Integrable Superhierarchy
of
Discretized 2d Supergravity

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Abstract

The hierarchical nonlinear super-differential equations are identified which describe universal behavior of the discretized model of 2d supergravity recently proposed. This is done by first taking a double scaling limit of the super Virasoro constraints (at finite $N$) of the model and by rederiving it from the $\tilde{G}_{-1/2}$ constraint and the two reduction of the super KP hierarchy discussed. The double-scaled constraints are found to be described by a twisted scalar and a Ramond fermion.

1Work supported in part by NSF Grant Phy 91-08054

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Matrix models have had successes recent years in providing us an opportunity to analyze nonperturbative effects in string theory. A deep connection to integrable system has been revealed, which we hope plays pivotal roles in the subsequent developments to come.

Somewhat surprisingly, there has been no substantial progress in constructing discretized models with supersymmetry which describe discretized 2d-supergravity and/or the one coupled to super-conformal matter. Going through this tantalizing period, a model has recently been found [1] whose properties at the $k$-th critical point agree with the results obtained from the continuum formulation of the 2d-supergravity coupled to the $(2, 4k)$ superconformal matter [2]. The partition function contains the integrations over Grassmann coordinates $\theta_i$ as well as the ones over the eigenvalue coordinates $\lambda_i$:

$$Z_N[g_p, \xi_p] = \int \prod_{i=1}^{N} d\lambda_i d\theta_i \Delta(\lambda_i, \theta_i) e^{-\frac{N}{2} \sum_{i=1}^{N} V(\lambda_i, \theta_i; g_p, \xi_{p+1/2})}.$$  \hspace{1cm} (1)

Here, $\Delta(\lambda_i, \theta_i)$ is a supersymmetric generalization of the Vandermonde determinant $\Delta(\lambda_i - \lambda_j - \theta_i \theta_j)$ and $V(\lambda, \theta; g_m, \xi_{m+1/2}) \equiv \sum_{p=0}^{\infty} g_p \lambda^p + \sum_{p=0}^{\infty} \xi_{p+1/2} \theta^p$. The degree of the potential at the lowest critical point ($k = 1$) alone tells us that the model contains more than just gravitational dressing of matter systems.

An essential point of the construction [1] is the super-Virasoro constraints at finite $N$ imposed on the model:

$$G_{n-1/2} Z_N[g_p, \xi_{p+1/2}] = 0 , \quad n = 0, 1, \ldots$$  \hspace{1cm} (2)

The generator $G_{n-1/2}$ is given through the Neveu-Schwarz supercurrent $G(p) = \sum_{\alpha, \beta} \frac{1}{2} G_{n-1/2} p^{-n-1} = \frac{1}{2} \alpha(p) b(p)$ and $\alpha(p) = \sum_{n \in \mathbb{Z}} \alpha_n p^{-n-1}$, $b(p) = \sum_{m \in \mathbb{Z}+1/2} b_m p^{-m-1/2}$. Here, $\alpha_p = -\frac{1}{N} \partial / \partial g_p$, $\alpha_{-p} = -\frac{1}{N} \partial g_p / \partial p$, $b_{p+1/2} = -\frac{1}{N} \partial / \partial \xi_{p+1/2}$, and $b_{-p-1/2} = -\frac{1}{N} \xi_{p+1/2}$ $p = 0, 1, \ldots$. The super-Vandermonde determinant has been obtained by implementing eq. (2) in the integrand. In view of the fact that the Dyson-Schwinger equation of the ordinary zero-dimensional matrix model is succinctly summarized as the Virasoro constraints [3][4] both in the double scaling limit [4] and at finite $N$, this construction albeit being semi-inductive can be regarded as a logical extension of the bosonic model which includes supersymmetry. The model may suggest a new avenue of thoughts toward supersymmetric triangulations and the combinatorially equivalent matrix integrals of some kind.
In this letter, we will identify a super-KP hierarchy attendant with our model in the double scaling limit. The physical significance of this lies in the fact that hierarchical equations are able to relate various correlation functions to one another and therefore reduce the problem into the one in which only the simplest correlator is involved. Our goal will be accomplished in two steps: first we take the double scaling limit of the super-Virasoro constraints of the model at finite $N$. Next, we will rederive these double-scaled constraints from the $\tilde{G}_{-1/2}$ constraint and two lemmas obtained from the super-KP hierarchy under consideration and its Baker-Akiezer wave functions. The procedure goes mostly in parallel to the one discussed in ref. [7], which owes some lemmas to ref. [8].

