PRODUCTS OF FUNCTIONS IN $\text{BMO}$ AND $\mathcal{H}^1$ SPACES ON SPACES OF HOMOGENEOUS TYPE

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Abstract. We give an extension to certain $RD$-space $\mathcal{X}$, i.e. space of homogeneous type in the sense of Coifman and Weiss, which has the reverse doubling property, of the definition and various properties of the product of functions in $\text{BMO}(\mathcal{X})$ and $\mathcal{H}^1(\mathcal{X})$, and functions in Lipschitz space $\Lambda_{\frac{1}{p}-1}(\mathcal{X})$ and $\mathcal{H}^p(\mathcal{X})$ for $p \in \left(\frac{n}{n+\theta}, 1\right]$, where $n$ and $\theta$ denote respectively the "dimension" and the order of $\mathcal{X}$.

1. Introduction

It is well known that $\text{BMO}(\mathbb{R}^n)$ is the dual space of $\mathcal{H}^1(\mathbb{R}^n)$ and that multiplication by $\varphi \in D(\mathbb{R}^n)$ is a bounded operator on $\text{BMO}(\mathbb{R}^n)$. Those facts allow Bonami, Iwaniec, Jones and Zinsmeister, to define in [2] a product $b \times h$ of $b \in \text{BMO}(\mathbb{R}^n)$ and $h \in \mathcal{H}^1(\mathbb{R}^n)$ as a distribution, operating on a test function $\varphi \in D(\mathbb{R}^n)$ by the rule

$$\langle b \times h, \varphi \rangle := \langle b \varphi, h \rangle.$$  

They proved that such distributions are sums of a function in $L^1(\mathbb{R}^n)$ and a distribution in a Hardy-Orlicz space $\mathcal{H}^p(\mathbb{R}^n, \nu)$ where

$$\varphi(t) = \frac{t}{\log(e + t)} \text{ and } d\nu(x) = \frac{dx}{\log(e + |x|)}.$$  

The idea of defining the above product is motivated among other things by the fact that for $1 < p < \infty$, the product $fg$ of $f \in L^p(\mathbb{R}^n)$ and $g$ in the dual space $L^p(\mathbb{R}^n)$ of $L^p(\mathbb{R}^n)$ is integrable (consequently is a distribution). The Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ being the right substitute of $L^1(\mathbb{R}^n)$ in many problems, it seems natural to look at its product with its dual space $\text{BMO}(\mathbb{R}^n)$. Following of the idea in [2], A. Bonami and J. Feuto in [1] extend results, replacing $\text{BMO}(\mathbb{R}^n)$ by $\text{bmo}(\mathbb{R}^n)$, defined as the space of locally integrable functions $b$ such that

$$\sup_{|B| \leq 1} \left(\frac{1}{|B|} \int_B |b(x) - b_B| dx\right) < \infty \quad \text{and} \quad \sup_{|B| \geq 1} \left(\frac{1}{|B|} \int_B |b(x)| dx\right) < \infty,$$

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where $B$ varies among all balls of $\mathbb{R}^n$, $|B|$ denotes the measure of the ball $B$ and $\mathbf{b}_B$ is the mean of $\mathbf{b}$ on $B$. They proved that in this case, the weight $x \mapsto \frac{dx}{\log(e+|x|)}$ is not necessary.

They also proved that for $\mathbf{b}$ in the Hardy space $H^p(\mathbb{R}^n)$ ($0 < p < 1$) the Hardy-Orlicz space is replaced by $H^p(\mathbb{R}^n)$ provided $\mathbf{b}$ belongs to the inhomogeneous Lipschitz space $\Lambda_{n\left(\frac{1}{p}-1\right)}(\mathbb{R}^n)$.

The space of homogeneous type introduced by R.R Coifman and G. Weiss in [4] being the right space for generalize results stated in the euclidean spaces, we give here the analogous of those results in this context. For this purpose, we consider a space of homogeneous type $(\mathcal{X}, d, \mu)$ (see Section 2 for more explanation about this space) in which all annuli are not empty, i.e. $B(x, R) \setminus B(x, r) \neq \emptyset$ for all $x \in \mathcal{X}$ and $0 < r < R < \infty$, where $B(x, r) = \{ y \in \mathcal{X} : d(x, y) < r \}$ is the ball centered at $x$ and with radius $r$. According to [24], the doubling measure $\mu$ then satisfies the reverse doubling property: there exist two positive constants $\kappa$ and a constant $c_\mu$ depending only on $\mu$, such that

\begin{equation}
\frac{\mu(B)}{\mu(\tilde{B})} \geq c_\mu \left( \frac{r(B)}{r(\tilde{B})} \right)^\kappa \text{ for all balls } \tilde{B} \subset B, \tag{4}
\end{equation}

where $r(B)$ denotes the radius of the ball $B$. This reverse doubling condition yields that $\mu(\mathcal{X}) = \infty$. Using the doubling condition (15) and the reverse condition (4), we have that

\begin{equation}
c_\mu \lambda^n \mu(B(x, r)) \leq \mu(B(x, \lambda r)) \leq C_\mu \lambda^n \mu(B(x, r)) \tag{5}
\end{equation}

for all $x \in \mathcal{X}$, $r > 0$ and $\lambda \geq 1$. We will refer to $n$ as the dimension of the space. We will also assume that there exists a positive non decreasing function $\varphi$ defined on $[0, \infty)$ such that for all $x \in \mathcal{X}$ and $r > 0$,

\begin{equation}
\mu(B(x, r)) \sim \varphi(r)^1 \tag{6}
\end{equation}

Notice that (4), (15) and (6) imply that

\begin{equation}
r^n \leq \varphi(r) \leq r^n \text{ if } 0 < r < 1 \tag{7}
\end{equation}

and

\begin{equation}
r^n \sim \varphi(r) \leq r^n \text{ if } 1 \leq r. \tag{8}
\end{equation}

These spaces are particular case of the class spaces of homogeneous type named $RD$-spaces in [8]. An example of such space is obtained by considering a Lie group $X$ with polynomial growth equipped with a left Haar measure $\mu$ and

\begin{itemize}
    \item[(1)] Hereafter we propose the following abbreviation $A \sim B$ for the inequalities $C^{-1}A \leq B \leq CB$, where $C$ is a positive constant not depending on not depending on the main parameters.
    \item[(2)] $A \leq B$ mean the ratio $A/B$ is bounded away from zero by a constant independent of the relevant variables in $A$ and $B$
\end{itemize}
the Carnot-Carathéodory metric $d$ associated with a Hörmander system of left invariant vector fields (see [10], [17] and [22]).

