Lamb shift in muonic helium ion

A.P. Martynenko

Samara State University, 443011, Pavlov street 1, Samara, Russia

The Lamb shift \((2P_{1/2} - 2S_{1/2})\) in the muonic helium ion \((\mu^4He)^+\) is calculated with the account of contributions of orders \(\alpha^3, \alpha^4, \alpha^5\) and \(\alpha^6\). Special attention is given to corrections of the electron vacuum polarization, the nuclear structure and recoil effects. The obtained numerical value of the Lamb shift 1379.028 meV can be considered as a reliable estimate for the comparison with experimental data.

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I. INTRODUCTION

The ion of muonic helium \((\mu^4He)^+\) is the bound state of the negative muon and alpha particle. This simple atom is short-lived. The lifetime is determined by the muon decay in a time \(\tau_\mu = 2.19703(4) \cdot 10^{-6}\) s. The lepton mass increases when passing from the electron to the muon hydrogenic atoms \((m_\mu/m_e = 206.7682838(54)\) \[1\]). As a result the role of the nuclear structure and polarizability effects, the electron vacuum polarization corrections and the recoil contributions to the fine and hyperfine structure of the energy spectrum essentially augments. Muonic atoms represent a unique laboratory for the determination of the nuclear properties. The effect of finite nuclear size is particularly important for muonic atoms, in which the muonic wave function has a significant overlap with the nucleus. Another nuclear effect is nuclear polarizability which refers to the contribution from intermediate excited states of the nucleus. So, the experimental investigation of the \((2P - 2S)\) Lamb shift in light muonic atoms (muonic hydrogen, muonic deuterium, muonic helium ions) can give more precise values of the nuclear charge radii (the proton, deuteron, helion, alpha particle) \[2, 3\]. In the case of muonic hydrogen the Lamb shift measurement is carried out at present at PSI (Paul Sherrer Institute) \[4, 5\]. The experiment in the field of laser spectroscopy of muonic helium ions has been performed many years ago at the muon beam of the CERN synchrocyclotron \[6, 7\]. The experiment has observed resonances in the ion of muonic helium \((\mu^4He)^+\) at 811.68(15) nm and 897.6(3) nm corresponding to the \((2P_{3/2} - 2S_{1/2})\) and \((2P_{1/2} - 2S_{1/2})\) transitions. The values of frequencies in meV are equal 1527.5 meV and 1381.29 (46) meV. At a later time the experimental study of the \((2P - 2S)\) splitting in the \((\mu^4He)^+\) \[8\] found no resonance effect at the wavelength interval 811.4 \(\leq \lambda \leq 812.0\) nm with a greater than 95% probability. So, at present there is the need of new experiment which could resolve the existing experimental problem.

Theoretical investigation of the Lamb shift \((2P - 2S)\) in muonic helium ions was performed many years ago in Refs.\[9, 12\] on the basis of the Dirac equation (see other references in

*Electronic address: a.p.martynenko@samsu.ru
review article [11]). Their calculation took into account different QED corrections with the accuracy 0.01 meV. High order corrections over the fine structure constant $\alpha$ to the Lamb shift ($2P - 2S$) in the electron hydrogenic atom were obtained in the last years in the analytical form. Modern status of these calculations is presented in Refs. [2, 3]. The aim of the present work is to calculate the Lamb shift ($2P - 2S$) in the ion of muonic helium ($\mu^4\text{He}^+$) with the account of contributions of orders $\alpha^3$, $\alpha^4$, $\alpha^5$ and $\alpha^6$ on the basis of quasipotential method in quantum electrodynamics [13–15]. We consider such effects of the electron vacuum polarization, the recoil and nuclear structure corrections which are crucial to attain the high accuracy. With the exception of the nuclear polarizability contribution, we calculate all corrections in the interval ($2P_{1/2} - 2S_{1/2}$) with a precision 0.001 meV.

Our purpose consists in the improvement of the earlier performed calculations [10, 11] and derivation the reliable estimate for the ($2P_{1/2} - 2S_{1/2}$) Lamb shift, which can be used for the comparison with experimental data. Modern numerical values of fundamental physical constants are taken from [1]: the electron mass $m_e = 0.510998918(44) \cdot 10^{-3}$ GeV, the muon mass $m_{\mu} = 0.1056583692(94)$ GeV, the fine structure constant $\alpha^{-1} = 137.03599911(46)$, the mass of alpha particle $m_\alpha = 3.72737917(32)$ GeV.

II. EFFECTS OF VACUUM POLARIZATION IN THE ONE-PHOTON INTERACTION

Our approach to the investigation of the Lamb shift ($2P - 2S$) in the muonic helium ion ($\mu^4\text{He}^+$) is based on the use of quasipotential method in quantum electrodynamics [15–17], where the two-particle bound state is described by the Schrödinger equation. The basic contribution to the muon and $\alpha$-particle interaction operator is determined by the Breit Hamiltonian [18]:

$$H_B = \frac{p^2}{2\mu} - \frac{Z\alpha}{r} - \frac{p_1^4}{8m_1^4} - \frac{p_2^4}{8m_2^4} + \frac{\pi Z\alpha}{2} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \delta(r) - \frac{Z\alpha}{2m_1m_2r} \left( \frac{p^2 + r \cdot (rp) p}{r^2} \right) + \frac{Z\alpha}{r^3} \left( \frac{1}{4m_1^2} + \frac{1}{2m_1m_2} \right) (L\sigma_1) = H_0 + \Delta V^B,$$

where $H_0 = p^2/2\mu - Z\alpha/r$, $m_1$, $m_2$ are the muon and $\alpha$-particle masses, $\mu = m_1m_2/(m_1 + m_2)$.

The wave functions of $2S$- and $2P$-states are equal:

$$\psi_{200}(r) = \frac{W^{3/2}}{2\sqrt{2\pi}} e^{-w r} \left( 1 - \frac{W r}{2} \right), \quad \psi_{2lm}(r) = \frac{W^{3/2}}{2\sqrt{6}} e^{-w r} Wr Y_{lm}(\theta, \phi), \quad W = \mu Z\alpha. \quad (2)$$

The ratio of the Bohr radius of muonic helium to the Compton wavelength of the electron $m_e/W = 0.34$, so, the basic contribution of the electron vacuum polarization (VP) to the Lamb shift is of order $\alpha (Z\alpha)^2$ (see Fig. (1a)).

