Non-Canonical Volume-Form Formalism of Modified Gravity Theories and Cosmology

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A concise description is presented of the basic features of the formalism of non-canonical spacetime volume-forms and its application in modified gravity theories and cosmology. The well known unimodular gravity theory appears as a very special case. Concerning the hot issues facing cosmology now, we specifically briefly outline the construction of: (a) unified description of dark energy and dark matter as manifestations of a single material entity – a special scalar field “darkon”; (b) quintessential models of universe evolution with a gravity-“inflaton”-assisted dynamical Higgs mechanism – dynamical suppression/generation of spontaneous electroweak gauge symmetry breaking in the “early”/“late” universe; (c) unification of dark energy and dark matter with diffusive interaction among them; (d) mechanism for suppression of 5-th force without fine-tuning.

1. Non-Riemannian Volume-Form Formalism

A broad class of actively developed modified/extended gravitational theories is based on employing alternative non-Riemannian spacetime volume-forms (metric-independent generally covariant volume elements) in the pertinent Lagrangian actions instead of, or alongside with, the canonical Riemannian volume element given by the square-root of the determinant of the Riemannian metric. This method was originally proposed in [1, 2] and for a concise geometric formulation using differential forms combined with canonical Hamiltonian formalism for systems with constraints (gauge symmetries), see [3, 4] (an earlier geometric formulation with a “quartet” of scalar fields appeared in [5]).

Volume-forms are fairly basic objects in differential geometry – they exist on arbitrary differentiable manifolds and define covariant (under general coordinate reparametrizations) integration measures. It is important to stress that the existence of volume-forms is completely independent of the presence or absence of additional geometric structures on the manifold – e.g., no Riemannian structure (purely metric or metric-affine) is a priori needed. Volume forms are defined [6] by nonsingular maximal rank differential forms ω:

\[ \int_M \omega(...) = \int_M dx^D \Omega(...) \]

\[ \omega = \frac{1}{D!} \omega_{\mu_1...\mu_D} dx^{\mu_1} \wedge ... \wedge dx^{\mu_D}, \]

\[ \omega_{\mu_1...\mu_D} = -\varepsilon_{\mu_1...\mu_D} \Omega, \]  

(1)

(our conventions for the alternating symbols \( \varepsilon^{\mu_1...\mu_D} \) and \( \varepsilon_{\mu_1...\mu_D} \) are: \( \varepsilon^{0...D-1} = 1 \) and \( \varepsilon_{0...D-1} = -1 \)).

The volume element density \( \Omega \) transforms as scalar density under general coordinate reparametrizations.

In standard generally-covariant theories (with action \( S = \int d^Dx \sqrt{-g} \mathcal{L} \)) the Riemannian spacetime volume-form is defined through the “D-bein” (frame-bundle) canonical one-forms \( e^A = e^A_\mu dx^\mu \) (\( A = 0, \ldots, D-1 \)):

\[ \omega = e^0 \wedge ... \wedge e^{D-1} = \det |e^A_\mu| \det dx^{\mu_1} \wedge ... \wedge dx^{\mu_D} \]

\[ \longrightarrow \quad \Omega = \det |e^A_\mu| \det dx^D x = \sqrt{-\det |g_{\mu\nu}|} \det dx^D x. \]  

(2)

Instead of, or alongside with, \( \sqrt{-g} \) we can employ one or several different alternative non-Riemannian volume elements as in [6] given by non-singular exact D-forms \( \omega^{(j)} = d\mathcal{B}^{(j)} \) where:

\[ B^{(j)} = \frac{1}{(D-1)!} B^{(j)}_{\mu_1...\mu_{D-1}} dx^{\mu_1} \wedge ... \wedge dx^{\mu_{D-1}} \]

\[ \longrightarrow \quad \Omega^{(j)} = \Phi(\mathcal{B}^{(j)}) = \frac{1}{(D-1)!} \varepsilon^{\mu_1...\mu_D} \partial_{\mu_1} B^{(j)}_{\mu_2...\mu_D}. \]  

(3)

In other words, the non-Riemannian volume elements are defined in terms of the dual field-strengths of auxiliary rank \( D - 1 \) tensor gauge fields \( B^{(j)}_{\mu_1...\mu_{D-1}} \).

Let us again strongly emphasize that the term “non-Riemannian” concerns only the nature of the non-canonical volume elements, which exist on the spacetime manifold with a standard Riemannian geometric structure, i.e., involving the metric \( g_{\mu\nu} \) and torsionless affine connection \( \Gamma^\lambda_{\mu\nu} \) either independent of \( g_{\mu\nu} \) (first-order metric-affine / Einstein-Palatini formalism) or as a Levi-Civita connection w.r.t. \( g_{\mu\nu} \) (second-order purely metric / Einstein-Hilbert formalism).

