Spectral stability of small amplitude solitary waves of the Dirac equation with the Soler-type nonlinearity

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Abstract

We study the point spectrum of the linearization at a solitary wave solution \( \phi_\omega(x)e^{-i\omega t} \) to the nonlinear Dirac equation in \( \mathbb{R}^n \), for all \( n \geq 1 \), with the nonlinear term given by \( f(\psi^*\beta\psi)\beta\psi \) (known as the Soler model). We focus on the spectral stability, that is, the absence of eigenvalues with positive real part, in the non-relativistic limit \( \omega \lesssim m \), in the case when \( f \in C^1(\mathbb{R}\setminus\{0\}) \), \( f(\tau) = |\tau|^k + O(|\tau|^K) \) for \( \tau \to 0 \), with \( 0 < k < K \). For \( n \geq 1 \), we prove the spectral stability of small amplitude solitary waves \( (\omega \lesssim m) \) for the charge-subcritical cases \( k \lesssim 2/n \) (in particular, \( 1 < k \leq 2 \) when \( n = 1 \)) and for the “charge-critical case” \( k = 2/n \) (with \( K > 4/n \)).

An important part of the stability analysis is the proof of the absence of bifurcations of nonzero-real-part eigenvalues from the embedded threshold points at \( \pm 2m \). Our approach is based on constructing a new family of exact bi-frequency solitary wave solutions in the Soler model, on using this family to determine the multiplicity of \( \pm 2\omega \) eigenvalues of the linearized operator, and on the analysis of the behaviour of “nonlinear eigenvalues” (characteristic roots of holomorphic operator-valued functions).

1 Introduction

We study stability of solitary waves in the nonlinear Dirac equation with the scalar self-interaction [Iva38, Sol70], known as the Soler model:

\[
i\partial_t \psi = D_m \psi - f(\psi^*\beta\psi)\beta\psi, \quad \psi(x,t) \in \mathbb{C}^N, \quad x \in \mathbb{R}^n, \quad n \geq 1.
\]

Here the Dirac operator is given by \( D_m = -i\alpha \cdot \nabla_x + \beta m \), with \( m > 0 \) and the self-adjoint \( N \times N \) Dirac matrices \( \alpha^j, 1 \leq j \leq n \), and \( \beta \) chosen so that \( D_m^2 = -\Delta + m^2 \); for details, see notations at the end of this section. We assume that \( f \in C^1(\mathbb{R}\setminus\{0\}) \) is real-valued, \( f(\tau) = |\tau|^k + O(|\tau|^K) \) as \( \tau \to 0 \), with \( 0 < k < K \). The structure of the nonlinearity, \( f(\psi^*\beta\psi)\beta\psi \), is such that the equation is \( U(1) \)-invariant and hamiltonian.

Given a solitary wave solution \( \psi(x,t) = \phi_\omega(x)e^{-i\omega t} \) to (1.1), with \( \omega \in \mathbb{R} \) and \( \phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N) \), we consider its perturbation, \( (\phi_\omega(x) + \rho(x,t))e^{-i\omega t} \), and study the spectrum of the linearized equation on \( \rho \) (that is, the spectrum of the linearization operator). We will say that this particular solitary wave is spectrally stable if the spectrum of the linearization operator has no points in the right half-plane. In the present work, we prove the spectral stability of small amplitude solitary waves corresponding to the nonrelativistic limit \( \omega \lesssim m \), in the case \( k \lesssim 2/n, K > k \) (“charge-subcritical”) and \( k = 2/n, K > 4 \) (“charge-critical”). This is the first rigorous result on spectral stability of solitary wave solutions of the nonlinear Dirac equation; it opens the way to the proofs of asymptotic stability in the nonlinear Dirac context.
The question of stability of solitary waves is answered in many cases for the nonlinear Schrödinger, Klein–Gordon, and Korteweg–de Vries equations (see e.g. the review [Str89]). In these systems, at the points represented by solitary waves, the hamiltonian function is of finite Morse index. In simpler cases, the Morse index is equal to one, and the perturbations in the corresponding direction are prohibited by a conservation law when the Vakhitov–Kolokolov condition [VK73] is satisfied. In other words, the solitary waves could be demonstrated to correspond to conditional minimizers of the energy under the charge constraint; this results not only in spectral stability but also in orbital stability [CL82, Wei85, SS85, Wei86, GSS87]. The nature of stability of solitary wave solutions of the nonlinear Dirac equation seems completely different from this picture [Ran83, Section V]. In particular, the hamiltonian function is not bounded from below, and is of infinite Morse index; the NLS-type approach to stability fails. As a consequence, we do not know how to prove the orbital stability but via proving the asymptotic stability first. The only known exception is the completely integrable massive Thirring model in (1+1)D, where the orbital stability was proved by means of a coercive conservation law [PS14, CPS16] coming from higher order integrals of motion.

The spectral stability of solitary waves to the cubic nonlinear Dirac equation in (1+1)D (known as the massive Gross–Neveu model) was demonstrated in [BC12a], where the spectrum of the linearization at solitary waves was computed via the Evans function technique; no nonzero-real-part eigenvalues have been detected; this result was confirmed by numerical simulations of the dynamics in [Lak18]. A similar model in dimension 1 is given by the nonlinear coupled-mode equations; the numerical analysis of spectral stability of solitary waves in such models has been done in [BPZ98, CP06, GW08]. In the absence of spectral stability, one expects to be able to prove orbital instability, in the sense of [GSS87]; in the context of the nonlinear Schrödinger equation, such instability is proved in e.g. [KS07, GO12]. If instead a particular solitary wave is spectrally stable, one hopes to prove the asymptotic stability. Let us give a brief account on asymptotic stability results in dispersive models with unitary invariance. The asymptotic stability for the nonlinear Schrödinger equation is proved using the dispersive properties; see for instance the seminal works [SW90, SW92] for small amplitude solitary waves bifurcating from the ground state of the linear Schrödinger equation (thus with a potential) and [BP92a, BP92b, BP92c, BP95] in the translation-invariant case, in dimension 1. Under ad hoc assumptions on the spectral stability some extensions to the nonlinear Schrödinger equation in any dimension can be expected, see [Cuc01, Cuc03, Cuc09]. The analysis of the dynamics of excited states and possible relaxation to the ground state solution for small solitary waves (in any dimension) was considered in [TY02a, TY02b, TY02c, TY02d, BS03, SW04, KS06, Sch09]. These have been improved in [PW97, Wed00, GNT04, GS05, CP06, GS06, GS07, KZ07, Miz07, Cuc08, CM08, CT09, KMz09, KZ09, Cuc11, CP14]. This path is also developed for the nonlinear Dirac equation in [Bou06, Bou08, BC12b, PS12, CT16, CPS17]. The needed dispersive properties of Dirac type models have been studied in these references and separately in [EV97, MNNO05, DF07, DF08, D’A08, BDF11, Kop11, CD13, Kop13, KT16, BG16, EGT16]. Note that the most famous class of dispersive estimates is the one of the Strichartz estimates; this class was commonly used as a major tool for well-posedness in some of the above references. We also refer, for the well-posedness problem, to the review [Pel11]. The question of the existence of stationary solutions, related to the indefiniteness of the energy, is discussed in the review [ELS08].

While the purely imaginary essential spectrum of the linearization operator is readily available via the Weyl’s theorem on the essential spectrum (see [BC16] for more details and for the background on the topic), the discrete spectrum is much more delicate. Our aim in this work is to investigate the presence of eigenvalues with positive real part, which would be responsible for the linear instability of a particular solitary wave. As \( \omega \) changes, such eigenvalues can bifurcate from the point spectrum on the imaginary axis or even from the essential spectrum. In [BC16], we have already shown that the bifurcations of eigenvalues from the essential spectrum into the half-planes with \( \text{Re} \lambda \neq 0 \) are only possible from the collisions of eigenvalues on the imaginary axis or from the embedded eigenvalues (let us mention that by [BC16, Theorem 2.2] there are no embedded eigenvalues beyond the embedded thresholds at \( \pm (m + |\omega|)i \)). There are also the following exceptional cases: the bifurcations could start at the embedded thresholds at \( \pm i(m + |\omega|) \) [BPZ98] or at the point of the collision of the edges of the continuous spectrum at \( \lambda = 0 \) when \( \omega = \pm m \) [CGG14] and at \( \lambda = \pm mi \) when \( \omega = 0 \) [KS02].
Let us mention that the linear instability in the nonrelativistic limit \( \omega \lesssim m \) in the “charge-supercritical” case \( k > 2/\omega \) (complementary to cases which we consider in this work) follows from [CGG14]; the restrictions in that article were \( k \in \mathbb{N} \) and \( n \leq 3 \), but they are easily removed by using the nonrelativistic asymptotics of solitary waves obtained in [BC17a]. By numerics of [CMKS+16], in the case of the pure-power nonlinearity \( f(\tau) = |\tau|^k \), \( k > 2/n \), the spectral instability disappears when \( \omega \in (0, m) \) becomes sufficiently small.

We note that quintic nonlinear Schrödinger equation in (1+1)D and the cubic one in (2+1)D are “charge critical” (all solitary waves have the same charge), and as a consequence the linearization at any solitary wave has a \( 4 \times 4 \) Jordan block at \( \lambda = 0 \), resulting in dynamic instability of all solitary waves; moreover, there is a blow-up phenomenon in the charge-critical as well as in the charge-supercritical cases; see in particular [ZSS71, ZSS75, Gla77, Wei83, Mer90]. On the contrary, for the nonlinear Dirac with the critical-power nonlinearity, the charge of solitary waves is no longer constant: by [BC17a], one has \( \partial_\omega Q(\phi_\omega) < 0 \) for \( \omega \lesssim m \), where \( Q(\phi_\omega) \) is the corresponding charge ((1.3) below). As a consequence, the linearization at solitary waves in the nonrelativistic limit has no \( 4 \times 4 \) Jordan block, which resolves into \( 2 \times 2 \) Jordan block (corresponding to the unitary invariance) and two purely imaginary eigenvalues.

Here is the plan of the present work. The results are stated in Section 2. The linearization operator is introduced in Section 3. In Sections 4 and 5, we study bifurcations of eigenvalues from the embedded thresholds at \( \pm (m + |\omega|)|i \) in the nonrelativistic limit \( \omega \to m - 0 \). In particular, we develop the theory of characteristic roots of operator-valued holomorphic functions, in the spirit of [Kel51, Kel71, MZS70, GS71] (this is done in Section 5). The bifurcations of eigenvalues from the origin are analyzed in Section 6.

In Appendix A, we construct the analytic continuation of the resolvent of the free Laplace operator, extending the three-dimensional approach of [Ra78] to all dimensions \( n \geq 1 \). In Appendix B we give details on the spectral theory for the nonlinear Schrödinger equation linearized at a solitary wave.

**Notations.** We denote \( \mathbb{N} = \{1, 2, \ldots\} \) and \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \). For \( \rho > 0 \), an open disc of radius \( \rho \) in the complex plane centered at \( z_0 \in \mathbb{C} \) is denoted by \( D_\rho(z_0) = \{ z \in \mathbb{C} ; |z - z_0| < \rho \} \); we also denote \( D_\rho = D_\rho(0) \). We denote \( r = |x| \) for \( x \in \mathbb{R}^n \), \( n \in \mathbb{N} \), and, abusing notations, we will also denote the operator of multiplication with \( |x| \) and \( \langle x \rangle = (1 + |x|^2)^{1/2} \), respectively.

We denote the standard \( L^2 \)-based Sobolev spaces of \( \mathbb{C}^N \)-valued functions by \( H^k(\mathbb{R}^n, \mathbb{C}^N) \). For \( s, k \in \mathbb{R} \), we define the weighted Sobolev spaces

\[
H^k_s(\mathbb{R}^n, \mathbb{C}^N) = \{ u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^N), \| u \|_{H^k_s} < \infty \}, \quad \| u \|_{H^k_s} = \| \langle r \rangle^s (-i\nabla)^k u \|_{L^2}.
\]

We write \( L^2_s(\mathbb{R}^n, \mathbb{C}^N) \) for \( H^0_s(\mathbb{R}^n, \mathbb{C}^N) \). For \( u \in L^2_s(\mathbb{R}^n, \mathbb{C}^N) \), we denote \( \| u \| = \| u \|_{L^2} \).

For any pair of normed vector spaces \( E \) and \( F \), let \( \mathcal{B}(E, F) \) denote the set of bounded linear maps from \( E \) to \( F \). For an unbounded linear operator \( A \) acting in a Banach space \( X \) with a dense domain \( D(A) \subset X \), the spectrum \( \sigma(A) \) is the set of values \( \lambda \in \mathbb{C} \) such that the operator \( A - \lambda : D(A) \to X \) does not have a bounded inverse. The generalized null space of \( A \) is defined by

\[
\mathcal{N}_g(A) := \bigcup_{k \in \mathbb{N}} \ker(A^k) = \bigcup_{k \in \mathbb{N}} \{ v \in D(A) ; A^j v \in D(A) \forall j < k, A^k v = 0 \}.
\]

The discrete spectrum \( \sigma_{\text{disc}}(A) \) is the set of isolated eigenvalues \( \lambda \in \sigma(A) \) of finite algebraic multiplicity, such that \( \dim \mathcal{N}_g(A - \lambda) < \infty \). The essential spectrum \( \sigma_{\text{ess}}(A) \) is the complementary set of discrete spectrum in the spectrum. The point spectrum \( \sigma_p(A) \) is the set of eigenvalues (isolated or embedded into the essential spectrum).

We denote the free Dirac operator by

\[
D_m = -i \alpha \cdot \nabla + \beta m = -i \sum_{j=1}^n \alpha^j \frac{\partial}{\partial x^j} + \beta m, \quad m > 0,
\]

and the massless Dirac operator by \( D_0 = -i \alpha \cdot \nabla \). Above, \( \alpha^j \) and \( \beta \) are self-adjoint \( N \times N \) Dirac matrices which satisfy \( (\alpha^j)^2 = \beta^2 = 1_N \), \( \alpha^j \alpha^k + \alpha^k \alpha^j = 2 \delta_{jk} 1_N \), \( \alpha^j \beta + \beta \alpha^j = 0 \), \( 1 \leq j, k \leq n \), so that \( D_m^2 = \ldots \)
\((-\Delta + m^2)1_N\). Here \(1_N\) denotes the \(N \times N\) identity matrix. The anticommutation relations lead to e.g. \(\text{Tr } \alpha^j = \text{Tr } \beta^{-1} \alpha^j \beta = - \text{Tr } \alpha^j = 0, 1 \leq j \leq n\), and similarly \(\text{Tr } \beta = 0\); together with \(\sigma(\alpha^j) = \sigma(\beta) = \{\pm 1\}\), this yields the conclusion that \(N\) is even. Let us mention that the Clifford algebra representation theory (see e.g. [Fed96, Chapter 1, §5.3]) shows that there is a relation \(N = 2^{(n+1)/2}M, M \in \mathbb{N}\). Without loss of generality, we may assume that \(\beta = \begin{bmatrix} 1_{N/2} & 0 \\ 0 & -1_{N/2} \end{bmatrix}\). Then the anticommutation relations \(\{\alpha^j, \beta\} = 0\) show that the matrices \((\alpha^j)_{1 \leq j \leq n}\) are block-antidiagonal, \(\alpha^j = \begin{bmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{bmatrix}, 1 \leq j \leq n\), where the matrices \((\sigma_j)_{1 \leq j \leq n}\) satisfy \(\sigma_j^2 = \delta_{jk}\), \(\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk}, 1 \leq j, k \leq n\); the Dirac operator is thus given by
\[
D_m = -\imath \alpha \cdot \nabla + \beta m = -\imath \sum_{j=1}^n \begin{bmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{bmatrix} \partial_j + m \begin{bmatrix} 1_{N/2} & 0 \\ 0 & -1_{N/2} \end{bmatrix}, \quad m > 0.
\] (1.2)

The charge functional, which is (formally) conserved due to the \(U(1)\)-invariance of the nonlinear Dirac equation (1.1), is denoted by
\[
Q(\psi) = \int_{\mathbb{R}^n} \psi^* (x, t) \psi(x, t) \, dx.
\] (1.3)

## 2 Main results

We consider the nonlinear Dirac equation (1.1), with the Dirac operator \(D_m\) of the form (1.2).

**Assumption 2.1.** One has \(f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})\), and there are \(k > 0\) and \(K > k\) such that
\[
|f(\tau) - |\tau|^k| = O(|\tau|^K), \quad |\tau f'(\tau) - k|\tau|^k| = O(|\tau|^K); \quad |\tau| \leq 1.
\]

If \(n \geq 3\), we additionally assume that \(k < 2/(n - 2)\).

In the nonrelativistic limit \(\omega \lesssim m\), the solitary waves to nonlinear Dirac equation could be obtained as bifurcations from the solitary wave solutions \(\varphi_\omega(y)e^{-\imath \omega t}\) to the nonlinear Schrödinger equation
\[
i \dot{\psi} = -\frac{1}{2m} \Delta \psi - |\psi|^{2k} \psi, \quad \psi(y, t) \in \mathbb{C}, \quad y \in \mathbb{R}^n.
\] (2.1)

By [Str77, BL83] and [BGK83] (for the two-dimensional case), the stationary nonlinear Schrödinger equation
\[
-\frac{1}{2m} u - \frac{1}{2m} \Delta u - |u|^{2k} u, \quad u(y) \in \mathbb{R}, \quad y \in \mathbb{R}^n, \quad n \geq 1
\]
has a strictly positive spherically symmetric exponentially decaying solution \(u_k \in C^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)\) (called the ground state) if and only if \(0 < k < 2/(n - 2)\) (any \(k > 0\) if \(n \leq 2\)). There are \(0 < c_{n,k} < C_{n,k} < \infty\) such that
\[
c_{n,k} |y|^{-(n-1)/2} e^{-|y|} \leq u_k(y) \leq C_{n,k} |y|^{-(n-1)/2} e^{-|y|}, \quad y \in \mathbb{R}^n;
\] (2.3)

see e.g. [BC17a, Lemma 4.5]. We set
\[
\hat{V}(t) := u_k(|t|), \quad \hat{U}(t) := -(2m)^{-1} \hat{V}'(t), \quad t \in \mathbb{R},
\] (2.4)

where \(u_k\) is considered as a function of \(r = |y|, y \in \mathbb{R}^n\). By (2.2), the functions \(\hat{V} \in C^2(\mathbb{R})\) and \(\hat{U} \in C^1(\mathbb{R})\) (which are even and odd, respectively) satisfy
\[
\frac{1}{2m} \hat{V} + \partial_t \hat{U} + \frac{n-1}{t} \hat{U} = |\hat{V}|^{2k} \hat{V}, \quad \partial_t \hat{V} + 2m \hat{U} = 0, \quad t \in \mathbb{R},
\] (2.5)

where \(\hat{U}(t)/t\) at \(t = 0\) is understood in the limit sense, \(\lim_{t \to 0} \hat{U}(t)/t = \hat{U}'(0)\).

In the nonrelativistic limit \(\omega \lesssim m\), the solitary wave solutions to (1.1) are obtained as bifurcations from \((\hat{V}, \hat{U})\) [BC17a]; we start with summarizing their asymptotics.
Theorem 2.1. Let $n \in \mathbb{N}$, $N = 2^{[(n+1)/2]}$. Assume that the function $f$ in (1.1) satisfies Assumption 2.1 with some $k$, $K$. There is $\omega_0 \in (m/2, m)$ such that for all $\omega \in (\omega_0, m)$ there are solitary wave solutions $\phi_\omega(x)e^{-i\omega t}$ to (1.1), such that $\phi_\omega \in H^2(\mathbb{R}^n, \mathbb{C}^N) \cap C(\mathbb{R}^n, \mathbb{C}^N)$, $\omega \in (\omega_0, m)$, with
\[
\phi_\omega(x) = \frac{\beta_\omega(x)}{|\phi_\omega(x)|^2/2} \quad \forall x \in \mathbb{R}^n, \quad \forall \omega \in (\omega_0, m),
\]
and
\[
\|\phi_\omega\|_{L^\infty(\mathbb{R}^n, \mathbb{C}^N)} = O((m^2 - \omega^2)^{1/2}), \quad \omega \lesssim m.
\]
More explicitly,
\[
\phi_\omega(x) = \begin{bmatrix} \frac{v(r, \omega)}{\sqrt{u(r, \omega) \xi}} \end{bmatrix}, \quad r = |x|, \quad \xi \in \mathbb{C}^{N/2}, \quad |\xi| = 1,
\]
where $v(r, \omega) = e^{\frac{t}{2r}}V(\epsilon r, \epsilon)$, $u(r, \omega) = e^{1+\frac{t}{2r}}U(\epsilon r, \epsilon)$, $r \geq 0$, with $\epsilon = \sqrt{m^2 - \omega^2}$, $\lim_{r \to 0} u(r, \epsilon) = 0$, and
\[
V(t, \epsilon) = \hat{V}(t) + \hat{V}(t, \epsilon), \quad U(t, \epsilon) = \hat{U}(t) + \hat{U}(t, \epsilon), \quad t \in \mathbb{R}, \quad \epsilon > 0,
\]
with $\hat{V}(t)$, $\hat{U}(t)$ defined in (2.5). There are $\gamma > 0$ and $\alpha < \infty$ such that $\hat{W}(t, \epsilon) = \begin{bmatrix} \hat{V}(t, \epsilon) \\ \hat{U}(t, \epsilon) \end{bmatrix}$ satisfies
\[
\|e^{\gamma(t)}\hat{W}\|_{H^1(\mathbb{R}, \mathbb{R}^2)} \leq \alpha e^{2x}, \quad \forall t \in (0, \epsilon_0),
\]
(2.8)
with $\epsilon_0 = \sqrt{m^2 - \omega_0^2}$ and
\[
\kappa := \min (1, K/k - 1).
\]
There is $b_0 < \infty$ such that
\[
|\hat{V}(t, \epsilon)| + |\hat{U}(t, \epsilon)| \leq b_0(t)^{-(n-1)/2}e^{-|t|}, \quad \forall t \in \mathbb{R}, \quad \forall \epsilon \in (0, \epsilon_0).
\]
(2.10)
There is a $C^1$ map $\omega \mapsto \phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N)$, with
\[
\partial_x \phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N), \quad \partial_x \tilde{W}(t, \epsilon) \in H^1(\mathbb{R}) \times H^1(\mathbb{R} \times \mathbb{R}/2),
\]
where $H^1(\mathbb{R})$ and $H^1(\mathbb{R} \times \mathbb{R}/2)$ denote functions from $H^1(\mathbb{R})$ which are even and odd, respectively;
\[
\|e^{\gamma(t)}\partial_x \tilde{W}(t, \epsilon)\|_{H^1(\mathbb{R}, \mathbb{R}^2)} = O(e^{2\kappa - 1}), \quad \epsilon \in (0, \epsilon_0),
\]
(2.11)
and there is $b > 0$ such that
\[
\|\partial_x \phi_\omega\|_{L^2(\mathbb{R}^n, \mathbb{C}^N)} = be^{-n+\frac{t}{2}}(1 + O(e^{2\kappa})), \quad \omega = \sqrt{m^2 - \kappa^2}, \quad \epsilon \in (0, \epsilon_0).
\]
Additionally, assume that $k$, $K$ from Assumption 2.1 satisfy either $k < 2/n$, or $k = 2/n$, $K > 4/n$. Then there is $\omega_1 < m$ such that $\partial_{\omega}Q(\phi_\omega) < 0$ for all $\omega \in (\omega_1, m)$, if instead $k > 2/n$, then there is $\omega_1 < m$ such that $\partial_{\omega}Q(\phi_\omega) > 0$ for all $\omega \in (\omega_1, m)$.

Above, $Q(\phi_\omega)$ is the charge functional defined in (1.3) evaluated at the solitary wave $\phi_\omega e^{-i\omega t}$.

Remark 2.2. If $f$ satisfies Assumption 2.1, then we may assume that there are $c$, $C < \infty$ such that
\[
|f(\tau) - |\tau|^k| \leq c|\tau|^K, \quad |f(\tau)| \leq (c+1)|\tau|^k, \quad \forall \tau \in \mathbb{R},
\]
(2.12)
\[
|\tau f'(\tau) - k|\tau|^k| \leq C|\tau|^K, \quad |\tau f'(\tau)| \leq (C + k)|\tau|^k, \quad \forall \tau \in \mathbb{R}.
\]
(2.13)
Indeed, we could achieve (2.12) by modifying $f(\tau)$ for $|\tau| > 1$, and since the $L^\infty$-norm of the resulting family of solitary waves goes to zero as $\omega \to m$ (cf. Theorem 2.1), we could then take $\omega_0 \lesssim m$ sufficiently close to $m$ so that $\|\phi_\omega\|_{L^\infty}$ remains smaller than one for $\omega \in (\omega_0, m)$. 

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By [BC16, Theorem 2.19], the eigenvalues of the linearization at solitary waves \( \phi_\omega e^{-i\omega t} \) with \( \omega = \omega_j \), \( \omega_j \to m \), can only accumulate to \( \lambda = \pm 2mi \) and \( \lambda = 0 \). We are going to relate the families of eigenvalues of the linearized nonlinear Dirac equation bifurcating from \( \lambda = 0 \) and from \( \lambda = \pm 2mi \) to the eigenvalues of the linearization of the nonlinear Schrödinger equation at a solitary wave. Given \( u_k(x) \), a strictly positive spherically symmetric exponentially decaying solution to (2.2), then \( u_k(x)e^{-i\omega t} \) with \( \omega = -\frac{1}{2m} \) is a solitary wave solution to the nonlinear Schrödinger equation (2.1). The linearization at this solitary wave is given by 
\[
\partial_t \rho = j \lambda \rho \text{ (see e.g. [VK73]), where}
\]
\[
j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \lambda = \begin{bmatrix} l_+ & 0 \\ 0 & l_- \end{bmatrix},
\]
\[
l_+ = \frac{1}{2m} - \frac{\Delta}{2m} - u_k^{2k}, \quad l_- = \frac{1}{2m} - \frac{\Delta}{2m} - (1 + 2k)u_k^{2k}, \quad D(l_\pm) = H^2(\mathbb{R}^n).
\]

**Theorem 2.2** (Bifurcations from \( \pm 2mi \) at \( \omega = m \)). Let \( n \geq 1 \). Let \( f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}) \) satisfy Assumption 2.1 with some values of \( k, K \). Let \( \phi_\omega(x)e^{-i\omega t} \), \( \omega \in (\omega_0, m) \), be a family of solitary wave solutions to (1.1) constructed in Theorem 2.1.

