Spherical subgroups of Kac-Moody groups and transitive actions on spherical varieties

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ABSTRACT. We define and study spherical subgroups of finite type of a Kac-Moody group. In analogy with the standard theory of spherical varieties, we introduce a combinatorial object associated with such a subgroup, its homogeneous spherical datum, and we prove that it satisfies the same axioms as in the finite-dimensional case. Our main tool is a study of varieties that are spherical under the action of a connected reductive group $L$, and come equipped with a transitive action of a group containing $L$ as a Levi subgroup.

1. Introduction

Let $G$ be a connected reductive complex algebraic group. A subgroup $H$ of $G$ is spherical if a Borel subgroup of $G$ has a dense orbit on $G/H$; in this case $G/H$ is called a spherical homogeneous space. Over the last 30 years the theory of spherical subgroups has been widely developed, as has the general theory of spherical varieties, i.e. normal $G$-varieties where a Borel subgroup of $G$ has a dense orbit.

The present paper introduces a new research project, aimed at bringing this theory to the infinite-dimensional setting of Kac-Moody groups. The first task is to choose a generalization of the definition of spherical subgroups, which in the classical case can be given in several different ways. Here we proceed as follows: we say that a subgroup $H$ of a Kac-Moody group $G$ is spherical of finite type if is contained in a parabolic subgroup $P \subseteq G$ of negative sign such that $P/H$ has a finite-dimensional Levi subgroup $L$ and $P/H$ is a (finite-dimensional) spherical variety under the action of $L$ (see Sections 9 and 10 and Definition 17.3 for details).

These assumptions are particularly effective because the intermediate subgroup $P$ provides a direct connection with the standard theory of spherical varieties. Our point of view is that the “symmetries” of $G/H$ given by the action of $G$ are essentially already encoded in the finite-dimensional variety $P/H$ viewed as a spherical $L$-variety equipped with the additional automorphisms given by the unipotent radical of $P$.

Our main results are the definition of a combinatorial object called the homogeneous spherical datum of $G/H$ (see Definition 16.3), generalizing the one given in [Lu01] in the finite-dimensional setting, and the proof that this datum satisfies the same combinatorial properties as in the finite-dimensional case (see Theorem 16.5). We recall that the classification of finite-dimensional spherical homogeneous spaces, according to the Luna conjecture, is established in terms of such objects, and that a proof of this conjecture has been proposed as the culmination of the work of several authors (see [Lu01, Lo09, Bra09, BP11a, BP11b], see also [Cu09] for a different approach).

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We also show that the homogeneous spherical datum is independent of the choice of \( P \), and invariant under conjugation of \( \mathcal{H} \) in \( \mathcal{G} \) (see Theorems 12.1 and 17.2). These crucial properties are not as obvious as in the classical theory.

We point out that a more encompassing study of the “big” homogeneous space \( \mathcal{G}/\mathcal{H} \) is possible, but goes beyond the scope of the present work. For example, using the ind-variety structure of \( \mathcal{G}/P \) it is possible to give a natural structure of ind-variety to \( \mathcal{G}/\mathcal{H} \). Here we do not pursue this line, nor the relationship between our definition of spherical subgroups of finite type and other possible generalizations. This will be the subject of a forthcoming paper.

This research project is motivated by various recent developments in the literature on infinite-dimensional groups. Symmetric subgroups of affine Kac-Moody groups have been classified (see e.g. [Ro94], [HG12] and references therein), the geometry of the corresponding symmetric spaces and of certain compactifications are a current research topic (see [Fr11], [So12]). Research on the space of loops on a spherical variety has led to a link between spherical varieties and Langlands duality (see [GN10]). On this subject see also [SV12]: although still mostly conjectural, the combinatorics of spherical varieties provide a way to analyze periods of automorphic forms and related problems in local harmonic analysis. Moreover, the techniques used for solving Delzant’s conjecture on multiplicity free hamiltonian manifolds (see [Kn11]), where affine smooth spherical varieties have a central role, can be applied also to multiplicity free Hamiltonian actions of loop groups (a paper by Friedrich Knop on this subject is in preparation). In addition, our results on \( P/\mathcal{H} \) intersect the study of the automorphism groups of spherical varieties, which is an ongoing research area (see [Br07], [LP14], [Pe12]).

In the first part of the paper, Sections 3–8, we develop the basic tools to study \( P/\mathcal{H} \). Here we work in even greater generality: we start with a connected linear algebraic group \( P \) with Levi subgroup \( L \), and we consider a homogeneous space \( P/H \) with the only assumption that it is a spherical \( L \)-variety. In this generality we define some invariants associated with the action of \( P \) on \( P/H \), the most relevant being its spherical roots (see Definition 5.11), and we establish relationships between the combinatorics of these objects and the geometry of \( P \)-equivariant compactifications of \( P/H \).

In the second part of the paper, Kac-Moody groups enter into the picture: we start now with \( \mathcal{H} \subseteq \mathcal{P} \subseteq \mathcal{G} \) as above, and identify \( \mathcal{P}/\mathcal{H} \) with a homogeneous space \( P/H \) where \( P \) is a finite-dimensional quotient of \( \mathcal{P} \). We discuss the invariants of \( P/H \) in this case and derive for them further combinatorial properties essentially from the structure of the roots of \( \mathcal{P} \) and the Weyl group of \( \mathcal{G} \). In Section 16 we collect the invariants we have defined and introduce the homogeneous spherical datum of \( \mathcal{G}/\mathcal{H} \), showing that it satisfies the same axioms Luna had introduced in [Lu01] in the finite-dimensional case. In Sections 17 we prove invariance of the homogeneous spherical datum under conjugation of \( \mathcal{H} \) in \( \mathcal{G} \), in Section 18 we discuss possible further developments, and finally Section 19 is devoted to some examples.

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2. Notation and basic assumptions

In this paper all varieties and algebraic groups are defined over the field \( \mathbb{C} \) of complex numbers, varieties are always assumed to be irreducible, and subgroups of algebraic groups are assumed to be closed.

If \( G \) is an affine algebraic group then we denote by \( \mathcal{X}(G) \) the group of its characters, by \( G^u \) its unipotent radical, by \( G^s \) the connected component containing the neutral element. If \( K \) is a subgroup of \( G \) and \( g \in G \), we set \( {}^g K = gKg^{-1} \). If \( V \) is a \( G \)-module then \( \bar{V}(G) \) denotes the set of common eigenvectors of all elements of \( G \) (in particular \( 0 \notin \bar{V}(G) \)), and if \( \chi \in \mathcal{X}(G) \) then we set

\[
\bar{V}_\chi = \{ v \in V \setminus \{0\} \mid gv = \chi(g)v \ \forall g \in G \}
\]

Unless otherwise stated, we denote the Lie algebra of a group with the corresponding lower case fraktur letter.

If \( X \) is a variety, by a discrete valuation on \( \mathbb{C}(X) \) we mean a map \( v: \mathbb{C}(X) \setminus \{0\} \to \mathbb{Q} \) such that \( v(fg) = v(f) + v(g) \) for all \( f, g \in \mathbb{C}(X) \setminus \{0\} \), such that \( v(f + g) \geq \min\{v(f), v(g)\} \) whenever \( f, g, f + g \in \mathbb{C}(X) \setminus \{0\} \), such that \( v(f) = 0 \) if \( f \) is constant, and such that the image of \( v \) is a discrete (additive) subgroup of \( \mathbb{Q} \). If \( G \) is a group and \( X \) a \( G \)-variety, we denote by \( \mathcal{D}(X)^G \) the set of \( G \)-stable prime divisors of \( X \).

We denote by \( P \) a connected affine algebraic group, we choose a Levi subgroup \( L \) of \( P \), a Borel subgroup \( B \) of \( L \) and a maximal torus \( T \) of \( B \). We denote by \( S_p \) the simple roots of \( L \) with respect to these choices, and we denote by \( B^- \) the Borel subgroup of \( L \) opposite to \( B \) with respect to \( T \).

For any character \( \alpha \) of \( T \) we denote by \( p_\alpha \) the associated root space of \( p \) if \( \alpha \) is a root of \( P \), and we set \( p_\alpha = 0 \) otherwise. The sum

\[
\bigoplus_{n > 0} p_{n\alpha}
\]

is a nilpotent Lie subalgebra of \( p \), and we denote by \( U_{\alpha} = \text{Exp}(\bigoplus_{n > 0} p_{n\alpha}) \) the corresponding unipotent subgroup of \( P \).

We come to our crucial assumption: unless otherwise stated, in the whole paper \( H \) denotes a subgroup of \( P \) such that \( P/H \) is a spherical \( L \)-variety, i.e. it has a dense \( B \)-orbit.

3. Invariants of spherical varieties

We begin recalling some notions and notations of the standard theory of spherical varieties. With any \( B \)-variety \( X \) one associates the lattice

\[
\Xi_B(X) = \left\{ \lambda \in \mathcal{X}(T) \mid \mathbb{C}(X)_\lambda(B) \neq \emptyset \right\},
\]

whose rank is by definition the rank of \( X \), and the vector space

\[
N_B(X) = \text{Hom}_\mathbb{Z}(\Xi_B(X), \mathbb{Q}).
\]

If \( X \) has an open \( B \)-orbit, then any discrete valuation \( v \) of \( \mathbb{C}(X) \) can be restricted to the subset \( \mathbb{C}(X)_v(B) \) and yields a well-defined element \( \rho(v) \) of \( N_B(X) \), by identifying \( \Xi_B(X) \) with the multiplicative group \( \mathbb{C}(X)_v(B) \) modulo the constant functions. If \( X \) is normal then this applies in particular to the valuation associated with any \( B \)-stable prime divisor.
D of X, and in this case we denote simply by $\rho(D)$ (or $\rho_X(D)$) the corresponding element of $N_B(X)$.

These definitions apply in particular if $X$ is a spherical $L$-variety. In this case the set $\mathcal{D}(X)^B \setminus \mathcal{D}(X)^K$ is called the set of colors (or $L$-colors) of $X$, and is denoted by $\Delta_L(X)$. If $K$ is the stabilizer in $L$ of a point in the open $L$-orbit of $X$, then $X$ is called an embedding of its open orbit $L/K$, and $K$ is called a spherical subgroup of $L$. Intersecting with $L/K$ gives a bijection between $\Delta_L(X)$ and $\mathcal{D}(L/K)^B$.

4. Toroidal embeddings

We recall that a spherical $L$-variety $X$ is toroidal (or $L$-toroidal) if no $L$-color contains an $L$-orbit. This, and the notion of an embedding, adapt quite directly to our setting.

4.1. Definition. A $P$-embedding of $P/H$ is a normal irreducible $P$-variety $X$ together with a $P$-equivariant open embedding $P/H \rightarrow X$. In this case we identify $P/H$ with its image in $X$. If, in addition, for all $D \in \mathcal{D}(P/H)^B$ the closure of $D$ in $X$ doesn’t contain any $P$-orbit, then $X$ is called a $P$-toroidal embedding of $P/H$.

4.2. Lemma. Let $Y_1$ and $Y_2$ be $P$-embeddings of $P/H$. Then there exist a $P$-toroidal embedding $X$ of $P/H$ and proper, $P$-equivariant maps $X \rightarrow Y_i$ for all $i$. In particular, if $Y_1$ and $Y_2$ are complete then $X$ is complete, and there exist complete $P$-toroidal embeddings of $P/H$.

Proof. The proof of [Kn91, Lemma 5.2] yields essentially the desired result; we need to adapt it here only to take $Y_1$ and $Y_2$ into account. Let $\pi : P \rightarrow P/H$ be the quotient map, and choose $f \in \mathbb{C}[P]^{(B \times H)}$ vanishing on $\pi^{-1}(D)$ for all $D \in \mathcal{D}(P/H)^B$. Let $V$ be the $P$-submodule of $\mathbb{C}[P]$ generated by $f$; this induces a $P$-equivariant morphism $\varphi : P/H \rightarrow \mathbb{P}(V^*)$. Let $X'$ be the closure of the image.

Regard $f$ as a linear function on $V^*$, and let $Z$ be its set of zeros in $\mathbb{P}(V^*)$. Since $V$ is generated by $f$ as a $P$-module, the elements $pf$ for $p$ varying in $P$ are not contained in any hyperplane of $V$. In other words they have no common zero as functions on $V^*$, so $\bigcap_{p \in P} pZ = \emptyset$. This implies that $Z$ doesn’t contain any $P$-orbit. Since $\varphi(D)$ is contained in $Z$ for all $D \in \mathcal{D}(P/H)^B$ in $\mathbb{P}(V^*)$, we deduce that the closure of $\varphi(D)$ in $X'$ doesn’t contain any $P$-orbit.

Now we identify $P/H$ with its image in $X' \times Y_1 \times Y_2$ under the map $(\varphi, \text{diag})$, and let $X$ be the normalization of the closure of the image. The map $\varphi$ extends to a proper $P$-equivariant morphism $X \rightarrow X'$, also denoted $\varphi$. The variety $X$ is a $P$-embedding of $P/H$ and dominates $Y_1$ and $Y_2$. It is $P$-toroidal: if the closure of any $D \in \mathcal{D}(P/H)^B$ in $X$ contained a $P$-orbit, then the closure of $\varphi(D)$ in $X'$ would contain a $P$-orbit: contradiction.

The second statement is obvious, and to show the last statement we take any two $P$-equivariant completions $Y_1$ and $Y_2$ of $P/H$ (see [Su74]), and apply the first statement.

4.3. Lemma. Let $K$ be a subgroup of $P$ containing $H$. There exist a $P$-toroidal embedding $X$ of $P/H$ and a proper $P$-equivariant morphism $X \rightarrow P/K$ extending the natural map $\pi : P/H \rightarrow P/K$.

Proof. It is enough to apply the proof of Lemma 4.2 with $P/K$ instead of $Y_1 \times Y_2$ and with $(\varphi, \pi)$ instead of $(\varphi, \text{diag})$. □
The next lemma is fundamental in this paper and carries most of the informations we will need about the action of $P^u$ on $P/H$. It resembles results by Demazure (see e.g. [Oda88, Section 3.4]) on automorphism groups of toric varieties, and also results by Bien and Brion in [BiBr96] on automorphism groups of complete toroidal spherical varieties. However, the results in [BiBr96] cannot be used here directly, since in general a $P$-toroidal embedding of $P/H$ is not $L$-toroidal.

4.4. Lemma. Let $X$ be a $P$-embedding of $P/H$, let $-\alpha$ be a root of $P$ and let $D \in \mathcal{D}(X)^B$. Let $U'_{-\alpha}$ be a one-dimensional subgroup of $U_{-\alpha}$ stable under conjugation by $T$, and suppose that $D$ is not stable under the action of $U'_{-\alpha}$, but it is stable under the action of the commutator $(U'_{-\alpha}, B^u)$. Then there exists a unique positive $n \in \mathbb{Z}$ such that $n\alpha \in \Xi_B(X)$ and $\langle \rho(D), n\alpha \rangle = 1$. Let also $E \in \mathcal{D}(X)^B$ be different from $D$. Then $E$ is stable under $U'_{-\alpha}$, and $\langle \rho(E), n\alpha \rangle \leq 0$. If $U_\alpha$ is one-dimensional then $n = 1$.

Proof. The uniqueness of $n$ is clear from the condition $\langle \rho(D), n\alpha \rangle = 1$, let us prove its existence.

First we claim that for some $v_0 \in U'_{-\alpha}$ the group $K = v_0^{-1}(\ker \alpha)B^uv_0$ has an open orbit on $D$. To prove this, consider $x \in D$ and $v_0 \in U'_{-\alpha}$ such that $v_0x$ is in the open $B$-orbit of $X$. Since $(\ker \alpha)B^u$ has codimension 1 in $B$, the orbit $(\ker \alpha)B^uv_0x$ has codimension at most 1 in $X$. Let $y$ be in this orbit, so there exist $t \in \ker \alpha$ and $b \in B^u$ such that $y = tbv_0x$. Then $y = v_0tbx$ with $c \in (U'_{-\alpha}, B^u)$. It follows that $v_0^{-1}y \in D$, whence the claim.

Consider now the morphism $\varphi: U'_{-\alpha} \times D \to X$ induced by the action. It is dominant, since $D$ has codimension 1 and not stable under $U'_{-\alpha}$. We claim that $\varphi$ is also generically injective, and for this it is enough to prove that for general $x \in D$ and $v \in U'_{-\alpha}$ the condition $vx \in D$ implies $v = e$.

Let $x$ be in the open $K$-orbit of $D$, suppose that $vx \in D$, let $b \in (\ker \alpha)B^u$ and consider the element $v(v_0^{-1}bv_0)x$. We have $v(v_0^{-1}bv_0) = cbv$ where $c \in (U'_{-\alpha}, B^u)$. It follows that $v_0^{-1}bv_0x = cbx \in cbD$, and the latter is equal to $D$ since $b$ and $c$ stabilize $D$. Since $Kx$ is open in $D$, we conclude that $vD = D$. The stabilizer of $D$ in $U'_{-\alpha} \cong \mathbb{G}_a$ is a proper subgroup, hence trivial; in other words $v = e$.

As a consequence the map $\varphi$ is birational. It is $U'_{-\alpha}$-equivariant, where we let this group act on $U'_{-\alpha} \times D$ by left translation on the first factor, and also $B$-equivariant, where we let $B$ act on $U'_{-\alpha}$ via the conjugation by $T$, and on $U'_{-\alpha} \times D$ diagonally. Notice that $B$ acts on $U'_{-\alpha} \cong \mathbb{C}$ linearly, via the character $-n\alpha$ for some positive integer $n$. If $U_\alpha$ is one-dimensional then $U'_{-\alpha} = U_{-\alpha}$ and $n = 1$.

The rational map $f_1: X \dashrightarrow U'_{-\alpha}$ induced by the first projection of $U'_{-\alpha} \times D$ onto $U'_{-\alpha}$ is also $U'_{-\alpha}$- and $B$-equivariant, hence $f_1$ is actually an element of $\mathbb{C}(X)_{(B)}^{(U'_{-\alpha})}$, which proves $n\alpha \in \Xi_B(X)$. On the other hand the rational map $f_2: X \dashrightarrow D$ induced by the second projection is also $B$-equivariant, therefore the pull-back of functions along $f_2$ gives an inclusion $\Xi_B(D) \subseteq \Xi_B(X)$ with $\mathbb{Z}\alpha \cap \Xi_B(D) = \{0\}$. Notice that $\varphi^{-1}$ is defined on the open subset $X_0 = U'_{-\alpha}D_0$ of $X$, where $D_0$ is the open $K$-orbit of $D$. Then $f|_{X_0}$ is a regular function on $X_0$ with $\text{div}(f|_{X_0}) = D_0$, which shows $\langle \rho(D), n\alpha \rangle = 1$.

Consider now a $B$-stable prime divisor $E \neq D$. We claim that $E$ doesn’t intersect $X_0$, hence it is a component of $X \setminus X_0$, therefore stable under $U'_{-\alpha}$. Indeed, if $E$ intersects $X_0$ then the restriction $f_2|_E: E \dashrightarrow D$ is defined and $B$-equivariant. Since $E \neq D$
the function $f_1$ restricts to a non-zero and $B$-semiinvariant rational function on $E$, which implies $n\alpha \in \Xi_B(E)$. On the other hand $f_2|_E$ is of finite degree, whence $\Xi_B(D)$ is identified (via the pull-back of functions along $f_2|_E$) with a finite index subgroup of $\Xi_B(E)$. This implies that a multiple of $\alpha$ is in $\Xi_B(D)$: contradiction.

Finally, any zero of the rational function $f_1$ on $X$ is mapped by $f_1$ to $e \in U^-_\alpha$, hence it is not $U^-_\alpha$-stable. This shows that $\langle \rho(E), n\alpha \rangle \leq 0$. \hfill \Box

We use now Lemma 4.4 to clarify some basic relationships between the action of $P^u$ on $P/H$ and its $B$-stable prime divisors.

4.5. Lemma. Suppose that $H$ doesn’t contain $P^u$ and $H$ is not contained in any proper parabolic subgroup of $P$. Then there exists an element $D \in D(P/H)^B$ not stable under $P^u$.

Proof. Since $H$ doesn’t contain $P^u$ and is not contained in any proper parabolic of $P$, we have $H^u \subseteq P^u$; replacing if necessary $H$ with a conjugate in $P$, we can suppose moreover that $H$ has a Levi subgroup $K$ contained in $L$ and such that $BK$ is open in $L$.

The quotient $P/H$ is then isomorphic to the $L$-equivariant vector bundle $L \times^K P^u/H^u$. Since $P^u/H^u$ is a non-trivial $B \cap K$-module and $B \cap K$ is solvable, the quotient $P^u/H^u$ has a $B \cap K$-stable prime divisor. The latter is not $P^u$-stable and therefore equal to the intersection of $P^u/H^u$ with a $B$-stable, non-$P^u$-stable prime divisor of $P/H$. \hfill \Box

4.6. Corollary. Let $X$ be a $P$-embedding of $P/H$. If $P^u$ doesn’t act trivially on $X$, then there exists $\beta \in \Xi_B(X)$ that is $\leq 0$ on $\rho_X(D(P)^P)$ but not on $\rho_X(D(X)^B)$.

