Noncommutative elliptic theory.

Examples

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Abstract

We study differential operators, whose coefficients define noncommutative algebras. As algebra of coefficients, we consider crossed products, corresponding to action of a discrete group on a smooth manifold. We give index formulas for Euler, signature and Dirac operators twisted by projections over the crossed product. Index of Connes operators on the noncommutative torus is computed.

Introduction

Noncommutative elliptic theory is the theory of differential operators, whose coefficients form algebras, which are in general noncommutative. Operators of this type first appeared in the work of A. Connes [1, 2], where on the real line the following operators were considered

\[ D = \sum_{\alpha + \beta \leq m} a_{\alpha\beta} x^\alpha \left(-i \frac{d}{dx}\right)^\beta. \]

Here the coefficients \( a_{\alpha\beta} \) belong to the algebra generated by operators \( U, V \)

\[ (Uf)(x) = f(x + 1), \quad (Vf)(x) = e^{-2\pi ix/\theta} f(x), \]

of unit shift and product by exponential (\( \theta \in (0, 1] \) is some fixed parameter). It is clear that the operators \( U \) and \( V \) do not commute. For such operators, the ellipticity condition and index formula were obtained.

Further examples of noncommutative differential operators were constructed on manifolds with torus action (see [3–5]) in connection with studying isospectral deformations and other questions of noncommutative geometry.

Specialists in the field of differential equations also studied noncommutative differential equations (e.g., see survey [6] and the references there). Namely, if a discrete group acts on a manifold, then one can consider operators of multiplication by functions and also operators of the group representation (which are called shift operators, or operators of change of variables) as coefficients of differential operators. This class of operators is known in the literature under various names: functional-differential operators, operators with shifts, nonlocal operators, noncommutative operators. We use the term “noncommutative”, because from our point of view it is most important in elliptic theory that
the coefficients of these operators form noncommutative algebras. This class of operators includes operators of A. Connes as well as operators on toric manifolds as special cases.

Although finiteness theorem (Fredholm property) for noncommutative differential equations was established relatively long ago [7], the index problem, i.e., the problem of expressing the index of a noncommutative elliptic operator in terms of its symbol, was open for a long time. In 2008 this problem was solved in [8,9]. The aim of the present paper is to apply this index formula to operators, which appear in geometry, and compute indices of some specific operators.

Let us briefly describe the contents of the paper.

In the first part of the paper (Sections 1-3) we review elliptic theory for noncommutative differential operators: definition of operators, formula for the symbol, finiteness theorem, construction of Chern character of elliptic symbol, index theorem. The proofs of all these results were published in [8], except Theorem 12 which is new. Then in Sections 4 and 5 we obtain index formulas for classical geometric operators: Euler, signature, Dirac operators. In Section 6, we compute the index of operators on the circle, while Section 7 deals with the index of operators of A. Connes on the noncommutative torus.

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1 Finiteness theorem

Noncommutative differential operators. Let a countable finitely generated group $\Gamma$ act on a smooth closed Riemannian manifold $M$.

Throughout the paper we consider only isometric actions of $\Gamma$ and suppose that the action extends to an action of a compact Lie group $G$. We shall also assume that the discrete groups under consideration are of polynomial growth.

On $M$ we choose a $G$-invariant Riemannian metric and volume form.

Definition 1. Noncommutative differential operator of order $\leq m$ is an operator of the form

$$D = \sum_{g \in \Gamma} T(g) D(g) : C^\infty(M) \rightarrow C^\infty(M),$$

which acts in the space of smooth functions, where

- $D(g)$ is a differential operator of order $\leq m$,
- $T(g)$ is shift operator (change of variables)

$$(T(g)u)(x) = u(g^{-1}(x)),$$

(corresponding to a diffeomorphism $g : M \rightarrow M$).
There is a question of convergence of the series in (1), if $\Gamma$ is infinite. Therefore, we shall first assume for simplicity that the operators $D(g)$ are equal to zero, except for finitely many elements $g$. Less restrictive condition will be given below.

**Symbol of noncommutative operator.** Obviously, operator (1) is determined modulo operators of order $\leq m - 1$ by the set of symbols $\sigma(D(g))$ of operators $D(g)$. This set can be naturally considered as a symbol of operator $D$ and is denoted by

$$\sigma(D) = \{\sigma(D(g))\}_{g \in \Gamma}. \quad (3)$$

Let us establish the composition formula, i.e., the rule of multiplication of symbols, corresponding to composition of operators.

The composition of operators $D = \sum_g T(g)D(g)$ and $Q = \sum_h T(h)Q(h)$ is equal to

$$DQ = \sum_{g,h} T(g)D(g)T(h)Q(h) = \sum_{g,h} T(gh)(T(h)^{-1}D(g)T(h))Q(h) = \sum_k T(k)\left(\sum_{gh=k} T(h)^{-1}D(g)T(h)Q(h)\right). \quad (4)$$

Here the operator $T(h)^{-1}D(g)T(h)$ is a differential operator with symbol

$$\sigma(T(h)^{-1}D(g)T(h)) = T(\partial h)\left(\sigma(D(g))\right), \quad (5)$$

where $\partial h : T^*M \rightarrow T^*M$ is the codifferential of diffeomorphism $h$. Note that (5) is just the usual formula for transformation of symbol under a change of variables. So, if we substitute (5) in (4), we see, that the product $DQ$ is an operator of the form (1), and the symbols of its components are equal to

$$\sigma(DQ)(k) = \sum_{gh=k} \left(T(\partial h)\sigma(D(g))\right)\sigma(Q(h)). \quad (6)$$

Algebras, whose elements are, similar to (3), functions on $\Gamma$ and the product is defined by convolution of the form (6) are called **crossed products** by $\Gamma$. Before we give the definition of the crossed product suitable for our purposes, we give two remarks.

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1. Recall that the symbol of a differential operator

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) \left( -i \frac{\partial}{\partial x} \right)^\alpha$$

of order $\leq m$ is a function

$$\sigma(P) = \sum_{|\alpha|=m} a_\alpha(x)x^\alpha, \quad (x, \xi) \in T^*M,$$

defined on the cotangent bundle minus the zero section $T^*M \setminus 0$ and homogeneous of order $m$.

2. The action of $G$ on $M$ extends to action on the cotangent bundle. Namely, an element $h \in G$ acts as the codifferential

$$\partial h = ((dh)^t)^{-1},$$

where $dh : TM \rightarrow TM$ is the differential of $h$, and $(dh)^t$ is the dual mapping of the cotangent bundle.
First, we shall consider restrictions of symbols to the cosphere bundle $S^*M \subset T^*M$, which consists of covectors of unit length. The action of $G$ restricts to the cosphere bundle, since the action is isometric.

Second, we shall consider classes of functions on $\Gamma$, which are larger than compactly supported functions, because in the class of functions with compact support one can not find inverse elements.\footnote{Indeed, consider operator $Id + T(g)/2$, where $g \neq e$. It is invertible. However, the inverse is given by the Neumann series \[(Id + T(g)/2)^{-1} = Id - T(g)/2 + T(g^2)/4 - \ldots\] and can not be written in general as a finite sum, unless $g$ is of finite order.}

**Definition 2.** *Smooth crossed product* of the Fréchet algebra $C^\infty(S^*M)$ and group $\Gamma$ is the algebra of $C^\infty(S^*M)$-valued functions on $\Gamma$, for which the following seminorms

$$\|f\|_{n,l} = \sup_{g \in \Gamma} \left( \|f(g)\|_n (1 + |g|)^l \right)$$

are finite for all $n$ and $l$, where $\{\| \cdot \|_n\}$ runs over some defining system of seminorms in $C^\infty(S^*M)$, with the product given by

$$(f_1 f_2)(k) = \sum_{gh=k} (T(\partial h) f_1(g)) f_2(h).$$

The smooth crossed product is denoted by $C^\infty(S^*M) \rtimes \Gamma$.