Let \( (\omega_j)_{j \in \mathbb{N}} \), \( \omega_j \in (\omega_0, m) \), be a sequence such that \( \omega_j \to m \) and assume that \( \lambda_j \) are eigenvalues of (1.1) linearized at \( \phi_\omega e^{-i\omega_j t} \) (see Section 4) such that \( \lambda_j \to 2mi \). Denote
\[
z_j = -\frac{2\omega_j + i\lambda_j}{\epsilon_j^2} \in \mathbb{C}, \quad \epsilon_j := (m^2 - \omega_j^2)^{1/2}, \quad j \in \mathbb{N},
\]
and let \( Z_0 \in \mathbb{C} \cup \{\infty\} \) be an accumulation point of the sequence \( (z_j)_{j \in \mathbb{N}} \). Then:

1. \( Z_0 \in \left\{ \frac{1}{2m} \right\} \cup \sigma_d(l_-). \) In particular, \( Z_0 \neq \infty \).
2. If the edge of the essential spectrum of \( l_- \) at \( 1/(2m) \) is a regular point of the spectrum of \( l_- \) (neither a resonance nor an eigenvalue), then \( Z_0 \neq 1/(2m) \).
3. If \( Z_0 = 0 \), then \( \lambda_j = 2\omega_j \epsilon_j \) for all but finitely many \( j \in \mathbb{N} \).

**Remark 2.3.** The definition (2.16) is chosen so that
\[
\lambda_j = i(2\omega_j + \epsilon_j^2 z_j), \quad j \in \mathbb{N}.
\]

**Remark 2.4.** For definitions and more details on resonances (known in this context as zero-energy resonances, virtual levels, and half-bound states), see e.g. [JK79] (for the case \( n = 3 \)), [JN01, Sections 5, 6] (for \( n = 1, 2 \), and [Yaf10, Sections 5.2 and 7.4] (for \( n = 1 \) and \( n \geq 3 \)). For the practical purposes, in the case \( V \neq 0 \), the resonances can be characterized as \( L^\infty \)-solutions to \(( -\Delta + V )\Psi = 0 \) which do not belong to \( L^2 \) (see e.g. [JN01, Theorem 5.2 and Theorem 6.2]).

**Remark 2.5.** We do not need to study the case \( \lambda_j \to -2mi \) since the eigenvalues of the linearization at solitary waves are symmetric with respect to real and imaginary axes; see e.g. [BC16].

In other words, as long as \( l_- \) has regular threshold points and no nonzero point spectrum, there can be no nonzero-real-part eigenvalues near \( \pm 2mi \) in the nonrelativistic limit \( \omega \lesssim m \). We prove Theorem 2.2 (1) and (2) in Section 4 and Theorem 2.2 (3) in Section 5.

**Theorem 2.3** (Bifurcations from the origin at \( \omega = m \)). Let \( n \geq 1 \). Let \( f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}) \) satisfy Assumption 2.1 with some values of \( k, K \). Let \( \phi_\omega e^{-i\omega t} \), \( \omega \in (\omega_0, m) \), be a family of solitary wave solutions to (1.1) constructed in Theorem 2.1.

Let \( (\omega_j)_{j \in \mathbb{N}} \), \( \omega_j \in (\omega_0, m) \), be a sequence such that \( \omega_j \to m \), and assume that \( \lambda_j \) are eigenvalues of (1.1) linearized at \( \phi_\omega e^{-i\omega_j t} \) (see Section 4) such that \( \lambda_j \to 0 \). Denote
\[
\Lambda_j := \frac{\lambda_j}{\epsilon_j} \in \mathbb{C}, \quad \epsilon_j := (m^2 - \omega_j^2)^{1/2}, \quad j \in \mathbb{N},
\]
and let \( \Lambda_0 \in \mathbb{C} \cup \{\infty\} \) be an accumulation point of the sequence \( (\Lambda_j)_{j \in \mathbb{N}} \). Then:
1. \( \Lambda_0 \in \sigma(jl) \cup \sigma(i l_-) \cup \sigma(-i l_-) \); in particular, \( \Lambda_0 \neq \infty \). If moreover \( N = 2 \), then \( \Lambda_0 \in \sigma(jl) \).

2. If \( \Re \lambda_j \neq 0 \) for all \( j \in \mathbb{N} \), then \( \Lambda_0 \in \sigma_p(jl) \cap \mathbb{R} \).

3. If \( \Re \lambda_j \neq 0 \) for all \( j \in \mathbb{N} \), then \( \Lambda_0 = 0 \) is only possible when \( k = 2/n \) and \( \partial_\omega Q(\phi_\omega) > 0 \) for \( \omega \in (\omega_*, m) \), with some \( \omega_* < m \). Moreover, in this case \( \lambda_j \in \mathbb{R} \) for all but finitely many \( j \in \mathbb{N} \).

**Remark 2.6.** The set \( \sigma(i l_-) \) appears in Theorem 2.3 (1) since the linearized operator has a certain degeneracy if \( N \geq 2 \); see Lemma 6.4 below.

In other words, as long as \( \partial_\omega Q(\phi_\omega) < 0 \) for \( \omega \lesssim m \) and some generic conditions on the point spectra of \( l_- \) and \( jl \) are satisfied, there can be no linear instability due to bifurcations from the origin: there would be no eigenvalues \( \lambda_j \) of the linearization at solitary waves with \( \omega_j \to m \) such that \( \Re \lambda_j \neq 0 \), \( \lambda_j \to 0 \). We prove Theorem 2.3 in Section 6.

Let us focus on the most essential point of our work. It is of no surprise that the behaviour of eigenvalues of the linearized operator near \( \lambda = 0 \), in the nonrelativistic limit \( \omega \lesssim m \), follows closely the pattern which one finds in the nonlinear Schrödinger equation with the same nonlinearity; this is the content of Theorem 2.3. In its proof in Section 6, we will make this rigorous by applying the rescaling and the Schur complement approach to the linearization of the nonlinear Dirac equation and recovering in the nonrelativistic limit \( \omega \to m \) the linearization of the nonlinear Schrödinger equation. Consequently, the absence of eigenvalues with nonzero real part in the vicinity of \( \lambda = 0 \) is controlled by the Vakhitov–Kolokolov condition \( \partial_\omega Q(\phi_\omega) < 0 \), \( \omega \lesssim m \). (cf. [VK73]). In other words, in the limit \( \omega \lesssim m \), the eigenvalue families \( \lambda_a(\omega) \) of the nonlinear Dirac equation linearized at a solitary wave which satisfy \( \lambda_a(\omega) \to 0 \) as \( \omega \to m \) are merely deformations of the eigenvalue families \( \lambda_a^{\text{NLS}}(\omega) \) of the nonlinear Schrödinger equation with the same nonlinearity (linearized at corresponding solitary waves).

On the other hand, by [BC16, Theorem 2.19], there could be eigenvalue families of the linearization of the nonlinear Dirac operator, which satisfy \( \lim_{\omega \to m} \lambda_a(\omega) = \pm 2 mi \). Could these eigenvalues go off the imaginary axis into the complex plane? Theorem 2.2 states that in the Soler model, under certain spectral assumptions, this scenario could be excluded. Rescaling and the Schur complement approach will show that there could be at most \( N/2 \) families of eigenvalues with nonnegative real part (with \( N \) being the number of spinor components) bifurcating from each of \( \pm 2 mi \) when \( \omega = m \); this essentially follows from Section 5 below (Lemmas 5.8, 5.11, and 5.10). At the same time, the linearization at a solitary wave has eigenvalues \( \lambda = \pm 2 \omega i \), each of multiplicity (at least) \( N/2 \); this follows from the existence of bi-frequency solitary waves [BC17b] in the Soler model. Namely, if there is a solitary wave solution of the form (2.7) to the nonlinear Dirac equation (1.1), then there are also bi-frequency solitary wave solutions of the form

\[
\psi(x, t) = \phi_{\omega, \xi}(x)e^{-i\omega t} + \chi_{\omega, \eta}(x)e^{i\omega t}, \quad \xi, \eta \in \mathbb{C}^{N/2}, \quad |\xi|^2 - |\eta|^2 = 1, \tag{2.18}
\]

where

\[
\phi_{\omega, \xi}(x) = \begin{bmatrix} v(r, \omega) \xi \\ i \frac{\sigma}{\tau} \cdot \sigma u(r, \omega) \xi \end{bmatrix}, \quad \chi_{\omega, \eta} = \begin{bmatrix} -i \frac{\sigma}{\tau} \cdot \sigma u(r, \omega) \eta \\ v(r, \omega) \eta \end{bmatrix}, \quad \xi, \eta \in \mathbb{C}^{N/2}, \tag{2.19}
\]

with \( v(r, \omega) \) and \( u(r, \omega) \) from (2.7). For more details and the relation to \( SU(1, 1) \) symmetry group of the Soler model, see [BC17b]. The form of these bi-frequency solitary waves allows us to conclude that \( \pm 2 \omega i \) are eigenvalues of the linearization at a solitary wave of multiplicities \( N/2 \) (see Lemma 3.4 below). Thus, we know exactly what happens to the eigenvalues which might bifurcate from \( \pm 2 mi \): they all turn into \( \pm 2 \omega i \) and stay on the imaginary axis.

**Remark 2.7.** The spectral stability properties of bi-frequency solitary waves (2.18) could be related to the spectral stability of standard, one-frequency solitary waves (2.7); see [BC17b].

As the matter of fact, since the points \( \pm 2 mi \) belong to the essential spectrum, the perturbation theory cannot be applied immediately for the analysis of families of eigenvalues which bifurcate from \( \pm 2 mi \). We use
the limiting absorption principle to rewrite the eigenvalue problem in such a way that the eigenvalue no longer appears as embedded. When doing so, we find out that the eigenvalues $+2\omega i$ become isolated solutions to the nonlinear eigenvalue problem, known as the characteristic roots (or, informally, nonlinear eigenvalues).

To make sure that we end up with isolated nonlinear eigenvalues, we need to be able to vary the spectral parameter to both sides of the imaginary axis. To avoid the jump of the resolvent at the essential spectrum, we use the analytic continuation of the resolvent in the exponentially weighted spaces. Finally, we show that under the circumstances of the problem the isolated nonlinear eigenvalues can not bifurcate off the imaginary axis. This part is based on the theory of the characteristic roots of holomorphic operator-valued functions [Kel51, Kel71, MS70, GS71]; more recent references are [Mar88] and [MM03, Chapter I]. Unlike in the above references, we have to deal with unbounded operators. As a result, we find it easier to develop our own approach; see Lemma 5.12 in Section 5. It is of utmost importance to us that we have the explicit description of eigenvectors corresponding to $+2\omega i$ eigenvalues (cf. Lemma 3.4). Knowing that $+2\omega i$ are eigenvalues of the linearization operator of particular multiplicity, we will be able to conclude that there could be no other eigenvalue families starting from $\pm 2mi$; in particular, no families of eigenvalues with nonzero real part.

We use Theorems 2.2, and 2.3 to prove the spectral stability of small amplitude solitary waves.

**Theorem 2.4** (Spectral stability of solitary waves of the nonlinear Dirac equation). Let $n \geq 1$. Let $f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$ satisfy Assumption 2.1, with $k, K$ such that either $0 < k < 2/n, K > k$ (charge-subcritical case), or $k = 2/n$ and $K > 4/n$ (charge-critical case). Further, assume that $\sigma_d(l_-) = \{0\}$, and that the threshold $\lambda = 1/(2m)$ of the operator $l_-$ is a regular point of its spectrum. Let $\phi_\omega(x) e^{-i\omega t}, \phi_\omega \in H^2(\mathbb{R}^n, \mathbb{C}^n), \omega \leq m$, be a family of solitary waves constructed in Theorem 2.1. Then there is $\omega_s \in (0, m)$ such that for each $\omega \in (\omega_s, m)$ the corresponding solitary wave is spectrally stable.

**Remark 2.8.** We note that, if either $k < 2/n, K > k$, or $k = 2/n, K > 4/n$, then, by Theorem 2.1, for $\omega \leq m$ one has $\partial_\omega Q(\phi_\omega) < 0$, which is formally in agreement with the Vakhitov–Kolokolov stability criterion [VK73].

**Proof.** We consider the family of solitary wave solutions $\phi_\omega(x) e^{-i\omega t}, \omega \leq m$, described in Theorem 2.1. Let us assume that there is a sequence $\omega_j \to m$ and a family of eigenvalues $\lambda_j$ of the linearization at solitary waves $\phi_\omega(x) e^{-i\omega t}$ such that $\text{Re}\, \lambda_j \neq 0$.

By [BC16, Theorem 2.19], the only accumulation points of the sequence $(\lambda_j)_{j \in \mathbb{N}}$ are $\lambda = \pm 2mi$ and $\lambda = 0$. By Theorem 2.2, as long as $\sigma_d(l_-) = \{0\}$ and the threshold of $l_-$ is a regular point of the spectrum, $\lambda = \pm 2mi$ can not be an accumulation point of nonzero-real-part eigenvalues; it remains to consider the case $\lambda_j \to \lambda = 0$. By Theorem 2.3 (2), if $\text{Re}\, \lambda_j \neq 0$ and $\Lambda_0$ is an accumulation point of the sequence $\Lambda_j := \lambda_j/(m^2 - \omega_j^2)$, then

$$\Lambda_0 \in \sigma_p(jl) \cap \mathbb{R},$$

where $jl$ is the linearization of the NLS in dimension $n$ (cf. (2.14), with the nonlinear term $-|\psi|^{2k}\psi$). For $k \leq 2/n$, the spectrum of the linearization of the corresponding NLS at a solitary wave is purely imaginary: $\sigma_p(jl) \subset i\mathbb{R}$. We conclude from (2.20) that one could only have $\Lambda_0 = 0$; by Theorem 2.3 (3), this would require that $k = 2/n$ and $\partial_\omega Q(\phi_\omega) > 0$ for $\omega \leq m$. On the other hand, as long as $k = 2/n$ and $K > 4/n$, Theorem 2.1 yields $\partial_\omega Q(\phi_\omega) < 0$ for $\omega \leq m$, hence $\Lambda_0 = 0$ would not be possible. We conclude that there is no family of eigenvalues $(\lambda_j)_{j \in \mathbb{N}}$ with $\text{Re}\, \lambda_j \neq 0$.

**Remark 2.9.** We can not claim the spectral stability for all subcritical values $k \in (0, 2/n)$: the nonlinear Schrödinger equation with nonlinearity of order $1+2k$ linearized at a solitary wave has a rich discrete spectrum for small values of $k$, and potentially any of its points could become a source of nonzero-real-part eigenvalues of linearization of the nonlinear Dirac. Such cases would require a more detailed analysis. (In particular, in one spatial dimension, we only prove the spectral stability for $1 < k \leq 2$; the critical, quintic case ($k = 2$) is included, but our proof formally does not cover the cubic case $k = 1$ because of the threshold resonance in the spectrum of one-dimensional cubic NLS.) Our numerics show that $\sigma_p(l_-) = \{0\}$ and the threshold $1/(2m)$
is a regular point of the spectrum of $L_-$, with $L_-$ corresponding to the nonlinear Schrödinger equation in $\mathbb{R}^n$ (thus the spectral hypotheses of Theorem 2.2 and Theorem 2.3 are satisfied) as long as

$$k > k_n, \quad \text{where} \quad k_1 = 1, \quad k_2 \approx 0.621, \quad k_3 \approx 0.461, \quad k_4 \approx 0.369.$$ 

3 The linearization operator

We assume that $f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$ satisfies Assumption 2.1 (recall Remark 2.2). Let $\phi_\omega(x)e^{-i\omega t}$ be a solitary wave solution to equation (1.1) of the form (2.7), with $\omega \in (\omega_0, m)$, where $\omega_0 \in (m/2, m)$ is from Theorem 2.1. Consider the solution to (1.1) in the form of the Ansatz $\psi(x, t) = (\phi_\omega(x) + \rho(x, t))e^{-i\omega t}$, so that $\rho(x, t) \in \mathbb{C}^N$ is a small perturbation of the solitary wave. The linearization at the solitary wave $\phi_\omega(x)e^{-i\omega t}$ (the linearized equation on $\rho$) is given by

$$i\partial_t \rho = \mathcal{L}(\omega)\rho, \quad \mathcal{L}(\omega) = D_m - \omega - f(\phi_\omega^* \beta \phi_\omega)\beta - 2f'(\phi_\omega^* \beta \phi_\omega) \text{Re}(\phi_\omega^* \beta \cdot \beta) \phi_\omega. \quad (3.1)$$

Remark 3.1. Even if $f'(\tau)$ is not continuous at $\tau = 0$, there are no singularities in (3.1) of solitary waves with $\omega \leq m$ constructed in Theorem 2.1: in view of the bound $f'(\tau) = O(|\tau|^{k-1})$ (cf. (2.13)) and the bound from below $\phi_\omega^* \beta \phi_\omega \geq |\phi_\omega|^2/2$ (cf. Theorem 2.1), the last term in (3.1) could be estimated by $O(|\phi_\omega|^{2k})$.

Since $\mathcal{L}(\omega)$ is not $\mathbb{C}$-linear, in order to work with $\mathbb{C}$-linear operators, we introduce the following matrices:

$$\alpha^j = \begin{bmatrix} \text{Re} \alpha^j & -\text{Im} \alpha^j \\ \text{Im} \alpha^j & \text{Re} \alpha^j \end{bmatrix}, \quad 1 \leq j \leq n; \quad \beta = \begin{bmatrix} \text{Re} \beta & -\text{Im} \beta \\ \text{Im} \beta & \text{Re} \beta \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1_N \\ -1_N & 0 \end{bmatrix}, \quad (3.2)$$

where the real part of a matrix is the matrix made of the real parts of its entries (and similarly for the imaginary part of a matrix). We denote

$$\phi_\omega(x) = \begin{bmatrix} \text{Re} \phi_\omega(x) \\ \text{Im} \phi_\omega(x) \end{bmatrix} \in \mathbb{R}^{2N}. \quad (3.3)$$

We mention that $J\alpha \cdot \nabla_x + m\beta$ is the operator which corresponds to $D_m$ acting on $\mathbb{R}^{2N}$-valued functions. Introduce the operator

$$\mathcal{L}(\omega) = J\alpha \cdot \nabla_x + m\beta - \omega - f(\phi_\omega^* \beta \phi_\omega)\beta - 2f'(\phi_\omega^* \beta \phi_\omega) \beta \phi_\omega. \quad (3.4)$$

By $\mathbb{C}$-linearity, we extend the operator $\mathcal{L}$ from its domain $H^1(\mathbb{R}^n; \mathbb{R}^{2N})$ onto $\mathcal{X} = H^1(\mathbb{R}^n; \mathbb{C}^{2N}) = H^1(\mathbb{R}^n; \mathbb{C} \otimes \mathbb{R}^{2N})$, where $\mathcal{L}$ is self-adjoint. The linearization at the solitary wave in (3.1) takes the form

$$\partial_t \rho = J\mathcal{L}(\omega)\rho, \quad \rho(x, t) = \begin{bmatrix} \text{Re} \rho(x, t) \\ \text{Im} \rho(x, t) \end{bmatrix} \in \mathbb{R}^{2N}, \quad (3.5)$$

with $J$ from (3.2) and with $\mathcal{L}$ from (3.4). By Weyl’s theorem on the essential spectrum, the essential spectrum of $J\mathcal{L}(\omega)$ is purely imaginary, with the edges at the thresholds $\pm (m - |\omega|)i$; see [BC16] for more details. There are also embedded thresholds $\pm (m + |\omega|)i$.

For the reader’s convenience, we record the results on the spectral subspace of $J\mathcal{L}(\omega)$ corresponding to the zero eigenvalue:

Lemma 3.2. Span $\{J\phi_\omega, \partial_x \phi_\omega ; 1 \leq j \leq n\} \subset \ker J\mathcal{L}(\omega),$

$$\text{Span} \{J\phi_\omega, \partial_x \phi_\omega, \partial_{xx} \phi_\omega, \alpha^j \phi_\omega - 2\omega x^j J\phi_\omega ; 1 \leq j \leq n\} \subset \mathcal{M}_g(J\mathcal{L}(\omega)).$$

The proof is in [BC16]. There are the following relations (see e.g. [BC16]):

$$\mathcal{L}(\omega)\phi_\omega = 0, \quad J\mathcal{L}(\omega) \partial_x \phi_\omega = J\phi_\omega. \quad (3.6)$$
Remark 3.3. This lemma does not give the complete characterization of the kernel of $\mathcal{J}L(\omega)$; for example, there are also eigenvectors due to the rotational invariance and purely imaginary eigenvalues passing through $\lambda = 0$ at some particular values of $\omega$ [CMKS+16]. We also refer to the proof of Proposition 6.13 below, which gives the dimension of the generalized null space for $\omega \lesssim m$.

Lemma 3.4. The operator $\mathcal{L}(\omega)$ from (3.1) corresponding to the linearization at a (one-frequency) solitary wave has the eigenvalue $-2\omega$ of geometric multiplicity (at least) $N/2$, with the eigenspace containing the subspace $\text{Span} \{ \chi_{\omega,\eta} : \eta \in \mathbb{C}^{N/2} \}$, with $\chi_{\omega,\eta}$ defined in (2.19). The operator $\mathcal{J}L(\omega)$ of the linearization at the solitary wave (cf. (3.5)) has eigenvalues $\pm 2\omega i$ of geometric multiplicity (at least) $N/2$.

Proof. This could be concluded from the expressions for the bi-frequency solitary waves (2.18) or verified directly. Indeed, one has $-2\omega \chi_{\omega,\eta} = (-i\alpha \cdot \nabla_x + (m - f)\beta - \omega)\chi_{\omega,\eta}$, and then one takes into account that $\phi_{\omega}(x)^*\beta\chi_{\omega,\eta}(x) = 0$, so that the last term in the expression (3.1) vanishes when applied to $\chi_{\omega,\eta}$. □

Let

$$
\pi_p = (1 + \beta)/2, \quad \pi_A = (1 - \beta)/2, \quad \pi^\pm = (1 \mp i\gamma)/2 \tag{3.7}
$$

be the projectors corresponding to $\pm 1 \in \sigma(\beta)$ (“particle” and “antiparticle” components) and to $\pm i \in \sigma(\gamma)$ (C-antilinear and C-linear). These projectors commute; we denote their compositions by

$$
\pi^\pm_p = \pi^\pm \pi_p, \quad \pi^\pm_A = \pi^\pm \pi_A. \tag{3.8}
$$

With $\xi \in \mathbb{C}^{N/2}, |\xi| = 1$, from Theorem 2.1 (cf. (2.7)), we denote

$$
\Xi = \begin{bmatrix}
\text{Re} \xi \\
0 \\
\text{Im} \xi \\
0
\end{bmatrix} \in \mathbb{C}^{2N}, \quad |\Xi| = 1. \tag{3.9}
$$

For the future convenience, we introduce the orthogonal projection onto $\Xi$:

$$
\Pi = \Xi (\Xi, \cdot)_{\mathbb{C}^{2N}} \in \text{End} (\mathbb{C}^{2N}). \tag{3.10}
$$

We note that, since $\beta \Xi = \Xi$,

$$
\Pi \circ \pi_p = \pi_p \circ \Pi = \Pi, \quad \Pi \circ \pi_A = \pi_A \circ \Pi = 0. \tag{3.11}
$$

Let $\psi_j \in H^1(\mathbb{R}^n, \mathbb{C}^{2N})$ be eigenfunctions corresponding to the eigenvalues $\lambda_j \in \sigma_p(\mathcal{J}L(\omega_j))$; thus, one has

$$
\mathcal{L}(\omega_j)\psi_j = (J \alpha \cdot \nabla_x + (m - f)\beta - \omega_j + \nu(\omega_j))\psi_j = -J\lambda_j \psi_j, \quad j \in \mathbb{N}, \tag{3.12}
$$

where (cf. (3.4))

$$
\nu(x, \omega)\psi(x) = -f(\Phi^* \beta \Phi_\omega)\beta \psi - 2\Phi^* \beta \psi f'(\Phi^* \beta \Phi_\omega)\beta \Phi_\omega. \tag{3.13}
$$

We will use the notations

$$
y = \epsilon_j x \in \mathbb{R}^n,
$$

where $\epsilon_j = \sqrt{m^2 - \omega_j^2}$, so that $J \alpha \cdot \nabla x = \epsilon_j J \alpha \cdot \nabla y =: \epsilon_j \mathcal{D}_0$ and $\Delta = \epsilon_j^2 \Delta_y$. For $\nu$ from (3.13), define the potential $V(y, \epsilon) \in \text{End} (\mathbb{C}^{2N})$ by

$$
V(y, \epsilon) = \epsilon^{-2}\nu(\epsilon^{-1}y, \omega), \quad \omega = \sqrt{m^2 - \epsilon^2}, \quad y \in \mathbb{R}^n, \quad \epsilon \in (0, \epsilon_0). \tag{3.14}
$$

We define $\Psi_j(y) = \epsilon_j^{-n/2}\psi_j(\epsilon_j^{-1}y)$. With $V$ from (3.14), $L$ is given by

$$
L(\omega_j) = \epsilon_j \mathcal{D}_0 + \beta m - \omega_j + J\lambda_j + \epsilon_j^2 V(\omega_j), \tag{3.15}
$$

10
and the relation (3.12) takes the form

$$
(\epsilon_j D_0 + \lambda m - \omega_j + J\lambda_j + \epsilon_j^2 V(\omega_j)) \Psi_j = 0.
$$

We need several estimates on the potential $V$. 

**Lemma 3.5.** There is $C < \infty$ such that for all $y \in \mathbb{R}^n$ and $\epsilon \in (0, \epsilon_0)$ one has

$$
\|V(y, \epsilon)\|_{\text{End}(C^2N)} \leq C u_k(y)^{2k},
$$

(3.17)

$$
\|\pi_P \circ V(y, \epsilon) \circ \pi_A\|_{\text{End}(C^2N)} + \|\pi_A \circ V(y, \epsilon) \circ \pi_P\|_{\text{End}(C^2N)} \leq C\epsilon u_k(y)^{2k},
$$

(3.18)

$$
\|\pi_A \circ (V(y, \epsilon) + u_k(2k)\beta) \circ \pi_A\|_{\text{End}(C^2N)} \leq C\epsilon^{2k} u_k(y)^{2k},
$$

(3.19)

$$
\|\pi_P \circ (V(y, \epsilon) + u_k(2k)\beta) \circ \pi_P\|_{\text{End}(C^2N)} \leq C\epsilon^{2k} u_k(y)^{2k}.
$$

(3.20)

Above, $u_k$ is the positive radially symmetric ground state of the nonlinear Schrödinger equation (2.2); $x = \min(1, K/k - 1) > 0$ was defined in (2.9).

**Proof.** The inequality (3.17) follows from (2.12) and (2.13):

$$
\|V(y, \epsilon)\|_{\text{End}(C^2N)} \leq C\epsilon^{-2} (|f'(v^2 - u^2)| + v^2 f'(v^2 - u^2)) \leq C\epsilon^{-2} v^{2k} \leq C u_k(y)^{2k},
$$

$\epsilon \in (0, \epsilon_0),

where $\omega = \sqrt{m^2 - \epsilon^2}$ and $|v(\epsilon^2 |y|, \omega)| \leq C\hat{V}(|y|)\epsilon^{2\pi}, |u(\epsilon^{-1} |y|, \omega)| \leq C\hat{V}(|y|)\epsilon^{1 + \frac{\beta}{2}}$ (in the notations from Theorem 2.1), with $\hat{V}(|y|) = u_k(y)$ (cf. (2.4)). The bound (3.18) follows from

$$
\|\pi_P \pi_A\|_{\text{End}(C^2N)} + \|\pi_A V \pi_P\|_{\text{End}(C^2N)} \leq C|\epsilon^{-2} f'(\phi^\beta \phi)vu|,
$$

where $|f'(\phi^\beta \phi)| = |f'(v^2 - u^2)| \leq C|v|^{2k - 2}$ by (2.6) and (2.13), with $v, u$ bounded as above.