Accounting the modification of the Coulomb potential due to vacuum polarization in the coordinate representation

$$V_{VP}^C(r) = \frac{\alpha}{3\pi} \int_1^\infty d\xi \rho(\xi) \left( -\frac{Z\alpha}{r} e^{-2m_e \xi r} \right), \quad \rho(\xi) = \frac{\sqrt{\xi^2 - 1} (2\xi^2 + 1)}{\xi^4}, \quad (3)$$

\[\text{end}^\text{macro}\]
we present one-loop VP contributions to the shifts of $2S$, $2P$-states and the Lamb shift $(2P - 2S)$ in the form:

$$\Delta E_{VP}(2S) = -\frac{\mu(Z\alpha)^2\alpha}{6\pi} \int_1^\infty \rho(\xi)d\xi \int_0^\infty xdx \left(1 - \frac{x}{2}\right)^2 e^{-x(1 + \frac{2me\xi}{W})} = -2077.231 \text{ meV},$$

$$\Delta E_{VP}(2P) = -\frac{\mu(Z\alpha)^2\alpha}{72\pi} \int_1^\infty \rho(\xi)d\xi \int_0^\infty x^3dx e^{-x(1 + \frac{2me\xi}{W})} = -411.449 \text{ meV},$$

$$\Delta E_{VP}(2P - 2S) = 1665.773 \text{ meV}.$$ (6)

The muon one-loop vacuum polarization correction is known in analytical form [2]. We included corresponding value to the total shift in section 5. The two-loop vacuum polarization effects in the one-photon interaction are shown in Fig. 1(b, c, d). To obtain the contribution of the amplitude in Fig. 1(b) to the interaction operator, it is necessary to use the following replacement in the photon propagator:

$$\frac{1}{k^2} \rightarrow \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi)d\xi \frac{1}{k^2 + 4m^2e\xi^2}.$$ (7)

In the coordinate representation the diagram with two sequential loops gives the following particle interaction operator:

$$V_{VP-VP}^C(r) = \frac{\alpha^2}{9\pi^2} \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \left(\frac{Z\alpha}{r}\right) \frac{1}{\left(\xi^2 - \eta^2\right)^2} \left(\xi^2 e^{-2me\xi r} - \eta^2 e^{-2me\eta r}\right).$$ (8)

Averaging (8) over the Coulomb wave functions (2), we find the contribution to the Lamb shift of order $\alpha^2(Z\alpha)^2$:

$$\Delta E_{VP-VP}(2P - 2S) = 3.800 \text{ meV}.$$ (9)

Higher order $\alpha^2(Z\alpha)^4$ correction is determined by the amplitude with two sequential electron (VP) and muon (MVP) loops. Corresponding potential can be written as:

$$\Delta V_{VP-MVP}(r) = -\frac{4(Z\alpha)^4\alpha^2}{45\pi^2m^2_1} \int_1^\infty \rho(\xi)d\xi \left[\frac{m^2e^2}{r} e^{-2me\xi r} - \pi\delta(r) - \frac{m^2e^2}{r} e^{-2me\xi r}\right].$$ (10)
FIG. 2: Effects of the three-loop vacuum polarization in the one-photon interaction (a,b) and in the third order perturbation theory (c). $\tilde{G}$ is the reduced Coulomb Green function (33).

Its contribution to the shift $(2P - 2S)$ is equal

$$\Delta E(2P - 2S) = 0.002 \text{ meV.} \quad (11)$$

The particle interaction potential, corresponding to two-loop amplitudes in Fig. 1(c,d) with the second order polarization operator, takes the form:

$$\Delta V_{2-\text{loop}}^{V_P} = -\frac{2}{3} Z \alpha \frac{\alpha}{r} \left(\frac{\pi}{\alpha}\right)^2 \int_0^1 \frac{f(v)dv}{1-v^2} e^{-\frac{2m\nu}{\sqrt{1-v^2}}}, \quad (12)$$

where the spectral function

$$f(v) = v \left\{ (3-v^2)(1+v^2) \right\} \left[ Li_2 \left( -\frac{1-v}{1+v} \right) + 2 Li_2 \left( \frac{1-v}{1+v} \right) + \frac{3}{2} \ln \frac{1+v}{1-v} \ln \frac{1+v}{1-v} - \ln \frac{1+v}{1-v} \ln v \right] + \left[ \frac{11}{16}(3-v^2)(1+v^2) + \frac{v^4}{4} \right] \ln \frac{1+v}{1-v} + \left[ \frac{3}{2} v(3-v^2) \ln \frac{1-v}{4} - 2(v(3-v^2) \ln v \right] + \frac{3}{8} v(5-3v^2) \right\}, \quad (13)$$

$Li_2(z)$ is the Euler dilogarithm. The potential $\Delta V_{2-\text{loop}}^{V_P}(r)$ gives larger contribution as compared with (8) both to the hyperfine structure and Lamb shift $(2P - 2S)$:

$$\Delta E_{2-\text{loop}}^{V_P}(2P - 2S) = 7.769 \text{ meV.} \quad (14)$$

Numerical value of corrections (9), (14) and an accuracy of the calculation show that it is important to consider three-loop contributions of the vacuum polarization (see Fig. 2). A part of corrections to the potential from the diagrams of three-loop vacuum polarization in the one-photon interaction can be derived as the relations (8), (12) (the sequential loops in Fig. 2(a,b)). Corresponding contributions to the potential and the splitting $(2P - 2S)$ are the following:

$$V_{V_P-V_P-V_P}^{C}(r) = -\frac{Z \alpha}{r} \frac{\alpha^3}{(3\pi)^3} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \int_1^\infty \rho(\zeta) d\zeta \times \quad (15)$$
\[
\times \left[ e^{-2m_e \zeta r} \frac{\zeta^4}{(\xi^2 - \zeta^2)(\eta^2 - \zeta^2)} + e^{-2m_e \xi r} \frac{\xi^4}{(\xi^2 - \zeta^2)(\eta^2 - \xi^2)} + e^{-2m_e \eta r} \frac{\eta^4}{(\xi^2 - \eta^2)(\zeta^2 - \eta^2)} \right],
\]

\[
V_{V\text{P}-2\text{-loop } V\text{P}} = -\frac{4\alpha^3(Z\alpha)}{9\pi^3} \int_1^\infty \rho(\xi)d\xi \int_1^\infty \frac{f(\eta)d\eta}{\eta r(\eta^2 - \xi^2)} \left( \eta^2 e^{-2m_e \eta r} - \xi^2 e^{-2m_e \xi r} \right),
\]

\[
\Delta E_{V\text{P}-V\text{P}-V\text{P}}(2P - 2S) = 0.008 \text{ meV},
\]

\[
\Delta E_{V\text{P}-2\text{-loop } V\text{P}}(2P - 2S) = 0.036 \text{ meV}.
\]

