RANDOM ATTRACTORS FOR 2D STOCHASTIC MICROPOLAR FLUID FLOWS ON UNBOUNDED DOMAINS

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Abstract. The asymptotic behavior of a model for 2D incompressible stochastic micropolar fluid flows with rough noise on a Poincaré domain is investigated. First, the existence and uniqueness of solutions to an evolution equation arising from the underlying stochastic micropolar fluid model is established via the Galerkin method and energy method. Then the existence of a random attractor is studied by using the theory of random dynamical systems for which the noise is dealt with by appropriate reproducing kernel Hilbert space.

1. Introduction. The micropolar fluid model is a qualitative generalization of the well-known Navier-Stokes model in the sense that it takes into account the microstructure of fluid [32]. It was first derived in 1966 by Eringen [23] to describe the motion of a class of non-Newtonian fluid with micro-rotational effects and inertia involved. The model can be described by the following equations:

\[
\begin{align*}
\nabla \cdot v &= 0, \\
\frac{\partial v}{\partial t} - (\mu + \mu_r)\Delta v - 2\mu_r \nabla \times w + (v \cdot \nabla)v + \nabla p &= f(t, x), \\
\frac{\partial w}{\partial t} - (c_a + c_d)\Delta w + 4\mu_r w + (v \cdot \nabla)w \\
&\quad - (c_0 + c_d - c_a)\nabla \text{div} w - 2\mu_r \nabla \times v &= g(t, x),
\end{align*}
\]

(1.1)

where \(v = (v_1, v_2, v_3)\) is the translational velocity, \(w = (w_1, w_2, w_3)\) is the angular velocity field of rotation of particles, \(p\) stands for the pressure, and \(f = (f_1, f_2, f_3)\) and \(g = (g_1, g_2, g_3)\) are the external force and moment, respectively. The positive parameters \(\mu\) and \(\mu_r\) represent the usual Newtonian viscosity and the microrotation viscosity, respectively, and \(c_0, c_a\) and \(c_d\) are angular viscous coefficients.

Micropolar fluid models play an important role in the fields of applied and computational mathematics. There is a rich literature on the mathematical theory of micropolar fluid model (1.1). In particular, the existence, uniqueness and regularity of solutions for the micropolar fluid flows have been investigated in [21, 31]. Extensive studies on long time behavior of solutions for the micropolar fluid flows

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have also been done. For example, in the case of 2D bounded domains, Chen, Chen and Dong proved the existence of an $H^2$-global attractor in [15] and verified the existence of uniform attractors in [16]. Lukaszewicz and Tarasińska proved the existence of an $H^1$-pullback attractor in [33]. Recently, Zhao et al. established the existence of $L^2$-pullback and $H^1$-pullback attractors, respectively, of solutions for the universe given by a temper condition in [43]. For the case of 2D unbounded domains, Dong and Chen [20] investigated the existence and regularity of global attractors. Zhao, Zhou and Lian [42] established the existence of an $H^1$-uniform attractor and further proved the $L^2$-uniform attractor belongs to the $H^1$-uniform attractor. Sun and Li [37] showed the existence of pullback attractors and discussed its tempered behavior and upper semi-continuity. Some efforts have also been spent on the study of 2D micropolar equations with partial dissipation. For example, Dong and Zhang [21] examined the micro-rotation viscosity, i.e., $c_{a} + c_{d} = 0$ in (1.1). The global regularity problem for this partial dissipation case is not trivial due to the presence of the term $\nabla \times w$ in the velocity equation. Dong and Zhang overcome the difficulty by making full use of the quantity $\nabla \times v - \frac{2\mu}{\rho + \rho_r} w$, which obeys a transport-diffusion equation. When the parameters $\mu = 0$ and $\mu_r \neq c_{a} + c_{d}$, the global well-posedness of the micropolar fluid equations were obtained in the framework of Besov spaces in [40]. Dong, Li and Wu studied the global regularity and large time behavior of solutions to the 2D micropolar equations with only angular viscosity dissipation [22]. More recently, Zhao, Li and Caraballo [45] investigated the micro-rotation viscosity, i.e., $\nabla \times w$ on the study of 2D micropolar equations with partial dissipation. For example, Dong proved the existence of an $H^2$-globally attracting measure [38]. Sun and Li [37] showed the existence of pullback attractors and discussed its tempered behavior and upper semi-continuity. Some efforts have also been spent on the study of 2D micropolar equations with partial dissipation. For example, Dong proved the existence of an $H^2$-globally attracting measure [38]. Sun and Li [37] showed the existence of pullback attractors and discussed its tempered behavior and upper semi-continuity. Some efforts have also been spent on the study of 2D micropolar equations with partial dissipation. For example, Dong proved the existence of an $H^2$-globally attracting measure [38]. Sun and Li [37] showed the existence of pullback attractors and discussed its tempered behavior and upper semi-continuity. Some efforts have also been spent on the study of 2D micropolar equations with partial dissipation. 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where $c := c_2 + 2c_d$ is the angular viscous coefficient, and the curl of $v$ and $w$ are
\[ \nabla \times v := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \nabla \times w := \left( \frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1} \right). \]

Here $W_1(t)$ and $W_2(t)$ are two-sided 2-dimensional and 1-dimensional cylindrical Wiener processes defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with their reproducing kernel Hilbert space (RKHS) satisfying some appropriate conditions to be specified in the main body of the paper.

During the past decades, analysis of infinite dimensional random dynamical system (RDS) has become an important branch in the study of qualitative behaviour of stochastic partial differential equations (PDEs). Since the papers of Brzeźniak et al. [2] and Crauel and Flandoli [17], the notions of random and attractors have been used in many papers to describe the asymptotic behavior of random, stochastic and non-autonomous PDEs (see, e.g., [1, 2, 13, 17, 18, 27, 28, 35]). Given a probability space, a random attractor is a compact random set, invariant for the associated RDS and attracting every element in its domain of attraction. There are two goals of this work: first to establish the existence and uniqueness of solutions to the evolution equation formulated from the problem (1.3), and second to prove the existence of random attractors for the problem (1.3).

It is worth mentioning that due to the lack of compact embedding in unbounded domains, the application of classical Galerkin methods to prove the existence of solutions is non trivial. Motivated by [25, 36], here we overcome this obstacle by using the Galerkin method combined with the technique of truncation function and decomposition of spatial domain. To prove the existence of random attractors for the problem (1.3), we follow the idea of [6, 7] using the corresponding Cameron-Martin (or reproducing kernel Hilbert) space to deal with the technical problems associated with the rough noise, and apply the concept of asymptotically compact cocycles introduced by [13]. The main technique employed here is the so-called energy equation method, which was first observed by Ball in 1922 for weakly damped, driven semilinear wave equations. It was then shown in [5] that such energy equations can be used to derive the asymptotic compactness of the semigroup generated by solutions of underlying evolution equations. Compared with the stochastic Navier-Stokes equations studied in [7], the angular velocity field $w$ of the micropolar particles in the flow leads to a different structure of abstract equation, that requires more complicated estimates and analysis in our studies.

The rest of this paper is organized as follows. In section 2 we first introduce a few function spaces and operators along with their properties. We then recall some concepts and basic theories of random dynamical systems and probability spaces required by the analysis in later sections. In Section 3 we prove the existence and uniqueness of solutions for the evolution equation arising from the system (1.3) and further show that the solution generates a random dynamical system. Then in Section 4 we show the existence of a random attractor for the random dynamical system obtained in Section 3.

2. Preliminaries. In this section we first introduce some notations and function spaces, and then recall some concepts and existing results on random dynamical systems. In the end, we set up the appropriate probability space in which later analysis will be conducted.
2.1. Notations. Given a Banach space \((X, \| \cdot \|_X)\), let \(L^p(I; X)\) be the space of strongly measurable functions on the closed interval \(I\), with values in \(X\), endowed with norm
\[
\| \varphi \|_{L^p(I; X)} := \left( \int_I \| \varphi(t) \|_X^p \, dt \right)^{1/p}, \quad \text{for } 1 \leq p < \infty,
\]
let \(C(I; X)\) be the space of continuous functions on the interval \(I\), with values in \(X\), endowed with the usual norm, let \(L^2_{\text{loc}}(I; X)\) be the space of locally integrable functions from the interval \(I\) to \(X\).

Given a Poincaré domain \(O \subset \mathbb{R}^2\), the basic functional spaces are the Lebesgue space \(L^p(O, \mathbb{R}^2)\) and Sobolev space \(W^{m,p}(O, \mathbb{R}^2)\) endowed with norms \(\| \cdot \|_{L^p}\) and \(\| \cdot \|_{W^{m,p}}\), respectively. For simplicity write \(L^p_O = L^p(O, \mathbb{R}^2)\), \(H^m_O = W^{m,2}(O, \mathbb{R}^2)\) and define
\[
\mathcal{V} = \mathcal{V}(O) := \{ \varphi \in C^\infty_0(O) \times C^\infty_0(O) \mid \varphi = (\varphi_1, \varphi_2), \nabla \cdot \varphi = 0 \},
\]
\[
\mathcal{V} = \mathcal{V}(O) := \mathcal{V} \times C^\infty_0(O),
\]
\[
\mathcal{X} = \mathcal{X}(O) := \text{closure of } \mathcal{V} \text{ in } L^2_O \times L^2_O, \text{ with norm } \| \cdot \|_\mathcal{X} \text{ and dual space } \mathcal{X}^*,
\]
\[
\mathcal{Y} = \mathcal{Y}(O) := \text{closure of } \mathcal{V} \text{ in } H^1_O \times H^1_O, \text{ with norm } \| \cdot \|_\mathcal{Y} \text{ and dual space } \mathcal{Y}^*,
\]
\[
\mathcal{X} = \mathcal{X}(O) := \text{closure of } \mathcal{V} \text{ in } L^2_O \times L^2_O \times L^2_O, \text{ with norm } \| \cdot \|_\mathcal{X} \text{ and dual space } \mathcal{X}^*,
\]
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\]
where \(\| \cdot \|_\mathcal{X}, \| \cdot \|_\mathcal{Y}, \| \cdot \|_{\mathcal{X}^*}\) and \(\| \cdot \|_{\mathcal{Y}^*}\) are defined by
\[
\| (u_1, u_2) \|_\mathcal{X} := (\| u_1 \|_{L^2}^2 + \| u_2 \|_{L^2}^2)^{1/2},
\]
\[
\| (u_1, u_2) \|_\mathcal{Y} := (\| u_1 \|_{H^1}^2 + \| u_2 \|_{H^1}^2)^{1/2},
\]
\[
\| (u_1, u_2, u_3) \|_{\mathcal{X}^*} := (\| (u_1, u_2) \|_{L^2}^2 + \| u_3 \|_{L^2}^2)^{1/2},
\]
\[
\| (u_1, u_2, u_3) \|_{\mathcal{Y}^*} := (\| (u_1, u_2) \|_{H^1}^2 + \| u_3 \|_{H^1}^2)^{1/2}.
\]
Denote by \(\langle \cdot, \cdot \rangle\) the inner product in \(L^2(\Omega)\), \(\mathcal{X}\) or \(\mathcal{X}^*\), and by \(\langle \cdot, \cdot \rangle\) the dual pairing between \(\mathcal{Y}\) and \(\mathcal{Y}^*\) or between \(\mathcal{Y}^*\) and \(\mathcal{Y}^*\). Throughout this article, we simplify the notations \(\| \cdot \|_{\mathcal{X}^*}, \| \cdot \|_\mathcal{X}\) and \(\| \cdot \|_{\mathcal{Y}^*}\) by the same notation \(\| \cdot \|\) when the context is clear.

Furthermore, for \(v(t, x), w(t, x), f(t, x)\) and \(g(t, x)\) in the system (1.3), setting \(u(t, x) := (v(t, x), w(t, x)), F(t, x) := (f(t, x), g(t, x))\), and \(\phi = (\phi_1, \phi_2, \phi_3)\), we define the operators \(A, B\) and \(\Lambda\) by
\[
[Au, \phi] := (\mu + \mu_r) (\nabla v, \nabla (\phi_1, \phi_2)) + c (\nabla w, \nabla \phi_3), \quad \forall u, \phi \in \hat{\mathcal{Y}},
\]
\[
[B(v, u), \phi] := ((v \cdot \nabla) u, \phi), \quad \forall v \in \mathcal{Y}, u, \phi \in \hat{\mathcal{Y}},
\]
\[
\Lambda(u) := (-2\mu_r \nabla \times w, -2\mu_r \nabla \times v + 4\mu_r w), \quad \forall u \in \hat{\mathcal{Y}}.
\]
Then we can formulate a weak version of equations in (1.3) as follows:
\[
\begin{align*}
\nabla \cdot v &= 0, & \text{in } (t_0, +\infty) \times \mathcal{O}, \\
\frac{\partial u}{\partial t} + Au + B(v, u) + \Lambda(u) &= F(t, x) + \frac{dW(t)}{dt}, & \text{in } (t_0, +\infty) \times \mathcal{O}, \\
u &= (v, w) = 0, & \text{on } (t_0, +\infty) \times \partial \mathcal{O}, \\
u(t_0, x) &= (v(t_0, x), w(t_0, x)) = (v_0(x), w_0(x)) = u_0(x), & \text{for } x \in \mathcal{O},
\end{align*}
\]
where \( W(t) := (W_1(t), W_2(t)) \) is a 3-dimensional two-sided Wiener process.