Let us prove (3.19). For any numbers $V > 0$ and $U, \hat{V}, \hat{V} \in \mathbb{R}$ which satisfy

$$
e_0 |U| \leq \frac{V}{2}, \quad |\hat{V}| \leq \min \left(1, \frac{1}{2}, C\epsilon^{2\pi}\right) \hat{V},
$$

(3.21)

with $V = \hat{V} + \hat{V}$ and $U = \hat{U} + \hat{U}$, there are the following bounds:

$$
|f(\epsilon^{2/2k}(V(\epsilon^2 U^2)) - \epsilon^{2k} \hat{V}^{2k})|

\leq |f(\epsilon^{2/2k}(V(\epsilon^2 U^2)) - \epsilon^{2k}(V^2 - \epsilon^2 U^2)^k) + \epsilon^{2k}(V^2 - \epsilon^2 U^2)^k - V^{2k}| + \epsilon^2 |V^{2k} - \hat{V}^{2k}|

\leq C \epsilon^{2k/2k}(V^2 - \epsilon^2 U^2)^k + O(\epsilon^2 V^{2k-1}U^2) + O(\epsilon^2 \hat{V}^{2k-1} \hat{V}) \leq C\epsilon^{2k+2\pi} \hat{V}^{2k},
$$

(3.22)

where we used (3.21) and also applied (2.12); similarly, using (2.13),

$$
|f'(\epsilon^{2/2k}(V(\epsilon^2 U^2)))\epsilon^{2k} V^2 - \epsilon^{2k} \hat{V}^{2k}| 

\leq \left|f'(\epsilon^{2/2k}(V(\epsilon^2 U^2))) - \epsilon^{2k}(V^2 - \epsilon^2 U^2)^k - k \epsilon^{2k}(V^2 - \epsilon^2 U^2)^k - V^{2k}| + \epsilon^2 |V^{2k} - \hat{V}^{2k}|

\leq C \epsilon^{2k/2k}(V^2 - \epsilon^2 U^2)^k + k \epsilon^2 |V^{2k-1}V^2 - \hat{V}^{2k}| + \epsilon^2 |V^{2k} - \hat{V}^{2k}|

\leq C\epsilon^{2k+2\pi} \hat{V}^{2k};
$$

(3.23)

$$
|f'((\epsilon^{2k}(V^2 - \epsilon^2 U^2))e^{2k}(V^2 - \epsilon^2 U^2))| \leq C|\epsilon^{2k}(V^2 - \epsilon^2 U^2)|^{k-1} e^{2k} \epsilon^{1 + \frac{\pi}{2}} |UV| \leq C\epsilon^3 \hat{V}^{2k}.
$$

(3.24)

By (2.3), (2.8), and (2.10), we may assume that $e_0 > 0$ in Theorem 2.1 is sufficiently small so that for $\epsilon \in (0, e_0)$ the functions $V(t, \epsilon), U(t, \epsilon), \hat{V}(t), \hat{V}(t, \epsilon)$ satisfy (3.21), pointwise in $t \in \mathbb{R}$. Then, by (3.22), one has $|\epsilon^{-2} f - \hat{V}^{2k}| \leq C\epsilon^{2k+2\pi} \hat{V}^{2k}$; so,

$$
\|\pi_A \circ (V + \hat{V}^{2k}(1 + 2k\Pi)\beta) \circ \pi_A\|_{\text{End}(C^2N)} \leq \|\pi_A \circ (V - \hat{V}^{2k}) \circ \pi_A\|_{\text{End}(C^2N)}

\leq C\epsilon^{-2} f - \hat{V}^{2k} + C\epsilon^{-2} f^{2+2/k} U^2 \leq C\epsilon^{2k+2\pi} \hat{V}^{2k};
$$

(3.25)
in the first inequality, we also took into account (3.11). The above is understood pointwise in $y \in \mathbb{R}^n$; $\hat{V}$ and $U$ are evaluated at $t = |y|$, $f$ and $f'$ are evaluated at $\phi^*\beta = V^2 - \epsilon^2U^2$ and are estimated with the aid of (2.12) and (2.13).

The proof of (3.19) is similar; we have:
\[
\|\pi_P \circ \left( V + \hat{V}^{2k}(1 + 2k\Pi)\beta \right) \circ \pi_P \|_{\text{End}(\mathbb{C}^{2N})} \\
\leq \| - \epsilon^2 f - \epsilon^2 \left( \phi^*\pi_P \right) f' \pi_P \Phi + \hat{V}^{2k}(1 + 2k\Pi) \|_{\text{End}(\mathbb{C}^{2N})} \\
\leq \| \epsilon^{-2}f - \hat{V}^{2k} \| + |\epsilon^{-2}f'\epsilon^{-2} - \hat{V}^{2k}| \leq C\epsilon^{-2}f - \hat{V}^{2k}| + C\epsilon^{-2}f'\epsilon^{-2} \leq C\epsilon^{-2}V^{2k}.
\]

Above, we took into account that $(\phi^*\beta \pi_P)\pi_P \Phi = (\pi_P \Phi)^* \pi_P \Phi = v^2\Pi$ since (cf. (2.7))
\[
\frac{1}{2}(1 + \beta)(\phi)_{\omega} = v \begin{bmatrix} \xi \\ 0 \end{bmatrix}, \quad \text{hence} \quad \pi_P \Phi = v\Xi.
\]

4 Bifurcations from embedded thresholds I

Now we proceed to the proof of Theorem 2.2 (I): we need to prove that the sequence (2.16),
\[
z_j = -\frac{2\omega_j + i\lambda_j}{\epsilon_j^2} \in \mathbb{C}, \quad j \in \mathbb{N},
\]

(4.1)
can only accumulate to either the discrete spectrum or the threshold of the operator $\mathbb{L}_-$ from (2.15).

We project (3.16) onto “particle” and “antiparticle” components and onto the $\mp i$ spectral subspaces of $J$ with the aid of projectors (3.7):
\[
e_jD_0\pi_A^{\mp}\Psi_j + (m - \omega_j - i\lambda_j)\pi_P^{\mp}\Psi_j + \epsilon_j^2\pi_P^{\mp}V\Psi_j = 0,
\]

(4.2)
\[
e_jD_0\pi_P^{\mp}\Psi_j - (m + \omega_j + i\lambda_j)\pi_A^{\mp}\Psi_j + \epsilon_j^2\pi_A^{\mp}V\Psi_j = 0,
\]

(4.3)
\[
e_jD_0\pi_A^{\mp}\Psi_j + (m - \omega_j + i\lambda_j)\pi_P^{\mp}\Psi_j + \epsilon_j^2\pi_P^{\mp}V\Psi_j = 0,
\]

(4.4)
\[
e_jD_0\pi_P^{\mp}\Psi_j - (m + \omega_j - i\lambda_j)\pi_A^{\mp}\Psi_j + \epsilon_j^2\pi_A^{\mp}V\Psi_j = 0.
\]

(4.5)

We will analyze the above relations with the aid of the limiting absorption principle.

Lemma 4.1. There is $C < \infty$ such that $\|u_k^\mp\pi^{\mp}\Psi_j\|_{L^2} \leq C\epsilon_j\|u_k^\mp\Psi_j\|_{L^2}$, for all $j \in \mathbb{N}$.

Proof. (4.4), (4.5) yield
\[
\left[ \begin{array}{cc} m - \omega_j + i\lambda_j \\ \epsilon_jD_0 \\
\end{array} \right] \pi_P^{\mp}\Psi_j = -\epsilon_j^2 \left[ \begin{array}{cc} \pi_P^{\mp}V\Psi_j \\
\pi_A^{\mp}V\Psi_j \\
\end{array} \right],
\]

hence
\[
\pi_P^{\mp}\Psi_j = \left[ \begin{array}{cc} m + \omega_j - i\lambda_j \\ \epsilon_jD_0 \\
\end{array} \right] \pi_A^{\mp}\Psi_j = \left[ \begin{array}{cc} \pi_A^{\mp}V\Psi_j \\
\pi_A^{\mp}V\Psi_j \\
\end{array} \right] - \epsilon_j D_0 (\Delta_y + \mu_j)^{-1} \left[ \begin{array}{cc} \pi_P^{\mp}V\Psi_j \\
\pi_A^{\mp}V\Psi_j \\
\end{array} \right],
\]

(4.6)

with
\[
\mu_j := ((\omega_j - i\lambda_j)^2 - m^2)/\epsilon_j^2 = (8m^2 + o(1))/\epsilon_j^2.
\]

The limiting absorption principle from Lemma A.1 with $\nu = 0$, 1 and $z = \mu_j$ gives
\[
\|u_k \circ (\Delta + \mu_j)^{-1} \circ u_k^\mp\| \leq C|\mu_j|^{-1/2}, \quad \|u_k^\mp \circ \epsilon_jD_0(\Delta + \mu_j)^{-1} \circ u_k^\mp\| \leq C\epsilon_j,
\]

(4.7)

where $\ldots \circ u_k^\mp$ denotes the composition with the operator of multiplication by $u_k^\mp$. (Note that since $\omega_j \to m$, $\text{Re}\lambda_j \neq 0$, and $\lambda_j \to 2mi$, one has $\text{Im}\mu_j = 0$ for all but finitely many $j \in \mathbb{N}$ which we discard.) Applying (4.7) to (4.6) leads to
\[
\|u_k^\mp\pi^{\mp}\Psi_j\| \leq C \left( \|u_k^\mp D_0(\Delta_y + \mu_j)^{-1} \pi^{\mp}V\Psi_j\| + \|u_k^\mp(\Delta_y + \mu_j)^{-1} \pi^{\mp}V\Psi_j\| \right)
\]
\[
\leq C \left( \|u_k^\mp \circ D_0(\Delta_y + \mu_j)^{-1} \circ u_k^\mp\| + \|u_k^\mp \circ (\Delta_y + \mu_j)^{-1} \circ u_k^\mp\| \right) \|u_k^\mp\Psi_j\| \leq C\epsilon_j \|u_k^\mp\Psi_j\|.
\]

(4.8)

We used the bound $\|V(y, \epsilon)\|_{\text{End}(\mathbb{C}^{2N})} \leq Cu_k(y)^{2k}$ from Lemma 3.5.
Lemma 4.2. \( Z_0 \neq \infty \).

Proof. We assume that there is \( \kappa > 0 \) is such that for all but finitely many \( j \) (which we discard), one has
\[
| (\omega_j + i\lambda_j)^2 - m^2 | \geq \kappa \epsilon_j^2; \tag{4.8}
\]
if no such \( \kappa > 0 \) exists, then \( (\omega_j + i\lambda_j)^2 = m^2 + O(\epsilon_j^2) \), hence \( -\omega_j - i\lambda_j = m + O(\epsilon_j^2) \) with the negative sign (except perhaps for finitely many terms which we discard) since \( \lambda_j \to -2mi \) as \( \omega_j \to m \); substituting \( z_j = -2j\omega/\epsilon_j^2 \), we arrive at \( \epsilon_j^2 z_j = m + \omega_j + O(\epsilon_j^2) \), which shows that \( |z_j| \) are uniformly bounded for all \( j \in \mathbb{N} \) hence there is nothing to prove.

We write (4.2), (4.3) as the following system:
\[
\begin{bmatrix}
  m - \omega_j - i\lambda_j \\
  \epsilon_j D_0
\end{bmatrix}
\begin{bmatrix}
  \pi_P^{-} \Psi_j \\
  \pi_A^{-} \Psi_j
\end{bmatrix}
= \begin{bmatrix}
  \pi_P^{-} V \Psi_j \\
  \pi_A^{-} V \Psi_j
\end{bmatrix}, \tag{4.9}
\]
which can then be rewritten as follows:
\[
\begin{bmatrix}
  \pi_P^{-} \Psi_j \\
  \pi_A^{-} \Psi_j
\end{bmatrix}
= \begin{bmatrix}
  m + \omega_j + i\lambda_j \\
  \epsilon_j D_0
\end{bmatrix}
\begin{bmatrix}
  \pi_P^{-} V \Psi_j \\
  \pi_A^{-} V \Psi_j
\end{bmatrix}, \tag{4.10}
\]
with
\[
\nu_j := (\omega_j + i\lambda_j)^2 - m^2 \epsilon_j^2. \tag{4.11}
\]
We notice that \( |\nu_j| \geq \kappa > 0 \) by (4.8) and that \( \text{Im } \nu_j \neq 0 \) except perhaps for finitely many values of \( j \), which we discard. Applying (4.7) to (4.10), we derive:
\[
\| u_k^r \pi^{-} \Psi_j \| \leq C(\epsilon_j + |\nu_j|^{-1/2}) \| u_k^r \Psi_j \|, \quad j \in \mathbb{N}. \tag{4.12}
\]
By Lemma 4.1 and (4.12), there is \( C < \infty \) such that
\[
\| u_k^r \pi^{-} \Psi_j \| \leq \| u_k^r \pi^{+} \Psi_j \| + \| u_k^r \pi^{-} \Psi_j \| \leq C(\epsilon_j + |\nu_j|^{-1/2}) \| u_k^r \Psi_j \|, \quad j \in \mathbb{N}. \tag{4.13}
\]
Assume that \( \limsup_{j \to \infty} |\nu_j| = +\infty \). Then the coefficient at \( \| u_k^r \Psi_j \| \) in the right-hand side of (4.13) would go to zero for an infinite subsequence of \( j \to \infty \); since \( \Psi_j \neq 0 \), we arrive at the contradiction.

Thus, \( \nu_j \) is uniformly bounded. From (4.11), we derive:
\[
\omega_j + i\lambda_j = -\sqrt{m^2 + \epsilon_j^2 \nu_j},
\]
where we have to choose the “positive” branch of the square root, \( \sqrt{m^2 + \epsilon_j^2 \nu_j} = m + O(\epsilon_j^2) \), since \( \lambda_j \to 2mi \) as \( j \to \infty \). This shows that
\[
z_j = -\frac{2\omega_j + i\lambda_j}{\epsilon_j^2} = -\omega_j - \sqrt{m^2 + \epsilon_j^2 \nu_j},
\]
therefore, \( z_j = O(1) \) are uniformly bounded and could not accumulate at infinity. \( \square \)

Substituting \(-m+\nu_j+1i\lambda_j/\epsilon_j^2\) into (4.7) gives
\[
\begin{bmatrix}
  m - \omega_j - i\lambda_j \\
  \epsilon_j D_0
\end{bmatrix}
\begin{bmatrix}
  \pi_P^{-} \Psi_j \\
  \pi_A^{-} \Psi_j
\end{bmatrix}
= \begin{bmatrix}
  \epsilon_j^2 \pi_P^{-} V \Psi_j \\
  \epsilon_j \pi_A^{-} V \Psi_j
\end{bmatrix}. \tag{4.14}
\]

We denote the matrix-valued operator in the left-hand side by
\[
A_j := \begin{bmatrix}
  m - \omega_j - i\lambda_j \\
  \epsilon_j D_0
\end{bmatrix}
\begin{bmatrix}
  \pi_P^{-} \Psi_j \\
  \pi_A^{-} \Psi_j
\end{bmatrix}, \tag{4.15}
\]
Lemma 4.3. For any \( \delta > 0 \), the operator
\[
  u_k^k \circ (l_\kappa - z)^{-1} \circ u_k^k : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad z \in \mathbb{C} \setminus \sigma(l_\kappa)
\]
is uniformly bounded for \( z \in \mathbb{C} \setminus \left( \sigma(l_\kappa) \cup \mathcal{D}_\delta(1/(2m)) \cup \mathcal{D}_\delta(\sigma_p(l_\kappa)) \right) \).

If, moreover, \( z = 1/(2m) \) is a regular point of the essential spectrum of \( l_\kappa \), then \( u_k^k \circ (l_\kappa - z)^{-1} \circ u_k^k : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is uniformly bounded for \( z \in \mathbb{C} \setminus \left( \sigma(l_\kappa) \cup \mathcal{D}_\delta(\sigma_p(l_\kappa)) \right) \).

Above, \( \mathcal{D}_\delta(\sigma_p(l_\kappa)) \) denotes an open \( \delta \)-neighborhood in \( \mathbb{C} \) of pure point spectrum of \( l_\kappa \).

Proof. The first part (bounds away from an open neighborhood of the threshold \( z = 1/(2m) \)) follows from [Agm75, Appendix A].

If the threshold \( z = 1/(2m) \) is a regular point of the essential spectrum of \( l_\kappa \) (neither an eigenvalue nor a resonance), then, by [Yaf10, Lemma 7.4.6] for \( n \neq 2 \) (with \( n = 3 \) already covered in [JK79]), there is \( \delta > 0 \) such that (4.22) is bounded uniformly for \( z \in \mathcal{D}_\delta(1/(2m)) \setminus [1/(2m), +\infty) \).

The case \( n = 2 \), which is left to prove, follows (together with the case \( n = 1 \)) from [JN01]. We recall the terminology from that article. Given \( H_0 = -\Delta \) and \( H = H_0 + V \), we denote
\[
  U = \begin{cases} 
    1, & V \geq 0; \\
    -1, & V < 0 
  \end{cases}, \quad v = |V|^{1/2}, \quad w = Uv, \quad M(\kappa) = U + v(H_0 + \kappa^2)^{-1}v
\]
where \( \text{Re} \kappa > 0 \). There is the identity
\[
  (1 - w(H + \kappa^2)^{-1}v)(1 + w(H_0 + \kappa^2)^{-1}v) = 1,
\]
hence, if \( M(\kappa) \) is invertible,
\[
  1 - w(H + \kappa^2)^{-1}v = (1 + w(H_0 + \kappa^2)^{-1}v)^{-1} = (U + v(H_0 + \kappa^2)^{-1}v)^{-1}U = M(\kappa)^{-1}U,
\]
\[
  U - M(\kappa)^{-1}w = M(\kappa)^{-1}, \quad w(H + \kappa^2)^{-1}w = U - M(\kappa)^{-1}
\]
(see [JN01, Equation (4.8)]). In the case at hand, \( V = -u_k^2 \), \( U = -1 \), \( v = -w = u_k^k \) (understood as operators of multiplication); thus,
\[
  M(\kappa) = -1 + u_k^k \circ \left( \frac{1}{2m} - \Delta \right)^{-1} + \kappa^2 \circ u_k^k,
\]
and, when \( M \left( i \sqrt{z + \frac{1}{2m}} \right) \) is invertible,
\[
  u_k^k \circ (l_\kappa - z)^{-1} \circ u_k^k = -1 \circ M \left( i \sqrt{z - \frac{1}{2m}} \right)^{-1}, \quad (4.17)
\]
with \( \text{Im} \sqrt{z - 1/(2m)} > 0 \) for \( z \in \mathbb{C} \setminus \mathbb{R}^+ \).

Using the kernel expression of \( (H_0 - z)^{-1} \), \( M(\kappa) \) extends to \( \{ \kappa \in \mathbb{C} \setminus \{0\} : \text{Re} \kappa \geq 0 \} \), see [JN01, Section 3, (3.14)], with a \( \frac{1}{2m} \log(\kappa) \) singularity. If \( -\Delta - u_k^2 \kappa \) has no zero energy resonance, then \( M(\kappa) \) is invertible in the orthogonal complement of \( v \), while \( M(\kappa)/\log(\kappa) \) is always invertible in the span of \( v \). We deduce that, as long as \( |\kappa| \) is small enough and \( \kappa \neq 0 \), \( M(\kappa) \) is invertible. Notice that in the limit \( \kappa \to 0 \) the inverse of \( M(\kappa) \) is bounded, with the kernel spanned by \( v \). The invertibility of \( M(\kappa) \) at the threshold \( \kappa = 0 \) of the essential spectrum is thus given, and hence by continuity in \( \kappa \), we deduce a uniform bound on \( \|M(\kappa)^{-1}\| \) in an open neighborhood of 0 in the half-plane \( \text{Re} \kappa \geq 0 \).

Now the conclusion of the lemma for the case \( n = 2 \) follows from (4.17).

The one-dimensional case can be dealt with similarly. \( \square \)
Lemma 4.4. Let \( \Lambda_j \to \Lambda_0 \in \mathbb{C} \cup \{ \infty \} \). Assume that \( \Lambda_0 \not\in \sigma_d(\imath l_-) \cup \{ 1/(2m) \} \) (if the threshold \( z = 1/(2m) \) is a regular point of the essential spectrum of \( l_- \), then it is enough to assume that \( \Lambda_0 \not\in \sigma_d(\imath l_-) \)). Then, for all but finitely many values of \( j \) (which we discard),

\[
u_j^k \circ A_j^{-1} \circ \nu_j^k : L^2(\mathbb{R}^n, \mathbb{C}^4) \to L^2(\mathbb{R}^n, \mathbb{C}^4)
\]
is bounded by \( C/(\Lambda_j)^{1/2} \), with \( C < \infty \) independent in \( j \in \mathbb{N} \).

Proof. We write the inverse of \( A_j = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \) (cf. (4.15)) in terms of the Schur complement. Assume that \( A_{11} \) is invertible. Its Schur complement is defined by \( T = A_{22} - A_{21} A_{11}^{-1} A_{12} \); if it is also invertible, then the inverse of \( A_j \) is given by

\[
A_j^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1} A_{12} T^{-1} A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} T^{-1} \\ -T^{-1} A_{21} A_{11}^{-1} & T^{-1} \end{bmatrix}.
\]

(4.18)

In our case, the Schur complement of \( A_{11} \) takes the form \( T = h_j \otimes I_{2N} \), with

\[
h_j = z_j - \frac{1}{m + \omega_j} + u_2^k + \frac{\Delta}{m - \omega_j - i\lambda_j}.
\]

It is enough to prove that the mapping \( \nu_j^k \circ h_j^{-1} \circ \nu_j^k : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is uniformly bounded (except perhaps at finitely many values of \( j \) which we discard).

We have:

\[
h_j = z_j - \frac{1}{m + \omega_j} + u_2^k + \frac{\Delta}{m - \omega_j - i\lambda_j}
\]

\[
= z_j + \frac{1}{m - \omega_j - i\lambda_j} \frac{1}{m + \omega_j} - \frac{1}{m - \omega_j - i\lambda_j} + \left( 1 - \frac{2m}{m - \omega_j - i\lambda_j} \right) u_2^k
\]

\[
= \frac{2m}{m - \omega_j - i\lambda_j} (1 - \zeta_j) + \left( 1 - \frac{2m}{m - \omega_j - i\lambda_j} \right) u_2^k,
\]

where the sequence

\[
\zeta_j = \frac{m - \omega_j - i\lambda_j}{2m} \left( z_j + \frac{1}{m - \omega_j - i\lambda_j} - \frac{1}{m + \omega_j} \right), \quad j \in \mathbb{N},
\]

has the same limit as the sequence \( (z_j)_{j \in \mathbb{N}} \). It follows that the mapping \( \nu_j^k \circ h_j \circ \nu_j^k : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) has a bounded inverse (for all except perhaps finitely many \( j \) which we discard, with the bound uniform in \( j \in \mathbb{N} \)) as long as so does \( \nu_j^k \circ (1 - \zeta_j) \circ \nu_j^k \), which in turn follows from Lemma 4.3. The improvement \( C/(\Lambda_j)^{1/2} \) comes from the limiting absorption principle for large values of the spectral parameter (cf. [Agm75, Remark 2 in Appendix A]).

Applying Lemma 4.4 to (4.14), we end up with

\[
\|\nu_j^k \pi_P \psi_j\| + \epsilon_j \|\nu_j^k \pi_A \psi_j\| \leq \frac{C}{(\Lambda_j)^{1/2}} \|\pi^k \psi_j\|; \quad \|\nu_j^k \pi^m \psi_j\| \leq \frac{C}{(\Lambda_j)^{1/2}} \epsilon_j \|\nu_j^k \psi_j\|. \tag{4.19}
\]

Lemma 4.5. \( Z_0 \in \sigma_d(\imath l_-) \cup \{ 1/(2m) \} \). If, moreover, the threshold \( z = 1/(2m) \) is a regular point of the essential spectrum of \( l_- \), then \( Z_0 \in \sigma_d(\imath l_-) \).
Proof. Writing explicitly the inverse of the matrix-valued operator in the left-hand side of (4.14), we have:

\[
\begin{bmatrix}
\pi_k P \Psi_j \\
\epsilon_j \pi_k A \Psi_j
\end{bmatrix}
= -\begin{bmatrix}
z_j - \frac{1}{m + \omega_j} + u_k^{2k} & -D_0 \\
\epsilon_j \pi_k A \Psi_j & -D_0 - \omega_j - i\lambda_j
\end{bmatrix}
j^{-1}_j \left[\begin{bmatrix}
\epsilon_j \pi_k A V \Psi_j \\
\epsilon_j \pi_k A \Psi_j - \epsilon_j u_k^{2k} \pi_k A \Psi_j
\end{bmatrix}
\right],
\]  
(4.20)

where

\[
h_j = \Delta_j + (m - \omega_j - i\lambda_j) \left(z_j - \frac{1}{m + \omega_j} + u_k^{2k}\right)
= \Delta_j + 2mu_k^{2k} + 2mz_j - 1 - (m + \omega_j + i\lambda_j)(u_k^{2k} + z_j) + \frac{2\omega_j + i\lambda_j}{m + \omega_j}
= 2m(z_j - L) - (m + \omega_j + i\lambda_j)(u_k^{2k} + z_j) + \frac{2\omega_j + i\lambda_j}{m + \omega_j}.
\]  
(4.21)

Due to the exponential decay of the potential represented by \(-u_k^{2k}\), the operator \(L\) from (2.15) has no embedded eigenvalues \(\lambda > 1/(2m)\) [RS78, Theorem XIII.56]. Moreover, the exponential decay of \(u_k^{2k}\) and [Yaf10, Theorem 6.2.1] provide the limiting absorption principle for \(L\) in the vicinity of any compact subset of \((0, +\infty)\). So, if we assume that the accumulation point \(Z_0\) of the sequence \((z_j)_{j \in \mathbb{N}}\) (which is finite by Lemma 4.2) satisfies either \(Z_0 \notin \sigma(L)\) or \(Z_0 \in (1/(2m), +\infty)\), the resolvent of \(L\) remains finite in the weighted spaces for \(z\) in the vicinity of \(Z_0\), arbitrarily close to the essential spectrum, as long as it stays away from an open neighborhood of the threshold at \(z = 1/(2m)\). In particular, the following mapping is continuous:

\[
u_k^L(\cdot - z)^{-1}u_k^L : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n), \quad z \in \mathbb{C} \setminus \sigma(L),
\]  
(4.22)

with the norm locally bounded in \(z\), and bounded uniformly for \(\text{Re } z \geq 1/(2m), z \notin \mathbb{D}_\delta(1/(2m))\) for any fixed \(\delta > 0\) (the restriction \(z \notin \mathbb{D}_\delta(1/(2m))\) is not needed if the threshold \(z = 1/(2m)\) is a regular point of the continuous spectrum of \(L\)). Therefore, if either \(z_j \to Z_0 \notin \sigma(L)\) or \(z_j \to Z_0 \in (1/(2m), +\infty)\) with \(\text{Im } z_j \neq 0\), taking into account that \(\lambda_j \to 2mi\) as \(\omega_j \to m\) (hence the last two terms in (4.21) go to zero as \(j \to \infty\)), we conclude that for \(h_j\) there is a bounded mapping \(u_k^L \circ h_j^{-1} \circ u_k^L : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\) (except perhaps for finitely many values of \(j\) which we discard), with the bound uniform in \(j \in \mathbb{N}\). Then the relation (4.20) yields

\[
\|\nu_k^L \pi \Psi_j\| + \epsilon_j \|\nu_k^L \pi \Psi_j\| \leq C(\epsilon_j^2 \|\nu_k^L \pi \Psi_j\| + \epsilon_j \|\nu_k^L \pi \Psi_j - \nu_k^{2k} \pi \Psi_j\|) 
\leq C(\epsilon_j^2 \|\nu_k^L \pi \Psi_j\| + \epsilon_j \|\nu_k^L \pi \Psi_j - \nu_k^{2k} \pi \Psi_j\|) + \epsilon_j \|\nu_k^L \pi \Psi_j - \nu_k^{2k} \pi \Psi_j\|),
\]  
(4.23)

with \(C = C(Z_0) < \infty\), and then, using bounds from Lemma 3.5, we obtain the estimate

\[
\|\nu_k^L \pi \Psi_j\| \leq C \epsilon_j^{\min(1, \omega)} \|\nu_k^L \Psi_j\|, \quad \forall j \in \mathbb{N}.
\]  
(4.24)

The inequality (4.24) and Lemma 4.1 lead to \(\|\nu_k^L \Psi_j\| = O(\epsilon_j^{\min(1, \omega)} \|\nu_k^L \Psi_j\|)\), in contradiction to \(\Psi_j \neq 0, j \in \mathbb{N}\). Thus, the assumptions that either \(Z_0 \notin \sigma(L)\) or \(Z_0 \in (1/(2m), +\infty)\) lead to a contradiction. 