The generic form of modified gravity actions involving (one or more) non-Riemannian volume-elements, called for short NRVF (Non-Riemannian Volume-Form) actions, read (henceforth \( D = 4 \), and we will use units...
with $16\pi G_{\text{Newton}} = 1$:

$$S = \int d^4 x \Phi(B^{(1)})(R + L^{(1)})$$

$$+ \int d^4 x \sum_{j \geq 2} \Phi(B^{(j)}) L^{(j)} + \int d^4 x \sqrt{-g} L^{(0)}, \quad (4)$$

where $R$ is the scalar curvature. The equations of motion of \( \Phi(B^{(j)}) \) w.r.t. the auxiliary tensor gauge fields \( B_{\mu
u}^{(j)} \) according to \( (4) \) yield:

$$\partial_\mu(R + L^{(1)}) = 0, \quad \partial_\mu L^{(j)} = 0 \quad (j \geq 2), \quad \rightarrow \quad R + L^{(1)} = M_1, \quad L^{(j)} = M_j, \quad (5)$$

where all \( M_j \) \( (j \geq 1) \) are free integration constants not present in the original NRVF gravity action \( (4) \).

A characteristic feature of the NRVF gravitational theories \( (4) \) is that when starting in the first-order (Palatini) formalism all non-Riemannian volume-elements \( \Phi(B^{(j)}) \) yield almost pure-gauge degrees of freedom, i.e. they do not introduce any additional physical (field-propagating) gravitational degrees of freedom except for few discrete degrees of freedom with conserved canonical momenta appearing as the arbitrary integration constants \( M_j \) in \( \Phi(B^{(j)}) \). The reason is that the NRVF gravity action \( (4) \) in Palatini formalism is linear w.r.t. the velocities of some of the components of the auxiliary gauge fields \( B_{\mu
u}^{(j)} \) defining the non-Riemannian volume-element densities, and does not depend on the velocities of the rest of auxiliary gauge field components. The (almost) pure-gauge nature of the latter is explicitly shown in \( \Phi(B^{(j)}) \) (appendices A) employing the standard canonical Hamiltonian treatment of systems with gauge potentials, i.e., systems with first-class Hamiltonian constraints a la Dirac \( [8, 9] \).

However, in the second-order formalism (where \( \Gamma^{(2)}_{\mu
u} \) is the usual Levi-Civita connection w.r.t. \( g_{\mu\nu} \)) the first non-Riemannian volume form \( \Phi(B^{(1)}) \) in \( (4) \) is not any more pure-gauge. The reason is that the scalar curvature \( R \) (in the metric formalism) contains second-order (time) derivatives (the latter amount to a total derivative in the ordinary case \( S = \int d^4 x \sqrt{-g} R + \ldots \)). Now defining \( \chi_1 = \Phi(B^{(1)})/\sqrt{-g} \), the latter field becomes physical degree of freedom as seen from the equations of motion of \( (4) \) w.r.t. \( g_{\mu\nu} \):

$$R_{\mu\nu} + \frac{1}{\chi_1}(g_{\mu\nu} \Box \chi_1 - \nabla_\mu \nabla_\nu \chi_1) + \ldots = 0. \quad (6)$$

As a final introductory remark let us note that the well-known covariant formulation of unimodular gravity \( [10] \) can be viewed as a simple particular case within the general class \( (4) \) of modified gravity actions based on the non-Riemannian volume-form formalism.

Indeed, the original action of unimodular gravity \( (1) \) reads:

$$S = \int d^4 x \sqrt{-g} (R + 2\Lambda) - \int d^4 x \Phi(2\Lambda) \quad (7)$$

with \( \Lambda \) being a dynamical field, and \( \Phi \equiv \partial_\mu F^\mu \) where the vector density \( F^\mu \) can be written as Hodge-dual \( F^\mu \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} B_{\nu\rho\sigma} \) w.r.t. rank 3 auxiliary gauge field \( B_{\nu\rho\sigma} \) (cf. \( (3) \) for \( D = 4 \)). Variation w.r.t. \( F^\mu \) implies \( \Lambda = \text{const} \), whereas variation w.r.t. \( \Lambda \) yields \( \Phi = \sqrt{-g} \), i.e., \( \chi_1 \equiv \frac{4}{\sqrt{-g}} = 1 \). As we will see in what follows, for general NRVF gravity models \( (4) \) the field ratio \( \chi_1 \) is either a non-trivial algebraic function of the matter fields in \( L^{(j)} \) within the first-order (Palatini) formalism (cf. Eq.\((22) \) below), or it becomes a new dynamical scalar field within the second-order (metric) formalism (cf. Eq.\((6) \)).

2. Simple Model of Unified Dark Energy and Dark Matter

A simple NRVF gravity model providing a unified description of dark energy and dark matter defined by an action, particular representative of the class \( (4) \), was proposed in \( \Phi(B^{(j)}) \) [11, 12]:

$$S = \int d^4 x \left[ \sqrt{-g}(R + X - V_1(\phi)) + \Phi(B)(X - V_2(\phi)) \right], \quad (8)$$

or equivalently:

$$S = \int d^4 x \sqrt{-g}(R - U(\phi)) + (\sqrt{-g} + \Phi(B))(X - V(\phi)) \quad (9)$$

using the notations: \( V \equiv V_2, \quad U \equiv V_1 - V_2, \quad X \equiv -\frac{2}{3} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \) and \( \Phi(B) \equiv 1/3! \varepsilon^{\mu\nu\rho\sigma} \partial_\mu B_{\nu\rho\sigma} \) (cf. \( (3) \)). Variation of the action \( (9) \) w.r.t. auxiliary gauge field \( B_{\nu\rho\sigma} \) yields (cf. the general Eqs.\((5) \)):

$$X - V(\phi) = -2M_0, \quad (10)$$

where \( M_0 \) is free integration constant. The variation of \( (9) \) w.r.t. scalar field \( \phi \) can be written in the following suggestive form:

$$\nabla_\mu J^\mu = -\sqrt{2} X U'(\phi), \quad (11)$$

$$J_\mu \equiv -(1 + \chi) \sqrt{2} X \partial_\mu \phi, \quad \chi \equiv \frac{\Phi(B)}{\sqrt{-g}}. \quad (12)$$

