The asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant

Oliver Coussaert\(^{(a)}\), Marc Henneaux\(^{(a,b)}\) and Peter van Driel\(^{(a)}\)

\(^{(a)}\) Université Libre de Bruxelles
CP 231, Bvd du Triomphe, B-1050 Brussels, Belgium

\(^{(b)}\) Centro de Estudios Científicos de Santiago,
Casilla 16443, Santiago 9 Chile

ULB-TH/95/08

Abstract

Liouville theory is shown to describe the asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant. This is because (i) Chern-Simons theory with a gauge group $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ on a space-time with a cylindrical boundary is equivalent to the non-chiral $SL(2, \mathbb{R})$ WZW model; and (ii) the anti-de Sitter boundary conditions implement the constraints that reduce the WZW model to the Liouville theory.

I. INTRODUCTION

Three-dimensional gravity theories have attracted considerable attention in the past fifteen years in the hope of getting a better understanding of the intricacies of their four-dimensional parents (see [1,2] and [3] for a recent review with an extensive list of references). In particular, the asymptotic structure of 3d gravity and 3d supergravity has been investigated in [4–6], with the following conclusions. In the case of a vanishing cosmological constant, the asymptotic behaviour of the gravitational field is quite constrained and does not allow one to define naturally translations and supersymmetries at spatial infinity.
property has been argued recently to explain how one could break supersymmetry without generating a cosmological constant [7]. By contrast, the asymptotic structure of 3d gravity with a negative cosmological constant \( \Lambda < 0 \), is much richer [6].

Indeed, in that case the asymptotic symmetry group turns out to be the group of conformal transformations in two dimensions, generated by the infinite dimensional Virasoro algebra. The appearance of this conformal symmetry can be understood either in terms of the Penrose description of infinity by means of a conformal compactification, where infinity appears as a timelike cylinder (\( \Lambda < 0 \)) and the asymptotic symmetry group is the group of its conformal symmetries [8]; or in terms of the Hamiltonian formulation where the canonical generators of the transformations preserving the boundary conditions are shown to close according to the conformal algebra [6].

The presence of the infinite-dimensional conformal group as asymptotic symmetry group strongly suggests that the asymptotic dynamics of the gravitational field in 3 dimensions, with a negative cosmological constant, is described by a two-dimensional conformal field theory. The purpose of this letter is to show that the conformal field theory in question is Liouville theory [9].

Our starting point is the crucial observation made in [11, 12] that three-dimensional Einstein gravity with \( \Lambda < 0 \) can be reformulated as a Chern-Simons gauge theory with gauge group \( SL(2, R) \times SL(2, R) \) and action

\[
S_E[A, \tilde{A}] = S_{CS}[A] - S_{CS}[\tilde{A}].
\]

Here \( A \) (respectively \( \tilde{A} \)) is the gauge field associated with the first (respectively the second) \( SL(2, R) \) factor and \( S_{CS} \) is the Chern-Simons action which in polar coordinates \( t, r, \varphi \) takes

\[1\] The assumption made in the footnote 11 of [4], that the contraction from \( \Lambda < 0 \) to \( \Lambda = 0 \) can be done smoothly, is thus incorrect.

\[2\] That there is a connection between three-dimensional gravity and Liouville theory is of course not new and has been discussed from a different perspective in [9] and [10].
the form,

\[ S_{CS} = \int dt dr d\varphi \; tr(\dot{A}_r A_\varphi - \dot{A}_\varphi A_r - A_0 F_{r\varphi}) \]  

(2)

The connections \( A \) and \( \tilde{A} \) are related to the triad \( e \) and spin connection \( \omega \) through \( A = e + \omega \), \( \tilde{A} = e - \omega \).

The surface at spatial infinity \((r = \infty)\) of anti-de-Sitter space is a timelike cylinder with coordinates \( t, \varphi \). We denote it by \( \Sigma_2 \). The boundary conditions on the metric given in [3] read, when translated in terms of the connection \( A \) and \( \tilde{A} \),

\[
A \sim \begin{bmatrix} \frac{dr}{2r} & O(1/r) \\ rdx^+ & -\frac{dr}{2r} \end{bmatrix}, \quad \tilde{A} \sim \begin{bmatrix} -\frac{dr}{2r} & rdx^- \\ O(1/r) & \frac{dr}{2r} \end{bmatrix}
\]

(3)

(to leading order). Here, \( x^\pm = t \pm \varphi \). The boundary conditions express that the metric approaches asymptotically the anti-de Sitter space and are in particular such that the triad \( e \) is non degenerate.

