STEEPEST DESCENT CURVES OF CONVEX FUNCTIONS 
ON SURFACES OF CONSTANT CURVATURE

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Abstract. Let \( S \) be a complete surface of constant curvature \( K = \pm 1 \), i.e. \( S^2 \) or \( \mathbb{I}^2 \), and \( \Omega \subset S \) a bounded convex subset. If \( S = S^2 \), assume also \( \text{diameter}(\Omega) < \frac{\pi}{2} \). It is proved that the length of any steepest descent curve of a quasi-convex function in \( \Omega \) is less than or equal to the perimeter of \( \Omega \). This upper bound is actually proved for the class of \( G \)-curves, a family of curves that naturally includes all steepest descent curves. In case \( S = S^2 \), it is also proved the existence of \( G \)-curves, whose length is equal to the perimeter of their convex hull, showing that the above estimate is indeed optimal. The results generalize theorems by Manselli and Pucci on steepest descent curves in the Euclidean plane.

1. Introduction

Let \( S \) be a complete surface of constant Gaussian curvature \( K = 0, +1 \) or \( -1 \), i.e. the Euclidean plane \( E^2 \), the unit sphere \( S^2 \) or the Lobachevskij plane \( \mathbb{I}^2 \). An absolutely continuous curve \( \gamma : [a, b] \to S \) (e.g. a rectifiable curve parameterized by arc length) is called \( G \)-curve if it satisfies the following condition (here and in the rest of the paper, we use the notation \( \gamma_{s_o} = \gamma(s) \)):

\[
\text{for any } s_o \in [a, b] \text{ for which } \dot{\gamma}_{s_o} \text{ exists and is different from 0, all points } \gamma_{s} \text{ with } s \leq s_o \text{ are in the same closed half space, bounded by the normal to } \gamma \text{ at } \gamma_{s_o}.
\]

Notice that the class of \( G \)-curves naturally includes all steepest descent curves of convex functions, that is the \( C^1 \)-curves \( \gamma_t \) satisfying equations of the form \( \dot{\gamma}_t = -\text{grad} f |_{\gamma_t} \) for some convex \( C^1 \)-function \( f : U \subset S \to \mathbb{R} \). The \( G \)-curves have been originally considered by Manselli and Pucci in [MP], where they determined an optimal upper bound for the length of \( G \)-curves contained in a given bounded convex subset of \( E^2 \).

Here we consider the problem of establishing similar upper bounds for the lengths of \( G \)-curves in \( S^2 \) and \( \mathbb{I}^2 \). Our first result consists of the following generalization of the upper bound determined in [MP]:

**Theorem 1.1.** Let \( S = S^2, E^2 \) or \( \mathbb{I}^2 \) and \( \gamma : [a, b] \to S \) a \( G \)-curve. In case \( S = S^2 \), assume also that \( \text{diameter}(\gamma([a, b])) < \frac{\pi}{2} \). Then the length \( \ell(\gamma) \) of \( \gamma \) is less than or equal to the perimeter \( p(\hat{\gamma}) \) of the convex hull \( \hat{\gamma} \) of \( \gamma \).
Notice that, as in Euclidean geometry, also in $S^2$ or $L^2$ the perimeter of a bounded convex set is less than or equal to the perimeter of any larger convex set (Prop. 2.3). Due to this, all $\mathcal{G}$-curves in a given convex set have length less than or equal to the perimeter of that set.

We also remark that the condition on diameter$(\gamma([a, b]))$ in case $S = S^2$ is a technical hypothesis, needed in our analysis of the growth of the perimeters $p(s)$ of the convex hulls $\gamma_{|[a, s]}$ of the arcs $\gamma_{|[a, s]}$, $s \in (a, b]$. It can be replaced by other assumptions, applicable when diameter$(\gamma([a, b])) \geq \frac{\pi}{2}$ (like e.g. that the tangent to $\gamma$ at $\gamma_s$ is a tangent line also to the convex hull $\gamma_{|[a, s]}$ for any $s$). But, at the moment, the previous statement is the best and most general one, which we have at our reach.

Secondly, for checking the optimality of this estimate, we consider the existence problem for $\mathcal{G}$-curves $\gamma : [a, b] \to S$ with maximal length property, i.e. such that the length $\ell(\gamma_{|[a, s]})$ equals to the perimeter of the convex hull $\gamma_{|[a, s]}$ for any $s \in [a, b]$. The solution of this problem in case $S = E^2$ was the second main result in [MP] (see also [MP1]), where the authors proved that, up to rigid motion, for given $L$ there exists a unique $\mathcal{G}$-curve, of length $L$ and $C^1$ outside the starting point, with the maximal length property. This curve is the unique curve in $E^2$ which is self-involute, i.e. it is equal to the initial arc of its own involute (see §4 for definitions; see also e.g. [Gu, Ra]).

From the proof of Theorem 1.1, it is natural to conjecture that a similar result should be true also when $S = S^2$ or $L^2$, at least for sufficiently small curves. Our second main result shows that for the case $S = S^2$ this expectation is indeed correct. In fact, we first prove that there exists on $S^2$ curves which are self-involute (Corollary 4.6) and then we prove that any self-involute of $S^2$ is also a $\mathcal{G}$-curve with maximal length property, showing in this way the optimality of the upper bound of Theorem 1.1.

The existence problem for $\mathcal{G}$-curves with maximal length property on $L^2$ remains open. It might be also interesting to know if the self-involuttes of $S^2$ share the same uniqueness properties of the self-involuttes of $E^2$ and whether there exists or not self-involuttes on $L^2$.

The structure of the paper is as follows. In §2, we collect a few basic facts on convex sets in surfaces of constant curvature. For reader’s convenience, we tried to make the exposition as much as possible self-contained. In §3, we introduce the notion of $\mathcal{G}$-curves and prove Theorem 1.1. In §4, we recall some well-known facts of the Differential Geometry of curves on surfaces, we introduce the definitions of involutes, almost self-involuttes and self-involuttes and prove our final results on the existence of self-involuttes and $\mathcal{G}$-curves with maximal length property on $S^2$.

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2. Preliminaries

2.1. Convex sets in surfaces of constant curvature.

Consider an (abstract) complete surface $S$ of constant curvature $K = +1, 0$ or $-1$, i.e. a complete 2-dimensional Riemannian manifold locally isometric to the unit sphere $S^2$, the Euclidean plane $E^2$ or the Lobachevskij plane $L^2$. It is well-known that either $S^2$ or $E^2$ or $L^2$ is also the universal covering space of $S$.

A line of $S$ is the trace of a maximal geodesic and, for any two points $x_1, x_2 \in S$, the (geodesic) segment joining $x_1$ and $x_2$ is an arc of minimal length between those points. We can now recall the definition of convex sets in $S$ (see e.g. [Al]).

**Definition 2.1.** A subset $Q \subset S$ is called convex if for any two points $x, x' \in Q$ there exists a unique segment joining $x$ to $x'$ and such segment is entirely in $Q$. Given a subset $A \subset S$, its convex hull is the intersection $\hat{A} = \bigcap A'$ of all convex sets $A' \subset S$ containing $A$.

Notice that, according to this definition, there are sets with no convex hull. For instance, if $A \subset S$ contains a pair of points with more than one segment joining them, there is no convex set that contains $A$.

We also recall that, according to how many segments might join two given points and whether or not they are all included in the subset, other notions of convex subsets in a surface can be given (see e.g. [Al, BZ, Ud]). However all these notions coincide when $S = E^2$ or $L^2$, while for the case $S = S^2$ the above choice turned out to be the most convenient one for establishing an upper bound for the length of steepest descent curves.

Let $\pi : \tilde{S} \to S$ be the universal covering space of $S$. If $U \subset S$ is an open convex set, then it is simply connected and the restriction $\pi|_{\tilde{U}} : \tilde{U} \to U$ of $\pi$ to a connected component $\tilde{U}$ of $\pi^{-1}(U)$ is an isometry between the convex subset $\tilde{U}$ of $\tilde{S}$ and $U$. Hence, for our purposes, with no loss of generality we may always reduce to the cases $S = S^2, E^2$ or $L^2$. In addition, the following lemma shows that the case $S = S^2$ can be always replaced by the assumption

\[ S = S^2_+ = S^2 \cap \{ (x^1, x^2, x^3) : x^3 > 0 \} . \]

**Lemma 2.2.** Let $U \subset S^2$ be open and convex. Then it is contained in a hemisphere.
Proof. Intuitively, the claim is a consequence of the fact that a convex subset of the unit sphere cannot contain a pair of antipodal points. But a short and precise proof can be obtained as follows. Let \( C \subset \mathbb{R}^3 \) be the cone given by the half-lines from the origin passing through the points of \( U \). One can check that \( U \) is convex if and only if \( C \) is convex. If we denote by \( \alpha \subset \mathbb{R}^3 \) a supporting plane of \( C \) through the origin, it follows immediately that \( U = S^2 \cap C \) is contained in a hemisphere bounded by \( \alpha \).

Consider the upper hemisphere \( S^2_+ \). The map \( \varphi : S^2_+ \to \{ x^3 = 1 \} \simeq \mathbb{R}^2 \), sending any \( x \in S^2_+ \) into its radial projection \( \bar{x} \in \{ x^3 = 1 \} \), maps the lines of \( S^2_+ \) into the lines of the Euclidean space \( E^2 = (\mathbb{R}^2, g_o) \) (here \( g_o \) is the standard Euclidean metric). In fact, the traces of geodesics in \( S^2_+ \) are great circles, i.e., intersections between \( S^2_+ \) and affine planes of \( \mathbb{R}^3 \) through the origin, and are mapped by \( \varphi \) into straight lines, given by the intersections of those planes with \( \{ x^3 = 1 \} \).

Similarly, consider the Lobachevskij plane \( L^2 = (\{ (x^1)^2 + (x^2)^2 - (x^3)^2 = -1, \ x^3 > 0 \}, g) \), where \( g \) is the Riemannian metric induced from the indefinite metric on \( \mathbb{R}^3 \)

\[
g = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 - dx^3 \otimes dx^3.
\]

Also in this case, the lines of \( L^2 \) are given by the intersections between \( L^2 \) and the affine planes of \( \mathbb{R}^3 \) through the origin. So, the map \( \psi : L^2 \to \{ x^3 = 1 \} \simeq \mathbb{R}^2 \), sending the points of \( L^2 \) into their radial projections from the origin on that plane, maps \( L^2 \) into the open unit disc \( \Delta \subset \mathbb{R}^2 \) and the lines of \( L^2 \) into the straight lines (chords) of \( \Delta \).

