Entanglement fidelity and measure of entanglement

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The notion of entanglement fidelity is to measure entanglement preservation through quantum channels. Nevertheless, the amount of entanglement present in a state of a quantum system at any time is measured by quantities known as measures of entanglement. Since there are different types of measures of entanglement, thus it is natural to expect an entanglement fidelity to solely depend on its own measure of entanglement counterpart. Here, we aim to investigate association between the so called entanglement fidelity and some different measures of entanglement, namely, entanglement of formation, concurrence and negativity.

I. INTRODUCTION

Entanglement as one of the main notion of quantum source of information has been always at the centre of attention in quantum sciences and technologies. The fundamental roles of quantum entanglement in quantum cryptography, superdense coding, quantum teleportation, quantum error correction, efficient quantum computation and many other applied and basic quantum sciences [1–4], have turned the study of entanglement into a major area of research.

Concerning the concept of quantum entanglement, the two relevant questions that naturally arise are: how to quantify and compare entanglement in quantum states? and how well entanglement of a quantum state is maintained and preserved through quantum channels during a quantum information processing? Although these questions have been dressed extensively in many research works, there are still much that remain to be explored. For the first question we encounter a concept known as measure of entanglement and the latter question leads us to a concept known as entanglement fidelity. So far different classes of measures of entanglement, such as entanglement of formation, concurrence, entanglement of distillation, relative entropy of entanglement, negativity, Bures metric, geometric measure of entanglement, etc., have been introduced. For a revive about measures of entanglement one may see Ref. [5]. The quantity of entanglement fidelity, introduced and discussed in pioneering works [5–7], is believed to provide a measure of how successfully the entanglement between a pair of quantum subsystems would be preserved through a quantum process. Despite that all measures of entanglement follow the same criteria [8–10], they do not all impose the same ordering in a set of states [11, 12]. In a word, not all the measures of entanglement behave mathematically in a same manner and this may imply that there are different types of entanglements present in a quantum system. From this point of view, as the entanglement fidelity measures entanglement preservation in a quantum state going through a quantum process, it is natural to ask what type of entanglement is of concern in the entanglement fidelity. To study correlations between the entanglement fidelity and measures of entanglement, here we consider some measures of entanglement, namely, the entanglement of formation, concurrence [13, 14] and the negativity [15], which have been shown to possess different entangling natures [12]. In fact, we investigate ordinal correlations between the so called entanglement fidelity discussed in Refs. [5–7] and these measures of entanglement. To this end, associated with entanglement of formation, concurrence and negativity we introduce fidelity type quantities and statistically compare them with the well known entanglement fidelity by using Kendall rank correlation coefficient [16]. Our analysis demonstrate that each measure of entanglement specifies its own entanglement fidelity and the entanglement fidelity in Refs. [5–7] is not related to any of the measures of entanglement, which the present work concerns. We notice that the entangling nature of the entanglement fidelity has been questioned before from different perspective and approach [17].

The paper is organized as it follows: In sec. II the entanglement fidelity is briefly reviewed. We discuss entanglement of formation and concurrence in sec. III and define associated fidelity type quantities. We recall the negativity and introduce an associated fidelity type quantity in sec. IV. In sec. V after introducing the statistical tool known as Kendall rank correlation coefficient and specifying the model system, we perform data analysis regarding ordinal correlations between the entanglement fidelity and the fidelities associated with entanglement of formation, concurrence and negativity. The paper ends with a summary in sec. VI.

II. ENTANGLEMENT FIDELITY

In this section we briefly recall the quantity of entanglement fidelity based on Refs. [5–7]

Consider a quantum system of combined two quantum subsystems labeled as $R$ and $Q$. Suppose the joint
system $RQ$ initially is prepared in a general pure state $\rho_{iRQ}^{R} = |RQ\rangle \langle RQ|$. Further assume that the subsystem $Q$ undergoes some evolutions described by a quantum operation $E$ while the subsystem $R$ is dynamically isolated. In this case, the overall dynamics of the joint system $RQ$ is described by the quantum operation $I \otimes E$, where $I$ here is the identity operator acting on the subsystem $R$. Thus the final state of the joint system is given by the density operator $\rho_{iRQ}^{R} = I \otimes E(\rho_{iRQ}^{R})$. For such a process, the quantity

$$F_e = \langle \rho_i^{RQ}, \rho_i^{RQ} \rangle_{HS} = \text{Tr}(\rho_i^{RQ} \rho_i^{RQ}) = \langle RQ | \rho_i^{RQ} | RQ \rangle$$