This finishes the proof of Theorem 2.2 (1) and (2).

5 Bifurcations from embedded thresholds II: characteristic roots of nonlinear eigenvalue problem

Let us prove Theorem 2.2 (3), showing that \(Z_0 = 0\) only when \(\lambda_j = 2\omega j\) for all but finitely many \(j \in \mathbb{N}\). First, we claim that the relations (4.4) and (4.5) allow one to express \(Y := \pi^+ \Psi_j\) in terms of \(X := \pi^- \Psi_j\).
Lemma 5.1. There is $\epsilon_1 \in (0, \epsilon_0)$ such that for any $\epsilon \in (0, \epsilon_1)$ and any $z \in \mathbb{D}_1$ the relations
\[
\epsilon \mathbf{D}_0 \pi_A Y - (\omega - i\lambda - m) \pi_p Y + \epsilon^2 \pi_p V(X + Y) = 0,
\]  
\[
\epsilon \mathbf{D}_0 \pi_p Y - (\omega - i\lambda + m) \pi_A Y + \epsilon^2 \pi_A V(X + Y) = 0,
\]
where $\omega = \sqrt{m^2 - \epsilon^2}$ and
\[
\lambda = \lambda(z) = (2\omega + \epsilon^2 z)i
\]  
(cf. (2.16)), define a linear map $\vartheta(\cdot, \epsilon, z) : L^{2-k}(\mathbb{R}^n, \text{Range } \pi^-) \to L^{2-k}(\mathbb{R}^n, \text{Range } \pi^+)$, $\vartheta(\cdot, \epsilon, z) : X \mapsto Y$, which is analytic in $z$, where for $\mu \in \mathbb{R}$ the exponentially weighted spaces are defined by
\[
L^{2,\mu}(\mathbb{R}^n) = \{ u \in L^2_{loc}(\mathbb{R}^n) ; \| \epsilon u \|_{L^2(\mathbb{R}^n)} \leq C \epsilon, \quad \epsilon \in (0, \epsilon_1) \}, \quad \epsilon \in (0, \epsilon_1), \quad z \in \mathbb{D}_1.
\]
Moreover, there is $C < \infty$ such that
\[
\| \vartheta(\cdot, \epsilon, z) \|_{L^{2-k}(\mathbb{R}^n, C^{2N}) \to L^{2-k}(\mathbb{R}^n, C^{2N})} \leq C\epsilon, \quad \epsilon \in (0, \epsilon_1), \quad z \in \mathbb{D}_1.
\]
Proof. By (4.6),
\[
\frac{\pi_p Y}{\pi_A Y} = \left[ \frac{\omega - i\lambda + m}{\epsilon \mathbf{D}_0} \right] \left( \Delta + \frac{(\omega - i\lambda)^2 - m^2}{\epsilon^2} \right) \left( \pi_p V(X + Y) \right). \quad \text{This leads to}
\]
\[
Y = \pi^+ \{ (\omega - i\lambda) + m(\pi_p - \pi_A) + \epsilon \mathbf{D}_0 \} \left( \Delta + \frac{(\omega - i\lambda)^2 - m^2}{\epsilon^2} \right)^{-1} V(X + Y).
\]
For $\epsilon > 0$ and $z \in \mathbb{C}$, $\text{Im } z < 0$ (so that $\text{Re } \lambda(z) > 0$; cf. (5.1)), we define the linear map
\[
\Phi(\cdot, \epsilon, z) : L^{2-k}(\mathbb{R}^n, C^{2N}) \to L^{2-k}(\mathbb{R}^n, \text{Range } \pi^+),
\]
\[
\Phi(\Psi, \epsilon, z) = \pi^+ \{ (\omega - i\lambda) + m(\pi_p - \pi_A) + \epsilon \mathbf{D}_0 \} \left( \Delta + \frac{(\omega - i\lambda)^2 - m^2}{\epsilon^2} \right)^{-1} V \Psi,
\]
where $\lambda = \lambda(z) = (2\omega + \epsilon^2 z)i$. Using the definition (5.3), the relation (5.2) takes the form
\[
Y = \Phi(X + Y, \epsilon, z).
\]
Since the norm of $\Phi(\cdot, \epsilon, z) \in \mathcal{B}(L^{2-k}(\mathbb{R}^n, C^{2N}), L^{2-k}(\mathbb{R}^n, C^{2N}))$, $z \in \mathbb{D}_1$, is small (as long as $\epsilon > 0$ is small enough), we will be able to use the above relation to express $Y$ as a function of $X \in L^{2-k}(\mathbb{R}^n, \text{Range } \pi^-)$.

Remark 5.2. The inverse of $\Delta + \frac{(\omega - i\lambda)^2 - m^2}{\epsilon^2}$ is not continuous in $\lambda$ at $\text{Re } \lambda = 0$; this discontinuity could result in two different families of eigenvalues bifurcating from an embedded eigenvalue even when its algebraic multiplicity is one. We will start with the resolvent corresponding to $\text{Re } \lambda > 0$ and then use its analytic continuation through $\text{Re } \lambda = 0$.

In view of Proposition A.2, it is convenient to change the variables so that in (5.3) we deal with $(-\Delta - \zeta^2)$; let $\zeta \in \mathbb{C}$ be defined by
\[
\zeta^2 = ((\omega - i\lambda)^2 - m^2)/\epsilon^2, \quad \text{Re } \zeta > 0
\]
the values of $\lambda$ with $\text{Re } \lambda > 0$ correspond to the values of $\zeta$ with $\text{Im } \zeta < 0$ (since $\lambda \in \mathbb{D}_{\epsilon_1}(2\omega i)$ and $\omega \in (\omega_1, m)$, where $\omega_1 = \sqrt{m^2 - \epsilon_1^2}$, with $\epsilon_1 \in (0, \epsilon_0)$ small enough).

Due to Lemma 3.5, $\| V(Y, \epsilon) \|_{\text{End } (C^{2N})} \leq C e^{-2k|\epsilon|}$; using the analytic continuation of the resolvent from Proposition A.2, the mapping (5.3) could be extended from $\{ \text{Im } \zeta < 0 \}$ to $\{ \zeta \in \mathbb{C}; \text{Im } \zeta < k \}$ \{k \in \mathbb{R}_+\}. For the uniformity, we require that
\[
| \text{Im } \zeta | < k,
\]
considering the resolvent $(-\Delta - \zeta^2)^{-1}$ for $\Im \zeta < 0$ (this corresponds to $\Re \lambda > 0$) and its analytic continuation into the strip $0 \leq \Im \zeta < k$ (this corresponds to $\Re \lambda \leq 0$). Due to our assumptions that $\omega \to m$ and $\lambda \to 2mi$, one has

$$\Re((\omega - i\lambda)^2 - m^2) = 8m^2 + O(\Re \lambda) + O(\Im \lambda - 2m) + O(\epsilon^2). \quad (5.7)$$

Therefore, by $(5.5)$,

$$\Re \zeta = \epsilon^{-1} \Re \sqrt{(\omega - i\lambda)^2 - m^2} = O(\epsilon^{-1}), \quad (5.8)$$

showing that for $\epsilon$ sufficiently small one has $\zeta \in \mathbb{C} \setminus \mathbb{D}_1$ (we take $\epsilon_1 > 0$ smaller if necessary). Since we only consider $z \in \mathbb{D}_1$, the relation (5.1) yields $|\Re \lambda| \leq |z| \epsilon^2 \leq \epsilon^2$, and then $(5.5)$ and $(5.8)$ lead to

$$\Im \zeta = \frac{1}{2 \Re \zeta} \frac{\Im((\omega - i\lambda)^2 - m^2)}{\epsilon^2} = O(\epsilon) \frac{\Re \lambda}{\epsilon^2} = O(\epsilon),$$

showing that the condition $(5.6)$ holds true for $\epsilon$ sufficiently small, satisfying assumptions of Proposition A.2. Then, by Proposition A.2, there is $C < \infty$ such that for any $\Psi \in L^{2,-k}(\mathbb{R}^n, \mathbb{C}^{2N})$ the map (5.3) satisfies

$$\|\Phi(\Psi, \epsilon, z)\|_{L^{2,-k}(\mathbb{R}^n, \mathbb{C}^{2N})} \leq C\epsilon \|\Psi\|_{L^{2,-k}(\mathbb{R}^n, \mathbb{C}^{2N})}, \quad \epsilon \in (0, \epsilon_1), \quad z \in \mathbb{D}_1. \quad (5.9)$$

We take $\epsilon_1$ smaller if necessary so that $\epsilon_1 \leq 1/(2C)$ (with $C < \infty$ from $(5.9)$); then the linear map

$$I - \Phi(\cdot, \epsilon, z) : Y \mapsto Y - \Phi(Y, \epsilon, z)$$

is invertible, with

$$\|(I - \Phi(\cdot, \epsilon, z))^{-1}\|_{L^{2,-k} \to L^{2,-k}} \leq 2, \quad \epsilon \in (0, \epsilon_1), \quad z \in \mathbb{D}_1. \quad (5.10)$$

Since $\Phi(\cdot, \epsilon, z)$ is linear, writing (5.4) in the form $Y - \Phi(Y) = \Phi(X)$, we can express $Y = (1 - \Phi)^{-1}\Phi(X)$. Thus, for each $\epsilon \in (0, \epsilon_1)$ and $z \in \mathbb{D}_1$, we may define the mapping $(X, \epsilon, z) \mapsto Y$, which we denote

$$\vartheta(\cdot, \epsilon, z) : L^{2,-k}(\mathbb{R}^n, \text{Range } \pi^-) \to L^{2,-k}(\mathbb{R}^n, \text{Range } \pi^+),$$

$$\vartheta(\cdot, \epsilon, z) : X \mapsto Y = (1 - \Phi(\cdot, \epsilon, z))^{-1}\Phi(X, \epsilon, z). \quad (5.11)$$

By (5.9) and (5.10), one has $\|\vartheta(\cdot, \epsilon, z)\|_{L^{2,-k} \to L^{2,-k}} \leq 2C\epsilon$, for $\epsilon \in (0, \epsilon_1)$ and $z \in \mathbb{D}_1$.

Finally, let us discuss the differentiability of $\vartheta$ with respect to $z$. The map $\Phi$ can be differentiated in the strong sense with respect to $z$. First, we notice that, by (5.1),

$$2|\zeta||\partial_z \zeta| = \left|\frac{\partial}{\partial z} \zeta^2\right| = \left|\frac{\partial}{\partial z} \left(\frac{(\omega - i\lambda)^2 - m^2}{\epsilon^2}\right)\right| = \left|\frac{2(\omega - i\lambda)}{\epsilon^2} \frac{\partial}{\partial z}(2\omega + \epsilon^2 z)\right| = 2|\omega - i\lambda|, \quad (5.12)$$

with the right-hand side bounded uniformly in $\epsilon \in (0, \epsilon_1)$ and $z \in \mathbb{D}_1$. Therefore, using the bound for the derivative of the analytic continuation of the resolvent (cf. Proposition A.2, which we apply with $\nu = 0$ and also with $\nu = 1$ to accommodate the operator $\epsilon D_0$ from the definition of $\Phi$ in (5.3)), we conclude that there is $C < \infty$ such that

$$\|\partial_z \vartheta(\cdot, \epsilon, z)\|_{L^{2,-k} \to L^{2,-k}} \leq \|\partial_z \Phi\|_{L^{2,-k} \to L^{2,-k}} \|\partial_z \zeta\| \leq \frac{C}{|\zeta|^2} \frac{|\omega - i\lambda|}{|\zeta|} \leq C\epsilon^3$$

for all $\epsilon \in (0, \epsilon_1)$ and $z \in \mathbb{D}_1$. Above, $\partial_z$ is considered as a gradient in $\mathbb{R}^2 \cong \mathbb{C}$; we used (5.12) and the estimate (5.8). Then it follows from (5.11) that there is $C < \infty$ such that one also has $\|\partial_z \vartheta(\cdot, \epsilon, z)\|_{L^{2,-k} \to L^{2,-k}} \leq C\epsilon^3$, for all $\epsilon \in (0, \epsilon_1)$ and $z \in \mathbb{D}_1$. One can see from (5.3) that $\Phi$ is analytic in the complex parameter $z$, hence so is $\vartheta$. \qed
As long as \( j \in \mathbb{N} \) is sufficiently large so that \( \epsilon_j \in (0, \epsilon_1) \) and \( z_j \in \mathbb{D}_1 \), by Lemma 5.1, the relations (4.4) and (4.5) allow us to express \( \pi^+ \Psi_j = \vartheta(\pi^- \Psi_j, \epsilon_j, z_j) \), with \( \vartheta(\cdot, \epsilon, z) \) a linear map from \( L^2, -k(\mathbb{R}^n, \mathbb{C}^{2N}) \) into itself, with

\[
\| \vartheta(\cdot, \epsilon, z) \|_{L^2, -k(\mathbb{R}^n, \mathbb{C}^{2N}) \to L^2, -k(\mathbb{R}^n, \mathbb{C}^{2N})} \leq C \epsilon.
\]

Now (4.2) and (4.3) can be written as

\[
e_jD_0 \pi_A \Psi_j + (m - \omega_j - i \lambda_j)\pi_P \Psi_j + \epsilon_j^2 \pi_P \vartheta(\epsilon_j, \epsilon, z) = 0, \tag{5.13}
\]

\[
e_jD_0 \pi_P \Psi_j - (m + \omega_j + i \lambda_j)\pi_P \Psi_j + \epsilon_j^2 \pi_P \vartheta(\epsilon_j, \epsilon, z) = 0, \tag{5.14}
\]

where \( \vartheta = \vartheta(\epsilon_j, \epsilon, z) \), with \( \vartheta(\epsilon_j, \epsilon, z) := \vartheta(\epsilon_j, \epsilon, z) \) satisfying

\[
\| \vartheta(\epsilon, \lambda) \|_{L^2 \to L^2} \leq \| \vartheta(\epsilon) \|_{L^2, -k \to L^2}(1 + \| \vartheta(\cdot, \epsilon, z) \|_{L^2, -k \to L^2, -k}) \leq C,
\]

so that \( \Psi(\epsilon_j) \Psi_j = \Psi(\epsilon_j)(\pi^- \Psi_j + \pi^+ \Psi_j) = \Psi(\epsilon_j)(\pi^- \Psi_j + \vartheta(\pi^- \Psi_j, \epsilon_j, z_j)) = \Psi^0 \pi^- \Psi_j \). We recall the definition \( z_j = -(2m + i\lambda_j)/\epsilon_j^2 \) (cf. (4.1)) and rewrite (5.13), (5.14), as the following system:

\[
\left[ \pi_P \left( \frac{m + \omega_j}{\epsilon_j^2} + z_j + \vartheta \right) \pi_P - \frac{\epsilon_j^2}{m + \omega_j} + \vartheta \right]_{\pi_P} \left[ \pi_A(z_j - \frac{1}{m + \omega_j} + \vartheta)_{\pi_A} \right] = 0, \quad j \in \mathbb{N}. \tag{5.15}
\]

We rewrite the above as the nonlinear eigenvalue problem

\[
T(\epsilon, z) \left[ \pi_P \Psi_j \right] = 0, \quad T(\epsilon, z) := \left[ \pi_P \left( \frac{m + \omega_j}{\epsilon_j^2} + z + \vartheta \right) \pi_P - \frac{\epsilon_j^2}{m + \omega_j} + \vartheta \right]_{\pi_P} \left[ \pi_A(z_j - \frac{1}{m + \omega_j} + \vartheta)_{\pi_A} \right], \tag{5.16}
\]

where we consider the operator \( T(\epsilon, z) : H^1(\mathbb{R}^n, \text{Range } \pi^-) \to L^2(\mathbb{R}^n, \text{Range } \pi^-) \) as

\[
T(\epsilon, z) : H^1(\mathbb{R}^n, \text{Range } \pi_P \times \text{Range } \pi_A) \to L^2(\mathbb{R}^n, \text{Range } \pi_P \times \text{Range } \pi_A).
\]

Note that the operator \( T(\epsilon, z) \) depends on \( z \) analytically via \( \vartheta \) (cf. Lemma 5.1). By Weyl’s theorem,

\[
\sigma_{\text{ess}}(T(\epsilon, z)) = (-\infty, -(m + \omega)^{-1} + z] \cup [(m - \omega)^{-1} + z, +\infty), \tag{5.17}
\]

so that \( 0 \notin \sigma_{\text{ess}}(T(\epsilon, z)) \). Thus, the values \( z_j \) defined in (4.1) are such that the kernel of \( T(\epsilon_j, z_j) \) is nontrivial; such values of \( z \) are called the characteristic roots (or, informally, nonlinear eigenvalues) of \( T(\epsilon, z) \).

**Nonlinear eigenvalue problem.** We will study the location of characteristic roots of \( T(\epsilon, z) \) using the theory developed by M. Keldysh [Kel51, Kel71]; see also [MS70, GS71]. Let us recall the standard terminology. Let \( H \) be a Hilbert space, \( \Omega \subset \mathbb{C} \) an open neighborhood of \( z_0 \in \mathbb{C} \), and let \( A(z) : H \to H \), \( z \in \Omega \), be an analytic family of closed operators: that is, we assume that for any \( u, v \in H \) and each \( z_0 \in \mathbb{C} \) and each \( \eta \) in the resolvent set of \( A(z_0) \), the function \( \langle u, (A(z) - \eta)^{-1} v \rangle \) is analytic in \( z \) in an open neighborhood of \( z_0 \in \Omega \).

**Remark 5.3.** By [Kat76, Theorem VII-1.3], this agrees with the definition of the analytic family of unbounded closed operators chosen by Kato [Kat76, Section VII-2].

The point \( z_0 \in \Omega \) is said to be regular for the operator-valued analytic function \( A(z) \) if the operator \( A(z_0) \) has a bounded inverse. If the equation \( A(z_0) \varphi = 0 \) has a non-trivial solution \( \varphi_0 \in H \), then \( z_0 \) is said to be a characteristic root of \( A \) and \( \varphi_0 \) an eigenvector of \( A \) corresponding to \( z_0 \).

The characteristic root \( z_0 \) of \( A \) is said to be normal if for some \( R > 0 \), all \( z \in \Omega \) satisfying \( 0 < |z - z_0| < R \) are regular points of \( A(z) \), and \( A(z_0) \) is a Fredholm operator.

**Assumption 5.4.** \( z_0 \in \Omega \) is a normal characteristic root of \( A(z) \), \( A(z) \) is resolvent-continuous in \( z \in \Omega \), and \( 0 \in \sigma_d(A(z_0)) \).

**Remark 5.5.** In the above references, it is assumed that \( A(z) : H \to H \) is bounded; we do not need this due to the assumption that \( 0 \in \sigma_d(A(z_0)) \) is an isolated point of the spectrum.
Assume that $z_0$ is an isolated characteristic root of $A(z)$ and that Assumption 5.4 is satisfied. There is $\delta > 0$ such that $\partial \mathbb{D}_\delta \in \rho(A(z_0))$. Due to the resolvent continuity of $A$ in $z$, there is an open neighborhood $U \subset \Omega$ of $z_0$ such that $\partial \mathbb{D}_\delta \in \rho(A(z))$ for all $z \in U$. Let

$$P_{\delta, z} = -\frac{1}{2\pi i} \oint_{|\eta| = \delta} (A(z) - \eta)^{-1} d\eta, \quad z \in U.$$  

Since $A(z_0)$ is Fredholm, we may assume that $\delta > 0$ is small enough so that $\text{rank} P_{\delta, z_0} < \infty$.

**Definition 5.6.** The multiplicity $\alpha \in \mathbb{N}$ of the characteristic root $z_0$ of $A(z)$ is the order of vanishing of $\det A(z)|_{\text{Range} P_{\delta, z}}$ at $z_0$.

**Remark 5.7.** The above definition does not depend on the choice of $\delta > 0$ (as long as $\partial \mathbb{D}_\delta \in \rho(A(z_0))$ and rank $P_{\delta, z_0} < \infty$).

**Lemma 5.8.** Let $z_0$ be a characteristic root of $A(z)$ of multiplicity $\alpha \in \mathbb{N}$. The geometric multiplicity of $0 \in \sigma_d(A(z_0))$ satisfies $g \leq \alpha$.

**Proof.** Denote $r = \dim X$. We choose the basis $\{\psi_i\}_{1 \leq i \leq r}$ in $\text{Range} P_{\delta, z_0}$ so that $\psi_i$, $1 \leq i \leq g$, are eigenvectors corresponding to zero eigenvalue of $A(z_0)$. Let $A(z)$ be the matrix representation of $A(z)$ in the basis $\{P_{\delta, z}\psi_i\}_{1 \leq i \leq r}$. Then the first $g$ columns of $A(z)$ vanish at $z = z_0$, hence $\det A(z) = O((z - z_0)^9)$. 

Let us show that the sum of multiplicities of characteristic roots is stable under perturbations (cf. [GS71, Theorem 2.2]).

**Lemma 5.9.** Let $z_0$ be a characteristic root of $A(z)$ of multiplicity $\alpha \in \mathbb{N}$. If $B(z) : H \to H$, $z \in U$ is an analytic family of operators and one has $\|(A(z) - \eta)^{-1}B(z)\| < 1$ for all $z \in U$ and $\eta \in \partial \mathbb{D}_\delta$, then the sum of multiplicities of all characteristic values of $A(z) + B(z)$ inside $U$ equals $\alpha$.

**Proof.** Denote

$$P_{\delta, \epsilon, z} = -\frac{1}{2\pi i} \oint_{|\eta| = \delta} (A(z) + \epsilon B(z) - \eta)^{-1} d\eta, \quad \epsilon \in [0, 1], \quad z \in U.$$  

By continuity in $\epsilon$ and $z$, one has rank $P_{\delta, \epsilon, z} = \text{rank} P_{\delta, 0, z_0} = r$, $\forall \epsilon \in [0, 1]$ and $\forall z \in U$. Let $\{\psi_i\}_{1 \leq i \leq r}$ be the basis in $\text{Range} P_{\delta, 0, z_0}$. Denote

$$\psi_i(\epsilon, z) = P_{\delta, \epsilon, z}\psi_i, \quad 1 \leq i \leq r, \quad \epsilon \in [0, 1], \quad z \in U;$$  

it is a basis in $\text{Range} P_{\delta, \epsilon, z}$. Let $M(\epsilon, z)$ be the matrix representation of $\left.\left(A(z) + \epsilon B(z)\right)\right|_{\text{Range} P_{\delta, \epsilon, z}}$ in this basis; now the statement of the lemma is an immediate consequence of the Rouché theorem applied to $\det M(\epsilon, z)$. 

