On the Possibility of Reflectionless Propagation of Plane Acoustic Waves in Continuously Stratified Media

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Abstract—The example of vertical propagation of acoustic plane waves in the atmosphere is used to establish that the reflectionless propagation of acoustic waves occurs only in continuously stratified media for which wave equations with variable pressure perturbation and vibrational velocity coefficients are reduced—via the same transformation for the same wave acoustic impedance profile, which is inversely proportional to the refractive index—to a wave comparison equation with constant coefficients. It is shown that the corresponding transformations of the wave equations are possible only for continuously stratified media with a constant wave acoustic impedance.

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INTRODUCTION

A quite detailed analysis of the results of studies on the problems considered in this article is presented in [1–5]. The authors of these papers believe they have proposed a quite new approach to describing the processes of reflectionless wave propagation in continuously stratified media: for any physical quantity, the initial wave equation with variable coefficients is, by certain transformations, reduced to the corresponding wave equation with constant coefficients. In this case, stratification of the wave propagation velocity must satisfy a certain ordinary differential equation, which is characteristic of the specific physical quantity under study, which satisfies the initial wave equation. After such transformations, using the causality principle as applied to the transformed wave equation with constant coefficients, the authors of [1–5] formulate a conclusion about the unidirectional, reflectionless nature of wave propagation, in which in a medium with the corresponding stratification, waves propagate in opposite directions from the source, independently of each other (i.e., without mutual influence).

If the approach proposed in [1–5] and the results obtained with its use are valid, then it should be expected that, e.g., for strictly vertical propagation of a plane pulsed acoustic wave in the atmosphere with a reflectionless sound velocity profile (see [2–5]), determined by the appropriate transformation of the wave equation for pressure perturbation, the propagation of both the vibrational velocity wave and its corresponding vibrational velocity divergence wave will also be reflectionless.

However, it follows from the results in [2–5] that for reflectionless propagation of a pulsed vibrational velocity wave in the atmosphere, the ordinary differential equation for the corresponding sound velocity profile significantly differs from the analogous equation for the sound velocity profile corresponding to the reflectionless propagation of the vibrational velocity divergence wave. From these two different differential equations (see [2–5]), one can quite naturally obtain essentially different solutions for the dependences of the sound velocity on the vertical coordinate (see [2–5]). At the same time, from fairly general physical considerations, they should obviously be the same for the same reflectionlessly propagating acoustic wave, since it should correspond to simultaneously reflectionlessly propagating vibrational velocity and vibrational velocity gradient waves, as well as pressure and density perturbations.

In addition, it is also important to note here that, earlier in [6], in order to obtain generalized WKB solutions, a more general method than used in [1–5] was applied for transforming second-order differential equations with variable coefficients to characteristic differential comparison equations [7], including those with constant coefficients (see [6]); at the same time, in [6], no reasonable were drawn conclusions about any reflectionless wave propagation.

Hence follows the obvious need for further research to determine the correct conditions for the realization of reflectionless processes of acoustic wave propagation in continuously stratified media.
The present work is devoted to just such research; it has, in a certain sense, a noticeably expressed methodological nature with a quite detailed presentation of relevant material, since the questions it raises have been discussed over a very significant time period in studies on the theory of wave processes in various media (see §6.3 in [8] and references therein).

**MEDIA WITH CONSTANT WAVE ACOUSTIC IMPEDANCE**

To simplify analytical calculations and analyze the results, we consider, as in [2–5], true-vertical (along the direction of the z) propagation of plane acoustic waves in a continuously stratified medium, but, in contrast to [2–5], ignoring the effect of gravity. In this case, from two equations (see §1.1 in [8])

\[
\frac{\partial v}{\partial t} = -\frac{1}{\rho} \frac{\partial p'}{\partial z} = \frac{1}{\rho c^2} \frac{\partial^2 p'}{\partial t^2},
\]

in which \(v\) is the vibrational velocity, \(p'\) is pressure perturbation in an acoustic wave, \(\rho(z)\) and \(c(z)\) are the equilibrium density and sound speed in the medium, \(t\) is time, and \(z\) is the vertical coordinate, we find the closed equations for \(p'\) and \(v\):

\[
\frac{\partial^2 p'}{\partial t^2} - \rho c^2 \frac{\partial}{\partial z} \left( \frac{1}{\rho} \frac{\partial p'}{\partial z} \right) = 0, \tag{2}
\]

\[
\frac{\partial^2 v}{\partial t^2} - \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho c^2 \frac{\partial v}{\partial z} \right) = 0. \tag{3}
\]

