Simple encoding of higher derivative gauge and gravity counterterms

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Invoking increasingly higher dimension operators to encode novel UV physics in effective gauge and gravity theories traditionally means working with increasingly more finicky and difficult expressions. We demonstrate that local higher derivative supersymmetric-compatible operators at four-points can be absorbed into simpler higher-derivative corrections to scalar theories, which generate the predictions of Yang-Mills and Gravity operators by suitable replacements of color-weights with color-dual kinematic weights as per Bern-Carrasco-Johansson double-copy. We exploit that Jacobisatisfying representations can be composed out of other Jacobi-satisfying representations, and show that at four-points only a small number of building blocks are required to generate the predictions of higher-derivative operators. We find that this construction saturates the higher-derivative operators contributing to the four-point supersymmetric open and closed-string tree amplitudes, presenting a novel representation of the four-point supersymmetric open string making this structure manifest, as well as identifying the only four additional gauge-invariant building blocks required to saturate the four-point bosonic open string.

Gravitational quantum scattering amplitudes—the invariant quantum evolution of what distance means in space and time, consistent in the classical limit with Einstein’s General Relativity (GR)—are much simpler than expected. This simplicity can be traced to the fact that their perturbative dynamics are completely encoded through a double-copy structure in the predictions of much simpler gluonic or gauge theories. In turn, these gauge theory predictions are strongly constrained by a similar structure relating kinematics and color-weight, entirely hidden in any standard ways of writing their actions.

While Yang-Mills (YM) theory is famously renormalizable in four dimensions, it ceases to be in higher-dimensions, requiring a completion in the UV. It is currently an open question as to whether any four-dimensional (pointlike) quantum field theory of gravity is perturbatively finite. The most promising case, maximally supersymmetric gravity, is a subject of much current research exploiting double-copy in the predictions of much simpler gluonic or gauge theories. Indepedent of perturbative finiteness, it is very possible that new physics in the UV could necessitate higher-order corrections, whose predictions using traditional methods are often exhaustive to produce. Our main result is that the predictions due to higher-derivative local gauge and gravity operators—encapsulating novel UV physics—can also be incredibly simple because of this very same double-copy structure.

Recent work has shown that at tree-level both the supersymmetric and bosonic open string amplitudes admit field theory double-copy descriptions, pulling the higher-derivative corrections to a putative effective scalar bi-colored theory, encapsulating all order $\alpha'$ corrections, called Z-theory. Inspired by the existence of Z-theory amplitudes, as a proof of concept, here we consider a bootstrap approach, asking simply what predictions are consistent with unitarity, double-copy structure, gauge invariance, and locality.

We find that all tree-level string corrections to supersymmetric YM and GR at four-points follow from simple field theory considerations. These corrections can be obtained through a simple composition rule that combines color-dual numerators into more complex numerators with the same algebraic properties, promoting color-weights to carry the higher-derivative corrections. This naturally introduces a new type of numerator, mixing color and kinematic factors to satisfy adjoint-type relations in concert. One might expect many possibilities even at four-points, yet we find only three distinct color building blocks. We see that they generate all four-point single-trace gauge-theory predictions consistent with maximal supersymmetry. Concordant corrections to maximal supergravity are even simpler, requiring only permutation invariant kinematic factors.

These considerations only specify the analytic form of higher-derivative corrections. One may choose to fix their coefficients by assuming the asymptotic uniqueness of the Veneziano amplitude (c.f. Ref. [21–23]). Similar ideas apply to the open bosonic string, where just five different gauge invariant building blocks apply. Our main result is that the predictions due to higher-derivative local gauge and gravity operators—encapsulating novel UV physics—can also be incredibly simple because of this very same double-copy structure.

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The color-weights graph weights and the kinematic weights satisfy Jacobi identi-
larization vectors. We emphasize that both the color
Lorentz products between external momenta and po-
n amplitudes invariant under linearized diffeomorphism:
Jacobi identities and anti-symmetry to generate gravity
the color weights with kinematic weights that also satisfy
s, where
c
n

As the
As the
Gauge invariance is maintained by the fact that the
appear in the full amplitude with coeffi-
cients that are independent color basis elements, they
must themselves be individually gauge invariant. Ex-
pressing these ordered amplitudes in a basis of kin-
matic weights \( n_g \), say by eliminating \( n_u \) via Equation \( 2 \), demonstrates that the distinct color orders are intimately related. Indeed one immediately identifies the permutation
invariant quantity:

\[
A_{YM}(s,t) = A_{YM}(s,u) = A_{YM}(u,t),
\]

with the identification of the permutation invariant product
\( stA_{YM}(s,t) = (stu)A_{YM}(s,t)/u \) a simple conse-
quence. This is the lowest multiplicity manifestation of the \((n-3)!\) (or the so called BCJ) ordered-amplitude
relations \( 1 \).

