ON THE KERNEL OF THE PUSH-FORWARD HOMOMORPHISM BETWEEN CHOW GROUPS.

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Abstract. In this note we prove that the kernel of the push-forward homomorphism on $k$-cycles modulo rational equivalence, induced by the closed embedding of an ample divisor linearly equivalent to some multiple of the theta divisor inside the Jacobian variety $J(C)$ is trivial. Here $C$ is a smooth projective curve of genus $g$.

CONTENTS

1. Introduction 1
2. Kummer variety of a hyperelliptic curve 3
3. Inclusion of theta divisor into the Jacobian 7
4. Special ample smooth divisors on $J(C)$ 10
5. Collino’s theorem for higher Chow groups 13
References 19

1. INTRODUCTION

In this note, we try to understand the nature of the kernel of the push-forward homomorphism at the level of Chow groups induced by the closed embedding of the theta divisor inside the Jacobian of a smooth projective curve. Our main aim is to prove that the kernel of the push-forward homomorphism is trivial in this case. This question is motivated by [Vo, Exercise 1, Chapter 10]. Let $S$ be a smooth, connected, complex, projective, algebraic surface embedded inside some $\mathbb{P}^N$. Let $C_t$ be a general smooth hyperplane section of $S$ and $j_t$ be the closed embedding of $C_t$ into $S$. Let $H_t$ be the Hodge structure

$$\ker(j_{t*} : H^1(C_t, \mathbb{Z}) \to H^3(S, \mathbb{Z}))$$

and $A_t$ be the abelian variety corresponding to $H_t$ inside $J(C_t)$. Then the kernel of the push-forward homomorphism $j_{t*}$ from $A_0(C_t)$ to $A_0(S)$ is a countable union of translates of an abelian subvariety $A_{0,t}$ of $A_t$. In the above $A_0$ denotes the group of algebraically trivial algebraic cycles modulo rational equivalence.

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For a very general $C_t$, the abelian variety $A_{0,t}$ is either 0 or $A_t$.

If the albanese map from $A_0(S)$ to $\text{Alb}(S)$ is not an isomorphism, then for a very general $t$, the kernel of the push-forward homomorphism $j_t\ast$ is countable.

In [BG], the first author and V. Guletskii extended the problem to even dimensional smooth projective varieties over any algebraically closed, uncountable ground field. For a smooth cubic fourfold in $\mathbb{P}^5$, and for a very general hyperplane section on it, it is shown that the kernel of the push-forward homomorphism on algebraically trivial algebraic 1-cycles modulo rational equivalence, induced by the closed embedding of the hyperplane section into the cubic fourfold, is countable.

In this paper, we investigate the situation on a Jacobian variety. However we restrict to special ample divisors, namely theta divisors and special smooth irreducible divisors linearly equivalent to $n\Theta$, for $n \geq 1$.

We consider the Jacobian variety of a smooth projective curve and the associated Kummer variety, and the theta divisor inside the Jacobian variety or the image of the theta divisor inside the Kummer variety. The question is what can we say about the kernel of the push-forward homomorphism induced by the above mentioned closed embeddings, on $A_k$ (the group of algebraically trivial $k$-cycles modulo rational equivalence) or more generally on $CH_k$.

We start by looking at the Kummer variety when the curve $C$ is a smooth projective hyperelliptic curve of genus 4. We look at the image of a symmetric theta divisor $\Theta$, i.e. $i(\Theta) = \Theta$, where $i$ is the inverse map on $J(C)$.

We show:

**Theorem 1.1.** Let $C$ be a hyperelliptic curve of even genus 4 and let $K(J(C))$ denote the Kummer variety associated to $J(C)$. Let $D$ denote the image of a symmetric theta-divisor $\Theta$ under the natural morphism $q : J(C) \to K(J(C))$. Let $j'_i$ denote the closed embedding of $D$ into $K(J(C))$. Then $A^2(D)$ is trivial and hence the kernel of the push-forward homomorphism $j'_i\ast$ from $A^2(D)$ to $A^3(K(J(C)))$ is trivial.

In the next section, we look at the more general situation of higher dimensional Jacobian varieties and smooth irreducible divisors linearly equivalent to multiples of theta divisor. Let $C$ be a smooth projective curve of genus $g$ and let $\Theta$ denote a theta divisor inside the Jacobian $J(C)$ of $C$.

Suppose $\pi : \tilde{C} \to C$ is a ramified finite Galois covering of degree $n$, for $n \geq 1$. Let $G$ denote the Galois group such that $C = \tilde{C}/G$. Then the induced morphism $\pi^* : J(C) \to J(\tilde{C})$ is injective. Furthermore, for a suitable translate $\Theta_{\tilde{C}}$ of the theta divisor in $J(\tilde{C})$, the restriction on $J(C)$ is a smooth, irreducible, ample divisor $H_C$ which is linearly equivalent to $n\Theta$.

Then we show the following.
Theorem 1.2. Let $C$ be a curve of genus $g$ and $H_C$ be as mentioned above. Let $j_C$ denote the closed embedding of $H_C$ inside $J(C)$. Then the kernel of the push-forward homomorphism $j_C^*: CH_k(H_C) \otimes \mathbb{Q} \to CH_k(J(C)) \otimes \mathbb{Q}$ is trivial, for $k \geq 0$.

Note that $H_C$ is a special ample divisor in the linear system $|n\Theta|$, since it is restriction of $\Theta$ from $J(\tilde{C})$. It will be interesting to look at the situation when $H_C$ is a general smooth divisor in $|n\Theta|$. However, as pointed out by C. Voisin, we cannot expect injectivity on $CH_0(H_C)_{\mathbb{Q}} \to CH_0(J(C))_{\mathbb{Q}}$, when $H_C$ is very general.

The proof utilises localization sequence of higher Chow groups, applied to $G$-fixed subvarieties of the Jacobian of $\tilde{C}$. An application of a theorem of Collino [Co, Theorem 1], which shows the injectivity, for $k$-cycles on inclusions of lower dimensional symmetric product $Sym^m(C)$ of a curve $C$ inside $Sym^n(C)$, for $m \leq n$, gives us the required injectivity. In the final section §5 we also extend his theorem for the pushforward map on higher Chow groups of symmetric powers of a curve, and for any of its open subset. This is crucial in proof of Theorem 1.2.

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Notations: In all of the text, $k$ is an uncountable, algebraically closed field and all the varieties are defined over $k$. Denote

$$CH_k(X)_{\mathbb{Q}} := CH_k(X) \otimes \mathbb{Q},$$

when $X$ is a variety (of pure dimension) defined over $k$ and $CH_k(X)$ denotes the Chow group of $k$-dimensional cycles modulo rational equivalence. We write

$$CH_k(X, s)_{\mathbb{Q}} := CH^{dim X-k}(X, s) \otimes \mathbb{Q}$$

the Bloch’s higher Chow groups with $\mathbb{Q}$-coefficients.

