse, namely, for \( f \in C^\infty \),

\[ g.f(x) = f \circ g(x) \]
\[ = \int e^{-i g(x) \xi} \hat{f}(\xi) d\xi, \]

which is also obviously a Fourier-integral operator, and the kernel of this operator is

\[ K_g(x, y) = \delta (y, g(x)) \]

where \( \delta \) is the Dirac \( \delta \)-function. This is the construction already used in the proof of Theorem 1.5.

2.2. \( \epsilon(D) \), its formal symbol and the splitting of \( FPDO \). The operator \( D = -iD_x \) splits \( C^\infty(S^1, \mathbb{C}^k) \) into three spaces:
- its kernel \( E_0 \), made of constant maps
- \( E_+ \), the vector space spanned by eigenvectors related to positive eigenvalues
- \( E_- \), the vector space spanned by eigenvectors related to negative eigenvalues.

The following elementary result will be useful for the sequel, see \[24\] for the proof, and e.g. \[27, 28\]:

**Lemma 2.1.** (i) \( \sigma(D) = \xi \)
(ii) \( \sigma(|D|) = |\xi| \) where \( |D| = \sigma \left( \int \lambda^{1/2}(\Delta - \lambda Id)^{-1} d\lambda \right) \), with \( \Delta = -D^2_x \).
(iii) \( \sigma(DD^{-1}) = \frac{\xi}{|\xi|} \), where \( DD^{-1} = |D|^{-1}D \) is the sign of \( D \), since \( |D|_{E_0} = Id_{E_0} \).
Lemma 2.5. Which means the following (see e.g. \[22\]):

Let us give another characterization of $p$ reads as

Definition 2.3. The case of non trivial bundle over $S^1$. Let $\pi: E \to S^1$ be a non trivial vector bundle over $S^1$ of rank $k$. Its bundle of frames is a $Gl(\mathbb{C}^k)$-principal bundle, which means the following (see e.g. \[22\]):

Lemma 2.5. Let $\varphi_1:a;b\times\mathbb{C}^k \to E$ and $\varphi_2:a';b'\times\mathbb{C}^k \to E$ be two local trivializations of $E$. Let $\mathcal{D} = \pi(\varphi_1(a;b\times\mathbb{C}^k) \cap \varphi_2(a';b'\times\mathbb{C}^k))$, let $\mathcal{D}_1 = \varphi_1^{-1}(\mathcal{D})$, and let $\mathcal{D}_2 = \varphi_2^{-1}(\mathcal{D})$. Then

\[ \varphi_2^{-1} \circ \varphi_1: \mathcal{D}_1 \times \mathbb{C}^k \to \mathcal{D}_2 \times \mathbb{C}^k \]

reads as

\[ \varphi_2^{-1} \circ \varphi_1 = \gamma \times M \]

where $\gamma$ is a smooth diffeomorphism from $\mathcal{D}_1$ to $\mathcal{D}_2$, and $M \in C^\infty(\mathcal{D}_1, Gl(\mathbb{C}^k))$. 

Proposition 2.4. \[24\] Let $a \in FPDO(S^1, \mathbb{C}^k)$. $p_+(a) = \sigma(p_{E_+}) \circ a = a \circ \sigma(a)$ and $p_-(a) = \sigma(p_{E_-}) \circ a = a \circ \sigma(p_{E_-})$.
Let us now turn to symbols of pseudo-differential operators acting on \( E \). We first assume that we work with a system of local trivializations such that the diffeomorphisms \( \gamma \) are translations, and let us now look at the transformations of the symbols read on local trivializations. Under these assumptions, and with the notations of the previous lemma, a formal symbol \( \sigma_1 \) read on \( D_1 \) reads on \( D_2 \) as

\[
\sigma_2(\gamma(x), \xi) = M(x)\sigma_1(x, \xi)M(x)^{-1}.
\]

This implies the following:

**Lemma 2.6.** The formal symbol \( \frac{\xi}{|\xi|} \) is a formal symbol of \( \mathcal{F}Cl^0(S^1, E) \).

Thus, all the results described in the last section applies also for \( \mathcal{F}Cl^0(S^1, E) \). We shall note by \( \epsilon \) any operator with formal symbol \( \frac{\xi}{|\xi|} \). Changing \( \epsilon \) into \( \frac{1}{2}(\epsilon^* \epsilon + \epsilon \epsilon^*) \), we assume it self-adjoint, and \( \epsilon^2 = \text{Id} + \text{smoothing operator} \).

**Proposition 2.7.** Let \( \nabla \) be a Riemannian covariant derivative on the bundle \( E \to S^1 \) and let \( \frac{\nabla}{dt} \) be the associated first order differential operator, given by the covariant derivative evaluated at the unit vector field over \( S^1 \). We modify the operator \( \frac{\nabla}{dt} \) into \( D = -i \frac{\nabla}{dt} + p_{ker} \frac{\nabla}{dt} \), and we set

\[
\epsilon(D) = D \circ |D|^{-1}.
\]

Then the formal symbol of \( \epsilon(D) \) is \( \frac{\xi}{|\xi|} \).

**Proof.** Let us use the holonomy trivialization over an interval \( I \). In this trivialization,

\[
\nabla \frac{\nabla}{dt} = \frac{d}{dt}
\]

and hence the formal symbol of \( \frac{\nabla}{dt} \) reads as \( i\xi \). Calculating exclusively on the algebra of formal operators on which composition and inversion governed by local formulas, we get \( \sigma(|D|) = |\xi| \) and, by the same arguments as those of [24], we get the result.

