RYSER’S THEOREM FOR SYMMETRIC $\rho$-LATIN SQUARES

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Abstract. Let $L$ be an $n \times n$ array whose top left $r \times r$ subarray is filled with $k$ different symbols, each occurring at most once in each row and at most once in each column. We establish necessary and sufficient conditions that ensure the remaining cells of $L$ can be filled such that each symbol occurs at most once in each row and at most once in each column, $L$ is symmetric with respect to the main diagonal, and each symbol occurs a prescribed number of times in $L$. The case where the prescribed number of times each symbol occurs is $n$ was solved by Cruse (J. Combin. Theory Ser. A 16 (1974), 18–22), and the case where the top left subarray is $r \times n$ and the symmetry is not required, was settled by Goldwasser et al. (J. Combin. Theory Ser. A 130 (2015), 26–41). Our result allows the entries of the main diagonal to be specified as well, which leads to an extension of the Andersen-Hoffman’s Theorem (Annals of Disc. Math. 15 (1982) 9–26, European J. Combin. 4 (1983) 33–35).

1. Introduction

Throughout this paper, $n$ and $k$ are positive integers, $[k] := \{1, 2, \ldots, k\}$, and $\rho := (\rho_1, \ldots, \rho_k)$ with $1 \leq \rho_\ell \leq n \leq k$ for $\ell \in [k]$ such that $\sum_{\ell \in [k]} \rho_\ell = n^2$. A $\rho$-latin square $L$ of order $n$ is an $n \times n$ array filled with $k$ different symbols, $[k]$, each occurring at most once in each row and at most once in each column, such that each symbol $i$ occurs exactly $\rho_i$ times in $L$ for $\ell \in [k]$, and $L$ is symmetric if $L_{ij} = L_{ji}$ for $i, j \in [n]$. An $r \times s$ $\rho$-latin rectangle on the set $[k]$ of symbols is an $r \times s$ array in which each symbol in $[k]$ occurs at most once in each row and in each column, and in which each symbol $i$ occurs at most $\rho_i$ times for $\ell \in [k]$. A latin square (or rectangle) is a $\rho$-latin square (or rectangle) with $\rho = (n, \ldots, n)$.

We are interested in the following problem.

Problem 1. Let $n > \min\{r, s\}$. Find necessary and sufficient conditions that ensure that a symmetric $r \times s$ $\rho$-latin rectangle $L$ can be extended to a symmetric $n \times n$ $\rho$-latin square $L'$.

We remark that even the case $\rho = (n, \ldots, n)$ of Problem 1 is far from being settled. The following result of Cruse which can be viewed as an analogue of Ryser’s theorem [13] for partial symmetric latin squares, resolves the case $r = s, \rho = (n, \ldots, n)$ of Problem 1. For $\ell \in [k]$, let $e_\ell$ be the number of occurrences of $\ell$ in $L$.

Theorem 1.1. [5, Theorem 1] An $r \times r$ symmetric latin rectangle $L$ on $[n]$ can be extended to an $n \times n$ symmetric latin square if and only if

(i) $e_\ell \geq 2r - n$ for $\ell \in [n]$;
(ii) $|\{\ell \in [n] \mid e_\ell \equiv n \pmod{2}\}| \geq r$.

Let $d := (d_1, \ldots, d_k)$ be the diagonal tail of $L'$ where $d_\ell$ is the number of occurrences of $\ell$ in $\{L'_{ii} \mid r + 1 \leq i \leq n\}$ and $\sum_{\ell \in [k]} d_\ell = n - r$ (we refer to the diagonal tail as the...
diagonal if \( r = s = 0 \). Observe that if \( L \) is extended to \( L' \), then by permuting rows and columns of \( L' \) one can obtain another \( \rho \)-latin square \( L'' \supseteq L \) with the same diagonal tail as \( L' \) such that the entries of the diagonal of \( L'' \backslash L \) are in a prescribed order. Hence, in order to extend \( L \) to \( L' \) whose diagonal entries are specified, it is enough to extend \( L \) to \( L' \) with a specified diagonal tail. Andersen, and independently, Hoffman, proved the following extension of Cruse’s Theorem.

**Theorem 1.2.** \([2, 10]\) An \( r \times r \) symmetric latin rectangle \( L \) on \( [n] \) can be extended to an \( n \times n \) symmetric latin square with a prescribed diagonal tail \( d \) if and only if

1. \( e_\ell \geq 2r - n + d_\ell \) for \( \ell \in [n] \);
2. \( e_\ell + d_\ell \equiv n \) (mod 2) for \( \ell \in [n] \).

Bryant and Rodger \([4]\) solved the case \( r \in \{1, 2\}, s = n, \rho = (n, \ldots, n) \) of Problem 1. Goldwasser et al. \([7]\) found necessary and sufficient conditions under which an \( r \times n \) \( \rho \)-latin rectangle can be extended to a \( \rho \)-latin square, generalizing a classical result of Hall \([8]\). For a survey of results on embedding of latin squares and related structures, cycle systems and graph designs, we refer the reader to \([12]\).

Let \( L \) be an \( r \times r \) symmetric \( \rho \)-latin rectangle. Let \( i \in [r] \) be a row of \( L \), \( \ell \in [k] \) be a symbol, \( I \subseteq [r] \) be a subset of rows, and \( K \subseteq [k] \) be a subset of symbols. Then \( \mu_K(i) \) and \( \mu_I(\ell) \) denote the number of occurrences of \( i \) in \( K \) that are missing in row \( i \), and the number of rows in \( I \) where symbol \( \ell \) is missing, respectively. Notice that

\[
\mu_{[r]}(\ell) = r - e_\ell, \quad 0 \leq \mu_I(\ell) \leq \min\{|I|, r - e_\ell\} \quad \forall \ell \in [k], I \subseteq [r],
\]

\[
\mu_{[k]}(i) = k - r, \quad 0 \leq \mu_K(i) \leq \min\{|K|, k - r\} \quad \forall i \in [r], K \subseteq [k].
\]

Suppose that an \( r \times r \) symmetric \( \rho \)-latin rectangle \( L \) is extended to an \( n \times n \) symmetric \( \rho \)-latin square \( L' \), and let us fix a symbol \( \ell \in [k] \). On the one hand, there are \( \rho_\ell - e_\ell \) occurrences of \( i \) outside the top left \( r \times r \) subsquare. On the other hand, there are at most \( n - r \) occurrences of \( i \) in the last \( n - r \) rows, and at most \( n - r \) occurrences of \( i \) in the last \( n - r \) columns. Therefore, we must have

\[
(1) \quad \rho_\ell - e_\ell \leq 2(n - r) \quad \forall \ell \in [k].
\]

Due to the symmetry of \( L \) and \( L' \), off-diagonal entries occur in pairs, and so, \( e_\ell - e_\ell^{(1)} \) and \( \rho_\ell - \rho_\ell^{(1)} \) are even, where \( e_\ell^{(1)} \) is the number of occurrences of \( i \) on the diagonal of \( L \) and \( \rho_\ell^{(1)} := e_\ell + d_\ell \). So, if for some \( \ell \in [k] \), \( \rho_\ell - e_\ell \) is odd, then \( d_\ell \) is also odd, and consequently, \( d_\ell \geq 1 \). Therefore, we have

\[
(2) \quad \left| \{\ell \in [k] \mid \rho_\ell \neq e_\ell \mod 2 \} \right| \leq n - r \quad \forall \ell \in [k].
\]

Moreover,

\[
(3) \quad n - r = \sum_{\ell \in [k]} d_\ell \equiv \sum_{\ell \in [k]} (\rho_\ell - e_\ell) \equiv \left| \{\ell \in [k] \mid \rho_\ell \neq e_\ell \mod 2 \} \right| \mod 2 \quad \forall \ell \in [k].
\]

We say that \( L \) is \( \rho \)-admissible if it meets conditions \((1)-(3)\).

In this paper, the complement of a set \( S \) is denoted by \( \overline{S} \), and \( x \lor y := \max\{0, x + y\} \). Observe that \( x \lor y \equiv z = (x - y) \lor z \). Whenever it is not ambiguous, we write \( x - y \lor z \) instead of \( (x - y) \lor z \).

