ON THE PICARD GROUP: TORSION AND THE KERNEL INDUCED BY A FAITHFULLY FLAT MAP

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Statement of results

For a homomorphism \( f : A \rightarrow B \) of commutative rings, let \( D(A, B) \) denote the kernel of the map \( \text{Pic}(A) \rightarrow \text{Pic}(B) \). Let \( k \) be a field and assume that \( A \) is a finitely generated \( k \)-algebra.

We prove a number of finiteness results for \( D(A, B) \). Here are four of them. [1]: Suppose \( B \) is a finitely generated and faithfully flat \( A \)-algebra which is geometrically integral over \( k \). If \( k \) is perfect, we find that \( D(A, B) \) is finitely generated. (In positive characteristic, we need resolution of singularities to prove this.) For an arbitrary field \( k \) of positive characteristic \( p \), we find that modulo \( p \)-power torsion, \( D(A, B) \) is finitely generated.

[2]: Suppose \( B = A \otimes_k k^{\text{sep}} \). We find that \( D(A, B) \) is finite.

[3]: Suppose \( B = A \otimes_k L \), where \( L \) is a finite, purely inseparable extension. We give examples to show that \( D(A, B) \) may be infinite.

[4]: Assuming resolution of singularities, we show that if \( K/k \) is any algebraic extension, there is a finite extension \( E/k \) contained in \( K/k \) such that \( D(A \otimes_k E, A \otimes_k K) \) is trivial.

The remaining results are absolute finiteness results for \( \text{Pic}(A) \). [5]: For every \( n \) prime to the characteristic of \( k \), \( \text{Pic}(A) \) has only finitely many elements of order \( n \).

[6]: Structure theorems are given for \( \text{Pic}(A) \), in the case where \( k \) is absolutely finitely generated.

In the body of the paper, all of these results are stated in a more general form, valid for schemes.

Notation and conventions

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Unless otherwise stated, all rings in this paper are commutative. Given a ring $A$, we denote by $A^*$ the group of units of $A$ and by $A_{\text{red}}$ the reduction of $A$ modulo its nilradical.

Given a set $A$ and a group $G$ acting on $A$, we let $A^G$ denote the subset of $A$ consisting of elements fixed by $G$. Given an abelian group $G$ and a positive integer $n$, we let $n^G$ denote the subgroup consisting of elements whose order divides $n$.

If $X$ is a scheme, we let $\Gamma(X)$ denote $\Gamma(X, \mathcal{O}_X)$. If $X$ is a $k$-scheme, where $k$ is a field, and $L$ is a $k$-algebra, we let $X_L$ denote $X \times_k L$. If $k$ is a field, $k^a$ denotes an algebraic closure of $k$, and $k^{\text{sep}}$ denotes a separable closure of $k$. We write $X^a$, $X^{\text{sep}}$ instead of $X_{k^a}$, $X_{k^{\text{sep}}}$.

An algebraic scheme is a scheme which is of finite type over a field.

1. The kernel under a separable extension

We start by recalling some material on Galois actions, leading up to an application of the Hochschild-Serre spectral sequence to the computation of Pic. This material is more or less standard, but it is not available in the literature in quite the form we need. In particular, we want the statement of (1.2) to be free of noetherian hypotheses. Later (see 1.9) this will be important, because a noetherian scheme $X$ may have $\Gamma(X)$ non-noetherian, and our proofs about $\text{Pic}(X)$ depend on understanding $\text{Pic} \Gamma(X)$.

If $Y$ is a scheme, and $G$ is a group (or just a set), we let $Y \times G$ denote the scheme which is a disjoint union of copies of $Y$, one for each $g \in G$.

**Definition 1.1.** Let $X$ be a scheme, and let $Y$ be a finite étale $X$-scheme. Suppose a finite group $G$ acts on the right of $Y$ as an $X$-scheme.\footnote{Hereafter we say simply that $G$ acts on $Y/X$.} Then this action is **Galois** if the map $Y \times G \to Y \times_X Y$ given by $(y, g) \mapsto (y, yg)$ is an isomorphism of schemes.\footnote{If $y \in Y$, then $yg \in Y$, and $g$ induces an isomorphism of $k(y)$ with $k(yg)$. Therefore we get maps $\sigma_1, \sigma_2 : \text{Spec}(k(y)) \to Y$, such that $\pi \circ \sigma_1 = \pi \circ \sigma_2$, where $\pi : Y \to X$ is the structure map. By the universal property of the fiber product, we obtain a morphism $\text{Spec}(k(y)) \to Y \times_X Y$, whose image is by definition the point $(y, yg)$.}

It is important to note that $G$ cannot be recovered from $Y \to X$ in general: for instance this is the case if $Y$ consists of a disjoint union of copies of $X$.

If we have a Galois action of $G$ on $Y/X$, and $X' \to X$ is any morphism, we get a Galois action of $G$ on $Y \times_X X'$ as an $X'$-scheme.

Let $A \subset B$ be rings, and let a finite group $G$ act on $B/A$, meaning that $G$ acts on (the left of) $B$ as an $A$-algebra. Then we shall call...
this action *Galois* if the action of $G$ on \( \text{Spec}(B)/\text{Spec}(A) \) is Galois with respect to definition (1.1). For some definitions, stated directly for rings, see [KO2, Chapter II, §5]. In the case where $A$ and $B$ are fields, the action of $G$ on $B/A$ is Galois if and only if $B/A$ is a Galois extension (in the usual sense), and $G = \text{Aut}_A(B)$.

We now recall the *Hochschild-Serre spectral sequence*. This may be found in Milne [Mi1, p. 105]. Although Milne refers only to locally noetherian schemes, the argument he presents is valid for any scheme. For clarity, however, we note that $X_{\text{ét}}$ as used here means the (small) étale site (on an arbitrary scheme $X$), as defined in [G2].

Let $X$ be a scheme, let $Y$ be a finite étale $X$-scheme, and let a Galois action of a finite group $G$ on $Y/X$ be given. Let $F$ be a sheaf (of abelian groups) for the étale topology on $X$. Then the Hochschild-Serre spectral sequence is:

$$E_2^{p,q} = H^p(G, H^q(Y_{\text{ét}}, \pi^*F)) \implies H^{p+q}(X_{\text{ét}}, F),$$

where $\pi : Y \to X$ is the structure morphism.

Apply this with $F = \mathbb{G}_m$. One has an exact sequence:

$$0 \to E_2^{1,0} \to H^1(X_{\text{ét}}, \mathbb{G}_m) \to E_2^{0,1}.$$  

The first term is $H^1(G, \Gamma(Y)^*)$. The middle term is $\text{Pic}(X)$. The last term is $H^0(G, \text{Pic}(Y))$, which embeds in $\text{Pic}(Y)$. Hence:

**Proposition 1.2.** Let $f : Y \to X$ be a finite étale morphism of schemes, and suppose we have a Galois action of a finite group $G$ on $Y/X$. Then

$$\text{Ker}[\text{Pic}(f)] \cong H^1(G, \Gamma(Y)^*).$$

In particular [Sw2, (4.2)], if $A \subset B$ are rings, and we are given a Galois action of a finite group $G$ on $B/A$, then $D(A, B) \cong H^1(G, B^*)$.

The following example shows that one has to be a bit careful about generalizing the proposition to the non-Galois case.

**Example 1.3.** Let $A$ be a complete discrete valuation ring with fraction field $K$, and let $L$ be a finite Galois extension of $K$ with Galois group $G$. Denote the integral closure of $A$ in $L$ by $B$. We would like there to be an exact sequence:

$$0 \to H^1(G, B^*) \to \text{Pic}(A) \to \text{Pic}(B),$$

and hence conclude that $H^1(G, B^*) = 0$. But this is not the case in general: $H^1(G, B^*)$ has order equal to the ramification index of $B/A$.

We recall some easy facts about cohomology of groups:

**Proposition 1.4.** Let $G$ be a finite group and $M$ a left $\mathbb{Z}G$-module.

1. If $M$ is finitely generated, then $H^n(G, M)$ is finite for every $n > 0$. 

2. If $M$ has trivial $G$-action then $H^1(G, M) \cong \text{Hom}_{\text{groups}}(G, M)$.

Proof. (1) Since $H^n(G, M) = \text{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$ and $\mathbb{Z}G$ is left Noetherian it is clear that all the cohomology groups are finitely generated modules (over $\mathbb{Z}G$ or, equivalently, over $\mathbb{Z}$). Moreover, they are annihilated by $|G|$ by [Br, Chap. III, (10.2)]. (2) is clear from the representation of $H^1(G, M)$ in terms of crossed homomorphisms (or, see [Bab, §23]). □

Proposition 1.5. Let a finite group $G$ act on a field $K$. Then $H^1(G, K^*)$ is finite, and (Hilbert’s Theorem 90) it is 0 if the group action is faithful.

Proof. Let $H$ be the subgroup of $G$ which acts trivially on $K$, and let $\overline{G} = G/H$. Let $k = K^{\overline{G}}$. Then the extension $K/k$ is Galois, with Galois group $\overline{G}$. By (1.2), $H^1(\overline{G}, K^*) \cong \ker[\text{Pic}(k) \to \text{Pic}(K)]$, which is 0. We are done if $G$ acts faithfully on $K$.

We have the inflation-restriction exact sequence

$$0 \to H^1(\overline{G}, K^*) \to H^1(G, K^*) \to H^1(H, K^*),$$

so it is enough to show that $H^1(H, K^*)$ is finite, which is clear from (1.4)(2). □

The next result is a variant of a well-known result due to Roquette [Ro]:

Proposition 1.6. Let $K$ be a field, $X$ a $K$-scheme of finite type, and $\Lambda$ the integral closure of $K$ in $A := \Gamma(X)_{\text{red}}$. Then $\Lambda$ is finite-dimensional as a $K$-vector space, and $A^*/\Lambda^*$ is a finitely generated free abelian group.

Proof. Let $\{U_1, \ldots, U_m\}$ be an affine open cover of $X$, and set $R_i = \Gamma(U_i)_{\text{red}}$. Each $R_i$ is a a reduced $K$-algebra of finite type. We have an embedding $A \to B := R_1 \times \cdots \times R_m$. Let $P_1, \ldots, P_n$ be the minimal prime ideals of $B$, and let $C_j$ be the normalization of the domain $B/P_j$. Then each $C_j$ is a normal domain of finite type over $K$, and $A \subset C_1 \times \cdots \times C_n$. Let $\Delta_j$ be the integral closure of $K$ in $C_j$. By the usual formulation of Roquette’s theorem (see [L, Chapter 2, (7.3)] or [Kr, (1.4)]) $C_j^*/\Delta_j^*$ is finitely generated for each $j$. We have $\Lambda = A \cap (\prod_j \Delta_j)$. Therefore $A^*/\Lambda^*$ embeds in the finitely generated group $\prod_j(C_j^*/\Delta_j^*)$. Obviously $A^*/\Lambda^*$ is torsion-free, and since it is finitely generated, it is free.

