1 INTRODUCTION

Application of building materials of high mechanical properties (i.e., steel), leads to the reduction of dimensions (thickness) of structural plate-like elements. This maintains the same structural capacity with reduced material cost, as well the price reduction. However, slender elements become vulnerable to buckling, leading to the use of stiffened plates.

Stiffened plates are widely used in civil engineering and other engineering fields, especially in design of long-span bridges with small cross-sections, construction of ship hulls, aircrafts, etc. During their working life, these structures are usually exposed to dynamic loading, thus the calculation of dynamic structural response is of high
克拉申ђеног обрадова овога зна-
жања у инжењерској прaksi. У таквим случајевима, неопходно је одредити основне динамичке карактеристике система, како су попут френквенице и облици осциловања.

Динамички одговор танких плоча може се одредити пременом Kirchhoff-ove класичне теорије плоча (classical plate theory - CPT). У случају деоблих плоча, ова теорија не дaje адекватне резултате zbog занемаривања деформације смicanja, па je потребно primeniti Mindlin-ovu теорију (first-order shear deformation theory - FSDT), koja uzima у обзир uticaj деформације смicanja, pretpostavljajući да je klizanje konstantno по deobли роље. Paralelno sa razvojem različитих теорија плоча, развијале су се и одговарајуће аналитичке методе [1]. Međutim, one se zasnivaju na tačном rešenju diferenцијалних jednačina kretanja плоча, sa specijalним uslovima oslanjanja. Leissa [2] je dao sveobuhvatan pregled аналитичких решења сlobodних вбрижача плоча разлиčитих облика, zasnovan на Kirchhoff-овoj класичној теорији плоча. Liew i ostali [3] анализирали су сlobodne вбрации debelih плоча sa proizvoljним konturnим uslovima, primenom Rayleigh-Ritz-ove методе. Primena pomenutih метода ограничена je на анализу slobodnih вбрација pojedinačних плоча и не може se lako proširiti на анализу složenijih система плоча sa različitim геометrijама и материјалним карактеристикама, какви најчешће jesu u inženjerskoj praksi (нпр. плоче sa ukrućenjima, sendvič-paneli, плоче promenljive deoblje). U таквим случајевима, у анализи se primenjuju numeričke методе, као што je метода konačnih еlementа (FEM) [4]. Poznata je иницијала da, у финалној анализи пременом MKE veličina konačног елемента zavisi од dužine talasa najviše френквенице од интереса за анализу. Prema [5, 6], однос између talasne dužine највише френквенице и већ најчешће елемента trebalo bi da se kreće у границима између 10 i 20, kako bi se dobili резултати zadovoljavajuće tačnosti. Minimalan broj konačних еlementа direktno je proporcionalan највишој разма-
tranoj френквеници, pa u случају složenih konstrukcija (gde je при анализи потrebno imati у виду и više тонове осциловања) potreban broj konačних еlementа постаје veliki, чиме se povećava ukupno trajanje proračuna.

У poslednje vreme, за анализу slobodnih вбрациja плоча се често koristi метода динамичке крутости (MDK) [7-15]. MDK kombinuje карактеристике MKE - kao što су физичка дискретизација и могућност povezuivanja еlementа u jedinstveni globalni sistem - sa rešenjem polja pomeranja, koje представља tačно rešenje дифerencijалне jednačine slobodних вбрациja. Како интерпо-
lacijе функције које се опишуе поле pomeranja u MDK представљају tačно rešenje дифerencijалне jednačine kretanja в frekventnom domену, грешке usled априкоси-
macije су eliminsane. Podela плоче на manje динамичке елементe неопходна je samo unutar plоče постоji neki геометријски и/или физички diskontinuitet. Time se smanjuju broj еlementа u анализи, broj stеpeni slobode, kao и време потребно за рад i могућност javljanja грешке, u poredenju s MKE. Za razliku od MKE - где je masa koncentrisана u čvorovima konačних еlementа, u MDK masa je kontinualno raspodeljena. Такоде, матери-
rijalno пругашење може se на jednostavan način uključiti u analizu putem kompleksnог модула elastičnosti $E_e = E(1+2\nu)$, gde je $\nu$ коэффент relativa пругашења.

