ON OKOUNKOV’S CONJECTURE CONNECTING HILBERT SCHEMES OF POINTS AND MULTIPLE $q$-ZETA VALUES

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Abstract. We compute the generating series for the intersection pairings between the total Chern classes of the tangent bundles of the Hilbert schemes of points on a smooth projective surface and the Chern characters of tautological bundles over these Hilbert schemes. Modulo the lower weight term, we verify Okounkov’s conjecture [Oko] connecting these Hilbert schemes and multiple $q$-zeta values. In addition, this conjecture is completely proved when the surface is abelian. We also determine some universal constants in the sense of Boissière and Nieper-Wisskirchen [Boi, BN] regarding the total Chern classes of the tangent bundles of these Hilbert schemes. The main approach of this paper is to use the set-up of Carlsson and Okounkov outlined in [Car, CO] and the structure of the Chern character operators proved in [LQW2].

1. Introduction

In the region $\text{Re } s > 1$, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. $$

The integers $s > 1$ give rise to a sequence of special values of the Riemann zeta function. Multiple zeta values are series of the form

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > \cdots > n_k} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

where $n_1, \ldots, n_k$ denote positive integers, and $s_1, \ldots, s_k$ are positive integers with $s_1 > 1$. Multiple $q$-zeta values are $q$-deformations of $\zeta(s_1, \ldots, s_k)$, which may take different forms (see [Bra1, Bra2, OT, Zud] for details). In [Oko], Okounkov proposed several interesting conjectures regarding multiple $q$-zeta values and Hilbert schemes of points. Motivated by these conjectures, we compute in this paper the generating series for the intersection pairings between the total Chern classes of the tangent bundles of the Hilbert schemes of points on a smooth projective surface and the Chern characters of tautological bundles over these Hilbert schemes.

2000 Mathematics Subject Classification. Primary 14C05; Secondary 11B65, 17B69.

Key words and phrases. Hilbert schemes of points on surfaces, multiple $q$-zeta values.

$^1$Partially supported by a grant from the Simons Foundation.

$^2$Partially supported by the Fundamental Research Funds for the Central Universities (No. 20720140526).
Let $X$ be a smooth projective complex surface, and let $X^{[n]}$ be the Hilbert scheme of $n$ points in $X$. A line bundle $L$ on $X$ induces a tautological rank-$n$ bundle $L^{[n]}$ on $X^{[n]}$. Let $\text{ch}_k(L^{[n]})$ be the $k$-th Chern character of $L^{[n]}$. Following Okounkov [Oko], we introduce the two generating series:

\[
\langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle = \sum_{n \geq 0} q^n \int_{X^{[n]}} \text{ch}_{k_1}(L_1^{[n]}) \cdots \text{ch}_{k_N}(L_N^{[n]}) \cdot c(T_X^{[n]}) \quad (1.1)
\]

\[
\langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle' = \langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle / \langle 1 \rangle = (q; q)^{\chi(X)} \cdot \langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle \quad (1.2)
\]

where $0 < q < 1$, $c(T_X^{[n]})$ is the total Chern class of the tangent bundle $T_X^{[n]}$, and the Euler class, and $(a; q)_n = \prod_{i=0}^n (1 - aq^i)$. In [Car], for $X = \mathbb{C}^2$ with a suitable $\mathbb{C}^*$-action and $L = O_X$, the series $\langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle$ in the equivariant setting has been studied. In [Oko], Okounkov proposed the following conjecture.

**Conjecture 1.1.** $\langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle'$ is a multiple $q$-zeta value of weight $\sum_{i=1}^N (k_i + 2)$.

In this paper, we study Conjecture 1.1. To state our result, we introduce some definitions. For integers $n_i > 0$, $w_i > 0$ and $p_i \geq 0$ with $1 \leq i \leq v$, define the weight of $\prod_{i=1}^v q_i^{n_i} p_i^{w_i}$ to be $\sum_{i=1}^v n_i w_i$. For $k \geq 0$ and $\alpha \in H^*(X)$, define $\Theta^k_\alpha(q, z)$ to be the weight-$(k + 2)$ multiple $q$-zeta value (with an additional variable $z$ inserted):

\[
\Theta^k_\alpha(q, z) = \sum_{a, s_1, \ldots, a, b, t_1, \ldots, t_k \geq 1} \sum_{s_1 + \sum_j b_j = 1} \prod_{i=1}^a (1 - q^{s_i})^{-1} \cdot \prod_{j=1}^{b} \frac{1}{f_j} \cdot z^{-m_j t_j} \cdot \prod_{m_1 \geq \cdots \geq m_k} \prod_{j=1}^{b} \frac{z^{-m_j t_j}}{(1 - q^{m_j})^{f_j}},
\]

where $K_X$ and $1_X$ are the canonical class and fundamental class of $X$ respectively. Let $\text{Coeff}_{\xi^0_1 \cdots \xi^0_N} (\cdot)$ denote the coefficient of $\xi^0_1 \cdots \xi^0_N$, $L$ also denote the first Chern class of the line bundle $L$, and $e_X$ be the Euler class of $X$.

**Theorem 1.2.** Let $L_1, \ldots, L_N$ be line bundles over $X$, and $k_1, \ldots, k_N \geq 0$. Then,

\[
\langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle' = \text{Coeff}_{\xi^0_1 \cdots \xi^0_N} \left( \prod_{i=1}^N \Theta^{k_i}_{\xi^1_i}(q, z_i) \right) + W, \quad (1.3)
\]

and the lower weight term $W$ is an infinite linear combination of the expressions:

\[
\prod_{i=1}^u \langle K_{X_i}^{r_i} L_{X_i}^{\ell_i} \rangle \cdot \prod_{i=1}^v \frac{q^{n_i w_i p_i}}{(1 - q^{n_i})^{w_i}}
\]

where $\sum_{i=1}^u w_i < \sum_{i=1}^N (k_i + 2)$, and the integers $u$, $v$, $r_i$, $\ell_{i,j} \geq 0$, $n_i > 0$, $w_i > 0$, $p_i \in \{0, 1\}$ depend only on $k_1, \ldots, k_N$. Furthermore, all the coefficients of this linear combination are independent of $q, L_1, \ldots, L_N$ and $X$. 

Theorem 1.2 proves Conjecture 1.1 modulo the lower weight term $W$. Note that the leading term $\text{Coeff}_{z_1^0 \cdots z_N^0} \left( \prod_{i=1}^{N} \Theta_{k_i}^1(q, z_i) \right)$ in $\langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle'$ has weight $\sum_{i=1}^{N} (k_i + 2)$, and is a multiple of $(K_X, K_X)^N$ whose coefficient depends only on $k_1, \ldots, k_N$ and is independent of the line bundles $L_1, \ldots, L_N$ and the surface $X$.

In general, it is unclear how to organize the lower weight term $W$ in Theorem 1.2 into multiple $q$-zeta values. On the other hand, we have the following result which together with Theorem 1.2 verifies Conjecture 1.1 when $X$ is an abelian surface.

**Theorem 1.3.** Let $L_1, \ldots, L_N$ be line bundles over an abelian surface $X$, and $k_1, \ldots, k_N \geq 0$. Then, the lower weight term $W$ in (1.3) is a linear combination of the coefficients of $z_1^0 \cdots z_N^0$ in some multiple $q$-zeta values (with additional variables $z_1, \ldots, z_N$ inserted) of weights $< \sum_{i=1}^{N} (k_i + 2)$. Moreover, the coefficients in this linear combination are independent of $q$.

We remark that some of the multiple $q$-zeta values mentioned in Theorem 1.3 are in the generalized sense, i.e., in the following form:

$$\sum_{n_1, \ldots, n_\ell \geq 0} \ell \prod_{i=1}^{\ell} \frac{(-n_i)^{w_i} q^{n_i} p_i f_i(z_1, \ldots, z_N)^{n_i}}{(1 - q^{n_i})^{w_i}}$$

where $0 \leq p_i \leq w_1$, and each $f_i(z_1, \ldots, z_N)$ is a monomial of $z_1^{\pm 1}, \ldots, z_N^{\pm 1}$. We refer to (1.34) in the proof of Theorem 1.10 for more details. As indicated in [Oko], the factors $(-n_i)^{w_i}$ in the above expression may be related to the operator $\frac{d}{dq}$.

The main idea in our proofs of Theorem 1.2 and Theorem 1.3 is to use the structure of the Chern character operators proved in [LQW2] and the set-up of Carlsson and Okounkov in [Car, CO]. Let $G_k(\alpha, n)$ be the degree-$(2k + |\alpha|)$ component of (2.1), and $\mathcal{G}_k(\alpha)$ be the Chern character operator acting on the Fock space $\mathbb{H}_X = \bigoplus_{n=0}^{\infty} H^*(X[n])$ via cup product by $\bigoplus_{n=0}^{\infty} G_k(\alpha, n)$. Then,

$$\text{ch}_k(L[n]) = G_k(1_X, n) + G_{k-1}(L, n) + G_{k-2}(L^2/2, n)$$

by the Grothendieck-Riemann-Roch Theorem. So $\langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle'$ is reduced to the series $F_{k_1, \ldots, k_N}^{\alpha_1, \ldots, \alpha_N}(q)$ defined by (2.2). Let $\mathcal{L}_1$ be the trivial line bundle on $X$ with a scaling action of $\mathbb{C}^*$ of character $1$\footnote{Throughout the paper, we implicitly set $t = 1$ for the generator $t$ of the equivariant cohomology $H^*_C(pt)$ of a point.} Using the set-up in [Car, CO], we get

$$F_{k_1, \ldots, k_N}^{\alpha_1, \ldots, \alpha_N}(q) = \text{Tr} q^\delta W(\mathcal{L}_1, z) \prod_{i=1}^{N} \mathcal{G}_k(\alpha_i)$$

where $W(\mathcal{L}_1, z)$ is the vertex operator constructed in [Car, CO], and $\delta$ is the number-of-points operator (i.e., $\delta|_{H^*(X[n])} = n \text{Id}$). The structure of the Chern character operators $\mathcal{G}_k(\alpha_i)$ is given by Theorem 2.3 which is proved in [LQW2].
It implies that the computation of $F^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)$ can be further reduced to

$$\text{Tr} \, q^0 \, W(\mathcal{L}_1, z) \prod_{i=1}^{N} \frac{a_{\lambda^{(i)}(\alpha_i)}}{\lambda^{(i)!}} \tag{1.4}$$

where $\lambda^{(i)}$ denotes a generalized partition which may also contain negative parts, and $\lambda^{(i)!}$ and $a_{\lambda^{(i)}(\alpha_i)}$ are defined in Definition 2.2 (ii). The trace (1.4) is investigated via some standard but rather lengthy calculations.

As an application, our results enable us to determine some of the universal constants in $\sum_{n} c(T_{X[n]}(i)) q^n$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ be the Catalan number, and $C_1 = \frac{(2i)}{i+1}$ be the Catalan number, and $\sigma_1(i) = \sum_{j|i} j$. By [Boi, BN], there exist unique rational numbers $b_\mu, f_\mu, g_\mu, h_\mu$ depending only on the (usual) partitions $\mu$ such that $\sum_{n} c(T_{X[n]}(i)) q^n$ is equal to

$$\exp \left( \sum_{\mu} q^{\mu} \left( b_\mu a_{-\mu}(1_X) + f_\mu a_{-\mu}(e_X) + g_\mu a_{-\mu}(K_X) + h_\mu a_{-\mu}(K_X^2) \right) \right) |0);$$

in addition, $b_{2i} = 0, b_{2i-1} = (-1)^{i-1} C_{i-1}/(2i-1), b_{(1^i)} = f_{(1^i)} = g_{(1^i)} = h_{(1^i)} = \sigma_1(i)/i$, and $h_{(1^i)} = 0$. In Theorem 6.3 we determine $b_{(i,1^j)}$ for $i \geq 2$ and $j \geq 0$.

The paper is organized as follows. In Sect. 2 we review the Heisenberg operators of Grojnowski and Nakajima, and the structure of the Chern character operators. In Sect. 3 we recall the vertex operator of Carlsson and Okounkov. In Sect. 4 we compute the trace (1.4). Theorem 1.2 and Theorem 1.3 are proved in Sect. 5. In Sect. 6 we determine the universal constants $b_{(i,1^j)}$ for $i \geq 2$ and $j \geq 0$.

**Convention.** All the (co)homology groups are in $\mathbb{C}$-coefficients unless otherwise specified. For $\alpha, \beta \in H^*(Y)$ where $Y$ is a smooth projective variety, $\alpha \beta$ and $\alpha \cdot \beta$ denote the cup product $\alpha \cup \beta$, and $\langle \alpha, \beta \rangle$ denotes $\int_Y \alpha \beta$.

**Acknowledgment.** The authors thank Professors Dan Edidin, Wei-ping Li and Weiqiang Wang for stimulating discussions and valuable helps. The second author also thanks the Mathematics Department of the University of Missouri for its hospitality during his visit in August 2015 as a Miller’s Scholar.

2. Basics on Hilbert schemes of points on surfaces

In this section, we will review some basic aspects of the Hilbert schemes of points on surfaces. We will recall the definition of the Heisenberg operators of Grojnowski and Nakajima, and the structure of the Chern character operators.

Let $X$ be a smooth projective complex surface, and $X^{[n]}$ be the Hilbert scheme of $n$ points in $X$. An element in $X^{[n]}$ is represented by a length-$n$ 0-dimensional closed subscheme $\xi$ of $X$. For $\xi \in X^{[n]}$, let $I_\xi$ be the corresponding sheaf of ideals. It is well known that $X^{[n]}$ is smooth. Define the universal codimension-2 subscheme:

$$\mathcal{Z}_n = \{ (\xi, x) \in X^{[n]} \times X \, | \, x \in \text{Supp}(\xi) \} \subset X^{[n]} \times X.$$
Denote by $p_1$ and $p_2$ the projections of $X^{[n]} \times X$ to $X^{[n]}$ and $X$ respectively. Let

$$\mathbb{H}_X = \bigoplus_{n=0}^{\infty} H^*(X^{[n]})$$

be the direct sum of total cohomology groups of the Hilbert schemes $X^{[n]}$. For $m \geq 0$ and $n > 0$, let $Q^{[m,m]} = \emptyset$ and define $Q^{[m+n,m]}$ to be the closed subset:

$$\{(\xi, x, \eta) \in X^{[m+n]} \times X \times X^{[m]} | \xi \supset \eta \text{ and } \text{Supp}(I_I/I_\xi) = \{x\}\}.$$

We recall Nakajima’s definition of the Heisenberg operators [Nak]. Let $\alpha \in H^*(X)$. Set $a_0(\alpha) = 0$. For $n > 0$, the operator $a_{-n}(\alpha) \in \text{End}(\mathbb{H}_X)$ is defined by

$$a_{-n}(\alpha)(a) = \tilde{p}_1 \cdot ([Q^{[m+n,m]}] \cdot \tilde{\rho}^* \alpha \cdot \tilde{p}_2 a)$$

for $a \in H^*(X^{[m]})$, where $\tilde{p}_1, \tilde{\rho}, \tilde{p}_2$ are the projections of $X^{[m+n]} \times X \times X^{[m]}$ to $X^{[m+n]}, X, X^{[m]}$ respectively. Define $a_n(\alpha) \in \text{End}(\mathbb{H}_X)$ to be $(-1)^n$ times the operator obtained from the definition of $a_{-n}(\alpha)$ by switching the roles of $\tilde{p}_1$ and $\tilde{p}_2$. We often refer to $a_{-n}(\alpha)$ (resp. $a_n(\alpha)$) as the creation (resp. annihilation) operator. The following is from [Nak, Gro]. Our convention of the sign follows [LQW2].

**Theorem 2.1.** The operators $a_n(\alpha)$ satisfy the commutation relation:

$$[a_m(\alpha), a_n(\beta)] = -m \delta_{m,-n} \cdot \langle \alpha, \beta \rangle \cdot \text{Id}_{\mathbb{H}_X}.$$

The space $\mathbb{H}_X$ is an irreducible module over the Heisenberg algebra generated by the operators $a_n(\alpha)$ with a highest weight vector $|0\rangle = 1 \in H^0(X^{[0]}) \cong \mathbb{C}$.

The Lie bracket in the above theorem is understood in the super sense according to the parity of the cohomology degrees of the cohomology classes involved. It follows from Theorem [21] that the space $\mathbb{H}_X$ is linearly spanned by all the Heisenberg monomials $a_{n_1}(\alpha_1) \cdots a_{n_k}(\alpha_k) |0\rangle$ where $k \geq 0$ and $n_1, \ldots, n_k < 0$.

**Definition 2.2.** (i) Let $\alpha \in H^*(X)$ and $k \geq 1$. Define $\tau_{k*} : H^*(X) \to H^*(X^k)$ to be the linear map induced by the diagonal embedding $\tau_k : X \to X^k$, and

$$(a_{m_1} \cdots a_{m_k})(\alpha) = a_{m_1} \cdots a_{m_k}(\tau_{k*}\alpha) = \sum_j a_{m_1}(\alpha_{j,1}) \cdots a_{m_k}(\alpha_{j,k})$$

when $\tau_{k*}\alpha = \sum_j \alpha_{j,1} \otimes \cdots \otimes \alpha_{j,k}$ via the Kînneth decomposition of $H^*(X^k)$.

(ii) Let $\lambda = (\cdots (-2)^{m-2}(-1)^{m-1}1^{m_1}2^{m_2} \cdots )$ be a generalized partition of the integer $n = \sum_i im_i$ whose part $i \in \mathbb{Z}$ has multiplicity $m_i$. Define $\ell(\lambda) = \sum_i m_i, |\lambda| = \sum_i im_i = n, s(\lambda) = \sum_i i^2 m_i, \lambda! = \prod_i m_i!,$ and

$$a_{\lambda}(\alpha) = \left(\prod_i (a_i)^{m_i}\right)(\alpha)$$

where the product $\prod_i (a_i)^{m_i}$ is understood to be $\cdots a_{-2} a_{-1} a_1 a_2 \cdots$.

