Proof of the Gaussian maximizers conjecture for the communication capacity of noisy heterodyne measurements

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Abstract

Basing on recently developed convex programming framework in the paper [arXiv:2204.10626], we provide a proof for a long-standing conjecture on optimality of Gaussian encodings for the ultimate communication rate of generalized heterodyne receivers under the oscillator energy constraint. Our results generalize previous ones (obtained under the assumption of validity of the energy threshold condition) and show a drastic difference in the structure of the optimal encoding within and beyond this condition. The core of the proof in the case beyond the threshold is a new log-Sobolev type inequality, which relates the generalized Wehrl entropy with the wavefunction gradient.

1 Introduction

Transmission of classical information encoded into quantum states is of great interest for both fundamental and practical reasons. Fundamentally, the classical capacity of a quantum channel defines the ultimate reliable communication rate [1, 2]. In reality, any physical signal has underlying quantum-mechanical nature which has to be taken into account. The latter fact is of central importance also for the measuring apparatus because of the quantum-mechanical complementarity, which sets up an upper bound for distinguishability of quantum signals.
In this paper we focus on the particular form of measurement — generalized heterodyne detection — which can be regarded as an approximate measurement of two non-commuting canonical variables (e.g., field quadratures playing the role of position and momentum). In the process of heterodyning, the signal density operator is mapped to a generalized Husimi function [3]. Quantum description of noisy heterodyne receivers and equivalent devices is given and reviewed, e.g., in Refs. [4, 5, 6, 7, 8, 9].

The generalized heterodyne measurement itself can be viewed as a specific quantum-classical Gaussian channel [10, 11], whose capacity is to be determined and the optimal encoding is to be found. This brings us to a long-standing Gaussian maximizer conjecture that the energy-constrained classical capacity of a general quantum Gaussian channel is always attained on a Gaussian encoding. Although many authors have evaluated the communication rate of Gaussian channels by using Gaussian encodings (e.g., in [12, 13, 14, 15]), the obtained evaluations give only a lower bound for the classical capacity unless the conjecture is proved.

The conjecture on optimality of Gaussian encodings was proved correct for gauge covariant and contravariant bosonic Gaussian channels (called phase-insensitive in quantum optics) [16, 17, 18], and later for a broader class of channels satisfying “threshold condition” under which the upper bound for the capacity as a difference between the maximum and the minimum output entropies is attainable [19, 20]. However, the conjecture remained open for a variety of other quantum and quantum-classical channels beyond the scope of the threshold condition [11] including the class of quantum-classical channels, such as noisy homodyne and heterodyne detection. Recently the conjecture was proved for a class of quantum-classical channels describing noisy homodyne detection [21]. In the proof of [21], a generalization of the celebrated log-Sobolev inequality [22, 23] appeared in the context of the convex optimization problem. In the present paper we make a step further and extend the proof to noisy heterodyne detection. Here we prove yet another log-Sobolev type inequality (Proposition 2), which enables us to extend the treatment beyond the scope of the threshold condition. Notably, the derived inequality is also a far-reaching generalization of the Wehrl inequality proved by Lieb [24]. Within the threshold condition we recover the result of Ref. [10] for the validity of the conjecture in this case. We also show that the optimal encodings significantly differ within and beyond the threshold condition, thus clarifying the physical meaning of the latter.
2 Capacity of noisy heterodyning: threshold condition

In this section we briefly summarize the results of Refs. [10, 11] concerning the unsharp joint position-momentum measurement (with the noisy optical heterodyning as the physical prototype). Statistics of the measurement outcome \((x, y) \in \mathbb{R}^2\) is described by the following positive operator-valued measure (POVM):

\[
M(dx dy) = D(x, y) \rho_\beta D(x, y)^* \frac{dx dy}{2\pi},
\]

where \(q\) and \(p\) are the canonical position and momentum operators, respectively, \(D(x, y) = \exp\left[i (yq - xp)\right]\) is a unitary position-momentum displacement operators, and \(\rho_\beta\) is a centered Gaussian density operator with the covariance matrix

\[
\beta = \begin{bmatrix} \beta_q & 0 \\ 0 & \beta_p \end{bmatrix}; \quad \beta_q \beta_p \geq \frac{1}{4}.
\]

Here \(\beta_q\) (\(\beta_p\)) is the noise power in position (momentum) quadrature. We denote

\[
m(x, y) = \frac{1}{2\pi} D(x, y) \rho_\beta D(x, y)^*.
\]

For a given system density operator \(\rho\) the output differential entropy\(^1\) reads

\[
h_M(\rho) = -\int \text{Tr}[\rho m(x, y)] \ln \text{Tr}[\rho m(x, y)] \, dx \, dy
\]

and represents a generalization of the Wehrl entropy. Eq. (4) reduces to the conventional Wehrl entropy when \(\beta_q = \beta_p = \frac{1}{2}\).