The fraction field $K_j$ of $C_j$ is a finitely generated field extension of $K$. Each $\Delta_j$, being 0-dimensional, reduced and connected, is a field algebraic over $K$. Since $K_j/K$ is finitely generated, so is $\Delta_j/K$. Therefore $\prod_j \Delta_j$ is a finite-dimensional $K$-algebra, and hence so is its subalgebra $\Lambda$. □
Theorem 1.7. Let $X$ be a scheme of finite type over a field $k$. Let $f : Y \to X$ be a finite, étale, surjective morphism of schemes. If $k$ has positive characteristic, assume that $X$ is reduced. Then $\text{Ker}[\text{Pic}(f)]$ is finite.

Remark 1.8. If $X$ is affine, $\text{Pic}(X) = \text{Pic}(X_{\text{red}})$, and so the assumption about $X$ being reduced (in positive characteristic) is not needed. In general it is: see (4.1).

Proof (of 1.7.) We may assume that $X$ is connected. Let $Y_0$ be a connected component of $Y$. Then $f|_{Y_0}$ is finite and étale, and since $X$ is connected, it is surjective. Therefore we may reduce to the case where $Y$ is connected. By [Mur, (4.4.1.8)], there exists a scheme $Y'$ over $Y$ such that $Y' \to X$ is finite étale surjective and the action of the finite group $\text{Aut}(Y'/X)$ on $Y'/X$ is Galois. Therefore we may assume that in fact there is a Galois action of a finite group $G$ on $Y/X$.

By (1.2), $\text{Ker}[\text{Pic}(f)] \cong H^1(G, \Gamma(Y)^*)$. We will complete the proof by showing that $H^1(G, \Gamma(Y)^*)$ is finite. This will depend only on the fact that we have a finite group $G$ acting on an algebraic scheme $Y$, which is reduced if the characteristic is positive.

Let $B = \Gamma(Y)$. Let $\Lambda$ be the integral closure of $k$ in $B_{\text{red}}$. By (1.6), $\Lambda$ is a finite-dimensional $k$-algebra and $B_{\text{red}}^*/\Lambda^*$ is finitely generated. By (1.4)(1) we know that $H^1(G, B_{\text{red}}^*/\Lambda^*)$ is finite.

We show that $H^1(G, \Lambda^*)$ is finite. Write $\Lambda = \prod_{j \in J} F_j$, where each $F_j$ is a finite-dimensional field extension of $k$ (and $J$ is a finite index set). Since the action of $G$ preserves idempotents, we can define an action of $G$ on $J$ by the rule $F_{gj} = gF_j$. Let $I$ be any orbit of $G$ on $J$, and look at $\Upsilon := \prod_{i \in I} F_i$. Since $\Upsilon^*$ is the direct sum of the groups $\Upsilon_i^*$ (over the various orbits of $G$ on $J$), it is enough to show that $H^1(G, \Upsilon^*)$ is finite. Fix $i \in I$, let $H$ be the isotropy subgroup of $i$, and put $F = F_i$. It follows from [Br, Chap. III, (5.3), (5.9), (6.2)] that $H^1(G, \Upsilon^*) \cong H^1(H, F^*)$, which is finite by (1.3). Hence $H^1(G, \Lambda^*)$ is finite.

Running the long exact sequence of cohomology coming from the exact sequence

$$1 \to \Lambda^* \to B_{\text{red}}^* \to B_{\text{red}}^*/\Lambda^* \to 1,$$

we conclude that $H^1(G, B_{\text{red}}^*)$ is finite. Thus we are done if $X$ is reduced.

We may assume that $\text{char}(k) = 0$. Let $N$ be a $G$-stable nilpotent ideal of $B$. (For instance, we might take $N$ to be the nilradical of $B$.) It is enough to show that the canonical map
$H^1(G, B^*) \rightarrow H^1(G, (B/N)^*)$ is injective. If $N^r = 0$, note that we can factor the map $B \rightarrow B/N$ as

$$B \rightarrow B/N^{r-1} \rightarrow B/N^{r-2} \rightarrow \cdots \rightarrow B/N,$$

and $G$ acts on everything in the sequence, so we can reduce to the case where $N$ has square zero. We have an exact sequence

$$0 \rightarrow N \rightarrow B^* \rightarrow (B/N)^* \rightarrow 1,$$

and it is enough to show that $H^1(G, N) = 0$. On the one hand, $H^1(G, N)$ is annihilated by $|G|$, and so is torsion. On the other hand, $H^1(G, N)$ is an $B[G]$-module, thus a $\mathbb{Q}$-vector space, and so is torsion-free. Hence $H^1(G, N) = 0$.

**Remark 1.9.** In the theorem, if $Y = X_L$ for some finite separable field extension $L/k$, we will show that there is no need to assume $X$ is reduced, even in positive characteristic. Indeed in that case, $\Gamma(X_L) = \Gamma(X)_L$, so we have a Galois action of $G$ on $\text{Spec } \Gamma(Y)/\text{Spec } \Gamma(X)$. Applying (1.2) twice, we conclude that $\text{Ker}[\text{Pic}(f)] = \text{Ker}[\text{Pic}(\Gamma(f))]$. But Pic of a ring is the same as Pic of its reduction, so if $C = \Gamma(X)$, we conclude that $\text{Ker}[\text{Pic}(f)] = \text{Ker}[\text{Pic}(C_{\text{red}}) \rightarrow \text{Pic}((C_{\text{red}})_L)]$. Applying (1.2) once again, we see that $\text{Ker}[\text{Pic}(f)] = H^1(G, (C_{\text{red}})_L) = H^1(G, B_{\text{red}})$. Now the remainder of the proof of the theorem goes through.

**Problem 1.10.** The proof of the theorem shows that if $k$ is a field, $A$ is a finitely generated $k$-algebra (reduced if $\text{char}(k) \neq 0$), and a finite group $G$ acts on $A$, then $H^1(G, A^*)$ is finite. Under the same hypotheses, for which $n \in \mathbb{N}$ is the set $H^1(G, \text{GL}_n(A))$ finite?

**Theorem 1.11.** Let $k$ be a field, and let $K/k$ be a Galois extension of fields, not necessarily finite. Let $S$ be a $k$-scheme of finite type, and assume that each connected component of $S$ is geometrically connected. Let $\Lambda$ be the integral closure of $K$ in $\Gamma(S_K)_{\text{red}}$. Assume that $\Gamma(S)^{\text{red}}_* \Lambda^* = \Gamma(S_K)^{\text{red}}_*$. Then the canonical map $\text{Pic}(S) \rightarrow \text{Pic}(S_K)$ is injective.

**Proof.** It is enough to show that for each finite Galois extension $L/k$ with $k \subset L \subset K$, the map $\text{Pic}(S) \rightarrow \text{Pic}(S_L)$ is injective. Let $G = \text{Gal}(L/k)$. Let $B_0 = \Gamma(S_L)_{\text{red}}$. Let $\Lambda_0$ be the integral closure of $L$ in $B_0$.

The condition of the theorem implies that $\Gamma(S)^{\text{red}}_* \Lambda_0^* = B_0^*$. Therefore the action of $G$ on $B_0^*/\Lambda_0^*$ is trivial. Hence $H^1(G, B_0^*/\Lambda_0^*) \cong \text{Hom}_{\leq \text{groups}}(G, B_0^*/\Lambda_0^*)$ by (1.4)(2). Since $G$ is finite and $B_0^*/\Lambda_0^*$ is torsion-free [by (1.5)], $H^1(G, B_0^*/\Lambda_0^*) = 0.$
Since $S$ is geometrically connected, $S_L$ is connected, and so $\Lambda_0$ is a field. Since $G$ acts faithfully on $\Lambda_0$, $H^1(G, \Lambda_0) = 0$ by (1.5).

Hence $H^1(G, B_0^*) = 0$. Arguing as in (1.9), one obtains the theorem.

**Theorem 1.12.** Let $k$ be a field and $S$ a scheme of finite type over $k$. Let $K/k$ be a separable algebraic field extension. Then the kernel of the map $\text{Pic}(S) \to \text{Pic}(S_K)$ (induced by the projection $\pi : S_K \to S$) is a finite group.

**Affine version of Theorem 1.12.** Let $k$ be a field, and let $A$ be a finitely generated $k$-algebra. Let $K/k$ be a separable algebraic field extension. Then the kernel of the induced map $\text{Pic}(A) \to \text{Pic}(A_K)$ is finite.

**Proof (of 1.12).** By passing to the normal closure, we may assume that $K/k$ is Galois (possibly infinite). Let $B = \Gamma(S_K)_{\text{red}}$, and let $\Lambda$ be the integral closure of $K$ in $B$. By (1.6), $B^*/\Lambda^*$ is finitely generated. Therefore we can find a finite Galois extension $L$ of $k$ (with $L \subset K$) such that the canonical map $(\Gamma(S_L)_{\text{red}})^* \to B^*/\Lambda^*$ is surjective. We can also choose $L$ so that each connected component of $S_L$ is geometrically connected. By (1.7) and (1.9), the kernel of $\text{Pic}(S) \to \text{Pic}(S_L)$ is finite, and by (1.11), the map $\text{Pic}(S_L) \to \text{Pic}(S_K)$ is injective, so we are done.

We will see in §4 that (1.12) can fail if $K/k$ is not assumed to be separable. On the other hand, assuming resolution of singularities, we will show in §4 that for any separated scheme $S$ of finite type over $k$, and any algebraic field extension $K/k$, there is an intermediate field $E/k$ of finite degree over $k$ such that $\text{Pic}(S_E) \to \text{Pic}(S_K)$ is one-to-one.

We conclude this section by showing that the kernel is not always trivial. In fact, any finite abelian group can occur. If, in the following construction, one takes $K/k = \mathbb{C}/\mathbb{R}$, one obtains the familiar example $A = \mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$). The Picard group of $A$ has order two (generated by the Möbius band), whereas $A \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}[U, U^{-1}]$, which has trivial Picard group.

**Example 1.13.** Let $K/k$ be a finite Galois extension with Galois group $G$. There is a domain $A$ of finite type over $k$ such that $\text{Pic}(A) \cong G/[G, G]$ but $\text{Pic}(A \otimes_k K)$ is trivial.

**Proof.** Make $\mathbb{Z}G$ into a $G$-module via the left regular action. We have an exact sequence of $G$-modules

$$0 \to \mathbb{Z} \xrightarrow{\sigma} \mathbb{Z}G \to \mathbb{L} \to \mathbb{V},$$
where \( G \) acts trivially on \( \mathbb{Z} \), \( \sigma \) takes 1 to \( \sum_{g \in G} g \), and \( L \) is defined by the sequence. Let \( F = \text{Im}(\sigma) \), which is the set of fixed points of \( \mathbb{Z}G \) under the action of \( G \). Clearly \( L \) is a free \( \mathbb{Z} \)-module of rank \( |G| - 1 \).