We recall that the abstract surface \( K^2 = (\Delta, \tilde{g} \overset{\text{def}}{=} \psi^{-1*}(g)) \) is usually called Klein disc.

Using the maps \( \varphi \) and \( \psi \), we may always identify \( S^2_+ \) and \( L^2 \) with (an open subset of) \( \mathbb{R}^2 \), endowed with a suitable metric \( g = g_{ij}dx^i \otimes dx^j \), whose geodesic segments coincide with the standard Euclidean segments. Under this identification, a subset \( Q \subset S \) is convex if and only if it is convex in the Euclidean sense.

For this reason, in the figures of this paper, the segments and convex subsets of \( S \) will be drawn as Euclidean segments and Euclidean convex subsets of \( \mathbb{R}^2 \). However, the reader should be aware that, when \( S \) is not the Euclidean space, these picture can be misleading for what concerns lengths and angles. For instance, for any \( v, w \in T_xS \simeq T_xE^2 \), the norm \( |v| \) and the cosine \( \cos(\hat{v}w) \) are in general different from the Euclidean values, since they are given in terms of the non-Euclidean metric \( g = g_{ij}dx^i \otimes dx^j \) of \( S \) by

\[
|v| = \sqrt{g_{ij}(x)v^i v^j}, \quad \cos(\hat{v}w) = \frac{g_{ij}(x)v^i w^j}{|v||w|}.
\]
A closed simple curve $P \subset S$ is called \textit{polygonal} if it is union of finitely many segments, called \textit{sides}. The endpoints of the sides are called \textit{vertices}.

The distance between two points $x_o, y_o \in S(\subseteq \mathbb{R}^2)$ is equal to the length of the segment between $x_o$ and $y_o$ w.r.t. $g$ i.e.

$$d_S(x_o, y_o) = \int_0^1 \sqrt{g_{ij}(x(t)) \dot{x}^i(t) \dot{x}^j(t)} dt,$$

where $x(t)$ is the line

$$x(t) = (x_o^1(1-t) + ty_o^1, x_o^2(1-t) + ty_o^2).$$

In particular, if $x_o, y_o$ are in a fixed convex compact subset $K \subset S$, there are $0 < C_K, C'_K$ such that

$$C_K |x_o - y_o| \leq d_S(x_o, y_o) \leq C'_K |x_o - y_o|.$$

The length $\ell_S(P)$ of a polygonal curve $P$ is the sum of the lengths of its sides. A curve $C \subset S$ is called \textit{rectifiable} if its length

$$\ell_S(C) = \sup \{ \ell_S(P) \mid P \text{ polygonal curve with vertices in } C \}$$

is finite. For an absolutely continuous parameterization $\gamma : [a, b] \to C \subset S$ of a rectifiable curve $C$, the tangent vector $\dot{\gamma}_{s_o}$ exists for almost any $s_o \in [a, b]$ and

$$\left. \frac{d\ell_S(\gamma|_{[a, s]})}{ds} \right|_{s_o} = |\dot{\gamma}_{s_o}| = \sqrt{g(\dot{\gamma}_{s_o}, \dot{\gamma}_{s_o})}.$$

By previous remarks, for any compact subset $K \subset S(\subset \mathbb{R}^2)$, a curve $C \subset K$ is rectifiable if and only if it is rectifiable as curve in the Euclidean plane and

$$C_K \cdot \ell(C) \leq \ell_S(C) \leq C'_K \cdot \ell(C) \quad (2.1)$$

where $\ell(C)$ is the length of $C$ in Euclidean sense.

The following proposition generalizes two well-known properties of convex sets in $\mathbb{R}^2$.

\textbf{Proposition 2.3.}

i) The boundary $\partial Q$ of a bounded convex set $Q \subset S$ is a rectifiable curve.

ii) If $Q \subset Q'$ are two bounded convex subsets of $S$, then $\ell_S(\partial Q) \leq \ell_S(\partial Q').$

\textbf{Proof.} (i) Since $Q \subset S \subset \mathbb{R}^2$ is convex also in the Euclidean sense, the claim follows immediately from known facts on Euclidean convex sets and (2.1).

(ii) Let $P$ be a polygonal curve with vertices $x_1, \ldots, x_n, x_{n+1} = x_1 \in \partial Q$ and $\hat{P}$ the (open) convex polygon with $\partial \hat{P} = P$. Denote also by $s_i$ the side of $P$ joining $x_i$ and $x_{i+1}$.

Being $Q'$ bounded and convex, the line $\hat{s}_1$ containing $s_1$ cuts $Q'$ into two convex subsets and the polygon $\hat{P}$ is entirely contained in one of them (see Fig.1). Call $Q'_{(1)}$ this convex set and observe that $\partial Q'_{(1)}$ is obtained by replacing a portion of $\partial Q'$ with the segment joining the endpoints of such
portion. Hence $\ell_S(\partial Q') \geq \ell_S(\partial Q'_{(1)})$. Next, consider the line $\hat{s}_2$ containing the side $s_2$ and the convex subset $Q'_2 \subset Q'_1$, which is cut by $\hat{s}_2$ and contains $P$. As before, we have that $\ell_S(\partial Q'_{(1)}) \geq \ell_S(\partial Q'_{(2)})$. Repeating the same construction for all the lines $\hat{s}_i$ containing the sides $s_i$, we end up with a nested sequence of convex sets $Q'_{(i)}$, $1 \leq i \leq n$, with $Q'_{(n)} = P$ and such that

$$\ell_S(\partial Q') \geq \ell_S(\partial Q'_{(1)}) \geq \ell_S(\partial Q'_{(2)}) \cdots \geq \ell_S(\partial P) = \ell_S(P)$$

By arbitrariness of $P$, it follows that $\ell_S(\partial Q') \geq \ell_S(\partial Q)$.

**Remark 2.4.** By a refinement of the proof, it is not hard to check that if $Q, Q'$ are as in Proposition 2.3 (ii) and if $Q' \setminus Q$ has non empty interior, then $\ell_S(\partial Q') > \ell_S(\partial Q)$.

Given a bounded convex set $Q \subset S$, we denote by $p(Q)$ the perimeter $\ell_S(\partial Q)$.

### 2.2. An auxiliary lemma.

We give here a lemma, needed in the proof of Theorem 1.1. It could be derived from the Gauss Lemma on derivatives of distance functions and it could be proved for Riemannian manifolds. However, we provide a self-contained proof, using only basic facts on surfaces of constant curvature.

In the following, given $y, z \in S = S^2_+, E^2$ or $JT^2$, we denote by $[y, z]$ the segment joining them and, given a curve $\eta : [a, b] \to S$, we denote by

$$d_{y, \eta} : [a, b] \to \mathbb{R}, \quad d_{y, \eta}(s) = \ell_S([y, \eta(s)]) = d_S(y, \eta_s).$$

**Lemma 2.5.** Let $\eta : [a, b] \to S$ be a curve in $S$, $s \in [a, b]$, such that the tangent vector $\dot{\eta}_s \neq 0$ exists at $\eta_s$, and $\{h_n\}$ a sequence of positive (negative) real numbers converging to 0 and $\{y_n\} \subset S$ a sequence of points, all of them different from $\eta_s$ and $\eta_{s+h_n}$, converging to a point $y_0$.

We denote by $\alpha_n$ and $\gamma_n$ the angles, with vertices in $\eta_s$ and $y_n$, respectively, formed by the tangent vectors of the oriented segments $[\eta_s, \eta_{s+h_n}]$ and

![Fig. 1](image-url)
and by the tangent vectors of the oriented segments \([y_n, \eta_s+h_n]\) and \([y_n, \eta_s]\). If \(y_o \neq \eta_s\)
\[
\lim_{n \to \infty} \frac{d_{y_n, \eta}(s + h_n) - d_{y_n, \eta}(s)}{h_n} = |\dot{\eta}_s| \cos \varphi ,
\] (2.2)
where \(\varphi = \lim_{n \to \infty} (\pi - \alpha_n)\). This equality holds also when \(y_o = \eta_s\), provided
that \(\lim_{n \to \infty} \alpha_n \) exists and \(\lim_{n \to \infty} \gamma_n = 0\).

Moreover, \(d'_{y_n, \eta}(s) = |\dot{\eta}_s| \cos \varphi\) for any \(y_o \in \mathcal{S}\),
where \(\varphi\) denotes the angle between \(\dot{\eta}_s\) and the tangent in \(\eta_s\) of the oriented segment
\([\eta_s, y_o]\) when \(y_o \neq \eta_s\) and \(\varphi = 0\) when \(y_o = \eta_s\).

Proof. First of all, let us recall the following well-known formulae of
Spherical and Hyperbolic Trigonometry (see e.g. [AVS] §I.3).