2. Kummer variety of a hyperelliptic curve

In this section we consider a hyperelliptic curve $C$ of genus 4 and the Kummer variety $K(J(C))$ associated to the Jacobian $J(C)$ of the curve $C$. Let $\Theta$ denote a symmetric theta divisor inside $J(C)$ and let $D$ denote the image of $\Theta$ inside $K(J(C))$, under the natural morphism from $J(C)$ to $K(J(C))$. We would like to investigate the kernel of the push-forward homomorphism at the level of Chow groups of one cycles, induced by the closed embedding of $D$ in $K(J(C))$.

First we prove the following two propositions which are true for any smooth projective curve of genus $g$. Define the map $i$ from $Pic^{g-1}C$ to itself, given by

$$\tilde{i}(\mathcal{O}(D)) = K_C \otimes \mathcal{O}(-D),$$

where for a divisor $D$, $\mathcal{O}(D)$ denote the line bundle associated to $D$ and $K_C$ is the canonical line bundle on $C$. Consider a theta characteristic $\tau$ such that $\tau^2 = K_C$. Consider the
following map

$$\otimes \tau^{-1} : \text{Pic}^{g-1} C \to J(C)$$
given by

$$\mathcal{O}(D) \mapsto \mathcal{O}(D) \otimes \tau^{-1}.$$  

**Lemma 2.1.** The following square is commutative.

\[
\begin{array}{ccc}
\text{Pic}^{g-1} C & \xrightarrow{i} & \text{Pic}^{g-1} C \\
\downarrow \otimes \tau^{-1} & & \downarrow \otimes \tau^{-1} \\
J(C) & \xrightarrow{i} & J(C)
\end{array}
\]

**Proof.** First observe that \(i \circ (\otimes \tau^{-1})\) is \(\mathcal{O}(-D) \otimes \tau\). On the other hand \(\otimes \tau^{-1} \circ \tilde{i}(\mathcal{O}(D))\) is equal to

\[K_C \otimes \mathcal{O}(-D) \otimes \tau^{-1}\]

that is nothing but

\[\mathcal{O}(-D) \otimes \tau^2 \otimes \tau^{-1}\]

which is equal to

\[\mathcal{O}(-D) \otimes \tau.\]

So the above diagram is commutative. 

\[\square\]

The commutativity of the above diagram gives us a map from \(\text{Pic}^{g-1} C\) to the Kummer variety \(K(J(C))\).

Now assume that \(C\) is hyperelliptic, that is there is a degree 2 morphism \(h : C \to \mathbb{P}^1\). So we have the following commutative triangle

\[
\begin{array}{ccc}
C & \xrightarrow{i} & C \\
\downarrow h & & \downarrow h \\
\mathbb{P}^1 & & \mathbb{P}^1
\end{array}
\]

where \(i\) is the hyperelliptic involution induced by the degree 2 morphism \(h\). Now we state the following theorem which will be used in the proof of the next proposition.

**Theorem 2.2.** ([Ha], IV, 5.4) Let \(D\) be an effective special divisor on a smooth curve \(C\), then

\[\dim(|D|) \leq \frac{1}{2} \deg(D).\]
Furthermore equality occurs if and only if \( D = 0 \) or \( D = K_C \) or \( C \) is hyperelliptic and \( D \) is a multiple of the unique \( g^1_2 \) on \( C \).

In the above theorem \( g^r_d \) denotes a linear system of dimension \( r \) and degree \( d \). Also by special divisor we mean a divisor \( D \) such that the dimension of the linear system associated to \( K_C - D \) is greater than zero.

Let \( l \) be the map from \( \text{Sym}^{g-1}C \) to \( \text{Pic}^{g-1}C \) given by

\[
l(D) = \mathcal{O}(D),
\]

where \( \mathcal{O}(D) \) denote the line bundle associated to the divisor \( D \). Consider \( i_C : \text{Sym}^{g-1}C \to \text{Sym}^{g-1}C \) given by

\[
i_C(P_1 + \cdots + P_n) = i(P_1) + \cdots + i(P_n),
\]

where \( i \) is the hyperelliptic involution on \( C \). With this definitions in hand we prove the following proposition.

**Proposition 2.3.** The following diagram is commutative.

\[
\begin{array}{ccc}
\text{Sym}^{g-1}C & \xrightarrow{l} & \text{Pic}^{g-1}C \\
\downarrow i_{C} & & \downarrow \tilde{i} \\
\text{Sym}^{g-1}C & \xrightarrow{l} & \text{Pic}^{g-1}C
\end{array}
\]

In other words the involution \( \tilde{i} \) lifts on the \((g-1)\)-st symmetric power of the curve.

**Proof.** First, the Riemann-Hurwitz formula tells us,

\[
K_C = h^*K_{\mathbb{P}^1} + \mathcal{O}(B)
\]

where \( B \) is the branched divisor of the morphism \( h \) and degree of \( B \) is \( 2g + 2 \). Now we have to show that \( \mathcal{O}(i_C(D)) \) is \( K_C - \mathcal{O}(D) \). In other words, we have to prove that

\[
\mathcal{O}(i_C(D)) \otimes \mathcal{O}(D) = K_C
\]

that is

\[
\mathcal{O}(D + i_C(D)) = K_C.
\]

Here \( i_C \) is the involution induced on the symmetric powers of \( C \) defined above, by the involution \( i \) on \( C \). Observe that \( D + i_C(D) \) is invariant under the involution \( i_C \).

Now consider the morphism \( h : C \to \mathbb{P}^1 \).

We compute \( h^0(K_C - D - i_C D) \), that is the dimension of the vector space of global sections of the line bundle \( K_C - \mathcal{O}(D + i_C D) \). By Riemann-Roch theorem we have that

\[
h^0(\mathcal{O}(D + i_C D)) - h^0(K_C - \mathcal{O}(D + i_C D)) = 2g - 2 - g + 1 = g - 1.
\]
Observe that \( \deg(K_C - \mathcal{O}(D + i_C D)) = 0 \). Now for a divisor \( D \) the degree is zero means that either the divisor is zero, in this case we have \( h^0(D) \) is one or \( D \) is non-zero. In the case \( D \) is non-zero, we have \( h^0(D) = 0 \), otherwise the line bundle associated to \( D \) would be trivial.