The dynamics of \( \phi \) is entirely determined by the dynamical constraint \( (10) \), completely independent of the potential \( U(\phi) \). On the other hand, the \( \phi \)-equation of motion written in the form \( (11) \) is in fact an equation determining the dynamics of \( \chi \). The energy-momentum tensor \( T_{\mu\nu} \) in the Einstein equations can be written in a relativistic hydrodynamical form as:

$$T_{\mu\nu} = \rho \delta_{\mu\nu} u_\nu + g_{\mu\nu} \tilde{p}, \quad J_\mu = \rho \delta_{\mu} u_\mu \quad (13)$$

where \( u_\mu \) is a fluid velocity unit vector:

$$u_\mu \equiv -\frac{\partial_\mu \phi}{\sqrt{2} X} \quad (\text{note } u^\mu u_\mu = -1), \quad (14)$$

and the energy density \( \tilde{\rho} \) and pressure \( \tilde{p} \) are given as:

$$\tilde{\rho} = \rho_0 + 2M_0 + U(\phi), \quad \tilde{p} = -2M_0 - U(\phi) \quad (15)$$
with \( \rho_0 \equiv (1 + \chi) 2X = \bar{\rho} + \bar{p} \). Energy-momentum conservation \( \nabla^\nu T_{\mu \nu} = 0 \) implies:
\[
\nabla^\nu (\rho_0 u_\mu) = -\sqrt{2X} U'(\phi), \quad u_\nu \nabla^\nu u_\mu = 0 ,
\]
the last Eq.(16) meaning that the matter fluid flows along geodesics. In Eqs.\((13, 15)\) the quantity \( \rho_0 \equiv 2M_0 + U(\phi) = -\bar{p} \) has the interpretation as dark energy density, whereas \( \rho_0 \) is the dark matter energy density. For \( U(\phi) = \text{const} \) or \( U(\phi) = 0 \) the model \((9)\) possesses a non-trivial hidden nonlinear Noether symmetry under:
\[
\delta_\phi \phi = \epsilon \sqrt{X}, \quad \delta_\mu \rho_\mu = 0 ,
\]
\[
\delta_\nu B^\mu = -\epsilon \frac{1}{2 \sqrt{X}} \phi^\mu (\Phi(B) + \sqrt{-g}) ,
\]
where \( B^\mu = \frac{1}{2} \epsilon^{\mu \nu \lambda \kappa} B_{\nu \lambda \kappa} \), with a Noether conserved current \( J^\mu = \rho_0 u_\mu \) according to \((12)\): \( \nabla_\mu J^\nu = 0 \). Specifically, for Friedmann-Lemaitre-Robertson-Walker scale factor \( a(t) \) Eq.\((12)\) with \( U(\phi) = 0 \) implies: \( \rho_0 = c_0 / a^3 \), \( c_0 \) being a free integration constant.

Thus, according to \((13, 15)\), the model provides an exact description of \( \Lambda \text{CDM} \) model, and for a non-trivial potential \( U(\phi) \), breaking the hidden Noether symmetry \((17)\), we have interacting dark energy and dark matter.

The above interpretation justifies the alias “darkon” for the scalar field \( \phi \). Let us specifically emphasize that both dark energy and dark matter components of the energy density \((15)\) have been dynamically generated thanks to the non-Riemannian volume element construction – both due to the appearance of the free integration constant \( M_0 \) and of the hidden nonlinear Noether symmetry \((17)\) ("darkon" symmetry). In Ref.\((13)\) the correspondence between \( \Lambda \text{CDM} \) model and the “darkon” Noether symmetry was exhibited up to linear order w.r.t. gravity-matter perturbations and the implications of the “darkon” symmetry breaking for possible explanation of the cosmic tensions was briefly discussed.