Following the lines of Ref. [21], an encoding \(E\) can be viewed as a probability distribution \(\pi(d\rho)\) on the set of quantum states \(\mathcal{S}\). The average state is

\[
\bar{\rho}_E = \int_{\mathcal{S}} \rho \pi(d\rho).
\]

The classical Shannon information between the input (encoded into quantum states \(\rho\) with probability distribution \(\pi(d\rho)\)) and the measurement out-

\(^1\)The differential entropy is well-defined in this case because the probability density \(\text{Tr}[\rho m(x, y)]\) is uniformly bounded [25].
come \((x, y)\) equals

\[
I(\mathcal{E}, M) = h_M(\bar{\rho}_{\mathcal{E}}) - \int_{\mathcal{S}} h_M(\rho)\pi(d\rho).
\]  

(6)

Let \(H = \frac{1}{2}(q^2 + p^2)\) be the system Hamiltonian and \(E \in [\frac{1}{2}, +\infty)\) be a maximally permissible average energy. Then the oscillator-energy-constrained classical capacity of the quantum-classical measurement channel under consideration is

\[
C(M, H, E) = \sup_{\mathcal{E}: \bar{\rho}_{\mathcal{E}} H \leq E} I(\mathcal{E}, M)
\]

(7)

\[
= \sup_{\mathcal{E}: \bar{\rho}_{\mathcal{E}} H \leq E} \left[ h_M(\bar{\rho}_{\mathcal{E}}) - e_M(\bar{\rho}_{\mathcal{E}}) \right],
\]

(8)

where the quantity \(e_M(\rho')\) is defined through

\[
e_M(\rho') = \inf_{\mathcal{E}: \bar{\rho}_{\mathcal{E}} = \rho'} \int h_M(\rho)\pi(d\rho)
\]

(9)

and represents an analogue of the convex closure of the output differential entropy for a quantum channel [26]. Since any measurement channel is entanglement-breaking, its classical capacity is additive [27, 28] and is given by the single-letter expression (7).

An important result of Ref. [11] is that for a general multi-mode Gaussian measurement channel and a quadratic Hamiltonian \(H\) the supremum in Eqs. (7), (8) is attained at a centered Gaussian density operator \(\rho_\alpha\) with the covariance matrix \(\alpha\). In the case of a single mode and the oscillator Hamiltonian \(H = \frac{1}{2}(q^2 + p^2)\) we are dealing with in the present paper, the covariance matrix \(\alpha = \begin{pmatrix} \alpha_q & 0 \\ 0 & \alpha_p \end{pmatrix}\), and the energy constraint reads as

\[
\alpha_q + \alpha_p \leq 2E.
\]

(10)

Thus, while the optimal average state \(\bar{\rho}_{\mathcal{E}}\) is available, the structure of the encoding \(\mathcal{E}\) itself is not known in general. The Gaussian maximizer conjecture states that the optimal encoding consists of squeezed coherent states with the displacement parameter having a Gaussian probability distribution. The conjecture was proved to be correct (in a much more general multi-mode situation) in [10] under the threshold condition which has a form of inequality between the matrix \(\alpha\) and the covariance matrix of the squeezed state.
minimizing the output entropy. In the single-mode case it takes the form (see [11]):

