NON-CUTOFF BOLTZMANN EQUATION WITH POLYNOMIAL DECAY PERTURBATION

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Abstract. The Boltzmann equation without an angular cutoff is considered when the initial data is a small perturbation of a global Maxwellian with an algebraic decay in the velocity variable. A well-posedness theory in the perturbative framework is obtained for both mild and strong angular singularities by combining three ingredients: the moment propagation, the spectral gap of the linearized operator, and the regularizing effect of the linearized operator when the initial data is in a Sobolev space with a negative index. A carefully designed pseudo-differential operator plays an central role in capturing the regularizing effect. Moreover, some intrinsic symmetry with respect to the collision operator and an intrinsic functional in the coercivity estimate are essentially used in the commutator estimates for the collision operator with velocity weights.

key words: moment propagation, coercivity, spectral gap, commutator estimates, regularizing effect.

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1. INTRODUCTION

This paper aims to present a complete well-posedness theory to the Boltzmann equation without an angular cutoff when the initial perturbation of a global equilibrium state is small and decays only algebraically in the velocity variable. Precisely, we consider the Cauchy problem for the non-cutoff Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q(F,F),$$

$$F|_{t=0} = F_0 \geq 0,$$  \hspace{1cm} (1.1)

where $(x,v) \in T^3 \times \mathbb{R}^3$ and the collision operator is given by

$$Q(F,F) = \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta) |v - v_\ast|^\gamma (F' F_\ast - F_\ast F') \, d\sigma \, dv_\ast.$$ \hspace{1cm} (1.2)

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Our analysis applies to the non-angular cutoff cross-section for hard potential, that is, when $b$ and $\gamma$ satisfy
\[ 0 < \gamma \leq 1, \quad b(\cos \theta) \sin \theta \sim \frac{1}{\theta^{1+2s}}, \quad 0 < s < 1. \] (1.3)
In the perturbative framework, let $\mu = (2\pi)^{-3/2}e^{-|v|^2/2}$ be the normalized equilibrium and $f$ be the perturbation by writing
\[ F = \mu + f. \]
Equation (1.1) becomes
\begin{align*}
\partial_t f + v \cdot \nabla_x f &= Q(\mu, f) + Q(f, \mu) + Q(f, f) = Lf + Q(f, f), \\
\left. f \right|_{t=0} &= f_0(x, v).
\end{align*}
To study equation (1.4) when the initial data only has an algebraic decay in the velocity variable, we first point out its main difference from the classical decomposition $F = \mu + \sqrt{\mu}f$ that implies a Gaussian tail in the perturbation. First of all, with the Gaussian tail decomposition, the corresponding linearized operator given by
\[ L^{(\mu)} f = \frac{1}{\sqrt{\mu}} (Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)) \]
\[ = \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta) |v - v_*|^\gamma \left( \sqrt{\mu} \partial^{\nu}_x f' + \sqrt{\mu} \partial^{\nu}_* f - \sqrt{\mu} \partial^{\nu}_* f \right) \, d\sigma \, dv_* \]
is self-adjoint and has the null space
\[ \text{Null} (L^{(\mu)}) = \text{Span} \left\{ \sqrt{\mu}, \sqrt{\mu} v, \sqrt{\mu} |v|^2 \right\}. \]
For this self-adjoint linear operator, one has the following strong coercivity estimate that implies the gain of both regularity and moment of order $s$ in the velocity variable (cf. [4,9,16]):
\[ \left\langle f, L^{(\mu)} f \right\rangle_{L^2_\sigma} \leq -c_0 \left( \| f \|_{H^{s/2}_x}^2 + \| f \|_{L^2_{x+v}}^2 \right), \quad f \in \text{Null}(L^{(\mu)})^\perp. \]
This coercivity property is essentially used in the well-posedness theory for the non-cutoff Boltzmann equation with Gaussian tail, cf. [4,9,16]. However, if we only assume an algebraic decay in the perturbation by writing $F = \mu + f$, then the corresponding linearized operator given by
\[ Lf = Q(\mu, f) + Q(f, \mu) \]
\[ = \int_{\mathbb{R}^3} \int_{S^2} (\partial^{\nu}_x f' + \partial^{\nu}_* f - \mu \partial^{\nu}_* f) b(\cos \theta) |v - v_*|^\gamma \, d\sigma \, dv_* , \]
is no longer self-adjoint. In addition, the coercivity only gains regularity rather than moments. Precisely, cf. [6], one has
\[ \langle Q(\mu, f), f \rangle = -c_1 J_1^T (f) + \text{mod} \left( \| f \|_{L^2_{x+v}}^2 \right) \leq -c_0 \| f \|_{H^{s/2}_x}^2 + C \| f \|_{L^2_{x+v}}^2, \]
where
\[ J_1^T (f) = \int_{\mathbb{R}^6} \int_{S^2} b(\cos \theta) |v - v_*|^\gamma \mu_* (f(v')) - f(v))^2 \, d\sigma \, dv_* \, dv. \]
Apparently, the linearized operator $L$ can no longer be used to control any moment gain. Therefore, in the commutator estimates for the collision operator with either weights or some pseudo-differential operators, the estimation becomes more subtle, especially in the strong singularity setting. For this, we will show that the functional $J_1^T (f)$ plays an important role. In fact, this function corresponds to the first component in the isotropic norm defined in [3] in the setting with a Gaussian tail. Note that even though $J_1^T (f)$ has a lower bound as $\| f \|_{H^{s/2}_x}^2$, its upper bound in the Sobolev norm can only be shown as $\| f \|_{H^{s/2}_x}^2$ because of the
Laplacian operator on a sphere. Therefore, in the commutator estimates for the collision operator and some pseudo-differential operators given in Section 5, we will keep the precise form of $J_{\gamma}^1$ rather than using the usual weighted Sobolev norms. On the other hand, we would like to mention that for mild singularity, that is, when $0 < s < 1/2$, using the lower bound in weighted Sobolev norm as $\|f\|_{H^{s/2}}^2$ is sufficient.

Now let us review some works related to our paper. First of all, many of the well-posedness theories on the Boltzmann equation established so far are based on Grad’s angular cutoff assumption. For this, there is the classical work on the renormalized solutions developed by DiPerna-Lions \[15\] for large initial data with finite mass, energy and entropy. In the perturbative framework, the pioneering work was obtained by Ukai \[31\] for $L^\infty$-solutions by using the spectrum of the linearized operator and a bootstrapping argument following the local existence result by Grad, \[19\]. And an $L^2$-framework by using the energy method and micro-macro decompositions was established in \[18,25,26\].

Without Grad’s angular cutoff assumption, the spectrum of the linearized operator around a global Maxwellian was studied by Pao \[30\] in 1970s. Later, the existences of weak and analytic (Gevrey) solutions were obtained by Arkeryd and Ukai in 1980s respectively, cf. \[11,32\].

In 1990s, P.-L. Lions used the entropy dissipation to show the gain of regularity:

$$\|\sqrt{F}\|_{H^s(|v|<M)}^2 \leq C_{M} \|F\|^\theta_{L^1}(\|F\|_{L^1} + D(F))^{1-\theta},$$

where

$$D(F) = -\int_{\mathbb{R}^3} Q(F,F) \log F dv,$$

for some constants $0 < \delta \leq \frac{1-s}{1+s}$ and $M > 0$ and $0 < \theta < 1$. Around the same time, Desvillettes firstly proved the regularization of solutions to some simplified kinetic models.

In early 2000s, the regularization induced by the grazing collisions was analyzed by using the entropy production and it was developed by many people, including Alexandre, Bouchut, Desvillettes, Golse, P.-L. Lions, Mouhot, Villani, Wennberg, cf. \[33\] and the references therein. In particular, some elegant formula were obtained in the work by Alexandre-Desvillettes-Villani-Wennberg \[1\] such as the cancellation lemma. In addition, it was proved that

$$\|\sqrt{F}\|_{H^s(|v|<M)}^2 \leq C_{F,M}(\|F\|_{L^2}^2 + D(F)),$$

which was later finalized in \[6\] Corollary 2.4 in the precise form as

$$\|\sqrt{F}\|_{H^s(\mathbb{R}^3)}^2 \leq C_{F}(\|F\|_{L^1}^2 + D(F)).$$

For the well-posedness theories of the Boltzmann equation without an angular cutoff, the existence of renormalized solutions was obtained by Alexandre-Villani in \[10\]. In 2011-12, two different approaches were introduced by Gressman-Strain \[16\] and Alexandre-Morimoto-Ukai-Xu-Yang \[3,9\] independently to obtain the well-posedness theory for small perturbations of a global equilibrium state with Gaussian tails. The regularizing effect was also obtained in our previous works, cf. \[4,22\]. Note that in the setting with a Gaussian tail decay, the well-posedness theories hold for both cases when the space variable is in torus and the whole space, because the self-adjoint linearized operator yields both gain of regularity and moments. However, it remains open to establish $L^\infty$-solutions to the Boltzmann equation without an angular cutoff in an analogous to Ukai’s result on the angular cutoff Boltzmann equation.

When the perturbation has only an algebraic decay in the velocity variable, there is a recent important progress made by Gualdani-Mischler-Mouhot in \[17\] on the spectral gap of the linearized operator around a global Maxwellian. Their result leads to the well-posedness theory on various kinetic equations with algebraic-decay perturbations when the space variable is in a torus, an example of which is the cutoff Boltzmann equation. In fact, the spectral gap in both the velocity variable in $\mathbb{R}^3$ and the space variable in
a torus was obtained in [29] under the cutoff assumption by analyzing the mixing between the convection and the coercivity in the velocity variable of the linearized operator.

Without an angular cutoff, a well-posedness theory was recently obtained in [21] for the case of the mild angular singularity where $0 < s < 1/2$. The main result of our paper gives a different approach to establish well-posedness that applies for both mild and strong angular singularity. There is also a recent work [20] that gives a well-posedness theory using yet a third method. We would like to mention that the spectral well-posedness that applies for both mild and strong angular singularity. There is also a recent work [20] where $0 < \gamma < \gamma_1$ and the coercivity in the velocity variable of the linearized operator.

To define the function spaces considered in this paper, we introduce the linear operator

$$\mathcal{L} = -v \cdot \nabla_x + L.$$  

Then the linearized equation for (1.4) is

$$\partial_t h = \mathcal{L} h, \quad h|_{t=0} = h_0(x, v).$$

Let $S_L$ be the associated semigroup on $L^2(dv; H^2(dx))$. Denote $W$ as the weight function such that

$$W(v) = \langle v \rangle^{m_0}, \quad m_0 > \max \{4s, 1\}. \quad (1.6)$$

Define a function space

$$Y_1 = \{ f \in L^2(dx \ dv) \big| W^{l-|\alpha|} \partial_x^\alpha f \in L^2(dx \ dv), \ |\alpha| = 0, 1, 2 \big\}, \quad l > 2.$$

For some $l_0 \in \mathbb{N}$ to be specified later, as in [17], define a norm to cope with the spectral gap by

$$\|h\| = \left( \|h\|_1^2 + A \int_0^\infty \|S_L(\tau)h\|_{L^2(W^{l_0}dv; H^2(dx))}^2 \, d\tau \right)^{1/2},$$

with $A$ being a large constant to be determined later. Here $h$ satisfies

$$\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} h \mu \, dv \, dx = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} v_i h \, mu \, dv \, dx = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |v|^2 h \mu \, dv \, dx = 0, \quad i = 1, 2, 3. \quad (1.7)$$

Note that the integral in the definition of the norm $\| \cdot \|$ is well-defined and equivalent to the $\| \cdot \|_{Y_1}$-norm if $\mathcal{L}$ has a spectral gap.

With these notations, we state the main theorems of this paper. The first one is the local existence result.

**Theorem 1.1.** Suppose $0 < s < 1$ and $0 < \gamma \leq 1$. Then there exists a sufficiently small constant $\epsilon_0 > 0$ such that if $f_0 \in Y_1$ with

$$\|f_0\|_{Y_1} \leq \epsilon_0,$$
then there exist constants $T$, $\epsilon_1 > 0$ such that the Cauchy problem (1.4) admits a unique solution

$$f \in L^\infty([0,T];Y_l)$$

satisfying

$$\|f\|_{L^\infty([0,T];Y_l)} \leq \epsilon_1,$$

(1.8)

and

$$A_l(f) = \int_0^T \sum_\alpha \|W^{l-|\alpha|} \partial^\alpha_x f(\tau)\|^2_{L^2(dx;H^s_{l/2}(dv))} d\tau < \epsilon_1^2.$$

(1.9)

Note that in the setting of this paper, the smallness assumption on the perturbation is needed even for local existence. The next result is about the global existence and large time behaviour of the solution.

**Theorem 1.2.** Suppose $0 < s < 1$ and $0 < \gamma \leq 1$. For some $l$ being suitably large and $\epsilon_0 > 0$ small enough, if $F^{in} = \mu + f^{in} \geq 0$ satisfies

$$\|f^{in}\|_{Y_l} < \epsilon_0,$$

$$\int_T^3 \int_{\mathbb{R}^3} f^{in} \phi(v) dv dx = 0$$

for any $\phi \in \text{Null}(L) = \text{Span} \{\mu, \mu v, \mu |v|^2\}$, then the non-cutoff Boltzmann equation has a unique non-negative solution $F \in L^\infty([0,\infty),Y_l)$ such that

$$\|F - \mu\|_{Y_l} \leq c e^{-\lambda t} \|f^{in}\|_{Y_l},$$

holds for some constants $\lambda > 0$ and $c > 0$.

**Remark 1.1.** The decay rate $\lambda$ of the perturbation can be made more precise as in [17,21]. In particular, it can be chosen as any constant less than the spectral gap of the linearized operator $L$. Since the idea is similar as in [17,21], we will not elaborate on it in this paper.

The rest of the paper is arranged as follows. In the next section, we will give some preliminary estimates for later use. Bounds related to the collision operator will be given in the Section 3. The spectral gap without an angular cutoff in the algebraic decay function space is given in Section 4. In Section 5, we prove a precise regularization estimate of the linearized collision operator with initial data in a Sobolev space with a negative index. The closed-form energy estimate will then be given in Section 6 and the proof of local and global existence with uniqueness and non-negativity will be given in the last section. Finally, in the Appendix we give some basic estimates about some differential operators and estimates related to the functional $J^\gamma_1(f)$.

2. SOME USEFUL ESTIMATES

In this section, we list some useful estimates that are needed for later estimation. For this, we introduce the notation

$$\|g\|_{H^\beta_{l}(dv)} = \left\|\left\langle v \right\rangle^\beta g \right\|_{H^\beta_{l}(dv)}, \quad \beta \in \mathbb{R}.$$ 

The first proposition is about the equivalence of weight and differential operators up to commutation.

**Proposition 2.1** (22). Suppose $\alpha, \theta > 0$. Then there exists a generic constant $C$ independent of $f$ such that

$$\frac{1}{C} \left\| \langle D_v \rangle^\theta \langle v \rangle^\alpha f \right\|_{L^2(dv)} \leq \left\| \langle v \rangle^\alpha \langle D_v \rangle^\theta f \right\|_{L^2(dv)} \leq C \left\| \langle D_v \rangle^\theta \langle v \rangle^\alpha f \right\|_{L^2(dv)},$$

that is, the above two norms are equivalent.

The second proposition is the trilinear estimate for hard potential with non-cutoff cross section.
Proposition 2.2 ([27]). Denote $a^+ = \max\{a, 0\}$. Then the bilinear operator $Q$ satisfies
\[
\left|\int_{\mathbb{R}^3} Q(f, g)h \, dv\right| \leq C \left( \|f\|_{L^1_{\gamma/2} + \gamma/2} + \|f\|_{L^2} \right) \left\| g \right\|_{H^{\gamma/2 + \sigma} + \gamma/2 + \gamma/2} \left\| h \right\|_{H^{-\sigma, \gamma/2 + m}}
\]
for any $\sigma \in [\min\{s - 1, -s\}, s]$, $m, \gamma, s \geq 0$. Here, $f, g, h$ are any functions so that the corresponding norms are well-defined. The constant $C$ is independent of $f, g, h$.

In later analysis, we often use two types of change of variables given in Proposition 2.3 ([1]). Suppose $f$ is smooth enough such that the integrals below are well-defined. Then
(a) (Regular change of variables)
\[
\int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)|v - v_s|^\gamma f(v') \, d\sigma dv = \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta) \frac{1}{\cos^{3+\gamma}(\theta/2)} |v - v_s|^\gamma f(v) \, d\sigma dv.
\]
(b) (Singular change of variables)
\[
\int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)|v - v_s|^\gamma f(v') \, d\sigma dv_s = \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta) \frac{1}{\sin^{3+\gamma}(\theta/2)} |v - v_s|^\gamma f(v_s) \, d\sigma dv_s.
\]

The proof of the main theorems relies on estimates of the solution in some weighted Sobolev spaces. For this, we need to consider the difference of the weight before and after collision, in particular, to seek for the cancellation of the angular singularity. Additional symmetry is also needed for strong singularity. To this end, we establish a technical lemma about the difference of the weights that is essential for the analysis. First, note that
\[
|v'|^2 = |v|^2 \cos^2 \frac{\theta}{2} + |v_s|^2 \sin^2 \frac{\theta}{2} + 2 \cos \frac{\theta}{2} \sin \frac{\theta}{2} |v - v_s| v \cdot \omega,
\]
and
\[
|v|^2 = |v_s|^2 \cos^2 \frac{\theta}{2} + |v_s|^2 \sin^2 \frac{\theta}{2} + 2 \cos \frac{\theta}{2} \sin \frac{\theta}{2} |v - v_s| v \cdot \omega,
\]
where $\omega = \frac{\sigma - (s-k)k}{|\sigma - (s-k)k|}$ with $k = \frac{v - v_s}{|v - v_s|}$. Here $\omega$ satisfies that $\omega \perp (v - v_s)$.

Remark 2.1. Since $\omega \perp (v - v_s)$, we have $v \cdot \omega = v_s \cdot \omega$. Hence, we have the freedom to choose when to use $v \cdot \omega$ or $v_s \cdot \omega$ in later estimates.

Lemma 2.1. Suppose $k > 3$ and $(v, v_s), (v', v'_s)$ are the velocity pairs before and after the collision. Let $\omega$ be the same vector as in (2.1) and (2.2). Then,
\[
\langle v' \rangle^{2k} - \langle v \rangle^{2k} = 2k \langle v \rangle^{2k-2} |v - v_s| (v \cdot \omega) \cos^{2k-1} \frac{\theta}{2} \sin \frac{\theta}{2} + \langle v_s \rangle^{2k} \sin^{2k} \frac{\theta}{2} + R_1 + R_2 + R_3,
\]
where there exists a constant $C_k$ only depending on $k$ such that
\[
|R_1| \leq C_k \langle v \rangle^{2k-2} \langle v_s \rangle^{2k-3} \sin^2 \frac{\theta}{2}, \quad |R_2| \leq C_k \langle v \rangle^{2k-2} \langle v_s \rangle^{2k} \sin^2 \frac{\theta}{2}, \quad |R_3| \leq C_k \langle v \rangle^{2k-4} \langle v_s \rangle^{4} \sin^2 \frac{\theta}{2}.
\]

Proof. By the Taylor expansion and (2.2), we have
\[
\langle v' \rangle^{2k} - \langle v \rangle^{2k} = \cos^{2k} \frac{\theta}{2} = k \left( \langle v \rangle^{2} \cos^{2} \frac{\theta}{2} \right)^{k-1} \langle v_s \rangle^{2} \sin^{2} \frac{\theta}{2} + 2k \left( \langle v \rangle^{2} \cos^{2} \frac{\theta}{2} \right)^{k-1} \cos \frac{\theta}{2} \sin \frac{\theta}{2} |v - v_s| (v \cdot \omega)
\]
\[
+ k(k - 1) \int_0^1 (1 - t) \left( \langle v \rangle^{2} \cos^{2} \frac{\theta}{2} + t \left( \langle v_s \rangle^{2} \sin^{2} \frac{\theta}{2} + 2 \cos \frac{\theta}{2} \sin \frac{\theta}{2} |v - v_s| (v \cdot \omega) \right) \right)^{k-2} \, dt
\]
\[
\triangleq D_1 + D_2 + D_3,
\]
\[
\triangleq D_1 + D_2 + D_3.
\]
Note that $\mathcal{D}_2$ gives the first term on the right hand side of (2.3) and $\mathcal{D}_1$ is part of $\mathcal{R}_1$. To estimate $\mathcal{D}_3$, we use the mean value theorem for the integrand in $\mathcal{D}_3$ such that
\[
\left(\langle v \rangle^2 \cos^2 \frac{\theta}{2} + t \langle \langle v \rangle^2 \sin^2 \frac{\theta}{2} + 2 \cos \frac{\theta}{2} \sin \frac{\theta}{2} |v - v_\ast| |v \cdot \omega| \right)^{k-2}
\]
\[= t^{k-2} \left( \langle v \rangle \sin \frac{\theta}{2} \right)^{2k-4} + (k - 2) \int_0^1 \left( t \langle \langle v \rangle^2 \sin^2 \frac{\theta}{2} + \tau \langle \langle v \rangle^2 \cos^2 \frac{\theta}{2} + 2 t \cos \frac{\theta}{2} \sin \frac{\theta}{2} |v - v_\ast| |v \cdot \omega| \right)^{k-3} d\tau
\]
\[\times \left( \langle \langle v \rangle^2 \cos^2 \frac{\theta}{2} + 2 t \cos \frac{\theta}{2} \sin \frac{\theta}{2} |v - v_\ast| |v \cdot \omega| \right)
\]
\[\Delta t^{k-2} \left( \langle v \rangle \sin \frac{\theta}{2} \right)^{2k-4} + \mathcal{D}_{3,1}.
\]  \hspace{1cm} (2.6)

By a direct estimate on $\mathcal{D}_{3,1}$, we have
\[
\mathcal{D}_{3,1} \leq C_k \left( \langle \langle v \rangle^2 \right)^{2k-4} + \langle v \rangle \left( \langle v \rangle \sin \frac{\theta}{2} \right)^{2k-5}.
\]  \hspace{1cm} (2.7)

Denoting $\mathcal{h} = \langle v \rangle^2 \sin^2 \frac{\theta}{2} + 2 \cos \frac{\theta}{2} \sin \frac{\theta}{2} |v - v_\ast| \langle v \cdot \omega \rangle$ and applying the bound on $\mathcal{D}_{3,1}$ in $\mathcal{D}_3$, we have
\[
\mathcal{D}_3 = (k - 1) \left( \int_0^1 (1 - t) t^{k-2} \left( \langle v \rangle \sin \frac{\theta}{2} \right)^{2k-4} dt \right) \mathcal{h}^2
\]
\[+ k(k - 1) \left( \int_0^1 (1 - t) \mathcal{D}_{3,1} dt \right) \mathcal{h}^2
\]
\[= \left( \langle v \rangle \sin \frac{\theta}{2} \right)^{2k-4} \mathcal{h}^2 + k(k - 1) \left( \int_0^1 (1 - t) \mathcal{D}_{3,1} dt \right) \mathcal{h}^2
\]
\[\Delta \mathcal{D}_{3,2} + \mathcal{D}_{3,3}.
\]

When estimating $\mathcal{h}^2$ in $\mathcal{D}_{3,2}$, we use $v \cdot \omega$ in its second term and obtain
\[
\mathcal{h}^2 = \left( \langle v \rangle \sin \frac{\theta}{2} \right)^4 + \mathcal{h}_1
\]
with
\[
\mathcal{h}_1 \leq C \langle v \rangle \sin^2 \frac{\theta}{2} \left( \langle \langle v \rangle \rangle^3 + \langle v \rangle \langle \langle v \rangle \rangle^2 + \langle v \rangle^3 \sin \frac{\theta}{2} \right).
\]

Therefore,
\[
\mathcal{D}_{3,2} = \left( \langle v \rangle \sin \frac{\theta}{2} \right)^{2k-4} + \left( \langle v \rangle \sin \frac{\theta}{2} \right)^{2k-4} \mathcal{h}_1,
\]
where
\[
\left( \langle v \rangle \sin \frac{\theta}{2} \right)^{2k-4} \mathcal{h}_1 \leq C \langle v \rangle^4 \langle v \rangle \sin \frac{\theta}{2} \langle \langle v \rangle \rangle^{2k-4} + C \langle \langle v \rangle \rangle^2 \langle v \rangle \sin \frac{\theta}{2} \langle \langle v \rangle \rangle^{2k-2} + C \langle v \rangle \langle v \rangle \sin \frac{\theta}{2} \langle \langle v \rangle \rangle^{2k-1}
\]
\[\leq C \langle v \rangle \langle v \rangle^{2k-1} \sin^2 \frac{\theta}{2} + C \langle v \rangle^{2k-2} \langle v \rangle \langle v \rangle^2 \sin \frac{\theta}{2} + C \langle v \rangle^{2k-4} \langle v \rangle \sin \frac{\theta}{2} \langle \langle v \rangle \rangle^4 \sin^2 \frac{\theta}{2}.
\]

Hence, the second term of $\mathcal{D}_{3,2}$ contributes only to the remainder term $\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3$ in (2.3). Finally, when estimating the term $\mathcal{D}_{3,3}$, we replace $v \cdot \omega$ by $v_\ast \cdot \omega$ (see Remark 2.1). Then $\mathcal{h}$ is directly bounded by
\[
\mathcal{h}^2 \leq C \langle v \rangle^2 \left( \langle v \rangle \sin \frac{\theta}{2} \right)^2 + C \langle v \rangle^2 \left( \langle v \rangle \sin \frac{\theta}{2} \right)^2.
\]

Together with (2.7), we obtain the bound of $\mathcal{D}_{3,3}$ as
\[
|\mathcal{D}_{3,3}| \leq C_k \langle v \rangle \langle v \rangle^{2k-1} \sin^2 k-3 \frac{\theta}{2} + C_k \langle v \rangle^{2k-2} \langle v \rangle^2 \sin^2 \frac{\theta}{2} + C_k \langle v \rangle^{2k-4} \langle v \rangle \sin^2 \frac{\theta}{2}.
\]

In summary, we have
\[
\mathcal{D}_2 = 2k \langle v \rangle^{2k-2} |v - v_\ast| \langle v \cdot \omega \rangle \cos^2 k-1 \frac{\theta}{2} \sin \frac{\theta}{2}, \quad \mathcal{D}_1 + \mathcal{D}_3 = \langle v \rangle^{2k} \sin^2 k \frac{\theta}{2} + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3,
\]
which completes the proof of the lemma.  \hfill \Box

Next we recall a coercivity estimate obtained in [9].
Proposition 2.4 ([9]). Suppose $F$ satisfies
\[
F \geq 0, \quad \|F\|_{L^1} \geq D_0, \quad \|F\|_{L^1} + \|F\|_{L_\infty} \leq E_0.
\]
Then there exist two constants $c_0$ and $C$ such that
\[
\int_{\mathbb{R}^3} Q(F, f) f \, dv \leq -c_0 \|f\|_{H^\eta/2}^2 + C \|f\|_{L^2}^2.
\]

Proof. Firstly of all, for $f \in H^s(\mathbb{R}^3)$, we have
\[
\|(-\Delta)^{s/2} f\|_{L^2}^2 = c_{d,s} \int_{\mathbb{R}^{2d}} \frac{|f(y) - f(x)|^2}{|y - x|^{d+2s}} \, dy dx,
\]
with $c_{d,s} = \frac{4^s \Gamma(d/2+s)}{\pi^{d/2} \Gamma(-s)}$. Observe that
\[
|f^\pm(y) - f^\pm(x)| \leq |f(y) - f(x)|, \quad \text{for any} \quad (y, x) \in \mathbb{R}^3 \times \mathbb{R}^3.
\]
As a consequence, it readily follows from (2.9) that
\[
\|(-\Delta)^{s/2} f^\pm\|_{L^2}^2 \leq \|(-\Delta)^{s/2} f\|_{L^2}^2.
\]
This leads to the second inequality in (2.8). Furthermore,
\[
|f(y) - f(x)| = |(f^+(y) - f^+(x)) - (f^-(y) - f^-(x))| \leq |f^+(y) - f^+(x)| + |f^-(y) - f^-(x)|.
\]
Thus,
\[
\|(-\Delta)^{s/2} f^\pm\|_{L^2}^2 \leq 2\|(-\Delta)^{s/2} f^+\|_{L^2}^2 + 2\|(-\Delta)^{s/2} f^-\|_{L^2}^2,
\]
which, after using (2.9), gives the first inequality in (2.8). \qed

Lemma 2.2. For any $h \in H^s_0(\mathbb{R}^3)$ with $s \in (0, 1)$, let $h^\pm$ denote the positive and negative parts of $h$. Then it holds
\[
\frac{1}{2} \|h\|_{H^s_{+}}^2 \leq \sum_{g \in \{h^\pm\}} \|g\|_{H^s_{+}}^2 \leq 2 \|h\|_{H^s_{+}}^2.
\]

Proof. Let $h \in H^s_0(\mathbb{R}^3)$ with $s \in (0, 1)$, and let $h^\pm$ denote the positive and negative parts of $h$. Then there exist constants $C > 0$ and $c > 0$ depending only on the mass and energy of $\mu$ such that for every $\varepsilon \in (0, 1)$,
\[
\int_{\mathbb{R}^6} \int_{\mathbb{S}^2} b_2(\cos \theta) \mu(v_\alpha) \mu(v) \left(H(v') - H(v)\right)^2 \geq c \sum_{g \in \{h^\pm\}} \left\|\tilde{g}(\xi) \xi^4 \mathbb{1}\{\xi \geq \frac{1}{\varepsilon}\}\right\|^2_{L^2} - C\|\theta^2 b_2\|_{L^4} \|h\|_{L^2}^2.
\]

Proof. As in [4], Proposition 1, we expand the square in the above integrand and then apply Bobylev’s identity together with the Cauchy-Schwarz inequality to obtain
\[
\int_{\mathbb{R}^6} \int_{\mathbb{S}^2} b(\cos \theta) \mu(v_\alpha) \mu(v) \left(H(v') - H(v)\right)^2 = 2 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) \left(\mu(v_\alpha)|h|^2(v) - \mu^{1/2}(v_\alpha)\mu^{1/2}(v')h'h\right)
\]
\[
\geq 2 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) \sum_{g \in \{h^\pm\}} \left(\mu(v_\alpha)g^2(v) - \mu(v')g(v')g(v)\right)
\]
\[
= 2 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) \sum_{g \in \{h^\pm\}} \left(\widehat{\mu}(0)|\tilde{g}(\xi)|^2 - \widehat{\mu}(\xi^-)\tilde{g}(\xi)^+\tilde{g}(\xi)\right).
\]
By applying the Cauchy-Schwarz inequality to the second term in the above summation, we obtain
\[
\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) \mu(v_*) \mu(v) (H(v') - H(v))^2 \geq \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) \sum_{\xi \in \{h^2\}} (\tilde{\mu}(0) - \tilde{\mu}(\xi^-)) |\tilde{g}(\xi)|^2 \\
+ \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) \sum_{\xi \in \{h^2\}} (\tilde{\mu}(0) - \tilde{\mu}(\xi^-)) |\tilde{g}(\xi^+)|^2 \\
+ \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) \sum_{\xi \in \{h^2\}} \tilde{\mu}(0) (|\tilde{g}(\xi)|^2 - |\tilde{g}(\xi^+)|^2). 
\]

For the last term on the right side, we apply the cancellation lemma from \cite[Lemma 1]{1} to obtain
\[
\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) \tilde{\mu}(0) (|\tilde{g}(\xi)|^2 - |\tilde{g}(\xi^+)|^2) \geq -C\|\theta^2 b\|_{L^1} \|\mu\|_{L^1} \|g\|^2_{L^2}, \tag{2.10}
\]
for some generic constant \( C > 0 \). The first and second terms are both positive that can be treated similarly. Consider the second term by applying the change of variables \( \xi \to \xi^+ \) and the fact that \( \tilde{\mu}(\xi) \) is decreasing in \( |\xi| \) to obtain
\[
\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) (\tilde{\mu}(0) - \tilde{\mu}(\xi^-)) |\tilde{g}(\xi^+)|^2 \geq \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \frac{2d-1}{\cos^d(\theta)} (\tilde{\mu}(0) - \tilde{\mu}(\xi^-)) |\tilde{g}(\xi)|^2 \tag{2.11}
\]
where \( \cos \theta = \hat{\xi} \cdot \sigma \). Now, set \( b = b_2 \) which is supported in \( \{ |\sin \theta| \leq \varepsilon \} \). Then,
\[
\int_{\mathbb{S}^2} 2^{d-1} \frac{b(\cos 2\theta)}{\cos^d(\theta)} (\tilde{\mu}(0) - \tilde{\mu}(\xi^-)) \geq c \int_0^{2\varepsilon} \left| \frac{1}{\theta^{1+2s}} (1 - \mu_\omega(\theta)) \right| \xi^{2s} \geq c \int_0^{2\varepsilon} \left| \frac{1}{\theta^{1+2s}} (1 - \mu_\omega(\theta)) \right| \xi^{2s} 1_{\{|\xi| \geq \frac{1}{2}\}}.
\]
Here \( \mu_\omega(\theta) = e^{-\theta^2/2} \) is the radial profile of the Fourier transform of \( \mu \). Similar estimate holds for the second term. Thus, (2.10) and (2.11) give the result of the lemma. \( \square \)

For the cross section \( B(\cos \theta, |v - v'|) \) with an angular cutoff, we will use the notation \( Q^\pm \) defined as follows throughout the paper:
\[
Q^+(f,g) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(\cos \theta, |v - v'|) f'_* g \, d\sigma \, dv_*, \quad Q^-(f,g) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(\cos \theta, |v - v'|) f g \, d\sigma \, dv_*.
\]
Note that \( Q(f,g) = Q^+(f,g) - Q^-(f,g) \).

3. UPPER BOUNDS ON \( Q \)

In this section, we will derive some bounds on the collision operator in some weighted \( L^2 \)-norms. For simplicity of notations, we denote \( d\Pi = d\sigma \, dv_* \, dv \).

The first estimate is about a commutator on the collision operator with a weight function.

**Proposition 3.1.** Suppose \( 0 < s < 1 \) and \( k > \frac{9}{2} + \frac{3}{2} + 2s \). Then
\[
\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) |v - v_*|^\gamma \left( \langle v' \rangle^k - \langle v \rangle^k \right) f'_* g \, d\Pi \\
\leq \left( \int_{\mathbb{S}^2} b(\cos \theta) \sin^{k-\frac{3}{2} - \frac{9}{2}} \, d\sigma \right) \|g\|_{L^1} \|\langle v \rangle^k f\|_{L^2} \|h\|_{L^2} + C_k \|g\|_{L^1} \|\langle v \rangle^k f\|_{L^2} \|h\|_{L^2} \\
+ C_k \|f\|_{L^1} \|\langle v \rangle^k g\|_{L^2} \|h\|_{L^2} + C_k \|f\|_{L^1} \|\langle v \rangle^k g\|_{L^2} \|h\|_{L^2} \|h\|_{L^2} \|h\|_{L^2} \|h\|_{L^2} \|h\|_{L^2} \|h\|_{L^2} \|h\|_{L^2} \|h\|_{L^2} \|h\|_{L^2} \tag{3.1}
\]
The parameters \( s', \gamma' \) satisfy the following conditions: if \( 0 < s < 1/2 \), then
\[
(s', \gamma') = (0, \gamma), \quad 0 < s < 1/2;
\]
if \( 1/2 \leq s < 1 \), then
\[
s' = 2s - 1 + \frac{\epsilon}{2} \in (0, s), \quad \epsilon \in (0, 2(1 - s)), \quad \frac{\gamma'}{2} = \frac{s'}{2} + (s' - 1) \in (0, \gamma/2).
\]

**Proof.** Denote
\[
\Gamma = \Gamma(f, g, h) = \int_{\mathbb{R}^3} b(\cos \theta) |v - v_s| \gamma \left( \langle v \rangle^k - \langle v \rangle^k \right) f_s \gamma f' h' \, d\mu.
\]
Then by Lemma 2.1,
\[
\Gamma = \int_{\mathbb{R}^3} b(\cos \theta) |v - v_s| \gamma \left( \langle v \rangle^k - \langle v \rangle^k \right) f_s \gamma f' h' \, d\mu.
\]

Here, we have replaced \( v \cdot \omega \) by \( v_s \cdot \omega \) in \( \Gamma_1 \). Now we estimate all the \( \Gamma_m \) separately. First, by the Cauchy-Schwarz inequality and the singular change of variables, we have
\[
|\Gamma_2| \leq \left( \int_{\mathbb{R}^3} b(\cos \theta) \sin^{k-\frac{3}{2} - \frac{\gamma}{2}} \left( \frac{\theta}{2} \right) |v - v_s| \gamma |g| \langle v_s \rangle^k f_s^2 \, dv, \, d\sigma \right)^{1/2}
\]
\[
\times \left( \int_{\mathbb{R}^3} b(\cos \theta) \sin^{k+\frac{3}{2} - \frac{\gamma}{2}} \left( \frac{\theta}{2} \right) |v' - v| \gamma |h(v')|^2 \, dv, \, d\sigma \right)^{1/2}
\]
\[
\leq \int_{\mathbb{R}^3} b(\cos \theta) \sin^{k-\frac{3}{2} - \frac{\gamma}{2}} \left( \frac{\theta}{2} \right) \, d\sigma \left( \int_{\mathbb{R}^3} |v - v_s| \gamma |g| \langle v_s \rangle^k f_s^2 \, dv, \, d\sigma \right)^{1/2}
\]
\[
\times \left( \int_{\mathbb{R}^3} |v - v'| \gamma |h(v')|^2 \, dv' \right)^{1/2}
\]
\[
\leq \left( \int_{\mathbb{R}^3} b(\cos \theta) \sin^{k-\frac{3}{2} - \frac{\gamma}{2}} \left( \frac{\theta}{2} \right) \, d\sigma \right) \|g\|_{L^1_\gamma} \left\| \langle v \rangle^k f \right\|_{L^2_\gamma} \|h\|_{L^2_\gamma},
\]
which holds when \( k > \frac{3}{2} + \frac{\gamma}{2} + 2s \). Similarly,
\[
|\Gamma_3| \leq \int_{\mathbb{R}^3} b(\cos \theta) |v - v_s| \gamma |g| \|f_s\| |h'| \, d\mu
\]
\[
\leq C_k \int_{\mathbb{R}^3} b(\cos \theta) |v - v_s| \gamma \langle v \rangle \langle v_s \rangle^k \sin^{k-3} \left( \frac{\theta}{2} \right) f_s |g| |h'| \, d\mu
\]
\[
\leq C_k \|g\|_{L^1_{1+\gamma}} \left\| \langle v \rangle^k f \right\|_{L^2} \|h\|_{L^2},
\]
(3.3)
where we need $k > \frac{9}{2} + \frac{7}{2} + 2s$. Next,

$$\begin{align*}
|\Gamma_4| & \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(\cos \theta) |v - v_\ast|^{\gamma} |\mathfrak{R}_2||f_\ast||g||h'| \, d\overline{m} \\
& \leq C_k \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(\cos \theta) |v - v_\ast|^{\gamma} \langle v \rangle^{k-2} \langle v_\ast \rangle^{2} \sin^2 \frac{\theta}{2} |f_\ast||g||h'| \, d\overline{m} \\
& \leq C_k \|f\|_{L^1_{\gamma+\gamma}} \left\| \langle v \rangle^k g \right\|_{L^2} \|h\|_{L^2}.
\end{align*}$$

Similarly,

$$\begin{align*}
|\Gamma_5| & \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(\cos \theta) |v - v_\ast|^{\gamma} |\mathfrak{R}_3||f_\ast||g||h'| \, d\overline{m} \\
& \leq C_k \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(\cos \theta) |v - v_\ast|^{\gamma} \langle v \rangle^{k-4} \langle v_\ast \rangle^{4} \sin^2 \frac{\theta}{2} |f_\ast||g||h'| \, d\overline{m} \\
& \leq C_k \|f\|_{L^1_{\gamma+\gamma}} \left\| \langle v \rangle^k g \right\|_{L^2} \|h\|_{L^2}.
\end{align*}$$

We can also estimate the bound on $\Gamma_6$ directly as

$$\begin{align*}
|\Gamma_6| & \leq C_k \|f\|_{L^1_{\gamma+\gamma}} \left\| \langle v \rangle^k g \right\|_{L^2} \|h\|_{L^2_{\gamma+\gamma}}.
\end{align*}$$

To estimate $\Gamma_1$, we rewrite $\omega$ as

$$\omega = \overline{\omega} \cos \frac{\theta}{2} + \frac{v' - v_\ast}{|v' - v_\ast|} \sin \frac{\theta}{2},$$

where $\overline{\omega} = (v' - v)/|v' - v|$. Note that $\overline{\omega} \perp (v' - v_\ast)$. Accordingly,

$$\begin{align*}
\Gamma_1 & = k \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(\cos \theta) |v - v_\ast|^{\gamma} \langle \langle v \rangle^{k-2} |v - v_\ast| (v_\ast \cdot \overline{\omega}) \cos^{k-4} \frac{\theta}{2} \sin^2 \frac{\theta}{2} \rangle |f_\ast|g|h'| \, d\overline{m} \\
& \quad + k \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(\cos \theta) |v - v_\ast|^{\gamma} \langle \langle v \rangle^{k-2} |v - v_\ast| (v_\ast \cdot \frac{v' - v_\ast}{|v' - v_\ast|}) \cos^{k-1} \frac{\theta}{2} \sin^2 \frac{\theta}{2} \rangle |f_\ast|g|h'| \, d\overline{m} \\
& \triangleq \Gamma_{1,1} + \Gamma_{1,2}.
\end{align*}$$

The second term $\Gamma_{1,2}$ is obviously bounded by

$$\begin{align*}
|\Gamma_{1,2}| & \leq C_k \|f\|_{L^1_{\gamma+\gamma}} \left\| \langle v \rangle^k g \right\|_{L^2} \|h\|_{L^2}.
\end{align*}$$

To estimate $\Gamma_{1,1}$, we consider the cases when $0 < s < 1/2$ and $1/2 \leq s < 1$ separately. In the case of mild singularity when $0 < s < 1/2$, we can directly bound $\Gamma_{1,1}$ by

$$\begin{align*}
|\Gamma_{1,1}| & \leq C_k \|f\|_{L^1_{\gamma+\gamma}} \left\| \langle v \rangle^k g \right\|_{L^2} \|h\|_{L^2}.
\end{align*}$$

Therefore, when $0 < s < 1/2$, we have

$$\begin{align*}
|\Gamma_1| & \leq C_k \|f\|_{L^1_{\gamma+\gamma}} \left\| \langle v \rangle^k g \right\|_{L^2} \|h\|_{L^2}, \quad s \in (0, 1/2).
\end{align*}$$

To treat the strong singularity when $1/2 \leq s < 1$, we denote $G(v) = g(v) \langle v \rangle^{k-2}$ and separate $\Gamma_{1,1}$ such that

$$\begin{align*}
\Gamma_{1,1} & = k \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(\cos \theta) |v - v_\ast|^{1+\gamma} (v_\ast \cdot \overline{\omega}) \cos^{k-4} \frac{\theta}{2} \sin^2 \frac{\theta}{2} |f_\ast|G'\overline{h}' \, d\overline{m} \\
& \quad + k \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(\cos \theta) |v - v_\ast|^{1+\gamma} (v_\ast \cdot \overline{\omega}) \cos^{k-1} \frac{\theta}{2} \sin^2 \frac{\theta}{2} |f_\ast|(G - G') \overline{h}' \, d\overline{m} \triangleq \Gamma_{1,1}^{(1)} + \Gamma_{1,1}^{(2)}.
\end{align*}$$
One key observation in this decomposition is that $\Gamma_{1,1}^{(1)} = 0$. Indeed, one can make the regular change of variables $v \to v'$ and take the new $v' - v_*$ as the north pole. Then

$$
\Gamma_{1,1}^{(1)} = k \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} b(\cos \theta) \frac{1}{\cos^{4+\gamma} \frac{\theta}{2}} |v' - v_*|^{1+\gamma} (v_* \cdot \vec{\omega}) \cos^k \theta \sin \frac{\theta}{2} f_* G' h' \sin \theta \, d\theta \, d\phi \, dv' \, dv_*,
$$

where $\vec{\omega} = (\cos \phi, \sin \phi, 0)$. Thus formally the integration in $\phi$ gives that $\Gamma_{1,1}^{(1)} = 0$. This can be made rigorous by first truncating the singularity of $b$ in $\theta$ and then passing the limit of truncation. Hence, if $1/2 \leq s < 1$, then

$$
|\Gamma_{1,1}| = |\Gamma_{1,1}^{(2)}| \leq C_k \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} b(\cos \theta) |v - v_*|^{2+\gamma} \theta^{2-2s-\epsilon} (\langle v_* \rangle |f_*|) |G - G'| |h'| \, d\theta \right)^{1/2}
= C_k \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} b(\cos \theta) \theta^{2s+\epsilon} \left( (\langle v_* \rangle^{1+\gamma} |f_*|) (\langle v' \rangle^{\gamma} |h'|^2) \right) \right)^{1/2}
\leq C_k \|f\|_{L_{1+\gamma}^1}^{1/2} \|h\|_{L_{5/2}^2} \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} b(\cos \theta) |v - v_*|^{2+\gamma} \theta^{2-2s-\epsilon} (\langle v_* \rangle |f_*|) |G - G'|^2 \, d\theta \right)^{1/2}
\leq C_k \|f\|_{L_{1+\gamma}^1}^{1/2} \|h\|_{L_{5/2}^2} \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} b(\cos \theta) |v - v_*|^{2+\gamma} \theta^{2-2s-\epsilon} (\langle v_* \rangle |f_*|) |G - G'|^2 \, d\theta \right)^{1/2}.
\tag{3.10}
$$

To bound the last factor in (3.10), we write

$$
|G - G'|^2 = \left( (G')^2 - 2G(G') + 2G(G') \right).
$$

Hence,

$$
\Gamma_{1,1}^{(3)} = \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} b(\cos \theta) |v - v_*|^{2+\gamma} \theta^{2-2s-\epsilon} (\langle v_* \rangle |f_*|) ((G')^2 - 2G) \, d\theta
+ 2 \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} b(\cos \theta) |v - v_*|^{2+\gamma} \theta^{2-2s-\epsilon} (\langle v_* \rangle |f_*|) G(G - G') \, d\theta
\leq C \|f\|_{L_{1+\gamma}^1} \left( \langle v' \rangle^{k-2} g \right)_{L_{1+\gamma}^2}^{1/2} + 2 \int_{\mathbb{R}^3} Q_k(\langle v \rangle f, G) G \, dv
\leq C \|f\|_{L_{1+\gamma}^1} \left( \langle v' \rangle^{k} g \right)_{L_{5/2}^2}^{1/2} + 2 \int_{\mathbb{R}^3} Q_k(\langle v \rangle f, G) G \, dv,
$$

where $Q_k$ denotes the bilinear operator with the cross section $\tilde{b} = b(\cos \theta) \theta^{2-2s-\epsilon} |v - v_*|^{2+\gamma}$. Hence, the singularity is given by

$$
\tilde{\theta} \tilde{b} \sim \frac{1}{\theta^{1+2s}}, \quad s' = 2s - 1 + \frac{\epsilon}{2}.
\tag{3.11}
$$

We choose $\epsilon > 0$ such that $s' < s$, that is,

$$
0 < \epsilon < 2(1 - s).
$$
By the trilinear estimate given in Proposition 2.2 for $Q_b$, we have
\[
\left| \int_{\mathbb{R}^3} Q_b(\langle v \rangle f, G) \, dv \right| \leq C \| \langle v \rangle f \|_{L^1_{2+\gamma+2s, \gamma/2}} \| G \|_{H^\gamma_{\gamma/2}}^2 \leq C \| \langle v \rangle f \|_{L^1_{2+\gamma+2s, \gamma/2}} \| \langle v \rangle^k g \|_{H^\gamma_{\gamma/2}},
\]
where the weight $\gamma'$ is given by
\[
\frac{\gamma'}{2} = \frac{2 + \gamma}{2} + s' - 2 = \frac{\gamma}{2} + (s' - 1) < \frac{\gamma}{2}.
\]
Altogether we have
\[
\left| \Gamma^{(3)}_{1,1} \right| \leq C \| f \|_{L^1_{2+\gamma+2s, \gamma/2}} \| \langle v \rangle^k g \|_{H^\gamma_{\gamma/2}},
\]
which, by (3.10), further gives
\[
|\Gamma_{1,1}| \leq C_k \| f \|_{L^1_{2+\gamma+2s, \gamma/2}} \| \langle v \rangle^k g \|_{H^\gamma_{\gamma/2}} \| h \|_{L^2_{\gamma/2}},
\]
where $s', \gamma'$ are defined in (3.11) and (3.12) respectively. Combining the estimates in (3.3)-(3.9) and (3.13), we obtain the desired estimate in (3.1).

