Method of moments with a choice of special basic functions for fourth-order partial differential equations

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Abstract. This paper presents an approximation method to solve the boundary value problem (BVP) for partial differential equations (PDEs) of a kind of the fourth order with the idea of discretization of a spatial variable by using the method of moments with a choice of special basic functions. A system of ordinary differential equations (ODEs) is obtained by multiplying the original equation by some auxiliary functions, followed by interpolation and integration over the spatial variable. Newton-Stirling, Hermite-Birkhoff and Hermite interpolations are flexibly applied to internal and pre-boundary nodes. In addition, boundary conditions are automatically satisfied without being approximated separately as in classical numerical methods (for example, method of grids, method of lines). Thus, the proposed schemes have a higher order of approximation. The main goal of the work is the construction of basic functions for a fourth-order differential operator and a possible increase in the order of the remainder term when passing to difference equations.

1. Introduction
In connection with the results achieved in the field of numerical solution of systems of ODEs [1, 2], it is expedient to construct differential-difference schemes of increased accuracy, which reduce the original problem to the Cauchy problem for the systems of ODEs, for boundary value problems in the case of PDEs.

In order to solve initial and BVP for PDEs, there are methods such as the method of grids, methods of integral relations, projection-variational methods, etc. The finite difference methods are a powerful tool to solve this kind of problem [3, 4, 5, 6, 7]. The order of approximation of the schemes constructed with the help of the method of lines [8] on a three-point stencil did not exceed \(O(h^4)\), since a further increase in the order of approximation by increasing the stencil of the difference scheme led to the difficulty of approximation near the boundary. The approach to a simple and effective method of a higher order of approximation is the goal of this study.

A class of fourth-order PDEs plays a crucial role in fields of applied science and engineering [9, 10]. It was used to study various problems of image processing such as image restoration and image enhancement [11, 12, 13, 14, 15, 16]. A class of second-order PDEs is widely studied and...
applied for the problems of the image processing field. However, the class of second-order PDEs can cause defects during restoring digital images such as blocky artifacts. This defect can cause many false-detection effects in computer vision. To avoid the artifact, a class of fourth-order PDEs is recommended to use [11, 12]. In this study, we do not focus on the image processing aspect, but focus on proposing a numerical method to solve the class of fourth-order PDEs.

There are several methods proposed to solve BVP of the fourth-order PDEs such as spline methods [17], higher-order difference method [18], decomposition method [19] and analysis-based method [20]. Spline functions are very common in the theory of approximation [21, 22, 23]. In this regard, we are concerned with the method of moments with the choice of special basic functions based on the spline functions to solve higher-order PDEs.

In the current article, the schemes of increased order of approximation are constructed, and the problem of approximation of the boundary points for some one-dimensional equations of mathematical physics is also solved. Computational algorithms reduce the initial problems for nonstationary equations of mathematical physics to solving the Cauchy problem for systems of first-order ODEs, for which computational methods have already been developed and implemented in [1, 2]. Especially, Hermite-Birkhoff and Hermite interpolations [24] are flexibly applied for the problems of the image processing field. However, the class of second-order PDEs can cause defects during restoring digital images such as blocky artifacts. This defect can cause many false-detection effects in computer vision. To avoid the artifact, a class of fourth-order PDEs is recommended to use [11, 12]. In this study, we do not focus on the image processing aspect, but focus on proposing a numerical method to solve the class of fourth-order PDEs.

2. Problem definition and construction principle of differential difference scheme

Using the method of moments with a special choice of basic functions, the problem reaches the problem of deflection of an elastic beam. Without loss of generality, in the domain $Q = \{(x, t): 0 \leq x \leq 1, 0 \leq t \leq T\}$, considering the following mathematical model:

$$\frac{\partial u (x, t)}{\partial t} = -\frac{\partial^4 u (x, t)}{\partial x^4} + f (x, t), \quad 0 < x < 1, \quad 0 < t \leq T, \quad (1)$$

with the conditions

$$u (x, 0) = \varphi (x), \quad 0 \leq x \leq 1,$$
$$u (0, t) = \mu_0 (t), \quad \frac{\partial u (0, t)}{\partial x} = \bar{\mu}_0 (t), \quad 0 \leq t \leq T, \quad (2)$$
$$u (1, t) = \mu_1 (t), \quad \frac{\partial u (1, t)}{\partial x} = \bar{\mu}_1 (t), \quad 0 \leq t \leq T.$$