On the sphere \(S^2\):
\[
\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a ,
\]
\[
\sin \alpha = \sin \beta \sin \gamma,
\]
\[
\sin a = \sin b = \sin c ,
\]
On the Lobachevskij plane \(H^2\):
\[
\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cosh a ,
\]
\[
\sin \alpha = \sin \beta \sin \gamma,
\]
\[
\sinh a = \sinh b = \sinh c ,
\]
where, \(a, b, c\) are the sides of a triangle and \(\alpha, \beta\) and \(\gamma\) the corresponding
opposite angles. Now, consider the case \(h_n > 0\) for all \(n\) (the case \(h_n < 0\)
is similar). For a given \(n\), consider the triangle of vertices \(y_n, \eta_s\) and \(\eta_s+h_n\)
and let us denote by
\[
\alpha_n = \ell_S([y_n, \eta_s+h_n]) , \quad \beta_n = \ell_S([y_n, \eta_s]) , \quad \gamma_n = \ell_S([\eta_s, \eta_s+h_n])
\]
and by \(\alpha_n, \beta_n\) and \(\gamma_n\) the corresponding opposite angles. Then, both the
hypotheses \(y_o \neq \eta_s\) and \(y_o = \eta_s\) with \(\lim_{n \to \infty} \alpha_n = \pi - \varphi\), \(\lim_{n \to \infty} \gamma_n = 0\),
imply that
\[
\lim_{n \to \infty} (a_n - b_n) = 0 , \quad \lim_{n \to \infty} c_n = 0 , \quad \lim_{n \to \infty} \gamma_n = 0 , \quad \lim_{n \to \infty} \alpha_n = \pi - \varphi , \quad \lim_{n \to \infty} \beta_n = \varphi . \quad (2.3)
\]
Observe that
\[
\lim_{n \to \infty} \frac{d_{y_n, \eta}(s + h_n) - d_{y_n, \eta}(s)}{h_n} = \lim_{n \to \infty} \frac{a_n - b_n}{c_n} \cdot \lim_{n \to \infty} \frac{c_n}{h_n} .
\]
The second limit is equal to \(|\dot{\eta}_s|\), while the first limit can be written as
\[
\lim_{n \to \infty} \frac{a_n - b_n}{c_n} = \lim_{n \to \infty} \frac{\sin(a_n - b_n)}{\sin(c_n)} = \lim_{n \to \infty} \frac{\sin a_n \cos b_n - \cos a_n \sin b_n}{\sin c_n}
\] (2.5)

or as
\[
\lim_{n \to \infty} \frac{a_n - b_n}{c_n} = \lim_{n \to \infty} \frac{\sinh(a_n - b_n)}{\sinh(c_n)} = \lim_{n \to \infty} \frac{\sinh a_n \cosh b_n - \cosh a_n \sinh b_n}{\sinh c_n}
\] (2.6)
Using (2.3), (2.4) and the above relations in (2.5) when \( S = S^2_+ \) or in (2.6) when \( S = J^2 \), we obtain that in both cases
\[
\lim_{n \to \infty} \frac{a_n - b_n}{c_n} = \lim_{n \to \infty} \frac{(\cos \beta_n - \cos \alpha_n)(1 - \cos \gamma_n)}{\sin^2 \gamma_n} = \cos \phi.
\]
In case \( S = E^2 \), using the Sine Law of Euclidean trigonometry and the standard property of Euclidean triangles \( \alpha_n + \beta_n + \gamma_n = \pi \), we have
\[
\lim_{n \to \infty} \frac{a_n - b_n}{c_n} = \lim_{n \to \infty} \frac{\sin \alpha_n - \sin \beta_n}{\sin \gamma_n} = \frac{2 \sin \left(\frac{\alpha_n - \beta_n}{2}\right) \cos \left(\frac{\alpha_n + \beta_n}{2}\right)}{\sin \gamma_n} = \frac{2 \sin \left(\frac{\alpha_n - \beta_n}{2}\right) \sin \left(\frac{\gamma_n}{2}\right)}{\sin \gamma_n} = \cos \phi,
\]
where the last equality follows from (2.4).

For what concerns the last claim, in case \( y_0 \neq \eta_0 \) it is a direct consequence of (2.2) for the sequence \( \{y_n = y_0\} \), while in case \( y_0 = \eta_0 \) it follows from the immediate observation that \( d_{y_0,\eta}(s) = \lim_{h \to 0} \frac{d_{S}(\eta_0, \eta_0 + h)}{h} = |\eta_s| \). \( \square \)

3. \( \mathcal{G} \)-curves

Let \( f : \mathcal{U} \subset S \longrightarrow \mathbb{R} \) be a map defined on a convex subset \( \mathcal{U} \) of \( S \). We say that \( f \) is quasi-convex if any level set \( \mathcal{U}_{\leq c} \triangleq \{ x \in \mathcal{U} : f(x) \leq c \} \) is convex in \( S \). Notice that, under the identification of \( S \) with \( \mathbb{R}^2 \) or \( \Delta \subset \mathbb{R}^2 \), \( f \) is quasi-convex as function on \( S \) if and only if it is quasi-convex in the usual Euclidean sense.

Let \( f : \mathcal{U} \subset S \longrightarrow \mathbb{R} \) be a \( C^1 \) quasi-convex function \( f \) with \( df \neq 0 \) at all points, so that the level sets \( \mathcal{U}_{\leq \xi} \) have \( C^1 \)-boundaries. Any curve \( \gamma_{\xi} \) of steepest descent for \( f \) (i.e. such that \( \dot{\gamma}_\xi = -\operatorname{grad} f|_{\gamma_\xi} \) intersects orthogonally all the boundaries of the level sets and, for given \( s_0 \), all points \( \gamma_s, s \geq s_0 \), are included in the level set \( \mathcal{U}_{\leq f(\gamma_{s_0})} \). This means that the class of steepest descent curves is naturally included in the following class of curves, which extends the class considered in [MP].

**Definition 3.1.** We say that an absolutely continuous curve \( \gamma : [a, b] \to S \) is in the class \( \mathcal{G} \) if, for any \( s_0 \in [a, b] \) such that \( \dot{\gamma}_{s_0} \) exists and is different from 0, all points \( \gamma_s \) with \( s \leq s_0 \) are in a same closed half space that is bounded by the normal line \( \ell \) to \( \gamma \) at \( \gamma_{s_0} \).

In the following, we denote by \( \gamma : [a, b] \to S \) a \( \mathcal{G} \)-curve of \( S \) and, for any \( s \in [a, b] \) we indicate by \( p(s) \) the perimeter of the convex hull of \( \gamma|_{[a, s]} \). By Proposition 2.3 (ii), the function \( p(s) \) is not decreasing and hence \( p'(s) \) exists for almost all \( s \in [a, b] \).

\(^1\) Warning. In [UD] and other places, “quasi-convex function” means a function on a totally convex set with totally convex level sets. Notice that if \( S = E^2 \), the notion of convexity and total convexity coincide.
Moreover, for any $x \in S$ and $v, w \in T_x S$, we will denote by $C(x; v, w)$ a closed convex sector bounded by the two geodesic rays originating from $x$ and tangent to $v$ and $w$. Identifying $S$ with (an open subset of) $\mathbb{R}^2$, the sector $C(x; v, w)$ is the convex angle with vertex $x$ and sides parallel to $v$ and $w$. It is clearly always uniquely determined except when it is a half-plane, i.e. when $\hat{v} w = \pi$.

For any point $\gamma_s$ of the curve $\gamma$, we call projecting sector of $\gamma$ at $\gamma_s$ the smallest closed convex sector containing $\gamma|_{[a, s]}$. For any point $\gamma_s$ of $\gamma$, we will denote by $v_i = v_i(s) \in T_{\gamma_s} S$, $i = 1, 2$, the tangent vectors of the boundary rays of the corresponding projecting sector, which will be therefore indicated by $C(\gamma_s; v_1, v_2)$.

**Theorem 3.2.** Let $\gamma : [a, b] \to S$ be a $G$-curve with convex hull $\hat{\gamma}$ and, in the case $S = S^2_+$, assume also that diameter$(\gamma) < \frac{\pi}{2}$. Then

$$\ell_S(\gamma) \leq p(\hat{\gamma}),$$

where the equality occurs only if for almost all $s \in [a, b]$ the projecting sector $C(\gamma_s; v_1, v_2)$ of $\gamma$ is such that $\hat{v_1} v_2 = \frac{\pi}{2}$ and either $v_1$ or $v_2$ is tangent to $\gamma$ at $\gamma_s$.

In particular, for any $G$-curve in a bounded convex set $Q \subset S$ (and satisfying the above condition when $S = S^2_+$) the length $\ell_S(\gamma)$ is less than or equal to $p(Q)$.

The proof of this result is based on the following two lemmata.

**Lemma 3.3.** Let $s$ be such that both $p'(s)$ and $\dot{\gamma}_s$ exist with $\dot{\gamma}_s \neq 0$. Then

$$\frac{p'(s)}{|\ddot{\gamma}_s|} \geq \cos \phi_1 + \cos \phi_2, \quad \text{with} \quad \phi_i \overset{\text{def}}{=} \pi - \hat{\gamma}_s v_i,$$

where $v_1, v_2$ are the vectors of the projecting sector $C(\gamma_s; v_1, v_2)$.

**Proof.** For $s \in [a, b]$ and $h > 0$, consider the following notations:

- $A_s = \gamma|_{[a, s]}$ and $\hat{A}_s$ is its convex hull;
- $A_s(h) = A_s \cup \{\gamma_{s+h}\}$ with convex hull $\hat{A}_s(h)$;
- $A_s(h) = C(\gamma_s; v_1, v_2) \cap \hat{A}_s(h)$.
Clearly,
\[ \hat{A}_s \subseteq A_{s}(h) \subseteq A_{s+h} \]
and hence, by Proposition 2.3 (ii),
\[ p(\hat{A}_s) \leq p(A_{s}(h)) \leq p(\hat{A}_{s+h}) \]
and
\[ p(s + h) - p(s) = p(\hat{A}_{s+h}) - p(\hat{A}_s) \geq p(\hat{A}_{s+h}) - p(\hat{A}_s). \] (3.2)

The boundary of \( \hat{A}_s(h) \) contains two segments, lying on two rays coming out from \( \gamma_{s+h} \), and these segments necessarily intersect the sides of \( C(\gamma_s; v_1, v_2) \) in two distinct points, which we call \( x_1^h \) and \( x_2^h \) (one of them might be \( \gamma_s \)). Moreover, since the set of points \( \{x_i^h, h > 0\} \) is bounded, there exists a sequence \( \{h_n\} \) with \( \lim_{n \to \infty} h_n = 0 \) and \( \lim_{n \to \infty} x_i^{h_n} = x_i \) for some \( x_i \) in one side of \( C(\gamma_s; v_1, v_2) \).

