Mixed Stochastic Differential Equations: Existence and Uniqueness Result

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Abstract In this paper we establish an existence and uniqueness result for solutions of multidimensional, time-dependent, stochastic differential equations driven simultaneously by a multidimensional fractional Brownian motion with Hurst parameter $H > 1/2$ and a multidimensional standard Brownian motion under a weaker condition than the Lipschitz one.

Keywords Fractional Brownian motion · Stochastic differential equations · Weak and strong solution · Bihari-type lemma

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1 Introduction

The fractional Brownian motion (fBm for short) $B^H = \{B^H(t), t \in [0, T]\}$ with Hurst parameter $H \in (0, 1)$ is a Gaussian self-similar process with stationary increments.

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This process was introduced by Kolmogorov [14] and studied by Mandelbrot and Van Ness [17], where a stochastic integral representation in terms of a standard Brownian motion (Bm for short) was established. The parameter $H$ is called Hurst index from the statistical analysis, developed by the climatologist Hurst [13]. The self-similarity and stationary increment properties make the fBm an appropriate model for many applications in diverse fields from biology to finance. From the properties of the fBm it follows that for every $\alpha > 0$

$$
\mathbb{E}\left(|B^H(t) - B^H(s)|^\alpha\right) = \mathbb{E}\left(|B^H(1)|^\alpha\right)|t-s|^\alpha H.
$$

As a consequence of the Kolmogorov continuity theorem, we deduce that there exists a version of the fBm $B^H$ which is a continuous process and whose paths are $\gamma$-Hölder continuous for every $\gamma < H$. Therefore, the fBm with Hurst parameter $H \neq \frac{1}{2}$ is not a semimartingale and then the Itô approach to the construction of stochastic integrals with respect to fBm is not valid. Two main approaches have been used in the literature to define stochastic integrals with respect to fBm with Hurst parameter $H$. Pathwise Riemann–Stieltjes stochastic integrals can be defined using Young’s integral [22] in the case $H > \frac{1}{2}$. When $H \in (\frac{1}{4}, \frac{1}{2})$, the rough path analysis introduced by Lyons [16] is a suitable method to construct pathwise stochastic integrals.

A second approach to develop a stochastic calculus with respect to the fBm is based on the techniques of Malliavin calculus. The divergence operator, which is the adjoint of the derivative operator, can be regarded as a stochastic integral, which coincides with the limit of Riemann sums constructed using the Wick product. This idea has been developed by Decreusefond and Üstünel [6], Carmona, Coutin and Montseny [5], Alòs, Mazet and Nualart [1,2], Alòs and Nualart [3] and Hu [12], among others. The integral constructed by this method has zero mean.

Let $T > 0$ be a fixed time and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ be a given filtered complete probability space with $(\mathcal{F}_t)_{t \in [0,T]}$ being a filtration that satisfies the usual hypotheses. The aim of this paper is to study the following stochastic differential equation (SDE for short) on $\mathbb{R}^n$

$$
X(t) = x_0 + \int_0^t b(s, X(s)) \, ds + \int_0^t \sigma_W(s, X(s)) \, dW(s) + \int_0^t \sigma_H(s, X(s)) \, dB^H(s),
$$

(1.1)

where $t \in [0, T]$, $x_0 \in \mathbb{R}^n$, $W$ is a $m$-dimensional standard $\mathcal{F}_t$-adapted Bm and $B^H$ a $d$-dimensional $\mathcal{F}_t$-adapted fBm with $H \in (\frac{1}{2}, 1)$. The main difficulty when considering Equation (1.1) lies in the fact that both stochastic integrals are dealt in different ways. Indeed, the integral with respect to the Bm is an Itô integral, while the integral with respect to the fBm has to be understood in the pathwise sense. Mixing the two integrals makes things difficult, forcing to consider very smooth coefficients to prove existence and uniqueness of solution to Eq. (1.1).

It is well known that, under suitable assumptions on the coefficients $b, \sigma_W, \sigma_H$ (see below), Equation (1.1) has a unique solution which is $(H - \varepsilon)$-Hölder continuous, for all $\varepsilon > 0$. This result was first considered in [15], where unique solvability was proved for time-independent coefficients and zero drift. In [24] the existence of solution to
(1.1) was proved under less restrictive assumptions, but only locally, i.e., up to a random time. In [9], global existence and uniqueness of solution to Eq. (1.1) was established under the assumption that $W$ and $B^H$ are independent. The latter result was obtained in [19,20] without the independence assumption. We stress on the fact that all these works consider the Lipschitz case. It should be noted, in addition, that the Lipschitz condition is the most used to establish the pathwise uniqueness for ordinary and SDEs via the Gronwall lemma. Thus, the following question appears naturally: Are there any weaker conditions than the Lipschitz continuity under which the SDE (1.1) has a unique strong solution?

In order to answer the above question our approach is to prove that the Euler’s polygonal approximations converge uniformly in $t \in [0,T]$, in probability, to a process, which we show to be the strong solution. The basic tools are the pathwise uniqueness for the SDE (1.1), tightness of the sequence of the laws of Euler’s approximations and the Skorokhod’s embedding theorem. It is important to note that the linear growth condition and the continuity of the coefficients are sufficient for the convergence of the Stieltjes and Itô integrals. However, the integral with respect to the fBm needs more regularity. To prove the convergence in probability we use an elementary result due to Gyöngy and Krylov [10]. It is worth mentioning that the pathwise uniqueness property for the SDE (1.1) is obtained under weaker assumption than the Lipschitz condition. More precisely our conditions are based on the modulus of continuity of the coefficients that achieve pathwise uniqueness using Bihari-type lemma. It should be noted that such conditions are considered by many authors for the existence and uniqueness of solutions of different kinds of equations where Bihari’s lemma is the cornerstone in the proof of these results, see for example [7,8,18].

The article is organized as follows. In Sect. 2, we state our assumptions on the coefficients $b, \sigma_W$ and $\sigma_H$ of Eq. (1.1), recall briefly the deterministic fractional calculus in order to define the integral with respect to fBm and introduce proper normed spaces. In addition, we give the definition of strong, weak solution and pathwise uniqueness of Eq. (1.1). In Sect. 3, the pathwise uniqueness property for the solutions of Eq. (1.1) is proved (see Theorem 7 below). Finally, in Sect. 4, we define the Euler approximations sequence and prove that it is tight. Moreover, we show that these approximations converge in probability to a process which turns out to be a strong solution of the SDE (1.1), cf. Theorem 9 below. In “Appendix,” we recall some technical results which play a great role in this work. We also show a version of Bihari’s lemma (cf. Lemma 14 in “Appendix”) which will be used in the proof of pathwise uniqueness to SDE (1.1).

2 Preliminaries

Throughout this paper we assume that the coefficients $b, \sigma_W$ and $\sigma_H$, which are continuous, satisfy, for all $x, y \in \mathbb{R}^n$ and $t \in [0,T]$, the following hypotheses (H.1) and (H.2):

**Hypothesis (H.1).** The functions $b$ and $\sigma_W$ have a linear growth and satisfy suitable modulus of continuity with respect to the variable $x$ uniformly in $t$.

Hypothesis (H.1) means that $b$ and $\sigma_W$ satisfy

\[
(H.1.1) \quad |b(t,x)| \leq K (1 + |x|),
\]
(H.1.2) \(|b(t, x) - b(t, y)|^2 \leq \varrho(|x - y|^2)\)

(H.1.3) \(|\sigma_W(t, x)| \leq K (1 + |x|),\)

(H.1.4) \(|\sigma_W(t, x) - \sigma_W(t, y)|^2 \leq \varrho(|x - y|^2),\)

where \(\varrho\) is a concave increasing function from \(\mathbb{R}_+\) to \(\mathbb{R}_+\) such that \(\varrho(0) = 0, \varrho(u) > 0\) for \(u > 0\) and for some \(q > 1\) we have

\[
\int_{0^+} \frac{du}{\varrho^q(u^{1/q})} = \infty. \quad (2.1)
\]

Hypothesis (H.2). The function \(\sigma_H\) is continuously differentiable in the second variable \(x\). Its derivative, with respect to \(x\), is bounded, Lipschitz with respect to the same variable uniformly with respect to the first variable \(t\). Moreover, both \(\sigma_H\) and its derivative are \(\beta\)-Hölder with respect to the first variable \(t\) uniformly with respect to the second variable.