We assume that on the interval $[0, 1]$, problem (1)–(2) has only an unique solution, and this solution is continuously differentiable up to fourth order inclusively. We perform partial discretization in the variable $x$: introduce a grid of nodes $\omega_h = \{x_i = ih, \quad i = 0, N, \quad h = 1/N\}$, the time variable $t$ is considered continuously. To solve problem (1)–(2) we use basic functions $E_{5,i}^{(3)} (x)$ in internal nodes $x_i, \quad i = 2, N - 2$ and $g_1 (x), g_{N-1} (x)$ in the respective pre-boundary nodes $x_1, x_{N-1}$. The idea of solving problem (1)–(2) is that we multiply the original equation by basic functions and integrate over $x$ from 0 to 1:

$$\int_0^1 g_1 (x) \frac{\partial u (x, t)}{\partial t} dx - \int_0^1 g_1 (x) f (x, t) dx = -\int_0^1 g_1 (x) \frac{\partial^4 u (x, t)}{\partial x^4} dx, \quad (3)$$

$$\int_0^1 E_{5,i}^{(3)} (x) \frac{\partial u (x, t)}{\partial t} dx - \int_0^1 E_{5,i}^{(3)} (x) f (x, t) dx = -\int_0^1 E_{5,i}^{(3)} (x) \frac{\partial^4 u (x, t)}{\partial x^4} dx, \quad i = 2, N - 2, \quad (4)$$

$$\int_0^1 E_{5,i}^{(3)} (x) \frac{\partial u (x, t)}{\partial t} dx - \int_0^1 E_{5,i}^{(3)} (x) f (x, t) dx = -\int_0^1 E_{5,i}^{(3)} (x) \frac{\partial^4 u (x, t)}{\partial x^4} dx, \quad i = 2, N - 2, \quad (4)$$
In this section, we will construct basic functions with special property for the 4th order differential operator. Possible to solve the system of first-order ODEs.

3. Construction of basic functions for the 4th order differential operator

In this section, we will construct basic functions with special property for the 4th order differential operator. A new question is what function can we use to obtain the following expression for a fourth-order differential operator

\[ \int_{x_{i-2}}^{x_{i+2}} g_i(x) u^{(4)}(x) dx = u(x_{i-2}) - 4u(x_{i-1}) + 6u(x_i) - 4u(x_{i+1}) + u(x_{i+2}), \quad i = \frac{2}{N-2}. \]  

(6)

We will find the functions \( g_i(x) \) as follows

\[ g_i(x) = \begin{cases} 
    g_{i-2}(x), & x \in [x_{i-2}, x_{i-1}], \\
    g_{i-1}(x), & x \in [x_{i-1}, x_i], \\
    g_{i+1}(x), & x \in [x_i, x_{i+1}], \\
    g_{i+2}(x), & x \in [x_{i+1}, x_{i+2}], \\
    0, & x \notin [x_{i-2}, x_{i+2}]. 
\end{cases} \]  

(7)

For the the left-hand side of (6), we take integration by parts

\[ \int_{x_{i-2}}^{x_{i+2}} g_i(x) u^{(4)}(x) dx = \int_{x_{i-1}}^{x_{i-2}} g_{i-2}(x) u^{(4)}(x) dx + \int_{x_{i-1}}^{x_i} g_{i-1}(x) u^{(4)}(x) dx + \int_{x_i}^{x_{i+1}} g_i(x) u^{(4)}(x) dx + \int_{x_{i+1}}^{x_{i+2}} g_{i+1}(x) u^{(4)}(x) dx + \int_{x_{i+1}}^{x_{i+2}} g_{i+2}(x) u^{(4)}(x) dx \]

\[ = g_{i-2}''(x_{i-2}) u(x_{i-2}) + \left[ g_{i-1}''(x_{i-1}) - g_{i-2}''(x_{i-1}) \right] u(x_{i-1}) + \left[ g_i''(x_i) - g_{i-1}''(x_{i-1}) \right] u(x_i) + \left[ g_{i+1}''(x_{i+1}) - g_{i+2}''(x_{i+1}) \right] u(x_{i+1}) + g_{i+2}''(x_{i+2}) u(x_{i+2}). \]