Form the group ring \( B = K[L] \). Since \( L \) is a free abelian group, \( B \) is isomorphic to the Laurent polynomial ring in \( |G| - 1 \) variables and hence has trivial Picard group. Note that \( G \) acts on \( B \) by acting as the Galois group on \( K \) and by the \( G \)-module structure on \( L \). Let \( A = B^G \). Then \( A \otimes_k K \cong B \) by [Sw1, (2.5)], and the action of \( G \) on \( A \) is Galois. Using (1.2) we get \( \text{Pic}(A) = D(A,B) = H^1(G,B^*) \). Let \( A \otimes_k K \cong B \) by [Sw1, (2.5)].

2. Torsion in Picard groups

We note that \( \text{Pic}(R) \) can be infinite even for \( R \) an domain finitely generated over a field. For example, take \( R = k[T^2, T^3] \), where \( k \) is an infinite field of characteristic \( p > 0 \). Then \( \text{Pic}(R) \) is isomorphic to the additive group of \( k \) and is therefore an infinite group of exponent \( p \). As long as we avoid the characteristic, however, this cannot happen. First we need the following lemma (cf. [Bas1, IX, (4.7)]):

**Lemma 2.1.** Let \( f : X \to S \) be a finite flat morphism of schemes, of constant degree \( d > 0 \). Then \( \text{Ker}[\text{Pic}(f)] \) is \( d \)-torsion.

**Proof.** Let \( M \in \text{Pic}(S) \). Then \( f_* f^* M \cong M \otimes \{sO_X \} \) as \( O_S \)-modules. If moreover \( M \in \text{Ker}[\text{Pic}()] \), then \( f_* f^* M \cong \{sO_X \} \). Hence \( M \otimes \{sO_X \} \cong \{sO_X \} \). Apply \( \wedge^d \), yielding \( M^{\otimes i} \otimes \wedge^i(\{sO_X \}) \cong \wedge^i(\{sO_X \}) \), and hence \( M^{\otimes i} \cong O_S \).

Now let us generalize to the proper case. First we need:

**Lemma 2.2.** Let \( f : X \to S \) be a proper flat morphism of noetherian schemes. Then \( f_* O_X \) is a locally free \( O_S \)-module.

**Proof.** We may assume that \( S \) is affine. Let \( W = \text{Spec}(f_* O_X) \) (cf. [Ha, II, exercise 5.17]), and label morphisms

\[
X \xrightarrow{\varphi} W \xrightarrow{h} S.
\]

Since \( f \) is proper, \( f_* O_X \) is coherent [EGA3$_1$, 3.2.1], and so it is enough to show that \( h \) is flat. Therefore it is enough to show that for any injection \( i \) of coherent \( O_S \)-modules, \( h^*(i) \) is also injective. By construction,
Corollary 2.3. Let \( f : X \to S \) be a surjective proper flat morphism of noetherian schemes. Then \( \text{Ker}[\text{Pic}(f)] \) is a bounded torsion group.

Proof. We may assume that \( S \) is connected. By (2.2), \( f_*\mathcal{O}_X \) is a locally free \( \mathcal{O}_S \)-module. Factor \( f \) as in the proof of (2.2). By the projection formula, \( \varphi_*\varphi^* \) is the identity, so \( \text{Pic}(\varphi) \) is injective. Apply (2.1).

Theorem 2.4. Let \( S \) be a scheme of finite type over a field \( k \) and let \( n \in \mathbb{N} \) be prime to the characteristic of \( k \). Then \( n\text{Pic}(S) \) is finite.

Affine version of Theorem 2.4. Let \( k \) be a field, and let \( A \) be a finitely generated \( k \)-algebra. Let \( n \in \mathbb{N} \) be prime to the characteristic of \( k \). Then \( n\text{Pic}(A) \) is finite.

Proof (of 2.4). Suppose first that \( k \) is separably closed. The argument in this case seems to be fairly well known and was pointed out to us several years ago by David Saltman and Tim Ford. We consider the Kummer sequence [SGA4 1/2, p. 21, (2.5)]:

\[
1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \stackrel{n}{\longrightarrow} \mathbb{G}_m \longrightarrow 1
\]

This is an exact sequence of sheaves for the étale topology on \( S \). Taking étale cohomology, we get an exact sequence

\[
H^1(S_{\text{ét}}, \mu_n) \longrightarrow H^1(S_{\text{ét}}, \mathbb{G}_m) \stackrel{n}{\longrightarrow} H^1(S_{\text{ét}}, \mathbb{G}_m). (\ast)
\]

Now \( H^1(S_{\text{ét}}, \mathbb{G}_m) \cong \text{Pic}(S) \) by [SGA4 1/2, p. 20, (2.3)]. Since \( k \) is separably closed, \( \mu_n \) is isomorphic to the constant sheaf \( \mathbb{Z}/n\mathbb{Z} \). (See [SGA4 1/2, p. 21, (2.4)].) Therefore \( \mu_n \) is constructible [SGA4 1/2, p. 43, (3.2)]. By [SGA4 1/2, p. 236, (1.10)] \( H^1(S_{\text{ét}}, \mu_n) \) is finitely generated. Hence the kernel of the map \( n \) in (\ast) is finite, so the theorem holds when \( k \) is separably closed. Apply (1.12).

As a consequence of (2.3) and (2.4), we have:

Corollary 2.5. Let \( S \) be a scheme of finite type over a field \( k \). Let \( f : X \to S \) be a proper flat surjective morphism of schemes. Assume that over each connected component of \( S \), the rank of the locally free sheaf \( f_*\mathcal{O}_X \) is invertible in \( k \). Then \( \text{Ker}[\text{Pic}(f)] \) is finite.

We can use (2.4) to answer a question of S. Montgomery about outer automorphism groups of Azumaya algebras:
Corollary 2.6. Let $k$ be a field. Let $R$ be a finitely generated $k$-algebra. Let $A$ be a (not necessarily commutative) Azumaya algebra over $R$ of degree $d$. Then $\text{Out}_R(A)$ is finite whenever $d$ is not a multiple of the characteristic of $k$.

Proof. There is an embedding (see e.g. [DI]) of $\text{Out}_R(A)$ in $\text{Pic}(R)$, and in fact the image of $\text{Out}_R(A)$ is contained in $d\text{Pic}(R)$. (See [KO1].)

In fact, one can show that the finiteness of $d\text{Pic}(R)$ for all $d$ is equivalent to the finiteness of $\text{Out}_R(M_{d^e}(R))$ for all $e$ (see [BG]).

Finally, we consider $n$-torsion in a normal algebraic scheme. It turns out that this is finite, even if $n$ is not prime to the characteristic, at least assuming that resolution of singularities holds. To prove this, we need to know what happens to $\text{Pic}$ of a normal scheme when its singular locus is deleted. The kernel is described by the following “folklore” lemma, which we state in greater generality for later application (6.5):

Lemma 2.7. Let $X$ be a noetherian $S_2$ scheme and let $C \subset X$ be a closed subset of codimension $\geq 2$. Let $U = X - C$. Then the canonical map $\text{Pic}(X) \to \text{Pic}(U)$ is injective.

Proof. Let $L$ be a line bundle on $X$ which becomes trivial on $U$. For any line bundle $M$ on $X$, the long exact sequence of local cohomology gives us

$$H^0_C(X, M) \to H'(X, M) \xrightarrow{\rho_M} H'(U, M) \to H^0_C(X, M).$$

Since $X$ is $S_2$, the end terms vanish [G1, (1.4), (3.7), (3.8)]. Let $\phi : \mathcal{O}_U \to L|_U$ be an isomorphism. Since $\rho_L$ is an isomorphism, we can lift $\phi$ to a morphism $\psi : \mathcal{O}_X \to L$. Since $\rho_{L^{-\infty}}$ is an isomorphism, we can lift $\phi^{-1}$ to a morphism $\psi' : L \to \mathcal{O}_X$. Since $\rho_{\mathcal{O}_X}$ and $\rho_L$ are isomorphisms, $\psi' \circ \psi$ and $\psi \circ \psi'$ are the identity maps. □

Lemma 2.8. Let $k$ be an algebraically closed field. Assume that resolutions of singularities exist for varieties over $k$. Let $X$ be a normal $k$-scheme of finite type. Let $n \in \mathbb{N}$. Then $^n\text{Pic}(X)$ is finite.

Sketch. We may assume that $X$ is connected. By (2.1), we may replace $X$ by $X_{\text{reg}}$ and so assume that $X$ is regular. If we further replace $X$ by a nonempty open subscheme, we kill a finitely generated subgroup of $\text{Pic}(X)$. In this way we may reduce to the case where $X$ is affine. Since we have resolution of singularities, we can embed $X$ as an open subscheme of a regular $k$-scheme $\overline{X}$. Then $\text{Pic}(X)$ is the

---

3Recall that a noetherian scheme $X$ is by definition $S_2$ if for every $x \in X$, depth $\mathcal{O}_{X,x} \geq \min \{\dim \mathcal{O}_{X,x}, 1\}$. 
quotient of Pic($X$) by a finitely generated subgroup. Now Pic$^0(X)$ is the group $A(k)$ of $k$-valued points of an abelian variety $A$ over $k$, and Pic($X$)/Pic$^0(X)$ is finitely generated (see e.g. [K, (5.1)]). Therefore it suffices to show that $nA$ is finite. This is well-known [Mum, p. 39]. □

3. Faithfully flat extensions

The main results of this section are (3.6), (3.8), and (3.10), which give information about the kernel of the map on Picard groups induced by a faithfully flat morphism of algebraic schemes. First we consider the case where the target of the morphism is normal, in which case we can weaken the hypothesis of faithful flatness.

**Theorem 3.1.** Let $k$ be a field. Let $X$ be a normal $k$-scheme of finite type. Let $f : Y \to X$ be a dominant morphism of finite type. Then Ker[Pic($f$)] is finitely generated (if char($k$) = 0) and is the direct sum of a finitely generated group and a bounded $p$-group (if char($k$) = $p > 0$). If $k$ is algebraically closed, and resolution of singularities holds, then Ker[Pic($f$)] is finitely generated.