Using the notation of Lemma 2.5, we may write
\[ p(\hat{A}_s(h_n)) - p(\hat{A}_s(h_n)) = \sum_{i=1}^{2} \left( \ell_S([x_i^{h_n}, \gamma_{s+h_n}]) - \ell_S([x_i^{h_n}, \gamma_s]) \right) = \sum_{i=1}^{2} \left( d_{x_i^{h_n}, \gamma}(s + h_n) - d_{x_i^{h_n}, \gamma}(s) \right). \] (3.3)

Recall that, for fixed \( i = 1, 2 \), all \( x_i^{h_n} \) lie in one of the two sides (call them \( r_i, i = 1, 2 \)) of \( C(\gamma_s; v_1, v_2) \). Hence, if there exists a subsequence \( \{x_i^{h_{nk}}\} \) converging to a point \( y_o \neq \gamma_s \), then \( y_o \in r_i \) and the angles \( \alpha_{nk} \), formed by the tangents in \( \gamma_s \) of the oriented segments \( [\gamma_s, \gamma_{s+h_{nk}}] \) and \( [\gamma_s, x_i^{h_{nk}}] \), converge to \( \pi - \phi_i \). By Lemma 2.5 we get (after replacing \( \{x_i^{h_n}\} \) by the above subsequence)
\[ \lim_{n \to \infty} \frac{d_{x_i^{h_n}, \gamma}(s + h_n) - d_{x_i^{h_n}, \gamma}(s)}{h_n} = |\gamma_s| \cos \phi_i. \] (3.4)

The same conclusion holds also if there exists a subsequence \( \{x_i^{h_{nk}}\} \) of points, all different from \( \gamma_s \), converging to \( y_o = \gamma_s \), because in such case, by possibly
taking another subsequence, the angles in $\hat{x}^n_{i,n}$, formed by the oriented segments $[x^h_{i,n}, \gamma_s+h_{nk}]$ and $[x^h_{i,n}, \gamma_s]$, tend to 0 and Lemma 2.5 applies.

To check this claim, consider the rays $r^o_{i,k}$ with origin in $\gamma_s+h_{nk}$ and containing $[\gamma_s+h_{nk}, x^h_{i,n}]$. By construction, any such ray lies in a line of support for $\gamma|[a,s]$ and contains a sub-ray, with origin in $x^h_{i,n}$, included in $C(\gamma_s; v_1, v_2)$.

By taking a suitable subsequence, the rays $r^o_{i,k}$ tend to a ray $r^o_i$, with origin in $\gamma_s$, entirely included in $C(\gamma_s; v_1, v_2)$ and lying in a line of support $\ell_o$ for $\gamma|[a,s]$.

Due to this, if $r^o_i$ intersected the interior of $C(\gamma_s; v_1, v_2)$, this would not be the smallest convex sector containing $\gamma|[a,s]$. So $r^o_i \subset \partial C(\gamma_s; v_1, v_2)$, that is $r^o_i = r_i$, and the angles in $x^h_{i,n}$, formed by $[x^h_{i,n}, \gamma_s+h_{nk}]$ and $[x^h_{i,n}, \gamma_s]$, tend to 0 as claimed.

Now, we claim that (3.4) is true also if there is no subsequence $\{x^h_{i,n}\}$, made of points all different from $\gamma_s$. In fact, in this case we may assume that $x^h_{i,n} = \gamma_s$ for any $n$ and hence that the projecting sector $\partial C(\gamma_s; v_1, v_2)$ lies in the intersection of half spaces, bounded by the lines $\ell_n$, which contain the boundary segment $[\gamma_s+h_{nk}, \gamma_s] \subset \partial A(\gamma_n)$. We remark that

a) the lines $\ell_n$ tend to the tangent line $\ell_o$ of $\gamma$ at $\gamma_s$ and $C(\gamma_s; v_1, v_2)$ is contained in a half-space bounded by $\ell_o$;

b) since the rays, with origin in $\gamma_s$ and containing $[\gamma_s, \gamma_s-h], h > 0$, are included in $C(\gamma_s; v_1, v_2)$ and tend to a ray $r_o \subset \ell_o$ when $h \to 0$, it follows that $r_o \subset C(\gamma_s; v_1, v_2)$.

From this, we infer that $r_o = r_i \subset \partial C(\gamma_s; v_1, v_2) \cap \ell_o$ and $\phi_o = \pi - \phi_s v_1 = 0$.

So, by Lemma 2.5

$$\lim_{n \to \infty} \frac{d_{x^h_{i,n}}(s + h_n) - d_{x^h_{i,n}}(s)}{h_n} = d'_{\gamma_s, \gamma}(s) = |\dot{\gamma}_s| \cos \theta = |\dot{\gamma}_s| \cos \phi_i$$

as claimed. From (3.2) and (3.3),

$$p'(s) \geq \sum_{i=1}^{2} \lim_{n \to \infty} \frac{d_{x^h_{i,n}}(s + h_n) - d_{x^h_{i,n}}(s)}{h_n} = |\dot{\gamma}_s|(\cos \phi_1 + \cos \phi_2). \quad \square$$

**Lemma 3.4.** Let $s$ be such that $\dot{\gamma}_s$ exists with $\dot{\gamma}_s \neq 0$ and let $C(\gamma_s; v_1, v_2)$ be the projecting sector of $\gamma$ at $\gamma_s$. If the curvature of $S$ is $K = +1$, assume also that $diam(\gamma|[a,s]) < \frac{\pi}{2}$. Then $\dot{\gamma}_s v_2 \leq \frac{\pi}{2}$ and hence, if $\phi_1 = \pi - \dot{\gamma}_s v_1$, $\cos \phi_1 + \cos \phi_2 \geq 1$.

The equality holds if and only if $\dot{\gamma}_s v_2 = \frac{\pi}{2}$ and one of the $\phi_i$’s is equal to 0.

**Proof.** First of all, we claim that, for any $s_0 \in [a,b]$, the function

$$d_{\gamma_{s_0}, \gamma} \colon [s_0, b] \to \mathbb{R}, \quad d'_{\gamma_{s_0}, \gamma}(s) = d_S(\gamma_{s_0}, \gamma_s)$$

is non-decreasing. In fact, since $\gamma$ is a $G$-curve, the vector $\dot{\gamma}_s$ and the tangent vector in $\gamma_s$ to the oriented segment $[\gamma_{s_0}, \gamma_s]$ points towards the same side
w.r.t. the normal line of \( \gamma \) in \( \gamma_s \). In particular, the angle \( \varphi \) between them is less than or equal to \( \frac{\pi}{2} \). By Lemma 2.5, \( d'_{\gamma_s, \gamma}(s) \) is non-negative for almost all \( s \) and the claim follows.

Secondly, we claim that for any \( a \leq s_1 < s_2 < s \leq b \), the angle \( \alpha \) formed by the oriented segments \([\gamma_{s_1}, \gamma_{s}]\) and \([\gamma_{s}, \gamma_{s_2}]\) is less than or equal to \( \frac{\pi}{2} \).

In case \( S = E^2 \) or \( J^2 \), it can be checked as follows. In the triangle with vertices \( \gamma_s, \gamma_{s_1}, \) and \( \gamma_{s_2} \), the sum of inner angles is less than or equal to \( \pi \) and hence \( \alpha > \frac{\pi}{2} \) only if it is the largest of these three angles. But this cannot be because the side \([\gamma_{s_1}, \gamma_{s_2}]\), opposite to \( \alpha \), is not the largest one (by the previous claim, it is shorter or equal to \([\gamma_{s_1}, \gamma_s]\)), in contrast with a well known fact of Euclidean and Hyperbolic Geometry.

Also in case the curvature of \( S \) is \( K = \pm 1 \) and diameter(\([\gamma_{s_1}, \gamma_s]\)) \( < \frac{\pi}{2} \), if \( \alpha \) were larger than \( \frac{\pi}{2} \), its opposite side in the triangle with vertices \( \gamma_s, \gamma_{s_1}, \) and \( \gamma_{s_2} \) would be the largest one, as it can be checked using the spherical law of cosines \( \cos a = \cos b \cos c + \sin a \sin b \cos \alpha \). Hence, also in this case we conclude that \( \alpha \leq \frac{\pi}{2} \) by the same argument as above.

Now, the first statement of the lemma follows immediately from the observation that \( v_1 v_2 = \phi_1 + \phi_2 \) is limit of angles delimited by segments \([\gamma_{s_1}, \gamma_s]\) and \([\gamma_s, \gamma_{s_2}]\) for some \( a \leq s_1 < s_2 < s \leq b \). To check the second statement, just look for the minimum of \( f(\phi_1, \phi_2) = \cos \phi_1 + \cos \phi_2 \) in the region \( \Omega = \{ 0 \leq \phi_i \leq \frac{\pi}{2} , \phi_1 + \phi_2 \leq \frac{\pi}{2} \} \).

Combining the results of these lemmata, \( p'(s) \geq |\dot{\gamma}_s| \) for almost all \( s \in [a, b] \), with equality only if the amplitude of the projecting sector is \( \frac{\pi}{2} \), and this implies the theorem.

4. Self-Involutes on Spheres

4.1. Basic facts on curves of surfaces of constant curvature.

In this section \( S \) is a simply connected, complete surface of constant curvature with a curvature \( K \) that might assume any real value. Recall that, when \( K = \pm 1/R^2 \), the surface \( S \) is either

\[
S_R^2 = \{ x \in \mathbb{R}^3 : x^T \cdot x = R^2 \}
\]

or

\[
J_R^2 = \left\{ x \in \mathbb{R}^3 : x^3 > 0 \text{ and } x^T \cdot I_{2,1} \cdot x = -R^2 \text{ where } I_{2,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\},
\]

respectively endowed with the Riemannian metric \( g \), which is induced either by the standard Euclidean metric \( g^E = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 \) or by the Lorentzian metric \( g^L = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 - dx^3 \otimes dx^3 \). For uniformity of notation, one can consider also \( E^2 \) as a surface in \( \mathbb{R}^3 \), namely setting \( E^2 = \{ x \in \mathbb{R}^3 : x^3 = 1 \} \subset \mathbb{R}^3 \), with the metric induced by the standard Euclidean metric \( g^E \) of \( \mathbb{R}^3 \).