Let $Q_s$ be a pseudo super-differential operator

$$Q_s = D + q_0(\{t_\ell\}) \sum_{\ell=1}^{\infty} q_\ell(\{t_\ell\}) D^{-\ell} , \quad D \equiv \partial/\partial x_s + x_s \partial/\partial x ,$$

satisfying $Dq_0(\{t_\ell\}) + 2q_1(\{t_\ell\}) = 0$. The coefficients $q_\ell$ depend on infinite number of parameters denoted by $\{t_\ell\}$ or $\{t\}$. Here, $x = t_1, x_s = t_{1/2}$. It is known that $Q_s$ can then be brought into the form $Q_s = K_s D K_s^{-1}$. We will find that the attendant super-KP hierarchy coincides with the one proposed by Mulase and Rabin [9],

$$\partial K_s/\partial t_\ell = - \left( Q_s^{2t_\ell} \right)_- K_s , \quad \partial K_s/\partial t_{\ell+1/2} = - \left( Q_s^{2t_{\ell+1}+1} - K_s x_s D^{2t_{\ell+1}+2} K_s^{-1} \right)_- K_s ,$$

confirming the conjecture made in [1]. We denote by $(\mathcal{O})_+$ and $(\mathcal{O})_-$ respectively the part containing the non-negative powers and the one containing the negative powers of $D$ in $\mathcal{O}$. We prepare the Baker-Akiezer wave function $w_\bullet(\lambda, \theta; \{t_\ell\})$ (with $\bullet = (\text{NS})$ or $(\text{R})$) which has two different local expressions:

$$\sum_{\ell=1}^{\infty} t_\ell \lambda^\ell + \sum_{\ell=0}^{\infty} t_{\ell+1/2} \lambda^{\ell+1/2} \theta$$

(NS) or

$$\sum_{\ell=1}^{\infty} t_\ell \lambda^\ell + \sum_{\ell=0}^{\infty} t_{\ell+1/2} \lambda^{-\ell+1/2} \theta$$

(R).

In either case, eq. (3) is equivalent to

$$\partial w(\lambda, \theta; \{t\}) / \partial t_\ell = \left( Q_s^{2t_\ell} \right)_+ w(\lambda, \theta; \{t\}) ,$$

$$\partial w(\lambda, \theta; \{t\}) / \partial t_{\ell+1/2} = \left( Q_s^{2t_{\ell+1}+1} - K_s x_s D^{2t_{\ell+1}+2} K_s^{-1} \right)_+ w(\lambda, \theta; \{t\}) .$$

Let us turn to the double scaling limit of the superstress tensor $G(p)$. Let $\zeta \equiv p^2 - 1$. For simplicity, we will restrict ourselves to the planar solution of ref. [1] which takes into account the even bosonic as well as the even and odd fermionic couplings to
all orders and the odd bosonic couplings only to the leading order. Change of variables we will make in what follows\textsuperscript{3} is dictated by the explicit construction of the scaling operators \textsuperscript{3,4} attendant with this planar solution\textsuperscript{4}. Let $\alpha(p) = \alpha^{(e)}(p) + \alpha^{(o)}(p) + \alpha^{(0)}(p) + Np^{-1}$, where the superscripts $e, o, +, -$ denote respectively the even, odd, positive and negative parts of the mode expansion of $\alpha(p)$. We have replaced the zero mode part\textsuperscript{3} by the result of its action on the partition function $Z_N$. Similarly, $b(p) = b^{(e)}(p) + b^{(o)}(p) + b^{(0)}(p) + Np^{-1}$. We will reexpand $\alpha^{(e)}(p)$ as

