HENRY HELSON MEETS OTHER BIG SHOTS – A BRIEF SURVEY

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Abstract. A theorem of Henry Helson shows that for every ordinary Dirichlet series \( \sum a_n n^{-s} \) with a square summable sequence \((a_n)\) of coefficients, almost all vertical limits \( \sum a_n \chi(n) n^{-s} \), where \( \chi : \mathbb{N} \to \mathbb{T} \) is a completely multiplicative arithmetic function, converge on the right half-plane. We survey on recent improvements and extensions of this result within Hardy spaces of Dirichlet series – relating it with some classical work of Bohr, Banach, Carleson-Hunt, Cesàro, Hardy-Littlewood, Hardy-Riesz, Menchoff-Rademacher, and Riemann.

1. Introduction

In his article \([20]\) from 1967 Henry Helson suggested that the classical theory of ordinary Dirichlet series should be combined with modern harmonic analysis and functional analysis, and since then this theory saw a remarkable comeback – in particular after the publication of the seminal article \([18]\).

The aim of this survey is to discuss various recent variants of a somewhat curious theorem of Helson dealing with Dirichlet series \( D = \sum a_n n^{-s} \) which have 2-summable coefficients. These Dirichlet series form the natural Hilbert space \( H_2 \), and to see a very first example look for \( \varepsilon > 0 \) at the following translated zeta series

\[
D = \sum \frac{1}{n^{\frac{1}{2} + \varepsilon}} n^{-s}.
\]

In general, each \( D \in H_2 \) converges for all \( s \in \mathbb{C} \) in the half-plane \([Re > 1/2]\). But if we multiply the \( a_n \)'s with some character \( \chi \), i.e. a completely multiplicative arithmetic functions \( \chi : \mathbb{N} \to \mathbb{T} \), and consider the new Dirichlet series \( D^\chi = \sum a_n \chi(n) n^{-s} \), then the convergence in general improves considerably. We refer to the following remarkable theorem from \([21, \text{Theorem, p. 140}]\) as 'Helson's theorem'.

**Theorem 1.1.** Let \( D = \sum a_n n^{-s} \) be a Dirichlet series with coefficients \( (a_n) \in \ell_2 \). Then for almost all characters \( \chi : \mathbb{N} \to \mathbb{T} \) we have that \( D^\chi = \sum a_n \chi(n) n^{-s} \) converges on all of \([Re > 0]\).

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What is here meant by 'almost all characters $\chi$'? A simple way to understand this is to identify the set $\Xi$ of all characters $\chi: \mathbb{N} \to \mathbb{T}$ with the infinitely dimensional torus $\mathbb{T}^\infty$, i.e. the countable product of $\mathbb{T} = \{w \in \mathbb{C}: |w| = 1\}$ which forms a natural compact abelian group, where the Haar measure is given by the normalized Lebesgue measure $dz$. We write $p = 2, 3, 5, \ldots$ for the sequence of prime numbers. If we consider pointwise multiplication on $\Xi$, then

$$(2) \quad \iota: \Xi \to \mathbb{T}^\infty, \quad \chi \mapsto \chi(p) = (\chi(p_n))_n,$$

is a group isomorphism which turns $\Xi$ into a compact abelian group. The Haar measure $d\chi$ is the push forward measure of $dz$ through $\iota^{-1}$.

Applying his theorem to the Dirichlet series from (1), Helson detects as a somewhat curious application that 'Riemann's conjecture holds true almost everywhere' in the following sense.

**Theorem 1.2.** For almost all $\chi \in \Xi$ the Dirichlet series $\zeta^\chi = \sum \chi(n)n^{-s}$ of the Riemann zeta series $\zeta = \sum n^{-s}$ have no zeros in the critical half-plane $[\Re > 1/2]$.

Let us come back to the Hilbert space $\mathcal{H}_2$, and look at it from a viewpoint originally invented by H. Bohr in [9]. To understand this recall first that the set of all characters on $\mathbb{T}^\infty$, so the dual group of $\mathbb{T}^\infty$, consists of all monomials $z \mapsto z^\alpha$, where $\alpha = (\alpha_k) \in \mathbb{Z}^{(\mathbb{N})}$ (all finite sequences of integers). Obviously $D = \sum a_n n^{-s} \in \mathcal{H}_2$ if and only if there is a (then unique) function $f \in H_2(\mathbb{T}^\infty)$ such that $a_n = \hat{f}(\alpha)$ for all $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ with $p^\alpha = n$. In other terms, the mapping

$$H_2(\mathbb{T}^\infty) \to \mathcal{H}_2, \quad f \mapsto D,$$

identifies two Hilbert spaces.

Now the following two questions appear naturally. If $D \in \mathcal{H}_2$ satisfies the assertion from Helson’s theorem, what does this mean for the associated function $f \in H_2(\mathbb{T}^\infty)$? And vice versa, if we have an appropriate theorem on pointwise convergence for functions in $H_2(\mathbb{T}^\infty)$, when does it transfer to a Helson-like theorem for ordinary Dirichlet?

Let us give four examples which indicate that it is worth to look at such an interplay more carefully. All four examples come along with some more precise questions.

**Ex 1:** Consider for $f \in H_2(\mathbb{T}^\infty)$ and $u > 0$ the 'translated' orthonormal series

$$\sum \hat{f}(\alpha)p^{-u\alpha}z^\alpha,$$

which clearly defines a function $f_u \in H_2(\mathbb{T}^\infty)$. Then an immediate translation of Helson’s theorem through (2) shows that

$$(3) \quad f_u(z) = \lim_{x \to \infty} \sum_{p^\alpha < x} \hat{f}(\alpha)p^{-u\alpha}z^\alpha$$

almost everywhere on $\mathbb{T}^\infty$.

Does this result even hold for $u = 0$, and if yes, what does this in turn then mean for Helson’s theorem? Note that in contrast to (3), only if $u > 1/2$, for all
\( f \in H_2(T^\infty) \)
\[
f_u(z) = \sum_{\alpha} \hat{f}(\alpha) p^{-u_\alpha} z^\alpha \quad \text{almost everywhere on } T^\infty
\]
(here the series only makes sense if we consider absolute convergence); see e.g. [13, Remark 11.3].

**Ex 2:** For each \( s \in \mathbb{C} \) we may interpret \( D^s = \sum (a_n n^{-s}) \chi(n) \) from Helson’s theorem as an orthonormal series in \( L_2(\Xi) \). This brings us to recall the Menchoff-Rademacher theorem, which is the fundamental theorem on almost everywhere convergence of general orthonormal series, and as most convergence theorems it is accompanied by a maximal inequality which was isolated by Kantorovitch; see e.g. the standard reference [1].

**Theorem 1.3.** Let \( \sum c_n x_n \) be an orthonormal sequence in some \( L_2(\mu) \), i.e. \( (c_n) \in \ell_2 \) and \( (x_n) \) an orthonormal sequence. Then \( \sum c_n x_n \) converges \( \mu \)-almost everywhere whenever \( (c_n \log n) \in \ell_2 \). Moreover, there is a constant \( C > 0 \) such that for all \( c_1, \ldots, c_\infty \in \mathbb{C} \)
\[
\left( \int \sup_{x} \left| \sum_{n=1}^{x} c_n x_n \right|^2 d\mu \right)^{\frac{1}{2}} \leq C \left\| (c_n \log n) \right\|_{2}.
\]

Does the Menchoff-Rademacher theorem imply Helson’s theorem, and does Kantorovitch’s inequality even add its relevant maximal inequality?

**Ex 3:** Of course, for special orthonormal sequences \( (x_n) \) the assertion of the Menchoff-Rademacher theorem improves considerably, and the most celebrated theorem in this direction is certainly due to Carleson. In this case it was Hunt who after a careful analysis of Carleson’s work, came up with what is now known as the Carleson-Hunt maximal inequality (see also Theorem 5.1 for \( 1 < p < \infty \)).

