THE JOINT PROBABILITY SIGNATURE OF TWO SYSTEMS

JEAN-LUC MARICHAL, PIERRE MATHONET, AND JORGE NAVARRO

ABSTRACT. The structure signature of a system made up of \( n \) components having continuous and i.i.d. lifetimes was defined in the eighties by Samaniego as the \( n \)-tuple whose \( k \)-th coordinate is the probability that the \( k \)-th component failure causes the system to fail. More recently, a bivariate version of this concept was considered as follows. The joint structure signature of a pair of systems built on a common set of components having continuous and i.i.d. lifetimes is a square matrix of order \( n \) whose \((k, l)\)-entry is the probability that the \( k \)-th failure causes the first system to fail and the \( l \)-th failure causes the second system to fail. This concept was successfully used to derive a signature-based decomposition of the joint reliability of the two systems. In this paper, we provide an explicit formula to compute the joint structure signature and extend this formula to the general non-i.i.d. case, assuming only that the distribution of the component lifetimes has no ties. We also provide and discuss a sufficient condition on this distribution for the joint reliability of two systems to have a signature-based decomposition.

1. INTRODUCTION

Consider a system \( S = (C, \phi, F) \), where \( C \) is the set \([n] = \{1, \ldots, n\}\) of components, \( \phi \) is the structure function, and \( F \) is the joint c.d.f. of the component lifetimes \( T_1, \ldots, T_n \), defined by

\[
F(t_1, \ldots, t_n) = \Pr(T_1 \leq t_1, \ldots, T_n \leq t_n), \quad t_1, \ldots, t_n \geq 0.
\]

We assume that the structure function is semicoherent, which means that the function \( \phi \) is nondecreasing in each variable and satisfies the conditions \( \phi(0, \ldots, 0) = 0 \) and \( \phi(1, \ldots, 1) = 1 \). We also assume throughout that the function \( F \) has no ties, which means that \( \Pr(T_i = T_j) = 0 \) for all distinct \( i, j \in C \).

Samaniego \([9]\) defined the signature of any system \( S \) whose components have continuous and i.i.d. lifetimes as the \( n \)-tuple \( s = (s_1, \ldots, s_n) \) whose \( k \)-th coordinate is the probability that the \( k \)-th component failure causes the system to fail. In other words, we have

\[
s_k = \Pr(T_S = T_{k:n}), \quad k \in [n],
\]

where \( T_S \) is the system lifetime and \( T_{k:n} \) is the \( k \)-th smallest component lifetime, that is, the \( k \)-th order statistic of the component lifetimes.

Boland \([1]\) showed that

\[
s_k = \sum_{A \subseteq C \atop |A| = n-k+1} \frac{1}{n} \phi(A) - \sum_{A \subseteq C \atop |A| = n-k} \frac{1}{n} \phi(A), \quad k \in [n].
\]

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Corresponding author: J.-L. Marichal, Mathematics Research Unit, FSTC, University of Luxembourg, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg, Luxembourg. Email: jean-luc.marichal[at]uni.lu.
Here and throughout, we identify Boolean vectors \( \mathbf{x} \in \{0,1\}^n \) and subsets \( A \subseteq [n] \) by setting \( x_i = 1 \) if and only if \( i \in A \). We thus use the same symbol to denote both a function \( f: (0,1)^n \to \mathbb{R} \) and the corresponding set function \( f: 2^{[n]} \to \mathbb{R} \) interchangeably. For instance, we write \( f(0,\ldots,0) = f(\emptyset) \) and \( f(1,\ldots,1) = f([n]) \).

Equation (1) clearly shows that the signature \( s \) is independent of the function \( F \). Thus, it depends only on \( n \) and \( \phi \). For this reason, it is often called the structure signature of the system.

The concept of signature can be easily extended to the general non-i.i.d. case where it is assumed only that \( F \) has no ties. The probability signature of a system \( S \) is the \( n \)-tuple \( \mathbf{p} = (p_1,\ldots,p_n) \) whose \( k \)-th coordinate is defined by

\[
p_k = \Pr(T_S = T_{k:n}), \quad k \in [n].
\]

By definition, this concept coincides with the structure signature whenever the component lifetimes are i.i.d. and continuous. Actually, it was shown \([5,6]\) that both concepts still coincide whenever the lifetimes are exchangeable (i.e., \( F \) is a symmetric function). However, it was also observed \([3,4,7,11]\) that in general the probability signature depends on the distribution \( F \) of the component lifetimes and that this dependence is captured by the relative quality function \( q: 2^{[n]} \to [0,1] \) defined as

\[
q(A) = \Pr\left( \text{max}_{i \in [n] \setminus A} T_i < \min_{j \in A} T_j \right), \quad A \subseteq C,
\]

with the convention that \( q(\emptyset) = q(C) = 1 \). The \( k \)-th coordinate of the probability signature is then given by the formula (see [3])

\[
p_k = \sum_{|A|=n-k+1} q(A) \phi(A) - \sum_{|A|=n-k} q(A) \phi(A), \quad k \in [n],
\]

which generalizes Boland’s formula (1) under the sole assumption that \( F \) has no ties.

