ISOMINKOWSKIAN FORMULATION OF GRAVITY

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Abstract

We submit the viewpoint that, perhaps, some of the controversies in gravitation occurred during this century are not due to insufficiencies of Einstein’s field equations, but rather to insufficiencies in the mathematics used for their treatment. For this purpose we treat the same equations with the novel, broader isomathematics and related isominkowskian geometry, and show an apparently final resolution in favor of existing relativities of controversies such as: the lack of invariance of the basic units of space and time; lack of compatibility between gravitational and relativistic conservation laws; lack of meaningful relativistic limit of gravitation; and others. However, an apparent necessary condition for the resolution of these controversies is the abandonment of the notion of curvature used in this century in favor of a conceptual and mathematical broader notion. A number of intriguing implications and experimental verifications are pointed out.

1. Introduction. One of the most majestic achievements of this century for mathematical beauty, axiomatic consistency and experimental verifications has been the special theory of relativity (STR)\(^1\). By comparison, despite equally outstanding achievements, the general theory of relativity (GTR)\(^2\) has remained afflicted by numerous problematic aspects at both classical and quantum levels.
The view submitted in this note is that, perhaps, some of the controversies in gravitation are not due to insufficiencies in current gravitational theories, but rather to insufficiencies in their mathematical treatment.

More specifically, we argue that the contemporary mathematics (consisting of conventional numbers and fields, vector and metric spaces, differential calculus and functional analysis, etc.) has produced an outstanding physical consistency when applied to relativistic theories, yet the same mathematics has produced unsettled problems when applied to gravitation.

As a concrete example, the unit $I = \text{diag. } ([1, 1, 1], 1)$ of the Minkowskian geometry representing in a dimensionless form the basic units of space and time, is invariant under the Poincaré symmetry, as well known. By comparison, we have the following

**Theorem 1**\(^{(30)}\). The fundamental units of space and time are not invariant for all geometries with non-null curvature.

In fact, the transition from the Minkowskian metric $\eta = \text{Diag}(1, 1, 1, -1)$ to a (3+1)-dimensional Riemannian metric $g(x)$ is characterized by a noncanonical transformation $x \rightarrow x' = U \times x, U \times U^t \neq I$, for which (by ignoring the dash) $g(x) = U \times \eta \times U^t$. Corresponding theories of quantum gravity are then generally nonunitary when formulated on conventional Hilbert spaces over conventional complex fields. The lack of the invariance of the basic units then follows at both classical and operator levers from the very definition of noncanonical and nonunitary transforms for all gravitational theories with curvature.

Theorem 1 implies rather serious ambiguities in the application of gravitational theories to actual measurements, evidently because we cannot possibly have a physically valid measure, say, of length, via a stationary meter varying in time. The hope that the problem is resolved by the joint change of the entire environment does not resolve the shortcoming because, e.g., the impasse remain for measures related to far away objects which, as such, are independent from our local environment.

We here argue that Theorem 1 is a specific manifestation of the insufficiency of the mathematics currently used for gravitation, because no corresponding shortcoming exists for the flat relativistic case.

We also argue that the shortcoming of Theorem 1 is at the foundation in a rather subtle way with a number of controversies in gravitation existing in the literature. For instance, as we shall see in this note, the achievement of a formulation of gravity with invariant basic units will automatically provide a novel unambiguous operator formulation of gravity as axiomatically consistent as relativistic quantum mechanics. After all, no axiomatically consistent operator theory of gravitation should be expected without the fundamental invariance of the basic units.

Since there is no conceivable possibility of achieving a formulation of gravitation with
invariant units based on conventional mathematics, our basic assumption is the use of a *generalized mathematics* which permits the preservation unchanged of Einstein’s field equations and related experimental verifications while verifying the uncompromisable conditions of invariant basic units of space and time.

The only known broader mathematics satisfying the above conditions is the novel *isomathematics*, first submitted by Santilli(3a) back in 1978, but which achieved sufficient operational maturity only recently in memoir(3f), and which includes: new numbers and fields, new vector and metric spaces, new algebras and geometries, etc. (an outline is also available in Page 18 of Web Site [4u]).

The *isotopies* are nowadays referred to maps (also called liftings) of any given linear, local-differential, and canonical or unitary theory into its most general known nonlinear, nonlocal-integral and noncanonical or nonunitary extensions, which are nevertheless capable of reconstructing linearity, locality and canonicity or unitarity on certain generalized isospaces over generalized isofields.

The new geometry capable of yielding the invariance of the basic unit is the *isominkowskian geometry* first submitted by Santilli(3h) in 1983, but which also reached operational maturity only recently following the advances of the preceding memoir(3f). The isominkowskian geometry was originally submitted for the most general possible realization of the Minkowskian axioms, but only recently has been understood(3u) also to embody jointly all the machinery of the Riemannian geometry, such as connections, covariant derivatives, etc., although expressed in a generalized way. In particular, the isominkowskian geometry admits all possible Riemannian metrics for exterior gravitational models in vacuum, as well as their extensions for interior gravitational models with a well behaved but otherwise unrestricted dependence on the velocities and other interior variables.

The isominkowskian geometry therefore appears to be ideally suited for our objective. In fact, on one side it can preserve Einstein’s (or any other) field equations, although formulated within the context of a broader mathematics, while achieving the uncompromisable invariance of the basic units of space and time.

The isotopies of classical and quantum mechanics were also submitted by Santilli(3a) back in 1978, but they too reached sufficient maturity only recently in memoir(3f) for the classical profile, memoir(3t) for the operator profile and memoir(3v) for applications and experimental verifications.

The delay in the achievement of operational maturity was due to the lack of invariance of preceding studies for reasons that escaped identification for years, and which resulted to rest where one would expect them the least, the use of the ordinary differential calculus under isotopies. Once the isotopic lifting of the differential calculus was identified in memoir(3f), all other problems of axiomatic consistency were easily solved, by reaching a generalized mathematics which yields the invariance of the units of space and time while being “directly universal”, that is, applicable to all well behaved, signature preserving broadening of a given
theory, such as the Minkowskian geometry (universality), directly in the fixed inertial frame of the observer (direct universality).

In summary, the novel isomathematics applies for the reformulation of all possible non-canonical or nonunitary theories while achieving the uncompromisable invariance of the basic units of space and time.

2. Isominkowskian geometry. The fundamental isotopy for relativistic theories is the lifting of the unit of conventional theories, the unit \( I = \text{diag.}(1, 1, 1, 1) \) of the Minkowski space and of the Poincaré symmetry, into a well behaved, nowhere singular, Hermitean and positive-definite 4 \( \times \) 4-dimensional matrix \( \hat{I} \) whose elements have an arbitrary dependence on local quantities and, therefore, can depend on the space–time coordinates \( x \) and other needed variables, \( I \to \hat{I} = \hat{I}(x,...) > 0 \).

The conventional associative product \( A \times B \) among generic quantities \( A, B \) is jointly lifted by the inverse amount, \( A \times B \to A\hat{\times}B = A \times \hat{T} \times B, \hat{I} = \hat{T}^{-1} \).

Under these assumptions \( \hat{I} \) is the (left and right) generalized unit of the new theory, \( \hat{\times}A = A\hat{\times}\hat{I} \equiv A, \forall A \), in which case (only) \( \hat{I} \) is called the isounit and \( \hat{T} \) is called the isotopic element.

For consistency, the totality of the original theory must be reconstructed to admit \( \hat{I} \) as the correct (left and right) unit. This implies the isotopies of numbers, angles, fields, spaces, differential calculus, functional analysis, geometries, algebras, symmetries, etc. (see ref.\(^3\) for a recent account).

We now study the possible application of the above isotopies to exterior gravitation in vacuum for which the dependence of the isounit is restricted to the \( x \)-dependence only. Let \( M(x, \eta, R) \) be the Minkowski space with space–time coordinates \( x = \{x^\mu\} = \{r, x^4\}, x^4 = c_0t \) (where \( c_0 \) is the speed of light in vacuum), and metric \( \eta = \text{diag.}(1, 1, 1, -1) \) over the reals \( R = R(n, +, \times) \). Let \( \mathcal{R}(x, g(x), R) \) be a (3 + 1)–dimensional Riemannian space with nowhere singular and symmetric metric \( g = g^t = U \times \eta \times U^t \). Let \( \mathcal{R}(x, g(x), R) \) be a (3 + 1)–dimensional Riemannian space with nowhere singular and symmetric metric \( g = g^t = U \times \eta \times U^t \).