**Theorem 1.4.** The Fourier series \( f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \) of every \( f \in H_2(T) \) converges almost everywhere on \( T \). Moreover, there is a (best) constant \( CH_2 > 0 \) such that
\[
\left( \int_{T} \sup_{x} \left| \sum_{n=x}^{\infty} \hat{f}(n) z^n \right|^2 dz \right)^{\frac{1}{2}} \leq CH_2 \left\| f \right\|_{2}.
\]

Does this deep result give new input to Helson’s theorem? Indeed, look at functions \( f \in H_2(T) \), or equivalently at functions in \( H_2(T^\infty) \) which only depend on the first variable, and \( u = 0 \). Then by Carleson’s theorem every Dirichlet series \( \sum a_n \chi(n) n^{-s} \), which is ‘thin’ in the sense that \( a_n \neq 0 \) only if \( n = 2^j \) for some \( j \), converges for almost all \( \chi \in \Xi \) in \( s = 0 \), and consequently for almost all \( \chi \) also on the right half-plane. Does this result extend to all \( D \in \mathcal{H}_2 \)? Equivalently, does (3) hold for \( u = 0 \)? Up to which extent is it possible to have some Carleson’s theorem for functions on the infinite dimensional torus?

**Ex 4:** Finally, we turn to another fundamental theorem on ordinary Dirichlet series. For its formulation we recall that the Banach space of all Dirichlet series
$D = \sum a_n n^{-s}$, which converge on the right half-plane to a bounded (and then necessarily holomorphic) function $f$, is denoted by $D_\infty$ (the norm given by the sup norm on the right half-plane). The following result due to Bohr [8] rules the theory of such series.

**Theorem 1.5.** Let $D = \sum a_n n^{-s} \in D_\infty$. Then $D$ for each $u > 0$ converges uniformly on $\{Re > u\}$. Even more, there is a constant $C > 0$ such that for every $x > 1$ we have

$$\sup_{Re>0} \left| \sum_{n=1}^{x} a_n n^{-s} \right| \leq C \log x \sup_{Re>0} |f(s)|.$$

We intend to explain in which sense this result can be seen as a sort of extreme case of a scale of Helson-like theorems.

We hope to convince our reader that the above apparently quite different theorems have a lot in common. Inspired through the work of Bayart [3], Bohr [8], Duy [14], Hardy-Riesz [17], Hedenmalm-Lindqvist-Seip [18], Hedenmalm-Saksman [19], and Helson [21], among others, we want to sketch that they in fact are intimately linked. And this picture gets visible if one looks at the above theorems within the more general scale of Hardy spaces $H_p$, $1 \leq p \leq \infty$, of ordinary Dirichlet series as invented by Bayart in [3] (see Section 2.2).

In particular, we want to discuss several results from our ongoing research project on general (!) Dirichlet series $\sum_n a_n e^{-\lambda_n s}$ (see [10], [11], [12], and [25]), although the present survey entirely focuses on ordinary Dirichlet series $\sum_n a_n n^{-s}$ only. Indeed, many of the results which we are going to discuss even hold in the much wider setting of general Dirichlet series and their related Fourier analysis on so-called Dirichlet groups.

Why do we here restrict ourself to the ordinary case? In fact we believe that our topic for ordinary Dirichlet series is interesting in itself, and more important, we hope that for a reader who is merely interested in the ordinary case, our presentation is particularly useful since it is not covered by technical difficulties which are unavoidable in the much wider framework of general Dirichlet series.

Our survey has eight sections: Helson meets Menchoff-Rademacher, Riemann, Carleson-Hunt, Bohr, Cesàro, Hardy-Riesz, Hardy-Littlewood, and Banach.

2. Preliminaries

For all information on Dirichlet series we refer to the monographs [13], [17], [22], or [23]. In the following we focus on a few facts of particular interest.

2.1. Kronecker flow. The continuous group homomorphism, the so-called Kronecker flow,

$$\beta : \mathbb{R} \to \mathbb{T}^\mathbb{N}, \ t \mapsto (p_k^{-it})_{k=1}^\infty$$

has dense range. Recall that the dual group $\hat{\mathbb{T}}^\mathbb{N}$ equals the group $\mathbb{Z}^{(\mathbb{N})}$ of all finite sequences $\alpha$ with entries from $\mathbb{Z}$ in the sense that every such $\alpha$ may be identified with the character $z^\alpha$. Then for every character $z^\alpha \in \hat{\mathbb{T}}^\mathbb{N}$ we have that $x = \log p^\alpha$
is the unique real number for which $e^{-ix} = z^\alpha \circ \beta$. In other words, $\hat{T}^\infty$ and 
$\{\log p^\alpha : \alpha \in \mathbb{Z}^{(N)}\}$ can be identified (as sets), and the natural order of $\mathbb{R}$ transfers to $\mathbb{T}^\infty$:

\begin{equation}
\alpha \in \mathbb{Z}^{(N)} \geq 0 \text{ if } \log p^\alpha \geq 0.
\end{equation}

Moreover, recall for $u > 0$ the definition of the Poisson kernel $P_u(t) = \frac{1}{\pi} \frac{u}{t^2 + u^2}$
on $\mathbb{R}$ which has $e^{-ut}$ as its Fourier transform. The push forward measure of $P_u$ 
under the Kronecker flow $\beta : \mathbb{R} \to \mathbb{T}^\infty$ is denoted by $p_u$. We have that 
$\hat{p_u}(\alpha) = \hat{P}_u(\log p^\alpha) = e^{-u|\log p^\alpha|}$ for every $\alpha \in \mathbb{Z}^{(N)} = \mathbb{T}^\infty$.

2.2. Hardy spaces. Bayart in [3] initiated an $\mathcal{H}_p$-theory of ordinary Dirichlet series. Recall that $H_p(\mathbb{T}^\infty), 1 \leq p \leq \infty$, is the closed subspace of all $f \in L_p(\mathbb{T}^\infty)$
which have a Fourier transforms $\hat{f} : \mathbb{Z}^{(N)} \to \mathbb{C}$ supported on $N_0^{(N)}$. Then the 
Banach spaces $\mathcal{H}_p, 1 \leq p \leq \infty$, consist of all ordinary Dirichlet series $\sum a_n n^{-s}$ for
which there is some (unique) $f \in H_p(\mathbb{T}^\infty)$ such that $a_n = \hat{f}(\alpha)$ for all $\alpha \in N_0^{(N)}$
with $p^s = n$. Together with the norm $\|D\|_p = \|f\|_p$ this leads to Banach spaces. Hence by the very definition the so-called Bohr transform

\begin{equation}
\mathfrak{B} : H_p(\mathbb{T}^\infty) \to \mathcal{H}_p, \ f \mapsto D,
\end{equation}

where $D = \sum a_n n^{-s}$ with $a_n = \hat{f}(\alpha)$ for all $\alpha \in N_0^{(N)}$ with $p^s = n$, is an isometric linear bijection. For every Dirichlet polynomial $D = \sum_{n \leq x} a_n n^{-s}$ we for $1 \leq p < \infty$ have that

\begin{equation}
\|D\|_p = \lim_{T \to \infty} \frac{1}{2T} \left( \int_{-T}^{T} \left| \sum_{n \leq x} a_n n^{-it} \right|^p dt \right)^{\frac{1}{p}},
\end{equation}

and hence in this case it turns out that the completion of the linear space of all Dirichlet polynomials under this norm gives precisely the Banach space $\mathcal{H}_p$.

Obviously, a Dirichlet series $D = \sum a_n n^{-s}$ belongs to the Hilbert space $\mathcal{H}_2$ if and only if the sequence $(a_n)$ of its Dirichlet coefficients belongs to $\ell_2$.

Moreover, as a consequence of Theorem 1.5 it can be shown that $\mathcal{H}_\infty$ and $D_\infty$ coincide as Banach spaces,

\begin{equation}
\mathcal{H}_\infty = D_\infty \text{ isometrically}.
\end{equation}

This fundamental fact was observed in [18] (see also [13, Corollary 5.3]), and we will come back to it in Section 6.