**Case C (Central).** The threshold condition on the parameters $\alpha_q, \alpha_p, \beta_q, \beta_p$ reduces to the inequalities:

$$\frac{1}{2\alpha_p} < \sqrt{\frac{\beta_q}{\beta_p}} < 2\alpha_q. \quad (11)$$

Assuming this condition to hold, the optimal encoding is Gaussian $E_0 = \{\pi_0(dx dy), \rho_0(x, y)\}$, where $\rho_0(x, y) = |x, y\rangle \langle x, y|$ is a squeezed coherent state with the vector $|x, y\rangle = D(x, y) |0\rangle$, and

$$\delta = \frac{1}{2} \sqrt{\frac{\beta_q}{\beta_p}} \quad (12)$$

and $\pi_0(dx dy)$ is the centered normal distribution with the nondegenerate covariance matrix

$$\gamma = \begin{bmatrix} \gamma_q & 0 \\ 0 & \gamma_p \end{bmatrix} = \begin{bmatrix} \alpha_q - \delta & 0 \\ 0 & \alpha_p - 1/(4\delta) \end{bmatrix}.$$  

The communication rate

$$I(E_0, M) = \frac{1}{2} \log \frac{(\alpha_q + \beta_q)(\alpha_p + \beta_p)}{\sqrt{\beta_q\beta_p} + 1/2}.$$  

An additional optimization over $\alpha_q, \alpha_p$ satisfying $\alpha_q + \alpha_p \leq 2E$ gives the energy-constrained capacity

$$C(M; H, E) = \log \left( \frac{E + (\beta_q + \beta_p)/2}{\sqrt{\beta_q\beta_p} + 1/2} \right)$$

for $E$ satisfying the energy threshold

$$E \geq \max \{E(\beta_p, \beta_q), E(\beta_q, \beta_p)\}, \quad E(\beta_1, \beta_2) = \frac{1}{2} \left( \beta_1 - \beta_2 + \sqrt{\frac{\beta_1}{\beta_2}} \right).$$

When the threshold condition is violated, then the Gaussian maximizer conjecture remained an open problem so far. To develop a general theory applicable both within and beyond the threshold condition we exploit a recently introduced framework of Ref. [21], which reformulates the optimization problem in terms of the convex programming.
3 Problem formulation in terms of convex programming

Following Ref. [21], we introduce the functional

\[ F(\mathcal{E}) = \int_{\mathcal{S}} h_M(\rho) \pi(d\rho) = \int_{\mathcal{S}} \text{Tr} \left[ K(\rho) \rho \right] \pi(d\rho), \]  

(13)

where

\[ K(\rho) = -\int m(x, y) \ln \text{Tr}[\rho m(x, y)] \, dx \, dy. \]  

(14)

For a fixed state \( \rho' \) the calculation of \( e_M(\rho') \) reduces to the optimization problem over distributions of density operators:

\[
\text{Minimize } \int_{\mathcal{S}} \text{Tr} \left[ K(\rho) \rho \right] \pi(d\rho) \quad \text{subject to } \int \rho \pi(d\rho) = \rho',
\]  

(15)

which is formally similar to a general Bayes problem studied in Refs. [29, 30]. Ref. [21] provides the following conditions for optimality of an encoding \( \mathcal{E}_0 \) with the distribution \( \pi_0(d\rho) \):

1. \( \Lambda_0 \leq K(\rho) \) for all \( \rho \in \mathcal{S} \);
2. \( [K(\rho) - \Lambda_0] \rho = 0 \) almost everywhere with respect to \( \pi_0(d\rho) \).

In Refs. [29, 30] mathematical theorems were proved giving precise regularity assumptions under which the conditions (i), (ii) are necessary and sufficient for the optimality. In solving our capacity problem for the Gaussian measurements we will use these conditions as sufficient in a broader context involving unbounded bosonic operators (see [21] for a justification).

By integrating (ii), we get an equation for determination of \( \Lambda_0 \), namely,

\[
\int_{\mathcal{S}} K(\rho) \rho \pi_0(d\rho) = \Lambda_0 \rho'.
\]  

(16)

However, a major difficulty may be to check the operator inequality (i).

Let us illustrate this in the case C (within the threshold condition) for noisy heterodyning (for which there is an alternative proof of optimality [10]). A candidate for the optimal encoding can be guessed within the class.
of Gaussian encodings. We will prove the optimality conditions (i) and (ii) for the Gaussian encoding $\mathcal{E}_0 = \{\pi_0(dx, dy), \rho_0(x, y)\}$, where

$$\rho_0(x, y) = |x, y\rangle_\delta \langle x, y| = D(x, y) |0\rangle_\delta \langle 0| D(x, y)^*.$$