Now we apply the above theory to the operator $T(\epsilon, z)$ defined in (5.16); first, we will do the reduction of $T$ using the Schur complement of its invertible block. By (5.17), we may assume that there is a sufficiently small open neighborhood $U$ of $z = 0$ and that $\omega_s \in (0, m)$ is sufficiently large so that

$$\sigma_{\text{ess}}(T(\epsilon, z)) \cap \mathbb{D}_{1/(4m)} = \emptyset \quad \forall \omega \in (\omega_s, m), \quad \forall \epsilon \in (0, \epsilon_s), \quad \forall z \in U,$$

where $\epsilon_s = \sqrt{m^2 - \omega_s^2}$. Since $0 \in \sigma_d(\mathbb{L})$, with $\mathbb{L}$ from (2.15), we may assume that the open neighborhood $U \ni \{0\}$ is small enough so that

$$U \cap \sigma(\mathbb{L}) = \{0\}. \quad (5.18)$$  

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Let $T_{ij}(\epsilon, z), i, j = 1, 2$, denote the operators which are the entries of $T(\epsilon, z)$ in (5.16), so that $T(\epsilon, z) = \begin{bmatrix} T_{11}(\epsilon, z) & T_{12}(\epsilon, z) \\ T_{21}(\epsilon, z) & T_{22}(\epsilon, z) \end{bmatrix}$. We then have:

\begin{equation}
\|T_{12}\|_{H^1 \to L^2} + \|T_{21}\|_{H^1 \to L^2} + \|T_{21}\|_{L^2 \to H^{-1}} = O(\epsilon^{-1}), \quad \|T_{11}\|_{L^2 \to L^2} = O(\epsilon^{-2}),
\end{equation}

and, taking $\epsilon_* > 0$ smaller if necessary, one has

\begin{equation}
T_{11}^{-1} = \frac{\epsilon_*^2}{2m} + O_{L^2 \to L^2}(\epsilon^4), \quad \epsilon \in (0, \epsilon_*), \quad z \in U,
\end{equation}

with the estimate $O_{L^2 \to L^2}(\epsilon^4)$ uniform in $z \in U$. This suggests that we study the invertibility of $T(\epsilon, z)$ in terms of the Schur complement of $T_{11}(\epsilon, z)$, which is defined by

\begin{equation}
S(\epsilon, z) = T_{22} - T_{21}T_{11}^{-1}T_{12} : H^1(\mathbb{R}^n, \text{Range } \pi_P^-) \to H^{-1}(\mathbb{R}^n, \text{Range } \pi_P^-),
\end{equation}

where $T_{11}, T_{12}, T_{21},$ and $T_{22}$ are evaluated at $(\epsilon, z) \in (0, \epsilon_*) \times U$. Let us derive the explicit expression for $S(\epsilon, z)$:

\begin{align*}
S(\epsilon, z) &= \pi_A^- \left( z - \frac{1}{m + \omega} + V^\theta \right) \pi_A^- \\
&\quad - \pi_A^- (D_0 + \epsilon V^\theta) \pi_P^- (m + \omega + \epsilon^2(V^\theta + z))^{-1} \pi_P^- (D_0 + \epsilon V^\theta) \pi_A^- \\
&\quad + \frac{1}{m + \omega} u_k^2 \pi_A^- (V^\theta - u_k^2) \pi_A^- \\
&\quad + \pi_A^- D_0^2 \pi_A^- - \pi_A^- (D_0 + \epsilon V^\theta) \pi_P^- \frac{1}{m + \omega} \pi_P^- (D_0 + \epsilon V^\theta) \pi_A^- \\
&\quad + \pi_A^- (D_0 + \epsilon V^\theta) \pi_P^- \frac{1}{m + \omega} \left( \left( 1 + \frac{\epsilon^2(V^\theta + z)}{m + \omega} \right)^{-1} - 1 \right) \pi_P^- (D_0 + \epsilon V^\theta) \pi_A^-.
\end{align*}

Above, $u_k = u_k(x)$ is the ground state of the nonlinear Schrödinger equation (2.2). We note that, by (3.13),

\begin{equation}
\pi_A^- V^\theta \pi_A^- = \pi_A V \circ (1 + \theta) \pi_A^- = \pi_A u_k^2 + O_{L^2 \to L^2}(\epsilon);
\end{equation}

we used the bounds $\|\pi_A \Phi_\omega\|_{L^\infty} = O(\epsilon^{1+\frac{2}{m}})$ (cf. Theorem 2.1) and $\|\theta\|_{L^2 \to L^2} = O(\epsilon)$ (cf. Lemma 5.1) which yield $\|V \circ \theta\|_{L^2 \to L^2} \leq \|V\|_{L^2 \to L^2} \|\theta\|_{L^2 \to L^2} = O(\epsilon)$ for $s > 1/2$. Thus, taking into account Lemma B.4, the operator $S(\epsilon, z)$ defined in (5.21) takes the form

\begin{equation}
S(\epsilon, z) = \pi_A^- \left( z - \frac{1}{2m} + u_k^2 + \frac{\Delta}{2m} + O_{H^1 \to H^{-1}}(\epsilon) \right) \pi_A^-, \quad \epsilon \in [0, \epsilon_*),
\end{equation}

with the estimate $O_{H^1 \to H^{-1}}(\epsilon)$ uniform in $z \in U$. Above, we extended $S(\epsilon, z)$ in (5.22) from $\epsilon \in (0, \epsilon_*)$ to $\epsilon \in [0, \epsilon_*)$ by continuity.

The following lemma allows us to reduce the problem of studying the characteristic roots of $T(\epsilon, z)$ (cf. (5.16)) to the characteristic roots of $S(\epsilon, z)$ (cf. (5.21)).

**Lemma 5.10.** If $\epsilon_* > 0$ is sufficiently small, then for all $\epsilon \in [0, \epsilon_*)$ the point $z_0 \in \mathbb{D}_{1/(2m)}$ is a characteristic root of $T(\epsilon, z)$ if and only if it is a characteristic root of $S(\epsilon, z)$.

**Proof.** Since $T_{11}(\epsilon, z) : L^2(\mathbb{R}^n, \text{Range } \pi_P^+) \to L^2(\mathbb{R}^n, \text{Range } \pi_P^+)$ is invertible for $\epsilon > 0$ small enough, the Schur complement approach allows us to factor the operator $T(\epsilon, z)$ (cf. (5.16)) as follows:

\begin{equation}
T(\epsilon, z) = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} 1_{2N} & 0 \\ T_{21}T_{11}^{-1} & 1_{2N} \end{bmatrix} \begin{bmatrix} T_{11} & 0 \\ T_{21}T_{11}^{-1} & 1_{2N} \end{bmatrix},
\end{equation}

with $S(\epsilon, z)$ from (5.21); the operators $T_{ij}$ depend on $\epsilon$ and $z$. We see that $T(\epsilon, z)$ is invertible if and only if so is $S(\epsilon, z)$. \[\square\]
Lemma 5.11. The multiplicity of the characteristic root \( z_0 = 0 \) of \( S(0, z) \) is \( \alpha = N/2 \).

Proof. By (5.22), \( S(0, 0) = (z - \lambda)\pi_A^{-} \). Since dim \( \ker \lambda = 1 \), one has dim \( \ker S(0, 0) = \text{rank} \pi_A^{-} = N/2 \). Let \( e_i \in \mathbb{C}^{2N}, 1 \leq i \leq N/2 \), be the basis in Range \( \pi_A^{-} \); then \( \{u_k e_i\}_{1 \leq i \leq N/2} \) is the basis in \( \ker S(0, 0) \). We do not need to use the Riesz projectors since the operator \( S(0, z) \) is invariant in this space, being represented by \( S(0, z) = z 1_{N/2} \); thus, det \( S(0, z) = z^{N/2} \).

Lemma 5.12. There is no sequence of characteristic roots \( z_j \neq 0 \) of \( T(\epsilon, J) \) such that \( z_j \to 0 \) as \( j \to \infty \).

Proof. Let \( \delta > 0 \) be such that \( \partial \mathbb{D}_\delta \subset \rho(T(\epsilon, z)), z_0 = 0 \). Due to the continuity of the resolvent in \( z \) and \( \epsilon \), there is \( \epsilon_* \in (0, \epsilon_1) \) and an open neighborhood \( U \subset \mathbb{D}_{1/(2m)}, z_0 = 0, \) such that \( \partial \mathbb{D}_\delta \subset \rho(T(\epsilon, z)) \) for all \( z \in U \) and \( \epsilon \in [0, \epsilon_*] \).

By (5.18) and Lemma 5.11, the sum of multiplicities of the characteristic roots of \( S(0, z) \) in \( U \) equals \( \alpha = N/2 \), and by Lemma 5.9 the same is true for \( S(\epsilon, z) \) for all \( \epsilon \in [0, \epsilon_*] \). At the same time, by Lemma 3.4 and Lemma 5.8, \( z = 0 \) is a characteristic root of \( S(\epsilon, z), \epsilon \in [0, \epsilon_*] \), of multiplicity at least \( \alpha = N/2 \). Hence, there can be no other, nonzero characteristic roots \( z \in U \) of \( S(\epsilon, z) \) for any \( \epsilon \in [0, \epsilon_*] \), and, in particular, given a sequence \( \epsilon_j \to 0 \), there is no sequence of characteristic roots \( z_j \) of \( S(\epsilon, z) \) such that \( z_j \neq 0 \) for \( j \in \mathbb{N} \), \( z_j \to 0 \) as \( j \to \infty \). By Lemma 5.10, the same conclusion holds true for \( T(\epsilon, z) \).

By Lemma 5.12, \( z_j = 0 \) for all but finitely many \( j \in \mathbb{N} \). By the definition \( z_j = -(2\omega_j + i\lambda_j)/\epsilon_j^2 \) (cf. (2.16)), \( \lambda_j = 2\omega_j \) for all but finitely many \( j \in \mathbb{N} \). This concludes the proof of Theorem 2.2 (3). The proof of Theorem 2.2 is now complete.

6 Bifurcations from the origin

We now prove Theorem 2.3. Let us first prove that \( \lambda_0 \neq \infty \); this is a consequence of the following lemma.

Lemma 6.1. Let \( \omega_j \in (0, m), j \in \mathbb{N} \); \( \omega_j \to m \). If there are eigenvalues \( \lambda_j \in \sigma_{p}(JL(\omega_j)), j \in \mathbb{N}, \) such that \( \lim_{j \to \infty} \lambda_j = 0, \) then the sequence

\[
\Lambda_j := \frac{\lambda_j}{\epsilon_j^2}, \quad j \in \mathbb{N}
\]

does not have an accumulation point at infinity.

Proof. Due to the exponential decay of solitary waves stated in Theorem 2.1, there is \( C < \infty \) and \( s > 1/2 \) such that

\[
\| (r)^{2s} V(\cdot, \omega) \|_{L^\infty(\mathbb{R}^n, End(\mathbb{C}^N))} \leq C, \quad \forall \omega \in (0, m).
\]

Let \( \Psi_j \in L^2(\mathbb{R}^n, \mathbb{C}^N), j \in \mathbb{N} \) be the eigenfunctions of \( JL(\omega_j) \) corresponding to \( \lambda_j \); we then have (cf. (3.16))

\[
(\epsilon_j \mathbf{D}_0 + \beta m - \omega_j + J\lambda_j)\Psi_j = -\epsilon_j^2 V(\omega_j)\Psi_j.
\]

Applying \( \pi^\pm = (1 \mp iJ)/2 \) to (6.2) and denoting \( \Psi_j^\pm = \pi^\pm \Psi_j \),

\[
(\epsilon_j \mathbf{D}_0 + \beta m - \omega_j + i\lambda_j)\Psi_j^+ = -\epsilon_j^2 \pi^+ V(\omega_j)\Psi_j, \quad (\epsilon_j \mathbf{D}_0 + \beta m - \omega_j - i\lambda_j)\Psi_j^- = -\epsilon_j^2 \pi^- V(\omega_j)\Psi_j.
\]

Since \( \omega_j \to m \), without loss of generality, we can assume that \( \omega_j > m/2 \) for all \( j \in \mathbb{N} \). Since the spectrum \( \sigma(JL) \) is symmetric with respect to real and imaginary axes, we may assume, without loss of generality, that \( \text{Im} \lambda_j \geq 0 \) for all \( j \in \mathbb{N} \), so that \( \text{Re}(i\lambda_j) \leq 0 \) (see Figure 1). Since \( \lambda_j \to 0 \), we can also assume that \( |\lambda_j| \leq m/2 \) for all \( j \in \mathbb{N} \). With \( \epsilon_j \mathbf{D}_0 + \beta m - \omega_j = D_m - \omega_j \) being self-adjoint, one has

\[
\| (D_m - \omega_j - i\lambda_j)^{-1} \| = 1/\text{dist}(i\lambda_j, \sigma(D_m - \omega_j)) = 1/|m - \omega_j - i\lambda_j|, \quad j \in \mathbb{N}.
\]
Remark 6.2. If \( \text{Re}(i\lambda) > m - \omega \), then dist\((i\lambda, \sigma(D_m - \omega)) < |m - \omega - i\lambda| \), and (6.4) does not hold.

From (6.3) and (6.4), using the bound (6.1) on \( V \), we obtain

\[
\| \Psi_j \|_{L^2} \leq \frac{\epsilon_j^2 \| \pi^- V \Psi_j \|}{|m - \omega_j - i\lambda_j|} \leq C \frac{\epsilon_j^2}{|m - \omega_j - i\lambda_j|} \| \langle r \rangle^{-s} \Psi_j \|, \quad \forall j \in \mathbb{N}. \tag{6.5}
\]

From (6.3) we have:

\[
\begin{bmatrix}
  m - \omega_j + i\lambda_j & \epsilon_j D_0 \\
  \epsilon_j D_0 & -m - \omega_j + i\lambda_j
\end{bmatrix}
\begin{bmatrix}
  \pi^+_P \Psi_j \\
  \pi^+_A \Psi_j
\end{bmatrix}
= -\epsilon_j^2 \begin{bmatrix}
  \pi^+_P V \Psi_j \\
  \pi^+_A V \Psi_j
\end{bmatrix}, \tag{6.6}
\]

hence

\[
\left( \frac{(-m - \omega_j + i\lambda_j)(m - \omega_j + i\lambda_j)}{\epsilon_j^2} + \Delta \right)
\begin{bmatrix}
  \pi^+_P \Psi_j \\
  \pi^+_A \Psi_j
\end{bmatrix}
\begin{bmatrix}
  m + \omega_j - i\lambda_j \\
  \epsilon_j D_0
\end{bmatrix}
= \begin{bmatrix}
  \epsilon_j D_0 \\
  -(m - \omega_j + i\lambda_j)
\end{bmatrix}
\begin{bmatrix}
  \pi^+_P V \Psi_j \\
  \pi^+_A V \Psi_j
\end{bmatrix}.
\]

Denote

\[
\mu_j = \frac{(-m - \omega_j + i\lambda_j)(m - \omega_j + i\lambda_j)}{\epsilon_j^2}. \tag{6.7}
\]

We may assume that \( |\mu_j| \geq \mu_0 > 0 \) for all \( j \in \mathbb{N} \), or else there would be nothing to prove: if \( \mu_j \to 0 \), we would have \( |\lambda_j - i(m - \omega_j)| = o(\epsilon_j^2) \), hence \( |\lambda_j| = O(\epsilon_j^2) \). Then, by the limiting absorption principle (cf. Lemma A.1),

\[
\| \langle r \rangle^{-s} \pi^+ G \| \leq \frac{C |\mu_j|^{-1/2}}{\| \langle r \rangle^{-s} \pi^+ V \Psi_j \|} + C\epsilon_j \| \langle r \rangle^{-s} \pi^+ V \Psi_j \|. \]

The above, together with (6.5) and the bound (6.1) on \( V \), leads to

\[
\| \langle r \rangle^{-s} \Psi_j \| \leq \| \langle r \rangle^{-s} \Psi_j \| + \| \langle r \rangle^{-s} \Psi_j \| \leq C\left( \frac{\epsilon_j^2}{|m - \omega_j - i\lambda_j|} + \frac{1}{|\mu_j|^{1/2}} + \epsilon_j \right) \| \langle r \rangle^{-s} \Psi_j \|. \]

If we had \( |\lambda_j|/\epsilon_j^2 \to \infty \), then \( |m - \omega_j - i\lambda_j| \geq |\lambda_j|/2 \) for \( j \) large enough, hence \( |\mu_j| \geq m|m - \omega_j - i\lambda_j|/\epsilon_j^2 \geq m|\lambda_j|/(2\epsilon_j^2) \) for \( j \) large enough (since \( \omega_j \to m \) and \( \lambda_j \to 0 \) in (6.7)),

\[
\| \langle r \rangle^{-s} \Psi_j \| \leq C\left( \frac{\epsilon_j^2}{|\lambda_j|} + \frac{\epsilon_j}{|\lambda_j|^{1/2}} + \epsilon_j \right) \| \langle r \rangle^{-s} \Psi_j \|. \]

Due to \( |\lambda_j|/\epsilon_j^2 \to \infty \), the above relation would lead to a contradiction since \( \Psi_j \neq 0, j \in \mathbb{N} \). We conclude that \( \lambda_j = \lambda_j/\epsilon_j^2 \) can not have an accumulation point at infinity.

\[ \Box \]

Lemma 6.3. For any \( \eta \in \mathbb{C} \setminus \overline{\mathbb{R}_-} \), there is \( s_0(\eta) \in (0, 1) \), lower semicontinuous in \( \eta \), such that the resolvent \( (-\Delta - \eta)^{-1} \) defines a continuous mapping

\[
(-\Delta - \eta)^{-1} : L^2_0(\mathbb{R}^n) \to H^2_0(\mathbb{R}^n), \quad 0 \leq s < s_0(\eta).
\]

Proof. Let \( f \in L^2_0(\mathbb{R}^n) \); define \( u = (-\Delta - \eta)^{-1} f \in H^2(\mathbb{R}^n) \). There is the identity

\[
(-\Delta - \eta) \langle r \rangle^s u + \langle r \rangle^s, -\Delta \rangle u = \langle r \rangle^s (-\Delta - \eta) u, \tag{6.8}
\]

which holds in the sense of distributions. Taking into account that

\[
\| \| \langle r \rangle^s, -\Delta \| u \| \leq C\| u \|_{H^2} O(s) \leq C\| f \|_{L^2} O(s),
\]

one concludes from (6.8) that \( (-\Delta - \eta) \langle r \rangle^s u \in L^2(\mathbb{R}^n) \) and hence \( \langle r \rangle^s u \in L^2(\mathbb{R}^n) \), both being bounded by \( C\| f \|_{L^2} \), with some \( C = C(\eta) < \infty \), thus so is \( \| u \|_{H^2} \). \[ \Box \]
It will be convenient to use the following operator:

\[
K = \frac{1}{2m} - \frac{\Delta}{2m} - u_k^2(1 + 2k\Pi).
\]  

(6.9)

Above, \(\Pi\) is the orthogonal projector onto \(\Xi \in \mathbb{C}^{2N}\); see (3.9), (3.10).

**Lemma 6.4.**

1. \(\sigma(JK|_{\text{Range } \pi_P}) \subset \sigma(jl) \cup \sigma(il-) \cup \sigma(-il-)\); the same is true for the point spectrum.

If, moreover, \(N = 2\), then \(\sigma(JK|_{\text{Range } \pi_P}) = \sigma(jl)\); the same is true for the point spectrum.

2. One has:

\[
\dim N_g(JK|_{\text{Range } \pi_P}) = \begin{cases} 
2n + N, & k \neq 2/n; \\
2n + N + 2, & k = 2/n.
\end{cases}
\]

(6.10)

**Proof.** We define the spaces

\[
\mathcal{X}_1 = L^2(\mathbb{R}^n, \text{Span}(\Xi, J\Xi)), \quad \mathcal{X}_2 = L^2(\mathbb{R}^n, \text{Span}(\Xi, J\Xi)\perp \text{Range } \pi_P);
\]

\(
\mathcal{X}_1 \oplus \mathcal{X}_2 = L^2(\mathbb{R}^n, \text{Range } \pi_P).
\)

(Note that both \(\Xi\) and \(J\Xi\) belong to \(\text{Range } \pi_P\).) The proof of Part 1 follows once we notice that \(JK\) is invariant in the spaces \(\mathcal{X}_i, 1 \leq i \leq 2\), and that \(JK|_{\mathcal{X}_1}\) is represented in \(L^2(\mathbb{R}^n, \text{Span}(\Xi, J\Xi))\) by

\[
j_1 = \begin{bmatrix} 0 & 1_n \\ -1_+ & 0 \end{bmatrix},
\]

while \(JK|_{\mathcal{X}_2}\) is represented in \(L^2(\mathbb{R}^n, \text{Span}(\Xi, J\Xi)\perp \text{Range } \pi_P)\) by

\[
1_{N/2 - 1} \otimes \mathbb{C} \begin{bmatrix} 0 & 1_n \\ -1_+ & 0 \end{bmatrix}.
\]

We also notice that if \(N = 2\), then \(\mathcal{X}_2 = \{0\}\).

The proof of Part 2 also follows from the above decomposition and the relations

\[
\dim N_g(JK|_{\mathcal{X}_1}) = \dim N_g(jl) = \begin{cases} 
2n + 2, & k \neq 2/n; \\
2n + 4, & k = 2/n.
\end{cases}
\]

We note that \(JK|_{\text{Range } \pi_A}\) is represented in \(L^2(\mathbb{R}^n, \text{Range } \pi_A)\) by \(1_{N/2} \otimes \mathbb{C} \begin{bmatrix} 0 & 1_2 \\ -1_+ & 0 \end{bmatrix} \).

Since \(\sigma(JL)\) is symmetric with respect to real and imaginary axes, we assume without loss of generality that \(\lambda_j\) satisfies

\[
\text{Im } \lambda_j \geq 0, \quad \forall j \in \mathbb{N}.
\]

(6.11)

Passing to a subsequence, we assume that

\[
\Lambda_j = \frac{\lambda_j}{\epsilon_j^2} \to \Lambda_0 \in \mathbb{C}.
\]

(6.12)

**Lemma 6.6.**

1. If \(\Lambda_0 \not\in \sigma(JK)\), then

\[
\|\pi_P \Psi_j\| + \epsilon_j^{-1}\|\pi_A \Psi_j\| \leq C\epsilon_j^{2\sigma}\|u_k^j \Psi_j\|, \quad \forall j \in \mathbb{N}.
\]
2. For $s > 0$ sufficiently small there is $C < \infty$ such that
\[
\left\| \pi_P^+ \Psi_j \right\|_{H^s} + \epsilon_j^{-1} \left\| \pi_A^+ \Psi_j \right\|_{H^s} \leq C \left\| u_k^j \Psi_j \right\|, \quad \forall j \in \mathbb{N} \text{ (except for finitely many)}.
\]

3. If $\Lambda_0 \not\in i[1/(2m), +\infty)$, then
\[
\left\| \pi_P^+ \Psi_j \right\|_{H^s} + \epsilon_j^{-1} \left\| \pi_A^+ \Psi_j \right\|_{H^s} \leq C \left\| u_k^j \Psi_j \right\|, \quad \forall j \in \mathbb{N}.
\]

4. If $\Lambda_0 \in i[1/(2m), +\infty)$, then
\[
\left\| u_k^j \pi_P^+ \Psi_j \right\| + \epsilon_j^{-1} \left\| u_k^j \pi_A^+ \Psi_j \right\| \leq C \left\| u_k^j \Psi_j \right\|, \quad \forall j \in \mathbb{N}.
\]

Proof. Let us prove Part 1. We divide (4.2), (4.4) by $\epsilon_j^2$ and (4.3), (4.5) by $\epsilon_j$, arriving at
\[
\left[ \frac{1}{m+\omega} + J \Lambda_j \quad D_0 \\
D_0 \\ -m - \omega + \epsilon_j^2 J \Lambda_j \right] \left[ \pi_P \Psi_j \right] = - \left[ \pi_P \Psi_j \right], \quad \forall j \in \mathbb{N}. \quad (6.13)
\]

We rewrite (6.13) as
\[
\left[ \frac{1}{m+\omega} - u_k^2 (1 + 2k) + J \Lambda_j \quad D_0 \\
D_0 \\ -m - \omega \right] \left[ \pi_P \Psi_j \right] = - \left[ \pi_P \Psi_j \right], \quad \forall j \in \mathbb{N}. \quad (6.14)
\]

The Schur complement of $T_{22}$ is given by
\[
S_j = \frac{1}{m+\omega} + \Lambda_j - u_k^2 (1 + 2k) - \frac{\Delta}{m+\omega}.
\]

If $\Lambda_j \to \Lambda_0 \not\in \sigma(JK)$, $S_j$ is invertible. The conclusion follows from (6.14) once we take into account the bounds from Lemma 3.5.

Let us prove Part 2. We apply $\pi^\pm$ to (6.13) and rewrite the result as
\[
\left[ \frac{1}{m+\omega} \pm i \Lambda_j \quad D_0 \\
D_0 \\ -m - \omega \right] \left[ \pi_P^\pm \Psi_j \right] = - \left[ \pi_P^\pm \Psi_j \right], \quad \forall j \in \mathbb{N}. \quad (6.15)
\]

Denote the matrix-valued operator in the left-hand side of (6.15) by $T^\pm$. The Schur complement of $T_{22}$ is given by
\[
S_j^\pm = T_{11} - T_{12} T_{22}^{-1} T_{21} = \frac{1}{m+\omega} \pm i \Lambda_j - \frac{\Delta}{m+\omega}. \quad (6.16)
\]

Since $\text{Im} \Lambda_0 \geq 0$ (cf. (6.11)), $S_j^-$ is invertible in $L^2$ (except perhaps at finitely many values of $j$ which we disregard); writing the inverse of $T^-$ in terms of $S_j^-$, we conclude from (6.15) that $\left\| \pi_P^- \Psi_j \right\|_{H^s} + \epsilon_j^{-1} \left\| \pi_A^- \Psi_j \right\|_{H^s} \leq C \left\| \Psi_j \right\|$. Moreover, by Lemma 3.3, for sufficiently small $s > 0$,
\[
\left\| \pi_P^- \Psi_j \right\|_{H^s} + \epsilon_j^{-1} \left\| \pi_A^- \Psi_j \right\|_{H^s} \leq C \left\| \Psi_j \right\| \leq C \left\| u_k^j \Psi_j \right\|, \quad \forall j \in \mathbb{N}.
\]

This proves Part 2. As long as $\Lambda_0 \not\in i[1/(2m), +\infty)$, Part 3 is proved in the same way as Part 2.

To prove Part 4, we write
\[
\left[ \frac{1}{m+\omega} + i \Lambda_j + \mu u_k^2 \quad D_0 \\
D_0 \\ -m - \omega \right] \left[ \pi_P^+ \Psi_j \right] = - \left[ \pi_P^+ \Psi_j \right], \quad \forall j \in \mathbb{N}. \quad (6.17)
\]
The Schur complement is

\[ S_j = \frac{1}{m + \omega_j} + i\Lambda_j + \mu u_k^{2k} - \frac{\Delta}{m + \omega_j}. \]

We pick \( \mu \geq 0 \) such that the threshold \( z = 1/(2m) \) is a regular point of the operator \( \frac{1}{2m} + \mu u_k^{2k} - \frac{\Delta}{2m} \), which is by [JN01] a generic situation; indeed, by (4.16), the resonances correspond to the situation when \( M = -1 + \mu K \) is not invertible, with \( K \) a compact operator. (For \( n \geq 3 \), enough to take \( \mu = 0 \) since \( -\Delta \) has no resonance at \( z = 0 \).) Then, by Lemma 4.3, \( u_k^* S_j u_k^* \) (for \( j \) large enough) is bounded in \( L^2 \) and the conclusion follows from (6.17).

The inclusion \( \Lambda_0 \in \sigma(JK) \) immediately follows from Lemma 6.6 (1) which shows that if the sequence \( \Lambda_j \) were to converge to a point away from \( \sigma(JK) \), then at most finitely many of \( \Psi_j \) could be different from zero. Together with the results on the spectrum of \( JK \) (cf. Lemma 6.4), this proves Theorem 2.3 (1).

**Proposition 6.7.** If \( \text{Re} \lambda_j \neq 0 \) for all \( j \in \mathbb{N} \), then \( \Lambda_0 \in \sigma_p(JK) \cap \mathbb{R} \).

**Proof.** From now on, we assume that the corresponding eigenfunctions \( \Psi_j \) (cf. (3.16)) are normalized:

\[ \|\Psi_j\|^2 = 1, \quad j \in \mathbb{N}. \]  

(6.18)

By (6.13), \( \varepsilon_j^{-1} D_0 \pi_A \Psi_j \) is uniformly bounded in \( L^2 \), while by Lemma 6.6 (3) and (4) so is \( u_k^j \varepsilon_j^{-1} \pi_A \Psi_j \). Again by (6.13), \( u_k^j D_0 \pi_p \Psi_j \) is uniformly bounded in \( L^2 \). It follows that both \( \varepsilon_j^{-1} \pi_p \Psi_j \) and \( \varepsilon_j^{-1} \pi_A \Psi_j \) belong to \( H^1_{\text{loc}}(\mathbb{R}^n, \mathbb{C}^{2N}) \) and contain weakly convergent subsequences; we denote their limits by

\[ \hat{\Psi} \in \tilde{H}^1_{\text{loc}}(\mathbb{R}^n, \mathbb{C}^{2N}), \quad \hat{A} \in H^1_{\text{loc}}(\mathbb{R}^n, \mathbb{C}^{2N}). \]  

(6.19)

Passing to the limit in (6.14) and using the bounds from Lemma 3.5, we arrive at the following system (valid in the sense of distributions):

\[ \left[ \frac{1}{2m} - u_k^{2k}(1 + 2k \Pi) + J\Lambda_0 \right] \begin{bmatrix} \hat{\Psi} \\ \hat{A} \end{bmatrix} = 0. \]  

(6.20)

Let us argue that if \( \text{Re} \lambda_j \neq 0 \) for all \( j \in \mathbb{N} \), then \( \begin{bmatrix} \hat{\Psi} \\ \hat{A} \end{bmatrix} \) is not identically zero. By Lemma 6.6 (2), using the compactness of the Sobolev embedding \( H^1_k \subset L^2 \), we conclude that there is an infinite subsequence (which we again enumerate by \( j \in \mathbb{N} \)) such that

\[ \pi_p \Psi_j \to \pi^- \hat{\Psi} \in \tilde{H}^1(\mathbb{R}^n, \mathbb{C}^{2N}), \quad \varepsilon_j^{-1} \pi_A \Psi_j \to \pi^- \hat{A} \in H^1(\mathbb{R}^n, \mathbb{C}^{2N}), \quad j \to \infty, \]  

(6.21)

with the strong convergence in \( L^2 \).

