Real forms of extended Kac–Moody symmetries and higher spin gauge theories

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Abstract We consider the relation between higher spin gauge fields and real Kac–Moody Lie algebras. These algebras are obtained by double and triple extensions of real forms $g_0$ of the finite-dimensional simple algebras $g$ arising in dimensional reductions of gravity and supergravity theories. Besides providing an exhaustive list of all such algebras, together with their associated involutions and restricted root diagrams, we are able to prove general properties of their spectrum of generators with respect to a decomposition of the triple extension of $g_0$ under its gravity subalgebra $gl(D, \mathbb{R})$. These results are then combined with known consistent models of higher spin gauge theory to prove that all but finitely many generators correspond to non-propagating fields and there are no higher spin fields contained in the Kac–Moody algebra.

Keywords Higher spin theory · (Super-)gravity · Hidden symmetries · Kac–Moody algebras · Duality
1 Introduction

The existence of infinite-dimensional Kac–Moody symmetries in models of matter-coupled gravity or supergravity theories extending the known symmetries of gravitational systems has been suggested repeatedly [1,2]. In the context of M-Theory (the putative non-perturbative formulation of superstring theory) the most widely discussed recent proposals involve $E_{11}$ [3] and $E_{10}$ [4], both of which are infinite-dimensional Kac–Moody extensions of the exceptional split Lie algebra $E_{8,8}$ arising in (ungauged) maximal supergravity in three dimensions. Being double and triple extensions of $E_{8,8}$, the split Lie algebras $E_{10}$ and $E_{11}$ are often denoted as $E_{8}^{++}$ and $E_{8}^{+++}$, respectively.

For any simple, complex and finite-dimensional Lie algebra $g$ there exists a chain of embeddings of complex Lie algebras

\[ g \subset g^+ \subset g^{++} \subset g^{+++} \]  

(1.1)

where $g^+$ is the affine extension of $g$, while the further extensions $g^{++}$ and $g^{+++}$ are Kac–Moody algebras of indefinite type (hence, the algebras $g^+$, $g^{++}$ and $g^{+++}$ are all infinite-dimensional). One of the objectives of this paper is to study the chain of embeddings of real Lie algebras

\[ g_0 \subset g_0^+ \subset g_0^{++} \subset g_0^{+++} \]  

(1.2)

associated to a real form $g_0$ of the complex $g$ in (1.1). The three extensions are defined and analysed in detail below. For the case $g_0 = E_{8,8}$ one obtains the split real Lie algebras $E_{10}$ and $E_{11}$ of [3,4].

More generally, real forms $g_0$ of a finite-dimensional Lie algebra $g$ often arise as infinitesimal symmetries of matter coupled gravity theories via dimensional reduction from $D \geq 4$ to $D = 3$ [5]. Well-known examples of the associated real Lie groups $G_0$ include the Ehlers group $SL(2, \mathbb{R})$ for pure gravity in $D = 4$ dimensions (and more generally $SL(D - 2, \mathbb{R})$ for higher dimensional pure gravity), and $SU(2, 1)$ for Maxwell-Einstein gravity in $D = 4$. Affine symmetries $g_0^+$ emerge upon further reduction to $D = 2$ (axisymmetric stationary or colliding plane wave solutions), the best known example being the Geroch group $A^{(1)}_1 \equiv SL(2, \mathbb{R})_{ce}$ [6,7]. One can also study $D = 3$ systems coupled to a maximal symmetric space of type $G_0/K(G_0)$ directly, where $\text{Lie}(G_0) = g_0$ and $K(G_0)$ is the maximal compact subgroup [8–11].

The Kac–Moody algebras $g_0^{++}$ and $g_0^{+++}$ occurring in the two proposals of [3,4] are both infinite-dimensional, and not fully understood. In fact, only a finite number of so-called ‘low level’ generators have found an interpretation in terms of physical degrees of freedom to date. Here, the term level refers to the decomposition of either algebra into an infinite ordered ‘stack’ of representations of some finite-dimensional subalgebra. In this paper, this subalgebra is always taken to be the $GL(D, \mathbb{R})$ symmetry associated to the $D$-bein describing gravity in $D$ space-time dimensions for $g_0^{+++}$, or the spatial $(D - 1)$-bein for $g_0^{++}$. Therefore, the Tits–Satake diagrams of $g_0^{++}$ and $g_0^{+++}$ always contain, respectively, an $A_{D-2}$ and an $A_{D-1}$ subdiagram. In the remainder we will refer to this subdiagram as the gravity line.