So we have two cases

\[
K_C = \mathcal{O}(D + i_C D)
\]

or

\[
h^0(K_C - \mathcal{O}(D + i_C D)) = 0.
\]

Suppose that \( h^0(K_C - \mathcal{O}(D + i_C D)) = 0 \). So by the Riemann-Roch theorem we get that

\[
h^0(\mathcal{O}(D + i_C D)) = 2g - 2 - g + 1 = g - 1.
\]

By the theorem 2.2 we get that \( \mathcal{O}(D + iD) \) is equal to \( L^{g-1} \) for a line bundle \( L \in g_2^1 \) on \( C \). We have

\[
K_C = h^*K_{\mathbb{P}^1} + \mathcal{O}(B)
\]

and also by 2.2 we get that any two divisors of degree \( 2g - 2 \) on a hyper-elliptic curve \( C \), are linearly equivalent, that is the corresponding line bundles on \( C \) are isomorphic. This tells us that \( h^*\mathcal{O}_{\mathbb{P}^1}(g - 1) \) and \( L^{g-1} \) are isomorphic. By the projection formula we get that

\[
h_*L^{g-1} = h_*h^*(\mathcal{O}_{\mathbb{P}^1}(g - 1))
\]

which is nothing but

\[
\mathcal{O}_{\mathbb{P}^1}(g - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(g - 1) \otimes \mathcal{O}(B).
\]

Since \( H^0(C, L^{g-1}) \) is isomorphic to \( H^0(\mathbb{P}^1, h_*L^{g-1}) \), we have \( h^0(L^{g-1}) \) is greater than \( g - 1 \) which is a contradiction. So the possibility that \( h^0(K_C - \mathcal{O}(D + i_C D)) = 0 \) is ruled out and we have the only possibility

\[
K_C = \mathcal{O}(D + i_C D).
\]

This gives us the commutativity of the diagram.

\[
\begin{array}{ccc}
\text{Sym}^{g-1}C & \xrightarrow{i} & \text{Pic}^{g-1}C \\
\downarrow i & & \downarrow \tilde{i} \\
\text{Sym}^{g-1}C & \xrightarrow{i} & \text{Pic}^{g-1}C
\end{array}
\]

This ends the proof. \( \square \)

Next, for Chow groups computations, we identify \( \text{Pic}^{g-1}C \) with \( J(C) \) using a base point \( P_0 \in C \). The image of \( \text{Sym}^{g-1}C \) in \( \text{Pic}^{g-1}C \) is denoted by \( \Theta \) and it is symmetric under \( \tilde{i} \) by the proposition 2.3.
Theorem 2.4. Let $C$ be a hyperelliptic curve of genus 4 and let $K(\text{Pic}^3C)$ denote the Kummer variety associated to $\text{Pic}^3 C$. Let $D$ denote the image of a symmetric theta-divisor $\Theta$ under the natural morphism from $\text{Pic}^3 C$ to $K(\text{Pic}^3 C)$. Let $j'$ denote the closed embedding of $D$ into $K(\text{Pic}^3 C)$. Then $A^2(D)$ is trivial and hence the kernel of the push-forward homomorphism $j'_* \colon A^2(D) \to A^3(K(\text{Pic}^3 C))$ is trivial.

Proof. The commutativity of the diagram in 2.3 gives us a map from $\text{Sym}^3 C/i$ to $\text{Pic}^3 C/\widetilde{i} \cong K(\text{Pic}^3 C)$, where the first morphism is birational and the second one is finite. Now $\text{Sym}^3 C/i$ is isomorphic to $\text{Sym}^3 \mathbb{P}^1$, which is isomorphic to the projective space $\mathbb{P}^3$. Note that $A^2(\mathbb{P}^3)$ is trivial hence weakly representable. Since weak representability of $A^2$ is a birational invariant, we get that $A^2(D)$ is isomorphic to an abelian variety $A$. By the proposition 6 in [BC] we get that the kernel of the push-forward homomorphism $j'_*$ from $A^2(D)$ to $A^3(K(\text{Pic}^3 C))$ is a countable union of translates of an abelian subvariety $A_0$ of the abelian variety $A$ representing $A^2(D)$. Since $H^3(\mathbb{P}^3, \mathbb{Z})$ is trivial, we get that the abelian variety $A$ is trivial. So the kernel of the push-forward homomorphism $j'_*$ is trivial. □

3. Inclusion of theta divisor into the Jacobian

In this section we investigate the kernel of the push-forward homomorphism, induced by the closed embedding of the theta divisor inside the Jacobian of a smooth projective curve $C$ of genus $g$. More precisely we prove the following theorem.

Theorem 3.1. Let $C$ be a smooth projective curve of genus $g$. Let $\Theta$ be a symmetric theta-divisor embedded inside $J(C)$ and let $j$ denote the embedding. Then the kernel of the push-forward homomorphism $j_*$ from $\text{CH}_k(\Theta)$ to $\text{CH}_k(J(C))$ is trivial.

Proof. It is well known that the map from $\text{Sym}^{g-1} C$ to $\Theta$ is surjective and birational. Let us fix a point $P$ in $C$. Consider the following map $j_C$ from $\text{Sym}^{g-1} C$ to $\text{Sym}^g C$ defined by

$$P_1 + \cdots + P_{g-1} \mapsto P_1 + \cdots + P_{g-1} + P.$$ 

Here the sum denotes the unordered set of points of lengths $(g - 1)$ and $(g)$.

With this definition of $j_C$ we observe that the following diagram is commutative.

\[
\begin{array}{ccc}
\text{Sym}^{g-1} C & \xrightarrow{q_{\Theta}} & \Theta \\
\downarrow j_C & & \downarrow j \\
\text{Sym}^g C & \xrightarrow{q} & \text{Pic}^g(C) \\
\end{array}
\]
We prove that commutativity of this diagram gives us the following formula at the level of $CH_k$.

$$j_* = q_* \circ j_{C*} \circ q_\Theta^*$$

To prove this we notice that for a prime $k$-cycle $V$ inside $\Theta$ we have

$$(q \circ j_C)(q_\Theta^{-1}(V)) = (j \circ q_\Theta)(q_\Theta^{-1}(V))$$

by the commutativity of the above diagram. Now $q_\Theta$ is surjective, so

$$q_\Theta(q_\Theta^{-1}(V)) = V.$$ 

Now suppose that $E$ is the exceptional locus of $q_\Theta$ in $\text{Sym}^{g-1}C$ such that $q_\Theta$ is injective on $\text{Sym}^{g-1}C \setminus E$ into $\Theta$. Now suppose $\alpha$ be a cycle class in $CH_k(\Theta)$ and it is non-zero. Suppose $q_\Theta^*(\alpha)$ is supported on $E$, $k \geq 0$, then by the localization exact sequence,

$$CH_k(E) \to CH_k(\text{Sym}^{g-1}C) \to CH_k(\text{Sym}^{g-1}C \setminus E) \to 0$$

we have $q_\Theta^*(\alpha)$ belongs to the image of the push-forward homomorphism from $CH_k(E)$ to $CH_k(\text{Sym}^{g-1}C)$, hence in the kernel of the pullback homomorphism from $CH_k(\text{Sym}^{g-1}C)$ to $CH_k(\text{Sym}^{g-1}C \setminus E)$. Therefore $\alpha$ goes to zero under the pullback homomorphism from $CH_k(\Theta)$ to $CH_k(\text{Sym}^{g-1}C \setminus E)$, but this homomorphism is injective, hence $\alpha$ is zero, we get a contradiction. So $q_\Theta^*(\alpha)$ is not supported on $E$ and hence it does not belong to the image of the push-forward homomorphism from $CH_k(E)$ to $CH_k(\text{Sym}^{g-1}C)$, induced by the embedding of $E$ into $\text{Sym}^{g-1}C$. Therefore we get that $q_\Theta^*(\alpha)$ does not belong to the kernel of the pull back homomorphism from $CH_k(\text{Sym}^{g-1}C)$ to $CH_k(\text{Sym}^{g-1}C \setminus E)$, induced by the embedding of $\text{Sym}^{g-1}C \setminus E$ into $\text{Sym}^{g-1}C$. In particular $q_\Theta^*(\alpha)$ is not zero, otherwise it will belong to the kernel of the pullback homomorphism described above. Since $j_{C*}$ is injective by theorem 1 in [Co] from $CH_k(\text{Sym}^{g-1}C)$ to $CH_k(\text{Sym}^{g}C)$, we get that $j_{C*}(q_\Theta^*(\alpha))$ is not zero. Since $q_\Theta^*(\alpha)$ is not supported on the exceptional locus $E$ of the morphism $q_\Theta$, we get that $j_{C*} \circ q_\Theta^*(\alpha)$ is not supported on the exceptional locus of $q$. This is because of the fact that $q_\Theta$ is the restriction of $q$. Now consider the following fiber square,