Proof. Suppose first that no proper parabolic subgroup of $P$ contains $H$. By Lemma 4.5 there exists $D \in D(P/H)^B$ not stable under $P^u$. The Lie algebra of $P^u$ admits a filtration $0 = u_0 \subset u_1 \subset u_2 \subset \ldots$ where the subspaces are $(\text{Ad} B)$-stable and the quotients are one-dimensional. Let $i > 0$ be such that the subset $\text{Exp}(u_i) \subseteq P^u$ stabilizes $D$ for all $j < i$, but $\text{Exp}(u_i)$ doesn’t. Let $\alpha$ be the root such that $T$ acts on $u_i/u_{i-1}$ with eigenvalue $-\alpha$, let $x \in u_i \setminus u_{i-1}$ be an $\text{Ad} T$-eigenvector, and $U^-_\alpha = \text{Exp}(\mathbb{C}x)$.

Then $U^-_\alpha$ is a $T$-stable subgroup of $U^-_\alpha$, it doesn’t stabilize $D$ but the commutator $(U^-_\alpha, B^u)$ does. Let us check this last assertion. The divisor $D$ intersects a $P$-stable quasi-projective open subset $X_0 \subseteq \mathbb{P}(V)$ of $X$ such that the action of $P$ on $V$ is linear (see [Su74]). Moreover, bracket with the Lie algebra of $B^u$ kills $u_i/u_{i-1}$ since the former is nilpotent and the latter is one-dimensional. Then one can compute the action of elements in $U^-_\alpha$ and $B^u$ on $X_0$ as unipotent operators in $\text{End}(V)$; a commutator of the form $(\text{Exp}(ax), \text{Exp}(y))$ for $a \in \mathbb{C}$ and $y$ in the Lie algebra of $B^u$ is equal in $\text{End}(V)$ to $\text{Exp}([ax, \text{Exp}(y)]) \text{Exp}(y)^{-1}$, which is easily seen to stabilize $D \cap X_0$. Lemma 4.4 implies the corollary with $\beta = n\alpha$.

Now let $Q$ be a proper parabolic subgroup of $P$ containing $H$. We may suppose that $Q$ is minimal with this property, therefore $H$ is not contained in any proper parabolic subgroup of $Q$. We may also suppose that $Q$ contains $T$ and that $B^uQ$ is open in $P$. We denote by $M$ the Levi subgroup of $Q$ containing $T$, and in this way $B_M = B \cap M = B \cap L$ is a Borel subgroup of $M$.

Since $P/Q$ is a complete $P$-orbit, the natural morphism $P/H \rightarrow P/Q$ extends to a $P$-equivariant map $X \rightarrow P/Q$, and the fiber $Y$ over the point $eQ$ is a $Q$-embedding of $Q/H$. The group $Q^u$ doesn’t act trivially on $Q/H$, otherwise $P^u$ being contained in $Q^u$ would
act trivially on $P/H$. The first part of the proof then yields $\beta \in \Xi_{B_M}(Y)$ that is $\leq 0$ on $\rho_Y(\mathcal{D}(Y)^Q)$ but not on $\rho_Y(\mathcal{D}(Y)^{B_M})$.

On the other hand $P/Q$ is a complete $L$-orbit, and the point $eQ$ is in the open $B^n$-orbit. It follows that restriction of functions from $X$ to $Y$ induces an isomorphism $\Xi_B(X) \cong \Xi_{B_M}(Y)$, that any $D \in \mathcal{D}(X)^B$ (resp. $D \in \mathcal{D}(X)^F$) intersects $Q/H$ in a $B_M$-stable (resp. $Q$-stable) prime divisor, and that this identifies $\rho_X(\mathcal{D}(X)^B)$ (resp. $\rho_X(\mathcal{D}(X)^F)$) with $\rho_Y(\mathcal{D}(Y)^{B_M})$ (resp. $\rho_Y(\mathcal{D}(Y)^Q)$) compatibly with the above isomorphism between the lattices. The corollary follows. \hfill $\square$

5. The spherical roots

In this section we use the theory of embeddings of spherical $L$-varieties as presented in [Kn91], we refer to loc.cit. for details.

Let us recall some basic facts of this theory. Given an spherical $L$-variety $X$, with any $L$-orbit $Y \subset X$ one associates the polyhedral (i.e. finitely generated) convex cone $\mathcal{C}_Y$ generated by $\rho(D)$ where $D$ varies in the set of all $B$-stable prime divisors of $X$ that contain $Y$. The colored cone associated with $Y$ is the couple $(\mathcal{C}_Y, \mathcal{D}_Y)$ where $\mathcal{D}_Y$ is the set of those elements in $\Delta_L(X)$ that contain $Y$.

The set of the colored cones of all $L$-orbits of $X$ is called the fan of $X$ and is denoted here by $\mathcal{F}_L(X)$. Thanks to [Kn91, Section 3, property CF2], any colored cone $(\mathcal{C}_Y, \mathcal{D}_Y)$ of $\mathcal{F}_L(X)$ is uniquely determined by $\mathcal{C}_Y$. Moreover, two $L$-orbits $Y, Z$ satisfy $\overline{Z} \supseteq Y$ if and only if $\mathcal{C}_Z$ is a face of $\mathcal{C}_Y$ and $\mathcal{D}_Z = \mathcal{D}_Y \cap \rho^{-1}(\mathcal{C}_Z)$.

The set of $L$-invariant valuations on $\mathbb{C}(X)$ is identified via the map $\rho$ with its image $V_L(X)$ in $N_B(X)$, and $V_L(X)$ is a polyhedral convex cone of maximal dimension called the valuation cone. If $X$ is complete and toroidal then $V_L(X)$ is equal to the union of all cones $\mathcal{C}_Y$ for $Y$ varying in the set of $L$-orbits of $X$. The dual cone of $-V_L(X)$ in $\Xi_B(X)$ is generated by a set $\Sigma_L(X)$ of primitive, linearly independent elements called the spherical roots of $X$. Finally, the equality $V_L(X) = N_B(X)$ holds if and only if $X$ is a horospherical $L$-variety, i.e. a maximal unipotent subgroup of $L$ stabilizes a point in the open $L$-orbit of $X$.

The purpose of this section is to adapt the above notions to our case, in particular to introduce a set of spherical roots related to the action on $P/H$ of $P$ instead of $L$. We accomplish this task by defining a set $V(P/H)$ that will play the role of the valuation cone. However, for this we must use $P$-toroidal compactifications of $P/H$, and several details need to be worked out to prove the two main results of this section: that what we define is indeed a convex cone (Theorem 5.9), and that it is independent from the chosen compactification (Corollary 5.10).

5.1. Definition. Let $X$ be a complete $P$-toroidal embedding of $P/H$. We define

$$V(X) = \bigcup_Y \mathcal{C}_Y$$

where $Y$ varies in the set of $L$-orbits of $X$ such that $\overline{Y}$ is $P$-stable.

Suppose that $X$ is as in the above definition. If $Y$ is an $L$-orbit with $P$-stable closure then $\overline{Y}$ contains a closed $P$-orbit $Z$. The latter is complete, therefore itself a closed $L$-orbit, hence $\mathcal{C}_Y$ is a face of the maximal cone $\mathcal{C}_Z \subset V(X)$. 

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It follows that in the definition of $V(X)$ the same set is obtained if we let $Y$ vary only in the set of closed $L$-orbits that are also $P$-orbits, because this corresponds exactly to taking only those $Y$ such that $\overline{Y}$ is $P$-stable and $C_Y$ is maximal in the fan $\mathcal{F}_L(X)$.

We provide now some general insight on $L$-orbits $Y \subseteq X$ whose closure is $P$-stable.

5.2. Proposition. Let $X$ be a complete $P$-toroidal embedding of $P/H$, let $Y \subseteq X$ be an $L$-orbit, and suppose that $C_Y \subseteq V(X)$. Then $\overline{Y}$ is $P$-stable, and $\mathcal{D}_Y = \emptyset$.

Proof. If $C_Y \subseteq V(X)$ then there exists a closed $P$-orbit $Z$ such that $C_Y$ is a face of the maximal cone $C_Z$. Since $X$ is toroidal, then any $B$-stable prime divisor containing $Z$ is $P$-stable, in particular $\mathcal{D}_Z = \emptyset$. The face $C_Y$ of $C_Z$ is itself the cone of a colored cone $(C_Y, D)$ of $\mathcal{F}_L(X)$, corresponding to an $L$-orbit $Y'$ of $X$, such that $\overline{Y'} \supseteq Z$ and $D = D_Z \cap \rho^{-1}(C_Y)$, i.e. $D = \emptyset$. Colored cones are uniquely determined by their cones, which implies $D = \emptyset$, and $Y' = Y$.

It follows that $\overline{Y} \supseteq Z$, hence any $B$-stable prime divisor containing $Y$ is also $P$-stable. As a consequence, the colored cone of $Y$ is equal to the colored cone of the intersection of all $P$-stable prime divisors containing $Y$. This implies that $\overline{Y}$ is equal to such intersection, and is $P$-stable.

5.3. Corollary. Let $X$ be a complete $P$-toroidal embedding of $P/H$, and let $Y \subseteq X$ be an $L$-orbit such that $\overline{Y}$ is $P$-stable. Then $Y$ has in $X$ an $L$-stable neighborhood that is an $L$-toroidal embedding of the open $L$-orbit of $P/H$.

Proof. By definition of $V(X)$ the cone $C_Y$ is in $V(X)$, and by Proposition 5.2 we have $\mathcal{D}_Y = \emptyset$. Consider the $L$-embedding $X_0$ of the open $L$-orbit of $X$ having fan

$$\{(C, \emptyset) \mid C \text{ is a face of } C_Y\}.$$ 

The embedding $X_0$ is $L$-toroidal, the fan of $X_0$ contains $(C_Y, \mathcal{D}_Y)$ and is contained in the fan of $X$. This implies that $X_0$ is $L$-equivariantly isomorphic to an open subset of $X$ (see [Kn91, Section 3]) containing $Y$. □

5.4. Lemma. Let $X$ be a normal $L$-variety, let $Y \subseteq X$ be an $L$-orbit and $Z \subseteq Y$ be a $B$-stable subvariety. Then $Z$ is contained in a $B$-stable prime divisor of $X$ not containing $Y$.

Proof. Thanks to [Su74], we may suppose that $X$ is quasi-projective, i.e. $X \subseteq \mathbb{P}(M)$ where $M$ is a finite-dimensional $L$-module. Let $X'$, $Y'$ and $Z'$ be the closures in $M$ of the cones over resp. $X$, $Y$ and $Z$. Choose a homogeneous function $f \in \mathbb{C}[Y']^{(B)}$ vanishing on $Z'$. By [Kn91, Theorem 1.1] we can extend $f$ to a $B$-semiinvariant homogeneous regular function $F$ on $X'$. Then

$$D = \{F = 0\} \cap X$$

contains $Z$ but not $Y$. Any irreducible component of $D$ containing $Z$ satisfies the desired properties. □

5.5. Proposition. Let $X$ be a $P$-toroidal embedding of $P/H$ and $Y \subseteq X$ an $L$-orbit such that $\overline{Y}$ is $P$-stable.

1. The closure $\overline{Y}$ is a $P$-toroidal embedding of an $L$-spherical homogeneous space $P/K$ for some subgroup $K \subseteq P$. 


(2) Restriction of rational functions from $X$ to $Y$ induces an identification between $\Xi_B(\overline{Y})$ and $C_Y(\subseteq \Xi_B(X))$, and correspondingly an identification between $N_B(\overline{Y})$ and $N_B(X)/(\text{span}_Q C_Y)$.

(3) The set $V(Y)$ is the projection on $N_B(\overline{Y})$ of the set $\bigcup Z C_Z$, where $Z$ varies in the set of $L$-orbits of $X$ such that $\overline{Z}$ is $P$-stable and $C_Z \supseteq C_Y$.

Proof. Let us prove 1. The closure $\overline{Y}$ is a spherical $L$-variety equipped with an action of $P$ extending the one of $L$. It follows that $\overline{Y}$ contains an open ($L$-spherical) $P$-orbit $P/K$. It remains to check that $\overline{Y}$ is $P$-toroidal. Let $D$ be an element of $\mathcal{D}(P/K)^B$ and suppose that $D \subseteq \overline{Y}$ contains a $P$-orbit. By Lemma 5.4 the closure $\overline{D}$ is contained in a $B$-stable prime divisor $E$ of $X$ not containing $\overline{Y}$. This implies that $\overline{D}$ is an irreducible component of $E \cap \overline{Y}$. Since $\overline{D}$ contains a $P$-orbit, then $E$ is $P$-stable, therefore $\overline{D}$ is also $P$-stable.

Part 2 is a standard result for spherical $L$-varieties, see e.g. [Kn91, Proof of Theorem 6.3]. To prove part 3, recall that for any $L$-orbit $Z \subseteq X$ the condition $Z \subseteq \overline{Y}$ is equivalent to $C_Z \supseteq C_Y$. It only remains to prove that the cone of $Z$ in the fan of $\overline{Y}$ is the projection of $C_Z$ in $N(\overline{Y})$, for any $L$-orbit $Z$ such that $Z \subseteq \overline{Y}$ and $\overline{Z}$ is $P$-stable. Thanks to Corollary 5.3, $Z$ has in $X$ an $L$-stable neighborhood $X_0$ that is an $L$-toroidal embedding. Then $X_0$ contains $Y$, since $\overline{Y} \supseteq Z$. At this point the cone of $Z$ in the fan of $\overline{Y}$ is indeed as desired by the standard theory of toric varieties, thanks to [Ti11, Theorem 29.1].

We are ready to prove that $V(X)$ is a convex cone. The key argument comes from Corollary 4.6, and the following three lemmas will enable us to apply it here.

5.6. Lemma. Let $X$ be a complete $P$-toroidal embedding of $P/H$, and let $Y \subseteq X$ be an $L$-orbit whose closure is $P$-stable. If $C_Y$ is contained in a cone of $\mathcal{F}_L(X)$ not contained in $V(X)$, then $P^u$ acts nontrivially on $\overline{Y}$.

Proof. Let $Z$ be an $L$-orbit whose cone contains $C_Y$ and is not contained in $V(X)$. Then $\overline{Z}$ is contained in $\overline{Y}$, it is $L$-stable but not $P$-stable. Hence $P^u$ doesn’t act trivially on $\overline{Y}$. □

5.7. Lemma. If $P^u$ acts non-trivially on a complete $P$-embedding $X$ of $P/H$ then $V(X)_R \setminus \{0\}$ is connected, where $V(X)_R$ is the convex cone generated by $V(X)$ in $N_B(X) \otimes_Q \mathbb{R}$.

Proof. Since $X$ is complete it contains some complete $P$-orbit, whose points are then fixed by $P^u$. Therefore the set $X_{P^u}$ is non-empty, and by [BB73, Theorem 1] it is connected. On the other hand, whenever an $L$-orbit $Y$ contains a fixed point under $P^u$ it is itself a $P$-orbit, thus has $P$-stable closure.

It follows that any two closed $L$-orbits $Y, Y'$ of $X$ that are also $P$-orbits are “connected” by a sequence of $L$-orbits $Y = Y_1, Y_2, \ldots, Y_k = Y'$ such that for all $i \in \{1, \ldots, k-1\}$ the orbit $Y_i$ is also a $P$-orbit, and we have either $\overline{Y}_i \supset Y_{i+1}$ or $Y_i \subseteq \overline{Y}_{i+1}$. Moreover, no orbit $Y_i$ is the open $L$-orbit of $X$, since $P^u$ doesn’t act trivially on the whole $X$.

This translates to the existence of a sequence of cones $C = C_1, C_2, \ldots, C_k = C'$ for any two maximal cones $C, C'$ in the union defining $V(X)$, such that for all $i \in \{1, \ldots, k-1\}$ the cone $C_i$ is contained in $V(X)$, and either is contained in $C_{i+1}$ or contains it. They are also different from $\{0\}$, and the lemma follows. □
5.8. Lemma. Let $V$ be a finite union of polyhedral convex cones of maximal dimension in a finite-dimensional real vector space, such that the intersection of any two of these cones is a face of both. If $V \setminus \{0\}$ is connected and locally convex, and $V$ is not convex, then $V$ is not contained in any half-space.

Proof. Suppose that $V$ is contained in one of the two half-spaces defined by a linear subspace $M$ of codimension 1. For dimension reasons $V$ is not contained in $M$. The interior $V'$ of $V$ is obtained from $V$ removing from each maximal cone all proper faces whose relative interior is not contained in the interior of $V$. Thanks to local convexity in all points except for 0, and because $V \setminus \{0\}$ is connected, we have that $V'$ is connected. On the other hand $V \setminus M$ is a subset of $V$ containing $V'$. It follows that $V \setminus M$ is connected, and also locally convex. Then $V \setminus M$ is convex, which implies that its closure, i.e. $V$, is convex: contradiction. □

5.9. Theorem. Let $X$ be a complete $P$-toroidal embedding of $P/H$. Then $V(X)$ is equal to the polyhedral convex cone generated by $\rho(D(X)^P)$, and has maximal dimension.

Proof. If it is a convex cone, then $V(X)$ has maximal dimension because $X$ contains a complete $P$-orbit, which is a complete $L$-orbit whose cone has maximal dimension and is contained in $V(X)$.

Denote by $\mathcal{Y}$ the set of $L$-orbits whose closure is $P$-stable. For $Y \in \mathcal{Y}$, the closure $\overline{Y}$ is contained only in $P$-stable prime divisors of $X$, because $X$ is $P$-toroidal. It follows from [Kn91, Lemma 3.2] that $C_Y$ is generated by the elements of $\rho(D(X)^P)$ it contains. Hence $V(X)$ contains $\rho(D(X)^P)$ and is contained in the convex cone generated by $\rho(D(X)^P)$. The theorem follows if we show that $V(X)$ is convex.

Let $Y \in \mathcal{Y}$. Consider the union of the projections on $N_B(\overline{Y})$ of all cones $C_Z$ such that $Z \in \mathcal{Y}$ and $C_Z \supseteq C_Y$. By Proposition 5.5, this union is equal to $V(\overline{Y})$. If $V(X)$ is not convex then there exists $Y \in \mathcal{Y}$ such that $V(\overline{Y})$ is not convex and $C_Y$ has maximal dimension. Maximality of the dimension of $C_Y$ implies that $V(\overline{Y})$ is locally convex, except for the origin.

On the other hand $V(\overline{Y})$ is contained in the valuation cone $V_L(\overline{Y})$ of $\overline{Y}$ as a spherical $L$-variety. So any point not in $V(\overline{Y})$ but in its convex hull is contained in some cone $C_Z$ of $F_L(\overline{Y})$ such that $Z \notin \mathcal{Y}$ and $C_Z \supseteq C_{Y_i}$. Supposing $x$ general we may also suppose that $C_Z$ is maximal. Then $C_Z$ is not contained in $V(\overline{Y})$, and by Lemma 5.6 we have that $P^u$ doesn’t act trivially on $\overline{Y}$. By Lemma 5.7, the set $V(\overline{Y})_R \setminus \{0\}$ is connected, and by Lemma 5.8 the set $V(\overline{Y})$ is not contained in any half-space. This contradicts Corollary 4.6, therefore $V(X)$ is convex and the proof is complete. □

5.10. Corollary. If $X$ and $Y$ are complete $P$-toroidal embeddings of $P/H$ then $V(X) = V(Y)$.

Proof. Thanks to Lemma 4.2 there exists a complete $P$-toroidal embedding equipped with $P$-equivariant maps to $X$ and to $Y$. Hence we may suppose that there is a $P$-equivariant morphism $\varphi: X \to Y$. In this situation [Kn91, Theorem 4.1] implies that each cone $C_Z$ of $F_L(X)$ is contained in the cone $C_{\varphi(Z)}$ of $F_L(Y)$.

Let $Z$ be a closed $L$-orbit of $X$. If $C_Z \subseteq V(X)$ then $\overline{Z}$ is $P$-stable by Proposition 5.2, and $P^u$ acts trivially on $Z$. Then it acts trivially on $\varphi(Z)$ too, which is therefore also a $P$-orbit. It follows that $(C_Z \subseteq) C_{\varphi(Z)} \subseteq V(Y)$, and $V(X) \subseteq V(Y)$.  

Suppose that $V(X) \subsetneq V(Y)$. In this case $V(Y) \setminus V(X)$ is a non-empty subset of $V_L(Y) = V_L(X)$, whence there is some maximal cone $C_Z$ of $\mathcal{F}_L(X)$ outside of $V(X)$ and with relative interior intersecting $V(Y)$. Therefore the relative interior of $C_{\varphi(Z)}$ intersects $V(Y)$, whence $C_{\varphi(Z)} \subseteq V(Y)$. By Proposition 5.2 the closure $\varphi(Z)$ is $P$-stable. Now $Z$ is contained in the closure in $X$ of some element $D$ of $\mathcal{D}(P/H)^B$, therefore the closure of $D$ in $Y$ contains $\varphi(Z)$, hence some $P$-orbit. This contradicts the fact that $Y$ is $P$-toroidal.

