Abstract

Common datasets have the form of elements with keys (e.g., transactions and products) and the goal is to perform analytics on the aggregated form of key and frequency pairs. A weighted sample of keys by (a function of) frequency is a highly versatile summary that provides a sparse set of representative keys and supports approximate evaluations of query statistics. We propose private weighted sampling (PWS): A method that ensures element-level differential privacy while retaining, to the extent possible, the utility of a respective non-private weighted sample. PWS maximizes the reporting probabilities of keys and improves over the state of the art also for the well-studied special case of private histograms, when no sampling is performed. We empirically demonstrate significant performance gains compared with prior baselines: 20%-300% increase in key reporting for common Zipfian frequency distributions and accuracy for ×2-8 lower frequencies in estimation tasks. Moreover, PWS is applied as a simple post-processing of a non-private sample, without requiring the original data. This allows for seamless integration with existing implementations of non-private schemes and retaining the efficiency of schemes designed for resource-constrained settings such as massive distributed or streamed data. We believe that due to practicality and performance, PWS may become a method of choice in applications where privacy is desired.

1 Introduction

Weighted sampling schemes are often used to obtain versatile summaries of large datasets. The sample constitutes a representation of the data and also facilitates efficient estimation of many statistics. Motivated by the increasing awareness and demand for data privacy, in this work we construct privacy preserving weighted sampling schemes. The privacy notion that we work with is that of differential privacy [25], a strong privacy notion that is considered by many researchers to be a gold-standard for privacy preserving data analysis.

Before describing our new results, we define our setting more precisely. Consider an input dataset containing \( m \) elements, where each element contains a key \( x \) from some domain \( \mathcal{X} \). For every key \( x \in \mathcal{X} \) we write \( w_x \) to denote the multiplicity of \( x \) in the input dataset. (We also refer to \( w_x \) as the frequency of \( x \) in the data.) With this notation, it is convenient to represent the input dataset in its aggregated form \( D = \{(x, w_x)\} \) containing pairs of a key and its frequency \( w_x \geq 1 \) in the data. Examples of such datasets are plentiful: Keys are search query strings and elements are search requests, keys are products and elements are transactions for the products, keys are locations and elements are visits by individuals, or keys are training examples and elements are activities that generate them. We aim here to protect the privacy of data elements. These example datasets tend to be very sparse, where the number of distinct keys in the data is much smaller than the size \( |\mathcal{X}| \) of the domain. Yet, the number of distinct keys can be very large and samples serve as
small summaries that can be efficiently stored, computed, and transmitted. We therefore aim for our private sample to retain this property and in particular only include keys that are in the dataset.

The (non-private) sampling schemes we consider are specified by a (non-decreasing) function \( q : \mathbb{N} \rightarrow [0, 1] \), where \( q(0) := 0 \). Such a sampling scheme takes an input dataset \( D = \{(x, w_x)\} \) and returns a sample \( S \subseteq D \), where each pair \((x, w_x)\) is included in \( S \) independently, with probability \( q(w_x) \). Loosely speaking, given a (non-private) sampling scheme \( A \), we aim in this paper to design a privacy preserving variant of \( A \) with the goal of preserving its "utility" to the extent possible under privacy constraints. We remark that an immediate consequence of the definition of differential privacy is that keys \( x \in \mathcal{X} \) with very low frequencies cannot be included in the private sample \( S \) (except with very small probability). On the other hand, keys with high frequencies can be included with probability (close to) 1. Private sampling schemes can therefore retain more utility when the dataset has many keys with higher frequencies or for tasks that are less sensitive to low frequency keys.

**Informal Problem 1.1.** Given a (non-private) sampling scheme \( A \), specified by a sampling function \( q \), design a private sampling scheme that takes a dataset \( D = \{(x, w_x)\} \) and outputs a “sanitized” sample \( S^* = \{(x, w_x^*)\} \). Informally, the goals are:

1. Each pair \((x, w_x)\) \( \in D \) is sampled with probability “as close as possible” to the non-private sampling probability \( q(w_x) \).
2. For every sampled pair \((x, w_x)\), the sanitized sample \( S \) contains a pair \((x, w_x^*)\) where \( w_x^* \) provides utility that is “as close as possible” to that of \( w_x \). In our constructions, \( w_x^* \) would be a random variable from which we can estimate linear statistics with (functions of) the frequency \( w_x \).

Informal Problem 1.1 generalizes one of the most basic tasks in the literature of differential privacy – privately computing histograms. Informally, algorithms for private histograms take a dataset \( D = \{(x, w_x)\} \) as input, and return, in a differentially private manner, a “sanitized” dataset \( D^* = \{(x, w_x^*)\} \), where the goal is to minimize \( \max_{x \in \mathcal{X}} | w_x - w_x^* | \). (It is often required that the output \( D^* \) is sparse, in the sense that if \( w_x = 0 \) then \( w_x^* = 0 \).) The works on private histograms date all the way back to the paper that introduced differential privacy [25], and it has received a lot of attention since then (e.g., [32, 28, 3, 4, 10, 8, 2, 7]). Observe that private histograms is a special case of Informal Problem 1.1 where \( q \equiv 1 \).

At first glance, one might try to solve Informal Problem 1.1 by a reduction to the private histograms problem. Specifically, we consider the baseline where the data is first “sanitized” using an algorithm for private histograms, and then a (non-private) weighted sampling algorithm is applied to the sanitized data (treating the sanitized frequencies as actual frequencies). This framework, of first sanitizing the data and then sampling it was also considered in [23]. We show that this baseline is sub-optimal, and improve upon it in several axes.

### 1.1 Our Contributions

Our proposed framework, Private Weighted Sampling (PWS), takes as input a non-private weighted sample \( S \) that is produced by a (non-private) weighted sampling scheme. We apply a “sanitizer” to the sample \( S \) to obtain a respective privacy-preserving sample \( S^* \). Our proposed solution has the following advantages.

**Scalability.** The private version is generated from the sample \( S \) as a post-processing step without the need to revisit the original dataset, which might be massive or unavailable. This means that we can augment existing implementations of non-private sampling schemes and retain their scalability and efficiency. This
is particularly appealing for sampling schemes designed for massive distributed or streamed data that use small sketches and avoid a resource-heavy aggregation of the data [27, 26, 17, 11, 16, 15, 30, 18, 22].

**Benefits of end-to-end privacy analysis** PWS achieves better utility compared to the baseline of first sanitizing the data and then sampling. In spirit, our gains follow from a well-known result in the literature of differential privacy stating that applying a differentially private algorithm on a random sample from the original data has the effect of boosting the privacy guarantees of the algorithm [11, 31, 9]. Our solution is derived from a precise end-to-end formulation of the privacy constraints that account for the benefits of the random sampling in our privacy analysis.

**Optimal reporting probabilities.** PWS is optimal in that it maximizes the probability that each key \( x \) is included in the private sample. Private reporting probabilities depends on the privacy parameters, frequency, and sampling rate and are at most the non-private sampling probabilities \( q(w_x) \). The derivation is provided in Section 3.

**Estimation of linear statistics.** Linear statistics according to a function of frequency have the form:

\[
s := \sum_x L(x)g(w_x),
\]

where \( g(w_x) \geq 0 \) is a non-decreasing function of frequency with \( g(0) := 0 \). The most common use case is when \( L(x) \) is a predicate and \( g(w) := w \) and the statistics is the sum of frequencies of keys that satisfy the selection \( L \). Our PWS sanitizer in Section 4 maintains optimal reporting probabilities and provides private information on frequencies of keys. We show that generally differential privacy does not allow for unbiased estimators without significantly compromising variance. We propose biased but nonnegative and low-variance estimators.

**Improvement over prior baselines.** We show analytically and empirically in Section 6 that we obtain orders of magnitude increase in reporting probability in low-frequency regimes. For estimation tasks, both PWS and prior schemes have lower error for higher frequencies but PWS obtains higher accuracy for lower frequencies than prior schemes. This is particularly helpful for datasets/selections with many mid-low frequency keys.

**Improvement for private histograms.** As an important special case of our results, we improve upon the state-of-the-art constructions for private (sparse) histograms [32, 8]. These existing constructions obtain privacy properties by adding Laplace or Gaussian noise to the frequencies of the keys whereas we directly formulate and solve elementary constraints. Let \( \pi_i^* \) denote the PWS reporting probability of a key with frequency \( i \), when applied to the special case of private histograms. Let \( \phi_i \) denote the reporting probability of the state-of-the-art solution for private (sparse) histograms of [32, 8]. Clearly \( \pi_i^* \) is always at least \( \phi_i \). We show that in low-frequency regimes we have \( \pi_i^*/\phi_i \approx 2i \). Similarly for estimation tasks, PWS provides more accurate estimates in these regimes. Qualitatively, PWS and private histograms have high reporting probabilities and low estimation error for high frequencies. But PWS significantly improves on low to medium frequencies, which is important for distributions with long tails. We empirically show gains of 20%-300% in overall key reporting for Zipf-distributed frequencies. As private histograms are one of the most important building blocks in the literature of differential privacy, we believe that our improvement is significant (both in theory and in practice).
Publicly available PWS implementation. We plan to make our code available with publication of the paper.

2 Preliminaries

We consider data in the form of a set of elements \( \mathcal{E} \), where each element \( e \in \mathcal{E} \) has a key \( e.\text{key} \in \mathcal{X} \). The frequency of a key \( x \), \( w_x := |\{ e \in \mathcal{E} \mid e.\text{key} = x \}| \), is defined as the number of elements with \( e.\text{key} = x \). The aggregated form of the data, known in the DP literature as its histogram, is the set of key and frequency pairs \( \{(x, w_x)\} \). We use the vector notation \( \mathbf{w} \) for the aggregated form. We will use \( m := |\mathcal{E}| \) for the number of elements and \( n \) for the number of distinct keys in the data.

2.1 Weighted Sampling

We consider without-replacement sampling schemes that are specified by non-decreasing probabilities \( (q_i)_{i \geq 1} \). The probability that a key is sampled depends on its frequency – a key with frequency \( i \) is sampled independently with probability \( q_i \).

