On Gevrey orders of power expansions of solutions to the third and fifth Painlevé equations

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Abstract

The question under consideration is Gevrey summability of power expansions of solutions to the third and fifth Painlevé equations near infinity. Methods of French and Japanese schools are used to analyse these properties of formal power-series solutions. The results obtained are compared with the ones obtained by means of Power Geometry.

1 General theory

Let $V$ be an open sector with a vertex in the infinity on extended complex plane or on Riemann surface of logarithm, i.e. $V = \{ z : |z| > R, \text{ Arg } z \in (\phi_1, \phi_2) \}$. $w$ is a holomorphic on $V$ function and $\hat{w} = \sum_{k=0}^{\infty} a_k z^{-k}$ is some formal series belonging to $\mathbb{C}[[1/z]]$.

A function $w$ is said to be asymptotically approximated by a series $\hat{w}$ on $V$, if for the points $z$ of any closed subsector $Y$ of $V$ and for any $n \in \mathbb{N}$ there exist the constants $M_{Y,n} > 0$:

$$|z^n||w(z) - \sum_{p=0}^{n-1} a_p z^{-p}| < M_{Y,n}.$$ 

If there exist the constants $A_Y$, i.e. $M_{Y,n} = C(n!)^{1/k}A_Y^n$ a series $\hat{w}$ is an asymptotical Gevrey series of order $1/k$ for a function $w$ on $\Omega_R(\phi_1, \phi_2)$. We define this like $\hat{w} \in \mathbb{C}[[z]]_{1/k}$.

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For $G(z,Y,Y_1,\ldots,Y_n)$ being an analytic function of $n+2$ variables. Let us consider an equation
\begin{equation}
G(z,w,Dw,\ldots,D^nw) = 0.
\end{equation}
Let $\hat{w} \in \mathbb{C}[[1/z]]$ be a formal series, being a formal solution of a differential equation (1), and $D$ — an operator $zd/dz$.

**Theorem 1.** (O. Perron, J.-P. Ramis, B. Malgrange, Y. Sibuya in different cases) Let $\hat{w} \in \mathbb{C}[[z]]^{1/k}$ be a solution to the equation (1). Then there exist $k' > 0$ i. e. for every open sector $V$ with the vertex in the infinity, having an angle less than $\min(\pi/k,\pi/k')$ and for a sufficiently big $R$ there exist a function $w$, being a solution to the equation (1) which is asymptotically approximated of Gevrey order $1/k$ by a series $\hat{w}$.

The next theorem contains conditions on the Newton polygon. We will describe the process of its construction.

Let us be given a linear differential operator
\begin{equation}
L = \sum_{k=0}^{n} a_k(z)D^k, \text{ where } a_k(z) \in \mathbb{C}[[z]][1/z], a_k(z) = \sum_{j_k=j_k,0}^{\infty} a_{j_k}z^{-j}_k,
\end{equation}
\begin{equation}
a_{j_k} = \text{const} \in \mathbb{C}.
\end{equation}
We will put in correspondence a set of points on the plane: $(k,j_k,0), k = 0, \ldots, n$ — a support of the operator $L$. We will define a set
\begin{equation}
N = \bigcup_{k=0}^{n} \{(q_1,q_2) : q_1 \leq k, q_2 \geq j_k,0\}
\end{equation}
and then we construct a convex hull of this set in half-plane $q_1 \geq 0$. A boundary of this set is called a Newton polygon of the linear differential operator $L$.

**Theorem 2.** (J.-P. Ramis) Let $\hat{w} \in \mathbb{C}[[z]]$ be a formal solution to the equation (1), then the series $\hat{w}$ converges or has a Gevrey order equal to $s$, where $s \in \{0,\frac{1}{k_1},\ldots,\frac{1}{k_N}\}$ and $0 < k_1 < \ldots < k_N < +\infty$ are all positive slopes of the edges of the $N(G,\hat{w})$ to the $X$-axis.

The previous theorem is formulated for a nonlinear differential equation and its formal solution $\hat{w}$.