The regaining of the invariance of the basic units is then permitted by assuming as basic isounit of the gravitational theory the quantity \( \hat{I} = U \times U^t = \hat{T} > 0 \) with explicit form derivable form a Riemannian metric via the isominkowskian factorization\(^{3o,3p}\)

\[
g(x) = \hat{T}(x) \times \eta, \hat{I}(x) = [T(x)]^{-1} = U \times U^t. \tag{1}
\]

As an example, for the case of the celebrated Schwarzschild’s metric\(^{2d}\), we have \( U \times U^t = \hat{I} = \text{Diag.}(1 – M/r), (1 – M/r), (1 – M/r), (1 – M/r)^{-1} \) and similarly for other metrics. It should however be indicated that a better representation of the Schwarzschild metric is that in isotropic coordinates\(^{2f}\) which requires a nondiagonal isounit.
We note however that \( \hat{I} \) is always positive-definite, assured by the locally Minkowskian character of Riemann. For simplicity but without loss of generality, the isounit can therefore be assumed herein as being diagonal.

An inspection of gravitational theories with conventional Riemannian metrics \( g(x) = \hat{T}(x) \times \eta \) yet referred to the generalized unit \( I = \hat{T}^{-1} \) reveals that its axiomatic structure is that of the isotopies of the Minkowski space \(^{(3)}g\) which are characterized by the dual lifting of \( \eta \) into \( \hat{\eta} = \hat{T} \times \eta \) and I into \( \hat{I} = \hat{T}^{-1} \). In fact, the isotopies of Riemannian spaces \(^{(3)}g\) are characterized by the different dual lifting of \( g(x) \) into \( \hat{g} = \hat{T} \times g \) and of I into \( \hat{I} = \hat{T}^{-1} \).

The main difference is that in the former case the entire functional dependence of the metric is absorbed in the isounit, while this is not the case for the latter case. As we shall see, the invariance of the basic unit is reached in the former but not in the latter case.

The studies of gravitational theories of type (1) must therefore be conducted within the context of the isotopies of the special relativity, also called isospecial relativity, first submitted by Santilli \(^{(3)}g\) in 1983 and then studied in a variety of works at both classical and operator levels (see Ref.\(^{(3r,3s)}g\) for a review of the studied up to 1995 and ref.\(^{(3d)}g\) for studies following the advent of the isodifferential calculus of memoir\(^{(3f)}g\)).

To construct the isospecial relativity we first need the lifting of numbers and fields \(^{(3g)}g\). For this we note that the conventional multiplicative unit I is lifted into the isounit, \( I \rightarrow U \times 1 \times U^t = \hat{I} \) while the additive unit 0 remains unchanged, \( 0 \rightarrow 0 = U \times 0 \times U^t = 0 \). The numbers are lifted into the so-called isonumbers, \( n \rightarrow \hat{n} = U \times n \times U^t = n \times \hat{I} \) with lifting of the product \( n \times m \rightarrow \hat{n} \times \hat{m} = \hat{n} \times \hat{T} \times \hat{m}, \hat{T} = \hat{I}^{-1} \).

The original field \( R = R(n, +, \times) \) is then lifted into the isofield \(^{(3g)}g\) \( \hat{R} = \hat{R}(\hat{n}, \hat{+}, \hat{\times}) \) for which all operations are isotopic. It is easy to see that \( \hat{R} \) is locally isomorphic to \( R \) by construction and, thus, the lifting \( R \rightarrow \hat{R} \) is an isotopy. Despite its simplicity, the lifting is not trivial, e.g., because the notion of primes and other properties of number theory depend on the assumed unit. For further aspects we refer to\(^{(5r)}g\) which also includes the isotopies of angles and functions analysis. Note for later needs the identity, \( \hat{n} \times A \equiv n \times A \).

Next, we need the lifting of the space \( M \) into the isominkowskian space \(^{(3h)}g\) \( \hat{M} = \hat{M}(\hat{x}, \hat{\eta}, \hat{R}) \) first proposed by Santilli in Ref.\(^{(3h)}g\) which is characterized by the isocoordinates \( x \rightarrow \hat{x} = U \times x \times U^t = x \times \hat{I} \), and isometric \( \eta \rightarrow \hat{\eta}(x) = U \times \eta \times U^t \equiv g(x) \) although, for consistency, the latter must be defined on \( \hat{R} \), thus having the structure \( \hat{N} = (\hat{N}_{\mu\nu}) = \hat{\eta} \times \hat{I} = (\hat{\eta}_{\mu\nu}) \times \hat{I} \).

The conventional interval on \( M \) is then lifted into the isointerval on \( \hat{M} \) over \( \hat{R}^{4h} \)

\[
(\hat{x} - \hat{y})^2 = (\hat{x} - \hat{y})^\mu \hat{\times} \hat{N}_{\mu\nu} \hat{\times} (\hat{x} - \hat{y})^\nu = [(x - y)^\mu \hat{\times} \hat{\eta}_{\mu\nu} \times (x - y)^\nu] \times \hat{I} =
\]

\[
= [(x^1 - y^1) \times \hat{T}_{11} \times (x^1 - y^1) + (x^2 - y^2) \times \hat{T}_{22} \times (x^2 - y^2) +
\]

5
\[(x^3 - y^3) \times T_{33} \times (x^3 - y^3) - (x^4 - y^4) \times T_{44} \times (x^4 - y^4) \times \hat{I}. \tag{2}\]

As one can see, the above interval coincides with the conventional Riemannian interval by conception, except for the factor \(\hat{I}\).

It is instructive to prove that the isoinvariant can also be obtained from the basic non-canonical transform according to the rule \(\hat{x}^2 = U \times x^2 \times U^t = [(x^t \times U^t) \times (U^{t-1} \times U^{-1}) \times \eta \times (U \times x)] \times (U \times U^t) = \hat{x}^t \times \hat{N} \times \hat{x}\) where "t" stands for transpose. This construction also clarifies that the coordinates \(x\) on \(M\) are lifted into the form \(U \times \hat{x}\) on \(\hat{M}\).

It is easy to see that \(\hat{M}\) is locally isomorphic to \(M\) and the lifting \(M \rightarrow \hat{M}\) is also an isotopy. In particular, the isospace \(\hat{M}\) is isoflat, i.e., it verifies the axiom of flatness in isospace over the isofields, that is, when referred to the generalized unit \(\hat{I}\), otherwise \(\hat{M}\) is evidently curved owing to the dependence \(\hat{\eta} = \hat{\eta}(x) = g(x)\). In other words, assumptions (1) eliminate the curvature in isospace while preserving the Riemannian metric.

Note that \(\hat{M}\) and \(\hat{R}\) have the same isounit \(\hat{I}\). The conventional Minkowskian setting admitted by the isospecial relativity for \(\hat{I} = \hat{i}\) is therefore that in which both, the minkowski space and ]the base field have the same unit \(I = \text{diag.}(1, 1, 1, 1)\), which implies a trivial redefinition of conventional fields hereon ignored.

Studies of isocotinuity properties on isospaces have been conducted by Kadeisvili\(^{(4r)}\) and those of the underlying novel isotopology by Tsagas and Sourlas\(^{(4s)}\).

The isominkowskian geometry\(^{(3r, 3u)}\) is the geometry of isospaces \(\hat{M}\), and incorporates in a symbiotic way both the Minkowskian and Riemannian geometries. In fact, it preserves all geometric properties of the conventional Minkowskian geometry, including the light cone and the maximal causal speed \(c_0\) (see below), while jointly incorporating the machinery of the Riemannian geometry in an isotopic form. As such, it is ideally suited for our objectives.