It follows that the conditions for finding the function \( g_i(x) \) have form

\[ g^{(4)}_{i,k}(x) = 0, \quad i = \frac{2}{N-2}, \quad k = -2, -1, 1, 2, \]  

(8)

\[ g^{(4)}_{i,k}(x_{i+k}) = 0, \quad i = \frac{2}{N-2}, \quad k \in \{-2, 2\}, \quad j = 0, \]  

(9)
From the conditions (8) it follows that we find the functions in the form

\[ g_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i, \quad i = \frac{2}{N}, \frac{N - 2}{2}, \]  

From conditions (10), we find functions

\[ g_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i, \quad i = \frac{2}{N}, \frac{N - 2}{2}, \]  

Using conditions (9), we obtain a system of linear equations for finding the remaining coefficients of the functions \( g_i \), \( i = \frac{2}{N}, \frac{N - 2}{2} \)

\[
\begin{align*}
\begin{cases}
    a_i - 1 &= \frac{1}{6}, \\
    6a_i - 2x - 2 + 2b_i - 2 &= 0, \\
    3a_i - 2x^2 - 2 + 2b_i - 2x - 2 + c_i - 2 &= 0, \\
    a_i - 2x^3 - 2 + b_i - 2x^2 - 2 + c_i - 2x - 2 + d_i - 2 &= 0,
\end{cases}
\end{align*}
\]

(14)

Using conditions (9), we obtain a system of linear equations for finding the remaining coefficients of the functions \( g_i \), \( i = \frac{2}{N}, \frac{N - 2}{2} \)

\[
\begin{align*}
\begin{cases}
    a_i + 1 &= \frac{1}{6}, \\
    6a_i + 2x + 2 + 2b_i + 2 &= 0, \\
    3a_i + 2x^2 + 2 + 2b_i + 2x + 2 + c_i + 2 &= 0, \\
    a_i + 2x^3 + 2 + b_i + 2x^2 + c_i + 2x + 2 + d_i + 2 &= 0,
\end{cases}
\end{align*}
\]

(15)

Solving the system (14), (15) we obtain

\[ g_i - 2(x) = \frac{1}{6}(x - x - 2)^3, \quad g_i + 2(x) = \frac{1}{6}(x - x + 2)^3, \quad i = \frac{2}{N}, \frac{N - 2}{2}. \]  

(16)

With the help of conditions (10), we find functions \( g_i \) due to the values \( g_i - 2(x - 1) = \frac{h^2}{2}, \quad g_i + 2(x - 1) = h \), i.e., we get a system of linear equations to find the remaining coefficients of the functions \( g_i \)

\[
\begin{align*}
\begin{cases}
    a_i - 1 &= -\frac{1}{2}, \\
    6a_i - 1x - 1 + 2b_i - 1 &= h, \\
    3a_i - 1x^2 - 1 + 2b_i - 1x - 1 + c_i - 1 &= h^2/2, \\
    a_i - 1x^3 - 1 + b_i - 1x^2 - 1 + c_i - 1x - 1 + d_i - 1 &= h^3/6,
\end{cases}
\end{align*}
\]

from which we obtain functions \( g_i \) of the form

\[ g_i(x) = -\frac{1}{2}(x - x - 1)^3 + \frac{h}{2}(x - x - 1)^2 + \frac{h^2}{2}(x - x - 1) + \frac{h^3}{6}, \quad i = \frac{2}{N}, \frac{N - 2}{2}. \]  

(17)
Similarly, using conditions (12), we find functions \( g_{i+1} (x) \) due to the values \( g_{i+2} (x_{i+1}) = h^3 / 6 \), \( g'_{i+2} (x_{i+1}) = -h^2 / 2 \), \( g''_{i+2} (x_{i+1}) = h \)

\[
g_{i+1} (x) = \frac{1}{2} (x - x_{i+1})^3 + \frac{h}{2} (x - x_{i+1})^2 - \frac{h^2}{2} (x - x_{i+1}) + \frac{3}{6}, \quad i = \frac{2}{N} - \frac{2}{2}.
\]  

(18)

It’s easy to check the correctness of the construction of functions \( g_{i-2} (x) \), \( g_{i-1} (x) \), \( g_{i+1} (x) \), \( g_{i+2} (x) \), due to the conditions (11). It is interesting that functions of the form (7) with formulas (16), (17), (18) can be rewritten compactly as follows

\[
E_{5,i}^{(3)} (x) = \frac{1}{3!} \left[ (x-x_{i-2})^3_+ - 4(x-x_{i-1})^3_+ + 6(x-x_i)^3_+ - 4(x-x_{i+1})^3_+ + (x-x_{i+2})^3_+ \right].
\]  