We start with computation of $K(\rho_0(x, y)) \rho_0(x, y)$ for an arbitrary $\delta$ and then focus on the case C where $\delta = \frac{1}{2} \sqrt{\delta_p}.$

Using Eq. (14) and the explicit formula

$$\delta_\langle x', y'| m(x, y)|x', y\rangle_\delta = \exp \left( -\frac{(x' - x)^2}{2(\beta_q + \delta)} - \frac{(y' - y)^2}{2(\beta_p + 1/4\delta)} \right) 2\pi \sqrt{\beta_q + \delta} (\beta_p + 1/4\delta),$$

as well as introducing $c = \ln 2\pi \sqrt{\beta_q + \delta} (\beta_p + 1/4\delta)$, we get

$$K(\rho_0(x', y')) = \int D(x, y) \rho_0 D(x, y)^* \left[ c + \frac{(x' - x)^2}{2(\beta_q + \delta)} + \frac{(y' - y)^2}{2(\beta_p + 1/4\delta)} \right] \frac{dx dy}{2\pi}.$$

Here we also used the formulas (see, e.g., [31])

$$\int D(x, y) \rho_0 D(x, y)^* \frac{dx dy}{2\pi} = I,$$

$$\int x^2 D(x, y) \rho_0 D(x, y)^* \frac{dx dy}{2\pi} = q^2 + \beta_q,$$

$$\int y^2 D(x, y) \rho_0 D(x, y)^* \frac{dx dy}{2\pi} = p^2 + \beta_p.$$

Hence

$$K(\rho_0(x, y)) \rho_0(x, y) = \left[ c + \frac{(q - x)^2 + \beta_q}{2(\beta_q + \delta)} + \frac{(p - y)^2 + \beta_p}{2(\beta_p + 1/4\delta)} \right] D(x, y) |0\rangle_\delta \langle x, y|$$

$$= D(x, y) \left[ c + \frac{q^2 + \beta_q}{2(\beta_q + \delta)} + \frac{p^2 + \beta_p}{2(\beta_p + 1/4\delta)} \right] |0\rangle_\delta \langle x, y|$$

$$= \left[ c + \frac{\beta_q}{2(\beta_q + \delta)} + \frac{\beta_p}{2(\beta_p + 1/4\delta)} \right] |x, y\rangle_\delta \langle x, y|$$

$$+ D(x, y) \frac{1}{2} \left[ \frac{q^2}{\beta_q + \delta} + \frac{p^2}{\beta_p + 1/4\delta} \right] |0\rangle_\delta \langle x, y|.$$
In the case C, we consider \( \delta \) given by Eq. (12). For this value of \( \delta \),

\[
\frac{\beta_q + \delta}{\beta_p + 1/4\delta} = \frac{2\delta}{1/2\delta},
\]

hence \( |0\rangle_\delta \) is the ground state of the Hamiltonian

\[
\frac{q^2}{\beta_q + \delta} + \frac{p^2}{\beta_p + 1/4\delta} = \frac{2\delta}{\beta_q + \delta} \left( \frac{q^2}{2\delta} + 2\delta p^2 \right)
\]

with the minimal eigenvalue

\[
\left( (\beta_q + \delta)(\beta_p + 1/4\delta) \right)^{-1/2} = \left( \sqrt{\beta_q \beta_p + 1/2} \right)^{-1}.
\]

Substituting the value (12) into Eq. (22), we finally obtain

\[
K(\rho_0(x,y))\rho_0(x,y) = \left[ c + \frac{\sqrt{\beta_q \beta_p}}{\sqrt{\beta_q \beta_p + 1/2}} + \frac{1}{2 \left( \sqrt{\beta_q \beta_p + 1/2} \right)} \right] |x,y\rangle_\delta \langle x,y|
\]

\[
= \ln 2\pi \left( \sqrt{\beta_q \beta_p + 1/2} + 1 \right) |x,y\rangle_\delta \langle x,y|
\]

\[
= \ln 2\pi e \left( \sqrt{\beta_q \beta_p + 1/2} \right) |x,y\rangle_\delta \langle x,y|.
\]

Integrating with respect to the probability distribution \( \pi_0(dx dy) \), we obtain (16) with the Hermitian operator

\[
\Lambda_0 = \ln 2\pi e \left( \sqrt{\beta_q \beta_p + 1/2} \right) I.
\]