In a particular case $\frac{\partial G}{\partial Y_n}(z,\hat{w},\ldots,\hat{w}^{(n)}) \neq 0$ a Newton polygon of this equation on a formal solution can be constructed (Remark A.2.4.3) as a polygon of an operator
\begin{equation}
L_0 = \sum_{k=0}^{n} \left( \frac{\partial G}{\partial Y_k}(z,\hat{w}, Dw, \ldots, D^nw) \right) D^k.
\end{equation}
We can show that this operator coincides with the operator $\mathcal{M}$ used to construct exponential expansions of solutions using the methods of Power Geometry [4].

Let $(a_{j,k})$ be a matrix of a transformation from the basis $D, D^2, \ldots, D^n$ to basis
\[
\frac{d}{dz}, z, \frac{d^2}{dz^2}, \ldots, z^n, \frac{d^n}{dz^n}
\]
in a vector space of linear differential operators. We can check that the element $a_{j,k}$ is equal to $a_{j,k} = S(j,k)$, where $S(j,k)$ is a Stirling number of the second kind. This assertion is easily proved by induction using the fact that the elements of the matrix satisfy an equation $a_{j+1,k+1} = a_{j,k} + (k + 1)a_{j,k+1}$. The transition matrix is lower triangular, the diagonal elements are equal to one ($S(j,j) = 1$). An inverse matrix $(a^{j,k})$ is also lower triangular, its elements are equal to $a^{i,k} = (-1)^{j-k}s(j,k)$, where $s(j,k)$ is a Stirling number of the first kind.

Let $G(z, w, Dw, \ldots, D^n w) = F(z, w, z \frac{d}{dz}, \ldots, z^n \frac{d^n}{dz^n}),$ i.e. $G(z, Y, Y_1, \ldots, Y_n) = F(z, Y, X_1, \ldots, X_n)$. An operator (2) can be rewritten using $z^k \frac{d^k}{dz^k}$ (we use designation $\delta_{j,l}$ for Kronecker delta):
\[
L_0 = \sum_{k=0}^{n} \sum_{l=0}^{n} \frac{\partial F}{\partial X_l} \frac{\partial X_l}{\partial Y_k} D^k = \sum_{k=0}^{n} \sum_{l=0}^{n} \frac{\partial F}{\partial X_l} \frac{d^l}{dz^l} \sum_{j=0}^{n} a_{k,j} z^j \frac{d^j}{dz^j} =
\]
\[
= \sum_{l=0}^{n} \frac{\partial F}{\partial X_l} \frac{d^l}{dz^l} \sum_{j=0}^{n} z^j \frac{d^j}{dz^j} \delta_{j,l} = \sum_{l=0}^{n} \frac{\partial F}{\partial X_l} z^l \frac{d^l}{dz^l},
\]
and this expression is a first variation of a differential sum $F$. Being calculated on a series $\hat{w}$, it coincides with an operator $\mathcal{M}$ from [4].

The conclusion is – to construct a Newton polygon of the equation $F(z, w, w', \ldots, w^{(n)}) = 0$ on a solution $\hat{w}$ we need to perform the following:

1) to calculate the first variation $\frac{\partial F}{\partial w}$ on a solution $\hat{w}$;
2) to perform a transformation expressed by a matrix $(-1)^{j-k}s(j,k)$;
3) to verify a condition $\frac{\partial G}{\partial Y_n}(z, \hat{w}, \ldots, D^n \hat{w}) \neq 0$;
4) to find the set $(k, j, k_0), k = 0, \ldots, n$;
5) to construct a convex hull of this set in half-plane $q_1 \geq 0$.

Remark 1. The steps 2 and 3 can be interchanged: i.e. instead of the condition
\[
\frac{\partial G}{\partial Y_n}(z, \hat{w}, \ldots, D^n \hat{w}) \neq 0
\]
we can check the condition
\[
\frac{\partial F}{\partial X_n}(z, \hat{w}, \hat{w}', \ldots, \hat{w}^{(n)}) \neq 0.
\]
Remark 2. We use a Newton polygon considered in [2] to calculate the Gevrey orders but it can be easily shown that Gevrey order of the solution can be calculated via constructing a polygon of the equation used in Power Geometry [3]. To calculate the order we should perform a transformation \( u = \ln y \) in the operator \( \mathcal{M}(z) \) [4], applied to \( u \), to reduce an expression by \( u \) and for the differential sum obtained (the sum depends on \( z \) and \( y \)) to construct a polygon. In conditions of the theorem 2 the tangents are replaced by cotangents.