(19)

By the way, for the pre-boundary nodes \( x_1 \), \( x_{N-1} \), we have basic functions \( g_1 (x) \), \( g_{N-1} (x) \) of the form

\[
g_1 (x) = \frac{1}{6} \left[ -11 (x-x_0)^3_+ + 18 (x-x_1)^3_+ - 9 (x-x_2)^3_+ + 2 (x-x_3)^3_+ \right] + 3h (x-x_0)^2_+,
\]

\[
g_{N-1} (x) = \frac{1}{6} \left[ 2 (x-x_{N-3})^3_+ - 9 (x-x_{N-2})^3_+ + 18F (x-x_{N-1})^3_+ - 11 (x-x_N)^3_+ \right] - 3h (x-x_N)^2_+,
\]

with following property

\[
\int_{x_0}^{x_3} g_1 (x) u^{(4)} (x) \, dx = 18u(x_1) - 9u(x_2) + 2u(x_3) - 11\mu_0(t) - 6h\bar{\mu}_0(t),
\]  

(20)

\[
\int_{x_{N-3}}^{x_N} g_{N-1} (x) u^{(4)} (x) \, dx = 2u(x_{N-3}) - 9u(x_{N-2}) + 18u(x_{N-1}) - 11\mu_1(t) + 6h\bar{\mu}_1(t).
\]  

(21)

Thus, the integrals on the right-hand sides of equalities (3)–(5) are calculated exactly by formulas (6), (20), (21). In the next Section, we will calculate the integrals on the left-hand sides of equalities (3)–(5) with the help of interpolation the function \( y(x,t) := \frac{\partial u(x,t)}{\partial t} \) with respect to the variable \( x \) based on assuming that the solution is sufficiently smooth.

4. Approximate calculation of integrals by interpolation methods

In this section, we calculate the integrals \( \int_0^1 g_1 (x) \frac{\partial u(x,t)}{\partial t} \, dx, \int_0^1 g_{N-1} (x) \frac{\partial u(x,t)}{\partial t} \, dx, \int_0^1 E_{5,i}^{(3)} (x) \frac{\partial u(x,t)}{\partial t} \, dx, \) \( i = \frac{2}{N} - \frac{2}{2} \) in (3)–(5) using Newton-Stirling interpolation and Hermite interpolation for a function \( y(x,t) := \frac{\partial u(x,t)}{\partial t} \) with respect to the variable \( x \).

4.1. Interpolation of function for three nodes

For internal nodes \( x_i, i = \frac{2}{N} - \frac{2}{2} \), the function \( y(x,t) \) is interpolated through three nodes using the Newton-Stirling formula

\[
y(x,t) = y(x_i + ph, t) = y(x_i, t) + \frac{p}{1!} \Delta y(x_i, t) + \frac{p^2}{2!} \Delta^2 y(x_i, t) + \Delta^3 y(x_i, t) + R_2 (x,t),
\]

\[\text{doi:10.1088/1742-6596/1803/1/012001}\]
\[ R_2(x,t) = R_2(x_i + ph,t) = \frac{h^3}{3!} p (p^2 - 1) y'''(x_i,t) + \frac{h^4}{4!} p^2 (p^2 - 1) y^{(4)}(\xi,t), \]

\[
\int_0^1 E_{N,i}^{(3)}(x) y(x,t) \, dx = h^4 \left[ y(x_i,t) + \frac{1}{3!} \Delta^2 y(x_{i-1},t) \right] + O(h^8), \quad i = 2, N-2,
\]

where \( R_2(x,t) \) is interpolation remainder, and \( \xi \) is a unknown fixed point in the interval \((x_{i-2}, x_{i+2})\). In the pre-boundary nodes, we integrate the function \( y(x,t) \) according to the Hermite formula. For \( i = 1 \), we put the following conditions on the interpolation polynomial \( P(x,t) : P(x_k,t) = y(x_k,t), \ k = 0, 1, 2, P'(x_0,t) = y'(x_0,t) \). As a result, we get \( P(x,t) = L_2(x,t) + H_0(t) \omega_3(x) \), where

\[
L_2(x,t) = \frac{y(x_0,t)}{2h^2} (x-x_1)(x-x_2) - \frac{y(x_1,t)}{h^2} (x-x_0)(x-x_2) + \frac{y(x_2,t)}{2h^2} (x-x_0)(x-x_1),
\]

\[
\omega_3(x) = (x-x_0)(x-x_1)(x-x_2), \quad H_0(t) = \frac{1}{\omega_3'(x_0)} [y'(x_0,t) - L_2'(x_0,t)].
\]