We prove an analogue of Cruse’s Theorem for \( \rho \)-latin squares.
Theorem 1.3. An $r \times r$ symmetric $\rho$-latin rectangle $L$ can be completed to an $n \times n$ symmetric $\rho$-latin square if and only if $L$ is $\rho$-admissible and the following two conditions

\begin{equation}
(n - r)|I| \leq \sum_{\ell \in [k]} \min \left\{ \left\lfloor \frac{\rho_\ell - e_\ell}{2} \right\rfloor, \mu_I(\ell) \right\} \quad \forall I \subseteq [r];
\end{equation}

\begin{equation}
\sum_{\ell \in K} (\rho_\ell - e_\ell + r - n) \leq \sum_{i \in [r]} \min \left\{ n - r, \mu_K(i) \right\} \quad \forall K \subseteq [k],
\end{equation}

or the following holds.

\begin{equation}
(n - r)(r - |I|) + \sum_{\ell \in K} \left\lfloor \frac{\rho_\ell - e_\ell}{2} \right\rfloor \geq \sum_{\ell \in K} (\rho_\ell - e_\ell - n + r - \mu_I(\ell)) + \sum_{i \in I} (n - r - \mu_K(i)) \quad \forall I \subseteq [r], K \subseteq [k].
\end{equation}

We also prove a similar statement for the case where the diagonal entries are specified. An $r \times r$ symmetric $\rho$-latin rectangle $L$ is $(\rho, d)$-admissible if it meets the following conditions.

\begin{align}
(7) & \quad \rho_\ell - e_\ell + d_\ell \leq 2(n - r) \quad \forall \ell \in [k]; \\
(8) & \quad \rho_\ell - e_\ell + d_\ell \equiv 0 \pmod{2} \quad \forall \ell \in [k].
\end{align}

Theorem 1.4. An $r \times r$ symmetric $\rho$-latin rectangle $L$ can be completed to an $n \times n$ symmetric $\rho$-latin square with a prescribed diagonal tail $d$ if and only if $L$ is $(\rho, d)$-admissible and the following two conditions

\begin{equation}
(n - r)|I| \leq \sum_{\ell \in [k]} \min \left\{ \left\lfloor \frac{\rho_\ell - e_\ell - d_\ell}{2} \right\rfloor, \mu_I(\ell) \right\} \quad \forall I \subseteq [r];
\end{equation}

\begin{equation}
\sum_{\ell \in K} (\rho_\ell - e_\ell + r - n) \leq \sum_{i \in [r]} \min \left\{ n - r, \mu_K(i) \right\} \quad \forall K \subseteq [k],
\end{equation}

or the following holds.

\begin{equation}
(n - r)(r - |I|) + \sum_{\ell \in K} \left\lfloor \frac{\rho_\ell - e_\ell - d_\ell}{2} \right\rfloor \geq \sum_{\ell \in K} (\rho_\ell - e_\ell - n + r - \mu_I(\ell)) + \sum_{i \in I} (n - r - \mu_K(i)) \quad \forall I \subseteq [r], K \subseteq [k].
\end{equation}

At first sight, it may not be obvious why Theorems 1.3 and 1.4 imply Theorems 1.1 and 1.2, respectively. See Remarks 5.5, 5.8, and 5.13.

Using detachments we reduce the completion of the desired partial $\rho$-latin square $A$ to finding a subgraph with prescribed degree sequences of an auxiliary graph associated with $A$. We complete the proof by applying the celebrated Lovász’s $(g, f)$-factor Theorem. Intuitively speaking, we use matching theory to decide which $n - r$ symbols among $k - r$ available symbols of each row should be chosen, and then we use detachment theory to arrange the chosen symbols in such a way that the latin property is maintained.

Further terminology along with the two main tools are discussed in Section 2. Theorems 1.3 and 1.4 are proven in Sections 3 and 4, respectively. We conclude the paper with a few corollaries and open problems.
2. Prerequisites

For a graph $G = (V, E)$, $u \in V$, $e \in E$, and $K \subseteq V$, $\deg_G(u)$, $\text{mult}_G(e)$, $\text{mult}_G(uS)$ denote the number of edges incident with $u$, the multiplicity of the edge $e$, and the number of edges between $u$ and $S$, respectively. If the edges of $G$ are colored with $k$ colors (the set of colors is always $[k]$), then $G(\ell)$ is the color class $\ell$ of $G$ for $\ell \in [k]$. A bigraph $G$ with bipartition $\{X, Y\}$ will be denoted by $G[X, Y]$, and if $S \subseteq X$, then $\bar{S}$ means $X \setminus S$.

For $i = 1, 2$, an edge that is incident with one vertex only is called an $i$-loop if it contributes $i$ to the degree of that vertex. If $e$ is a 1-loop incident with vertex $u$, we write $e = u$, and if $e$ is a 2-loop incident with vertex $u$, we write $e = u^2$. We denote the set of 1-loops of a graph $G$ by $E^1(G)$, and $E^2(G) := E(G) \setminus E^1(G)$.

Let $\mathbb{K}_n$ denote the $n$-vertex graph in which every pair of distinct vertices are adjacent such that each vertex is incident with a 1-loop. Observe that $\text{mult}_{\mathbb{K}_n}(u) = \text{mult}_{\mathbb{K}_n}(uv) = 1$ for $u, v \in V(\mathbb{K}_n)$ with $u \neq v$. There is a one-to-one correspondence between symmetric latin squares of order $n$ and 1-factorizations of $\mathbb{K}_n$.

Let $G$ be a graph whose edges are colored, and let $\alpha \in V(G)$. By splitting $\alpha$ into $\alpha_1, \ldots, \alpha_p$, we obtain a new graph $F$ whose vertex set is $(V(G) \setminus \{\alpha\}) \cup \{\alpha_1, \ldots, \alpha_p\}$ so that each edge $\alpha u$ in $G$ becomes $\alpha_i u$ for some $i \in [p]$ in $F$, and each 1-loop $\alpha$ in $G$ becomes $\alpha_i$ for some $i \in [p]$ in $F$. Intuitively speaking, when we split a vertex $\alpha$ into $\alpha_1, \ldots, \alpha_p$, we share the edges incident with $\alpha$ among $\alpha_1, \ldots, \alpha_p$. In this manner, $F$ is a detachment of $G$, and $G$ is an amalgamation of $F$ obtained by identifying $\alpha_1, \ldots, \alpha_p$ by $\alpha$. We need the following detachment lemma. Here, $x \approx y$ means $|y| \leq x \leq |y|$.

**Lemma 2.1.** [3, Theorem 4.1] Let $G$ be a graph whose edges are colored with $k$ colors, and let $\alpha \in V(G)$. There exists a graph $F$ obtained by splitting $\alpha$ into $\alpha_1, \ldots, \alpha_p$ such that

(i) $\deg_F(\alpha_i) \approx \deg_G(\alpha_i)/p$ for $i \in [p], \ell \in [k]$;
(ii) $\text{mult}_F(\alpha_i) \approx \text{mult}_G(\alpha)/p$ for $i \in [p]$;
(iii) $\text{mult}_F(\alpha_iu) \approx \text{mult}_G(\alpha u)/p$ for $i \in [p], u \in V(G) \setminus \{\alpha\}$;
(iv) $\text{mult}_F(\alpha_i\alpha_j) \approx \text{mult}_G(\alpha^2)/\binom{\ell}{2}$ for $i, j \in [p], i \neq j$.

Using this detachment lemma, it is easy to construct symmetric $\rho$-latin squares.