$$\begin{array}{ccc}
\text{Sym}^{g}C \setminus E' & \longrightarrow & \text{Sym}^{g}C \\
\downarrow & & \downarrow q \\
U & \longrightarrow & J(C)
\end{array}$$

where $E'$ is the exceptional locus of the map $\text{Sym}^{g}C \to J(C)$, and $U$ be the open subscheme of $J(C)$ such that $\text{Sym}^{g} \setminus E'$ is isomorphic to $U$. This fiber square gives us the
following commutative square at the level of Chow groups.

\[
\begin{array}{c}
CH_k(\text{Sym}^g C) \\
\downarrow q_* \\
CH_k(J(C)) \\
\downarrow \\
CH_k(U)
\end{array}
\rightarrow
\begin{array}{c}
CH_k(\text{Sym}^g C \setminus E') \\
\downarrow \\
CH_k(U)
\end{array}
\]

Since \(j_* q^*_\Theta(\alpha)\) is not supported on \(E'\), we get that the image of \(j_* q^*_\Theta(\alpha)\) under the pull back homomorphism \(CH_k(\text{Sym}^g C) \to CH_k(\text{Sym}^g C \setminus E')\) is nonzero. Also observe that the right vertical homomorphism above is an isomorphism, so \(j_* q^*_\Theta(\alpha)\) is mapped to some non-zero element in \(CH_k(U)\). By the commutativity of the above diagram, we get that \(q_*(j_* q^*_\Theta(\alpha))\) is non-zero. In other words \(j_* (\alpha)\) is non-zero. So \(j_*\) is injective. \(\square\)

3.2. **Finite group quotients of \(J(C)\).** Now we try to prove that the kernel of the push-forward homomorphism from \(CH_k(D)\) to \(CH_k(K(J(C)))\) is trivial. More generally let \(G\) be a finite group acting on \(J(C)\), where \(C\) is a smooth projective curve of genus \(g\). Let \(\Theta\) denote the theta divisor of \(J(C)\) such that \(G(\Theta) = \Theta\). Then we prove the following.

**Proposition 3.3.** Let \(j_G\) denote the embedding of \(\Theta/G\) into \(J(C)/G\). Then the kernel of the push-forward homomorphism \(j_{G*}\) from \(CH_k(\Theta/G)_\mathbb{Q}\) to \(CH_k(J(C)/G)_\mathbb{Q}\) is trivial.

**Proof.** By theorem 3.1 it suffices to check that the action of \(G\) intertwines with \(j_*\). That is we have to show that

\[
g.j_*(a) = j_*(g.a)
\]

for any \(a\) in \(CH_k(\Theta)\) and for any \(g \in G\). For that write \(a\) as \(\sum n_i V_i\). Then

\[
g.(j_*(a)) = g.(\sum n_i j(V_i)) = \sum n_i g(V_i)
\]

since \(j\) is a closed embedding, we have

\[
\sum n_i g(V_i) = j_*(\sum n_i g(V_i))
\]

that is same as

\[
j_*(g.a)
\]

By [Fu, Example 1.7.6], we have

\[
CH_k(\Theta/G)_\mathbb{Q} = CH_k(\Theta)^G_\mathbb{Q}
\]

where \(CH_k(\Theta)^G_\mathbb{Q}\) denotes the \(G\)-invariants in \(CH_k(\Theta)_\mathbb{Q}\). By the above intertwining of the group action of \(G\), we get that \(j_{G*}|CH_k(\Theta)^G_\mathbb{Q}\) takes it values in \(CH_k(J(C))^G_\mathbb{Q}\). Since \(j_*\) is injective we get that \(j_{G*}|CH_k(\Theta)^G_\mathbb{Q}\) is injective and \(j_{G*}|CH_k(\Theta)_\mathbb{Q}\) is nothing but \(j_{G*}\). So we get that \(j_{G*}\) is injective. \(\square\)
4. Special ample smooth divisors on $J(C)$

Let $n\Theta$ denote the $n$-th multiple of $\Theta_C$, that is

$$\Theta + \cdots + \Theta$$

$n$ times, inside the Jacobian of a genus $g$ smooth projective curve $C$. Since $h^0(n\Theta_C) = n^g$, we can choose $H_C$, a smooth, irreducible, ample divisor on $J(C)$, linearly equivalent to $n\Theta$. We are interested to investigate the kernel of the push-forward homomorphism at the level of Chow groups with rational coefficients, induced by the closed embedding of $H_C$ into $J(C)$. Consider a Galois covering

$$\pi: \tilde{C} \longrightarrow C$$

of degree $n$ branched along $r$ points where $r \geq 1$. In particular let $G$ be a finite group acting on $\tilde{C}$ such that $C = \tilde{C}/G$.

Let $\pi^*$ denote the morphism induced by $\pi$ from $J(C)$ to $J(\tilde{C})$. Since $\pi^*$ is injective by [BL, Corollary 11.4.4], we identify the image of $\pi^*$ with the polarized pair $(J(C), H_C)$.

Let us denote genus of $\tilde{C}$ by $\tilde{g}$. Note that for a general translate of $\Theta_{\tilde{C}}$, the restriction of the translate to $J(C)$ is smooth and irreducible. Since by [BL, Lemma 12.3.1],

$$(\pi^*)^*(\Theta_{\tilde{C}}) \equiv n\Theta_C \equiv H_C,$$

we have $H_C$ is equal to $J(C) \cap \Theta_{\tilde{C}}$, and it is smooth and irreducible.

Note that $H_C$ is special in the linear system $|n\Theta_C|$ since it is restriction of $\Theta_{\tilde{C}}$ and for a general member of $n\Theta_C$, this does not happen.

Denote $CH_*(H_C)_\mathbb{Q} := CH_*(H_C) \otimes \mathbb{Q}$ and $CH_*(J(C))_\mathbb{Q} := CH_*(J(C)) \otimes \mathbb{Q}$. In the following, we identify $Pic^g(C) = J(C)$ and $Pic^g(\tilde{C}) = J(\tilde{C})$ (without specifying a choice of base point).