It should be indicated that this author has studied until now the interior gravitational problem via the isotopies of the Riemannian geometry. The use of the isominkowskian geometry for the characterization of the exterior gravitational problem was briefly indicated in note\(^{(3s)}\) and it is studied in more details in this work. Also, the main line of the isominkowskian geometry inclusive of the machinery of the Riemannian geometry are presented in this note for the first time with detailed study in paper\(^{(3u)}\).

To outline the new geometry, one must know that, as indicated in Sect. 1, the use of the ordinary differential calculus leads to inconsistencies under isotopies (e.g., lack of invariance) because dependent on the assumption of the trivial unit \(1\) in a hidden way. The central tool of the isominkowskian geometry is therefore the isodifferential calculus on \(\hat{M}(\hat{x}, \hat{v}, \hat{R})\), first introduced in\(^{(3s)}\), which is characterized by the isodifferentials, isoderivatives and related
properties $\hat{dx}^\nu = \hat{T}_\mu^\nu \times dx^\nu, \hat{dx}_\mu = \hat{\partial}_\mu \times dx^\nu, \hat{\partial}_\mu = \partial/\partial x^\mu, \hat{\partial}^\mu = \partial/\partial x^\mu = \hat{T}_\mu^\nu \times \partial/\partial x^\nu, \hat{\partial}^\mu = \partial/\partial x^\mu = \hat{T}_\mu^\nu \times \partial/\partial x^\nu, \hat{\partial}^\mu = \partial/\partial x^\mu$. 

Note that the original axioms must be preserved for an isotopy. Thus, the isodifferential calculus is isocommutative, i.e., commutative on $\hat{M}$ over $\hat{R}, \hat{\partial}_\alpha \hat{\partial}_\beta = \hat{\partial}_\beta \hat{\partial}_\alpha$. However, the same isocalculus is not, in general, commutative in its projection on $M$ over $R$.

Note also the hidden isoquotient A/B = (A/B) × $\hat{I}$ and isoproduct $\hat{\partial} \hat{\times} \hat{\partial}$. Thus, by including the isoquotient, the quantity $\hat{\partial} \hat{\times} \hat{\partial}$ should be more rigorously written $\hat{\partial} \hat{\times} \hat{\partial}$. This results in an inessential final multiplication of the expression considered - by $\hat{I}$ and, as such, it will be ignored hereon for simplicity.

The entire formalism of the Riemannian geometry can then be formulated on the isominkovskian space via the isodifferential calculus. This aspect is studied in details elsewhere. We here mention: isochristoffel's symbols $\hat{\Gamma}_{\alpha\beta\gamma} = \frac{1}{2} \hat{\times} (\hat{\partial}_\alpha \hat{\eta}_{\beta\gamma} + \hat{\partial}_\gamma \hat{\eta}_{\alpha\beta} - \hat{\partial}_\beta \hat{\eta}_{\alpha\gamma}) \times \hat{I}$, isocovariant differential $\hat{D}\hat{X}^\beta = \hat{d}\hat{X}^\beta + \hat{\Gamma}_{\alpha\beta\gamma}^\beta \hat{\times} \hat{X}^\alpha \hat{\times} \hat{d}\hat{x}^\gamma$, isocovariant derivative $\hat{X}_\mu^\beta = \hat{\partial}_\mu \hat{X}^\beta + \hat{\Gamma}_{\alpha\mu\beta}^\beta \hat{\times} \hat{X}^\alpha$, isocurvature tensor $\hat{R}_{\alpha\beta\gamma}^\delta = \hat{\partial}_\beta \hat{\Gamma}_{\alpha\gamma\delta} - \hat{\partial}_\gamma \hat{\Gamma}_{\alpha\beta\delta} + \hat{\Gamma}_{\rho\beta\delta} \hat{\times} \hat{\Gamma}_{\alpha\gamma\rho} - \hat{\Gamma}_{\rho\gamma\delta} \hat{\times} \hat{\Gamma}_{\alpha\beta\rho}$, etc.

The verification, this time, of the Riemannian properties is shown by the fact that (under the assumed conditions) the isocovariant derivatives of all isometrics on $\hat{M}$ over $\hat{R}$ are identically null, $\hat{\eta}_{\alpha\beta\gamma} = 0, \alpha, \beta, \gamma = 1, 2, 3, 4$. This illustrates that the Ricci Lemma also holds under the Minkowskian axioms.

A similar occurrence holds for all other properties, including the five identities of the Riemannian geometry (where the firth is the forgotten Freud identity, as studied in details elsewhere).

In summary, the study of the isominkovskian geometry reveals the emergence of the new notion of isocurvature here introduced apparently for the first time, here referred to the redefinition of curvature via the use of isomathematics based on rules (1).

3. Classical unification of the special and general relativities. We are now equipped to present, apparently for the first time, the classical equations of our isominkovskian formulation of gravity, here called iso einstein equations on $\hat{M}$ over $\hat{R}$, which can be written

$$\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2} \hat{\times} \hat{N}_{\mu\nu} \times \hat{R} = \hat{k} \hat{\times} \hat{\tau}_{\mu\nu},$$

(3)

where $\hat{\tau}_{\mu\nu}$ is the source isotensor on $\hat{M}, \hat{\frac{1}{2}} = \frac{1}{2} \hat{\times} \hat{I}, \hat{N}_{\mu\nu} = \hat{\eta}_{\mu\nu} \times \hat{I} = g_{\mu\nu} \times \hat{I}, \hat{k} = k \times \hat{I}$ and $k$ is the usual constant.

Despite apparent differences, it should be indicated that Eqs. (3) coincide numerically with Einstein’s equations both in isospace as well as in their projection in ordinary spaces for all diagonal Riemannian metrics.
The preservation in isospace of the numerical value of the conventional field equations stems from a general property of the isotopies of preserving all original numerical values (see later on the preservation of the speed of light as the maximal causal speed on $\hat{M}$). In fact, the isoderivative $\hat{\partial}_\mu = \hat{T}_\alpha^\mu \times \partial_\alpha$ deviates from the conventional derivative $\partial_\mu$ by the isotopic factor $\hat{T}$. But its numerical value must be referred to $\hat{I} = \hat{T}^{-1}$, rather than $I$. This implies the preservation in isospace of the original value of $\partial_\mu$ and, consequently, of the original field equations.

For the case of the projection of in ordinary spaces, the isoequations are reducible to the conventional equations multiplied by common isotopic factors which, as such, are inessential and can be eliminated. In fact, the isochristoffel's symbols deviate from the conventional symbols by the same factor $\hat{T}$ (again, because $\hat{\eta} \equiv g$), and the same happens with other terms, except for possible redefinition of the source when needed, thus preserving again the conventional field equations and related experimental verifications also in our space-time.

Note that the isominkowskian formulation of gravity permits a geometric unification of the special and general relativities into one single relativity, the isospecial relativity where for $\hat{I} = I = \text{diag.}(1,1,1,1)$ we have the special and for $\hat{I} = \hat{I}(x) = U \times U^\dagger$ we have the general. The invariance of the isounit is illustrated below.

4. Operator unification of the special and general relativities. We now indicate that the above classical unification admit a step–by–step operator counterpart, called operator isogravity (OIG), first submitted by Santilli at the VII M. Grossmann Meeting on General Relativity of 1993\(^{(30)}\).

It should be indicated from the outset that OIG is structurally different than the conventional quantum gravity (QG)\(^6\) on numerous grounds, e.g., because OIG and QM have different units, Hilbert spaces, and fields. In particular, the word “operator” in OIG is suggested to keep in mind the differences with “quantum” mechanics (as it should also be for QG).

To identify the explicit form of OIG, we note that the original noncanonical transform $U \times U^\dagger = \hat{I} \neq I$ is mapped into a nonunitary transform on a conventional Hilbert space $\mathcal{H}$ over the complex field $C(c,+,\times)$. The isounit of the operator theory is therefore $\hat{I} = U \times U^\dagger = \hat{I}^\dagger, \hat{T} = (U \times U^\dagger)^{-1} = T^\dagger = \hat{I}^{-1}$, where the representation of gravity occurs as per Eqs. (1). Then, OIG requires the isotopies of the totality of relativistic quantum mechanics (RQM) resulting in a formulation known as relativistic hadronic mechanics (RHM)\(^{(34,35)}\).

