Quantum equations for massless particles of any spin in curved spacetime

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Abstract. Quantum equations for massless particles of any spin in curved spacetime are presented. These equations can be extended to a global spacetime from the ones derived in a flat spacetime. Here we demonstrate that these equations can be derived also from a scalar Lagrangian, and they form a basis for quantum theory of spins 1 and 2 in the vicinity of massive objects. The conserved current is calculated by applying Noether theorem and a complete set of commutative observables is determined. The solutions of these equations are considered in a flat Minkowski spacetime and in axially symmetric spacetimes.

1. Introduction
In previous contributions [1]-[4], it was demonstrated that free massless particles of any spin in flat Minkowski spacetime satisfy the following quantum wave equations,

$$\left( \frac{\hat{E}}{c} \gamma^0 - \gamma^k \hat{p}_k \right) \Phi^{(4s)} = 0,$$

where $s$ is the spin, the gamma’s $(\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ denote $4s \times 4s$ Hermitian matrices, $\hat{E} = i\hbar \partial_t$ and $\hat{p}_k = -i\hbar \partial_k$ are the energy and momentum operators. Properties of these matrices are given in the Appendix. Here we note that the gamma matrices form representations of the Pauli matrices and factorize the d’Almbertian operator. For spin 1 and 2 they are chosen to reproduce the main equations and the subsidiary conditions [1]-[3] and have eigenvalues of $\pm 1$ only, corresponding to forward and backward helicities. These are the only possible helicity values for massless particles. It is to be noted that the gamma matrices satisfy the Pauli algebra with $\gamma^0$ being the identity matrix. The wave functions $\Phi^{(4s)}$ in the expression above form bases for $D(s-1/2,1/2)$ representations of the Lorentz group, which we write as a $4s$ column matrix with $(2s - 1)$ components of spin $(s - 1)$ and $(2s + 1)$ components of spin $s$, i.e.,
Using natural units with light velocity \( c = 1 \) and Planck constant \( \hbar = 1 \) Eq.1 can be rewritten as,

\[
\gamma^b \eta_{bc} \partial_c \Phi^{(4s)} = 0,
\]

where \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) stands for the Minkowski metric.

Our main purpose in the present work is to discuss the properties of the solutions of a covariant form of the equations above. Such a covariant form was reported already in [5]. We recall that the derivation reported is based on the Principle of Equivalence which we may phrase as follows: A curved spacetime is locally a Minkowski spacetime in a sufficiently small region around any point of a curved spacetime. Accordingly, Eqs.(1) should be valid in sufficiently small regions around any point of a curved spacetime and can be used as our starting point. In order to reformulate the above equations in a global curved spacetime some modifications are in order. First the Minkowski metric must be replaced by a curved spacetime tensor metric \( g_{\mu\nu} \). Secondly, the partial local derivatives must be replaced by the covariant derivatives \( \partial_\mu + \Omega_\mu \), where \( \Omega_\mu \) is the connection coefficient for the wave function \( \Phi^{(4s)} \). Thirdly, the gamma matrices in Eq.(3) carry tangent space indices so that they maintain a flat spacetime form and must be transformed to global gamma matrices by means of vierbein fields i.e., \( \gamma^\mu = E^\mu_a \gamma_a \) [6]. Here \( E^\mu_a(x) \) denotes an inverse vierbein field. A vierbein field is denoted by \( e^a_\mu(x) \).

As shown in [5], the connection coefficient of the wave function is related to the spin connection,

\[
\Omega_\mu = \frac{1}{4} \omega_{\mu bc} \gamma_{b}^c; \quad \omega_{\mu}^a = \frac{1}{2} \gamma^b \partial_\sigma E_\sigma^b \ln e + \partial_\sigma E_\sigma^b ,
\]

where \( e = \sqrt{-\text{det} g} \) is the square root of the determinant of the global metric tensor \( g_{\mu\nu} \). Inserting these modifications in Eq.(3) gives the covariant form of Eq.(1),

\[
E^\mu_a \gamma^a \partial_\mu \left( \sqrt{e} \Phi^{(4s)} \right) = -\frac{1}{2} \gamma^b \partial_\sigma E_\sigma^b \Phi^{(4s)},
\]

as reported in [5]. We note that the Hermitian conjugate of \( \Phi^{(4s)} \) is defined as \( [\Phi^{(4s)}]^H = [\Phi^{(4s)}]^T \). Clearly, \( \Phi^H \Phi \) is positive definite which we interpret as probability density.

