HELICES IN THE EUCLIDEAN 5-SPACE $E^5$

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Abstract. In this study, we have identified $V_3$ slant helix ($2^{nd}$ type slant helix), $V_5$ slant helix ($3^{rd}$ type slant helix) and attained some characteristic properties in the Euclidean 5-Space $E^5$. In addition to this, we have proven that there are no other helices other than $V_1$ helix (inclined curve), $V_3$ slant helix and $V_5$ slant helix in the 5-dimensional Euclidean space $E^5$.

1. Introduction

The theory of curves is one of the fundamental topics of differential geometry. Some specific curves play important roles in a variation of sciences. As an example, the helix curves can often be seen in the fields of biology and computer technologies along with the daily life. [2]. The classic results in $R^3$ that are related to the helix curves and have so many fields of application, were given by M.A. Lancet in 1802 and by B. de Saint Venant in 1845, [4]. There have been many studies related to the slant helices and darboux helices in the Euclidean 3-Space, [2,7,8,14] and some results have been achieved that are related to the helices and $B_3$ slant helix (3rd type slant helix) in the Euclidean 4-Space, [9,11,12]. Apart from that, while different characterizations have been given for the inclined and non-null inclined curves in the Euclidean 5-Space and Lorentzian space, [1,3]. Moreover the non-null helices have been examined in the Lorentzian 6-Space and new characterizations have been reached for the $V_n$ slant helix in the n-dimensional Euclidean Space, [5,13]. In this study, $V_3$ slant helix and $V_5$ slant helix have been identified and some results have been obtained in the five dimensional Euclidean space $E^5$. Then we have proven that there are no other helices other than $V_1$-helix, $V_3$ slant helix and $V_5$ slant helix in $E^5$.

2. Preliminaries

Let $\alpha: I \subset R \to E^5$ be an arbitrary curve in $E^5$. Recall that the curve $\alpha$ is said to be of unit speed curve if $\langle \alpha'(s), \alpha'(s) \rangle = 1$ where $\langle , \rangle$ is the standard scalar product in the Euclidean space $E^5$ given by

$$\langle X, Y \rangle = \sum_{i=1}^{5} x_i y_i$$

for each $X = (x_1, x_2, x_3, x_4, x_5), Y = (y_1, y_2, y_3, y_4, y_5) \in E^5$. In particular, the norm of a vector $X$ is given by $\|X\| = \sqrt{\langle X, X \rangle}$, [6].

Let $\{V_1, V_2, V_3, V_4, V_5\}$ be the moving frame along $\alpha$. The Frenet equations of the curve $\alpha$ are given by

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where $V_i$ are called $i^{th}$ Frenet vectors and the functions $k_i$ are called $i^{th}$ curvatures of the curve $\alpha$. [6].

A regular curve is called a W-curve if it has constant Frenet curvatures. [10].

3. $V_1$ Helices in the Five Dimensional Euclidean Space $E^5$

**Definition 3.1.** Let $\alpha: I \subset \mathbb{R} \to E^5$ be a unit speed curve. If the tangent vector $V_1$ of the curve $\alpha$ makes a constant angle with the fixed direction $U$, then $\alpha$ is either a $V_1$-helix or inclined curve, [1].

Here $\langle V_1, U \rangle = \cos \theta = \text{const}$, $\tan t \neq 0$ expression can be written from the definition 3.1.

**Theorem 3.1.** Let $\alpha$ be a unit speed regular curve in $E^5$. Then $\alpha$ is a $V_1$-helix if and only if the function,

$$\left( \frac{k_1}{k_2} \right)^2 + \frac{1}{k_3^2} \left[ \left( \frac{k_1}{k_2} \right)' \right]^2 + \frac{1}{k_4^2} \left[ \frac{k_1k_3}{k_2} + \left[ \frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' \right] ' \right]^2$$

is constant.