**Proposition 2.8.** For each \( A \in PDO(S^1, E) \), \( [A\epsilon(D)] \in PDO^{-\infty}(S^1; E) \).

**Proof.** We remark that, for any multiindex \( \alpha \) such that \( |\alpha| > 0 \), \( D_\alpha^2 \sigma(\epsilon(D)) = 0 \) and \( D_\alpha^2 \sigma(\epsilon(D)) = 0 \). Hence, in \( FPDO(S^1, E) \),

\[
\sigma([A, \epsilon(D)]) = [\sigma(A), \sigma(\epsilon(D))] = 0
\]

so that \( [A, \epsilon(D)] \in PDO^{-\infty}(S^1, E) \).

**2.4. The splitting read on the phase function.** The fiber bundle \( T^* S^1 - S^1 \) has two connected components and the phase function is positively homogeneous, so that we can make the same procedure as in the case of the symbols. But we remark that we can split

\[
\phi = \phi_+ + \phi_-
\]

where \( \phi_+ = 0 \) if \( \xi < 0 \) and \( \phi_- = 0 \) if \( \xi > 0 \). Unfortunately, \( \phi_+ \) and \( \phi_- \) are not phase functions of Fourier integral operators because there are some points where
Proof. Since \( \sigma \), see e.g. [27], we know that \( C \) and hence it would be true on \( H \) where
\[\begin{align*}
\sigma^2_{\delta_1} & = 0 \quad \text{or} \quad \sigma^2_{\delta_2} = 0. \quad \text{However, we can have the following identities:}
\int_{\mathbb{R}} e^{i\phi(x,\xi)}\sigma(x,\xi) \hat{f}(\xi) d\xi &= \int_{\xi > 0} e^{i\phi(x,\xi)}\sigma(x,\xi) \hat{f}(\xi) d\xi + \int_{\xi < 0} e^{i\phi(x,\xi)}\sigma(x,\xi) \hat{f}(\xi) d\xi \\
&= \int_{\xi > 0} e^{i\phi(x,\xi)}\sigma(x,\xi) \hat{f}(\xi) d\xi + \int_{\xi < 0} e^{i\phi(x,\xi)}\sigma(x,\xi) \hat{f}(\xi) d\xi \\
&= \int_{\mathbb{R}} e^{i\phi(x,\xi)}\sigma_+(x,\xi) \hat{f}(\xi) d\xi \\
&= \int_{\mathbb{R}} e^{i\phi(x,\xi)}\sigma_+(x,\xi) \hat{f}(\xi) d\xi \\
&= \int_{\mathbb{R}} e^{i\phi(x,\xi)}\sigma_+(x,\xi) \hat{f}(\xi) d\xi \\
&= \int_{\mathbb{R}} e^{i\phi(x,\xi)}\sigma_+(x,\xi) \hat{f}(\xi) d\xi
\end{align*}\]

2.5. The Schwinger cocycle on \( PDO(S^1, E) \). Let us first precise which polarization we choose on \( L^2(S^1, E) \). We can choose independently two polarizations:
- one setting \( H^{(1)}_+ = E_+ \) and \( H^{(1)}_- = E_0 \oplus E_- \),
- or another one setting \( H^{(2)}_+ = E_+ \oplus E_0 \) and \( H^{(2)}_- = E_- \).
Since \( E_0 \) is of dimension \( k \), the orthogonal projection on \( E_0 \) is a smoothing operator. Hence,

\[\sigma(p_{H^{(1)}_+}) = \sigma(p_{H^{(2)}_+}) = 1_{\xi > 0}\]
and
\[\sigma(p_{H^{(1)}_-}) = \sigma(p_{H^{(2)}_-}) = 1_{\xi < 0} \cdot \]

We introduce the notation, for \( A \in PDO(S^1, E) \),
\[A_{++} = p_{H_+} Ap_{H_+},\]
where \( H_+ \) denotes \( H^{(1)}_+ \) or \( H^{(2)}_+ \), and we set \( p(D) = p_{H_+} - p_{H_-} \). We notice that \( \sigma(A_{++}) = \sigma_+(A) \), and extend the main result of [27, 28]:

**Theorem 2.9.** For any \( A \in PDO(S^1, E) \), \([A, \epsilon(D)] \in PDO^{-\infty}(S^1, E) \). Consequently,
\[c^D_s : A, B \in PDO(S^1, E) \mapsto \frac{1}{2} \text{tr} (\epsilon(D)[\epsilon(D), A][\epsilon(D), B])\]
is a well-defined 2-cocycle on \( PDO(S^1, E) \). Moreover, \( c^D_s \) is non trivial on any Lie algebra \( A \) such that \( C^\infty(S^1, \mathbb{C}) \subset A \subset PDO(S^1, E) \).

Notice that \( C^\infty(S^1, \mathbb{C}) \) is understood as an algebra acting on \( C^\infty(S^1, \mathbb{C}) \) by scalar multiplication fiberwise. The proof follows the same arguments as in [28].

**Proof.** Since [45], see e.g. [27], we know that \( c^D_s \) is non trivial on \( C^\infty(S^1, \mathbb{C}) \). If \( c^D_s \) was trivial, there would have a 1-form \( \nu : A \rightarrow \mathbb{C} \) such that
\[c^D_s = \nu([\cdot, \cdot]),\]
and hence it would be true on \( C^\infty(S^1, \mathbb{C}) \) which is a commutative algebra. Hence, since \( c^D_s \neq 0 \) on \( C^\infty(S^1, \mathbb{C}) \), it is non trivial on \( A \). \( \square \)
3. Sets of Fourier Integral operators

3.1. The set $FIO(S^1, E)$. Let us now define

$$FIO_{res}(S^1; E) = \{ A \in FIO(S^1, E) \text{ such that } [A; \epsilon(D)] \in PDO^{-\infty}(S^1, E) \}.$$ 

**Proposition 3.1.** $FIO_{res}(S^1, E)$ is a set, stable under composition, with unit element.

**Proof.** $FIO(S^1, E)$ is stable under composition [16]. Since $Cl(S^1, E)$ is contained in $FIO_{res}(S^1, E)$ by Theorem 2.4, so that $FIO_{res}(S^1, E)$ contains the identity map. Let $A, B \in FIO_{res}(S^1, E)$,

$$[AB; \epsilon(D)] = A[B; \epsilon(D)] + [A; \epsilon(D)]B$$

Since $[A; \epsilon(D)]$ and $[B; \epsilon(D)]$ are smoothing, we get that $[AB; \epsilon(D)]$ is smoothing.