Threshold sampling is a class of weighted sampling schemes. A threshold sampling scheme (see Algorithm 1) is specified by \( (D, f, \tau) \), where \( D \) is a distribution, \( f \) is a function of frequency, and \( \tau \) a numeric threshold value that specifies the sampling rate. For each key we draw i.i.d. \( u_x \sim D \). The two common choices are \( D = \text{Exp}[1] \) for a probability proportional to size without replacement (ppswor) sample \[35\] and \( D = U[0, 1] \) for a Poisson Probability Proportional to Size (PPS) sample \[33,34,24\]. A key \( x \) is included in the sample if \( u_x \leq \tau f(w_x) \). The probability that a key with frequency \( i \) is sampled is

\[
q_i := \Pr_{u \sim D}[u < f(i)\tau].
\]  

(2)

Samples of different datasets are coordinated when the same \( \{u_x\} \) is used to sample both. Coordinated samples generalize MinHash sketches and support estimation of similarity measures and aggregates over multiple datasets \[5,37,12,6,13,14\].

Threshold sampling is related to bottom-k (order) sampling \[36,33,24,19,20\] but instead of specifying the sample size \( k \) we specify an inclusion threshold \( \tau \). Ppswor is equivalent to drawing keys sequentially with probability proportional to \( f(w_x) \). The bottom-k version stops after \( k \) keys and the threshold version has a stopping rule that corresponds to the threshold. The bottom-k version of Poisson PPS sampling is known as sequential Poisson or Priority sampling.

Algorithm 1: Threshold Sampling

// Threshold Sampler:
Input: Dataset \( w \) of key frequency pairs \( (x, w_x) \); distribution \( D \), function \( f \), threshold \( \tau \)
Output: Sample \( S \) of key-frequency pairs from \( w \)
begin
  \( S \leftarrow \emptyset \)
  \( \text{foreach } (x, w_x) \in w \text{ do} \)
    \( \text{Draw independent } u_x \sim D \)
    \( \text{if } u_x < f(w_x)\tau \text{ then} \)
      \( S \leftarrow S \cup \{(x, w_x)\} \)
return \( S \)
Threshold sampling (via the respective bottom-\(k\) schemes) can be implemented efficiently using small sketches (of size expected sample size) on aggregated data that can be distributed or streamed \cite{24, 36, 34, 19}. It can also be implemented using small sketches for various functions of frequency including the moments \(f(w) = w^p\) for \(p \in [0, 2]\) \cite{16, 15, 18, 21}.

In the sequel we assume that the threshold \(\tau\) is fixed. But the treatment extends to when the threshold is privately determined from the data. If we have a private approximation of the total count \(\|f(w)\|_1 := \sum_x f(w_x)\), we can set \(\tau \approx k/\|f(w)\|_1\). This provides (from the non-private sample that corresponds to the threshold) estimates with additive error \(\|f(w)\|_1/\sqrt{k}\) for statistics with function of frequency \(g = f\) and when \(L\) is a predicate.

### 2.2 Differential Privacy

The privacy requirement we consider is element-level differential privacy. Two datasets with aggregated forms \(w\) and \(w'\) are neighbors if the frequency of one key differs by 1 and is the same for all other keys. Equivalently, the datasets are neighbors if \(\|w - w'\|_1 = 1\). The privacy requirements are specified using two parameters \(\varepsilon, \delta \geq 0\).

**Definition 2.1** \cite{25}. A mechanism \(M\) is \((\varepsilon, \delta)\)-differentially private if for any two neighboring inputs \(w, w'\) and set of potential outputs \(T\),

\[
\Pr[M(w) \in T] \leq e^\varepsilon \Pr[M(w') \in T] + \delta.
\]  

**Algorithm 2:** Private Weighted Samples

```plaintext
// Sanitized Keys:
Input: \((\varepsilon, \delta)\), weighted sample \(S\), taken with non-decreasing probabilities \((q_i)_{i \geq 1}\)
Output: Private sample of keys \(S^*\)
begin
    Compute probabilities \((p_i)_{i \geq 1}\) according to \((\varepsilon, \delta)\) and \((q_i)_{i \geq 1}\)
    \(S^* \leftarrow \emptyset\)
    foreach \((x, w_x) \in S\) do
        With probability \(p_{w_x}\), \(S^* \leftarrow S^* \cup \{x\}\)
    return \(S^*\)

// Sanitized keys and frequencies:
Input: \((\varepsilon, \delta)\), weighted sample \(S\), taken with non-decreasing probabilities \((q_i)_{i \geq 1}\)
Output: Sanitized sample \(S^*\)
begin
    Compute \\(\{p_{ij}\}\) for \(i \geq 1\) and \(0 \leq j \leq i\), according to \((\varepsilon, \delta)\) and \((q_i)_{i \geq 1}\). The rows \(p_i\) are probability vectors.
    \(S^* \leftarrow \emptyset\)
    foreach \((x, w_x) \in S\) do
        Draw \(j\) according to probability vector \(p_{w_x}\).
        if \(j > 0\) then
            \(S^* \leftarrow S^* \cup \{(x, j)\}\)
    return \(S^*\)

// Estimator:
Input: Sanitized sample \(S^* = \{(x, j_x)\}, \{\pi_{ij}\}\) (where \(\pi_{ij} := p_{ij}q_i\), functions \(g(i), L(x)\)
Output: Estimate of the linear statistics \(\sum_x L(x)g(x)\)
begin
    Compute \((a_j)_{j \geq 1}\) using \(\{\pi_{ij}\}\) and \(g(i)\)
    return \(\sum_{(x,j_x) \in S^*} L(x)a_{j_x}\) // Per-key estimates for \(g()\)
```

5
2.3 Private Weighted Samples

Given a (non-private) weighted sample $S$ of the data in the form of key and frequency pairs and (a representation) of the sampling probabilities $(q_i)_{i \geq 1}$ that guided the sampling, our goal is to release as much of $S$ as we can without violating element-level differential privacy.

We consider two utility objectives. The basic objective, sanitized keys, is to maximize the reporting probabilities of keys in $S$. The private sample in this case is simply a subset of the keys in $S$. The refined objective is to facilitate estimates of linear statistics. The private sample includes sanitized keys from $S$ together with information on their frequencies. The formats of the sanitizers and estimators are provided as Algorithm 2.

3 Sanitized Keys

Algorithm 3: Compute $\pi_i$ for Sanitizing Keys

| Input: $(\epsilon, \delta)$, non-decreasing sampling probabilities $(q_i)_{i \geq 1}$ |
|-----------------------------------------------|
| Output: $\{\pi_i\}$                          |
| $\pi_0 \leftarrow 0$                          |
| foreach $i = 1, \ldots, n$ do                 |
| $\pi_i \leftarrow \min\{q_i, e^{\epsilon \pi_{i-1} + \delta}, 1 + e^{-\epsilon}(\pi_{i-1} + \delta - 1)\}$ |

The sanitizer $C$ uses a (representation of a) non-decreasing $(q_i)_{i \geq 1}$ and computes (a representation of) corresponding reporting probabilities $(p_i)_{i \geq 1}$. A sample is then sanitized by considering each pair $(x, w_x) \in S$ and reporting the key $x$ independently with probability $p_{w_x}$.

We find it convenient to express constraints on $(p_i)_{i \geq 1}$ in terms of $\pi_i := p_i q_i$, which is the overall reporting probability $\Pr[x \in C(A(w))]$ of a key $x$ with frequency $i$ – the probability that $x$ is sampled and then reported. Keys of frequency 0 are not sampled or reported and we have $q_0 = 0$ and $\pi_0 := 0$. The objective of maximizing $p_i$ corresponds to maximizing $\pi_i$. We establish the following (the proof is provided in Appendix A):

Lemma 3.1. Consider weighted sampling scheme $A$ where keys are sampled independently according to a non-decreasing $(q_i)_{i \geq 1}$ and a key sanitizer $C$ (Algorithm 2) is applied to the sample. Then the probabilities $p_i \leftarrow \pi_i / q_i$, where $\pi_i$ are the iterates computed in Algorithm 3 are at the maximum under the DP constraints for $C(A(\cdot))$. Moreover, $(\pi_i)_{i \geq 1}$ is non-decreasing.

3.1 Structure and Properties of $(\pi_i)_{i \geq 1}$

The solution as computed in Algorithm 3 applies with any non-decreasing $q_i$. We explore properties of the solution that allow us to compute and store it more efficiently and understand the reporting loss (reduction in reporting probabilities) that is due to the privacy requirement. Proofs are provided in Appendix A.

We provide closed-form expressions of the solution $\pi_i^*$ that corresponds to $q_i = 1$ for all $i$ (aka, the private histogram problem). We will use the following definition of $L(\epsilon, \delta)$. To simplify the presentation, we assume that $\epsilon$ and $\delta$ are such that $L$ is an integer (this assumption can be removed).

$$L(\epsilon, \delta) := \frac{1}{\epsilon} \ln \left( \frac{e^\epsilon - 1 + 2\delta}{\delta (e^\epsilon + 1)} \right) \approx \frac{1}{\epsilon} \ln \left( \frac{\epsilon}{2\delta} \right)$$ (4)
Lemma 3.2. When \( q_i = 1 \) for all \( i \), the sequence \((\pi_i)_{i \geq 1}\) computed by Algorithm 3 has the form:

\[
\pi_i^* = \begin{cases} 
\delta e^{\frac{1}{\varepsilon}} \frac{L - 1}{e^{\delta} - 1} & i \leq L + 1 \\
1 - \delta e^{\frac{1}{\varepsilon}} \frac{(2L + 1 - i) - 1}{e^{\delta} - 1} & L \leq i \leq 2L \\
1 & i \geq 2L + 1
\end{cases}
\]  

(5)

For the general case where the \( q_i \)'s can be smaller than 1, we bound the number of frequency values for which \( \pi_i < q_i \). On these frequencies, the private reporting probability is strictly lower than that of the original non-private sample, and hence there is reporting loss due to privacy.

Lemma 3.3. There are at most \( 2L(\varepsilon, \delta) \) values \( i \) such that \( \pi_i < q_i \), where \( L \) is as defined in (4).

We now consider the structure of the solution for threshold sampling. The solution has a particularly simple form that can be efficiently computed and represented.