Interpolation remainder is

\[
R(x,t) = (x-x_0)^2 (x-x_1)(x-x_2) \frac{y^{(4)}(\xi,t)}{4!},
\]

with \( \xi \) is a unknown fixed point in the interval \((x_0,x_2)\). Then, we obtain

\[
\int_0^1 g_1(x) y(x,t) \, dx = h^4 \left[ \frac{8}{7} y(x_1,t) + \frac{19}{56} y(x_2,t) \right] + \frac{1}{56} h^4 \mu_0'(t) - \frac{3}{140} h^5 \bar{\mu}_0'(t) + O(h^8).
\]

In a similar way, for \( i = N-1 \), we put the following conditions on the interpolation polynomial \( P(x,t) : P(x_k,t) = y(x_k,t), \ k = N-2, N-1, N, P'(x_N,t) = y'(x_N,t) \). Then, we calculate the integral \( \int_0^1 g_{N-1}(x) y(x,t) \, dx \) with the interpolation polynomial \( P(x,t) \) as follow

\[
\int_0^1 g_{N-1}(x) y(x,t) \, dx = h^4 \left[ \frac{19}{56} y(x_{N-2},t) + \frac{8}{7} y(x_{N-1},t) \right] + \frac{1}{56} h^4 \mu_1'(t) + \frac{3}{140} h^5 \bar{\mu}_1'(t) + O(h^8).
\]

### 4.2. Interpolation of function for five nodes

In this case, we increase the number of nodes to interpolate the function at the internal as well as the pre-boundary nodes. The interpolation of function is done the same as above, but for the pre-boundary nodes we use Hermite-Birkhoff interpolation. Specifically, for the internal nodes \( x_i, i = 2, N-2 \), the function \( y(x,t) \) is interpolated through five nodes using the Newton-Stirling formula

\[
y(x,t) = y(x_i + ph,t) = y(x_i,t) + \frac{p}{1!} \Delta y(x_i,t) + \frac{p^2}{2!} \Delta^2 y(x_{i-1},t) + \frac{p^3}{3!} \Delta^3 y(x_{i-2},t) + \frac{p^4}{4!} \Delta^4 y(x_{i-3},t) + R_4(x,t),
\]

with

\[
R_4(x,t) = R_4(x_i + ph,t) = \frac{h^5}{5!} p \left( p^4 - \frac{2^4}{3} p^2 + \frac{2^4}{3} \right) y^{(5)}(x_i,t)
\]
Then we obtain
\[ \int_{0}^{1} E^{(3)}_{5,t}(x) y(x,t) \, dx = h^4 \left[ y(x_1, t) + \frac{1}{3!} \Delta^2 y(x_{i-1}, t) - \frac{1}{6!} \Delta^4 y(x_{i-2}, t) \right] + O(h^{10}). \]

For the pre-boundary node \( x_1 \), the interpolation polynomial \( P(x,t) \) need to satisfy conditions \( P(x_k,t) = y(x_k,t) \), \( k \in \{0, 1, 2, 3\} \), \( P'(x_0,t) = y'(x_0,t) \), \( P^{(4)}(x_0,t) = y^{(4)}(x_0,t) \). From this we have
\[
\int_{0}^{1} g_1(x) y(x,t) \, dx = h^4 \left[ \frac{185}{168} y(x_1, t) + \frac{121}{336} y(x_2, t) - \frac{1}{216} y(x_3, t) \right] + \frac{131}{3024} h^4 \mu'_0(t) - \frac{19}{2520} h^5 \tilde{\mu}_0(t) - \frac{1}{630} h^8 \left( \frac{\partial f(0,t)}{\partial t} - \mu''_0(t) \right) + O(h^{10}).
\]