It is straightforward to guess that the Lagrangian density is given by,

\[
\mathcal{L} = \Phi^H \left( E^\mu_a \gamma^a \partial_\mu \left( \sqrt{e} \Phi^{(4s)} \right) \right) + \frac{1}{2} \gamma^b \partial_\sigma E_\sigma^b \Phi^{(4s)},
\]

The variation w.r.t. \( \Phi^H \) obviously yields Eq.(5). Variation w.r.t. \( \Phi \) gives the Hermitian conjugate equation,

\[
\left( \partial_\mu \left( \sqrt{e} \Phi^H \right) \right) \gamma^a E^\mu_a + \frac{1}{2} \partial_\sigma E_\sigma^b \Phi^H \gamma^b = \left( E^\mu_a \gamma^a \partial_\mu \left( \sqrt{e} \Phi^* \right) \right) + \frac{1}{2} \gamma^b \partial_\sigma E_\sigma^b \Phi^* = 0.
\]
The solutions Φ and Φ∗ correspond to forward and backward helicity, each with two eigenvalues of energy one positive and one negative. In what follows we disregard negative energy solutions and use positive energy only but with opposite helicities. Our main interest in the present note is to demonstrate that Eqs.(5) and (7) form a basis for a quantum theory of massless particles of spin 1 (the photon) and spin 2 (the graviton). The conserved current density of Eq.(5) above can be obtained from the Lagrangian, Eq.(6), using Noether’s Theorem. Let Φ be a solution of Eq.(5), and let δΦb be the variation of the b component of Φ of some conserved quantity, δΦb = iΦbδα and δ(Φb)∗ = −i(Φb)∗δα, then,

\[ j^\mu = \frac{\delta L}{\delta (\partial_\mu \Phi_b)} \frac{\delta \Phi^b}{\delta \alpha} - J^\mu_0 = \Phi^H_b E^\mu_a \gamma^a \Phi^b = \Phi^H \gamma^\mu \Phi. \] (8a)

With \( j^0 = \Phi^H \gamma^0 \Phi = \Phi^H_b \Phi^b \) being the probability density, the current above satisfies the continuity equation, i.e.,

\[ \partial_\mu j^\mu = \partial_\mu [\Phi^H \gamma^\mu \Phi]. \] (9)

The conserved current above is independent of the spin connection. A more intuitive derivation of this expression given in the Appendix, explains why the conserved current does not depend on the spin connection. In what follows we consider Eq.(5) in Minkowski flat spacetime, and in axially symmetric spacetimes, namely, the Friedman-Walker-Robertson (FRW) and Schwarzschild spacetimes.

2. Free Particle Equations In Minkowski Spacetime

In Minkowski flat spacetime, \( g_{\mu\nu} = \text{diag}(1, -1, -1, -1), \sqrt{e} = 1 \) so that Eqs.(5) and (7) reduce to,

\[ i\partial_t \Phi = -i \gamma \cdot \nabla \Phi, \] (10)

\[ i\partial_t \Phi^* = i \gamma \cdot \nabla \Phi^*, \] (11)

where \( \nabla \) is the del operator, and \( \hat{E} = i\partial_t \) and \( \hat{p} = -i\nabla \) are the energy and momentum operators, respectively. We identify \(-i \gamma \cdot \nabla \) with the Hamiltonian \( H_0 \) of the unperturbed free space. In spherical coordinates,

\[ \nabla = \hat{r} \gamma^r \partial_r + \hat{\theta} \gamma^\theta \partial_\theta + \hat{\phi} \gamma^\phi \partial_\phi, \] (12)

and the angular momentum operator is defined as,

\[ L = -i\hat{r} \times \nabla. \] (13)