**Theorem 2.2.** For every $n, k, \rho = (\rho_1, \ldots, \rho_k)$ with $1 \leq \rho_1, \ldots, \rho_k \leq n \leq k$ and $\sum_{\ell \in [k]} \rho_\ell = n^2$, there exists a symmetric $\rho$-latin square of order $n$ if and only if the following conditions hold.

\[
\begin{align*}
|\{\ell \in [k] \mid \rho_\ell \equiv 1 \pmod{2}\}| &\leq n, \\
|\{\ell \in [k] \mid \rho_\ell \equiv 1 \pmod{2}\}| &\equiv n \pmod{2}.
\end{align*}
\]

**Proof.** The proof of necessity is very similar to those of (2) and (3). To prove the sufficiency, let $G$ be a graph with $V(G) = \{\alpha\}$, and $\text{mult}_G(\alpha) = n, \text{mult}_G(\alpha^2) = \binom{n}{2}$. We color the 1-loops of $G$ such that $\text{mult}_{G(\ell)}(\alpha) = 1$ whenever $\rho_\ell$ is odd for $\ell \in [k]$. Since $|\{\ell \in [k] \mid \rho_\ell \equiv 1 \pmod{2}\}| \leq n$, this is possible. The number of uncolored 1-loops is even for $n - |\{\ell \in [k] \mid \rho_\ell \equiv 1 \pmod{2}\}|$ is even. Moreover, $\rho_\ell - \text{mult}_{G(\ell)}(\alpha)$ is even for $\ell \in [k]$. Therefore, we can color the uncolored 1-loops by coloring an even number of 1-loops with each color. Then, we color the remaining edges of $G$ such that $\text{mult}_{G(\ell)}(\alpha^2) = (\rho_\ell - \text{mult}_{G(\ell)}(\alpha))/2$ for $\ell \in [k]$. This is possible, because the coloring of 1-loops ensures $\rho_\ell - \text{mult}_{G(\ell)}(\alpha)$ is even for
\( \ell \in [k], \) and
\[
\sum_{\ell \in [k]} (\rho_\ell - \text{mult}_{G(\ell)}(\alpha))/2 = (n^2 - n)/2 = \text{mult}_G(\alpha^2).
\]

Applying the detachment lemma with \( p = n \) yields the graph \( F \cong K_n \) whose colored edges corresponds to symbols in the desired \( \rho \)-latin square \( L \). More precisely, \( L \) is obtained by placing symbol \( \ell \) in \( L_{ij} \) and \( L_{ji} \) whenever the edge \( \alpha_i \alpha_j \) is colored \( \ell \) for \( i \neq j \), and placing symbol \( \ell \) in \( L_{ii} \) whenever the 1-loop \( \alpha_i \) is colored \( \ell \).

Let \( f, g \) be integer functions on the vertex set of a graph \( G \) such that \( 0 \leq g(x) \leq f(x) \) for all \( x \). A \((g, f)\)-factor is a spanning subgraph \( F \) of \( G \) with the property that \( g(x) \leq \deg_F(x) \leq f(x) \) for each \( x \). Let \( G[X, Y] \) be a bipartite graph. By [6, Theorem 5] and [9, Theorem 1], \( G \) has a \((g, f)\)-factor if and only if one of the following two conditions hold.

\[
\sum_{a \in A} g(a) \leq \sum_{a \in N_G(A)} \min \left\{ f(a), \text{mult}_G(aA) \right\} \quad \forall A \subseteq X, A \subseteq Y,
\]
\[
\sum_{a \in A} f(a) \geq \sum_{a \notin A} \left( g(a) - \deg_{G-A}(a) \right) \quad \forall A \subseteq X \cup Y.
\]

Here, \( N_G(A) \) is the neighborhood of \( A \) in \( G \). We remark that both of these results are special cases of the Lovász’s \((g, f)\)-factor Theorem [11].

### 3. Cruse’s Theorem for \( \rho \)-latin Squares

In this section, we prove Theorem 1.3. We established the necessity of (1)–(3) in the introduction. The necessity of the remaining conditions will be evident at the end of the proof. To prove the sufficiency, suppose that \( L \) is a \( \rho \)-admissible \( r \times r \) symmetric \( \rho \)-latin rectangle. Let \( F = K_n \) with \( V(F) = \hat{X} := \{x_1, \ldots, x_n\} \), and let \( X = \{x_1, \ldots, x_r\} \). In \( F \), the 1-loop \( x_i \) is colored \( \ell \) if \( L_{ii} = \ell \) for \( i \in [r] \), and the edge \( x_i x_j \) is colored \( \ell \) if \( L_{ij} = \ell \) for distinct \( i, j \in [r] \). Since \( L \) is symmetric, this coloring is well-defined. Observe that some edges of \( F \) are uncolored. We have \( \deg_{F(\ell)}(u) \leq 1 \) for \( u \in \hat{X}, \ell \in [k] \). Let \( G \) be the graph obtained by amalgamating \( x_{r+1}, \ldots, x_n \) of \( F \) into a single vertex \( \alpha \), so \( \text{mult}_G(\alpha) = \text{mult}_G(\alpha x_i) = n - r \) for \( i \in [r] \), and \( \text{mult}_G(\alpha^2) = \binom{n-r}{2} \). Let \( \Gamma[X, [k]] \) be the simple bigraph whose edge set is
\[
\{ u\ell \mid u \in X, \ell \in [k], \deg_{F(\ell)}(u) = 0 \}.
\]

For \( i \in [r] \), \( \sum_{\ell \in [k]} \deg_{F(\ell)}(x_i) = r \), and so we have

\[
\text{(12)} \quad \begin{cases} 
\deg_{\Gamma}(x_i) = k - r & \text{if } i \in [r], \\
\deg_{\Gamma}(\ell) = r - e_\ell & \text{if } \ell \in [k].
\end{cases}
\]

Observe that \( L \) can be completed to an \( n \times n \) symmetric \( \rho \)-latin square if and only if the uncolored edges of \( F \) can be colored so that

\[
\text{(13)} \quad \forall \ell \in [k], \begin{cases} 
\deg_{F(\ell)}(u) \leq 1 & \text{if } u \in \hat{X}, \\
|E^1(F(\ell))| + 2|E^2(F(\ell))| = \rho_\ell.
\end{cases}
\]
We show that the coloring of $F$ can be completed such that (13) holds if and only if the coloring of $G$ can be completed such that

\[
(14) \quad \forall \ell \in [k] \quad \begin{cases} 
\deg_{G(\ell)}(u) \leq 1 & \text{if } u \in X, \\
\deg_{G(\ell)}(\alpha) \leq n - r, \\
|E^1(G(\ell))| + 2|E^2(G(\ell))| = \rho_\ell.
\end{cases}
\]

To see this, first assume that the coloring of $F$ can be completed so that (13) holds. Identifying all the vertices in $\bar{X}\setminus \bar{X}$ by $\alpha$, we will get the graph $G$ satisfying (14). Conversely, suppose that we have a coloring of $G$ such that (14) holds. Applying the detachment lemma to $G$, we get a graph $F'$ obtained by splitting $\alpha$ into $\alpha_1, \ldots, \alpha_{n-r}$, such that

(i) $\deg_{F'(\ell)}(\alpha_i) \approx \deg_{G(\ell)}(\alpha)/(n-r) \leq 1$ for $i \in [n-r], \ell \in [k]$;
(ii) $\mult_{F'}(\alpha_i) = \mult_{G(\ell)}(\alpha)/(n-r) = 1$ for $i \in [n-r]$;
(iii) $\mult_{F'}(\alpha_i, u) = \mult_{G(\ell)}(\alpha u)/(n-r) = 1$ for $i \in [n-r], u \in X$;
(iv) $\mult_{F'}(\alpha_i, \alpha_j) = \mult_{G(\ell)}(\alpha^2)/(n-r) = 1$ for distinct $i, j \in [n-r]$.

Since $F' \cong F$ and the coloring of $F'$ satisfies (13), we are done.