There exists a number of the diagrams that express three-loop corrections in the polarization operator. They were first calculated for the \((2^P - 2^S)\) Lamb shift in Refs.\cite{19, 20}. The largest contribution to the energy spectrum comes from the sixth-order vacuum polarization diagrams with one electron loop (\(\Pi^{(6)}\) corrections \cite{19}). The estimate of their contribution to the Lamb shift in \((\mu/4H)^+\) is included in Table I. The analysis of the contribution of three-loop vacuum polarization in the third order perturbation theory in Fig. 2(c) shows that we can neglect it accounting the declared accuracy of the calculation.

![FIG. 3: The Wichmann-Kroll correction. The wave line shows the Coulomb photon.](image)

Additional one-loop vacuum polarization diagram is presented in Fig. 3. In the energy spectrum it gives the correction of the fifth order over \(\alpha\) (the Wichmann-Kroll correction) \cite{21, 22}. The particle interaction potential can be written in this case in the integral form:

\[
\Delta V^{WK}(r) = \frac{\alpha(Z\alpha)^3}{\pi r} \int_0^\infty \frac{d\zeta}{\zeta^4} e^{-2m_e \zeta r} \left[ -\frac{\pi^2}{12} \sqrt{\zeta^2 - 1}\theta(\zeta - 1) + \int_0^\zeta dx \sqrt{\zeta^2 - x^2} f^{WK}(x) \right].
\]

The exact form of the spectral function \(f^{WK}\) is presented in Refs.\cite{2, 21, 22}. Numerical integration in Eq.(19) with the wave functions (2) gives the following contribution to the Lamb shift:

\[
\Delta E^{WK}(2P - 2S) = -0.020 \text{ meV}.
\]

**III. RELATIVISTIC CORRECTIONS WITH THE VACUUM POLARIZATION EFFECTS**

The electron vacuum polarization effects lead not only to corrections in the Coulomb potential \cite{3}, but also to the modification of the other terms of the Breit Hamiltonian.
The one-loop vacuum polarization corrections in the Breit interaction were obtained in Refs. [23, 24]:

\[ \Delta V^{B}_{VP}(r) = \frac{\alpha}{3\pi} \int_{1}^{\infty} \rho(\xi) d\xi \sum_{i=1}^{4} \Delta V_{i,VP}^{B}(r), \] (21)

\[ \Delta V_{1,VP}^{B} = \frac{Z\alpha}{8m_{1}^{2}} \left[ 4\pi \delta(r) - \frac{4m_{e}^{3}\xi^{2}}{r} e^{-2m_{e}r} \right], \] (22)

\[ \Delta V_{2,VP}^{B} = -\frac{Z\alpha\mu^{2}\xi^{2}}{m_{1}m_{2}r} e^{-2m_{e}r} (1 - m_{e}r), \] (23)

\[ \Delta V_{3,VP}^{B} = -\frac{Z\alpha}{2m_{1}m_{2}} \left[ \frac{e^{-2m_{e}r}}{r} \right] \left[ \delta_{ij} + \frac{r_{j}r_{j}}{r^{2}} (1 + 2m_{e}r) \right] p_{j}, \] (24)

\[ \Delta V_{4,VP}^{B} = \frac{Z\alpha}{r^{3}} \left[ \frac{1}{4m_{1}^{2}} + \frac{1}{2m_{1}m_{2}} \right] e^{-2m_{e}r} (1 + 2m_{e}r) (\mathbf{L}\sigma_{1}). \] (25)

In the first order perturbation theory (PT) the potentials \( \Delta V_{i,VP}^{B}(r) \) give necessary contributions of order \( \alpha(Z\alpha)^{4} \) to the shift (2P - 2S):

\[ \Delta E_{1,VP}^{B}(2P - 2S) = -0.894 \text{ meV}, \] (26)

\[ \Delta E_{2,VP}^{B}(2P - 2S) = 0.012 \text{ meV}, \] (27)

\[ \Delta E_{3,VP}^{B}(2P - 2S) = 0.022 \text{ meV}, \] (28)

\[ \Delta E_{4,VP}^{B}(2P - 2S) = -0.088 \text{ meV}. \] (29)

The potentials \( \Delta V_{2,VP}^{B}, \Delta V_{3,VP}^{B}, \Delta V_{4,VP}^{B} \) take into account the recoil effects over the ratio \( m_{1}/m_{2} \). We have included in Table I the summary correction of order \( \alpha(Z\alpha)^{4} \), which is determined by the relations (26)-(29). Next to leading order correction of order \( \alpha^{2}(Z\alpha)^{4} \) appears in the energy spectrum from two-loop modification of the Breit Hamiltonian. We consider the term of the leading order over \( m_{1}/m_{2} \) in the potential (the function \( f(v) \) is determined by Eq.(13)):

\[ \Delta V_{2\text{-loop } VP}^{B}(r) \approx \frac{\alpha^{2}(Z\alpha)}{12\pi^{2}} \left[ \frac{1}{m_{1}^{2}} + \frac{1}{m_{2}^{2}} \right] \int_{0}^{1} f(v) dv \left[ 4\pi \delta(r) - \frac{4m_{e}^{3}}{(1 - v^{2})^{2}} e^{-2m_{e}r} \right]. \] (30)

Corresponding (2P - 2S) shift is the following:

\[ \Delta E_{2\text{-loop } VP}^{B}(2P - 2S) = -0.003 \text{ meV}. \] (31)

Other two-loop contributions to the Breit potential are omitted because they give the energy corrections which lie outside the accuracy of the calculation in this work.

In the second order perturbation theory (SOPT) we have a number of the electron vacuum polarization contributions in the leading orders \( \alpha^{2}(Z\alpha)^{2} \) and \( \alpha(Z\alpha)^{4} \), shown in the diagrams of Fig.4 (b,c):
FIG. 4: Effects of the one-loop and two-loop vacuum polarization in the second order perturbation theory (SOPT). The dashed line shows the Coulomb photon. $\tilde{G}$ is the reduced Coulomb Green function (33). The potentials $\Delta V^B$, $\Delta V^C_P$ and $\Delta V^B_{VP}$ are determined respectively by relations (1), (3) and (21).