### 3. Quintessential Inflationary Model with Dynamical Higgs Effect in Metric-Affine Formulation

The starting point is the following specific NRVF gravity action from the class \((4)\) involving coupling to a scalar “inflaton” \( \phi \) and to the bosonic sector of the standard electroweak particle model where, following the remarkable Bekenstein’s idea from 1986 \((14)\) about gravity-assisted dynamical spontaneous symmetry breakdown, the Higgs-like \( SU(2) \times U(1) \) iso-doublet scalar \( \sigma_a \) enters with a standard positive mass-squared and without self-interaction in sharp distinction w.r.t. standard particle model. The pertinent NRVF action reads explicitly \((21)\) [15 16]:
\[
S = \int d^4x \Phi_1(A) \left[ R(g, \Gamma) - 2\Lambda_0 \frac{\Phi_1(A)}{\sqrt{-g}} + L^{(1)}(\phi, \sigma) \right] + \int d^4x \Phi_2(B) \left[ f_2 e^{2\alpha \phi} + L_{\text{EW-gauge}} - \frac{\Phi_0(C)}{\sqrt{-g}} \right] \quad \text{(18)}
\]
with notations:

- \( \Phi_1(A) = \frac{1}{\sqrt{2}} \epsilon^{\mu \nu \lambda \kappa} \partial_\mu A_{\nu \lambda \kappa} \) and similarly for \( \Phi_2(B), \Phi_0(C) \) according to \((3)\):
- The scalar curvature \( R(g, \Gamma) = g^{\mu \nu} R_{\mu \nu}(\Gamma) \) is given in terms of the Ricci tensor \( R_{\mu \nu}(\Gamma) \) in the first-order (Palatini) formalism;
- The matter Lagrangian reads:
\[
L^{(1)}(\varphi, \sigma) = X_{\varphi} + f_1 e^{\alpha \varphi} + X_\sigma - m_0^2 g_0^g \sigma_a e^{\alpha \sigma} , \quad \text{(19)}
\]
\[
X_{\varphi} = -\frac{1}{2} g^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi , \quad X_\sigma = -g^{\mu \nu} \partial_\mu \sigma_a \partial_\nu \sigma_a ;
\]
- \( L_{\text{EW-gauge}} \) denotes the Lagrangian of the \( SU(2) \times U(1) \) gauge fields.
- \( \Lambda_0 \) is small dimensional constant which will be identified in the sequel with the “late” universe cosmological constant in the dark energy dominated accelerated expansion’s epoch.

The equations of motion w.r.t. auxiliary tensor gauge fields in \( \Phi_1(A), \Phi_2(B) \) and \( \Phi_0(C) \) yield (cf. \((5)\)):
\[
g^{\mu \nu} \left[ R_{\mu \nu}(\Gamma) - \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi - \partial_\mu \sigma_a \partial_\nu \sigma_a \right] - 4\Lambda_0 \frac{\Phi_1(A)}{\sqrt{-g}} + (f_1 - m_0^2 g_0^g \sigma_a) e^{\alpha \varphi} = M_1(20)
\]
\[
f_2 e^{2\alpha \phi} + L_{\text{EW-gauge}} - \frac{\Phi_0(C)}{\sqrt{-g}} = -M_2, \quad \frac{\Phi_2(B)}{\sqrt{-g}} = \chi(21)
\]
where \( M_1, \chi_2 \) are integration constants. The \( g^{\mu \nu} \)-equations of motion together with \((20, 21)\) imply that the ratio \( \chi_1 \equiv \frac{\Phi_1(A)}{\sqrt{-g}} \) is an algebraic function of the matter fields:
\[
\chi_1(\varphi, \sigma) = \frac{\Phi_1(A)}{\sqrt{-g}} = \frac{2\chi_1(f_2 e^{2\alpha \varphi} + M_2)}{M_1 + (m_0^2 g_0^g \sigma_a - f_1)} e^{\alpha \varphi} . \quad \text{(22)}
\]

The equation of motion w.r.t. \( \Gamma^{\lambda}_{\mu \nu} \), following analogous derivation in \((1)\), yields a solution for \( \Gamma^{\lambda}_{\mu \nu} \) as a Levi-Civita connection w.r.t. to a Weyl-conformally rescaled metric:
\[
\tilde{g}_{\mu \nu} = \chi_1(\varphi, \sigma) g_{\mu \nu} \quad \text{(23)}
\]
with \( \chi_1(\varphi, \sigma) \) as in \((22)\). The conformal transformation \( g_{\mu \nu} \to \tilde{g}_{\mu \nu} \) on the NRVF action \((18)\) converts the latter into the physical Einstein-frame action (objects in the Einstein-frame are indicated by a bar):
\[
S_{\text{EF}} = \int d^4x \sqrt{-\tilde{g}} \left[ R(\tilde{g}) - \frac{1}{2} \tilde{g}^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \right] - \tilde{g}^{\mu \nu} \partial_\mu \sigma_a \partial_\nu \sigma_a - U_{\text{eff}}(\varphi, \sigma) + L_{\text{EW-gauge}}(\tilde{g}) . \quad \text{(24)}
\]
Here the interesting object is the effective Einstein-frame scalar potential:
\[
U_{\text{eff}}(\varphi, \sigma) = \frac{\left[ M_1 + e^{\alpha \sigma} (m_0^2 g_0^g \sigma_a - f_1) \right]^2}{4\chi_2(f_2 e^{2\alpha \varphi} + M_2)^2} + 2\Lambda_0 , \quad \text{(25)}
\]
which is entirely dynamically generated due to the appearance of the free integration constants $M_1, \chi_2$ \cite{20, 21}. $U_{\text{eff}}(\varphi, \sigma)$ exhibits a number of remarkable features:

- $U_{\text{eff}}(\varphi, \sigma)$ possesses two (infinitely) large flat regions as a function of $\varphi$ at $\sigma_a$ = fixed.