In contrast to the situation for $\mathcal{H}_\infty$, arbitrary Dirichlet series in $\mathcal{H}_p, 1 \leq p < \infty$, only converge pointwise on $[\text{Re} > 1/2]$ (even absolutely), and in general this half-plane can not be replaced by a bigger one (see e.g [13, Remark 12.13 and Theorem 12.11]). We remark that by a result from [19] each Dirichlet series $D$ in $\mathcal{H}_2$ converges almost everywhere on the abscissa $[\text{Re} = 1/2]$ (and consequently also every $D \in \mathcal{H}_p, 2 \leq p < \infty$), whereas Bayart in [4] gave an example of a Dirichlet series in $\mathcal{H}_\infty$ that diverges at every point of the imaginary axis. It seems that for Dirichlet series in $\mathcal{H}_p, 1 \leq p < 2$, there is no such precise knowledge on pointwise convergence on the abscissa $[\text{Re} = 1/2]$. 


The horizontal translation of a Dirichlet series \( D = \sum a_n n^{-s} \) about \( u > 0 \) is defined to be the Dirichlet series

\[
D_u := \sum \frac{a_n}{n^u} n^{-s}.
\]

Given \( D \in \mathcal{H}_p \), then \( \mathcal{B}(f * p_u) = D_u \), which in particular shows that \( D_u \in \mathcal{H}_p \).

But more can be said: For each \( 1 \leq p, q < \infty \) and \( u > 0 \) there is a constant \( E = E(u, p, q) \) such that for each \( D \in \mathcal{H}_p \) we have

\[
(10) \quad D_u \in \mathcal{H}_q \quad \text{and} \quad ||D_u||_q \leq E||D||_p.
\]

This result is basically due to Bayart [3], for a self-contained proof see [13, Theorem 12.9]; we refer to this fact as the 'hypercontractivity' of the Hardy spaces \( \mathcal{H}_p \).

### 2.3. Vertical limits.

Given a Dirichlet series \( D = \sum a_n n^{-s} \), then we call the Dirichlet series \( D_z(s) := \sum \frac{a_n}{n^u} n^{-s} \) the translation of \( D \) about \( z \in \mathbb{C} \), and each Dirichlet series of the form

\[
D^\chi = \sum a_n \chi(n) n^{-s} \quad \chi \in \Xi
\]

is said to be a vertical limits of \( D \). Examples are vertical translations \( D_{it} = \sum a_n n^{i\tau} n^{-s} \) with \( \tau \in \mathbb{R} \), and the terminology is explained by the fact that each vertical limit may be approximated by vertical translates. More precisely, given \( D = \sum a_n n^{-s} \) which converges absolutely on the right half-plane, for every \( \chi \in \Xi \) there is a sequence \( (\tau_k)_k \subset \mathbb{R} \) such that \( D_{i\tau_k} \) converges to \( D^\chi \) uniformly on \( \{Re > \varepsilon\} \) for all \( \varepsilon > 0 \). Assume conversely that for \( (\tau_k)_k \subset \mathbb{R} \) the vertical translations \( D_{i\tau_k} \) converge uniformly on \( \{Re > \varepsilon\} \) for every \( \varepsilon > 0 \) to a holomorphic function \( f \) on \( \{Re > 0\} \). Then there is \( \chi \in \Xi \) such that \( f(s) = \sum_{n=1}^\infty a_n \chi(n) n^{-s} \) for all \( s \in \{Re > 0\} \). For this see [10, Section 4.1].

It is simple to show that each vertical limit \( D^\chi \) belongs to \( \mathcal{H}_p \) if and only if \( D \) does, and the norms remain the same (apply Bohr transform and use the rotation invariance of the Lebesgue measure on \( \mathbb{T}^\infty \)).

Finally, we recall that every function \( f \in L_1(\mathbb{T}^\infty) \) for almost all \( z \in \mathbb{T}^\infty \) allows a locally Lebesgue integrable 'restriction' \( f_z : \mathbb{R} \to \mathbb{C} \) such that \( f_z(t) = f(z \beta(t)) \) for almost all \( t \in \mathbb{R} \) (see [10, Lemma 3.10]). More explicitly, for almost all \( z \in \mathbb{T}^\infty \) the function

\[
(11) \quad f_z : \mathbb{R} \to \mathbb{C}, \quad f_z(t) = f(p^{-it}z)
\]

is locally integrable.

Given \( f \in H_1(\mathbb{T}^\infty) \), the family \( (f_z)_{z \in \mathbb{T}^\infty} \) of functions on \( \mathbb{R} \) form a sort of bridge to tools from Fourier analysis on \( \mathbb{R} \). The following simple lemma (see [12]) shows how pointwise convergence on \( \mathbb{T}^\infty \) is related with pointwise convergence on \( \mathbb{R} \).

**Lemma 2.1.** Let \( f_n, f \in H_1(\mathbb{T}^\infty) \). Then the following are equivalent:

1. \( \lim_{n \to \infty} f_n(z) = f(z) \) for almost all \( z \in \mathbb{T}^\infty \)

2. \( \lim_{n \to \infty} f_n(z)(t) = f_z(t) \) for almost all \( z \in \mathbb{T}^\infty \) and for almost all \( t \in \mathbb{R} \).

In particular, if all \( f_n \) are polynomials and \( D_n \in \mathcal{H}_1 \) are the Dirichlet series associated to \( f_n \) under Bohr's transform, then (1) and (2) are equivalent to each of the following two further statements:
\( \lim_{n \to \infty} D_n^\chi(0) = f(\chi(p)) \) for almost all \( \chi \in \Xi \)

\( \lim_{n \to \infty} D_n^\chi(it) = f\left(\frac{\chi(p)}{p^t}\right) \) for almost all \( \chi \in \Xi \) and for almost all \( t \in \mathbb{R} \).

3. Helson meets Menchoff-Rademacher

Let us come back to Helson’s theorem from Theorem 1.1 which states that almost all vertical limits \( D^\chi \) for Dirichlet series \( D \in H_2 \) converge on the right half-plane. Helson’s original proof is mainly based on classical tools from Fourier analysis, as e.g. the Fourier inversion theorem or the Hausdorff-Young inequality.

How does the relevant maximal inequality for Helson’s theorem look like? Inspired by an idea of Bayart [3] we isolate this maximal inequality using Kantorovitch’s maximal inequality (4) from the Menchoff-Rademacher theorem.

**Theorem 3.1.** For every \( u > 0 \) there is a constant \( C = C(u) > 0 \) such that for every \( D \in H_2 \) we have

\[
\left( \int_\Xi \sup_x \left| \sum_{n=1}^x a_n \chi(n) \chi(n) \right|^2 \right)^{1/2} d\chi \leq C \|D\|_2.
\]

Equivalently, for every \( f \in H_2(\mathbb{T}^\infty) \)

\[
\left( \int_{\mathbb{T}^\infty} \sup_{x} \left| \sum_{\alpha, p^\alpha < x} \frac{\hat{f}(\alpha)}{p^\alpha} z^\alpha \right|^2 dz \right)^{1/2} \leq C \|f\|_2.
\]

**Proof.** Clearly, the functions \( \Xi \to \mathbb{C}, \chi \mapsto \chi(n) \) form an orthonormal system in \( L_2(\Xi) \). Hence by Theorem 1.3 for all \( a_1, \ldots, a_x \in \mathbb{C} \)

\[
\left( \int_\Xi \sup_x \left| \sum_{n=1}^x a_n \chi(n) \right|^2 \right)^{1/2} \ll \left\| \left( a_n \frac{\log n}{n^u} \right) \right\|_2 \ll \|a_n\|_2 = \|D\|_2.
\]

The second assertion is an obvious reformulation through Bohr’s transform (7). □

But applying the hypercontractivity estimate from (10) we even get all this for the much larger class of Dirichlet series in \( H_1 \) (apart from the maximal inequality this was observed in [3, Theorem 6]).

