VSR symmetries in the DKP algebra: the interplay between Dirac and Elko spinor fields

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VSR symmetries are naturally incorporated in the DKP algebra, departing from the Lorentz symmetries in particular on the spin-0 and the spin-1 DKP sectors. The Elko (dark) spinor fields play an essential role, unravelling hidden symmetries on the bosonic DKP fields, which manifest the intrinsic VSR symmetries of Elko fields.

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I. INTRODUCTION

Elko (dark) spinor fields (dual-helicity eigenspinors of the charge conjugation operator) are spin one-half matter fields, whose unexpected properties make it to be a good candidate to describe dark matter. Recent attempts to scrutinize Elko fields structure, incorporating the very special relativity (VSR) paradigm and dark matter as well have been reported. In this context, dynamical constraints are turned into a dark spinor mass generation mechanism in the VSR framework through a natural coupling to the kink solution of a $\lambda\phi^4$ field theory, with the encrypted symmetries maintained by the VSR. These developments regard still exotic couplings among Elko spinor fields and scalar field topological solutions. Furthermore, some attempts to detect them in the LHC have been proposed, as well as promising cosmological applications.
The aim of this article is to evince hidden symmetries and further structures underlying bosonic fields described by the spacetime DKP algebra formed by the tensor product of two Clifford algebras that comprise Dirac and Elko spinor fields. First, the unexpected behavior of discrete symmetries acting on the so-called DKP spinor fields\(^1\) reveals that the Lorentz group is not the symmetry group associated to the DKP algebra in the present case. Instead, merely the subgroups HOM(2) and SIM(2), related to VSR symmetries act in the DKP algebras, when it is generated by Elko spinor fields. The incorporation of, for instance, the parity and the time reversal operators widens the VSR subgroups to the full Lorentz group. It must be stressed that the HOM(2) is both sufficient and necessary to encompass the negative outcome of the Michelson-Morley experiment. Although the well known DKP algebra obtained by the tensor product between Dirac spinor fields has Lorentz group symmetries, it is usually constructed via the standard tensor product. Therefore, it is difficult to probe some different character concerning the DKP algebra when it is generated by the tensor product of two Dirac spinor fields. The Clifford algebra, however, is a superalgebra, allowing, then, the use of the graded tensor product. The VSR symmetries regarding Elko spinor fields\(^2,^3\) are here shown to manifest further in the DKP algebra, when correctly the graded tensor product is employed.

This paper is organized as follows: in the next Section we review the basic features of the DKP algebra, while in Section III it is studied the DKP algebra as a product of two Clifford algebras. In Section IV, Elko spinor fields are briefly revisited. The class preserving spinor fields are discussed in Section V, and the conditions that the Elko and Dirac spinor fields must satisfy for the correct definition of the DKP fields, in terms of the tensor product of spinor fields, are obtained. This is accomplished in order to assure that the spinor fields classes, under the Lounesto spinor field classification, are preserved. The investigation of Elko and Dirac spinors as building blocks of the DKP algebra is performed in Section VI, revealing how the Elko unusual features reflects at the bosonic level of the symmetries of the DKP algebra. In the last Section we conclude, pointing out some important remarks.

**II. THE DKP ALGEBRA**

The study of elementary particles by means of classical wave theory depends on the charged (electric charge or moment dipolar charge) or uncharged aspect of the particle under consideration.

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\(^1\) They are not legitimate spinor fields, and their features are concisely exposed in Section III.
Following Dirac, Kemmer and Schrödinger, a particle of mass \( m \) is said to be a meson if it is described by the wave equation

\[
(\eta^{\mu\nu} \beta_\mu \partial_\nu + im) \psi = 0,
\]

where \( \beta_\mu \) satisfy the commutation rules proposed in [7]:

\[
\beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu = \eta_{\nu\rho} \beta_\mu + \eta_{\nu\mu} \beta_\rho.
\]

By denoting the set \( \{ \gamma_\mu \} \) of Dirac matrices in Eq.(9), the set \( \{ 1, \gamma_\mu, \gamma_\mu \gamma_\nu, \gamma_\mu \gamma_\nu \gamma_\rho, \gamma_\mu \gamma_1 \gamma_2 \gamma_3 \} \) \((\mu, \nu, \rho = 0, 1, 2, 3, \text{ and } \mu < \nu < \rho)\) is hereon considered to be a basis for \( M(4, \mathbb{C}) \), such that \( \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \). The Clifford product is denoted by juxtaposition.

The Dirac equation is usually written in a form that can be immediately compared with the meson wave equation, by means of the prescription \( \beta^\mu \rightarrow \gamma^\mu \) and \( \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \). The massless DKP theory can not be obtained as a zero mass limit of the massive DKP case. Some authors consider the Harish-Chandra Lagrangian density for the massless DKP theory in Minkowski spacetime given by [8]

\[
L = i \bar{\psi} \tau \beta^\mu \partial_\mu \psi - i \partial_\mu \bar{\psi} \beta^\mu \tau \psi - \bar{\psi} \tau \psi,
\]

where \( \tau \) is a singular idempotent matrix satisfying \( \beta^\mu \tau + \tau \beta^\mu = \beta^\mu \), when one chooses a representation in which \( \beta_0^\dagger = \beta_0, \beta_i^\dagger = -\beta_i \) and \( \tau^\dagger = \tau \). Besides, the angular momentum can be written as \( S^{\mu\nu} = \beta^\mu \beta^\nu - \beta^\nu \beta^\mu \) [7].

By denoting hereon a \( n \times n \) matrix \( B \) by \( \mathbb{B}_n \), the spin 0 sector is realized when a representation of the DKP algebra is used, where \( \tau = \{ 0 \} \oplus \mathbb{I}_4 \), reproducing the massless Klein-Gordon-Fock field [9]. The scalar sector of the massless DKP theory associated to the 5-dimensional representation of massless DKP algebra provides

\[
\psi = (\varphi, A^\mu)^\dagger,
\]

where \( \varphi \) and \( A^\mu \) are scalar and 4-vector under Lorentz transformations, respectively.