Since $L$ is $p$-admissible, for $\ell \in [k]$ we have $\rho_\ell - e_\ell \leq 2(n-r)$, and so $\rho_\ell - e_\ell - n + r \leq (\rho_\ell - e_\ell)/2$. Moreover, $\rho_\ell - e_\ell - n + r \leq r - e_\ell$ for $\ell \in [k]$. We show that the coloring of $G$ can completed such that (14) is satisfied if and only if there exists a subgraph $\Theta$ of $\Gamma$ with $r(n-r)$ edges so that

\[
(15) \quad \begin{cases} 
\deg_\Theta(x_i) = n - r & \text{if } i \in [r], \\
\rho_\ell - e_\ell - n + r \leq \deg_\Theta(\ell) \leq \frac{\rho_\ell - e_\ell}{2} & \text{if } \ell \in [k].
\end{cases}
\]

To prove this, suppose that the coloring $G$ can be completed such that (14) is satisfied. Let $\Theta[X, [k]]$ be the bigraph whose edge set is

$$\{u\ell \mid u \in X, \ell \in [k], \alpha u \in E(G(\ell))\}.$$

It is clear that $\Theta \subseteq \Gamma$. For $i \in [r]$, $\deg_\Theta(x_i) = \mult_G(\alpha x_i) = n - r$, and for $\ell \in [k],
\rho_\ell = |E^1(G(\ell))| + 2|E^2(G(\ell))|
= e_\ell + \mult_G(\alpha) + 2\mult_G(\alpha, X) + 2\mult_G(\alpha^2)
\geq e_\ell + 2\deg_\Theta(\ell).

Thus, $\deg_\Theta(\ell) \leq (\rho_\ell - e_\ell)/2$ for $\ell \in [k]$. Moreover,

$$n - r \geq \deg_{G(\ell)}(\alpha) = \deg_\Theta(\ell) + \mult_{G(\ell)}(\alpha) + 2\mult_{G(\ell)}(\alpha^2)
= \deg_\Theta(\ell) + \mult_{G(\ell)}(\alpha) + (\rho_\ell - e_\ell - \mult_{G(\ell)}(\alpha) - 2\deg_\Theta(\ell))
= \rho_\ell - e_\ell - \deg_\Theta(\ell),$$

and so $\deg_\Theta(\ell) \geq \rho_\ell - e_\ell - n + r$. Conversely, suppose that $\Theta \subseteq \Gamma$ satisfying (15) exists. For each $\ell \in [k]$, if $\ell x_i \in E(\Theta)$ for some $i \in [r]$, we color an $\alpha x_i$-edge in $G$ with $\ell$. Since $\deg_\Theta(x_i) = n - r$ for $i \in [r]$, all the edges between $\alpha$ and $X$ can be colored this way. Since $\Theta$ is simple, $d_{G(\ell)}(u) \leq 1$ for $\ell \in [k]$ and $u \in X$. Let $O$ be the set of colors $\ell \in [k]$ such that $\rho_\ell - e_\ell$ is odd. By (2) and (3), $(n - r - |O|)/2$ is a non-negative integer. By (15),
\[(\rho_\ell - e_\ell)/2 - \deg_\Theta(\ell) \geq 0 \text{ for } \ell \in [k]\setminus O \text{ and } (\rho_\ell - e_\ell - 1)/2 - \deg_\Theta(\ell) \geq 0 \text{ for } i \in O. \]

\[
\frac{n - r - |O|}{2} \leq \frac{(n - r)^2 - |O|}{2} = \frac{n^2 - r^2 - r(n - r) - |O|}{2} \\
= \sum_{\ell \in [k]} \frac{\rho_\ell - e_\ell}{2} - \sum_{\ell \in [k]} \deg_\Theta(\ell) - \frac{|O|}{2} \\
= \sum_{\ell \in [k]\setminus O} \left(\frac{\rho_\ell - e_\ell}{2} - \deg_\Theta(\ell)\right) + \sum_{i \in O} \left(\frac{\rho_\ell - e_\ell - 1}{2} - \deg_\Theta(\ell)\right),
\]

there exists a sequence of integers \(a_1, \ldots, a_k\) such that

\[
\begin{cases}
    a_1 + \cdots + a_k = \frac{n - r - |O|}{2}, \\
    0 \leq a_\ell \leq \frac{\rho_\ell - e_\ell}{2} - \deg_\Theta(\ell) & \text{for } \ell \in [k]\setminus O, \\
    0 \leq a_\ell \leq \frac{\rho_\ell - e_\ell - 1}{2} - \deg_\Theta(\ell) & \text{for } \ell \in O.
\end{cases}
\]

Now let

\[
d_\ell = \begin{cases}
    2a_\ell & \text{for } \ell \in [k]\setminus O, \\
    2a_\ell + 1 & \text{for } \ell \in O.
\end{cases}
\]

Observe that the non-negative sequence \(d_1, \ldots, d_k\) satisfies the following.

\[
\begin{cases}
    d_1 + \cdots + d_k = n - r, \\
    d_\ell = \rho_\ell - e_\ell (\text{mod} 2) & \text{for } \ell \in [k], \\
    \deg_\Theta(\ell) \leq \frac{\rho_\ell - e_\ell - d_\ell}{2} & \text{for } \ell \in [k].
\end{cases}
\]

We color the loops of \(G\) such that \(\mult_{G(\ell)}(\alpha) = d_\ell\) for \(\ell \in [k]\) and

\[
\mult_{G(\ell)}(\alpha^2) = \frac{1}{2}(\rho_\ell - e_\ell - d_\ell) - \deg_\Theta(\ell) \quad \forall \ell \in [k].
\]

This is possible for (16) and

\[
\sum_{\ell \in [k]} (\rho_\ell - e_\ell - 2\deg_\Theta(\ell) - d_\ell)) = n^2 - r^2 - 2r(n - r) - (n - r) \\
= 2\left(\frac{n - r}{2}\right) = 2\mult_G(\alpha^2).
\]

For \(\ell \in [k]\),

\[
|E^1(G(\ell))| + 2|E^2(G(\ell))| = e_\ell + 2\deg_\Theta(\ell) + \mult_{G(\ell)}(\alpha) + 2\mult_{G(\ell)}(\alpha^2) = \rho_\ell.
\]

Finally, we have the following for \(\ell \in [k]\), and so (14) holds.

\[
\deg_{G(\ell)}(\alpha) = \mult_{G(\ell)}(\alpha, X) + \mult_{G(\ell)}(\alpha) + 2\mult_{G(\ell)}(\alpha^2) \\
= \deg_\Theta(\ell) + \mult_{G(\ell)}(\alpha) + (\rho_\ell - e_\ell - 2\deg_\Theta(\ell) - \mult_{G(\ell)}(\alpha)) \\
= \rho_\ell - e_\ell - \deg_\Theta(\ell) \\
\leq n - r.
\]
Let
\[
\begin{cases}
g, f : V(\Gamma) \rightarrow \mathbb{N} \cup \{0\}, \\
g(u) = f(u) = n - r \quad \text{for } u \in X, \\
g(\ell) = \max\{\rho_\ell - e_\ell - n + r, 0\} \quad \text{for } \ell \in [k], \\
f(\ell) = \lfloor (\rho_\ell - e_\ell)/2 \rfloor \quad \text{for } \ell \in [k].
\end{cases}
\]

Clearly, \( \Theta \subseteq \Gamma \) exists if and only if \( \Gamma \) has a \((g, f)\)-factor, but by \([6, \text{Theorem 5}]\), \( \Gamma \) has a \((g, f)\)-factor if and only if the following conditions hold.

\[
\sum_{i \in I} g(i) \leq \sum_{\ell \in [k]} \min\left\{ f(\ell), \text{mult}_\Gamma(\ell I) \right\} \quad \forall I \subseteq X,
\]

\[
\sum_{\ell \in K} g(\ell) \leq \sum_{i \in [r]} \min\left\{ f(i), \text{mult}_\Gamma(i K) \right\} \quad \forall K \subseteq [k].
\]

Equivalently, we must have

\[
(n - r)|I| \leq \sum_{\ell \in [\rho]} \min\left\{ \left\lfloor \frac{\rho_\ell - e_\ell}{2} \right\rfloor, \mu_I(\ell) \right\} \quad \forall I \subseteq [\rho],
\]

\[
\sum_{\ell \in K} (\rho_\ell - e_\ell + r - n) \leq \sum_{i \in [r]} \min\left\{ n - r, \mu_K(i) \right\} \quad \forall K \subseteq [k].
\]

By \([9, \text{Theorem 1}]\), \( \Gamma \) has a \((g, f)\)-factor if and only if

\[
\sum_{a \in A} f(a) \geq \sum_{a \in A} \left( g(a) - \deg_{G - \hat{A}}(a) \right) \quad \forall A \subseteq X \cup Y,
\]

or equivalently,

\[
\sum_{i \in I} (n - r) + \sum_{\ell \in K} \left\lfloor \frac{\rho_\ell - e_\ell}{2} \right\rfloor \geq \sum_{\ell \in K} (\rho_\ell - e_\ell - n + r - \mu_I(\ell))
\]

\[
+ \sum_{i \in I} \left( n - r - \mu_K(i) \right) \quad \forall I \subseteq [\rho], K \subseteq [k].
\]

This completes the proof of Theorem 1.3.