The lemmata below include some useful estimates and properties for the operators defined in (2.1)–(2.3), which can be easily verified using results established in works [32, 34, 38, 42, 44].

**Lemma 2.1.** The operator \( A \) defined by (2.1) is linear continuous both from \( \hat{Y} \) to \( \hat{Y}^* \) and from \( \hat{Y} \cap (H^1_0)^3 \) to \( \hat{R} \); the operator \( \Lambda(\cdot) \) defined by (2.3) is linear continuous from \( \hat{Y} \) to \( \hat{R} \); the operator \( B(\cdot, \cdot) \) defined by (2.2) is continuous from \( \mathcal{Y} \times \hat{Y} \) to \( \hat{Y}^* \) and satisfies
\[
\|[B(v, \psi), \phi]\| = -\|[B(v, \phi), \psi]\|, \quad \forall v \in \mathcal{Y}, \; \psi, \phi \in \hat{Y}.
\]  

**Lemma 2.2.** Given any \( v \in \mathcal{Y}, \phi, \psi \in \hat{Y} \), there exist positive constants \( \kappa_1, \kappa_2, \kappa_3 = \kappa_3(\mathcal{O}) \), and \( \kappa_4(\mu_r) \) such that
\[
\kappa_1\|[A\phi, \phi]\| \leq \|[\phi]\|^2_3 \leq \kappa_2\|[A\phi, \phi]\|, 
\]  
(2.6)
\[
\|v\|_{L^2_0} \leq 2^{1/2} \|v\|_{L^2_0} \|\nabla v\|^2_{L^2_0}, 
\]  
(2.7)
\[
\|[B(v, \psi), \phi]\| \leq \begin{cases} 
\|v\|_{L^2_0} \|\phi\|_{L^2_0} \|\nabla \phi\| \leq \kappa_3 \|v\|^{1/2} \|\nabla v\|^{1/2} \|\phi\|^{1/2} \|\nabla \phi\|^{1/2}, \\
\|v\|_{L^2_0} \|\psi\|_{L^2_0} \|\nabla \phi\| \leq \kappa_3 \|v\|^{1/2} \|\nabla v\|^{1/2} \|\psi\|^{1/2} \|\nabla \phi\|^{1/2}, 
\end{cases}
\]  
(2.8)
\[
\|\Lambda(\phi)\| \leq \kappa_4(\mu_r)\|\phi\|_{\hat{Y}}, \quad \|\Lambda(\phi)\|_{\hat{Y}} \leq \kappa_4(\mu_r)\|\phi\|. 
\]  
(2.9)

In addition, let \( \kappa := \min\{\mu, c\} \), then
\[
\kappa\|\phi\|^2_{\hat{Y}} \leq \|[A\phi, \phi]\| + \|[\Lambda(\phi), \phi]\|, \quad \forall \phi \in \hat{Y}. 
\]  
(2.10)

### 2.2. Random dynamical systems

In this subsection we recall some basic concepts and theory of random dynamical systems needed in the sequel. For a comprehensive presentation of the theory of random dynamical system, see, e.g., monographs [1, 7, 9, 24]. Note that for a general stochastic non-autonomous system, the framework of nonautonomous random dynamical systems (see, e.g., [10, 11, 39]) is required. For the particular system under consideration in this work, the framework of random dynamical systems suffices as shown later in Subsection 3.3.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, and let \((\mathcal{X}, d_{\mathcal{X}})\) be a Polish space, i.e., a metrizable complete separable topological space, equipped with its Borel sigma algebra \( \mathcal{B} \). For any two non-empty sets \( X \subseteq \mathcal{X} \) and \( Y \subseteq \mathcal{X} \), denote by \( \text{dist}_{\mathcal{X}}(X, Y) \) the Hausdorff semidistance between defined by
\[
\text{dist}_{\mathcal{X}}(X, Y) = \sup_{x \in X} \inf_{y \in Y} d_{\mathcal{X}}(x, y).
\]

Denote by \( \mathcal{P}(\mathcal{X}) \) the family of all non-empty closed and bounded subsets of \( \mathcal{X} \). Then \( \text{dist}_{\mathcal{X}}(X, Y) \) restricted to \( \mathcal{P}(\mathcal{X}) \) is a distance [14].
Definition 2.1. The quadruplet $Q := (\Omega, F, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is called a metric dynamical system if (i) $\theta : \mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \theta_t \omega$ is $(B(\mathbb{R}) \times F, F)$-measurable, (ii) $\theta_0$ is identity, i.e., $\theta_0 \omega = \omega$ for all $\omega \in \Omega$, (iii) $\theta_t \circ \theta_s = \theta_{t+s}$ for all $s, t \in \mathbb{R}$, and (iv) $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2.2. A map $\Psi : \mathbb{R}^+ \times \Omega \times \mathcal{X} \to \mathcal{X}$, $(t, \omega, x) \mapsto \Psi(t, \omega, x)$ is called a continuous random dynamical system (RDS) on $\mathcal{X}$ (over $Q$), if (i) $\Psi$ is $(B([0, \infty) \times F \times B(\mathcal{X}), B(\mathcal{X}))$-measurable, (ii) $\Psi(0, \omega) = \text{Id}$ on $\mathcal{X}$ for all $\omega \in \Omega$, (iii) $\Psi$ is a cocycle under $\{\theta_t\}_{t \in \mathbb{R}}$, i.e., $\Psi(t+s, \omega, x) = \Psi(t, \theta_s \omega, \Psi(s, \omega, x))$ for all $x \in \mathcal{X}$ and $\omega \in \Omega$; and (iv) the mapping $\Psi(t, \omega, \cdot) : \mathcal{X} \to \mathcal{X}$ is continuous. Similarly, $\Psi$ is said to be time continuous if for all $\omega \in \Omega$ and $x \in \mathcal{X}$, the mapping $\Psi(\cdot, \omega, x) : \mathbb{R}^+ \to \mathcal{X}$ is continuous.

Definition 2.3. A random set $D(\omega)$ is a multi-valued map $D : \Omega \to \mathcal{P}(\mathcal{X})$ such that for every $x \in \mathcal{X}$, the mapping $\omega \mapsto \text{dist}_\mathcal{X}(x, D(\omega))$ is measurable. A random set $D(\omega)$ is said to be bounded (resp. closed or compact) if $D(\omega)$ is bounded (resp. closed or compact) for almost all $\omega \in \Omega$.

In the definitions and proposition below, let $\Psi$ be a continuous random dynamical system (RDS) on $\mathcal{X}$ and denote by $\mathcal{D}$ a particular collection of subsets of $\mathcal{X}$.

**Definition 2.4.** A random compact set $\mathcal{A} : \Omega \to \mathcal{P}(\mathcal{X})$ is called a random $\mathcal{D}$-attractor (or pullback $\mathcal{D}$-attractor) for the RDS $\Psi$ if

(i) $\mathcal{A}$ is $\Psi$-invariant, i.e., $\Psi(t, \omega) \mathcal{A}(\omega) = \mathcal{A}(\theta_t \omega)$ for all $t \geq 0$ and $\mathbb{P}$-a.s.;

(ii) $\mathcal{A}(\omega)$ attracts all sets in $\mathcal{D}$, i.e., for all $D \in \mathcal{D}$, and a.e. $\omega \in \Omega$

$$\lim_{t \to \infty} \text{dist}_\mathcal{X}(\Psi(t, \theta_{-t} \omega) D(\theta_{-t} \omega), \mathcal{A}(\omega)) = 0.$$ 

**Remark 2.1.** The collection $\mathcal{D}$ is called the domain of attractor for $\mathcal{A}$. Examples of $\mathcal{D}$ in the studies of random attractors include the collection of all finite deterministic subsets of $\mathcal{X}$, the collection of all compact deterministic subsets of $\mathcal{X}$, the collection of closed and bounded random subsets of $\mathcal{X}$, and the collection of all tempered random subsets of $\mathcal{X}$ [12].

The following well-known results on the existence of a random $\mathcal{D}$-attractor can be found in e.g., [7, 13].

**Proposition 2.1.** The RDS $\Psi$ possesses a measurable random $\mathcal{D}$-attractor provided

(i) there exists a $\mathcal{D}$-absorbing set $K \in \mathcal{D}$ for $\Psi$, i.e., given any $D \in \mathcal{D}$, there exists $T(D)$ such that

$$\Psi(t, \theta_{-t} \omega) D(\theta_{-t} \omega) \subset K(\omega) \quad \forall \ t \geq T(D);$$

(ii) $K$ is $\mathcal{D}$-asymptotically compact under $\Psi$, i.e., for a.e. $\omega \in \Omega$ and any positive sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$, every sequence $\{x_n\}$ with $x_n \in \Psi(t_n, \theta_{-t_n} \omega, K(\theta_{-t_n} \omega))$ has a convergent subsequence.

More precisely, the random $\mathcal{D}$-attractor is given by

$$\mathcal{A}(\omega) = \bigcap_{t \geq T(K)} \bigcup_{t \geq \tau} \varphi(t, \theta_{-t} \omega, K(\theta_{-t} \omega)), \ \omega \in \Omega.$$ 

2.3. **Probability space setup.** Consider a real separable Hilbert space $\mathcal{H}$ and a real Banach space $\mathcal{X}$, let $\vartheta = \vartheta_\mathcal{H}$ be the canonical cylindrical finitely additive set-valued function, or a Gaussian distribution on $\mathcal{H}$.
Definition 2.5. A bounded linear operator \( \ell : \mathcal{H} \to \mathcal{X} \) is called \( \vartheta \)-radonifying if \( \ell(\vartheta) \) is sigma additive.

Definition 2.6. The reproducing kernel Hilbert space (RKHS) of a centered Gaussian measure \( \nu \) on a separable Banach space \( \mathcal{X} \) is a (unique) Hilbert space \( (\mathcal{G}, \| \cdot \|_\mathcal{G}) \) such that \( \mathcal{G} \to \mathcal{X} \) continuously and for each \( X \in \mathcal{X}^* \) the random variable \( X \) on probability space \( (\mathcal{X}, \nu) \) is normal with mean 0 and variance \( |X|^2 \).

Note for a \( \vartheta \)-radonifying operator \( \ell : \mathcal{H} \to \mathcal{X} \), \( \ell(\vartheta\beta) \) has a unique extension to a sigma additive Borel probability measure \( \nu_\beta \) on \( \mathcal{X} \), and moreover, \( \nu_\beta \) is a centered Gaussian measure on \( \mathcal{X} \) whose RKHS is \( \mathcal{H} \). Throughout this paper it is assumed that

\[ \text{(A)} \quad \mathbb{H} \subset \hat{\mathcal{X}} \cap \mathcal{L}^1_\mathcal{O} \] is a Hilbert space such that for some \( \gamma \in (0, 1/2) \),

\[ A^{-\gamma} : \mathbb{H} \to \hat{\mathcal{X}} \cap \mathcal{L}^1_\mathcal{O} \text{ is } \vartheta - \text{radonifying}. \]

Set \( \mathcal{X} = \hat{\mathcal{X}} \cap \mathcal{L}^1_\mathcal{O} \) and let \( \mathcal{B} \) be the completion of \( A^{-\gamma}(\mathcal{X}) \) with respect to the image norm \( \|x\|_\mathcal{B} = \|A^{-\gamma}x\|_\mathcal{X} \) for \( x \in \mathcal{X} \). Then \( \mathcal{B} \) is a separable Banach space. Now for \( \beta \in (0, \frac{1}{2}) \), define

\[ C_\mathcal{B}^\beta(\mathbb{R}, \mathcal{B}) := \{ \omega \in C(\mathbb{R}, \mathcal{B}) | \omega(0) = 0, \sup_{t,s \in \mathbb{R}} \frac{\|\omega(t) - \omega(s)\|_\mathcal{B}}{|t-s|^\beta(1 + |t| + |s|)^{\frac{3}{2}}} < \infty \}, \]

endowed with the norm

\[ \|\omega\|_{C_\mathcal{B}^\beta(\mathbb{R}, \mathcal{B})} = \sup_{t,s \in \mathbb{R}} \frac{\|\omega(t) - \omega(s)\|_\mathcal{B}}{|t-s|^\beta(1 + |t| + |s|)^{\frac{3}{2}}} < \infty. \]

In particular, let \( C_\mathcal{B}^{\frac{1}{2}}(\mathbb{R}, \mathcal{B}) \) denote the space of all continuous functions \( \omega : \mathbb{R} \to \mathcal{B} \) following a linear growth condition, i.e., there exists some \( C = C(\omega) > 0 \) such that

\[ \|\omega(t)\|_\mathcal{B} \leq C(1 + |t|^{\frac{3}{2}}), \quad t \in \mathbb{R}. \]

It is straightforward to check that the space \( C_\mathcal{B}^{\frac{1}{2}}(\mathbb{R}, \mathcal{B}) \) endowed with the norm \( \|\omega\|_{C_\mathcal{B}^{\frac{1}{2}}(\mathbb{R}, \mathcal{B})} = \sup_{t \in \mathbb{R}} \|\omega(t)\|_\mathcal{B} / (1 + |t|^{\frac{3}{2}}) \) is also a separable Banach space.