We use the natural notations,

$$FIO^0_{res} = FIO^0 \cap FIO_{res}.$$ 

Then, as in the context of the restricted general linear group described in [45], a Fourier integral operator $A \in FIO_{res}(S^1, E)$ can be written blockwise:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the decomposition $H = H_+ \oplus H_-.$

3.2. The groups $FIO^*_{res}(S^1, E)$ and $FIO^{0,*}_{res}(S^1, E)$. We shall note by $FIO^*_{res}(S^1, E)$ the group of units of this set, and by $FIO^{0,*}_{res}(S^1, E)$ the group of units of the set $FIO^0_{res}(S^1, E)$.

**Proposition 3.2.** $FIO^*_{res}(S^1, E) = FIO^*(S^1, E) \cap FIO_{res}(S^1, E)$ and $FIO^{0,*}_{res}(S^1, E) = FIO^{0,*}(S^1, E) \cap FIO_{res}(S^1, E)$.

**Proof.** We already have trivially $FIO^*_{res}(S^1, E) \subset (S^1, E) \cap FIO_{res}(S^1, E)$. Let $A \in FIO^*(S^1, E) \cap FIO_{res}(S^1, E)$. We have to check that $A^{-1} \in FIO_{res}(S^1, E)$,

$$A[A^{-1}, \epsilon(D)] = [AA^{-1}, \epsilon(D)] - [A, \epsilon(D)]A^{-1}$$

$$= [Id, \epsilon(D)] - [A, \epsilon(D)]A^{-1}$$

$$= -[A, \epsilon(D)]A^{-1}$$

$$\in PDO^{-\infty}(S^1, E)$$

So that

$$[A^{-1}, \epsilon(D)] = A^{-1}A[A^{-1}, \epsilon(D)]$$

$$\in PDO^{-\infty}(S^1, E)$$

The proof is the same for 0–order operators.

By the way, since $FIO^{0,*}(S^1, E)$ is a ”generalized Lie group” in the sense of Omori, it is a Frölicher Lie group. By the trace property of Frölicher spaces, using the last proposition, $FIO^0_{res}(S^1, E)$ is a Frölicher Lie group [29].
3.3. The Determinant bundle and the Schwinger cocycle. Now, since we have that
\[ FIO_{res}^{0,*} \subset GL_{res}, \]
the determinant bundle defined over \( GL_{res} \) can be pulled-back on \( FIO_{res}^{0,*} \). The same way, it is shown in [26, 28] that the Schwinger cocycle extends to the Lie algebra \( PDO^{0}(S^1, E) + PDO^{1}(S^1, \mathbb{C}) \otimes \text{Id}_{E} \).

3.4. Yet another Lie group of Fourier integral operators in \( GL_{res} \). We now specialize to \( M = S^1 \). Let us first gather and reformulate many known results:

**Lemma 3.3.** \( Diff^{+}(S^1) \times C^{\infty}(S^1, \mathbb{C}^{*}) \subset FIO_{res}^{0,*}(S^1, \mathbb{C}). \)

**Proof.** First, we have that
\[ C^{\infty}(S^1, \mathbb{C}^{*}) \subset C^0(S^1, \mathbb{C}) \]
so that
\[ C^{\infty}(S^1, \mathbb{C}^{*}) \subset FIO_{res}^{0,*}(S^1, \mathbb{C}). \]

Let \( g \in Diff^{+}(S^1) \). Following [45], the map \( f \mapsto |g'|^{1/2} \cdot (f \circ g) \) describes an operator in \( U_{res} \subset GL_{res} \). Since the map \( f \mapsto |g'|^{1/2} \cdot f \) is a multiplication operator in \( C^{\infty}(S^1, \mathbb{C}^{*}) \), we get that
\[ f \mapsto f \circ g = \int e^{-ig(-\cdot) \cdot \xi} \hat{f}(\xi) d\xi \in GL_{res} \cap FIO_{res}^{0,*}(S^1, \mathbb{C}). \]

\[ \square \]

**Theorem 3.4.** Assume that \( E \) be a trivial vector bundle over \( S^1 \). Let \( \tilde{\pi} \) be the projection \( FIO_{Diff}^{0}(S^1, E) \rightarrow Diff(S^1) \). Then
\[ \pi^{-1}(Diff^{+}(S^1)) \subset FIO_{res}(S^1, E). \]

This is a simple consequence of the previous results.

**Theorem 3.5.** Assume that \( E \) is non trivial and let \( \epsilon(D) \) defined as before. Let \( \tilde{\pi} \) be the projection \( FIO_{Diff}^{0}(S^1, E) \rightarrow Diff(S^1) \). Then
\[ FIO_{Diff}^{+}(S^1, E) = \pi^{-1}(Diff^{+}(S^1)) \subset FIO_{res}(S^1, E), \]
and there is a global smooth section (in the sense of Frölicher spaces)
\[ Diff^{+}(S^1) \rightarrow FIO_{res}(S^1, E) \]
of the short exact sequence:
\[ 0 \rightarrow PDO^{*}(S^1, E) \rightarrow FIO_{Diff}^{0}(S^1, E) \cap FIO_{res}(S^1, E) \rightarrow Diff^{+}(S^1) \rightarrow 0. \]

