SOLVING KEPLER’S EQUATION VIA SMALE’S $\alpha$-THEORY

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Abstract. We obtain an approximate solution $\tilde{E} = \tilde{E}(e, M)$ of Kepler’s equation $E - e \sin(E) = M$ for any $e \in [0, 1)$ and $M \in [0, \pi]$. Our solution is guaranteed, via Smale’s $\alpha$-theory, to converge to the actual solution $E$ through Newton’s method at quadratic speed, i.e. the $n$-th iteration produces a value $E_n$ such that $|E_n - E| \leq (\frac{1}{2})^{2^{n-1}}|\tilde{E} - E|$. The formula provided for $\tilde{E}$ is a piecewise rational function with conditions defined by polynomial inequalities, except for a small region near $e = 1$ and $M = 0$, where a single cubic root is used. We also show that the root operation is unavoidable, by proving that no approximate solution can be computed in the entire region $[0, 1) \times [0, \pi]$ if only rational functions are allowed in each branch.

1. Introduction

Kepler’s laws describe the way planets move in their orbits about the Sun. Geometrically, they say that the planets move in planar elliptical orbits with eccentricity $e \in [0, 1)$, and that the area swept by the line joining the planet and the Sun increases linearly with time, which leads immediately to Kepler’s equation $E - e \sin(E) = M$, relating mean and eccentric anomalies: the mean anomaly is a fictitious angle $M$ that increases linearly with time at a rate $M = 2\pi t/T$, where $T$ is the orbital period, and the eccentric anomaly $E$ gives the coordinates of the planet in its orbit plane as $(x, y) = (a \cos(E), b \sin(E))$. Here, the $xy$-plane has origin at the center of the ellipse with the $x$-axis pointing to the perihelion, and the values $a$ and $b$ are the semi-major and semi-minor axis of the ellipse. Therefore, finding the exact location of a planet at a given time requires solving an instance of Kepler’s equation for some $M$, assuming that the values $a$, $b$, $e$ and $T$ are known (actually, only $a$ and $e$ are needed, since $b = a\sqrt{1 - e^2}$ and $T$ can be obtained from $a$ using the third law). For a derivation of these formulas, and a detailed introduction to Kepler’s equation, see [1].

By a symmetry argument, the equation can be easily reduced to the case $M \in [0, \pi]$. The existence and uniqueness of solution $E \in [0, \pi]$ follows from the fact that the function $f_{e,M} : [0, \pi] \to [0, \pi]$ given by $f_{e,M}(E) = E - e \sin(E) - M$ is strictly increasing.

Key words and phrases. Kepler’s equation, Newton’s method.

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Several solutions to the problem have been proposed since it was stated 400 years ago. Some authors have tried non-iterative methods to solve the equation up to a fixed predetermined accuracy ([3, Ch. 1]). However, we want to calculate the solution with arbitrary precision, hence our interest in iterative techniques.

Kepler himself proposed to use a fixed-point iteration to solve the equation, i.e. guess $E_0$, an approximation of the exact solution $E$, and then iterate $E_{n+1} = M + e \sin(E_n)$. This sequence converges to $E$, since $|E_{n+1} - E| = |M + e \sin(E_n) - E| = e|\sin(E_n) - \sin(E)| \leq e|E_n - E|$, which implies that $|E_n - E| \leq e^n|E_0 - E|$ as $n \to \infty$. The problem with this approach is that the convergence is slow for values of $e$ near 1. For the orbit of Mercury, which has $e \approx 0.2$, about 5 iterations are needed to reduce the error by a factor of $10^{-3}$, while for values of eccentricity $e > 0.5$ the fixed-point iteration is even slower than a bisection method.

Although the fixed-point iteration does not provide an efficient solution to Kepler’s equation, it exhibits the structure of most of the current methods to solve it: first, guess an approximation $\tilde{E}$ of the solution (called starter), and then use some iterative technique to produce a sequence quickly converging to the actual solution (see [1, 2, 9, 13]). For the second part, Newton’s method seems to be the most used iteration, mainly due to its conceptual simplicity, generality and fast convergence. The guessing part, however, requires some specific understanding on the equation and has been the subject of many recent papers ([2, 7, 10, 11, 12, 13]).

Starters have been compared (and optimized) using different criteria, such as the number of iterations needed to reach certain precision, the distance to the actual solution, the number of floating point operations needed for its computation, etc. For this purpose, we adopt a criterion which is very specific to Newton’s method and guarantees that the iterations reduce the error at quadratic speed. More precisely, we will only accept an approximate solution $\tilde{E}$ of the equation $f_{e,M}(E) = 0$ if Newton’s method starting at $E_0 = \tilde{E}$ produces a sequence $E_n$ such that $|E_n - E| \leq \left(\frac{1}{2}\right)^n|\tilde{E} - E|$ for all $n \geq 0$.

Taking one of these starters satisfying $\tilde{E} \in [0, \pi]$, the initial error is at most $\pi$, so we obtain an accuracy $10^{-N}$ after only $n = [\log_2 (1 + \log_2(\pi) + \log_2(10)N)]$ iterations. In particular, ten iterations of Newton’s method starting from $\tilde{E}$ give an error less than $10^{-307}$ for any input value of $e$ and $M$.

We will use a simple test, due to Smale [14] and later improved by Wang and Han [16], which depends only on the starter $\tilde{E}$, and guarantees the speed of convergence that we claim.

**Definition 1.1** (Smale’s $\alpha$-test). We say that $\tilde{E}$ is an approximate zero of $f_{e,M}$ if it satisfies the following condition

$$\alpha(f_{e,M}, \tilde{E}) = \beta(f_{e,M}, \tilde{E}) \cdot \gamma(f_{e,M}, \tilde{E}) < \alpha_0,$$

where

$$\beta(f_{e,M}, \tilde{E}) = \frac{|f_{e,M}(\tilde{E})|}{|f'_{e,M}(\tilde{E})|}, \quad \gamma(f_{e,M}, \tilde{E}) = \sup_{k \geq 2} \left| \frac{f^{(k)}_{e,M}(\tilde{E})}{k!f'_{e,M}(\tilde{E})} \right|^{\frac{1}{k-1}}$$
and \( \alpha_0 = 3 - 2\sqrt{2} \approx 0.1715728 \).

Odell and Gooding [12] compiled a list of starters that have been proposed in the literature by many authors. The following table provides a formula for those that will be studied in this paper.

| Starter | Formula |
|---------|---------|
| \( S_1 \) | \( M \) |
| \( S_2 \) | \( M + e \sin(M) \) |
| \( S_3 \) | \( M + e \sin(M)(1 + e \cos(M)) \) |
| \( S_4 \) | \( M + e \) |
| \( S_5 \) | \( M + \frac{e \sin(M)}{1 - \sin(M) + e \sin(M)} \) |
| \( S_6 \) | \( M + \frac{e(\pi - M)}{1 + e} \) |
| \( S_7 \) | \( \min \left\{ \frac{M}{1 + e}, S_4, S_6 \right\} \) |
| \( S_8 \) | \( S_3 + e^{\sin(\pi - S_3)} \) |
| \( S_9 \) | \( M + e \sin(M)(1 - 2e \cos(M) + e^2)^{-1/3} \) |
| \( S_{10} \) | \( s - \frac{q}{r}, \) where \( r = \frac{1M}{e}, q = \frac{2(1-e)}{1+e} \) and \( s = \sqrt{(r^2 + q^2)^{1/2} + r^{1/2}} \) |

Table 1. Classical starters.