5.11. Definition. Define $V(P/H) = V(X)$ where $X$ is any complete $P$-toroidal embedding of $P/H$. Also, define the set $\Sigma(P/H)$ of spherical roots as the minimal set of primitive elements of $\Xi_B(P/H)$ such that

$$V(P/H) = \{ v \in N_B(P/H) \mid \langle v, \sigma \rangle \leq 0 \forall \sigma \in \Sigma(P/H) \}.$$ 

5.12. Remark. To avoid confusion between the sets of spherical roots of $P/H$ as an $L$-variety and the above set of spherical roots “with respect to the action of $P$”, from now on we’ll use the term spherical roots exclusively for the elements of $\Sigma(P/H)$. We will refer to the other set only using the notation $\Sigma_L(P/H)$.

5.13. Proposition. We have $V(P/H) = N_B(P/H)$ if and only if $\Sigma(P/H) = \emptyset$ if and only if $H$ contains a maximal unipotent subgroup of $P$.

Proof. The first equivalence is obvious. For the second, let $X$ be a $P$-toroidal embedding of $P/H$. If $H$ contains a maximal unipotent subgroup of $P$ then it contains $P_u$, therefore $P_u$ acts trivially on $P/H$. It follows that all $L$-orbits of $X$ are also $P$-orbits, and $X$ is also $L$-toroidal. As a consequence, we have $\mathcal{D}(X)^P = \mathcal{D}(X)^L$, and $V(X)$ is equal to the valuation cone $V_L(X)$ of $X$ as a spherical $L$-variety. Since $X$ is horospherical by our assumption, we have $V_L(X) = N_B(X)(= N_B(P/H))$.

Suppose now that $V(X) = N_B(X)$. By Corollary 4.6, the unipotent radical $P_u$ acts trivially on $P/H$. This implies as above that all $L$-orbits of $X$ are also $P$-orbits, and $V(X) = V_L(X)$. Finally, the equality $V_L(X) = N_B(X)$ implies that $X$ is a horospherical $L$-variety, whence $H$ also contains a maximal unipotent subgroup $U$ of $L$. The product $UP_u$ is contained in $H$ and is a maximal unipotent subgroup of $P$. □

For any spherical variety $X$, the elements of $\Sigma_L(X)$ are linear combinations of simple roots of $L$ with non-negative rational coefficients (actually, integer in most cases). The analogue in our setting is provided by the following theorem.

5.14. Theorem. Let $\sigma \in \Sigma(P/H)$. Then $\sigma \in \Sigma_L(P/H)$, or $-\sigma$ is a root of $P_u$.

Proof. Let $X$ be a complete $P$-toroidal embedding of $P/H$. Choose an $L$-orbit $Y$ such that $C_Y$ is contained in $V(X)$, has codimension 1 in $N_B(X)$ and is orthogonal to $\sigma$. Then $\overline{Y}$ is $P$-stable by Proposition 5.2, and by Proposition 5.5 we have $\Xi_B(\overline{Y}) = \mathbb{Z}\sigma$ and $V(\overline{Y})$ is the half-line of $N_B(\overline{Y})$ defined by $\langle -, \sigma \rangle \leq 0$.

Since $N_B(\overline{Y})$ has dimension 1, there are only two possibilities for $V_L(\overline{Y})$, namely $V(\overline{Y})$ and $N_B(\overline{Y})$. In the first case $C_Y$ is contained not only in a face of codimension 1 of $V(X)$ but also of $V_L(X)$, therefore $\sigma \in \Sigma_L(P/H)$. Hence we may suppose that $V_L(\overline{Y}) = N_B(\overline{Y})$, i.e. $\overline{Y}$ is $L$-horospherical. Moreover $\overline{Y}$ is complete, hence there are cones in $\mathcal{F}_L(\overline{Y})$ not in $V(\overline{Y})$. This implies that not all closures of $L$-orbits of $\overline{Y}$ are $P$-stable, therefore $P_u$ doesn’t act trivially on $\overline{Y}$. □
Let $P/K$ be the open $P$-orbit of $\overline{Y}$, and consider the natural morphism $\pi: P/K \to P/KP^u$. We may suppose that $eKP^u$ is in the open $B$-orbit of $P/KP^u$. Then we have an exact sequence of abelian groups

$$1 \to \frac{\mathbb{C}(KP^u/K)(B\cap KP^u)}{\mathbb{C}^\times} \to \frac{\mathbb{C}(P/K)(B)}{\mathbb{C}^\times} \to \frac{\mathbb{C}(P/KP^u)(B)}{\mathbb{C}^\times} \to 1$$

where $\mathbb{C}^\times$ denotes the constant non-zero functions, the second map is given by pull-back of functions, and the third is given by restriction of functions to the fiber $\pi^{-1}(eKP^u)$.

The quotient $KP^u/K = P^u/K \cap P^u$ is affine and connected, and not equal to a single point because $P^u$ doesn’t act trivially on $\overline{Y}$. Therefore the group $\mathbb{C}(KP^u/K)(B\cap KP^u)/\mathbb{C}^\times$ is infinite, which implies that $\mathbb{C}(P/KP^u)(B)/\mathbb{C}^\times$ is finite. So it is trivial and the quotient $P/KP^u$ has rank 0, i.e. it is a complete $L$-homogeneous space, and $KP^u/P^u$ is a parabolic subgroup of $P/P^u \cong L$. This also implies that $K$ is connected.

Then we may suppose that $KP^u/P^u$ contains $B^-$, that a Levi component $M$ of $K$ contains $T$ and is also a Levi component of a parabolic subgroup of $L$, which also implies that $B \cap M$ is equal to $B \cap KP^u$ and is a Borel subgroup of $M$. With this choice, the above considerations amount to the equality $\Xi_M(P^u/K \cap P^u) = \Xi_L(P/K)$.

Recall that $P^u$ as a variety is $L$-equivariantly isomorphic to its Lie algebra $\mathfrak{n}$ via the exponential map, which restricts to an $M$-equivariant isomorphism between the intersection $K \cap P^u$ and its own Lie algebra $\mathfrak{n}$. Then $Z = KP^u/K \cong \mathfrak{n}/\mathfrak{i}$ is a spherical module under the adjoint action of $M$. Moreover $Z$ has rank 1, which implies that it is an irreducible $M$-module whose lattice $\Xi_M(Z)$ is generated by the highest weight $\gamma$ of the dual module $Z^*$. With this notation, the opposite $-\gamma$ is the lowest weight of $Z$, therefore a root of $P^u$, and it is also either $\sigma$ or $-\sigma$. Moreover, being the eigenvalue of a regular function of $Z$, the character $\gamma$ is non-negative on the valuations of any $B \cap M$-stable prime divisor of $Z$.

Consider the closure $\overline{Z}$ of $Z$ in $\overline{Y}$. It is connected and complete, so a function $f \in \mathbb{C}(Z)(B\cap M)$, being non-constant, cannot be regular on $\overline{Z}$. This implies that there exists a $B \cap M$-invariant prime divisor $D$ of $\overline{Z}$ (contained in $\overline{Z} \setminus Z$) where $f$ has a pole. As a consequence, the closure $\overline{Y}$ has a $B$-stable prime divisor $D$ that intersects the closure $\overline{Z}$ of $Z$ in $E$, and such that $\langle \rho(D), \gamma \rangle < 0$. Now $D$ cannot intersect $P/K$: if it did, it would be either the inverse image of a $B$-stable prime divisor of $P/KP^u$, in which case it wouldn’t intersect $\overline{Z}$ (moreover its valuation would be 0), or it would intersect $Z$, which is false. Hence $D$ is $P$-stable, and we conclude that $\gamma = \sigma$, i.e. $-\sigma$ is a root of $P^u$. □

6. Morphisms

Given a spherical $L$-homogeneous space $X$, equivariant morphisms with connected fibers $X \to Y$ are classified in [Kn91, Theorem 4.4] by means of colored subspaces of $N_B(X)$, i.e. couples $(\mathcal{C}, \mathcal{D})$ such that $\mathcal{D} \subseteq \Delta_L(X)$, the set $\mathcal{C}$ is a linear subspace of $N_B(X)$ generated as a convex cone by $\mathcal{D}$ and by $\mathcal{C} \cap V_L(X)$. Precisely, with any morphism $\varphi: X \to Y$ one associates the colored subspace $(\mathcal{C}_\varphi, \mathcal{D}_\varphi)$, where $\mathcal{C}_\varphi$ is the subspace $\Xi_B(Y)^\perp$ of $N_B(X)$, and $\mathcal{D}_\varphi$ is the set of colors of $X$ mapped dominantly onto $Y$.

In this section we describe how this correspondence can be used to describe $P$-equivariant morphisms with domain $P/H$. 12
6.1. Lemma. Let $K$ be a subgroup of $P$ and $\eta: P/H \rightarrow P/K$ a $P$-equivariant map. Then there exist a $P$-toroidal embedding $Z$ of $P/H$ and a set $D_0$ such that:

1. $D_{\eta} \subseteq D_0 \subseteq D_{\eta} \cup D(Z)^L$, where $X$ is the open $L$-orbit of $Z$;
2. the convex cone generated by $\rho(D_0)$ is a vector subspace,
3. $\Xi_B(P/K)$ is a subgroup of finite index of $\rho(D_0)^\perp(\subseteq \Xi_B(P/H))$.

Moreover, the map $\eta$ has connected fibers if and only if $\Xi_B(P/K)$ is a saturated sublattice of $\Xi_B(P/H)$.

Proof. We may assume that $K$ contains $H$ and that $P/H \rightarrow P/K$ is the usual associated map. Using Lemma 4.3 we extend the map $P/H \rightarrow P/K$ to a proper $P$-equivariant map $\psi: Z \rightarrow P/K$ where $Z$ is a $P$-toroidal embedding of $P/H$. Denote by

$$Z \xrightarrow{\mu} Z' \xrightarrow{\psi'} P/K$$

the Stein factorization of $\psi$.

Denote by $D'_0 \subseteq D(Z')^B$ and by $D_0 \subseteq D(Z)^B$ the subsets of the elements mapped dominantly onto $P/K$ by $\psi'$ and $\psi$ respectively. Then $D_{\eta} \subseteq D_0 \subseteq D_{\eta} \cup D(Z)^L$, and $\rho(D_0) = \rho(D'_0)$ up to replacing some elements with positive rational multiples.

For a $B$-semiinvariant rational function on $Z'$, being a pull-back is equivalent to being constant on the fibers of $\psi'$. Since $\psi'$ is proper and has connected fibers, being a pull-back is also equivalent to having no poles along the $B$-stable prime divisors of $Z'$ mapped dominantly onto $P/K$. This shows that the subgroup $\Xi_B(P/K)$ of $\Xi_B(Z')$ is the set of elements that are $\geq 0$ on $\rho(D'_0)$. Since it is a subgroup of $\Xi_B(Z')$, then all elements of $\rho(D'_0)$ are actually 0 on it, whence the convex cone generated by $\rho(D'_0)$ is a vector subspace.

The map $\mu$ is finite, therefore $\Xi_B(Z')$ is a subgroup of $\Xi_B(Z)$ of finite index. Moreover $Z$ is equal to $Z'$ if and only if $\Xi_B(Z) = \Xi_B(Z')$, and the proof is complete. □

6.2. Definition. A colored subspace $(C,D)$ of $N_B(P/H)$ is called $P$-equivariant if $C$ is generated as a convex cone by $\rho(D)$ and by finitely many elements of $\rho(D(P/H)^L) \cup V(P/H)$.

6.3. Theorem. Let $X$ be the open $L$-orbit of $P/H$, let $Y$ be an $L$-homogeneous space, and $\varphi: X \rightarrow Y$ be an $L$-equivariant map with connected fibers. Then $(C_\varphi,D_\varphi)$ is $P$-equivariant if and only if there exists a subgroup $K$ of $P$ containing $H$ such that $P/K$ is an $L$-embedding of $Y$ and the corresponding map $P/H \rightarrow P/K$ extends $\varphi$. If this is the case, a subgroup $K$ with these properties is unique.

Proof. Suppose that $\varphi: X \rightarrow Y$ extends to a $P$-equivariant map $P/H \rightarrow P/K$. Then $(C_\varphi,D_\varphi)$ is $P$-equivariant thanks to Lemma 6.1.

We prove now the converse statement. Suppose that $C_\varphi$ fulfills the above requirements. First we construct a map $P/H \rightarrow P/K$ associated with $C_\varphi$ as in the proof of [Kn91, Theorem 4.4] (see [Kn91, proof of Theorem 2.3] for more details), then we check it extends $\varphi$.

The dual cone $\Lambda$ of $C_\varphi$ in $\Xi_B(P/H)$ is actually a subgroup and is equal to $C_\varphi^\perp$; by Gordan’s Lemma the monoid $\Lambda$ is finitely generated. Let $\chi_1,\ldots,\chi_n$ be generators of $\Lambda$ and for all
We claim now that $P \subseteq P/K$ dominantly onto $P/K$. For all $H$ is a torus and a simply connected semisimple group, which implies that $C[P]$ is factorial. Choose a non-zero function $f_0 \in C[P]^{(B \times H)}$ that vanishes exactly on the inverse image of $D_0$ in $P$, with enough multiplicity so that $f_1 = f_0 g_i$ is a regular function on $P$.

The $P$-module $M$ generated by $f_0, f_1, \ldots, f_n$ in $C[P]$ defines naturally a $P$-equivariant map $\psi: P/H \to \mathbb{P}(M^*)$. The image is a $P$-orbit, and we denote it by $P/K$. Notice that $P/K = P \cdot X_0$ where $X_0 = \psi(P/H) \setminus \{f_0 = 0\}$.

We claim now that

1. $\exists B(P/K) = \Lambda$;
2. $\psi$ has connected fibers;
3. $\mathcal{D}_\phi$ is the set of $L$-colors of $P/H$ mapped dominantly by $\psi$ onto $P/K$.

From these three properties we conclude that $\psi$, seen as an $L$-equivariant map with connected fibers between $X$ and the open $L$-orbit of $P/K$, has colored equal to $(C_\phi, \mathcal{D}_\phi)$, hence $\psi|_X = \varphi$.

Let us show (1). It follows if we prove that $\mathcal{M} = \mathbb{C}[X_0]^{(B)}$, where

$$\mathcal{M} = \{f \in C(P/H)_{\chi}^{(B)} \mid \lambda \in \Lambda\}.$$ 

The inclusion $\mathcal{M} \subseteq \mathbb{C}[X_0]^{(B)}$ stems from the fact that the functions $g_1, \ldots, g_n$ are regular on the quasi-affine, $B$-stable set $X_0$. To show the other inclusion, we show that $\langle c, f \rangle \geq 0$ for any $c$ among generators of $C_\phi$ as in Definition 6.2 and any $f \in \mathbb{C}[X_0]^{(B)}$.

If $c = \rho(D)$ where $D \in \mathcal{D}_\phi \cup \mathcal{D}(P/H)^L$ then $\langle c, \chi \rangle$, which is equal to $\text{ord}_D(f)$, is non-negative because the poles of $f$ as a rational function on $P/H$ are contained in $D_0$. Otherwise $c \in V(P/H)$. By Theorem 5.9 the element $c$ is a linear combination with non-negative coefficients $c = a_1 \rho(v_1) + \ldots + a_m \rho(v_m)$ where $v_1, \ldots, v_m$ are $P$-invariant discrete valuations of $C(P/H)$. By [Kn91, Corollary 1.5] the valuation $v_i$ can be lifted to a $P$-invariant valuation on $C(P)$; then $v_i$ is non-negative on $f_0, \ldots, f_n$, hence on $M$, for all $i$. The latter generates the algebra $\mathbb{C}[X_0]$, whence $c$ is non-negative on $\chi$.

Part (2) stems from part (1) and Lemma 6.1, let us show part (3). Any $L$-color of $P/H$ not in $\mathcal{D}_\phi$ is mapped to the proper subset $\{f_0 = 0\}$ of $P/K$, so it is not mapped dominantly onto $P/K$. Suppose now that an $L$-color $D \in \mathcal{D}_\phi$ is not mapped dominantly onto $P/K$. Then $f_0$ is non-zero on the inverse image of $D$ on $P$, so $\psi(D)$ intersects $X_0$ in a $B$-stable prime divisor $E$. By part (1), and since $\rho(\mathcal{D}_\phi) \subseteq C_\phi$, no non-zero $B$-semiinvariant rational function of $X_0$ vanishes on $E$. This is absurd since $X_0$ is quasi-affine, and part (3) is shown.

It remains to show uniqueness. Suppose that $\varphi$ has extensions $\psi_i: P/H \to P/K_i$ for $i \in \{1, 2\}$. Then we may also suppose that $H$ is in the open $B$-orbit of $P/H$, and that $\psi_i(eH) = eK_i$, so in particular $K_i \supseteq H$. From the first part of the proof we deduce that $\mathbb{C}(P/K_1)^{(B)} = \mathbb{C}(P/K_2)^{(B)}$ as subsets of $\mathbb{C}(P/H)$. Denote by $Y_i^0$ the open $B$-orbit of
We may restate more concisely the above theorem in the following way.

6.4. Corollary. The bijection \( \varphi \mapsto (C_\varphi, D_\varphi) \) of [Kn91, Theorem 4.4] induces a bijection between the set of surjective \( P \)-equivariant maps with domain \( P/H \) and connected fibers and \( P \)-equivariant colored subspaces of \( N_B(P/H) \).

Let \( K \) be a subgroup of \( P \) and \( \varphi: P/H \to P/K \) a \( P \)-equivariant morphism, denote by \( \varphi^*: \Xi_B(P/K) \to \Xi_B(P/H) \) the homomorphism induced by the pull-back of functions, and by \( \varphi_*: N_B(P/H) \to N_B(P/K) \) the dual homomorphism. If \( \varphi \) has connected fibers then \( \varphi_* \) can be identified with the quotient map \( N_B(P/H) \to N_B(P/H)/C_\varphi \).

6.5. Proposition. Let \( \varphi: P/H \to P/K \) be as above. Then \( \varphi_*(V(P/H)) = V(P/K) \).

Proof. Let \( Y \) be a \( P \)-toroidal complete embedding of \( P/K \). Thanks to Lemma 4.3, we can find a \( P \)-toroidal embedding \( X \) of \( P/H \) such that the map \( \varphi \) extends to a \( P \)-equivariant map \( \varphi: X \to Y \). Let \( Z \) be a closed \( P \)-orbit of \( X \), so that \( C_Z \) is a maximal cone contained in \( V(X) \). Then \( \varphi(Z) \) is a complete \( P \)-orbit of \( Y \), and \( C_{\varphi(Z)} \) is contained in \( V(Y) \). On the other hand \( C_{\varphi(Z)} \supseteq \varphi^*(C_Z) \), which implies \( \varphi^*(V(X)) \subseteq V(Y) \).

Let now \( Z \) be a \( P \)-stable prime divisor of \( Y \). The inverse image \( \varphi^{-1}(Z) \) contains a \( P \)-stable prime divisor \( Z' \) of \( X \) mapped dominantly onto \( Z \). Then \( \mathbb{Q}_{\geq 0}\varphi^*(\rho_X(Z')) = \mathbb{Q}_{\geq 0}\rho_Y(Z) \), which shows \( \varphi^*(V(X)) \supseteq V(Y) \).

We expect that it is possible to develop a theory of \( P \)-equivariant embeddings of \( P/H \) based on the same ideas of [Kn91], as already suggested by the proof of Theorem 6.3. In particular, one can show as in [Kn91, Lemma 5.1] that \( V(P/H) \) is the image in \( N_B(P/H) \) of the set of \( P \)-invariant discrete valuations of \( C(P/H) \), in analogy with the standard theory of spherical varieties. This goes beyond the scope of the present work.

7. Localization via one-parameter subgroups

The goal of this section is to adapt some of the results of [Kn14, Section 4] to our setting; several proofs here follow the guidelines of those in loc.cit. with only minimal modifications. We start recalling some well-known results on the action of one-parameter subgroups on normal, complete varieties, and we refer to loc.cit. for more details.

Let \( X \) be a complete normal \( \mathbb{G}_m \)-variety. Then there is a unique connected component \( Y \) of \( X^\mathbb{G}_m \) such that

\[
\lim_{t \to 0} t \cdot x \in Y
\]
for a general point \( x \in X \), and \( Y \) is also complete. Denote by \( \pi_\lambda(x) \) the above limit and by \( Z \) the subset of \( X \) such that \( \pi_\lambda(x) \in Y \). Then \( Z \) is open in \( X \), the map \( \pi : Z \to Y \) is a categorical quotient by \( \mathbb{G}_m \) and its general fibers are irreducible. In particular \( Y \) is irreducible and normal.