**Proof.** Let \( g \in Diff^{+}(S^1) \). We fix on \( E \) a connection \( \nabla \) and we set \( n = \text{rank}(E) \). Since \( Diff^{+}(S^1) \) is the connected component of \( \text{Id}_{S^1} \) in \( Diff(S^1) \), given \( \eta \) the unit vector field defined by orientation on \( S^1 \), we can choose a path
\[ \gamma \in C^{\infty}([0, 1], Diff^{+}(S^1)) \subset C^{\infty}([0, 1] \times S^1, S^1) \]
such that
\[ \gamma(0) = \text{Id}_{S^1}, \gamma(1) = g \]
and
\[ \forall x \in S^1, \forall t \in [0; 1], \left( \frac{d\gamma}{dt}(t)(x), \eta(x) \right)_{\gamma(t), S^1} > 0. \]
This path is unique up to parametrization since we impose also the condition of minimal length. Let
\[ H_x = Hol(\gamma(\cdot)(x)) \in GL((E_x, E_{g(x)})) \]
be the induced parallel transport map. We get, for each \( g \in Diff_+(S^1) \), a map
\[ H_g : N_f \rightarrow N_f \]
which is smooth by the properties of parallel transport, linear on the fibers, invertible, and which projects on \( S^1 \) to \( g \). Thus, \( H_g \in Aut(E) \), and it easy to see that it is a bijection on the collection of smooth trivializations of \( E \). Now, turning to the map
\[ g \mapsto H_g, \]
is appears as a smooth map \( Diff(M) \rightarrow Aut(E) \), but it is not a group morphism since \( E \) is non trivial. We have moreover that
\[ \forall g \in Diff^+(S^1), [\nabla, H_g] = 0 \]
since \( \dim(S^1) = 1 \) and \( H_g \) is a parallel transport map. So that, since \( \epsilon \) is the sign operator of \( i\nabla \eta \), we get that \( H_g \in GL_{res} \).

Now, an operator in \( FI\Omega_{Diff^+}(S^1, E) \) reads as
\[ H_g \circ A, \]
where \( A \in PDO^*(S^1, E) \subset GL_{res} \). Then \( H_g \circ A \in FI\Omega_{res} \). \( \Box \)

**Theorem 3.6.** The group
\[ FC\Sigma_{Diff^+}^0(S^1, E) = FI\Omega_{Diff^+}^*(S^1, E) \cap FC\Sigma^0(S^1, E) \]
is a regular Frölicher Lie group.

**Proof.** We get the obvious exact sequence of Lie groups:
\[ 0 \rightarrow Cl^{*, 0}(S^1, E) \rightarrow FC\Sigma_{Diff^+}^*(S^1, E) \rightarrow Diff^+(S^1) \rightarrow 0. \]
Both \( Cl^{*, 0}(S^1, E) \) and \( Diff^+(S^1) \) are regular, and \( Aut(E) \subset Cl^{*, 0}(S^1, E) \), so that the smooth section \( Diff^+(S^1) \rightarrow Aut(E) \) described in the proof of the previous theorem gives the result by Theorem 1.14. \( \Box \)

4. MANIFOLDS OF EMBEDDINGS

**Notation :** Let \( E \rightarrow M \) be a smooth vector bundle over \( M \) with typical fiber \( x \). For \( k \in \mathbb{N}^* \), we denote by
- \( E^{x^k} \) the product bundle, of basis \( M \), with typical fiber \( F^{x^k} \);
- \( \Omega^k(E) \) the space of \( k \)-forms on \( M \) with values in \( E \), that is, the set of smooth maps \( (TM)^{x^k} \rightarrow E \) that are fiberwise \( k \)-linear and skew-symmetric \( (T_x M)^{x^k} \rightarrow E_x \) for any \( x \in M \). If \( F = M \times F \), we note \( \Omega^k(M, F) \) the space of \( k \)-forms instead of \( \Omega^k(E) \).

Let \( M \) be a compact manifold without boundary; let \( N \) be a Riemannian manifold, equipped with the metric (\( , \)). Let \( Emb(M, N) \) be the manifold of smooth embeddings \( M \rightarrow N \).

The group of diffeomorphisms of \( M \), \( Diff(M) \), acts smoothly and on the right on \( Emb(M, N) \), by composition. Moreover,
\[ B(M, N) = Emb(M, N)/Diff(M) \]
is a smooth manifold \[21\], and \( \pi : \text{Emb}(M, N) \to B(M, N) \) is a principal bundle with structure group \( \text{Diff}(M) \) (see \[21\]). Then, \( g \in \text{Emb}(M, N) \) is in the \( \text{Diff}(M) \)-orbit of \( f \) if and only if \( g(M) = f(M) \). Let us now precise the vertical tangent space and a normal vector space of the orbits of \( \text{Diff}(M) \) on \( \text{Emb}(M, N) \). \( T_fP \text{Emb}(M, N) \), the tangent space at \( f \), is identified with the space of smooth sections of \( f^*TN \), which is the pull-back of \( TN \) by \( f \). \( \text{VT}_fP \), the vertical tangent space at \( f \) is the space of smooth sections of \( Tf(M) \). Let \( \mathcal{N}_f \) be the normal space to \( f(M) \) with respect to the metric \((.,.)\) on \( N \). For any \( x \in M \), \( T_{f(x)}N = Tf(x)f(M) \oplus \mathcal{N}_f(M) \). Hence, denoting \( f \circ \mathcal{N}_f \) the pull back of \( \mathcal{N}_f \) by \( f \), we have that

\[
C^\infty(f^*TN) = C^\infty(TM) \oplus f^*\mathcal{N}_f.
\]