**Theorem 3.2.** For every \( u > 0 \) there is a constant \( C = C(u) > 0 \) such that for every \( D \in H_1 \) we have

\[
\int_\Xi \sup_x \left| \sum_{n=1}^x a_n \chi(n) \right| d\chi \leq C \|D\|_1.
\]

Equivalently, for every \( f \in H_1(\mathbb{T}^\infty) \)

\[
\int_{\mathbb{T}^\infty} \sup_{x} \left| \sum_{\alpha, p^\alpha < x} \frac{\hat{f}(\alpha)}{p^\alpha} z^\alpha \right| dz \leq C \|f\|_1.
\]
Proof. Take \( u > 0 \) and \( D \in H_1 \). Then we know from (10) that \( D_u \in H_2 \) and 
\[ \|D_u\|_2 \leq E(u)\|D_u\|_1. \]
Hence by Theorem 3.1
\[
\int_{\Xi} \sup_x \left| \sum_{n=1}^{x} \frac{a_n}{n^u} \chi(n) \right| d\chi \leq \left( \int_{\Xi} \sup_x \left| \sum_{n=1}^{x} \frac{a_n}{n^u} \chi(n) \right|^2 d\chi \right)^{\frac{1}{2}} \\
\leq C(u)\|D_u\|_2 \leq C(u)E(u)\|D\|_1,
\]
which is the inequality we aimed for. The second assertion again follows from Bohr’s transform (7).
\[\square\]

To derive from the preceding maximal inequalities results on pointwise convergence is standard, and an argument is formalized in [12, Lemma 3.6]. Then we together with Lemma 2.1 conclude the following improvement of Theorem 1.1 (taken from [12, Corollary 2.3]).

**Theorem 3.3.** Let \( f \in H_1(\mathbb{T}^\infty) \) and \( D = \sum a_n n^{-s} \in H_1 \) its associated Dirichlet series under the Bohr transform. Then for all \( u > 0 \)
\[
f(z) = \lim_{x \to \infty} \sum_{p^\alpha < x} \hat{f}(\alpha) p^{\alpha u} z^\alpha
\]
almost everywhere on \( \mathbb{T}^\infty \). Equivalently, almost all vertical limits \( D^x \) converges on the right half-plane. More precisely, there is a null set \( N \subset \Xi \) such that, if \( \chi \notin N \), then we for all \( u + it \in [\text{Re} > 0] \) have
\[
D^x(u + it) = f_{\chi(p)} * P_u(t).
\]

There is another interesting aspect of this theorem. A result of Bayart from [3] (see e.g. [13, Theorem 12.11] for an explicit formulation) states that the best \( u > 0 \) such that \( \sum_\alpha \frac{|\hat{f}(\alpha)|}{p^{\alpha u}} < \infty \) for all \( f \in H_1(\mathbb{T}^\infty) \), equals 1/2. In particular, for all \( u \leq 1/2 \) there is \( f \in H_1(\mathbb{T}^\infty) \) such that the equality \( f(z) = \sum_\alpha \frac{\hat{f}(\alpha)}{p^{\alpha u}} z^\alpha \) a.e.’ fails. Note that in contrast to this, the limit in Theorem 3.3 exists for all \( u > 0 \).

4. HELSON MEETS Riemann

Let us come back to Theorem 1.2 which shows that Riemann’s conjecture holds for almost all vertical limits of the zeta series. The following result from [18, Theorem 4.6] (here even proved for Dirichlet series in \( H_1 \)) is an improvement.

**Theorem 4.1.** Let \( D \in H_1 \) have completely multiplicative coefficients \( a_n \). Then almost all vertical limits \( D^x \) converge on [\text{Re} > 0] to a zero free function.

**Proof.** Assume first \( D \in H_2 \), and recall the definition of the Möbius function \( \mu : \mathbb{N} \to \{1, -1, 0\} \)
\[
\mu(n) = \begin{cases} 
1 & \text{if } n = 1 \\
(-1)^k & \text{if } n = p_1 \ldots p_k \\
0 & \text{else}
\end{cases}
\]
Then for every character $\chi$ a formal(!) calculation shows that
\[
\left( \sum a_n \chi(n) n^{-s} \right) \ast \left( \sum a_n \chi(n) \mu(n) n^{-s} \right) = 1,
\]
where $\ast$ stands for the Cauchy product. Almost all vertical limits of these two Dirichlet series in $H_2$ converge on the right half-plane, hence they are zero free. In a second step, let $D \in H_1$. Then, given $n \in \mathbb{N}$, by hypercontractivity (10) the horizontal translation $D^{1/n} \in H_2$, and hence there is a null set $\Xi_n$ such that all $(D^{1/n})^\chi, \chi \in C(\bigcup \Xi_n)$ all Dirichlet series $(D^{1/n})^\chi, n \in \mathbb{N}$ are zero free on the right half-plane which implies that all $D^\chi$ are.

If we apply this result to the translated zeta series from (1), then the proof of Theorem 1.2 is immediate.

## 5. Helson meets Carleson-Hunt

It is well-known that Theorem 1.4 extends to the case $1 < p < \infty$, a result then usually called Carleson-Hunt theorem.

**Theorem 5.1.** Let $1 < p < \infty$. The Fourier series $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ of every $f \in H_p(\mathbb{T})$ converges almost everywhere on $\mathbb{T}$. Moreover, there is a (best) constant $C_{H_p} > 0$ such that
\[
\left( \int_{\mathbb{T}} \sup_x \left| \sum_{n<x} \hat{f}(n) z^n \right|^p dz \right)^{\frac{1}{p}} \leq C_{H_p} \|f\|_p.
\]

And all this fails for $p = 1$. A reasonable question would be to ask for almost everywhere pointwise convergence on $\mathbb{T}^\infty$ of the Fourier series $\sum_{\alpha} \hat{f}(\alpha) z^\alpha$ whenever $f \in H_p(\mathbb{T}^\infty)$. But how do we order the multi indices? If we consider absolute convergence of $\sum_{\alpha} \hat{f}(\alpha) z^\alpha$, then we necessarily need to assume $(\hat{f}(\alpha)) \in \ell_1$.

For $p = 2$ the following result was proved by Hedenmalm-Saksman in [19, Theorem 1.5], and later it was extended by Duy in [14] to the scale of $1 < p < \infty$ and arbitrary general Dirichlet series (see also [11])

**Theorem 5.2.** Let $1 < p < \infty$. Then for every $f \in H_p(\mathbb{T}^\infty)$ we have
\[
\left( \int_{\mathbb{T}^\infty} \sup_x \left| \sum_{\alpha < x} \hat{f}(\alpha) z^\alpha \right|^p dz \right)^{\frac{1}{p}} \leq C_{H_p} \|f\|_p.
\]

Equivalently, for every $D = \sum a_n n^{-s} \in H_p$
\[
\left( \int_{\mathbb{T}^\infty} \sup_x \left| \sum_{n=1}^{x} a_n \chi(n) \right|^p d\chi \right)^{\frac{1}{p}} \leq C_{H_p} \|D\|_p.
\]

Clearly, here the particular case $p = 2$ proves Theorem 3.1 for $u = 0$ which means a strong improvement. In fact the proof from [14] (and also [11]) follows the strategy invented by Hedenmalm-Saksman in [19]; this strategy applies the original Carleson-Hunt theorem for functions in one variable through a magic trick invented by Fefferman in [15]. Observe, that if the function $f$ in the preceding...
theorem only depends on the first variable, then we recover the full one variable
case from Theorem 5.1. But note that Theorem 5.2 does not cover Theorem 3.2,
since for \( p = 1 \) it definitely fails.

Again we may deduce convergence theorems. Combining Theorem 5.2 and
adding Lemma 2.1, we conclude the following result from [11] (see again [19, Theorem 1.4] for the case \( p = 2 \)).

**Theorem 5.3.** Let \( f \in H_p(\mathbb{T}^\infty), 1 < p < \infty \) and \( D = \sum a_n n^{-s} \in \mathcal{H}_p \) its associated Dirichlet series under the Bohr transform. Then

\[
f(z) = \lim_{x \to \infty} \sum_{p^s < x} \hat{f}(\alpha) z^\alpha
\]

almost everywhere on \( \mathbb{T}^\infty \). In particular, for almost all \( \chi \in \Xi \)

1. \( D^X \) converges in \( s = 0 \) to \( f(\chi(p)) \),
2. \( D^X \) converges almost everywhere on the imaginary axis, and for almost all \( t \in \mathbb{R} \) we have \( D^X(it) = f_{\chi(p)}(t) \),
3. \( D^X \) converges everywhere on the right half-plane, and we for all \( u > 0 \)
almost everywhere on \( \mathbb{R} \) have \( D^X(u + it) = f_{\chi(p)} * P_u(t) \).