Given a \( C^3 \) curve \( \eta : I \subset \mathbb{R} \to S \subset \mathbb{R}^3 \), parameterized by arc length, the Frenet frame of \( \eta \) at \( s = s_o \) is the orthonormal basis \((t_s, n_s) \subset T_{\eta_o}S, \) given
by \( t_s = \dot{\eta}_s \) and the unit vector \( n_s \), tangent to \( S \), orthogonal to \( t_s \) and so that \((\eta_s, t_s, n_s)\) is a positively oriented basis for \( \mathbb{R}^3 \). By a simple generalization of the classical theory of curves in \( E^2 \), one can derive the following “Frenet formulae”

\[
\nabla_{\dot{\eta}_s} t_s = \kappa_s n_s, \quad \nabla_{\dot{\eta}_s} n_s = -\kappa_s t_s, \tag{4.1}
\]

for some smooth function \( \kappa_s \), called (geodesic) curvature of \( \eta \). We also recall the Levi-Civita connection \( \nabla \) of \( S \) is such that, for any smooth curve \( \eta \) and any vector field \( Y_{\eta_s} \), tangent to \( S \) and defined at the points of \( \eta \),

\[
\nabla_{\dot{\eta}_s} Y = \dot{Y}_{\eta_s} + Kg(\dot{\eta}_s, Y_{\eta_s})\eta_s
\]

(here, \( \dot{\eta}_s \) and plus sign denote the standard first order derivative and the sum of maps with values in \( \mathbb{R}^3 \)). Also, given a point \( x_o \in S \subset \mathbb{R}^3 \) and a unit vector \( v \in T_{x_o}S \), the geodesic \( \gamma_s \) with \( \gamma_0 = x_o \) and \( \dot{\gamma}_s = v \) is of the form

\[
\gamma_s = \cos \left( \frac{s}{R} \right) x_o + \sin \left( \frac{s}{R} \right) (Rv) \quad \text{or} \quad \gamma_s = \cosh \left( \frac{s}{R} \right) x_o + \sinh \left( \frac{s}{R} \right) (Rv)
\]

or

\[
\gamma_s = x_o + sv
\]

according to the value of \( K \).

We conclude this subsection, stating the following simple generalization of the classical “Fundamental Theorems of Plane Curves”. It is an immediate consequence of the Existence and Uniqueness Theorem for O.D.E.’s.

**Theorem 4.1.**

1) Let \( \eta : [0, L] \subset \mathbb{R} \to S \) and \( \eta' : [0, L'] \subset \mathbb{R} \to S \) be two \( C^2 \) curves parameterized by arc length and with curvature functions \( \kappa \) and \( \kappa' \), respectively. There exists an isometry \( g : S \to S \) such that \( \eta' = g \circ \eta \) if and only if \( L = L' \) and \( \kappa_s = \kappa'_s \) for any \( s \in [0, L] \).

2) Let \( \kappa : [0, L] \to \mathbb{R} \) be \( C^0 \). For any \( x_o \in S, v \in T_{x_o}S \) with \( |v| = 1 \) and \( L \leq L \) sufficiently small, there exists a unique \( C^2 \) curve \( \eta : [0, L] \to S \) parameterized by arc length and with curvature function \( \kappa \), such that \( \eta_0 = x_o \) and \( \dot{\eta}_0 = v \).

4.2. Involutes and (almost) self-involutes.

**Definition 4.2.** Let \( \eta : [0, L] \to S \subset \mathbb{R}^3 \) a curve, parameterized by arc length (hence \( \eta_s = t_s \) and \( C^1 \) on \( (0, L) \)). The involute of \( \eta \) is the curve \( \tilde{\eta} : [0, L] \to S \) such that, for any \( s \in (0, L) \), the point \( \tilde{\eta}_s \in S \) is determined by the formula

\[
\tilde{\eta}_s = \gamma_{\eta_s - t_s}, \tag{4.3}
\]

where \( \gamma_{\eta_s - t_s} \) is the geodesic of \( S \) with \( \gamma_{\eta_0 - t_s} = \eta_s \) and \( \dot{\gamma}_{\eta_0 - t_s} = -t_s \).

The curve \( \eta_s \) is called almost self-involute if there exists an isometry of \( S \), which maps \( \eta \) into the initial arc of length \( L \) of its own involute \( \tilde{\eta} \) (re-parameterized by arc length). In case \( \eta_s \) coincides with such initial arc of \( \tilde{\eta} \), we call it self-involute.
According to the value of $K$, one immediately obtains that
\[
\tilde{\eta}_s = \cos \left( \frac{s}{R} \right) \eta_s - \sin \left( \frac{s}{R} \right) (R\dot{\eta}_s) \quad \text{or} \quad \tilde{\eta}_s = \cosh \left( \frac{s}{R} \right) \eta_s - \sinh \left( \frac{s}{R} \right) (R\dot{\eta}_s).
\]

or
\[
\tilde{\eta}_s = \eta_s - s\dot{\eta}_s.
\] (4.4)

Simple computations show that, in case $\eta$ is $C^2$, the Frenet frames $(\mathbf{t}_s, \mathbf{n}_s, \mathbf{b}_s)$, the arc length parameter $\tilde{s}$ and the curvature $\tilde{\kappa}_s$ at the points $\tilde{\eta}_s = \tilde{\eta}$ are (only for $s < \pi/2$ when $K > 0$ and up to changes $\tilde{s} \to -\tilde{s}$)

\[
\mathbf{t}_s = -\mathbf{n}_s, \quad \mathbf{n}_s = \begin{cases} \sin \left( \frac{s}{R} \right) \frac{\eta_s}{R} + \cos \left( \frac{s}{R} \right) \dot{\eta}_s & \text{if } K = \frac{1}{R^2} \\ \sinh \left( \frac{s}{R} \right) \frac{\eta_s}{R} + \cosh \left( \frac{s}{R} \right) \dot{\eta}_s & \text{if } K = -\frac{1}{R^2} \\ \dot{\eta}_s & \text{if } K = 0 \end{cases},
\]

(4.5)

\[
\dot{\tilde{s}} = \begin{cases} R \sin \left( \frac{s}{R} \right) \kappa_s & \text{if } K = \frac{1}{R^2} \\ R \sinh \left( \frac{s}{R} \right) \kappa_s & \text{if } K = -\frac{1}{R^2} \\ s\kappa_s & \text{if } K = 0 \end{cases}, \quad \tilde{\kappa}_s = \begin{cases} \frac{1}{R} \cot \left( \frac{s}{R} \right) & \text{if } K = \frac{1}{R^2} \\ \frac{1}{R} \coth \left( \frac{s}{R} \right) & \text{if } K = -\frac{1}{R^2} \\ \frac{1}{s} & \text{if } K = 0 \end{cases}.
\]

Using Theorem 4.1, one can infer that a curve $\eta : [0, L] \to S^2_R$, parameterized by arc length, smooth and with strictly positive curvature function $\kappa_s$ on $(0, L)$, is congruent to the initial arc of length $L$ of its own involute if and only if there exists a strictly increasing smooth function $\tau : (0, L) \subset (0, \tilde{L}) \to (0, L)$, with $\lim_{s \to 0^+} \tau_s = 0$ and satisfying the following equations

\[
\begin{cases} 
\dot{\tau}_s = R \sin \left( \frac{s}{R} \right) \kappa_s \\
\kappa_{\tau_s} = \frac{1}{R} \cot \left( \frac{s}{R} \right)
\end{cases}.
\]

(4.6)

Moreover, for any pair of smooth functions $\kappa : (0, L) \to \mathbb{R}$ and $\tau : (0, \tilde{L}) \subset (0, L) \to (0, L)$ with $\kappa_s, \tau_s > 0$ for any $s \in (0, L)$, $\lim_{s \to 0^+} \tau_s = 0$ and satisfying (4.6), there exists a curve $\eta : [0, L] \to S^2_R$ with curvature $\kappa_s$ and
an almost self-involute \( \eta \) related function

An existence result for the almost self-involutes on spheres.

4.3. Of the solutions to those systems.

of the self-involutes of a given length is clearly bounded by the cardinality of the solutions to those systems.

We want to show that it is possible to determine \( t \) such that the functions \( \vartheta_n \) converge uniformly on \( [0,t_0) \) to a solution of (4.9). This is a direct consequence of the following claims.

Claim 1. There exist \( t_0 \in (0,1) \) and \( K < 1 \) such that \( \vartheta_n(t) \) is defined for any \( n \) and any \( t \in [0,t_0) \) and satisfies

\[
0 < \vartheta_n(t) \leq Kt \quad \text{for any } t > 0.
\]

Claim 2. \( \lim_{t \to 0^+} \vartheta_n'(t) = A \) for any \( n \).

Claim 3. \( \vartheta_n''(t) \geq 0 \) for any \( n \) and any \( t \in (0,t_0) \).
Claim 4. \( \vartheta_{n-1}(t) \leq \vartheta_n(t) \) for any \( n \geq 1 \) and any \( t \in [0, t_0] \).

Claim 5. The sequence \( \{\vartheta_n\} \) converges uniformly to a function \( \vartheta : [0, t_0] \to \mathbb{R} \) which is a solution of (4.9).

Let us now proceed with the proofs of such claims.

**Proof of Claim 1.** Consider a value \( B > 0 \) such that
\[
A^5 + \frac{A^3}{2} + A^2 B - \frac{3B}{2} < 0
\]  
(4.12)
and notice that the function
\[
F(x) = \tan \left( Ax + \frac{B}{3} x^3 \right) - \left( A + \frac{B}{2A^2} x^2 \right) \sin (x)
\]
is such that
\[
F(0) = F'(0) = F''(0) = 0
\]
and
\[
F'''(0) = \frac{2}{A^2} \left( A^5 + \frac{A^3}{2} + A^2 B - \frac{3B}{2} \right) < 0
\]  
(4.13)
Therefore there exists an open interval \( I = (0, \varepsilon) \), on which \( F(x) < 0 \) and consequently
\[
\tan \left( Ax + \frac{B}{3} x^3 \right) \sin (x) < \left( A + \frac{B}{2A^2} x^2 \right).
\]  
(4.14)
Now, let \( t_0 > 0 \) satisfy:

i) \( t_0 \leq \varepsilon \);

ii) \( \frac{1}{2} + \frac{B}{3A} t_0^2 + \frac{B^2}{18A^2} t_0^4 < 1 \);

iii) \( A + B t_0^2 < 1 \).