Now consider two systems \( S_1 = (C, \phi_1, F) \) and \( S_2 = (C, \phi_2, F) \), with a common set \( C \) of components and a common distribution \( F \) of lifetimes, and assume that both structure functions \( \phi_1 \) and \( \phi_2 \) are semicoherent.

**Example 1.** As a basic example consider the systems \( S_1 = (C, \phi_1, F) \) and \( S_2 = (C, \phi_2, F) \), where \( C \) is a set of four components and the structure functions \( \phi_1 \) and \( \phi_2 \) are depicted in Figures 1 and 2 respectively. Since \( S_1 \) is a series system made up of components 1 and 2, the function \( \phi_1 \) is constant with respect to its third and fourth variables. Similarly, we see that \( \phi_2 \) is constant with respect to its first variable.

![Figure 1. System S1](image1.png)

Under the assumption that the lifetimes are continuous and i.i.d., Navarro et al. \([8]\) defined the joint structure signature of the systems \( S_1 \) and \( S_2 \) as the square matrix \( s \) of order \( n \) whose \((k,l)\)-entry is the probability

\[
s_{k,l} = \Pr(T_{S_1} = T_{k:n} \text{ and } T_{S_2} = T_{l:n}), \quad k,l \in [n].
\]
Thus, $s_{k,l}$ is the probability that the $k$-th failure causes the system $S_1$ to fail and that the $l$-th failure causes the system $S_2$ to fail.

Just as the structure signature was used by Samaniego \[9,10\] to derive a signature-based decomposition of the reliability

$$T_S(t) = \Pr(T_S > t), \quad t \geq 0,$$

of the system $S$, the concept of joint probability signature was designed for two systems sharing the same set of components, they can be easily and naturally extended to an arbitrary number of systems.

In this section we mainly show how Equation (1) can be extended to joint structure signatures and from this result we derive an alternative proof of the signature-based decomposition of the joint reliability

$$T_{S_1,S_2}(t_1,t_2) = \Pr(T_{S_1} > t_1 \text{ and } T_{S_2} > t_2), \quad t_1,t_2 \geq 0,$$

of the systems $S_1$ and $S_2$.

In this paper we provide a generalization of Boland’s formula (1) to joint structure signatures and from this result we derive an alternative proof of the signature-based decomposition of the joint reliability of two systems with continuous and i.i.d. components. We also extend the concept of joint structure signature to the general dependent setting, assuming only that the function $F$ has no ties. Thus, we define the joint probability signature of two systems $S_1 = (C, \phi_1, F)$ and $S_2 = (C, \phi_2, F)$ as the square matrix $p$ of order $n$ whose $(k,l)$-entry is the probability

$$p_{k,l} = \Pr(T_{S_1} = T_{k_{\text{kn}}} \text{ and } T_{S_2} = T_{l_{\text{kn}}}), \quad k,l \in [n].$$

In general this matrix depends on both the structures of the systems and the distribution $F$ of the component lifetimes. We introduce a bivariate version of the relative quality function and use it to provide an extension of Equation (2) to joint probability signatures. We also provide and discuss a sufficient condition on the distribution function $F$ for the joint reliability of two systems to have a signature-based decomposition.

\textbf{Remark 1.} It is noteworthy that, even though our definitions and results on the concept of joint probability signature are designed for two systems sharing the same set of components, they can be easily and naturally extended to an arbitrary number of systems.

\section{The Joint Probability Signature}

In this section we mainly show how Equation (1) can be extended to joint structure signatures and how Equation (2) can be extended to joint probability signatures.

Recall first that the tail structure signature of a system $S$ is the $(n+1)$-tuple $\overline{s} = (\overline{s}_0, \ldots, \overline{s}_n)$ defined by

$$\overline{s}_k = \sum_{i=k+1}^{n} s_i = \sum_{\substack{A \subseteq C \ni A = n \setminus k \ni \phi(A)}} \frac{1}{\binom{n}{|A|}} \phi(A), \quad k = 0, \ldots, n,$$

where the second expression immediately follows from (1). (Here and throughout we use the natural convention that $\sum_{i=k+1}^{n} x_i = 0$ when $k = n$.) Conversely, the structure signature $s$ can also be easily retrieved from $\overline{s}$ by using (1), that is, by computing $s_k = \overline{s}_{k-1} - \overline{s}_k$ for every $k \in [n]$. Thus, assuming that the component lifetimes are continuous and i.i.d., we see that $\overline{s}_k = \Pr(T_S > t_{k_{\text{kn}}})$ is the probability that the system $S$ survives beyond the $k$-th failure (with the usual convention that $T_{0_{\text{kn}}} = 0$).