We use the maximal characterization of Hardy spaces in space of homogeneous type as developed by Grafakos, Lu and Yang in [8]. It is proved that this maximal characterization of $\mathcal{H}^p(\mathcal{X}, d, \mu)$ agrees with the atomic characterization of Coifman and Weiss in [5] if $p \in \left(\frac{n}{n+\theta}, 1\right]$, where $\theta$ is as in relation (17).

We recall that for $p \in (0, 1]$ and $q \in [1, \infty] \cap (p, \infty]$, a function $a \in L^q(\mathcal{X}, d, \mu)$ is said to be a $(p, q)$-atom if the following conditions are fulfilled:

1. $a$ is supported in a ball $B$;
2. $\|a\|_{L^q(\mathcal{X}, d, \mu)} \leq \frac{1}{\mu(B)}$ if $q < \infty$ and $\|a\|_{L^\infty(\mathcal{X}, d, \mu)} \leq \frac{1}{\mu(B)}$ if $q = \infty$;
3. $\int_{\mathcal{X}} a(x) d\mu(x) = 0$.

It is proved in Corollary 4.19 of [8] that for $p \in \left(\frac{n}{n+\theta}, 1\right]$ and $q \in (p, \infty] \cap [1, \infty]$, $f \in \mathcal{H}^p(\mathcal{X}, d, \mu)$ if and only if there is a sequence $(a_i)_{i \geq 0}$ of $(p, q)$-atoms, each $a_i$ supported in a ball $B_i$, and a sequence $(\lambda_i)_{i \geq 0}$ of scalars such that

\begin{align}
\mathcal{H} &= \sum_{i=1}^{\infty} \lambda_i a_i \text{ and } \sum_{i=1}^{\infty} |\lambda_i|^p < \infty,
\end{align}

where the first series is considered in the sense of distribution as defined in [8], and $\|\mathcal{H}\|_{\mathcal{H}^p(\mathcal{X})} \sim \inf \left\{ \left( \sum_{i \geq 0} |\lambda_i|^p \right)^{\frac{1}{p}} \right\}$, the infimum being taken over all the decomposition of $f$ as above and $\|\mathcal{H}\|_{\mathcal{H}^p(\mathcal{X})}$ as in [22]. For $b \in BMO(\mathcal{X}, d, \mu)$ and $\mathcal{H} \in \mathcal{H}^1(\mathcal{X}, d, \mu)$ as in [9], the series $\sum_{i=1}^{\infty} \lambda_i (b - b_{B_i}) a_i$ and $\sum_{i=1}^{\infty} \lambda_i b_{B_i} a_i$ converge in the sense of distribution as we can see in the proof of Theorem 1.1. Thus we define the product of $b \times \mathcal{H}$ as the sum of both series, i.e. we put

\begin{align}
b \times \mathcal{H} &:= \sum_{i=1}^{\infty} \lambda_i (b - b_{B_i}) a_i + \sum_{i=1}^{\infty} \lambda_i b_{B_i} a_i.
\end{align}

Our main result can be stated as follows.

**Theorem 1.1.** For $\mathcal{H} \in \mathcal{H}^1(\mathcal{X}, d, \mu)$ and $b \in BMO(\mathcal{X}, d, \mu)$, the product $b \times \mathcal{H}$ can be given a meaning in the sense of distributions. Moreover, if $x_0$ is a fixed element of $\mathcal{X}$ then we have the inclusion

\begin{align}
b \times \mathcal{H} &\in L^1(\mathcal{X}, d, \mu) + \mathcal{H}^p(\mathcal{X}, d, \nu),
\end{align}

where

\begin{align}
d\nu(x) &= \frac{d\mu(x)}{\log(e + d(x_0, x))}.
\end{align}

This result is a generalization of Theorem A of [2]. In Proposition 4.1, we prove that the estimate is valid without weight for $b$ in $BMO(\mathcal{X}, d, \mu)$, while in Theorem 4.2 we obtain that Hardy-Orlicz class is replaced by the classical
weight Hardy space $H^p(X, d, \tau) \ (d\tau(x) = w(x)d\mu(x)$ for some appropriate weight) when $f \in H^p(X, d, \mu)$ and $b \in A_{1-p}(X, d, \mu)$. This result is new even in the Euclidean case, since in $[1]$ there was only a remark on the possibility of such estimate.

Section 2 is devoted to notations and definitions. We recall in this paragraph the definition of spaces of homogeneous type and the grand maximal characterization of Hardy space as introduced in $[8]$. In section 3, we give a prerequisite on Hardy-Orlicz space and prove some lemmas we need for our main result. We prove our main result in the last section, as well as its extensions.

Throughout the paper, $C$ will denotes constants that are independent of the main parameters involved, with values which may differ from line to line.

2. Notations and definitions

A quasimetric $d$ on a set $X$, is a function $d : X \times X \to [0, \infty)$ which satisfies

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y$ in $X$;
(iii) there exists a finite constant $K_0 \geq 1$ such that

\[
d(x, y) \leq K_0 (d(x, z) + d(z, y))
\]

for all $x, y, z$ in $X$.

The set $X$ equipped with a quasimetric $d$ is called quasimetric space.

Let $\mu$ be a positive Borel measure on $(X, d)$ such that all balls defined by $d$ have finite and positive measure. We say that the triple $(X, d, \mu)$ is a space of homogeneous type if there exists a constant $C \geq 1$ such that for all $x \in X$ and $r > 0$, we have

\[
\mu(B(x, 2r)) \leq C \mu(B(x, r)).
\]

This property is known as the doubling property. If $C_0$ is the smallest constant for which (14) holds, then by iterating (14), we have

\[
\frac{\mu(B)}{\mu(B')} \leq C' \left( \frac{r(B)}{r(B')} \right)^n
\]

for all balls $B \subset B'$ where $n = \log_2(C_0)$ and $C' = C_0(2K_0)^n$.

Notice that from the reverse doubling property, $\mu(\{x\}) = 0$ for all $x \in X$. We also have that

\[
\mu(B(x, r + d(x, y))) \sim \mu(B(y, r)) + \mu(B(y, d(x, y)))
\]

for $x, y \in X$ and $r > 0$.

In this paper, $X = (X, d, \mu)$ is a space of homogeneous type in which relations (4) and (6) are satisfy. We also assume (see (15)) that there exist two constants $A'_0 > 0$ and $0 < \theta \leq 1$ such that

\[
|d(x, z) - d(y, z)| \leq A'_0 d(x, y)\theta [d(x, z) + d(y, z)]^{1-\theta}.
\]
The space is saying to be of order $\theta$. We will refer to the constants $K_0, C_0, \mathbf{n}, \kappa, C_\mu, C^\prime_\mu, A^\prime_0$ and $\theta$ mentioned above, as the constants of the space. We will not mention the measure and the quasimetric when talking about the space $(X, d, \mu)$. But if we use another measure than $\mu$, this will be mentioned explicitly. The following abbreviation for the measure of balls will be also used

\[(18) \quad V_\epsilon(x) = \mu(B(x, r)) \quad \text{and} \quad V(x, y) = \mu(B(x, d(x, y))), \]

for all $x, y \in X$ and $\epsilon > 0$.