Besides the preceding isotopies $R \rightarrow \hat{R}$ and $\hat{M} \rightarrow \hat{M}$, RHM is based on the lifting of the Hilbert space $\mathcal{H}$ with states $|\Psi >, |\Phi >, \ldots$ and inner product $< \Phi |\Psi > \in C(c,+,\times)$ into the isohilbert space $\hat{\mathcal{H}}^{(4t)}$ with isostetes $|\hat{\Psi} > = U \times |\Psi >, |\hat{\Phi} > = U \times |\Phi >, \ldots$ inner product $< \hat{\Phi} |\hat{\Psi} > = U \times < \Phi |\Psi > \times U^\dagger = < \hat{\Phi} |\hat{T} \times |\hat{\psi} > \times \hat{I}$ defined on the isofield $\hat{C}(\hat{c},+,\hat{\times})$ with isonormalization $< \hat{\Psi} |\hat{T} \times |\hat{\psi} > = \hat{I}$.
We then have the iso-four-momentum operator \( p_\mu \hat{x} \vec{|Ψ} > = -i \hat{x} \partial_\mu |Ψ > = -i \hat{T}_\mu \times ∂_\mu |Ψ > \), with fundamental isocommutation rules \([\hat{x}_\mu, \hat{p}_\nu] = U \times [x_\mu, p_\nu] \times U^\dagger = \hat{x}_\mu \times \hat{T} \times \hat{p}_\nu - \hat{p}_\nu \times \hat{T} \times \hat{x}_\mu = i \hat{x} \hat{N}_{\mu \nu} \). The (nonrelativistic) isoheisenberg equations \( i \partial_t |Ψ > = A \times |Ψ > = (E \times |I_\hat{s} >) \times \hat{T}_s \times |Ψ > \equiv E \times |Ψ > \). (4)

Note that the final numbers of the theory are conventional. We also have the lifting of expectation values into the form \(<A> = <\hat{Ψ} | \times T \times A \times \hat{T} \times |\hat{Ψ}> / <\hat{Ψ} | \times \hat{T} \times |\hat{Ψ}> \), and the compatible liftings of the remaining aspects of RQM \(^{(3s)} \). In particular, \( I \) is the fundamental invariant of the isotheory, \( id\hat{I}/dt = I \hat{x} H - H \hat{x} I \equiv 0 \).

It is easy to prove that RHM preserves all conventional properties of RQM \(^{(3t)} \). In particular: isohermiticity coincides with conventional Hermiticity, \( H^\dagger \equiv H^\dagger \) (all quantities which are originally observables remain, therefore, so under isotopies); the isoeigenvalues of isohermitean operators are isoreal (thus conventional because of the identity \( \hat{E} \hat{x} \vec{|Ψ} > \equiv E \times |Ψ > \)); RHM is form invariant under isounitary transforms \( \hat{U} \hat{x} \hat{U}^\dagger = \hat{U}^\dagger \hat{x} \hat{U} = \hat{I} \). In fact, we have the invariance of the isounit \( I \rightarrow \tilde{I} = \hat{U} \hat{x} \hat{I} \hat{x} \hat{U}^\dagger \equiv \hat{I} \), of the isoassociative product \( \hat{U} \hat{x} (A \hat{x} B) \hat{x} \hat{U}^\dagger = A \prime \hat{x} \); etc; and the same occurs for all other properties (including causality). Note that nonunitary transforms on \( \mathcal{H} \) can always be identically rewritten as isounitary transforms on \( \mathcal{H}, U = \hat{U} \times \hat{T}^{1/2}, U \times U^\dagger \equiv \hat{U} \hat{x} \hat{U}^\dagger = \hat{U}^\dagger \hat{x} \hat{U} = \hat{I} \), under which RHM is invariant \(^{(3t)} \).

It should be stressed that RHM is not a new theory, but merely a new realization of the abstract axioms of RQM. In fact, RHM and RQM coincide at the abstract, realization-free level where all distinctions are lost between \( I \) and \( \tilde{I}, R \) and \( \tilde{R}, M \) and \( \tilde{M}, \mathcal{H} \) and \( \tilde{\mathcal{H}} \), etc. Yet, RHM is inequivalent to RQM evidently because the two theories are related by a nonunitary transform. Also, RHM is broader than RQM, it recovers the latter identically for \( \hat{I} = \hat{I} \) and can approximate the latter as close as desired for \( \hat{I} \approx \hat{I} \).

On summary, the entire formulation of RHM of memoir \(^{(3s,3t)} \) can be consistently specialized for the gravitational isounit \( \hat{I}(x) \) yielding the proposed OIG.

5. The Poincaré-Santilli isosymmetry. An important property of the isominkowskian formulation of gravity, which is lacking for conventional formulations, is that of admitting
a universal, classical and operator symmetry for all possible Riemannian formulations of gravitation first identified by Santilli\cite{3h--3l} under the name of *isopoincaré symmetry* $\hat{P}(3.1)$, and today called *Poincaré-Santilli isosymmetry*\cite{5,6}, which results to be locally isomorphic to the conventional symmetry $P(3.1)$.

The isosymmetry $\hat{P}(3.1)$ is the invariance of isointerval (2) and can be easily constructed via the *isotopies of Lie’s theory* first proposed by Santilli\cite{3a,3d} via the lifting of universal enveloping algebras, Lie algebras, Lie group, transformation and representation theories, etc., and today called *Lie-Santilli isotheory*\cite{5,6}. The latter theory essentially consists in the reconstruction of all branches of Lie’s theory for the generalized unit $\hat{I} = [\hat{T}]^{-1}$. Since $\hat{I} > 0$, one can see from the inception that the Poincaré-Santilli isosymmetry is isomorphic to the conventional one, $\hat{P}(3.1) \approx P(3.1)$ (see ref.\cite{44} for a recent accounts).

Note that all simple Lie algebras are known from cartan’s classification. Therefore, the Lie-Santilli isotheory cannot produce new Lie algebras, but only *new realizations* of known Lie algebras of nonlinear, nonlocal and nonhamiltonian type.

Moreover, a primary function of the Lie-Santilli isotheory is that of reconstructing as exact conventional space-time and internal symmetries when believed to be conventionally broken. In particular, one of the primary functions of the Poincaré-Santilli isosymmetry is to establish that the abstract axioms of the conventional Poincaré symmetry remain exact under nonlinear, nonlocal and nonhamiltonian interactions, evidently when properly treated.

In this section we shall show in particular that, contrary to a rather popular belief, the rotational, Lorenzt and Poincaré symmetry do indeed remain exact for all possible *gravitational* models.

The operator version of the isosymmetry $\hat{P}(3.1)$ is characterized by the conventional generators and parameter only reformulated on isospaces over isofields $X = \{X_k\} = \{M_{\mu\nu} = x_{\mu}p_{\nu} - x_{\mu}p_{\nu}, p_\alpha\} \rightarrow \hat{X} = \{M_{\mu\nu} = \hat{x}_{\mu} \times \hat{p}_{\nu} - \hat{x}_{\nu} \times \hat{p}_{\mu}, \hat{p}_\alpha\}, k = 1, 2, ..., 10, \mu, \nu = 1, 2, 3, 4, \text{ and } w = \{w_k\} = \{(\theta, v), a\} \in R \rightarrow \hat{w} = \hat{w} \times \hat{I} \in \hat{R}(\hat{n}, +, \hat{x})$. Since the generators of space-time symmetries represents conventional total conservation laws, the preservation under isotopies of conventional generators ensured *ab initio* the preservation for the isominkowskian formulation of gravity of conventional total conservation laws.