Moreover, for any volume form \( dx \) on \( M \), if

\[
<.,.>: X, Y \in C^\infty(f^*TN) \mapsto \int_M (X(x), Y(x))dx
\]

is a \( L^2 \)-inner product on \( C^\infty(f^*TN) \), this splitting is orthogonal for \( <.,.> \). We get here a fundamental difference between the inclusion \( \text{Emb}(M, N) \subset C^\infty(M, N) \), where the model space of the type \( C^\infty(f^*TN) \), and \( \text{Emb}(M, N) \) as a \( \text{Diff}(M) \)-principal bundle: sections of the vertical tangent vector bundle read as order 1 differential operators, just like the structure group of \( T\text{C}(M) \). To be more precise, let \( X \in C^\infty(f^*TN) \) and let \( p : f^*TN \to Tf(M) \) be the orthogonal projection. The vector field \( p(X) \in C^\infty(Tf(M)) \) is seen as a differential operator acting on smooth functions \( f(M) \sim M \to \mathbb{R} \), and the normal component \((Id - p)(X)\) is a smooth section on \( \mathcal{N}_f \). In the sequel we shall note

\[
\mathcal{N} = \prod_{f \in \text{Emb}(M, N)} \mathcal{N}_f.
\]

We turn now to local trivializations. Let \( f \in C^\infty_b(M, N) \). We define the map \( \text{Exp}_f : C^\infty_b(M, f^*TN) \to C^\infty_b(M, N) \) defined by \( \text{Exp}_f(v) = \text{exp}_{f(I_f)}v(.) \) where \( \text{exp} \) is the exponential map on \( N \). Then \( \text{Exp}_f \) is a smooth local diffeomorphism. Restricting \( \text{Exp}_f \) to a \( C^\infty \) - neighborhood \( \tilde{U}_f \) of the 0-section of \( f^*TN \), we define a diffeomorphism, setting

\[
(\text{Exp}_f)_\tilde{U}_f : \tilde{U}_f \to V_f = \text{Exp}_f(\tilde{U}_f) \subset C^\infty_b(M, N).
\]

Then, setting \( U_f = I_f^{-1}\tilde{U}_f \), we can define a chart \( \Xi^f \) on \( V_f \) by:

\[
\Xi^f(g) = (I_f^{-1} \circ (\text{Exp}_f|_{\tilde{U}_f})^{-1})(g) \in U_f \subset C^\infty_b(M, E).
\]

Given \( f, g \in C^\infty_b(M, N) \) such that \( V_{f,g} = V_f \cap V_g \neq \emptyset \), we compute the changes of charts \( \Xi^f \) from \( U^f_{f, g} = \Xi/V_{f, g} \) to \( U^g_{f, g} = \Xi/gV_{f, g} \). Let \( u \in U^f_{f, g}, v = (\Xi^f)^{-1}(u) \in V_{f, g} \).

\[
\Xi^f(u) = \Xi^g \circ (\Xi^f)^{-1}(u) = (I_g^{-1} \circ (\text{Exp}_g)^{-1} \circ \text{Exp}_f \circ I_f)(u).
\]

Since, \( \forall x \in M \), the transition maps

\[
\Xi^{f,g}(u)(x) = (I_g^{-1} \circ (\text{exp}_{g(x)})^{-1} \circ \text{exp}_{f(x)} \circ I_f)(u(x))
\]
are smooth, \((V_f, \Xi^f, U_f)_{f \in C^\infty(M,N)}\) is a smooth atlas on \(C^\infty_b(M,N)\). Moreover, let \(w \in C^\infty_0(M,E)\), setting \(v = (\Xi^f)^{-1}(u)\), the evaluation of the differential at \(x \in M\) reads:

\[
D_u \Xi^f g (w)(x) = (I_g \circ D_v (\exp_g(x))^{-1} \circ D_u (\exp_f(x) \circ I_f))(w(x)).
\]

Hence, for \(u \in C^\infty\), \(D_u \Xi^f g\) is a multiplication operator acting on smooth sections of \(E\) for any isomorphism \(I_f\) and \(I_g\) we can choose. Since \(I_f\) and \(I_g\) are fixed, the family \(u \mapsto D_u \Xi^f g\) is a smooth family of 0-order differential operators; this construction is described carefully in [10]. Now, let \(f \in \text{Emb}(M,N)\) and let us consider the map

\[
\Phi^U_f : (f, v, X) \in T^*_U (1-p)TU \oplus pTU \mapsto \Xi^f(v) \exp_{\text{Diff}(M)}(X) \in \text{Emb}(M,N).
\]

This map gives a local (fiberwise) trivialization of the principal bundles \(\text{Emb}(M,N) \to B(M,N)\) following [14, 21, 33], and we see that the changes of local trivializations have \(\text{Aut}(N)\) as a structure group. We have also to remark that, since we are working all along the construction with the exponential map of the Riemannian metric of \(TN\), with Riemannian connections, and hence with \(\text{so}(\text{dim}(N))\)–valued connections, we can restrict the “minimal” structure group to a group of “orthogonal automorphisms” \(O_{\text{Diff}}\) that can be identified as a subgroup if \(\text{Aut}(E)\) through the restricted short exact sequence:

\[
0 \to O(N_f) \to O_{\text{Diff}} \to \text{Diff}(M) \to 0,
\]

where \(O(N_f)\) is the orthogonal gauge group of \(N_f\).

