Bayesian Linear Bandits for Large-Scale Recommender Systems

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Abstract—Potentially, taking advantage of available side information boosts the performance of recommender systems; nevertheless, with the rise of big data, the side information has often several dimensions. Hence, it is imperative to develop decision-making algorithms that can cope with such a high-dimensional context in real-time. That is especially challenging when the decision-maker has a variety of items to recommend. In this paper, we build upon the linear contextual multi-armed bandit framework to address this problem. We develop a decision-making policy for a linear bandit problem with high-dimensional context vectors and several arms. Our policy employs Thompson sampling and feeds it with reduced context vectors, where the dimensionality reduction follows by random projection. Our proposed recommender system follows this policy to learn online the item preferences of users while keeping its runtime as low as possible. We prove a regret bound that scales as a factor of the reduced dimension instead of the original one. For numerical evaluation, we use our algorithm to build a recommender system and apply it to real-world datasets. The theoretical and numerical results demonstrate the effectiveness of our proposed algorithm compared to the state-of-the-art in terms of computational complexity and regret performance.

Index Terms—Recommender systems, decision-making, online learning, multi-armed bandit

I. INTRODUCTION

Over the past decade, recommender systems have flourished to improve the economy by guiding decision-makers in different roles such as service providers, consumers, and producers, toward cost-effective and time-saving actions while retaining the constraints such as safety, privacy, and quality-of-service satisfaction. Famous examples of success stories include the recommendation systems deployed in online shopping or streaming websites that provide personalized suggestions to the users [1]–[3].

A widely-used metric to evaluate a recommender system is the returned payoff, measured in terms of the users’ responses to recommended items. One famous example is the Click-Through Rate (CTR). Therefore, the decision-making algorithms driving a recommender system aim at maximizing the payoffs over time [4]–[6].

Due to the ever-increasing demand for online services, recommender systems must serve a large and diverse group of users by providing a fast and accurate recommendation among a large group of available items. To deliver real-time services that fit the users’ interests recommender systems take advantage of side information. Building such efficient recommender systems becomes more challenging in a large-scale scenario with high-dimensional side information [4]. In this paper, we take advantage of an online framework, namely Multi-Armed Bandit (MAB), to build a recommender system and address the above-mentioned challenges.

MAB is a robust framework to model and solve sequential decision-making problems [7]. The seminal MAB problem portrays a finite set of arms and a player. The player sequentially pulls one arm at each time $t$. Upon pulling each arm, the player receives a random reward produced by an unknown generating process. The goal is to maximize the total accumulated reward over a finite time horizon. The Contextual Multi-Armed Bandit (CMAB) problem is one of the extensions of the seminal MAB problem [8]. In the CMAB framework, each arm associates with a context vector. At each round of decision-making, the player observes these contexts before selecting an arm.

Contextual bandits serve as a solution to address the challenges in a variety of domains, including healthcare [9], information retrieval [10], recommender systems [11], and anomaly detection [12]; nonetheless, there remain some open issues. Often, real-world problems have a large scale with high-dimensional contextual information. For example, in modern recommendation systems, several items are available to recommend to a massive number of users where a high-dimensional context vector represents each item and/or each user [13]. The majority of the current contextual bandit-based solutions are, however, not appropriately scalable. Thus, they are not capable of dealing with such scenarios.

To this end, we consider a multi-armed bandit problem where the number of arms is large and the context vectors have high dimensions. The state-of-the-art methods that address such a problem suffer from two main shortcomings: (i) excessive computational complexity and (ii) weak regret performance. The currently proposed approaches [14]–[17] achieve a regret bound that depends directly on the dimension of the context vectors; consequently, the regret scales as a factor of the context vectors’ dimension.

We confine our attention to the CMAB problem with linear payoffs; that is, we assume that there is an unknown parameter vector $\theta_a \in \mathbb{R}^n$ so that the reward of each arm $a$ is given by $r_{a,t} = \theta_a^\top x_{a,t} + \eta_t$, where $x_{a,t} \in \mathbb{R}^n$ is the associated context to arm $a$ at time $t$ and $\eta_t$ is the noise term. The goal is to learn the unknown parameters as the player sequentially
observes the new context vectors, pulls a new arm, and obtains a reward. To achieve this goal, we develop a novel algorithm that maps the context vectors into space $\mathbb{R}^d$ with a lower dimension $d < n$, thereby reducing the complexity as we update the model parameters. In contrast to the state-of-the-art research, our proposed algorithm guarantees an upper regret bound that depends on the lower dimension $d$ instead of the original dimension $n$. Thus, our policy is specifically suitable for the case of CMAB problems with high-dimensional context vectors.

To perform dimensionality reduction, we use Random Projection (RP), as it is computationally efficient [18]–[20]. RP works by projecting the data points in the original space $\mathbb{R}^n$ to a random lower-dimensional space $\mathbb{R}^d$, where $d < n$, using a randomly designed projection matrix $P \in \mathbb{R}^{d \times n}$ whose columns are scaled to have unit length. As we are now working in the lower-dimensional space $\mathbb{R}^d$, the player’s goal is to learn the unknown parameter $\psi_\star = P\theta_\star$. After obtaining the reduced context vector in the space $\mathbb{R}^d$, we use Thompson Sampling (TS) [21] due to its lower computational complexity compared to other methods such as those based on the Upper Confidence Bound (UCB). Moreover, TS is particularly beneficial in the CMAB problems with infinite arms [17], thereby making our policy suitable for scenarios with large number of arms.

As stated above, our proposed decision-making policy leverages two well-known tools; nevertheless, our work is not a straightforward extension of the existing methods, and the regret analysis of our algorithm is not derivative. The proposed method has a wide range of applications, e.g., to design online recommender engines, medical decision support systems, and mobile computing platforms. Among these, we consider recommendation systems for our numerical experiments. We use two real-world datasets to evaluate our proposed algorithm. Numerical results demonstrate the superiority of our proposed algorithm compared to the state-of-the-art contextual bandit algorithms.