(1)

defined as the Hilbert-Schmidt inner product between two states $\rho_{iRQ}^{R}$ and $\rho_{iRQ}^{R}$, is believed to quantify the entanglement fidelity indicating the variation of the entanglement in the quantum process $[3,7]$. The $F_e$, in fact, takes its value in the interval $[0,1]$, where the values close to 1 imply that the entanglement is well preserved while the values close to 0 indicates that the entanglement is mostly destroyed.

Although the quantity $F_e$ given in Eq. (1) is the state fidelity (squared) between joint initial state $\rho_i^{RQ}$ and final state $\rho_{iRQ}^{R}$, it has been shown in Ref. [5] that $F_e$ actually is an intrinsic property of the subsystem $Q$ itself and depends solely on the initial state of the subsystem $Q$ given by the reduced density operator

$$\rho_i^{Q} = \text{Tr}_{R} \rho_{iRQ}^{R},$$

(2)

where $\text{Tr}_{R}$ indicates partial trace over the subsystem $R$, and the quantum channel $E$ to which the subsystem $Q$ is subjected. It has been further shown in Ref. [5] that the $F_e$ does not in general agree with the state fidelity (squared) between initial and final states of the subsystem $Q$, i.e., $F(\rho_i^{Q}, \rho_i^{Q}) = \text{Tr}(\rho_i^{Q} \rho_i^{Q})$, where $\rho_i^{Q} = \text{Tr}_{R} \rho_i^{RQ}$. In fact, the following general relation holds

$$F_e \equiv F(\rho_i^{Q}, E) \leq F(\rho_i^{Q}, \rho_i^{Q}).$$

(3)

An interesting question, which may arise here, is if $F_e$ and $F$ are both kind of state fidelities depending only on the initial state and the quantum channel, why $F_e$ and $F$ do not in general agree? Is it simply because of the mathematical fact that the two operators $\text{Tr}_{R}$ and $E$ do not in general commute, i.e., $\text{Tr}_{R}(I \otimes E(\rho_i^{RQ})) \neq E(\text{Tr}_{R}(\rho_i^{RQ}))$ or $F_e$ and $F$ are in principle related to different quantum concepts and structures? As argued in Refs. [5,7], it is believed that $F_e$ is related to how well the quantum entanglement between two subsystems $R$ and $Q$ present in the state $\rho_i^{RQ}$ is preserved through the quantum process $E$. However, the $F$ is a useful measure of how far the two state $\rho_i^{RQ}$ and $\rho_i^{RQ}$ are. From this point of view, since there are different measures of entanglement or in some sense different types of entanglement, one may ask what type of entanglement the $F_e$ concerns? Below we examine correlations between the fidelity $F_e$ and two types of entanglement given by concurrence, a measure of the entanglement of formation, and negativity, a measure of the entanglement cost.

III. ENTANGLEMENT OF FORMATION AND CONCURRENCE

One of the fundamental measure of entanglement, which is in some sense defined based on the amount of resources needed to form a given entangled state, is known as entanglement of formation $[13]$. An explicit mathematical formulation of the entanglement of formation for a pair of qubits has been established in Refs. [13, 14]. This explicit formula is given based on a quantity called concurrence $[13, 14]$, which in its own right introduces a measure of entanglement as well. These measures are defined as follows.

For a given mixed state density operator $\rho^{RQ}$ of a pair of quantum subsystems $R$ and $Q$, the entanglement of formation is defined as $[13]$

$$E(\rho^{RQ}) = \min_{k} \sum_{k} p_k E(|\psi_k\rangle),$$

(4)

where $E(|\psi_k\rangle)$ is the pure state entanglement given by the von Neumann’s entropy of either of the two subsystems $R$ and $Q$, i.e.,

$$E(|\psi_k\rangle) = -\text{Tr}(\rho_k^{R} \log_2 \rho_k^{R}) = -\text{Tr}(\rho_k^{Q} \log_2 \rho_k^{Q}).$$