If \( \lambda : \mathbb{G}_m \to T \) is a one-parameter subgroup of \( T \), then one denotes by \( L^\lambda \) the centralizer of \( \lambda(\mathbb{G}_m) \) in \( L \) and by \( L_\lambda \) the parabolic subgroup of \( L \) of those \( l \in L \) such that \( \lim_{t \to 0} \lambda(t)l\lambda(t)^{-1} \) exists. Then \( L^\lambda \) is a Levi subgroup of \( L_\lambda \), and the map \( l \mapsto \pi_{L}(l) = \lambda(t)l\lambda(t)^{-1} \) is the natural homomorphism \( L_\lambda \to L^\lambda \).

**7.1. Definition.** A one-parameter subgroup \( \lambda \) of \( T \) is \( P \)-dominant if \( \langle \lambda, \alpha \rangle \leq 0 \) for all roots \( \alpha \) of \( P^u \) and of \( (B^-)^u \). If this is the case we denote by \( P^\lambda \) the centralizer in \( P \) of \( \lambda(\mathbb{G}_m) \), and put

\[
P_\lambda = \{ p \in P \mid \lim_{t \to 0} \lambda(t)p\lambda(t)^{-1} \text{ exists} \}.
\]

For all \( p \in P_\lambda \) we also set \( \pi_\lambda(p) = \lim_{t \to 0} \lambda(t)p\lambda(t)^{-1} \).

**7.2. Lemma.** Let \( \lambda \) be a \( P \)-dominant one-parameter subgroup of \( T \). The groups \( P_\lambda \) and \( P^\lambda \) have the same Levi subgroup \( L^\lambda \); we have \( (P^\lambda)^u = P^\lambda \cap P^u \) and \( P^\lambda = (P^\lambda)^u L^\lambda \). The intersection \( B^\lambda = L^\lambda \cap B \) is a Borel subgroup of \( L^\lambda \) and is equal to \( \pi_P(B) \); the intersection \( B^-\lambda = L^\lambda \cap B^- \) is the Borel subgroup of \( L^\lambda \) opposite to \( B^\lambda \) with respect to \( T \). Finally, a root \( \alpha \) of \( P^u \) (resp. \( B^- \)) is not a root of \( (P^\lambda)^u \) (resp. \( B^\lambda \)) if and only if \( \langle \lambda, \alpha \rangle < 0 \).

Proof. All statements are deduced easily from the properties of \( L_\lambda \) and \( L^\lambda \) recalled above, once one observes that for any \( p \in P \) the limit

\[
\lim_{t \to 0} \lambda(t)p\lambda(t)^{-1}
\]

exists in \( P \) if and only if \( \lim_{t \to 0} \lambda(t)l\lambda(t)^{-1} \) exists in \( L \) and \( \lim_{t \to 0} \lambda(t)u\lambda(t)^{-1} \) exists in \( P^u \), where \( p = ul \) with \( u \in P^u \) and \( l \in L \).

**7.3. Definition.** A one-parameter subgroup \( \lambda \) of \( T \) induces an element \( \lambda^\vee \in N_B(P/H) \) by restricting to \( \Xi_B(P/H) \) the natural pairing of \( \lambda \) with \( X(B) \).

**7.4. Lemma.** If a one-parameter subgroup \( \lambda \) of \( T \) is \( P \)-dominant then \( -\lambda^\vee \in V(P/H) \).

Proof. This stems from Theorem 5.14.

**7.5. Definition.** Let \( X \) be a complete \( P \)-toroidal embedding of \( P/H \) and \( \lambda \) a \( P \)-dominant one-parameter subgroup of \( T \). We define \( X_\lambda = Z \) and \( X^\lambda = Y \), where \( Z \) and \( Y \) are defined as above considering \( X \) as a \( \mathbb{G}_m \)-variety via the action of \( \lambda(\mathbb{G}_m) \subseteq P \). The variety \( X^\lambda \) is called the localization of \( X \) with respect to \( \lambda \).

**7.6. Lemma.** Let \( X \) and \( \lambda \) be as in Definition 7.5. Then \( X_\lambda \) is \( P_\lambda \)-stable and \( X^\lambda \) is \( P^\lambda \)-stable and spherical as a \( L^\lambda \)-variety. Moreover, the map \( \pi_\lambda : X_\lambda \to X^\lambda \) is \( P_\lambda \)-equivariant, if we let \( P_\lambda \) act on \( X^\lambda \) via the quotient \( \pi_P : P_\lambda \to P^\lambda \).

Proof. The centralizer \( P^\lambda \) of \( \lambda(\mathbb{G}_m) \) acts on \( X^\lambda(\mathbb{G}_m) \); since \( P^\lambda \) is connected it preserves the connected components hence also \( X^\lambda \). The other statements follow, except for the fact that \( X^\lambda \) is spherical. The open subset \( X_\lambda \) of \( X \) intersects the open \( B \)-orbit of \( X \), which implies easily that the open \( B \)-orbit is contained in \( X_\lambda \) and its image on \( X^\lambda \) is then an open \( B^\lambda \)-orbit of \( X^\lambda \).

**7.7. Lemma.** Let \( X \) and \( \lambda \) be as above, and let \( \alpha \) be a root of \( P \) with \( \langle \lambda, \alpha \rangle < 0 \). Then \( U_\alpha \) acts trivially on \( X^\lambda \).

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Proof. This is shown as in [Lu97, Proof of Proposition 1.3].

Thanks to the above lemma $X^λ$ has an open $B^λ$-orbit, therefore also an open $P^λ$-orbit.

7.8. Proposition. Let $X$, $λ$ be as above and $P^λ/K$ be the open $P^λ$-orbit of $X^λ$. For all $D \in \mathcal{D}(P^λ/K)^{B^λ}$ there is a unique $D^* \in \mathcal{D}(P/H)^B$ such that $π_λ(D^* \cap X^λ) = D$. The map $D \mapsto D^*$ is a bijection between $\mathcal{D}(P^λ/K)^{B^λ}$ and the set of those elements of $\mathcal{D}(P/H)^B$ that are not stable under $P^λ$. Moreover, for any root $α$ of $P^λ$, an element $D \in \mathcal{D}(P^λ/K)^{B^λ}$ is not stable under $U_α \subseteq P^λ$ if and only if the same holds for $D^*$.

Proof. We have that $E = π_λ^\ast(D)$ is irreducible, because $π_λ$ is $P^λ$-equivariant, its general fibers are irreducible, and $P^λ/K$ is $P^λ$-homogeneous and dense in $X^λ$. Moreover $E$ cannot intersect the open $B$-orbit of $X$, otherwise $D$ would intersect the open $B^λ$-orbit of $X^λ$. Hence the closure $\overline{E}$ of $E$ in $X$ is an element of $\mathcal{D}(X)^B$. Since $D$ is not $P^λ$-stable then $\overline{E}$ is not $P^λ$-stable. This also implies that $\overline{E}$ is not $P$-stable, therefore the intersection $D^* = \overline{E} \cap P/H$ is non-empty and obviously an element of $\mathcal{D}(P/H)^B$.

The map $D \mapsto D^*$ is injective, so it remains only to prove that its image is as claimed. Let $F \in \mathcal{D}(P/H)^B$ be not $P^λ$-stable. Then $F$ intersects $X_λ$, otherwise it would be $P_λ$-stable. But $F$ is anyway $λ(\mathbb{G}_m)$-stable because $λ(\mathbb{G}_m) \subseteq B$, therefore $D_0 = π_λ(F \cap X_λ)$ is not $P^λ$-stable. It follows that $D = D_0 \cap P^λ/K$ is non-empty and a $B^λ$-stable prime divisor of $P^λ/K$ satisfying $D^* = F$.

The last assertion stems from the fact that $π_λ: X_λ \to X^λ$ is in particular $P^λ$-equivariant. □

7.9. Lemma. Let $X$ and $λ$ be as above, and let $Z$ be a closed $P^λ$-orbit of $X^λ$. Then $Z$ contains a point fixed by $P^uB^−$, and $PZ$ is a closed $P$-orbit of $X$ with $PZ \cap X^λ = Z$.

Proof. Since $Z$ is a complete $P^λ$-orbit then $(P^λ)^u$ acts trivially on $Z$, which is also a complete $L^λ$-orbit. So $Z$ contains a point $z$ fixed by $B^λ$ and by $(P^λ)^u$. By Lemma 7.2 and Lemma 7.7 we have that $z$ is fixed also by $P^u$ and by $B^−$, whence $PZ = PZ$ is a closed $P$-orbit of $X$. Finally $z \in (PZ)^λ$, hence $PZ \cap X^λ = (Pz)^λ = P^λz = Z$. □

7.10. Definition. Let $X$ and $λ$ be as above. We denote by $C(λ)$ the cone of $\mathcal{F}_L(X)$ such that the relative interior of $C(λ)$ contains $−λ^v$.

7.11. Theorem. Under the assumptions and notations of Proposition 7.8, the variety $X^λ$ is a $P^λ$-toroidal complete embedding of $P^λ/K$. Moreover $Ξ_{B^λ}(X^λ)$ is equal to $C(λ)^\perp(\subseteq Ξ_B(X))$, we have $\Sigma(X^λ) = Σ(X) \cap C(λ)^\perp = Σ(X) \cap λ^\perp$, and for all $D \in \mathcal{D}(P^λ/K)^{B^λ}$ the element $ρ_{X^λ}(D)$ is the restriction of $ρ_X(D^*)$ on $Ξ_{B^λ}(P^λ/K)$.

Proof. We have already observed that $X^λ$ is normal and complete, with open $P^λ$-orbit $P^λ/K$. Let us show it is $P^λ$-toroidal, so suppose that the closure in $X^λ$ of some $D \in \mathcal{D}(P^λ/K)^{B^λ}$ contains a $P^λ$-orbit $Z$. We may suppose that $Z$ is closed, and by Lemma 7.9 we have that $Z$ contains a point fixed by $P^uB^−$, so $PZ$, which is a closed $P$-orbit, is also equal to $BZ$. It follows that the closure of $D^*$ in $X$ contains the $P$-orbit $PZ$, contradicting the fact that $X$ is toroidal.

To show the statement on $Ξ_{B^λ}(X^λ)$, we observe that $λ$ is dominant as a weight of the maximal torus $T$ of $L$. In the same spirit as the proof of Corollary 5.3, consider now the fan of colored cones

$$\mathcal{F} = \{(C \cap V_L(X), \varnothing)\}$$
where $\mathcal{C}$ varies in the set of cones of $\mathcal{F}_L(X)$. It is the fan of an $L$-toroidal embedding $\tilde{X}$ of the open $L$-orbit $X_0$ of $X$, such that the identity on $X_0$ extends to an $L$-equivariant map $\tilde{X} \to X$. Moreover, if a cone of $\mathcal{F}_L(X)$ is entirely contained in $V_L(X)$ then it is also a cone of $\mathcal{F} = \mathcal{F}_L(\tilde{X})$.

The cone $\mathcal{C}(\lambda)$ is also a cone of $\mathcal{F}_L(\tilde{X})$. Let $Y$ (resp. $\tilde{Y}$) be the $L$-orbit of $X$ (resp. $\tilde{X}$) with cone $\mathcal{C}(\lambda)$. By Proposition 5.2, we have $\mathcal{C}_Y = \mathcal{C}_{\tilde{Y}}$ and $\mathcal{D}_Y = \mathcal{D}_{\tilde{Y}} (= \emptyset)$. Then by [Kn91, Sections 3 and 4] we have that $\tilde{Y}$ is mapped to $Y$, and there exist $L$-equivariant neighborhoods $\tilde{X}_\mathcal{V}$ of $\tilde{Y}$ in $\tilde{X}$ and $X_\mathcal{V}$ of $Y$ in $X$ such that $\tilde{X}_\mathcal{V}$ is mapped $L$-isomorphically onto $X_\mathcal{V}$. It follows that the map $\tilde{Y} \to Y$ is an isomorphism.

Consider now the localization $\tilde{X}^{\lambda}$ of $\tilde{X}$ with respect to $\lambda$ as defined in [Kn14, Section 4]. Thanks to [Kn14, Proposition 4.8] we have that an open subset of $\tilde{X}^{\lambda}$ is contained in $\tilde{Y}$. Therefore $\tilde{X}^{\lambda}$ intersects the open set $\tilde{X}_\mathcal{V}$. Then it follows easily from the definition of $\tilde{X}^{\lambda}$ and $X^{\lambda}$ that $\tilde{X}^{\lambda}$ is mapped birationally onto $X^{\lambda}$. The equality $\Xi_{B^\lambda}(X^{\lambda}) = \mathcal{C}(\lambda)^\perp$ is now a consequence of [Kn14, Theorem 4.6]. It also follows that $N_{B^\lambda}(X^{\lambda})$ is identified with the quotient of $N_{B^\lambda}(X)$ by its subspace $\text{span}_Q \mathcal{C}(\lambda)$. The assertion on $\rho_{X^{\lambda}}(D)$ stems from the fact that $D^\ast = \pi^{-1}_\lambda(D) \cap P/H$.

To show that $\Sigma(X^{\lambda}) = \Sigma(X) \cap \mathcal{C}(\lambda)^\perp$, it is enough to show that $V(X^{\lambda}) \subseteq N_{B^\lambda}(X^{\lambda})$ is the union of the projections of the maximal cones of $\mathcal{F}_L(X)$ contained in $V(X)$ and containing $\mathcal{C}(\lambda)$.

A maximal cone of $\mathcal{F}_L(X)$ contained in $V(X)$ is of the form $\mathcal{C}_Z$ for $Z$ a closed $P$-orbit of $X$. As above, we have that $Z$ has an $L$-stable neighborhood $X_Z$ equivariantly isomorphic to an $L$-stable neighborhood $\tilde{X}_Z$ of $\tilde{Z}$, where the latter is a closed $P$-orbit of $\tilde{X}$ mapped isomorphically onto $Z$. If $\mathcal{C}_Y \ni \mathcal{C}(\lambda)$ then $\tilde{X}^{\lambda} \cap \tilde{Z} \neq \emptyset$, and $\tilde{X}_Z$ (resp. $X_Z$) contains the open set $\tilde{X}_\mathcal{V}$ (resp. $X_\mathcal{V}$) considered before. This implies that $\tilde{X}^{\lambda}$ and $X^{\lambda}$ are isomorphic in a neighborhood of $\tilde{X}^{\lambda} \cap \tilde{Z} = X^{\lambda} \cap Z$. Thanks to [Kn14, Theorem 4.6] the projection of $\mathcal{C}_Z$ in $N_{B^\lambda}(X^{\lambda})$ is the maximal cone $\mathcal{C}_{X^{\lambda} \cap Z}$ associated with $X^{\lambda} \cap Z$, which is a closed $L$-orbit of $X^{\lambda}$. Since $P^\lambda$ acts trivially on $Z$, we have that $(P^\lambda)^n$ acts trivially on $X^{\lambda} \cap Z$. Therefore the latter is a closed $P^\lambda$-orbit of $X^{\lambda}$ and $\mathcal{C}_{X^{\lambda} \cap Z} \subseteq V(X^{\lambda})$. Lemma 7.9 implies that all closed $P^\lambda$-orbits of $X^{\lambda}$ are obtained in this way, and the proof of $\Sigma(X^{\lambda}) = \Sigma(X) \cap \mathcal{C}(\lambda)^\perp$ is complete.

It remains to show that the latter intersection is also equal to $\Sigma(X) \cap \lambda^\perp$, and this follows from the fact that any $\sigma \in \Sigma(X)$ is $\leq 0$ on $V(X)$. Then $\sigma \in \mathcal{C}(\lambda)^\perp$ if and only if $\sigma \in \lambda^\perp$. □

7.12. Corollary. We have $\Sigma_L(P/H) = \Sigma(P/H) \cap \text{span}_Q S_P$.

Proof. Let $\lambda$ be a $P$-dominant one-parameter subgroup of $T$ such that $S \cap \lambda^\perp = S_P$, and let $X$ be a $P$-toroidal complete embedding of $P/H$. Then $\Sigma(X^{\lambda}) = \Sigma(X) \cap \lambda^\perp = \Sigma(P/H) \cap \text{span}_Q S_P$ by Theorem 7.11. On the other hand $P^\lambda = L^\lambda = L$ and $B^\lambda = B$, therefore $\Sigma(X^{\lambda}) = \Sigma_L(X^{\lambda})$.

In the proof of Theorem 7.11 we have shown that $X^{\lambda}$ is $L$-equivariantly birational to the localization in $\lambda$ of an $L$-toroidal complete embedding of the open $L$-orbit of $P/H$. By [Kn14, Corollary 4.7] we conclude $\Sigma_L(X^{\lambda}) = \Sigma_L(P/H) \cap \lambda^\perp = \Sigma_L(P/H)$. □
8. Examples

8.1. Example. Consider the case \( L = T \) and \( \dim P^u = 1 \), and denote by \(-\alpha\) the root of \( P^u \). Notice that \( \alpha \) extends to a character of \( P \), trivial on \( P^u \), and that \( H^u \subseteq P^u \). We may also assume that \( H = M \times H^u \), where \( M \subseteq T \). Here the quotient \( P/H = T \times M \) \( P^u/H^u \) is spherical under the action of \( L(= T) \) if and only if \( H^u = P^u \) or \( M^o \) acts nontrivially on \( P^u/H^u \). The last condition amounts to requiring that no multiple of \( \alpha \) is trivial on \( M \). If \( H^u = P^u \) then \( \mathcal{D}(P/H)^B = \emptyset \) \( \Sigma(P/H) \), and \( \Xi_B(P/H) = \mathcal{X}(T/M) \).

Let us discuss the case where \( M^o \) acts nontrivially on \( P^u/H^u \), which implies that \( H^u \) is trivial and \( H = M \). Moreover \( \mathcal{D}(P/H)^B \) has one element \( D \), not stable under \( P^u = U_{-\alpha} \), and \( \alpha \) is an element of \( \Xi_B(P/H) \) where \( \rho(D) \) takes value 1. Any \( B \)-semistable rational function on \( P/H \) with order 0 along \( D \) is constant on the fibers of \( P/H \to T/H \), hence it is the pull-back of a function on \( T/H \). This implies \( \Xi_B(P/H) = \mathcal{X}(T/H) \oplus \mathbb{Z}\alpha \).

To compute \( \Sigma(P/H) \), which amounts to compute \( V(P/H) \), we may suppose that \( H \) is connected and a direct factor of \( T \), so \( T = H \times R \). In this case \( P/H \) is \( P \)-equivariantly isomorphic to \( R \times P^u \) where \( H \) acts only on \( P^u \) by conjugation, \( R \) acts only on itself by left multiplication, and \( P^u \) acts only on itself by left multiplication. We also have \( D = R \times \{ e \} \). Then \( \mathbb{P}^{\dim R} \times \mathbb{P}^1 \) is a \( P \)-toroidal compactification of \( P/H \), where we embed \( R \cong (\mathbb{G}_m)^{\dim R} \) naturally in \( \mathbb{C}^{\dim R} \subset \mathbb{P}^{\dim R} \) and \( P^u \cong \mathbb{C} \) in \( \mathbb{P}^1 \). It is elementary to compute \( V(\mathbb{P}^{\dim R} \times \mathbb{P}^1) \), which is defined by the equation \( \langle -, \alpha \rangle \leq 0 \). We conclude that \( \Sigma(P/H) = \{ \{ \alpha \} \} \).

8.2. Example. We want to discuss the fundamental example where \( P/H \cong \mathbb{C}^n \) with \( n \geq 1 \) and \( P \) is the semidirect product of \( \text{GL}(n) \) with the group of translations of \( \mathbb{C}^n \). The special case where \( n = 1 \) was essentially analyzed already in Example 8.1. It is convenient to work even more in general, so our assumptions are precisely the following: we require that \( L \subseteq H \), with \( (L, L) \cong \text{SL}(n) \) or \( \text{Sp}(n) \) (the latter with \( n \) even), and that \( H^u \) is a normal subgroup of \( P^u \) with \( P/H \cong P^u/H^u = M \) abelian and isomorphic to \( \mathbb{C}^n \) or \( (\mathbb{C}^n)^* \).

The rank of \( P/H \) as a spherical \( L \)-variety is 1, and \( \Xi_B(P/H) \) is generated by the \( B \)-eigenvalue \( \omega \) of a regular function \( f \in \mathbb{C}[M]^{(B)} \) that is actually linear and a highest weight vector of \( M^* \). There is exactly one \( B \)-stable prime divisor of \( P/H \), namely \( \text{div}(f) \). It is \( L \)-stable if \( \dim M = 1 \) and an \( L \)-color if \( \dim M > 1 \). We claim that \( \Sigma(P/H) = \{ \{ \omega \} \} \).

Indeed, we can embed \( P/H \) equivariantly in \( \mathbb{P}^n \), which is a complete \( P \)-toroidal embedding of \( P/H \). Its unique \( P \)-stable prime divisor is \( \mathbb{P}^n \setminus M \) where \( f \) has a pole, whence the claim.