The dynamic stiffness method (DSM) - 7-15 - is increasingly used in the free vibration analysis of plates. DSM combines the properties of the FEM (such as physical discretization and the possibility of assembly of single elements into the unique global system) with the solution of the displacement field in the form of exact solution of differential equation of the free vibration problem. Having in mind that interpolation functions which describe the displacement field in the DSM represent the exact solution of the differential equation of motion in the frequency domain, the approximation errors are eliminated. The plate discretization is necessary only if some geometrical and/or physical discontinuity is present. This reduces both the numbers of elements in the analysis and degrees of freedom, as well as computational time and error possibility, in comparison with the FEM. In contrary to the FEM where the mass is lumped in nodes of finite elements, in the DSM the mass is continuously distributed. In addition, material damping can be easily included in the analysis using the complex elasticity modulus $E_e = E(1+2\nu)$, where $\nu$ is the relative damping coefficient. Using this
Na taj način, omogućeno je da različiti elementi konstrukcije imaju različito prigušenje, što je još jedna od prednosti MDK u poređenju s MKE.

U okviru ovog rada, prikazan je numerički model za analizu slobodnih neprigušenih vibracija Mindlin-ovih ploča sa ukrcuvenjima sa proizvoljnim grančnim uslovima, primenom MDK. Na osnovu dinamičkih matrica krutosti za analizu slobodnih poprečnih vibracija i vibracija u ravni, izvedena je matrica rotacije za različite položaje ploča koje su pod pravim uglom u odnosu na referentnu ravan [16]. Primijenjen je sličan postupak kao u MKE za formiranje globalne dinamičke matrice krutosti ploče sa ukrcuvenjima i razvijen je računarski program u MATLAB-u [17] za analizu slobodnih vibracija sistema ploča. Prikazani postupak verifikovan je upoređivanjem tih rezultata sa rezultatima dobijenim primenom programskog paketa Abaqus [18].

2 POSTUPAK FORMIRANJA DINAMIČKE MATRICE KRUTOSTI PRAVOUGAONE PLOČE

Postupak formiranja dinamičke matrice krutosti pravougaonog elementa ploče za poprečne i vibracije u ravni detaljno je prikazan u radovima [13, 15], dok će ovdje biti prikazani osnovni koraci u postupku formiranja dinamičke matrice krutosti. Polaznu tačku predstavljaju jednačine kretanja elementa Mindlin-ove ploče u vremenskom domenu:

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + (1-\frac{v}{2}) \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial y} + a_1 \frac{\partial^2 u}{\partial x^2} + a_2 \frac{\partial^2 v}{\partial x^2} + k \frac{\partial^2 w}{\partial x^2} &= \rho \frac{h}{E} \frac{\partial^2 u}{\partial t^2} \\
\frac{\partial^2 v}{\partial y^2} + (1-\frac{v}{2}) \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y} + a_1 \frac{\partial^2 v}{\partial y^2} + a_2 \frac{\partial^2 w}{\partial y^2} + k \frac{\partial^2 w}{\partial y^2} &= \rho \frac{h}{E} \frac{\partial^2 v}{\partial t^2}
\end{align*}
\]