In section 2 we present an analytical study of the starters \( \tilde{E} = 0, \pi, M, \frac{M}{1-e} \) using the notion of approximate zero. More precisely, for each of these starters, we obtain in Theorems 2.2, 2.3, 2.4 and 2.5 regions where they satisfy Smale’s \( \alpha \)-test, thus providing approximate solutions. We also show in Theorem 2.6 that Ng’s starter \( S_{10} \) [10, Eq. 9], which is obtained by solving a cubic equation, gives an approximate solution on the entire domain.

Similarly, in section 3 we compare the remaining starters \( S_2, \ldots, S_9 \), and the improved \( S_7 \) starter obtained by Calvo et al. in [2, Prop. 1]. More precisely, we check numerically where those starters satisfy Smale’s \( \alpha \)-test on a very fine grid of points in \([0, 1) \times [0, \pi]\).

In section 4 we show a simple starter \( \tilde{E} = \tilde{E}(e, M) \) which satisfies the \( \alpha \)-test for all \( e \in [0, 1) \) and \( M \in [0, \pi] \). The starter is a piecewise-defined function that requires a single cubic root in a small part of the region close to the corner \( e = 1, M = 0 \). Apart from that root, the rest of the expressions involved are constant or rational functions that can be computed with at most two arithmetic operations. The highlights of this starter are its computational simplicity and the fact that it is formally proven to converge at quadratic speed since the first iteration, thus providing arbitrary precision with a very few Newton’s method steps. It should be noted that reducing the initial error (i.e. the distance from the starter to the exact solution) is not our design goal.
Theorem 1.2. The starter

\[
\tilde{E}(e, M) = \begin{cases} 
M & \text{if } e \leq \frac{1}{2} \text{ or } M \geq \frac{2\pi}{3} \\
\frac{2\pi}{3} & \text{if } e \geq \frac{1}{2} \text{ and } \frac{\pi}{4} \leq M \leq \frac{2\pi}{3} \\
\frac{\pi}{2} & \text{if } e \geq \frac{1}{2} \text{ and } \frac{\pi}{7} \leq M \leq \frac{\pi}{4} \\
\frac{M}{1-e} & \text{if } e \geq \frac{1}{2}, M \leq \frac{\pi}{7} \text{ and } M < \frac{\sqrt[3]{12\pi}(1-e)^{3/2}}{\sqrt[3]{e}^{2}} \\
\frac{\sqrt[3]{6Me^2}}{e} - \frac{2(1-e)}{\sqrt[3]{6Me^2}} & \text{otherwise}
\end{cases}
\]

is an approximate zero of \( f_e,M \) for all \( e \in [0, 1) \) and \( M \in [0, \pi] \).

Figure 1. The points where \( \tilde{E} = M \), \( \tilde{E} = \frac{3\pi}{2} \) and \( \tilde{E} = \frac{\pi}{2} \) satisfy the \( \alpha \)-test for \( f_e,M(E) \) are shown in blue, red and green respectively. The ones of \( \tilde{E} = \frac{M}{1-e} \) and \( \tilde{E} = \frac{\sqrt[3]{6Me^2}}{e} - \frac{2(1-e)}{\sqrt[3]{6Me^2}} \) appear in yellow and orange.

This way of constructing an approximate solution by a piecewise function can be compared to Ng’s approach (see Figure 2 of [10]). However, our function is computationally simpler because Ng’s formula outside the corner uses rational functions involving many terms and near the corner uses \( S_{10} \), which requires at least a cubic and a square root for its computation.

The region near the \((1, 0)\) corner where a cubic root is needed can be reduced as much as desired but cannot be completely avoided, as the following two results show. Other authors have found similar obstructions in handling values of the eccentricity near 1 ([7]; [10]; [11]).

Theorem 1.3. For any \( \varepsilon > 0 \), there is a piecewise constant function \( \tilde{E} \) defined in \((0, 1) \times [0, \pi] \) \( \setminus ([1 - \varepsilon, 1] \times [0, \arccos(1 - \varepsilon)]) \) that satisfies the \( \alpha \)-test.

Theorem 1.4. Let \( \tilde{E} \) be a piecewise rational function in \([0, 1] \times [0, \pi] \) with a finite number of branches defined by polynomial inequalities. Then there exists \((e_0, M_0)\) such that \( \tilde{E}(e_0, M_0) \) is not an approximate zero of \( f_{e_0,M_0} \).

The starter defined in Theorem 1.3 can be extended if \( \varepsilon < 1 - \cos(\pi/7) \) to the whole region by using \( \frac{M}{1-e} \) and \( \frac{\sqrt[3]{6Me^2}}{e} - \frac{2(1-e)}{\sqrt[3]{6Me^2}} \) in the corner, as in Theorem 1.2. This result is the basis for a constructing lookup tables of starters.
Finally, Theorem 1.4 and Remark 5.1 show that the classical starters $S_1, \ldots, S_8$ and the improved $S_7$ of [2] will necessarily fail near the corner $(1, 0)$, as Figures 3, 4, 5 and 6 will later illustrate. Our theorem also excludes the possibility of using truncated power series (with integer exponents) for approximate zeros near the corner.

2. Analytical study of classical starters via $\alpha$-theory

In this section we find regions where the starters $\tilde{E} = 0, \pi, M, \frac{M}{1-e}$ are approximate zeros of Kepler’s equation in Theorems 2.2, 2.3, 2.4 and 2.5. We compare these with the regions computed numerically on a fine grid in Figures 2 and 3. We also show that Ng’s starter $S_{10}$ works in the entire region in Theorem 2.6.

Throughout the paper, we will need the following technical result.

Lemma 2.1. Let $n \geq 2$ and $x \geq \frac{n!}{(n+1)!} = \frac{1}{n+1}$. Then, the sequence $\left\{ \left( \frac{x}{(k+1)!} \right)^{\frac{1}{k-1}} \right\}_{k \geq n}$ is decreasing.

Proof. It is enough to show that $\left( \frac{x}{(k+1)!} \right)^{\frac{1}{k-1}} \geq \left( \frac{x}{(k+1)!} \right)^{\frac{1}{k-1}}$ for all $k \geq n$, which is equivalent to the inequality $\left( \frac{x}{(k+1)!} \right)^{k-1} \geq \left( \frac{x}{(k+1)!} \right)^{k-1}$, or more simply $x \geq \frac{k!}{(k+1)!}$. Note that the sequence $\frac{k!}{(k+1)!}$ is decreasing, since $\frac{(k+1)!}{(k+1)!} \frac{x}{6(1-e)^{1/2}}$. In particular, $x \geq \frac{n!}{(n+1)!} \geq \frac{k!}{(k+1)!}$ for all $k \geq n$, as we needed.