The second order perturbation theory corrections in the energy spectrum of hydrogen-like system are determined by the reduced Coulomb Green function $\tilde{G}$ (RCGF), whose partial expansion has the form [25]:

$$\tilde{G}_n(r, r') = \sum_{l,m} \tilde{g}_{nl}(r, r') Y_{lm}(\mathbf{n}) Y_{lm}^*(\mathbf{n'}).$$  \hspace{1cm} (33)

The radial function $\tilde{g}_{nl}(r, r')$ was presented in [25] in the form of the Sturm expansion in the Laguerre polynomials. For the calculation of the Lamb shift ($2P - 2S$) in muonic helium it is convenient to use the compact representation for the RCGF of $2S$- and $2P$- states, which was obtained in [23]:

$$\tilde{G}(2S) = -\frac{Z\alpha \mu^2}{4x_1x_2} e^{-\frac{x_1+x_2}{2}} \frac{1}{4\pi} g_{2S}(x_1, x_2),$$  \hspace{1cm} (34)

$$g_{2S}(x_1, x_2) = 8 x_2 - 4x_2^2 + 8x_2 + 12x_2x_2 - 26x_2x_2 + 2x_2x_2 - 4x_2 - 26x_2x_2 + 23x_2x_2 -$$  \hspace{1cm} (35)

$$- x_2^2 + 2x_2x_2 - x_2^2 + 4e^x (1 - x_2)(x_2 - 2)x_2 + 4(x_2 - 2)x_2(x_2 - 2)x_2 \times$$

$$\times [\ln(x_2) - \ln(x_2) - \ln(x_2)],$$

$$\tilde{G}(2P) = -\frac{Z\alpha \mu^2}{36x_1^2x_2^2} e^{-\frac{x_1+x_2}{2}} \frac{3}{2} \frac{\mathbf{x}_1 \mathbf{x}_2}{x_1x_2} g_{2P}(x_1, x_2),$$  \hspace{1cm} (36)

$$g_{2P}(x_1, x_2) = 12x_2^3 + 36x_2x_2 + 36x_2x_2 + 24x_2 + 36x_2x_2 + 36x_2x_2 + 36x_2x_2 + 49x_2x_2 - 3x_2x_2 -$$  \hspace{1cm} (37)

$$- 12e^x(2 + x_2x_2)x_2 - 3x_2x_2 + 12x_2x_2 [-2C + Ei(x_2) - \ln(x_2) - \ln(x_2)],$$
where \( x_\prec = \min(x_1, x_2) \), \( x_\succ = \max(x_1, x_2) \), \( C = 0.57721566\ldots \) is the Euler constant. As a result the two-loop vacuum polarization contributions in the first term of Eq.(32) can be presented originally in the integral form (Fig.4(c)). The subsequent numerical integration gives the following results:

\[
\Delta E_{SOPT}^{V,V,V}(2S) = -\frac{\mu \alpha^2 (Z \alpha)^2}{72\pi^2} \int_1^\infty \rho(x) dx \int_1^\infty \rho(y) dy \times \int_0^\infty e^{-x(1-\frac{2m_w}{W})} dx \int_0^\infty e^{-y(1-\frac{2m_w}{W})} dy g_{2S}(x,x') = -1.901 \text{ meV},
\]

\[
\Delta E_{SOPT}^{V,V,V}(2P) = -\frac{\mu \alpha^2 (Z \alpha)^2}{7776\pi^2} \int_1^\infty \rho(x) dx \int_1^\infty \rho(y) dy \times \int_0^\infty e^{-x(1+\frac{2m_e}{W})} dx \int_0^\infty e^{-y(1+\frac{2m_e}{W})} dy g_{2P}(x,x') = -0.194 \text{ meV},
\]

The second term in (32) has the similar structure (see Fig.4(b)). The transformation of the different matrix elements is carried out with the use of the algebraic relation of the form:

\[
<\psi| (2\mu)^2 \sum_m |\psi_m><\psi_m| \Delta V_{V,P}^C|\psi >=<\psi| (E_2 + \frac{Z \alpha}{r}) (\tilde{H}_0 + \frac{Z \alpha}{r}) \sum_m |\psi_m><\psi_m| \Delta V_{V,P}^C|\psi >=<\psi| (E_2 + \frac{Z \alpha}{r})^2 \tilde{G} \Delta V_{V,P}^C|\psi > - <\psi| \frac{Z \alpha}{r} \Delta V_{V,P}^C|\psi > + <\psi| \frac{Z \alpha}{r} |\psi > |\psi| \Delta V_{V,P}^C|\psi >.
\]

Omitting further details of the calculation of numerous matrix elements in Eq.(40), we present here the summary numerical contribution from the second term in Eq.(32) to the shift \( (2P - 2S) \):

\[
\Delta E_{SOPT}^{B,V,V}(2P - 2S) = 1.434 \text{ meV}.
\]

The three-loop vacuum polarization contributions to the energy spectrum in the second order perturbation theory are presented in Fig.5. Respective potentials are obtained earlier in relations (3), (8), (12). Considering the accuracy of the calculation we can restrict our analysis by the shifts of \( 2S \)- level, which can be written in the form:

\[
\Delta E_{SOPT}^{V,V,V}(2S) = -\frac{\mu \alpha^2 (Z \alpha)^2}{108\pi^3} \int_1^\infty \rho(x) dx \int_1^\infty \rho(y) dy \int_1^\infty \rho(z) dz \int_0^\infty dx (1-\frac{x}{2})^3.
\]
\[ \int_0^\infty \, dx' \left(1 - \frac{x'}{2}\right) e^{-x'(1 + \frac{2m_e^2}{W})} \frac{1}{\xi^2 - \eta^2} \left[ \xi^2 e^{-x(1 + \frac{2m_e^2}{W})} - \eta^2 e^{-x(1 + \frac{2m_e^2}{W})} \right] \frac{g_2(x, x')}{g_2(x, x')} = -0.011 \text{ meV}, \]