- The first one – the (-) flat “inflaton” region for large negative values of $\varphi$ (and $\sigma_a$ = finite) corresponds to the “slow-roll” inflationary evolution of the “early” universe driven by $\varphi$ where:

$$
U_{\text{eff}}(\varphi, \sigma) \simeq U(-) = \frac{M^2}{4\chi_2 M_2} + 2\Lambda_0,
$$

\cite{26} reduces to (an almost) constant value independent of the finite value of $\sigma_a$, which is energy scale of the inflationary epoch. Thus, in the “early” universe there is no spontaneous breaking of electroweak $SU(2) \times U(1)$ symmetry. Moreover, $\sigma_a$ does not participate in the “slow-roll” inflationary evolution, so $\sigma$ stays constant there equal to the “false” vacuum value $\sigma = 0$ \cite{10}.

- The second flat region is the (+) flat “inflaton” region for large positive values of $\varphi$ (and $\sigma_a$ = finite) which corresponds to the evolution of the post-inflationary (“late”) universe. Here:

$$
U_{\text{eff}}(\varphi, \sigma) \simeq U(+)(\sigma) = \left(\frac{m_0^2 \sigma_a \sigma_0 - f_1}{4\chi_2 f_2}\right)^2 + 2\Lambda_0
$$

\cite{27} becomes a dynamically induced $SU(2) \times U(1)$ spontaneous symmetry breaking Higgs-like potential with a Higgs “vacuum” at $|\sigma_{\text{vac}}| = \frac{1}{m_0}\sqrt{f_1}$.

- Relations \cite{26, 27} allow the following natural identification of the scales of the parameters: $\Lambda_0 \sim 10^{-122} M_{\text{Pl}}^4$ (current epoch observable cosmological constant); $f_1 \sim f_2 \sim M_{\text{EW}}$ and $m_0 \sim M_{\text{EW}}$ (M$_{\text{EW}}$ being the electroweak mass scale); $M_1 \sim M_2 \sim 10^{-8} M_{\text{Pl}}$ corresponding to the “early” universe’s energy scale of inflation being of order $10^{-2} M_{\text{Pl}}$.

Concerning confrontation with the observational data, the viability of the present model (in a slightly simplified form without the Higgs scalar, which as already mentioned does not influence the slow-roll inflationary dynamics) has been analyzed and confirmed numerically in Ref. \cite{17}. The results for the tensor-to-scalar ratio $r \simeq 10^{-3}$ and for the scalar spectral index $n_s \simeq 0.96$ which are in a good agreement with the latest PLANCK data \cite{21}. Further detailed numerical studies on the NRVF models have been presented in Refs. \cite{22, 23}.

Let us also note that Ref. \cite{17} (for an earlier version, see \cite{18}) exhibits an explicit realization of the cosmological “seesaw” mechanism through the NRVF formulation, as well as it yields an additional “emergent universe” cosmological solution without a “Big-Bang” initial singularity. For a brief illustration of the latter effects let us consider the “inflaton-only” NRVF action studied in \cite{17} (for simplicity we skip the $R^2$ term):

$$
S = \int d^4x \Phi_1(A) \left[ R + X_\varphi - f_1 e^{-\alpha \varphi} \right]
\int d^4x \Phi_2(B) \left[ -b e^{-\alpha \varphi} X_\varphi + f_2 e^{-2\alpha \varphi} - \frac{\Phi_0(C)}{\sqrt{-g}} \right],
$$

where $b$ is an additional dimensionless parameter.

The “inflaton” potential in the Einstein frame (analog of \cite{25}) is $U_{\text{eff}}(\varphi) = \frac{1}{4\chi_2} \left( f_1 e^{-\alpha \varphi} + M_1 \right)^2 \left( f_2 e^{-2\alpha \varphi} + M_2 \right)^{-1}$, so that on the (-) and (+) “inflaton” flat regions $U_{\text{eff}}(\varphi)$ reduces to: $U(-) \simeq \frac{f_1^2}{4\chi_2 f_2}$ and $U(+) \simeq \frac{M_1^2}{4\chi_2 M_2}$, accordingly. Therefore, choosing $f_1 \sim f_2 \sim 10^{-8} M_{\text{Pl}}$ conforming to the inflationary scale, and taking $M_1 \sim M_{\text{EW}}$ and $M_2 \sim M_{\text{EW}}$ we achieve $U(+) \sim 10^{-122} M_{\text{Pl}}$ vastly smaller than $U(-)$. If we take $\alpha \rightarrow -\alpha$ in \cite{28} the roles of $f_1, f_2$ and $M_1, M_2$ are interchanged.

Similar “seesaw” effect is found in Refs. \cite{19, 20} where the scalar potential is extracted from the slow-roll parameters. $^1$

Furthermore, the NRVF model \cite{28} yields in Einstein-frame “emergent universe” solution for the range of the $b$-parameter: $-4(2 - \sqrt{3}) < b \frac{f_1}{f_2} < -1$.