Furthermore;

$$U = \cos \theta \left[ V_1 + \frac{k_1}{k_2} V_3 + \frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' V_4 + \frac{1}{k_4} \left( \frac{k_1k_3}{k_2} + \left[ \frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' \right] ' \right) V_5 \right]$$

where $\theta$ is the angle between the vectors $V_1$ and $U$, [1].

**Theorem 3.2.** Let $\alpha$ be a unit speed regular curve in $E^5$. Then, $\alpha$ is a $V_1$-helix if and only if the following equalities is satisfied

$$k_4 f'(s) = \frac{k_1k_3}{k_2} + \left[ \frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' \right]'$$

and

$$\frac{1}{k_4} \frac{d}{ds} f(s) = -\frac{1}{k_3} \left( \frac{k_1}{k_2} \right)'$$

where $f$ is $C^2$-function, [1].

**Theorem 3.3.** $\alpha$ is a unit speed curve in $E^5$. Then $\alpha$ is a $V_1$-helix if and only if the equation is satisfied.

$$\frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' = \left( A - \int \left[ \frac{k_1k_3}{k_2} \sin \int k_4 ds \right] ds \right) \sin \int k_4 ds - \left( B + \int \frac{k_1k_3}{k_2} \cos \int k_4 ds \right) \cos \int k_4 ds$$

for some constant $A$ and $B$, [1].

Now, we will examine the helices other than $V_1$-helix in the five dimensional Euclidean space $E^5$. 
4. V₃ slant helix (2ⁿᵈ type slant helix) in the five dimensional Euclidean space \(E^5\)

**Definition 4.1.** Let \(\alpha\) be a unit speed curve with nonzero curvatures \(k_i,\ (1 \leq i \leq 4)\) in \(E^5\). If the third unit Frenet vector field \(V₃\) of the curve \(\alpha\) makes a constant angle \(\phi\) with the fixed direction \(U\), then \(\alpha\) is called a \(V₃\)-slant helix (or 2ⁿᵈ type slant helix). (Suppose that \(\langle U,U \rangle = 1\)).

**Theorem 4.1.** Let \(\alpha\) be a unit speed curve in \(E^5\). Then \(\alpha\) is a \(V₃\) slant helix if and only if \(\frac{k₂}{k₁} = \text{const} \tan t\) and \(\frac{k₃}{k₄} = \text{const} \tan t\).

**Proof:** If \(\alpha\) is a \(V₃\)-slant helix and \(U\) is a fixed unit vector, then the following equality can be written as

\[
\langle V₃, U \rangle = \cos \phi = \text{const} \tan t \neq 0.
\]

Taking the differential of equation (4.1) with respect to \(s\) and using the Frenet equations, we obtain

\[
\langle -k₂V₂ + k₃V₄, U \rangle = 0.
\]

Therefore, \(U\) lies on the hyperplane spanned by the Frenet vectors \(V₁, V₃\) and \(V₅\). Then, we reach

\[
U = u₁V₁ + u₃V₃ + u₅V₅
\]

where, \(u₁ = u₁(s)\) and \(u₃ = \cos \phi = \text{const} \tan t\).

The differential of equation (4.2), we have

\[
u₁'V₁ + (u₁k₁ - u₃k₂)V₂ + u₃'V₃ + (u₃k₃ - u₅k₄)V₄ + u₅'V₅ = 0.
\]

By the above equality, the coefficients \(Vᵢ\) are zero for \(1 \leq i \leq 5\). So we get

\[
u₁' = 0
\]
\[
u₁k₁ - u₃k₂ = 0
\]
\[
u₃' = 0
\]
\[
u₃k₃ - u₅k₄ = 0
\]
\[
u₅' = 0
\]

Thus, it is easy to obtain that the coefficients \(u₁, u₃\) and \(u₅\) are given by

\[
u₁ = \cos \phi \frac{k₂}{k₁} = \text{const} \tan t
\]
\[
u₃ = \cos \phi = \text{const} \tan t
\]
\[
u₅ = \cos \phi \frac{k₃}{k₄} = \text{const} \tan t
\]