**Remark 3.** Interested reader can find detailed exposition of smooth crossed products in [10]. In particular, it is proved in the cited paper that the smooth crossed product is an algebra. Note also that in Definition 2 one can replace $C^\infty(S^*M)$ by any Fréchet algebra, on which group $\Gamma$ acts.

**Definition 4.** *Symbol* of noncommutative operator (1) is the collection (3), which is considered as an element of the smooth crossed product:

$$\sigma(D) \in C^\infty(S^*M) \rtimes \Gamma.$$ 

A noncommutative differential operator is called *elliptic*, if its symbol is invertible in the algebra $C^\infty(S^*M) \rtimes \Gamma$.

The equality (6) means that for noncommutative operators $D, Q$ one has composition formula

$$\sigma(DQ) = \sigma(D)\sigma(Q). \quad (7)$$

**Finiteness theorem.**

**Theorem 5.** An elliptic noncommutative operator $D$ is Fredholm as an operator acting in Sobolev spaces

$$D : H^s(M) \rightarrow H^{s-m}(M)$$

for all $s$. Moreover, the kernel and cokernel consist of smooth functions.
Sketch of the proof. 1. Using classical theory of pseudodifferential operators (e.g., see [11]), for any \( a \in C^\infty(S^*M) \rtimes \Gamma \) we can define operator \( A \) of the form (1), which has symbol \( a \). In addition, composition formula (7) remains valid.

2. Let \( Q \) denote an operator with symbol \( \sigma(D)^{-1} \in C^\infty(S^*M) \rtimes \Gamma \). Then the operators

\[
Id - QD \quad \text{and} \quad Id - DQ
\]

have zero symbols by composition formula (7) and, therefore, are compact operators of order \(-1\). Then the desired properties follow from the standard considerations. \( \square \)

Operators in subspaces. The theory of scalar operators, which we considered so far, has the following natural generalization.

Let \( D \) be a \( N \times N \) matrix operator with matrix components, which are noncommutative operators, and \( P_1 \) and \( P_2 \) be projections in the algebra \( \text{Mat}_N(C^\infty(M) \rtimes \Gamma) \) of \( N \times N \) matrices over the crossed product \( C^\infty(M) \rtimes \Gamma \). Suppose that the following equality holds

\[
D = P_2DP_1.
\]

Then we can consider the restriction

\[
D : P_1(C^\infty(M, \mathbb{C}^N)) \longrightarrow P_2(C^\infty(M, \mathbb{C}^N)) \tag{8}
\]

of \( D \) to subspaces in the space of vector-functions on \( M \), which are defined as ranges of projections \( P_1 \) and \( P_2 \).

The operator (8) is denoted by \( \mathbf{D} = (D, P_1, P_2) \) and called noncommutative operator acting in spaces defined by projections \( P_1, P_2 \). Operator (8) is called elliptic, if there exists a matrix symbol \( r \in \text{Mat}_N(C^\infty(S^*M) \rtimes \Gamma) \) such that

\[
r\sigma(D) = \sigma(P_1), \quad \sigma(D)r = \sigma(P_2).
\]

In this case a result similar to Theorem 5 holds. Namely, if operator (8) is elliptic, then it has Fredholm property and its kernel and cokernel consist of smooth functions.

2 Topological invariants

To describe index formula, we need to introduce topological invariants, which describe contributions of the symbol of elliptic operator and the manifold. There are two such invariants. The first is the Chern character, which is defined on a certain \( K \)-group. The second invariant is a suitable modification of the Todd class. These topological invariants are constructed in the present section.

2.1 \( K \)-theory class of elliptic symbol

For a noncommutative elliptic operator \( \mathbf{D} = (D, P_1, P_2) \), we define the element

\[
[\sigma(D)] \in K_0(C^\infty_0(T^*M) \rtimes \Gamma)
\]
of the even $K$-group of the crossed product of the algebra $C^\infty_0(T^*M)$ of compactly supported functions on the cotangent bundle by $\Gamma$. To this end, consider the projections

$$q_2 = \frac{1}{2} \begin{pmatrix} (1 - \sin \psi)\sigma(P_1) & \sigma(D)^{-1}\cos \psi \\ \sigma(D)\cos \psi & (1 + \sin \psi)\sigma(P_2) \end{pmatrix}, \quad q_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma(P_2) \end{pmatrix}$$

(9)

over the algebra $C^\infty_0(T^*M) \rtimes \Gamma$ with adjoint unit. Here $\psi \in C^\infty(T^*M)$ denotes a $\Gamma$-invariant function on the cotangent bundle $T^*M$ with canonical coordinates $x, \xi$, which depends only on $|\xi|$, for small $|\xi|$ is equal to $-\pi/2$, then increases as $|\xi|$ increases, while for large $|\xi|$ this function is equal to $+\pi/2$. In addition, the symbols of operators are considered as homogeneous functions of degree zero on the cotangent bundle $T^*M \setminus 0$ minus the zero section. Finally, we assume that the projection $P_2$ is equal to $\text{diag}(1, 1, ..., 1, 0, 0, ..., 0)$.

Let us now set

$$[\sigma(D)] = [q_2] - [q_1].$$

2.2 Chern character for crossed products

Let $X$ be a compact smooth manifold, on which $\Gamma$ acts and the action satisfies conditions formulated at the beginning of Section 1.

To define the Chern character on the $K$-group of the crossed product $C^\infty(X) \rtimes \Gamma$, we embed the crossed product algebra in a certain differential graded algebra (algebra of noncommutative differential forms), and then construct a differential trace on this algebra, which takes noncommutative differential forms to de Rham forms on the manifold.

Noncommutative differential forms. Denote by $\Omega(X)$ the algebra of differential forms on $X$ with differential $d$. The action of $\Gamma$ on smooth functions extends to the action on forms.

Consider the smooth crossed product $\Omega(X) \rtimes \Gamma$. This algebra is graded and has the differential

$$(d\omega)(g) = d(\omega(g)), \quad \omega \in \Omega(X) \rtimes \Gamma.$$ 

Equality $d^2 = 0$ is obvious, while the Leibniz rule

$$d(\omega_1\omega_2) = (d\omega_1)\omega_2 + (-1)^{\deg \omega_1}\omega_1d\omega_2, \quad \omega_1, \omega_2 \in \Omega(X) \rtimes \Gamma,$$

follows from the invariance of the exterior differential with respect to diffeomorphisms.

We shall refer to $\Omega(X) \rtimes \Gamma$ as the algebra of noncommutative differential forms on the $\Gamma$-manifold $X$.

Differential trace. Given $g \in \Gamma$, denote by $X^g$ the fixed point set of the action of $g$ on $X$. Since by assumption $g$ acts isometrically, the set $X^g \subset X$ is a smooth submanifold (e.g., see [12]).