The spin-1 DKP field \( \psi \) can be provided by choosing \( \tau = 0_4 \oplus \mathbb{I}_6 \), therefore the DKP field is thus a 10-component column vector

\[
\psi = (\psi^\mu, F^{\mu\nu})^\dagger,
\]

where \( \psi^\mu \) and \( F^{\mu\nu} \) are respectively a 4-vector and some antisymmetric tensor in Minkowski spacetime. It can be related to the tensor product of two Dirac spinor fields, and endows their symmetries.
under charge conjugation \( C \), parity \( P \), and time reversal \( T \). Although in [10] the \( A^\mu \) and the \( F^{\mu\nu} \) are respectively interpreted as the electromagnetic potential and field strength, the formalism is thoroughly general. It means that for the standard DKP field, the charge conjugation, parity, and time reversal are involutions, when acting on the scalar, the 4-vector, and the antisymmetric tensor of the DKP field components in (4, 5). In particular, \( P^2 = I = (CPT)^2 \).

In the next section the DKP fields (4) and (5) are shown to be expressed as the tensor product of two algebraic spinor fields, namely, elements of a minimal left ideal in the Clifford-Dirac algebra \( \mathcal{C}_{1,3}(\mathbb{C}) \). Usually such fields are written as the tensor product of two Dirac spinor fields. Hereupon we shall depart from this assumption and evince the unexpected role of introducing Elko spinor fields as building blocks composing the tensor product of spinor fields. We shall prove that the interplay between Dirac and Elko spinor fields manifest VSR symmetries in the spin-0 and spin-1 DKP algebra sectors with the aid of the graded tensor product.

**III. THE DKP ALGEBRA AS THE TENSOR PRODUCT OF TWO CLIFFORD ALGEBRAS**

In order to realize the relationship between Clifford algebras and DKP algebras [11, 12], let us emphasize that given a quadratic space \((V, g)\), which can be thought as being the tangent space on a Lorentzian manifold at a point on the manifold, consider \( \mathcal{A} \) an associative algebra with unity \( 1_\mathcal{A} \) and let \( \gamma \) be the linear application \( \gamma : V \to \mathcal{A} \). The pair \((\mathcal{A}, \gamma)\) is a Clifford algebra \( \mathcal{C}(V, g) \) for the quadratic space \((V, g)\) when \( \mathcal{A} \) is generated as an algebra by \( \{\gamma(v) | v \in V\} \) and \( \{a 1_\mathcal{A} | a \in \mathbb{R}\} \), satisfying \( \gamma(v)\gamma(u) + \gamma(u)\gamma(v) = 2g(v, u)1_\mathcal{A} \), for all \( v, u \in V \). In an implicit notation, \( vu + uv = 2g(v, u) \).

In the context of the Clifford algebras \( \mathcal{C}(V, g) \), spinors are well known to be elements of a minimal ideal of \( \mathcal{C}(V, g) \) [13]. The composed spinor describing an element of the DKP algebra is shown to be, thus, an element of \( \mathcal{C}(V, g) \otimes \mathcal{C}(V, g) \). This tensor product is not graded: the product \((u \otimes v)(u' \otimes v')\) equals \( uu' \otimes vv' \), even if \( u' \) and \( v \) are odd. By considering the mapping \( \delta : V \to \mathcal{C}(V, g) \otimes \mathcal{C}(V, g) \) defined by

\[
\delta(v) = \frac{1}{2}(v \otimes 1 + 1 \otimes v) \tag{6}
\]

(here “1” denotes the identity in \( \mathcal{C}(V, g) \)), it can be shown that [12]

\[
\delta(u)\delta(v)\delta(u) = \frac{1}{8}(uvu \otimes 1 + u^2 \otimes v + (uv + vu) \otimes u + u \otimes (uv + vu) + v \otimes u^2 + 1 \otimes uvu). \tag{7}
\]
Using the Clifford relation $uv + vu = 2g(u,v)$, which implies that $uvu = 2g(u,v)u - g(u,u)v$, one gets

$$\delta(u)\delta(v)\delta(u) = g(u,v)\delta(u).$$

By the universal property of the DKP (meson) algebra $B(V,g)$, the map $\delta$ extends to an algebra monomorphism $\Delta : B(V,g) \rightarrow \mathcal{C}\ell(V,g) \otimes \mathcal{C}\ell(V,g)$, mapping every $\psi \in B(V,g)$ in $\frac{1}{2}(\psi \otimes 1 + 1 \otimes \psi)$. It is worth mentioning that, for every $v \in V$, $\Delta(2v^2 - g(v,v)) = v \otimes v$. The DKP (or meson) algebra $B(V,g)$, is defined as the subalgebra generated in $\mathcal{C}\ell(V,g) \otimes \mathcal{C}\ell(V,g)$ by all elements $\delta(v)$.

**IV. ELKO SPINOR FIELDS**

In this Section some the formal properties of Elko spinor fields are briefly revised. An Elko can be expressed in general as

$$\lambda(p) = \left( i\Phi \phi^*(p) \phi(p) \right),$$  \hspace{1cm} (8)

where $\phi(p)$ denotes a left-handed Weyl spinor, and given the rotation generators denoted by $J$, the Wigner’s spin-$1/2$ time reversal operator $\Phi$ satisfies $\Phi J \Phi^{-1} = -J^*$. Hereon, as in [1], the Weyl representation of $\gamma^\mu$ is used, i.e.,

$$\gamma^0 = \gamma^0 = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix}, \quad -\gamma^k = \gamma^k = \begin{pmatrix} 0_2 & -\sigma_k \\ \sigma_k & 0_2 \end{pmatrix}, \quad \gamma^5 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i\gamma^{0123} = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix},$$  \hspace{1cm} (9)