The set of all generalized partitions is denoted by $\widehat{P}$.

(iii) A generalized partition becomes a partition in the usual sense if the multiplicity $m_i = 0$ for all $i < 0$. The set of all partitions is denoted by $P$. 


For \( n > 0 \) and a homogeneous class \( \alpha \in H^s(X) \), let \( |\alpha| = s \) if \( \alpha \in H^s(X) \), and let \( G_k(\alpha, n) \) be the homogeneous component in \( H^{|\alpha|+2k}(X^n) \) of
\[
G(\alpha, n) = p_1(\text{ch}(O_Z \cdot p_2^*\alpha \cdot p_2^*\text{td}(X))) \in H^s(X^n)
\] (2.1)
where \( \text{ch}(O_Z) \) denotes the Chern character of \( O_Z \) and \( \text{td}(X) \) denotes the Todd class. We extend the notion \( G_k(\alpha, n) \) linearly to an arbitrary class \( \alpha \in H^s(X) \), and set \( G(\alpha, 0) = 0 \). It was proved in \[LQW1\] that the cohomology ring of \( X^n \) is generated by the classes \( G_k(\alpha, n) \) where \( 0 \leq k < n \) and \( \alpha \) runs over a linear basis of \( H^s(X) \). The Chern character operator \( \mathfrak{S}_k(\alpha) \in \text{End}(H_X) \) is the operator acting on \( H^s(X^n) \) by the cup product with \( G_k(\alpha, n) \). The following is from \[LQW2\].

**Theorem 2.3.** Let \( k \geq 0 \) and \( \alpha \in H^s(X) \). Then, \( \mathfrak{S}_k(\alpha) \) is equal to
\[
- \sum_{\ell(\lambda) = k + 2, |\lambda| = 0} \frac{1}{\lambda!} a_\lambda(\alpha) + \sum_{\ell(\lambda) = k, |\lambda| = 0} \frac{s(\lambda) - 2}{24\lambda!} a_\lambda(e_X(\alpha))\]
\[
+ \sum_{\ell(\lambda) = k + 1, |\lambda| = 0} \frac{g_{1,\lambda}}{\lambda!} a_\lambda(K_X^2(\alpha)) + \sum_{\ell(\lambda) = k, |\lambda| = 0} \frac{g_{2,\lambda}}{\lambda!} a_\lambda(K_X(\alpha))
\]
where all the numbers \( g_{1,\lambda} \) and \( g_{2,\lambda} \) are independent of \( X \) and \( \alpha \).

For \( \alpha_1, \ldots, \alpha_N \in H^s(X) \) and integers \( k_1, \ldots, k_N \geq 0 \), define the series
\[
F_{k_1, \ldots, k_N}^{\alpha_1, \ldots, \alpha_N}(q) = \sum_n q^n \int_{X^n} \left( \prod_{i=1}^N G_{k_i}(\alpha_i, n) \right) c(T_X(n)).
\] (2.2)

In view of Götsche’s Theorem in \[Got\], we have \( F(q) = (q; q)_\infty^{-\chi(X)} \).

The following is from \[LQW3\] and will be used throughout the paper.

**Lemma 2.4.** Let \( k, s \geq 1 \), \( n_1, \ldots, n_k, m_1, \ldots, m_s \in \mathbb{Z} \), and \( \alpha, \beta \in H^s(X) \).

(i) The commutator \( [[(a_{n_1} \cdots a_{n_k})(\alpha), (a_{m_1} \cdots a_{m_s})(\beta)] \] is equal to
\[
- \sum_{l=1}^k \sum_{j=1}^s n_l \delta_{n_l - m_j} \cdot \left( \prod_{\ell=1}^{j-1} a_{m_{\ell}} \prod_{1 \leq u < k, u \neq \ell} a_{n_u} \prod_{\ell=j+1}^s a_{m_{\ell}} \right) (\alpha \beta).
\]

(ii) Let \( j \) satisfy \( 1 \leq j < k \). Then, \( (a_{n_1} \cdots a_{n_k})(\alpha) \) is equal to
\[
\left( \prod_{1 \leq s < j} a_{n_s} \cdot a_{n_{j+1}} \cdots a_{n_{j+s}} \prod_{j+1 < s \leq k} a_{n_s} \right) (\alpha) - n_j \delta_{n_j - n_{j+1}} \left( \prod_{1 \leq s < k} a_{n_s} \right) (e_X(\alpha)).
\]

3. The vertex operators of Carlsson and Okounkov

In this section, we will recall the vertex operators constructed in \[CO\] and \[Car\], and use them to rewrite the generating series \( F_{k_1, \ldots, k_N}^{\alpha_1, \ldots, \alpha_N}(q) \) defined in (2.2).

Let \( L \) be a line bundle over the smooth projective surface \( X \). Let \( E_L \) be the virtual vector bundle on \( X^k \times X^\ell \) whose fiber at \( (I, J) \in X^k \times X^\ell \) is given by
\[
E_L|_{(I,J)} = \chi(O, L) - \chi(J, I \otimes L).
\]
Let $\mathcal{L}_m$ be the trivial line bundle on $X$ with a scaling action of $\mathbb{C}^*$ of character $m$, and let $\Delta_n$ be the diagonal in $X^{[n]} \times X^{[n]}$. Then,

$$E_{\mathcal{L}_m}|_{\Delta_n} = T_{X^{[n]},m}, \quad (3.1)$$

the tangent bundle $T_{X^{[n]}}$ with a scaling action of $\mathbb{C}^*$ of character $m$. By abusing notations, we also use $L$ to denote its first Chern class. Put

$$\Gamma_{\pm}(L, z) = \exp \left( \sum_{n>0} \frac{z^n}{n} a_{\pm n}(L) \right). \quad (3.2)$$

**Remark 3.1.** There is a sign difference between the Heisenberg commutation relations used in [Car] (see p.3 there) and in this paper (see Theorem 2.1). So for $n > 0$, our Heisenberg operators $a_{-n}(L)$ and $a_{n}(-L)$ are equal to the Heisenberg operators $a_{-n}(L)$ and $a_{n}(L)$ in [Car]. Accordingly, our operators $\Gamma_{-}(L, z)$ and $\Gamma_{+}(L, z)$ are equal to the operators $\Gamma_{-}(L, z)$ and $\Gamma_{+}(L, z)$ in [Car].

The following commutation relations can be found in [Car] (see Remark 3.1):

$$[\Gamma_{+}(L, x), \Gamma_{+}(L', y)] = [\Gamma_{-}(L, x), \Gamma_{-}(L', y)] = 0, \quad (3.3)$$

$$\Gamma_{+}(L, x) \Gamma_{-}(L', y) = (1 - y/x)^{(L,L')} \Gamma_{-}(L', y) \Gamma_{+}(L, x). \quad (3.4)$$

Let $W(L, z) : \mathbb{H}_X \to \mathbb{H}_X$ be the vertex operator constructed in [CO, Car] where $z$ is a formal variable. By [Car], $W(L, z)$ is defined via the pairing

$$\langle W(L, z) \eta, \xi \rangle = \int_{X^{[k]} \times X^{[\ell]}} (\eta \otimes \xi) c_{k+\ell} (E_L) \quad (3.5)$$

for $\eta \in H^*(X^{[k]})$ and $\xi \in H^*(X^{[\ell]})$. The main result in [Car] is (see Remark 3.1):

$$W(L, z) = \Gamma_{-}(L - K_X, z) \Gamma_{+}(-L, z). \quad (3.6)$$

**Lemma 3.2.** Let $\partial$ be the number-of-points operator, i.e., $\partial|_{H^*(X^{[n]})} = n \text{Id}$. Then,

$$F_{k_1, \ldots, k_N}^{\alpha_1, \ldots, \alpha_N}(q) = \text{Tr} q^\partial W(\mathcal{L}_1, z) \prod_{i=1}^{N} \mathfrak{g}_{k_i}(\alpha_i). \quad (3.7)$$

**Proof.** We will show that the coefficients of $q^n$ on both sides of (3.7) are equal. Let $\{e_j\}_j$ be a linear basis of $H^*(X^{[n]})$. Then the fundamental class of the diagonal $\Delta_n$ in $X^{[n]} \times X^{[n]}$ is given by $[\Delta_n] = \sum_j (-1)^{\lfloor e_j \rfloor} e_j \otimes e_j^*$ where $\{e_j^*\}_j$ is the linear basis of $H^*(X^{[n]})$ dual to $\{e_j\}_j$ in the sense that $\langle e_j, e_j^* \rangle = \delta_{j,j'}$. By the definitions
of $W(L, z)$ and $\mathfrak{g}_k(\alpha)$, \( \text{Tr} q^n W(\mathfrak{L}_1, z) \prod_{i=1}^{N} \mathfrak{g}_{k_i}(\alpha_i) \) is equal to

\[
q^n \sum_{j} (-1)^{|e_j|} \left\langle W(\mathfrak{L}_1, z) \left( \prod_{i=1}^{N} \mathfrak{g}_{k_i}(\alpha_i) \right) e_j, e_j^* \right\rangle 
\]

\[
= q^n \sum_{j} (-1)^{|e_j|} \int_{X^{[n]} \times X^{[n]}} \left( \prod_{i=1}^{N} \mathfrak{g}_{k_i}(\alpha_i) \right) e_j \otimes e_j^* \ c_{2n}(E_{\mathfrak{L}_1}) 
\]

\[
= q^n \sum_{j} (-1)^{|e_j|} \int_{X^{[n]} \times X^{[n]}} \left( \prod_{i=1}^{N} G_{k_i}(\alpha_i, n) \right) e_j \otimes e_j^* \ c_{2n}(E_{\mathfrak{L}_1}) 
\]

\[
= q^n \sum_{j} (-1)^{|e_j|} \int_{X^{[n]} \times X^{[n]}} \left( e_j \otimes e_j^* \right) p_1^* \left( \prod_{i=1}^{N} G_{k_i}(\alpha_i, n) \right) c_{2n}(E_{\mathfrak{L}_1}) 
\]

\[
= q^n \int_{X^{[n]} \times X^{[n]}} [\Delta_n] p_1^* \left( \prod_{i=1}^{N} G_{k_i}(\alpha_i, n) \right) c_{2n}(E_{\mathfrak{L}_1}) 
\]

where \( p_1 : X^{[n]} \times X^{[n]} \rightarrow X^{[n]} \) denotes the first projection. By (3.1), we have \( c_{2n}(E_{\mathfrak{L}_1})|_{\Delta_n} = c(T_{X^{[n]}}) \). Here and below, we implicitly set \( t = 1 \) for the generator \( t \) of the equivariant cohomology \( H^*_c(pt) \) of a point. Therefore,

\[
\text{Tr} q^n W(\mathfrak{L}_1, z) \prod_{i=1}^{N} \mathfrak{g}_{k_i}(\alpha_i) = q^n \int_{X^{[n]}} \left( \prod_{i=1}^{N} G_{k_i}(\alpha_i, n) \right) c(T_{X^{[n]}}). \quad \square
\]

4. The trace \( \text{Tr} q^n W(\mathfrak{L}_1, z) \prod_{i=1}^{N} \frac{a_{\lambda(i)}(\alpha_i)}{\lambda(i)!} \) and the series \( F_{k_1, \ldots, k_N}(q) \)

In this section, we will first determine the structure of \( \text{Tr} q^n W(\mathfrak{L}_1, z) \prod_{i=1}^{N} \frac{a_{\lambda(i)}(\alpha_i)}{\lambda(i)!} \). Then, the structure of the generating series \( F_{k_1, \ldots, k_N}(q) \) will follow from Lemma 3.2 and Theorem 2.3 and the structure of \( \text{Tr} q^n W(\mathfrak{L}_1, z) \prod_{i=1}^{N} \frac{a_{\lambda(i)}(\alpha_i)}{\lambda(i)!} \).

We begin with four technical lemmas. To explain the ideas behind these lemmas, note from (3.6) that \( \text{Tr} q^n W(\mathfrak{L}_1, z) \prod_{i=1}^{N} \frac{a_{\lambda(i)}(\alpha_i)}{\lambda(i)!} \) is equal to

\[
\text{Tr} q^n \Gamma_-(\mathfrak{L}_1 - K_X, z) \Gamma_+(-\mathfrak{L}_1, z) \prod_{i=1}^{N} \frac{a_{\lambda(i)}(\alpha_i)}{\lambda(i)!}. \quad (4.1)
\]
Lemma 4.1 deals with the commutator between \( \frac{a_{\lambda}(\alpha)}{\lambda!} \) and \( \exp \left( \frac{z^n}{n} a_{-n}(\gamma) \right) \). It enables us in Lemma 4.2 to eliminate \( \Gamma_{-}(\mathcal{L}_1 - K_X, z) \) from (4.1), and allows us in Lemma 4.3 to eliminate \( \Gamma_{+}(-\mathcal{L}_1, z) \) from (4.1). Lemma 4.4 determines the structure of \( \text{Tr} q^\varphi \prod_{i=1}^{N} \frac{a_{\lambda(i)}(\alpha_i)}{\lambda^{(i)!}} \). The proofs of these lemmas are standard but lengthy.

Recall from Definition 2.2 (ii) that \( \mathcal{P} \) denotes the set of generalized partitions. If \( \lambda = (\cdots (-2)^{s-2}(-1)^{s-1}1^s 2^s \cdots) \) and \( \mu = (\cdots (-2)^{t-2}(-1)^{t-1}1^t 2^t \cdots) \), let

\[
\lambda - \mu = (\cdots (-2)^{s-2-t-2}(-1)^{s-1-t-1}1^{s-t}2^{s-t} \cdots)
\]

with the convention that \( \lambda - \mu = \emptyset \) if \( s_i < t_i \) for some \( i \).

**Lemma 4.1.** Let \( \lambda \in \mathcal{P} \). Assume that \( \gamma \in H^{\text{even}}(X) \). Then,

\[
\frac{a_{\lambda}(\alpha)}{\lambda!} \exp \left( \frac{z^n}{n} a_{-n}(\gamma) \right) = \exp \left( \frac{z^n}{n} a_{-n}(\gamma) \right) \sum_{i \geq 0} \frac{(-z^n)^i}{i!} \frac{a_{\lambda-(n^i)}(\gamma^i \alpha)}{(\lambda - (n^i))!}, \tag{4.2}
\]

\[
\exp \left( \frac{z^n}{n} a_{n}(\gamma) \right) \cdot \frac{a_{\lambda}(\alpha)}{\lambda!} = \sum_{i \geq 0} \frac{(-z^n)^i}{i!} \frac{a_{\lambda-i(-n^i)}(\gamma^i \alpha)}{(\lambda - ((-n^i)^i))!} \exp \left( \frac{z^n}{n} a_{n}(\gamma) \right). \tag{4.3}
\]

**Proof.** Note that the adjoint of \( a_m(\beta) \) is equal to \((-1)^m a_{-m}(\beta)\). So (4.3) follows from (4.2) by taking adjoint on both sides of (4.2) and by making suitable adjustments. To prove (4.2), put

\[
A = \frac{a_{\lambda}(\alpha)}{\lambda!} \exp \left( \frac{z^n}{n} a_{-n}(\gamma) \right).
\]

Then,

\[
A = \frac{a_{\lambda}(\alpha)}{\lambda!} \sum_{t \geq 0} \frac{1}{t!} \left( \frac{z^n}{n} a_{-n}(\gamma) \right)^t
\]

\[
= \frac{1}{\lambda!} \sum_{t \geq 0} \frac{1}{t!} \left( \frac{z^n}{n} \right)^t \sum_{i=0}^{t} \binom{t}{i} \left( a_{-n}(\gamma) \right)^{t-i} \left[ a_{\lambda}(\alpha), a_{-n}(\gamma) \right]_{i \text{ times}}.
\]

Let \( \lambda = (\cdots (-2)^{s-2}(-1)^{s-1}1^s 2^s \cdots) \). We conclude from Lemma 2.4 (i) that the commutator \( \left[ a_{\lambda}(\alpha), a_{-n}(\gamma) \right] \) is equal to

\[
s_n(s_n - 1) \cdots (s_n + 1 - i) (-n)^i a_{\lambda-(n^i)}(\gamma^i \alpha)
\]

where by our convention, \( \lambda - (n^i) = \emptyset \) if \( s_n < i \). So \( A \) is equal to

\[
\frac{1}{\lambda!} \sum_{t \geq 0} \frac{1}{t!} \left( \frac{z^n}{n} \right)^t \sum_{i=0}^{t} \binom{t}{i} \left( a_{-n}(\gamma) \right)^{t-i} \cdot s_n(s_n - 1) \cdots (s_n + 1 - i) (-n)^i a_{\lambda-(n^i)}(\gamma^i \alpha).
\]

Simplifying this, we complete the proof of our formula (4.2). \( \square \)

Let \( \mathcal{P}_+ = \mathcal{P} \) be the subset of \( \mathcal{P} \) consisting of the usual partitions, and \( \mathcal{P}_- \) be the subset of \( \mathcal{P} \) consisting of generalized partitions of the form \((\cdots (-2)^{s-2}(-1)^{s-1})\).
Lemma 4.2. Let $\lambda^{(1)}, \ldots, \lambda^{(N)} \in \tilde{\mathcal{P}}$ be generalized partitions, and $\alpha_1, \ldots, \alpha_N \in H^*(X)$. Then, the trace $\text{Tr} q^\delta W(\mathfrak{L}_1, z) \prod_{i=1}^N \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$ is equal to

$$\sum_{\mu^{(i,s)} \in \tilde{\mathcal{P}}^+, 1 \leq i \leq N, s \geq 1} \sum_{1 \leq i \leq N, n \geq 1} \prod_{i,s \geq 1} \frac{(-zq^n)^{m_{n,i,s}}}{m_{n,i,s}!} \cdot \text{Tr} q^\delta \Gamma_+(-\mathfrak{L}_1, z) \prod_{i=1}^N \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \left(1 \pm \frac{1}{\lambda^{(i)} - \sum_{s \geq 1} \mu^{(i,s)}} \right).$$

(4.4)

where $\mu^{(i,s)} = (1^{m_{1,i,s}} \cdots n^{m_{n,i,s}} \cdots) \in \tilde{\mathcal{P}}_+$ for $1 \leq i \leq N$ and $s \geq 1$.

