Color Magnetic Flux Tubes in Dense QCD

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Abstract

QCD is expected to be in the color-flavor locking phase in high baryon density, which exhibits color superconductivity. The most fundamental topological objects in the color superconductor are non-Abelian vortices which are topologically stable color magnetic flux tubes. We present numerical solutions of the color magnetic flux tube for diverse choices of the coupling constants based on the Ginzburg-Landau Lagrangian. We also analytically study its asymptotic profiles and find that they are different from the case of usual superconductors. We propose the width of color magnetic fluxes and find that it is larger than naive expectation of the Compton wave length of the massive gluon when the gluon mass is larger than the scalar mass.

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1 Introduction

One of the important problems for understanding the strong interaction is to determine the phase diagram of QCD. In the very high density region with a large chemical potential at low temperature, it is expected that QCD is in the color-flavor locking (CFL) phase which exhibits color superconductivity \[1, 2\]. There is the $SU(3)_L \times SU(3)_R$ flavor symmetry acting on light quarks $u, d, s$ when their masses are very small compared to the chemical potential. The $SU(3)_C$ color symmetry is locked with the flavor symmetry by the condensations

$$
\Phi_R \sim \langle \psi_R \sigma_2 \psi_R \rangle, \quad \Phi_L \sim \langle \psi_L \sigma_2 \psi_L \rangle,
$$

where $\psi_{L,R}$ are the quark fields. Apart from the discrete symmetry the symmetry of the system is

$$
G = SU(3)_C \times SU(3)_L \times SU(3)_R \times U(1)_B
$$

where the first element and the rests stand for gauge and flavor symmetries, respectively. We assume that the ground state is the positive parity state $\Phi_R = \Phi_L \equiv \Phi$ which would be determined by the instanton effect, so the chiral symmetry is broken to $SU(3)_{L+R}$ which we write $SU(3)_F$. The color superconductivity is expected to be realized in the core of a neutron star.

In the usual superconductors the gauge group $U(1)$ for the electro-magnetic force is broken in the vacuum where electrons condensate to make a Cooper pair. When the superconductor is in the external magnetic fields which are above the critical value, magnetic fields inside the superconductor are squeezed and quantized. Consequently there appear Abrikov-Nielsen-Olesen(ANO) vortices as magnetic flux tubes which topologically wind the spontaneously broken $U(1)$ \[3, 4\]. The stability of the superconductor under the external magnetic fields is determined by static forces among vortices. If the force is repulsive (type II) the superconductor is stable whereas if the force is attractive (type I) it is unstable. In the former the vortices constitute the so-called Abrikosov lattice \[3\]. Therefore understanding the interaction between vortices is very important to study stability of superconductors.

The types of color superconductor (type I or type II), which characterize a response to external magnetic fields, were studied in \[5\] by considering a domain wall separating the normal phase and the superconducting phase. They concluded that it is of type I in the weak coupling region.
It is, however, more important to study vortices in color superconductor, in order to understand its stability or a response to (external) color magnetic fields and/or rotation of the media which exhibits superfluidity too. Color magnetic vortices were studied in [6, 7, 8] but these are unstable to decay because they are not topologically protected. On the other hand the superfluid vortices which wind the broken $U(1)_B$ appear in a response to rotation [9, 7]. They are topologically stable but are dynamically unstable as we discuss below.

The most fundamental objects are non-Abelian vortices carrying color magnetic fluxes [10, 11, 12], which were called the semi-superfluid vortices. Let us explain these objects. The order parameter space, parameterized by (would-be) Nambu-Goldstone(NG) modes of the symmetry breaking in the CFL phase, is

$$M \simeq \frac{SU(3) \times U(1)}{\mathbb{Z}_3} = U(3).$$

(1.3)

These NG modes are eaten by the $SU(3)$ gauge fields (gluons) except for the $U(1)$ part. This has a non-trivial first homotopy group;

$$\pi_1 (M) = \pi_1 [U(3)] = \mathbb{Z}.$$  

(1.4)

The $U(1)_B$ vortex found in [9, 7] is of the form $\Phi = v f(r)1_3$ with the boundary conditions $f(r \to \infty) = 1$ and $f(r = 0) = 0$. However this does not have minimum winding number but has the triple of the minimum element. The fundamental vortex-string with the minimum number is a non-Abelian vortex [10, 11, 12], for instance given as

$$\Phi = v \text{diag}(f(r)e^{i\theta}, g(r), g(r)),$$  

$$A_i \sim \frac{\epsilon_{ij} x^j}{r^2} [1 - h(r)] \text{diag}(2, -1, -1)$$

(1.5)

with the boundary conditions $f(r \to \infty), g(r \to \infty) = 1, h(r \to \infty) = 0$ and $f(r = 0) = 0, g'(r = 0) = 0, h(r = 0) = 1$. The asymptotic behavior $(r \to \infty)$ of the scalar field of the non-Abelian vortex is

$$\Phi \to v \text{diag}(e^{i\theta}, 1, 1) = ve^{i\theta} \text{diag}(e^{i\phi}, e^{-i\phi}, e^{-i\phi}) \sim ve^{i\phi}1_3,$$  

(1.6)

1 In a realistic situation such as a neutron star, the external color magnetic field cannot be considered because it is in the confining phase outside the core of a neutron star. Our statement about a response to color magnetic field is of a theoretical interest. The vortex here should be created under a rapid rotation of superfluids, or under a phase transition in a rapid cooling of a neutron star by the Kibble-Zurek mechanism [10].
where the last denotes a gauge transformation by $U(r, \theta) = \text{diag} \left( e^{-i \frac{2}{3} \theta E(r)}, e^{i \frac{2}{3} \theta E(r)}, e^{\frac{2}{3} \theta E(r)} \right)$ with an arbitrary function $E(r)$ satisfying the boundary conditions $E(r = 0) = 0$ and $E(r \to \infty) = 1$.

From Eq. (1.6) one can understand that the non-Abelian vortex has the $U(1)_B$ winding $1/3$ of that of the $U(1)_B$ superfluid vortex. The $U(1)_B$ symmetry is global and so the tension (the energy per unit length) of a vortex-string is logarithmically divergent in infinite space. Since the tension is proportional to the square of the winding number, the non-Abelian vortex has the tension $1/9$ of the one of the $U(1)_B$ superfluid vortex. One $U(1)_B$ superfluid vortex has the triple amount of the sum of the energies of three well-separated non-Abelian semi-superfluid vortices. Therefore the decay of $U(1)_B$ superfluid vortex into three non-Abelian semi-superfluid vortices is inevitable from a point of view of the energetics.

