Superfluidity and pairing phenomena in ultracold atomic Fermi gases in one-dimensional optical lattices, Part II: Effects of population imbalance

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(Dated: January 15, 2020)

In this paper, we study the effect of population imbalance and its interplay with pairing strength and lattice effect in atomic Fermi gases in a one-dimensional optical lattice. We compute various phase diagrams as the system undergoes BCS-BEC crossover, using the same pairing fluctuation theory as in Part I. We find widespread pseudogap phenomena beyond the BCS regime and intermediate temperature superfluid states for relatively low population imbalances. The Fermi surface topology plays an important role in the behavior of $T_c$. For large $d$ and/or small $t$, which yield an open Fermi surface, superfluidity can be readily destroyed by a small amount of population imbalance $p$. The superfluid phase, especially in the BEC regime, can exist only for a highly restricted volume of the parameter space. Due to the continuum-lattice mixing, population imbalance gives rise to a new mechanism for pair hopping, as assisted by excessive majority fermions, which may lead to significant enhancement of $T_c$ on the BEC side of the Feshbach resonance, and also render $T_c$ approaching a constant asymptote in the BEC limit, when it exists. Furthermore, we find that not all minority fermions will be paired up in BEC limit, unlike the 3D continuum case. These predictions can be tested in future experiments.

I. INTRODUCTION

With multiple experimentally tunable parameters, ultracold atomic Fermi gases and optical lattices have attracted enormous attention [1,3]. Fermions in optical lattices are often described by a Hubbard model [2,4]. Among them, the one-dimensional (1D) optical lattices have been realized experimentally for a long time [5–7]. However, a proper treatment of fermions in 1D optical lattices is not yet available, since most theoretical in this regard addresses pure lattice cases [8–13]. Theoretical studies on such a true 1D optical lattice in the experimental sense have been scarce. Devreese et al. studied possible Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) states [14,15] in such a 1D optical lattice [16,18], but mostly restricted to the BCS and crossover regimes. Indeed, the superfluid and pairing physics in a 1D optical lattice has not been adequately studied thus far. In Part I of the present work [19], we have systematically studied the behavior of BCS–BEC crossover of atomic Fermi gases in a 1D optical lattice in the absence of a population (and mass) imbalance. In particular, we have found widespread pseudogap phenomena, which bear strong signatures in single particle excitation spectrum and the superfluid density.

In this paper, we continue from Part I [19] and study the effects of population imbalance and its interplay with lattice constant $d$ and lattice hopping parameter $t$, besides the interaction strength and temperature, within the framework of the same pairing fluctuation theory. We find that the exponential behaviors of the fermionic chemical potential $\mu$ and the pairing gap $\Delta$ as a function of pairing strength in the BEC regime remain the same as in the balanced case. The behavior of the superfluid transition temperature $T_c$ is largely governed by the Fermi surface topology. For large $d$ and/or small $t$, which lead to an open Fermi surface, a small amount of population imbalance $p$ may readily destroy superfluidity. Furthermore, the mixing between continuum and discrete lattice dimensions has more profound consequences than in the balanced case; the excessive majority fermions can assist pair hopping, providing a new pair hopping mechanism, which dominates the hopping via virtual pair unbinding [20] in the BEC regime. Together with the quasi-two dimensionality, which yields a constant ratio $\Delta^2/\mu$ in the BEC limit, this new mechanism leads to a constant asymptote for $T_c$ for a BEC superfluid (when a BEC solution exists) in the presence of population imbalance. We shall present detailed $T_c$ versus interaction strength $1/k_B T_c$ with varying lattice constants $d$, population imbalances $p$ and hopping integrals $t$. As these phase diagrams reveal, (i) the superfluid phase exists only in a very restricted volume of the multi-dimensional parameter space; (ii) the pseudogap phenomena widely exist; (iii) intermediate temperature superfluidity is also a widespread phenomenon in the presence of population imbalance, similar to the homogeneous case [23].

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[21, 22]. We shall also present $T_c$ versus interaction strength $1/k_B T_c$ with varying lattice constants $d$, population imbalances $p$ and hopping integrals $t$. As these phase diagrams reveal, (i) the superfluid phase exists only in a very restricted volume of the multi-dimensional parameter space; (ii) the pseudogap phenomena widely exist; (iii) intermediate temperature superfluidity is also a widespread phenomenon in the presence of population imbalance, similar to the homogeneous case [23].
irrespective of the lattice constraint; (iv) a small population imbalance may greatly enhance the superfluidity by raising $T_c$ on the BEC side of the Feshbach resonance; (v) a BEC superfluid exists only for a limited small volume in the parameter space of $(t, d, p)$, and (vi) Not all minority fermions will be paired when a BEC superfluid does exist.

II. THEORETICAL FORMALISM

In this section, we present briefly the theory, adapted for the population imbalanced case, with spin dependent chemical potential $\mu_\sigma$ and Green’s functions $G_{0\sigma}(K)$ and $G_\sigma(K)$, with the (pseudo)spins $\sigma = \uparrow, \downarrow$. We keep the same notations as in Part I [19].

A. Pairing fluctuation theory with a population imbalance

The single particle dispersion is given by $\xi_{k\sigma} = k^2/2m + 2t[1 - \cos(kz)] - \mu_\sigma \equiv \xi_k - \mu_\sigma$. The bare Green’s function is given by $G_{0\sigma}(K) = i\omega_n - \xi_{k\sigma}$, with the self-energy $\Sigma_\sigma(K) = \sum_Q t(Q)G_{0\bar{\sigma}}(Q - K)$, where $\bar{\sigma} = -\sigma$. The $T$-matrix $t(Q) = t_{sc}(Q) + t_{pg}(Q)$, where $t_{sc}(Q) = -\Delta_\sigma^2/\delta \mu_\sigma$ vanishes for $T > T_c$ and $t_{pg}(Q) = U/[1 + U\chi(Q)]$, with the pair susceptibility $\chi(Q) = \sum_{K,\sigma} G_{0\bar{\sigma}}(Q - K)G_{\sigma}(K)/2$. The self-energy is given by $\Sigma_\sigma(K) = \Sigma_{sc,\sigma}(K) + \Sigma_{pg,\sigma}(K)$, where $\Sigma_{sc,\sigma}(K) = -\Delta_\sigma^2 G_{0\bar{\sigma}}(-K)$, and $\Sigma_{pg,\sigma}(K) = \sum_Q t_{pg}(Q)G_{\bar{\sigma}}(Q - K)$. At $T < T_c$, the BEC condition remains $t_{pg}^{-1}(Q = 0) = U^{-1} + \chi(0) = 0$, and $\Sigma_{pg,\sigma}(K) \approx -\Delta_\sigma^2 G_{0\bar{\sigma}}(-K)$, with $\Delta_\sigma^2 \equiv -\sum_Q t_{pg}(Q)$. Then the total self-energy $\Sigma_\sigma(K) \approx -\Delta_\sigma^2 G_{0\bar{\sigma}}(-K)$, where $\Delta^2 = \Delta_{sc}^2 + \Delta_{pg}^2$. Finally, the full Green’s function becomes more complex due to population imbalance,

$$G_\sigma(K) = \frac{\frac{n_k^2}{i\omega_n - E_{k\sigma}}}{\omega_n - E_{k\sigma}} + \frac{\frac{\xi_k^2}{E_{k\bar{\sigma}}}}{\omega_n + E_{k\bar{\sigma}}},$$

where $n_k^2 = (1 + \xi_k/E_k)^2$, $\xi_k^2 = (1 - \xi_k/E_k)^2$, $E_{k\bar{\sigma}} = E_k - \mu$, $E_{k\bar{\sigma}} = E_k + \mu$, and $E_k = \sqrt{\xi_k^2 + \Delta^2}$. From the number constraint $n_\sigma = \sum_K G_{\sigma}(K)$, we can get the total fermion number density $n = n_{\uparrow} + n_{\downarrow}$ and the density difference $\delta n = n_{\uparrow} - n_{\downarrow} \equiv m$.