4. Dynamical Generation of Inflation in Metric Formulation

Let us now consider a substantially truncated version of the model \cite{18} without any matter fields, i.e. a pure gravity model involving few non-Riemannian volume elements \cite{25}:

$$
S = \int d^4x \left\{ \Phi_1(A) \left[ R(g) - 2\Lambda_0 \frac{\Phi_1(A)}{\sqrt{-g}} \right] + \Phi_2(B) \left[ \frac{\Phi_0(C)}{\sqrt{-g}} \right] \right\},
$$

where now unlike \cite{15} $R(g) \equiv g^\mu\nu R_{\mu\nu}(g)$ is the scalar curvature in the second order (metric) formalism ($\Gamma^\lambda_{\mu
u}(g)$ being the Levi-Civita connection w.r.t. $g_{\mu\nu}$).

The equations of motion w.r.t. auxiliary tensor gauge fields $A_{\mu\nu,\lambda}$, $\Phi_2(B)$ and $\Phi_1(A)$ are special cases of the dynamical constraint Eqs. \cite{20, 21} with all matter field terms being zero, which again introduce the three free integration constants $M_{1,2}, \chi_2$.

Passage to the physical Einstein frame is again realized via the conformal transformation \cite{23}, however this time we have to use the well-known formulas for conformal transformations within the metric formalism (e.g. \cite{20};

\footnote{The paper \cite{20} was awarded second prize in the 2020 Essay Competition of the Gravity Research Foundation.}
bars indicate magnitudes in the $\tilde{g}_{\mu\nu}$-frame):

$$R_{\mu\nu}(g) = R_{\mu\nu}(\tilde{g}) - \frac{3}{\chi_1} \tilde{g}_{\mu\nu} \tilde{\epsilon}^\lambda \partial_\lambda \chi_1^{1/2} \partial_\lambda \chi_1^{1/2}$$

$$+ \chi_1^{-1/2} (\nabla_\mu \nabla_\nu \chi_1^{1/4} + \tilde{g}_{\mu\nu} \nabla_\lambda \chi_1^{1/2}),$$

with $\chi_1 \equiv \frac{\tilde{g}_{\mu\nu}}{\sqrt{\tilde{g}}}$. Redefining $\chi_1$ as $\chi_1 = \exp (u/\sqrt{3})$ allows to write the Einstein-frame NRVF action in the form:

$$S_{\text{EF}} = \int d^4x \sqrt{-g} \left[ R(g) - \frac{1}{2} \tilde{g}_{\mu\nu} \partial_\mu u \partial_\nu u - U_{\text{eff}}(u) \right]$$

$$U_{\text{eff}}(u) = 2 \Lambda_0 - M_1 \exp (-\frac{u}{\sqrt{3}}) + \chi_2 M_2 \exp (-2 \frac{u}{\sqrt{3}}).$$

Thus, from the original pure-gravity NRVF action we derived a physical Einstein-frame action containing a dynamically created scalar field $u$ with a non-trivial effective scalar potential $U_{\text{eff}}(u)$ entirely dynamically generated by the initial non-Riemannian volume elements in [29] because of the appearance of the free integration constants $M_{1,2}$, $\chi_2$ in their respective equations of motion. There are two main features of the effective potential [33] which are relevant for cosmological applications with the dynamically created field $u$ as an “inflaton”:

- $U_{\text{eff}}(u)$ [33] possesses one flat region for large positive values of $u$ where $U_{\text{eff}}(u) \approx 2 \Lambda_0$, which corresponds to ‘early’ universe’ inflationary evolution with energy scale $2 \Lambda_0$.

- $U_{\text{eff}}(u)$ [33] has a stable minimum for a small finite value $u = u_*$ where $e^{u_* - \sqrt{3}} = M_1/(2 \chi_2 M_2)$.

- The region around the stable minimum at $u = u_*$ correspond to “late” universe’ evolution where the minimum value of the potential:

$$U_{\text{eff}}(u_*) = 2 \Lambda_0 - \frac{M_1^2}{4 \chi_2 M_2} = 2 \Lambda_{\text{DE}}$$

is the dark energy density value.

**Remark.** The effective potential $U_{\text{eff}}$ [33] generalizes the well-known Starobinsky inflationary potential [27] [25] reduces to Starobinsky potential upon taking the following special values for the parameters: $\Lambda_0 = \frac{1}{4} M_1 = \frac{1}{2} \chi_2 M_2$.

In Ref. [25] a thorough analysis has been performed of the slow-roll inflationary dynamics driven by the dynamically created “inflaton” $u$ with its dynamically generated effective potential [33], including explicit calculation of the standard slow-roll parameters $\epsilon$ and $\eta$, as well we have obtained explicit expressions for the tensor-to-scalar ratio $r$ and the scalar spectral index $n_s$ of density perturbations as functions of the number of e-folds $N = \log a$ (a being the Friedmann scale factor):

$$r \approx \frac{12}{N^2} \frac{\chi_0}{4}, \quad n_s \approx 1 - \frac{r}{4} - \sqrt{\frac{r}{3}},$$

with $\chi_0 \equiv \frac{\tilde{g}_{\mu\nu}}{\sqrt{\tilde{g}}} - \frac{1}{2} \log \left(2(1 + 2/\sqrt{3})\right)$. $u(N)$ is the value of the “unflaton” at the start of inflation as function of $N$.