A. Related Works

Online methods such as reinforcement learning and multi-armed bandit algorithms are popular bases to design recommender systems. Some examples include [22]–[25]. The core concept is to design algorithms that balance exploration and exploitation to maximize the total payoff over time. In the context of recommender systems, exploration means learning the payoff of new items by recommending those items to users. Exploitation involves recommending the best item to users using the collected data. Besides exploration-exploitation balance, another important criterion is to maximize the total reward while keeping the runtime as low as possible. That results in faster services, and thereby a higher users’ satisfaction level.

Specifically, the contextual bandit framework serves as a conventional model to formalize and solve recommendation problems. Some recent works include [11], [13], [25]–[28]. Despite being designed to solve large-scale problems, the performance of the state-of-the-art methods depends strongly on the number of items and the dimension of the context vectors. For instance, in [25], the authors consider the linear contextual bandit problem. They propose the BallExplore algorithm to model and solve a recommendation problem with high-dimensional context vectors. They prove a regret bound that is proportional to the original dimension of context vectors. Also, the proposed algorithm runs in quadratic time regarding the original dimension $n$. As another example, in [24], the authors propose an algorithm for personalized news article recommendation that also runs in quadratic time w.r.t the original dimension $n$. In contrast, the time complexity of our proposed algorithm is linear concerning the original dimension of contextual data.

As mentioned before, we extend the state-of-the-art research in the area of contextual bandits. In the following, we review notably-related research works and highlight the novelty of our approach. In [15], the authors study the CMAB problem with linear payoffs. They propose a UCB-based algorithm, namely LinUCB. The proposed algorithm achieves a regret bound of order $O(\sqrt{\ln(A)T \ln(T)})$, where $A$ represents the number of arms. Likewise, in [16], the authors develop the decision-making policy LinRel. This algorithm achieves a regret bound similar to that in [15]. Reference [14] proposes OFUL, a UCB-based algorithm for the CMAB problem. The upper regret bound is of order $O(n\log(T)\sqrt{T} + n\log(n))$. The development and the analysis are based on the assumption that the set of context vectors, that represents the set of arms, is the unit ball in $\mathbb{R}^n$. In comparison with the research works described above, the authors in [29] make several additional assumptions, e.g., on the expected covariance matrix of the samples and on the distribution of the context vectors. In return, their proposed policy achieves a regret bound of order $O(S^2 \log(T) + \log(n)^2)$.
Besides, in [34], the authors study the high-dimensional linear contextual bandit problem assuming that the set of contexts are sparse; i.e., only a subset of contexts is correlated with the reward. The proposed algorithm achieves a regret bound that scales logarithmically with the original dimension $n$. Reference [35] proposes SOFUL and Sketched linear TS policies. These methods use matrix sketching to boost the computational efficiency of the algorithms developed in [14] and [17]. However, the achieved regret bounds directly depend on the original dimension $n$. Finally, [36] uses a combinatorial bandit algorithm as a subroutine to select $d$ entries of context vectors out of $n$, thereby reducing the dimension of each context vector at each time $t$. They use the reduced context vectors to update the posterior distribution on the reward parameter. This work does not provide any theoretical analysis for the regret bound.

B. Organization

In Section II we formulate the problem. In Section III we introduce our proposed algorithm, namely BCMAB-RP. Section IV includes the theoretical analysis of the regret performance of BCMAB-RP. Section V is dedicated to numerical evaluation. Section VI concludes the paper.

II. Problem Formulation

We denote the set of arms by $A = \{1, 2, \ldots, A\}$. For each arm $a \in A$, $x_{a,t} \in \mathbb{R}^n$ represents its corresponding random context vector at time $t$. By $I_{d \times d}$ and $0_d$, we denote an identity matrix of size $d \times d$ and a zero vector of dimension $d$, respectively. $\mu_{\text{min}}(Z)$ represents the minimum eigenvalue of the positive definite matrix $Z$. Moreover, for a positive definite matrix $Z \in \mathbb{R}^{d \times d}$ and any vector $y \in \mathbb{R}^d$, we define $\|y\|_Z = \sqrt{y^\top Z y}$.

Let $r_{a,t} \in [0, 1]$, $\forall a \in A, \forall t$, represent the random reward corresponding to the arm $a$ at time $t$. The instantaneous rewards of each arm $a$ at each time $t$ are independent random variables drawn from an unknown probability distribution. The reward $r_{a,t}$ for each arm $a \in A$ is linear with respect to the context vector $x_{a,t}$; that is, there exists an unknown parameter vector $\theta_\star \in \mathbb{R}^n$ such that

$$r_{a,t} = \theta_\star^\top x_{a,t} + \eta_t,$$

where $\eta_t$ is a zero-mean random noise. We only assume that $\|x_{a,t}\|_2 \leq 1$, $\forall a \in A$, and $\|\theta_\star\|_2 \leq 1$. Consider the $\sigma$-algebra $\mathcal{F}_{t-1} = \sigma(\{x_{a,t}\}_{a=1}^t, \{\eta_t\}_{t=1}^{t-1})$. As it is conventional [14], [17], and [29], we assume that $\eta_t$ is conditionally $R$-sub-Gaussian, where $R \geq 0$ is a fixed constant. Formally,

$$\mathbb{E}[e^{c \eta_t} | \mathcal{F}_{t-1}] \leq e^{\frac{c^2 R^2}{2}}, \quad \forall \lambda \in \mathbb{R}.$$  

At each time $t$, we reduce the dimension of each context vector using the Random Projection (RP) method. More precisely, we use a random projection matrix $P \in \mathbb{R}^{d \times n}$ to project the original context vectors in the space $\mathbb{R}^n$ on a lower-dimensional space $\mathbb{R}^d$, i.e., $d < n$. It is common to design the matrix $P$ such that each entry of $P$ is a realization of independent and identically distributed (i.i.d.) zero-mean variables with Gaussian distribution [37]. Therefore,

$$z_{a,t} = P x_{a,t}, \quad \forall a \in A.$$

At each time $t$, let $\rho_t$ represent the noise term caused by the random projection performed on the context vector and the parameter $\theta_\star$. Formally, $\rho_t = \theta_\star^\top x_{a,t} - \psi_\star^\top z_{a,t}, \forall a \in A, \forall t$. In the space $\mathbb{R}^d$, we have $\psi_\star = P \theta_\star$. Thus, we can rewrite (1) as

$$r_{a,t} = \psi_\star^\top z_{a,t} + \rho_t + \eta_t.$$  

Based on (4), we define a $\sigma$-algebra as $\mathcal{F}_{t-1} = \sigma(\{x_{a,t}\}_{a=1}^t, \{\eta_t\}_{t=1}^{t-1}, \{\rho_t\}_{t=1}^{t-1})$. In our theoretical analysis, we use the filtration $\{\mathcal{F}_{t}\}_{t \geq 1}$ to derive the regret bound.

