Electric–magnetic duality as a quantum operator and more symmetries of \( U(1) \) gauge theory

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Received: 22 January 2021 / Revised: 18 February 2021 / Accepted: 19 February 2021 / Published online: 17 May 2021
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Abstract
We promote the Noether charge of the electric–magnetic duality symmetry of \( U(1) \) gauge theory, \("G\)" , to a quantum operator. We construct ladder operators, \( D^\dagger_{\pm k}(\pm) \) and \( D_{\pm k}(\pm) \), which create and annihilate the simultaneous quantum eigenstates of the quantum Hamiltonian (or number) and the electric–magnetic duality operators, respectively. Therefore, all the quantum states of the \( U(1) \) gauge fields can be expressed in the form of \( |E, g\rangle \), where \( E \) is the energy of the state, and \( g \) is the eigenvalue of the quantum operator \( G \), where the \( g \) is quantized in the unit of 1. We also show that ten independent bilinears comprised of the creation and the annihilation operators can form \( SO(2, 3) \), which is as demonstrated in Dirac’s paper published in 1962. The number operator and the electric–magnetic duality operator are members of the \( SO(2, 3) \) family of generators. We note that there are two more generators that commute with the number operator(or Hamiltonian). We prove that these generators are, indeed, symmetries of the action in \( U(1) \) gauge field theory.

Keyword Electric–magnetic duality

1 Motivation and summary
Maxwell equations without any electric sources enjoy an interesting symmetry called electric–magnetic duality symmetry [1–3]. Under a transformation of \( (\vec{E}, \vec{B}) \to (\vec{B}, -\vec{E}) \), Maxwell equations are invariant. In fact, the symmetry can be realized in the form of the (infinitesimal) canonical transformation given by

\[
\delta E = \theta \nabla \times \vec{A} \quad \text{and} \quad \delta \vec{A} = \theta \nabla^{-2} \nabla \times \vec{E},
\]

(1)

where \( \theta \) is an infinitesimal rotation angle [4, 5]. Such a symmetry appears not only in Maxwell theory but also in linearized Einstein gravity [6], Bosonic and Fermionic gauge field theories [7], partially massless (gravity) theories [8], and as approximate symmetries in non-Abelian gauge theories in a few different contexts [9, 10]. In the real world, electric–magnetic duality symmetry is not respected, because no magnetic monopoles exist once interactions with charged matter are considered. However, the symmetry can be approximately restored by experiment in a certain material system [11].

This symmetry is also able to be realized as a rotation of the electric and magnetic couplings, which is called “S-duality” [12–14]. The symmetry group is widely known to be \( SL(2, \mathbb{R}) \), but if electric and magnetic charges are introdced, it is broken to \( SL(2, \mathbb{Z}) \) by Dirac’s quantization condition. In this note, however, we discuss the symmetry in the Maxwell theory without charges.

The Noether theorem implies the existance of a corresponding conserved charge due to the electric–magnetic duality symmetry, which is given by

\[
G = \frac{1}{2} \int d^3x [\vec{E} \cdot \nabla^{-2} \vec{\nabla} \times \vec{E} - \vec{A} \cdot \vec{\nabla} \times \vec{A}].
\]

(2)

Because this is a symmetry generator, \( [H, G] = 0 \), where \( H \) is the Hamiltonian given by

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\[ H(\vec{E}, \vec{A}) = \frac{1}{2} \int d^3x [(\vec{E})^2 + (\nabla \times \vec{A})^2] \quad (3) \]

and \( \{ C, D \} = \frac{\delta C}{\delta x_a} \frac{\delta D}{\delta x_b} - \frac{\delta C}{\delta x_b} \frac{\delta D}{\delta x_a} \) is the Poisson bracket. \( \vec{E} \) represents the electric field, and the \( \vec{A} \) is the vector potential. The Maxwell action that we discuss is

\[ S = \int dt \left[ \int d^3x \vec{E}(t, x) \cdot \dot{\vec{A}}(t, x) - H(\vec{E}, \vec{A}) \right], \quad (4) \]

where the “·” between the two fields represents a scalar product between two vectors. The “·” on top of a field is the time derivative.

In this note, we rewrite the vector potential and the electric field in terms of certain creation and annihilation operators, which provide a Hilbert space of simultaneous eigenstates of the quantum operators of \( H \) and \( G \). Assumption that no negative norm state exist gives a quantization of the “G”-charge with unit 1. In fact, the charge “G” is the generator of the rotation of the polarization \([15, 16]\).

## 2 Quantization of the \( U(1) \) gauge fields with the electric–magnetic duality generator

In the first part of this note, we quantize the \( U(1) \) gauge field theory and construct quantum states labeled by their energy eigenvalues and electric magnetic duality charge. The traditional way to quantize in classical field theory is to solve the classical field equations, find positive and negative frequency modes, and promote the coefficient of each mode to an annihilation or creation operators satisfying a certain commutation relation between the operators. If one follows this standard process, the creation or the annihilation operators are, indeed, not the ladder operators for the electric–magnetic duality generator, \( G \), even though they are the ones for the Hamiltonian.

We start with definition of the creation and the annihilation operators, which are given by

\[ A_a(k) = \frac{1}{\sqrt{2|k|}} \left( A_a(k) + \hat{A}_a(k) \right), \quad \text{and} \]

\[ E_a(k) = -i \sqrt{\frac{|k|}{2}} \left( A_a(k) - \hat{A}_a(k) \right), \quad (5) \]

where \( A_a \) and \( E_a \) are the gauge fields and the electric fields, respectively, and \( \hat{A}_a \) and \( \hat{A}_a^\dagger \) are the creation and the annihilation operators in momentum space. The index \( a \) runs from 1 to 3, so the \( k_a \) are 3-momenta.