If we introduce a new spatial coordinate (see §1.2 in [8])

\[
\xi = \frac{1}{\rho_0} \int_{z_0}^z \rho(z)dz, \tag{4}
\]

then the equations (2), (3) are brought to the following form:

\[
\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - a^2 \frac{\partial^2 p'}{\partial \xi^2} = 0, \tag{5}
\]

\[
\frac{1}{c_0^2} \frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial \xi} \left( a^2 \frac{\partial v}{\partial \xi} \right) = 0. \tag{6}
\]

In expressions (4)–(6) \(\rho_0 = \rho(z_0)\) and \(c_0 = c(z_0)\) are the values of the equilibrium density and sound speed at an arbitrarily chosen boundary \(z = z_0\);

\[
a = \rho c / \rho_0 c_0 \tag{7}
\]

is a value characterizing the relative change in the wave acoustic impedance of the medium.

For a constant wave acoustic impedance of the medium (7), i.e., at \(a(\xi) \equiv 1\) for \(-\infty < \xi < +\infty\), equations (5), (6) are reduced to the same wave equation with constant coefficients. The latter means that if \(z = z_0 (\xi = 0)\), the perturbation pulse of pressure \(p' = p_0 \Phi(t, \xi = 0)\) or vibrational velocity \(v = v_0 \Phi(t, \xi = 0)\) is given, then the corresponding solutions to equations (5), (6) will describe propagation in opposite directions without the mutual influence of both pulse pressure waves and vibrational velocity waves:

\[
p'(t, \xi) = p_0^{(+)} \Phi \left( t - \frac{\xi}{c_0} \right) + p_0^{(-)} \Phi \left( t + \frac{\xi}{c_0} \right), \tag{8}
\]

\[
v(t, \xi) = v_0^{(+)} \Phi \left( t - \frac{\xi}{c_0} \right) + v_0^{(-)} \Phi \left( t + \frac{\xi}{c_0} \right). \tag{9}
\]

In expressions (8), (9) \(p_0, p_0^{(+)}, p_0^{(-)}\) and \(v_0, v_0^{(+)}, v_0^{(-)}\) are constant amplitude factors with the corresponding dimension; \(p_0 = p_0^{(+)} + p_0^{(-)}, v_0 = v_0^{(+)} + v_0^{(-)}\). Consequently, for a constant wave acoustic impedance of a continuously stratified medium, the reflectionless propagation of acoustic waves is realized in it, in which the spatiotemporal profiles of pressure and vibrational velocity waves (8) remain unchanged; they are also described by the same functions, which differ from each other only (which is fundamentally important) by constant amplitude factors (see (9)).

Let us now show that the presence of any dependence \(a(\xi) \neq \text{const}\) leads to reflections during the propagation of an acoustic wave and, thus, to changes in the spatiotemporal profiles of pressure and vibrational velocity waves due to the natural manifestation of topological dispersion.

For this, we use the Fourier transforms

\[
p'(t, \xi) = \int_{-\infty}^{+\infty} p(\omega, \xi) e^{i\omega t} d\omega, \quad v(t, \xi) = \int_{-\infty}^{+\infty} v(\omega, \xi) e^{i\omega t} d\omega, \tag{10}
\]

with the help of which, in addition to the expressions (4), from (1), we obtain for the dimensionless quantities

\[
\Pi = i \bar{\rho}(\omega, \xi) \bar{c}_0^2, \quad U = \tau(\omega, \xi) / \bar{c}_0 \tag{11}
\]

the following equations:

\[
U = 1 / k_0 d\Pi/d\xi, \quad dU/d\xi = -k_0 n^2 \Pi. \tag{12}
\]