For the pre-boundary node \( x_{N-1} \), the interpolation polynomial \( P(x,t) \) is constructed similarly:
\[
\int_{0}^{1} g_{N-1}(x) y(x,t) \, dx = h^4 \left[ -\frac{1}{216} y(x_{N-3}, t) + \frac{121}{336} y(x_{N-2}, t) + \frac{185}{168} y(x_{N-1}, t) \right] + \frac{131}{3024} h^4 \mu'_1(t) + \frac{19}{2520} h^5 \tilde{\mu}'_1(t) - \frac{1}{630} h^8 \left( \frac{\partial f(1,t)}{\partial t} - \mu''_1(t) \right) + O(h^{10}).
\]

5. Matrix form of system of ODE with respect to time variable

We have transferred the solution of the original problem to the solution of the Cauchy problem for the system of the first-order ODEs for the time variable \( t \). We write this system in the vector-matrix form as follows
\[
\begin{cases}
0 \quad P U'(t) = -QU(t) + F(t), \\
U(0) = \Phi,
\end{cases}
\]
or in the form:
\[
\begin{cases}
U'(t) = -h^{-4} P^{-1} Q U(t) + h^{-4} P^{-1} F(t), \\
U(0) = \Phi,
\end{cases}
\]
where
\[
U(t) = (u_1(t), \ldots, u_{N-1}(t))^T, \quad U'(t) = \left( \frac{du_1(t)}{dt}, \ldots, \frac{du_{N-1}(t)}{dt} \right)^T, \quad \Phi = (\varphi(x_1), \ldots, \varphi(x_{N-1}))^T,
\]
\[
Q = \begin{pmatrix}
18 & -9 & 2 & 0 & 0 & \cdots & 0 \\
-4 & 6 & -4 & 1 & 0 & \cdots & 0 \\
1 & -4 & 6 & -4 & 1 & \cdots & 0 \\
0 & 1 & -4 & 6 & -4 & \cdots & 0 \\
0 & \cdots & 0 & 1 & -4 & 6 & -4 \\
0 & \cdots & 0 & 0 & 2 & -9 & 18
\end{pmatrix}_{(N-1) \times (N-1)}.
\]

In the case of interpolation over five nodes, we obtain
\[
P = \begin{pmatrix}
\frac{185}{168} & \frac{121}{336} & -\frac{1}{216} & 0 & \cdots & 0 \\
\frac{185}{168} & \frac{121}{336} & -\frac{1}{216} & 0 & \cdots & 0 \\
-\frac{720}{180} & \frac{720}{180} & \frac{720}{180} & \cdots & \cdots & \cdots \\
0 & \cdots & -\frac{1}{720} & \frac{31}{180} & \frac{79}{180} & \frac{31}{180} \\
0 & \cdots & 0 & -\frac{276}{336} & \frac{336}{336} & \frac{336}{336}
\end{pmatrix}_{(N-1) \times (N-1)}.
\]
well-known scheme of the second order of approximation: let us compare the 5-point scheme with three and five nodes of interpolation (22) with the exact solution on an example of the test problem (1)–(2) with the

6. Numerical experiments

Using a matrix consisting of eigenvectors of the matrix $P^{-1}Q$, scheme (22) can be reduced to a system of decayed differential equations [3], which can be solved exactly or approximately, and then by inverse transformation one can find the solution of the original scheme. Let $B = \{b_{ij}\}_{i,j=1}^{N-1}$ be the matrix of which columns are the eigenvectors of the matrix $P^{-1}Q$. We have the following representation $B^{-1}P^{-1}QB = \Lambda$, where $\Lambda = diag \{\lambda_i\}_{i=1}^{N-1}$ is diagonal matrix, on the diagonal of which are the eigenvalues of the matrix $P^{-1}Q$. Replacing the functions $V(t) = B^{-1}U(t)$ and multiplying both vector equations of the problem (22) by the matrix $B^{-1} = \{b_{ij}\}_{i,j=1}^{N-1}$ on the left, we get the problem

$$
\begin{align*}
V'(t) &= -h^{-4}A V(t) + h^{-4}B^{-1}P^{-1}F(t), \\
V(0) &= B^{-1}\Phi,
\end{align*}
$$

which is split into $(N-1)$ independent Cauchy problems

$$
\begin{align*}
v_i'(t) &= -h^{-4}\lambda_i v_i(t) + \tilde{f}_i(t), \\
v_i(0) &= \sum_{j=1}^{N-1} b_{ij} \varphi(x_j), \quad i = 1, N-1.
\end{align*}
$$