4.1. Oriented embeddings. If \(M\) is oriented, we note by \(\text{Diff}^+(M)\) the group of orientation preserving diffeomorphisms and we have the following trivial lemma:

**Lemma 4.1.**

\[
\frac{\text{Diff}(M)}{\text{Diff}^+(M)} = \mathbb{Z}_2.
\]

Then, defining

\[
B^+(M,N) = \frac{\text{Emb}(M,N)}{\text{Diff}^+(M)}
\]

we get:

**Proposition 4.2.** \(B^+(M,N)\) is a 2-cover of \(B(M,N)\).

4.2. Almost complex structure on oriented knots. Here, we consider the base manifold \(B^+(S^1, N) = \text{Emb}(S^1, N)/\text{Diff}^+(S^1)\) of oriented knots. In this example, the tangent space \(TB^+(S^1, N)\) is obviously a vector bundle with structure group \(FCl^0_{\text{Diff}^+}(S^1, N)\). Let us consider the operator

\[
J = i \epsilon(D)
\]

where \(D = -i \nabla\). We get that \(J^2 = -1d\), so that \(J\) is an almost complex structure of \(TB^+(S^1, N)\).
5. Chern-Weil forms on principal bundle of embeddings and homotopy invariants

5.1. Chern forms in infinite dimensional setting. Let $P$ be a principal bundle, of basis $M$ and with structure group $G$. Let $\mathfrak{g}$ be the Lie algebra of $G$. Recall that $G$ acts on $P$, and also on $P \times \mathfrak{g}$ by the action $((p, v), g) \in (P \times \mathfrak{g}) \times G \mapsto (p, g, Ad_{g^{-1}}(v)) \in (P \times \mathfrak{g})$. Let $AdP = P \times Adg = (P \times \mathfrak{g})/G$ be the adjoint bundle of $P$, of basis $M$ and of typical fiber $\mathfrak{g}^k$. We do not know how to generalize the classical definition of Chern forms. However, in the case of infinite dimensional matrix groups $Gl_n$ with Lie algebra $\mathfrak{gl}_n$, the set $\mathfrak{pol}(P)$ is generated by the polynomials $A \in \mathfrak{gl}_n \mapsto \text{tr}(A^k)$, for $k = 0, ..., n$. This leads to classical definition of Chern forms. In the case of infinite dimensional structure groups, most situations are still unknown and we do not know how to define a set of generators for $\mathfrak{pol}(P)$.

**Definition 5.1.** Let $k$ in $\mathbb{N}^*$. We define $\mathfrak{pol}^k(P)$, the set of smooth maps $Ad^kP \to \mathbb{C}$ that are $k$-linear and symmetric on each fiber, equivalently as the set of smooth maps $P \times \mathfrak{g}^k \to \mathbb{C}$ that are $k$-linear symmetric in the second variable and $G$-invariants with respect to the natural coadjoint action of $G$ on $\mathfrak{g}^k$.

Let $C(P)$ be the set of connections on $P$. For any $\theta \in C(P)$, we denote by $F(\theta)$ its curvature and $\nabla^\theta$ (or $\nabla$ when it carries no ambiguity) its covariant derivation. Given an algebra $A$, in this section, we study the maps, for $k \in \mathbb{N}^*$,

\begin{align}
Ch : C(P) \times \mathfrak{pol}^k(P) & \to \Omega^{2k}(M, \mathbb{C}) \\
(\theta, f) & \mapsto \text{Alt}(f(F(\theta), ..., F(\theta)))
\end{align}

where $\text{Alt}$ denotes the skew-symmetric part of the form. Notice that, in the case of the finite dimensional matrix groups $Gl_n$ with Lie algebra $\mathfrak{gl}_n$, the set $\mathfrak{pol}(P)$ is generated by the polynomials $A \in \mathfrak{gl}_n \mapsto \text{tr}(A^k)$, for $k = 0, ..., n$. This leads to classical definition of Chern forms. However, in the case of infinite dimensional structure groups, most situations are still unknown and we do not know how to define a set of generators for $\mathfrak{pol}(P)$.

**Lemma 5.2.** Let $f \in \mathfrak{pol}^k(P)$. Then

\[ f([a_1, v], a_2, ..., a_k) + f(a_1, [a_2, v], ..., a_k) + \]

\[ ... + f(a_1, a_2, ..., [a_k, v]) = 0. \]

**Proof.** Let us notice first that $f$ is symmetric. Let $v \in \mathfrak{g}$, and $c_t$ a path in $G$ such that $\left\{ \frac{d}{dt}c_t \right\}_{t=0} = v$. Let $a_1, ..., a_k \in \mathfrak{g}^k$.

\[ \left\{ \frac{d}{dt} \{ f(ad_{c_t^{-1}}a_1, ..., ad_{c_t^{-1}}a_k) \} \right\}_{t=0} = f([a_1, v], a_2, ..., a_k) + f(a_1, [a_2, v], ..., a_k) + \]

\[ ... + f(a_1, a_2, ..., [a_k, v]) \]

Since $f$ in $G$-invariant, we get

\[ f([a_1, v], a_2, ..., a_k) + f(a_1, [a_2, v], ..., a_k) + \]

\[ ... + f(a_1, a_2, ..., [a_k, v]) = 0. \]

**Lemma 5.3.** Let $f \in \mathfrak{pol}^k(P)$ such that $f$, as a smooth map $P \times \mathfrak{g}^k \to \mathbb{C}$, satisfies $d^M f = 0$ on a system of local trivializations of $P$. Then, the map

\[ Ch^f : \theta \in C(P) \mapsto Ch^f(\theta) = Ch(\theta, f) \in \Omega^*(P, \mathbb{C}) \]

takes values into closed forms on $P$. Moreover,
(i) it is vanishing on vertical vectors and defines a closed form on $M$.
(ii) the cohomology class of this form does not depend on the choice of the chosen connexion $\theta$ on $P$.