Two things should be emphasised about (3). (i) First, the lightlike components \( A_- \) of \( A \) and \( \tilde{A}_+ \) of \( \tilde{A} \) are set equal to zero asymptotically. (ii) Second, \( A_+^{(3)} \) and \( \tilde{A}_-^{(3)} \) are not functions of the variables \( t \) and \( \varphi \) to leading order in \( r \). At the same time, \( A_+^{(3)} \) and \( \tilde{A}_-^{(3)} \) are set to vanish. Here, the indices in parentheses are Lie algebra indices. We shall examine in turn the respective implications of (i) and (ii). We start by showing that (i) reduces the Chern-Simons theory to the \( SL(2, R) \) non-chiral Wess-Zumino-Witten model. To that end, we closely follow the work of [13], adapted to the boundary conditions at hand. In section III we show that the implication of (ii) is to reduce this WZW model to the Liouville model.

II. FROM THE EINSTEIN ACTION TO THE NON-CHIRAL \( SL(2, R) \) THE CASE WESS-ZUMINO-WITTEN MODEL

The action (1) is not an extremum on-shell when \( A_- \) and \( \tilde{A}_+ \) are required to vanish on the boundary. Rather, \( \delta S \) is then equal to the surface term \( \delta[\int_{\Sigma_2} dt d\varphi \; tr(A_+^{2} + \tilde{A}_-^{2})] \)
on the surface $\Sigma_2$ at infinity. [ We shall examine the terms that arise at $t_1$ and $t_2$ when discussing (ii)]. In order to have $\delta S = 0$, one must therefore add to the action the surface term $-\int_{\Sigma_2} dt d\varphi \; tr(A_\varphi^2 + \tilde{A}_\varphi^2)$, leading to the improved action

$$S[A, \tilde{A}] = S_{CS}[A] - \int_{\Sigma_2} dt d\varphi tr(A_\varphi^2) - S_{CS}[\tilde{A}] - \int_{\Sigma_2} dt d\varphi tr(\tilde{A}_\varphi^2)$$

(4)

The temporal component $A_0$ and $\tilde{A}_0$ of the vector potential appears as Lagrange multiplier implementing the constraints $F_{r\varphi} = \tilde{F}_{r\varphi} = 0$. One can solve these constraints as

$$A_i = G_1^{-1} \partial_i G_1, \quad \tilde{A}_i = G_2^{-1} \partial_i G_2$$

(5)

where $G_1$ and $G_2$ are asymptotically given by

$$G_1 \sim g_1(t, \varphi) \begin{bmatrix} \frac{1}{\sqrt{r}} & 0 \\ 0 & \frac{1}{\sqrt{r}} \end{bmatrix}, \quad G_2 \sim g_2(t, \varphi) \begin{bmatrix} \frac{1}{\sqrt{r}} & 0 \\ 0 & \sqrt{r} \end{bmatrix}$$

(6)

and where $g_1(t, \varphi)$ and $g_2(t, \varphi)$ are arbitrary elements of $SL(2, R)$. With (6), the asymptotic behaviour of the radial components of $A$ and $\tilde{A}$ coincide with the one of (3), while the tangential components behave as

$$A_\alpha \sim \begin{bmatrix} a_\alpha^{(3)} & \frac{1}{r} a_\alpha^{(+)} \\ a_\alpha^{(-)} & -a_\alpha^{(3)} \end{bmatrix}, \quad \tilde{A}_\alpha \sim \begin{bmatrix} \tilde{a}_\alpha^{(3)} & \tilde{a}_\alpha^{(+)} r \\ \tilde{a}_\alpha^{(-)} r^{-1} & -\tilde{a}_\alpha^{(3)} \end{bmatrix}$$

(7)

where $a_\alpha = g_1^{-1} \partial_\alpha g_1$ and $\tilde{a}_\alpha = g_2^{-1} \partial_\alpha g_2$. The group elements $g_1(t, \varphi)$ and $g_2(t, \varphi)$ will be restricted below so that $A_\alpha$ and $\tilde{A}_\alpha$ fulfill the remaining boundary conditions (ii).

