A CANONICAL CONNECTION ON BUNDLES ON RIEMANN SURFACES AND QUILLEN CONNECTION ON THE THETA BUNDLE

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ABSTRACT. We investigate the symplectic geometric and also the differential geometric aspects of the moduli space of connections on a compact connected Riemann surface $X$. Fix a theta characteristic $K_{X}^{1/2}$ on $X$; it defines a theta divisor on the moduli space $\mathcal{M}$ of stable vector bundles on $X$ of rank $r$ degree zero. Given a vector bundle $E \in \mathcal{M}$ lying outside the theta divisor, we construct a natural holomorphic connection on $E$ that depends holomorphically on $E$. Using this holomorphic connection, we construct a canonical holomorphic isomorphism between the following two:

1. the moduli space $\mathcal{C}$ of pairs $(E, D)$, where $E \in \mathcal{M}$ and $D$ is a holomorphic connection on $E$, and
2. the space $\text{Conn}(\Theta)$ given by the sheaf of holomorphic connections on the line bundle on $\mathcal{M}$ associated to the theta divisor.

The above isomorphism between $\mathcal{C}$ and $\text{Conn}(\Theta)$ is symplectic structure preserving, and it moves holomorphically as $X$ runs over a holomorphic family of Riemann surfaces.

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1. INTRODUCTION

Let $X$ be a compact connected Riemann surface of genus at least two. Let $K_{X}^{1/2}$ be a square-root of the canonical line bundle $K_X$ of $X$; it is called a theta characteristic of
Let $M$ denote the moduli space of stable vector bundles on $X$ of rank $r$ and degree zero. It has the theta divisor $D_\Theta$ defined by all $E$ such that $H^0(X, E \otimes K_X^{1/2}) \neq 0$; the holomorphic line bundle on $M$ corresponding to the divisor $D_\Theta$ is denoted by $\Theta$. The moduli space $M$ has a natural Kähler structure. The Kähler 2-form on $M$ coincides with the symplectic form on the $U(r)$ character variety for $X$, [Go], [AB], once we identify this character variety with $M$ using [NS] (see the map $\psi_U$ below).

Let $C$ denote the moduli space of holomorphic connections on $X$ of rank $r$ such that the underlying holomorphic vector bundle is stable; it projects to $M$ by mapping elements to the underlying holomorphic vector bundle. This $C$ is a holomorphic torsor on $M$ for the holomorphic cotangent bundle $T^*M$ (this means that the fibers of $T^*M$ act freely transitively on the fibers of $C$ over $M$). This moduli space $C$ is equipped with a natural holomorphic symplectic structure [Go], [AB]. There is a natural $C^\infty$ section $\psi_U : M \to C$ that sends any $E \in M$ to the unique unitary flat connection on $E$ [NS].

Let $Conn(\Theta)$ denote the holomorphic fiber bundle on $M$ given by the sheaf of holomorphic connections on the line bundle $\Theta$. There is a tautological holomorphic connection on $Conn(\Theta)$ is also a holomorphic torsor on $M$ for $T^*M$. Although the line bundle $\Theta$ depends on the choice of the theta characteristic $K_X^{1/2}$, the $T^*M$-torsor $Conn(\Theta)$ does not depend on the choice of the theta characteristic (see Remark 2.2).

There is a unique Hermitian connection on $\Theta$ whose curvature is the Kähler form on $M$ [Qu]. Let

$$\psi_Q : M \to Conn(\Theta)$$

be the corresponding $C^\infty$ section of the projection $Conn(\Theta) \to M$.

Since both $C$ and $Conn(\Theta)$ are torsors over $M$ for $T^*M$, and they are equipped with the $C^\infty$ sections $\psi_U$ and $\psi_Q$ respectively, there is a unique $C^\infty$ isomorphism

$$F : C \to Conn(\Theta)$$

satisfying the following two conditions:

- $F$ takes the section $\psi_U$ to $\psi_Q$, and
- $F$ preserves the $T^*M$-torsor structure up to the multiplicative factor $2r$, meaning $F(E, D+v) = F(E, D) + 2r \cdot v$, where $E \in M$ with $D$ a holomorphic connection on $E$ and $v \in T^*_E M = H^0(X, \text{End}(E) \otimes K_X)$.

The following was proved in [BH] (recalled here in Theorem 2.1):

The above isomorphism $F$ is holomorphic, and it preserves the holomorphic symplectic forms up to the factor $2r$, meaning the pullback, by $F$, of the holomorphic symplectic form on $Conn(\Theta)$ coincides with $2r$ times the holomorphic symplectic form on $C$.

Take any holomorphic vector bundle $E \in M$ such that $H^0(X, E \otimes K_X^{1/2}) = 0$ (so $E$ lies outside the theta divisor $D_\Theta$). We construct a natural holomorphic connection on $E$; see Section 3.1. Unlike the unitary connection, it moves holomorphically as $E$ moves in a holomorphic family of vector bundles. In fact, this connection moves holomorphically as
the pair \((X, E)\) moves in a holomorphic family. Let

\[ \phi : \mathcal{M} \setminus D_\Theta \longrightarrow \mathcal{C}\big|_{\mathcal{M} \setminus D_\Theta} \]

be the holomorphic section given by this natural holomorphic connection.

The holomorphic line \(\Theta\) has a canonical trivialization outside the theta divisor \(D_\Theta\). This trivialization produces a holomorphic section of the fiber bundle \(\text{Conn}(\Theta) \longrightarrow \mathcal{M}\) outside \(D_\Theta\). Let

\[ \tau : \mathcal{M} \setminus D_\Theta \longrightarrow \text{Conn}(\Theta)\big|_{\mathcal{M} \setminus D_\Theta} \]

be the section given by this canonical trivialization. Unlike the section \(\psi_Q\), this section \(\tau\) is holomorphic.

Since both \(\mathcal{C}\big|_{\mathcal{M} \setminus D_\Theta}\) and \(\text{Conn}(\Theta)\big|_{\mathcal{M} \setminus D_\Theta}\) are torsors over \(\mathcal{M} \setminus D_\Theta\) for the holomorphic cotangent bundle \(T^*(\mathcal{M} \setminus D_\Theta)\), and \(\phi\) and \(\tau\) are holomorphic sections, there is a unique holomorphic isomorphism

\[ G : \mathcal{C}\big|_{\mathcal{M} \setminus D_\Theta} \longrightarrow \text{Conn}(\Theta)\big|_{\mathcal{M} \setminus D_\Theta} \]

satisfying the following two conditions:

- \(G\) takes the section \(\phi\) to \(\tau\), and
- \(G\) preserves the \(T^*(\mathcal{M} \setminus D_\Theta)\)-torsor structures up to the multiplicative factor \(2r\); this means that \(G(E, D + v) = G(E, D) + 2r \cdot v\), where \(E \in \mathcal{M} \setminus D_\Theta\), \(D\) is a holomorphic connection on \(E\) and \(v \in T^*_E \mathcal{M} = H^0(X, \text{End}(E) \otimes K_X)\).

Our main result says the following (see Theorem 1.1):

**Theorem 1.1.** The above isomorphism \(G\) coincides with the restriction of the isomorphism \(F\) to the open subset \(\mathcal{C}\big|_{\mathcal{M} \setminus D_\Theta}\).

Theorem 1.1 has the following consequence (see Corollary 4.5):

**Corollary 1.2.** The above holomorphic isomorphism \(G\) extends to a holomorphic isomorphism

\[ G' : \mathcal{C} \sim \longrightarrow \text{Conn}(\Theta) \]

over entire \(\mathcal{M}\).

**Remark 1.3.** We note that the isomorphism \(G\) in Theorem 1.1 is constructed purely algebro-geometrically. Hence the construction of its closure \(G'\) in Corollary 1.2 is purely algebro-geometric. On the other hand, the two \(C^\infty\) sections \(\psi_U\) and \(\psi_Q\) mentioned earlier are not algebro-geometric. Theorem 1.1 implies that given the input of the algebro-geometric isomorphism \(G'\), any one of the two sections \(\psi_U\) and \(\psi_Q\) determines the other uniquely.

As mentioned before, both \(\mathcal{C}\) and \(\text{Conn}(\Theta)\) are equipped with holomorphic symplectic structures. Let \(\Phi_1\) and \(\Phi_2\) denote the holomorphic symplectic forms on \(\mathcal{C}\) and \(\text{Conn}(\Theta)\) respectively. We prove the following relationship between these two symplectic forms (see Corollary 4.6):

**Corollary 1.4.** For the isomorphism \(G'\) in Corollary 1.2,

\[ (G')^* \Phi_2 = 2r \cdot \Phi_1. \]
Both $\phi$ and $\tau$ move holomorphically as $X$ moves in a holomorphic family of Riemann surfaces. Therefore, Theorem 1.1 has the following consequence (see Proposition 5.1):

**Proposition 1.5.** The isomorphism $F$ moves holomorphically as $X$ moves in a holomorphic family of Riemann surfaces.

It may be mentioned that Proposition 1.5 not immediate from the isomorphism of [BH]. This is particularly useful as one of the questions inspiring this investigation is the omnipresence of the determinantal line in questions involving deformations of connections, that is isomonodromy; this manifests itself in the role of tau-functions. The role of this line is somewhat surprising; it is as if in a linear algebra problem, the main issue was the determinant. Several papers have been devoted to this issue, notably by Malgrange [Ma]. This paper and its predecessor [BH] can be viewed as a further exploration of this issue; one is comparing one torsor ($C$), defined over the moduli space in terms of connections on a full Riemann surface, and another (Conn) which is simply the natural locus for connections on the determinant line; the first should contain much more information, but for certain things, it does not.

2. Moduli space of stable vector bundles

2.1. Two torsors on a moduli space. Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 2$. We recall that a holomorphic vector bundle $V$ on $X$ is called stable if

\[
\frac{\text{degree}(F)}{\text{rank}(F)} < \frac{\text{degree}(V)}{\text{rank}(V)}
\]

for all holomorphic subbundles $F \subset V$ of positive rank. This condition implies that any stable vector bundle is simple. Fix a positive integer $r$. Let $\mathcal{M}$ denote the moduli space of stable vector bundles on $X$ of rank $r$ and degree zero (see [Ne], [Si1] for the construction of this moduli space).

The holomorphic cotangent bundle of $X$ will be denoted by $K_X$. A holomorphic connection on a holomorphic vector bundle $E$ on $X$ is a holomorphic differential operator of order one

\[
D_E : E \rightarrow E \otimes K_X
\]

satisfying the Leibniz identity, which says that $D(f \cdot s) = f \cdot D(s) + s \otimes df$, where $s$ is any locally defined holomorphic section of $E$ and $f$ is any locally defined holomorphic function on $X$ [At]. Holomorphic connections on $X$ are flat because there are no nonzero $(2, 0)$–forms on a Riemann surface.