To construct the simultaneous eigenstates of the quantum operators of \( H \) and \( G \), we introduce other creation and annihilation operators given by (Fig. 1)

\[ D_{(\pm)c}^\dagger(k) \equiv A_c^\dagger(k) \pm i e_{abc} \frac{k_b}{|k|} A_a^\dagger(k), \quad (6) \]

\[ D_{(\mp)c}(k) \equiv A_c(k) \pm i e_{abc} \frac{k_b}{|k|} A_a(k). \quad (7) \]

The new operators have the following properties: \( D_{(\pm)c}^\dagger(k) \) are the creation operators in a sense that they increase the energy of the states by an amount \(|k|\), but \( D_{(\mp)c}^\dagger(k) \) increase the electric–magnetic duality charge with a unit of 1, whereas \( D_{(\mp)c}(k) \) decreases that charge by the same amount. Likewise, \( D_{(\mp)c}(k) \) are the annihilation operators for the Hamiltonian, but they change the electric–magnetic duality charge in amounts of ±1, respectively. Therefore, the quantum states are labeled using the two quantum numbers, energy, \( E \) and electric–magnetic duality charge \( g \), as \(|E, g\).

Interesting properties of the states are listed in order. First, the vacuum state is the \( g = 0 \) state. Therefore, we express the vacuum as \(|0, 0\). Second, for the \( N \)-particle eigen states of \( H \) and \( G \), \(|E, g\), if \( N \) is even, then \( g \) is an even number, and if \( N \) is odd, then \( g \) is an odd number.

The emergent \( SO(2, 3) \) from the bilinear operators of \( D_{(\mp)c}^\dagger(k) \) or \( D_{(\mp)c}(k) \). The second issue that we deal with is the bilinear operators and their group structure. We use ten independent bilinear operators made of the \( D_{(\mp)c}(k) \) and the \( D_{(\pm)c}^\dagger(k) \) operators. Four of them are given by
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\[
S_3 = \frac{1}{16} \sum_{\xi=\pm} \int d^3 k \left( D_{\xi \alpha}(k) D_{\xi \alpha}(k) \right) + D_{\xi \alpha}(k) D_{\xi \alpha}^\dagger(k) = \frac{1}{2} \hat{N}
\]

\[
= \frac{1}{4} \int d^3 k \left( A_{\alpha}(k) A_{\alpha}(k) + A_{\alpha}(k) A_{\alpha}^\dagger(k) \right),
\]

\[
L_2 = -\frac{1}{8} \int d^3 k \left( D_{(+)\alpha}(k) D_{(+)\alpha}(k) - D_{(-)\alpha}(k) D_{(-)\alpha}(k) \right) = -\frac{1}{2} \hat{G}
\]

\[
= -\frac{1}{2i} \int d^3 k \epsilon_{\alpha \beta \gamma} k_{\gamma} A_{\alpha}^\dagger(k) A_{\beta}(k),
\]

\[
K_2 = \frac{i}{8} \int d^3 k \left( D_{(+)\alpha}(k) D_{(+)\alpha}(k) - D_{(-)\alpha}(k) D_{(-)\alpha}(k) \right)
\]

\[
= \frac{i}{4} \int d^3 k \left( A_{\alpha}^\dagger(k) A_{\alpha}^\dagger(k) - A_{\alpha}(k) A_{\alpha}(k) \right),
\]

\[
Q_2 = -\frac{1}{8} \int d^3 k \left( D_{(+)\alpha}(k) D_{(+)\alpha}(k) + D_{(-)\alpha}(k) D_{(-)\alpha}(k) \right)
\]

\[
= -\frac{1}{4} \int d^3 k \left( A_{\alpha}^\dagger(k) A_{\alpha}^\dagger(k) + A_{\alpha}(k) A_{\alpha}(k) \right).
\]

These operators commute with the electric–magnetic duality generator, \( G \). The electric–magnetic duality generator in terms of the primitive annihilation and creation operators is given by

\[
G \equiv -i \int d^3 k \frac{\epsilon_{\alpha \beta \gamma} k_{\gamma}}{|k|} A_{\alpha}^\dagger(k) A_{\beta}(k).
\]

By employing commutation relations between \( G \) and \( A_{\alpha}^\dagger(k), A_{\alpha}(k) \), one can realize that \( G \) is a \( SO(2) \) rotation generator and \( \vec{A} = (A_1(k), A_2(k)) \) and \( \vec{\bar{A}} = (A_1^\dagger(k), A_2^\dagger(k)) \) are vectors under such a transform, where we have the direction of the momentum of the gauge fields to be \( \vec{k} = (0, 0, k) \). In fact, they transform as

\[
\left( \begin{array}{c} \mathcal{A}_1(k) \\ \mathcal{A}_2(k) \end{array} \right) \rightarrow \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} \mathcal{A}_1(k) \\ \mathcal{A}_2(k) \end{array} \right),
\]

as do the creation operators. Manifestly the above 4 bilinear operators are invariant under the \( SO(2) \) rotation, because they are either inner products or fully anti symmetric combination of the vector components of \( \mathcal{A} \) or \( \bar{\mathcal{A}} \). Therefore, they commute with one another.