(1a)

\[
\begin{align*}
D \left( \frac{\partial^2 \phi_x}{\partial x^2} - 2 \frac{\partial^2 \phi_x}{\partial x \partial y} + 2 \frac{\partial^2 \phi_x}{\partial y^2} \right) + k \frac{\partial^2 \phi_x}{\partial x^2} &= \rho \frac{h^3}{12} \frac{\partial^2 \phi_x}{\partial t^2} \\
D \left( \frac{\partial^2 \phi_y}{\partial y^2} - 2 \frac{\partial^2 \phi_y}{\partial x \partial y} + 2 \frac{\partial^2 \phi_y}{\partial y^2} \right) - k \frac{\partial^2 \phi_y}{\partial y^2} &= \rho \frac{h^3}{12} \frac{\partial^2 \phi_y}{\partial t^2}
\end{align*}
\]

(1b)

gde \( u, v, w, \phi_x, \phi_y \) predstavljaju komponentalna pomeranja i rotacije, \( h \) je debljina ploče, \( \rho \) je gustina, \( E \) je modul elastičnosti, \( G \) je modul smicanja, \( v \) je Poissonov koeficijent, \( D=Eh^3/12(1-\nu^2) \) je krutost ploče na savijanje, \( D=\rho E h(1-\nu^2) \) je krutost ploče u ravni, \( a_1=(1-\nu)/2, a_2=(1+\nu)/2 \) i \( k=5/6 \) je faktor korekcije smicanja.

Pretpostavlja se da su pomeranja harmonijske funkcije frekvencije \( \omega \), odnosno:

\[
\mathbf{u}(x,y,t) = \mathbf{\hat{u}}(x,y,\omega) e^{i\omega t}
\]

(2)

gde \( \mathbf{\hat{u}}(x,y,\omega) \) predstavlja amplitudu polja pomeranja \( u, v, w, \phi_x, \phi_y \) u frekventnom domenu. Na osnovu ove pretpostavke, jednačine kretanja (1) iz vremenskog domena transformišu se u frekventni domen.

Na slici 1a prikazano je polje pomeranja pravougaone ploče po Mindlin-ovoj teoriji.

2 PROCEDURE FOR DEVELOPMENT OF DYNAMIC STIFFNESS MATRIX OF RECTANGULAR PLATE

Development of the dynamic stiffness matrix for rectangular plate element undergoing transverse and in-plane vibrations has been given in detail in [13, 15]. In the paper, only the basic steps in the procedure will be presented. The procedure starts from with equations of motion of Mindlin plate element in the time domain:

where \( u, v, w, \phi_x, \phi_y \) are the displacement components, \( h \) is the plate thickness, \( \rho \) is the mass density, \( E \) is the modulus of elasticity, \( G \) is the shear modulus, \( v \) is the Poisson's coefficient, \( D=\rho E h(1-\nu^2) \) is the flexural plate stiffness, \( D_1=\rho E h(1-\nu^2) \) is the in-plane plate stiffness, \( a_1=(1-\nu)/2, a_2=(1+\nu)/2, \) and \( k=5/6 \) is the shear correction factor.

It is assumed that the displacements are the harmonic functions of frequency \( \omega \), i.e.:

\[
\mathbf{u}(x,y,t) = \mathbf{\hat{u}}(x,y,\omega) e^{i\omega t}
\]

(2)

where \( \mathbf{\hat{u}}(x,y,\omega) \) is the amplitude of the displacement field \( u, v, w, \phi_x, \phi_y \) in the frequency domain. Using the above assumption, equations of motion (1) are transformed from time to frequency domain.

Figure 1 shows the displacement field of the rectangular plate based on Mindlin plate theory.
Proizvoljno polje pomeranja može se prikazati kao superpozicija rešenja za četiri slučaja simetrije u odnosu na x i y koordinate ose: simetrija-simetrija (SS), simetrija-antimetrija (SA), antimetrija-simetrija (AS) i antimetrija-antimetrija (AA):

\[ \hat{u}(x, y, \omega) = \hat{u}^{SS}(x, y, \omega) + \hat{u}^{SA}(x, y, \omega) + \hat{u}^{AS}(x, y, \omega) + \hat{u}^{AA}(x, y, \omega) \]  

(3)