In the low-dimensional space $\mathbb{R}^d$, the goal is to learn the unknown parameter $\psi_\star$. To this end, we utilize Thompson Sampling (TS). First, we assume a set of parameters $\psi$ and a prior distribution on these parameters. Then, at each time $t$, the player pulls an arm according to its posterior probability of being the best arm. The posterior distribution yields

$$\mathbb{P}(\psi | r_{a,t}) \propto \mathbb{P}(r_{a,t} | \psi) \mathbb{P}(\psi).$$

At each time $t$, we define

$$Z_t = \lambda I_{d \times d} + \sum_{t=1}^{t-1} z_{a,t}\tau z_{a,t}\tau^\top,$$

$$b_t = \sum_{t=1}^{t-1} r_{a,t}\tau z_{a,t},$$

$$\hat{\psi}_t = Z_t^{-1} b_t.$$  

Let $\|z_{a,t}\|_2 \leq L_z$, $\forall a \in A$, and $\|\psi_\star\|_2 \leq L_\psi$ for some constants $L_z$, $L_\psi \geq 1$. Define $\nu_t = R_t \sqrt{4d \log \left(\frac{2+2LT\sqrt{L}}{\delta}\right)} + \lambda \frac{L_\psi}{\sqrt{t}} + \varepsilon \sqrt{t}$, with $\delta, \varepsilon \in (0, 1)$, which are the parameters of our algorithm. Parameter $\varepsilon$ determines the probability that $\psi_\star^\top z_{a,t}$ is close to the expected reward of arm $a$, i.e., $\theta_\star^\top x_{a,t}, \forall a \in A$.

Similar to [17], we use Gaussian likelihood function and Gaussian prior to obtain the posterior distribution. At each time $t$, if we consider the prior to be $\mathcal{N}(\hat{\psi}_t, \nu_t^2 Z_t^{-1})$ and the likelihood of reward $r_{a,t}$ to be $\mathcal{N}(\psi_\star^\top z_{a,t}, \nu_t^2)$, we obtain the posterior distribution as [17]

$$\mathcal{N}(\hat{\psi}_{t+1}, \nu_{t+1}^2 Z_{t+1}^{-1}).$$

As mentioned previously, the player’s goal is to maximize its total reward over the time horizon $T$. Alternatively, the player aims at minimizing its regret, defined as the difference between the cumulative reward of the optimal policy and the cumulative reward of the applied policy. Formally, the regret is defined as

$$R(T) = \sum_{t=1}^{T} \theta_\star^\top x_{a_t\star, t} - \sum_{t=1}^{T} \theta_\star^\top x_{a_t, t},$$

where $a_t^\star = \arg\max_{a \in A} \theta_\star^\top x_{a, t}$ is the optimal arm at time $t$, and $a_t$ denotes the played arm at time $t$ under the applied policy.
Remark 1. Based on our definition of reward in (1), we have $E[r_{a,t}|F_{t-1}] = \theta_a x_{a,t}$. Therefore, the regret in (10) is well-defined considering $F_{t-1}$. Moreover, this definition coincides with the pseudo-regret defined in [14] and [38].

In the following section, we describe our proposed algorithm, namely BCMAB-RP, to solve the formulated problem under the condition that for each arm $a \in A$, the player can only observe the $d$-dimensional vector $z_{a,t}$ instead of the full context vector $x_{a,t}$.

III. BCMAB-RP ALGORITHM

We propose Algorithm 1 to solve the problem (10). In the initial phase, BCMAB-RP constructs the random projection matrix $P$. We construct $P$ as a random matrix whose elements are drawn from a normal distribution $\mathcal{N}(0, \kappa^2)$, where $\kappa$ is a parameter of the algorithm. At each time $t$, BCMAB-RP utilizes this matrix to compute the reduced context vector $z_{a,t}$ for each arm $a \in A$. It then generates a sample $\tilde{\psi}_t$ from the posterior distribution and selects the arm that has the highest value of $\tilde{\psi}_t z_{a,t}$. Then, it observes the corresponding reward value. Finally, it updates the posterior distribution of $\psi_t$ using the reduced context vector of the selected arm and the corresponding reward.

Remark 2. The computational complexity of BCMAB-RP is polynomial w.r.t. the lower dimension $d$. We observe that for a fixed $d$, the computational complexity of BCMAB-RP scales linearly w.r.t. the original dimension $n$. This is an improvement over the previous methods, such as the works proposed in [17], [15], [14], and [25].

IV. THEORETICAL ANALYSIS

In this section, we theoretically analyze the BCMAB-RP algorithm by stating an upper bound on its regret. At each time $t$, we define

$$s_{a,t} = \sqrt{z_{a,t}^\top Z_t^{-1} z_{a,t}}, \quad \forall a \in A.$$  \hfill (13)

Moreover, we define $E_{\psi_t}(t)$ and $E_{\tilde{\psi}_t}(t)$ as the events that $\psi_t^\top z_{a,t}$ and $\tilde{\psi}_t^\top z_{a,t}$ are concentrated around their corresponding expected values. Formally, we define the event $E_{\psi_t}(t)$ as

$$|\psi_t^\top z_{a,t} - \psi_*^\top z_{a,t}| \leq \alpha_t s_{a,t}, \quad \forall a \in A,$$  \hfill (14)

where $\alpha_t = R \sqrt{d \log \left(\frac{2n^2 + 2t/\Delta}{\kappa^2}\right)} + \lambda \frac{1}{\sqrt{t}} L \psi + \varepsilon \sqrt{t}$. Moreover, we define the event $E_{\tilde{\psi}_t}(t)$ as

$$|\tilde{\psi}_t^\top z_{a,t} - \psi_*^\top z_{a,t}| \leq \beta_t s_{a,t}, \quad \forall a \in A,$$  \hfill (15)

where $\beta_t = \min \{4d \log (t), 4\sqrt{\log (tA)}\} \nu_t$.