$$
\sum_{n=0}^{\infty} t_n^{B^+} \left( -\frac{d}{dp} \sigma_n^{B^+} \right).$

Here $\sigma_n^{B^+} = -c_n \sum_{k=0}^{n} \left( \binom{2n}{k} c_k^{-1} p^{2k} \right)$, with $c_n \equiv (-1/4)^n \left( \binom{2n}{n} \right)$ is, aside from the normalization, taken from the form of the scaling operator $\textsuperscript{3}$. By checking $[\alpha^{(e)}(p), \alpha^{(o)}(p)]$, one can prove $\alpha^{(e)}(p) = \sum_{n=1}^{\infty} \xi^{n-1/2} \partial / t_n^{B^+}$ and $\sigma_n^{B^+} = \frac{(-1)^n}{\pi} \sum_{n=0}^{\infty} B(n-\ell-1/2, 3/2)(-\xi)^{\ell} \equiv B_n(\xi)$. The remaining operators in $\alpha(p), b(p)$ can be reexpanded in a similar fashion. We find that

$$
\alpha^{(e)} = 2 \left( \frac{A}{N} \right)^{-1} \sqrt{1 + \xi} \sum_{\ell=0}^{\infty} t_{\ell}^{B+} \frac{dB_{e,\ell}(\xi)}{d\xi}, \quad \alpha^{(e)} = \left( -\frac{A}{N} \right)^{-1} \sum_{\ell=0}^{\infty} \frac{\partial}{\partial \xi^{\ell-3/2}}
$$

$$
\alpha^{(o)} = 2 \left( \frac{A}{N} \right)^{-1} \sum_{\ell=0}^{\infty} (\ell + \frac{1}{2}) t_{\ell}^{B^+} B_{\ell}(\xi), \quad \alpha^{(o)} = \left( -\frac{A}{N} \right)^{-1} \sum_{\ell=0}^{\infty} \frac{\partial}{\partial \xi^{\ell-3/2}}
$$

$$
b^{(e)} = \left( \frac{A}{N} \right)^{-1} \sum_{\ell=0}^{\infty} t_{\ell}^{B^+} B_{\ell}(\xi), \quad b^{(e)} = \left( -\frac{A}{N} \right)^{-1} \sum_{\ell=0}^{\infty} \frac{\partial}{\partial \xi^{\ell-1/2}}
$$

$$
b^{(o)} = \left( \frac{A}{N} \right)^{-1} \sum_{\ell=0}^{\infty} (\ell + \frac{1}{2}) t_{\ell}^{B^+} \xi^{\ell} \frac{d\xi^{\ell}}{\sqrt{1+\xi^{\ell}}} B_{\ell}(\xi^{\ell}), \quad b^{(o)} = \left( -\frac{A}{N} \right)^{-1} \sum_{\ell=0}^{\infty} \frac{\partial}{\partial \xi^{\ell}}
$$

\(\xi^{\ell-3/2}\) are consistent change of bases appropriate to taking the double scaling limit.

We now go on to derive the double-scaled super-Virasoro constraints. Let us imagine evaluating $\mathfrak{g} \frac{d\xi}{2\pi i} \xi^n G(p) \equiv \xi^n G(p)$, $n = 0, 1, \cdots$. Inside the $\xi$ integration, the following manipulation is permitted for $B_{\ell}(\xi)$:

$$
B_{\ell}(\xi) = (-\xi)^\ell \int \frac{d\ell'}{2\pi i} \sum_{k=0}^{\infty} \xi^{(\ell-k-1/2)-1} (1 - \xi)^{3/2-1} (-\xi)^k = -\xi^{\ell-1/2} (1 + \xi)^{1/2}.
$$

\textsuperscript{3}We have developed a systematic method of taking a double scaling limit of the constraints by making a change of variables based on the form of the scaling operators. The details will be discussed elsewhere together with other points.