The symmetric tensor parts of the bilinear combination of the vectors \( \vec{A} \) or \( \vec{\bar{A}} \) are given as

\[
L_1 = \frac{1}{8} \int d^3 k \left[ D_{+ (1)}^\dagger(k) D_{+ (1)}(k) + D_{+ (1)}^\dagger(k) D_{+ (1)}(k) \right],
\]

\[
L_3 = -\frac{i}{8} \int d^3 k \left[ \bar{D}_{+ (1)}^\dagger(k) D_{+ (1)}^\dagger(k) - \bar{D}_{+ (1)}^\dagger(k) D_{+ (1)}(k) \right],
\]

\[
K_1 = \frac{i}{16} \int d^3 k \left[ D_{+ (1)}^\dagger(k) D_{+ (1)}(k) - D_{+ (1)}^\dagger(k) D_{+ (1)}(k) \right]
\]

\[
+ D_{+ (1)}(k) D_{+ (1)}(k) - D_{+ (1)}(k) D_{+ (1)}(k),
\]

\[
K_3 = \frac{1}{8} \sum_{\xi=\pm} \int d^3 k \left[ D_{\xi (1)}^\dagger(k) D_{\xi (1)}^\dagger(k) + D_{\xi (1)}^\dagger(k) D_{\xi (1)}^\dagger(k) \right],
\]

\[
Q_1 = -\frac{1}{16} \int d^3 k \left[ D_{+ (1)}^\dagger(k) D_{+ (1)}^\dagger(k) - D_{+ (1)}^\dagger(k) D_{+ (1)}^\dagger(k) \right]
\]

\[
+ D_{+ (1)}(k) D_{+ (1)}^\dagger(k) + D_{+ (1)}(k) D_{+ (1)}^\dagger(k),
\]

\[
Q_3 = -\frac{i}{8} \sum_{\xi=\pm} \int d^3 k \left[ D_{\xi (1)}^\dagger(k) D_{\xi (1)}^\dagger(k) - D_{\xi (1)}^\dagger(k) D_{\xi (1)}^\dagger(k) \right].
\]

These bilinear operators are the symmetric traceless components of the \( SO(2) \) tensors as \( \mathcal{A}_{\alpha} A_{\alpha}^\dagger, A_{\alpha}^\dagger A_{\alpha}, \) or \( A_{\alpha} A_{\alpha}^\dagger \). Absolutely these will change under the \( SO(2) \) rotation, consequently, they do not commute with \( G \), i.e., \( L_2 \). The commutators between the bilinear operators are given in Table 1.

In fact, the tensor components transform under the \( SO(2) \) rotation as

\[
L'_1 = L_1 \cos(2\theta) + L_3 \sin(2\theta),
\]

\[
L'_3 = -L_1 \sin(2\theta) + L_3 \cos(2\theta),
\]

\[
K'_1 = K_1 \cos(2\theta) + K_3 \sin(2\theta),
\]

\[
K'_3 = -K_1 \sin(2\theta) + K_3 \cos(2\theta),
\]

\[
Q'_1 = Q_1 \cos(2\theta) + Q_3 \sin(2\theta),
\]

\[
Q'_3 = -Q_1 \sin(2\theta) + Q_3 \cos(2\theta).
\]
This means that combinations of the tensor components such as $L_1 + L_2$, $K_1 + K_2$ and $Q_1 + Q_2$ are invariant under electric–magnetic duality rotation.

3 More electric–magnetic duality-like symmetries

Once one observes the SO(2, 3) bilinear generators, one may realize that the operators $L_1$ and $L_3$ commute with the number operator (also with the Hamiltonian) and with $L_2$, which is the electric–magnetic duality generator. This means that $L_1$ and $L_3$ are candidates for the symmetry generator of the $U(1)$ gauge theory. The Hamiltonian is obtained using the Legendre transformation. Then, a quantity $\int d^3x \tilde{E}(x, t) \cdot \tilde{A}(x, t)$ is invariant under $L_1$ or $L_3$, the Lagrangian will be, too. In the following discussion, we formulate the symmetry transformation in the language of a SO(2) rotation, because the operators $L_1$ and $L_2$ are the tensor components of the SO(2) rotation along the direction of the momentum, $\tilde{k}$, which is generated from $G$. We set $\tilde{k} = |k| \hat{x}_3$, where $\hat{x}_3$ is the third directional unit vector in three-dimensional flat space.

The symmetry generator $L_1$ and its transformation To check if $L_1$ and $L_3$ are indeed a symmetry of the $U(1)$ gauge field theory action, we examine the $L_1$ operator first. We start with the relation between the creation and the annihilation operators and between the electric fields and the gauge fields, which are given by

$$A_a(k) = -\frac{1}{\sqrt{2|k|}} (A_a(k) + A_a^\dagger(k)), \quad \text{and}$$

$$E_a(k) = -i \sqrt{\frac{|k|}{2}} (A_a(k) - A_a^\dagger(k)).$$

Using the above expression, we get the transformation of the fields $A_a(k)$ and $E_a(k)$ under $L_1$, which are given by

$$\delta A_a = e[L_1, A_a(k)] = -i \frac{e}{2|k|} e^+_{3cd} \Pi_{ca}(k) E_d(k),$$

$$\delta E_a = e[L_1, E_a(k)] = i \frac{e}{2} |k| e^+_{3cd} \Pi_{ca}(k) A_d(k),$$

where $e^+_{abc}$ are symmetric and off-diagonal symbols defined as

$$e^+_{abc} = 0 \quad \text{if any of the indices are the same with an(other) other(s)},$$

$$e^+_{abc} = 1 \quad \text{if all the indices are different from one another}.$$