(5)

for reduced density operators $\rho_k^{R} = \text{Tr}_{Q}(|\psi_k\rangle \langle \psi_k|)$ and $\rho_k^{Q} = \text{Tr}_{R}(|\psi_k\rangle \langle \psi_k|)$. The minimum in Eq. (4) is taken over all possible pure-state decompositions of

$$\rho^{RQ} = \sum_{k} p_k |\psi_k\rangle \langle \psi_k|. \quad (6)$$

In two qubits case, the entanglement of formation can be explicitly expressed as a computable function of the state density operator $\rho^{RQ} [13, 14]$. This computable mathematical description make use of the spin flip transformation, which for general two-qubit mixed state $\rho^{RQ}$ reads

$$\tilde{\rho}^{RQ} = \sigma_y \otimes \sigma_y \rho^{RQ} \sigma_y \otimes \sigma_y.$$ 

(7)

Here $\tilde{\rho}^{RQ}$ is the complex conjugate of $\rho^{RQ}$ taken in the standard two-qubit computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, and

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(8)

is the second component of Pauli matrices in the single-qubit computation basis $\{|0\rangle, |1\rangle\}$. Considering $\lambda_1, \ldots, \lambda_4$ to be the eigenvalues of the hermitian matrix $\sqrt{\tilde{\rho}^{RQ} \rho^{RQ} \tilde{\rho}^{RQ}}$ in decreasing order, the entanglement of formation of the two-qubit mixed state $\rho^{RQ}$ can be written as $[14]$

$$E(\rho^{RQ}) = -\xi \log_2 \xi - (1 - \xi) \log_2 (1 - \xi).$$

(9)
where \( \xi = \frac{1 + \sqrt{1 - |C(\rho^{RQ})|^2}}{2} \) for the concurrence \( C(\rho^{RQ}) \) defined as

\[
C(\rho^{RQ}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}. \tag{10}
\]

As mentioned above the concurrence \( C(\rho^{RQ}) \) by itself is also identified as a measure of entanglement [14].

Associated with the above two measures of entanglement, entanglement of formation and concurrence, we may consider the following fidelity type quantities

\[
F_e = 1 - |E(\rho^{RQ}) - E(I \otimes \mathcal{E}(\rho^{RQ}))|, \quad F_c = 1 - |C(\rho^{RQ}) - C(I \otimes \mathcal{E}(\rho^{RQ}))|. \tag{11}
\]

Similar to the entanglement fidelity \( F_e \), these quantities also have their values in the interval \([0, 1]\) and allow us to evaluate how the amount of entanglements quantified by entanglement of formation or concurrence are preserved, when the system is subjected to the quantum channel \( I \otimes \mathcal{E} \). The values of \( F_{ef} \) and \( F_c \) close to 1 indicate that the entanglement of formation and concurrence are well preserved, and the values close to 0 imply that the entanglement of formation and concurrence are mainly lost. In fact, \( F_{ef} \) and \( F_c \) can be, respectively, regarded as fidelity of entanglement of formation and fidelity of concurrence in a quantum process, where the subsystem \( Q \) undergoes some evolutions described by a quantum operation \( \mathcal{E} \) while the subsystem \( R \) is dynamically isolated.

Note that from Eq. (10), we have the entanglement of formation as an increasing function of concurrence. This implies that any ordinal correlation between \( F_e \) and \( F_c \) will hold true between \( F_{ef} \) and \( F_{ef} \) as well and vice versa. Therefore in the following we only focus on the ordinal correlation between \( F_e \) and \( F_c \).

IV. NEGATIVITY

Another measure of entanglement that we consider here is known as negativity. Negativity, which in a sense measures the entanglement cost of a quantum state [13], can be regarded as a quantitative version of the Peres-Horodecki criterion [19, 20]. Although, in the case of two-qubit pure states, negativity coincide with concurrence, the two measures of entanglement behave very differently in general [12] and are believed to reflect different types of entanglement present in a quantum system. The negativity for a general bipartite mixed state \( \rho^{RQ} \) reads [12, 13]

\[
N(\rho^{RQ}) = ||[\rho^{RQ}]^{Tr_R}||_1 - 1, \tag{12}
\]

where \( T_R \) denotes the partial transpose with respect to subsystem \( R \) and thus