Lemma 3.4. When the sampling probabilities \((q_i)_{i \geq 1}\) are those of threshold ppswor sampling with \( f(i) = i \) then the solution has the form \( \pi_i = \pi_i^* \) for \( i < \ell \) and \( \pi_i = q_i \) for \( i \geq \ell \), where \( \ell = \min\{i : \pi_i^* > q_i\} \) is the lowest position with \( \pi_i^* > q_i \).

4 Sanitized Keys and Frequencies

Algorithm 4: Compute \( \pi_{i,j} \) for Sanitized keys and frequencies

```
Input: \((\varepsilon, \delta)\), non-decreasing \((q_i)_{i \geq 1}\)
Output: \(\pi_{i,j}\) for \(j \leq i\)
\pi_0, 0 ← 1, \pi_0 = 0
foreach \(i = 1, \ldots, \text{Max Frequency}\) do // Iterate over rows
    \pi_i ← \min\{q_i, e^{\varepsilon} \pi_{i-1} + \delta, 1 + e^{\varepsilon} (\pi_{i-1} + \delta - 1)\} // Total probability for key with frequency \(i\) to be in output
    \pi_{i,0} ← 1 - \pi_i
foreach \(j = 1, \ldots, i - 1\) do // Set minimum values
    \pi_{i,j} ← \max\{0, e^{\varepsilon} (\sum_{h=0}^{j-1} \pi_{i-1,h} - \delta) - \sum_{h=j+1}^{i-1} \pi_{i,h}\} // Min values allowed for \(\pi_{i,j}\)
    R ← \pi_i - \sum_{h=1}^{i-1} \pi_{i,h} // Remaining probability to assign
foreach \(j = i, \ldots, 1\) do // Set final values for \(\pi_{i,j}\)
    if \(R = 0\) then Break
    \(U ← e^{\varepsilon} \sum_{h=j}^{i-1} \pi_{i-1,h} + \delta - \sum_{h=j+1}^{i-1} \pi_{i,h}\) // Max value allowed for \(\pi_{i,j}\)
    if \(U - \pi_{i,j} \leq R\) then
        \(R ← R - (U - \pi_{i,j})\)
        \(\pi_{i,j} ← U\)
    else
        \(R ← 0\)
```

Our refined sanitizer \(C\) returns keys \(x\) and information on their frequency in the form of a number \(j \in [1, w_x]\). We use \(p_{i,j}\) for the probability that \(C\) reports \"j\" for a sampled key that has frequency \(i\). Note that \(\sum_{j=1}^{i} p_{i,j}\) is the total probability that a key with frequency \(i\) is reported by \(C\). We use

\[
\pi_{i,j} ← q_ip_{i,j}
\]
for the probability that a key with frequency $i$ is sampled and included in the private sample with reported value $j$.

The reader can interpret the returned value $j$ as a token, so that token $j$ can be returned only for keys with frequency that is at least $j$. The domain that we chose for $j$ is not significant as the estimators we propose treat the outputs as coming from a sequence of tokens. Intuitively, we set $\pi_{ij}$ so that the output distribution for a frequency $i$ overlaps to the extent possible with that of frequencies that are close to $i$ and is as disjoint as possible with outputs of frequencies that are much smaller or larger than $i$.

We express constraints in terms of $\{\pi_{i,j}\}$. A solution for $\pi$ is realizable if and only if for all $i$,

$$
\sum_{j=1}^{i} \pi_{i,j} \leq q_i .
$$

Note that the DP constraints imply that $\sum_{j=1}^{i} \pi_{i,j} \leq \pi_i$, where $(\pi_i)_{i \geq 1}$ are the solution for sanitized keys (Algorithm 3). This holds because we have a superset of the constraints – we obtain the DP constraints for sanitized key by grouping all outputs with a key $x$ with all possible values of $j$. We will propose solutions $\{\pi_{i,j}\}$ that have equality, that is, we do not lose key reporting when we also report information on frequency. For notation convenience, we will use $\pi_{i,0} := \pi_i$ for the probability that a key is not reported.

Algorithm 4 constructs a solution for $\pi_{i,j}$ in order of increasing $i$, where the $i$th row is set so that the probability mass of $\pi_i$ is greedily pushed to the extent possible to higher $j$ values. More formally, we establish the following (The proof is provided in Section B):

**Lemma 4.1.** The values $\pi_{i,j}$ computed in Algorithm 4 satisfy:

1. $\forall i$, $\sum_{j=1}^{i} \pi_{i,j} = \pi_i$, and thus are also realizable (satisfy (6))

2. The DP constraints

3. For each $i, j$, the value of $\pi_{ij}$ is maximum subject to: The above, the rows $\pi_h$, for $h < i$, and $\pi_{ih}$ for $h > j$.

**4.1 Structure of the $\pi_{i,j}$ solution**

We express the solution $\pi_{i,j}^*$ that corresponds to $q_i = 1$ for all $i$. This form also applies to rows $i \leq h$ in any solution for that satisfies $\pi_i = \pi_i^*$ for all $i \leq h$.

**Lemma 4.2.** Let $\pi_{ij}^*$ be the solution computed in Algorithm 4 for $q_i = 1$ for all $i$. Let $L(\varepsilon, \delta)$ be integral and as in (4). Then the matrix has a lower triangular form, with the non-zero entries as follows:

$$
\text{For } j \in \{\max\{1, i - 2L + 1\}, \ldots, i\}, \pi_{ij}^* = \pi_{i-j+1}^* - \pi_{ij}^*. 
$$

Equivalently,

$$
\pi_{i,j}^* = \begin{cases} 
\delta \varepsilon^{(i-j)} & \text{if } i-j \in [0, L] \\
\delta \varepsilon^{(2L-1-(i-j))} & \text{if } i-j \in [L+1, 2L-1]. 
\end{cases}
$$
5 Estimation of Linear Statistics

The objective is to estimate, from the sample, statistics of the form

\[ s := \sum_x L(x) g(w_x) . \] (7)

We briefly review estimators for the non-private setting where the sample consists of pairs \((x, w_x)\) of keys and their frequency. We use per-key inverse-probability estimators [29] (also known as importance sampling). The estimate \(\hat{g}(w_x)\) of \(g(w_x)\) is 0 if key \(x\) is not included in the sample and otherwise the estimate is

\[ a_{w_x} := \frac{g(w_x)}{q_{w_x}} . \] (8)

These estimates are nonnegative, a desired property for nonnegative values, and are also unbiased when \(q_{w_x} > 0\). The estimate for a query statistics (7) will be the sum

\[ \hat{s} := \sum_{(x, w_x)} L(x) \hat{g}(w_x) = \sum_{(x, w_x) \in S} L(x) a_{w_x} . \] (9)

Since the estimate is 0 for keys not represented in the sample, it can be computed from the sample. The variance of a per-key estimate for a key with frequency \(i\) is \(g(i)^2 \left( \frac{1}{q_i} - 1 \right)\) and the variance of the sum estimator (9) is

\[ \text{Var}[\hat{s}] = \sum_x L(x)^2 g(w_x)^2 \left( \frac{1}{q_{w_x}} - 1 \right) . \]

These inverse-probability estimates are optimal for the sampling scheme in that they minimize the sum of per-key variance under unbiasedness and non-negativity constraints. We note that the quality of the estimates depends on the match between \(g(i)\) and \(q_i\): Probability Proportional to Size (PPS), where \(q_i \propto g(i)\) is most effective and minimizes the sum of per-key variance for the sample size. Our aim here is to optimize what we can do privately when \(q\) and \(g\) are given.

5.1 Estimation with Sanitized Samples

We now consider estimation from sanitized samples \(S^*\). We specify our estimators \((a_j)_{j \geq 1}\) in terms of the reported sanitized frequencies \(j\). The estimate is 0 for keys that are not reported and are \(a_j\) when reported with value \(j\). The estimate of a statistics is

\[ \hat{s} := \sum_{(x, j) \in S^*} L(x) a_j . \] (10)

We now consider choices of \((a_j)_{j \geq 1}\). A first attempt is to use the unique unbiased estimator: The unbiasedness constraints

\[ \forall i, \sum_{j=1}^{i} \pi_{ij} a_j = g(i) \]

form a triangular system with a unique solution \((a_j)_{j \geq 1}\):

\[ a_i \leftarrow \frac{g(i) - \sum_{j=1}^{i-1} \pi_{i,j} a_j}{\pi_{i,i}} . \]
However, \((a_j)_{j \geq 1}\) may include negative values and estimates have high variance. Private samples necessitate biased estimation. First, the inclusion probability of keys with frequency \(w_x = 1\) cannot exceed \(\delta\). Therefore, the variance contribution of the key to any unbiased estimate is at least \(1/\delta\) which is typically large with respect to the support size. Second, we show in Appendix D that even for the special case of \(q = 1\), any unbiased estimator applied to (any) sanitized keys and frequencies scheme with optimal reporting probabilities must assume negative values. That is, the DP schemes do not admit unbiased nonnegative estimators without compromising reporting probabilities.

We propose estimators that are biased but are nonnegative and have other desirable properties.

- **Maximum Likelihood (ML)**

\[
a_j \leftarrow \frac{g(i)}{\pi_i}, \text{ where } i = \arg \max_h \pi_{ih} .
\]  

(11)

This estimate providing the "right" estimate for the frequency \(i\) for which the probability of reporting \(j\) is maximized. The estimate can be biased up or down.

- **Biased-down (BD)**

\[
a_j \leftarrow \min_{i|\pi_{ij} > 0} \frac{g(i) - \sum_{h=1}^{j-1} a_h \pi_{ih}}{\pi_i - \sum_{h=1}^{j-1} \pi_{ih}} .
\]  

(12)

The sequence \((a_j)\) is non-decreasing and is guaranteed not to over-estimate (see details in Appendix C). Note (from the structure of \(\pi_{ij}\)) that the minimum is over at most \(2L((\varepsilon, \delta))\) values of \(i\) and each sum can be over at most \(2L(\varepsilon, \delta)\) positive \(\pi_{ih}\) entries (between \(i \pm L(\varepsilon, \delta)\)).