Let $\mathcal{C}$ denote the moduli space of all holomorphic connections on $X$ of rank $r$ such that the underlying holomorphic vector bundle is stable [Si1], [Si2]. In other words, $\mathcal{C}$ parametrizes all isomorphism classes of pairs of the form $(E, D_E)$, where $E$ is a stable holomorphic vector bundle on $X$ of rank $r$ and degree zero and $D_E$ is a holomorphic connection on $E$.

Any indecomposable holomorphic vector bundle on $X$ of degree zero admits a holomorphic connection [At, p. 203, Proposition 19], [We], in particular, any stable vector bundle on $X$ of degree zero admits a holomorphic connection. Let

\[
\varphi : \mathcal{C} \rightarrow \mathcal{M}, \quad (E, D_E) \mapsto E
\] (2.1)
be the forgetful map that forgets the holomorphic connection; as noted above, the map \( \varphi \) is surjective. Any two holomorphic connections on \( E \in \mathcal{M} \) differ by an element of

\[
H^0(X, \text{End}(E) \otimes K_X) = T^*_E \mathcal{M}.
\]

In fact, \( C \) is a holomorphic torsor over \( \mathcal{M} \) for the holomorphic cotangent bundle \( T^* \mathcal{M} \). This means that there is a holomorphic action

\[
\delta : C \times_{\mathcal{M}} T^* \mathcal{M} \to C
\]

of \( T^* \mathcal{M} \) on \( C \) such that the map of fiber products

\[
C \times_{\mathcal{M}} T^* \mathcal{M} \to C \times_{\mathcal{M}} C, \quad (a, b) \mapsto (\delta(a, b), a)
\]

is an isomorphism.

Any stable holomorphic vector bundle on \( X \) of degree zero admits a unique holomorphic connection whose monodromy representation is unitary [NS, p. 560–561, Theorem 2]. Therefore, the projection \( \varphi \) in \((2.1)\) has a \( C^\infty \) section

\[
\psi_U : \mathcal{M} \to C
\]

that sends any stable vector bundle \( E \in \mathcal{M} \) to the unique holomorphic connection on \( E \) whose monodromy representation is unitary. This section \( \psi_U \) is not holomorphic.

There is a natural holomorphic symplectic form on \( C \)

\[
\Phi_1 \in H^0(C, \Omega^2_C)
\]

[Go], [AB]. It is known that this holomorphic 2–form \( \Phi_1 \) is algebraic [Bi2].

We shall now construct another holomorphic torsor over \( \mathcal{M} \) for the holomorphic cotangent bundle \( T^* \mathcal{M} \).

Fix a theta characteristic \( K^{1/2}_X \) on \( X \). So \( K^{1/2}_X \) is a holomorphic line bundle on \( X \) of degree \( g - 1 \) such that \( K^{1/2}_X \otimes K^{1/2}_X \) is holomorphically isomorphic to the holomorphic cotangent bundle \( K_X \). Fix a holomorphic isomorphism of \( K^{1/2}_X \otimes K^{1/2}_X \) with \( K_X \). Let

\[
D_\Theta := \{ E \in \mathcal{M} \mid H^0(X, E \otimes K^{1/2}_X) \neq 0 \} \subset \mathcal{M}
\]

be the theta divisor on \( \mathcal{M} \) (see [La]). Note that by Riemann–Roch we have

\[
dim H^0(X, E \otimes K^{1/2}_X) - \dim H^1(X, E \otimes K^{1/2}_X) = \text{degree}(E \otimes K^{1/2}_X) - r(g - 1)
\]

\[
= r(g - 1) - r(g - 1) = 0.
\]

So \( H^1(X, E \otimes K^{1/2}_X) \neq 0 \) if and only if \( E \in D_\Theta \). The holomorphic line bundle \( \mathcal{O}_\mathcal{M}(D_\Theta) \) on \( \mathcal{M} \) will be denoted by \( \Theta \).

Let \( \text{At}(\Theta) \to \mathcal{M} \) be the Atiyah bundle for \( \Theta \). It fits in the short exact sequence of holomorphic vector bundles

\[
0 \to \mathcal{O}_\mathcal{M} \to \text{At}(\Theta) \to T\mathcal{M} \to 0
\]

over \( \mathcal{M} \) (see [At, p. 187, Theorem 1]). For \( i \geq 0 \), let \( \text{Diff}^i_\mathcal{M}(\Theta, \Theta) \) be the holomorphic vector bundle on \( \mathcal{M} \) given by the sheaf of holomorphic differential operators from \( \Theta \) to itself. We note that \( \text{At}(\Theta) = \text{Diff}^1_\mathcal{M}(\Theta, \Theta) \), and the exact sequence in \((2.5)\) coincides with the sequence

\[
0 \to \text{Diff}^0_\mathcal{M}(\Theta, \Theta) = \mathcal{O}_\mathcal{M} \to \text{Diff}^1_\mathcal{M}(\Theta, \Theta) \to T\mathcal{M} \to 0,
\]
where the projection to $T\mathcal{M}$ is the symbol map. Consider the dual exact sequence of (2.5)

$$0 \rightarrow T^*\mathcal{M} \rightarrow \text{At}(\Theta)^* \xrightarrow{\alpha} \mathcal{O}^*_\mathcal{M} = \mathcal{O}_\mathcal{M} \rightarrow 0.$$  

Let $1_\mathcal{M} : \mathcal{M} \rightarrow \mathcal{O}_\mathcal{M}$ be the section given by the constant function 1 on $\mathcal{M}$. Now define

$$\text{At}(\Theta)^* \supset \alpha^{-1}(1_\mathcal{M}(\mathcal{M})) =: \text{Conn}(\Theta) \xrightarrow{q} \mathcal{M},$$  

where $\alpha$ is the projection in (2.6). Holomorphic sections of $\text{Conn}(\Theta)$ over an open subset $U \subset \mathcal{M}$ are identified with the holomorphic connections on $\Theta|_U$. From (2.6) it follows immediately that $\text{Conn}(\Theta)$ is a holomorphic torsor over $\mathcal{M}$ for the holomorphic cotangent bundle $T^*\mathcal{M}$.

For any open subset $U \subset \mathcal{M}$, all $C^\infty$ maps $s : U \rightarrow \text{Conn}(\Theta)$ such that $q \circ s = \text{Id}_U$, where $q$ is the projection in (2.7), are in bijection with the $C^\infty$ connections $D_U$ on the holomorphic line bundle $\Theta|_U$ that is equivalent to the condition that $D_U(\gamma)$ is a $C^\infty$ section of $(\Theta \otimes K_X)|_U$, for every holomorphic section $\gamma$ of $\Theta|_U$. Such a connection $D_U$ on $\Theta|_U$ is holomorphic if and only if the corresponding section $s$ of the projection is holomorphic.

There is a natural Kähler form $\omega_\mathcal{M}$ on $\mathcal{M}$ [AB]; this form $\omega_\mathcal{M}$ coincides with the symplectic form on the irreducible unitary character variety Hom$^{ir}(\pi_1(X), U(r))/U(r)$ that was constructed by Goldman [Go]; here Hom$^{ir}(\pi_1(X), U(r))$ denotes the space of all homomorphisms $\rho : \pi_1(X) \rightarrow U(r)$ such that the standard action of $\rho(\pi_1(X))$ on $\mathbb{C}^r$ does not preserve any nonzero proper subspace of $\mathbb{C}^r$. More precisely, we have

$$\omega_\mathcal{M} = \psi_U^* \Phi_1,$$

where $\psi_U$ and $\Phi_1$ are as in (2.2) and (2.3) respectively.

Quillen constructed an explicit Hermitian structure on the line bundle $\Theta$ with the property that the curvature of the corresponding Chern connection on $\Theta$ coincides with the Kähler form $\omega_\mathcal{M}$ in (2.8) [Qu] (see also [BGS1], [BGS2], [BGS3]). As a corollary, the de Rham cohomology class for $\omega_\mathcal{M}$ is integral. We note that there is at most one Hermitian connection on $\Theta$ whose curvature is $\omega_\mathcal{M}$. In other words, the Chern connection of the Hermitian structure on $\Theta$ constructed in [Qu] is the unique Hermitian connection whose curvature is $\omega_\mathcal{M}$. It should be clarified that this condition — that the curvature is $\omega_\mathcal{M}$ — does not determine the Hermitian structure on $\Theta$ uniquely; any two Hermitian structures on $\Theta$ satisfying this condition differ by a constant scalar multiplication. However, the Hermitian connection is unique. Let $\nabla_Q$ denote the unique Hermitian connection on $\Theta$ whose curvature is $\omega_\mathcal{M}$. So $\nabla_Q$ produces a $C^\infty$ section

$$\psi_Q : \mathcal{M} \rightarrow \text{Conn}(\Theta)$$

of the holomorphic fibration $q$ in (2.7).

There is a holomorphic symplectic form

$$\Phi_2 \in H^0(\text{Conn}(\Theta), \Omega^2_{\text{Conn}(\Theta)})$$

on $\text{Conn}(\Theta)$ which can be described as follows. The holomorphic line bundle $q^*\Theta$, where $q$ is the projection in (2.7), has a tautological holomorphic connection (see [BHS, p. 372, Proposition 3.3], [BB]). The curvature of this tautological holomorphic connection on $q^*\Theta$ is the 2–form $\Phi_2$ in (2.10).
2.2. An isomorphism of torsors. In this subsection a result from [BH] will be recalled.

Let
\[ \delta : \mathcal{C} \times_{\mathcal{M}} T^*\mathcal{M} \longrightarrow \mathcal{C} \quad \text{and} \quad \eta : \text{Conn}(\Theta) \times_{\mathcal{M}} T^*\mathcal{M} \longrightarrow \text{Conn}(\Theta) \]
be the holomorphic \( T^*\mathcal{M} \)-torsor structures on \( \mathcal{C} \) and \( \text{Conn}(\Theta) \) respectively. Let
\[ m : T^*\mathcal{M} \longrightarrow T^*\mathcal{M}, \quad v \mapsto 2r \cdot v \]
be the multiplication by \( 2r \).

**Theorem 2.1** ([BH Proposition 2.3, Theorem 3.1]). There is a unique holomorphic isomorphism
\[ F : \mathcal{C} \longrightarrow \text{Conn}(\Theta) \]
such that
\begin{enumerate}
  \item \( \varphi = q \circ F \), where \( \varphi \) and \( q \) are the projections in (2.11) and (2.7) respectively,
  \item \( F \circ \psi_U = \psi_Q \), where \( \psi_U \) and \( \psi_Q \) are the sections in (2.2) and (2.9) respectively, and
  \item \( F \circ \delta = \eta \circ (F \times m) \) as maps from \( \mathcal{C} \times_{\mathcal{M}} T^*\mathcal{M} \) to \( \text{Conn}(\Theta) \), where \( \delta \), \( \eta \) and \( m \) are the maps in (2.11) and (2.12).
\end{enumerate}

Moreover, \( F^*\Phi_2 = 2r \cdot \Phi_1 \), where \( \Phi_1 \) and \( \Phi_2 \) are the symplectic forms in (2.3) and (2.10) respectively.