By multiplying this expression on the left side by \( \hat{r} \) we can write the \( \nabla \) as,

\[ \nabla = -\hat{r} \partial_r - \frac{1}{r} \hat{r} \times L. \] (14)

It is simple to show, using the gamma matrices properties listed in the Appendix, that for any three vectors \( A \) and \( B \),

\[ (\gamma \cdot A)(\gamma \cdot B) = \gamma^0 A \cdot B + \frac{i}{2} \gamma \cdot (A \times B). \] (15)

Then taking \( A = \hat{r} \) and \( B = L \), one obtains,

\[ \gamma \cdot \hat{r} \times L = -i (\gamma \cdot \hat{r}) (2\gamma \cdot L). \] (16)
With the above relation we can rewrite the Hamiltonian in the form,
\[ H_0 = -i\gamma \cdot \nabla = -i\gamma \cdot \hat{r} \partial_r - \frac{1}{r} \gamma \cdot \hat{r} (2\gamma \cdot \mathbf{L}). \] (17)

Finally we identify the spin \( S \) with \( \gamma \) and define an angular operator,
\[ K = 2\gamma \cdot \mathbf{L} + 1 = 2\mathbf{S} \cdot \mathbf{L} + 1 = J^2 - L^2 - S^2 + 1, \] (18)
where \( J = L + S \) is the total angular momentum. Substituting this in Eq.(17) yields,
\[ H_0 = -i\gamma \cdot \hat{r} \left[ \partial_r + \frac{1}{r} - \frac{1}{r} K \right]. \] (19)

This expression of the Hamiltonian suggests that we may construct the solution \( \Phi \) by separation of variables. In the Appendix we show that \( H_0, J_2, J_3, K \) and parity \( P \) form a complete set of commuting operators. Then the angular part of the spinor \( \Phi \) is a spinor of spherical harmonics which we write as,
\[ Y_{lm}^j (\Omega) = \sum_{m_s} C (l, s, j; m_j - m_s, m_s, m_j) Y_{lm}^{m_l} (\theta, \varphi) \chi_{m_s}^{m_j}, \] (20)
where \( C (l, s, j; m_j - m_s, m_s, m_j) \) is a Clebsch-Gordan coefficient for combining angular momenta \( l \) and \( s \) to total angular momentum \( j \) with magnetic quantum numbers \( m_j - m_s, m_s, m_j \), respectively. \( Y_{lm}^{m_l} (\theta, \varphi) \) are the usual spherical harmonics which are eigenfunctions of \( L^2, L_3 \) and \( \chi_{m_s}^{m_j} \) are eigenfunctions of \( S^2 \) and \( S_3 \). The spinor spherical harmonics are orthonormal, i.e.,
\[ \int d\Omega \left( Y_{lm,j'}^{m_{l,j}} (\Omega) \right)^\dagger Y_{lm,j}^j (\Omega) = \delta_{j'j} \delta_{ll} \delta_{m_j m_j}. \] (21)

Further more, the function \( \Phi \) is an eigenfunction of the operators \( J_2, J_3 \) angular operator \( K \) and parity \( (P) \),
\[ J_2 \Phi_{m_j, \kappa}^j = j (j + 1) \Phi_{m_j, \kappa}^j, \] (22)
\[ J_3 \Phi_{m_j, \kappa}^j = m_j \Phi_{m_j, \kappa}^j, \] (23)
\[ K \Phi_{m_j, \kappa}^j = \kappa_j \Phi_{m_j, \kappa}^j. \] (24)

