UNIQUE SOLVABILITY OF ELLIPTIC PROBLEMS ASSOCIATED WITH TWO-PHASE INCOMPRESSIBLE FLOWS IN UNBOUNDED DOMAINS

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Abstract. This paper shows the unique solvability of elliptic problems associated with two-phase incompressible flows, which are governed by the two-phase Navier-Stokes equations with a sharp moving interface, in unbounded domains such as the whole space separated by a compact interface and the whole space separated by a non-compact interface. As a by-product, we obtain the Helmholtz-Weyl decomposition for two-phase incompressible flows.

1. Introduction and main results.

1.1. Introduction. Let Ω+ be a bounded domain in the N-dimensional Euclidean space \( \mathbb{R}^N \), \( N \geq 2 \), with boundary \( \Sigma \), and let \( \Omega_- = \mathbb{R}^N \setminus (\Omega_+ \cup \Sigma) \). Let us define \( \rho = \rho_+ \mathbb{1}_{\Omega_+} + \rho_- \mathbb{1}_{\Omega_-} \) for positive constants \( \rho_\pm \), where \( \mathbb{1}_A \) is the indicator function of \( A \subset \mathbb{R}^N \). We set for an open set \( G \) of \( \mathbb{R}^N \) and for \( q \in (1, \infty) \)

\[
\tilde{H}_q^1(G) = \{ f \in L_{q,\text{loc}}(G) : \nabla f \in L_q(G)^N \},
\]

and define a solenoidal space \( J_q(\mathbb{R}^N \setminus \Sigma) \) and a space \( G_q(\mathbb{R}^N \setminus \Sigma) \) as follows:

\[
J_q(\mathbb{R}^N \setminus \Sigma) = \{ u \in L_q(\mathbb{R}^N \setminus \Sigma)^N : (u, \nabla \varphi)_{\mathbb{R}^N \setminus \Sigma} = 0 \text{ for any } \varphi \in \tilde{H}_q^1(\mathbb{R}^N) \},
\]

\[
G_q(\mathbb{R}^N \setminus \Sigma) = \{ v \in L_q(\mathbb{R}^N \setminus \Sigma)^N : v = \rho^{-1} \nabla \psi \text{ for some } \psi \in \tilde{H}_q^1(\mathbb{R}^N) \},
\]

where \( q' = q/(q-1) \) and

\[
(u, \nabla \varphi)_{\mathbb{R}^N \setminus \Sigma} = \int_{\mathbb{R}^N \setminus \Sigma} u(x) \cdot \nabla \varphi(x) \, dx.
\]

Here the central dot denotes the scalar product of \( \mathbb{R}^N \).

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Let \( f \in L_q(\mathbb{R}^N \setminus \Sigma)^N \). One of the purposes of this paper is to show the unique solvability of the following weak elliptic problem: Find \( u \in \tilde{H}_q^1(\mathbb{R}^N) \) such that
\[
(\rho^{-1}\nabla u, \nabla \varphi)_{\mathbb{R}^N \setminus \Sigma} = (f, \nabla \varphi)_{\mathbb{R}^N \setminus \Sigma} \quad \text{for any } \varphi \in \tilde{H}_q^1(\mathbb{R}^N). \tag{1.1}
\]
This weak elliptic problem arises from the study of two-phase incompressible flows governed by the two-phase Navier-Stokes equations with a sharp moving interface. The momentum equation of the two-phase Navier-Stokes equations is linearized as
\[
\rho \partial_t u - \mu \Delta u + \nabla p = g \quad \text{in } \mathbb{R}^N \setminus \Sigma,
\]
where \( g \) is a given function and \( \mu = \mu_+ \mathbb{1}_{\Omega_+} + \mu_- \mathbb{1}_{\Omega_-} \) for positive constants \( \mu_\pm \) describing the viscosity coefficients, and then the unique solvability of (1.1) enables us to eliminate the pressure \( p \) from the linearized equation. This elimination of pressure plays an important role in applications such as the generation of analytic \( C_0 \)-semigroups, the maximal regularity, and the local and global solvability of the two-phase Navier-Stokes equations (cf. [18, 14, 20]). Another very important application of the unique solvability of (1.1) is a two-phase version of the Helmholtz-Weyl decomposition as follows:
\[
L_q(\mathbb{R}^N \setminus \Sigma)^N = J_q(\mathbb{R}^N \setminus \Sigma) \oplus G_q(\mathbb{R}^N \setminus \Sigma), \tag{1.2}
\]
where \( \oplus \) denotes the direct sum. Note that this decomposition is equivalent to the unique solvability of (1.1).

Prüss and Simonett [18, Proposition 8.6.2] proved the unique solvability of a weak elliptic problem associated with two-phase incompressible flows in the case where \( \Omega_\pm \) are both bounded domains, while the case of unbounded domains is not very well known to the best of our knowledge. This motivates us to study the unique solvability of (1.1). Furthermore, as examples of unbounded domains with non-compact interface \( \Sigma \), we also treat the whole space with a flat interface and the whole space with a bent interface in the present paper. The former is in Subsection 3.3 below, while the latter is in Subsection 4.2 below.

At this point, we introduce a short history of one-phase case for the unique solvability of weak elliptic problems and the Helmholtz-Weyl decomposition.

We first introduce the classical weak Neumann problem: Find \( u \in \tilde{H}_q^1(D) \) such that
\[
(\nabla u, \nabla \varphi)_D = (f, \nabla \varphi)_D \quad \text{for any } \varphi \in \tilde{H}_q^1(D), \tag{1.3}
\]
where \( f \in L_q(D)^N \) and \( D \) is a domain in \( \mathbb{R}^N \). It is well known that the unique solvability of (1.3) is equivalent to the following Helmholtz-Weyl decomposition:
\[
L_q(D)^N = L_{q,\sigma}(D) \oplus G_q(D), \tag{1.4}
\]
where \( L_{q,\sigma}(D) \) and \( G_q(D) \) are given by
\[
L_{q,\sigma}(D) = C_{0,\sigma}^\infty(D) = \{ u \in C_0^\infty(D)^N : \text{div } u = 0 \text{ in } D \},
\]
\[
G_q(D) = \{ v \in L_q(D)^N : v = \nabla \psi \text{ for some } \psi \in \tilde{H}_q^1(D) \}.
\]

The investigation of (1.4) (or (1.3)) can be traced back to Weyl [31]. Although (1.4) holds for any domain \( D \) in \( \mathbb{R}^N \) when \( q = 2 \) (cf. [28]), the general \( L_q \)-framework is more involved. According to [29, 10, 16, 26, 8, 17, 9, 1, 12], we can conclude that (1.4) is valid for any \( q \in (1, \infty) \) whenever \( D \) is \( \mathbb{R}^N \) itself, the half space, a bounded or an exterior domain in \( \mathbb{R}^N \) with smooth boundary, a perturbed half space, a flat layer, an aperture domain, or a bounded convex domain. One also knows in [5] that (1.4) holds only when \( 3/2 - \varepsilon < q < 3 + \varepsilon \) for some \( \varepsilon = \varepsilon(D) > 0 \), assuming \( D \) is
a bounded Lipschitz domain in $\mathbb{R}^N$ with $N \geq 3$. Note that (1.4) may fail for some unbounded domain and $q \in (1, \infty)$, which is pointed out in [15]. One however has a chance to obtain (1.4) for any $q \in (1, \infty)$ and for general unbounded domains, called uniform $C^1$ domains, by introducing mixed $L_q$-spaces due to [6, 7].

The classical (1.4) is widely used for one-phase problems with non-slip boundary condition. On the other hand, to handle one-phase incompressible flows with a free surface, we make use of the weak Dirichlet problem as follows: Let $\Gamma$ be a connected component of the boundary of $D$ and

$$\hat{H}^1_{q,\Gamma}(D) = \{ u \in \hat{H}^1_q(D) : u = 0 \text{ on } \Gamma \} \quad (1 < q < \infty).$$

One then says that the weak Dirichlet problem is uniquely solvable in $\hat{H}^1_{q,\Gamma}(D)$ if and only if for any $f \in L_q(D)$ there is a unique solution $u \in \hat{H}^1_{q,\Gamma}(D)$ to

$$\langle \nabla u, \nabla \varphi \rangle_D = \langle f, \nabla \varphi \rangle_D \quad \text{for any } \varphi \in \hat{H}^1_{q,\Gamma}(D)$$

and there holds the estimate: $\| \nabla u \|_{L_q(D)} \leq C \| f \|_{L_q(D)}$ for some positive constant $C$ independent of $u$, $f$, and $\varphi$. In [27, 23, 18], it is proved that the weak Dirichlet problem is uniquely solvable in $\hat{H}^1_{q,\Gamma}(D)$ when $D$ is $\mathbb{R}^N$, the half space, a bounded or an exterior domain with smooth boundary (cf. also [2, 19]).

Finally, we introduce the strong elliptic problem associated with (1.1). For an open set $G$ of $\mathbb{R}^N$ and $q \in (1, \infty)$, one sets

$$\hat{H}^2_q(G) = \{ f \in L_{1,\text{loc}}(G) : \nabla f \in \hat{H}^1_q(G)^N \}.$$ 

The strong elliptic problem is then stated as follows: Find $v_\pm \in \hat{H}^2_q(\Omega_\pm) \cap \hat{H}^1_q(\Omega_\pm)$ such that

$$\begin{cases}
\Delta v_\pm = \text{div} f_\pm & \text{in } \Omega_\pm, \\
\rho_+ v_+ = \rho_- v_- & \text{on } \Sigma, \\
n \cdot \nabla (v_+ - v_-) = n \cdot (f_+ - f_-) & \text{on } \Sigma,
\end{cases} \quad (1.5)$$

where $n$ is a unit normal vector on $\Sigma$ pointing from $\Omega_+$ into $\Omega_-$. Throughout this paper, $n$ is seen as an $N$-vector of function defined on $\mathbb{R}^N$ (cf. [21, Corollary A.3] and Assumption 1 below). In this paper, we first prove the unique solvability of (1.5), and then we prove the unique solvability of (1.1) by using the result of the strong elliptic problem. This approach is also applied to the problems with non-compact interfaces in Sections 3 and 4 below.

**Notation.** Let $G$ be an open set in $\mathbb{R}^N$, and let $u = u(x)$ and $v = v(x) = (v_1(x), \ldots, v_N(x))^T$ be respectively a scalar-valued function on $G$ and a vector-valued function on $G$, where $x = (x_1, \ldots, x_N)$. Then, for $\partial_j = \partial / \partial x_j$,

$$\nabla u = (\partial_1 u, \ldots, \partial_N u)^T, \quad \nabla v = \{ \partial_j v_k : j, k = 1, \ldots, N \},$$

and also $\nabla v = \sum_{j=1}^N \partial_j v_k$. Furthermore, for $u = u(x) = (u_1(x), \ldots, u_N(x))^T$ and $v = v(x)$ defined on $G$,

$$(u, v)_G = \int_G u(x)v(x) \, dx, \quad (u, v)_G = \int_G u(x) \cdot v(x) \, dx = \sum_{j=1}^N \int_G u_j(x)v_j(x) \, dx.$$

Let $X$ be a Banach space. Then $X^m$, $m \geq 2$, denotes the $m$-product of $X$, while the norm of $X^m$ is usually denoted by $\| \cdot \|_X$ instead of $\| \cdot \|_{X^m}$ for the sake

$^{1}$ $M^T$ denotes the transpose of $M$. 

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Suppose that Assumption 1 holds and \( \rho_{\pm} \) are positive constants. Let \( q \in (1, \infty) \) and \( q' = q/(q-1) \) with \( \max(q, q') \leq r \).

1. Existence. Let \( f \in E_q(\Omega_{\pm}) \) with \( n \cdot f \in H^1_q(\Omega_{\pm}) \). Then the strong elliptic problem (1.5) admits solutions \( v_{\pm} \in \tilde{H}^1_q(\Omega_{\pm}) \cap \tilde{H}^2_q(\Omega_{\pm}) \) satisfying

\[
\| \nabla^2 v_{\pm} \|_{L^q(\Omega_{\pm})} \leq C \sum_{s \in \{+, -\}} \left( \| f_s \|_{E_q(\Omega_s)} + \| n \cdot f \|_{H^1_q(\Omega_s)} \right), \tag{1.6}
\]

\[
\| \nabla v_{\pm} \|_{L^q(\Omega_{\pm})} \leq C \sum_{s \in \{+, -\}} \| f_s \|_{L^q(\Omega_s)}, \tag{1.7}
\]

with some positive constant \( C = C(N, q, r, \rho_{\pm}, \rho_{\pm}) \). Additionally, if \( n \cdot (f_+ - f_-) = 0 \) on \( \Sigma \), then \( v_{\pm} \) satisfy

\[
\| \nabla^2 v_{\pm} \|_{L^q(\Omega_{\pm})} \leq C \sum_{s \in \{+, -\}} \| f_s \|_{E_q(\Omega_s)} \tag{1.8}
\]

for some positive constant \( C = C(N, q, r, \rho_{\pm}, \rho_{\pm}) \).

2. Uniqueness. If \( v_{\pm} \in \tilde{H}^1_q(\Omega_{\pm}) \cap \tilde{H}^2_q(\Omega_{\pm}) \) satisfies

\[
\Delta v_{\pm} = 0 \text{ in } \Omega_{\pm}, \quad \rho_{\pm} v_+ = \rho_{\pm} v_- \text{ and } n \cdot \nabla(v_+ - v_-) = 0 \text{ on } \Sigma,
\]

then \( v_{\pm} = \rho_{\pm}^{-1} c \) for some constant \( c \).

For the weak elliptic problem (1.1), our main result reads as
Theorem 1.2. Suppose that Assumption 1 holds and \( \rho = \rho_+ \mathbb{1}_{\Omega_+} + \rho_- \mathbb{1}_{\Omega_-} \) for positive constants \( \rho_\pm \). Let \( q \in (1, \infty) \) and \( q' = q/(q-1) \) with \( \max(q, q') \leq r \).

(1) Existence. Let \( f \in L_q(\mathbb{R}^N \setminus \Sigma)^N \). Then the weak elliptic problem \((1.1)\) admits a solution \( u \in \tilde{H}^1_q(\mathbb{R}^N) \) satisfying the estimate: \( \|\nabla u\|_{L_q(\mathbb{R}^N)} \leq C\|f\|_{L_q(\mathbb{R}^N \setminus \Sigma)} \) for some positive constant \( C = C(N, q, r, \rho_+, \rho_-) \).

(2) Uniqueness. If \( u \in \tilde{H}^1_q(\mathbb{R}^N) \) satisfies
\[
(\rho^{-1} \nabla u, \nabla \varphi)_{\mathbb{R}^N \setminus \Sigma} = 0 \quad \text{for any } \varphi \in \tilde{H}^1_q(\mathbb{R}^N),
\]
then \( u = c \) for some constant \( c \).

Remark 1. In this paper, we construct the solution of the weak elliptic problem \((1.1)\) by means of the solution to the strong elliptic problem \((1.5)\). Theorem 1.2, therefore, requires the same regularity of \( \Sigma \) as Theorem 1.1. It may however be relaxed to \( C^1 \) if we construct more directly the solution of \((1.1)\) in a different way than this paper.

Furthermore, we have by Theorem 1.2 the two-phase version of the Helmholtz-Weyl decomposition as follows:

Theorem 1.3. Suppose that the same assumption as in Theorem 1.2 holds. Then the decomposition \((1.2)\) holds.

This paper is organized as follows: The next section introduces function spaces and lemmas used in Section 3. Section 3 treats strong elliptic problems with and without resolvent parameter \( \lambda \), and also a weak elliptic problem in the whole space with a flat interface. Section 4 treats strong and weak elliptic problems similar to Section 3 in the whole space with a bent interface, and proves the unique solvability of the problems by using results obtained in Section 3. In Section 5, we first introduce the unique solvability of a strong elliptic problem with resolvent parameter \( \lambda \) in a bounded domain, which is proved by the standard localization technique together with a result given in Section 4. Next, we prove the unique solvability of the strong elliptic problem without \( \lambda \) by using the result with \( \lambda \) and the Riesz-Schauder theory. Section 6 proves our main results as stated above, i.e. Theorems 1.1 and 1.2, by the main result of Section 5 with a cut-off technique.
The following lemma is proved in [25, Lemma 5.4].

Lemma 2.2. Let \( q \in (1, \infty) \). Assume that \( m(\xi') \) is defined on \( \mathbb{R}^{N-1} \setminus \{0\} \), which is many times differentiable with respect to \( \xi' \) and satisfies for any multi-index \( \alpha' \in \mathbb{N}^{N-1} \) and \( \xi' \in \mathbb{R}^{N-1} \setminus \{0\} \)

\[
|\partial_{\xi'}^\alpha m(\xi')| \leq c(\alpha')|\xi'|^{-|\alpha'|},
\]

where \( c(\alpha') \) is a positive constant independent of \( \xi' \). Furthermore, define the operators \( M_j \) \((j = 1, 2)\) by the formulas:

\[
[M_1 f_\pm](x) = \int_0^\infty F_{\xi'}^{-1} \left[ m(\xi')|\xi'| e^{-|\xi'|((x_N + y_N) f_\phi(\xi', \pm y_N))} \right] (x') \, dy_N
\]

for \( x = (x', x_N) \in \mathbb{R}^N_+ \); \( M_2 f_\pm \) as above.

Let \( \text{sign}(a) \) be the sign function of \( a \), that is, \( \text{sign}(a) = 1 \) when \( a > 0 \), \( \text{sign}(a) = -1 \) when \( a < 0 \), and \( \text{sign}(a) = 0 \) when \( a = 0 \). Then we have

Lemma 2.3. Let \( \xi' \in \mathbb{R}^{N-1} \setminus \{0\} \) and \( \varepsilon > 0 \). Then

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\varepsilon|\xi'|} \frac{|\xi'| e^{i\varepsilon \xi_N}}{|\xi'|^2} \, d\xi_N = \frac{\text{sign}(a)}{2} e^{-|\xi'| |a|}. \tag{2.2}
\]

Proof. This formula follows from the residue theorem immediately, so that the detailed proof may be omitted.