The non-Abelian semi-superfluid vortex (1.5) carries a color magnetic flux. Color magnetic fluxes also exist in quark gluon plasma [14] but they are unstable. Contrary to those, color magnetic flux tube (1.5) is topologically (and dynamically) stable. The vortex solution (1.5) breaks the color-flavor locked symmetry $SU(3)_{C+F}$ down to its subgroup $[SU(2) \times U(1)]_{C+F}$. Consequently there appear further NG zero modes [11]

$$\mathbb{C}P^2 = \frac{SU(3)_{C+F}}{[SU(2) \times U(1)]_{C+F}}. \tag{1.7}$$

This space parameterizes a continuous family of the vortex solutions. These zero modes are called orientational zero modes. All solutions of the continuous family in Eq. (1.7) have the same tension and the same boundary condition (1.6) up to a regular gauge transformation. Therefore the orientational zero modes of $\mathbb{C}P^2$ (1.7) are normalizable and can be regarded as the genuine moduli (collective coordinates) of the vortex [18]. It corresponds one-to-one to the color

\begin{itemize}
\item[2] This transformation is well-defined in the whole space because of the triviality of the first homotopy group: $\pi_1[SU(N)] = 0$.
\item[3] The authors in [13] studied different kind of magnetic flux tubes in dense QCD. Although the authors called them “gluon vortices”, those are magnetic for usual electro-magnetic force of $U(1)_{EM}$ (mixed with one color component of $SU(3)_C$). The important difference between their flux tubes and our color magnetic flux tubes is that their vortices are generated by the unbroken generator and therefore topologically trivial, while our vortices are generated by the broken generators and consequently topologically non-trivial. We will give some comments in the discussion.
\item[4] Those flux tubes are topological unstable but it was discussed in [15] that they are dynamically metastable.
\item[5] The idea was brought from non-Abelian vortices in supersymmetric $U(N)$ gauge theories. In this case the overall phase $U(1)_B$ is also gauged and the vortices are local vortices with finite tension [16], unlike non-Abelian semi-superfluid vortices. See [17] for a review.
\end{itemize}
magnetic flux which the vortex carries.

As in usual superconductors, the (in)stability of the color superconductor in the presence of the (external) color flux is determined by the interaction between non-Abelian semi-superfluid vortices (color magnetic fluxes) given in Eq. (1.5). The asymptotic interaction between two well-separated non-Abelian semi-superfluid vortices has been calculated [11, 12] in which the universal repulsion has been found. This calculation is valid when the distance between them is much larger than the Compton wave lengths of massive particles, which are essentially the penetration depth and the coherence length. In this region, semi-superfluid vortices are essentially $U(1)_B$ global vortices with the winding number $1/3$ as in Eq. (1.6) and, in fact, the static force between them is $1/3$ of that between two $U(1)_B$ vortices [11, 12]. This result implies important consequences. First the color superconductor is stable in the presence of the (external) color magnetic fields. Non-Abelian semi-superfluid vortices will constitute a lattice, at least when the lattice spacing is much larger than the penetration depth and the coherence length. Second the $U(1)_B$ superfluid vortex is unstable to decay into three non-Abelian semi-superfluid vortices because of the repulsion among them, at least for large fluctuations. Each semi-superfluid vortex carries different color magnetic flux with the total color being cancelled out. In the core of neutron star, one can expect that the $U(1)_B$ superfluid vortices are first created in a response to rotation of the star. Then each of them must be broken into three semi-superfluid vortices which will constitute the lattice of color magnetic flux tubes. Phase boundary of the CFL and hadronic phase was studied in [19] in an application to the neutron star physics.

However the interaction obtained in [11, 12] is not valid when two vortices are closer such that the distance between them is of the order of the Compton wave lengths. We need to know the core structure of the vortex in order to study short range interactions. In the analysis of Balachandran et. al [10], they used an approximation for profile functions in (1.5): they assumed a constant $g$ and solved equation for $f$ only. This is not a good approximation around the core of the vortex. In fact, as we show in this paper, the profile of $g$ significantly differs from the constant for some parameter region.

In this paper we study the profile functions of non-Abelian semi-superfluid vortices in detail as a first step to study the short range interaction of them. First we study the asymptotic tails of the profile functions by analytically studying the equations of motion, and find that they are different from the case of the ANO vortex in usual superconductors where the tails of the
$U(1)$ gauge field with mass $m_e$ and the scalar field with mass $m_H$ decay exponentially as $e^{-m_e}$ and $e^{-m_H}$, respectively; on the other hand the both tails of the scalar and gauge fields of a non-Abelian superfluid vortex behave as $e^{-m}$ with the lighter mass $m$ among the masses $m_G$ for massive gluons and $m_\chi$ for massive traceless scalar fields. We then construct numerical solutions for diverse choices of the coupling constants, by using the relaxation method with the appropriate boundary conditions. By the numerical solutions we determine the coefficients in the asymptotic solutions. We propose the width of the color magnetic flux by the diffractiveness weighted by the magnetic flux. We calculate it numerically and confirm the above estimation of the tails. Therefore we conclude that the width of the color magnetic flux is significantly different from the naive expectation of the Compton wave length (the penetration depth) of the massive gluons when the gluon mass $m_G$ is larger than the mass $m_\chi$ of the traceless scalar fields; In general the width approaches to a constant depending of $m_\chi$ not behaving as $m_G^{-1}$.

Throughout this paper we turn off the electro-magnetic interaction $U(1)_{EM}$ in order to study purely non-Abelian aspects of vortices. The inclusion of it does not change the properties found in this paper. However the electro-magnetic interaction explicitly breaks the flavor symmetry $SU(3)_F$, since $U(1)_{EM}$ is embedded into $SU(3)_F$. Consequently the orientation modes of $CP^2$ get masses. This aspect remains as a future problem.