\textsuperscript{4}A multiplicity of the constraints is expected once odd bosonic couplings are taken to all orders. See ref. \textsuperscript{10}.

\textsuperscript{5}This part only renormalizes a cosmological constant subtractively and will not be discussed hereafter.
In the first equality, we have used the integral representation of the Beta function and extended the sum to infinity by analytic continuation. The original $\zeta'$ contour encloses the origin and one. In the second equality, we have made the $\zeta'$ contour large ($|\zeta'|>|\zeta|$) to sum the geometric series and changed integration variables to $x \equiv \zeta'/\zeta$ and $\zeta$ to pick a pole at $x = -1$. (This is allowed as the integrand does not have a cut extending from $x = -1$.) Finally we rescale as

$$\zeta = a^{2/m} \zeta_{sc}, \quad t^B_\ell = a^{2(1-\ell/m)} t_\ell^\pm, \quad t^{BS}_\ell = a^{2(1-\ell/m \pm 1/2m)} t_\ell^{S \pm},$$

$A^{-1} = 1 + a^2 t$, $1/N = \kappa a^{2+1/m}$, and take $a \rightarrow 0$ limit. After some calculation, we find $G(p) \stackrel{a \rightarrow 0}{\longrightarrow} \sqrt{2}a^{-3/m} \tilde{G}(\zeta_{sc})$

$$2\tilde{G}(\zeta_{sc}) = \sum_{n \in \mathbb{Z}} \tilde{G}_{n-1/2} \zeta_{sc}^{-n-1} = \alpha^{(tw)}(\zeta_{sc}) \psi_R(\zeta_{sc}),$$

$$\alpha^{(tw)}(\zeta_{sc}) = \sum_{m \in \mathbb{Z}+1/2} \alpha_m \zeta_{sc}^{-m-1}, \quad \psi_R(\zeta_{sc}) = \sum_{m \in \mathbb{Z}} b_m \zeta_{sc}^{-m-1/2},$$

$$\alpha_{\ell+1/2} = \left( \frac{\Lambda}{2} \right) \left( \frac{\partial}{\partial t^\ell_\ell} + \frac{\partial}{\partial t^{\ell+1}} \right) \equiv \frac{\partial}{\partial t^\ell_\ell}, \quad \alpha_{\ell-1/2} = \left( \frac{\Lambda}{2} \right) \left( t^\ell_\ell + t^{\ell-1} \right) \equiv \left( \ell + \frac{1}{2} \right) j^\ell,$$

$$b_\ell = \left( \frac{\Lambda}{2} \right) \left( \frac{\partial}{\partial t^\ell_\ell} + \frac{\partial}{\partial t^{\ell-1}} \right) \equiv \frac{\partial}{\partial t^\ell_\ell}, \quad b_{-\ell} = \left( \frac{1}{\sqrt{2}\Lambda} \right) \left( t^{S +} + t^{S -} \right) \equiv j^S, \quad j^S_0 = \left( \frac{1}{\sqrt{2}\Lambda} \right) t_0^{S +} + \left( \frac{\Lambda}{2} \right) \frac{\partial}{\partial t_0^{S +}}, \quad \ell = 1, 2, \ldots.$$