**Remark 6.8.** If additionally \( \Lambda_0 \notin i[1/(2m), +\infty) \), then \( S_j^+ \) from (6.16) is also invertible; just like above, one concludes that there is an infinite subsequence (which we again enumerate by \( j \in \mathbb{N} \)) such that

\[ \pi_p^+ \Psi_j \to \pi^+ \hat{\Psi} \in \tilde{H}^1(\mathbb{R}^n, \mathbb{C}^{2N}), \quad \varepsilon_j^{-1} \pi_A^+ \Psi_j \to \pi^+ \hat{A} \in H^1(\mathbb{R}^n, \mathbb{C}^{2N}), \quad j \to \infty, \]  

(6.22)

with the strong convergence in \( L^2 \).

**Lemma 6.9** (Krein’s theorem). Let \( J \in \text{End}(\mathbb{C}^{2N}) \) be skew-adjoint and invertible and let \( L \) be self-adjoint on \( L^2(\mathbb{R}^n, \mathbb{C}^{2N}) \). If \( \lambda \in \sigma_p(JL) \setminus i\mathbb{R} \) and \( \Psi \) is a corresponding eigenvector, then \( \langle \Psi, L\Psi \rangle = 0 \) and \( \langle \Psi, JL\Psi \rangle = 0 \).

**Proof.** If \( \text{Re} \lambda \neq 0 \), the identity \( \langle \Psi, L\Psi \rangle = \lambda \langle \Psi, J^{-1}\Psi \rangle \) equals zero since \( \langle \Psi, L\Psi \rangle \in \mathbb{R}, \langle \Psi, J^{-1}\Psi \rangle \in i\mathbb{R} \). 

\[ \square \]
Lemma 6.10. If Re $\lambda_j \neq 0$ for all $j \in \mathbb{N}$, then $\hat{P}^- \neq 0$.

Proof. Krein’s theorem (cf. Lemma 6.9) yields $0 = \langle \Psi_j, J \Psi_j \rangle = i \|\Psi_j^+\|^2 - i \|\Psi_j^-\|^2$, $j \in \mathbb{N}$; thus, by (6.18),

$$\|\Psi_j^+\|^2 = \|\Psi_j^-\|^2 = \|\Psi_j\|^2/2 = 1/2.$$  

(6.23)

Therefore,

$$\|\hat{P}^-\|^2 = \lim_{j \to \infty} \|\pi\hat{P}^-\Psi_j\|^2 = \lim_{j \to \infty} (\|\pi\hat{P}^-\Psi_j\|^2 + \|\pi\hat{A}^-\Psi_j\|^2) = 1/2,$$

for all $j \in \mathbb{N}$. (6.24)

Above, in the first two relations, we took into account (6.21).

Thus, \[ \hat{P}^-, \hat{A} \in L^2(\mathbb{R}^n, \mathbb{C}^{2N} \times \mathbb{C}^{2N}) \] is not identically zero, hence $\Lambda_0 \in \sigma_p(JK)$. It remains to prove that

$$\Lambda_0 \in \mathbb{R}. \tag{6.25}$$

Let us assume that, on the contrary, $\Lambda_0 \in \sigma_p(JK) \cap (i\mathbb{R} \setminus \{0\})$. By (6.11), it is enough to consider

$$\Lambda_0 = ia, \quad a > 0. \tag{6.26}$$

By (6.24), $\|\hat{P}^-\|^2 = 1/2$. Since $\|\hat{P}^+\|^2 \leq \lim_{j \to \infty} \|\pi\hat{P}^+\Psi_j\|^2 \leq \lim_{j \to \infty} (\|\pi\hat{P}^+\Psi_j\|^2 + \|\pi\hat{A}^+\Psi_j\|^2) = 1/2$, we arrive at the inequality

$$\|\hat{P}^+\|^2 \leq \|\hat{P}^-\|^2 \leq 0. \tag{6.27}$$

From the above and from $JK\hat{P} = ia\hat{P}$ it follows that

$$\langle \hat{P}, K\hat{P} \rangle = \langle \hat{P}, -iaJ\hat{P} \rangle = a\langle \hat{P}^+, \hat{P}^+ \rangle - a\langle \hat{P}^-, \hat{P}^- \rangle \leq 0. \tag{6.28}$$

Remark 6.11. If $\Lambda_0$ belongs to the spectral gap of $JK$ ($\Lambda_0 \in i\mathbb{R}$, $|\Lambda_0| < 1/(2m)$), then both $\pi\hat{P}^\pm\Psi_j$ and $\epsilon_j^\pm \pi\hat{A}^-\Psi_j$ (up to choosing a subsequence) converge to $\pi\hat{P}^\pm \Psi \in H^1(\mathbb{R}^n, \mathbb{C}^{2N})$ and $\pi\hat{A}^\pm \Psi \in H^1(\mathbb{R}^n, \mathbb{C}^{2N})$ strongly in $L^2$ (cf. Remark 6.8). Then, by the above arguments, $\|\hat{P}^\pm\|^2 = 1/2$ and hence $\langle \hat{P}, J\hat{P} \rangle = 0 = \langle \hat{P}, K\hat{P} \rangle$.

Lemma 6.12. If $\Lambda_0 \in i\mathbb{R} \setminus \{0\}$, $\Lambda_0 \in \sigma_p(J1)$, and $z$ is a corresponding eigenvector, then $\langle z, 1z \rangle > 0$.

Proof. Let $z$ be an eigenfunction which corresponds to $\Lambda_0 \in \sigma_p(J1) \cap i\mathbb{R}$, $\Lambda_0 \neq 0$. Let $p, q \in L^2(\mathbb{R}^n, \mathbb{C})$ be such that $z = \begin{bmatrix} p \\ iq \end{bmatrix}$ and let $\Lambda_0 = ia$ with $a \in \mathbb{R} \setminus \{0\}$. Then $ia \begin{bmatrix} p \\ iq \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p \\ iq \end{bmatrix}$ results in $ap = l_q$ and $aq = l_p$ (note that $q \notin \text{ker} l_-$; otherwise one would conclude that $p \equiv 0$ and then also $q \equiv 0$, so that $z \equiv 0$, hence not an eigenvector). These relations lead to

$$\langle p, l_{1p} \rangle = a\langle q, p \rangle = \langle q, ap \rangle = \langle q, l_q \rangle = \langle q, l_{-q} \rangle,$$

hence

$$\langle z, 1z \rangle = \langle \begin{bmatrix} p \\ iq \end{bmatrix}, \begin{bmatrix} l_{-p} & 0 \\ 0 & l_{-q} \end{bmatrix} \begin{bmatrix} p \\ iq \end{bmatrix} \rangle = \langle p, l_{1p} \rangle + \langle q, l_{-q} \rangle = 2\langle q, l_{-q} \rangle > 0,$$

where we took into account that $l_-$ is semi-positive-definite and that $q \notin \text{ker} l_-$. \hfill \Box

Since $K$ is invariant in the subspaces $\mathcal{X}_1$ (where it is represented by $1$) and $\mathcal{X}_2$ (where it is represented by a positive-definite operator $l_-$), it follows from Lemma 6.12 that the quadratic form

$$\langle \cdot, K\cdot \rangle = \langle \cdot, K\cdot \rangle |_{\mathcal{X}_1} + \langle \cdot, K\cdot \rangle |_{\mathcal{X}_2} = \langle \cdot, K\cdot \rangle |_{\mathcal{X}_1} + \langle \cdot, (1_{\mathcal{X}_2} \otimes l_-) \cdot \rangle |_{\mathcal{X}_2}$$

is strictly positive-definite on any eigenspace of $JK$ corresponding to $\Lambda_0 = ia \in \sigma_p(JK)$, $a > 0$. Therefore,

$$\langle \hat{P}, K\hat{P} \rangle > 0. \tag{6.29}$$

The relations (6.28) and (6.29) lead to a contradiction; we conclude that (6.25) is satisfied. \hfill \Box
Proposition 6.13 concludes the proof of Theorem 2.3 (2).

The case \( \Lambda_0 = 0 \). Now we turn to Theorem 2.3 (3), which treats the case \( \Lambda_0 = 0 \). Let us find the dimension of the spectral subspace of \( JL(\omega) \) corresponding to all eigenvalues which satisfy \( |\lambda| = o(\epsilon^2) \).

Proposition 6.13. There is \( \delta > 0 \) sufficiently small and \( \epsilon_1 > 0 \) such that \( \partial \Sigma_{\delta, \epsilon} \subset \rho(JL) \) for all \( \epsilon \in (0, \epsilon_1) \), and for the Riesz projector

\[
P_{\delta, \epsilon} = -\frac{1}{2\pi i} \int_{|\eta| = \delta \epsilon^2} (JL(\omega) - \eta)^{-1} \, d\eta, \quad \omega = \sqrt{m^2 - \epsilon^2}
\]

(6.30)
o one has rank \( P_{\delta, \epsilon} = 2n + N \) if \( k \neq 2/n \), and \( 2n + N + 2 \) otherwise. One also has \( \dim \ker JL(\omega) = n + N - 1 \).

Remark 6.14. Let us first give an informal calculation of rank \( P_{\delta, \epsilon} \), which is the dimension of the generalized null space of \( JL \). By Lemma 3.2, due to the unitary and translation invariance, the null space is of dimension (at least) \( n + 1 \), and there is (at least) a \( 2 \times 2 \) Jordan block corresponding to each of these null vectors, resulting in \( \dim \mathcal{N}_g(JL(\omega)) \geq 2n + 2 \). Moreover, the ground states of the nonlinear Dirac equation from Theorem 2.1 have additional degeneracy due to the choice of the direction \( \xi \in \mathbb{C}^{N/2} \), \( |\xi| = 1 \) (cf. (2.7)). The tangent space to the sphere on which \( \xi \) lives is of complex dimension \( N/2 - 1 \). (Let us point out that the real dimension is \( N - 2 \), as it should be; we did not expect to have the real dimension \( N - 1 \) since we have already factored out the action of the unitary group.) Thus,

\[
\dim \mathcal{N}_g(JL(\omega)) \geq 2(n + 1) + 2(N/2 - 1) = 2n + N, \quad \omega \leq m.
\]

(6.31)

Whether this is a strict inequality, depends on the Vakhtov–Kolokolov condition \( \partial_\omega Q(\phi_\omega) = 0 \) which indicates the jump by 2 in size of the Jordan block corresponding to the unitary symmetry, and on the energy vanishing \( E(\phi_\omega) = 0 \), which indicates jumps in size of Jordan blocks corresponding to the translation symmetry [BCS15].

Proof of Proposition 6.13. Let \( \delta > 0 \) be such that \( \overline{\Sigma}_\delta \cap \sigma(jl) = \{0\} \); we recall that \( j, l \) are defined in (2.14). Let us define the operator

\[
L(\omega) = \epsilon^2 L(\omega) = \epsilon^{-1} D_0 + \epsilon^2 (\beta m - \omega) + V(y, \omega)
\]

(6.32)
(cf. (3.15)), where \( y = \epsilon x, \epsilon = \sqrt{m^2 - \omega^2} \), and \( D_0 \) is the Dirac operator in the variables \( y = \epsilon x \) (we recall that \( \epsilon D_0 = \epsilon j \alpha \cdot \nabla_y = j \alpha \cdot \nabla_x \)). We rewrite (6.30) as follows:

\[
P_{\delta, \epsilon} = -\frac{1}{2\pi i} \int_{|\eta| = \delta} (JL(\omega) - \eta)^{-1} \, d\eta, \quad \omega = \sqrt{m^2 - \epsilon^2}.
\]

Lemma 6.15. Let

\[
p_\delta = -\frac{1}{2\pi i} \int_{|\eta| = \delta} (JK - \eta)^{-1} \pi_P \, d\eta
\]

be the Riesz projector onto the generalized null space of \( JK|_{\text{Range } \pi_P} \). Then:

1. \( \| \pi_P P_{\delta, \epsilon} \pi_P \pi_P P_{\delta, \epsilon} \pi_P \|_{L^2(\mathbb{R}^n, \mathbb{C}^{4N})} \rightarrow L^2(\mathbb{R}^n, \mathbb{C}^{4N}) \rightarrow 0 \) as \( \epsilon \rightarrow 0 \);

2. There is \( \epsilon_1 > 0 \) such that, for any \( \epsilon \in (0, \epsilon_1) \), one has rank \( P_{\delta, \epsilon} = \text{rank } p_\delta \).

Proof. By Lemma 6.4, \( \sigma(JK) \subset \sigma(jl) \cup \sigma(i l_-) \cup \sigma(-i l_-) \), hence \( (JK - \eta)|_{\text{Range } \pi_P} \) has a bounded inverse \( (JK - \eta)^{-1} : H^{-1}(\mathbb{R}^n, \text{Range } \pi_P) \rightarrow H^1(\mathbb{R}^n, \text{Range } \pi_P) \) on the circle \( |\eta| = \delta \), \( \eta \in \mathbb{C} \), with \( \delta > 0 \) sufficiently small (cf. Lemma B.4).
On the direct sum \((\text{Range } \pi_P) \oplus (\text{Range } \pi_A)\), the operator \(J\mathcal{L}(\omega) - \eta\) is represented by the matrix

\[
\begin{bmatrix}
A_{11}(\epsilon) - \eta & A_{12}(\epsilon) \\
A_{21}(\epsilon) & A_{22}(\epsilon) - \eta
\end{bmatrix} := \begin{bmatrix}
\pi_P J\mathcal{L} \pi_P - \eta & \pi_P J\mathcal{L} \pi_A \\
\pi_A J\mathcal{L} \pi_P & \pi_A J\mathcal{L} \pi_A - \eta
\end{bmatrix}.
\]

According to (6.32),

\[
\|A_{12}(\epsilon)\|_{H^1 \rightarrow L^2} + \|A_{12}(\epsilon)\|_{L^2 \rightarrow H^{-1}} = O(\epsilon^{-1}),
\]

\[
(A_{22}(\epsilon) - \eta)^{-1}|_{\text{Range } \pi_A} = -\frac{\epsilon^2}{2m} J^{-1} + O(L^2 \rightarrow L^2(\epsilon^4)).
\]

In the last relation, we used the following (cf. (6.32)):

\[
A_{22}(\epsilon) - \eta = \pi_A J\mathcal{L} \pi_A - \eta = -\epsilon^{-2}(m + \omega) J + \pi_A J\mathcal{V}(y, \omega) \pi_A - \eta.
\]

The Schur complement of \(A_{22}(\epsilon) - \eta\) is given by

\[
S(\epsilon, \eta) = (A_{11}(\epsilon) - \eta, A_{12}(\epsilon) - \eta) A_{21},
\]

\[
= \pi_P \left( \frac{J}{m + \omega} + J\mathcal{V} - \eta \right) \pi_P - \pi_P (\epsilon^{-1} J D_0 + J\mathcal{V} ) \pi_A (\epsilon^{-1} J D_0 + J\mathcal{V}) \pi_P,
\]

which we consider as an operator \(S(\epsilon, \eta): H^1(\mathbb{R}^n, \mathbb{C}^{2N}) \rightarrow H^{-1}(\mathbb{R}^n, \mathbb{C}^{2N})\). With the above expression for \((A_{22}(\epsilon) - \eta)^{-1}\), the Schur complement (6.34) takes the form

\[
S(\epsilon, \eta) = \pi_P \left( \frac{J}{m + \omega} + J\mathcal{V} - \eta - \frac{J\Delta}{2m} + O(H^1 \rightarrow H^{-1}(\epsilon^2)) \right) \pi_P.
\]

Using the expression (6.35), we can write the inverse of \(J\mathcal{L}(\omega) - \eta\), considered as a map

\[
(J\mathcal{L}(\omega) - \eta)^{-1}: L^2(\mathbb{R}^n, \text{Range } \pi_P \oplus \text{Range } \pi_A) \rightarrow L^2(\mathbb{R}^n, \text{Range } \pi_P \oplus \text{Range } \pi_A),
\]

as follows (cf. (4.18)):

\[
(J\mathcal{L} - \eta)^{-1} = \begin{bmatrix}
S(\epsilon, \eta)^{-1} & S(\epsilon, \eta)^{-1} A_{12}(A_{22} - \eta)^{-1} \\
-(A_{22} - \eta)^{-1} A_{21} S(\epsilon, \eta)^{-1} & (A_{22} - \eta)^{-1} + (A_{22} - \eta)^{-1} A_{21} S(\epsilon, \eta)^{-1} A_{12}(A_{22} - \eta)^{-1}
\end{bmatrix}.
\]

Since

\[
\|(S(\epsilon, \eta) - (JK - \eta))|_{\text{Range } \pi_P}\|_{H^1 \rightarrow H^{-1}} = O(\epsilon),
\]

uniformly in \(|\eta|\), while \(JK - \eta\) has a bounded inverse from \(H^{-1}(\mathbb{R}^n, \mathbb{C}^{2N})\) to \(H^1(\mathbb{R}^n, \mathbb{C}^{2N})\) for \(|\eta| = \delta\), the operator \(S(\epsilon, \eta)|_{\text{Range } \pi_P}\) is also invertible for \(|\eta| = \delta\) as long as \(\epsilon > 0\) is sufficiently small, with its inverse being a continuous map \(H^{-1}(\mathbb{R}^n, \text{Range } \pi_P) \rightarrow H^1(\mathbb{R}^n, \text{Range } \pi_P)\). Using (6.33), we conclude that the matrix (6.36) has all its entries, except the top left one, of order \(O(\epsilon)\) (when considered in the \(L^2 \rightarrow L^2\) operator norm). Hence, it follows from (6.36) and (6.37) that, considering \(P_{\delta, \epsilon}\) as an operator on Range \(\pi_P \oplus \text{Range } \pi_A\),

\[
\left\| P_{\delta, \epsilon} - \begin{bmatrix}
p_\delta & 0 \\
0 & 0
\end{bmatrix} \right\|_{L^2 \rightarrow L^2} = \left\| \frac{1}{2\pi i} \oint_{|\eta| = \delta} \begin{bmatrix}
S(\epsilon, \eta)^{-1} - (JK - \eta)^{-1} & 0 \\
0 & 0
\end{bmatrix} d\eta \right\|_{L^2 \rightarrow L^2} + O(\epsilon) = O(\epsilon).
\]

This proves Lemma 6.15 (1). The statement (2) follows since both \(P_{\delta, \epsilon}\) and \(p_\delta\) are projectors. \(\square\)

The statement of Proposition 6.13 on the rank of \(P_{\delta, \epsilon}\) follows from Lemma 6.15 and Lemma 6.4 (2). The dimension of the kernel of \(J\mathcal{L}(\omega)\) follows from considering the rank of the projection onto the neighborhood of the eigenvalue \(\lambda = 0\) of the self-adjoint operator \(\mathcal{L}\):

\[
\hat{P}_{\delta, \epsilon} = -\frac{1}{2\pi i} \oint_{|\eta| = \delta} (\mathcal{L}(\omega) - \eta)^{-1} d\eta, \quad \omega = \sqrt{m^2 - \epsilon^2},
\]

similarly to how it was done for \(P_{\delta, \epsilon}\), and from the relation ker \(J\mathcal{L}(\omega) = \ker \mathcal{L}(\omega) = \text{rank } \hat{P}_{\delta, \epsilon}, \epsilon \in (0, \epsilon_1)\). Above, \(\delta > 0\) is small enough so that \(\mathbb{D}_\delta \cap \sigma(1) = \{0\}\). This finishes the proof of Proposition 6.13. \(\square\)
Now we return to the proof of Theorem 2.3 (3). If there is an eigenvalue family \((\lambda_j)_{j \in \mathbb{N}}, \lambda_j \in \sigma_p(JL(\omega_j))\), such that \(\Lambda_j \neq 0\) and \(\Lambda_j = \frac{\lambda_j}{m^2 - \omega_j^2} \to 0\) as \(\omega_j \to m\), then the dimension of the generalized kernel of the nonrelativistic limit of the rescaled system jumps up, so that \(\dim \mathcal{N}_g(JL(\omega))|_{\omega < m} + 1 \geq 2n + N + 1\), or, taking into account the symmetry of \(\sigma(JL(\omega))\) with respect to reflections relative to the axes \(\mathbb{R}\) and \(\mathbb{i}\mathbb{R}\), we see that there is at least one more eigenvalue family, hence the dimension of the generalized kernel of the nonrelativistic limit jumps up by at least two:

\[
\dim \mathcal{N}_g(JL(\omega))|_{\omega < m} + 2 \geq 2n + N + 2.
\]

Comparing this inequality to Lemma 6.4 (2) shows that the assumption \(\Lambda_j \neq 0\) for \(j \in \mathbb{N}, \Lambda_j \to 0\) leads to \(\dim \mathcal{N}_g(jl) \geq 2n + 4\). By Lemma B.3 (see Appendix B), this is only possible in the charge-critical case \(k = 2/n\).

Thus, we know that \(k = 2/n\). The remaining part of the argument further develops the approach from [CP03, Com11] to show that there could be no subsequence \(\Lambda_j \to 0\) with \(\text{Re} \Lambda_j \neq 0\) in the case when \(\partial_\omega Q(\phi_\omega) < 0\) for \(\omega \lesssim m\), in a formal agreement with the Vakhitov–Kolokolov stability condition [VK73]. We define

\[
\Phi(y, \omega) = e^{-\frac{1}{T} \phi_\omega(\epsilon^{-1})}, \quad e_1(y, \omega) = e^{-\frac{1}{T} J \phi_\omega(\epsilon^{-1})}, \quad e_2(y, \omega) = e^{2 - \frac{1}{T} (\partial_\omega \phi_\omega)(\epsilon^{-1})};
\]

here and below, \(\epsilon = \sqrt{m^2 - \omega^2}\). Noting the factor \(\epsilon^{-2}\) in the definition of \(\mathcal{L}\) in (6.32), we deduce from (3.6) the relations

\[
J \mathcal{L}(\omega) e_1(\omega) = 0, \quad J \mathcal{L}(\omega) e_2(\omega) = e_1(\omega), \quad \omega \in (\omega_0, m).
\]

Let \(\omega_1 = \sqrt{m^2 - \epsilon_1^2}\), with \(\epsilon_1\) from Proposition 6.13. With

\[
\theta(y) = -\frac{m}{k} u_k(y) - m y \cdot \nabla u_k(y) \quad \theta \in H^1(\mathbb{R}^n)
\]

and real-valued \(\alpha, \beta \in H^2(\mathbb{R}^n)\) such that

\[
1_+ \theta(y) = u_k(y), \quad 1_- \alpha(y) = \theta(y), \quad 1_+ \beta(y) = \alpha(y)
\]

(cf. (B.7), (B.8), and (B.10) in the proof of Lemma B.3), we define \(E_3(y) = -J \mathcal{E} \alpha(y), E_4(y) = -\mathcal{E} \beta(y)\), so that \(E_3, E_4 \in H^2(\mathbb{R}^n, \mathbb{R}^{2n})\) satisfy

\[
JKE_3(y) = \mathcal{E} \theta(y), \quad JKE_4(y) = E_3(y).
\]

Lemma 6.16. The functions \(e_1(\omega), e_2(\omega), e_4(\omega) = P_\omega E_4\), and \(e_3(\omega) = JL(\omega) e_4(\omega)\), defined for \(\omega \in (\omega_1, m]\), can be extended to continuous maps \(e_i : (\omega_1, m] \to L^2(\mathbb{R}^n, \mathbb{R}^{2n})\), \(1 \leq i \leq 4\), with \(e_1(m) = J \mathcal{E} u_k, e_2(m) = \mathcal{E} \theta\), and \(e_i(m) = \lim_{\omega \to m} e_i(\omega) = E_i, i = 3, 4\), so that

\[
JKE_1(\omega) = 0, \quad JKE_2(\omega) = e_1(\omega), \quad \omega \in (\omega_1, m]; \quad JKE_3(m) = e_2(m), \quad JKE_4(m) = e_3(m).
\]

Proof. By Theorem 2.1, \(e_1(y, \omega) = e^{-\frac{1}{T} J \phi_\omega(\epsilon^{-1})} = J \mathcal{E} u_k(y) + O_{H^1(\mathbb{R}^n, \mathbb{C}^{2n})}(\epsilon^{2r})\), so \(\lim_{\omega \to m} e_1(y, \omega)\) is defined in \(H^1(\mathbb{R}^n, \mathbb{C}^{2n})\). Since

\[
v(r, \omega) = e^{1/k} (\hat{V}(er) + \hat{V}(er, \epsilon)) \quad \text{and} \quad u(r, \omega) = e^{1+1/k} (\hat{U}(er) + \hat{U}(er, \epsilon)),
\]

with \(\hat{V} , \hat{U}\) from (2.4), one has \(\partial_\omega v(x, \omega) = \partial_\omega \frac{\partial}{\partial \omega} (e^{1/k} \hat{V}(ex) + e^{1/k} \hat{V}(ex, \epsilon)), \) so that

\[
e^{2-1/k} \partial_\omega v(-1, \omega) = -\omega \left(\frac{\hat{V}(y)}{k} + y \cdot \nabla \hat{V}(y) + \frac{\hat{V}(y, \epsilon)}{k} + y \cdot \nabla \hat{V}(y, \epsilon) + e_\omega \hat{V}(y, \epsilon)\right).
\]
Using (2.8) from Theorem 2.1 to bound the $y \cdot \nabla \hat{V}$-term, one has $\|y|\nabla_y \hat{V}(y, \epsilon)\|_{L^2(\mathbb{R}^n)} = O(\epsilon^{2\lambda})$; due to (2.11) from Theorem 2.1, $\|\partial_y \hat{V}(\cdot, \epsilon)\|_{H^1(\mathbb{R}^n, \mathbb{R}^2)} = O(\epsilon^{2\lambda-1})$. Taking into account these estimates in (6.44), we arrive at

$$
epsilon^2 \frac{1}{2} (\partial_y v)(e^{-1} y, \epsilon) = -\omega \left( \frac{1}{k} \hat{V}(y) + y \cdot \nabla \hat{V}(y) \right) + O_{L^2(\mathbb{R}^n)}(\epsilon^{2\lambda}),$$

with a similar expression for $\epsilon^2 - \frac{1}{2} \partial_y u$. This leads to

$$
epsilon^2 \frac{1}{2} (\partial_y \Phi)(e^{-1} y) = -\omega \left( \frac{1}{k} \hat{V}(y) + y \cdot \nabla \hat{V}(y) \right) \Xi + O_{L^2(\mathbb{R}^n)}(\epsilon^{2\lambda}).$$

Taking into account that $e_2(y, \omega) = \epsilon^2 - \frac{1}{2} (\partial_y \Phi)(e^{-1} y)$ (cf. (6.38)), the relation (6.45) allows us to define

$$e_2(m) := \lim_{\omega \to m} e_2(\omega) = \lim_{\omega \to m} e^2 \frac{1}{2} (\partial_y \Phi)(e^{-1} \cdot) = \Xi \theta,$$

with $\theta(y)$ from (6.40). By (6.45), the convergence in (6.46) is in $L^2(\mathbb{R}^n, \mathbb{C}^{2N})$.