Similarly, the tail probability signature of a system $S$ is the $(n+1)$-tuple $\overline{p} = (\overline{p}_0, \ldots, \overline{p}_n)$, where

$$\overline{p}_k = \sum_{i=k+1}^{n} p_i = \sum_{\substack{A \subseteq C \ni A = n \setminus k \ni \phi(A)}} q(A) \phi(A), \quad k = 0, \ldots, n,$$

with $q(A) = \Pr(T_S > t_{k_{\text{kn}}})$.
Conversely, the probability signature \( p \) can be easily retrieved from \( \overline{P} \) by using (2), that is, by computing \( p_k = \overline{P}_{k-1} - \overline{P}_k \) for every \( k \in [n] \). Thus, assuming only that \( F \) has no ties, we see that \( \overline{P}_k = \Pr(T_S > T_{k:n}) \) is the probability that the system \( S \) survives beyond the \( k \)-th failure.

Since the tail signatures proved to be much easier to handle than the standard signatures in many computation problems, it is natural to extend these concepts to the bivariate case.

Given two systems \( S_1 = (C, \phi_1, F) \) and \( S_2 = (C, \phi_2, F) \) for which the component lifetimes are continuous and i.i.d., we define the \textit{tail bivariate structure signature} as the square matrix \( \overline{S} \) of order \( n + 1 \) whose \((k,l)\)-entry is the probability

\[
\overline{S}_{k,l} = \Pr(T_{S_1} > T_{k:n} \text{ and } T_{S_2} > T_{l:n}), \quad k, l = 0, \ldots, n.
\]

Similarly, assuming only that \( F \) has no ties, the \textit{tail bivariate probability signature} is the square matrix \( \overline{P} \) of order \( n + 1 \) whose \((k,l)\)-entry is the probability

\[
\overline{P}_{k,l} = \Pr(T_{S_1} > T_{k:n} \text{ and } T_{S_2} > T_{l:n}), \quad k, l = 0, \ldots, n.
\]

Thus, \( \overline{P}_{k,l} \) is the probability that the system \( S_1 \) survives beyond the \( k \)-th failure and the system \( S_2 \) survives beyond the \( l \)-th failure. In particular,

\[
\begin{align*}
\overline{P}_{k,l} &= 0 \quad \text{if } k = n \text{ or } l = n, \\
\overline{P}_{k,0} &= \Pr(T_{S_1} > T_{k:n}), \quad k = 0, \ldots, n, \\
\overline{P}_{0,l} &= \Pr(T_{S_2} > T_{l:n}), \quad l = 0, \ldots, n, \\
\overline{P}_{0,0} &= 1.
\end{align*}
\]

The following proposition provides the conversion formulas between the matrices \( p \) and \( \overline{P} \). Clearly, the corresponding formulas between the matrices \( s \) and \( \overline{S} \) also hold.

**Proposition 2.** We have

\[
\overline{P}_{k,l} = \sum_{i=k+1}^{n} \sum_{j=l+1}^{n} p_{i,j}, \quad k, l = 0, \ldots, n,
\]

and

\[
p_{k,l} = \overline{P}_{k-1,l-1} - \overline{P}_{k,l-1} - \overline{P}_{k-1,l} + \overline{P}_{k,l}, \quad k, l \in [n].
\]

**Proof.** The first formula follows from the definition. Indeed, the probability that \( S_1 \) survives beyond the \( k \)-th failure and that \( S_2 \) survives beyond the \( l \)-th failure is the probability that \( S_1 \) fails due to the \( i \)-th failure for some \( i > k \) and that \( S_2 \) fails due to the \( j \)-th failure for some \( j > l \).

The second formula is straightforward. \( \square \)

**Corollary 3.** We have

\[
\overline{S}_{k,l} = \sum_{i=k+1}^{n} \sum_{j=l+1}^{n} s_{i,j}, \quad k, l = 0, \ldots, n,
\]

and

\[
s_{k,l} = \overline{S}_{k-1,l-1} - \overline{S}_{k,l-1} - \overline{S}_{k-1,l} + \overline{S}_{k,l}, \quad k, l \in [n].
\]

The conversion formulas given in Proposition 2 show that all the information contained in the matrix \( p \) is completely encoded the matrix \( \overline{P} \) and vice versa. To ease the presentation of our results, we will show how to compute the matrix \( \overline{P} \) from \( \phi_1, \phi_2, \) and \( F \) (see Theorem 8) and similarly for the matrix \( \overline{S} \) (see Corollary 9).