At each time $t$, we divide the arms into the following two sets: (i) set of saturated arms, and (ii) set of unsaturated arms. The set of saturated arms includes any arm for which the estimated reward’s standard deviation is smaller than that of the optimal arm $a_*$, Any arm whose estimated reward’s standard deviation is greater than that of the optimal arm belongs to the set of unsaturated arms. Formally, at each time $t$, the set of arms $A$ is divided into two following sets.

- Saturated arms:
  $$C(t) = \{a \in A \mid \psi_*^\top z_{a,t} - \psi_*^\top z_{a,t} > \gamma_t s_{a,t}\},$$  \hfill (16)

- Unsaturated arms:
  $$\overline{C}(t) = \{a \in A \mid \psi_*^\top z_{a,t} - \psi_*^\top z_{a,t} \leq \gamma_t s_{a,t}\},$$  \hfill (17)

where $\gamma_t = \alpha_t + \beta_t$. Finally, we define $\Delta_n(t) = \psi_*^\top z_{a_*,t} - \psi_*^\top z_{a,t}, \forall a \in A$.

The following theorem states an upper bound on the regret of the decision policy BCMAB-RP, summarized in Algorithm 1.

Theorem 1. Let $\kappa^2 = \frac{1}{\alpha^2}$. For any $\delta, \varepsilon \in (0, 1)$ and $\lambda \geq 1$, with probability $\left(1 - \delta\right)\left(1 - 4\varepsilon^2\exp\left(-\frac{d \log (tA)}{8}\right)\right)$, the regret of BCMAB-RP is upper bounded as

$$R(T) = O\left(T d \sqrt{\ln (T) \ln \left(\frac{1}{\delta}\right) \min \{\sqrt{d}, \sqrt{\log (tA)}\}} \left(\sqrt{\log (T)} + \sqrt{\lambda + \varepsilon \sqrt{T}}\right)\right).$$  \hfill (18)

Proof. See Appendix VII-B1.
The original dimension $n$ does not appear in our regret bound, which is an improvement over the previous works that directly scale with $n$. Note that although the regret bound depends on the reduced dimension $d$, choosing a small $d$ does not necessarily reduce the regret as in this case, the obtained regret bound holds with a low probability. Indeed, choosing $d$ to be too small might even increase the regret due to the excessive information loss. In our numerical experiments, we show that by reducing the dimension of context vectors appropriately, we can achieve good regret performance compared to some benchmark algorithms. On the other hand, choosing a large $d$ expands the regret bound while that holds with a higher probability; that is, a large $d$ decreases the uncertainty of the provided regret bound, as expected intuitively. Also, as mentioned before, choosing a smaller value of $d$ improves the running time of our proposed algorithm. Therefore, selecting a suitable value for the reduced parameter $d$ is crucial for achieving a low computational complexity while ensuring a negligible regret. We elaborate on this trade-off in our numerical analysis in Section V.

V. Numerical Analysis

In this section, we study the following issues through numerical results:

- The performance of our proposed decision-making policy compared to benchmark algorithms in terms of the runtime and accumulated reward;
- The effect of the reduced dimension $d$ on the performance of our algorithm;
- The trade-off between computational complexity and regret bound in a real-world scenario together with the balance found by our algorithm, in particular, in comparison with the theoretical results.

We evaluate the performance of our algorithm using two real-world datasets, as described below.

**MovieLens:**

This dataset contains users’ ratings and tag applications applied to a set of movies from the MovieLens web site. The ratings have a 5-star scale, with half-star increments. Thus, the possible values for rating are 0.5, 1, 1.5, ..., 5. In our experiment, we select the top $\mathcal{A} = 1000$ movies based on the ratings given by the users. We form the context vector for each user by using the movies that user has watched together with the tags he applied to those movies. Afterward, we extract latent context vectors for each arm, i.e., movie, using a low-rank matrix factorization with 150 latent contexts. We then concatenate these context vectors to create the context vector of each joke-user pair. The dimension of the final context vectors is $n = 300$. The user receives a reward 1 if the corresponding rating is greater than 0. If the rating is less than 0 or no rating for a joke by a user exists, then the user collects a reward equal to 0.

We evaluate the performance of our proposed decision-making policy (BCMAB-RP) regarding two performance metrics: (i) the total payoff, i.e., total accumulated reward, and (ii) the runtime. We compare our algorithm with the following policies as benchmarks:

- **Linear TS:** It has a Bayesian approach similar to our proposed algorithm; however, it utilizes the original context vector with dimension $n$ to select arms. At each time $t$, it selects the arm with the highest value of $\tilde{\theta}_t^\top x_{a,t}$, where $\tilde{\theta}_t \in \mathbb{R}^n$ is the parameter drawn from the posterior distribution $\mathcal{N}((\tilde{\theta}_t, \sqrt{X_t^{-1}})$ over the set of candidate parameters $\tilde{\theta}$.
- **LinUCB:** It learns the parameter vector $\theta_s \in \mathbb{R}^n$ by solving a least square problem. At each time $t$, it selects an arm $a \in \mathcal{A}$ with the highest UCB index $\tilde{\theta}_t^\top x_{a,t} + \sqrt{X_t^{-1} x_{a,t}^\top x_{a,t}}$.
- **CBRAP:** It has a similar approach to LinUCB. At each time $t$, it selects an arm $a \in \mathcal{A}$ with the highest UCB index $\psi_t^\top z_{a,t} + \alpha \sqrt{z_{a,t}^\top Z_t^{-1} z_{a,t}}$, where $z_{a,t} \in \mathbb{R}^n$ is the reduced context vector obtained via the random projection approach from the original context vector $x_{a,t}$.
- **$\epsilon$-greedy:** At each time $t$, $\epsilon$-greedy chooses an arm uniformly at random with probability $\epsilon$ and the best arm so far with probability $1 - \epsilon$.
- **Random:** At each time $t$, it chooses an action $a \in \mathcal{A}$ uniformly at random.