Hypothesis (H.2) means that \(\sigma_H\) and its derivative satisfy

(H.2.1) \(\left| \partial_x \sigma_H(t, x) \right| \leq K\)

(H.2.2) \(\left| \partial_x \sigma_H(t, x) - \partial_x \sigma_H(t, y) \right| \leq K |x - y|\)

(H.2.3) \(\left| \sigma_H(t, x) - \sigma_H(s, x) \right| + \left| \partial_x \sigma_H(t, x) - \partial_x \sigma_H(s, x) \right| \leq K |s - t|^{\beta} .\)

Example 1 Let us give two examples of such function \(\varrho\). Let \(q > 1\) and \(\delta\) be sufficiently small. Define

\[
\varrho_1(u) := \begin{cases} 
  u \log^{1/q}(u^{-1}), & 0 \leq u \leq \delta \\
  \delta \log^{1/q}(\delta^{-1}) + \varrho'_1(\delta)(u - \delta), & u > \delta
\end{cases}
\]

\[
\varrho_2(u) := \begin{cases} 
  u \log^{1/q}(u^{-1}) \log^{1/q}(\log(u^{-1})), & 0 \leq u \leq \delta \\
  \delta \log^{1/q}(\delta^{-1}) \log^{1/q}(\log(\delta^{-1})) + \varrho'_2(\delta)(u - \delta), & u > \delta
\end{cases}
\]

It is easy to see that, for \(i = 1, 2\), the function \(\varrho_i\) is concave nondecreasing function satisfying \((2.1)\).

We begin by a brief review of the deterministic fractional calculus. We start with the definition of the integral with respect to fBm as a generalized Lebesgue–Stieltjes integral, following the work of Zähle [23]. We fix \(\alpha \in (0, 1)\). The Weyl–Marchaud derivatives of \(f : [a, b] \rightarrow \mathbb{R}^n\) are given by:

\[
D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{f(x)}{(x - a)^\alpha} + \alpha \int_a^x \frac{f(y) - f(y)}{(y - x)^{\alpha + 1}} dy \right) \mathbb{I}_{(a, b)}(x)
\]

and

\[
D_{b-}^\alpha f(x) = \frac{(-1)^\alpha}{\Gamma(1 - \alpha)} \left( \frac{f(x)}{(b - x)^\alpha} + \alpha \int_x^b \frac{f(y) - f(y)}{(y - x)^{\alpha + 1}} dy \right) \mathbb{I}_{(a, b)}(x),
\]
where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ is the Gamma function. Assuming that $D_{a+}^\alpha f \in L^1[a, b]$ and $D_{b-}^{1-\alpha} g_{b-} \in L^\infty[a, b]$, where $g_{b-}(x) = g(x) - g(b-)$, the generalized (fractional) Lebesgue–Stieltjes integral of $f$ with respect to $g$ is defined as

$$
\int_a^b f \, dg := (-1)^\alpha \int_a^b D_{a+}^\alpha f(x) \, D_{b-}^{1-\alpha} g_{b-}(x) \, dx.  \tag{2.2}
$$

If $a \leq c < d \leq b$ then we have

$$
\int_c^d f \, dg = \int_a^b \mathbb{1}_{(c,d)} f \, dg.
$$

It follows from the Hölder continuity of $B^H$ that $D_{b-}^{1-\alpha} B^H_{b-} \in L^\infty[a, b]$ almost surely (a.s. for short). Then, for a function $f$ with $D_{a+}^\alpha f \in L^1[a, b]$, we can define the integral with respect to $B^H$ through (2.2).

Let $0 < \alpha < 1/2$ and $\mu \in (0, 1]$ be fixed. We will consider the following normed spaces:

1. $C^\mu$ is the space of $\mu$-Hölder continuous functions $f : [0, T] \to \mathbb{R}^d$, equipped with the norm

$$
\|f\|_\mu := \|f\|_\infty + \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t-s)^\mu} < \infty,
$$

where

$$
\|f\|_\infty := \sup_{0 \leq t \leq T} |f(t)|.
$$

2. $C^\mu_0$ denotes the space of $\mu$-Hölder continuous functions $f : [0, T] \longrightarrow \mathbb{R}^d$ such that

$$
\lim_{\varepsilon \to 0} \left( \sup_{0 < |t-s| < \varepsilon} \frac{|f(t) - f(s)|}{(t-s)^\mu} \right) = 0.
$$

This space equipped with the norm $\|\cdot\|_\mu$ is complete and separable [11].

3. $W_{0}^{\alpha, \infty}$ is the space of measurable functions $f : [0, T] \longrightarrow \mathbb{R}^d$ such that

$$
\|f\|_{\alpha, \infty} := \sup_{0 \leq t \leq T} \|f\|_{\alpha, t} < \infty,
$$

where

$$
\|f\|_{\alpha, t} := |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} \, ds.
$$

4. Finally, $W_{T}^{1-\alpha, \infty}$ denotes the space of measurable functions $f : [0, T] \longrightarrow \mathbb{R}^m$ such that
\[ \| f \|_{1-\alpha, T} := \sup_{0 \leq t \leq T} \| f \|_{1-\alpha, \infty, t} < \infty, \]

where

\[ \| f \|_{1-\alpha, \infty, t} := \sup_{0 \leq u < v < t} \left( \frac{|f(v) - f(u)|}{(v-u)^{1-\alpha}} + \int_u^v \frac{|f(y) - f(u)|}{(y-u)^{2-\alpha}} \, dy \right). \]

Hence, it is clear that

\[ \sup_{0 \leq u < v < t} \left| D^{1-\alpha} B^H_v(u) \right| \leq \frac{1}{\Gamma(\alpha)} \| B^H \|_{1-\alpha, \infty, t} < \infty, \]

where the last inequality is a consequence of the fact that the random variable \( \| B^H \|_{1-\alpha, \infty, t} \) has moments of all orders, see Lemma 7.5 in Nualart and Rascanu [21]. Thus, the stochastic integral with respect to the fBm admits the following estimate

\[ \left| \int_0^t f(s) \, dB^H(s) \right| \leq \frac{1}{\Gamma(\alpha)} \| B^H \|_{1-\alpha, \infty, t} \| f \|_{1,1,t}, \quad (2.3) \]

where

\[ \| f \|_{1,1,t} := \int_0^t |f(s)| \, ds + \int_0^t \int_0^s \frac{|f(s) - f(y)|}{(s-y)^{\alpha+1}} \, dy \, ds. \]

We give the definition of strong and weak solution as well as pathwise uniqueness for Eq. (1.1).

**Definition 2** (Strong solution). By a strong solution of Eq. (1.1) we mean an \( \mathcal{F}_t \)-adapted continuous process \( X(t), t \in [0, T] \) such that there exists an increasing sequence of stopping times \( (T_R)_{R \in \mathbb{N}} \) satisfying \( \lim_{R \to \infty} T_R = T \) a.s. and for any \( R \in \mathbb{N}^* \), we have

1. \( \sup_{t \in [0,T]} \mathbb{E} \left[ \| X(\cdot \wedge T_R) \|_{1-\alpha,t}^2 \right] < \infty. \)
2. The equation

\[ X(t \wedge T_R) = x_0 + \int_0^{t \wedge T_R} b(s, X(s)) \, ds + \int_0^{t \wedge T_R} \sigma_W(s, X(s)) \, dW(s) \]

\[ + \int_0^{t \wedge T_R} \sigma_H(s, X(s)) \, dB^H(s), \quad (2.4) \]

holds a.s.