Proof. For simplicity, put $Q_1 = \text{Tr} q^\delta W(\mathfrak{L}_1, z) \prod_{i=1}^N \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$. By (3.6),

$$Q_1 = \text{Tr} q^\delta \Gamma_-(-\mathfrak{L}_1 - K_X, z) \prod_{i=1}^N \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \text{Tr} \Gamma_-(-\mathfrak{L}_1 - K_X, zq) q^\delta \Gamma_+(-\mathfrak{L}_1, z) \prod_{i=1}^N \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \Gamma_-(-\mathfrak{L}_1 - K_X, zq).$$

(4.5)

By (3.2) and applying (4.2) repeatedly, we obtain

$$\prod_{i=1}^N \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \Gamma_-(-\mathfrak{L}_1 - K_X, zq)$$

$$= \prod_{i=1}^N \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \exp \left( \sum_{n>0} \frac{(zq^n)}{n} a_{-n}(\mathfrak{L}_1 - K_X) \right)$$

$$= \Gamma_-(-\mathfrak{L}_1 - K_X, zq) \sum_{\mu^{(i,1)} \in \tilde{\mathcal{P}}^+, 1 \leq i \leq N, n \geq 1} \prod_{1 \leq i \leq N, n \geq 1} \frac{(-zq^n)^{m_{n,i,1}}}{m_{n,i,1}!} \cdot \prod_{i=1}^N \frac{a_{\lambda^{(i)} - \mu^{(i,1)}}(1 \pm \frac{1}{\lambda^{(i)} - \mu^{(i,1)}})}{(\lambda^{(i)} - \mu^{(i,1)})!} \cdot \prod_{i=1}^N \frac{(-zq^n)^{m_{n,i,1}}}{m_{n,i,1}!}.$$ 

where $\mu^{(i,1)} = (1^{m_{1,i,1}} \cdots n^{m_{n,i,1}} \cdots)$. Therefore, $Q_1$ is equal to

$$\text{Tr} q^\delta \Gamma_+(-\mathfrak{L}_1, z) \Gamma_-(-\mathfrak{L}_1 - K_X, zq) \sum_{\mu^{(i,1)} \in \tilde{\mathcal{P}}^+, 1 \leq i \leq N, n \geq 1} \prod_{1 \leq i \leq N, n \geq 1} \frac{(-zq^n)^{m_{n,i,1}}}{m_{n,i,1}!}.$$
Lemma 4.3. Let \( \mathfrak{L}_1, \mathfrak{L}_1 - K_X \) where \( \tilde{\mu}_{\mathfrak{L}} \in \widetilde{\mathcal{P}} \) is equal to
\[
\prod_{i=1}^{N} \frac{a_{\lambda(i) - \mu(i,t)}((1_X - K_X) \sum_{n \geq 1} m_{n}^{-1} \alpha_i)}{\lambda(i) - \mu(i,t)!}.
\]
Since \( \langle \mathfrak{L}_1, \mathfrak{L}_1 - K_X \rangle = 0 \), we see from (3.4) that \( Q_1 \) is equal to
\[
\text{Tr} \, q^\theta \Gamma_-(\mathfrak{L}_1 - K_X, zq) \Gamma_+(\mathfrak{L}_1, z) \cdot \sum_{\mu^{(i)} \in \bar{\mathcal{P}}_+} \prod_{1 \leq s \leq N} \frac{(-zq)^{m_{n}^{(i,s)}}}{m_{n}^{(i,s)}!}.
\]
Repeat the above process beginning at line (4.5) \( s \) times. Then, \( Q_1 \) is equal to
\[
\text{Tr} \, q^\theta \Gamma_-(\mathfrak{L}_1 - K_X, zq^s) \Gamma_+(\mathfrak{L}_1, z) \cdot \sum_{\mu^{(i,r)} \in \bar{\mathcal{P}}_+} \prod_{1 \leq s \leq N} \frac{(-zq)^{m_{n}^{(i,s)}}}{m_{n}^{(i,s)}!}.
\]
where \( \mu^{(i,r)} = (1^{m_1} \cdots n^{m_n}) \). Letting \( s \to +\infty \) proves our lemma. \( \square \)

Lemma 4.3. Let \( \tilde{\lambda}^{(1)}, \ldots, \tilde{\lambda}^{(N)} \in \widetilde{\mathcal{P}} \) be generalized partitions, and \( \tilde{\alpha}_1, \ldots, \tilde{\alpha}_N \in H^*(X) \). Then, the trace \( \text{Tr} q^\theta \Gamma_+(-\mathfrak{L}_1, z) \prod_{i=1}^{N} \frac{a_{\tilde{\lambda}(i)}(\tilde{\alpha}_i)}{\tilde{\lambda}(i)!} \) is equal to
\[
\sum_{\sum_{i=1}^{N} (|\tilde{\lambda}^{(i)}| - \sum_{t \geq 1} |\tilde{\mu}^{(i,t)}|) = 0} \prod_{1 \leq i \leq N} \frac{(-1) q^{t-1} m_{\tilde{\lambda}^{(i)}}^{(i,t)}}{m_{\tilde{\lambda}^{(i)}}^{(i,t)}!} \cdot \text{Tr} q^\theta \prod_{i=1}^{N} \frac{a_{\tilde{\lambda}(i)}(\tilde{\mu}(i,t))}{\tilde{\lambda}(i) - \tilde{\mu}(i,t)!}.
\]
where \( \tilde{\mu}^{(i,t)} = (\cdots (-n)^{\tilde{m}_i^{(i,t)}} \cdots (-1)^{\tilde{m}_i^{(i,t)}}) \in \bar{\mathcal{P}}_+ \) for \( 1 \leq i \leq N \) and \( t \geq 1 \).

Proof. For simplicity, put \( Q_2 = \text{Tr} q^\theta \Gamma_+(-\mathfrak{L}_1, z) \prod_{i=1}^{N} \frac{a_{\tilde{\lambda}(i)}(\tilde{\alpha}_i)}{\tilde{\lambda}(i)!} \). By (3.2),
\[
Q_2 = \text{Tr} q^\theta \exp \left( \sum_{n > 0} \frac{z^{-n}}{n} a_n(-\mathfrak{L}_1) \right) \prod_{i=1}^{N} \frac{a_{\tilde{\lambda}(i)}(\tilde{\alpha}_i)}{\tilde{\lambda}(i)!}.
\]
Applying (4.3) repeatedly, we see that \( Q_2 \) is equal to
\[
\sum_{\tilde{\mu}^{(i,t)} \in \bar{\mathcal{P}}_+} \prod_{1 \leq i \leq N} \frac{(-1) q^{t-1} m_{\tilde{\lambda}^{(i)}}^{(i,t)}}{m_{\tilde{\lambda}^{(i)}}^{(i,t)}!} \cdot \text{Tr} q^\theta \prod_{i=1}^{N} \frac{a_{\tilde{\lambda}(i)}(\tilde{\mu}(i,t))}{\tilde{\lambda}(i) - \tilde{\mu}(i,t)!} \cdot \Gamma_+(-\mathfrak{L}_1, z).
\]
where \( \tilde{\mu}^{(i,1)} = (\cdots (-n)^{\tilde{n}^{(i,1)}} \cdots (-1)^{\tilde{n}^{(i,1)}}) \in \tilde{\mathcal{P}}_- \). Now \( Q_2 \) is equal to

\[
\sum_{\tilde{\mu}^{(i,1)} \in \tilde{\mathcal{P}}_-} \frac{1}{\tilde{m}^{(i,1)}} \cdot q^{\sum_{i=1}^{N} (|\tilde{\lambda}^{(i)}| - |\tilde{\mu}^{(i,1)}|)} \cdot \text{Tr} \prod_{i=1}^{N} \frac{a_{\tilde{\lambda}^{(i)} - \tilde{\mu}^{(i,1)}} (\tilde{\alpha}_i)}{(\lambda^{(i)} - \tilde{\mu}^{(i,1)})}.
\]

By degree reason, \( \text{Tr} q^N \Gamma_+ (-\Sigma_1, z) a^{\mu}_\beta = 0 \) if \( |\mu| > 0 \). If \( |\mu| = 0 \), then we have \( \text{Tr} q^N \Gamma_+ (-\Sigma_1, z) a^{\mu}_\beta = \text{Tr} q^N a^{\mu}_\beta \). So \( Q_2 \) is equal to

\[
\sum_{\sum_{i=1}^{N} (|\tilde{\lambda}^{(i)}| - |\tilde{\mu}^{(i,1)}|) > 0} \frac{1}{\tilde{m}^{(i,1)}} \cdot q^{\sum_{i=1}^{N} (|\tilde{\lambda}^{(i)}| - |\tilde{\mu}^{(i,1)}|)} \cdot \text{Tr} \prod_{i=1}^{N} \frac{a_{\tilde{\lambda}^{(i)} - \tilde{\mu}^{(i,1)}} (\tilde{\alpha}_i)}{(\lambda^{(i)} - \tilde{\mu}^{(i,1)})}.
\]

Repeating the process in the previous paragraph \( t \) times, we conclude that

\[
Q_2 = U(t) - V(t)
\]

where \( U(t) \) is given by

\[
\sum_{\sum_{i=1}^{N} (|\tilde{\lambda}^{(i)}| - \sum_{r=1}^{t} |\tilde{\mu}^{(i,r)}|) < 0} \prod_{r=1}^{t} \frac{1}{\tilde{m}^{(i,r)}} \cdot q^{\sum_{i=1}^{N} (|\tilde{\lambda}^{(i)}| - \sum_{r=1}^{t} |\tilde{\mu}^{(i,r)}|)} \cdot \text{Tr} \prod_{i=1}^{N} \frac{a_{\tilde{\lambda}^{(i)} - \sum_{r=1}^{t} \tilde{\mu}^{(i,r)}} (\tilde{\alpha}_i)}{(\lambda^{(i)} - \sum_{r=1}^{t} \tilde{\mu}^{(i,r)})}.
\]

\[ \text{(4.6)} \]

with \( \tilde{\mu}^{(i,r)} = (\cdots (-n)^{\tilde{n}^{(i,r)}} \cdots (-1)^{\tilde{n}^{(i,r)}}) \in \tilde{\mathcal{P}}_- \), and \( V(t) \) is given by

\[
\sum_{\sum_{i=1}^{N} (|\tilde{\lambda}^{(i)}| - \sum_{r=1}^{t} |\tilde{\mu}^{(i,r)}|) > 0} \prod_{r=1}^{t} \frac{1}{\tilde{m}^{(i,r)}} \cdot q^{\sum_{i=1}^{N} (|\tilde{\lambda}^{(i)}| - \sum_{r=1}^{t} |\tilde{\mu}^{(i,r)}|)} \cdot \text{Tr} \prod_{i=1}^{N} \frac{a_{\tilde{\lambda}^{(i)} - \sum_{r=1}^{t} \tilde{\mu}^{(i,r)}} (\tilde{\alpha}_i)}{(\lambda^{(i)} - \sum_{r=1}^{t} \tilde{\mu}^{(i,r)})}.
\]

\[ \text{(4.7)} \]

Denote line \( U(t) \) by \( \tilde{U}(t) \). Since \( \sum_{i=1}^{N} (|\tilde{\lambda}^{(i)}| - \sum_{r=1}^{t} |\tilde{\mu}^{(i,r)}|) < 0 \) and \( |\tilde{\mu}^{(i,r)}| < 0 \), \( \tilde{U}(t) \) is a polynomial in \( q \) with coefficients being bounded in terms of \( -\sum_{i=1}^{N} |\tilde{\lambda}^{(i)}| \).
Moreover, $q^t \tilde{U}(t)$. Line (4.7) is bounded in terms of the generalized partitions $\tilde{\lambda}^{(i)}$.
Since $0 < q < 1$, $U(t) \to 0$ as $t \to +\infty$. Letting $t \to +\infty$, we see that $Q_2$ equals

$$
\sum_{\sum_{i=1}^{N} (|\tilde{\lambda}^{(i)}| - \sum_{t \geq 1} |\tilde{\mu}^{(i,t)}|)} \prod_{t \geq 1} \left( \prod_{1 \leq i \leq N} \frac{z^{-n_{\tilde{\mu}^{(i,t)}}}}{\tilde{m}_{n_{\tilde{\mu}^{(i,t)}}}} \cdot q^{\sum_{i=1}^{N} \sum_{t \geq 1} |\tilde{\mu}^{(i,t)}| - |\tilde{\lambda}^{(i)}|} \prod_{n \geq 1} \prod_{1 \leq i \leq N} \frac{z^{-n_{\tilde{\mu}^{(i,t)}}}}{\tilde{m}_{n_{\tilde{\mu}^{(i,t)}}}} \right). 
$$

Replacing $q^{\sum_{i=1}^{N} \sum_{t \geq 1} |\tilde{\mu}^{(i,t)}| - |\tilde{\lambda}^{(i)}|}$ by $q^{-\sum_{i=1}^{N} \sum_{t \geq 1} |\tilde{\mu}^{(i,t)}|}$ proves our lemma. \qed

**Lemma 4.4.** Let $\lambda^{(1)}, \ldots, \lambda^{(N)} \in \tilde{P}$ be generalized partitions, and $\alpha_1, \ldots, \alpha_N \in H^*(X)$ be homogeneous. Then, $\text{Tr} q^a \prod_{i=1}^{N} \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)!}}$ can be computed by an induction on $N$, and is a linear combination of expressions of the form:

$$
(q; q)_\infty^{-\chi(X)} \cdot \text{Sign}(\pi) \cdot \prod_{i=1}^{u} \langle e_{\alpha_i}^{m_i}, \prod_{j \in \pi_i} \alpha_j \rangle \cdot \prod_{i=1}^{v} \frac{n_i^k q^{n_i}}{1 - q^{n_i}} \quad (4.8)
$$

where $0 \leq v \leq \sum_{i=1}^{N} \ell(\lambda^{(i)})/2$, $m_i \geq 0$, $n_i > 0$, all the integers involved and the partition $\{\pi_1, \ldots, \pi_u\}$ of $\{1, \ldots, N\}$ depend only on $\lambda^{(1)}, \ldots, \lambda^{(N)}$, and $\text{Sign}(\pi)$ is the sign compensating the formal difference between $\prod_{i=1}^{u} \prod_{j \in \pi_i} \alpha_j$ and $\alpha_1 \cdots \alpha_N$. Moreover, the coefficients of this linear combination are independent of $q, \alpha_i, n_i, X$.

**Proof.** For simplicity, put $A_N = \text{Tr} q^a \prod_{i=1}^{N} \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)!}}$. Since $a_{\lambda^{(i)}}(\alpha_i)$ has conformal weight $|\lambda^{(i)}|$, $A_N = 0$ unless $\sum_{i=1}^{N} |\lambda^{(i)}| = 0$. In the rest of the proof, we will assume $\sum_{i=1}^{N} |\lambda^{(i)}| = 0$. We will divide the proof into two cases.

**Case 1:** $|\lambda^{(i)}| = 0$ for every $1 \leq i \leq N$. Then, $\ell(\lambda^{(i)}) \geq 2$ for every $i$. Since $a_{\lambda^{(i)}}(\alpha_i)$ has degree $2(\ell(\lambda^{(i)}) - 2) + |\alpha_i|$, $A_N = 0$ unless $\ell(\lambda^{(i)}) = 2$ and $|\alpha_i| = 0$ for all $1 \leq i \leq N$. Assume that $\ell(\lambda^{(i)}) = 2$ and $|\alpha_i| = 0$ for all $1 \leq i \leq N$. Then for every $1 \leq i \leq N$, we have $\lambda^{(i)} = ((-n_i) n_i)$ for some $n_i > 0$. We further assume that $n_1 = \ldots = n_r$ for some $1 \leq r \leq N$ and $n_i \neq n_1$ if $r < i \leq N$. Let $\alpha_1 = a1_X$
and $\tau_{2*1}X = \sum_j (-1)^{|\beta_j|} \beta_j \otimes \gamma_j$ with $\langle \beta_j, \gamma_{j'} \rangle = \delta_{j,j'}$. Then, $A_N$ is equal to

$$a \sum_j (-1)^{|\beta_j|} \text{Tr} \ q^\beta \ a_{-n_1}(\beta_j) \ a_{n_1}(\gamma_j) \prod_{i=2}^N a_{\lambda(i)}(\alpha_i)$$

$$= aq^{n_1} \sum_j (-1)^{|\beta_j|} \text{Tr} \ a_{-n_1}(\beta_j) q^\beta a_{n_1}(\gamma_j) \prod_{i=2}^N a_{\lambda(i)}(\alpha_i)$$

$$= aq^{n_1} \sum_j \text{Tr} \ q^\beta a_{n_1}(\gamma_j) \prod_{i=2}^N a_{\lambda(i)}(\alpha_i) \cdot a_{-n_1}(\beta_j)$$

$$= aq^{n_1} \sum_j \text{Tr} \ q^\beta a_{n_1}(\gamma_j) a_{-n_1}(\beta_j) \prod_{i=2}^N a_{\lambda(i)}(\alpha_i)$$

$$+ aq^{n_1} \sum_j \sum_{i=2}^r \text{Tr} \ q^\beta a_{n_1}(\gamma_j) \prod_{i=2}^{i-1} a_{\lambda(i)}(\alpha_i) \cdot [a_{\lambda(i)}(\alpha_i), a_{-n_1}(\beta_j)] \cdot \prod_{k=i+1}^N a_{\lambda(k)}(\alpha_k).$$

By Lemma 2.4(i), $A_N$ is equal to the sum of the expressions

$$\left\langle e_X, \alpha_1 \prod_{i=1}^{k_1} \alpha_{\lambda(i)} \right\rangle \cdot \frac{(-n_1)^{k_1} q^{n_1}}{1 - q^{n_1}} \cdot \text{Tr} \ q^\beta \prod_{i \in \{2, \ldots, N\} \setminus \{j_1, \ldots, j_{k_1}\}} a_{\lambda(i)}(\alpha_i) \tag{4.9}$$

where $0 \leq k_1 \leq r - 1$, $\{j_1, \ldots, j_{k_1}\} \subset \{2, \ldots, r\}$, every factor in $(-n_1)^{k_1}$ comes from a commutator of type $[a_{n_1}(\cdot), a_{-n_1}(\cdot)]$, and the coefficients of this linear combination depend only on $\lambda^{(1)}, \ldots, \lambda^{(N)}$. In particular, we have

$$A_1 = \text{Tr} \ q^\beta a_{\lambda(1)}(\alpha_1) = (q; q)_\infty^{-\lambda(X)} \cdot \langle e_X, \alpha_1 \rangle \cdot \frac{(-n_1) q^{n_1}}{1 - q^{n_1}}. \tag{4.10}$$

Combining with (4.9), we see that our lemma holds in this case.