The isotopies preserve the original connectivity properties\cite{3r}. The connected component of $\hat{P}(3.1)$ is then given by $\hat{P}_0(3.1) = S\hat{O}(3.1) \times \hat{T}(3.1)$, where $S\hat{O}(3.1)$ is the connected Lorentz-Santilli isosymmetry first submitted in Ref.\cite{3h} and $\hat{T}(3.1)$ is the group of *isotranslations*\cite{3k}. $\hat{P}_0(3.1)$ can be written via the *isoexponentiation* $\hat{e}^A = \hat{I} + A/1! + A \times A/2! + ... = (e^{A \times \hat{T}}) \times \hat{I}$ characterized by the *isotopic Poincaré–Birkhoff–Witt theorem*\cite{3a,3d,5} of the underlying isoenveloping associative algebra

\[
\hat{P}_0(3.1) : \hat{A}(\hat{w}) = \Pi_k e^{i \times X \times w} = (\Pi_k e^{i \times X \times \hat{T} \times w}) \times \hat{I} = \hat{A}(w) \times \hat{I}.
\] (5)
Note the appearance of the gravitational isotopic element $\hat{T}(x)$ in the *exponent* of the group structure. This illustrates the nontriviality of the lifting and its *nonlinear* character, as evidently necessary for any symmetry of gravitation. What is intriguing is that the isosymmetry $\hat{P}(3.1)$ recovers linearity on $\hat{M}$ over $\hat{R}$, a property called *isolinearity*\textsuperscript{34}.

Conventional linear transforms on $M$ violate isolinearity on $\hat{M}$ and must then be replaced with the *isotransforms* $\hat{x}' = \hat{A}(\hat{w}) \hat{X} \hat{A}(\hat{w}) = \hat{A}(\hat{w}) \hat{T}(x) \times \hat{x}$ which can be written from (5) for computational purposes (only) $\hat{x}' = \hat{A}(\hat{w}) \times \hat{x}$. The preservation of the original dimension is ensured by the *isotropic Baker–Campbell–Hausdorff Theorem*\textsuperscript{3a,3d,5,6}. Structure (5) then forms a connected *Lie–Santilli isogroup*\textsuperscript{5} with laws $\hat{A}(\hat{w}) \hat{X} \hat{A}(\hat{w}') = \hat{A}(\hat{w}') \hat{A}(\hat{w}) = \hat{A}(\hat{w} + \hat{w}')$, $\hat{A}(\hat{w}) \hat{X} (\hat{A}(\hat{w})^{-1}) = \hat{A}(0) = \hat{I}(x) = [T(x)]^{-1}$.

As one can see, $\hat{P}_0(3.1)$ is noncanonical on $M$ over $R$ (e.g., because it does not preserve the conventional unit I), but it is canonical on $\hat{M}$ over $\hat{R}$, a property called *isocanonicity* (because it leaves invariant by construction the isounit). This confirms the achievement, apparently for the first time, of an operator theory of gravity verifying the fundamental invariance of its unit. The invariance at the classical level is consequential.

One should be aware that a rather The use of the isodifferential calculus on $\hat{M}$ then yields the Poincaré-Santilli isalgebra $\hat{p}_0(3.1)$\textsuperscript{3k}

$$[\hat{M}_{\mu\nu}, \hat{M}_{\alpha\beta}] = i \times (\hat{\eta}_{\alpha\alpha} \times \hat{M}_{\mu\beta} - \hat{\eta}_{\mu\alpha} \times \hat{M}_{\nu\beta} - \hat{\eta}_{\mu\beta} \times \hat{M}_{\mu\alpha} + \hat{\eta}_{\mu\alpha} \times \hat{M}_{\alpha\beta}),$$

$$[\hat{M}_{\mu\nu}, \hat{p}_\alpha] = i \times (\hat{\eta}_{\mu\alpha} \times \hat{p}_\nu - \hat{\eta}_{\alpha\alpha} \times \hat{p}_\mu), [\hat{p}_\alpha, \hat{p}_\beta] = 0, \hat{\eta}_{\mu\nu} = g_{\mu\nu}(x),$$

where $[A,B] = A \times \hat{T}(x) \times B - B \times \hat{T}(x) \times A$ is the *isoproduct* (originally proposed in \textsuperscript{(3b)}), which does indeed satisfy the Lie axioms in isospace, as one can verify. Note the appearance of the Riemannian metric $\hat{\eta}_{\mu\nu} = g_{\mu\nu}(x)$, this time, as the "structure functions" $\hat{\eta}_{\mu\nu}$ of the isalgebra\textsuperscript{3a,3d,5}. Note also that the *momentum components isocommute* (while they are notoriously non–commutative for QG). This confirms the achievement of an isoflat representation of gravity.

The local isomorphism $\hat{p}_0(3.1) \approx p_0(3.1)$ is ensured by the positive–definiteness of $\hat{T}$. In fact, the use of the generators in the form $\hat{M}_\mu = \hat{x}^\mu \times \hat{p}_\mu - \hat{x}^\nu \times \hat{p}_\mu$ would yield conventional structure constants under a *generalized* Lie product, as one can verify. The above local isomorphism is sufficient, per se', to guarantee the axiomatic consistency of OIG.

The *isocasimir invariants* of $\hat{p}_0(3.1)$ are simple isotopic images of the conventional ones $C^0 = \hat{I} = [\hat{T}(x)]^{-1}, C^{(2)} = \hat{p}^2 = \hat{p}_\mu \hat{p}^\mu = \hat{\eta}^{\mu\nu} \times \hat{p}_\mu \times \hat{p}_\nu, C^{(4)} = \hat{W}_\mu \hat{W}^\mu, \hat{W}_\mu = \epsilon_{\mu\alpha\beta} \hat{M}^\alpha_{\beta} \hat{p}^\nu$. From them, one can construct any needed *gravitational relativistic equation*, such as the *isodirac equation*  

$$(\hat{\gamma}^\mu \hat{\gamma}^\tau \times \hat{p}_\mu + i \hat{\gamma}^{\hat{m}}) \times \hat{T}(x) \times \hat{p}^\tau - i \times m \times \hat{I}(x) \times \hat{T}(x) \times | >= 0,$$
\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = \hat{\gamma}^\mu \times \hat{T} \times \hat{\gamma}^\nu + \hat{\gamma}^\nu \times \hat{T} \times \hat{\gamma}^\mu = 2 \times \hat{\gamma}^{\mu\nu} \equiv 2 \times g^{\mu\nu}, \hat{\gamma}^\mu = \hat{T}_{\mu\mu}^{1/2} \times \gamma^\mu \times I \ (\text{no sum}), \ (7)

where \(\gamma^\mu\) are the conventional gammas and \(\hat{\gamma}^\mu\) are the isogamma matrices. Note that the anti-isocommutators of the isogamma matrices yield (twice) the Riemannian metric \(g(x)\), thus confirming the representation of Einstein’s (or other) gravitation in the structure of Dirac’s equation. As an illustration, we have the Dirac–Schwarzschild equation given by Eqs. (7) with \(\hat{\gamma}_k = (1 - 2M/r)^{-1/2} \times \gamma_k \times I\) and \(\hat{\gamma}_4 = (1 - 2M/r)^{1/2} \times \gamma^4 \times I\), although, as indicated in Sect. 1, the representation in isotropic coordinates and a nondiagonal isounit \(E(\gamma)\) would be preferable. Similarly one can construct the isogravitational version of all other equations of RQM.

These equations are not a mere mathematical curiosity because they establish the compatibility of OIG with experimental data in particle physics in view of the much smaller contribution of gravitational over electromagnetic, weak and strong contributions. Our unification of the special and general relativities is, therefore, compatible with experimental evidence at both classical and operator levels.

The Poincaré-Santilli isotransforms are given by:

1) Isorotations. The space components \(S\hat{O}(3)\), called isorotations\(^{(3i)}\), can be computed from isoexponentiations (5) with the explicit form in the \((x,y)\)-plane (were we ignore again the factorization of \(I\) for simplicity)

\[
\begin{align*}
    x' &= x \times \cos(\hat{T}_x^1 \times \hat{T}_y^2 \times \theta_3) - y \times \hat{T}_y^1 \times \hat{T}_2^3 \times \sin(\hat{T}_x^1 \times \hat{T}_2^3 \times \theta_3), \\
    y' &= x \times \hat{T}_y^1 \times \hat{T}_2^3 \times \sin(\hat{T}_x^1 \times \hat{T}_2^3 \times \theta_3) + y \times \cos(\hat{T}_x^1 \times \hat{T}_2^3 \times \theta_3),
\end{align*}
\]

(see\(^{(3e)}\) for general isorotations in all there Euler angles). Isotransforms (8) leave invariant all ellipsoidal deformations \(x \times \hat{T}_x^1 \times \hat{T}_y^2 \times \hat{T}_2^3 \times \hat{T}_z^3 \times z = R\) of the sphere \(x \times x + y \times y + z \times z = r\). Such ellipsoid become perfect spheres \(\hat{r}^2 = (\hat{r}^t \times \hat{\delta} \times \hat{r}) \times I_s\) in isoeuclidean spaces\(^{(3h,3r)}\) \(E(\hat{r}, \hat{\delta}, R), \hat{r} = \{\hat{r}^k\} = \{\hat{r}^k\} \times I_s, \hat{\delta} = \hat{T}_s \times \hat{\delta}, \delta = \text{diag.}(1,1,1), \hat{T}_s = \text{diag.}(\hat{T}_1, \hat{T}_2, \hat{T}_3), I_s = T_s^{-1}\), called isospheres.