$$n = \sum_k \left[\left(1 - \frac{\xi_k}{E_k}\right) + 2f(E_k)\frac{\xi_k}{E_k}\right],$$

$$p_m = \sum_k \left[f(E_{k\uparrow}) - f(E_{k\downarrow})\right],$$

where $\tilde{f}(x) = [f(x + h) + f(x - h)]/2$. Similar to the $p = 0$ case, the extended gap equation is given by

$$\frac{m}{4\pi a} = \sum_k \left[\frac{1}{2\xi_k} - \frac{1 - 2\tilde{f}(E_k)}{2E_k}\right] + a_0\mu_p,$$

with $\mu_p = 0$ at $T \leq T_c$. The inverse $T$-matrix expansion [11] remains formally the same as in the $p = 0$ case, and all the coefficients are determined automatically in the expansion process. Their concrete expressions are given by Eqs. (A4), (A5) and (A7) in the Appendix of Part I with the Fermi distribution functions $f(x)$ and $f'(x)$ replaced by $\tilde{f}(x)$ and $\tilde{f}'(x)$, respectively. The pseudogap equation is the same,

$$a_0\Delta_{pg}^2 = \sum_q \frac{b(\tilde{\Omega}_q)}{\sqrt{1 + 4\frac{a_1}{a_0}(\tilde{\Omega}_q - \mu_p)}},$$

with the pair dispersion

$$\tilde{\Omega}_q = \frac{a_0^2 + 4a_1a_0(\Omega_q - \mu_p) - a_0}{2a_1}.$$

The pair density is given by $n_p = a_0\Delta_\sigma^2$.

Equations (2)-(5) form a closed set of self-consistent equations, which can be used to solve for $(\mu_\uparrow, \mu_\downarrow, T^*)$ with $\Delta = 0$, for $(\mu_\uparrow, \mu_\downarrow, \Delta_{pg}, T_c)$ with $\Delta = 0$, and for $(\mu_\uparrow, \mu_\downarrow, \Delta, \Delta_{pg})$ at $T < T_c$. Here the pair formation temperature $T^*$ is approximated by the mean-field $T_c$, and the order parameter $\Delta_{sc}$ is derived from $\Delta_{sc}^2 = \Delta^2 - \Delta_{pg}^2$.

B. Stability analysis

In the presence of population imbalance, not all solutions of Eqs. (2)-(5) are stable. The stability analysis can be done following Ref. [24], as we summarize here. Consider the thermodynamic potential $\Omega_5$, which consists of the fermionic ($\Omega_F$) and bosonic ($\Omega_B$) contributions,

$$\Omega_5 = \Omega_F + \Omega_B,$$

$$\Omega_F = -\frac{\Delta^2}{U} + \sum_k (\xi_k - E_k) - T\sum_{k,\sigma} \ln \left(1 + e^{-E_k/T}\right),$$

$$\Omega_B = a_0\mu_p\Delta_{pg}^2 + T\sum_q \ln \left(1 + e^{-\tilde{\Omega}_q/T}\right).$$

The stability condition of population imbalanced Sarma phase [25] against phase separation (PS) can be simply expressed as

$$\frac{\partial^2 \Omega_5}{\partial \Delta^2} = 2\sum_k \frac{\Delta_k^2}{E_k^2} \left[\frac{1 - 2\tilde{f}(E_k)}{2E_k} + \tilde{f}'(E_k)\right] > 0,$$

where $\tilde{f}'(x) = d\tilde{f}(x)/dx$. This condition is equivalent to the positive definiteness of the particle number susceptibility matrix $\left\langle \frac{\partial n_{\sigma}}{\partial \mu_\sigma} \right\rangle$ [24-26], which represents a form of generalized compressibility.

C. Superfluid density

Similar to the $p = 0$ case, the superfluid “density” ($n_s/m$), can also be derived using the linear response theory, following earlier works [23][24][27].
For the present contact potential, the superfluid density is given by
\[ \left( \frac{n_s}{m} \right)_i = 2 \sum_k \frac{\Delta^2_k}{E_k^2} \left[ \frac{1 - \tilde{f}(E_k)}{2E_k} + \tilde{f}'(E_k) \right] \left( \frac{\partial \xi_k}{\partial k_i} \right)^2, \quad (8) \]
where \(i = x, y, z\) and \(\tilde{f}'(x) = d\tilde{f}(x)/dx\).

As we will see below, the behavior of the superfluid density can become very usual for \(p \neq 0\). Nevertheless, we expect the \(T\) dependence of both \((n_s/m)_i\) and \((n_s/m)_z\) are close to each other.

**D. Asymptotic behavior in the deep BEC regime**

Unlike the \(p = 0\) case \([19]\), in the presence of a population imbalance \(p \neq 0\), the BEC limit is more complicated, as one can no longer obtain a complete analytical solution without resorting to numerics. However, one can still greatly reduce the complexity of the equations, as follows.

For \(p = (n_\uparrow - n_\downarrow)/n\), we consider \(p > 0\), without loss of generality. The excessive majority fermions require \(\mu^\uparrow > 0\) throughout the BCS–BEC crossover, whereas \(\mu\) to leading order is roughly given by its balanced counterpart in the BEC limit, where the two-body physics dominates. Then \(\mu_\downarrow\) is given by \(\mu_\downarrow = 2\mu - \mu^\uparrow\). The size of \(\mu^\uparrow > 0\) is determined by \(p\), and \(\mu^\downarrow \approx 2\mu \to -\infty\), so that \(f(E_k^\uparrow) = f(E_k^\downarrow) = 0\).

The Fermi function \(f(E_k^\uparrow)\) no longer vanishes exponentially, and will lead to corrections to the equations above. Nevertheless, this Fermi function places a small finite energy and momentum cutoff, so that we have \(E_k \approx |\mu|\) to the leading order in many occasions. Thus to leading order corrections, the equation for total number density now becomes
\[ (1 - p)n = -\frac{m\Delta^2}{4\mu d} - \frac{np\Delta^2}{2\mu} \quad (9) \]
\[ \Delta = \sqrt{\frac{4\pi |\mu| d(1 - p)n}{m} \left( 1 - \frac{\pi dn_p}{\mu m} \right)}. \quad (10) \]

Interestingly, the leading correction to \(\Delta^2\) is independent of \(1/k_{F}\alpha\), given by \(8(\pi dn/m)^2(1 - p)p\), which vanishes when \(p = 0\). So is the correction term in Eq. \[(9)\].

Expanding \(E_k^\downarrow\), we have
\[ E_k^\downarrow = E_k - h \approx \xi_k^\downarrow - \frac{\Delta^2}{2\mu} \approx \xi_k^\downarrow + \frac{4\pi dn_\downarrow}{m}. \quad (11) \]

Note that the second term is again a constant for given \(p\), independent of \(1/k_{F}\alpha\), precisely because \(\Delta^2/\mu \to \text{const}\). For this reason, the difference \(E_k^\downarrow - \xi_k^\downarrow \approx 4\pi dn_\downarrow / m\) will not approach 0 in the BEC limit, unlike the case in 3D continuum.

The equation of number difference is given by
\[ pn = \sum_k f(E_k^\uparrow) - \sum_k f(\xi_k^\downarrow + 4\pi dn_\downarrow/m) \equiv \frac{mt}{\pi^2 d} I_1. \quad (12) \]

Here the dimensionless integral \(I_1\) depends on \(\mu^\uparrow\) and \(T\).