We will first be concerned with how we can express higher-derivative corrections to Yang-Mills by only mod-
ifying the color-weights in a manner consistent with this
adjoint-type structure. Let us now introduce the no-
tion of a Jacobi identity satisfying composition. If we
have functional maps \( j(a,b,c) \) and \( k(a,b,c) \) that satisfy
Jacobi identities \( (X_s = X_t + X_u) \) and antisymmetry
\( X(a,b,c) = -X(a,c,b) \), then we can define a new an-
tisymmetric and Jacobi-satisfying representation \( n_s \) as a
composition of \( j \) and \( k \) by

\[
n_s = J(j,k) \equiv j_s k_t - j_t k_s.
\]

At four points, it is natural to ask if there exists a scalar
color-dual function that is only linear in the Mandelstam
invariants. Indeed one does, which we will refer to as the
simple scalar numerator:

\[
j^{ss}(a,b,c) = c - b.
\]

This corresponds to a scalar charged in the adjoint me-
diated by a massless vector, e.g. with interaction term
\( f^{abc}A^\mu(\partial_\mu \phi)\phi \). What happens when we compose the sim-
ple scalar with itself? We find the Jacobi-satisfying kine-
matic numerator associated with the NLSM,

\[
j^{ss}_s = s(u - t) = sj^{ss}_s \propto J(j^{ss},j^{ss}).
\]

Any further compositions between \( j^{ss} \) and \( j^{nl} \) only differ from these numerators by appropriate powers of permu-
tation invariant combinations of the Mandelstam invari-
ants, \( (s^2 + t^2 + u^2) \) and \( (stu) \). It is perhaps not surpris-
ing that a gauge-invariant color-dual kinematic numera-
tor representation for Yang-Mills can be written \( 22 \) as:

\[
n_{YM}^{\pm} = \frac{1}{4}(A_{YM}(u,t)/s) j^{\pm}_s = \frac{1}{4}A_{YM}(u,t) j^{ss}_s.
\]

The most straightforward modification of the color-
weights that preserves anti-symmetry and Jacobi involves
simple products of permutation invariant scalar combina-
tions:

\[
\tilde{c}_g(X,Y) = (stu)^Y (s^2 + t^2 + u^2)^X c_g \alpha^{3X+2Y},
\]

where we introduce a dimensional parameter \( \alpha' \) to track
mass-dimension. This results in an ordered \( s-t \) channel
scattering contribution proportional to:

\[
\hat{A}^{(X,Y)}(s,t) = \frac{1}{st} (stu)^X (s^2 + t^2 + u^2)^Y \times \\
(\hat{c}_X + \hat{c}_{Y,ss})
\]

(14)

As all such modifications result in manifestly permutation invariant scalings of the bi-adjoint ordered amplitude, all field theory relations are automatically preserved. One might be surprised by the appearance of simple scalar numerator appearing in the expression above, but recall that \(stA(s,t)\) must be permutation invariant for 4-point ordered amplitudes. Namely

\[
A_s = j_s \times \text{[local permutation invariants]}
\]

and

\[
j_s = j_s \times \text{[local permutation invariants]}
\]

using the following general decomposition,

\[
j(s,t,u) = j^{ss}(s,t,u) \left( \frac{\text{tr}(s,t,u)}{t-u} + \frac{\text{tr}(s,t,u)}{u-s} \right) + j^{nl}(s,t,u) \left( \frac{\text{tr}(s,t,u)}{u-s} - \frac{\text{tr}(s,t,u)}{t-u} \right),
\]