**Case 2:** $\sum_{i=1}^N |\lambda^{(i)}| = 0$ but $|\lambda^{(i_0)}| \neq 0$ for some $i_0$. Then, $N \geq 2$, and we may assume that $|\lambda^{(i_0)}| < 0$. To simplify the expressions, we further assume that every $\alpha_i$ has an even degree. Note that $A_N$ can be rewritten as

$$\text{Tr} \ q^\beta \frac{a_{\lambda(i_0)}(\alpha_{i_0})}{\lambda^{(i_0)}!} \prod_{1 \leq i \leq N; i \neq i_0} \frac{a_{\lambda(i)}(\alpha_i)}{\lambda^{(i)}!}$$

$$+ \sum_{r=1}^{i_0-1} \text{Tr} \ q^\beta \prod_{i=1}^{r-1} \frac{a_{\lambda(i)}(\alpha_i)}{\lambda^{(i)}!} \cdot \left[ \frac{a_{\lambda(i_0)}(\alpha_{i_0})}{\lambda^{(i_0)}!} \cdot \frac{a_{\lambda(i_0)}(\alpha_{i_0})}{\lambda^{(i_0)}!} \right] \cdot \prod_{r+1 \leq i \leq N; i \neq i_0} \frac{a_{\lambda(i)}(\alpha_i)}{\lambda^{(i)}!}. $$
Since \( q^N a_{\lambda}^{(i_0)}(\alpha_{i_0}) = q^{-|\lambda|} a_{\lambda}^{(i_0)}(\alpha_{i_0}) q^N \), we see that \( A_N \) is equal to

\[
q^{-|\lambda|} \text{Tr} a_{\lambda}^{(i)}(\alpha_{i_0}) q^N \prod_{1 \leq i < N, i \neq i_0} a_{\lambda}^{(i)}(\alpha_{i_0}) \lambda(i)!
\]

\[
+ \sum_{r=1}^{i_0-1} \text{Tr} q^r \prod_{i=1}^{r-1} a_{\lambda}^{(i)}(\alpha_{i_0}) \lambda(i)!
\]

\[
= \prod_{1 \leq i < N, i \neq i_0} a_{\lambda}^{(i)}(\alpha_{i_0}) \lambda(i)!
\]

\[
+ \sum_{r=1}^{i_0-1} \text{Tr} q^r \prod_{i=1}^{r-1} a_{\lambda}^{(i)}(\alpha_{i_0}) \lambda(i)!
\]

Note that \( \text{Tr} q^r \prod_{1 \leq i < N, i \neq i_0} a_{\lambda}^{(i)}(\alpha_{i_0}) \lambda(i)! \) is equal to

\[
A_N + \sum_{r=i_0+1}^{N} q^{-|\lambda|} \prod_{1 \leq i < r, i \neq i_0} a_{\lambda}^{(i)}(\alpha_{i_0}) \lambda(i)!
\]

Therefore, we conclude that \( (1 - q^{-|\lambda|}) A_N \) is equal to

\[
q^{-|\lambda|} \sum_{r=i_0+1}^{N} q^{r} \prod_{1 \leq i < r, i \neq i_0} a_{\lambda}^{(i)}(\alpha_{i_0}) \lambda(i)!
\]

\[
+ \sum_{r=1}^{i_0-1} \text{Tr} q^r \prod_{i=1}^{r-1} a_{\lambda}^{(i)}(\alpha_{i_0}) \lambda(i)!
\]

\[
\prod_{r+1 \leq i < N, i \neq i_0} a_{\lambda}^{(i)}(\alpha_{i_0}) \lambda(i)!
\]

Put \( n_0 = -|\lambda| > 0 \). It follows that \( A_N \) is equal to

\[
\frac{q^{n_0}}{1 - q^{n_0}} \sum_{r=i_0+1}^{N} q^{r} \prod_{1 \leq i < r, i \neq i_0} a_{\lambda}^{(i)}(\alpha_{i_0}) \lambda(i)!
\]

\[
+ \frac{1}{1 - q^{n_0}} \sum_{r=1}^{i_0-1} \text{Tr} q^r \prod_{i=1}^{r-1} a_{\lambda}^{(i)}(\alpha_{i_0}) \lambda(i)!
\]

\[
\prod_{r+1 \leq i < N, i \neq i_0} a_{\lambda}^{(i)}(\alpha_{i_0}) \lambda(i)!
\]

By Lemma 2.4 (i) and (ii), our lemma holds in this case as well. □

The following theorem provides the structure of the trace

\[
\text{Tr} q^N W(\mathcal{L}, z) \prod_{i=1}^{N} a_{\lambda}^{(i)}(\alpha_{i_0}) \lambda(i)!
\]
Theorem 4.5. For $1 \leq i \leq N$, let $\lambda^{(i)} = (\cdots (-n)^{m_n^{(i)}} \cdots (-1)^{\tilde{m}_n^{(i)}} 1^{m_1^{(i)}} \cdots n^{m_n^{(i)}} \cdots)$ and $\alpha_i \in H^*(X)$ be homogeneous. Then, $\text{Tr} q^\delta W(\mathcal{O}_1, z) \prod_{i=1}^N \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$ is equal to

$$z \sum_{i=1}^N |\lambda^{(i)}| \cdot (q;q)_\infty \chi(X) \cdot \prod_{i=1}^N \langle (1_X - K_X) \sum_{n \geq 1} m_n^{(i)}, \alpha_i \rangle \cdot \prod_{1 \leq i \leq N, n \geq 1} \left( \frac{(-1)^{m_n^{(i)}}}{m_n^{(i)}!} \frac{q^{n m_n^{(i)}}}{(1 - q^n)^{m_n^{(i)}}} \right) + \widetilde{W},$$

and the lower weight term $\widetilde{W}$ is a linear combination of expressions of the form:

$$z \sum_{i=1}^N |\lambda^{(i)}| \cdot (q;q)_\infty \chi(X) \cdot \text{Sign}(\pi) \cdot \prod_{i=1}^u \left( K_{\lambda^{(i)}}^{r_i} \prod_{j \in \pi_i} \alpha_j \right) \cdot \prod_{i=1}^v \frac{q_{\mu_{\lambda^{(i)}}} \cdot \mu_{\lambda^{(i)}}}{(1 - q^{\alpha_i})^{-\alpha_i}} \quad (4.12)$$

where $\sum_{i=1}^u w_i < \sum_{i=1}^N \ell(\lambda^{(i)})$, the integers $u, v, r_i, r'_i \geq 0$, $n_i > 0, w_i > 0, p_i \in \{0, 1\}$ and the partition $\pi = \{\pi_1, \ldots, \pi_u\}$ of $\{1, \ldots, N\}$ depend only on the generalized partitions $\lambda^{(1)}, \ldots, \lambda^{(N)}$, and $\text{Sign}(\pi)$ is the sign compensating the formal difference between $\prod_{i=1}^u \prod_{j \in \pi_i} \alpha_j$ and $\alpha_1 \cdots \alpha_N$. Moreover, the coefficients of this linear combination are independent of $q, \alpha_1, \ldots, \alpha_N$ and $X$.

Proof. For simplicity, put $\text{Tr}_\lambda = \text{Tr} q^\delta W(\mathcal{O}_1, z) \prod_{i=1}^N \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$. Combining Lemma 4.2 and Lemma 4.3, we conclude that $\text{Tr}_\lambda$ is equal to

$$\sum_{\sum_{i=1}^N |\lambda^{(i)}| - \sum_{s \geq 1} |\mu^{(i,s)}| - \sum_{t \geq 1} |\tilde{\mu}^{(i,t)}| = 0} \prod_{1 \leq i \leq N, s \geq 1} \frac{(-z q^n)^{m_n^{(i,s)}}}{m_n^{(i,s)}!} \cdot \prod_{t \geq 1} \frac{(-q t^{-1})^{n \tilde{m}_n^{(i,t)}}}{n \tilde{m}_n^{(i,t)}!} \cdot \text{Tr} q^\delta \prod_{i=1}^N \frac{a_{\lambda^{(i)} - \sum_{s \geq 1} \mu^{(i,s)} - \sum_{t \geq 1} \tilde{\mu}^{(i,t)}}(1_X - K_X)^{\sum_{s,n \geq 1} m_n^{(i,s)}}{\lambda^{(i)} - \sum_{s \geq 1} \mu^{(i,s)} - \sum_{t \geq 1} \tilde{\mu}^{(i,t)}!}$$

where $\mu^{(i,s)} = (1^{m_n^{(i,s)}} \cdots n^{m_n^{(i,s)}} \cdots)$ and $\tilde{\mu}^{(i,t)} = (\cdots (-n)^{\tilde{m}_n^{(i,t)}} \cdots (-1)^{\tilde{m}_n^{(i,t)}} \cdots)$. The sum of all the exponents of $z$ is $\sum_{i=1}^N |\lambda^{(i)}|$. So $\text{Tr}_\lambda$ is equal to

$$z \sum_{\sum_{i=1}^N |\lambda^{(i)}| - \sum_{s \geq 1} |\mu^{(i,s)}| - \sum_{t \geq 1} |\tilde{\mu}^{(i,t)}| = 0} \prod_{1 \leq i \leq N, s \geq 1} \frac{(-z q^n)^{m_n^{(i,s)}}}{m_n^{(i,s)}!} \cdot \prod_{t \geq 1} \frac{(-q t^{-1})^{n \tilde{m}_n^{(i,t)}}}{n \tilde{m}_n^{(i,t)}!} \cdot \text{Tr} q^\delta \prod_{i=1}^N \frac{a_{\lambda^{(i)} - \sum_{s \geq 1} \mu^{(i,s)} - \sum_{t \geq 1} \tilde{\mu}^{(i,t)}}(1_X - K_X)^{\sum_{s,n \geq 1} m_n^{(i,s)}}{\lambda^{(i)} - \sum_{s \geq 1} \mu^{(i,s)} - \sum_{t \geq 1} \tilde{\mu}^{(i,t)}!} \quad (4.13)$$

By our convention, $\sum_{s \geq 1} \mu^{(i,s)} + \sum_{t \geq 1} \tilde{\mu}^{(i,t)} = \lambda^{(i)}$ for every $1 \leq i \leq N$. We now divide the rest of the proof into Case A and Case B.
of generality, we may assume that
\[ \sum_{s \geq 1} \mu(s) + \sum_{t \geq 1} \bar{\mu}(t) = \lambda(i) \]
for every \( 1 \leq i \leq N \). Then line (4.13) is
\[ \text{Tr} q^s \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{s \geq 1} m_{i,s}} \rangle = (q; q)_{\infty}^{-\chi(X)} \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{s \geq 1} m_{i,s}} \rangle. \]

Therefore, the contribution \( C_1 \) of this case to \( \text{Tr}_\lambda \) is equal to
\[ z \sum_{i=1}^N |\lambda(i)| \cdot (q; q)^{-\chi(X)} \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{s \geq 1} m_{i,s}} \rangle \cdot \prod_{1 \leq i \leq N, n \geq 1} \left( \frac{q^n}{m_n !} \right). \]

Case A: \( \sum_{s \geq 1} \mu(s) + \sum_{t \geq 1} \bar{\mu}(t) \leq 1 \) for every \( 1 \leq i \leq N \). Then line (4.13) is
\[ \text{Tr} q^s \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{s \geq 1} m_{i,s}} \rangle = (q; q)_{\infty}^{-\chi(X)} \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{s \geq 1} m_{i,s}} \rangle. \]

Therefore, the contribution \( C_1 \) of this case to \( \text{Tr}_\lambda \) is equal to
\[ z \sum_{i=1}^N |\lambda(i)| \cdot (q; q)^{-\chi(X)} \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{s \geq 1} m_{i,s}} \rangle \cdot \prod_{1 \leq i \leq N, n \geq 1} \left( \frac{q^n}{m_n !} \right). \]

Rewrite \( q^n \) as \( q^{(s-1)n} q^n \). Then the contribution \( C_1 \) is equal to
\[ z \sum_{i=1}^N |\lambda(i)| \cdot (q; q)^{-\chi(X)} \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{s \geq 1} m_{i,s}} \rangle \cdot \prod_{1 \leq i \leq N, n \geq 1} \left( \frac{q^n}{m_n !} \right). \]

Since
\[ \prod_{i \geq 1} \left( \frac{q^{s(n)} i_{s,n}}{i_{s,n} !} \right) = \prod_{n \geq 1} \left( \frac{1}{i_n (1 - q^n)^{i_n}} \right), \]
\( C_1 \) is equal to
\[ z \sum_{i=1}^N |\lambda(i)| \cdot (q; q)^{-\chi(X)} \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{s \geq 1} m_{i,s}} \rangle \cdot \prod_{1 \leq i \leq N, n \geq 1} \left( \frac{1}{m_n ! (1 - q^n)^{m_n}} \right). \]

Case B: \( \sum_{s \geq 1} \mu(s) + \sum_{t \geq 1} \bar{\mu}(t) < \lambda(i) \) for some \( 1 \leq i \leq N \). Without loss of generality, we may assume that \( \sum_{s \geq 1} \mu(s) + \sum_{t \geq 1} \bar{\mu}(t) = \lambda(i) \) for every \( 1 \leq i \leq N \). Then line (4.13) is
\[ \text{Tr} q^s \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{s \geq 1} m_{i,s}} \rangle = (q; q)_{\infty}^{-\chi(X)} \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{s \geq 1} m_{i,s}} \rangle. \]

Therefore, the contribution \( C_2 \) of this case to \( \text{Tr}_\lambda \) is equal to
\[ z \sum_{i=1}^N |\lambda(i)| \prod_{i=1}^{N_1} \langle (1_X - K_X)^{\sum_{s \geq 1} m_{i,s}} \rangle \cdot \prod_{n \geq 1} \left( \frac{(-1)^{m_n (i)}}{m_n !} \frac{q^{m_{n} (i)}}{1 - q^n} \right) \frac{1}{\bar{m}_n ! (1 - q^n)^{\bar{m}_n (i)}}. \]
\[
\cdot \prod_{N+1 \leq i \leq N} \left( \frac{(-1)^{p_n(i)}}{p_n(i)!} \frac{qn^{p_n(i)}}{(1 - q^n)^{p_n(i)}} \frac{1}{\bar{p}_n(i)!} \frac{1}{(1 - q^n)^{\bar{p}_n(i)}} \right)
\]

\[
\cdot \text{Tr } q^\theta \prod_{i = N_1 + 1}^N a_{\lambda(i) - \lambda(i)} \left( (1 - K_X) \sum_{n > 1} p_n(i) \alpha_i \right) \left( (\lambda(i) - \lambda(i))! \right)
\]

(4.15)

By Lemma 4.4, \( C_2 \) is a linear combination of expressions of the form:

\[
\sum_{i=1}^{N} \left \langle (1 - K_X) \sum_{n \geq 1} m_n(i), \alpha_i \right \rangle \prod_{1 \leq i \leq N} \left( \frac{(-1)^{m_n(i)}}{m_n(i)!} \frac{q^{m_n(i)}}{(1 - q^n)^{m_n(i)}} \frac{1}{\bar{m}_n(i)!} \frac{1}{(1 - q^n)^{\bar{m}_n(i)}} \right)
\]

\[
\prod_{N+1 \leq i \leq N} \left( \frac{(-1)^{p_n(i)}}{p_n(i)!} \frac{qn^{p_n(i)}}{(1 - q^n)^{p_n(i)}} \frac{1}{\bar{p}_n(i)!} \frac{1}{(1 - q^n)^{\bar{p}_n(i)}} \right)
\]

where \( v < \sum_{i = N_1 + 1}^N \ell(\lambda(i) - \bar{\lambda}(i)), m_i > 0, \{\pi_1, \ldots, \pi_u\} \) is a partition of \( \{N_1 + 1, \ldots, N\} \), and \( \text{Sign}(\pi) \) compensates the formal difference between \( \prod_{i = 1}^u \prod_{j \in \pi_i} \alpha_j \) and \( \alpha_{N_1 + 1} \cdots \alpha_N \). The coefficients of this linear combination are independent of \( q, \alpha_1, \ldots, \alpha_N \) and \( X \), and depend only on the partitions \( \lambda(i) - \bar{\lambda}(i) \). Note that for nonnegative integers \( a \) and \( b \), the pairing \( \langle e_X^a, (1 - K_X)^b \rangle = (e_X^a, (1 - K_X)^b, \beta) \) is a linear combination of \( \langle e_X^c K_X^c, \beta \rangle, 0 \leq c \leq b \). In addition, we have

\[
\sum_{1 \leq i \leq N_1, n \geq 1} (m_n(i) + \bar{m}_n(i)) + \sum_{N_1 + 1 \leq i \leq N, n \geq 1} (p_n(i) + \bar{p}_n(i)) + v < \sum_{i=1}^{N} \ell(\lambda(i))
\]

regarding the weights in \( C_2 \). It follows that \( C_2 \) is a linear combination of the expressions (4.12). Combining with (4.14) completes the proof of our theorem. \( \square \)

Remark 4.6. When \( N = 1 \), we can work out the lower weight term \( \widetilde{W} \) in Theorem 4.5 by examining its proof more carefully and by using (4.13). To state the result, let \( \lambda = (\cdots (-n)^{m_n} \cdots (-1)^{m_1} m_1 \cdots m_{n_1} \cdots) \in \mathcal{P} \). For \( n_1 \geq 1 \) with \( m_{n_1} \cdot \bar{m}_{n_1} \geq 1 \), define \( m_{n_1}(n_1) = m_{n_1} - 1, \bar{m}_{n_1}(n_1) = \bar{m}_{n_1} - 1 \), and \( m_n(n_1) = m_n \).
Lemma 4.7. Denote the \( \tilde{m}_n(n_1) = \tilde{m}_n \) if \( n \neq n_1 \). Then, \( \text{Tr} q^b W(\mathcal{L}_1, z) \frac{a_{\lambda}(\alpha)}{\lambda!} \) is equal to the sum

\[
\sum_{n \geq 1} \left( -1 \right)^{m_n} q^{m_n} n_1 q^{n_1} \left( 1 - q^n \right)^{m_n} \left( 1 - q^n \right)^{m_n(n_1)} \frac{1}{m_n! (1 - q^n)^{m_n(n_1)} (1 - q^n)^{m_n(n_1)}} \cdot \prod_{n \geq 1} \left( \frac{1}{m_n!} \left( 1 - q^n \right)^{m_n} \tilde{m}_n! \left( 1 - q^n \right)^{\tilde{m}_n} \right).
\]

The next lemma is used to organize the leading term in Theorem 4.5.