where $\sigma_i$ are the Pauli matrices. Elko spinor fields are eigenspinors of the charge conjugation operator $C$, namely, $C\lambda(p) = \pm \lambda(p)$. The plus [minus] sign regards self-conjugate, [anti self-conjugate] spinor fields, denotes by $\lambda^S(p) [\lambda^A(p)]$. Explicitly, the complete form of Elko spinor fields can be found by solving the equation of helicity $(\sigma \cdot \hat{p})\phi^\pm(0) = \pm \phi^\pm(0)$ in the rest frame and subsequently performing a boost, in order to recover the result for any $p$ [1]. Elko spinor fields are given by

$$\lambda^{S/A}_{\{\pm,\pm\}}(p) = \sqrt{\frac{E + m}{2m}} \left(1 \mp \frac{p}{E + m}\right) \lambda^{S/A}_{\{\pm,\pm\}}(0),$$

where $\lambda^{S/A}_{\{\pm,\pm\}}(0) = \begin{pmatrix} \pm i\Phi[\phi^\pm(0)]^* \\ \phi^\pm(0) \end{pmatrix}$. Since $\Phi[\phi^\pm(0)]^*$ and $\phi^\pm(0)$ present opposite helicities, by choosing $i\Phi = \sigma_2$, as in [1], it is possible to write

$$\lambda^S_{\{\mp,\mp\}}(0) = \begin{pmatrix} \sigma_2\phi^*(0) \\ \phi(0) \end{pmatrix}.$$  \hspace{1cm} (11)
Furthermore, we can conveniently express the other three types of Elko on Eq. (10) by fixing one of them, as:

\[
\lambda_S^{\{\pm, \mp\}}(0) = \gamma^5 \lambda_A^{\{\pm, \mp\}}(0), \quad \lambda_A^{\{\pm, \mp\}}(0) = \pm i \gamma^0 \lambda^S_{\{\pm, \mp\}}(0), \quad (12)
\]

By defining the antisymmetric tensor \( \varepsilon^{\{-+, +\}}_{\{+, -\}} = -1 \), Elko spinors are straightforwardly shown to obey the following pair of algebraic equations [1]

\[
\begin{align*}
(i \gamma^\mu \nabla_\mu) \delta_\alpha^\beta + m \varepsilon_\alpha^\beta \lambda^S_\beta(x) &= 0, \quad (13) \\
(i \gamma^\mu \nabla_\mu) \delta_\alpha^\beta - m \varepsilon_\alpha^\beta \lambda^A_\beta(x) &= 0. \quad (14)
\end{align*}
\]

Indeed, Eqs. (12) imply that

\[
\gamma^5 \left( (i \gamma^\mu \nabla_\mu) \delta_\alpha^\beta + m \varepsilon_\alpha^\beta \right) \lambda^S_\beta(x) = \left( (i \gamma^\mu \nabla_\mu) \delta_\alpha^\beta - m \varepsilon_\alpha^\beta \right) (\gamma^5 \lambda^S_\beta(x))
\]

\[
= \left( (i \gamma^\mu \nabla_\mu) \delta_\alpha^\beta - m \varepsilon_\alpha^\beta \right) \lambda^A_\beta(x) = 0.
\]

Hence, the equations (13) and (14) are equivalent, and it is possible to write down the above constraints as

\[
\gamma^\mu \nabla_\mu \lambda^S_{\{\pm, \mp\}} = \mp im \lambda^S_{\{\pm, \mp\}}. \quad (15)
\]

There are several interesting and unusual aspects concerning Elko theory. Most importantly for our purposes is to remember the Elko behavior under discrete symmetries. Note that the covariant part of (15) is the same of the Dirac equation. Hence, some similarities between the covariance condition for Elko and Dirac spinors are expected. In fact, as \( \lambda \) (hereon we suppress the labels \( S \) and \( A \), for simplicity) belongs to a linear representation of the Lorentz group, or, more precisely, a subgroup of the Lorentz group, it shall exist \( P(\Lambda) \) such that \( \lambda' = P(\Lambda)\lambda \). As the Lorentz transformed Levi-Civita symbol is the original one and the mass term is the eigenvalue of a Lorentz Casimir invariant, the covariance condition resulting from the transformed equation (15) is given by \( P \gamma^\mu P^{-1} \Lambda^\nu = \gamma^\nu \). For the particular case of parity transformation, \( \Lambda^\nu = P(\Lambda) \lambda \). As we shall see, this property is crucial to unveil the VSR hidden symmetry of DKP algebra.
V. CLASS PRESERVING SPINOR FIELDS

This Section is devoted to concisely revisit the Lounesto spinor field classification and to introduce the class-preserving Elko and Dirac spinor fields, in order to express DKP field in terms of the tensor product between Elko and Dirac spinor fields — and subsequently reveal the VSR symmetries encoded in the DKP algebra. Consider \( \{ e_\mu \} \) is a section of the frame bundle \( P_{\text{SO}(1,3)}(M) \) and \( \{ \theta^\mu \} \) be respectively its associated dual basis. Classical spinor fields carrying a \( D(1/2,0) \oplus D(0,1/2) \) representation of \( \text{SL}(2,\mathbb{C}) \) are sections of the vector bundle \( P_{\text{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4 \), where \( \rho \) stands for the \( D(1/2,0) \oplus D(0,1/2) \) representation of \( \text{SL}(2,\mathbb{C}) \) in the complex \( 4 \times 4 \) matrices. Within this formalism, it is possible to make use of the multivector structure and write down a mother spinor field given by

\[
\psi \sim (\sigma + J + iS - i\gamma_{0123}K + \gamma_{0123}\omega)\eta,
\]

where \( \eta \) is a spinor and \( \sigma, J, S, K, \) and \( w \) are the bilinear covariants provided by:

\[
\begin{align*}
\sigma &= \psi^\dagger \gamma_0 \psi, \\
J &= J_\mu \theta^\mu = \psi^\dagger \gamma_0 \gamma_\mu \psi \theta^\mu, \\
S &= S_{\mu\nu} \theta^\mu \theta^\nu = \frac{1}{2} \psi^\dagger \gamma_0 \gamma_\mu \gamma_\nu \psi \theta^\mu \wedge \theta^\nu, \\
K &= K_\mu \theta^\mu = \psi^\dagger \gamma_0 i\gamma_{0123} \gamma_\mu \psi \theta^\mu, \\
\omega &= -\psi^\dagger \gamma_0 \gamma_{0123} \psi.
\end{align*}
\]