2 The fifth Painlevé equation

We consider the fifth Painlevé equation

\[
\frac{d^2 w}{dz^2} = \left( \frac{1}{2w} + \frac{1}{w-1} \right) \left( \frac{dw}{dz} \right)^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2} \left( \alpha w + \frac{\beta}{w} \right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1},
\]

where \( \alpha, \beta, \gamma, \delta \) are complex parameters, \( z \) is an independent complex variable, \( w \) is a dependent one, and we consider its formal power series solutions near infinity. Such solution are obtained in a work [1].

If \( \alpha \beta \gamma \delta \neq 0 \) there exist the following five expansions:

\[
(-1)^l \sqrt{\frac{\beta}{\delta}} \frac{1}{z} + \left( -\frac{2\beta}{\delta} + (-1)^l \frac{\gamma}{2\delta} \right) \sqrt{\frac{\beta}{\delta}} \frac{1}{z} + \sum_{s=3}^{\infty} \frac{c_{-s,l}}{z^s}, l = 1, 2,
\]

\[
-1 + \frac{2\gamma}{\delta z} + \sum_{s=2}^{\infty} \frac{c_{-s}}{z^s},
\]

\[
(-1)^l \sqrt{-\frac{\delta}{\alpha}} \frac{1}{z} + 2 + (-1)^l \frac{\gamma}{2\sqrt{-\alpha \delta}} + \sum_{s=1}^{\infty} \frac{c_{-s,l}}{z^s}, l = 1, 2.
\]

If \( \alpha \beta \gamma \neq 0, \delta = 0 \) there exist the following four expansions:

\[
(-1)^l \sqrt{-\frac{\beta}{\gamma} \frac{1}{z}} + \frac{\beta}{\gamma z} + \sum_{s=3}^{\infty} \frac{c_{-s,l}}{z^{s/2}}, l = 3, 4,
\]

\[
(-1)^l \sqrt{-\frac{\gamma}{\alpha} \frac{1}{z}} + 1 + \sum_{s=1}^{\infty} \frac{c_{-s,l}}{z^{s/2}}, l = 3, 4.
\]

The coefficients \( c_s, c_{s,l} \in \mathbb{C}, l = 1, 2, 3, 4 \) are uniquely determined constants, i.e.: if we fix the values of the parameters \( \alpha, \beta, \gamma, \delta \) these coefficients are uniquely determined as solutions of a non-degenerate system of linear equations.
Theorem 3. The series (5), (6) and a regular part of a series (4) are of Gevrey order 1. The series (7), (8) considered as the series in a new variable $\sqrt{z}$ are also of Gevrey order 1.

Proof. We apply the theorem 2 taking as an equation (1) an equation (3) multiplied by $z^2 w(w - 1)$ with all the terms of the equation put into the right part:

$$f(z, w) \overset{\text{def}}{=} -z^2 w(w - 1)w'' + z^2 \left(\frac{3}{2} w - \frac{1}{2}\right)(w')^2 - zw(w - 1)w' + (w - 1)^3(\alpha w^2 + \beta) + \gamma z w^2(w - 1) + \delta z^2 w^2(w + 1) = 0,$$

we take instead $\hat{w}$ the series (4), (5), (6) in course. If the principal part is not equal to zero, we can easily obtain the case of a zero principal part using a transformation; we speak about Gevrey order of a regular part of the series.