Theorem 2.2. $\tilde{E} = 0$ is an approximate zero of $f_{e,M}(E)$ in the region $R_1 \cup R_2$, where

$$R_1 = \left\{ 0 \leq M \leq 4\alpha_0(1-e), 0 \leq e \leq \frac{3}{11} \right\},$$

$$R_2 = \left\{ 0 \leq M \leq \frac{\sqrt{6}\alpha_0(1-e)^{3/2}}{\sqrt{e}}, \frac{3}{11} \leq e < 1 \right\}.$$

Proof. It is enough to show that $\alpha(f_{e,M}, 0) < \alpha_0$, which is equivalent to

$$\frac{M}{1-e} \sup_{k \geq 3, k \text{ odd}} \left( \frac{e}{k!(1-e)} \right)^{\frac{1}{k-1}} < \alpha_0,$$

since $f(0) = -M, f'(0) = 1-e, f^{(\text{even})}(0) = 0$ and $f^{(\text{odd})}(0) = \pm e$. When $e \in \left[ \frac{3}{11}, 1 \right)$, we have $\frac{e}{1-e} \geq \frac{3}{8}$, and by Lemma 2.1

$$\sup_{k \geq 3, k \text{ odd}} \left( \frac{e}{k!(1-e)} \right)^{\frac{1}{k-1}} = \sqrt{\frac{e}{6(1-e)}}.$$
\[ \frac{e}{1 - e} \leq \frac{3}{8}, \] so
\[ \left( \frac{e}{k!(1 - e)} \right)^{\frac{1}{k-1}} \leq \left( \frac{1}{16} \right)^{\frac{1}{k-1}} \leq \frac{1}{4} \quad \forall k \geq 3. \]

This means that Smale’s condition is implied by \( \frac{M}{4(1-e)} < \alpha_0 \), which corresponds to the region \( R_1 \).

**Theorem 2.3.** \( \tilde{E} = \pi \) is an approximate zero of \( f_{e,M}(E) \) in the region \( R_3 \cup R_4 \), where
\[
R_3 = \left\{ \pi - 4\alpha_0(1 + e) < M \leq \pi, \; 0 \leq e \leq \frac{3}{5} \right\}, \\
R_4 = \left\{ \pi - \sqrt{6\alpha_0(1 + e)^{3/2}} < M \leq \pi, \; \frac{3}{5} \leq e < 1 \right\}.
\]

**Proof.** Since \( f(\pi) = \pi - M, \; f'(\pi) = 1 + e, \; f^{(\text{even})}(\pi) = 0 \) and \( f^{(\text{odd})}(\pi) = \pm e \), Smale’s \( \alpha \)-test is equivalent to
\[
\frac{\pi - M}{1 + e} \sup_{k \geq 3, \; k \, \text{odd}} \left( \frac{e}{k!(1 + e)} \right)^{\frac{1}{k-1}} < \alpha_0.
\]
For any \( e \in [0, \frac{3}{5}] \), we have \( \frac{e}{1 + e} \leq \frac{3}{8} \). This gives the following estimate for the supremum:
\[
\left( \frac{e}{k!(1 + e)} \right)^{\frac{1}{k-1}} \leq \left( \frac{3/8}{k!} \right)^{\frac{1}{k-1}} \leq \left( \frac{1}{16} \right)^{\frac{1}{k-1}} \leq \frac{1}{4}, \quad \forall k \geq 3.
\]
This means that Smale’s condition is implied by \( \frac{\pi - M}{\sqrt{6}(1 - e)} < \alpha_0 \), which corresponds exactly to the region \( R_3 \). For the other case, where \( e \in [\frac{3}{5}, 1) \), the supremum is \( \sqrt{\frac{e}{6(1-e)}} \) by Lemma 2.1, so the \( \alpha \)-condition is reduced to
\[
\frac{(\pi - M)\sqrt{e}}{\sqrt{6}(1 + e)^{3/2}} < \alpha_0,
\]
which corresponds to the region \( R_4 \). \( \square \)

**Theorem 2.4.** \( \tilde{E} = M \) is an approximate zero of \( f_{e,M}(E) \) in the region
\[
\left\{ 0 \leq e \leq \frac{1}{2} \right\} \cup \left\{ \frac{2\pi}{3} \leq M \leq \pi \right\} \cup R_2,
\]
where \( R_2 \) is defined as in Theorem 2.2.

**Proof.** Consider first the strip \( M \geq \frac{2\pi}{3} \).
\[
\beta(f_{e,M}, M) = \left| \frac{e \sin(M)}{1 - e \cos(M)} \right| \leq \left| \frac{\sin(M)}{1 - \cos(M)} \right| = \cot \left( \frac{M}{2} \right) \leq \cot \left( \frac{\pi}{3} \right) = \frac{1}{\sqrt{3}}.
\]
By Lemma 2.1 we have that for any even integer \( k \geq 2 \),
\[
\left| \frac{e \sin(M)}{k!(1 - e \cos(M))} \right|^{\frac{1}{k-1}} \leq \left| \frac{1/\sqrt{3}}{k!} \right|^{\frac{1}{k-1}} \leq \frac{1}{2\sqrt{3}}.
\]
and for any odd integer $k \geq 3$,
\[
\left| \frac{e \cos(M)}{k!(1 - e \cos(M))} \right|^{\frac{1}{k-1}} \leq \left| \frac{1}{2k!} \right|^{\frac{1}{k-1}} \leq \frac{1}{2\sqrt{3}}.
\]

The last two inequalities together imply $\gamma(f_{e,M}, M) \leq \frac{1}{2\sqrt{3}}$ and $\alpha(f_{e,M}, M) \leq \frac{1}{6} < \alpha_0$. This proves that the starter $\tilde{E} = M$ satisfies $\alpha$-test in the strip $M \geq \frac{2\pi}{3}$.

In the region \(\left\{ \frac{\pi}{2} \leq M \leq \frac{2\pi}{3}, 0 \leq e \leq \frac{1}{2} \right\}\), we have that $\sin(M) \in \left[\frac{\sqrt{3}}{2}, 1\right]$ and $\cos(M) \in \left[-\frac{1}{2}, 0\right]$, so
\[
\beta(f_{e,M}, M) = \left| \frac{f(M)}{f'(M)} \right| = \frac{e \sin(M)}{1 - e \cos(M)} \leq \frac{1}{2}.
\]

On the other hand, using Lemma 2.1 gives us
\[
\sup_{k \geq 3, k \text{ even}} \left| \frac{f^{(k)}(M)}{k! f'(M)} \right|^{\frac{1}{k-1}} \leq \sup_{k \geq 4, k \text{ even}} \left| \frac{1}{2k!} \right|^{\frac{1}{k-1}} = \max \left\{ \frac{1}{4}, \frac{1}{\sqrt{48}} \right\} = \frac{1}{\sqrt{48}} \approx 0.2752,
\]
\[
\sup_{k \geq 3, k \text{ odd}} \left| \frac{f^{(k)}(M)}{k! f'(M)} \right|^{\frac{1}{k-1}} \leq \sup_{k \geq 4, k \text{ odd}} \left| \frac{1}{4k!} \right|^{\frac{1}{k-1}} = \max \left\{ \frac{1}{\sqrt{24}}, \frac{1}{\sqrt{480}} \right\} = \frac{1}{\sqrt{480}} \approx 0.2136.
\]

Therefore, $\gamma(f_{e,M}, M) \leq \frac{1}{\sqrt{48}}$ and the $\alpha$-test holds because $\frac{1}{2} - \frac{1}{\sqrt{48}} < \alpha_0$.