In comparison with the \(p = 0\) case, the gap equation now also contains an extra term which is of the same order as the leading term in the BEC limit, namely,
\[ \sum_k f(E_k^\uparrow) \approx \frac{1}{2|\mu|} \sum_k f(E_k^\uparrow) = \frac{pm}{2|\mu|}, \quad (13) \]

Thus without this term, the leading order chemical potential is given by \(\mu_0 = -t e^{d/a}\), the same as in the \(p = 0\) case, since the two-body physics dominates the deep BEC regime. The gap equation can now be simplified in a fashion similar to the \(p = 0\) case, and we obtain
\[ \mu = \mu_0 + 2t + \frac{2\pi dn_\uparrow}{m}, \quad (14) \]
formally identical to the expression for \(p = 0\). Plugging Eq. \[(14)\] into Eq. \[(10)\], we can obtain the gap \(\Delta\). Note that for given \((t, d, p)\) in the deep BEC regime, Eqs. \[(14)\] and \[(10)\] completely determines \(\mu\) and \(\Delta\) as a function of \(1/k_{F}\alpha\).

As discussed in Part I, the exponential behavior of \(\mu\) and \(\Delta\) as a function of \(1/k_{F}\alpha\) is an important feature of the quasi-two-dimensionality of the present system; the ratio \(\Delta^2/\mu\) approaches a constant, independent of pairing strength. As we shall see below, this has important consequences. The (2nd and 3rd) correction terms in Eq. \[(14)\] are also constants.

Finally, to solve for \(T_c\) (and \(\mu^\uparrow\)), we need to simplify the expressions for the dispersion of the pairs. Defining \(\sum_k f(\xi_k^\downarrow) \equiv \frac{m t}{\pi^2 d} I_2\), then the coefficient \(a_0\) is given by
\[ n_p = a_0 \Delta^2 = n - \frac{1}{2} \sum_k f(E_k^\uparrow) + \frac{1}{2} \sum_k [f(E_k^\uparrow) - f(\xi_k^\downarrow)] \]
\[ = n_\downarrow + \frac{mt}{2\pi^2 d} (I_2 - I_1). \quad (15) \]

Note here both the integrals \(I_1\) and \(I_2\) depend only on \(\mu^\uparrow\) and \(T\), which are independent of the pairing strength in the BEC limit. Both will vanish when \(p = 0\). However, in the presence of population imbalance, \(I_2 - I_1\) will not vanish in the BEC limit due to Eq. \[(11)\]. Therefore, the pair density, \(n_p\), will approach a constant BEC asymptote, which is smaller than \(n_\downarrow\) for \(p > 0\). Namely, \(n_p\), not all minority fermions will be paired up.

The coefficient \(a_1\) is now given by
\[ a_1 \Delta^2 = \frac{m^2 t}{8\pi^3 d^2 n_\downarrow} (I_2 - I_1) + \frac{1}{4|\mu|} \left( n_\downarrow + \frac{m^2 t^2}{\pi^3 d^2 n_\downarrow} I_3 \right), \quad (16) \]
where the integral \(I_3 = \frac{\pi^2 d}{2mt^2} \sum_k \xi_k^\downarrow[f(\xi_k^\downarrow) - f(E_k^\downarrow)]\). Again, for \(p = 0\), all the \(I\)'s vanish, so that Eq. \[(16)\] recovers the \(p = 0\) result, \(a_1 \Delta^2 = n/8|\mu|\). It is a dramatic difference that a finite population imbalance contributes a finite, constant, first term on the right hand side of Eq. \[(16)\].

After some lengthy but straightforward derivation, we obtain
\[ B_\parallel = \frac{1}{4m} + \frac{1}{4m} \left[ \frac{t}{2\pi^2 d} (3I_2 + I_1) - \frac{mt^2}{2\pi^2 d^2 n_\downarrow} \right], \quad (17) \]
where \( I_4 = \frac{\pi^2 d}{2m^2 t^2} \sum_k |f(\xi^\downarrow_k) - f(\xi^\uparrow_k)| k^2 \). The first term is the \( p = 0 \) result, while the rest is the contribution of population imbalance. Here we have kept only the leading order terms and dropped terms of order \( 1/\mu \) or higher. The pair density \( n_p \) is to be replaced with Eq. (15).

The pair hopping integral \( t_B \) is given by

\[
t_B = \frac{t^2}{n_p} \left\{ \frac{m}{2\pi^2 d} \left( I_5 - I_6 + I_7 - \frac{m t}{\pi d n_d} I_8 \right) + \frac{n_p}{2|\mu|} \left( 1 - \frac{8}{\pi} I_5 - \frac{4t^2 m^2}{\pi^3 d^2 n_d^2} I_9 \right) \right\}
\]

where

\[
I_5 = \frac{\pi^2 d}{m t} \sum_k f(E^\uparrow_k) \cos(k_d d),
\]
\[
I_6 = \frac{\pi^2 d}{m t} \sum_k f(\xi^\downarrow_k) \cos(k_d d),
\]
\[
I_7 = -\frac{4\pi^2 d}{m} \sum_k f'(\xi^\uparrow_k) \sin^2(k_d d),
\]
\[
I_8 = \frac{\pi^2 d}{m t} \sum_k [f(\xi^\uparrow_k) - f(E^\uparrow_k)] \sin^2(k_d d),
\]
\[
I_9 = \frac{\pi^2 d}{2m t^2} \sum_k \xi_k [f(\xi^\uparrow_k) - f(E^\uparrow_k)] \sin^2(k_d d).
\]

For \( p = 0 \), all integral \( I' \)’s vanish so that Eq. (18) reduces to the \( p = 0 \) result, \( t_B = t^2/2|\mu| \). As in Eq. (17), here \( n_p \) is to be replaced with Eq. (15). Once again, population imbalance leads to the first term in the brackets in Eq. (18), which is a constant of interaction strength and thus becomes the dominant term. This will dramatically change the behavior of the \( T_c \) solution.

Equation (14) completely determines \( \mu \), and then Eq. (10) is used to fully fix the gap \( \Delta \), for given \( 1/k_F a \) in the deep BEC regime. Since the quantities \( n_p,\ a_1,\ B_1 \) and \( t_B \) rely only on \( \mu \) and \( T \) (with corrections of order \( O(1/\mu) \)), then \( \mu_T \) and \( T_c \) can be obtained via solving the pseudogap equation (5) along with the number difference Eq. (12) with \( \Delta_{PG} = \Delta \). Note that Eq. (5) depends only on the product \( a_0 \Delta^2 \) and the ratio \( a_0/a_1 \), but not on the value of \( \Delta \). The fact that the leading terms of \( a_0 \Delta^2,\ a_1 \Delta^2,\ a_0/a_1,\ B_1 \) and \( t_B \) are all independent of \( \mu \) or \( \Delta \) in the presence of a population imbalance implies that \( \mu_T \) and \( T_c \), along with these quantities, all approach their respective interaction-independent independent BEC asymptotes, which depend only on \( (t, d, p) \).

III. NUMERICAL RESULTS AND DISCUSSIONS

In this subsection, we present our results in the presence of a population imbalance, while the parameters \( (t, d, 1/k_F a) \) vary.

For our numerical calculations, we define Fermi momentum \( k_F = (3\pi^2 n)^{1/3} \) and Fermi energy \( E_F = k_B T_F = h^2 k_F^2/2m \), as given by a homogeneous, balanced, noninteracting Fermi gas with the same total number density \( n \) in 3D.