Let us evaluate the eigenvalues of the operator \( \kappa = (1 + 2\mathbf{L} \cdot \mathbf{S})^2 \). From addition of angular momenta \( j = l \pm s \). Then,
\[ \kappa = (2\mathbf{S} \cdot \mathbf{L} + 1)^2 = (J^2 - L^2 - S^2 + 1)^2 \]
\[ = [j(j + 1) - l(l + 1) - s(s + 1) + 1] = (2ls + 1)^2, \] (25)
so that for \( j = l + s \), the eigenvalues are,
\[ \kappa_j = (2ls + 1), \] (26)
and similarly, for \( j = l - s \),
\[ \kappa_j = -2s (l + 1). \] (27)
We can use \( \kappa_j \) to label \( \Phi \) since once it is set so are \( l \) and \( j \).
2.1. Quantization of the Function $\Phi$

The free particle solutions of Eqs.(10-11) correspond respectively to forward and backward helicity. Let $u(p, s) \exp(-ipx)$ denote the positive energy and positive helicity solution and let $v(p, s) \exp(ipx)$ denote the positive energy and negative helicity solution. Inserting these into Eqs.(10-11) gives,

$$ (\omega_p - \gamma \cdot p) u(p) = 0, \quad (\omega_p + \gamma \cdot p) v(p) = 0. \quad (28) $$

Here $\omega_p = p_0 = \pm \sqrt{p \cdot p}$. The spinors $u(p)$ and $v(p)$ are orthogonal so that we add the conditions,

$$ u^a (p) v^\dagger_a (p') = v^a (p) u^\dagger_a (p') = 0, \quad (29) $$

and normalize the spinors according to,

$$ u^a (p) u^\dagger_a (p') = \omega_p; \quad v^a (p) v^\dagger_a (p') = \omega_p. \quad (30) $$

We then write,

$$ \Phi (x) = \int \frac{d^3 p}{(2\pi)^{3/2}} \sqrt{\frac{1}{\omega_p}} \left[ b(p) u(p) \exp(-ipx) + d^\dagger(p) v(p) \exp(ipx) \right], $$

$$ \Phi^H (x) = \int \frac{d^3 p}{(2\pi)^{3/2}} \sqrt{\frac{1}{\omega_p}} \left[ b^\dagger(p) u^\dagger(p) \exp(ipx) + d(p) v^\dagger(p) \exp(-ipx) \right], \quad (31)-(32) $$

where $b(p)$ and $b^\dagger(p)$ are annihilation and creation operators for a positive energy and forward helicity, and $d(p)$, $d^\dagger(p)$ are annihilation and creation operators for a positive energy and backward helicity. Next in order to quantize $\Phi$ we impose the following commutators,

$$ \left[ \Phi (x), \Phi (x')^H \right] = i\delta^3 (x - x'), $$

$$ \left[ \Phi (x), \Phi (x') \right] = 0, $$

$$ \left[ \Phi^H (x), \Phi^H (x') \right] = 0. \quad (33) $$

These are equivalent to the following commutators for creation and annihilation operators,

$$ \left[ b(p), b^\dagger (p') \right] = \left[ d(p), d^\dagger (p') \right] = \delta_{pp'} \delta_{\sigma \sigma'}, \quad (34) $$

$$ \left[ b^\dagger (p), b^\dagger (p') \right] = \left[ d^\dagger (p), d^\dagger (p') \right] = \left[ b(p), b (p') \right] = \left[ d(p), d (p') \right] = 0. \quad (35) $$

$$ \left[ b(p), b (p') \right] = \left[ d(p), b^\dagger (p') \right] = \left[ d^\dagger (p), b (p') \right] = 0. \quad (36) $$

where $\sigma$ stands for the helicity. This can be shown rather straightforward by inserting Eq.(31) and Eq.(32) in Eqs.(33). Then substituting in the double integral obtained the normalization conditions Eq.(30), the orthogonality conditions Eq.(29), the commutator Eqs.(34-36), and by integrating Eqs.(33) are recovered.
3. Particle Equations In Axially Symmetric Spacetime