8.3. Example. Let us consider a variation of Example 8.2, where \( P/H \cong \mathbb{C}^m \), but now \( L \) is equal to \((\mathbb{G}_m)^m \) acting on \( \mathbb{C}^m \) by multiplication on each coordinate, and \( P^u \) is the group of translations of \( \mathbb{C}^m \). Here \( \Xi(P/H) = \mathcal{X}(L) \cong \mathbb{Z}^m \), the colors of \( P/H \) are the coordinate hyperplanes of \( \mathbb{C}^m \), and a \( P \)-toroidal compactification of \( P/H \) is \( \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 \).

It follows that \( \Sigma(P/H) = \{ \epsilon_1, \ldots, \epsilon_m \} \) where \( \epsilon_1, \ldots, \epsilon_m \) is the standard basis of \( \mathcal{X}(L) \).

Notice that \( P/H \) and its \( P \)-equivariant embeddings are toric varieties, which raises the question of the relationship between our invariants and the standard theory of toric varieties and their automorphisms. Here we do not develop this theme further.

8.4. Example. For later convenience we discuss another variation of Example 8.2: we assume \( n > 1 \) and we enlarge \( P^u/H^u \), assuming that \( P^u/H^u \cong M \oplus \mathbb{C} \) as \( L \)-modules. Denote again by \( \omega \) and \( f \) resp. the highest weight and a highest weight vector of \( M^* \),
and by $\beta$ the highest weight of the dual of $C$, so that $\Xi_B(P/H) = \text{span}_\mathbb{Z}\{\omega, \beta\}$. In this case $P/H$ has exactly one $L$-stable prime divisor $E = M \oplus \{0\}$, and exactly one $L$-color $F = M' \oplus C$, where $M' = \text{div}(f)$. The variety $\mathbb{P}^n \times \mathbb{P}^1$ is a complete $P$-toroidal embedding of $P/H$, and we conclude that $\Sigma(P/H) = \{\omega, \beta\}$.

Maintain now all the above assumptions except for the fact that $H^u$ is allowed to be non-normal in $P^u$. The descriptions of $\mathcal{D}(P/H)^B$ and of $\Xi_B(P/H)$ remain the same. Moreover, the summand $\mathbb{C}$ is $T$-stable so it is the image in $P^u/H^u$ of a one-dimensional $T$-stable subgroup $U'_\beta$ of $U_\beta$. Since $U'_\beta$ moves $E$, Lemma 4.5 implies that $\beta$ is non-negative on $\rho(D)$ for any $P$-stable prime divisor in any $P$-toroidal embedding of $P/H$. It follows that $\beta$ is a linear combination of spherical roots of $P/H$ with non-negative rational coefficients.

The normalizer $K$ of $H^u$ in $P^u$ is strictly bigger than $H^u$ and stable under conjugation by $L$, so $K/H^u$ in $P^u/H^u$ is $M \oplus \mathbb{C}$ or $M$ or $\mathbb{C}$. It is easy to deduce that it must contain $\mathbb{C}$, so there is a subgroup $R$ of $P^u$ contained in $K$ and containing $H^u$, such that $R$ is normal in $P^u$ with $P^u/R$ abelian and isomorphic to $M$, and $H^u$ is normal in $R$ with $R/H^u$ abelian and isomorphic to $\mathbb{C}$. The natural map $P/H \to P/RL$ is $P$-equivariant, so the spherical root of $P/RL$, which is $\omega$ by Example 8.2, is also a linear combination of spherical roots of $P/H$ with non-negative rational coefficients.

On the other hand $X = M \times \mathbb{P}^1$ is a $P$-toroidal embedding of $P/H$, and thanks to Lemma 4.2 there exists a complete $P$-toroidal embedding $Y$ of $P/H$ mapping surjectively onto a $P$-equivariant completion $\overline{X}$ of $X$. The valuation of the $P$-stable invariant divisor $D = M \times \{\infty\}$ of $X$ is 0 on $\omega$ and $<0$ on $\beta$, so the same holds for a $P$-stable prime divisor $E$ of $Y$ mapping surjectively onto $D$. This implies that $\langle -, \omega \rangle = 0$ defines a face of $V(P/H)$ of dimension 0 or 1, and of dimension 1 if and only if $\omega \in \Sigma(P/H)$.

9. Kac-Moody groups and parabolic subgroups

We denote by $\mathcal{G}$ a minimal Kac-Moody group with set of simple roots $S$, as constructed in [Ku02, Section 7.4] (denoted $\mathcal{G}^{\text{min}}$ in loc.cit.) following Kac-Peterson (see e.g. [PK83]). We denote by $T$ the standard maximal torus of $\mathcal{G}$. If $\alpha$ is a real root, we denote by $\mathcal{U}_\alpha$ the one-parameter root subgroup of $\mathcal{G}$ associated with $\alpha$, by $\mathcal{U}$ (resp. $\mathcal{U}^-$) the subgroup generated by $\mathcal{U}_\alpha$ for all positive (resp. negative) real roots $\alpha$. We denote by $\mathcal{B}$ (resp. $\mathcal{B}^-$) the standard Borel subgroup containing $T$ and $\mathcal{U}$ (resp. $\mathcal{U}^-$). We denote by $\mathcal{W}$ the Weyl group of $\mathcal{G}$.

A subset of simple roots is called of finite type if the corresponding Dynkin subdiagram has connected components only of finite type. A standard parabolic subgroup of $\mathcal{G}$ containing $\mathcal{B}$ or $\mathcal{B}^-$ is called of finite type if the corresponding set of simple roots is of finite type.

For any standard parabolic subgroup $\mathcal{P} \subseteq \mathcal{G}$ of finite type and of negative sign, i.e. containing $\mathcal{B}^-$, we denote the corresponding subset of simple roots by $S_\mathcal{P}$. We denote by $L$ the standard Levi subgroup of $\mathcal{P}$ containing $T$.

We also consider the “completed” Kac-Moody group as constructed in [Ku02, Section 7.1], and we denote it here by $\hat{\mathcal{G}}$. Correspondingly, we denote by $\hat{\mathcal{B}}$ the standard Borel subgroup of $\hat{\mathcal{G}}$, so $\mathcal{B} = \mathcal{B} \cap \mathcal{G}$. We denote by $\hat{\mathcal{P}}^+$ the standard parabolic subgroup of $\hat{\mathcal{G}}$. Correspondingly, we denote by $\hat{\mathcal{L}}$ the standard Levi subgroup of $\hat{\mathcal{P}}^+$ containing $\hat{T}$.
Denote by \( \tilde{\omega} : G \rightarrow G \) an involution of \( G \) as in [Ku02, 7.4.E.1] sending \( U_\alpha \) onto \( U_{-\alpha} \) for all real roots \( \alpha \). We recall that \( \tilde{\mathcal{P}}^+ \) is equipped with the structure of a pro-group, and we fix a subgroup \( N \subseteq \tilde{\mathcal{P}}^+ \) in its defining family. Then \( \tilde{\omega}(\mathcal{P}) \) is mapped surjectively onto \( \tilde{\mathcal{P}}^+/N \) (this can be shown as in the proof of [Ku02, Corollary 7.3.8]).

Therefore we can identify \( \tilde{\mathcal{P}}^+/N \) with the quotient \( \mathcal{P} = \mathcal{P}/\tilde{\omega}^{-1}(N \cap G) \), and equip \( \mathcal{P} \) with the corresponding structure of (finite-dimensional) algebraic group. We identify the subgroups \( L \) and \( T \) of \( \mathcal{P} \) with their (isomorphic) images in \( P \), and we denote by \( B \) the Borel subgroup of \( L \) containing \( T \) and associated with the set of simple roots \( S_\mathcal{P} \) (i.e. \( B = L \cap B \)).

10. Spherical subgroups of Kac-Moody groups

We fix from now on a parabolic subgroup \( \mathcal{P} \) of finite type of \( G \) containing \( B^- \) and a finite-dimensional quotient \( P \) of \( \mathcal{P} \) constructed as in the previous section. We denote by \( H \) a closed subgroup of \( P \) and by \( H \) its inverse image in \( \mathcal{P} \); we identify the quotient \( \mathcal{P}/H \) with \( P/H \).

10.1. Definition. If \( P/H \) is a spherical \( L \)-variety then \( H \) is called a standard spherical subgroup of finite type of \( G \). We define:

1. the lattice \( \Xi(G/H) = \Xi_B(P/H) \),

   whose rank is by definition the rank of \( G/H \) and is denoted by \( \text{rk} G/H \);

2. the rational vector space \( N(G/H) = \text{Hom}_{\mathbb{Z}}(\Xi(G/H), \mathbb{Q}) \);

3. the finite set \( \Delta(G/H) = \mathcal{D}(P/H)^B \cup (S \setminus S_\mathcal{P}) \)

   (considered as an abstract disjoint union), whose elements are the colors of \( G/H \).

4. We say that a simple root \( \alpha \) of \( G \) moves a color \( D \) of \( G/H \) if \( D \) is not stable under the action of \( U_{-\alpha} \) in the case where \( D \in \mathcal{D}(P/H)^B \), and if \( D = \alpha \) in the case where \( D \in S \setminus S_\mathcal{P} \).

5. The map \( \rho \) is defined on \( \mathcal{D}(P/H)^B \) with values in \( N(G/H) \), we extend it to the set \( \Delta(G/H) \). Let \( D \in \Delta(G/H) \setminus \mathcal{D}(P/H)^B \), thus \( D = \alpha \in S \setminus S_\mathcal{P} \). By Lemma 4.4 all elements of \( \mathcal{D}(P/H)^B \) are \( U_{-\alpha} \)-stable, except possibly for one element \( E \), and we define

\[
\rho(D) = \begin{cases} 
\alpha^\vee|\Xi(G/H) - \rho(E) & \text{if } E \text{ exists,} \\
\alpha^\vee|\Xi(G/H) & \text{otherwise.} 
\end{cases}
\]

We denote this extended map also by \( \rho_{G/H} \).

6. We define the valuation cone \( V(G/H) = V(P/H) \) and the set of spherical roots \( \Sigma(G/H) = \Sigma(P/H) \).

7. If \( H \) contains a maximal unipotent subgroup of \( P \), then \( G/H \) is called horospherical.
Unless otherwise stated, from now on we assume that $\mathcal{H}$ is a standard spherical subgroup of finite type.

The above data associated with $\mathcal{G}/\mathcal{H}$ actually do not depend on $\mathcal{P}$. This is not clear a priori, and will be shown in Section 12.

Conjugating $\mathcal{H}$ with an element $p \in \mathcal{P}$ does not change the above objects, since $P/H$ and $P/(\overline{p}H)$, where $\overline{p}$ is the class of $p$ in $P$, are $P$-equivariantly isomorphic. Conjugating $\mathcal{H}$ in $\mathcal{G}$ also behaves well but again requires a detailed treatment, see Section 17.

If $\mathcal{G}$ is finite-dimensional (and thus also $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{H}$) then all of the above coincide with the standard invariants of the spherical homogeneous space $\mathcal{G}/\mathcal{H}$ with respect to the action of the reductive group $\mathcal{G}$ and its Borel subgroup $\mathcal{B}$. This follows e.g. from Theorem 12.1, because in this case one can choose $\mathcal{P} = \mathcal{G}$.

We end this section rewriting in the setting of Kac-Moody groups two results obtained previously.

10.2. Proposition. Any spherical root of $\mathcal{G}/\mathcal{H}$ is a linear combination of simple roots of $\mathcal{G}$ with non-negative rational coefficients.

Proof. This is Theorem 5.14. □

10.3. Proposition. We have $\Sigma(\mathcal{G}/\mathcal{H}) = \emptyset$ if and only if $\mathcal{G}/\mathcal{H}$ is horospherical.

Proof. This is Proposition 5.13. □

10.4. Example. The case where $\mathcal{H}$ is a parabolic subgroup of $\mathcal{G}$ (containing $\mathcal{B}^-$ and contained in $\mathcal{P}$) can be regarded as the most basic one. Here $\Sigma(\mathcal{G}/\mathcal{H}) = \emptyset$ and $\Delta(\mathcal{G}/\mathcal{H})$ can be identified with $S \setminus S_\mathcal{H}$ thanks to the natural identification between $\mathcal{D}(P/H)^B$ and $S_\mathcal{P} \setminus S_\mathcal{H}$. It is straightforward to show that $\mathcal{H}$ has this form (up to conjugation in $\mathcal{P}$) if and only if $\mathcal{G}/\mathcal{H}$ has rank 0, since $P/H$ has rank 0 as a spherical $L$-variety if and only if it is a complete $L$-homogeneous space.

11. Types of simple roots

11.1. Proposition. Let $\alpha \in S \setminus S_\mathcal{P}$. Then at most one element of $\mathcal{D}(P/H)^B$ is not stable under $U_{-\alpha}$. In particular, a simple root $\alpha \in S$ moves at most two colors of $\mathcal{G}/\mathcal{H}$, and it moves two colors if and only if $\alpha \in \Sigma(\mathcal{G}/\mathcal{H})$. In the latter case $\langle \rho(D), \alpha \rangle = 1$ for $D$ equal to either color moved by $\alpha$, and $\langle \rho(E), \alpha \rangle \leq 0$ for any color $E \in \mathcal{D}(P/H)^B$ not moved by $\alpha$.

Proof. The first assertion stems from Lemma 4.4, since in this case $U_{-\alpha}$ is one-dimensional and commutes with $B^\alpha$. The fact that $\alpha \in S$ moves at most two colors follows, in the case where $\alpha \in S \setminus S_\mathcal{P}$. Otherwise $\alpha$ is a simple root of $L$, so it moves no $L$-stable prime divisor of $P/H$ and it moves not more than two $L$-colors of $P/H$ by [Lu97, Section 2.7].

We show the remaining assertions. Assume $\alpha \in S_\mathcal{P}$: in this case $\alpha$ moves two colors if and only if $\alpha \in \Sigma_L(P/H)$; this is equivalent to being in $\Sigma(\mathcal{G}/\mathcal{H})$ thanks to Corollary 7.12. The value on $\alpha$ of any $L$-color of $P/H$ moved by $\alpha$ is computed in [Lu97, Section 3.2]. This concludes the proof for $\alpha \in S_\mathcal{P}$, except for the case where $E$ is not an $L$-color, i.e. if $E$ is $L$-stable. In this case $\rho(E)$ is in $V_L(P/H)$, and $\alpha \in \Sigma_L(P/H)$ assures $\langle \rho(E), \alpha \rangle \leq 0$. 

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It remains to consider $\alpha \in S \setminus S_P$: then $\alpha$ moves two colors if and only if it moves an element $D \in \mathcal{D}(P/H)^{(B)}$.

Suppose that $\alpha \in S \setminus S_P$ moves indeed an element $D \in \mathcal{D}(P/H)^{(B)}$, and let $X$ be a $P$-toroidal complete embedding of $P/H$. Lemma 4.4 implies that $\alpha \in \Xi_B(X)$, and that $\langle \rho(E), \alpha \rangle \leq 0$ for all $P$-stable prime divisor $E$ of $X$. Then $\langle V(X), \alpha \rangle \leq 0$, and $\alpha$ is then a linear combination with non-negative coefficients of spherical roots of $G/H$. From Proposition 10.2 we deduce that $\alpha \in \Sigma(G/H)$.

Suppose now that $\alpha \in S \setminus S_P$ belongs to $\Sigma(G/H)$, and choose a $P$-toroidal complete embedding $X$ of $P/H$. Choose also a one parameter subgroup $\lambda$ adapted to $S' = \{\alpha\}$ and consider the localization $X^\lambda$. In this case $P^\lambda$ has one-dimensional unipotent radical, equal to $U_{-\alpha}$, and $L^\lambda = T$. Moreover, the localization $X^\lambda$ is a $P^\lambda$-toroidal complete embedding of an $L^\lambda$-spherical homogeneous space $P^\lambda/K$. By Theorem 7.11 we have $\Sigma(X^\lambda) = \{\alpha\}$, hence $P^\lambda/K$ occurs in Example 8.1 as the case that has a $B^\lambda$-stable prime divisor not stable under $U_{-\alpha}$. We conclude thanks to Proposition 7.8 that $X$ has a $B$-stable prime divisor not stable under $U_{-\alpha}$, i.e. $\alpha$ moves an element of $\mathcal{D}(P/H)^{(B)}$.

Finally, suppose that $\alpha$ is in $S \setminus S_P$ and is a spherical root, denote by $D$ the color in $\mathcal{D}(P/H)^{(B)}$ moved by $\alpha$ and by $E$ the other color moved by $\alpha$ (so formally $E = \alpha$). The equality $\langle \rho(D), \alpha \rangle = 1$ follows from Lemma 4.4, and it implies $\langle \rho(E), \alpha \rangle = \langle \alpha^\vee, \alpha \rangle - \langle \rho(D), \alpha \rangle = 2 - 1 = 1$. □

The last statement of the above proposition holds in greater generality, namely for all $E \in \Delta(G/H)$ not moved by $\alpha$. We will prove this fact later in Corollary 15.7.

11.2. Definition. We denote $S^a(G/H) = S \cap \Sigma(G/H)$ and $S^{2a}(G/H) = S \cap \frac{1}{2} \Sigma(G/H)$, we denote by $S^b(G/H)$ the set of simple roots of $G$ that move exactly one color of $G/H$ and are not in $S^{2a}(G/H)$, and we denote by $S^p(G/H)$ the set of simple roots of $G$ that move no color of $G/H$. The type of a simple root $\alpha$ is $(a)$, $(2a)$, $(b)$ or $(p)$ according to the set where it belongs. If $\alpha \in S^a(G/H)$ then we denote by $D_\alpha^+$ and $D_\alpha^-$ the two colors moved by $\alpha$ (with $D_\alpha^+ \in \mathcal{D}(P/H)^{(B)}$). If $\alpha$ moves only one color of $G/H$, then we denote this color by $D_\alpha$.

11.3. Definition. We denote by $\mathbf{A}(G/H)$ the set of all colors of $G/H$ moved by simple roots that are also spherical roots. For such a root $\alpha$, we denote by $\mathbf{A}(G/H, \alpha)$ or simply $\mathbf{A}(\alpha)$ the set $\{D_\alpha^+, D_\alpha^-\}$ of the two colors moved by $\alpha$.

11.4. Proposition. The set $S$ is the disjoint union

$$S = S^a(G/H) \cup S^{2a}(G/H) \cup S^b(G/H) \cup S^p(G/H).$$

Any simple root in $S^{2a}(G/H)$ moves exactly one color, and $S^{2a}(G/H) \cup S^p(G/H) \subseteq S_P$.

For any $\alpha \in S$, according to its type we have the following formulae for the colors moved by $\alpha$:

| type | $\rho(D_\alpha^+) + \rho(D_\alpha^-) = \alpha^\vee|_{\Xi(G/H)}$ |
|------|------------------------------------------------|
| $(a)$ | $\rho(D_\alpha) = \frac{1}{2} \alpha^\vee|_{\Xi(G/H)}$ |
| $(2a)$ | $\rho(D_\alpha) = \alpha^\vee|_{\Xi(G/H)}$ |

Proof. The four subsets of $S$ are disjoint by definition, except for $S^{2a}(G/H)$ that may intersect $S^p(G/H)$. Let $\alpha \in S^{2a}(G/H)$. If $\alpha \in S_P$ then $\alpha \in S_P \cap \Sigma_L(P/H)$, so $\alpha$ moves exactly one color of the spherical $L$-variety $P/H$ (see e.g. [Lu97, Section 2.7]).
Suppose now that $\alpha \in S \setminus S_\varphi$, and let $X$ be a complete $P$-toroidal embedding of $P/H$. As in the proof of Proposition 11.1, a localization of $X$ at $S' = \{\alpha\}$ occurs in Example 8.1, but no variety in this example has spherical root $2\alpha$: contradiction.

It follows that $S^{2\alpha}(G/H) \cup S^{\alpha}(G/H)$ is the set of simple roots that move exactly one color each. Therefore $S^{2\alpha}(G/H)$ and $S^{\alpha}(G/H)$ are disjoint, and the union of the four subsets is $S$. Moreover, any root in $S \setminus S_\varphi$ moves at least one color by definition, and we have also proved $S^{2\alpha}(G/H) \cup S^{\alpha}(G/H) \subseteq S_\varphi$.

It remains the formulae for the colors moved by a simple root $\alpha$. If $\alpha \in S_\varphi$ then they are proved in [Lu97, Proposition 3.4]. Otherwise $\alpha$ is in $S \setminus S_\varphi$ and it is either in $S^{\alpha}(G/H)$ or $S^{\alpha}(G/H)$ thanks to the first part of the proof. In this case the corresponding formula holds by definition. 

\[\square\]

12. Independence from $\mathcal{P}$

Suppose that $\mathcal{H}$ is contained in another parabolic subgroup $\mathcal{Q}$ of finite type containing $\mathcal{B}^-$. Then we can apply to $\mathcal{Q}$ the same construction of Section 9, based on the pro-group structure of $\hat{\mathcal{Q}}^\times$. It is easy to check that the defining families of $\hat{\mathcal{P}}^\times$ and of $\hat{\mathcal{Q}}^\times$ have non-empty intersection, which shows that $\mathcal{H}$ is also the inverse image in $\mathcal{Q}$ of an algebraic subgroup of $\mathcal{Q}$, the latter being a finite-dimensional quotient of $\mathcal{Q}$. In other words $\mathcal{Q}$ can play the same role of $\mathcal{P}$ in our constructions. However, choosing $\mathcal{Q}$ instead of $\mathcal{P}$ essentially doesn’t change the objects introduced in Definition 10.1.