In fact, the deformation of the semi-axes \(1_k \to \hat{T}_{kk}\) while the related units are deformed of the inverse amounts \(1_k \to \hat{T}_{kk}^{-1}\), preserves the perfect sphericity (because the invariant in isospace is \((\text{Length})^2 \times (\text{Unit})^2\)). Note that this perfect sphericity in \(E\) is the geometric origin of the isomorphism \(\hat{O}(3) \equiv O(3)\), with consequential preservation of the exact rotational symmetry for the space–components \(g(r)\) of all possible Riemannian metrics (becomes the isogeodesics are perfect circles).

2) Isoboosts. The connected Lorentz-Santilli isosymmetry \(S\hat{O}(3.1)\) is characterized by the isorotations and the isoboosts\(^{(3h)}\) which can be written in the \((3,4)\)-plane.
\[ x^{3'} = x^3 \times \sinh(\hat{T}^{33}_{33} \times \hat{T}^{44}_{44} \times v) - x^4 \times \hat{T}^{33}_{33} \times \hat{T}^{44}_{44} \times \cosh(\hat{T}^{33}_{33} \times \hat{T}^{44}_{44} \times v) = \]
\[ = \tilde{\gamma} \times (x^3 - \hat{T}^{33}_{33} \times \hat{T}^{44}_{44} \times \hat{\beta} \times x^4) \]
\[ x^{4'} = -x^3 \times \hat{T}^{33}_{33} \times c_0^{-1} \times \hat{T}^{44}_{44}^{-1} \times \sinh(\hat{T}^{33}_{33} \times \hat{T}^{44}_{44} \times v) + x^4 \times \cosh(\hat{T}^{33}_{33} \times \hat{T}^{44}_{44} \times v) = \]
\[ = \tilde{\gamma} \times (x^4 - \hat{T}^{33}_{33} \times \hat{T}^{44}_{44}^{-1} \times \hat{\beta} \times x^3) \]
\[ \tilde{\beta}^2 = v_k \times \hat{T}_{kk} \times v_k/c_0 \times \hat{T}_{44} \times c_0, \quad \tilde{\gamma} = (1 - \tilde{\beta}^2)^{-\frac{1}{2}}. \quad (9) \]

Note that the above isotransforms are formally similar to the Lorentz transforms, as expected from their isotopic character. Isotransforms (9) characterize the light isocone\(^{(3s)}\), i.e., the perfect cone in isospace \(\hat{M}\). In a way similar to the isosphere, we have the deformation of the light cone axes \(1_\mu \rightarrow \hat{T}^{\mu \mu}_{12}\) while the corresponding units are deformed of the inverse amount \(1_\mu \rightarrow \hat{T}^{-1}_{\mu \nu}\), thus preserving the perfect cone in isospace.

In particular, the isolight cone also has the conventional characteristic angle, as a necessary condition for an isotopy (the proof of the latter property requires the use of isotrigonometric and isohyperbolic functions). Thus, the maximal causal speed in isominkowski space is the conventional speed in vacuum \(c_0\). The identity of the light cone and isocones is the geometric origin of the isomorphism \(\hat{SO}(3,1) \approx SO(3,1)\) and, thus, of the exact validity of the Lorentz symmetry for all possible Riemannian metrics \(g(x)\).

3) Isotranslations. The isotopies of translations can be written

\[ x' = (e^{ix \hat{p} \times a}) \hat{x} \hat{x} = [x + a \times A(x)] \times \hat{I}, \hat{p}' = (e^{ix \hat{p} \times a}) \hat{\hat{p}} = \hat{\hat{p}}, A_\mu = \hat{T}^{1/2}_{\mu \mu} + a^\alpha \times [\hat{T}^{1/2}_{\mu \mu}, \hat{p}_\alpha]/1! + \ldots. \quad (10) \]

and they are also nonlinear, as expected.

4) Isoselftransforms. Intriguingly, the isotopies identify one additional symmetry which is absent in the conventional case. It is here called isoselfscalar invariance and it is given by

\[ \hat{I} \rightarrow \hat{I}' = n^2 \times \hat{I}, \hat{\eta} \rightarrow \hat{\eta} = n^2 \times \eta, \quad (11) \]

and

\[ x^\prime = (e^{ix \hat{p} \times a}) \hat{x} \hat{x} = (x + a \times A(x)) \times \hat{I}, \hat{p}' = (e^{ix \hat{p} \times a}) \hat{\hat{p}} = \hat{\hat{p}}, A_\mu = \hat{T}^{1/2}_{\mu \mu} + a^\alpha \times [\hat{T}^{1/2}_{\mu \mu}, \hat{p}_\alpha]/1! + \ldots. \quad (10) \]
where $n$ is an 11-th parameter, under which the interval remains invariant, \( \hat{x}^2 = (x^\mu \times \hat{T}_\mu \times \eta_{\alpha\nu} \times x^\nu) \times \hat{I} \equiv [x_\mu \times (n^{-2} \times \hat{T}_\mu) \times \eta_{\alpha\nu} \times x^\nu] \times (n^2 \times \hat{I}) \).

Note that, even though $n^2$ is factorizable, the corresponding isosymmetry is not trivial, e.g., because $n^2$ enters into the argument of the isosoretz transforms (9). Note also that the isominkowskian representation of gravity is permitted precisely by the latter isoinvariance. In fact, isoinvariance (11) holds also for the conventional Poincaré symmetry, by introducing the generalized unit at the foundation of the isominkowskian gravity.

The same symmetry also holds for the isoinner product (whenever $n$ does not depend on the integration variable),
\[
\langle \hat{\Phi} \times \hat{T} \times |\hat{\Psi} \times \hat{I} \equiv < \hat{\Phi} \times (n^{-2} \times \hat{T}) \times |\hat{\Psi} \times (n^2 \times \hat{I}) >.
\]

Note finally that the latter symmetries have remained undetected throughout this century because they required the prior discovery of new numbers, those with an arbitrary unit.

5) Isoinversions. The isodiscrete transforms (3) are
\[
\pi \times x = (-r, x^4), \hat{\tau} \times x = \tau \times x = (r, -x^4), \hat{\pi} = \pi \times \hat{I}, \hat{\tau} = \tau \times \hat{I},
\]

where $\pi$, $\tau$ are the conventional inversion operators. Despite their simplicity, the physical implications of isoinversions are nontrivial because of the possibility of reconstructing as exact discrete symmetries when believed to be broken, which is studied by embedding all symmetry breaking terms in the isounit.

The general Poincaré-Santilli isosymmetry is usually defined as the 11-dimensional set of isorotations, isoboosts, isotranslations, isoinversions, isoselftransforms and isoinversions. The restricted Poincaré-Santilli isosymmetry is the general isosymmetry in which the isounit is averaged into constants.

6. Inclusion of interior gravitation. The attentive reader may have noted that the isotopies leave unrestricted the functional dependence of the isometric. Its sole dependence on the coordinates is therefore a restriction which has been used so far for a representation of exterior gravitation in vacuum.

In the general case we have isometrics with an unrestricted functional dependence, \( \hat{\eta} = \hat{T}(x, v, \Psi, \partial\Psi, ...) \times \eta, \hat{T} > 0, v = dx/dt \), which, as such, can represent interior gravitation problems with a well behaved but otherwise unrestricted nonlinearity in the velocities, wave functions and their derivatives, as expected in realistic interior models, e.g., of neutron stars, quasars, black holes and all that.