\footnote{This can be achieved, if we consider direct sum of the initial operator and the invertible operator $(Id - P_2, Id - P_2, Id - P_2)$.}
Let us define the **differential trace**

$$\tau_g : \Omega(X) \rtimes \Gamma \rightarrow \Omega(X^g).$$  \hfill (10)

This means that $\tau_g$ has to be linear mapping, which satisfies the following properties

$$\tau_g(\omega_2 \omega_1) = (-1)^{\deg \omega_1 \deg \omega_2} \tau_g(\omega_1 \omega_2), \quad \text{for all } \omega_1, \omega_2 \in \Omega(X) \rtimes \Gamma,$$  \hfill (11)

$$d(\tau_g(\omega)) = \tau_g(d\omega), \quad \text{for any form } \omega \in \Omega(X) \rtimes \Gamma.$$  \hfill (12)

To define the trace (10), we introduce some notation. Let $C_g \subset G$ be the centralizer of $g$. The centralizer is a closed Lie subgroup in $G$. Denote elements of centralizer by $h$ and the induced Haar measure on the centralizer by $dh$.

Denote by $\langle g \rangle \subset \Gamma$ the conjugacy class of $g$, which is the set of elements, which can be written in the form $zgz^{-1}$ for some $z \in \Gamma$. We also fix for any element $g' \in \langle g \rangle$ some element $z = z(g, g')$, which conjugates $g$ and $g' = zgz^{-1}$. Any such element defines a diffeomorphism $z : X^g \rightarrow X^{g'}$.

Let us now define the trace (10) as

$$\tau_g(\omega) = \sum_{g' \in \langle g \rangle} \int_{C_g} h^*(z^*\omega(g')) \bigg|_{X^{g'}} dh.$$  \hfill (13)

One can show that this expression does not depend on the choice of elements $z$ and satisfies conditions (11), (12).

**Example 6.** Suppose that $X$ is a one-point space, and $\Gamma$ is finite. Then the crossed product $\Omega(X) \rtimes \Gamma$ coincides with the group algebra $\mathbb{C}\Gamma$, while the trace (13) is equal to

$$\tau_g(f) = \sum_{g' \in \langle g \rangle} f(g').$$

**Example 7.** Suppose now that $X$ is arbitrary, and $\Gamma$ is finite. Consider the unit element $g = e \in \Gamma$. In this case we can set $G = \Gamma$. Then the trace $\tau_e$ is equal to

$$\tau_e(\omega) = \frac{1}{|\Gamma|} \sum_{h \in \Gamma} h^*(\omega(e)), \quad \omega \in \Omega(X) \rtimes \Gamma,$$

i.e., its value is equal to the averaging over $\Gamma$ of the value of $\omega$ on the unit element of the group.

**Chern character.** We are now in a position to define the Chern character. Let $P \in \text{Mat}_N(C^\infty(X) \rtimes \Gamma)$ be a matrix projection.

Consider the restriction of the differential $d$ to the range of $P$: \n
$$\nabla_P = P \circ d \circ P : \Omega(X, \mathbb{C}^N) \rtimes \Gamma \rightarrow \Omega(X, \mathbb{C}^N) \rtimes \Gamma.$$  \hfill (14)

Here $P$ acts in the space $\Omega(X, \mathbb{C}^N) \rtimes \Gamma$ by left multiplication. An easy computation shows that the curvature form $\nabla_P^2$ is actually operator of multiplication by a noncommutative matrix-valued 2-form.

\[5\text{Note that the centralizer of an element } g \text{ is the set of elements of the group, which commute with } g.\]
Definition 8. The Chern character, corresponding to element \(g \in \Gamma\), of projection \(P\) is the following de Rham form

\[
\text{ch}_g(P) = \text{tr} \tau_g \left(P \exp \left(-\frac{\nabla^2_P}{2\pi i}\right)\right) \in \Omega(X^g),
\]

(14)
defined on the fixed point set \(X^g\), where tr is the matrix trace.

A straightforward computation shows that the form (14) is closed, and its cohomology class does not change under stable homotopies of projection \(P\). Thus, we obtain the well-defined homomorphism

\[
\text{ch}_g : K_0(C^\infty(X) \rtimes \Gamma) \longrightarrow H^{ev}(X^g),
\]

(15)

\[
P \mapsto \text{ch}_g(P).
\]

Remark 9. If the group is finite, then the Chern character (14) is equal to the one constructed in [13, 14].

Chern character for projections in bundles. It is important in applications to have formulas for the Chern character of projections defined in nontrivial vector bundles. Let us show how the formulas given above have to be modified in this case.

Let \(E\) be a finite-dimensional complex vector bundle over \(X\). First, we define the algebra of noncommutative endomorphisms \(C^\infty(X, \text{End} E)\) as the algebra of functions on \(\Gamma\), such that the value of a function at point \(g \in \Gamma\) is an element of the space \(\text{Hom}(E, g^*E)\). The product in this algebra is defined by the formula

\[
(f_1f_2)(k) = \sum_{gh=k} h^*f_1(g)(h^*)^{-1}f_2(h),
\]

(cf. (6)) where \(h^* : E \to h^*E\) is isomorphism. End \(E\)-valued noncommutative differential forms are defined similarly and denoted by \(\Omega(X, \text{End} E)\). The trace in this case is equal to

\[
\tau_g(\omega) = \sum_{g' \in \langle g \rangle} \int_{C_g} h^* \text{tr}(z^*\omega(g')) \bigg|_{X^g} dh,
\]

where tr is the trace of endomorphism of bundle \(E|_{X^g}\).

Then the Chern character of any projection \(P \in C^\infty(X, \text{End} E)\) is computed by the same formula (14), where we use the following noncommutative connection

\[
\nabla_P = P \circ \nabla_E \circ P : \Omega(X, \text{End} E) \longrightarrow \Omega(X, \text{End} E).
\]

(Here \(\nabla_E\) is some connection in \(E\).)

2.3 Todd class

Given \(g \in \Gamma\), denote by \(M^g\) the fixed point manifold of this element. Let \(N^g\) be the normal bundle of \(M^g\) in \(M\).
The differential of $g$ induces orthogonal endomorphism of $N^g$ and the exterior form bundle
\[ \Omega(N^g \otimes \mathbb{C}) = \Omega^{ev}(N^g \otimes \mathbb{C}) \oplus \Omega^{odd}(N^g \otimes \mathbb{C}). \]

Consider the Chern character\(^6\)
\[ \text{ch} \Omega^{ev}(N^g \otimes \mathbb{C})(g) - \text{ch} \Omega^{odd}(N^g \otimes \mathbb{C})(g) \in H^{ev}(M^g). \tag{16} \]

The zero degree component of this expression is nonzero [16]. Therefore, the class (16) is invertible and the following class
\[ \text{Td}_g(T^*M \otimes \mathbb{C}) = \text{Td}(T^*M^g \otimes \mathbb{C}) \frac{\text{ch} \Omega^{ev}(N^g \otimes \mathbb{C})(g) - \text{ch} \Omega^{odd}(N^g \otimes \mathbb{C})(g)}{\text{Td}(T^*M^g \otimes \mathbb{C})}, \tag{17} \]
is well defined, where Td in the right hand side of the equality is the standard Todd class of a complex vector bundle, and the division is well defined, because both numerator and denominator are even degree cohomology classes.