In the region \(\left\{ 0 \leq M \leq \frac{\pi}{2}, 0 \leq e \leq \frac{1}{2} \right\}\),

\[
(2.1) \quad \frac{e \sin(M)}{1 - e \cos(M)} \leq \frac{1}{2} \frac{\sin(M)}{1 - \frac{1}{2} \cos(M)} \leq \frac{1}{\sqrt{3}}.
\]
and using Lemma 2.4 we obtain that

\[
\sup_{k \geq 2} \left| \frac{f^{(k)}(M)}{k!f'(M)} \right|^{\frac{1}{k-1}} \leq \max \left\{ g_2, g_4, \sup_{k \geq 2} \left| g_k \right|^{\frac{1}{k-1}} \right\}
\]

where \( g_k = \left( \frac{\frac{1}{2} \sin(M)}{k!(1 - \frac{1}{2} \cos(M))} \right)^{\frac{1}{k-1}} \) for \( k = 2, 4 \). Similarly,

\[
\sup_{k \geq 3} \left| \frac{f^{(k)}(M)}{k!f'(M)} \right|^{\frac{1}{k-1}} \leq \max \left\{ g_3, g_5, \sup_{k \geq 3} \left| g_k \right|^{\frac{1}{k-1}} \right\}
\]

where \( g_k = \left( \frac{\frac{1}{2} \cos(M)}{k!(1 - \frac{1}{2} \cos(M))} \right)^{\frac{1}{k-1}} \) for \( k = 3, 5 \). Therefore,

\[
\gamma(f_{e,M}, M) \leq \max \left\{ g_2, g_3, g_4, g_5, \sqrt[5]{6!} \sqrt[7]{7!} \right\} = \max \left\{ g_2, g_3, g_4, g_5, \sqrt[3]{6!} \sqrt[5]{7!} \right\}.
\]

As an immediate consequence of the second inequality in (2.1), we get \( g_2 < g_4 \), \( \frac{1}{2} \sin(M) g_4 < \alpha_0 \), and \( \frac{1}{2} \sin(M) \sqrt[5]{6!} < \alpha_0 \). It remains to see that

\[
\frac{1}{2} \sin(M) g_k \leq \alpha_0 \text{ for } k = 3, 5,
\]

which is equivalent to proving

\[
\frac{\sin^3(M) \cos(M)}{(1 - \frac{1}{2} \cos(M))^3} < 48\alpha_0^2 \approx 1.41, \text{ and } \frac{\sin^4(M) \cos(M)}{(1 - \frac{1}{2} \cos(M))^5} < 3840\alpha_0^4 \approx 3.33.
\]

In both cases, the left-hand side function has a maximum and the inequalities are true at it.

Finally, note that \( f_{e,M}(M) = -e \sin M \leq 0 \) and \( f_{e,M} \) is increasing, so \( 0 \leq M \leq E \), where \( E \) represents the exact solution of Kepler’s equation. In particular, \( M \) is always closer to \( E \) than 0, hence for any point in \( R_2 \), the starter \( E = M \) gives an approximate solution.

**Theorem 2.5.** \( \tilde{E} = \frac{M}{1-e} \) is an approximate zero of \( f_{e,M}(E) \) in the region \( R_5 \cup R_6 \), where

\[
R_5 = \left\{ 0 \leq M < \min \left\{ \sqrt[3]{\frac{1-e}{e^{3/2}}} \frac{1}{\alpha_0^{3/2}}, \sqrt[5]{\frac{1-e}{e^{3/2}}} \frac{1}{\alpha_0^{4/5}} \right\}, 0 \leq e \leq \frac{3}{11} \right\},
\]

\[
R_6 = \left\{ 0 \leq M < \sqrt[3]{\frac{1-e}{e^{3/2}}} \frac{1}{e^{3/2}}, \frac{3}{11} \leq e < 1 \right\}.
\]

This region contains the region of Theorem 2.2.
Proof. In this case we have
\[ |f(\tilde{E})| = e \frac{M}{1 - e} - \sin \left( \frac{M}{1 - e} \right) \leq \frac{eM^3}{6(1 - e)^3}, \]
\[ |f'(\tilde{E})| \geq 1 - e \quad \text{and} \quad |f^{(k)}(\tilde{E})| \leq e \quad \text{for all} \quad k \geq 2. \] Besides,
\[ \gamma \left( f_{e,M}, \frac{M}{1 - e} \right) \leq \max \left\{ \frac{eM}{2 - e} \frac{\frac{e}{3!}}{k \geq 3}, \frac{e}{k!} \right\}. \]
In particular, Smale’s \( \alpha \)-test is satisfied if
\[ \frac{M^4e^2}{12(1 - e)^6} < \alpha_0 \quad \text{and} \quad \frac{eM^3}{6(1 - e)^4} \sup_{k \geq 3} \frac{e}{k!} < \alpha_0. \]
The first condition is equivalent to \( M < \frac{\sqrt[3]{120} \alpha_0 (1 - e)^{3/2}}{e^{1/2}} \), which is true in both
\( R_5 \) and \( R_6 \). The second inequality needs to be discussed depending on the
value of \( e \).

When \( e \in [\sqrt{3}/11, 1] \), we have by Lemma 2.1 that
\[ \sup_{k \geq 3} \left| \frac{e}{k!} \right|^{1/2} = \frac{\sqrt{6}}{6(1 - e)^{1/2}}, \]
so the second inequality becomes \( M < \sqrt{6} \sqrt[3]{120} \alpha_0 (1 - e)^{3/2} \), which is automati-
\[ \text{cally true in } R_6 \text{ since } \sqrt{6} \sqrt[3]{120} > \sqrt[3]{120} \alpha_0. \]
In the other case, i.e. when \( e \in [0, \sqrt{3}/11] \), we have \( \frac{e}{12} \leq \sqrt{3}/8. \) In particular,
we can estimate the supremum from above as follows:
\[ \sup_{k \geq 3} \left| \frac{e}{k!} \right|^{1/2} \leq \sup_{k \geq 3} \left| \frac{3}{8k!} \right|^{1/2} = \frac{1}{4}, \]
where we have used Lemma 2.1. Therefore, in the case \( e \in [0, \sqrt{3}/11] \), the
\( \alpha \)-test is satisfied when
\[ M < \frac{\sqrt[3]{120} \alpha_0 (1 - e)^{3/2}}{e^{1/2}} \quad \text{and} \quad M < \frac{\sqrt[3]{240} \alpha_0 (1 - e)^{3/2}}{e^{3/2}}, \]
which is the definition of the region \( R_5 \).

Finally, the inclusion \( R_2 \subseteq R_6 \) follows immediately from \( \sqrt{6} \alpha_0 < \sqrt[3]{120} \alpha_0 \)
and \( R_1 \subseteq R_5 \) from the fact that \( 4 \alpha_0 (1 - e) < \sqrt[3]{120} \alpha_0 (1 - e)^{3/2} \) and \( 4 \alpha_0 (1 - e) < \sqrt[3]{240} \alpha_0 (1 - e)^{3/2} \) for all \( e \in [0, \sqrt{3}/11] \).

\[ \square \]

**Theorem 2.6.** The exact solution of the cubic equation \( \tilde{E}(1 - e) + e \frac{E^3}{3} - M = 0 \) is an approximate zero of \( f_{e,M}(E) \) in the entire region \([0, 1] \times [0, \pi]\).