Now, just like in [8] (see §8.3), we find the solutions to system of equations (12) in a form convenient...
for comparative analysis with the geometrical acoustic approximation:

$$
\Pi = \frac{1}{\sqrt{n}} \left( \alpha_1 \exp \left\{ -i k_0 \int_0^\xi n d\xi \right\} + \alpha_2 \exp \left\{ i k_0 \int_0^\xi n d\xi \right\} \right),
$$

$$
U = -i \sqrt{n} \left( \alpha_1 \exp \left\{ -i k_0 \int_0^\xi n d\xi \right\} - \alpha_2 \exp \left\{ i k_0 \int_0^\xi n d\xi \right\} \right),
$$

where $n = 1/a(\xi)$ is the effective refractive index of acoustic waves, $\alpha_1$ and $\alpha_2$ are functions to be defined. Substituting representation (13) into Eqs. (12), we obtain the following system of equations to determine coefficients $\alpha_1$ and $\alpha_2$:

$$
\frac{d\alpha_1}{d\xi} = \frac{dn/d\xi}{2n} \alpha_2 \exp \left\{ 2i k_0 \int_0^\xi n d\xi \right\},
$$

$$
\frac{d\alpha_2}{d\xi} = \frac{dn/d\xi}{2n} \alpha_1 \exp \left\{ -2i k_0 \int_0^\xi n d\xi \right\}.
$$

As seen from (13), (14), it is for a constant value of the wave acoustic impedance of the medium ($dn/d\xi = 0$) that acoustic waves propagate in opposite directions without mutual influence, which is a consequence of the absence of their continuous reflection by the corresponding inhomogeneities $p(x)$ and $c(z)$.

Since only in this case do the coefficients $\alpha_1$ and $\alpha_2$ take constant values (see (14)), from (10), (11), (13) it follows that only for reflectionless propagation of acoustic perturbations do the spatiotemporal profiles of pressure and vibrational velocity waves remain unchanged. Obviously, if $a(\xi) \neq \text{const}$, then in the case of sufficient smallness of the quantity $(dn/d\xi)/n \ll 1$ in (14) in the corresponding zero, geometrical acoustic approximation for $\alpha_1 = \text{const}$ and $\alpha_2 = \text{const}$ expressions (10), (11), (13) also describe the propagation of pulse pressure and vibrational velocity waves without changes in their spatiotemporal profiles, but with varying amplitudes and propagation times. Thus, the reflectionless sound velocity profiles are determined by the following from (7) for the $a = 1$ obvious equality $c(z) = c_0 n_0/p(z)$, and, as noted in [8] (see §6.3), in this case, the absence of reflection for all frequencies $0 < \omega < \infty$ is caused by the fact that it does not result from wave interference.

From all of the above, it follows that determining the reflectionless profiles of wave acoustic impedance in continuously stratified media requires a joint, consistent analysis of wave equations with variable coefficients for pressure and vibrational velocity perturbation in an acoustic wave. In this case, the corresponding equations should be reduced by the same transformation and the same profile of the wave acoustic impedance (which is reflectionless) to a comparison wave equation with constant coefficients, since a reflectionless propagating acoustic wave should correspond (“simultaneously”) to reflectionless propagating pressure and vibrational velocity waves. Naturally, the acoustic reflection coefficient $V$ determined with the continuity conditions for quantities $p'(\xi, t)$ and $v(\xi, t)$ at an arbitrary depth $\xi$ will take a zero value only for a constant wave acoustic impedance of the medium.

In concluding this section, it seems important to note that when plane waves propagate from a homogeneous medium into a medium with a certain anisotropy of its acoustic characteristics, there may be no reflection in the entire admissible range of angles of incidence (see [10] and §6.3 in [8]).

**ANALOG OF THE REFERENCE EQUATION METHOD FOR DETERMINING REFLECTIONLESS WAVE ACOUSTIC IMPEDANCE PROFILES**

As noted in the Introduction, it follows from analysis of the results obtained in [1–5] that even with a changing value of the wave acoustic impedance of the medium $a(z) \neq \text{const}$, certain dependences $a(\xi) = a_p(\xi)$ and $a(\xi) = a_v(\xi)$ can exist for which reflectionless propagation of acoustic waves is realized for all radiation frequencies. Since such a conclusion clearly contradicts the results in the previous section, below, we consider the existence of the corresponding dependences $a_p(\xi)$ and $a_v(\xi)$, which should correspond to the correct (finite values with physical meaning) solutions for $p'(t, \xi)$ and $v(t, \xi)$ describing the reflectionless propagation of pressure and vibrational velocity waves simultaneously for each of the two dependences $a_p(\xi)$ and $a_v(\xi)$.