Yet another contributions of the second order PT exist (see Fig.4(d,e,f)), which have the general structure similar to Eqs.(38), (39). They appear after the replacements \( \Delta V_{CP} \rightarrow \Delta V_B \) and \( \Delta V_{CP} \rightarrow \Delta V_{VP,VP} \) in the basic amplitude shown in Fig.4(c). The estimate of this contribution of order \( \alpha^2(Z\alpha)^4 \) to the shift \( (2P - 2S) \) can be derived if we take into account in the Breit potential the leading order term in the ratio \( m_1/m_2 \). Its numerical value is

\[ \Delta E^{V_{VP,VP};\Delta V_B}_S(2P - 2S) = 0.008 \text{ meV}. \]  

The two-loop vacuum polarization contribution is determined also by the amplitude in Fig.4(a). To obtain its numerical value in the energy spectrum we have to use Eqs.(3) and (21). In the leading order in the ratio \( m_1/m_2 \) we take again the potential (22), which leads to the following correction of order \( \alpha^2(Z\alpha)^4 \):

\[ \Delta E^{V_{VP};\Delta V_B}_S(2P - 2S) = -0.006 \text{ meV}. \]  

**IV. NUCLEAR STRUCTURE AND VACUUM POLARIZATION EFFECTS**

![Diagram](image)

**FIG. 6:** The leading order nuclear structure and vacuum polarization corrections. The thick point represents the nuclear vertex operator.

The influence of the nuclear structure on the muon motion in the ion \((\mu \frac{1}{2}He)^+\) is determined in the leading order by the charge radius of alpha particle \( r_\alpha = 1.676(8) \text{ fm} \) (Fig.6(a)):

\[ \Delta E_{str}[2P - 2S] = -\frac{\mu^3(Z\alpha)^4}{12} < r_\alpha^2 > = -295.848 \text{ meV}. \]  

Next to leading order correction of order \((Z\alpha)^5\) is described by one-loop exchange diagrams (Fig.7). Introducing the charge form factor \( F(k^2) \) of the alpha particle, we can express
it in the integral form:

$$\Delta E_{\text{str}}^{2\gamma}(nS) = -\frac{\mu^3(Z\alpha)^5}{\pi n^3} \delta_0 \int_0^{\infty} \frac{dk}{k} V(k),$$  \hspace{1cm} (47)

$$V(k) = \frac{2(F^2 - 1)}{m_1 m_2} + \frac{8m_1[-F(0) + 4m_2^2 F'(0)]}{m_2(m_1 + m_2)k} + \frac{k^2}{2m_1^3 m_2^3} \times$$

$$\times [2(F^2 - 1)(m_1^2 + m_2^2) - F^2 m_1^2] + \frac{\sqrt{k^2 + 4m_1^2}}{2m_1^3 m_2(m_1^2 - m_2^2)k} \times$$

$$\times \left\{ k^2 \left[ 2(F^2 - 1)M_1 - F^2 M_2 \right] + 8m_1^4 F^2 + \frac{16m_1^4 M_2^2 (F^2 - 1)}{k^2} \right\} -$$

$$- \frac{\sqrt{k^2 + 4m_1^2 m_2}}{2m_1^3 (m_1^2 - m_2^2)k} \left\{ k^2 \left[ 2(F^2 - 1) - F^2 \right] + 8m_2^4 F^2 + \frac{16m_2^4 (F^2 - 1)}{k^2} \right\}.$$

To perform numerical integration in Eq.(47) we use the dipole and Gaussian parameterizations of the charge form factor:

$$F(k^2) = \frac{\Lambda^4}{(k^2 + \Lambda^2)^2}, \quad \Lambda^2 = \frac{12}{<r_\alpha^2>}, \quad F(k^2) = \exp[-\frac{1}{6}k^2 r_\alpha^2].$$  \hspace{1cm} (49)

Numerical values of contributions to the Lamb shift $(2P - 2S)$ are equal

$$\Delta E_{D,\text{str}}^{2\gamma}(2P - 2S) = 7.196 \text{ meV}, \quad \Delta E_{G,\text{str}}^{2\gamma}(2P - 2S) = 6.605 \text{ meV}.$$  \hspace{1cm} (50)

FIG. 7: The nuclear structure corrections of order $(Z\alpha)^5$. The thick point is the nuclear vertex operator.

The essential increase of corrections (46), (50) as compared with muonic hydrogen is conditioned by two reasons. In the first place, the charge radius of the $\alpha$ - particle increases so that $\mu^2 < r_\alpha^2 > = 0.76$. Secondly, the charge of the $\alpha$-particle $Z = 2$, resulting the additional factor $2^5$ in Eq.(50). The particle interaction amplitudes containing the nuclear structure and vacuum polarization effects must be considered to obtain total value of the
Lamb shift. In the leading order such amplitude is presented in Fig.6(b). Corresponding interaction operator can be written as

$$\Delta V^{VP}_{\text{str}}(r) = \frac{2}{3} \pi Z \alpha < r^2 > \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) d\xi \left[ \delta(r) - \frac{m_e^2 \xi^2}{\pi r} e^{-2m_e r} \right].$$  \hspace{1cm} (51)

Its contributions to the shifts of the $2S$- and $2P$- levels are determined by the following expressions:

$$\Delta E^{VP}_{\text{str}}(2S) = \frac{\alpha(Z\alpha)^4 < r^2 > \mu^3}{36\pi} \int_1^\infty \rho(\xi) d\xi [1 - \frac{4m_e^2 \xi^2}{W^2} \int_0^\infty x dx (1 - \frac{x}{2}) e^{-x(1 + \frac{2m_e}{W})}] = 0.937 \text{ meV},$$  \hspace{1cm} (52)

$$\Delta E^{VP}_{\text{str}}(2P) = -\frac{\alpha(Z\alpha)^4 \mu^3 < r^2 > m_e^2}{108\pi} \int_1^\infty \xi^2 \rho(\xi) d\xi \int_0^\infty x^3 e^{-x(1 + \frac{2m_e}{W})} dx = -0.023 \text{ meV},$$  \hspace{1cm} (53)

$$\Delta E^{VP}_{\text{str}}(2P - 2S) = -0.960 \text{ meV}. \hspace{1cm} (54)$$

The contribution of the same order $\alpha(Z\alpha)^4$ is given by the amplitude in the second order perturbation theory in Fig.6(c):

$$\Delta E^{VP}_{\text{str,SOPT}}(2P - 2S) = -\frac{\alpha(Z\alpha)^4 \mu^3 < r^2 > }{36\pi} \int_1^\infty \rho(\xi) d\xi \times  \hspace{1cm} (55)$$

$$\times \int_0^\infty dx e^{-x(1 + \frac{2m_e}{W})} (1 - \frac{x}{2}) [4x(x - 2)(\ln x + C) + x^3 - 13x^2 + 6x + 4] = -1.506 \text{ meV}.$$  

![Diagram](image)

**FIG. 8**: The nuclear structure and two-loop vacuum polarization effects in the one-photon interaction. The thick point is the nuclear vertex operator.