Note also that the isometric can also contain nonlocal–integral terms, e.g., representing wave-overlapplings. Nevertheless, the theory verifies the condition of locality in isospace, called isolocality, because its topology is everywhere local except at the unit.

Note that the addition of interior gravitational problems occurs without altering the axioms of the exterior problem in vacuum, yet gaining an arbitrary functional dependence.
for more realistic treatments of interior conditions. This evidently permits a geometric unification of exterior and interior gravitational problems which are solely differentiated by the functional dependence of the isounits.

A first illustration of the extension of the exterior axioms to realistic interior conditions is offered by the isoselfscalar transforms (11) which permit the representation of electromagnetic waves propagating within physical media with local varying speed \( c = c_0 / n \).

This allows the construction, apparently for the first time, of Schwarzschild’s and other gravitational models within atmospheres and chromospheres with a locally varying speed of light. Applications to specific cases, such as the study of gravitational horizons via the light isocone, are then expected to permit refinements of current studies evidently due to deviations from the value in vacuum of the speed of light in the hyperdense chromospheres outside gravitational horizons.

Note that the general Poincaré-Santilli isosymmetry is used for the local speed of light within the interior of a given atmosphere or chromosphere, while the restricted isosymmetry is used when the average speed of light is needed.

As an example, the characterization of the speed of electromagnetic waves traveling within a planetary atmosphere or chromosphere requires the general Poincaré-Santilli isosymmetry because changing with the density, temperature, etc. On the contrary, the characterization of the average speed of electromagnetic waves propagating through an entire given atmosphere or chromosphere requires the use of the restricted isosymmetry.

We finally note that the realization of the isotopic element \( \hat{T}_{\mu\nu} = n^{-2}, \mu = 1, 2, 3, 4 \) is a particular case of the broader realization \( \hat{T}_{\mu\nu} = n^{-2}n_{\mu} \) in which the index of refraction is \( n_4 \) and the \( n_{\mu} \)'s provide its “space-time symmetrization”. The latter realization is particularly suited for the direct geometrization of the anisotropy of astrophysical atmospheres or chromospheres caused by intrinsic angular momenta, as well as their inhomogeneity caused by the radial change of density and other characteristics.

7. Direct universality of the Poincaré-Santilli isosymmetry for exterior and interior gravitations. The results of this note imply the following:

**Theorem 2.** The 11-dimensional, general, Poincaré-Santilli isosymmetry on isominkowski spaces over real isofields with well behaved, positive-definite isounits is the largest possible isolinear, isolocal and isocanonical invariance of isoseparation (2) (universality) in the fixed x-frame of the experimenter (direct universality).

The verification of the invariant under the Poincaré-Santilli isotransforms of all possible separation (2) is instructive. The maximal character of the isosymmetry can be proved as in the conventional case. Note that for any arbitrarily given (diagonal) Riemannian metric \( g(x) \) (such as Schwarzschild, Krasner, etc.) there is nothing to compute because one merely
plots the $\hat{T}_{\mu\nu}$ terms of the decomposition $g_{\mu\nu} = \hat{T}_{\mu\nu} \times \eta_{\mu\nu}$ (no sum) in the above given isotransforms. The invariance of the separation $x^t \times g \times x$ is then ensured.

The $(2 + 2)$–de Sitter or other cases can be derived from the theorem via mere changes of signature or dimension of the isounit. The extension to positive-definite yet nondiagonal isounit is trivial and will be implied hereon.

Note finally that isosymmetry $\hat{P}(3.1)$ cannot be even defined, let alone constructed in conventional Riemannian spaces and all their possible isotopies, thus rendering the isominkowskian formulation of gravity rather unique for our purposes.

8. Resolution of some of the controversies in gravitation. In summary, in this note we have presented, apparently for the first time, a geometric unification of the special and general relativities for both classical and operator profiles, as well as for both exterior and interior problems. The results are centrally dependent on the use of the isominkowskian geometry introduced in this note and Ref.\(^{(3u)}\), rather than the use of the isoriemannian form as studied in Ref.\(^{(3s)}\).

The above unifications are centrally dependent on the achievement of a universal symmetry for gravitation which, by conception and construction, is locally isomorphic to the Poincaré symmetry of the special relativity. This eliminates the historical difference between the special and general relativities whereby the former admits a universal symmetry, while the latter does not\(^{(1,2)}\). Note the necessity of the representation of gravity in isominkowski space for the very formulation of the above unifications.

These results have a number of implications. First, they allow to illustrate the viewpoint expressed in Sect. 1 to the effect that some of controversies in gravitation may well be due to insufficiencies in the used mathematics.

The first illustration is given by the physical shortcoming of conventional formulation of gravitation of being without invariant basic units of space and time (Theorem 1). This shortcoming can now be rigorous verified via Theorem 2. In fact, it is easy to see that, when formulated on conventional spaces over conventional fields, the isosymmetry $\hat{P}(3.1)$ does not leave invariant conventional units.

Theorem 2 also allows to resolve the shortcoming. In fact, the space-time isounit is indeed invariant under the isosymmetry $\hat{P}(3.1)$ by conception and construction. Moreover, it has ben proved in the adjoint analytic study [\(3x\)] that the isosymmetry $\hat{P}(3.1)$ also leaves invariant the conventional unit $I = \text{diag. } ([1, 1, 1, 1])$ when interpreted as isocanonical transforms on isospaces over isofields. This is an expected consequence of the mechanism of isotopies according to which the joint lifting of a metric while the basic unit is lifted by the inverse amount preserves all original properties.

Theorem 2 also permits the resolution of the controversy whether the total conservation laws of general relativity are compatible with those of the special relativity via a mere visual examination.
Recall that the generators of all space-time symmetries characterize total conserved quantities. The compatibility of the total conservation laws of the general and special relativities is therefore established by the visual observation that the generators of the conventional and isotopic Poincare’ symmetries coincide. In fact, only the mathematical operations on them are changed in the transition from the relativistic to the gravitational case.

Yet another controversy which appears to be resolved by our isominkowskian treatment of gravity is the apparent lack of a meaningful relativist limit in conventional gravitational theories. In fact, such a limit is now clearly and unequivocally established by $I \rightarrow I$ under which the special relativity is recovered identically in all its aspects.

The isominkowskian treatment of gravity also permits a resolution of some of the limitations of conventional gravitational models, such as their insufficiency to provide an effective representation of interior gravitational problems. In fact, conventional formulations of gravity admit only a limited dependence on the velocities, while being strictly local-differential and derivable from a first-order Lagrangian (variationally self-adjoint). These characteristics are evidently exact for exterior problems in vacuum.

By comparison, interior gravitational problems, such as all forms of gravitational collapse, are constituted by extended and hyperdense hadrons in conditions of total mutual penetration in large numbers into small regions of space. It is well known that these conditions imply effects which are arbitrarily nonlinear in the velocities as well as in the wavefunctions, nonlocal-integral on various quantities and variationally nonselfadjoint, (i.e. not representable via first-order Lagrangians). It is evident that the latter conditions are beyond any scientific expectation of quantitative treatment via conventional gravitational theories.

The isominkowskian formulation of gravity resolve this limitation too and shows that it is equally due to insufficiencies in the underlying mathematics. In fact, isogravitation extends the applicability of Einstein’s axioms to a form which is ”directly universal” for exterior and interior gravitations.

As indicated earlier, this extension is due to the fact that the functional dependence of the metric in Riemannian treatments is restricted to the sole dependence on the local coordinates, $g = g(x)$, while under isotopies the same dependence becomes unrestricted, $g = g(x, v, \phi, \partial \psi, ...)$ without altering the original geometric axioms. This results in geometric unification of exterior and interior problems, despite their sizable structural differences of topological, analytic and other characters. The latter unification was studied in details in ref.(3s) under the isoriemannian geometry and it is studied with the isominkowskian geometry in this note for the reasons indicated earlier.