Since the coefficients \(u₁\) and \(u₅\) constants, the ratios \(\frac{k₂}{k₁}\) and \(\frac{k₃}{k₄}\) are constant, respectively. Therefore, if we substitute \(u₁, u₃\) and \(u₅\) in (4.2), we have

\[
U = \cos \phi \frac{k₂}{k₁} V₁ + \cos \phi V₃ + \cos \phi \frac{k₃}{k₄} V₅.
\]

where \(\phi \neq \frac{\pi}{2}\).

Conversely, while the ratios \(\frac{k₂}{k₁}\) and \(\frac{k₃}{k₄}\) are constant, we can define the vector \(U\).

Since the differential of \(U\) is \(\frac{dU}{ds} = 0\), \(U\) is a fixed vector. Furthermore, since \(\langle V₃, U \rangle = \cos \phi = \text{const} \tan t\), the curve \(\alpha\) become \(V₃\) slant helix.

This completes the proof.

**Result 4.1.** \(\alpha\) is a \(V₃\) slant helix if and only if the ratios \(\frac{V₃'}{V₁'}\) and \(\frac{V₃'}{V₅'}\) are constant.
Proof: If $\alpha$ is a $V_3$ slant helix, then from the theorem 4.1 we can write
\[
\frac{k_2}{k_1} = \text{const} \tan t
\]
and
\[
\frac{k_3}{k_4} = \text{const} \tan t.
\]
From the Frenet equations, it is easily to see that
\[
\frac{\|V'_2\|}{\|V'_1\|} = \sqrt{1 + \left(\frac{k_2}{k_1}\right)^2}
\]
and
\[
\frac{\|V'_4\|}{\|V'_5\|} = \sqrt{1 + \left(\frac{k_3}{k_4}\right)^2}.
\]
So we have the ratios
\[
\frac{\|V'_2\|}{\|V'_1\|}
\]
and
\[
\frac{\|V'_4\|}{\|V'_5\|}
\]
are constant.
Conversely, if $\frac{\|V'_2\|}{\|V'_1\|} = \text{const} \tan t$ and $\frac{\|V'_4\|}{\|V'_5\|} = \text{const} \tan t$, then the ratios $\frac{k_2}{k_1}$ and $\frac{k_3}{k_4}$ are constant. Thus, from Theorem 4.1 $\alpha$ is $V_3$ slant helix.

Result 4.2. If $\alpha$ is a W-curve, then $\alpha$ is $V_3$ slant helix.

Proof: If $\alpha$ is a W-curve, then the curvatures $k_i, \ 1 \leq i \leq 4$ are constants. Thus, the ratios $\frac{k_2}{k_1}$ and $\frac{k_3}{k_4}$ constant. This shows that $\alpha$ is $V_3$ slant helix from Theorem 4.1.

5. $V_5$ slant helix ($3^{rd}$ type slant helix) in five dimensional Euclidean space $E^5$

Definition 5.1. A unit speed curve $\alpha : I \subset \mathbb{R} \to E^5$ is said to be a $V_5$ slant helix ($3^{rd}$ type slant helix) if the fifth unit Frenet vector field $V_5$ makes a constant angle $\Psi$ with the unit and fixed direction $U$.

Theorem 5.1. Let $\alpha$ be a unit speed curve in $E^5$.
\(i\) $\alpha$ is a $V_5$ slant helix if and only if the function
\[
\frac{1}{k_1^2} \left[ \frac{k_4k_2}{k_3} + f'(s) \right]^2 + [f(s)]^2 + \left( \frac{k_4}{k_3} \right)^2
\]
is constant, where $f(s) = \left( \frac{k_3}{k_2} \right)' \frac{1}{k_2}$.
\(ii\) $\alpha$ is a $V_5$ slant helix if and only if the following equation is satisfied
\[
\frac{k_4k_2}{k_1k_3} + \frac{f'}{k_1} + f k_1 = 0
\]
where $f(s) = \left( \frac{k_3}{k_2} \right)' \frac{1}{k_2}$. 

