Continuous Strategy Replicator Dynamics for Multi–Agent Learning

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The problem of multi–agent learning and adaptation has attracted a great deal of attention in recent years. It has been suggested that the dynamics of multi agent learning can be studied using replicator equations from population biology. Most existing studies so far have been limited to discrete strategy spaces with a small number of available actions. In many cases, however, the choices available to agents are better characterized by continuous spectra. This paper suggests a generalization of the replicator framework that allows to study the adaptive dynamics of Q–learning agents with continuous strategy spaces. Instead of probability vectors, agents strategies are now characterized by probability measures over continuous variables. As a result, the ordinary differential equations for the discrete case are replaced by a system of coupled integral–differential replicator–equations that describe the mutual evolution of individual agent strategies. We derive a set of functional equations describing the steady state of the replicator dynamics, examine their solutions for several two–player games, and confirm our analytical results using simulations.

I. INTRODUCTION

The notion of autonomous agents that learn by interacting with the environment, and possibly with other agents, is a central concept in modern distributed AI [18]. Of particular interests are systems where multiple agents learn concurrently and independently by interacting with each other. This multi–agent learning problem has attracted a great deal of attention due to number of important applications. Among existing approaches, multi–agent reinforcement learning (MARL) algorithms have become increasingly popular due to their generality [2, 3, 6, 13]. Although MARL does not hold the same convergence guarantees as in single–agent case, it has been shown to work well in practice.

From the analysis standpoint, MARL represents a complex dynamical system, where the learning trajectories of individual agents are coupled with each other via a collective reward mechanism. Thus, it is desirable to know what are the possible long–term behaviors of those trajectories. Specifically, one is usually interested whether, for a particular game structure, those trajectories converge to a desirable steady state (called fixed points), or oscillate indefinitely between many (possibly infinite) meta–stable states. While answering this question has proven to be very difficult in the most general settings, there has been some limited progress for specific scenarios. In particular, it has been established that for simple, stateless Q–learning with finite number of actions, the learning dynamics can be examined using the so called replicator equations from population biology [5]. Namely, if one associates a particular biological trait with each pure strategy, then the adaptive learning of (possibly mixed) strategies in multi–agent settings is analogous to competitive dynamics of mixed population, where the species evolve according to their relative fitness in the population. This framework has been used successfully to study various interesting features of adaptive dynamics of learning agents [12, 13, 16, 19].

Most existing studies so far have focused on discrete action spaces, which has limited the full analysis of the learning dynamics to games with very few actions. On the other hand, in many practical scenarios, strategic interactions between agents are better characterized by continuous spectra of possible choices. For instance, modeling an agent’s bid in an auction with a continuous rather than discrete variable is more natural. In such situations, agents strategies are represented as a probability density functions defined over a continuous set of strategies. Of course, in reality all the decisions are made over a discretized subset. However, the rationale for using the continuous approximation is that it makes the dynamics more amenable to mathematical analysis.

In this paper we consider simple Q–learning agents that play repeated continuous–strategy games. The agents use Boltzmann action–selection mechanism that controls the exploration/exploitation tradeoff by a single, temperature–like parameter. The reward functions for the agents are assumed to be functions of continuous variables instead of tensors, and the agent strategies are represented as probability distribution over those variable. In contrast to the finite strategy spaces where the learning dynamics is captured by a set of coupled ordinary differential equations, the replicator dynamics for the continuous–strategy games are described by functional–differential equations for each agent, with coupling across different agents/equations.

The long–term behavior of those equations define the steady–state, or equilibrium, profiles of the agent strategies. It is shown that, in general, the steady state strategy profiles of the replicator dynamics do not correspond to the Nash equilibria of the game. This discrepancy can be attributed to the limited–rationality of the agents due to the non–zero temperature (i.e., exploration). Furthermore, it is shown that the correspondence with the Nash equilibria is often recovered if one gradually decreases the temperature. However, for certain games, the replicator dynamics might converge to strategy profiles that have non–zero entropy.
even if the limit of perfectly rational agents, and for which there is no corresponding Nash equilibria in the zero–temperature limit.