Let us first introduce the bivariate version of the relative quality function.
In particular, for every Proposition 5.

whenever of permutations of events labeled by the permutations of everywhere of the sample space

We now show that this expression can be easily computed in the continuous and i.i.d. case, or more generally whenever the events almost everywhere. We thus have

with the convention that \( q(A, \emptyset) = q(A, C) = q(A) \) for every \( A \subseteq C \) and \( q(\emptyset, B) = q(C, B) = q(B) \) for every \( B \subseteq C \).

By definition, the bivariate relative quality function satisfies the following properties:

for every \( A, B \subseteq C \), we have \( q(A, B) = q(B, A) \) and \( q(A, A) = q(A) \). Moreover, the number \( q(A, B) \) is the probability that the best \( |A| \) components are precisely those in \( A \) and that the best \( |B| \) components are precisely those in \( B \). In particular, we have \( q(A, B) = 0 \) whenever \( A \not\subseteq B \) and \( B \not\subseteq A \).

We then easily derive the following proposition.

Proposition 5. For every \( A \subseteq C \) and every \( l \in \{0, \ldots, |A|\} \), we have

In particular, for every \( k, l \in [n] \) we have

The bivariate relative quality function can also be computed in terms of probabilities of events labeled by the permutations of \([n] = \{1, \ldots, n\}\). Indeed, denote by \( \mathcal{S}_n \) the group of permutations of \([n]\) and define the event

Since \( F \) has no ties, the collection of events \( \{E_\sigma : \sigma \in \mathcal{S}_n\} \) forms a partition almost everywhere of the sample space \([0, +\infty]^n\). Moreover, it is clear that

almost everywhere. We thus have

We now show that this expression can be easily computed in the continuous and i.i.d. case, or more generally whenever the events \( E_\sigma \) (\( \sigma \in \mathcal{S}_n \)) are equally likely, for which we have \( \Pr(E_\sigma) = 1/|n|! \) since \( F \) has no ties.

Definition 6. Define the function \( q_0 : 2^{[n]} \times 2^{[n]} \to [0, 1] \) as

\[
q_0(A, B) = \begin{cases} 
\frac{(n-|A|)! 
(1 + (|B| - |A|))! |A|!}{n!} & \text{if } B \subseteq A, \\
\frac{(n-|B|)! (1 + (|A| - |B|))! |B|!}{n!} & \text{if } A \subseteq B, \\
0 & \text{otherwise.}
\end{cases}
\]

Proposition 7. If the events \( E_\sigma \) (\( \sigma \in \mathcal{S}_n \)) are equally likely, then \( q = q_0 \).

Proof. We simply use (4) and count the relevant permutations. \qed
We are now able to present our main result, which shows how to compute the matrix $\mathbf{P}$ from $\varphi_1, \varphi_2,$ and $q.$

**Theorem 8.** For every $k, l \in \{0, \ldots, n\}$ we have

\[
\overline{P}_{k,l} = \sum_{|A|=n-k} \sum_{|B|=n-l} q(A, B)\varphi_1(A)\varphi_2(B).
\]

**Proof.** We just consider the event

\[
E_{k,l} = (T_{S_1} > T_{kn} \text{ and } T_{S_2} > T_{ln})
\]

that defines $\overline{P}_{k,l}.$ The fact that a given observed tuple of lifetimes $(t_1, \ldots, t_n)$ belongs to $E_{k,l}$ depends only on the ordering of lifetimes. Therefore, for each permutation $\sigma \in \mathfrak{S}_n,$ we have either $E_{\sigma} \subseteq E_{k,l}$ or $E_{\sigma} \cap E_{k,l} = \emptyset.$ Moreover, the former case occurs if and only if

\[
\varphi_1(\{\sigma(k+1), \ldots, \sigma(n)\}) = \varphi_2(\{\sigma(l+1), \ldots, \sigma(n)\}) = 1.
\]

We thus have

\[
\overline{P}_{k,l} = \Pr(E_{k,l}) = \Pr\left(\bigcup_{\sigma \in \mathfrak{S}_n : \varphi_1(\{\sigma(k+1), \ldots, \sigma(n)\})=1} E_{\sigma}\right).
\]

that is,

\[
\overline{P}_{k,l} = \sum_{\sigma \in \mathfrak{S}_n} \Pr(E_{\sigma})\varphi_1(\{\sigma(k+1), \ldots, \sigma(n)\})\varphi_2(\{\sigma(l+1), \ldots, \sigma(n)\}).
\]

