Cosmic slowing down of acceleration using $f_{\text{gas}}$

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Abstract

We investigate the recent – low-redshift – expansion history of the universe using the most recent observational data. Using only data from 42 measurements of $f_{\text{gas}}$ from Allen et al. in clusters, we found that cosmic acceleration could have already peaked and we are witnessing now its slowing down. This effect, found previously by Shafieloo, Sahni and Starobinsky in 2010 using supernova data (at that time the Constitution supernovae Type Ia sample) appears again using an independent observational probe. We also discuss the result using the most recent Union 2.1 data set.

Key words: circumstellar matter – galaxies: clusters: general – cosmological parameters – dark energy – large-scale structure of Universe – infrared: stars.

1 INTRODUCTION

Dark energy is the name of the mysterious component which drives the current accelerated expansion of the universe (Frieman, Turner & Huterer 2008; Li et al. 2011a). In its simplest form, this can be described by a fluid with constant equation-of-state parameter $w = -1$, which corresponds to a cosmological constant, leading to the successful Λ cold dark matter (ΛCDM) model: the simplest model that fits a varied set of observational data.

This model has several unnatural properties that lead us to look for models whose description is less forced. For example, the current value for $\Omega_\Lambda$ and $\Omega_M$ are of the same order of magnitude, a fact highly improbable, because the dark matter contribution decreases with $a^{-3}$, where $a(t)$ is the scalefactor, meanwhile the cosmological constant contribution has always had the same value. This problem in particular is known as the cosmic coincidence problem.

Thus, it seems necessary to assume the existence of a dynamical cosmological constant or a theoretical model with a dynamical equation-of-state parameter $w(z) = p/\rho$. The source of this dynamical dark energy could either be a new field component filling the universe, as a quintessence scalar field (Wetterich 1988; Ratra & Peebles 1988; Frieman et al. 1995; Turner & White 1997; Caldwell, Dave & Steinhardt 1998; Steinhardt et al. 1999) or be produced by modifying gravity (Tsujikawa 2010; Capozziello & De Laurentis 2011; Starkman 2011).

In Shafieloo, Sahni & Starobinsky 2009, the authors suggested that the current observational data favour a scenario in which the acceleration of the expansion has passed a maximum value and is now decelerating. The key point in deriving this conclusion is the use of the Chevalier–Polarski–Linder (CPL) parametrization (Chevallier & Polarski 2001; Linder 2003) for a dynamical equation-of-state parameter:

$$w(a) = w_0 + (1 - a)w_1,$$

where $w_0$ and $w_1$ are constants to be fixed by observations, in that case the Constitution data set (Hicken et al. 2009b). Using the Union 2 data set (Amanullah et al. 2010) the authors in Li et al. (2011b) found similar conclusions, under the assumption of a flat universe. In Cárdenas & Rivera (2011), we revisit this problem using the Union 2 data set and we extend the analysis to curved space–times. We found the three observational tests, supernovae Type Ia (SNIa), baryon acoustic oscillation (BAO) and cosmic microwave background (CMB), that can all be accommodated in the same trend, assuming a very small value for the curvature parameter, $\Omega_k \simeq -0.08$. The best-fitting values found suggested that the acceleration of the universe has already reached its maximum and is currently moving towards a decelerating phase.

In this paper, we investigate the recent (low-redshift) expansion history of the universe in light of recent data. We use the gas mass fraction in clusters – 42 measurements of $f_{\text{gas}}$ extracted from (Allen et al. 2008) – and the latest Union 2.1 data set (Suzuki et al. 2011) of SNIa. We also consider the constraints from BAO (Reid et al. 2010a) through the distance scale and also from CMB from the Wilkinson Microwave Anisotropy Probe (WMAP) 9-yr distance posterior (Hinshaw et al. 2012).

The paper is organized as follows: in the next section, we restudy the flat universe case using both SNIa data sets (Hicken et al. 2009b; Amanullah et al. 2010). Then, we present the results using the $f_{\text{gas}}$ data. After that, we discuss the result using the latest Union 2.1 SNIa data. We end with a discussion of the results.