We express the expected value, bias, Mean Squared Error (MSE), and variance of the per-key estimate for a key with frequency \(i\):

\[
E_i := \sum_{j=1}^i \pi_{ij} a_j
\]

\[
\text{Bias}_i := E_i - g(i)
\]

\[
\text{MSE}_i := \pi_i g(i)^2 + \sum_{j=1}^i \pi_{ij} (a_j - g(i))^2
\]

\[
\text{Var}_i := \text{MSE}_i - \text{Bias}_i^2
\]

For the sum estimate \((\hat{s})\) we get:

\[
\text{Bias}[\hat{s}] = \sum_x L(x) \text{Bias}_{w_x}
\]

\[
\text{Var}[\hat{s}] = \sum_x L(x)^2 \text{Var}_{w_x}
\]

\[
\text{MSE}[\hat{s}] = \text{Var}[\hat{s}] + \text{Bias}[\hat{s}]^2
\]

\[
\text{NRMSE}[\hat{s}] = \frac{\sqrt{\text{MSE}[\hat{s}]}}{s}
\]  

(13)

Note that the variance component of the normalized squared error \(\text{MSE}[\hat{s}]/s^2\) decreases linearly with support size whereas the bias component may not. We therefore consider both the variance and bias of the per-key estimators and qualitatively seek low bias and "bounded" variance. We measure quality of statistics estimators using the Normalized Root Mean Squared Error (NRMSE).
6 Performance Analysis

We study the performance of PWS on the key reporting and estimation objectives and compare with a baseline method that provides the same privacy guarantees. We use precise expressions (not simulations) to compute probabilities, bias, variance, and MSE of the different methods.

6.1 Private Histograms Baseline

We review the Stability-based Histograms (SbH) method of \[8, 32, 38\], which we use as a baseline. SbH, provided as Algorithm 5, is designed for the special case when \(q_i = 1\) for all frequencies. The input \(S\) is the full data of pairs of keys and positive frequencies \((x, w_x)\). The private output \(S^*\) is a subset of the keys in the data with positive sanitized frequencies \((x, w^*_x)\).

**Algorithm 5: Stability-based Histograms (SbH) \[8, 32, 38\]**

| Input: \((\varepsilon, \delta)\), \(S = \{(x, w_x)\}\) where \(w_x > 0\) |
| Output: Key value pairs \(S^*\) |
| \(S^* \leftarrow \emptyset\) // Initialize private histogram |
| \(T \leftarrow (1/\varepsilon) \ln(1/\delta) + 1\) // Threshold |
| \(\text{foreach} (x, w_x) \in S \text{ do} \)
| \(w^*_x \leftarrow w_x + \text{Lap}(\frac{1}{\varepsilon})\) // Add Laplace random variable |
| \(\text{if} \ w^*_x \geq T \text{ then} \)
| \(S^* \leftarrow S^* \cup (x, w^*_x)\) |

The SbH method is considered the state of the art for sparse histograms (only keys with \(w_x > 0\) can be reported). The method returns non-negative \(w^*_x > 0\) sanitized frequencies. For the case of no sampling, we compare PWS (with \(q \equiv 1\)) with SbH. We use the SbH sanitized frequencies directly for estimation. For sampling, our baseline is sampled-SbH: The data is first sanitized using SbH and then sampled, using a weighted sampling algorithm with \(q\), while treating the sanitized frequencies as actual frequencies. For estimation, we apply the estimator (9) (which in this context is biased). To facilitate comparison with SbH and sampled-SbH we express the reporting probabilities, bias, and variance in Appendix E.

6.2 Reporting Probabilities: No Sampling

We first consider the case where there is no sampling and the objective of maximizing the number of privately reported keys. We compare the PWS (optimal) probabilities \(\pi^*_i\) (5) to the baseline SbH \[8, 32, 38\] reporting probabilities \(\phi_i\) (41). Figure 1 shows reporting probability per frequency for selected DP parameters. We can see that with both private methods the reporting probability reaches 1 for high frequencies but PWS (Opt) reaches the maximum earlier and is significantly higher than \(\phi\) along the way. Analytically from the expressions we can see that for \(i \leq L(\varepsilon, \delta), \pi^*_i/\phi_i \in [2, 2/\varepsilon]\) and for lower \(i\) we have \(\pi^*_i/\phi_i \approx 2i\). We can also see that \(\pi^*_i\) reaches 1 at \(2L \approx (2/\varepsilon) \ln(1/\delta)\) whereas \(\phi_i > 1 - \delta\) for \(i \approx \frac{1}{\varepsilon} \ln(1/(2\delta^2))\). The ratio between the frequency values when maximum reporting is reached is \(\approx \ln(1/\delta)/\ln(\varepsilon/\delta)\). Figure 2 shows the expected numbers of reported keys with PWS (Opt) and SbH for frequency distributions that are \(\text{Zipf}[\alpha]\) with \(\alpha = 1, 2\) as we sweep the privacy parameter \(\delta\). Overall we see that PWS gains 20%-300% in the number of keys reported over baseline. Note that as expected, the optimal PWS reports all keys when \(\delta = 1\) (i.e., no privacy guarantees) but SbH incurs reporting loss.
Figure 1: Key reporting probability for frequency with Opt and SbH for $(\varepsilon, \delta) = (0.1, 0.01), (0.01, 10^{-6})$

Figure 2: Expected fraction of keys that are privately reported with PWS (Opt) and SbH for Zipf[\(\alpha\)] frequency distributions. For $\alpha = 1, 2$ and privacy parameters $\varepsilon = 0.1$ and sweeping $\delta$ between 1 and $10^{-8}$. Left: The respective ratio of PWS to SbH.

### 6.3 Reporting probabilities with Sampling

Figure 3 shows, for representative sampling rates and privacy parameters, the reporting probabilities for PWS (optimal reporting probabilities), sampled-SbH, and respective non-private sampling. As expected,

for sufficiently large frequencies both private methods have reporting probabilities that match the sampling probabilities $q$ of the non-private scheme. But PWS reaches $q$ at a lower frequency than sampled-SbH and has significantly higher reporting probabilities for lower frequencies. Figure 4 shows the fraction of keys reported for Zipf distributions as we sweep the sampling rate (threshold $\tau$). PWS reports more keys than sampled-SbH and the gain persists also with low sampling rates. We can see that with PWS, thanks to end-
to-end privacy analysis, the reporting loss due to sampling mitigates the reporting loss needed for privacy – reporting approaches that of the non-private sampling when the sampling rate $\tau$ approaches $\delta$. Sampled-SbH, on the other hand, incurs reporting loss due to privacy on top of the reporting loss due to sampling.

### 6.4 Estimation of Linear Statistics

We evaluate estimation quality for linear statistics (7) when $g(w) = w$ and $L(x)$ is a selection predicate. The statistics is simply the sum of frequencies of selected keys. We compare performance of PWS with the MLE estimator (11), the baseline sampled-SbH, and for reference, the estimator of the respective non-private sample (9). Figure 5 (top) shows normalized bias $\text{Bias}_i / i$ as a function of the frequency $i$ for the two private methods (the non-private estimator is unbiased and not shown). With both methods, the bias decreases with frequency and diminishes for $i \gg 2\varepsilon^{-1} \ln(1/\delta)$. PWS has lower bias at lower frequencies than SbH, allowing for more accurate estimates on a broader range. We can see that with PWS, the bias decreases when the sampling rate ($\tau$) decreases and diminishes when $\tau$ approaches $\delta$. This is a benefit of the end-to-end privacy analysis. The bias of the baseline method does not change with sampling rate.

Figure 6 shows the normalized variance $\text{Var}_i / i^2$ per frequency $i$ for representative parameter settings. The private methods PWS and sampled-SbH maintain low variance across frequencies: The value is fractional with no sampling and is of the order of that of the non-private unbiased estimator with sampling. In particular this means that the bias is a good proxy for performance and that the improvement in bias of PWS with respect to baseline does not come with a hidden cost in variance. For high frequencies (not shown), keys with all methods are included with probability (close to) 1. The non-private method that reports exact frequencies have 0 variance whereas the private methods maintain a low variance, but the normalized variance diminishes for all methods.

For statistics estimation, the per-key performance suggest that when the selection has many high frequency keys, the private methods perform well and are similar to non-private sampling. When the selection is dominated by very low frequencies, the private methods perform poorly and well below the respective non-private sample. But for low to medium frequencies, PWS can provide drastic improvements over SbH and the gain increases with lower sampling rates. Figure 5 (bottom) shows the NRMSE as a function of sampling rate for estimating the sum of frequencies on a selection of $2 \times 10^5$ keys with frequencies uniformly drawn between 1 and 200. We can see that the error of non-private sampling and of sampled-SbH decreases with higher sampling rate. Note the perhaps counter-intuitive phenomenon that PWS (MLE) hits its sweet spot midway: This is due to a balance of the two components of the error, the variance which increases and
the bias that decreases when the sampling rate decreases. Also note that PWS significantly improves over SbH also with no sampling ($\tau = 1$).

![Graphs showing normalized bias and NRMSE for PWS and SbH](image)

**Figure 5:** Top: Normalized bias for PWS (MLE) and sampled-SbH as a function of frequency, for different sampling rates. The bias of the sampled-SbH estimates (shown once) does not change with sampling rate. Bottom: NRMSE as a function of sampling rate for a selection of $2 \times 10^5$ keys with frequencies drawn uniformly [1, 200].

**Conclusion**

We presented Private Weighted Sampling (PWS), a method to post-process a weighted sample and produce a version that is differentially private. Our private samples maximize the number of reported keys subject to the privacy constraints and support estimation of linear statistics. We demonstrate significant improvement over prior methods for both reporting and estimation tasks, even for the well studied special case of private histograms (when there is no sampling).
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A Proofs: Sanitized Keys

We establish that \((\pi_i)_{i \geq 1}\) as computed by Algorithm \ref{alg:3} are maximum under the DP constraints and are non-decreasing.