**Proof.** First, note that the derivative of the left-hand side of the equation is
\( (1 - e) + e \tilde{E}^2/2 > 0 \), so the expression is increasing. This means that the
cubic has only one real root. Moreover, the values of the cubic at 0 and \( \pi \)
are \( -M \leq 0 \) and \( \pi (1 - e) + e \frac{E^3}{3} - M \geq \pi - M \geq 0 \) respectively, so the real
root \( \tilde{E} \) must be in \([0, \pi]\). In particular, we have that \( \tilde{E} < \sqrt[3]{120} \), so
\[ |f(\tilde{E})| = |\tilde{E} - e \sin(\tilde{E}) - M| = |\tilde{E}(1 - e) + e \left( \frac{\tilde{E}^3}{3!} - \frac{\tilde{E}^5}{5!} + \cdots \right) - M| \leq \frac{\tilde{E}^5}{120}. \]
Let us now consider two different cases depending on the value of \( \tilde{E} \).

If \( \tilde{E} \leq \frac{\pi}{2} \), we have that

\[
\gamma(f_{e,M}, \tilde{E}) \leq \sup_{k \geq 2} \frac{1}{k!(1 - e \cos(\tilde{E}))} \leq \frac{\pi^2}{4k!E^2} \leq \frac{\pi^2}{8E^2}
\]

by Lemma 2.1. Therefore, the \( \alpha \)-test follows if we prove

\[
\frac{e^{\tilde{E}}}{120} \frac{\pi^2}{E^2 8E^2} < \frac{\pi^4 \tilde{E}}{3840} < \alpha_0 \iff \tilde{E} < \frac{3840\alpha_0}{\pi^4} \approx 6.76,
\]

which is always true in this region.

If \( \tilde{E} > \frac{\pi}{2} \), then

\[
\gamma(f_{e,M}, \tilde{E}) = \max\{g_2, g_3, g_4, g_5\},
\]

where

\[
g_2 = \frac{1}{2(1 - e \cos(\tilde{E}))}, \quad g_3 = \sqrt{\frac{\cos(\tilde{E})}{6(1 - e \cos(\tilde{E}))}},
\]

\[
g_4 = \sup_{k \geq 4, k \text{ even}} \left| \frac{1}{k!(1 - e \cos(\tilde{E}))} \right| \leq \frac{1}{24(1 - e \cos(\tilde{E}))},
\]

\[
g_5 = \sup_{k \geq 5, k \text{ odd}} \left| \frac{1}{k!(1 - e \cos(\tilde{E}))} \right| \leq \frac{1}{120(1 - e \cos(\tilde{E}))} \leq g_4.
\]

Therefore, the \( \alpha \)-test is satisfied if

\[
\frac{e^{\tilde{E}}}{120(1 - e \cos(\tilde{E}))} g_i < \alpha_0 \text{ for } i = 2, 3, 4.
\]

Since \( g_2, g_3 \) and \( g_4 \), are increasing in \( M \), it is enough to prove the inequalities when \( M = \pi \). Moreover, \( \tilde{E}(e, \pi) \) is decreasing, so \( \tilde{E}(e, \pi) \in \left[ \sqrt[3]{6\pi}, \pi \right] \) and \( 1 - e \cos(\tilde{E}(e, \pi)) \geq 1 - e \cos(\sqrt[3]{6\pi}) \).

We also have that \( \pi = e^{\tilde{E}(e, \pi)} + (1 - e)\tilde{E}(e, \pi) \geq e^{\tilde{E}(e, \pi)} + (1 - e)\sqrt[3]{6\pi} \), hence

\[
\tilde{E}(e, \pi) \leq \sqrt[3]{\frac{6(\pi - (1 - e)^3\sqrt[3]{6\pi})}{e}}.
\]
Let us now study the three different cases. When \( i = 2 \), it is enough to prove that

\[
\frac{e\tilde{E}^5}{120(1 - e \cos(\tilde{E}))} g^2 < \frac{e\tilde{E}(e, \pi)^5}{240(1 - e \cos(\sqrt{6}\pi))^2} < \alpha_0,
\]

which is true using that \( \tilde{E} \leq \pi \) in \( e \in [0, 0.17] \), \( \tilde{E}(e, \pi) \leq 2.92 \) in \( e \in [0.17, 0.3] \), \( \tilde{E}(e, \pi) \leq 2.84 \) in \( e \in [0.3, 0.4] \) and Eq. (2.2) in \( e \in [0.4, 1] \).

When \( i = 3 \), it suffices to show that

\[
\frac{e\tilde{E}^5}{120(1 - e \cos(\tilde{E}))} g^3 < \frac{e\tilde{E}^5(e, \pi)\sqrt{|\cos(\tilde{E}(e, \pi))|}}{120\sqrt{6}(1 - e \cos(\sqrt{6}\pi))^{3/2}} < \alpha_0,
\]

which is true using that

- \( \tilde{E} \leq \pi \) and \( \sqrt{|\cos(\tilde{E}(e, \pi))|} \leq 1 \) in \( e \in [0, 0.2] \),
- Eq. (2.2) and \( \sqrt{|\cos(\tilde{E}(e, \pi))|} \leq 1 \) in \( e \in [0.2, 0.7] \),
- Eq. (2.2) and \( \sqrt{|\cos(\tilde{E}(e, \pi))|} < 0.91 \) in \( e \in [0.7, 1] \).

Lastly, the case \( i = 4 \) follows by using \( \tilde{E} \leq \pi \) in \( e \in [0, 0.2] \) and Eq. (2.2) in \( e \in [0.2, 1] \).

3. Numerical comparison of classical starters via \( \alpha \)-theory

We tested numerically the \( \alpha \)-condition on a fine grid (dividing each axis in 1000 points) for the starters \( S_2, \ldots, S_9 \), defined in [12], and the improved \( S_7 \) starter obtained in [2, Prop. 1], which we denote \( S_{CEMR} \). Note that none of the starters produce approximate zeros near the corner \((1, 0)\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The regions of \( S_2 \), \( S_3 \) and \( S_4 \).}
\end{figure}
4. A SIMPLE NEW STARTER THAT COVERS THE ENTIRE REGION

We devote this section to proving Theorem 1.2. We study each branch separately.