Proof. The proof runs as in the finite dimensional case, see e.g. [22]. First, it is vanishing on vertical vectors and $G$-invariant because the curvature of a connexion vanishes on vertical forms and is $G$-covariant for the coadjoint action. Let us now fix $f \in \mathfrak{pol}^k(P)$. We compute $df(F(\theta),...,F(\theta))$. We notice first that it vanishes on vertical vectors trivially. Let us fix $Y_1^h,\ldots,Y_{2k}^h,X^h$ horizontal vectors on $P$ at $p \in P$. On a local trivialization of $P$ around $p$, these vectors read as

$$
Y_1^h = Y_1 - \tilde{\theta}(Y_1) \\
(...)
Y_{2k}^h = Y_{2k} - \tilde{\theta}(Y_{2k})
X^h = X - \tilde{\theta}(X)
$$

where $\tilde{\theta}$ stands here for the expression of $\theta$ in the local trivialization, and $Y_1,\ldots,Y_{2k},X$ tangent vectors on $M$ at $\pi(p) \in M$. We extend these vector fields on a neighborhood of $p$-
- by the action of $G$ in the vertical directions
- setting the vector fields constant on $U \times p$, where $U$ is a local chart on $M$ around $\pi(p)$.

Then, we have

$$f(F(\theta),...,F(\theta))(Y_1^h,\ldots,Y_{2k}^h) = f(F(\theta),...,F(\theta))(Y_1,...,Y_{2k})$$

since $F(\theta)$ is vanishing on vertical vectors.

Then, on a local trivialization with the notations defined before (the sign $\text{Alt}$ is omitted for easier reading), and writing $d^M$ for the differential of forms on any open subset of $M$,

$$d^M f(F(\tilde{\theta}),...,F(\tilde{\theta})) = \sum_{i=1}^k f(d^M F(\tilde{\theta}),F(\tilde{\theta}),...,F(\tilde{\theta})) + f(F(\tilde{\theta}),d^M F(\tilde{\theta}),...,F(\tilde{\theta})) + ... + f(F(\tilde{\theta}),F(\tilde{\theta}),...,d^M F(\tilde{\theta}))$$

and then, using Lemma 5.2

$$\nabla^\theta f(F(\tilde{\theta}),...,F(\tilde{\theta})) = \sum_{i=1}^k f(\nabla^\theta F(\tilde{\theta}),F(\tilde{\theta}),...,F(\tilde{\theta})) + f(F(\tilde{\theta}),\nabla^\theta F(\tilde{\theta}),...,F(\tilde{\theta})) + ... + f(F(\tilde{\theta}),F(\tilde{\theta}),...,\nabla^\theta F(\tilde{\theta}))$$

Then, by Bianchi identity, we get that

$$d^M Ch(f,\theta) = \nabla^\theta Ch(f,\theta) = 0$$

This proves (i) Then, following e.g. [22], if $\theta$ and $\theta'$ are connections, fix $\mu = \theta' - \theta$ and $\theta_t = \theta + t\nu$ for $t \in [0;1]$. We have

$$\frac{dF(\theta_t)}{dt} = \nabla^{\theta_t} \mu$$
Moreover, $\mu$ is $G$-invariant and vanishes on vertical vectors. Thus,
\[
\frac{d\text{Ch}(f, \theta_t)}{dt} = k f(F(\theta_t), ..., F(\theta_t), \nabla^\theta_1 \mu) = kd^M(f(F(\theta_t), ..., F(\theta_t), \mu)).
\]

Integrating in the $t$-variable, we get
\[
\text{Ch}(f, \theta_0) - \text{Ch}(f, \theta_1) = -kd^M \int_0^1 f(F(\theta_t), ..., F(\theta_t), \mu) dt.
\]

Even if these computations are local, the two sides are global objects and do not depend on the chosen trivialization, which ends the proof. \hfill $\square$

**Important remark.** The condition $d^M f = 0$ is a **local** condition, checked in an (adequate) system of trivializations of the principal bundle, because it has to be checked on the vector bundle $\text{Ad}(P)^{\times k}$. This is in particular the case when we can find a 0-curvature connection $\theta$ on $P$ such that
\[
[\nabla^\theta, f] = 0
\]

In that case, since the structure group $G$ is regular, we can find a system of local trivializations of $P$ defined by $\theta$ and such that, on any local trivialization, $\nabla^\theta = d^M$ (see e.g. [21], [25] for the technical tools that are necessary for this).

This technical remark can appear rather unsatisfactory first because it restricts the ability of application of the previous lemma, secondly because we need have a local (and rather unelegant) condition. This is why we give the following theorem, from Lemma 5.3

**Theorem 5.4.** Let $f \in \mathfrak{pol}(P)$ for which there exists $\theta \in \mathcal{C}(P)$ such that $[\nabla^\theta, f] = 0$. We shall note this set of polynomials by $\mathfrak{pol}_{\text{reg}}(P)$. Then, the map
\[
\text{Ch}^f : \theta \in \mathcal{C}(P) \mapsto \text{Ch}^f(\theta) = \text{Ch}(\theta, f) \in \Omega^*(P, \mathbb{C})
\]
takes values into closed forms on $P$. Moreover,

(i) it is vanishing on vertical vectors and defines a closed form on $M$.

(ii) the cohomology class of this form does not depend on the choice of the chosen connexion $\theta$ on $P$.

Moreover, $\forall (\theta, f) \in \mathcal{C}(P) \times \mathfrak{pol}_{\text{reg}}(P), [\nabla^\theta, f] = 0$.