There is a unique \( C^\infty \) isomorphism \( \mathcal{C} \longrightarrow \text{Conn}(\Theta) \) that satisfies the three conditions in the first part of Theorem 2.1. The content of the first part of Theorem 2.1 is that this \( C^\infty \) isomorphism is actually holomorphic. The second part of Theorem 2.1 says that this isomorphism is compatible, up to the factor \( 2r \), with the symplectic structures on \( \mathcal{C} \) and \( \text{Conn}(\Theta) \).

**Remark 2.2.** For a different choice of the theta characteristic on \( \mathcal{X} \), the corresponding theta line bundle on \( \mathcal{M} \) differs from \( \Theta \) by a line bundle of order two on \( \mathcal{M} \). Any line bundle of finite order has a tautological flat holomorphic connection. This implies that the \( T^*\mathcal{M} \)-torsor \( \text{Conn}(\Theta) \) is actually independent of the choice that theta characteristic on \( \mathcal{X} \).

3. Another isomorphism of torsors

3.1. A canonical connection. For \( i = 1, 2 \), let
\[ p_i : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \]
be the projection to the \( i \)-th factor. Let
\[ \Delta := \{ (x, x) \mid x \in \mathcal{X} \} \subset \mathcal{X} \times \mathcal{X} \]
be the diagonal divisor. We shall identify \( \Delta \) with \( \mathcal{X} \) via the map \( x \mapsto (x, x) \). Using the Poincaré adjunction formula, the restriction of the holomorphic line bundle \( \mathcal{O}_{\mathcal{X} \times \mathcal{X}}(\Delta) \) to \( \Delta \) is identified with the normal bundle of the divisor \( \Delta \subset \mathcal{X} \times \mathcal{X} \), which in turn is identified with \( TX \) using the identification of \( \Delta \) with \( \mathcal{X} \). However this isomorphism between \( \mathcal{O}_{\mathcal{X} \times \mathcal{X}}(\Delta)|_{\Delta} \) and \( TX \) changes by multiplication by \( -1 \) under the involution \( (x, y) \mapsto (y, x) \) of \( \mathcal{X} \times \mathcal{X} \). In other words, this involution acts by multiplication by \( -1 \) on \( \mathcal{O}_{\mathcal{X} \times \mathcal{X}}(\Delta)|_{\Delta} \).
Using the isomorphism between $\mathcal{O}_{X \times X}(\Delta)\mid_{\Delta}$ and $TX$, the restriction of $(p_1^*K_{X}^{1/2}) \otimes (p_2^*K_{X}^{1/2}) \otimes \mathcal{O}_{X \times X}(\Delta)$ to $\Delta$ is identified with $K_{X} \otimes TX = \mathcal{O}_X$. It should be clarified that this isomorphism changes by multiplication by $-1$ under the involution $(x, y) \mapsto (y, x)$ of $X \times X$.

Take any $E \in \mathcal{M} \setminus D_\Theta$, where $D_\Theta$ is constructed in (2.4). Since 

$$H^0(X, E \otimes K_{X}^{1/2}) = 0 = H^1(X, E \otimes K_{X}^{1/2}) ,$$

using Serre duality, we have

$$H^0(X, E^* \otimes K_{X}^{1/2}) = H^1(X, E \otimes K_{X}^{1/2})^* = 0 = H^0(X, E \otimes K_{X}^{1/2})^* = H^1(X, E^* \otimes K_{X}^{1/2}) .$$

(3.2)

Consider the short exact sequence of coherent sheaves on $X \times X$

$$0 \to (p_1^*(E \otimes K_{X}^{1/2})) \otimes (p_2^*(E^* \otimes K_{X}^{1/2})) \to (p_1^*(E \otimes K_{X}^{1/2})) \otimes (p_2^*(E^* \otimes K_{X}^{1/2})) \otimes \mathcal{O}_{X \times X}(\Delta) \to \text{End}(E)\mid_{\Delta} \to 0;$$

recall that $(p_1^*K_{X}^{1/2}) \otimes (p_2^*K_{X}^{1/2}) \otimes \mathcal{O}_{X \times X}(\Delta)\mid_{\Delta} = \mathcal{O}_X$, and note that $(p_1^*E) \otimes (p_2^*E^*)\mid_{\Delta} = \text{End}(E)$ (the identification between $X$ and $\Delta$ is being used). Let

$$H^0(X \times X, (p_1^*(E \otimes K_{X}^{1/2})) \otimes (p_2^*(E^* \otimes K_{X}^{1/2})))$$

$$\to H^0(X \times X, (p_1^*(E \otimes K_{X}^{1/2})) \otimes (p_2^*(E^* \otimes K_{X}^{1/2})) \otimes \mathcal{O}_{X \times X}(\Delta))$$

$$\xrightarrow{h_E} H^0(X, \text{End}(E)) \to H^1(X \times X, (p_1^*(E \otimes K_{X}^{1/2})) \otimes (p_2^*(E^* \otimes K_{X}^{1/2})))$$

(3.3)

be the corresponding long exact sequence of cohomologies. Using Künneth formula,

$$H^0(X \times X, (p_1^*(E \otimes K_{X}^{1/2})) \otimes (p_2^*(E^* \otimes K_{X}^{1/2}))) = H^0(X, E \otimes K_{X}^{1/2}) \otimes H^0(X, E^* \otimes K_{X}^{1/2})$$

and

$$H^1(X \times X, (p_1^*(E \otimes K_{X}^{1/2})) \otimes (p_2^*(E^* \otimes K_{X}^{1/2})))$$

$$= \bigoplus_{j=0}^{1} H^j(X, E \otimes K_{X}^{1/2}) \otimes H^{1-j}(X, E^* \otimes K_{X}^{1/2}) .$$

(3.4)

Hence invoking (3.2) we conclude that

$$H^k(X \times X, (p_1^*(E \otimes K_{X}^{1/2})) \otimes (p_2^*(E^* \otimes K_{X}^{1/2}))) = 0$$

(3.5)

for $k = 0, 1$. Consequently, the homomorphism $h_E$ in the exact sequence in (3.3) is an isomorphism. Now define

$$\beta_E := h_E^{-1}(\text{Id}_E) \in H^0(X \times X, (p_1^*(E \otimes K_{X}^{1/2})) \otimes (p_2^*(E^* \otimes K_{X}^{1/2})) \otimes \mathcal{O}_{X \times X}(\Delta)) ,$$

(3.6)

where $\text{Id}_E \in H^0(X, \text{End}(E))$ is the identity endomorphism.

It was noted earlier that $(p_1^*K_{X}^{1/2}) \otimes (p_2^*K_{X}^{1/2}) \otimes \mathcal{O}_{X \times X}(\Delta)\mid_{\Delta} = \mathcal{O}_X$. We shall now show that

$$((p_1^*K_{X}^{1/2}) \otimes (p_2^*K_{X}^{1/2}) \otimes \mathcal{O}_{X \times X}(\Delta))\mid_{2\Delta} = \mathcal{O}_{2\Delta} .$$

(3.7)

To prove (3.7), take any holomorphic coordinate function $z : U \to \mathbb{C}$ on some analytic open subset $U$ of $X$. Take a holomorphic section

$$s_z \in H^0(U, K_{X}^{1/2})$$
such that $s \otimes s = dz \in H^0(U, K_U)$; note that there are exactly two such sections, and they differ by multiplication by $-1$. Now we have

$$1 = \frac{1}{z \circ p_1 - z \circ p_2} \sigma \big( (p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \big) \in H^0(U \times U, ((p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \otimes O_{X \times X}(\Delta))|_{U \times U}$$

Of course, this section $\frac{1}{z \circ p_1 - z \circ p_2} \sigma \big( (p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \big)$ depends on the coordinate function $z$. However, it is straightforward to check that the restriction of the section to $(2\Delta) \cap (U \times U)$

$$\frac{1}{z \circ p_1 - z \circ p_2} \sigma \big( (p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \big) \in (2\Delta) \cap (U \times U)$$

is actually independent of the choice of the holomorphic coordinate function $z$. Consequently, the locally defined sections of the form $\frac{1}{z \circ p_1 - z \circ p_2} \sigma \big( (p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \big)$ patch together compatibly to define a canonical section of $(p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \otimes O_{X \times X}(\Delta)|_{2\Delta}$. Let

$$\sigma_0 \in H^0(2\Delta, (p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \otimes O_{X \times X}(\Delta))$$

be this canonical section. This section $\sigma_0$ produces the isomorphism in (3.7) between

$$(p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \otimes O_{X \times X}(\Delta)|_{2\Delta}$$

and $O_{2\Delta}$ by sending any locally defined section $f$ of $O_{2\Delta}$ to the locally defined section $f \cdot \sigma_0$ of $(p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \otimes O_{X \times X}(\Delta)|_{2\Delta}$. We note that the restriction of the section $\sigma_0$ in (3.8) to $\Delta \subset 2\Delta$ coincides with the section given by the trivialization of $(p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \otimes O_{X \times X}(\Delta)|_{\Delta}$ (the trivialization of $(p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \otimes O_{X \times X}(\Delta)|_{\Delta}$ was obtained earlier using the Poincaré adjunction formula). Consider the section $\beta_E$ in (3.9). It is easy to see that there is a unique section

$$\hat{\beta}_E \in H^0(2\Delta, (p_1^* E) \otimes (p_2^* E^*))$$

over $2\Delta$, such that

$$\beta_E|_{2\Delta} = \hat{\beta}_E \otimes \sigma_0 .$$

Indeed, $(\sigma_0)^{-1}$ is a section of $((p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \otimes O_{X \times X}(\Delta))^*|_{2\Delta}$ over $2\Delta$. Now define

$$\tilde{\beta}_E := (\beta_E|_{2\Delta}) \otimes (\sigma_0)^{-1} ,$$

and consider it as a section of $((p_1^* E) \otimes (p_2^* E^*))|_{2\Delta}$ using the duality pairing

$$\big((p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \otimes O_{X \times X}(\Delta)|_{2\Delta}\big) \otimes \big((p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \otimes O_{X \times X}(\Delta)|_{2\Delta}\big)^* \rightarrow O_{2\Delta} .$$

Since $h_E(\beta_E) = Id_E$ (see (3.9)), and the restriction of $\sigma_0$ to $\Delta \subset 2\Delta$ coincides with the section of $((p_1^* K_{X}^{1/2}) \otimes (p_2^* K_{X}^{1/2}) \otimes O_{X \times X}(\Delta)|_{\Delta}$ given by its trivialization, we conclude that

$$\hat{\beta}_E|_{\Delta} = Id_E ,$$

using the natural identification of $((p_1^* E) \otimes (p_2^* E^*))|_{\Delta} \rightarrow \Delta$ with $\text{End}(E) \rightarrow X$. Consequently, the section $\hat{\beta}_E$ in (3.9) defines a holomorphic connection on the holomorphic vector bundle $E$, following the idea of Grothendieck of defining a connection as an extension, to the first order neighborhood of the diagonal, of the isomorphism of the two pullbacks on the diagonal (see [De] p. 6, 2.2.4). This holomorphic connection on $E$ defined by $\hat{\beta}_E$ will be denoted by

$$\hat{\beta}_E .$$

(3.10)
For notational convenience, let
\[ M^0 := \mathcal{M} \setminus D_\Theta \] (3.11)
denote the complement of \( D_\Theta \) in \( \mathcal{M} \). Define
\[ C^0 := \varphi^{-1}(M^0) \subset \mathcal{C} , \]
where \( \varphi \) is the projection in (2.1). Let
\[ \hat{\varphi} : C^0 \longrightarrow \mathcal{M} \] (3.12)
be the restriction of the map \( \varphi \) to \( C^0 \). We note that \( C^0 \) is a holomorphic torsor over \( M^0 \) for \( T^*\mathcal{M}^0 \).