A. Effect of population imbalance on BCS–BEC crossover

1. An unphysical nearly isotropic case: \( t/E_F = 1 \) and \( k_F d = 1 \)

First, we consider the case \( t/E_F = 1 \) and \( k_F d = 1 \), which is not physically accessible, but provides a nearly spherical Fermi surface in the noninteracting limit [28], and thus may serve to make contact with the 3D homogeneous case [24]. Shown in Fig. 1 is the evolution of the phase diagram in the \( T - p \) plane for three representative pairing strengths in the (a) near-BCS, (b) unitary, and (c) near-BEC regimes, respectively. The phase diagram in each case consists of a small intermediate temperature, Sarma (i.e., polarized) superfluid phase (yellow shaded, labeled “SF”), a large pseudogapped normal phase (“PG”), an unpaired normal Fermi gas phase (“Normal”), as well as an unstable phase (“Unstable”), which often gives way to phase separation [29]. Considering the different vertical scales, the superfluid phase has roughly comparable phase space volumes for the three cases, more or less
similar to its homogeneous counterpart in 3D free space, as shown in Figs. 6 and 7 in Ref. [24]. Here the (in)stability condition (green line) is given by Eq. (7). Indeed, For $k_F d = 1$, we have $\pi / d \gg k_F$, so that the confinement in $k_z$ has only a minor impact on the momentum distribution. In addition, similar to the 3D homogeneous situation, the unitary case has the highest $T_c$ at $p = 0$ among all three cases, and there exists no stable Sarma superfluid at $T = 0$ when $p \neq 0$ for the cases considered ($1 / k_F a \leq 0.5$). At $T = 0$, the $p = 0$ and $p \neq 0$ cases are not continuously connected in the BCS and unitary regimes. A zero $T$ polarized superfluid solution exists only in the deep BCS regime [23, 24]. At the same time, the (red) $T_c$ curve intersects with the (green) instability boundary for the near-BEC case. And in the deep BEC regime, the instability line intersects with the $p$ axis at a finite value, indicating the existence of a stable zero $T$ polarized Sarma superfluid.

Now we turn to the effect of population imbalance on the behavior of $T_c$ throughout the BCS-BEC crossover. Keeping $T_c$ as the function, there are still four independent control variables, $p$, $1 / k_F a$, $t$, and $d$, which can yield many different facets of the very rich phase space. In this section, we shall only present a few very informative phase diagrams.

Shown in Fig. 2 is the calculated $T_c - 1 / k_F a$ phase diagram for different $p$ from 0.01 to 0.99 at fixed $k_F d = 1$ and $t / E_F = 1$. For comparison, we also plot the $p = 0$ curve (black dashed). This figure bears a lot of similarity with that for the simple 3D homogeneous case, shown in Ref. [23]. For both cases, there exist intermediate temperature superfluids from the BCS to the near-BEC regime. This unusual phase has a higher and a lower $T_c$ for a given $1 / k_F a$. At the same time, for intermediate levels of $p$ (0.1 and 0.13 shown here), the $T_c$ curve splits into two branches, and the left branch shrinks to zero and disappears as $p$ further increases. The $T_c$ solutions inside the yellow shaded region do not satisfy the stability condition of Eq. (7), and hence are unstable. The difference comes mainly on the BEC side. As $1 / k_F a$ increases into the BEC regime, for our present case, $T_c$ decreases, which is qualitatively consistent with the $p = 0$ cases shown in Figs. 1 and 2 of Part I [19], reflecting the lattice effect on pair hopping.

The most surprising feature in Fig. 2 is that $T_c$ for $p = 0$ decreases faster, and thus intersects with the $p \neq 0$ curves. This means we can get a higher $T_c$ by allowing a small population imbalance on the BEC side of the Feshbach resonance. Indeed, as we have shown analytically in Eq. (18), due to population imbalance, an additional mechanism for pair hopping kicks in; a pair can hop to its neighboring site via exchanging only the majority fermion component of a pair with an excessive majority fermion that is already present on the neighboring site, leaving the previous majority fermion component behind. In this way, the minority fermion component glides through the sites whereas the majority fermions do not necessarily have to hop. Note here that a “site” in the lattice dimension corresponds actually to a 2D plane, which guarantees that there are always excessive majority fermions available on the neighboring “site”, when $p \neq 0$, in the thermodynamic limit. This is a consequence of lattice-continuum dimensional mixing. The presence of a transverse continuum dimension is crucial for this to happen. Due to this new pair hopping mechanism, $t_0$ approaches a constant in the BEC limit, and so does $T_c$. Indeed, as one can see, the $T_c$ curves already flatten out towards BEC.

2. Realistic cases with smaller $2mtd^2 < 1$

Now we consider more realistic cases which are accessible experimentally, as constrained by the condition $2mtd^2 < 1$. Shown in Fig. 3 are the $T - p$ phase diagrams with $(t / E_F, k_F d) = (0.05, 2)$, for the same values of $1 / k_F a$ as in Fig. 2. In comparison, we observe that the reduced $(t, d)$ or $td^2$ has led to significant reduction on $T_c$ and the phase space volumes of the superfluid (“SF”) and paired (“PG” and “Unstable”) phases. This reduction reveals that the small $t$ and relatively large $d$ are detrimental to both superfluidity and pairing. The most dramatic effect is the rapid shrink of the SF phase as $1 / k_F a$ increases towards the BEC regime. Further more, the $T_c$ curve no longer intersects with the instability line. This suggests that for finite $p > 0$, there is no superfluidity at $T = 0$ even in the deep BEC regime, for the present choice of $(t, d)$. On the other hand, the superfluid solution for $p = 0$ always exist [19], in that case, the area of the SF phase does not completely vanish even though it may become very small. Here one may also notice that the unitary case no longer has the highest $T_c$. This is because the maximum $T_c$ for $k_F d = 2$ has shifted away from unitarity towards the BEC side in the 1D optical lattice [19]. As one can expect, the smaller $t$ and larger $d$ make the system quasi-2D, giving rise to stronger pairing fluctuations and thus reduced $T_c$.

In analogy to Fig. 1 we show in Fig. 3 a realistic case with $t / E_F = 0.1$ and $k_F d = 0.5$. With this reduced $t$ and $d$, the Fermi surface is an elongated ellipsoid in the noninteracting limit, as shown in the inset. Plotted here is $T_c$ as a function of $1 / k_F a$ for different $p$ from 0 to 0.132, as labeled next to the color coded curves. Also labeled on the top axis is the effective parameter $1 / k_F a_{\text{eff}} = \sqrt{2mtd / k_F a}$, as defined in
Figure 3. Evolution of the T – p phase diagram with t/E_F = 0.05 and k_F d = 2 for different pairing strengths. Other parameters are the same as in Fig. 1

Part I [19] and Ref. [20]. This parameter is certainly closer to the 1/k_F a parameter of the 3D homogeneous case [23]. Similar to that in Fig. 1 the superfluid T_c solution within the small yellow shaded area is unstable. In addition, the lower branch T_c vanishes somewhere close to but on the BEC side of unitarity. In comparison with Fig. 1, however, the overall T_c is strongly suppressed by a factor of 4. This reduced T_c is mainly caused by the small t and small d, which brings the noninteracting chemical potential down dramatically to μ ≈ 0.276E_F ≈ E_F/4. The other main difference is that the population imbalance p cannot go to a high value as it does in Fig. 1 before T_c disappears completely. While the T_c curve can still persists into the BEC limit for p ≤ 0.1, it bends back for p = 0.115 and forms a superfluid dome in the near-BEC regime. The superfluid phase quickly shrinks when p increases further, and then disappears for p ≥ 0.132.

To understand the difference between Figs. 3 and 1 we note that the elliptical Fermi surface in Fig. 3 can be rescaled more or less into a sphere; this allows for some similarities in the T_c curves. However, as pairing strength increases and the pairing gap becomes large, the pair occupation number ν^2_k (and hence the fermion momentum distribution) will soon feel the confinement of the limited momentum space in the lattice direction. As a consequence, the excessive majority fermions will no longer be evenly distributed in all directions (after the rescaling). This causes pairing more difficult in the BEC regime and thus leads to a dome shape of the superfluid phase. It also explains why p cannot be large before superfluidity disappears.