To this end, in analogy with [6–8], to solve the formulated problem, we use the well-known method of comparing equations (5), (6) characteristic of their model comparison equations. Since in this case it is of interest to solve equations (5), (6) describing for $a(\xi) \neq \text{const}$ only reflectionless acoustic wave propagation over the entire radiation frequency range, the model for (5) and (6) will obviously be a wave equation with constant coefficients [8]. Quite naturally, the corresponding transformations of equations (5), (6) to such a model equation can be implemented under certain conditions that can be satisfied only for the selected specific dependences $a_p(\xi) = a_p(\xi)$ and $a_v(\xi) = a_v(\xi)$.

Just like in [1–5], when determining these dependences $a_p(\xi)$ and $a_v(\xi)$, we use the representation of solutions to equations (5) and (6) in the following form:

$$
p'(t, \xi) = p_0 A(\xi) \Psi(t, \tau(\xi)),
$$

$$
v(t, \xi) = v_0 B(\xi) \Phi(t, \tau(\xi)),
$$

for comparative analysis with the geometrical acoustic approximation.
in which the variables $\tau(\xi)$, $A(\xi)$, and $B(\xi)$ should satisfy certain conditions so that functions $\Psi(t, \tau(\xi))$ and $\Phi(t, \tau(\xi))$ are the solutions to the corresponding wave equations with constant coefficients:

\[ \frac{1}{c_0^2} \frac{d^2 \Psi}{dt^2} - \frac{d^2 \Psi}{\xi^2} = 0, \]  
\[ \frac{1}{c_0^2} \frac{d^2 \Phi}{dt^2} - \frac{d^2 \Phi}{\xi^2} = 0. \]  

Using the relations following from (5), (15), (17)

\[ \tau = \int_0^\xi \frac{d\xi}{a(\xi)}, \quad A(\xi) = \sqrt{\frac{a(\xi)}{2}}, \]  

we find the sought dependence

\[ a(\xi) = a_p(\xi) = (1 + \xi/H)^2, \]  

in which $|H|$ is the characteristic scale of the inhomogeneity of the medium.

Thus, taking into account (1), (4) and the causality principle, as well as the dependences obtained above (19), (20) the solutions describing the propagation of acoustic pressure and vibrational velocity waves can be represented in quite simple analytical form:

\[ p'(t, \xi) = p_+ \left( \eta_p^{(+)} \right) + p_- \left( \eta_p^{(-)} \right), \]  
\[ p_+ \left( \eta_p^{(+)} \right) = p_0^{(+)} \sqrt{a_p} \Psi \left( \eta_p^{(+)} \right), \]  
\[ p_- \left( \eta_p^{(-)} \right) = p_0^{(-)} \sqrt{a_p} \Psi \left( \eta_p^{(-)} \right); \]  
\[ v(t, \xi) = \frac{1}{\rho_0 c_0 a_p} \times \left\{ p_+ \left( \eta_p^{(+)} \right) - p_- \left( \eta_p^{(-)} \right) - \frac{c_0}{2} \int p'(t, \xi) dt \right\}. \]  

Here,

\[ \eta_p^{(+)} = t - \tau_p/c_0, \quad \eta_p^{(-)} = t + \tau_p/c_0, \]  
\[ \tau_p = \xi/(1 + \xi/H). \]  

Similarly, using the relations following from (6), (16), (18)

\[ \tau = \int_0^\xi \frac{d\xi}{\sqrt{a(\xi)}}, \quad B(\xi) = 1/\sqrt{a(\xi)}, \]  

we find the dependence significantly differing from $a(\xi) = a_p(\xi)$ (20)

\[ a(\xi) = a_s(\xi) = (1 + \xi/H)^{2/3}. \]  

Therefore, taking into account (1), (4) and the causality principle, as well as the obtained dependences (24), (25) the solutions describing the propagation of acoustic vibrational velocity and pressure waves can be represented in an analytical markedly differing from the solutions (21), (22):

\[ v(t, \xi) = v_+ \left( \eta_v^{(+)} \right) + v_- \left( \eta_v^{(-)} \right), \]  
\[ v_+ \left( \eta_v^{(+)} \right) = \frac{\eta_v^{(+)} - \eta_v^{(-)}}{\sqrt{a_v}}, \]  
\[ v_- \left( \eta_v^{(-)} \right) = \frac{\eta_v^{(+)} - \eta_v^{(-)}}{\sqrt{a_v}}; \]  
\[ p'(t, \xi) = \rho_0 c_0 a_c \times \left\{ v_+ \left( \eta_v^{(+)} \right) - v_- \left( \eta_v^{(-)} \right) + \frac{c_0}{2} \int v(t, \xi) dt \right\}. \]  