2 $q(z)$ AT LOW REDSHIFT FROM SNIa

In this section, we describe the results obtained by Shafieloo et al. (2009) and Li et al. (2011b) for the deceleration parameter at small

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redshift using the Constitution (Hicken et al. 2009b) and Union 2 (Amanullah et al. 2010) data sets. The analysis of this section is restricted to flat universes. The extension to curved space–time was performed first in Cardenas & Rivera (2011), and can be implemented directly.

The first step is to define a cosmological model to test. The comoving distance from the observer to redshift \( z \) is given by

\[
r(z) = \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')},
\]

(2)

where

\[
E^2(z) = \Omega_m (1 + z)^3 + \Omega_k f(z),
\]

(3)

where \( \Omega_k = 1 - \Omega_m \) and in general

\[
f(z) = \exp \left\{ 3 \int_0^z \frac{1 + w(z')}{1 + z'} \frac{dz'}{z'} \right\}.
\]

(4)

In this case, the cosmological model is defined through a selection of the equation-of-state parameter \( w(z) \). Using the CPL parametrization (1), \( w = w_0 + w_1 z/(1 + z) \), the function defined in (4) takes the form

\[
f(z) = \exp \left( \frac{3w_1 z}{1 + z} \right).
\]

(5)

Performing the analysis, we find the best values of the parameters that fit the observations (the details of the statistical analysis is described in Appendix ), which in turn give us the best Hubble function \( E(z) \) as a function of redshift that agrees with the data. From it, we can compute the deceleration parameter as

\[
q(z) = - (1 + z) \frac{1}{E(z)} \frac{dE(z)}{dz} - 1.
\]

(6)

Using the Constitution sample consisting of 397 SNIa data points leads to the results shown in Table 1, which are very similar to the ones informed in Shafieloo et al. (2009) and Wei (2010).

In particular, using only the SNIa data, we plot the deceleration parameter as a function of redshift in Fig. 1. We observe that even at 1\( \sigma \), the expansion of the universe peaks at redshift \( z \approx 0.2 \) and then evolves towards a non-decelerating regime today. If we add the BAO constraints, this behaviour is still observed. However, as was mentioned in the original papers, incorporating the CMB constraints leads to a smooth deceleration–acceleration transition for \( q(z) \), typically of \( \Lambda \)CDM (\( w_0 \approx -1 \) and \( w_1 \approx 0 \)). The values of \( \chi^2_{min} \) show that something is wrong with the CPL parametrization in trying to fit the three observational probes. The large negative value of \( w_1 \) is responsible of this anomalously low-redshift behaviour using supernovae shown in Fig. 1. Although the error in the determination of \( w_1 \) is larger, using only SNIa compared to the other cases, even at 1\( \sigma \) the low-redshift transition is still observable.

Using the Union 2 set (Amanullah et al. 2010) consisting of 557 SNIa, the analysis leads to the results shown in Table 2 (see also Fig. 2).

Table 1. The best-fitting values for the free parameters using the Constitution data set in the case of a flat universe model using only SNIa data (1), SNIa+BAO (2) and SNIa+BAO+CMB (3). See the related Fig. 1.

| Set       | \( \chi^2_{min} / \text{dof} \) | \( \Omega_m \) | \( w_0 \)  | \( w_1 \)  |
|-----------|---------------------------------|----------------|-----------|-----------|
| (1)       | 461.2/394                       | 0.452 ± 0.043  | −0.22 ± 0.59 | −11 ± 7   |
| (2)       | 466.6/400                       | 0.303 ± 0.034  | −0.89 ± 0.25 | −1 ± 2    |
| (3)       | 468.2/403                       | 0.281 ± 0.014  | −0.99 ± 0.12 | −0.1 ± 0.5 |

Again, we found large negative values for \( w_1 \) in the SNIa and SNIa+BAO fit, but in the case with CMB data, the value of this parameter appears smaller and positive.