The rest of this paper is organized as follows: In the next section we provide a brief overview of relevant literature. In Section III we introduce our model, derive the replicator equations for the continuos strategy spaces, and a set of coupled non–linear functional equations that describe the steady state strategy profile. In Section IV we illustrate the framework on several examples of two–agent games, and provide some detailed results for general bi–linear and quadratic payoffs. Finally, we conclude the paper with a discussion of our results and possible future directions in Section V.

II. BACKGROUND AND RELATED WORK

Reinforcement learning (RL) \[8, 18\] is a powerful framework in which an agent learns to behave optimally through a trial and error exploration of the environment. At each step of interaction with the environment, the agent chooses an action based on the current state of the environment, and receives a scalar reinforcement signal, or a reward, for that action. The agent’s overall goal is to learn to act in a way that will increase the long–term cumulative reward. Although RL was originally developed for single–agent learning in stationary environment, it has been also generalized for multi–agent scenario. In the multi-agent setting, the environment is highly dynamic because of the presence of other learning agents, and the usual conditions for convergence to an optimal policy do not necessarily hold \[3, 7, 17\]. Nevertheless, various generalizations of single agent learning algorithms have been successfully applied to multi–agent settings.

Despite some empirical success, theoretical advances in multi–agent reinforcement learning have been rather scarce. Recent work has suggested to utilize the link between MARL and replicator dynamics from the population biology. Those equations has demonstrated very rich and complex behavior, such as sensitivity to initial conditions, and even Hamiltonian chaos \[15, 16\]. A similar approach was used in \[19\], where the Cross Learning model of Ref. \[1\] was extended to \(Q\)–learning, and where the link between multi–agent replicator dynamics and Evolutionary Game Theory was reiterated. The replicator dynamics framework was also used in \[12\] to study advantages of lenient learners in repeated coordination games, where some convergence guarantees on certain games were obtained.

In addition to discrete strategy spaces, recent work has addressed games with continuous strategy spaces. For instance, continuous strategy version of the prisoner dilemma has been considered in \[9, 10, 20, 21\]. Replicator equations in continuous strategy spaces have been studied in the context of evolutionary and dynamical stability in \[1, 4, 11\]. The corresponding replicator equation is similar to the one studied in the present paper, except there is no entropy (mutation) term. This is an important distinction since without the entropy term, the allowed steady state solution are confined to a set of Dirac’s \(\delta\)–measures at distinct points. In other words, even if one starts with a continuous population, the replicator dynamics will converge to monomorphic fixed points.

The work that is closest to one presented here is Ref \[14\], which studied continuous strategy replicator system with and without mutation. They established that for specific games, mutations resulted in non–trivial modification to the equilibrium structure. The difference between their work and the model presented here is the origin of the mutation term – while in Ref \[14\] the mutation was added as a generic diffusive term, in our model it has a very intuitive entropic meaning that results from the Boltzmann action selection mechanism. Specifically, the replicator equation in our case results from minimizing a certain functional that is reminiscent of the so called free energy from statistical physics. This fact provides a very intuitive picture of the steady state structure.

III. MODEL

We consider a system of \(N\) agents that are playing repetitive games with each other. In the present paper, we assume a stateless model, so that the reward for each agent depends only on the collective action of the agents. Let \(x_i\) denote the action taken by the \(i\)-th agent. Also, let \(x_i\) denote the collective action profile of all the agents except \(i\). Without a loss of generality, we assume that the actions are restricted to the unit interval, \(x_i \in [0, 1], 1 = 1, 2, ..N\).