(18)

a fact easily verified by recalling the definitions of \(j^{ss}(s,t,u) = (u-t)\) and \(j^{nl}(s,t,u) = s(u-t)\). What is particularly notable is that their coefficients in Equation (15) are each permutation invariant under all \(S_3(s,t,u)\) by virtue of the adjoint-type properties of \(j(a,b,c)\). One might be concerned about potential poles, but it is straightforward to see that both must be local expressions. The simplest argument is to realize \(b = c\) is always a zero of the polynomial \(j(a,b,c)\) by virtue of antisymmetry, and thus \((b-c)\) must be a factor of \(j(a,b,c)\). Defining local polynomials \(k(a,b,c) = j(a,b,c)/(b-c)\), now manifestly symmetric in \(b \leftrightarrow c\), we are left with:

\[
j(s,t,u) = j^{ss}(s,t,u) \left( \frac{k(s,t,u) - sk(t,s,u)}{s-t} \right) + j^{nl}(s,t,u) \left( \frac{k(s,t,u) - sk(t,s,u)}{s-t} \right)
\]

(19)

taking care of all divisors except for \((s-t)\). But \(s = t\) is manifestly a zero of each numerator in these expressions, and thus the remaining divisor \((s-t)\) must be a factor of both. In summary we see that our two building blocks can reproduce every scalar polynomial adjoint-type numerator involving \(c_s, c_t,\) and \(c_u\).

We have not yet exhausted all potential local operators. Namely, we have not yet considered the possibility that the color-weight information may not be in the adjoint, and could itself be permutation invariant, as per the symmetric symbol:

\[
d_{abcd} = \frac{1}{3!} \sum_{\sigma \in S_3(b,c,d)} \text{Tr}(T^aT^{\sigma_1}T^{\sigma_2}T^{\sigma_3}).
\]

(20)

This could be dressed with color-dual scalar weights and permutation symmetric kinematics to generate the predictions of additional distinct operators. Due to redundancy between building blocks, we need only consider adding to our repertoire of globally consistent building
blocks at four-points the contributions of scalar weights of the non-linear sigma model,
\[ c_s^{(X,Y,d,nl)} = d^{abcd} f_s^{nl}(stu) X (s^2 + t^2 + u^2)^Y \alpha^{(2+3X+2Y)} \text{,} \]
where, due to the propagator canceling prefactor in every \( f_s^{nl} \), we are now free to include the cases where \( X \geq 0 \). These building blocks result in \((s,t)\) ordered scattering amplitudes proportional to:
\[ stA_{d,nl}^{(X,Y)} (s,t) = (stu)^{X+1}(s^2 + t^2 + u^2)^Y d^{abcd} \text{,} \]
again manifestly satisfying the usual field-theory relations by construction. Putatively distinct weights proportional to the simple scalar numerator,
\[ c_{sym,ss} \propto d^{abcd}(stu) X (s^2 + t^2 + u^2)^Y j^{ss} \text{,} \]
can be seen to be redundant, building equivalent amplitudes to those generated from \( c_s^{(X^{-1},Y+1,d,nl)} \) in Equation (21).

With only three building blocks: \( c_s(X,Y) \), \( c_s(X,Y,ss) \), and \( c_s(X,Y,d,nl) \) we have exhausted all single-trace higher-derivative modifications of color-weight, and so we find that the generic form of such single-trace higher-derivative corrections to Yang-Mills to be encapsulated by:
\[ \hat{c}_s = \sum_{i=2}^{\infty} \alpha_i t^i \times \left\{ \sum_{X,Y \geq 0, 3X+2Y = i-2} X^{A,Y} c_s^{(X,Y,d,nl)} \right\} + \sum_{X \geq 1, Y \geq 0, 3X+2Y = i-1} X^{A,Y} c_s^{(X,Y,ss)} \text{,} \]
where \( X, Y \) are integers, and the \( a_i \) are free parameters encoding distinct operator Wilson-coefficients. All local higher-derivative SUSY-compatible gauge corrections to the four point tree-level amplitude, consistent with adjoint representations, will be given by such \( \hat{c} \) simply as:
\[ \mathcal{A}_{4}^{YM+HD} = \frac{\hat{c}_s n_s^{YM}}{s} + \frac{\hat{c}_t n_t^{YM}}{t} + \frac{\hat{c}_u n_u^{YM}}{u} \text{,} \]
In Tab. [1] we provide corresponding higher derivative scalar and gauge operators associated with the various \( A \) through mass dimension four.

The supersymmetric open string is a known UV completion to (super) Yang-Mills. It is a fair question to ask whether our simple color-modified building blocks for higher-derivative amplitudes are sufficient to capture the open superstring low-energy expansion [18, 26, 27]. We will see that the answer is yes, but will delay this discussion until after we have introduced the permutation invariant stration in the next section.