3. Problems in the whole space with flat interface. Let us introduce the flat interface:

\[
\mathbb{R}^N_0 = \{ x = (x', x_N) : x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}, x_N = 0 \}.
\]

This section mainly considers the following strong elliptic problem:

\[
\begin{cases}
\Delta v_\pm = \text{div} f_\pm & \text{in } \mathbb{R}^{N}_+, \\
\rho_+ v_+ = \rho_- v_- & \text{on } \mathbb{R}^{N}_0, \\
\partial_N v_+ - \partial_N v_- = f_+ - f_- & \text{on } \mathbb{R}^{N}_0, \tag{3.1}
\end{cases}
\]

where the constants \( \rho_\pm > 0 \). Concerning (3.1), we prove the following result.

Theorem 3.1. Let \( q \in (1, \infty) \).

1. Existence. Let \( f_\pm = (f_{\pm 1}, \ldots, f_{\pm N})^T \in L_q(\mathbb{R}^{N}_\pm) \). Then the strong problem (3.1) admits solutions \( v_\pm \in H^1_q(\mathbb{R}^{N}_\pm) \setminus H^1_q(\mathbb{R}^{N}_\pm) \) satisfying

\[
\sum_{s \in \{+,-\}} \|\nabla^2 v_s\|_{L_q(\mathbb{R}^N)} \leq C_2 \sum_{s \in \{+,-\}} \|\text{div} f_s, \nabla f_s\|_{L_q(\mathbb{R}^N)}, \tag{3.2}
\]
Let us emphasize that the solution of (3.1) is unique up to some constant from Theorem 3.1. However, when we study the problem
\[
\begin{aligned}
\rho_\pm \lambda v_\pm - \Delta v_\pm &= -\text{div} f_\pm + g_\pm & \text{in } \mathbb{R}^N_\pm, \\
\rho_\pm v_\pm &= \rho_\mp v_\mp & \text{on } \mathbb{R}^N_0, \\
\partial_N v_+ - \partial_N v_- &= f_{\mp} - f_{\pm} + h_+ - h_- & \text{on } \mathbb{R}^N_0,
\end{aligned}
\]  
with the parameter \( \lambda \) varying in the sector
\[\Sigma_{\sigma,\lambda_0} = \{z \in \mathbb{C} : |\arg z| < \pi - \sigma, |\lambda| > \lambda_0\} \quad (0 < \sigma < \pi/2, \lambda_0 \geq 0),\]
(3.6) has the standard uniqueness by the following theorem.

**Theorem 3.2.** Let \( \sigma \in (0, \pi/2) \) and \( q \in (1, \infty) \). Then, for any
\[f_\pm = (f_{\pm 1}, \ldots, f_{\pm N})^T \in L_q(\mathbb{R}^N_\pm), \quad g_\pm \in L_q(\mathbb{R}^N_\pm), \quad h_\pm \in H^1_q(\mathbb{R}^N_\pm)\]
and for any \( \lambda \in \Sigma_{\sigma,0} \), the strong elliptic problem (3.6) admits unique solutions \( v_\pm \in H^2_q(\mathbb{R}^N_\pm) \). In addition, the solutions \( v_\pm \) satisfy
\[
\sum_{s \in \{+, -\}} \| (\lambda v_s, \lambda^{1/2} \nabla v_s, \nabla^2 v_s) \|_{L_q(\mathbb{R}^N_\pm)} 
\leq C_1 \sum_{s \in \{+, -\}} \| (\text{div} f_s, g_s, \lambda^{1/2} f_{sN}, \nabla f_{sN}, \lambda^{1/2} h_s, \nabla h_s) \|_{L_q(\mathbb{R}^N_\pm)},
\]
and also
\[
\sum_{s \in \{+, -\}} \| (\lambda^{1/2} v_s, \nabla v_s) \|_{L_q(\mathbb{R}^N_\pm)} 
\leq C_1 \sum_{s \in \{+, -\}} \left( \| f_s \|_{L_q(\mathbb{R}^N_\pm)} + |\lambda|^{-1/2} \| g_s \|_{L_q(\mathbb{R}^N_\pm)} + \| h_s \|_{L_q(\mathbb{R}^N_\pm)} + |\lambda|^{-1/2} \| \nabla h_s \|_{L_q(\mathbb{R}^N_\pm)} \right),
\]
where \( C_1 = C_1(N, q, \sigma, \rho_+, \rho_-) \) is a positive constant independent of \( \lambda \). Additionally, if \( f_{\pm} - f_{\mp} = 0 \) on \( \mathbb{R}^N_0 \), then \( v_\pm \) satisfy
\[
\sum_{s \in \{+, -\}} \| (\lambda v_s, \lambda^{1/2} \nabla v_s, \nabla^2 v_s) \|_{L_q(\mathbb{R}^N_\pm)} 
\leq C_1 \sum_{s \in \{+, -\}} \| (\text{div} f_s, g_s, \lambda^{1/2} h_s, \nabla h_s) \|_{L_q(\mathbb{R}^N_\pm)}.
\]

Theorem 3.2 is essentially proved in the appendix of [13] (cf. also [24, Appendix B]), so that the detailed proof may be omitted. In the rest of this section, we focus on the proof and the application of Theorem 3.1.
3.1. Auxiliary problem. Let us consider

\[ \Delta U_\pm = \text{div} f_\pm \quad \text{in} \quad \mathbb{R}^N_\pm. \tag{3.7} \]

Concerning the equations (3.7), we prove

**Lemma 3.3.** Let \( q \in (1, \infty) \). Then, for any \( f_\pm = (f_{\pm 1}, \ldots, f_{\pm N})^T \in C_0^\infty(\mathbb{R}^N) \), there exist \( U_\pm \in \dot{H}_q^1(\mathbb{R}^N) \cap \dot{H}_q^2(\mathbb{R}^N) \), satisfying (3.7), such that the following assertions hold.

1. There hold the estimates:

\[ \| \nabla^2 U_\pm \|_{L_q(\mathbb{R}^N)} \leq C \| \text{div} f_\pm \|_{L_q(\mathbb{R}^N)} \quad \text{and} \quad \| \nabla U_\pm \|_{L_q(\mathbb{R}^N)} \leq C \| f_\pm \|_{L_q(\mathbb{R}^N)}, \]

with a positive constant \( C = C(N, q, \rho_+, \rho_-) \).

2. The traces of \( \partial_N U_\pm \) on \( \mathbb{R}^N_0 \) are given by

\[ \partial_N U_\pm(x', 0) = \sum_{j=1}^{N-1} \int_0^\infty \mathcal{F}_\xi^{-1} \left[ i \xi_0 e^{-|\xi|^q y_0} \mathcal{F}_\xi (\xi, \pm y_N) \right] (x') dy_N, \]

where the symbols \( \mathcal{F}_\xi^{-1} \) and \( \tilde{\cdot} \) are defined as (2.1).

**Proof.** Let us introduce the Fourier transform of \( u = u(x) \) on \( \mathbb{R}^N \) and the inverse Fourier transform of \( v = v(\xi) \) on \( \mathbb{R}^N \) as follows:

\[ \mathcal{F}[u](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} u(x) \, dx, \quad \mathcal{F}_\xi^{-1}[v](x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} v(\xi) \, d\xi. \]

Let \( u_\pm = u_\pm(x', x_N) \) be functions defined on \( \mathbb{R}^N_\pm \), and define the odd extensions \( E^o_\pm u_\pm \) of \( u_\pm \) and the even extensions \( E^e_\pm u_\pm \) of \( u_\pm \) as follows:

\[
\begin{align*}
(E^o_+ u_+)(x) &= \begin{cases} u_+(x', x_N) & (x_N > 0), \\
u_+(x', -x_N) & (x_N < 0), \end{cases} \\
(E^o_- u_-)(x) &= \begin{cases} u_-(x', -x_N) & (x_N > 0), \\
u_-(x', x_N) & (x_N < 0), \end{cases} \\
(E^e_+ u_+)(x) &= \begin{cases} u_+(x', x_N) & (x_N > 0), \\
u_+(x', -x_N) & (x_N < 0), \end{cases} \\
(E^e_- u_-)(x) &= \begin{cases} u_-(x', -x_N) & (x_N > 0), \\
u_-(x', x_N) & (x_N < 0). \end{cases}
\end{align*}
\]

In addition, we set

\[ \mathbf{F}_\pm = (F_{\pm 1}, \ldots, F_{\pm N-1}, F_{\pm N})^T = (E^o_\pm f_{\pm 1}, \ldots, E^e_\pm f_{\pm N-1}, E^o_\pm f_{\pm N})^T. \]

It then holds that

\[ \partial_j F_{\pm j} = E^o_\pm \partial_j f_{\pm j} \quad (j = 1, \ldots, N - 1), \quad \partial_N F_{\pm N} = E^o_\pm \partial_N f_{\pm N}, \]

which imply

\[ \text{div} \mathbf{F}_\pm = E^o_\pm z_\pm \quad \text{for} \quad z_\pm = \text{div} f_\pm. \tag{3.8} \]

We thus see that

\[ \mathcal{F} [\text{div} \mathbf{F}_\pm](\xi) = i \xi \cdot \mathcal{F} \mathbf{F}_\pm(\xi) = \mathcal{F} [E^o_\pm z_\pm](\xi). \tag{3.9} \]
Let us define for $\varepsilon > 0$

$$U^\varepsilon = -\mathcal{F}^{-1}_\xi \left[ e^{-\varepsilon|\xi|^2} \frac{i\xi \cdot \mathcal{F}[F^\varepsilon](\xi)}{|\xi|^2} \right] (x),$$

$$F^\varepsilon = \mathcal{F}^{-1}_\xi \left[ e^{-\varepsilon|\xi|^2} \mathcal{F}[\mathcal{F}_\pm](\xi) \right] (x),$$

$$F_{\pm,\text{div}} = \mathcal{F}^{-1}_\xi \left[ e^{-\varepsilon|\xi|^2} \mathcal{F}[^{\text{div}} F^\varepsilon](\xi) \right] (x).$$

Then $U^\varepsilon$ solve by (3.9)

$$\Delta U^\varepsilon = F_{\pm,\text{div}} \quad \text{in } \mathbb{R}^N.$$

In addition, the derivatives of $U^\varepsilon$ are formulated by (3.9),

$$\partial_j U^\varepsilon(x) = \mathcal{F}^{-1}_\xi \left[ \frac{\xi_j \mathcal{F}[F^\varepsilon](\xi)}{|\xi|^2} \right] (x) \quad (j = 1, \ldots, N),$$

$$\partial_k \partial_l U^\varepsilon(x) = \mathcal{F}^{-1}_\xi \left[ \frac{\xi_k \xi_l \mathcal{F}[F_{\pm,\text{div}}](\xi)}{|\xi|^2} \right] (x) \quad (k, l = 1, \ldots, N).$$

Then the $L_q$ boundedness of the Riesz operators yields that

$$\|\nabla U^\varepsilon\|_{L_q} \leq C\|F^\varepsilon\|_{L_q}, \quad \|\nabla^2 U^\varepsilon\|_{L_q} \leq C\|F^\varepsilon_{\pm,\text{div}}\|_{L_q},$$

$$\|\nabla(U^\varepsilon - U^\varepsilon')\|_{L_q} \leq C\|F^\varepsilon_{\pm} - F^\varepsilon_{\pm}'\|_{L_q},$$

$$\|\nabla^2(U^\varepsilon - U^\varepsilon')\|_{L_q} \leq C\|F^\varepsilon_{\pm,\text{div}} - F^\varepsilon_{\pm,\text{div}}'\|_{L_q},$$

(3.11)

where $\varepsilon, \varepsilon' > 0$ and $C$ is a positive constant independent of $\varepsilon$ and $\varepsilon'$.

Next, we compute the formulas of $\partial N U^\varepsilon$ on $\mathbb{R}_0^N$. It holds that

$$\mathcal{F}[E^\varepsilon_{\pm,\varepsilon}](\xi) = \int_0^\infty \left( \pm e^{-i\varepsilon yN\xi} + e^{i\varepsilon yN\xi} \right) \hat{z}_{\pm}(\xi', \pm yN) \, dyN,$$

while by (3.9)

$$\partial N U^\varepsilon = -\mathcal{F}^{-1}_\xi \left[ e^{-\varepsilon|\xi|^2} \frac{i\xi \cdot \mathcal{F}[E^\varepsilon_{\pm,\varepsilon}](\xi)}{|\xi|^2} \right] (x).$$

These formulas yield

$$\partial N U^\varepsilon = -\int_0^\infty \mathcal{F}^{-1}_\xi \left[ \hat{z}_{\pm}(\xi', \pm yN) \right] \left( \frac{1}{2\pi} \int_{-\infty}^\infty e^{-\varepsilon|\xi|^2} \xi N \left( \pm e^{i(xN-\varepsilon yN)\xi} \mp e^{i(xN+\varepsilon yN)\xi} \right) \, d\xiN \right) (x') \, dyN.$$ 

Inserting (2.2) into the above formula of $\partial N U^\varepsilon$ furnishes

$$\left(\partial N U^\varepsilon(x', 0) = \mp \int_0^\infty \mathcal{F}^{-1}_\xi \left[ e^{-\varepsilon|\xi|^2} \hat{z}_{\pm}(\xi', \pm yN) \right] (x') \, dyN. \right. \quad \text{(3.12)}$$

We now consider the limit: $\varepsilon \to 0$. By the well-known property of the heat kernel $\mathcal{F}^{-1}_\xi [e^{-\varepsilon|\xi|^2}](x)$, we have

$$\lim_{\varepsilon \to 0} \|F^\varepsilon_{\pm} - F_{\pm}\|_{L_q} = 0, \lim_{\varepsilon \to 0} \|F_{\pm,\text{div}} - \text{div} F_{\pm}\|_{L_q} = 0.$$

Combining these property with (3.11) shows that there exist $U_{\pm} \in \tilde{H}^1_q(\mathbb{R}^N) \cap \tilde{H}^2_q(\mathbb{R}^N)$ such that

$$\lim_{\varepsilon \to 0} \|\nabla(U^\varepsilon_{\pm} - U_{\pm})\|_{L_q} = 0, \lim_{\varepsilon \to 0} \|\nabla^2(U^\varepsilon_{\pm} - U_{\pm})\|_{L_q} = 0.$$
We thus see that $U_\pm$ satisfy, by (3.8), (3.10), and (3.11), the equations:

$$
\Delta U_\pm = \text{div} \mathbf{F}_\pm = \text{div} f_\pm \quad \text{in} \quad \mathbb{R}^N_\pm
$$

and the estimates:

$$
\|\nabla U_\pm\|_{L^q(\mathbb{R}^N)} \leq C\|\mathbf{F}_\pm\|_{L^q(\mathbb{R}^N)} \leq C\|f_\pm\|_{L^q(\mathbb{R}^N)},
\|\nabla^2 U_\pm\|_{L^q(\mathbb{R}^N)} \leq C\|\text{div} \mathbf{F}_\pm\|_{L^q(\mathbb{R}^N)} \leq C\|\text{div} f_\pm\|_{L^q(\mathbb{R}^N)},
$$

while there holds by (3.12)

$$(\partial_N U_\pm)(x',0) = \mp \int_0^\infty \mathcal{F}_\xi^{-1} \left[ e^{-|\xi'|y_N} \tilde{z}_\pm(\xi', \pm y_N) \right] (x') dy_N.
$$

Combining these formulas with

$$
\tilde{z}_\pm(\xi', \pm y_N) = \sum_{j=1}^{N-1} i\xi_j \hat{f}_{\pm j}(\xi', \pm y_N) + (\partial_N \hat{f}_{\pm N})(\xi', \pm y_N)
$$

and with integration by parts furnishes the desired formulas of $\partial_N U_\pm$ on $\mathbb{R}^N_0$. This completes the proof of Lemma 3.3. \qed

3.2. **Proof of Theorem 3.1.** We prove Theorem 3.1 in this subsection. In view of Lemma 2.1, it suffices to consider

$$
f_\pm = (f_{\pm 1}, \ldots, f_{\pm N})^T \in C_0^\infty(\mathbb{R}^N_0)^N.
$$

Let $U_\pm$ be the solutions of (3.7) constructed in Lemma 3.3, and set $v_\pm = U_\pm + w_\pm$ in (3.1). Then,

$$
\begin{cases}
\Delta w_\pm = 0 & \text{in} \quad \mathbb{R}^N_\pm, \\
\rho_+ w_+ - \rho_- w_- = g_1 & \text{on} \quad \mathbb{R}^N_0, \\
\partial_N w_+ - \partial_N w_- = g_2 & \text{on} \quad \mathbb{R}^N_0,
\end{cases}
$$

where

$$
g_1 = -(\rho_+ U_+ - \rho_- U_-), \quad g_2 = -(\partial_N U_+ - \partial_N U_-) + f_{+N} - f_{-N}.
$$

We apply the partial Fourier transform, given in (2.1), to (3.13) in order to obtain

$$
\begin{cases}
(\partial_N^2 - |\xi'|^2) \hat{w}_\pm(x_N) = 0, \quad x_N > 0, \\
\rho_+ \hat{w}_+ (0) - \rho_- \hat{w}_- (0) = \hat{g}_1 (0), \\
\partial_N \hat{w}_+ (0) - \partial_N \hat{w}_- (0) = \hat{g}_2 (0).
\end{cases}
$$

Solving these ordinary differential equations with respect to $x_N$, we have

$$
\hat{w}_\pm(x_N) = \left( \frac{1}{\rho_+ + \rho_-} \right) \left( \pm \hat{g}_1 (0) - \rho_+ \frac{\hat{g}_2 (0)}{|\xi'|} \right) e^{\mp |\xi'| x_N}.
$$

Thus, setting $w_\pm = \mathcal{F}_{\xi}^{-1}[\hat{w}_\pm(x_N)](x')$, we see that $w_\pm$ are solutions to (3.13).