The game proceeds as follows: At each time step, each agent chooses an action, receives a reward that depends on the collective action of all the agents, and update his strategies accordingly. Each agent has a \(Q\)–function that encodes the relative utility of actions. Those \(Q\)–functions are updated after each time an agent selects an action, according to the following reinforcement rule \[22\]:

\[
Q_i(x_i, t + \delta t) = Q_i(x_i, t) + \alpha[f_i(x_i; x_{-i}) - Q_i(x_i, t)]
\]

(1)

where \(f_i(x_i; x_{-i})\) is the reward of the agent \(i\) when he takes the action \(x_i\) and the rest of the agents take the collective action \(x_{-i}\).
Next, we have to specify how agents choose actions based on $Q$–functions. There are several action–selection mechanisms. Here we focus on the so called Boltzmann exploration, where the probability of selecting a particular action $x_i$ is given as
\[ p_i(x, t) = C(t)e^{\beta Q_i(x, t)} \]  
(2)
where $C(t)$ is simply a (time–dependent) normalization constant:
\[ C(t) = \left[ \int dx e^{\beta Q_i(x, t)} \right]^{-1}, \]  
(3)
and where $\beta \equiv 1/T$ is a parameter that controls the exploration/exploitation tradeoff, and has a meaning of inverse temperature owing to the analogy with statistical–mechanical systems.

We now assume that the agents interact many times between two consecutive updates of their strategies. In this case, the reward of the $i$–th agent in Equation 1 should be understood in terms of the average reward, where the averaging is done over the strategies of other agents in the system. Specifically, taking the limit $\delta t \to 0$, we can rewrite Equation 1 as follows:
\[ \frac{\partial Q_i}{\partial t} = \alpha \left[ r_i(x_i) - Q_i(x_i, t) \right] \]  
(4)
where $r_i(x_i)$ is the average reward “felt” by the $i$–th agent:
\[ r_i(x) = \int \ldots \int \prod_{j \neq i} dx_j p_j(x_j) f_i(x_i, x - i) \]  
(5)

We want to eliminate $Q$ from the dynamics, so that the learning trajectories are expressed solely through the agent strategies. To achieve this, let us take the derivative of Equation 2 in respect to time:
\[ \frac{\partial p_i}{\partial t} = \frac{\partial C}{\partial t} e^{\beta Q} + C \beta e^{\beta Q} \frac{\partial Q_i}{\partial t} \]  
(6)

Note that
\[ \frac{dC}{dt} = -\frac{\int dx \beta e^{\beta Q} \frac{\partial Q}{\partial t}}{\left[ \int dx e^{\beta Q} \right]^2} = -\beta C(t) \frac{\int dx p_i(x, t) \frac{\partial Q}{\partial t}}{\int dx e^{\beta Q}} \]  
(7)

Combining Equations 2, 4, and 7 we arrive at the following:
\[ \frac{1}{p_i(x, t)} \frac{\partial p_i(x, t)}{\partial (\alpha t)} = \left[ r_i(x) - \int d\tilde{x} r_i(\tilde{x}) p_i(\tilde{x}, t) \right] - T \left[ \ln p_i(x, t) - \int d\tilde{x} p_i(\tilde{x}, t) \ln(p_i(\tilde{x}, t)) \right] \]  
(8)

Equations 8 have a very simple interpretation. Indeed, the first term suggests that a probability of a playing a particular pure strategy increases with a rate proportional to the overall efficiency of that strategy. This is reminiscent of of fitness-based selection mechanism in population biology. The second term, on the other hand, does not have a direct analogue in population biology, and describes the agents’ tendency to randomize over the strategies.

### A. Steady State Solution

We are interested in the asymptotic behavior of the Equations 8 after sufficiently long time, $P_i(x) = p_i(x, t \to \infty)$. The steady–state equation is obtained by setting the time–derivative to zero, which yields a set of equations (for each agent)
\[ P_i(x) \left[ R_i(x, t) - T \ln P_i(x) + \text{const} \right] = 0, \]  
(9)
where $R_i(x)$ depend on the collective strategy profile of all the agents as described by Equation 5 with $p_i(x)$ replaced by $P_i(x)$. Let us focus on solutions $P_i(x)$ that have continuous support on the interval $[0, 1]$, i.e., $P_i(x)$ is strictly positive for all $x \in [0, 1]$, except maybe a finite number of poinst. Since Equation 9 should be satisfied for all the values $x_i \in [0, 1]$, then the expression in the parenthesis should nullify for all $x$, which yields for $i = 1, 2, \ldots N$ :
\[ P_i(x) = A_i e^{\beta R_i(x)} \]  
(10)
Equations are coupled, highly non-linear integral equations, whose solution determine the steady state strategy profiles of the agents. The coupling enters non-trivially through the average rewards received by the agents. Note, that in the single-agent learning scenario, \( R_i(x) \) can be viewed as the reward received by the agent for playing the strategy \( x \). In this case, Equation suggests that the steady state strategy profile is given by the Gibbs-like measure for a system with energy \(-R_i(x)\) and temperature \(1/\beta\), and can be derived from the following considerations. Let us define the following functional:

\[
\Phi[p(x)] = \int dx R(x)p(x) - T \int dx p(x) \ln(1/p(x))
\]