Proof: i) From the above definition 5.1, we give the following
\begin{equation}
\langle V_5, U \rangle = \cos \psi = \text{cons} \tan t
\end{equation}
If we take the differential of equation (5.1) with respect to \( s \), we obtain
\[ \langle -k_4 V_4, U \rangle = 0. \]
Therefore, we may express
\begin{equation}
U = u_1 V_1 + u_2 V_2 + u_3 V_3 + u_5 V_5.
\end{equation}
we know that \( u_i = u_i(s) \) and \( u_5 = \cos \psi = \text{cons} \tan t \)
The differentiation (5.2) gives
\begin{equation}
(u'_1 - u_2 k_1) V_1 + (u_1 k_1 + u'_2 - u_3 k_2) V_2 + (u_2 k_2 + u'_3) V_3 + (u_3 k_3 - u_5 k_4) V_4 + u'_5 V_5 = 0
\end{equation}
and from this equation we find
\begin{align*}
u'_1 - u_2 k_1 &= 0 \\
u_1 k_1 + u'_2 - u_3 k_2 &= 0 \\
u_2 k_2 + u'_3 &= 0 \\
u_3 k_3 - u_5 k_4 &= 0 \\
u'_5 &= 0.
\end{align*}
Using the above equations, we can form
\begin{equation}
u_1 = \frac{\cos \psi}{k_1} \left\{ k_4 k_2 k_3 + \left[ \left( \frac{k_4}{k_3} \right) \frac{1}{k_2} \right]' \right\}
\end{equation}
\begin{align*}
u_2 &= -\cos \psi \left( \frac{k_4}{k_3} \right)' \frac{1}{k_2} \\
u_3 &= \cos \psi \frac{k_4}{k_3} \\
u_5 &= \cos \psi = \text{cons} \tan t.
\end{align*}
and
\begin{equation}
u'_1 = u_2 k_1.
\end{equation}
If we define \( f = f(s) \) by
\begin{equation}
\left( \frac{k_4}{k_3} \right)' \frac{1}{k_2} = f(s)
\end{equation}
then the equation (5.3) writes as
\begin{equation}
u_1 = \frac{\cos \psi}{k_1} \left( \frac{k_4 k_2}{k_3} + f' \right)
\end{equation}
\begin{align*}
u_2 &= -\cos \psi f \\
u_3 &= \cos \psi \frac{k_4}{k_3} \\
u_5 &= \cos \psi = \text{cons} \tan t.
\end{align*}
and equation (5.4) can be written as
\begin{equation}
\left[ \frac{k_4 k_2}{k_1 k_3} + \frac{f'}{k_1} \right]' + f k_1 = 0.
\end{equation}
Therefore, equation (5.2) takes the following form:
\begin{equation}
U = \cos \psi \left( \frac{k_4 k_2}{k_1 k_3} + \frac{f'}{k_1} \right) V_1 - \cos \psi f V_2 + \cos \psi \frac{k_4}{k_3} V_3 + \cos \psi V_5.
\end{equation}
Since $U$ is a fixed vector, the following expression

\[(5.8) \quad \frac{1}{k_1^2} \left[ \frac{k_2 k_4}{k_1 k_3} + f' \right]^2 + f'^2 + \left( \frac{k_4}{k_3} \right)^2\]

is obtained as constant.