For instance, $e^+_{112} = e^+_{333} = 0$ and $e^+_{123} = e^+_{132} = 1$. Obviously that the Hamiltonian is invariant under the transformation. Because the action is obtained from the Hamiltonian using a Legendre transform as

$$S = \int d^3k dt E_a(k, t) \dot{A}_a(k, t) - \int H(E_a, A_a) dt,$$

and because the Hamiltonian is understood to be invariant under the above transformation, to prove that the action, $S$, is invariant, we need to show that $\int d^3k dt E_a(k, t) \dot{A}_a(k, t)$ does not change up to a total derivative under it. The change of the term is given by

$$\delta \left( \int d^3k dt E_a(k, t) \dot{A}_a(k, t) \right)$$

$$= \int d^3k dt \delta E_a(k, t) \dot{A}_a(k, t) + \int d^3k dt E_a(k, t) \delta \dot{A}_a(k, t)$$

$$= \int d^3k dt \left( i \frac{e}{4} |k| e^+_{3cd} \Pi_{ca}(k) \frac{\partial (A_d(k) A_a(-k))}{\partial t} - i \frac{e}{4} \frac{1}{|k|} e^+_{3cd} \Pi_{ca}(k) \frac{\partial (E_d(k) E_a(-k))}{\partial t} \right).$$
Therefore, manifestly $L_1$ is a symmetry generator.

One probably asks if a finite version of the transformation by exponentiating an infinitesimal one. One realizes that the creation and annihilation operators transform as

\[
\begin{pmatrix}
A_1' \\
A_2'
\end{pmatrix}
= 
\begin{pmatrix}
\cosh \theta & 0 \\
0 & \cosh \theta
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
A_1^{\dagger} \\
A_2^{\dagger}
\end{pmatrix}
= 
\begin{pmatrix}
\cosh \theta & \sinh \theta \\
\sinh \theta & \cosh \theta
\end{pmatrix}
\begin{pmatrix}
A_1^{\dagger} \\
A_2^{\dagger}
\end{pmatrix}.
\]

A pseudo rotation exists between them. The annihilation operators transform with the rotation angle $\theta$, whereas the creation operators transform with $-\theta$. The gauge and the electric fields change under the symmetry generator operation as

\[
\begin{pmatrix}
A_1' \\
A_2' \\
E_1' \\
E_2'
\end{pmatrix}
= 
\begin{pmatrix}
\cosh \theta & 0 & -i |k| \sinh \theta & 0 \\
0 & \cosh \theta & 0 & -i |k| \sinh \theta \\
i |k| \sinh \theta & 0 & \cosh \theta & 0 \\
0 & i |k| \sinh \theta & 0 & \cosh \theta
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2 \\
E_1 \\
E_2
\end{pmatrix}.
\]

Under such a transformation, the Hamiltonian (the number operator) is manifestly invariant. What matters is the tranform of the term $\int \dot{E}(k) \cdot \dot{\hat{A}}(k) \, d^3k$ in the action. This changes as

\[
\begin{align*}
\int \dot{E}(k) \cdot \dot{\hat{A}}(k) \, d^3k & \to \int \dot{E}(k) \cdot \dot{\hat{A}}(k) \, d^3k \\
& + \int \, d^3k \frac{\partial}{\partial t} \left( \sinh \frac{\theta}{2} (A_1 E_1 + A_2 E_2) \right) \\
& + \sinh \frac{\theta}{2} \cos \theta \left( i |k| A_1 A_2 - i |k| E_1 E_2 \right)
\end{align*}
\]

The symmetry generator $L_2$ and its transformation. The 2nd and last operator that we examine is the $L_2$ operator. The transformations of the fields $A_{\mu}(k)$ and $E_{\mu}(k)$ when we act on them with this operator are given by

\[
\begin{align*}
\delta A_{\mu} &= e[L_2, A_{\mu}(k)] = -\frac{i e}{2} \frac{\Pi_{\mu \nu}(-k)}{|k|} E_{\nu}(k), \\
\delta E_{\mu} &= e[L_2, E_{\mu}(k)] = \frac{i e}{2} |k| \Pi_{\mu \nu}(k) A_{\nu}(k),
\end{align*}
\]

where $\Pi_{\mu \nu}(k)$ is given by

\[
\Pi_{\mu \nu}(k) = -e_{3\alpha} e_{\nu \beta}.
\]

The Hamiltonian is invariant under the transformation. Again, we show that $\int d^3k \, d\beta A_4(k) \hat{A}_4(k, t)$ does not change up to the total derivative under this transform as

\[
\delta \left( \int d^3k \, d\beta A_4(k) \hat{A}_4(k, t) \right) = \int d^3k d\beta \left( \delta E_4(k, t) \hat{A}_4(k, t) + \delta \hat{A}_4(k, t) \hat{A}_4(k, t) \right) \\
= \int d^3k d\beta \left( \frac{e}{4} |k| \frac{\partial \hat{A}_4(k, t)}{\partial t} \right) \\
= \frac{e}{4} \int d^3k \left( \frac{|k|}{|k|} \frac{\partial \hat{A}_4(k, t)}{\partial t} \right).
\]

One probably asks if a finite version of the transformation can be obtained by exponentiating the infinitesimal one. One realizes that the creation and annihilation operators transform as

\[
\begin{pmatrix}
A_1' \\
A_2'
\end{pmatrix}
= 
\begin{pmatrix}
e^{-\frac{\theta}{2}} & 0 \\
0 & e^{-\frac{\theta}{2}}
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
A_1^{\dagger} \\
A_2^{\dagger}
\end{pmatrix}
= 
\begin{pmatrix}e^{-\frac{\theta}{2}} & 0 \\
0 & e^{-\frac{\theta}{2}}
\end{pmatrix}
\begin{pmatrix}
A_1^{\dagger} \\
A_2^{\dagger}
\end{pmatrix}.
\]