**Definition 2.1.** Let $x_0 \in X$, $r > 0$, $0 < \beta \leq 1$ and $\gamma > 0$. A complex values function $\varphi$ on $X$ is called a test function of type $(x_0, r, \beta, \gamma)$ if the following hold:

\[(i) \quad |\varphi(x)| \leq C \frac{1}{\mu(B(x, r + d(x, x_0)))} \left( \frac{r}{r + d(x, x_0)} \right)^\gamma \quad \text{for all } x \in X, \]

\[(ii) \quad |\varphi(x) - \varphi(y)| \leq C \left( \frac{d(x, y)}{r + d(x, x_0)} \right)^\beta \frac{1}{\mu(B(x, r + d(x, x_0)))} \left( \frac{r}{r + d(x, x_0)} \right)^\gamma \quad \text{for all } x, y \in X \]

satisfying $d(x, y) \leq \frac{r + d(x_0, x)}{2K_0}$. We denote by $\mathcal{G}(x_0, r, \beta, \gamma)$ the set of all test functions of type $(x_0, r, \beta, \gamma)$, equipped with the norm

\[(19) \quad \|\varphi\|_{\mathcal{G}(x_0, r, \beta, \gamma)} = \inf \{ C : (i) \text{ and (ii) hold} \}. \]

In the sequel, we will fix an element $x_0$ in $X$ and put $\mathcal{G}(\beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$. It is easy to prove that

\[(20) \quad \mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma), \]

with equivalent norms for all $x_1 \in X$ and $r > 0$. Furthermore, it is easy to check that $\mathcal{G}(\beta, \gamma)$ is a Banach space.

For a given $\epsilon \in (0, \theta]$ and $\beta, \gamma \in (0, \epsilon]$, $\mathcal{G}_0(\beta, \gamma)$ denotes the completion of $\mathcal{G}(\epsilon, \epsilon)$ in $\mathcal{G}(\beta, \gamma)$. Equipp $\mathcal{G}_0(\beta, \gamma)$ with the norm $\|\varphi\|_{\mathcal{G}_0(\beta, \gamma)} = \|\varphi\|_{\mathcal{G}(\beta, \gamma)}$, and denote $(\mathcal{G}_0(\beta, \gamma))'$ its dual space; that is the set of linear functionals $f$ from $\mathcal{G}_0(\beta, \gamma)$ to $\mathbb{C}$ with the property that there exists a constant $C > 0$ such that for all $\varphi \in \mathcal{G}_0(\beta, \gamma)$, $|\langle f, \varphi \rangle| \leq C \|\varphi\|_{\mathcal{G}(\beta, \gamma)}$. This dual space will be refer to as a distribution space.

For $f \in (\mathcal{G}_0(\beta, \gamma))'$, the grand maximal function $f^\ast$ of $f$ in the sense of Grafakos, Liu and Yang is defined for $x \in X$ by

\[(21) \quad f^\ast(x) = \sup \left\{ |\langle f, \varphi \rangle| : \varphi \in \mathcal{G}_0(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x, x, \beta, \gamma)} \leq 1 \text{ for some } r > 0 \right\}. \]

The corresponding Hardy space $\mathcal{H}^p(X)$ is defined for $p \in (0, \infty]$ to be the set of $h \in (\mathcal{G}_0(\beta, \gamma))'$ for which

\[(22) \quad \|h\|_{\mathcal{H}^p(X)} := \|h^\ast\|_{L^p(X)} < \infty. \]
It is proved in Proposition 3.15 and Theorem 4.17 of [8] that for \( \epsilon \in (0, \theta] \) and \( p \in \left( \frac{n}{n+\epsilon}, 1 \right] \), the definition of \( \mathcal{H}^p(\mathcal{X}) \) as stated above is independent of the choice of the underlying space of distribution, i.e. if \( f \in (\mathcal{G}_0(\beta_1, \gamma_1))' \) with
\[
(23) \quad n(1/p - 1) < \beta_1, \gamma_1 < \epsilon
\]
and \( \|f\|_{\mathcal{H}^p(\mathcal{X})} < \infty \) then \( f \in (\mathcal{G}_0(\beta_2, \gamma_2))' \) for every \( \beta_2 \) and \( \gamma_2 \) satisfying (23).

In the rest of the paper \( 0 < \epsilon \leq \theta \) is fixed and \( p \in \left( \frac{n}{n+\epsilon}, 1 \right] \). We also fix the underline space of distribution \( \mathcal{G}_0(\beta, \gamma) \) with \( \beta \) and \( \gamma \) as in (23).

As mentioned in the introduction, the dual space of \( \mathcal{H}^1(\mathcal{X}) \) is \( \text{BMO}(\mathcal{X}) \) (space of bounded mean oscillation function), defined as the set of locally integrable functions \( b \) satisfying
\[
\frac{1}{\mu(B)} \int_B |b(x) - b_B| \, d\mu(x) \leq A, \quad \text{for all ball } B,
\]
where \( b_B = \frac{1}{\mu(B)} \int_B b(x) \, d\mu(x) \), and \( A \) a constant depending only on \( b \) and the space constant. We put
\[
(25) \quad \left\| b \right\|_{\text{BMO}(\mathcal{X})} = \sup_{B:\text{ball}} \frac{1}{\mu(B)} \int_{B} |b(x) - b_B| \, d\mu(x)
\]
and
\[
(26) \quad \left\| b \right\|_{\text{BMO}^+} = \left\| b \right\|_{\text{BMO}(\mathcal{X})} + \left| f_{\mathbb{B}} \right|,
\]
where \( \mathbb{B} \) is the ball center at \( x_0 \) and with radius 1. When the measure of \( \mathcal{X} \) is finite, \( (\text{BMO}(\mathcal{X}), \left\| \cdot \right\|_{\text{BMO}}) \) is a Banach space. The set of equivalence classes of functions under the relation “\( b_1 \) and \( b_2 \) in \( \text{BMO}(\mathcal{X}) \) are equivalent if and only if \( b_1 - b_2 \) is constant” which we still denote by \( \text{BMO}(\mathcal{X}) \) equipped with \( \left\| \cdot \right\|_{\text{BMO}(\mathcal{X})} \) is a Banach space.

As proved in [5], we have that for every \( 1 \leq q < \infty \)
\[
(27) \quad \left\| b \right\|_{\text{BMO}(\mathcal{X})} \lesssim \sup_{B:\text{ball}} \left( \frac{1}{\mu(B)} \int_{B} |b - b_B|^q \, d\mu \right)^{\frac{1}{q}} \lesssim \left\| b \right\|_{\text{BMO}(\mathcal{X})},
\]
for all \( b \) in \( \text{BMO}(\mathcal{X}) \), where the supremum is taken over all balls of \( \mathcal{X} \).