\( \Phi \) is so called free energy from statistical physics. It is easy to check that the Equation minimizes \( \Phi \), subject to the condition that \( p(x) \) is normalized. Indeed, introducing a Lagrange multiplier \( \lambda \) to account for the normalization constraint, and taking the functional derivative with respect to \( p \), we obtain

\[
\delta \Phi[p(x)] = \int dx [R(x) - T \ln p(x)] - \lambda \delta p(x) = 0
\]

which again yields 10. This suggests that for any non-zero temperature, the replicator dynamics will lead to a steady state that has non-zero entropy. In the terminology of population biology, non-zero temperature guarantees population diversity, and the monomorphic solutions are not allowed. The actual degree of diversity is governed by the temperature. Below we examine this question in more details.

IV. EXAMPLES

Owing to the highly non-linear nature of the steady state equations, they cannot be solved analytically in the most general case of arbitrary payoffs. However, one can still establish important results for certain class of games. In the reminder of the paper we focus on several such examples. We will limit our consideration to two-player games. Let \( x \) and \( y \) denote the actions of each agent. The system then has the following form:

\[
P_1(x) = A_1 e^{\beta R_1(x)} \equiv A_1 e^{\beta \int dy f_1(x,y) P_2(y)} \quad (13)
\]

\[
P_2(y) = A_2 e^{\beta R_2(y)} \equiv A_2 e^{\beta \int dx f_2(x,y) P_1(x)} \quad (14)
\]

where \( A_1, A_2 \) are normalization constants.

A. Bi-Linear Payoff

We first consider games with symmetric bi-linear payoffs structure, which, in the general case, can be written as follows:

\[
f_1(x, y) = a_1 xy + b_1 x
\]

\[
f_2(x, y) = a_2 xy + b_2 y
\]

It is simple to see that the average reward of the agents \( x \) is a linear function of \( x \):

\[
R_1(x) = x(a_1 \overline{y} + b_1) \quad (16)
\]

\[
R_2(x) = y(a_2 \overline{x} + b_2) \quad (17)
\]

where \( \overline{x}, \overline{y} \) are the means of respective distributions. This suggests the following form for the steady state solutions:

\[
P_1(x) \propto e^{\gamma_1 x}, P_2(y) \propto e^{\gamma_2 y} \quad (18)
\]

where \( \gamma_1, \gamma_2 \) can be found from a self-consistency condition. For the symmetric case, \( a_1 = a_2 = a \) and \( b_1 = b_2 = b \), this condition yields

\[
\frac{\gamma_1}{\beta} = g(\gamma_2), \frac{\gamma_2}{\beta} = g(\gamma_1)
\]

where the function \( g(\gamma) \) is defined as follows:

\[
g(\gamma) = b + a \left[ \frac{1}{1 - e^{-\gamma}} - \frac{1}{\gamma} \right]
\]
As we show below, the nature of the solution strongly depends on the choice of the parameters. We now examine this dependence in more details. First, let us focus on the symmetric solutions, $\gamma_1 = \gamma_2 \equiv \gamma$, in which case Equation 19 becomes

$$\frac{\gamma}{\beta} = g(\gamma)$$

(21)