\[
\langle i_R j_Q | [\rho^{RQ}]^{Tr_R} | k_R l_Q \rangle = \langle k_R j_Q | \rho^{RQ} | i_R l_Q \rangle \tag{13}
\]

for a given orthonormal product basis \( |i_R j_Q \rangle = |i_R \rangle \otimes |j_Q \rangle \in \mathcal{H}_R \otimes \mathcal{H}_Q \). For any operator \( A \), \( \|A\|_1 = \text{Tr}\sqrt{A^\dagger A} \) is the trace norm, which is equal to the sum of the absolute values of the eigenvalues of \( A \) in the case of hermitian operator \( A \). Since \( \text{Tr}[|\rho^{RQ}|^{Tr_R}] = 1 \), the negativity in Eq. (12) is actually twice the sum of the absolute values of the negative eigenvalues of \( [\rho^{RQ}]^{Tr_R} \).

Similar to the previous section, we may consider the quantity,

\[
F_n = 1 - |N(\rho^{RQ}) - N(I \otimes \mathcal{E}(\rho^{RQ}))|, \tag{14}
\]

in order to evaluate the negativity type of entanglement fidelity or in short the fidelity of negativity. Indeed, the \( F_n \) provides a measure of how well the negativity between subsystems \( R \) and \( Q \) is preserved by the quantum process \( \mathcal{E} \). Here also we have \( F_n \in [0, 1] \), where the values close to 1 or 0, respectively, indicate that the negativity is mainly preserved or lost.

As shown in Ref. [12], concurrence and negativity do not impose the same ordering in a set of states, which may in a sense imply these two measures of entanglement refer independently to different types of entanglement. Therefore \( F_c \) and \( F_n \) are independent quantities and an ordinal correlation between \( F_c \) and \( F_n \) would be independent of an ordinal correlation between \( F_e \) and \( F_n \).

V. ENTANGLEMENT FIDELITY VS. CONCURRENCE AND NEGATIVITY

Having introduced the fidelities \( F_e \), \( F_{ef} \), \( F_c \), and \( F_n \), in previous sections, here we examine correlations between \( F_e \) and the other fidelities to see if the entangling nature of \( F_e \) is the entanglement of formation and concurrence type or the negativity type. We explore ordinal correlations and as mentioned in sec. III any ordinal correlation between \( F_c \) and \( F_n \) would lead to the same ordinal correlation between \( F_e \) and \( F_{ef} \). Hence, below we only examine ordinal correlations among the three quantities \( F_e \), \( F_c \), and \( F_n \).

We employ the statistical tool known as Kendall rank correlation coefficient or in short Kendall’s tau coefficient, which measures ordinal associations between two observed quantities [16]. We evaluate the fidelities \( F_e \), \( F_c \), and \( F_n \) for a specific family of two-qubit states of a model system subjected to some quantum noise channels. We then compare the collected data sets related to these fidelities pairwise for ordinal correlations via Kendall’s tau coefficient.

A. Kendall’s tau coefficient

For a set of paired observations \( \{(x_i, y_i)\}_{i=1}^n \) of two real-valued quantities \( X \) and \( Y \), the Kendall’s tau coefficient is defined as

\[
\tau = \frac{2}{n(n-1)} \sum_{i<j} \text{sgn}(x_i - x_j)\text{sgn}(y_i - y_j), \tag{15}
\]

where \( \text{sgn} \) denotes the sign function. Note that the coefficient is in the range \(-1 \leq \tau \leq 1\), and has the following properties
• If \( \tau = 1 \), the two quantities \( X \) and \( Y \) perfectly follow the same ordering, i.e.,
  \[
  x_i \geq x_j \iff y_i \geq y_j,
  \]
  and thus there exist a direct correlation between the two quantities.

• If \( \tau = -1 \) the two quantities \( X \) and \( Y \) perfectly follow the opposite ordering, i.e.,
  \[
  x_i \geq x_j \iff y_i \leq y_j,
  \]
  and thus there exist a reverse correlation between the two quantities.

• The two quantities \( X \) and \( Y \) are independent if the Kendall’s tau coefficient is approximately zero.