Na slici 1b prikazana je simetrična deformacija ploče (SS) po jednom rešenju za kretanje. Podelom polja pomeranja na četiri slučaja simetrije, moguće je analizirati samo jednu četvrtinu ploče, čime se znatno umanjuje red dinamičkih matrica krutosti i ubrzava proračun. Rešenje jednačina kretanja u frekvencijskom domenu pretpostavlja se u obliku beskonačnog Fourier-ovog reda u sledećem obliku:

\[ \hat{u}^{ij}(x, y, \omega) = \sum_{m=0,1} C_m^{ij} f_m^{ij}(x) g_m^{ij}(y) \]  

(4)

Arbitrary displacement field can be represented as a superposition of four symmetry contributions with respect to x and y coordinate axes: symmetric-symmetric (SS), symmetric-anti-symmetric (SA), anti-symmetric-symmetric (AS) and anti-symmetric-anti-symmetric (AA):

where \( f_m^{ij}(x), g_m^{ij}(y) \) are the basis trigonometric functions which depend both on symmetry case \((i,j)\) and the solution of corresponding equations of motion, \( C_m^{ij} \) are the integration constants, while \( i, j = S, A \).

Based on the well-known kinematic and constitutive relations of the plate, as well as the equation (4), the force vector in an arbitrary point of the plate can be written as:

\[ \hat{f}^{ij}(x, y, \omega) = \sum_{m=0,1} C_m^{ij} f_m^{ij}(x) g_m^{ij}(y) \]  

(5)

where \( f_m^{ij}(x), g_m^{ij}(y) \) are derivations of the base functions depending on assumed plate theory. For practical purposes, the infinite series in equations (4) and (5) is truncated to point \( M \) (the number of terms in the series expansion), thus the accuracy of the solution practically depends only on \( M \).

The next step is the definition of the displacement vectors \( \hat{q}^{ij} \) along boundaries \( x = a \) and \( y = b \) of the quarter of the plate, for each symmetry contribution, which are derived by substituting the boundary coordinates in the equation (4):

\[ \hat{q}^{ij} = \begin{bmatrix} \hat{u}^{ij}(a, y, \omega) \\ \hat{u}^{ij}(x, b, \omega) \end{bmatrix} \]  

(6)
In a similar manner, the force vector $\mathbf{\hat{Q}}$ on the plate boundaries is obtained by replacing the coordinates of the boundaries in equation (5):

$$
\mathbf{\hat{Q}} = \begin{bmatrix}
\hat{\mathbf{Q}}^i(x, b, \omega) \\
\hat{\mathbf{Q}}^j(x, b, \omega)
\end{bmatrix}
$$

(7)

Since the components of the displacement and force vectors along plate boundaries are functions of spatial coordinates $x$ and $y$, it is impossible to establish a direct relation between these vectors on one side, and the vector of integration constants $\mathbf{C}$ on the other side. This problem can be solved by using the projection method, which is based on the representation of the displacement and force functions along the plate boundary in the Fourier series form:

$$
\mathbf{q}^i = \frac{2}{L_s} \int H^i \hat{\mathbf{q}} ds = D^i \mathbf{C}
$$

$$
\mathbf{\hat{Q}}^i = \frac{2}{L_s} \int H^i \mathbf{\hat{Q}} ds = F^i \mathbf{C}
$$

(8)

gде $H^i$ матрица базних функций за одговарајући симетрије, $L=a$ за контуру паралелну са x-осом, док je $L=b$ за контуру паралелну са y-осом. Елиминациjом вектора интеграциjних константи из jedнаčina (8) добија се динамиčка матрица крутеће четвртине пločе $\mathbf{D}_K$ за сваки од четири симетриjе:

$$
\mathbf{q}^i = F^i (D^i)^{-1} \mathbf{q}^6 = \mathbf{K}_D^i \mathbf{q}^6
$$

(9)

Динамиčка матрица крутеће четвртине пločе $\mathbf{K}_D$ може се добити примењом transfer матрице, као што je покazano u radovima [13, 15].