Different spinors arise from different bilinear covariants combinations, but not every combination give rise to a different spinor, since the bilinear covariants satisfy the Fierz identities

\[
\begin{align*}
J^2 &= \omega^2 + \sigma^2, & K^2 + J^2 &= 0, & J \cdot K &= 0, & J \wedge K &= -(\omega + \sigma \gamma_{0123})S.
\end{align*}
\]

It turns out that the Lounesto spinor field classification provides just the following spinor field classes [13], where in the first three classes clearly \( J, S, K \neq 0 \):

1) \( \sigma \neq 0, \quad \omega \neq 0 \)

2) \( \sigma \neq 0, \quad \omega = 0 \)

3) \( \sigma = 0, \quad \omega \neq 0 \)

4) \( \sigma = 0 = \omega, \quad K \neq 0, \quad S \neq 0 \)

5) \( \sigma = 0 = \omega, \quad K = 0, \quad S \neq 0 \)

6) \( \sigma = 0 = \omega, \quad K \neq 0, \quad S = 0 \)

Lounesto spinor field classes are well known not preserved by sum of spinor fields. For instance, the sum of two Weyl spinor fields (type-(6)) are usually Dirac spinor fields under this classification.

In Section VI we shall analyze the space generated by tensor product of Elko linear spaces and Dirac linear spaces. Thus, in what follows we aim to find the vector space structure underlying Elko and Dirac spinor fields spaces. This is necessary to guarantee that the linear combination of spinor fields in the same class under Lounesto classification is in the same spinor field class, still.
To accomplish this for Elko spinor fields, we need to find a linear structure such that every spinor field must be a type-(5) spinor field or the zero vector. Therefore an eigenspinor of the charge conjugation operator restricts to those corresponding to real eigenvalues. A way to find this structure is consider the eigenspaces of the charge conjugation operator associated to positive or negative eigenvalues, namely, the self-conjugate class preserving and anti self-conjugate class preserving spaces, respectively. They are 4-dimensional real linear spaces:

\[
\mathcal{E}^S = \text{span}_\mathbb{R}(\{(-i, 0, 0, 1)^T, (-1, 0, 0, i)^T, (0, i, 1, 0)^T, (0, 1, i, 0)^T\}),
\]

\[
\mathcal{E}^A = \text{span}_\mathbb{R}(\{(i, 0, 0, 1)^T, (1, 0, 0, i)^T, (0, -i, 1, 0)^T, (0, -1, i, 0)^T\}).
\]

By explicit calculations we can show that merely a few combinations mixing elements of both spaces are of type-(5), but no one corresponds to Elko.

For Dirac spinor fields cases (types-(1), (2), (3)), at least one of \(\sigma\) or \(\omega\) must be non null. Given two Dirac spinors

\[
\psi(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x))^T, \quad \zeta(x) = (\zeta_1(x), \zeta_2(x), \zeta_3(x), \zeta_4(x))^T,
\]

where \(\psi_\mu(x)\) and \(\zeta_\mu(x)\) are complex functions, the conditions for class-preserving sum are to choose at least one condition among the eleven ones below (for type-(1) Dirac spinor fields):

\[
\pm \psi_1^* \psi_3 \pm \psi_2^* \psi_4 \pm \psi_3^* \psi_1 \pm \psi_4^* \psi_2 \neq \pm \zeta_1^* \zeta_3 \pm \zeta_2^* \zeta_4 \pm \zeta_3^* \zeta_1 \pm \zeta_4^* \zeta_2,
\]

\[
\pm \psi_1^* \psi_4 \pm \psi_2^* \psi_3 \pm \psi_3^* \psi_2 \pm \psi_4^* \psi_1 \neq \pm \zeta_1^* \zeta_4 \pm \zeta_2^* \zeta_3 \pm \zeta_3^* \zeta_4 \pm \zeta_4^* \zeta_1,
\]

\[
||\psi_1||^2 \pm ||\psi_2||^2 \mp ||\psi_3||^2 - ||\psi_4||^2 \neq ||\zeta_1||^2 \pm ||\zeta_2||^2 \mp ||\zeta_2||^2 - ||\zeta_4||^2.
\]

For type-(2) Dirac spinors, one of the conditions in (21), namely

\[
\psi_1^* \psi_3 + \psi_2^* \psi_4 + \psi_3^* \psi_1 + \psi_4^* \psi_2 \neq \zeta_1^* \zeta_3 + \zeta_2^* \zeta_4 + \zeta_3^* \zeta_1 + \zeta_4^* \zeta_2,
\]

can be relaxed, as well as one of the conditions in (23), namely

\[
||\psi_1||^2 + ||\psi_2||^2 - ||\psi_3||^2 - ||\psi_4||^2 \neq ||\zeta_1||^2 + ||\zeta_2||^2 - ||\zeta_2||^2 - ||\zeta_4||^2,
\]

for type-(3) Dirac spinor fields.