Graphical illustration of Equation 21 is presented in Figure 1, where we plot $g(\gamma)$ together with the line $\gamma/\beta$ for different values of $b$. Let us first consider the case $a > 0$ (Figure 1(a)). In this case, $g(\gamma)$ is a monotonically increasing function of its argument, which tends to $b$ as $\gamma \to -\infty$, and to $b + a$ as $\gamma \to \infty$. A simple inspection shows that for $b > 0$, the equation has a unique solution that increases almost linearly with the inverse temperature $\beta$. Similarly, for $b \leq -a$, the equation again has a unique solution, which decreases linearly with increasing $\beta$. Thus, the steady state strategies are peaked around $x, y = 0$ ($x, y = 1$) for $b < -a$, ($b > 0$), with the width of the distribution shrinking linearly with $\beta$. The situation is different for the intermediate values of $b$. Indeed, it can be seen from Figure 1(a) that for $-a < b < 0$, there is a critical value $\beta_c$ so that for any $\beta > \beta_c$, there are three distinct solutions. A simple analysis shows that the solution in the middle is unstable, while two
other solutions, one always positive and the other negative (encircled in Figure 1(a)) are stable. Furthermore, it is easy to check that asymptotic behavior of the solutions for large values of $\beta$ is $\gamma \approx \pm \beta |b|$. Thus, in zero–temperature limit $\beta \to 0$ the steady state strategy profiles are $\delta$–functions at either 0 or 1, depending on the initial conditions. This prediction was confirmed in our simulations in Figure 1(b), where we show the steady–state strategy profile for a game with $a = 1$ and $b = 0$. The open symbols are the results of simulations, which agree very well with the analytical prediction. For this particular case, the steady–state strategy is peaked around the Nash equilibrium point $x = y = 1$. Following to the discussion above, it is clear that while increasing $\beta$, $p(x)$ will tend to a point mass at $x = 1$.

Now consider the case $a < 0$. The corresponding $g(\gamma)$ is shown in Figure 2(a). Since $g(\gamma)$ is a strictly decreasing function, Equation 21 has only one solution, for any values of $b$ and $\beta$. However, a simple analysis reveals that depending on the value of $b$, the asymptotic behavior of the solution for large $\beta$ is different. Indeed, for $|b| > |a|$, the solution behaves asymptotically as $\gamma \approx \pm |b|/\beta$. Thus, the zero–temperature steady state strategy profile again again corresponds to point mass at $x = 0$ or $x = 1$, depending on the initial conditions. For the intermediate values of $|b| < -a$, on the other hand, the solution is almost independent of $\beta$, as it can be seen from the graph. In this case, the steady state strategies have continuous support for any $\beta$, and do not converge to the monomorphic state that corresponds to the Nash equilibrium at $x^* = y^* = 0$. This behavior is depicted in Figure 2(b) where we plot the steady state strategy profile for different values of $\beta$. Note that increasing $\beta$ makes the distribution more peaked initially, but it saturates for sufficiently large $\beta$. This is demonstrated in the inset of Figure 2(b) which shows the dependence of $\gamma$ on the inverse temperature. Thus, even in the limit $\beta \to \infty$, the solution will always have a finite (non–zero) entropy, and will not converge to a monomorphic state.

So far our analysis has focused on the symmetric solutions to the steady state equations 19. For sufficiently small $\beta$, the symmetric solution is the only one. However, above some critical value of $\beta$, another, asymmetric solution appears (strictly speaking, there are two solutions related by $x \leftrightarrow y$ symmetry). This is shown in Figure 3(a) where we compare analytical and simulation results for a particular value of $\beta$. In simulations, we had to start from asymmetric initial strategies in order to arrive at the asymmetric steady state. The structure of the solution can be summarized as follows: While increasing $\beta$, the strategy around 0 narrows much faster compared to the strategy around 1. This is also shown in Figure 3(b) where we plot the bifurcation diagram of the solution to Equations 19. For small $\beta$, there is only a symmetric solution, $\gamma_1 = \gamma_2 \equiv \gamma$. Starting from a critical $\beta$, the asymmetric solution appears, with $\gamma_1$ and $\gamma_2$ diverging from each other. We note that the symmetric solution exists at any value of $\beta$, as shown by the dashed line.