Yet another controversy which appears to be resolved by the isominkowskian formulation of gravity is the achievement of an axiomatically consistent operator version of gravity, that is with: invariance of the basic units; preservation of the original Hermiticity at all times; uniqueness and invariance of the numerical predictions; consistent PCT and other theorems; etc.
Even though far from being a complete theory, our OIG does indeed offer realistic hopes of achieving such an axiomatically consistent operator form of gravity, as expected from the validity of the conventional axioms of RQM.

The resolution of other controversies cannot be studied in this introductory note and are contemplated for study in subsequent works.

**Concluding Remarks.** By keeping in mind that Einstein’s field equations are preserved unchanged by conception, an important issue is whether the isominkowskian reformulation of gravity coincides with conventional gravity on physical grounds or it predicts novel physical features.

It is easy to see that a number of new features are indeed predicted. To begin, the isominkowskian formulation of gravity predicts that the maximal causal speed in our space-time is a local quantity which can be arbitrarily smaller or bigger 'than the speed of light in vacuum.

In fact, except for being well behaved (and non-null), the parameter $n$ of isoselftransformation (11) remains unrestricted by the isotopies. Therefore, we have $n = 1$ in vacuum for which $c = c_0/n = c_0$, but otherwise we can have $n > 1(c < c_0)$ or $n < 1(c > c_0)$. As a result, the Poincaré-Santilli isosymmetry is a natural invariance for arbitrary causal speeds, whether equal, smaller or bigger than the speed of light in vacuum.

The case $c < c_0$ is known since Lorentz’s\(^{(7a)}\) who was the first to investigate the lack of applicability of his celebrated transforms for electromagnetic waves propagating in our atmosphere or other transparent material media (see also the related quotation by Pauli\(^{(7b)}\)).

The case $c > c_0$ has been predicted since some time in interior problems only, but experimentally detected only recently, e.g., for the speed of photons traveling in certain guides\(^{(8a,8b)}\) or for the speed of matter in astrophysical explosions\(^{(8c−8e)}\). The recent Ref.\(^{(8f)}\) has identified solutions of conventional relativistic equations with arbitrary speeds in vacuum of which $\hat{P}(3.1)$ is evidently the natural invariance). If confirmed, these waves would be the first case of speeds $c > c_o$ in exterior conditions in vacuum.

It should be noted that, despite the local variation of $c$, the maximal causal speed on $\hat{M}$ over $\hat{R}$ remain $c_0$, again, because the change $c \to c_0/n$ is compensated by an inverse change of the unit $1 \to n$. By recalling that the STR is evidently inapplicable (and not "violated") for locally varying causal speeds, we can therefore say that the isotopies render the STR universally applicable for relativistic and gravitational, classical and operator, as well as exterior or interior problems, under local speeds of electromagnetic waves.

Yet other novel predictions are related to gravitational singularities. In fact, we have the following property of self-evident proof.

**Theorem 3.** Gravitational singularities (horizons) are the zeros of the space (time) component of the isounit.
The above novel interpretation of gravitational singularities and horizons is trivially equivalent to the conventional one for the exterior case in vacuum. However, gravitational collapse is a typical interior case for which the isotopic representation becomes nontrivial, e.g., because it includes the nonlinear, nonlocal and noncanonical effects indicated earlier. The isominkowskian formulation of gravity therefore implies a re-examination of gravitational singularities on both mathematical and physical grounds which will be done elsewhere.

Note that the zeros of the isounit have been excluded from Theorem 2 because of their yet unknown topological structure.

Another important implication of the isominkowskian formulation of gravity is that it offers realistic possibilities for an axiomatically consistent inclusion of gravitation in grand unified gauge theories, as studied in the recent contribution to the VIII M. Grossmann Meeting on General Relativity. By comparison, no such inclusion is possible for the Riemannian treatment of gravitation.

Intriguingly, the Isotopic Grand Unification is permitted precisely by the elimination of the conventional notion of curvature, as an evident necessary condition to bring gravitation into a form axiomatically compatible with electroweak interactions.

We should also indicate the novel prediction of the isodoppler shift, namely, a shift due to the inhomogeneity and anisotropy of the medium in which electromagnetic waves propagate, which is suitable for experimental verifications.

But perhaps the most intriguing novel feature of the isominkowskian formulation of gravity is that of introducing a novel notion of space-time, where the novelty rests in the units of space-time itself. For instance, we have a novel local notion of time, as illustrated by the dependence of its unit from the gravitational field in the isotopic reformulation of the Schwarzschild metric, \( \hat{T}_t = (1 - 2M/r) \).

As one can see, time is predicted to have novel different flows for different gravitational fields, according to a behaviour which is different than that predicted by conventional gravitational theories. Space has a behaviour which is the inverse of that of time. Conventional space-time is recovered in empty space for \( M = 0 \) (or for infinite distances from a gravitational field).

This novel notion of space-time admits additional intriguing and far reaching implication. For instance, by recalling that the structure of the isotopic invariant is \([\text{Length}]^2 \times [\text{Unit}]^2 = \text{Inv.}\) we have a new form of geometric propulsion called isolocomotion in which distances are covered by their geometric reduction due to the local increase of the energy, rather than via Newtonian displacement. In fact, the space-isounit increases with the local energy, thus implying a decrease of the distance.

Similarly, we have a novel cosmology called isocosmology which is the first on scientific record being characterized by a universal symmetry, the Poincaré-Santilli isosymmetry \( \hat{P}(3,1) \), the first to admit a realistic representation of interior gravitational problem and the first to admit arbitrary local values of the maximal causal speed, with a number of conse-
quences, such as the elimination of the need for the “missing mass” in the universe (see ref. [3s] for brevity).

It should be also indicated that the isotopies with basic lifting $I \rightarrow \hat{I}(x, \Psi, ...)$ = $\hat{I}^\dagger$ constitute only the first step of a chain of generalized methods (3f). The second class is given by the genotopies (3a) in which the isounit is no longer Hermitean. This broader class geometrizes in a natural way the irreversibility of interior gravitational problems and it has been used, e.g., for the black hole model of ref. (4d). A third class of methods is given by the (multi–valued) hyperstructures (3f), in which the generalized unit is constituted by a set of non-Hermitean quantities. The latter class appears to be significant for quantitative studies of biological structures with their typical irreversibility and variation of physical characteristics for which the conventional RQM is manifestly inapplicable due to its reversibility as well as sole characterization of conservation laws.

Also, the isotopies, genotopies and hyperstructures admit antiautomorphic images, called isodualities (3r,3s), and characterized by the map $\hat{I} \rightarrow \hat{I}^d = -\hat{I}^\dagger$ which are currently under study for antimatter (3q). In this case the energy–momentum tensor of antimatter becomes negative–definite, thus removing a problem of compatibility between the current representations of antimatter in classical and particle physics. The gravitational treatment of antimatter via the isodualities of the isominkowskian geometry is studied elsewhere and it is another necessary condition for the axiomatically consistent inclusion of gravity in unified gauge theories of electroweak interactions (3q).

On historical grounds, we note that, as studied in detail in memoir (3t) for the general case of RHM, our OIG can be interpreted as a nonunitary completion of RQM considerably along the historical $E - P - R$ argument (9a) for which von Neumann’s theorem (9b) and Bell’s inequalities (9c) do not apply evidently because of their nonunitary structure.

Moreover, from the abstract identity of the right modular associative action $H \times |\Psi >$ and its isotopic image $\hat{H} \hat{\times} |\hat{\Psi} >$, one can see that the isoeigenvalue equation $\hat{H} \hat{\times} |\hat{\Psi} > = E_{\hat{T}} \times |\hat{\Psi} >$ characterizes an explicit and concrete operator realization of the ”hidden variable” $\lambda = \lambda(x, ...)$ $\equiv \hat{T}$. Our isotopic formulation of gravity can therefore be interpreted as a realization of the theory of hidden variables. After all, the ”hidden” character of gravitation in our theory is illustrated by the recovering of the conventional unit under the isoexpectation value $\langle \hat{I} \rangle = I$.

In conclusion, the viewpoint we have attempted to convey in this note is that an alternative formulation of gravity always existed. It did creep in un–noticed until now because embedded where nobody looked for, in the unit of the classical and quantum version of the special relativity.

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