We also have by the doubling condition of the measure \( \mu \), that for \( b \in \text{BMO}(\mathcal{X}) \), and \( B \) a ball in \( (\mathcal{X}, d) \),
\[
(28) \quad |b_B - b_{2^kB}| \leq C(1 + k) \left\| f \right\|_{\text{BMO}(\mathcal{X})} \quad \text{for all non negative integer } k,
\]

Theorem B of [5] (see also Theorem 5.3 of [11]) stated that for \( \frac{n}{n+\epsilon} < p < 1 \), the dual space of Hardy space \( \mathcal{H}^p(\mathcal{X}) \) is the Lipschitz space \( \Lambda_{\frac{1}{p-1}}(\mathcal{X}) \). We
recall that for $0 < \gamma$, the Lipschitz space $\Lambda_\gamma(\mathcal{X})$ is the set of those functions $f$ on $\mathcal{X}$ for which

\[(29) \quad |f(x) - f(y)| \leq A\mu(B)^\gamma,\]

where $B$ is any ball containing both $x$ and $y$ and $A$ is a constant depending only on $f$.

We can see that this definition of Lipschitz recovers the Euclidean case only when $0 < \gamma < \frac{1}{n}$. In fact, unless $\gamma$ is sufficiently small, it can happen that the only functions satisfying (29) are the constants. But, as shown in [4] there are situations where these spaces are not trivial. However, we are going to consider only $0 < \gamma < \frac{\epsilon}{n}$, since it is the range in which the atomic definition of Hardy coincides with the maximal function characterization. Let put

\[(30) \quad \|f\|_{\Lambda_\gamma(\mathcal{X})} = \inf \{ A : (29) \text{ holds} \}\]

then $\|\cdot\|_{\Lambda_\gamma(\mathcal{X})}$ is a norm on the set of equivalence classes of functions under the relation "$b_1$ and $b_2$ in $\Lambda_\gamma(\mathcal{X})$ are equivalent if and only if $b_1 - b_2$ is constant", which we still denote $\Lambda_\gamma(\mathcal{X})$.

3. A prerequisite about Orlicz spaces

Let

\[(31) \quad \varphi(t) = \frac{t}{\log (e + t)} \text{ for all } t > 0.\]

A $\mu$-measurable function $f : \mathcal{X} \to \mathbb{R}$ is said to belong to the Orlicz space $L^\varphi(\mathcal{X})$ if

\[(32) \quad \|f\|_{L^\varphi} := \inf \left\{ k > 0 : \int_{\mathcal{X}} \varphi \left( k^{-1} |f(x)| \right) d\mu(x) \leq 1 \right\} < \infty.\]

It is easy to see that $L^1(\mathcal{X}) \subset L^\varphi(\mathcal{X})$. More precisely, we have

\[(33) \quad \|f\|_{L^\varphi(\mathcal{X})} \leq \|f\|_{L^1(\mathcal{X})}.\]

We are going to recall some results involved Orlicz spaces mention in [2], which are also valid in the context of space of homogeneous type.

(i) If $\exp L(\mathcal{X})$ is the Orlicz space associated to the Orlicz function $t \mapsto e^t - 1$ and $\log L(\mathcal{X})$ the one associated to $t \mapsto t \log(e + t)$ then we have the following Hölder type inequality

\[(34) \quad \|fg\|_{L^\varphi(\mathcal{X})} \leq 4 \|f\|_{L^1(\mathcal{X})} \|g\|_{\exp L(\mathcal{X})}\]

for all $f \in L^\varphi(\mathcal{X})$ and $g \in \exp L(\mathcal{X})$ using the elementary inequality

\[(35) \quad \frac{ab}{\log(e + ab)} \leq a + e^b - 1 \text{ for all } a, b \geq 0.\]
We also have the duality between $\exp L(X)$ and $L \log L(X)$, that is
\begin{equation}
\|fg\|_{L^1(X)} \leq 2 \|f\|_{L \log L(X)} \|g\|_{\exp L(X)},
\end{equation}
using the following inequalities
\begin{equation}
abla \leq a \log(1 + a) + e^b - 1 \text{ for all } a, b \geq 0.
\end{equation}
(ii) Since the Orlicz function $\varphi$ we consider is not convex, the triangular inequality does not hold for $\|\cdot\|_{L^\varphi(X)}$. But we have the following substitute
\begin{equation}
\|f + g\|_{L^\varphi(X)} \leq 4 \|f\|_{L^\varphi(X)} + 4 \|g\|_{L^\varphi(X)},
\end{equation}
for $f, g \in L^\varphi(X)$. This relation remain valid if we replace the measure $\mu$ by any one absolutely continuous compared to $\mu$.

(iii) $L^\varphi(X)$ equipped with the metric
\begin{equation}
\delta(f, g) := \inf \left\{ \delta > 0 : \int_X \varphi(\delta^{-1}|f(x) - g(x)|) \, d\mu(x) \leq \delta \right\}
\end{equation}
is a complete linear metric space.

(iv) If $\delta(f, g) \leq 1$, then
\begin{equation}
\|f - g\|_{L^\varphi} \leq \delta(f, g) \leq 1.
\end{equation}

(v) A sequence $(f_n)_{n>0}$ converge in $L^\varphi(X)$ to $f$ if and only if $\lim_{n \to \infty} \|f_n - f\|_{L^\varphi} = 0$.

We define the Hardy-Orlicz space $H^\varphi(X)$, to be the subset of $G_\varphi(\beta, \gamma)'$ consists of distributions $f$ such that $f^* \in L^\varphi(X)$, and we put
\begin{equation}
\|f\|_{H^\varphi(X)} := \|f^*\|_{L^\varphi(X)}.
\end{equation}
In [21], it is proved that this characterization of Hardy-Orlicz spaces coincide with some atomic characterization.