The double-scaled super-Virasoro constraints are expressible in terms of the twisted scalar $\alpha^{(tw)}(\zeta_{sc})$ and the Ramond fermion $\psi_R(\zeta_{sc})$, each of which is made of a combination of couplings with positive parity and the ones with negative parity. The double-scaled super-Virasoro constraints are stated as

$$\tilde{G}_{n-1/2} \text{lim } Z_N = 0, \quad \tilde{G}_{-1/2} = \sum_{\ell=0}^{\infty} \left( \ell + \frac{1}{2} \right) j^\ell \frac{\partial}{\partial t^\ell_\ell} + \sum_{\ell=1}^{\infty} j^\ell \frac{\partial}{\partial t^{\ell+1}} + \frac{1}{2} b_0 j_0,$$

$$\tilde{G}_{n-2} = \sum_{\ell=0}^{\infty} \left( \ell + \frac{1}{2} \right) j^\ell \frac{\partial}{\partial t^{\ell+1}} + \sum_{\ell=1}^{\infty} j^\ell \frac{\partial}{\partial t^{\ell+1}} + b_0 \frac{\partial}{\partial t_{n-1}^{S +}} + \sum_{\ell=0}^{n-2} \frac{\partial}{\partial t_{\ell+1}} \frac{\partial^2}{\partial t_{n-1-\ell}^{S +}}. \quad \text{(6)}$$

In the remainder of this letter, we will reobtain eq. (6) from the $\tilde{G}_{-1/2}$ constraint and eq. (5). We will prove that

Claim: There exists an operator $T$ such that the matrix elements (not just the residue) of $T_s \equiv [K_s T K_s^{-1}]_-$ vanish once the $\tilde{G}_{-1/2}$ constraint is invoked.

In particular, we will show that the $T_s$ is given by

$$T_s \delta(x-x')(x_s - x'_s) = -\tau_{NS}^{-1} \{ \{ t \} \} \tau_{NS}^{-1} \{ \{ t' \} \} \times$$

$$\text{res}_{\nu, \theta} \left[ \left( V(\nu, \theta; \{ t \}) S \right) \text{res}_\lambda \left( \frac{1}{\lambda} \right) G_{SKP}(\lambda) \right] \tau_{R} \{ \{ t \} \} \left( V(\nu, \theta; \{ t' \}) \tau_{NS} \{ \{ t' \} \} \right) \quad \text{(7)}$$
Here, $\alpha = 1/2$ and $\{t\}$ and $\{t\}'$ differ by $x \neq x'$, $x_s \neq x_s'$ only. We have introduced $\tilde{X}(\lambda, \theta) \equiv X(\lambda, \theta; \{t\}) - X(\lambda^{-1}, \theta; \{\partial/\partial t\})$, where

$$
\begin{align*}
X(\lambda, \theta; \{t\}) &= \left\{ \sum_{\ell=1}^{\infty} t_{\ell} \lambda^\ell + \sum_{\ell=0}^{\infty} t_{\ell+1} \lambda^{\ell/2} \theta \right\} \\
X(\lambda^{-1}, \theta; \{\partial/\partial t\}) &= \left\{ \sum_{\ell=1}^{\infty} \frac{1}{\ell} \frac{\partial}{\partial t_{\ell}} \lambda^{-\ell} - \sum_{\ell=0}^{\infty} \frac{\partial}{\partial t_{\ell+1/2}} \lambda^{-\ell-1/2} \theta \right\} (\text{NS}) \\
&\quad + \left\{ \sum_{\ell=1}^{\infty} \frac{1}{\ell} \frac{\partial}{\partial t_{\ell}} \lambda^{-\ell} - \sum_{\ell=0}^{\infty} \left(1 - \frac{1}{2} \delta_{\ell,0}\right) \frac{\partial}{\partial t_{\ell+1/2}} \lambda^{-\ell-1/2} \theta \right\} (\text{R})
\end{align*}
$$