We repeat the experiments by running the algorithms for $T = 100,000$ over 5 repetitions and report the results by averaging over the repetitions. There are 2,406 and 47,983 unique users in the experiments on the MovieLens and Jester dataset, respectively. Table I lists the average cumulative reward and the average runtime of each policy corresponding to different datasets. For BCMAB-RP and CBRAP, we list the results for different reduced dimensions. We reduce the context dimension to 5%, 10%, and 20% of the original context dimension to analyze the effect of the reduced dimension $d$ on the algorithm’s performance.

The experimental results show the trade-off between the time consumption and the cumulative reward. As we increase the reduced dimension $d$, the regret of BCMAB-RP decreases, i.e., the average cumulative reward increases while its runtime increases. BCMAB-RP performs better than Linear TS and LinUCB in terms of runtime. Although $\epsilon$-greedy and Random algorithms have a lower runtime than BCMAB-RP, they show a suboptimal performance concerning the average cumulative reward. As expected, our algorithm surpasses CBRAP in
TABLE I
COMPARISON OF AVERAGE CUMULATIVE REWARD (CR) AND TIME CONSUMPTION (TC) OF DIFFERENT POLICIES CORRESPONDING TO DIFFERENT DATASETS AND CONTEXT DIMENSIONS (CD).

| Dataset | Policy   | CD | TC (second) | CR    |
|---------|----------|----|-------------|-------|
| MovieLens | Linear TS | 120 | 4529.2 | 57563.8 |
|          | BCMAB-RP  | 6  | 1710.8 | 46739.8 |
|          |           | 12 | 3693.7 | 48694.2 |
|          |           | 24 | 3813.8 | 58610.2 |
|          | LinUCB    | 120 | 6254.8 | 57600.0 |
|          | CBRAP     | 6  | 2230.0 | 43739.0 |
|          |           | 12 | 5047.1 | 46307.0 |
|          |           | 24 | 5157.5 | 47391.0 |
|          | ε-greedy  | –  | 1463.4 | 36645.4 |
|          | Random    | –  | 1432.6 | 6551.4 |
| Jester   | Linear TS | 300 | 5384.6 | 52668.4 |
|          | BCMAB-RP  | 15 | 331.3  | 52929.4 |
|          |           | 30 | 796.1  | 52147.8 |
|          |           | 60 | 972.8  | 53111.4 |
|          | LinUCB    | 300 | 2051.7 | 52883.0 |
|          | CBRAP     | 15 | 388.2  | 45651.0 |
|          |           | 30 | 881.0  | 46222.0 |
|          |           | 60 | 964.9  | 45806.0 |
|          | ε-greedy  | –  | 241.0  | 37414.0 |
|          | Random    | –  | 223.9  | 14379.8 |

The reduced dimension $d$ makes a trade-off between the computational complexity of our algorithm and its regret performance. As evident from the theoretical and numerical results, large $d$ increases the runtime; however, choosing a small $d$ might yield information loss, thereby harming the performance w.r.t. the accumulated rewards. For example, for MovieLens dataset, as we increase the reduced dimension $d$, the runtime increases while the regret decreases. Although for larger values of $d$ regret bound expands, this does not necessarily mean that the achieved cumulative reward will be different in practice. That is the case for our experiment on Jester dataset, where the cumulative reward is not affected much as we decrease the value of reduced dimension $d$.

VI. CONCLUSION

We developed a decision-making policy, namely BCMAB-RP, for the linear CMAB problem that is implementable in recommender systems. BCMAB-RP is specifically suitable for scenarios with several recommendation items and high-dimensional side information. The policy utilizes Thompson Sampling at its core and also takes advantage of random projection to reduce the dimension of the context vectors. BCMAB-RP guarantees a low computational complexity while achieving significantly good performance in terms of the total accumulated reward. We theoretically analyzed BCMAB-RP and proved an upper regret bound that depends on the reduced dimension of the context vector. For numerical evaluation,
we apply BCMAB-RP on real-world datasets for content recommendation. The results demonstrate its superiority compared to several state-of-the-art CMAB algorithms. Besides developing online content recommender systems, our work fits several real-world problems, such as edge computing, medical decision support, and stock trading.

VII. APPENDIX

A. Auxiliary Results

Theorem 2. ([14])
Let \( \{F_i\}_{t=0}^{\infty} \) be a filtration. Let \( \{\eta_t\}_{t=1}^{\infty} \) be a real-valued stochastic process such that \( \eta_t \) is \( F_t \)-measurable and \( \eta_t \) is conditionally \( R \)-sub-Gaussian for some \( R \geq 0 \), i.e.,

\[
\mathbb{E}[\exp(\lambda \eta_t) | F_{t-1}] \leq \exp\left(\frac{\lambda^2 R^2}{2}\right), \quad \forall \lambda \in \mathbb{R}. \tag{19}
\]

Let \( \{x_t\}_{t=1}^{\infty} \) be an \( \mathbb{R}^n \)-valued stochastic process such that \( x_t \) is \( F_{t-1} \)-measurable. Assume that \( V \) is a positive definite matrix with dimension \( n \times n \). For any \( t \geq 0 \), define

\[
\bar{V}_t = V + \sum_{\tau=1}^{t} x_{\tau} x_{\tau}^\top,
\]
and

\[
S_t = \sum_{\tau=1}^{t} \eta_t x_{\tau}. \tag{21}
\]

Then, for any \( \delta > 0 \), with probability at least \( 1 - \delta \), for all \( t \geq 0 \),

\[
\|S_t\|_V^{-2} \leq 2R^2 \log \left( \frac{\det(\bar{V})^{1/2}\det(V)^{-1/2}}{\delta} \right). \tag{22}
\]

Lemma 1. ([42])
For a Gaussian distributed random variable \( Z \) with mean \( m \) and variance \( \sigma^2 \), for any \( z \), we have

\[
\frac{1}{4\sqrt{\pi}} \exp\left(-\frac{z^2}{2}\right) \leq \mathbb{P}(|Z - m| > z\sigma) \leq \frac{1}{2} \exp\left(-\frac{z^2}{2}\right). \tag{23}
\]

B. Main Results

Before we proceed to the proof of Theorem 1, we prove Lemmas 2-6.