Next, we keep t/E_F = 0.1 but increase the lattice spacing d to k_F d = 1.5 so that the pairs feel more strongly the restriction of |k_z| ≤ π/d. Shown in Fig. 4(a,c) are the behaviors of (a,e) T_c, the coefficients (b) B_|| and (c) B_z, and (d,f) the pair fraction n_p/n_{1p} (all at T_c) as a function of 1/k_F a for a series of p from 0 to 0.1. The Fermi surface now has open ends at k_z = ±π/d, as shown in the inset of panel (b). It can no longer become nearly spherical by momentum rescaling. This inevitably shall lead to a bigger difference from Fig. 1 The curves in panels (a)-(d) are plotted in a semi-log scale, making the exponential dependence of T_c on 1/k_F a for p = 0 in the BCS regime self-evident as a straight line (orange dashed). It turns out that the coefficients B_||, B_z and pair density n_p all bear similar exponential dependencies. Panels (e) and (f) are plotted in linear scales. In the presence of a finite imbalance p, as the interaction strength decreases, T_c follows the p = 0 curve until it hits the lower threshold, at which it curves back into a lower branch of T_c. Similar behaviors happen to B_||, B_z and n_p as well. On the other hand, on the BEC side of the Feshbach resonance, B_z approaches a constant for p ≠ 0, (and B_|| differs substantially from its p = 0 value). Accordingly, T_c approaches a constant BEC asymptote, and so does n_p. All superfluid solutions in Fig. 4 are stable. Panels (d,f) reveal that the pair density n_p is higher along the lower branch of T_c than the upper branch, as expected. We note that n_p/n_{1p} < 1, indicating that not all minority fermions form pairs even in the deepest BEC limit, in contrast to the 3D continuum case. The BEC asymptotic behaviors are governed by Eqs. (15)-(18).

Similar to Fig. 2 in both Figs. 4 and 5 the p = 0 curve for T_c quickly drops with increasing 1/k_F a and intersects with the
Figure 5. Behavior of (a, e) $T_c/T_F$, (b) $B_0$, (c) $B_c$ (in units of $1/2m$) and (d, f) $n_a/n_↓$ as a function of $1/k_Fa$ for different $p$ from 0 to 0.1 at fixed $k_Fd = 1.5$ and $t/E_F = 0.1$. The color coding for panels (a)-(d) are the same.

$p \neq 0$ curves. Namely, in these physically accessible cases, our earlier finding about the enhancement of $T_c$ by population imbalance remains valid.

In comparison with Fig. 2, a big qualitative difference is that there is no moderate level of $p$ in Fig. 5 such that the $T_c$ curve splits into a left and a right branch. In addition, due to the big difference between Fermi surfaces of these two cases, the lower $T_c$ here does not vanish in the neighborhood of unitarity, but rather either extends all the way to the BEC limit (for small $p \leq 0.002$) or curls up and joins the upper $T_c$ before it enters the deep BEC regime (for $p \geq 0.003$). The $T_c$ curve for $p = 0.003$ can extends into the BEC regime up to $1/k_Fa = 2.854$ or $1/k_Fa_{eff} = 1.354$. Furthermore, here we do not find the counter-T$c$ curve that is similar to the $t/E_F = 0.115$ case in Figs. 4. Therefore, while one may find a BEC superfluid for large $p$ up to nearly unity in Fig. 2, it is not possible for the quasi-2D case in Fig. 5. Indeed, the superfluid solution will disappear from the entire phase space when $p > 0.124$ for the present parameters $(t/E_F, k_Fd) = (0.1, 1.5)$. In other words, superfluidity now exists only in a small portion of the phase space; for small $t$ and relatively not so small $d$, the superfluid phase can be easily destroyed by a small amount of population imbalance. In addition, a deep BEC superfluid exists only for very low $p$ as well. Reducing $t$ and/or increasing $d$ further may destroy completely the superfluid phase even in the deepest BEC limit. Therefore, one needs to reduce $d$ and/or increase $t$ to have a superfluid with a relatively large $p$, as will be shown soon below.

We notice that the enhancement of $T_c$ or superfluidity by population imbalance occurs mainly on the BEC side of unitarity. To show this more explicitly, we plot in Fig. 5(a) the behavior of $T_c$ as a function of $p$ at a series of pairing strengths for fixed $(t/E_F, k_Fd) = (0.2, 2)$. While one may find a maximum allowable range of $p$ around $1/k_Fa = -0.7$, and a maximum $T_c$ at unitarity, these two cases do not see the enhancement effect, since for both cases, $T_c$ reaches its maximum at $p = 0$. In contrast, for $1/k_Fa = 1$, 1.5 and 2, as $p$ increases from 0, $T_c$ experiences an initial rapid jump from its $p = 0$ value to a much higher value at $p > 0$, and then slowly drops down and bends back towards $p = 0$. There exists a significant range of $p$ in which $T_c$ is larger than its $p = 0$ counterpart. The back-bending behavior of $T_c$ versus $p$ is consistent with the intermediate temperature superfluidity with an upper and lower $T_c$. The much reduced maximum $p$ for these case demonstrates that a superfluid solution exists only for small $p$ on the BEC side of unitarity for the current $(t, d)$ combination.

B. Influence of $t$ and $d$ on the superfluid phase diagrams

1. $T - 1/k_Fa$ phase diagrams for different $t$ and $d$

The effect of increasing $t/E_F$ on this phase diagram is shown in Fig. 5(b), where $T_c$ vs $p$ at unitarity is plotted for a series of $t$ at $k_Fd = 2$. The maximum $T_c$ at $p = 0$ increases with $t$, but the maximum reachable $p$ seems to saturate for $t/E_F > 0.1$.

The evolution of superfluid phase from Fig. 2 to Fig. 5 and Fig. 6 tells that in the presence of a population imbalance, the superfluid phase volume decreases quickly and then disappears completely as the system evolves into the quasi-2D regime.

If we allow ourselves to use somewhat larger range of $t$, we will obtain the $T_c$ curves shown in Fig. 7 as a function of
Figure 6. $T_c - p$ phase diagram with $k_d = 2$ (a) for different $1/k_F a$ from -0.7 to 2 at fixed $t/E_F = 0.2$, and for (b) different values of $t/E_F$ from 0.005 to 0.3 (as labeled) at unitarity.

Figure 7. Behavior of $T_c$ as a function of $1/k_F a$ at fixed $k_d = 2$ and $p = 0.01$, but for different values of $t/E_F$, as labeled. It becomes more 3D-like as $t$ increases.

Figure 8. $T_c - 1/k_F a$ phase diagram for different $k_d$ from 0.1 to 8 at fixed $p = 0.01$ and $t/E_F = 0.05$. The inset shows the small $d$ cases, which share the same axis labels as the main figure. Increasing $d$ destroys the superfluid in the deep BEC regime.