Let \( \Omega^\beta(\mathcal{B}) \) be the closure of \( \{ \omega \in C_0^\infty(\mathbb{R}, \mathcal{B}) : \omega(0) = 0 \} \) in \( C_\mathcal{B}^{\frac{1}{2}}(\mathbb{R}, \mathcal{B}) \), which is a separable Banach space. Denote by \( \mathcal{F} \) the Borel sigma-algebra on \( \Omega^\beta(\mathcal{B}) \), one can then show by methods from [3] that for \( \beta \in (0, \frac{1}{2}) \), there exists a Borel probability measure \( \mathcal{P} \) on \( \Omega^\beta(\mathcal{B}) \) such that the canonical process \( \{W(t)\}_{t \in \mathbb{R}} \), defined by

\[ W(t, \omega) = \omega(t), \quad \omega \in \Omega^\beta(\mathcal{B}), \quad (2.11) \]

is a two-sided Wiener process such that the RKHS of the Gaussian measure \( \nu(W) \) on \( \mathcal{B} \) is equal to \( \mathbb{H} \).

For \( t \in \mathbb{R} \), let \( \tilde{\mathcal{G}}_t := \sigma\{W(s) : s < t\} \) and let \( e_t : \Omega^\beta(\mathcal{B}) \to \mathcal{B}, \vartheta \mapsto \vartheta(t) \) be the evaluation map at time \( t \). Then for each \( t \in \mathbb{R} \) and any map \( z : e_t : \mathcal{B}^* \to L^2(\Omega^\beta(\mathcal{B}), \tilde{\mathcal{G}}_t, \mathcal{P}) \) satisfying \( \mathbb{E}|zoe_t|^2 = t\|z\|_{\mathcal{O}^\beta}^2 \), there exists a unique extension of \( zoe_t \) to a bounded linear map \( W_t : \mathbb{H} \to L^2(\Omega^\beta(\mathcal{B}), \tilde{\mathcal{G}}_t, \mathcal{P}) \). Moreover, the family \( \{W_t\}_{t \in \mathbb{R}} \) is an \( \mathcal{F} \)-cylindrical Wiener process on a filtered probability space \( (\Omega^\beta(\mathcal{B}), (\tilde{\mathcal{G}}_t))_{t \in \mathbb{R}}, \mathcal{P} \) (see [4]).

In the end consider a flow on \( C_{1/2}(\mathbb{R}, \mathcal{X}) \) defined by

\[ \theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}. \quad (2.12) \]
It is straightforward to check that the spaces $\mathcal{C}^2_{1/2}(\mathbb{R}, \mathbb{B})$ and $\Omega_{1/2}(\mathbb{B})$ are invariant with respect to the above flow $\theta$. Unless otherwise specified, the notation $\theta_t$ is used for the flow $\theta_t$ restricted to one of these spaces. Note also that for each $t \in \mathbb{R}, \theta_t$ preserves $\mathbb{P}$.

3. Existence and uniqueness of solutions. The aim of this section is to study the existence and uniqueness of solutions to the following functional analytic version of problem (2.4):

$$
\begin{align*}
\begin{cases}
    du + (Au + B(v, u) + \Lambda(u))dt = F(t)dt + dW(t), & t \geq t_0, \\
    u(t_0) = u_0 \in \tilde{X},
\end{cases}
\end{align*}
$$

in which $F \in L^2(t_0, T; \mathcal{Y})$ and $\{W(t)\}_{t \in \mathbb{R}}$ is a two-sided cylindrical Wiener process in $\tilde{X}$ with its reproducing kernel Hilbert space RKHS satisfying Assumption (A), on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$ defined in Subsection 2.3.

**Definition 3.1.** Let Assumption (A) hold and let $\{W(t)\}_{t \in \mathbb{R}}$ be a two-sided Wiener process introduced in Subsection 2.3 such that the RKHS of the Gaussian measure is equal to $\mathcal{H}$. Then for $t_0 \in \mathbb{R}, T > t_0, u_0 \in \tilde{X}, F \in L^2_{loc}(t_0, \infty; \mathcal{Y})$, a process $u(t) \in \mathcal{C}([t_0, T]; \tilde{X}) \cap L^2(t_0, T; \mathcal{Y})$ is said to be a solution to problem (3.1) if $u(t_0) = u_0$ and for any $\phi \in \tilde{Y}$, it holds that

$$
\langle u(t), \phi \rangle + \int_{t_0}^t \|Au(\tau), \phi\|d\tau + \int_{t_0}^t \|B(v(\tau), u(\tau)), \phi\|d\tau + \int_{t_0}^t \|\Lambda(u(\tau)), \phi\|d\tau
$$

$$
= \langle u(t_0), \phi \rangle + \int_{t_0}^t \|F, \psi\|d\tau + \int_{t_0}^t \|\phi, dW(\tau)\|, \quad \forall t > t_0.
$$

The section consists of three parts, in the first part we rewrite system (3.1) as a random evolution equation with random parameters but no white noise, in the second part we establish the existence and uniqueness of solutions to the resulting random equation, and in the third part we prove that the solution generates a random dynamical system.

3.1. Transformation from the stochastic to random problem. First consider the stochastic evolution equation on $\mathcal{O}$:

$$
d\delta_\alpha(t) + (A + \alpha)\delta_\alpha(t)dt = dW(t), \quad t \in \mathbb{R}. \quad (3.2)
$$

Then due to [6, Proposition 6.10], for $\gamma$ under Assumption (A), $\alpha \geq 0, \beta \in (\gamma, 1/2)$ and $\omega \in \mathcal{C}^2_{1/2}(\mathbb{R}, \mathbb{B})$ the process

$$
\delta_\alpha(t, \omega) = \int_{-\infty}^t (A + \alpha)^{t-\tau}e^{-(t-\tau)(A+\alpha)}((A+\alpha)^{-\gamma}\theta_{\tau, \omega})(t - \tau)d\tau \quad (3.3)
$$

is a stationary Ornstein-Uhlenbeck process that solves the stochastic equation (3.2). Properties of $\delta_\alpha$ will be recalled as needed. Note that the parameter $\alpha$ will be used for the existence of attractors, but not necessary for the existence of solutions. Through the rest of this section we write $\delta_\alpha$ as $\delta$ for simplicity when the calculations do not depend on $\alpha$.

Now let $y(t) = u(t) - \delta(t)$. Recall that $u(t) = (v(t), w(t))$ and denote by $z(t)$ and $\eta(t)$ the first two components of $y(t)$ and $\delta(t)$, respectively. Then the equation
Theorem 3.1. (3.1) can be rewritten as the following path-wise random evolution equations:

\[
\begin{aligned}
\frac{dy(t, \omega)}{dt} + Ay + B(z + \eta, y + \delta) + \Lambda(y + \delta) - \alpha \delta &= F(t), \\
y(t_0) &= u(t_0) - \delta(t_0) := y_0.
\end{aligned}
\tag{3.4}
\]

3.2. Existence and uniqueness of solutions. This subsection is devoted to establishing the existence and uniqueness of a solution to the random equation (3.4), defined below (see e.g., [6, 38]).

**Definition 3.2.** Let Assumption (A) hold, \( t_0 \in \mathbb{R} \) and \( T > t_0 \). For \( y_0 \in \hat{X} \), \( \delta \in L^4(t_0, T; L^2(t_0, T; \hat{X})) \) and \( F \in L^2(t_0, T; \hat{Y}^*) \), a process \( \{y(t)\}_{t \geq 0} \) with trajectories in \( C([t_0, T]; \hat{X}) \cap L^2(t_0, T; \hat{Y}) \) is called a solution to problem (3.4) if \( y(t_0) = y_0 \) and for any \( \phi \in \hat{Y} \) the equation

\[
\frac{d}{dt} \langle y, \phi \rangle + \|Ay, \phi\| + \|B(z + \eta, y + \delta), \phi\| + \|\Lambda(y + \delta), \phi\| - \alpha \|\delta, \phi\| = \|F, \phi\|
\]

holds in the distribution sense on \((t_0, T)\).

**Theorem 3.1.** (Existence) Let Assumption (A) hold, \( t_0 \in \mathbb{R} \), \( T > t_0 \), and \( \alpha \geq 0 \). Then given any \( y_0 \in \hat{X} \), \( \delta \in L^4(t_0, T; L^2(t_0, T; \hat{X})) \) and \( F \in L^2(t_0, T; \hat{Y}^*) \), problem (3.4) possesses at least one solution.

**Proof.** First, observe that \( B(z + \eta, y + \delta) = B(z, y) + B(\eta, y) + B(z, \delta) + B(\eta, \delta) \).

Set

\[
G(t, x) = \alpha \delta + F(t, x) - B(\eta, \delta) - \Lambda(\delta).
\]

Since \( \delta \in L^4(t_0, T; L^2(t_0, T; \hat{X})) \) and \( F \in L^2(t_0, T; \hat{Y}^*) \), then it follows Lemma 2.2 that \( G(t, x) \in L^2(t_0, T; \hat{Y}^*) \). The proof will be proceeded in three steps.

**Step I.** Local existence and uniqueness of the Galerkin approximate solutions.

Consider an orthonormal basis \( \{\xi_j\}_{j=1}^\infty \subset \hat{Y} \) or \( \hat{X} \) such that \( \text{span}\{\xi_1, \xi_2, \ldots\} \) is dense in \( \hat{Y} \). Denote

\[
\hat{Y}_m := \text{span}\{\xi_1, \xi_2, \ldots, \xi_m\} \quad \text{and} \quad P_my := \sum_{j=1}^m \langle y, \xi_j \rangle \xi_j, \ y \in \hat{X} \text{ or } \hat{Y}.
\]

For \( t \geq t_0 \), define \( y^m(t) := \sum_{j=1}^m \hat{y}_{m,j}(t)\xi_j \), where the coefficients \( \hat{y}_{m,j}(t) \) are desired to satisfy the following Cauchy problem of ordinary differential equations:

\[
\begin{cases}
\frac{d}{dt} \langle y^m(t), \xi_j \rangle + \|Ay^m(t), \xi_j\| + \|B(z^m + \eta, y^m + \delta) - B(\eta, \delta), \xi_j\| \\
+ \|\Lambda(y^m(t)), \xi_j\| = \|G(t), \xi_j\|, & 1 \leq j \leq m, \ t \geq t_0,
\end{cases}
\tag{3.5}
\]

where \( z^m \) is the first two components of \( y^m \). By classical theory of ordinary differential equations, there exists a time \( T^* > t_0 \) and a Galerkin approximate solution \( y^m(t) \) on the time interval \([t_0, T^*)\).

**Step II.** Apriori estimates of the Galerkin approximate solutions.
Based on Lemma 2.1 and Lemma 2.2, multiplying the first equation of (3.5) by $\hat{y}_{m,j}(t)$, and summing from $j = 1$ to $m$ gives

$$
\frac{1}{2} \frac{d}{dt} \|y^m(t)\|^2 + \kappa \|y^m(t)\|_{\tilde{Y}}^2 + \|B(z^m, \delta), y^m(t)\| \\
\leq \frac{1}{2} \frac{d}{dt} \|y^m(t)\|^2 + \|Ay^m(t), y^m(t)\| + \|\Lambda(y^m(t)), y^m(t)\| + \|B(z^m, \delta), y^m(t)\| \\
= \|G(t), y^m(t)\|. 
$$

(3.6)

Then by Hölder’s inequality, Young’s inequality and Lemma 2.2 again, we deduce that

$$
\|B(z^m, \delta), y^m(t)\| \leq \|z^m\|_{\mathcal{L}_b^3} \|\nabla y^m\|_{\mathcal{L}_b^3} \leq 2^{1/4} \|z^m\|^{1/4} \|\nabla z^m\|^{1/4} \|\nabla y^m\| \leq \frac{k}{4} \|y^m\|_{\tilde{Y}}^2 + \frac{27}{2k^3} \|\delta\|_{\mathcal{L}_b^4}^2 \|y^m\|^2,
$$

$$
\|G(t), y^m(t)\| \leq \|G(t)\|_{\tilde{Y}}. \|y^m(t)\|_{\tilde{Y}} \leq \frac{k}{4} \|y^m(t)\|_{\tilde{Y}}^2 + \frac{1}{k} \|G(t)\|_{\tilde{Y}}^2.
$$

Substituting the above two inequalities into (3.6) to obtain

$$
\frac{d}{dt} \|y^m(t)\|^2 + \kappa \|y^m(t)\|_{\tilde{Y}}^2 \leq \frac{27}{k^3} \|\delta(t)\|_{\mathcal{L}_b^4}^4 \|y^m(t)\|^2 + \frac{2}{k} \|G(t)\|_{\tilde{Y}}^2, \ \forall t \in [t_0, T^*),
$$

(3.7)

and applying Gronwall’s inequality to the above inequality yields

$$
\|y^m(t)\|^2 \leq \|y^m(t_0)\|^2 e^{\frac{27}{k^3} \int_{t_0}^{T^*} \|\delta(s)\|_{\mathcal{L}_b^4}^4 ds} + \frac{2}{k} \int_{t_0}^{t} \|G(s)\|_{\tilde{Y}}^2 e^{\frac{27}{k^3} \int_{t}^{T^*} \|\delta(s)\|_{\mathcal{L}_b^4}^4 ds} ds < +\infty, \ \forall t \in [t_0, T^*).
$$