For a plausible assumption about the scales of $M_{1,2}$, $\chi_2$ and taking $N = 60$ e-folds till end of inflation the observables are predicted to be: $n_s \approx 0.969$, $r \approx 0.0026$, which conform to the PLANCK constraints [21] (0.95 < $n_s < 0.97$, $r < 0.064$).

### 5. Dynamical Spacetime Formulation

Let us now observe that the non-Riemannian volume element $\Omega = \Phi(B)$ [3] on a Riemannian manifold can be rewritten using Hodge duality (here $D = 4$) in terms of a vector field $\chi^\mu = \frac{1}{3} \sqrt{-g} \varepsilon^{\mu
u\alpha\lambda} B_{\nu\alpha\lambda}$ so that $\Omega$ becomes $\Omega(\chi) = \partial_\mu (\sqrt{-g} \chi^\mu)$, i.e. it is a non-canonical volume element different from $\sqrt{-g}$, but involving the metric. It can be represented alternatively through a Lagrangian multiplier action term yielding covariant conservation of a specific energy-momentum tensor of the form $T^{\mu\nu} = g^{\mu\nu} L$:

$$S_{(\chi)} = \int d^4x \sqrt{-g} \chi_{\mu\nu} T^{\mu\nu} = \int d^4x \partial_\mu (\sqrt{-g} \chi^\mu) (-L),$$

where $\chi_{\mu\nu} = \partial_\mu \chi_\nu - \Gamma^\lambda_{\mu\nu} \chi_\lambda$. The vector field $\chi_\mu$ is called “dynamical space time vector”, because of the energy density of $T^{00}$ is a canonically conjugated momentum w.r.t. $\chi_0$, which is what we expected from a dynamical time.

In what follows we will briefly consider a new class of gravity-matter theories based on the ordinary Riemannian volume element $\sqrt{-g}$ but involving action terms of the form [30] where now $T^{\mu\nu}$ is of more general form than $T^{\mu\nu} = g^{\mu\nu} L$. This new formalism is called “dynamical spacetime formalism” [28, 29] due to the above remark on $\chi_0$.

Ref. [30] describes a unification between dark energy and dark matter by introducing a quintessential scalar field in addition to the dynamical time action. The total Lagrangian reads:

$$L = \frac{1}{2} \sqrt{\tilde{g}} \chi_{\mu\nu} T^{\mu\nu} - \frac{1}{2} g^{\alpha\beta} \phi, \phi, \phi, \phi - V(\phi),$$

with energy-momentum tensor $T^{\mu\nu} = -\frac{1}{2} \phi^{\mu} \phi^{\nu}$. From the variation of the Lagrangian term $\chi_{\mu\nu} T^{\mu\nu}$ with respect to the vector field $\chi_\mu$, the covariant conservation of the energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$ is implemented. The latter within the FLRW framework forces the kinetic term of the scalar field to behave as a dark matter component:

$$\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \dot{\phi}^2 = \frac{2 \Omega_{m0}}{\rho^3}.$$
For constant potential $V(\phi) = \Omega_\Lambda = \text{const}$ the current is covariantly conserved. In the FLRW setting, where the dynamical time ansatz introduces only a time component $\chi_\mu = (\chi_0, 0, 0, 0)$, the variation (39) gives:

$$\chi_0 - 1 = \xi a^{-3/2},$$

(40)

where $\xi$ is an integration constant. Accordingly, the FLRW energy density and pressure read:

$$\rho = \left(\chi_0 - \frac{1}{2}\right) \dot{\phi}^2 + V, \quad p = \frac{1}{2} \dot{\phi}^2 (\chi_0 - 1) - V.$$  

(41)

Plugging the relations (38,40) into the density and the pressure terms (41) yields the following simple form of the latter:

$$\rho = \Omega_\Lambda + \frac{\xi \Omega_{m0}}{a^{3/2}} + \frac{\Omega_{m0}}{a^3}, \quad p = -\Omega_\Lambda + \frac{\xi \Omega_{m0}}{2 a^{3/2}}.$$  

(42)

In (42) there are 3 components for the ”dark fluid”: dark energy with $\omega_\Lambda = -1$, dark matter with $\omega_m = 0$ and an additional equation of state $\omega_\xi = 1/2$. For non-vanishing and negative $\xi$ the additional part introduces a minimal scale parameter, which avoids singularities. If the dynamical time is equivalent to the cosmic time $\chi_0 = t$, we obtain $\xi = 0$ from Eq. (40), whereupon the density and the pressure terms (42) coincide with those from the $\Lambda$CDM model precisely. The additional part (for $\xi \neq 0$) fits to the late time accelerated expansion data [31], with $\Omega_{m0} = 0.305_{-0.023}^{+0.031}$, $\xi = 0.183_{-0.125}^{+0.143}$ and with the Hubble parameter $H_0 = 62.57_{-1.52}^{+1.28}$Mpc/(km/sec). [32] shows that with higher dimensions, the solution derived from the Lagrangian (37) describes inflation, where the total volume oscillates and the original scale parameter exponentially grows.