To check the condition (i) it is sufficient to prove

\[
\langle \psi | \Lambda_0 | \psi \rangle \leq \langle \psi | K(\rho) | \psi \rangle
\]

(23)

for an arbitrary density operator \( \rho \) and a dense subset of \( \psi \) in the system Hilbert space. We can assume that \( \psi \) is a unit vector. Due to nonnegativity of the classical relative entropy of two probability densities (the Kullback-
Leibler divergence) we have

\[ \langle \psi | K(\rho) | \psi \rangle = -\int \langle \psi | m(x, y) | \psi \rangle \ln \text{Tr}[\rho m(x, y)] \, dx \, dy \]

\[ = -\int \langle \psi | m(x, y) | \psi \rangle \ln \frac{\text{Tr}[\rho m(x, y)]}{\langle \psi | m(x, y) | \psi \rangle} \, dx \, dy \]

\[ - \int \langle \psi | m(x, y) | \psi \rangle \ln \langle \psi | m(x, y) | \psi \rangle \, dx \, dy \]

\[ \geq -\int \langle \psi | m(x, y) | \psi \rangle \ln \langle \psi | m(x, y) | \psi \rangle \, dx \, dy \]

\[ = h_M(|\psi \rangle \langle \psi|), \]

where \( h_M(|\psi \rangle \langle \psi|) \) is the output differential entropy for the considered measurement channel (the generalized Wehrl entropy), which is bounded from below by \( \langle \psi | \Lambda_0 | \psi \rangle = \ln 2\pi e \left( \sqrt{\beta_q\beta_p} + 1/2 \right) \) due to the following result (that completes the proof of inequality (23)).

**Proposition 1.**

\[ \min_{\|\psi\|=1} h_M(|\psi \rangle \langle \psi|) = \ln 2\pi e \left( \sqrt{\beta_q\beta_p} + 1/2 \right). \]  

(24)

Let us outline the proof since it seems was not explained sufficiently before. Proposition 1 in [17], applied to the entropy function, implies that the output differential entropy of the measurement channel is minimized by coherent states (a generalization of Wehrl’s inequality). This proof is for the gauge-invariant case i.e. for the standard complex structure of multiplication by \( i \), but since all the complex structures are isomorphic via a symplectic conjugation, the result is valid for any complex structure with the correspondingly squeezed states as the minimizers. The result (24) which can be found in Sec. IV of the paper [20] (see also Example in [10]) corresponds to the following complex structure

\[ J = \begin{bmatrix} 0 & -\sqrt{\frac{\beta_p}{\beta_q}} \\ \sqrt{\frac{\beta_q}{\beta_p}} & 0 \end{bmatrix}. \]

in the one-mode case. The Wehrl inequality proved by Lieb [21] corresponds to \( \beta_q = \beta_p = 1/2 \) and gives the minimal entropy \( \ln 2\pi e \) attained at the coherent states.
4 Capacity of noisy heterodyning: beyond threshold condition

The threshold condition (11) can be violated in two ways:

- **Case L (Left).** \[ \frac{1}{2\alpha_p} \geq \sqrt{\frac{\beta_p}{\beta_q}}. \]

- **Case R (Right).** \[ \sqrt{\frac{\beta_p}{\beta_q}} \geq 2\alpha_q. \]

Since these cases are completely symmetric with respect to exchange between \( q \) and \( p \), we will restrict ourselves to the case L.

The Gaussian maximizer conjecture for the noisy heterodyning in the case L is formulated in Ref. [11]. Suppose \( \rho_0 \) is the average state in the optimal encoding, then the conjecture states that the optimal encoding itself is Gaussian and takes the form \( E_0 = \{ \pi_0(dx), \rho_0(x) \} \), where

\[
\pi_0(dx) = \frac{1}{\sqrt{2\pi\gamma}} \exp \left[ -\frac{x^2}{2\gamma} \right] dx, \quad \rho_0(x) = |x\rangle \delta \langle x| \tag{25}
\]

with

\[
\delta = \frac{1}{4\alpha_p}, \quad \gamma = \alpha_q - \frac{1}{4\alpha_p}.
\]

This encoding is the same as in the case of unsharp position measurement (noisy homodyning in quantum optics) discussed in Ref. [21]. If the conjecture is correct, then the oscillator-energy-constrained capacity reads

\[
C(M, H, E) = \log \left( \frac{\sqrt{1 + 8E\beta_q + 4\beta_q^2} - 1}{2\beta_q} \right) \tag{26}
\]

provided \( \beta_q \leq \beta_p \) and \( E < E(\beta_p, \beta_q) \).