From now on, we estimate $w_\pm$. To this end, we decompose $w_\pm$ as follows:

$$
w_\pm = \pm \left( \frac{1}{\rho_+ + \rho_-} \right) w_{1\pm} + \left( \frac{\rho_+}{\rho_+ + \rho_-} \right) w_{2\pm},
$$

where

$$
w_{1\pm} = \mathcal{F}_{\xi}^{-1} \left[ \hat{g}_1 (0) e^{\mp |\xi'| x_N} \right] (x'), \quad w_{2\pm} = \mathcal{F}_{\xi}^{-1} \left[ \frac{\hat{g}_2 (0)}{|\xi'|} e^{\mp |\xi'| x_N} \right] (x').$$

Let us consider \( \| \nabla w^1_\perp \|_{L^\infty(\mathbb{R}^N_\perp)} \) and \( \| \nabla^2 w^1_\perp \|_{L^\infty(\mathbb{R}^N_\perp)} \). One has

\[
\widehat{g}_1(0) e^{\mp |\xi'|_N} = - \int_0^\infty \frac{\partial}{\partial y_N} \left( e^{-|\xi'|_N} \widehat{g}_1(\xi', \pm y_N) \right) \, dy_N
\]

\[
= \int_0^\infty |\xi'| e^{-|\xi'|_N} \widehat{g}_1(\xi', \pm y_N) \, dy_N
\]

\[
\mp \int_0^\infty e^{-|\xi'|_N} \partial_N \widehat{g}_1(\xi', \pm y_N) \, dy_N,
\]

which, combined with \( |\xi'| = - \sum_{j=1}^{N-1} (i\xi_j)^2 / |\xi'| \), furnishes

\[
\widehat{g}_1(0) e^{\mp |\xi'|_N} = - \sum_{j=1}^{N-1} \int_0^\infty \frac{i\xi_j}{|\xi'|} e^{-|\xi'|_N} \partial_j \widehat{g}_1(\xi', \pm y_N) \, dy_N
\]

\[
\mp \int_0^\infty e^{-|\xi'|_N} \partial_N \widehat{g}_1(\xi', \pm y_N) \, dy_N.
\]

Inserting these relations into \( w^1_\perp \) yields

\[
w^1_\perp = - \sum_{j=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[ \frac{i\xi_j}{|\xi'|} e^{-|\xi'|_N} \partial_j \widehat{g}_1(\xi', \pm y_N) \right] (x') \, dy_N
\]

\[
\mp \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[ e^{-|\xi'|_N} \partial_N \widehat{g}_1(\xi', \pm y_N) \right] (x') \, dy_N.
\]

Thus, for \( k, l = 1, \ldots, N - 1, \)

\[
\partial_k w^1_\perp = \sum_{j=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[ \frac{i\xi_k \xi_j}{|\xi'|^2} e^{-|\xi'|_N} \partial_j \widehat{g}_1(\xi', \pm y_N) \right] (x') \, dy_N
\]

\[
\mp \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[ \frac{i\xi_k}{|\xi'|} e^{-|\xi'|_N} \partial_N \widehat{g}_1(\xi', \pm y_N) \right] (x') \, dy_N,
\]

\[
\partial_N w^1_\perp = \pm \sum_{j=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[ \frac{i\xi_j}{|\xi'|} e^{-|\xi'|_N} \partial_j \widehat{g}_1(\xi', \pm y_N) \right] (x') \, dy_N
\]

\[
\mp \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[ e^{-|\xi'|_N} \partial_N \widehat{g}_1(\xi', \pm y_N) \right] (x') \, dy_N,
\]

\[
\partial_l \partial_k w^1_\perp = \partial_N \partial_k w^1_\perp
\]

\[
= \pm \sum_{j=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[ \frac{i\xi_j}{|\xi'|} e^{-|\xi'|_N} \partial_k \partial_j \widehat{g}_1(\xi', \pm y_N) \right] (x') \, dy_N
\]

\[
+ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[ e^{-|\xi'|_N} \partial_k \partial_N \widehat{g}_1(\xi', \pm y_N) \right] (x') \, dy_N.
\]
Analogously, for $\Delta'g_1 = \sum_{j=1}^{N-1} \partial_j^2 g_1$, 

$$ \partial_N^2 w_{\pm}^1 = - \int_0^{\infty} F_{\xi'}^{-1} \left[ |\xi'| e^{-|\xi'|(|\pm x_N+y_N|)} \widehat{\Delta g_1}(\xi', \pm y_N) \right] (x') \, dy_N$$

$$ \pm \sum_{j=1}^{N-1} \int_0^{\infty} F_{\xi'}^{-1} \left[ \frac{\xi_j}{|\xi'|} |\xi'| e^{-|\xi'|(|\pm x_N+y_N|)} \partial_j \partial_N g_1(\xi', \pm y_N) \right] (x') \, dy_N. $$

By [25, Lemma 5.2] and the Leibniz formula, we have for any $s \in \mathbb{R}$ and any multi-index $\alpha', \beta' \in \mathbb{N}_0^{N-1}$

$$ |\partial_{\xi'}^s \{(\xi')^{\beta'}|\xi'|^s\}| \leq C|\xi'|^{-|\alpha'|+|\beta'|+s} \quad (\xi' \in \mathbb{R}^{N-1} \setminus \{0\}),$$

with some positive constant $C = C(N, s, \alpha', \beta')$. Combining this inequality with Lemma 2.2 furnishes

$$ \|\nabla^1 w_{\pm} \|_{L^q(\mathbb{R}^N_\pm)} \leq C \sum_{s \in \{+,-\}} \|\nabla U_s\|_{L^q(\mathbb{R}^N)},$$

$$ \|\nabla^2 w_{\pm} \|_{L^q(\mathbb{R}^N_\pm)} \leq C \sum_{s \in \{+,-\}} \|\nabla^2 U_s\|_{L^q(\mathbb{R}^N)}. $$

We similarly see that

$$ \|\nabla^2 w_{\pm} \|_{L^q(\mathbb{R}^N_\pm)} \leq C \sum_{s \in \{+,-\}} \left( \|\nabla \partial_N U_s\|_{L^q(\mathbb{R}^N)} + \|f_{sN}\|_{L^q(\mathbb{R}^N)} \right)$$

and that, if $f_{+N} - f_{-N} = 0$ on $\mathbb{R}^N$,

$$ \|\nabla^2 w_{\pm} \|_{L^q(\mathbb{R}^N_\pm)} \leq C \sum_{s \in \{+,-\}} \|\nabla \partial_N U_s\|_{L^q(\mathbb{R}^N)}. $$

Next, we estimate $\|\nabla w_{\pm} \|_{L^q(\mathbb{R}^N)}$. By Lemma 3.3,

$$ \tilde{g}_2(0) = \sum_{s \in \{+,-\}} \int_0^{\infty} e^{-|\xi'|y_N} \left( i\xi' \cdot \widehat{f}_N(\xi', sy_N) + s|\xi'| \widehat{f}_{sN}(\xi', sy_N) \right) \, dy_N,$$

which implies that

$$ w_{\pm}^2 = \sum_{s \in \{+,-\}} \left\{ \sum_{j=1}^{N-1} \int_0^{\infty} F_{\xi'}^{-1} \left[ \frac{\xi_j}{|\xi'|} |\xi'| e^{-|\xi'|(|\pm x_N+y_N|)} \widehat{f}_{sj}(\xi', sy_N) \right] (x') \, dy_N 

+ s \int_0^{\infty} F_{\xi'}^{-1} \left[ e^{-|\xi'|(|\pm x_N+y_N|)} \widehat{f}_{sN}(\xi', sy_N) \right] (x') \, dy_N \right\}. $$

Then, for $k = 1, \ldots, N - 1$,

$$ \partial_k w_{\pm}^2 = \sum_{s \in \{+,-\}} \left\{ - \sum_{j=1}^{N-1} \int_0^{\infty} F_{\xi'}^{-1} \left[ \frac{\xi_j \xi_k}{|\xi'|} |\xi'| e^{-|\xi'|(|\pm x_N+y_N|)} \widehat{f}_{sj}(\xi', sy_N) \right] (x') \, dy_N 

+ s \int_0^{\infty} F_{\xi'}^{-1} \left[ \frac{\xi_k}{|\xi'|} |\xi'| e^{-|\xi'|(|\pm x_N+y_N|)} \widehat{f}_{sN}(\xi', sy_N) \right] (x') \, dy_N \right\}, $$

$$ \partial_N w_{\pm}^2 = \sum_{s \in \{+,-\}} \left\{ \sum_{j=1}^{N-1} \int_0^{\infty} F_{\xi'}^{-1} \left[ \frac{\xi_j}{|\xi'|} |\xi'| e^{-|\xi'|(|\pm x_N+y_N|)} \widehat{f}_{sj}(\xi', sy_N) \right] (x') \, dy_N 

\pm s \int_0^{\infty} F_{\xi'}^{-1} \left[ |\xi'| e^{-|\xi'|(|\pm x_N+y_N|)} \widehat{f}_{sN}(\xi', sy_N) \right] (x') \, dy_N \right\}. $$
Combining these formulas with (3.15) and Lemma 2.2 furnishes
\[ \| \nabla w^2 \|_{L^q(R^N)} \leq C \left( \| f_+ \|_{L^q(R^N)} + \| f_- \|_{L^q(R^N)} \right). \]

Recalling \( v_\pm = U_\pm + w_\pm \) and (3.14), we obtain solutions \( v_\pm \) of (3.1) satisfying (3.2)-(3.4) by the last estimate, (3.16)-(3.18), and Lemma 3.3.

Finally, we prove the uniqueness of solutions to (3.1). Let \( v_\pm \) satisfy (3.5) and \( f = f_+ \mathbb{1}_{R^N_+} + f_- \mathbb{1}_{R^N_-} \) for some \( f_\pm \in C_0^\infty(R^N_\pm) \). By the discussion above, there exists \( w_\pm \in \hat{H}^1_q(R^N_\pm) \cap \hat{H}^2_q(R^N_\pm) \) with \( q' = q/(q-1) \) satisfying
\[
\begin{cases}
\Delta w_\pm = \text{div } f_\pm & \text{in } R^N_\pm, \\
\rho_+ w_+ = \rho_- w_- & \text{on } R^N_0, \\
\partial_N w_+ = \partial_N w_- & \text{on } R^N_0.
\end{cases}
\]

At this point, we introduce the following lemma (cf. e.g. [11, Section II.6], [22], [23, Proof of Theorem A.4]).

**Lemma 3.4.** Let \( q \in (1, \infty) \) and set
\[ d_q(x) = \begin{cases} (1 + |x|^2)^{1/2} & \text{when } q \neq N, \\ (1 + |x|^2)^{1/2} \log(2 + |x|^2)^{1/2} & \text{when } q = N. \end{cases} \]

Then, for any \( u \in \hat{H}^1_q(R^N) \), there exists a constant \( c_u \) such that
\[ \left\| \frac{u - c_u}{d_q} \right\|_{L^q(R^N)} \leq C \| \nabla u \|_{L^q(R^N)}, \]
where \( C = C(N, q) \) is a positive constant independent of \( u \) and \( c_u \).

Now we set
\[ z = \rho_+ z_+ \mathbb{1}_{R^N_+} + \rho_- z_- \mathbb{1}_{R^N_-} \]
for \( z \in \{ v, w \} \).

Then \( v \in \hat{H}^1_q(R^N) \) with \( \nabla v = \rho_+ (\nabla v_+) \mathbb{1}_{R^N_+} + \rho_- (\nabla v_-) \mathbb{1}_{R^N_-} \) by \( \rho_+ v_+ = \rho_- v_- \) on \( R^N_0 \), while \( w \in \hat{H}^1_q(R^N) \) with \( \nabla w = \rho_+ (\nabla w_+) \mathbb{1}_{R^N_+} + \rho_- (\nabla w_-) \mathbb{1}_{R^N_-} \) by \( \rho_+ w_+ = \rho_- w_- \) on \( R^N_0 \). By Lemma 3.4, there exist constants \( c_v \) and \( c_w \) such that
\[ \left\| \frac{v - c_v}{d_q} \right\|_{L^q(R^N)} \leq C \| \nabla v \|_{L^q(R^N)}, \quad \left\| \frac{w - c_w}{d_{q'}} \right\|_{L^{q'}(R^N)} \leq C \| \nabla w \|_{L^{q'}(R^N)}. \]

Let \( \psi_R \) be a cut-off function of Sobolev’s type as follows\(^2\): For \( R > 0 \) large enough and for \( \psi \in C_0^\infty(R) \) satisfying \( \psi(t) = 1 \) when \( |t| \leq 1/2 \) and \( \psi(t) = 0 \) when \( |t| \geq 1 \),
\[ \psi_R(x) = \begin{cases} 1 & \text{when } |x| \leq e^{\sqrt[4]{\log R}}, \\ \psi \left( \frac{\log \log |x|}{\log \log R} \right) & \text{when } |x| \geq e^{\sqrt[4]{\log R}}. \end{cases} \]

Note that for \( D_R = \{ x \in R^N : e^{\sqrt{\log R}} \leq |x| \leq R \} \)
\[ |\nabla \psi_R(x)| \leq \frac{C}{\log \log R} \frac{1}{|x| \log |x|}, \quad \text{supp } \nabla \psi_R \subset D_R, \quad (3.19) \]
where \( C \) is a positive constant independent of \( R \).

\(^2\)See e.g. [11, Proof of Theorem II.7.1].
Choose $R > 0$ large enough so that $\psi_R = 1$ on $\text{supp} f$. It then holds that
\[
(\nabla v, f)_{R_N} = (\nabla (v - c_v), f)_{R_N} = -(v - c_v, \text{div} f)_{R_N} = -(v - c_v, \psi_R \Delta w^+)_{R_N}^+ - (v - c_v, \psi_R \Delta w^-)_{R_N}^-
= -(v - c_v, \text{div}(\psi_R \nabla w^+) - \nabla \psi_R \cdot \nabla w^+)_{R_N}^+ - (v - c_v, \text{div}(\psi_R \nabla w^-) - \nabla \psi_R \cdot \nabla w^-)_{R_N}^-
= (\rho_+ \nabla v^+, \psi_R \nabla w^+)_{R_N}^+ + (v - c_v, \psi_R \nabla \cdot w^+)_{R_N}^+ + (\rho_+ \nabla v^+, \nabla \cdot w^+)_{R_N}^+ + (v - c_v, \psi_R \nabla \cdot w^-)_{R_N}^+.
\]

On the other hand,
\[
\begin{align*}
(\rho_+ \nabla v^+, \psi_R \nabla w^+)_{R_N}^+ + (\rho_- \nabla v^-, \psi_R \nabla w^-)_{R_N}^- &= (\psi_R \nabla v^+, \nabla (\rho_+ w^+ - c_w))_{R_N}^+ + (\psi_R \nabla v^-, \nabla (\rho_- w^- - c_w))_{R_N}^-
= -(\text{div}(\psi_R \nabla v^+), \rho_+ w^+ - c_w)_{R_N}^+ - (\text{div}(\psi_R \nabla v^-), \rho_- w^- - c_w)_{R_N}^-
= -(\nabla \psi_R \cdot \nabla v^+, w - c_w)_{R_N}^+ - (\nabla \psi_R \cdot \nabla v^-, w - c_w)_{R_N}^-,
\end{align*}
\]
and thus
\[
(\nabla v, f)_{R_N} = (v - c_v, \nabla \psi_R \cdot \nabla w^+)_{R_N}^+ + (v - c_v, \nabla \psi_R \cdot \nabla w^-)_{R_N}^-
- (\nabla \psi_R \cdot \nabla v^+, w - c_w)_{R_N}^+ - (\nabla \psi_R \cdot \nabla v^-, w - c_w)_{R_N}^-.
\]

We here see that by Lemma 3.4 and (3.19)
\[
\begin{align*}
|v - c_v, \nabla \psi_R \cdot \nabla w^+|_{R_N^2} &\leq \frac{|v - c_v|_{L_q(R_N^2)} \|\nabla \psi_R\|_{L_\infty(R_N^2)} \|\nabla w^+\|_{L_q'(R_N^2)}}{d_q}
\leq \frac{C}{\log \log R} \|\nabla \psi_R\|_{L_q(R_N^2)} \|\nabla w^\pm\|_{L_q'(R_N^2)} \to 0 \text{ as } R \to \infty,
\end{align*}
\]
and also
\[
\lim_{R \to \infty} (\nabla \psi_R \cdot \nabla v^+, w - c_w)_{R_N^+} = 0.
\]
By these properties, letting $R \to \infty$ in (3.20) furnishes
\[
(\rho_+ \nabla v^+, f^+)_{R_N^+} + (\rho_- \nabla v^-, f^-)_{R_N^-} = (\nabla v, f)_{R_N} = 0,
\]
which implies $\rho_+ v^+ = \rho_- v^-$ for some constants $c_\pm$. Since $\rho_+ v^+ = \rho_- v^-$ on $R_0^N$, we have $c_+ = c_-$. Thus setting $c = c_+ = c_-$ yields $v^\pm = \rho_\pm^{-1} c$, which implies the uniqueness of solutions to (3.1). This completes the proof of Theorem 3.1.

3.3. Weak elliptic problem with flat interface. For $f \in L_q(R_N^N \setminus R_0^N)^N$ with $q \in (1, \infty)$, this subsection considers the unique solvability of the following weak elliptic problem: Find $u \in \tilde{H}^1_q(R_N^N)$ such that
\[
(\rho^{-1} \nabla u, \nabla \varphi)_{R_N^N \setminus R_0^N} = (f, \nabla \varphi)_{R_N^N \setminus R_0^N} \text{ for any } \varphi \in \tilde{H}^1_q(R_N^N),
\]
where $q' = q/(q - 1)$ and $\rho = \rho_+ \mathbb{1}_{R_+^N} + \rho_- \mathbb{1}_{R_-^N}$ for positive constants $\rho_\pm$. More precisely, we prove

**Theorem 3.5.** Let $q \in (1, \infty)$ and $q' = q/(q - 1)$.

1. **Existence.** Let $f \in L_q(R_N^N \setminus R_0^N)^N$. Then the weak elliptic problem (3.21) admits a solution $u \in \tilde{H}^1_q(R_N^N)$ satisfying
\[
\|\nabla u\|_{L_q(R_N^N)} \leq 2C_2(\rho_+ + \rho_-)\|f\|_{L_q(R_N^N \setminus R_0^N)},
\]
where $C_2$ is the same constant as in Theorem 3.1.
(2) **Uniqueness.** If \( u \in \tilde{H}^1_q(\mathbb{R}^N) \) satisfies
\[
(\rho^{-1}\nabla u, \nabla \varphi)_{\mathbb{R}^N \setminus R_0^N} = 0
\]
for any \( \varphi \in \tilde{H}^1_q(\mathbb{R}^N), \)
then \( u = c \) for some constant \( c. \)

**Proof.** (1). Since \( C_0^\infty(\mathbb{R}^N) \) is dense in \( L_q(\mathbb{R}^N \setminus R_0^N) \), it suffices to consider \( f \in C_0^\infty(\mathbb{R}^N)^N \). By Theorem 3.1, we have \( v_\pm \in \tilde{H}^1_q(\mathbb{R}^N_\pm) \cap \tilde{H}^2_q(\mathbb{R}^N_\pm) \) satisfying (3.1) and (3.3) with \( f_+ = f \) and \( f_- = f \), where we note that \( f_+ - f_- = 0 \) on \( R_0^N \) in this case. Setting \( u_\pm = \rho_\pm v_\pm \) and \( u = u_+ \mathbf{1}_{R_+^N} + u_- \mathbf{1}_{R_-^N} \) furnishes that
\[
\begin{cases}
\rho_-^{-1}\Delta u_+ = \text{div} f_+ & \text{in } \mathbb{R}_+^N, \\
u_+ = u_- & \text{on } \partial \mathbb{R}_0^N, \\
\rho_+^{-1}\partial_N u_+ = \rho_-^{-1}\partial_N u_- & \text{on } \partial \mathbb{R}_0^N,
\end{cases}
\]
and that \( u \in \tilde{H}^1_q(\mathbb{R}^N) \) with \( \nabla u = \rho_+ (\nabla v_+ \mathbf{1}_{R_+^N} + \rho_- (\nabla v_-) \mathbf{1}_{R_-^N} \) by \( \rho_+ v_+ = \rho_- v_- \) on \( R_0^N \). Combining the last property with (3.3) yields
\[
\|\nabla u\|_{L_q(\mathbb{R}^N)} \leq C_2 (\rho_+ + \rho_-) \sum_{z \in \{+, -\}} \|f_z\|_{L_q(\mathbb{R}^N \setminus R_0^N)} \leq 2C_2 (\rho_+ + \rho_-) \|f\|_{L_q(\mathbb{R}^N \setminus R_0^N)},
\]
which implies that \( u \) satisfies (3.22).