The two-loop vacuum polarization corrections with the account of the nuclear structure are presented in Fig.8(a,b,c). The interaction potentials constructed by means of Eqs.(7), (8), (12), (51), are determined by the integral relations:

$$\Delta V^{VP-VP}_{\text{str}}(r) = \frac{2}{3} Z \alpha < r^2 > \left( \frac{\alpha}{3\pi} \right)^2 \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \times \hspace{1cm} (56)$$
\[
\times \left[ \pi \delta(r) - \frac{m^2_e}{r(\xi^2 - \eta^2)} \left( \xi^4 e^{-2m_e \xi r} - \eta^4 e^{-2m_e \eta r} \right) \right],
\]

\[
\Delta V_{str}^{2\text{-}loop, VP}(r) = \frac{4}{9} Z \alpha < r_\alpha^2 > \left( \frac{\alpha}{\pi} \right)^2 \int_0^1 \frac{f(v)dv}{1-v^2} \left[ \pi \delta(r) - \frac{m^2_e}{r(1-v^2)} e^{-\frac{2m_er}{\sqrt{1-v^2}}} \right]. \tag{57}
\]

The sum of corrections (56) and (57) to the Lamb shift \((2P - 2S)\) is equal:

\[
\Delta E_{str}^{VP, VP}(2P - 2S) = -0.008 \text{ meV}. \tag{58}
\]

\[\text{FIG. 9: The nuclear structure and two-loop vacuum polarization effects in the second order perturbation theory. The sick point is the nuclear vertex operator. } \tilde{G} \text{ is the reduced Coulomb Green function.} \]

We have included in the Table I the two-loop vacuum polarization and nuclear structure contribution of order \(\alpha^2 (Z \alpha)^4 \Delta E_{str, SOPPT}^{VP, VP}\) in the second order PT, shown in Fig.9(a,b,c,d). In the sixth order over \(\alpha\) there exists also the nuclear structure correction coming from the two-photon exchange diagrams with the electron vacuum polarization insertion (see Fig.10). We find the analytical expression of this correction and the numerical value by means of Eq.(47) in the form:

\[
\Delta E_{str, VP}^{2\gamma}(nS) = -\frac{2\mu^3 \alpha (Z \alpha)^5}{\pi^2 n^3} \int_0^\infty kV(k)dk \int_0^1 \frac{v^2(1-v^2)dv}{k^2(1-v^2) + 4m^2_e}, \tag{59}
\]

\[
\Delta E_{str, VP}^{2\gamma}(2P - 2S) = 0.127 \text{ meV}. \tag{60}
\]

\[\text{FIG. 10: The nuclear structure and electron vacuum polarization effects in the two-photon exchange diagrams. The thick point is the nuclear vertex operator.}\]
V. RECOIL CORRECTIONS, MUON SELF-ENERGY AND VACUUM POLARIZATION EFFECTS

The investigation of the different order corrections to the Lamb shift \((2P - 2S)\) of electronic hydrogen has been performed for many years. Modern analysis of the advances in the solution of this problem is presented in a review articles [2, 3]. The most part of the results was obtained in the analytical form, so they can be used directly in the muonic helium ion. In this section we analyse different contributions up to the sixth order over \(\alpha\) in the energy spectrum \((\mu^4He)^+\) and derive their numerical values in the Lamb shift \((2P - 2S)\).

The recoil correction of order \((Z\alpha)^4\) in the Lamb shift appears in the matrix element of the Breit potential with the functions (2) [2, 28, 29]:

\[
\Delta E_{\text{rec}}(2S) = \frac{\mu^3(Z\alpha)^4}{12m_2^2} = 0.295 \text{ meV.} \tag{61}
\]

The recoil correction of the fifth order over \((Z\alpha)\) is determined by the relation [2, 28]:

\[
\Delta E_{\text{rec}}^{(Z\alpha)^5} = \frac{\mu^3(Z\alpha)^5}{m_1m_2\pi n^3} \left[ \frac{2}{3} \delta l_0 \ln \frac{1}{Z\alpha} - \frac{8}{3} \ln k_0(n,l) - \frac{1}{9} \delta l_0 - \frac{2}{m_2 - m_1} \delta l_0 (m_2^2 \ln \frac{m_1}{\mu} - m_1^2 \ln \frac{m_2}{\mu}) \right], \tag{62}
\]

where \(\ln k_0(n,l)\) is the Bethe logarithm:

\[
\ln k_0(2S) = 2.811769893120563, \tag{63}
\]

\[
\ln k_0(2P) = -0.030016708630213, \tag{64}
\]

\[a_n = -2 \left[ \ln \frac{2}{n} + (1 + \frac{1}{2} + \ldots + \frac{1}{n} + 1 - \frac{1}{2n}) \delta l_0 + \frac{(1 - \delta l_0)}{l(l + 1)(2l + 1)} \right]. \tag{65}
\]

The expression (62) gives the following numerical result:

\[
\Delta E_{\text{rec}}^{(Z\alpha)^5}(2P - 2S) = -0.433 \text{ meV.} \tag{66}
\]

The recoil correction of the sixth order over \((Z\alpha)\) was calculated analytically in [30]:

\[
\Delta E_{\text{rec}}^{(Z\alpha)^6}(2P - 2S) = \frac{(Z\alpha)^6m_1^2}{8m_2^2} \left( \frac{23}{6} - 4 \ln 2 \right) = 0.004 \text{ meV.} \tag{67}
\]