We have only, thus far, modified color-weights. As global supersymmetry is satisfied by the unmodified Yang-Mills kinematic weights, this exhausts a discussion consistent with the global supersymmetry inherent in YM amplitudes at tree-level. Composition between the above scalar weights and kinematic Yang-Mills weights always satisfies Jacobi, but it is easy to see that the only composition that maintains gauge invariance is redundant with the trivial modification of color-weights with permutation invariant prefactors that we have already considered in Equation (14).

What about non-supersymmetric operators that can be applied to gauge theory? The same discussion carries through, essentially unchanged, by replacing \( n^{YM} \) with other gauge invariant 4-point adjoint color-dual graph weights not trivially related to \( n^{YM} \), such as the \( \hat{n}_j^{YM} \) numerator weights identified in Ref. [27], or more broadly with the type of \( n^{(DF)} \) weights responsible for the ordered amplitudes defined in the context of the bosonic open string as per Ref. [16, 21]. We will return to this discussion in a later section discussing the open bosonic string where we make it clear that only four additional building blocks are required.

Next let us consider counterterms to gravity consistent with an adjoint double-copy representation. From a color-kinematic perspective, it is natural to consider replacing the color weights with Yang-Mills weights. Let us first treat the familiar replacements of the \( f^{abc} \) based color weights \( c_g \rightarrow n_g^{YM} \), so that
\[ \hat{n}^{(X,Y)} = e^{c_s(X,Y)} |_{c_s \rightarrow n_g^{YM}} \text{,} \]
and similarly for \( c_s(X,Y,ss) \). In the case of \( n_s^{(X,Y,ss)} \) we encounter no obstacle in the resulting ordered amplitudes; indeed, quite simply one finds \( A^{bi-adj}(s,t) \rightarrow A^{YM}(s,t) \) in Equation (14). Gauge-invariance, however, immediately excludes the amplitudes generated from \( n_s^{(X,Y,ss)} \), for essentially the same reason that we could only include trivial (permutation invariant) higher-derivative modifications to \( n^{YM} \). We are left only with the question as to what permutation invariant quantity to replace \( e^{abcd} \) with in \( c_s^{(X,Y,d,n=nl)} \) to generate gravity amplitudes \emph{without} introducing unphysical poles. There are two distinct candidates: \( stA^{YM}(s,t) \) and \( A^{YM}(s,t) \). It turns out that both choices are redundant with the amplitudes generated by \( n_s^{(X,Y)} \), leaving us with only the trivial building block \( n_s^{(X,Y)} \) for adjoint higher-derivative operators consistent with local supersymmetry for \( N > 4 \) in four-dimensions. As such, we have a simple argument that the only such higher derivative local operators available to gravity at 4-point give predictions simply proportional to the 4-point graviton amplitude:
\[ \mathcal{A}_{4}^{HD,GR} = \sum_{i} \alpha_i t^i \sum_{X \geq 1, Y \geq 0, 3X+2Y = i} X^{A,Y} (stu)^{X} (s^2 + t^2 + u^2)^Y \text{.} \]
As these modifications amount to simple factors of permutation invariant kinematics, all of these higher-dimensional corrections are consistent with local supersymmetric Ward identities. For operators restricted to
on-shell local supersymmetry consistent with $\mathcal{N} \leq 4$, one can also consider similar arguments where at least one copy has the adjoint color-weights replaced with non-supersymmetric gauge-invariant adjoint-type graph ones. Such an example is $F_3^3$, whose double-copy to gravity was considered in [27, 28]. These are of particular interest because of the possibility of removing anomalies in associated supergravity theories [29–31].

II. STRIATING BY $d^{abcd}$ STRUCTURES.

In addition to admitting an adjoint-type double-copy structure, these corrections admit an alternative decomposition into permutation invariant quantities. This is not the first opportunity to see that a single amplitude may admit multiple distinct double copy descriptions, depending on which algebra color-kinematics duality makes manifest. Indeed, the dimensional reduction of four-dimensional supergravity theories to three-dimensions admits both the adjoint-type double-copy construction of three-dimensional super-Yang-Mills amplitudes, as well as the three-algebra type double-copy construction of BLG amplitudes [32–34]. As we see here, the ability to striate along different algebras may be quite general.