Next, we prove that \( u \) is a solution to (3.21). Let \( \varphi \in \tilde{H}^1_q(\mathbb{R}^N) \). Then there exists \( \{\varphi_j\}_{j=1}^\infty \subset C_0^\infty(\mathbb{R}^N) \) such that \( \lim_{j \to \infty} \|\nabla (\varphi_j - \varphi)\|_{L_q(\mathbb{R}^N)} = 0 \). Thus,
\[
\begin{align*}
(f, \nabla \varphi)_{\mathbb{R}^N \setminus R_0^N} &= \lim_{j \to \infty} (f, \nabla \varphi_j)_{\mathbb{R}^N \setminus R_0^N} = - \lim_{j \to \infty} \left\{ (\text{div} f_+, \varphi_j)_{\mathbb{R}^N_+} + (\text{div} f_-, \varphi_j)_{\mathbb{R}^N_-} \right\} \\
&= - \lim_{j \to \infty} \left\{ (\rho_+^{-1}\Delta u_+, \varphi_j)_{\mathbb{R}^N_+} + (\rho_-^{-1}\Delta u_-, \varphi_j)_{\mathbb{R}^N_-} \right\} \\
&= \lim_{j \to \infty} \left\{ (\rho_+^{-1}\nabla u_+, \nabla \varphi_j)_{\mathbb{R}^N_+} + (\rho_-^{-1}\nabla u_-, \nabla \varphi_j)_{\mathbb{R}^N_-} \right\} \\
&= \lim_{j \to \infty} (\rho^{-1}\nabla u, \nabla \varphi)_{\mathbb{R}^N \setminus R_0^N} = (\rho^{-1}\nabla u, \nabla \varphi)_{\mathbb{R}^N \setminus R_0^N}.
\end{align*}
\]
This completes the proof of (1).

(2). Let \( u \in \tilde{H}^1_q(\mathbb{R}^N) \) satisfy (3.23), and let \( f \in C_0^\infty(\mathbb{R}^N)^N \). By (1), we have \( v \in \tilde{H}^1_q(\mathbb{R}^N) \) satisfying
\[
(\rho^{-1}\nabla v, \nabla \psi)_{\mathbb{R}^N \setminus R_0^N} = (f, \nabla \psi)_{\mathbb{R}^N \setminus R_0^N}
\]
for any \( \psi \in \tilde{H}^1_q(\mathbb{R}^N) \). Choosing \( \psi = u \) in this equality yields
\[
(f, \nabla u)_{\mathbb{R}^N} = (\rho^{-1}\nabla v, \nabla u)_{\mathbb{R}^N \setminus R_0^N} = (\rho^{-1}\nabla u, \nabla v)_{\mathbb{R}^N \setminus R_0^N} = 0,
\]
which implies \( u = c \) for some constant \( c. \) This completes the proof of (2). \( \square \)

4. **Problems in the whole space with bent interface.** Let \( \Phi : \mathbb{R}^N_+ \to \mathbb{R}^N_y \) be a bijection of class \( C^1 \) and \( \Phi^{-1} \) the inverse mapping of \( \Phi \), where the subscripts \( x \) and \( y \) denote their variables. Let \( (\mathcal{V}_x \Phi)(x) = A + B(x) \) and \( (\mathcal{V}_y \Phi^{-1})(\Phi(x)) = A_{-1} + B_{-1}(x) \). Assume that \( A, A_{-1} \) are orthonormal matrices with constant coefficients and \( \det A = \det A_{-1} = 1 \), and also assume that \( B(x), B_{-1}(x) \) are matrix-valued functions of \( H^1_p(\mathbb{R}^N) \), \( r \in (N, \infty) \), satisfying
\[
\|
\begin{pmatrix} B & B_{-1} \end{pmatrix} \|_{L_\infty(\mathbb{R}^N)} \leq M_1 \quad (0 < M_1 \leq 1/2),
\|
\begin{pmatrix} \nabla B & \nabla B_{-1} \end{pmatrix} \|_{L_r(\mathbb{R}^N)} \leq M_2 \quad (M_2 \geq 1).
\]
(4.1)
In what follows, we will choose $M_1$ small enough eventually.

Let $\Omega_{\pm} = \Phi(R_{\pm}^N)$, $\Sigma = \Phi(R_0^N)$, and $\bar{n} = \bar{n}(y)$ be the unit normal vector on $\Sigma$ pointing from $\Omega_+$ into $\Omega_-$. Let $A_{ij}$ and $B_{ij}(x)$ be respectively the $(i,j)$-component of $A_{-1}$ and the $(i,j)$-component of $B_{-1}(x)$. In addition, setting $\Phi^{-1} = (\Phi_{-1,1}, \ldots, \Phi_{-1,N})$, we see that $\Sigma$ is represented as $\Phi_{-1,N}(y) = 0$. This representation implies that

$$\bar{n}(\Phi(x)) = -\frac{(\nabla_y \Phi_{-1,N})(\Phi(x))}{|\nabla_y \Phi_{-1,N}(\Phi(x))|} = -\frac{(A_{N1} + B_{N1}(x), \ldots, A_{NN} + B_{NN}(x))^\top}{\sqrt{\sum_{j=1}^N (A_{Nj} + B_{Nj}(x))^2}}$$

for $n_0 = (0, \ldots, 0, -1)^\top$. Especially, $\bar{n}$ is defined on $\mathbb{R}^N$ through the relation (4.2).

Remark 2. Let $J$ be the Jacobian of $\Phi$, i.e. $J =\det(\nabla \Phi)$, and let $d = |(A_{-1} + B_{-1}(x))^\top n_0|$. Since $M_1$ is small enough, there are positive constants $C_3$ and $C_4$, independent of $M_1$, $M_2$, and $r$, such that the following inequalities hold:

$$C_3 \leq J(x), d(x) \leq C_4 \quad (x \in \mathbb{R}^N), \quad \sup_{x \in \mathbb{R}^N} |1 - J(x)| \leq C_4 M_1,$$

$$\|\nabla J\|_{L^r(\mathbb{R}^N)} \leq C_4 M_2, \quad \|\nabla d\|_{L^r(\mathbb{R}^N)} \leq C_4 M_2. \quad (4.3)$$

This section mainly considers the strong elliptic problem as follows:

$$\begin{align*}
\Delta \bar{v}_{\pm} &= \text{div} \bar{f}_{\pm} \quad \text{in } \Omega_{\pm}, \\
r_+ \bar{v}_+ &= r_- \bar{v}_- \quad \text{on } \Sigma, \\
\bar{n} \cdot \nabla (\bar{v}_+ - \bar{v}_-) &= \bar{n} \cdot (\bar{f}_+ - \bar{f}_-) \quad \text{on } \Sigma.
\end{align*} \quad (4.4)$$

Assume that $\rho_{\pm}$ are positive constants throughout this section and set $F_q(\Omega_{\pm})$ as

$$F_q(\Omega_{\pm}) = \{ \bar{f}_{\pm} \in E_q(\Omega_{\pm}) : \bar{n} \cdot \bar{f}_{\pm} \in H^1_q(\Omega_{\pm}) \}.$$ 

Concerning (4.4), we prove the following theorem:

Theorem 4.1. Let $M_1, M_2, r, \Omega_{\pm}$, and $\Sigma$ be as above, and let $q \in (1, \infty)$. Assume that $\max(q, q') \leq r$ for $q' = q/(q-1)$. Then there exists $M_1 \in (0, 1/2)$, depending only on $N, q, r, \rho_{\pm}$, and $\rho_{\pm}$, such that the following assertions hold.

1. Existence. Let $\bar{f}_{\pm} \in F_q(\Omega_{\pm})$. Then the strong elliptic problem (4.4) admits solutions $\bar{v}_{\pm} \in \tilde{H}^1_q(\Omega_{\pm}) \cap \tilde{H}^2_q(\Omega_{\pm})$ satisfying

$$\|\nabla^2 \bar{v}_{\pm}\|_{L_q(\tilde{\Omega}_{\pm})} \leq C \sum_{s \in \{0, \ldots, 1\}} \left( \|\bar{f}_{\pm}\|_{E_q(\tilde{\Omega}_{s})} + \|\bar{n} \cdot \bar{f}_{\pm}\|_{H^1_q(\tilde{\Omega}_{s})} \right), \quad (4.5)$$

$$\|\nabla \bar{v}_{\pm}\|_{L_q(\tilde{\Omega}_{\pm})} \leq C' \sum_{s \in \{0, \ldots, 1\}} \|\bar{f}_{\pm}\|_{L_q(\tilde{\Omega}_{s})}, \quad (4.6)$$

with positive constants $C = C(M_2, r, N, q, r_{\pm}, \rho_{\pm})$ and $C' = C'(N, q, r_{\pm} \rho_{\pm})$ independent of $M_1$. Additionally, if $\bar{n} \cdot (\bar{f}_+ - \bar{f}_-)$ $= 0$ on $\Sigma$, then $\bar{v}_{\pm}$ satisfy

$$\|\nabla^2 \bar{v}_{\pm}\|_{L_q(\tilde{\Omega}_{\pm})} \leq C \sum_{s \in \{0, \ldots, 1\}} \|\bar{f}_{\pm}\|_{E_q(\tilde{\Omega}_{s})}. \quad (4.7)$$
Theorem 4.2. For some nonhomogeneous system.

Uniqueness (2)

Then there exist $M_1 \in (0,1/2)$ and $\Lambda_1 \geq 1$ such that, for any $\lambda \in \Sigma_{\sigma,\lambda_1}$, the strong elliptic problem (4.8) admits unique solutions $\tilde{v}_\pm \in \tilde{H}_q^2(\tilde{\Omega}_\pm)$. In addition, the solutions $\tilde{v}_\pm$ satisfies

\[
\|(\lambda \tilde{v}_\pm, \lambda^{1/2} \nabla \tilde{v}_\pm, \nabla^2 \tilde{v}_\pm)\|_{L_q(\tilde{\Omega}_\pm)} \leq C \sum_{\sigma \in \{+, -\}} \|\left( \div \tilde{f}_s, \tilde{g}_s, \lambda^{1/2} (\tilde{n} \cdot \tilde{f}_s), \nabla (\tilde{n} \cdot \tilde{f}_s), \lambda^{1/2} \tilde{n}_s, \nabla \tilde{n}_s \right)\|_{L_q(\tilde{\Omega}_s)},
\]

and also

\[
\|(\lambda^{1/2} \tilde{v}_\pm, \nabla \tilde{v}_\pm)\|_{L_q(\tilde{\Omega}_\pm)} \leq C \sum_{\sigma \in \{+, -\}} \left( \|\tilde{f}_s\|_{L_q(\tilde{\Omega}_s)} + |\lambda|^{-1/2} \|\tilde{g}_s\|_{L_q(\tilde{\Omega}_s)} + \|\tilde{n}_s\|_{L_q(\tilde{\Omega}_s)} + |\lambda|^{-1/2} \|\nabla \tilde{n}_s\|_{L_q(\tilde{\Omega}_s)} \right),
\]

where $C = C(M_2, r, N, q, \sigma, \rho_+, \rho_-)$ is a positive constant independent of $M_1$, $\Lambda_1$, and $\lambda$. Additionally, if $\tilde{n} \cdot (\tilde{f}_+ - \tilde{f}_-) = 0$ on $\tilde{\Sigma}$, then $\tilde{v}_\pm$ satisfy

\[
\|(\lambda \tilde{v}_\pm, \lambda^{1/2} \nabla \tilde{v}_\pm, \nabla^2 \tilde{v}_\pm)\|_{L_q(\tilde{\Omega}_\pm)} \leq C \sum_{\sigma \in \{+, -\}} \left( \|\div \tilde{f}_s, \tilde{g}_s, \lambda^{1/2} \tilde{n}_s, \nabla \tilde{n}_s \right)\|_{L_q(\tilde{\Omega}_s)}.
\]

Here, $M_1$ depends only on $N, q, \sigma, \rho_+$, and $\rho_-$, while $\Lambda_1$ depends only on $M_2, r, N, q, \sigma, \rho_+$, and $\rho_-$. The proof of Theorem 4.2 is similar to Theorem 4.1 in view of Theorem 3.2, so that the detailed proof may be omitted (cf. also [13, Appendix]). At this point, we introduce the following lemma, see [23, Lemma 2.4], which plays an important role in proving Theorem 4.1 in the following subsections.

Lemma 4.3. Let $q \in (1,\infty)$ and $r \in (N,\infty)$. Assume that $q \leq r$. Then there exists a positive constant $C = C(N, q, r)$ such that, for any $\epsilon > 0$, $a \in L_{r}(R^N_+)$, and $b \in H^1_{\epsilon}(R^N_+),$

\[
\|ab\|_{L_q(R^N_+)} \leq \epsilon \|\nabla b\|_{L_q(R^N_+)} + C \epsilon^{-\frac{N}{r'}} \|a\|_{L^r_{\epsilon}(R^N_+)} \|b\|_{L_q(R^N_+)}.
\]
4.1. **Proof of Theorem 4.1.** To prove Theorem 4.1, we first reduce (4.4) to a problem in the whole space with the flat interface by the change of variables: $y = \Phi(x)$. Let $D_j = \partial / \partial y_j$ and $\partial_j = \partial / \partial x_j$. One then notes the following fundamental relations: For $j, k = 1, \ldots, N$,

$$D_j = \sum_{l=1}^N (A_{lj} + B_{lj}(x)) \partial_l, \quad \nabla_y = (A_{-1} + B_{-1}(x))^T \nabla_x,$$

$$D_jD_k = \sum_{l,m=1}^N A_{lj}A_{mk} \partial_l \partial_m$$

$$+ \sum_{l,m=1}^N (A_{lj}B_{mk}(x) + A_{mk}B_{lj}(x) + B_{lj}(x)B_{mk}(x)) \partial_l \partial_m$$

$$+ \sum_{l,m=1}^N (A_{lj} + B_{lj}(x))(\partial_l B_{mk}(x)) \partial_m.$$

Let $\tilde{u}$ be a vector function on $\mathbb{R}^N_\Sigma = \tilde{\Omega}_+ \cup \tilde{\Omega}_-$, and let $u = \tilde{u} \circ \Phi$. Then, for any $\tilde{\varphi} \in C^\infty_0(\mathbb{R}^N_\Sigma)$ and for $\varphi = \tilde{\varphi} \circ \Phi$,

$$(\text{div}_y \tilde{u}, \tilde{\varphi})_{\mathbb{R}^N_\Sigma} = -(\tilde{u}, \nabla_y \tilde{\varphi})_{\mathbb{R}^N_\Sigma} = -(J\tilde{u}, (A_{-1} + B_{-1})^T \nabla_x \varphi)_{\mathbb{R}^N_\Sigma}$$

$$= (\text{div}_x \{J(A_{-1} + B_{-1})\tilde{u}\}, \varphi)_{\mathbb{R}^N_\Sigma}$$

$$= ([J^{-1} \text{div}_x \{J(A_{-1} + B_{-1})u\}] \circ \Phi^{-1}, \tilde{\varphi})_{\mathbb{R}^N_\Sigma},$$

which implies

$$(\text{div}_y \tilde{u}) \circ \Phi = J^{-1} \text{div}_x \{J(A_{-1} + B_{-1})u\}. \quad (4.9)$$

Analogously, we observe for a scalar function $\tilde{u}$ on $\mathbb{R}^N_\Sigma$ and $u = \tilde{u} \circ \Phi$

$$(\Delta_y \tilde{u}, \tilde{\varphi})_{\mathbb{R}^N_\Sigma} = (\nabla_y \tilde{u}, \nabla_y \tilde{\varphi})_{\mathbb{R}^N_\Sigma}$$

$$= (J(A_{-1} + B_{-1})^T \nabla_x u, (A_{-1} + B_{-1})^T \nabla_x \varphi)_{\mathbb{R}^N_\Sigma}$$

$$= (\text{div}_x \{J(A_{-1} + B_{-1})(A_{-1} + B_{-1})^T \nabla_x u\}, \varphi)_{\mathbb{R}^N_\Sigma}$$

and thus

$$(\Delta_y \tilde{u}) \circ \Phi = J^{-1} \text{div}_x \{J(A_{-1} + B_{-1})(A_{-1} + B_{-1})^T \nabla_x u\}. \quad (4.10)$$

For simplicity, one sets

$$C_{-1}(x) = A_{-1}B_{-1}(x)^T + B_{-1}(x)A_{-1}^T + B_{-1}(x)B_{-1}(x)^T,$$

and then

$$(A_{-1} + B_{-1}(x))(A_{-1} + B_{-1}(x))^T = I + C_{-1}(x). \quad (4.11)$$

Furthermore, let us define $v_\pm = \tilde{v}_\pm \circ \Phi$ and $f_\pm = \tilde{f}_\pm \circ \Phi$. Then, by (4.2) and (4.9)-(4.11), we obtain an equivalent system of (4.4) as follows:

$$\Delta v_\pm = \text{div}(F_\pm + \mathcal{F}(v_\pm)) \quad \text{in } \mathbb{R}^N_\pm,$$

$$\rho_+ v_+ = \rho_- v_- \quad \text{on } \mathbb{R}^N_0,$$

$$n_0 \cdot \nabla(v_+ - v_-) = n_0 \cdot \{(F_+ + \mathcal{F}(v_+)) - (F_- + \mathcal{F}(v_-))\} \quad \text{on } \mathbb{R}^N_0, \quad (4.12)$$

where we have set

$$F_\pm = J(A_{-1} + B_{-1})f_\pm, \quad \mathcal{F}(v_\pm) = (1 - J)\nabla v_\pm - JC_{-1} \nabla v_\pm. \quad (4.13)$$
Remark 3. (1) The above $F$ in (4.13) is linear.