We have the holomorphic map
\[ \phi : \mathcal{M}^0 \longrightarrow C^0 , \ E \mapsto (E, \hat{\beta}_E^0) , \] (3.13)
where \( \hat{\beta}_E^0 \) is the holomorphic connection in (3.10). So \( \phi \) is a holomorphic section of the projection \( \hat{\varphi} \) in (3.12), meaning \( \hat{\varphi} \circ \phi = \text{Id}_{\mathcal{M}^0} \).

3.2. A holomorphic isomorphism of torsors. Define
\[ \text{Conn}(\Theta)^0 := q^{-1}(M^0) \subset \text{Conn}(\Theta) , \]
where \( q \) is the projection in (2.1), and \( M^0 \) is the Zariski open subset of \( \mathcal{M} \) in (3.11). Let
\[ \hat{q} : \text{Conn}(\Theta)^0 \longrightarrow \mathcal{M}^0 \] (3.14)
be the restriction of the map \( q \) to \( \text{Conn}(\Theta)^0 \). We note that \( \text{Conn}(\Theta)^0 \) is a holomorphic torsor over \( \mathcal{M}^0 \) for \( T^*\mathcal{M}^0 \).

The restriction of the line bundle \( \Theta = \mathcal{O}_M(D_\Theta) \) to \( \mathcal{M}^0 \) has a tautological isomorphism with the trivial line bundle \( \mathcal{O}_{\mathcal{M}^0} \). Therefore, the trivial holomorphic connection on \( \mathcal{O}_{\mathcal{M}^0} \), defined by the de Rham differential, produces a holomorphic connection on the restriction \( \Theta|_{\mathcal{M}^0} \). Let
\[ \tau : \mathcal{M}^0 \longrightarrow \text{Conn}(\Theta)^0 \] (3.15)
be the holomorphic section of the projection \( \hat{q} \) in (3.14) given by this tautological connection on \( \Theta|_{\mathcal{M}^0} \).

Let
\[ \delta^0 : C^0 \times_{\mathcal{M}^0} T^*\mathcal{M}^0 \longrightarrow C^0 \quad \text{and} \quad \eta^0 : \text{Conn}(\Theta)^0 \times_{\mathcal{M}^0} T^*\mathcal{M}^0 \longrightarrow \text{Conn}(\Theta)^0 \] (3.16)
be the restrictions of the maps \( \delta \) and \( \eta \) in (2.11). So \( \delta^0 \) and \( \eta^0 \) give the \( T^*\mathcal{M}^0 \)-torsor structures on \( C^0 \) and \( \text{Conn}(\Theta)^0 \) respectively. Similarly,
\[ \mathbf{m}^0 : T^*\mathcal{M}^0 \longrightarrow T^*\mathcal{M}^0 , \ v \mapsto 2r \cdot v \] (3.17)
is the restriction of the map in (2.12).

**Lemma 3.1.** There is a unique holomorphic isomorphism
\[ G : C^0 \longrightarrow \text{Conn}(\Theta)^0 \]
such that

1. \( \hat{\varphi} = \hat{q} \circ G \), where \( \hat{\varphi} \) and \( \hat{q} \) are the projections in (3.12) and (3.14) respectively,
2. \( G \circ \phi = \tau \), where \( \phi \) and \( \tau \) are the sections in (3.13) and (3.15) respectively, and
(3) $G \circ \delta^0 = \eta^0 \circ (G \times m^0)$ as maps from $C^0 \times_M T^*M^0$ to $\text{Conn}(\Theta)^0$, where $\delta^0$, $\eta^0$ and $m^0$ are the maps in (3.16) and (3.17).

**Proof.** This is straightforward. For any stable vector bundle $E \in M^0$ and any 
\[ \nu \in H^0(X, \text{End}(E) \otimes K_X) = T^*_{E}M^0, \]
define 
\[ G(\delta^0(\phi(E)), \nu) = \eta^0(\tau(E), 2r \cdot \nu). \]
Then $G$ is evidently a well defined map from $C^0$ to $\text{Conn}(\Theta)^0$. It is holomorphic because $\phi$, $\tau$, $\delta^0$ and $\eta^0$ are all holomorphic maps. This map $G$ satisfies all the three conditions in the lemma. The uniqueness of $G$ is evident. \qed

4. THE TWO ISOMORPHISMS OF TORSORS COINCIDE

The following theorem is the main result proved here.

**Theorem 4.1.** The restriction of the isomorphism $F$ in Theorem 2.1 to the open subset $C^0 \subset C$ coincides with the isomorphism $G$ in Lemma 3.1.

**Proof.** In view of the first condition in both Theorem 2.1 and Lemma 3.1 we get a map 
\[ \Gamma_0 : C^0 \rightarrow T^*M^0, \quad z \mapsto F(z) - G(z). \] (4.1)
In other words, $F(z) = \eta^0(G(z), \Gamma_0(z))$. This map $\Gamma_0$ is holomorphic because both $F$ and $G$ are so. Take any $E \in M^0$ and any 
\[ \alpha \in \tilde{\varphi}^{-1}(E) \subset C^0, \]
where $\tilde{\varphi}$ is the projection in (3.12), and also take any 
\[ \nu \in H^0(X, \text{End}(E) \otimes K_X) = T^*_{E}M^0. \]
Now from the third condition in both Theorem 2.1 and Lemma 3.1, we have 
\[ \Gamma_0(\alpha + \nu) = F(\alpha + \nu) - G(\alpha + \nu) = F(\alpha) - G(\alpha) + 2r \cdot \nu - 2r \cdot \nu = \Gamma_0(\alpha). \]
Consequently, the map $\Gamma_0$ in (4.1) produces a holomorphic 1–form 
\[ \Gamma \in H^0(M^0, T^*M^0) \] (4.2)
that sends any $E \in M^0$ to $\Gamma_0(\alpha) \in T^*_{E}M^0$ with $\alpha \in \tilde{\varphi}^{-1}(E)$; as shown above, $\Gamma_0(\alpha)$ is independent of the choice of $\alpha$. In other words, $\Gamma_0 = \Gamma \circ \tilde{\varphi}$, where $\tilde{\varphi}$ is the projection in (3.12).

The following proposition would be used in the proof of Theorem 4.1.

**Proposition 4.2.** The holomorphic 1–form $\Gamma$ on $M^0$ in (4.2) is a meromorphic 1–form on $M$, and its order of pole at the divisor $D_\Theta = M \setminus M^0$ is at most one, or equivalently, 
\[ \Gamma \in H^0(M, (T^*M) \otimes \Theta) = H^0(M, (T^*M) \otimes \mathcal{O}_M(D_\Theta)). \]

**Proof of Proposition 4.2.** Let $W \rightarrow M$ be a holomorphic torsor for $T^*M$ and $s$ a holomorphic section of $W$ over the open subset $M^0 = M \setminus D_\Theta$. Then the meromorphicity of $s$ is defined by choosing holomorphic trivializations of $W$ on open neighborhoods, in $M$, of points of $D_\Theta$ (a trivialization of a torsor is just a holomorphic section of it). Such a trivialization of $W$ over $U \subset M$ turns $s$ into a holomorphic 1–form on $U \cap M^0$; define $s|_U$ to be meromorphic if this holomorphic 1–form on $U \cap M^0$ is meromorphic near
$D_\Theta \cap U \subset U$. Since any two holomorphic trivializations, over $U$, of the torsor $\mathbb{W}$ differ by a holomorphic 1–form on $U$, this definition of the meromorphicity of $s|_U$ does not depend on the choice of the trivialization of $\mathbb{W}|_U$. For the same reason, the order of pole at $D_\Theta$ of a meromorphic section $s$ of $\mathbb{W}$ of the above type is also well-defined.

Let $\varpi_1$ be the smooth $(1,0)$–form on $\mathcal{M}^0$ given by $\phi - \psi_U|_{\mathcal{M}^0}$, where $\phi$ (respectively, $\psi_U$) is the section of the $T^*\mathcal{M}^0$–torsor $\mathcal{C}^0$ (respectively, $T^*\mathcal{M}$–torsor $\mathcal{C}$) constructed in (3.13) (respectively, (2.2)). Let $\varpi_2$ be the smooth $(1,0)$–form on $\mathcal{M}^0$ given by $\tau - \psi_Q|_{\mathcal{M}^0}$, where $\tau$ (respectively, $\psi_Q$) is the section of the $T^*\mathcal{M}^0$–torsor $\text{Conn}(\Theta)^0$ (respectively, $T^*\mathcal{M}$–torsor $\text{Conn}(\Theta)$) constructed in (3.13) (respectively, (2.2)). It can be shown that

$$\Gamma = 2r \cdot \varpi_1 - \varpi_2.$$  \hfill (4.3)

Indeed, using the third property in Theorem 2.1 and the third property in Lemma 3.1 we have

$$2r \cdot \varpi_1 - \varpi_2 = 2r(\phi - \psi_U|_{\mathcal{M}^0}) - (\tau - \psi_Q|_{\mathcal{M}^0}) = G(\phi) - G(\psi_U|_{\mathcal{M}^0}) - \tau + \psi_Q|_{\mathcal{M}^0}$$

$$= \psi_Q|_{\mathcal{M}^0} - G(\psi_U|_{\mathcal{M}^0}) = F(\psi_U|_{\mathcal{M}^0}) - G(\psi_U|_{\mathcal{M}^0}) = \Gamma.$$  \hfill (4.3)

Both $\psi_U$ and $\psi_Q$ are smooth sections over entire $\mathcal{M}$. The holomorphic section $\tau$ of $\text{Conn}(\Theta)^0$ is a meromorphic section of $\text{Conn}(\Theta)$ with a pole of order one at $D_\Theta$. Indeed, this follows immediately from the fact that the holomorphic connection on the line bundle $\Theta|_{\mathcal{M}^0} = \mathcal{O}_{\mathcal{M}^0}$ over $\mathcal{M}^0$, given by the canonical holomorphic trivialization of $\Theta|_{\mathcal{M}^0}$ (the holomorphic connection is defined by the de Rham differential), is actually a logarithmic connection on $\Theta$ over $\mathcal{M}$. In view of these, using (4.3) we conclude the following:

- $\Gamma$ is a meromorphic 1–form on $\mathcal{M}$ if and only if the section $\phi$ of $\mathcal{C}^0$ in (3.13) is meromorphic, and
- if $\Gamma$ is meromorphic, and the order of its pole at $D_\Theta$ is more than one, then the order of the pole of $\Gamma$ at $D_\Theta$ coincides with the order of pole of $\phi$ at $D_\Theta$, in particular, the order of the pole of $\phi$ at $D_\Theta$ is more than one.