This is a considerably strong extension of Theorem 1.1. Since the Carleson-Hunt Theorem 5.1 on pointwise convergence does not hold for \( p = 1 \), for this case the first two conclusions are false (to see this for the second one use Lemma 2.1) whereas the last one holds by Theorem 3.3.

**6. Helson meets Bohr**

Are Helson’s Theorem 1.1 and Bohr’s Theorem 1.5 linked? Can Helson-like arguments be used to prove the important Banach space identity \( \mathcal{H}_\infty = \mathcal{D}_\infty \) from (9)? Both questions have a positive answer, and this will be a consequence of the following maximal inequality from [11] which is a variant of Theorem 5.2.

**Theorem 6.1.** For every \( u > 0 \) there is a constant \( C > 0 \) such that for every \( 1 \leq p < \infty \) there and every \( f \in H_p(\mathbb{T}^\infty) \) we have

\[
\left( \int_{\mathbb{T}^\infty} \sup_x \left| \sum_{p^s \leq x} \frac{\hat{f}(\alpha)}{p^{n\alpha}} z^\alpha \right|^p dz \right)^\frac{1}{p} \leq C \| f \|_p.
\]

Equivalently, for every \( D = \sum a_n n^{-s} \in \mathcal{H}_p \)

\[
(12) \quad \left( \int_{\Xi} \sup_x \left| \sum_{n=1}^x \frac{a_n}{n^u} \chi(n) \right|^p d\chi \right)^\frac{1}{p} \leq C \| D \|_p.
\]

What are the differences of Theorem 6.1 and Theorem 5.2? There are two. Admitting translations along some \( u > 0 \), Theorem 6.1 holds for \( p = 1 \), whereas Theorem 5.2 does not. So in this case the preceding result recovers Theorem 3.2. But secondly, and more important, the constant in the maximal inequality from Theorem 6.1 does not depend on \( p \). We remark that another application of Lemma 2.1 again allows us, now with a different argument, to recover Theorem 3.3.
Let us indicate what is happening whenever in Theorem 6.1 the parameter $p$ tends to $\infty$. Obviously, we get that there is a constant $C > 0$ such that for each $u > 0$ and $D = \sum a_n n^{-s} \in \mathcal{H}_\infty$

\[
\|\sup_x \left| \sum_{n=1}^{x} \frac{a_n}{n^u} \chi(n) \right| \|_{L_\infty(\mathcal{E})} \leq C\|D\|_{\mathcal{H}_\infty},
\]

The following corollary of this inequality recovers part of Theorem 1.5 and the identity from (9). It shows up to which amount the preceding Helson type Theorem 6.1 still reflects Bohr’s original ideas.

**Corollary 6.2.** For each $D = \sum a_n n^{-s} \in \mathcal{H}_\infty$ we have

\[D \in \mathcal{D}_\infty \quad \text{and} \quad \|D\|_{\mathcal{D}_\infty} \leq \|D\|_{\mathcal{H}_\infty},\]

and moreover $D$ for each $u > 0$ converges uniformly on $[\Re > u]$.

**Proof.** We fix some $D = \mathfrak{B}(f) \in \mathcal{H}_\infty$. From the density of the Kronecker flow from (5) (i.e. Kronecker’s approximation theorem) and (13) we deduce that for each $y > 0$ and each $\psi \in \Xi$

\[
\sup_{t \in \mathbb{R}} \left| \sum_{n<y} \frac{a_n}{n^u} \psi(n) n^{-it} \right| = \sup_{z \in \mathbb{T}} \left| \sum_{p^\alpha \leq y} \frac{a_n}{n^u} \psi(p^\alpha) z^\alpha \right| = \left\| \sum_{n=1}^{y} \frac{a_n}{n^u} \psi(n) \chi(\cdot) \right\|_{L_\infty(\Xi)} = C\|D\|_{\mathcal{H}_\infty}.
\]

Hence by the so-called Bohr-Cahen formula on uniform convergence (see e.g. [13, Proposition 1.6]) we see that each $\chi \in \Xi$ and not just a contractive inclusion.

Indeed, denote the $N$th partial sum of $D$ by $D_N$, and fix some $u > 0$. Since we already checked that $D$ and $D^x$ converge uniformly on $[\Re > u]$, another application of (5) yields

\[
\|D_u\|_{\mathcal{D}_\infty} = \lim_{N \to \infty} \|(D_N)_{u}\|_{\mathcal{D}_\infty} = \lim_{N \to \infty} \|(D^x_N)_{u}\|_{\mathcal{D}_\infty} = \|D_u\|_{\mathcal{D}_\infty}.
\]

Alltogether we get that there is some $\chi$ such that

\[
\|D\|_{\mathcal{D}_\infty} = \|D^x\|_{\mathcal{D}_\infty} = \|f_{\chi(p)} \cdot P_u\|_{L_\infty(\mathbb{R})} \leq \|f_{\chi(p)}\|_{L_\infty(\mathbb{R})} = \|f\|_{H_\infty(\mathbb{T})} = \|D\|_{H_\infty},
\]

the conclusion. 

We emphasize that this ‘outcome’ of Theorem 6.1 is - in two respects - weaker than what is known from Theorem 1.5 and (9). Indeed, (9) is an isometric identity and not just a contractive inclusion.
We refer to [13, Corollary 5.3] and [18], where the Banach space identity (9) is proved using Bohr’s Theorem 1.5 and going an (independently interesting) 'detour’ through bounded holomorphic functions on the open unit ball of $c_0$. In the following we indicate an argument for (9) which avoids this detour, and uses instead Corollary 6.2 (so a 'Helson-like argument') and Theorem 1.5 (first statement). This idea goes back to [10, Proof of Theorem 4.1].

A proof of (9) using no infinite dimensional holomorphy: From Corollary 6.2 we know that $H_\infty \subset D_\infty$, and $\|D\|_{H_\infty} \leq \|D\|_{D_\infty}$.

Conversely, take $D \in D_\infty$ with its associated sequence of Nth partial sums $D^N$, and look again for each $u > 0$ at the sequence $(f^N_u)$ in $H_\infty(T^\infty)$ which corresponds to the sequence $(D^N_u)$ under the Bohr transform. By (5) and Theorem 1.5 (first statement) this is a Cauchy sequence in the Banach space $H_\infty$ which has a limit $f_u$ satisfying $\|f_u\|_{H_\infty} \leq \|D\|_{D_\infty}$. Now recall that the unit ball of $L_\infty(T^\infty)$ endowed with its weak star topology is metrizable and compact. Hence $(f_1/n)_n$ has a weak star convergent subsequence with limit $f \in L_\infty(T^\infty)$ and $\|f\|_{H_\infty} \leq \|D\|_{D_\infty}$. Then a simple argument shows that $f \in H_\infty(T^\infty)$ and $B(f) = D$, which finishes the argument. □

7. Helson meets Cesàro

For $f \in H_1(T)$ is well-known that

\[ f(z) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n \leq k} \hat{f}(n) z^n = \lim_{N \to \infty} \sum_{n=0}^{N-1} \hat{f}(n) \left(1 - \frac{n}{x}\right) z^n \]

holds in the $H_1$-norm as well as pointwise almost everywhere on $T$. In other words, Cesàro means (arithmetic means) are perfectly adapted to the summation of Fourier series of integrable functions in one variable.