(2) By (4.1), there is a constant $C_5 > 0$, independent of $M_1, M_2,$ and $r$, such that

$$
\|C_{-1}\|_{L_\infty(\mathbb{R}^N)} \leq C_5 M_1, \quad \|\nabla C_{-1}\|_{L_r(\mathbb{R}^N)} \leq C_5 M_2. \tag{4.14}
$$

In what follows, $F_{\pm N}$ stands for the $N$th component of $F_{\pm}$, while $F_N(v_\pm)$ the $N$th component of $F(v_\pm)$. By (4.3) and (4.14),

$$
\|\text{div} F_\pm\|_{L_2(\mathbb{R}^2)} \leq C_4^{1-1/\epsilon} \|\text{div} \tilde{f}_\pm\|_{L_2(\tilde{N}_\pm)}. \tag{4.15}
$$

In addition, since it holds by (4.2) that $F_{\pm N}$ is linear,

$$
F_{\pm N} = -n_0 \cdot \{J(A_{-1} + B_{-1})f_\pm\} = -Jd(\tilde{n} \cdot \tilde{f}_\pm), \tag{4.16}
$$

one has by (4.3) and Lemma 4.3 with $\epsilon = 1$

$$
\|F_{\pm N}\|_{H^j_q(\mathbb{R}^N)} \leq C_{M_2, r} \|\tilde{n} \cdot \tilde{f}_\pm\|_{H^j_q(\tilde{N}_\pm)} \quad (j = 0, 1). \tag{4.17}
$$

Here and subsequently, $C_{M_2, r}$ stands for generic positive constants depending on $M_2, N, q,$ and $r$, but independent of $M_1$. Similarly to (4.15) and (4.17),

$$
\|F_\pm\|_{L_q(\mathbb{R}^N)} \leq C_0 \|\tilde{f}_\pm\|_{L_q(\tilde{N}_\pm)}, \tag{4.18}
$$

where $C_0$ is a positive constant independent of $M_1, M_2,$ and $r$. Concerning $F(v_\pm)$, we have by Lemma 4.3, (4.3), and (4.14).

Lemma 4.4. Let $q \in (1, \infty), r \in (N, \infty), \text{ and } \epsilon > 0.$ Assume that $q \geq r$. Then there exist a positive constant $\alpha_{M_2, r, \epsilon}$ depending on $M_2, r,$ and $\epsilon$, but independent of $M_1$, and a positive constant $C_7$ independent of $M_1, M_2, r,$ and $\epsilon$, such that for any $w_\pm \in H^j_q(\mathbb{R}^N)$

$$
\|(\text{div} F(w_\pm), \nabla F(w_\pm))\|_{L_q(\mathbb{R}^N)} \leq C_7(M_1 + \epsilon) \|\nabla^2 w_\pm\|_{L_q(\mathbb{R}^N)} + \alpha_{M_2, r, \epsilon} \|\nabla w_\pm\|_{L_q(\mathbb{R}^N)},
$$

and also

$$
\|F(w_\pm)\|_{L_q(\mathbb{R}^N)} \leq C_7 M_1 \|\nabla w_\pm\|_{L_q(\mathbb{R}^N)}.
$$

By Theorem 3.1, one sees for $w_\pm \in \tilde{H}^j_q(\mathbb{R}^N) \cap \tilde{H}^2_q(\mathbb{R}^N)$ that there is a unique solution $z_\pm \in \tilde{H}^j_q(\mathbb{R}^N) \cap \tilde{H}^2_q(\mathbb{R}^N)$ to

$$
\begin{cases}
\Delta z_\pm = \text{div}(F_\pm + F(w_\pm)) & \text{in } \mathbb{R}^N, \\
\rho_+ z_+ = \rho_- z_- & \text{on } \mathbb{R}^N, \\
n_0 \cdot \nabla (z_+ - z_-) = n_0 \cdot \{(F_+ + F(w_+)) - (F_- + F(w_-))\} & \text{on } \mathbb{R}^N. \tag{4.19}
\end{cases}
$$

In addition, by (3.2) and (3.3),

$$
\sum_{s \in \{+, -\}} \|\nabla^2 z_\pm\|_{L_q(\mathbb{R}^N)} \leq C_2 \sum_{s \in \{+, -\}} \|\text{div} F_s, \nabla F_{s N}, \text{div} F_N(w_s), \nabla F_N(w_s)\|_{L_q(\mathbb{R}^N)};
$$

$$
\sum_{s \in \{+, -\}} \|\nabla z_\pm\|_{L_q(\mathbb{R}^N)} \leq C_2 \sum_{s \in \{+, -\}} \|F_s, F_N(w_s)\|_{L_q(\mathbb{R}^N)};
$$

$^3$See Remark 2 for the definitions of $J$ and $d.$
which, combined with Lemma 4.4, furnish apriori estimates for the solutions of (4.19) as follows:

\[
\sum_{s \in \{+,-\}} \left| \frac{\nabla^2 v_s}{L_q(R^N_s)} \right| \leq C_2 \sum_{s \in \{+,-\}} \left( \left| \frac{\text{div}(F_s, \nabla F_s N)}{L_q(R^N_s)} \right| + C_7(M_1 + \epsilon) \left| \frac{\nabla^2 w_s}{L_q(R^N_s)} \right| + \alpha M_2, r, \epsilon \left| \frac{\nabla w_s}{L_q(R^N_s)} \right| \right),
\]

(4.20)

\[
\sum_{s \in \{+,-\}} \left| \frac{\nabla v_s}{L_q(R^N_s)} \right| \leq C_2 \sum_{s \in \{+,-\}} \left( \left| \frac{F_s}{L_q(R^N_s)} \right| + C_7 M_1 \left| \frac{\nabla w_s}{L_q(R^N_s)} \right| \right). 
\]

(4.21)

**Remark 4.** If \( \tilde{n} \cdot (\tilde{f}_+ - \tilde{f}_-) = 0 \) on \( \tilde{\Sigma} \), then \( n_0 \cdot (F_+ - F_-) = 0 \) on \( R^N_0 \) by (4.16). When \( \tilde{n} \cdot (\tilde{f}_+ - \tilde{f}_-) = 0 \) on \( \tilde{\Sigma} \), one thus has by (3.4)

\[
\sum_{s \in \{+,-\}} \left| \frac{\nabla^2 v_s}{L_q(R^N_s)} \right| \leq C_2 \sum_{s \in \{+,-\}} \left| \frac{\text{div}(F_s, \text{div}(w_s))}{L_q(R^N_s)} \right|,
\]

and then the a priori estimate (4.20) is replaced by

\[
\sum_{s \in \{+,-\}} \left| \frac{\nabla^2 v_s}{L_q(R^N_s)} \right| \leq C_2 \sum_{s \in \{+,-\}} \left( \left| \frac{\text{div}F_s}{L_q(R^N_s)} \right| + C_7(M_1 + \epsilon) \left| \frac{\nabla^2 w_s}{L_q(R^N_s)} \right| + \alpha M_2, r, \epsilon \left| \frac{\nabla w_s}{L_q(R^N_s)} \right| \right).
\]

(4.22)

From now on, we prove the existence of solutions \( v_\pm \) to (4.12). Let \( v_\pm^{(0)} = 0 \) and \( v_\pm^{(j)} \in \tilde{H}^2_0(R^N_s) \cap \tilde{H}^1_0(R^N_s) \), \( j \geq 1 \), be the unique solutions to

\[
\begin{aligned}
\Delta v_\pm^{(j)} &= \text{div}(F_\pm + F(v_\pm^{(j-1)})) \quad \text{in } R^N_s, \\
\rho_0 v_\pm^{(j)} &= \rho_- v_\pm^{(j)} \quad \text{on } R^N_0, \\
n_0 \cdot \nabla (v_\pm^{(j)} - v_\pm^{(j-1)}) &= n_0 \cdot (F_+ + F(v_\pm^{(j-1)})) - (F_- + F(v_\pm^{(j-1)})) \quad \text{on } R^N_0.
\end{aligned}
\]

(4.23)

By (4.20) and (4.21), \( v_\pm^{(j)} \) satisfy

\[
\sum_{s \in \{+,-\}} \left| \frac{\nabla^2 v_\pm^{(j)}}{L_q(R^N_s)} \right| \leq C_2 \sum_{s \in \{+,-\}} \left( \left| \frac{\text{div}(F_s, \nabla F_s N)}{L_q(R^N_s)} \right| + C_7(M_1 + \epsilon) \left| \frac{\nabla^2 w_s}{L_q(R^N_s)} \right| + \alpha M_2, r, \epsilon \left| \frac{\nabla w_s}{L_q(R^N_s)} \right| \right),
\]

(4.24)

\[
\sum_{s \in \{+,-\}} \left| \frac{\nabla v_\pm^{(j)}}{L_q(R^N_s)} \right| \leq C_2 \sum_{s \in \{+,-\}} \left( \left| \frac{F_s}{L_q(R^N_s)} \right| + C_7 M_1 \left| \frac{\nabla v_\pm^{(j-1)}}{L_q(R^N_s)} \right| \right). 
\]

(4.25)

Inductively, it follows from (4.25) and \( v_\pm^{(0)} = 0 \) that

\[
\sum_{s \in \{+,-\}} \left| \frac{\nabla v_\pm^{(j)}}{L_q(R^N_s)} \right| \leq X_j \sum_{s \in \{+,-\}} \left| \frac{F_s}{L_q(R^N_s)} \right|, \quad X_j = C_2 \sum_{k=0}^{j-1} (C_2 C_7 M_1)^k,
\]

(4.26)

which, inserted into (4.24), furnishes

\[
\sum_{s \in \{+,-\}} \left| \frac{\nabla^2 v_\pm^{(j)}}{L_q(R^N_s)} \right| \leq C_2 \sum_{s \in \{+,-\}} \left( \left| \frac{\text{div}(F_s, \nabla F_s N)}{L_q(R^N_s)} \right| \right)
\]
+ C_7(M_1 + \varepsilon)\|\nabla^2 u_s^{(j-1)}\|_{L_q(\mathbb{R}_+^N)} + \alpha_{M_2, r, \varepsilon} Y_j \|F_s\|_{L_q(\mathbb{R}_+^N)}.  \\
Inductively, it follows from this inequality and \(v_s^{(0)} = 0\) that  \\
\[
\sum_{s \in \{+, -\}} \|\nabla^2 v_s^{(j)}\|_{L_q(\mathbb{R}_+^N)} \leq \sum_{s \in \{+, -\}} \|(\text{div} \ F_s, \nabla F_s)\|_{L_q(\mathbb{R}_+^N)} \\
+ Z_j \sum_{s \in \{+, -\}} \|F_s\|_{L_q(\mathbb{R}_+^N)}, \quad (4.27)
\]

where  \\
\[
Y_j = C_2 \sum_{k=0}^{j-1} (C_2 C_7(M_1 + \varepsilon))^k, \quad Z_j = C_2 \alpha_{M_2, r, \varepsilon} \sum_{k=0}^{j-1} (C_2 C_7(M_1 + \varepsilon))^k X_{j-k}.
\]

Let \(u^{(j)} = \rho_+ v_+^{(j)} 1_{\mathbb{R}_+^N} + \rho_- v_-^{(j)} 1_{\mathbb{R}_-^N}\). One then sees by \(\rho_+ v_+^{(j)} = \rho_- v_-^{(j)}\) on \(\mathbb{R}_0^N\) that \(u^{(j)} \in \hat{H}_q^1(\mathbb{R}^N)\) with \(\nabla u^{(j)} = \rho_+ (\nabla v_+^{(j)}) 1_{\mathbb{R}_+^N} + \rho_- (\nabla v_-^{(j)}) 1_{\mathbb{R}_-^N}\). On the other hand, recalling Remark 3 (1), we have by (4.21)

\[
\sum_{s \in \{+,-\}} \|\nabla (v_s^{(j)} - v_s^{(j-1)})\|_{L_q(\mathbb{R}_+^N)} \leq C_2 C_7 M_1 \sum_{s \in \{+,-\}} \|\nabla (v_s^{(j-1)} - v_s^{(j-2)})\|_{L_q(\mathbb{R}_+^N)},
\]

which, combined with

\[
\frac{1}{C_8} \sum_{s \in \{+,-\}} \|\nabla (v_s^{(j)} - v_s^{(j-1)})\|_{L_q(\mathbb{R}_+^N)} \leq \|\nabla (u^{(j)} - u^{(j-1)})\|_{L_q(\mathbb{R}_+^N)} \leq C_8 \sum_{s \in \{+,-\}} \|\nabla (v_s^{(j)} - v_s^{(j-1)})\|_{L_q(\mathbb{R}_+^N)}
\]

for some positive constants \(C_8 \geq 1\) depending only on \(q, \rho_+, \text{ and } \rho_-\), furnishes

\[
\|\nabla (u^{(j)} - u^{(j-1)})\|_{L_q(\mathbb{R}_+^N)} \leq C_2 C_7 (C_8)^2 M_1 \|\nabla (u^{(j-1)} - u^{(j-2)})\|_{L_q(\mathbb{R}_+^N)}.
\]

Choose \(M_1\) and \(\varepsilon\) small enough so that

\[
C_2 C_7 (C_8)^2 M_1 \leq \frac{1}{2}, \quad C_2 C_7 (M_1 + \varepsilon) \leq \frac{1}{2}.
\]

Then the above inequality for \(u^{(j)} - u^{(j-1)}\) implies that \(\{u^{(j)}\}_{j=1}^{\infty}\) is a Cauchy sequence in \(\hat{H}_q^1(\mathbb{R}_+^N)\), and thus there exists \(u \in \hat{H}_q^1(\mathbb{R}_+^N)\) such that

\[
\lim_{j \to \infty} \|\nabla (u^{(j)} - u)\|_{L_q(\mathbb{R}_+^N)} = 0. \quad (4.28)
\]

Furthermore, we see that

\[
X_j \leq C_2 \sum_{k=0}^{\infty} 2^{-k} = 2C_2, \quad Y_j \leq 2C_2, \quad Z_j \leq C_2 \alpha_{M_2, r, \varepsilon} \cdot 2 \cdot 2C_2,
\]

which, combined with (4.26) and (4.27), furnishes

\[
\sum_{s \in \{+,-\}} \|\nabla v_s^{(j)}\|_{L_q(\mathbb{R}_+^N)} \leq 2C_2 \sum_{s \in \{+,-\}} \|F_s\|_{L_q(\mathbb{R}_+^N)}, \quad (4.29)
\]

\[
\sum_{s \in \{+,-\}} \|\nabla^2 v_s^{(j)}\|_{L_q(\mathbb{R}_+^N)} \leq 2C_2 K, \quad (4.30)
\]
where we have set
\[ K = \sum_{s \in \{+,-\}} \| (\text{div} \, F_s, \nabla F_s) \|_{L_q(\mathbb{R}_+^N)} + 2C_2 \alpha_{M_2, r, \varepsilon} \sum_{s \in \{+,-\}} \| F_s \|_{L_q(\mathbb{R}_+^N)}. \]

Let \( u_+ \) and \( u_- \) be respectively the restriction of \( u \) on \( \mathbb{R}_+^N \) and the restriction of \( u \) on \( \mathbb{R}_0^N \). It then holds that \( u_+ \in \dot{H}_q^1(\mathbb{R}_+^N) \) and
\[
\nabla u = (\nabla u_+) \mathbb{1}_{\mathbb{R}_+^N} + (\nabla u_-) \mathbb{1}_{\mathbb{R}_0^N}, \quad u_+ = u_- \text{ on } \mathbb{R}_0^N. \tag{4.31}
\]
Let us define \( v_\pm = \rho_\pm^{-1} u_\pm \). By the second property of (4.31),
\[
\rho_+ v_+ = \rho_- v_- \quad \text{on } \mathbb{R}_0^N. \tag{4.32}
\]
In addition, we see by the definition of \( u^{(j)} \) and the first property of (4.31) that
\[
\| (\nabla u_\pm^{(j)} - v_\pm) \|_{L_q(\mathbb{R}_+^N)} = \rho_\pm^{-1} \| (\nabla u^{(j)} - u) \|_{L_q(\mathbb{R}_+^N)}
\]
which, combined with (4.28), furnishes
\[
\lim_{j \to \infty} \| (\nabla u_\pm^{(j)} - v_\pm) \|_{L_q(\mathbb{R}_+^N)} = 0. \tag{4.33}
\]
Taking the the limit: \( j \to \infty \) in (4.29) thus implies that \( v_\pm \) satisfy
\[
\sum_{s \in \{+,-\}} \| \nabla v_s \|_{L_q(\mathbb{R}_+^N)} \leq 2C_2 \sum_{s \in \{+,-\}} \| F_s \|_{L_q(\mathbb{R}_+^N)}. \tag{4.34}
\]

Next, we consider the higher regularity of \( v_\pm \). Let \( k, l = 1, \ldots, N \). By (4.30), there exist \( v_{\pm,kl} \in L_q(\mathbb{R}_+^N) \) such that \( \partial_k \partial_l u^{(j)}_\pm \to v_{\pm,kl} \) weakly in \( L_q(\mathbb{R}_+^N) \) as \( j \to \infty \). Then we can prove \( \partial_k \partial_l v_\pm = v_{\pm,kl} \in L_q(\mathbb{R}_+^N) \) by using the convergence in distribution. Consequently, \( \partial_k \partial_l u^{(j)}_\pm \to \partial_k \partial_l v_\pm \) weakly in \( L_q(\mathbb{R}_+^N) \) as \( j \to \infty \). It thus follows from (4.30) that
\[
\| \partial_k \partial_l v_\pm \|_{L_q(\mathbb{R}_+^N)} \leq \liminf_{j \to \infty} \| \partial_k \partial_l u^{(j)}_\pm \|_{L_q(\mathbb{R}_+^N)} \leq 2C_2 K. \tag{4.35}
\]

Let us prove that \( v_\pm \) satisfies (4.12) by the limit: \( j \to \infty \) in (4.23). The first line of (4.12) is satisfied immediately, and the third line of (4.12) can be proved by Remark 3 (1), Lemma 4.4, and the inequality (cf. [4, Proposition 16.2]):
\[
\| f_\pm \|_{L_q(\mathbb{R}_+^N)} \leq q^{1/q} \| f_\pm \|_{L_q(\mathbb{R}_+^N)}^{1-1/q} \| \nabla f_\pm \|_{L_q(\mathbb{R}_+^N)}^{1/q} \quad \text{for any } f_\pm \in H^1_q(\mathbb{R}_+^N),
\]
and the second line of (4.12) is already obtained in (4.32).