The energy contributions obtained from the radiative corrections in the lepton line, the Dirac and Pauli form factors and the muon vacuum polarization have the form [2, 31]:

\[
\Delta E_{\text{MVP,MSE}}(2S) = \frac{\alpha(Z\alpha)^4}{8\pi} \frac{\mu^3}{m_1^2} \left[ \frac{4}{3} \ln \frac{m_1}{\mu(Z\alpha)^2} - \frac{4}{3} \ln k_0(2S) + \frac{38}{45} + \frac{\alpha}{\pi} \left( -\frac{9}{4} \zeta(3) + \frac{3}{2} \pi^2 \ln 2 - \frac{10}{27} \pi^2 - \frac{2179}{648} \right) + 4\pi Z\alpha \left( \frac{427}{384} - \frac{\ln 2}{2} \right) \right] = 10.939 \text{ meV}, \tag{68}
\]

\[
\Delta E_{\text{MVP,MSE}}(2P) = \frac{\alpha(Z\alpha)^4}{8\pi} \frac{\mu^3}{m_1^2} \left[ -\frac{4}{3} \ln k_0(2P) - \frac{m_1}{6\mu} \right]. \tag{69}
\]
\[
-\frac{\alpha}{3\pi} \frac{m_1}{\mu} \left( \frac{3}{4} \zeta(3) - \frac{\pi^2}{2} \ln 2 + \frac{\pi^2}{12} + \frac{197}{144} \right) = -0.168 \text{ meV}.
\]

Omitting explicit form of the radiative-recoil corrections of orders \( \alpha (Z\alpha)^5 \) and \((Z^2\alpha)(Z\alpha)^4\) from the Table 9 [2], we give their numerical value in the Lamb shift \((2P - 2S)\) of muonic helium \((\mu \frac{1}{2}He)^+\):

\[
\Delta E_{\text{rad-rec}}(2P - 2S) = -0.038 \text{ meV}.
\]

The nuclear structure corrections of orders \((Z\alpha)^6\) and \(\alpha(Z\alpha)^5\) were studied in Refs. [32, 33] for arbitrary hydrogenic atom. Let us present their numerical values in the case of muonic helium:

\[
\Delta E_{\text{str}}^{(Z\alpha)^6}(2P - 2S) = \frac{(Z\alpha)^6}{12} \mu^3 \left\{ r_\alpha^2 \left[ (\ln Z\alpha r) + C - \frac{3}{2} \right] - \frac{1}{2} r_\alpha^2 + \frac{1}{3} (r_\alpha^3) \left( \frac{1}{r} \right) - \frac{1}{40} \mu^2 (r_\alpha^4) \right\} = -0.12576 \cdot r_\alpha^2 + 0.047 = -0.306 \text{ meV},
\]

where the quantities \(I_{2,3}^{\text{rel}}, F_{\text{NR}}\) are written explicitly in [32],

\[
\Delta E_{\text{str}}^{(Z\alpha)^5}(2P - 2S) = 0.070 \text{ meV}.
\]

The diagram in Fig. 11(b) gives the contribution to the energy spectrum, which can be expressed in terms of the slope of the Dirac form factor \(F_1'\) and the Pauli form factor \(F_2\):

\[
\Delta E_{\text{rad+VP}}(nS) = \frac{\mu^3}{m_1^2} \frac{(Z\alpha)^4}{n^6} \left[ 4m_1^2 F_1'(0) \delta_{l0} + F_2(0) \frac{C_{jl}}{2l + 1} \right], \quad (73)
\]

\[
C_{jl} = \delta_{l0} + (1 - \delta_{l0}) \frac{j(j + 1) - l(l + 1) - \frac{3}{4}}{l(l + 1)}. \quad (74)
\]

The two-loop contribution to the form factors \(F_1'(0)\) and \(F_2(0)\) was calculated in [34]:

\[
m_1^2 F_1'(0) = \left( \frac{\alpha}{\pi} \right)^2 \left[ \frac{1}{9} \ln^2 \frac{m_1}{m_e} - \frac{29}{108} \ln \frac{m_1}{m_e} + \frac{1}{9} \zeta(2) + \frac{395}{1296} \right], \quad (75)
\]
\[ F_2(0) = \left( \frac{\alpha}{\pi} \right)^2 \left[ \frac{1}{3} \ln \frac{m_1}{m_e} - \frac{25}{36} + \frac{\pi^2}{4} \frac{m_e}{m_1} - \frac{4}{m_1} \ln \frac{m_1}{m_e} + \frac{3}{m_1^2} \right]. \] (76)

Then the correction to the Lamb shift \((2P - 2S)\) is equal

\[ \Delta E_{\text{rad+VP}}(2P - 2S) = -0.031 \text{ meV}. \] (77)

To estimate the muon self-energy and electron vacuum polarization contribution in Fig. 11(a), we use the relation obtained in [23]:

\[ \Delta E_{\text{MSE}}^{\text{VP}}(2P - 2S) = -0.107 \text{ meV}. \] (79)

To estimate the muon self-energy and electron vacuum polarization contribution in Fig. 11(a), we use the relation obtained in [23]:

\[ \Delta E_{\text{MSE}}^{\text{VP}}(2P - 2S) = -0.107 \text{ meV}. \] (79)

The hadron vacuum polarization (HVP) contribution can be taken into account on the basis of numerical result obtained for muonic hydrogen in Refs. [35–37]. The HVP correction and the contribution of the nuclear polarizability calculated in Refs. [38, 39] are included in Table I.

**VI. SUMMARY AND CONCLUSION**

In this work, various corrections of orders \(\alpha^3, \alpha^4, \alpha^5\) and \(\alpha^6\) have been calculated for the Lamb shift \((2P - 2S)\) in muonic helium ion \((\mu^4He)^+\). Contrary to earlier performed investigations of the energy spectra of light muonic atoms in Refs. [9, 10], we have used the three-dimensional quasipotential approach for the description of two-particle bound state. Our analysis of the different contributions to the Lamb shift accounts for the terms of two groups. The first group contains the specific corrections for muonic helium ion, connected with the electron vacuum polarization effects, nuclear structure and recoil effects in the first and second order perturbation theory. The contributions of this group are calculated numerically for the first time. The necessary order corrections of the second group include the analytical results known from the corresponding calculation in the electronic hydrogen Lamb shift. Recent advances in the physics of the energy spectra of simple atoms are presented in the review articles [2, 3] which we use in this study. Numerical values of all corrections are written in the Table I, which contains also basic references on the earlier performed investigations (other references can be found in [2]). Total numerical value 1379.028 meV of the Lamb shift \((2P - 2S)\) in muonic helium ion from the Table I is in the agreement with theoretical results 1380.9 meV obtained in Refs. [9, 10], and 1378.71 meV [41], and with the CERN experimental value 1381.29 meV [6]. The difference of our results from Refs. [9, 10] is connected both to the calculation of new contributions of higher order and slightly different numerical value of the charge radius of \(\alpha\)-particle used in this work. The authors of Refs. [9, 10] used the value of charge radius \(r_{\alpha} = 1.674(12) \text{ fm}\). As has been mentioned above the correction values were obtained with a 0.001 meV accuracy because certain contributions to the Lamb shift \((2P - 2S)\) of order \(\alpha^6\) attain the value of several \(\mu\text{eV}\). The theoretical error caused by the uncertainties in the fundamental parameters (fine structure constant, particle masses) entering the Eq.(6) is around \(10^{-5}\) meV. Other part of the theoretical
error is related to QED corrections of higher order. This part can be estimated from the leading contribution of higher order over \( \alpha \): 

\[
m_{e}\alpha(Z\alpha)^6 \ln(Z\alpha) / \pi n^3 \approx 0.001 \text{ meV}.
\]

Finally, the biggest theoretical uncertainty connected with the nuclear structure and polarizability contributions is discussed below.