**Definition 3** (Weak solution). By a weak solution of Eq. (1.1) we mean a triplet \( (X, W, B^H), (\Omega, \mathcal{F}, P) \) and \( (\mathcal{F}_t)_{t \in [0,T]} \), such that

1. \( (\Omega, \mathcal{F}, P) \) is a probability space, and \((\mathcal{F}_t)_{t \in [0,T]}\) is a filtration, of sub-\( \sigma \)-algebra of \( \mathcal{F} \), satisfying the usual conditions.
2. $W = (W_t, \mathcal{F}_t)_{t \in [0,T]}$ is a Bm, $B^H_t = (B^H_t, \mathcal{F}_t)_{t \in [0,T]}$ is a fBm and $X = (X_t, \mathcal{F}_t)_{t \in [0,T]}$ is a continuous and $\mathcal{F}_t$-adapted process satisfying a.s. Eq. (2.4) for some increasing sequence of stopping times $(T_R)_{R \in \mathbb{N}^*}$ such that $\lim_{R \to \infty} T_R = T$ a.s.

**Definition 4** (Pathwise uniqueness). We say that pathwise uniqueness holds for Eq. (1.1) if whenever $(X, W, B^H)$ and $(\tilde{X}, W, B^H)$ are two weak solutions of Eq. (1.1) defined on the same probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ then $X$ and $\tilde{X}$ are indistinguishable.

### 3 Pathwise Uniqueness

In this section we investigate the pathwise uniqueness of a solution for Eq. (1.1), cf. Theorem 7 below, where we make use of the so-called Bihari-type lemma (see Lemma 14 in “Appendix”).

Since $(\|B^H\|^{1-\alpha, \infty, t})_{t \in [0,T]}$ is a continuous $\mathcal{F}_t$-adapted process, then for any integer $R > 0$ we define the following stopping time

$$T_R := \inf \left\{ t \geq 0 : \|B^H\|^{1-\alpha, \infty, t} \geq R \right\} \wedge T.$$

**Remark 3.1** We should note here that the family $(T_R)_{R \in \mathbb{N}^*}$ satisfies the following property which will be useful in the sequel

$$P \left[ \bigcup_{R \in \mathbb{N}^*} \{ T_R = T \} \right] = 1.$$

Indeed, $P \left[ \bigcap_{R \in \mathbb{N}^*} \{ T_R < T \} \right] \leq P \left[ \|B^H\|^{1-\alpha, \infty, T} = \infty \right] = 0$ since by Lemma 7.5 in Nualart and Rascanu [21] the random variable $\|B^H\|^{1-\alpha, \infty, t}$ has moments of all orders.

Let $X$ be a solution of Eq. (1.1). For every $R \in \mathbb{N}^*$, we define the stochastic processes $X_R$ by

$$X_R(t) := X(t \wedge T_R), \quad t \in [0, T].$$

Then it is easy to see that the following equation

$$X_R(t) = x_0 + \int_0^{t \wedge T_R} b(s, X(s)) \, ds + \int_0^{t \wedge T_R} \sigma_W(s, X(s)) \, dW(s)$$

$$+ \int_0^{t \wedge T_R} \sigma_H(s, X(s)) \, dB^H(s)$$

holds almost surely. We have the following Lemma.
Lemma 5 For any integers $N \geq 1$ and $R \geq 1$, there exists a positive constant $C_N$ such that

$$\sup_{s \in [0,t]} \mathbb{E}\left[ \|X_R\|_{\alpha,s}^{2N} \right] \leq C_N |x_0|^{2N} d_\alpha \exp\left( c_\alpha t \left[ C_N \left( 1 + R^{2N} \right) \left( \sqrt{T} + 1 \right) \right]^{\frac{2}{1-2\alpha}} \right),$$

where $d_\alpha$ and $c_\alpha$ are positive constants depending only on $\alpha$, for all $t \in [0, T]$.

Proof Along the proof $C_N$ will denote a generic positive constant, which may vary from line to line and may depend on $N$ and other parameters of the problem. It follows from the convexity of $x^{2N}$ that

$$\mathbb{E}\left[ \|X_R\|_{\alpha,t}^{2N} \right] \leq C_N \left( |x_0|^{2N} + \mathbb{E}\left[ \left\| \int_0^{\wedge T_R} b(s, X(s)) \, ds \right\|_{\alpha,t}^{2N} \right] + \mathbb{E}\left[ \left\| \int_0^{\wedge T_R} \sigma_W(s, X(s)) \, dW(s) \right\|_{\alpha,t}^{2N} \right] + \mathbb{E}\left[ \left\| \int_0^{\wedge T_R} \sigma_H(s, X(s)) \, dB^H(s) \right\|_{\alpha,t}^{2N} \right] \right) = C_N \left( |x_0|^{2N} + A_1 + A_2 + A_3 \right).$$

Furthermore we have

$$\left\| \int_0^{\wedge T_R} b(s, X(s)) \, ds \right\|_{\alpha,t} \leq \int_0^{\wedge T_R} |b(s, X(s))| \, ds + \int_0^t (t-s)^{-\alpha-1} \int_s^{\wedge T_R} |b(u, X(u))| \, du \, ds \leq \int_0^t |b(s \wedge T_R, X(s \wedge T_R))| \, ds + \int_0^t (t-s)^{-\alpha-1} \int_s^t |b(u \wedge T_R, X(u \wedge T_R))| \, du \, ds \leq \int_0^t |b(s \wedge T_R, X(s \wedge T_R))| \, ds + \frac{1}{\alpha} \int_0^t (t-r)^{-\alpha} |b(r \wedge T_R, X(r \wedge T_R))| \, dr \leq C_{\alpha,T} \int_0^t (t-r)^{-\alpha} |b(r \wedge T_R, X(r \wedge T_R))| \, dr$$

where $C_{\alpha,T}$ is a constant depending on $\alpha$ and $T$. Using the linear growth assumption in (H.1.1), Hölder’s inequality and the fact that $\alpha < \frac{1}{2}$, we obtain

$$A_1 \leq C_N \mathbb{E}\left[ \left( 1 + \int_0^t |X_R(s)| \, ds \right)^{2N} \right].$$
\[ \leq C_N \mathbb{E} \left[ \left( 1 + \int_0^t |X_R(s)|^2 \, ds \right)^N \right] \]
\[ \leq C_N \left( 1 + \int_0^t \mathbb{E} \left[ |X_R(s)|^{2N} \right] \, ds \right). \]

We have also that
\[
A_2 \leq C_N \mathbb{E} \left[ \left| \int_0^{t \wedge T_R} \sigma_w(s, X(s)) \, dW(s) \right|^{2N} \right] \\
+ C_N \mathbb{E} \left[ \left( \int_0^t (t - s)^{-\alpha - 1} \left| \int_{s \wedge T_R}^{t \wedge T_R} \sigma_w(u, X(u)) \, dW(u) \right| \, ds \right)^{2N} \right] \\
= A_{21} + A_{22}. 
\]

For \( A_{21} \), using the linear growth assumption in (H.1.3), the Burkholder and Hölder inequalities, we obtain
\[
A_{21} \leq C_N \mathbb{E} \left[ \int_0^{t \wedge T_R} |\sigma_w(s, X(s))|^{2N} \, ds \right] \\
\leq C_N \mathbb{E} \left[ \int_0^t |\sigma_w(s \wedge T_R, X(s \wedge T_R))|^{2N} \, ds \right] \\
\leq C_N \left( 1 + \int_0^t \mathbb{E} \left[ |X_R(s)|^{2N} \right] \, ds \right)
\]

For \( A_{22} \), again the Burkholder and Hölder inequalities give
\[
A_{22} \leq C_N \left( \int_0^t \frac{ds}{(t - s)^{\alpha + \frac{1}{2}}} \right)^{2N - 1} \int_0^t (t - s)^{-\alpha - \frac{1}{2} - N} \mathbb{E} \left[ \int_{s \wedge T_R}^{t \wedge T_R} |\sigma_w(u, X(u))|^{2N} \, du \right] \, ds \\
\leq C_N \int_0^t (t - s)^{-\alpha - \frac{3}{2}} \mathbb{E} \left[ \int_{s \wedge T_R}^{t \wedge T_R} |\sigma_w(u, X(u))|^{2N} \, du \right] \, ds \\
\leq C_N \int_0^t (t - s)^{-\alpha - \frac{3}{2}} \mathbb{E} \left[ \int_s^{t \wedge T_R} |\sigma_w(u \wedge T_R, X(u \wedge T_R))|^{2N} \, du \right] \, ds.
\]