To understand the nature of the asymmetric solution, we make the following observation: Consider the discrete map

$$ z_{t+1} = \beta g(z_t) \quad (22) $$

Clearly, the symmetric solutions to Equation 19 correspond to the fixed points of this map, $z^* = \beta g(z^*)$. Furthermore, it is easy to see that asymmetric solutions of Equation 19, if they exist, correspond to the attractors of the map 22 with period $T = 2$. And conversely, if Equation 19 allows an asymmetric solution, then the map 22 necessarily has an attractor with period $T = 2$. A simple inspection shows that for $a < 0$, $|b| < |a|$ such attractors exist, as shown schematically in Figure 4.
B. Quadratic Payoff

We now consider another important class of games where the payoffs can be expressed through the following quadratic forms:

\[ f_1(x, y) = -(x + y - 2a_1)^2 \]
\[ f_2(x, y) = -(x + y - 2a_2)^2 \]  (23)

Here \(0 < a_1, a_2 < 1\), and generally speaking, \(a_1 \neq a_2\). Let \(P_1(x)\) and \(P_2(y)\) be the steady state strategy profiles corresponding to this payoff. It is easy to check that the average rewards are given as follows:

\[ R_1(x) = -(x + y - 2a_1)^2 - (y^2 - y^2) \]  (24)
\[ R_2(y) = -(y + x - 2a_2)^2 - (x^2 - x^2) \]  (25)

This suggests the following form for the steady state strategies:

\[ P_1(x) = c(x_0) e^{-\beta(x - x_0)^2} \]  (26)
\[ P_2(y) = c(y_0) e^{-\beta(y - y_0)^2} \]  (27)

where \(c(x_0), c(y_0)\), are the respective normalization factors, and the function \(c(z)\) can be expressed through the error functions as follows:

\[ c(z) = 2\sqrt{\beta} \left[ erf(\sqrt{\beta z}) + erf(\sqrt{\beta(1 - z)}) \right]^{-1} \]  (28)

Combining Equations 14, 25 and 27, we find the following transcendental equations for the parameters \(x_0\) an \(y_0\):

\[ x_0 = 2a_1 - \mu(y_0) \]  (29)
\[ y_0 = 2a_2 - \mu(x_0) \]  (30)

where the function \(\mu(z)\) is the mean of a truncated Gaussian distribution centered at \(z\), and is given as follows:

\[ \mu(z) = z - c(z) \frac{e^{-\beta(1-z)^2} - e^{-\beta z^2}}{2\beta} \]  (31)

Let us first analyze the symmetric case \(a_1 = a_2 \equiv a\). Note that in this case the game becomes a pure coordination game, which has continuously many Nash equilibria given by \(x^* + y^* = 2a\). Furthermore, it is easy to check that there is no mixed Nash equilibrium. To see why this is the case, assume the contrary, and let \(P_1(x)\) be the mixed strategy of the first agent. Then, using Equation 25 it is straightforward to show that the best response of the second agent is to play a pure strategy \(P_1'(x) = \delta(x - x^*)\)
with \( x^* = 2a - \bar{y} \). However, if the second agent plays according to this pure strategy, then the first agent will do better by playing a pure strategy as well, \( P_2(y) = \delta(y - \bar{y}) \).

Let us now consider the corresponding steady state structure within the replicator framework. The steady state strategy profiles are (truncated) normal distributions centered at points \((x_0, y_0)\) which need to be found from the system of transcendental Equations 29–31. We now analyze this system in more details. First of all, we check for a symmetric solution \( x_0 = y_0 \), for which the system of equations reduces to the following equation:

\[
x_0 = 2a - \mu(x_0) \equiv a + c(x_0) \frac{e^{-\beta(1-x_0)^2} - e^{-\beta x_0^2}}{4\beta}
\]

Graphical representation of Equation 32 is shown in Figure 5(a). An inspection confirms that a symmetric solution is present for any value of the inverse temperature \( \beta \). The actual steady state strategy profiles corresponding to different values of \( \beta \) are shown in Figure 5(b), together with simulation results. Note that in the limit \( \beta \to \infty \) one has \( x_0 = y_0 = a \), suggesting that the strategy profiles of both agents become peaked around \( a \) in the limit of large \( \beta \).