Let us summarize the basic particularities of the calculation performed above.

1. Numerical value of specific parameter \( m_{e}/\mu Z\alpha = 0.34 \) in muonic helium \((\mu ^4He)^+\) is sufficiently large, so the electron vacuum polarization effects play essential role in the interaction operator. We have considered one-loop, two-loop and three-loop vacuum polarization contributions.

2. Nuclear structure effects are expressed in the Lamb shift of muonic helium ion both in terms of the charge radius of \( \alpha \)-particle in the leading and next to leading orders and by means of the charge form factor of the \( \alpha \)-particle in two-photon exchange amplitudes.

3. An estimate of the alpha particle polarizability contribution to the Lamb shift is taken from Refs.\[38, 39\]. Nuclear structure and polarizability effects give the largest theoretical uncertainty in the total value of the Lamb shift \((2P−2S)\). For instance, in the leading order \((Z\alpha)^4\) the theoretical error, connected with the uncertainty in the value of the alpha particle charge radius \( r_{\alpha} = 1.676(8) \text{ fm} \) comprises \(±2.8 \text{ meV} \). Corresponding theoretical error of the nuclear polarizability contribution amounts to 0.6 meV \[38, 39\]. Total numerical result of this work for the \((2P−2S)\) Lamb shift can be used for a comparison with the future experimental data and determination more precise value of the alpha particle charge radius.

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TABLE I: Lamb shift \((2P_{1/2} - 2S_{1/2})\) in muonic helium ion \((\mu ^4He)^+\).

| Contribution to the splitting | \(\Delta E(2P - 2S)\), meV | Formula, Reference |
|-------------------------------|-------------------------------|-------------------|
| 1                             | 2                             | 3                 |
| VP contribution of order \(\alpha(Z\alpha)^2\) in one-photon interaction | 1665.773 | (6) |
| Two-loop VP contribution of order \(\alpha^2(Z\alpha)^2\) in \(\gamma\gamma\) interaction | 11.569 | (9), (14) |
| VP and MVP contribution in \(\gamma\gamma\) interaction | 0.002 | (11) |
| Three-loop VP contribution in \(\gamma\gamma\) interaction | 0.046 | (17), (18), [19] |
| The Wichmann-Kroll correction | -0.020 | (20) |
| Additional LBL correction | 0.006 | [41, 42] |
| Relativistic and VP corrections of order \(\alpha(Z\alpha)^4\) in the first order PT | -0.948 | (26)-(29) |
| Relativistic and two-loop VP corrections of order \(\alpha^2(Z\alpha)^4\) in the first order PT | -0.003 | (31) |
| Two-loop VP contribution of order \(\alpha^2(Z\alpha)^2\) in the second order PT | 1.707 | (38)-(39) |
| Relativistic and one-loop VP corrections of order \(\alpha(Z\alpha)^4\) in the second order PT | 1.434 | (41) |
| Three-loop VP contribution in the second order PT of order \(\alpha^3(Z\alpha)^2\) | 0.028 | (42)-(43) |
| Relativistic and two-loop VP corrections of order \(\alpha^2(Z\alpha)^4\) in the second order PT | 0.002 | (44)-(45) |
| Nuclear structure contribution of order \((Z\alpha)^4\) | -295.848 | (46),[2] |
| Nuclear structure contribution of order \((Z\alpha)^5\) from \(2\gamma\) amplitudes | 6.605 (7.196) | (50), [41] |
| Nuclear structure and VP contribution in \(\gamma\gamma\) interaction of order \(\alpha(Z\alpha)^4\) | -0.960 | (54) |
| Nuclear structure and VP contribution in the second order PT of order \(\alpha(Z\alpha)^4\) | -1.506 | (55) |
| Nuclear structure and two-loop VP contribution in \(\gamma\gamma\) interaction of order \(\alpha^2(Z\alpha)^4\) | -0.008 | (58) |
| Nuclear structure and two-loop VP contribution in the second order PT of order \(\alpha^2(Z\alpha)^4\) | -0.018 | Fig.9 |
| Nuclear structure contribution of order \(\alpha(Z\alpha)^5\) from \(2\gamma\) amplitudes with VP insertion | 0.127 | (60) |
Table I (continued).

| 1                                              | 2      | 3                                      |
|------------------------------------------------|--------|----------------------------------------|
| Recoil correction of order \((Z\alpha)^4\)       | 0.295  | (61), [2, 28, 29]                      |
| Recoil correction of order \((Z\alpha)^5\)       | -0.433 | (66), [2, 28]                         |
| Recoil correction of order \((Z\alpha)^6\)       | 0.004  | (67), [2]                             |
| Muon self-energy and MVP contribution            | -11.107| (68)-(69), [2]                        |
| Radiative-recoil corrections                     | -0.038 | (70), Table 9, [2]                    |
| of orders \(\alpha(Z\alpha)^5\), \((Z^2\alpha)(Z\alpha)^4\) |        |                                        |
| Nuclear structure corrections of orders \(Z\alpha)^6\), \(\alpha(Z\alpha)^5\) | -0.236 | (71), (72), [2, 32, 33]               |
| Muon form factor \(F_1'(0), F_2(0)\) contributions| -0.031 | (79), [2, 23, 34]                     |
| Muon self-energy and VP contribution             | -0.107 (-0.065) | (81), [2, 23, 43]               |
| HVP contribution                                 | 0.223  | [35–37]                               |
| Nuclear polarizability contribution              | 2.470  | [38–40]                               |
| Total contribution                               | 1379.028|                                        |