This paper is organized as follows. In Sec. 2 the Ginzburg-Landau Lagrangian is studied. We explain the symmetry structure in detail especially paying attention to the discrete symmetries. We also give the mass spectra of the CFL vacuum. In Sec. 3 we study the equations of motion in the cylindrical coordinates in the presence of a vortex. In Sec. 4 asymptotic behaviors of the profile functions are studied analytically. In Sec. 5 we give numerical solutions of the profile functions and determine the coefficients of the asymptotic profile functions. We numerically evaluate the width of the color magnetic flux and compare it with the Compton wave length of the massive gluons. Sec. 6 is devoted to conclusion and discussions. We discuss the force between two semi-superfluid vortices at the short distance. We discuss that the inter-vortex force mediated by the exchange of the massive particles becomes comparable with the inter-vortex force mediated by the exchange of the massless $U(1)_B$ NG boson. In Appendix we discuss general ansatz of the vortex in the diagonal entries.
2 The Non-Abelian Landau-Ginzburg Model

Our starting point is the Ginzburg-Landau Lagrangian \[20, 21, 6\]

\[
L = \text{Tr} \left[ -\frac{1}{4} F_{ij} F^{ij} + \nabla_i \Phi^\dagger \nabla^i \Phi - \lambda_2 (\Phi^\dagger \Phi)^2 + \mu^2 \Phi^\dagger \Phi \right] - \lambda_1 \left( \text{Tr} [\Phi^\dagger \Phi] \right)^2 - \frac{3 \mu^4}{4 (3 \lambda_1 + \lambda_2)} \tag{2.1}
\]

where the last constant term is introduced for the vacuum energy to vanish. Our notation is \( \nabla_i = \partial_i - i g_s A_i \), \( F_{ij} = \partial_i A_j - \partial_j A_i - i g_s [A_i, A_j] \) and \( \text{Tr} [T^a T^b] = \delta^{ab} \) for \( a = 1, 2, \cdots, 8 \). \( g_s \) is \( SU(3)_C \) gauge coupling constant. For the stability of vacua, we consider the parameter region \( \mu^2 > 0, \lambda_2 > 0 \) and \( 3 \lambda_1 + \lambda_2 > 0 \). The action of color, flavor and baryon symmetries on \( \Phi \) is given by

\[ \Phi \rightarrow e^{i \theta} U_C \Phi U_F, \quad U_C \in SU(3)_C, \: U_F \in SU(3)_F, \: e^{i \theta} \in U(1)_B. \tag{2.2} \]

There is some redundancy of the action of these symmetries. The actual symmetry is given by \( G \equiv \frac{SU(3)_C \times SU(3)_F \times U(1)_B}{(Z_3)^{C+B} \times (Z_3)^{F+B}} \), \( (Z_3)^{C+B} \): \( (w^k 1_3, w^{-k} 1_3) \in SU(3)_C \times SU(3)_F \times U(1)_B \), \( (Z_3)^{F+B} \): \( (1_3, w^k 1_3, w^{-k}) \in SU(3)_C \times SU(3)_F \times U(1)_B \), \( w = e^{2 \pi i/3}, \quad k = 0, 1, 2. \tag{2.6} \)

For later use let us redefine the discrete symmetry in the denominator as \( (Z_3)^{C+B} \times (Z_3)^{F+B} \simeq (Z_3)^{C+F} \times (Z_3)^{C-F+B} \) with

\[
(Z_3)^{C+F} : \quad (w^k 1_N, w^{-k} 1_N, 1) \in SU(3)_C \times SU(3)_F \times U(1)_B, \tag{2.7}
\]

\[
(Z_3)^{C-F+B} : \quad (w^k 1_N, w^k 1_N, w^{-2k}) \in SU(3)_C \times SU(3)_F \times U(1)_B. \tag{2.8}
\]

Next we discuss symmetry breaking in the vacua. By using the symmetry \( G \), one can choose a vacuum expectation value (VEV) as

\[
\langle \Phi \rangle = v 1_3, \quad v^2 \equiv \frac{\mu^2}{2 (3 \lambda_1 + \lambda_2)} > 0 \tag{2.9}
\]

\( ^6 \) The stiffness parameter in front of the kinetic term of \( \Phi \) is set to be 1 by a field redefinition.
without loss of generality. By this condensation the gauge symmetry $SU(3)_C$ is completely broken, and the full symmetry $G$ is spontaneously broken down to

$$H = \frac{SU(3)_{C+F} \times (\mathbb{Z}_3)_{C-F+B}}{(\mathbb{Z}_3)_{C+B} \times (\mathbb{Z}_3)_{F+B}} = SU(3)_{C+F}$$

(2.10)

with

$$SU(3)_{C+F} : (U, U^\dagger, 1) \in SU(3)_C \times SU(3)_F \times U(1)_B.$$  

(2.11)

Therefore as denoted in Eq. (1.3) the order parameter space (the vacuum manifold) is given by

$$M = G/H = \frac{SU(3)_{C-F} \times U(1)_B}{(\mathbb{Z}_3)_{C-F+B}} \approx U(3)$$

(2.12)

with

$$SU(3)_{C-F} : (U, U, 1) \in SU(3)_C \times SU(3)_F \times U(1)_B.$$  

(2.13)

This space is parameterized by $SU(3)$ would-be NG bosons, which are eaten by $SU(3)_C$ gauge bosons (gluons), and one massless NG boson of the spontaneously broken $U(1)_B$.

The mass spectra around the Higgs vacuum (2.9) can be found by perturbing $\Phi$ as

$$\Phi = v1_3 + \frac{\phi + i\varphi}{\sqrt{2}}1_3 + \frac{\chi^a + i\zeta^a}{\sqrt{2}}T^a.$$  

(2.14)

The trace parts $\phi$ and $\varphi$ belong to the singlet of the color-flavor locked symmetry whereas the traceless part $\chi$ and $\zeta^a$ belong to the adjoint representation of it. The $SU(3)$ gauge fields (gluons) get mass with eating $\zeta^a$ by the Higgs mechanism. The masses of fields are given by

$$m^2_G = 2g^2 v^2, \quad m^2_\phi = 2\mu^2, \quad m^2_\varphi = 0, \quad m^2_\chi = 4\lambda_2 v^2,$$

(2.15)

where $m_G$ is the mass of the $SU(3)$ massive gauge bosons (gluons) and $\varphi$ is the NG boson associated with the spontaneously broken $U(1)_B$ symmetry. The trace part $\phi$ and the traceless part $\chi$ of $\Phi$ are massive bosons. At the weak coupling regime these masses are shown to be

$$m_G \gg m_\phi = 2m_\chi.$$  

(2.16)

We would like to stress that this system has four different mass scales. As in usual superconductors, $m_G^{-1}$ is the penetration depth and $m_\chi^{-1}$ is the coherence length. On the other hand,

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\footnote{We thank N. Yamamoto for this point.}
the massive boson $\phi$ with mass $m_\phi$ and the massless NG mode $\varphi$ for the spontaneously broken $U(1)_B$ are typical for the superfluid system with a superfluid vortex of the size $m_\phi^{-1}$. Therefore the system is mixed up with the (non-Abelian) superconductor and the superfluid. This is why the authors of [10] called this system the semi-superfluid.