These relations are also satisfied by two 4-dimensional complementary class preserving vector spaces:

\[
\mathcal{D}^+ = \text{span}_\mathbb{R}(\{(1, 0, 1, 0)^T, (i, 0, i, 0)^T, (0, i, 0, 1)^T, (0, -1, 0, i)^T\})
\]

\[
\mathcal{D}^- = \text{span}_\mathbb{R}(\{(-1, 0, 1, 0)^T, (-i, 0, i, 0)^T, (0, -i, 0, 1)^T, (0, 1, 0, i)^T\}).
\]
These spaces contain all Dirac spinor fields types. The subspaces generated by \( \{(1,0,1,0)^\top, (i,0,i,0)^\top, (-1,0,1,0)^\top, (-i,0,i,0)^\top\} \) are subspaces of type-(2) spinor fields and the generated by \( \{(0,i,0,1)^\top, (0,-1,0,i)^\top, (0,-i,0,1)^\top, (0,1,0,i)^\top\} \) are subspaces of type-(3) spinor fields. At this point we should notice that for Dirac fields the constraints \( K \neq 0 \) or \( S \neq 0 \) is not necessary because of completeness of Lounesto classification, there are no classes with \( \sigma \) or \( \omega \) non null and \( K \) or \( S \) null. This fact is result of the implications \( K = 0 \Rightarrow \sigma = 0 = \omega \) and \( S = 0 \Rightarrow \sigma = 0 = \omega \).

VI. ELKO AND DIRAC SPINOR FIELDS AS AN UNIQUE OBJECT: THE DKP ALGEBRA

For Elko spinor fields space the charge conjugation, the parity, and the time reversal symmetries satisfy the following expressions [1]:

\[
\mathcal{C}^2 = -\mathcal{P}^2 = -\mathcal{T}^2 = \mathbb{I},
\]

\[
[\mathcal{C}, \mathcal{P}] = [\mathcal{C}, \mathcal{T}] = [\mathcal{P}, \mathcal{T}] = \mathbb{O}, \quad (\mathcal{C}\mathcal{P})^2 = (\mathcal{C}\mathcal{T})^2 = (\mathcal{P}\mathcal{T})^2 = (\mathcal{C}\mathcal{P}\mathcal{T})^2 = -\mathbb{I}.
\]

Besides, for Dirac spinor fields they read

\[
\mathcal{C}^2 = \mathcal{P}^2 = \mathcal{T}^2 = \mathbb{I},
\]

\[
\{\mathcal{C}, \mathcal{P}\} = \{\mathcal{C}, \mathcal{T}\} = \{\mathcal{P}, \mathcal{T}\} = \mathbb{O}, \quad (\mathcal{C}\mathcal{P})^2 = (\mathcal{C}\mathcal{T})^2 = (\mathcal{P}\mathcal{T})^2 = (\mathcal{C}\mathcal{P}\mathcal{T})^2 = \mathbb{I}.
\]

Hereon we shall denote by \( \mathcal{D}[\mathcal{E}] \) any of the spaces (24, 25) \([19, 20]\). Operators acting on the space \( S_1 \otimes S_2, S_i \in \{\mathcal{D}, \mathcal{E}\}, i = 1, 2 \), are constructed as follows:

\[
\mathcal{O} = \frac{1}{\sqrt{2}} (\mathcal{O}_{S_1} \otimes \mathbb{I}_{S_2} + \mathbb{I}_{S_1} \otimes \mathcal{O}_{S_2}),
\]

where \( \mathcal{O}_{S_i} \in \text{End}(S_i) \) denote general operators in the endomorphism group acting on the Dirac or Elko class preserving spinor fields space. We choose the factor \( \frac{1}{\sqrt{2}} \) above in order that the discrete symmetries act on the DKP algebra (to be identified to the tensor product of spinor fields) as involutions or anti-involutions, or nilpotent operators, in some cases. Hereupon our notation is devoid of the subindexes \( (\cdot)_{S_i}, (\cdot)_{\mathcal{D}}, \) or \( (\cdot)_{\mathcal{E}} \), which shall be implicitly realizable.

Given \( \chi_i \in S_i \), the actions of the operator \( \mathcal{O} \in \text{End}(S_1 \otimes S_2) \) and its square are respectively given by

\[
\mathcal{O}(\chi_1 \otimes \chi_2) = \frac{1}{\sqrt{2}} (\mathcal{O}_{\chi_1} \otimes \chi_2 + \chi_1 \otimes \mathcal{O}_{\chi_2}),
\]

\[
\mathcal{O}^2(\chi_1 \otimes \chi_2) = \frac{1}{2} \mathcal{O}^2 \chi_1 \otimes \chi_2 + \mathcal{O}_{\chi_1} \otimes \mathcal{O}_{\chi_2} + \frac{1}{2} \chi_1 \otimes \mathcal{O}^2 \chi_2,
\]
and in general it is straightforward to show that
\[ O^n(\chi_1 \otimes \chi_2) = 2^{-n} \sum_{p=0}^{n} \binom{n}{p} O^{n-p} \chi_1 \otimes O^p \chi_2. \] (33)

The commutator and the anti-commutator of two operators \( O_i = \frac{1}{\sqrt{2}} (O_i \otimes \mathbb{I} + \mathbb{I} \otimes O_i) \), acting on the space \( S_1 \otimes S_2 \) are thus provided by
\[
\begin{align*}
[O_1, O_2] &= \frac{1}{2} ([O_1, O_2] \otimes \mathbb{I} + \mathbb{I} \otimes [O_1, O_2]), \\
\{O_1, O_2\} &= \frac{1}{2} ([O_1, O_2] \otimes \mathbb{I} + 2O_1 \otimes O_2 + 2O_2 \otimes O_1 + \mathbb{I} \otimes \{O_1, O_2\}).
\end{align*}
\] (34)