**Lemma 3.1.** Let $b$ be in $\BMO(X)$. There exists a constant $C$ such that for every $(1, q)$-atom $a$ supported in a ball $B$,
\begin{equation}
\|(b - b_B)a^*\|_{L^1(X)} \leq C \|b\|_{\BMO(X)}.
\end{equation}

*Proof.* Let $b \in \BMO(X)$ and $a$ a $(1, q)$-atom supported in $B = B(x_0, R)$. We have
\begin{equation}
\|(b - b_B)a^*\|_{L^1(X)} = \int_{B(x_0, 2K_0R)} |b(z) - b_B| a^*(z) d\mu(z) + \int_{B^c(x_0, 2K_0R)} |b(z) - b_B| a^*(z) d\mu(z),
\end{equation}
where $B^c(x_0, 2K_0R) = X \setminus B(x_0, 2K_0R)$. Furthermore we have
\begin{equation}
a^*(z) \leq C M a(z) \text{ for all } z \in X,
\end{equation}
where $M a(z)$ is the maximal function of $a$.
where \( \mathcal{M}a(z) = \sup_{B: B \ni z} \frac{1}{\mu(B)} \int_B |a(x)| \, d\mu(x) \) denote the Hardy-Littlewood maximal function of \( a \), according to Proposition 3.10 of [8]. We also have

\[
M^*(z) \leq C \left( \frac{R}{d(z, x_0)} \right)^\beta \frac{1}{\mu(B(z, d(z, x_0)))}, \quad \text{for all } z \notin B(x_0, 2K_0 R),
\]

as it is shown in the proof of Lemma 4.4 of [8]. If we take (44) into first term of the sums (43) and use Hölder inequality with \( 1 < q < \infty \), then we have

\[
\left( \int_{B(x_0, 2K_0 R)} |b(z) - b_B| M^*(z) \, d\mu(z) \right) \leq \left( \int_{B(x_0, 2K_0 R)} |b(z) - b_B|^q \, d\mu(z) \right)^{\frac{1}{q}} \left( \int_{X} \mathcal{M}a(z)^q \, d\mu(z) \right)^{\frac{1}{q}}.
\]

Since the Hardy-Littlewood maximal operator \( \mathcal{M} \) is bounded in \( L^q(X) \), there exists a constant \( C \) such that

\[
\int_{B(x_0, 2K_0 R)} |b(z) - b_B| M^*(z) \, d\mu(z) \leq C \|b\|_{BMO(X)}
\]

according to relation (27).

On the other hand if we take (44) in the second term of (43) we have

\[
\left( \int_{B^c(x_0, 2K_0 R)} |b(z) - b_B| M^*(z) \, d\mu(z) \right) \leq \left( \int_{B^c(x_0, 2K_0 R)} |b(z) - b_B|^q \, d\mu(z) \right)^{\frac{1}{q}} \left( \int_{X} \mathcal{M}a(z)^q \, d\mu(z) \right)^{\frac{1}{q}}.
\]

Since the series \( \sum_{k=1}^{\infty} (2K_0)^{-k\beta} \) converges, we also have that there exists a constant \( C \) not depending on \( b \) and \( a \), such that

\[
\left( \int_{B^c(x_0, 2K_0 R)} |b(z) - b_B| M^*(z) \, d\mu(z) \right) \leq C \|b\|_{BMO(X)},
\]

which end the proof. \( \Box \)

It is well known that the John-Nirenberg inequality is valid in the context of space of homogeneous type (see [14]). This inequality states that there exist
constants $K_1$ and $K_2$ such that for any $b \in BMO(\mathcal{X})$ with $\|b\|_{BMO(\mathcal{X})} \neq 0$ and any ball $B \subset \mathcal{X}$, we have

\begin{equation}
\mu(\{x \in B : |b(x) - b_B| > \lambda\}) \leq K_1 \exp\left(-\frac{K_2 \lambda}{\|b\|_{BMO(\mathcal{X})}}\right) \mu(B) \text{ for all } \lambda > 0.
\end{equation}

An immediate consequence of this inequality is that there is a constant $K_3$ depending only on the space constants, such that

\begin{equation}
\frac{1}{\mu(B)} \int_B \exp\left(\frac{|b - b_B|}{K_3 \|b\|_{BMO(\mathcal{X})}}\right) \leq 2.
\end{equation}

for all balls $B$ in $\mathcal{X}$.

Notice that we can choose $K_3$ as big as we like.

**Lemma 3.2.** Let $B$ be the ball centered at $x_0$ with radius 1. There exists a positive constant $K_4$ such that for any $b \in BMO(\mathcal{X})$ with $\|b\|_{BMO(\mathcal{X})} \neq 0$ we have

\begin{equation}
\int_{\mathcal{X}} \frac{|b(x) - b_B|}{(1 + d(x_0, x))^{2n}} d\mu(x) \leq 1.
\end{equation}

**Proof.** Let $b \in BMO(\mathcal{X})$ with $\|b\|_{BMO(\mathcal{X})} \neq 0$. We have

\begin{equation}
\int_{\mathcal{X}} e^{-\frac{|b(x) - b_B|}{K_4 \|b\|_{BMO(\mathcal{X})}}} \frac{1}{(1 + d(x, x_0))^{2n}} d\mu(x) = \int_{B} e^{-\frac{|b(x) - b_B|}{K_4 \|b\|_{BMO(\mathcal{X})}}} \frac{1}{(1 + d(x, x_0))^{2n}} d\mu(x) + \int_{B^c} e^{-\frac{|b(x) - b_B|}{K_4 \|b\|_{BMO(\mathcal{X})}}} \frac{1}{(1 + d(x, x_0))^{2n}} d\mu(x),
\end{equation}

where $B^c = \mathcal{X} \setminus B$. The first term in the right hand side is less that $\mu(B)$, for the second term, we have

\begin{equation}
\int_{B^c} e^{-\frac{|b(x) - b_B|}{K_4 \|b\|_{BMO(\mathcal{X})}}} \frac{1}{(1 + d(x, x_0))^{2n}} d\mu(x) = \sum_{k=0}^{\infty} \int_{B(x_0, 2^{k+1})} \left( e^{-\frac{|b(x) - b_B|}{K_4 \|b\|_{BMO(\mathcal{X})}}} - 1 \right) d\mu(x).
\end{equation}

Using the fact that $|b_B - b_{B(x_0, 2^{k+1})}| \leq \log(2\frac{C_0 (k+1)}{\log 2}) \|b\|_{BMO(\mathcal{X})}$ and $\mu(B(x_0, 2^{k+1})) \leq 2^{(k+1)\log 2} C_0 \mu(B)$, we have the term we are estimated less than

\begin{equation}
C \mu(B) \sum_{k=0}^{\infty} 2^{-n+\frac{C_0}{K_4 \log 2}} k.
\end{equation}
Take \( K_3 > \frac{C_0}{n \log 2} \). Then the series (54) converges. Therefore,

\[
\int_X e^{K_3|\hat{b}(x) - \hat{b}|} \frac{1}{(1 + d(x_0, x))^{2n}} \, d\mu(x) \leq C \mu(B).
\]

The result follows. \(\square\)

Let us introduce the following measures

\[
d\nu := \frac{d\mu(x)}{\log(e + d(x_0, x))}, \quad d\sigma(x) := \frac{d\mu(x)}{(1 + d(x_0, x))^{2n}},
\]

where \( n \) is the dimension of \( \mathcal{X} \). It follows from the above lemma that for \( b \in \text{BMO}(\mathcal{X}) \) we have

\[
\|b - b_B\|_{\text{Exp} L(\mathcal{X}, \sigma)} \leq C \|b\|_{\text{BMO}(\mathcal{X})}.
\]

We can also see that for a \( \nu \)-measurable function \( f \), we have

\[
\|f\|_{L^\nu(\mathcal{X}, \nu)} \leq \|f\|_{L^1(\mathcal{X})}.
\]

The next result is the analogous of Lemma 3.2 of [2] in the context of spaces of homogeneous type, and its proof is just an adaptation of the one given in that paper.