The interactions between two semi-superfluid vortices at large distance $r \gg \max\{m_\chi^{-1}, m_\phi^{-1}, m_G^{-1}\}$ are interpolated by the $U(1)_B$ NG mode. As a result they repel each other by the long range force [11].

Once the vortices are placed at the distance of the scale $\mathcal{O}(m_\chi^{-1}, m_\phi^{-1}, m_G^{-1})$, we cannot ignore exchange of the massive particles, and the interaction must depend on mass ratios

$$\gamma_1 \equiv \frac{m_G}{m_\chi}, \quad \gamma_2 \equiv \frac{m_\chi}{m_\phi} \tag{2.17}$$

Therefore, the type of relatively short range interactions would not be so simple, unlike the typeI/II classification for usual superconductors. For instance, there is no critical coupling limit where the interactions are completely cancelled out in the case of color superconductor. The actual interactions at intermediate distance may be quite complicated. See Secs. 4 and 5 for further discussion.

For later convenience, let us rewrite the Landau-Ginzburg potential in terms of the dimensionful parameters $\mu^2 = m_\phi^2/2$, $\lambda_1 = (m_\phi^2 - m_\chi^2)/(12v^2)$ and $\lambda_2 = m_\chi^2/(4v^2)$:

$$V = \frac{m_\phi^2}{12v^2} (\text{Tr} [\Phi^\dagger \Phi - v^2 1_3])^2 + \frac{m_\chi^2}{4v^2} \text{Tr} [\langle \Phi^\dagger \Phi \rangle^2] \tag{2.18}$$

where $\langle A \rangle$ stands for the traceless part of an $N$ by $N$ matrix $A$: $\langle A \rangle \equiv A - (1/N)\text{Tr}A$.

## 3 Equations of Motion for a Vortex

We would like to consider topologically stable vortex configurations supported by the first homotopy group in Eq. (1.4). Here we are interested in the minimally wound vortex solution. We make a diagonal ansatz for the minimally winding vortex in the cylindrical coordinates $(r, \theta, x_3)$,

$$\Phi(r, \theta) = \begin{pmatrix} e^{i\theta} f(r) & 0 & 0 \\ 0 & g(r) & 0 \\ 0 & 0 & g(r) \end{pmatrix}, \quad A_i(r, \theta) = \frac{1}{g_s} \frac{\epsilon_{ij} x^j}{r^2} [1 - h(r)] \begin{pmatrix} -2/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \tag{3.1}$$
All other solutions are generated by the color-flavor symmetry. For later convenience let us rewrite the solutions (3.1) as

\[
\Phi(r, \theta) = e^{i\theta} \left( \frac{1}{\sqrt{3}} T_0 - \sqrt{\frac{2}{3}} G(r) T_8 \right)
\]  
(3.2)

\[
A_\alpha(r, \theta) = \frac{1}{g_s} \epsilon_{ij} \frac{x^j}{r^2} [1 - h(r)] \sqrt{\frac{2}{3}} T_8
\]  
(3.3)

with the \(U(3)\) generators

\[
T_0 = \frac{1}{\sqrt{3}} \text{diag}(1, 1, 1), \quad T_8 = \frac{1}{\sqrt{6}} \text{diag}(-2, 1, 1)
\]

and the redefined profile functions

\[
F \equiv f + 2g, \quad G \equiv f - g.
\]

Note that the first term proportional to \(T_0\) in \(\Phi\) in Eq. (3.2) is invariant under the color-flavor locking symmetry \(H\) in Eq. (2.10) while the second term proportional to \(T_8\) (traceless part) breaks \(H\) down to \(U(2)\). This symmetry breaking by \(T_8\) gives rise to the NG zero modes \(SU(3)/U(2) \simeq \mathbb{C}P^2\) associated with the vortices as in Eq. (1.7) \[11\]. Therefore the non-zero profile function \(G(r)\) plays a role of the order parameter for this breaking of the color-flavor symmetry. With the ansatz (3.3), the color-magnetic flux is given by

\[
F_{12} = \frac{\sqrt{6} h'}{3g_s r} T_8.
\]

Equations of motion for the profile function \(f(r), g(r), h(r)\) are of the form

\[
f'' + \frac{f'}{r} - \frac{(2h + 1)^2}{9r^2} f - \frac{m^2}{6} \left( f^2 + 2g^2 - 3 \right) - \frac{m^2}{3} f (f^2 - g^2) = 0,
\]

(3.7)

\[
g'' + \frac{g'}{r} - \frac{(h - 1)^2}{9r^2} g - \frac{m^2}{6} g (f^2 + 2g^2 - 3) + \frac{m^2}{6} g (f^2 - g^2) = 0,
\]

(3.8)

\[
h'' - \frac{h'}{r} - \frac{m^2}{3} \left( g^2 (h - 1) + f^2 (2h + 1) \right) = 0.
\]

(3.9)

We solve these differential equations with the following boundary conditions

\[
\begin{cases}
(f, g, h) \rightarrow (1, 1, 0) & \text{as} & r \rightarrow \infty, \\
(f, g', h) \rightarrow (0, 0, 1) & \text{as} & r \rightarrow 0.
\end{cases}
\]

(3.10)

The third terms in the left hand side of Eqs. (3.7) and (3.8) are typical for global vortex configurations which leads logarithmic divergence of the tension.
The tension of the vortex-string is given by

\[
T = 2\pi v^2 \int dr r \left[ f'^2 + \frac{(2h + 1)^2}{9r^2} f^2 + 2 \left( g'^2 + \frac{(h - 1)^2}{9r^2} g^2 \right) + \frac{h'^2}{3m_G^2 r^2} + \mathcal{V} \right], \tag{3.11}
\]

\[
\mathcal{V} = \frac{1}{12} v^2 \left[ (f^2 + 2g^2 - 3)^2 m_\phi^2 + 2 (f^2 - g^2)^2 m_\chi^2 \right]. \tag{3.12}
\]

This diverges as \( T \sim \frac{2\pi v^2}{3} \log L/r_0 \) with \( L \) being an IR cutoff scale, the size of the system, and \( r_0 \) being a typical scale \( r_0 \sim m^{-1} \). The factor 1/3 reflects the fact that the minimum winding vortex winds 1/3 of \( U(1)_B \).

We will see that the color-magnetic flux in Eq. (3.6) is well squeezed and becomes well localized tube even though the energy of the vortex itself logarithmically diverges.