• If \( |\tau| \neq 1 \), the two quantities \( X \) and \( Y \) do not in principle follow certain correlated ordinal patterns. Therefore \( X \) and \( Y \) cannot both refer to some physical observations with the same physical properties and nature at each instance.

B. Model system and quantum channels

We focus on two-qubit systems prepared initially in a general two-qubit pure state \( \rho_i^{RQ} = |RQ\rangle \langle RQ| \), where
  \[
  |RQ\rangle = \alpha |00\rangle + \beta |01\rangle + \gamma |10\rangle + \delta |11\rangle
  \]
with
  \[
  \alpha = \cos \psi
  \]
  \[
  \beta = \sin \psi \cos \theta e^{i \zeta}
  \]
  \[
  \gamma = \sin \psi \sin \theta \cos \varphi e^{i \eta}
  \]
  \[
  \delta = \sin \psi \sin \theta \sin \varphi e^{i \xi},
  \]
Here the complex amplitudes \( \alpha, \beta, \gamma \) and \( \delta \) are specified with hyperspherical coordinates \( (\psi, \theta, \varphi) \) on a 3-sphere and phase factors \( \zeta, \eta \) and \( \xi \).

Since the initial state \( \rho_i^{RQ} \) is a two-qubit pure state the initial concurrence and negativity are the same \(^{12}\) and simply read
  \[
  C(\rho_i^{RQ}) = N(\rho_i^{RQ}) = 2|\alpha \delta - \beta \gamma|. \tag{20}
  \]

For our purpose, we assume the qubit \( Q \) in our model system is subjected to some quantum noise channels. The channels that we have in mind are "amplitude damping", "bit flip" and "phase flip" channels. The corresponding quantum operations are described by the following operation elements

• Amplitude damping:
  \[
  \begin{pmatrix}
  1 & 0 \\
  0 & \sqrt{1-p}
  \end{pmatrix}, \quad \begin{pmatrix}
  0 & \sqrt{p} \\
  0 & 0
  \end{pmatrix},
  \tag{21}
  \]

• Bit flip:
  \[
  \sqrt{1-p} \begin{pmatrix}
  1 & 0 \\
  0 & 1
  \end{pmatrix}, \quad \sqrt{p} \begin{pmatrix}
  0 & 1 \\
  1 & 0
  \end{pmatrix},
  \tag{22}
  \]

• Phase flip:
  \[
  \sqrt{1-p} \begin{pmatrix}
  1 & 0 \\
  0 & 1
  \end{pmatrix}, \quad \sqrt{p} \begin{pmatrix}
  1 & 0 \\
  0 & -1
  \end{pmatrix},
  \tag{23}
  \]
where \( p \in [0,1] \).

C. Data analysis and discussion

Having introduced the tool and the model system, here we discuss the data collection and analysis. A randomly selected initial joint two-qubit state of type Eq. \(^{18}\) is sent through one of the quantum channels introduced above for \( M = 2 \times 10^2 \) different values of \( p \) uniformly distributed in \([0,1]\). We emphasis that in each channel only the qubit \( Q \) is affected by quantum noise operations and the qubit \( R \) is left alone. For each value of \( p \) the fidelities \( F_c, F_e, F_n \) are evaluated to produce corresponding three \( M \)-data sets. Then the \( M \)-data sets are mutually compared by Kendall’s tau coefficient to detect ordinal correlations among the fidelities \( F_c, F_e, F_n \). This procedure is repeated for \( M = 5 \times 10^3 \) numbers of normally distributed random initial states of type Eq. \(^{18}\). The results for the three channels are as follow.

We use the same sample of initial states throughout the analysis. The concurrence and negativity distributions of the randomly selected sample of initial states are shown in Fig. 1. As the initial states are two-qubit pure states, concurrence and negativity distributions are the same.
FIG. 2. (Color online). Kendalls $\tau$ coefficient between $F_e$ and $F_c$ against state index in randomly selected sample of $M = 5 \times 10^3$ initial states. The "a", "b" and "p" panels, respectively, correspond to "amplitude damping", "bit flip" and "phase flip" channels. Each initial state is sent through the given quantum channel for $M = 2 \times 10^2$ different values of $p$ uniformly distributed in $[0, 1]$ and a $M$-set of paired observations of quantities $F_e$ and $F_c$ is produced. The $\tau$ coefficient is plotted for the produced $M$-set of paired observations.