**Theorem 4.1.** Let $C$ be a curve of genus $g$ and $H_C$ be as mentioned above. Let $j_C$ denote the closed embedding of $H_C$ inside $J(C)$. Then the kernel of the push-forward homomorphism $j_{C*}$ from $CH_k(H_C)_\mathbb{Q}$ to $CH_k(J(C))_\mathbb{Q}$ is trivial, for $k \geq 1$.

**Proof.** By the above discussion we have the following commutative diagram

\[
\begin{array}{ccc}
H_C & \longrightarrow & \Theta_{\tilde{C}} \\
| & & | \\
J(C) & \longrightarrow & J(\tilde{C}).
\end{array}
\]
This diagram gives us the following commutative diagram at the level of $\text{CH}_k$.

\[
\begin{array}{cccc}
\text{CH}_k(H_C)_{\mathbb{Q}} & \rightarrow & \text{CH}_k(\Theta_{\overline{C}})_{\mathbb{Q}} \\
\downarrow{j_{C,*}} & & \downarrow{j_{\overline{C},*}} \\
\text{CH}_k(J(C))_{\mathbb{Q}} & \rightarrow & \text{CH}_k(J(\overline{C}))_{\mathbb{Q}}
\end{array}
\]

Using [3.1] we get that $j_{\overline{C},*}$ is injective. To prove that the homomorphism $j_{C,*}$ is injective we use the localization exact sequence of Bloch’s higher Chow groups [31]. First note that $\text{Sym}^\delta_{\overline{C}}$ is birational to $\Theta_{\overline{C}}$, and $\text{Sym}^\delta C$ is birational to $J(\overline{C})$. Consider the natural morphism from $\text{Sym}^\delta_{\overline{C}}$ to $J(\overline{C})$. Let $H'_C, J(C)'$ denote the scheme theoretic inverse images of $H_C, J(C)$ in $\text{Sym}^\delta_{\overline{C}}$. Now fix a base-point $P_0$ in $\overline{C}$ and we consider the inclusion $\text{Sym}^\delta_{\overline{C}} \hookrightarrow \text{Sym}^\delta C$ given by

\[P_1 + \cdots + P_{\delta-1} \mapsto P_1 + \cdots + P_{\delta-1} + P_0.\]

Then by using the localization exact sequence at the level of higher Chow groups we have the following commutative diagram:

\[
\begin{array}{cccc}
\text{CH}_k(\text{Sym}^\delta_{\overline{C}}, 1) & \rightarrow & \text{CH}_k(\text{Sym}^\delta_{\overline{C}} \setminus H'_C, 1) & \rightarrow & \text{CH}_k(H'_C) & \rightarrow & \text{CH}_k(\text{Sym}^\delta_{\overline{C}}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{CH}_k(\text{Sym}^\delta C, 1) & \rightarrow & \text{CH}_k(\text{Sym}^\delta C \setminus J(C)', 1) & \rightarrow & \text{CH}_k(J(C)') & \rightarrow & \text{CH}_k(\text{Sym}^\delta C)
\end{array}
\]

Since $C = \overline{C}/G$, consider the induced action of $G$ on symmetric powers of $\overline{C}$. The group $G$ acts component wise and we have $\text{Sym}^\delta_{\overline{C}}/G$ is isomorphic to $\text{Sym}^\delta_{\overline{C}}$ and similarly $\text{Sym}^\delta C/G$ is isomorphic to $\text{Sym}^\delta C$. Since $G$ acts trivially on $C$, we get that $G$ acts trivially on $J(C)'$ and $H'_C$ respectively. So $G$ acts trivially on $\text{CH}_k(H'_C)$ and $\text{CH}_k(J(C)')$. Now consider the $G$-invariant part of the above $G$-equivariant commutative diagram with $\mathbb{Q}$-coefficients. That is consider

\[
\begin{array}{cccc}
\text{CH}_k(\text{Sym}^\delta_{\overline{C}}, 1)_{\mathbb{Q}}^G & \rightarrow & \text{CH}_k(\text{Sym}^\delta_{\overline{C}} \setminus H'_C, 1)_{\mathbb{Q}}^G & \rightarrow & \text{CH}_k(H'_C)_{\mathbb{Q}}^G & \rightarrow & \text{CH}_k(\text{Sym}^\delta_{\overline{C}})_{\mathbb{Q}}^G \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{CH}_k(\text{Sym}^\delta C, 1)_{\mathbb{Q}}^G & \rightarrow & \text{CH}_k(\text{Sym}^\delta C \setminus J(C)', 1)_{\mathbb{Q}}^G & \rightarrow & \text{CH}_k(J(C)')_{\mathbb{Q}}^G & \rightarrow & \text{CH}_k(\text{Sym}^\delta C)_{\mathbb{Q}}^G
\end{array}
\]
have that the homomorphism is injective. Also by Collino’s theorem in [Co, Theorem 1] we have that the homomorphism \( CH(\text{Sym}^g C, 1)_Q \to \text{non-zero element} \), then the element we started with from \( Q \) goes to a nonzero element, then the element we started with from \( \text{CH}^k(\text{Sym}^g C, 1)_Q \). Now suppose that we start with some non-zero element in \( CH^k(C) \), and suppose that it goes to something non-zero in \( CH^k(\text{Sym}^g C, 1)_Q \), since the homomorphism from \( CH^k(\text{Sym}^g C, 1)_Q \) to \( CH^k(\text{Sym}^g C, 1)_Q \) is injective, we get that the non-zero element we started with \( CH^k(H'_C)_Q \) goes to some non-zero element in \( CH^k(J(C'))_Q \). Then by the localization exact sequence, it follows that the element in \( CH^k(H'_C)_Q \) is in the image of the homomorphism \( CH^k(\text{Sym}^g C, 1)_Q \to CH^k(H'_C)_Q \). Suppose the element in \( CH^k(\text{Sym}^g C, 1)_Q \) is non-zero. Since the map \( CH^k(\text{Sym}^g C, 1)_Q \to CH^k(\text{Sym}^g C, 1)_Q \) is injective, the image of that element in \( CH^k(C, 1)_Q \) is non-zero. Either this element goes to zero or it is mapped to a non-zero element in \( CH^k(J(C'))_Q \). If it goes to a non-zero element, then the element we started with from \( CH^k(H'_C)_Q \) goes to a
non-zero element in $CH_k(J(C)')_\mathbb{Q}$. Suppose the element in $CH_k(Sym^{g-g}C \setminus J(C)', 1)_\mathbb{Q}$ goes to zero in $CH_k(J(C)')_\mathbb{Q}$. Then the element is in the image of the map

$$CH_k(Sym^{g-g}C, 1)_\mathbb{Q} \rightarrow CH_k(Sym^{g-g}C \setminus J(C)', 1)_\mathbb{Q}.$$  

Then by using the isomorphism $CH_k(Sym^{g-g-1}C, 1)_\mathbb{Q}$ with $CH_k(Sym^{g-g}C, 1)_\mathbb{Q}$, we get that this element, comes from an element in $CH_k(Sym^{g-g-1}C, 1)_\mathbb{Q}$, then composing the two maps

$$CH_k(Sym^{g-g-1}C, 1)_\mathbb{Q} \rightarrow CH_k(Sym^{g-g-1}C \setminus H'_C, 1)_\mathbb{Q} \rightarrow CH_k(H'_C)_\mathbb{Q}$$

we get that the element we started with in $CH_k(H'_C)_\mathbb{Q}$ is zero, which is a contradiction to the fact that we started with a non-zero element from $CH_k(H'_C)_\mathbb{Q}$. So we prove that the map from $CH_k(H'_C)_\mathbb{Q}$ to $CH_k(J(C)')_\mathbb{Q}$ is injective.