Finally, setting \( \tilde{v}_\pm = v_\pm \circ \Phi^{-1} \), we observe that \( \tilde{v}_\pm \) are solutions to (4.4) and that by (4.1), (4.3), and Lemma 4.3 with \( \varepsilon = 1 \)
\[
\| \nabla \tilde{v}_\pm \|_{L_q(\tilde{\Omega}_\pm)} \leq C \| \nabla v_\pm \|_{L_q(\mathbb{R}_+^N)}, \quad \| \nabla^2 \tilde{v}_\pm \|_{L_q(\tilde{\Omega}_\pm)} \leq C \| \nabla^2 v_\pm \|_{L_q(\mathbb{R}_+^N)}
\]
for some positive constant \( C = C(M_2, N, q, r) \) independent of \( M_1 \). Combining these inequalities with (4.34) and (4.35), together with (4.15), (4.17), and (4.18), furnishes (4.4) and (4.5). When \( \hat{n} \cdot (\hat{e}_\pm - \hat{e}_-) = 0 \) on \( \hat{\Sigma} \), we can obtain (4.7) by using (4.22) instead of (4.20) in the above argument. The uniqueness of solutions of (4.4) follows from the existence of solutions of (4.4) similarly to the proof of Theorem 3.1 (2). This completes the proof of the Theorem 4.1.
4.2. Weak elliptic problem with bent interface. Throughout this subsection, we assume that \( r \in (N, \infty) \) and \( \Omega_{\pm}, \Sigma \) are given in Theorem 4.1. For \( f \in L_q(\mathbb{R}^N \setminus \Sigma)^N \) with \( q \in (1, \infty) \), this subsection considers the unique solvability of the following weak elliptic problem: Find \( u \in \tilde{H}^1_q(\mathbb{R}^N) \) such that
\[
(\rho^{-1}\nabla u, \nabla \varphi)_{\mathbb{R}^N/N} = (f, \nabla \varphi)_{\mathbb{R}^N/N} \quad \text{for any} \quad \varphi \in \tilde{H}^1_q(\mathbb{R}^N),
\]
where \( q' = q/(q - 1) \) and \( \rho = \rho_{+} \mathbb{1}_{\tilde{\Omega}_{+}} + \rho_{-} \mathbb{1}_{\tilde{\Omega}_{-}} \) for positive constants \( \rho_{\pm} \). In the same manner that we have obtained Theorem 3.5 from Theorem 3.1, we can obtain the following theorem from Theorem 4.1.

**Theorem 4.5.** Let \( r, \tilde{\Omega}_{\pm}, \) and \( \tilde{\Sigma} \) be as above, and let \( q \in (1, \infty) \). Assume that \( \max(q, q') \leq r \) for \( q' = q/(q - 1) \) and \( \rho = \rho_{+} \mathbb{1}_{\tilde{\Omega}_{+}} + \rho_{-} \mathbb{1}_{\tilde{\Omega}_{-}} \) for constants \( \rho_{\pm} > 0 \).

1. **Existence.** Let \( f \in L_q(\mathbb{R}^N \setminus \Sigma)^N \). Then the weak elliptic problem (4.36) admits a solution \( u \in \tilde{H}^1_q(\mathbb{R}^N) \) satisfying
\[
\|\nabla u\|_{L_q(\mathbb{R}^N)} \leq 2C'(\rho_{+} + \rho_{-})\|f\|_{L_q(\mathbb{R}^N \setminus \Sigma)},
\]
where \( C' \) is the same constant as in Theorem 4.1.

2. **Uniqueness.** If \( u \in \tilde{H}^1_q(\mathbb{R}^N) \) satisfies
\[
(\rho^{-1}\nabla u, \nabla \varphi)_{\mathbb{R}^N/N} = 0 \quad \text{for any} \quad \varphi \in \tilde{H}^1_q(\mathbb{R}^N),
\]
then \( u = c \) for some constant \( c \).

5. Strong problems in bounded domains. Throughout this section, we assume

**Assumption 2.** (a) \( r \) is a real number satisfying \( r > N \).
(b) \( G \) is a bounded domain with boundary \( \Gamma \) of class \( W^{2-1/r}_r \).
(c) \( G_{+} \) is a subdomain of \( G \) with boundary \( \Sigma \) of class \( W^{2-1/r}_r \) satisfying \( \Sigma \cap \Gamma = \emptyset \).
(d) \( G_{-} = G \setminus (G_{+} \cup \Sigma) \).

Let \( n \) be a unit normal vector on \( \Sigma \) pointing from \( G_{+} \) into \( G_{-} \), and set
\[
F_q(G_{\pm}) = \{f_{\pm} \in E_q(G_{\pm}) : n \cdot f_{\pm} \in H^1_q(G_{\pm})\}.
\]

In this section, we consider the following strong elliptic problem:
\[
\begin{aligned}
\Delta v_{\pm} &= \text{div} f_{\pm} & & \text{in} \ G_{\pm}, \\
\rho_{+} v_{+} &= \rho_{-} v_{-} & & \text{on} \ \Sigma, \\
\mathbf{n} \cdot \nabla (v_{+} - v_{-}) &= \mathbf{n} \cdot (f_{+} - f_{-}) & & \text{on} \ \Sigma, \\
v_{-} &= 0 & & \text{on} \ \Gamma.
\end{aligned}
\]

(5.1)

More precisely, we prove

**Theorem 5.1.** Suppose that Assumption 2 holds and \( \rho_{\pm} \) are positive constants. Let \( q \in (1, \infty) \) with \( \max(q, q') \leq r \) for \( q' = q/(q - 1) \).

1. **Existence.** Let \( f_{\pm} \in F_q(G_{\pm}) \). Then the strong elliptic problem (5.1) admits solutions \( v_{\pm} \in H^2_q(G_{\pm}) \) satisfying
\[
\|\nabla^2 v_{\pm}\|_{L_q(G_{\pm})} \leq C \sum_{s \in \{+,-\}} \left(\|f_{s}\|_{E_q(G_{s})} + \|n \cdot f_{s}\|_{H^1_q(G_{s})}\right),
\]
\[
\|v_{\pm}\|_{H^1_q(G_{\pm})} \leq C \sum_{s \in \{+,-\}} \|f_{s}\|_{L_q(G_{s})},
\]

(5.2) (5.3)
with a positive constant $C = C(N, q, r, \rho_+, \rho_-)$. Additionally, if $\mathbf{n} \cdot (\mathbf{f}_+ - \mathbf{f}_- ) = 0$ on $\Sigma$, then $v_\pm$ satisfy
\[
\|\nabla^2 v_\pm\|_{L^q(G_\pm)} \leq C \sum_{s \in \{-, +\}} \|f_s\|_{L^q(G_s)}.
\]

(2) **Uniqueness.** If $v_\pm \in H^2_q(G_\pm)$ satisfy
\[
\Delta v_\pm = 0 \text{ in } G_\pm; \quad \rho_+ v_+ = \rho_- v_-, \quad \mathbf{n} \cdot \nabla(v_+ - v_-) = 0 \text{ on } \Sigma; \quad v_- = 0 \text{ on } \Gamma,
\]
then $v_\pm = 0$.

5.1. **Strong elliptic problem with $\lambda$ in $G_\pm$.** To prove Theorem 5.1, we consider
\[
\begin{cases}
\rho_+ \lambda v_+ - \Delta v_+ = -\text{div } \mathbf{f}_+ + g_+ & \text{in } G_+, \\
\rho_- v_- = \rho_+ v_+ & \text{on } \Sigma, \\
\mathbf{n} \cdot \nabla(v_+ - v_-) = \mathbf{n} \cdot (\mathbf{f}_+ - \mathbf{f}_-) + h_+ - h_- & \text{on } \Sigma, \\
v_- = 0 & \text{on } \Gamma.
\end{cases}
\]

Note that [24, Appendix B] already studies the strong elliptic problem in $\mathbb{R}^N$ with the Dirichlet boundary condition in the case where $\lambda$ is taken into account, so that resolvent estimates of solutions in the bent half-space case are also available similarly to the proof of Theorem 4.2. Combining that result with Theorem 4.2 and the standard localization technique yields

**Theorem 5.2.** Suppose that the same assumption as in Theorem 5.1 holds, and let $\sigma \in (0, \pi/2)$. Then there is a constant $\lambda_2 \geq 1$ such that, for any $\lambda \in \Sigma_{\sigma, \lambda_2}$,
\[
g_\pm \in L^q(G_\pm), \quad f_s \in F^q(G_\pm), \quad h_\pm \in H^1_q(G_\pm),
\]
the strong elliptic problem (5.5) admits unique solutions $v_\pm \in H^2_q(G_\pm)$. In addition, the solutions $v_\pm$ satisfy
\[
\| (\lambda v_\pm, \lambda^{1/2} \nabla v_\pm, \nabla^2 v_\pm) \|_{L^q(G_\pm)} \leq C \sum_{s \in \{-, +\}} \| (\text{div } f_s, g_s, \lambda^{1/2} (\mathbf{n} \cdot f_s), \nabla (\mathbf{n} \cdot f_s), \lambda^{1/2} h_s, \nabla h_s) \|_{L^q(G_s)},
\]
and also
\[
\| (\lambda^{1/2} v_\pm, \nabla v_\pm) \|_{L^q(G_\pm)} \leq C \sum_{s \in \{-, +\}} \left( \|f_s\|_{L^q(\Omega_s)} + |\lambda|^{-1/2} \|g_+\|_{L^q(G_s)} + \|h_s\|_{L^q(G_s)} + |\lambda|^{-1/2} \|\nabla h_s\|_{L^q(G_s)} \right),
\]
where $C = C(N, q, r, \rho_+, \rho_-, \sigma)$ is a positive constant independent of $\lambda$. Additionally, if $\mathbf{n} \cdot (\mathbf{f}_+ - \mathbf{f}_-) = 0$ on $\Sigma$, then $v_\pm$ satisfy
\[
\| (\lambda v_\pm, \lambda^{1/2} \nabla v_\pm, \nabla^2 v_\pm) \|_{L^q(G_\pm)} \leq C \sum_{s \in \{-, +\}} \| (\text{div } f_s, g_s, \lambda^{1/2} h_s, \nabla h_s) \|_{L^q(G_s)}.
\]
5.2. **Proof of Theorem 5.1.** We start with the following strong elliptic problem:

\[
\begin{cases}
-\Delta v_\pm = d_\pm & \text{in } G_\pm, \\
\rho_+ v_+ = \rho_- v_- & \text{on } \Sigma, \\
\mathbf{n} \cdot \nabla(v_+ - v_-) = 0 & \text{on } \Sigma, \\
v_- = 0 & \text{on } \Gamma.
\end{cases}
\] (5.6)

Concerning this problem, we have

**Theorem 5.3.** Suppose that the same assumption as in Theorem 5.1 holds. Then, for any \(d_\pm \in L_q(G_\pm)\), the strong elliptic problem (5.6) admits unique solutions \(v_\pm \in H^2_q(G_\pm)\), which satisfy

\[
\sum_{s \in \{+,-\}} \|v_s\|_{H^2_q(G_s)} \leq C \sum_{s \in \{+,-\}} \|d_s\|_{L_q(G_s)}
\]

for some positive constant \(C = C(N,q,r,\rho_+,\rho_-)\).

**Proof.** The proof is based on the Riesz-Schauder theory together with Theorem 5.2, so that the detailed proof may be omitted.

Now we prove Theorem 5.1. Let \(\mu = 2\lambda^2\) for \(\lambda^2\) introduced in Theorem 5.2. Theorem 5.2 yields \(w_\pm \in H^2_q(G_\pm)\) such that

\[
\begin{cases}
\rho_\pm \mu w_\pm - \Delta w_\pm = - \text{div} f_\pm & \text{in } G_\pm, \\
\rho_+ w_+ = \rho_- w_- & \text{on } \Sigma, \\
\mathbf{n} \cdot \nabla(w_+ - w_-) = \mathbf{n} \cdot (f_+ - f_-) & \text{on } \Sigma, \\
w_- = 0 & \text{on } \Gamma,
\end{cases}
\]

while Theorem 5.3 yields \(z_\pm \in H^2_q(G_\pm)\) such that

\[
\begin{cases}
-\Delta z_\pm = \rho_\pm \mu w_\pm & \text{in } G_\pm, \\
\rho_+ z_+ = \rho_- z_- & \text{on } \Sigma, \\
\mathbf{n} \cdot \nabla(z_+ - z_-) = 0 & \text{on } \Sigma, \\
z_- = 0 & \text{on } \Gamma.
\end{cases}
\]

Thus \(v_\pm = w_\pm + z_\pm\) become solutions to (5.1) and satisfy (5.2)-(5.4) by Theorems 5.2 and 5.3. The uniqueness of solutions of (5.1) is already proved in Theorem 5.3. This completes the proof of Theorem 5.1.

6. **Proof of Theorems 1.1 and 1.2.** Throughout this section, we assume that \(\Omega_\pm\) satisfy Assumption 1.

First, let us introduce the notation used in this section. For \(L > 0\), we set

\[
B_L = \{x \in \mathbb{R}^N : |x| < L\}, \quad S_L = \{x \in \mathbb{R}^N : |x| = L\}.
\]

Fix \(R > 0\) such that \(\Omega_+ \cup \Sigma \subset B_R\). We then define \(G, \Gamma,\) and \(G_\pm\) as follows:

\[
G = B_{4R}, \quad \Gamma = S_{4R}, \quad G_+ = \Omega_+, \quad G_- = B_{4R} \setminus (\Omega_+ \cup \Sigma).
\]

Let \(\varphi, \psi_0,\) and \(\psi_\infty\) be functions in \(C^\infty(\mathbb{R}^N)\) satisfying \(0 \leq \varphi, \psi_0, \psi_\infty \leq 1\) and

\[
\varphi(x) = \begin{cases} 1 & \text{for } x \in B_{2R}, \\ 0 & \text{for } x \in \mathbb{R}^N \setminus B_{3R}, \end{cases},
\]

\[
\psi_0(x) = \begin{cases} 1 & \text{for } x \in B_{(3+1/3)R}, \\ 0 & \text{for } x \in \mathbb{R}^N \setminus B_{(3+2/3)R}, \end{cases},
\]

\[
\psi_\infty(x) = \begin{cases} 1 & \text{for } x \in B_{(3+2/3)R}, \\ 0 & \text{for } x \in \mathbb{R}^N \setminus B_{(3+1/3)R}, \end{cases}.
\]
followed from Theorem 3.1. to the proof of Theorem 3.1 (2), while Theorem 1.2 follows from Theorem 1.1 as Theorem 3.5 has

In addition, we set

\[ \varphi(x) = \begin{cases} 0 & \text{for } x \in B(2-2/3)_R, \\ 1 & \text{for } x \in \mathbb{R}^N \setminus B(2-1/3)_R. \end{cases} \]

One sees that

\[ f \in \mathbb{R} \to \frac{\psi(x)}{\partial x} \in \mathbb{R} \]

we consider the following equivalent system of (1.5) for simplicity of notation:

\[ \| f \|_{E_q(D \setminus \Sigma)} = \| f \|_{D_{+}} + \| f \|_{D_{-}} \]

where \( f \in D_{+} \) stands for the restriction of \( f \) on \( D_{+} \), respectively. On the other hand,

\[ F_q(D \setminus \Sigma) = \{ f \in E_q(D \setminus \Sigma) : n \cdot f \}_{D_{+}} \in H^1_q(D_{+}) \]

\[ \| f \|_{F_q(D \setminus \Sigma)} = \| f \|_{E_q(D \setminus \Sigma)} + \| n \cdot f \|_{H^1_q(D_{+})}. \]

**Remark 5.** One sees that \( f \in F_q(D \setminus \Sigma) \) is equivalent to \( f = f_{+} \mathbb{1}_{D_{+}} + f_{-} \mathbb{1}_{D_{-}} \) for some \( f_{\pm} \in E_q(D_{\pm}) \) with \( n \cdot f_{\pm} \in H^1_q(\Omega_{\pm}) \).

The aim of this section is to prove Theorems 1.1 and 1.2. To this end, it suffices to prove the existence of solutions of (1.5) satisfying (1.6)-(1.8). Instead of (1.5), we consider the following equivalent system of (1.5) for simplicity of notation:

\[ \begin{cases} \Delta v = \text{div } f & \text{in } \mathbb{R}^N \setminus \Sigma, \\ [\rho v] = 0 & \text{on } \Sigma, \\ [n \cdot \nabla v] = [n \cdot f] & \text{on } \Sigma, \end{cases} \]

where \( \rho = \rho_{+} \mathbb{1}_{\Omega_{+}} + \rho_{-} \mathbb{1}_{\Omega_{-}} \) for positive constants \( \rho_{\pm} \) and

\[ \| f \|_{(x_0)} = \lim_{\varepsilon \to 0^+} (f(x_0 - \varepsilon n) - f(x_0 + \varepsilon n)) \quad (x_0 \in \Sigma). \]

Note that \( n \) points from \( \Omega_{+} \) into \( \Omega_{-} \). We here recall the assumption for \( q \in (1, \infty) \).

