Scattering theory of the hyperbolic $BC_n$ Sutherland and the rational $BC_n$ Ruijsenaars–Schneider–van Diejen models

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Abstract

In this paper, we investigate the scattering properties of the hyperbolic $BC_n$ Sutherland and the rational $BC_n$ Ruijsenaars–Schneider–van Diejen many-particle systems with three independent coupling constants. Utilizing the recently established action-angle duality between these classical integrable models, we construct their wave and scattering maps. In particular, we prove that for both particle systems the scattering map has a factorized form.

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1 Introduction

The Calogero–Moser–Sutherland (CMS) many-particle systems (see e.g. [1], [2], [3], [4]) and their relativistic deformations, the Ruijsenaars–Schneider–van Diejen (RSvD) models (see e.g. [5], [6]) are among the most actively studied integrable systems. They appear in several branches of mathematics and physics, with numerous applications ranging from symplectic geometry, Lie theory and harmonic analysis to solid state physics and Yang–Mills theory. The intimate connection with the theory of soliton equations is a particularly important and appealing feature of these finite dimensional integrable systems. It is a remarkable fact that the CMS and the RSvD models associated with the $A_n$ root system can be used to describe the soliton interactions of certain integrable field theories defined on the whole real line (see e.g. [5], [7], [8], [9], [10]).

In particular, these particle systems are characterized by conserved asymptotic momenta and factorized scattering maps (see e.g. [7], [11], [12], [13]). That is, in perfect analogy with the behavior of the solitons, the $n$-particle scattering is completely determined by the 2-particle processes. However, apart from some heuristic arguments [14], the link between the integrable boundary field theories and the particle models associated with the non-$A_n$-type root systems has not been developed yet. In our paper [15] we have proved that the classical hyperbolic $C_n$ Sutherland model also has a factorized scattering map, but to our knowledge analogous results for the other non-$A_n$-type root systems are not available. Motivated by this fact, in this paper we work out in detail the scattering theory of the hyperbolic $BC_n$ Sutherland and the rational $BC_n$ RSvD models with the maximal number of independent coupling constants.

Upon introducing the subset

$$\mathcal{c} = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > \ldots > x_n > 0 \} \subset \mathbb{R}^n, \quad (1.1)$$

let us recall that the classical hyperbolic $BC_n$ Sutherland and the rational $BC_n$ RSvD models live on the phase spaces

$$\mathcal{P}^S = \{ (q, p) \mid q \in \mathcal{c}, p \in \mathbb{R}^n \} \quad \text{and} \quad \mathcal{P}^R = \{ (\lambda, \theta) \mid \lambda \in \mathcal{c}, \theta \in \mathbb{R}^n \}, \quad (1.2)$$

respectively. By endowing these spaces with the natural symplectic forms

$$\omega^S = \sum_{a=1}^n dq_a \wedge dp_a \quad \text{and} \quad \omega^R = \sum_{a=1}^n d\lambda_a \wedge d\theta_a, \quad (1.3)$$

we may think of $\mathcal{P}^S$ and $\mathcal{P}^R$ as two different copies of the cotangent bundle $T^*\mathcal{c}$. The hyperbolic $BC_n$ Sutherland model is characterized by the interacting many-body Hamiltonian

$$H^S = \sum_{a=1}^n \left( \frac{p_a^2}{2} + g_1^2 w(q_a) + g_2^2 w(2q_a) \right) + \sum_{1 \leq a < b \leq n} \left( g_2^2 w(q_a - q_b) + g_2^2 w(q_a + q_b) \right), \quad (1.4)$$

with potential function $w(x) = \sinh(x)^{-2}$. To ensure the repulsive nature of the interaction, on the real coupling parameters $g$, $g_1$, and $g_2$ we impose the constraints $g^2 > 0$ and $g_1^2 + g_2^2 > 0$. As for the rational $BC_n$ RSvD model, the dynamics is governed by the Hamiltonian

$$H^R = \sum_{a=1}^n \cosh(2\theta_a) u_a(\lambda) + \frac{\nu \kappa}{4\mu^2} \prod_{a=1}^n \left( 1 + \frac{4\mu^2}{\lambda_a^2} \right) - \frac{\nu \kappa}{4\mu^2}, \quad (1.5)$$

2
where the real parameters $\mu$, $\nu$ and $\kappa$ appear in the function

$$v_a(\lambda) = \left(1 + \frac{\nu^2}{\lambda_a^2}\right)^{\frac{1}{2}} \left(1 + \frac{\kappa^2}{\lambda_a^2}\right)^{\frac{1}{2}} \prod_{b=1 \atop (b \neq a)}^{n} \left(1 + \frac{4\mu^2}{(\lambda_a - \lambda_b)^2}\right)^{\frac{1}{2}} \left(1 + \frac{4\mu^2}{(\lambda_a + \lambda_b)^2}\right)^{\frac{1}{2}},$$

(1.6)

too. Let us note that on the RSvD coupling parameters we impose the conditions $\mu \neq 0$, $\nu \neq 0$ and $\nu \kappa \geq 0$.

Working in a symplectic reduction framework, in our paper [16] we established the action-angle duality between the hyperbolic $BC_n$ Sutherland and the rational $BC_n$ RSvD models, provided the coupling parameters satisfy the relations

$$g_2^2 = \mu^2, \quad g_1^2 = \frac{1}{2} \nu \kappa, \quad g_2^2 = \frac{1}{2} (\nu - \kappa)^2.$$ 

(1.7)

By continuing our work [16], in this paper we explore the scattering theory of these particle systems. More precisely, after a brief overview in section 2 on the necessary background material, in section 3 we examine the temporal asymptotics of the trajectories of the Sutherland and the RSvD many-body models. The outcome of our analysis is formulated in lemmas 1 and 3.

As expected, the asymptotics naturally lead to the wave and the scattering maps, too. Quite remarkably, the symplecticity of the wave maps can be seen as an immediate consequence of the duality between these integrable many-body models. We also prove that the scattering maps have factorized forms, which is the characteristic property of the soliton systems. Our main results on the wave and the scattering maps are summarized in theorems 2 and 4. To conclude the paper, in section 4 we discuss our results and offer some open problems related to the scattering theory of the CMS and the RSvD systems.