We now show by induction that for any \( n \) and \( t \in (0, t_0] \)
\[
0 < \vartheta_n(t) \leq At + \frac{B}{3} t^3.
\]  
(4.15)
First of all, for \( n = 0 \), we have that for any \( t > 0 \)
\[
0 < \vartheta_0(t) = At \leq At + \frac{B}{3} t^3.
\]  
(4.16)
Secondly, assume that \( \vartheta_{n-1}(t) \) is defined for any \( t \in [0, t_0] \) and that \( 0 < \vartheta_{n-1}(t) \leq At + B t_0^3 \) for any \( t \in (0, t_0] \). Then, by (iii), we have that \( \vartheta_{n-1}(t) \leq At + \frac{Bt_0^3}{3} \leq t \) in \( [0, t_0] \) and hence that \( \vartheta_{n-1}([0, t_0]) \subset [0, t_0] \). It follows that the function \( \vartheta_n \) is well defined in \( [0, t_0] \) and \( C^1 \) with \( \vartheta_n' > 0 \) for all \( t > 0 \).

From (i) and (ii) and inductive hypothesis, it follows that for any \( t \in (0, t_0) \)
\[
\vartheta_n'(t) = \frac{\tan (\vartheta_{n-1}(\vartheta_n(t)))}{\sin (\vartheta_{n-1}(t))} \leq \frac{\tan (A \vartheta_{n-1}(t) + B \vartheta_{n-1}(t))}{\sin (\vartheta_{n-1}(t))} \leq A + \frac{B}{2A^2} \vartheta_{n-1}(t) \leq A + \frac{B}{2A^2} \left( A^2 t^2 + \frac{2}{3} AB t^4 + B^2 t^6 \right) \leq A + B t^2 \left( \frac{1}{2} + \frac{1}{3} \frac{B t^2}{A} + \frac{B^2 t^4}{18A^2} \right) \leq A + B t^2
\]  
(4.17)
From this, the inequality (4.15) follows by integration. Now, by (4.15) and the assumption (iii) on \( t_o \) and setting \( K = A + \frac{Bt_o^2}{3} < 1 \), we obtain the inequality (4.11), namely \( \vartheta_n(t) \leq t \left( A + \frac{Bt_o^2}{3} \right) = Kt \).

**Proof of Claim 2.** By construction, \( \vartheta'_0(0) = A \) and if \( \lim_{t \to 0^+} \vartheta'_{n-1}(t) = A \), then

\[
\lim_{t \to 0^+} \vartheta'_n(t) = \lim_{t \to 0^+} \frac{\vartheta'_n(t) - \vartheta'_n(0)}{t} = \lim_{t \to 0^+} \frac{\vartheta'_n(t) - \vartheta'_n(0)}{\vartheta'_{n-1}(t)} = \lim_{t \to 0^+} \frac{\vartheta'_{n-1}(t) - \vartheta'_{n-1}(0)}{\vartheta'_{n-1}(t)} = \frac{\vartheta'_{n-1}(t)}{\vartheta'_{n-1}(t)} = 1.
\]

The claim follows by induction on \( n \).

**Proof of Claim 3.** Also this claim is proved by induction. When \( n = 0 \), the claim is trivial since \( \vartheta'_0(t) = 0 \). Assume now that \( \vartheta''_{n-1} \geq 0 \) on \((0, t_o)\) and observe that

\[
\vartheta''_n(t) = \frac{d}{dt} \frac{\tan(\vartheta_{n-1}(\vartheta_{n-1}(t)))}{\sin(\vartheta_{n-1}(t))} = \frac{1}{\cos^2(\vartheta_{n-1}(\vartheta_{n-1}(t)))} \vartheta'_{n-1}(\vartheta_{n-1}(t)) \vartheta''_{n-1}(t) - \frac{\sin(\vartheta_{n-1}(t))}{\cos(\vartheta_{n-1}(t))} \vartheta'_{n-1}(t) = \frac{\vartheta'_{n-1}(t) \cos(\vartheta_{n-1}(t))}{\cos^2(\vartheta_{n-1}(\vartheta_{n-1}(t))) \sin^2(\vartheta_{n-1}(t))} \cdot \mathcal{F}^{(n-1)}(\vartheta_{n-1}(s)) \quad (4.18)
\]

where, for any \( m \), we denote

\[
\mathcal{F}^{(m)}(s) \overset{\text{def}}{=} \vartheta'_m(s) \tan(s) - \sin(\vartheta_m(s)) \cos(\vartheta_m(s)). \quad (4.19)
\]

Using the assumption \( \vartheta''_{n-1}(s) \geq 0 \) for \( s \in (0, t_o) \) and the fact that \( \vartheta''_{n-1}(s) > 0 \), we have that

\[
\mathcal{F}^{(n-1)'}(s) = \vartheta''_{n-1}(s) \tan(s) + \vartheta'_{n-1}(s) \frac{1}{\cos^2(s)} - \vartheta''_{n-1}(s) (\cos^2(\vartheta_{n-1}(s)) - \sin^2(\vartheta_{n-1}(s))) = \vartheta''_{n-1}(s) \tan(s) + \frac{\vartheta'_{n-1}(s)}{\cos^2(s)} (1 - \cos^2(s) \cos(2 \vartheta_{n-1}(s))) \geq 0.
\]

Since \( \mathcal{F}^{(m)}(0) = 0 \) for any \( m \), it follows that \( \mathcal{F}^{(n-1)}(s) \geq 0 \) for any \( s \in (0, t_o) \) and hence that \( \mathcal{F}^{(n-1)}(\vartheta_{n-1}(t)) \geq 0 \) for any \( t \in (0, t_o) \). From this and (4.18), we get \( \vartheta''_n(t) \geq 0 \) as needed.

**Proof of Claim 4.** First of all, notice that for any \( t \in (0, t_o) \)

\[
\vartheta'_1(t) = \frac{\tan(\vartheta_0(\vartheta_0(t)))}{\sin(\vartheta_0(t))} = \frac{\tan(A^2t)}{\sin(At)} \geq \frac{A^2t}{At} = A = \vartheta'_0(t).
\]
Hence, by integration, \( \vartheta_1(t) \geq \vartheta_0(t) \) for any \( t \in (0, t_0) \). Let us assume that \( \vartheta_{n-2} \leq \vartheta_{n-1} \) at all points of \([0, t_0]\). In order to prove the claim by an inductive argument, we only need to check that \( \vartheta'_{n-1} \leq \vartheta'_n \) on the same interval \([0, t_0]\). To see this, we notice that

\[
\vartheta'_n(t) - \vartheta'_{n-1}(t) = \frac{\tan(\vartheta_{n-1}(\vartheta_{n-1}(t)))}{\sin(\vartheta_{n-1}(t))} - \frac{\tan(\vartheta_{n-2}(\vartheta_{n-2}(t)))}{\sin(\vartheta_{n-2}(t))} \geq \vartheta_{n-2} - \vartheta_{n-3}
\]

for some \( \tilde{s} \in (\vartheta_{n-2}(t), \vartheta_{n-1}(t)) \). On the other hand

\[
d \frac{\tan(\vartheta_{n-2}(s))}{\sin(s)} = \frac{\cos s}{\cos^2(\vartheta_{n-2}(s)) \sin^2 s} \left\{ \vartheta'_{n-2}(s) \tan(s) - \sin(\vartheta_{n-2}(s)) \cos(\vartheta_{n-2}(s)) \right\} = \frac{\cos s}{\cos^2(\vartheta_{n-2}(s)) \sin^2 s} \mathcal{F}^{(n-2)}(s)
\]

where \( \mathcal{F}^{(n-2)}(s) \) is as defined in (4.19). In the proof of the previous claim, we showed that \( \mathcal{F}^{(m)} \geq 0 \) for any \( m \geq 1 \) and from this we conclude that \( \vartheta'_{n-1} \leq \vartheta'_n \) as needed.

**Proof of Claim 5.** For the proof of this claim, we need the following properties which are consequences of the previous claims:

i) \( \vartheta'_{n-1}(\vartheta_{n-1}(t)) \leq \vartheta'_{n-1}(t) \);

ii) \( \cos^2(\vartheta_{n-1}(\vartheta_{n-1}(t))) \geq \cos^2(\vartheta_{n-1}(t)) \);

iii) \( \vartheta'_{n-1}(t) \leq 1 \);

iv) \( \frac{1}{\cos^2(\vartheta_{n-1}(t_0))} = K \).

Using these relations, for any \( t \leq t_0 \) we have that (here “\( \vartheta^2_m(t) \)” stands for “\( \vartheta_m(\vartheta_m(t)) \)”):

\[
\vartheta'_n(t) = \frac{\vartheta'_{n-1}(t) \cos(\vartheta_{n-1}(t))}{\cos^2(\vartheta_{n-1}(t)) \sin^2(\vartheta_{n-1}(t))} \cdot \{ \vartheta'_{n-1}(\vartheta_{n-1}(t)) \tan(\vartheta_{n-1}(t)) - \sin(\vartheta_{n-1}(t)) \cos(\vartheta_{n-1}(t)) \} \leq \frac{K}{\sin^2 \vartheta_{n-1}(t)} \left\{ \vartheta'_{n-1}(\vartheta_{n-1}(t)) \tan(\vartheta_{n-1}(t)) - \sin(\vartheta_{n-1}(t)) \cos(\vartheta_{n-1}(t)) \right\} = \frac{1}{\sin(\vartheta_{n-1}(t))} \left\{ \vartheta'_{n-1}(\vartheta_{n-1}(t)) \cos(\vartheta_{n-1}(t)) - \vartheta'_n(t) \cos^2(\vartheta_{n-1}(t)) \right\} = \frac{K}{\sin(\vartheta_{n-1}(t))} \left\{ \vartheta'_{n-1}(\vartheta_{n-1}(t)) - \vartheta'_n(t) \cos(\vartheta_{n-1}(t)) \cos^2(\vartheta_{n-1}(t)) \right\} \leq \frac{K'}{\sin(\vartheta_{n-1}(t))} \left\{ \vartheta'_{n-1}(\vartheta_{n-1}(t)) - \vartheta'_n(t) \cos(\vartheta_{n-1}(t)) \right\} \leq
\]
C. i.e. the functions $\vartheta_n$ are uniformly bounded. From this we get that the $\vartheta_n|_{[0,t_o]}$ are uniformly bounded and equicontinuous. Since the sequence $\{\vartheta_n|_{[0,t_o]}\}$ is monotone, the sequence $\{\vartheta_n|_{[0,t_o]}\}$ uniformly converges to a $C^1$-function $\vartheta : [0, t_o] \to \mathbb{R}$ satisfying

$$\lim_{n \to \infty} \vartheta_n'(t) = \vartheta'(t) \quad \text{for any } t \in [0, t_o].$$