**Proof of Lemma 3.1** We use the notation \(\pi^*_i \coloneqq 1 - \pi_i\) for the probability of a key not being included in \(C(A())\). Since key inclusions in the (original or sanitized) sample are independent, the probability of a particular sanitized sample \(Z\) (set of keys, possibly empty) has the product form

\[
P_w(Z) := \prod_{x \in Z} \pi_w x \prod_{x \notin Z} \pi^*_w x.
\]

Since \(\pi_i = q_i p_i\) is the product of two probabilities, one that is given \((q_i)\) and one that we set \((p_i)\) then our solution for \((\pi_i)_{i \geq 1}\) is realizable if and only if it satisfies the constraints for all \(i\):

\[
\pi_i \leq q_i. \quad (14)
\]

We now consider the DP constraints. Consider two neighboring datasets \(w\) and \(w'\) and the two cases (i) For some \(i \geq 1\) there is a key \(x\) such that \(w_x = i\) and \(w'_x = i - 1\) (ii) For some \(i \geq 0\) there is a key \(x\) such that \(w_x = i\) and \(w'_x = i + 1\).
We consider sets $T$ of possible outputs that we partition to outputs $T^+$ that include the key $x$ and outputs $T^-$ that do not include $x$. We use the notation

$$Q^+ = \sum_{Z \in T^+} P_{w-x}(Z \setminus \{x\})$$
$$Q^- = \sum_{Z \in T^-} P_{w-x}(Z)$$

for the respective sums over these sets of outputs of the probability projected on keys other than $x$.

The general DP constraints on a set of outputs $T$ have one of the following form corresponding to our two cases:

$$Q^+ \pi_i + Q^- \overline{\pi_i} \leq e^\varepsilon (Q^+ \pi_{i-1} + Q^- \overline{\pi_{i-1}}) + \delta$$  \hspace{1cm} (15)
$$Q^+ \pi_i + Q^- \overline{\pi_i} \leq e^\varepsilon (Q^+ \pi_{i+1} + Q^- \overline{\pi_{i+1}}) + \delta$$  \hspace{1cm} (16)

We observe, assuming monotonicity of $\pi_i$, that constraints (15) are strictest when $Q^-$ is as small as possible. This because $\overline{\pi_i} \leq \pi_{i-1}$ implies $Q^- \overline{\pi_i} \leq e^\varepsilon Q^- \overline{\pi_{i-1}}$ (using $e^\varepsilon \geq 1$), so the larger $Q^-$ is, the less strict the inequality becomes. This is achieved when $T^- = \emptyset$ and thus $Q^- = 0$, that is, $T$ only includes outputs that include $x$. Similarly, assuming $\pi_i \geq \pi_{i-1}$, the constraints are strictest when $Q^+$ is as large as possible. That is $T^+$ includes all possible outcomes on keys other than $x$ and thus $Q^+ = 1$. A similar argument shows that constraints (16) are strictest when $Q^+$ is as large as possible. That is $T^+$ includes all possible outcomes on keys other than $x$ and thus $Q^+ = 1$. We obtain that the DP constraints simplify to

$$i \geq 1: \pi_i \leq e^\varepsilon \pi_{i-1} + \delta$$  \hspace{1cm} (17)
$$i \geq 0: \overline{\pi_i} \leq e^\varepsilon \overline{\pi_{i+1}} + \delta$$  \hspace{1cm} (18)

We show below that the solution $(\pi_i)_{i \geq 1}$ of the simplified constraints (as subset of all constraints) is non-decreasing and hence any solution to the full set of constraints must also be non-decreasing and thus the assumption that led to the simplification is valid.

We observe that the constraints (17) and (18) are upper bounds on $\pi_i$ in terms of $\pi_{i-1}$ and the feasibility constraint (14) is also an upper bound on $\pi_i$. Therefore, each iterate $\pi_i$ computed in Algorithm 3 attains the maximum possible value by the constraints, provided that $\pi_{i-1}$ is at its maximum value. The claim that each $\pi_i$ is maximized follows by induction.

Finally, to establish that $(\pi_i)_{i \geq 1}$ are non-decreasing we show that each term in the minimum that determines $\pi_{i+1}$ is at least $\pi_i$:

- (14): $q_{i+1} \geq q_i \geq \pi_i$
- (17): $e^\varepsilon \pi_i + \delta \geq \pi_i$
- (18): $1 + e^{-\varepsilon} (\pi_i + \delta - 1) = (1 - e^{-\varepsilon}) + e^{-\varepsilon} (\pi_i + \delta) \geq (1 - e^{-\varepsilon}) \pi_i + e^{-\varepsilon} (\pi_i + \delta) = \pi_i + e^{-\varepsilon} \delta \geq \pi_i$

Consider the iterates $\pi_i$ and the corresponding constraints sequence of the minimum among the three constraints: (14), (17), and (18). We will slightly abuse notation and use these references to constraints in expressions. We first consider the relation between (17) and (18):
Lemma A.1. The constraints sequence has all positions with \((17)\) preceding all positions with \((18)\). In the typical settings of \(\varepsilon \ll 1\), the highest position \(i\) before the transition has \(\pi_i \leq (1 - \delta)/2\).

Proof. The ratio of constraints as a function of \(x = \pi_i\) is:
\[
\frac{17}{18} = \frac{e^\varepsilon x + \delta}{1 + e^{-\varepsilon} (x + \delta - 1)}.
\]
This is an increasing function for \(x \in (0, 1]\). Therefore the iterates \(\pi_i\) are such that initially \((17)\) is smaller (ratio is lower than 1) and then \((18)\) is smaller (ratio is larger than 1). Solving for the crossing point (ratio equal to one) we get
\[
x = 1 - \frac{\delta}{1 + e^\varepsilon} \approx \frac{1 - \delta}{2 + \varepsilon},
\]
using the first order approximation \(e^z \approx 1 + z\) which holds when \(z \ll 1\). Also note that \(x \leq (1 - \delta)/2\).

A subsequence of \((\pi_i)\) where all constraints are \((17)\) or all \((18)\) has a compact form:

Lemma A.2. The iterates \(\pi_{i+1} = e^\varepsilon \pi_i + \delta\) on a sub-sequence with only \((17)\) that starts at \(i_0\) can be compactly expressed for \(i > i_0\):
\[
\pi_i = \pi_{i_0} e^{(i-i_0)\varepsilon} + \delta \frac{e^{(i-i_0)\varepsilon} - 1}{e^\varepsilon - 1}.
\]
Similarly, for a sub-sequence with only \((18)\) constraints where \(\pi_{i+1} = e^{-\varepsilon} (\pi_i - \delta)\) we get:
\[
\pi_i = \pi_{i_0} e^{-(i-i_0)\varepsilon} - \delta e^{-\varepsilon} \frac{1 - e^{-(i-i_0)\varepsilon}}{1 - e^{-\varepsilon}}.
\]

Proof. The iterates form a geometric series.

We now establish the closed-form expressions of the solution \(\pi_i^*\) that corresponds to \(q_i = 1\) for all \(i\).

Proof of Lemma 3.2. Since \(q_i = 1\) for all \(i\), the constraint sequence includes only \((17)\) and \((18)\) constraints (until the minimum exceeds 1, in which \(\pi_i = 1\) at this position and all subsequent positions). From Lemma A.1 we know it has the form \((17)^+ (18)^*\). We have \(\pi_i^* = \delta\).

We have \(\pi_1^* = \delta\) and hence while the constraint \((17)\) holds. Using \((19)\) (Lemma A.2) we have \(\pi_i^* = \delta \frac{e^{i\varepsilon} - 1}{e^\varepsilon - 1}\). From the proof of Lemma A.1 we have \((17)\) in the constraint sequence until \(\pi_i^* > \frac{1 - \delta}{1 + e^\varepsilon}\). From our choice of \(L\), we have
\[
\pi_L^* = \delta \frac{e^{L\varepsilon} - 1}{e^\varepsilon - 1} = \frac{1 - \delta}{1 + e^\varepsilon}.
\]
Therefore, both \((17)\) and \((18)\) hold at position \(L\). We have \(\pi_{L+1}^* = e^\varepsilon \pi_L^* + \delta\). From our choice of \(L\) we have \(\pi_{L+1}^* = \pi_L^*\) and \(\pi_L^* = \pi_{L+1}^*\). We apply \((20)\) (Lemma A.2) with \(i_0 = L\) to obtain the claim. Note the symmetry of the solution where for \(1 \leq i \leq L\), \(\pi_{2L+1-i}^* = \pi_i^*\).

We are now ready to bound the number of positions \(i\) where \(\pi_i < q_i\):
Proof of Lemma 3.3: We have $\pi_i < q_i$ if and only if the $i$th position in the constraint sequence has (17) or (18).

The sequence of $\pi_i$ is non-decreasing with at most one transition from (17) to (18). For $i$ such that $q_i \leq \delta$ we have the constraint (14). Hence $\pi_i \geq \delta = \pi_i^*$ at the first position with $\pi_i < q_i$. Each application of the minimum of (17) and (18) increases $\pi_i$. The total increase is larger when the initial value is larger. Let $h$ be the $i$th position with (17) or (18). Because of the monotone increase we can show by induction that $\pi_h \geq \pi_i^*$.

With threshold sampling (say by moments of frequency) we have a closed form for $q_i$. Using this and Lemma A.2 we can express the solution $\pi_i$ with computation that depends on the number of transitions in the constraint sequence. The following Lemma bounds the number of such transitions. The proof of Lemma 3.4 follows as a special case:

Lemma A.3. For threshold ppswor with $p = 1$ the constraints sequence has the regular-expression form

$$(17)^* (18)^* (14)^* .$$

For priority sampling with $p \leq 1$ and for priority with $p \geq 1$ when $\tau \geq \delta$, all (14) constraints must follow all (17) constraints in the constraint sequence.

Proof. From Lemma A.1 there is at most one transition from (17) to (18).

We now consider the relation between (14) and (17). When $\pi_i = q_i$, we will have (14) $\leq$ (17) if

$$\rho(i, \delta) := \frac{q_{i+1} - \delta}{q_i} \leq e^\epsilon . \quad (21)$$

We need to establish that once (21) holds for $i = i_0$, it continues to hold for $i \geq i_0$. Equivalently, establishing the claim for all $\epsilon > 0$ is equivalent to establishing that $\rho(i, \delta)$ is non-increasing with $i$ when $\rho(i, \delta) > 1$. That is, that $\rho(i, \delta)$ is non-increasing, equivalently, that the partial derivative satisfies

$$\frac{\partial \rho(i, \delta)}{\partial i} \leq 0 \quad (22)$$

when

$$q_{i+1} - q_i \geq \delta . \quad (23)$$

In some of the derivations i will be convenient to work with the continuous form of (23):

$$\frac{\partial q_i}{\partial i} \geq \delta . \quad (24)$$

Consider ppswor threshold sampling with $p = 1$. Recall that $q_i = 1 - e^{-\tau i}$. Therefore, the condition (23) is $e^{-\tau i}(1 - e^{-\tau}) \geq \delta$. It suffices to check (22) when $\delta + e^{-\tau} \leq 1$. Substituting and solving (22) we obtain that the derivative is negative when $\delta + e^{-\tau} \leq 1$. Therefore, there can be at most one transition from (17) to (14) for ppswor with $p = 1$.