### 3 Index theorem

**Theorem 10.** Let $D$ be a noncommutative elliptic operator on a closed manifold $M$. Then
\[ \text{ind} \ D = \sum_{\langle g \rangle \subset \Gamma} \langle \text{ch}_g[D] \rangle \text{Td}_g(T^*M \otimes \mathbb{C}), [T^*M^g] \tag{18} \]
where $\langle g \rangle$ runs over the set of conjugacy classes of $\Gamma$; $[T^*M^g] \in H^{ev}(T^*M^g)$ is the fundamental class of $T^*M^g$, the Todd class is lifted from $M_g$ to the corresponding $T^*M^g$ by the natural projection and the angular brackets denote the pairing between homology and cohomology. The series in (18) is absolutely convergent.

For matrix operators (systems of equations), the index formula is written in terms of integrals over the corresponding cosphere bundles. To formulate the result, we introduce the corresponding odd Chern character.

**Definition 11.** The odd Chern character of matrix elliptic symbol
\[ \sigma \in \text{Mat}_N(C^\infty(S^*M) \rtimes \Gamma) \]
is the collection
\[ \text{ch}^{odd}(\sigma) \in \bigoplus_{\langle g \rangle \subset \Gamma} H^{odd}(S^*M^g) \]
of cohomology classes, which are defined as
\[ \text{ch}^{odd}_g(\sigma) = \text{tr} \tau_g \left[ \sum_{n \geq 0} \frac{1}{(2\pi i)^{n+1}(2n+1)!} (\sigma^{-1}d\sigma)^{2n+1} \right]. \tag{19} \]

\(^6\)Recall (see [15]) the definition of the class $\text{ch} E(g)$ of a $G$-vector bundle $E$ on a trivial $G$-space $X$. If we decompose the bundle $E$ as the direct sum $E = \bigoplus \lambda E_\lambda$ of eigensubbundles with regard to the action of $g$, then we have
\[ \text{ch} E(g) = \sum \lambda \text{ch} E_\lambda \in H^{ev}(X). \]
The index formula (18) in this case becomes
\[
\text{ind } D = \sum_{\langle g \rangle \subset \Gamma} \langle \text{ch}^{\text{odd}}(\sigma(D)) \rangle T_d(g(T^*M \otimes \mathbb{C}), [S^*M_g]) .
\] (20)

**Theorem 12** (on index contribution of trivial element of the group). *Suppose that either the action of \( \Gamma \) on \( M \) is free or \( \Gamma \) is torsion free. Then for any elliptic operator \( D \) one has*
\[
\text{ind } D = \langle \text{ch}_e[\sigma(D)] \rangle T_d(T^*M \otimes \mathbb{C}), [T^*M] .
\] (21)

*Proof.* If the action is free, Eq.(21) follows from (18), since \( M^g \) is empty for any \( g \neq e \).

Suppose now that \( \Gamma \) is torsion free (i.e., has no elements of finite order). Denote by \( C^\infty(\Gamma) \) the smooth crossed product \( C \rtimes \Gamma \), which is just the algebra of rapidly decaying functions on \( \Gamma \) with convolution product.

1. It was shown in [8], that the index mapping can be included as a side of the commutative triangle
\[
\begin{array}{c}
K_0(C^\infty_0(T^*M) \rtimes \Gamma) \\
\bigg\downarrow \text{ind}
\end{array} \xrightarrow{p_t} K_0(C^\infty(\Gamma)),
\]
\[
\bigg\downarrow \tau \quad Z
\] (22)

where \( p_t \) stands for the direct image mapping in \( K \)-theory, which is induced by the projection \( p : M \to pt \) to the one-point space, while
\[
\tau = \sum_{\langle g \rangle \subset \Gamma} \text{ch}_g,
\] (23)

where we recall that \( \text{ch}_g : K_0(C^\infty(\Gamma)) \to H^{\text{ev}}(pt) = \mathbb{C} \). In addition, the number \( \text{ch}_g(p_t[\sigma(D)]) \) is equal to the contribution of the conjugacy class \( \langle g \rangle \) to the index formula (18).

2. The mapping \( \tau \) is induced by the tracial state
\[
\tau : C^*(\Gamma) \longrightarrow \mathbb{C},
\]
\[
f \longmapsto \sum_{g \in \Gamma} f(g),
\]
where \( C^*(\Gamma) \) stands for the group \( C^* \)-algebra. Consider now the tracial state
\[
\tau_e = \text{ch}_e,
\]
i.e., \( \tau_e(f) = f(e) \). By [17, Theorem 1] all tracial states induce the same mapping on the \( K \)-group. The desired equality
\[
\text{ind } D = \tau(p_t[\sigma(D)]) = \tau_e(p_t[\sigma(D)])
\]
now follows, i.e., the index is equal to the contribution of the trivial conjugacy class.

The proof of the theorem is now complete. \( \square \)
4 Index of twisted operators

In this section we show how $G$-invariant elliptic operators can be twisted by certain projections to produce noncommutative elliptic operators.

**Twisted operators.** Suppose that we have a $G$-invariant elliptic operator

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

acting in spaces of sections of two $G$-bundles $E, F$. The $G$-invariance condition means that for any $g \in G$ one has

$$DT_E(g) = T_F(g)D,$$

where $T_E(g), T_F(g)$ denote actions of shift operators on sections of $E$ and $F$, respectively.

Given a projection

$$P \in \text{Mat}_n(C^\infty(M) \rtimes \Gamma),$$

let us define projection $\tilde{P}_E : C^\infty(M, E \otimes \mathbb{C}^n) \to C^\infty(M, E \otimes \mathbb{C}^n)$ by the formula

$$\tilde{P}_E = \sum_{g \in \Gamma} T_E(g) \otimes P(g). \quad (24)$$

One defines a similar projection $P_F$, corresponding to $F$.

The direct sum of $n$ copies of $D$ is denoted by $D \otimes 1_n$.

**Definition 13.** The operator

$$P_F(D \otimes 1_n)P_E : P_E(C^\infty(M, E \otimes \mathbb{C}^n)) \to P_F(C^\infty(M, F \otimes \mathbb{C}^n)) \quad (25)$$

is denoted by $D \otimes 1_P$ and called operator $D$ twisted by projection $P$.

**Proposition 14.** The operator $[25]$ is elliptic.

**Proof.** Consider the symbol

$$\sigma(P_E)(\sigma(D)^{-1} \otimes 1_n)\sigma(P_F). \quad (26)$$

Here the inverse $\sigma(D)^{-1}$ exists, since we assumed that $D$ is elliptic.

We claim that the symbol $[26]$ is the inverse of the symbol of the operator $[25]$. Indeed, it follows from the $G$-invariance of $D$ that $\sigma(D \otimes 1_n)$ intertwines projections $\sigma(P_E)$ and $\sigma(P_F)$. Thus, we get

$$\sigma(P_F)(\sigma(D) \otimes 1_n)\sigma(P_E)(\sigma(D)^{-1} \otimes 1_n)\sigma(P_F) = \sigma(P_F)(\sigma(D) \otimes 1_n)(\sigma(D)^{-1} \otimes 1_n)\sigma(P_F) = \sigma(P_F).$$

Similar computation can be done for the composition in inverse order. \qed
Multiplicative property of Chern character. The symbol of the $G$-invariant elliptic operator $D$ defines the class

$$[\sigma(D)] \in K_{G,c}^0(T^*M)$$

in the equivariant $K$-group (with compact supports) of the cotangent bundle, while projection $P$ defines the class

$$[P] \in K_0(C^\infty(M) \rtimes \Gamma)$$

in the $K$-group of the crossed product.