to define $V(\lambda, \theta; \{t\}) := e^{\tilde{X}(\lambda, \theta)}$ as well as $\bar{V}(\lambda, \theta; \{t\}) := e^{-\tilde{X}(\lambda, \theta)}$. Later, we will need $V(\lambda, \theta, \lambda'; \{t\}, \{t\}') := e^{-X(\lambda, \theta)+X(\lambda', \theta')}$ : We denote by $G_{\text{SKP}}(\lambda)$ a supercurrent $-1/2D\tilde{X}(\lambda, \theta)\partial \tilde{X}(\lambda, \theta)|_{\theta=0}$ with $D \equiv \partial/\partial \theta + \theta \partial/\partial \lambda$. The tau function $\tau_*[[\{t\}]]$ (where $\bullet = (\text{NS})$ or (R)) is introduced through $w_* (\lambda, \theta; \{t\}) = \frac{1}{\tau_*[[\{t\}]]} V_* (\lambda, \theta; \{t\}) \tau_*[[\{t\}]]$. The operator $S \equiv \lim_{\lambda_0 \to 0} S(\lambda_0)$ is a spin operator $[11]$ which creates a cut and interpolates between (NS) and (R) sectors : $\tau_R [[\{t\}]] = S \tau_{NS} [[\{t\}]]$.

By a straightforward calculation, one establishes

$$
\text{Lemma 1} : \quad \text{res}_{\nu, \theta} \left[ \left( \mathcal{P}(\{t\}) e^{\lambda+x, \lambda', \theta} \mathcal{R}(\{t\}') e^{-x', \lambda', \theta} \right) \right] = \left[ \mathcal{P}(\{t\}) \mathcal{R}(\{t\}') \right] \times
\delta(x-x')(x_s-x_s')
$$

where $\mathcal{P}(\{t\}) = \sum_{i=-\infty}^{\infty} p_i(\{t\})D^i$, $\mathcal{R}(\{t\}') = \sum_{i'=\infty}^{\infty} r_{i'}(\{t\}')(-D')^{i'}$

We find a supersymmetric extension of the bilinear identity $\int d\lambda d\theta w(\lambda, \theta; \{t\})^* w(\lambda, \theta; \{t\}') = 0$, which is a simple consequence of Lemma 1 and eq. (3). This can be readily seen by expanding the left-hand side of the equation which we want to prove. Let us now prove

$$
\text{Lemma 2} : \quad \tau_{NS}^{-1}[[\{t\}]] \tau_{NS}[[\{t\}']] \text{res}_{\nu, \theta} \left[ \left( \mathcal{V} (\nu, \theta; \{t\}) \mathcal{V} (\lambda, \theta'', \mu, \theta''; \{t\}) \tau_{NS}[[\{t\}]] \right) \right] \times
$$

$$(\mathcal{V} (\nu, \theta; \{t\}') \tau_{NS}[[\{t\}]] \right) = (\theta'' - \theta') w(\mu, \theta''; \{t\}) w^*(\lambda, \theta''; \{t\})
$$

$$
+ (\lambda - \mu - \theta''') \tau_{NS}^{-1}[[\{t\}]] \tau_{NS}^{-1}[[\{t\}']]
$$

$$(\mathcal{D} \mathcal{X}(\lambda, \theta', \{t - t'\}) - \mathcal{D} \mathcal{X}(\lambda^{-1}, \theta''; \{\partial/\partial t - \partial/\partial t'\}) \mathcal{V} (\mu, \theta''; \{t\}) \tau_{NS}[[\{t\}]] \mathcal{V} (\lambda, \theta''; \{t\}') \tau_{NS}[[\{t\}']].
$$