**Proof.** Let $f \in \mathfrak{pol}_{\text{reg}}(P)$ and let $\theta \in \mathcal{C}(P)$ such that $[\nabla^\theta, f] = 0$. Let $\theta' \in \mathcal{J}(P)$ and let $\nu = \theta' - \theta \in \Omega^1(M, \mathfrak{g})$. Let $(\alpha_1, ..., \alpha_k) \in (\Omega^2(M, \mathfrak{g}))^k$.

\[
[\nabla^\theta', f](\alpha_1, ..., \alpha_k) = [\nabla^\theta, f](\alpha_1, ..., \alpha_k) + f([\alpha_1, \nu], ..., \alpha_n) + ... + f(\alpha_1, ..., [\alpha_n, \nu])
\]

\[
= f([\alpha_1, \nu], ..., \alpha_n) + ... + f(\alpha_1, ..., [\alpha_n, \nu]) = 0.
\]

Then, $\forall (\theta, f) \in \mathcal{C}(P) \times \mathfrak{pol}_{\text{reg}}(P), [\nabla^\theta, f] = 0$. By the way, $\forall \theta' \in \mathcal{C}(P)$,
\[
d^M f(\alpha_1, ..., \alpha_k) = f(\nabla^\theta\alpha_1, ..., \alpha_k) + ... + f(\alpha_1, ..., \nabla^\theta\alpha_k).
\]

Applying this to $\alpha_1 = ... = \alpha_k = F(\theta')$, we get
\[
d\text{Ch}(f, \theta') = f(\nabla^\theta' F(\theta'), ..., F(\theta')) + ... + f(F(\theta'), ..., \nabla^\theta' F(\theta')) = 0
\]
by Bianchi identity. Thus \(Ch(f, \theta')\) is closed. Then, mimicking the end of the proof of Lemma 5.3 we get that the difference \(Ch(f, \theta) - Ch(f, \theta')\) is an exact form, which ends the proof.

**Proposition 5.5.** Let \(\phi : \mathfrak{g}^k \to \mathbb{C}\) be a \(k\)-linear, symmetric, \(Ad\)-invariant form. Let \(f : P \times \mathfrak{g}^k \to \mathbb{C}\) be the map induced by \(\phi\) by the formula: \(f(x, g) = \phi(g)\). Then \(f \in \mathfrak{Pol}_{reg}\).

**Proof.** Obviously, \(f \in \mathfrak{Pol}\). Let \(\varphi : U \times G \to P\) and \(\varphi' : U \times G \to P\) be a local trivialisations of \(P\), where \(U\) is an open subset of \(M\). Then there exists a smooth map \(g : U \to G\) such that \(\varphi'(x, e_G) = \varphi(x, e_G)g(x)\). Then we remark that \(\varphi^*f = \varphi'^*f\) is a constant map on horizontal slices since \(\phi\) is \(Ad\)-invariant. Moreover, since \(\varphi^*f\) is a constant (polynomial-valued) map on \(\varphi(x, e_G)\) we get that \([\nabla^\theta, f] = 0\) for the (flat) connection \(\theta\) such that \(T\varphi(x, e_G)\) spans the horizontal bundle over \(U\). \(\square\)

5.2. **Application to \(Emb(M, N)\).** Mimicking the approach of [27], the cohomology classes of Chern-Weil forms should give rise to homotopy invariants. Applying Theorem 5.3 we get:

**Theorem 5.6.** The Chern-Weil forms \(Ch^f\) is a \(H^*(B(M, N))\)-valued invariant of the homotopy class of an embedding, \(\forall k \in \mathbb{N}^*\).

When \(M = S^1\), \(Emb(S^1, N)\) is the space of (parametrized) smooth knots on \(N\), and \(B(S^1, N)\) is the space of non parametrized knots. Its connected components are the homotopy classes of the knots, through classical results of differential topology, see e.g. [15]. We now apply the material of the previous section to manifolds of embeddings. For this, we can define invariant polynomials of the type of those obtained in [27] (for mapping spaces) by a field of linear functionals \(\lambda\) with “good properties” that ensures that

\[
A \mapsto \lambda(A^k) \in \mathfrak{Pol}_{reg}^k.
\]

This approach is a straightforward generalization of the description of Chern-Weil forms on finite dimensional principal bundles where polynomials are generated by functionals of the type \(A \mapsto \text{tr}(A^k)\) (\(\text{tr}\) is the classical trace) but as we guess that we can consider other classes of polynomials for spaces of embeddings. In this paper, let us describe how to replace the classical trace of matrices \(\text{tr}\) by a renormalized trace \(\text{tr}^Q\). In the most general case, it is not so easy to define a family of weights \(f \in Emb(M, N) \mapsto Q_f\) which satisfy the good properties. Indeed, we have two examples of constructions which match the necessary assumptions for \(\mathfrak{Pol}_{reg}\) when \(M = S^1\), and the first one is derived from the following example:

**Knot invariant through Kontsevich and Vishik trace.** The Kontsevich and Vishik trace is a renormalized trace for which \(\text{tr}^Q([A, B]) = 0\) for each differential operator \(A, B\) and does not depend on the weight chosen in the odd class. For example, one can choose \(Q = Id + \nabla^*\nabla\), where \(\nabla\) is a connection induced on \(\mathcal{N}_f\) by the Riemannian metric, as described in [27]. It is an order 2 injective elliptic differential operator (in the odd class), and the coadjoint action of \(Aut(\mathcal{N}_f)\) will give rise to another order 2 injective elliptic differential operator [13]. When \(Q = Id + \nabla^*\nabla\), this only changes \(\nabla\) into another connection on \(E\). Thus, setting

\[
\phi(A, ..., A) = \text{tr}^Q(A^k),
\]
we have
\[ f \in \mathcal{P}ol_{\text{reg}}. \]

Let us now consider a connected component of \( B(M, N) \), i.e. a homotopy class of an embedding among the space of embeddings. We apply now the construction to \( M = S^1 \). The polynomial
\[ \phi : A \mapsto \text{tr}^Q(A^k) \]
is \( \text{Diff}(S^1) \)-invariant, and gives rise to an invariant of non oriented knots, i.e. a Chern form on the base manifold
\[ B(S^1, N) = \text{Emb}(S^1, N)/\text{Diff}(S^1) \]
by theorem [5,6]. This approach can be extended to invariants of embeddings, replacing \( S^1 \) by another odd-dimensional manifold.

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