(3.8)

In addition, for $t > t_0$ integrating equation (3.7) over $[t_0, t]$ and using (3.8), we get

$$
\|y^m(t)\|^2 + \kappa \int_{t_0}^{t} \|y^m(\tau)\|_{\tilde{Y}}^2 d\tau \\
\leq \|y^m(t_0)\|^2 + \frac{27}{k^3} \int_{t_0}^{t} \|\delta(\tau)\|_{\mathcal{L}_b^4}^4 \|y^m(\tau)\|^2 d\tau + \frac{2}{k} \int_{t_0}^{t} \|G(\tau)\|_{\tilde{Y}}^2 d\tau \\
\leq \frac{27}{k^3} \|y^m(t_0)\|^2 e^{\frac{27}{k^3} \int_{t_0}^{T^*} \|\delta(s)\|_{\mathcal{L}_b^4}^4 ds} \int_{t_0}^{T^*} \|\delta(s)\|_{\mathcal{L}_b^4}^4 ds + \frac{2}{k} \int_{t_0}^{T^*} \|G(s)\|_{\tilde{Y}}^2 e^{\frac{27}{k^3} \int_{t_0}^{T^*} \|\delta(s)\|_{\mathcal{L}_b^4}^4 ds} ds \int_{t_0}^{T^*} \|\delta(s)\|_{\mathcal{L}_b^4}^4 ds \\
+ \frac{54}{k^4} \int_{t_0}^{t} \|G(\tau)\|_{\tilde{Y}}^2 e^{\frac{27}{k^3} \int_{t}^{T^*} \|\delta(s)\|_{\mathcal{L}_b^4}^4 ds} ds \int_{t_0}^{T^*} \|\delta(s)\|_{\mathcal{L}_b^4}^4 ds \\
+ \frac{2}{k} \int_{t_0}^{T^*} \|G(\tau)\|_{\tilde{Y}}^2 ds + \|y^m(t_0)\|^2, \ \forall t \in [t_0, T^*).
$$

Therefore, we can take $T^* = T$ and obtain that

$$
\{y^m\}_{m \in \mathbb{N}} \text{ is bounded in } L^\infty(t_0, T; \tilde{X}) \cap L^2(t_0, T; \tilde{Y}),
$$

(3.9)

which, together with the local existence of solution obtained in Step I, imply the global existence of the Galerkin approximate solution for all time $t \in [t_0, T]$.

**Step III.** Existence of the global weak solutions.

In the following, we will prove that the limit function of the Galerkin approximate solutions is indeed a solution of (3.4). First by using a diagonal procedure, we can
deduce from (3.9) that there exists a subsequence of \( \{y^m\} \), still denoted by \( \{y^m\} \) and an element \( \bar{y} \in L^\infty(t_0, T; \mathfrak{X}) \cap L^2(t_0, T; \mathfrak{Y}) \) such that
\[
\begin{align*}
    y^m & \rightharpoonup^{*} \bar{y} \text{ weakly star in } L^\infty(t_0, T; \mathfrak{X}) \text{ as } m \to \infty, \\
    y^m & \to \bar{y} \text{ weakly in } L^2(t_0, T; \mathfrak{Y}) \text{ as } m \to \infty.
\end{align*}
\] (3.10)

Then due to (3.9) in [36], for any bounded open set \( E \subset \mathcal{O} \), there exists a subsequence (depending on \( E \)) of \( \{y^m\} \), denoted by \( \{y^{m_k}_E\} \) such that
\[
y^{m_k}_E(t) \to \bar{y}(t) \text{ strongly in } L^2(t_0, T; \mathfrak{Y}(E)) \text{ as } k \to \infty.
\] (3.11)

Note that for \( \phi \in C^1([t_0, T]) \) with \( \phi(T) = 0 \) and each fixed \( \xi_j \), it follows from (3.5) that
\[
\begin{align*}
    &- \int_{t_0}^{T} \langle y^m(t), \xi_j \phi'(t) \rangle \, dt + \int_{t_0}^{T} \| Ay^m(t), \xi_j \phi(t) \| \, dt \\
    &+ \int_{t_0}^{T} \| B(z^m + \eta, y^m + \delta) - B(\eta, \delta), \xi_j \phi(t) \| \, dt + \int_{t_0}^{T} \| \Lambda(y^m(t)), \xi_j \phi(t) \| \, dt \\
    = &\langle y^m(t_0), \xi_j \rangle \phi(t_0) + \int_{t_0}^{T} \| G(t), \xi_j \phi(t) \| \, dt.
\end{align*}
\] (3.12)

We next concentrate on finding a solution by passing to the limit in (3.12). To that end, let \( E_j \subset \mathcal{O} \) be a sequence of regular bounded open sets that contains all supports of functions \( \xi_j \) of the basis. Then, for each \( E_j \) following the procedure above, we choose a subsequence \( y^{m_k}_{E_j} \) satisfying (3.11).

Since the span \( \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\} \) is dense in \( \mathfrak{Y} \), for every \( \xi \in \mathfrak{Y} \) and any \( \varepsilon > 0 \), there exists \( N_\varepsilon > 0 \) such that
\[
\| \xi_j - \xi \|_{\mathfrak{Y}} < \varepsilon \text{ for all } j \geq N_\varepsilon.
\] (3.13)

Set \( \bar{\kappa} = \max\{\mu + \mu_r, \alpha\} \). For the above \( \varepsilon \), based on Lemma 2.1 and Lemma 2.2, we can deduce from (3.10), (3.11) and (3.13) that there exists a \( K_\varepsilon > 0 \) such that for \( k \geq K_\varepsilon \) and \( j \geq N_\varepsilon \), the following estimates (i) – (iii) hold, where \( C > 0 \) is a generic constant.

(i) \[
\begin{align*}
    &\left| \int_{t_0}^{T} \| Ay^{m_k}_{E_j}(t), \xi_j \phi(t) \| \, dt - \int_{t_0}^{T} \| Ay(\bar{y}), \xi_j \phi(t) \| \, dt \right| \\
    \leq & \bar{\kappa} \left( \int_{t_0}^{T} \int_{\mathcal{O}} \| \nabla (y^{m_k}_{E_j} - \bar{y}) \cdot \xi_j \phi(t) \, dx \, dt \right) + \int_{t_0}^{T} \int_{\mathcal{O}} \nabla \bar{y} : \nabla (\xi_j - \xi) \phi(t) \, dx \, dt \\
    \leq & \bar{\kappa} \left( \int_{t_0}^{T} \int_{\mathcal{O}} \| (y^{m_k}_{E_j} - \bar{y}) \cdot \xi_j \phi(t) \, dx \, dt \right) + \| \xi_j - \xi \| \int_{t_0}^{T} \nabla \bar{y} : \phi(t) \, dx \, dt \\
    \leq & C \varepsilon;
\end{align*}
\]

(ii) \[
\begin{align*}
    &\left| \int_{t_0}^{T} \| \Lambda(y^{m_k}_{E_j}(t)), \xi_j \phi(t) \| \, dt - \int_{t_0}^{T} \| \Lambda(\bar{y}(t)), \xi_j \phi(t) \| \, dt \right| \\
    = & \left| \int_{t_0}^{T} \int_{\mathcal{O}} \Lambda(y^{m_k}_{E_j}(t) - \bar{y}(t)) \cdot \xi_j \phi(t) \, dx \, dt \right|
\end{align*}
\]
where $\hat{y}$ is the third component of $y$, and thus

\begin{align*}
& (ii) \leq 4\mu_r \|y_{m_k}^{m_k}(t) - \hat{y}(t)\|_{L^2(t_0, T; \gamma(E_J))} \|\xi_j \|_{\gamma(E_J)} \|\phi(t)\|_{L^2(0, T)} \\
& + \kappa_4(\mu_r) \|\xi_j - \xi\| \|\hat{y}(t)\|_{L^2(t_0, T; \gamma(J))} \|\phi(t)\|_{L^2(0, T)} \\
& \leq C\varepsilon;
\end{align*}

\begin{align*}
(iii) \quad & \left| \int_{t_0}^T \left[ \frac{\partial}{\partial t} \left( B(z_{E_j}^{m_k} + \eta, y_{E_j}^{m_k} + \delta) - B(\eta, \delta), \xi_j \phi(t) \right) \right] dt - \int_{t_0}^T \left[ B(\hat{z} + \eta, \hat{y} + \delta) - B(\eta, \delta), \xi_j \phi(t) \right] dt \right| \\
& = \int_{t_0}^T \left[ \frac{\partial}{\partial t} \left( \left[ B(z_{E_j}^{m_k} - \hat{z}, y_{E_j}^{m_k} - \hat{y}) , \xi_j \phi(t) \right] + \left[ B(z_{E_j}^{m_k} - \hat{z}, \hat{y}) , \xi_j \phi(t) \right] \right) dt \\
& + \int_{t_0}^T \left[ B(\hat{z}, \hat{y}) , \xi_j - \xi \right] \phi(t) dt \right] \\
& \leq \kappa_3 \sup_{t \in [t_0, T]} \|\phi(t)\| \left\{ \int_{t_0}^T \left( \|z_{E_j}^{m_k} - \hat{z}\|_{H^2(E_J)} \|\nabla(z_{E_j}^{m_k} - \hat{z})\| \|y_{m_k}^{m_k}\| \|y_{E_j}^{m_k} - \hat{y}\| \|\nabla(y_{m_k}^{m_k} - \hat{y})\| \|\nabla(y_{E_j}^{m_k} - \hat{y})\| \|\nabla\xi_j\| \right) \right\} \\
& \leq C\varepsilon,
\end{align*}

where $\hat{z}$ is the first two components of $\hat{y}$. In addition, it follows from (3.10), (3.11) and (3.13) that

\begin{align*}
(iv) \quad & \left| \int_{t_0}^T \|G(t, \xi_j \phi(t))\| dt - \int_{t_0}^T \|G(t, \xi \phi(t))\| dt \right| = \left| \int_{t_0}^T \|G(t, (\xi_j - \xi) \phi(t))\| dt \right| \\
& \leq \|\xi_j - \xi\| \int_{t_0}^T \|G(t)\| \|\phi(t)\| dt \\
& \leq \|\xi_j - \xi\| \|G(t)\|_{L^2(t_0, T; \gamma(J))} \|\phi(t)\|_{L^2(0, T)} \leq C\varepsilon \quad \text{for } j \geq N_\varepsilon.
\end{align*}
Following similar computations we can obtain for \( k \geq K_\varepsilon \) and \( j \geq N_\varepsilon \) that
\[
\begin{align*}
(v) \quad & \left| \int_{t_0}^{T} \langle y_{E_j}^m(t), \hat{y}_j \phi'(t) \rangle dt - \int_{t_0}^{T} \langle \hat{y}(t), \hat{y}_j \phi'(t) \rangle dt \right| \leq C\varepsilon, \\
(vi) \quad & \left| \langle y_{E_j}^m(t), \hat{y}_j \phi(t) - \langle \hat{y}_0, \hat{y} \rangle \phi(t) \rangle \right| \leq C\varepsilon.
\end{align*}
\]

Now, setting \( j = N_\varepsilon + 1 \) in the estimates (i)–(vi), picking a common sequence from \( \{y_{E_j}^m\} \), using Lemma 5.2 of [6], letting \( k \to \infty \) and passing to the limit in (3.12) we obtain
\[
\begin{align*}
\int_{t_0}^{T} [A\hat{y}(t), \hat{y}_j \phi(t)] dt - \int_{t_0}^{T} \langle \hat{y}(t), \hat{y}_j \phi'(t) \rangle dt + \int_{t_0}^{T} \|B(\hat{z} + \eta, \hat{y} + \delta) - B(\eta, \delta), \hat{y}_j \phi(t)\| dt \\
+ \int_{t_0}^{T} \|\Lambda(\hat{y}(t)), \hat{y}_j \phi(t)\| dt = \langle \hat{y}_0, \hat{y} \rangle \phi(t_0) + \int_{t_0}^{T} \|G(t), \hat{y}_j \phi(t)\| dt, \quad \forall \xi \in \tilde{Y},
\end{align*}
\]
where \( \hat{y} \) satisfies the first equation of (3.4) in the distribution sense. Moreover, since \( \hat{y} \in L^2(t_0, T; \tilde{Y}) \), it is not difficult to check that \( \hat{y}' \in L^2(t_0, T; \tilde{Y}^*) \). Furthermore, it follows from a trace theorem (see e.g. Theorem 1.3.1 in [29] or Lemma III.1.2 in [38]) that \( \hat{y} \in C([t_0, T]; \tilde{X}) \). In addition, from the first equation of (3.4), we can obtain an analogous expression to (3.14) with \( \langle \hat{y}(t_0), \hat{y} \rangle \) instead of \( \langle \hat{y}_0, \hat{y} \rangle \). So \( \langle \hat{y}(t_0) - \hat{y}_0, \hat{y} \rangle = 0 \) for all \( \hat{y} \in \tilde{Y} \), which implies \( \hat{y}(t_0) = \hat{y}_0 \). The proof is complete.