Therefore, to prove the proposition it suffices to show the following two:

1. the section $\phi$ of $\mathcal{C}^0$ is meromorphic, and
2. the order of pole of $\phi$ at $D_\Theta$ is one.

These will be proved by giving a global construction of $\phi$.

It is known that there is no Poincaré vector bundle over $X \times \mathcal{M}$ [Ra, p. 69, Theorem 2]. However, there is a canonical algebraic vector bundle over $X \times X \times \mathcal{M}$ whose fiber over $X \times X \times \{E\}$ is $E \boxtimes E^* = (p_1^*E) \otimes (p_2^*E^*)$ for every $E \in \mathcal{M}$, where $p_1$ and $p_2$ are the projections in (3.11). This canonical vector bundle on $X \times X \times \mathcal{M}$, which we shall denote by $\mathcal{E}$, can be constructed as a descended bundle from the product of $X \times X$ with the quot scheme. The reason that the corresponding vector bundle descends is that the action of the multiplicative group $\mathbb{C}^*$ on $E \boxtimes E^*$, induced by the scalar multiplications on $E$, is the trivial action. The restriction of the vector bundle $\mathcal{E} \rightarrow X \times X \times \mathcal{M}$ to

$$\Delta \times \mathcal{M} \subset X \times X \times \mathcal{M}$$

coincides with the universal endomorphism bundle over $X \times \mathcal{M}$. Let

$$\mathcal{V} := \mathcal{E}|_{\Delta \times \mathcal{M}} \rightarrow \Delta \times \mathcal{M} = X \times \mathcal{M}$$  \hfill (4.4)
be the universal endomorphism bundle. So we have $V|_{X \times \{E\}} = \text{End}(E)$ for all $E \in \mathcal{M}$. Let $q_{12} : X \times X \times \mathcal{M} \to X \times X$ be the projection to the first two factors in the Cartesian product. Let

$$q_2 : X \times X \times \mathcal{M} \to X, \quad (x, y, E) \mapsto y$$

be the projection to the second factor. Let

$$J : X \times X \times \mathcal{M} \to \mathcal{M}, \quad (x, y, E) \mapsto E$$

(4.5)

be the projection to the third factor. For notational convenience, the holomorphic line bundle $q_{12}^*(p_1^*K_X^{1/2} \otimes p_2^*K_X^{1/2})$ on $X \times X \times \mathcal{M}$ will be denoted by $\mathcal{K}$; recall that $K_X^{1/2}$ is a theta characteristic on $X$.

Consider the vector bundle

$$\mathcal{E} \otimes \mathcal{K} \otimes q_{12}^*\mathcal{O}_{X \times X}(\Delta) \to X \times X \times \mathcal{M}.$$ (4.6)

It fits in the following short exact sequence of coherent sheaves on $X \times X \times \mathcal{M}$:

$$0 \to \mathcal{E} \otimes \mathcal{K} \to \mathcal{E} \otimes \mathcal{K} \otimes q_{12}^*\mathcal{O}_{X \times X}(\Delta) \to \mathcal{V} \to 0,$$ (4.7)

where $\mathcal{V}$ is defined in (4.4), and it is supported on $\Delta \times \mathcal{M} = X \times \mathcal{M} \subset X \times X \times \mathcal{M}$. Recall from Section 3.1 that the restriction of $(p_1^*K_X^{1/2}) \otimes (p_2^*K_X^{1/2}) \otimes \mathcal{O}_{X \times X}(\Delta)$ to $\Delta \subset X \times X$ is identified with $\mathcal{O}_X$; so, the restriction of $\mathcal{E} \otimes \mathcal{K} \otimes q_{12}^*\mathcal{O}_{X \times X}(\Delta)$ to $\Delta \times \mathcal{M}$ is identified with $\mathcal{E}|_{\Delta \times \mathcal{M}} = \mathcal{V}$. Now consider the long exact sequence of direct images, for the projection $J$ in (4.5), corresponding to the short exact sequence of sheaves in (4.7):

$$0 \to J_*(\mathcal{E} \otimes \mathcal{K}) \to J_*(\mathcal{E} \otimes \mathcal{K} \otimes q_{12}^*\mathcal{O}_{X \times X}(\Delta)) \to J_*\mathcal{V} \to 0,$$ (4.8)

$$\to R^1J_*(\mathcal{E} \otimes \mathcal{K}) \to \ldots.$$

First note that $J_*(\mathcal{E} \otimes \mathcal{K}) = 0$, because for every $E \in \mathcal{M}^0$, we have

$$H^0(X \times X, (p_1^*(E \otimes K_X^{1/2})) \otimes (p_2^*(E^* \otimes K_X^{1/2}))) = 0$$

(see (3.5)). Also, $J_*\mathcal{V} = \mathcal{O}_\mathcal{M}$, because every stable vector bundle is simple. Consequently, from (4.8) we have the exact sequence

$$0 \to J_*(\mathcal{E} \otimes \mathcal{K} \otimes q_{12}^*\mathcal{O}_{X \times X}(\Delta)) \to \mathcal{O}_\mathcal{M} \to R^1J_*(\mathcal{E} \otimes \mathcal{K}).$$ (4.9)

Next we note that

$$H^1(X \times X, (p_1^*(E \otimes K_X^{1/2})) \otimes (p_2^*(E^* \otimes K_X^{1/2}))) = 0$$

for all $E \in \mathcal{M}^0$ (see (3.5)). Also, for a general point $E \in D_\Theta$, using (3.4) it follows that

$$\dim H^1(X \times X, (p_1^*(E \otimes K_X^{1/2})) \otimes (p_2^*(E^* \otimes K_X^{1/2}))) = 1.$$

Consequently, the support of $R^1J_*(\mathcal{E} \otimes \mathcal{K})$ is the divisor $D_\Theta$, and the rank of the sheaf

$$R^1J_*(\mathcal{E} \otimes \mathcal{K}) \to D_\Theta$$

is one.

Let $1_\mathcal{M}$ be the section of $\mathcal{O}_\mathcal{M}$ given by the constant function 1 on $\mathcal{M}$. Since $R^1J_*(\mathcal{E} \otimes \mathcal{K})$ is supported on $D_\Theta$, from (3.11) we conclude the following:
The restriction $1_M$ to $\mathcal{M}^0 = \mathcal{M} \setminus D_\Theta \subset \mathcal{M}$ is a holomorphic section of

$$(J_*(\mathcal{E} \otimes \mathcal{K} \otimes q_{12}^*\mathcal{O}_{X \times X}(\Delta)))|_{\mathcal{M}^0} \longrightarrow \mathcal{M}^0$$

(more precisely, $1_M|_{\mathcal{M}^0}$ is the image of a holomorphic section of $(J_*(\mathcal{E} \otimes \mathcal{K} \otimes q_{12}^*\mathcal{O}_{X \times X}(\Delta)))|_{\mathcal{M}^0}$); this section of $(J_*(\mathcal{E} \otimes \mathcal{K} \otimes q_{12}^*\mathcal{O}_{X \times X}(\Delta)))|_{\mathcal{M}^0}$ given by $1_M$ will be denoted by $1'_M$.

The above defined $1'_M$ is a meromorphic section of $J_*(\mathcal{E} \otimes \mathcal{K} \otimes q_{12}^*\mathcal{O}_{X \times X}(\Delta))$ with a pole of order one on $D_\Theta$.

In other words, we have

$$1'_M \in H^0(\mathcal{M}, J_*(\mathcal{E} \otimes \mathcal{K} \otimes q_{12}^*\mathcal{O}_{X \times X}(\Delta)) \otimes \mathcal{O}_M(D_\Theta)). \quad (4.10)$$

Now using the projection formula we have

$$J_*(\mathcal{E} \otimes \mathcal{K} \otimes q_{12}^*\mathcal{O}_{X \times X}(\Delta)) \otimes \mathcal{O}_M(D_\Theta) = J_*(\mathcal{E} \otimes \mathcal{K} \otimes (q_{12}^*\mathcal{O}_{X \times X}(\Delta)) \otimes J^*\mathcal{O}_M(D_\Theta)),$$

and hence $1'_M$ in $(4.10)$ defines a section

$$1''_M \in H^0(X \times X \times \mathcal{M}, \mathcal{E} \otimes \mathcal{K} \otimes (q_{12}^*\mathcal{O}_{X \times X}(\Delta)) \otimes J^*\mathcal{O}_M(D_\Theta)). \quad (4.11)$$

For every $E \in \mathcal{M}^0$, the section $\beta_E$ in $(3.6)$ coincides with the restriction $1''_M|_{X \times X \times \{E\}}$, where $1''_M$ is the section in $(4.11)$. Now from the construction in $(3.13)$ of the section $\phi$ of the $T^*\mathcal{M}^0$–torsor $\mathcal{C}^0$ it follows that

1. $\phi$ is meromorphic, and
2. the order of pole, at $D_\Theta$, of $\phi$ is one.

As noted before, Proposition 4.2 follows from these two.

Continuing with the proof Theorem 4.1, let

$$\Psi : \mathcal{M} \longrightarrow J(X) = \text{Pic}^0(X) \quad (4.12)$$

be the determinant map $E \longmapsto \bigwedge^r E$. The image of the pullback homomorphism

$$(d\Psi)^* : \Psi^*T^*J(X) \longrightarrow T^*\mathcal{M},$$

where $d\Psi$ is the differential of $\Psi$, has a canonical direct summand; we shall now recall a description of this direct summand.