Here let us make comments on related vortices in the CFL phase. Another color magnetic vortex in the CFL phase studied in [8] is generated by only \( T_8 \) (mixed with electro magnetic \( U(1)_{EM} \)) without the part of \( T_0 \) in Eq. (3.2). For the single-valuedness of fields the minimum winding is the triple of our color magnetic flux, and consequently it carries the triple amount of ours. It is, however, unstable because \( \pi_1[SU(3)_C] \) is trivial. On the other hand, the global \( U(1)_B \) vortex which is made by only the generator of global symmetry \( T_0 \) has been studied in [9, 7]. It carries neither a color magnetic flux nor internal orientations. Since such configuration cannot be combined with \( SU(3) \), it is necessary to wind 2\( \pi \) along \( T_0 \), and consequently it has the triple winding of our color magnetic flux. As stated in the introduction, it decays into three semi-superfluid vortices which have different color fluxes with the total flux cancelled out.

Before closing this section, let us give a comment on a relation to a non-Abelian global vortex which appears when the chiral symmetry is spontaneously broken in the \( U(N) \) linear sigma model [22, 23, 24, 25]. The linear sigma model is realized from our Lagrangian by just ungauging \( SU(N)_C \), namely turning off the gauge coupling \( g_s = 0 \) \((m_G = 0)\). At the level of equations of motion, it is enough to set \( h(r) = 1 \). Then Eq. (3.9) becomes trivial and we are left with Eqs. (3.7) and (3.8) (with \( h = 1 \)) which are nothing but equations of motion for the non-Abelian global vortices [23, 25]. However \( h(r) = 1 \) is inconsistent with the boundary condition (3.10), and so the non-Abelian global vortices cannot be obtained in a continuous limit of the non-Abelian semi-superfluid vortex.
4 Asymptotics

Let us next study the asymptotics in the region $r \gg \max\{m^{-1}_\chi, m^{-1}_\phi, m^{-1}_G\}$. The configuration is almost vacuum in such region, so it is useful for us to use the profile function $F, G$ rather than $f, g$ in Eqs. (3.7) and (3.8). Let us perturb the fields by

$$F = 3 + \delta F, \quad G = \delta G, \quad h = \delta h, \quad (|\delta F|, |\delta G|, |\delta h|) \ll 1,$$

and linearize the equations of motion as

$$\left(\triangle - m^2_\phi - \frac{1}{9r^2}\right)\delta F = \frac{1}{3r^2}, \quad \left(\triangle - m^2_\chi - \frac{1}{9r^2}\right)\delta G = \frac{2}{3r^2} \delta h,$$

$$\delta h'' - \frac{\delta h'}{r} - m^2_G\delta h = \frac{2}{3} m^2_G \delta G,$$

where $\triangle \equiv \frac{1}{r} \partial_r (r \partial_r)$. The last equation can be rewritten with $\tilde{\delta}h = \delta h/(m_Gr)$ as

$$\left(\triangle - m^2_G - \frac{1}{r^2}\right)\tilde{\delta}h = \frac{2}{3} m_G \frac{\delta G}{r}.$$

The equations for $\delta F$ is the same with the one for a global $U(1)$ vortex with the winding number 1/3.

First let us solve Eq. (4.2). The equation with the right hand side being zero, $[\triangle - m^2_\phi - 1/(9r^2)]\delta F = 0$, has a solution $q_\phi K_{1/3}(m_\phi r)$ with $q_\phi$ being an integration constant. Here $K_{1/3}(r)$ is one of the modified Bessel function $K_n(r)$ of the 2nd class, which solves

$$\left(\triangle - m^2 - \frac{n^2}{r^2}\right) K_n(mr) = 0.$$ 

It is known that $K_n(r)$ with $0 \leq n \leq 1$ is well approximated by $K_{1/2}(r) = \sqrt{\frac{\pi}{2r}} e^{-r}$ at $r \gg 1$. So we get

$$\delta F = q_\phi \sqrt{\frac{\pi}{2m_\phi r}} e^{-m_\phi r} + \left(-\frac{1}{3m^2_\phi r^2} + \mathcal{O}\left(\frac{1}{(m_\phi r)^4}\right)\right).$$

Since $K_n(r)$ is much smaller than $1/r^2$ for $r \gg 1$, the first term can be neglected as in the case of the $U(1)$ global (superfluid) vortex. We will not discuss the coefficient $q_\phi$ below.

---

8 If we consider local vortex by gauging $U(1)_B$, all the linearized equations at asymptotic region become the form of $(\triangle - m^2_\chi)\delta X = 0$. Here symbol $X$ stands for all the fields, namely Abelian, non-Abelian gauge fields and all the scalar fields.
Let us next solve Eqs. (4.3) and (4.5). Clearly, the solution cannot have a tail decreasing with a polynomial. They have exponentially small tails similarly to the ANO vortex in the superconductor. The equations with the right hand sides of Eqs. (4.3) and (4.5) being zero can be solved by

\[
\delta G \simeq q_\chi K_{1/3}(m_\chi r) \simeq q_\chi \sqrt{\frac{\pi}{2m_\chi r}} e^{-m_\chi r}, \quad \delta h \simeq q_G m_G r K_1(m_G r) \simeq q_G \sqrt{\frac{\pi m_G r}{2}} e^{-m_G r},
\]

where \(q_\chi\) and \(q_G\) are integration constants, which are determined numerically in the next section. Let us take into account the corrections from the right hand sides of Eqs. (4.3) and (4.5). When the inequality \(m_G \not\simeq m_\chi\) holds, we can ignore \(\delta h \sim e^{-m_G r}\) in Eq. (4.3). So \(\delta G \sim e^{-m_\chi r}\) in Eq. (4.8) is a good approximation. However, we cannot discard \(\delta G\) in Eq. (4.4) since \(\delta G \not\simeq e^{-m_\chi r} \gg e^{-m_G r}\).

We thus find the following approximations

\[
\begin{align*}
\delta G &= q_\chi \sqrt{\frac{\pi}{2m_\chi r}} e^{-m_\chi r}, & \text{for } m_G \not\simeq m_\chi (\gamma_1 \not\simeq 1). \\
\delta h &= -\frac{2q_\chi}{3} \frac{m_G^2}{m_\chi^2 - m_G^2} \sqrt{\frac{\pi}{2m_\chi r}} e^{-m_\chi r}
\end{align*}
\]

On the other hand, when the other inequality \(m_\chi \not\simeq m_G\) holds, we should reconsider approximation for \(\delta G\). In the same way we find the approximations

\[
\begin{align*}
\delta G &\simeq -\frac{2q_G}{3} \frac{1}{(m_G^2 - m_\chi^2)r^2} \sqrt{\frac{\pi}{2}} \frac{m_G r}{2} e^{-m_G r}, & \text{for } m_\chi \not\simeq m_G (\gamma_1 \not\simeq 1). \\
\delta h &\simeq q_G \sqrt{\frac{\pi m_G r}{2}} e^{-m_G r}
\end{align*}
\]

These are not good approximation for \(m_\chi \approx m_G\). For the regions \(m_\chi \not\simeq m_G\), we find from Eqs. (4.9) and (4.10) that both the massive traceless scalars \(\chi\) and the massive gauge fields (gluons) have the same asymptotic behavior with the exponential tails \(e^{-mr}\) with the common mass \(m \equiv \min\{m_\chi, m_G\}\). These asymptotic behaviors are quite different from the ANO vortex in usual superconductors.