As Dirac and Elko spinor fields are elements of minimal left ideals in the Dirac-Clifford algebra \( \mathcal{C}\ell_{1,3}(\mathbb{C}) \), they are not \textit{a priori} even elements under the graded involution — that defines the \( \mathbb{Z}_2 \)-grading, turning \( \mathcal{C}\ell_{1,3}(\mathbb{C}) \) into a superalgebra. Therefore, the graded tensor product of (graded) algebras \( A \) and \( B \) is now defined. Here it suffices to consider \( A \simeq B \simeq \mathcal{C}\ell_{1,3}(\mathbb{C}) \). The alternating tensor product \( A \hat{\otimes} B \) between those algebras is defined as the algebra generated by the product \( a \hat{\otimes} b, a \in A, b \in B \), defined by
\[ (a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{\deg(b_1)\deg(a_2)}a_1a_2 \hat{\otimes} b_1b_2. \] (36)

Hence the analogue of Eq. (31) in this context provided by the graded tensor product reads
\[
\begin{align*}
O(\chi_1 \hat{\otimes} \chi_2) &= \frac{1}{\sqrt{2}} \left[ (O \hat{\otimes} \mathbb{I})(\chi_1 \hat{\otimes} \chi_2) + (\mathbb{I} \hat{\otimes} O)(\chi_1 \hat{\otimes} \chi_2) \right] \\
&= \frac{1}{\sqrt{2}} [O\chi_1 \hat{\otimes} \chi_2 + \chi_1 \hat{\otimes} O\chi_2].
\end{align*}
\] (37)

Heretofore there is no difference between this case and the former one presented in (31). Notwithstanding, it does not hold in general, when higher order compositions of \( O \) are regarded. Indeed
\[
\begin{align*}
O^2(\chi_1 \hat{\otimes} \chi_2) &= \frac{1}{\sqrt{2}} O(\chi_1 \hat{\otimes} \chi_2 + \chi_1 \hat{\otimes} O\chi_2) \\
&= \frac{1}{2} [O^2\chi_1 \hat{\otimes} \chi_2 + \chi_1 \hat{\otimes} O\chi_2 + (-1)^{\deg O}\chi_1 \hat{\otimes} O^2\chi_2 + \chi_1 \hat{\otimes} O\chi_2].
\end{align*}
\] (38)

The parity, charge conjugation, and time-reversal operators are respectively defined in the Dirac algebra as
\[
P = e^{i\phi \gamma^0} R, \quad C = i\gamma^2 K, \quad T = i\gamma^1 \gamma^3 C,
\] (39)
where in spherical coordinates the action of \( R \) is to make \( \{\theta \mapsto \pi - \theta, \phi \mapsto \phi + \pi, r \mapsto r\} \). The operator \( K \) is the complex conjugation operator. At this point it is worth to call attention to
the fact that although spinors are assumed to be in $\mathcal{C}\ell_{1,3}(\mathbb{C})$, the operators are defined on the representation space and need to be written in the $\mathcal{C}\ell_{1,3}(\mathbb{C})$. Here, solely $K$ need to be adapted.

It acts on an element $\chi$ of the algebraic spinor space as $K\chi = \gamma^{013}\chi^*(\gamma^{013})^{-1}$. Consequently, $\mathcal{C}, \mathcal{P},$ and $\mathcal{T}$ are all odd operators under the graded involution, and therefore

$$2\mathcal{C}^2 = C^2 \otimes I + I \otimes C^2,$$

$$2\mathcal{P}^2 = \mathcal{P}^2 \otimes I + I \otimes \mathcal{P}^2,$$

$$2\mathcal{T}^2 = \mathcal{T}^2 \otimes I + I \otimes \mathcal{T}^2,$$

hold for the alternating tensor product.

In order to analyze the behavior of class preserving spaces obtained above under actions of combinations of $\mathcal{C}, \mathcal{P}, \mathcal{T}$ operators, on the next sections we will compute explicitly the action of these operators on all combinations for the tensor product between Dirac and Elko class-preserving spaces.

**A. The tensor product between Dirac spinor fields**

Starting with the parity operator $\mathcal{P} \in \text{End}(\mathcal{D} \otimes \mathcal{D})$, Eq. (32) implies that

$$\mathcal{P}^2(\psi_1 \otimes \psi_2) = \frac{1}{2}(\mathcal{P}^2\psi_1 \otimes \psi_2 + 2\mathcal{P}\psi_1 \otimes \mathcal{P}\psi_2 + \psi_1 \otimes \mathcal{P}^2\psi_2)
= \psi_1 \otimes \psi_2 + \mathcal{P}\psi_1 \otimes \mathcal{P}\psi_2,$$

(43)

since $\mathcal{P}^2 = I_D$, as indicated in Eqs. (29). Therefore, for all $n = 1, 2, \ldots$, a periodicity mod 2 for the operator $\mathcal{P} \in \text{End}(\mathcal{D} \otimes \mathcal{D})$, given by

$$\mathcal{P}^{2n-1} = 2^{n-3/2}(\mathcal{P} \otimes I + I \otimes \mathcal{P}), \quad \mathcal{P}^{2n} = 2^{n-1}(I \otimes I + \mathcal{P} \otimes \mathcal{P}),$$

(44)

holds. Moreover, the anti-commutators and commutators of the charge conjugation and the parity operators in the DKP algebra are respectively given by

$$\{\mathcal{C}, \mathcal{P}\} = \mathcal{C} \otimes \mathcal{P} + \mathcal{P} \otimes \mathcal{C},$$

$$[\mathcal{C}, \mathcal{P}] = \frac{1}{2}([\mathcal{C}, \mathcal{P}] \otimes I + I \otimes [\mathcal{C}, \mathcal{P}]).$$

(45)

Analogously, when one takes the time reversal operator the following relations are obtained:

$$\{\mathcal{C}, \mathcal{T}\} = \mathcal{C} \otimes \mathcal{T} + \mathcal{T} \otimes \mathcal{C},$$

$$[\mathcal{C}, \mathcal{T}] = \frac{1}{2}([\mathcal{C}, \mathcal{T}] \otimes I + I \otimes [\mathcal{C}, \mathcal{T}]).$$

(46)
For the alternating tensor product, according to Eqs. (29) and (40-42) the analogue of Eq.(32) are given by

\[ C^2 = P^2 = T^2 = I \otimes I, \] (47)

One can see from the equation above that the discrete symmetries act on the DKP algebra — generated by the (alternate) tensor product of two Dirac spinor fields — in the same way as they act on the Dirac spinor fields in Eq.(28). Heretofore this algebra can be used to emulate the classical construction for the DKP algebra in Section II in this paper, reproducing all results there contained, as the discrete symmetries are the same. Hereon we shall show that this is not the case when at least one of the Dirac spinor fields in the tensor product, constituting a DKP field, is substituted by an Elko spinor field.