Fig. 2 illustrates our analysis of the Kendall’s tau coefficient for the pair $F_e$ and $F_c$ in the three quantum channels. As seen from the figure, not only the absolute value of Kendall’s tau coefficient in the most of the cases is not one but it is zero in many cases particularly in the bit flip and phase flip channels, where we just get zero Kendall’s tau coefficient for each initial state. This provide a solid evidence for the entanglement fidelity $F_e$ to be independent of the fidelity $F_c$. Therefore, the entangling nature of the entanglement fidelity $F_e$ is not of type concurrence. Consequently, the $F_e$ is independent of $F_{ef}$ and the entangling nature of $F_e$ is not of type entangle-
ment of formation either.

In Fig. 3 we evaluate the Kendall’s tau coefficient for the pair $F_c$ and $F_n$ in the three quantum channels. Similarly, we see that in the most of the cases the absolute value of Kendall’s tau coefficient between $F_c$ and $F_n$ is not one and even in many cases is zero. In the bit flip and phase flip channels, we notice that the Kendall’s tau coefficient totally vanishes for the whole sample of initial states. Therefore, the entanglement fidelities $F_c$ and $F_n$ behave differently and indeed are independent from each other in principle. In a word, the entangling nature of the entanglement fidelity $F_c$ can not be of type negativity.

At the end of sec. IV we point out that based on Ref. [12] $F_c$ and $F_n$ are independent quantities, which consequently imply that our analysis in Figs. 2 and 3 are independent. To further clarify this we compare $F_c$ and $F_n$ through amplitude damping channel in Fig. 4. The figure shows that the Kendall’s tau coefficient between $F_c$ and $F_n$ vanishes for a number of corresponding paired observations. This approves that $F_c$ and $F_n$ are independent quantities and thus have different entangling natures. Moreover, the absolute value of Kendall’s tau coefficient is not one for most of the paired observations, which further confirms the different behaviors and natures of the two quantities $F_c$ and $F_n$.

![Figure 4](image)

**FIG. 4.** (Color online) Kendalls $\tau$ coefficient between $F_c$ and $F_n$ against state index in randomly selected sample of $M = 5 \times 10^3$ initial states. Each initial state is sent through the amplitude damping channel for $M = 2 \times 10^2$ different values of $p$ uniformly distributed in [0, 1] and a $M$-set of paired observations of quantities $F_c$ and $F_n$ is produced. The $\tau$ coefficient is plotted for the produced $M$-set of paired observations.

We conclude this section with some remarks. Our analysis above demonstrate that all the three entanglement fidelities $F_c$, $F_e$ and $F_n$ are mutually independent quantities and therefore they cannot refer to the same entangling nature of a quantum system. Each measure of entanglement defines individually its own entanglement fidelity and thus we expect the entanglement fidelity $F_e$ to associate with a measure of entanglement. So, If it is not entanglement of formation, concurrence or negativity then this question of what is the entangling nature of the entanglement fidelity $F_e$? or explicitly what is the measure of entanglement corresponding to the entanglement fidelity $F_e$? remains still open and requires further investigation.

VI. SUMMARY

In summary, we studied correlations between the entanglement fidelity and measures of entanglement, namely, entanglement of formation, concurrence and negativity. Related to each measure of entanglement we introduced a fidelity type quantity and compared it with the so called quantity of entanglement fidelity, introduced in Refs. [5–7], through the statistical tool known as Kendall rank correlation coefficient. With the analysis of Kendall rank correlation coefficient in two-qubit quantum systems subjected to three different quantum processes, we showed that there are no ordinal correlations between the entanglement fidelity and the three measures of entanglement and indeed they are independent. This confirms that entangling nature of the entanglement fidelity introduced in Refs. [5–7] is neither of type entanglement of formation and concurrence nor of type negativity.

Moreover, we examined Kendall rank correlation coefficient for ordinal correlations between the fidelity type of quantities associated to measures of entanglement. Our analyses demonstrate that the fidelities associated to entanglement of formation and concurrence are independent form the one associated to negativity. This, in fact, indicates that each measure of entanglement defines merely its own entanglement fidelity.

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[21] Note that we here use the negativity defined in Ref. [12], which is twice the negativity introduced in Ref. [15].