We next consider priority threshold sampling $q_i = \min\{1, \tau i^p\}$. For $i^p \tau \geq 1$, $q_i = 1$ and (23) does not hold. Therefore it suffices to consider $q_i = \tau i^p < 1$ and

$$\rho(i, \delta) = \frac{(i+1)^p - \delta}{i^p} .$$

20
When \( p \leq 1 \), \((i + 1)^p - i^p \leq 1\) and thus condition (23) does not hold when \( \tau \leq \delta \). Hence it suffices to consider \( p \geq 1 \) or \( \delta < \tau \). Using the continuous form (24) with \( q_i = \tau i^p \) we get that it is satisfied when

\[
i^{p-1} \geq \delta/(p\tau)
\]

By solving (22) we get

\[
(i + 1)^{p-1} > \frac{\delta}{\tau}
\]

This holds for all \( i \geq 1 \) and \( p \geq 1 \) when \( \delta \leq \tau \). Since it suffices to for the solution to hold for (25), we obtain the claim for \( p \leq 1 \). Combining, we obtain that there can be at most one transition from (17) to (14) for priority sampling with \( p \leq 1 \) and for \( p \geq 1 \) when \( \tau \geq \delta \).

We next consider the relation between (14) and (18). We define

\[
\rho(i, \delta) := \frac{q_i - \delta}{q_{i+1}}
\]

When \( \pi_i = q_i \), we will have (14) \( \leq \) (18) if \( \rho(i, \delta) \leq e^\varepsilon \).

To establish the claim for all \( \varepsilon > 0 \) it is equivalent to establish that \( \rho(i, \delta) \) is non-increasing when it is greater than 1. Equivalently, that \( \frac{q_i - q_{i+1}}{q_{i+1}} = q_{i+1} - q_i > \delta \) (same as (23) and (24)) implies that \( \rho(i, \delta) \) is non-increasing. That is,

\[
\frac{\partial \rho(i, \delta)}{\partial i} \leq 0.
\]

For ppswor with \( p = 1 \) we get

\[
\rho(i, \delta) = \frac{e^{-\tau i} - \delta}{e^{-\tau(i+1)}}
\]

and that (26) holds for all \( \delta \geq 0 \) and \( \tau > 0 \).

\[\square\]

### B Proofs: Sanitized Keys and Frequencies

We establish properties of the values \( \{\pi_{i,j}\} \) computed by Algorithm A4.

**Proof of Lemma 4.1** We write the DP constraints in terms of sets \( T \) of potential outputs on pairs of neighboring datasets \( w \) and \( w' \). We follow the proof of Lemma 3.1. Consider two neighboring datasets \( w \) and \( w' \) and the two cases: (i) For some \( i \geq 1 \) there is a key \( x \) such that \( w_x = i \) and \( w'_x = i - 1 \). (ii) For some \( i \geq 0 \) there is a key \( x \) such that \( w_x = i \) and \( w'_x = i + 1 \).

We consider a set of outputs \( T \). A potential output \( Z \in T \) is a set of key value pairs \((y, j)\) where key \( y \) is reported with value \( j \). For purposes of this proof we partition \( T \) to sets \( T_j \) according to the output on key \( x \). If \((x, j) \in Z \) we place \( Z \in T_j \) and if key \( x \) is not in \( Z \) we place \( Z \) in \( T_0 \).

We denote by \( Q_j \) the respective combined probability of outputs \( T_j \) when projected on all keys other than \( x \) (equivalently, the probability of \( T_j \) when key \( x \) is removed from the dataset). The general DP constraints have the form

\[
\Pr[C(A(w)) \in T] \leq e^\varepsilon \Pr[C(A(w')) \in T] + \delta.
\]

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We have

$$\Pr[C(A(w)) \in T] = \sum_{j=0}^{i} Q_j \pi_{i,j}.$$  

From the two choices of the neighboring dataset $w'$ we have one of:

$$\Pr[C(A(w')) \in T] = \sum_{j=0}^{i-1} Q_j \pi_{i-1,j}$$

$$\Pr[C(A(w')) \in T] = \sum_{j=0}^{i+1} Q_j \pi_{i+1,j}.$$  

The corresponding DP constraints are:

$$i \geq 1: \sum_{j=0}^{i} Q_j \pi_{i,j} \leq e^\varepsilon \sum_{j=0}^{i-1} Q_j \pi_{i-1,j} + \delta.$$  

(27)

$$i \geq 1: \sum_{j=0}^{i} Q_j \pi_{i,j} \leq e^\varepsilon \sum_{j=0}^{i+1} Q_j \pi_{i+1,j} + \delta.$$  

(28)

Considering any particular set of values $\pi_{i,j}$, the strictest constraints of the form (27) would have $Q_j = 1$ for $j$ where $\pi_{i,j} > \pi_{i-1,j}$ and $Q_j = 0$ otherwise. Similarly for constraints of the form (28), the strictest would have $Q_j = 1$ for $j$ where $\pi_{i,j} > \pi_{i+1,j}$ and $Q_j = 0$ otherwise. By "strictest" constraints we mean that if the values $\{\pi_{i,j}\}$ satisfy these selected constraints, the satisfy all DP constraints.

Therefore, taking the union of all these "strictest" sets of constraints over all $\{\pi_{i,j}\}$ we obtain that without loss of generality it suffices to solve for constraints of the following form: For $i \geq 1$ and all $J \subset \{0, \ldots, i\}$:

$$\sum_{j \in J} \pi_{i,j} \leq e^\varepsilon \sum_{j \in J} \pi_{i-1,j} + \delta.$$  

(29)

$$\sum_{j \in J} \pi_{i,j} \leq e^\varepsilon \sum_{j \in J} \pi_{i+1,j} + \delta.$$  

(30)

We re-arrange the set of constraints (30) and get the equivalent set for all $i$ and $J \subset \{0, \ldots, i\}$:

$$\sum_{j \in J} \pi_{i,j} \geq -e^{-\varepsilon} \left(\sum_{j \in J} \pi_{i-1,j} - \delta\right).$$  

(31)

For sets $J$, the constraints (29) and (6) determine upper bounds on $\sum_{j \in J} \pi_{i,j}$ as determined by $\sum_{j \in J} \pi_{i-1,j}$ for $j \in J$. The constraints (31) determine lower bounds.

The solution is constructed by increasing $i$, so that row $i$ is set after rows $h < i$ are set. For the $i$th row, solutions that push probability mass to higher $j$ have the form where for some $h > 0$, $\pi_{i-1,j} \leq \pi_{i,j}$ for $j \geq h$ and $\pi_{i-1,j} \geq \pi_{i,j}$ for $j \leq h$. Formally, the sequence $(\pi_{i,j} - \pi_{i-1,j})_{j=1}^{i}$ is all non-negative or is first non-positive and then non-negative. When we seek this form, the strictest constraints of the type (29) would have $J = \{h, \ldots, i\}$ for some $h$ and the strictest constraints of the type (31) would have $J = \{0, \ldots, h\}$.
for some $h$. We will verify that the solution we construct for this pruned set of constraints has this form and therefore it satisfies all constraints.

We construct the solution so that $\sum_{j=1}^{i} \pi_{ij} = \pi_i$ and $\pi_{i,0} = 1 - \pi_i$. Note that these setting imply that (6) are satisfied. We show inductively as we construct the solution that it satisfies this. To set $\pi_i$, we first compute the minimum values we can have given $\pi_{i-1}$, so that (31) are satisfied. The minimum values are set by increasing $j$ using the constraint (31) on $J = \{0, \ldots, j\}$. The minimum values for $j = 1, \ldots, i - 1$ are:

$$D_{i,j} \leftarrow \max \left\{ 0, e^{-\varepsilon} \left( \sum_{h=0}^{j} \pi_{i-1,h} - \delta \right) - \sum_{h=0}^{j-1} \pi_{i,h} \right\}$$

Setting $\pi_{i,j} \geq D_{i,j}$ ensures that (31) are satisfied for all $J$ of the prefix form $J = \{0, \ldots, h\}$ for some $h$. Note that

$$\forall j, D_{i,j} \leq \pi_{i-1,j} \quad (32)$$

Let

$$D_i \leftarrow \sum_{h=1}^{i-1} D_{i,h}$$

be the sum of the minimum values. From (32) we get

$$D_i \leq \sum_{h=1}^{i-1} \pi_{i-1,h} = \pi_{i-1} \leq \pi_i \quad (33)$$

The next step is to set $\pi_{i,j}$ right-to-left in decreasing $j$ order to the maximum extent possible above $D_{i,j}$ so that (6) and (29) are satisfied. We set the initial remaining probability mass to allocate to $R \leftarrow \pi_i - D_i$. From (33) we get that $R \geq 0$. While allocating we maintain that $\sum_{j=1}^{i} (\pi_{i,j} - D_{i,j}) \leq R$.

This part is done in order of decreasing $j$ using the constraint (29) for $J = \{j, \ldots, i\}$. At each $j$ we allocate the maximum value $U$ we can have for $\pi_{i,j}$ given values we already set for $\pi_{i,h}$ for $h > j$:

$$U \leftarrow e^\varepsilon \sum_{h=j}^{i-1} \pi_{i-1,h} + \delta - \sum_{h=j+1}^{i} \pi_{i,h}$$

We then compute the increase $\Delta \leftarrow U - D_{i,j}$. If $\Delta \leq R$ we set $\pi_{i,j} \leftarrow U$ and $R \leftarrow R - \Delta$. Otherwise, we set $\pi_{i,j} \leftarrow D_{i,j} + R$. This step ends when $R = 0$ (remaining $\pi_{i,j}$ are set to $D_{i,j}$).