**Assumption 3.** Let \( q \in (1, \infty) \) and \( \max(q, q') \leq r \) for \( q' = q/(q - 1) \).

As mentioned above, it suffices to prove in this section

**Theorem 6.1.** Suppose that Assumption 3 holds. Then, for any \( f \in F_q(\mathbb{R}^N \setminus \Sigma) \), the system (6.1) admits a solution \( v \in \tilde{H}^1_q(\mathbb{R}^N \setminus \Sigma) \cap \tilde{H}^2_q(\mathbb{R}^N \setminus \Sigma) \) satisfying

\[ \| \nabla^2 v \|_{L_q(\mathbb{R}^N \setminus \Sigma)} \leq C \| f \|_{E_q(\mathbb{R}^N \setminus \Sigma)}, \quad \| \nabla v \|_{L_q(\mathbb{R}^N \setminus \Sigma)} \leq C \| f \|_{L_q(\mathbb{R}^N \setminus \Sigma)}, \]

with some positive constant \( C = C(N, q, r, \rho_{+}, \rho_{-}) \). Additionally, if \( n \cdot (f_{+} - f_{-}) = 0 \) on \( \Sigma \), then \( v \) satisfies

\[ \| \nabla^2 v \|_{L_q(\mathbb{R}^N \setminus \Sigma)} \leq C \| f \|_{E_q(\mathbb{R}^N \setminus \Sigma)} \]

for some positive constant \( C = C(R, N, q, r, \rho_{+}, \rho_{-}) \).

---

4The uniqueness of solutions of (1.5) follows from the existence of solutions of (1.5) similarly to the proof of Theorem 3.1 (2), while Theorem 1.2 follows from Theorem 1.1 as Theorem 3.5 has followed from Theorem 3.1.
6.1. **Solution operators.** First, let us consider the following problem in the whole space:

$$\Delta V = \text{div} F \quad \text{in} \quad \mathbb{R}^N. \quad (6.4)$$

Similarly to [24, Page 1700], we obtain

**Lemma 6.2.** Let \( q \in (1, \infty) \). Then there exists a linear operator

$$S_\infty : E_q(\mathbb{R}^N) \rightarrow \tilde{H}_q^1(\mathbb{R}^N) \cap \tilde{H}_q^2(\mathbb{R}^N)$$

such that \( V = S_\infty F \) is a solution to (6.4). In addition,

$$\|S_\infty F\|_{L_q(G)} + \|\nabla S_\infty F\|_{L_q(\mathbb{R}^N)} \leq C\|F\|_{L_q(\mathbb{R}^N)},$$

$$\|\nabla^2 S_\infty F\|_{L_q(\mathbb{R}^N)} \leq C\|\text{div} F\|_{L_q(\mathbb{R}^N)},$$

with some positive constant \( C = C(R, N, q) \).

We next consider the following problem in a bounded domain:

$$\begin{cases}
\Delta V = \text{div} F & \text{in} \ G \setminus \Sigma, \\
\|\rho V\| = 0 & \text{on} \ \Sigma, \\
[n \cdot \nabla V] = [n \cdot F] & \text{on} \ \Sigma, \\
V = 0 & \text{on} \ \Gamma.
\end{cases} \quad (6.5)$$

Then, by Theorem 5.1, we have

**Lemma 6.3.** Suppose that Assumption 3 holds. Then there is a linear operator

$$S_0 : F_q(G \setminus \Sigma) \rightarrow \tilde{H}_q^2(G \setminus \Sigma)$$

such that, for any \( F \in F_q(G \setminus \Sigma) \), \( V = S_0 F \) is a solution to (6.5). In addition,

$$\|\nabla^2 S_0 F\|_{L_q(G \setminus \Sigma)} \leq C\|F\|_{F_q(G \setminus \Sigma)}, \quad \|S_0 F\|_{H_q^3(G \setminus \Sigma)} \leq C\|F\|_{L_q(G \setminus \Sigma)},$$

with some positive constant \( C = C(N, q, r, \rho_+, \rho_-) \). Additionally, if \( [n \cdot F] = 0 \) on \( \Sigma \), then it holds that

$$\|\nabla^2 S_0 F\|_{L_q(G \setminus \Sigma)} \leq C\|F\|_{E_q(G \setminus \Sigma)}$$

for some positive constant \( C = C(N, q, r, \rho_+, \rho_-) \).

Let us define an operator \( \mathcal{S} : F_q(\mathbb{R}^N \setminus \Sigma) \rightarrow \tilde{H}_q^1(\mathbb{R}^N \setminus \Sigma) \cap \tilde{H}_q^2(\mathbb{R}^N \setminus \Sigma) \) by

$$\mathcal{S}f = \psi_\infty S_\infty(\varphi_\infty f) + \psi_0 S_0(\varphi_0 f) \quad (f \in F_q(\mathbb{R}^N \setminus \Sigma)). \quad (6.6)$$

Note that \( \varphi_\infty f \in E_q(\mathbb{R}^N) \) and \( \varphi_0 f \in F_q(G \setminus \Sigma) \) when \( f \in F_q(\mathbb{R}^N \setminus \Sigma) \). By Lemmas 6.2 and 6.3, the above \( \mathcal{S}f \) satisfies the following estimates:

$$\|\nabla^2 \mathcal{S}f\|_{L_q(\mathbb{R}^N \setminus \Sigma)} \leq C\|f\|_{E_q(\mathbb{R}^N \setminus \Sigma)}, \quad \|\nabla \mathcal{S}f\|_{L_q(\mathbb{R}^N \setminus \Sigma)} \leq C\|f\|_{L_q(\mathbb{R}^N \setminus \Sigma)}, \quad (6.7)$$

Additionally, if \( n \cdot (f_+ - f_-) = 0 \) on \( \Sigma \), then it holds that

$$\|\nabla^2 \mathcal{S}f\|_{L_q(\mathbb{R}^N \setminus \Sigma)} \leq C\|f\|_{E_q(\mathbb{R}^N \setminus \Sigma)}. \quad (6.8)$$

At this point, we introduce a function space \( L_q(\mathbb{R}^N) \) for \( q \in (1, \infty) \). Let \( R_1 = (2 - 2/3)R \) and \( R_2 = (3 + 2/3)R \), and then

$$L_q(\mathbb{R}^N) = \{ f \in L_q(\mathbb{R}^N) : \text{supp} f \subset D_{R_1,R_2}, \langle f, 1 \rangle_{L_q(\mathbb{R}^N)} = 0 \}$$

and \( \|f\|_{L_q(\mathbb{R}^N)} = \|f\|_{L_q(\mathbb{R}^N)} \), where \( D_{R_1,R_2} = \{ x \in \mathbb{R}^N : R_1 \leq |x| \leq R_2 \} \).

Now we consider for \( f \in L_q(\mathbb{R}^N) \) the following problem in the whole space:

$$\Delta v = f \quad \text{in} \ \mathbb{R}^N. \quad (6.9)$$

For this problem, we have
Lemma 6.4. Let $q \in (1, \infty)$. Then there exists a linear operator
\[ \mathcal{T}_\infty : L_q(\mathbb{R}^N) \to \tilde{H}^1_q(\mathbb{R}^N) \cap \tilde{H}^2_q(\mathbb{R}^N) \]
such that, for any $f \in L_q(\mathbb{R}^N)$, $v = \mathcal{T}_\infty f$ is a solution to (6.9). In addition,
\[
\begin{align*}
\|\mathcal{T}_\infty f\|_{L_q(G)} + \sup_{|x| \geq 4R} |x|^{N-1} \|\mathcal{T}_\infty f(x)\| & \leq C \|f\|_{L_q(\mathbb{R}^N)}, \\
\|\nabla \mathcal{T}_\infty f\|_{L_q(\mathbb{R}^N)} + \sup_{|x| \geq 4R} |x|^N |\nabla \mathcal{T}_\infty f(x)| & \leq C \|f\|_{L_q(\mathbb{R}^N)}, \\
\|\nabla^2 \mathcal{T}_\infty f\|_{L_q(\mathbb{R}^N)} & \leq C \|f\|_{L_q(\mathbb{R}^N)},
\end{align*}
\]
with some positive constant $C = C(R, N, q)$.

Proof. See e.g. [24, pp.1703–1704] and [30, Lemma 1].

Next, we consider the following problem in a bounded domain:
\[
\begin{cases}
\Delta v = f & \text{in } G \setminus \Sigma, \\
[\rho v] = 0 & \text{on } \Sigma, \\
[n \cdot \nabla v] = 0 & \text{on } \Sigma, \\
v = 0 & \text{on } \Gamma.
\end{cases}
\tag{6.10}
\]

By Theorem 5.3, we have

Lemma 6.5. Suppose that Assumption 3 holds. Then there is a linear operator
\[ \mathcal{T}_0 : L_q(G \setminus \Sigma) \to H^2_q(G \setminus \Sigma) \]
such that, for any $f \in L_q(G \setminus \Sigma)$, $v = \mathcal{T}_0 f$ is a solution to (6.10). In addition,
\[ \|\mathcal{T}_0 f\|_{H^2_q(G \setminus \Sigma)} \leq C \|f\|_{L_q(G \setminus \Sigma)} \]
for some positive constant $C = C(N, q, r, \rho_+, \rho_-)$.

Finally, we introduce $\mathcal{T} : L_q(\mathbb{R}^N) \to \tilde{H}^1_q(\mathbb{R}^N \setminus \Sigma) \cap \tilde{H}^2_q(\mathbb{R}^N \setminus \Sigma)$ as follows: For $f \in L_q(\mathbb{R}^N)$, we choose a constant $c_f$ so that
\[ \int_{D_{R_3, R_4}} (\tilde{T}_\infty f + c_f) \, dx = 0, \tag{6.11} \]
where $D_{R_3, R_4} = \{ x \in \mathbb{R}^N : R_3 \leq |x| \leq R_4 \}$ for $R_3 = (3 + 1/3)R$ and $R_4 = (4 - 1/3)R$. Let us define
\[ \mathcal{T}_\infty f = \tilde{T}_\infty f + c_f, \quad \mathcal{T} f = \varphi_\infty \mathcal{T}_\infty f + \varphi_0 \mathcal{T}_0 f. \tag{6.12} \]

We then have by (6.11)
\[ \int_{D_{R_3, R_4}} \mathcal{T} f \, dx = \int_{D_{R_3, R_4}} \mathcal{T}_\infty f \, dx = 0. \tag{6.13} \]
In addition, since $c_f = -|D_{R_3, R_4}|^{-1} \int_{D_{R_3, R_4}} \tilde{T}_\infty f \, dx$, it holds by Lemma 6.4 that
\[ |c_f| \leq \frac{1}{|D_{R_3, R_4}|} \int_{D_{R_3, R_4}} |\tilde{T}_\infty f| \, dx \leq C \|\tilde{T}_\infty f\|_{L_q(G)} \leq C \|f\|_{L_q(\mathbb{R}^N)}. \]

Thus $\mathcal{T} f$ satisfies, together with Lemmas 6.4 and 6.5, the following estimate:
\[ \|\nabla \mathcal{T} f\|_{L_q(\mathbb{R}^N \setminus \Sigma)} + \|\nabla^2 \mathcal{T} f\|_{L_q(\mathbb{R}^N \setminus \Sigma)} \leq C \|f\|_{L_q(\mathbb{R}^N)}. \tag{6.14} \]
6.2. An auxiliary problem. In this subsection, we consider for \( f \in \mathcal{L}_q(\mathbb{R}^N) \) the following auxiliary problem:

\[
\begin{cases}
    \Delta v = f & \text{in } \mathbb{R}^N \setminus \Sigma, \\
    [\rho v] = 0 & \text{on } \Sigma, \\
    [n \cdot \nabla v] = 0 & \text{on } \Sigma.
\end{cases}
\]

(6.15)

Concerning this system, we prove

**Lemma 6.6.** Suppose that Assumption 3 holds. Let \( q \in [2, \infty) \) additionally and \( f \in \mathcal{L}_q(\mathbb{R}^N) \). Then (6.15) admits a solution \( v \in \tilde{H}^1_q(\mathbb{R}^N \setminus \Sigma) \cap \tilde{H}^2_q(\mathbb{R}^N \setminus \Sigma) \) satisfying

\[
\| \nabla v \|_{\mathcal{L}_q(\mathbb{R}^N \setminus \Sigma)} + \| \nabla^2 v \|_{\mathcal{L}_q(\mathbb{R}^N \setminus \Sigma)} \leq C \| f \|_{\mathcal{L}_q(\mathbb{R}^N)}
\]

(6.16)

for some positive constant \( C = C(R, N, q, \rho_+, \rho_-) \).

**Proof.** Let \( T \) be the operator defined as (6.12). Then, in \( \mathbb{R}^N \setminus \Sigma \),

\[
\Delta T f = \varphi_\infty \Delta T_\infty f + 2\nabla \varphi_\infty \cdot \nabla T_\infty f + (\Delta \varphi_\infty) T_\infty f \\
+ \varphi_0 \Delta T_0 f + 2\nabla \varphi_0 \cdot \nabla T_0 f + (\Delta \varphi_0) T_0 f.
\]

Since

\[
\varphi_\infty \Delta T_\infty f = \varphi_\infty f, \quad \varphi_0 \Delta T_0 f = \varphi_0 f, \quad \varphi_0 + \varphi_\infty = 1,
\]

it holds that

\[
\Delta T f = f + G f \quad \text{in } \mathbb{R}^N \setminus \Sigma,
\]

where

\[
G f = 2\nabla \varphi_\infty \cdot \nabla T_\infty f + (\Delta \varphi_\infty) T_\infty f + 2\nabla \varphi_0 \cdot \nabla T_0 f + (\Delta \varphi_0) T_0 f.
\]

Thus we have achieved

\[
\begin{cases}
    \Delta T f = f + G f & \text{in } \mathbb{R}^N \setminus \Sigma, \\
    [\rho T f] = 0 & \text{on } \Sigma, \\
    [n \cdot \nabla T f] = 0 & \text{on } \Sigma.
\end{cases}
\]

(6.18)

The following lemma is proved in the appendix A below.

**Lemma 6.7.** Suppose that Assumption 3 holds.

1. Then \( G \) is a compact operator on \( \mathcal{L}_q(\mathbb{R}^N) \).
2. Let \( q \in [2, \infty) \) additionally. Then \((I + G)^{-1}\) exists in \( \mathcal{L}(\mathcal{L}_q(\mathbb{R}^N)) \).

Setting \( v = T(I + G)^{-1} f \) for \( f \in \mathcal{L}_q(\mathbb{R}^N) \), we see that \( v \) is a solution to (6.15) by (6.18) and satisfies (6.16) by (6.14). This completes the proof of Lemma 6.6 \( \square \)

6.3. Proof of Theorem 6.1. Let \( S \) be the operator defined as (6.6). Then we observe that in \( \mathbb{R}^N \setminus \Sigma \)

\[
\Delta S f = \psi_\infty \Delta S_\infty (\varphi_\infty f) + 2\nabla \psi_\infty \cdot \nabla S_\infty (\varphi_\infty f) + (\Delta \psi_\infty) S_\infty (\varphi_\infty f) \\
+ \psi_0 \Delta S_0 (\varphi_0 f) + 2\nabla \psi_0 \cdot \nabla S_0 (\varphi_0 f) + (\Delta \psi_0) S_0 (\varphi_0 f).
\]

Note that in \( \mathbb{R}^N \setminus \Sigma \)

\[
\psi_\infty \Delta S_\infty (\varphi_\infty f) + \psi_0 \Delta S_0 (\varphi_0 f) = \psi_\infty \text{div}(\varphi_\infty f) + \psi_0 \text{div}(\varphi_0 f) = \text{div } f.
\]

This relation implies

\[
\Delta S f = \text{div } f + R f \quad \text{in } \mathbb{R}^N \setminus \Sigma,
\]

where

\[
R f = 2\nabla \psi_\infty \cdot \nabla S_\infty (\varphi_\infty f) + (\Delta \psi_\infty) S_\infty (\varphi_\infty f)
\]
On the other hand, by (6.7), (6.8), and (6.22). This completes the proof of Theorem 6.1 for Lemma A.1.

Proof. Step 1: \( f \in L_q(\mathbb{R}^N) \). We start with

\[ [\rho \mathcal{S}f] = 0, \quad [\mathbf{n} \cdot \nabla \mathcal{S}f] = [\mathbf{n} \cdot f] \quad \text{on } \Sigma, \]

and also one has for \( f \in F_q(\mathbb{R}^N \setminus \Sigma) \)

\[ \mathcal{R}f \in \mathbb{L}_q(\mathbb{R}^N), \quad \| \mathcal{R}f \|_{\mathbb{L}_q(\mathbb{R}^N)} \leq C \| f \|_{\mathbb{L}_q(\mathbb{R}^N \setminus \Sigma)}. \] (6.20)

Next, we consider

\[ \begin{cases} \Delta w = -\mathcal{R}f & \text{in } \mathbb{R}^N \setminus \Sigma, \\ [\rho w] = 0 & \text{on } \Sigma, \\ [\mathbf{n} \cdot \nabla w] = 0 & \text{on } \Sigma. \end{cases} \] (6.21)

In the following, the discussion of \( w \) is divided into two cases.

**Case 1:** \( q \in [2, \infty) \). By Lemma 6.6 and (6.20), the system (6.21) admits a solution \( w \in \tilde{H}^1_q(\mathbb{R}^N \setminus \Sigma) \cap \tilde{H}^2_q(\mathbb{R}^N \setminus \Sigma) \) satisfying

\[ \| \nabla w \|_{\mathbb{L}_q(\mathbb{R}^N \setminus \Sigma)} + \| \nabla^2 w \|_{\mathbb{L}_q(\mathbb{R}^N \setminus \Sigma)} \leq C \| f \|_{\mathbb{L}_q(\mathbb{R}^N \setminus \Sigma)}. \] (6.22)

Thus \( v = \mathcal{S}f + w \) solves (6.1) and satisfies (6.2)-(6.3) by (6.7), (6.8), and (6.22). This completes the proof of Theorem 6.1 for \( q \in [2, \infty) \).

**Case 2:** \( q \in (1, 2) \). Since Theorem 6.1 is already proved for \( q \in [2, \infty) \) in Case 1 as above, one can prove by the result of Case 1 the following lemma.

**Lemma 6.8.** Suppose that Assumption 3 holds. Let \( q \in (1, 2) \) additionally and \( f \in \mathbb{L}_q(\mathbb{R}^N) \). Then (6.15) admits a solution \( v \in \tilde{H}^1_q(\mathbb{R}^N \setminus \Sigma) \cap \tilde{H}^2_q(\mathbb{R}^N \setminus \Sigma) \) satisfying

\[ \| \nabla v \|_{\mathbb{L}_q(\mathbb{R}^N \setminus \Sigma)} + \| \nabla^2 v \|_{\mathbb{L}_q(\mathbb{R}^N \setminus \Sigma)} \leq C \| f \|_{\mathbb{L}_q(\mathbb{R}^N)} \]

for some positive constant \( C = C(R, N, q, \rho, \rho_+, \rho_-) \).