12.1. Theorem. Let $\mathcal{Q}$ be a parabolic subgroup of $\mathcal{G}$ of finite type containing $\mathcal{B}^-$ and $\mathcal{H}$. Then $S^\alpha(G/H)$, $\Sigma(G/H)$ and $\Xi(G/H)$ defined with respect to $\mathcal{P}$ are equal to the same objects defined with respect to $\mathcal{Q}$. The same holds for $\Delta(G/H)$, $A(G/H)$ and $\rho_{G/H}$ if we consider the first two simply as “abstract” finite sets, in the sense that $\Delta(G/H)$ defined with respect to $\mathcal{P}$ can be identified with $\Delta(G/H)$ defined with respect to $\mathcal{Q}$ compatibly with the map $\rho_{G/H}$ and respecting the fact that a color is moved by a given simple root.

Proof. The intersection $\mathcal{Q} \cap \mathcal{P}$ is also a parabolic subgroup of $\mathcal{G}$ of finite type containing $\mathcal{B}^-$ and $\mathcal{H}$, therefore we may suppose that $\mathcal{Q}$ is contained in $\mathcal{P}$, and also that the finite-dimensional quotient $\mathcal{Q}$ of $\mathcal{Q}$ as above is the projection of $\mathcal{Q}$ in $\mathcal{P}$. More precisely, the projection $\mathcal{Q}$ is a parabolic subgroup of $\mathcal{P}$, and $P/Q$ is a spherical $L$-variety. But $Q^u$ contains $P^u$, so $P/Q$ is isomorphic to $(P/P^u)/(Q/P^u)$. The quotient $P/P^u$ is isomorphic to $L$ and $Q/P^u$ is a parabolic subgroup, so the rank of $P/Q$ as a spherical $L$-variety is 0 and $P/Q \cong L/R$ where $R$ is a parabolic subgroup of $L$.

Denote by $L_Q$ the Levi subgroup of $\mathcal{Q}$ containing $T$, and by $B_Q$ the Borel subgroup $B \cap L_Q$ of $L_Q$. Consider the natural map $\varphi \colon P/H \to P/Q$. Since $P/Q \cong L/R$ then $\Xi_B(P/H) = \Xi_{B_Q}(Q/H)$, since $Q/H$ is the fiber of $\varphi$ over $eQ$ and $BQ$ is open in $P$. Let $X$ be a complete $P$-toroidal embedding of $P/H$, then $\varphi$ extends to a $P$-equivariant map $X \to P/Q$ also denoted by $\varphi$, and denote by $X_Q$ the fiber $\varphi^{-1}(eQ)$ in $X$.

Since $BQ$ is open in $P$ we have that intersecting with $\varphi^{-1}(eQ)$ induces a bijection between $\mathcal{D}(X_Q)^{B_Q}$ and the subset of $\mathcal{D}(X)^B$ of those elements mapped dominantly onto $P/Q$. On the other hand any $P$-orbit of $X$ intersects $X_Q$ in a $Q$-orbit, and if $Y$ is a closed $Q$-orbit of $X_Q$ then $Q^u \supseteq P^u$ acts trivially on $Y$ and $BY$ is a closed $P$-orbit of $X$. It follows that $X_Q$ is a complete $Q$-toroidal embedding of $Q/H$, and that $\Sigma(P/H) = \Sigma(Q/H)$. 

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In other words the invariants $\Xi(\mathcal{G}/\mathcal{H})$ and $\Sigma(\mathcal{G}/\mathcal{H})$ are equal if defined with $\mathcal{P}$ or with $\mathcal{Q}$. Let us consider now the set of colors. The elements of $\mathcal{D}(P/H)^B$ mapped not dominantly onto $P/Q$ are precisely the inverse images via $\varphi$ of the $B_M$-stable prime divisors of $P/Q$. These are in natural bijection with $S_P \setminus S_Q$. We have thus a natural identification of the sets of colors defined with respect to $\mathcal{P}$ and the set

$$\{D \in \mathcal{D}(P/H)^B \text{ mapped dominantly onto } P/Q\} \cup (S_P \setminus S_Q) \cup (S \setminus S_P)$$

which is identified with the set

$$\mathcal{D}(Q/H)^{B_Q} \cup (S \setminus S_Q)$$

that is the set of colors defined with respect to $\mathcal{Q}$. This identification respects the property of being moved by a simple root, it remains to prove that it is compatible with the map $\rho$. For any $D$ color defined with respect to $\mathcal{P}$ let us denote for a while $D'$ the same color defined with respect to $\mathcal{Q}$. If $D \in \mathcal{D}(Q/H)^{B_Q}$ then $\rho(D) = \rho(D')$ because $D'$ is actually the intersection of $D$ with $Q/H$. If $D \in S \setminus S_Q$ instead, then $\rho(D)$ is either equal to $\alpha \gamma|_{\mathcal{G}(\mathcal{G}/\mathcal{H})}$ for some simple root $\alpha$ moving $D$, or to $\frac{1}{|\alpha|} \alpha \gamma|_{\mathcal{G}(\mathcal{G}/\mathcal{H})}$, or to $\rho(E) - \alpha \gamma|_{\mathcal{G}(\mathcal{G}/\mathcal{H})}$ for some $E \in \mathcal{D}(Q/H)^{B_Q}$ also moved by $\alpha$. Which is the case is determined by $\Sigma(\mathcal{G}/\mathcal{H})$ and which simple roots move which colors, therefore $\rho(D')$ falls in the same case of $D$. This assures $\rho(D) = \rho(D')$ in the first two cases, and also in the third since $E \in \mathcal{D}(Q/H)^{B_Q}$ implies $\rho(E) = \rho(E')$. 

13. Localization at simple roots

13.1. Definition. Let $S' \subseteq S$. A one-parameter subgroup $\lambda$ of $T$ is adapted to $S'$ if $\langle \lambda, \alpha \rangle = 0$ for all $\alpha \in S'$ and $\langle \lambda, \beta \rangle > 0$ for all $\beta \in S \setminus S'$. In this case, if $X$ is a complete $P$-toroidal embedding of $P/H$, then $X^\lambda$ is called a localization of $X$ at $S'$.

For any $S' \subseteq S$ we denote by $\mathcal{G}_{S'}$ the minimal Kac-Moody group with maximal torus $T$ and set of simple roots $S'$; it is naturally a subgroup of $\mathcal{G}$. As for $\mathcal{G}$, denote by $\mathcal{B}_{S'}$ and $\mathcal{B}_{S'}^-$ the standard Borel subgroups of $\mathcal{G}_{S'}$ containing $T$.

13.2. Lemma. Let $S' \subseteq S$ and $\lambda$ a one parameter subgroup of $T$ adapted to $S'$. Then $P^\lambda$ is a finite-dimensional quotient of $\mathcal{P}_{S'}$, where the latter is the parabolic subgroup of $\mathcal{G}_{S'}$ associated with $S' \cap S_P$ and containing $\mathcal{B}_{S'}^-$.

Proof. This follows from Lemma 7.2. 

13.3. Theorem. In the hypotheses of Lemma 13.2, let $X$ be a complete $P$-toroidal embedding of $P/H$. Denoting by $P^\lambda/K$ the open $P^\lambda$-orbit of $X^\lambda$ and by $K$ the inverse image of $K$ in $\mathcal{P}_{S'}$, the subgroup $K$ of $\mathcal{G}_{S'}$ is standard spherical of finite type, and we have:

$$S^p(\mathcal{G}_{S'}/K) = S^p(\mathcal{G}/\mathcal{H}) \cap S'$$
$$\Sigma(\mathcal{G}_{S'}/K) = \Sigma(\mathcal{G}/\mathcal{H}) \cap \text{span}_Q S'$$
$$\Xi(\mathcal{G}_{S'}/K) = \Xi(\mathcal{G}/\mathcal{H}) \cap C(\lambda)^\perp.$$

Moreover, the set $\Delta(\mathcal{G}_{S'}/K)$ can be identified with the subset of $\Delta(\mathcal{G}/\mathcal{H})$ of elements moved by some $\alpha \in S'$, in such a way that the identification respects the property of being moved by a simple root in $S'$ and is compatible with the map $\rho$, in the sense that

$$\rho_{\mathcal{G}_{S'}/K}(D) = \rho_{\mathcal{G}/\mathcal{H}}(D)|_{\Xi(\mathcal{G}_{S'}/K)}$$

for any $D \in \Delta(\mathcal{G}_{S'}/K)$. 

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Proof. Everything follows from Proposition 7.8 and Theorem 7.11.

13.4. Remark. If $S'$ in the above theorem is of finite type, then $\mathcal{G}_{S'}$ and $\mathcal{K}$ are finite-dimensional groups. As already remarked after Definition 10.1, in this case the invariants computed in the theorem above are the usual invariants of the finite-dimensional spherical homogeneous space $\mathcal{G}_{S'}/\mathcal{K}$ under the action of $\mathcal{G}_{S'}$. In particular, if $S'$ is of finite type then the set of spherical roots $\Sigma(\mathcal{G}/\mathcal{H}) \cap \text{span}_Q S'$ is the set of spherical roots of a usual finite-dimensional spherical variety.

13.5. Corollary. If $\sigma \in \Sigma(\mathcal{G}/\mathcal{H})$ is a multiple of a simple root $\alpha$, then $\sigma = \alpha$ or $\sigma = 2\alpha$.

Proof. Let $G$ be a connected reductive group and $X$ a spherical $G$-homogeneous space. Then $X$ admits a geometric quotient $X/Z(\mathcal{G})^\circ$, which is a spherical $(G, G)$-homogeneous space. It has the same spherical roots as $X$, thanks to [Kn91, Theorem 4.4]. The corollary follows now from the well-known classification of spherical $\text{SL}(2)$-varieties, thanks to Remark 13.4 applied to the subset of simple roots $S' = \{\alpha\}$.

14. Localization at spherical roots

14.1. Theorem. Let $Y$ be an $L$-orbit of a $P$-toroidal embedding $X$ of $P/H$. Suppose that $X' = Y$ is $P$-stable and let $P/K$ be the open $P$-orbit of $X'$. Then $K$ is the projection in $P$ of a standard spherical subgroup of finite type $K$ of $G$ contained in $P$, and

\[
S^p(\mathcal{G}/K) = S^p(\mathcal{G}/H), \\
\Sigma(\mathcal{G}/K) = \Sigma(\mathcal{G}/H) \cap C^\perp_Y, \\
\Xi(\mathcal{G}/K) = \Xi(\mathcal{G}/H) \cap C^\perp_Y.
\]

Moreover, the set $A(\mathcal{G}/K)$ can be identified with the subset

\[
\bigcup_{\alpha \in \Sigma(\mathcal{G}/K)} \{ D_{\alpha}^+, D_{\alpha}^- \}
\]

of $A(\mathcal{G}/H)$ in such a way that the identification respects the property of being moved by a simple root in $S$ and is compatible with the map $\rho$, in the sense that

\[\rho_{\mathcal{G}/K}(D) = \rho_{\mathcal{G}/H}(D)|_{\Xi(\mathcal{G}/K)}\]

for any $D \in A(\mathcal{G}/K)$.

Proof. The homogeneous space $P/K$ is $L$-spherical, therefore the inverse image $K$ of $K$ in $P$ is a spherical subgroup of finite type of $G$. Thanks to Proposition 5.5, the variety $X'$ is a $P$-toroidal embedding of $P/K$ with lattice and spherical roots as desired.

Let us prove the assertion on $S^p$. First observe that $S^p(\mathcal{G}/K)$ is equal to the set of simple roots of $L$ moving no $L$-color of $X'$. Then the proof of Theorem 7.11 implies that the open $L$-orbit of $X$ has in $X$ an $L$-stable neighborhood $Z$ that is an $L$-toroidal embedding and contains the open $L$-orbit of $X'$. At this point the assertion on $S^p$ follows from [Kn14, Proposition 6.1].

The last assertion of the theorem follows if we prove an analogue of [Kn14, Lemma 6.2], i.e.:

(1) we have $S^a(\mathcal{G}/H) \cap C^\perp_Y = S^a(\mathcal{G}/K)$;
(2) if \( D \in \mathcal{D}(P/H)^B \) is a color of \( G/H \) moved by \( \alpha \in S^a(G/H) \cap C^+_D \), then \( \overline{D} \cap P/K \) contains a color \( E \) of \( G/K \) moved by \( \alpha \) so that \( \rho_{G/K}(E) \) is the restriction of \( \rho_{G/H}(D) \) to \( \Xi(G/K) \).

Part (1) follows from \( \Sigma(G/K) = \Sigma(G/H) \cap C^+_D \), let us prove part (2). If \( \alpha \in S_P \) then \( D \) is an \( L \)-color of \( X \), and in this case we conclude using [Kn14, Lemma 6.2] applied to an \( L \)-toroidal neighborhood of \( Y \) in \( X \) thanks to Corollary 5.3. We may suppose \( \alpha \in S \setminus S_P \), so \( \alpha \) moves exactly one element \( D \in \mathcal{D}(P/H)^B \) and exactly one element \( E \in \mathcal{D}(P/K)^B \). The two functionals associated with \( D \) and \( E \) are both 1 on \( \alpha(\in \Xi(G/K) \subseteq \Xi(G/H)) \), in particular they are non-zero.

By Lemma 5.4 there is an element \( \tilde{D} \in \mathcal{D}(X)^B \) that contains \( E \) but not \( X' \). Since \( E \) is moved by \( \alpha \) but \( X' \) is not, we deduce that \( \tilde{D} \) is moved by \( \alpha \), so \( \tilde{D} = \overline{D} \). In other words \( E \) is an irreducible component of \( \overline{D} \cap P/K \). Also, since \( \overline{D} \) is the only \( B \)-stable prime divisor of \( P/H \) moved by \( \alpha \), we have that \( \overline{D} \) is the only \( B \)-stable prime divisor of \( X \) containing \( E \) but not \( X' \). As a consequence, if a \( B \)-seminvariant rational function \( f \) on \( X' \) vanishes on \( \overline{D} \) once extended to \( X \), then \( f \) vanishes on \( E \), because in \( X \) its poles are \( B \)-stable prime divisors of \( X \) different from \( \overline{D} \), so they do not contain \( E \). This implies that \( \rho_{G/K}(E) \) is a positive rational multiple of \( \rho_{G/H}(D)|_{\Xi(G/K)} \). Since these two functionals are both 1 on \( \alpha \), they are equal. \( \square \)

14.2. Definition. In the assumptions of Theorem 14.1, the variety \( X' \) is called a localization of \( X \) in \( \Sigma' = \Sigma(G/K) \).

15. Other relations between spherical roots and colors

15.1. Proposition. Two different simple roots \( \alpha \) and \( \beta \) move the same color of \( G/H \) if and only if one of the two following mutually exclusive situations occur.

(1) Both \( \alpha \) and \( \beta \) are spherical roots, and they move the same color.

(2) They satisfy \( \langle \alpha^\vee, \beta \rangle = 0 \), and \( \alpha + \beta \) or \( \frac{1}{2}(\alpha + \beta) \) is in \( \Sigma(G/H) \).

Moreover, if condition (2) holds, then both \( \alpha \) and \( \beta \) are in \( S^b(G/H) \).

Proof. We prove the “only if” part: supposing that condition (1) is not satisfied, we show that (2) holds.

If \( \alpha \) and \( \beta \) move the same color \( D \) then \( D \in \mathcal{D}(P/H)^B \). Suppose that \( \alpha \notin \Sigma(G/H) \), then it is either in \( S^a(G/H) \) or in \( S^b(G/H) \). In either case, and since it moves a color \( D \) in \( \mathcal{D}(P/H)^B \), the root \( \alpha \) is in \( S_P \). Then \( D \) is not \( L \)-stable so it is an \( L \)-color of \( P/H \), and \( \rho(D) \) is a positive rational multiple of \( \alpha^\vee|_{\Xi(G/H)} \) by Proposition 11.4. Suppose now \( \beta \in \Sigma(G/H) \): then \( \langle \rho(D), \beta \rangle \) is equal to 1 by Proposition 11.1 and is equal to a positive multiple of \( \langle \alpha^\vee, \beta \rangle \), which is impossible. Hence \( \beta \notin \Sigma(G/H) \), and for the same reasons as \( \alpha \) we conclude \( \beta \in S_P \). Then the rest follows from [Lu01, Proposition 3.2] applied to the \( L \)-spherical variety \( P/H \).

The “if” part amounts to the fact that if condition (2) is met, then \( \alpha \) and \( \beta \) move the same color. This follows from Remark 13.4 and Theorem 13.3 applied to the subset of simple roots \( S' = \{ \alpha, \beta \} \) together with [Lu01, Proposition 3.2]. We recall that the argument in loc.cit. is essentially the following: since \( \langle \alpha^\vee, \beta \rangle = 0 \), one reduces the problem as in the proof of Corollary 13.5 to the study of spherical \( SL(2) \times SL(2) \)-homogeneous spaces, which are well-known. For more details see also [Kn14, Proof of Proposition 5.4].
The last assertion and the fact that (1) and (2) are mutually exclusive follow in the same way. \qed

The rest of this section is devoted to adapting to our situation the result [Kn14, Proposition 6.5].

15.2. Definition. A set of spherical roots $\Sigma' \subseteq \Sigma(G/H)$ is called a set of neighbors if there exists $v \in V(G/H)$ such that

$$\Sigma' = \{ \sigma \in \Sigma(G/H) \mid v(\sigma) = 0 \}.$$ 

Two spherical roots $\sigma, \tau \in \Sigma(G/H)$ are neighbors if $\{\sigma, \tau\}$ is a set of neighbors.

Notice that any two spherical roots that are multiples of simple roots are neighbors: this is shown exactly as in [Kn14, Lemma 6.4], thanks to Proposition 10.2.

15.3. Remark. Let $X$ be a complete $P$-toroidal embedding of $P/H$. We remark that localizations in $\Sigma'$ of $X$ exist for any set of neighbors $\Sigma' \subseteq \Sigma(G/H)$. Indeed, the set $F = V(X) \cap \{ \sigma = 0 \text{ for all } \sigma \in \Sigma' \}$ is a face of $V(X)$. Then it is enough to consider $X' = \overrightarrow{Y}$ where $Y$ is an $L$-orbit such that $C_Y$ is contained in $F$ and has the same dimension of $F$. Notice that the rank of such localization is $\text{rk} X - \dim F$.

We prepare for the proof of the main result of this section, Theorem 15.6, with two lemmas.

15.4. Lemma. Let $G$ be a connected reductive group and $M$ be a spherical $G$-module of rank 1. If $M$ has dimension 1 then it has exactly one $G$-stable prime divisor and no $G$-color, otherwise it has exactly one $G$-color and no $G$-stable prime divisor.

Proof. Let us denote by $B_G$ a Borel subgroup of $G$. If $M$ has dimension 1 then $\{0\}$ is the only $B_G$-stable prime divisor of $M$ and it is $G$-stable. If $M$ has dimension greater than 1 but only rank 1, then $(G, G)$ acts non-trivially on it. On the other hand $M$ is factorial, so any $B_G$-stable prime divisor $D$ has a global equation. Considering once again that $M$ has rank 1, there can only be one $B_G$-stable prime divisor, which is a $G$-color and precisely the zero set of a highest weight vector of $M^*$.

15.5. Lemma. Suppose that $H$ is not contained in any proper parabolic subgroup of $P$, that $P/H$ has rank 2, exactly one $L$-stable prime divisor $E$ and exactly one $L$-color $F$. If $|\Sigma_L(P/H)| = 0$ then $H$ contains a Levi subgroup of $P$ and $P/H$ is $L$-equivariantly isomorphic to a spherical $L$-module of the form $M \oplus \mathbb{C}$ as in Example 8.4. If $|\Sigma_L(P/H)| = 1$, then either the element of $\Sigma_L(P/H)$ is 0 on $\rho(E)$, or $H$ contains a Levi subgroup of $P$ and $P/H$ is $L$-equivariantly isomorphic to a spherical $L$-module of dimension $> 1$.

Proof. We may assume that $H$ has a Levi subgroup $L_H$ contained in $L$: since $H$ is not contained in any proper parabolic of $P$, it follows that $H^u \subseteq P^u$ and $P/H$ is $L$-equivariantly isomorphic to the smooth affine variety $L \times L_H^u P^u/H^u$.

We proceed to prove the lemma under the assumption that $|\Sigma_L(P/H)| = 0$.