Lemma 4.7. For \( \alpha \in H^*(X) \) and \( k \geq 0 \), define \( \Theta_k^\alpha(q) \) to be

\[
\sum_{\ell(\lambda) = k + 2, |\lambda| = 0} \langle (1 - K_X) \Sigma_{n \geq 1} i_n \alpha \rangle \cdot \prod_{n \geq 1} \left( -1 \right)^{i_n} \frac{q^{i_n}}{i_n!} \frac{1}{(1 - q^n)^{i_n}}
\]

where \( \lambda = (\cdots (-n)^i \cdots (1)^i \cdots n^i) \). Then, \( \Theta_k^\alpha(q) = \text{Coeff}_{z^0} \Theta_k^\alpha(q, z) \) which denotes the coefficient of \( z^0 \) in \( \Theta_k^\alpha(q, z) \) defined by

\[
\sum_{a, s_1, \ldots, s_a, b, t_1, \ldots, t_b \geq 1 \atop \sum_{i=1}^a s_i + \sum_{j=1}^b t_j = k + 2} \langle (1 - K_X) \Sigma_{n \geq 1} s_i \alpha \rangle \prod_{i=1}^a \left( -1 \right)^{s_i} \frac{q^{s_i}}{s_i!} \prod_{j=1}^b \frac{1}{t_j!}
\]

\[
\cdot \sum_{\sum_{n > \cdots > n_a \geq 1} a \geq 1} \sum_{\sum_{m > \cdots > m_b \geq 1} b \geq 1} \prod_{i=1}^a \frac{(z^2)^{n_i s_i}}{(1 - q^{n_i})^{s_i}} \cdot \prod_{j=1}^b \frac{z^{-m_j t_j}}{(1 - q^{m_j})^{t_j}}.
\]

Proof. Put \( A = \langle (1 - K_X) \Sigma_{n \geq 1} i_n \alpha \rangle \) which implicitly depends on \( \sum_{n \geq 1} i_n \). Rewrite |\lambda| and \( \ell(\lambda) \) in terms of the integers \( i_n \) and \( \tilde{i}_n \). Then, \( \Theta_k^\alpha(q) \) is equal to

\[
\sum_{\sum_{n \geq 1} i_n + \sum_{n \geq 1} \tilde{i}_n = k + 2 \atop \sum_{n \geq 1} n_i = \sum_{n \geq 1} n_i > 0} A \prod_{n \geq 1} \left( -1 \right)^{i_n} \frac{q^{i_n}}{i_n!} \frac{1}{(1 - q^n)^{i_n}} \frac{1}{\tilde{i}_n!} \frac{1}{(1 - q^n)^{\tilde{i}_n}}.
\]

Denote the positive integers in the ordered list \( \{i_1, \ldots, i_n, \ldots\} \) by \( s_a, \ldots, s_1 \) respectively (e.g., if the ordered list \( \{i_1, \ldots, i_n, \ldots\} \) is \( \{2, 0, 5, 4, 0, \ldots\} \), then \( a = 3 \) with \( s_3 = 2, s_2 = 5, s_1 = 4 \)). We have \( a \geq 1 \). Similarly, denote the positive integers in the ordered list \( \{\tilde{i}_1, \ldots, \tilde{i}_n, \ldots\} \) by \( t_b, \ldots, t_1 \) respectively. Then \( b \geq 1 \). Since \( \sum_{n \geq 1} i_n = \sum_{i=1}^a s_i \), we get \( A = \langle (1 - K_X) \Sigma_{n \geq 1} s_i \alpha \rangle \). Rewriting \( (18) \) in terms of \( s_a, \ldots, s_1 \) and \( t_b, \ldots, t_1 \), we see that \( \Theta_k^\alpha(q) = \text{Coeff}_{z^0} \Theta_k^\alpha(q, z) \).

We remark that the multiple q-zeta value \( \Theta_k^\alpha(q, z) \) has weight \( (k + 2) \).
Theorem 4.8. For $1 \leq i \leq N$, let $k_i \geq 0$ and $\alpha_i \in H^*(X)$ be homogeneous. Then,
\[
F_{k_1, \ldots, k_N}^{\alpha_1, \ldots, \alpha_N}(q) = (q; q)_{\infty}^{\chi(X)} \cdot \text{Coeff}_{z_1^1 \cdots z_N^0} \left( \prod_{i=1}^{N} \Theta_{k_i}^{\alpha_i}(q, z_i) \right) + W_1,
\]
and the lower weight term $W_1$ is an infinite linear combination of the expressions:
\[
(q; q)_{\infty}^{\chi(X)} \cdot \text{Sign}(\pi) \cdot \prod_{i=1}^{u} \left( K_{X}^{\rho_i} e_{X}, \prod_{j \in \pi_i} \alpha_j \right) \cdot \prod_{i=1}^{v} \frac{q^{n_i p_i}}{(1 - q^{n_i})^{w_i}}
\]
where $\sum_{i=1}^{u} w_i < \sum_{i=1}^{N} (k_i + 2)$, and the integers $u, v, r_i, r'_i \geq 0, n_i > 0, w_i > 0, p_i \in \{0, 1\}$ and the partition $\pi = \{\pi_1, \ldots, \pi_u\}$ of $\{1, \ldots, N\}$ depend only on the integers $k_i$. Moreover, the coefficients of this linear combination are independent of $q, \alpha_i, X$.

Proof. By Lemma 4.7, $F_{k_1, \ldots, k_N}^{\alpha_1, \ldots, \alpha_N}(q) = \text{Tr} q^0 W(K_1, z) \prod_{i=1}^{N} \Theta_{k_i}^{\alpha_i}(q)$. Combining with Theorem 4.8 and Theorem 4.5, we conclude that
\[
F_{k_1, \ldots, k_N}^{\alpha_1, \ldots, \alpha_N}(q) = \tilde{F}_{k_1, \ldots, k_N}^{\alpha_1, \ldots, \alpha_N}(q) + W_{1,1}
\]
where $W_{1,1}$ is an infinite linear combination of the expressions (4.20), and
\[
\tilde{F}_{k_1, \ldots, k_N}^{\alpha_1, \ldots, \alpha_N}(q) := (-1)^N \sum_{\epsilon(\lambda^{(i)}) = k_i + 2, |\lambda^{(i)}| = 0}^{1 \leq i \leq N} \text{Tr} q^0 W(K_1, z) \prod_{i=1}^{N} \frac{\alpha_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}.
\]
Applying Theorem 4.5 again, we see that $\tilde{F}_{k_1, \ldots, k_N}^{\alpha_1, \ldots, \alpha_N}(q)$ is equal to
\[
(-1)^N (q; q)_{\infty}^{\chi(X)} \cdot \sum_{\epsilon(\lambda^{(i)}) = k_i + 2, |\lambda^{(i)}| = 0}^{1 \leq i \leq N} \prod_{i=1}^{N} \langle (1_X - K_X) \sum_{n \geq 1} m_n^{(i)}, \alpha_i \rangle.
\]
\[
= \prod_{1 \leq i \leq N, n \geq 1} \left( \frac{(-1)^{m_n^{(i)}}}{m_n^{(i)}!} \frac{q^{nm_n^{(i)}}}{(1 - q^n)^{m_n^{(i)}}} \frac{1}{(1 - q^{n})^{\hat{m}_n^{(i)}}(1 - q^{n})^{\bar{m}_n^{(i)}}} \right) + W_{1,2}
\]
where the lower weight term $W_{1,2}$ is an infinite linear combination of the expressions (4.20), and we have put $\lambda^{(i)} = \{\ldots (-n)^{\hat{m}_n^{(i)}} \ldots (-1)^{\hat{m}_1^{(i)}} m_1^{(i)} \ldots n_{n_i^{(i)}} \ldots \}$. So
\[
\tilde{F}_{k_1, \ldots, k_N}^{\alpha_1, \ldots, \alpha_N}(q) = (q; q)_{\infty}^{\chi(X)} \cdot \prod_{i=1}^{N} \Theta_{k_i}^{\alpha_i}(q) + W_{1,2}
\]
\[
= (q; q)_{\infty}^{\chi(X)} \cdot \text{Coeff}_{z_1^1 \cdots z_N^0} \left( \prod_{i=1}^{N} \Theta_{k_i}^{\alpha_i}(q, z_i) \right) + W_{1,2}
\]
by Lemma 4.7. Putting $W_1 = W_{1,1} + W_{1,2}$ completes the proof of (4.19). □

Our next goal is to relate the lower weight term $W_{1,2}$ in (4.23) and (4.24) to multiple $q$-zeta values (with additional variables $z_1, \ldots, z_N$ inserted). We will assume $e_X^{\alpha_i} = 0$ for all $1 \leq i \leq N$. We begin with a lemma strengthening Lemma 4.4.
Lemma 4.9. Let $\lambda^{(1)}, \ldots, \lambda^{(N)} \in \tilde{P}$, and $\alpha_1, \ldots, \alpha_N \in H^*(X)$ be homogeneous. Assume that $e_X \alpha_i = 0$ for every $1 \leq i \leq N$, and $\sum_{i=1}^N |\lambda^{(i)}| = 0$. Put

$$A_N = \text{Tr} q^b \prod_{i=1}^N \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}.$$ 

(i) If $\ell(\lambda^{(i)}) \geq 2$ for every $1 \leq i \leq N$, then $A_N = 0$.

(ii) If $A_N \neq 0$, then $A_N$ is a linear combination of the expressions:

$$\begin{align*}
(q;q)_\infty \cdot \text{Sign}(\pi) \cdot \prod_{i=1}^{u} \left( 1_X, \prod_{j \in \pi_i} \alpha_j \right) & \cdot \prod_{i=1}^{\tilde{\ell}} \frac{(-\tilde{n}_i)q^{\tilde{n}_i\tilde{p}_i}}{1 - q^{\tilde{n}_i}} \\
= (q;q)_\infty \cdot \text{Sign}(\pi) \cdot \prod_{i=1}^{u} \left( 1_X, \prod_{j \in \pi_i} \alpha_j \right) & \cdot \prod_{i=1}^{\ell} \frac{(-n_i^{(i)})w_iq^{n_i^{(i)p_i}}}{(1 - q^{n_i^{(i)}})^{w_i}}
\end{align*}$$

(4.25)

where $\tilde{\ell} = \sum_{i=1}^N \ell(\lambda^{(i)})/2 = \sum_{i=1}^{\ell} w_i, \tilde{p}_i \in \{0, 1\}, 0 \leq p_i \leq w_i$, the partition $\pi = \{\pi_1, \ldots, \pi_u\}$ of $\{1, \ldots, N\}$ depend only on $\lambda^{(1)}, \ldots, \lambda^{(N)}$, the integers $\tilde{n}_1, \ldots, \tilde{n}_{\tilde{\ell}}$ are the positive parts (repeated with multiplicities) in $\lambda^{(1)}, \ldots, \lambda^{(N)}$, the integers $n_1^{(i)}, \ldots, n_{\ell}^{(i)}$ denote the different integers in $\tilde{n}_1, \ldots, \tilde{n}_{\tilde{\ell}}$, and each $n_i^{(i)}$ appears $w_i$ times in $\tilde{n}_1, \ldots, \tilde{n}_{\tilde{\ell}}$.

Proof. (i) As in the proof of Lemma 4.4, $A_N = 0$ unless $\ell(\lambda^{(i)}) = 2$ and $|\alpha_i| = 0$ for every $1 \leq i \leq N$. Assume $\ell(\lambda^{(i)}) = 2$ and $|\alpha_i| = 0$ for every $1 \leq i \leq N$. To prove $A_N = 0$, we will use induction on $N$. If $N = 1$, then $A_1 = 0$ by (4.11). Let $N \geq 2$. If $|\lambda^{(i)}| = 0$ for every $1 \leq i \leq N$, then $A_N = 0$ by (4.9). Assume $|\lambda^{(i_0)}| \neq 0$ for some $1 \leq i_0 \leq N$. Since $\sum_{i=1}^N |\lambda^{(i)}| = 0$, we may further assume that $|\lambda^{(i_0)}| < 0$. By (4.11), Lemma 2.3 (i) and (ii), and induction, we conclude that $A_N = 0$.

(ii) Note that (4.26) follows from (4.25) since each integer $n_i^{(i)}$ appears $w_i$ times among the integers $\tilde{n}_1, \ldots, \tilde{n}_{\tilde{\ell}}$. In the following, we will prove (4.26). To simplify the signs, we will assume that $|\alpha_i|$ is even for every $i$.

Since $A_N \neq 0$, we conclude from (i) that $\ell(\lambda^{(i_0)}) = 1$ for some $1 \leq i_0 \leq N$. If $\lambda^{(i_0)} = (-n_0)$ for some $n_0 > 0$, then by (4.11), $A_N$ is equal to

$$\frac{1}{1 - q^{n_0}} \sum_{r=1}^{i_0-1} \text{Tr} q^b \prod_{i=1}^{r-1} \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \frac{a_{\lambda^{(r)}}(\alpha_r)}{\lambda^{(r)}!} \cdot \prod_{r+1 \leq i \leq N, i \neq i_0} \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \prod_{r+1 \leq i \leq N, i \neq i_0} \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$$

$$+ \frac{q^{n_0}}{1 - q^{n_0}} \sum_{r=i_0+1}^N \text{Tr} q^b \prod_{1 \leq r \leq r-1, i \neq i_0} \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \frac{a_{\lambda^{(r)}}(\alpha_r)}{\lambda^{(r)}!} \cdot \prod_{i=r+1}^N \frac{a_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}.$$
where 0

Note that $[a\alpha_r(\alpha_r)/\lambda(\alpha_r)] = (-n_0) a\alpha_r(\alpha_r)/(\lambda(\alpha_r) - (n_0))!$, and $[a\alpha_r(\alpha_r)/\lambda(\alpha_r)] = (-n_0) a\alpha_r(\alpha_r)/(\lambda(\alpha_r) - (n_0))!$. Therefore, by induction, $A_N$ is a linear combination of the expressions (4.23). We remark that the negative parts (repeated with multiplicities) in $\lambda(1), \ldots, \lambda(N)$ are $-\tilde{n}_1, \ldots, -\tilde{n}_k$. □

**Theorem 4.10.** For $1 \leq i \leq N$, let $k_i \geq 0$ and $\alpha_i \in H^*(X)$ be homogeneous. Assume that $e_X\alpha_i = 0$ for every $1 \leq i \leq N$. Then,

$$\mathcal{F}_{k_1 \ldots k_N}^{\alpha_1 \ldots \alpha_N}(q) = (q; q)_\infty^{\lambda(X)} \cdot \text{Coeff}_{z_1 \ldots z_N}( \prod_{i=1}^N \Theta_k^{\alpha_i}(q, z_i)) + W_{1,2}, \tag{4.27}$$

and $(q; q)_\infty^{\lambda(X)} \cdot W_{1,2}$ is a linear combination of the coefficients of $z_0^1 \cdots z_N^0$ in some multiple $q$-zeta values (with variables $z_1, \ldots, z_N$ inserted) of weights $< \sum_{i=1}^N (k_i + 2)$. Moreover, the coefficients in this linear combination are independent of $q$.