**Theorem 4.1.** $\tilde{E} = \frac{2\pi}{3}$ is an approximate zero of $f_{e,M}(E)$ in the region

$$\left\{ \frac{\pi}{4} \leq M \leq \frac{2\pi}{3}, \frac{1}{2} \leq e < 1 \right\}.$$  

**Proof.** First of all, we have that

$$\beta \left( f_{e,M}, \frac{2\pi}{3} \right) \leq \frac{5\pi}{12} - \frac{\sqrt{3}}{2} e - \frac{\pi}{4} \frac{1 + e/2}{1 + e/2}.$$  

On the other hand,

$$\gamma \left( f_{e,M}, \frac{2\pi}{3} \right) = \max \left\{ \sup_{k \geq 2, k \text{ even}} \left| \frac{e^{\sqrt{3}/2}}{k!(1 + e/2)} \right|^{1/4}, \sup_{k \geq 3, k \text{ odd}} \left| \frac{e/2}{k!(1 + e/2)} \right|^{1/4} \right\}.$$  

Since $\frac{e/2}{1 + e/2} \in [1/3, 1/5]$, we can apply Lemma 2.1 for $n = 4$ and $n = 5$:

$$\sup_{k \geq 2, k \text{ even}} \left| \frac{e^{\sqrt{3}/2}}{k!(1 + e/2)} \right|^{1/4} = \max \left\{ \frac{e^{\sqrt{3}/2}}{2!(1 + e/2)}, \frac{e^{3/2}}{4!(1 + e/2)} \right\};$$

$$\sup_{k \geq 3, k \text{ odd}} \left| \frac{e/2}{k!(1 + e/2)} \right|^{1/4} = \max \left\{ \left( \frac{e/2}{3!(1 + e/2)} \right)^{1/2}, \left( \frac{e/2}{5!(1 + e/2)} \right)^{1/2} \right\}.$$
Comparing the four functions, we obtain
\[ \gamma \left( f_e, M, \frac{2\pi}{3} \right) = \left( \frac{e^{\sqrt{3}/2}}{4!(1 + e/2)} \right)^{\frac{1}{3}}. \]
Therefore, the \( \alpha \)-test is satisfied if
\[ \frac{5\pi}{12} - \frac{\sqrt{3}/2}{1 + e/2} \left( \frac{e^{\sqrt{3}/2}}{4!(1 + e/2)} \right)^{\frac{1}{3}} < \alpha_0. \]
Taking derivatives, it can be shown that the left-hand side of the inequality is a decreasing function of \( e \). Also, its value at \( e = 1/2 \) is approximately 0.1706, which is less than \( \alpha_0 \).

Theorem 4.2. \( \tilde{E} = \frac{\pi}{2} \) is an approximate zero of \( f_e, M(E) \) in the region
\[ \left\{ \frac{\pi}{7} \leq M \leq \frac{\pi}{4}, \frac{1}{2} \leq e < 1 \right\}. \]

Proof. We have that \( f_e, M(\frac{\pi}{2}) = \frac{\pi}{2} - e - M \leq \frac{\pi}{2} - e - \frac{\pi}{7} = \frac{5\pi}{14} - e \) and \( f'_e, M(\pi/2) = 1 \). Moreover, \( f^{(odd)}(\pi/2) = 0 \), hence
\[ \gamma(f_e, M, \pi/2) = \sup_{k \geq 2, k \text{ even}} \left| \frac{e^k}{k!} \right|^{\frac{1}{k-1}} = \max \left\{ \frac{e}{2}, \sup_{k \geq 4, k \text{ even}} \left| \frac{e^k}{k!} \right|^{\frac{1}{k-1}} \right\} = \max \left\{ \frac{e}{2}, \sqrt[24]{\frac{e}{24}} \right\} \]
by Lemma 2.1. The \( \alpha \)-test is satisfied because
\[ \left( \frac{5\pi}{14} - e \right) \frac{e}{2} \leq \left( \frac{5\pi}{14} - \frac{5\pi}{28} \right) \frac{5\pi/28}{2} \approx 0.1573 < \alpha_0, \]
\[ \left( \frac{5\pi}{14} - e \right) \sqrt[24]{\frac{e}{24}} \left( \frac{5\pi}{14} - \frac{1}{2} \right) \sqrt[24]{\frac{5\pi/14}{24}} \approx 0.1711 < \alpha_0, \]
which ends the proof. \( \square \)

Theorem 4.3. \( \tilde{E} = \frac{3\sqrt{6}Me^2}{e} - \frac{2(1-e)^{3/2}}{\sqrt{6}Me^2} \) is an approximate zero of \( f_e, M(E) \) in the region \( R_7 \), where
\[ R_7 = \left\{ \frac{8(1 - e)^{3/2}}{27\sqrt{6}Me^1/2} < M \leq \frac{\pi}{7}, \frac{3}{11} \leq e < 1 \right\}. \]

Proof. The first condition we have to impose is that \( \tilde{E} \geq 0 \), which is equivalent to \( M \geq \frac{\sqrt{2(1-e)^{3/2}}}{\sqrt{6}Me^1/2} \) and true in \( R_7 \). We also show that \( \tilde{E} \leq \pi/2 \) in \([0, \pi/7] \times [0, 1]\), which includes \( R_7 \).

Indeed, \( \tilde{E} \leq \frac{\pi}{7} \) is equivalent to
\[ h(e, M) = \frac{3\sqrt{6}Me^{1/3}M^{2/3}}{2(1-e)} - \frac{\pi^{3/2}e^{2/3}M^{1/3}}{4(1-e)} \leq 1. \]
For a fixed \( e \), the function \( h \) has a minimum at \( M = \frac{\pi^3}{384}e \) and no other critical points. Therefore, the inequality (4.1) holds if and only if \( h(e, 0) \leq 1 \).
and $h(e, \pi/2) \leq 1$. The first one is trivial since $h(e, 0) = 0$ and the second one is equivalent to

$$2\frac{\sqrt{36}}{36} \left(\frac{\pi}{7}\right)^{\frac{3}{2}} e^3 - \pi \sqrt{6} \left(\frac{\pi}{7}\right)^{\frac{3}{2}} e^3 - 4(1 - e) < 0.$$

The substitution $e = x^3$ transforms the inequality above into

$$2\frac{\sqrt{36}}{36} \left(\frac{\pi}{7}\right)^{\frac{3}{2}} x - \pi \sqrt{6} \left(\frac{\pi}{7}\right)^{\frac{3}{2}} x^2 - 4(1 - x^3) < 0,$$

which is verified for all $x \in [0, 1]$ since the expression in $x$ is increasing and the inequality is true at $x = 1$.

Substituting the expression for $\tilde{E}$ and using the Taylor expansion of $\sin \tilde{E}$, we obtain

$$|f(\tilde{E})| = |\tilde{E} - e\sin(\tilde{E}) - M| = |\tilde{E}(1 - e) + e \left(\frac{\tilde{E}^3}{3!} - \frac{\tilde{E}^5}{5!} + \cdots\right) - M|$$

$$\leq \left|\tilde{E}(1 - e) + e \frac{\tilde{E}^3}{6} - M\right| + \left|\frac{\tilde{E}^5}{120}\right| = \frac{2(1 - e)^3}{9eM} + \frac{\tilde{E}^5}{120},$$

where we have bounded the alternating series using Leibniz’s criterion (possible because $\tilde{E} < \sqrt{42}$).

Since $\tilde{E} \leq \pi/2$, we have both $f'(\tilde{E}) \geq 1 - e$ and $f'(\tilde{E}) \geq 1 - \cos(\tilde{E}) = 2\sin^2(\tilde{E}/2) \geq \frac{4}{\pi^2}\tilde{E}^2$. Therefore, the $\alpha$-test follows if we prove the stronger conditions

$$(4.2) \quad \frac{2(1 - e)^2}{9eM}\gamma(f_e, M, \tilde{E}) < \frac{3\alpha_0}{4} \quad \text{and} \quad \left|\frac{\tilde{E}^3\pi^2}{480}\right| \gamma(f_e, M, \tilde{E}) < \frac{\alpha_0}{4}.$$ 

The second one holds because

$$\gamma(f_e, M, \tilde{E}) \leq \sup_{k \geq 2} \left|\frac{1}{k!(1 - \cos(\tilde{E}))}\right|^{\frac{1}{k-1}} \leq \sup_{k \geq 2} \left|\frac{\pi^2}{4k!\tilde{E}^2}\right|^{\frac{1}{k-1}} = \frac{\pi^2}{8E^2},$$

by Lemma 2.1, and

$$\left|\frac{\tilde{E}^3\pi^2}{480}\right| \frac{\pi^2}{8E^2} = \frac{\pi^4\tilde{E}}{3840} < \frac{\alpha_0}{4} \iff \tilde{E} < \frac{960\alpha_0}{\pi^4} \approx 1.69,$$

which is true since $\tilde{E} \leq \pi/2$ in $R_7$.