The picture above suggests that in the zero temperature limit \( \beta \to \infty \), the steady state strategy profiles should correspond to the symmetric pure strategy Nash equilibrium \( x^* = y^* = a \). Indeed, our results suggests that if one starts in the vicinity of the symmetric equilibrium, then this picture holds. However, further analysis reveals that the convergence to this symmetric equilibrium is not trivial. To understand why this is the case, let us consider again the steady state equations 29 and 30, and look for the solutions of form \( x_0 + y_0 = 2a \). This leads to the following equation:

\[
x_0 = 2a - \mu(2a - x_0)
\]

The graphical representation of Equation 33 is shown in Figure 6. An inspection of this equation for various \( \beta \) yields the following observation: the symmetric solution \( x_0 = y_0 = a \) is the only solution, and for relatively small values of \( \beta \), the replicator dynamics settles into this symmetric solution relatively quickly. However, increasing \( \beta \) leads to the appearance of continuously many metastable states that are characterized by the condition \( x + y \approx 2a \). Thus, when the system starts close to those metastable states, it can get trapped there for very long times, converging very slowly to the actual steady state. In fact, the convergence times diverges as \( \beta \to \infty \). Also, note that the metastable state that the system will fall into will depend on the initial conditions.

Finally, we consider the asymmetric case \( a_1 \neq a_2 \). It can be shown that any asymmetry, however small, drastically changes the structure of the Nash equilibria. Consider, for instance, the following perturbation of the game considered above: \( a_1 = a - \delta a, \ a_2 = a + \delta a, \) with \( a = 0.5 \). It is clear that for any \( \delta a \), no matter how small, the asymmetry leads to a single pure Nash equilibrium at \( \{x^*, y^*\} = \{0, 1\} \). Our simulations show that similar behavior persists for finite \( \beta \), and that the agents’ strategies drift towards the deterministic equilibrium points when one increases \( \beta \), as shown in Figure 7.
FIG. 6: Graphical representation of Equation (33) shown for three different values of the parameter $\beta$.

FIG. 7: Steady–state strategy profiles for the asymmetric quadratic game with parameters $a_1 = 0.45$, $a_2 = 0.55$, and for two values of the inverse temperature $\beta = 10, 50$.

C. The Political Advertisement Game

In addition to simple bi–linear and quadratic games, where analytical examination of the steady state structure was possible, we considered other games for which the solution of the steady state equation cannot be obtained analytically, so one has to use numerical techniques. As an example, we consider the so called political advertisement game, were two political parties decide their expenditure levels $x$ and $y$ for campaign advertisement. The total number of votes participating in the election is proportional to the collective expenditure, and the fraction of votes each party gets is proportional to individual expenditures. Thus, the payoff has the following structure:

$$f_1(x, y) = \frac{x}{x+y} - x, \quad f_2(x, y) = \frac{y}{x+y} - y,$$

(34)

It is easy to show that this game has a single non–trivial Nash equilibrium at $x^* = y^* = 1/4$. Furthermore, a simple inspections shows that there are no mixed Nash equilibria.

We studied the above game using both numerical solution to the replicator dynamics equations, as well as actual simulations of the game. In Figure 8(a) we plot the steady state strategy profile of the agents the advertisement game. Again, the solution is symmetric. The strategy profile of each agent seems to centered around the pure Nash equilibrium, with the exact shape of the density depending on the parameter $\beta$. The average strategy of an agent, defined as $x_{avg} = \int dx P_1(x)x$ is close, but not equal to
the Nash equilibrium point \( x^* \). However, the strategy average tends asymptotically to this value as one increases the parameter \( \beta \), as demonstrated in Figure 8(b).

![Graphs showing steady-state strategy profiles for the political advertisement game.](image)

**FIG. 8**: Steady–state strategy profiles for the political advertisement game: (a) Average strategy (expenditure) of an agent plotted against the inverse temperature \( \beta \). The horizontal line corresponds to the Nash equilibrium \( x^* = 1/4 \).