1/$k_F a$. Here we fix $p = 0.01$ and $k_d = 2$, but vary $t/E_F$ from 0.0001 up to 0.5, as labeled. For small $t/E_F \leq 0.1$, we have a simple closed loop. Both $T_c$ and the size of the loop increases as $t$ grows. For $t/E_F = 0.15$ (red) and 0.205 (green), the $T_c$ loop extends into the BEC regime, but still cannot reach the deep BEC limit; the $T_c$ curve turns back somewhere on the BEC side of unitarity, and form a closed cycle. As $t$ increases further, for $t/E_F \geq 0.21$, the $T_c$ curves successfully extend all the way into the BEC limit. For $t/E_F = 0.21$ (orange curve), both the upper and lower $T_c$ branches extend to $1/k_F a = +\infty$. However, for $t/E_F \geq 0.25$ (black and pink curves), the lower $T_c$ branch bends downwards around unitarity and vanishes at an intermediate pairing strength, somewhere on the BEC side of unitarity. In such a case, there exists a stable homogeneous polarized superfluid in the BEC regime at $T = 0$, similar to the case for a simple 3D continuum. For $k_d = 2$, our calculation reveals that the Fermi surface has two open ends at $k_z = \pm \pi/d$ for $t/E_F \leq 0.21$, whereas it becomes a closed ellipsoid again for the large $t/E_F \geq 0.25$ cases. The corresponding $T_c$ behavior for the latter cases is similar to that found in Fig. 3.

So far, we have restricted ourselves to fairly small $d$, with $d \leq 2$. In Fig. 3, we show the behavior of $T_c$ for a large range of $d$, from $k_d = 0.1$ to 8, with a fixed $p = 0.01$ and $t/E_F = 0.05$. For $k_d \geq 4$, we have the range $|k_z| < \pi/d < k_F$, which makes the lattice effect much stronger. Note that for $t/E_F = 0.05$, the $k_d = 6$ and 8 cases are physically inaccessible. Nonetheless, these curves show a clear trend, namely, with increasing $d$, the maximum $T_c$ increases and the $T_c$ loop becomes narrower in terms of $1/k_F a$, more concentrated near unitarity. On the other hand, for small $k_d/d$, $\pi/d$ becomes very large. With a small $t$ (shown in the inset), the lattice band will be fully occupied, giving rise to an elongated open-end Fermi cylinder (for $k_d \geq 0.5$) in the momentum space. Due to this small $d$, except for the $k_d = 0.1$ case (which has a closed ellipsoid Fermi surface), other $T_c$ curves in the figure cannot
access the deep BEC limit. Starting from a small \( d \), this set of curves reveal that increasing \( d \) leads to the formation of a closed curve of \( T_c \) so that the superfluid phase in the deep BEC regime is destroyed.

From Figs.\[1\] to \[8\] we find that the behavior \( T_c \) has a close connection to the topology of the Fermi surface. For a closed Fermi surface, it can be brought into a nearly spherical shape by momentum rescaling. For small \( p \), the situation for pairing is very much like in the 3D homogeneous case. Therefore, the \( T_c \) curve for low \( p \) is similar to the 3D homogeneous case; the lower \( T_c \) vanishes in the near BEC regime, and there exists a superfluid ground state in the BEC regime. For open Fermi surfaces, pairing and superfluidity become more difficult, making a ground state superfluid impossible. Note that for a simple tight-binding band in the lattice dimension with nearest-neighbor approximation, the Fermi surface topology changes from closed below half filling to open above half filling. Above half filling, the fermion motion on the Fermi surface becomes more hole-like in the \( k_z \) direction. While the in-plane motion is always particle-like, this change of character may have detrimental effect on the pairing and superfluidity.

\[2\]
Continuous evolution of the superfluid phase with \( t \) and \( d \)

Now, we investigate how \( T_c \) evolves continuously with lattice spacing \( d \). Plotted in Fig.\[9\] are a series of \( T_c \) curves as a function of \( k_d \), for fixed \( p = 0.01 \) and \( 1/k_F a = 1 \) but different \( t/E_F \) from 0.001 to 0.5. Except for the large \( t/E_F \) (\( \geq 0.3 \)) cases, which are unphysical or hard to realize experimentally, \( T_c \) curves form a series of loops. This agrees with the existence of two branches at this interaction strength. The superfluid phase space area shrinks with decreasing \( t \). This means that, for small \( t \) at the particular \( 1/k_F a = 1 \), a large \( d \) will not be able to maintain the superfluid phase. This pairing strength, the largest reachable value of \( k_d \) is highly nonmonotonic as a function of \( t \), with a minimum of 1.13 for \( t/E_F = 0.05 \). This also confirms that the ground state at \( 1/k_F a = 1 \) is not a superfluid for \( t/E_F \leq 0.2 \) and \( p = 0.01 \). For larger \( t \), the interaction parameter \( 1/k_F a \) at which the lower \( T_c \) would vanish becomes smaller than 1, as can be seen from Fig.\[7\]. This explains why for \( t/E_F = 0.3 \) and 0.5 in Fig.\[9\] there is no longer a lower \( T_c \) solution for \( k_d \) \( \leq 2.2 \) and \( 4.1 \), respectively.

The evolution of \( T_c \) with continuously varying \( t \) is presented in Fig.\[10\] for a series of interaction parameter \( 1/k_F a \) from 0 at unitarity to 7.0 in the BEC regime. Here \( p = 0.01 \) and \( k_d = 1 \) are fixed. Logarithmic and linear scales are used for the horizontal axis in the main figure and the inset, respectively. The log scale serves to magnify the small \( t \) regime. For \( 1/k_F a \leq 1.1 \), the curves have an upper and a lower branch, which joins at the small \( t \) end. Indeed, from Figs. \[1\]-\[8\], we find that no matter whether the Fermi surface is closed or open, there are always two \( T_c \) branches in the unitary and BCS regimes. For \( 1/k_F a \geq 1.2 \), we find that the \( T_c \) curves pinch together and then split into two parts around \( t/E_F = 0.04 \). The left part forms a loop, which shrinks quickly as \( 1/k_F a \) moves towards BEC. This left loop is the same superfluid phase as the left loop in Fig.\[4\]; they are just different cuts of the superfluid region in the multidimensional phase diagram. For stronger interactions in the BEC regime, either a large \( t \) or a very tiny \( t \) is needed to maintain a superfluid phase. While the former case allows a closed Fermi surface and thus a superfluid solution in the BEC regime, the latter case will allow two branches of \( T_c \) which persist into the BEC regime. One can also tell from this figure that, for small \( t/E_F < 0.12 \), either there is no \( T_c \) at all or there is a lower \( T_c > 0 \), so that the ground state (with \( p = 0.01 \) and \( k_d = 1 \)) is not a superfluid for \( 1/k_F a \leq 7 \).

Due to the high complexity of the multidimensional phase diagram, the counterpart of the above figures would look somewhat different when \( (t, d, p, 1/k_F a) \) changes.
C. Gaps in the superfluid phase

In Fig. 11 we present, as an example for intermediate temperature superfluidity, the behavior of the order parameter $\Delta_{sc}$, the pseudogap $\Delta_{pg}$ and the total gap $\Delta$ and a few relevant quantities as a function of temperature in the superfluid phase. Also plotted is the solution above the upper $T_c$, especially for the pair chemical potential $\mu_p$. Shown in the figure is for the case of $k_Fd = 2$, $t/E_F = 0.2$ and $p = 0.01$ at unitarity. It is close to the case of $t/E_F = 0.205$ in Fig. 7. Near the upper $T_c$, the behavior of the gaps look similar to regular superfluid Fermi gases in the pseudogap regime; The order parameter $\Delta_{sc}$ turns on with decreasing $T$, while the pseudogap $\Delta_{pg}$ starts to decrease, leaving the total gap roughly constant or slightly increasing. Above the upper $T_c$, the pair chemical potential $\mu_p$ starts to decrease from zero with increasing $T$. The vanishing of $\Delta_{sc}$ at the upper $T_c$ is the same as in BEC of ideal Bose gases. As the temperature decreases towards the lower $T_{c,L}$, $\Delta_{pg}$ increases again, which suppresses $\Delta_{sc}$ quickly down to zero. This can be understood from the highly decreased value of $B_z = t_0d^2$ at $T_{c,L}$ in panel (c); As $B_z$ decreases, pairs become heavy in the lattice direction, leading to reduced energy cost for exciting finite momentum pairs and hence an rapid increase in $\Delta_{pg}$, which then exhausts the order parameter via $\Delta_{sc}^2 = \Delta^2 - \Delta_{pg}^2$. We note that there are no other sharp changes in $B_{||}$, $a_0$ and $a_1$. Further lowering $T$ below $T_{c,L}$ would enter again a normal state. However, the trend of $B_z$ at $T_{c,L}$ suggests that this normal state may soon become unstable against pair density wave (or stripe order) formation in the lattice direction (with a negative $B_z$ at lower $T$). Other possible solutions in this low $T$ normal state include phase separation and possible FFLO-like solutions with a wavevector along the $\hat{z}$ direction, in fact, the pair density wave solution is similar to an FFLO state, except that it may not exhibit superfluidity.

