Black hole thermodynamics and generalized uncertainty principle with higher order terms in momentum uncertainty

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Abstract

In this paper we study the modification of thermodynamic properties of Schwarzschild and Reissner-Nordström black hole in the framework of generalized uncertainty principle with correction terms up to fourth order in momentum uncertainty. The mass-temperature relation and the heat capacity for these black holes have been investigated. These have been used to obtain the critical and remnant masses. The entropy expression using this generalized uncertainty principle reveals the area law up to leading order logarithmic corrections and subleading corrections of the form $\frac{1}{A^n}$. The mass output and radiation rate using Stefan-Boltzmann law have been computed which show deviations from the standard case and the case with the simplest form for the generalized uncertainty principle.

1 Introduction

The consistent unification of quantum mechanics (QM) with general relativity (GR) is one of the major tasks in theoretical physics. GR deals with the definition of world-lines of particles, which is in contradiction with QM since it does not allow the notion of trajectory due to the presence of an uncertainty in the determination of the momentum and position of a quantum particle. It has been the aim to unify these two theories into one theory known as quantum gravity. It is quite interesting that all approaches towards quantum gravity such as black hole physics [1]-[3], string theory [4, 5] or even Gedanken experiment [6] predict the existence of a minimum measurable length. The occurrence of such a minimal length also arises in various theories of quantum gravity phenomenology, namely, the generalized uncertainty principle (GUP) [7]-[9], modified dispersion relation (MDR) [10]-[13], deformed special relativity (DSR) [14], to name a few. It is now widely accepted that Heisenberg uncertainty principle would involve corrections from gravity at energies close to the Planck scale. Thus emergence of a minimal length seems to be inevitable when gravitational effects are taken into account. There has been a lot of work incorporating the existence of a minimal length scale in condensed matter and atomic physics experiments such as Lamb Shift, Landau levels and the scanning Tunneling Microscope [15]-[20], loop quantum gravity [21]-[23], noncommutative geometry [24], computing

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Planck scale corrections to the phenomena of superconductivity and quantum Hall effect \[25\] and understanding its consequences in cosmology \[26, 27\].

The incorporation of the GUP to study black hole thermodynamics has been another interesting area of active research \[28]-[39\]. It has been observed that the GUP reveals a self-complete characteristic of gravity which basically amounts to hiding any curvature singularity behind an event horizon as a consequence of matter compression at the Planck scale \[40]-[42\]. Further, the effects of the GUP have also been considered in the tunneling formalism for Hawking radiation to evaluate the quantum-corrected Hawking temperature and entropy of a Schwarzschild black hole \[43]-[47\]. In our earlier findings \[33]-[37\], we have studied the modification of thermodynamic properties, namely the temperature, heat capacity and entropy of black holes due to the simplest form of the GUP. Interestingly the correction to the Schwarzschild black hole temperature due to quadratic and linear-quadratic GUP has also been compared with the corrections from the quantum Raychaudhuri equation \[48\]. Very recently the Lorentz-invariance-violating class of dispersion relations have been applied to study the thermodynamics of black holes \[49\]. It would therefore be interesting to compare these results with those coming from the GUP.

The above studies motivate us to investigate the modification of thermodynamic properties for Schwarzschild and Reissner-Nordström (RN) black holes using the form of the GUP proposed in \[31\]. This GUP involves higher order terms in the momentum uncertainty. We compute the remnant and critical masses analytically for these black holes below which the temperature becomes ill-defined. We then use the Stefan-Boltzmann law to estimate the mass and the energy output as a function of time. We finally compute the entropy and obtain the well known area theorem containing corrections from the GUP with higher order terms in momentum uncertainty.

The paper is organized as follows. In section 2, we study the thermodynamics of Schwarzschild black hole taking into account the effect of the GUP, with higher order terms in momentum uncertainty. In sub-section 2.1, we also obtain the mass and radiation rate characteristics for the Schwarzschild black hole as a function of time by using the Stefan-Boltzmann law. In section 3, we study the thermodynamics of Reissner-Nordström black holes taking into account the effect of the GUP. Finally, we conclude in section 4.

## 2 Thermodynamics of Schwarzschild black hole

In this paper we work with the following form of the GUP \[50\]

\[
\Delta x \Delta p \geq \frac{\hbar}{2} \sum_{i=0}^{\infty} a_i \left( \frac{l_p \Delta p}{\hbar} \right)^{2i}; \quad [a_0 = 1; a_i > 0, i = 1, 2, ...]
\]

where \( l_p \) is the Planck length \((\sim 10^{-35}m)\). Keeping terms upto fourth order in momentum uncertainty, we have

\[
\Delta x \Delta p \geq \frac{\hbar}{2} \left\{ a_0 + a_1 \left( \frac{l_p \Delta p}{\hbar} \right)^2 + a_2 \left( \frac{l_p \Delta p}{\hbar} \right)^4 \right\}.
\]

We now consider a Schwarzschild black hole of mass \( M \). In the vicinity of the event horizon of the black hole, let a pair (particle-antiparticle) production occur. For simplicity we consider the particle to be massless. The particle with negative energy falls inside the horizon and that with positive energy escapes outside the horizon and gets observed by some observer at infinity. The momentum of the emitted particle \( (p) \), which also characterizes the temperature \( (T) \), is of the order of its uncertainty in momentum \( \Delta p \). Consequently

\[
T = \frac{(\Delta p)c}{k_B}
\]
where $c$ is the speed of light and $k_B$ is the Boltzmann constant.