A kind of chiral rotation exists between them. The creation operators rotate with an angle opposite that of the annihilation operators. The gauge and the electric fields change under the symmetry generator operation as

\[
\begin{pmatrix}
A_1' \\
A_2' \\
E_1' \\
E_2'
\end{pmatrix}
= 
\begin{pmatrix}
\cosh \theta & 0 & -i |k| \sinh \theta & 0 \\
0 & \cosh \theta & 0 & -i |k| \sinh \theta \\
i |k| \sinh \theta & 0 & \cosh \theta & 0 \\
0 & i |k| \sinh \theta & 0 & \cosh \theta
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2 \\
E_1 \\
E_2
\end{pmatrix}.
\]

Under such a transformation, the Hamiltonian (the number operator) is manifestly invariant. What matters is the transform of the term $\int \dot{E}(k) \cdot \dot{\hat{A}}(k) \, d^3k$ in the action. This changes as

\[
\begin{align*}
\int \dot{E}(k) \cdot \dot{\hat{A}}(k) \, d^3k & \to \int \dot{E}(k) \cdot \dot{\hat{A}}(k) \, d^3k \\
& + \int \, d^3k \frac{\partial}{\partial t} \left( \sinh \frac{\theta}{2} (A_1 E_1 + A_2 E_2) \right) \\
& + \sinh \frac{\theta}{2} \cos \theta \left( i |k| A_1 A_2 - i |k| E_1 E_2 \right)
\end{align*}
\]

4 Discussion

In this note, we discuss electric–magnetic duality symmetry. Its generator is a member of the group SO(2, 3). The group SO(2, 3) is not a symmetry group of Maxwell theory.

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defined in $\mathbb{R}^{1,3}$. Under the transformations generated by $L_1$, $L_3$, $K_1$, $K_3$, $Q_1$, and $Q_3$, the Maxwell action changes. The set of $\{S_i, L_i\}$, where $i = 1, 2, 3$, forms a Cartan subgroup of $SO(2, 3)$, which is indeed the symmetry group of the Maxwell theory. This is $SL(2, \mathbb{R})$.

Free Maxwell theory enjoys a novel spacetime symmetry, which is the conformal symmetry group $SO(2, 4)$, which is ensured by the fact that stress–energy tensor of Maxwell theory is traceless. The Poincare and the Lorentz groups, $SO(1, 4) \supset SO(1, 3)$, are the subgroups of the conformal symmetry group. The Poincare group is isomorphic to the group of $SO(2, 3)$ that we obtained in this note taking account of Wick rotation of one of the non-compact direction to a compact one. The structures are similar, but their physical origins are different.

We also see this as follows: we note that the number operator and the electric–magnetic duality operator have definite physical meanings. Especially, the electric–magnetic duality operator is the helicity operator of photon states. Because Maxwell theory is translationally invariant, one may consider the corresponding Noether charges as symmetry generators. Four generators, the Hamiltonian density and the momentum density operators, are $\mathcal{H} = |k|A_a^0(k)A_a(k)$ and $\mathcal{P} = kA_a^0(k)A_a(k)$. However, in $SO(2, 3)$, the symmetric combinations of $A_a^0(k)$ and $A_a(k)$ with their index summation, i.e., $\sum_a k A_a^0(k)A_a(k)$, is the number operator only. Therefore, no one-to-one correspondence between the spacetime symmetry generators and the SO(2, 3) generators is found.

Acknowledgements J.H.O thanks his $\mathcal{W}/\mathcal{J}/\mathcal{D}$ and $\mathcal{D}/\mathcal{D}$. He also thanks Hyun Seok Yang for useful discussions. This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP) (No.2016R1C1B1010107) and Research Institute for Natural Sciences, Hanyang University.

Appendix A: Quantization with the electric–magnetic duality generator and the quantum operators and states

The electric–magnetic duality generator in momentum space is given by

$$G = \frac{i}{2} \int d^3 k \epsilon_{abc} \left[ E_a(k) \frac{k_b}{k^2} E_c(-k) + A_a(k) k_b A_c(-k) \right],$$

where $E_a(k)$ are the electric fields and $A_a(k)$ are the gauge fields. They are a canonical pair and so satisfy the following Poisson bracket relation:

$$\{A_b(x), E_a(y)\} = \delta_{ab} \delta^{(3)}(x - y). \quad (38)$$

The two fields satisfy the Gauss constraint $\partial_\mu E^\mu = 0$ and this ensures that they are transverse fields. Therefore, the Poisson bracket relation can be modified as

$$\{A^\mu_a(x), E^\nu_a(y)\} = \Pi_{ab}^{\mu \nu}(x - y), \quad (39)$$

where the $\Pi_{ab}^{\mu \nu}(x) = \left( \frac{\delta_{ab} - \frac{\partial_k}{\sqrt{k^2}} \right)$ is the project operator.

Quantization Maxwell’s theory is mathematically a collection of two independent harmonic oscillators:

$$A_a(x) = \frac{1}{2\pi} \int d^3 k \exp(ik_p x_p) \sum_{A=1,2} q_A(k)e^{iA}(k),$$

$$E_a(x) = \frac{1}{2\pi} \int d^3 k \exp(ik_p x_p) \sum_{A=1,2} p^{iA}(k)e^{iA}(k), \quad (40)$$

where the indices $a, b ...$ are three-dimensional spatial indices and $A, B ...$ are the polarization indices. The $e^{iA}(k)$ is the polarization vector, $e^{iA}(k)k_a = 0$ to ensure that the fields $A_a(x)$ and $E_a(x)$ are transverse and $e^{iA}(k)e^{iA}(k) = \delta^{ii}, p^{iA}(k) = p^{iA}(k)^*$, $q^{iA}(k) = q^{iA}(k)^*$ and $e^{iA}(k) = e^{iA}(k)^*$, because the fields are real. $e^{iA}(k_1)e^{iA}(k_2) \equiv \Pi^{ii}_{A}(k_1 - k_2) = \delta_{ii} - \frac{k_k k_{k}}{k^2}$.