Strictly speaking (5) is valid only if the spatial sections have no hole. In general, one should allow for holonomies, which appear as additional ”zero mode terms” in (5). Such additional terms are necessary to describe black holes in 3 dimensions which can be obtained from anti-de Sitter space by making appropriate identifications [14,15]. Furthermore, there are then also additional inner boundaries with their own surface dynamics. The dynamics on a black hole horizon, has been treated in [16]. Since we are interested here only in the asymptotic dynamics of the gravitational field, we shall however drop the holonomies and ignore the inner surfaces. A full treatment will be given in [17].
Now, if one inserts (6) in the action (5), one gets

\[ S[A, \tilde{A}] = S^R_{ZW}[g_1] - S^L_{ZW}[g_2] \]  

(8)

where \( S^R_{ZW}[g_1] \) and \( S^L_{ZW}[g_2] \) are the two-dimensional chiral Wess-Zumino-Witten (WZW) actions \([13,18,19,21,20]\). These first order action generalise the abelian actions of \([22]\) and respectively describe a right-moving group element \( g_1(x^+) \) and a left moving group element \( g_2(x^-) \),

\[ S^R_{ZW}[g_1] = \int_{\Sigma_2} dtd\varphi \, tr(\dot{g}_1 g'_1 - (g'_1)^2) + \Gamma[g_1] \]  

(9)

\[ S^L_{ZW}[g_2] = \int_{\Sigma_2} dtd\varphi \, tr(\dot{g}_2 g'_2 + (g'_2)^2) + \Gamma[g_2] \]  

(10)

where \( \dot{g} = g^{-1} \frac{\partial}{\partial t} g \), \( g' = g^{-1} \frac{\partial}{\partial \varphi} g \) and \( \Gamma[g] \) is the usual three-dimensional part of the WZW-action. As shown in \([18–20]\), the actions (9) and (10) each lead to a single chiral Kac-Moody symmetry (of opposite chirality). One expects the sum (8) of the left and right chiral actions (9,10) to be equivalent to the standard, non chiral, WZW action \([23]\) with dynamical variable \( g = g_1^{-1} g_2 \) since in that model the right moving and left moving sectors are indeed decoupled \([23,24]\). This expectation turns out to be true.

One to establish the equivalence is to rewrite the standard WZW action in Hamiltonian form, since (9) and (10) are linear and of first order in the time derivatives. We denote by \( \Pi_g \) the momentum conjugate to \( g \) and by \( u \) the function of \( g \) and \( \Pi_g \) which is equal to \( \dot{g} \) when the equations of motion hold. One may take \( g \) and \( u \) as independent variables. The change of variables

\[ g = g_1^{-1} g_2, \quad u \equiv \dot{g}|_{onshell} = -g_2^{-1} \frac{\partial}{\partial \varphi} g_1 g_1^{-1} g_2 - g_2^{-1} \frac{\partial}{\partial \varphi} g_2 \]  

(11)

brings (8) to the standard WZW action in first order form or, after elimination of the auxiliary field \( u \), to the standard, non chiral \( SL(2, R) \) WZW action in second order form,

\[ S[A, \tilde{A}] = S^{ZW}[g], \quad S^{ZW}[g] = \int_{\Sigma_2} dtd\varphi (tr(g_+ g_-) - \Gamma[g]) \]  

(12)

where \( g_{\pm} \equiv g^{-1} \frac{\partial}{\partial x^{\pm}} g \). We omit the details here, leaving them for the complete treatment \([17]\) where, in particular the zero modes and holonomies will be included also. We simply
note that the transformation (11) is the direct generalisation of the transformation analysed in [18], [25] in the $U(1)$ case, which establishes the equivalence of the sum of left moving chiral boson and a right moving chiral boson to a massless Klein-Gordon field.

We have thus shown so far that the asymptotic dynamics of the gravitational field in three dimensions with $\Lambda < 0$ is described by the (non-chiral) $SL(2, R)$ WZW action. We have not yet incorporated, however, all the boundary conditions on the connection. This missing step is taken now.