We are now ready to show a key coercivity estimate for $Q(F, f)$ stated in

**Proposition 3.2.** Suppose $F = \mu + g$ with $F$ satisfying the conditions in Proposition 2.4. For $k > \frac{9}{2} + \frac{\gamma}{2} + 2s$, we have
\[
\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} Q(F, f) f \langle v \rangle^{2k} \, dv \, dx \\
\leq -\frac{\gamma_0}{2} \int_{\mathbb{T}^3} \| \langle v \rangle^k f \|_{L^2_{\gamma/2}}^2 \, dx - \left( \frac{c_0}{4} \delta_2 - C_k \sup_{\mathbb{T}^3} \| g \|_{L^1_{3+\gamma+2s, \gamma/2}} \right) \int_{\mathbb{T}^3} \| \langle v \rangle^k g \|_{L^2_{\gamma/2}}^2 \, dx \\
+ C_k \int_{\mathbb{T}^3} \| \langle v \rangle^k f \|_{L^2_{\gamma/2}}^2 \, dx + C_k \int_{\mathbb{T}^3} \| \langle v \rangle^k g \|_{L^2_{\gamma/2}} \| \langle v \rangle^k f \|_{L^1_{3+\gamma}} \, dx \\
+ \left( \int_{S^2} b(\cos \theta) \sin^{k-2-\frac{\gamma}{2}-\frac{\delta_2}{2}} \, d\sigma \right) \int_{\mathbb{T}^3} \| f \|_{L^1_{\gamma/2}} \| \langle v \rangle^k g \|_{L^2_{\gamma/2}} \| \langle v \rangle^k f \|_{L^2_{\gamma/2}} \, dx,
\]
where $\gamma_0$ is defined in (3.17), $c_0$ is the coefficient in Proposition 2.4 and $\delta_2$ is a small enough constant (which may depend on $k$).

**Proof.** We will give two different estimates on $\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} Q(F, f) f \langle v \rangle^{2k} \, dv \, dx$. The first one contains dissipation in terms of $\| \langle v \rangle^k f \|_{L^2_{\gamma/2}}^2$ while the second one contains dissipation of $\| \langle v \rangle^k f \|_{H^\gamma_{\gamma/2}}^2$. First, by the definition of $Q$, we have
\[
\int_{\mathbb{R}^3} Q(F, f) f \langle v \rangle^{2k} \, dv = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2_{+}} b(\cos \theta) |v - v_*|^\gamma (F^* f' - F_* f) f \langle v \rangle^{2k} \, d\Omega
\]
\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2_{+}} b(\cos \theta) |v - v_*|^\gamma F_* f \left( f' \langle v \rangle^{2k} - f \langle v \rangle^{2k} \right) \, d\Omega
\]
\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2_{+}} b(\cos \theta) |v - v_*|^\gamma F_* \left( f f' \langle v \rangle^{2k} - |f|^2 \langle v \rangle^{2k} \right) \, d\Omega
\]
\[
\leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2_{+}} b(\cos \theta) |v - v_*|^\gamma F_* \left( |f| \left( |f'| \langle v \rangle^{k} \right) \langle v \rangle^{k} - |f|^2 \langle v \rangle^{2k} \right).
Hence, \[
\int_{\mathbb{R}^3} Q(F, f) f \langle v \rangle^{2k} \, dv \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2_2} b(\cos \theta) \left| v - v_* \right| F_* \left( |f| \left( |f'| \langle v' \rangle^k \right) \langle v \rangle^k \cos^k \frac{\theta}{2} - |f|^2 \langle v \rangle^{2k} \right) \, d\Omega \\
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2_2} b(\cos \theta) \left| v - v_* \right| F_* |f| |f'| \langle v' \rangle^k \left( \langle v' \rangle^k - \langle v \rangle^k \cos^k \frac{\theta}{2} \right) \, d\Omega \\
\triangleq T_1 + T_2. \tag{3.15}
\]

We estimate \( T_1 \) and \( T_2 \) separately. Firstly, \( T_1 \) is a dissipative term. Indeed, by Cauchy-Schwarz,
\[
|f| \left( |f'| \langle v' \rangle^k \right) \langle v \rangle^k \cos^k \frac{\theta}{2} - |f|^2 \langle v \rangle^{2k} \leq \frac{1}{2} \left( |f'| \langle v' \rangle^k \right)^2 \cos^{2k} \frac{\theta}{2} - |f|^2 \langle v \rangle^{2k} .
\]

Therefore, using a regular change of variables, we have
\[
T_1 \leq \frac{1}{2} \int_{\mathbb{R}^3} \int_{S^2_2} b(\cos \theta) \left| v - v_* \right| F_* \left( \left( |f'| \langle v' \rangle^k \right)^2 \cos^{2k} \frac{\theta}{2} - |f|^2 \langle v \rangle^{2k} \right) \, d\Omega \\
= \frac{1}{2} \int_{\mathbb{R}^3} \int_{S^2_2} b(\cos \theta) \left| v - v_* \right| F_* |f|^2 \langle v \rangle^{2k} \left( \cos^{2k-3-\gamma} \frac{\theta}{2} - 1 \right) \, d\Omega . \tag{3.16}
\]

Let \( 0 < \gamma_1 < \gamma_2 \) be the coefficients such that
\[
\gamma_1 \langle v \rangle^{\gamma} \leq \int_{\mathbb{R}^3} \left| v - v_* \right| \mu_* \, dv_* \leq \gamma_2 \langle v \rangle^{\gamma} .
\]

Denote \( \gamma_0 \) as the constant given by
\[
\gamma_0 = -\frac{\gamma_1}{2} \int_{S^2_2} b(\cos \theta) \left( \cos^{2k-3-\gamma} \frac{\theta}{2} - 1 \right) \, d\sigma . \tag{3.17}
\]

Note that for \( k \geq \frac{5 + \gamma}{2} \), the constant \( \gamma_0 \) has a strict lower bound that is independent of \( k \). Hence, \[
\int_{T_3} T_1 \, dx \leq - \left( \gamma_0 - C_k \sup_{T_3} \| g \|_{L^2_{\gamma}(dv)} \right) \left\| \langle v \rangle^k f \right\|^2_{L^2_{\gamma/2}(dx \, dv)} . \tag{3.18}
\]

The bound of the second term \( T_2 \) can be obtained by a direct application of Proposition 3.1. We note that \( T_2 \) only contains \( \Gamma_1 \sim \Gamma_5 \) in Proposition 3.1 since the difference in \( T_2 \) is \( \langle v' \rangle^k - \langle v \rangle^k \cos^k \frac{\theta}{2} \) instead of \( \langle v' \rangle^k - \langle v \rangle^k \). Hence, using the bounds for \( \Gamma_1 \sim \Gamma_5 \) in Proposition 3.1 we have
\[
T_2 \leq C_k \| f \|_{L^1_{\gamma/2}} \left\| \langle v \rangle^k f \right\|_{L^2_{\gamma/2}} + \left( \int_{S^2_2} b(\cos \theta) \sin^{k-\frac{3}{2} - \frac{\gamma}{2}} \frac{\theta}{2} \, d\sigma \right) \| f \|_{L^1_{\gamma/2}} \left\| \langle v \rangle^k g \right\|_{L^2_{\gamma/2}} \left\| \langle v \rangle^k f \right\|_{L^2_{\gamma/2}} \\
+ C_k \| f \|_{L^1_{1+\gamma}} \left\| \langle v \rangle^k g \right\|_{L^2} \left\| \langle v \rangle^k f \right\|_{L^2} + C_k \left\| \langle v \rangle^k f \right\|^2_{L^2} + C_k \| g \|_{L^1_{1+\gamma}} \left\| \langle v \rangle^k f \right\|^2_{L^2} \\
+ C_k \left\| \langle v \rangle^k f \right\|_{H^s_{\gamma/2}} \left\| \langle v \rangle^k f \right\|_{L^2_{\gamma/2}} + C_k \| g \|_{L^1_{1+2s}} \left\| \langle v \rangle^k f \right\|_{H^s_{\gamma/2}} \left\| \langle v \rangle^k f \right\|_{L^2_{\gamma/2}} .
\]

By the interpolation of \( H^s_{\gamma/2} \) between \( L^2 \) and \( H^s_{\gamma/2} \), we have if \( 1/2 \leq s < 1 \), then
\[
T_2 \leq C_k \| f \|_{L^1_{\gamma/2}} \left\| \langle v \rangle^k f \right\|_{H^s_{\gamma/2}} + \left( \int_{S^2_2} b(\cos \theta) \sin^{k-\frac{3}{2} - \frac{\gamma}{2}} \frac{\theta}{2} \, d\sigma \right) \| f \|_{L^1_{\gamma/2}} \left\| \langle v \rangle^k g \right\|_{L^2_{\gamma/2}} \left\| \langle v \rangle^k f \right\|_{L^2_{\gamma/2}} \\
+ C_k \| f \|_{L^1_{1+\gamma}} \left\| \langle v \rangle^k g \right\|_{L^2} \left\| \langle v \rangle^k f \right\|_{L^2} + C_k \left\| \langle v \rangle^k f \right\|^2_{L^2} + C_k \| g \|_{L^1_{1+\gamma}} \left\| \langle v \rangle^k f \right\|^2_{L^2} \\
\leq \left( \delta_1 + C_k \| g \|_{L^1_{1+2s}} \right) \left\| \langle v \rangle^k f \right\|^2_{H^s_{\gamma/2}} + C_k \| f \|_{L^1_{1+\gamma}} \left\| \langle v \rangle^k g \right\|_{L^2} \left\| \langle v \rangle^k f \right\|_{L^2} + C_k \left\| \langle v \rangle^k f \right\|^2_{L^2} \\
+ \left( \int_{S^2_2} b(\cos \theta) \sin^{k-\frac{3}{2} - \frac{\gamma}{2}} \frac{\theta}{2} \, d\sigma \right) \| f \|_{L^1_{\gamma/2}} \left\| \langle v \rangle^k g \right\|_{L^2_{\gamma/2}} \left\| \langle v \rangle^k f \right\|_{L^2_{\gamma/2}} .
\]
Combining (3.18) and (3.19) gives
\[ I_T \]
Applying Proposition 2.4 to \( T \)
Note that the second term
\[ c_2 + \int b(\cos \theta) \sin^{k-\frac{3}{2}} \frac{\theta}{2} d\sigma \int \int \int f \|L_1^1 \| \langle v \rangle^k g \|L_2^2 \| \langle v \rangle^k f \|L_{1/2}^2 \| dx \]
\[ + C_k \int \int \| \langle v \rangle^k f \|L_2^2 \| \langle v \rangle^k f \|L_{1/2}^2 \| dx \]
\[ + \int \int \int b(\cos \theta) \sin^{k-\frac{3}{2}} \frac{\theta}{2} d\sigma \int \int \int \langle v \rangle^k \|L_1^1 \| g \|L_{1/2}^1 \| \langle v \rangle^k f \|L_{1/2}^2 \| dx \]
\[ + C_k \int \| \langle v \rangle^k f \|L_2^2 \| \langle v \rangle^k f \|L_{1/2}^2 \| dx \].

Next, we give the second estimate on \( \int T_3 \int \int Q(F, f) \langle v \rangle^k dv dx \) by firstly rewriting it as
\[ \int T_3 \int \int Q(F, f) \langle v \rangle^k dv dx = \int T_3 \int \int Q(F, \langle v \rangle^k f) \langle v \rangle^k f dv dx \]
\[ + \int \int \int \left( \langle v \rangle^k Q(F, f) - Q(F, \langle v \rangle^k f) \right) \langle v \rangle^k f dv dx \]
\[ \Delta = T_3 + T_4. \]

Applying Proposition 2.4 to \( T_3 \) yields
\[ T_3 \leq -c_0 \int \| \langle v \rangle^k f \|_{L_{1/2}^2}^2 dx + C_1 \int \| \langle v \rangle^k f \|_{L_{1/2}^2}^2 dx. \]
Note that the second term \( T_4 \) has the form
\[ T_4 = \int \int \int b(\cos \theta) |v - v_4|^\gamma \left( \langle v' \rangle^k - \langle v \rangle^k \right) F_v f' \langle v' \rangle^k d\Pi dx. \]

Applying the commutator estimate in Proposition 3.1 to \( T_4 \) gives
\[ T_4 \leq \left( \int \int b(\cos \theta) \sin^{k-\frac{3}{2}} \frac{\theta}{2} d\sigma \int \int \int f \|L_1^1 \| \langle v \rangle^k F \|L_{1/2}^2 \| \langle v \rangle^k f \|L_{1/2}^2 \| dx \]
\[ + C_k \int \|f\|_{L_1^1} \langle v \rangle^k F \|L_{1/2}^2 \| \langle v \rangle^k f \|L_{1/2}^2 \| dx \]
\[ + C_k \int \|F\|_{L_1^1} \langle v \rangle^k f \|L_{1/2}^2 \| \langle v \rangle^k f \|L_{1/2}^2 \| dx \]
\[ \leq \left( \frac{c_0}{2} + C_k \sup \|g\|_{L_1^1} \right) \int \int \| \langle v \rangle^k f \|_{L_{1/2}^2}^2 dx + C_k \int \| \langle v \rangle^k f \|_{L_{1/2}^2}^2 dx. \]

Combining (3.21) and (3.22), we obtain
\[ \int T_3 \int \int Q(F, f) \langle v \rangle^k dv dx \]
\[ \leq \left( \frac{c_0}{2} - C_k \sup \|g\|_{L_1^1} \right) \int \int \| \langle v \rangle^k f \|_{L_{1/2}^2}^2 dx + C_k \int \| \langle v \rangle^k f \|_{L_{1/2}^2}^2 dx. \] (3.23)
Let $\delta_2 > 0$ be a small number to be determined. Multiply $\delta_2$ to (3.23) and add it to (3.20). This gives
\[
\int_{T^3} \int_{\mathbb{R}^3} Q(F, f) f (\langle v \rangle)^{2k} \, dv \, dx \\
\leq -\left( \frac{C_0}{2} \delta_2 - \delta_1 - C_k \sup_{T^3} \| g \|_{L^{1+\gamma/2}\cap L^2} \right) \int_{T^3} \| \langle v \rangle^k f \|_{H^{\gamma/2}}^2 \, dx \\
- (\gamma_0 - C_k \delta_2) \int_{T^3} \| \langle v \rangle^k f \|_{L^{1+\gamma/2}}^2 \, dx + C_k \int_{T^3} \| \langle v \rangle^k f \|_{L^2}^2 \, dx \\
+ \left( \int_{S^2} b(\cos \theta) \sin^{k-\frac{\gamma}{2}-\frac{\gamma}{2}} \frac{\theta}{2} \, d\sigma \right) \int_{T^3} \| f \|_{L^1} \| \langle v \rangle^k g \|_{L^{1+\gamma/2}} \| \langle v \rangle^k f \|_{L^{1+\gamma/2}} \, dx \\
+ C_k \int_{T^3} \| \langle v \rangle^k g \|_{L^2} \| \langle v \rangle^k f \|_{L^{1+\gamma/2}} \, dx.
\]
Then the dissipation given in the inequality (3.14) follows from the fact by first taking $\delta_2$ small enough such that $C_k \delta_2 < \frac{\gamma_0}{4}$ and then taking $\delta_1 > 0$ small enough such that $\delta_1 < \frac{\gamma_0}{4} \delta_2$. \hfill \square

Remark 3.1. We keep the second term on the right hand side of the inequality in Proposition 3.2 in the current form since in later sections we may apply the supremum in $x \in T^3$ to either the $g$-term or the $f$-term depending on the need.

Now we state the proposition for the bound of $Q(g, \mu)$.

**Proposition 3.3.** Let $k > \frac{\gamma}{2} + \frac{\gamma}{2} + 2s$. Then
\[
\int_{T^3} \int_{\mathbb{R}^3} Q(g, \mu) f (\langle v \rangle)^{2k} \, dv \, dx \leq \left( \int_{S^2} b(\cos \theta) \sin^{k-\frac{\gamma}{2}-\frac{\gamma}{2}} \frac{\theta}{2} \, d\sigma \right) \int_{T^3} \| \langle v \rangle^k g \|_{L^{1+\gamma/2}} \| \langle v \rangle^k f \|_{L^{1+\gamma/2}} \, dx \\
+ C_k \int_{T^3} \| \langle v \rangle^k g \|_{L^2} \| \langle v \rangle^k f \|_{L^{1+\gamma/2}} \, dx.
\]

**Proof.** First of all,
\[
\int_{T^3} \int_{\mathbb{R}^3} Q(g, \mu) f (\langle v \rangle)^{2k} \, dv \, dx = \int_{T^3} \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)|v - v_*| \gamma g_* \mu (f' (\langle v \rangle)^{2k} - f (\langle v \rangle)^{2k}) \, d\sigma \, d\Pi \, dx \\
= \int_{T^3} \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)|v - v_*| \gamma g_* \mu (f' (\langle v \rangle)^k - f (\langle v \rangle)^k) \, d\sigma \, d\Pi \, dx \\
+ \int_{T^3} \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)|v - v_*| \gamma g_* \mu (f' (\langle v \rangle)^k) \, d\sigma \, d\Pi \, dx \\
= T_5 + T_6.
\]

Note that
\[
T_5 = \int_{T^3} \left( Q(g, \mu (\langle v \rangle)^k), f (\langle v \rangle)^k \right) \, dx.
\]

Taking $m = \gamma/2$ and $\sigma = s$ in the trilinear estimate in Proposition 2.2, we have
\[
|T_5| \leq C \int_{T^3} \left( \| g \|_{L^{1+\gamma/2}\cap L^2} \right) \| \langle v \rangle^k f \|_{L^{1+\gamma/2}} \, dx \leq C \| \langle v \rangle^k g \|_{L^{1+\gamma/2}} \| \langle v \rangle^k f \|_{L^{1+\gamma/2}}, \quad k > 2 + 2s + \gamma. \tag{3.24}
\]
Applying Proposition 3.1 to $T_6$, we obtain
\[
T_6 \leq \left( \int_{S^2} b(\cos \theta) \sin^{k-\frac{\gamma}{2}-\frac{\gamma}{2}} \frac{\theta}{2} \, d\sigma \right) \| \langle v \rangle^k g \|_{L^{1+\gamma/2}} \| \langle v \rangle^k f \|_{L^{1+\gamma/2}} + C_k \| \langle v \rangle^k g \|_{L^2} \| \langle v \rangle^k f \|_{L^{1+\gamma/2}}. \tag{3.25}
\]
The desired bound is then obtained by combining (3.24) and (3.25). \hfill \square

The next proposition is about the bounds on the commutators with respect to the spatial derivatives.
Proposition 3.4. Let \( \alpha \) be any multi-index such that \( |\alpha| = 2 \). Suppose \( l \geq 2 + \frac{6}{m_0} \) with \( m_0 \) being the exponent in (1.6). Let \( F = \mu + f \). Then
\[
\left| \int_{\mathbb{R}^3} (\partial_x^2 Q(F, g) - Q(F, \partial_x^2 g)) W^{2(l-|\alpha|)}(\partial_x^2 g) \, dx \right| \\
\leq C_l \| f \|_{Y_l} \left( \sum_{\alpha} \| W^{l-|\alpha|} \partial_x^2 g \|_{L^2(dx; H^{s}_{\gamma/2}(dv))} \right) \left( \sum_{\alpha} \| W^{l-|\alpha|} \partial_x^2 h \|_{L^2(dx; H^{s}_{\gamma/2}(dv))} \right) \\
+ C_l \| g \|_{Y_l} \left( \sum_{\alpha} \| W^{l-|\alpha|} \partial_x^2 f \|_{L^2(dx; H^{s}_{\gamma/2}(dv))} \right) \left( \sum_{\alpha} \| W^{l-|\alpha|} \partial_x^2 h \|_{L^2(dx; H^{s}_{\gamma/2}(dv))} \right).
\]
(3.26)

Proof. By the Leibniz rule for the bilinear operator, the commutator satisfies
\[
\partial_x^2 Q(F, g) - Q(F, \partial_x^2 g) = \sum_{|\alpha_1| \neq 0} C^{\alpha_1, \alpha_2} Q(\partial_x^{\alpha_1} F, \partial_x^{\alpha_2} g) = \sum_{|\alpha_1| \neq 0} C^{\alpha_1, \alpha_2} Q(\partial_x^{\alpha_1} f, \partial_x^{\alpha_2} g).
\]

For each \((\alpha_1, \alpha_2) \neq (0, 2)\), we have
\[
\left| \int_{\mathbb{R}^3} Q(\partial_x^{\alpha_1} f, \partial_x^{\alpha_2} g) \left( W^{2(l-|\alpha|)} \partial_x^2 h \right) \, dv \right| \\
\leq \left| \int_{\mathbb{R}^3} \left( W^{l-|\alpha|} Q(\partial_x^{\alpha_1} f, \partial_x^{\alpha_2} g) - Q(\partial_x^{\alpha_1} f, W^{l-|\alpha|} \partial_x^2 g) \right) \left( W^{l-|\alpha|} \partial_x^2 h \right) \, dv \right| \\
+ \left| \int_{\mathbb{R}^3} Q(\partial_x^{\alpha_1} f, W^{l-|\alpha|} \partial_x^2 g) \left( W^{l-|\alpha|} \partial_x^2 h \right) \, dv \right| \leq T_{7,1} + T_{7,2}.
\]

By the trilinear estimate given in Proposition 2.2 with \((m, \sigma) = (0, 0)\), we have
\[
T_{7,2} \leq C_l \| \partial_x^{\alpha_1} f \|_{L^2_{\gamma,2}} \left\| W^{l-|\alpha|} \partial_x^{\alpha_2} g \right\|_{H^{s}_{\gamma/2} + 2s} \left\| W^{l-|\alpha|} \partial_x^2 h \right\|_{H^{s}_{\gamma/2}} \\
\leq C_l \left\| (v)^{1+\gamma} \partial_x^{\alpha_1} f \right\|_{L^2} \left\| W^{l-|\alpha|} \partial_x^{\alpha_2} g \right\|_{H^{s}_{\gamma/2} + 2s} \left\| W^{l-|\alpha|} \partial_x^2 h \right\|_{H^{s}_{\gamma/2}}.
\]

To bound \( \int_{T^3} T_{7,2} \, dx \), we consider the two cases: \(|\alpha_1| = |\alpha_2| = 1\) and \( |\alpha_1| = |\alpha_2| = 1\). If \(|\alpha_1| = |\alpha_2| = 1\), then let
\[
q = \frac{3}{2} + \frac{2}{m_0}, \quad 2 = \frac{2}{q} = \frac{2}{q}.
\]

Note that in this case, \( q > 3 \) and \( p \in (2, 6) \) because \( m_0 > 4s \). Recall that in \( \mathbb{R}^3 \), we have the Sobolev embeddings
\[
H^1(\mathbb{R}^3) \hookrightarrow H^{\frac{3}{2} - \frac{2}{p}}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3), \quad H^{\frac{3}{2} - \frac{2}{q}}(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)
\]
with \( \frac{3}{2} - \frac{2}{q} = 1 - \frac{2a}{m_0} \). Hence,
\[
\int_{T^3} \left\| (v)^{1+\gamma} \partial_x^{\alpha_1} f \right\|_{L^2}^2 \left\| W^{l-|\alpha|} \partial_x^{\alpha_2} g \right\|_{H^{s}_{\gamma/2} + 2s}^2 \, dx \\
= \int_{T^3} \left\| (v)^{1+\gamma} \partial_x^{\alpha_1} f \right\|_{L^2}^2 \left\| W^{l-1-(1-2s/m_0)} \partial_x^{\alpha_2} g \right\|_{H^{s}_{\gamma/2}}^2 \, dx \\
\leq \left( \int_{T^3} \left\| (v)^{1+\gamma} \partial_x^{\alpha_1} f \right\|_{L^p}^p \, dx \right)^{\frac{2}{p}} \left( \int_{T^3} \left\| W^{l-1-(1-2s/m_0)} \partial_x^{\alpha_2} g \right\|_{H^{s}_{\gamma/2}}^q \, dx \right)^{\frac{2}{q}} \leq C \left\| (v)^{1+\gamma} \partial_x^{\alpha_1} f \right\|_{L^p}^2 \left( \int_{T^3} \left\| W^{l-(2-2s/m_0)} \partial_x^{\alpha_2} g \right\|_{H^{s}_{\gamma/2}}^2 \, dx \right)^{\frac{2}{q}} \leq C \| f \|_{Y_l}^2 \sum_{\alpha} \left\| W^{l-|\alpha|} \partial_x^2 g \right\|_{L^2(dx; H^{s}_{\gamma/2}(dv))}^2.
\]
(3.27)
The bound for $T_{7.2}$ with $(|\alpha_1|, |\alpha_2|) = (2, 0)$ also follows from the Sobolev embedding. In this case, we have $2 - \frac{2a}{m_0} > 1/2$, which implies that

$$H^{2\frac{-2a}{m_0}}(\mathbb{R}^d) \hookrightarrow L^{\infty}(\mathbb{R}^d).$$

Hence,

$$\sup_{T_3} \|W^{l-|\alpha|}D_{x}^{2}g\|_{H^{s}_{\star/2+2s}}^{2} = \sup_{T_3} \|W^{l-(2\frac{-2a}{m_0})}g\|_{H^{s}_{\star/2}}^{2} \leq \int_{T_3} \|W^{l-(2\frac{-2a}{m_0})}(D_{x})^{2\frac{-2a}{m_0}}g\|_{H^{s}_{\star/2}}^{2} \, dx$$

$$\leq \sum_{\alpha} \|W^{l-|\alpha|}D_{x}^{2}g\|_{L^{2}(dx; H^{s}_{\star/2}(dv))}^{2}.$$ 

Therefore, the bound in (3.28) also holds when $(\alpha_1, \alpha_2) = (2, 0)$. Applying such bound, we obtain that

$$\int_{T_3} T_{7.2} \, dx \leq C \|f\|_{Y_1} \left( \sum_{\alpha} \|W^{l-|\alpha|}D_{x}g\|_{L^{2}(dx; H^{s}_{\star/2}(dv))} \right) \left( \sum_{\alpha} \|W^{l-|\alpha|}D_{x}h\|_{L^{2}(dx; H^{s}_{\star/2}(dv))} \right).$$

By the definition of $Q$, the term $T_{7.1}$ has the form

$$T_{7.1} = \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} b(\cos \theta) |v - v| \gamma \left( W^{l-|\alpha|}(v') - W^{l-|\alpha|}(v) \right) \left( \partial_{x}^{\alpha}f \right) \left( \partial_{x}^{\alpha}g \right) \left( W^{l-|\alpha|}D_{x}^{2}h \right) \, dv \right|.$$ 

By the commutator estimate in Proposition 3.1, we have

$$T_{7.1} \leq C_l \|D_{x}^{\alpha}f\|_{L^{1+\gamma}_{\alpha}L^{2}} \|W^{l-|\alpha|}D_{x}^{2}g\|_{H^{s}_{\star/2}(dv)} \|W^{l-|\alpha|}D_{x}^{2}h\|_{H^{s}_{\star/2}(dv)}$$

$$+ C_l \|f\|_{Y_1} \left( \sum_{\alpha} \|W^{l-|\alpha|}D_{x}g\|_{L^{2}(dx; H^{s}_{\star/2}(dv))} \right) \left( \sum_{\alpha} \|W^{l-|\alpha|}D_{x}h\|_{L^{2}(dx; H^{s}_{\star/2}(dv))} \right).$$

The upper bound of $T_{7.1}$ is derived in a similar way as that for $T_{7.2}$ by Sobolev embeddings in $\mathbb{R}^3$. Therefore, we have

$$\int_{T_3} T_{7.1} \, dx \leq C_l \|f\|_{Y_1} \left( \sum_{\alpha} \|W^{l-|\alpha|}D_{x}g\|_{L^{2}(dx; H^{s}_{\star/2}(dv))} \right) \left( \sum_{\alpha} \|W^{l-|\alpha|}D_{x}h\|_{L^{2}(dx; H^{s}_{\star/2}(dv))} \right)$$

$$+ C_l \|g\|_{Y_1} \left( \sum_{\alpha} \|W^{l-|\alpha|}D_{x}g\|_{L^{2}(dx; H^{s}_{\star/2}(dv))} \right) \left( \sum_{\alpha} \|W^{l-|\alpha|}D_{x}h\|_{L^{2}(dx; H^{s}_{\star/2}(dv))} \right).$$

The estimate in (3.26) is then obtained by combining (3.28) with (3.29). □

4. Spectral properties of $\mathcal{L}$ in $L^2_{x,v}$

In this section, we establish the spectral analysis of the linearized operator $\mathcal{L}$ defined in (1.5). This will play a key role in controlling the linear growth of the nonlinear equation when performing energy estimates.

4.1. Spectral Analysis of Linearized operator in Gaussian-weighted $L^2_{x,v}$. First we study the spectrum of the operator $\mathcal{L}(\mu) : \mathcal{D}(\mathcal{L}(\mu)) \to L^2_{x,v}$, defined on a dense subset of $\mathcal{D}(\mathcal{L}(\mu)) \subseteq L^2_{x,v}$, where

$$\mathcal{L}(\mu)(h) = \mu^{-1/2}(Q(\mu^{1/2}h, \mu) + Q(\mu, \mu^{1/2}h))$$

$$= \mu^{1/2} \int_{\mathbb{R}^3} |u|^\gamma b(\bar{u} \cdot \sigma) \mu_s \int_{\mathbb{R}^2} \left( \frac{h'}{\mu^{1/2}} + \frac{h_s'}{\mu_s^{1/2}} - \frac{h}{\mu^{1/2}} - \frac{h_s}{\mu_s^{1/2}} \right) \, d\sigma d\nu_s,$$

where $\mu$ is the normalized global Maxwellian. The kernel of $\mathcal{L}(\mu)$ in $L^2_v$ is given by

$$\text{Ker}(\mathcal{L}(\mu)) = \text{Span}\{ \sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu} \}.$$
Thus, the generators of the kernel are in the Schwartz space $\mathcal{S}$. The projection onto $\text{Ker}(L^{(\mu)})$ is defined as

$$\pi(h) \triangleq \sum_{\varphi \in \text{Ker}(L^{(\mu)})} \left( \int_{\mathbb{R}^3} h \varphi \, dv \right) \varphi.$$ 

Let us first address the decomposition of $L^{(\mu)}$ which is based on truncations of small and large velocities, and grazing angles. This decomposition is a bit different from the classical decomposition made in the spectral analysis in the cutoff case. Recall that the scattering kernel $b$ satisfies (1.3). We use the decomposition

$$b(\cos \theta) = b(\cos \theta) \left( 1_{|\sin(\theta)| \geq \epsilon} + 1_{|\sin(\theta)| < \epsilon} \right) \overset{\Delta}{=} b_1(\cos \theta) + b_2(\cos \theta).$$

(4.1)

For the kinetic potential write $|\cdot|^\gamma \overset{\Delta}{=} \Phi_1 + \Phi_2$, with $\gamma \in (0,1]$, where

$$\Phi_1(|u|) \overset{\Delta}{=} |u|^\gamma \chi_{\delta \leq |u| \leq \delta^{-1}}, \quad \Phi_2(|u|) \overset{\Delta}{=} |u|^\gamma \left( 1 - \chi_{\delta \leq |u| \leq \delta^{-1}} \right).$$

(4.2)

Here $\chi_{\delta \leq |u| \leq \delta^{-1}}$ is a smooth version of the indicator $1_{|u| \leq \delta^{-1}}$. Also, denote $L^{(\mu)}$ by $L^{(\mu)}_{\Phi,b}$ to emphasize the kernel dependence and then decompose it as

$$L^{(\mu)}_{\Phi,b} = L^{(\mu)}_{\Phi_1,b_1} + L^{(\mu)}_{\Phi_2,b_2} = L^{(\mu)}_{\Phi_1,b_1} + L^{(\mu)}_{\Phi_2,b_1} + L^{(\mu)}_{\Phi_2,b_2}$$

$$= \left( L^{(\mu)}_{\Phi_1,b_1} + \mu^{-1/2}Q_{\Phi_1,b_1}(\mu, \mu^{1/2} h) \right) + \left( 1 - L^{(\mu)}_{\Phi_2,b_1} + \mu^{-1/2}Q_{\Phi_2,b_1}(\mu, \mu^{1/2} h) + L^{(\mu)}_{\Phi_2,b_2} \right)$$

(4.3)

The operators $K_{\delta,\epsilon}$ and $\Lambda_{\delta,\epsilon}$ are self-adjoint in $L^2_\delta$ since $L^{(\mu)}$ is self-adjoint in $L^2_\epsilon$ for any reasonable kinetic kernel $\Phi(u)b(\cos \theta)$ (see [13, Chapter 7]) and $\mu^{-1/2}Q(\mu, \mu^{1/2} h)$ is a multiplication operator. The operator $\Lambda_{\delta,\epsilon}$ include all the singular features of $L^{(\mu)}$ in terms of tails and regularization.

The linearization that we make in this subsection is $f = \mu + \mu^{1/2} h$. The full equation for $h$ reads

$$\partial_t h = \mu^{-1/2}Q(\mu^{1/2} h, \mu^{1/2} h) + L^{(\mu)}(h) - v \cdot \nabla_x h.$$ 

(4.4)

In this way, we want to study the $L^2_{x,v}$ spectral properties of the operator

$$L^{(\mu)}(h) - v \cdot \nabla_x h.$$ 

A spectral gap in $H^1_{x,v}$ was found for [29] for this operator in the cutoff case using a combination of spectral theory and energy estimates. The proof follows after checking some structural conditions and a priori estimates satisfied by $L^{(\mu)}$. This approach does not seem to apply directly to the non-cutoff case. Here we give a more “perturbation-type” argument that works in both cutoff and non-cutoff cases.

4.1.1. Dissipative part. Let us prove that for $\delta > 0$ and $\epsilon > 0$ sufficiently small, the operator $- (\Lambda_{\delta,\epsilon} + v \cdot \nabla_x)$ is dissipative in $L^2_{x,v}$. The operator $\Lambda_{\delta,\epsilon}$ is composed of two singular parts such that $\Lambda_{\delta,\epsilon} = \Lambda_1 + \Lambda_2$, where $\Lambda_1$ is related to the growth in velocity (tails)

$$-\Lambda_1 \overset{\Delta}{=} L^{(\mu)}_{\Phi_2,b_1} - \mu^{-1/2}Q_{\Phi_1,b_1}(\mu, \mu^{1/2} h),$$

and $\Lambda_2$ is related to regularity

$$-\Lambda_2 \overset{\Delta}{=} L^{(\mu)}_{\Phi_2,b_2}.$$ 

Proposition 4.1 (Singular part $\Lambda_2$). There exist constants $c > 0$ and $C > 0$ depending only on mass and energy of $\mu$, and $\kappa_0 > 0$ depending only on $\Phi = |\cdot|^\gamma$, such that for any $s \in (0,1)$ and $\epsilon \in (0,1/5]$, we have

$$\langle L^{(\mu)}_{\Phi_2,b_2}(h), h \rangle \leq -c\kappa_0 \sum_{g \in \{h^{\pm}\}} \left\| \left( \gamma / 2 g(\xi) \left| \xi \right|^2 \mathbf{1} \left\{ \left| \xi \right| \geq \frac{1}{\epsilon} \right\} \right) \right\|_{L^2_x}^2 + C\| \theta^2 b_2 \|_{L^2_x} \| v \| / 2^2 h \|_{L^2_v}^2,$$

where $h^{\pm}$ are the positive and negative parts of $h$ respectively. We remark that the constants $c$, $C$ and $\kappa_0$ are independent of $\epsilon$. 
Proof. Note that

\[
\langle L_{\Phi, b_2}^{(\mu)}(h), h \rangle = \mu^{-1/2}Q_{\Phi, b_2}(\mu, \mu^{1/2}h), h \rangle + \mu^{-1/2}Q_{\Phi, b_2}(\mu^{1/2}h, \mu), h \rangle.
\] (4.5)

For the first term in the right side of (4.5) it follows

\[
\langle \mu^{-1/2}Q_{\Phi, b_2}(\mu, \mu^{1/2}h), h \rangle = \int_{\mathbb{R}^6} \int_{S^2} \Phi(u)b_2(\cos \theta)\mu(v_s)\mu(v)H(v)(H(v') - H(v))
\]

\[
= - \frac{1}{2} \int_{\mathbb{R}^6} \int_{S^2} \Phi(u)b_2(\cos \theta)\mu(v_s)\mu(v)(H(v') - H(v))^2 + \frac{1}{2} \int_{\mathbb{R}^6} \int_{S^2} \Phi(u)b_2(\cos \theta)\mu(v_s)\mu(v)(H(v')^2 - H(v)^2),
\] (4.6)

where \( u = v - v_s \) and \( H(v) = \mu^{-1/2}(v)h(v) \). Since \( \mu v_s = \mu' v'_s \), the last term in the right side of (4.6) is zero. Using the technique of proof of [9, Proposition 2.1] and Lemma 2.3, we have

\[
- \frac{1}{2} \int_{\mathbb{R}^6} \int_{S^2} \Phi(u)b_2(\cos \theta)\mu(v)(H(v') - H(v))^2 
\]

\[
\leq -c\kappa_0 \sum_{g \in \{h^{\pm}\}} \left\| \hat{\gamma}/\|g\| \right\| \|v\| \|1\{\|v\| \geq \frac{1}{2}\} \right\|_{L^2_{\hat{\gamma}}}^2 + C\|\theta b_2\|_{L^2_{\hat{\gamma}}} \|v\|^{\gamma/2}h \|^2_{L^2_{\hat{\gamma}}},
\] (4.7)

where the constants \( c, C > 0 \) depend only on mass and energy of \( \mu \), and \( \kappa_0 > 0 \) only on the potential \( \Phi \). For the second term in (4.5) we can use [9, Lemma 2.15] with \( B \) replaced by \( \Phi b_2 \), so that we obtain

\[
\langle \mu^{-1/2}Q_{\Phi, b_2}(\mu^{1/2}h, \mu), h \rangle \leq C\|\theta b_2\|_{L^2_{\hat{\gamma}}} \|\mu^{1/10}h\|^2_{L^2_{\hat{\gamma}}},
\] (4.8)

The proposition follows from (4.7) and (4.8). \( \square \)

**Proposition 4.2 (Singular part \( \Lambda_1 \)).** For every \( \varepsilon > 0 \) there exist constants \( c > 0 \) and \( C > 0 \), depending only on the mass and energy of \( \mu \), such that for any \( s \in (0, 1), \varepsilon \in (0, 1), \) and \( \delta^\gamma \in (0, \frac{c}{2C}\varepsilon^{2\gamma}) \), it follows that

\[
\langle L_{\Phi_1, b_1}^{(\mu)} h - \mu^{-1/2}Q_{\Phi_1, b_1}(\mu^{1/2}h), h \rangle \leq -\frac{\varepsilon}{2}\|b_1\|_{L^1_{\hat{\gamma}}} \|v\|^{\gamma/2}h \|^2_{L^2_{\hat{\gamma}}}.
\]

The constants are independent of both \( \delta > 0 \) and \( \varepsilon > 0 \).

Proof. Note that

\[
\Lambda_1 h = L_{\Phi_1, b_1}^{(\mu)} h - \mu^{-1/2}Q_{\Phi_1, b_1}(\mu^{1/2}h) = \mu^{-1/2}Q_{\Phi_1, b_1}^+(\mu^{1/2}h, \mu) - \mu^{-1/2}Q_{\Phi_1, b_1}^-(\mu^{1/2}h, \mu) + \mu^{-1/2}Q_{\Phi_1, b_1}^+(\mu, \mu^{1/2}h) - \mu^{-1/2}Q_{\Phi_1, b_1}^-(\mu, \mu^{1/2}h).
\]

The first three terms in the right side are treated similarly. Let us proceed with one of them and leave the other two to the reader. Note that

\[
\langle \mu^{-1/2}Q_{\Phi_1, b_1}^+(\mu, \mu^{1/2}h), h \rangle = \int_{\mathbb{R}^6} \int_{S^2} \Phi_1(u)b_1(\cos \theta)\mu^{1/2}v_s h(v)\mu^{1/2}v'_s h(v').
\] (4.9)

Since \( b_1 \) is supported in \( |\sin(\theta)| \geq \varepsilon \) one has

\[
|v'_s| \geq |u||\sin\frac{\theta}{2} - |v_s| \geq \frac{\varepsilon}{2}|v| - 2|v_s|,
\]

thus,

\[
\mu^{1/32}(\varepsilon v/2) \geq \mu^{1/16}(v'_s)\mu^{1/4}(v_s).
\]
Plugging this inequality in (4.9), and recalling that $\Phi_2$ is supported on $\{|u| \leq \delta\} \cup \{|u| \geq \delta^{-1}\}$ (so that $\Phi_2(|u|) \leq \delta\gamma \langle v^* \rangle^{2\gamma} \langle v \rangle^{2\gamma}$), one concludes that the right side of (4.9) is controlled by

$$
\int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \Phi_2(u) b_1(\cos \theta) \mu^{1/4}(v^*) \mu^{1/32}(\varepsilon v/2) h(v) h(v')
\leq \delta\gamma \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} b_1(\cos \theta) \langle v^* \rangle^{2\gamma} \mu^{1/4}(v) \langle v \rangle^{2\gamma} \mu^{1/32}(\varepsilon v/2) h(v) h(v')
\leq \frac{\delta\gamma}{\varepsilon^{2\gamma}} \sup_{x \in \mathbb{R}^3} \langle x \rangle^{2\gamma} \mu^{1/32}(x/2) \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} b_1(\cos \theta) \langle v^* \rangle^{2\gamma} \mu^{1/4}(v^*) h(v) h(v').
$$

As a consequence,

$$
\langle \mu^{-1/2} Q_{\Phi_2,b_1}^+(\mu, \mu^{1/2} h), h \rangle \leq \frac{C\delta}{\varepsilon^{2\gamma}} \langle Q_{\Phi_1,b_1}^+(\langle \cdot \rangle^{2\gamma} \mu^{1/4}, h), h \rangle \leq \frac{C\delta}{\varepsilon^{2\gamma}} \langle b_1 \| L^\pi_b \| \langle \cdot \rangle^{2\gamma} \mu^{1/4} \| L^\pi_b \| h \| L^2_h \rangle.
$$

Now, for the last term, it follows readily

$$
\langle \mu^{-1/2} Q_{\Phi_2,b_1}^-(\mu, \mu^{1/2} h), h \rangle \geq c \| b_1 \| L^\pi_b \| \langle v \rangle^{\gamma/2} h \| L^2_h \rangle,
$$

with $c > 0$ depending only on mass and energy of $\mu$. The result follows from (4.10) and (4.11).

\[\Box\]

**Theorem 4.1.** Let $h \in H^{s/2}_\gamma (dv)$ with $s \in (0,1)$. There exist constants $c > 0$, $C > 0$, $c_0 > 0$, and $\varepsilon_0 > 0$ depending only on mass and energy of $\mu$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $\delta_0 \in (0, \frac{1}{\varepsilon_0^{2\gamma}})$, the operator $-\Lambda_{\delta,\varepsilon}$ satisfies the dissipative estimate

$$
\langle -\Lambda_{\delta,\varepsilon}(h), h \rangle \leq -c_0 \kappa_0 \| \langle v \rangle^{\gamma/2} h \|_{H^s_\gamma}.
$$

The constant $\kappa_0 > 0$ was introduced in Proposition [4.7].

**Proof.** Using Propositions 4.1 and 4.2 we have that

$$
-\langle \Lambda_{\delta,\varepsilon}(h), h \rangle \leq -c_0 \sum_{g \in \{h^\pm\}} \| \langle \cdot \rangle^{\gamma/2} g(\xi) \|_{L^\pi_h} \{ |\xi| \geq \frac{1}{\varepsilon} \} \| b_1 \| L^\pi_b \| \langle v \rangle^{\gamma/2} h \|_{L^2_h}^2 + C \| \theta^2 b_2 \|_{L^\pi_h} \| \langle v \rangle^{\gamma/2} h \|_{L^2_h}^2.
$$

Note that $\| b_1 \|_{L^\pi_b} \sim \kappa_0 \varepsilon^{-2s}$ and $\| \theta^2 b_2 \|_{L^\pi_b} \sim \kappa_0 \varepsilon^{2-2s}$. Thus, we may choose any $\varepsilon$ such that

$$
\varepsilon \leq \min \left\{ \left( \frac{1}{\varepsilon_0} \right)^{1/2}, \left( \frac{\varepsilon_0^{1/\gamma}}{\varepsilon_0^{1/\gamma}} \right)^{1/2} \right\} \triangleq \varepsilon_0,
$$

and obtain

$$
-\langle \Lambda_{\delta,\varepsilon}(h), h \rangle \leq -2c_0 \kappa_0 \sum_{g \in \{h^\pm\}} \| \langle \cdot \rangle^{\gamma/2} g(\xi) \|_{L^\pi_h} \{ |\xi| \geq \frac{1}{\varepsilon} \} \| b_1 \| L^\pi_b \| \langle v \rangle^{\gamma/2} h \|_{L^2_h}^2 + \| \langle v \rangle^{\gamma/2} h \|_{L^2_h}^2.
$$

This proves the result after using Lemma 2.2 in the last inequality.