Applying now Fubini’s theorem and using the growth assumption in (H.1.3), we obtain
\[
A_{22} \leq C_N \left( \int_0^t (t - s)^{-\alpha - \frac{1}{2}} \left( 1 + \mathbb{E} \left[ |X_R(s)|^{2N} \right] \right) \, ds \right). 
\]

\( \sum \) Springer
Thus

\[ A_2 \leq C_N \left( 1 + \int_0^t (t-s)^{-\alpha - \frac{1}{2}} \mathbb{E} \left[ |X_R(s)|^{2N} \right] ds \right). \]

Let us remark that, for \( t \in [0, T] \), we have

\[ \int_0^{t \land T_R} \sigma_H(s, X(s)) dB^H(s) = \int_0^t \sigma_H(s \land T_R, X(s \land T_R)) dB^H(s \land T_R). \]  (3.1)

Then it follows from Proposition 12 (jj), in “Appendix,” that

\[ A_3 \leq C_N R^{2N} \int_0^t \left( (t-s)^{-2\alpha} + s^{-\alpha} \right) \left( 1 + \mathbb{E} \left[ \|X_{R(t)}\|_{\alpha,s}^{2N} \right] \right) ds. \]

Putting all the estimates obtained for \( A_1, A_2 \) and \( A_3 \) together, we obtain

\[ \mathbb{E} \left[ \|X_{R(t)}\|_{\alpha,t}^{2N} \right] \leq C_N |x_0|^{2N} + C_N (1 + R^{2N}) \int_0^t \varphi(t, s) \mathbb{E} \left[ \|X_{R(t)}\|_{\alpha,s}^{2N} \right] ds, \]  (3.2)

where

\[ \varphi(t, s) := s^{-\alpha} + (t-s)^{-\alpha - \frac{1}{2}}. \]

Therefore, since the right-hand side of Eq. (3.2) is an increasing function of \( t \), we have

\[ \sup_{0 \leq s \leq t} \mathbb{E} \left[ \|X_{R(t)}\|_{\alpha,s}^{2N} \right] \leq C_N |x_0|^{2N} + C_N (1 + R^{2N}) \int_0^t \varphi(s, t) \sup_{0 \leq u \leq s} \mathbb{E} \left[ \|X_{R(t)}\|_{\alpha,u}^{2N} \right] ds. \]

By a simple calculation the r.h.s. of the above inequality is bounded by

\[ C_N |x_0|^{2N} + C_N (1 + R^{2N}) \left( \sqrt{T} + 1 \right) \int_0^t (t-s)^{-\alpha - \frac{1}{2}} s^{-\alpha - \frac{1}{2}} \sup_{0 \leq u \leq s} \mathbb{E} \left[ \|X_{R(t)}\|_{\alpha,u}^{2N} \right] ds. \]

As a consequence, by the Gronwall-type lemma (Lemma 7.6 in [21]), we deduce the desired estimate. \( \square \)

Let \( X \) and \( Y \) be two solutions of Eq. (1.1) defined on the same probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)\). For \( M \in \mathbb{N}^* \), we define the following stopping time

\[ \tau_M := \inf \left\{ t > 0 : \|X\|_{\alpha,t} \lor \|Y\|_{\alpha,t} > M \right\} \land T. \]

Now for every \( R, M \in \mathbb{N}^* \), we define the stochastic processes \( X_{R,M} \) (resp. \( Y_{R,M} \)) by

\[ X_{R,M}(t) := X(t \land T_R \land \tau_M), \quad t \in [0, T], \]

(resp. \( Y_{R,M}(t) := Y(t \land T_R \land \tau_M), \quad t \in [0, T] \)).
Lemma 6 Under Hypotheses (H.1) and (H.2), there exists a positive constant $C_{R,M}$ such that for $t \in [0, T],$

$$
E\left[\|X_{R,M} - Y_{R,M}\|_{\alpha,t}^2\right] 
\leq C_{R,M} \int_0^t \varphi(s, t) \left[ E\left[\|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2\right] + \varrho \left( E\left[\|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2\right]\right)\right] ds.
$$

(3.3)

Proof The proof of this result is long and technical. It is divided into several parts. First we have

$$
X_{R,M}(t) - Y_{R,M}(t) = \int_0^{t \wedge T_R \wedge \tau_M} (b(s, X(s)) - b(s, Y(s))) ds 
+ \int_0^{t \wedge T_R \wedge \tau_M} (\sigma_W(s, X(s)) - \sigma_W(s, Y(s))) dW(s) 
+ \int_0^{t \wedge T_R \wedge \tau_M} (\sigma_H(s, X(s)) - \sigma_H(s, Y(s))) dB^H(s)
= B_1(t \wedge T_R \wedge \tau_M) + B_2(t \wedge T_R \wedge \tau_M) + B_3(t \wedge T_R \wedge \tau_M).
$$

It follows that

$$
\|X_{R,M} - Y_{R,M}\|_{\alpha,t}^2 
\leq 3 \left( \|B_1(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2 + \|B_2(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2 + \|B_3(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2 \right).
$$

We have to estimate $\|B_i(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2, i \in \{1, 2, 3\}.$ For the sake of conciseness, we define

$$
\Delta(f)(s) = f(s, X(s)) - f(s, Y(s)), \quad f \in \{b, \sigma_W, \sigma_H\}.
$$

Step 1: $B_1.$ Using simple estimations it is easy to see that

$$
\|B_1(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t} \leq \int_0^{t \wedge T_R \wedge \tau_M} |\Delta(b)(s)| ds 
+ \int_0^t (t - s)^{-\alpha-1} \int_s^{t \wedge T_R \wedge \tau_M} |\Delta(b)(u)| du ds 
\leq \int_0^t |\Delta(b)(s \wedge T_R \wedge \tau_M)| ds 
+ \int_0^t (t - s)^{-\alpha-1} \int_s^t |\Delta(b)(u \wedge T_R \wedge \tau_M)| du ds 
\leq C_{\alpha,T} \int_0^t (t - r)^{-\alpha} |\Delta(b)(r \wedge T_R \wedge \tau_M)| dr.
$$
We use the fact that $\alpha < \frac{1}{2}$, Hölder inequality and hypothesis \((H.1.2)\) to obtain

$$\|B_1(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2 \leq C_{\alpha,T}^2 \int_0^t |\Delta(b)(s \wedge T_R \wedge \tau_M)|^2 ds$$

$$\leq C_{\alpha,T}^2 \int_0^t \varrho \left( |X_{R,M}(s) - Y_{R,M}(s)|^2 \right) ds$$

$$\leq C_{\alpha,T}^2 \int_0^t \varrho \left( \|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2 \right) ds.$$

**Step 2: B3.** If $1 - H < \alpha < \min(\beta, 1/2)$, we have from Proposition 4.3 in [21] (see Proposition 11 (ii) in “Appendix”) that

$$\|B_3(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2 \leq CR^2 \left( \int_0^t (t-s)^{-2\alpha} \right.$$  

$$+ s^{-\alpha} \|\Delta(\sigma_H)(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,s} ds \big)^2.$$