2 Preliminaries

One of the main ingredients of the scattering theory we wish to develop in this paper is the action-angle duality between the hyperbolic $BC_n$ Sutherland and the rational $BC_n$ RSvD systems [16]. To keep the paper self-contained, in this section we gather some relevant facts about the symplectic reduction derivation of these models with emphasis on their duality properties. Up to some minor changes in the conventions, our overview closely follows [16]. For convenience, we formulate our results using the three independent RSvD coupling parameters $\mu$, $\nu$ and $\kappa$. Without loss of generality, we may assume from the outset that $\nu > 0 > \mu$ and $\kappa \geq 0$.

2.1 Background material from symplectic geometry

Upon taking an arbitrary positive integer $n \in \mathbb{N}$, we introduce the set

$$\mathbb{N}_n = \{1, \ldots, n\} \subset \mathbb{N}.$$ 

(2.1)

In the following we also frequently use the notation $N = 2n$. Now, with the aid of the unitary $N \times N$ matrix

$$C = \begin{bmatrix} 0_n & 1_n \\ 1_n & 0_n \end{bmatrix},$$

(2.2)
we define the non-compact real reductive matrix Lie group
\[ G = U(n, n) = \{ y \in GL(N, \mathbb{C}) \mid y^* C y = C \} \]  
(2.3)
with Lie algebra
\[ \mathfrak{g} = u(u, n) = \{ Y \in \mathfrak{gl}(N, \mathbb{C}) \mid Y^* C + CY = 0 \} \].
(2.4)

Turning to \( \mathcal{P} = G \times \mathfrak{g} \), at each point \((y, Y) \in \mathcal{P}\) we define the 1-form
\[ \vartheta(y, Y) \delta y \otimes \delta Y = \text{tr} (Y y^{-1} \delta y) \quad (\delta y \in T_y G, \delta Y \in T_Y G \cong \mathfrak{g}). \]  
(2.5)

It is clear that the product manifold \( \mathcal{P} \) equipped with the symplectic form \( \omega = -d\vartheta \) provides a convenient model for the cotangent bundle \( T^* G \).

Next, for each column vector \( V \in \mathbb{C}^N \) we define the \( N \times N \) matrix
\[ \xi(V) = i \mu (VV^* - 1_N) + i(\mu - \nu)C. \]  
(2.6)

Also, by taking the unitary elements, in \( G \) (2.3) we choose the maximal compact subgroup
\[ K = U(n, n) \cap U(N) \cong U(n) \times U(n). \]  
(2.7)

In its Lie algebra \( \mathfrak{k} = u(n, n) \cap u(N) \) the subset
\[ \mathcal{O} = \{ \xi(V) \mid V \in \mathbb{C}^N, V^* V = N, CV + V = 0 \} \subset \mathfrak{k} \]  
(2.8)
forms an orbit under the adjoint action of \( K \). Consequently, it comes naturally equipped with the Kirillov–Kostant–Souriau symplectic structure \( \omega^\mathcal{O} \) having the form
\[ \omega^\mathcal{O}_\rho([X, \rho], [Z, \rho]) = \text{tr} (\rho[X, Z]), \]  
(2.9)
where \( \rho \in \mathcal{O} \) and \([X, \rho], [Z, \rho] \in T_\rho \mathcal{O} \) with some \( X, Z \in \mathfrak{k} \).

Our study of the CMS and the RSvD particle systems is based on the symplectic reduction of the extended phase space \( \mathcal{P}^{\text{ext}} = \mathcal{P} \times \mathcal{O} \) endowed with the symplectic form
\[ \omega^{\text{ext}} = \frac{1}{2} (\omega + \omega^\mathcal{O}). \]  
(2.10)

In passing we mention that the factor 1/2 is inserted into this definition purely for convenience. Now, let us note that the symplectic left action of the product Lie group \( K \times K \) on \( \mathcal{P}^{\text{ext}} \) defined by the formula
\[ (k_L, k_R) \cdot (y, Y, \rho) = (k_L y k_R^{-1}, k_R Y k_R^{-1}, k_L \rho k_L^{-1}) \quad ((k_L, k_R) \in K \times K) \]  
(2.11)
admits a \( K \times K \)-equivariant momentum map. Making use of the identification \((\mathfrak{k} \oplus \mathfrak{k})^* \cong \mathfrak{k} \oplus \mathfrak{k} \) induced by an appropriate multiple of the \( \text{tr} \) functional, the momentum map can be written as
\[ J^{\text{ext}} : \mathcal{P}^{\text{ext}} \to \mathfrak{k} \oplus \mathfrak{k}, \quad (y, Y, \rho) \mapsto ((y Y y^{-1})_t + \rho) \oplus (-Y_t - i\kappa C), \]  
(2.12)
where \( Y_t \) denotes the anti-Hermitian part of the matrix \( Y \).

The point of the above discussion is that by starting with certain \( K \times K \)-invariant Hamiltonians on \( \mathcal{P}^{\text{ext}} \), both the hyperbolic \( BC_n \) Sutherland and the rational \( BC_n \) RSvD models with three independent coupling constants can be derived by reducing the symplectic manifold \( (\mathcal{P}^{\text{ext}}, \omega^{\text{ext}}) \) at the zero value of the momentum map \( J^{\text{ext}} \).
2.2 The Sutherland model from Marsden–Weinstein reduction

One of the key objects in the symplectic reduction derivation of the hyperbolic $BC_n$ Sutherland model is its Lax matrix $L : P^S \to g$. It has the form

$$L = \begin{bmatrix} A & B \\ -B & -A \end{bmatrix} - i\kappa C,$$

(2.13)

where $A$ and $B$ are appropriate $n \times n$ matrices. More precisely, their entries are

$$A_{a,b} = \frac{-i\mu}{\sinh(q_a - q_b)}, \quad A_{c,c} = p_c, \quad B_{a,b} = \frac{i\mu}{\sinh(q_a + q_b)}, \quad B_{c,c} = i\nu + \kappa \cosh(2q_c) \sinh(2q_c),$$