To compute this sum we group the terms for which $\{\sigma(k+1), \ldots, \sigma(n)\}$ is a given set $A$ of cardinality $n-k$ and $\{\sigma(l+1), \ldots, \sigma(n)\}$ is a given set $B$ of cardinality $n-l$ (with a necessary inclusion relation) and then sum over all the possibilities for such $A$’s and $B$’s. The result then follows from Eq. (4). \(\square\)

It is clear that Eq. (5) generalizes Eq. (3). Moreover, if $\mathbf{P}^1$ (resp. $\mathbf{P}^2$) denotes the tail probability signature of the system $S_1$ (resp. $S_2$), from Eqs. (3) and (5) it follows that

\[
\begin{align*}
\overline{P}_{k,0} &= \mathbf{P}^1_k \quad \text{for } k = 0, \ldots, n, \\
\overline{P}_{0,l} &= \mathbf{P}^2_l \quad \text{for } l = 0, \ldots, n.
\end{align*}
\]

Also, if $p^1$ (resp. $p^2$) denotes the probability signature of the system $S_1$ (resp. $S_2$), we clearly have

\[
\begin{align*}
\sum_{l=1}^{n} p_{k,l} &= p^1_k \quad \text{for } k = 1, \ldots, n, \\
\sum_{k=1}^{n} p_{k,l} &= p^2_l \quad \text{for } l = 1, \ldots, n.
\end{align*}
\]

As far as the matrix $\mathbf{S}$ is concerned, we have the following immediate corollary.

**Corollary 9.** For every $k, l \in \{0, \ldots, n\}$ we have

\[
\overline{S}_{k,l} = \sum_{A:B} q_0(A, B)\varphi_1(A)\varphi_2(B),
\]
Applying Corollary 9 and then Proposition 2 to Example 1 we obtain the following matrices

\[
\Sigma = \frac{1}{12} \begin{bmatrix}
12 & 9 & 4 & 0 & 0 \\
6 & 3 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
s = \frac{1}{12} \begin{bmatrix}
0 & 3 & 3 & 0 \\
2 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

**Remark 2.** Formula (6) shows that, by adding the entries in each row of the matrix \(s\), we obtain the structure signature \(s^1\) of the system \(S_1\). Applying this fact to the example above, we immediately obtain \(s^1 = \frac{1}{12}(6, 4, 2, 0)\). However, it is important to remember that \(S_1\) has four components, two of which are null or irrelevant. As detailed in [9], from this signature it is not difficult to compute the signature of the (coherent) system obtained from \(S_1\) by ignoring the irrelevant components.

It is clear that Theorem 8 easily generalizes to the case of \(m\) systems

\[
S_1 = (C, \phi_1, F), \ldots, S_m = (C, \phi_m, F).
\]

Indeed, for \(k_1, \ldots, k_m \in \{0, \ldots, n\}\) we then have

\[
\mathcal{T}_{k_1, \ldots, k_m}^{s^1} = \sum_{|A_1|=n-k_1} \cdots \sum_{|A_m|=n-k_m} q(A_1, \ldots, A_m) \phi_1(A_1) \cdots \phi_m(A_m).
\]

Moreover, if there exists a permutation \(\sigma \in \mathfrak{S}_m\) such that \(A_{\sigma(1)} \subseteq \cdots \subseteq A_{\sigma(1)}\), then

\[
q_0(A_1, \ldots, A_m) = \frac{(n-|A_{\sigma(1)}|)!}{n!} \left(\frac{|A_{\sigma(1)}|}{|A_{\sigma(2)}|}! \cdots \frac{|A_{\sigma(m)}|}{n!}\right).
\]

Otherwise, \(q_0(A_1, \ldots, A_m) = 0\).

3. **Signature-based decomposition of the joint reliability**

A signature-based decomposition of the joint reliability function of two systems \(S_1 = (C, \phi_1, F)\) and \(S_2 = (C, \phi_2, F)\) whose components have continuous and i.i.d. lifetimes was given in Navarro et al. [8]. In this section we recall this signature-based decomposition and give an alternative proof of it. More precisely, we provide a sufficient condition on the distribution of the component lifetimes (much more general than being continuous and i.i.d.) for this signature-based decomposition to hold. Note that, even in the non-i.i.d. case, the signature considered here in the decomposition is the bivariate structure signature as a combinatorial object defined in Corollaries 3 and 9. Our result is presented in Proposition 12.

