Lectures on Hilbert Schemes of Points on Surfaces

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Lectures on Hilbert Schemes of Points on Surfaces
Lectures on Hilbert Schemes of Points on Surfaces

Hiraku Nakajima
Abstract. In this book, the author discusses the Hilbert scheme of points $X^\bullet$ on a complex surface $X$ from various points of view. It inherits structures of $X$, e.g. it is a nonsingular complex manifold, it has a holomorphic symplectic form if $X$ has one, it has a hyper-Kähler metric if $X = \mathbb{C}^2$, and so on. A new structure is revealed when we study the homology group of $X^\bullet$. The generating function of Poincaré polynomials has a very nice expression. The direct sum $\bigoplus_n H_\bullet(X^\bullet)$ is a representation of the Heisenberg algebra.

Part of this book was written while the author enjoyed the hospitality of the Institute for Advanced Study. His stay was supported by National Science foundation Grant #DMS97-29992.
Contents

Preface ix

Introduction 1

Chapter 1. Hilbert scheme of points 5
  1.1. General Results on the Hilbert scheme 5
  1.2. Hilbert scheme of points on the plane 7
  1.3. Hilbert scheme of points on a surface 12
  1.4. Symplectic structure 13
  1.5. The Douady space 15

Chapter 2. Framed moduli space of torsion free sheaves on $\mathbb{P}^2$ 17
  2.1. Monad 18
  2.2. Rank 1 case 24

Chapter 3. Hyper-Kähler metric on $(\mathbb{C}^2)^{[n]}$ 29
  3.1. Geometric invariant theory and the moment map 29
  3.2. Hyper-Kähler quotients 37

Chapter 4. Resolution of simple singularities 47
  4.1. General Statement 47
  4.2. Dynkin diagrams 49
  4.3. A geometric realization of the McKay correspondence 52

Chapter 5. Poincaré polynomials of the Hilbert schemes (1) 59
  5.1. Perfectness of the Morse function arising from the moment map 59
  5.2. Poincaré polynomial of $(\mathbb{C}^2)^{[n]}$ 63

Chapter 6. Poincaré polynomials of Hilbert schemes (2) 73
  6.1. Results on intersection cohomology 73
  6.2. Proof of the formula 75

Chapter 7. Hilbert scheme on the cotangent bundle of a Riemann surface 79
  7.1. Morse theory on holomorphic symplectic manifolds 79
  7.2. Hilbert scheme of $T^*\Sigma$ 80
  7.3. Analogy with the moduli space of Higgs bundles 85

Chapter 8. Homology group of the Hilbert schemes and the Heisenberg algebra 89
  8.1. Heisenberg algebra and Clifford algebra 89
  8.2. Correspondences 91
  8.3. Main construction 93
  8.4. Proof of Theorem 8.13 96
Chapter 9. Symmetric products of an embedded curve, symmetric functions
and vertex operators

9.1. Symmetric functions and symmetric groups 105
9.2. Grojnowski’s formulation 109
9.3. Symmetric products of an embedded curve 110
9.4. Vertex algebra 114
9.5. Moduli space of rank 1 sheaves 121

Bibliography 125

Index 131
Preface

This book is based on courses of lectures which I delivered at University of Tokyo, Nagoya University, Osaka University and Tokyo Institute of Technology between 1996 and 1998.

The purpose of the lectures was to discuss various properties of the Hilbert schemes of points on surfaces. This object is originally studied in algebraic geometry, but as it has been realized recently, it is related to many other branches of mathematics, such as singularities, symplectic geometry, representation theory, and even to theoretical physics. The book reflects this feature of Hilbert schemes. The subjects are analyzed from various points of view. Thus this book tries to tell the harmony between different fields, rather than focusing attention on a particular one.

These lectures were intended for graduate students who have basic knowledge on algebraic geometry (say chapter 1 of Hartshorne: “Algebraic Geometry”, Springer) and homology groups of manifolds. Some chapters require more background, say spectral sequences, Riemannian geometry, Morse theory, intersection cohomology (perverse sheaves), etc., but the readers who are not comfortable with these theories can skip those chapters and proceed to other chapters. Or, those readers could get some idea about these theories before learning them in other books.

I have tried to make it possible to read each chapter independently. I believe that my attempt is almost successful. The interdependence of chapters is outlined in the next page. The broken arrows mean that we need only the statement of results in the outgoing chapter, and do not need their proof.

Sections 9.1, 9.4 are based on A. Matsuo’s lectures at the University of Tokyo. His lectures contained Monster and Macdonald polynomials. I regret omitting these subjects. I hope to understand these by Hilbert schemes in the future.

The notes were prepared by T. Gocho and N. Nakamura. I would like to thank them for their efforts. I am also grateful to A. Matsuo and H. Ochiai for their comments throughout the lectures. A preliminary version of this book has been circulating since 1996. Thanks are due to all those who read and reviewed it, in particular to V. Baranovsky, P. Deligne, G. Ellingsrud, A. Fujiki, K. Fukaya, M. Furuta, V. Ginzburg, I. Grojnowski, K. Hasegawa, N. Hitchin, Y. Ito, A. King, G. Kuroki, M. Lehn, S. Mukai, I. Nakamura, G. Segal, S. Strømme, K. Yoshioka, and M. Verbitsky. Above all I would like to express my deep gratitude to M. A. de Cataldo for his useful comments throughout this book.