### D. Investment Game

Our last example is a game for which the steady state of the replicator dynamics coincides with the Nash Equilibrium. Consider a model of two–firm competition where each firm chooses an investment level from the unit interval. The firm with the highest investment wins the market, which has unit value, with ties broken randomly. Denoting the investment levels of the players as \( x \) and \( y \), the payoff structure has the following form:

\[
  f_1(x, y) = \begin{cases} 
  1-x & \text{if } x > y; \\
  -x & \text{if } x < y; \\
  \frac{1}{2} - x & \text{if } x = y.
\end{cases}
\]  

with a similar (symmetric) payoff for the second agent. Note that the game does not have a pure Nash equilibrium. Indeed, assume the contrary, and let \( x^* \) be the equilibrium strategy of the first agent. If \( x^* < 1 \), then the second agent will do better by playing \( y = x^* + \varepsilon < 1 \), while if \( x^* = 1 \), then he will be better off playing \( y = 0 \). At the same time, one can show that there is a Nash equilibrium where both players mix uniformly over the pure strategies. It is straightforward to show that the same strategy profiles are also the steady state solutions of the replicator equation. Indeed, note that for a strategy profile \( P(y) \) of the second agent, the average reward of the first one is given as given as

\[
  R_1(x) = -x + \int_0^x P(y),
\]  

Then a simple inspection shows that choosing \( P(x) = P(y) = 1 \) solves the steady state equations. This is shown in Figure 9 where we also show the simulation results.

Note that there is an intuitive argument why the steady state of the replicator dynamics corresponds to the Nash equilibrium for this particular game. Indeed, recall that the Nash equilibrium minimizes the energy, while the replicator dynamics minimizes the free energy functional [11] which, at non–zero temperature, is different from the energy. Those two minimization objectives will generally yield different strategy profiles. For this particular game, however, the strategy profile that minimizes the energy also happens to maximize the entropy (and thus, minimize the entropic term in the free energy), so that minimization of either objective yields the same result.
V. CONCLUSION

In this paper we presented a generalization of the replicator dynamics framework to study the dynamics of multi-agent reinforcement learning with continuous strategy spaces. We presented a set of differential–functional equation that describe the adaptive learning dynamics in multi-agent settings. We also derived a set of equations that characterize the steady state strategy profile of the learning agents. We demonstrated the analytical framework on several examples, and obtained an excellent agreement with the simulations results.

It was shown, both theoretically and through simulations, that for the Boltzmann exploration mechanism, the long–term limit of the replicator dynamics does not necessarily correspond to the Nash equilibria of the corresponding game–theoretical system. The reason for this is the bounded rationality of the agent as characterized by the exploration noise in their strategies. Specifically, the replicator dynamics at non–zero temperature minimizes the free energy, and not the energy, which corresponds to the Nash equilibria for rational agents. We demonstrated on several examples that the Nash equilibrium is often recovered from the replicator dynamics in the zero temperature limit $\beta \rightarrow \infty$. In this limit, the strategies generally become a $\delta$–measures peaked at the Nash equilibria. However, as it was shown on the example of the bi–linear game, this is not always the case. In particular, for some games the replicator equations might yield a steady state solutions that have continuous support even if the limit $\beta \rightarrow \infty$, and which do not have a corresponding Nash equilibrium in that limit. Finally, note that for one of election game considered here, the steady state of the replicator dynamics coincides with the Nash equilibria for arbitrary $\beta$: The underlying reason for this is the strategy profile that minimizes the energy (NE) also happens to maximize the entropy, so that the the same strategy profile minimizes both energy and free energy. More generally, one can say that any uniform Nash equilibrium also serves as a steady state for the replicator dynamics.

There are several important directions to pursue this work further. First of all, it will be worthwhile to perform more formal analysis of the steady state equations, and examine the issues of existence and uniqueness of the solutions depending on particular payoff structure. Furthermore, we intend to extend our analysis beyond two–player games, and specifically, consider games with a very large number of agents, $N \gg 1$, where statistical–mechanical approaches for analyzing the solution structure might be appropriate. Finally, another direction of further research is to generalize the stateless Q–learning model considered here to a more general model which will account for different states and probabilistic transitions between them. We believe that such generalization will be possible by complementing the replicator equations with equations that describe the Markovian evolution of the states.
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