For every \(j \in C\) and every \(t \geq 0\), let us denote by \(X_j(t) = \text{Ind}(T_j > t)\) the state variable of component \(j\) at time \(t \geq 0\). Let us also consider the state vector \(X(t) = (X_1(t), \ldots, X_n(t))\) at time \(t \geq 0\).

It was proved [4] (see also [2, 5, 9] for earlier works) that, the reliability function of any system \(S = (C, \phi, F)\) is given by

\[
\mathcal{T}_S(t) = \sum_{x \in \{0,1\}^n} \phi(x) \Pr(X(t) = x),
\]

and, if the state variables \(X_1(t), \ldots, X_n(t)\) are exchangeable, which means that

\[
\Pr(X(t) = x) = \Pr(X(t) = \sigma(x)), \quad x \in \{0,1\}^n, \quad \sigma \in \mathfrak{S}_n,
\]

we immediately obtain the structure signature as a combinatorial object defined in Corollaries 3 and 9. Our result is presented in Proposition 12.
where \( \sigma(x) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) \), then the reliability function admits the following signature-based decomposition:

\[
F_S(t) = \sum_{k=1}^{n} s_k F_{kn}(t),
\]

where \( s_k \) is given by Eq. (1) and \( F_{kn}(t) = \Pr(T_{kn} > t) \).

We now generalize Eqs. (9) and (10) to the joint reliability function of two systems \( S_1 = (C, \phi_1, F) \) and \( S_2 = (C, \phi_2, F) \). The extension of Eq. (9) is given in the following proposition.

**Proposition 10.** We have

\[
F_{S_1, S_2}(t_1, t_2) = \sum_{x, y \in \{0, 1\}^n} \phi_1(x) \phi_2(y) \Pr(X(t_1) = x \text{ and } X(t_2) = y).
\]

**Proof.** We have

\[
F_{S_1, S_2}(t_1, t_2) = \Pr(\phi_1(X(t_1)) = 1 \text{ and } \phi_2(X(t_2)) = 1) = \sum_{x \in \{0, 1\}^n} \sum_{y \in \{0, 1\}^n} \Pr(X(t_1) = x \text{ and } X(t_2) = y),
\]

which proves the result. \( \square \)

By applying Proposition 10 to the joint reliability function of a \( k \)-out-of-\( n \) system and a \( l \)-out-of-\( n \) system, namely

\[
F_{kn, ln}(t_1, t_2) = \Pr(T_{kn} > t_1 \text{ and } T_{ln} > t_2), \quad k, l \in [n],
\]

we immediately obtain the following corollary.

Recall first that, for every \( x \in \{0, 1\}^n \), we set \( |x| = \sum_{i=1}^{n} x_i \).

**Corollary 11.** We have

\[
F_{kn, ln}(t_1, t_2) = \sum_{x \in \{0, 1\}^n} \sum_{y \in \{0, 1\}^n} \Pr(X(t_1) = x \text{ and } X(t_2) = y).
\]

We are now able to extend Eq. (10) to the joint reliability function of two systems \( S_1 = (C, \phi_1, F) \) and \( S_2 = (C, \phi_2, F) \). The result is given in the next proposition. The proof mainly uses combinatorial arguments and therefore constitutes an alternative proof of the decomposition reported in Navarro et al. [8]. Moreover, it enables us to relax the i.i.d. assumption by considering a much more general condition.

**Proposition 12.** If the distribution function \( F \) satisfies the condition

\[
\Pr(X(t_1) = x \text{ and } X(t_2) = y) = \Pr(X(t_1) = \sigma(x) \text{ and } X(t_2) = \sigma(y))
\]

for any \( x, y \in \{0, 1\}^n \), any \( t_1, t_2 \geq 0 \), and any permutation \( \sigma \in S_n \), then we have

\[
F_{S_1, S_2}(t_1, t_2) = \sum_{k=1}^{n} \sum_{l=1}^{n} s_{k,l} F_{kn, ln}(t_1, t_2),
\]

where \( s_{k,l} \) is defined in Corollaries 3 and 7.

**Proof.** For any sequence \( x_k \), define the sequence \( (\Delta x)_k = x_{k+1} - x_k \). For two sequences \( x_k \) and \( y_k \) such that \( x_0 y_0 = x_n y_n = 0 \), we then clearly have

\[
\sum_{k=1}^{n} (\Delta x)_{k-1} y_k = \sum_{k=0}^{n-1} x_k (\Delta y)_k.
\]
For any double sequence \( x_{k,l} (k, l = 0, \ldots, n) \), define \((\Delta_1 x)_{k,l} = x_{k+1,l} - x_{k,l}\) and \((\Delta_2 x)_{k,l} = x_{k,l+1} - x_{k,l}\). By Proposition \[2\] we then clearly have \( s_{k,l} = (\Delta_1 \Delta_2)_{k-1,l-1} \) for any \( k, l \in [n] \).