The equations (5) assume relatively simpler form in axially symmetric spacetime. Following refs.[9],[10] we assume that the wave function $\Phi$ factorizes as,

$$\Phi(x^0, x^1, x^2, x^3) = \psi(x^0, x^1, x^2, x^3) f(x^1, x^2),$$

and require that the function $\psi$ satisfies an equation independent of the spin connection, i.e.,

$$E^a_\mu \gamma^a \partial_\mu \psi = 0.$$  \hspace{1cm} (38)

Then by inserting Eq.(37) in Eq.(5) and using Eq.(38) one finds that the function $f(x^1, x^2)$ satisfies the equation,

$$\partial_\mu \ln (f(x^1, x^2) \sqrt{e}) = -\frac{1}{2} e^b_\mu \partial_\sigma E^\sigma_b.$$ \hspace{1cm} (39)

The above expression depends on the metric tensor and contains the dependence on the spin connection. Obviously, axial symmetry restricts the form of the metric tensor and consequently the corresponding vierbein fields. In what follows we show that in FRW and Schwarzschild spacetimes Eq.(39) is integrable and its solution assumes a simple analytic form. We note that black holes are stationary axis-symmetric solutions of the Einstein equation[7],[8], so that the factorization procedure proposed above may serve to study the behavior of free massless particles such as photons and gravitons in the vicinity of black holes and massive objects. The Schwarzschild and the FRW metric tensors are diagonal and Eq.(39) assumes simple forms. Explicitly,

$$\partial_\mu \ln (f(x^1, x^2) \sqrt{e}) = -\frac{1}{2} e^t_t \partial_t E^t_t + e^r_r \partial_r E^r_r + e^\theta_\theta \partial_\theta E^\theta_\theta + e^\phi_\phi \partial_\phi E^\phi_\phi.$$ \hspace{1cm} (40)

The last term on the r.h.s. vanishes because of symmetry. In what follows we consider the equations above in static FRW and Schwarzschild spacetimes.

3.1. Particle Equations In Static FRW Spacetime

In FRW spacetime the metric is $g_{\mu\nu} = diag (1, -a^2/F^2, -a^2r^2, -a^2r^2 \sin^2 \theta)$, where $a$ representing the size of the universe with dimension of length, $r$ is a dimensionless parameter and $F = \sqrt{1 - kr^2}$. Thus the vierbein and inverse vierbein fields are respectively,

$$e^a_\mu = diag(1, a/F, ar, ar \sin \theta),$$ \hspace{1cm} (41)

and,

$$E^\mu_a = diag (1, F/a, 1/(ar), 1/(ar \sin \theta)).$$ \hspace{1cm} (42)

Inserting these into Eq.(40) and integrating gives,

$$\partial_\mu \ln (f(x^1, x^2) \sqrt{e}) = \partial_\mu \ln (1 - kr^2)^{-1/4},$$ \hspace{1cm} (43)

and,

$$f(r, \theta) = a^{3/2} r^{1/2} \sin^{1/2} \theta.$$ \hspace{1cm} (44)

Consider now Eq.(38). The FRW spacetime is spherically symmetric so that we can apply separation of variables as in the Minkowski case. In fact by inserting the vierbeins fields Eq.(38) can be written in the form,

$$i \gamma^\mu \partial_\mu \psi = H_0 \psi = -i \gamma \cdot \hat{r} \left[ \frac{F}{a} \partial_r + \frac{1}{ar} - \frac{1}{ar} K \right] \psi,$$ \hspace{1cm} (45)
so that the angular part is identical up to a factor $a$ to that found for the Minkowski case. The radial function is different because of the factor $F/a$. It is to be indicated that in the expression above the vierbein axes are chosen in the directions of $t,x,y,z$, so that the gamma matrices which were determined in [1]-[3], must be transformed to,

$$
\begin{pmatrix}
\gamma^t \\
\gamma^r \\
\gamma^\theta \\
\gamma^\phi
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & F & 0 & 0 \\
0 & 0 & \frac{1}{r} & 0 \\
0 & 0 & 0 & \frac{1}{r \sin \theta}
\end{pmatrix}
\begin{pmatrix}
\gamma^0 \\
\gamma^1 \\
\gamma^2 \\
\gamma^3
\end{pmatrix}.
$$

(46)

Here $(\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ are the gamma matrices worked out in refs.[1]-[3] to reproduce the subsidiary conditions along with the main equation.