**Proof.** See the appendix C below. \( \square \)

By Lemma 6.8 and (6.20), the system (6.21) admits a solution \( w \in \tilde{H}^1_q(\mathbb{R}^N \setminus \Sigma) \cap \tilde{H}^2_q(\mathbb{R}^N \setminus \Sigma) \) satisfying (6.22). Thus \( v = \mathcal{S}f + w \) solves (6.1) and satisfies (6.2)-(6.3) by (6.7), (6.8), and (6.22). This completes the proof of Theorem 6.1 for \( q \in (1, 2) \), which furnishes the proof of Theorem 6.1.

**Appendix A. Proof of Lemma 6.7.** This appendix proves Lemma 6.7 for \( \mathcal{G} \) defined as (6.17). We start with

**Lemma A.1.** For \( q \) satisfying Assumption 3, \( \mathcal{G} \) is a compact operator on \( \mathbb{L}_q(\mathbb{R}^N) \).

**Proof.** Step 1: \( \mathcal{G} f \in \mathbb{L}_q(\mathbb{R}^N) \) for any \( f \in \mathbb{L}_q(\mathbb{R}^N) \). It is clear that

\[ \text{supp } \mathcal{G} f \subset D_{R_1, R_2}. \] (A.1)

In addition, by \( \nabla \varphi_\infty = -\nabla \varphi_0 \) and \( \Delta \varphi_\infty = -\Delta \varphi_0 \),

\[ \begin{align*}
\mathcal{G} f &= -2 \nabla \varphi_0 \cdot \nabla T_\infty f - (\Delta \varphi_0) T_\infty f + 2 \nabla \varphi_0 \cdot \nabla T_0 f + (\Delta \varphi_0) T_0 f \\
&= \varphi_0 \Delta T_\infty f - \text{div}(\nabla \varphi_0) T_\infty f + \varphi_0 \nabla T_\infty f \\
&\quad - \varphi_0 \Delta T_0 f + \text{div}(\nabla \varphi_0) T_0 f + \varphi_0 \nabla T_0 f \quad \text{in } \mathbb{R}^N \setminus \Sigma, \end{align*} \] (A.2)

The first property of (6.20) is proved in the appendix B below, while the second property of (6.20) follows from Lemmas 6.2 and 6.3 immediately.
which, combined with \( \varphi_0 \Delta T_\infty f = \varphi_0 f \) and \( \varphi_0 \Delta T_0 f = \varphi_0 f \) in \( \mathbb{R}^N \setminus \Sigma \), furnishes

\[
Gf = -\text{div}((\nabla \varphi_0) T_\infty f + \varphi_0 \nabla T_\infty f) \\
+ \text{div}((\nabla \varphi_0) T_0 f + \varphi_0 \nabla T_0 f) \quad \text{in} \; \mathbb{R}^N \setminus \Sigma.
\]

It then holds that

\[
(Gf, 1)_{B_{4R} \setminus \Sigma} = (\text{div}((\nabla \varphi_0) T_\infty f + \varphi_0 \nabla T_\infty f), 1)_{B_{4R} \setminus \Sigma} \\
+ \text{div}((\nabla \varphi_0) T_0 f, 1)_{B_{4R} \setminus \Sigma} + (\text{div}(\varphi_0 \nabla T_0 f), 1)_{B_{4R} \setminus \Sigma} =: I_1 + I_2 + I_3.
\]

By Gauss’s divergence theorem, we see that

\[
I_1 = I_2 = 0, \quad I_3 = \int_\Sigma \|n \cdot \nabla T_0 f\| \, d\sigma = 0,
\]

where \( d\sigma \) is the surface element of \( \Sigma \). Hence \((Gf, 1)_{\mathbb{R}^N} = (Gf, 1)_{B_{4R} \setminus \Sigma} = 0\), which, combined with \((A.1)\), implies \(Gf \in L_q(\mathbb{R}^N)\).

**Step 2:** \(G\) is a compact operator on \(L_q(\mathbb{R}^N)\).

Let \( \{f^{(j)}\}_{j=1}^\infty \) be a bounded sequence in \(L_q(\mathbb{R}^N)\), i.e. there exists a positive constant \(M\) such that \(\|f^{(j)}\|_{L_q(\mathbb{R}^N)} \leq M\) for any \(j \in \mathbb{N}\). Note that by (6.17), Lemmas 6.4 and 6.5, and \(\text{supp} \nabla \varphi_0, \text{supp} \nabla \varphi_\infty \subset \{x \in \mathbb{R}^N : 2R \leq |x| \leq 3R\}\)

\[
G \in L(L_q(\mathbb{R}^N), H^1_0(\mathbb{R}^N)). \tag{A.3}
\]

Since \(H^1_0(\mathbb{R}^N)\) is compactly embedded into \(L_q(D_{R_1,R_2})\) by the Rellich-Kondrachov theorem (cf. [3, Theorem 6.3]), \(G\) can be regarded as a compact operator from \(L_q(\mathbb{R}^N)\) into \(L_q(D_{R_1,R_2})\) by \((A.3)\). Thus there exists \(G_j \in L_q(D_{R_1,R_2})\) such that

\[
\lim_{j \to \infty} \|Gf^{(j)} - G_j\|_{L_q(D_{R_1,R_2})} = 0, \tag{A.4}
\]

up to some extraction. Let us define

\[
\tilde{G}_f = \begin{cases} 
G_f & \text{in } D_{R_1,R_2}, \\
0 & \text{in } \mathbb{R}^N \setminus D_{R_1,R_2},
\end{cases}
\]

and then \(\text{supp} \tilde{G}_f \subset D_{R_1,R_2}\). In addition, by \((A.4)\),

\[
(Gf, 1)_{\mathbb{R}^N} = (Gf, 1)_{D_{R_1,R_2}} = \lim_{j \to \infty} (Gf^{(j)}, 1)_{D_{R_1,R_2}}
\]

which, combined with \(Gf^{(j)} \in L_q(\mathbb{R}^N)\) as was proved in Step 1, implies \((\tilde{G}_f, 1)_{\mathbb{R}^N} = 0\). Hence \(\tilde{G}_f \in L_q(\mathbb{R}^N)\). On the other hand, by \((A.4)\) and \(Gf^{(j)} \in L_q(\mathbb{R}^N)\),

\[
\lim_{j \to \infty} \|Gf^{(j)} - \tilde{G}_f\|_{L_q(\mathbb{R}^N)} = \lim_{j \to \infty} \|Gf^{(j)} - Gf\|_{L_q(D_{R_1,R_2})} = 0,
\]

and therefore \(G\) is a compact operator on \(L_q(\mathbb{R}^N)\). This completes the proof. \(\square\)

Now we prove Lemma 6.7 (2). In view of Lemma A.1 and the Riesz-Schauder theory, it suffices to prove that the kernel of \(I+G\) is trivial in what follows. Let us begin with some property of \(T\) defined in (6.12).

**Lemma A.2.** Suppose that Assumption 3 holds, and let \(f \in L_q(\mathbb{R}^N)\) with \(Tf = 0\). Then \(f = 0\).
Proof. By the assumption $T f = 0$, we have
\[ T_\infty f = 0 \quad \text{when } |x| \geq 3R, \quad T_0 f = 0 \quad \text{when } |x| \leq 2R. \tag{A.5} \]
Here we set
\[ w = \begin{cases} T_0 f & (x \in B_{3R} \setminus B_{(3/2)R}), \\ 0 & (x \in B_{(3/2)R}). \end{cases} \]
Then $w$ satisfies by the second property of (A.5) and $T_0 f = 0$ on $\Gamma$
\[ \Delta w = f \quad \text{in } G, \quad w = 0 \quad \text{on } \Gamma. \tag{A.6} \]
On the other hand, $T_\infty f$ also satisfies (A.6) by the first property of (A.5). The uniqueness of solutions of (A.6) thus implies $T_0 f = T_\infty f$ in $G$, which, combined with the assumption $T f = 0$ and $\varphi_\infty = 1 - \varphi_0$, furnishes
\[ 0 = T f = (1 - \varphi_0)T_\infty f + \varphi_0 T_0 f = T_\infty f \quad \text{in } G. \]
Combining this property with (A.5) yields $T_\infty f = 0$ in $\mathbb{R}^N$. Since
\[ \Delta T_\infty f = f \quad \text{in } \mathbb{R}^N, \]
one concludes $f = 0$. This completes the proof of the lemma. \qed

Next we complete the proof of Lemma 6.7 (2) by the following result.

Lemma A.3. Suppose that Assumption 3 holds. Let $q \in [2, \infty)$ additionally and $f \in L_q(\mathbb{R}^N)$ with $(I + G) f = 0$. Then $f = 0$.

Proof. Let $\omega(x)$ be an element of $C_0^\infty(\mathbb{R}^N)$ satisfying
\[ \omega(x) = \begin{cases} 1 & (|x| \leq 1), \\ 0 & (|x| \geq 2), \end{cases} \]
and set $\omega_L(x) = \omega(x/L)$ for $L > 0$ large enough. Note that $T f \in H^2_{q', \text{loc}}(\mathbb{R}^N \setminus \Sigma)$ for $q' = q/(q - 1)$ by $q \in [2, \infty)$. Then, by the assumption $(I + G) f = 0$, (6.18), and integration by parts,
\[ 0 = ((I + G) f, \omega_L \rho T f)_{\mathbb{R}^N \setminus \Sigma} = (\Delta T f, \omega_L \rho T f)_{\mathbb{R}^N \setminus \Sigma} \]
\[ = -\langle \nabla T f, \omega_L \rho \nabla T f \rangle_{\mathbb{R}^N \setminus \Sigma} - \langle \nabla T f, (\nabla \omega_L) \rho T f \rangle_{\mathbb{R}^N}. \tag{A.7} \]
Since $L > 0$ is large enough and supp$(\nabla \omega)(\cdot / L) \subset D_{L, 2L} = \{ x \in \mathbb{R}^N : L \leq |x| \leq 2L \}$, one sees that on supp$(\nabla \omega)(\cdot / L)$
\[ T f = \tilde{T}_\infty f + c_f, \quad \nabla T f = \nabla \tilde{T}_\infty f, \quad \rho = \rho_- . \]
We thus observe that
\[ |(\nabla T f, (\nabla \omega_L) \rho T f)_{\mathbb{R}^N}| = \frac{\rho_-}{L} \int_{\mathbb{R}^N} \nabla \tilde{T}_\infty f \cdot (\nabla \omega) \left( \frac{x}{L} \right) \tilde{T}_\infty f + c_f \, dx \]
\[ \leq \frac{\rho_-}{L} \int_{D_{L, 2L}} |\nabla \tilde{T}_\infty f| \left| (\nabla \omega) \left( \frac{x}{L} \right) \right| |\tilde{T}_\infty f| \, dx \]
\[ + \frac{\rho_- |c_f|}{L} \int_{D_{L, 2L}} |\nabla \tilde{T}_\infty f| \left| (\nabla \omega) \left( \frac{x}{L} \right) \right| \, dx \]
\[ =: I_1 + I_2. \]
By Lemma 6.4,
\[ I_1 \leq \frac{C}{L} \left( \sup_{x \in \mathbb{R}^N} |\nabla \omega(x)| \right) \| f \|_{L_q(\mathbb{R}^N)}^2 \int_{D_{L, 2L}} |x|^{-(2N - 1)} \, dx , \]
Proof.

Since $\text{supp } f \in B$. Suppose that Assumption Lemma B.1. holds by (6.13) that $\rho = 0$. Thus the monotone convergence theorem yields

$$\int_{D_{L,2L}} |x|^{-N} dx.$$ 

Hence $\lim_{L \to \infty} (\nabla Tf, (\nabla \omega_L) \rho \Gamma f)_{\mathbb{R}^N} = 0$.

Now we take the limit: $L \to \infty$ in (A.7), i.e.

$$\lim_{L \to \infty} \int_{\mathbb{R}^N \setminus \Sigma} \rho(x) |(\nabla Tf)(x)|^2 \omega \left(\frac{x}{L}\right) dx = 0.$$ 

Then the monotone convergence theorem yields

$$\int_{\mathbb{R}^N \setminus \Sigma} \rho(x) |(\nabla Tf)(x)|^2 = 0.$$ 

Thus $\nabla Tf = 0$ in $\mathbb{R}^N \setminus \Sigma$, which implies that there are constants $c_{\pm}$ so that $\rho \Gamma f = c_{\pm}$ in $\Omega_{\pm}$. Since $[\rho \Gamma f] = 0$ on $\Sigma$, one has $c_{\pm} = c_{\pm}$. On the other hand, it holds by (6.13) that

$$c_+ |D_{R_1,R_2}| = \int_{D_{R_3,R_4}} \rho \Gamma f dx = \rho_+ \int_{D_{R_3,R_4}} \Gamma f dx = 0,$$ 

which implies $c_+ = c_- = 0$. Hence $\Gamma f = 0$, which, combined with A.2, furnishes $f = 0$. This completes the proof of the lemma. \qed

Appendix B. Study of $\mathcal{R}f$. In this appendix, we prove for $\mathcal{R}f$ defined as (6.19)

Lemma B.1. Suppose that Assumption 3 holds. Then $\mathcal{R}f \in \mathcal{L}_q(\mathbb{R}^N)$ for any $f \in \mathcal{F}_q(\mathbb{R}^N \setminus \Sigma)$.

Proof. Since $\text{supp } \mathcal{R}f \subset D_{R_1,R_2}$, it suffices to verify that $(\mathcal{R}f,1)_{\mathbb{R}^N} = 0$ for $f \in \mathcal{F}_q(\mathbb{R}^N \setminus \Sigma)$ in what follows. Let $\psi = 1 - \psi_\infty$. Similarly to (A.2), we write

$$\mathcal{R}f = \psi \Delta S_\infty(\varphi \infty f) - \text{div}(\nabla \psi) S_\infty(\varphi \infty f) + \psi \nabla S_\infty(\varphi \infty f)$$

$$- \psi_0 \Delta S_0(\varphi \infty f) + \text{div}(\nabla \psi_0) S_0(\varphi \infty f) + \psi_0 \nabla S_0(\varphi \infty f)$$

in $\mathbb{R}^N \setminus \Sigma$, which, combined with the facts:

$$\psi \Delta S_\infty(\varphi \infty f) = \psi \text{div}(\varphi \infty f) = 0,$$ 

$$\psi_0 \Delta S_0(\varphi \infty f) = \psi_0 \text{div}(\varphi \infty f) = \text{div}(\varphi \infty f),$$

furnishes that

$$\mathcal{R}f = - \text{div}(\varphi \infty f) - \text{div}(\nabla \psi) S_\infty(\varphi \infty f) + \psi \nabla S_\infty(\varphi \infty f)$$

$$+ \psi_0 \nabla S_0(\varphi \infty f)$$

in $\mathbb{R}^N \setminus \Sigma$.

It then holds that

$$(\mathcal{R}f,1)_{\mathcal{F}_q(\mathbb{R}^N \setminus \Sigma)} = - (\text{div}(\varphi \infty f),1)_{\mathcal{F}_q(\mathbb{R}^N \setminus \Sigma)}$$

$$- (\text{div}(\nabla \psi) S_\infty(\varphi \infty f) + \psi \nabla S_\infty(\varphi \infty f),1)_{\mathcal{F}_q(\mathbb{R}^N \setminus \Sigma)}$$

$$+ (\text{div}(\nabla \psi_0) S_0(\varphi \infty f),1)_{\mathcal{F}_q(\mathbb{R}^N \setminus \Sigma)} + (\text{div}(\psi_0 \nabla S_0(\varphi \infty f),1)_{\mathcal{F}_q(\mathbb{R}^N \setminus \Sigma)}$$

$$= I_1 + I_2 + I_3 + I_4.$$ 

By Gauss's divergence theorem, we see that

$$I_1 = - \int_{\Sigma} [n \cdot f] d\sigma, \quad I_2 = I_3 = 0,$$

$$I_4 = \int_{\Sigma} \psi_0 [n \cdot \nabla S_0(\varphi \infty f)] d\sigma = \int_{\Sigma} \varphi \infty [n \cdot f] d\sigma = \int_{\Sigma} [n \cdot f] d\sigma,$$

where $d\sigma$ is the surface element of $\Sigma$. Hence $(\mathcal{R}f,1)_{\mathbb{R}^N} = (\mathcal{R}f,1)_{\mathcal{F}_q(\mathbb{R}^N \setminus \Sigma)} = 0$, which completes the proof. \qed
Appendix C. Proof of Lemma 6.8. In this appendix, we prove Lemma 6.8. Similarly to Subsection 6.2 and the appendix A, it suffices to prove the following lemma.

Lemma C.1. Suppose that Assumption 3 holds. Let \( q \in (1, 2) \) additionally and \( f \in L^q_{\infty}(\mathbb{R}^N) \) with \( (I + G)f = 0 \). Then \( f = 0 \).

Proof. Since \( (I + G)f = 0 \), we have by (6.18)
\[
\begin{align*}
\Delta T f &= 0 \quad \text{in } \mathbb{R}^N \setminus \Sigma, \\
[\rho T f] &= 0 \quad \text{on } \Sigma, \\
[n \cdot \nabla T f] &= 0 \quad \text{on } \Sigma.
\end{align*}
\]

On the other hand, one sees by Theorem 6.1 that for any \( \Phi \in C^\infty_0(\mathbb{R}^N \setminus \Sigma) \subset F_q'(\mathbb{R}^N \setminus \Sigma), q' = q/(q-1) \in (2, \infty) \), there exists \( w \in H^1_q(\mathbb{R}^N \setminus \Sigma) \cap H^2_q(\mathbb{R}^N \setminus \Sigma) \) such that
\[
\begin{align*}
\Delta w &= \text{div } \Phi \quad \text{in } \mathbb{R}^N \setminus \Sigma, \\
[\rho w] &= 0 \quad \text{on } \Sigma, \\
[n \cdot \nabla w] &= 0 \quad \text{on } \Sigma.
\end{align*}
\]

Similarly to the proof of uniqueness in Subsection 3.2, it holds that \( \rho T f = c \) for some constant \( c \). One then concludes \( c = 0 \) by (6.13), i.e. \( T f = 0 \), which, combined with Lemma A.2 furnishes \( f = 0 \). This completes the proof of Lemma C.1. □

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