**Theorem 3.2. (Uniqueness)** Under the conditions of Theorem 3.1, problem (3.4) has at most one solution.

**Proof.** Let \( y^{(i)} = (z^{(i)}, y_3^{(i)}) = u^{(i)} - \delta = (u^{(i)} - \eta, w^{(i)} - \delta_3), \) \( i = 1, 2 \), be two solutions of (3.4) with the same initial data. Set \( U = y^{(1)} - y^{(2)} = (V, U_3) \), where \( V \) and \( U_3 \) are the first two and the third components of \( U \), respectively. Then it holds that
\[
\begin{align*}
\frac{d}{dt} U(t) & + AU + B(z^{(1)} + \eta, y^{(1)} + \delta) - B(z^{(2)} + \eta, y^{(2)} + \delta) + \Lambda(U) = 0, \\
U(t_0) & = 0.
\end{align*}
\]
Testing (3.15) by \( U(t) \) gives
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|U(t)\|^2 & + \|AU(t), U(t)\| + \|B(z^{(1)} + \eta, y^{(1)} + \delta) - B(z^{(2)} + \eta, y^{(2)} + \delta), U(t)\| \\
& + \|\Lambda(U(t)), U(t)\| = 0.
\end{align*}
\]
Then by (2.7), (2.10), the fact \( \|V(t)\| \leq \|U(t)\| \) and \( \|\nabla V(t)\| \leq \|\nabla U(t)\| \), and Young’s inequality we have
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|U(t)\|^2 & + \kappa \|U(t)\|^2 \leq \frac{1}{2} \frac{d}{dt} \|U(t)\|^2 + \|AU(t), U(t)\| + \|\Lambda(U(t)), U(t)\| \\
& = -\|B(V, y^{(1)}) + B(V, \delta), U(t)\| \\
& \leq \frac{1}{4} \|V\|^2 \|\nabla V\|^2 \|\nabla U(t)\| (\|y^{(1)}\|_{L^\infty_B} + \|\delta\|_{L^\infty_B}) \\
& \leq \frac{\kappa}{2} \|U(t)\|^2 + \frac{27}{16\kappa^3} (\|y^{(1)}\|_{L^\infty_B} + \|\delta\|_{L^\infty_B})^4 \|U(t)\|^2.
\end{align*}
\]
Since \( U(t_0) = 0 \) and \( \int_{t_0}^{T} (\|y^{(1)}(t)\|_{L^\infty_B} + \|\delta(t)\|_{L^\infty_B})^4 dt < \infty \) and, applying Gronwall’s inequality to the above inequality, we can conclude that \( \|U(t)\|^2 \equiv 0 \) for all \( t \in [t_0, T] \), i.e., \( y^{(1)}(t) = y^{(2)}(t), \) for all \( t \in [t_0, T] \). The proof is complete.
At this point we have proved that under the Assumption (A), for $t_0 \in \mathbb{R}, T > t_0, y_0 \in \widehat{X}, \delta \in L^1([t_0, T], L^2([0, T]; \mathbb{R}))$ and $F \in L^2([t_0, T]; \mathbb{R})$, the problem (3.4) has a unique solution $y(t_0, \delta(t, \omega); t_0, y_0)$ that belongs to $C([t_0, T]; \widehat{X}) \cap L^2([t_0, T]; \mathbb{R})$ and satisfies $y(t_0, \delta(t, \omega), y_0) = y_0$. It remains to show that $u(t) = y(t) + \delta(t)$ is the unique solution to the problem (3.1).

**Theorem 3.3.** Suppose that Assumption (A) holds. In the framework of Definition 3.1 let $y_a(t)$ be the unique solution of the problem (3.4), then the process $u(t) = y_a(t) + \delta_a(t)$ is the unique solution to (3.1) if $\{u(t)\}_{t \geq t_0}$ has trajectories in $C([t_0, T]; \widehat{X}) \cap L^2([t_0, T]; \mathbb{R})$.

**Proof.** Since $\delta_a$ solves the stochastic equation (3.2), then based on Definition 3.1 and Definition 3.2, the existence of solutions to problem (3.1) follows from Theorem 3.1. In addition, by using the same proof as that of [6, Proposition 6.16], it can be obtained that

$$y_a(t) + \delta_a(t) = y_\beta(t) + \delta_\beta(t), \quad \forall \alpha, \beta \geq 0,$$

which together with Theorem 3.2 yield the uniqueness of the solution to problem (3.1). The proof is complete. \qed

3.3. **Generation of a random dynamical system.** In this subsection we show that the solution $u(t)$ to the problem (3.1) generates a random dynamical system. Let $\gamma, \beta \in (\gamma, 1/2)$ and $\Omega := \mathbb{R}^\mathbb{Z}$ be the same as defined in Subsection 2.3. Define a mapping $\Psi = \Psi_\alpha : \mathbb{R}_+ \times \Omega \times \widehat{X} \rightarrow \widehat{X}$ by

$$\Psi(t, \omega, x) = y(t; 0, \delta(t, \omega), x - \delta(0, \omega)) + \delta(t, \omega),$$

(3.16)

where $y(t; t_0, \delta(t, \omega), y_0)$ is the solution to problem (3.4) with initial condition $y(t_0) = y_0$. Then the mapping $\Psi(\cdot, \cdot, x) : \widehat{X} \rightarrow \widehat{X}$ is continuous in the weak topologies. More precisely, if $x_n \rightarrow x$ weakly in $\widehat{X}$, then for any $\phi \in \widehat{X}$, $\langle \Psi(\cdot, \cdot, x_n), \phi \rangle \rightarrow \langle \Psi(\cdot, \cdot, x), \phi \rangle$ uniformly in $[t_0, T]$ as $n \rightarrow \infty$ (see [6, Lemma 7.1, Lemma 7.2]). Moreover, the following theorem holds.

**Theorem 3.4.** The mapping $\Psi : \mathbb{R}_+ \times \Omega \times \widehat{X} \rightarrow \widehat{X}$ defined in (3.16) is a random dynamical system.

**Proof.** First note that all properties in Definition 2.2 except the cocycle property follow naturally from the above arguments. It remains to verify the following cocycle property of $\Psi$ under $\{\theta_t\}_{t \in \mathbb{R}}$:

$$\Psi(t + s, \omega, x) = \Psi(t, \theta_s \omega, \Psi(s, \omega, x)), \quad \forall t, s \in \mathbb{R}_+, \quad x \in \widehat{X}.$$  

(3.17)

From (2.12) and (3.3), we see that for any $\alpha > 0$

$$\delta_\alpha(t, \theta_s \omega) = \int_{-\infty}^t (A + \alpha)^{1+\gamma}e^{-(t-\tau)(A+\alpha)}((A + \alpha)^{-\gamma}\theta_\tau(\theta_s \omega))(t - \tau)d\tau$$

$$= \int_{-\infty}^t (A + \alpha)^{1+\gamma}e^{-(t-\tau)(A+\alpha)}((A + \alpha)^{-\gamma}(\omega(t + s) - \omega(s + \tau)))d\tau$$

$$= \int_{-\infty}^{t+s} (A + \alpha)^{1+\gamma}e^{-(t-\tau)(A+\alpha)}((A + \alpha)^{-\gamma}(\omega(t + s) - \omega(t)))d\tau'$$

$$= \delta_\alpha(t + s, \omega) \quad \forall s \in \mathbb{R}.$$  

(3.18)
Then observing that $\Psi(t + s, \omega, x) = y(t + s; 0, \delta(t + s, \omega), x - \delta(0, \omega)) + \delta(t + s, \omega)$, and

$$\Psi(t, \theta_s \omega, \Psi(s, \omega, x)) = y(t; 0, \delta(t, \theta_s \omega), \Psi(s, \omega, x) - \delta(0, \theta_s \omega)) + \delta(t, \theta_s \omega)$$

$$= y(t; 0, \delta(t, \theta_s \omega), y(s; 0, \delta(s, \omega), x - \delta(0, \omega)) + \delta(s, \omega) - \delta(s, \omega)) + \delta(t, \theta_s \omega)$$

$$= y(t; 0, \delta(t, \theta_s \omega), y(s; 0, \delta(s, \omega), x - \delta(0, \omega))) + \delta(t + s, \omega),$$

it suffices to prove that, for any $t, s \in \mathbb{R}_+$,

$$y(t + s; 0, \delta(t + s, \omega), x - \delta(0, \omega)) = y(t; 0, \delta(t, \theta_s \omega), y(s; 0, \delta(s, \omega), x - \delta(0, \omega))).$$

(3.19)

Now fix $s \in \mathbb{R}_+$ and let

$$y_1(t) = y(t + s; 0, \delta(t + s, \omega), x - \delta(0, \omega)), \quad t \in \mathbb{R}_+,$$

$$y_2(t) = y(t; 0, \delta(t, \theta_s \omega), y(s; 0, \delta(s, \omega), x - \delta(0, \omega))), \quad t \in \mathbb{R}_+. $$

In particular,

$$y_1(0) = y(s; 0, \delta(s, \omega), x - \delta(0, \omega))$$

$$= y(0; 0, \delta(0, \theta_s \omega), y(s; 0, \delta(s, \omega), x - \delta(0, \omega))) = y_2(0).$$

Since $y$ is the solution of (3.4) and

$$y_1'(t) = \frac{dy(t + s; 0, \delta(t + s, \omega), x - \delta(0, \omega))}{dt},$$

$$y_2'(t) = \frac{dy(t; 0, \delta(t, \theta_s \omega), y(s; 0, \delta(s, \omega), x - \delta(0, \omega)))}{dt}.$$  

Then $y_1(t)$ and $y_2(t)$ satisfy, respectively,

$$\begin{align*}
y_1'(t) &= -Ay(t + s; 0, \delta(t + s, \omega), x - \delta(0, \omega)) - B(z + \eta, y + \delta) \\
&\quad -\Lambda(y(t + s; 0, \delta(t + s, \omega), x - \delta(0, \omega)) + \delta(t + s, \omega)) \\
&\quad + \alpha \delta(t + s, \omega) + F \\
&= -Ay_1(t) - B(z_1(t) + \eta(t + s, \omega), y_1(t) + \delta(t + s, \omega)) \\
&\quad -\Lambda(y_1(t) + \delta(t + s, \omega)) + \alpha \delta(t + s, \omega) + F, \\
y_1(0) &= y(s; 0, \delta(s, \omega), x - \delta(0, \omega)),
\end{align*}$$

$$\begin{align*}
y_2'(t) &= -Ay(t; 0, \delta(t, \theta_s \omega), y(s; 0, \delta(s, \omega), x - \delta(0, \omega))) - B(z + \eta, y + \delta) \\
&\quad -\Lambda(y(t; 0, \delta(t, \theta_s \omega), y(s; 0, \delta(s, \omega), x - \delta(0, \omega))) + \delta(t, \theta_s \omega)) \\
&\quad + \alpha \delta(t, \theta_s \omega) + F \\
&= -Ay_2(t) - B(z_2(t) + \eta(t, \theta_s \omega), y_2(t) + \delta(t, \theta_s \omega)) \\
&\quad -\Lambda(y_2(t) + \delta(t, \theta_s \omega)) + \alpha \delta(t, \theta_s \omega) + F, \\
y_2(0) &= y(s; 0, \delta(s, \omega), x - \delta(0, \omega)),
\end{align*}$$

where $z_i(t)$ is the first two components of $y_i(t), i = 1, 2$, respectively. Further by (3.18), we can conclude that both $y_1$ and $y_2$ are the solutions to problem (3.4) with the same initial data $y(s; 0, \delta(s, \omega), x - \delta(0, \omega))$. It then follows from the uniqueness of solutions that $y_1(t) = y_2(t), \forall t \in \mathbb{R}_+$. Finally, since $s$ is arbitrary in $\mathbb{R}_+$, the equality (3.19) holds. The proof is complete. \(\square\)
Notice that for any \( t_0 \in \mathbb{R} \), \( \omega \in \Omega \) and \( u_0 \in \hat{X} \), the solution process of the problem (3.1) satisfies
\[
u(t; t_0, \delta(t, \omega), u_0) = \gamma(t; t_0, \delta(t, \omega), u_0 - \delta(t_0, \omega)) + \delta(t, \omega) = \Psi(t - t_0, \theta_{t_0} \omega, u_0).
\] (3.20)

From now on we refer to \( \{\Psi(t, \omega)\}_{t \in \mathbb{R}, \omega \in \Omega} \) defined in (3.16) as the RDS generated by the problem (3.1).

4. Existence of random attractors. This section is devoted to the existence of a random attractor of the RDS \( \{\Psi(t, \omega)\}_{t \in \mathbb{R}, \omega \in \Omega} \) defined in (3.16). In particular, we first define the appropriate probability space and domain of attraction \( \mathcal{P} \).

Then we construct an absorbing set \( \mathcal{K} \) and prove the asymptotic compactness of \( \{\Psi(t, \omega)\}_{t \in \mathbb{R}, \omega \in \Omega} \).