The Fourier transform

$$\phi(x) = \frac{1}{2\pi} \int d^3 k \exp(ik_p x_p)\phi(k), \quad (41)$$

defines the fields in momentum space as

$$A_a(k) = q_A(k)e^{iA}(k) \quad \text{and} \quad E^a(k) = p^{iA}(k)e^{iA}(k). \quad (42)$$

Their Poisson brackets are given by

$$\{A_b(k_1), E^a(k_2)\} = \Pi^{ab}_{ii}(k_1 + k_2),$$

$$\{q_A(k_1), p^{iA}(k_2)\} = \delta^{ii}_{A}(k_1 + k_2). \quad (43)$$

We define creation and annihilation operators as

$$a_A(k) = \frac{1}{\sqrt{|k|}} q_A(k) + \frac{i}{\sqrt{|k|}} p_A(k),$$

$$a^\dagger_A(k) = -i \sqrt{|k|} \left( q_A(-k) - \frac{i}{|k|} p_A(-k) \right),$$

and the inverse relations are given by

$$q_A(k) = \frac{1}{\sqrt{2|k|}} (a_A(k) + a^\dagger_A(-k)),$$

$$p_A(k) = -i \sqrt{|k|} (a_A(k) - a^\dagger_A(-k)). \quad (45)$$

The final forms of the Poisson bracket are given by
\[\{A^j_a(k),G\} = \epsilon_{abc} \frac{k_b}{|k|} \Pi_{ab}(k)A^l_a(k),\]
\[\{A_a(k),G\} = \epsilon_{abc} \frac{k_b}{|k|} \Pi_{ab}(k)A_a(k),\]
\[\{A_a(k),A^l_a(k')\} = -i \Pi_{ab}(k)\delta^{(3)}(k-k'),\]
\[\{H,A_a(k)\} = i|k|A_a(k),\]
\[\{H,A^l_a(k)\} = -i|k|A^l_a(k),\]

where \(A_a(k) = e^a(-k)A_a(k)\) and \(A^l_a(k) = e^a(k)a_A^l(k)\). The quantization of the fields is performed by switching the Poisson brackets to the commutators as \(\{\} \to -i[\] and promoting all fields to quantum operators. The forms of the Hamiltonian and electric–magnetic duality operators are given by

\[H = \int d^3k |k|a_A^\dagger(k)a^l(k) \equiv \int d^3k |k|A^l_a(k)A_a(k),\]

and

\[G = -i \int d^3k \epsilon_{abc}e^a(k)e^b(-k)\frac{k_b}{|k|}a^l_A(k)a_b(k)\]
\[\equiv -i \int d^3k \epsilon_{abc}e^a(k)A^l_a(k)A_a(k).\]

The annihilation and creation operators are ladders for the Hamiltonian, but those do not increase or decrease the eigenvalues of the operator, \(G\). To find simultaneous ladders for the \(H\) and \(G\), we define

\[D^j_{(\pm)c}(k) = A^j_c(k) \pm i\epsilon_{abc}\frac{k_b}{|k|}A^l_a(k),\]

and

\[D^j_{(\mp)c}(k) = A^j_c(k) \mp i\epsilon_{abc}\frac{k_b}{|k|}A_a(k).\]

Then, the commutation relations can be modified to

\[\left[D^j_{(\pm)c}(k),G\right] = \pm D^j_{(\mp)c}(k), \quad [D^j_{(\mp)c}(k),G] = \mp D^j_{(\pm)c}(k),\]
\[\left[H,D^j_{(\pm)c}(k)\right] = i|k|D^j_{(\pm)c}(k), \quad [H,D^j_{(\pm)c}(k)] = -i|k|D^j_{(\mp)c}(k),\]
\[\left[D^j_{(\pm)c}(k),D^j_{(\mp)d}(k')\right] = 2\delta^{(3)}(k-k')\left(\Pi_{cd}(k) \pm i\epsilon_{ced}\frac{k_e}{|k|}\right).\]

**Appendix B: Construction of the \([SO(2, 3)]\) group from bilinear operators from \(D^j_{(\pm)a}(k)\) and \(D^j_{(\pm)a}(k)\)**

We start with new definitions of \(D^j_{(\pm)a}(k)\) and \(D^j_{(\pm)a}(k)\) for further convenience, which are given by

\[D^j_{(\pm)c}(k) = A^j_c(k) - i\xi\epsilon_{abc}\frac{k_b}{|k|}A^l_a(k),\]

\[D^j_{(\pm)c}(k) = A^j_c(k) + i\xi\epsilon_{abc}\frac{k_b}{|k|}A_a(k),\]

where for \(\xi = \pm 1\), \(D^j_{(\pm)c}(k)\) represents \(D^j_{(\pm)c}(k)\) and for \(\xi' = \pm 1\), \(D_{(\pm)c}(k)\) represents \(D_{(\pm)c}(k)\), respectively. The bilinear operators constructed out of the operators \(D^j_{(\pm)c}(k)\) and \(D^j_{(\mp)c}(k)\) have the following forms:

\[\int d^3k D^j_{(\pm)c}(k)D^j_{(\mp)c}(k)d^3k,\]

\[\int d^3k D^j_{(\mp)c}(k)D^j_{(\pm)c}(k)d^3k,\]

\[\int d^3k D^j_{(\pm)c}(k)D^j_{(\mp)c}(k)d^3k,\]

\[\int d^3k D^j_{(\mp)c}(k)D^j_{(\pm)c}(k)d^3k,\]

where the third and the fourth operators are the same up to a constant(a-c-number). Because we are interested in their commutation relations, we regard the third and the fourth operators as the same. One can also classify the operators into their trace, anti-symmetric and traceless-symmetric parts.