Here

\[ \eta_v^{(+)} = t - \tau_v/c_0, \quad \eta_v^{(-)} = t + \tau_v/c_0, \]  
\[ \tau_v = 3H \left\{(1 + \xi/H)^{1/3} - 1 \right\}. \]  

Thus, analysis of the equation (5) (see (15), (17)), which describes the propagation of acoustic pressure waves, leads (see (19), (20)) to the possibility of their reflectionless propagation in a medium with varying wave acoustic impedance, only with a certain dependence $a(\xi) = a_p(\xi)$ (20). At the same time, analysis of the equation (6) (see (16), (18)), which describes the propagation of acoustic vibrational velocity waves leads to the possibility of their reflectionless propagation in a medium with varying wave acoustic impedance with its dependence $a(\xi) = a_s(\xi)$ (25), which differs significantly from $a(\xi) = a_p(\xi)$ (20). However, as shown in the previous section, if an acoustic wave propagates without reflections in a continuously stratified medium, then the mode of reflectionless propagation should be simultaneously inherent to the pressure perturbation wave and corresponding vibrational velocity wave for the same dependence of the wave acoustic impedance of the medium $a(\xi)$ for which the equations (5) and (6) with the relevant transformations (15) and (16) should reduce to model comparison equations (17) and (18). Therefore, in the situation considered here, it is necessary to require fulfillment of the obvious equality $a_v(\xi) = a_s(\xi)$, which takes place only when $|H| = 0$, i.e., in a continuously stratified medium with a constant wave acoustic impedance, in which only reflectionless propagation of acoustic waves is possible in the entire range of radiation frequencies.

Let us now analyze what processes actually describe the solutions obtained using the approach proposed in [1–5] (21), (22) and (26), (27), which, as is shown in the next section, is only a special case of the relatively long-known method for finding generalized WKB solutions [6, 11]. In this case, attention should be paid to the following main regularities.
First, in a medium with the dependence of the wave acoustic impedance \( a_p(\xi) \) (20), the shape of the pressure wave profile remains self-similar (in analogy with the self-similar solution), since only their amplitude and propagation time change (see Fig. (21)). Moreover, if the asymptotic behavior \( \xi \to \infty \) is valid for \( z \to \infty \), then, as follows from the expression for \( \tau(\xi) \) (23), the pulse pressure wave propagating upward over a finite time \( t = t_0 = H/c_0 \) \( (H > 0) \) will reach the infinitely distant turning horizon \( \xi = \infty \), after reflection from which it will return within the same time \( t = t_0 \) to the initial horizon \( \xi = 0 \). The shape of the vibrational velocity wave profile is not self-similar, and the corresponding differences increase at \( H > 0 \) (see (22)). In the downward departing wave at \( H > 0 \), the differences in the shape of the vibrational velocity profile from the self-similar one will decrease; however, such a wave will not be reflected from the infinitely distant turning horizon \( \xi = -\infty \) because it cannot even reach the horizon \( \xi = -H \) due to the fact that \( \tau_p \to \infty \) for \( \xi \to -H \) (see (23)). It is of interest to note that such situations, characterized by an increase in the wave propagation time to infinity in a limited spatial region of the medium, occur in certain technical devices, the wave processes in which were first described in [12–15] using the WKB approximation and exact analytical solutions. Clearly, for \( H < 0 \) (20) upwardly and downwardly propagating acoustic waves will exhibit the patterns described above for acoustic waves propagating down and up, respectively, for \( H > 0 \) in (20).

Second, in a medium with the dependence of the wave acoustic impedance \( a_v(\xi) \) (25), the shape of the vibration velocity wave profile remains self-similar (see Fig. (26)); the differences in the shape of the profile of an upwardly propagating pressure wave from a self-similar one decrease at \( H > 0 \) (see (27)), and such a wave does not return to the initial horizon \( \xi = 0 \), since \( \tau \to \infty \) for \( \xi \to \infty \) (see (28)). However, a downwardly propagating acoustic wave in which the differences in the shape of the pressure perturbation profile increase at \( H > 0 \) (see (27)) will experience reflection when within a time \( t = 3H/c_0 \) it reaches the horizon \( \xi = -H \), where the derivative \( da_v/\xi \) undergoes a discontinuity. Naturally, when \( H < 0 \) in \( a_v(\xi) \) (25) upwardly and downwardly propagating acoustic waves will exhibit the patterns described above for downwardly and upwardly propagating acoustic waves, respectively, for \( H > 0 \) at \( a_v(\xi) \) (25).