One would need to include the $q^4$ order in the inverse $T$ matrix expansion in order to obtain a meaningful solution below $T_{c,L}$, which is beyond the scope of current work.

It is interesting to note that while $\Delta$ is roughly a constant in $T$, $\mu^+$ and $\mu^-$ becomes far apart at low $T$. This large separation, with $h = 0.346E_F$, is comparable to the Clogston limit for pair breaking [31], $\Delta/\sqrt{2} = 0.388E_F$, where $\Delta/E_F = 0.549$ at $T_{c,L}$. In other words, the disappearance of superfluidity at the lower $T_{c,L}$ is compatible with the Clogston picture as well. The small difference between $h$ and $\Delta/\sqrt{2}$ may be attributable to the deviation of the Fermi surface from an isotropic 3D sphere [32]. In addition, the gap is large (beyond the BCS regime) so that self-consistent calculations are important. On the other hand, at the upper $T_c$, $h$ is much smaller than $\Delta/\sqrt{2}$, implying that the vanishing of the superfluid order at the upper $T_c$ is not associated with the Clogston picture but rather driven by pairing fluctuations.

D. Superfluid density

In this section, we show the behavior of the superfluid density. Here we choose to show only cases of intermediate temperature superfluidity, with both an upper $T_c$ and a lower $T_{c,L}$, as in Subsec. III C Cases without a lower $T_c$ (for large $t$) are more qualitatively similar to their balanced counterpart shown in Part I [19].

Plotted in Fig. 12 are the temperature dependence of both the in-plane (black curves) and lattice components (red curves) of $(n/m)$ as a function of $T/T_c$, for $k_Fd = 2$ and $p = 0.01$. Panels (a-c) are for the BCS, unitary and BEC cases, respectively, for $t/E_F = 0.2$. The corresponding curve of $T_c$ versus $1/k_Fa$ is close to the green one for $t/E_F = 0.205$ in Fig. 7. These results suggest that both components decreases as $1/k_Fa$ in-
creases. The suppression of the lattice component, \((n_s/m)_z\), can be attributed more to the effect that the system becomes more 2D and \(t_B\) decreases with increasing pairing strength. However, the reduction of the in-plane \((n_s/m)_t\) is likely due to the increase of the pseudogap \(\Delta_{pg}\) with decreasing \(T\) towards \(T_{c,L}\), leading to premature shut-off of the superfluid density before it fully reaches its maximum possible value (normally) at \(T = 0\).

Shown for comparison in Fig. 12(d) is the case of \(t/E_F = 0.1\) at unitarity, with other parameters the same as in Fig. 12(b). As can be seen, the in-plane curves are very close to each other for these two cases. However, the lattice component is drastically suppressed by the smaller \(t\) in panel (d). This can be understood qualitatively from the increased fermion band mass and hence the pair mass in the \(\hat{z}\) direction.

For all panels in Fig. 12 the temperature dependencies of both components are close to each other, despite their rather different magnitudes. This is because the main \(T\) dependence comes from the common prefactor \(\Delta_{sc}^2\) in Eq. (8).

**E. BEC asymptotic behavior with \(p \neq 0\)**

Finally, we show in Fig. 13 the asymptotic behavior of \(\mu\), \(\Delta\), \(B_z\) and \(T_c\) in the BEC limit. Plotted in Fig. 13 are \(1 - \mu\) and \(\Delta\) in units of \(E_F\) versus \(1/k_Fa\) in a semi-log scale. The straight lines confirm their exponential dependence on \(1/k_Fa\).

The dashed lines are analytical asymptotic solution, in perfect agreement with full numerical solutions (solid lines). The red dashed lines in panels (b) and (c) present the solution obtained using the asymptotic expansions, while the cyan dotted-dashed lines represent the deepest BEC asymptotes. Clearly, the asymptotic expansions and the BEC asymptotes are all in quantitative agreement with the full numerical solutions. This provides direct support of our analytical derivations in the BEC regime. These plots demonstrate that in the deep BEC limit, \(B_z\) and \(T_c\) approach a constant asymptote, as also shown in Fig. 5. Similar constant asymptotic behaviors are found for \(B_1\), \(\alpha_0\Delta^2\) and \(\alpha_1\Delta^2\) as well, a plot of which can be seen in Ref. [80].

Given these BEC asymptotic behaviors, we can investigate the phase diagrams in the BEC limit as a function of \(t\), \(d\) and \(p\). Shown in Fig. 14(a) is the BEC asymptote of \(T_c\) with \(t/E_F = 0.1\) as a function of \(p\) with different \(k_Fd\) from 0.25 to 4, 0.5, 0.75 and 0.95. The Fermi surface topology at \(p = 0\) changes from closed to open, as the lattice spacing increases across \(k_Fd = 0.942\). Therefore, nearly all cases shown here have a closed Fermi surface. It can be readily seen that for \(k_Fd = 0.95\), the maximum \(p\) is only about 0.01; there is no BEC superfluid solution for larger \(p\). The maximum \(p\) increases as \(d\) decreases. For smaller \(k_Fd = 0.25\), \(p\) survives up to about 0.475. This may largely have to do with the fact that a smaller \(d\) places less restrictive confinement for pair motion in the \(k_z\) direction, and thus the system is closer to the 3D case, so that it can accommodate a larger population imbalance.

Plotted in Fig. 14(b) are \(T_c\) curves with \(p = 0.01\) as a function of \(t/E_F\) with different \(k_Fd\) = 4, 2, 1, 0.5 and down to 0.25. These curves demonstrate that the lowest threshold of \(t\) for having a BEC superfluid solution increases with \(d\).
For \( k_F d = 4 \), we need \( t/E_F \gtrsim 0.42 \). For \( k_F d = 2 \), the threshold drops to about 0.21, in agreement with Fig. 7. For \( k_F d = 0.25 \), the threshold becomes \( t/E_F \approx 0.03 \). In particular, for \( k_F d = 1 \), the threshold is about 0.105 (\( > 0.1 \)). This explains why there is no \( k_F d = 1 \) curve in panel (a), calculated for \( t/E_F = 0.1 \). As \( d \) increases, the overall \( T_c \) also increases, since the 2D planar density \( n_{2D} \) increases and so does the noninteracting chemical potential. In reality, \( t \) is normally small. This requires a small \( d \) in order to have a BEC superfluid, as one can see from Fig. 8 as an example, where only the \( k_F d = 0.1 \) curve persists into the BEC limit for small \( t/E_F = 0.05 \). Our calculations show that these thresholds roughly correspond to half filling of the lattice band, where the Fermi surface topology changes.