3.2. Particle Equations In Static Schwarzschild Spacetime

In Schwarzschild spacetime the metric is $g_{\mu \nu} = \text{diag} \left( F, -1/F, -r^2, -r^2 \sin^2 \theta \right)$, where $F = \left( 1 - \frac{2GM}{r} \right)$. Here $G$ is the gravitation constant and $M$ is a constant with dimension of mass. This metric approaches the Minkowski metric as $M$ approaches zero. Similarly, when $r$ goes to infinity, the Schwarzschild metric approaches the Minkowski metric. The corresponding vierbein fields and their inverse are

$$
e^a_{\mu} = \text{diag}(\sqrt{F}, 1/\sqrt{F}, r, r \sin \theta)
$$

and

$$
E^a_\mu = \text{diag}(1/\sqrt{F}, \sqrt{F}, 1/r, 1/(r \sin \theta)).
$$

Inserting these into Eq.(40) and integrating gives,

$$
f(r, \theta) = \left( 1 - \frac{2MG}{r} \right)^{1/4} r^{1/2} \theta.
$$

(47)

The equation for $\psi$ will be,

$$
H_0 \psi = -i\gamma \cdot \nabla \left[ \left( 1 - \frac{2MG}{r} \right) \frac{1}{r} - \frac{1}{r} K \right] \psi.
$$

(48)

Here as well the angular part is given by spherical harmonics as in the Minkowski case. However the radial function is different because of the factor $\left( 1 - \frac{2MG}{r} \right)$. We may thus conclude that as in the Minkowski spacetime the solutions of Eq. (5) in the axially symmetric FRW and Schwarzschild spacetimes are simple modes which are specified by the orbital angular momentum and its z component ($l = 0, 1, 2, \cdots$), ($m_l = -l, -l+1, \cdots, l-1, l$), the spin and its z component ($s$ , $m_s = -s, s$), the total angular momentum ($j = l + s$) and its z component ($m_j = -j, \cdots, j$), and $\kappa_j = (2s + 1)$ and $-2s (l + 1)$ for $J = L \pm S$, respectively. To completely specify the solution $\Phi$ we have to add of course the radial quantum number ($n_r = 0, 1, 2, \cdots$).

3.3. Quantization of the Function $\psi$

The quantization procedure which we have applied to the function $\Phi$ in Minkowski spacetime is applicable also in axially symmetric spacetime. In fact Eqs. (38) for the function $\psi$ and its complex conjugate in axially symmetric spacetime are similar to Eqs.(10-11),

$$
i\partial_t \psi = -i\gamma \cdot \nabla \psi,
$$

(49)
\[ i \partial_t \psi^* = i \gamma \cdot \nabla \psi^*. \]  

(50)

Then we may quantize \( \psi \) following exactly the same procedure of quantizing \( \Phi \) in the Minkowski case. We first expand \( \psi \) as,

\[ \psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\omega_p} \left[ b(p) u(p) \exp(-ipx) + d^\dagger(p) v(p) \exp(ipx) \right], \]

(51)

and require that \( \psi \) satisfies the following commutators,

\[ \left[ \psi(x), \psi(x')^H \right] = i\delta^3(x-x'), \]

\[ \left[ \psi(x), \psi(x') \right] = 0, \]

\[ \left[ \psi^H(x), \psi^H(x') \right] = 0. \]

(53)

By substituting Eqs.(51-52) in the above expressions and imposing the orthogonality and normalization conditions Eqs.(29-30) we obtain exactly the same set of the commutators Eqs.(34-36).