For $i = 4$, one has: $\lim_{\omega \to m} e_4(\omega) = \lim_{\omega \to m} P_{\delta, \omega}(\omega) E_4 = E_4 + \lim_{\omega \to m} (P_{\delta, \omega} - P_\delta) E_4 = E_4$, with the limit holding in $L^2$ norm. In the last relation, we used the relation $p_\delta E_i = E_i$ and Lemma 6.15.

For $i = 3$, the result follows from $e_3(\omega) = JL(\omega) e_4(\omega) = JL(\omega) P_{\delta, \omega} e_4(\omega)$ since $JL(\omega) P_{\delta, \omega}$ is a bounded operator.

We also point out that not only $e_1(\omega)$ and $e_2(\omega)$, but also $e_3(\omega)$ and $e_4(\omega)$ are real-valued; this follows from the observation that $E_4 \in L^2(\mathbb{R}^n, \mathbb{R}^{2N})$ is real-valued, while $P_{\delta, \omega}$ commutes with the operator $K : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ of complex conjugation since $JL$ has real coefficients.

The vector space $\text{Range} \, P_{\delta, \epsilon}$ is spanned by the vectors

$$\{ e_i(\omega), \, 1 \leq i \leq 4; \, \partial_j \Phi, \, \alpha_j \Phi, \, -2 \omega y j \Phi, \, 1 \leq j \leq n; \, \Theta_i(\omega), \, 1 \leq l \leq N - 2 \},$$

where $\Theta_i(\omega)$ are certain vectors from $\ker JL(\omega)$, with $1 \leq l \leq N - 2$ due to Proposition 6.13 (which states that $\dim P_{\delta, \omega} = 2n + N + 2$, $\dim \ker JL(\omega)|_{P_{\delta, \omega}} = n + N - 1$).

**Remark 6.17.** When $n = 3$ and $N = 4$, there are three vectors $\Theta_i(\omega)$ corresponding to infinitesimal rotations around three coordinate axes, but, as it was mentioned in [BCS15], the span of these vectors, $\text{span} \{ \Theta_i; \, 1 \leq l \leq 3 \}$, turns out to contain the null eigenvector $e_1(\omega)$.

In the basis (6.47) of the space $\text{Range} \, P_{\delta, \epsilon}$, the operator $(JL(\omega) - \lambda_1 N)|_{\text{Range} \, P_{\delta, \omega}}$ is represented by

$$M_\omega - \lambda_1 N = \begin{bmatrix}
-\lambda & 1 & \sigma_1(\omega) & 0 & 0 & 0 \\
0 & -\lambda & \sigma_2(\omega) & 0 & 0 & 0 \\
0 & 0 & \sigma_3(\omega) - \lambda & 1 & 0 & 0 \\
0 & 0 & \sigma_4(\omega) & -\lambda & 0 & 0 \\
\vdots & \vdots & \vdots & -\lambda_1 n & 1_n & 0 \\
\vdots & \vdots & \vdots & 0 & -\lambda_1 n & 0 \\
\vdots & \vdots & \vdots & 0 & 0 & -\lambda_1 N_{-2}
\end{bmatrix}.$$  

(6.48)

We used (6.43). Above, vertical dots denote columns of irrelevant coefficients, while $\sigma_i(\omega), \, 1 \leq i \leq 4$, are certain continuous functions. Considering (6.48) at $\lambda = 0$ and $\epsilon = 0$, one concludes from (6.43) that

$$\sigma_1(m) = \sigma_3(m) = \sigma_4(m) = 0, \quad \sigma_2(m) = 1.$$

(6.49)

From (6.48), we also have

$$\det(M_\omega - \lambda) = (-\lambda)^{2n+N}(\lambda^2 - \lambda \sigma_3(\omega) - \sigma_4(\omega)).$$

(6.50)
Lemma 6.18. For any solitary wave \( \phi(x) e^{-i\omega t} \) with \( \phi \in H^{1/2}({\mathbb{R}}^n) \) and any \( 1 \leq j \leq n \), one has \( \langle \phi, \alpha^{j}\phi \rangle = 0 \).

Proof. The local version of the charge conservation, \( \partial_{\mu} J^{\mu} = 0 \), with \( J^{\mu}(x,t) = \bar{\psi}(x,t)\gamma^{\mu}\psi(x,t) \), when applied to a solitary wave with stationary charge and current densities, \( J^{\mu}(x,t) = \bar{\phi}(x)\gamma^{j}\phi(x) \), yields the desired identity:

\[
0 = \partial_{t} \int_{\mathbb{R}^n} J^{0}(x,t) x^{j} \, dx = -\int_{\mathbb{R}^n} \left( \partial_{t} J^{j}(x) \right) x^{j} \, dx = \int_{\mathbb{R}^n} J^{j}(x) \, dx, \quad 1 \leq j \leq n.
\]

Expanding \( J\mathcal{L}e_{3}(\omega) \) over the basis in Range \( P_{\delta,\omega} \), we conclude that for some continuous functions \( \gamma_{j}(\omega) \) and \( \rho_{j}(\omega), 1 \leq j \leq n \), and \( \tau_{l}(\omega), 1 \leq l \leq N - 2 \), there is a relation

\[
\mathcal{J}e_{3}(\omega) = \sum_{i=1}^{4} \sigma_{i}(\omega)e_{i}(\omega) + \sum_{j=1}^{n} \left( \gamma_{j}(\omega)\partial_{j} \Phi_{\omega} + \rho_{j}(\omega)(\alpha^{4}\Phi_{\omega} - 2\omega x^{j}\Phi_{\omega}) \right) + \sum_{l=1}^{N-2} \tau_{l}(\omega)\Theta_{l}(\omega), \tag{6.51}
\]

for \( \omega_{1} < \omega \leq m \). Pairing (6.51) with \( \Phi_{\omega} = J^{-1}e_{1}(\omega) \), we get:

\[
0 = \sigma_{2}(\omega)\langle J^{-1}e_{1}(\omega), e_{2}(\omega) \rangle + \sigma_{4}(\omega)\langle J^{-1}e_{1}(\omega), e_{4}(\omega) \rangle, \quad \omega_{1} < \omega \leq m. \tag{6.52}
\]

We took into account that one has \( \langle \Phi, \nu \rangle = \langle \mathcal{L}e_{2}, \nu \rangle = \langle e_{2}, \mathcal{L}\nu \rangle = 0 \) for any \( \nu \in \ker \mathcal{L} \), the identities

\[
\langle J^{-1}e_{1}, J\mathcal{L}e_{3}(\omega) \rangle = -\langle \mathcal{L}e_{1}, e_{3}(\omega) \rangle = 0,
\]

and also the identity \( \langle \Phi_{\omega}, \alpha^{4}\Phi_{\omega} - 2\omega x^{j}\Phi_{\omega} \rangle = 0 \) which holds due to Lemma 6.18 and due to \( \Phi_{\omega}^{-1}J\Phi_{\omega} \equiv 0 \) (the left-hand side is skew-adjoint while all the quantities are real-valued). Since

\[
\langle J^{-1}e_{1}(\omega), e_{2}(\omega) \rangle = e^{-\frac{2}{\omega}}\langle \Phi_{\omega}(\omega^{-1} \cdot), (\partial_{\omega} \Phi_{\omega})(\omega^{-1} \cdot) \rangle = e^{-\frac{2}{\omega}+n}\langle \Phi_{\omega}, \partial_{\omega} \Phi_{\omega} \rangle = \partial_{\omega}Q(\phi_{\omega})/2, \tag{6.53}
\]

the relation (6.52) takes the form

\[
\sigma_{2}(\omega)\partial_{\omega}Q(\phi_{\omega})/2 = \mu(\omega)\sigma_{4}(\omega), \quad \omega_{1} < \omega \leq m, \tag{6.54}
\]

where \( \mu(\omega) := -\langle J^{-1}e_{1}(\omega), e_{4}(\omega) \rangle \) is a continuous function of \( \omega \in (\omega_{1}, m] \).

Remark 6.19. By (6.53) and Lemma 6.16, \( \partial_{\omega}Q(\phi_{\omega}) \) is a continuous function of \( \omega \in (\omega_{1}, m] \).

Lemma 6.20. There is \( \omega_{2} \in (\omega_{1}, m) \) such that \( \mu(\omega) > 0 \) for \( \omega_{2} < \omega \leq m \).

Proof. We have \( \mu(\omega) = -\langle \Phi_{\omega}, e_{4} \rangle = -\langle \Phi_{\omega}, P_{\delta,\omega}(e_{4}) \rangle = -\langle J^{-1}e_{1}(m), e_{4}(m) \rangle + O(\epsilon) \), while (6.43) yields

\[
-\langle J^{-1}e_{1}(m), e_{4}(m) \rangle = -\langle Ke_{2}(m), e_{4}(m) \rangle = -\langle Je_{3}(m), J^{-1}e_{3}(m) \rangle = \langle \Xi_{\alpha}, K\Xi_{\alpha} \rangle > 0.
\]

Above, we used (6.42) and the explicit form of \( E_{2} \) and \( E_{3} \).

Lemma 6.21. There is \( \omega_{3} \in (\omega_{2}, m) \) such that \( \sigma_{3}(\omega) \equiv 0 \) for \( \omega \in [\omega_{3}, m] \).

Proof. Applying \( (J\mathcal{L}(\omega))^{2} \) to (6.51), we get

\[
(J\mathcal{L})^{3}e_{3}(\omega) = \sigma_{3}(\omega)(J\mathcal{L})^{2}e_{3}(\omega) + \sigma_{4}(\omega)(J\mathcal{L})^{2}e_{4}(\omega).
\]

Coupling this relation with \( J^{-1}e_{4} \) and using the identities \( (J^{-1}e_{4}, (J\mathcal{L})^{3}e_{3}) = \langle e_{3}, \mathcal{L}J\mathcal{L}e_{3} \rangle = 0 \) and \( (J^{-1}e_{4}, (J\mathcal{L})^{2}e_{4}) = -\langle e_{4}, \mathcal{L}J\mathcal{L}e_{4} \rangle = 0 \) (both of these due to skew-adjointness of \( J\mathcal{L} \)), taking into account that \( e_{i}(\omega), 1 \leq i \leq 4 \), are real-valued by Lemma 6.16, while \( J \) and \( \mathcal{L} \) have real coefficients, we have

\[
\sigma_{3}(\omega)(J^{-1}e_{4}, (J\mathcal{L})^{2}e_{3}) = 0, \tag{6.55}
\]

The factor at \( \sigma_{3}(\omega) \) is nonzero for \( \omega < m \) sufficiently close to \( m \). Indeed, using (6.49),

\[
\langle J^{-1}e_{4}, (J\mathcal{L})^{2}e_{3} \rangle|_{\omega=m} = \langle J^{-1}e_{4}, \sigma_{2}e_{1} + \sigma_{3}J\mathcal{L}e_{3} + \sigma_{4}e_{3} \rangle|_{\omega=m} = \langle J^{-1}e_{4}, e_{1} \rangle|_{\omega=m} = -\langle e_{4}, \Phi \rangle|_{\omega=m},
\]

which is positive due to Lemma 6.20. Due to continuity in \( \omega \) of the coefficient at \( \sigma_{3}(\omega) \) in (6.55), we conclude that \( \sigma_{3}(\omega) \) is identically zero for \( \omega \in [\omega_{3}, m] \), with some \( \omega_{3} < m \). \( \square \)

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Since $\sigma_3(\omega)$ is identically zero for $\omega \in [\omega_3, m]$, we conclude from (6.50) that the nonzero eigenvalues of $J.L(\omega)$ satisfy $\lambda^2 - \sigma_4(\omega) = 0$, $\omega \in [\omega_3, m]$. By (6.49) and Lemma 6.20, the relation (6.54) shows that $\sigma_4(\omega)$ is of the same sign as $\partial_\omega Q(\phi_{\omega})$. Thus, if $\partial_\omega Q(\phi_{\omega}) > 0$ for $\omega \lesssim m$, then for these values of $\omega$ there are two nonzero real eigenvalues of $J.L(\omega)$, one positive (indicating the linear instability) and one negative, both of magnitude $\sim \sqrt{\partial_\omega Q(\phi_{\omega})}$ for $\omega \lesssim m$; hence, there are two real eigenvalues of $JL$, of magnitude $\sim \epsilon^2 \sqrt{\partial_\omega Q(\phi_{\omega})}$. This completes the proof of Theorem 2.3.
Appendix: Analytic continuation of the free resolvent

Let us remind the limiting absorption principle for the free resolvent [Agm75, Remark 2 in Appendix A] and [JK79, Theorem 8.1].

Lemma A.1 (Limiting absorption principle for the Laplace operator). Let \( n \geq 1 \). For any \( k \in \mathbb{N}_0, \nu \leq 2 + 2k, s > 1/2 + k, \) and \( \delta > 0 \), there is \( C = C(n, s, k, \nu, \delta) < \infty \) such that

\[
\| \partial_z^k (-\Delta - z)^{-1} \|_{L^2(\mathbb{R}^n) \to \mathcal{H}^\nu(\mathbb{R}^n)} \leq C|z|^{-(k+1-\nu)/2}, \quad z \in \mathbb{C} \setminus (\mathbb{D}_\delta \cup \mathbb{R}_+).
\]

Proof. For \( \nu = 0 \), the lemma rephrases [JK79, Theorem 8.1] (stated for \( n = 3 \)) or [Agm75, Theorem A.1 and Remark 2 in Appendix A]. The recurrence based on the identities

\[
-\Delta(-\Delta - z)^{-1} = 1 + z(-\Delta - z)^{-1} \quad \text{and} \quad \partial_z^k(-\Delta - z)^{-1} = k!(-\Delta - z)^{-k-1}, \quad k \geq 0,
\]

provides all other cases. \( \square \)

Let \( E_\mu : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) denote the operator of multiplication by \( e^{-\mu(r)} \), \( \mu \in \mathbb{R} \). Following [Rau78], not to confuse the regularized resolvent

\[
R^0_\mu(\zeta^2) := E_\mu R^0(\zeta^2) E_\mu = E_\mu(-\Delta - \zeta^2)^{-1} E_\mu, \quad \zeta \in \mathbb{C}, \quad \text{Im} \zeta > 0,
\]

with its analytic continuation through the line \( \text{Im} \zeta = 0 \); we will denote the latter by \( F^0_\mu(\zeta) \).

Proposition A.2 (Analytic continuation of the resolvent). Let \( n \geq 1 \).

1. There is an analytic function \( F^0_\mu(\zeta) \),

\[
F^0_\mu : \{ \text{Im} \zeta > -\mu \} \setminus (-i\mathbb{R}_+) \to \mathcal{B}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n)),
\]

such that \( F^0_\mu(\zeta) = R^0_\mu(\zeta^2) \) for \( \text{Im} \zeta > 0 \), and for any \( k \in \mathbb{N}_0, \nu \leq 2 + 2k, \delta > 0 \), there is \( C = C(n, k, \nu, \mu, \delta) < \infty \) such that

\[
\| \partial_\zeta^k F^0_\mu(\zeta) \|_{L^2 \to \mathcal{H}^\nu} \leq \frac{C}{(1 + |\zeta|)^{k+1-\nu}}, \quad \zeta \in \mathbb{C} \cap \{ \text{Im} \zeta \geq -\mu + \delta \}, \quad \text{dist}(\zeta, -i\mathbb{R}_+) > \delta. \quad (A.1)
\]

2. If \( n \) is odd and satisfies \( n \geq 3 \), then (A.1) holds for all \( \zeta \in \mathbb{C} \cap \{ \text{Im} \zeta \geq -\mu + \delta \} \).

Remark A.3. This result in dimension \( n = 3 \) was stated and proved in [Rau78, Proposition 3], as a consequence of the explicit expression for the integral kernel of \( R^0_\mu(\zeta^2) \),

\[
-\frac{e^{-\mu(y)} e^{i|y-x|} e^{-\mu(x)}}{4\pi |y-x|}, \quad \text{Im} \zeta > 0, \quad x, y \in \mathbb{R}^3,
\]

which could be extended analytically to the region \( \text{Im} \zeta > -\mu \) as a holomorphic function of \( \zeta \) with values in \( L^2(\mathbb{R}^3 \times \mathbb{R}^3) \). In [Rau78], the restriction on \( \zeta \) was stronger: \( \text{Im} \zeta > -\mu/2 + \delta \) (with any \( \delta > 0 \)); this was a pay-off for using an elegant argument based on the Huygens principle. (We note that our signs and inequalities are often the opposite to those of [Rau78] since we consider the resolvent of \( -\Delta \) instead of \( \Delta \).)

Proof. Let us define the analytic continuation of \( F^0_\mu(\zeta) \). For \( u, v \in L^2(\mathbb{R}^n) \) we define \( u_\mu(x) = e^{-\mu(x)} u(x), v_\mu(x) = e^{-\mu(x)} v(x) \) and consider

\[
I(\zeta) = \langle v, F^0_\mu(\zeta) u \rangle = \int_{\mathbb{R}^n} \tilde{v}_\mu(\xi) \frac{1}{\xi^2 - \zeta^2} \tilde{u}_\mu(\xi) \frac{d^n \xi}{(2\pi)^n}, \quad (A.2)
\]

which is an analytic function in \( \zeta \in \mathbb{C}^+ := \{ z \in \mathbb{C} : \text{Im} \ z > 0 \} \).
Let us prove analyticity in $\zeta$ for $\Im \zeta > -\mu$, $\Re \zeta > 0$ (the case $\Re \zeta < 0$ is considered similarly). It is enough to prove that for any $a > 0$ and any $\delta > 0$, $\delta \leq a/2$, $I(\zeta)$ extends analytically into the rectangular neighborhood

$$K_\delta^a = \{ \zeta \in \mathbb{C} : a - \delta \leq \Re \zeta \leq a + \delta, -\mu + \delta \leq \Im \zeta \leq 0 \}$$

(see Figure 2), satisfying there the bounds (A.1) with constants $c_j$ independent of $a$. We pick $a > 0$ and $\delta > 0$, with $a \geq 3 \delta$, and break the integral (A.2) into two:

$$I(\zeta) = I_1^{(\delta)}(\zeta) + I_2^{(\delta)}(\zeta) = \int_{|\xi - a| > 2 \delta} + \int_{|\xi - a| < 2 \delta}. \quad \text{(A.4)}$$

The first integral in (A.4) is finite, being bounded by

$$\int_{|\xi - a| > 2 \delta} |\hat{v}_\mu(\xi)||\hat{\mu}(\xi)| \left| \frac{1}{2|\xi|} - \frac{1}{|\xi - \zeta| + \zeta} \right| \frac{d^n \xi}{(2\pi)^n} \leq \int_{\mathbb{R}^n} |\hat{v}_\mu(\xi)||\hat{\mu}(\xi)| \left( \frac{2}{|\xi|} - \frac{\delta}{(2\pi)^n} \right) \leq \frac{\|u_\mu\|\|v_\mu\|}{|\zeta|\delta},$$

and therefore is analytic in $\zeta$ and is bounded by $C/|\zeta|$. Above, to estimate the denominators, we took into account that for $\zeta \in K_\delta^a$ and $|\xi - a| > 2 \delta$,

$$|\xi| + |\zeta| \geq |(\xi - a)| + (a + \Re \zeta) \geq |\xi - a| - |a + \Re \zeta| > 2 \delta - \delta = \delta.$$

To analyze the second integral in (A.4), we will deform the contour of integration in $\xi$. Let $g_0 \in C^\infty_{\text{comp}}(\mathbb{R})$ be even, $g_0 \leq 0$, $\text{supp} g_0 \subseteq [-2 \delta, 2 \delta]$, with $g_0(0) = -\mu + \delta/2$ and non-decreasing away from the origin. Moreover, we may assume that $|g_0'| < 4 \mu/\delta$ and that dist($\gamma_0, K_\delta^a$) $\geq \delta/2$, where $K_\delta^a$ is defined in (A.3) and $\gamma_0 = \{(\lambda, g_0(\lambda)) : |\lambda| \leq 2 \delta\}$; see Figure 2. Define $g_0(t) = g_0(t - a)$.

**Lemma A.4.** Assume that $u \in L^{2,\mu}(\mathbb{R}^n)$, so that $\|u\|_{L^2,\mu(\mathbb{R}^n)} := \|e^{ix(\nu)}u\|_{L^2(\mathbb{R}^n)} < \infty$. Then its Fourier transform, $\hat{u}(\xi)$, can be extended analytically into the $\mu$-neighborhood of $\mathbb{R}^n \subset \mathbb{C}^n$, which we denote by

$$\Omega_\mu(\mathbb{R}^n) = \{ \xi \in \mathbb{C}^n : |\Im \xi| < \mu \} \subset \mathbb{C}^n,$$

and there is $C_\mu < \infty$ such that

$$\|\hat{u}\|_{L^2(\Omega_\mu(\mathbb{R}^n))} \leq C_\mu \|u\|_{L^2,\mu(\mathbb{R}^n)}, \quad \text{(A.5)}$$

where $\Omega_\mu(\mathbb{R}^n)$ is interpreted as a region in $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

By Lemma A.4, the functions $U(\xi) = \hat{u}_\mu(\xi)$ and $V(\xi) = \hat{v}_\mu(\xi)$ could be extended analytically in $\xi \in \mathbb{R}^n$ into the strip $\xi \in \mathbb{C}^n$, $|\Im \xi| < \mu$. We rewrite the second integral in (A.4) in polar coordinates, denoting $\lambda = |\xi| \in [a - 2 \delta, a + 2 \delta]$, and then deform the contour of integration in $\lambda$, arriving at

$$I_2^{(\delta)}(\zeta) = \int_{\gamma_a \times S^{n-1}} \frac{V(\theta \lambda) U(\theta \lambda)}{\lambda^2 - \zeta^2} \lambda^{n-1} d\lambda d\Omega_\theta \frac{d\Omega_\theta}{(2\pi)^n}.$$

Figure 2: The set $K_\delta^a$ and the contour $\gamma_a = \{ \lambda : \Im \lambda = g_a(\Re \lambda), a - 2 \delta \leq \Re \lambda \leq a + 2 \delta \}$: dist($K_\delta^a, \gamma_a$) $\geq \delta/2$. 

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with $\gamma_a$ as on Figure 2. Clearly, (A.6) is analytic for $\Re \zeta > 0$ and $\Im \zeta > 0$ (since $\Im \lambda^2 \leq 0$ while $\Im \zeta^2 > 0$).

Let us argue that (A.6) can also be extended analytically into the box $K^\delta_a$. For $\lambda \in \gamma_a$ and $\zeta \in K^\delta_a$, taking into account that

$$|\lambda - \zeta| \geq \delta/2, \quad |\lambda + \zeta| \geq \Re \lambda + \Re \zeta \geq (a - 2\delta) + (a - \delta) = 2a - 3\delta \geq a$$

(recall that $\delta \leq a/3$), we see that (A.6) defines an analytic function which is bounded by

$$|I_2^{(\delta)}(\zeta)| \leq \frac{2}{a\delta} \int_{\gamma_a \times S^{n-1}} |V(\theta \lambda)|^2 |\lambda|^{n-1} |d\lambda| \left( \frac{d\Omega_\theta}{(2\pi)^n} \right) \int_{\gamma_a \times S^{n-1}} |U(\theta \zeta)|^2 |\lambda|^{n-1} |d\lambda| \left( \frac{d\Omega_\theta}{(2\pi)^n} \right)^{1/2}. \quad \text{(A.7)}$$

Our assumption that $a \geq 3\delta$ allows us to bound the first factor in (A.7) by $\frac{2}{a\delta} \leq \frac{2}{3\delta^2}$. Moreover, if $|\zeta| \geq 2(\mu + \delta)$, the first factor in (A.7) is also bounded by

$$\frac{2}{a\delta} \leq \frac{2}{(|\Re \zeta| - \delta)\delta} \leq \frac{2}{(|\zeta| - \mu - \delta)\delta} \leq \frac{4}{|\zeta|^2}, \quad \forall \zeta \in K^\delta_a \setminus \mathbb{D}_{2(\mu + \delta)}.$$

Therefore, that factor is bounded by $c/(1 + |\zeta|)$ with certain $c = c(\mu, \delta) < \infty$. To study the integrals in (A.7), we parametrize $\xi$ as follows:

$$\xi = \eta + iG(\eta), \quad \eta \in \mathbb{R}^n, \quad ||\eta|| - a| \leq 2\delta, \quad G(\eta) := \left\{ \eta \right\}_{\eta/|\eta|} g_a(|\eta|) \rho(|\eta|/\delta),$$

where $\rho \in C^\infty(\mathbb{R})$ satisfies $\rho(t) \equiv 1$ for $|t| \geq 1$, $\rho(t) \equiv 0$ for $|t| \leq 1/2$. We have:

$$\int_{\gamma_a \times S^{n-1}} |U(\theta \zeta)|^2 |\lambda|^{n-1} |d\lambda| \left( \frac{d\Omega_\theta}{(2\pi)^n} \right)^{1/2} \int_{||\eta|| - a| \leq 2\delta} |U(\eta + iG(\eta))|^2 d^n\eta,$$

where we took into account that both $|\lambda/\Re \lambda|$ and $|d\lambda/\Re d\lambda|$ are bounded by $\sqrt{1 + (\mu/\delta)^2} \leq \sqrt{1 + (4\mu/\delta)^2}$. One has:

$$U(\eta + iG(\eta)) = A_g u(\eta) = \int_{\mathbb{R}^n} e^{-ix \cdot \eta} e^{G(\eta)} e^{-\mu(x) \cdot u(x)} \, dx.$$

Above, $A_g$ an oscillatory integral operator with the non-degenerate phase function $\phi(x, \eta) = x \cdot \eta$ and bounded smooth symbol $a(x, \eta) = e^{x \cdot G(\eta) - \mu(x)}$. By the van der Corput-type arguments applied to $A_g A^*_g$ [Ste93, Chapter IX], $A_g$ is continuous in $L^2(\mathbb{R}^n)$, so that there is $c = c(\mu, \delta) < \infty$ such that

$$\int_{\gamma_a \times S^{n-1}} |U(\theta \zeta)|^2 |\lambda|^{n-1} |d\lambda| \left( \frac{d\Omega_\theta}{(2\pi)^n} \right) \leq c(\mu, \delta) ||u||^2.$$

There is a similar bound for $V$. Thus, there is $C = C(\mu, \delta) < \infty$ such that $|I_2^{(\delta)}(\zeta)| \leq \frac{C(\mu, \delta) ||u||^2}{||v||^2}$, which is the desired bound.