4. Andersen-Hoffman’s Theorem for \( \rho \)-Latin Squares

First, we show that symmetric \( \rho \)-Latin squares with prescribed diagonals exist, and then, we prove Theorem 1.4.

**Theorem 4.1.** For every \( n, k, d = (d_1, \ldots, d_k), \rho = (\rho_1, \ldots, \rho_k) \) with \( 1 \leq \rho_1, \ldots, \rho_k \leq n \leq k \) and \( \sum_{\ell \in [k]} \rho_\ell = n^2, \sum_{\ell \in [k]} d_\ell = n \), there exists a symmetric \( \rho \)-Latin square of order \( n \) with diagonal \( d \) if and only if \( \rho_\ell - d_\ell \) is even for \( \ell \in [k] \).

**Proof.** Let \( G \) be a \( k \)-edge-colored graph with \( V(G) = \{\alpha\} \) such that

\[
\text{mult}_G(\alpha) = n, \quad \text{mult}_G(\alpha^2) = \binom{n-2}{2},
\]

\[
\text{mult}_G(\alpha^\ell) = \frac{\rho_\ell - d_\ell}{2} \quad \forall \ell \in [k],
\]

or equivalently,

\[
\text{mult}_G(\alpha) = n, \quad \text{mult}_G(\alpha^2) = \binom{n}{2}, \quad \text{mult}_G(\alpha^\ell) = \frac{\rho_\ell - d_\ell}{2} \quad \forall \ell \in [k].
\]
Applying the detachment lemma with \( p = n \) yields the graph \( F \cong K_n \) whose colored edges corresponds to symbols in the desired \( p \)-latin square \( L \). More precisely, \( L \) is obtained by placing symbol \( \ell \) in \( L_{ij} \) and \( L_{ji} \) whenever the edge \( \alpha_i\alpha_j \) is colored \( \ell \) for \( i \neq j \), and placing symbol \( \ell \) in \( L_{ii} \) whenever the 1-loop \( \alpha_i \) is colored \( \ell \).

**Proof of Theorem 1.4.** To prove the necessity, suppose that an \( r \times r \) symmetric \( p \)-latin rectangle \( L \) is extended to an \( n \times n \) symmetric \( p \)-latin square \( L' \) with a prescribed diagonal tail \( d \), and let us fix a symbol \( \ell \in [k] \). Without loss of generality we assume that \( L_{ii} = \ell \) for \( r + 1 \leq i \leq r + d_{\ell} \). On the one hand, there are \( \rho_{\ell} - e_{\ell} - d_{\ell} \) occurrences of \( i \) outside the top left \( (r + d_{\ell}) \times (r + d_{\ell}) \) subsquare. On the other hand, there are at most \( n - r - d_{\ell} \) occurrences of \( i \) in the last \( n - r - d_{\ell} \) rows, and at most \( n - r - d_{\ell} \) occurrences of \( i \) in the last \( n - r - d_{\ell} \) columns. Therefore, we must have

\[
\rho_{\ell} + d_{\ell} - e_{\ell} \leq 2(n - r) \quad \forall \ell \in [k].
\]

Due to the symmetry of \( L \) and \( L' \), off-diagonal entries occur in pairs, and so

\[
\rho_{\ell} - e_{\ell} + d_{\ell} \equiv 0 \pmod{2} \quad \forall \ell \in [k].
\]

To prove the sufficiency, let \( F = K_n \) with \( V(F) = X := \{x_1, \ldots, x_n\} \), and let \( X = \{x_1, \ldots, x_r\} \). Similar to Section 3, using \( L \) we color the edges of \( F \), but in addition, for \( \ell \in [k] \), we color \( d_{\ell} \) arbitrary uncolored 1-loops (incident with vertices of \( X \setminus X \)) with color \( \ell \). Let \( G \) and \( \Gamma \) be defined as they were defined in Section 3. Note that here, \( \text{mult}_{G(\ell)}(\alpha) = d_{\ell} \) for \( \ell \in [k] \), and \( \Gamma \) satisfies (12). Again, \( L \) can be completed if and only if the coloring of \( G \) can be completed such that (14) is satisfied.

Since \( L \) is \((p,d)\)-admissible, for \( \ell \in [k] \) we have \( 2(n - r) \geq \rho_{\ell} - e_{\ell} + d_{\ell} \), and so \( \rho_{\ell} - e_{\ell} - n + r \leq (\rho_{\ell} - e_{\ell} - d_{\ell})/2 \). Now, we show that the coloring of \( G \) can be completed such that (14) is satisfied if and only if there exists a subgraph \( \Theta \) of \( \Gamma \) with \( r(n - r) \) edges so that

\[
\begin{aligned}
\text{deg}_\Theta(x_i) &= n - r & \text{if } i \in [r], \\
\rho_{\ell} - e_{\ell} - n + r &\leq \text{deg}_\Theta(\ell) \leq \frac{\rho_{\ell} - e_{\ell} - d_{\ell}}{2} & \text{if } \ell \in [k].
\end{aligned}
\]

(17)

To prove this, suppose that the coloring \( G \) can be completed such that (14) is satisfied. Define \( \Theta[X, [k]] \subseteq \Gamma \) with edge set

\[\{u\ell \mid u \in X, \ell \in [k], \alpha u \in E(G(\ell))\}\]

For \( i \in [r] \), \( \text{deg}_\Theta(x_i) = \text{mult}_G(\alpha x_i) = n - r \), and for \( \ell \in [k] \),

\[\begin{aligned}
\rho_{\ell} &= |E^1(G(\ell))| + 2|E^2(G(\ell))| \\
&= e_{\ell} + d_{\ell} + 2 \text{mult}_{G(\ell)}(\alpha, X) + 2 \text{mult}_{G(\ell)}(\alpha^2) \\
&\geq e_{\ell} + d_{\ell} + 2 \text{deg}_\Theta(\ell).
\end{aligned}\]

Thus, \( \text{deg}_\Theta(\ell) \leq (\rho_{\ell} - e_{\ell} - d_{\ell})/2 \) for \( \ell \in [k] \). Moreover,

\[n - r \geq \text{deg}_{G(\ell)}(\alpha) = d_{\ell} + \text{deg}_\Theta(\ell) + 2 \text{mult}_{G(\ell)}(\alpha^2) = d_{\ell} + \text{deg}_\Theta(\ell) + (\rho_{\ell} - e_{\ell} - d_{\ell} - 2 \text{deg}_\Theta(\ell)) = \rho_{\ell} - e_{\ell} - n + r.
\]

and so \( \text{deg}_\Theta(\ell) \geq \rho_{\ell} - e_{\ell} - n + r \). Conversely, suppose that \( \Theta \subseteq \Gamma \) satisfying (17) exists. For each \( \ell \in [k] \), if \( \ell x_i \in E(\Theta) \) for some \( i \in [r] \), we color an \( \alpha x_i \)-edge in \( G \) with \( \ell \). Since \( \text{deg}_\Theta(x_i) = n - r \) for \( i \in [r] \), all the edges between \( \alpha \) and \( X \) can be colored this way. Since
Θ is simple, $d_{\Gamma}(u) \leq 1$ for $\ell \in [k]$ and $u \in X$. By (8) and (17), $\rho_\ell - e_\ell - 2 \deg_\Theta(\ell) - d_\ell$ is a non-negative even integer for $\ell \in [k]$. Since

$$\sum_{\ell \in [k]} \left( \rho_\ell - e_\ell - 2 \deg_\Theta(\ell) - d_\ell \right) = 2 \binom{n-r}{2},$$

we can color the $\alpha^2$-edges so that for $\ell \in [k]$, $\mult_{\Gamma}(\alpha^2) = (\rho_\ell - e_\ell - 2 \deg_\Theta(\ell) - d_\ell) / 2$. Finally, for $\ell \in [k]$,

$$|E^1(\Gamma)| + 2|E^2(\Gamma)| = e_\ell + d_\ell + 2 \deg_\Theta(\ell) + 2 \mult_{\Gamma}(\alpha^2) = \rho_\ell,$$

and

$$\deg_{\Gamma(\ell)}(\alpha) = \mult_{\Gamma(\ell)}(\alpha, X) + \mult_{\Gamma(\ell)}(\alpha) + 2 \mult_{\Gamma(\ell)}(\alpha^2)$$

$$= \deg_\Theta(\ell) + d_\ell + (\rho_\ell - e_\ell - 2 \deg_\Theta(\ell) - d_\ell)$$

$$= \rho_\ell - e_\ell - \deg_\Theta(\ell)$$

$$\leq n - r,$$

and so (14) holds.