4. Concluding Remarks
Quantum equations for free massless particles of any spin were derived in curved spacetime. These equations can be obtained by extending the ones presented in Refs.[1]-[3], to a global curved spacetime as demonstrated in Ref. [5]. Here we have shown that the same equations can be derived from a scalar Lagrangian and a complete set of commuting variables is determined. By applying the Noether theorem to this Lagrangian a conserved current which satisfies the continuity equation was found. The free particle equations assume a simple form in Minkowski spacetime and can be solved by variable separation. The solution of the equations in curved spacetime is more involved. We have shown though that in stationary and axially symmetric spacetime a solution can be obtained by factorizing the solutions into factors \( \psi(x,t,x^r,x^\theta,x^\phi) \) and \( f(x^r,x^\theta) \) each satisfying a manageable equation. In particular the factor \( f(x^r,x^\theta) \) which contains the spin connection can be calculated analytically for both the FRW and Schwarzschild space time. The angular part of \( \psi(x,t,x^r,x^\theta,x^\phi) \) is practically equal to that of \( \Phi \) in the Minkowski case but the radial part needs to be calculated numerically. We believe that the scheme presented above may serve to study the eigenvalues and degeneracies of massless particles on the 3-sphere.

Appendix A. Properties of the Gamma Matrices
We recall that the gamma matrices satisfy the Pauli algebra and that \( \gamma^0 \) is the identity matrix. We recall that the gamma matrices satisfy the Pauli algebra and that \( \gamma^0 \) is the identity matrix. The gamma matrices satisfy the following relations,

\[ (\gamma^a)^\dagger = (\gamma^a); \quad (\gamma^a)^2 = \gamma^0, \]

(A.1)

\[ \{\gamma^a, \gamma^b\} = 2\gamma^0 \delta^{ab}; \quad a, b = 0, 1, 2, 3, \]

(A.2)

\[ [\gamma^a, \gamma^b] = i\epsilon^{abc}\gamma_c; \quad a, b, c = 1, 2, 3, \]

(A.3)
trace(γ^a) = 0;  \ a = 1, 2, 3.  \ (A.4)

We stress that although the anticommutators Eqs.(A.2) are common features of the Clifford algebra. Our gamma matrices do not form a Clifford algebra and are different from the Dirac’s gamma matrices.

The gamma matrices satisfy the following identity: For any two three vectors \( A \) and \( B \),

\[
(\gamma \cdot A)(\gamma \cdot B) = A \cdot B + \frac{i}{2} \gamma \cdot A \times B. \tag{A.5}
\]

Proof:

\[
(\gamma_i A_i)(\gamma_j B_j) = \gamma_i \gamma_j A_i B_j \\
= \delta_{ij}(\gamma^i)^2 A_i B_j + \frac{i}{2} \epsilon^{ijk} A_i B_j = \gamma^0 A_i B_i + \frac{i}{2} \gamma_k (A \times B)^k, \tag{A.6}
\]

Where we have used Eqs.(A.1),(A.2) and (A.3).

Appendix B. Conserved Current

By factorizing the d’Alembertian operator we obtain two equations (see ref. [5] for more details),

\[
\gamma^\nu (x) \left( \partial_\nu + \Omega_\nu \right) \Phi (x) = 0, \tag{B.1}
\]

\[
\gamma^\mu (x) \left( \partial_\mu + \Omega_\mu^* \right) \Phi^* (x) = 0, \tag{B.2}
\]

where \( \Phi^* \) is the complex conjugate of \( \Phi \), \( \Omega_\mu (x) \) and \( \Omega_\mu^* (x) \) are the connection coefficients for the functions \( \Phi (x) \) and \( \Phi^* (x) \). Here we note that \( \Omega (x) = -\Omega^* (x) \). We use the equations above to prove our claim for the conserved current. We multiply Eq.(B.1) by the Hermitian conjugate \( \Phi^H \) and Eq.(B.2) by the transposed \( \Phi^T \) of \( \Phi \), and sum both equations to obtain,

\[
E^\nu_{\mu \psi_1} \gamma^a \partial_\nu \psi_1 + \cdots + E^\nu_{\psi_{(4s)}} \gamma^a \partial_\nu \psi_{(4s)} + E^\nu_{\nu \psi_1} \gamma^a \partial_\nu \psi^H_1 + \cdots + E^\nu_{\nu \psi_{(4s)}} \gamma^a \partial_\nu \psi^H_{(4s)} = 0. \tag{B.3}
\]