**Lemma 3.3.** Let \( f \in \text{Exp} L(\mathcal{X}, \sigma) \). Then for \( g \in L^1(\mathcal{X}) \) we have \( g \cdot f \in L^\nu(\mathcal{X}, \nu) \) and

\[
\|g \cdot f\|_{L^\nu(\mathcal{X}, \nu)} \leq C \|g\|_{L^1(\mathcal{X})} \|f\|_{\text{Exp} L(\mathcal{X}, \sigma)}.
\]

If moreover \( f \in \text{BMO}(\mathcal{X}) \) then

\[
\|g \cdot f\|_{L^\nu(\mathcal{X}, \nu)} \leq C \|g\|_{L^1(\mathcal{X})} \|f\|_{\text{BMO}(\mathcal{X})}.
\]

**Proof.** Let \( f \in \text{Exp} L(\mathcal{X}, \sigma) \) and \( g \in L^1(\mathcal{X}) \). If \( \|g\|_{L^1(\mathcal{X})} = 0 \) or \( \|f\|_{\text{Exp} L(\mathcal{X}, \sigma)} = 0 \) then there is nothing to prove. Thus we assume that \( \|g\|_{L^1(\mathcal{X})} \|f\|_{\text{Exp} L(\mathcal{X}, \sigma)} \neq 0 \).

Let us put \( A = 8n \|g\|_{L^1(\mathcal{X})} \) and \( B = 8n \|f\|_{\text{Exp} L(\mathcal{X}, \sigma)} \). We are going to prove that the constant \( C \) is \( 64n^2 \). For this it is sufficient to prove that

\[
\int_X \log \left( e + \frac{1}{AB} |fg| \right) \frac{d\mu(x)}{\log(e + d(x_0, x))} \leq 1.
\]

For this purpose, we will use the following elementary inequality:

\[
2n \log(e + d(x_0, x)) > \log(e + (1 + d(x_0, x))^{2n}) \quad \text{for all } x \in \mathcal{X},
\]

and for all \( a, b > 0 \),

\[
\log(e + a) \log(e + b) > \frac{1}{2} \log(e + ab).
\]
It comes from the relation (62) that
\[
\frac{1}{AB} |fg| \leq \frac{2n}{AB} |fg| \log(e + d(x_0, x))^2n
\]
so that applying relation (63) to the left hand side of the inequality, yields
\[
\frac{1}{AB} |fg| \leq \frac{4n}{AB} |fg| (1 + d(x_0, x))^{2n}
\]
according to relation (65). Taking the integral of both sides we obtain inequality (59), since
\[
4n \left( \frac{e^{|f|}}{2} - 1 \right)
\]
\[
\frac{1}{2} (1 + d(x_0, x))^{2n}
\]
and
\[
4n \frac{|g|}{B} = \frac{1}{2} \frac{|g|}{\|g\|_{L^1(X)}}.
\]
The inequality (60) is also trivial if \( \|f\|_{BMO(X)} = 0 \). Thus we assume that \( f \) is not constant almost everywhere and we put \( f \cdot g = (f - f_B) \cdot g + f_B \cdot g \), so that using relation (38), relation (59) and (57), we have
\[
\|f \cdot g\|_{L^1(X, \nu)} \leq C \left( \|f - f_B\|_{BMO(X)} \|g\|_{L^1(X)} + |f_B| \|g\|_{L^1(X)} \right)
\]
\[
\leq C \|g\|_{L^1(X, \nu)} \|f\|_{BMO^+(X)},
\]
which complete our proof. \(\square\)

4. Proof of our main result

**Proof of Theorem 1.1** Let \( b \in BMO(\mathcal{X}) \) and \( h = \sum_{i=1}^{\infty} \lambda_i a_i \in \mathcal{H}^1(\mathcal{X}) \), where \( (a_i)_{i \geq 1} \) is a sequence of \( (p, \infty) \)-atoms, with \( a_i \) supported in the ball \( B_i \), and \( (\lambda_i)_{i \geq 1} \) a sequence of scalars such that \( \sum_{i=1}^{\infty} |\lambda_i| < \infty \). To prove our theorem, it is enough to show that the series
\[
\sum_{i=1}^{\infty} \lambda_i (b - b_{B_i}) a_i \quad \text{and} \quad \sum_{i=1}^{\infty} \lambda_i b_{B_i} a_i
\]
are convergent in \( L^1(\mathcal{X}) \) and \( \mathcal{H}^p(X, \nu) \) respectively, since the product \( b \times h \) by definition is the sum of both series.
The convergence of the first series in $L^1(\mathcal{X})$ is immediate, since for all index $i$ we have

$$\| \lambda_i (b - b_{B_i}) a_i \|_{L^1(\mathcal{X})} \leq |\lambda_i| \|b\|_{BMO(\mathcal{X})} \quad \text{and} \quad \sum_{i=1}^{\infty} |\lambda_i| < \infty,$$

according to Lemma 3.1. For the second series, we consider the partial sum

$$S_k^\ell := \sum_{i=k}^{\ell} \lambda_i a_i b_{B_i} \quad \text{for} \quad k < \ell.$$

Our series converges in $\mathcal{H}^p(\mathcal{X}, \nu)$ if and only if $\lim_{k \to \infty} \| (S_k^\ell)^* \|_{L^p(\mathcal{X}, \nu)} = 0$. But we have

$$(S_k^\ell)^* \leq \sum_{i=k}^{\ell} |\lambda_i| (a_i b_{B_i})^* \leq \sum_{i=k}^{\ell} |\lambda_i| |b - b_{B_i}| (a_i)^* + \left( \sum_{i=k}^{\ell} |\lambda_i| (a_i)^* \right) |b|,$$

so that

$$\left\| (S_k^\ell)^* \right\|_{L^p(\mathcal{X}, \nu)} \leq C \left[ \left\| \sum_{j=k}^{\ell} |\lambda_i| |b - b_{B_i}| (a_i)^* \right\|_{L^p(\mathcal{X}, \nu)} + \left\| \left( \sum_{i=k}^{\ell} |\lambda_i| (a_i)^* \right) |b| \right\|_{L^p(\mathcal{X}, \nu)} \right]$$

$$\leq C \left[ \left\| \sum_{i=k}^{\ell} |\lambda_i| |b - b_{B_i}| (a_i)^* \right\|_{L^1(\mathcal{X})} + \left\| \left( \sum_{i=k}^{\ell} |\lambda_i| (a_i)^* \right) |b| \right\|_{L^p(\mathcal{X}, \nu)} \right]$$

$$\leq C \|b\|_{BMO(\mathcal{X})} \sum_{i=k}^{\ell} |\lambda_i|,$$

where the last inequality come from Lemma 3.1 and Lemma 3.3. It comes out that,

$$\lim_{k \to \infty} \left\| (S_k^\ell)^* \right\|_{L^p} \leq C \|b\|_{BMO(\mathcal{X})} \lim_{k \to \infty} \sum_{i=k}^{\ell} |\lambda_i| = 0,$$

since $\sum_{i=1}^{\infty} |\lambda_i| < \infty$.