Now using assumptions \((H.2)\) and Lemma 7.1 in Nualart Rascanu [21] we obtain

$$|\sigma_H(t, x_1) - \sigma_H(s, x_2) - \sigma_H(t, y_1) + \sigma_H(s, y_2)|$$

$$\leq K |x_1 - x_2 - y_1 + y_2| + K |x_1 - y_1| |t-s|^\beta$$

$$+ K |x_1 - y_1| (|x_1 - x_2| + |y_1 - y_2|).$$

Therefore

$$\left| \sigma_H(t \wedge T_R \wedge \tau_M, X_{R,M}(t)) - \sigma_H(s \wedge T_R \wedge \tau_M, X_{R,M}(s)) - \sigma_H(t \wedge T_R \wedge \tau_M, Y_{R,M}(t)) + \sigma_H(s \wedge T_R \wedge \tau_M, Y_{R,M}(s)) \right|$$

$$\leq K \left[ |X_{R,M}(t) - X_{R,M}(s) - Y_{R,M}(t) + Y_{R,M}(s)| \right.$$

$$+ K |X_{R,M}(t) - Y_{R,M}(t)| |t-s|^\beta$$

$$+ |X_{R,M}(t) - Y_{R,M}(t)| \left( |X_{R,M}(t) - X_{R,M}(s)| + |Y_{R,M}(t) - Y_{R,M}(s)| \right) \big].$$

Thus we have

$$\|\Delta(\sigma_H)(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}$$

$$\leq K \left[ |X_{R,M}(t) - Y_{R,M}(t)| \right.$$  

$$+ \int_0^t \frac{|X_{R,M}(t) - X_{R,M}(s) - Y_{R,M}(t) + Y_{R,M}(s)|}{(t-s)^{\alpha+1}} ds.$$
\[ + |X_{R,M}(t) - Y_{R,M}(t)| \left( \int_0^t \frac{ds}{(t-s)^{\alpha-\beta+1}} + \int_0^t \frac{|X_{R,M}(t) - X_{R,M}(s)|}{(t-s)^{\alpha+1}} ds \right) \]

\[ + \int_0^t \frac{|Y_{R,M}(t) - Y_{R,M}(s)|}{(t-s)^{\alpha+1}} ds \] \right].

Now it is easy to see that

\[ \|B_3(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2 \leq C R^2 \int_0^t \left( (t-s)^{-2\alpha} + s^{-\alpha} \right) \left( 1 + \|X_{R,M}\|_{\alpha,s}^2 + \|Y_{R,M}\|_{\alpha,s}^2 \right) \|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2 ds \]

\[ \leq C R^2 M^2 \int_0^t \varphi(s,t) \|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2 ds. \]

**Step 3: B_2.** Till now we have made estimates for pathwise integrals. As \( B_2 \) is a stochastic integral we need to use martingale-type inequality. First we have

\[ \|B_2(\cdot \wedge T_R \wedge \tau_M)\|_{\alpha,t}^2 \leq 2 \left( |B_2(t \wedge T_R \wedge \tau_M)|^2 + (\tilde{B}_2(t))^2 \right), \]

where

\[ \tilde{B}_2(t) := \int_0^t \left| \int_{s \wedge T_R \wedge \tau_M} \frac{\Delta(\sigma_W)(u) dW(u)}{(t-s)^{\alpha+1}} \right| ds. \]

It then follows from Burkholder inequality and assumption (H.1.4) that

\[ \mathbb{E}(|B_2(t \wedge T_R \wedge \tau_M)|^2) \leq \mathbb{E} \left( \int_0^t |\Delta(\sigma_W)(s \wedge T_R \wedge \tau_M)|^2 ds \right) \]

\[ \leq C \mathbb{E} \left( \int_0^t \varphi \left( |X_{R,M}(s) - Y_{R,M}(s)|^2 \right) ds \right) \]

\[ \leq C \mathbb{E} \left( \int_0^t \varphi \left( \|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2 \right) ds \right). \]

\( \tilde{B}_2: \) Using Hölder’s inequality and Fubini’s theorem we have

\[ \mathbb{E} \left[ (\tilde{B}_2(t))^2 \right] \leq C \mathbb{E} \left[ \int_0^t (t-s)^{-\frac{3}{2} - \alpha} \left( \int_{s \wedge T_R \wedge \tau_M} \Delta(\sigma_W)(u) dW(u) \right)^2 ds \right] \]

\[ \leq C \int_0^t (t-s)^{-\frac{3}{2} - \alpha} \mathbb{E} \left[ \left( \int_{s \wedge T_R \wedge \tau_M} \Delta(\sigma_W)(u) dW(u) \right)^2 \right] ds. \]
Using the same techniques as in the estimation of $I_2$ we have

$$
\mathbb{E} \left[ \left| \int_{t \wedge T \wedge \tau} \Delta(\sigma_W)(u) \, dW(u) \right|^2 \right] \leq \mathbb{E} \left[ \int_s^t |\Delta(\sigma_W)(u \wedge T \wedge \tau)|^2 \, du \right] 
\leq C \mathbb{E} \left[ \int_s^t \varrho \left( \|X_{R,M} - Y_{R,M}\|_{\alpha,u}^2 \right) \, du \right].
$$

Then, it follows that

$$
\mathbb{E} \left[ \left| \tilde{B}_2(t) \right|^2 \right] \leq C \int_0^t (t - s)^{-\frac{3}{2} - \alpha} \mathbb{E} \left[ \int_s^t \varrho \left( \|X_{R,M} - Y_{R,M}\|_{\alpha,u}^2 \right) \, du \right] \, ds.
$$

Consequently

$$
\mathbb{E} \left[ \|B_2\|_{\alpha,t}^2 \right] \leq C \int_0^t \varphi(s,t) \mathbb{E} \left[ \varrho \left( \|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2 \right) \right] \, ds.
$$

**Step 4:** Combining all estimates leads to

$$
\mathbb{E} \left[ \|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2 \right] 
\leq C_{M,R} \int_0^t \varphi(s,t) \mathbb{E} \left[ \|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2 + \varrho \left( \|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2 \right) \right] \, ds.
$$

Since $\varrho$ is concave, Jensen’s inequality gives

$$
\mathbb{E} \left[ \|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2 \right] 
\leq C_{M,R} \int_0^t \varphi(s,t) \left[ \mathbb{E} \left[ \left( \|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2 \right) \right] + \varrho \left( \mathbb{E} \left[ \|X_{R,M} - Y_{R,M}\|_{\alpha,s}^2 \right] \right) \right] \, ds.
$$

This concludes the proof.

**Theorem 7 (Pathwise uniqueness).** Let $1 - H < \alpha < \min (\beta, 1/2)$. Then, under Hypotheses (H.1) and (H.2), the pathwise uniqueness property holds for Eq. (1.1).

**Proof** It is simple to see that the function $\tilde{\varrho}(u) = u + \varrho(u)$ is a concave increasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\tilde{\varrho}(0) = 0$ and $\tilde{\varrho}(u) > 0$ for $u > 0$. On the other hand, we have $\varrho(u) \geq \varrho(1)u$ for $0 \leq u \leq 1$. Then

$$
\int_0^1 \frac{du}{\tilde{\varrho}^q(u^{1/q})} \geq \left( \frac{\varrho(1)}{1 + \varrho(1)} \right)^q \int_0^1 \frac{du}{\varrho^q(u^{1/q})} = \infty.
$$

Therefore, condition (2.1) is satisfied for the function $\tilde{\varrho}$. Consequently, we can apply Lemma 14 in “Appendix” to the inequality (3.3) to obtain

$$
\|X_{R,M} - Y_{R,M}\|_{\alpha,t}^2 = 0, \text{ a.s.}
$$
This implies $X(t) = Y(t)$ a.s. for all $t < T_R \land \tau_M$. By letting $M \to \infty$ we get, by Lemma 5, $X(t) = Y(t)$ a.s. for all $t < T_R$. The proof is finished by using Remark 3.1.

4 Euler Approximation Scheme

In this section, we apply the Euler approximation procedure in order to obtain a weak solution of Eq. (1.1). Under the condition that pathwise uniqueness holds for Eq. (1.1) we prove that the Euler approximation converges to a process which is a strong solution of the SDE (1.1), see Theorem 9 below.