For the first inequality in (4.2), we need

$$\gamma(f_e, M, \tilde{E}) \leq \max\left\{\frac{e\sin(\tilde{E})}{2!(1 - e)} \sup_{k \geq 3} \left|\frac{e}{k!(1 - e)}\right|^{\frac{1}{k-1}}\right\}$$

$$\leq \max\left\{\frac{e\tilde{E}}{2!(1 - e)} \left|\frac{e}{3!(1 - e)}\right|^{\frac{1}{2}}\right\},$$

true by Lemma 2.1 when $e \geq \frac{3}{11}$. Therefore,

$$\frac{2(1 - e)^2}{9eM} \left|\frac{e}{3!(1 - e)}\right|^{\frac{1}{2}} < \frac{3\alpha_0}{4} \iff M > \frac{8(1 - e)^{3/2}}{27\sqrt{6}\alpha e^{1/2}},$$

which is one of the conditions of the region $R_7$. 

It only remains to show that
\[
\frac{2(1-e)^2}{9eM} \frac{e\tilde{E}}{2!(1-e)} = \frac{\tilde{E}(1-e)}{9M} < \frac{3\alpha_0}{4},
\]
which is equivalent to
\[
M - \frac{4}{27\alpha_0}(1-e)\tilde{E} > 0 \quad \text{or} \quad \frac{\sqrt[3]{6}e^{2/3}}{(1-e)^{2/3}}M^{4/3} - \frac{4\sqrt[3]{36}e^{1/3}}{27\alpha_0(1-e)}M^{2/3} > -\frac{8}{27\alpha_0}.
\]
This is true for every \(e \in [0, 1)\) and \(M \in [0, \pi]\) because, if we fix \(e\), the function \(g\) has a minimum at \(M = \sqrt{\frac{48}{27\alpha_0^3}}\) and
\[
g \left( e, \sqrt{\frac{48}{27\alpha_0^3}} \right) = -\frac{24}{27\alpha_0^2} > -\frac{8}{27\alpha_0}.
\]

Proof of Theorem 1.2. It follows immediately from Theorems 2.4, 4.1, 4.2, 2.5 and 4.3, and the inequality \(\frac{4\sqrt[3]{12}}{\sqrt[3]{6}}\alpha_0 > \frac{8}{27\alpha_0}\) that implies that the “otherwise” region is included in the one from Theorem 4.3.

5. Approximate solutions near \(e = 1\) and \(M = 0\)

In this section we will prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Given \(\varepsilon > 0\), let us take a natural number \(N\) such that \(N > \frac{\pi + 2}{2\alpha_0^2}\). Given two integers \(i \in \{0, \ldots, N-1\}\) and \(j \in \{0, \ldots, N\}\), we define the constants \(E_{ij}^{\text{low}} = \frac{\pi j}{N}\) and \(E_{ij}^{\text{up}} = \pi\), which satisfy
\[
E_{ij}^{\text{low}} - \frac{i}{N} \sin(E_{ij}^{\text{low}}) - \frac{\pi j}{N} = -\frac{i}{N} \sin \left( \frac{\pi j}{N} \right) \leq 0,
\]
\[
E_{ij}^{\text{up}} - \frac{i}{N} \sin(E_{ij}^{\text{up}}) - \frac{\pi j}{N} = \pi - \frac{\pi j}{N} \geq 0,
\]
respectively. By the bisection method, we can thus find \(E_{ij}\) such that
\[
\frac{\pi j}{N} = E_{ij}^{\text{low}} \leq E_{ij} \leq E_{ij}^{\text{up}} = \pi \quad \text{and} \quad \left| E_{ij} - \frac{i}{N} \sin(E_{ij}) - \frac{\pi j}{N} \right| < \frac{1}{N}.
\]

Given \((e, M) \in ([0, 1) \times [0, \pi]) \setminus ([1 - \varepsilon, 1] \times [0, \arccos(1 - \varepsilon)])\), we now define \(\tilde{E}(e, M) = E_{ij}\), where \(i = \lfloor Ne \rfloor \in \{0, \ldots, N-1\}\) and \(j = \lceil \frac{MN}{\pi} \rceil \in \{0, \ldots, N\}\). Therefore, \(\tilde{E}\) is a piecewise constant function and it only remains to show that it satisfies the \(\alpha\)-test.

Indeed, we have that
\[
|f(\tilde{E})| = |E_{ij} - e \sin(E_{ij}) - M| = \left| \left( E_{ij} - \frac{i}{N} \sin(E_{ij}) - \frac{\pi j}{N} \right) - \left( e - \frac{i}{N} \right) \sin(E_{ij}) - \left( M - \frac{\pi j}{N} \right) \right| < \frac{1}{N} + \left| e - \frac{i}{N} \right| + \left| M - \frac{\pi j}{N} \right| \leq \frac{\pi + 2}{N}.
\]
On the other hand, \( |f'(\tilde{E})| = 1 - e \cos(\tilde{E}) \geq \varepsilon \) because

\[
|f'(\tilde{E})| \geq \begin{cases} 
1 - e \geq \varepsilon & \text{if } e \in [0, 1 - \varepsilon], \\
1 - \cos(E_{ij}) \geq 1 - \cos(M) \geq \varepsilon & \text{if } \tilde{E} \in [0, \pi/2], M \geq \arccos(1 - \varepsilon), \\
1 \geq \varepsilon & \text{if } \tilde{E} \in [\pi/2, \pi],
\end{cases}
\]

where we have used that \( E_{ij} \geq E_{ij}^{\text{low}} = \frac{\pi}{N} = \frac{\pi(MN)}{N} \geq M. \)

Since \( |f^{(k)}(\tilde{E})| \leq 1 \), we obtain using Lemma 2.1 and the hypothesis over \( N \) that

\[
\alpha(f_{e,M}, \tilde{E}) \leq \frac{\pi + 2}{N \varepsilon} \sup_{k \geq 2} \left\{ \frac{1}{k!} \left| \frac{1}{\sqrt[k]{\varepsilon}} \right| \right\} \leq \frac{\pi + 2}{2N \varepsilon^2} < \alpha_0,
\]

which ends the proof. \( \square \)

**Proof of Theorem 1.2** We proceed by contradiction, i.e., we assume that \( \tilde{E}(e, M) \) is an approximate zero of \( f_{e,M} \) for all \( e \in [0, 1) \) and \( M \in [0, \pi]. \)

Since the branches of \( \tilde{E} \) are given by polynomial inequalities, there is an open set \( U \subseteq \mathbb{R}^2 \) and \( \varepsilon > 0 \) such that \( U \supseteq \{1\} \times [0, \varepsilon] \) and \( \tilde{E} \) is a rational function on \( U \cap ([0, 1] \times [0, \pi]). \) We also assume that \( U \subseteq [1/2, 1] \times [0, 0.0001]. \)