This feature – the inversion of \( q(z) \) at \( z \approx 0.2 \) – can be interpreted in two ways: first, we can assume that the origin of the feature is the use of an inappropriated parametrization, in this case we can say that the CPL parametrization is inadequate to fit data at large and small redshift. Clearly, a large negative value for \( w_1 \) spoils the large \( z \) behaviour of \( w(z) \). The other way, is to consider the data from SNIa and also from BAO that are telling us something about our recent – low-redshift – expansion history of the universe, which the CMB data cannot detect.
This is the kind of small redshift transitions in $w(z)$ that were discussed first in Bassett et al. (2002) and also by Mortonson et al. (2009). In Bassett et al. (2002), they used a form,

$$w(z) = w_0 + \frac{w(z_f) - w_0}{1 + \exp[(z - z_0)/\Delta]},$$

that captures the essence of a single transition at $z_0$ in a range $\Delta$, from $w_0$ initially to a final value $w_f$ in the future. In Mortonson et al. (2009), the authors discussed the possibility of a fast change in $w(z)$ at $z < 0.02$ and its implication for a standard scalar field model. A more recent work focused on small redshift transitions in parametrizations of $q(z)$ instead of $w(z)$ (Gong & Wang 2006, 2007; Cunha 2009; Cai & Tuo 2011), the so called cosmographic approach. Even some authors have claimed that the universe has already entered into a decelerated phase using only low-redshift ($z < 0.1$) SNIa data (Guimaraes & Lima 2011).

To settle this dilemma, independent evidence is required to identify whether this effect is real or not. In the next section, we study this problem using data from gas mass fraction in clusters, an observational probe completely independent of the SNIa.

### 3 RESULTS USING THE $f_{\text{gas}}$ DATA

Sasaki (1996) was the first who described the method of using measurements of the apparent dependence on redshift of the baryonic mass fraction to constrain cosmological parameters. The method is based on theoretical arguments and simulation results indicating that this mass fraction, in the largest clusters, must be constant with redshift $z$ (Eke, Navarro & Frenk 1998). This null evolution with $z$ is only possible if the reference cosmology used in making the measurements of the baryonic mass fraction matches the true cosmology.

In this section, we use the data from Allen et al. (2008) which consist of 42 measurements of the X-ray gas mass fraction $f_{\text{gas}}$ in relaxed galaxy clusters spanning the redshift range $0.05 < z < 1.1$. The $f_{\text{gas}}$ data are quoted for a flat $\Lambda$CDM reference cosmology with $h = H_0/100 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1} = 0.7$ and $\Omega_M = 0.3$.

To determine constraints on cosmological parameters, we use the model function (Allen et al. 2004)

$$f_{\text{gas}}^{\Lambda\text{CDM}}(z) = \frac{b \Omega_b}{(1 + 0.19 \sqrt{h}) \Omega_M} \left[ \frac{d_A^{\Lambda\text{CDM}}(z)}{d_A(z)} \right]^{3/2},$$

where $b$ is a bias factor which accounts that the baryon fraction is slightly lower than for the universe as a whole. From Eke et al. (1998), it is obtained that $b = 0.824 \pm 0.0033$. In the analysis we also use standard Gaussian priors on $\Omega_b h^2 = 0.0214 \pm 0.0020$ (Kirkman et al. 2003) and $h = 0.72 \pm 0.08$ (Freedman et al. 2001).

The results of the analysis are shown in Table 3, where we have omitted the best-fitting values for $\Omega_b$, $b$ and $h$ because they do not show significant variations.

### 4 $q(z)$ FROM THE UNION 2.1 DATA ALONE

The latest sample of SNIa is the Union 2.1 (Suzuki et al. 2011). It consists of 580 data points spanning a redshift range $0.015 < z < 1.41$. The results, performing the same analysis as before, are displayed in Table 4 and also in Fig. 4.

In Fig. 3, we plot the decelerated parameter as a function of redshift in the case of $f_{\text{gas}}$+BAO. We get the same trend at small $z$ previously found using the Constitution and Union 2 samples of SNIa within $1\sigma$. This is an independent confirmation that the data are telling us something about our local universe.

![Figure 3](https://example.com/figure3.png)

**Figure 3.** Using the $f_{\text{gas}}$ data from Allen et al. (2008), we plot the deceleration parameter reconstructed using the best-fitting values for the $f_{\text{gas}}$+BAO case. We consider the error propagation at $1\sigma$ in the best-fitting parameters.