**Proof.** We may assume that $X$ is connected. By (2.7), we may assume that $X$ is regular. Then we may replace $X$ by any nonempty open subscheme. In particular, we may assume that $X$ is affine. Moreover, by replacing $Y$ by a suitable open subscheme, we may assume that $Y$ is affine too. We may assume that $Y$ is a regular integral scheme. We may embed $Y$ as an open subscheme of an $X$-scheme $\overline{Y}$ which is projective over $X$ and is an integral scheme as well. Replace $\overline{Y}$ be its normalization. Now by again replacing $X$ by a nonempty open subscheme, we may assume (by generic flatness) that the morphism $\overline{Y} \to X$ is flat; certainly we may assume that it is surjective. Call this morphism $\varphi$. By (2.3), Ker[Pic($\varphi$)] is a bounded torsion group. Hence by (2.7), Ker[Pic($\varphi$)] is finite (if char($k$) = 0) and is the direct sum of a finite group and a bounded $p$-group (if char($k$) = $p > 0$). [By (2.8), if $k$ is algebraically closed, then Ker[Pic($\varphi$)] is always finite.] By (2.7) and [Ha, II, (6.5c), (6.16)], Ker[Pic($\overline{Y}$) $\to$ Pic($Y$)] is finitely generated. The theorem follows. □

Now we want to see what happens when we consider a faithfully flat morphism $Y \to X$, where $X$ is not necessarily normal. The issue is complicated by nilpotents, even in the affine case. The problem [see examples (3.3), (3.4) below] is that one can have a domain $A$, and a finitely generated faithfully flat $A$ algebra $B$, such that $B_{\text{red}}$ is not flat over $A$. 
Lemma 3.2. Let $A$ be a reduced ring, with total ring of fractions $K$. Let $B$ be a faithfully flat $A$-algebra. Then (inside $B \otimes_A K$) $B \cap K = A$.

Proof. Let $b \in B \cap K$, so we have an equation of the form $bu = v$, for some $u, v \in A$ with $u$ a non-zero-divisor. By the equational criterion for faithful flatness [Bour, Ch. I, §3, n° 7, Prop. 13], $b \in A$.

Example 3.3. Let $k$ be a field of characteristic 2. Let $A = k[s, t]/(s^2 - t^3)$. Let $B = A[x, y]/(x^2 - s, y^2 - t)$. Then $B$ is a faithfully flat $A$-algebra. Now $B \cong k[x, y]/(x^4 - y^6)$, so the nilradical of $B$ is generated by $(x^2 - y^3)$. Hence $B_{\text{red}} = A[x, y]/(x^2 - s, y^2 - t, x^2 - y^3)$. Since $s = ty$ in $B_{\text{red}}$, we have $y \in B_{\text{red}} \cap A_{\text{nor}}$ and $y \notin A$, so it follows from (3.2) that $B_{\text{red}}$ is not flat over $A$. One can also prove this directly by taking $M = A/(t)$ and checking that the map $M \to M \otimes_A B_{\text{red}}$ is not injective.

Example 3.4 (shown to us by Bill Heinzer and Sam Huckaba). It is known that there is a smooth curve $C \subset \mathbb{P}^3$ of degree 8 and genus 5 which is set-theoretically the intersection of two surfaces $S, T$, but which is not arithmetically Cohen-Macaulay (see [Bar], [Hu]). Let $\tilde{C} = S \cap T$, scheme-theoretically. Choose lines $L, L'$ which are noncoplanar and do not meet $C$. Then projection from $L$ onto $L'$ defines a nonconstant morphism $\tilde{C} \to \mathbb{P}^1$. Let $A$ be the homogeneous coordinate ring of $\mathbb{P}^1$, and let $B$ be the homogeneous coordinate ring of $\tilde{C}$. Then $A = \mathbb{C}[\sim, \sim]$, $B$ is Cohen-Macaulay, $A \subset B$, and $B$ is module-finite over $A$, so $B$ is faithfully flat over $A$. (See [Ma, p. 140].) On the other hand, $B_{\text{red}}$ is not Cohen-Macaulay, so $B_{\text{red}}$ is not flat over $A$.

It is not clear if the behavior illustrated by the characteristic $p$ example can be mimicked in characteristic zero:

Problem 3.5. Let $k$ be an algebraically closed field of characteristic zero. Let $A$ be a finitely generated, reduced $k$-algebra. Let $B$ be a faithfully flat and finitely generated $A$-algebra. Do we have $B_{\text{red}} \cap A_{\text{nor}} = A$?

Theorem 3.6. Let $k$ be a perfect field. Assume that resolutions of singularities exist for varieties over $k^a$. Let $X$ and $Y$ be geometrically integral $k$-schemes of finite type. Let $f : Y \to X$ be a faithfully flat morphism of $k$-schemes. Then $\text{Ker}[\text{Pic}(f)]$ is finitely generated.

We are not sure to what extent the hypothesis “geometrically integral” can be relaxed. Certainly if $X$ is not affine, nonreduced, and $Y$ is disconnected, one can have trouble (4.3). In positive characteristic, the
assumption that $Y$ is reduced is needed (4.2). In characteristic zero, we do not know if it is necessary to assume that $Y$ is reduced. [If the answer to (3.5) is yes, then we do not need to assume $Y$ reduced.] We do not know if it is necessary to assume that $X$ and $Y$ are geometrically irreducible. Cf. (3.10).

Affine version of Theorem 3.6. Let $k$ be a perfect field. Assume that resolutions of singularities exist for varieties over $k$. Let $A$ be a finitely generated $k$-algebra. Let $B$ be a finitely generated and faithfully flat $A$-algebra, which is geometrically integral over $k$. Then $\text{Ker}[\text{Pic}(A) \to \text{Pic}(B)]$ is finitely generated.

Proof (of 3.6). By (1.12), $\text{Ker}[\text{Pic}(X) \to \text{Pic}(X^\text{nor})]$ is finite, so we may assume that $k$ is algebraically closed. By (3.1), the theorem holds when $X$ is normal. To complete the proof, we need to show that $\text{Ker}[\text{Pic}(f) \cap \text{Pic}(X) \to \text{Pic}(X^\text{nor})]$ is finitely generated. If we relax the assumptions on $Y$, by assuming only that each connected component of $Y$ is geometrically integral, it is enough to do the two cases:

(i) $Y$ is an “open cover” of $X$;
(ii) $X$, $Y$ are both affine.

Let $P$ be the fiber product of $X^\text{nor}$ and $Y$ over $X$. Then $P$ is reduced. Let $\pi : X^\text{nor} \to X$ and $\tau : P \to Y$ be the canonical maps. Let $C$ be the quotient sheaf $(\pi_*\mathcal{O}^*_X^\text{nor})/\mathcal{O}^*_X$, and similarly let $D = (\tau_*\mathcal{O}^*_P)/\mathcal{O}^*_Y$. (Note that the canonical map $\mathcal{O}^*_Y \to \tau_*\mathcal{O}_P$ is injective.) We have a commutative diagram with exact rows (and some maps labelled):

$$
\begin{array}{ccc}
1 & \rightarrow & \Gamma(X^\text{nor})^*/\Gamma(X)^* \\
\downarrow & & \downarrow \lambda \\
1 & \rightarrow & \Gamma(C) \\
\uparrow & & \uparrow \delta \\
1 & \rightarrow & \Gamma(D) \\
& & \downarrow \\
& & \Gamma[\text{Pic}(Y) \to \text{Pic}(P)] \\
\end{array}
$$

We will be done with the proof if we can show that $\delta$ is injective and that $\text{Coker}(\lambda)$ is finitely generated.

We show that $\delta$ is injective. In case (i) this is clear, since $C$ is a sheaf. So we may assume that $X$, $Y$ are both affine. Let $A = \Gamma(X)$, $B = \Gamma(Y)$. Let $\alpha \in \Gamma(C)$. Then there exist elements $f_1, \ldots, f_n \in A$ with $(f_1, \ldots, f_n) = (1)$ and elements $\alpha_i \in (A^\text{nor})^*_f$ $(i = 1, \ldots, n)$, $\beta_{ij} \in A^*_{f_i f_j}$ such that $\alpha_i = \beta_{ij} \alpha_j$ in $(A^\text{nor})^*_{f_i f_j}$ and $\alpha$ induces $\alpha_1, \ldots, \alpha_n$. Suppose $\alpha$ maps to $1 \in \Gamma(D)$. Then for each $i$ we have elements $b_i \in B^*_{f_i}$ such that $b_i = \alpha_i$ in $B^*_{f_i} \otimes_A A^\text{nor}$ for each $i$. By (3.2), applied with $A_{f_i}$ substituted for $A$, we see that $\alpha_i \in A_{f_i}$. Hence $\alpha = 1$. Hence $\delta$ is injective.
To complete the proof, we need to show that
\[ \text{Coker}(\lambda) = \frac{\Gamma(P)^*}{\Gamma(Y)^* \Gamma(X_{\text{nor}})^*} \]
is finitely generated. For this, we may assume that \( Y \) is connected. By (1.6), it is enough to show that the map \( P \to X_{\text{nor}} \) is bijective on connected components. This is true since \( X \) and \( Y \) are integral schemes.

We note that the proof breaks down if we do not assume that \( Y \) is geometrically irreducible. Indeed, without this hypothesis, the map \( P \to X_{\text{nor}} \) may not be bijective on connected components:

**Example 3.7.** Let \( k \) be a field of characteristic \( \neq 2 \). Let \( A \) be the subring \( k[t^2, t + t^{-1}] \) of \( k[t, t^{-1}] \). Let \( B = A[x]/(x^2 - t^2) \). Then \( B \) is a faithfully flat \( A \)-algebra. We will see that (i) \( \text{Spec}(B) \) is connected, but (ii) \( \text{Spec}(B \otimes_A A_{\text{nor}}) \) is not connected. First note that \( t = (t^2 + 1)/(t + t^{-1}) \), from which it follows that \( A_{\text{nor}} = k[t, t^{-1}] \). Then \( B \otimes_A A_{\text{nor}} \cong A_{\text{nor}} \times A_{\text{nor}} \), so (ii) holds. Let \( z = t^2 + 1, y = t + t^{-1} \). Then \( A \cong k[y, z]/(y^2 + z^2 - y^2z) \), so
\[ B \cong k[x, y, z]/(x^2 - z + 1, y^2 + z^2 - y^2z) \cong k[x, y]/((x^2 + 1)^2 - x^2y^2). \]
From this one sees that \( \text{Spec}(B) \) has two smooth components, meeting at the two points \((\pm i, 0)\). Hence \( \text{Spec}(B) \) is connected.

**Theorem 3.8.** Let \( k \) be a field of characteristic \( p > 0 \). Let \( X \) and \( Y \) be geometrically irreducible \( k \)-schemes of finite type. Let \( f : Y \to X \) be a faithfully flat morphism of \( k \)-schemes. Then \( \text{Ker}[\text{Pic}(f)] \) is the direct sum of a finitely generated group and a \( p \)-group.

**Affine version of Theorem 3.8.** Let \( k \) be a field of characteristic \( p > 0 \). Let \( A \) be a finitely generated \( k \)-algebra. Let \( B \) be a finitely generated and faithfully flat \( A \)-algebra such that \( \text{Spec}(A \otimes_k k^a) \) is irreducible. Then the group \( \text{Ker}[\text{Pic}(A) \to \text{Pic}(B)] \) is the direct sum of a finitely generated group and a \( p \)-group.