(2.14)

where $a, b, c \in \mathbb{N}^n$ and $a \neq b$. Next, to each real $n$-tuple $q = (q_1, \ldots, q_n)$ we associate the matrix

$$Q = \text{diag}(q_1, \ldots, q_n, -q_1, \ldots, -q_n),$$

(2.15)

and we also introduce the column vector $E \in \mathbb{C}^N$ with components

$$E_a = -E_{n+a} = 1 \quad (a \in \mathbb{N}_n).$$

(2.16)

Finally, let $U(1)_s$ denote the diagonal embedding of $U(1)$ into $K \times K$ and define the product manifold

$$\mathcal{M}^S = P^S \times (K \times K)/U(1)_s.$$  

(2.17)

Having equipped with the above objects, we can start the reduction procedure by analyzing the level set

$$\Sigma_0 = \{(y, Y, \rho) \in \mathcal{P}^\text{ext} | J^\text{ext}(y, Y, \rho) = 0\}.$$  

(2.18)

It turns out to be an embedded submanifold of $\mathcal{P}^\text{ext}$, and the diffeomorphism

$$\Upsilon_0^S : \mathcal{M}^S \to \Sigma_0, \quad (q, p, (\eta_L, \eta_R)U(1)_s) \mapsto (\eta_L e_Q^{\eta_R^{-1}} \eta_L L(q, p) \eta_R^{-1} \eta_L \xi(E) \eta_L^{-1})$$

(2.19)

gives rise to the identification $\mathcal{M}^S \cong \Sigma_0$. By inspecting the (residual) $K \times K$-action (2.11) on $\mathcal{M}^S$, it is clear that the base manifold of the trivial principal $(K \times K)/U(1)_s$-bundle

$$\pi^S : \mathcal{M}^S \to \mathcal{P}^S, \quad (q, p, (\eta_L, \eta_R)U(1)_s) \mapsto (q, p)$$

(2.20)

provides a realization of the reduced manifold. Since $(\pi^S)^*\omega^S = (\Upsilon_0^S)^*\omega^\text{ext}$, we conclude that the Sutherland phase space $\mathcal{P}^S$ (1.2) does serve as a convenient model for the reduced symplectic manifold $\mathcal{P}^\text{ext}/\!\!/0(K \times K)$. Moreover, making use of the $K \times K$-invariant Hamiltonian

$$F : \mathcal{P}^\text{ext} \to \mathbb{R}, \quad (y, Y, \rho) \mapsto F(y, Y, \rho) = \frac{1}{4} \text{tr}(Y^2),$$

(2.21)

we find the relationship $(\pi^S)^*H^S = (\Upsilon_0^S)^*F$ as well. In other words, the reduced Hamiltonian induced by the quadratic function $F$ coincides with the Hamiltonian (1.4) of the hyperbolic $BC_n$ Sutherland model with coupling constants displayed in equation (1.7).
2.3 The symplectic reduction derivation of the RSvD model

Compared to the Sutherland model, the reduction derivation of the rational $BC_n$ RSvD model requires a slightly longer preparation. As a first step, for each $a \in \mathbb{N}_n$ we define the function

$$c \ni \lambda \mapsto z_a(\lambda) = -\left(1 + \frac{i\nu}{\lambda_a}\right) \prod_{\substack{b=1 \atop b \neq a}}^{n} \left(1 + \frac{2i\mu}{\lambda_a - \lambda_b}\right) \left(1 + \frac{2i\mu}{\lambda_a + \lambda_b}\right) \in \mathbb{C}.$$  \hfill (2.22)

Also, we introduce the function $A: \mathcal{P}_R \rightarrow G$ with matrix entries

$$A_{a,b} = e^{-\theta_{a-b} \cdot z_a z_b} \frac{2i\mu}{2\mu + \lambda_a - \lambda_b}, \quad A_{n+a,n+b} = e^{\theta_{a-b} \cdot z_a z_b} \frac{2i\mu}{|z_a z_b|^\frac{1}{2} 2i\mu - \lambda_a + \lambda_b},$$  \hfill (2.23)

where $a, b \in \mathbb{N}_n$. As we have shown in our earlier paper [17], the positive definite $N \times N$ matrix $A$ provides a Lax matrix for the rational $C_n$ RSvD model with parameters $\mu$ and $\nu$.

Next, with the aid of the $\kappa$-dependent functions

$$\alpha(x) = \frac{\sqrt{x + \sqrt{x^2 + \kappa^2}}}{\sqrt{2x}} \quad \text{and} \quad \beta(x) = i\kappa \frac{1}{\sqrt{2x + \sqrt{x^2 + \kappa^2}}}$$  \hfill (2.24)

defined for $x > 0$, for each $\lambda \in c$ we introduce the $N \times N$ matrix

$$h(\lambda) = \begin{bmatrix} \text{diag}(\alpha(\lambda_1), \ldots, \alpha(\lambda_n)) & \text{diag}(\beta(\lambda_1), \ldots, \beta(\lambda_n)) \\ -\text{diag}(\beta(\lambda_1), \ldots, \beta(\lambda_n)) & \text{diag}(\alpha(\lambda_1), \ldots, \alpha(\lambda_n)) \end{bmatrix} \in G.$$  \hfill (2.25)

Utilizing $h(\lambda)$, the Lax matrix of the rational $BC_n$ RSvD model can be written as

$$A^{BC}: \mathcal{P}_R \rightarrow G, \quad (\lambda, \theta) \mapsto A^{BC}(\lambda, \theta) = h(\lambda)^{-1}A(\lambda, \theta)h(\lambda)^{-1}.$$  \hfill (2.26)

Heading toward our goal, we still need the column vector $F(\lambda, \theta) \in \mathbb{C}^N$ with components

$$F_a(\lambda, \theta) = e^{-\theta_a \cdot z_a(\lambda)} \frac{1}{2} \quad \text{and} \quad F_{n+a}(\lambda, \theta) = e^{\theta_a \cdot z_a(\lambda)} |z_a(\lambda)|^{-\frac{1}{2}},$$  \hfill (2.27)

where $a \in \mathbb{N}_n$, together with the column vector

$$V(\lambda, \theta) = A(\lambda, \theta)^{-\frac{1}{2}}F(\lambda, \theta) \in \mathbb{C}^N.$$  \hfill (2.28)

Also, for each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ we define the diagonal matrix