Now $H'_C$ is birational to $H_C$ and $J(C)'$ is birational to $J(C)$. So we have the commutative diagram

$$
\begin{array}{ccc}
CH_k(H'_C)_\mathbb{Q} & \rightarrow & CH_k(J(C)')_\mathbb{Q} \\
\downarrow & & \downarrow \\
CH_k(H_C)_\mathbb{Q} & \rightarrow & CH_k(J(C))_\mathbb{Q}
\end{array}
$$

Then arguing as in proposition 3.1 and noting that the support of a cycle on $H'_C$ does not lie on the exceptional locus of the birational map from $J(C)'$ to $J(C)$, we prove that the homomorphism $j_{C*}$ at the level of Chow groups of $k$-cycles with rational coefficients is injective.

\[\square\]

5. Collino’s theorem for higher Chow groups

Let $C$ be a smooth projective curve over an algebraically closed field. Let $Sym^n C$ denote the $n$-th symmetric power of $C$. Let us fix a point $p$ in $C$. Consider the closed embedding $i_{m,n}$ of $Sym^m C$ to $Sym^n C$, given by

$$[x_1, \cdots, x_m] \mapsto [x_1, \cdots, x_m, p, \cdots, p]$$

where $[x_1, \cdots, x_m]$ denote the unordered $m$-tuple of points in $Sym^n C$. Then the push-forward homomorphism $i_{m,n*}$ from $CH_*(Sym^n C)$ to $CH_*(Sym^n C)$ is injective as proved in [Co, Theorem 1]. In this section we prove that the same holds for the higher Chow groups. That is the push-forward homomorphism $i_{m,n*}$ from $CH_*(Sym^n C, s)$ to $CH_*(Sym^n C, s)$ is injective. To prove that we follow the approach by Collino in [Co], the argument present here is a minor modification of the arguments in [Co], but we write it for our convenience.

Let $\Gamma^s$ be the correspondence given by

$$\pi_n \times \pi_m (\Gamma^s)$$
supported on \((\text{Sym}^m C \times \text{Spec}(k) \Delta^s) \times \text{Spec}(k) (\text{Sym}^n C \times \text{Spec}(k) \Delta^s)\) where \(\Gamma'\) is the graph of the projection \(pr_{n,m}^s\) from \((C^n \times \text{Spec}(k) \Delta^s)\) to \((C^m \times \text{Spec}(k) \Delta^s)\) and \(\pi_n\) is the natural morphism from \(C^n \times \text{Spec}(k) \Delta^s\) to \(\text{Sym}^n C \times \text{Spec}(k) \Delta^s\). Let \(g^s_\ast\) be the homomorphism induced by \(\Gamma^s\) at the level of algebraic cycles.

First we prove the following lemma.

**Lemma 5.1.** The homomorphism \(g^s_\ast \circ i^s_{m,n*}\) at the level of the group of algebraic cycles, is induced by the cycle \((i^s_{m,n} \times \text{id})^*\Gamma^s\) on \((\text{Sym}^m C \times \text{Spec}(k) \Delta^s) \times (\text{Sym}^n C \times \text{Spec}(k) \Delta^s)\).

**Proof.** Let’s denote \(i^s_{m,n*}\) as \(i^s_\ast\). We have

\[
g^s_\ast i^s_\ast(Z) = pr_{(\text{Sym}^m C \times \Delta^s)\ast}(i^s_\ast(Z) \times \text{Sym}^m C \times \Delta^s.\Gamma^s).
\]

The above expression can be written as

\[
pr_{(\text{Sym}^m C \times \Delta^s)\ast}(i^s_\ast(Z) \times \text{Sym}^m C \times \Delta^s).\Gamma^s)
\]

By the projection formula the above is equal to

\[
pr_{(\text{Sym}^m C \times \Delta^s)\ast} \circ (i^s_\ast \times \text{id})\ast((Z \times \text{Sym}^m C \times \Delta^s).(i^s_\ast \times \text{id})^*\Gamma^s).
\]

Since \(pr_{(\text{Sym}^m C \times \Delta^s)\ast} \circ (i^s_\ast \times \text{id})\) is the projection \(pr_{(\text{Sym}^m C \times \Delta^s)\ast}\) we get that the above is equal to

\[
pr_{(\text{Sym}^m C \times \Delta^s)\ast}((Z \times \text{Sym}^m C \times \Delta^s).(i^s_\ast \times \text{id})^*\Gamma^s).
\]

Here the above two projections are taken respectively on \((\text{Sym}^n C \times \Delta^s) \times (\text{Sym}^m C \times \Delta^s)\) and on \((\text{Sym}^n C \times \Delta^s) \times (\text{Sym}^m C \times \Delta^s)\). So we get that \(g^s_\ast \circ i^s_\ast\) is induced by \((i^s_\ast \times \text{id})^*\Gamma^s\).

\(\square\)

Now consider a closed subscheme \(W\) of \(\text{Sym}^n C\). Let \(i_{m,n}\) denote the embedding of \(\text{Sym}^m C\) into \(\text{Sym}^n C\). Consider the morphism \(i^s_{m,n}\) from \((\text{Sym}^m C \setminus i_{m,n}^{-1}W) \times \Delta^s\) to \((\text{Sym}^n C \setminus W) \times \Delta^s\). Consider the restriction of \(\Gamma^s\) to \(((\text{Sym}^n C \setminus W) \times \Delta^s) \times ((\text{Sym}^m C \setminus i_{m,n}^{-1}W) \times \Delta^s)\). Denote it by \(\Gamma^{st}\). Let \(g^{st}_\ast\) denote the homomorphism induced by \(\Gamma^{st}\). Then arguing as in the previous lemma we get the following.

**Corollary 5.2.** The homomorphism \(g^{st}_\ast \circ i^s_{m,n*}\) is induced by the cycle \((i^s_{m,n} \times \text{id})^*\Gamma^{st}\) on \(((\text{Sym}^m C \setminus i_{m,n}^{-1}W) \times \Delta^s) \times ((\text{Sym}^m C \setminus i_{m,n}^{-1}W) \times \Delta^s)\).

**Proof.** It follows by arguing as in lemma with \(g^s_\ast, \Gamma^s\) replaced by \(g^{st}_\ast, \Gamma^{st}\). \(\square\)

Now let us consider the closed embedding \(\text{Sym}^{m-1} C \times \Delta^s\) into \(\text{Sym}^m C \times \Delta^s\), induced by the embedding \(\text{Sym}^{m-1} C\) into \(\text{Sym}^m C\). Let \(\rho^s\) be the embedding of the complement of \(\text{Sym}^{m-1} C \times \Delta^s\) in \(\text{Sym}^m C \times \Delta^s\). Then we have the following proposition.