**Proof** (of 3.8). It follows from (2.1) and (2.4) that \( \text{Ker}[\text{Pic}(X) \to \text{Pic}(X^a)] \) is the direct sum of a finitely generated group and a \( p \)-group, so we may assume that \( k \) is algebraically closed. Since \( \text{Ker}[\text{Pic}(X) \to \text{Pic}(X_{\text{red}})] \) is a \( p \)-group, we may assume that \( X \) is reduced.

If \( Y \) is reduced, we are done by (3.6). In the general case, we modify the argument of the proof of (3.6). Since the scheme \( P \) in that argument may be nonreduced, we do not get that \( \text{Coker}(\lambda) \) is finitely generated,
but rather (∗) it is finitely generated mod $p$-power torsion. We still get that
\[ M := \frac{\Gamma(P)_{\text{red}}^*}{\Gamma(Y)_{\text{red}}^* \Gamma(X_{\text{nor}})^*} \]
is finitely generated. Let $N$ be the nilradical of $\Gamma(P)$. Then the kernel of the canonical map $\text{Coker} (\lambda) \rightarrow M$ is a quotient of $1 + N$, and hence is a bounded $p$-group, so (∗) follows.

**Remark 3.9.** By (5.4), it will follow that the $p$-group of the theorem is actually a bounded $p$-group, provided that resolution of singularities is valid.

**Theorem 3.10.** Let $X$ be a reduced scheme of finite type over a perfect field. Let $U_1, \ldots, U_n$ be an open cover of $X$. Then the canonical map
\[ \text{Pic}(X) \rightarrow \text{Pic}(U_1) \times \cdots \times \text{Pic}(U_n) \]
has finitely generated kernel.

For the case where $X$ is normal (in fact, any noetherian normal scheme), this is easy, following roughly from (2.7). For the general case, one can follow the proof of (3.6), taking $Y$ to be the disjoint union of the $U_i$’s; one need only adjust the last sentence.

4. **Examples where the kernel is not finitely generated**

Let $f : Y \rightarrow X$ be a faithfully flat morphism of noetherian schemes. The results 1.7, 1.12, 2.3, 3.1, 3.6, and 3.10 all give conditions under which $\text{Ker}[\text{Pic}(f)]$ is finitely generated. While it is reasonable to think that there are unifying and more general results with the same conclusion, we do not know what form such results should take. With this in mind, we give in this section a varied collection of examples in which $\text{Ker}[\text{Pic}(f)]$ is not finitely generated.

We mention an obviously related question, about which we know very little. For which faithfully flat proper morphisms $f : Y \rightarrow X$ is it the case that the map
\[ \{\text{iso. classes of vector bundles on } X\} \rightarrow \{\text{iso. classes of vector bundles on } Y\} \]
is finite-to-one? Cf. (1.10), (5.3).

Returning to Pic, first we see that there are examples with $X$ algebraic of positive characteristic, and $f$ a finite étale morphism. For these, by (1.7), $X$ must be nonreduced.
Example 4.1. Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $\lambda : F \to E$ be a finite étale morphism of varieties over $k$, of degree $p$, and suppose we have a Galois action of a finite group $G$ on $F/E$. Let $X = E \times_k R$, $Y = F \times_k R$, and let $f : Y \to X$ be the induced morphism. Then $\text{Ker}[\text{Pic}(f)] \cong H^1(G, \Gamma(Y)^*)$ by (1.2). Since $\Gamma(Y)$ is just $R$, and $G$ acts trivially on it, we have by (1.4)(2) that $\text{Ker}[\text{Pic}(f)] \cong \text{Hom}(\mathbb{Z}/\mathbb{Z}, R^*)$, which is not finitely generated, since $(1 + c\epsilon)^p = 1$ in $R$ for each $c \in k$.

To get examples of such morphisms $\lambda$, let $E$ be an elliptic curve over $k$ which is not supersingular. Then the Tate module $T_p(E)$ is isomorphic to the $p$-adic integers $\mathbb{Z}_l$. According to a theorem of Serre-Lang [SGA1, XI, (2.1)], the $p$-primary part of the algebraic fundamental group $\pi_1(E)$ is $T_p(E)$. Therefore there exists a surjective homomorphism $\pi_1(E) \to \mathbb{Z}/\mathbb{Z}$, and so there exists a variety $F$ and a morphism $\lambda$ as indicated. More concretely, this may be seen as follows. The multiplication by $p$ map $E^p \to E$ factors through $E^{(p)}$, the scheme defined in essence by raising the coefficients in the equation defining $E$ to the $p^{th}$ power. The induced map $E^{(p)} \to E$, called the Verschiebung, is exactly $\lambda$.

Now we see that there are examples with $X$ reduced and algebraic, even over an algebraically closed field (of positive characteristic), and $f$ a finite flat morphism:

Example 4.2. Take example (3.3), and use $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$. Since the map $A \to B_{\text{red}}$ factors through $A_{\text{nor}}$, it follows that $\text{Ker}[\text{Pic}(f)] = k$, which is not finitely generated if $k$ is infinite.

Now we see that there are examples with $X$ algebraic and $Y$ étale, even in characteristic zero. Indeed, one may take $Y$ to be an “affine open cover” of $X$:

Example 4.3. Let $T$ be a projective variety over an algebraically closed field $k$, and let $F$ be a coherent sheaf on $T$ with $H^1(T, F) \neq 0$. Make $\mathcal{A} := \mathcal{O}_T \oplus F$ into an $\mathcal{O}_T$-algebra by forcing $\mathcal{F} \cdot \mathcal{F} = 0$. Let $X = \text{Spec}(\mathcal{A})$. Let $Y$ be an affine open cover of $X$, i.e. the disjoint union of the schemes in such a cover. Then $\text{Ker}[\text{Pic}(X) \to \text{Pic}(X_{\text{red}})] \subset \text{Ker}[\text{Pic}(X) \to \text{Pic}(Y)]$, and $\text{Ker}[\text{Pic}(X) \to \text{Pic}(X_{\text{red}})]$ is a nonzero vector space (over $k$), so $\text{Ker}[\text{Pic}(X) \to \text{Pic}(Y)]$ is not finitely generated.

Now we give families of examples in which $X$ and $Y$ can be chosen to be reduced noetherian schemes of characteristic zero, but the morphism is not of finite type:

Example 4.4. Given any ring $A$ there exists a faithfully flat extension $B$ with $\text{Pic}(B) = 1$: take $B = A[x]$ localized at the set of primitive
polynomials (i.e. polynomials such that the coefficients generate the unit ideal). For the fact that $\text{Pic}(B) = 1$, see [EG, (5.4), (3.5), (2.6 with $R = S$)].

**Example 4.5.** Let $k$ be a field and let $X = \text{Spec}(k[t^2, t^3])$. Let

$$Y = \text{Spec}(k[t, t^{-1}] ) \times k[t^2, t^3]_{(t^2,t^3)},$$

which is the disjoint union of $X_{\text{reg}}$ and $\text{Spec } \mathcal{O}_{X, \delta}$, where $x$ is the singular point of $X$. Then $\text{Ker}[\text{Pic}(f)] = k$.

Now we will give examples based on purely inseparable base extension. For purely inseparable extensions we cannot use Galois cohomology to control the Picard group. Instead, we use differentials, following the ideas in Samuel’s notes on unique factorization domains [Sam]. We build a purely inseparable form of the affine line whose Picard group is infinite.

We need the following version of a result of Samuel [Sam, 2.1, p. 62]:

**Lemma 4.6.** Let $B$ be a domain of characteristic $p > 0$ with fraction field $Q(B)$. Let $\delta$ be a $\mathbb{Z}$-linear derivation of $B$ with $A$ the subring of invariants of $\delta$ (i.e. the elements with $\delta(a) = 0$). Let $\Delta$ be the logarithmic derivative of $\delta$ defined on $Q(B)^*$ (i.e. $\Delta(b) = \delta(b)/b$). If $M_1, M_2$ are invertible ideals of $A$ with $M_iB = b_iB$ for each $i$ ($b_1, b_2 \in B$), then $M_1 \cong M_2$ if and only if $\Delta(b_1) - \Delta(b_2) \in \Delta(B^*)$.

**Proof.** If $M_1 \cong M_2$, then $b_1 = aub_2$ for some $a$ in the quotient field of $A$ and $u \in B^*$. Thus, $\Delta(b_1) = \Delta(b_2) + \Delta(u)$.

Conversely, if $\Delta(b_1) - \Delta(b_2) = \Delta(u)$ for some $u \in B^*$, then replacing $b_1$ by $b_1u$ allows us to assume that $\Delta(b_1) = \Delta(b_2)$ whence $\delta(b_1^{-1}b_2) = 0$. Set $a = b_1^{-1}b_2$. Thus, $a$ is in the quotient field of $A$. Replacing $M_1$ by $aM_1$ allows us to assume that $M_1B = M_2B$. We claim that this implies $M_1 = M_2$. It suffices to check this locally and so we may assume that each $M_i$ is principal. Let $a_i$ be a generator for $M_i$. It follows that $a_1/a_2$ is a unit in $B$. We have $\delta(a_1/a_2) = 0$ and $a_1/a_2 \in B$, so $a_1/a_2 \in A$. Similarly, $a_2/a_1 \in A$, so $a_1/a_2 \in A^*$, and hence $M_1 = M_2$. 

**Example 4.7.** Let $k$ be a separably closed imperfect field of characteristic $p$. Let $\alpha \in k \setminus k^p$. Let $q$ be a power of $p$ with $q > 2$, and let $K = k(\alpha^{1/q})$. Let $A = k[X, Y]/(X^q - X - \alpha Y^q)$. Set $r = q/p$. Then $A$ is a Dedekind domain and $P = \text{Pic}(A)$ is a group of exponent $q$ with $P/rP$ infinite. Also, $A_K \cong K[Z]$. Hence the kernel of the map $\text{Pic}(A) \to \text{Pic}(A_K)$ is infinite of exponent $q$.

**Proof.** Let $\beta = \alpha^{1/q}$. In the polynomial ring $B = K[Z]$, put $x = Z^q$ and $y = \beta^{-1}(Z^q - Z)$. Note that $Z = x - \beta y$, so that $K[Z] = K[x, y]$. 

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Moreover, we can identify $A$ with the subring $k[x, y]$ of $B$. Thus $A_K = B$. Thus $A$ is a Dedekind domain and $\text{Pic}(A_K)$ is trivial. It follows that $\text{Pic}(A)$ has exponent dividing $q$.

Let $V = \{ a \in k : a^q - a \in \alpha k^q \}$. Since $k$ is separably closed, for every $b \in k$ there is an element $a \in V$ such that

$$a^q - a - \alpha b^q = 0.$$ 

Thus $V$ is infinite. Given $a \in V$, define $b \in k$ by the above equation. Let $M(a)$ be the maximal ideal $(x-a, y-b)$ of $A$. Set $c = a - \beta b = a^{1/q}$. Note that $M(a)B = (Z - c)B$.