As in $(4.4)$, let $\mathcal{V} \longrightarrow X \times \mathcal{M}$ be the universal endomorphism bundle, and let

$$\mathcal{V}^0 \subset \mathcal{V}$$

be the universal endomorphism bundle of trace zero. There is a natural decomposition into traceless and trace components:

$$\mathcal{V} = \mathcal{V}^0 \oplus \mathcal{O}_{X \times \mathcal{M}}; \quad (4.13)$$

the above inclusion map $\mathcal{O}_{X \times \mathcal{M}} \hookrightarrow \mathcal{V}$ is defined by $f \longmapsto f \cdot \text{Id}$. Let

$$P : X \times \mathcal{M} \longrightarrow \mathcal{M} \quad \text{and} \quad p : X \times \mathcal{M} \longrightarrow X \quad (4.14)$$

be the natural projections. Then we have

$$T^*\mathcal{M} = P_*(\mathcal{V} \otimes p^*K_X),$$
where \( P \) and \( p \) are the projections in (4.14). Consequently, the decomposition in (4.13) produces a holomorphic decomposition
\[
T^*\mathcal{M} = P_*(\mathcal{V}^0 \otimes p^*K_X) \oplus P_*p^*K_X; \tag{4.15}
\]
we note that \( P_*p^*K_X \) is the trivial holomorphic vector bundle
\[
\mathcal{M} \times H^0(X, K_X) \rightarrow \mathcal{M}
\]
with fiber \( H^0(X, K_X) \). Tensoring (4.15) with \( \Theta \) we obtain
\[
(T^*\mathcal{M}) \otimes \Theta = P_*(\mathcal{V}^0 \otimes p^*K_X) \otimes \Theta \oplus (\mathcal{M} \times H^0(X, K_X)) \otimes \Theta.
\]
This produces a decomposition
\[
H^0(\mathcal{M}, (T^*\mathcal{M}) \otimes \Theta) = H^0(\mathcal{M}, P_*(\mathcal{V}^0 \otimes p^*K_X) \otimes \Theta) \oplus (H^0(\mathcal{M}, \Theta) \otimes H^0(X, K_X))
\]
\[
= H^0(\mathcal{M}, P_*(\mathcal{V}^0 \otimes p^*K_X) \otimes \Theta) \oplus H^0(X, K_X); \tag{4.16}
\]
the last equality follows from the fact that \( H^0(\mathcal{M}, \Theta) = \mathbb{C} \) [BNR, p. 169, Theorem 2].

We note that the inclusion map
\[
H^0(X, K_X) = H^0(J(X), T^*J(X)) \hookrightarrow H^0(\mathcal{M}, (T^*\mathcal{M}) \otimes \Theta)
\]
in (4.16) coincides with the pullback of 1–forms on \( J(X) \) to \( \mathcal{M} \) by the projection \( \Psi \) in (4.12).

The following proposition would be used in the proof of Theorem 4.1.

**Proposition 4.3.** For the projections \( P \) and \( p \) in (4.14),
\[
H^0(\mathcal{M}, P_*(\mathcal{V}^0 \otimes p^*K_X) \otimes \Theta) = 0,
\]
where \( \mathcal{V}^0 \) is the subbundle in (4.13).

**Proof of Proposition 4.3.** If \( r = 1 \), then \( \mathcal{V}^0 = 0 \), and hence in this case the proposition is obvious. Hence in the proof we assume that \( r \geq 2 \). The proof proceeds by showing that the sections must vanish on the projective spaces lying inside \( \mathcal{M} \) given by Hecke transforms.

Let
\[
H : \mathbb{P} \longrightarrow X \times \mathcal{M} \tag{4.17}
\]
be the universal projective bundle; so for any \((x, E) \in X \times \mathcal{M}\), the inverse image \( H^{-1}(x, E) \) is the space of all hyperplanes in the fiber \( E_x \); in particular, \( \mathbb{P} \) is a holomorphic fiber bundle over \( X \times \mathcal{M} \) with the projective space \( \mathbb{C}\mathbb{P}^{r-1} \) as the typical fiber. Let
\[
T_H = \text{kernel}(dH) \rightarrow \mathbb{P}
\]
be the (holomorphic) relative tangent bundle for the projection \( H \) in (4.17), where \( dH \) is the differential of the map \( H \). We note that \( \mathcal{V}^0 \) in (4.13) is the following direct image:
\[
H_*T_H = \mathcal{V}^0 \longrightarrow X \times \mathcal{M}.
\]

Given any element \((x, E) \in X \times \mathcal{M}\), along with a hyperplane \( S \subset E_x \), let \( F(x, E, S) \) be the holomorphic vector bundle over \( X \) whose sheaf of sections fits in the short exact sequence of coherent sheaves on \( X \)
\[
0 \longrightarrow F(x, E, S) \longrightarrow E \longrightarrow E_x/S \longrightarrow 0;
\]
the above sheaf \( E_x/S \) is the torsion sheaf supported at the point \( x \) and its stalk at \( x \) is the quotient line \( E_x/S \).
Let \( \mathcal{N} \) denote the moduli space of stable vector bundles over \( X \) of rank \( r \) and degree \(-1\). Using the above construction of \( F(x, E, S) \), we get a rational map
\[
\xi : \mathbb{P} \rightarrow X \times \mathcal{N}, \quad (x, E, S) \mapsto (x, F(x, E, S)),
\]
which is called the Hecke morphism \([NR1], [NR2]\). It is known that there is a nonempty Zariski open subset \( U \subset \mathbb{P} \) such that the pair \((\xi, U)\) satisfies the following conditions:

1. The rational map \( \xi \) is actually defined as a map on \( U \); the restriction of \( \xi \) to \( U \) will be denoted by \( \hat{\xi} \).
2. The codimension of the complement \( \mathbb{P} \setminus U \) is at least two (see the proof of \([NR2]\), Proposition 5.4).
3. The map \( \hat{\xi} : U \rightarrow \xi(U) \) defines a holomorphic fiber bundle over \( \xi(U) \) with the projective space \( \mathbb{CP}^{r-1} \) as the typical fiber \([NR2]\) p. 411, Proposition 6.8).
4. The relative tangent bundle \( T_H \) on \( U \) coincides with \( \Omega_{\hat{\xi}} \otimes (p \circ H)^* K_X \), where \( \Omega_{\hat{\xi}} \) is the relative cotangent bundle for the map \( \hat{\xi} \) (see \([Bi1]\) p. 265, (2.7)).

It may be clarified that the above vector bundle \( \Omega_{\hat{\xi}} \) is the cokernel of the pullback homomorphism
\[
d(\hat{\xi})^* : \hat{\xi}^* T^*(\xi(U)) \rightarrow T^* U.
\]

Take a point
\[
z = (x, W) \in \hat{\xi}(U) \subset X \times \mathcal{N}.
\]
Let
\[
\mathbb{F}_z = \hat{\xi}^{-1}(z) \subset U
\]
be the fiber of \( \hat{\xi} \) over \( z \); as mentioned in (3) above, this fiber is isomorphic to \( \mathbb{CP}^{r-1} \). We will compute the restriction of the line bundle \((P \circ H)^* \Theta\) to \( \mathbb{F}_z \cong \mathbb{CP}^{r-1}\), where \( P \) and \( H \) are the projections in (4.14) and (4.17) respectively.

Let \( P(W_x) \) be the projective space that parametrizes all the lines in the fiber \( W_x \) of the vector bundle \( W \) in (4.18). Let \( L_0 \rightarrow P(W_x) \) be the tautological line bundle of degree \(-1\); the fiber of \( L_0 \) over any line \( \zeta \subset W_x \) is \( \zeta \) itself. The inverse image \( \mathbb{F}_z = \hat{\xi}^{-1}(z) \) is identified with this projective space \( P(W_x) \). For any \( \zeta \in P(W_x) \), the corresponding element \( (x, E, S) \in \mathbb{F}_z \subset \mathbb{P} \) is uniquely determined by the following condition: The holomorphic vector bundle \( E \) fits in the short exact sequence of sheaves on \( X \)
\[
0 \rightarrow W \rightarrow E \rightarrow \mathcal{Q} \rightarrow 0,
\]
where \( \mathcal{Q} \) is a torsion sheaf of degree one supported at \( x \), and the kernel of the homomorphism of fibers \( W_x \rightarrow E_x \), given by the above homomorphism \( W \rightarrow E \) of sheaves, is the line \( \zeta \), while the subspace \( S \subset E_x \) is the image of this homomorphism \( W_x \rightarrow E_x \).

To describe the fiber \( \mathbb{F}_z \) globally, let \( \Pi_1 \) (respectively, \( \Pi_2 \)) be the projection of \( X \times P(W_x) \) to \( X \) (respectively, \( P(W_x) \)). On \( X \times P(W_x) \) we have the holomorphic vector bundle \( \mathcal{W} \) which is defined by the short exact sequence of sheaves on \( X \times P(W_x) \)
\[
0 \rightarrow \mathcal{W}^* \rightarrow \Pi_1^* W^* \rightarrow \iota^*_x L_0^* \rightarrow 0,
\]
where \( \iota^x : P(W_x) \to X \times P(W_x) \) is the embedding defined by \( y \mapsto (x, y) \). From this exact sequence it follows that \( \mathcal{W} \) fits in the short exact sequence of sheaves
\[
0 \to \Pi_1^*W \to \mathcal{W} \to \iota^x_0L_0 \to 0 \tag{4.19}
\]
on \( X \times P(W_x) \). The map \( H|_{\mathbb{F}_z} \) coincides with the classifying map
\[
\mathbb{F}_z \to \mathcal{U} \subset \mathcal{M}
\]
for the above holomorphic family of vector bundles \( \mathcal{W} \) on \( X \) parametrized by \( P(W_x) = \mathbb{F}_z \).

Now, tensoring the exact sequence in (4.19) with \( \Pi_1^*K^{1/2}_X \), and then taking the long exact sequence of direct images with respect to the projection \( \Pi_2 \), we have the exact sequence of sheaves on \( P(W_x) \)
\[
0 \to \Pi_2^*(\Pi_1^*(W \otimes K^{1/2}_X)) \to \Pi_2^*(\mathcal{W} \otimes \Pi_1^*K^{1/2}_X) \to L_0 \to 0
\]
\[
\to R^1\Pi_2^*(\Pi_1^*(W \otimes K^{1/2}_X)) \to R^1\Pi_2^*(\mathcal{W} \otimes \Pi_1^*K^{1/2}_X) \to 0 ; \tag{4.20}
\]
and also note that
\[
R^1\Pi_2^*((\iota^x_0L_0) \otimes \Pi_1^*K^{1/2}_X) = 0
\]
because the support of \( (\iota^x_0L_0) \otimes \Pi_1^*K^{1/2}_X \) is finite over \( P(W_x) \).