The mapping $\sigma(D), P \mapsto [\sigma(D) \otimes 1_P]$, where $D \otimes 1_P$ is the twisted operator, defines the product on $K$-groups

$$K_0(C^\infty(M) \rtimes \Gamma) \times K_{G,c}^0(T^*M) \longrightarrow K_0(C^\infty_0(T^*M) \rtimes \Gamma),$$

$$[P], [\sigma(D)] \longmapsto [\sigma(D \otimes 1_P)].$$

To formulate the multiplicative property of the Chern character with respect to this product, we denote by $G(g) \subset G$ the compact subgroup generated by element $g$, and by $ev_g : R(G(g)) \to \mathbb{C}$ the evaluation mapping for characters of representations at point $g$. Let us define the homomorphism

$$ch(g) : K^0_G(X) \to H^{ev}(X^g)$$

as the composition

$$K^0_G(X) \to K^0_{G(g)}(X^g) \simeq K^0(X^g) \otimes R(G(g)) \overset{ev_g}{\to} K^0(X^g) \overset{ch}{\to} H^{ev}(X^g),$$

(27)

where the first mapping is the restriction to the fixed point set and the isomorphism describes the equivariant $K$-group of a trivial $G(g)$-space in terms of usual (nonequivariant) $K$-groups.

It was proved in [8] that the diagram

$$
\begin{array}{ccc}
K_0(C^\infty(M) \rtimes \Gamma) \times K_{G,c}^0(T^*M) & \longrightarrow & K_0(C^\infty_0(T^*M) \rtimes \Gamma) \\
\text{ch}_g \times \text{ch}(g) & \downarrow & \text{ch}_g \\
H^{ev}(M^g) \times H^{ev}_c(T^*M^g) & \longrightarrow & H^{ev}_c(T^*M^g)
\end{array}
$$

(28)

is commutative for any $g \in \Gamma$.

Index formula for twisted operators. We shall assume for simplicity that all submanifolds of fixed points $M^g$ are orientable.

Since the diagram (28) is commutative, we have

$$ch_g(\sigma(D \otimes 1_P)) = ch[\sigma(D)](g) \cdot ch(P).$$

\footnote{If the fixed-point manifolds are nonorientable, then the formulas below can be modified using local coefficient systems as in [15]. We shall omit this standard procedure here.}
Substituting this equality in the index formula (18), and integrating over the fibers of the cosphere bundles, we obtain the formula

\[ \text{ind}(D \otimes 1_P) = \sum_{\langle g \rangle \subset \Gamma} \langle \psi_g^{-1}(\text{ch}[\sigma(D)](g)) \rangle \text{Td}_g(T^*M \otimes \mathbb{C}) \text{ch}_g(P), [M^g] \]  

(29)

for the index of twisted operators, where \( \psi_g : H^*(M^g) \longrightarrow H_c^*(T^*M^g) \) stands for the Thom isomorphism defined by the orientation of \( M^g \).

5 Index of geometric operators

Let us give index formulas for geometric operators twisted by noncommutative projections.

**Euler operator.** Let \( E \) be the Euler operator

\[ d + d^*: \Omega^{\text{ev}}(M) \longrightarrow \Omega^{\text{odd}}(M), \]

acting between spaces of forms of even and odd degrees.

Consider the twisted Euler operator

\[ E \otimes 1_P : P\Omega^{\text{ev}}(M, \mathbb{C}^n) \longrightarrow P\Omega^{\text{odd}}(M, \mathbb{C}^n), \]

where \( P \in \text{Mat}_n(C^\infty(M) \rtimes \Gamma) \) is a projection.

**Theorem 15.** The index of twisted Euler operator is equal to

\[ \text{ind}(E \otimes 1_P) = \sum_{\langle g \rangle \subset \Gamma} \chi(M^g) \text{ch}_g^0(P), \]

where \( \chi \) stands for the Euler characteristic, the number \( \text{ch}_g^0(P) \) is equal to the zero degree component of the Chern character. In addition, \( \chi(M^g) \) and \( \text{ch}_g^0(P) \) are treated as locally-constant functions on the fixed point set \( M^g \).

**Proof.**

1. The restriction of the symbol \( \sigma(E) \) to \( T^*M^g \subset T^*M \) has the decomposition\(^8\)

\[ \sigma(E)|_{T^*M^g} \simeq \left( \sigma(E_M^g) \otimes 1_{\Omega^{\text{ev}}(N)} \right) \oplus \left( \sigma(E_M^g) \otimes 1_{\Omega^{\text{odd}}(N)} \right), \]

(30)

where \( E_M^g \) denotes the Euler operator on \( M^g \) and \( E_M^g \) is its adjoint.

2. It follows from (30) that

\[ \text{ch} \sigma(E)(g) = \text{ch} \sigma(E_M^g) \cdot \text{ch}(\Omega^{\text{ev}}(N) - \Omega^{\text{odd}}(N))(g). \]

---

\(^8\)Hereinafter, lifts (see (24)) of projection \( P \) to spaces of vector bundle sections will be denoted for short as \( P \).

\(^9\)This decomposition follows from the orthogonal decomposition of tangent bundle

\[ TM|_{M^g} \simeq TM^g \oplus N, \]

and the corresponding graded decompositions \( \Omega(TM)|_{M^g} \simeq \Omega(TM^g) \otimes \Omega(N) \) of the exterior forms.
Hence, we obtain
\[ \text{ch} \sigma(E)(g) \, \text{Td}_g(T^*M \otimes \mathbb{C}) = \text{ch} \sigma(E_{M^g}) \, \text{Td}(T^*M^g \otimes \mathbb{C}). \]
Substituting this formula in Eq. (29), we see that the contribution of \( M^g \) to the index is equal to
\[ \int_{M^g} \psi_g^{-1} \left( \text{ch} \sigma(E_{M^g})(g) \right) \, \text{Td}(T^*M^g \otimes \mathbb{C}) \, \text{ch}_g(P) = \int_{M^g} e(TM^g) \, \text{ch}_g(P) = \chi(M^g) \, \text{ch}_g^0(P), \]
where \( e(TM^g) \) stands for the Euler class. Here in the first equality we use the classical result
\[ \psi_g^{-1} \left( \text{ch} \sigma(E_{M^g})(g) \right) \, \text{Td}(T^*M^g \otimes \mathbb{C}) = e(TM^g), \]
and in the second equality the fact that the Euler class is a top degree class.

The proof of the theorem is now complete. \( \square \)

**Signature operator.** Suppose that \( M \) is a 2\( l \)-dimensional oriented manifold. We define the involution on the space of differential forms
\[ \alpha = i^{k(k-1)+l} : \Omega^k(M) \rightarrow \Omega^{2l-k}(M), \quad \alpha^2 = \text{Id}, \]
where \( * \) stands for the Hodge operator. The subspace of forms satisfying equality \( \alpha \omega = \omega \) is called the space of self-dual forms, and satisfying equality \( \alpha \omega = -\omega \) is called anti-self-dual forms. The corresponding subspaces in \( \Omega(M) \) are denoted by \( \Omega^+(M) \) and \( \Omega^-(M) \).