We leave to the reader the simple task of checking that the sequence $\vartheta_n(\vartheta_n(t))$ uniformly converges to $\vartheta(\vartheta(t))$, from which follows that the limit function $\vartheta(t)$ is indeed a solution to the differential problem (4.9). \(\square\)

4.4. Existence of self-involutes on the spheres.

**Theorem 4.4.** Let $\eta : [0, L] \to S^2_R$ be an almost self-involute on $S^2_R$ of class $C^3$, parameterized by arc length, such that

$$\lim_{s \to 0^+} \tau_s = 0, \quad \lim_{s \to 0^+} \kappa_s = a, \quad \lim_{s \to 0^+} \kappa_s = +\infty. \quad (4.21)$$

Then $\eta$ is self-involute if and only if $a$ is the unique solution of the equation

$$a = e^{\frac{3\pi}{2\eta_s}}. \quad (4.22)$$

**Proof.** Given an almost self-involute $\eta : [0, L] \to S^2_R$, let $\tilde{\eta}$ be the initial arc of the involute that is congruent to $\eta$. With no loss of generality, we assume that $\eta_0 = (0, 0, R)$. By definition, we have that $\tilde{\eta}_0 = \eta_0 = (0, 0, R)$ and that there exists an orthogonal matrix

$$A^{(n)} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad A^{(n)} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ \sin \phi & -\cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.23)$$

such that $\tilde{\eta}_s = A^{(n)} \cdot \eta_s$ for any $s \in [0, L]$.

We also identify $E^2$ with the plane $\{ x^3 = 1 \}$ in $\mathbb{R}^3$ and also in this case for any almost self-involute $\eta : [0, L] \to E^2 = \{ x^3 = 1 \}$ with $\eta_0 = (0, 0, 1)$, we denote by $\tilde{\eta}$ its involute and by $A^{(n)}$ the orthogonal matrix as in (4.23) such that $\tilde{\eta}_s = A^{(n)} \cdot \eta_s$.

Finally, for any almost self-involute $\eta$ on $S^2_R$ or $E^2$, we call fundamental pair of $\eta$ the couple $(A, a)$ given by $A = A^{(n)}$ and $a = \lim_{s \to 0^+} \tau_s$, where $\tau$ is defined in (4.6) or (4.7), respectively. Clearly, $\eta$ is self-involute if and only if $A = I$.

The proof is a consequence of the existence of a canonical correspondence between any almost self-involute $\eta$ on $S^2_R$ with fundamental pair $(a, A)$ and
an almost self-involute \( \eta_s^{(\infty)} \) on \( E^2 \), determined up to Euclidean isometries, having the same fundamental pair of \( \eta \) and congruent to the curve
\[
\eta_s^{(\infty)} = \left( \frac{s}{\sqrt{1+a^2}} \cos \left( a \log \left( \frac{s}{\sqrt{1+a^2}} \right) \right), -\frac{s}{\sqrt{1+a^2}} \sin \left( a \log \left( \frac{s}{\sqrt{1+a^2}} \right) \right), 1 \right) \tag{4.24}
\]
which is a parameterization by arc length of the curve
\[
\gamma_t = (e^{-\frac{t}{3}} \cos t, e^{-\frac{t}{3}} \sin t, 1),
\]
studied by Manselli and Pucci in [MP1]. In that paper, it is proved that \( \eta^{(\infty)} \) is self-involute (i.e. with a fundamental pair \((I, a)\)) if and only if \( a \) is solution of (4.22). Since \( \eta^{(\infty)} \) and \( \eta \) have the same fundamental pair, the conclusion follows.

Let us prove the existence of the correspondence \( \eta \rightarrow \eta^{(\infty)} \) described above. For any \( 0 \neq \lambda \in \mathbb{R} \), let
\[
\eta^{(\lambda)} : [0, L] \rightarrow S_{\lambda R}^2, \quad \eta_s^{(\lambda)} \overset{\text{def}}{=} \lambda \eta \left( \frac{s}{\lambda} \right), \tag{4.25}
\]
which is the initial arc of length \( L \) (parameterized by arc length) of the dilatation of \( \eta \) by \( \lambda \). From definitions, one can check that \( \eta^{(\lambda)} \) is an almost self-involute of \( S_{\lambda R}^2 \), with involute \( \eta^{(\lambda)} = \tilde{\eta}^{(\lambda)} = A \cdot \eta^{(\lambda)} \) and with functions \( \kappa_s^{(\lambda)} \) and \( \tau_s^{(\lambda)} \) given by
\[
\kappa_s^{(\lambda)} = \frac{1}{\lambda} \kappa_{\frac{s}{\lambda}}, \quad \tau_s^{(\lambda)} = \lambda \tau_{\frac{s}{\lambda}}. \tag{4.26}
\]
In particular, \( \lim_{s \rightarrow 0} \tau_s^{(\lambda)} = \lim_{s \rightarrow 0} \tilde{\tau}_s = a \) and the fundamental pair \((A, a)\) is the same for all curves \( \eta^{(\lambda)} \). For all \( \lambda \) sufficiently large, \( \eta^{(\lambda)} : [0, L] \rightarrow S_{\lambda R}^2 \) is included in the upper hemisphere and it can be identified with its image in \( \{ x^3 = \lambda R \} \) by the projection \( \pi^{(\lambda)} : S_{\lambda R}^2 \rightarrow \{ x^3 = \lambda R \} \simeq E^2 \) of center the origin.

The correspondence we are looking for is based on the following lemma.

**Lemma 4.5.** There exists a sequence \( \lambda_n \rightarrow +\infty \) such that the curves \( \eta_s^{(\lambda_n)} \) converge on \( (0, L] \), uniformly on compacta together with their first and second derivatives, to a \( C^2 \)-curve \( \eta^{(\infty)} \).

**Proof.** To prove this, first of all notice that, being of length \( L \) on the sphere and obtained via the projection \( \pi^{(\lambda)} \), any curve \( \eta^{(\lambda)} \) starts from \( x_0 = (0, 0, \lambda R) \simeq (0, 0, 1) \in E^2 \) and it is contained in the closed disk \( D_r(x_0) \) of radius \( r = \lambda R \tan \left( \frac{L}{\lambda R} \right) < 2L \) for all \( \lambda \) sufficiently large. Secondly, let us denote by \( g^{(\lambda)} \) the metric on \( E^2 \simeq \{ x^3 = \lambda R \} \) defined as push-forward by the projection \( \pi^{(\lambda)} \) of the metric of \( S_{\lambda R}^2 \). Notice that on any closed disc \( D_{\eta_0}(x_0) \), the metric \( g^{(\lambda)} \) converges uniformly to the Euclidean metric \( g_o \) together with all derivatives. Now, by construction, for any \( \lambda \) and \( s \in [0, L] \), we have that \( g^{(\lambda)}(\dot{\eta}_s^{(\lambda)}, \dot{\eta}_s^{(\lambda)}) = 1 \) and hence \( |\dot{\eta}_s^{(\lambda)}| = \sqrt{g_o(\dot{\eta}_s^{(\lambda)}, \dot{\eta}_s^{(\lambda)})} \) is
uniformly bounded for all $\lambda$ sufficiently large. A similar argument shows that
also the normal vectors $n_s^{(\lambda)}$ of the curve $\eta^{(\lambda)}$ (orthogonal to the $t_s^{(\lambda)} = \tilde{n}_s$ w.r.t. $g^{(\lambda)}$) are uniformly bounded. On the other hand, from (4.6), the fact
that $\tau$ is monotone and that $\lim_{s \to 0^+} \tau_s = 0$, one has that for any fixed
$0 < \varepsilon_o < L$ and any $s \in [\varepsilon_o, L]$

$$\lim_{\lambda \to +\infty} \kappa_s^{(\lambda)} = \lim_{\lambda \to +\infty} \frac{1}{\lambda} \kappa_s^{(\lambda)} = \lim_{\lambda \to +\infty} \frac{1}{\lambda R} \cot \left( \frac{\tau^{-1}(\lambda s)}{R} \right) =$$

$$= \lim_{\mu \to 0^+} \frac{\mu}{\tau^{-1}(\mu s)} \lim_{s \to 0^+} \frac{\alpha}{s} \leq \frac{a}{\varepsilon_o} \tag{4.27}$$

and with similar computations

$$\lim_{\lambda \to +\infty} \dot{\kappa}_s^{(\lambda)} = -\frac{a}{s^2} \geq -\frac{a}{\varepsilon_o^2} .$$

From this and (4.1), it follows that the covariant derivatives $\nabla_{\ddot{g}^{(\lambda)}} t_s^{(\lambda)}$ and
$\nabla_{\ddot{g}^{(\lambda)}} s^{(\lambda)}$ are uniformly bounded in any given interval $[\varepsilon_o, L]$. Considering
the explicit expression of such covariant derivatives in terms of the Christof-
fel symbols of $g^{(\lambda)}$ and of the derivatives $\ddot{n}_s^{(\lambda)}$ and $\ddot{n}_s^{(\lambda)}$, one can directly check
that on any given interval $[\varepsilon_o, L]$, the curves $\eta^{(\lambda)}$ are uniformly bounded in
$C^3$-norm. From this, the lemma follows. □

Using definitions and convergence in $C^2$, one can check that if $(A, a)$ is
the fundamental pair of $\eta : [0, L] \to S^2$, then the limit curve $\eta^{(\infty)}$ and
$\tilde{\eta}^{(\infty)} = A \cdot \eta^{(\infty)}$ satisfy the relation (4.4), i.e. $\eta^{(\infty)}$ is almost self-involute.
Moreover, by (4.4), we see that the curvature of $\eta^{(\infty)}$ is given by $\kappa_s^{(\infty)} = \frac{2}{s}$
and the associated function is $\tau_s^{(\infty)} = as$. In particular, the fundamental
pair of $\eta^{(\infty)}$ is $(A, a)$, the same of $\eta$. Notice also that since the curve (4.24)
has curvature function given by $\kappa_s = \frac{2}{s}$, by the Fundamental Theorem of
Plane Curves, $\eta^{(\infty)}$ is congruent to (4.24) and this concludes the proof that
the correspondence $\eta \mapsto \eta^{(\infty)}$ has all stated properties. □

From this and Theorem 4.3, the next corollary follows immediately.