B. The tensor product between Dirac and Elko spinor fields

For the parity operator \( \mathcal{P} \in \text{End}(D \otimes E) \) defined by (30) it follows that

\[ \mathcal{P}^2 (\psi \otimes \lambda) = \frac{1}{2} (P^2 \psi \otimes \lambda + 2P \psi \otimes P \lambda + \psi \otimes P^2 \lambda) \]
\[ = P \psi \otimes P \lambda \] (48)

since \( P^2_D = I_D \) , and \( P^2_E = -I_E \) as evinced by Eqs.(27) and (29). Furthermore, for all \( n = 0, 1, 2 \ldots \), a periodicity mod 4 for the operator \( \mathcal{P} \in \text{End}(D \otimes E) \)

\[ \mathcal{P}^{4n} = (-1)^{n} I \otimes I, \] (49)
\[ \mathcal{P}^{4n \pm 1} = (-1)^{n} 2^{-1/2} (P \otimes I \pm I \otimes P), \] (50)
\[ \mathcal{P}^{4n+2} = (-1)^{n} P \otimes P, \] (51)

holds as well as the following relations:

\[ \{ C, \mathcal{P} \} = \frac{1}{2} I \otimes \{ C, \mathcal{P} \} + (C \otimes \mathcal{P} + \mathcal{P} \otimes C), \] (52)
\[ [ C, \mathcal{P} ] = \frac{1}{2} [ C, \mathcal{P} ] \otimes I. \] (53)

As the previous case, exactly the same relations provided by Eqs.(46) hold if one substitutes the parity operator for the time reversal operator. Based on the properties of Elko and Dirac spinor fields, under the parity operator, respectively provided by Eqs.(26) and (28), we show now that the DKP field, formed by those two spinor fields, reflects both properties in a very unique way.
Indeed, by taking into account the alternating tensor product we have

\[ C^2 = \mathbb{I} \otimes \mathbb{I}, \quad (54) \]
\[ \mathcal{P}^2 = \mathcal{P}^2 = \mathbb{O} \otimes \mathbb{O}, \quad (55) \]

Those results evince that the DKP field constructed by a tensor product between Dirac and Elko spinor fields is not invariant under the full Lorentz group, as the parity operator is nilpotent. It shows how the VSR symmetry in the Elko field manifests in the DKP field constituted by it. The same shall be in the case analyzed in the next Subsection.

C. The tensor product between Elko and Dirac spinor fields

In this case \( \mathcal{P} \in \text{End}(\mathcal{E} \otimes \mathcal{D}) \) and, analogously, we can show that Eqs. (52, 53) hold. What makes this case to differ from the one in the last Subsection is merely the commutator and the anti-commutator, provided by

\[ \{ C, \mathcal{P} \} = \frac{1}{2} \{ C, \mathcal{P} \} \otimes \mathbb{I} + C \otimes \mathcal{P} + \mathcal{P} \otimes C, \]
\[ [C, \mathcal{P}] = \frac{1}{2} \mathbb{I} \otimes [C, \mathcal{P}], \quad (56) \]

instead of (52, 53). Analogously, exactly the same relations hold if we substitute the parity operator for the time-reversal operator. The expressions for \( C^2, \mathcal{P}^2, \mathcal{P}^2, (C \mathcal{P} \mathcal{P})^2 \) and involving alternating tensor product equals to the case \( \mathcal{D} \otimes \mathcal{E} \).

D. The tensor product between Elko spinor fields

For \( \mathcal{P} \in \text{End}(\mathcal{E} \otimes \mathcal{E}) \) the following results can be straightforwardly obtained:

\[ \mathcal{P}^2(\lambda_1 \otimes \lambda_2) = \mathcal{P} \lambda_1 \otimes \mathcal{P} \lambda_2 - \lambda_1 \otimes \lambda_2. \]

For all \( n = 1, 2, \ldots \), there is a periodicity mod 2 for the operator \( \mathcal{P} \in \text{End}(\mathcal{E} \otimes \mathcal{E}) \), as follows:

\[ \mathcal{P}^{2n-1} = (-2)^{n-3/2} (\mathcal{P} \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{P}), \]
\[ \mathcal{P}^{2n} = (-2)^{n-1} (\mathcal{P} \otimes \mathcal{P} - \mathbb{I} \otimes \mathbb{I}). \quad (57) \]

Besides,

\[ \{ C, \mathcal{P} \} = \frac{1}{2} (\{ C, \mathcal{P} \} \otimes \mathbb{I} + \mathbb{I} \otimes \{ C, \mathcal{P} \}) + C \otimes \mathcal{P} + \mathcal{P} \otimes C, \]
\[ [C, \mathcal{P}] = \mathbb{O} \otimes \mathbb{O}. \quad (58) \]
On the other hand, when one uses the alternating tensor product it reads

$$C^2 = -P^2 = -T^2 = \mathbb{I} \hat{\otimes} \mathbb{I},$$

(59)

The DKP algebra generated by the alternate tensor product of two Elko reflects the same properties of discrete symmetries acting on Elko spinor fields in Eq. (26).