3 PLOČE SA UKRUČENJIMA

Попредне вибрације и вибрације у равни за једну изотропну пloču представљају два независна стана. Стога, динамиčка матрица крутеће пloчe може се написати као:

$$
\mathbf{K}_D = \begin{bmatrix}
\mathbf{K}_D^T & 0 \\
0 & \mathbf{K}_D^B
\end{bmatrix}
$$

(10)

gде je $\mathbf{K}_D^T$ динамиčка матрица крутеће пloчe изложене попредним вибрациjама, док je $\mathbf{K}_D^B$ динамиčка матрица крутеће пloчe за вибрације у равни. Сагласно jedнаčini (10), вектор проекциjа pомерanja и sila на konturи пloчe може се приказати u следећем obлику:

Динамиčка матрица крутеће пloчe за вибрацијe у равни $\mathbf{K}_D^B$ може се добити примењом transfer матрице, као што je показано u radovima [13, 15].
U slučaju ploča sa ukrućenjima, gde su ploče međusobno spojene pod pravim uglom, poprečne vibracije jedne ploče izazivaju vibracije u ravnim drugih ploča i obrnuto. Zbog toga je potrebno uspostaviti vezu između vektora projekcija pomeranja i sila lokalnog i vektora $q^i$ i $Q^i$ u globalnom koordinatnom sistemu (slike 2 i 3), pomoću matrice rotacije $T$:

$$q^i = Tq^i, \quad Q^i = TQ^i$$

For stiffened plates (where the plates are perpendicular to each other), transverse vibrations of a single plate cause the in-plane vibrations of a corresponding perpendicular plate, and vice versa. Consequently, it is necessary to establish the relation between the projection vectors of displacements $\vec{q}^i$ and forces $\vec{Q}^i$ in the local, and the corresponding projection vectors $\vec{q}^i$ and $\vec{Q}^i$ in the global c. s. (Figures 2 and 3). This is accomplished by using the rotation matrix $T$:

$$q^i = Tq^i, \quad Q^i = TQ^i$$
According to the established relations between the projection vectors in the local and global coordinate systems, dynamic stiffness matrix of the plate in global coordinate system is derived as:

\[
\mathbf{K}_D^G = \mathbf{T}^T \mathbf{K}_D^T \mathbf{T}
\]

Dynamic stiffness matrices of individual plates are assembled in the global dynamic stiffness matrix of the plate assembly, using similar assembly procedure as in the conventional FEM. Note that the plates are connected along the boundaries instead at nodes. In the analysis, arbitrary boundary conditions can be applied.

The dynamic stiffness matrix is the square, frequency-dependent matrix whose size depends on the number of terms \( M \) in the general solution. The natural frequencies can be determined from the following equation:

\[
\det \left[ \mathbf{K}_{DG,nn}^G (\omega) \right] = 0
\]

where \( \mathbf{K}_{DG,nn}^G \) is the global dynamic stiffness sub-matrix of the plate assembly related to the unknown generalized displacement projections.

Since the elements of the dynamic stiffness matrix \( \mathbf{K}_{DG,nn}^G \) contain transcendental functions, the solutions can be obtained applying some of the search methods. To avoid numerical difficulties when calculating the zeroes of equation (14), the natural frequencies can be determined as maxima of the following expression:

\[
g(\omega) = \log \frac{1}{\det \left[ \mathbf{K}_{DG,nn}^G (\omega) \right]}
\]

4 NUMERICAL EXAMPLES

Application of the dynamic stiffness method in the free vibration analysis of stiffened plates is illustrated in the following examples. In order to validate the presented method based on the DSM, a computer code in MATLAB [17] was developed for the free vibration analysis of stiffened plates with arbitrary boundary conditions. The results were compared with the results obtained using Abaqus [18].