First of all, we discuss their trace parts, which are listed below:

\[\int d^3k D^j_{(\pm)c}(k)D^j_{(\pm)c}(k) = (1 - \xi')\int d^3k A^j_a(k)A^l_a(k),\]

\[\int d^3k D^j_{(\mp)c}(k)D^j_{(\mp)c}(k) = (1 - \xi')\int d^3k A^j_a(k)A_a(k),\]

and

\[\int d^3k D^j_{(\pm)c}(k)D^j_{(\mp)c}(k) = (1 + \xi')\int d^3k A^j_a(k)A_a(k)\]
\[-i(\xi + \xi')\int d^3k \epsilon_{ced}\frac{k_e}{|k|}A^l_a(k)A_a(k).\]

When \(\xi\) and \(\xi'\) take the same sign, the bilinear operators Eqs. (58) and (59) become null identically. If they take different signs as equations \((\xi, \xi') = (+1, -1)\) or \((\xi, \xi') = (-1, +1)\), then (58) and (59) are proportional to the trace of the primitive creation and annihilation operators,
respectively. Because the operators are not Hermitian, we constitute their appropriate linear combinations to be Hermitian operators. They are nothing but the $K_2$ and the $Q_2$ operators listed below.

\[
K_2 = \frac{i}{8} \int d^3k (D^{(\alpha)}_{+:+}(k)D^{(+\alpha)}_{-}(k) - D^{(+\alpha)}_{-}(k)D^{(\alpha)}_{+:+}(k))
= \frac{i}{4} \int d^3k (A^{(+\alpha)}_{\alpha}(k)A^{(-\alpha)}_{\alpha}(k) - A^{(-\alpha)}_{\alpha}(k)A^{(+\alpha)}_{\alpha}(k)), \tag{61}
\]

\[
Q_2 = -\frac{1}{8} \int d^3k (D^{(+\alpha)}_{-}(k)D^{(+\alpha)}_{-}(k) + D^{(+\alpha)}_{-}(k)D^{(+\alpha)}_{+}(k))
= -\frac{1}{4} \int d^3k (A^{(+\alpha)}_{\alpha}(k)A^{(+\alpha)}_{\alpha}(k) + A^{(+\alpha)}_{\alpha}(k)A^{(+\alpha)}_{\alpha}(k)). \tag{62}
\]

Only when $\xi$ and $\xi'$ take the same sign, does Eq. (60) become non-trivial and is a linear combination of the number operator and the electric–magnetic duality operator. Together with this, we examine the anti-symmetric parts of the bilinear operators, which are given by

\[
\int d^3k D^{(\alpha)}_{(\xi)\alpha} D^{(\xi')\alpha}(k) = i(-\xi + \xi') \int d^3k \frac{k_f}{|k|} (\epsilon_{\xi\alpha\beta} A^{\alpha}_{\beta} A^{\alpha}_{\beta}) + \epsilon_{\xi\alpha\beta} A^{\alpha}_{\beta} A^{\alpha}_{\beta}), \tag{63}
\]

\[
\int d^3k D^{(\alpha)}_{(\xi)\alpha} D^{(\xi')\alpha}(k) = i(-\xi + \xi') \int d^3k \frac{k_f}{|k|} (\epsilon_{\xi\alpha\beta} A^{\alpha}_{\beta} A^{\alpha}_{\beta} - \epsilon_{\xi\alpha\beta} A^{\alpha}_{\beta} A^{\alpha}_{\beta}), \tag{64}
\]

\[
\int d^3k D^{(\alpha)}_{(\xi)\alpha} D^{(\xi')\alpha}(k) = \int d^3k \left[ A^{\alpha\alpha} A^{\alpha}_{\alpha} - A^{\alpha\alpha} A^{\alpha}_{\alpha} + \frac{\xi' \epsilon_{\xi\alpha\beta} k_f}{|k|} A^{\alpha}_{\beta} A^{\alpha}_{\beta}(\epsilon_{\xi\alpha\beta} e_{\xi\alpha\beta} - \epsilon_{\xi\alpha\beta} e_{\xi\alpha\beta}) + \frac{k_f}{|k|} (i \xi' A^{\alpha}_{\alpha} A^{\alpha}_{\alpha} + i \xi A^{\alpha}_{\alpha} A^{\alpha}_{\alpha}) - \frac{k_f}{|k|} (i \xi' A^{\alpha}_{\alpha} A^{\alpha}_{\alpha} + i \xi A^{\alpha}_{\alpha} A^{\alpha}_{\alpha}) \right]. \tag{65}
\]

Equations (63) and (64) turns out not to be independent operators. Once we contract the operators with a tensor $\frac{k_f}{|k|} \epsilon_{\alpha\beta\gamma}$, they become proportional to Eqs. (58) and (59), respectively. The same contractions acting on Eq. (65) leads another linear combination of the number and electric–magnetic duality operators. Therefore, appropriate combinations of Eqs. (65) and (60) provide the operators:

\[
S_3 = \frac{1}{16} \int d^3k (D^{(\alpha)}_{+:+}(k)D^{(+\alpha)}_{-}(k) + D^{(+\alpha)}_{-}(k)D^{(\alpha)}_{+:+}(k))
+ D^{(+\alpha)}_{-}(k)D^{(+\alpha)}_{-}(k) + D^{(+\alpha)}_{-}(k)D^{(\alpha)}_{+:+}(k))
= \frac{1}{4} \int d^3k (A^{(+\alpha)}_{\alpha}(k)A^{(+\alpha)}_{\alpha}(k) + A^{(+\alpha)}_{\alpha}(k)A^{(+\alpha)}_{\alpha}(k)) = \frac{1}{2} \mathcal{N},
\]

\[
L_2 = -\frac{1}{8} \int d^3k (D^{(+\alpha)}_{-}(k)D^{(+\alpha)}_{-}(k) - D^{(-\alpha)}_{-}(k)D^{(-\alpha)}_{-}(k))
= -\frac{1}{2} \int d^3ke^{\alpha}_{\alpha}(k) A^{(\alpha)}_{(\alpha)}(k) = -\frac{1}{2} \mathcal{G}.
\]