Lemma 2. Consider the linear model for reward variables defined in (1). At each time \( t \), for any \( \lambda > 0 \) and \( \delta, \varepsilon \in (0, 1) \), we have

\[
\mathbb{P}(E_{\psi}(t)) \geq (1 - \frac{\delta}{2})(1 - 2 \exp(-\frac{d^2}{8})). \tag{24}
\]

Moreover, for all possible filtrations \( F'_{t-1} \), we have

\[
\mathbb{P}(E_{\psi}(t)|F'_{t-1}) \geq 1 - \frac{1}{t^2}. \tag{25}
\]

Proof. Based on the definitions in (5), (7), and (8), we have

\[
Z_t(\hat{\psi}_t - \psi_*) = \sum_{\tau=1}^{t-1} r_{a_{\tau},\tau} z_{a_{\tau},\tau} - \sum_{\tau=1}^{t-1} z_{a_{\tau},\tau}^\top \psi_* - \lambda \psi_*
\]

\[
= \sum_{\tau=1}^{t-1} z_{a_{\tau},\tau}^\top (r_{a_{\tau},\tau} - \theta_*^\top x_{a_{\tau},\tau})
\]

\[
+ \sum_{\tau=1}^{t-1} z_{a_{\tau},\tau}^\top (\theta_*^\top x_{a_{\tau},\tau} - \psi_*^\top z_{a_{\tau},\tau}) - \lambda \psi_*.
\]

Moreover, since \( \lambda \leq \mu_{\min}(Z_t) \), we observe that ([14])

\[
\|\psi_*\|_Z^{-1} \leq \frac{1}{\lambda_{\min}(Z_t)} \|\psi_*\|_Z^2 \leq \frac{L^2}{\lambda}. \tag{27}
\]

In addition, for any arm \( a \in A \), we have ([43])

\[
\mathbb{P}(\psi_*^\top z_{a,t} < \theta_*^\top x_{a,t}|dk^2 - \varepsilon dk^2 |x_{a,t}||\theta_*|_2)
\]

\[
< \exp\left(-\frac{d^2}{8}\right), \tag{28}
\]

\[
\mathbb{P}(\psi_*^\top z_{a,t} > \theta_*^\top x_{a,t}|dk^2 + \varepsilon dk^2 |x_{a,t}||\theta_*|_2)
\]

\[
< \exp\left(-\frac{d^2}{8}\right). \tag{29}
\]

Based on our assumption on the construction of the random matrix \( P \), it holds \( dk^2 = 1 \). Moreover, we have \( \|x_{a,t}\|_2 \leq 1 \) and \( \|\theta_*\|_2 \leq 1 \). Therefore, based on (28) and (29), with probability at least \( 1 - 2 \exp\left(-\frac{d^2}{8}\right) \), the following holds.

\[
\left| \sum_{\tau=1}^{t-1} z_{a_{\tau},\tau}^\top (\theta_*^\top x_{a_{\tau},\tau} - \psi_*^\top z_{a_{\tau},\tau}) \right|_Z^{-1}
\]

\[
\leq \sum_{\tau=1}^{t-1} \left( \theta_*^\top x_{a_{\tau},\tau} - \psi_*^\top z_{a_{\tau},\tau} \right) \|z_{a,t}\| \leq \varepsilon \sqrt{t}. \tag{30}
\]

Summarizing the above results, with probability at least \( 1 - \left(1 - \frac{1}{2}\right) \left(1 - 2 \exp\left(-\frac{d^2}{8}\right)\right) \), we observe that

\[
|\hat{\psi}_t^\top z_{a,t} - \psi_*^\top z_{a,t}| = |(\hat{\psi}_t - \psi_*)^\top z_{a,t}|
\]

\[
\leq \left( \sum_{\tau=1}^{t-1} z_{a_{\tau},\tau}^\top (r_{a_{\tau},\tau} - \theta_*^\top x_{a_{\tau},\tau}) \|Z_t^{-1}\| + \lambda \|\psi_*\|_Z \|Z_t^{-1}\|ight)
\]

\[
+ \sum_{\tau=1}^{t-1} z_{a_{\tau},\tau}^\top (\theta_*^\top x_{a_{\tau},\tau} - \psi_*^\top z_{a_{\tau},\tau}) \|z_{a,t}\| \|Z_t^{-1}\| \leq \left( R\sqrt{d \log \left( \frac{2 + 2tL^2/\lambda}{\delta} \right) + \lambda^2 L^2 + \varepsilon \sqrt{t} \right) \|z_{a,t}\| \|Z_t^{-1}\| = \alpha \sqrt{s_{a,t}}. \tag{31}
\]

where (a) follows from (26) and (b) follows from (27), (30), and Theorem 2.

For the second event \( \mathbb{P}(E_{\hat{\psi}}(t)|F'_{t-1}) \), the result follows directly from (17).

Lemma 3. Let \( p = \frac{1}{\sqrt{N}} \). For any filtration \( F'_{t-1} \) such that \( E_{\hat{\psi}}(t) \) is true, we have

\[
\mathbb{P}(\hat{r}_{a_{\tau},t} > \psi_*^\top z_{a_{\tau},t}|F'_{t-1}) \geq p. \tag{32}
\]
By definition of the saturated arms, we have $|\hat{\psi}_t^T z_{a_i^*,t} - \psi_t^T z_{a_i^*,t}| \leq \alpha_t s_{a_i^*,t}$. Moreover, Gaussian random variable $\hat{\psi}_t^T z_{a_i^*,t}$ has mean $\hat{\psi}_t^T z_{a_i^*,t}$ and standard deviation $\nu_t s_{a_i^*,t}$. Therefore, using Lemma 1 we observe that