The first example is concerned with a reinforced concrete girder of a box cross-section, \((E = 31.5 \text{ GPa}, \nu = 0.2\) and \(\rho = 2500 \text{ kg/m}^3\), \(\epsilon_1 = 200 \text{ mm}^2/\text{m}^4\)) with geometrical and concrete density parameters.

4 NUMERICAL EXAMPLES
• uključena ivica paralelna sa y-osom (C): 
  \[ u=v=w=\phi_z=0, \]
• slobodna ivica paralelna sa y-osom (F): \( \phi_z=0 \).

Prvih deset sopstvenih frekvencija \( f = \omega/2\pi \) sračunato je primenom različitih broja članova reda, kako bi se utvrdila konvergencija rešenja. Rezultati su prikazani u Tabeli 1 i upoređeni sa numeričkim rešenjem dobijenim primenom 46250 konačnih elemenata tipa S4R u Abaqus-u (dimenzija elementa 0.1 m). Za rešenje s \( M=7 \) članova reda, sračunato je odstupanje \( \Delta \) od rešenja u Abaqus-u.

\[
\begin{array}{cccccccc}
\text{M} & \text{f} & \text{Abaqus} & \Delta \% \\
\hline
1 & S - F & S - F & 9.9 & 0.35 \\
2 & S - F & S - F & 16.6 & 0.29 \\
3 & S - F & S - F & 23.7 & 0.52 \\
4 & S - F & S - F & 27.2 & 0.45 \\
5 & S - F & S - F & 27.7 & 0.05 \\
6 & S - F & S - F & 28.9 & 0.40 \\
7 & S - F & S - F & 37.6 & 0.59 \\
8 & S - F & S - F & 37.9 & 0.24 \\
9 & S - F & S - F & 38.7 & 0.41 \\
10 & S - F & S - F & 43.7 & 0.04 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{M} & \text{f} & \text{Abaqus} & \Delta \% \\
\hline
1 & S - C & S - C & 16.8 & 4.06 \\
2 & S - C & S - C & 29.2 & 5.31 \\
3 & S - C & S - C & 32.9 & 1.01 \\
4 & S - C & S - C & 38.8 & 0.85 \\
5 & S - C & S - C & 41.5 & 2.35 \\
6 & S - C & S - C & 45.5 & 4.32 \\
7 & S - C & S - C & 50.5 & 0.35 \\
8 & S - C & S - C & 51.2 & 2.33 \\
9 & S - C & S - C & 55.4 & 2.79 \\
10 & S - C & S - C & 61.9 & 3.32 \\
\end{array}
\]

The first ten natural frequencies computed using the proposed method are in excellent agreement with the results obtained using the FEM (the average difference is 0.33% for S-F-S-F and 2.67% for S-C-S-C case), which confirms the exceptional possibilities of the application of the dynamic stiffness method in the free vibration analysis of reinforced concrete plates, even for a small number of terms in the general solution (M=7).

The computational time has been significantly decreased in comparison with the FEM.

\[ \text{Table 1: Sopstvene frekvencije [Hz] armirano-betonskog nosača sandučastog preseka i (b) armirano-betonske korube} \]

\[ \text{Figure 4. Geometry and boundary conditions of: (a) reinforced concrete girder with box cross-section and (b) reinforced concrete ribbed slab} \]

\[ \text{Tabeli 1. Sopstvene frekvencije [Hz] armirano-betonskog nosača sandučastog preseka} \]

\[ \text{Table 1. Dimensionless natural frequencies [Hz] of reinforced concrete girder of a box cross-section} \]

\[ \text{Table 1. Sopstvene frekvencije [Hz] armirano-betonskog nosača sandučastog preseka} \]