$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n).$$  \hfill (2.29)

At this point we can put the RSvD particle system into the context of symplectic reduction. Upon introducing the smooth product manifold

$$M^R = \mathcal{P}_R \times (K \times K)/U(1),$$  \hfill (2.30)
the diffeomorphism \( \Upsilon^R_0: \mathcal{M}^R \to \mathcal{L}_0 \) defined by the formula

\[
(\lambda, \theta, (\eta_L, \eta_R)U(1)_*) \mapsto (\eta_L A(\lambda, \theta) h(\lambda)^{-1} \eta_R^{-1}, \eta_R h(\lambda) A h(\lambda)^{-1} \eta_R^{-1}, \eta_L \xi(V(\lambda, \theta)) \eta_L^{-1}) \tag{2.31}
\]

provides an alternative parametrization of the level set \( \mathcal{L}_0 \) \((2.18)\). Moreover, the explicit form of the (residual) \( K \times K \)-action \((2.11)\) on the model space \( \mathcal{M}^R \cong \mathcal{L}_0 \) permits us to identify the reduced manifold with the base manifold of the trivial principal \( (K \times K)/U(1)_* \)-bundle

\[
\pi^R: \mathcal{M}^R \to \mathcal{P}^R, \quad (\lambda, \theta, (\eta_L, \eta_R)U(1)_*) \mapsto (\lambda, \theta). \tag{2.32}
\]

Since \((\pi^R)^* \omega^R = (\Upsilon^R_0)^* \omega^{ext}\), we conclude that the reduced space \( \mathcal{P}^{ext}/\!/_0(K \times K) \) can be naturally identified with the RSvD phase space \( \mathcal{P}^R \) \((1.2)\). Finally, let us consider the function

\[
f: \mathcal{P}^{ext} \to \mathbb{R}, \quad (y, Y, \rho) \mapsto f(y, Y, \rho) = \frac{1}{2} \text{tr}(yy^*). \tag{2.33}
\]

From the relationship \((\pi^R)^* H^R = (\Upsilon^R_0)^* f\) we infer immediately that the reduced Hamiltonian function corresponding to the \( K \times K \)-invariant function \( f \) coincides with the Hamiltonian \((1.5)\) of the rational \( BC_n \) RSvD model.

### 2.4 Action-angle duality

The two different parameterizations \( \Upsilon^S_0 \) \((2.19)\) and \( \Upsilon^R_0 \) \((2.31)\) of the level set \( \mathcal{L}_0 \) \((2.18)\) induce two different models, \( \mathcal{P}^{S} \) and \( \mathcal{P}^{R} \) \((1.2)\), of the symplectic quotient \( \mathcal{P}^{ext}/\!/_0(K \times K) \). Therefore there is a symplectomorphism

\[
\mathcal{S}: \mathcal{P}^S \to \mathcal{P}^R \tag{2.34}
\]

between \( \mathcal{P}^S \) and \( \mathcal{P}^R \), uniquely characterized by the equation

\[
\mathcal{S} \circ \pi^S \circ (\Upsilon^S_0)^{-1} = \pi^R \circ (\Upsilon^R_0)^{-1}. \tag{2.35}
\]

Making use of this purely geometric observation we can easily construct canonical action-angle variables for both the Sutherland and the RSvD models. Indeed, by pulling back the canonical positions and momenta of the RSvD system, the global coordinates \( \mathcal{S}^* \lambda_a \) and \( \mathcal{S}^* \theta_a \) on \( \mathcal{P}^S \) give rise to canonical action-angle variables for the Sutherland model. Similarly, by pulling back the canonical positions and momenta of the Sutherland system, the global coordinates \( (\mathcal{S}^{-1})^* q_a \) and \( (\mathcal{S}^{-1})^* p_a \) on \( \mathcal{P}^R \) provide canonical action-angle variables for the RSvD model. This remarkable phenomenon goes under the name of action-angle duality between the Sutherland and the RSvD particle systems.

### 3 Scattering theory

In this section we provide a thorough analysis of the time evolution of the hyperbolic \( BC_n \) Sutherland and the rational \( BC_n \) RSvD dynamics for the large positive and negative values of time \( t \). Based on the resulting temporal asymptotics we construct the wave and the scattering maps as well.
3.1 Scattering properties of the Sutherland model

Take an arbitrary point \((q, p) \in P_S\) and consider the Hamiltonian flow of the Sutherland model
\[
\mathbb{R} \ni t \mapsto (q(t), p(t)) \in P_S
\] (3.1)
satisfying \((q(0), p(0)) = (q, p)\). Recalling the Sutherland Lax matrix (2.13), we let \(L = L(q, p)\). Keeping the notation (2.15) in effect, for each \(t \in \mathbb{R}\) we also define
\[
Q(t) = \text{diag}(q_1(t), \ldots, q_n(t), -q_1(t), \ldots, -q_n(t)).
\] (3.2)

Now, it is clear that the (complete) ‘geodesic’ flow
\[
\mathbb{R} \ni t \mapsto (e^{Q(t)}L, \xi(E)) \in \mathcal{L}_0
\] (3.3)
generated by the unreduced free Hamiltonian \(F\) (2.21) projects onto the (complete) reduced flow (3.1). In particular, utilizing the parametrization \(\Upsilon_0^S\) (2.19), for all \(t \in \mathbb{R}\) we can write
\[
e^{Q(t)}L = k_L(t)e^{Q(t)}k_R(t)^{-1}
\] (3.4)
with some \(k_L(t), k_R(t) \in K\). It entails the spectral identification
\[
\sigma(e^{2Q(t)}) = \sigma(e^{2Q(t)}e^{tL}e^{tL^*}),
\] (3.5)
which can be seen as the starting point of a purely algebraic solution algorithm of the Sutherland model based on matrix diagonalization.