**Theorem 1.** The Gaussian encoding described above is optimal for the constrained classical capacity of generalized heterodyning in the case L.

We will prove the theorem by using the formalism developed in Sec. 3. To check the condition (ii), we can begin with the computation of \( \Lambda_0 \) as in the Case C and then set \( y = 0 \) in Eq. (17) so that \( D(x, 0) = D(x) \). We then obtain

\[
K(\rho_0(x)) = c + \frac{(q - x)^2 + \beta_q}{2(\beta_q + \delta)} + \frac{p^2 + \beta_p}{2(\beta_p + 1/\delta)}
\]
and

\[
K(\rho_0(x))\rho_0(x) = \left[ c + \frac{\beta_q}{2(\beta_q + \delta)} + \frac{\beta_p}{2(\beta_p + 1/4\delta)} \right] |x\rangle \langle x| + D(x) \frac{\delta}{\beta_q + \delta} \left( \frac{q^2}{2\delta} + 2\delta p^2 \right) |0\rangle_\delta \langle x| - D(x) \frac{1}{2} \left[ \frac{4\delta^2}{\beta_q + \delta} - \frac{1}{\beta_p + 1/4\delta} \right] p^2 |0\rangle_\delta \langle x|.
\]

Taking into account that the squeezed vacuum \(|0\rangle_\delta\) is the ground state of the corresponding oscillator Hamiltonian, i.e.,

\[
\left( \frac{q^2}{2\delta} + 2\delta p^2 \right) |0\rangle_\delta = |0\rangle_\delta,
\]

and \(D(x)\) commutes with \(p^2\), we finally get

\[
K(\rho_0(x))\rho_0(x) = \Lambda_0 \rho_0(x),
\]

where

\[
\Lambda_0 = \ln 2\pi \sqrt{(\beta_q + \delta)(\beta_p + 1/4\delta)} + \frac{\beta_q + 2\delta}{2(\beta_q + \delta)} + \frac{\beta_p}{2(\beta_p + 1/4\delta)} - \frac{1}{2} \left[ \frac{4\delta^2}{\beta_q + \delta} - \frac{1}{\beta_p + 1/4\delta} \right] p^2.
\]  

(27)

Note that

\[
\frac{4\delta^2}{\beta_q + \delta} - \frac{1}{\beta_p + 1/4\delta} = \frac{4\delta^2\beta_p - \beta_q}{(\beta_q + \delta)(\beta_p + 1/4\delta)} \geq 0
\]

for \(\delta \geq \frac{1}{2} \sqrt{\frac{\beta_q}{\beta_p}}\).

To check the condition (i) it is sufficient to prove

\[
\langle \psi | \Lambda_0 | \psi \rangle \leq \langle \psi | K(\rho) | \psi \rangle
\]

(28)

for an arbitrary density operator \(\rho\) and a dense subset of \(\psi\) in the system Hilbert space. Arguing as in the Sec. 3, the inequality \(28\) will follow if we prove

\[
\langle \psi | \Lambda_0 | \psi \rangle \leq -\int \langle \psi | m(x, y) | \psi \rangle \ln \langle \psi | m(x, y) | \psi \rangle \, dx \, dy
\]

(29)
for a unit vector $\psi$. With $\Lambda_0$ given by (27) it amounts to
\[
\int \langle \psi | m(x, y) | \psi \rangle \ln \langle \psi | m(x, y) | \psi \rangle \, dxdy
+ \ln 2\pi \sqrt{\beta_q + \delta} (\beta_p + 1/4\delta) + \frac{\beta_q + 2\delta}{2 (\beta_q + \delta)} + \frac{\beta_p}{2 (\beta_p + 1/4\delta)}
\leq \frac{4\delta^2 \beta_p - \beta_q}{2 (\beta_q + \delta) (\beta_p + 1/4\delta)} \int |\psi'(x)|^2 \, dx,
\] (30)
where $\psi'(x)$ is the derivative of $\psi(x)$, the wavefunction in position representation.