\[ \text{Table 1. Dimensionless natural frequencies [Hz] of reinforced concrete girder of a box cross-section} \]
The second example is concerned with the reinforced concrete ribbed slab, \( E = 31.5 \text{ GPa}, \nu = 0.2 \) and \( r = 2500 \text{ kg/m}^2), whose geometry and boundary conditions are presented in Figure 4b. In this case the number of dynamic stiffness elements is equal to 5, while the number of boundary lines is equal to 16. The first ten natural frequencies \( f = \omega/2\pi \) have been calculated using different number of terms in the general solution. The results are presented in Table 2 and compared with the FEM numerical solutions obtained using 23520 Abaqus S4R type plate element (element size 0.025 m). For the solutions with \( M = 7 \) terms in the series expansion, the differences \( \Delta \) from the solutions obtained using Abaqus are calculated. As in the previous example, the natural frequencies computed using the proposed method are in excellent agreement with the FEM (the average difference is 0.22% for S-F-S-F and 0.14% for S-C-S-C case).

| Table 2. Dimensionless natural frequencies [Hz] of reinforced concrete ribbed slab |
|---------------------------------|-----------------|-----------------|
| \( n \) | S – F – S – F | S – C – S – C |
| \( n \) | DSM | Abaqus | \( \Delta \) [%] | DSM | Abaqus | \( \Delta \) [%] |
| 1 | 33.9 | 33.0 | 32.9 | 32.9 | 32.8 | 0.34 | 34.0 | 57.7 | 57.5 | 57.5 | 57.5 | 0.03 |
| 2 | 36.4 | 35.5 | 35.4 | 35.3 | 35.3 | 0.14 | 40.7 | 85.7 | 84.9 | 84.7 | 84.5 | 0.20 |
| 3 | 40.7 | 55.4 | 58.0 | 57.9 | 57.9 | -0.08 | 52.3 | 110.8 | 110.7 | 110.7 | 110.7 | -0.02 |
| 4 | 52.7 | 77.8 | 76.9 | 76.7 | 76.5 | 0.32 | 52.7 | 116.6 | 115.7 | 115.4 | 115.1 | 0.22 |
| 5 | 52.6 | 87.2 | 86.2 | 85.9 | 85.5 | 0.43 | 58.4 | 129.8 | 141.6 | 141.2 | 141.0 | 0.13 |
| 6 | 60.0 | 91.7 | 90.5 | 90.1 | 89.8 | 0.29 | 59.7 | 134.9 | 145.7 | 145.4 | 145.1 | 0.21 |
| 7 | 67.1 | 96.9 | 96.3 | 96.1 | 95.8 | 0.27 | 67.3 | 137.3 | 155.1 | 164.6 | 164.5 | 0.03 |
| 8 | 78.4 | 109.7 | 108.6 | 108.3 | 108.3 | 0.05 | 78.8 | 144.4 | 189.0 | 188.9 | 189.0 | -0.04 |
| 9 | 79.4 | 129.3 | 131.3 | 132.7 | 132.3 | 0.29 | 79.6 | 146.7 | 200.3 | 199.6 | 199.1 | 0.27 |
| 10 | 82.8 | 133.9 | 136.1 | 135.6 | 135.6 | 0.01 | 82.7 | 155.8 | 201.6 | 201.6 | 202.0 | -0.22 |

Figure 5 illustrates the comparison of some mode shapes and the corresponding natural frequencies (modes 2, 4, 7 and 8) of the reinforced-concrete ribbed slab, for the combination of boundary conditions S-C-S-C and \( M=7 \) terms in the series expansion. Mode shapes have been calculated using the original MATLAB code based on the dynamic stiffness method, as well as by using the commercial software Abaqus. Excellent agreement has been obtained.

To illustrate the advantage of the DSM in comparison with the FEM by means of the necessary computational time, number of degrees of freedom in the numerical model of the girder with the box cross-section, formulated using the DSM or the FEM, will be compared.