The integrand of the left-hand side is not normal-ordered. Putting everything normal ordered, we find that it equals (for $|\nu| > |\mu|, |\lambda|$)

$$
e^{\mathcal{X}(\mu, \theta'; \{t\}-\mathcal{X}(\lambda, \theta''; \{t\})} \text{res}_{\nu, \theta} \left[ \left( e^{-\mathcal{X}(\nu^{-1}, \theta'; \{\partial/\partial t'\}) + \mathcal{X}(\lambda^{-1}, \theta''; \{\partial/\partial t\}) - \mathcal{X}(\mu^{-1}, \theta'; \{\partial/\partial t\})} \tau_{NS}[[\{t\}]] \right) \right]
$$

$$
\left( \mathcal{X}(\nu, \theta; \{t-t'\}) \tau_{NS}[[\{t\}]] \right) e^{\mathcal{X}(\nu, \theta; \{t-t'\}) \frac{\nu - \mu - \theta''}{\nu - \lambda - \theta''}}
$$

$^6$ The existence of such nonlocal operator is strongly supported by its appearance in Ising model and the equivalent description in terms of the GSO projected free Majorana fermion. The operator $S$ establishes a correspondence between $\tau_{NS}[[\{t\}]]$ and $\tau_R [[\{t\}]]$. 

The bilinear identity ensures that this expression has no singularity around the origin. One can, therefore, simply pick the pole at \( \nu = \lambda + \theta \theta'' \), \( \theta = \theta'' \) to evaluate the integrand. We easily find the right-hand side. Lemma 2 follows.

We finally proceed to the computation of the right-hand side of eq. (7). First, we are concerned with reexpressing the part \( S \res \lambda \left[ \frac{1}{\lambda^2} G(\lambda) \right] \tau_R \{ \{ t \} \} \). As the operator product goes as \( G(\lambda) S = \frac{1}{2} \lambda^{-3/2} \hat{S} + o(\lambda^{-1/2}) \), \( \hat{S} \) is another spin operator conjugate to \( S \) \(^{(1)}\), we find that the above expression can be written as \( \frac{1}{2} \res \lambda \left[ \frac{1}{\lambda^{3/2}} B(\lambda) \right] \tau_{NS} \{ \{ t \} \} \). Here \( B(\lambda) \) is a normal-ordered local operator acting on the Neveu-Schwarz state \( S_{\res} \). We find that any normal-ordered local operator consisting of products of \( \hat{X}(\lambda, \theta) \) and its (super)-derivative is obtained from \( V(\lambda, \theta'', \mu, \theta'; \{ t \}) \) by successive (super)-differentiations. The right-hand side of eq. (7) can in principle be evaluated by using two lemmas established, leading to a position space expression as given by the left-hand side. Let us see this by evaluating the contribution due to \( G(\lambda) \). Let \( B(\lambda) \) be \( \lambda^m G(\lambda) + \cdots \) with \( m \) an integer. We find that the right-hand side of eq. (7) equals

\[
- \tau_{NS}^{-1} \{ \{ t \} \} \tau_{NS}^{-1} \{ \{ t' \} \} \res \lambda \res \nu, \theta \left[ \left\{ \frac{1}{2 \lambda^\beta} \right\} D^\nu V(\lambda, \theta'', \mu, \theta'; \{ t \}) \right] \bigg|_{\mu = \lambda, \theta'' = 0} \\
- \frac{\beta^2}{2 \lambda^{\beta+1}} D^\nu V(\lambda, \theta'', \mu, \theta'; \{ t \}) \bigg|_{\mu = \lambda, \theta'' = 0} + \cdots \tau_{NS} \{ \{ t \} \} \left( \hat{V}(\nu, \theta; \{ t' \}) \right) \tau_{NS} \{ \{ t' \} \} \right].
\]

Here, \( \beta = 2 - m \). We can further convert this expression, applying formulas obtained by expanding Lemma 2 around \( (\mu, \theta') = (\lambda, \theta'') \). We find

\[
\res \lambda, \theta'' \left[ \frac{\theta''}{2 \lambda^\beta} \partial_{\lambda} w(\lambda, \theta'', \{ t \}) \right] w^*(\lambda, \theta', \{ t' \}) + \frac{\theta''}{2 \lambda^\beta} D'' w(\lambda, \theta''; \{ t \}) D'^\nu w^*(\lambda, \theta''; \{ t' \}) \\
- \frac{\beta \theta''}{2 \lambda^{\beta+1}} w(\lambda, \theta''; \{ t \}) w^*(\lambda, \theta''; \{ t' \}) + \cdots.
\]