Let

$$\left\{ g, f : V(\Gamma) \to \mathbb{N} \cup \{0\}, \right. $$

$$\left. g(u) = f(u) = n - r \quad \text{for} \quad u \in X, \right. $$

$$\left. g(\ell) = \rho_\ell - e_\ell + r - n \quad \text{for} \quad \ell \in [k], \right. $$

$$\left. f(\ell) = (\rho_\ell - e_\ell - d_\ell)/2 \quad \text{for} \quad \ell \in [k]. \right. $$

The existence of $\Theta \subseteq \Gamma$ is equivalent to the existence of a $(g, f)$-factor in $\Gamma$. By [6, Theorem 5], $\Gamma$ has a $(g, f)$-factor if and only if the following conditions hold.

$$\sum_{i \in I} g(i) \leq \sum_{\ell \in [k]} \min \left\{ f(\ell), \mult_{\Gamma}(\ell I) \right\} \quad \forall I \subseteq X,$$

$$\sum_{\ell \in K} g(\ell) \leq \sum_{i \in [r]} \min \left\{ f(i), \mult_{\Gamma}(i K) \right\} \quad \forall K \subseteq [k].$$

Equivalently, we must have

$$(n - r)|I| \leq \sum_{\ell \in [k]} \min \left\{ \frac{\rho_\ell - e_\ell - d_\ell}{2}, \mu_{\Gamma}(\ell) \right\} \quad \forall I \subseteq [r],$$

$$\sum_{\ell \in K} (\rho_\ell - e_\ell + r - n) \leq \sum_{i \in [r]} \min \left\{ n - r, \mu_K(i) \right\} \quad \forall K \subseteq [k].$$

By [9, Theorem 1], $\Gamma$ has a $(g, f)$-factor if and only if

$$\sum_{a \notin A} f(a) \geq \sum_{a \in A} \left( g(a) - \deg_{\Gamma}(a) \right) \quad \forall A \subseteq X \cup Y,$$
or equivalently,
\[
\sum_{i \in I} (n - r) + \sum_{i \in K} \frac{\rho_\ell - e_\ell - d_\ell}{2} \geq \sum_{\ell \in K} \left( \rho_\ell - e_\ell - n + r - \mu_I(\ell) \right) \\
+ \sum_{i \in I} \left( n - r - \mu_K(i) \right) \quad \forall I \subseteq [r], K \subseteq [k].
\]

5. Corollaries

**Corollary 5.1.** An \( r \times r \) symmetric \( \rho \)-latin rectangle with \( e_\ell \geq r - n + \rho_\ell \) for \( \ell \in [k] \) can be embedded into an \( n \times n \) symmetric \( \rho \)-latin square with a prescribed diagonal tail \( d \) if and only if \( \rho_\ell + e_\ell + d_\ell \) is even for \( \ell \in [k] \), and any of the following conditions is satisfied.

\[
\sum_{\ell \in K} \frac{\rho_\ell - e_\ell - d_\ell}{2} \geq \sum_{i \in I} \left( n - r - \mu_K(i) \right) \quad \forall I \subseteq [r], K \subseteq [k];
\]

\[
(n - r)|I| \leq \sum_{\ell \in [k]} \min \left\{ \frac{\rho_\ell - e_\ell - d_\ell}{2}, \mu_I(\ell) \right\} \quad \forall I \subseteq [r].
\]

**Proof.** Suppose that \( e_\ell \geq r - n + \rho_\ell \) for \( \ell \in [k] \). Since \( d_\ell \leq n - r \) for \( \ell \in [k] \), we have

\[
\rho_\ell - e_\ell + d_\ell \leq (n - r) + d_\ell \leq 2(n - r) \quad \forall \ell \in [k],
\]

and so (7) holds. Moreover, \( \rho_\ell - e_\ell + r - n = 0 \) for \( \ell \in [k] \), and consequently, (10) is satisfied. Slight modification to the proof of [9, Theorem 1], implies that the graph \( \Gamma[X, [k]] \) of the proof of Theorem 1.4 has a \((g, f)\)-factor with \( g(\ell) = 0 \) for \( \ell \in [k] \) if and only if

\[
\sum_{\ell \in K} f(\ell) \geq \sum_{i \in I} \left( g(i) - \deg_{G-K}(i) \right) \quad \forall I \subseteq X, K \subseteq [k],
\]

or equivalently,

\[
\sum_{\ell \in K} \frac{\rho_\ell - e_\ell - d_\ell}{2} \geq \sum_{i \in I} \left( n - r - \mu_K(i) \right) \quad \forall I \subseteq [r], K \subseteq [k].
\]

\[
(n - r)|I| \geq \sum_{\ell \in [k]} \left( \rho_\ell - e_\ell - n + r - \mu_I(\ell) \right) \quad \forall I \subseteq [r];
\]

\[
\sum_{\ell \in K} f(\ell) \geq \sum_{i \in I} \left( g(i) - \deg_{G-K}(i) \right) \quad \forall I \subseteq [r], K \subseteq [k].
\]

\[
\sum_{\ell \in K} f(\ell) \geq \sum_{i \in I} \left( g(i) - \deg_{G-K}(i) \right) \quad \forall I \subseteq [r], K \subseteq [k].
\]

**Remark 5.2.** If \( e_\ell \geq r - n + \rho_\ell \) for \( \ell \in [k] \), then in particular we must have that \( k \geq n + r \) for

\[
k(n - r) = \sum_{\ell \in [k]} (n - r) \geq \sum_{\ell \in [k]} (\rho_\ell - e_\ell) = (n + r)(n - r).
\]

**Corollary 5.3.** If \( e_\ell \geq 2r + d_\ell - \rho_\ell \) for \( \ell \in [k] \), then an \( r \times r \) symmetric \( \rho \)-latin rectangle can be embedded into an \( n \times n \) symmetric \( \rho \)-latin square with a prescribed diagonal tail \( d \) if and only if \( \rho_\ell + e_\ell + d_\ell \) is even for \( \ell \in [k] \), and any of the following conditions is satisfied.

\[
(n - r)|I| \geq \sum_{\ell \in [k]} \left( \rho_\ell - e_\ell - n + r - \mu_I(\ell) \right) \quad \forall I \subseteq [r];
\]

\[
\sum_{\ell \in K} f(\ell) \geq \sum_{i \in I} \left( g(i) - \deg_{G-K}(i) \right) \quad \forall I \subseteq [r], K \subseteq [k].
\]
Proof. Suppose that \( e_\ell \geq 2r + d_\ell - \rho_\ell \) for \( \ell \in [k] \). Since \( \rho_\ell \leq n \) for \( \ell \in [k] \), we have
\[
\rho_\ell - e_\ell + d_\ell \leq 2\rho_\ell - 2r \leq 2(n - r) \quad \forall \ell \in [k].
\]
Therefore, (7) is satisfied. Moreover,
\[
r - e_\ell \leq \frac{\rho_\ell - e_\ell - d_\ell}{2}.
\]
Thus, the following confirms that condition (9) holds.
\[
(n - r)|I| \leq (k - r)|I| = \sum_{\ell \in I} \deg_\Gamma(u) = \sum_{\ell \in N_\Gamma(I)} \mult_\Gamma(\ell I)
\]
\[
= \sum_{\ell \in N_\Gamma(I)} \min \left\{ \mult_\Gamma(\ell I), \deg_\Gamma(\ell) \right\}
\]
\[
\leq \sum_{\ell \in N_\Gamma(I)} \min \left\{ \mult_\Gamma(\ell I), \frac{\rho_\ell - e_\ell - d_\ell}{2} \right\}
\]
\[
= \sum_{\ell \in [k]} \min \left\{ \frac{\rho_\ell - e_\ell - d_\ell}{2}, \mu_\Gamma(\ell) \right\} \quad \forall I \subseteq [r].
\]