**Proposition 5.3.** At the level of the group of algebraic cycles we have

\[
\rho^{ss} \circ g^s_\ast \circ i^s_\ast = \rho^{ss}.
\]
Proof. To prove the proposition we prove that

\[(i^s \times \text{id})^{-1}\Gamma^s = \Delta \cup D\]

where \(\Delta\) means the diagonal in \((\text{Sym}^m C \times \Delta^s) \times (\text{Sym}^m C \times \Delta^s)\) and \(D\) is a closed subscheme of \((\text{Sym}^m C \times \Delta^s) \times (\text{Sym}^{m-1} C \times \Delta^s)\). For that we write out

\[(i^s \times \text{id})^{-1}\Gamma^s ,\]

that is equal to

\[(i^s \times \text{id})^{-1}(\pi_n \times \pi_m)\text{Graph}(pr_{n,m}^s) .\]

The above is equal to

\[(i^s \times \text{id})^{-1}(\pi_n \times \pi_m)\{(x_1, \ldots, x_n, \delta_s, x_1, \ldots, x_m, \delta^s) | x_i \in C, \delta^s \in \Delta^s\}\]

that is

\[(i^s \times \text{id})^{-1}\{(x_1, \ldots, x_n, \delta^s, x_1, \ldots, x_m, \delta^s) | x_i \in C, \delta^s \in \Delta^s\} .\]

Call the set

\[\{(x_1, \ldots, x_n, \delta^s, x_1, \ldots, x_m, \delta^s) | x_i \in C, \delta^s \in \Delta^s\}\]

as \(B\), and the set

\[(i^s \times \text{id})^{-1}\{(x_1, \ldots, x_n, \delta^s, x_1, \ldots, x_m, \delta^s) | x_i \in C, \delta^s \in \Delta^s\} .\]

as \(A\). The set \(A\) is of the form

\[\{(x_1', \ldots, x_m', \delta^s, y_1', \ldots, y_m', \delta^s) | (x_1', \ldots, x_m', p, \ldots, p, \delta^s, y_1', \ldots, y_m', \delta^s) \in B\} .\]

So the set \(A\) can be written as the union of

\[\{(x_1', \ldots, x_m', \delta^s, x_1', \ldots, x_m', \delta^s) | x_i \in C, \delta^s \in \Delta^s\}\]

and

\[\{(x_1', \ldots, x_m', \delta^s, x_1' \ldots, p, x_m', \delta^s) | x_i \in C, \delta^s \in \Delta^s\} ,\]

that is the union

\[\Delta \cup D\]

where \(\Delta\) is the diagonal in the scheme \((\text{Sym}^m C \times \Delta^s) \times (\text{Sym}^m C \times \Delta^s)\) and \(D\) is a closed subscheme in \((\text{Sym}^m C \times \Delta^s) \times (\text{Sym}^{m-1} C \times \Delta^s)\). Therefore we get that

\[(i^s \times \text{id})^*(\Gamma) = \Delta + Y\]

where \(Y\) is supported on \((\text{Sym}^m C \times \Delta^s) \times (\text{Sym}^{m-1} C \times \Delta^s)\). So \(g_*i_*^s(Z)\) is equal to

\[\text{pr}_{\text{Sym}^m C \times \Delta^s}[(\Delta + Y)].(Z \times \text{Sym}^m C \times \Delta^s)] = Z + Z_1\]

where \(Z_1\) is supported on \(\text{Sym}^{m-1} C \times \Delta^s\). So

\[\rho_*^s g_*i_*^s = \rho_*^s(Z + Z_1) = \rho_*^s(Z)\]

since \(\rho_*^s(Z_1) = 0\). Hence the proposition is proved.
Now we want to run the same argument as in proposition 5.3 but for open varieties. That is let $W$ be a closed subscheme in $\text{Sym}^n C$. Let us consider the embedding of $(\text{Sym}^{m-1} C \setminus i^{-1}_{m-1,n} W) \times \Delta^s$ into $(\text{Sym}^m C \setminus i^{-1}_{m,n} W) \times \Delta^s$, induced by the embedding $\text{Sym}^{m-1} C$ into $\text{Sym}^m C$. Let $\rho^{st}$ be the embedding of the complement of $(\text{Sym}^{m-1} C \setminus i^{-1}_{m-1,n} W) \times \Delta^s$ in $(\text{Sym}^m C \setminus i^{-1}_{m,n} W) \times \Delta^s$. Then arguing as in proposition 5.3 we prove that

**Corollary 5.4.** At the level of algebraic cycles we have

$$\rho^{st} \circ g^{st} \circ i_{m,n}^* = \rho^{st}.$$

**Proof.** We argue as in proposition 5.9 with $g^s$ replaced by $g^{st}$ and $\Gamma^s$ by $\Gamma^{st}$ and noting that

$$(i_{m,n}^s \times \text{id})^s(\Gamma^{st}) = ((i_{m,n}^s \times \text{id})^s \Gamma^s) \cap ((\text{Sym}^m C \setminus i_{m,n}^{-1} W \times \Delta^s) \times (\text{Sym}^m C \setminus i_{m,n}^{-1} W \times \Delta^s)).$$

Now we prove that the push-forward homomorphism $i^s$ from $CH^*(\text{Sym}^m C, s)$ to $CH^*(\text{Sym}^n C, s)$ is injective. This involves many steps. The first step would be to verify that the push-forward homomorphism $i^s$ is defined at the level of higher Chow groups. Here $\mathcal{Z}$ denotes the group of admissible cycles, as defined by S. Bloch [Bl].

**Lemma 5.5.** $i^s_{m,n}$ is well defined from $CH^*(\text{Sym}^m C, s)$ to $CH^*(\text{Sym}^n C, s)$.

**Proof.** The morphism $i_{m,n}$ is defined from $\text{Sym}^m C$ to $\text{Sym}^n C$. That will give us a morphism $i^s_{m,n}$ from $\text{Sym}^m C \times \Delta^s$ to $\text{Sym}^n C \times \Delta^s$. So consider the face morphisms

$$\partial_i : \Delta^{s-1} \to \Delta^s$$

given by

$$(t_0, \ldots , t_{s-1}) \mapsto (t_0, \ldots , t_{i-1}, 0, t_i, \ldots , t_{s-1}).$$

This face morphisms give rise to the morphisms from $\text{Sym}^m C \times \Delta^{s-1}$ to $\text{Sym}^m C \times \Delta^s$, continue to call these morphisms as $\partial_i$. Consider the following commutative diagram

$$\begin{array}{ccc}
\text{Sym}^m C \times \Delta^{s-1} & \xrightarrow{\partial_i} & \text{Sym}^m C \times \Delta^s \\
\text{i}^{s-1}_{m,n} \downarrow & & \text{i}^s_{m,n} \downarrow \\
\text{Sym}^n C \times \Delta^{s-1} & \xrightarrow{\partial_i} & \text{Sym}^n C \times \Delta^s
\end{array}$$

From the above commutative diagram we get the commutativity of the following diagram:

$$\begin{array}{ccc}
\mathcal{Z}^*(\text{Sym}^m C \times \Delta^s) & \xrightarrow{\partial_i^*} & \mathcal{Z}^*(\text{Sym}^m C \times \Delta^{s-1}) \\
\text{i}^s_{m,n,*} \downarrow & & \text{i}^{s-1}_{m,n,*} \downarrow \\
\mathcal{Z}^*(\text{Sym}^n C \times \Delta^s) & \xrightarrow{\partial_i^*} & \mathcal{Z}^*(\text{Sym}^n C \times \Delta^{s-1})
\end{array}$$
The commutativity of this diagram and induced maps on admissible cycles shows that $i_{m,n}^*$ is well defined at the level of higher Chow groups.