On the other hand, for the sequence \( f_{k,l} = \overline{F}_{k,n,t,n}(t_1, t_2) \), by Corollary \[1\] we observe that
\[
(\Delta_1 \Delta_2 f)_{k,l} = \sum_{u \in \{0,1\}^n} \sum_{v \in \{0,1\}^n} \Pr(X(t_1) = u \text{ and } X(t_2) = v).
\]

Now, observing that \( S_{k,l} = 0 \) whenever \( k = n \) or \( l = n \) and defining \( \overline{F}_{k,n,t,n}(t_1, t_2) = 0 \) whenever \( k = 0 \) or \( l = 0 \), by two applications of \[13\] we can rewrite the right-hand side of \[12\] as
\[
\sum_{k=1}^{n-1} \sum_{l=1}^{n-1} (\Delta_1 \Delta_2 S)_{k-1,l-1} f_{k,l} = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S_{k,l} (\Delta_1 \Delta_2 f)_{k,l}.
\]

Using Eqs. \[8\] and \[14\], this double sum immediately becomes
\[
\sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \left( \sum_{x \in \{0,1\}^n} \sum_{y \in \{0,1\}^n} q_0(x, y) \phi_1(x) \phi_2(y) \right) \left( \sum_{u \in \{0,1\}^n} \sum_{v \in \{0,1\}^n} \Pr(X(t_1) = u \text{ and } X(t_2) = v) \right)
\]
or equivalently (observing that \( \phi_1(x) \phi_2(y) = 0 \) whenever \( |x| = |y| = 0 \)),
\[
\sum_{x \in \{0,1\}^n} \sum_{y \in \{0,1\}^n} q_0(x, y) \phi_1(x) \phi_2(y) \sum_{u \in \{0,1\}^n} \sum_{v \in \{0,1\}^n} \Pr(X(t_1) = u \text{ and } X(t_2) = v).
\]

To prove our result, by Proposition \[10\] it is now enough to show that the equality
\[
q_0(x, y) \sum_{u \in \{0,1\}^n} \sum_{v \in \{0,1\}^n} \Pr(X(t_1) = u \text{ and } X(t_2) = v) = \Pr(X(t_1) = x \text{ and } X(t_2) = y)
\]
holds for any \( x, y \in \{0,1\}^n \) and any \( t_1, t_2 \geq 0 \).

If \( x \) and \( y \) are not ordered by inclusion, then both sides of \[15\] are zero. So we may assume that \( x \preceq y \) or \( y \preceq x \) (recall that \( x \preceq y \) means that \( x_i \leq y_i \) for \( i = 1, \ldots, n \)). Up to relabeling the components, we may assume w.l.o.g. that \( y \preceq x \). If \( y = x \), then by \[11\] the sum in the left-hand side of \[15\] is equal to
\[
\sum_{u \in \{0,1\}^n} \Pr(X(t_1) = u \text{ and } X(t_2) = u) = \binom{n}{|x|} \Pr(X(t_1) = x \text{ and } X(t_2) = x)
\]
and \[15\] holds true by definition of \( q_0 \).

If \( y < x \) (i.e., \( x \not\preceq y \) and \( x \not\preceq y \)) and \( t_2 \geq t_1 \), then both sides of \[15\] are zero since the components of the system are nonrepairable. If \( y < x \) and \( t_1 \geq t_2 \), then using Eq. \[11\] we obtain
\[
\Pr(X(t_1) = u \text{ and } X(t_2) = v) = \begin{cases} 0, & \text{if } v \not\subseteq u, \\ \Pr(X(t_1) = x \text{ and } X(t_2) = y), & \text{if } v \subseteq u, \end{cases}
\]
for any \( u \) and \( v \) such that \(|u| = |x|\) and \(|v| = |y|\).

The result then follows by counting the number of tuples \( u \) and \( v \) for which \(|u| = |x|\), \(|v| = |y|\), and \( v \subseteq u \).