$$
\mathbb{P}(\tilde{r}_{a_i^*,t} > \hat{\psi}_t^T z_{a_i^*,t}|F_{t-1}'') \\
= \mathbb{P}\left(\frac{\tilde{r}_{a_i^*,t} - \hat{\psi}_t^T z_{a_i^*,t}}{\nu_t s_{a_i^*,t}} > \frac{\hat{\psi}_t^T z_{a_i^*,t} - \psi_t^T z_{a_i^*,t}}{\nu_t s_{a_i^*,t}}|F_{t-1}'\right) \\
\geq \frac{1}{4\sqrt{\pi}} \exp\left(-\frac{7Y_t^2}{2}\right),
$$

where

$$
|Y_t| = \left|\frac{\hat{\psi}_t^T z_{a_i^*,t} - \psi_t^T z_{a_i^*,t}}{\nu_t s_{a_i^*,t}}\right| \leq \frac{\alpha_t s_{a_i^*,t}}{\nu_t s_{a_i^*,t}},
$$

$$
= \frac{R}{4\sqrt{\pi}} \left[\log\left(\frac{2+2L^2/\lambda}{\delta}\right) + \lambda \frac{1}{2} L\psi_t + \varepsilon \sqrt{t}\right] \leq 1.
$$

Hence, we conclude the proof.

**Lemma 4.** For any filtration $F_{t-1}'$ such that $E_{\tilde{q}}(t)$ is true, we have

$$
\mathbb{P}(a_t \in C(t)|F_{t-1}') \geq p - \frac{1}{t^2},
$$

where $p = \frac{1}{4\pi^{1/2}}$.

**Proof.** Our proof is inspired by [17]. The difference here is that we are working in the low-dimensional space $\mathbb{R}^d$ instead of the original space $\mathbb{R}^a$.

First, note that our proposed algorithm selects arm $a$ at each time $t$ which has the highest index $\tilde{r}_{a_i^*,t} = \hat{\psi}_t^T z_{a_i^*,t}$. Hence, if we assume that $\tilde{r}_{a_i^*,t} > \tilde{r}_{a_i^*,t}$, $\forall a \in C(t)$, i.e., the index for the optimal arm is greater than the indices of all the saturated arms, then one of the unsaturated arms, which also include the optimal arm, must be played at time $t$. Therefore

$$
\mathbb{P}(a_t \in C(t)|F_{t-1}') \geq \mathbb{P}(\tilde{r}_{a_i^*,t} > \tilde{r}_{a_i^*,t}, \forall a \in C(t)|F_{t-1}').
$$

By definition of the saturated arms, we have $\hat{\psi}_t^T z_{a_i^*,t} > \psi_t^T z_{a_i^*,t}, \forall a \in C(t)$. Moreover, if both the events $E_{\tilde{q}}(t)$ and $E_{\hat{q}}(t)$ are true, we get $\tilde{r}_{a_i^*,t} \leq \psi_t^T z_{a_i^*,t} + \gamma_t s_{a_i^*,t}, \forall a \in C(t)$. Therefore, given a filtration $F_{t-1}'$ such that $E_{\tilde{q}}(t)$ is true, either $E_{\hat{q}}(t)$ is false, or else we have for all $a \in C(t)$

$$
\tilde{r}_{a_i^*,t} \leq \psi_t^T z_{a_i^*,t} + \gamma_t s_{a_i^*,t} \leq \psi_t^T z_{a_i^*,t},
$$

where (a) follows from the definition of the saturated arms. Therefore, for any $F_{t-1}'$ such that $E_{\tilde{q}}(t)$ is true, we have

$$
\mathbb{P}(\tilde{r}_{a_i^*,t} > \tilde{r}_{a_i^*,t}, \forall a \in C(t)|F_{t-1}') \\
\geq \mathbb{P}(\tilde{r}_{a_i^*,t} > \psi_t^T z_{a_i^*,t}|F_{t-1}') - \mathbb{P}(E_{\tilde{q}}(t)|F_{t-1}') \geq p - \frac{1}{t^2},
$$

by Lemma 3 and where the last inequality follows from Lemma 2.}

**Lemma 5.** Let $\lambda \geq 1$. For any filtration $F_{t-1}'$ such that $E_{\tilde{q}}(t)$ is true,

$$
\mathbb{E}[\hat{\psi}_t^T z_{a_i^*,t} - \psi_t^T z_{a_i^*,t}|F_{t-1}'] \leq \frac{3\gamma_t}{p} \mathbb{E}[s_{a_i^*,t}|F_{t-1}'] + \frac{4L\psi_t L\gamma t}{p t^2},
$$

where $p = \frac{1}{4\pi^{1/2}}$.

**Proof.** The proof is inspired by [17]. The difference here is that we are working in the low-dimensional space $\mathbb{R}^d$ instead of the original space $\mathbb{R}^a$.

Let $\pi_t$ denote the unsaturated arm with the smallest $s_{a_i^*,t}$. Formally,

$$
\pi_t = \arg \min_{a \in C(t)} s_{a_i^*,t}.
$$

Note that $\pi_t$ is fixed since $C(t)$ and $s_{a_i^*,t}, \forall a \in A$, are fixed for a fixed $F_{t-1}'$. Hence, for any $F_{t-1}'$ such that $E_{\tilde{q}}(t)$ is true, we have

$$
\mathbb{E}[s_{a_i^*,t}|F_{t-1}'] \geq \mathbb{E}[s_{\pi_t^*,t} F_{t-1}'] \geq \mathbb{P}(a_t \in C(t)|F_{t-1}') \geq \mathbb{P}(\tilde{r}_{a_i^*,t} > \tilde{r}_{a_i^*,t}, \forall a \in C(t)|F_{t-1}'),
$$

by the definition of unsaturated arms. Hence, for any $F_{t-1}'$ such that $E_{\tilde{q}}(t)$ is true, either $E_{\hat{q}}(t)$ holds or $E_{\tilde{q}}(t)$ is false. Therefore,