In the sequel, we fix $\varepsilon = \varepsilon_0$ and $\delta_0 := \left( \frac{\varepsilon_0^{1/\gamma}}{\varepsilon_0} \right)^{1/2}$. We also set $\Delta \triangleq \Lambda_{\delta_0,\varepsilon_0}$ and denote the dissipative operator as

$$
\mathcal{L}^{(\mu)}_D \triangleq -\left( \Delta + v \cdot \nabla_x \right).
$$

This operator is closed in $L^2_{x,v}$.

**Proposition 4.3.** The spectrum of $\mathcal{L}^{(\mu)}_D$, as operator on $L^2_{x,v}$, lies in $\{ z \in \mathbb{C} : \Re z \leq -c_0 \kappa_0 \}$. 

Proof. Note that the domain $\mathcal{D}(L_D^{(\mu)})$ is dense in $L^2_{x,v}$, for instance, it contains $C^1_{x,v}$-functions with strong velocity decay. We now prove the existence and uniqueness of the problem

$$(-L_D^{(\mu)} + \lambda \mathbf{1})u = f \in L^2_{x,v}, \quad \lambda \in \mathbb{C}. \quad (4.12)$$

Writing $u = u_R + i u_I$, $f = f_R + i f_I$, and $\lambda = \Re \lambda_R + i \Re \lambda_I$, problem $\text{(4.12)}$ is equivalent to the 2-system of real valued problems

$$\left((-L_D^{(\mu)} + \lambda_R) \mathbf{1}_{2 \times 2} + \lambda_I \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\begin{bmatrix} u_R \\ u_I \end{bmatrix} = \begin{bmatrix} f_R \\ f_I \end{bmatrix}. \quad (4.13)$$

We start perturbing this problem to

$$\left((-L_D^{(\mu)} - \epsilon L_P + \lambda_R) \mathbf{1}_{2 \times 2} + \lambda_I \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\begin{bmatrix} u_R \\ u_I \end{bmatrix} = \begin{bmatrix} f_R \\ f_I \end{bmatrix}, \quad (4.14)$$

where

$$L_P := \left(\nabla_x \cdot (v)^{\gamma+2} \nabla_x + (v)^{\gamma+2} \Delta_x - (v)^{\gamma+2}\right) \mathbf{1}_{2 \times 2}.$$ 

This leads us to introduce the bilinear form $B^*[\cdot, \cdot]: \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ with Hilbert space given by

$$\mathcal{H} := H^1_{x,v}(v)^{\gamma/2+1}) \times H^1_{x,v}(v)^{\gamma/2+1}),$$

and, using the definition of $L_D^{(\mu)}$ and $L_P$, weak formulation

$$B^*[u, w] := \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left((Au) \cdot w - u \cdot (v \cdot \nabla_x w) + \epsilon (v)^{\gamma+2} \nabla_x u \cdot \nabla_x w + \nabla_x u \cdot \nabla_x w + u \cdot w + \lambda_R u \cdot w + \lambda_I \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u \cdot w \right) dv dx.$$

Thanks to Proposition $\text{2.2}$ and the arguments given previously in this section, it follows that

$$|B^*[u, w]| \leq (C(\mu) + |\lambda| + \epsilon)\|u\|_{\mathcal{H}}\|w\|_{\mathcal{H}}.$$

In addition, thanks to Theorem $\text{4.1}$ as long as $\lambda_R + c_0\kappa_0 > 0$ it follows that

$$B^*[u, w] \geq (c_0\kappa_0 + \lambda_R) \int_{\mathbb{T}^3} \|v\|^{\gamma/2} u^2_{H^s_x \times H^s_v} dx + \epsilon \|u\|_{\mathcal{H}}^2 \geq \epsilon \|u\|_{\mathcal{H}}^2. \quad (4.15)$$

Note that the antisymmetric term related to $\lambda_I$ vanishes. As a consequence, invoking Lax-Milgram theorem, for any $f$ in the dual of $\mathcal{H}$ (in particular, for any $f \in L^2_{x,v} \times L^2_{x,v}$) one has a unique $u \in \mathcal{H}$ such that

$$B^*[u, w] = \langle f, w \rangle, \quad \forall w \in \mathcal{H}, \quad \forall \epsilon > 0.$$

This provides existence and uniqueness for problem $\text{(4.14)}$ as long as $\lambda_R + c_0\kappa_0 > 0$. Furthermore, using the first inequality in estimate $\text{(4.15)}$, one concludes that for $f \in L^2_{x,v} \times L^2_{x,v}$ the weak solution to problem $\text{(4.14)}$ satisfies

$$(c_0\kappa_0 + \lambda_R) \int_{\mathbb{T}^3} \|v\|^{\gamma/2} u^2_{H^s_x \times H^s_v} dx \leq B^*[u, w] = \langle f, u \rangle \leq \|f\|_{L^2_{x,v} \times L^2_{x,v}} \|u\|_{L^2_{x,v} \times L^2_{x,v}};$$

that is,

$$\left(\int_{\mathbb{T}^3} \|v\|^{\gamma/2} u^2_{H^s_x \times H^s_v} dx\right)^{1/2} \leq (c_0\kappa_0 + \lambda_R)^{-1} \|f\|_{L^2_{x,v} \times L^2_{x,v}}. \quad (4.16)$$

Now for a fixed $f \in L^2_{x,v} \times L^2_{x,v}$ take the sequence of solutions $\{u^\epsilon\}$ with $\epsilon \to 0$ to problem $\text{(4.14)}$. By previous estimate, there exists (up to a subsequence) a weak limit $u_f \in L^2_{x,v} \times L^2_{x,v}$. Clearly, such limit satisfies problem $\text{(4.13)}$ in the sense of distributions with estimate $\text{(4.16)}$. Furthermore, any solution to

$\text{L}_D^{(\mu)} \mathbf{1}_{2 \times 2} u_f = -f + \lambda R u_f \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u_f \in L^2_{x,v} \times L^2_{x,v}$. 

\text{We note here that each term in the evaluation $L_D^{(\mu)} \mathbf{1}_{2 \times 2} u_f$ is not, in general, an $L^2_{x,v}$ function. However, one has $L_D^{(\mu)} \mathbf{1}_{2 \times 2} u_f = -f + \lambda R u_f \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u_f \in L^2_{x,v} \times L^2_{x,v}$.}
Proof. Proposition 4.4. The eigenvectors associated to the eigenvalues of \( L^{(\mu)} \) in \( L^2_{x,v} \times L^2_{x,v} \) satisfies estimate (4.16). Therefore, by linearity, solutions are unique in this space. Additionally, estimate (4.16) shows that \( D(L^{(\mu)}_o) \subset H^0_{x,v}(\{v\gamma/2\}) \subset L^2_{x,v} \). This proves that any \( \lambda \in \mathbb{C} \) such that \( \lambda_R > -c_0\kappa_0 \) belongs to the range of \( L^{(\mu)}_o \). \( \square \)

Remark 4.1. By the same token, the spectrum of \(-\Lambda\), as operator on \( L^2_v \), lies in \( \{ z \in \mathbb{C} : \Re z \leq -c_0\kappa_0 \} \). Since \( \Lambda \) is self-adjoint we conclude that Spectrum \( (-\Lambda) \subset (-\infty, -c_0\kappa_0] \).

4.1.2. Localization of the spectrum. We know that \( \mathcal{K} \triangleq \mathcal{K}_{\delta_o,\varepsilon_o} \) is continuous in \( L^2_v \), that is,
\[
\|\mathcal{K}(h)\|_{L^2_v} \leq C(\delta_o,\varepsilon_o)\|h\|_{L^2_v},
\]
with \( C(\delta_o,\varepsilon_o) \) depending only on mass and energy of \( \mu \). Let us prove \( \mathcal{K} \) is \( \Lambda \)-compact by taking a sequence \( \{h_n\} \subseteq \mathcal{D}(\Lambda) \subseteq L^2_v \) such that both \( \{h_n\} \) and \( \{\Lambda h_n\} \) are bounded in \( L^2_v \). Then, by Theorem 4.1,
\[
c_0\kappa_0\|\langle \nu\rangle^{\gamma/2} h_n\|_{L^2_v}^2 \leq \langle \Lambda h_n, h_n \rangle \leq \|\Lambda h_n\|_{L^2_v} \|h_n\|_{L^2_v}.
\]
Thus, \( \{h_n\} \) is compact in \( L^2_v \). Using Weyl’s theorem for stability of essential spectrum under relative compact (self-adjoint) perturbations we just proved the following result.

Corollary 4.1. The essential spectrum of \( L^{(\mu)} = \mathcal{K} - \Lambda \), as an operator on \( L^2_v \), lies in \( (-\infty, -c_0\kappa_0] \). In particular, if \( 0 \in \text{Spec}(L^{(\mu)}) \), it will be a discrete eigenvalue and, thus, the kernel of \( L^{(\mu)} \) will be finite dimensional.

Recall that the Dirichlet form of \( L^{(\mu)} \) is non positive, \( \langle L^{(\mu)} h, h \rangle \leq 0 \). This implies, since \( L^{(\mu)} \) is self-adjoint, that the discrete spectrum of \( L^{(\mu)} \) lies in \( (-\infty, 0] \). As a consequence, the restriction of \( L^{(\mu)} \) to \( \overset{o}{L}^2_v \setminus \text{Ker}(L^{(\mu)}) \), denoted by \( L^{(\mu)}_o \), is invertible with inverse \( (L^{(\mu)}_o)^{-1} \) and with domain \( D((L^{(\mu)}_o)^{-1}) = \overset{o}{L}^2_v \setminus \text{Ker}(L^{(\mu)}) \). This observation together with Corollary 4.1 show that, in fact, \( L^{(\mu)}_o \) has a spectral gap (denoted by \( \lambda_o > 0 \))
\[
\langle L^{(\mu)}_o h, h \rangle \leq -\lambda_o \|h\|_{L^2_v}^2, \quad h \in \overset{o}{L}^2_v \setminus \text{Ker}(L^{(\mu)}).
\]
This leads to the following additional feature in the spectrum.

Proposition 4.4. The eigenvectors associated to the eigenvalues of \( L^{(\mu)} \) form a basis in \( \overset{o}{L}^2_v \).

Proof. Note that \( (L^{(\mu)}_o)^{-1} \) is compact. Indeed, take \( \{g_n\} \) a bounded sequence in \( \overset{o}{L}^2_v \) and set \( h_n = (L^{(\mu)}_o)^{-1} g_n \). Then, thanks to Theorem 4.1 and the continuity of \( \mathcal{K} \), it follows that
\[
-\langle g_n, h_n \rangle = -\langle L^{(\mu)}_o(h_n), h_n \rangle = \langle \Lambda(h_n), h_n \rangle - \langle \mathcal{K}(h_n), h_n \rangle \geq c_o\kappa_0 \|h_n\|_{H^0_{x,v/2}}^2 - C \|h_n\|_{L^2_v}^2.
\]
We conclude by Cauchy-Schwarz inequality that
\[
c_o\kappa_0 \|h_n\|_{H^0_{x,v/2}}^2 \leq \|h_n\|_{L^2_v} \|g_n\|_{L^2_v} + C \|h_n\|_{L^2_v}^2 \leq \bar{C}(\|g_n\|_{L^2_v}^2 + \|h_n\|_{L^2_v}^2).
\]
(4.17)
Furthermore, since \( L^{(\mu)}_o \) has spectral gap in \( \overset{o}{L}^2_v \setminus \text{Ker}(L^{(\mu)}) \), one has that
\[
-\langle g_n, h_n \rangle = -\langle L^{(\mu)}_o(h_n), h_n \rangle \geq \lambda_o \|h_n\|_{L^2_v}^2.
\]
As a consequence, again by Cauchy-Schwarz inequality,
\[
\|h_n\|_{L^2_v}^2 \leq \frac{1}{\lambda_o} \|g_n\|_{L^2_v}^2.
\]
(4.18)
 Gathering the estimates (4.17) and (4.18) lead to
\[
\|h_n\|_{H^0_{x,v/2}(\text{div})}^2 \leq \frac{\bar{C}}{c_o\kappa_0} \left( 1 + \frac{1}{\lambda_o^2} \right) \|g_n\|_{L^2_v}^2.
\]
which proves that \( (L^{(μ)}_0)^{-1} \) is compact as an operator onto \( L^2_\mathbb{X} \setminus \text{Ker}(L^{(μ)}) \). Being the inverse of a self-adjoint operator, it is also self-adjoint. Therefore, by the Divergence theorem for compact self-adjoint operators, its eigenvectors, and hence the eigenvectors of \( L^{(μ)}_0 \), form a basis of \( L^2_\mathbb{X} \setminus \text{Ker}(L^{(μ)}) \). Together with the eigenvectors of the null space one obtains a basis of \( L^2_\mathbb{X} \) composed of the eigenvectors of \( L^{(μ)} \).

\[ \text{Remark 4.2.} \text{ Recall that the discrete spectrum of compact operators accumulate at 0. As a consequence, the proof of Proposition 4.4 shows a difference between the discrete spectrum of } \mathcal{L}^{(μ)} \text{ in the cutoff and non cutoff cases. In the later, the discrete spectrum decreases up to } -\infty. \]

In this final part of the section we localize the spectrum of the operator

\[ L^{(μ)} - v \cdot \nabla x = \mathcal{K} + \mathcal{L}^{(μ)}_D. \]

\[ \text{Lemma 4.1.} \text{ The operator } \mathcal{K} \text{ is relative compact with respect to } \mathcal{L}^{(μ)}_D. \]

\[ \text{Proof.} \text{ Take a sequence } \{h_n\} \subset \mathcal{D}(\mathcal{L}^{(μ)}_D) \subset L^2_{\mathbb{X},v} \text{ such that both } \{h_n\} \text{ and } \{\mathcal{L}^{(μ)}_D(h_n)\} \text{ are bounded in } L^2_{\mathbb{X},v}. \]

Then, by the Divergence theorem and Theorem 4.1,

\[ c_o\kappa_0 \int_{\mathbb{T}^3} \|\langle v \rangle^{1/2} h_n \|_{L^2_{\mathbb{X},v}}^2 \leq \int_{\mathbb{T}^3} \langle \mathcal{L}^{(μ)}_D(h_n) \rangle_{\mathbb{X},v} \leq \int_{\mathbb{T}^3} \|\mathcal{L}^{(μ)}_D(h_n)\|_{L^2_{\mathbb{X},v}} \|h_n\|_{L^2_{\mathbb{X},v}} \leq \|\mathcal{L}^{(μ)}_D(h_n)\|_{L^2_{\mathbb{X},v}} \|h_n\|_{L^2_{\mathbb{X},v}}. \]

As a consequence, using \[ 12 \] Proposition 1.1\[ 1 \] it also follows that

\[ \|(-\Delta_x) \rtimes h_n \|_{L^2_{\mathbb{X},v}}^2 \leq C_{d,s} \|(\Delta_x) \frac{1}{2} h_n \|_{L^2_{\mathbb{X},v}} \| \mathcal{L}^{(μ)}_D(h_n) \|_{L^2_{\mathbb{X},v}} \leq C_{d,s}(c_o\kappa_0)^{-\frac{1}{2}} \| \mathcal{L}^{(μ)}_D(h_n) \|_{L^2_{\mathbb{X},v}} \| h_n \|_{L^2_{\mathbb{X},v}}. \]

Thus,

\[ \sup \left\{ \int_{\mathbb{T}^3} \|\langle v \rangle^{1/2} h_n \|_{H^s_{\mathbb{X}}}^2 + \|(-\Delta_x) \rtimes h_n \|_{L^2_{\mathbb{X},v}}^2 \right\} \leq C \left( \| \mathcal{L}^{(μ)}_D(h_n) \|_{L^2_{\mathbb{X},v}}^2 + \| h_n \|_{L^2_{\mathbb{X},v}}^2 \right), \]

which implies that \( \{h_n\} \) is compact in \( L^2(\mathbb{T}^3 \times \mathbb{R}^3) \). That is, \( \mathcal{K} \) is \( \mathcal{L}^{(μ)}_D \)-compact.

\[ \text{□} \]

\[ \text{Proposition 4.5.} \text{ The essential spectrum of } L^{(μ)} - v \cdot \nabla x \text{ lies in } \{ z \in \mathbb{C} : \Re z < -c_o\kappa_0 \}. \text{ Furthermore, the set } \{ z \in \mathbb{C} : \Re z > -c_o\kappa_0 \} \text{ is contained in the resolvent of } L^{(μ)} - v \cdot \nabla x \text{ except, possibly, for countably many eigenvalues.} \]

\[ \text{Proof.} \text{ We use a similar argument given in } [28] \text{ proof of Proposition 3.4] using relative compact perturbations in Banach spaces. More precisely, one uses } [23] \text{ Chapter IV - Theorem 5.35 and footnote} \text{ that asserts that, given Lemma 4.1}, \text{ } \mathcal{K} + \mathcal{L}^{(μ)}_D \text{ and } \mathcal{L}^{(μ)}_D \text{ have the same complementary of the Fredholm domain. Using Corollary 4.3} \text{ this implies that}

\[ \text{Complementary of the Fredholm domain of } L^{(μ)} - v \cdot \nabla x \subset \{ z \in \mathbb{C} : \Re z \leq -c_o\kappa_0 \}. \]

\[ \text{(4.19)} \]

Now, the Fredholm set is composed by a countable number of connected open set in which

\[ \text{null}(z) := \text{ dimension of null space of } T - z, \]

\[ \text{def}(z) := \text{ codimension of the range of } T - z \]

are finite and constant, and, a countable set of isolated values points (the eigenvalues). It is known that the boundary of each of these components belong to the complementary of the Fredholm domain. As a consequence, given \[ 4.19 \], the intersection of the Fredholm set and \( \{ z \in \mathbb{C} : \Re z > -c_o\kappa_0 \} \) is composed of, only, one component and a countable number of eigenvalues. Since \( (a, \infty) \), for any \( a \geq \|K\|_2 \), belongs to the resolvent of \( L^{(μ)} - v \cdot \nabla x \) one concludes that this component is part of the resolvent, that is, \( \text{null}(z) = 0 \) and \( \text{def}(z) = 0 \) in \( \{ z \in \mathbb{C} : \Re z > -c_o\kappa_0 \} \) except for a countably many eigenvalues. Thus, the essential spectrum

\[ ^2 \text{This proposition is shown for } x \in \mathbb{R}^3. \text{ The same argument applies for } x \in \mathbb{T}^3 \text{ using Fourier series instead of Fourier transform.} \]
lies in \( \{ z \in \mathbb{C} : \Re z \leq -c_0 \kappa_0 \} \) and the set \( \{ z \in \mathbb{C} : \Re z > -c_0 \kappa_0 \} \) is contained in the resolvent of \( L^{(\mu)} - v \cdot \nabla_x \) except for a countably many eigenvalues.

**Theorem 4.2.** The operator \( L^{(\mu)} - v \cdot \nabla_x \), as an operator in \( L^2_{\mu,v} \), has essential spectrum localized in \( \{ z \in \mathbb{C} : \Re z \leq -c_0 \kappa_0 \} \). Furthermore, its eigenpairs are identical to those of \( L^{(\mu)} \) as an operator in \( L^2_v \).

**Proof.** It remains to prove that the eigenpairs of \( L^{(\mu)} - v \cdot \nabla_x \) and \( L^{(\mu)} \) are identical. Take first an \((\lambda, \varphi(v))\) eigenpair of \( L^{(\mu)} \). Then,

\[
(L^{(\mu)} - v \cdot \nabla_x) \varphi(v) = L^{(\mu)}(\varphi(v)) = \lambda \varphi(v).
\]

Therefore, \((\lambda, \varphi(v))\) is also an eigpair of \( L^{(\mu)} - v \cdot \nabla_x \). Now, take \((\lambda, \varphi(x,v))\) an eigenpair of \( L^{(\mu)} - v \cdot \nabla_x \).

Since the set of eigenvectors \( \{ \varphi_i \} \) of \( L^{(\mu)} \) form a base in \( L^2_v \) by Proposition 4.4 we can write the separation of variables

\[
\varphi(x,v) = \sum_{i \in \mathbb{N}} a_i(x) \varphi_i(v).
\]

Plugging this expression into the equation

\[
(L^{(\mu)} - v \cdot \nabla_x) \varphi(x,v) = \lambda \varphi(x,v),
\]

we conclude that

\[
a_i(x)(\lambda_i - \lambda) = v \cdot \nabla_x a_i(x), \quad \forall v \in \mathbb{R}^3, \quad i \in \mathbb{N},
\]

for eigenpairs \((\lambda_i, \varphi_i)\) of \( L^{(\mu)} \). For the set \( \mathcal{I}_1 = \{ i \in \mathbb{N} : \lambda_i \neq \lambda \} \), the right side of (4.20) is a function of velocity and the left side is not. We conclude that \( a_i(x) = a_i = 0 \) for every \( i \in \mathcal{I}_1 \). Note that \( \mathcal{I}_1^c \neq \emptyset \), otherwise \( \varphi(x,v) = 0 \). In \( \mathcal{I}_1^c \) we conclude immediately that \( \lambda \) is eigenvalue of \( L^{(\mu)} \), \( a_i(x) = a_i \) for any \( i \in \mathcal{I}_1^c \), and

\[
\varphi(x,v) = \sum_{i \in \mathcal{I}_1^c} a_i \varphi_i(v) \triangleq \varphi(v) = \text{eigenvector of } L^{(\mu)} \text{ associated to } \lambda.
\]

\[ \square \]

### 4.2. Localization of the spectrum in polynomially weighted \( L^2_{\mu,v} \)

In this section we want to “enlarge” the localization of the spectrum of the linearized Boltzmann operator from the space \( E = L^2(\mu^{-1/2}, \mathbb{T}^3 \times \mathbb{R}^3) \) to the space \( \mathcal{E} = L^2_{\mu}(\langle v \rangle^k, \mathbb{T}^3 \times \mathbb{R}^3) \) with \( k \geq 2 \). The idea of the enlargement of space in the Boltzmann context was introduced in [28] to study rate of convergence of the homogeneous problem. In fact, we will use a later development [17] Theorem 2.1 to facilitate the discussion, although, the argument could be accomplished with classical perturbation theory, as done in [28]. Let us first introduce the operators we work with in this section

\[
L(h) \triangleq Q(\mu, h) + Q(h, \mu),
\]

which is the operator that naturally appears after the linearization \( f = \mu + h \) in the nonlinear problem. We will consider \( L \) as a closed operator in \( L^2_{\mu}(\langle v \rangle^k, \mathbb{R}^3) \), with \( k \geq 2 \). The final objective is then to study the spectral properties in \( L^2_{\mu,v}(\langle v \rangle^k, \mathbb{T}^3 \times \mathbb{R}^3) \) of the (closed) operator

\[ L - v \cdot \nabla_x. \]

Given [17] Theorem 2.1 and Remark 2.2 (1)], we will be able to localize the spectrum in the larger space \( \mathcal{E} \) by knowing the following:

(i) The localization of the spectrum of \( L - v \cdot \nabla_x \) in the smaller space \( E \).

(ii) The operator decomposes as \( L = A - B \) where \( B \) is a (closed) dissipative operator and \( A : \mathcal{E} \to E \) is bounded.
Regarding item (i), this is exactly what we did in previous section. Regarding the decomposition in (ii), we use the analogous decomposition adding the advection operator

\[
L_{\Phi,b} - v \cdot \nabla x = L_{\Phi,b_1} + L_{\Phi,b_2} - v \cdot \nabla x = L_{\Phi_1,b_1} + L_{\Phi_2,b_1} + L_{\Phi,b_2} - v \cdot \nabla x
\]

\[
= \left( L_{\Phi_1,b_1} + Q_{\Phi_1,b_2}(\mu, h) \right) + \left( L_{\Phi_2,b_1} - Q_{\Phi_2,b_1}(\mu, h) \right) + L_{\Phi,b_2} - v \cdot \nabla x \]

(4.21)

4.2.1. Dissipative part. We already pointed out in the previous section that \( B_\delta \) involves all singular part of the operator and decomposes in the tail associated component

\[
-B_1 \triangleq L_{\Phi,b_1} - Q_{\Phi_1,b_1}(\mu, h),
\]

and the regularity associated component

\[
-B_2 \triangleq L_{\Phi,b_2} - v \cdot \nabla x.
\]

Let us proceed, as we did previously, and prove that \( B \) is indeed a dissipative operator for suitable choices of \( \delta > 0 \) and \( \varepsilon > 0 \) depending only on the mass of energy of \( \mu \).

**Proposition 4.6** (Singular part \( L_{\Phi,b_2} \)). Let \( s \in (0, 1) \). For any \( k > \frac{9}{2} + \frac{2}{\varepsilon} + 2s \) there exist constants \( c > 0 \), depending only on mass and energy of \( \mu \), and \( C_k > 0 \) such that

\[
\langle L_{\Phi,b_2}(h), h\langle v \rangle^{2k} \rangle \leq -c_k \| \langle \cdot \rangle^{\gamma/2+k} h(\xi) \|_a^2 \left\{ \| \xi \| \geq \frac{1}{2} \right\}^2 + C_k \| \theta^2 b_2 \nu^\theta \| L^2 \| \langle v \rangle^{\gamma/2+k} h \| L^2.
\]

The constants are independent of \( \varepsilon \).

**Proof.** Let us compute

\[
\langle L_{\Phi,b_2}(h), h\langle v \rangle^{2k} \rangle = \langle Q_{\Phi,b_2}(\mu, h), h\langle v \rangle^{2k} \rangle + \langle Q_{\Phi,b_2}(h, \mu), h\langle v \rangle^{2k} \rangle
\]

\[
= \int_{\mathbb{R}^d} \int_{S^2} \Phi(u)b_2(\cos \theta) \mu(v_s)h(v)\langle h(v')\langle v' \rangle^{2k} - h(v)\langle v \rangle^{2k} \rangle d\sigma dv_s dv
\]

\[
+ \int_{\mathbb{R}^d} \int_{S^2} \Phi(u)b_2(\cos \theta) \mu(v_s)\mu(v)\langle h(v')\langle v' \rangle^{2k} - h(v)\langle v \rangle^{2k} \rangle d\sigma dv_s dv.
\]

Let us consider each of these integral on the right side separately. For the first integral,

\[
\langle Q_{\Phi,b_2}(\mu, h), h\langle v \rangle^{2k} \rangle = \int_{\mathbb{R}^d} \int_{S^2} \Phi(u)b_2(\cos \theta) \mu(v_s)h(v)h(v')\langle v' \rangle^{2k}(\langle v' \rangle^{k} - \langle v \rangle^{k}) d\sigma dv_s dv
\]

\[
+ \int_{\mathbb{R}^d} \int_{S^2} \Phi(u)b_2(\cos \theta) \mu(v_s)\mu(v)\langle h(v')\langle v' \rangle^{2k} - h(v)\langle v \rangle^{2k} \rangle d\sigma dv_s dv = \Gamma^{b_2}(\mu, h, \langle \cdot \rangle^k) + \langle Q_{\Phi,b_2}(\mu, h, \langle \cdot \rangle^k), h\langle v \rangle^{2k} \rangle.
\]

(4.22)

where \( \Gamma^{b_2} \) is defined by \([3.2]\) with \( b \) replaced by \( b_2 \). Now, a similar (and simpler) argument given in the proof of Proposition \([3.1]\) recalling \([3.9]\) Proposition 2.1], shows

\[
\langle Q_{\Phi,b_2}(\mu, h, \langle \cdot \rangle^k), h\langle v \rangle^{2k} \rangle \leq -c_k \| \langle \cdot \rangle^{\gamma/2+k} h(\xi) \|_a^2 \left\{ \| \xi \| \geq \frac{1}{2} \right\}^2 + C_k \| \theta^2 b_2 \nu^\theta \| L^2 \| \langle v \rangle^{\gamma/2+k} h \| L^2.
\]

(4.23)

The estimation for \( \Gamma^{b_2}(\mu, h, \langle \cdot \rangle^k) \) is almost the same as in the proof of Proposition \([3.1]\). If one splits

\[
\Gamma^{b_2}(\mu, h, \langle \cdot \rangle^k) = \Gamma^{b_2}_{1,1} + \Gamma^{b_2}_{1,2} + \sum_{m=2}^{6} \Gamma^{b_2}_{m},
\]

by the same way, then it follows from the proof of Proposition \([3.1]\) that

\[
|\Gamma^{b_2}_{1,2}| + \sum_{m=2}^{6} |\Gamma^{b_2}_{m}| \leq C_k \| \theta^2 b_2 \nu^\theta \| L^2 \| \langle v \rangle^{\gamma/2+k} h \| L^2.
\]

(4.24)
By the same observation as for $\Gamma_{1,1}$ in the proof of Proposition 3.1 for any $0 < s < 1$ we have

\[ \Gamma_{1,1}^{b_2}(\mu, h, h(\cdot)^k) = k \int_{R^3 \times R^3 \times S^2_+} b_2(\cos \theta) \left| v - v_s \right|^{1+\gamma} (v_s \cdot \bar{\omega}) \cos^k \frac{\phi}{2} \sin \frac{\theta}{2} \mu_s \left( \frac{H}{v} - H' \right) H' d\pi, \]

where $H = h(v)^k$. Therefore

\[ \Gamma_{1,1}^b(\mu, h, h(\cdot)^k) = k \int_{R^3 \times R^3 \times S^2_+} b_2(\cos \theta) \left| v - v_s \right|^{1+\gamma} (v_s \cdot \bar{\omega}) \cos^k \frac{\phi}{2} \sin \frac{\theta}{2} \mu_s \left( H - H' \right) H' d\pi \]

\[ + k \int_{R^3 \times R^3 \times S^2_+} b_2(\cos \theta) \left| v - v_s \right|^{1+\gamma} \left( \frac{1}{v^2} - \frac{1}{(v')^2} \right) (v_s \cdot \bar{\omega}) \cos^k \frac{\phi}{2} \sin \frac{\theta}{2} \mu_s \left| H' \right|^2 d\pi \]

\[ \Delta \Gamma_{1,1,main}^{b_2}(\mu, h, h(\cdot)^k) + \Gamma_{1,1,rest}^{b_2}(\mu, h, h(\cdot)^k). \]

Since it follows from the mean value theorem that

\[ \left| v - v_s \right|^{1+\gamma} \left( \frac{1}{v^2} - \frac{1}{(v')^2} \right) \leq \left( \frac{\sqrt{2}}{2} \right)^3 |v_s|^3 \sin \frac{\theta}{2}, \]

\[ |\Gamma_{1,1,rest}^{b_2}| \text{ is estimated by } C_k \left\| \theta^2 b_2 \right\|_{L^1_v} \left\| \langle v \rangle^{\gamma/2+k} h \right\|_{L^2_v}. \] It follows from the Cauchy-Schwarz inequality that

\[ \Gamma_{1,1,main}^{b_2}(\mu, h, h(\cdot)^k) \leq C_k \left( \int_{R^3 \times R^3 \times S^2_+} b_2(\cos \theta) \left| v - v_s \right|^{\gamma+1} \mu_s \left( H - H' \right)^2 d\pi \right)^{1/2} \]

\[ \times \left( \int_{R^3 \times R^3 \times S^2_+} b_2(\cos \theta) \left| v - v_s \right|^{\gamma+1} \mu_s \left( H - H' \right)^2 d\pi \right)^{1/2} \]

\[ \leq C_k \left( -2 \langle Q_{\Phi, b_2} \left( \mu, h, h(\cdot)^k \right), h(\cdot)^k \rangle + \left\| \theta^2 b_2 \right\|_{L^1_v} \left\| \langle v \rangle^{\gamma/2+k} h \right\|_{L^2_v} \right)^{1/2} \left( \left\| \theta^2 b_2 \right\|_{L^2_v} \left\| \langle v \rangle^{\gamma/2+k} h \right\|_{L^2_v} \right)^{1/2}, \]

(4.25)

where we used the formula

\[ \int_{S^2_+} b_2(\cos \theta) \Phi \mu_s \left( H - H' \right)^2 d\pi = \left( -2 \langle Q_{\Phi, b_2} \left( \mu, h, h(\cdot)^k \right), h(\cdot)^k \rangle \right) + \int_{R^3 \times R^3 \times S^2_+} b_2(\cos \theta) \Phi \mu_s \left( H^2 - H'^2 \right) d\pi \]

and the cancellation lemma in 1. Summing up the above estimates we obtain

\[ \langle Q_{\Phi, b_2} \left( \mu, h, h(\cdot)^k \right), h(\cdot)^{2k} \rangle \leq -cn \left( |\cdot|^{\gamma/2+k} h(\xi) |\xi| \mathbb{1} \left\{ |\xi| \geq \frac{1}{\varepsilon} \right\} \right)^2 + C_k \left\| \theta^2 b_2 \right\|_{L^1_v} \left\| \langle v \rangle^{\gamma/2+k} h \right\|_{L^2_v}. \]

(4.26)

Let us move to the second integral

\[ \langle Q_{\Phi, b_2} \left( \mu, h, h(\cdot)^k \right), h(\cdot)^{2k} \rangle = \int_{R^3} \int_{R^3} \Phi(u) b_2(\cos \theta) \left( h(v_s) \mu(v) h(v') \langle v' \rangle^k - \langle v \rangle^k \right) d\sigma dv, dv \]

\[ + \int_{R^3} \int_{R^3} \Phi(u) b_2(\cos \theta) \left( h(v_s) \mu(v) h(v') \langle v' \rangle^k - h(v') \langle v \rangle^k \right) d\sigma dv, dv \]

\[ = \Gamma_{1,1}^{b_2}(h, \mu, h(\cdot)^k) + \langle Q_{\Phi, b_2} \left( \mu, h, h(\cdot)^k \right), h(\cdot)^k \rangle. \]

As for $\Gamma_{1,1}(h, \mu, h(\cdot)^k)$, we need only to consider

\[ \Gamma_{1,1}^{b_2}(h, \mu, h(\cdot)^k) = k \int_{R^3 \times R^3 \times S^2_+} b_2(\cos \theta) \left| v - v_s \right|^{1+\gamma} (v_s \cdot \bar{\omega}) \cos^k \frac{\phi}{2} \sin \frac{\theta}{2} \mu_s \mu(v(k-2) - \mu'(v') \langle v' \rangle^k \left( H(v') \right) d\pi, \]

because the other terms of the decomposition has the same bound as the right hand side of (4.24). Since it follows from the mean value theorem that

\[ \left| v - v_s \right|^{1+\gamma} |\mu(v) \langle v \rangle^k - \mu'(v') \langle v' \rangle^k| \leq C_k \int_0^1 |v'' - v_s|^{2+\gamma} \mu(v'' \langle v'' \rangle^k)^{1/2} d\tau \sin \frac{\theta}{2}, \]

\[ v'' = v_s + \tau(v' - v_s) \]

\[ \leq C_k \int_0^1 \mu(v'' \langle v'' \rangle^k)^{1/2} d\tau \sin \frac{\theta}{2}, \]
we have

$$|\Gamma_{b_1}^{h_2}(h,\mu,h)\rangle \leq C_k \|\theta^2 b_2\|_{L^1}\|\langle v\rangle^{\gamma+2k}h\|_{L^2}.$$  

If we put $\tilde{\mu} = \langle v\rangle^k \mu$, then

$$\langle Q_{\Phi,b_2}(h,\mu\langle v\rangle^k),h\rangle = \int_{\mathbb{R}^6} \int_{S^2} \Phi(u)b_2(\cos \theta) h(v_*) (\tilde{\mu}(v) - \tilde{\mu}(v')) H' \, d\tau$$

$$+ \int_{\mathbb{R}^6} \int_{S^2} \Phi(u)b_2(\cos \theta) h(v_*) (\tilde{\mu}(v') H' - \tilde{\mu}(v) H) \, d\tau.$$  

Using the Taylor expansion

$$\tilde{\mu}(v) - \tilde{\mu}(v') = \nabla \tilde{\mu}(v') \cdot (v - v') + \int_0^1 (1 - \tau) \nabla \tilde{\mu}(v'_\tau) d\tau (v - v')^2$$

to the first term and applying the cancellation lemma to the second term, we obtain that

$$|\langle Q_{\Phi,b_2}(h,\mu\langle v\rangle^k),h\rangle| \leq C_k \|\theta^2 b_2\|_{L^1}\|\langle v\rangle^{\gamma+2k}h\|_{L^2}.$$  

**Proposition 4.7** (Singular part $B_1$). There exist constant $c > 0$ depending only on the mass and energy of $\mu$, such that for any $s \in (0,1)$, $k > 4 + \frac{3+\gamma}{2}, \varepsilon \in (0,1)$ and $\delta^\gamma := \delta(k,\varepsilon)^\gamma \in (0,C_k,\varepsilon^{2s})$, it follows that for any $0 < \nu \leq 1$,

$$\langle L_{\Phi_{b_2}} h - Q_{\Phi_{b_1}}^\nu, (\mu, h) \rangle \leq -\nu \|\cos^{2k-(3+\gamma)}(\theta/2) - 2(1 + \nu) \sin^{2k-\frac{3+\gamma}{2}}(\theta/2)\|_{L^1}\|\langle v\rangle^{k+\gamma/2}h\|_{L^2}.$$  

In addition to mass and energy of $\mu$ and $k$, the constant $C_k$ also depends on $b,s,\gamma$.

Before starting with the proof observe that for $0 < \theta \leq \pi/2$ and $k > 4 + \frac{3+\gamma}{2}$,

$$1 - \cos(\theta/2)^{2k-(3+\gamma)} - 2(1 + \nu) \sin^{2k-\frac{3+\gamma}{2}}(\theta/2) \geq \sin^2(\theta/2) \left(1 - 2(1 + \nu) \sin^{2k-\frac{3+\gamma}{2}}(\theta/2)\right) \geq \sin^2(\theta/2) \left(1 - \frac{2(1 + \nu)}{2}\right) > 0.$$  

**Proof.** The proof is similar to those for Propositions 3.2 and 3.3. However, since the coefficients involved need to be more explicit, we show the full details here. Note that

$$L_{\Phi_{b_2}} h - Q_{\Phi_{b_1}}^\nu(h, \mu) = Q_{\Phi_{b_2}}(h, \mu) + Q_{\Phi_{b_2}}^+ (h, \mu) - Q_{\Phi_{b_1}}^-(h, \mu).$$

Let us first control the term $Q_{\Phi_{b_2}}(h, \mu)$. As before,

$$\langle Q_{\Phi_{b_2}}(h, \mu), h(\langle v\rangle^2) \rangle = \int_{\mathbb{R}^6} \int_{S^2} \Phi_2 (u)b_1(\cos \theta) h(v_*) \mu(v) \langle h(v') \langle v\rangle^{2k} - h(v) \langle v\rangle^{2k} \rangle d\sigma dv, dv$$

$$+ \int_{\mathbb{R}^6} \int_{S^2} \Phi_2 (u)b_1(\cos \theta) h(v_*) \mu(v) \langle h(v') \langle v\rangle^{k} - \langle v\rangle^{k} \rangle d\sigma dv, dv$$

Since $\Phi_2(u) \leq \delta^\gamma \langle v\rangle^{2\gamma}$, the second integral in the right side is controlled by

$$\int_{\mathbb{R}^6} \int_{S^2} \Phi_2 (u)b_1(\cos \theta) |h(v_*)| \mu(v) \langle v\rangle^{k} \langle h(v') \langle v\rangle^{k} - h(v) \langle v\rangle^{k} \rangle d\sigma dv, dv$$

$$\leq \delta^\gamma C \|b_1\|_{L^1}\|\langle v\rangle^{2\gamma}\|_{L^2}\|\mu\|_{L^\infty}\|\langle v\rangle^{k+3\gamma/2}\|_{L^\infty}\|\langle v\rangle^{k+\gamma/2}\|_{L^\infty}.$$  

(4.27)
Indeed, for any $m > 0$ we have

$$
(1 + X)^m \leq (1 + \nu)X^m + \left(1 + \frac{1}{(1 + \nu)^{1/m} - 1}\right)^m,
$$

for any $X > 0$, \hspace{1cm} (4.29)

because $(1 + X)^m > (1 + \nu)X^m$ implies $X < 1/((1 + \nu)^{1/m} - 1)$. If we put $v' = v + \tau(v' - v)$ for $\tau \in [0, 1]$, then it follows from the mean value theorem that

$$
\left| \langle v \rangle^k - \langle v' \rangle^k \right| \leq k \int_0^1 \langle v' \rangle^{k-1} \frac{d\tau}{v' - v} \leq k \int_0^1 \left(1 + |v'|\right)^{k-1} \frac{d\tau}{v' - v}
$$

$$
\leq k \int_0^1 (\sqrt{2} \langle v \rangle + \tau |v' - v|)^{k-1} d\tau |v' - v|.
$$

Apply (4.29) with $m = k - 1$ and $X = |v' - v|/(\sqrt{2} \langle v \rangle)$. Then

$$
\left| \langle v \rangle^k - \langle v' \rangle^k \right| \leq (1 + \nu)k \int_0^1 \tau^{k-1} d\tau |v' - v| + k \left(1 + \frac{1}{(1 + \nu)^{1/(k-1)} - 1}\right)^{k-1} \left(\sqrt{2} \langle v \rangle\right)^{k-1} |v - v'|
$$

$$
= (1 + \nu)|v - v'|^k + C_{k, \nu} |v - v'| \langle v \rangle^{k-1},
$$

which proves (4.28).