Now we can roughly estimate the width of the color magnetic flux tube given in Eq. (3.6). It can be read from the tail \(\delta h\) of the gauge fields. Namely it can be estimated by the Compton

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9 Remember that the exponential tails of the ANO vortices are like \(e^{-m_H r}\) for the massive scalar field \(H\) with mass \(m_H\) and \(e^{-m_e r}\) for the massive \(U(1)\) gauge boson with mass \(m_e\). The layer structures are exchanged for \(m_e > m_H\) (type II) and \(m_e < m_H\) (type I). Then their interactions depend on the ratio \(m_e/m_H\) giving classification of type I and II. However for the strong type II region \((m_e > 2m_H)\), the tail of the gauge field becomes \(e^{-2m_H r}\) [27]. The width of the gauge field cannot become smaller than the half of that \(1/m_H\) of the scalar field.
wave lengths \( m^{-1} \) with \( m = \min\{m_\chi, m_G\} \). It is the Compton wave length of massive gluons (penetration depth) as expected when \( m_\chi > m_G (\gamma_1 < 1) \), while it is no longer the case when \( m_G > m_\chi (\gamma_1 > 1) \): contrary to the naive expectation the tail of gauge field decays at Compton wave length of massive traceless scalar field \( \chi \) (coherence length)\(^{10}\). In Sec. 5 these estimations are confirmed by the numerical calculations.

The \( U(1)_B \) NG boson makes the power-like tails in the traceless part of \( \Phi \) which is color-flavor singlet as seen in Eq. (4.7). So the power-like tail is dominated at the large distance from the core of the non-Abelian semi-superfluid vortex. It yields the logarithmic divergence in the tension of the vortex and leads to the long range repulsive forces between two vortices \(^{11}\). As we approach the vortex core from the large distance, we see the exponential tails behaving as \( e^{-m_\phi r}, e^{-m_\chi r}, e^{-m_G r} \). We will first come across the tail of the fields with the lightest mass.

When \( m_\phi \) is the lightest mass, the semi-superfluid vortex behaves as the usual superfluid vortex (with \( 1/3 \) \( U(1)_B \) winding), since the massive singlet scalar field \( \phi \) exists also in a usual \( U(1) \) superfluid vortex. In general, the exchange of the massive scalar field leads to an attractive interaction (as in usual superconductors), and so we expect that the short range interaction is attractive. In this case non-Abelian properties are somewhat hidden inside the core of the superfluid vortex.

More interesting is the case that \( m_\chi \) (or \( m_G \)) is the smallest, since the asymptotic tail which we first come across is those in Eq. (4.9) (or Eq. (4.10)). There are two remarkable points. i) The asymptotic behaviors are quite different from the ANO vortex in usual superconductors. The interactions of ANO vortices depend on the ratio of the masses of the massive gauge boson and the massive scalar field, giving classification of type I and II as denoted in footnote \(^9\). In Eqs. (4.9) and (4.10) we find that both the scalar and vector fields have the asymptotic tails of the same order \( e^{-mr} \) with \( m = \min\{m_\chi, m_G\} \), as already pointed out. So we may have to consider two contributions equivalently. ii) The interactions may depend on the internal orientation moduli \( \mathbb{C}P^2 \). In fact, in the case of the local \( U(N) \) vortices for which \( U(1)_B \) is gauged, the exchanging of the particles which are not singlet but in the adjoint representation of the color-flavor symmetry leads to interactions depending on the internal orientations \(^{28}\). In order to get better understanding about the short range interactions between non-Abelian semi-superfluid

\(^{10}\) Similar phenomenon is known for usual superconductors as denoted in the footnote\(^9\) However the threshold is \( m_e = 2m_H \) in that case.
vortices, we need more qualitative and quantitative studies which is beyond the scope of this paper. However see further discussion in Sec. 6.

5 Numerics: Solutions and Width of Color Magnetic Flux

In this section we provide numerical solutions. We use the relaxation method with appropriate boundary conditions, see Fig. 1. We see that the interiors of the vortices depend on the ratios $\gamma_1$ and $\gamma_2$.

Some numerical solutions were previously found in [10] with the approximation of $g(r) = 1$ in Eq. (3.2). Although this approximation makes numerical calculation easier, the authors of [10] did not evaluate the validity of the approximation. Note that $g(r) = 1$ is never true unlike the case of non-Abelian global vortex [25] and the non-Abelian local vortex [26], in whose cases...
the unwinding fields can be constant for particular values of coupling constants. Moreover, $G = f - g$ plays a role of the order parameter for the breaking of the $SU(3)_{C+F}$, so we need the profile function $g(r)$. As we can see from the boundary condition \( q \), $g(0)$ is not fixed and this is a typical parameter to characterize the solutions. We find $0 \leq g(0) \leq \sqrt{3/2}$ where $g(0) \to 0$ as $m_\chi \to \infty$ and $g(0) \to \sqrt{3/2}$ as $m_\phi \to \infty$. This can be understood by considering $m_\chi \to \infty$ in Eq. \( (2.18) \) which forces $f^2 + 2g^2 = 3$ while $m_\phi \to \infty$ imposes $f = g$. Qualitatively speaking, we find $g(0) < 1$ when $m_\phi < m_G$, and $1 < g(0)$ for $m_\chi < m_\phi$. In the case of $m_\phi = m_\chi$, the profile function $g(r)$ is almost 1 everywhere\(^{11}\). These properties can be seen in Fig. \( 1 \).