It is worth mentioning that the number of degrees of freedom in the DSM depends on the number of contours and number of terms in series expansion \( M \), while in the FEM it depends on the number of nodes and the element type (in the considered example, the finite element has 6 nodal degrees of freedom). Number of degrees of freedom in the DSM (for \( M=7 \)) and the FEM is therefore:

\[
\text{NDOF}_{\text{DSM}} = N_{\text{contours}} \cdot (9M + 18) = 26 \cdot (9 \cdot 7 + 18) = 1924
\]

\[
\text{NDOF}_{\text{FEM}} = N_{\text{nodes}} \cdot 6 = 46250 \cdot 6 = 277500
\]

that confirms the potential of the presented method by means of the necessary computational time.
5 CONCLUSION

This paper deals with the application of the dynamic stiffness method in the free vibration analysis of stiffened plates. The procedure for the development of dynamic stiffness matrix of rectangular plate, as well as rotation matrices for different plate orientations and perpendicularly oriented with respect to the reference plane, has been provided. The similar procedure as in the FEM for the development of the global dynamic stiffness matrix of the stiffened plate, has been applied. Computer program in MATLAB has been developed for the free vibration analysis of plate assemblies. The natural frequencies and mode shapes were determined for different combinations of boundary conditions.
Dobijeni rezultati verifikovani su upoređivanjem sa rezultatima dobijenih primenom programskog paketa Abaqus zasnovanog na MKE. Tačnost rezultata dobijenih primenom MDK zavisi isključivo od broja članova reda usvojenog rešenja. Uočena je brza konvergencija rezultata dobijenih po MDK. Već sa tri člana do pet članova reda, dobijaju se rezultati visoke tačnosti. Međutim, za više tonove oscilovanja - povećava se broj potrebnih članova reda.

Na osnovu dobijenih rezultata, može se zaključiti da MDK poseduje veliki potencijal u dinamičkoj analizi konstrukcija, koji se može proširiti na analizu ploča i ljudski zasnovane na teoriji višeg reda, ploče spojene pod proizvoljnim uglom, kao i konstruktivne elemente sačinjene od modernih kompozitnih materijala.

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REZIME

PRIMENA METODE DINAMIČKE KRUTOSTI U NUMERIČKOJ ANALIZI SLOBODNIH VIBRACIJA PLOČA SA UKRUČENJIMA

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U okviru ovog rada, analizirane su slobodne vibracije ploča sa ukručenjima, primenom metode dinamičke krutosti. Formulisan je pravougaoni element Mindlin-ove ploče, zasnovan na metodi dinamičke krutosti. Primenom matrice rotacije, formirane su dinamičke matrice krutosti pojedinačnih ploča u globalnom koordinatnom sistemu. Korišćenjem sličnog postupka kao u metodi konačnih elemenata, izvedena je globalna dinamička matrica krutosti sistema ploča. Određene su sopstvene frekvencije ploča sa ukručenjima za različite konturne uslove i upoređene sa vrednostima dobijenim u komercijalnom programskom paketu Abaqus. Dobijeni su rezultati visoke tačnosti.

Ključne reči: slobodne vibracije, dinamička matrica krutosti, ploče sa ukručenjima

SUMMARY

APPLICATION OF DYNAMIC STIFFNESS METHOD IN NUMERICAL FREE VIBRATION ANALYSIS OF STIFFENED PLATES

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The free vibration analysis of stiffened plate assemblies has been performed in this paper by using the dynamic stiffness method. Rectangular Mindlin plate dynamic stiffness element has been formulated. Using the rotation matrices, dynamic stiffness matrices of single plates have been derived in global coordinate system. The global dynamic stiffness matrix of plate assembly has been derived by using similar assembly procedure as in the finite element method. The natural frequencies of stiffened plate assemblies with different boundary conditions have been computed and validated against the results obtained by using the commercial software package Abaqus. High accuracy of the results has been demonstrated.

Key words: free vibrations, dynamic stiffness matrix, stiffened plates