Note that due to [6, section 6], the solution \( \delta_\alpha \) to the stochastic differential equation (2.12) is an \( \mathbb{X} \)-valued stationary and ergodic process, where \( \mathbb{X} = \hat{X} \cap L^\beta_{\text{loc}} \) as defined in Section 2.3. Moreover, \( E\|\delta_\alpha(0)\|_{\mathbb{X}}^4 \to 0 \) as \( \alpha \to \infty \). By the Strong Law of Large Numbers (see [19] for a similar argument),
\[
\lim_{t \to \infty} \frac{1}{t} \int_{-t}^{0} \|\delta_\alpha(\tau)\|_{\mathbb{X}}^4 \, d\tau = E\|\delta_\alpha(0)\|_{\mathbb{X}}^4, \quad \text{P-a.s. on } C^2_t(\mathbb{R}, \mathbb{X}).
\] (4.1)

Therefore, we can conclude that there exists \( \alpha_0 \geq 0 \) such that
\[
E\|\delta_\alpha(0)\|_{\mathbb{X}}^4 < \frac{\kappa^4(1 + \lambda)}{54}, \quad \forall \alpha \geq \alpha_0,
\] (4.2)
where \( \kappa \) and \( \lambda \) are as defined in (2.10) and (1.2), respectively.

Let \( \Omega^\beta(\mathbb{B}) \) be defined as in Section 2.3, and for any \( \alpha > 0 \) denote by \( \Omega^\beta_\alpha(\mathbb{B}) \) the set of those \( \omega \in \Omega^\beta(\mathbb{B}) \) for which the equality (4.1) holds true. Then it follows from [6, Corollary 6.8] that the set \( \Omega^\beta_\alpha(\mathbb{B}) \) is invariant with respect to the flow \( \theta_\alpha \) defined by (2.12), i.e.,
\[
\theta_{t\alpha} \omega \in \Omega^\beta_\alpha(\mathbb{B}), \quad \forall \alpha > 0, \ t \in \mathbb{R}, \ \omega \in \Omega^\beta_\alpha(\mathbb{B}).
\]

Now fix \( \beta \in (\gamma, \frac{1}{2}) \) and set \( \hat{\Omega} = \bigcap_{n=1}^{\infty} \Omega^\beta_n(\mathbb{B}) \). Let \( \hat{F}, \hat{P} \) and \( \hat{\theta} \) are the natural restrictions of \( F, P \) and \( \theta \) to \( \hat{\Omega} \). Then the quadruple \( \hat{Q} = (\hat{\Omega}, \hat{F}, \hat{P}, \hat{\theta}) \) defines a metric dynamical system, and for each \( \omega \in \hat{\Omega} \) the limit in (4.1) exists. Moreover, it follows from (4.1) and (4.2) that

**Lemma 4.1.** For each \( \omega \in \hat{\Omega} \), there exist \( T = t^*(\omega) \geq 0 \) and \( \alpha_0 \geq 0 \) such that
\[
\frac{27}{\kappa^3} \int_{-t}^{0} \|\delta_\alpha(\tau)\|_{\mathbb{X}}^4 \, d\tau < \frac{\kappa^4(1 + \lambda)}{2}, \quad \forall \ t \geq t^*, \ \alpha \geq \alpha_0.
\]

**Lemma 4.2.** Let Assumption (A) hold and \( t_0 \in \mathbb{R} \). For any \( T > t_0 \), the solution \( y(t; t_0, \delta(t, \omega), y_0) \) of the problem (3.4) with \( \delta \in L^4(t_0, T; \mathcal{L}^\beta_{\text{loc}}) \cap L^2(t_0, T; \hat{X}) \) and \( F \in L^2(t_0, T; \hat{Y}^\alpha) \) satisfies
\[
\|y(t; t_0, \delta(t, \omega), y_0)\|^4 \leq \|y_0\|^4 \cdot e^{-\kappa(1 + \lambda)(t - t_0)} + \frac{27}{\kappa^4} \int_{t_0}^{t} \|\delta(s, \omega)\|_{\mathbb{X}}^4 \, ds
\]
\[
+ \frac{2}{\kappa} \int_{t_0}^{t} \|G(s)\|_{\mathbb{X}}^2 \cdot e^{-\kappa(1 + \lambda)(t - s)} + \frac{27}{\kappa^4} \int_{t_0}^{t} \|\delta(\tau, \omega)\|_{\mathbb{X}}^4 \, d\tau \, ds,
\] (4.3)
where \( G = \alpha \delta + F - B(\eta, \delta) - \Lambda(\delta) \) and \( \alpha \geq \alpha_0 \).
Proof. Testing the first equation in (3.4) by \(y(t, x)\), similar to (3.6) we have
\[
\frac{1}{2} \frac{d}{dt} \|y(t)\|^2 + \kappa \|y(t)\|^2_3 + \|B(z, \delta), y(t)\| \leq \|G(t), y(t)\|. \tag{4.4}
\]
Observe that \(\|z(t)\| \leq \|y(t)\|\) and \(\|\nabla z(t)\| \leq \|\nabla y(t)\|\), then from (2.7), Hölder inequality and Young’s inequality, it follows that
\[
\|B(z, \delta), y(t)\| \leq \|z(t)\| \|\nabla y(t)\| \|\delta(t)\| \leq 2^{1/4} \|z(t)\|^{1/2} \|\nabla y(t)\|^{1/2} \|\delta(t)\| \leq 2^{1/4} \|y(t)\|^{1/2} \|\nabla y(t)\|^{1/2} \|\delta(t)\| \leq \frac{\kappa}{4} \|y(t)\|^2 + \frac{27}{2\kappa^2} \|\delta(t)\|^2 \|y(t)\|^2. \tag{4.5}
\]
In addition,
\[
\|G(t), y(t)\| \leq \|G(t)\|_3, \|y(t)\|_3 \leq \frac{\kappa}{4} \|y(t)\|^2 + \frac{1}{\kappa} \|G(t)\|^2_3. \tag{4.6}
\]
Inserting the inequalities (4.5) and (4.6) into (4.4), and using the fact \((1+\lambda)\|y(t)\|^2 \leq \|y(t)\|^2_3\) we obtain
\[
\frac{d}{dt} \|y(t)\|^2 \leq \frac{27}{\kappa^2} \|\delta(t)\|^2 \|y(t)\|^2 - \kappa \|y(t)\|^2_3 + \frac{2}{\kappa} \|G(t)\|^2_3,
\]
\[
\leq \left(-\kappa(1+\lambda) + \frac{27}{\kappa^2} \|\delta(t)\|^2 \right) \|y(t)\|^2 + \frac{2}{\kappa} \|G(t)\|^2_3. \tag{4.7}
\]
Applying the Gronwall inequality to (4.7) yields (4.3). The proof is complete. \(\square\)

Definition 4.1. A function \(r : \Omega \to (0, +\infty)\) is said to belong to the class \(\mathcal{R}\) iff
\[
\lim_{t \to \infty} r^2(\theta_{-t}\omega) \cdot e^{-\kappa(1+\lambda)t + \frac{27}{\kappa^2} \int_0^\tau \|\delta(s, \omega)\|^4 \|y(t)\|^4 \|y(t)\|^2_3 \, ds} = 0. \tag{4.8}
\]

The class \(\mathcal{R}\) is a special case of the class of tempered processes. In particular, the constant functions belongs to \(\mathcal{R}\). Moreover, the class \(\mathcal{R}\) is closed with respect to sum and scalar multiplication. Furthermore, if \(r \in \mathcal{R}\) and \(0 \leq \tilde{r} \leq r\), then \(\tilde{r} \in \mathcal{R}\).

The following results can be shown from similar proof for Proposition 2.24 of [7].

Proposition 4.1. For any \(\omega \in \Omega\), the functions \(r_i : \Omega \to (0, +\infty), \quad i = 1, 2, 3, 4, 5\), defined respectively by
\[
\begin{align*}
    r_1^2(\omega) &:= \|\delta(0, \omega)\|^2, \\
r_2^2(\omega) &:= \sup_{t \leq 0} \|\delta(t, \omega)\|^2 \cdot e^{\kappa(1+\lambda)t + \frac{27}{\kappa^2} \int_0^\tau \|\delta(s, \omega)\|^4 \|y(t)\|^4_3 \, ds}, \\
r_3^2(\omega) &:= \int_{-\infty}^0 \|\delta(t, \omega)\|^2 \cdot e^{\kappa(1+\lambda)t + \frac{27}{\kappa^2} \int_0^\tau \|\delta(s, \omega)\|^4 \|y(t)\|^4_3 \, dt}, \\
r_4^2(\omega) &:= \int_0^\infty \|\delta(t, \omega)\|^4_3 \cdot e^{\kappa(1+\lambda)t + \frac{27}{\kappa^2} \int_0^\tau \|\delta(s, \omega)\|^4 \|y(t)\|^4_3 \, dt}, \\
r_5^2(\omega) &:= \int_{-\infty}^0 e^{\kappa(1+\lambda)t + \frac{27}{\kappa^2} \int_0^\tau \|\delta(s, \omega)\|^4 \|y(t)\|^4_3 \, dt} \, dt,
\end{align*}
\]
all belong to class \(\mathcal{R}\).
Lemma 4.3. (\(\mathcal{D}_\mathcal{R}\) – absorbing set) Let Assumption (A) hold and suppose that \(F(t,x) \in L^2(-\infty, 0; Y^*)\). Then the RDS \(\{\Psi(t,\omega)\}_{t \in \mathbb{R}, \omega \in \Omega}\) has a \(\mathcal{D}_\mathcal{R}\)-absorbing set \(\mathcal{K}(\omega)\) defined by \(\mathcal{K}(\omega) := \{ u \in \tilde{X} : \|u\|^2 \leq R^2(\omega) \}\) with

\[
R^2(\omega) = 2(1 + r_1^2(\omega) + r_2^2(\omega)) + \frac{4}{\kappa} \int_{-\infty}^0 \|G(s)\|^2_{\tilde{Y}^*} e^{\kappa(1+\lambda)s + \frac{2\pi}{\kappa} \int_0^s \|\delta(\tau,\omega)\|^4_{\tilde{C}_0} d\tau} ds.
\]

Proof. First by the definition (3.16), for any \(D(\omega) \in \mathcal{D}_\mathcal{R}\) and \(u_0 \in D(\omega)\),

\[
\| \Psi(t,\omega, u_0) \|^2 \leq 2\| y(t; 0, \delta(t,\omega), u_0 - \delta(0,\omega) \|^2 + 2\| \delta(t,\omega) \|^2.
\]

(4.9)

Then by Lemma 4.2 we have

\[
\| y(t; 0, \delta(t,\omega), u_0 - \delta(0,\omega)) \|^2 \leq \left( \|u_0\|^2 + \| \delta(0,\omega) \|^2 \right) e^{-\kappa(1+\lambda)t + \frac{2\pi}{\kappa} \int_0^s \|\delta(s,\omega)\|^4_{\tilde{C}_0} ds + \frac{2}{\kappa} \int_0^t \|G(s)\|^2_{\tilde{Y}^*} e^{-\kappa(1+\lambda)(t-s) + \frac{2\pi}{\kappa} \int_s^t \|\delta(t,\omega)\|^4_{\tilde{C}_0} d\tau} ds + 2\| \delta(t,\omega) \|^2.
\]

(4.10)

Inserting the above inequality in (4.9), and replacing \(\omega\) by \(\theta - t\omega\) results in

\[
\| \Psi(t,\theta - t\omega, u_0) \|^2 \leq 2\left( \|u_0\|^2 + \| \delta(0,\theta - t\omega) \|^2 \right) e^{-\kappa(1+\lambda)t + \frac{2\pi}{\kappa} \int_0^s \|\delta(s,\theta - t\omega)\|^4_{\tilde{C}_0} ds + \frac{4}{\kappa} \int_0^t \|G(s)\|^2_{\tilde{Y}^*} e^{-\kappa(1+\lambda)(t-s) + \frac{2\pi}{\kappa} \int_s^t \|\delta(s,\omega)\|^4_{\tilde{C}_0} d\tau} ds + 2\| \delta(0,\theta - t\omega) \|^2.
\]

(4.11)

Since \(u_0 \in D(\theta - t\omega) \in \mathcal{D}_\mathcal{R}\), there exists \(T_D > 0\) such that for any \(t \geq T_D\),

\[
\|u_0\|^2 e^{-\kappa(1+\lambda)t + \frac{2\pi}{\kappa} \int_0^s \|\delta(s,\omega)\|^4_{\tilde{C}_0} ds} < 1.
\]

(4.12)

Then by (4.11), (4.12) and definitions of \(r_1(\omega)\) and \(r_2(\omega)\), we obtain that

\[
\| \Psi(t,\theta - t\omega, u_0) \|^2 < 2(1 + r_1^2(\omega) + r_2^2(\omega)) + \frac{4}{\kappa} \int_{-\infty}^0 \|G(s)\|^2_{\tilde{Y}^*} e^{\kappa(1+\lambda)s + \frac{2\pi}{\kappa} \int_0^s \|\delta(\tau,\omega)\|^4_{\tilde{C}_0} d\tau} ds := R^2(\omega), \quad \forall u_0 \in D(\theta - t\omega), \ t > T_D.
\]

Due to Proposition 4.1, \(R(\omega) \in \mathcal{D}_\mathcal{R}\). Therefore the random set \(\mathcal{K}(\omega) := \{ u \in \tilde{X} : \|u\|^2 \leq R^2(\omega) \}\) absorbs every element of \(\mathcal{D}_\mathcal{R}\) and thus is a \(\mathcal{D}_\mathcal{R}\)-absorbing set for the RDS \(\{\Psi(t,\omega)\}_{t \in \mathbb{R}, \omega \in \Omega}\). The proof is complete. \(\square\)

Next, we focus on investigating the asymptotic compactness of the random dynamical system \(\{\Psi(t,\omega)\}_{t \in \mathbb{R}, \omega \in \Omega}\) defined by (3.16), over the metric dynamical system \(Q = (\Omega, \bar{F}, \bar{P}, \bar{\theta})\).
Lemma 4.4. ($\mathcal{D}$-asymptotic compactness) Let Assumption (A) hold and suppose that $F(t, x) \in L^2(-\infty, 0; \mathcal{Y}^*)$. Then the RDS $\{\Psi(t, \omega)\}_{t \in \mathbb{R}, \omega \in \Omega}$ is $\mathcal{D}$-asymptotically compact.