In Fig. 4, we plot the decelerated parameter as a function of redshift in the case of $f_{\text{gas}}$+BAO. We get the same trend at small $z$ previously found using the Constitution and Union 2 samples of SNIa within $1\sigma$. This is an independent confirmation that the data are telling us something about our local universe.

### Table 3. The best-fitting values for the free parameters using only $f_{\text{gas}}$ data (1), $f_{\text{gas}}$+BAO (2) and $f_{\text{gas}}$+BAO+CMB (3) in the case of a flat universe model. See also Fig. 3. We have omitted the best-fitting values for $\Omega_b$, $b$ and $h$ because they do not show significant variations.

| Data set | $\chi^2_{\text{min}}$/dof | $\Omega_m$ | $w_0$ | $w_1$ |
|----------|-------------------------|-----------|-------|-------|
| (1)      | 41.63/39                | 0.271 ± 0.019 | -0.83 ± 0.65 | -2.4 ± 5.8 |
| (2)      | 43.40/45                | 0.280 ± 0.016 | -0.68 ± 0.41 | -3.3 ± 3.2 |
| (3)      | 43.91/48                | 0.270 ± 0.010 | -1.34 ± 0.12 | 1.0 ± 0.4 |

### Table 4. The best-fitting values for the free parameters using the Union 2.1 data set in the case of a flat universe model. See also Fig. 4.

| Set | $\chi^2_{\text{min}}$/dof | $\Omega_m$ | $w_0$ | $w_1$ |
|-----|-------------------------|-----------|-------|-------|
| (1) | 562.23/577              | 0.27 ± 0.47 | -1.00 ± 0.43 | 0.5 ± 5 |
| (2) | 563.64/583              | 0.295 ± 0.032 | -1.02 ± 0.18 | -0.2 ± 1.1 |
| (3) | 564.65/586              | 0.279 ± 0.014 | -1.04 ± 0.11 | 0.1 ± 0.5 |

![Figure 4](https://example.com/figure4.png)

**Figure 4.** Using the Union 2.1 data from Suzuki et al. (2011), we plot the deceleration parameter reconstructed using the best-fitting values for the SNIa+BBAO case.
following change: a correction for the host-mass SNIa luminosity relation.

The fact that this low-redshift effect has been detected in the previous SNIa data sets, and that from now on, after some correction (from systematics) it disappears completely, can at least be considered disturbing. This is even more intriguing considering the findings in this paper, which point towards independent evidence of such a low-redshift behaviour.

A thorough study of this low-redshift behaviour of SNIa will be discussed elsewhere.

5 DISCUSSION

In this paper, using $f_{\text{gas}}$ data from clusters, we have found evidence for a low-redshift transition of the deceleration parameter, indicating that the acceleration has passed a maximum around $z \approx 0.2$ and now evolves towards a decelerating phase in the near future.

This finding is in agreement with previous studies using the Constitution (Hicken et al. 2009b) and Union 2 (Amanullah et al. 2010) data sets for SNIa, but they are in intriguing disagreement with the latest Union 2.1 set (Suzuki et al. 2011).

Our analysis then strongly suggests that low-redshift measurements have reached the precision to scrutinize the local universe, this time not only with SNIa, but also with galaxy clusters.

Regarding the intriguing disagreement mentioned, the only significant change in the analysis performed between the Union 2.1 data set and the previous ones was the correction for the host-mass SNIa luminosity relation.

This is exactly the case of a probable related issue considered in the past: the so-called Hubble bubble. In Zehavi et al. (1998) using around 40 SNIa, they found evidence for a local void or Hubble bubble as was named after. In Jha et al. (2007), almost 10 years after, the authors found new evidence pointing to similar conclusions.

Their existence has been related to the details of the treatment of extinction and reddening by dust (Conley et al. 2007). Hicken et al. demonstrated (Hicken et al. 2009a) that no such Hubble bubble exists if a reduced value $R_V = 1.7$ is used instead.