We use Lemma 1 to convert this expression into a position space operator acting on \( \delta(x - x')(x_s - x'_s) \). In this way, we obtain the operator \( T_s = [K_s T K_s^{-1}] \) such that

\[
T \equiv 1/2 \sum_{\ell=1}^\infty t_{\ell} (\partial/\partial x)^{\ell-1-\beta} \partial/\partial x_s + 1/2 \sum_{\ell=0}^\infty t_{\ell+1/2} (\partial/\partial x)^{\ell-\beta} \\
- (\beta/2) (\partial/\partial x)^{-1-\beta} \partial/\partial x_s + \cdots.
\]

We are now convinced that the pseudo-differential operator \( T_s = [K_s T K_s^{-1}] \) in Eq. (7) exists.

\(^7\) In the bosonic case, the corresponding operator is simply a stress-energy tensor and explicit computation has been given in \(^3\). This is not the case here. We will not need it however.
The two reduction of the hierarchy is stated as

\[ \left( \frac{\partial}{\partial t_2} \right) \tau [\{ t \}] = 0, \quad \left( \frac{\partial}{\partial t_{2\ell+1/2}} \right) \tau [\{ t \}] = 0, \quad \ell = 1, 2, \cdots \]  

(8)

We observe that \( \frac{1}{\sqrt{2}} \text{res}_\lambda \left( \frac{1}{\sqrt{\lambda}} \text{GSKP}(\lambda) \right) \tau_R [\{ t \}] \) coincides with \( \tilde{G}_{-1/2} \lim_{\alpha \to 0} \mathcal{Z}_N \) if the identification

\[ \sqrt{2t_{2\ell+1}} = j_{\ell} (\ell = 0, 1, \cdots), \quad t_{2\ell+1/2} = j_{\ell}^s (\ell = 1, 2, \cdots), \quad (\sqrt{2} \Lambda_s) t_{1/2} = t_0^{S+} \]

is made and if the partition function is identified as the \( \tau \) function. Eq. (7), therefore, implies the vanishing of the matrix elements of \( T_s \) under the \( \tilde{G}_{-1/2} \) constraint. The Claim has now been proven.

Let \( \hat{\lambda} = K_s \frac{\partial}{\partial x} K_s^{-1} \). We find, under the constraint,

\[ \left( \hat{\lambda}^{2\ell} K_s^s T K_s^{-1} \right) (-) = \hat{\lambda}^{2\ell} T_s = 0 \]  

(9)

as \( \hat{\lambda}^2 \) is a differential (as opposed to a pseudo-differential) operator under eq. (8). On the other hand, \( \left( \hat{\lambda}^{2\ell} K_s^s T K_s^{-1} \right) (-) \delta(x - x')(x_s - x'_s) \) is equal to the right-hand side of eq. (7) with \( \alpha (= 1/2) \) replaced by \( \alpha - 2\ell \). We conclude that eq. (8) implies

\[ \text{res}_\lambda \left( \lambda^{-\alpha+2\ell} \text{GSKP}(\lambda) \right) \tau_R [\{ t \}] = 0 \]

But this, together with eq. (9), is nothing but the \( \tilde{G}_{\ell-1/2} \) constraint in eq. (6), which we wanted to rederive.

It is straightforward to extend this proof to the \( p \)-th reduction of the super KP hierarchy discussed here and will be reported elsewhere.

The author thanks Luis Alvarez-Gaumé and Juan Mañes for continuing discussions on this subject. They have independently obtained the double scaling limit of the super-Virasoro constraints. The author has been informed that they carried out an explicit computation up to genus two together with K. Becker and M. Becker [12].
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