Let $0 = t^n_0 < t^n_1 < \cdots < t^n_i < \cdots < t^n_n = T$ be a sequence of partitions of $[0, T]$. For every integer $R$ we define the family of stochastic processes by

$$X^n_R(t) := X^n(t \land T_R), \quad t \in [0, T].$$

Then it is easy to see that the process $X^n_R$ satisfies, a.s., the following statement

$$X^n_R(t) = x_0 + \int_0^{t \land T_R} b(k_n(s), X^n(k_n(s))) \, ds + \int_0^{t \land T_R} \sigma_W(k_n(s), X^n(k_n(s))) \, dW(s) + \int_0^{t \land T_R} \sigma_H(k_n(s), X^n(k_n(s))) \, dB^H(s).$$

Lemma 8 Suppose that Assumptions (H.1) and (H.2) hold. Then, for all $N \in \mathbb{N}^*$ and $R \in \mathbb{N}^*$, there exists a positive constant $C_{N,R}$ such that

$$\sup_{t \in [0,T]} \mathbb{E} \left[ \left\| X^n_R \right\|_{\alpha,t}^{2N} \right] \leq C_{N,R},$$

for all $n \in \mathbb{N}$. Moreover, we also have

$$\mathbb{E} \left[ \left| X^n_R(t) - X^n_R(s) \right|^{2N} \right] \leq C_{N,R} |t - s|^N,$$

for all $s, t \in [0, T]$ and $n \in \mathbb{N}$. \hfill \(\square\)
Proof It follows from the convexity of $x^{2N}$ that
\[
\mathbb{E}\left[ \|X^n_R\|_{2N,\alpha,t}^2 \right] \leq C_N \left( |x_0|^{2N} + \mathbb{E} \left[ \left\| \int_0^{\cdot \wedge T_R} b(k_n(s), X^n(k_n(s))) \, ds \right\|_{\alpha,t}^{2N} \right] \right.
\]
\[
+ \mathbb{E} \left[ \left\| \int_0^{\cdot \wedge T_R} \sigma_W(k_n(s), X^n(k_n(s))) \, dW(s) \right\|_{\alpha,t}^{2N} \right]
\]
\[
+ \mathbb{E} \left[ \left\| \int_0^{\cdot \wedge T_R} \sigma_H(k_n(s), X^n(k_n(s))) \, dB_H(s) \right\|_{\alpha,t}^{2N} \right] \right)
\]
\[= C_N \left( |x_0|^{2N} + I_1 + I_2 + I_3 \right).
\]

Using the same estimations as in the proof of Lemma 5, we obtain
\[
I_1 \leq C_N \left( 1 + \int_0^t \mathbb{E} \left[ |X^n(k_n(s) \wedge T_R)|^{2N} \right] \, ds \right)
\]
\[\leq C_N \left( 1 + \int_0^t \mathbb{E} \left[ |X^n_R(k_n(s))|^{2N} \right] \, ds \right).
\]
\[
I_2 \leq C_N \mathbb{E} \left[ \int_0^{t \wedge T_R} \sigma_W(k_n(s), X^n(k_n(s))) \, dW(s) \right]^{2N}
\]
\[+ C_N \mathbb{E} \left[ \left( \int_0^t (t-s)^{-\alpha-1} \left\| \int_s^{t \wedge T_R} \sigma_W(k_n(s), X^n(k_n(s))) \, dW(u) \right\|_{\alpha,t}^{2N} \right] \right)
\]
\[= I_{21} + I_{22}.
\]

For $I_{21}$, using the linear growth assumption in (H.1.3), Burkholder’s and Hölder’s inequalities, we obtain
\[
I_{21} \leq C_N \mathbb{E} \left[ \int_0^{t \wedge T_R} |\sigma_W(k_n(s), X^n(k_n(s)))|^{2N} \, ds \right]
\]
\[\leq C_N \mathbb{E} \left[ \int_0^{t \wedge T_R} \left| \sigma_W(k_n(s \wedge T_R), X^n(k_n(s \wedge T_R))) \right|^{2N} \, ds \right]
\]
\[\leq C_N \left( 1 + \int_0^t \mathbb{E} \left[ |X^n_R(k_n(s))|^{2N} \right] \, ds \right).
\]

For $I_{22}$, again the Burkholder and Hölder inequalities give
\[
I_{22} \leq C_N \left( \int_0^t \frac{ds}{(t-s)^{\alpha+\frac{1}{2}}} \right)^{2N-1}
\]
\[\times \int_0^t (t-s)^{-\alpha-\frac{1}{2}-N} \mathbb{E} \left[ \left\| \int_{s \wedge T_R}^{t \wedge T_R} \sigma_W(k_n(s), X^n(k_n(s))) \, dW(u) \right\|_{\alpha,t}^{2N} \right] \, ds
\]
\[ \leq C_N \int_0^t (t-s)^{-\alpha - \frac{3}{2}} \mathbb{E} \left[ \int_{s \wedge T_R}^{t \wedge T_R} \left| \sigma_W(k_n(s), X^n(k_n(s))) \right|^{2N} \, du \right] \, ds \]

\[ \leq C_N \int_0^t (t-s)^{-\alpha - \frac{3}{2}} \mathbb{E} \left[ \int_s^T \left| \sigma_W(k_n(u) \wedge T_R, X^n(k_n(u) \wedge T_R)) \right|^{2N} \, du \right] \, ds. \]

Applying now Fubini’s theorem and using the growth assumption in (H.1.3), we obtain

\[ I_{22} \leq C_N \int_0^t (t-s)^{-\alpha - \frac{1}{2}} \left( 1 + \mathbb{E} \left[ \left| X^n_R(k_n(s)) \right|^{2N} \right] \right) \, ds. \]

Thus

\[ I_2 \leq C_N \left( 1 + \int_0^t (t-s)^{-\alpha - \frac{1}{2}} \mathbb{E} \left[ \left| X^n_R(k_n(s)) \right|^{2N} \right] \, ds \right). \]

Let us remark that

\[ \int_0^{t \wedge T_R} \sigma_H(k_n(s), X^n(k_n(s))) \, dB^H(s) = \int_0^t \sigma_H(k_n(s) \wedge T_R, X^n(k_n(s) \wedge T_R)) \, dB^H(s \wedge T_R) \]

\[ = \int_0^t \sigma_H(k_n(s) \wedge T_R, X^n_R(k_n(s))) \, dB^H(s \wedge T_R) \quad (4.4) \]

Using (4.4) and Proposition 12 (jj) in “Appendix” we obtain

\[ I_3 \leq C_N R^{2N} \left( \int_0^t \left( (t-s)^{-2\alpha} + s^{-\alpha} \right) \left( 1 + \mathbb{E} \left[ \| X^n_R(k_n(\cdot)) \|_{\alpha,s} \right] \right) \, ds \right)^{2N}. \]

By Hölder’s inequality we have

\[ I_3 \leq C_N R^{2N} \int_0^t \varphi(s, t) \left( 1 + \mathbb{E} \left[ \| X^n_R(k_n(\cdot)) \|_{\alpha,s} \right] \right) \, ds. \]

Putting all the estimates obtained for \( I_1, I_2 \) and \( I_3 \) together, we obtain

\[ \mathbb{E} \left[ \| X^n_R \|_{\alpha,t}^{2N} \right] \leq C_N |x_0|^{2N} + C_N \left( 1 + R^{2N} \right) \int_0^t \varphi(s, t) \mathbb{E} \left[ \| X^n_R(k_n(\cdot)) \|_{\alpha,s}^{2N} \right] \, ds. \]

(4.5)

Therefore, since the right-hand side of Eq. (4.5) is an increasing function of \( t \), we have

\[ \sup_{0 \leq s \leq t} \mathbb{E} \left[ \| X^n_R \|_{\alpha,s}^{2N} \right] \leq C_N |x_0|^{2N} + C_N \left( 1 + R^{2N} \right) \int_0^t \varphi(s, t) \mathbb{E} \left[ \sup_{0 \leq u \leq s} \| X^n_R \|_{\alpha,u}^{2N} \right] \, ds. \]
As a consequence, by the Gronwall-type lemma (cf. Lemma 7.6 in [21]), we deduce the first estimate (4.2) of the lemma. Let us now prove the second estimate (4.3). We have

\[
X^n_R(t) - X^n_R(s) = \int_{s \wedge T_R}^{t \wedge T_R} b(k_n(r), X^n(k_n(r))) \, dr + \int_{s \wedge T_R}^{t \wedge T_R} \sigma_W(k_n(r), X^n(k_n(r))) \, dW(r) + \int_{s \wedge T_R}^{t \wedge T_R} \sigma_H(k_n(r), X^n(k_n(r))) \, dB^H(r).
\]