Let $a_1$ and $a_2$ be distinct elements of $V$. Let $b_i$ and $c_i$ be the corresponding elements defined above. It suffices to show that $M(a_1)^r$ and $M(a_2)^r$ are nonisomorphic.

Let $\gamma = \alpha^{1/p}$. Define a derivation $\delta$ on $k[\gamma]$ with $\delta(k) = 0$ and $\delta(\gamma) = \gamma$. Extend this derivation to $A_0 = A[\gamma]$ (and to its quotient field) by taking $\delta$ trivial on $A$. Set $W = Z^r$. Since $Z = x - \beta y$, $W = x^r - \gamma y^r$. Thus, $\delta(W) = -\gamma y^r = W - W^q$. Let $\Delta$ denote the logarithmic derivative of $\delta$.

The following observation will be useful: $\delta(c_i^r) = -\gamma b_i^r = c_i^r - c_i^q$. Thus

$$\Delta(W - c_i^r) = \frac{W - W^q - c_i^r + c_i^q}{W - c_i^r} = 1 - (W - c_i^r)^{q-1}.$$ 

Thus,

$$\Delta(W - c_1^r) - \Delta(W - c_2^r) = (W - c_2^r)^{q-1} - (W - c_1^r)^{q-1}$$

is a polynomial in $Z$ of degree $r(q - 2) > 0$ as long as $q > 2$.

Let $J = (W - c_1^r)A_0$ and $I = M(a_i)^rA_0$. Then $I^rA_0 = J^rA_0$, since after extension to $B$, they become equal. Now $A_0$ is a normal domain, so its group of invertible fractional ideals is torsion-free. Hence $IA_0 = JA_0$, i.e. $M(a_i)^rA_0 = (W - c_i^r)A_0$. Therefore, it suffices to show (by [1,6]) that $\Delta(W - c_1^r) - \Delta(W - c_2^r) \neq \Delta(u)$ for any $u \in A_0^*$. Since $\Delta(u)$ is a constant in $Z$, the result follows by the previous paragraph. \qed

Now we give an alternate (more geometric) explanation of the preceding result. It will show that the kernel of $\text{Pic}(A) \to \text{Pic}(A_K)$ is infinite and $q$-torsion, but not that the kernel has exponent $q$.

Let $U = \text{Spec}(A)$, and let $V = \text{Proj}(k[X, Y, T]/(X^q - XT^{q-1} - \alpha Y^q))$. Then $U$ is an open subscheme of $V$. We have a commutative diagram

$$
\begin{array}{ccc}
\text{Pic}(V) & \longrightarrow & \text{Pic}(U) \\
\downarrow & & \downarrow \\
\text{Pic}(V_K) & \longrightarrow & \text{Pic}(U_K).
\end{array}
$$
Of course Pic($U_K$) = 0 since $A_K$ is a polynomial ring. Since $U$ is obtained from $V$ by deleting the single regular point $[X, Y, T] = [1, 0, 0]$, Pic($V$) $\to$ Pic($U$) is surjective with cyclic kernel, and in fact one sees that the kernel is infinite cyclic. On the other hand, the map Pic($V$) $\to$ Pic($V_K$) is injective, as is well-known.\footnote{More generally, it is even true that if $K/k$ is any field extension, $V$ is a projective $k$-scheme, and one has two coherent $O_V$-modules which become isomorphic over $V_K$, then they were already isomorphic over $V$ – see [Wi, (2.3)].} Hence the kernel of Pic($U$) $\to$ Pic($U_K$) is exactly Pic$^0(V)$.

By results of Grothendieck [FGA, (2.1), (3.1)], there exists a commutative group scheme $P$ of finite type over $k$ such that Pic$^0(V) = P(k)$. Since $\dim(V) = 1, H^2(V, O_V) = 0$, so $P$ is smooth and $\dim(P) = h^1(V, O_V)$ by [FGA, #236, 2.10(ii, iii)]. Now $h^1(V, O_V)$ is just the arithmetic genus of a plane curve of degree $q$, which is $(q - 1)(q - 2)/2$. In particular, since $q \geq 3$, we have $h^1(V, O_V) > 0$. Hence $P$ is positive-dimensional. Now since $P$ is a geometrically integral scheme of finite type over a separably closed field $k$, $P(k)$ is Zariski dense in $P(k^a)$ – see [Se, discussion on p. 107]. In particular, since $P$ is positive-dimensional, it follows that $P(k)$ is infinite. Hence Pic$^0(V)$ is infinite. Hence the kernel of the map Pic($U$) $\to$ Pic($U_K$) is infinite; it is $q$-torsion by (2.1). This completes the alternate proof of (4.7), except that we have not shown that the kernel has exponent $q$.

We will see in the next section (at least assuming resolution of singularities) that for a separated scheme $X$ of finite type over a field $k$, the kernel of the map Pic($X$) $\to$ Pic($X_{k^a}$) is always bounded (i.e., torsion with finite exponent). The Picard group itself, however, can have infinite exponent. For example, if $X$ is any smooth affine or projective curve of positive genus over an algebraically closed field, then the torsion subgroup of Pic($X$) is unbounded, because then Pic($X$) is the quotient of an abelian variety by a finitely generated subgroup.

5. EVENTUAL VANISHING OF THE KERNEL OF Pic

Let $k$ be a field. The main theorem of this section (5.2) asserts that if $X$ is a separated $k$-scheme of finite type, then there exists a finite field extension $k^+$ of $k$, such that for every algebraic field extension $L$ of $k^+$, the canonical map Pic($X_L$) $\to$ Pic($X_{L^a}$) is injective.

This statement has a sheaf-theoretic formulation, which we consider, in part because it figures in the proof. Let $F$ be an (abelian group)-valued $k$-functor, meaning a functor from $\leq k$-algebras $\geq$ to $\leq$ abelian groups $\geq$. Let $p : B \to C$ be a faithfully flat homomorphism of $k$-algebras. There are maps $i_1, i_2 : C \to C \otimes_B C$, given by
The sequence
$$0 \to F(B) \xrightarrow{F(p)} F(C) \xrightarrow{F(i_1) - F(i_2)} F(C \otimes_B C) \quad (\ast)$$
is exact for all $p$, then one says that $F$ is a sheaf (for the fpqc [faithfully flat quasi-compact] topology).

This is often too much to ask, and so one may look only at certain maps $p$, or ask only that $F(p)$ be injective. To relate this to the theorem (5.2), let $F$ be given by $B \mapsto \text{Pic}(X_B)/\text{Pic}(B)$. Of course if $B$ is a field, we have $F(B) = \text{Pic}(X_B)$. What the theorem says is that if we enlarge $k$ sufficiently (replacing it by a finite extension), then $F(L \to L^a)$ is injective, for all algebraic extensions $L$ of $k$.

In this sense, $F$ becomes close to being a sheaf, if we allow for the enlargement of $k$. However, in general, one cannot get $(\ast)$ exact in an analogous manner. More precisely, for suitable $k$ and $X$, one cannot find a finite extension $k^+$ of $k$ such that for every algebraic extension $L$ of $k^+$, if $p : L \to L^a$ is the canonical map, then $(\ast)$ is exact. As an example, let $k$ be a separable closure of $\mathbb{F}_q(\approx)$, for some prime $q$, and let $X = \text{Spec}(k[x, y]/(y^2 - x^3))$. Then $\text{Pic}(X_L) = L$ for every extension field $L$ of $k$. For any finite extension $k^+$ of $k$, there is some $a \in k^+ - (k^+)^q$. If $L = k^+$, one finds that $a^{1/q}$ lies in the kernel of $F(i_1) - F(i_2)$, but not in the image of $F(p)$.

**Definition 5.1.** Let $k$ be a field. A $k$-scheme $S$ is geometrically stable if (1) it is of finite type, and (2) every irreducible component of $S_{\text{red}}$ is geometrically integral and has a rational point.

**Theorem 5.2.** Let $k$ be a field; assume that resolutions of singularities exist for varieties over $k^a$. Let $X$ be a separated $k$-scheme of finite type. Then there exists a finite field extension $k^+$ of $k$, such that for every algebraic field extension $L$ of $k^+$, the canonical map $\text{Pic}(X_L) \to \text{Pic}(X_{L^a})$ is injective.

**Proof of (5.2)** In the course of the proof, we will refer to enlarging $k$, by which we mean that $k$ is to be replaced by a suitably large finite field extension, contained in $k^a$. This is done only finitely many times. Then, at the end of the proof, the $k$ we have is really the $k^+$ of which the theorem speaks. We may treat $L$ as an extension of $k$ which is contained in $k^a$.

By induction on $\dim(X)$, we may assume that the theorem holds when $X$ is replaced by a scheme of strictly smaller dimension. (The case of dimension zero is trivial.)

**Step 1. The case of a geometrically normal scheme**
If $T$ is a $k$-scheme, we have let $T^a$ denote $T_k$. However, in some places in the next paragraph we shall define a $k^a$-scheme $T^a$, even though $T$ has not yet been defined; we will then proceed to construct a $k$-scheme $T$ such that $T^a = T_k$.

Suppose $X$ is geometrically normal. By [N], there is a proper $k^a$-scheme $X^a$ which contains $X^a$ as a dense open subscheme. (Note that $X$ does not yet been defined.) After normalizing $X^a$ we may assume that $X^a$ is normal. Let $\pi^a : Y^a \to X^a$ be a resolution of singularities, by which we mean that $Y^a$ is regular, $\pi^a$ is a proper morphism, and $\pi^a$ is an isomorphism over $X^a - \text{Sing}(X^a)$. (Note that $Y$ and $\pi$ have not yet been defined.)

By looking at the equations defining $X^a$, $Y^a$, and $\pi^a$, we can (after enlarging $k$ if necessary) find a $k^a$-scheme $X$ containing $X^a$ as a dense open subscheme, a $k^a$-scheme $Y$, and a morphism $\pi : Y \to X$ such that $\pi^a \times_k k^a = \pi^a$. Then $X$ is geometrically normal, $Y$ is geometrically regular, and (by faithfully flat descent [EGA4, (2.7.1)(vii)]) $\pi$ is proper.

Let $Y = \pi^{-1}(X)$.

By enlarging $k$ if necessary, we may assume that if $C_1, \ldots, C_n$ are the irreducible codimension one components of $Y - Y^a$, and if $p : Y^a \to Y$ is the natural map, then $p^{-1}(C_1), \ldots, p^{-1}(C_n)$ are the irreducible codimension one components of $Y^a - Y^a$.

Let $d : \text{Pic}(Y^a) \to \text{Pic}(Y^a)$, $e : \text{Pic}(Y) \to \text{Pic}(Y^a)$, $f : \text{Pic}(Y) \to \text{Pic}(Y^a)$ and $h : \text{Pic}(Y) \to \text{Pic}(Y^a)$ be the canonical maps.