From what has been said about the behavior of solutions (21), (22) and (26), (27) it follows that they do not describe reflectionless acoustic waves propagation for \( a(\xi) \neq \text{const} \); nevertheless, these solutions are of obvious interest, since in an explicit analytical form in the class of elementary functions they describe traveling waves in which the shape of the wave profile of either the pressure perturbation or vibrational velocity remains self-similar, but only for certain dependences of the wave acoustic impedance of the medium \( a_p(\xi) \) (20) or \( a_v(\xi) \) (25), respectively.

In concluding this section, it is important to note the following. In [16], where the transformation of long-wavelength surface waves was studied in a zone of variable depth, the reflectionless profile of which \( h_\theta(x) \) was determined using the approach proposed in [1–5], analysis of the wave equation only for elevation \( \eta(x,t) \) of the water surface established the existence, nevertheless, of the distributed nature of reflection of a pulse wave from the bottom relief with the corresponding reflectionless dependence of the depth of the water layer \( h_\theta(x) = h_\theta(1 + x/H)^{4/3} \) on the horizontal distance \( x \geq 0 \) \( (H > 0) \). Based on the above results (see (20)–(23) and (25)–(28)), this statement in [16], which contradicts the conclusions of [1–5], is quite natural. Indeed, using an approach similar to [1–5] in analyzing the wave equation for the horizontal velocities of liquid particles \( w(x,t) \) in a surface wave, one can obtain the reflectionless dependence \( h_\theta(x) = h_\theta(1 + x/H)^{4/3} \) significantly different from \( h_\theta(x) \). Since the reflection coefficient \( V \) of surface waves is determined (in analogy with acoustic waves) using the continuity conditions for quantities \( \eta(x,t) \) and \( w(x,t) \) (in this case for \( x = 0 \), see [16]), it is quite obvious that its value will differ from zero for \( h(x) = h_\theta(x) \) or \( h(x) = h_\theta(x) \) with natural asymptotic behavior \( V \to 0 \) for \( H \to \infty \).

**GENERALIZED WKB METHOD FOR DETERMINING REFLECTIONLESS WAVE ACOUSTIC IMPEDANCE PROFILES**

As shown in [8] (see §3.1 and §6.3), with correct formulation of the problem of finding reflectionless profiles of the refractive index of acoustic waves, in this case \( n(\xi) = 1/a(\xi) \), when solving equations (5), (6), it is necessary to specify the correct boundary conditions for \( \xi \to \pm \infty \) for quantities \( p'(t,\xi) \) and \( v(t,\xi) \), which must take finite values in the entire domain of definition \( -\infty < \xi < +\infty \). Therefore, in analogy with [8] (see §6.3), it is necessary to assume that each sought dependence \( a(\xi) \) is described by a smooth function with a continuous first derivative and a characteristic asymptotic behavior \( a(\xi \to \pm \infty) = a_\pm = \text{const} \), \( a(\xi \to -\infty) = a_- = \text{const} \), \( da/\xi(\xi \to \pm \infty) \to 0 \), which makes it possible to state a priori the reflectionless propagation of acoustic waves for \( \xi \to \pm \infty \) when such waves propagate independently of each other in opposite directions.

Given the introduced restrictions on the possible behaviors of dependences \( a(\xi) \), when solving the equa-
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As in [6], setting, without loss of generality, the quantity \( \varepsilon_j = 0 \) (see [7, 17]), we bring equations (39) to their characteristic model comparison equations (34) with exact solutions for \( \overline{p}(\omega, \xi) \) and \( \overline{v}(\omega, \xi) \) corresponding to (35):

\[
\overline{p}(\omega, \xi) = \overline{p}_1(\omega, \xi) + \overline{p}_2(\omega, \xi),
\]
\[
\overline{v}(\omega, \xi) = \frac{p^{(s)}(\omega)}{\sqrt{q_1(\omega, \xi)}} \exp \left\{ \mp i k_0 \int_0^{\xi} q_1(\omega, \xi) d\xi \right\};
\]
\[
\overline{v}_1(\omega, \xi) = \frac{v^{(s)}(\omega)}{\sqrt{q_2(\omega, \xi)}} \exp \left\{ \mp i k_0 \int_0^{\xi} q_2(\omega, \xi) d\xi \right\}.
\]