Therefore

\[
\mathbb{E} \left[ |X^n_R(t) - X^n_R(s)|^{2N} \right] \leq C_N \left\{ \mathbb{E} \left[ \left| \int_{s \wedge T_R}^{t \wedge T_R} b(k_n(r), X^n(k_n(r))) \, dr \right|^{2N} \right] + \mathbb{E} \left[ \left| \int_{s \wedge T_R}^{t \wedge T_R} \sigma_W(k_n(r), X^n(k_n(r))) \, dW(r) \right|^{2N} \right] + \mathbb{E} \left[ \left| \int_{s \wedge T_R}^{t \wedge T_R} \sigma_H(k_n(r), X^n(k_n(r))) \, dB^H(r) \right|^{2N} \right] \}
= C_N (J_1 + J_2 + J_3).
\]

Applying Hölder’s inequality, the growth assumption (H.1.1) and (4.2), we have

\[
J_1 \leq \mathbb{E} \left[ \left( \int_{s \wedge T_R}^{t \wedge T_R} |b(k_n(r) \wedge T_R, X^n(k_n(r) \wedge T_R))| \, dr \right)^{2N} \right] \leq C_N (t - s)^{2N - 1} \int_{s \wedge T_R}^{t \wedge T_R} \mathbb{E} \left[ |b(k_n(r) \wedge T_R, X^n_R(k_n(r)))|^{2N} \right] \, dr \leq C_N (t - s)^{2N}.
\]

By the Hölder and Burkholder inequalities and using (4.2), we obtain

\[
J_2 \leq C_N (t - s)^{N - 1} \mathbb{E} \left[ \int_{s \wedge T_R}^{t \wedge T_R} |\sigma_W(k_n(r), X^n(k_n(r)))|^{2N} \, dr \right] \leq C_N (t - s)^{N - 1} \mathbb{E} \left[ \int_{s \wedge T_R}^{t \wedge T_R} |\sigma_W(k_n(r) \wedge T_R, X^n_R(k_n(r)))|^{2N} \, dr \right] \leq C_N (t - s)^N.
\]

Let us note that we obtain from (2.3) and the Hölder inequality

\[
\left( \int_{s \wedge T_R}^{t \wedge T_R} f(u) \, dB^H(u) \right)^{2N} \leq C_N R^{2N(t - s)^{2N(1 - \alpha) + 2\alpha - 1}} \int_{s \wedge T_R}^{t \wedge T_R} \frac{\|f(r)\|_{\alpha}^N}{(r - s)^{2\alpha}} \, dr.
\]
Combining this estimate and (4.4) we obtain
\[ J_3 \leq CN R^{2N} (t-s)^{2N(1-\alpha)+2\alpha-1} \mathbb{E} \left[ \int_s^t \| \sigma_H(k_n(r) \wedge T_R, X^n_R(k_n(r))) \|_{2^\alpha}^{2N} \frac{dr}{(r-s)^{2\alpha}} \right]. \]

Using the Hölder inequality, assumption (H.2) and (4.2), we arrive at
\[ J_3 \leq CN R^{2N} (t-s)^{2N(1-\alpha)+2\alpha-1} \mathbb{E} \left[ \int_s^t \frac{1 + \| X^n_R(k_n(r)) \|_{2^\alpha}^{2N}}{(r-s)^{2\alpha}} \, dr \right] \]
\[ \leq CN (t-s)^N. \]

All these estimates allow us to obtain
\[ \mathbb{E} \left[ |X^n_R(t) - X^n_R(s)|^{2N} \right] \leq C_{N,R} |t-s|^N. \]

The proof of Lemma 8 is then completed. \( \square \)

Now we are able to give the convergence result.

**Theorem 9** Assume that \( \sigma_W \) and \( b \) are continuous satisfying the linear growth condition. Suppose moreover that \( \sigma_H \) satisfies assumption (H.2) and that for Eq. (1.1) the pathwise uniqueness holds. Then Euler’s approximations \( X^n(t) \) converge to a process \( X(t) \) in probability, uniformly in \( t \) in \([0, T]\). Furthermore \( X(t) \) is the unique strong solution of Eq. (1.1).

**Proof** Fix \( \eta < 1/2 \). We have from (4.3) in Lemma 8 that \( X^n_R \) is weakly relatively compact in \( C^\eta_0 \) for every \( R \). We want to deduce from this the weak compactness in \( C^\eta_0 \) of \( X^n \). Clearly it suffices to show that
\[ \limsup_{j \to \infty} P \left( T_R \leq T \right) = 0. \]

This is a consequence of the fact that the random variable \( \| B^H \|_{1-\alpha, \infty, t} \) has moments of all orders (see Lemma 7.5 in [21]). We now take two subsequences \( X^l, X^m \) of the Euler’s approximations \( X^n \). Then obviously \( (X^l, X^m) \) is a tight family of processes in \( C^\eta_0 \times C^\eta_0 \). By Skorokhod’s embedding theorem there exist a probability space \( (\tilde{\Omega}, \mathcal{F}, \tilde{P}) \) and a sequence \( (\tilde{X}^{l,n}, \tilde{X}^{m,n}, \tilde{B}^n, \tilde{W}^n) \) with values in \( C^\eta_0 \) such that

1. The law of \( (\tilde{X}^{l,n}, \tilde{X}^{m,n}, \tilde{B}^n, \tilde{W}^n) \) and \( (X^l, X^m, B^H, W) \) coincides for every \( n \in \mathbb{N} \).
2. There exists a subsequence \( (\tilde{X}^{l(j)}, \tilde{X}^{m(j)}, \tilde{B}^{n(j)}, \tilde{W}^{n(j)}) \) converging in \( C^\eta_0 \) to \( (\hat{X}, \hat{Y}, \hat{B}, \hat{W}) \) uniformly in \( t \), \( \tilde{P} \) a.s., that is
\[ \lim_{j \to \infty} \left( \| \tilde{X}^{m(j)} - \hat{X} \|_{\eta} + \| \tilde{X}^{l(j)} - \hat{Y} \|_{\eta} + \| \tilde{B}^{n(j)} - \hat{B} \|_{\eta} + \| \tilde{W}^{n(j)} - \hat{W} \|_{\eta} \right) = 0. \]
We obtain from Lemma 3.1 in Gyöngy and Krylov [10] and the convergence of integrals with respect to fBms (5.7) in Guerra and Nualart [9] that

\[
\lim_{j \to \infty} \int_0^t b(k_{l(j)}(s), \tilde{X}^{l(j)}(k_{l(j)}(s))) \, ds = \int_0^t b(s, \hat{X}(s)) \, ds
\]

\[
\lim_{j \to \infty} \int_0^t \sigma_W(k_{l(j)}(s), \tilde{X}^{l(j)}(k_{l(j)}(s))) \, d\hat{W}^{n(j)}(s) = \int_0^t \sigma_W(s, \hat{X}(s)) \, d\hat{W}(s)
\]

\[
\lim_{j \to \infty} \int_0^t \sigma_H(k_{l(j)}(s), \tilde{X}^{l(j)}(k_{l(j)}(s))) \, d\hat{B}^{n(j)}(s) = \int_0^t \sigma_H(s, \hat{X}(s)) \, d\hat{B}(s),
\]

and

\[
\lim_{j \to \infty} \int_0^t b(k_{m(j)}(s), \tilde{X}^{m(j)}(k_{m(j)}(s))) \, ds = \int_0^t b(s, \hat{Y}(s)) \, ds
\]

\[
\lim_{j \to \infty} \int_0^t \sigma_W(k_{m(j)}(s), \tilde{X}^{m(j)}(k_{m(j)}(s))) \, d\hat{W}^{n(j)}(s) = \int_0^t \sigma_W(s, \hat{Y}(s)) \, d\hat{W}(s)
\]

\[
\lim_{j \to \infty} \int_0^t \sigma_H(k_{m(j)}(s), \tilde{X}^{m(j)}(k_{m(j)}(s))) \, d\hat{B}^{n(j)}(s) = \int_0^t \sigma_H(s, \hat{Y}(s)) \, d\hat{B}(s),
\]

in probability, and uniformly in \( t \in [0, T] \). Therefore, the processes \( \hat{X}, \hat{Y} \) satisfy the same SDE (1.1), on \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\), with the driving noises \( \hat{W}, \hat{B} \) and the initial condition \( x_0 \) on the time interval \([0, \hat{T}_R]\) with

\[
\hat{T}_R := \inf \left\{ t \geq 0 : \| \hat{B} \|_{1-\alpha, \infty, t} \geq R \right\} \wedge T, \quad R \in \mathbb{N}.
\]