For any normal proper $k$-scheme $V$ of finite type, the canonical map $\text{Pic}(V) \to \text{Pic}(V^a)$ is injective. (See [Mi3, (6.2)].) In particular, $e$ is injective. Since $[p^{-1}(C_1)], \ldots, [p^{-1}(C_n)]$ generate Ker($d$), it follows that $[C_1], \ldots, [C_n]$ generate Ker($de$). Since $f$ is surjective, it follows that $h$ is injective.

A slight modification of the argument above shows that the canonical map $h_L : \text{Pic}(Y_L) \to \text{Pic}(Y^a)$ is injective for every algebraic extension $L/k$. Let $L$ be a line bundle on $X_L$ that becomes trivial on $X^a$. Since $h_L$ is injective, $L$ becomes trivial on pullback to $Y_L$. Let $r : X^a \to X_L$ be the canonical map. Then the restriction of $L$ to $X_L - r(\text{Sing}(X^a))$ is trivial. By (2.7) $L$ is trivial. This completes the proof when $X$ is geometrically normal.

**Step 2. The case of a geometrically reduced scheme**

If $X$ is geometrically reduced we may assume, by enlarging $k$ if need be, that the normalization $X_{\text{nor}}$ is geometrically normal. Let $\pi : X_{\text{nor}} \to X$ be the canonical map. Let $\mathcal{I} = [\mathcal{O}_X : \pi_* \mathcal{O}_{X_{\text{nor}}}]$ be the
conductor of $X_{nor}$ into $X$. This is a coherent sheaf of ideals in $O_X$. Let $X/I$ denote $\text{Spec}(O_X/I)$, and let $X_{nor}/I$ denote $(X/I) \times_X X_{nor}$. By further enlarging $k$, we may assume that the pullback of $I$ to $X^a$ is the conductor of $(X_{nor})^a$ into $X^a$. Enlarging $k$ still further, we may assume that $X_{nor}$, $X/I$, and $X_{nor}/I$ are all geometrically stable.

Let $F_1$ and $F_2$ be the (abelian group)-valued $k$-functors defined by

$$F_1(B) = \frac{\Gamma((X_{nor})_B)^*}{\Gamma(X_B)^*} \quad \text{and} \quad F_2(B) = \frac{\Gamma((X_{nor}/I)_B)^*}{\Gamma((X/I)_B)^*}.$$ 

Let $G = F_2/F_1$, the quotient in $\ll (\text{abelian group})$-valued $k$-functors $\gg$. We have a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & F_1(L) & \rightarrow & F_2(L) & \rightarrow & G(L) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & F_1(L^a) & \rightarrow & F_2(L^a) & \rightarrow & G(L^a) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & F_1(L^a \otimes_L L^a) & \rightarrow & F_2(L^a \otimes_L L^a) & \rightarrow & G(L^a \otimes_L L^a) & \rightarrow & 0
\end{array}
\]

with exact rows. Now we use [J2, (4.5)]:

**Theorem** Let $S$ and $T$ be geometrically stable $k$-schemes, and let $f : S \rightarrow T$ be a dominant morphism of $k$-schemes. Let $Q$ be the (abelian group)-valued $k$-functor given by $Q(A) = \Gamma(S_A)^*/\Gamma(T_A)^*$. Let $p : B \rightarrow C$ be a faithfully flat homomorphism of reduced $k$-algebras. Then the sequence

$$0 \rightarrow Q(B) \rightarrow Q(C) \rightarrow Q(C \otimes_B C)$$

is exact.

This implies that the first two columns are exact. It follows that the canonical map $G(L) \rightarrow G(L^a)$ is injective.

We need a scheme-theoretic version of Milnor's Mayer-Vietoris sequence [Bas1, Chap. IX, (5.3)], which may be found in [We, (7.8)(i)], and which implies that there is an exact sequence

$$0 \rightarrow F_1(L) \rightarrow F_2(L) \rightarrow \text{Pic}(X_L) \rightarrow \text{Pic}((X_{nor})_L) \times \text{Pic}((X/I)_L) \rightarrow \text{Pic}((X/I)^a)_L$$

for each algebraic extension $L/k$. By the induction hypothesis announced near the beginning of the proof, we may assume that $\text{Pic}((X/I)_L) \rightarrow \text{Pic}((X/I)^a)_L$ is injective. Also, by Step 1, $\text{Pic}((X_{nor})_L) \rightarrow \text{Pic}((X_{nor})^a)_L$ is injective. Since $G(L) \rightarrow G(L^a)$ is injective, it follows that $\text{Pic}(X_L) \rightarrow$
Pic($X^a$) is injective. Thus the theorem is true for geometrically reduced schemes.

**Step 3. Deal with the nonreduced case**

This step may of course be ignored if $X$ is affine. Otherwise, we use the following result [J2,(5.2)(a)]:

**Theorem** Let $k$ be a field and $X$ a geometrically stable $k$-scheme. Let $i : X_0 \to X$ be a nilimmersion. For any $k$-algebra $A$, let $\kappa(A)$ be the kernel of the natural map $\text{Pic}(X \times_k A) \to \text{Pic}(X_0 \times_k A)$. Let $A \to B$ be a faithfully flat homomorphism of reduced $k$-algebras. Then the induced map $\kappa(A) \to \kappa(B)$ is injective.

To complete the proof of (5.2), we may assume, by enlarging $k$ if need be, that $X$ is geometrically stable and that $k$ is big enough so that the conclusion is valid for the (geometrically reduced) scheme $X_{\text{red}}$. We have the following commutative diagram, for any algebraic extension $L/k$:

$$
\begin{array}{c}
\text{Pic}(X_L) \xrightarrow{\alpha} \text{Pic}((X_{\text{red}})_L) \\
\downarrow \gamma \qquad \downarrow \delta \\
\text{Pic}(X^a) \xrightarrow{\beta} \text{Pic}(X^a_{\text{red}})
\end{array}
$$

Taking $(A \to B) = (L \to k^a)$ in [J2,(5.2)(a)], cited above, we see that $\gamma$ is one-to-one on $\text{Ker}(\alpha)$. Since $\delta$ is injective, so is $\gamma$. \qed

**Remark 5.3.** Let $F$ be a finite field of characteristic different from 5 and containing a primitive fifth root of unity $\zeta$. Let $Y \subset \mathbb{P}^3_F$ be the Fermat quintic given by the equation:

$$x^5 + y^5 + z^5 + t^5 = 0.$$

Then the group of fifth roots of unity acts on $Y$ by sending $(x, y, z, t)$ to $(x, \zeta y, \zeta^2 z, \zeta^3 t)$. This action has no fixed points, and the quotient $X$ is a smooth projective surface which is called the Goedeaux surface. For any finite extension $E/F$, $\text{CH}^2(X_E)_{\text{tors}} = \mathbb{Z}/\ell\mathbb{Z}$ [KS1, Proposition 9]. On the other hand, over an algebraic closure $\overline{F}$, we have $\text{CH}^2(X_{\overline{F}})_{\text{tors}} = 0$ [Mi2]. Thus for each $E$ there exists a finite extension $H$ over which the $\mathbb{Z}/\ell\mathbb{Z}$ dies, but a new one comes to take its place. Now for any smooth surface over a field, the natural map $K_0(X) \to \text{CH}^*(X)$ is an isomorphism between the Grothendieck ring and the Chow ring. Hence we produce an element of $K_0(X)$ with similar properties. Thus the theorem does not hold with $K_0$ in place of Pic.
Corollary 5.4. Let $k$ be a field; assume that resolutions of singularities exist for varieties over $k^a$. Let $X$ be a $k$-scheme of finite type. Then the kernel of the map $\text{Pic}(X) \to \text{Pic}(X^a)$ is a bounded torsion group.

Proof. One can (details omitted) use (3.10) to reduce to the case where $X$ is separated. Let $K = k^a$. Let $k^+$ be as in (5.2). Then $\text{Pic}(X) \to \text{Pic}(X_K)$ and $\text{Pic}(X) \to \text{Pic}(X_{k^+})$ have the same kernel. By (2.1) the kernel has exponent dividing $[k^+: k]$. 

6. Finitely Generated Fields

We say a field $k$ is absolutely finitely generated if it is finitely generated over its prime subfield. In this section we will study the structure of $\text{Pic}(X)$, where $X$ is a scheme of finite type over an absolutely finitely generated field. We begin with a result that is presumably well known, but for which we have found no reference.

Proposition 6.1. Let $X$ be a normal scheme of finite type over $\mathbb{Z}$ or over an absolutely finitely generated field $k$. Then $\text{Pic}(X)$ is finitely generated.

Proof. A theorem due to Roquette [Ro], [L, Chap. 2, (7.6)] handles the case of schemes of finite type over $\mathbb{Z}$. Suppose now that $X$ is of finite type over the absolutely finitely generated field $k$. By (2.7) the map $\text{Pic}(X) \to \text{Pic}(X - \text{Sing}(X))$ is injective. Therefore we may assume that $X$ is regular. There exists a finitely generated $\mathbb{Z}$-algebra $A \subset k$ and an $A$-scheme $X_0$ of finite type such that $X \cong X_0 \times_A k$. We have $X \subset X_0$. Since $X$ is regular, $\mathcal{O}_{X,x}$ is regular for every $x \in X$. Since $X_0$ is excellent, its regular locus is open, so there exists an open subscheme of $X_0$ which is regular and contains $X$. By replacing $X_0$ by this subscheme, we may assume that $X_0$ is regular. The map on divisor class groups $\text{Cl}(X_0) \to \text{Cl}(X)$ is certainly surjective, and since both $X_0$ and $X$ are regular, the map $\text{Pic}(X_0) \to \text{Pic}(X)$ is surjective. Since $\text{Pic}(X_0)$ is finitely generated (by Roquette’s theorem), so is $\text{Pic}(X)$.

The following examples show, in contrast, that the torsion subgroup of $\text{Pic}(X)$ need not be finite if $X$ is not normal.

Example 6.2. Let $B = \mathbb{Q}[\zeta, \zeta^{-1}]$, and put $A = \mathbb{Q} + (\zeta - 1)\mathbb{B}$. Then $A$ is a one-dimensional domain, finitely generated as a $\mathbb{Q}$-algebra, and $\text{Pic}(A) \cong \mathbb{Q}/\mathbb{Z}$. In particular, $\text{Pic}(A)$ is an unbounded torsion group.