Consider the full four-point amplitudes for Yang-Mills and gravity expressed in terms of Jacobi-satisfying structures along different algebras may be quite general. The three-algebra type double-copy construction of three-dimensional super-Yang-Mills amplitudes, as well as the tree-algebra type double-copy construction of BLG amplitudes [32–34]. As we see here, the ability to striate along different algebras may be quite general.

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where we use Tr[aj] to denote $\text{Tr}[T^{\alpha_1}_{\beta_1} T^{\alpha_2}_{\beta_2} T^{\alpha_3}_{\beta_3} T^{\alpha_4}_{\beta_4}]$. The above is invariant under exchange of any channels, and by expressing the Chan-Paton trace factors in terms of $c_\alpha$, $c_{\alpha'}$, and $d^{abcd}$ (and explicitly symmetrizing as appropriate), we find the following simple color-dual permutation invariant form for the full Chan-Paton dressed open superstring,

$$ -A^{OSS} = \Gamma_{\{s,t,u\}}(Z_{\text{adj}} + d^{abcd}Z_{\text{sym}})[stA^{YM}(s,t)]_{stu} $$

(36)

where the manifestly permutation symmetric $\Gamma_{\{s,t,u\}}$ corresponds to a series of higher mass-dimension combinations of Mandelstam invariants with coefficients responsible for familiar $\xi$ contributions to the low-energy expansion, $Z_{\text{adj}}$ contains all expressions involving Chan-Paton trace combinations $c_\alpha$, $c_{\alpha'}$, or $c_{u_\alpha}$, and $Z_{\text{sym}}$ contains all terms proportional to $d^{abcd}$. These are given as follows:

$$ \Gamma_{\{s,t,u\}} = \frac{\pi^2}{\alpha^2} \frac{\csc(\pi \alpha') \csc(\pi \alpha')}{\Gamma(-\alpha') \Gamma(-\alpha') \Gamma(-\alpha')} $$

(37)

$$ Z_{\text{sym}} = 2 \left[ \sin(\pi \alpha') + \sin(\pi \alpha') + \sin(\pi \alpha') \right] $$

(38)

$$ Z_{\text{adj}} = c_\alpha z_\alpha + c_{\alpha'} z_{\alpha'} + c_{u_\alpha} z_{u_\alpha} $$

(39)

Note that even $Z_{\text{adj}}$ is manifestly permutation invariant, as the $z_\alpha$ satisfy anti-symmetry and Jacobi-identities in concordance with $c_\alpha$. The $z_\alpha$ take a particularly simple form, with

$$ z_\alpha = \frac{1}{3}(\sin(\pi \alpha') - \sin(\pi \alpha')) $$

(40)

and the rest following from relabeling: $z_t = z_\alpha|_{s=t}$ and $z_u = z_\alpha|_{s=tt}$. In this form, all the coefficients for $c_{(X,Y,d,n)}$ may be easily identified already from the low-energy expansion of $Z_{\text{sym}}$. The remaining two building blocks only require a little teasing out from $Z_{\text{adj}}$, which may be achieved by using Equation (36) to rewrite the $z_\alpha$. This allows us to separate $z_\alpha$ into terms proportional to $j_s^{ss} = (u-t)$, and terms proportional to $j_s^{ss} = (u-t)$,

$$ z_\alpha = \frac{1}{3}(S_p - S_t) = (u-t)Z^{bi-adj} + s(u-t)Z^{ss} $$

(41)

where $S_p$ denotes $\sin(\pi \alpha')$, and the $Z^{bi-adj}$ and $Z^{ss}$ higher derivative corrections are given as follows:

$$ Z^{bi-adj} = \frac{1}{3} \left( s(t-u)S_p + t(u-s)S_t + u(s-t)S_u \right) $$

(42)

$$ Z^{ss} = \frac{1}{3} \left( s(t-u)S_p + t(u-s)S_t + u(s-t)S_u \right) $$

(43)