$$
\mathbb{E}[\hat{\psi}_t^T z_{a_i^*,t} - \psi_t^T z_{a_i^*,t}|F_{t-1}'] \\
\leq \mathbb{E}[\hat{\psi}_t^T z_{\pi_t^*,t} - \psi_t^T z_{\pi_t^*,t} + \gamma_t s_{\pi_t^*,t}] + \gamma_t \mathbb{E}[s_{\pi_t^*,t} F_{t-1}'] + 2L\psi_t L\gamma t t^2 \mathbb{P}(E_{\tilde{q}}(t)|F_{t-1}') \\
\leq \mathbb{E}[\hat{\psi}_t^T z_{\pi_t^*,t} - \psi_t^T z_{\pi_t^*,t} + \gamma_t s_{\pi_t^*,t} F_{t-1}'] + 2L\psi_t L\gamma t t^2 \mathbb{P}(E_{\hat{q}}(t)|F_{t-1}') \\
\mathbb{E}[\hat{\psi}_t^T z_{a_i^*,t} - \psi_t^T z_{a_i^*,t}] + 4L\psi_t L\gamma t t^2 \mathbb{P}(E_{\tilde{q}}(t)|F_{t-1}'),
$$

by Lemma 3 and where (a) follows from (42) and the fact that $\Delta_t(t) \leq 2L\psi_t L\gamma t,$ $\forall a$. Moreover, (b) follows from (25) in Lemma 2 and (41). Finally, (c) follows from the fact that $0 \leq s_{a_i^*,t} \leq L\gamma t$ since $\lambda \geq 1$. 

$$
\text{Proof.}
$$
Lemma 6. Let $I(E)$ represent the indicator function which is equal to 1 if the event $E$ happens, and is 0 otherwise. Define $\Delta_a(t) = \psi_t^\top z_{a,t} - \psi_t^\top z_{a,t}^*$ and $\Delta'_a(t) = \Delta_a(t) I(E_{\hat{a}}(t))$. Moreover, define

$$U_t = \Delta'_a(t) - \frac{3\gamma_t}{p} s_{a,t} - \frac{4L\psi L_z \gamma_t}{pt^2},$$

$$W_t = \sum_{t=1}^T U_t,$$

where $p = \frac{1}{4e^2\sqrt{n}}$. Then, $\{W_t\}_{t=0}^T$ is a super-martingale process with respect to filtration $F_t$.

Proof. Our proof is inspired by [16]. The main difference here is that we are working with the filtration $F'_t$ instead of the filtration $F_t$.

We need to prove that $\mathbb{E}[W_t - W_{t-1} | F'_{t-1}] \leq 0$ for all $t = 1, \ldots, T$ and any $F'_{t-1}$. In other words, we need to prove the following.

$$\mathbb{E}[\Delta'_a(t) | F'_{t-1}] \leq \frac{3\gamma_t}{p} \mathbb{E}[s_{a,t} | F'_{t-1}] + \frac{4L\psi L_z \gamma_t}{pt^2}.$$  \hfill (46)

We observe that $E_{\hat{a}}(t)$ is completely determined by $F'_{t-1}$ whether it is true or not. If $F'_{t-1}$ is such that $E_{\hat{a}}(t)$ is not true, then $\Delta'_a(t) = 0$, and (46) holds trivially. If $F'_{t-1}$ is such that $E_{\hat{a}}(t)$ is true, then the result follows from Lemma 5.

1) Proof of Theorem 7

Proof. Let us denote the regret at time $t$ by $\text{regret}(t)$. Hence,

$$\text{regret}(t) = \theta^\top x_{a,t} - \theta^\top x_{a,t}^*.$$  \hfill (47)

Therefore, with probability $1 - 2 \exp(-\frac{d\epsilon^2}{n})$, we have

$$\text{regret}(t) \leq (\psi_t^\top z_{a,t} - \psi_t^\top z_{a,t}^*) + 2\epsilon,$$  \hfill (48)

where (a) follows from (28) and (29) and (b) follows from the definition of $\Delta_a(t)$.

Now, we observe that $|W_t - W_{t-1}| = |U_t| < \frac{9L\psi L_z \gamma_t}{p}$. Therefore, $U_t$ is bounded. Hence, we can apply the Azuma-Hoeffding inequality. Consequently, with probability $1 - \frac{d}{2}$, we achieve

$$\sum_{t=1}^T \Delta'_a(t) \leq \frac{3\gamma_t}{p} s_{a,t} + \frac{4L\psi L_z \gamma_t}{pt^2} + \sqrt{2 \left( \left( \sum_{t=1}^T \frac{(9L\psi L_z \gamma_t)^2}{p^2} \right) \ln \left( \frac{2}{\delta} \right) \right)}.$$  \hfill (49)

Combining all the above results, with probability $(1 - \delta)(1 - 4T \exp(-\frac{d\epsilon^2}{n}))$, we have

$$R(T) = \sum_{t=1}^T \text{regret}(t) \leq \sum_{t=1}^T (\Delta_a(t) + 2\epsilon)$$

$$\leq \sum_{t=1}^T \Delta'_a(t) + 2T\epsilon$$

$$\leq \sum_{t=1}^T \frac{3\gamma_t}{p} s_{a,t} + \frac{4L\psi L_z \gamma_t}{p} \sum_{t=1}^T \frac{1}{t^2}$$

$$+ \frac{9L\psi L_z \gamma_T}{p} \sqrt{2T \ln \left( \frac{2}{\delta} \right) + 2T\epsilon}$$

$$\leq \left( \sqrt{\log(T)} + \sqrt{\lambda + \epsilon \sqrt{T}} \right) \min \{ \sqrt{d}, \log(A) \}.$$  \hfill (50)

where (a) follows from (48) and (b) follows from the following fact. As proved in Lemma 2, $E_{\hat{a}}(t)$ holds for all $t$ with probability at least $(1 - \frac{\delta}{2})(1 - 2 \exp(-\frac{d\epsilon^2}{n}))$. Therefore, $\Delta'_a(t) = \Delta_a(t)$ with probability at least $1 - \frac{\delta}{2}(1 - 2 \exp(-\frac{d\epsilon^2}{n}))$ for all $t$. Moreover, (c) follows from (48) and (d) follows from the fact that $\sum_{t=1}^T s_{a,t} < 5\sqrt{dT \ln(T)}$ and $\gamma_T = \mathcal{O} \left( \left( \sqrt{\log(T)} + \sqrt{\lambda + \epsilon \sqrt{T}} \right) \min \{ \sqrt{d}, \log(A) \} \right)$.

Therefore, we conclude the proof.\hfill $\blacksquare$

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