Proof. We note that $I := (x - 1)^2B$ is the conductor of $B$ into $A$. Therefore by Milnor’s Mayer-Vietoris exact sequence [Bas1, Chap. IX,
(5.3)] [or see (♦, p. 22) for the scheme-theoretic version], we have Pic(A) \cong (B/I)^*/((A/I)^*U), where \( U \) is the image of \( B^* \) in \( (B/I)^* \). But \( (A/I)^* = \mathbb{Q}^* \), so Pic(A) \cong (B/I)^*/U. Now

\[ B^* = \{ sx^j : s \in \mathbb{Q}^*, j \in \mathbb{Z} \} \cong \mathbb{Q}^* \oplus \mathbb{Z}, \]

and \( (B/I)^* \cong \mathbb{Q}^* \oplus W \), where \( W = \{ 1 + s(x - 1) : s \in \mathbb{Q} \} \cong \mathbb{Q} \). By keeping track of these identifications, one easily gets Pic(A) \cong \mathbb{Q}/\mathbb{Z}.

By a slight modification we get an example of finite type over \( \mathbb{Z} \):

**Example 6.3.** Fix a positive integer \( m \), put \( B = \mathbb{Z} \langle \cup, \cup^{−m}, \frac{1}{s} \rangle \), and let \( A = \mathbb{Z} \langle \cup, \cup^{−m} \rangle + (\cup^{−m})^k \mathbb{B} \). Then \( A \) is a two-dimensional domain finitely generated as a \( \mathbb{Z} \)-algebra, and Pic(A) \cong \bigoplus_{p|m} \mathbb{Z}/\mathbb{Z}.

The pathology in the examples above stems from the fact that \( B/I \) is not reduced.

Before stating our main finiteness theorems [(6.5) and (6.6)] we record the following result from [CGW, (7.4)]:

**Theorem 6.4.** Let \( k \) be an absolutely finitely generated field and let \( \Lambda \) be a finite-dimensional reduced \( k \)-algebra. Let \( E_1 \) and \( E_2 \) be intermediate subalgebras of \( \Lambda/k \).

1. If \( k \) has positive characteristic \( p \), then \( \Lambda^*/E_1^*E_2^* \) is a direct sum of a countably generated free abelian group, a finite group, and a bounded \( p \)-group.
2. If \( \Lambda/k \) is separable, then \( \Lambda^*/E_1^*E_2^* \) is a direct sum of a countably generated free abelian group and a finite group.

**Theorem 6.5.** Let \( k \) be a field finitely generated over \( \mathbb{Q} \) and let \( X \) be a reduced \( k \)-scheme of finite type which is seminormal and \( S_2 \). Then Pic(X) is isomorphic to the direct sum of a free abelian group and a finite abelian group.

**Proof.** Let \( \mathcal{I} \) be the conductor of \( X_{\text{nor}} \) into \( X \) (see §3, Step 2). Let \( X/\mathcal{I} := \text{Spec}(\mathcal{O}_X/\mathcal{I}) \) and \( X_{\text{nor}}/\mathcal{I} := (X/\mathcal{I}) \times_{X} X_{\text{nor}} \) denote the corresponding closed subschemes. Since \( X \) is seminormal, it follows [T, (1.3)] that \( X/\mathcal{I} \) is reduced. Let \( Q \) be the non-normal locus of \( X/\mathcal{I} \), which has codimension \( \geq 2 \) in \( X \). By [2.7], the canonical map Pic(X) \rightarrow Pic(X - Q) is injective. Therefore we may replace \( X \) by \( X - Q \) and start the proof over, with the added assumption that \( X/\mathcal{I} \) is normal.

Let \( D \) be the kernel of the map \( \phi : \text{Pic}(X) \rightarrow \text{Pic}(X/\mathcal{I}) \times \text{Pic}(X_{\text{nor}}) \). By [6.4], the target of this morphism is finitely generated. Let \( \Lambda \) be the integral closure of \( k \) in \( \Gamma(X_{\text{nor}}/\mathcal{I}) \), \( E_1 \) the integral closure of \( k \) in
Γ(X/I), and E₂ the image in Λ of the integral closure of k in Γ(X_{nor}). Using (1.6) and the exact sequence (♦, p. 22) with L = k, we see that there is an exact sequence

\[ \text{finitely generated} \to \Lambda^*/E_1^*E_2^* \to D \to \text{finitely generated}. \]

By (6.4)(2), \( \Lambda^*/E_1^*E_2^* \) is free ⊕ finite. It follows that D and thence Pic(X) is free ⊕ finite.

\[ \text{Theorem 6.6. Let } k \text{ be an absolutely finitely generated field of positive characteristic } p, \text{ and let } X \text{ be a } k\text{-scheme of finite type. Then Pic}(X) \text{ has the form} \]

\[ (\text{countably generated free abelian group}) \oplus (\text{bounded } p\text{-group}) \oplus (\text{finite group}). \]

\[ \text{Proof. Let } \mathcal{C} \text{ be the class of abelian groups having the form ascribed to Pic}(X) \text{ in the theorem. We leave to the reader to verify that } \mathcal{C} \text{ is closed under formation of subgroups and extensions. Induct on } \text{dim}(X); \text{ the case where } \text{dim}(X) = 0 \text{ is trivial. We will reduce to the case where } X \text{ is reduced. For this, it is enough to show (\ast) that if } J \subset \mathcal{O}_X \text{ is a square-zero ideal, } X_0 = \text{Spec}(\mathcal{O}_X/J), \text{ and } \text{Pic}(X_0) \in \mathcal{C}, \text{ then } \text{Pic}(X) \in \mathcal{C}. \text{ The standard exact sequence of sheaves on } X \]

\[ 0 \to J \xrightarrow{a \mapsto 1+a} \mathcal{O}_X^* \to (\mathcal{O}_X)/J \to 1 \]

yields on taking cohomology an exact sequence

\[ H^1(X, J) \to \mathcal{H}^\infty(X, \mathcal{O}_X^*) \to \mathcal{H}^\infty(X, (\mathcal{O}_X/J)^*), \]

from which (\ast) follows, since \( H^1(X, J) \) is an \( \mathbb{F}_p \)-vector space. Therefore we may assume that \( X \) is reduced.

Let \( \mathcal{I}, \mathcal{D}, \phi, \text{ etc. } \) be as in the proof of (6.5). (Here we do not know that \( X/\mathcal{I} \) is normal.) By induction and (6.1), the target of \( \phi \) is in \( \mathcal{C} \), and therefore \( \text{Im}(\phi) \) is also. Since \( \mathcal{C} \) is closed under extensions, it suffices to show that \( D \in \mathcal{C} \). Arguing as in the proof of (6.5), with (6.4)(2) replaced by (6.4)(1), we see that this is the case. \[ \square \]

\[ \text{Remark 6.7. For } k = \mathbb{F}_p; \text{ it was shown in } [\text{J1, (10.11)}] \text{ that Pic}(X) \text{ has the form} \]

\[ (\oplus_{n=1}^\infty F) \oplus (\text{finitely generated abelian group}), \]

where \( F \) is a finite \( p\)-group.
7. Complements on \(K_0(X)\)

Let \(k\) be a field, and let \(X\) be a \(k\)-scheme of finite type. Let \(K_0(X)\) denote the Grothendieck group of vector bundles on \(X\). In this section, which is purely expository, we consider the analog for \(K_0(X)\) of the absolute finiteness results for \(\text{Pic}(X)\) proved in sections 2 and 6. Consider the following table:

| \(k\) alg. | \(k\) abs. f.g. | \(k\) finite | \(k\) abs f.g. |
|------------|-----------------|--------------|----------------|
| closed     | of char. \(p>0\) | \(\text{finite}\) | of char. 0     |

\begin{itemize}
\item \(X\) arbitrary:  
  - \([1]\) finite
  - \([3]\) finite
  - \([4]\) f.g.
  - \([5]\) f.g.
\item \(X\) regular:  
  - \([2]\) finite
  - \([5]\) f.g.
\end{itemize}

If we view this as a collection of statements about \(\text{Pic}(X)\), then all five statements are true\(^5\), as we have seen in the preceding sections. From now on, regard the table as a table of five conjectures about \(K_0(X)\). The numbering of these conjectures is not related to the numbering of results in the introduction.

It would be surprising if all five of these conjectures held. There is no field over which any of them are known to hold, even if one restricts attention to smooth projective or smooth affine schemes \(X\). Conjecture [5] would follow from a conjecture of Bass [Bas2, §9.1] to the effect that \(K_i(X)\) is finitely generated for all \(i \geq 0\) and all regular schemes \(X\) which are of finite type over \(\mathbb{Z}\).

Over an arbitrary field \(k\), it is not clear what sort of finiteness statement might hold for \(K_0(X)\): there are examples of infinite \(n\)-torsion in the Chow groups of smooth projective varieties over a field of characteristic zero [KM].

If \(X\) is smooth of dimension \(n\), then the operation of taking Chern classes defines a homomorphism of graded rings

\[
\text{Gr}[K_0(X)] \to \text{CH}^*(X),
\]

which becomes an isomorphism after tensoring by \(\mathbb{Z}[\mathbb{P}/(\mathbb{A}^n - \mathbb{A}^n)]\). (See [F, (15.3.6)].) It follows that for any almost any question about \(K_0(X)\), there is a parallel question about the groups \(\text{CH}^q(X), q = 1, \ldots, n\). Note that when \(q = 1\), \(\text{CH}^1(X) = \text{Pic}(X)\).

\(^5\)For statement [2], we have used resolution of singularities.
For the remainder of this section, suppose that $X$ is smooth, projective, and of dimension $n$; we give a partial discussion of results and conjectures about $\text{CH}^q(X)$, for $q \geq 2$.

First suppose that $k$ is algebraically closed. Some things are known when $q \in \{2, n\}$: (i) The group $m\text{CH}^2(X)$ is finite if $m$ is invertible in $k$ [Ra, (3.1)]; (ii) the group $m\text{CH}^n(X)$ is finite for every $m$. This follows from Roitman’s theorem (See e.g. [Ra, (3.2)].) Hence [2] holds for smooth projective surfaces.

Now suppose that $k$ is a number field. Bloch has conjectured that $\text{CH}^q(X)$ is finitely generated for all $q$. All results in this direction assume at least that $H^2(X, \mathcal{O}_X) = 0$. With this hypothesis, it has been shown that the torsion subgroup of $\text{CH}^2(X)$ is finite [CR]. Moreover, if $X$ is a surface which is not of general type, it is known [Sal], [CR] that $\text{CH}^2(X)$ is finitely generated. Hence conjecture [5] holds for a smooth projective surface over a number field which is not of general type.

Finally, suppose that $k$ is a finite field. Again, it is conjectured that $\text{CH}^q(X)$ is finitely generated for all $q$. What is known is that $\text{CH}^0(X)$ is finitely generated (in fact $\text{CH}_0(X)$ is finitely generated for any scheme $X$ of finite type over $\mathbb{Z}$ [KS2]), and that the torsion subgroup of $\text{CH}^2(X)$ is finite [CSS], cf. [CR, (3.7)]. The first assertion implies that conjecture [5] holds for any smooth projective surface over a finite field.

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