Denote by \( S \) the signature operator
\[ d + d^* : \Omega^+(M) \rightarrow \Omega^-(M), \] (31)
acting from self-dual to anti-self-dual forms. Let us assume that \( G \) acts on \( M \) by orientation-preserving diffeomorphisms. In this case \( \Omega^\pm(M) \) are \( G \)-invariant subspaces. Consider the twisted signature operator
\[ S \otimes 1_P : P\Omega^+(M, \mathbb{C}^n) \rightarrow P\Omega^-(M, \mathbb{C}^n), \] (32)
where \( P \in \text{Mat}_n(C^\infty(M) \rtimes \Gamma) \) is a projection.

To give explicit index formula in this case, let us introduce following [15] special cohomology class. Choose an element \( g \in \Gamma \). Then \( g \) acts on the normal bundle \( N^g \) of the embedding \( M^g \subset M \) by orthogonal transformations. Therefore, \( N^g \) decomposes as the orthogonal sum
\[ N^g = N^g(-1) \bigoplus_{0<\theta<\pi} N^g(\theta), \quad \text{where } g|_{N^g(-1)} = -\text{Id}, \quad g|_{N^g(\theta)} = e^{i\theta} \text{Id}, \]
of eigensubbundles. The bundles \( N^g(\theta) \) have complex structure. Let us note that there is no eigenvalue 1 in this decomposition. The bundle \( N^g(-1) \), as well as the manifold \( M^g \) are even-dimensional. Denote
\[ t = \dim M^g/2, \quad r = \dim N^g(-1)/2, \quad s(\theta) = \dim N^g(\theta)/2. \]
Let $L_g(M)$ be the cohomology class

$$2^{t-r} \prod_{0<\theta<\pi} \left( \left( \frac{i \theta}{2} \right)^{-s(\theta)} \right) L(TM^g)L(N^g(-1))^{-1} e(N^g(-1)) \prod_{0<\theta<\pi} \mathcal{M}^\theta(N^g(\theta)) \in H^{ev}(M^g),$$

where $L$ is the Hirzebruch $L$-class of a real vector bundle, defined in the Borel-Hirzebruch formalism by the function

$$\frac{x/2}{\text{th} x/2},$$

$e(N^g(-1))$ is the Euler class of an oriented vector bundle and $\mathcal{M}^\theta(N^g(\theta))$ is the characteristic class of the complex vector bundle $N^g(\theta)$, which is defined by the function

$$\frac{\text{th} \frac{i \theta}{2}}{\text{th} \frac{x + i \theta}{2}}.$$

**Theorem 16.** The index of twisted signature operator is equal to

$$\text{ind}(S \otimes 1_P) = \sum_{(g) \subset \Gamma} \int_{M^g} L_g(M) \text{ch}_g(P).$$

**Proof.** Atiyah and Singer in [15] proved that

$$\psi_g^{-1}(\text{ch} \sigma(S)(g)) \text{Td}_g(T^* M \otimes \mathbb{C}) = L_g(M).$$

Substituting this equality in Eq. (29), we see that the contribution of $M^g$ to the index is equal to the desired expression. \hfill \Box

**Dirac operator.** Let $M$ be an even-dimensional oriented manifold, which is endowed with a $G$-invariant spin-structure (i.e., the action of $G$ on $M$ lifts to the action on the spinor bundle $S(M)$). Let $\mathcal{D}$ be the Dirac operator [15]

$$\mathcal{D} : S^+(M) \longrightarrow S^-(M),$$

acting between sections of half-spinor bundles.

Consider the twisted Dirac operator

$$\mathcal{D} \otimes 1_P : PS^+(M, \mathbb{C}^n) \longrightarrow PS^-(M, \mathbb{C}^n),$$

where $P \in \text{Mat}_n(C^\infty(M) \ltimes \Gamma)$ is some projection.

Let us consider for simplicity the case when either $\Gamma$ is torsion free, or its action is free. In this case, Eqs. (21) and (29) give the index formula. \hfill 10

\footnote{The orientation of the bundle $N^g(-1)$ is determined by the orientation of $M^g$ (see [15]).}
Theorem 17. The index of twisted Dirac operator is equal to

$$\text{ind}(D \otimes 1_P) = \int_M A(TM) \text{ch}_e(P),$$

(33)

where $A(TM)$ is the $A$-class of the tangent bundle, which is defined in Borel-Hirzebruch formalism by the function

$$\frac{x/2}{\sinh x/2}.$$

6 Index of operators on the circle

Reflection group. Consider the operator

$$D = \frac{d}{d\varphi} : C^\infty_+(S^1) \longrightarrow C^\infty_- (S^1),$$

(34)

acting in the spaces

$$C^\infty_\pm (S^1) = \{ u \in C^\infty (S^1) \mid u(-\varphi) = \pm u(\varphi) \}$$

of even and odd functions on the circle. On the one hand, the index of this operator is obviously equal to one.

On the other hand, the operator (34) is equivalent to the Euler operator

$$\mathcal{E} \otimes 1_P : P\Omega^0(S^1) \longrightarrow P\Omega^1(S^1),$$

$$f \longmapsto df,$$

twisted by the projection $P = (1 + g^*)/2$, where $g(\varphi) = -\varphi$ denotes the action of $\mathbb{Z}_2$ by reflections.

The mapping $g$ has 2 fixed points: $\varphi = 0$ and $\varphi = \pi$. Thus, by virtue of Theorem 15 we get the index formula

$$\text{ind} D = \text{ind}(\mathcal{E} \otimes 1_P) = \chi(S^1) \text{ch}^0_e(P) + \chi(\{0\}) \text{ch}^0_y(P) + \chi(\{\pi\}) \text{ch}^0_y(P) =$$

$$= 0 + \frac{1}{2} + \frac{1}{2} = 1,$$

where $\text{ch}^0_y$ stands for the zero component of the Chern character.

This example shows that the contributions to the index formula of the conjugacy classes of the group can be fractional numbers.

Rotation group. Given a nonzero vector

$$\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{T}^n,$$

let us consider the equation

$$\sum_{g \in \mathbb{Z}^n, |l| \leq m} a_{g,l}(x) \left( -i \frac{d}{dx} \right)^l u(x - g\alpha) = f(x)$$

(35)
on the circle \( S^1 \), where \( u, f \in C^\infty(S^1) \), and \( g\alpha = g_1\alpha_1 + \ldots + g_n\alpha_n \). Here we assume that the coefficients \( a_{g,l}(x) \) are rapidly decaying as \( |g| \to \infty \).