The dynamical spacetime Lagrangian can be generalized to yield a diffusive energy-momentum tensor. Ref. [33] shows that the diffusion equation has the form:

$$\nabla_\mu T^{\mu \nu} = 3 \sigma j^\nu, \quad j^\mu = 0,$$  

(43)

where $\sigma$ is the diffusion coefficient and $j^\mu$ is a current source. The covariant conservation of the current source indicates the conservation of the number of the particles. By introducing the vector field $\chi_\mu$ in a different part of the Lagrangian:

$$\mathcal{L}_{(\chi,A)} = \chi_\mu \psi T^{\mu \nu} + \frac{\sigma}{2} (\chi_\mu + \partial_\mu A)^2,$$  

(44)

the energy-momentum tensor $T^{\mu \nu}$ gets a diffusive source. From a variation with respect to the dynamical space time vector field $\chi_\mu$ we obtain:

$$\nabla_\mu T^{\mu \nu} = \sigma (\chi^\mu + \partial^\mu A) = f^\mu,$$  

(45)

a current source $f^\mu = \sigma (\chi^\mu + \partial^\mu A)$ for the energy-momentum tensor. From the variation with respect to the new scalar $A$, a covariant conservation of the current emerges $j_\mu = 0$. The latter relations correspond to the diffusion equation (43). Refs. [34,57] study the cosmological solution using the energy-momentum tensor $T^{\mu \nu} = -\frac{1}{2} g^{\mu \nu} \phi^4$. The total Lagrangian reads:

$$\mathcal{L} = \frac{1}{2} R - \frac{1}{2} g^{\alpha \beta} \phi_\alpha \phi_\beta - V(\phi) + \chi_\mu \psi T^{\mu \nu} + \frac{\sigma}{2} (\chi_\mu + \partial_\mu A)^2.$$  

(46)

The FLRW solution unifies the dark energy and the dark matter originating from one scalar field with possible diffusion interaction. Ref. [72] investigates more general energy-momentum tensor combinations and shows that asymptotically all of the combinations yield $\Lambda$CDM model as a stable fixed point. Ref. [33] shows that the DST theories and Diffusive extensions can describe a Lagrangian formulation for Running Vacuum Models.

6. Scale Invariance, Fifth Force and Fermionic Matter

The originally proposed theory with two volume elements (integration measure densities) [12], where at least one of them was a non-canonical one and short-termed “two-measure theory” (TMT), has a number of remarkable properties if fermions are included in a self-consistent way [2]. In this case, the constraint that arises in the TMT models in the Palatini formalism can be represented as an equation for $\chi = \Phi / \sqrt{-g}$, in which the left side has an order of the vacuum energy density, and the right side (in the case of non-relativistic fermions) is proportional to the fermion density. Moreover, it turns out that even cold fermions have a (non-canonical) pressure $P_{\text{noncan}}$ and the corresponding contribution to the energy-momentum tensor has the structure of a cosmological constant term which is proportional to the fermion density. The remarkable fact is that the right hand side of the constraint coincide with $P_{\text{noncan}}$. This allows us to construct a cosmological model [38] of the late universe in which dark energy is generated by a gas of non-relativistic neutrinos without the need to introduce into the model a specially designed scalar field.

In models with a scalar field, the requirement of scale invariance of the initial action [1] plays a very constructive role. It allows to construct a model [39] where without fine tuning we have realized: absence of initial singularity of the curvature; $k$-essence; inflation with graceful exit to zero cosmological constant.

Of particular interest are scale invariant models in which both fermions and a dilaton scalar field $\phi$ are present. Then it turns out that the Yukawa coupling of fermions to $\phi$ is proportional to $P_{\text{noncan}}$. As a result, it follows from the constraint, that in all cases when fermions are in states which constitute a regular barionic matter, the Yukawa coupling of fermions to dilaton has an order of ratio of the vacuum energy density to the fermion energy density [40]. Thus, the theory provides a solution of the 5-th force problem without any fine tuning or a special design of the model. Besides, in the described states, the regular Einstein’s equations are reproduced. In
the opposite case, when fermions are very deluded, e.g. in the model of the late Universe filled with a cold neutrino gas, the neutrino dark energy appears in such a way that the dilaton $\phi$ dynamics is closely correlated with that of the neutrino gas [11].

A scale invariant model containing a dilaton $\phi$ and dust (as a model of matter) [11] possesses similar features. The dilaton to matter coupling "constant" $f$ appears to be dependent of the matter density. In normal conditions, i.e. when the matter energy density is many orders of magnitude larger than the dilaton contribution to the dark energy density, $f$ becomes less than the ratio of the "mass of the vacuum" in the volume occupied by the matter to the Planck mass. The model yields this kind of "Archeimedes law" without any special (intended for this) choice of the underlying action and without fine tuning of the parameters. The model not only explains why all attempts to discover a scalar force correction to Newtonian gravity were unsuccessful so far but also predicts that in the near future there is no chance to detect such corrections in the astronomical measurements as well as in the specially designed fifth force experiments on intermediate, short (like millimeter) and even ultrashort (a few nanometer) ranges. This prediction is alternative to predictions of other known models.

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