### III. FROM THE WZW MODEL TO LIOUVILLE THEORY

The conditions that we have not taken into account at this stage are the conditions (ii) which read, in terms of the group element $g$

$$J_{-}^{(+)} \equiv (g^{-1}\partial_- g)^{(+)}) = 1, \quad \tilde{J}_{-}^{(+) \equiv (\partial_+ gg^{-1})^{(-)}} = 1$$

and $J_{-}^{(3)} = 0, \quad J_{+}^{(3)} = 0$. Since the $J$’s are just the Kac-Moody currents of the WZW model, we recognise (13) as the conditions implementing the familiar Hamiltonian reduction of the WZW model to the Liouville theory. The conditions $J_{+}^{(3)} = J_{-}^{(3)} = 0$ appear as “gauge condition”. This reduction has been discussed at length in the literature so that we do not need to recall the details here.

Let us simply point out the perhaps less familiar fact that the reduction can be carried out directly at the level of the action. As shown in [26], the WZW action reads, if one parametrises $g$ according to the Gauss decomposition

$$g = \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \exp(\frac{1}{2}\phi) & 0 \\ 0 & \exp(-\frac{1}{2}\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix},$$

$$S_{WZW}[g] = \int dt d\phi [\frac{1}{2}\partial_+ \phi \partial_- \phi + 2(\partial_- X)(\partial_+ Y) \exp(-\phi)]$$

Now the action (12) is defined on the cylinder $[t_1, t_2] \times S_1$ of finite height $t_2 - t_1$ and is stationnary on the classical history provided one fixes $\phi, X$ and $Y$ at the time boundary
$t_1$ and $t_2$. However, since we want to implement the constraints (13), it is not $\phi, X$ and $Y$ that we want to fix at the boundaries, but rather, $\phi$ and the momentum $\partial_- Y$ and $\partial_+ X$ conjugate to $X$ and $Y$, since $J_+^{(+)} = \partial_+ X \exp(-\phi)$ and $\tilde{J}_-^{(-)} = \partial_- Y \exp(-\phi)$. Hence the constraints (13) cannot be simply plugged in inside (13). The action appropriate to the new set of boundary conditions differs from (13) by a boundary term at $t_1$ and $t_2$,

$$S_{\text{impr}}^{ZW}[g] = S^{ZW}[g] - 2 \oint d\phi (X \partial_+ Y + Y \partial_- X) \exp(-\phi)|_{t_1}^{t_2}$$  \hspace{1cm} (16)

With the “improved” term, it is legitimate to insert the constraints (13) in the action (16). If one does so, one ends up with the Liouville action for $\phi$

$$S[A, \tilde{A}] = S_{\text{Liouville}}[\phi] = \int dtd\phi \left( \frac{1}{2} \partial_+ \phi \partial_- \phi + 2 \exp(\phi) \right)$$  \hspace{1cm} (17)

Hence, we have established that the asymptotic dynamics of the gravitational field in three dimensions, with $\Lambda < 0$, is indeed described by the Liouville theory. As it is well known, this theory is conformally invariant and possesses two sets of Virasoro generators $L_n$ and $\tilde{L}_n$. These can be viewed as generating the residual Kac-Moody symmetries preserving the constraints and are the asymptotic generators found by a totally different approach in [6]. [See also [27] and [28] for a related analysis of the surface terms in the Chern-Simons theories.]

Remark: one can actually substitute the constraints $J_+^{(+)} = \mu$ and $\tilde{J}_-^{(-)} = \nu$ in the action (16), provided one observes that the constants $\mu$ and $\nu$ are functionals of the fields and varies them accordingly in the action principle. A very similar situation occurs when one treats the cosmological constant as a dynamical variable. The subtleties of the variational principle are explained in that case in [29].

IV. CONCLUSIONS

In this letter, we have completed the analysis of the asymptotic dynamics of 3d Einstein gravity with a negative cosmological constant. We have shown that the Virasoro symmetry
generators found in \[6\] arise because the asymptotic dynamics is described by a conformally invariant theory, namely the Liouville theory.