Making use of the symplectomorphism \(S\) (2.34), the above matrix flows can be parametrized with the dual variables \((\lambda, \theta) = S(q, p)\) as well. First, recalling the matrices (2.23) and (2.25), we introduce the shorthand notations
\[
A = A(\lambda, \theta) \quad \text{and} \quad h = h(\lambda).
\]
Second, using the abbreviation defined in (2.29), notice that the parametrization \(\Upsilon_0^S\) (2.31) together with the defining property of \(S\) (2.35) immediately lead to the relationships
\[
e^{Q} = \eta_L^{-1}A^*h^{-1}\eta_R^{-1} \quad \text{and} \quad L = \eta_RhA\eta_L^{-1}
\] (3.6)
with some \(\eta_L, \eta_R \in K\). It readily follows that
\[
e^{2Q}e^{tL}e^{tL^*} = \eta_R^{-1}Ae^{tA}h^{-2}e^{tA}\eta_R^{-1}.
\] (3.7)

However, since the functions (2.24) appearing in the definition of \(h\) (2.25) obey the identities
\[
\alpha(x)^2 + \beta(x)^2 = 1, \quad \alpha(x)^2 - \beta(x)^2 = \sqrt{1 + \kappa^2/x^2}, \quad 2\alpha(x)\beta(x) = \frac{ik}{x},
\] (3.8)
we find easily that
\[
h^{-2} = \sqrt{1_N + \kappa^2\Lambda^{-2}} + i\kappa C\Lambda^{-1}.
\] (3.9)
Therefore, by plugging the above formulae into (3.5), we obtain
\[
\sigma(e^{2Q(t)}) = \sigma\left(A\sqrt{1_N + \kappa^2\Lambda^{-2}}e^{2tA} + i\kappa AC\Lambda^{-1}\right).
\] (3.10)
By exploiting the consequences of this formula, now we can work out the scattering theory of the Sutherland model. Before going into the details, for each \( a \in \mathbb{N}_n \) and \( \lambda \in \mathfrak{c} \) we define

\[
\Delta_a(\lambda) = -\frac{1}{2} \sum_{b=1}^{n-1} \ln \left( 1 + \frac{4\mu^2}{(\lambda_a - \lambda_b)^2} \right) + \frac{1}{2} \sum_{b=a+1}^{n} \ln \left( 1 + \frac{4\mu^2}{(\lambda_a - \lambda_b)^2} \right) + \frac{1}{2} \sum_{b \neq a} \ln \left( 1 + \frac{4\mu^2}{(\lambda_a + \lambda_b)^2} \right) + \frac{1}{2} \ln \left( 1 + \frac{\nu^2}{\lambda_a^2} \right) + \frac{1}{2} \ln \left( 1 + \frac{\kappa^2}{\lambda_a^2} \right). \tag{3.11}
\]

**Lemma 1.** Take an arbitrary \((q, p) \in \mathcal{P}^S\) and let \((\lambda, \theta) = \mathcal{S}(q, p) \in \mathcal{P}^R\). Consider the Hamiltonian flow of the Sutherland model

\[
\mathbb{R} \ni t \mapsto (q(t), p(t)) \in \mathcal{P}^S \tag{3.12}
\]
satisfying \((q(0), p(0)) = (q, p)\). Then there are some positive constants \(T > 0, r > 0\) and \(C > 0\), such that for all \( a \in \mathbb{N}_n \) and for all \( t > T \) we have

\[
\left| q_a(t) - \left( t\lambda_a - \theta_a + \frac{1}{2} \Delta_a(\lambda) \right) \right| \leq Ce^{-tr} \quad \text{and} \quad |p_a(t) - \lambda_a| \leq Ce^{-tr}, \tag{3.13}
\]

whereas for all \( a \in \mathbb{N}_n \) and for all \( t < -T \) we can write

\[
\left| q_a(t) - \left( t(-\lambda_a) + \theta_a + \frac{1}{2} \Delta_a(\lambda) \right) \right| \leq Ce^{tr} \quad \text{and} \quad |p_a(t) - (-\lambda_a)| \leq Ce^{tr}. \tag{3.14}
\]

**Proof.** To prove the lemma, our guiding principle is Ruijsenaars’ result on the temporal asymptotics of the spectra of exponential matrix flows (see Theorem A2 in [7]). First, we introduce the \( n \times n \) matrix \( \mathcal{R}_n \) with entries

\[
(\mathcal{R}_n)_{a,b} = \delta_{a+b,n+1}. \tag{3.15}
\]

By conjugating with

\[
\mathcal{W} = \begin{bmatrix} 1_n & 0_n \\ 0_n & \mathcal{R}_n \end{bmatrix} \in GL(N, \mathbb{C}), \tag{3.16}
\]

we also define the \( N \times N \) matrices

\[
M = \mathcal{W} \mathcal{A} \sqrt{1_N + \kappa^2 A^{-2}} \mathcal{W}^{-1}, \quad D = \mathcal{W} \Lambda \mathcal{W}^{-1}, \quad X = i\kappa \mathcal{W} \Lambda \mathcal{A} \Lambda^{-1} \mathcal{W}^{-1}. \tag{3.17}
\]

Let us observe that the diagonal matrix \( D \) has the property that its eigenvalues are in strictly decreasing order on the diagonal.

By inspecting the matrix entries of \( M \) (3.17), notice that for all \( a, b \in \mathbb{N}_n \) we have

\[
M_{a,b} = \mathcal{A}_{a,b}(1 + \kappa^2 \lambda_b^{-2})^{\frac{1}{2}}. \tag{3.18}
\]

For each \( a \in \mathbb{N}_n \) let \( M^{(a)} \) denote the leading principal \( a \times a \) submatrix taken from the upper-left-hand corner of \( M \). Since \( M^{(a)} \) is essentially a Cauchy matrix, for its determinant we have

\[
\det(M^{(a)}) = \prod_{b=1}^{a} e^{-2b_k} |z_b(\lambda)| (1 + \kappa^2 \lambda_b^{-2})^{\frac{1}{2}} \prod_{1 \leq c < d \leq a} (1 + 4\mu^2(\lambda_c - \lambda_d)^{-2})^{-1}. \tag{3.19}
\]
By taking the quotients of the consecutive leading principal minors of $M$, we also define

$$m_1 = M_{1,1} \quad \text{and} \quad m_a = \frac{\det(M^{(a)})}{\det(M^{(a-1)})} \quad (2 \leq a \leq n). \tag{3.20}$$