The $j_s^{ss}$ terms can be seen to correspond within $Z_{\text{adj}}$ to corrections of the form $stA^{X,Y}(s,t)$ (c.f. Equation (43)), and $j_s^{in}$ terms are likewise associated with $stA^{X,Y}(s,t)$ (c.f. Equation (44)). Both $Z^{bi-adj}$ and $Z^{ss}$ are manifestly permutation symmetric and local in all orders of an $\alpha' \to 0$ expansion, meaning they are completely spanned at any mass-dimension, MD, by a basis in $(stu)^N$ and $(s^2 + t^2 + u^2)^N$ such that $3X + 2Y = MD$ as per our building blocks. We have therefore exposed within the 4-point open superstring the three unique Jacobi-identity satisfying modifications to the color-weights of Yang-Mills. This can now be written in terms of

$$ stA^Z(s,t) = stA^{bi-adj}(s,t) + stA_{ss}(s,t) f_{ss}(s,t,u) + stA_{ss}(s,t) f_{ss}(s,t,u) + d^{abc} \bar{a} \bar{s} \bar{a} \bar{a} f_{id}(s,t,u), $$

(44)

where the higher-derivative expressions through mass dimension six are given by:

$$ f_{bi}(s,t,u) = \Gamma_{\{s,t,u\}}Z^{bi-adj} = 1 - \sigma_3 \xi \alpha' $$

$$ - \frac{1}{2} (\sigma_2 \sigma_3 \xi \alpha')^3 + \frac{1}{16} \sigma_3^2 (8 \xi^2 + 5 \xi) \alpha' + \ldots $$

$$ f_{ss}(s,t,u) = \Gamma_{\{s,t,u\}}Z^{ss} = \frac{1}{4} \sigma_3 \xi \alpha' + \frac{5}{32} \sigma_2 \sigma_3 \xi \alpha' + \ldots $$

$$ f_{id}(s,t,u) = \Gamma_{\{s,t,u\}}Z^{sym} = 6 \sigma_3 \xi \alpha' + \frac{15}{4} \sigma_2 \sigma_3 \xi \alpha' $$

$$ - 6 (\sigma_2 \xi \alpha')^2 + \frac{63}{32} \sigma_2 \sigma_3 \xi \alpha' + \ldots $$

(45)

We use $\sigma_2$ and $\sigma_3$ to denote $(s^2 + t^2 + u^2)$ and $(stu)$ respectively. The individual $\sigma_{X,Y}$ coefficients that fix Equation (24) to the low-energy expansion of the open superstring up through mass-dimension thirteen are given in supplementary Tab. II and through mass-dimension sixteen in a machine readable auxiliary Mathematica file.

We now turn to the open bosonic string amplitude at four-point tree-level. It was shown in Refs. [16, 20] that this amplitude also obeys a field theoretic adjoint-type double-copy description with $Z$ amplitudes as follows:

(open bosonic string) = $(Z\text{-theory}) \otimes (YM + (DF)^2)$,

(46)

where $(DF)^2$ is a massive higher derivative YM theory, compatible with the usual BCJ relations but in violation of supersymmetric Ward identities. It is straightforward to identify that only four new higher mass-dimension gauge-invariant adjoint-type vector building blocks are required to build this amplitude upon dressing with the permutation invariant objects $\sigma_2$ and $\sigma_3$, compactly encoded in the following denominator:

$$ A_{4}^{YM + (DF)^2} - A_{4}^{YM} = \alpha' A_{4}^{F^3} + \alpha'' A_{4}^{(F^3)^2 + F^4} + \alpha''' A_{4}^{D^F + D^2 F^4} + \alpha'''' A_{4}^{D^4 F^4} \frac{1}{(1 - \alpha' s)(1 - \alpha' t)(1 - \alpha' u)} $$

(47)

Machine readable expressions for the four new gauge-invariant building blocks of these amplitudes are included in an auxiliary Mathematica file.
IV. DISCUSSION

We have shown that at four-points there are simple building blocks, manifest in two algebraic striaations of four-point scattering amplitudes, that encode higher derivative corrections to effective gauge and gravity theories. We have demonstrated that these building blocks can be exposed to all orders in $\alpha'$ in the open supersymmetric and bosonic string amplitudes. Preliminary exploration confirms [33] that the pattern of identifying color-dual building blocks that admit composition continues at higher-multiplicity, a topic that merits detailed study. Gaining all-multiplicity control would mean that, through unitarity methods, one could build relatively easy to construct higher loop-order scalar integrands that trivially recycle, through double copy, known gauge and gravity integrands to their higher-derivative corrections. It is noteworthy that, at four-points, compatibility with adjoint double-copy structure involving Yang-Mills building blocks ensures compatibility with supersymmetry.