**Corollary 5.6.** Let $W$ be a closed subscheme of $\text{Sym}^nC$. Consider the morphism $i$ from $\text{Sym}^mC \setminus i^{-1}(W)$ to $\text{Sym}^nC \setminus W$. Then the homomorphism $i_{m,n}^*$ is well defined from $CH^*(\text{Sym}^mC \setminus i^{-1}(W), s)$ to $CH^*(\text{Sym}^nC \setminus W, s)$.

**Proof.** Proof follows by arguing similarly as in lemma 5.5 with $\text{Sym}^mC, \text{Sym}^nC$ replaced by $\text{Sym}^mC \setminus i^{-1}(W), \text{Sym}^nC \setminus W$.

**Lemma 5.7.** Let $\rho$ be the inclusion from $\text{Sym}^mC \setminus \text{Sym}^mC = C_0(m)$ to $\text{Sym}^nC$. Then the homomorphism $\rho^{**}$ is well defined at the level of higher Chow groups.

**Proof.** To prove this consider the diagram at the level of schemes.

\[
\begin{array}{ccc}
C_0(m) \times \Delta^{s-1} & \xrightarrow{\rho^{**}} & \text{Sym}^mC \times \Delta^{s-1} \\
\partial_i & & \partial_i \\
C_0(m) \times \Delta^s & \xrightarrow{\rho^*} & \text{Sym}^mC \times \Delta^s
\end{array}
\]

That gives the following commutative diagram at the level of $\mathcal{Z}^*$.

\[
\begin{array}{ccc}
\mathcal{Z}^*(\text{Sym}^mC \times \Delta^s) & \xrightarrow{\rho^{**}} & \mathcal{Z}^*(C_0(m) \times \Delta^{s-1}) \\
\partial_i^* & & \partial_i^* \\
\mathcal{Z}^*(\text{Sym}^mC \times \Delta^{s-1}) & \xrightarrow{\rho^{**-1}} & \mathcal{Z}^*(C_0(m) \times \Delta^{s-1})
\end{array}
\]

Therefore we have $\rho^{**}$ is well defined at the level of higher Chow groups.

**Corollary 5.8.** Let $W$ be a closed subscheme in $\text{Sym}^mC$. Denote the complement of $\text{Sym}^{m-1}C \setminus i_{m,n}^{-1}W$ in $\text{Sym}^mC \setminus W$ as $W_0(m)$. Let $\rho$ be the inclusion of $W_0(m)$ into $\text{Sym}^mC \setminus W$. Then the homomorphism $\rho^{**}$ is well defined from $CH^*(\text{Sym}^mC \setminus W, s)$ to $CH^*(W_0(m), s)$.

**Proof.** Proof follows by arguing as in lemma 5.7 with $C_0(m), \text{Sym}^mC$ replaced by $W_0(m), \text{Sym}^mC \setminus W$.

**Proposition 5.9.** The push-forward homomorphism $i_0^*$ from $CH^*(\text{Sym}^mX, s)$ to $CH^*(\text{Sym}^nX, s)$ is injective.

**Proof.** We prove this by induction. First $\text{Sym}^0C$ is a single point and the morphism $i_{0,n}^* = (p, \cdots, p)$, so the push-forward induced by this morphism is injective. Assume now
that $i^*_s$ is injective for $m - 1$ and any $n$ greater than or equal to $m - 1$. Then consider the following commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & CH^*(\text{Sym}^{m-1}C, s) & \overset{i^{m-1}_{m,n*}}{\longrightarrow} & CH^*(\text{Sym}^{m}C, s) & \overset{\rho^*}{\longrightarrow} & CH^*(C_0(m), s) \\
\downarrow & & \downarrow i^{m,n*}_{m,n} & & \downarrow & & \\
0 & \longrightarrow & CH^*(\text{Sym}^{m-1}C, s) & \overset{i^{m-1}_{m,n*}}{\longrightarrow} & CH^*(\text{Sym}^{n}C, s) & \longrightarrow & CH^*((\text{Sym}^{m-1}C)^c, s)
\end{array}
\]

In the above $(\text{Sym}^{m-1}C)^c$ is the complement of $\text{Sym}^{m-1}C$ in $\text{Sym}^{n}C$. In this diagram the left part of the two rows are exact by the induction hypothesis and the middle part is exact by the localization exact sequence for higher Chow groups. Now suppose that $z$ belongs to $CH^*(\text{Sym}^{m}C, s)$, such that $i^{m,n*}_s(z) = 0$ and let $Z$ be the cycle such that the cycle class of $Z$ is $z$. Let $cl(Z)$ denote the cycle class in the Higher Chow group, corresponding to the algebraic cycle $Z$.

Then we have

\[cl(\rho^*(\rho^* g^s_s i^*_s(Z))) = 0\]

which means by the proposition 5.3

\[cl(\rho^*(Z)) = 0 ,\]

hence

\[\rho^*(cl(Z)) = \rho^*(z) = 0 .\]

So by the localization exact sequence there exists $z'$ in $CH^*(\text{Sym}^{m-1}C, s)$, such that

\[z = i^{m-1}_{m,n*}(z') .\]

By the commutativity of the left square of the above commutative diagram we get that

\[i^{m-1}_{m,n*}(z') = 0 .\]

By the injectivity of $i^{m-1}_{m,n*}$ we get that $z' = 0$, so $z = 0$, hence $i^{m,n*}_s$ is injective.

**Corollary 5.10.** Let $W$ be a closed subscheme inside $\text{Sym}^{n}C$. Consider the embedding $i_{m,n}$ from $\text{Sym}^{m}C \setminus i_{m,n}^{-1}(W)$ to $\text{Sym}^{n}C \setminus W$. Then the homomorphism $i^{m,n*}_s$ from $CH_*(\text{Sym}^{m}C \setminus i_{m,n}^{-1}(W), s)$ to $CH_*(\text{Sym}^{n}C \setminus W, s)$ is injective.

**Proof.** The proof follows by arguing as in proposition 5.9 with $\text{Sym}^{m}C, \text{Sym}^{n}C$ replaced by $\text{Sym}^{m}C \setminus i_{m,n}^{-1}(W), \text{Sym}^{n}C \setminus W$ respectively and by corollary 5.4.
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