Now we can sum pairs of terms like,

\[
E^\mu_{\mu \psi_1} \gamma^a \partial_\mu \psi_1 + E^\mu_{\mu \psi_{(4s)}} \gamma^a \partial_\mu \psi_{(4s)} = \partial_\mu (\psi^H \gamma^a \psi_n). \tag{B.4}
\]

Inserting Eq.(B.4) in Eq.(B.3) yields,

\[
\partial_0 (\Phi^H \Phi) + \partial_k \left( \Phi^H \gamma^k \Phi \right) = 0; k = 1, 2, 3. \tag{B.5}
\]

This explains intuitively why the conserved current does not depend on the spin connection. Note that \( \Phi^H \Phi \) is positive definite and is interpreted as probability density.
Appendix C. Commuting Variables

In what follows we show that the total angular momentum $J = L + S$, $J^2$, $J_3$, $\kappa = 2\gamma \cdot L - 1$ and parity $P$ form complete set of commuting observables. To show that we recall that the Hamiltonian of the free field and the $i^{th}$ component of the orbital and spin angular momentum are given by,

$$H_0 = -i\gamma^i \partial_i,$$  

$$L_i = -i\epsilon^{ijk} x_j \partial_k,$$  

and,

$$S_i = \epsilon^{ijk} \gamma_j \partial_k.$$  

Then by simple algebraic manipulations one finds that,

$$[L_i, H_0] = +\epsilon^{ijk} \gamma_j \partial_k,$$  

and,

$$[S_i, H_0] = -\epsilon^{ijk} \gamma_j \partial_k.$$  

Then from the expressions Eq.(C.4) and Eq.(C.5), The $i^{th}$ component of the total angular momentum commute with the Hamiltonian so that,

$$[J_3, H_0] = 0,$$  

$$[J^2, H_0] = 0.$$

The operator $\kappa = 2\gamma \cdot L - 1$ also commutes with $J_i$ since,

$$[L_i, K] = [L_i, 2\gamma \cdot L] = 2 \left( L_i \gamma^j L_j - \gamma^j L_j L_i \right) = 2i\gamma^j \epsilon^{ijk} L_k,$$  

$$[S_i, K] = [\gamma_i, 2\gamma \cdot L] = 2 \left( \gamma_i \gamma^j L_j - \gamma^j L_j \gamma_i \right) = 2 \left[ \gamma_i, \gamma^j \right] L_j = 2i\epsilon^{ijk} \gamma_j L_k.$$  

These two expressions above sum to zero and hence,

$$[J_i, K] = 0.$$  

Now we show that the parity operator commutes with the Hamiltonian. By definition,

$$P = \beta P; \quad \beta = \begin{pmatrix} I_{2s} & 0 \\ 0 & -I_{2s} \end{pmatrix},$$  

where $\mathcal{P}$ and $\mathcal{P}$ are the parity operators acting on the spinors and on coordinates, respectively. Indeed,

$$[\mathcal{P}, H_0] = [\beta P, -i\gamma^k \partial_k] = -i\beta P \gamma^k \partial_k + \gamma^k \partial_k i\beta P = -i\beta \gamma^k P \partial_k + i\gamma^k \beta \partial_k P$$

$$= -i\beta P \gamma^k (-\partial_k P) + i\gamma^k \beta \partial_k P = i \left\{ \beta, \gamma^k \right\} \partial_k P = 0.$$  

We have used the fact that $P$ does not depend on the coordinates and therefore $\partial_k P = 0$. Furthermore the parity operator commutes with $J_i$ since $[\mathcal{P}, J_i] = \beta [P, J_i]$. The coordinate parity operator $P$ commutes with $J_i$ since the angular momentum is a pseudovector. To conclude $H_0, J^2, J_3, K$ and $P$ form a complete set of commuting operators.
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