To construct the symmetric parts of the bilinear operators, we utilize the following identities:

\[
D^{(\alpha)}_{(\xi)\alpha} D^{(\xi')\alpha}(k) = 2A^{(\alpha)}_{\alpha} A^{\alpha}_{\alpha} - 2\xi \epsilon_{\xi\alpha\beta} k_f \frac{|k|}{k_f^{\alpha\beta}} A^{(\alpha)}_{\beta} A^{(\alpha)}_{\beta}
- i(\xi + \xi') \frac{k_f}{|k|} (\epsilon_{\xi\alpha\beta} A^{(\alpha)}_{\beta} + \epsilon_{\xi\alpha\beta} A^{(\alpha)}_{\beta}).
\]

\[
D^{(\alpha)}_{(\xi)\alpha} D^{(\xi')\alpha}(k) = 2A^{(\alpha)}_{\alpha} A^{\alpha}_{\alpha} - 2\xi \epsilon_{\xi\alpha\beta} k_f \frac{|k|}{k_f^{\alpha\beta}} A^{(\alpha)}_{\beta} A^{(\alpha)}_{\beta}
+ i(\xi + \xi') \frac{k_f}{|k|} (\epsilon_{\xi\alpha\beta} A^{(\alpha)}_{\beta} A^{(\alpha)}_{\beta}).
\]

\[
D^{(\alpha)}_{(\xi)\alpha} D^{(\xi')\alpha}(k) = A^{(\alpha)}_{\alpha} A^{\alpha}_{\alpha} + A^{(\alpha)}_{\alpha} A^{\alpha}_{\alpha}
+ \xi \epsilon_{\xi\alpha\beta} k_f \frac{|k|}{k_f^{\alpha\beta}} A^{(\alpha)}_{\beta} A^{(\alpha)}_{\beta}
+ \epsilon_{\xi\alpha\beta} A^{(\alpha)}_{\beta} A^{(\alpha)}_{\beta} - i\xi A^{(\alpha)}_{(\alpha)} A^{(\alpha)}_{\alpha}
+ \epsilon_{\xi\alpha\beta} A^{(\alpha)}_{\beta} A^{(\alpha)}_{\alpha} - i\xi A^{(\alpha)}_{(\alpha)} A^{(\alpha)}_{\alpha}.
\]

Because they are tensor components, they change under $SO(3)$ spatial rotation. Therefore, we choose our frame as $k = (0, 0, k)$. then, the spatial index $a$ in $D^{(\alpha)}_{a\xi}(k)$ and $D^{(\alpha)}_{a\xi}(k)$ can take 1 or 2. All possible (linearly independent of one another) choices are listed below:

\[
L_1 = \frac{1}{8} \int d^3k [D^{(\alpha)}_{+:+}(k)D^{(+\alpha)}_{-}(k) + D^{(+\alpha)}_{-}(k)D^{(\alpha)}_{+:+}(k)],
\]

\[
L_3 = -\frac{i}{8} \int d^3k [D^{(\alpha)}_{+:+}(k)D^{(+\alpha)}_{-}(k) - D^{(+\alpha)}_{-}(k)D^{(\alpha)}_{+:+}(k)],
\]

\[
K_1 = \frac{i}{16} \int d^3k [D^{(\alpha)}_{+:+}(k)D^{(+\alpha)}_{-}(k) - D^{(+\alpha)}_{-}(k)D^{(\alpha)}_{+:+}(k)]
+ D^{(+\alpha)}_{-}(k)D^{(+\alpha)}_{-}(k) - D^{(+\alpha)}_{-}(k)D^{(\alpha)}_{+:+}(k)],
\]

\[
K_3 = \frac{1}{8} \int d^3k [D^{(\alpha)}_{+:+}(k)D^{(+\alpha)}_{-}(k) + D^{(+\alpha)}_{-}(k)D^{(\alpha)}_{+:+}(k)]
+ D^{(+\alpha)}_{-}(k)D^{(+\alpha)}_{-}(k) + D^{(+\alpha)}_{-}(k)D^{(\alpha)}_{+:+}(k)],
\]

\[
Q_1 = -\frac{1}{16} \int d^3k [D^{(\alpha)}_{+:+}(k)D^{(+\alpha)}_{-}(k) - D^{(+\alpha)}_{-}(k)D^{(\alpha)}_{+:+}(k)]
- D^{(+\alpha)}_{-}(k)D^{(+\alpha)}_{-}(k) + D^{(+\alpha)}_{-}(k)D^{(\alpha)}_{+:+}(k)],
\]

\[
Q_3 = \frac{i}{8} \int d^3k [D^{(\alpha)}_{+:+}(k)D^{(+\alpha)}_{-}(k) + D^{(+\alpha)}_{-}(k)D^{(\alpha)}_{+:+}(k)]
- D^{(+\alpha)}_{-}(k)D^{(+\alpha)}_{-}(k) - D^{(+\alpha)}_{-}(k)D^{(\alpha)}_{+:+}(k)].
\]
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