From the solutions in Fig. \( 1 \) we determine the integration constants $q_\chi$ and $q_G$ in the asymptotic solutions in Eqs. \( (4.9) \) and \( (4.10) \) as summarized in Table \( 1 \).

| $m_G$ ($m_\chi = m_\phi = 1$) | 2  | 3  | 4  | 5  |
|-----------------------------|----|----|----|----|
| $q_\chi (\delta G)$        | 1.9 (±0.1) | 1.60 (±0.03) | 1.46 (±0.03) | 1.38 (±0.02) |
| $q_\chi (\delta h)$        | 2.0 (±0.1) | 1.62 (±0.01) | 1.47 (±0.02) | 1.38 (±0.01) |

| $m_\chi$ ($m_G = m_\phi = 1$) | 2  | 3  | 4  | 5  |
|-----------------------------|----|----|----|----|
| $q_G (\delta G)$            | 1.9 (±0.1) | 1.65 (±0.1) | 1.56 (±0.05) | 1.54 (±0.05) |
| $q_G (\delta h)$            | 1.68 (±0.03) | 1.53 (±0.02) | 1.48 (±0.02) | 1.47 (±0.02) |

Table 1: Several numerical values of $q_\chi$ and $q_G$. The case of $m_G = 2, 3, 4, 5 > m_\chi = 1$ \( \text{in Eq. \( (4.9) \)} \) and the case of $m_\chi = 2, 3, 4, 5 > m_G = 1$ \( \text{in Eq. \( (4.10) \)} \) are written in the upper and lower boxes, respectively. The parentheses beside $q_\chi$ and $q_G$ imply that we evaluate these values by using the the asymptotic tails $\delta G$ or $\delta h$ in Eq. \( (4.9) \) or \( (4.10) \). For $q_\chi$ we find a good agreement between $q_\chi(\delta G)$ and $q_\chi(\delta h)$. However for $q_G$ it is not so good for smaller $m_\chi$.

Let us next focus on the color magnetic flux by looking at $h(r)$. The function $h(r)$ behaves almost similar among the cases of \{ $m_G, m_\phi, m_\chi$ \} = \{1, 1, 1\}, \{1, 5, 1\}, \{1, 1, 5\}, \{1, 5, 5\} in Fig. \( 1 \) because of the common choice of $m_G = 1$. When the gluon mass is larger, $m_G = 5$, $h(r)$ becomes more sharp as can been seen in the cases of \{5, 1, 1\} and \{5, 5, 1\} in Fig. \( 1 \). The flux becomes far more sharp when we take $m_\chi \sim m_G$ as can been seen in the case of \{5, 1, 5\} in Fig. \( 1 \). Qualitatively these behaviors match the result from the asymptotic tails in the previous section.

\(^{11}\) The approximation $g = 1$ used in Ref. \( 10 \) may be justified in the region $m_\phi \sim m_\chi$. However, since the relation $\lambda_1 = \lambda_2$ holds at weak coupling regime \( 10 \) the masses are related by $m_\phi = 2m_\chi$. 

15
In order to get a more quantitative width, let us define a width of the color magnetic flux by

\[ \langle r \rangle \equiv \sqrt{\frac{\int dx^2 F_{12}^2 r^2}{\int dx^2 F_{12}^2}} = \sqrt{\int_0^\infty dr \, r^2 h'}. \]  

(5.1)

The numerical values of this quantity are shown in Fig. 2 for various values of \( m_G \) and \( m_\chi \). The left-upper panel of Fig. 2 shows that the width rapidly decreases for relatively small \( m_G (\lesssim 4) \) and then approaches to a constant value for larger \( m_G (\gtrsim 4) \). Contrary to this the Compton wave length \( m_G^{-1} \) of the massive gluon goes down to zero as \( m_G \) increases. These behaviors are significantly different. Furthermore the width approaches to smaller constant at large \( m_G \) for larger \( m_\chi \) as can be seen in the upper-right panel of Fig. 2. We thus find that the size of the color magnetic flux cannot become smaller than some value determined by the mass \( m_\chi \) of the traceless scalar fields for the larger gluon mass \( m_G \). This is consistent with the asymptotic behavior found in Eq. (4.9).

On the other hand, the size of the color magnetic flux is well approximated by the Compton wave length \( m_G^{-1} \) when \( m_G \) changes with keeping \( m_\chi = m_G \) as shown in the lower panel in Fig. 2. The result in this parameter region is complement to the last section, where it was difficult to study analytically.

6 Conclusion and Discussion

In this paper we have studied the topologically stable vortex in the color superconductor. By using the relaxation method with the appropriate boundary conditions, we have presented numerical solutions for diverse choices of the coupling constants (Fig. 1). We have found the asymptotic tails of the profile functions in Eqs. (4.9) and (4.10) and have determined their coefficients \( q_\chi \) and \( q_G \) in Table 1. Furthermore we have proposed the width \( \langle r \rangle \) of the color flux tube as in Eq. (5.1) and have calculated it numerically. Contrary to the naive expectation, the width of the color-magnetic flux tube does not behave as the Compton wave length (the penetration depth) \( m_G^{-1} \) of the massive gluons for the gluon mass \( m_G \) larger than the mass \( m_\chi \) of the traceless scalar \( \chi \) as shown in Fig. 2.

\[ \text{12 The definition of this width has been given first in study of a local } U(N) \text{ vortex } [26] \text{ for which the phase } U(1)_B \text{ is gauged.} \]
Fig. 2: The width $\langle r \rangle$ of the color flux tube in Eq. (5.1) v.s the Compton wave length $m_G^{-1}$. In the upper-left panel we plot the width $\langle r \rangle$ as dots in the cases of $m_\chi = 1$ and various $m_G$ (with $m_\phi = 1$). The Compton wave length $2.2/m_G$ is drawn by the broken curve in which the coefficient is determined by trying to fit it to the numerical points. (However they fit well nowhere). The unbroken curve is $2.78 \times m_G^{-0.64}$ made by MATHMATIC which fits very well at $m_G < 1$. In the upper-right panel the width $\langle r \rangle$ in the cases $m_\chi = 1, 2, 3$ (disk, box, diamond) are plotted (with $m_\phi = 1$). In the lower panel we plot the width $\langle r \rangle$ in the cases of $m_G = m_\chi$ (with $m_\phi = 1$). The broken curve denotes the Compton wave length $2.4/m_G$ which fits to the data better than the case of upper-left panel. The unbroken curve denotes $2.72 \times m_G^{-0.82}$ which fits well to the data everywhere.
Here let us discuss the application of our result to the analysis of the interaction between two semi-superfluid vortices. As shown in [11], the long range interaction is mediated by the massless NG mode of the broken $U(1)_B$. The universal repulsion between two vortices at distance $R(\gg m^{-1}_{\phi,\chi,G})$ was found [11] to be

$$F_{\text{long}}(R) \simeq \frac{4\pi}{N_C R}, \quad N_C = 3. \quad (6.1)$$

This force comes from the overlapping of the long power tails of the two vortices. When the vortices are placed at a relatively short distance $R \gtrsim m^{-1}_{\phi,\chi,G}$, we may find the exponential tails $e^{-m\phi r}, e^{-m\chi r}, e^{-mG r}$. The overlapping of these tails yields the inter-vortex potential. For instance, when the tail of $\chi$ is overlapped for $m_\chi < m_G$, it can be written as

$$V_\chi(R) = C_\chi \frac{2\pi q_\chi^2}{m_\chi} K_0(m_\chi R). \quad (6.2)$$

Here $R$ is the relative distance and $q_\chi$ is the parameter in Eq. (4.9). The function $C_\chi$ is unknown and should depend on the relative orientation of the two vortices in the internal space.\(^{13}\) Thus the corresponding inter-vortex force is given by

$$F_\chi(R) = -\frac{\partial V_\chi}{\partial R} = C_\chi \frac{2\pi q_\chi^2 m_\chi}{m_\chi} K_1(m_\chi R). \quad (6.3)$$