**Proof.** To simplify the signs, we will assume that $|\alpha_i|$ is even for every $i$. Recall that $\mathcal{F}_{k_1 \ldots k_N}^{\alpha_1 \ldots \alpha_N}(q)$ is defined in (4.22), and that (4.27) is just (4.24). From the proofs of (4.24) and Theorem 4.5, we see that the lower weight term $W_{1,2}$ in (4.27) is the contributions of Case B in the proof of Theorem 4.5 to the right-hand-side of (4.22). By (4.15) and Lemma 4.7, up to a re-ordering of the set $\{1, \ldots, N\}$, these contributions are of the following form, denoted by $C_{2,N-N_1}$:

$$\text{Coeff}_{z_1 \ldots z_{N_1}} \left( \prod_{i=1}^{N_1} \Theta_k^{\alpha_i}(q, z_i) \right) \cdot (-1)^{N-N_1} \sum_{\ell(\lambda(i))=k_{i+2}+\ell(\lambda(i))=0} \sum_{\lambda(i) < \lambda(i)} \prod_{N_1+1 \leq i \leq N} \left( (-1)^{p_n(i)} \prod_{n \geq 1} \frac{q^{n p_n(i)}}{p_n(i)! (1 - q^n)^{p_n(i)!}} \frac{1}{1 - q^n} \right)$$

$$\cdot \text{Tr} q^0 \prod_{i=N_1+1}^N a_{\lambda(i) - \lambda(i)} \left( (1 - K_X) \sum_{n \geq 1} p_n(i) \alpha_i \right) \left( \lambda(i) - \lambda(i) \right)!$$

where $0 \leq N_1 < N$, $\lambda(i)$ is denoted by $(\cdots (-n)^{p_n(i)} \cdots (-1)^{p_n(i)} p_1^{(i)} \cdots n^{p_n(i)} \cdots )$, and $\sum_{i=N_1+1}^N |\lambda(i) - \lambda(i)| = 0$. We may let $N_1 = 0$. Put $\mu(i) = \lambda(i) - \lambda(i)$. Then $C_{2,N}$ is

$$(-1)^N \cdot \sum_{\ell(\lambda(i))=\ell(\mu(i))=k_{i+2}+\ell(\mu(i))=0} \prod_{1 \leq n \leq N} \left( (-1)^{p_n(i)} \prod_{n \geq 1} \frac{q^{n p_n(i)}}{p_n(i)! (1 - q^n)^{p_n(i)!}} \frac{1}{1 - q^n} \right)$$

$$\cdot \text{Tr} q^0 \prod_{i=1}^N a_{\mu(i)} \left( (1 - K_X) \sum_{n \geq 1} p_n(i) \alpha_i \right) \mu(i)! \tag{4.28}$$

$$\cdot \text{Tr} q^0 \prod_{i=1}^N a_{\mu(i)} \left( (1 - K_X) \sum_{n \geq 1} p_n(i) \alpha_i \right) \mu(i)! \tag{4.29}$$
where $\mu^{(i)} \neq \emptyset$ for every $1 \leq i \leq N$, and $\sum_{i=1}^{N} |\mu^{(i)}| = 0$. By Lemma (4.29) (ii), the trace on line (4.29) is a linear combination of the expressions:

$$
(q; q)_{\infty}^{-\chi(X)} \cdot \prod_{i=1}^{W} \left( 1 \sum_{\pi_i} \left( (1 - KX)^{\sum_{n \geq 1} p_{n}^{(i)} \alpha_j} \right) \right) \cdot \prod_{i=1}^{\ell} \frac{(-n_i)^{w_i} q^n p_i}{(1 - q^n)^{w_i}}
$$

(4.30)

where $\sum_{i=1}^{\ell} w_i = \sum_{i=1}^{N} \ell(\mu^{(i)})/2$, $0 \leq p_i \leq w_i$, and the mutually distinct integers $n'_1, \ldots, n'_\ell$ appear $w_1, \ldots, w_\ell$ times respectively as the positive parts (repeated with multiplicities) of $\mu^{(1)}, \ldots, \mu^{(N)}$ (so the negative parts, repeated with multiplicities, of $\mu^{(1)}, \ldots, \mu^{(N)}$ are $-n'_1, \ldots, -n'_\ell$ with multiplicities $w_1, \ldots, w_\ell$ respectively).

We now fix the type of the $N$-tuple $(\mu^{(1)}, \ldots, \mu^{(N)})$. Define $\Xi$ to be the set consisting of all the $N$-tuples $(\tilde{\mu}^{(1)}, \ldots, \tilde{\mu}^{(N)})$ obtained from $(\mu^{(1)}, \ldots, \mu^{(N)})$ as follows: take $N$ mutually distinct positive integers $n_1, \ldots, n_\ell$, and obtain $\tilde{\mu}^{(i)}, 1 \leq i \leq N$ from $\mu^{(i)}, 1 \leq i \leq N$ by replacing every part $\pm n'_j$ in $\mu^{(i)}$ by $\pm n_j$. Denote the contribution of the type $\Xi$ to $C_{2,N}$ by $C_{2,N}^\Xi$. Then, $C_{2,N} = \sum_{\Xi} C_{2,N}^\Xi$. Thus, to prove the statement about $(q; q)_{\infty}^{-\chi(X)}$. We, in our theorem, it remains to study $C_{2,N}^\Xi$. For $1 \leq i \leq N$, let $\ell_{i,+}$ (resp. $\ell_{i,-}$) be the sum of the multiplicities of the positive (resp. negative) parts in $\mu^{(i)}$. Denote the parts (repeated with multiplicities) of $\mu^{(i)}$ by $-n'_{j_1}, \ldots, -n'_{j_{\ell_{i,-}}}$, $n_{h_1}, \ldots, n_{h_{\ell_{i,+}}}$. By the definition of $\Xi$, the following data are the same for every $N$-tuple $(\tilde{\mu}^{(1)}, \ldots, \tilde{\mu}^{(N)}) \in \Xi$:

- the indexes $j_1, \ldots, j_{\ell_{i,-}}$ (1 ≤ $i$ ≤ $N$) up to re-ordering
- the indexes $h_1, \ldots, h_{\ell_{i,+}}$ (1 ≤ $i$ ≤ $N$) up to re-ordering
- the partition $\{\pi_1, \ldots, \pi_w\}$ of $\{1, \ldots, N\}$ and integers $w_i, p_i$ in (4.30)
- the coefficient of line (4.30) in the linear combination.

So by (4.28), (4.29) and (4.30), $C_{2,N}^\Xi$ is a linear combination of the expressions

$$
(q; q)_{\infty}^{-\chi(X)} \sum_{n_1, \ldots, n_\ell > 0, n_i \neq n_j \text{ if } i \neq j} \prod_{i=1}^{\ell} \frac{(-n_i)^{w_i} q^n p_i}{(1 - q^n)^{w_i}}
$$

$$
\prod_{i=1}^{\ell} \frac{(-1)^{p_i} q^{p_i} p_i}{p_i !} \frac{1}{(1 - q^n)^{p_i} p_i !}
$$

(4.31)

and the following data are the same for every $N$-tuple $(\tilde{\mu}^{(1)}, \ldots, \tilde{\mu}^{(N)}) \in \Xi$:

$$
\begin{align*}
\prod_{i=1}^{w} \left( 1 \sum_{\pi_i} \left( (1 - KX)^{\sum_{n \geq 1} p^{(i)}_{n} \alpha_j} \right) \right) \cdot \prod_{1 \leq i \leq N} \left( \frac{(-1)^{p_i}}{p_i !} \frac{q^{p_i}}{(1 - q^n)^{p_i} p_i !} \right)
\end{align*}
$$

(we have moved the factor $(-1)^N$ into the coefficients of the linear combination).

Inserting the variables $z_1, \ldots, z_N$, we conclude that $(q; q)_{\infty}^{-\chi(X)} \cdot C_{2,N}^\Xi$ is a linear combination of the coefficients of $z_1^0 \cdots z_N^0$ in the expressions

$$
\left( \sum_{n_1, \ldots, n_\ell > 0, n_i \neq n_j \text{ if } i \neq j} \prod_{i=1}^{\ell} \frac{(-n_i)^{w_i} q^n p_i}{(1 - q^n)^{w_i}} \cdot \prod_{1 \leq i \leq N} \frac{z_i^{-1} \sum_{r=1}^{\ell_{i,+}} n_{h_{i,r}} - \sum_{r=1}^{\ell_{i,-}} n_{h_{i,r}}}{1 \leq i \leq N} \right)
$$

(4.31)
Indeed, the sum of the terms with \( 2 \leq i < s \)

\[ \ell \]

We claim that line (4.31) is the sum of \( \ell! \) multiple \( q \)-zeta values of weight \( \sum_{i=1}^\ell w_i \).

Indeed, the sum of the terms with \( n_1 > \cdots > n_\ell \) in line (4.31) is equal to:

\[
\sum_{n_1 > \cdots > n_\ell} \prod_{i=1}^\ell \frac{(-n_i)^{w_i} q^{n_i} \prod_{r=1}^\ell z_N^{n_i}}{(1 - q^{n_i})^{w_i}} \prod_{1 \leq i \leq \ell} z_i^{\sum_{r=1}^\ell n_{i,r} + \sum_{r=1}^\ell n_{i,r} - \ell_{i,+} - \ell_{i,-}}
\]

where each \( f_i(z_1, \ldots, z_N) \) is a suitable monomial of \( z_1^{\pm 1}, \ldots, z_N^{\pm 1} \). So line (4.31) is the sum of the following \( \ell! \) multiple \( q \)-zeta values:

\[
\sum_{n_1 > \cdots > n_\ell} \prod_{i=1}^\ell \frac{(-n_i)^{w_i} q^{n_i} \prod_{r=1}^\ell f_i(z_1, \ldots, z_N)}{(1 - q^{n_i})^{w_i}}
\]

where \( \sigma \) runs in the symmetric group \( S_\ell \). Furthermore, as in the proof of Lemma 4.7, the product of lines (4.32) and (4.33) is equal to

\[
\prod_{1 \leq i \leq \ell} \frac{1}{\ell_i} \cdot \prod_{1 \leq i \leq N} \left( \sum_{a_i, b_i \geq 0, s_i(i)} a_i \frac{q z_i^{s_i(i)}}{(1 - q^{n_i})^{s_i(i)}} \cdot \sum_{m_i > \cdots > m_i} b_i \frac{z_i^{-m_i t_i(i)}}{(1 - q^{m_i})^{t_i(i)}} \right).
\]

Combining with lines (4.31) and (4.35), we see that \( (q, q)_{\infty}(X) \cdot C_{2, N}^X \) is a linear combination of the coefficients of \( z_1^{0} \cdots z_N^{0} \) in some multiple \( q \)-zeta values of weights

\[
w \ := \ \sum_{i=1}^\ell w_i + \sum_{i=1}^N \left( \sum_{r=1}^{a_i} s_i^{(i)} + \sum_{r=1}^{b_i} t_i^{(i)} \right)
\]

\[
= \ \sum_{i=1}^\ell w_i + \sum_{i=1}^N \left( k_i + 2 - \ell_{i,+} - \ell_{i,-} \right).
\]

Note from (4.30) that \( \sum_{i=1}^\ell w_i = \sum_{i=1}^N \ell(\mu(i))/2 = \sum_{i=1}^N (\ell_{i,+} + \ell_{i,-})/2 \). So we have \( w < \sum_{i=1}^N (k_i + 2) \). This completes the proof of our theorem. \( \square \)
Proposition 4.11. The generating series $F_0^\alpha(q)$ is equal to

\[(q;q)_\infty^{-\chi(X)} \cdot \langle 1_X - K_X, \alpha \rangle \cdot \sum_n \frac{q^n}{(1 - q^n)^2} + (q;q)_\infty^{-\chi(X)} \cdot \langle e_X, \alpha \rangle \cdot \sum_n \frac{nq^n}{1 - q^n}. \quad (4.36)\]

Proof. By Lemma 3.2, $F_0^\alpha(q) = \text{Tr} q^d W(\mathcal{L}_1, z) \mathcal{G}_k(\alpha)$. By Theorem 2.3, we have $\mathcal{G}_0(\alpha) = -\sum_{n>0}(a_{-n}a_n)(\alpha)$ Now (4.36) follows from Remark 4.6 \qed

Remark 4.12. By (4.36), $F_0^{1x}(q) = (q;q)_\infty^{-\chi(X)} \cdot \chi(X) \cdot \sum_n \frac{nq^n}{1 - q^n} = q \frac{d}{dq}(q;q)_\infty^{-\chi(X)}$.

Proposition 4.13. Let $\alpha \in H^*(X)$ be a homogeneous class satisfying $e_X\alpha = 0$. Then, the generating series $F_1^\alpha(q)$ is the coefficient of $z^0$ in

\[
(q;q)_\infty^{-\chi(X)} \cdot \langle K_X - K_X^2, \alpha \rangle \cdot \left( \sum_n \frac{(n-1)q^n}{(1 - q^n)^2} + \sum_n (qz)^n \cdot \left( \sum_m \frac{z^{-2m}}{(1 - q^m)^2} + 2 \sum_{m_1>m_2} \frac{z^{-m_1}}{1 - q^{m_1}} \cdot \frac{z^{-m_2}}{1 - q^{m_2}} \right) \right) \cdot 
\]

Proof. We have $F_1^\alpha(q) = \text{Tr} q^d W(\mathcal{L}_1, z) \mathcal{G}_1(\alpha)$. It is known that $\mathcal{G}_1(\alpha) = -\sum_{\lambda(\alpha) = 3, |\lambda| = 0} \frac{\lambda!}{\lambda!} - \sum_{n>0} \frac{n-1}{2}(a_{-n}a_n)(K_X\alpha). \quad (4.37)$

Applying Remark 4.6 to $-\sum_{n>0} \frac{n-1}{2}\text{Tr} q^d W(\mathcal{L}_1, z) (a_{-n}a_n)(K_X\alpha)$ yields the weight-2 terms in our proposition. Again by Remark 4.6, the trace $\text{Tr} q^d W(\mathcal{L}_1, z) \frac{a_\lambda(\alpha)}{\lambda!}$ with $\lambda(\lambda) = 3$ and $|\lambda| = 0$ contains only weight-3 terms (i.e., does not contain lower weight terms). So the proof of Theorem 4.8 shows that

\[-\sum_{\lambda(\lambda) = 3, |\lambda| = 0} \text{Tr} q^d W(\mathcal{L}_1, z) \frac{a_\lambda(\alpha)}{\lambda!} = (q;q)_\infty^{-\chi(X)} \cdot \text{Coeff}_q \Theta_1^\alpha(q, z). \]

Expanding $\text{Coeff}_q \Theta_1^\alpha(q, z)$ yields the weight-3 terms in our proposition. \qed

Proposition 4.14. Let $\alpha \in H^*(X)$ be homogeneous satisfying $K_X\alpha = e_X\alpha = 0$.

(i) If $|\alpha| < 4$, then $F_k^\alpha(q) = 0$ for every $k \geq 0$;

(ii) Let $|\alpha| = 4$ and $k \geq 0$. Then, $F_k^\alpha(q)$ is the coefficient of $z^0$ in

\[-(q;q)_\infty^{-\chi(X)} \cdot \langle 1_X, \alpha \rangle \cdot \sum_{a, s_1, \ldots, a, b, t_1, \ldots, t_k \geq 1} \prod_{i=1}^a \frac{(-1)^{s_i}}{s_i!} \cdot \prod_{j=1}^b \frac{1}{t_j^{k+2}}. \]
\[ \sum_{n_1 \geq \cdots \geq n_a} \prod_{i=1}^{a} \frac{(qz)^{n_is_i}}{(1 - q^{n_is_i})} \cdot \sum_{m_1 \geq \cdots \geq m_b} \prod_{j=1}^{b} \frac{z^{-m_jt_j}}{(1 - q^{m_j})^{t_j}}. \]  

(4.38)

In particular, if \( 2 \nmid k \), then \( F_k^\alpha(q) = 0 \).

**Proof.** Since \( K_X \alpha = e_X \alpha = 0 \), we conclude from Theorem 2.3 that

\[ \Theta_k(\alpha) = - \sum_{\ell(\lambda) = k+2, |\lambda| = 0} \frac{a_\lambda(\alpha)}{\lambda!}. \]

As in the proof of Proposition 4.13, Remark 4.6 and the proof of Theorem 4.8 yield

\[ F_k^\alpha(q) = (q; q)^{-\chi(X)} \cdot \text{Coeff}_{z^0} \Theta_k^\alpha(q, z). \]

By the definition of \( \Theta_k^\alpha(q, z) \) in (4.17), we see that (i) holds and that our formula for \( F_k^\alpha(q) \) with \(|\alpha| = 4 \) and \( k \geq 0 \) holds. Note that line (4.38) can be rewritten as

\[ \sum_{n_1 \geq \cdots \geq n_a} \prod_{i=1}^{a} \frac{(qz^2)^{n_is_i/2}}{(1 - q^{n_is_i})^s_i} \cdot \sum_{m_1 \geq \cdots \geq m_b} \prod_{j=1}^{b} \frac{(q^2z^{-2})^{m_jt_j/2}}{(1 - q^2m_j)^{t_j}}. \]

Therefore, if \(|\alpha| = 4 \) and \( 2 \nmid k \), then the role of \( a, s_1, \ldots, s_a \) and the role of \( b, t_1, \ldots, t_b \) in the above formula of \( F_k^\alpha(q) \) are anti-symmetric; so \( F_k^\alpha(q) = 0 \). \qed

5. The reduced series \( \langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle' \)

In this section, we will prove Conjecture 1.1 modulo the lower weight term. Moreover, for abelian surfaces, we will verify Conjecture 1.1.

Let \( L \) be a line bundle on the smooth projective surface \( X \). It induces the tautological rank-\( n \) bundle \( L^{[n]} \) over the Hilbert scheme \( X^{[n]} \):

\[ L^{[n]} = p_{1*}(p_2^*L|_{Z_n}) \]

where \( Z_n \) is the universal codimension-2 subscheme of \( X^{[n]} \times X \), and \( p_1 \) and \( p_2 \) are the projections of \( X^{[n]} \times X \) to \( X^{[n]} \) and \( X \) respectively. By the Grothendieck-Riemann-Roch Theorem and (2.1), we obtain

\[ \text{ch}(L^{[n]}) = p_{1*}(\text{ch}(O_{Z_n}) \cdot p_2^*\text{ch}(L) \cdot p_2^*\text{td}(X)) = G(1_X, n) + G(L, n) + G(L^2/2, n). \]

(5.1)

Since the cohomology degree of \( G_i(\alpha, n) \) is \( 2i + |\alpha| \), we have

\[ \text{ch}_k(L^{[n]}) = G_k(1_X, n) + G_{k-1}(L, n) + G_{k-2}(L^2/2, n). \]

(5.2)

Following Okounkov [Oko], we have defined the generating series \( \langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle \) and its reduced version \( \langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle' \) in (1.1) and (1.2) respectively.