The $L$-variety $P/H$ has an $L$-equivariant map $P/H \to L/L_H$. We identify via taking inverse images the set $\Delta_L(L/L_H)$ with a subset of $\Delta_L(P/H)$. The fiber $P^u/H^u$ over $eL_H$ is an $L_H$-spherical module, of rank equal to the difference $\text{rk} P/H - \text{rk} L/L_H$. Moreover, one can choose $L_H$ and a Borel subgroup of $L_H$ in such a way that intersecting with $P^u/H^u$ induces a bijection between $\Delta_L(P/H) \setminus \Delta_L(L/L_H)$ and $\Delta_{L_H}(P^u/H^u)$, and also...
a bijection between $D(P/H)^L$ and $D(P^n/H^n)^{L_H}$. Since $\Sigma_L(P/H) = \emptyset$ we also have $\Sigma_L(L/L_H) = \emptyset$, i.e. the quotient $L/L_H$ is a torus and has no $L$-colors. It also follows that $\Sigma_L(P^n/H^n) = \emptyset$. To summarize, the spherical module $P^n/H^n$ has no spherical root for the action of $L_H$, it has exactly one $L_H$-stable prime divisor and one $L_H$-color.

If $L/L_H$ has rank 2 or 1 then $P^n/H^n$ has rank 0 (i.e. is trivial) or 1. The first possibility is excluded because the trivial module has no $L_H$-color nor any $L_H$-stable prime divisor, and the possibility of $P^n/H^n$ having rank 1 is excluded by Lemma 15.4. Therefore $L/L_H$ has rank 0, so $L = L_H$ and $P'/H \cong P^n/H^n$. We refer now to the classification of spherical modules (see e.g. [Kn98]). It implies that $P^n/H^n$ under the action of $(L, L)$ is a sum of standard representations (or their duals) of special linear groups and of symplectic groups. Each summand has rank 1; if it has dimension 1 then it has one $L$-stable prime divisor and no $L$-color, otherwise it has one $L$-color and no $L$-stable prime divisors. It follows that $P^n/H^n \cong M \oplus \mathbb{C}$ with $\dim M > 1$.

Let us examine now the case $|\Sigma_L(P/H)| = 1$, and denote by $\sigma$ the element of $\Sigma_L(P/H)$. We discuss separately the cases where $\Sigma_L(L/L_H)$ is empty and where it is not.

If $\Sigma_L(L/L_H)$ is empty then $L/L_H$ is a torus, it has no $L$-stable prime divisor nor any $L$-color. If it has rank 1, then $P_n/H^n$ has rank 1 and this is excluded by Lemma 15.4. So $L = L_H$ and $P'/H \cong P^n/H^n$. In this case the fact that $\Sigma(P^n/H^n) \neq \emptyset$ implies that $(L, L)$ acts non-trivially on $P^n/H^n$, which has then dimension $> 1$.

Finally, suppose that $\Sigma_L(L/L_H)$ is not empty. This implies that $L/L_H$ has some $L$-color, and the colored subspace associated with the $L$-equivariant map $P/H \to L/L_H$ is $(\sigma^+, \emptyset)$. Moreover, $L/L_H$ has no $L$-stable prime divisor, so the pull-back of any $B$-semiinvariant function from $L/L_H$ to $P/H$ has neither a zero nor a pole on $E$. In other words $\langle \rho(E), \sigma \rangle = 0$.

15.6. Theorem. Let $\alpha \in S^n(G/H)$ and $D \in A(\alpha)$. If $\sigma \in \Sigma(G/H)$ is a neighbor of $\alpha$ and $\langle \rho(D), \sigma \rangle > 0$ then $\sigma \in S^n(G/H)$ and $D \in A(\sigma)$.

Proof. Let $X$ be a complete $P$-toroidal embedding of $P/H$. Replacing if necessary $X$ with a suitable localization in $\{\sigma, \alpha\}$, we may assume that $\Sigma(G/H) = \{\sigma, \alpha\}$ and that $X$ has rank 2. In addition, by making $P$ smaller if necessary, we may assume that no proper parabolic of $P$ contains $H$.

Denote by $E$ the other color moved by $\alpha$ and suppose that $D$ or $E$ is also moved by another simple root $\beta$. By Proposition 15.1 the simple root $\beta$ is a spherical root, so $\beta = \sigma$ and we are done. Therefore we assume that $\alpha$ is the only simple root moving $D$ or $E$. Moreover, we may assume that $\sigma$ is not a simple root, otherwise it would be non-positive on $\rho(D)$.

At this point the proof bifurcates in two parts, depending on whether $D$ is in $D(P/H)^B$ or not. In the first part we assume that $D \notin D(P/H)^B$, so $(D =) \alpha \in S \setminus S_P$ and $E \in D(P/H)^B$. Since no other simple roots move $E$, we even have $E \in D(P/H)^L$. Let us also suppose that $\langle \rho(E), \sigma \rangle \leq 0$: if this is not the case we refer to the second part of the proof switching the roles of $D$ and $E$. Our goal now is to show that all these assumptions lead to a contradiction, which will eventually emerge from a detailed analysis of $H^n$.

Recall that $E$ is the unique $L$-stable element of $D(P/H)^B$, and also the unique element such that $\langle \rho(E), \alpha \rangle > 0$ (this value is not only positive, but actually 1). The variety $P/H$ admits a $P$-equivariant morphism to a point, let us denote it by $\psi: P/H \to P/P$, with
associated colored subspace equal to \((N(X), \Delta_L(P/H))\) (recall that \(\Delta_L(P/H)\) is the set of \(L\)-colors of \(P/H\)). This implies that the whole space \(N(X)\) is generated as a convex cone by \(V(X)\) together with \(\rho(E)\) and \(\rho(\Delta_L(P/H))\).

Consider now the element
\[
\gamma = \sigma - \langle \rho(E), \sigma \rangle \alpha.
\]
Since \(\langle \rho(E), \sigma \rangle \leq 0\), the element \(\gamma\) is \(\leq 0\) on \(V(X)\). On \(\rho(E)\) it is zero, indeed:
\[
\langle \rho(E), \gamma \rangle = \langle \rho(E), \sigma \rangle - \langle \rho(E), \sigma \rangle \langle \rho(E), \alpha \rangle = 0,
\]
so there must be an element \(F \in \Delta_L(P/H)\) where \(\gamma\) is positive. Then it is elementary to show that \(N(X)\) is also generated as a convex cone by \(V(X)\) together with \(\rho(E)\) and the only \(\rho(F)\). In other words also \((N(X), \{F\})\) is a \(P\)-equivariant colored subspace. Denote by \(\psi': P/H \to P/K\) the associated morphism: then \(P/K\) is an \(L\)-spherical variety of rank 0, i.e. a complete \(L\)-homogeneous space. So \(K\) is a parabolic subgroup of \(P\) containing \(H\), whence \(K = P\), \(\psi' = \psi\) and \(\Delta_L(P/H) = \{F\}\).

In other words \(P/H\) has exactly one \(B\)-stable prime divisor different from \(E\), namely \(F\). Let us assume in addition that \(\Sigma_L(P/H) = \emptyset\). By Lemma 15.5 the homogeneous space \(P/H\) occurs in Example 8.4 as the case of the spherical \(L\)-module \(M \oplus C\) with \(M\) of dimension \(> 1\). Thanks to Example 8.4, one of the spherical roots of \(P/H\) is the highest weight of \(M^*\), and it takes value 0 on \(\rho(E)\) and 1 on \(\rho(F)\). Only \(\sigma\) can be this spherical root, in particular
\[
\langle \alpha^\vee, \sigma \rangle = \langle \rho(D), \sigma \rangle + \langle \rho(E), \sigma \rangle > 0.
\]

Example 8.4 also shows that there is a normal subgroup \(R\) of \(P^u\) containing \(H^u\), such that \(P^u/R \cong M\). Then there must exist \(\beta \in S \setminus \{\alpha\}\) such that \(U_{-\beta}\) is not in \(R\), otherwise \(R\) would contain all negative root spaces except for \(U_{-\alpha}\) and \(P^u/R\) would be one-dimensional. Then a non-trivial element of the projection of \(U_{-\beta}\) in \(P^u/R\) is a highest weight vector of \(P^u/R\). In other words \(-\beta\) is the highest weight of \(M\), and the lowest weight of \(M\) is an element of \(-\beta + \text{span}_{\mathbb{Z}_{\geq 0}} S_P\). So the highest weight \(\sigma\) of \(M^*\) is in \(-\beta + \text{span}_{\mathbb{Z}_{\geq 0}} S_P\). It follows that \(\langle \alpha^\vee, \sigma \rangle \leq 0\), contradicting (1).

The case \(\Sigma_L(P/H) = \emptyset\) is completed, and we examine the only other possibility for \(\Sigma_L(P/H)\), namely \(\Sigma_L(P/H) = \{\sigma\}\). In this case \(\langle \alpha^\vee, \sigma \rangle \leq 0\) (because \(\sigma \in \text{span}_{\mathbb{Q}_{>0}} S_P\)), so \(\langle \rho(D), \sigma \rangle > 0\) implies \(\langle \rho(E), \sigma \rangle < 0\). This excludes the possibility that \(\langle \rho(E), \sigma \rangle = 0\), so by Lemma 15.5 the group \(H\) contains a Levi subgroup of \(P\) (we may suppose it is \(L\)), and \(P/H \cong P^u/H^u\) is \(L\)-equivariantly isomorphic to a spherical \(L\)-module of dimension \(> 1\). As above, there exists a simple root \(\beta \in S \setminus S_P\) such that \(U_{-\beta}\) is not contained in \(H^u\), otherwise \(P/H\) would have dimension 1. Moreover \(\beta\) is the lowest weight of the \(L\)-module \(P^u/H^u\). The \(T\)-weights of \(P^u/H^u\) are now in the set \(\text{span}_{\mathbb{Z}}\{\beta, S_P\}\), so the latter contains also \(\Xi_B(P/H)\). But \(\text{span}_{\mathbb{Z}}\{\beta, S_P\}\) doesn’t contain \(\alpha\): contradiction.

We proceed now to the final part of the proof, where we assume \(D \in D(P/H)^B\). Here the proof is similar to the one of [Kn14, Proposition 6.5].

If \(\sigma\) is a simple root then it moves \(D\), because otherwise we would have \(\langle \rho(D), \sigma \rangle \leq 0\) thanks to Proposition 11.1, and we are done. Therefore we can assume that \(\sigma\) is not a simple root, so \(\alpha\) is the only simple root of \(G\) moving two colors. Since \(\rho(D)\) is positive both on \(\alpha\) and on \(\sigma\), the the convex cone generated by \(\rho(D)\) together with \(V(P/H)\) is the whole \(N(P/H)\). In other words \(N(P/H)\) is the first component of a \(P\)-equivariant colored
vector space, equal to either \((N(P/H), \emptyset)\) or \((N(P/H), \{D\})\) depending from whether \(D\) is also \(L\)-invariant or not. By Theorem 6.3 this colored vector space is associated with a \(P\)-equivariant morphism \(\varphi: P/H \to P/Q\). Moreover the \(L\)-spherical variety \(P/Q\) has rank 0, i.e. it is a complete \(L\)-homogeneous space and \(Q\) is a parabolic subgroup of \(P\).

The morphism \(\varphi\) is the key tool in our argument, and our first goal is to show that \(S^p(G/Q) = S^p(G/H)\), where \(Q\) is the inverse image of \(Q\) in \(P\).

If \(\alpha \in S_P\), then both \(D\) and \(E\) are \(L\)-colors of \(P/H\). There is no \(L\)-color of \(X\) other than \(D\) mapped dominantly to \(P/Q\), therefore \(E\) is mapped to a color of \(P/Q\) moved by \(\alpha\). In other words \(\alpha \notin S^p(G/Q)\), and up to replacing if necessary \(P\) with \(Q\), we may assume that \(\alpha \notin S_P\).

At this point \(D\) is \(L\)-stable, it is the unique element of \(D(P/H)^B\) moved by \(\alpha\) and the unique element of \(D(P/H)^B\) mapped dominantly onto \(P/Q\), and all simple roots in \(S \setminus S_P\) different from \(\alpha\) do not move any element of \(D(P/H)^B\). We also conclude that \(D\) is the only element of \(D(P/H)^B\) that is \(L\)-stable, because any other \(L\)-stable prime divisor would have to be moved by some simple root in \(S \setminus S_P\). In other words \(D(P/H)^B\) is the disjoint union \(\Delta_L(P/H) \cup \{D\}\), where we denote as usual by \(\Delta_L(P/H)\) the set of \(L\)-colors of \(P/H\).

We claim that no two different simple roots move the same color of \(G/H\). Otherwise such color would be an \(L\)-color of \(P/H\) and thus equal to the inverse image of an \(L\)-color of \(P/Q\) moved by two different simple roots, which is impossible. Adding the consideration that \(\alpha\) is the only one simple root moving two colors of \(G/H\), we have \(|\Delta(G/H)| = |S \setminus S^p(G/H)| + 1\). Notice that \(S \setminus S^p(G/H)\) is also in natural bijection with \((S \setminus S_P) \cup \Delta_L(P/H)\), again because no root in \(S_P\) moves more than one color. By “natural” we mean here a map respecting the fact that a color is moved by a simple root.

Maintaining the same terminology, we turn to the colors of \(G/Q:\) since \(Q\) is contained in \(P\) we have that \(\Delta(G/Q)\) is in natural bijection with \((S \setminus S_P) \cup D(P/Q)^B\).

By taking inverse images along \(\varphi\), the set \(D(P/Q)^B\) is naturally identified with a subset of \(\Delta_L(P/H)\). Any element in the difference \(\Delta_L(P/H) \setminus D(P/Q)^B\) is an \(L\)-color that must be mapped dominantly onto \(P/Q\), so such element cannot exist. Therefore \(D(P/Q)^B\) is naturally in bijection with the whole \(\Delta_L(P/H)\). The conclusion is the desired equality \(S^p(G/Q) = S^p(G/H)\).

Now we consider again a complete \(P\)-toroidal embedding \(X\) of \(P/H\), and by Lemma 4.3 we may suppose that \(\varphi: P/H \to P/Q\) extends to a \(P\)-equivariant map \(X \to P/Q\), denoted again \(\varphi\). Let \(X' \subset X\) be a localization of \(X\) in \(\{\sigma\}\), with open \(P\)-orbit \(P/K\), and let \(K\) be the inverse image of \(K\) in \(P\). By Theorem 14.1, we have \(S^p(G/K) = S^p(G/H)\). This equality together with \(S^a(G/K) = \emptyset\), which is a consequence of \(\Sigma(G/K) = \{\sigma\}\), implies that \(D(P/K)^B\) is in natural bijection with \(D(P/Q)^B\). In other words no \(B\)-stable prime divisor of \(P/K\) is mapped surjectively onto \(P/Q\).

We are ready to prove the final step. The above considerations lead to the fact that \((N(P/K), \emptyset)\) is the colored cone of the restriction \(\varphi|_{P/K}: P/K \to P/Q\), and that \(N(P/K)\) is generated by \(V(P/K)\). But the latter is equal to the half-space defined by the condition \(\langle -, \sigma \rangle \leq 0\). We obtain the desired contradiction, and the proof is complete.
15.7. Corollary. The last statement of Proposition 11.1 holds for any $E \in \Delta(\mathcal{G}/\mathcal{H})$ not moved by $\alpha$.

Proof. This stems from the fact that two spherical roots that are simple roots are neighbors. \qed

15.8. Corollary. Theorem 15.6 holds even if $\sigma$ is not assumed to be a neighbor of $\alpha$.

Proof. Since $V(\mathcal{G}/\mathcal{H})$ has maximal dimension in $N(\mathcal{G}/\mathcal{H})$, the cone $C = \text{span}_{\mathbb{R}_{\geq 0}} \Sigma$ (inside $\Xi = \mathbb{R} \otimes_{\mathbb{Z}} \Xi(\mathcal{G}/\mathcal{H})$) is strictly convex. Consider now the vector subspace $M = \text{span}_{\mathbb{R}} C$ of $\Xi$, and extend $\mathbb{R}$-linearly the functional $\rho(D)$ to $M$. Since $C$ is strictly convex, there exists an affine hyperplane $M'$ of $M$ such that $A = M' \cap C$ is a convex polytope (of maximal dimension in $M'$) whose vertices are the spherical roots of $\mathcal{G}/\mathcal{H}$, up to positive multiples. Denote by $A_+$ the union of all proper faces of $A$ where $\rho(D)$ is non-negative. Since $\alpha$ and $\sigma$ are both positive on $\rho(D)$, the polytope $A_+$ is not empty.

The main observation is that $A_+$ is connected. This holds if $\rho(D)$ is non-negative on all vertices of $A$, because then $A_+$ is the whole boundary $\partial A$, and it is also true if $\rho(D)$ is negative on some vertex of $A$, because in this case $A_+$ is a deformation retract of the intersection $\partial A \cap \{(\rho(D), -) \geq 0\}$, which is itself homeomorphic to a disc of dimension equal to the dimension of $M'$ minus 1. Take now the two vertices of $A$ corresponding resp. to $\sigma$ and $\alpha$. They are contained in $A_+$, hence they are connected by a chain of vertices of $A_+$ such that any two successive elements of the chain are adjacent vertices of $A$, i.e. the corresponding spherical roots are neighbors. The corollary is now a direct consequence of Theorem 15.6. \qed

16. Spherical systems and Luna’s axioms

16.1. Definition. The homogeneous spherical datum of $\mathcal{G}/\mathcal{H}$ is defined as the quintuple $(S^p(\mathcal{G}/\mathcal{H}), \Sigma(\mathcal{G}/\mathcal{H}), \mathbf{A}(\mathcal{G}/\mathcal{H}), \Xi(\mathcal{G}/\mathcal{H}), \rho_{\mathcal{G}/\mathcal{H}|_{A(\mathcal{G}/\mathcal{H})}})$. We also define the spherical system of $\mathcal{G}/\mathcal{H}$ as the triple $(S^p(\mathcal{G}/\mathcal{H}), \Sigma(\mathcal{G}/\mathcal{H}), \mathbf{A}(\mathcal{G}/\mathcal{H}))$, where $\mathbf{A}(\mathcal{G}/\mathcal{H})$ is considered implicitly equipped with the map $\mathbf{A}(\mathcal{G}/\mathcal{H}) \rightarrow \text{Hom}_\mathbb{Z}(\text{span}_\mathbb{Z} \Sigma(\mathcal{G}/\mathcal{H}), \mathbb{Q})$ induced by $\rho$.

16.2. Definition. An element $\sigma \in \mathcal{X}(\mathcal{T})$ is a spherical root for $\mathcal{G}$ if there exists a standard spherical subgroup of finite type $\mathcal{K}$ of $\mathcal{G}$ with $\sigma \in \Sigma(\mathcal{G}/\mathcal{K})$. The set of spherical roots for $\mathcal{G}$ is denoted by $\Sigma(\mathcal{G})$. A spherical root $\sigma$ for $\mathcal{G}$ is compatible with a subset $S^p \subseteq S$ if there exists a standard spherical subgroup of finite type $\mathcal{K}$ of $\mathcal{G}$ with $S^p(\mathcal{G}/\mathcal{K}) = S^p$ and $\sigma \in \Sigma(\mathcal{G}/\mathcal{K})$.

Thanks to Remark 15.3 and Theorem 14.1, a spherical root $\sigma$ is compatible with $S^p$ if and only if there exists a spherical subgroup of finite type $\mathcal{K}$ of $\mathcal{G}$ with $S^p(\mathcal{G}/\mathcal{K}) = S^p$ and $\{\sigma\} = \Sigma(\mathcal{G}/\mathcal{K})$.

16.3. Definition. An (abstract) homogeneous spherical datum$^1$ of $\mathcal{G}$ is a quintuple $(S^p, \Sigma, \mathbf{A}, \Xi, \rho)$, where $S^p$ is a subset of $S$, $\Sigma$ is finite subset of primitive elements of $\Xi$, which is a subgroup of $\mathcal{X}(\mathcal{T})$, $\mathbf{A}$ is a finite set, $\rho$ is a map $\rho: \mathbf{A} \rightarrow \text{Hom}_\mathbb{Z}(\Xi, \mathbb{Z})$, such that the following properties are satisfied:

(A1) For all $\delta \in \mathbf{A}$ and $\sigma \in \Sigma$ we have $\delta(\sigma) \leq 1$, and $\delta(\sigma) = 1$ implies $\gamma \in \Sigma \cap S$.

$^1$See Theorem 16.5.
(A2) For all $\alpha \in \Sigma \cap S$, the set $A(\alpha) = \{ \delta \in A \mid \delta(\alpha) = 1 \}$ contains exactly two elements $\delta^{+}_\alpha$, $\delta^{-}_\alpha$, and they satisfy $\rho(\delta^{+}_\alpha) + \rho(\delta^{-}_\alpha) = \alpha^{\vee}|_\Xi$.

(A3) The set $A$ is the union of all $A(\alpha)$ for $\alpha \in \Sigma \cap S$.

($\Sigma$1) If $2\alpha \in \Sigma \cap 2S$ then $\langle \alpha^{\vee}, \Xi \rangle \subseteq 2\mathbb{Z}$.