The asymptotic reduction of the Einstein action -equivalent to $SL(2, R) \times SL(2, R)$ Chern-Simons action- to the Liouville action follows a two-step procedure. First, one imposes conditions of opposite chiralities on each $SL(2, R)$ factor, namely $A_- = 0$ and $\tilde{A}^+ = 0$. This leads to the sum of two chiral $SL(2, R)$ WZW actions of opposite chiralities or, what is the same, to the non chiral $SL(2, R)$ WZW action. Next one imposes the constraints on the Kac-Moody currents that reduce the $SL(2, R)$ WZW action to the Liouville theory. The two steps are precisely incorporated in the boundary conditions on the triads and connection expressing asymptotic approach to the anti-de Sitter space, and thus, they have a direct geometrical interpretation. Our analysis also exemplifies very clearly how the asymptotic dynamics is sensitive to the boundary conditions.

A further account of this work, with extension to supersymmetry (important for proving positivity of the energy theorems) will be reported elsewhere.

Acknowledgements.

One of us (M.H.) is grateful to Lars Brink and Steve Carlip for discussions. This work has been partly supported by research funds from the FNRS and the Commission of the European Communities. O.C. is Aspirant FNRS.
REFERENCES

[1] S. Deser, R. Jackiw and G. ’t Hooft, Ann. Phys. (NY) 153 (1984) 220; S. Deser and R. Jackiw, Ann. Phys (NY) 153 (1984) 405; R. Jackiw, Nucl. Phys. B252 (1985) 343.

[2] J.D. Brown, ”Lower dimensional gravity”, World Scientific (Singapore 1988).

[3] S. Carlip, ”Lectures on (2+1)-Dimensional Gravity”, UC Davies preprint UCD-95-6, grqc/9503024.

[4] M.Henneaux, Phys. Rev. D29 (1984) 2766.

[5] S. Deser, Class. Quant. Grav. 2 (1985) 489.

[6] J.D. Brown and M. Henneaux, Comm. Math. Phys. 104, 207 (1986).

[7] E. Witten, ”Is supersymmetry really broken?”, IAS preprint IASSNS-HEP-94-72 (1994).

[8] R. Penrose, in: Relativity, groups and tology. De Witt C., De Witt B. (eds) New York: Gordon and Breach 1964.

[9] S. Carlip, Nucl. Phys. B362 (1991) 111.

[10] H. Verlinde, Nucl. Phys. B337 (1990) 652.

[11] A. Achuracco and P.K. Towsend, Phys. Lett. B180 (1986).

[12] E. Witten, Nucl. Phys. B311 (1988/89) 46.

[13] G. Moore and N. Seiberg, Phys. Lett. B220 (1989) 422; S. Elitzur, G. Moore, A. Schwinisner and N. Seiberg, Nucl. Phys. B326 (1989) 108.

[14] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849.

[15] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D48 (1993) 1506.

[16] S. Carlip, ”The statistical mechanics of the (2+1)-dimensional black hole”, preprint UCD-94-32, grqc/9409052.
[17] O. Coussaert, M. Henneaux and P. van Driel, in preparation.

[18] J. Sonneschein, Nucl. Phys. B309 (1988) 752.

[19] P. Salomonson, B.S. Skazerstein and A. Stern, Phys. Rev. Lett. 62 (1989) 1817.

[20] M. Stone, Phys. Rev. Lett. 63 (1989) 731.

[21] M.F. Chu, P. Goddard, I. Halliday, D. Olive and A. Schwimmer, Phys. Lett. B266 (1991) 71.

[22] R. Floreanini, R. Jackiw, Phys. Rev. Lett. 59 (1987) 1873.

[23] E. Witten, Comm. Math. Phys. 92 (1984) 455.

[24] A.M Polyakov and P.B. Wiegmann, Phys. Lett. B131 (1983) 121.

[25] M. Henneaux and C. Teitelboim, in "Quantum mechanics of Fundamental Systems 2", Proceedings of the meeting held in Santiago in December 1987, pp 79-112, C. Teitelboim and J. Zanelli, Plenum Press (New York 1989).

[26] P. Forgacs, A. Wipf, J. Balog, L. Feher and L. O’Raifeartaigh, Phys. Lett. B227 (1989) 213.

[27] M. Bañados, preprint hepth 9405171.

[28] A. P. Balachandran, G. Bimonti, K.S. Gupta, A. Stern, Int. J. Mod. Phys A7 (1992) 4655.

[29] M. Henneaux and C. Teitelboim, "Quantisation of Gauge Systems", Princeton University Press (Princeton 1992), Chapter 1, exercices 1.4 and 1.5.