What about infinitely many variables? We know from Theorem 3.3 that for any $D \in H_1$ the sequence of Cesàro means

\[ \left( \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n \leq k} a_n \chi(n) n^{-s} \right)_{N \in \mathbb{N}} \]

for almost all $\chi \in \Xi$ converge on $[Re > 0]$, i.e. the Cesàro means of almost all vertical limits $D^N$ converge on $[Re > 0]$. And in Theorem 5.3 we even saw that for $D \in H_p, 1 < p < \infty$, almost all vertical limits converge on the imaginary axis. But since this is false in the case $p = 1$, the following question seems natural:

**Question:** Is it true that, given $D \in H_1$, the limits

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n \leq k} a_n \chi(n) n^{-it} \]
exist for almost all $\chi \in \Xi$ and $t \in \mathbb{R}$? Or, by Lemma 2.1 equivalently, do we for $f \in H_1(\mathbb{T}^\infty)$ have that

\begin{equation}
(15) \quad f(z) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{p^k \leq k} \hat{f}(\alpha) z^\alpha
\end{equation}

almost everywhere on $\mathbb{T}^\infty$?

Let us show, with an idea going back to Hardy and Riesz from [17, Thorem 21, p. 36], that the answer is negative. Indeed, for $f \in H_1(\mathbb{T}^\infty)$ and every $N$

$$\frac{1}{N} \sum_{k=0}^{N-1} \sum_{p^k \leq k} \hat{f}(\alpha) z^\alpha = \sum_{p^k < N} \hat{f}(\alpha) \left(1 - \frac{p^\alpha}{N}\right) z^\alpha.$$ 

Hence, if the answer to the above question is affirmative, then for every $f \in H_1(\mathbb{T})$ we get that

\begin{equation}
(16) \quad \lim_{N \to \infty} \sum_{2^j < N} \hat{f}(j) \left(1 - \frac{2^j}{N}\right) z^j = f(z) \quad \text{almost everywhere on } \mathbb{T},
\end{equation}

and a straight forward calculation shows

$$\sum_{2^j < 2^{N+1}} \hat{f}(j) z^j (2^{N+1} - 2^j) - \sum_{2^j < 2^N} \hat{f}(j) z^j (2^N - 2^j) = 2^N \left( \hat{f}(N) z^N + \sum_{j=0}^{N-1} \hat{f}(j) z^j \right).$$

Consequently,

$$\sum_{j=0}^{N-1} \hat{f}(j) z^j = 2 \sum_{2^j < 2^{N+1}} \hat{f}(j) z^j \left(1 - \frac{2^j}{2^{N+1}}\right) - \sum_{2^j < 2^N} \hat{f}(j) z^j \left(1 - \frac{2^j}{2^N}\right) - \hat{f}(N) z^N,$n

and so using (16) we conclude that for almost all $z \in \mathbb{T}$

$$\sum_{j=0}^{\infty} \hat{f}(j) z^j = f(z),$$

a contradiction.

8. Helson meets Hardy-Riesz

A proper substitute for Cesàro summation (even within the setting of general Dirichlet series) was already suggested by Hardy and Riesz in [17] where the first author writes: ...it appeared from the investigations of Riesz that these arithmetic means are not so well adapted to the study of the series as certain other means in a somewhat different manner. These 'logarithmic means', ..., have generalisations especially adapted to the study of general series $\sum a_n e^{-\lambda_n s}$. 
Let \( D = \sum a_n n^{-s} \in \mathcal{H}_1 \) and \( f \in H_1(T^\infty) \) be associated under the Bohr transform \( \mathfrak{B} \). Then the first Riesz mean (or logarithmic mean) of \( D \) of length \( x > 0 \) is given by the Dirichlet polynomial

\[
R_x(D) := \sum_{\log n < x} a_n \left(1 - \frac{\log n}{x}\right) n^{-s},
\]

and analogously the analytic polynomial

\[
R_x(f) = \mathfrak{B}^{-1}(R_x(D)) = \sum_{\log p^\alpha < x} \hat{f}(\alpha) \left(1 - \frac{\log p^\alpha}{x}\right) z^\alpha
\]
is the first Riesz mean (logarithmic mean) of length \( x > 0 \) of \( f \).

The following maximal inequality is the main result from [12, Theorem 2.1] (even for Riesz means of general Dirichlet series of arbitrary order \( k > 0 \)), and it rules the logarithmic summation of functions on the infinite dimensional torus and ordinary Dirichlet series. Compare this with Theorem 5.2 which handles usual summation – but only for \( p > 1 \).

Recall that the weak \( L_1(T^\infty) \)-space \( L_{1,\infty}(T^\infty) \) consists of all measurable functions \( f: T^\infty \to \mathbb{C} \) for which there is a constant \( C > 0 \) such that for all \( \alpha > 0 \)

\[
m \{ z \in T^\infty \mid |f(z)| > \alpha \} \leq \frac{C}{\alpha}.
\]

With \( \|f\|_{1,\infty} := \inf C \), the space \( L_{1,\infty}(T^\infty) \) becomes a quasi Banach space.

**Theorem 8.1.** The sub-linear operator

\[
T(f)(w) := \sup_{x > 0} |R_x(f)(w)| = \sup_{x > 0} \left| \sum_{\log p^\alpha < x} \hat{f}(\alpha) \left(1 - \frac{\log p^\alpha}{x}\right) z^\alpha \right|
\]
is bounded from \( H_1(T^\infty) \) to \( L_{1,\infty}(T^\infty) \), and from \( H_p(T^\infty) \) to \( L_p(T^\infty) \), whenever \( 1 < p \leq \infty \).

In the following section we briefly sketch the proof of this result which seems to have some facets of independent interest.

But again we first want to formulate a consequence on pointwise Cesàro summation of functions \( f \in H_1(T^\infty) \) as well as a Helson type theorem on logarithmic summation of vertical limits of ordinary Dirichlet series. From Theorem 8.1 and some standard arguments (which as mentioned above were formalized in [12, Lemma 3.6]) the next two results are given in [12, Corollary 2.2] and [12, Corollary 2.7].

**Corollary 8.2.** Let \( f \in H_1(T^\infty) \).

(1) For almost all \( z \in T^\infty \)

\[
\lim_{x \to \infty} \sum_{\log p^\alpha < x} \hat{f}(\alpha) \left(1 - \frac{\log p^\alpha}{x}\right) z^\alpha = f(z).
\]
There is a null set $N \subset T^\infty$ such that for all $u > 0$ and all $z \notin N$

\[
\lim_{x \to \infty} \sum_{n \leq x} \hat{f}(\alpha) e^{-u \log p^\alpha} \left(1 - \frac{\log p^\alpha}{x}\right) z^\alpha = f \ast p_u(z)
\]

Note that $f \ast p_u \in H_1(T^\infty)$ for all $u > 0$ and $f \in H_1(T^\infty)$, and hence by (1) we know that $\lim_{x \to \infty} R_x(f \ast p_u) = f \ast p_u$ almost everywhere. But the point of (2) is that the null set $N$ works for all $u$ simultaneously. In [12, Proposition 2.4] it is proved that $\lim_{u \to \infty} f \ast p_u = f$ almost everywhere, so (1) is definitely the border case of (2).

Lemma 2.1 transports these results on almost everywhere pointwise limit of Riesz means of functions on the infinite dimensional torus to almost everywhere pointwise convergence on the imaginary axis of almost all the vertical limits of their associated Dirichlet series.

**Corollary 8.3.** Let $D \in \mathcal{H}_1$ and $f \in H_1(T^\infty)$ its associated function under the Bohr transform. Then there is a null set $N \subset \Xi$ such that for all $\chi \notin N$

\begin{align*}
\lim_{x \to \infty} \sum_{n \leq x} a_n \chi(n) \left(1 - \frac{\log n}{x}\right) & = f(\chi(p)) \\
\lim_{x \to \infty} \sum_{n \leq x} a_n \chi(n) \left(1 - \frac{\log n}{x}\right) n^{-it} & = f(\frac{\chi(p)}{p^it}) \text{ for almost all } t \in \mathbb{R} \\
\lim_{x \to \infty} \sum_{n \leq x} a_n \chi(n) \left(1 - \frac{\log n}{x}\right) n^{-(u+it)} & = f_\chi \ast P_u(t) \text{ for all } u + it \in [Re > 0]
\end{align*}

9. Helson meets Hardy–Littlewood

Here we want to sketch the proof of Theorem 8.1, since we feel that it has some features which are independently interesting.