From equation (3.19) it follows that for all $a \in \mathbb{N}$, we can write

$$m_a = e^{-2\theta_a} |z_a(\lambda)| (1 + \kappa^2 \lambda_a^{-2})^{\frac{a-1}{2}} \prod_{b=1}^{a-1} (1 + 4\mu^2(\lambda_a - \lambda_b)^{-2})^{-1}. \tag{3.21}$$

Therefore, recalling the formulae (3.11) and (3.12), we end up with the concise expression

$$\ln(m_a) = -2\theta_a + \Delta_a(\lambda). \tag{3.22}$$

Utilizing the above objects, we are now in a position to analyze the asymptotic properties of the trajectory (3.12). By combining equations (3.10) and (3.17), it is clear that for all $t \in \mathbb{R}$ we can write

$$\sigma(e^{2Q(t)}) = \sigma(M e^{2tD} + X). \tag{3.23}$$

By slightly generalizing Ruijsenaars’ aforementioned theorem, one can easily verify that the exponentially growing large eigenvalues of the matrices $Me^{2tD}$ and $Me^{2tD} + X$ have essentially the same asymptotic properties as $t \to \infty$. More precisely, there are some positive constants $T > 0$, $r > 0$ and $C > 0$, such that for all $a \in \mathbb{N}$ and for all $t > T$ we have

$$e^{2q_a(t)} = m_a e^{2t\lambda_a (1 + \varepsilon_a(t))}, \tag{3.24}$$

where the error term $\varepsilon_a(t)$ obeys the estimation

$$\max\{|\varepsilon_a(t)|, |\dot{\varepsilon}_a(t)|\} \leq Ce^{-tr} \leq \frac{1}{2}. \tag{3.25}$$

By taking the logarithm of equation (3.24), it readily follows that

$$\left|q_a(t) - t\lambda_a - \frac{1}{2} \ln(m_a)\right| = \frac{1}{2} \left|\ln(1 + \varepsilon_a(t))\right| \leq |\varepsilon_a(t)| \leq Ce^{-tr}. \tag{3.26}$$

Due to the formula (3.22), for the large positive values of $t$ the control over $q_a(t)$ is complete. Next, by taking the derivative of equation (3.24), we also find

$$\dot{q}_a(t) = \lambda_a + \frac{1}{2 + \varepsilon_a(t)} \frac{\dot{\varepsilon}_a(t)}{1 + \varepsilon_a(t)}. \tag{3.27}$$

However, the Hamiltonian equations of motion yield $p = \dot{q}$, whence we obtain

$$|p_a(t) - \lambda_a| = \frac{1}{2} \frac{|\dot{\varepsilon}_a(t)|}{|1 + \varepsilon_a(t)|} \leq |\dot{\varepsilon}_a(t)| \leq Ce^{-tr}. \tag{3.28}$$
To conclude the proof, let us observe that by conjugating with the $N \times N$ matrix $R_N$ (3.15), the spectral identification (3.23) can be rewritten as

$$
\sigma(e^{2Q(t)}) = \sigma \left( R_N M R_N^{-1} e^{2t R_N D R_N^{-1}} + R_N X R_N^{-1} \right).
$$

Since the eigenvalues of the diagonal matrix $R_N D R_N^{-1} = -D$ are in strictly increasing order on the diagonal, the asymptotic relationships (3.14) for $t \to -\infty$ can be established by the same techniques that we used for the $t \to \infty$ case.

Keeping the notations introduced in lemma 1, the asymptotics can be rewritten as

$$
q_a(t) \sim t p_a^+ + q_a^+ \quad \text{and} \quad p_a(t) \sim p_a^+ \quad (t \to \pm \infty),
$$

where the asymptotic phases and momenta have the form

$$
q_a^\pm = \mp \theta_a + \frac{1}{2} \Delta_a(\lambda) \quad \text{and} \quad p_a^\pm = \pm \lambda_a,
$$

respectively. Notice that the asymptotic states $(q^\pm, p^\pm)$ belong to the manifolds

$$
\mathcal{P}^\pm = \{(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} | y_1 \gtrless \ldots \gtrless y_n \gtrsim 0\},
$$

that we endow with the natural symplectic forms

$$
\omega^\pm = \sum_{a=1}^n dx_a \wedge dy_a.
$$

One of the principal goals of scattering theory is to study the wave maps

$$
W_\pm^S : \mathcal{P}^S \to \mathcal{P}^\pm, \quad (q, p) \mapsto (q^\pm, p^\pm).
$$

**Theorem 2.** The wave maps $W_\pm^S$ of the Sutherland model are symplectomorphisms from $\mathcal{P}^S$ onto $\mathcal{P}^\pm$. The scattering map $S^S = W_+^S \circ (W_-^S)^{-1}$ is also a symplectomorphism of the form

$$
\mathcal{P}^- \ni (x, y) \mapsto S^S(x, y) = (-x_1 + \Delta_1(-y), \ldots, -x_n + \Delta_n(-y), -y_1, \ldots, -y_n) \in \mathcal{P}^+.
$$

**Proof.** Let us introduce the maps

$$
T_\pm^S : \mathcal{P}^R \to \mathcal{P}^\pm, \quad (\lambda, \theta) \mapsto \left( \mp \theta_1 + \frac{1}{2} \Delta_1(\lambda), \ldots, \mp \theta_n + \frac{1}{2} \Delta_n(\lambda), \pm \lambda_1, \ldots, \pm \lambda_n \right).
$$

Recalling $\Delta_a$ (3.11), it is evident that $T_\pm^S$ are symplectomorphisms with inverses

$$
(T_\pm^S)^{-1}(x, y) = \left( \pm y_1, \ldots, \pm y_n, \mp x_1 + \frac{1}{2} \Delta_1(\pm y), \ldots, \mp x_n + \frac{1}{2} \Delta_n(\pm y) \right).
$$

Moreover, remembering the asymptotic phases and momenta (3.31), it readily follows that the wave maps (3.34) are symplectomorphisms of the form

$$
W_\pm^S = T_\pm^S \circ S.
$$

Since $S^S = T_+^S \circ (T_-^S)^{-1}$, the explicit formula (3.35) is also immediate. \qed
3.2 Scattering properties of the RSvD model