Conversely, if equation (5.8) holds, then the fixed vector $U$ can be defined as

\[U = \cos \psi \left( \frac{k_4 k_2}{k_1 k_3} + \frac{f'}{k_1} \right) V_1 - \cos \psi f V_2 + \cos \psi \frac{k_4}{k_3} V_3 + \cos \psi V_5\]

In this case $\frac{dU}{ds} = 0$ and $\langle V_5, U \rangle = \cos \psi = \text{const} \tan t$. It is clear that $\alpha$ is $V_5$ slant helix.

ii) If $\alpha$ is $V_5$ slant helix, then from the proof of theorem 5.1 i), we have

\[\left[ \frac{k_4 k_2}{k_1 k_3} + \frac{f'}{k_1} \right]' + f k_1 = 0\]

Conversely, if the equation (5.6) holds, then the following can be written

\[U = \cos \psi \left( \frac{k_4 k_2}{k_1 k_3} + \frac{f}{k_1} \right) V_1 - \cos \psi f V_2 + \cos \psi \frac{k_4}{k_3} V_3 + \cos \psi V_5\]

Since $\frac{dU}{ds} = 0$ and $\langle V_5, U \rangle = \cos \psi = \text{const} \tan t$, $\alpha$ becomes $V_5$ slant helix.

**Result 5.1.** Let $\alpha$ be $V_5$ slant helix. If the ratio $\frac{k_4}{k_3}$ is constant, then the ratio $\frac{k_2}{k_1}$ is constant. (Namely, $\alpha$ is $V_3$ slant helix)

**Proof:** Suppose that $\alpha$ is $V_5$ slant helix and the ratio $\frac{k_4}{k_3}$ is constant. So the equation (5.8) becomes constant, that is, the ratio $\frac{k_2}{k_1}$ is found as constant. This means that the curve $\alpha$ is $V_3$ slant helix from theorem 4.1.

**Result 5.2.** If $\alpha$ is $V_5$ slant helix and $\frac{\|V_3\|}{\|V_2\|}$ is constant, then $\frac{\|V_2\|}{\|V_1\|}$ is constant. (Namely, $\alpha$ is $V_3$ slant helix)

**Proof:** It is obvious from result 5.1.

Also it can be examined if there are any other helices other than $V_1$ helix, $V_3$ slant helix and $V_5$ slant helix in $E^5$.

**Theorem 5.2.** Let $\alpha$ be a unit speed curve in $E^5$.

i) There is no fixed direction making a constant angle with the second Frenet vector $V_2$ of the curve $\alpha$.

ii) There is no fixed direction making a constant angle with the fourth Frenet vector $V_4$ of the curve $\alpha$.

**Proof:** Let us assume that the second Frenet vector $V_2$ of the unit speed curve $\alpha$ in $E^5$ makes a constant angle with the fixed direction $U$. So, we can write

\[(5.9) \quad \langle V_2, U \rangle = \cos \beta\]

where $\beta$ is a constant angle between $V_2$ and $U$. Differentiating equation (5.9) with respect to $s$, we obtain

\[\langle -k_1 V_1 + k_2 V_3, U \rangle = 0\]

This shows that the vector $U$ is perpendicular to the Frenet vectors $V_1$ and $V_3$, so the following can be written

\[(5.10) \quad U = u_2 V_2 + u_4 V_4 + u_5 V_5, \quad u_i = u_i(s)\]
Differentiating the equation (5.10), we have
\((-u_2k_1) V_1 + u'_2 V_2 + (u_2 k_2 - u_4 k_3) V_3 + (u'_4 - u_5 k_3) V_4 + (u_4 k_4 + u'_5) V_5 = 0\)
which leads to the following system
\[
\begin{align*}
-u_2 k_1 &= 0 \\
 u'_2 &= 0 \\
 u_2 k_2 - u_4 k_3 &= 0 \\
 u'_4 - u_5 k_4 &= 0 \\
 u_4 k_4 + u'_5 &= 0 
\end{align*}
\]
(5.11)

By taking account of the equations (5.11) we get \(u_2 = u_4 = u_5 = 0\) which gives us that \(\overrightarrow{U} = 0\).
Moreover, this shows that there is no fixed direction \(U\) that makes a constant angle with the Frenet vector \(V_2\).

ii) The proof is similar to the proof of theorem 5.2.i.

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