The first integral is controlled by

$$
\int_{\mathbb{R}^6} \int_{S^2} \Phi_2(u)b_1(\cos(\theta))|h(v_*)|\mu(v)|h(v')|\langle v' \rangle^k \langle v' \rangle^k - \langle v \rangle^k \left| d\sigma d\nu, dv
$$

$$
\leq (1 + \nu) \int_{\mathbb{R}^6} \int_{S^2} \Phi_2(u) \sin(\theta/2)b_1(\cos(\theta))|h(v_*)|\langle v_* \rangle^k \mu(v)|h(v')|\langle v' \rangle^k \left| d\sigma d\nu, dv
$$

$$
+ C_{k, \nu} \delta^{\gamma/2} \int_{\mathbb{R}^6} \int_{S^2} \sin(\theta/2)b_1(\cos(\theta))|h(v_*)|\langle v_* \rangle^{1+3\gamma/2} \mu(v)|h(v')|\langle v' \rangle^k d\sigma d\nu, dv.
$$

The last integral in this inequality is controlled, for any $k > 1 + \frac{3\gamma+2\gamma}{2}$, as

$$
C_{k, \nu} \delta^{\gamma/2} \int_{\mathbb{R}^6} \int_{S^2} \sin(\theta/2)b_1(\cos(\theta))|h(v_*)|\langle v_* \rangle^{1+3\gamma/2} \mu(v)|h(v')|\langle v' \rangle^k d\sigma d\nu, dv
$$

$$
\leq \delta^{\gamma/2} C_{k, \nu} \|b_1\|_{L^1_\theta} \|h\|_{L^1}^{1+3\gamma/2} \|h\|_{L^2}^{k+3\gamma/2} \|h\|_{L^2}^{3\gamma+2\gamma/2} \leq \delta^{\gamma/2} C_{k, \nu} \|b_1\|_{L^1_\theta} \|h\|_{L^\infty}^{k+\gamma/2} \|h\|_{L^2}^{3\gamma+2\gamma/2}.
$$

For the first integral, one uses Cauchy–Schwarz inequality

$$
h(v_*)\langle v_* \rangle^k h(v')\langle v' \rangle^k \leq \frac{|h(v_*)|^2\langle v_* \rangle^{2k}}{2\sin(\theta/2)} + \frac{1}{2}|h(v')|^2\langle v' \rangle^{2k} \sin(\theta/2)^{\frac{3\gamma+2\gamma}{2}},
$$

the bound $\Phi_2(u) \leq |u|^\gamma$, and the change of variables $v' \to v_*$ in the second of the above terms to conclude that

$$
\int_{\mathbb{R}^6} \int_{S^2} \Phi_2(u) \sin(\theta/2)b_1(\cos(\theta))|h(v_*)|\langle v_* \rangle^k \mu(v)|h(v')|\langle v' \rangle^k d\sigma d\nu, dv
$$

$$
\leq \int_{\mathbb{R}^6} \int_{S^2} |u|^\gamma \sin(\theta/2)^{k-\frac{3\gamma+2\gamma}{2}} b_1(\cos(\theta)) |h(v_*)|^2 \langle v_* \rangle^{2k} \mu(v) d\sigma d\nu, dv.
$$

\hspace{1cm} (4.31)
Let us move now to the term \(Q_{\Phi_2, b_1}^+(\mu, h)\). We use the ideas of [14] Prop. 2.1 of page 131 [1] which give us
\[
\langle Q_{\Phi_2, b_1}^+(\mu, h), h(v)^{2k} \rangle \\
\leq \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \Phi_2 \left( \frac{u}{\cos(\theta/2)} \right) b_1 \left( \cos(\theta) \right) \cos^{-3}(\theta/2) \nu^{-1}(\theta) \mu(v_s)|h(v)|^2 \langle v \rangle^{2k} d\sigma dv, \\
+ \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \Phi_2(u)b_1(\cos(\theta))\nu(\theta)\mu(v_s)|h(v)|^2 \langle v' \rangle^{2k} d\sigma dv, \\
\leq \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \Phi_2(u)b_1(\cos(\theta))\cos^{-3-\gamma}(\theta/2) \nu^{-1}(\theta) \mu(v_s)|h(v)|^2 \langle v' \rangle^{2k} d\sigma dv, \\
+ \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \Phi_2(u)b_1(\cos(\theta))\nu(\theta)\mu(v_s)|h(v)|^2 \langle v' \rangle^{2k} d\sigma dv,
\]
for any \(\nu(\theta) > 0\). As a consequence,
\[
\langle Q_{\Phi_2, b_1}^+(\mu, h) - Q_{\Phi, b_1}^+(\mu, h), h(v)^{2k} \rangle \leq \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \Phi_2(u)b_1(\cos(\theta)) \\
\times \left( \cos^{-3-\gamma}(\theta/2)\nu(\theta)^{-1} - 1 \right) \mu(v_s)|h(v)|^2 \langle v \rangle^{2k} d\sigma dv \\
+ \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \Phi_2(u)b_1(\cos(\theta))\nu(\theta)\mu(v_s)|h(v)|^2 \langle v' \rangle^{2k} \nu(\theta) - \langle v \rangle^{2k} d\sigma dv.
\]
At this point one chooses \(\nu(\theta) := \cos^{-2k}(\theta/2)\). For the second integral in (4.32), one uses the classical formula for \(\omega \in \mathbb{S}^1\) with \(\omega \perp u\),
\[
\left| \langle v \rangle^{2k} - \cos(\theta/2)\langle v \rangle^{2k} \right|^2 \leq \frac{1}{2} \cos(\theta/2)^2 \langle v \rangle^{2k} \langle v \rangle^{2k} + 2 \cos(\theta/2)\sin(\theta/2)|u| v_s \cdot \omega.
\]
Thus, for any \(k \geq 2\) it follows from the Taylor expansion of the second order that
\[
\langle v \rangle^{2k} - \cos(\theta/2)\langle v \rangle^{2k} = k \left( \langle v \rangle^{2k} \cos(\theta/2) \right)^{k-1} \left( \langle v \rangle^{2k} \sin(\theta/2) + 2 \cos(\theta/2)\sin(\theta/2)|u| v_s \cdot \omega \right)
\]
\[
+ k(k-1) \int_0^1 (1-t) \left( \langle v \rangle^{2k} \cos(\theta/2) + t(\langle v \rangle^{2k} \sin(\theta/2) + 2 \cos(\theta/2)\sin(\theta/2)|u| v_s \cdot \omega \right)^k dt
\]
\[
\times \left( \langle v \rangle^{2k} \sin(\theta/2) + 2 \cos(\theta/2)\sin(\theta/2)|u| v_s \cdot \omega \right)^2
\]
\[
= 2k \langle v \rangle^{2k-2} \cos^{-1}(\theta/2)\sin(\theta/2)|u| v_s \cdot \omega \pm c_k \sin^2(\theta/2) \langle v \rangle^{2k} \langle v \rangle^{2k}.
\]
Therefore,
\[
\left| \int_{\mathbb{S}^1} \left( \langle v \rangle^{2k} - \cos(\theta/2)\langle v \rangle^{2k} \right) d\omega \right| \leq c_k \sin^2(\theta/2) \langle v \rangle^{2k} \langle v \rangle^{2k}.
\]
In this way, the second integral in (4.32) is estimated by
\[
\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \Phi_2(u) b_1(\cos(\theta/2)) \frac{1}{\cos(\theta/2)^{2k}} \left( \langle v \rangle^{2k} - \cos(\theta/2)\langle v \rangle^{2k} \right) \mu(v_s)|h(v)|^2 d\sigma dv \\
\leq c_k \delta \left\| b_1 \sin^2(\theta/2) \right\|_{L^1_\delta} \left\| \mu(x)^{2(\gamma+\beta)} \right\|_{L^2_\delta} \left\| h(v)^k \right\|_{L^2_\delta} \leq c_k \delta \left\| b_1 \sin^2(\theta/2) \right\|_{L^1_\delta} \left\| h(v)^k \right\|_{L^2_\delta}.
\]
Gathering (4.27), (4.30), (4.31), (4.32) and (4.33) one gets,
\[
\langle L_{\Phi_2, b_1} h - Q_{\Phi, b_1}^+(\mu, h), h(v)^{2k} \rangle \\
\leq -\frac{1}{2} \left\| (1-\cos^{2k-3(\gamma+\beta)}(\theta/2) - 2(1+\nu)\sin^{2k+\beta}(\theta/2))b_1 \right\|_{L^1_\delta} |h(v)|^2 \langle v \rangle^{2k} \left( \int_{\mathbb{R}^3} \mu(v_s)|u|\gamma dv_s \right) \\
+ \delta \gamma \left| b_1 \right|_{L^1_\delta} \left| h(v)^{k+\gamma/2} \right|_{L^2_\delta}, \quad k > 2 + \frac{3+\gamma}{2}.
\]
Now, one has the estimates
\[
\int_{\mathbb{R}^3} \mu(v_s)|u|\gamma dv_s \geq c \langle v \rangle^\gamma, \quad \left| b_1 \right|_{L^1_\delta} \sim \frac{\kappa_0}{2s \tilde{e}^{2s}},
\]
as a consequence, the proposition follows taking any \( \delta := \delta(k, \varepsilon) > 0 \) such that
\[
\frac{\delta^{1/2} C_{k, \varepsilon} \kappa_0}{2s \varepsilon^{2s}} \leq \frac{1}{4} \left(1 - \cos(\theta/2)^{2k-(3+\gamma)} - 2(1 + \nu) \sin^{\frac{3+\gamma}{2}} (\theta/2)\right) b_1 \| L^1_{\beta^0} .
\]

**Theorem 4.3.** Let \( k > \frac{9}{2} + \frac{3}{2} + 2s \) and \( h \in H^{k+\gamma/2}_{\alpha}(\mathbb{R}^3) \) with \( s \in (0,1) \). There exist constants \( c_0 > 0, \varepsilon_0 > 0 \) depending on the mass and energy of \( \mu \), the scattering kernel \( b \) and \( k \), such that for any \( \delta^{1/2} := \delta(k, \varepsilon_0) > 0 \in (0, C_k \varepsilon_0) \) with \( C_k > 0 \) given in the Proposition 4.7, the operator \( -B_1 + L_{\Phi_1, b_1} \) (with \( \varepsilon = \varepsilon_0 \)) satisfies the dissipative estimate
\[
\langle (-B_1 + L_{\Phi_1, b_1})(h), h(v)^{2k} \rangle \leq -c_0 \kappa_0 \| \langle v \rangle^{k+\gamma/2} h \|^2_{L^2_{\alpha}} . \tag{4.34}
\]

Furthermore, it follows from (4.34) that
\[
\langle -B_{3, \varepsilon} h, h(v)^{2k} \rangle_{L^2_{\alpha}} \leq -c_0 \kappa_0 \| \langle v \rangle^{k+\gamma/2} h \|^2_{L^2_{\alpha}} . \tag{4.35}
\]

**Proof.** Using Propositions 4.6 and 4.7 one has
\[
\langle (-B_1 + L_{\Phi_1, b_1})(h), h(v)^{2k} \rangle \leq -c_1 \kappa_0 \left\| \langle \rangle^{\gamma/2+k} h(\xi) \right\| \left\{ \left\| \xi \right\| \geq \frac{1}{2} \right\} \|^2_{L^2_{\alpha}} + \left( C_k \| \theta^2 b_2 \|_{L^1_{\beta^0}} - c_2 \| (1 - \cos(\theta/2)^{2k-(3+\gamma)} - 2(1 + \nu) \sin^{\frac{3+\gamma}{2}} (\theta/2) \| b_1 \| L^1_{\beta^0} \right) \| h \|_{L^2_{\alpha}} \]

for positive constants \( c_1, c_2 \) that depend only on the mass and energy of \( \mu \) and positive constant \( C_k \) that depends additionally on \( k \). Since \( \| \theta^2 b_2 \|_{L^1_{\beta^0}} \sim \frac{\kappa_0 \varepsilon^2}{2 - 2s} \), one can choose \( \varepsilon = \varepsilon_0 > 0 \) sufficiently small such that
\[
C_k \| \theta^2 b_2 \|_{L^1_{\beta^0}} \leq \frac{\kappa_0}{2} \left(1 - \cos(\theta/2)^{2k-(3+\gamma)} - 2(1 + \nu) \sin(\theta/2)^{k-\frac{3+\gamma}{2}} \right) b_1 \| L^1_{\beta^0} .
\]

Clearly \( \varepsilon_0 \) has the aforementioned dependence on the parameters. The choice
\[
c_0 \Delta \min \left\{ c_1, \frac{c_2 \varepsilon^2}{4k_0} \left(1 - \cos(\theta/2)^{2k-(3+\gamma)} - 2(1 + \nu) \sin(\theta/2)^{k-\frac{3+\gamma}{2}} \right) b_1 \| L^1_{\beta^0} \right\}
\]

proves the first statement (4.34). Using the divergence theorem one has \( \langle -v \cdot \nabla_x h, h(v)^{2k} \rangle_{x,v} = 0 \), which proves the second statement (4.35).

4.2.2. Bounded part. Let us consider the operator
\[
A_{3, \varepsilon} h = L_{\Phi_1, b_1} + Q_{\Phi_1, b_1} (\mu, h) = Q_{\Phi_1, b_1} (h, \mu) + Q_{\Phi_1, b_1}^+ (\mu, h) .
\]

The following bound for \( A_{3, \varepsilon} \) holds:

**Proposition 4.8.** For any \( \delta > 0, \varepsilon > 0, k > \frac{3}{2} \) the operator \( A_{3, \varepsilon} : L^2 ((v)^k, \mathbb{R}^3) \rightarrow L^2 (\mu^{-1/2}(v), \mathbb{R}^3) \) is a bounded operator with norm estimated as
\[
\| A_{3, \varepsilon} \| \leq 3 \delta^{-\gamma} \varepsilon^{\frac{3+\gamma}{2}} \| b_1 \|_{L^1_{\beta^0}} \max \left\{ \| \mu^{1/4} \|_{L^1}, \| \mu^{1/4} \|_{L^2} \right\} .
\]

In particular, the operator \( A_{3, \varepsilon} : L^2 ((v)^k, T^3 \times \mathbb{R}^3) \rightarrow L^2 (\mu^{-1/2}(v), T^3 \times \mathbb{R}^3) \) is bounded with the same norm.

**Proof.** Noticing that Cauchy Schwarz inequality implies that for any \( \epsilon > 0 \)
\[
|v|^2 = |v - u|^2 \leq (1 + \epsilon^-1) |v|^2 + (1 + \epsilon) |u|^2 ,
\]
\[
|v|^2 = |v^+|^2 \leq (1 + \epsilon^-1) |v^+|^2 + (1 + \epsilon) |u|^2 , \quad u^\pm := \frac{1}{2} (u \pm |u|) ,
\]

one can choose \( \epsilon = 2 \) to prove that \( |v|^2 \leq \frac{3}{2} \min \{|v|^2, |v^+|^2\} + 3 |u|^2 \). Therefore,
\[
\mu(v) \mu^{-1/2}(v') \leq \mu^{-1/4}(v) \mu^{3/4}|v^2| \quad \text{and} \quad \mu(v) \mu^{-1/2}(v') \leq \mu^{-1/4}(v) \mu^{3/4}|v|^2 .
\]
Indeed, take an arbitrary $\varphi \geq 0$ and compute using this estimate
\[
\int_{\mathbb{R}^3} |Q_{b_1,b_2}(h)\varphi| \, dv \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \Phi_1(|u|)b_1(\cos(\theta)) |h(v)\varphi(v)\mu^{-1/2}(v')\varphi(v')\, d\sigma dv, dv \\
\leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \epsilon^{3/4}|u|^2 \Phi_1(|u|)b_1(\cos(\theta)) |h(v)\mu^{1/4}(v)\varphi(v')\, d\sigma dv, dv = \int_{\mathbb{R}^3} Q_{b_1,b_2}(h) \varphi \, dv.
\]
Since $\varphi$ is arbitrary, the first estimate in (4.36) follows. A similar argument gives the second estimate. As a consequence, using Young’s inequality for the gain collision operator in estimate (4.36) it follows that
\[
\|Q_{b_1,b_2}(h)\mu^{-1/2}\|_{L^2}^2 \leq C\delta^{-\gamma}L^2b_1\|h\|_{L^4}^2 \leq C\delta^{-\gamma}L^2b_1\|h\|_{L^4}^2,
\]
and
\[
\|Q_{b_1,b_2}(h)\mu^{-1/2}\|_{L^2}^2 \leq C\delta^{-\gamma}L^2b_1\|h\|_{L^4}^2,
\]
with $C = \max\{\|\mu^{1/4}\|_{L^1}, \|\mu^{1/4}\|_{L^2}\}$ and $k > \frac{3}{2}$. Furthermore,
\[
\|Q_{b_1,b_2}(h)\mu^{-1/2}\|_{L^2}^2 \leq C\delta^{-\gamma}b_1\|h\|_{L^4}^2 \leq C\delta^{-\gamma}b_1\|h\|_{L^4}^2, \quad k > \frac{3}{2},
\]
for the same aforementioned constant $C$. \hfill \Box

4.2.3. Enlargement of the Spectrum. We are in position now to extend Theorem 4.2 to the larger space $L^2(|v|^k, \mathbb{T}^3 \times \mathbb{R}^3)$.

**Theorem 4.4.** Let $s \in (0,1)$. The operator $\mathcal{L} = -\nabla \cdot \nabla_x$ defined on $L^2(|v|^k, \mathbb{T}^3 \times \mathbb{R}^3)$, with $k > \frac{3}{2} + \frac{3}{2} + 2s$, has essential spectrum localized in $\{ z \in \mathbb{C} : \Re z < -c_o\kappa_0 \}$. Furthermore, its eigenpairs are identical to those of $\mathcal{L}(v)$ (as an operator in $L^2(|v|^{-1/2}(v), \mathbb{R}^3)$) in $\{ z \in \mathbb{C} : \Re z > -c_o\kappa_0 \}$. Thus, $\mathcal{L}$ has the same spectral gap as $\mathcal{L}(v)$ and its null space is given by
\[
\text{Null}(\mathcal{L}) = \text{Span}\{\mu, v\mu, |v|^2\mu\}.
\]

**Proof.** Set the spaces $\mathcal{E} = L^2(|v|^k, \mathbb{T}^3 \times \mathbb{R}^3)$ and $E = L^2(|v|^{-1/2}(v), \mathbb{T}^3 \times \mathbb{R}^3)$. Note that $E \subset \mathcal{E}$. Theorem 4.2 gives us the localization of $\mathcal{L}$ as an operator in $E$. Using Theorem 4.3 and Proposition 4.8, we know that we can decompose $\mathcal{L} = \mathcal{A} - \mathcal{B}$ with $\mathcal{B} : \mathcal{E} \to \mathcal{E}$ closed and dissipative and $\mathcal{A} : \mathcal{E} \to E$ bounded (for a suitable choice of the parameters $\delta > 0$ and $\epsilon > 0$). This fulfills the hypothesis (H1) and (H2) of [17] Theorem 2.1] which implies the result. \hfill \Box

A direct consequence of Theorem 4.4 and [17] Theorem 2.1] is

**Corollary 4.2.** Let $s \in (0,1)$. The operator $\mathcal{L}$ generates a strongly continuous semigroup $e^{\mathcal{L}t}$ in $\mathcal{E} = L^2(|v|^k, \mathbb{T}^3 \times \mathbb{R}^3)$, for any $k > \frac{3}{2} + \frac{3}{2} + 2s$. Moreover, if $h(t)$ is the solution to the initial value problem
\[
\frac{dh}{dt} = \mathcal{L}h, \quad h_0 \in \mathcal{E},
\]
that is, $h(t) = e^{\mathcal{L}t}h_0$, then
\[
||h(t) - \pi h_0||_{\mathcal{E}} \leq Ce^{-\lambda t}||1 - \pi||h_0||_{\mathcal{E}}.
\]
Here $\pi$ is the projection onto $\text{Null}(\mathcal{L})$ such that
\[
\pi g \triangleq \sum_{\varphi \in \{v_1, \ldots, v_n, |v|^2\}} \left(\int_{\mathbb{T}^3 \times \mathbb{R}^3} g \varphi \, dv \, dx\right) \varphi \mu
\]

\[3\text{Note that the fact that } \mathcal{A} \text{ is bounded from the large space } \mathcal{E} \text{ to the small space } E \text{ ensures both (H2) (ii) and (iii).} \]
and \( \lambda > 0 \) is the spectral gap of \( \mathcal{L}(\nu) \) as an operator in \( L^2(\mu^{-1/2}(v), \mathbb{R}^3) \).

5. Regularization of \( \mathcal{L} \)

Recall the linearized operator \( \mathcal{L} \) is

\[
\mathcal{L} h = -v \cdot \nabla_x h + L h, \quad L h = Q(h, \mu) + Q(\mu, h).
\]

In this section, we will show the regularization of \( \mathcal{L} \) in both \( x, v \). The main result is

**Theorem 5.1.** Let \( \mathcal{L} \) be the linearized operator and let \( k_0 \in \mathbb{R} \) satisfy

\[
k_0 > \frac{5\gamma + 37}{2}, \tag{5.1}
\]

so that the spectral gap of \( \mathcal{L} \) holds on the space \( L^2((v)^{k_0} \, dv \, dx) \). Let \( h \) be the solution to the linear equation

\[
\partial_t h = \mathcal{L} h, \quad h|_{t=0} = h^{in}(x, v), \tag{5.2}
\]

where the Fourier transform of \( h^{in} \) in both \( x, v \) satisfies

\[
\sum_{\ell \in \mathbb{Z}^3} \int_{\mathbb{R}^3} \left| \frac{1}{\langle \xi \rangle^s + \langle \ell \rangle} \langle v \rangle^k h^{in}(\xi) \right|^2 \, d\xi < \infty.
\]

Then for any \( t > 0 \), we have \( h(t, \cdot, \cdot) \in L^2((v)^{k_0} \, dv \, dx) \) with the bound

\[
\int_{\mathbb{T}^3 \times \mathbb{R}^3} (v)^{2k_0} |h(t, x, v)|^2 \, dx \, dv \leq C_{k_0} \sum_{\ell \in \mathbb{Z}^3} \int_{\mathbb{R}^3} \left| \frac{1}{\langle \xi \rangle^s + \langle \ell \rangle} \langle v \rangle^k h^{in}(\xi) \right|^2 \, d\xi.
\]

The proof of Theorem 5.1 relies on various commutator estimates related to the collision operator. These estimates are the subjects of the following subsections.

5.1. **Definition of \( \mathcal{M} \).** The regularization of \( \mathcal{L} \) will be shown by applying a Fourier multiplier \( \mathcal{M} \) to the linearized equation (5.2). To define the operator \( \mathcal{M} \), we use the Fourier series with respect to \( x \) and write

\[
h(t, x, v) = \sum_{\ell \in \mathbb{Z}^3} e^{-2\pi i \ell \cdot x} h_\ell(t, v).
\]

Thus equation (5.2) is reduced to

\[
(\partial_t + v \cdot (-2\pi i \ell)) h_\ell = L h_\ell, \quad h_\ell|_{t=0} = h^{in}_\ell(v). \tag{5.3}
\]

Let \( \delta > 0 \) be small to be chosen later. For a fixed \( T > 0 \) and any \( t \in [0, T] \), we define a symbol of \( \mathcal{M}(t, \ell, D_v) \) as

\[
\mathcal{M}(t, \ell, \xi) = \left(1 + \delta \int_t^T \langle \xi + 2\pi(t - \rho)\ell \rangle^{2s} \, d\rho \right)^{-1/2-\varepsilon} = \left(1 + \delta \int_0^{T-t} \langle \xi - 2\pi \tau \ell \rangle^{2s} \, d\tau \right)^{-1/2-\varepsilon} \tag{5.4}
\]

with \( 0 < \varepsilon < \frac{1-2s}{2s} \). For brevity, we write \( 2\pi \ell = \eta \) and \( \mathcal{M} = \mathcal{M}(t, \eta, \xi) \) sometimes in the following.

The basic estimate for the symbol \( \mathcal{M} \) is

**Lemma 5.1.** Let \( \mathcal{M}(t, \eta, \xi) \) be the symbol defined in (5.4) with \( \eta = 2\pi \ell \). Then for any \( \alpha \in \mathbb{Z}^3 \) there exists a constant \( C \) that only depends on \( s, \varepsilon, \alpha \) such that

\[
\left| \nabla_\xi^\alpha \mathcal{M}(t, \eta, \xi) \right| / \mathcal{M}(t, \eta, \xi) \leq C \begin{cases} 
(\langle \xi \rangle + (T-t)\eta)^{-1} & \text{if } s < 1/2, \\
(\langle \xi \rangle + (T-t)\eta)^{-\min(|\alpha|,2)} & \text{if } s > 1/2, \\
(\langle \xi \rangle + (T-t)\eta)^{-\min(|\alpha|,2-\varepsilon_1)} & \forall 0 < \varepsilon_1 < 1, \text{ if } s = 1/2,
\end{cases}
\]
Proposition 5.2. Let
\[ By Corollary A.1 and Lemma A.2 in the appendix, we have \]

Commutator estimate for 
\[ R > \]
where the choice of the constant \( C \) will be shown by dividing the collision kernel in
\[ \] where
\[ χ \] and
\[ \] on \([0, 1]\).
\[ \] satisfies that for any \( R \)
\[ \] \( M = \) \( \nabla \) \( \ast \)
\[ δ \int_0^T \nabla \xi \langle (\xi - \eta \tau)^{2s} \rangle \, d\tau \]
\[ \leq C_s \frac{δ(T-t)(\langle \xi \rangle + (T-t)|\eta|)^{2s-\epsilon}}{1 + δ(T-t)\langle \xi \rangle + (T-t)|\eta|} \leq C_s (\langle \xi \rangle + (T-t)|\eta|)^{-1} \]

Furthermore, we have
\[ \frac{∂_ξ \nabla_ξ M}{M} = \frac{∂_ξ M \nabla_ξ M}{M} \]
\[ - \left( \frac{1}{2} + \epsilon \right) \left[ \delta \int_0^{T-t} \partial_ξ \nabla_ξ \langle (\xi - \eta \tau)^{2s} \rangle \, d\tau \right] - \left( \frac{1}{2} + \epsilon \right) \left[ \delta \int_0^{T-t} \partial_ξ \nabla_ξ \langle (\xi - \eta \tau)^{2s} \rangle \, d\tau \right] \]

Applying Corollary A.1 and Lemma A.2 in view of (5.5), we obtain the inequalities for the case of the second order derivatives. The cases for higher order derivatives can be obtained inductively.

We are mainly concerned with the commutator estimate of \( M \) with the collision operator \( Q \). The result will be shown by dividing the collision kernel in \( Q \) into the bounded and unbounded domains in terms of \(|v - v_*|\). More precisely, let \( 0 ≤ \chi_R(r) ≤ 1 \) be a smooth cutoff function such that
\[ \chi_R(r) = \begin{cases} 1, & 0 ≤ r ≤ R, \\ 0, & r > 2R, \end{cases} \]
and \( \chi_R \) satisfies that for any \( k ∈ \mathbb{N} \)
\[ |D^k \chi_R(r)| ≤ C (r)^{-k}, \]
where \( C \) is independent of \( R \). Such \( \chi_R \) exists, for example, by rescaling a smooth cutoff function supported on \([0, 2]\). Denote
\[ \Phi_R(v - v_*) = |v - v_*|^{γ} \chi_R(|v - v_*|), \quad \Phi_R(v - v_*) = |v - v_*|^{γ} (1 - \chi_R(|v - v_*|)) \]
Decompose \( Q \) such that
\[ Q(f, g) = Q_R(f, g) + Q_\Pi(f, g) \]
\[ = \int_{\mathbb{R}^3} \int_{S^2} b(\cos θ) \Phi_R(f'_* g' - f_* g) \, dσ \, dv_* + \int_{\mathbb{R}^3} \int_{S^2} b(\cos θ) \Phi_\Pi(f'_* g' - f_* g) \, dσ \, dv_* \]
where the choice of the constant \( R > 0 \) will be specified later.

5.2. Commutator estimate for \( Q_R \). The first commutator estimate is for \( Q_R \) with \( M \). The estimate is proved in a similar way as in \([\text{9}] \) Proposition 3.4. 

Proposition 5.2. Let \( R > 0 \) and \( Q_R \) be defined as in (5.8). Then there exists a constant \( C_R \) such that
\[ \left| \left( M Q_R(f, g) - Q_R(f, Mg), h \right) \right| \leq C_R \| f \|_{L^1} \| Mg \|_{H^\nu} \| h \|_{H^\nu} \]
for any \( s' \) satisfying
\[
\begin{align*}
  s' &\geq (s - 1/2)^+ & \text{if } s \neq 1/2, \\
  s' &> 0 & \text{if } s = 1/2.
\end{align*}
\]

**Proof.** We use a decomposition
\[
1 = 1_{\{\xi_* \geq \sqrt{2}|\xi|\}} + 1_{\{\xi_* \leq |\xi|/2\}} + 1_{\sqrt{2}|\xi| \geq |\xi|/2}.
\]

The following property holds in each of the following subdomains:
\[
\begin{align*}
  &\langle \xi \rangle \lesssim \langle \xi - \xi_* \rangle, \quad \text{on supp } 1_{\{\xi_* \geq \sqrt{2}|\xi|\}}; \\
  &\langle \xi \rangle \sim \langle \xi - \xi_* \rangle, \quad \text{on supp } 1_{\{\xi_* \leq |\xi|/2\}}; \\
  &\langle \xi \rangle \gtrsim \langle \xi - \xi_* \rangle, \quad \text{on supp } 1_{\sqrt{2}|\xi| \geq |\xi|/2}.
\end{align*}
\]

(5.9)

Noting the Ukai formula given in Lemma 5.1, we put
\[
F(x; a) = \frac{x}{\left(1 + \delta(ax^{2s} + a^{2s+1}|\eta|^{2s})\right)^{1/2 + \varepsilon}}.
\]

It is easy to check that \( F(x; a) \) is an increasing function in \( x \in (0, \infty) \) because of \( \varepsilon < \frac{1-s}{2s} \). Therefore, it follows from the first formula of (5.9) that, on supp \( 1_{\{\xi_* \geq \sqrt{2}|\xi|\}} \), we have
\[
\langle \xi \rangle \mathcal{M}(t, \xi, \eta) \sim F(\langle \xi \rangle; T - t) \leq F(\langle \xi \rangle; T - t) \lesssim \langle \xi \rangle \mathcal{M}(t, \xi, \eta) \sim \langle \xi \rangle \mathcal{M}(t, \xi - \xi_*, \eta).
\]

Hence,
\[
\mathcal{M}(t, \xi, \eta) \lesssim \frac{\langle \xi \rangle}{\langle \xi \rangle} \mathcal{M}(t, \xi - \xi_*, \eta), \quad \text{if } \langle \xi \rangle \geq \sqrt{2}|\xi|.
\]

(5.10)

Combining the mean value theorem, Lemma 5.1 and the second formula in (5.9) on supp \( 1_{\{\xi_* \leq |\xi|/2\}} \), we have
\[
|M(t, \xi, \eta) - \mathcal{M}(t, \xi - \xi_*, \eta)| \leq \int_0^1 |(\nabla_\xi \mathcal{M})(t, \tau, \xi - \xi_*, \eta)|d\tau \langle |\xi| \rangle \\
\lesssim \mathcal{M}(t, \xi - \xi_*, \eta) \frac{\langle \xi \rangle}{\langle |\xi| \rangle}, \quad \text{if } \langle \xi \rangle \leq |\xi|/2.
\]

(5.11)

Here and in what follows, we abbreviate \( \mathcal{M}(\xi) = \mathcal{M}(t, \xi, \eta) \) to show its dependence on \( \xi \). On supp \( 1_{\{\xi_* \leq |\xi|/2\}} \) we have \( \langle \xi \rangle \sim \langle \xi - \xi_* \rangle \) and hence
\[
|M(\xi) - \mathcal{M}(\xi - \xi_*)| \lesssim \mathcal{M}(\xi - \xi_*)
\]

which together with (5.11) gives
\[
|M(\xi) - \mathcal{M}(\xi - \xi_*)| \lesssim \frac{\langle \xi \rangle^\kappa}{\langle |\xi| \rangle^\kappa} \mathcal{M}(\xi - \xi_*)
\]

(5.12)

for any \( 0 \leq \kappa \leq 1 \) and \( \langle \xi_* \rangle \leq |\xi|/2 \).

On supp \( 1_{\sqrt{2}|\xi| \geq |\xi|/2} \), by means of Proposition 5.1 and the third formula from (5.9), we have
\[
\mathcal{M}(\xi) \sim \mathcal{M}(\xi_*), \quad \mathcal{M}(\xi - \xi_*) \sim \mathcal{M}(\xi - \xi_*) \frac{1 + \delta \left( (T - t)(\xi - \xi_*)^{2s} + (T - t)^{2s+1}|\eta|^{2s} \right)}{1 + \delta \left( (T - t)(\xi_*)^{2s} + (T - t)^{2s+1}|\eta|^{2s} \right)} \lesssim \mathcal{M}(\xi - \xi_*).
\]

(5.13)

It follows from (5.10), (5.12) and (5.13) that
\[
|M(\xi) - \mathcal{M}(\xi - \xi_*)| \leq \mathcal{M}(\xi - \xi_*) \left\{ \frac{\langle \xi \rangle}{\langle \xi_* \rangle} 1_{\{\xi_* \geq \sqrt{2}|\xi|\}} + 1_{\sqrt{2}|\xi| > |\xi|/2} + \frac{\langle \xi \rangle^\kappa}{\langle \xi_* \rangle^\kappa} 1_{|\xi|/2 > \langle \xi_* \rangle} \right\},
\]

which corresponds to [3] (3.4).
We shall follow an almost same procedure as in the proof of \cite[Proposition 3.4]{9}. By using the Bobylev formula from the Appendix of \cite{1}, we have
\[
(Q_R(f, g), h) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} b\left(\frac{\xi}{|\xi|}, \sigma\right) [\Phi_R(\xi - \xi^-, 0) - \Phi_R(\xi^-, 0)] \hat{f}(\xi, \sigma) \bar{g}(\xi - \xi^+) \bar{h}(\xi) d\xi d\sigma,
\]
where $\xi^- = \frac{1}{2} (\xi - |\xi| \sigma)$. Therefore,
\[
\begin{align*}
\left(\mathcal{M}(D) Q_R(f, g) - Q_R(f, \mathcal{M}(D) g), h\right) &= \iint b\left(\frac{\xi}{|\xi|}, \cdot\right) \Phi_R(\xi - \xi^- - \Phi_R(\xi^-)) \\
&\quad \times \left(\mathcal{M}(\xi) - \mathcal{M}(\xi - \xi^-)\right) \hat{f}(\xi) \bar{g}(\xi - \xi^-) \bar{h}(\xi) d\xi d\sigma \\
&= \iint_{|\xi^-| \leq \frac{1}{2} |\xi|} \cdots d\xi d\sigma + \iint_{|\xi^-| \geq \frac{1}{2} |\xi|} \cdots d\xi d\sigma \\
&= A_1(f, g, h) + A_2(f, g, h).
\end{align*}
\]
For $A_1$, we use the Taylor expansion of $\Phi_R$ of order 2 to have
\[
A_1 = A_{1,1}(f, g, h) + A_{1,2}(f, g, h),
\]
where
\[
A_{1,1} = \iint b\left(\frac{\xi}{|\xi|}, \sigma\right) \left(\mathcal{M}(\xi) - \mathcal{M}(\xi - \xi^-)\right) \hat{f}(\xi) \bar{g}(\xi - \xi^-) \bar{h}(\xi) d\xi d\sigma,
\]
and $A_{1,2}(f, g, h)$ is the remaining term corresponding to the second order Taylor expansion of $\Phi_R$.

We first consider $A_{1,1}$. By writing
\[
\xi^- = \frac{|\xi|}{2} \left(\left(\frac{\xi}{|\xi|} \cdot \frac{\xi}{|\xi|}ight) - \frac{|\xi|}{2}\sigma\right),
\]
we see that the integral corresponding to the first term on the right hand side vanishes because of the symmetry on $S^2$. Hence, we have
\[
A_{1,1} = \int_{g^6} K(\xi, \xi^-) \left(\mathcal{M}(\xi) - \mathcal{M}(\xi - \xi^-)\right) \hat{f}(\xi) \bar{g}(\xi - \xi^-) \bar{h}(\xi) d\xi d\sigma,
\]
where
\[
K(\xi, \xi^-) = \int_{g^2} b\left(\frac{\xi}{|\xi|}, \cdot\right) \left(1 - \left(\frac{\xi}{|\xi|} \cdot \sigma\right)\right) \frac{\xi}{2} \cdot (\nabla \Phi_R)(\xi^-) 1_{|\xi^-| \leq \frac{1}{2} |\xi|} d\sigma.
\]
Note that $|\nabla \Phi_R(\xi^-)| \lesssim \frac{1}{\langle \xi^- \rangle^{3+\gamma}}$, from the Appendix of \cite{9}. If $\sqrt{2}|\xi| \leq \langle \xi^+ \rangle$, then $\theta \frac{\xi}{2} |\xi| = |\xi^-| \leq |\xi|/2$, and we have
\[
|K(\xi, \xi^-)| \lesssim \int_0^{\pi/2} \theta^{1-2s} d\theta \frac{\langle \xi \rangle}{\langle \xi^- \rangle^{3+\gamma+1}} \lesssim \frac{1}{\langle \xi^+ \rangle^{3+\gamma}} \frac{\langle \xi \rangle}{\langle \xi^- \rangle}. \]

On the other hand, if $\sqrt{2}|\xi| \geq \langle \xi^+ \rangle$, then
\[
|K(\xi, \xi^-)| \lesssim \int_0^{\pi/2} \theta^{1-2s} d\theta \frac{\langle \xi \rangle}{\langle \xi^- \rangle^{3+\gamma+1}} \lesssim \frac{1}{\langle \xi^+ \rangle^{3+\gamma}} \frac{\langle \xi \rangle}{\langle \xi^- \rangle}^{2s-1}.
\]
Hence, we obtain
\[
|K(\xi, \xi^-)| \lesssim \frac{1}{\langle \xi^+ \rangle^{3+\gamma}} \left\{ \left(\frac{\langle \xi \rangle}{\langle \xi^- \rangle}\right)^{1_{\langle \xi^- \rangle \geq \sqrt{2}|\xi|} + 1_{\sqrt{2}|\xi| \geq \langle \xi \rangle / 2}} + \left(\frac{\langle \xi \rangle}{\langle \xi^- \rangle}\right)^{2s-1} 1_{|\xi|/2 \geq \langle \xi \rangle} \right\}.
\]
Similar to $A_{1,1}$, we can also write
\[
A_{1,2} = \int_{g^6} \tilde{K}(\xi, \xi^-) \left(\mathcal{M}(\xi) - \mathcal{M}(\xi - \xi^-)\right) \hat{f}(\xi) \bar{g}(\xi - \xi^-) \bar{h}(\xi) d\xi d\sigma,
\]
where
\[
\tilde{K}(\xi, \xi_*) = \int_{\mathbb{R}^2} b\left(\frac{\xi}{|\xi|}, \sigma\right) \int_0^1 (1 - \tau) (\nabla^2 \tilde{\Phi}_R)(\xi_* - \tau \xi^-) : (\xi^- \otimes \xi^-) 1_{|\xi^-| \leq \frac{1}{2} (\xi_*)} d\tau d\sigma.
\]
Again from the Appendix of [9], we have
\[
|(\nabla^2 \tilde{\Phi}_R)(\xi_* - \tau \xi^-)| \lesssim \frac{1}{(\xi_* - \tau \xi^-)^{3+\gamma+2}} \approx \frac{1}{(\xi_*)^{3+\gamma+2}},
\]
because $|\xi^-| \leq (\xi_*)/2$. This leads to
\[
|\tilde{K}(\xi, \xi_*)| \lesssim \frac{1}{(\xi_*)^{3+\gamma}} \left\{ \left( \frac{\xi}{\xi_*} \right)^2 1_{(\xi_* - \xi) \geq \sqrt{2}|\xi|} + 1_{\sqrt{2}|\xi| \geq (\xi_*)/2} + \left( \frac{\xi}{\xi_*} \right)^{2s} 1_{|\xi|/2 \geq (\xi_*)} \right\}.
\]
It follows from (5.14), (??) and (??) that we have
\[
|A_1| \lesssim |A_{1,1}| + |A_{1,2}| \lesssim A_1 + A_2 + A_3,
\]
where
\[
A_1 = \int_{\mathbb{R}^6} \left| \frac{\tilde{f}(\xi)}{(\xi_*)^{3+\gamma}} \right| |\mathcal{M}(\xi - \xi_*) \tilde{g}(\xi - \xi_*)| |\tilde{h}(\xi)| 1_{(\xi_*) \geq \sqrt{2}|\xi|} d\xi, \tag{5.15}
\]
and
\[
A_2 = \int_{\mathbb{R}^6} \left| \frac{\tilde{f}(\xi)}{(\xi_*)^{3+\gamma}} \right| |\mathcal{M}(\xi - \xi_*) \tilde{g}(\xi - \xi_*)| |\tilde{h}(\xi)| 1_{\sqrt{2}|\xi| > (\xi_*)/2} d\xi,
\]
\[
A_3 = \int_{\mathbb{R}^6} \left| \frac{\tilde{f}(\xi)}{(\xi_*)^{3+\gamma}} \right| |\mathcal{M}(\xi - \xi_*) \tilde{g}(\xi - \xi_*)| |\tilde{h}(\xi)| \left( \frac{|\xi|^{2s - \kappa}}{(\xi_*)^{2s - \kappa}} \right) 1_{|\xi|/2 > (\xi_*)} d\xi.
\]
For $\gamma > 0$, we can obtain
\[
|A_1|^2 \lesssim \|\tilde{f}\|_L^\infty \left( \int_{\mathbb{R}^3} \frac{d\xi_*}{(\xi_*)^{3+\gamma}} \int_{\mathbb{R}^2} |\tilde{h}(\xi)|^2 d\xi \right)
\]
\[
\times \left( \int_{\mathbb{R}^3} \frac{d\xi_*}{(\xi_*)^{3+\gamma}} \int_{\mathbb{R}^3} \left( \frac{\xi}{\xi_*} \right)^{3+\gamma} 1_{(\xi_*) \geq \sqrt{2}|\xi|} |\mathcal{M}(\xi - \xi_*) \tilde{g}(\xi - \xi_*)|^2 d\xi \right)
\]
\[
\lesssim \|f\|_p^2 \|Mg\|_L^2 \|h\|_L^2.
\]
Noticing the third formula of (5.9), we get
\[
|A_2|^2 \lesssim \left( \int_{\mathbb{R}^3} \left| \frac{\tilde{f}(\xi)}{(\xi_*)^{3+2\gamma}} \right|^2 d\xi \int_{|\xi^-| \leq (\xi_*)} d\xi \right) \left( \int_{\mathbb{R}^6} |\mathcal{M}(\xi - \xi_*) \tilde{g}(\xi - \xi_*)|^2 |\tilde{h}(\xi)|^2 d\xi \right)
\]
\[
\lesssim \int_{\mathbb{R}^3} \left| \frac{\tilde{f}(\xi)}{(\xi_*)^{3+2\gamma}} \right|^2 d\xi \| \mathcal{M}g \|_L^2 \|h\|_L^2 \lesssim \|f\|_p^2 \|Mg\|_L^2 \|h\|_L^2.
\]
Setting $\tilde{G}(\xi) = (\xi)^{s'} \mathcal{M}(\xi) \tilde{g}(\xi)$ and $\tilde{H}(\xi) = (\xi)^{s'} \tilde{h}(\xi)$ with $s' = s - \kappa/2$, we have when $\kappa \leq 2s$,
\[
|A_3|^2 \lesssim \left( \int_{\mathbb{R}^3} \left| \frac{\tilde{f}(\xi)}{(\xi_*)^{3+\gamma+2s - \kappa}} \right|^2 d\xi \int_{\mathbb{R}^3} |\tilde{h}(\xi)|^2 d\xi \right) \left( \int_{\mathbb{R}^3} \left| \frac{\tilde{f}(\xi)}{(\xi_*)^{3+\gamma+2s - \kappa}} \right|^2 d\xi \int_{\mathbb{R}^3} 1_{|\xi|/2 \geq (\xi_*)^2} |\tilde{G}(\xi - \xi_*)|^2 d\xi \right)
\]
\[
\lesssim \|f\|_p^2 \|Mg\|_{H_{s'}}^2 \|h\|_{H_{s'}}^2.
\]
The above three estimates yield the desired estimate for $A_1(f, g, h)$. 

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Next consider $A_2(f, g, h)$. Write $A_2(f, g, h)$ as
\[
A_2 = \iiint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) K_{|\xi| \geq \frac{1}{2} (\xi, h)}(\xi) d\xi d\sigma d\xi d\sigma
- \iiint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) K_{|\xi| \geq \frac{1}{2} (\xi, h)}(\xi) d\xi d\sigma
= A_{2.1}(f, g, h) - A_{2.2}(f, g, h).
\]
Since $|\xi| = |\xi| \sin(\theta/2) \geq (\xi, h)/2$ and $\theta \in [0, \pi/2]$, we have $\sqrt{2}|\xi| \geq (\xi, h)$. Write
\[
A_{2,j} = \iiint K_j(\xi, \xi)(M(\xi) - M(\xi - \xi)) \hat{f}(\xi, h) \hat{g}(\xi, \xi) d\xi d\xi.
\]
Then by $\left|\hat{\Phi}_R(\xi)\right| \lesssim \frac{1}{(\xi, h)^{\frac{1}{2} + \gamma}}$, we have
\[
|K_2(\xi, \xi)| = \int b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \hat{\Phi}_R(\xi) 1_{|\xi| \geq \frac{1}{2} (\xi, h)} d\sigma \lesssim \frac{1}{(\xi, h)^{\frac{1}{2} + \gamma}} \left\{ 1 \frac{1}{\sqrt{2}(|\xi|) \geq |\xi|/2} + \left(\frac{|\xi|}{(\xi, h)}\right)^{\frac{1}{2}} 1_{|\xi|/2 \geq (\xi, h)} \right\},
\]
which shows the desired estimate for $A_{2.2}$ in exactly the same way as the estimation on $A_2$ and $A_3$.

As for $A_{2.1}$, it suffices to work under the condition $|\xi, h| \geq \frac{1}{2} |\xi|^2$. In fact, on the complement of this set, we have $|\xi, h| > |\xi|$, and $\hat{\Phi}_R(\xi, h)$ is the same as $\hat{\Phi}_R(\xi)$. Therefore, we can consider $A_{2.1,p}$ in which $K_1(\xi, \xi)$ is replaced by
\[
K_{1,p}(\xi, \xi) = \int_{\mathbb{R}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \hat{\Phi}_R(\xi, h) 1_{|\xi| \geq \frac{1}{2} (\xi, h)} 1_{|\xi| \geq \xi} |\xi, h| |\xi| d\sigma.
\]
By writing
\[
1 = 1_{(\xi, h) \geq |\xi|/2} + 1_{(\xi, h) < |\xi|/2},
\]
we decompose respectively
\[
A_{2.1,p} = B_1 + B_2.
\]
On the set of integration in $K_{1,p}$, we have $|\xi, h| \lesssim |\xi|$, because $|\xi| \leq |\xi|/2$ by $|\xi|^2 \leq 2 |\xi, h| \leq |\xi| ||\xi|$. Furthermore, on the set for $B_1$ we have $|\xi| \sim (\xi, h)$, so that $|\xi| \lesssim |\xi|$ and $b 1_{|\xi| \geq \frac{1}{2} (\xi, h)} 1_{(\xi, h) \geq |\xi|/2}$ is bounded. By means of \((5.14)\) and Cauchy-Schwarz inequality, we have
\[
|B_1|^2 \lesssim \|f\|_{L^2}^2 \iint \left|\hat{\Phi}_R(\xi, h)\right| |M(\xi - h)\hat{g}(\xi - h)|^2 d\sigma d\xi
\times \iint \left|\hat{\Phi}_R(\xi, h)\right| |\hat{g}(\xi)|^2 d\sigma d\xi
\lesssim \|f\|_{L^2}^2 \|Mg\|_{L^2}^2 \|h\|_{L^2}^2.
\]
Here, noting $\xi - \xi = \xi^+ - u$ with $u = \xi - \xi^-$, it holds for the first integral factor
\[
\iint \left|\hat{\Phi}_R(\xi, h)\right| |M(\xi - h)\hat{g}(\xi - h)|^2 d\sigma d\xi \lesssim \int_{\mathbb{R}^2} \left(\int (u)^{3-\gamma} |M(\xi - h)\hat{g}(\xi - h)|^2 d\xi d\sigma \right) d\sigma,
\]
we have used the change of variables $(\xi, h) \to (\xi^+, u)$ whose Jacobian is
\[
\left|\frac{\partial (\xi^+, u)}{\partial (\xi, h)}\right| \equiv \left|\frac{\partial \xi^+}{\partial \xi}\right| = \left|\frac{\partial \xi - \xi}{\partial \xi}\right| = \left|\frac{\xi + \xi}{\xi} \otimes \sigma\right| = \frac{|1 + \frac{\xi}{|\xi|} \cdot \sigma|}{|\xi|} = \frac{\cos^2(\theta/2)}{4} \geq \frac{1}{8}, \quad \theta \in [0, \pi/2].
\]
On the set of the integration for $B_2$, we recall $\langle \xi \rangle \sim \langle \xi - \xi_* \rangle$ and (5.12) with $\kappa \in [0, 1]$. Setting $\hat{G}(\xi) = \mathcal{M}(\xi)\hat{g}(\xi)$, we have

$$|B_2|^2 \lesssim \|f\|_{L^1}^2 \iint b \mathbf{1}_{|\xi| \geq \frac{1}{2} |\xi_*|} \frac{\hat{F}_R(\xi_* - \xi^-)(\xi_*)^\kappa}{\langle \xi \rangle^\kappa} |\hat{G}(\xi - \xi_*)|^2 d\sigma d\xi_* \times \frac{\hat{F}_R(\xi_* - \xi^-)(\xi_*)^\kappa}{\langle \xi \rangle^\kappa} |\hat{h}(\xi)|^2 d\sigma d\xi_* .$$

We use the change of variables $\xi_* \to u = \xi_* - \xi^-$. Note that $|\xi^-| \geq \frac{1}{2} (u + \xi^-)$ implies $|\xi^-| \geq \langle u \rangle / \sqrt{10}$, and that

$$\langle \xi_* \rangle^\kappa \lesssim \langle \xi - \xi^- \rangle^\kappa + |\xi|^\kappa \sin^\kappa \theta / 2 .$$

If $\kappa - 2s \neq 0$, then we have

$$\iint b \mathbf{1}_{|\xi| \geq \frac{1}{2} |\xi_*|} \frac{\hat{F}_R(\xi_* - \xi^-)(\xi_*)^\kappa}{\langle \xi \rangle^\kappa} d\sigma d\xi_* \lesssim \frac{1}{\langle u \rangle^{3+\gamma}} \left( \frac{\langle u \rangle^\kappa}{\langle \xi \rangle^\kappa} \right) \int b \mathbf{1}_{|\xi| \geq \langle u \rangle} d\sigma + \int b(\cos \theta) \sin^\kappa(\theta / 2) \mathbf{1}_{|\xi| \geq \langle u \rangle} d\sigma du \lesssim \langle \xi \rangle^{(2s-\kappa)^+} ,$$

which yields

$$|B_2| \lesssim \|f\|_{L^1} \|\mathcal{M}g\|_{H^{(s-k/2)^+}} \|h\|_{H^{(s-k/2)^+}} ,$$

for any $\kappa \in [0, 1]$ with $\kappa \neq 2s$. Summing above estimates we complete the proof of the proposition.

The second commutator estimate is for $Q_R(\mu, g)$ with the weight $\langle v \rangle^k$.

**Proposition 5.3.** Let $k, R > 0$ and $Q_R$ be defined as in (5.8). Then

$$\left( \langle v \rangle^k Q_R(\mu, g) - Q_R(\mu, \langle v \rangle^k g), h \right) \leq C_{k,R} \min \left\{ \|g\|_{L^2}, \|h\|_{H^{s'}} , \|g\|_{H^{s'}} , \|h\|_{L^2} \right\} ,$$

where $C_{k,R}$ may depend on $k, R$ and $s' = 0$ if $s < 1/2$, $s' > 2s - 1$ if $s \geq 1/2$.