Using Remark 3.1 with \( \hat{B} \) instead \( B^H \) we conclude that \( \hat{X}, \hat{Y} \) satisfy the same SDE (1.1) on \([0, T]\). Then by pathwise uniqueness, we conclude that \( \hat{X}(t) = \hat{Y}(t) \) for all \( t \in [0, T] \) \( \hat{P} \) a.s. Hence, by applying Lemma 13 in “Appendix” we obtain the convergence of Euler’s approximations \( X^n(t) \) to a process \( X(t) \) in probability, uniformly in \( t \) in \([0, T]\). Therefore, \( \{X(t), t \in [0, T]\} \) satisfy Eq. (1.1).

As a consequence we obtain the following existence result.

**Theorem 10** Assume that \( b, \sigma_W \) and \( \sigma_H \) satisfy hypotheses (H.1) – (H.2). If \( 1 - H < \alpha < \min (\beta/2, 1) \), then Eq. (1.1) has a unique strong solution.

**Remark 4.1** It should be noted that when the Hurst parameter \( H \leq \frac{1}{2} \) a different approach is used to construct the integral with respect to the fBm and then many estimates we have used in the paper fail.

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Appendix

In this appendix, we recall some results which play a great role in this work. We also show a technical lemma that has been used in the proof of pathwise uniqueness. We begin with some a priori estimates from the paper of Nualart and Rascanu [21].

**Proposition 11** We have

(i) \( \left\| \int_0^t f(s) \, ds \right\|_{\alpha,t} \leq C \int_0^t \frac{|f(s)|}{(t-s)^{\alpha}} \, ds. \)

(ii) \( \left\| \int_0^t f(s) \, dB^H(s) \right\|_{\alpha,t} \leq C \left\| B^H \right\|_{1-\alpha,\infty,t} \int_0^t \left( (t-s)^{-2\alpha} + s^{-\alpha} \right) \| f \|_{\alpha,s} \, ds. \)

Moreover, under the linear growth assumption, we have from Nualart and Rascanu [21], the following

**Proposition 12** Assume \((H.1)\) and \((H.2)\). The following estimates hold

(j) \( \left\| \int_0^t b(s, f(s)) \, ds \right\|_{\alpha,t} \leq C \left( \int_0^t \frac{|f(s)|}{(t-s)^{\alpha}} \, ds + 1 \right). \)

(jj) \( \left\| \int_0^t \sigma_H(s, f(s)) \, dB^H(s) \right\|_{\alpha,t} \leq C \left\| B^H \right\|_{1-\alpha,\infty,t} \int_0^t \left( (t-s)^{-2\alpha} + s^{-\alpha} \right) \left( 1 + \| f \|_{\alpha,s} \right) \, ds. \)

We recall the following characterization of the convergence in probability in terms of weak convergence, see Gyöngy and Krylov [10].

**Lemma 13** Let \((Z_n)_{n \in \mathbb{N}}\) be a sequence of random elements in a Polish space \((\mathcal{E}, d)\) equipped with the Borel \(\sigma\)-algebra. Then \((Z_n)_{n \in \mathbb{N}}\) converges in probability to an \(\mathcal{E}\)-valued random element if and only if for every pair of subsequences \((Z_{m})_{m \in \mathbb{N}}\) and \((Z_{k})_{k \in \mathbb{N}}\) there exists a subsequence \((Z_{m(p)}, Z_{k(p)})_{p \in \mathbb{N}}\) converging weakly to a random element \(v\) supported on the diagonal \(\{(x, y) \in \mathcal{E} \times \mathcal{E} : x = y\}\).

Finally, let us give a version of Bihari’s lemma.

**Lemma 14** Fix \(1/2 < \alpha < 1, a, b \geq 0\). Let \(f : [0, \infty) \rightarrow [0, \infty)\) be a continuous function such that

\[ f(t) \leq a + bt^{\alpha} \int_0^t (t-s)^{-\alpha} s^{-q} \varphi (f(s)) \, ds, \]

where \(\varphi\) is a concave increasing function from \(\mathbb{R}_+\) to \(\mathbb{R}_+\) such that \(\varphi(0) = 0, \varphi(u) > 0\) for \(u > 0\) and satisfying (2.1) for some \(q > 1\). Then for any \(1 < p < 2\) such that \(\alpha < 1/p\) and \(q > 1\) with \(1/p + 1/q = 1\) we have

\[ f(t) \leq \left[ F^{-1} \left( F(2^{q-1}a^q) + 2^{q-1}b^q C_{\alpha,p}^q t^{q((1/p)-\alpha)+1} \right) \right]^{1/q}, \]

for all \(t \in [0, T]\) such that

\[ F(2^{q-1}a^q) + 2^{q-1}b^q C_{\alpha,p}^q t^{q((1/p)-\alpha)+1} \in \text{Dom}(F^{-1}), \]
where
\[ F(x) = \int_1^x \frac{du}{\varrho^q(u^{1/q})}, \quad \text{for } x \geq 0, \]
and \( F^{-1} \) is the inverse function of \( F \). In particular, if moreover, \( a = 0 \) then \( f(t) = 0 \) for all \( 0 < t < T \).

**Proof** Let \( 1 < p < 2 \) be such that \( \alpha < 1/p \). Using the Hölder inequality we obtain
\[ f(t) \leq a + bt^\alpha \left( \int_0^t (t-s)^{-p\alpha}s^{-p\alpha} \, ds \right)^{1/p} \left( \int_0^t \varrho^q(f(s)) \, ds \right)^{1/q}. \]

For the first integral, using \( s = tu \), we have
\[ \int_0^t (t-s)^{-p\alpha}s^{-p\alpha} \, ds = t^{1-2p\alpha} \int_0^1 (1-u)^{-p\alpha}u^{-p\alpha} \, du = C_{\alpha,p} t^{1-2p\alpha} \]
where \( C_{\alpha,p} = B(1-p\alpha, 1-p\alpha) \) is the beta function. It follows that
\[ f(t) \leq a + b C_{\alpha,p}^{1/p} t^{(1/p)-\alpha} \left( \int_0^t \varrho^q(f(s)) \, ds \right)^{1/q}. \]

This yields
\[ f^q(t) \leq 2^{q-1} a^q + 2^{q-1} b^q C_{\alpha,p}^{q/p} t^{q((1/p)-\alpha)} \left( \int_0^t \varrho^q(f(s)) \, ds \right). \]

Then it follows from Bihari’s Lemma, see [4], that
\[ f(t) \leq \left[ F^{-1} \left( F(2^{q-1} a^q) + 2^{q-1} b^q C_{\alpha,p}^{q/p} t^{q((1/p)-\alpha)+1} \right) \right]^{1/q}, \]
for all \( t \in [0, T] \) such that
\[ F(2^{q-1} a^q) + 2^{q-1} b^q C_{\alpha,p}^{q/p} t^{q((1/p)-\alpha)+1} \in \text{Dom}(F^{-1}). \]

Now, it is simple to see from Eq. (2.1) that \( \lim_{x \to 0} F(x) = \infty \); therefore \( \lim_{x \to \infty} F^{-1}(x) = 0 \) which implies that \( f(t) = 0 \) when \( a = 0 \) for \( t \in [0, T] \).

\[ \square \]

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