Let
\[
\left\{ \begin{array}{l}
g, f : V(\Gamma) \to N \cup \{0\},
g(u) = f(u) = n - r \quad \text{for } u \in X, 
g(\ell) = \rho_\ell - e_\ell + r - n \quad \text{for } \ell \in [k], 
f(\ell) = \beta \quad \text{for } \ell \in [k],
\end{array} \right.
\]
where \( \beta \) is a sufficiently large number. The graph \( \Gamma \) of the proof of Theorem 1.4 has a \((g, f)\)-factor if and only if
\[
(n - r)(r - |I|) + \sum_{\ell \in K} \beta \geq \sum_{\ell \in K} \left( \rho_\ell - e_\ell - n + r - \mu_\Gamma(\ell) \right)
\]
\[
+ \sum_{i \in I} \left( n - r - \mu_\Gamma(i) \right) \quad \forall I \subseteq X, K \subseteq [k].
\]
(19)

For \( K \neq [k] \), (19) is trivial, and for \( K = [k] \), it simplifies to the following.
\[
(n - r)(r - |I|) \geq \sum_{\ell \in [k]} \left( \rho_\ell - e_\ell - n + r - \mu_\Gamma(\ell) \right) + \sum_{i \in I} \left( n - r - \mu_{[k]}(i) \right)
\]
\[
= \sum_{\ell \in [k]} \left( \rho_\ell - e_\ell - n + r - \mu_\Gamma(\ell) \right) + \sum_{i \in I} \left( n - r - (k - r) \right)
\]
\[
= \sum_{\ell \in [k]} \left( \rho_\ell - e_\ell - n + r - \mu_\Gamma(\ell) \right) \quad \forall I \subseteq [r].
\]
\[
\square
\]

Remark 5.4. If \( e_\ell \geq 2r + d_\ell - \rho_\ell \) for \( \ell \in [k] \), then we have
\[
r^2 = \sum_{\ell \in [k]} e_\ell \geq \sum_{\ell \in [k]} (2r + d_\ell - \rho_\ell) = 2kr + (n - r) - n^2,
\]
and so the number of symbols meets the following upper bound.
\[ k \leq \frac{n^2 + r^2 - n + r}{2r}. \]

**Remark 5.5.** We show that Corollary 5.3 implies the Andersen-Hoffman’s Theorem. Let \( \rho_1 = \cdots = \rho_k = n = k \). The hypothesis of Corollary 5.3 will be that \( e_\ell \geq 2r + d_\ell - n \) for \( \ell \in [k] \) which is the same as condition (i) of Theorem 1.2. We claim that the remaining conditions of Corollary 5.3 are trivial. Since for \( \ell \in [k] \),
\[
\rho_\ell - e_\ell - n + r = n - e_\ell - n + r = \mu_I(\ell) \\
= r - e_\ell - \mu_I(\ell) \\
= r - e_\ell - \mu_I(\ell) \\
= \text{mult}_I(\ell),
\]
for \( I \subseteq [r] \), we have
\[
\sum_{\ell \in [k]} (\rho_\ell - e_\ell - n + r - \mu_I(\ell)) = \sum_{\ell \in [k]} \text{mult}_I(\ell) = |I|(n - r).
\]

For \( K \subseteq [k] \),
\[
\sum_{\ell \in K} (\rho_\ell - e_\ell + r - n) = \sum_{\ell \in K} (n - e_\ell + r - n) \\
= \sum_{\ell \in K} (r - e_\ell) \\
= \sum_{i \in [r]} \mu_K(i) \\
= \sum_{i \in [r]} \min \left\{ n - r, \mu_K(i) \right\}.
\]

**Corollary 5.6.** If
\[
2r + d_\ell - e_\ell \leq \rho_\ell \leq n - r + e_\ell \quad \ell \in [k],
\]
then an \( r \times r \) symmetric \( \rho \)-latin rectangle can be embedded into an \( n \times n \) symmetric \( \rho \)-latin square with a prescribed diagonal tail \( d \) if and only if \( \rho_\ell + e_\ell + d_\ell \) is even for \( \ell \in [k] \).

**Proof.** Condition (7) holds, and the inequality in condition (17) is trivial, and so \( \Theta \subseteq \Gamma \) always exists. \( \square \)

**Corollary 5.7.** Suppose that
\[
(n - r)(r - e_\ell) \geq (k - r)(\rho_\ell - e_\ell - n + r) \quad \forall \ell \in [k];
\]
\[
(k - r)(\rho_\ell - e_\ell - d_\ell) \geq 2(n - r)(r - e_\ell) \quad \forall \ell \in [k];
\]
\[
(r - e_\ell)(\rho_\ell - e_\ell - d_\ell) \geq 2(r - e_\ell')(\rho_\ell - e_\ell - n + r) \quad \forall \ell, \ell' \in [k].
\]
Then an \( r \times r \) symmetric \( \rho \)-latin rectangle can be completed to an \( n \times n \) symmetric \( \rho \)-latin square with a prescribed diagonal tail \( d \) if and only if \( \rho_\ell + e_\ell + d_\ell \) is even for \( \ell \in [k] \).

**Proof.** Using the first two inequalities above, we show that (7) is satisfied. Let us fix \( \ell \in [k] \). If \( \rho_\ell = e_\ell + d_\ell \), then \( \rho_\ell - e_\ell + d_\ell = 2d_\ell \leq 2(n - r) \) and so (7) holds. If \( e_\ell = r \), then
\( \rho - e_\ell + d_\ell = \rho_\ell - r + d_\ell \leq n - r + d_\ell \leq 2(n - r) \) and so again, (7) holds. So we assume that \( \rho_\ell > e_\ell + d_\ell \) and \( e_\ell < r \). We have

\[
(n - r)(r - e_\ell) \geq (k - r)(\rho_\ell - e_\ell - n + r) \\
\geq \frac{2(n - r)(r - e_\ell)}{\rho_\ell - e_\ell - d_\ell} (\rho_\ell - e_\ell - n + r).
\]

Therefore, \( \rho_\ell - e_\ell - d_\ell \geq 2(\rho_\ell - e_\ell - n + r) \) which implies (7).

By [9, Corollary 2], if for all pairs of vertices \( x, y \) of the bigraph \( \Gamma[X, [k]] \),

\[
(20) \quad f(x) \deg_{\Gamma}(y) \geq g(y) \deg_{\Gamma}(x),
\]

then \( \Gamma \) has a \((g, f)\)-factor (and so \( \Theta \subseteq \Gamma \) satisfying (17) exists). Recall that \( \deg_{\Gamma}(u) = k - r \), \( g(u) = f(u) = n - r \) for \( u \in X \), and \( \deg_{\Gamma}(i) = r - e_\ell \), \( g(\ell) = \rho_\ell - e_\ell + r - n \), \( f(\ell) = (\rho_\ell - e_\ell - d_\ell)/2 \) for \( \ell \in [k] \). Since for \( x, y \in X \) (20) is trivial, the proof is complete. \( \square \)

**Remark 5.8.** Corollary 5.7 is another generalization of the Andersen-Hoffman’s Theorem. To see this, let \( \rho_1 = \cdots = \rho_k = n = k \). Then the first inequality of Corollary 5.7 is trivial, and each of the remaining two are equivalent to condition (i) of Theorem 1.2.

An \( r \times r \) symmetric \( \rho \)-latin rectangle is nearly \( \rho \)-admissible if it meets conditions (2) and (3).