Since \( H|_{\mathbb{F}_z} \) coincides with the classifying map for the above holomorphic family of vector bundles \( \mathcal{W} \) on \( X \) parametrized by \( P(W_x) \), it follows that the pulled back line bundle \( ((P \circ H)^*\Theta)|_{\mathbb{F}_z} \), where \( P \) is the projection in (4.14), is identified with the line bundle
\[
((P \circ H)^*\Theta)|_{\mathbb{F}_z} = \det(\Pi_2^*(\mathcal{W} \otimes \Pi_1^*K^{1/2}_X))^* \otimes \det(R^1\Pi_2^*(\mathcal{W} \otimes \Pi_1^*K^{1/2}_X)) \tag{4.21}
\]
(see \cite{Ko} Ch. V, § 6 for the construction of determinant bundle). For any exact sequence of coherent sheaves
\[
0 \to A_1 \to A_2 \to \ldots \to A_m \to 0
\]
on a complex manifold \( Y \), we have \( \bigotimes_{i=1}^m (\det(A_i))^{(-1)^i} = \mathcal{O}_Y \) \cite{Ko} p. 165, Proposition (6.9)]. Consequently, from (4.20) and (4.21) we conclude that
\[
((P \circ H)^*\Theta)|_{\mathbb{F}_z} = L_0^*,
\]
because both \( \Pi_2^*(\Pi_1^*(W \otimes K^{1/2}_X)) \) and \( R^1\Pi_2^*(\Pi_1^*(W \otimes K^{1/2}_X)) \) are trivial vector bundles. In other words, the degree of the line bundle \( (P \circ H)^*\Theta \) restricted to \( \mathbb{F}_z = P(W_x) \) is 1.

Using the above properties of \( (\xi, \mathcal{U}) \) we are in a position to complete the proof of the proposition.

We have
\[
H^0(\mathcal{M}, P_*(\mathcal{V} \otimes p^*K_X) \otimes \Theta) = H^0(X \times \mathcal{M}, \mathcal{V} \otimes (p^*K_X) \otimes (P^*\Theta)) , \tag{4.22}
\]
because \( P_*(\mathcal{V} \otimes (p^*K_X) \otimes (P^*\Theta)) = P_*(\mathcal{V} \otimes p^*K_X) \otimes \Theta \) by the projection formula. Next we have
\[
H^0(X \times \mathcal{M}, \mathcal{V} \otimes (p^*K_X) \otimes (P^*\Theta)) = H^0(\mathbb{P}, T_H \otimes ((P \circ H)^*K_X) \otimes (P \circ H)^*\Theta)
\]
\[
= H^0(\mathcal{U}, T_H \otimes ((P \circ H)^*K_X) \otimes (P \circ H)^*\Theta) ; \tag{4.23}
\]
the first equality follows from the fact that 
\[ H_*(T_H \otimes ((p \circ H)^*K_X) \otimes (P \circ H)^*\Theta) = \Gamma^0 \otimes (p^*K_X) \otimes (P^*\Theta) \] 
(by the projection formula), and the second equality follows from the fact that the codimension of the complement \( \mathbb{P} \setminus U \) is at least two.

As before, take a fiber \( F_z = \hat{\xi}^{-1}(z) \) of the map \( \hat{\xi} \). As shown above, \( F_z \) is identified with the projective space \( \mathbb{P}(W_x) \), and the restriction of \( T_H \) (respectively, \( (P \circ H)^*\Theta \)) to \( F_z \) is isomorphic to \( T^*F_z \) (respectively, \( O_{F_z}(1) \)); note that the restriction of \( (p \circ H)^*K_X \) to \( \hat{\xi}^{-1}(z) \) is a trivial line bundle. Consequently, the restriction of \( T_H \otimes ((p \circ H)^*K_X) \otimes (P \circ H)^*\Theta \) to \( F_z \) is isomorphic to \( T^*F_z \otimes O_{F_z}(1) \).

Next we note that the holomorphic vector bundle \( T^*F_z \otimes O_{F_z}(1) \) on the projective space \( F_z = \mathbb{P}(W_x) \) is semistable of negative degree (its degree is \( -1 \)), and hence the vector bundle \( T^*F_z \otimes O_{F_z}(1) \) does not have any nonzero holomorphic section. This implies that \( H^0(U, T_H \otimes ((p \circ H)^*K_X) \otimes (P \circ H)^*\Theta) = 0 \).

Consequently, from (4.23) and (4.22) we now conclude that 
\[ H^0(M, P_*(\Gamma^0 \otimes p^*K_X) \otimes \Theta) = 0. \]

This completes the proof of Proposition 4.3.

We continue with the proof of Theorem 4.1. Combining (4.16) with Proposition 4.3, it follows that we are reduced to the trace component:
\[ H^0(M, (T^*M) \otimes \Theta) = \{ \Psi^*\omega \mid \omega \in H^0(J(X), T^*J(X)) \} = H^0(X, K_X), \]
where \( \Psi \) is the projection in (4.12).

Let
\[ \Gamma' \in H^0(J(X), T^*J(X)) = H^0(X, K_X) \]
be the 1–form corresponding to the section \( \Gamma \) in Proposition 4.2 for the isomorphism in (4.24).

The proof of Theorem 4.1 will be completed using the following lemma.

**Lemma 4.4.** The 1–form \( \Gamma' \) on \( J(X) \) in (4.25) is invariant under the holomorphic involution
\[ \iota_J : J(X) \longrightarrow J(X) \]
declared by \( L \mapsto L^* \).

**Proof of Lemma 4.4.** Let \( \iota_M : M \longrightarrow M \) be the holomorphic involution defined by \( E \mapsto E^* \). Note that
\[ \iota_J \circ \Psi = \Psi \circ \iota_M, \]
where \( \Psi \) is constructed in (4.12) and \( \iota_J \) is defined in the statement of the lemma. By Serre duality,
\[ H^k(X, E^* \otimes K_X^{1/2}) = H^{1-k}(X, E \otimes K_X^{1/2})^* \]
for \( k = 0, 1 \). This implies that the above involution \( \iota_M \) preserves the divisor \( D_\Theta \) defined in (2.4). Since \( D_\Theta \) is preserved by \( \iota_M \), the involution \( \iota_M \) has a tautological lift to the line bundle \( \Theta = O_M(D_\Theta) \). Let
\[ \iota_\Theta : \Theta \longrightarrow \Theta \]
be the resulting involution of $\Theta$ over the involution $\iota_{M}$ of $\mathcal{M}$. This involution $\iota_{\Theta}$ of $\Theta$ produces a holomorphic involution of the complex manifold Conn($\Theta$) constructed in (2.7). The involution of Conn($\Theta$) constructed this way will be denoted by $\iota_{T}$. We note that

$$\iota_{M} \circ q = q \circ \iota_{T},$$

where $q$ is the projection in (2.7). This implies that the involution $\iota_{T}$ preserves the open subset Conn($\Theta$)$^{0} = q^{-1}(M \setminus D_{\Theta})$ in (3.14).

The involution $\iota_{T}$ (respectively, $\iota_{M}$) defines an action of $\mathbb{Z}/2\mathbb{Z}$ on Conn($\Theta$) (respectively, $\mathcal{M}$). The section $\tau$ in (3.15) is evidently $\mathbb{Z}/2\mathbb{Z}$–equivariant, for the actions of $\mathbb{Z}/2\mathbb{Z}$ on Conn($\Theta$)$^{0}$ and $\mathcal{M}^{0}$.

For any $E \in \mathcal{M}$, the fiber $\Theta_{E}$ of $\Theta$ over $E$ is the line $\wedge^{top} H^{0}(X, E \otimes K_{X}^{1/2}) \otimes \wedge^{top} H^{1}(X, E \otimes K_{X}^{1/2})$. Using (4.26) we get an isomorphism of $\Theta_{E}$ with the fiber $\Theta_{E^{*}}$. Also, the involution $\iota_{\Theta}$ of $\Theta$ in (4.27) produces an isomorphism of $\Theta_{E}$ with $\Theta_{E^{*}}$. These two isomorphisms between $\Theta_{E}$ and $\Theta_{E^{*}}$ actually coincide.

The Kähler form $\omega_{\mathcal{M}}$ on $\mathcal{M}$ (see (2.8)) is clearly preserved by the involution $\iota_{M}$ of $\mathcal{M}$. From this it can be deduced that the section $\psi_{Q}$ in (2.9) is $\mathbb{Z}/2\mathbb{Z}$–equivariant, for the above actions of $\mathbb{Z}/2\mathbb{Z}$ on Conn($\Theta$) and $\mathcal{M}$. Indeed, the section $\psi_{Q}$ corresponds to the unique Hermitian connection on $\Theta$ whose curvature is the Kähler form $\omega_{\mathcal{M}}$. In other words, the section $\psi_{Q}$ is uniquely determined by $\omega_{\mathcal{M}}$. Therefore, the section $\psi_{Q}$ in (2.9) is $\mathbb{Z}/2\mathbb{Z}$–equivariant, because $\omega_{\mathcal{M}}$ is preserved by $\iota_{M}$.

Given a holomorphic connection $\nabla$ on a holomorphic vector bundle $E$, the dual vector bundle $E^{*}$ is equipped with the dual connection $\nabla^{*}$. Therefore, we have a holomorphic involution

$$\iota_{C} : C \rightarrow C, \ (E, \nabla) \mapsto (E^{*}, \nabla^{*}).$$

The involution $\iota_{C}$ gives an action of $\mathbb{Z}/2\mathbb{Z}$ on $C$. The projection $\varphi$ in (2.11) is clearly $\mathbb{Z}/2\mathbb{Z}$–equivariant, for the actions of $\mathbb{Z}/2\mathbb{Z}$ on $C$ and $\mathcal{M}$. In particular, $\iota_{C}$ preserves the Zariski open subset $C^{0}$ in (3.12). Since the dual of a unitary connection on $E$ is a unitary connection on $E^{*}$, the section $\psi_{U}$ in (2.2) is $\mathbb{Z}/2\mathbb{Z}$–equivariant, for the actions of $\mathbb{Z}/2\mathbb{Z}$ on $C$ and $\mathcal{M}$.

Let $\hat{\iota} : X \times X \times \mathcal{M} \rightarrow X \times X \times \mathcal{M}$ be the holomorphic involution defined by $(x, y, E) \mapsto (y, x, \iota_{M}(E)) = (y, x, E^{*})$. This involution naturally lifts to an involution of the vector bundle $\mathcal{E} \otimes \mathcal{K} \otimes q_{12}^{*}\mathcal{O}_{X \times X}(\Delta)$ in (4.16). The earlier mentioned involution $\iota_{\Theta}$ of the line bundle $\Theta$ produces an action of $\mathbb{Z}/2\mathbb{Z}$ on the pullback $J^{*}\Theta$, where $J$ is the projection in (1.5). These actions of $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{E} \otimes \mathcal{K} \otimes q_{12}^{*}\mathcal{O}_{X \times X}(\Delta)$ and $J^{*}\Theta$ together produce an action of $\mathbb{Z}/2\mathbb{Z}$ on the tensor product

$$\mathcal{E} \otimes \mathcal{K} \otimes (q_{12}^{*}\mathcal{O}_{X \times X}(\Delta)) \otimes J^{*}\Theta \rightarrow X \times X \times \mathcal{M}.$$

The section $1_{M}'$ in (4.11) of this tensor product is anti-invariant for the above action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{E} \otimes \mathcal{K} \otimes (q_{12}^{*}\mathcal{O}_{X \times X}(\Delta)) \otimes J^{*}\Theta$ (meaning the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts as multiplication by $-1$). It follows from this that the section $\phi$ in (3.13) is $\mathbb{Z}/2\mathbb{Z}$–equivariant, for the actions of $\mathbb{Z}/2\mathbb{Z}$ on $C^{0}$ and $\mathcal{M}^{0}$.