It is worth remarking on a striking fact that Eqs. [21] and [22] make manifest. Consider the SUSY-compatible $F^4$ amplitude:

$$A_{4}^{F_{\text{SUSY}}} = d^{a_1a_2a_3a_4} s t A_{\text{YM}}^{y}(s,t).$$

(48)

It was observed [27] that the kinematic factor accompanying individual trace terms, $s t A_{\text{YM}}^{y}(s,t)$, does not satisfy the $(n - 3)!$ field theory relations associated with adjoint color-kinematic structure. It is possible to misconstrue this result to show that $F^4$ is incompatible with color-kinematics duality in some broad sense, a question we can address.

We should emphasize two points clear now from a perspective informed by many examples [32, 36–38] of non-adjoint color-kinematics duality satisfying representations. First, even as written, there is a manifest completely-symmetric color-kinematics duality at work for $F^4$: both the color term, $d^{abcd}$, and the kinematic (Born-Infeld) term, $s t A_{\text{YM}}^{y}(s,t)$, are invariant under all permutations. This seemingly trivial duality even has teeth: there is an associated double-copy construction. Replacing the $d^{abcd}$ term with the permutation invariant kinematic weight $s t A_{\text{YM}}^{y}(s,t)$ generates the gravitational $R^4$ amplitude consistent with maximal local supersymmetry:

$$A_{4}^{R_{\text{SUSY}}} = (s t A_{\text{YM}}^{y}(s,t))^2 = s t u (A^{GR}),$$

(49)

with the relationship to the four-graviton scattering amplitude clear from comparison to Equation [29].

Second, we learn from Eqs. [21] and [22] that both $A_{4}^{R_{\text{SUSY}}}$ and $A_{4}^{F_{\text{SUSY}}}$ also manifest a non-trivial adjoint color-dual double-copy structure at four-points:

$$A_{4}^{F_{\text{SUSY}}} = \sum_{p \in \{s,t,u\}} \frac{d_{abcd} n_{p}^{\text{NLSM}}(n_{p}^{\text{YM}})}{p},$$

(50)

$$A_{4}^{R_{\text{SUSY}}} = \sum_{p \in \{s,t,u\}} \frac{((s t u) n_{p}^{\text{YM}})(n_{p}^{\text{YM}})}{p}.$$ 

(51)

The key to realizing this adjoint-type color-dual representation is to allow both color and scalar kinematics to conspire to satisfy the adjoint algebraic relations within the same adjoint-type color-dual weight—a lesson driven home top-down by abelian $Z$-theory [17], and constructively presented here.

Not all effective particles are massless, and not all such particles are single-trace in the adjoint (c.f. QCD, Einstein-Yang-Mills, and the standard model more generally), yet many admit color-dual representations [36–40]. It will be fascinating to see if such simple constructive building blocks are available for higher-derivative corrections to their predictions. Even in the adjoint, we have only focused here on structures involving gauge-kinematics in at least one copy. Generalizations of these building blocks should be relevant to exploring higher-derivative corrections to more phenomenological effective field theories [11–13].

These results also hold promise for the development of a more general field-theoretic approach towards understanding double-copy structure. The non-trivial relationship between relatively simple scalar operators and their double-copy with Yang-Mills to generate non-trivial gauge-operators provides a wealth of data at the level of the action towards building maps between operators, even at relatively low mass-dimension just for four field interactions. We expect that both the adjoint-type triangulation and permutation-invariant striation of these operators will prove instructive.

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TABLE I. Scalar and gauge operators corresponding to $\hat{c}$ through $\alpha'^4$. For the bi-colored scalar operators we suppress their second color-indices and color-factors $\tilde{c}_s$. The symmetrized trace operators generate the $d^{abc\prime}$ symbol as per \cite{26}.