**Theorem 5.1.** Let \( L_1, \ldots, L_N \) be line bundles over \( X \), and \( k_1, \ldots, k_N \geq 0 \). Then,

\[ \langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle' = \text{Coeff}_{z^0} \left( \prod_{i=1}^{N} \Theta_{k_i}^{1x}(q, z_i) \right) + W, \]

(5.3)
and the lower weight term $W$ is an infinite linear combination of the expressions:

$$
\prod_{i=1}^{u} \left\langle K_{X}^{i} e_{X}^{r_{i}}, L_{1}^{\ell_{1}} \cdots L_{N}^{\ell_{N}} \right\rangle \cdot \prod_{i=1}^{v} \frac{q^{n_{i} w_{i} p_{i}}}{(1 - q^{n_{i}})^{w_{i}}}
$$

where $\sum_{i=1}^{u} w_{i} < \sum_{i=1}^{N} (k_{i} + 2)$, and the integers $u, v, r_{i}, r_{i}', \ell_{i,j} \geq 0, n_{i} > 0, w_{i} > 0, p_{i} \in \{0, 1\}$ depend only on $k_{1}, \ldots, k_{N}$. Furthermore, all the coefficients of this linear combination are independent of $q, L_{1}, \ldots, L_{N}$ and $X$.

**Proof.** We conclude from (1.1), (1.2), (5.2) and (2.2) that

$$
\langle \text{ch}^{L}_{k_{1}} \cdots \text{ch}^{L}_{k_{N}} \rangle' = (q; q)_{\infty} \cdot F_{k_{1}, \ldots, k_{N}}^{1} (q) + (q; q)_{\infty} \cdot A
$$

where $A$ is the sum of the series $F_{k_{1}', \ldots, k_{N}'}^{\alpha_{1}, \ldots, \alpha_{N}}$ such that for every $1 \leq i \leq N$,

$$
(\alpha_{i}, k_{i}') \in \{(1_{X}, k_{i}), (L_{i}, k_{i} - 1), (L_{i}^{2}/2, k_{i} - 2)\},
$$

and $\sum_{i=1}^{N} k_{i}' < \sum_{i=1}^{N} k_{i}$. Now our result follows from Theorem 4.8. \qed

**Theorem 5.2.** Let $L_{1}, \ldots, L_{N}$ be line bundles over an abelian surface $X$, and $k_{1}, \ldots, k_{N} \geq 0$. Then, the lower weight term $W$ in (5.3) is a linear combination of the coefficients of $z_{1}^{0} \cdots z_{N}^{0}$ in some multiple $q$-zeta values (with additional variables $z_{1}, \ldots, z_{N}$ inserted) of weights $< \sum_{i=1}^{N} (k_{i} + 2)$. Moreover, the coefficients in this linear combination are independent of $q$.

**Proof.** Since $e_{X} = K_{X} = 0$, $F_{k_{1}, \ldots, k_{N}}^{1} = F_{k_{1}, \ldots, k_{N}}^{\alpha_{1}, \ldots, \alpha_{N}}$ by Lemma 3.2. Theorem 2.3 and (1.22). By Theorem 4.10 and the proof of Theorem 5.1 our theorem follows. \qed

Our next two propositions compute the series $\langle \text{ch}^{L}_{k} \rangle$ completely, and should offer some insight into the lower weight term $W$ in Theorem 5.1. Proposition 5.3 calculates $\langle \text{ch}^{L}_{k} \rangle$ by assuming $e_{X} = 0$, while Proposition 5.4 deals with the series $\langle \text{ch}^{L}_{k} \rangle, k \geq 2$ by assuming $e_{X} = K_{X} = 0$ (i.e., by assuming that $X$ is an abelian surface). Note from (5.2) that when $\chi(X) = 0$, we have

$$
\langle \text{ch}^{L}_{k} \rangle' = \langle \text{ch}^{L}_{k} \rangle = F_{k}^{1} (q) + F_{k-1}^{L} (q) + \frac{1}{2} \cdot F_{k-2}^{L} (q).
$$

**Proposition 5.3.** Let $L$ be a line bundle over a smooth projective surface $X$ with $e_{X} = 0$. Then, the series $\langle \text{ch}^{L}_{k} \rangle$ is the coefficient of $z^{0}$ in

$$
-\langle K_{X}, L \rangle \cdot \sum_{n} \frac{q^{n}}{(1 - q^{n})^{2}} - \frac{\langle K_{X}, K_{X} \rangle}{2} \cdot \sum_{n} \frac{(n - 1) q^{n}}{(1 - q^{n})^{2}}
$$

$$
- \frac{\langle K_{X}, K_{X} \rangle}{2} \cdot \sum_{n} \frac{(q z)^{n}}{1 - q^{n}} \cdot \left( \sum_{m} \frac{z^{-2m}}{(1 - q^{m})^{2}} + 2 \sum_{m_{1} > m_{2}} \frac{z^{-m_{1}} - z^{-m_{2}}}{1 - q^{m_{1}} - 1 - q^{m_{2}}} \right).
$$

**Proof.** Our formula follows from (5.4), (4.36) and Proposition 4.13. \qed
Proposition 5.4. Let $L$ be a line bundle over an abelian surface $X$. If $2 \nmid k$, then $\langle ch^k_L \rangle = 0$. If $2 \mid k$, the generating series $\langle ch^k_L \rangle$ is the coefficient of $z^0$ in

\[-\frac{\langle L, L \rangle}{2} \cdot \sum_{a, s_1, \ldots, s_a, t_1, \ldots, t_b \geq 1} \prod_{i=1}^a (-1)^{s_i} \cdot \prod_{j=1}^b \frac{1}{t_j!} \cdot \sum_{m_1 > \cdots > m_a} \prod_{n=1}^a (q^n)^{m_i} \cdot \sum_{m_1 > \cdots > m_b} \prod_{j=1}^b \frac{z^{-m_j t_j}}{(1 - q^{m_j})^{t_j}}.\]

Proof. Follows immediately from (5.4) and Proposition 4.14.

\[\square\]

6. Applications to the universal constants in $\sum_n c(T_{X[n]}) q^n$

Let $x \in H^4(X)$ be the cohomology class of a point in the surface $X$. In this section, we will compute $F_{k_1, \ldots, k_N}(x)$ in terms of the universal constants in the expression of $\sum_n c(T_{X[n]}) q^n$ formulated in [Boi, BN]. Comparing with Proposition 4.14 (ii) enables us to determine some of these universal constants.

Let $C_i = \binom{2i}{i}/(i + 1)$ be the Catalan number and $\sigma_1(i) = \sum_{j \mid i} j$. Recall that $\mathcal{P} = \mathcal{P}_+$ is the set of all the usual partitions. The following lemma is from [Boi, BN].

Lemma 6.1. There exist unique rational numbers $b_\mu, f_\mu, g_\mu, h_{\mu}$ depending only on the partitions $\mu \in \mathcal{P}$ such that $\sum_n c(T_{X[n]}) q^n$ is equal to

$$\exp \left( \sum_{\mu \in \mathcal{P}} q^{\mu[i]} \left( b_\mu a_{-\mu}(1_X) + f_\mu a_{-\mu}(e_X) + g_\mu a_{-\mu}(K_X) + h_\mu a_{-\mu}(K_X^2) \right) \right) |0\rangle.$$ 

In addition, for $i \geq 1$, we have $b_{2i} = 0$, $b_{2i-1} = (-1)^{i-1} C_{i-1}/(2i-1)$, $b_{1^i} = f_{1^i} = g_{1^i} = h_{1^i} = 0$.

Our goal is to compute $F_{k_1, \ldots, k_N}(x)$ in terms of the universal constants $b_\mu, f_\mu, g_\mu$ and $h_\mu$. Using the definition of the operators $\mathfrak{G}_{k_i}(\alpha_i)$, we see that

\[F_{k_1, \ldots, k_N}^{\alpha_1, \ldots, \alpha_N}(q) = \sum_n q^n \left\langle \left( \prod_{i=1}^N G_{k_i}(\alpha_i, n) \right) c(T_{X[n]}), 1_{X[n]} \right\rangle = \sum_n q^n \left\langle \left( \prod_{i=1}^N \mathfrak{G}_{k_i}(\alpha_i) \right) c(T_{X[n]}), 1_{X[n]} \right\rangle = \left\langle \left( \prod_{i=1}^N \mathfrak{G}_{k_i}(\alpha_i) \right) \sum_n c(T_{X[n]} q^n), 1 \right\rangle \]

where we have put $|1\rangle = \sum_n 1_{X[n]} = \exp (a_{-1}(1_X)) \cdot |0\rangle$. 

(6.1)
Lemma 6.2. Let \( w \in \mathbb{H}_X \), and \( \mathcal{G} \) be a (possibly infinite) sum of monomials of Heisenberg creation operators. Then, \( \langle \mathcal{G} w, |1\rangle = \langle \mathcal{G}|0\rangle , |1\rangle \cdot \langle w, |1\rangle \).

Proof. By linearity, it suffices to prove that
\[
\left\langle \prod_{i=1}^s a_{-n_i}(\alpha_i) \cdot \prod_{j=1}^t a_{-m_j}(\beta_j) |0\rangle , |1\rangle \right\rangle = \left\langle \prod_{i=1}^s a_{-n_i}(\alpha_i) |0\rangle , |1\rangle \right\rangle \cdot \left\langle \prod_{j=1}^t a_{-m_j}(\beta_j) |0\rangle , |1\rangle \right\rangle
\]
where \( n_1, \ldots, n_s, m_1, \ldots, m_t > 0 \), and \( \alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t \) are homogeneous. Indeed, if \( a_{-n_i}(\alpha_i) \not\in \mathcal{A}_a(x) \) for some \( i \) or if \( a_{-m_j}(\beta_j) \not\in \mathcal{A}_a(x) \) for some \( j \), then both sides are equal to 0. Otherwise, letting \( a_{-n_i}(\alpha_i) = u_i a_{-1}(x) \) for every \( i \) and \( a_{-m_j}(\beta_j) = v_j a_{-1}(x) \) for every \( j \), we see that both sides are \( u_1 \cdots u_s v_1 \cdots v_t \). \( \square \)

Lemma 6.3. Let \( b_{(1)} \) and \( b_{(i+1)} \) be from Lemma 6.1. Let \( \tilde{b}_{(i,1)} = (j+1)b_{(i+1)} \) if \( i = 1 \), and \( \tilde{b}_{(i,1)} = b_{(i,1)} \) if \( i > 1 \). Then, \( F_{k_1, \ldots, k_N}(q) \) is equal to
\[
(q;q)_\infty^{\chi(X)} \cdot (-1)^N \sum_{\sum_{i=1}^N m_{i,s} = k_{i+2}} \frac{1}{\prod_{1 \leq i \leq N}(\sum_{s \geq 1} s m_{i,s})!} \cdot \prod_{s \geq 1} \left( \frac{(-s)^{m_{s,s}!}}{\prod_{i=1}^N m_{i,s}!} \sum_{t_0 + t_1 + \cdots + t_j = m_s} \prod_{j=0}^{+\infty} \frac{(\tilde{b}_{(i,1)} q^{s+j})^{t_j}}{t_j!} \right)
\]
where \( m_{i,s} \geq 0 \) for every \( i \) and \( s \), and \( m_s = \sum_{i=1}^N m_{i,s} \) for every \( s \geq 1 \).

Proof. By (6.1) and Theorem 2.3, we obtain
\[
F_{k_1, \ldots, k_N}(q) = \left\langle \left( \prod_{i=1}^N \mathcal{G}_{k_i}(x) \right) \sum_n c(T_X[n]) q^n , |1\rangle \right\rangle ,
\]
(6.2)
\[
\prod_{i=1}^N \mathcal{G}_{k_i}(x) = (-1)^N \sum_{\ell(\lambda(i)) = k_{i+2}} \prod_{i=1}^N \frac{a_{\lambda(i)}(x)}{(\lambda(i))!}.
\]
(6.3)
Note that \( \tau_{\ell_s} x = \underbrace{x \otimes \cdots \otimes x}_{\ell \text{ times}}, K_X^2 = \langle K_X, K_X \rangle x, e_X = \chi(X) x, \) and
\[
\tau_{\ell_s} 1_X = \underbrace{1_X \otimes x \otimes \cdots \otimes x}_{(\ell - 1) \text{ times}} + \cdots + \underbrace{x \otimes \cdots \otimes x \otimes 1_X + w}_{(\ell - 1) \text{ times}}
\]
(6.4)
where \( w \) is a sum of cohomology classes of the form \( \alpha_1 \otimes \cdots \otimes \alpha_{\ell} \) with \( 0 < |\alpha_i| < 4 \) for some \( i \). So for a generalized partition \( \lambda \), positive integers \( n_1, \ldots, n_s \) and homogeneous classes \( \alpha_1, \ldots, \alpha_s \in H^*(X) \), we have
\[
\langle a_{\lambda}(x) a_{-n_1}(\alpha_1) \cdots a_{-n_s}(\alpha_s) |0\rangle , |1\rangle \rangle = 0
\]
(6.5)
if $0 < |\alpha_i| < 4$ for some $i$, or if $a_{-n_i}(\alpha_i) \in \mathbb{C} a_{-j}(x)$ for some $i$ and for some $j > 1$. Combining with (6.2), (6.3) and Lemma 6.1 we see that $F_{k_1, \ldots, k_N}^{x \ldots x}(q)$ equals

$$
\left\langle \left( \prod_{i=1}^{N} \mathfrak{g}_{k_i}(x) \right) \exp \left( \sum_{\mu \in P} b_\mu a_{-\mu}(1_X)q^{\mu |} + \sum_{i} \tilde{f}_{(1)} a_{-(1)}(x)q^{i} \right) \right| 0, 1 \right\rangle
$$

$$
= \left\langle \exp \left( \sum_{i} \tilde{f}_{(1)} a_{-(1)}(x)q^{i} \right) \prod_{i=1}^{N} \mathfrak{g}_{k_i}(x) \cdot \exp \left( \sum_{\mu \in P} b_\mu a_{-\mu}(1_X)q^{\mu |} \right) \left| 0, 1 \right\rangle
$$

where $\tilde{f}_{(1)} = \chi(X) \cdot f_{(1)}$. By Lemma 6.2 $F_{k_1, \ldots, k_N}^{x \ldots x}(q)$ is equal to

$$
\left\langle \exp \left( \sum_{i} \tilde{f}_{(1)} a_{-(1)}(x)q^{i} \right) \right| 0, 1 \right\rangle \cdot \left\langle \prod_{i=1}^{N} \mathfrak{g}_{k_i}(x) \cdot \exp \left( \sum_{\mu \in P} b_\mu a_{-\mu}(1_X)q^{\mu |} \right) \right| 0, 1 \right\rangle. \quad (6.6)
$$

In particular, setting $N = 0$, we conclude that

$$
\left\langle \exp \left( \sum_{i} \tilde{f}_{(1)} a_{-(1)}(x)q^{i} \right) \right| 0, 1 \right\rangle = F(q) = (q; q)_{\infty}^{-\chi(X)}. \quad (6.7)
$$

It follows from (6.6) and (6.4) that $F_{k_1, \ldots, k_N}^{x \ldots x}(q)$ is equal to

$$(q; q)_{\infty}^{-\chi(X)} \left\langle \prod_{i=1}^{N} \mathfrak{g}_{k_i}(x) \cdot \exp \left( \sum_{\mu \in P} b_\mu a_{-\mu}(1_X)q^{\mu |} \right) \right| 0, 1 \right\rangle
$$

$$
= (q; q)_{\infty}^{-\chi(X)} \left\langle \prod_{i=1}^{N} \mathfrak{g}_{k_i}(x) \cdot \exp \left( \sum_{\mu \in P} \tilde{b}_{(i,1)} a_{-i}(1_X) a_{-1}(x)^{q^{i+j}} \right) \right| 0, 1 \right\rangle. \quad (6.8)
$$

where $\tilde{b}_{(i,1)} = (j + 1) b_{(1)}$ if $i = 1$, and $\tilde{b}_{(i,1)} = b_{(i,1)}$ if $i > 1$. Let $\lambda^{(1)}, \ldots, \lambda^{(N)}$ be from the right-hand-side of (6.3). In order to have a nonzero pairing

$$
\left\langle \prod_{i=1}^{N} a_{\lambda^{(i)}}(x) \cdot \exp \left( \sum_{i \geq 1, j \geq 0} \tilde{b}_{(i,1)} a_{-i}(1_X) a_{-1}(x)^{q^{i+j}} \right) \right| 0, 1 \right\rangle, \quad (6.9)
$$

each $\lambda^{(i)}$ with $1 \leq i \leq N$ must be of the form $((-1)^{n_i} 1^{m_{i,1}} 2^{m_{i,2}} \cdots)$; since $\ell(\lambda^{(i)}) = k_i + 2$ and $|\lambda^{(i)}| = 0$, we get $n_i + \sum_{s \geq 1} m_{i,s} = k_i + 2$ and $n_i = \sum_{s \geq 1} s m_{i,s}$; so

$$
\sum_{s \geq 1} (s + 1) m_{i,s} = k_i + 2. \quad (6.10)
$$
In this case, using Lemma 6.2 we see that (6.9) is equal to

\[
\left\langle a_{-1}(x) \sum_{i} \prod_{s} a_{s}(x)^{m_{i,s}} \cdot \exp \left( \sum_{i \geq 1, j \geq 0} \tilde{b}_{i,1}(X) a_{-1}(x)^i q^{i+j} \right) \left| 0 \right>, \left| 1 \right> \right.
\]

\[
\left. = \left\langle \prod_{1 \leq i \leq N, s \geq 1} a_{s}(x)^{m_{i,s}} \cdot \exp \left( \sum_{i \geq 1, j \geq 0} \tilde{b}_{i,1}(X) a_{-1}(x)^i q^{i+j} \right) \left| 0 \right>, \left| 1 \right> \right. \]

Putting \( m_s = \sum_{i=1}^{N} m_{i,s} \) for every \( s \geq 1 \). Then, (6.9) is equal to

\[
\left\langle \prod_{s \geq 1} a_{s}(x)^{m_s} \cdot \prod_{i \geq 1, j \geq 0} \sum_{t} \frac{1}{t!} \left( \tilde{b}_{i,1}(X) a_{-1}(x)^i q^{i+j} \right)^{t} \left| 0 \right>, \left| 1 \right> \right. \]

\[
\left. = \prod_{s \geq 1} \left( (-s)^{m_s} m_s! \sum_{t_0 + t_1 + \ldots + t_j = m_s} \prod_{j=0}^{+\infty} \frac{\tilde{b}_{i,1}(X)^{s+j} t_j}{t_j!} \right) \right. \]

Combining this with (6.8), (6.3), (6.9) and (6.10), \( F_{x_{1}, \ldots, x_{N}}^{x}(q) \) is equal to

\[
(q; q)^{\chi(X)} \cdot (-1)^N \sum_{\sum_{s \geq 1} 1(s+1) m_{i,s} = i + 1} \prod_{i=1}^{N} \frac{1}{(\sum_{s \geq 1} 1 s m_{i,s})!} \cdot \prod_{s \geq 1} \left( \prod_{i \geq 1, s \geq 1} \frac{(-s)^{m_{i,s}} t_{j}}{t_{j}!} \sum_{t_0 + t_1 + \ldots + t_j = m_s} \prod_{j=0}^{+\infty} \frac{\tilde{b}_{i,1}(X)^{s+j} t_j}{t_j!} \right) \]

where \( m_{i,s} \geq 0 \) for every \( i \) and \( s \), and \( m_s = \sum_{i=1}^{N} m_{i,s} \) for every \( s \geq 1 \).