To show that we can always exhaust $R$ (and thus have the property that $\sum_{j=1}^{i} \pi_{i,j} = \pi_i$), note that if the situation is that current values have $\sum_{j=1}^{i} \pi_{i,j} < \pi_i$ we must have slack and $U > \pi_{i,j}$ in the constraints. This since $\sum_{j=1}^{i} \pi_{i,j} < \pi_i \leq e^\varepsilon \pi_{i-1}$. We obtain $U = \pi_i - \sum_{j=2}^{i} \pi_{i,j} > \pi_{i,1}$ when processing $j = 1$.

It follows from the construction of the solution that it satisfies the constraints (6) (since $\pi_i = \sum_{j=1}^{i} \pi_{i,j}$), constraints (29) for $J$ of a suffix form $\{h, \ldots, i\}$, and constraints (31) for $J$ of a prefix form $\{0, \ldots, h\}$.

It remains to show that these constraints are sufficient. That is, the sequence $(\pi_{i,j} - \pi_{i-1,j})_{j=1}^{i}$ is all non-negative or first non-positive then non-negative. We first consider $\pi_{i,i}$. From the construction $\pi_{i,i} = \min\{\delta, \delta\} > 0$.

Note that the constraints (29) always allow for setting $\pi_{i,j} \geq e^\varepsilon \pi_{i-1,j}$ after setting $\pi_{i,h}$ for $h > j$ that satisfy (29) for $J = \{h, \ldots, i\}$. Since we put $\pi_{i,j}$ at the maximum possible value until exhausting $R$ we will have the suffix until $R$ is exhausted with $\pi_{i,j} > \pi_{i-1,j}$. Then possible one value at the position $j$ that exhausts $R$ where the inequality can be either way. And then a prefix where $\pi_{i,j} \leftarrow D_{i,j} \leq \pi_{i-1,j}$.  

\[ \Box \]
C Properties of the Biased-Down Estimator

Lemma C.1. The estimator expressed by the sequence (12)

\[ a_j \leftarrow \min_{i \mid \pi_{i,j} > 0} \frac{g(i) - \sum_{h=1}^{j-1} a_h \pi_{i,h}}{\pi_i - \sum_{h=1}^{j-1} \pi_{i,h}}. \]  

(34)

is biased-down and non-decreasing.

Proof. The estimate \( a_j \) is always set to be at most the value needed to have an unbiased estimate for \( g(i) \) when \( a_i = a_j \) for \( i \geq j \). Therefore, the estimate can only be biased down.

Let \( r_{i,j} = \frac{g(i) - \sum_{h=1}^{j-1} a_h \pi_{i,h}}{\pi_i - \sum_{h=1}^{j-1} \pi_{i,h}} \) and recall that \( a_j \) is set to the minimum over applicable \( i \) of \( r_{i,j} \). Now note that \( r_{i,j} \) is non-decreasing with \( j \) because \( a_j \) is always set to be at most \( r_{i,j} \). Since for each \( j \) we take a minimum over a set of values that can only be larger, \( a_j \) may only increase.

D Limits on private non-negative unbiased estimation

We show that private weighted sampling schemes with optimal key reporting generally do not admit non-negative and unbiased estimation of frequencies. The lemma below considers the case when there is no sampling, but the argument extends to sampling schemes where \( \pi_i = \pi_i^* \) for an appropriate prefix of the sequence.

Lemma D.1. Consider \( q \equiv 1 \) and (any) keys and frequencies sanitizer with optimal \( \pi_i^* \) reporting of keys. Then there is no unbiased and nonnegative estimator for frequencies.

Proof. Consider a sanitized keys and frequencies scheme for \( q = 1 \). The scheme reports a key with frequency \( i \) with (optimal) probability \( \pi_i^* \). When a key is reported, the scheme reports a token as a sanitized frequency. Let \( T[i] \) be the distribution on output tokens for a key with frequency \( i \). To make this a distribution we use the special output \( \perp \) for the (probability \( \pi_i^* \)) event that the key is not reported. Using our notation we have

\[ \Pr_{t \sim T[i]} [t \neq \perp] = \pi_i^*. \]

Consider a token \( t \) that has positive probability to be reported with \( i = 1 \), that is, \( \Pr_{z \sim T[1]} [z = t] > 0 \). We argue that for any \( h \leq L(\varepsilon, \delta) \) (where \( L \) is as defined in (4))

\[ \Pr_{z \sim T[h]} [z = t] = e^{(h-1)e} \Pr_{z \sim T[1]} [z = t]. \]  

(35)

The argument follows from the privacy constraints for maintaining the maximum key reporting probabilities of \( \pi_i^* \). The maximum probability with frequency \( h \) for tokens that are not reported for frequency \( h - 1 \) is \( \delta \). Therefore, to have \( \pi_i^* = \pi_{i-1}^* e^\varepsilon + \delta \) the reporting probability of each token reported for \( h - 1 \) must increase by a factor of at least \( e^\varepsilon \).

We now consider estimation. A general estimator for this scheme returns an estimate with expected value \( a_t \) for output token \( t \). Note that any unbiased estimator must be 0 when a key is not reported \( (a_{\perp} = 0) \) and for all \( h \) we have:

\[ h = \mathbb{E}_{z \sim T[h]} a_z \]

(36)
Let $T_1$ be the set of possible output tokens $t \neq \perp$ such that $\Pr_{z \sim T[1]}[z = t] > 0$. We have $\Pr_{z \sim T[1]}[z \in T_1] = \pi^*_1 = \delta$ and from unbiasedness (36) with $h = 1$:

$$E_{z \sim T[1]} a_z = \frac{1}{\delta}.$$ 

Consider now the estimates for a key with frequency $h \leq L(\varepsilon, \delta)$. We use (36) to obtain:

$$h = E_{z \sim T[h]} a_z = E_{z \sim T[h]} I_{z \notin T_1} a_z + E_{z \sim T[h]} I_{z \in T_1} a_z$$

We will show that we can have that the second term is larger than $h$ which will mean that the first term is negative. We use (35):

$$E_{z \sim T[h]} I_{z \in T_1} a_z = \sum_{t \in T_1} \Pr_{z \sim T[1]}[z = t] a_t = e^{\varepsilon(h-1)} \sum_{t \in T_1} \Pr_{z \sim T[1]}[z = t] a_t = e^{\varepsilon(h-1)} E_{z \sim T[1]} a_z = e^{\varepsilon(h-1)}.$$ 

We obtain that when $h < e^{\varepsilon(h-1)}$ holds, which is the case for example when $\varepsilon = 1$ and $h = 2$, we have $a_t < 0$ on some tokens $t$. This because the contribution to the expectation of the estimate of frequency $h$ that is only due to outputs $T_1$ already exceed the value $h$. Therefore, we must have negative values $a_t < 0$ on at least some tokens $t \notin T_1$.

## E SbH baseline: Expressions

In this section we derive expressions for inclusion probabilities, bias, and error for the baseline method of Stability-based Histograms [8] (SbH) (see Section 6.1). We use these expressions in our empirical and analytical evaluation.

In this section we treat the weighted sampling probabilities $q_i$ as a continuous function for $i \geq 0$ and use estimators that are continuous functions of a reported $j$. For consistency with other parts of the paper we maintain the discrete indices notation $i, j$. For a key with frequency $i$, we express the probability density $\phi_{i,j}$ that the key is sampled and reported with frequency $j \geq T$. The distribution $\text{Lap}[1/\varepsilon]$ is a combination of $(1/2)\text{Exp}[\varepsilon]$ and $(-1/2)\text{Exp}[\varepsilon]$.

$$j \geq i : \quad \phi_{i,j} = \frac{1}{2} q_j \varepsilon e^{-\varepsilon(j-i)}$$

$$j \leq i : \quad \phi_{i,j} = \frac{1}{2} q_j \varepsilon e^{-\varepsilon(i-j)}.$$ 

The respective overall reporting probability for a key with frequency $i$ is

$$\phi_i = \int_{T}^{\infty} \phi_{i,j} dj.$$
For estimation, we follow (8). For reported frequency \( j \) to estimate \( g(i) \) for a key with frequency \( i \) we use:

\[
a_j := \frac{g(j)}{q_j}.
\]

Note that since this is applied after the privacy transform, the estimator is biased. But for keys with frequencies where \( g(j) \) is likely to be close to \( g(i) \) and \( q_j \) close to \( q_i \) this estimate would be closer to a direct use of (8) on the original data. The expected value and MSE of the estimate for a key with frequency \( i \) are:

\[
E_i = \int_T^\infty a_j \phi_{i,j} dj
\]

\[
\text{MSE}_i = (g(i))^2 \cdot (1 - \phi_i) + \int_T^\infty (a_j - g(i))^2 \phi_{i,j} dj.
\]

Using these per-frequency expressions, we can express the MSE and bias of sum estimators for linear statistics as in Section 5.1.