The following example shows that representation \[12\] cannot be extended to arbitrary distribution functions \( F \).
Example 13. Consider the systems $S_1 = (C, \phi_1, F)$ and $S_2 = (C, \phi_2, F)$, where $\phi_1(x_1, x_2) = x_1$ and $\phi_2(x_1, x_2) = x_2$ and assume that the component lifetimes $T_1$ and $T_2$ are independent and have exponential distributions with reliability functions $F_1(t) = e^{-t}$ and $F_2(t) = e^{-2t}$, respectively. Then the joint reliability function of these systems is given by

$$F_{S_1,S_2}(t_1,t_2) = \begin{cases} e^{-3t_2}, & \text{if } t_1 \leq t_2, \\ e^{-t_1-2t_2}, & \text{if } t_2 \leq t_1. \end{cases}$$

We also have

$$F_{1:2,1:2}(t_1,t_2) = \begin{cases} e^{-3t_2}, & \text{if } t_1 \leq t_2, \\ e^{-3t_1}, & \text{if } t_2 \leq t_1, \end{cases}$$

$$F_{1:2,2:2}(t_1,t_2) = \begin{cases} e^{-2t_1-t_2} + e^{-t_1-2t_2} - e^{-3t_2}, & \text{if } t_1 \leq t_2, \\ e^{-3t_1}, & \text{if } t_2 \leq t_1, \end{cases}$$

$$F_{2:2,1:2}(t_1,t_2) = \begin{cases} e^{-3t_2}, & \text{if } t_1 \leq t_2, \\ e^{-2t_1-t_2} + e^{-t_1-2t_2} - e^{-3t_1}, & \text{if } t_2 \leq t_1, \end{cases}$$

$$F_{2:2,2:2}(t_1,t_2) = \begin{cases} e^{-t_1+t_2} - e^{-3t_2}, & \text{if } t_1 \leq t_2, \\ e^{-t_1} + e^{-2t_1} - e^{-3t_1}, & \text{if } t_2 \leq t_1. \end{cases}$$

Since the exponential functions involved in the expressions above are linearly independent, one can easily show that there cannot exist real numbers $w_{1,1}$, $w_{1,2}$, $w_{2,1}$, and $w_{2,2}$ such that

$$F_{S_1,S_2} = w_{1,1}F_{1:2,1:2} + w_{1,2}F_{1:2,2:2} + w_{2,1}F_{2:2,1:2} + w_{2,2}F_{2:2,2:2}.$$
Proof. (a) Assume that the component lifetimes are exchangeable. Then, for any intervals
\(I_1 = [a_1, b_1], \ldots, I_n = [a_n, b_n]\) we have
\[
Pr(T_1 \in I_1, \ldots, T_n \in I_n) = Pr(T_{\sigma(1)} \in I_1, \ldots, T_{\sigma(n)} \in I_n).
\]
Now suppose w.l.o.g. that \(t_1 \leq t_2\). Then the event \((X(t_1) = x \text{ and } X(t_2) = y)\) is empty if \(y \not\leq x\). Otherwise, it is equal to
\[
\left( \bigcap_{x_i = 0} (t_i \leq t_1) \right) \cap \left( \bigcap_{x_i = 1} (t_i < T_i) \right) \cap \left( \bigcap_{y_i = 1} (t_2 < T_i) \right).
\]
Using (16), we see that the probability of the latter event remains unchanged if we replace
\(T_1, \ldots, T_n\) with \(T_{\sigma(1)}, \ldots, T_{\sigma(n)}\). This shows that condition (11) holds for any \(x, y \in \{0, 1\}^n\) and any permutation \(\sigma \in S_n\).
(b) Since \(t_1 < t_2\), we obtain
\[
Pr(X(t_1) = x) = \sum_{y \in \{0, 1\}^n \atop y \leq x} Pr(X(t_1) = x \text{ and } X(t_2) = y)
\]
\[
= \sum_{y \in \{0, 1\}^n \atop y \leq x} Pr(X(t_1) = \sigma(x) \text{ and } X(t_2) = \sigma(y)).
\]
Using the fact that \(y \leq x\) if and only if \(\sigma(y) \leq \sigma(x)\), we then have
\[
Pr(X(t_1) = x) = \sum_{y \in \{0, 1\}^n \atop \sigma(y) \leq \sigma(x)} Pr(X(t_1) = \sigma(x) \text{ and } X(t_2) = \sigma(y))
\]
\[
= Pr(X(t_1) = \sigma(x)).
\]
This completes the proof. □

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**MATHEMATICS RESEARCH UNIT, FSTC, UNIVERSITY OF LUXEMBOURG, 6, RUE COUDENHOVE-KALERGI, L-1359 LUXEMBOURG, LUXEMBOURG**

*E-mail address: jean-luc.marichal[at]uni.lu*

**UNIVERSITY OF LIÈGE, DEPARTMENT OF MATHEMATICS, GRANDE TRAVERSE, 12 - B37, B-4000 LIÈGE, BELGIUM**

*E-mail address: p.mathonet[at]ulg.ac.be*

**FACULTAD DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, 30100 MURCIA, SPAIN**

*E-mail address: jorgenav[at]um.es*