The first variation of the equation $P_5$ represented in a form of a differential sum (2) is equal to

$$-z^2 w(w - 1) \frac{d^2}{dz^2} + (z^2(3w - 1)w' - zw(w - 1)) \frac{d}{dz} - z^2 (2w - 1)w'' + \frac{3z^2 (w')^2}{2} - z(2w - 1)w' + (w - 1)^2(5\alpha w^2 - 2\alpha w + 3\beta) + \gamma z(3w^2 - 2w) + \delta z^2(3w^2 + 2w).$$

We substitute a series (6) to the expression (2) and write only coefficients of $\frac{d^2}{dz^2}, \frac{d}{dz}$ and identity operator with the maximum degree in $z$:

$$( -1)^l \sqrt{B} \delta \left( z \frac{d^2}{dz^2} + 2 \frac{d}{dz} + 2 \delta z \right), l = 1, 2,$$

a support of such an operator consists of the points (0, $-1$), (1, 1), (2, 1), the Newton polygon is shown in Fig. 1.

Analogous calculations can be performed for the series (5). We obtain an operator

$$-2z^2 \frac{d^2}{dz^2} - z \frac{d}{dz} + \delta z^2,$$

the support of it consists of the points (0, $-2$), (1, 0), (2, 0), its Newton polygon (brought down by a vector (0, 1)) is shown in Fig. 1.

For the series (4) we obtain an operator

$$\frac{\delta}{\alpha} z^4 \frac{d^2}{dz^2} + 2 \sqrt{-\delta} \frac{\delta}{\alpha} z^3 \frac{d}{dz} - 3 \frac{\delta^2}{\alpha} z^4, l = 1, 2.$$
Its support consists of the points \((0, -4), (1, -2), (2, -2)\), its Newton polygon (brought down by a vector \((0, 3)\)) is shown in Fig. 1.

As we see in Fig. 1, the unique positive tangent of the Newton polygon is equal to 1, using the theorem 2 we obtain that the series (6), the series (5) and a regular part of the series (4) are of Gevrey order 1.

Let us calculate Gevrey order of the series (7) and (8) obtained if \(\delta = 0\). To transform the fifth Painlevé equation to the form (1) we perform substitute \(t = \sqrt{z}\) and consider the series (7) and (8) as the series in decreasing half-integer degrees of \(t\). We calculate the first variation (we define differentiation with the respect to \(t\) as a dot):

\[
-t^2w(w-1) \frac{d^2}{dt^2} + \frac{1}{4} \left( t^2(3w-1)w + tw(w-1) - w(w-1) \right) \frac{d}{dt} - \\
\frac{(2w-1)(t^2\dot{w} + \ddot{w})}{4} + \frac{3t^2w^2}{8} + \\
+(w-1)^2(5\alpha w^2 - 2\alpha w + 3\beta) + \gamma^2(3w^2 - 2w).
\]

We substitute a series (8) to the expression (2) and write only coefficients of \(\frac{d^2}{dt^2}, \frac{d}{dt}\) and identity operator with the maximum degree in \(t\):

\[
\frac{c_{-1/2} t^2}{4} \frac{d^2}{dt^2} + \frac{c_{-1/2} - 3c_{-1/2}}{4t} \frac{d}{dt} + 3\gamma c_{-1/2} t = \frac{c_{-1/2}}{4t} D_t^2 + \frac{-c_{-1/2}}{t} D_t + 3\gamma c_{-1/2} t,
\]

where \(D_t = t \frac{d}{dt}, c_{-1/2} = (-1)^l \sqrt{\frac{\beta}{\gamma}}, l = 1, 2\).

The support of the operator consists of the points \((0, -1), (1, 1), (2, 1)\), its Newton polygon is shown in Fig. 1. We obtain again that the series (8) is of Gevrey order 1.

An operator corresponding to a series (7) has the form

\[
\frac{c_{1/2}^2 t^4}{4} \frac{d^2}{dt^2} + \frac{c_{1/2}^2 t^3}{4} \frac{d}{dt} - 3\gamma c_{1/2}^2 t^4 = \frac{c_{1/2}^2 t^2}{4} D_t^2 - \frac{3c_{1/2}^2 t^2}{4} D_t - 3\gamma c_{1/2}^2 t^4,
\]

where \(c_{1/2} = (-1)^l \sqrt{\frac{\beta}{\alpha}}, l = 1, 2\).