**Proof.** By the definition of $Q_R$, we have

$$\left( \langle v \rangle^k Q_R(\mu, g) - Q_R(\mu, \langle v \rangle^k g), h \right) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)|v - v_*|^7 \chi_R(|v - v_*|) \mu_* g \left( \langle v' \rangle^k h' - \langle v \rangle^k h \right)$$

$$- \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)|v - v_*|^7 \chi_R(|v - v_*|) \mu_* g \langle v \rangle^k \langle h' - h \rangle$$

$$= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)|v - v_*|^7 \chi_R(|v - v_*|) \left( \langle v' \rangle^k - \langle v \rangle^k \right) \mu_* gh' .$$

Note that by the cutoff function on $|v - v_*|$, we have

$$\langle v \rangle \sim \langle v_* \rangle \sim \langle v' \rangle \sim \langle v'_* \rangle ,$$

where the equivalence constants may depend on $R$. It follows from Lemma 2.1 that

$$\langle v' \rangle^k - \langle v \rangle^k = k\langle v \rangle^{k-2} |v - v_*| (v_* \cdot \omega) \cos^{k-1} \frac{\theta}{2} \sin \frac{\theta}{2} + \mathcal{R} ,$$

where $|\mathcal{R}| \leq C_{k,R} \langle v_* \rangle^k \sin \frac{\theta}{2}$ and $S^2 \ni \omega \perp (v - v_*)$. As in the proof of Proposition 3.1, one can replace $\omega$ by $\bar{\omega} \in S^2$ with $\bar{\omega} \perp (v' - v_*')$. The term coming from $\mathcal{R}$ can be easily estimated by $C_{k,R} \|g\|_{L^2} \|h\|_{L^2}$. In
Indeed, it suffices to write because of \([9, \text{Proposition 2.2}]\). Similarly, we have another bound since \(\omega \) side of (5.17). On the other hand, when \(0 < s < 4\) RICARDO ALONSO, YOSHINORI MORIMOTO, WEIRAN SUN, AND TONG YANG

and to use the symmetry on \(S\). The first term on the right hand side vanishes because of the symmetry on \(S^2\). The third is estimated by \(C\|g\|_{L^2}\|h\|_{L^2}\). It follows from Cauchy-Schwarz inequality that the second is estimated by

\[
C_{k,R} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)|v-v_\star|^{2s+\epsilon} \langle v_\star \rangle^k \mu \, g \, 2\omega d\sigma dv d\mu \right)^{1/2} \\
\times \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)\theta^{2(2s+\epsilon)} \langle v_\star \rangle^k \mu \left( \frac{h'}{\langle v' \rangle} - \frac{h}{\langle v \rangle} \right)^2 d\sigma dv d\mu \right)^{1/2} \\
\leq C_{k,R} \|g\|_{L^2} \|\langle v \rangle^{-1} \omega\|_{H^{2s-1+\epsilon}} \text{ for any } \epsilon > 0,
\]

because of \([9, \text{Proposition 2.2}]\). Similarly, we have another bound since \(\omega \) in (5.17) can be replaced by \(\overline{\omega}\). Indeed, it suffices to write

\[
|\langle v \rangle^{-2} g h'| = g'(\langle v \rangle^{-2} h') + \left( \frac{g}{\langle v \rangle} - \frac{g'}{\langle v' \rangle} \right) \langle \langle v \rangle^{-1} h \rangle' + \langle \langle v \rangle^{-k} - \langle v' \rangle^{-k} \rangle \frac{g}{\langle v \rangle} h'
\]

and to use the symmetry on \(S^2\) with the north pole \(v' - v_\star\) for the first term and Lemma 2.1 for the third term again. This completes the proof of the proposition. \(\square\)

We also have the following direct bound on \(Q_R(f, \mu)\).

**Proposition 5.4.** Let \(k, R > 0\) and \(Q_R\) be defined as in (5.8). Then

\[
\left| \left( \langle v \rangle^k Q_R(f, \mu), \ h \right) \right| \leq C_{k,R} \|f\|_{L^2} \|h\|_{L^2},
\]

where \(C_{k,R}\) may depend on \(k, R\).

**Proof.** By the definition of \(Q_R\), we have

\[
\left( \langle v \rangle^k Q_R(f, \mu), \ h \right) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)|v-v_\star|^{2s} \chi_R(|v-v_\star|) f_\star \mu \left( \langle v' \rangle^k h' - \langle v \rangle^k h \right) \\
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)|v-v_\star|^{2s} \chi_R(|v-v_\star|) f_\star \mu \left( \langle v' \rangle^k - \langle v \rangle^k \right) h' \\
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)|v-v_\star|^{2s} \chi_R(|v-v_\star|) f_\star \mu \langle v \rangle^k (h' - h) \\
\Delta = I + II.
\]

For the first term \(I\) we use (5.17) with \(\omega\) replaced by \(\overline{\omega}\) and split \(I = I_{\text{main}} + I_\mathcal{R}\). In view of (5.16), we have the following bound for the term \(I_\mathcal{R}\) coming from \(\mathcal{R}\),

\[
|I_\mathcal{R}| \leq C_{k,R} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)\theta^2 \langle v_\star \rangle^{-2} f_\star \langle v \rangle^{k+2} \mu |h'| d\sigma dv d\mu \leq C_{k,R} \|f\|_{L^2} \|h\|_{L^2}.
\]
The main term is written as
\[
\mathcal{I}_{\text{main}} = k \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) |v - v_*|^\gamma \chi_R (|v - v_*|)(v_* \cdot \tilde{\omega}) \cos \frac{\theta}{2} \sin \frac{\theta}{2} f_* ((v)^{k-2} \mu) h' d\sigma dv dv_*
\]
\[
= k \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) |v - v_*|^\gamma \chi_R (|v - v_*|)(v_* \cdot \tilde{\omega}) \cos \frac{\theta}{2} \sin \frac{\theta}{2} f_* \left( ((v)^{k-2} \mu) - ((v')^{k-2} \mu') \right) h' d\sigma dv dv_*
\]
because of the symmetry on \( \mathbb{S}^2 \) with the north pole \( v' \rightarrow v_* \). Since it follows from the mean value theorem that
\[
\chi_R (|v - v_*|) \left( |(v)^{k-2} \mu| - |(v')^{k-2} \mu'| \right) \leq C_{k,R} \int_0^1 (v_* - \tau (v' - v))^{1/2} d\tau \sin \frac{\theta}{2},
\]
we have
\[
|\mathcal{I}_{\text{main}}| \leq C_{k,R} \int_0^1 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) \mu (v + \tau (v' - v))^{1/2} h' |d\sigma|dv dv_* \leq C_{k,R} \|f\|_{L^2} \|h\|_{L^2}.
\]
To bound the second term \( \mathcal{II} \) on the right hand side of \( (5.18) \), we use Proposition 2.1 in [1] to have
\[
|\mathcal{II}| = \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) |v - v_*|^\gamma \chi_R (|v - v_*|) f_* |(v)^{k} (h' - h)| \right|
\]
\[
= \left| \left( Q_R (f, \mu (v)^{k}), h \right) \right| \leq C_{k,R} \|f\|_{L^2} \|h\|_{L^2}.
\]
Then the desired bound is a combination of above three estimates.

5.3. **Commutator estimate for** \( Q_{\mathcal{F}} \). **Now we consider the commutator between** \( \mathcal{M}(t, D_v, \eta) \) **and the regular part of the kinetic factor** \( \Phi_{\mathcal{F}}(v - v_*) \) **defined in** \( (5.7) \). **First, by the property of** \( \chi_R \) **in** \( (5.6) \), **we have**
\[
\Phi_{\mathcal{F}}(z) \in C^\infty (\mathbb{R}^3), \quad |\partial_z^\alpha \Phi_{\mathcal{F}}(z)| \leq C(z)^{\gamma - |\alpha|},
\]
where \( C \) is independent of \( R \). **From now on, we use the notation** \( \lesssim \) **only when the constant involved is independent of** \( R, \delta, t, T \).

**Lemma 5.2.** **Let** \( \gamma \leq 1 \) **and denote**
\[
\Phi_*(v) = \Phi_{\mathcal{F}}(v - v_*),
\]
regarding \( v_* \) **as a parameter. Denote**
\[
[\mathcal{M}(D_v), \Phi_*(v)] = \mathcal{M}(D_v) \Phi_*(v) - \Phi_*(v) \mathcal{M}(D_v).
\]
**Then for any real number** \( r \in \mathbb{R} \) **there exists a constant** \( C = C(r) > 0 \) **independent of** \( \delta, R > 0 \) **and** \( 0 \leq t \leq T \) **such that the following estimate holds for any** \( f(v) \in \mathcal{S}(\mathbb{R}^3) \).
\[
\|\| [\mathcal{M}(D_v), \Phi_*(v)] f \|_{H^r} \leq C \|\| \mathcal{M}(D_v) f \|_{H^{r-1}}.
\]
**In other words, we can write** \( [\mathcal{M}(D_v), \Phi_*(v)] = a(v, D_v) \mathcal{M}(D_v)^{-1} \mathcal{M}(D_v) \) **with** \( a(v, \xi) \) **belonging to the symbol class** \( S^0_{0,0} \) **uniformly with respect to** \( v_* \), **and in addition to** \( \delta, R, T, t \).

**Proof.** **The lemma follows from the calculus of pseudo-differential operators ( for example, see Kumano-go [24]).** **For the sake of completeness and for the later use in some arguments, we give a brief proof. It follows from Theorem 3.1 of [24] that for any** \( N \in \mathbb{N}, \)
\[
\mathcal{M}(D_v) \Phi_* = \Phi_* \mathcal{M}(D_v) + \sum_{1 \leq |\alpha| < N} \frac{1}{\alpha!} \Phi_{*,(\alpha)}(v) \mathcal{M}^{(\alpha)}(D_v) + r_N(v, D_v),
\]
where \( \Phi_{*,(\alpha)}(v) = D_v^\alpha \Phi_*(v), \mathcal{M}^{(\alpha)}(\xi) = \partial_\xi^\alpha \mathcal{M}_0(\xi) \) **and**
\[
r_N(v, \xi) = N \sum_{|\alpha| = N}^{N} \int_0^{1} \frac{(1 - \tau)^{N-1}}{\alpha!} r_N(v, D_v) d\tau.
\]
Here

\[ r_{N,\tau,\alpha}(v, \xi) = \text{Os} - \int \int e^{-i\xi \cdot z} M^{(\alpha)} (\xi + \tau \zeta) \Phi_{*,(\alpha)}(v + z) \frac{dz d\zeta}{(2\pi)^3}, \]

where “Os” means the oscillating integral (24). Fix \( N \geq 3 \). Since \( \Phi_{*,(\alpha)} \) satisfies (5.19) \((|\alpha| \neq 0)\) and it follows from Lemma [5.1] that

\[ |M^{(\alpha)}(\xi)| \leq C_\alpha |\xi|^{-1} M(\xi) \]  
(5.23)

with a constant \( C_\alpha > 0 \) depending only on \( \alpha \), the estimate [5.21] with \( r = 0 \) is obvious if we show that \( r_{N,\tau,\alpha}(v, \xi)/M(\xi) \) belongs to the symbol class \( S_{\alpha}^{-1} \) uniformly with respect to \( \tau \in [0,1] \). That is, cf. [24],

\[ |D_\xi^\beta \partial_\xi^\beta (r_{N,\tau,\alpha}(v, \xi)/M(\xi))| \leq C_{\beta,\beta'} |\xi|^{-1}, \]  
(5.24)

because it follows from the product formula of pseudodifferential operators that

\[ Op\left(r_{N,\tau,\alpha}(v, \xi)/M(\xi)\right)Op(M(\xi)) = r_{N,\tau,\alpha}(v, D_v). \]

In view of (5.23), it suffices to show

\[ |D_\xi^\beta \partial_\xi^\beta r_{N,\tau,\alpha}(v, \xi)| \leq C_{\beta,\beta'} |\xi|^{-1} M(\xi), \]  
(5.25)

instead of (5.24). We only consider (5.25) with \( \beta = \beta' = 0 \) because the proof for the general case is similar by taking the derivatives of the integrand. Firstly, using the elementary identities

\[ e^{-iz \cdot \zeta} = (1 - \Delta z)^{-2} e^{-iz \cdot \zeta}, \quad e^{iz \cdot \zeta} = (1 - \Delta z)^{-m} z^{m} e^{-iz \cdot \zeta}, \]

we have, for \( l, m \in \mathbb{N} \) with \( l \geq 4, m \geq 2 \),

\[
\begin{align*}
  r_{N,\tau,\alpha}(v, \xi) &= \int \left( \int \int e^{-iz \cdot \zeta} (1 - \Delta z)^{-m} (1 - \Delta z)^m \{ (\xi)^{-2} (1 - \Delta z)^{\Phi_{*,(\alpha)}(v + z)} \} \frac{d\zeta}{(2\pi)^3} \right) dz \\
  &= \int \left( (1 - \Delta z)^{\Phi_{*,(\alpha)}(v + z)} \right) \left( \int e^{-iz \cdot \zeta} (1 - \Delta z)^m \{ (\xi)^{-2} M^{(\alpha)}(\xi + \tau \zeta) \} \frac{d\zeta}{(2\pi)^3} \right) dz \\
  &= \int \left( \{ (1 - \Delta z)^{\Phi_{*,(\alpha)}(v + z)} \right) \left( \int_{|\zeta| \leq \frac{\epsilon}{2}} \frac{d\zeta}{(2\pi)^3} + \int_{|\zeta| \geq \frac{\epsilon}{2}} \{ \ldots \right) \frac{d\zeta}{(2\pi)^3} \right) dz \\
  &= \int \left( (1 - \Delta z)^{\Phi_{*,(\alpha)}(v + z)} \right) \left( I_1(\xi; z) + I_2(\xi; z) \right) \frac{dz}{(z)^{2m}}. \\
\end{align*}
\]

Since \( \langle \xi \rangle \) and \( \langle \xi + \tau \zeta \rangle \) are equivalent in \( I_1 \), it follows that

\[ |I_1| \leq C \langle \xi \rangle^{-1} M(\xi). \]

Moreover, the same bound for \(|I_2|\) holds because

\[
\begin{align*}
  |I_2| &\lesssim \int_{|\zeta| \geq \frac{\epsilon}{2}} \frac{\langle \xi \rangle^{-1} \langle \xi + \tau \zeta \rangle^{-1} \left( 1 + (\delta(T-t)) (\xi + \tau \zeta) \right) \frac{d\zeta}{(2\pi)^3}}{\langle \xi \rangle^{\ell-1} \left( (1 + (\delta(T-t)) (\xi + \tau \zeta)^{2s} + (T-t)^{2s} |\eta|^{2s}) \right)^{1/2+\epsilon}} \\
  &\lesssim \int_{|\zeta| \geq \frac{\epsilon}{2}} \frac{\langle \xi \rangle^{-1} \left( 1 + (\delta(T-t)) (\zeta^{2s(\ell-1)/(1+2s)} + (T-t)^{2s} |\eta|^{2s}) \right)^{1/2+\epsilon}}{\langle \xi \rangle^{\ell}} \frac{d\zeta}{(2\pi)^3} \lesssim \langle \xi \rangle^{-1} M(\xi). \\
\end{align*}
\]

These estimates lead to (5.25). The estimate [5.21] with \( r \neq 0 \) also follows by considering the expansion formula with \( (D_v)^r \) similar to (5.22).

\[ \square \]

**Corollary 5.5.** Let \( \phi_{*,k} \) be defined as in (5.31) and \( \Phi_{*,k}(v) = \Phi_{v}(v - v_*) \phi_{*,k}(v) \). Then we have

\[ [M(D_v), \Phi_{*,k}(v)] = A_{*,k}(v, D_v) M(D_v) \langle D_v \rangle^{-1}, \]  
(5.27)
where $2^k \langle v - v_\ast \rangle^{-\gamma} A_{*,k}(v, D_v)$ is a pseudo-differential operator with a symbol belonging to $S^0_{0,0}$ and its operator norm from $L^2$ to $L^2$ is uniformly bounded with respect to $k, v_\ast, \delta, t, R$. Furthermore, for any real $a, b$ and $c$ satisfying $a + \gamma - 1 = b + c$, there exists a constant $C > 0$ depending only on $a, b$ and $c$ such that

$$\|\langle v - v_\ast \rangle^a A_{*,k}(v, D_v)f\|_{L^2} \leq C 2^k \|\langle v - v_\ast \rangle^c f\|_{L^2}.$$  \hspace{1cm} (5.28)

**Proof.** It suffices to note in the similar formula as in (5.26) that for any real $b$ we have

$$|\Phi_{*,k,(\alpha)}(v + z)| \leq C \langle v - v_\ast + z \rangle^{-\gamma} \langle |v| + b \rangle^{-k} \leq C \langle v - v_\ast \rangle^{-\gamma} \langle |v| + b \rangle^{-k} \langle z \rangle^{\gamma} \langle |v| + b \rangle,$$

where $C$ is independent of $R$. The power of $\langle z \rangle$ can be handled by the factor $\langle z \rangle^{2m}$ by taking a sufficiently large $m$. \hfill $\Box$

**Corollary 5.6.** For any $k \in \mathbb{R}$ there exists an $L^2$-bounded operator $A(v, D_v)$ whose symbol belongs to $S^0_{0,0}$ uniformly with respect to $\eta, \delta, \epsilon, T - t$ such that

$$\langle v \rangle^{k} [\mathcal{M}, \langle v \rangle^{-k}] = [\langle v \rangle^{k}, \mathcal{M}] \langle v \rangle^{-k} = A(v, D_v)\mathcal{M},$$

with $\|\langle v \rangle^{1 + |D_v|^2} + |T - t|^2\eta^2\|_{L^2} \leq C \|g\|_{L^2}$ for $g \in \mathcal{S}(\mathbb{R}^3)$, where $C > 0$ is independent of $\eta, \delta, \epsilon, T - t$.

**Proof.** The proof is almost the same as the one of Lemma 5.2, with $\Phi_\ast$ replaced by $\langle v \rangle^{-k}$. We thus omit the details. \hfill $\Box$

**Proposition 5.7.** Let $0 < s < 1$ and $0 < \gamma \leq 1$. For any $a \in \mathbb{R}$ we have

$$\left| \left( \mathcal{M}Q_{\bar{\Pi}}(f, g) - Q_{\bar{\Pi}}(f, \mathcal{M}g), h \right) \right| \leq \|f\|_{H^s+\epsilon} \|h\|_{H^{s+\epsilon}} \left\{ \|f\|_{L^1_{2\alpha+\gamma+2}} \|\mathcal{M}g\|_{L^2_{2\alpha+\gamma+2}} + \int_{\mathbb{R}^6 \times S^2} b(\cdot) |f_\ast| |v_\ast|^{2\alpha+\gamma+2} \left( \langle \cdot \rangle^{\alpha+\gamma/2} \mathcal{M}g \right)(v) - \left( \langle \cdot \rangle^{\alpha+\gamma/2} \mathcal{M}g \right)(v) \right\}^{1/2}$$

$$+ R^{\max\{s-\gamma/2,1\}} \left\{ \|f\|_{L^1_{2\alpha+\gamma+2}} \|\mathcal{M}g\|_{L^2_{2\alpha+\gamma+2}} + \int_{\mathbb{R}^6 \times S^2} b(\cdot) |f_\ast| |v_\ast|^{2\alpha+\gamma+2} \left( \langle \cdot \rangle^{\alpha+\gamma/2} \mathcal{M}g \right)(v) - \left( \langle \cdot \rangle^{\alpha+\gamma/2} \mathcal{M}g \right)(v) \right\}^{1/2}$$

where $0 < s' < s$ is arbitrary, and $\epsilon'$ is non-negative and can be chosen as zero if $s \neq 1/2$.

**Proof.** For the proof we introduce the Littlewood-Paley decomposition in $\mathbb{R}^3$ as follows:

$$\sum_{k=0}^{\infty} \phi_k(v) = 1, \quad \phi_k(v) = \phi(2^{-k} v) \text{ for } k \geq 1 \text{ with } 0 \leq \phi_0, \phi \in C^\infty_0(\mathbb{R}^3),$$

and

$$\text{supp } \phi_0 \subset \{ |v| < 2 \}, \quad \text{supp } \phi \subset \{ |v| < 3 \}.$$

Take also $\bar{\phi}_0$ and $\bar{\phi} \in C^\infty_0$ such that

$$\bar{\phi}_0 = 1 \text{ on } \{ |v| \leq 2 \}, \quad \text{supp } \bar{\phi}_0 \subset \{ |v| < 3 \},$$

$$\bar{\phi} = 1 \text{ on } \{ 1/2 \leq |v| \leq 3 \}, \quad \text{supp } \bar{\phi} \subset \{ 1/3 < |v| < 4 \}.$$
Furthermore, we assume that all these functions are radial. It follows from the equivalence \(|v' - v_\ast| \leq |v - v_\ast| \leq \sqrt{2}v' - v_\ast|\) that

\[
\bar{\phi}_k(v' - v_\ast)\phi_k(v - v_\ast) = \phi_k(v - v_\ast) = \bar{\phi}_k(v - v_\ast)\phi_k(v - v_\ast), \quad k \geq 0.
\]  

(5.30)

Write

\[
\phi_{s,k}(v) = \phi_k(v - v_\ast), \quad \bar{\phi}_{s,k}(v) = \bar{\phi}(v - v_\ast), \quad \Phi_{s,k}(v) = \Phi_s(v)\phi_{s,k}(v).
\]

By the definition of \(\Phi = \Phi_\pi(v - v_\ast)\), we have

\[
\Phi_s(v) = \sum_{2^k \geq R} \Phi_s(v)\phi_{s,k}(v).
\]  

(5.32)

If we set \(\Phi_{s,k}(v) := \Phi_s(v)\phi_{s,k}(v)\), then for any real number \(r\) we have

\[
|D^n_0 \Phi_{s,k}(v)| \leq C(v - v_\ast)\gamma^{-|\alpha| + r}2^{-kr}\bar{\phi}_{s,k}(v),
\]

where \(C\) is independent of \(R\). It follows from (5.30) and (5.32) that

\[
(Q_\pi(f, g), h) = \sum_{k=1}^{\infty} \int_{\mathbb{R}^9} \int_{S^2} b \left( \frac{v - v_\ast}{|v - v_\ast|} \cdot \sigma \right) f(v_\ast)\Phi_{s,k}(v)g(v) \left\{ \bar{\phi}_{s,k}(v')h(v') - \bar{\phi}_{s,k}(v)h(v) \right\} d\sigma dv_s dv.
\]

Therefore, writing \(f_0 = f(v_\ast), g = g(v_\ast), h' = h(v')\) and so on, we have

\[
\left( M(D)Q_\pi(f, g) - Q_\pi(f, M(D)v), h \right)
\]

\[
= \sum_{2^k \geq R} \int_{\mathbb{R}^9} \int_{S^2} b \left( \frac{v - v_\ast}{|v - v_\ast|} \cdot \sigma \right) \Phi_{s,k}g(\bar{\phi}'_{s,k}(Mh) - \bar{\phi}_{s,k}(Mh)) d\sigma dv_s dv
\]

\[
- \sum_{2^k \geq R} \int_{\mathbb{R}^9} \int_{S^2} f_0 b \left( \frac{v - v_\ast}{|v - v_\ast|} \cdot \sigma \right) \Phi_{s,k}(Mg) \left( \bar{\phi}'_{s,k}h' - \bar{\phi}_{s,k}h \right) d\sigma dv_s dv
\]

\[
= \sum_{2^k \geq R} \left[ \int_{\mathbb{R}^9} \int_{S^2} f_0 b \left( \frac{v - v_\ast}{|v - v_\ast|} \cdot \sigma \right) \left( \Phi_{s,k}g(M\bar{\phi}_{s,k}h) - (M\Phi_{s,k})g(M\bar{\phi}_{s,k}h) \right) d\sigma dv_s dv
\]

\[
+ \int_{\mathbb{R}^9} \int_{S^2} f_0 b \left( \frac{v - v_\ast}{|v - v_\ast|} \cdot \sigma \right) \Phi_{s,k}g(M[\bar{\phi}_{s,k}h]) \left( [\Phi_{s,k}, M]h \right) d\sigma dv_s dv
\]

\[
- \int_{\mathbb{R}^9} \int_{S^2} f_0 b \left( \frac{v - v_\ast}{|v - v_\ast|} \cdot \sigma \right) \left( [\Phi_{s,k}, M]g(M\bar{\phi}_{s,k}h') \right) d\sigma dv_s dv
\]

\[
:= \sum_{2^k \geq R} [I_k + II_k - III_k].
\]

**Bound of \(I_k\)** If we set \(G_{s,k} = \Phi_{s,k}g\) and \(h_{s,k} = \bar{\phi}_{s,k}h\), then it follows from the Bobylev formula that

\[
I_k = \int_{\mathbb{R}^9} \int_{S^2} f_0 b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left( M(\xi) - M(\xi^+) \right) e^{iv' \cdot \xi} \hat{G}_{s,k}(\xi^+) e^{iv' \cdot \xi} h_{s,k}(\xi) d\sigma d\xi dv_s (2\pi)^3.
\]

Indeed, this follows from the substitution of

\[
Mh_{s,k}(v') = \int_{\mathbb{R}^3} e^{iv' \cdot \xi} M(\xi) h_{s,k}(\xi) \frac{d\xi}{(2\pi)^3}, \quad h_{s,k}(v') = \int_{\mathbb{R}^3} e^{iv' \cdot \xi} \hat{h}_{s,k}(\xi) \frac{d\xi}{(2\pi)^3}.
\]
and the exchange of \( \frac{\nu-a}{|\nu-a|} \cdot \sigma \) and \( \frac{\xi}{|\xi|} \cdot \sigma \) on the integrand. We decompose \( \mathcal{I}_k \) into

\[
\mathcal{I}_k = \int_{\mathbb{R}^3} \left\{ \int_{\mathbb{R}^3 \times S^2} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \frac{\mathcal{M}(\xi) - \mathcal{M}(\xi^*)}{\mathcal{M}(\xi)} e^{iv \cdot \xi^*} \mathcal{M}(\xi) \hat{G}_{\ast,k}(\xi) e^{iv \cdot \xi \hat{h}_{\ast,k}(\xi)} d\sigma d\xi \right\} dv_*
\]

\[
+ \int_{\mathbb{R}^3 \times S^2} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left( e^{iv \cdot \xi^*} \mathcal{M}(\xi^*) \hat{G}_{\ast,k}(\xi^*) - e^{iv \cdot \xi} \mathcal{M}(\xi) \hat{G}_{\ast,k}(\xi) \right) \times \frac{\mathcal{M}(\xi) - \mathcal{M}(\xi^*)}{\mathcal{M}(\xi^*)} e^{iv \cdot \xi \hat{h}_{\ast,k}(\xi)} d\sigma d\xi \hat{h}_{\ast,k}(\xi) \cdot \mathcal{M}(\xi^*) d\sigma d\xi
\]

\[
+ \int_{\mathbb{R}^3 \times S^2} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left( \frac{\mathcal{M}(\xi) - \mathcal{M}(\xi^*)}{\mathcal{M}(\xi^*)} \right)^2 e^{iv \cdot \xi^*} \mathcal{M}(\xi) \hat{G}_{\ast,k}(\xi) e^{iv \cdot \xi \hat{h}_{\ast,k}(\xi)} d\sigma d\xi \hat{h}_{\ast,k}(\xi) \cdot \mathcal{M}(\xi^*) d\sigma d\xi
\]

\[
\Delta \int f_* \left\{ T_k^{(1)}(v_*) + T_k^{(2)}(v_*) + T_k^{(3)}(v_*) \right\} dv_*.
\]

First we estimate \( T_k^{(1)}(v_*) \) in the case \( 1/2 < s < 1 \) by using the Taylor expansion

\[
\mathcal{M}(\xi) - \mathcal{M}(\xi^*) = \nabla \mathcal{M}(\xi) \cdot \xi^* - \int_0^1 (1 - \tau) \left( \nabla \otimes \nabla \mathcal{M}(\xi - \tau \xi^*) \right) d\tau \xi^* \cdot \xi^*.
\]

Noting

\[
\xi^* = \frac{|\xi|}{2} \left( \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \frac{\xi}{|\xi|} - \sigma \right) + \left( 1 - \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \right) \frac{\xi}{|\xi|}
\]

we see that the integral corresponding to the first term on the right hand side vanishes because of the symmetry on \( S^2 \). By means of the Cauchy-Schwarz inequality, it follows from Lemma 5.1 that

\[
|T_k^{(1)}(v_*)| \lesssim \|M^{2k(a-\gamma/2)} \Phi_{\ast,k} g\|_{L^2} \|g^{2k(-a+\gamma/2)} h_{\ast,k}\|_{L^2} \tag{5.34}
\]

\[
\lesssim \langle v_\ast \rangle^{2(a+\gamma)} \|\phi_{\ast,k}(v)\|_{L^2} + 2^{-k} \|\nu_a\|_{L^2} \|M g\|_{L^2} \|\tilde{\phi}_{\ast,k}(v)\|_{L^2} \|h_{\ast,k}\|_{L^2},
\]

where we have used Lemma 5.2 and its proof. When \( s = 1/2 \), in view of Lemma 5.1 we see that (5.34) holds with \( L^2 \) norm of one of the factor replaced by \( H^s(\mathbb{R}^3) \). If \( 0 < s < 1/2 \), then one can get (5.34) directly by means of the mean value theorem instead of the Taylor expansion of the second order. The mean value theorem can be applied to \( T_k^{(3)}(v_*) \) and we get the same bound as the right-hand side of (5.34) for \( T_k^{(3)}(v_*) \).

We consider \( T_k^{(2)}(v_*) \). Since it follows from the same manipulation as the Bobylev formula (see Proposition 1) and its proof that

\[
\int_{\mathbb{R}^3 \times S^2} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left| e^{iv \cdot \xi^*} \mathcal{M}(\xi^*) \hat{G}_{\ast,k}(\xi^*) - e^{iv \cdot \xi} \mathcal{M}(\xi) \hat{G}_{\ast,k}(\xi) \right|^2 d\sigma d\xi \hat{h}_{\ast,k}(\xi) \cdot \mathcal{M}(\xi^*) \hat{h}_{\ast,k}(\xi) \cdot \mathcal{M}(\xi^*)
\]

\[
= \int_{\mathbb{R}^3 \times S^2} b \left( \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) \left| (MG_{\ast,k})(v') - (MG_{\ast,k})(v) \right|^2 d\sigma dv,
\]

we have

\[
|T_k^{(2)}(v_*)| \lesssim \left( \int_{\mathbb{R}^3 \times S^2} b(\cdot) 2^{k(2a-\gamma)} \left| (MG_{\ast,k})(v') - (MG_{\ast,k})(v) \right|^2 d\sigma dv \right)^{1/2} \|2^{k(-a+\gamma/2)} h_{\ast,k}\|_{L^2}. \tag{5.35}
\]

Note that

\[
\left| (MG_{\ast,k})(v') - (MG_{\ast,k})(v) \right|^2 \leq 2 \left| \Phi_{\ast,k}(v')(M g)(v') - \Phi_{\ast,k}(v)(M g)(v) \right|^2
\]

\[
+ 2 \left| ([M, \Phi_{\ast,k}](v')(M g)(v') - ([M, \Phi_{\ast,k}](v)(M g)(v) \right|^2.
\]

If we put \( \Phi_{\ast,k}(v) = 2^{k(a-\gamma/2)} \Phi_{\ast,k}(v)(v)^{-a+\gamma/2} \) then \( \Phi_{\ast,k}(v) \lesssim \langle v_\ast \rangle \langle v'_\ast \rangle \Phi_{\ast,k}(v) \) and

\[
\left| \Phi_{\ast,k}(v') - \Phi_{\ast,k}(v) \right| \lesssim \langle v_\ast \rangle \langle v'_\ast \rangle \Phi_{\ast,k}(v) \sin \theta/2.
\]
Therefore,
\[
2^{k(2a-\gamma)} |\Phi_{*,k}(v')(\mathcal{M}g)(v') - \Phi_{*,k}(v)(\mathcal{M}g)(v)|^2 \\
\lesssim (v_*)^{2|a|+\gamma} \phi_{*,k}(v)^2 \langle (\cdot)^{a+\gamma/2}\mathcal{M}g(\cdot) \rangle(v')^2 - \langle (\cdot)^{a+\gamma/2}\mathcal{M}g(\cdot) \rangle(v)^2 \\
+ \theta^2 (v_*)^{2|a|+\gamma} \phi_{*,k}(v')^2 \langle (\cdot)^{a+\gamma/2}\mathcal{M}g(\cdot) \rangle(v')^2 ,
\]
and
\[
\int_{\mathbb{R}^3 \times S^2} b(\cdot) 2^{k(2a-\gamma)} |\Phi_{*,k}(v')(\mathcal{M}g)(v') - \Phi_{*,k}(v)(\mathcal{M}g)(v)|^2 \ d\sigma dv \\
\lesssim (v_*)^{2|a|+\gamma} \left\{ \int_{\mathbb{R}^3 \times S^2} b(\cdot) \phi_{*,k}^2(v) \langle (\cdot)^{a+\gamma/2}\mathcal{M}g(\cdot) \rangle(v')^2 - \langle (\cdot)^{a+\gamma/2}\mathcal{M}g(\cdot) \rangle(v)^2 \ d\sigma dv \\
+ \int_{\mathbb{R}^3} \phi_{*,k}^2(v) \langle (\cdot)^{a+\gamma/2}\mathcal{M}g(\cdot) \rangle(v')^2 \ dv \right\}.
\]

In view of Corollary 5.5, we put $[\mathcal{M}, \Phi_{*,k}]g = A_{*,k}(D_\cdot)^{-1}\mathcal{M}g = A_{*,k}\tilde{g}$. Dividing the interval $[0, \pi/2]$ into $[0, 2^{-k})$, $[2^{-k}, \pi/2]$ we have
\[
\int_{\mathbb{R}^3 \times S^2} b(\cdot) 2^{k(2a-\gamma)} |[\mathcal{M}, \Phi_{*,k}]g| \langle (\cdot)^{a+\gamma/2}\mathcal{M}g(\cdot) \rangle(v') - \langle [\mathcal{M}, \Phi_{*,k}]g(\cdot) \rangle(v)|^2 \ d\sigma dv \\
\lesssim 2^{k(2a-\gamma+2\delta)} \int_{\mathbb{R}^3} \left| \nabla_v(A_{*,k}\tilde{g}) \right|^2 + \left| A_{*,k}\tilde{g} \right|^2 \ d\sigma dv \\
\lesssim 2^{-2k(1-s)} \int_{\mathbb{R}^3} |v - v_+|^{a+\gamma/2}\mathcal{M}g|^2 \ dv \lesssim 2^{-2k(1-s)} (v_*)^{2|a|+\gamma} \int_{\mathbb{R}^3} |(\cdot)^{a+\gamma/2}\mathcal{M}g(\cdot)|^2 \ dv ,
\]
where we have used (5.28) and the fact that $\nabla_v(A_{*,k}\tilde{g}) = (\nabla_vA_{*,k})\tilde{g} + A_{*,k}(\nabla_v\tilde{g})$ and the regular change of variable $v \to v + \tau(v' - v)$ for $\tau \in [0, 1]$.

Above two estimates together with (5.35) lead us to
\[
|\mathcal{I}_k^{(2)}(v_*)| \lesssim (v_*)^{2|a|+\gamma} \phi_{*,k}(v)^{-a+\gamma/2} h_{L^2} \\
\times \left[ \left( \int_{\mathbb{R}^3 \times S^2} b(\cdot) \phi_{*,k}(v)^2 \langle (\cdot)^{a+\gamma/2}\mathcal{M}g(\cdot) \rangle(v') - \langle (\cdot)^{a+\gamma/2}\mathcal{M}g(\cdot) \rangle(v)^2 \ d\sigma dv \right)^{1/2} \\
+ \left( \int_{\mathbb{R}^3} \phi_{*,k}(v)^2 \langle (\cdot)^{a+\gamma/2}\mathcal{M}g(\cdot) \rangle(v')^2 \ dv \right)^{1/2} \right] \right].
\]

Summing up estimates for $\mathcal{I}_k^{(j)}(v_*)$, $j = 1, 2, 3$, we have
\[
\sum_{2^k \geq R} \mathcal{I}_k \lesssim \|f\|_{L^2(|a|+\gamma+2)} \|h\|_{L^2_{-a+\gamma/2}} \left\{ \|f\|_{L^2(|a|+\gamma+2)} \|\mathcal{M}g\|_{L^2_{a+\gamma/2}} + \\
\left( \int_{\mathbb{R}^3 \times S^2} b(\cdot) \|f\|_{L^2(|a|+\gamma+2)} \langle (\cdot)^{a+\gamma/2}\mathcal{M}g(\cdot) \rangle(v') - \langle (\cdot)^{a+\gamma/2}\mathcal{M}g(\cdot) \rangle(v)^2 \ d\sigma dv \right)^{1/2} \right) ,
\]
provided that $s \neq 1/2$. When $s = 1/2$, the term $\|f\|_{L^2(|a|+\gamma+2)} \|\mathcal{M}g\|_{L^2_{a+\gamma/2}}$ is replaced by $\|f\|_{L^2_{a+\gamma+2}} \|\mathcal{M}g\|_{H_{a+\gamma/2}}$.

In order to have a small factor we go back to above procedure. All $\mathcal{I}_k^{(j)}(v_*)$ contain a factor $\mathcal{M}(\xi^+) - \mathcal{M}(\xi)$. Note that for any $0 < s' < s$ we have
\[
\left| \mathcal{M}(\xi^+) - \mathcal{M}(\xi) \right| \lesssim \frac{\delta(T-t)(1 + |\xi| + (T-t)|\eta|)^{2s-1}|\xi|^{s'}}{1 + \delta(T-t)(1 + |\xi| + (T-t)|\eta|)^{2s}} \left( \delta(T-t)^{s'/2s} |\xi|^{s'} \right)^	heta
\]
because of $\sup_{X \in [0,\infty]} X^{(2s-s')/2s}/(1 + X) < 1$, and moreover
\[
\left| \frac{1}{\mathcal{M}(\xi)} \int_0^1 (1 - \tau) \left( \nabla \otimes \nabla \mathcal{M}(\xi - \tau \xi^-) \right) d\tau \xi^- \cdot \xi^- \right| \lesssim (\delta(T-t))^{s'/2s} |\xi|^{s'} \theta^2 ,
\]
with a convention that $|\xi|^s$ is replaced by $|\xi|^s+\varepsilon$ when $s = 1/2$.

Using the above observation we can show that
\[
\sum_{2^k \geq R} I_k \lesssim \delta^{\varepsilon'/2(2s)} \| f \|_{L_2^{|a|_{a+\gamma+2}}} ^{1/2} \| h \|_{H^{\varepsilon'/2}_{a+\gamma+2}} \left\{ \| f \|_{L_2^{|a|_{a+\gamma+2}}} \| M_g \|_{L_2^{|a|_{a+\gamma+2}}} \right\}^{1/2} + \left( \int_{\mathbb{R}^3 \times \mathbb{S}^2} b(\cdot) |f_s| |v_s|^2 |\langle \cdot \rangle^{a+\gamma/2} M_g(v') - |\langle \cdot \rangle^{a+\gamma/2} M_g(v)\rangle |^2 d\sigma dv_s \right)^{1/2},
\]
where $\varepsilon' > 0$ can be chosen to be zero if $s \neq 1/2$.

**Bound of $II_k$** Put $H_{*,k}(v) = [\tilde{\phi}_{*,k}, M] h$. Then as for $I_k$, we have
\[
II_k = \int_{\mathbb{R}^3} f_s \left( \int_{\mathbb{R}^3 \times \mathbb{S}^2} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) e^{iv \cdot \xi} \frac{M(\xi) - M(\xi^+)}{M(\xi)} e^{iv \cdot \xi} \frac{\tilde{H}_{*,k}(\xi)}{M(\xi)} d\sigma d\xi \right) dv_s.
\]

We decompose this into
\[
II_k = \int_{\mathbb{R}^3} f_s \left\{ \int_{\mathbb{R}^3 \times \mathbb{S}^2} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) e^{iv \cdot \xi} \frac{M(\xi) - M(\xi^+)}{M(\xi)} e^{iv \cdot \xi} \frac{\tilde{H}_{*,k}(\xi)}{M(\xi)} d\sigma d\xi \right\} dv_s.
\]

By means of Corollary 5.5 (with $\gamma = 0$), we have
\[
H_{*,k}(v) = -(\tilde{\phi}_{*,k} M)^* h = -(D_v)^{-1} M (A_{*,k}(v, D_v))^* h
\]
and
\[
2^{-ka} \| \langle \xi \rangle^s M(\xi)^{-1} \tilde{H}_{*,k}(\xi) \|_{L_2^2} \lesssim 2^{-ka} \| (\langle D \rangle^{-1} + \gamma (A_{*,k}(v, D_v))^* h) \|_{L_2^2} \lesssim 2^{-k} \| (v - v_*)^{-\alpha} h \|_{L_2^2}.
\]

Similar to the argument for $I_k^{(1)}$, we can obtain
\[
|II_k^{(1)}(v_s)| \lesssim |v_s|^2 (\gamma/2 + |a|_{a+\gamma+2}) \times \| \tilde{\phi}_{*,k} M g \|_{L_2^2} + 2^{-k} \| \langle v \rangle^{a+\gamma/2} M g \|_{L_2^2} \lesssim 2^{k(\gamma/2 - 1)} \| (v - v_*)^{-\alpha} h \|_{L_2^2},
\]
where the right hand side becomes obviously the upper bound for $|II_k^{(3)}(v_s)|$ too. By the almost same procedure as for $I_k^{(2)}(v_s)$, we also have
\[
|II_k^{(2)}(v_s)| \lesssim |v_s|^2 (\gamma/2 + |a|_{a+\gamma+2}) \| (v - v_*)^{-\alpha} h \|_{L_2^2}
\]
and
\[
\int_{\mathbb{R}^3} f_s \left\{ II_k^{(2)}(v_s) + II_k^{(3)}(v_s) + II_k^{(4)}(v_s) \right\} dv_s.
\]

(5.38)
Putting
\[ \tilde{G}_k(v; v_*) = 2^{k(a-\gamma/2)} M(D_v) \Phi([v - v_*]) \phi_k(v - v_*) g(v), \]
\[ \tilde{H}_k(v; v_*) = M^{-1}(D_v) [\tilde{\phi}_k(v - v_*) M(D_v)] 2^{k(-a+\gamma/2)} h(v), \]
we consider \( \mathcal{I}_k^{(3)}(v_*) \) by first writing it as
\[
\mathcal{I}_k^{(3)}(v_*) = \int_{\mathbb{R}^3 \times S^2} b(\cdot) \tilde{G}_k(v; v_*) (\tilde{H}_k(v'; v_*) - \tilde{H}_k(v; v_*)) d\sigma dv
\]
\[
= \int_{\mathbb{R}^3 \times S^2} b(\cdot) (\tilde{G}_k(v; v_*) - \tilde{G}_k(v'; v_*)) (\tilde{H}_k(v'; v_*) - \tilde{H}_k(v; v_*)) d\sigma dv
\]
\[
+ \int_{\mathbb{R}^3 \times S^2} b(\cdot) \tilde{G}_k(v'; v_*) (\tilde{H}_k(v'; v_*) - \tilde{H}_k(v; v_*)) d\sigma dv
\]
\[
+ \int_{\mathbb{R}^3 \times S^2} b(\cdot) \tilde{G}_k(v; v_*) (\tilde{H}_k(w; v_*) - \tilde{H}_k(v; v_*)) d\sigma dv
\]
\[
\triangleq S^k(v_*) + M_0^k(v_*) + R^k_1(v_*) + R^k_2(v_*),
\]
where (see Figure 1 of [AMUYX10-2013])
\[ w = v_* + \left( \cos^2 \frac{\theta}{2} \right) (v - v_*) = \frac{v' + v_*}{2} + \frac{|v' - v_*|}{2} \omega, \]
\[ \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma = \frac{v' - v_*}{|v' - v_*|} \cdot \omega. \]

It follows from the Cauchy-Schwarz inequality that
\[
|S^k(v_*)|^2 \leq \left( \int_{\mathbb{R}^3 \times S^2} b(\cdot) \left| \tilde{G}_k(v; v_*) - \tilde{G}_k(v'; v_*) \right|^2 d\sigma dv \right) \left( \int_{\mathbb{R}^3 \times S^2} b(\cdot) \left| \tilde{H}_k(v'; v_*) - \tilde{H}_k(v; v_*) \right|^2 d\sigma dv \right).
\]
In the estimation for \( \mathcal{I}_k^{(2)}(v_*), \) we have already shown
\[
\int_{\mathbb{R}^3 \times S^2} b(\cdot) \left| \tilde{G}_k(v; v_*) - \tilde{G}_k(v'; v_*) \right|^2 d\sigma dv
\]
\[
\lesssim \langle v_* \rangle^{\gamma+2|\alpha|+2} \left\{ \int_{\mathbb{R}^3 \times S^2} b(\cdot) \tilde{\phi}_{\alpha,k}(v)^2 \left[ \langle \cdot \rangle^{a+\gamma/2} M g(v') - \langle \cdot \rangle^{a+\gamma/2} M g(v) \right]^2 d\sigma dv \right.
\]
\[
+ \int_{\mathbb{R}^3} \tilde{\phi}_{\alpha,k}(v)^2 \left[ \langle \cdot \rangle^{a+\gamma/2} M g(v) \right]^2 dv + \int_{\mathbb{R}^3} 2^{-2k(1-s)} \left[ \langle \cdot \rangle^{a+\gamma/2} M g(v) \right]^2 dv \right\}.
\]
Similar argument as for (5.36) yields
\[
\int_{\mathbb{R}^3 \times S^2} b(\cdot) \left| \tilde{H}_k(v'; v_*) - \tilde{H}_k(v; v_*) \right|^2 d\sigma dv \lesssim 2^{-2k(1-s)} \langle v_* \rangle^{2|\alpha|+\gamma+2} \int_{\mathbb{R}^3} \left| \langle v \rangle^{-a+\gamma/2} h(v) \right|^2 dv.
\]
Therefore, we have
\[
\sum_{2^k \geq h} \int_{\mathbb{R}^3} f(v_*) |S^k(v_*)| dv_*
\]
\[
\lesssim R^{a-1} \| f \|_{L^2_{\gamma+2|\alpha|+2}} \left\| \langle v \rangle^{-a+\gamma/2} h \right\|_{L^2} \left\| f \|_{L^1_{\gamma+2|\alpha|+2}} \right\| M g \|_{L^2_{a+\gamma/2}}^2
\]
\[
+ \int_{\mathbb{R}^3 \times S^2} b(\cdot) |f_*(v_*)|^{\gamma+2|\alpha|+2} \left[ \langle \cdot \rangle^{a+\gamma/2} M g(v') - \langle \cdot \rangle^{a+\gamma/2} M g(v) \right]^2 d\sigma dv dv_* \right)^{1/2}. \]
The same observation as for $M_1$ in the proof of Lemma B.4 shows

\begin{equation}
M_0^k(v_*) = -2\int_{k}^{(3)}(v_*) + R_3^k(v_*),
\end{equation}

\begin{equation}
R_3^k(v_*) = \int_{\mathbb{R}^3 \times \mathbb{R}^2} b(\cdot)\tilde{G}_k(v'; v_*) (\tilde{H}_k(v; v_*) - \tilde{H}_k(v'; v_*)) \left( \frac{1}{\cos^4(\theta'/2)} - 1 \right) d\sigma dv.
\end{equation}

The term coming from $R_3^k$ is bounded as

\begin{equation}
\sum_{2^k \geq R} \int_{\mathbb{R}^3} |f(v_*)| R_3^k(v_*)|dv_*| \lesssim R^{\gamma/2 - 1} \|f\|_{L_{1/2+2|\omega|+2}} \|Mg\|_{L_{3+\gamma/2}} \|v\|^{-a} h_{L^2}.
\end{equation}