In (41) and (42) functions \( q_1(\omega, \xi) \) are the solutions to the corresponding differential equations:

\[
q_1^2 - \frac{q_1^{1/2}}{k_0^{1/2}} \frac{d^2}{d\xi^2} \left( q_1^{1/2} \right) = n^2,
\]
\[
q_2^2 - \frac{1}{k_0^2} \left[ q_2^{1/2} \frac{d^2}{d\xi^2} \left( q_2^{1/2} \right) - b(\xi) \right] = n^2,
\]

dependences \( p^{(s)}(\omega) \) (or \( v^{(s)}(\omega) \)) determine the initial spectral content of upwardly and downwardly propagating pressure (or vibrational velocity) waves, respectively.

Thus, as follows from solutions (41), (42), the amplitude of each spectral component of pressure and vibrational velocity waves depends not only on the spatial variable \( \xi \), but also on the radiation frequency \( \omega \), and the dependence of the phase on \( \omega \) differs from linear, because \( q_j = q_j(\omega, \xi) \) (see (43), (44)). Nevertheless, in the considered case of a continuously stratified medium, in contrast to the statement made in [18], it is still possible to have traveling waves propagating in opposite directions. Naturally, even with the same spectral content of pressure and vibrational velocity waves, i.e., when the relations \( p^{(s)}(\omega) / v^{(s)}(\omega) = \text{const} \), are met, in general, due to differences between functions \( q_1(\omega, \xi) \) (43) and \( q_2(\omega, \xi) \) (44), transformation of the spatiotemporal profile of pulse pressure waves (10), (41) and vibrational velocity waves (10), (42) in their distribution will occur in different ways. When the following from (43), (44) equality \( d^2a/d\xi^2 = 0 \) is met, from which the obvious dependences

\[
a(\xi) = a_0(\xi) = \left( 1 + \frac{\xi}{H} \right),
\]
\[
m(\xi) = n_0(\xi) = \left( 1 + \frac{\xi}{H} \right)^{-1},
\]
follow, transformation of the shape of the profile of pulse pressure waves (10), (41) and vibrational velocity waves...
waves (10), (42) will proceed in the same way, since for $q_0(\omega, \xi) = q_2(\omega, \xi)$ the following relations are satisfied:

$$ p_2(\xi) = \frac{\int p_2(\omega, \xi) e^{i\omega t} d\omega}{\int v_2(\omega, \xi) e^{i\omega t} d\omega}, $$

in which

$$ p_2(\xi) = \int p_2(\omega, \xi) e^{i\omega t} d\omega, $$

$$ v_2(\xi) = \int v_2(\omega, \xi) e^{i\omega t} d\omega. $$

(46)

However, dependences $a_n(\xi)$ and $n(\xi)$ (45) do not correspond to reflectionless profiles $a(\xi)$ and $n(\xi)$, since they do not satisfy the conditions of possible behavior $a(\xi)$ and $n(\xi)$ for $-\infty < \xi < +\infty$ formulated at the beginning of this section.

In addition, it is of interest to note two special cases in which it is possible to obtain simple analytical solutions in the class of elementary functions for pulse pressure waves and vibrational velocity waves.

First, assuming the joint satisfaction of the equalities following from (43) (see [6])

$$ \left\{ \begin{align*} q_1(\xi) &= n(\xi), \\
\frac{d^2}{d\xi^2} q_1^{-1/2} &= 0, \end{align*} \right. $$

(47)

which are satisfied by the a priori known dependence $a(\xi) = a_n(\xi)$ (see (20)), from (10), (41), (46) we automatically obtain analytical solutions coinciding with (21) for pressure waves with a self-similar shape of the profile, propagating in mutually opposite directions. In this case, the standard WKB approximation for solving equation (30) is an exact analytical solution (see [6] and §8 in [8]), since all subsequent terms of the series (beginning from the third) in the corresponding asymptotic expansion are identically equal to zero. However, dependence $n(\xi) = n_n(\xi) = 1/a_n(\xi)$ (47), as well as the dependence $n(\xi) = n_n(\xi)$ (45), does not satisfy the conditions for the possible behavior of reflectionless profiles $n(\xi)$.