Let us compare these forces $F_{\text{long}}$ and $F_\chi$ by extrapolating them near the vortex core. In the following we assume $C_\chi = O(1)$ for simplicity. As an example we choose $(m_\chi, m_\phi, m_G) = (1, 1, 3)$, see Fig. 3. The width $\langle r \rangle$ can be read as $r = 1.58$ from Fig. 2. The approximation $G = q_\chi K_0(m_\chi r)$, initially obtained for the large distance, is still valid up to $r \gtrsim 2$ as can be seen in Fig. 3: The profile functions for $(m_\chi, m_\phi, m_G) = (1, 1, 3)$ in the left-most panel $[h(r), f(r), g(r)] = [R \text{ (solid line)}, G \text{ (short dash line)}, B \text{ (long dash line)}]$. The middle panel shows the ratio $g/(q_\chi K_0(m_\chi r)) \sim 1$. A rough estimation of the long range and short range interactions are shown in the right-most panel.

\(^{13}\) For usual superconductors, $C_\chi = -1$ and this gives the attractive force.
the middle panel of Fig. 3. Therefore we can safely use Eq. (6.3) in this region. As one can see in
the right panel of Fig. 3, the two kinds of the inter-vortex forces $F_{\text{long}}$ and $F_{\chi}$ extrapolated to the
near region of the core are comparable around $R = 2$. We conclude that the inter-vortex force
mediated by the exchange of the massive particles like the massive gluons becomes comparable
with the inter-vortex force mediated by the exchange of the massless $U(1)_B$ NG boson. More
precise discussion remains as a future problem. The detailed study of it is necessary to understand
the lattice structure of non-Abelian semi-superfluid vortices when the lattice spacing is of the
order of the penetration depth or the coherence length.

If the short range force is attractive the vortex lattice collapses for high vortex density. There may appear composite vortices. The internal structure of a composite vortex may be ample which was studied in the case of local $U(N)$ vortices \[29\]. Other interesting aspects are
dynamical collision (for instance the reconnection) of non-Abelian vortex strings \[30\] and a gas
of non-Abelian vortices at finite temperature \[31\] (lower than transition temperature to quark gluon plasma phase), both of which were also studied in the case of local $U(N)$ vortices.

Let us make brief comments on possible application to the neutron star physics. In a response
to the rapid rotation of a neutron star, superfluid vortices are created along the rotating axis in
the outer core (the hadronic phase) where neutrons exhibit superfluidity. It was pointed out in
the seminal paper by Anderson and Itoh \[32\] that those vortices may explain the pulsar glitch
phenomenon. If the CFL phase is realized in the core of a neutron star, those superfluid vortices
are somehow connected to $U(1)_B$ superfluid vortices in the CFL phase. Each $U(1)_B$ superfluid
vortex must be divided into three semi-superfluid vortices (color flux tubes). Therefore color
flux tubes are necessary ingredients in the study of neutron stars if the CFL phase is realized in
the core. On the other hand, pulsars are accompanied with the large amount of magnetic fields
of $U(1)_{\text{EM}}$. Protons exhibit superconductivity in the hadronic phase, and so there should exist
magnetic flux tubes along the axis of the magnetic fields. If those magnetic fields penetrate into
the CFL phase realized in the core, there is no topological reason for those fluxes to be squeezed,
because the $U(1)_{\text{EM}}$ symmetry (with mixed with one color component) is not broken there. It
was, however, pointed out that fluxes are squeezed into flux tubes there \[13\] where the CFL phase
is modified by large magnetic fields \[33\]. In general, the axes of the rotation and the magnetic
fields of the pulsar are different with some angle, the interaction between the (electro-)magnetic
fluxes and the color magnetic fluxes studied in this paper should be important in the study of
the neutron stars.

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A General Diagonal Solutions

In this appendix we study more general diagonal vortex solutions. They can be obtained as follows:

\[
\Phi(r, \theta) = e^{i\theta \left( \frac{T_0}{\sqrt{3}} - \sqrt{2} \frac{\nu_3 T_3 + \nu_8 T_8}{\sqrt{3}} \right)} \left( \frac{F(r)}{\sqrt{3}} T_0 - \frac{\sqrt{2} G(r)}{\sqrt{3}} (\nu_3 T_3 + \nu_8 T_8) \right),
\]

\[
A_i(r, \theta) = \frac{1}{g_s} \frac{\epsilon_{ij} x^j}{r^2} \left[ 1 - h(r) \right] \sqrt{\frac{2}{3}} (\nu_3 T_3 + \nu_8 T_8),
\]

where we have defined the Cartan subalgebra \{T_0, T_3, T_8\} of U(3) by

\[
T_0 = \frac{1}{\sqrt{3}} \text{diag}(1, 1, 1), \quad T_3 = \frac{1}{\sqrt{2}} \text{diag}(0, 1, -1), \quad T_8 = \frac{1}{\sqrt{6}} \text{diag}(-2, 1, 1).
\]

The parameters \(\nu_3\) and \(\nu_8\) are determined by the requirement of the single valuedness of the fields, \(e^{i2\pi \left( \frac{T_0}{\sqrt{3}} - \sqrt{2} \frac{\nu_3 T_3 + \nu_8 T_8}{\sqrt{3}} \right)} = 1\). Three solutions are found: \((\nu_3, \nu_8) = (0, 1), (\pm \sqrt{3}/2, -1/2)\).

For instance in the case of \((\nu_3, \nu_8) = (0, 1)\) we find the solutions for Eq. (3.1) with

\[
f = \frac{F + 2G}{3}, \quad g = \frac{F - G}{3}.
\]

This solution with \((\nu_3, \nu_8) = (\pm \sqrt{3}/2, -1/2)\) constitute the weight vectors of 3.

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