From all these it follows that $\Gamma_{0}$ (constructed in (4.11)) is $\mathbb{Z}/2\mathbb{Z}$–equivariant, for the actions of $\mathbb{Z}/2\mathbb{Z}$ on $C^{0}$ and $T^{*}\mathcal{M}^{0}$; the action of $\mathbb{Z}/2\mathbb{Z}$ on $T^{*}\mathcal{M}^{0}$ is induced by the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{M}^{0}$ constructed using the above involution $\iota_{M}$. Since $\Gamma_{0}$ is $\mathbb{Z}/2\mathbb{Z}$–equivariant, it follows that $\Gamma$ in Proposition 4.2 is $\mathbb{Z}/2\mathbb{Z}$–invariant for the action on $(T^{*}\mathcal{M}) \otimes \Theta$.
constructed using the actions of $\mathbb{Z}/2\mathbb{Z}$ on $T^*\mathcal{M}$ and $\Theta$ (given by $\iota_\Theta$ in (4.27)). This immediately implies that $\Gamma'$ in (4.25) is left invariant under the involution $\iota_J$ of $J(X)$. This completes the proof of Lemma 4.4.

Continuing with the proof of Theorem 4.1, we note that $\iota^*_J\alpha = -\alpha$ for all $\alpha \in H^0(J(X), T^*J(X))$, where $\iota_J$ is the involution in Lemma 4.4. Hence from Lemma 4.4 it follows immediately that $\Gamma' = 0$. In view of (4.24), this implies that $\Gamma$ in Proposition 4.2 vanishes identically. Hence $\Gamma_0$ in (4.1) vanishes identically. Therefore, we conclude that the restriction, to the open subset $\mathcal{C}^0 \subset \mathcal{C}$, of the isomorphism $F$ in Theorem 2.1 coincides with the isomorphism $G$ in Lemma 3.1. This completes the proof of Theorem 4.1.

Theorem 2.1 and Theorem 4.1 together give the following:

**Corollary 4.5.** The holomorphic isomorphism $G : \mathcal{C}^0 \rightarrow \text{Conn}(\Theta)^0$ in Lemma 3.1 extends to a holomorphic isomorphism $G' : \mathcal{C} \rightarrow \text{Conn}(\Theta)$.

**Proof.** Since $F$ in Theorem 2.1 is a holomorphic isomorphism from $\mathcal{C}$ to $\text{Conn}(\Theta)$, this follows from Theorem 4.1.

The isomorphism $G'$ in Corollary 4.5 has the following property:

**Corollary 4.6.** For the isomorphism $G'$ in Corollary 4.5,

$$(G')^*\Phi_2 = 2r \cdot \Phi_1,$$

where $\Phi_1$ and $\Phi_2$ are the symplectic forms in (2.3) and (2.10) respectively.

**Proof.** The final part of Theorem 2.1 says that $F^*\Phi_2 = 2r \cdot \Phi_1$. Since $F = G'$, the result follows from this.

**Corollary 4.7.** The image of the section $\phi$ in (3.13) is a Lagrangian submanifold of $\mathcal{C}^0$ equipped with the symplectic form $\Phi_1|_{\mathcal{C}^0}$ in (2.3).

**Proof.** In Corollary 4.6 we saw that $G'$ is symplectic structure preserving (up to the factor $2r$). The image of the section $\tau$ in (3.15) is clearly a Lagrangian submanifold of $\text{Conn}(\Theta)^0$ with respect to the symplectic form $\Phi_2|_{\text{Conn}(\Theta)^0}$ in (2.10) (the trivial connection is flat). Since $G(\phi(\mathcal{M}^0)) = \tau(\mathcal{M}^0)$, and $G$ is symplectic structure preserving up to the factor $2r$, from the above observation — that the image of $\tau$ is a Lagrangian submanifold of $\text{Conn}(\Theta)^0$ with respect to the symplectic form $\Phi_2|_{\text{Conn}(\Theta)^0}$ — it follows immediately that $\phi(\mathcal{M}^0)$ is a Lagrangian submanifold of $\mathcal{C}^0$ with respect to the symplectic form $\Phi_1|_{\mathcal{C}^0}$ in (2.3).

5. **Family of Riemann surfaces**

Let $\mathcal{T}$ be a connected complex manifold, and let

$$F : \mathcal{X}_\mathcal{T} \rightarrow \mathcal{T}$$
be a holomorphic family of compact connected Riemann surfaces of genus $g$, with $g \geq 2$, parametrized by $\mathcal{T}$, and equipped with a theta characteristic $L$. This means that $L$ is a holomorphic line bundle over $\mathcal{X}_T$, and there is a given holomorphic isomorphism

$$I : L \otimes L \to K_F,$$

where $K_F \to \mathcal{X}_T$ is the relative holomorphic cotangent bundle for the project $F$; in other words, $K_F$ is the cokernel of the dual of the differential $dF$

$$(dF)^* : F^* T^* \to T^* \mathcal{X}_T.$$

For each point $t \in \mathcal{T}$, the compact Riemann surface $F^{-1}(t)$ will be denoted by $\mathcal{X}_t$. The holomorphic line bundle $L_{\mid \mathcal{X}_t}$ on $\mathcal{X}_t$ will be denoted by $L_t$.

Let

$$\gamma : \mathcal{M}_T \to \mathcal{T}$$

be the relative moduli space of stable vector bundles of rank $r$ and degree zero. So for any $t \in \mathcal{T}$, the fiber $\gamma^{-1}(t)$ is the moduli space of stable vector bundles on $\mathcal{X}_t$ of rank $r$ and degree zero. Let

$$\Theta_T \to \mathcal{M}_T$$

be the relative theta bundle constructed using the relative theta characteristic $L$. So $\Theta_T$ corresponds to the reduced effective divisor on $\mathcal{M}_T$ defined by all $(t, E)$, where $t \in \mathcal{T}$ and $E \in \gamma^{-1}(t)$, such that $H^0(\mathcal{X}_t, E \otimes L_t) \neq 0$.

Let

$$q_T : \text{Conn}^r(\Theta_T) \to \mathcal{M}_T$$

be the holomorphic fiber bundle over $\mathcal{M}_T$ defined by the sheaf of relative holomorphic connections on $\Theta_T$. So for any $t \in \mathcal{T}$, the fiber $(q_T)^{-1}(t)$ is $\text{Conn}(\Theta)$ in (2.7) for $X = \mathcal{X}_t$. The holomorphic fiber bundle in (5.1) has a $C^\infty$ section

$$\hat{\psi}_Q : \mathcal{M}_T \to \text{Conn}^r(\Theta_T)$$

given by the Chern connection associated to the Quillen metric on $\Theta_T$ [Qu]; so for each $t \in \mathcal{T}$, the restriction of $\hat{\psi}_Q$ to $\gamma^{-1}(t)$ is the section $\psi_Q$ in (2.9) for the Riemann surface $X = \mathcal{X}_t$.

Let

$$\varphi_T : \mathcal{C}_T \to \mathcal{M}_T$$

be the moduli space of relative holomorphic connections; the fiber of $\varphi_T$ over any $(t, E)$, where $t \in \mathcal{T}$ and $E \in \gamma^{-1}(t)$, is the space of all holomorphic connections on the stable vector bundle $E \to \mathcal{X}_t$, in particular, this fiber is an affine space for $H^0(\mathcal{X}_t, \text{End}(E) \otimes T^* \mathcal{X}_t)$.

The holomorphic fiber bundle in (5.3) has a $C^\infty$ section

$$\hat{\psi}_U : \mathcal{M}_T \to \mathcal{C}_T$$

that sends any stable vector bundle of degree zero to the unique holomorphic connection on it whose monodromy is unitary; so for each $t \in \mathcal{T}$, the restriction of $\hat{\psi}_U$ to $\gamma^{-1}(t)$ is the section $\psi_U$ in (2.2) for the Riemann surface $X = \mathcal{X}_t$.

For each $t \in \mathcal{T}$, there is a natural holomorphic isomorphism

$$F_t : (\gamma \circ \varphi_T)^{-1}(t) \to (\gamma \circ q_T)^{-1}(t)$$
\((\varphi_T \text{ and } q_T \text{ are the projections in (5.3) and (5.1) respectively})\) that takes the image of the section \(\hat{\psi}_U\) (constructed in (5.4)) to the image of the section \(\hat{\psi}_Q\) constructed in (5.2) (see Theorem 2.1). These isomorphisms \(\{F_t\}_{t \in T}\) together define a \(C^\infty\) isomorphism
\[
\hat{F} : C_T \longrightarrow \text{Conn}^r(\Theta_T) ;
\]
the restriction of \(\hat{F}\) to \((\gamma \circ \varphi_T)^{-1}(t)\) is the above holomorphic isomorphism \(F_t\) for every \(t \in T\).

**Proposition 5.1.** The \(C^\infty\) isomorphism \(\hat{F}\) in (5.5) is holomorphic.

**Proof.** For every \(t \in T\), consider the holomorphic isomorphism
\[
G'_t : (\gamma \circ \varphi_T)^{-1}(t) \longrightarrow (\gamma \circ q_T)^{-1}(t)
\]
in Corollary 4.5, so \(G'_t\) is \(G'\) in Corollary 4.5 for \(X_t = X\). These isomorphisms combine together to define an isomorphism
\[
\hat{G}' : C_T \longrightarrow \text{Conn}^r(\Theta_T) ;
\]
the restriction of \(\hat{G}'\) to \((\gamma \circ \varphi_T)^{-1}(t)\) is the above holomorphic isomorphism \(G'_t\) for every \(t \in T\). From the construction of the isomorphism \(G\) in Lemma 3.1 it follows immediately that \(G\) depends holomorphically on the Riemann surface \(X\). Note that both the sections \(\phi\) and \(\tau\), constructed in (3.13) and (3.15) respectively, depend holomorphically on the Riemann surface. Therefore, its extension \(G'\) also depends holomorphically on the Riemann surface \(X\). Consequently, the above isomorphism \(\hat{G}'\) is holomorphic.

Now, Theorem 4.1 implies that \(\hat{G}'\) coincides with \(\hat{F}\) in (5.5). Hence the map \(\hat{F}\) is holomorphic. \(\square\)

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