**Corollary 4.6.** There exists $L_o > 0$ such that for any $L < L_o$ there exists
a self-involute curve of length $L$ on the unit sphere $S^2$.

5. $G$-CURVES ON SPHERES WITH THE “MAXIMAL LENGTH PROPERTY”

This section is devoted to show the existence of $G$-curves $\gamma : [0, L] \to S = S^2_+$ realizing the “maximal length property”, i.e. such that for any $s \in [0, L]$, the length $\ell_S(\gamma|_{[0,s]}) = s$ coincides with the largest possible value according
to Theorem 3.2, namely $\ell(\gamma|_{[0,s]}) = p(s)$. 

Theorem 5.1. If \( \eta : [0, L] \rightarrow S = S^2_+ \) is a self-involute curve, then it is a \( G \)-curve with the maximal length property, i.e. such that
\[
P(s) = \ell_S(\gamma|_{[0,s]}) , \quad \text{for any } s \in [0, L] .
\]

Proof. In the following, we constantly identify the self-involute \( \eta \) with its image in \( \{ x^3 = 1 \} \simeq E^2 \), determined by the projection \( \pi : S^2_+ \rightarrow \{ x^3 = 1 \} \) of center the origin. We also denote by \( g \) the metric on \( E^2 \) defined as push-forward by the projection \( \pi \) of the metric of \( S^2 \) and by \( g_o = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 \) the standard Euclidean metric.

A simple computation shows
\[
g = g_{ij} dx^i \otimes dx^j \quad \text{with} \quad \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & -1/(1+x^2)^2 \\ -1/(1+x^2)^2 & 1 \end{pmatrix} \cdot
\]
From this, by well-known formulae, we may express in term of \( \eta_s = (\eta_s^1, \eta_s^2) \) and its derivatives the following objects:
- the Christoffel symbols \( \Gamma_{ij}^k \big|_{\eta_s} = \frac{1}{2} g^{km} \left( \frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right) \big|_{\eta_s} \); 
- the covariant derivatives \( \nabla_{t_s} t_s = \left( \eta_s^i + \Gamma_{jk}^i \big|_{\eta_s} \eta_s^j \eta_s^k \right) \frac{\partial}{\partial x^j} \); 
- the geodesic curvature \( k_s = \sqrt{g(\nabla_{t_s} t_s, \nabla_{t_s} t_s)} = g(\nabla_{t_s} t_s, n_s) \).

Since \( \kappa_s > 0 \), also the Euclidean curvature \( \kappa_s^E \) of \( \eta_s \) is positive. This can be checked as follows. Recall that \( \dot{\eta}_s = g_o(\dot{\eta}_s, \dot{\eta}_s) \kappa_s^E n_s^E + \lambda_s \dot{\eta}_s \) for some function \( \lambda_s \) and with \( n_s^E \) denoting the unit normal vector in the Euclidean sense. Since \( g(\dot{\eta}_s, n_s) = g(t_s, n_s) = 0 \) and using the expression for \( \nabla_{t_s} t_s \),
\[
g_o(\dot{\eta}_s, \dot{\eta}_s) \kappa_s^E g(n_s^E, n_s) = g(\dot{\eta}_s, n_s) = \kappa_s - g \left( \Gamma_{jk}^i \big|_{\eta_s} \eta_s^j \eta_s^k \frac{\partial}{\partial x^j}, n_s \right).
\]
Using properties of projectively flat connections or just by a direct computation, one can see that the vector \( v_s = \Gamma_{jk}^i \big|_{\eta_s} \eta_s^j \eta_s^k \frac{\partial}{\partial x^j} \) is proportional to \( \dot{\eta}_s = t_s \). From this, it follows that \( \kappa_s^E = \frac{g_o(\dot{\eta}_s, \dot{\eta}_s) \kappa_s}{g(\dot{\eta}_s, n_s)} \). Since \( n_s^E \) and \( n_s \) lie on the same side w.r.t. \( t_s \), \( g(n_s^E, n_s) > 0 \) and \( \kappa_s^E \) is positive.

Being \( \kappa^E > 0 \), the “angle” function (i.e. the Euclidean angle between \( \dot{\eta}_s \) and the \( x^3 \)-axis), computable by
\[
\varphi : [0, L] \rightarrow \mathbb{R} , \quad \varphi \overset{\text{def}}{=} \int_L^s \kappa_{o}^E du + \varphi_L , \quad \text{with} \quad \varphi_L = \frac{\dot{\eta}_L}{\dot{\eta}_3} \frac{\partial}{\partial x^3} ,
\]
is monotone increasing.

Now, for any \( s \in (0, L] \), let us consider the following notation:
- \( \ell_1^{(s)} \) denotes the tangent line to \( \eta \) at the point \( x = \eta_s \), while \( \ell_2^{(s)} \) denotes the line through \( x \) and parallel to the vector \( n_s \); recall that these lines, up to reparameterizations, are geodesics for both the Euclidean metric \( g_o \) and the spherical metric \( g \);
- \( \gamma(s) \) is the closed, piecewise \( C^1 \) curve, formed by the arc \( \eta[[\tau^{-1}(s),s]] \) and the segment joining \( \eta_s \) and \( \eta_{\tau^{-1}(s)} \);
- \( \rho(s) \) is the total rotation of \( \gamma(s) \), i.e. the multiple of \( 2\pi \) defined by
  \[
  \rho(s) = \varphi_s - \varphi_{\tau^{-1}(s)} + \mathbf{n}_s \mathbf{t}_s \tag{5.2}
  \]
  and \( m_s = \rho(s)/2\pi \). Being \( \varphi \) monotone increasing, \( \rho(s) \) coincides with the total curvature of the curve \( \gamma(s) \). From this, by Fenchel’s theorem for piecewise differentiable curves ([MP]; see also [Mi, Ac]), \( \gamma(s) \) is a simple convex curve if and only if \( m_s = 1 \).

We claim that for any \( s \), \( \ell_1(s) \) and \( \ell_2(s) \) are support lines for the arc \( \eta[[0,s]] \) and that \( \gamma(s) \) is a simple and convex curve. This immediately implies that \( \eta \) is a \( G \)-curve and that \( \gamma(s) \) is the boundary of the convex hull of \( \eta[[0,s]] \). Being \( \eta_s \) self-involute, it follows that

\[
 p(s) = \ell_S(\eta[[\tau^{-1}(s),s]]) + \ell_S([\eta_{\tau^{-1}(s)}, \eta_s]) = s - \tau^{-1}(s) + \tau^{-1}(s) = s
\]

i.e. \( \eta_s \) satisfies (5.1) at any \( s \).

The proof of the theorem is therefore a direct consequence of the following three claims.

**Claim 1:** \( m_s = 1 \) for any \( s \), i.e. any closed curve \( \gamma(s) \) is simple and convex.

In fact, from (5.2), the map \( s \mapsto m_s \) is continuous and therefore constant. Moreover, for any \( \lambda > 1 \), the arc \( \eta[[0,\frac{\lambda}{\lambda}]] \) is homothetic to the curve \( \eta^{(\lambda)} \) described in (4.25) (which we also identify with the corresponding projected curve on \( \mathbb{E}^2 \)) and, by the proof of Theorem 4.4, we may choose \( \lambda \) so large that \( \eta^{(\lambda)} \) is arbitrarily close in \( C^2 \)-norm to the self-involute \( \eta^{(\infty)} \). By the results in [MP], \( \eta^{(\infty)} \) is a \( G \)-curve of \( \mathbb{E}^2 \). So, if we denote by \( \gamma^{(s)(\infty)} \) the closed curve formed by the segment joining \( \eta_s^{(\infty)} \) and \( \eta^{(\infty)}_{\tau^{-1}(s)} \) and the arc \( \eta^{(\infty)}[[\tau^{-1}(s),s]] \), it is immediate to realize that \( \gamma^{(s)(\infty)} \) is simple and convex, that is \( m_s^{(\infty)} = 1 \). Since \( \gamma(s) \) is homothetic to the piecewise \( C^2 \) closed curve \( \gamma^{(s)(\lambda)} \), close to \( \gamma^{(s)(\infty)} \) in \( C^2 \)-norm, it follows that also \( m_s = 1 \) for any \( 0 < s \leq \frac{\lambda}{\lambda} \) and hence for all values of \( s \).

**Claim 2:** \( \ell_1(s) \) is a support line for \( \eta[[0,s]] \).

To see this, notice that, being \( \gamma(s) \) closed and convex, \( \ell_1(s) \) is a support line for \( \gamma(s) \) and hence for \( \eta[[\tau^{-1}(s),s]] \). On the other hand, the spherical distance \( d_S(\tau^{-1}(s), \ell_1(s)) \) is equal to the length of the segment joining \( \eta_{\tau^{-1}(s)} \) and \( \eta_s \), because it lies in a line which is \( g \)-orthogonal to \( \ell_1(s) \). The length of this segment is equal to \( \tau^{-1}(s) \) by the definition of self-involute. This length is also equal to the length of the arc \( \eta[[0,\tau^{-1}(s)]] \). Hence, this arc lies in the same half-plane of \( \eta[[\tau^{-1}(s),s]] \) and \( \ell_1(s) \) is a support line for the whole curve \( \eta[[0,s]] \).
Claim 3: $\ell_2^{(s)}$ is a support line for $\eta|_{[0,s]}$.

As before, being $\gamma^{(s)}$ closed and convex, $\ell_2^{(s)}$ is a support line for $\eta|_{[\tau^{-1}(s),s]}$. On the other hand, by definition of self-involute, $\ell_2^{(s)} = \ell_1^{(\tau^{-1}(s))}$ and hence, by Claim 2, it is also a support line for the arc $\eta|_{[\tau^{-1}(s),s]}$. The two arcs lie in the same half-plane, because the Euclidean curvature of $\eta$ is strictly positive at all points and the claim follows. □

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