Proof. This proof employs the same techniques as those in [6, 7], with different computations. For completeness we provide all necessary details. Let $\omega \in \Omega$ be fixed and $D \in \mathcal{D}$. Consider a positive sequence $\{t_n\}_{n=0}^{\infty}$ satisfying $t_n \to +\infty$ as $n \to +\infty$ and a sequence $\{x_n\}_{n=0}^{\infty}$ satisfying $x_n \in D(\theta_{-t_n}, \omega)$ for all $n \in \mathbb{N}$. It suffice to show that the sequence $\{\Psi(t_n, \theta_{-t_n}, \omega, x_n)\}_{n \in \mathbb{N}}$ is compact in $\hat{\mathcal{X}}$, i.e., there exist a subsequence $\{\Psi(t_{n_k}, \theta_{-t_{n_k}}, \omega, x_{n_k})\}_{k \in \mathbb{N}}$ and some $x_0 \in \hat{\mathcal{X}}$ such that

$$\psi(t_{n_k}, \theta_{-t_{n_k}}, \omega, x_{n_k}) \to x_0 \quad \text{strongly in } \hat{\mathcal{X}} \text{ as } k \to \infty. \quad (4.13)$$

The proof is proceeded in three steps below.

**Step I.** Since $\hat{\mathcal{X}}$ is a reflexive Banach space, then by Lemma 4.3 there exist a subsequence of $\{\Psi(t_n, \theta_{-t_n}, \omega, x_n)\}_{n \in \mathbb{N}}$ (denoted by the same symbol) such that, for some $x_0 \in \hat{\mathcal{X}}$,

$$\psi(t_n, \theta_{-t_n}, \omega, x_n) \to x_0 \text{ weakly in } \hat{\mathcal{X}} \text{ as } n \to \infty. \quad (4.14)$$

Also, because $\delta(0, \omega) \in \hat{\mathcal{X}}$, it holds that

$$\psi(t_n, \theta_{-t_n}, \omega, x_n) - \delta(0, \omega) \to x_0 - \delta(0, \omega) \text{ weakly in } \hat{\mathcal{X}} \text{ as } n \to \infty. \quad (4.15)$$

Then from the lower semi-continuity of norm, it follows that

$$\|x_0 - \delta(0, \omega)\| \leq \liminf_{n \to \infty} \|\psi(t_n, \theta_{-t_n}, \omega, x_n) - \delta(0, \omega)\|. \quad (4.16)$$

**Step II.** Due to Lemma 4.3, there exists a $\mathcal{D}$-absorbing set $\mathcal{K}$ which absorbs $D$. We next construct a backward sequence $\{u_{-n}\}_{n=-\infty}^{0}$ such that

$$u_{-n} \in \mathcal{K}(\theta_{-n}\omega), n \in \mathbb{N} \quad \text{and} \quad u_{-m} = \psi(-m + n, \theta_{-n}\omega, u_{-n}), 0 \leq m \leq n.$$ 

First because $\mathcal{K}(\theta_{-1}\omega)$ absorbs $D$, there exists a constant $N_1(\omega) \in \mathbb{N}$, such that

$$\{\psi(-1 + t_n, \theta_{1-t_n}, \theta_{-1}\omega, x_n) : n \geq N_1(\omega)\} \subset \mathcal{K}(\theta_{-1}\omega).$$

Hence, we can find a subsequence $\{n_k\} \subset \mathbb{N}$ and $u_{-1} \in \mathcal{K}(\theta_{-1}\omega)$ such that

$$\psi(-1 + t_{n_k}, \theta_{-t_{n_k}}\omega, x_{n_k}) \to u_{-1} \text{ weakly in } \hat{\mathcal{X}} \text{ as } k \to \infty. \quad (4.17)$$

On the other hand, the cocycle property of $\psi$ implies that

$$\psi(t_{n_k}, \theta_{-t_{n_k}}\omega, \cdot) = \psi(1, \theta_{-1}\omega, \psi(-1 + t_{n_k}, \theta_{-t_{n_k}}\omega, \cdot)), \quad k \in \mathbb{N}.$$ 

Therefore it follows from (4.14), (4.17) and the continuity of $\psi$ that

$$x_0 = \psi(1, \theta_{-1}\omega, u_{-1}).$$

Proceed in the same way, for each $n \in \mathbb{N}$, we can construct a diagonal subsequence of $\{n_k\}$, still denoted by $\{n_k\}$, and $u_{-k} \in \mathcal{K}(\theta_{-k}\omega)$ such that

$$\begin{cases} 
\psi(-n + t_{n_k}, \theta_{-t_{n_k}}\omega, x_{n_k}) \to u_{-n} \text{ weakly in } \hat{\mathcal{X}}; \\
\psi(u_{-(n-1)}) = \psi(1, \theta_{-n}\omega, u_{-n}).
\end{cases} \quad (4.18)$$
The cocycle property $\Psi$ again implies that
\[
\Psi(t_{nk}, \theta_{-t_{nk}} \omega, \cdot) = \Psi(n, \theta_{-n} \omega, \Psi(-n + t_{nk}, \theta_{-t_{nk}} \omega, \cdot)), \quad n \in \mathbb{N},
\tag{4.19}
\]
which, along with (4.18) and the continuity of $\Psi$ give
\[
x_0 = \lim_{k \to \infty} \Psi(t_{nk}, \theta_{-t_{nk}} \omega, x_{nk}) \\
= \lim_{k \to \infty} \Psi(n, \theta_{-n} \omega, \Psi(t_{nk} - n, \theta_{-t_{nk}} \omega, x_{nk})) \\
= \Psi(n, \theta_{-n} \omega, \lim_{k \to \infty} \Psi(t_{nk} - n, \theta_{-t_{nk}} \omega, x_{nk})) \\
= \Psi(n, \theta_{-n} \omega, u_{-n}),
\tag{4.20}
\]
where “$\lim$” represents the limit in the weak topology on $\hat{X}$. The above equality (4.20) means precisely that $x_0 = u(0, -k; \omega, u_{-k})$, where $u$ is the solution of problem (3.1). With the same procedure as above, we can deduce that
\[
\Psi(m, \theta_{-n} \omega, u_{-n}) = u_{m-n}, \quad \text{if} \quad 0 \leq m \leq n.
\tag{4.21}
\]

**Step III.** In this step we prove (4.16). To this end, fix $n \in \mathbb{N}$ and consider problem (3.1) on the time interval $[-n, 0]$. By (3.20) and (4.19), we have
\[
\|\Psi(t_{nk}, \theta_{-t_{nk}} \omega, x_{nk}) - \delta(0, \omega)\|^2 \\
= \|\Psi(n, \theta_{-n} \omega, \Psi(t_{nk} - n, \theta_{-t_{nk}} \omega, x_{nk})) - \delta(0, \omega)\|^2 \\
= \|y(0, -n, \delta(0, \omega), \Psi(t_{nk} - n, \theta_{-t_{nk}} \omega, x_{nk}) - \delta(-n, \omega))\|^2,
\]
where $y(\cdot; -n, \delta(\cdot, \omega), \Psi(t_{nk} - n, \theta_{-t_{nk}} \omega, x_{nk}) - \delta(-n, \omega))$ is the solution of (3.4) with $\delta(\cdot) = \delta_n(\cdot, \omega)$ and the initial value $y(-n) = \Psi(t_{nk} - n, \theta_{-t_{nk}} \omega, x_{nk}) - \delta(-n, \omega)$.

Now, testing the first equation of (3.4) by $y(t)$ to get
\[
\frac{1}{2} \frac{d}{dt} \|y(t)\|^2 = -\frac{\kappa(1 + \lambda)}{2} \|y(t)\|^2 + \|G(t, y(t)) - \|B(z(t), \delta(t))\|, y(t)\| \\
- \|A g(t, y(t)) - \|A (y(t))\|, y(t)\| + \frac{\kappa(1 + \lambda)}{2} \|y(t)\|^2,
\tag{4.22}
\]
where $G(t) = \alpha \delta(t) + F(t) - B(\eta(t), \delta(t)) - \Lambda(\delta(t))$. Integrating the above equation from $-n$ to $0$ results in
\[
\|y(0)\|^2 = \|\Psi(t_{nk}, \theta_{-t_{nk}} \omega, x_{nk}) - \delta(0, \omega)\|^2 \\
= e^{-\kappa(1+\lambda)n} \|\Psi(t_{nk} - n, \theta_{-t_{nk}} \omega, x_{nk}) - \delta(-n, \omega)\|^2 \\
- 2 \int_{-n}^{0} e^{\kappa(1+\lambda)s} \|B(z(s), \delta(s))\|, y(s)\| + \|A y(s), y(s)\| + \|\Lambda(y(s)), y(s)\|)ds \\
+ 2 \int_{-n}^{0} e^{\kappa(1+\lambda)s} \|G(s), y(s)\|ds + \kappa(1 + \lambda) \int_{-n}^{0} e^{\kappa(1+\lambda)s} \|y(s)\|^2 ds.
\tag{4.23}
\]
Therefore, if there exists a function $h(\cdot) \in L^1(-\infty, 0)$ and
\[
\lim_{k \to \infty} \|\Psi(t_{nk}, \theta_{-t_{nk}} \omega, x_{nk}) - \delta(0, \omega)\|^2 \leq \int_{-\infty}^{-n} h(s)ds + \|x_0 - \delta(0, \omega)\|^2,
\tag{4.24}
\]
then (4.16) follows from taking limit $k \to \infty$ in (4.24). What remains is to prove (4.24).

For any $t \in (-n, 0)$, set
\[
y_{n_k}(t) = y(t; -n, \delta(t, \omega), \Psi(t_{nk} - n, \theta_{-t_{nk}} \omega, x_{nk}) - \delta(-n, \omega)) \\
y_{n}(t) = y(t; -n, \delta(t, \omega), u_{-n} - \delta(-n, \omega)).
\]
By the continuity of $\Psi$ and (4.18), we have

$$y_{nk}(\cdot) \rightharpoonup y_n(\cdot) \text{ weakly in } L^2(-n, 0; \hat{Y}) \text{ as } k \to \infty,$$

and consequently,

$$\lim_{k \to \infty} \int_{-n}^{0} e^{\kappa(1+\lambda)s} \| G(s), y_{nk}(s) \| ds = \int_{-n}^{0} e^{\kappa(1+\lambda)s} \| G(s), y_n(s) \| ds. \tag{4.26}$$

From Lemma 5.2 of [6], Conclusion (3.9) in [36] and (4.25), we deduce that there exists a subsequence of (denoted still by) $\{ y_{nk} \}$ satisfying

$$\lim_{k \to \infty} \int_{-n}^{0} e^{\kappa(1+\lambda)s} \| B(z_{nk}, \delta(s, \omega)), y_{nk}(s) \| ds = \int_{-n}^{0} e^{\kappa(1+\lambda)s} \| B(z_n, \delta(s, \omega)), y_n(s) \| ds, \tag{4.27}$$

where $z_{nk}$ and $z_n$ are the first two components of $y_{nk}$ and $y_n$, respectively. Set

$$\| y(t) \| := \| Ay(t), y(t) \| + \| \Lambda(y(t)), y(t) \| - \frac{\kappa(1+\lambda)}{2} \| y(t) \|^2.$$

Then $\| \cdot \|$ is equivalent to the norm $\| \cdot \|_{\hat{Y}}$. Thus, by the lower semi-continuity of norm, (4.25) implies that

$$\limsup_{k \to \infty} \left\{ - \int_{-n}^{0} e^{\kappa(1+\lambda)s} \| y_{nk}(s) \|^2 ds \right\} \leq - \int_{-n}^{0} e^{\kappa(1+\lambda)s} \| y_n(s) \|^2 ds. \tag{4.28}$$