Note that for $\delta = \frac{1}{2} \sqrt{\frac{\beta_q}{\beta_p}}$ the right-hand side vanishes and the inequality turns into the valid inequality (24) for the case C. However, if $\delta > \frac{1}{2} \sqrt{\frac{\beta_q}{\beta_p}}$, then the inequality (30) represents a new type of log-Sobolev inequalities as compared to the known in literature [23, 21]. It relates the generalized Wehrl entropy $h_M(\langle \psi | \psi \rangle)$ with the wavefunction gradient. We present a proof of the inequality (30) in the next section. This proof completes the proof of the of the Gaussian maximizer conjecture in the case L. The results can be readily extended to the case R by a proper change of variables.

5 Proof of the log-Sobolev type inequality (30)

To simplify the notation denote $\rho_{\beta_q \beta_p}(x, y) := \text{Tr}[\rho m(x, y)]$, where $m(x, y)$ is defined through the Gaussian state $\rho_{\beta}$ with the covariance matrix
\[
\begin{pmatrix}
\beta_q & 0 \\
0 & \beta_p
\end{pmatrix}
\]
via formula (3). By $h(P||Q)$ we denote the the classical relative entropy (Kullback–Leibler divergence of a probability density $P$ from a probability
Lemma 1. The following equality takes place
\[
\begin{align*}
\text{h} \left( \rho_p^{\beta_q \beta_p} \left| \rho_{0;0}^{\beta_q \beta_p} \right. \right) &= \int \text{Tr}[\rho_m(x,y)] \ln \text{Tr}[\rho_m(x,y)] \, dx \, dy \\
&+ \ln 2\pi \sqrt{(\beta_q + \delta)(\beta_p + 1/4\delta)} \\
&+ \beta_q + \langle q^2 \rangle_\rho + \frac{\beta_p + \langle p^2 \rangle_\rho}{2(\beta_q + \delta)} \\
&\leq \int \text{Tr}[\rho_m(x,y)] \ln \text{Tr}[\rho_m(x,y)] \, dx \, dy \\
&+ \ln 2\pi \sqrt{(\beta_q + \delta)(\beta_p + 1/4\delta)} \\
&+ \beta_q + \langle q^2 \rangle_\rho + \frac{\beta_p + \langle p^2 \rangle_\rho}{2(\beta_q + \delta)} \\
&\leq \frac{4\delta^2 \beta_p - \beta_q}{2(\beta_q + \delta)(\beta_p + 1/4\delta)} \langle p^2 \rangle_\rho.
\end{align*}
\]
(31)

where \( \langle q^2 \rangle_\rho = \text{Tr}[\rho q^2] \) and \( \langle p^2 \rangle_\rho = \text{Tr}[\rho p^2] \).

Proof. Using the explicit form
\[
p^{\beta_q \beta_p}_{0;0}(x,y) = \frac{\exp \left( -\frac{x^2}{2(\beta_q + \delta)} - \frac{y^2}{2(\beta_p + 1/4\delta)} \right)}{2\pi \sqrt{(\beta_q + \delta)(\beta_p + 1/4\delta)}},
\]
(32)

and properties (19), (20), (21), we readily get Eq. (31) by the definition of the relative entropy.

Proposition 2. For all density operators \( \rho \) and real positive parameters \( \beta_q, \beta_p, \delta \) satisfying \( \beta_q \beta_p \geq 1 + \frac{1}{4} \) and \( \delta \geq \frac{1}{2} \sqrt{\frac{\beta_q}{\beta_p}} \), the following inequality holds:
\[
\begin{align*}
&\int \text{Tr}[\rho_m(x,y)] \ln \text{Tr}[\rho_m(x,y)] \, dx \, dy \\
&+ \ln 2\pi \sqrt{(\beta_q + \delta)(\beta_p + 1/4\delta)} \\
&+ \beta_q + \frac{\beta_p + \langle p^2 \rangle_\rho}{2(\beta_q + \delta)} \\
&\leq \frac{4\delta^2 \beta_p - \beta_q}{2(\beta_q + \delta)(\beta_p + 1/4\delta)} \langle p^2 \rangle_\rho.
\end{align*}
\]
(33)

Proof. Using Lemma 1, we rewrite the desired inequality (33) in the form
\[
h \left( \rho_p^{\beta_q \beta_p} \left| \rho_{0;0}^{\beta_q \beta_p} \right. \right) \leq \frac{\langle q^2 \rangle_\rho + 4\delta^2 \langle p^2 \rangle_\rho - 2\delta}{2(\beta_q + \delta)}.
\]
(34)