The support of the operator consists of the points \((0, -4), (1, -2), (2, -2)\), its Newton polygon (brought down by a vector \((0, 3)\)) is shown in Fig. 1. The regular part of the series (7) considered as the Laurent series in a variable \(t\) is of Gevrey orser 1.

This statement accomplishes the proof of the theorem 3.
Assertion 1. There exist \( k' \geq 1 \) and \( R_0 \in \mathbb{R}_+ \) i.e. for every open sector \( \{ z : |z| > R \geq R_0, \text{Arg} z \in (\varphi_1, \varphi_2) \} \), \( \varphi_2 - \varphi_1 < \pi/k' \leq \pi \) there exist a solution to the fifth Painlevé equation approximated by this Gevrey-1 series by each of the power series (series (6), series (5) and regular part of the series (4)) obtained.

There exist \( k' \geq 1/2 \) and \( R_0 \in \mathbb{R}_+ \) i.e. for every domain \( \{ z : |\sqrt{z}| > R \geq R_0, \text{Arg} z \in (\varphi_1, \varphi_2) \} \), \( \varphi_2 - \varphi_1 < \pi/k' \leq 2\pi \) there exist a solution to the fifth Painlevé equation (with a parameter \( \delta = 0 \)) approximated by these Gevrey-1 series (8) and (7) correspondingly.

This assertion is consequence of the theorems 1 and 3.

The results of this section are partially published as a preliminary version in [5].

Other results concerning the fifth Painlevé equation are published in [7], [8].

3 The third Painlevé equation

Let us pass on to the consideration of the analogous questions for the power expansions of solutions to the third Painlevé equation:

\[
  w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w},
\]

which can also be found in a book [1].

When \( \alpha \beta \gamma \delta \neq 0 \) they form four power expansions:

\[
  \left( -i \frac{d}{dz} \right)^l \left( -\frac{i}{\gamma} \delta - \left( \frac{(-1)^l \beta}{4 \sqrt{-\gamma \delta}} + \frac{\alpha}{4 \gamma} \right) \frac{1}{z} + \sum_{k=2}^{\infty} \frac{c_{l,k}}{z^k} \right), l = 1, 2, 3, 4,
\]

where \( i^2 = -1 \), and we consider the main branch of the root while speaking about the forth root.

We apply the theorem 2 taking an equation (12) multiplied by \( zw \) with all the terms of the equation put into the right part as an equation (1):

\[
  -zw w'' + z (w')^2 - w w' + w(\alpha w^2 + \beta) + \gamma z w^4 + \delta z = 0,
\]

and taking the series (13) as a formal power series solution \( \hat{w} \).

The first variation of the equation (14) is equal to

\[
  -zw \frac{d^2}{dz^2} + (2zw - w) \frac{d}{dz} - zw'' - w' + 3\alpha w^2 + \beta + 4\gamma z w^3.
\]
We substitute a series \((13)\) to the expression \((15)\) and write only coefficients of \(\frac{d^2}{dz^2}, \frac{d}{dz}\) and identity operator with the maximum degree in \(z\):
\[
i^k \sqrt{-\delta} \frac{d^2}{dz^2} + B \frac{d}{dz} + 4z^3 + \sqrt{\delta^3}, \quad l = 1, 2, 3, 4,
\]
a support of such an operator consists of the points \((0, -1), (1, 1), (2, 1)\), the Newton polygon is shown in Fig. 1.

As we see in Fig. 1, the unique positive tangent of the Newton polygon is equal to 1, using the theorem [2] we obtain that the series \((13)\) are of Gevrey order 1.

So, we obtain the following theorem

**Theorem 4.** The series \((13)\) are of Gevrey order equal to one. There exist \(k' \geq 1\) \(R_0 \in \mathbb{R}^+\) i.e. for any open sector \(\{z : |z| > R_0, \text{Arg} z \in (\phi_1, \phi_2)\}\), \(\phi_2 - \phi_1 < \pi/k' \leq \pi\) there exist a solution to the third Painlevé equation approximated by this Gevrey-1 series.

![Fig. 1](image-url)

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