Take an arbitrary \((\lambda, \theta) \in \mathcal{P}^R\) and consider the Hamiltonian flow of the RSvD model

\[
\mathbb{R} \ni t \mapsto (\lambda(t), \theta(t)) \in \mathcal{P}^R
\]

(3.39)

passing through the point \((\lambda, \theta)\) at \(t = 0\). Remembering the matrices (2.23), (2.25), (2.26), and the column vector (2.28), we introduce the abbreviations

\[
\mathcal{A} = \mathcal{A}(\lambda, \theta), \quad h = h(\lambda), \quad \mathcal{A}^{BC} = \mathcal{A}^{BC}(\lambda, \theta), \quad \mathcal{V} = \mathcal{V}(\lambda, \theta).
\]

(3.40)

Keeping the notation displayed in equation (2.29), for all \(t \in \mathbb{R}\) we also define the diagonal matrix

\[
\Lambda(t) = \text{diag}(\lambda_1(t), \ldots, \lambda_n(t), -\lambda_1(t), \ldots, -\lambda_n(t)).
\]

(3.41)

Now, bearing in mind the symplectic reduction derivation of the RSvD model, let us note that the (complete) 'linear' flow

\[
\mathbb{R} \ni t \mapsto \left(\mathcal{A}^{\frac{1}{2}} h^{-1}, h \Lambda h^{-1} - t \left(\mathcal{A}^{BC} - (\mathcal{A}^{BC})^{-1}\right), \xi(\mathcal{V})\right) \in \mathfrak{L}_0
\]

(3.42)

generated by the unreduced Hamiltonian \(f (2.33)\) projects onto the (complete) reduced RSvD flow (3.39). Recalling the parametrization \(\Upsilon^R (2.31)\), it is evident that for all \(t \in \mathbb{R}\) we have

\[
h \Lambda h^{-1} - t \left(\mathcal{A}^{BC} - (\mathcal{A}^{BC})^{-1}\right) = k_R(t) h(\lambda(t)) \Lambda(t) h(\lambda(t))^{-1} k_R(t)^{-1}
\]

(3.43)

with some \(k_R(t) \in K\). In particular, we obtain the spectral identification

\[
\sigma(\Lambda(t)) = \sigma \left(h \Lambda h^{-1} - t \left(\mathcal{A}^{BC} - (\mathcal{A}^{BC})^{-1}\right)\right).
\]

(3.44)

The point is that each trajectory of the RSvD dynamics can be recovered by simply diagonalizing a linear matrix flow.

Utilizing the duality symplectomorphism \(S (2.34)\), our next goal is to parametrize the above linear matrix flow with the dual variables \((q, p) = S^{-1}(\lambda, \theta) \in \mathcal{P}^S\). Recalling the Sutherland Lax matrix (2.13), we first define \(L = L(q, p)\). At this point, remembering the notation (2.15), it is clear that the form of \(T^R_0 (2.19)\) and the defining property of \(S (2.35)\) lead to the relationships

\[
\mathcal{A}^{\frac{1}{2}} h^{-1} = \eta_L e^Q \eta_R^{-1} \quad \text{and} \quad h \Lambda h^{-1} = \eta_R L \eta_R^{-1}
\]

(3.45)

with some \(\eta_L, \eta_R \in K\). It entails \(\mathcal{A}^{BC} = \eta_R e^{2Q} \eta_R^{-1}\), therefore the matrix

\[
\mathcal{A}^{BC} - (\mathcal{A}^{BC})^{-1} = 2\eta_R \sinh(2Q) \eta_R^{-1}
\]

(3.46)

has a simple spectrum. Moreover, recalling the relationship (3.44), for the spectrum of \(\Lambda(t)\) we obtain the particularly useful formula

\[
\sigma(\Lambda(t)) = \sigma \left(L - 2t \sinh(2Q)\right).
\]

(3.47)
Lemma 3. Take an arbitrary point \((\lambda, \theta) \in \mathcal{P}^R\) and let \((q, p) = S^{-1}(\lambda, \theta) \in \mathcal{P}^S\). Consider the Hamiltonian flow of the RSvD model
\[
\mathbb{R} \ni t \mapsto (\lambda(t), \theta(t)) \in \mathcal{P}^R
\]
satisfying \((\lambda(0), \theta(0)) = (\lambda, \theta)\). Then there are some positive constants \(T > 0\) and \(C > 0\), such that for all \(a \in \mathbb{N}_n\) and for all \(t > T\) we have
\[
|\lambda_a(t) - (2t \sinh(2q_a) - p_a)| \leq Ct^{-1} \quad \text{and} \quad |\theta_a(t) - q_a| \leq Ct^{-2},
\]
whereas for all \(a \in \mathbb{N}_n\) and for all \(t < -T\) we can write
\[
|\lambda_a(t) - (2t \sinh(-2q_a) + p_a)| \leq C|t|^{-1} \quad \text{and} \quad |\theta_a(t) - (-q_a)| \leq C|t|^{-2}.
\]

Proof. Making use of \(\mathcal{R}_n (3.15)\), we define
\[
\mathcal{W} = \begin{bmatrix} 0_n & 1_n \\ \mathcal{R}_n & 0_n \end{bmatrix} \in GL(N, \mathbb{C}).
\]
With the aid of the \(N \times N\) matrices
\[
D = -2\mathcal{W} \sinh(2Q)\mathcal{W}^{-1} \quad \text{and} \quad M = \mathcal{W}L\mathcal{W}^{-1},
\]
the spectral identification \((3.47)\) can be cast into the form
\[
\sigma(\Lambda(t)) = \sigma(tD + M),
\]
where \(D\) is a diagonal matrix with its eigenvalues in strictly decreasing order on the diagonal. Therefore elementary perturbation theory can be readily applied to analyze the properties of \(\Lambda(t)\) as \(t \to \infty\). (For a short account on the relevant facts from perturbation theory see e.g. Theorem A1 in \([7]\).) More precisely, there are some positive constants \(T_0 > 0\) and \(C > 0\), such that for all \(a \in \mathbb{N}_n\) and for all \(t > T_0\) the eigenvalue \(\lambda_a(t)\) has the form
\[
\lambda_a(t) = tD_{a,a} + M_{a,a} + \varepsilon_a(t) = 2t \sinh(2q_a) - p_a + \varepsilon_a(t),
\]
where the error term \(\varepsilon_a(t)\) satisfies
\[
|\varepsilon_a(t)| \leq Ct^{-1} \quad \text{and} \quad |\dot{\varepsilon}_a(t)| \leq Ct^{-2}.
\]
Since the asymptotic property of \(\lambda(t)\) is under control, now we can turn our attention to the asymptotic analysis of \(\theta(t)\). Making use of the equations of motion generated by the RSvD Hamiltonian \(H^R (1.5)\), we find
\[
\dot{\lambda}_a(t) = 2\sinh(2\theta_a(t))v_a(\lambda(t)).
\]
From the asymptotic behavior of \(\lambda(t) (3.54)\) it is clear that there are some constants \(T_1 \geq T_0\) and \(K > 0\), such that for all \(a, b, c \in \mathbb{N}_n, a \neq b, a \neq c, a \neq c\), and for all \(t > T_1\) we have
\[
|\lambda_a(t) - \lambda_b(t)| \geq Kt \quad \text{and} \quad \lambda_c(t) \geq Kt.
\]
By inspecting $v_a(1.6)$, it is also evident that there are some constants $T \geq T_1$ and $\mathcal{H} > 0$, such that for all $a \in \mathbb{N}_n$ and for all $t > T$ we can write