February, 1999

Hiraku Nakajima
Interdependence of the Chapters
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Index

\((C^2)^{(n)}\), 8, 24, 36, 41, 47, 59, 63, 81
\(e_n\), 106
\(\bigoplus_n H_n(X^{[n]})\)
is a graded Hopf algebra, 110
\(H_n(X^{[n]})\)is a representation of the Heisenberg superalgebra, 94
\(Hilb_X\), 5
\(h_n\), 106
\(L^\ast \Sigma\), 85, 111
\(L\), 106
\(M(r,n)\), 17, 45
\(\mathcal{M}_U(r,n)\), 43, 45
\(\mathcal{M}_{W}(r,n)\), 43, 45
\(m_n\), 106
\(P[i]\), 93
\(P_{\nu}[i]\), 94
\(S^n(C^2)\), 26, 34, 41
\(S^nX\), 75
\(S^nX\), 7
\(S^nX\), 6
\(X^{[n]}\), 6

ADHM datum, 43
affine algebro-geometric quotient, 29
ALE space, 18, 46, 47, 52, 83, 124
anti-self-dual connection, 42, 43
Bellinison spectral sequence, 18
Borel-Moore homology, 91
Calogero-Moser system, 42
Chern class, 107
Clifford algebra, 89
complete symmetric function, see \(h_n\)
conformal vector, 115
coupling classes of symmetric groups, 108
coproduct, 109
correspondence, 92
cotangent bundle of a Riemann surface, 80
crepant resolution, 56
C\(^\ast\) -action, 79, 80
decomposable diagonal class, 26, 53
decomposition theorem, 74
douady space, 15
douady-barelet morphism, 15
dynkin diagram, 49
elementary symmetric function, see \(e_n\)
equivariant K-group, 110
equivariant cohomology, 62
Fock space, 89
framed moduli space
of anti-self-dual connections, see \(\mathcal{M}_{W}(r,n)\)
of ideal instantons, see \(\mathcal{M}_U(r,n)\)
of torsion-free sheaves, see \(\mathcal{M}(r,n)\)
Göttsche’s formula, 69, 73, 84, 90, 94
grothendieck group
of algebraic vector bundles, 54
of coherent sheaves, 54, 103
of complexes of algebraic vector bundles, 54
of equivariant topological vector bundles, 110
Heisenberg algebra, 89
Heisenberg superalgebra, 90
hilbert scheme:
functor, see \(Hilb_X\)
grothendieck’s theorem, 5
of points
definition, 6
on the cotangent bundle, 80
on the plane, see \((C^2)^{(n)}\)
Hilbert-Chow morphism, 7, 10, 75
Hodge numbers of \(X^{[n]}\), 77, 95
holomorphic symplectic form, 10, 13, 79, 85
hyper-Kähler
manifold, 38
moment map, 38
quotient, 37, 39
structure, 11, 14, 37, 47
ideal instanton, 45
instanton, 42, 46
integrable system, 42, 86
intersection cohomology, 73
intersection pairing, 92
Lagrangian, 79
Macdonald’s formula, 73, 95
McKay correspondence, 50, 52, 110
minimal resolution, 47
moduli space of Higgs bundles, 38, 79, 85
moment map, 32, 59
monad, 18
Morse theory, 59, 79
Néron-Severi group, 122
orbit sum, see $m_\nu$
Poincaré polynomial
definition, 59
of $(\mathbb{C}^2)^{[n]}$, 69
of $X^{[n]}$, see Göttsche’s formula
of $T^*\Sigma^{[n]}$, 84
of $S^n X$, see Macdonald’s formula
quiver variety, 18, 79
resolution of singularities, 12
ring of symmetric functions, see $\Lambda$
semistable, 35
simple singularity, 47
spectral curve, 87
stability, 7, 17, 36, 49, 65, 85
stable manifold, 60
symmetric function, 105
symmetric group, 6, 57, 108
symmetric products of an embedded curve, 80, 86, 110
symplectic quotient, 33
tautological vector bundle, 52
unstable manifold, 60, 79
vacuum vector, 115
vertex algebra, 114
vertex operator, 118, 123
Virasoro algebra, 115, 124
Young diagram, 65, 81
This beautifully written book deals with one shining example: the Hilbert schemes of points on algebraic surfaces ... The topics are carefully and tastefully chosen ... The young person will profit from reading this book.

—Mathematical Reviews

The Hilbert scheme of a surface $X$ describes collections of $n$ (not necessarily distinct) points on $X$. More precisely, it is the moduli space for 0-dimensional subschemes of $X$ of length $n$. Recently it was realized that Hilbert schemes originally studied in algebraic geometry are closely related to several branches of mathematics, such as singularities, symplectic geometry, representation theory—even theoretical physics. The discussion in the book reflects this feature of Hilbert schemes.

One example of the modern, broader interest in the subject is a construction of the representation of the infinite-dimensional Heisenberg algebra, i.e., Fock space. This representation has been studied extensively in the literature in connection with affine Lie algebras, conformal field theory, etc. However, the construction presented in this volume is completely unique and provides an unexplored link between geometry and representation theory.

The book offers an attractive survey of current developments in this rapidly growing subject. It is suitable as a text at the advanced graduate level.