One of the central tools is given by the following Hardy-Littlewood maximal operator. If $f \in L_1(T^\infty)$, then we define for $z \in T^\infty$

\[
\mathcal{M}(f)(z) := \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \left| \int_I f\left(\frac{z}{p^it}\right) \, dt \right|
\]

where $I$ stands for any interval in $\mathbb{R}$ with Lebesgue measure $|I|$. Recall from (11) that $f\left(\frac{z}{p^it}\right) = f(\beta(t)z) = f_\chi(t)$ for almost all $z \in T^\infty$ defines a locally integrable function on $\mathbb{R}$, and so $\mathcal{M}(f)(z)$ is defined almost everywhere.

**Theorem 9.1.** The sublinear operator $\mathcal{M}$ is bounded from $L_1(T^\infty)$ to $L_{1,\infty}(T^\infty)$ and from $L_p(T^\infty)$ to $L_p(T^\infty)$, whenever $1 < p \leq \infty$.

The details of the proof are given in [12, Theorem 2.1], and it is not too surprising that the first part of the proof uses Vitali’s covering lemma, whereas the second part then follows applying the Marcinkiewicz interpolation theorem.
But before we discuss how to apply the preceding theorem, let us give the following direct consequence, which is of independent interest (see [12, Corollary 2.11], and compare the second equality with (8)).

**Corollary 9.2.** Let $f \in L_1(\mathbb{T}^\infty)$. Then for almost all $z \in \mathbb{T}^\infty$ we have

$$\lim_{T \to 0} \frac{1}{2T} \int_{-T}^{T} f\left(\frac{z}{p^t}\right) dt = f(z),$$

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f\left(\frac{z}{p^t}\right) dt = \int_{\mathbb{T}^\infty} f(w) dw.$$

The next theorem reduces the proof of Theorem 8.1 to Theorem 9.1.

**Theorem 9.3.** Let $f \in H_1(\mathbb{T}^\infty)$. Then for almost all $w \in \mathbb{T}^\infty$ have

$$T(f)(w) = \sup_{x > 0} \left| \sum_{\log p^v < x} \hat{f}(\alpha) \left(1 - \frac{\log p^v}{x}\right) z^\alpha \right| \leq C \mathcal{M}(f)(z),$$

where $C > 0$ is an absolute constant.

We finish this section indicating how to prove this inequality. The proof starts with two concrete integrals. The first integral follows from a standard application of Cauchy’s theorem (it is a particular case of [17, Lemma 10, p.50]),

$$(18) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{ys}}{s^2} ds = \begin{cases} y & \text{for } y \geq 0 \\ 0 & \text{for } y < 0 \end{cases} \text{ and } c > 0$$

whereas the second one is a consequence of an elementary calculation:

$$(19) \int_{\mathbb{R}} \frac{P_v(y-t)}{v^2 + y^2} dy = \frac{2}{4v^2 + t^2} \text{ for } t, v > 0.$$ 

From these two integrals we deduce some sort of Perron formula for logarithmic means.

**Proposition 9.4.** Let $f \in H_1(\mathbb{T}^\infty)$. Then there is a null set $N$ in $\mathbb{T}^\infty$ such that for all $z \notin N$ and for all $x > 0$

$$\frac{2\pi x}{e} R_x(f)(z) = \int_{\mathbb{R}} f_x(a) \mathcal{F}_{L_1(\mathbb{R})} \left( \frac{P_a(\cdot - a)}{1 + (\cdot - a)^2} \right) (-x) da,$$

where $\mathcal{F}_{L_1(\mathbb{R})}$ stands for the Fourier transform on $L_1(\mathbb{R})$. 
Proof. Let us first assume that \( f = \sum_{p^\alpha < N} \hat{f}(\alpha)z^\alpha \in H_1(\mathbb{T}^\infty) \). Then by (18) for \( c > 0 \) and \( z \in \mathbb{T}^\infty \)

\[
e^{-xc} \int_{\mathbb{R}} f_z(a) \int_{\mathbb{R}} \frac{P_z(t-a)}{(c+it)^2} e^{itz} dt \, da = e^{-xc} \int_{\mathbb{R}} \frac{\sum_{\log p^\alpha < x} \hat{f}(\alpha)z^\alpha e^{-\log p^\alpha (c+it)}}{(c+it)^2} e^{itz} dt
\]

and the choice \( c = 1/x \) leads to the conclusion. To prove this for arbitrary \( f \in H_1(\mathbb{T}^\infty) \), observe that for all \( v > 0 \) the operator

\[
A : L_1(\mathbb{T}^\infty) \rightarrow L_1(\mathbb{T}^\infty, L_1(\mathbb{R})), \quad f \mapsto \left[z \mapsto \frac{f_z \ast P_v}{(v+\cdot)^2}\right]
\]

is bounded. Indeed, by (19) and Fubini’s theorem for every \( f \in L_1(\mathbb{T}^\infty) \)

\[
\|A(f)\|_1 = \int_{\mathbb{T}^\infty} \left| \int_{\mathbb{R}} f_z \ast P_v(y) \frac{dy}{(v+iy)^2} \right| dz \\
\leq \int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f_z(t)| \int_{\mathbb{R}} \frac{P_v(y-t)}{v^2 + y^2} dy dt dz \\
\leq \int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f_z(t)| \frac{2}{4v^2 + t^2} dt dz = C_1(v)\|f\|_1.
\]

Additionally, this shows, that for some null set \( N \) in \( \mathbb{T}^\infty \) we have that \( \frac{f_z \ast P_v}{(v+\cdot)^2} \in L_1(\mathbb{R}) \) for all \( z \notin N \), and so we in particular deduce that for \( z \notin N \) and \( x > 0 \)

\[
F_{L_1(\mathbb{R})} \left( \frac{f_z \ast P_{\frac{x}{z}}}{(\frac{x}{z} + \cdot)^2} \right) (-x) = \int_{\mathbb{R}} f_z(a) F_{L_1(\mathbb{R})} \left( \frac{P_{\frac{x}{z}}(\cdot - a)}{(\frac{x}{z} + \cdot)^2} \right) (-x) da.
\]

Now let \( (Q^n) \) be a sequence of polynomials in \( H_1(\mathbb{T}^\infty) \) converging to \( f \) in \( H_1(\mathbb{T}^\infty) \) (see e.g. [13, Proposition 5.5]). Then, by the continuity of \( A \) and \( F_{L_1(\mathbb{R})} \), we for all \( z \notin N \) obtain some subsequence \( (Q^{n_k}) \) such that under uniform convergence on \( \mathbb{R} \)

\[
F_{L_1(\mathbb{R})} \left( \frac{f_z \ast P_{\frac{x}{z}}}{(\frac{x}{z} + \cdot)^2} \right) = \lim_{k \to \infty} F_{L_1(\mathbb{R})} \left( \frac{Q^{n_k} \ast P_{\frac{x}{z}}}{(\frac{x}{z} + \cdot)^2} \right).
\]

So, knowing that the claim holds true for polynomials, by (21) for all \( z \notin N \) and \( x > 0 \)

\[
F_{L_1(\mathbb{R})} \left( \frac{f_z \ast P_{\frac{x}{z}}}{(\frac{x}{z} + \cdot)^2} \right) (-x) = \lim_{k \to \infty} F_{L_1(\mathbb{R})} \left( \frac{Q^{n_k} \ast P_{\frac{x}{z}}}{(\frac{x}{z} + \cdot)^2} \right) (-x)
\]

\[
= \frac{2\pi x}{e} \lim_{k \to \infty} R_x(Q^{n_k})(z) = \frac{2\pi x}{e} R_x(f)(z),
\]

which looking again at (21) finishes the argument. \( \square \)

Finally, we may combine all this to get the
Proof of Theorem 9.3. For $a \in \mathbb{R}$ and $x > 0$ we define

$$K(a) := \frac{1}{|1 + ia|^2} \text{ and } K_x(a) := xK(ax) = \frac{x}{|1 + iax|^2}.$$  