**Corollary 5.9.** If \( e_\ell \geq r - n + \rho_\ell \) for \( \ell \in [k] \), then an \( r \times r \) symmetric \( \rho \)-latin rectangle \( L \) can be embedded into an \( n \times n \) symmetric \( \rho \)-latin square if and only if \( L \) is nearly \( \rho \)-admissible, and any of the following conditions is satisfied.

\[
\sum_{\ell \in K} \left\lfloor \frac{\rho_\ell - e_\ell}{2} \right\rfloor \geq \sum_{i \in I} \left( n - r - \mu_K(i) \right) \quad \forall I \subseteq [r], K \subseteq [k];
\]

\[
(n - r)|I| \leq \sum_{\ell \in [k]} \min \left\{ \left\lfloor \frac{\rho_\ell - e_\ell}{2} \right\rfloor, \mu_I(\ell) \right\} \quad \forall I \subseteq [r].
\]

**Proof.** If \( e_\ell \geq r - n + \rho_\ell \) for \( \ell \in [k] \), then (1) holds, and (5) is satisfied for \( \rho_\ell - e_\ell + r - n = 0 \) for \( \ell \in [k] \). The remaining argument is very similar to that of Corollary 5.1. \( \square \)

**Corollary 5.10.** If \( \left\lfloor \frac{\rho_\ell - e_\ell}{2} \right\rfloor \geq r - e_\ell \) for \( \ell \in [k] \), then an \( r \times r \) symmetric \( \rho \)-latin rectangle \( L \) can be embedded into an \( n \times n \) symmetric \( \rho \)-latin square if and only if \( L \) is nearly \( \rho \)-admissible, and any of the following conditions is satisfied.

\[
\sum_{\ell \in K} (\rho_\ell - e_\ell + r - n) \leq \sum_{i \in [k]} \min \left\{ n - r, \mu_K(i) \right\} \quad \forall K \subseteq [k];
\]

\[
(n - r)(r - |I|) \geq \sum_{\ell \in [k]} \left( \rho_\ell - e_\ell - n + r - \mu_I(\ell) \right) \quad \forall I \subseteq [r].
\]

**Proof.** Since \( \rho_\ell \leq n \) and \( (\rho_\ell - e_\ell)/2 \geq [(\rho_\ell - e_\ell)/2] \geq r - e_\ell \) for \( \ell \in [k] \), (1) holds. Moreover,

\[
(n - r)|I| \leq \sum_{u \in I} \deg_{\Gamma}(u) = \sum_{\ell \in N_I(\ell)} \min \left\{ \text{mult}_{\Gamma}(\ell I), \deg_{\Gamma}(\ell) \right\}
\]

\[
\leq \sum_{\ell \in [k]} \min \left\{ \text{mult}_{\Gamma}(\ell I), \left\lfloor \frac{\rho_\ell - e_\ell}{2} \right\rfloor \right\}, \quad \forall I \subseteq [r],
\]
corollaries 5.10 and 5.12 generalize Cruse’s Theorem. Let

**Remark 5.13.** Both Corollaries 5.10 and 5.12 generalize Cruse’s Theorem. Let $n = k, \rho_1 = \cdots = \rho_k = n$. By the following, (2) is equivalent to condition (ii) of Theorem 1.1.

$$\left| \{ \ell \in [k] \mid \rho_\ell \not\equiv e_\ell \mod 2 \} \right| \leq n - r \iff \left| \{ \ell \in [n] \mid e_\ell \equiv n \mod 2 \} \right| \geq r.$$ 

Let $A = \left| \{ \ell \in [n] \mid e_\ell \not\equiv n \mod 2 \} \right|, B = [n] \setminus A$. The following confirms that (3) holds.

$$\left| \{ \ell \in [k] \mid \rho_\ell \not\equiv e_\ell \mod 2 \} \right| = \left| \{ \ell \in [n] \mid e_\ell \not\equiv n \mod 2 \} \right|$$

$$\equiv \sum_{\ell \in A} (n - e_\ell)$$

$$\equiv \sum_{\ell \in A} (n - e_\ell) + \sum_{\ell \in B} (n - e_\ell)$$

$$= \sum_{\ell = 1}^{n} (n - e_\ell)$$

$$= n^2 - r^2 \equiv n - r \mod 2.$$ 

The first inequality of Corollary 5.12 is trivial, and the remaining two are equivalent to condition (i) of Theorem 1.1. To see why Corollary 5.10 implies Cruse’s Theorem, see Remark 5.5.
A $\rho$-latin square $L$ is **diagonal** if each $\ell \in [k]$ occurs at most once on the diagonal, and is **idempotent** if $L_{ii} = i$ for $i \in [n]$. Completing partial idempotent latin squares has a rich history (see [1]). Theorem 1.4 in particular settles the necessary and sufficient conditions that ensure an $r \times r$ symmetric diagonal (or idempotent) $\rho$-latin rectangle $L$ can be embedded into an $n \times n$ symmetric diagonal (or idempotent) $\rho$-latin square, but the following problem remains unsolved.

**Problem 2.** Find necessary and sufficient conditions that ensure that an idempotent $r \times s$ $\rho$-latin rectangle can be extended to an idempotent $n \times n$ $\rho$-latin square.

We remark that Problem 2 is open even for latin squares [1].

Let $\rho = (\rho_1, \ldots, \rho_k)$ with $1 \leq \rho_\ell \leq n \leq k$ for $\ell \in [k]$ such that $\sum_{\ell \in [k]} \rho_\ell = n^2$. A **partial $\rho$-latin square** $L$ of order $n$ is an $n \times n$ array that is partially filled using $k$ different symbols each occurring at most once in each row and at most once in each column, such that each symbol $i$ occurs at most $\rho_\ell$ times in $L$ for $\ell \in [k]$. We say that $L$ is **critical** if it can be extended to exactly one $\rho$-latin square of order $n$, but removal of any entry of $L$ destroys the uniqueness of the extension, and the number of non-empty cells of $L$ is the **size** of the critical partial $\rho$-latin square.

**Problem 3.** Find good bounds for the smallest and largest sizes of critical partial $\rho$-latin squares.

**References**

[1] Atif A. Abueida and C. A. Rodger. Completing a solution of the embedding problem for incomplete idempotent Latin squares when numerical conditions suffice. Discrete Math., 312(22):3328–3334, 2012.
[2] Lars D. Andersen. Embedding Latin squares with prescribed diagonal. In Algebraic and geometric combinatorics, volume 65 of North-Holland Math. Stud., pages 9–26. North-Holland, Amsterdam, 1982.
[3] A. Bahmanian. Detachments of hypergraphs I: The Berge-Johnson problem. Combin. Probab. Comput., 21(4):483–495, 2012.
[4] Darryn Bryant and C. A. Rodger. On the completion of Latin rectangles to symmetric Latin squares. J. Aust. Math. Soc., 76(1):109–124, 2004.
[5] Allan B. Cruse. On embedding incomplete symmetric Latin squares. J. Combinatorial Theory Ser. A, 16:18–22, 1974.
[6] R. Cymer and Mikio Kano. Generalizations of marriage theorem for degree factors. Graphs Combin., 32(6):2315–2322, 2016.
[7] J. L. Goldwasser, A. J. W. Hilton, D. G. Hoffman, and Sibel Ózkan. Hall’s theorem and extending partial Latinized rectangles. J. Combin. Theory Ser. A, 130:26–41, 2015.
[8] Marshall Hall. An existence theorem for Latin squares. Bull. Amer. Math. Soc., 51:387–388, 1945.
[9] Katherine Heinrich, Pavol Hell, David G. Kirkpatrick, and Gui Zhen Liu. A simple existence criterion for $(g < f)$-factors. Discrete Math., 85(3):313–317, 1990.
[10] D. G. Hoffman. Completing incomplete commutative Latin squares with prescribed diagonals. European J. Combin., 4(1):33–35, 1983.
[11] L. Lovász. The factorization of graphs. II. Acta Math. Acad. Sci. Hungar., 23:223–246, 1972.
[12] Chris A. Rodger. Recent results on the embedding of Latin squares and related structures, cycle systems and graph designs. volume 47, pages 295–311 (1993). 1992. Combinatorics 92 (Catania, 1992).
[13] H. J. Ryser. A combinatorial theorem with an application to latin rectangles. Proc. Amer. Math. Soc., 2:550–552, 1951.

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