The previous work cite (Jha et al. 2007) used an MLCS2k2-fitting method with a reddening parameter $R_V = 3.1$. The Constitution and Union 2 samples were derived using a Milky Way extinction $R_V = 3.1$ from Cardelli et al. (1989), meanwhile the Union 2.1 uses a lower value $R_V = 1.7$.

As it was stressed also in Wiltshire et al. (2012) there is a danger in considering the $R_V$ parameter as an adjustable parameter in the light-curve reduction. Actually, independent studies have found average value of $R_V = 2.77 \pm 0.41$ (Finkelman et al. 2008, 2011) using galaxies, see also Motta et al. (2002) and Mediavilla et al. (2005) for a lens galaxy at $z = 0.88$, a very much the Milky Way value, so it is an important issue to take care of the treatment of the $R_V$ parameter in SNIa studies.

Our results suggest to perform a careful study at small redshift to elucidate the low-redshift – apparently anomalously–behaviour of $q(z)$.

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APPENDIX A: STATISTICAL ANALYSIS

The SN Ia data give the luminosity distance \( d_L(z) = (1 + z)r(z) \). We fit the SN Ia with the cosmological model by minimizing the \( \chi^2 \) value defined by

\[
\chi^2_{\text{SN Ia}} = \sum_{i=1}^{s} \frac{(\mu_i(z) - \mu_{\text{obs}}(z_i))^2}{\sigma^2_{\mu i}},
\]

where \( \mu(z) = 5 \log_{10}[d_L(z) \text{ Mpc}^{-1}] + 25 \) is the theoretical value of the distance modulus and \( \mu_{\text{obs}} \) is the corresponding observed one.

The BAO measurements considered in our analysis are obtained from the WiggleZ experiment (Blake et al. 2011), the Sloan Digital Sky Survey (SDSS) Third data release (DR3) BAO distance measurements (Percival et al. 2010) and Six degree field Galaxy Survey (6dFGS) BAO data (Beutler et al. 2011).

The \( \chi^2 \) for the WiggleZ BAO data is given by

\[
\chi^2_{\text{WiggleZ}} = (\vec{A}_{\text{obs}} - \vec{A}_0)^T \mathbf{C}_{\text{WiggleZ}}^{-1} (\vec{A}_{\text{obs}} - \vec{A}_0),
\]

where the data vector is \( \vec{A}_{\text{obs}} = (0.474, 0.442, 0.424) \) for the effective redshift \( z = 0.44, 0.6 \) and 0.73. The corresponding theoretical value \( \vec{A}_0 \) denotes the acoustic parameter \( A(z) \) introduced by Eisenstein et al. (2005):

\[
A(z) = \frac{D_V(z) \sqrt{\Omega_m H_0^2}}{cz},
\]

and the distance scale \( D_V \) is defined as

\[
D_V(z) = \frac{1}{H_0} \left[ (1 + z)^2 D_A(z) \frac{cz}{E(z)} \right]^{1/3},
\]

where \( D_A(z) \) is the Hubble-free angular diameter distance which relates to the Hubble-free luminosity distance through \( D_A(z) = D_L(z)/(1 + z)^2 \). The inverse covariance \( \mathbf{C}_{\text{WiggleZ}}^{-1} \) is given by

\[
\mathbf{C}_{\text{WiggleZ}}^{-1} = \begin{pmatrix}
1040.3 & -807.5 & 336.8 \\
-807.5 & 3720.3 & -1551.9 \\
336.8 & -1551.9 & 2914.9
\end{pmatrix}.
\]