Then with Proposition 9.4 and (19) we obtain for almost all $z \in \mathbb{T}^\infty$ and $x > 0$

$$|R_x(f)(z)| \leq \frac{C_1}{x} \int_{\mathbb{R}} |f_z(a)| \left\| \frac{P_a(-a)}{(\frac{1}{x} + ia)^2} \right\|_{L_1(\mathbb{R})} \, da \leq \frac{C_2}{x} \int_{\mathbb{R}} |f_z(a)| \frac{1}{|\frac{1}{x} + ia|^2} \, da = C_3 \int_{\mathbb{R}} |f_z(a)| K_x(a) \, da = C_3(|f_z| * K_x)(0).$$

Now by [16, Theorem 2.1.10, p.91] we have for almost all $z \in \mathbb{T}^\infty$

$$\sup_{x>0} |f_z| * K_x(0) \leq \|K\|_{L_1(\mathbb{R})} \sup_{T>0} \frac{1}{2T} \int_{-T}^T |f_z(t)| \, dt \leq \|K\|_{L_1(\mathbb{R})} \overline{M}(f)(z),$$

which then all in all gives the conclusion. \hfill \Box

10. Helson meets Banach

A sequence $(x_n)$ in a Banach space $X$ is a Schauder basis whenever every $x \in X$ has a unique series representation $x = \sum_{n=1}^{\infty} a_n x_n$. For $H_p$'s of Dirichlet series the following result was first realized in [2].

**Theorem 10.1.** Let $1 < p < \infty$. Then for every $D = \sum a_n n^{-s} \in H_p$

$$D = \lim_{x \to \infty} \sum_{n=1}^{x} a_n n^{-s}$$

converges in $H_p$, i.e. the Dirichlet series $n^{-s}, n \in \mathbb{N}$, form a Schauder basis of $H_p$. Equivalently, for every $f \in H_p(\mathbb{T}^\infty)$

$$f = \lim_{x \to \infty} \sum_{p^\alpha < x} \hat{f}(\alpha) z^{\alpha}$$

converges in $H_p(\mathbb{T}^\infty)$. In other words, the monomials $z^{\alpha}$ ordered as in (6) (i.e. $z^{\alpha} \leq z^{\beta} :\iff \log p^\alpha \leq \log p^\beta$) form a Schauder basis of $H_p(\mathbb{T}^\infty)$.

An equivalent formulation of all this is that for $1 < p < \infty$ all projections

$$S^p_x : H_p \to H_p, \sum a_n n^{-s} \mapsto \sum_{n < x} a_n n^{-s}$$

are uniformly bounded, and this is an immediate consequence of Theorem 5.2. As explained in Section 7 the proof of Theorem 5.2 is based on the deep Carleson-Hunt Theorem 1.4.

But let us remark that Theorem 10.1 is also an almost straightforward consequence of Rudin’s work from [24, Theorem 8.7.2]. There Rudin proves (as a
particular case of a more general result on compact abelian groups with ordered duals, see (6) for the order on \( \hat{T}^\infty \) that the so-called Riesz projection

\[ L_2(\hat{T}^\infty) \to L_2(\hat{T}^\infty), \quad f \mapsto \sum_{\alpha \geq 0} \hat{f}(\alpha) z^\alpha \]

for every \( 1 < p < \infty \) extends to a bounded operator on \( L_p(\hat{T}^\infty) \). For the border cases \( p = \infty \) and \( p = 1 \) this is false. Indeed, from Theorem 1.5 one may deduce that

\[ \log \log x \ll \| S^\infty_x : H_\infty \to H_\infty \| \ll \log x, \]

whereas a very recent estimate from [6] states that

\[ \log \log x \ll \| S^1_x : H_1 \to H_1 \| \ll \frac{\log x}{\log \log x} \]

(in both cases the lower estimates are simple consequences of the one variable case and Bohr’s transform from (7)).

Finally, we ask a question for the Banach spaces \( H_1 \) and \( H_1(\hat{T}^\infty) \) which is analog to the one we posed in Section 7.

**Question:** Is it true that for all \( D \in H_1 \) we have that

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n \leq k} a_n n^{-s} = D \quad \text{in} \quad H_1, \]

or, equivalently, do we for \( f \in H_1(\hat{T}^\infty) \) have that

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{p^\alpha \leq k} \hat{f}(\alpha) z^\alpha = f \quad \text{in} \quad H_1(\hat{T}^\infty)? \]

The answer is again no. Otherwise the same argument as in Section 7 would show that the Fourier series of every \( f \in H_1(\hat{T}) \) converges in \( H_1(\hat{T}) \) which we know is false.

**References**

[1] G. Alexits: *Convergence problems of orthogonal series*, International Series of Monographs in Pure and Applied Mathematics, Vol. 20, Pergamon Press, New York (1961)
[2] A. Aleman, J.F. Olsen, and E. Saksman: *Fourier multipliers for Hardy spaces of Dirichlet series*, International Mathematics Research Notices (16) (2014) 4368-4378
[3] F. Bayart: *Hardy spaces of Dirichlet series and their composition operators*, Monatsh. Math. 136(3) (2002) 203-236
[4] F. Bayart: *Opérateurs de composition sur des espaces de séries de Dirichlet et problèmes d’hypercyclicité simultanée*, PhD thesis, Université des Sciences et Technologie Lille, Lille, France, 2002
[5] H. F. Bohnenblust and E. Hille: *On the absolute convergence of Dirichlet series*, Ann. of Math. 32(3) (1931) 600-622
[6] A. Bondarenkov, O. Brevig, E. Saksman, and K. Seip: *Linear properties of \( H^p \) spaces of Dirichlet series*, arXiv:1801.06515v2
[7] A. S. Besicovitch: *Almost periodic functions*, Dover publications (1954)
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[8] H. Bohr, Über die gleichmäßige Konvergenz Dirichletscher Reihen, J. Reine Angew. Math. 143 (1913) 203-211
[9] H. Bohr: Über die Bedeutung der Potenzreihen endlich vieler Variabeln in der Theorie der Dirichletschen Reihen \( \sum \frac{a_n}{x}\), Nachr. Ges. Wiss. Gött. Math. Phys. Kl. 4 (1913) 441-488
[10] A. Defant and I. Schoolmann: \( H_p \)-theory of general Dirichlet series, preprint 2019
[11] A. Defant and I. Schoolmann: On a theorem for Helson for general Dirichlet series, preprint 2019
[12] A. Defant and I. Schoolmann: Riesz means in Hardy spaces on Dirichlet groups, preprint 2019
[13] A. Defant, D. García, M. Maestre, and P. Sevilla Peris: Dirichlet series and holomorphic functions in high dimensions, to appear in: New Mathematical Monographs Series, Cambridge University Press 2019
[14] T.K. Duy: On convergence of Fourier series of Besicovitch almost periodic functions, Lithuanian Math. J. 53,3 (2013) 264-279
[15] C. Fefferman: On the convergence of multiple Fourier series, Bull. Amer. Math. Soc. 77 (1971) 744-745
[16] L. Grafakos: Classical Fourier analysis, Graduate Texts in Mathematics 249, (2014)
[17] G. H. Hardy and M. Riesz: The general theory of Dirichlet series, Cambridge Tracts in Mathematics and Mathematical Physics 18 (1915)
[18] H. Hedenmalm, P. Lindqvist, and K. Seip: A Hilbert space of Dirichlet series and systems of dilated function in \( L^2(0,1) \), Duke Math. J. 86 (1997) 1-37
[19] H. Hedenmalm and E. Saksman: Carleson’s convergence theorem for Dirichlet series, Pacific J. of Math. 208 (2003) 85-109
[20] H. Helson: Foundations of the modern theory of Dirichlet series, Acta Math. 118 (1967) 61-77
[21] H. Helson: Compact groups and Dirichlet series, Ark. Mat. 8 (1969) 139-143
[22] H. Helson: Dirichlet series: Regent Press, 2005
[23] H. Queffélec and M. Queffélec: Diophantine approximation and Dirichlet series, Hindustan Book Agency, Lecture Note 2 (2013)
[24] W. Rudin, Fourier analysis on groups, Interscience Publishers (1962)
[25] I. Schoolmann: On Bohr’s theorem for general Dirichlet series, preprint 2019

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