We consider $R_3^k(v_*)$, by taking the change of variables $v \rightarrow z + v_*$. Putting

$$
\tilde{G}_k(z + v_*; v_*) = 2^{k(a-\gamma/2)} \mathcal{M}(D_z)\Phi(|z|)\phi_k(z)g(z + v_*) \triangleq G^0_{*,k}(z),
$$

$$
\tilde{H}_k(z + v_*; v_*) = M^{-1}(D_z)\tilde{\phi}_k(z), \mathcal{M}(D_z)2^{k(-a+\gamma/2)} h(z + v_*) \triangleq H^0_{*,k}(z),
$$

we have

\begin{equation}
R_3^k(v_*) = \int_{0}^{2\pi} b(\cos \theta) \sin \theta \int_{\mathbb{R}^3} G_{*,k}(z) \left( \frac{H^0_{*,k}(z \cos^2(\theta/2)) - H^0_{*,k}(z)}{\sin^2(\theta/2)} \right) dz d\theta
\end{equation}

\begin{equation}
= \int_{0}^{2\pi} b(\cos \theta) \sin \theta \int_{\mathbb{R}^3} \tilde{G}_{*,k}(\xi) \left( \frac{1}{\cos^4(\theta/2)} \right) \tilde{H}^0_{*,k}(\frac{\xi}{\cos^2(\theta/2)}) - \tilde{H}^0_{*,k}(\xi) d\xi d\theta
\end{equation}

\begin{equation}
= \int_{0}^{2\pi} b(\cos \theta) \sin \theta \left( \frac{1}{\cos^4(\theta/2)} \right) \int_{\mathbb{R}^3} \tilde{G}_{*,k}(\xi) \tilde{H}^0_{*,k}(\frac{\xi}{\cos^2(\theta/2)}) - \tilde{H}^0_{*,k}(\xi) d\xi d\theta
\end{equation}

\begin{equation}
+ \int_{0}^{2\pi} b(\cos \theta) \sin \theta \int_{\mathbb{R}^3} \tilde{G}_{*,k}(\xi) \left( \tilde{H}^0_{*,k}(\frac{\xi}{\cos^2(\theta/2)}) - \tilde{H}^0_{*,k}(\xi) \right) d\xi d\theta
\end{equation}

\begin{equation}
\triangleq R_{1,1}^k(v_*) + R_{1,2}^k(v_*).
\end{equation}

We have

\begin{equation}
\sum_{2^k \geq R} \int_{\mathbb{R}^3} |f(v_*)| R_{1,1}^k(v_*)|dv_*| \lesssim R^{\gamma/2 - 1} \|f\|_{L_{1/2+2|\omega|+2}} \|Mg\|_{L_{3+\gamma/2}} \|v\|^{-a} h_{L^2}.
\end{equation}

Write

\begin{equation}
R_{1,2}^k(v_*) = \int_{\mathbb{R}^3} \tilde{G}_{*,k}(\xi) \left( \int_{0}^{2-k/2(\xi)^{-1/2}} b(\cos \theta) \sin \theta \tan^2 \frac{\theta}{2}
\end{equation}

\begin{equation}
\times \left( \int_{0}^{1} \xi \cdot \nabla_{z} \tilde{H}^0_{*,k}(\xi(1 + \tau \tan^2 \theta/2)) d\theta d\tau \right) d\xi
\end{equation}

\begin{equation}
+ \int_{\mathbb{R}^3} \tilde{G}_{*,k}(\xi) \left( \int_{2-k/2(\xi)^{-1/2}}^{\pi/2} b(\cos \theta) \sin \theta \left( \tilde{H}^0_{*,k}(\frac{\xi}{\cos^2(\theta/2)}) - \tilde{H}^0_{*,k}(\xi) \right) d\theta \right)
\end{equation}

\begin{equation}
\triangleq B_{1}^k(v_*) + B_{2}^k(v_*).
\end{equation}

By Cauchy-Schwarz inequality, we have

\begin{equation}
|B_{1}^k(v_*)|^2 \lesssim \int_{\mathbb{R}^3} |(\xi)^{1-s} |G^0_{*,k}(\xi)|^2 \int_{0}^{2-k/2(\xi)^{-1/2}} b(\cos \theta) \sin \theta \tan^2 \frac{\theta}{2} d\theta d\xi
\end{equation}

\begin{equation}
\times \int_{\mathbb{R}^3} |(\xi)^{1+s} |\nabla_{z} \tilde{H}^0_{*,k}(\xi)|^2 \int_{0}^{2-k/2(\xi)^{-1/2}} b(\cos \theta) \sin \theta \tan^2 \frac{\theta}{2} d\theta d\xi,
\end{equation}

where, in the second factor, we have used the change of variable

$$
\xi \rightarrow (1 + \tau \tan^2 \frac{\theta}{2})\xi
$$
after exchanging $d\theta d\xi$ by $d\xi d\theta$. Therefore,

$$|B_k^{(1)}(v_*)|^2 \lesssim \left( \int_{\mathbb{R}^3} |\tilde{G}_{s,k}^0(\xi)|^2 d\xi \right) \left( \int_{\mathbb{R}^3} 2^{2ks}(\xi)^{2s} |\nabla \xi^2 \hat{h}_{s,k}^0(\xi)|^2 d\xi \right).$$

On the other hand,

$$|B_k^{(2)}(v_*)|^2 \lesssim \left( \int_{\mathbb{R}^3} |\tilde{G}_{s,k}^0(\xi)|^2 d\xi \right) \left( \int_{\mathbb{R}^3} 2^{2ks}(\xi)^{2s} |\hat{h}_{s,k}^0(\xi)|^2 d\xi \right).$$

Both estimates lead us to

$$\sum_{2^k \geq R} \int_{\mathbb{R}^3} f(v_*) R_{1,2}^k(v_*) |dv_*| \lesssim R^{s-1} \|f\|_{L^2_{\gamma+2|a|+2}} \|\mathcal{M}g\|_{L^2_{a+\gamma/2}} \|\langle D \rangle^{s-1}(v)^{-a} \hat{h}\|_{L^2_{\gamma/2}}. \quad (5.44)$$

As for $R_{R}^k(v_*)$, it follows from Cauchy-Schwarz inequality that

$$|R_{R}^k(v_*)|^2 \leq \left( \int_{\mathbb{R}^3 \times \mathbb{S}^2} b(\cdot) \left| \tilde{G}_k(w' ; v_*) - \tilde{G}_k(v ; v_*) \right|^2 |d\sigma dv| \right) \left( \int_{\mathbb{R}^3 \times \mathbb{S}^2} b(\cdot) \left| \tilde{H}_k(w; v_*) - \tilde{H}_k(v; v_*) \right|^2 |d\sigma dv| \right).$$

The first factor of the right hand side is exactly the same as the one of (5.35). In view of the proof of Lemma B.4 for $M_1$, the second factor is bounded by

$$2 \left\{ \int_{\mathbb{R}^3 \times \mathbb{S}^2} b(\cdot) \left| \tilde{H}_k(w; v_*) - \tilde{H}_k(v; v_*) \right|^2 |d\sigma dv| + \int_{\mathbb{R}^3 \times \mathbb{S}^2} b(\cdot) \left| \tilde{H}_k(v'; v_*) - \tilde{H}_k(v; v_*) \right|^2 |d\sigma dv| \right\} \leq 2^{5/2} \int_{\mathbb{R}^3 \times \mathbb{S}^2} b(\cdot) \left| \tilde{H}_k(v'; v_*) - \tilde{H}_k(v; v_*) \right|^2 |d\sigma dv|,$$

which concludes

$$\sum_{2^k \geq R} \int_{\mathbb{R}^3} f(v_*) R_{2,2}^k(v_*) |dv_*| \lesssim R^{s-1} \|f\|_{L^2_{\gamma+2|a|+2}} \|\langle v \rangle^{-a+\gamma/2} \hat{h}\|_{L^2_{a+\gamma/2}} \left\{ \|f\|_{L^2_{\gamma+2|a|+2}} \|\mathcal{M}g\|_{L^2_{\gamma+2|a|+2}} \right\}^{1/2}. \quad (5.45)$$

Summing up estimates from (5.40) to (5.45) we have

$$\sum_{2^k \geq R} \left| \int_{\mathbb{R}^3} f(v_*) T_k^{(3)}(v_*) |dv_*| \right| \lesssim R^{\max(s-1,\gamma/2-1)} \|f\|_{L^2_{\gamma+2|a|+2}} \|\langle v \rangle^{-a+\gamma/2} \hat{h}\|_{L^2_{a+\gamma/2}} \left\{ \|f\|_{L^2_{\gamma+2|a|+2}} \|\mathcal{M}g\|_{L^2_{\gamma+2|a|+2}} \right\}^{1/2}.$$  

From this together with (5.38) and (5.39), we obtain the upper bound for

$$\sum_{2^k \geq R} \left| \int_{\mathbb{R}^3} f(v_*) T_k(v_*) |dv_*| \right|.$$

**Bound of $T_k$** Putting $F_{s,k}(v) = [\Phi_{s,k}, \mathcal{M}]g$, we have

$$T_k = \int_{\mathbb{R}^3} \left( \int_{\mathbb{S}^2} f(b \frac{\xi}{|\xi|} \cdot \sigma) \left( e^{iv_* \cdot \xi^+} \hat{F}_{s,k}(\xi^+) - e^{iv_* \cdot \xi^-} \hat{F}_{s,k}(\xi) \right) e^{iv_* \cdot \xi^+} \hat{h}_{s,k}(\xi) |d\sigma| \xi \right) \frac{|dv_*|}{(2\pi)^3}$$

$$\Delta \int_{\mathbb{R}^3} f(v_*) J^k(v_*) |dv_*|.$$
The estimation for $J_k(v_*)$ is almost same as for $II_k^3(v_*)$. Indeed, we decompose it into

$$J_k(v_*) = \int_{\mathbb{R}^3 \times S^2} b(\cdot) F_{*,k}(v)(h_{*,k}(v') - h_{*,k}(v)) d\sigma dv$$

$$= \int_{\mathbb{R}^3 \times S^2} b(\cdot) (F_{*,k}(v) - F_{*,k}(v'))(h_{*,k}(v') - h_{*,k}(v)) d\sigma dv$$

$$+ \int_{\mathbb{R}^3 \times S^2} b(\cdot) F_{*,k}(v')(h_{*,k}(v') - h_{*,k}(v)) d\sigma dv$$

$$+ \int_{\mathbb{R}^3 \times S^2} b(\cdot) (F_{*,k}(v) - F_{*,k}(v))(h_{*,k}(w) - h_{*,k}(v)) d\sigma dv$$

$$\triangleq \tilde{S}_k(v_*) + \tilde{M}_k(v_*) + \tilde{R}_k(v_*) + \tilde{R}_k(v_*)$$

By using a similar procedure with the role of $g$ and $h$ exchanged, we can show

$$\sum_{\eta \leq R} \int_{\mathbb{R}^3} f_{\eta}(v_*) J_k(v_*) d\nu_{\eta} \lesssim R^{\max\{s-1, \gamma/2-1\}} \|f\|^{1/2}_{L^1_{t+k=|\eta|+2}} \|Mg\|_{L^2_{t+k+\gamma/2}} \left\{ \|f\|^{1/2}_{L^1_{t+k=|\eta|+2}} \|h\|^{2}_{L^2_{t-k-\gamma/2}} \right\}^{1/2}.$$ 

And the proof of (5.29) is completed. □

5.4. Proof of Theorem 5.1. Now we apply Propositions 5.2-5.7 and the spectral gap estimate to obtain a closed form energy estimate for the linearized equation (5.2). The statement of the theorem is

Theorem 5.8. Let $f$ be the solution to (5.1). Let $k_0$ satisfy (5.1) so that the spectral gap theorem holds. Let $\epsilon, \delta > 0$ (in the definition of $M$) be small enough. Then there exists $T_0 > 0$ such that

$$\int_0^{T_0} \left\| (v)^{k_0} f(t, \cdot) \right\|_{L^2_{v,s}}^2 dt \leq \frac{C}{\epsilon^2 \delta} \sum_{\ell \in \mathbb{Z}^3} \int_{\mathbb{R}^3} \frac{1}{\langle \xi \rangle^s + \langle \ell \rangle^{\gamma/2}} \left\| \langle v \rangle^{k_0} \tilde{h}^m_{\ell}(\xi) \right\|^2 d\xi$$

and for any $t \geq T_0$,

$$\left\| (v)^{k_0} f(t, \cdot) \right\|_{L^2_{v,s}} \leq C \left( \frac{1}{\sqrt{T_0}} + \frac{C_R}{\sqrt{\epsilon^2 \delta}} \right) e^{-\lambda t} \left( \sum_{\ell \in \mathbb{Z}^3} \int_{\mathbb{R}^3} \frac{1}{\langle \xi \rangle^s + \langle \ell \rangle^{\gamma/2}} \left\| \langle v \rangle^{k_0} \tilde{h}^m_{\ell}(\xi) \right\|^2 d\xi \right)^{1/2}.$$ 

Here $\lambda > 0$ is the same decay rate as in the spectral gap estimate.

Proof. Let $T_0 < 1$ be small enough whose size will be specified later. First we consider the bound over any finite time interval $[0, T]$ with $T \leq T_0$. The equation for $f_{\ell}$ is

$$\partial_t f_{\ell} - i\eta \cdot \nabla x f_{\ell} = Q(\mu, f_{\ell}) + Q(f_{\ell}, \mu), \quad (5.46)$$

where $f_{\ell}$ is the $\ell$-th Fourier mode in $x$. We multiply (5.46) by $M \left( (v)^{2k_0} M f_{\ell} \right)$, integrate over $\mathbb{R}^3_v$, and then take the real part of the equation. Noting that

$$\left( \partial_t - \eta \cdot \nabla_x \right) M(t, \xi, \eta) = \left( \frac{1}{2} + \varepsilon \right) M(t, \xi, \eta) \frac{\delta \langle \xi \rangle^{2s}}{1 + \delta \int_0^{t-t_0} \langle \xi - \tau \rangle^{2s} d\tau}, \quad (5.47)$$

the left hand side gives

$$LHS \geq \frac{1}{2} \frac{d}{dt} \left\| (v)^{k_0} M f_{\ell} \right\|_{L^2_v}^2 - \left( \frac{1}{2} + \varepsilon \right) c_0 \delta \left\| (v)^{k_0} M f_{\ell} \right\|_{H^1_v}^2. \quad (5.48)$$
The contribution from each term on the right hand side is estimated as follows. First, we decompose $Q$ as $Q_R + Q_{\infty}$. Putting $h_\ell = \langle v \rangle^k_0 M f_\ell$, we consider

$$
\left(Q_R(\mu, f_\ell), M(\langle v \rangle^{2k_0} M f_\ell)\right) = \left(Q_R(\mu, f_\ell), [M, (\langle v \rangle^{k_0})h_\ell]\right) + \langle v \rangle^{k_0} Q_R(\mu, f_\ell) - Q_R(\mu, (\langle v \rangle^{k_0} f_\ell), Mh_\ell)
$$

$$+ \left(MQ_R(\mu, (\langle v \rangle^{k_0} f_\ell) - Q_R(\mu, M(\langle v \rangle^{k_0} f_\ell)), h_\ell) + \left(Q_R(\mu, [M, (\langle v \rangle^{k_0} f_\ell)], h_\ell)
$$

$$+ \left(Q_R(\mu, (\langle v \rangle^{k_0} M f_\ell) - (\langle v \rangle^{k_0} Q_R(\mu, M f_\ell), h_\ell)
$$

$$\Delta = R_1 + R_2 + R_3 + R_4 + R_5.
$$

By means of Proposition 5.3, we have

$$|R_2| + |R_5| \leq C_{k_0, R} \| f_\ell \|_{L^2(\mathbb{S})} \| \langle v \rangle^{k_0} M f_\ell \|_{H^{s'}(\mathbb{S})}.
$$

It follows from Proposition 5.2 that

$$|R_3| \leq C_R \langle v \rangle^{k_0} M f_\ell \|_{H^{s'}(\mathbb{S})}^2.
$$

By a similar argument used in Corollary 5.6, there exists $A(\nu, D_\nu) \in O(p(S^{-1}_0))$ such that

$$[M, (\langle v \rangle^{k_0}) = A(\nu, D_\nu) \langle v \rangle^{k_0} M, \|A(\nu, D_\nu) g\|_{H^{2s'}(\mathbb{S})} \leq \|g\|_{H^{2s'}(\mathbb{S})}.
$$

It follows from [27] Proposition 6.11 that

$$|R_1| \leq C_{k_0, R} \| f_\ell \|_{H^{2s'}(\mathbb{S})} \| h_\ell \|_{L^2} \leq C_{k_0, R} \| \langle v \rangle^{k_0} M f_\ell \|_{H^{2s'}(\mathbb{S})} \| \langle v \rangle^{k_0} M f_\ell \|_{L^{2}(\mathbb{S})}.
$$

By means of Corollary 5.6, we write $[M, (\langle v \rangle^{k_0}) = \langle v \rangle^{k_0} A(\nu, D_\nu)$ and

$$R_1 = \left(\langle v \rangle^{k_0} Q_R(\mu, f_\ell) - Q_R(\mu, (\langle v \rangle^{k_0} f_\ell), A M h_\ell) + \left(Q_R(\mu, (\langle v \rangle^{k_0} f_\ell), A M h_\ell). \right.
$$

Apply Proposition 5.3 to the first term and [27] Proposition 6.11 to the second one. Then

$$|R_1| \leq C_{k_0, R} \left(\| f_\ell \|_{L^{2}(\mathbb{S})} \| \langle v \rangle^{k_0} M f_\ell \|_{L^{2}(\mathbb{S})} + \| \langle v \rangle^{k_0} f_\ell \|_{L^{2}(\mathbb{S})} \| \langle v \rangle^{k_0} M f_\ell \|_{H^{2s'}(\mathbb{S})}^2 \right).$$

Summing up estimates for $R_j, j = 1, \cdots, 5$, we see that there exists a $0 \leq s'' < s$ such that

$$\left(Q_R(\mu, f_\ell), M(\langle v \rangle^{2k_0} M f_\ell)\right) \leq \left(Q_R(\mu, M f_\ell), (\langle v \rangle^{2k_0} M f_\ell)\right)
$$

$$+ C_{k_0, R} \left(\| \langle v \rangle^{k_0} M f_\ell \|_{H^{s''}(\mathbb{S})}^2 + \| \langle v \rangle^{k_0} f_\ell \|_{L^{2}(\mathbb{S})}^2 \right),$$

where $s'' = 0$ if $0 < s < 1/2$.

Apply Proposition 5.7 with $f = \mu, g = f_\ell, h = \langle v \rangle^{2k_0} M f_\ell, a = k_0$. Then there exists a $C_{k_0} > 0$ independent of $R > 0$ such that

$$\left(Q_{\infty}(\mu, f_\ell), M(\langle v \rangle^{2k_0} M f_\ell)\right) \leq \left(Q_{\infty}(\mu, M f_\ell), (\langle v \rangle^{2k_0} M f_\ell)\right)
$$

$$+ C_{k_0} \left[\tilde{s}^{s/(2a_2)} \| M f_\ell \|_{H^{s'/a_2} + \gamma/2} \left\{\| M f_\ell \|_{L^{2}_{a_2} + \gamma/2}^2 \right. \right.
$$

$$+ \int_{\mathbb{R}^d} \int_{\mathbb{S}^2} b(\cdot) \mu_g(v_\nu)^{2k_0 + \gamma/2} \left(\| (\langle v \rangle^{k_0 + \gamma/2} M f_\ell)(v) \| - \langle (\cdot)^{k_0 + \gamma/2} M f_\ell(v) \| \right)^{2} d\sigma dv d\nu \left. \right. \right]
$$

$$+ R^{\max\{s, s''\} - 1/2} \left\{\| M f_\ell \|_{L^{2}_{a_2 + \gamma/2}}^2 \right. \right.
$$

$$+ \int_{\mathbb{R}^d} \int_{\mathbb{S}^2} b(\cdot) \mu_g(v_\nu)^{2k_0 + \gamma/2} \left(\| (\langle v \rangle^{k_0 + \gamma/2} M f_\ell)(v) \| - \langle (\cdot)^{k_0 + \gamma/2} M f_\ell(v) \| \right)^{2} d\sigma dv d\nu \right\},$$

where $0 < \tilde{s} < s$ is arbitrary, and $\varepsilon'$ is non-negative and can be chosen to be zero if $s \neq 1/2$. 


By Proposition 3.2 with $F = \mu$ and its proof, we notice that
\[
\left( Q(\mu, M_f), \langle v \rangle^{2k_0} M_f \right) \leq -c_0 \left\{ \| M_f \|_{L^{2k_0}([0, T] \times R^d)}^2 \right. \\
+ \int_{R^d \times S^2} b(\cdot) \mu_\ast \langle v \rangle^{2k_0 + \gamma} \left( \langle \cdot \rangle^{k_0+\gamma/2} M_f(\cdot) - \langle \cdot \rangle^{k_0+\gamma/2} M_f(v) \right) \langle v \rangle^{2\gamma} d\sigma dv \right\} \quad (5.51)
\]
where the integral part on the right-hand side is obtained by using Lemma B.2 and B.3.

Choosing a small $\delta > 0$ and a large $R$, by means of (5.49), (5.50) and (5.51), we obtain for any $\epsilon' > 0$
\[
\left( Q(\mu, f_\ell), M(\langle v \rangle^{2k_0} M_f) \right) \leq -c_0 \left\{ \| M_f \|_{L^{2k_0}([0, T] \times R^d)}^2 \right. \\
- \frac{\gamma_0}{2} \| M_f \|_{L^{2k_0}([0, T] \times R^d)}^2 + C \| M_f \|_{L^{2k_0}([0, T] \times R^d)}^2,
\]
where it follows from (3.17) that
\[
\gamma_0 \geq \left( \int_{R^d} b(\cos \theta) \sin^2 \frac{\theta}{2} d\sigma \right) \int_{R^d} |v - v_\ast|^\gamma \mu_\ast dv_\ast. \quad (5.53)
\]
Notice that
\[
\gamma_1 \triangleq \min_{v} \int |v - v_\ast|^\gamma \mu_\ast dv_\ast > 2^{-\gamma - \frac{28}{3\sqrt{2\pi}}}, \quad (5.54)
\]
because if $|v| \geq 2$, then
\[
\frac{\int |v - v_\ast|^\gamma \mu_\ast dv_\ast}{\langle v \rangle^\gamma} \leq 2^{-\gamma} \int_{|v_\ast| \leq 1/2} \mu_\ast dv_\ast \geq 2^{-\gamma} (2\pi)^{-3/2} e^{-2^{-3/2} 4\pi / 3} \geq 2^{-\gamma - \frac{28}{3\sqrt{2\pi}}},
\]
where we have used $|v - v_\ast| \geq |v|/2 + 1 - |v|$. Moreover if $|v| < 2$ then for any $0 < \epsilon'' < 1$
\[
\int |v - v_\ast|^\gamma \mu_\ast dv_\ast \geq 2^{-\gamma} \int_{|v_\ast| \leq 1/2} \mu_\ast dv_\ast \geq 2^{-\gamma} \left( 1 - \int_{|v_\ast| \leq \epsilon''} \mu_\ast dv_\ast \right) \\
\geq 2^{-\gamma} \left( 1 - (2\pi)^{-3/2} \frac{4\pi}{3} \left( \epsilon'' \right)^3 \right) \geq 2^{\gamma},
\]
where the last inequality follows by choosing $\epsilon'' = 3/5$.

It follows from Corollary 5.6 that $M(\langle v \rangle^{k_0} M_f) = \langle v \rangle^{k_0} M + [M, \langle v \rangle^{k_0} (Id + A)]$ and
\[
\left| \left( Q_R(f_\ell, \mu), M(\langle v \rangle^{2k_0} M_f) \right) \right| = \left| \left( Q_R(f_\ell, \mu), \langle v \rangle^{k_0} (Id + A) M(\langle v \rangle^{k_0} M_f) \right) \right| \\
\leq C_{k_0,R} \| M \|_{L^2([0, T] \times R^d)} \| \langle v \rangle^{k_0} M_f \|_{L^2([0, T] \times R^d)}, \quad (5.55)
\]
where we have used Proposition 5.4 in the last inequality.

We now consider
\[
\left( Q_{\Pi}(f_\ell, \mu), M(\langle v \rangle^{2k_0} M_f) \right) - \left( Q_{\Pi}(f_\ell, \langle v \rangle^{k_0} \mu), \langle v \rangle^{-k_0} M(\langle v \rangle^{2k_0} M_f) \right) \\
= \int_{R \times R^d \times S^2} b(\cdot) \Pi \mu_\ast \langle v \rangle^{k_0} - \langle v \rangle^{k_0} \Pi M(\langle v \rangle^{k_0} M_f) \langle v \rangle^{2\gamma} d\sigma dv_\ast. \quad (5.56)
\]
It follows from Proposition 2.2 (Theorem 2.1) that
\[
\left| \left( Q_{\Pi}(f_\ell, \langle v \rangle^{k_0} \mu), \langle v \rangle^{-k_0} M(\langle v \rangle^{2k_0} M_f) \right) \right| \leq C_{k_0,R} \| f_\ell \|_{L^1_{T+2} \times R^d} \| \langle v \rangle^{k_0} \mu \|_{H^{2k_0}_{T+2} \times R^d} \| \langle v \rangle^{k_0} M_f \|_{L^2}.
\]
Note that
\[
\langle v \rangle^{-k_0} M(\langle v \rangle^{k_0} \mu) = M - [M, \langle v \rangle^{-k_0}] \langle v \rangle^{k_0} = M + (\langle v \rangle^{k_0} [M, \langle v \rangle^{-k_0}])^+. 
\]
Apply Corollary 5.6 to the second term. Then we have \( (v)^{-k_0} \mathcal{M}(v)^{k_0} = \mathcal{M}(Id + A^*) \) with \( A \in \operatorname{Op}(S^0_{0,0}) \).

Therefore, if we put \( g_\ell = (Id + A^*)(v)^{k_0} \mathcal{M}f_\ell \), then the right hand side of (5.56) is written as

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2_+} b \left( \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) \Phi_f(v - v_*) \mu \left( \langle v' \rangle^{k_0} - \langle v \rangle^{k_0} \right) (\mathcal{M}g_\ell)^{\gamma} dv_* d\sigma.
\]

We apply Lemma 2.1 to the decomposition of the factor \( \langle v' \rangle^{k_0} - \langle v \rangle^{k_0} \) with \( 2k = k_0 \). To estimate the part coming from the first term of the right hand side of (2.3), we rewrite \( \omega \) as

\[
\omega = \tilde{\omega} \cos \frac{\theta}{2} + \frac{v' - v_*}{|v' - v_*|} \sin \frac{\theta}{2},
\]

with \( \tilde{\omega} = (v' - v)/|v' - v| \) satisfying \( \tilde{\omega} \perp (v' - v_*) \). Then, by the almost same procedure for \( \Gamma_1 \) in the proof of Proposition 3.1, one can show that it is bounded by a constant times

\[
\|f_\ell\|_{L^1_{\gamma+2,(\mu\gamma)\cap L^2(\mu)}} \|\mathcal{M}g_\ell\|_{L^2(\mu)} \leq C_{k_0} \|\langle v \rangle^{k_0} f_\ell\|_{L^2(\mu)} \|\langle v \rangle^{k_0} \mathcal{M}f_\ell\|_{L^2(\mu)}.
\]

The part coming from \( \mathcal{R}_1 \) is estimated by

\[
\int_{\mathbb{R}^3} \langle v \rangle^{\gamma+1} \mu(v) \left( \int_{\mathbb{R}^3} \int_{S^2_+} b \left( \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) \langle v_* \rangle^{k_0+\gamma-1} |f_\ell| |\sin^{k_0-3} \frac{\theta}{2} (\mathcal{M}g_\ell)^{\gamma} dv_* d\sigma \right) dv
\]

\[
\leq C_{k_0} \|\langle v \rangle^{k_0} f_\ell\|_{L^2(\mu)} \|\langle v \rangle^{k_0} \mathcal{M}f_\ell\|_{L^2(\mu)},
\]

where we have used the singular change of variable \( v' \to v_* \) and the \( L^2 \) boundedness of \( \mathcal{M}(Id + A^*) \). The parts coming from \( \mathcal{R}_2, \mathcal{R}_3 \) are easily estimated by \( \|f_\ell\|_{L^1_{\gamma+4(\mu\gamma)\cap L^2(\mu)}} \|\langle v \rangle^{k_0} \mathcal{M}f_\ell\|_{L^2(\mu)} \). It remains to estimate the part coming from the second term of (2.3), that is,

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2_+} b \left( \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) \Phi_f(v - v_*) \mu(v) \langle v_* \rangle^{k_0} \sin^{k_0-3} \frac{\theta}{2} (\mathcal{M}g_\ell)^{\gamma} dv_* d\sigma
\]

\[
= \int_{\mathbb{R}^3} \mu(v) \left( \int_{\mathbb{R}^3} \int_{S^2_+} b \left( \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) \sin^{k_0-3} \frac{\theta}{2} F(v_*; v) \left( \frac{v' - v}{\sin \theta/2} \right)^{\gamma/2} (\mathcal{M}g_\ell)^{\gamma} dv_* d\sigma \right) dv
\]

\[
= \int_{\mathbb{R}^3} \mu(v) \left( \int_{\mathbb{R}^3} \int_{S^2_+} b \left( \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) \sin^{k_0-3} \frac{\theta}{2} F(v_*; v) \mathcal{M}(Dv') \left( \frac{v' - v}{\sin \theta/2} \right)^{\gamma/2} g_\ell(v') dv_* d\sigma \right) dv
\]

\[
\tilde{\Gamma} = \Gamma_{\ell,1} + \Gamma_{\ell,2},
\]

where \( \theta \) is determined from \( \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma \) and

\[
F(v_*; v) = \frac{\Phi_f(v - v_*)}{(v - v_*)^{\gamma/2} f_\ell (v_*)^{k_0}}.
\]

Since for \( \alpha \neq 0 \) we have

\[
\left| D^\alpha_z \left( \frac{z}{\sin \theta/2} \right)^{\gamma/2} \right| \leq C_{\alpha} (\sin \theta/2)^{-|\alpha|} \left( \frac{z}{\sin \theta/2} \right)^{\gamma/2 - |\alpha|} \leq C_{\alpha} (\sin \theta/2)^{-|\alpha|} |z|^{\gamma/2 - |\alpha|},
\]

\[
\left| D^\alpha_z \left( \frac{z}{\sin \theta/2} \right)^{\gamma/2} \right| \leq C_{\alpha} (\sin \theta/2)^{-|\alpha|} \left( \frac{z}{\sin \theta/2} \right)^{\gamma/2 - |\alpha|} \leq C_{\alpha} (\sin \theta/2)^{-|\alpha|} |z|^{\gamma/2 - |\alpha|},
\]

\[
\left| D^\alpha_z \left( \frac{z}{\sin \theta/2} \right)^{\gamma/2} \right| \leq C_{\alpha} (\sin \theta/2)^{-|\alpha|} \left( \frac{z}{\sin \theta/2} \right)^{\gamma/2 - |\alpha|} \leq C_{\alpha} (\sin \theta/2)^{-|\alpha|} |z|^{\gamma/2 - |\alpha|},
\]
the singular change of variable $v_+ \to v'$ and the Calderón-Vaillancourt theorem yield

$$| \Gamma_{\ell,2} | \leq \int_{\mathbb{R}^3} \mu(v) \langle v \rangle^\gamma \left( \int_{\mathbb{S}^2} b(\cos \theta) \sin^{k_0-3} \frac{\theta}{2} \left( \int_{\mathbb{R}^3} |f_{\ell_+}(v_+)^{k_0} | dv_+ \right)^{1/2} \right. \\
\times \left. \left( \int_{\mathbb{R}^3} \frac{\Phi_\mu(v' - v/\sin \theta/2)}{|v'|^\gamma (|v' - v/\sin \theta/2|^{\gamma/2} - 1) \sqrt{2}} \left[ \mathcal{M}(D_{v'}) \cdot \left( \frac{v' - v}{\sin \theta/2} \right)^{\gamma/2} g_r(v') \right]^2 dv \right)^{1/2} d\sigma \right) dv \\
\leq C_{R, k_0} \int_{\mathbb{R}^3} \mu(v) \langle v \rangle^\gamma dv \int_{\mathbb{S}^2} b(\cos \theta) \sin^{k_0-3-5} \frac{\theta}{2} d\sigma \| f_{\ell} \|_{L^2(d\sigma)} \| (v)^{k_0} \mathcal{M} f_{\ell} \|_{L^2(d\sigma)}.$$

Put

$$G(w; v, \theta) = \left\langle \frac{w - v}{\sin \theta/2} \right\rangle^{\gamma/2} g_{\ell}(w).$$

Then the exchange of $\xi \cdot \sigma/|\xi|$ and $(v - v_+) \cdot \sigma/|v - v_+|$ as similar as in the derivation of the Bobylev formula leads us to

$$\Gamma_{\ell,1} = \int_{\mathbb{R}^3} \mu(v) \left( \int_{\mathbb{S}^2} b(v - v_+) \cdot \sigma \cdot \sigma F(v_+; v) \int_{\mathbb{R}^3} e^{i v_+ \xi} \mathcal{M}(\xi, \ell) G(\xi, v, \theta) \frac{d\xi}{(2\pi)^3} \sin^{k_0} \frac{\theta}{2} dv_+ d\sigma \right) dv \\
= \int_{\mathbb{R}^3} \mu(v) \left( \int_{\mathbb{S}^2} b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) F(\xi; v) \mathcal{M}(\xi, \ell) e^{2\xi \cdot \ell} G(\xi, v, \theta) \sin^{k_0} \frac{\theta}{2} \frac{d\xi d\sigma}{(2\pi)^3} \right) dv \\
\overset{\Delta}{=} \int_{\mathbb{R}^3} \mu(v) \langle v \rangle^\gamma K(v) dv ,$$

where $\theta$ on the second formula is determined by $\cos \theta = \xi \cdot \sigma/|\xi|$. It follows from Cauchy-Schwarz inequality and the singular change of variable $\xi \to \xi^-$ that

$$|K(v)|^2 \leq \int_{\mathbb{S}^2} b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) |\mathcal{M}(\xi^-, \ell) (v)^{-1/2} F(\xi^-; v)|^2 \sin^{k_0-\gamma/2+3/2} \frac{\theta}{2} \frac{d\xi d\sigma}{(2\pi)^3} \\
\times \left( \int_{\mathbb{S}^2} b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) |\mathcal{M}(\xi^-, \ell) (v)^{-1/2} G(\xi, v, \theta)|^2 \sin^{k_0+\gamma/2-3/2} \frac{\theta}{2} \frac{d\xi d\sigma}{(2\pi)^3} \right)^2 \\
= (2\pi)^2 \int_0^{\pi/2} b(\cos \theta) \sin^{k_0-\gamma/2-3/2} \frac{\theta}{2} \sin \theta d\theta \| \mathcal{M}(v)^{\gamma/2} F_k(\cdot; v) \|_{L^2}^2 \\
\times \int_0^{\pi/2} b(\cos \theta) \sin^{k_0+\gamma/2-3/2} \frac{\theta}{2} \sin \theta \| (v)^{\gamma/2} G(\cdot, v, \theta) \|_{L^2}^2 d\theta \\
\leq \left( \int_{\mathbb{S}^2} b(\cos \theta) \sin^{k_0-\gamma/2-3/2} \frac{\theta}{2} d\sigma \left( \| \mathcal{M} f_{\ell} \|_{L^2_{k_0+\gamma/2}}^2 + C_{R, k_0} \| f_{\ell} \|_{L^2_{k_0}}^2 \right) \right)^2 ,$$

because $(v)^{-1/2} \left( \frac{w - v}{\sin \theta/2} \right)^{\gamma/2} \leq \frac{1}{\sin^{\gamma/2} \theta/2} (w)^{\gamma/2}$. Noting $|\sin \theta/2| \leq 1/2$ for $\theta \in [0, \pi/2]$, we have

$$| \Gamma_{\ell,1} | \leq 2^{-(2k_0-\gamma-5)/4} \int_{\mathbb{R}^3} \mu(v) \langle v \rangle^\gamma dv \int_{\mathbb{S}^2} b(\cos \theta) \sin^2 \frac{\theta}{2} d\sigma \left( \| \mathcal{M} f_{\ell} \|_{L^2_{k_0+\gamma/2}}^2 + C_{R, k_0} \| f_{\ell} \|_{L^2_{k_0}}^2 \right) .$$

As a consequence, if we define

$$\gamma_3 \overset{\Delta}{=} 2^{-(2k_0-\gamma-5)/4} \int_{\mathbb{R}^3} \mu(v) \langle v \rangle^\gamma dv \int_{\mathbb{S}^2} b(\cos \theta) \sin^2 \frac{\theta}{2} d\sigma , \tag{5.58}$$

then we have

$$\left| (Q_{\mu}(f_{\ell}, \mu), \mathcal{M}(v)^{2k_0} \mathcal{M} f_{\ell}) \right| \leq \gamma_3 \| \mathcal{M} f_{\ell} \|_{L^2_{k_0+\gamma/2}(d\sigma)}^2 + C_{R, k_0} \| f_{\ell} \|_{L^2_{k_0}(d\sigma)}^2 , \tag{5.59}$$

where we notice that
\[
\int \langle v \rangle^\gamma \mu(v) dv < \int \left(1 + \sum_{j=1}^{3} |v_j| \right) \mu(v) dv = 1 + \frac{6}{\sqrt{2\pi}} < \frac{26}{3\sqrt{2\pi}}. \tag{5.60}
\]

Combine (5.48) with (5.52), (5.55) and (5.59). If \( \gamma_0 / 2 > \gamma_3 \), which is verified for \( k_0 \) satisfying (5.1) in view of (5.54) and (5.60), then we have
\[
\frac{d}{dt} \left\| \left( \langle v \rangle^{k_0} M f \right) \right\|_{L_x^2}^2 \leq -\frac{c_0}{2} \left\| \left( \langle v \rangle^{k_0} M f_\ell \right) \right\|_{H_x^s}^2 + C_R \left\| f_\ell \right\|_{L_x^2(\langle v \rangle^{k_0} dv)}^2 \leq C_R \left\| \langle v \rangle^{k_0} f_\ell \right\|_{L_x^2}^2.
\]

Integrating \( t \) on \([0, T]\), by Corollary 5.6 we get
\[
\left\| \langle v \rangle^{k_0} f_\ell(t, \cdot) \right\|_{L_x^2}^2 \leq \left\| \left( \langle v \rangle^{k_0} M(0, D_v, \ell) f^{in}_\ell \right) \right\|_{L_x^2}^2 + C_R \int_0^T \left\| \langle v \rangle^{k_0} f_\ell(s, \cdot) \right\|_{L_x^2}^2 ds
\leq \left\| (I + A(0, v, D_v, \ell)) M(0, D_v, \ell) \left( \langle v \rangle^{k_0} f^{in}_\ell \right) \right\|_{L_x^2}^2 + C_R \int_0^T \left\| \langle v \rangle^{k_0} f_\ell(s, \cdot) \right\|_{L_x^2}^2 ds
\leq C_{k_0} \int_{\mathbb{R}_x^2} \frac{(\xi)^{2s}}{(1 + c_0 \delta T \langle \xi \rangle^{2s})^{1+2s}} \left| \langle \xi \rangle^{-s} \langle v \rangle^{k_0} f_\ell(0, \xi) \right|^2 d\xi + C_R \int_0^T \left\| \langle v \rangle^{k_0} f_\ell(s, \cdot) \right\|_{L_x^2}^2 ds. \tag{5.61}
\]

Integrating \( T \) on \([0, T_0]\), we obtain that
\[
\int_0^{T_0} \left\| \langle v \rangle^{k_0} f_\ell(t, \cdot) \right\|_{L_x^2}^2 dt \leq \frac{C_{k_0}}{2c_0 \delta} \left\| \langle v \rangle^{k_0} f^{in}_\ell \right\|_{H_x^s}^2 + C_R T_0 \int_0^{T_0} \left\| \langle v \rangle^{k_0} f_\ell(s, \cdot) \right\|_{L_x^2}^2 ds.
\]

Therefore, by choosing \( T_0 \) small such that \( C_R T_0 < 1/2 \), we have
\[
\int_0^{T_0} \left\| \langle v \rangle^{k_0} f_\ell(t, \cdot) \right\|_{L_x^2}^2 dt \leq \frac{C_{k_0}}{c_0 \delta} \left\| \langle v \rangle^{k_0} f^{in}_\ell \right\|_{H_x^s}^2,
\]

Adding all the modes \( \ell \) then gives
\[
\int_0^{T_0} \left\| \langle v \rangle^{k_0} f_\ell(t, \cdot) \right\|_{L_x^2}^2 dt \leq \frac{C_{k_0}}{c_0 \delta} \left\| \langle v \rangle^{k_0} f^{in}_\ell \right\|_{L_x^2(\langle v \rangle^{k_0} dv)}^2,
\]

where \( T_0 \) is small enough. In particular, this implies at \( t = T_0 \), it holds that
\[
\left\| \langle v \rangle^{k_0} f_\ell(T_0, \cdot) \right\|_{L_x^2}^2 \leq \left( \frac{C_{k_0}}{T_0 \delta} + \frac{C_{k_0} C_{k_0}}{c_0 \delta} \right) \left\| \langle v \rangle^{k_0} f^{in}_\ell \right\|_{L_x^2(\langle v \rangle^{k_0} dv)}^2. \tag{5.62}
\]

Similar to (5.61), we have
\[
\sum_{\ell \in \mathbb{Z}^3} \left\| \langle v \rangle^{k_0} f_\ell(T, \cdot) \right\|_{L_x^2}^2 \leq C_{k_0} \sum_{\ell \in \mathbb{Z}^3} \left\| M(0, D_v, \ell) \left( \langle v \rangle^{k_0} f^{in}_\ell \right) \right\|_{L_x^2}^2 + C_R \int_0^T \sum_{\ell \in \mathbb{Z}^3} \left\| \langle v \rangle^{k_0} f_\ell(s, \cdot) \right\|_{L_x^2}^2 ds
\leq C_{k_0} \sum_{\ell \in \mathbb{Z}^3} \int_{\mathbb{R}_x^2} \frac{(\eta)^{2s/(2s+1)}}{(1 + c_0 \delta T \langle \eta \rangle^{2s+1})^{1+2s}} \left| \langle \eta \rangle^{-s/(2s+1)} \langle v \rangle^{k_0} f_\ell(0, \xi) \right|^2 d\xi
\leq C_R \int_0^T \sum_{\ell \in \mathbb{Z}^3} \left\| \langle v \rangle^{k_0} f_\ell(s, \cdot) \right\|_{L_x^2}^2 ds.
\]

Integrating \( T \) on \([0, T_0]\), we obtain that if then \( C_R T_0 < 1/2 \)
\[
\int_0^{T_0} \sum_{\ell \in \mathbb{Z}^3} \left\| \langle v \rangle^{k_0} f_\ell(T, \cdot) \right\|_{L_x^2}^2 dt \leq \frac{C_{k_0} (2c_0 \delta + 1)}{4c_0 \delta} \sum_{\ell \in \mathbb{Z}^3} \int_{\mathbb{R}_x^2} \left| \langle \eta \rangle^{-s/(2s+1)} \langle v \rangle^{k_0} f_\ell(0, \xi) \right|^2 d\xi
\]

because \( \int_0^\infty \frac{dr}{(1 + c_0 \delta t^{2s+1})^{1+2s}} < 1 + \int_0^\infty \frac{dr}{c_0 \delta t^{2s+1}} \). Noting
\[
\langle \xi \rangle^{-2s} \mathbf{1}_{\langle \xi \rangle \geq |\eta|^{1/(2s+1)}} + \langle \eta \rangle^{-2s/(2s+1)} \mathbf{1}_{\langle \xi \rangle < |\eta|^{1/(2s+1)}} \leq \frac{2}{\langle \xi \rangle^{2s} + \langle \eta \rangle^{2s/(2s+1)}},
\]
and splitting \( \sum_{t \in \mathbb{Z}^3} \int_{\mathbb{R}^3} \) into two regions \( \{ \xi \geq \langle \eta \rangle^{1/(2s+1)} \} \) and \( \{ \xi < \langle \eta \rangle^{1/(2s+1)} \} \), we obtain the desired estimates in Theorem 5.8.

Starting from \( t = T_0 \), we can apply the spectral gap estimate and obtain that

\[
\left\| \langle e \rangle^{k_a} f(t, \cdot) \right\|_{L^6} \leq C \left( \frac{1}{\sqrt{T_0}} + \frac{1}{\sqrt{t_0 \delta}} \right) e^{-\lambda M} \left( \sum_{t \in \mathbb{Z}^3} \int_{\mathbb{R}^3} \left| \langle \xi \rangle^s + \langle \tau \rangle^{\frac{3}{2s+1}} \hat{h}_1^0(\xi) \right|^2 d\xi \right)^{1/2}, \quad t \geq T_0,
\]

where \( T_0 \) depends on \( R \).