Similarly, for the SDSS DR3 BAO distance measurements, the \( \chi^2 \) can be expressed as (Percival et al. 2010)

\[
\chi^2_{\text{SDSS}} = (\vec{d}_{\text{obs}} - \vec{d}_{\text{th}})^T \mathbf{C}_{\text{SDSS}}^{-1} (\vec{d}_{\text{obs}} - \vec{d}_{\text{th}}),
\]

where \( \vec{d}_{\text{obs}} = (0.1905, 0.1097) \) are the data points at \( z = 0.2 \) and 0.35. \( \vec{d}_{\text{th}} \) denotes the distance ratio,

\[
d_L = \frac{r_s(z)}{D_V(z)},
\]

where, \( r_s(z) = c \int_{z}^{\infty} \frac{c_s(z')}{H(z')} dz' \),

\[
\text{with the sound speed} \ c_s(z) = 1/\sqrt{3(1 + \frac{\Omega_b}{\Omega_m}(1 + z))}, \text{ with} \ R_0 = 31500 \Omega_m h^2 (T_{\text{CMB}}/2.7 \text{ K})^{-1} \text{ and} \ T_{\text{CMB}} = 2.726 \text{ K}.
\]

The redshift \( z_d \) at the baryon drag epoch is fitted with the formula proposed by Eisenstein & Hu (1998),

\[
z_d = \frac{129(\Omega_m h^2)^{0.251}}{1 + 0.659(\Omega_m h^2)^{0.828}}[1 + b_1(\Omega_m h^2)^{0.74}],
\]

where

\[
b_1 = 0.313(\Omega_m h^2)^{-0.419}[1 + 0.607(\Omega_m h^2)^{0.674}],
\]

\[
b_2 = 0.238(\Omega_m h^2)^{0.223}.
\]

\( \mathbf{C}_{\text{SDSS}}^{-1} \) in equation (12) is the inverse covariance matrix for the SDSS data set given by

\[
\mathbf{C}_{\text{SDSS}}^{-1} = \begin{pmatrix}
30124 & -17227 \\
-17227 & 86977
\end{pmatrix}.
\]

For the 6dFGS BAO data (Beutler et al. 2011), there is only one data point at \( z = 0.106 \), the \( \chi^2 \) is easy to compute:

\[
\chi^2_{\text{6dFGS}} = \left( \frac{d_L - 0.336}{0.015} \right)^2.
\]

The total \( \chi^2 \) for all the BAO data sets thus can be written as

\[
\chi^2_{\text{BAO}} = \chi^2_{\text{WiggleZ}} + \chi^2_{\text{SDSS}} + \chi^2_{\text{6dFGS}}.
\]

We also include CMB information by using the WMAP 9-yr data (Hinshaw et al. 2012) to probe the expansion history up to the last scattering surface. The \( \chi^2 \) for the CMB data is constructed as

\[
\chi^2_{\text{CMB}} = \mathbf{X}^T \mathbf{C}_{\text{CMB}}^{-1} \mathbf{X},
\]

where

\[
\mathbf{X} = \begin{pmatrix}
l_A - 302.40 \\
R - 1.7246 \\
z_s - 1090.88
\end{pmatrix}.
\]

Here \( l_A \) is the ‘acoustic scale’ defined as

\[
l_A = \frac{\pi d_L(z_s)}{(1 + z)r_s(z_s)},
\]

where \( d_L(z_s) = D_L(z)/H_0 \) and the redshift of decoupling \( z_s \) is given by (Hu & Sugiyama 1996),

\[
z_s = 1048 \left[ 1 + 0.00124 \left( \Omega_{m} h^2 - 0.738 \right) \right] [1 + g_t(\Omega_m h^2)]^2.
\]

\[
g_1 = \frac{0.0783 (\Omega_m h^2)^{-0.238}}{1 + 39.5 (\Omega_m h^2)^{0.765}}, \quad g_2 = \frac{0.560}{1 + 21.1 (\Omega_m h^2)^{1.81}}.
\]

The ‘shift parameter’ \( R \) is defined as (Bond, Efstathiou & Tegmark 1997)

\[
R = \frac{\sqrt{\Omega_b}}{c(1 + z_s)} D_L(z_s).
\]

\( \mathbf{C}_{\text{CMB}}^{-1} \) in equation (A15) is the inverse covariance matrix,

\[
\mathbf{C}_{\text{CMB}}^{-1} = \begin{pmatrix}
3.182 & 18.253 & -1.429 \\
18.253 & 1188.789 & -193.808 \\
-1.429 & -193.808 & 4.556
\end{pmatrix}.
\]

This paper has been typeset from a TeX/LaTeX file prepared by the author.