If we replace $BMO(\mathcal{X})$ by $bmo(\mathcal{X})$, then we obtain that the Hardy-Orlicz space does not depend on a weight. More precisely, we obtain the following result

**Proposition 4.1.** For $b$ in $bmo(\mathcal{X})$ and $h$ in $\mathcal{H}^1(\mathcal{X})$, we can give a meaning to the product $b \times h$ in the sense of distribution. Furthermore,

$$b \times h \in L^1(\mathcal{X}) + \mathcal{H}^p(\mathcal{X}).$$
Proof. The proof is almost similar to the one of Theorem 1.1. Let \( h \in \mathcal{H}^1(\mathcal{X}) \) be as in the previous theorem. We have for all \( i \)

\[
\|(b - B_i) a_i\|_{L^1(\mathcal{X})} \leq 2 \|b\|_{\text{bmo}(\mathcal{X})},
\]

so that \( \sum_{i=1}^{\infty}(b - B_i) a_i \) converge normally in \( L^1(\mathcal{X}) \).

Since for all \( i \) we have

\[
(b_i, a_i)^* \leq |b - B_i| a_i^* + |b| a_i^*,
\]

it follows that if

\[
|b| \left( \sum_{i=1}^{\infty} \lambda_i a_i^* \right)
\]

belongs to \( L^p(\mathcal{X}, d, \mu) \), then

\[
\sum_{i=1}^{\infty} \lambda_i b_i a_i
\]

converge in \( \mathcal{H}^p(\mathcal{X}) \), since according to Lemma 3.1 \( \sum \lambda_i (b - B_i) a_i^* \) converge normally in \( L^1(\mathcal{X}) \) and therefore in \( L^p(\mathcal{X}) \). Let us put \( \psi = \left| \sum_{i=1}^{\infty} \lambda_i a_i^* \right| \in L^1(\mathcal{X}) \), and consider a ball \( B \) such that \( \mu(B) = 1 \). We have, as proved in [1]

\[
\int_B \varphi(|b| \psi) d\mu = \int_B \frac{|b| \psi}{\log(e + |b| \psi)} d\mu \leq C \|b\|_{\text{bmo}(\mathcal{X})} \int_B \psi d\mu.
\]

In fact, we have

\[
\int_B \frac{|b| \psi}{\log(e + |b| \psi)} d\mu \leq \int_{B \cap \{|b| \leq 1\}} \psi d\mu + \int_{B \cap \{|b| > 1\}} |b| \frac{\psi}{\log(e + \psi)} d\mu.
\]

Since \( b \in \text{bmo}(\mathcal{X}, d, \mu) \) implies by the John-Nirenberg inequality [3] that there is a constant \( C \) depending only on the space constant, such that \( \|b\|_{\text{Exp L}(B)} \leq C \|b\|_{\text{bmo}(\mathcal{X})} \) and \( \left\| \frac{\psi}{\log(e + \psi)} \right\|_{L^1(\mathcal{X})} \leq \|\psi\|_{L^1(\mathcal{X})} \), the result follow from the duality between \( \text{Exp L}(B) \) and \( L \log L(B) \). This being true for all ball \( B \) of measure 1, we take the sum over all such ball which are almost disjoint.

\[
\square
\]

Let us consider now the Hardy space \( \mathcal{H}^p(\mathcal{X}) \), with \( p < 1 \). We have the following result

**Theorem 4.2.** Let \( \frac{2}{p+1} < p < 1 \). For \( f \in \Lambda^1_{p-1}(\mathcal{X}) \) and \( g \in \mathcal{H}^p(\mathcal{X}) \) we can give a meaning to the product \( f \times g \) as a distribution. Moreover, we have the inclusion

\[
f \times g \in L^1(\mathcal{X}) + \mathcal{H}^p(\mathcal{X}, d, \tau), \quad \text{where } d\tau(x) = \frac{d\mu(x)}{(2K_0^2 + K_0 d(x_0, x))^{(1-p)n}}.
\]
Proof. Let \( f \in \Lambda_{\frac{1}{p}-1}(\mathcal{X}) \) and \( g \in \mathcal{H}^p \). We assume that \( g \) has the following atomic decomposition
\[
(77) \quad g = \sum_{i=1}^{\infty} \lambda_i a_i,
\]
where \( a_i \)'s are atoms supported respectively in the balls \( B_i \). All we have to prove is that the series
\[
(78) \quad \sum_{i=0}^{\infty} \lambda_i (f - f_{B_i}) a_i
\]
and
\[
(79) \quad \sum_{i=0}^{\infty} \lambda_i f_{B_i} a_i
\]
converge respectively in \( L^1(\mathcal{X}) \) and in \( \mathcal{H}^p(\mathcal{X}, d, \tau) \). Arguing as in the previous theorem, we have that series \( (78) \) converges normally in \( L^1(\mathcal{X}) \). It remain to prove that \( (79) \) converge in \( \mathcal{H}^p(\mathcal{X}, d, \tau) \). As in Theorem 1.1, we have
\[
(80) \quad (S_k^\ell)^* \leq \sum_{i=k}^{\ell} |\lambda_i| (a_i f_{B_i})^* \leq \sum_{i=k}^{\ell} |\lambda_i| |f - f_{B_i}| (a_i)^* + \left( \sum_{i=k}^{\ell} |\lambda_i| (a_i)^* \right) |f|,
\]
where \( S_k^\ell = \sum_{i=k}^{\ell} \lambda_i a_i b_{B_i} \) for \( k < \ell \). We claim that Lemma 3.1 remain true if we replace the space \( \text{BMO}(\mathcal{X}) \) by \( \Lambda_{\frac{1}{p}-1}(\mathcal{X}) \) and the \((1, q)\)-atoms by \((p, q)\)-atoms \( q \geq 1 \), i.e. for \( f \in \Lambda_{\frac{1}{p}-1}(\mathcal{X}) \) and \( a \) a \((p, q)\)-atom supported in the ball \( B \),
\[
(81) \quad \| (f - f_B) a^* \|_{L^1} \leq C \| f \|_{\Lambda_{\frac{1}{p}-1}}.
\]
In fact, by the definition of Lipschitz space \( \Lambda_{\frac{1}{p}-1}(\mathcal{X}) \), we have
\[
(82) \quad \int_{B(x_0, 2K_0 R)} |f(z) - f_B| a^*(z) d\mu(z) \leq C \| f \|_{\Lambda_{\frac{1}{p}-1}(\mathcal{X})}.
\]
In other respect
\[
(83) \quad a^*(z) \leq C \mu(B(x_0, R))^{1 - \frac{1}{p}} \left( \frac{R}{d(z, x_0)} \right)^{\beta} \frac{1}{\mu(B(z, d(z, x_0)))},
\]
for all \( z \notin B(x_0, 2K_0 R) \) according to Lemma 4.4 of [8].