6. Energy Estimates

In this section we close the a-priori estimate based on the estimates in Sections 2.3. The main result is

**Theorem 6.1.** Let \( f \) be the solution to \( (1.4) \) with initial data \( f_0 \in Y_l \) and \( l_0 > \frac{5s+37}{2} \). Then

\[
\frac{1}{2} \frac{d}{dt} \left( \| f \|_{Y_l}^2 + \eta C_l \int_0^\infty \| S^2(\tau) f(t, \cdot) \|_{L^2(W^1 H^1_{\mu}(dz))}^2 d\tau \right) \\
\leq -\left( \frac{\gamma_0}{4} - C_l \| f \|_{Y_l} \right) \sum_{\alpha} \int_{T_3} \| W^{2(1-|\alpha|)} \partial^\alpha_x f \|_{L^2_{\gamma}}^2 dx \\
- \left( \frac{\gamma_0}{2} \delta_2 - C_l \| f \|_{Y_l} \right) \sum_{\alpha} \int_{T_3} \| W^{2(1-|\alpha|)} \partial^\alpha_x f \|_{H^{\gamma}_{\gamma}}^2 dx - C_l \| f \|_{Y_l}^2.
\]

\[ (6.1) \]

**Proof.** The proof is divided into two steps.

**Step 1. Evolution of \( \| f \|_{Y_l} \).** For any \( |\alpha| = 0, 2 \), the equation for \( \partial^\alpha_x f \) is

\[
(\partial_t + v \cdot \nabla_x) \partial^\alpha_x f = Q \left( F, \partial^\alpha_x f \right) + Q \left( \partial^\alpha_x f, \mu \right) + (\partial^\alpha_x Q(F, f) - Q(F, \partial^\alpha_x f))
\]

\[ = \Gamma_1 + \Gamma_2 + \Gamma_3. \]  

\[ (6.2) \]

Multiply \( W^{2(1-|\alpha|)} \partial^\alpha_x f \) to \( (6.2) \), integrate in \( x, v \), and sum over \( |\alpha| = 0, 2 \). The left hand side gives

\[
\frac{1}{2} \frac{d}{dt} \sum_{\alpha} \int_{T_3} \int_{\mathbb{R}^3} W^{2(1-|\alpha|)} \partial^\alpha_x f \|_{L^2_{\gamma}}^2 dx \right) dv = \frac{1}{2} \frac{d}{dt} \| f \|_{Y_l}^2.
\]

\[ (6.3) \]

By Proposition 3.2 the first term \( \Gamma_1 \) on the right hand side satisfies

\[
\int_{T_3} \int_{\mathbb{R}^3} Q \left( F, \partial^\alpha_x f \right) \left( \partial^\alpha_x f \right) W^{2(1-|\alpha|)} dv dx \\
\leq -\frac{\gamma_0}{4} \int_{T_3} \| W^{2(1-|\alpha|)} \partial^\alpha_x f \|_{L^2_{\gamma}}^2 dx - \left( \frac{\gamma_0}{2} \delta_2 - C_l \| f \|_{Y_l} \right) \int_{T_3} \| W^{2(1-|\alpha|)} \partial^\alpha_x f \|_{H^{\gamma}_{\gamma}}^2 dx \\
+ C_l \int_{T_3} \| W^{2(1-|\alpha|)} \partial^\alpha_x f \|_{L^2} dx + C_l \int_{T_3} \| W^{2(1-|\alpha|)} \partial^\alpha_x f \|_{L^2} \| W^{1-|\alpha|} \partial^\alpha_x f \|_{L^2} \| \partial^\alpha_x f \|_{L^2} dx.
\]

\[ (6.4) \]

It follows from the proof of Proposition 3.2 that the first term of \( (6.4) \) can be replaced by

\[
-\frac{1}{2} \left( \int_{S^2} b(\cos \theta) \sin \frac{\theta}{2} d\sigma \right) \int_{T_3} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \mu(v_*) |v - v_*|^7 d_{v_*} \right) |W^{1-|\alpha|} \partial^\alpha_x f(x, v)|^2 dx dv.
\]

\[ (6.5) \]

We treat the two cases \( |\alpha| = 0 \) and \( |\alpha| = 2 \) separately. If \( |\alpha| = 0 \), then the last two terms in \( (6.4) \) satisfy

\[
\int_{T_3} \| W^{1} f \|_{L^2} \| f \|_{L^4_{1+}} dx + \int_{T_3} \| f \|_{L^4} \| W^{1} f \|_{L^2} dx \leq \left( \sup_{T_3} \| f \|_{L^2_{1+}} \right) \| W^{1} f \|_{L^2} \| f \|_{L^2} (dx, dv) \\
\leq C \| f \|_{Y_l} \| W^{1} f \|_{L^2} (dx, dv).
\]
If $|\alpha| = 2$, then the second last term in (6.4) satisfies
\[
\int_{T^3} \left\| W^{l-2} f \right\|_{L^2_x} \left\| W^{l-2} \partial_x^\alpha f \right\|_{L^2_x} \left\| \partial_x^\alpha f \right\|_{L^1_{[-\gamma]}(dv)} \, dx \leq \left( \sup_{T^3} \left\| W^{l-2} f \right\|_{L^2_x} \right) \left\| W^{l-2} \partial_x^\alpha f \right\|_{L^2_{x,v}} \leq \| f \|_{Y_1} \sum_{\alpha} \left\| W^{l-2} \partial_x^\alpha f \right\|_{L^2_{x,v}}^2.
\]

Similarly, the last term in (6.4) satisfies
\[
\int_{T^3} \left\| \partial_x^\alpha f \right\|_{L^1} \left\| W^{l-2} f \right\|_{L^{1/2}_x} \left\| W^{l-2} \partial_x^\alpha f \right\|_{L^{1/2}_x} \, dx \leq \left( \sup_{T^3} \left\| W^{l-2} f \right\|_{L^{1/2}_x} \right) \left\| W^{l-2} \partial_x^\alpha f \right\|_{L^{1/2}_{x,v}} \leq \| f \|_{Y_1} \sum_{\alpha} \left\| W^{l-2} \partial_x^\alpha f \right\|_{L^{1/2}_{x,v}}^2.
\]

Hence for both cases we have
\[
\int_{T^3} \int_{R^3} Q(F, \partial_x^\alpha f) (\partial_x^\alpha f) W^{2(l-|\alpha|)} \, dv \, dx \leq -\frac{3\gamma_0}{8} \int_{T^3} \left\| W^{l-|\alpha|} \partial_x^\alpha f \right\|_{L^2_{x,v}}^2 \, dx - \left( \frac{C_0}{4} \delta_2 - C_l \| f \|_{Y_1} \right) \int_{T^3} \left\| W^{l-|\alpha|} \partial_x^\alpha f \right\|_{H^{1/2}_{x,v}}^2 \, dx + C_l \| f \|_{Y_1}^2. \tag{6.6}
\]

By Proposition 3.3 the second term $\Gamma_2$ on the right hand side satisfies
\[
\int_{T^3} \int_{R^3} Q(\partial_x^\alpha f, \mu) (\partial_x^\alpha f) W^{2(l-|\alpha|)} \, dv \, dx \leq C_l \left\| W^{l-|\alpha|} \partial_x^\alpha f \right\|_{L^2_{x,v}}^2 + C_l \left\| W^{l-|\alpha|} \partial_x^\alpha f \right\|_{L^{1/2}_{x,v}}^2, \tag{6.7}
\]
where
\[
C_0 = \int_{S^2} b(\cos \theta) \sin^{m_0(l-|\alpha|)} - \frac{3+\gamma}{2} \theta^2 \, d\theta \int_{R^3} \langle \nu \rangle^\gamma \mu(\nu) \, d\nu, \quad m_0 > \max\{4s, 1\}.
\]

The second term on the right hand side of (6.7) has the following more accurate estimate
\[
\left( \int_{S^2} b(\cos \theta) \sin^{m_0(l-|\alpha|)} - \frac{3+\gamma}{2} \theta^2 \, d\theta \right) \int_{T^2} \left( \int_{R^3} \mu(v_*) |v - v_*|^{\gamma} \, dv_* \right) \left\| W^{l-|\alpha|} \partial_x^\alpha f(x, v) \right\|_{L^2_{x,v}}^2 \, dx. \tag{6.8}
\]

The bound related to $\Gamma_3$ is obtained by using $g = h = f$ in Proposition 3.4, which gives
\[
\left\| \int_{T^3} \int_{R^3} \Gamma_3 \left( W^{2(l-|\alpha|)} \partial_x^\alpha f \right) \, dv \, dx \right\| \leq C_l \| f \|_{Y_1} \sum_{\alpha} \left\| W^{l-|\alpha|} \partial_x^\alpha f \right\|_{L^{2}(dx,H^{1/2}_{x,v}(dv))}^2. \tag{6.9}
\]

Combine all the bounds in (6.6)-(6.9) in considering the accurate version. If $\ell$ satisfies
\[
\frac{1}{2} \left( m_0(\ell - 2) - \frac{3+\gamma}{2} - 2 \right) \geq 2 \times 2^2
\]
then we derive that
\[
\frac{1}{2} \frac{d}{dt} \| f \|_{Y_1}^2 \leq -\left( \frac{\gamma_0}{4} - C_l \| f \|_{Y_1} \right) \sum_{\alpha} \int_{T^3} \left\| W^{l-|\alpha|} \partial_x^\alpha f \right\|_{L^2_{x,v}}^2 \, dx - \left( \frac{C_0}{4} \delta_2 - C_l \| f \|_{Y_1} \right) \sum_{\alpha} \int_{T^3} \left\| W^{l-|\alpha|} \partial_x^\alpha f \right\|_{H^{1/2}_{x,v}}^2 \, dx + C_l \| f \|_{Y_1}^2. \tag{6.10}
\]

**Step 2. Evolution of the semigroup part.** In the second step, we derive the evolution of the semigroup part \( \int_0^\infty \| S_{\tau}(h) \|_{L^2(dx; H^2_{x,v}(dx))}^2 \). This part will provide a linear damping term that can be used to control
the linear growth term in \([6,10]\). Let \(f\) be the solution to \([1,4]\). Then the semigroup term satisfies

\[
\frac{1}{2} \frac{d}{dt} \int_0^\infty \| \mathcal{S}_\tau(f(t, \cdot)) \|^2_{L^2(W^{1,0}_0; H^2(dx))} \, d\tau
\]

\[
= \frac{1}{2} \frac{d}{dt} \int_0^\infty \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\mathcal{S}_\tau(D_x)^2 f(t, x, v)|^2 W^{2l_0} \, dv \, dx \, d\tau
\]

\[
= \int_0^\infty \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left( \mathcal{S}_\tau(D_x)^2 f(t, x, v) \right) \left( \mathcal{S}_\tau(D_x)^2 f(t, x, v) \right) W^{2l_0} \, dv \, dx \, d\tau
\]

\[
= \int_0^\infty \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left( \mathcal{S}_\tau(D_x)^2 f(t, x, v) \right) \left( \mathcal{S}_\tau(D_x)^2 f(t, x, v) \right) W^{2l_0} \, dv \, dx \, d\tau
\]

\[
+ \int_0^\infty \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left( \mathcal{S}_\tau(D_x)^2 f(t, x, v) \right) \left( \mathcal{S}_\tau(D_x)^2 f(t, x, v) \right) W^{2l_0} \, dv \, dx \, d\tau
\]

\[
\Delta = E_1 + E_2.
\]

We will show that \(E_1\) provides a linear damping term and \(E_2\) can be well controlled. First, since \(\mathcal{S}_\tau\) is the semigroup generated by \(\mathcal{L}\), we have

\[
\mathcal{S}_\tau(D_x)^2 f(t, x, v) = \partial_\tau \left( \mathcal{S}_\tau(D_x)^2 f(t, x, v) \right).
\]

Therefore,

\[
E_1 = \int_0^\infty \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left( \mathcal{S}_\tau(D_x)^2 f(t, x, v) \right) \partial_\tau \left( \mathcal{S}_\tau(D_x)^2 f(t, x, v) \right) W^{2l_0} \, dv \, dx \, d\tau
\]

\[
= \frac{1}{2} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_0^\infty \partial_\tau \left( \left( \mathcal{S}_\tau(D_x)^2 f(t, x, v) \right)^2 \right) W^{2l_0} \, dv \, dx \, d\tau
\]

\[
= -\frac{1}{2} \left\| W^{l_0} f \right\|_{L^2(W^{1,0}_0; H^2(dx))}^2.
\]

(6.11)

The bound of \(E_2\) is given by the exponential decay and regularization of \(\mathcal{S}_\tau\). By Theorem \([5,8]\), there exist constants \(C\) and \(T_0\) such that

\[
\int_0^{T_0} \| \mathcal{S}_\tau(Q(f, f)) \|^2_{L^2(W^{1,0}_0; H^2(dx))} \, d\tau \leq C \| Q(f, f) \|^2_{H^{-\gamma}(W^{1,0}_0; H^2(dx))},
\]

and

\[
\| \mathcal{S}_\tau(Q(f, f)) \|_{L^2(W^{1,0}_0; H^2(dx))} \leq C e^{-\lambda \tau} \| Q(f, f) \|_{H^{-\gamma}(W^{1,0}_0; H^2(dx))}, \quad \tau \geq T_0.
\]

Applying these bounds, we have

\[
|E_2| \leq \int_0^{T_0} \| \mathcal{S}_\tau(f) \|^2_{L^2(W^{1,0}_0; H^2(dx))} \| \mathcal{S}_\tau(Q(f, f)) \|^2_{L^2(W^{1,0}_0; H^2(dx))} \, d\tau
\]

\[
+ \int_{T_0}^\infty \| \mathcal{S}_\tau(f) \|^2_{L^2(W^{1,0}_0; H^2(dx))} \| \mathcal{S}_\tau(Q(f, f)) \|^2_{L^2(W^{1,0}_0; H^2(dx))} \, d\tau
\]

\[
\leq C \int_0^{T_0} \| f \|_{L^2(W^{1,0}_0; H^2(dx))} \| \mathcal{S}_\tau(Q(f, f)) \|_{L^2(W^{1,0}_0; H^2(dx))} \, d\tau
\]

\[
+ C \int_{T_0}^\infty e^{-\lambda \tau} \| f \|_{L^2(W^{1,0}_0; H^2(dx))} \| Q(f, f) \|_{H^{-\gamma}(W^{1,0}_0; H^2(dx))} \, d\tau
\]

\[
\leq C \| f \|_{L^2(W^{1,0}_0; H^2(dx))} \| Q(f, f) \|_{H^{-\gamma}(W^{1,0}_0; H^2(dx))}
\]

\[
\leq C \sum_{|\alpha| \leq 2} \sum_{\alpha_1 + \alpha_2 = \alpha} \| f \|_{Y_1} \| Q(\partial_{x_1}^{\alpha_1} f, \partial_{x_2}^{\alpha_2} f) \|_{H^{-\gamma}(W^{1,0}_0; L^2(dx))}.
\]
Lemma 7.1

Local existence of linear equation and non-negativity.

where \( F \) as in [9]. It follows from the proof of ([9], Lemma 2.17) that if

\[
\text{multiplying (6.12) by 4}
\]

by taking

\[
\frac{37 + 5\gamma}{2m_0} \leq t_0 < t - \frac{2s + \gamma}{m_0}.
\]

Using such \( t_0 \) we have the bound of \( E_2 \) as

\[
|E_2| \leq C \|f\|_{Y_1} \sum_{\alpha} \left\| W^{l-|\alpha|, \alpha} f \right\|^2_{L^2(dx; H^{s/2}_{0/2}(dv))},
\]

which gives

\[
\frac{1}{2} \frac{d}{dt} \int_0^\infty \| S_{E}(t, \cdot, \cdot) \|^2_{L^2(W^{0;0;0}; H^{s/2}(dx))} \, dt \\
\leq -\frac{1}{2} \left\| W^{l0,0,0} f \right\|^2_{L^2(dx; H^{s/2}(dx))} + C \|f\|_{Y_1} \sum_{\alpha} \left\| W^{l-|\alpha|, \alpha} f \right\|^2_{L^2(dx; H^{s/2}_{0/2}(dv))}.
\]

Multiplying (6.12) by \( 4C_t \) where \( C_t \) is the constant in front of the linear growth in [6.10] and adding the resulting equation with (6.10), we obtain the closed energy estimate stated in (6.11). This estimate extends the local-in-time bound of \( \|f\|_{Y_1} \) to the global one provided the initial norm is small. \( \square \)

7. Local existence and non-negativity

Recall the notation

\[
J^p_1(f) = \iiint b\Phi(|v - v_s|)\mu_s(f' - f)^2 \, dv \, dv_s \, ds,
\]

as in [9]. It follows from the proof of ([9], Lemma 2.17) that if

\[
C_0(F, f) = \iiint bF_s \left( f' - f \right)^2 \, dv \, dv_s \, ds \quad \text{for } F \geq 0 \text{ with } \|F\|_{L^1} \geq D_0 \text{ and } \|F\|_{L^2} + \|F\|_{L^{\infty}} \leq E_0,
\]

then there exist \( C, C' > 0 \) independent of \( F \) and \( C_F > 0 \) such that

\[
C_0(F, f) \leq C \|F\|_{L^2} \left( J^p_1(f) + \|f\|^2_{L^2} \right),
\]

\[
\|F\|_{L^1} \cdot J^p_1(f) \leq 2C_0(F, f) + C' \|F\|_{L^2} + \|f\|^2_{H^s} \leq C_F \left( C_0(F, f) + \|f\|^2_{L^2} \right),
\]

where \( F = \mu + f \) and \( C_F \) depends only on \( D_0 \) and \( E_0 \).

7.1. Local existence of linear equation and non-negativity.

Lemma 7.1 (A linear equation). There exist some \( C_0 > 1, \tau_0 > 0, T_0 > 0, l_0 > 0 \) such that for all \( 0 < T \leq T_0, l \geq l_0, f_0 \in Y_1 \) with \( \mu + f_0 \geq 0, g \in L^\infty([0, T]; Y_1) \) satisfying

\[
\|g\|_{L^\infty([0, T]; Y_1)} \leq \tau_0, \mu + g \geq 0, \text{ and } \int_0^T \sum_\alpha \left\| W^{l-|\alpha|, \alpha} g(t) \right\|^2_{L^2(dx; H^{s/2}_{0/2}(dv))} \, dt < \tau_0^2,
\]

the Cauchy problem

\[
\begin{cases}
\partial_t f + v \cdot \nabla_x f - Q(\mu + g, f) = Q(g, \mu), \\
f|_{t=0} = f_0(x, v),
\end{cases}
\]

(7.1)
admits a weak solution satisfying

\[ f \in L^\infty([0, T]; Y_1), \quad (v)^{1/2} \nabla_v f \in L^2(0, T; Y_1), \quad \mu + f \geq 0 \]

with the energy bound

\[
\|f\|^2_{L^\infty([0, T]; Y_1)} + \int_0^T \sum_\alpha \left\| W^{-[\alpha]} \partial_x^\alpha f(\tau) \right\|^2_{L^2(dx; H^\gamma_{1/2}(\partial_v))} d\tau \\
\leq 2 \left( \|f_0\|^2_{Y_1} + \delta_4 \int_0^T \sum_\alpha \left\| W^{-[\alpha]} \partial_x^\alpha g(\tau) \right\|^2_{L^2(dx; H^\gamma_{1/2}(\partial_v))} d\tau \right).
\]

(7.2)

Here \(\delta_4 > 0\) is defined in (7.12). Moreover,

\[
J^f_1(f) \lesssim \|f_0\|^2_{Y_1} + \delta_4 \int_0^T \sum_\alpha \left\| W^{-[\alpha]} \partial_x^\alpha g(\tau) \right\|^2_{L^2(dx; H^\gamma_{1/2}(\partial_v))} d\tau.
\]

(7.3)

**Proof.** To be rigorous, we regularize the \(f\)-equation and consider the following system

\[
\begin{cases}
\partial_t f_\kappa + v \cdot \nabla_x f_\kappa - Q(\mu + g, f_\kappa) = Q(g, \mu) - 2\kappa W^0(v)(\lambda_0 I - \Delta_v)f_\kappa, \\
f_\kappa |_{\tau = 0} = f_0(x, v),
\end{cases}
\]

where \(\lambda_0 > 0\) is large enough such that for any \(\psi \in H^2(W^{l+2} \partial_v dx)\), it holds that

\[
(2W^0(v)(\lambda_0 I - \Delta_v)\psi, \psi)_{Y_1} \geq \sum_{|\alpha| \leq 2} \left\| W^{-[\alpha]} [\psi, \partial_x^{[\alpha]} v] \right\|^2_{L^2(dx; H^1(dx))}.
\]

(7.5)

Let \(Q\) be the linear operator given by

\[
Q = -\partial_t + (v \cdot \nabla_x - Q(\mu + g, \cdot)) + \kappa W^0(v)(\lambda_0 I - \Delta_v)\),
\]

where the adjoint operator \((\cdot)^*\) is taken with respect to the scalar product in \(Y_1\). Then, for all \(h \in C^\infty([0, T] \times \mathbb{T}^3_3, \mathcal{S}(\mathbb{R}^3_3))\), with \(h(T) = 0\) and \(0 \leq t \leq T\), and for any \(0 < \delta_3 \ll 1\), we have

\[
\text{Re}(h(t), Qh(t))_{Y_1} = -\frac{1}{2} \frac{d}{dt} \|h\|^2_{Y_1} + \text{Re}(v \cdot \nabla_x h, h)_{Y_1} - \text{Re}(Q(\mu + g, h), h)_{Y_1}
\]

\[
+ 2\kappa \left( W^0(v)(\lambda_0 I - \Delta_v)h, h \right)_{Y_1}
\]

\[
\geq -\frac{1}{2} \frac{d}{dt} \|h(t)\|^2_{Y_1} + \left( \frac{\gamma_0}{2} - C_1 \|g\|_{Y_1} \right) \sum_\alpha \left\| W^{([\alpha] - 1)} \partial_x^{\alpha} h \right\|^2_{L^2(dx; H^\gamma_{1/2}(\partial_v))}
\]

\[
- C_1 \|h\|_{Y_1} \left\| W^{t-[\alpha]} \partial_x^{\alpha} h \right\|^2_{L^2(dx; H^\gamma_{1/2}(\partial_v))}
\]

\[
- C_1 \|g\|_{Y_1} \|h\|_{Y_1} \left\| W^{t-[\alpha]} \partial_x^{\alpha} h \right\|^2_{L^2(dx; H^\gamma_{1/2}(\partial_v))}
\]

\[
= \frac{\gamma_0}{4} \delta_2 \sum_\alpha \left\| W^{t-[\alpha]} \partial_x^{\alpha} h \right\|^2_{L^2(dx; H^\gamma_{1/2}(\partial_v))} - C_1 \|h\|_{Y_1} \left\| W^{t-[\alpha]} \partial_x^{\alpha} h \right\|^2_{L^2(dx; H^\gamma_{1/2}(\partial_v))}
\]

by (7.5) and the similar method as for \(\Gamma_1\) and \(\Gamma_3\) in the proof of Theorem 6.1. Putting

\[
a(t) = \sum_\alpha \left\| W^{t-[\alpha]} \partial_x^{\alpha} g(t) \right\|^2_{L^2(dx; H^\gamma_{1/2}(\partial_v))}
\]

Putting
and taking a sufficiently small $\delta_3$, we get
\[
\text{Re}(h(t), Qh(t))_{Y_t} \geq -\frac{1}{2} \frac{d}{dt} \left( ||h(t)||^2_{Y_t} \right) - C_l (1 + a(t)) ||h(t)||_{Y_t}^2 + \kappa \sum_{\alpha} \left| W^{l-|\alpha|+3} \partial_x^\alpha h \right|_{L^2_{Y_t}}^2 + \left( \frac{c_0}{4} \delta_2 - C_l \|g\|_{Y_t} \right) \sum_{\alpha} \left| W^{l-|\alpha|} \partial_x^\alpha h \right|_{L^2_{(dx; H_{Y_t}^s/2; (dv))}}^2.
\]

Therefore, by (7.6) we have
\[
\begin{align*}
&- \frac{d}{dt} \left( ||h(t)||^2_{Y_t} \right) + \frac{c_0 \delta_2}{8} \sum_{\alpha} \left| W^{l-|\alpha|} \partial_x^\alpha h \right|_{L^2_{(dx; H_{Y_t}^s/2; (dv))}}^2 \\
\leq & \ 2e^{V(t)} \left| (h(t), Qh(t))_{Y_t} \right| - \kappa e^{V(t)} \sum_{\alpha} \left| W^{l-|\alpha|+3} \partial_x^\alpha h \right|_{L^2_{Y_t}}^2 \\
\end{align*}
\]

If $\varepsilon_0 < \frac{c_0 \delta_2}{8}$ and $V(t) = 2C_l \int_t^T (1 + a(\tau)) d\tau$ then we get
\[
\begin{align*}
&\frac{d}{dt} \left( \frac{1}{2} ||h(t)||^2_{Y_t} \right) + \frac{c_0 \delta_2}{8} \sum_{\alpha} \left| W^{l-|\alpha|} \partial_x^\alpha h \right|_{L^2_{(dx; H_{Y_t}^s/2; (dv))}}^2 \\
\leq & \ 2e^{V(t)} \left| (h(t), Qh(t))_{Y_t} \right| - \kappa e^{V(t)} \sum_{\alpha} \left| W^{l-|\alpha|+3} \partial_x^\alpha h \right|_{L^2_{Y_t}}^2. \\
\end{align*}
\]

Since $h(T) = 0$, for all $t \in [0, T]$ we have
\[
\begin{align*}
&||h(t)||^2_{Y_t} + \kappa \int_0^T \sum_{\alpha} \left| W^{l-|\alpha|+3} \partial_x^\alpha h(\tau) \right|_{L^2_{Y_t}}^2 d\tau \leq 2 \int_t^T e^{V(\tau)} \left| (h(\tau), Qh(\tau))_{Y_t} \right| d\tau \\\n\leq & \ 2e^{V(0)} \int_0^T \left| (h(t), Qh(t))_{Y_t} \right| d\tau. \\
\end{align*}
\]

We estimate the right hand side of the above inequality in two different ways. First,
\[
\int_0^T \left| (h(t), Qh(t))_{Y_t} \right| d\tau \leq \int_0^T \|h\|_{Y_t} \|Qh\|_{Y_t} d\tau \leq \|h\|_{L^\infty([0, T]; Y_t)} \|Qh\|_{L^1([0, T]; Y_t)},
\]
which implies
\[
\|h\|_{L^\infty([0, T]; Y_t)} \leq 2e^{V(0)} \|Qh\|_{L^1([0, T]; Y_t)},
\]
and
\[
\kappa \int_0^T \sum_{\alpha} \left| W^{l-|\alpha|+3} \partial_x^\alpha h(\tau) \right|_{L^2_{Y_t}}^2 d\tau \leq 2e^{V(0)} \|h\|_{L^\infty([0, T]; Y_t)} \|Qh\|_{L^1([0, T]; Y_t)} \leq 4e^{2V(0)} \|Qh\|_{L^1([0, T]; Y_t)},
\]
Second, we have
\[
2e^{V(0)} \int_0^T \left| (h(t), Qh(t))_{Y_t} \right| d\tau \leq \frac{\kappa}{2} \int_0^T \sum_{\alpha} \left| W^{l-|\alpha|+3} \partial_x^\alpha h(\tau) \right|_{L^2_{Y_t}}^2 d\tau \\
+ \frac{1}{2\kappa} e^{2V(0)} \int_0^T \sum_{\alpha} \left| W^{l-|\alpha|+3} \partial_x^\alpha Qh(\tau) \right|_{L^2_{H_{Y_t}^s}}^2 d\tau.
\]

Therefore, by (7.6) we have
\[
\begin{align*}
&||h(t)||^2_{Y_t} + \frac{\kappa}{2} \int_0^T \sum_{\alpha} \left| W^{l-|\alpha|+3} \partial_x^\alpha h(\tau) \right|_{L^2_{Y_t}}^2 d\tau \leq \frac{1}{2\kappa} e^{2V(0)} \int_0^T \sum_{\alpha} \left| W^{l-|\alpha|+3} \partial_x^\alpha Qh(\tau) \right|_{L^2_{H_{Y_t}^s}}^2 d\tau. \\
&\end{align*}
\]

Denote $L^2([0, T]; \tilde{Y}_t)$ as the space such that $w \in L^2([0, T]; \tilde{Y}_t)$ if and only if
\[
\int_0^T \sum_{\alpha} \left| W^{l-|\alpha|-3} \partial_x^\alpha w(\tau) \right|_{L^2_{H_{Y_t}^s}}^2 d\tau < \infty.
\]

Consider the vector subspace
\[
\mathbb{W} = \{ w = Qh : h \in C^\infty([0, T] \times T_x^3, S(\mathbb{R}^3)) , h(T) = 0 \} \subseteq L^1([0, T], Y_t) \cap L^2([0, T]; \tilde{Y}_t).
\]
This inclusion holds because it follows from Proposition 2.2 that for any $\varphi \in Y_t$ we have
\[
|\varphi, Q(\mu + g, \cdot) h|_{Y_t} = \sum_\alpha \langle Q(\mu + g), \varphi \rangle, W^{2(l - |\alpha|)} \partial_x^\alpha h \rangle_{L^2(dx dv)} \leq \sum_\alpha \left( 1 + \langle \nu \rangle^2 \right) \| \varphi \|_{L^{2/3}(dx dv)} \| W^{2l} \partial_x^\alpha h \|_{L^2(dx; H^{\gamma/2}_{\gamma/2}(dv))}.
\]
Since $f_0 \in Y_t$, we define a linear functional
\[
G : \mathbb{W} \rightarrow \mathbb{C}
\]
\[
w = Qh \mapsto (f_0, h(0))_{Y_t} - \langle Q(g, \mu), h \rangle_{L^2([0,T]; Y_t)},
\]
where $h \in C^\infty([0,T] \times \mathbb{T}_2^3, S(\mathbb{R}_2^3))$, with $h(T) = 0$. According to (7.7), the operator $Q$ is injective. The linear functional $G$ is therefore well defined. It follows from Proposition 3.3 and (7.7) that $G$ is a continuous linear form on $(\mathbb{W}, \| \cdot \|_{L^1([0,T]; Y_t) \cap L^2([0,T]; \hat{Y}_t)})$. Due to the estimates that
\[
|G(w)| \leq \| f_0 \|_{Y_t} \| h(0) \|_{Y_t} + C_T \sum_\alpha \left| W^{l-|\alpha|} \partial_x^\alpha g \right|_{L^2([0,T]; L^2_{\gamma/2}(dx dv))} \| W^{l-|\alpha|} \partial_x^\alpha h \|_{L^2([0,T]; L^2_{\gamma/2}(dx dv))}
\]
\[
\leq C_T \| Qh \|_{L^1([0,T]; Y_t) \cap L^2([0,T]; \hat{Y}_t)}.
\]
By using the Hahn-Banach theorem, $G$ may be extended as a continuous linear form on $L^1([0,T]; Y_t) \cap L^2([0,T]; \hat{Y}_t)$ with a norm smaller than $C_T$. It follows that there exists $f_\kappa \in L^\infty([0,T]; Y_t) \cap L^2([0,T]; \hat{Y}_t)$ such that
\[
G(w) = \int_0^T (f_\kappa(t), w(t))_{Y_t} dt, \quad \forall w \in L^1([0,T]; Y_t).
\]
Hence for all $h \in C^\infty(( -\infty, T]; C^\infty(\mathbb{T}_2^3; S(\mathbb{R}_2^3)))$,
\[
G(Ch) = \int_0^T (f_\kappa(t), Ch(t))_{Y_t} dt = (f_0, h(0))_{Y_t} - \int_0^T \langle Q(g(t), \mu(t), h(t))_{Y_t} dt.
\]
This shows that $f_\kappa \in L^\infty([0,T]; Y_t) \cap L^2([0,T]; \hat{Y}_t)$ is a weak solution of the Cauchy problem (7.4) because $\sum_{|\alpha| \leq 2} W^{2(l-|\alpha|)} \partial_x^\alpha$ is bijective in $C^\infty(( -\infty, T]; C^\infty(\mathbb{T}_2^3; S(\mathbb{R}_2^3)))$. Note that $f_\kappa \in L^2([0,T]; \hat{Y}_t)$ if and only if
\[
\int_0^T \sum_\alpha \left| W^{l-|\alpha|+3} \partial_x^\alpha f_\kappa(\tau) \right|^2_{L^2_{\gamma/2}(dx H^\alpha_x)} d\tau < \infty.
\]
In particular, this implies
\[
\int_0^T \sum_\alpha \left| W^{l-|\alpha|} \partial_x^\alpha f_\kappa(\tau) \right|^2_{L^2(dx; H^{\gamma/2}_{\gamma/2}(dv))} d\tau < \infty. \quad (7.10)
\]
Equipped with the regularity of $f_\kappa$ in (7.10), we are ready to show the energy bound in (7.2). Similar to the proof for Theorem 5.1, we have
\[
\frac{d}{dt} \| f_\kappa \|_{Y_t} \leq - \left( \frac{\gamma_0}{4} - C_l \| g \|_{Y_t} \sum_\alpha \left| W^{l-|\alpha|} \partial_x^\alpha f_\kappa \right|^2_{L^2_{\gamma/2}(dx dv)} + C_l \| f_\kappa \|_{Y_t}^2
\]
\[
- \left( \frac{C_0}{4} - C_l \| g \|_{Y_t} \sum_\alpha \left| W^{l-|\alpha|} \partial_x^\alpha f_\kappa \right|^2_{L^2(dx; H^{\gamma/2}_{\gamma/2}(dv))}
\]
\[
+ C_l \| f \|_{Y_t} \sum_\alpha \left| W^{l-|\alpha|} \partial_x^\alpha g \right|^2_{L^2(dx; H^{\gamma/2}_{\gamma/2}(dv))} \right)
\]
\[
+ \delta_2 \sum_\alpha \left| W^{l-|\alpha|} \partial_x^\alpha g \right|^2_{L^2_{\gamma/2}(dx dv)}.
\]

where the small coefficient $\delta_4$ is given by
\[
\delta_4 = \frac{4}{\gamma_0} \int_{\mathbb{S}^2} b(\cos \theta) \sin^{m_0-3-\gamma/2} \frac{\theta}{2} \, d\theta; \quad m_0 > \max\{4s, 1\}. \tag{7.12}
\]
By the definition of $\gamma_0$ in [3.17], we have
\[
\delta_4 \leq \frac{8}{\gamma_1} 2^{-\frac{4-\gamma/2}{1-\gamma/2}}. \tag{7.13}
\]
Rigorously speaking, due to the transport term $v \cdot \nabla_x f_\kappa$, one should regularize $f_\kappa$ in $x$ as well to justify (7.11). The procedures are the same as in the proof of Theorem 1.1 in [8] and we refer the reader there for the details. Putting
\[
E_1(f_\kappa) = \int_0^T \sum_\alpha \left\| W^{l-|\alpha|} \partial_x^\alpha f_\kappa(\tau) \right\|^2_{L^2(dx; H^\gamma_\kappa(du))} \, d\tau,
\]
we have
\[
(1 - C_1 T) \| f_\kappa \|_{L^\infty([0,T];Y_1)}^2 + \frac{3C_0}{16} \delta_2 E_1(f_\kappa) \leq \| f_0 \|_{Y_1}^2 + C_1 \| f_\kappa \|_{L^\infty([0,T];Y_1)} E_1(f_\kappa)^{1/2} E_1(g)^{1/2} + \delta_4 E_1(g),
\]
which shows
\[
\frac{C_0}{8} \delta_2 E_1(f_\kappa) \leq \| f_0 \|_{Y_1}^2 + C_1 \| f_\kappa \|_{L^\infty([0,T];Y_1)} E_1(g) + \delta_4 E_1(g),
\]
and hence
\[
(1 - C_1 T - C_1 E_1(g)) \| f_\kappa \|_{L^\infty([0,T];Y_1)}^2 \leq \frac{3}{2} \| f_0 \|_{Y_1}^2 + \frac{3}{2} \delta_4 E_1(g).
\]
This concludes
\[
\| f_\kappa \|_{L^\infty([0,T];Y_1)} + E_1(f_\kappa) \leq \| f_0 \|_{Y_1}^2 + 2 \delta_4 E_1(g), \tag{7.14}
\]
under the assumption that $\gamma_0$ and $T_0$ are sufficiently small. Moreover, by Corollary B.1 and (7.14), we obtain (7.3) by using the term $-(Q(\mu + g, W^{l-|\alpha|} \partial_x^\alpha f_\kappa), W^{l-|\alpha|} \partial_x^\alpha f_\kappa)_{L^2(dx; du)}$ to control
\[
\int_{T^2} \left( \iint B(\mu + g) \left( (W^{l-|\alpha|} \partial_x^\alpha f_\kappa)' - W^{l-|\alpha|} \partial_x^\alpha f_\kappa \right)^2 \, d\tau \, d\sigma \right) \, dx.
\]
The existence of a weak solution to (7.1) is then obtained by using the uniform estimate in (7.2) and passing $\kappa \to 0$.

**Proof of the non-negativity** We put $G = \mu + g$ and $F = \mu + f$, and follow the method developed in [4]. It follows from (7.1) that
\[
\begin{cases}
\partial_t F + v \cdot \nabla_x F = Q(G, F), \\
F_{|t=0} = F_0 = \mu + f_0 \geq 0.
\end{cases}
\tag{7.15}
\]
Thanks to (7.2), we have
\[
\int_0^T \left\| J^{\Phi_0}_1(W^i F(t)) \right\|_{L^1(T^2)} \, dt < \infty,
\]
and hence, if $F_\pm = \pm \max\{ \pm F, 0 \}$ then we have
\[
\int_0^T \left\| J^{\Phi_0}_1(W^i F_-(t)) \right\|_{L^1(T^2)} \, dt + \int_0^T \left\| J^{\Phi_0}_1(W^i F_+(t)) \right\|_{L^1(T^2)} \, dt < \infty,
\]
because for $\bar{F} = W^i F$
\[
J^{\Phi_0}_1(\bar{F}) = J^{\Phi_0}_1(\bar{F}_+) + 2 \int \int b(\bar{F}_+ - \bar{F}_- + \bar{F}_+ \bar{F}_-) \, d\tau \, d\sigma,
\]
and the third term is non-negative. Take the convex function $\beta(s) = \frac{1}{2} (s^-)^2 - \frac{1}{2} s (s^-)$ with $s^- = \min\{s, 0\}$, and notice that
\[
\beta_s(F) := \left( \frac{d}{ds} \beta \right)(F) = F_- \in L^2([0,T] \times T^2; L^2_{2+\gamma/2}(\mathbb{R}^3)).
\]
Let $m$ be sufficiently large but $W^m(t)^s \leq W^l$. Multiply the first equation of (7.15) by $\beta_s(F)W^{2m} = F_-W^{2m}$ and integrate over $[0,t] \times T_2^3 \times \mathbb{R}_0^3, \, (t \in (0,T))$. Then, in view of $\beta(F(0)) = F_0^2_2 - 2 = 0$ and

$$
\int_0^t \int_{T_2^3 \times \mathbb{R}_0^3} W^{2m} \cdot \nabla_x \left( \beta(F(\tau)) \right) dxdv \tau = 0,
$$

we have

$$
\int_{T_2^3 \times \mathbb{R}_0^3} \beta(F(t))W^{2m} dxdv = \int_0^t \left( \int_{T_2^3 \times \mathbb{R}_0^3} Q(G(\tau), F(\tau)) \beta_s(F(\tau)) W^{2m} dxdv \right) d\tau,
$$

where the right hand side is well defined because

$$
G \in L^\infty([0,T] \times T_2^3; L^2_{10}(\mathbb{R}_0^3)), \quad W^lF, W^lF_x \in L^2([0,T] \times T_2^3; H^s_{4/2}(\mathbb{R}_0^3)).
$$

The integrand on the right hand side is equal to

$$
\int_{T_2^3 \times \mathbb{R}_0^3} Q(G, F_-)F_-W^{2m} dxdv + \int_{T_2^3 \times \mathbb{R}_0^3 \times \mathbb{R}^2} BG'_s(F_+)F_-W^{2m} dxdv dxdx.
$$

From the induction hypothesis, the second term is non-positive. Therefore it follows from Proposition 3.2 with $l = m$ that

$$
\|W^mF_-(t)\|^2_{L^2(dxdv)} = \int_{T_2^3 \times \mathbb{R}_0^3} \beta(W^mF(t)) dxdv
\leq \int_0^t \left( \int_{T_2^3 \times \mathbb{R}_0^3} Q(G(\tau), F_-(\tau)) F_-(\tau) W^{2m} dxdv \right) d\tau
\leq -c_0(1 - C_i \|g\|_{L^\infty([0,T]; Y_i)}) \int_0^t \|W^mF_-(\tau)\|^2_{L^2_{4/2}(dxdv)} d\tau
+ C_i(1 + \|g\|_{L^\infty([0,T]; Y_i)}) \int_0^t \|W^mF_-(\tau)\|^2_{L^2(dxdv)} d\tau,
$$

which implies that $F = \mu + f \geq 0$ for $(t,x,v) \in [0,T] \times T_2^3 \times \mathbb{R}_0^3$. $\square$

7.2. Local solution for non-linear equation and its uniqueness.

**Theorem 7.1** (Local Existence). There exist $\epsilon_0, \epsilon_1$ and $T > 0$ such that if $f_0 \in Y_1$ and

$$
\|f_0\|_{Y_1} \leq \epsilon_0,
$$

then the Cauchy problem (1.4) admits a unique solution

$$
f \in L^\infty([0,T]; Y_1) \text{ satisfying } \|f\|_{L^\infty([0,T]; Y_1)} \leq \epsilon_1,
$$

and

$$
E_l(f) \triangleq \int_0^T \sum_\alpha \|W^{l-|\alpha|} \partial_\alpha f(\tau)\|^2_{L^2(dx; H^s_{4/2}(d\nu))} d\tau < \epsilon_1^2.
$$

**Proof.** Consider the sequence of approximate solutions defined by $f^0 = 0$ and

$$
\begin{cases}
\partial_\alpha f^{n+1} + v \cdot \nabla_x f^{n+1} - Q(\mu + f^n, f^{n+1}) = Q(f^n, \mu),
\end{cases}
$$

$$
f^{n+1}|_{t=0} = f_0(x,v).
$$

Use Lemma 7.1 with $f = f^{n+1}, \, g = f^n$ and choose $T, \delta$ sufficiently small. Then it follows from (7.2) that

$$
\|f^n\|_{L^\infty([0,T]; Y_1)} \leq \epsilon_1, \quad E_l(f^n) \leq \epsilon_1^2,
$$

inductively, if $\epsilon_0$ and $\delta_4$ are taken such that

$$
2(\epsilon_0^2 + \delta_4 \epsilon_1^2) \leq \epsilon_1^2.
$$
A sufficient condition for (7.20) to hold is by choosing
\[ \delta_4 < \frac{1}{2}, \quad \epsilon_1 \geq 2\epsilon_0. \]
It remains to prove the convergence of the sequence \{f^n\}. Setting \( w^n = f^{n+1} - f^n \), from (7.18) we have
\[ \partial_t w^n + v \cdot \nabla_x w^n - Q(\mu + f^n, w^n) = Q(w^{n-1}, \mu + f^n), \]
with \( w^n|_{t=0} = 0 \). Repeating the estimates leading to (7.2), we have
\[
\begin{align*}
\|w^n\|_{L^{\infty}(0,T;Y_{l'})}^2 + E_{l'}(w^n) & \\
& \leq 2\delta_4 E_{l'}(w^{n-1}) + 2\sum_{\alpha} \int_0^T \int_{\mathbb{R}^3} \partial_x^\alpha Q(w^{n-1}, f^n) (\partial_x^\alpha w^n) \, \mathbf{w}^{2l'-|\alpha|} \, dv \, dx \, dt,
\end{align*}
\]
where \( E_{l'} \) is defined in (7.17). We claim that if we choose \[ l' = l - 2, \quad l \geq 11 + \gamma, \]
then for each \(|\alpha| \leq 2\), the last term of the right hand side of the above inequality is bounded by
\[
\begin{align*}
\left| \int_0^T \int_{\mathbb{R}^3} \partial_x^\alpha Q(w^{n-1}, f^n) (\partial_x^\alpha w^n) \, \mathbf{w}^{2l'-|\alpha|} \, dv \, dx \, dt \right| & \\
& \leq C_l \left( \|w^{n-1}\|_{L^{\infty}(0,T;Y_{l'})}^2 + E_{l'}^{1/2}(w^{n-1}) \right) \left( \|f^n\|_{L^{\infty}(0,T;Y_l)} + E_{l'}^{1/2}(f^n) \right) E_{l''}^{1/2}(w^n).
\end{align*}
\]
The proof is similar to the one for Proposition 3.3. Indeed, for any \( \alpha_1 + \alpha_2 = \alpha \),
\[
\begin{align*}
\int_{\mathbb{R}^3} \partial_x^\alpha w^n \, dv & \\
& = \int_{\mathbb{R}^3} \left( \partial_x^\alpha w^n \right) \cdot \left( \partial_x^\alpha f^n \right) \, dv \\
& = \int_{\mathbb{R}^3} \left( \partial_x^\alpha w^n \right) \cdot \left( \partial_x^\alpha f^n \right) \, dv \\
& = \int_{\mathbb{R}^3} \left( \partial_x^\alpha w^n \right) \cdot \left( \partial_x^\alpha f^n \right) \, dv \\
& = \int_{\mathbb{R}^3} \left( \partial_x^\alpha w^n \right) \cdot \left( \partial_x^\alpha f^n \right) \, dv \\
& = \int_{\mathbb{R}^3} \left( \partial_x^\alpha w^n \right) \cdot \left( \partial_x^\alpha f^n \right) \, dv \\
& \Delta T_8 + T_9.
\end{align*}
\]
Applying the trilinear estimate in Proposition 2.2 to \( T_8 \) gives
\[
|T_8| \, dx \, dt \leq C_{l'} \left( \|\partial_x^\alpha w^{n-1}\|_{L^{l+2} \cap L^2} \|\partial_x^\alpha f^n \|_{W^{l'-|\alpha|} \cap L^2} \right) \left( \|\partial_x^\alpha w^n \|_{W^{l'-|\alpha|} \cap L^2} \right) \left( \|\partial_x^\alpha w^n \|_{W^{l'-|\alpha|} \cap L^2} \right) \left( \|\partial_x^\alpha w^n \|_{W^{l'-|\alpha|} \cap L^2} \right) \left( \|\partial_x^\alpha w^n \|_{W^{l'-|\alpha|} \cap L^2} \right),
\]
if we choose \( l' \) such that
\[ \frac{\gamma}{2} + 2s + 4 \leq l' \leq l - 2s, \quad l \geq 8 + \gamma/2. \]
By Proposition 3.1 we bound $T_9$ as

$$
|T_9| \leq C_0 \|\partial_x^2 f^n\|_{L^1_T} \left( \|W^{l'-|\alpha|} \partial_x^{\alpha_1} w^{n-1}\|_{L^2_{t,x}} \|W^{l'-|\alpha|} \partial_x^{\alpha} w^n\|_{L^2_{t,x}} 
+ C_l \|\partial_x^2 f^n\|_{L^1_{t,x}} \left( \|W^{l'-|\alpha|} \partial_x^{\alpha_1} w^{n-1}\|_{L^2_T} \|W^{l'-|\alpha|} \partial_x^{\alpha} w^n\|_{L^2_T} 
+ C_k \|\partial_x^{\alpha} w^{n-1}\|_{L^1_{t,x}} \left( \|W^{l'-|\alpha|} \partial_x^{\alpha} f^n\|_{L^2_T} \|W^{l'-|\alpha|} \partial_x^{\alpha} w^n\|_{L^2_T} 
+ C_l \|\partial_x^{\alpha} w^{n-1}\|_{L^1_{t,x}} \left( \|W^{l'-|\alpha|} \partial_x^{\alpha} f^n\|_{L^2_T} \|W^{l'-|\alpha|} \partial_x^{\alpha} w^n\|_{L^2_T} 
+ C_l \|\partial_x^{\alpha} w^{n-1}\|_{L^1_{t,x}} \right) \right) \right)
\right)
\right).