$$v_a(\lambda(t)) \leq 1 + \mathcal{H}t^{-2}.$$  

(3.58)

By combining equations (3.54) and (3.56), for all $a \in \mathbb{N}_n$ and for all $t > T$ we obtain

$$\sinh(2\theta_a(t)) - \sinh(2q_a) = \left(1 - v_a(\lambda(t))\right) \sinh(2q_a) + \frac{1}{2} \ddot{\epsilon}_a(t),$$

(3.59)

whence the estimation

$$|\theta_a(t) - q_a| \leq \left|\frac{\sinh(2\theta_a(t)) - \sinh(2q_a)}{2}\right| \leq \left(\frac{\mathcal{H} \sinh(2q_1)}{2} + \frac{C}{4}\right) \frac{1}{t^2}$$

(3.60)

is immediate. Thus the asymptotic relationships (3.49) are established.

To conclude, let us note that by applying the same techniques on the matrix flow

$$t \mapsto \mathcal{R}_N(tD + M)\mathcal{R}_N^{-1},$$

(3.61)

one can easily prove the remaining relationships (3.50) as well.

Keeping the notations of lemma 3, our results on the asymptotics can be rephrased as

$$\lambda_a(t) \sim 2t \sinh(2\theta_a^-) + \lambda_a^\pm \quad \text{and} \quad \theta_a(t) \sim \theta_a^\pm \quad (t \to \pm \infty),$$

(3.62)

where for all $a \in \mathbb{N}_n$ we have

$$\lambda_a^\pm = \mp p_a \quad \text{and} \quad \theta_a^\pm = \pm q_a.$$  

(3.63)

Notice that the asymptotic states $(\lambda^\pm, \theta^\pm)$ belong to the phase spaces $\mathcal{P}^\pm$ (3.32), therefore the RSvD model can be characterized by the wave maps

$$W^R_\pm : \mathcal{P}^R \to \mathcal{P}^\pm, \quad (\lambda, \theta) \mapsto (\lambda^\pm, \theta^\pm).$$

(3.64)

**Theorem 4.** The wave maps $W^R_\pm$ of the RSvD model are symplectomorphisms from $\mathcal{P}^R$ onto $\mathcal{P}^\pm$. The scattering map is also a symplectomorphism of the form

$$S^R = W^R_+ \circ (W^R_-)^{-1} : \mathcal{P}^- \to \mathcal{P}^+, \quad (x, y) \mapsto S^R(x, y) = (-x, -y).$$

(3.65)

**Proof.** With the aid of the symplectomorphisms

$$\mathcal{T}^R_\pm : \mathcal{P}^S \to \mathcal{P}^\pm, \quad (q, p) \mapsto (\mp p, \pm q)$$

(3.66)

the wave maps (3.64) can be realized as compositions of symplectomorphisms of the form

$$W^R_\pm = \mathcal{T}^R_\pm \circ S^{-1}.$$  

(3.67)

Due to the relationship $S^R = \mathcal{T}^R_+ \circ (\mathcal{T}^R_-)^{-1}$, the formula (3.65) also follows.  

14
4 Discussion

For many physical systems it is the scattering theory that provides the main tool to investigate the properties of the constituent particles and the nature of their interaction. At the same time, scattering theory is a notoriously difficult subject heavily relying on non-trivial techniques from hard mathematical analysis. However, by exploiting the action-angle duality between the Sutherland and the RSvD models, a careful analysis of their algebraic solution algorithms led us to rigorous temporal asymptotics of the trajectories. Having a glance at the structure of the resulting wave maps $W^S_\pm$ (3.38) and $W^R_\pm$ (3.67), it is transparent that they are made of two strikingly different building blocks. The explicitly defined maps $T^S_\pm$ (3.36) and $T^R_\pm$ (3.66) are manifestly symplectic, meanwhile the construction and the symplecticity of $S$ (2.31) hinges on the symplectic reduction derivation of the Sutherland and the RSvD systems. That is, beside providing canonical action-angle variables, the geometric ideas culminating in the duality symplectomorphism $S$ (2.34) prove to be fundamental in the scattering theory as well.

Looking at the scattering maps $S^S$ (3.35) and $S^R$ (3.65), we see that, up to an overall sign, the asymptotic momenta of both the hyperbolic $BC_n$ Sutherland and the rational $BC_n$ RSvD models are preserved. Following Ruijsenaars’ terminology [13], we can say that these integrable systems are pure soliton systems of type $BC_n$. Moreover, for the Sutherland model the classical phase shifts are completely determined by the 2-particle processes and the 1-particle scattering on the external field. As for the rational RSvD system, the phase shifts are trivial. In other words, in both cases the scattering map has a factorized form. Due to the puzzling connection with the theory of solitons, it would be of considerable interest to develop an analogous theory at the quantum level, too. Relatedly, we find it a much more ambitious, but equally motivated research problem to classify the pure soliton systems associated with arbitrary root systems. We wish to come back to these issues in later publications.

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