Arguing as in the proof of Lemma 3.1, we have that \( \sum |\lambda_i| |f - f_{B_i}| (a_i)^* \) converges in \( L^1(\mathcal{X}) \). The proof of the theorem will be complete if we establish that for any ball \( B \) of radius 1, we have for \( f \in \Lambda_{\frac{1}{p}-1}(\mathcal{X}) \) and \( \psi \in L^p(B) \)
\[
(84) \quad \int_{B} (|f(x)\psi(x)|)^p d\tau(x) \leq C \| f \|_{\Lambda_{\frac{1}{p}-1}(\mathcal{X})}^{p} \int_{B} |\psi(x)|^p d\mu(x),
\]
where \( \|f\|_{\Lambda_{\frac{1}{p}}^+(x)}^p = \|f\|_{\Lambda_{\frac{1}{p}}^-(x)}^p + \max(\|f(x_0)\|, 1)^p \). Following the method in [1], we have
\[
\int_{B \cap \{|f| > 1\}} \frac{|f(x)\psi(x)|^p}{(2K_0^2 + K_0d(x_0, x))^n(1-p)} d\mu(x) \leq \int_{B \cap \{|f| < 1\}} |\psi(x)|^p d\mu(x)
+ \int_{B \cap \{|f| > 1\}} \frac{|f(x)|^p |\psi(x)|^p}{(2K_0^2 + K_0d(x_0, x))^n(1-p)} d\mu(x).
\]
Furthermore,
\[
\int_{B \cap \{|f| > 1\}} \frac{|f(x)|^p |\psi(x)|^p}{(2K_0^2 + K_0d(x_0, x))^n(1-p)} d\mu(x)
\leq \int_{B \cap \{|f| > 1\}} |f(x) - f(x_0)|^p \frac{|\psi(x)|^p}{(2K_0^2 + K_0d(x_0, x))^n(1-p)} d\mu(x)
+ |f(x_0)|^p \int_{B \cap \{|f| > 1\}} \frac{|\psi(x)|^p}{(2K_0^2 + K_0d(x_0, x))^n(1-p)} d\mu(x).
\]
Since \( B \subset B(x_0, 2K_0^2 + K_0d(x_0, x)) \) for all \( x \) in the ball \( B \) of radius 1, it comes from the definition of Lipschitz space \( \Lambda_{\frac{1}{p}}^-(\mathcal{X}) \) that the first term in the right hand side of the above inequality is less or equal to
\[
\|f\|_{\Lambda_{\frac{1}{p}}^-}(x) \int_{B} \frac{\mu(B(x_0, 2K_0^2 + K_0d(x_0, x)))^{1-p}}{(2K_0^2 + K_0d(x_0, x))^n(1-p)} |\psi(x)|^p d\mu(x).
\]
But, from (85) and (86) we have that \( \mu(B(x_0, 2K_0^2 + K_0d(x_0, x))) \lesssim (2K_0^2 + K_0d(x_0, x))^n \).
Thus
\[
\int_{B \cap \{|f| > 1\}} \frac{|f(x)|^p |\psi(x)|^p}{(2K_0^2 + K_0d(x_0, x))^n(1-p)} d\mu(x) \lesssim \left( \|f\|_{\Lambda_{\frac{1}{p}}^-}(x) + |f(x_0)|^p \right) \int_{B} |\psi(x)|^p d\mu(x).
\]
The result follow by covering the hold space by almost disjoint balls of radius 1. \( \square \)

**Remark 4.3.** Let \( \frac{n}{p+\epsilon} < p < 1 \) and \( \gamma := \frac{1}{p} - 1 \). Then, for \( h \in \mathcal{H}^p(\mathcal{X}) \) and \( f \in \Lambda_{\gamma}(\mathcal{X}) \cap L^{\infty}(\mathcal{X}) \), the product \( h \times f \) can be given a meaning in the sense of distributions. Moreover, we have the inclusion
\[
(87) \quad h \times f \in L^1(\mathcal{X}) + \mathcal{H}^p(\mathcal{X}).
\]
Proof. Let $h \in \mathcal{H}^p(X)$ be as in (9), where the atoms involved are $(p, \infty)$-atoms, and $f \in \Lambda_\gamma(X)$. From Theorem 4.2 we have that

$$(88) \quad \sum_{i=1}^{\infty} \lambda_i (f - f_{B_i}) a_i$$

converge in $L^1(X)$. For the series $\sum_{i=1}^{\infty} \lambda_i f_{B_i} a_i$, we just have to remark that the functions $\frac{1}{\|f\|_{L^\infty(X)}} f_{B_i} a_i$ are $(p, \infty)$-atoms. In fact,

(i) $\text{supp} f_{B_i} a_i \subset B_i$, since $\text{supp} a_i \subset B_i$

(ii) $\int_X f_{B_i} a_i(x) dx = 0$

(iii) $|f_{B_i} a_i(x)| \leq \|f\|_{L^\infty(X)} \mu(B_i)^{-\frac{1}{p}}$

and this end the proof, since $\sum_{i=1}^{\infty} |A_i|^p < \infty$ \hfill $\square$

Remark 4.4. In the case $\mu(X) < \infty$, all our results remain valid, provided we consider the constant function $\mu(X)^{-\frac{1}{p}}$ as an atom, and put

$$(89) \quad \|b\|_{\text{BMO}(X)} = \sup_{B: \text{ball}} \frac{1}{\mu(B)} \int_B |b(x) - b_B| d\mu(x) + \|b\|_{L^1(X)}$$

and

$$(90) \quad \|f\|_{\Lambda_\gamma(X)} = \sup \left\{ \frac{|f(x) - f(y)|}{\mu(B)}, \text{ for all ball } B \ni x, y \right\} + \left| \int_X f(x)d\mu(x) \right| .$$

In this case the reverse doubling condition [4], need to be satisfied just for small balls.

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