(7.24)

Integrating in $t, x$ and using Hölder’s inequality, we have

$$
\int_0^T \int_{\mathbb{T}^d} |T_8| \, dx \, dt \leq C_{l'} \|w^{n-1}\|_{L^\infty([0,T];Y_{l'})} E_{l'}^{1/2}(f^n) E_{l'}^{1/2}(w^n).
$$

Similarly, we have the bound for $T_9$ as

$$
\int_0^T \int_{\mathbb{T}^d} |T_9| \, dx \, dt \leq C_{l'} \left( \|f^n\|_{L^\infty([0,T];Y_{l'})} + E_{l'}^{1/2}(f^n) \right) E_{l'}^{1/2}(w^{n-1}) E_{l'}^{1/2}(w^n),
$$

if we choose $l'$ such that $9 + \gamma \leq l'$. In summary, (7.22) holds if

$$
9 + \gamma \leq l' \leq l - 2s.
$$

Applying (7.22) and Hölder’s inequality in (7.21), we have

$$
\|w^n\|^2_{L^\infty([0,T];Y_{l'})} + E_{l'}(w^n) \leq \left( 4\delta_4 + C_1 \left( \|f^n\|^2_{L^\infty([0,T];Y_{l'})} + E_{l'}(f^n) \right) \right) \left( \|w^{n-1}\|^2_{L^\infty([0,T];Y_{l'})} + E_{l'}(w^{n-1}) \right)
\leq \left( 4\delta_4 + 2C_1\epsilon_1^2 \right) \left( \|w^{n-1}\|^2_{L^\infty([0,T];Y_{l'})} + E_{l'}(w^{n-1}) \right).
$$

(7.25)

Hence, if we choose $\delta, \epsilon_1$ small enough such that

$$
4\delta_4 + 2\epsilon_1^2 < 1,
$$

(7.26)

then the series $\sum_{n=0}^{\infty} \left( \|w^{n-1}\|^2_{L^\infty([0,T];Y_{l'})} + E_{l'}(w^{n-1}) \right)$ converges. With the smallness condition (7.26), there exists a function $f \in L^\infty(0, T; Y_{l'})$ with $E_{l'}(f) < \infty$ such that

$$
f^n \to f \quad \text{strongly in } L^\infty(0, T; Y_{l'}), \quad E_{l'}(f^n - f) \to 0.
$$

(7.27)

Moreover, by (7.19) we also have that $f \in L^\infty([0,T];Y_l)$ and

$$
\|f\|_{L^\infty([0,T];Y_l)} \leq \epsilon_1, \quad E_l(f) \leq \epsilon_1^2.
$$

(7.28)

To complete the proof of the local existence of the solution to the nonlinear equation, we only need to show that

$$
Q(\mu + f^n, f^{n+1}) \to Q(\mu + f, f) \quad \text{in } \mathcal{D}'([0,T] \times \mathbb{T}^3 \times \mathbb{R}^3).
$$

(7.29)
To this end, let \( \phi \in C_{c}^{\infty}([0,T] \times \mathbb{T}^{3} \times \mathbb{R}^{3}) \). Then by letting \((\sigma, m) = (-s, 0)\) in Proposition 2.2 and using the uniform bounds in (7.19) and (7.28), we have

\[
\left| \int_{\mathbb{T}^{3}} \int_{\mathbb{R}^{3}} (Q(\mu + f^{n}, f^{n+1}) - Q(\mu + f, f)) \phi \, dv \, dx \right| \\
\leq \int_{\mathbb{T}^{3}} \int_{\mathbb{R}^{3}} Q(f^{n} - f, f^{n+1}) \phi \, dv \, dx + \left| \int_{\mathbb{T}^{3}} \int_{\mathbb{R}^{3}} Q(\mu + f, f^{n+1} - f) \phi \, dv \, dx \right| \\
\leq C_{\phi} \left( \int_{\mathbb{T}^{3}} \| f^{n} - f \|_{L_{t}^{2} \cap L_{\mathbb{T}^{3}}} \| f^{n+1} + f^{n} \|_{L_{t}^{2} \cap L_{\mathbb{T}^{3}}} \right) \\
\leq C_{\phi} \left( \| f^{n} - f \|_{L_{t}^{2} \cap L_{\mathbb{T}^{3}}} + \| f^{n+1} - f \|_{L_{t}^{2} \cap L_{\mathbb{T}^{3}}} \right) \\
\rightarrow \infty
\]

as long as \( l' \geq 2 + \gamma + 2s \). Hence (7.29) holds. The other terms in equation (7.18) are all linear, therefore they all converge to the corresponding terms in \( f \) in the sense of distribution. We thereby complete the proof of the existence of a weak solution \( f \) to (1.4) with the desired bounds in (7.16) and (7.17). Given the bounds for \( f \), the uniqueness follows from the estimate in (7.25). \( \square \)

**Remark 7.1.** By the definition of \( \delta_{4} \) in the proof of Theorem 7.1 one sufficient condition for \( \delta_{4} < 1/8 \) is

\[
\frac{8}{\gamma} 2^{\frac{1 - \gamma - 2s}{2}} < 1/8,
\]

which gives \( l \geq 33 \).

**APPENDIX A. TWO LEMMAS**

In the first part of the appendix, we give two lemmas that have been used and can be useful for future study.

**Lemma A.1** (Ukai estimate). For any \( \alpha > 0 \), there exists a constant \( c_{\alpha} > 0 \) such that

\[
\int_{0}^{1} |t - s\eta|^{\alpha} \, ds \geq c_{\alpha} (|t|^{\alpha} + t^{\alpha+1}|\eta|^{\alpha}) \,. \tag{A.1}
\]

**Remark**: If \( \alpha = 2 \), estimate follows from a direct calculation. The following simple proof in general case is due to Seiji Ukai.

**Proof.** Setting \( s = t\tau \) and \( \bar{\eta} = t\eta \), we see that the estimate is equivalent to

\[
\int_{0}^{1} |\xi - \tau\bar{\eta}|^{\alpha} \, d\tau \geq c_{\alpha} (|\xi|^{\alpha} + |\bar{\eta}|^{\alpha})
\]

Since this is trivial when \( \bar{\eta} = 0 \), we may assume \( \bar{\eta} \neq 0 \). If \( |\xi| < |\bar{\eta}| \) then

\[
\int_{0}^{1} |\xi - \tau\bar{\eta}|^{\alpha} \, d\tau \geq |\bar{\eta}|^{\alpha} \int_{0}^{1} \left| \frac{\tau}{|\bar{\eta}|} \right|^{\alpha} \, d\tau
\]

\[
= |\bar{\eta}|^{\alpha} \left\{ \int_{0}^{1} \left( \frac{|\xi|}{|\bar{\eta}|} - \tau \right)^{\alpha} \, d\tau + \int_{1}^{0} \left( \tau - \frac{|\xi|}{|\bar{\eta}|} \right)^{\alpha} \, d\tau \right\}
\]

\[
\geq \frac{|\bar{\eta}|^{\alpha}}{\alpha + 1} \min_{0 \leq \theta \leq 1} \left( \theta^{\alpha+1} + (1 - \theta)^{\alpha+1} \right) = \frac{|\bar{\eta}|^{\alpha}}{2^{\alpha}(\alpha + 1)}
\]

\[
\geq \frac{1}{2^{\alpha+1}(\alpha + 1)} (|\xi|^{\alpha} + |\bar{\eta}|^{\alpha})
\]

Thus, the proof is completed.
If $|\xi| \geq |\tilde{\eta}|$ then
\[
\int_0^1 |\xi - \tau \tilde{\eta}|^\alpha d\tau \geq |\xi|^{\alpha} \int_0^1 \left(1 - \tau \frac{\tilde{\eta}}{|\xi|}\right)^\alpha d\tau \geq |\xi|^{\alpha} \int_0^1 (1 - \tau)^\alpha d\tau
\]
\[= \frac{|\xi|^{\alpha}}{\alpha + 1} \geq \frac{1}{2(\alpha + 1)}(|\xi|^{\alpha} + |\tilde{\eta}|^{\alpha}).\]
Hence we obtain (A.3). \[\Box\]

**Corollary A.1.** For any $\alpha > 0$, we have
\[
\int_0^t (\xi - s\eta)^{\alpha} ds \sim t (1 + |\xi|^2 + t^2 |\eta|^2)^{\alpha/2}. \tag{A.2}
\]

**Lemma A.2.** For any $0 < \beta < 1$, there exists a constant $C_\beta > 0$ such that
\[
\int_0^t (\xi - s\eta)^{-\beta} ds \leq C_\beta \frac{t}{(1 + |\xi|^2 + |\eta|^2)^{\beta/2}}. \tag{A.3}
\]

**Proof.** Setting $s = t\tau$ and $\tilde{\eta} = t\eta$ as in the preceding proof, we see that the estimate is equivalent to
\[
\int_0^1 \frac{d\tau}{(\xi - \tau \tilde{\eta})^\beta} \leq C_\beta \frac{1}{(1 + |\xi|^2 + |\eta|^2)^{\beta/2}}.
\]
Note $1 + |\xi - \tau \tilde{\eta}|^2 \geq 1 + (|\xi| - |\tau \tilde{\eta}|)^2$ and put $a = |\xi|, b = |\tilde{\eta}|$. Then, it suffices to show
\[
\int_0^1 \frac{d\tau}{(1 + (a - \beta r)^2)^{\beta/2}} \leq C_\beta (1 + a^2 + b^2)^{-\beta/2},
\]
when $\max(a, b) > 1$. If $b \leq a/2$ then this holds with $C_\beta = 5^\beta$ because $(a - br)^2 \geq a^2/4 \geq (a^2 + b^2)/5$. If $a/2 < b \leq 2a$ then by the change of variable $u = a - br$, we have
\[
\int_0^1 \frac{d\tau}{(1 + (a - \beta r)^2)^{\beta/2}} = \frac{1}{b} \int_{a-b}^a \frac{du}{(1 + u^2)^{\beta/2}}
\]
\[\leq \frac{1}{b} \int_{-a}^a \frac{du}{(1 + u^2)^{\beta/2}} \leq \frac{2}{b} \int_0^a \frac{du}{u^\beta} = \frac{2a^{1-\beta}}{(1-\beta)b} \leq \frac{4}{(1-\beta)a^\beta}.
\]
If $b > 2a$, then
\[
\int_0^1 \frac{d\tau}{(1 + (a - \beta r)^2)^{\beta/2}} = \frac{1}{b} \int_{a-b}^a \frac{du}{(1 + u^2)^{\beta/2}}
\]
\[\leq \frac{2}{b} \int_0^{b-a} \frac{du}{(1 + u^2)^{\beta/2}} \leq \frac{2}{b} \int_0^b \frac{du}{u^\beta} = \frac{2}{(1-\beta)b^\beta}.
\]
\[\Box\]

**Appendix B. Some estimates on $Q$.**

The second part of the appendix is about some estimates on the nonlinear collision operator $Q$.

**Lemma B.1.** Let $b$ satisfy $b(\cos \theta) \sin \theta \sim \frac{1}{\theta^{1+\alpha}}$ with $0 < s < 1$. Then there exists a constant $C > 0$ such that
\[
\iint bF_\alpha(g - g')^2 d\sigma dv_* \leq C \left\| F \right\|_{L^1_{\nu}} \left( J^0_{\nu}(g) + \|g\|^2_{L^2_{\nu}} \right). \tag{B.1}
\]
If $F \in L^1$ satisfies $F \geq 0$ and there exist constants $D_0, E_0 > 0$ such that
\[
\|F\|_{L^1} \geq D_0 \text{ and } \|F\|_{L^1_{\nu}} + \|F\|_{L^1_{\log \nu}} \leq E_0,
\]
then there exists a $C_F > 0$ depending only on $D_0$ and $E_0$ such that
\[
J^0_{\nu}(g) \leq C_F \left( \iint bF_\alpha(g - g')^2 d\sigma dv_* + \|g\|^2_{L^2_{\nu}} \right). \tag{B.2}
\]
Proof. For the proof of (B.1) we may assume \( F \geq 0 \). It follows from [1 Proposition 1] that

\[
J_{1}^{g_{0}}(g) = \iiint bM_{s}(g' - g)^{2}d\sigma dv d\nu,
\]

\[
= \frac{1}{(2\pi)^{3}} \iiint b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) \{ \tilde{M}(0) |\hat{\varphi}(\xi) - \hat{\varphi}(\xi^{+})|^{2} \}
+ 2Re\left( (\tilde{M}(0) - \tilde{M}(\xi^{-}))\hat{\varphi}(\xi^{+})\overline{\hat{\varphi}(\xi)} \right) d\xi d\sigma,
\]

(B.3)

and

\[
C_{0}(F, g) \triangleq \iiint bF_{s}(g' - g)^{2}d\sigma dv d\nu,
\]

\[
= \frac{1}{(2\pi)^{3}} \iiint b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) \{ \tilde{F}(0) |\hat{\varphi}(\xi) - \hat{\varphi}(\xi^{+})|^{2} \}
+ 2Re\left( (\tilde{F}(0) - \tilde{F}(\xi^{-}))\hat{\varphi}(\xi^{+})\overline{\hat{\varphi}(\xi)} \right) d\xi d\sigma.
\]

(B.4)

Since \( \tilde{F}(0) = \| F \|_{L^{1}} \) and \( \tilde{M}(0) = c_{0} > 0 \), we obtain

\[
c_{0}C_{0}(F, g) - \| F \|_{L^{1}}J_{1}^{g_{0}}(g) = -\frac{2}{(2\pi)^{3}}\| F \|_{L^{1}} \iiint b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) Re\left( (\tilde{M}(0) - \tilde{M}(\xi^{-}))\hat{\varphi}(\xi^{+})\overline{\hat{\varphi}(\xi)} \right) d\xi d\sigma
+ \frac{2c_{0}}{(2\pi)^{3}} \iiint b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) Re\left( (\tilde{F}(0) - \tilde{F}(\xi^{-}))\hat{\varphi}(\xi^{+})\overline{\hat{\varphi}(\xi)} \right) d\xi d\sigma \]

\[\triangleq D_{1} + D_{2}.
\]

Write

\[
D_{2} = \frac{2c_{0}}{(2\pi)^{3}} \int |\hat{\varphi}(\xi)|^{2} \left( \int b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) Re\left( (\tilde{F}(0) - \tilde{F}(\xi^{-}))d\sigma \right) d\xi
+ \int b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) Re\left( (\tilde{F}(0) - \tilde{F}(\xi^{-}))\hat{\varphi}(\xi^{+})\overline{\hat{\varphi}(\xi)} \right) d\xi d\sigma \}
\]

\[\triangleq D_{2,1} + D_{2,2}.
\]

By the Cauchy-Schwarz inequality, we have

\[
|D_{2,2}|^{2} \lesssim \int b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) |\tilde{F}(0) - \tilde{F}(\xi^{-})|^{2} |\hat{\varphi}(\xi)|^{2} d\xi d\sigma \int b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) |\hat{\varphi}(\xi^{+}) - \hat{\varphi}(\xi)|^{2} d\xi d\sigma
\]

\[\triangleq D_{2,2}^{(1)} \times D_{2,2}^{(2)}.
\]

Since

\[
|\hat{F}(0) - \hat{F}(\xi^{-})| \leq \int F(v)|1 - e^{-iv\xi^{-}}|dv,
\]

we have

\[
D_{2,2}^{(1)} \leq \frac{1}{2} \iiint |\hat{\varphi}(\xi)|^{2} F(v) F(u) \left( \int b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) (|1 - e^{-iv\xi^{-}}|^{2} + |1 - e^{-iw\xi^{-}}|^{2})d\sigma \right) dv du d\xi
\]

\[\leq C\| g \|_{H^{s}}^{2} \| F \|_{L^{1}} \| F \|_{L^{1}}^{2},
\]

because

\[
\int b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) |1 - e^{-iv\xi^{-}}|^{2}d\sigma \leq C \int_{0}^{(v(\xi))^{-1}} \theta^{-1 - 2s} \| v \|_{\xi}^{2}\theta^{2} d\theta + \int_{(v(\xi))^{-1}}^{\pi/2} \theta^{-1 - 2s} d\theta
\]

\[\leq C(v)^{2s}(\xi)^{2s}.
\]
Then we have $|A_{2,1}| \leq C\|g\|^2_{H^s} \|F\|_{L^1_{\gamma}}$ because
\[
\int b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \Re (\tilde{F}(0) - \tilde{F}(\xi^-)) d\sigma = \int F(v) \left( \int b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) (1 - \cos(v \cdot \xi^-)) d\sigma \right) dv
\leq C|\xi|^{2s} \int F(v)v^{2s} dv.
\]

Since $\tilde{M}(\xi)$ is real valued, it follows that
\[
\Re \left( \tilde{M}(0) - \tilde{M}(\xi^-) \right) \tilde{g}(\xi) = \left( \int (1 - \cos(v \cdot \xi^-)) M(v) dv \right) \Re \tilde{g}^+(\xi) \tilde{g}(\xi)
\leq \min \{ |\langle \xi \rangle^2 \theta^2, 1 \} |\tilde{g}(\xi)\tilde{g}(\xi)|.
\]

Therefore, by using Cauchy-Schwarz inequality and the change of variables $\xi \to \xi^+$, we obtain
\[
|D_1| \leq C\|F\|_{L^2} \|g\|^2_{H^s}.
\]

Furthermore, it follows from (B.3) that
\[
D_{2,2}^{(2)} = \int b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \Re (\tilde{G}(\xi) - \tilde{G}(\xi^+)) d\xi d\sigma \leq C \left( J_1^{\Phi_0}(g) + \|g\|^2_{H^s} \right),
\]
which yields $|D_{2,2}| \leq C\|F\|_{L^1_{\gamma}}^{1/2} \|F\|_{L^2_{\gamma}}^{1/2} \|g\|_{H^s} \left( J_1^{\Phi_0}(g) + \|g\|^2_{H^s} \right)^{1/2}$.

Hence
\[
|D_2| \leq C\|F\|_{L^1_{\gamma}} \|g\|_{H^s} \left( J_1^{\Phi_0}(g) + \|g\|^2_{H^s} \right)^{1/2}.
\]

Finally, we have
\[
|c_0c_0(F,g) - \|F\|_{L^2} J_1^{\Phi_0}(g)| \leq C\|F\|_{L^1_{\gamma}} \|g\|_{H^s} \left( J_1^{\Phi_0}(g) + \|g\|^2_{H^s} \right)^{1/2} \leq \frac{1}{2} \|F\|_{L^1_{\gamma}} J_1^{\Phi_0}(g) + C_F \|g\|^2_{H^s},
\]
which completes the proof of the lemma because it follows from [1] Proposition 1 and 2 (see also the proof of [6] Proposition 2.1) that
\[
\|g\|^2_{H^s} \leq C_F \left( \Phi_0(F,g) + \|g\|^2_{L^2_{\gamma/2}} \right).
\]

The following lemma is essentially the same as [9] Lemma 3.2.

**Lemma B.2.** Let $0 \leq \gamma \leq 2$ and $0 < s < 1$. Then for $0 \leq F \in L^1_{2s+\gamma}$ we have
\[
\iint BF_s(g' - g)^2 d\sigma dv, dv \lesssim \|F\|_{L^1_{\max(2s,2s+\gamma)}} \left( J_1^{\Phi_0}((v)\gamma/2)g + \|g\|^2_{L^2_{\gamma/2}} \right).
\]

Since the estimate
\[
-2(Q(F,g),g) = \iint BF_s(g' - g)^2 d\sigma dv, dv + \iint BF_s(g^2 - g^2) d\sigma dv, dv,
\]
holds and the cancellation lemma in [1] Lemma 1] shows that the second term on the right hand side is bounded above from $C\|F\|_{L^1_{\gamma}} \|g\|^2_{L^2_{\gamma/2}}$, we have the following;

**Corollary B.1.**
\[
-(Q(F,g),g)_{L^2} \lesssim \|F\|_{L^1_{\max(2s,2s+\gamma)}} \left( J_1^{\Phi_0}((v)\gamma/2)g + \|g\|^2_{L^2_{\gamma/2}} \right).
\]
Proof of Lemma B.2. Since \(|v - v_\star| \lesssim (v')^\gamma + (v_\star)^\gamma\), we have
\[
\iint \int b(\cos \theta)|v - v_\star|^\gamma F_\star(g' - g)^2 d\sigma dv_\star \lesssim \iint \int b(\cos \theta)F_\star\left((v')^{\gamma/2}g' - (v)^{\gamma/2} g\right)^2 d\sigma dv_\star
+ \iint \int b(\cos \theta)\left((v_\star)^{\gamma/2}F_\star(g' - g)^2 d\sigma dv_\star
+ \iint \int b(\cos \theta)F_\star\left((v)^{\gamma/2} - (v')^{\gamma/2}\right)^2 |g|^2 d\sigma dv_\star.
= A^{(1)} + A^{(2)} + A^{(3)}.
\]
By the mean value theorem we have, for a suitable \(v'_\star = v + \tau(v' - v)\) with \(0 < \tau < 1\),
\[
\left|(v)^{\gamma/2} - (v')^{\gamma/2}\right| \leq C_\gamma (v'_\star)^{(\gamma/2-1)}|v - v_\star|\theta
\leq \sqrt{2}C_\gamma (v'_\star - v_\star)^{(\gamma/2-1)}(v'_\star - v_\star)\theta
\leq \sqrt{2}C_\gamma (v - v_\star)^{(\gamma/2)}(v_\star)^{(\gamma/2-1)}\theta \leq \sqrt{2}C_\gamma (v)^{(\gamma/2)}(v_\star)^{(\gamma/2-1)}\theta.
\]
Therefore, we have
\[
A^{(3)} \lesssim \|F\|_{L_2^1}\|g\|_{L_2^{\gamma/2}}^2.
\]
It follows from (B.1) that
\[
A^{(1)} \lesssim \|F\|_{L_2^1}\left(J_{1}^B((v)^{\gamma/2}g) + \|g\|_{L_2^{\gamma/2}}^2\right),
A^{(2)} \lesssim \|F\|_{L_2^{1+\gamma}}\left(J_{1}^B(g) + \|g\|_{L_2^{\gamma}}^2\right).
\]
Note that
\[
J_{1}^B(g) \leq 2J_{1}^B((v)^{\gamma/2}g) + C\|g\|_{L_2^{\gamma/2}}^2
\quad \text{(B.6)}
\]
holds because
\[
(g' - g)^2 \leq (v')^{\gamma}(g' - g)^2 \leq 2(v')^{\gamma/2}g' - (v)^{\gamma/2}g)^2 + 2(v')^{\gamma/2} - (v)^{\gamma/2}\|g\|^2.
\]
Then the lemma follows immediately. \(\square\)

By (B.2) we have the following opposite bound corresponding to Lemma B.2.

Lemma B.3. If \(F \in L^1\) satisfies \(F \geq 0\) and there exist constants \(D_0, E_0 > 0\) such that
\[
\|F\|_{L^1} \geq D_0 \quad \text{and} \quad \|F\|_{L_{\max(2,2+\gamma)}^1} + \|F\|_{L^{\log L}} \leq E_0,
\]
then there exists a \(C_F > 0\) depending only on \(D_0\) and \(E_0\) such that
\[
J_{1}^B((v)^{\gamma/2}g) \leq C_F\left(\iint \int BF_\star(g' - g)^2 d\sigma dv_\star + \|g\|_{L_2^{\gamma/2}}^2\right),
\]
\[
\leq C_F\left(-\langle Q(F,g),g \rangle + \|g\|_{L_2^{\gamma/2}}^2\right).
\quad \text{(B.7)}
\]
Proof. As in [6], for \(D_0, E_0 > 0\) we put
\[
\mathcal{U}(D_0, E_0) = \{F \in L_{\max(2,2+\gamma)}^1 \cap L^{\log L}; F \geq 0, \|F\|_{L^1} \geq D_0, \|F\|_{L_{\max(2,2+\gamma)}^1} + \|F\|_{L^{\log L}} \leq E_0\}.
\]
Set \(B(R) = \{v \in \mathbb{R}^3; |v| \leq R\}\) for \(R > 0\) and \(B_0(R, r) = \{v \in B(R); |v - v_0| \geq r\}\) for a \(v_0 \in \mathbb{R}^3\) and \(r \geq 0\).
It follows from the definition of \(\mathcal{U}(D_0, E_0)\) that there exist positive constants \(R > 1 > r_0\) depending only on \(D_0, E_0\) such that
\[
g \in \mathcal{U}(D_0, E_0) \quad \text{implies} \quad \chi_{B_0(R, r_0)}g \in \mathcal{U}(D_0/2, E_0),
\]
where \(\chi_A\) denotes a characteristic function of the set \(A \subset \mathbb{R}^3\). We denote
\[
C_\gamma(F, g) = \iint \int BF_\star(g' - g)^2 d\sigma dv_\star.
\]

Let $\varphi_R$ be a non-negative smooth function not greater than one, which is 1 for $|v| \geq 4R$ and 0 for $|v| \leq 2R$. In view of

\[
\frac{\langle v \rangle}{4} \leq |v - v_*| \leq 2\langle v \rangle \text{ on supp } (\chi_{B(R)})_* \varphi_R,
\]

we have

\[
4^{\gamma/2} |v - v_*|^2 F_* (g' - g)^2 \geq (F \chi_{B(R)})_* (\langle v \rangle^{\gamma/2} \varphi_R)^2 (g' - g)^2
\]

\[
\geq (F \chi_{B(R)})_* \left[ \frac{1}{2} \left( (\langle v \rangle^{\gamma/2} \varphi_R g) - (\langle v \rangle^{\gamma/2} \varphi_R g) \right)^2 - \left( (\langle v \rangle^{\gamma/2} \varphi_R) - (\langle v \rangle^{\gamma/2} \varphi_R) \right)^2 g^2 \right].
\]

It follows from the mean value theorem that for a suitable $v'_r = v + \tau (v' - v)$ with $0 < \tau < 1$,

\[
\left| \langle v \rangle^{\gamma/2} - \langle v' \rangle^{\gamma/2} \right| \lesssim \langle v \rangle^{\gamma/2} v_* \max \{1, \gamma^{-1}\} \theta.
\]

Therefore, we have

\[
C_\gamma (F, g) \geq 2^{-1-2|\gamma|} C_0 (F \chi_{B(R)}, \varphi_R (\langle v \rangle^{\gamma/2} g) - C_R \| F \|_{L^1} g \|^2_{L^2_{\gamma/2}}, \tag{B.10}
\]

for a positive constant $C_R \sim R^{\max \{2, 2(\gamma^{-1})\}}$. For a set $B(4R)$ we take a finite covering

\[
B(4R) \subset \bigcup_{v_j \in B(4R)} A_j, \quad A_j = \{ v \in \mathbb{R}^3 : |v - v_j| \leq \frac{r_0}{4} \}, \quad j \in \{1, 2, \ldots, N\}, \quad N = N(R, r_0).
\]

For each $A_j$ we choose a non-negative smooth function $\varphi_{A_j}$ which is 1 on $A_j$ and 0 on $\{|v - v_j| \geq r_0/2\}$. Note that

\[
\frac{r_0}{2} \leq |v - v_*| \leq 6R \text{ on supp } (\chi_{B_j(R, r_0)})_* \varphi_{A_j}.
\]

Then we have

\[
|v - v_*|^2 F_* (g' - g)^2 \geq r_0^2 (F \chi_{B_j(R, r_0)})_* \varphi_{A_j} (g' - g)^2
\]

\[
\geq r_0^2 (F \chi_{B_j(R, r_0)})_* \left[ \frac{1}{2} \left( (\langle v \rangle^{\gamma/2} \varphi_{A_j} g) - (\langle v \rangle^{\gamma/2} \varphi_{A_j} g) \right)^2 - \left( (\langle v \rangle^{\gamma/2} \varphi_{A_j}) - (\langle v \rangle^{\gamma/2} \varphi_{A_j}) \right)^2 g^2 \right].
\]

Since $\left| (\langle v \rangle^{\gamma/2} \varphi_{A_j})' - (\langle v \rangle^{\gamma/2} \varphi_{A_j}) \right| \lesssim R^{\gamma/2 + \max \{1, \gamma^{-1}\}} r_0^{-1} \theta$ if $|v_*| \leq R$, we obtain

\[
C_\gamma (F, g) \gtrsim r_0^2 C_0 (F \chi_{B_j(R, r_0)}, \varphi_{A_j} (\langle v \rangle^{\gamma/2} g) - C_{R, r_0} \| F \|_{L^1} g \|^2_{L^2_{\gamma/2}}, \tag{B.11}
\]

for a positive constant $C_{R, r_0} \sim R^{2+\max \{1, \gamma^{-1}\}} r_0^{-2}$. Writing $\varphi_0$ and $\varphi_j (j \geq 1)$ instead of $\varphi_R$ and $\varphi_{A_j}$, and summing up (B.10) and (B.11) we have

\[
C_\gamma (F, g) \gtrsim r_0^2 \sum_{j=0}^N C_0 (F \chi_{B_j(R, r_0)}, \varphi_j (\langle v \rangle^{\gamma/2} g) - C_{R, r_0} \| F \|_{L^1} g \|^2_{L^2_{\gamma/2}},
\]

where $\chi_{B_0(R, r_0)} = \chi_{B(R)}$. Apply \[B.2\] to each $C_0(\cdot, \cdot)$ term. Then

\[
\sum_{j=0}^N r_0^2 \langle (\varphi_j (\langle v \rangle^{\gamma/2} g) \rangle \leq r_0^{-2} C_\gamma (F, g) + C_F \| g \|^2_{L^2_{\gamma/2}}.
\]

Since for $G = \langle v \rangle^{\gamma/2} g$ we have

\[
(G - G')^2 \lesssim \left( \sum_{j=0}^N \langle \varphi_j \rangle^2 \right) (G - G')^2 \lesssim 2 \sum_{j=0}^N (\varphi_j G' - \varphi_j G)^2 + 2 \sum_{j=0}^N (\varphi_j' - \varphi_j)^2 G^2.
\]

Then we obtain \[B.7\] because

\[
|\varphi_j' - \varphi_j| \lesssim |(\nabla \varphi_j)(v'_r)| |v - v_*| \sin \frac{\theta}{2} \lesssim |(\nabla \varphi_j)(v'_r)| |v'_r - v_*| \theta \lesssim R r_0^{-1} \langle v_* \rangle \theta.
\]
Remark B.1. Taking $F = M$ in Lemma \[B.2\] and \[B.3\] we have the equivalence

$$J_1^{\Phi_{\nu}}(g) \sim J_1^{\Phi_{\nu}}((\nu)^{\gamma/2}g), \mod \|g\|_{L_2^{\gamma/2}}^2,$$

which is a sharper version of the formula given above \[9\], Lemma 2.17.

Lemma B.4. Let \[9\], Lemma 2.17.

\[\frac{\left|\left(Q(F, g), h\right)_{L^2}\right|}{\left|Q(F, g)\right|_{L_2}} \lesssim \|F\|_{L_2} \|g\|_{H^s} \|h\|_{H^s} \lesssim \|F\|_{L_2} (J_1^{\Phi_{\nu}}(g) + \|g\|_{L_2}^2)^{1/2} (J_1^{\Phi_{\nu}}(h) + \|h\|_{L_2}^2)^{1/2}.

Proof. We recall the decomposition of \[\text{B.6}\] such that

$$Q(F, g) = Q_R(F, g) + Q_{\mathcal{R}}(F, g)$$

$$= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) \Phi_R(F_\ast g - F_\ast g) \, d\sigma \, dv_\ast + \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos \theta) \Phi_{\mathcal{R}}(F_\ast g - F_\ast g) \, d\sigma \, dv_\ast.$$ 

It follows from \[4\] Proposition 2.1 (see also \[27\] Prop.6.11]) that

$$\left|\left(Q_R(F, g), h\right)_{L^2}\right| \lesssim \|F\|_{L_2} \|g\|_{H^s} \|h\|_{H^s} \lesssim \|F\|_{L_2} (J_1^{\Phi_{\nu}}(g) + \|g\|_{L_2}^2)^{1/2} (J_1^{\Phi_{\nu}}(h) + \|h\|_{L_2}^2)^{1/2}.$$ 

In view of \[\text{B.6}\], it suffices to consider only $\left(Q_{\mathcal{R}}(F, g), h\right)_{L^2}$, whose estimation is the almost same as in the proof of \[7\] Lemma 3.2. We write

$$A = (Q_{\mathcal{R}}(F, g), h) = \int\int\int b(\cos \theta) \Phi_{\mathcal{R}} F_\ast g(\nu(v')) - h(v) \, dv_\ast \, d\sigma$$

$$= \int\int\int b(\cos \theta) \Phi_{\mathcal{R}} F_\ast (g(v) - g(v'))(h(v') - h(v)) \, dv_\ast \, d\sigma$$

$$+ \int\int\int b(\cos \theta) \Phi_{\mathcal{R}} F_\ast g(v')(h(v') - h(w)) \, dv_\ast \, d\sigma$$

$$+ \int\int\int b(\cos \theta) \Phi_{\mathcal{R}} F_\ast g(v)(h(w) - h(v)) \, dv_\ast \, d\sigma$$

$$+ \int\int\int b(\cos \theta) \Phi_{\mathcal{R}} F_\ast (g(v') - g(v))(h(w) - h(v)) \, dv_\ast \, d\sigma$$

$$\Delta S + M_0 + R_1 + R_2,$$

where (see also Figure 1 in \[7\] given below)

$$w = v_\ast + \left(\cos^2 \frac{\theta}{2}\right)(v - v_\ast) = \frac{v' + v_\ast}{2} + \frac{v' - v_\ast}{2} \omega, \quad \cos \theta = \frac{v - v_\ast}{|v - v_\ast|} \cdot \sigma = \frac{v' - v_\ast}{|v' - v_\ast|} \cdot \omega.$$
and we let the inverse transformation be $v$.

It is not difficult to see that

$$|S| \leq \iint BF_{\ast}(g' - g)^2 d\sigma d\nu \times \iint BF_{\ast}(h' - h)^2 d\sigma d\nu$$

Using the Cauchy-Schwarz inequality, we have

$$\|F\|_{L^2_{\max\{2,2\}}(\sigma)}^2 \left( \int g'(v)^{\gamma/2} + \|g\|_{L_{\gamma/2}^2}^2 \right) \left( \int g(v)^{\gamma/2} + \|h\|_{L_{\gamma/2}^2}^2 \right)$$

by means of Lemma B.2.

For $M_0$, we write

$$M_0 = \int \Phi_{\gamma}(v - v_s) b(f(v) - h(v)) d\sigma d\nu,$$

Since

$$\Phi_{\gamma}(v) = \Phi_{\gamma}(v) - \Phi_{\gamma}(v - v_s),$$

it is not difficult to see that

$$|R_3| \lesssim \|F\|_{L^2_{\gamma/2}} \|g\|_{L^2_{\gamma/2}} \|h\|_{L^2_{\gamma/2}}.$$

For each fixed $(\sigma, v_s)$, we perform the change of variables $v \rightarrow v'$, as in [1]. Recall that

$$dv = \left| \frac{dv'}{dv} \right| dv' = \frac{2^{3-1}}{(\frac{v' - v_s}{|v' - v_s|} \cdot \sigma)^2} dv',$$

and we let the inverse transformation be $v' \rightarrow v = v_s(v', v_s)$. Hence

$$M_1 = \int_{R^3_{v_s}} F(v_s) \left( \int_{\mathbb{R}^3} \Phi_{\gamma}(v' - v_s) b(f(v) - h(v)) d\sigma \right) dv_s$$

$$= \int_{R^3_{v_s}} F(v_s) \left( \int_{\mathbb{R}^3} \Phi_{\gamma}(v' - v_s) b(2 \left( \frac{v' - v_s}{|v' - v_s|} \cdot \sigma \right)^2 - 1) \right.$$

$$\times g(v') h(v') - h(v_s) + \left. \left( \frac{v' - v_s}{|v' - v_s|} \cdot \sigma \right)^2 (v_s - v_s) \right) \left( \frac{2^{3-1}}{(\frac{v' - v_s}{|v' - v_s|} \cdot \sigma)^2} d\nu' d\sigma \right) dv_s.$$
Take polar coordinates $\sigma = (\vartheta, \phi) \in [0, \pi] \times [0, 2\pi]$ with pole $v' - v_*$. Then we have

$$
\int_{S^2 \setminus \{\sigma = 1/\sqrt{2}\}} \cdots - h(v_*) + \left( \frac{v' - v_*}{|v' - v_*|} \cdot \sigma \right)^2 (v_\sigma - v_*) \cdot d\sigma
$$

$$
= \int_{0}^{2\pi} d\phi \int_{0}^{\pi/4} \cdots h(v_* + \cos^2 \vartheta (v(\vartheta, \phi) - v_*)) \cdot \frac{2^{3-1} \sin \vartheta d\vartheta}{\cos^2 \vartheta}
$$

because $\vartheta = \theta/2$ and $\cos \theta = \frac{v' - v_*}{|v' - v_*|} : \omega$ (see Figure 1). Therefore, writing $v$ and $\sigma$ instead of $v'$ and $\omega$, we have

$$
M_1 = \int b\Phi \overline{F}_* g(v) (h(v) - h(v')) \frac{d\sigma}{\cos^3(\theta/2)} dv dv_* = -A + R_4,
$$

where

$$
R_4 = \int b\Phi \overline{F}_* g(v) (h(v) - h(v')) \left( \frac{1}{\cos^3(\theta/2)} - 1 \right) dv dv_* d\sigma.
$$

It is easy to check that

$$
|R_4| \lesssim \|F\|_{L^1} \|g\|_{L^2_{\gamma/2}} \|h\|_{L^2_{\gamma/2}}.
$$

Now we concentrate on the term

$$
R_1 = \int b\Phi \overline{F}_* g(v) (h(w) - h(v)) dv dv_* d\sigma,
$$

where

$$
w = v_* + \frac{1}{2} \left( 1 + \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) (v - v_*).
$$

Note that

$$
dw = (\cos^2(\theta/2))^3 dv, \quad |w - v| = |v - v_*| \sin^2(\theta/2).
$$

Take the dyadic decomposition

$$
B = \sum_{\ell=1}^{\infty} |v - v_*| \varphi(2^{-\ell} |v - v_*|) b(\cos \theta).
$$

We choose $\psi$ such that $\varphi \subset \subset \psi$. Writing $\varphi_\ell(z) = \varphi(2^{-\ell} z)$ and $\psi_\ell(z) = (2^{-\ell} |z|) \psi(2^{-\ell} z)$, we obtain by the change of variables $v \to v_* + z$,

$$
\tilde{R}_1(v_*) = \sum_{\ell \geq R} 2^{2\ell} \int_{0}^{\pi/2} b(\cos \theta) \sin \theta \int_{\mathbb{R}^2} \varphi_\ell(z) \left( h(v_* + \cos^2 \frac{\theta}{2} z) - h(v_* + z) \right) \psi_\ell(z) g(v_* + z) dz d\theta
$$

$$
\Delta \sum_{\ell} 2^{2\ell} J_\ell.
$$

We divide

$$
J_\ell = \int_{0}^{\pi/2} b(\cos \theta) \sin \theta \int_{\mathbb{R}^2} \left( \varphi_\ell(z) - \varphi_\ell(\cos^2 \frac{\theta}{2} z) \right) h(v_* + \cos^2 \frac{\theta}{2} z) \psi_\ell(z) g(v_* + z) dz d\theta
$$

$$
+ \int_{0}^{\pi/2} b(\cos \theta) \sin \theta \int_{\mathbb{R}^2} \left( \varphi_\ell(\cos^2 \frac{\theta}{2} z) h(v_* + \cos^2 \frac{\theta}{2} z) - \varphi_\ell(z) h(v_* + z) \right) \psi_\ell(z) g(v_* + z) dz d\theta
$$

$$
\Delta J_\ell^{(1)} + J_\ell^{(2)}.
$$
Let $H_\ell(z; v_\ast) = \varphi_\ell(z) h(v_\ast + z)$, $G_\ell(z; v_\ast) = \psi_\ell(z) g(v_\ast + z)$ and denote the Fourier transforms of $G_\ell, H_\ell$ with respect to $z$ by $\widehat{G}_\ell(\xi; v_\ast), \widehat{H}_\ell(\xi; v_\ast)$, respectively. Then Plancherel formula gives
\[
J^{(2)}_\ell = \int_0^{\pi/2} b(\cos \theta) \sin \theta \left( \int_{\mathbb{R}^2_\ell} \left( \frac{1}{(\cos^2 \frac{\theta}{2})^3} \widehat{H}_\ell\left( \frac{\xi}{\cos^2 \frac{\theta}{2}}; v_\ast \right) - \widehat{H}_\ell(\xi; v_\ast) \right) \overline{\widehat{G}_\ell}(\xi; v_\ast) d\xi \right) d\theta
= \int_0^{\pi/2} b(\cos \theta) \sin \theta \left( \int_{\mathbb{R}^2_\ell} \left( \frac{1}{(\cos^2 \frac{\theta}{2})^3} - 1 \right) \left( \widehat{H}_\ell\left( \frac{\xi}{\cos^2 \frac{\theta}{2}}; v_\ast \right) G_\ell(\xi; v_\ast) d\xi \right) d\theta
+ \int_0^{\pi/2} b(\cos \theta) \sin \theta \left( \int_{\mathbb{R}^2_\ell} \left( \widehat{H}_\ell\left( \frac{\xi}{\cos^2 \frac{\theta}{2}}; v_\ast \right) - \widehat{H}_\ell(\xi; v_\ast) \right) \overline{\widehat{G}_\ell}(\xi; v_\ast) d\xi \right) d\theta
= J^{(2,1)}_\ell + J^{(2,2)}_\ell.
\]

It is easy to see that
\[
|J^{(2,1)}_\ell| \lesssim \|\varphi_\ell(\cdot - v_\ast) h\|_{L^2} \|\psi_\ell(\cdot - v_\ast) g\|_{L^2}.
\] (B.12)

A similar estimate is also true for $J^{(1)}_\ell$, by replacing $\varphi_\ell$ by $\tilde{\varphi}_\ell$ which is defined from a suitable $\tilde{\varphi}$ satisfying $\varphi \subset \subset \tilde{\varphi}$. Write
\[
J^{(2,2)}_\ell = \int_{\mathbb{R}^2_\ell} \overline{G}_\ell(\xi; v_\ast) \left( \int_0^{2^{-\ell/2}(\xi)^{-1/2}} b(\cos \theta) \sin \theta \tan^{\frac{\theta}{2}} \int_0^1 \xi \cdot \nabla_{\xi} \widehat{H}_\ell(\xi + \tau \tan^{\frac{\theta}{2}} \xi; v_\ast) d\tau \right) d\xi
+ \int_{\mathbb{R}^2_\ell} \overline{G}_\ell(\xi; v_\ast) \left( \int_0^{\pi/2} b(\cos \theta) \sin \theta \left( \widehat{H}_\ell(\xi; v_\ast) - \widehat{H}_\ell(\xi; v_\ast) \right) d\xi \right) d\xi
= A_\ell + B_\ell.
\]

By Cauchy-Schwarz inequality, we have
\[
|A_\ell|^2 \lesssim \int_{\mathbb{R}^2_\ell} |\xi|^{1+s} |\overline{G}_\ell(\xi; v_\ast)|^2 \int_0^{2^{-\ell/2}(\xi)^{-1/2}} b(\cos \theta) \sin \theta \tan^{\frac{\theta}{2}} d\theta d\xi
\times \int_{\mathbb{R}^2_\ell} |\overline{\nabla_{\xi} \widehat{H}_\ell(\xi; v_\ast)}|^2 \int_0^{2^{-\ell/2}(\xi)^{-1/2}} b(\cos \theta) \sin \theta \tan^{\frac{\theta}{2}} d\theta d\xi ,
\]
where, in the second factor, we have used the change of variable
\[
\xi \to (1 + \tau \tan^{\frac{\theta}{2}}) \xi
\]
after exchanging $d\theta d\xi$ by $d\xi d\theta$. Therefore, we get
\[
|A_\ell|^2 \lesssim \left\{ \left( \int_{\mathbb{R}^2_\ell} 2^{2s\ell} |\overline{\nabla_{\xi} \widehat{H}_\ell(\xi; v_\ast)}|^2 d\xi \right) \left( \int_{\mathbb{R}^2_\ell} |\xi|^{1+s} |\overline{G}_\ell(\xi; v_\ast)|^2 d\xi \right), \right. \right.
\]
\[
\left. \left. \left( \int_{\mathbb{R}^2_\ell} |\overline{\nabla_{\xi} \widehat{H}_\ell(\xi; v_\ast)}|^2 d\xi \right) \left( \int_{\mathbb{R}^2_\ell} 2^{2s\ell} |\overline{G}_\ell(\xi; v_\ast)|^2 d\xi \right) \right\}. \right\}
\] (B.13)

On the other hand, we have
\[
|B_\ell|^2 \lesssim \left\{ \left( \int_{\mathbb{R}^2_\ell} 2^{2s\ell} |\overline{G}_\ell(\xi; v_\ast)|^2 d\xi \right) \left( \int_{\mathbb{R}^2_\ell} |\xi|^{s} |\overline{H}_\ell(\xi; v_\ast)|^2 d\xi \right), \right. \right.
\]
\[
\left. \left. \left( \int_{\mathbb{R}^2_\ell} |\xi|^{s} |\overline{G}_\ell(\xi; v_\ast)|^2 d\xi \right) \left( \int_{\mathbb{R}^2_\ell} 2^{2s\ell} |\overline{H}_\ell(\xi; v_\ast)|^2 d\xi \right) \right\}. \right\}
\] (B.14)
Summing up the above estimates of the first case we obtain
\[ \sum 2^{\ell}|J_\ell| \lesssim \sum (v_\ell)^{\gamma+\epsilon} \| \tilde{\varphi}_\ell (v)^{\gamma+\epsilon/2} g(v) \|_{L^2} \| \tilde{\varphi}_\ell (D)^s h(v) \|_{L^2} \]
\[ \lesssim (v_\ell)^{\gamma+\epsilon} \left( \sum \| \tilde{\varphi}_\ell (v)^{\gamma+\epsilon/2} g(v) \|_{L^2}^2 \right)^{1/2} \left( \sum \| \tilde{\varphi}_\ell (D)^s h(v) \|_{L^2}^2 \right)^{1/2} \]
because
\[ \sum (\tilde{\varphi}_\ell F, \tilde{\varphi}_\ell F)_{L^2} = \left( \sum (\tilde{\varphi}_\ell)^2 F, F \right)_{L^2} \lesssim \| F \|_{L^2}^2. \]
Here it should be noted that the commutator
\[ \| [(D)^s, \varphi_\ell] g \|_{L^2} \lesssim 2^{-\ell} \| g \|_{L^2} \]
is harmless to the above summation process. Since \( \| h \|_{L^2_{\gamma/2}} \lesssim J_0^\Phi (\langle v \rangle)^{\gamma/2} h + \| h \|_{L^2_{\gamma/2}} \), we obtain
\[ |R_1| \lesssim \| F \|_{L^1_{\gamma/2}} \| g \|_{L^2_{\gamma/2}} \left( J_0^\Phi (\langle v \rangle)^{\gamma/2} h + \| h \|_{L^2_{\gamma/2}} \right)^{1/2}. \]
Another case for \( R_1 \) is now obvious.

As for \( R_2 \), it follows from Cauchy-Schwarz inequality that
\[ |R_2|^2 \leq \left( \int BF_\ell (g(v') - g(v))^2 dv dv' d\sigma \right) \left( \int b \Phi F_\ell (h(w) - h(v))^2 dv dw d\sigma \right). \]
The first factor is estimated by using Lemma B.2. Note that the second factor is estimated above from
\[ 2 \left( \int b \Phi F_\ell (h(w) - h(v'))^2 dv dw d\sigma + \int b \Phi F_\ell (h(v') - h(v))^2 dv dv' d\sigma \right). \]
The first term can be handled in the same way as for \( M_0 \), regarding \( (w, v') \) as \( (v', v) \), and it is estimated by \( J_1^\Phi (\langle v \rangle)^{\gamma/2} h + \| h \|_{L^2_{\gamma/2}} \) up to a constant factor. And this completes the proof of the lemma. \( \square \)

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