Second, assuming satisfaction of the equalities following from (44),

$$ \left\{ \begin{align*} q_2(\xi) &= n(\xi), \\
q_2^{1/2} \frac{d^2}{d\xi^2} q_2^{-1/2} - b(\xi) &= 0, \end{align*} \right. $$

(48)

which are satisfied by the dependence $a(\xi) = a_n(\xi)$ (25), we automatically obtain from (10), (42), (46) analytical solutions coinciding with (26) for vibrational velocity waves with a self-similar shape of the profile, traveling in mutually opposite directions. In this case, the standard WKB approximation for solving equation (31) is also an exact analytical solution, since all subsequent terms of the series (starting from the third) in the corresponding asymptotic expansion are identically equal to zero. Naturally, the dependence $n(\xi) = n_n(\xi) = 1/a_n(\xi)$ following from (48) does not satisfy the above conditions for the possible behavior of reflectionless profiles $n(\xi)$.

Clearly, only for joint satisfaction of the equalities following from (43), (44)

$$ \left\{ \begin{align*} q_1(\xi) &= q_2(\xi) = n(\xi), \\
\frac{d^2}{d\xi^2} q_1^{-1/2} &= 0, \\
q_2^{1/2} \frac{d^2}{d\xi^2} q_2^{-1/2} - b(\xi) &= 0, \end{align*} \right. $$

(49)

which are satisfied only for a constant wave acoustic impedance of the medium $a(\xi) = \text{const}$ ($n(\xi) = \text{const}$) (i.e., for $a_p(\xi) = a_r(\xi) = 1$ at $1/H = 0$), we automatically obtain from (21), (22) and (10), (41), (42) analytical solutions coinciding with (8), (9) that describe the pulse pressure and vibrational velocity waves with a constant shape of the profile, propagating in mutually opposite directions. In this case, the reflectionless propagation of such waves is possible if the obvious equality $a(\xi) = a_s = a$ is satisfied.

Here, it seems important to note that quite a long time ago (see [11]), a very efficient iterative scheme was developed for analytical solution of equations (43), (44), which makes it possible to find generalized WKB functions (41) and (42) satisfying the corresponding equations (30) and (31) with any desired accuracy, and thereby to obtain approximate analytical solutions with a certain accuracy describing pulse pressure and vibrational velocity waves traveling in opposite directions (see (46)).

**CONCLUSIONS**

Using the transformation of wave equations for pressure and vibrational velocity perturbation proposed in [8], which describe the acoustic wave propagation in media with continuously stratified sound speed and density, the study applied the example of vertical propagation of plane acoustic waves in the atmosphere (ignoring the effect of gravity) to demonstrate that the mode of their reflectionless propagation in the entire range of radiation frequencies takes place only at a constant value of the wave acoustic impedance of the medium.

It was found that the reflectionless acoustic wave propagation occurs only in continuously stratified media for which the wave equations with variable coefficients for pressure and vibrational velocity perturbation are by the same transformations reduced to a comparison wave equation with constant coefficients for the same wave acoustic impedance profile, which is inversely proportional to the refractive index. Therefore, it has been established that the approach pro-
posed in [1–5], similar to the reference equation method [8], to transforming wave equations with variable coefficients for various physical quantities—e.g., pressure and vibrational velocity perturbations, or, like in [2–5], perturbations of the vibrational velocity and its divergence—to a comparison wave equation with constant coefficients for different refractive index profiles, allows no unambiguous conclusions about the reflectionless nature of acoustic wave propagation in media with the corresponding (see [2–5]) refractive index profiles.

Note that the approach used in [1–5] for the analytical description of wave processes in continuously stratified media with certain refractive index profiles (see [2–5]) is just a special case of the more general method, previously proposed in [6, 11], for obtaining generalized WKB solutions, which in specifically considered situations (see [2–5]) are exact analytical solutions in the class of elementary functions, which, however, do not satisfy correct physical conditions in the entire range of variation of the independent spatial variable (see Sect. 3.1 and Sect. 6.3 in [8]). It was also demonstrated that the generalized WKB method [6, 11] makes it possible, in the absence of wave turning horizons in continuously stratified media, to unambiguously identify waves propagating in opposite directions, despite their frequency-dependent influence on each other [18].

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