Our next result determines the universal constants \( \tilde{b}_{i,1} \) with \( i \geq 2 \) and \( j \geq 0 \).

**Theorem 6.4.** Let the numbers \( b_{i,1} \) and \( \tilde{b}_{i,1} \) be from Lemma 6.1. Let \( \tilde{b}_{i,1} = (j+1)b_{i,1+1} = \sigma_1(j+1) \) if \( i = 1 \), and \( \tilde{b}_{i,1} = b_{i,1} \) if \( i > 1 \).

(i) If \( i \) is an even positive integer, then \( \tilde{b}_{i,1} = 0 \) for all \( j \geq 0 \).

(ii) Let \( i > 1 \) be odd. Then, \( \sum_{j \geq 0} b_{i,1}(j+1) q^{i+j} \) is equal to

\[
\sum_{\sum_{s \geq 1} 1(s+1) m_{s} = i+1} \frac{1}{(\sum_{s \geq 1} 1 s m_{s})!} \prod_{2s} \left( \sum_{j \geq 0} \prod_{j=0}^{+\infty} \frac{((-s)^{m_{s}} t_{j})}{t_{j}!} \sum_{t_0 + t_1 + \ldots + t_j = m_s} \prod_{j=0}^{+\infty} \frac{\tilde{b}_{i,1}(X)^{s+j} t_j}{t_j!} \right) \]

\[ - \sum_{\sum_{u=1}^{a} n_u s_u = \sum_{v=1}^{b} m_v t_v} \prod_{u=1}^{a} \frac{(-1)^{n_u} t_u! \cdot (1 - q^{n_u} s_u)}{s_u!} \cdot \prod_{v=1}^{b} \frac{1}{t_v! \cdot (1 - q^{m_v} t_v)}. \]
Proof. (i) Setting \( N = 1 \) in Lemma 6.3, we see that \( P^x_k(q) \) is equal to

\[
-(q;q)_\infty \chi(x) \cdot \sum_{s \geq 1} \frac{1}{(\sum_{s \geq 1} s m_s)!} \prod_{s \geq 1} \left( \sum_{\sum_{j=0}^{+\infty} t_j = m_s} \prod_{j=0}^{+\infty} \frac{((-s)\tilde{b}(s,1)q^{s+j})^{t_j}}{t_j!} \right).
\]

Comparing this with (6.12) which holds for all \( k \geq 0 \), we obtain

\[
\sum_{s \geq 1} \frac{1}{(\sum_{s \geq 1} s m_s)!} \prod_{s \geq 1} \left( \sum_{\sum_{j=0}^{+\infty} t_j = m_s} \prod_{j=0}^{+\infty} \frac{((-s)\tilde{b}(s,1)q^{s+j})^{t_j}}{t_j!} \right)
= \sum_{a \geq 1} \prod_{u=1}^{a} \frac{(-1)^{s_u} q^{n_u s_u}}{s_u!} \cdot \prod_{v=1}^{b} \frac{1}{t_v!} \cdot (1 - q^{m_v})^{t_v}.
\]

The largest value of \( s \) satisfying \( \sum_{s \geq 1} (s+1)m_s = k + 2 \) is given by \( s = k + 1 \) together with \( m_{k+1} = 1 \). So the above identity can be rewritten as

\[
\frac{1}{k!} \prod_{j=0}^{(k+1)q^{(k+1)+j}} 
= \sum_{s \geq 1} \frac{1}{(\sum_{s \geq 1} s m_s)!} \prod_{s \geq 1} \left( \sum_{\sum_{j=0}^{+\infty} t_j = m_s} \prod_{j=0}^{+\infty} \frac{((-s)\tilde{b}(s,1)q^{s+j})^{t_j}}{t_j!} \right)
- \sum_{a \geq 1} \prod_{u=1}^{a} \frac{(-1)^{s_u} q^{n_u s_u}}{s_u!} \cdot \prod_{v=1}^{b} \frac{1}{t_v!} \cdot (1 - q^{m_v})^{t_v}.
\]

Replacing \( k + 1 \) by \( i \), we conclude that \( \frac{1}{(i-1)!} \sum_{j=0}^{+\infty} \tilde{b}(i,1)v^{i+j} \) is equal to

\[
\sum_{s \leq i<s+1} \frac{1}{(\sum_{s \geq 1} s m_s)!} \prod_{s \geq 1} \left( \sum_{\sum_{j=0}^{+\infty} t_j = m_s} \prod_{j=0}^{+\infty} \frac{((-s)\tilde{b}(s,1)q^{s+j})^{t_j}}{t_j!} \right)
- \sum_{a \geq 1} \prod_{u=1}^{a} \frac{(-1)^{s_u} q^{n_u s_u}}{s_u!} \cdot \prod_{v=1}^{b} \frac{1}{t_v!} \cdot (1 - q^{m_v})^{t_v}. \tag{6.12}
\]

Note that (6.12) is equal to 0 if \( 2|i \). Letting \( i = 2 \), we get \( \sum_{j=0}^{+\infty} \tilde{b}(2,1)v^{2+j} = 0 \). Therefore, \( \tilde{b}(2,1) = 0 \) for every \( j \geq 0 \). Hence we have \( b(2,1) = 0 \) for every \( j \geq 0 \).
Next, let \( i > 2 \) and \( 2 | i \). Assume inductively that \( b_{(s,1^i)} = 0 \) for every \( j \geq 0 \) whenever \( 2 \leq s < i \) and \( 2 | s \). Since (6.12) is 0, \( \frac{1}{(i-1)!} \sum_{j \geq 0} b_{(i,1^j)} q^{i+j} \) is equal to

\[
\sum_{1 \leq s < (s+1)m = i+1} \frac{1}{(\sum s^m_s)!} \prod_{s \geq 1} \left( \sum_{\sum j_0 t_j = m_s} \prod_{j \geq 0} (-s)^{b_{(s,1^j)} q^{s+j} t_j} \right)
\]

The condition \( \sum_{1 \leq s < i} (s+1)m_s = i+1 \) implies that \( m_s > 0 \) for some even integer \( s < i \). Hence \( \sum_{j \geq 0} b_{(i,1^j)} q^{i+j} = 0 \) by induction. So \( b_{(i,1^j)} = 0 \) for all \( j \geq 0 \).

(ii) Follows immediately from (i), (6.11) and (6.12). \( \square \)

Note that \( b_{(2i)} = 0, i \geq 1 \) has been proved in [Boi] [BN] (see Lemma 6.1). Next, using the universal constants \( f_{(2,1^i)} \), \( g_{(2,1^i)} \) and \( h_{(2,1^i)} \), we compute the generating series \( F^\alpha_1(q) \) for a cohomology class \( \alpha \) with \( |\alpha| < 4 \).

**Lemma 6.5.** Let \( f_{(2,1^i)}, g_{(2,1^i)} \) and \( h_{(2,1^i)} \) be from Lemma 6.1, and let \( \alpha \in H^*(X) \) be a homogeneous class with \( |\alpha| < 4 \). Then,

(i) \( F^1\chi_\alpha(q) = \sum_{j \geq 0} \tilde{f}_{(2,1^i)} q^{2+j} \) where \( \tilde{f}_{(2,1^i)} = \chi(X) \cdot f_{(2,1^i)} + \langle K_X, K_X \rangle \cdot h_{(2,1^i)} \);

(ii) \( F^\alpha_1(q) = (q; q)_{\infty}^{-\chi(X)} \sum_{j \geq 0} g_{(2,1^i)} q^{2+j} \cdot \langle \alpha, K_X \rangle \).

**Proof.** (i) Let \( \alpha \in H^*(X) \) be an arbitrary cohomology class. Note that for all \( n \geq 1 \) and \( A \in \mathbb{H}_X \), we have \( \langle (n-1)(a_{-n} a_n)(K_X \alpha)A, |1| \rangle = 0 \). By (6.1) and (4.37),

\[
F^\alpha_1(q) = \left\langle \mathbf{G}_1(\alpha) \sum_n c(T_X^{[n]} \alpha) q^n, |1| \right\rangle = -\sum_{\ell(\lambda) = 3, |\lambda| = 0} \frac{1}{\lambda!} \left\langle a_\lambda(\alpha) \sum_n c(T_X^{[n]} \alpha) q^n, |1| \right\rangle = -\frac{1}{2} \left\langle (a_{-1} a_{-1} a_2)(\alpha) \sum_n c(T_X^{[n]} \alpha) q^n, |1| \right\rangle = \frac{1}{2} \left\langle a_2(\alpha) \sum_n c(T_X^{[n]} \alpha) q^n, |1| \right\rangle.
\]

(6.13)
Set $\alpha = 1_X$. Put $\tilde{f}_\mu = \chi(X) \cdot f_\mu + \langle K_X, K_X \rangle \cdot h_\mu$. By Lemma 6.1 $F_1^{1_X}(q)$ equals

$$-rac{1}{2} \left\langle a_2(1_X) \exp \left( \sum_{\mu \in \mathcal{P}} q^{\mu} \tilde{f}_\mu a_{-\mu}(x) \right) \right| \langle 0 \rangle, |1 \rangle \right\rangle \right.$$ 

$$= -\frac{1}{2} \left\langle a_2(1_X) \exp \left( \sum_{j \geq 0} q^{2+j} \tilde{f}_{(2,1)^j} a_{-2}(x) a_{-1}(x)^j \right) \right| \langle 0 \rangle, |1 \rangle \right\rangle \right.$$ 

$$= -\frac{1}{2} \left\langle a_2(1_X) \left( \sum_{j \geq 0} q^{2+j} \tilde{f}_{(2,1)^j} a_{-2}(x) a_{-1}(x)^j \right) \right| \langle 0 \rangle, |1 \rangle \right\rangle \right.$$ 

$$= \sum_{j \geq 0} \tilde{f}_{(2,1)^j} q^{2+j}.$$ 

(ii) Let $0 < |\alpha| < 4$. Again by (6.13) and Lemma 6.1 $F_1^\alpha(q)$ is equal to

$$-\frac{1}{2} (q; q)^{-\chi(X)}_\infty \left\langle a_2(\alpha) \exp \left( \sum_{\mu \in \mathcal{P}} q^{\mu} \alpha \chi(K_X) \right) \right| \langle 0 \rangle, |1 \rangle \right\rangle \right.$$ 

$$= -\frac{1}{2} (q; q)^{-\chi(X)}_\infty \left\langle a_2(\alpha) \exp \left( \sum_{j \geq 0} q^{2j} \alpha \chi(g_{(2,1)^j} a_{-2}(K_X) a_{-1}(x)^j) \right) \right| \langle 0 \rangle, |1 \rangle \right\rangle \right.$$ 

Therefore, $F_1^\alpha(q) = (q; q)^{-\chi(X)}_\infty \sum_{j \geq 0} g_{(2,1)^j} q^{2+j} \langle \alpha, K_X \rangle$ when $0 < |\alpha| < 4$. \hfill $\square$

**Proposition 6.6.** Let the numbers $g_{(2,1)^j}$ and $h_{(2,1)^j}$ be from Lemma 6.1. Then, $g_{(2,1)^j} = -h_{(2,1)^j}$. Moreover, $\sum_{j \geq 0} g_{(2,1)^j} q^{2j}$ is the coefficient of $z^0$ in

$$\frac{1}{2} \left( \sum_{n} \frac{(n-1)q^n}{(1-q^n)^2} + \sum_{n} (qz)^n \left( \sum_{m \geq 0} z^{-2m} + 2 \sum_{m > 0} \frac{z^{-m_1} z^{-m_2}}{1-q^{m_1} 1-q^{m_2}} \right) \right).$$

**Proof.** For simplicity, denote the previous line by $A(z)$. Let $X$ be a smooth projective surface with $\chi(X) = 0$ and $\langle K_X, K_X \rangle \neq 0$. On one hand, applying Lemma 6.3 (i) and Proposition 4.13 to $F_1^{1_X}(q)$, we conclude that $\sum_{j \geq 0} h_{(2,1)^j} q^{2+j}$ is the coefficient of $z^0$ in $-A(z)$. On the other hand, applying Lemma 6.3 (ii) and Proposition 4.13 to $F_1^{K_X}(q)$, we see that $\sum_{j \geq 0} g_{(2,1)^j} q^{2+j}$ is the coefficient of $z^0$ in $A(z)$. It follows that $g_{(2,1)^j} = -h_{(2,1)^j}$ for every $j \geq 0$. \hfill $\square$

**Remark 6.7.** Let $N \geq 1$. Let $\alpha_1, \ldots, \alpha_N \in H^*(X)$ be homogeneous classes such that $K_X \alpha_i = e_X \alpha_i = 0$ for all $1 \leq i \leq N$, and let $k_1, \ldots, k_N \geq 0$.

(i) As in the proof of Lemma 6.3 we have

$$F_{k_1, \ldots, k_N}^{\alpha_1 \ldots \alpha_N}(q) = (q; q)^{-\chi(X)}_\infty \left\langle \prod_{i=1}^N \psi_{k_i}(\alpha_i) \right| \exp \left( \sum_{\mu \in \mathcal{P}} b_\mu a_{-\mu}(1_X) q^{\mu} \right) \right| \langle 0 \rangle, |1 \rangle \right\rangle \right.$$
In principle, together with Theorem 4.8 this allows us to determine many of the universal constants $b_\mu$ in Lemma 6.1.

(ii) In particular, $F_{k_1}^{\alpha_1}(q) = 0$ if $|\alpha_1| < 4$. This matches with Proposition 4.14(i).

References

[Boi] S. Boissi`ere, Chern classes of the tangent bundle on the Hilbert scheme of points on the affine plane. J. Alg. Geom. 14 (2005), 761-787.

[BN] S. Boissi`ere, M.A. Nieper-Wisskirchen, Generating series in the cohomology of Hilbert schemes of points on surfaces. LMS J. Comput. Math. 10 (2007), 254-270 (electronic).

[Bra1] D. Bradley, Multiple $q$-zeta values. J. Algebra 283 (2005), 752-798.

[Bra2] D. Bradley, On the sum formula for multiple $q$-zeta values. Rocky Mountain J. Math. 37 (2007), 1427-1434.

[Car] E. Carlsson, Vertex operators and moduli spaces of sheaves. Ph.D thesis, Princeton University, 2008.

[CO] E. Carlsson, A. Okounkov, Exts and Vertex Operators. Duke Math. J. 161 (2012), 1797-1815.

[Got] L. G¨ottsche, The Betti numbers of the Hilbert scheme of points on a smooth projective surface, Math. Ann. 286 (1990) 193–207.

[Gro] I. Grojnowski, Instantons and affine algebras I: the Hilbert scheme and vertex operators, Math. Res. Lett. 3 (1996) 275–291.

[LQW1] W.-P. Li, Z. Qin and W. Wang, Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces. Math. Ann. 324 (2002), 105-133.

[LQW2] W.-P. Li, Z. Qin and W. Wang, Hilbert schemes and $W$ algebras. Intern. Math. Res. Notices 27 (2002), 1427-1456.

[LQW3] W.-P. Li, Z. Qin, W. Wang, Stability of the cohomology rings of Hilbert schemes of points on surfaces. J. reine angew. Math. 554 (2003), 217-234.

[Nak] H. Nakajima, Heisenberg algebra and Hilbert schemes of points on projective surfaces, Ann. Math. 145 (1997) 379–388.

[Oko] A. Okounkov, Hilbert schemes and multiple $q$-zeta values. Funct. Anal. Appl. 48 (2014), 138-144.

[OT] J. Okuda, Y. Takeyama, On relations for the multiple $q$-zeta values. Ramanujan J. 14 (2007), 379-387.

[Zud] W. Zudilin, Algebraic relations for multiple zeta values. Russian Math. Surveys 58 (2003), 1-29.

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