Presented in Fig. 14(c) is the BEC asymptote of \( T_c \) calculated for \( t/E_F = 0.1 \), as a function of \( k_F d \) with different population imbalances from \( p = 0.005 \) to 0.6. The maximum allowed \( k_F d \) decreases quickly with increasing \( p \). For \( p = 0.005 \), \( k_F d \) goes up to 1.2. For \( p = 0.01 \), \( k_F d \) is allowed up to about 0.96. For \( p = 0.6 \), one needs a small \( k_F d < 0.22 \) to have a BEC superfluid. Figure 14(c) also reveals that for a given \( d \), there is a maximum allowed \( p \), beyond which the BEC superfluid solution no longer exists, in agreement with Fig. 14(a).

Shown in Fig. 14(d) is the BEC asymptote of \( T_c \) calculated for \( p = 0.01 \), as a function of \( k_F d \) with different tunneling \( t/E_F \) from 0.05 to 0.5. As is shown, the maximum possible \( d \) increases with \( t \). While for \( t/E_F = 0.5 \) this maximum is about 4.8, it decreases down to about 0.49 for \( t/E_F = 0.05 \). If one wants to have a larger \( d \) for the same small \( t \), one will need to use a smaller \( p \), as indicated by Fig. 14(c). The lower end \( t/E_F = 0.05 \) is more realistic. It means that for a typical \( k_F d \approx 1 \), a small amount of population imbalance will be sufficient to destroy the superfluid solutions in the BEC regime.

We point out that in all four panels of Fig. 14 there exists a narrow range of the parameters where the \( T_c \) curve bends back and thus is double-valued, which correspond to the two branches such as the low \( p \) curves shown in Fig. 5(e), with an open Fermi surface. For the rest part of the curves, there is only one (upper) \( T_c \), corresponding to, e.g., the low \( p \) curves in Fig. 4 with a closed Fermi surface.

F. Further Discussions

From the numerical results presented above, we see that the behavior of \( T_c \) and the phase diagrams are very complex, in the presence of a population imbalance. In the physically accessible scope of the parameters, e.g., constrained by the condition \( 2m \hbar^2 t^2 < 1 \), the superfluid phase occupies only a very restricted small volume in the multi-dimensional phase space. Superfluidity can be easily destroyed by small amount of population imbalance when the lattice constant \( d \) becomes large and/or the tunneling matrix element \( t \) becomes small. To understand this destruction of superfluidity, we notice that large \( d \) and small \( t \) put the system in the quasi-2D regime, such that the lattice band is essentially fully occupied, and in-plane chemical potential (in the noninteracting limit) is much higher than the lattice band width \( 2t \), leaving almost no dispersion on the Fermi surface along the lattice direction. Excessive fermions will necessarily have to occupy high in-plane momentum states and thus cost a lot of excitation energy. In this case, a small population imbalance will create a substantial mismatch \( h \) in chemical potentials that is sufficient to destroy pairing.

On the other hand, we find that smaller \( d \) is more benign in the behavior of \( T_c \), e.g., the \( k_F d = 0.1 \) case in Fig. 8. For small \( d \), the momentum space constraint \( |k_F| < \pi/d \) in the lattice direction becomes less restrictive so that the Fermi surface becomes an ellipsoid, which can be mapped back into a sphere via momentum rescaling. Whether closed or open, the Fermi surface topology in the noninteracting limit plays an important role in classifying the behavior of the \( T_c \) curves. With a closed Fermi surface, the superfluid solution in the BEC regime (if it exists) has only one (upper) \( T_c \). In contrast, with an open Fermi surface, the superfluid has both an upper and a lower \( T_c \). Further careful analysis may involve different Fermi surfaces for the two spin components and how their influence evolves with \( p \).

More surprisingly, when the superfluid solution exists in the BEC regime or on the BEC side of unitarity, \( T_c \) can be substantially enhanced by a small amount of population imbalance with respect to the balanced case. Via analytical analysis in the BEC regime, we show that this enhancement is associated with contributions to \( t_b \) from excessive unpaired majority atoms. These contributions lead to a constant BEC asymptote for \( t_b \) and a few other quantities, and hence a constant BEC asymptote for \( T_c \) via the pseudogap equation. These contributions to \( t_b \) constitute a new pair hopping mechanism assisted by excessive majority atoms. For this mechanism to work, it is important that there is at least one transverse continuum dimension. In the present case of 1D optical lattices, there are two transverse continuum dimensions, i.e., the 2D \( xy \) plane. This guarantees that there are always excessive majority atoms available on a neighboring lattice “site”. Therefore, lattice-continuum mixing is crucial for this unusual behavior.

Another important difference between 1D optical lattices and the 3D continuum case is the pair fraction in the BEC limit. For the latter case, all minority atoms will form pairs, namely, \( n_p/n_\perp = 1 \) in the BEC limit. In contrast, for the present case, we always have \( n_p/n_\perp < 1 \) for nonzero \( p \), as can be seen from Eq. (15). The difference can be attributed to the quasi-two-dimensionality in the present case, which leads to a constant ratio of \( \Delta^2/|\mu| \) in the BEC limit, in contrast to vanishing as \( 1/\sqrt{|\mu|} \) in 3D continuum.

Our calculations are based on the assumption that the 2D planes are homogeneous. In real experiments, they are always finite and confined in a shallow trapping potential. At the same time, the lattice direction is confined by a trapping potential as well. The finite size and trap effects are beyond the scope of the current work and will be left for future investigations. We note that recent progress in implementing uniform box trapping potential \([44,36]\) can greatly reduce the complexity.
IV. CONCLUSIONS

In summary, we have studied the ultracold atomic Fermi gases in a 1D optical lattice in the presence of population imbalance with a pairing fluctuation theory, as they undergo a BCS-BEC crossover. We find that superfluidity exists only for a very restricted range of parameters, while it can be readily destroyed by a small amount of imbalance \( p \) at large \( d \) and small \( t \). When the superfluid solution does exist on the BEC side of the Feshbach resonance, \( T_c \) can be enhanced substantially by even a tiny amount of population imbalance, via the new pair hopping mechanism assisted by excessive majority atoms. In general, when \( t \) is small, the \( T_c \) curve bends back on the BEC side and the superfluidity disappears in the deep BEC regime. Meanwhile, the superfluid phase shrinks as \( p \) increases. For fixed \( d \) and \( p \), the superfluid region in the \( T - 1/k_F a \) plane shrinks as \( t \) decreases, while for fixed \( p \) and \( t \), the \( T_c \) curve forms a closed loop in the \( T - 1/k_F a \) plane and becomes narrower near unitarity as \( d \) increases. In general, whether there is only one (upper) \( T_c \) or there are both an upper and a lower \( T_c \) in the BEC regime depends largely on the Fermi surface topology. The former occurs with a closed ellipsoidal Fermi surface, while the latter happens when the Fermi surface has two open ends at the Brillouin zone boundaries. Further more, due to the quasi-two-dimensionality, only part of the minority atoms will be paired even if superfluidity exists in the BEC limit.

Our results demonstrate that experimentally one needs to be careful to maintain a good population balance to stay in the superfluid phase. On the other hand, a perfect balance may not always be desirable. A small amount of imbalance may be good for enhancing \( T_c \), making the superfluid phase easier to access. It may take some trial and error to find the optimal parameters in experiment.

These predicted behaviors of fermions on a 1D optical lattice are very different from pure 3D continuum or 3D lattices, and have not been reported in the literature. Since optical lattices have been realized experimentally for a long time, these predictions should be tested in future experiments.

V. ACKNOWLEDGMENTS

We thank the useful discussions with Chenchao Xu and Yanming Che. This work was supported by the NSF of China (Grant No. 11774309 and No. 11674283), and the NSF of Zhejiang Province of China (Grant No. LZ13A040001). C. Lee was supported by the Key-Area Research and Development Program of GuangDong Province under Grants No. 2019B030330001, the NSF of China under Grants No. 11874434 and No. 11574405, and the Science and Technology Program of Guangzhou (China) under Grant No. 201904020024.

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