**VII. FINAL REMARKS: VSR STRUCTURE IN THE DKP FIELDS**

It was shown the definition of the DKP (or meson) algebra via Clifford algebras. Dirac spinor fields are regular under Lounesto spinor field classification, and they are a reliable arena to test the formalism. Elko (dark) spinor fields are used to incorporate the VSR symmetries, departing from the Lorentz paradigm. Cohen and Glashow have shown that time dilation, the law of velocity addition, and the universal and isotropic velocity, do not demand the entire Poincaré group. Instead, those properties can be evinced by VSR subgroups [2]. If any of the discrete symmetries of $P, T, CP, \text{ or } CT$ is violated, the symmetry group is isomorphic to VSR subgroups and the largest one is obtained by adjoining the four spacetime translation generators to the 4-parameter subgroup $\text{SIM}(2)$.

We have proved that $C, P, \text{ and } T$ symmetries acting on Dirac spinor fields (Eqs. (28)) manifest in the corresponding DKP field, by Eq. (47), naturally, when one uses the alternating tensor product. Solely in this way the correct properties of DKP fields can be manifested. It also occurs for Elko spinor fields. In fact, Eqs. (29), evincing how the square of $C, P, \text{ and } T$ act on Elko, are induced in the corresponding DKP algebra, by Eq. (59), naturally, when one uses the alternating tensor product.

Unexpectedly, the DKP fields $D \hat{\otimes} E$ and $E \hat{\otimes} D$ induce the charge conjugation operator to be an involution (Eq. (54)). Notwithstanding, the parity and the time reversal operator are nilpotent operators (Eqs. (55)).

As a DKP algebra is a subalgebra of $\mathcal{C}(\mathbb{C}) \otimes \mathcal{C}(\mathbb{C})$ satisfying (6), it can happen that the tensor products between Elko and Dirac spinor fields are a subset of the DKP algebra, but disjoint with respect to the standard DKP field, inside $\mathcal{C}(\mathbb{C}) \otimes \mathcal{C}(\mathbb{C})$. Therefore, the discrete symmetries have a different behavior. When the standard tensor product is taken into account some results regarding the DKP algebra are recovered. Instead, when the one takes into account the tensor product between Elko and Dirac spinor fields $\mathcal{C}(\mathbb{C})$, the parity and the time reversal operators are shown to be idempotents (see Eq. (55)). It might be a signal that either the subsets
$\mathcal{D} \otimes \mathcal{E}$ and $\mathcal{E} \otimes \mathcal{D}$ are different from the standard DKP subalgebra or the symmetries on the spinor fields are not induced [and do not induce] by [on] the DKP algebra.

As Elko is ruled by VSR $\text{HOM}(2)$ and $\text{SIM}(2)$ symmetries, and as the DKP algebra is a subalgebra in $\mathcal{C}_\ell_1(\mathbb{C})$, it reflects exactly that in particular the parity symmetry is broken, and do not correspond to the standard DKP algebra. It indicates a manifestation of the Elko symmetries on the spin-0 and spin-1 bosonic sectors of DKP algebra.

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[1] D. V. Ahluwalia and D. Grumiller, *Spin Half Fermions, with Mass Dimension One: Theory, Phenomenology, and Dark Matter*, JCAP 07 (2005) 012 [arXiv:hep-th/0412080]; D. V. Ahluwalia and D. Grumiller, *Dark matter: A spin one half fermion field with mass dimension one?*, Phys. Rev. D 72 (2005) 067701 [arXiv:hep-th/0410192].

[2] A. G. Cohen and S. L. Glashow, *Very special relativity*, Phys. Rev. Lett. 97 (2006) 021601 [arXiv:hep-ph/0601236].

[3] D. V. Ahluwalia, S. P. Horvath, *Very special relativity as relativity of dark matter: the Elko connection*, JHEP 11 (2010) 078 [arXiv:1008.0436 [hep-ph]].

[4] A. E. Bernardini and R. da Rocha, *Dynamical dispersion relation for ELKO dark spinor fields*, Phys. Lett. B 717 (2012) 238 [arXiv:1203.1049 [hep-th]].

[5] M. Dias, F. de Campos, J. M. Hoff da Silva, *Exploring light Elkos signal at accelerators*, Phys. Lett. B 706 (2012) 352 [arXiv:1012.4642 [hep-ph]].

[6] R. da Rocha, A. E. Bernardini and J. M. Hoff da Silva, *Exotic Dark Spinor Fields*, JHEP 04 (2011) 110 [arXiv:1103.4759 [hep-th]].

[7] R. J. Duffin, *On The Characteristic Matrices of Covariant Systems*, Phys. Rev. 54 (1938) 1114; N. Kemmer, *The particle aspect of meson theory*, Proc. Roy. Soc. Lond. A 173 (1939) 91; N. Kemmer, Proc. Roy. Soc. Lond. A 166 (1938) 127; G. Petiau, University of Paris thesis, Académie Royale De Belgique. Classe Des Sciences. Mémoires. Collection 16 (1936) 1114.

[8] Harish-Chandra, *The Correspondence between the Particle and the Wave Aspects of the Meson and the Photon*, Proc. Roy. Soc. Lond. A 186 (1946) 502.
[9] R. Casana, V. Ya. Fainberg, J. T. Lunardi, B. M. Pimentel and R. G. Teixeira, *Massless DKP fields in Riemann-Cartan space-times*, *Class. Quantum Grav.* 20 (2003) 2457.

[10] P. K. Jena, P. C. Naik and T. Pradhan, *Photon As The Zero Mass Limit Of Dkp Field*, *J. Phys. A* 13 (1980) 2975.

[11] N. Jacobson, “Structure and representations of Jordan algebras”, Amer. Math. Soc., Colloquium Publications 39, Providence RI, 1968.

[12] A. Micali, M. Rachidi, *On Meson Algebras*, *Adv. Appl. Clifford Algebras* 18 (2008) 875.

[13] P. Lounesto, *Clifford Algebras and Spinors*, Cambridge Univ. Press, Cambridge, 2002.