The vector \( \alpha \) defines the action of the group \( \mathbb{Z}^n \) on \( S^1 \):

\[
g(\varphi) = \varphi + g\alpha.
\]

Denote the corresponding smooth crossed product by \( C^\infty(S^1) \rtimes \mathbb{Z}^n \). Eq. \( \text{(35)} \) corresponds to the following noncommutative operator

\[
D = \sum_{g \in \mathbb{Z}^n, l \leq m} T(g) b_{g,l}(x) \left( -i \frac{d}{dx} \right)^l : C^\infty(S^1) \to C^\infty(S^1). \tag{36}
\]

The symbol of \( D \) is equal to

\[
\sigma(D) = \{ b_{g,m}(x)\xi^m \} \in C^\infty(S^*S^1) \rtimes \mathbb{Z}^n, \quad (x, \xi) \in S^*S^1. \tag{37}
\]

The decomposition \( S^*S^1 = S^1 \sqcup S^1 \) of the cosphere bundle induces the decomposition

\[
C^\infty(S^*S^1) \rtimes \mathbb{Z}^n = C^\infty(S^1) \rtimes \mathbb{Z}^n \oplus C^\infty(S^1) \rtimes \mathbb{Z}^n.
\]

The corresponding components of the symbol are denoted by \( \sigma(D) = \sigma_+ \oplus \sigma_- \).

If the symbol is invertible in \( C^\infty(S^*S^1) \rtimes \mathbb{Z}^n \), then the operator is elliptic. Since the group \( \mathbb{Z}^n \) is torsion free, there are no index contributions of nontrivial conjugacy classes and we get the index formula

\[
\text{ind } D = -\frac{1}{2\pi i} \int_{S^1} \left[ \sigma_+^{-1}d\sigma_+(0) - \sigma_-^{-1}d\sigma_-(0) \right]
\]

in terms of the symbol. Here we applied the index formula for matrix operators \( \text{(20)} \).

The obtained index formula is similar to Nöther-Muskhelishvili formula, which expresses the index of singular integral operators on the circle in terms of the winding number of the symbol.

Remark 18. If the components of \( \alpha \) in \( \text{(35)} \) linearly dependent over the field of rational numbers, then there are relations among the corresponding shifts. In this case, it is natural to formulate the ellipticity condition in terms of the crossed product \( C^\infty(S^*S^1) \rtimes (\mathbb{Z}^n / L) \), where \( L \) is a certain lattice and there may be nontrivial contributions of fixed point sets to the index formula.

7 Index of operators on the real line

A. Connes in his note \([1]\) considered elliptic operators on the noncommutative torus. Later, such operators were studied in a number of papers (e.g., see \([2, 18–20]\)). Let us show how the results presented in the preceding sections enable one to compute index in this situation.

\[^{11}\text{The coefficients in } \text{(35)} \text{ and } \text{(36)} \text{ are related by the equality: } b_{g,l}(x) = a_{g,l}(x + g\alpha).\]
Noncommutative torus $A^\infty_{1/\theta}$. Given some number $\theta$ such that $0 < \theta \leq 1$, consider operators
\[(Uf)(x) = f(x + 1), \quad (Vf)(x) = e^{-2\pi i x/\theta} f(x)\]of unit shift and multiplication by exponential, which act on functions on the real line with coordinate $x$. These operators satisfy the commutation relation
\[VU = e^{2\pi i /\theta} UV.\]Let $A^\infty_{1/\theta}$ denote the algebra of operators
\[
\sum_{j,k=\infty}^\infty a_{jk} U^j V^k,
\]where coefficients $a_{jk} \in \mathbb{C}$ are rapidly decaying: for any $l$ there exists a constant $C$ such that
\[|a_{jk}| \leq C(1 + |j| + |k|)^{-l}, \quad j, k = 0, \pm 1, \pm 2, \ldots\]

**Example 19.** For $\theta = 1$ the algebra $A^\infty_{1/\theta}$ is commutative and isomorphic to the algebra of smooth functions on the torus $T^2$. The isomorphism is defined on generators as
\[U \mapsto e^{i\varphi}, \quad V \mapsto e^{i\psi},\]where $\varphi, \psi$ denote coordinates on $T^2$. In addition, the rapid decay condition on $a_{jk}$ is transformed to the condition of $C^\infty$-smoothness of the corresponding function on the torus.

For $\theta \neq 1$ the algebra $A^\infty_{1/\theta}$ is noncommutative and is called the algebra of functions on the noncommutative torus.

**Remark 20.** The fact that $A^\infty_{1/\theta}$ is indeed an algebra, i.e., the product of two series (40) enjoys the rapid decay property, follows from the isomorphism of $A^\infty_{1/\theta}$ and the smooth crossed product $C^\infty(S^1) \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts by rotations by angle $2\pi/\theta$ (so-called irrational rotation algebra).

**Operators on the noncommutative torus. Reduction to operators on $T^2$.** In the Schwartz space $S(\mathbb{R})$ of functions on the real line, A. Connes proposed to consider differential operators
\[D = \sum_{\alpha + \beta \leq m} a_{\alpha\beta} x^\alpha \left(-i \frac{d}{dx}\right)^\beta : S(\mathbb{R}) \longrightarrow S(\mathbb{R}),\]

The reader might naturally have a question, why operators of the form (41) are called operators on noncommutative torus? More generally, what are operators on a “noncommutative space”? We refer the reader to [2] for detailed exposition of these questions. Here we note only, that operators (41) have compact commutators with operators $U', V'$, which generate the noncommutative torus $A^\infty_{\theta}$:
\[(U'f)(x) = f(x + \theta), \quad (V'f)(x) = e^{-2\pi ix} f(x).\]In addition, if the operator has the Fredholm property, then it defines an element in the Kasparov $K$-homology group of the algebra $A^\infty_{\theta}$. It is shown in the cited book that the mentioned duality between the algebras $A^\infty_{1/\theta}$ and $A^\infty_{\theta}$ is actually a special case of Poincaré duality in noncommutative geometry.
whose coefficients contain polynomials and elements $a_{\alpha\beta} \in A_{i/\theta}^\infty$.

Let us show how operators (41) can be reduced to operators on a compact manifold. Consider the real line as the total space of the spiral covering

$$\mathbb{R} \longrightarrow S^1$$

over the circle of length $\theta$ (see Fig. 1).

Then the Schwartz space on $\mathbb{R}$ is isomorphic to the space of smooth sections of a (nontrivial!) bundle over $S^1$, whose fiber is the Schwartz space $S(\mathbb{Z})$ of rapidly-decaying sequences (i.e., functions on fibers of the covering). If we now make Fourier transform $\mathcal{F}$

$$S(\mathbb{Z}) \overset{\mathcal{F}}{\longrightarrow} C^\infty(S^1)$$

fiberwise, then the obtained space is just the space of smooth sections of a one-dimensional complex vector bundle over the torus $\mathbb{T}^2$. Collecting these transformations, we obtain the following result.

On $\mathbb{T}^2$ choose coordinates $0 \leq \varphi \leq \theta$, $0 \leq \psi \leq 1$.

**Lemma 21.** One has isomorphism

$$S(\mathbb{R}) \simeq C^\infty(\mathbb{T}^2, \gamma)$$

of the Schwartz space on the real line and the space

$$C^\infty(\mathbb{T}^2, \gamma) = \{ g \in C^\infty(\mathbb{R} \times S^1) \mid g(\varphi + \theta, \psi) = g(\varphi, \psi)e^{-2\pi i\psi} \}$$

of smooth sections of the line bundle $\gamma$ on $\mathbb{T}^2$. This isomorphism is defined as

$$f(x) \mapsto \sum_{n \in \mathbb{Z}} f(\varphi + \theta n)e^{2\pi in\psi}.$$
The inverse mapping is
\[ g(\varphi, \psi) \mapsto \frac{1}{2\pi} \int_{S^1} g(\theta \{x/\theta\}, \psi) e^{-2\pi i [x/\theta] \psi} d\psi, \]
where \([a]\) and \(\{a\}\) are, respectively, the integer and fractional part of a real number \(a\).