For arbitrarily fixed \( \beta_q > 0 \) and \( \delta > 0 \) define \( \tilde{\beta}_p := \frac{\beta_q}{4\delta^2} \). To prove the proposition we need to prove the validity of (34) for all \( \beta_p \geq \tilde{\beta}_p \).
If $\beta_p = \tilde{\beta}_p$, then the original inequality (33) reduces to
\[
\int \text{Tr}[\rho m(x, y)] \ln \text{Tr}[\rho m(x, y)] \, dx \, dy + \ln 2\pi(\sqrt{\beta_q\beta_p} + \frac{1}{2}) + 1 \leq 0, \quad (35)
\]
which was proved to be correct in the earlier work [20] (see also Proposition 1 in Sec. 3). Therefore, the equivalent inequality (34) holds true if $\beta_p = \tilde{\beta}_p$.

If $\beta_p > \tilde{\beta}_p$, then $p^p_{\beta q \beta_p} = T_{\beta_p - \tilde{\beta}_p} p^\beta_{\tilde{\beta}_p}$, where the Markov operator $T_t$ acts on a probability density $P(x, y)$ as follows:
\[
T_t P(x, y) = \frac{1}{\sqrt{2\pi t}} \int \exp \left( -\frac{(y - v)^2}{2t} \right) P(x, v) \, dv. \quad (36)
\]
Due to monotonicity of the classical relative entropy under Markov operators (classical channels), we have
\[
\begin{align*}
&h \left( p^p_{\beta q \beta_p} \left| \left| p^\beta_{\tilde{\beta}_p} \right| \left| \left| p^\beta_{\tilde{\beta}_p} |0\rangle \langle 0| \right. \right. \right) \\
&\quad \leq h \left( p^\beta_{\tilde{\beta}_p} \left| \left| \left| p^\beta_{\tilde{\beta}_p} |0\rangle \langle 0| \right. \right. \right) \\
&\quad \leq h \left( p^\beta_{\tilde{\beta}_p} \left| \left| \left| p^\beta_{\tilde{\beta}_p} |0\rangle \langle 0| \right. \right. \right) \\
&\quad \leq \langle q^2 \rangle_{\rho} + 4\delta^2 \langle p^2 \rangle_{\rho} - 2\delta
\end{align*}
\]
where the last inequality is just the same as (35). □

Note that equality in (34) and, consequently, in (33) takes place if $\rho = |0\rangle \langle 0|$ because both parts of (34) are equal to zero in this case. Substituting $\rho = |\psi\rangle \langle \psi|$ for $\rho$ in (33), we get the log-Sobolev type inequality (30) and, hence, the optimality of the Gaussian encoding in the case L. This concludes the proof of Theorem 1.

6 Conclusions

We have resolved the Gaussian maximizer conjecture for noisy heterodyne measurement channel in the affirmative. We treated both threshold scenarios (within the threshold and beyond it) on an equal footing by using the recently developed reformulation of the problem in terms of the convex programming [21]. This allowed us to reproduce the known results within the threshold and achieve new results beyond the threshold. The optimal encodings to attain the oscillator-energy-constrained capacity are shown to be
Gaussian in both scenarios, however, the structure of the optimal encoding is different: one should use a two-parameter family of states (17) within the threshold condition and a one-parameter family of states (25) beyond the threshold.

Physical meaning of such a structure is clear: when the ratio of noises $\beta_p/\beta_q$ is greater than certain threshold value, the optimal choice for the encoding becomes to invest all the data into position $q$, leaving momentum $p$ ignored. The optimal encoding is then the same as in the homodyne case [21] which can be regarded as a limiting case $\beta_p \to +\infty$.

Note that beyond the threshold $e_M(\tilde{\rho}_E) \neq \min_{\rho} h_M(\rho)$, which means that the previously known methods (e.g., those used in Refs. [16] [17] [18]) are not applicable to this case. The conjecture was finally resolved by proving a new inequality (30) that relates the generalized Wehrl entropy with the wavefunction gradient. The new inequality can be considered as another generalization of the log-Sobolev inequality in addition to the recently proposed in Ref. [21].

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