6. Numerical experiments

On an example of the test problem (1)–(2) with the exact solution $u(x,t) = e^{-2t} \sin t \cos(5x)$, let us compare the 5-point scheme with three and five nodes of interpolation (22) with the well-known scheme of the second order of approximation:

$$
\begin{align*}
U'(t) &= -h^{-4}\tilde{Q}U(t) + \bar{F}(t), \\
U(0) &= \Phi,
\end{align*}
$$

where

$$
\bar{Q} = \begin{pmatrix}
7 & -4 & 1 & 0 & 0 & \ldots & 0 \\
-4 & 6 & -4 & 1 & 0 & \ldots & 0 \\
1 & -4 & 6 & -4 & 1 & \ldots & 0 \\
0 & 1 & -4 & 6 & -4 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 0 & 1 & -4 & 6 & -4 \\
0 & \ldots & 0 & 0 & 1 & -4 & 7 \\
\end{pmatrix}^{(N-1)\times(N-1)}
$$
\[
\vec{F}(t) = \begin{pmatrix}
4h^{-4}\mu_0(t) + 2h^{-3}\bar{\mu}_0(t) + f_1(t) \\
-h^{-4}\mu_0(t) + f_2(t) \\
f_3(t) \\
\vdots \\
f_N-3(t) \\
-h^{-4}\mu_1(t) + f_{N-2}(t) \\
4h^{-4}\mu_1(t) - 2h^{-3}\bar{\mu}_1(t) + f_{N-1}(t)
\end{pmatrix}.
\]

The basic functions \(g_1(x), g_{N-1}(x), E_{5,4}^{3}(x)\) with spatial step \(h = 0.1\) are showed in Fig. 1, 2, 3, respectively. The observed absolute errors for above methods (22), (23) are calculated and given in Table 1. The numerical results are presented in Fig. 4. As one can see that the proposed method worked very well in the terms of accuracy.

**Figure 1.** Graph of the basic function \(g_1(x)\) on \([0, 1]\) with \(h = 0.1\)

**Figure 2.** Graph of the basic function \(g_{N-1}(x)\) on \([0, 1]\) with \(h = 0.1\)

**Figure 3.** Graph of the basic function \(E_{5,4}^{(3)}(x)\) on \([0, 1]\) with \(h = 0.1\)

**Figure 4.** Presentation of \(u(x,t)\) in \(\bar{Q}\) with \(N = 16\)

7. Conclusions

As a result of this work, using the method of moments with special basic functions, a five-point scheme of sixth order of approximation was constructed for the fourth-order partial differential equation. When constructing the schemes, special interpolation was used in internal and pre-boundary nodes. Comparing with scheme (23) and schemes presented in [17, 18, 21] it can be seen that the proposed scheme (22) is simple and efficient with high order of approximation. It
should be noted that the approximation error arising in finding the solution appears both due to interpolation of function and due to the numerical method for solving the system of first-order ODEs.

Table 1. Comparison of approximation error between methods at the moment \( t = 0.1 \)

| Nodes | Grid method (23) | Method of moments (22) |
|-------|------------------|------------------------|
|       |                  | Interpolation for 3 nodes | Interpolation for 5 nodes |
| 0.1   | 0.00034309       | 1.03881 * 10^{-9}       | 6.16208 * 10^{-11}       |
| 0.2   | 0.00130647       | 4.98148 * 10^{-8}       | 2.89842 * 10^{-10}       |
| 0.3   | 0.00264062       | 1.07032 * 10^{-8}       | 6.19832 * 10^{-10}       |
| 0.4   | 0.00398207       | 1.63000 * 10^{-8}       | 9.42072 * 10^{-10}       |
| 0.5   | 0.00495038       | 1.97674 * 10^{-8}       | 1.14138 * 10^{-10}       |
| 0.6   | 0.00524877       | 1.97040 * 10^{-8}       | 1.13732 * 10^{-10}       |
| 0.7   | 0.00474379       | 1.58685 * 10^{-8}       | 9.16119 * 10^{-10}       |
| 0.8   | 0.00350449       | 9.45159 * 10^{-8}       | 5.46268 * 10^{-10}       |
| 0.9   | 0.00179171       | 2.99861 * 10^{-8}       | 1.74060 * 10^{-10}       |

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