### E.1 Explicit expressions

Substituting \( \phi_{i,j} \) and \( a_j \) we obtain explicit expressions in terms of \( (q_i)_{i \geq 1} \) and \( g() \):

\[
\phi_i = \begin{cases} 
  i > T : & \frac{1}{2} \varepsilon \left( \int_i^\infty q_j e^{-\varepsilon(j-i)} dj + \int_T^i q_j e^{-\varepsilon(i-j)} dj \right) \\
  i \leq T : & \frac{1}{2} \varepsilon e^{\varepsilon i} \int_T^\infty q_j e^{-\varepsilon j} dj 
\end{cases}
\]

(37)

The expected value \( E_i \) of the estimate of \( g(i) \) is:

\[
E_i = \begin{cases} 
  i > T : & \frac{1}{2} \varepsilon \left( e^{\varepsilon i} \int_i^\infty q_j a_j e^{-\varepsilon j} dj + e^{-\varepsilon i} \int_T^i q_j a_j e^{\varepsilon j} dj \right) \\
  i \leq T : & \frac{1}{2} \varepsilon e^{\varepsilon i} \int_T^\infty g(j) e^{-\varepsilon j} dj + e^{-\varepsilon i} \int_i^T g(j) e^{\varepsilon j} dj 
\end{cases}
\]

(38)

The expected value \( E_i \) (and hence the bias \( E_i - g(i) \)) does not depend on the sampling \( q \). We express the expected value for the special case when \( g(i) = i \):

\[
E_i = \begin{cases} 
  i > T : & \frac{1}{2} \varepsilon e^{\varepsilon i} e^{-\varepsilon i} (\varepsilon i + 1) \\
  & + \frac{1}{2} \varepsilon e^{-\varepsilon} e^{-\varepsilon i} (e^{\varepsilon i} (\varepsilon i - 1) - e^{\varepsilon T} (\varepsilon T - 1)) \\
  & = i - \frac{1}{2} \varepsilon e^{-\varepsilon(i-T)} (T - \frac{1}{2}) \\
  i \leq T : & \frac{1}{2} \varepsilon e^{\varepsilon i} e^{-\varepsilon T} (\varepsilon T + 1) \\
  & = \frac{1}{2} e^{-\varepsilon(T-i)} (T + \frac{1}{2}) 
\end{cases}
\]

(39)
The MSE (general $q$ and $g$) is:

\[
\text{MSE}_i = (1 - \phi_i)g(i)^2 + \int_T^\infty q_j(a_j - g(i))^2 \text{PDF}_{\text{Lap}[1/\varepsilon]}(j - i) dj
\]

\[
= -2g(i)\xi_i + (g(i))^2 \phi_i + (g(i))^2(1 - \phi_i) + \int_T^\infty \frac{(g(j))^2}{q_j} \text{PDF}_{\text{Lap}[1/\varepsilon]}(j - i) dj
\]

\[
= (g(i))^2 - 2g(i)\xi_i + \int_T^\infty \frac{(g(j))^2}{q_j} \text{PDF}_{\text{Lap}[1/\varepsilon]}(j - i) dj
\]

\[
\begin{cases}
  i > T : & \frac{1}{2} \xi e^\varepsilon i \int_i^\infty \frac{g(j)^2}{q_j} e^{-\varepsilon j} dj + \frac{1}{2} \xi e^{-\varepsilon i} \int_T^1 \frac{g(j)^2}{q_j} e^{\varepsilon j} dj \\
  i \leq T : & \frac{1}{2} \xi e^\varepsilon i \int_0^\infty \frac{g(j)^2}{q_j} e^{-\varepsilon j} dj
\end{cases}
\]

E.2 Expressions for Private histograms ($q \equiv 1$)

We now express the reporting probabilities $\phi_i$ for the case where no sampling is subsequently performed ($q \equiv 1$). From (37) we obtain:

\[
\phi_i = 1 - \text{CDF}_{\text{Lap}[1/\varepsilon]}(T - i) = \begin{cases}
  i \geq T : & 1 - \frac{1}{2\xi} e^{-(i - 1)\varepsilon} \\
  i < T : & \frac{1}{2} \nabla \xi (i - 1)
\end{cases}
\]

By substituting $q \equiv 1$ and $g(i) = i$ in (40) we get

\[
\text{MSE}_i = \begin{cases}
  i > T : & \frac{1}{2\xi} - e^{-\varepsilon i(T - i)} (\frac{1}{2\xi} T^2 - iT + \frac{i}{\xi} + \frac{1}{2\xi}) \\
  i \leq T : & i^2 + e^{-\varepsilon (T - i)} (\frac{1}{2\xi} T^2 - iT - \frac{i}{\xi} + \frac{1}{2\xi})
\end{cases}
\]

E.3 Expressions with sampling

The sampling schemes we consider are parameterized by $\tau > 0$. For threshold ppswor sampling $q_j = 1 - e^{-\tau f(j)}$ or threshold Poisson $q_j = \min\{1, \tau f(j)\}$. We express $\phi_i$ for ppswor threshold sampling and function of frequency $f(w) = w$:

\[
\phi_i = \begin{cases}
  i \geq T : & 1 - \frac{1}{2\xi} e^{-\varepsilon i(T - i)} - \frac{e^{-\varepsilon i(T - i)} + e^{-\varepsilon T}}{2\xi} e^{-\varepsilon i} + I_{x \neq \tau} \frac{e^{-\varepsilon i}}{2(\tau - \xi)} (e^{-\varepsilon (i - T) - \tau T} - e^{-\varepsilon t}) - I_{x = \tau} \frac{e^{-\varepsilon i}}{2\xi} (i - T) e^{-\varepsilon i} \\
  i \leq T : & \frac{1}{2\xi} e^{-\varepsilon (T - i)} (1 - \frac{e^{-\varepsilon i}}{\tau + e^{-\varepsilon T}})
\end{cases}
\]

We now consider priority sampling with threshold $\tau$ and $f(w) = w$. We start from expressing the inclusion probability $\phi_i$. Recall that in the non-private case, the inclusion probability of $i$ is $q_i = \min\{\tau i, 1\}$. 27
If $T \geq \frac{1}{\tau}$,

$$
\phi_i = \int_T^\infty q_j \text{PDF}_{\text{Lap}[\frac{1}{\mathcal{E}}]}(j - i) \, dj
$$

$$
= \int_T^\infty \text{PDF}_{\text{Lap}[\frac{1}{\mathcal{E}}]}(j - i) \, dj
$$

$$
= 1 - \text{CDF}_{\text{Lap}[\frac{1}{\mathcal{E}}]}(T - i)
$$

$$
= \begin{cases} 
  i \geq T : & 1 - \frac{1}{2} e^{-(i - \frac{1}{\mathcal{E}})} \\
  i < T : & \frac{1}{2} \delta e^{(i - \frac{1}{\mathcal{E}})}
\end{cases}
$$

Otherwise, $T < \frac{1}{\tau}$.

$$
\phi_i = \int_T^\infty q_j \text{PDF}_{\text{Lap}[\frac{1}{\mathcal{E}}]}(j - i) \, dj
$$

$$
= \int_T^{\frac{1}{\tau}} q_j \text{PDF}_{\text{Lap}[\frac{1}{\mathcal{E}}]}(j - i) \, dj
$$

$$
+ \int_{\frac{1}{\tau}}^\infty \text{PDF}_{\text{Lap}[\frac{1}{\mathcal{E}}]}(j - i) \, dj
$$

$$
= \tau \int_T^{\frac{1}{\tau}} \cdot \frac{1}{2} \delta e^{-(j - \frac{1}{\mathcal{E}})} \, dj + 1 - \text{CDF}_{\text{Lap}[\frac{1}{\mathcal{E}}]}(1 - i)
$$

To compute the inclusion probability, we consider three cases:

1. $i \leq T$. In that case,

$$
\phi_i = \frac{\tau}{2} \left( \left( T + \frac{1}{\mathcal{E}} \right) e^{(i - T)\mathcal{E}} - \frac{1}{\mathcal{E}} e^{(i - \frac{1}{\mathcal{E}})\mathcal{E}} \right).
$$

2. $T < i < 1/\tau$. In that case,

$$
\phi_i = \tau \left( i - \frac{1}{2} \mathcal{E} e^{(i - \frac{1}{\mathcal{E}})} - \frac{1}{\mathcal{E}} \left( T - \frac{1}{\mathcal{E}} \right) e^{(T - i)} \right).
$$

3. $i \geq 1/\tau$. In that case,

$$
\phi_i = 1 - \frac{\tau}{2} \mathcal{E} e^{(i - \frac{1}{\mathcal{E}})} - \frac{\tau}{2} \left( T - \frac{1}{\mathcal{E}} \right) e^{(T - i)}.
$$

To compute the MSE, we use Eq. (40), and need to compute $\int_T^{\infty} \frac{(g(j))^2}{q_j} \text{PDF}_{\text{Lap}[1/\mathcal{E}]}(j - i) \, dj$. In our implementation, we wrote functions that evaluate the integrals:

$$
\int x e^{x} \, dx = \frac{1}{\mathcal{E}^2} e^{x}(\mathcal{E} x - 1) + C
$$

$$
\int x e^{-x} \, dx = -\frac{1}{\mathcal{E}^2} e^{-x}(\mathcal{E} x + 1) + C.
$$

$$
\int x^2 e^{x} \, dx = \frac{1}{\mathcal{E}^3} e^{x}(e^2 x^2 - 2 \mathcal{E} x + 2) + C
$$
\[ \int x^2 e^{-\varepsilon x} dx = -\frac{1}{\varepsilon^3} e^{-\varepsilon x} (\varepsilon^2 x^2 + 2\varepsilon x + 2) + C.\]

Then we considered the following cases in order to compute the MSE. If \( i \leq T \), we need to compute
\[ \frac{1}{2} \varepsilon e^{\varepsilon i} \int_T^\infty \frac{j^2}{\min\{\tau_j, 1\}} e^{-\varepsilon j} dj, \]
and we have two cases:

1. \( 1/\tau < T \). In that case, the integral
   \[ \int_T^\infty \frac{j^2}{\min\{\tau_j, 1\}} e^{-\varepsilon j} dj \]
   becomes
   \[ \int_T^\infty j^2 e^{-\varepsilon j} dj. \]

2. \( 1/\tau \geq T \). In that case, the integral
   \[ \int_T^\infty \frac{j^2}{\min\{\tau_j, 1\}} e^{-\varepsilon j} dj \]
   becomes
   \[ \frac{1}{\tau} \int_T^{1/\tau} j e^{-\varepsilon j} dj + \int_1^\infty j^2 e^{-\varepsilon j} dj. \]

Similarly, if \( i > T \), we need to compute
\[ \frac{1}{2} \varepsilon e^{\varepsilon i} \int_i^\infty \frac{j^2}{\min\{\tau_j, 1\}} e^{-\varepsilon j} dj + \frac{1}{2} \varepsilon e^{-\varepsilon i} \int_i^T \frac{j^2}{\min\{\tau_j, 1\}} e^{\varepsilon j} dj, \]
and consider the three cases: (i) \( 1/\tau < T \), (ii) \( T \leq 1/\tau < i \), and (iii) \( i \leq 1/\tau \).
Figure 6: Normalized variance $\frac{\text{Var}_i}{i^2}$ and variance $\text{Var}_i$ for PWS (MLE) and sampled-SbH as a function of the frequency $i$. 