The estimates on $\partial^j_x F^0_m(\xi)$, $j \in \mathbb{N}$, are proved similarly, writing out the derivatives of $(\xi^2 - \zeta^2)^{-1}$ and proceeding with the same decomposition as in (A.4); the only difference is the contribution from higher powers of $\xi^2 - \zeta^2$ in the denominator.

This settles the first part of Proposition A.2.

Before we prove the second part of Proposition A.2, we need the following technical lemma.
Lemma A.5. Let $\rho > 0$ and let $N \in \mathbb{N}$ be odd and satisfy $N \geq 3$. The analytic function

$$F_{N, \rho}(\zeta) = \int_{0}^{\rho} \frac{\lambda^{N-1} d\lambda}{\lambda^2 - \zeta^2}, \quad \zeta \in \mathbb{C}, \quad \text{Im} \, \zeta > 0,$$

extends analytically into an open disc $\mathbb{D}_\rho$. Moreover, one has

$$|F_{N, \rho}(\zeta)| \leq \frac{\rho^{N-2}}{2} \left( 2 + \ln N + \pi + \ln \frac{\rho + |\zeta|}{\rho - |\zeta|} \right), \quad \zeta \in \mathbb{D}_\rho. \quad (A.8)$$

Proof. Using the identity $\frac{x^2}{x^2 - \zeta^2} = 1 + \frac{\zeta^2}{x^2 - \zeta^2}$ (note that the denominator is nonzero since $\lambda \geq 0$ and $\text{Im} \, \zeta > 0$) and remembering that $N$ is odd, we have:

$$F_{N, \rho}(\zeta) = \int_{0}^{\rho} \frac{\lambda^{N-1} d\lambda}{\lambda^2 - \zeta^2} = \int_{0}^{\rho} \left( \frac{\zeta^2}{\lambda^2 - \zeta^2} + \frac{\zeta^2 (\rho - \lambda) N - 5 + \zeta^2 N - 3 + \zeta^2 N - 1}{\lambda^2 - \zeta^2} \right) d\lambda$$

$$= \frac{\rho^N}{N - 2} + \frac{\zeta^2 \rho^{N-4}}{N - 4} + \cdots + \zeta^2 N - 3 + \frac{\zeta^2 N - 2}{2} \left[ \ln \left( \frac{\rho - \zeta}{\rho + \zeta} \right) + \pi i \right]. \quad (A.9)$$

Above, $\ln$ denotes the analytic branch of the natural logarithm on $\mathbb{C} \setminus \mathbb{R}_-$ specified by $\ln(1) = 0$. Note that, since $\text{Im} \, \zeta > 0$,

$$\lim_{\lambda \to 0^+} \ln \frac{\lambda - \zeta}{\lambda + \zeta} = \lim_{\lambda \to 0^+} \ln \left( -1 + \frac{2\lambda}{\zeta} \right) = \ln(-1 - 0i) = -\pi i.$$ 

Due to the assumption $N \geq 3$, the right-hand side of (A.9) extends to an analytic function of $\zeta$ as long as $\zeta \in \mathbb{D}_\rho$. The bound (A.8) immediately follows from the inequalities

$$|\zeta| < \rho, \quad 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{N - 2} < 1 + \frac{1}{2} \sum_{j=2}^{N-2} \frac{1}{j} < 1 + \frac{1}{2} \ln(N - 2),$$

and the bound

$$\left| \ln \left( \frac{\rho - \zeta}{\rho + \zeta} \right) + \pi i \right| \leq \pi + \ln \frac{\rho + |\zeta|}{\rho - |\zeta|}$$

valid for $\zeta \in \mathbb{D}_\rho$.

Remark A.6. Note that the conclusion of the lemma would not hold if $N$ were even: in that case, one arrives at functions which have a branching point at $\zeta = 0$; e.g.

$$\int_{0}^{\rho} \frac{\lambda d\lambda}{\lambda^2 - \zeta^2} = \frac{1}{2} \ln \left( 1 - \frac{\rho^2}{\zeta^2} \right), \quad \int_{0}^{\rho} \frac{\lambda^3 d\lambda}{\lambda^2 - \zeta^2} = \int_{0}^{\rho} \left( \frac{\zeta^2 \lambda}{\lambda^2 - \zeta^2} \right) d\lambda = \frac{\rho^2}{2} + \frac{\zeta^2}{2} \ln \left( 1 - \frac{\rho^2}{\zeta^2} \right),$$

which behave like $\ln \left( -\frac{\rho^2}{\zeta} \right)$ and $\zeta^2 \ln \left( -\frac{\rho^2}{\zeta} \right)$ when $|\zeta| \ll \rho$ (hence they have a branching point at $\zeta = 0$).

Now let us prove the second part of Proposition A.2; from now on, we assume that $n$ is odd and satisfies $n \geq 3$. It is enough to prove that the function $I(\zeta)$ defined in (A.2) is analytic inside the disc $\mathbb{D}_\mu \subset \mathbb{C}$.

We pick $\rho \in (0, \mu)$ and break the integral (A.2) into two parts:

$$I(\zeta) = \int_{\mathbb{R}^n} \frac{V(\xi)U(\xi)}{\xi^2 - \zeta^2} \, d^n \xi = I^{(1)}_1(\zeta) + I^{(2)}_2(\zeta) = \int_{|\xi| \leq \rho} \frac{V(\xi)U(\xi)}{\xi^2 - \zeta^2} \, d^n \xi + \int_{|\xi| > \rho} \frac{V(\xi)U(\xi)}{\xi^2 - \zeta^2} \, d^n \xi. \quad (A.10)$$

The function $I^{(1)}_2(\zeta)$ in (A.10) is analytic in the disc $\zeta \in \mathbb{D}_\rho$, and moreover for any $r \in (0, \rho)$ one has

$$\sup_{\zeta \in \mathbb{D}_r} |I^{(1)}_2(\zeta)| \leq \sup_{\zeta \in \mathbb{D}_r} \left| \int_{|\xi| > \rho} \frac{V(\xi)U(\xi)}{\xi^2 - \zeta^2} \, d^n \xi \right| \leq \frac{1}{\rho^2 - r^2} \|v\|_{L^2} \|u\|_{L^2}.$$
Let us consider $I^{(r)}_1(\zeta)$. Since both $V(\xi)$ and $U(\xi)$ are analytic for $\xi \in \mathbb{C}^n$, $|\xi| < \mu$, we have the power series expansions

$$V(\xi) = \sum_{\alpha \in \mathbb{N}_0^n} V_\alpha \xi^\alpha, \quad U(\xi) = \sum_{\alpha \in \mathbb{N}_0^n} U_\alpha \xi^\alpha, \quad V(\xi)U(\xi) = \sum_{\alpha \in \mathbb{N}_0^n} C_\alpha \xi^\alpha,$$

which are absolutely convergent for $|\xi| < \mu$. Denote $\lambda = |\xi|$, $\theta = \xi / |\xi| \in S^{n-1}$. Then

$$I^{(r)}_1(\zeta) = \sum_{\alpha \in \mathbb{N}_0^n} C_\alpha \int_{S^{n-1}} \theta^\alpha d\Omega_\theta \int_0^\rho \frac{\lambda^{\alpha + n - 1} d\lambda}{\lambda^2 \theta^2 - \zeta^2}.$$  \hspace*{1cm} (A.11)

We note that, by parity considerations, the terms corresponding to at least one $\alpha_j$ being odd are equal to zero, hence the summation in the right-hand side is only over $\alpha \in (2\mathbb{N}_0)^n$. We claim that the series (A.11) defines an analytic function in $\mathbb{D}_\rho$, and moreover for each $r \in (0, \rho)$ there is $C < \infty$ such that

$$\sup_{\zeta \in \mathbb{D}_r} |I^{(r)}_1(\zeta)| \leq C \|v\|_{L^2(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)}.$$  \hspace*{1cm} (A.12)

We have:

$$I^{(r)}_1(\zeta) = \sum_{\alpha \in (2\mathbb{N}_0)^n} C_\alpha \int_{S^{n-1}} \theta^\alpha d\Omega_\theta \frac{1}{\theta^2} F_{\alpha+n,\rho}(\zeta \sqrt{\theta^2})$$

$$= \sum_{\alpha \in (2\mathbb{N}_0)^n} C_\alpha \int_{S^{n-1}} \theta^\alpha d\Omega_\theta R^{|\alpha|} \frac{F_{\alpha+n,\rho}(\zeta \sqrt{\theta^2})}{\theta^2 R^{|\alpha|}},$$  \hspace*{1cm} (A.12)

where $R \in (\rho, \mu)$. By Lemma A.5,

$$\left| (|\alpha| + n) F_{\alpha+n,\rho}(\zeta) \right| \leq (|\alpha| + n) \frac{\rho^{|\alpha| - 2} |\alpha| - 2}{2R^{|\alpha|}} \left( 2 + \ln(|\alpha| + n) + \pi + \ln \frac{\rho + |\alpha|}{\rho - |\alpha|} \right)$$

are analytic functions of $\zeta \in \mathbb{D}_r$, $r \in (0, \rho)$, which are bounded uniformly in $\alpha \in \mathbb{N}_0^n$ and $\zeta \in \mathbb{D}_r$, by some $c_{r,\rho,\mu} < \infty$, $0 < r < \rho < R < \mu$. Using this bound in (A.12), one has:

$$|I^{(r)}_1(\zeta)| \leq c_{r,\rho,\mu} \sum_{\alpha \in (2\mathbb{N}_0)^n} |C_\alpha \theta^\alpha R^{|\alpha|}|, \quad \zeta \in \mathbb{D}_r. \hspace*{1cm} (A.13)$$

Now we can argue that the series (A.12) is absolutely convergent. To bound the right-hand side in (A.13), we use the following lemma which makes the use of Cauchy estimates.

**Lemma A.7.** For any $0 < R < \mu$ there is $C_{R,\mu} < \infty$ such that for any analytic function $U(\xi) = \sum_{\alpha \in \mathbb{N}_0^n} u_\alpha \xi^\alpha$, $\xi \in \mathbb{D}_\mu \subset \mathbb{C}^n$, which has finite norm in $L^1(\mathbb{B}_\mu^{2n})$, where $\mathbb{B}_\mu^{2n} \subset \mathbb{B}^{(2n)}$, is identified with $\mathbb{D}_\mu \subset \mathbb{C}^n$, one has

$$\sup_{\xi \in \mathbb{D}_R} \sum_{\alpha \in \mathbb{N}_0^n} |u_\alpha \xi^\alpha| \leq C_{R,\mu} \|U\|_{L^1(\mathbb{B}_\mu^{2n})}.$$  \hspace*{1cm} (A.5)

This lemma, together with the estimate (A.5) from Lemma A.4, shows that, for $\zeta \in \mathbb{D}_r$, (A.13) is bounded by

$$|I^{(r)}_1(\zeta)| \leq c_{r,\rho,\mu} \sup_{\xi \in \mathbb{D}_R} \sum_{\alpha \in (2\mathbb{N}_0)^n} |C_\alpha \xi^\alpha| \leq c_{r,\rho,\mu} C_{R,\mu} \|VU\|_{L^1(\mathbb{B}_\mu^{2n})} \hspace*{1cm} (A.14)$$

$$\leq c_{r,\rho,\mu} C_{R,\mu} \|V\|_{L^2(\mathbb{B}_\mu^{2n})} \|U\|_{L^2(\mathbb{B}_\mu^{2n})} \leq c_{r,\rho,\mu} C_{R,\mu}^2 \|v\|_{L^2(\mathbb{R}^{2n})} \|u\|_{L^2(\mathbb{R}^{2n})}.$$  \hspace*{1cm} (A.14)

where $V(\xi)$ and $U(\xi)$, $\xi \in \mathbb{R}_\mu(\mathbb{R}^n) \subset \mathbb{C}^n$, denote the analytic continuations of $\hat{v}(\xi)$ and $\hat{u}(\xi)$, $\xi \in \mathbb{R}^n$, into the $\mu$-neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$. We conclude that the series (A.11) is absolutely convergent and therefore defines an analytic function.

Thus, $I^{(r)}_1(\zeta)$ (and hence $I(\zeta)$ in (A.10)) has an analytic continuation into the disc $\mathbb{D}_\rho$ for arbitrary $\rho \in (0, \mu)$, and for any $r \in (0, \rho)$ $I^{(r)}_1(\zeta)$ (and hence $I(\zeta)$) is bounded by $C(r)\|v\|\|u\|$ as long as $\zeta \in \mathbb{D}_r$. This concludes the proof of Proposition A.2. \qed
Appendix: Spectrum of the linearized nonlinear Schrödinger equation

For the nonlinear Schrödinger equation and several similar models, real eigenvalues could only emerge from the origin, and this emergence is controlled by the Vakhitov–Kolokolov stability condition [VK73]. Let us give the essence of the linear stability analysis on the example of the (generalized) nonlinear Schrödinger equation,

$$i\partial_t \psi = -\frac{1}{2m}\Delta \psi - f(|\psi|^2)\psi, \quad \psi(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}^n, \quad n \geq 1, \quad t \in \mathbb{R},$$

where the nonlinearity satisfies $f \in C^\infty(\mathbb{R})$, $f(0) = 0$. One can easily construct solitary wave solutions $\phi(x)e^{-i\omega t}$, for some $\omega \in \mathbb{R}$ and $\phi \in H^1(\mathbb{R}^n)$; $\phi(x)$ satisfies the stationary equation $\omega \phi = -\frac{1}{2m}\Delta \phi - f(\phi^2)\phi$, and can be chosen strictly positive, even, and monotonically decaying away from $x = 0$. The value of $\omega$ can not exceed 0; we will only consider the case $\omega < 0$. We use the Ansatz $\psi(x,t) = (\phi(x) + \rho(x,t))e^{-i\omega t}$, with $\rho(x,t) \in \mathbb{C}$. The linearized equation on $\rho$ is called the linearization at a solitary wave:

$$\partial_t \rho = \frac{1}{\mu}(-\frac{1}{2m}\Delta \rho - \omega \rho - f(\phi^2)\rho - 2f'(\phi^2)\phi \Re \rho). \quad (B.1)$$

Remark B.1. Because of the term with $\Re \rho$, the operator in the right-hand side is $\mathbb{R}$-linear but not $\mathbb{C}$-linear.

To study the spectrum of the operator in the right-hand side of (B.1), we first write it in the $\mathbb{C}$-linear form, considering its action onto $\rho(x,t) = \begin{bmatrix} \Re \rho(x,t) \\ \Im \rho(x,t) \end{bmatrix}$:

$$\partial_t \rho = j l \rho, \quad \rho(x,t) = \begin{bmatrix} \Re \rho(x,t) \\ \Im \rho(x,t) \end{bmatrix};$$

$$j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad l = \begin{bmatrix} l_+ & 0 \\ 0 & l_- \end{bmatrix}, \quad \text{with} \quad l_- = -\frac{1}{2m}\Delta - \omega - f(\phi^2), \quad l_+ = l_- - 2\phi^2 f'(\phi^2). \quad (B.2)$$

If $\phi \in \mathcal{S}(\mathbb{R}^n)$, then by Weyl’s theorem on the essential spectrum one has

$$\sigma_{\text{ess}}(l_-) = \sigma_{\text{ess}}(l_+) = [\omega, +\infty).$$

Lemma B.2. $\sigma(jl) \subset \mathbb{R} \cup i\mathbb{R}$.

Proof. We consider $(jl)^2 = -\begin{bmatrix} l_- l_+ & 0 \\ 0 & l_+ l_- \end{bmatrix}$. Since $l_-$ is positive-definite ($\phi \in \ker l_-$, being nowhere zero, corresponds to the smallest eigenvalue), we can define the self-adjoint square root of $l_-$. Then

$$\sigma_d((jl)^2) \setminus \{0\} = \sigma_d(l_- l_+) \setminus \{0\} = \sigma_d(l_+ l_-) \setminus \{0\} = \sigma_d(l_-^{1/2} l_+ l_-^{1/2}) \setminus \{0\} \subset \mathbb{R},$$

with the inclusion due to $l_-^{1/2} l_+ l_-^{1/2}$ being self-adjoint. Thus, any eigenvalue $\lambda \in \sigma_d(jl)$ satisfies $\lambda^2 \in \mathbb{R}$. \qed

Given the family of solitary waves, $\phi_\omega(x)e^{-i\omega t}$, $\omega \in \mathcal{O} \subset \mathbb{R}$, we would like to know at which $\omega$ the eigenvalues of the linearized equation with $\Re \lambda > 0$ appear. Since $\lambda^2 \in \mathbb{R}$, such eigenvalues can only be located on the real axis, having bifurcated from $\lambda = 0$. One can check that $\lambda = 0$ belongs to the discrete spectrum of $jl$, with

$$jl \begin{bmatrix} 0 \\ \phi_\omega \end{bmatrix} = 0, \quad jl \begin{bmatrix} -\partial_\omega \phi_\omega \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \phi_\omega \end{bmatrix},$$

for all $\omega$ which correspond to functions which are spherically symmetric in $x$, the dimension of the generalized null space of $jl$ is at least two. Hence, the bifurcation follows the jump in the dimension of the generalized null space of $jl$. Such a jump happens
at a particular value of \( \omega \) if one can solve the equation \( j1\alpha = \begin{bmatrix} \partial_\omega \phi_\omega \\ 0 \end{bmatrix} \). This leads to the condition that 
\[
\begin{bmatrix} \partial_\omega \phi_\omega \\ 0 \end{bmatrix}
\] is orthogonal to the null space of the adjoint to \( j1 \), which contains the vector \( \begin{bmatrix} \phi_\omega \\ 0 \end{bmatrix} \); this results in 
\[
\langle \phi_\omega, \partial_\omega \phi_\omega \rangle = \partial_\omega \|\phi_\omega\|_2^2 / 2 = 0.
\]
A slightly more careful analysis [CP03] based on construction of the moving frame in the generalized eigenspace of \( \lambda = 0 \) shows that there are two real eigenvalues \( \pm \lambda \in \mathbb{R} \) that have emerged from \( \lambda = 0 \) when \( \omega \) is such that \( \partial_\omega \|\phi_\omega\|_2^2 \) becomes positive, leading to a linear instability of the corresponding solitary wave. The opposite condition, \( \partial_\omega \|\phi_\omega\|_2^2 < 0 \), is the Vakhitov–Kolokolov stability criterion which guarantees the absence of nonzero real eigenvalues for the nonlinear Schrödinger equation. It appeared in [VK73, CL82, Sha83, Wei86, GSS87, BP92b] in relation to linear and orbital stability of solitary waves.

For the applications to the nonrelativistic limit of the nonlinear Dirac equation, we need to consider the linearization of the nonlinear Schrödinger equation with pure power nonlinearity: \( f(\tau) = |\tau|^k, k > 0 \):
\[
i\dot{\psi} = -\frac{1}{2m} \Delta \psi - |\psi|^{2k} \psi, \quad \psi(x, t) \in \mathbb{C}, \quad x \in \mathbb{R}^n.
\]

We need the detailed knowledge of the spectrum of the linearization at the solitary wave \( u_k(x)e^{-i\omega t} \), with \( u_k \) a strictly positive spherically symmetric solution to (2.2) and \( \omega = -\frac{1}{2m} \) (cf. (2.14), (2.15)):
\[
j1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} l_+ & 0 \\ 0 & l_- \end{bmatrix} = \begin{bmatrix} 0 & l_- \\ -l_+ & 0 \end{bmatrix},
\]
\[
l_- = \frac{1}{2m} - \frac{\Delta}{2m} - u_k^{2k}, \quad l_+ = \frac{1}{2m} - \frac{\Delta}{2m} - (1 + 2k)u_k^{2k}.
\]

**Lemma B.3.** The dimension of the null space and the generalized null space of \( j1 = \begin{bmatrix} 0 & 1_- \\ -l_+ & 0 \end{bmatrix} \) is given by
\[
\mathcal{N}(j1) = n + 1, \quad \mathcal{N}_g(j1) = \begin{cases} 2n + 2, & k \neq 2/n; \\ 2n + 4, & k = 2/n. \end{cases}
\]

**Proof.** Such computations have appeared in many articles. The relation (2.2) shows that \( l_-u_k = 0 \). Taking the derivatives of this relation with respect to \( x^j \), one also gets \( l_+ \partial_j u_k = 0, 1 \leq j \leq n \). From [Kwo89] or [CGNTO8, Lemma 2.1] we have that \( \dim \ker l_+ = n + 1 \), hence there are no other vectors in the kernel of \( l \).

Now let us study the generalized eigenvectors. The relation \( l_-u_k = 0 \) leads to
\[
l_-(x^j u_k) = -\frac{1}{m} \partial_j u_k, \quad 1 \leq j \leq n.
\]
This shows that
\[
\begin{bmatrix} 0 & 1_- \\ -l_+ & 0 \end{bmatrix} \begin{bmatrix} \partial_j u_k \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} 0 & l_- \\ -l_+ & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x^j u_k \end{bmatrix} = -\frac{1}{m} \begin{bmatrix} \partial_j u_k \\ 0 \end{bmatrix}, \quad 1 \leq j \leq n. \tag{B.3}
\]

We cannot extend this sequence: there is no \( v \) such that
\[
\begin{bmatrix} 0 & 1_- \\ -l_+ & 0 \end{bmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ x^j u_k \end{bmatrix},
\]
since \( x^j u_k \) is not orthogonal to the kernel of \( l_+ \). Indeed, as follows from the identity
\[
\langle x^j u_k, \partial_j u_k \rangle = \langle (-u_k - x^3 \partial_j u_k), u_k \rangle, \tag{B.4}
\]
one has \( \langle x^j u_k, \partial_j u_k \rangle = -\frac{1}{2} \langle u_k, u_k \rangle < 0 \).
By (2.2), the function \( u_{k,\lambda}(x) = \lambda^{1/k} u_k(\lambda x) \) satisfies the identity

\[
-\frac{\lambda^2}{2m} u_{k,\lambda} = -\frac{1}{2m} \Delta u_{k,\lambda} - u_{k,\lambda}^{1+2k}.
\]

Differentiating this identity with respect to \( \lambda \) at \( \lambda = 1 \) yields

\[
0 = l_+ (\partial_\lambda |_{\lambda=1} u_{k,\lambda}) + \frac{1}{m} u_k = l_+ \left( \frac{1}{k} u_k + x \cdot \nabla u_k \right) + \frac{1}{m} u_k. \tag{B.5}
\]

This shows that

\[
\begin{bmatrix}
0 & l_- \\
-l_+ & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
u_k
\end{bmatrix} = 0,
\begin{bmatrix}
0 & l_- \\
-l_+ & 0
\end{bmatrix}
\begin{bmatrix}
-\theta \\
u_k
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

with

\[
\theta = -\frac{m}{k} u_k - m x \cdot \nabla u_k, \quad l_+ \theta = u_k. \tag{B.6}
\]

The relations (B.3) and (B.6) show that \( \dim \mathcal{M}_q(\Omega) \geq 2n + 2 \). The dimension jumps above \( 2n + 2 \) in the case when one can find \( \alpha \) such that

\[
l_- \alpha = \theta, \quad \text{hence} \quad \begin{bmatrix}
0 & l_- \\
-l_+ & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
\alpha
\end{bmatrix} = \begin{bmatrix}
\theta \\
0
\end{bmatrix}. \tag{B.8}
\]

This happens when \( \theta \) in (B.7) is orthogonal to \( \ker l_- = \text{Span}(u_k) \). Using the identity (B.4), we see that

\[
\frac{1}{m} \langle \theta, u_k \rangle = \left\langle -\frac{1}{k} u_k - x \cdot \nabla u_k, u_k \right\rangle = -\frac{1}{k} \langle u_k, u_k \rangle + \frac{n}{2} \langle u_k, u_k \rangle. \tag{B.9}
\]

The right-hand side of (B.9) vanishes when \( k = 2/n \) (that is, when the nonlinear Schrödinger equation is charge-critical). We may choose \( \alpha \) to be spherically symmetric so that it is orthogonal to \( \ker l_+ = \text{Span}(\partial_j u_k \; : 1 \leq j \leq n) \); then, by the Fredholm alternative, there is \( \beta \in L^2(\mathbb{R}^n) \) such that

\[
l_+ \beta = \alpha, \quad \text{hence} \quad \begin{bmatrix}
0 & l_- \\
-l_+ & 0
\end{bmatrix}
\begin{bmatrix}
-\beta \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
\alpha
\end{bmatrix}. \tag{B.10}
\]

(Let us also point out that \( \alpha \) and \( \beta \) can be chosen real-valued.) This process can not be continued: there is no \( w \in L^2(\mathbb{R}^n) \) such that \( l_- w = \beta \) since \( \beta \) is never orthogonal to \( \ker l_- \); indeed, due to semi-positivity of \( l_- \), one has

\[
\langle \beta, u_k \rangle = \langle \beta, l_+ \theta \rangle = \langle \beta, l_+ l_- \alpha \rangle = \langle \alpha, l_- \alpha \rangle > 0.
\]

We also need the following technical result.

**Lemma B.4.** For \( z \in \rho(l_-) \), the operator \( (l_- - z)^{-1} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n) \) extends to a continuous mapping

\[
(l_- - z)^{-1} : H^{-1}(\mathbb{R}^n) \to H^1(\mathbb{R}^n).
\]

**Proof.** Set \( a = \sup_{x \in \mathbb{R}^n} u_k(x)^{2k} \). Then there is \( C < \infty \) such that for any \( \varphi \in C^\infty_{\text{comp}}(\mathbb{R}^n) \)

\[
C \| \varphi \|^2_{H^1} \geq |\langle \varphi, (l_- + a) \varphi \rangle| \geq \langle \varphi, \left( -\frac{1}{2m} \Delta + \frac{1}{2m} \right) \varphi \rangle = \frac{1}{2m} \| \varphi \|^2_{H^1}, \quad \forall \varphi \in C^\infty_{\text{comp}}(\mathbb{R}^n),
\]

hence the self-adjoint operator

\[
l_- + a : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \tag{B.11}
\]

is positive-definite and invertible. We can extract its square root, which is a positive-definite bounded invertible operator

\[
(l_- + a)^{1/2} : H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n);
\]

\[
41
\]
then \((B.11)\) also defines a bounded invertible operator \((l_+ + a)^{1/2} : H^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n)\), and by duality there is also a bounded invertible mapping \((l_+ + a)^{1/2} : L^2(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)\). We fix \(z \in \rho(l_-)\); then the mapping 

\[
(l_- + a)^{1/2}(l_- - z)(l_- + a)^{-1/2} : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)
\]

is bounded and invertible. Since \(l_- + a\) and its square root commute with \(l_- - z\) (when restricted e.g. to the space of Schwartz functions), we apply the density argument to conclude that \(l_- - z\) extends to a bounded invertible mapping \(H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)\).

\[\square\]

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