Next, we estimate the first term on the right hand side of (4.23). In fact, for $t_{nk} - n > 0$, it follows from (3.20) and (4.3) that

$$\begin{align*}
& e^{-\kappa(1+\lambda)n} \| y(t_{nk} - n; \theta-t_{nk} \omega, x_{nk}) - \delta(-n, \omega) \|^2 \\
& = e^{-\kappa(1+\lambda)n} \| y(-n; t_{nk}, \delta(-n, \omega), x_{nk} - \delta(-t_{nk}, \omega)) \|^2 \\
& \leq e^{-\kappa(1+\lambda)n} \left\{ \| x_{nk} - \delta(-t_{nk}, \omega) \|^2 e^{-\kappa(1+\lambda)(t_{nk} - n) + \frac{27}{\kappa} \int_{-t_{nk}}^{n} \| \delta(s, \omega) \|^4 ds} \\
& + \frac{28}{\kappa} \int_{-t_{nk}}^{n} \| G(s) \|^2 \gamma \cdot e^{-\kappa(1+\lambda)(-n-s) + \frac{27}{\kappa} \int_{s}^{n} \| \delta(t, \omega) \|^4 ds} ds \right\} \\
& \leq 2I_{nk}^{(1)} + 2I_{nk}^{(2)} + \frac{2}{\kappa} f_{nk}^{(3)}, \tag{4.29}
\end{align*}$$

where

$$\begin{align*}
I_{nk}^{(1)} &= \| x_{nk} \|^2 \cdot e^{-\kappa(1+\lambda)t_{nk} + \frac{27}{\kappa} \int_{-t_{nk}}^{n} \| \delta(s, \omega) \|^4 ds}, \\
I_{nk}^{(2)} &= \| \delta(-t_{nk}, \omega) \|^2 \cdot e^{-\kappa(1+\lambda)t_{nk} + \frac{27}{\kappa} \int_{-t_{nk}}^{n} \| \delta(s, \omega) \|^4 ds}, \\
I_{nk}^{(3)} &= \int_{-t_{nk}}^{n} \| G(s) \|^2 \gamma \cdot e^{-\kappa(1+\lambda)s + \frac{27}{\kappa} \int_{s}^{n} \| \delta(t, \omega) \|^4 ds} ds.
\end{align*}$$

Following computations similar to those in [6, 7] we can derive that

$$\lim_{k \to \infty} I_{nk}^{(1)} = 0, \quad \lim_{t \to \infty} \| \delta(-t, \omega) \|^2 \cdot e^{-\kappa(1+\lambda)t + \frac{27}{\kappa} \int_{-t}^{0} \| \delta(s, \omega) \|^4 ds} ds = 0,$$

$$\int_{-\infty}^{0} \left( \| \delta(s, \omega) \|^2 + \| \delta(s, \omega) \|^4 \right) \cdot e^{\kappa(1+\lambda)s + \frac{27}{\kappa} \int_{s}^{0} \| \delta(t, \omega) \|^4 ds} ds < \infty.$$
Thus, based on the above arguments, we can conclude that there exists a non-negative function $h(s) \in L^2(\mathbb{R}^1, \mathcal{F}, \mathbb{P})$ such that
\[
\limsup_{k \to \infty} e^{-\kappa(1+\lambda)n} \|\Psi(t_{n_k} - n, \theta_{n_k} \omega, x_{n_k}) - \delta(-n, \omega)\|^2 \leq \int_{-\infty}^{-n} h(s)ds, \quad n \in \mathbb{N}.
\]  
(4.30)

Finally, taking (4.23), (4.26)-(4.28) and (4.30) into account, we obtain
\[
\|\Psi(t_{n_k}, \theta_{n_k} \omega, x_{n_k}) - \delta(0, \omega)\|^2 \leq 2 \int_{-n}^{0} e^{\kappa(1+\lambda)s} \left(\|G(s), y_n(s)\| - \|B(z_n(s), \delta(s)), y_n(s)\|\right) ds
\]
\[
\quad - 2 \int_{-n}^{0} e^{\kappa(1+\lambda)s} \left(\|A_{y_n(s)}, y_n(s)\| + \|A_{y_n(s)}, y_n(s)\| + \frac{\kappa(1+\lambda)}{2} \|y_n(s)\|^2\right) ds
\]
\[
\quad + \int_{-\infty}^{-n} h(s)ds.
\]  
(4.31)

On the other hand, integrating (4.22) from $-n$ to 0 with initial value $u_{-n} - \delta(-n, \omega)$ and thanks to (4.20) we have that
\[
\|x_0 - \delta(0, \omega)\| = \|\Psi(n, \theta_{-n} \omega, u_{-n}) - \delta(0, \omega)\|^2 = \|y(0; -n, \omega, u_{-n} - \delta(-n))\|^2
\]
\[
= 2 \int_{-n}^{0} e^{\kappa(1+\lambda)s} \left(\|G(s), y_n(s)\| - \|B(z_n(s), \delta(s), y_n(s))\|\right) ds
\]
\[
\quad - 2 \int_{-n}^{0} e^{\kappa(1+\lambda)s} \left(\|A_{y_n(s)}, y_n\| + \|A_{y_n(s)}, y_n\| + \frac{\kappa(1+\lambda)}{2} \|y_n\|^2\right) ds
\]
\[
\quad + \|u_{-n} - \delta(-n)\|^2 e^{-\kappa(1+\lambda)n},
\]
which together with (4.31) gives (4.24). Hence (4.16) holds. Further, we have (4.13).

The proof is complete.

Based on Lemma 2.1, as a consequence of Lemma 4.3 and Lemma 4.4, we have

**Theorem 4.1.** Let the domain $\mathcal{O} \subset \mathbb{R}^2$ be a Poincaré domain and the Assumption (A) hold. Then for $F(t, x) \in L^2(-\infty, T; \mathcal{Y})$, $T \in \mathbb{R}$, the RDS $\{\Psi(t, \omega)\}_{t \in \mathbb{R}, \omega \in \Omega}$ over the metric dynamical system $\hat{Q} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\theta})$ generated by the problem (3.1) possesses an $\hat{\mathcal{F}}$- measurable $\hat{\mathcal{G}}$- attractor $\mathcal{A}$.

**REFERENCES**

[1] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, Berlin, 1998.
[2] Z. Brzeźniak, M. Capiński and F. Flandoli, Pathwise global attractors for stationary random dynamical systems, *Probability Theory and Related Fields*, 95 (1993), 87–102.
[3] Z. Brzeźniak, On Sobolev and Besov spaces regularity of Brownian paths, *Stochastics and Stochastics Reports*, 56 (1996), 1–15.
[4] Z. Brzeźniak and S. Peszat, Stochastic two dimensional Euler equations, *Ann. Probab.*, 29 (2001), 1796–1832.
[5] J. Ball, Global attractors for damped semilinear wave equations, *Discrete Contin. Dyn. Sys.*, 10 (2004), 31–52.
[6] Z. Brzeźniak and Y. Li, Asymptotic compactness and absorbing sets for 2D stochastic Navier-Stokes equations on some unbounded domains, *Trans. Amer. Math. Soc.*, 358 (2006), 5587–5629.
[7] Z. Brzeźniak, T. Caraballo, J. A. Langa, Y. Li, G. Łukaszewicz and J. Real, Random attractors for stochastic 2D-Navier-Stokes equations in some unbounded domains, *J. Differential Equations*, 255 (2013), 3897–3919.
[8] T. Caraballo, The long-time behaviour of stochastic 2D-Navier-Stokes equations, Probabilistic Methods in Fluids, (2003), 70–83.

[9] T. Caraballo and X. Han, Applied Nonautonomous and Dynamical Systems, SpringerBriefs in Mathematics, Springer, Switzerland, 2016.

[10] T. Caraballo and J. A. Langa, On the upper semicontinuity of cocycle attractors for non-autonomous and random dynamical systems, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 10 (2003), 491–513.

[11] T. Caraballo, J. A. Langa, V. S. Melnik and J. Valero, Pullback attractors of nonautonomous and stochastic multivalued dynamical systems, Set-Valued Anal., 10 (2003), 153–201.

[12] T. Caraballo and K. Lu, Attractors for stochastic lattice dynamical systems with a multiplicative noise, Front. Math. China, 3 (2008), 317–335.

[13] T. Caraballo, G. Lukaszewicz and J. Real, Pullback attractors for asymptotically compact non-autonomous dynamical systems, Nonlinear Anal. - TMA, 64 (2006), 484–498.

[14] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics, 580, Springer, Berlin.

[15] J. Chen, Z. Chen and B. Dong, Existence of H2-global attractors of two-dimensional micropolar fluid flows, J. Math. Anal. Appl., 322 (2006), 512–522.

[16] J. Chen, B. Dong and Z. Chen, Uniform attractors of non-homogeneous micropolar fluid flows in non-smooth domains, Nonlinearity, 20 (2007), 1619–1635.

[17] H. Crauel and F. Flandoli, Attractors for random dynamical systems, Probability Theory and Related Fields, 100 (1994), 365–393.

[18] H. Crauel, Random Probability Measures on Polish Spaces, Stochastics Monographs, 11, Taylor & Francis, London, 2002.

[19] G. Da Prato and J. Zabczyk, Ergodicity for Infinite Dimensional System, Cambridge University Press, Cambridge, 1996.

[20] B. Dong and Z. Chen, Global attractors of two-dimensional micropolar fluid flows in some unbounded domains, Appl. Math. Comp., 182 (2006), 610–620.

[21] B. Dong and Z. Zhang, Global regularity of the 2D micropolar fluid flows with zero angular viscosity, J. Differential Equations, 249 (2010), 200–213.

[22] B. Dong, J. Li and J. Wu, Global well-posedness and large-time decay for the 2D micropolar equations, J. Differential Equations 262 (2017), 3488–3523.

[23] A. C. Eringen, Theory of micropolar fluids, J. Math. Mech., 16 (1966), 1–18.

[24] F. Flandoli and B. Schmalfuss, Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative white noise, Stochastics and Stochastics Reports, 59 (1996), 21–45.

[25] M. J. Garrido-Atienza and P. Marín-Rubio, Navier-Stokes equations with delays on unbounded domains, Nonlinear Analysis, 64 (2006), 1100–1118.

[26] X. Han and P. E. Kloeden, Random Ordinary Differential Equations and their Numerical Solutions, Springer Nature, Singapore, 2017.

[27] X. Han, W. Shen and S. Zhou, Random attractors for stochastic lattice dynamical systems in weighted spaces, J. Differential Equations, 250 (2011), 1235–1266.

[28] P. E. Kloeden and B. Schmalfuss, Asymptotic behavior of nonautonomous difference inclusions, Systems Control Lett. 33 (1998), 275–280.

[29] J.-L. Lions and E. Magenes, Non-Homegeneous Boundaryy Value Problem and Applications, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[30] L. Liu and T. Caraballo, Analysis of a stochastic 2D-Navier-Stokes model with infinite delay, J. Dyn. Diff. Eqns., 31 (2019), 2249–2274.

[31] G. Lukaszewicz, Micropolar Fluids: Theory and Applications, Modeling and Simulation in Science, Engineering and Technology, Birkhäuser, Boston, 1999.

[32] G. Lukaszewicz, Long time behavior of 2D micropolar fluid flows, Math. Comput. Modelling, 34 (2001), 487–509.

[33] G. Lukaszewicz and A. Tarasińska, On H1-pullback attractors for nonautonomous micropolar fluid equations in a bounded domain, Nonlinear Analysis, 71 (2009), 782–788.

[34] P. Marín-Rubio and J. Real, Attractors for 2D-Navier-Stokes equations with delays on some unbounded domains, Nonlinear Anal., 67 (2007), 2784–2799.

[35] B. Schmalfuss, Attractors for non-autonomous dynamical systems, International Conference on Differential Equations, 99 (2000), 684–689.

[36] W. Sun, Micropolar fluid flows with delay on 2D unbounded domains, Journal of Applied Analysis and Computation, 8 (2018), 356–378.
[37] W. Sun and Y. Li, Asymptotic behavior of pullback attractors for non-autonomous micropolar fluid flows in 2D unbounded domains, *Electronic Journal of Differential Equations*, 2018 (2018), 1–21.

[38] R. Temam, *Navier-Stokes Equations*, North-Holland Publish Company, Amsterdam, 1979.

[39] B. Wang, Random attractors for non-autonomous stochastic wave equations with multiplicative noise, *Disc. Cont. Dyn. Sys.*, 34 (2014) 269–300.

[40] L. Xue, Well posedness and zero microrotation viscosity limit of the 2D micropolar fluid equations, *Math. Methods Appl. Sci.*, 34 (2011), 1760–1777.

[41] C. Zhao and T. Caraballo, Asymptotic regularity of trajectory attractor and trajectory statistical solution for the 3D globally modified Navier-Stokes equations, *J. Differential Equations*, 266 (2019) 7205–7229.

[42] C. Zhao, S. Zhou and X. Lian, $H^1$-uniform attractor and asymptotic smoothing effect of solutions for a nonautonomous micropolar fluid flow in 2D unbounded domains, *Nonlinear Anal.-RWA*, 9 (2008), 608–627.

[43] C. Zhao, W. Sun and C. Hsu, Pullback dynamical behaviors of the non-autonomous micropolar fluid flows, *Dynamics of Partial Differential Equations*, 12 (2015), 265–288.

[44] C. Zhao and W. Sun, Global well-posedness and pullback attractors for a two-dimensional non-autonomous micropolar fluid flows with infinite delays, *Commun. Math. Sci.*, 15 (2017), 97–121.

[45] Caidi Zhao, Yanjiao Li and Tomás Caraballo, Trajectory trajectory statistical solutions and Liouville type equations for evolution equations: Abstract results and applications, *J. Differential Equations*, 269 (2020), 467–494.

[46] Caidi Zhao, Yanjiao Li and Yanmiao Sang, Using trajectory attractor to construct trajectory statistical solution for the 3D incompressible micropolar flows, *Z. Angew. Math. Mech.*, 100 (2020), e201800197.

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