By definition of approximate zero, we have that

\[
|f(\tilde{E})| \max \left\{ \frac{e \sin(\tilde{E})}{2(1 - e \cos \tilde{E})^2}, \sqrt{\frac{e \cos(\tilde{E})}{6(1 - e \cos \tilde{E})^3}}, \sqrt[3]{\frac{e \sin(\tilde{E})}{24(1 - e \cos \tilde{E})^4}} \right\} < \alpha_0.
\]

It can be readily verified that \( B \geq 0.14433 \) for all \( e \in [1/2, 1) \) and any \( \tilde{E} \in \mathbb{R}, \) so \( |f(\tilde{E})| \leq 0.14433 \leq 1.1888 \) in \( U. \) By the triangle inequality, this implies that \( |\tilde{E}| < 1.1888 + e + M < 2.1889 \) in \( U. \) Repeating the argument, but using that \( |\tilde{E}| < 2.1889, \) it can be shown that \( B \geq 0.176, \) so \( |\tilde{E}| < \frac{0.10}{\sqrt[3]{\varepsilon}} + e + M \leq 1.975 \) in \( U. \) Doing this one more time, gives \( B \geq 0.2368 \) and the estimate \( |\tilde{E}| < 1.725 \) in \( U. \)

Since \( \tilde{E} \) is bounded in \( U, \) it can be extended analytically to \( \{1\} \times (0, \delta) \) for some \( 0 < \delta < \varepsilon \leq 0.0001. \) To show this, recall that \( \tilde{E}(e, M) = \frac{g(e, M)}{q(e, M)} \) for some polynomials \( p \) and \( q \) with no common factors. Now, if \( q(1, M) \) were zero (as a polynomial), then \( q \) would be divisible by \( e - 1 \) and \( p \) would not, so \( \tilde{E} \) would not be bounded, in contradiction with our previous result. This proves that \( q(1, M) \neq 0, \) so we can take \( \delta > 0 \) small enough to ensure that \( q(1, M) \) has no roots in \( (0, \delta), \) hence \( \tilde{E}(1, M) \) is well defined.

Denote \( \tilde{E}_1(M) = \tilde{E}(1, M) \) for \( M \in (0, \delta). \) Using that \( B \geq \frac{e |\sin(\tilde{E})|}{2(1 - e \cos \tilde{E})^2}, \)
we get

\[
|\tilde{E} - e \sin \tilde{E} - M| \leq \frac{2\alpha_0(1 - e \cos \tilde{E})^2}{e |\sin(\tilde{E})|}.
\]

Taking limit as \( e \to 1^- \), we obtain

\[
|\tilde{E}_1 - \sin \tilde{E}_1 - M| \leq \frac{2\alpha_0(1 - \cos \tilde{E}_1)^2}{|\sin(\tilde{E}_1)|} = \frac{4\alpha_0 |\sin(\tilde{E}_1)|^3}{|\cos(\tilde{E}_1/2)|} \leq \frac{\alpha_0 |\tilde{E}_1|^3}{2 |\cos(\tilde{E}_1/2)|} < 0.133 |\tilde{E}_1|^3
\]
for all $M \in (0, \delta)$. By the power series expansion of $\sin(\tilde{E}_1)$,

$$\left| \frac{\tilde{E}_1^3}{3!} - \frac{\tilde{E}_1^5}{5!} + \ldots - M \right| < 0.133|\tilde{E}_1^3|.$$ 

By the triangle inequality,

$$\left| \frac{\tilde{E}_1^3}{6} - M \right| \leq 0.133|\tilde{E}_1^3| + \left| \frac{\tilde{E}_1^5}{5!} - \frac{\tilde{E}_1^7}{7!} + \ldots \right|$$

$$\leq |\tilde{E}_1^3| \left( 0.133 + \frac{\tilde{E}_1^2}{120} \left( 1 + \frac{\tilde{E}_1^2}{6 \cdot 7} + \frac{\tilde{E}_1^4}{6 \cdot 7 \cdot 8 \cdot 9} + \ldots \right) \right)$$

$$\leq |\tilde{E}_1^3| \left( 0.133 + \frac{1.725^2}{120} \left( 1 + \frac{1.725^2}{6^2} + \frac{1.725^4}{6^4} + \ldots \right) \right)$$

$$\leq 0.161|\tilde{E}_1|^3,$$

for all $M \in (0, \delta)$. This implies that $(1/6 - 0.161)|\tilde{E}_1^3| \leq M$, or equivalently,

$$|\tilde{E}_1| \leq \sqrt[3]{\frac{|M|}{1/6 - 0.161}} \xrightarrow{M \to 0^+} 0.$$ 

This shows that $\tilde{E}_1$ has a removable singularity at $M = 0$, so it can be extended analytically to $[0, \delta]$ with $\tilde{E}_1(0) = 0$. Moreover, $\tilde{E}_1(M) = Mr(M)$ for some analytic function $r(M)$ in $[0, \delta)$, since the power series of $\tilde{E}_1$ cannot have a non-zero constant term.

Finally, by definition of approximate zero,

$$\alpha_0 > \frac{|f(\tilde{E})|}{1 - e \cos(\tilde{E})} \max \left\{ \frac{|\sin(\tilde{E})|}{2(1 - e \cos(\tilde{E}))}, \frac{|\cos(\tilde{E})|}{\sqrt{6(1 - e \cos(\tilde{E}))}} \right\} \left( \max \{|\sin \tilde{E}_1|, |\cos \tilde{E}_1|\} \right) \geq \frac{|f(\tilde{E})|}{\sqrt[3]{12(1 - e \cos(\tilde{E}))^{3/2}}}.$$

and taking limit as $e \to 1^-$,

$$|\tilde{E}_1 - \sin \tilde{E}_1 - M| \leq \sqrt{12} \alpha_0 (1 - \cos(\tilde{E}_1))^{3/2} = \sqrt{96} \alpha_0 |\sin^3 \left( \frac{\tilde{E}_1}{2} \right)| \leq \frac{\sqrt{96} \alpha_0 |\tilde{E}_1^3|}{8} = \sqrt[3]{\frac{3}{2} \alpha_0 |\tilde{E}_1|^3}.$$

Dividing by $M$, using that $\tilde{E}_1(M) = Mr(M)$ and taking limits as $M \to 0^+$,

$$\left| r(M) - \frac{\sin(Mr(M))}{M} - 1 \right| \leq \sqrt{\frac{3}{2} \alpha_0 M^2 |r(M)|^3},$$

which gives us the contradiction $1 \leq 0$. \hfill \Box

Remark 5.1. Note that in the proof of Theorem 1.4, we use the rationality of the function only to show that it can be analytically extended to a small segment $\{1\} \times [0, \varepsilon]$ for some $\varepsilon > 0$. If we start with an analytic function
defined on $[0, 1] \times [0, \pi]$, this step is not necessary and the same contradiction is obtained.

This shows that the classical starters $S_1, \ldots, S_8$, as well as $S_{CEMR}$, are not approximate zeros in the entire domain, as Figures 3, 4, 5, and 6 illustrate.

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