($\Sigma$2) If $\alpha$ and $\beta$ are orthogonal simple roots such that $\alpha + \beta$ or $\frac{1}{2}(\alpha + \beta)$ is in $\Sigma$, then $\alpha^{\vee}|_\Xi = \beta^{\vee}|_\Xi$.

(S) Any $\sigma \in \Sigma$ is compatible with $S^p$.

An (abstract) spherical system of $G$ is a triple $(S^p, \Sigma, A)$, where $S^p$ is a subset of $S$, $\Sigma$ is a finite subset of $\mathcal{X}(\mathcal{T})$, $A$ is a finite set implicitly equipped with a map $\rho: A \to \text{Hom}_\mathbb{Z}(\text{span}_\mathbb{Z} \Sigma, \mathbb{Z})$, such that $(S^p, \Sigma, A, \text{span}_\mathbb{Z} \Sigma, \rho)$ is an abstract spherical homogeneous datum.

The above definition is the same given in [Lu01] in the finite-dimensional case. In the next, we give the combinatorial counterpart of the condition that $\mathcal{H} \subseteq \mathcal{P}$ for some $\mathcal{P}$.

16.4. Definition. An abstract homogeneous spherical datum or an abstract spherical system of $G$ is of finite type if there exist a subset $A_1$ of $A$, a subset $S_2$ of $S \setminus (\Sigma \cup S^p)$, and an element

$$\eta \in \text{span}_{\mathbb{Q}_{>0}} (\rho(A_1) \cup \{ \alpha^{\vee}|_\Xi \mid \alpha \in S_2 \})$$

such that $\langle \eta, \sigma \rangle > 0$ for all $\sigma \in \Sigma$, and such that $S_1 \cup S_2 \cup S^p$ is a subset of $S$ of finite type, where

$$(2) \quad S_1 = \{ \alpha \in \Sigma \cap S \mid A(\alpha) \subseteq A_1 \}.$$ 

Finally, in the following theorem we collect our main results and check that the objects associated with $\mathcal{H}$ indeed satisfy Luna’s axioms.

16.5. Theorem. The homogeneous spherical datum (resp. spherical system) of $G/\mathcal{H}$ is an abstract homogeneous spherical datum (resp. spherical system) of finite type of $G$.

Proof. The assertion on the spherical system follows from the one on the homogeneous spherical datum, because the required conditions on values of functionals on $\text{span}_\mathbb{Z} \Sigma(G/\mathcal{H})$ are implied by the same conditions holding on the bigger lattice $\Xi$.

We prove the assertion on the homogeneous spherical datum. Axiom (A1) follows from Corollary 15.8, axiom (A2) stems from Propositions 11.1 and 11.4, axiom (A3) holds by definition of $A(G/\mathcal{H})$, axiom ($\Sigma$1) stems from Proposition 11.4, axiom ($\Sigma$2) from Proposition 15.1, and axiom (S) is obvious.

To show that the homogeneous spherical datum of $G/\mathcal{H}$ is of finite type, consider the $P$-equivariant morphism $\varphi: P/\mathcal{H} \to P/P$. The associated $P$-equivariant colored subspace $(C_\varphi, D_\varphi)$ satisfies $C_\varphi = N(P/\mathcal{H})$, so the latter is generated as a convex cone by $\rho(D(P/\mathcal{H})^B) \cup V(P/\mathcal{H})$. In other words the cone generated by $\rho(D(P/\mathcal{H})^B)$ meets the relative interior of $-V(P/\mathcal{H})$. Now define $A_1 = \rho(D(P/\mathcal{H})^B) \cap A(G/\mathcal{H})$ and $S_2$ to be the set of simple roots moving some element of $\rho(D(P/\mathcal{H})^B) \setminus A(G/\mathcal{H})$.

For each $\alpha \in S_2$, the color $D_\alpha \in \rho(D(P/\mathcal{H})^B) \setminus A(G/\mathcal{H})$ moved by $\alpha$ satisfies $\rho(D_\alpha) \in \mathbb{Q}_{>0}\alpha^{\vee}|_\Xi$. The existence of an $\eta$ with the required property follows, up to replacing if necessary $A_1$ and $S_2$ with smaller subsets (i.e. those of the elements actually involved in the linear combination $\eta$). The set $S_1$ defined as in (2) is a subset of $S^p$, because
A_1 \subseteq D(P/H)^B$ and if $A(\alpha)$ contains only elements in $D(P/H)^B$ then $\alpha$ is in $S_P$. Also $S_2$ is a subset of $S_P$: indeed, if a simple root $\alpha \notin S_P$ moves an element of $D(P/H)^B$ then it is a spherical root and the colors it moves are both in $A(G/H)$. We conclude that $S_1 \cup S_2 \cup S^p$ is a subset of $S_P$, therefore it is of finite type and the proof is complete. 

17. Invariance under conjugation

Recall that a subgroup of a reductive group is called very reductive if it’s not contained in any proper parabolic.

17.1. Lemma. Let $R$ be a very reductive subgroup of $L$, choose a representative $w \in G$ of an element of $W$, and suppose we have $wR \subseteq Q$ for some parabolic subgroup $Q \subseteq G$ of finite type containing $B^-$. Also suppose that $wR$ is a very reductive subgroup of the standard Levi subgroup $M$ of $Q$. Then $M = wL$.

Proof. The group $N = wL \cap Q$ is a parabolic subgroup of the reductive group $wL$. Since $N$ contains the very reductive subgroup $wR$ of $wL$, it follows that $N = wL$. In other words $wL \subseteq Q$, so $wL \subseteq M$. Now $P$ and (therefore) $L$ contain $w^{-1}M$ by symmetry: the lemma follows.

In the next theorem we show that the homogeneous spherical datum is invariant under conjugation of $H$ in $G$. Here, for brevity, we consider again $A(G/H)$ as an “abstract” finite set in the same sense of Theorem 12.1.

17.2. Theorem. Let $g \in G$ and suppose that $gH$ is a standard spherical subgroup of finite type of $G$, thus contained in some parabolic $Q$ of finite type containing $B^-$. Then the spherical homogeneous datum of $G/gH$ (defined with respect to $Q$) is equal to the spherical homogeneous datum of $G/H$ (defined with respect to $P$).

Proof. Denote by $S_Q$ the set of simple roots associated with $Q$, and set $K = gH$. Up to conjugating $H$ inside $P$ and $K$ inside $Q$, and up to replacing $P$ and $Q$ with smaller parabolic subgroups (thanks to Theorem 12.1), we can assume that $H$ has a Levi subgroup $L_H$ that is a very reductive subgroup of $L$, and that $gL_H$ is a very reductive subgroup of the standard Levi subgroup $M$ of $Q$. By the Bruhat decomposition, and since both $P$ and $Q$ contain $B^-$, we can also assume that $g = w \in W$; here we have implicitly fixed a lift of each simple reflection of $W$ in $G$. Lemma 17.1 implies that $gL = M$, and we can also assume that $wB$ is the Borel subgroup $B_M = M \cap B$ of $M$ (recall that $B = L \cap B$).

Write

$w = s_1 \cdots s_m$

where $s_k$ is the simple reflection corresponding to the simple root $\alpha_k$ for all $k \in \{1, \ldots, m\}$, and $m$ is the length of $w$. We proceed by induction on $m$, the base of the induction being the obvious case $m = 0$.

We can assume that $\alpha_m \in S \setminus S_P$, otherwise we replace $H$ with $gH$, which is also in $P$, and conclude by induction. The group $\tilde{Q} = w^{-1}Q$ contains $H$ and $L$, and either $\alpha_m$ or $-\alpha_m$ is a root of $\tilde{Q}$. But if $-\alpha_m$ is a root of $\tilde{Q}$ then $w(\alpha_m)$ is negative. This contradicts the fact that the expression (3) is reduced, so $\alpha_m$ is a root of $\tilde{Q}$ and $w(\alpha_m)$ is a root of $Q$ (in particular it is negative).
Consider now the set of simple roots $S_P \cup \{\alpha_m\}$. If the corresponding Dynkin subdiagram is not of finite type, then $w_0^M w$ sends infinitely many positive roots to negative ones, where $w_0^M$ is the longest element of the Weyl group of $M$. Indeed, $w_0^M w(S_P) = -S_Q$ and $w_0^M w(\alpha_m)$ is negative. This is impossible, therefore $S_P \cup \{\alpha_m\}$ is of finite type. We denote by $P_0$ the corresponding parabolic subgroup containing $B^-$ of $G$, and by $L_0$ the standard Levi subgroup of $P_0$.

Let $w_0^L$ and $w_0^{L_0}$ be the longest elements of the Weyl groups of resp. $L$ and $L_0$, and consider the subgroup $H_1 = w_0^{L_0} w_0^L H$. It is conjugated to $K$ by the element $w w_0^L w_0^{L_0}$, and $G/H_1$ has the same homogeneous spherical datum of $G/H$ by Theorem 12.1 since $H$ and $H_1$ are conjugated in $P_0$.

Denote by $P_1$ the parabolic subgroup of finite type of $G$ contained in $P_0$, associated with the set of simple roots $-w_0^{L_0}(S_P)$ and containing $B^-$. We have that $L_1 = w_0^{L_0} L$ is the standard Levi subgroup of $P_1$. Notice that the projection of $P \cap \tilde{Q}$ in $L_0$ is equal to $L$, therefore $L$ contains the projection of $H$ in $L_0$. It follows that $H_1$ is contained in $P_1$, and has Levi subgroup equal to a very reductive subgroup of $L_1$. In addition, the Borel subgroup $B_1 = L_1 \cap B$ of $L_1$ is sent onto $B_M$ by $w w_0^L w_0^L$.

Finally, we observe that $w w_0^L w_0^{L_0}$ is shorter than $w$, because it sends all negative roots of $L_1$ to the negative roots of $M$ (they are in bijection with the negative roots of $L$, that are sent to negative roots by $w$), it sends all roots of $P_1^u \cap L_0$ to negative roots (they are in bijection with the roots of $P_1^u \cap L_0$, which contain $-\alpha_m$ and are all sent to positive roots by $w$), and $w_0^L w_0^{L_0}$ permutes the roots of $P_0^u$. We can apply induction to the groups $H_1$ and $K$, and the proof is complete. □

17.3. Definition. Let $H$ be a subgroup of $G$. If it is conjugated to a standard spherical subgroup of finite type, then $H$ is called a spherical subgroup of finite type of $G$. In this case, the homogeneous spherical datum of $G/H$ is defined as the homogeneous spherical datum of $G/K$, where $K$ is any standard spherical subgroup of $G$ of finite type conjugated to $K$.

18. Further developments

It is natural to pose the question whether homogeneous spherical data are as relevant here as in the classical theory of spherical varieties.

First of all, in this setting they are not entirely combinatorial objects, because axiom (S) involves the notion of a spherical root compatible with $S_P$. To make this into a purely combinatorial condition one could classify all spherical subgroups of finite type with only one spherical root. It has been done in the finite-dimensional setting (see e.g. [Br89]), and completing this task for $G$ infinite-dimensional seems to be attainable (see Example 19.4), as it has also been communicated to us by Friedrich Knop. It will be included in a forthcoming paper.

Then we come to the question whether the classification, i.e. the Luna conjecture, can be extended to our situation.

18.1. Conjecture. Mapping $H \subseteq G$ to the homogeneous spherical datum of $G/H$ induces a bijection between the set of conjugacy classes of spherical subgroups of finite type of $G$ and homogeneous spherical data of finite type of $G$. 

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Evidence supporting this generalization is still limited to a direct verification for a few cases of \( \mathcal{G} \), such as \( \mathcal{G} \) of affine type with Cartan matrix of size 2 or 3.

19. Examples

19.1. Example. Let \( \mathcal{P} = B^- \) and \( \mathcal{H} = T(U_-, U_-) \). Then \( P/H \) is isomorphic to an affine space, where \( L = T \) acts with characters equal to the simple roots, and \( P^u \) acts as the full group of translations. In other words it occurs in Example 8.3. Therefore \( \Sigma(\mathcal{G}/\mathcal{H}) = S \), the lattice \( \Xi(\mathcal{G}/\mathcal{H}) \) is equal to the root lattice, the set \( \mathcal{D}(P/H)^B \) contains \( n = |S| \) elements \( D_1^+, \ldots, D_n^+ \), each moved by exactly one simple root \( \alpha_i \), and
\[
\langle \rho(D_i^+), \alpha_j \rangle = \delta_{ij}.
\]
The whole set \( \Delta(\mathcal{G}/\mathcal{H}) \) is equal to \( \{D_0^-, D_0^+ - D_1^-, D_1^+, D_1^-\} \), with
\[
\langle \rho(D_i^-), \alpha_j \rangle = \langle \alpha_i^\vee, \alpha_j \rangle - \delta_{ij}.
\]
The corresponding Luna diagram for \( \mathcal{G} \) e.g. of type \( A_1^{(1)} \) is the following:

\[
\begin{array}{ll}
\alpha_0 & \alpha_1 \\
D_0^- & 1 \quad -1 \\
D_0^+ & 1 \quad -1 \\
D_1^- & 0 \quad 1 \\
D_1^+ & -2 \quad 1 \\
\end{array}
\]
and the Luna diagram is

This \( \mathcal{H} \) is also contained in \( \mathcal{P} \) where \( S_P = \{\alpha_0\} \). The Levi subgroup \( L \) has type \( A_1 \), and \( H \cap L \) is a maximal torus of \( L \). If we call \( N \) the normalizer of \( H \cap L \) in \( L \), then \( H^u \) is stable under conjugation by \( N \), so \( K = NH^u \) is a (disconnected) subgroup of \( P \) with \( K^\circ = H \). Call \( K \) the inverse image of \( K \) in \( \mathcal{P} \); then \( \Sigma(\mathcal{G}/K) = \{2\alpha_0, \alpha_1\} \) (with \( \Sigma_L(P/K) = \{2\alpha_0\} \)) and \( \Delta(\mathcal{G}/K) = \{D_0, D_1^+, D_1^-\} \) with pairing
\[
\begin{array}{ll}
\langle \rho(-,-) \rangle & 2\alpha_0 \quad \alpha_1 \\
D_0 & 2 \quad -1 \\
D_1^+ & 0 \quad 1 \\
D_1^- & -4 \quad 1 \\
\end{array}
\]
and Luna diagram

We refer to [BP11c, Section 2] for details on Luna diagrams.

19.2. Example. With \( \mathcal{G} \) of type \( A_1^{(1)} \), let us call \( \alpha_0 \) and \( \alpha_1 \) the two simple roots. We modify a bit Example 19.1 and consider \( \mathcal{H} \) to be the subgroup generated by \( T \) and \( U_- \alpha \) for \( \alpha \) any positive real root different from \( \alpha_0 \). In other words \( H \) contains \( T \), and contains \( U_- \beta \) if and only if \( \beta \) is a positive root different from \( \alpha_0 \) and from \( \delta = \alpha_0 + \alpha_1 \). We have again \( \Sigma(\mathcal{G}/\mathcal{H}) = S = \{\alpha_0, \alpha_1\} \) and \( \Delta(\mathcal{G}/\mathcal{H}) = \{D_0^-, D_0^+, D_1^+, D_1^-\} \), but the pairing between colors and spherical roots is the following:
\[
\begin{array}{ll}
\langle \rho(-,-) \rangle & 2\alpha_0 \quad \alpha_1 \\
D_0^- & 2 \quad -1 \\
D_1^+ & 0 \quad 1 \\
D_1^- & -4 \quad 1 \\
\end{array}
\]
and Luna diagram

This \( \mathcal{H} \) is also contained in \( \mathcal{P} \) where \( S_P = \{\alpha_0\} \). The Levi subgroup \( L \) has type \( A_1 \), and \( H \cap L \) is a maximal torus of \( L \). If we call \( N \) the normalizer of \( H \cap L \) in \( L \), then \( H^u \) is stable under conjugation by \( N \), so \( K = NH^u \) is a (disconnected) subgroup of \( P \) with \( K^\circ = H \). Call \( K \) the inverse image of \( K \) in \( \mathcal{P} \); then \( \Sigma(\mathcal{G}/K) = \{2\alpha_0, \alpha_1\} \) (with \( \Sigma_L(P/K) = \{2\alpha_0\} \)) and \( \Delta(\mathcal{G}/K) = \{D_0, D_1^+, D_1^-\} \) with pairing
\[
\begin{array}{ll}
\langle \rho(-,-) \rangle & 2\alpha_0 \quad \alpha_1 \\
D_0 & 2 \quad -1 \\
D_1^+ & 0 \quad 1 \\
D_1^- & -4 \quad 1 \\
\end{array}
\]
and Luna diagram

We refer to [BP11c, Section 2] for details on Luna diagrams.
19.3. Example. We give an example of the situation of Theorem 17.2. Let $\mathcal{G}$ be of type $A_2^{(1)}$ with simple roots $\alpha_0, \alpha_1$ and $\alpha_2$. Define $\Sigma = \{\alpha_0, \alpha_1 + \alpha_2\}$ with $\langle \rho(D_0^+), \alpha_1 + \alpha_2 \rangle = 0$; the corresponding Luna diagram is the following:

```
\[ 0 \circ \quad 1 \quad 2 \]
```

Setting $\Xi = \text{span}_\mathbb{Z} \Sigma$, these invariants are obtained with $\mathcal{H}$ in $\mathcal{P}$, where $S_P = \{\alpha_1\}$ and $\mathcal{H}$ is generated by $T$ together with $U_{\alpha_1}$ and $U_{\beta}$ for $\beta$ any negative real root different from $-\alpha_2$ and $-\alpha_1 - \alpha_2$. They are also obtained with $\mathcal{K}$ and $\mathcal{Q}$, if we define them as $\mathcal{H}$ and $\mathcal{P}$ but exchanging $\alpha_1$ and $\alpha_2$. Indeed $\mathcal{H}$ and $\mathcal{K}$ are conjugated in $\mathcal{G}$.

19.4. Example. We give an example of a spherical root that is “new”, i.e. not appearing in the classical theory of finite-dimensional spherical varieties. For this, we choose $\mathcal{H}$ in an infinite-dimensional $\mathcal{G}$ so that $\Sigma_L(P/H) = \emptyset \neq \Sigma(P/H)$ and $\text{rk} \mathcal{G}/\mathcal{H} = 1$. These properties are achieved imposing $H \supseteq L$, so that $P/H = P^u/H^u$, and choosing $H^u$ so that $P^u/H^u$ is an irreducible spherical $L$-module of rank 1. Notice that this is only possible if $L$ has semisimple type $A_n$ or $C_n$ for some $n$, thanks to the classification of spherical modules.

For example, let us consider $\mathcal{G}$ of type $G_2^{(1)}$, with simple roots $\alpha_0, \alpha_1, \alpha_2$ numbered as follows:

```
\[ 0 \quad 1 \quad 2 \]
```

Set $S_P = \{\alpha_0, \alpha_1\}$, consider the corresponding parabolic subgroup $\mathcal{P}$, its Levi subgroup $L$, and set $\mathcal{H} = LH^u$ where $H^u$ is the group generated by $U_\alpha$ where $\alpha$ is any positive real root with $S_P^H^u$ at least 2. Then $P/H = P^u/H^u$ is isomorphic to $\mathbb{C}^3$ and equal to the defining representation of SL(3) if one chooses an appropriate isomorphism $(L, L) \cong \text{SL}(3)$. The group $P^u$ acts on $P/H$ as the full group of translations. As in Example 8.2, we have $\Sigma(P/H) = \{\omega\}$ where $\omega$ is the highest weight of $(\mathbb{C}^3)^*$. The highest weight of $P^u/H^u$ is $-\alpha_2$, therefore

$$\omega = -s_0s_1s_0(\alpha_2) = \alpha_0 + \alpha_1 + \alpha_2$$

is the spherical root of $\mathcal{G}/\mathcal{H}$.

19.5. Example. Not all abstract spherical systems are of finite type. For example, the following Luna diagram

```
\[ \bigcirc \bigcirc \bigcirc \]
```

corresponds to the spherical system $(S^p, \Sigma, \mathcal{A})$ given by $S^p = \emptyset$, $\Sigma = S = \{\alpha_0, \alpha_1\}$, $\mathcal{A} = \{D_0^+, D_0^-, D_1^+, D_1^-\}$ with the values

| $\langle \rho(-), - \rangle$ | $\alpha_0$ | $\alpha_1$ |
|--------------------------|----------|----------|
| $D_0^+$                 | 1        | -1       |
| $D_0^-$                 | 1        | -1       |
| $D_1^+$                 | -1       | 1        |
| $D_1^-$                 | -1       | 1        |

For this system the sets $A_1$, $S_2$ and the linear combination $\eta$ needed in Definition 16.4 don’t exist, because no linear combination (with non-negative coefficients) of the rows of the above table has only positive entries.

\[ ^2 \text{Recall that, for } S' \subseteq S, \text{ the } S'^{-}\text{-height of a positive root is defined as the usual height but where only simple roots not in } S' \text{ count.} \]
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