Applying isomorphism (42), we obtain the following correspondence between operators on the real line and on the torus

| operators on the real line | operators on the torus |
|---------------------------|------------------------|
| \(-i \frac{d}{dx}\)       | \(-i \frac{\partial}{\partial \varphi}\) |
| \(x\)                     | \(-i \frac{\theta}{2\pi} \frac{\partial}{\partial \psi} + \psi\) |
| \(e^{-2\pi i x/\theta}\)  | \(e^{-2\pi i \varphi/\theta}\) |
| \(f(x) \to f(x+1)\)      | \(g(\varphi, \psi) \mapsto g(\varphi+1, \psi)\) |

It follows from this table that we obtain noncommutative operators on the torus, which can be studied using Theorems 5 and 10. This was done in [8] (including the computation of the index in terms of the symbol of operator). In the present paper, we restrict ourselves by considering one important example.

**Index of twisted Dirac operators.** Let \(P \in A_{1/\theta}^\infty\) be a projection. Consider the noncommutative operator
\[ D = P \left( x + \frac{d}{dx} \right) P : P(S(\mathbb{R})) \to P(S(\mathbb{R})). \] (43)

The corresponding operator on the torus is equal to
\[ P \left( -i \frac{\theta}{2\pi} \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \varphi} + \psi \right) P : P(C^\infty(\mathbb{T}^2, \gamma)) \to P(C^\infty(\mathbb{T}^2, \gamma)) \] (44)
and is the adjoint of the Dirac operator
\[ -i \frac{\theta}{2\pi} \frac{\partial}{\partial \psi} - \frac{\partial}{\partial \varphi}, \]
twisted by projection \(P\). Therefore, operator (44) is Fredholm for any projection \(P\).

Let us note that the operator (44) acts in sections of a nontrivial bundle \(\gamma\). Thus, to compute the index, we shall apply index formula (33).

To this end, we choose the following connection in \(\gamma\)
\[ \nabla_\gamma = d + \frac{2\pi i}{\theta} \varphi d\psi : C^\infty(\mathbb{T}^2, \gamma) \to C^\infty(\mathbb{T}^2, \gamma \otimes \Omega^1(\mathbb{T}^2)). \]

Then we define the corresponding noncommutative connection \(\nabla_P = P \nabla_\gamma P\) and its curvature form
\[ F_P = (\nabla_P)^2 = \left( PdP + \frac{2\pi i}{\theta} d\psi P \varphi P \right)^2. \]
Straightforward computation shows that the curvature form is the operator of multiplication by the noncommutative 2-form

\[ F_P = PdPdP + \frac{2\pi i}{\theta} d\psi (P - P([\varphi, P], \frac{dP}{d\varphi})]. \]

Since the coefficients of \( P \) do not depend on \( \psi \), we have \( dP = d\varphi \frac{dP}{d\varphi} \) and thus

\[ F_P = \frac{2\pi i}{\theta} d\varphi d\psi \left( P - P \left[ [\varphi, P], \frac{dP}{d\varphi} \right] \right). \]

The index formula (53) in the special case of \( \mathbb{T}^2 \) gives us

\[ \text{ind } D = - \int_{\mathbb{T}^2} A(T^*\mathbb{T}^2) \text{ch}_e(P). \]

(There is minus sign, because we deal with the adjoint of the Dirac operator on \( \mathbb{T}^2 \).) The cotangent bundle of the torus is trivial, thus \( A(T^*\mathbb{T}^2) = 1 \) and, therefore, the index is determined by the degree two component of the Chern character:

\[ \text{ind } D = \frac{1}{2\pi i} \int_{\mathbb{T}^2} \tau_e(F_P). \]

Substituting the curvature form in this equation, we obtain

\[ \text{ind } D = \frac{1}{2\pi i} \int_{\mathbb{T}^2} \tau_e(F_P) = \frac{1}{2\pi i} \int_{\mathbb{T}^2} F_P(0) = \]

\[ = \frac{1}{2\pi i} \int_{0 \leq \varphi \leq \theta, 0 \leq \psi \leq 1} F_P(0) = \frac{1}{\theta} \int_{0 \leq \varphi \leq \theta} \left( P - P \left[ [\varphi, P], \frac{dP}{d\varphi} \right] \right)(0) d\varphi. \] (45)

Here the second equality follows from the fact the the integral over \( \mathbb{T}^2 \) already contains averaging over \( \varphi \), therefore, additional averaging, which is contained in \( \tau_e \), is superfluous, and can be omitted.

Let now \( P \) be the Rieffel projection [21]:

\[ P = U^{-1}g + f + gU, \]

where \( f, g \) are smooth functions with period \( \theta \), which are defined as follows. Let us choose \( \varepsilon > 0 \) small enough, so that the intervals \([0, \varepsilon]\) and \([1, 1 + \varepsilon]\) of the circle \( \mathbb{R}/\theta\mathbb{Z} \) are disjoint. Then we set

\[
\begin{align*}
  f(x) &= \begin{cases} 
  1, & \text{if } x \in [\varepsilon, 1], \\
  0, & \text{if } x > 1 + \varepsilon, \\
  1 - f(1 + x), & \text{if } (0, \varepsilon),
  \end{cases} \\
  g(x) &= \begin{cases} 
  \sqrt{f(x) - f^2(x)}, & \text{if } x \in [0, \varepsilon], \\
  0, & \text{if } x \notin [0, \varepsilon].
  \end{cases}
\end{align*}
\]
Since \([\varphi, U] = -U\), \([\varphi, U^{-1}] = U^{-1}\), we get
\[
[\varphi, P] = U^{-1}g - gU.
\]
for the Rieffel projection. Let us substitute this expression in \(P[[\varphi, P], P']\), multiply the forms and retain only terms containing only zero power of \(U\). We get
\[
-U^{-1}g^2Uf' + U^{-1}gf'gU + 2fU^{-1}gg'U - 2fg'g + g^2f' - gUf'U^{-1}g.
\]
Since we have to apply trace to this expression, we can multiply the first three summands in the formula by \(U\) on the left and by \(U^{-1}\) on the right. The resulting element is equal to
\[
-g^2Uf'U^{-1} + gf'g + 2UfU^{-1}gg' - 2fg'g + g^2f' - gUf'U^{-1}g.
\]
Because \(UfU^{-1}\) is equal to \(1 - f\) on the support of \(g\), the last expression is equal to
\[
f'g^2 + fg'g(1 - 2f) + f'g^2 + g^2f' = 4g^2f' + (1 - 2f)(g^2)'
\]
\[
= f'(4f - 4f^2 + (1 - 2f)^2) = f'.
\]
Let us substitute this formula in (45) and obtain finally
\[
\text{ind } D = \frac{1}{\theta} \left( \int_0^\theta f(\varphi)d\varphi - f(\varphi)|_0^\theta \right) = \left\{ \frac{1}{\theta} \right\} - \frac{1}{\theta} = -\left[ \frac{1}{\theta} \right],
\]
where \(\{\cdot\}\) and \([\cdot]\) stand for the fractional and integer part of a real number.

It follows from (46) that for \(\theta\) sufficiently small operator (43) has large cokernel.

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