| Mass Dim. | $\tilde{c}_g$ | Scalar operator/\tilde{c}_s | Gauge operator \cite{26} |
|-----------|----------------|----------------------------|--------------------------|
| 2         | $c_g^{(0,0,d,n)}$ | $d^{abcd}(\partial_\mu \varphi_a \varphi_b \partial_\nu \varphi_c \varphi_d)$ | symTr $\left(F_{\mu \nu} F_{\rho \sigma} F_{\sigma \mu} - \frac{1}{2} (F_{\mu \nu} F_{\rho \sigma})^2 \right)$ |
| 3         | $c_g^{(1,0)}$ | $\hat{f}_{abcd} f_{cde} (\partial_\mu \varphi_a \varphi_b \partial_\nu \varphi_c \varphi_d)$ | $c_t \left\{ F_\mu^a D_\lambda \partial_\nu (F_\rho^\alpha F_\sigma^\beta D_\lambda D_\sigma D_\nu \partial_\mu) + F_\mu^a D_\lambda D_\nu (F_\rho^\alpha F_\sigma^\beta D_\lambda D_\sigma D_\lambda) \right\}$ |
| 4         | $c_g^{(0,1,d,n)}$ | $d^{abcd}(\partial_\mu \varphi_a \varphi_b \partial_\nu \varphi_c \varphi_d)$ | symTr $\left[F_{\mu \nu} D^\lambda D^\nu D_\rho D_\lambda D_\sigma D_\rho D_\sigma D_\mu \right]$ |
| 4         | $c_g^{(1,0,s)}$ | $\hat{f}_{abcd} f_{cde} (\partial_\mu \varphi_a \varphi_b \partial_\nu \varphi_c \varphi_d)$ | $(c_s + c_s) \times \left[ F_\mu^a D^\lambda D^\nu D_\rho D_\lambda D_\sigma D_\rho D_\sigma D_\mu \right]$ |

TABLE II. Values of the $a_{X,Y}^{[MD]}$ coefficients appearing in $\hat{c}$ that match $\mathcal{A}_{YM+1D}$ to the low energy expansion of the open superstring amplitude through mass dimension 13.

$|a_{0,0}^{[0]}| = 1$  $|a_{0,0}^{[2]}| = 2\zeta_2$  $|a_{0,0}^{[4]}| = \frac{4}{3}$  $|a_{0,0}^{[6]}| = -\frac{8}{3}$  $|a_{0,0}^{[10]}| = \frac{341}{2048}$
$|a_{0,1}^{[0]}| = \frac{8}{3}$  $|a_{0,1}^{[2]}| = \frac{2\zeta_2}{3}$  $|a_{0,1}^{[4]}| = \frac{2\zeta_2}{3}$  $|a_{0,1}^{[6]}| = -\frac{8}{3}$  $|a_{0,1}^{[10]}| = \frac{341}{2048}$
$|a_{1,0}^{[0]}| = \frac{4}{3}$  $|a_{1,0}^{[2]}| = \frac{2\zeta_2}{3}$  $|a_{1,0}^{[4]}| = \frac{2\zeta_2}{3}$  $|a_{1,0}^{[6]}| = -\frac{8}{3}$  $|a_{1,0}^{[10]}| = \frac{341}{2048}$
$|a_{1,1,2}^{[0]}| = \frac{2\zeta_2}{3}$  $|a_{1,1,2}^{[2]}| = \frac{2\zeta_2}{3}$  $|a_{1,1,2}^{[4]}| = \frac{2\zeta_2}{3}$  $|a_{1,1,2}^{[6]}| = -\frac{8}{3}$  $|a_{1,1,2}^{[10]}| = \frac{341}{2048}$
$|a_{2,0,0}^{[0]}| = \frac{8}{3}$  $|a_{2,0,0}^{[2]}| = \frac{2\zeta_2}{3}$  $|a_{2,0,0}^{[4]}| = \frac{2\zeta_2}{3}$  $|a_{2,0,0}^{[6]}| = -\frac{8}{3}$  $|a_{2,0,0}^{[10]}| = \frac{341}{2048}$
$|a_{2,0,1}^{[0]}| = \frac{8}{3}$  $|a_{2,0,1}^{[2]}| = \frac{2\zeta_2}{3}$  $|a_{2,0,1}^{[4]}| = \frac{2\zeta_2}{3}$  $|a_{2,0,1}^{[6]}| = -\frac{8}{3}$  $|a_{2,0,1}^{[10]}| = \frac{341}{2048}$
$|a_{2,0,2}^{[0]}| = \frac{8}{3}$  $|a_{2,0,2}^{[2]}| = \frac{2\zeta_2}{3}$  $|a_{2,0,2}^{[4]}| = \frac{2\zeta_2}{3}$  $|a_{2,0,2}^{[6]}| = -\frac{8}{3}$  $|a_{2,0,2}^{[10]}| = \frac{341}{2048}$

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