The Hawking temperature of the black hole will be equal to the temperature of the particle when thermodynamic equilibrium is reached. The uncertainty in the position of a particle near the event horizon of the Schwarzschild black hole will be of the order of the Schwarzschild radius of the black hole

$$\Delta x = \epsilon r_s ; \quad r_s = \frac{2GM}{c^2} \quad (4)$$

where $\epsilon$ is a calibration factor, $r_s$ is the Schwarzschild radius and $G$ is the Newton’s universal gravitational constant.

To relate the Hawking temperature of the black hole with the mass of the black hole, we consider the saturated form of the GUP (2)

$$\Delta x \Delta p = \frac{\hbar}{2} \left\{ a_0 + a_1 \left( \frac{l_p \Delta p}{\hbar} \right)^2 + a_2 \left( \frac{l_p \Delta p}{\hbar} \right)^4 \right\} . \quad (5)$$

Substituting eq.(s) (3) and (4) in eq. (5) gives

$$M = \frac{M_p^2 c^2}{4\epsilon k_B T} \left\{ 1 + a_1 \frac{k_B^2 T^2}{M_p^2 c^4} + a_2 \frac{k_B^4 T^4}{M_p^4 c^8} \right\} \quad (6)$$

where the relations $\frac{c \hbar}{l_p} = M_p c^2$ and $M_p = \frac{c^2 l_p}{G}$, ($M_p$ being the Planck mass) have been used. In the absence of corrections due to GUP, eq.(6) reduces to

$$M = \frac{M_p^2 c^2}{4\epsilon k_B T} . \quad (7)$$

The value of $\epsilon$ now gets fixed to $2\pi$ by comparing this expression with the semi-classical Hawking temperature $T = \frac{M_p^2 c^2}{8\pi M k_B}$ \text{[51, 52]}. The mass-temperature relation \text{[6]} finally takes the form

$$M = \frac{M_p^2 c^2}{8\pi} \left\{ \frac{1}{k_B T} + a_1 \frac{k_B T}{M_p^2 c^4} + a_2 \frac{k_B^3 T^3}{M_p^4 c^8} \right\} . \quad (8)$$

The heat capacity of the black hole therefore reads

$$C = \frac{c^2 dM}{dT} \quad (9)$$

$$= \frac{M_p^2 c^4}{8\pi} \left\{ -\frac{1}{k_B T^2} + a_1 \frac{k_B}{M_p^2 c^4} + 3a_2 \frac{k_B^3 T^2}{M_p^4 c^8} \right\}$$

$$= \frac{k_B}{8\pi} \left\{ \frac{M_p c^2}{k_B T} \right\}^2 + a_1 + 3a_2 \left( \frac{k_B T}{M_p c^2} \right)^2 \} . \quad (10)$$

The above expression takes the form

$$C = \frac{k_B}{8\pi} \left\{ -\frac{1}{T'^2} + a_1 + 3a_2 T'^2 \right\} \quad (11)$$

after the following notations are introduced

$$M' = \frac{8\pi M}{M_p} ; \quad T' = \frac{k_B T}{M_p c^2} . \quad (12)$$
The mass of the black hole decreases due to radiation from the black hole. This leads to an increase in the temperature of the black hole. It can be observed from eqs. (7) and (10) that there exists a finite temperature at which the heat capacity vanishes. To find out this temperature, we set \( C = 0 \). This gives

\[
3a_2T^4 + a_1T^2 - 1 = 0.
\]

Solving this, we get

\[
T^2 = \frac{1}{6a_2} \left\{ -a_1 + \sqrt{a_1^2 + 12a_2} \right\}
\]

where the positive sign before the square root has been taken so that the above result reduces to corresponding result when \( a_2 = 0 \) [34]. Finally we get the expression for \( T' \) to be

\[
T' = \frac{1}{\sqrt{6a_2}} \left\{ -a_1 + \sqrt{a_1^2 + 12a_2} \right\}^{\frac{3}{2}}.
\]

Now in terms of \( T' \), \( M' \) the mass-temperature relation (8) can be represented as

\[
a_2T'^4 + a_1T'^2 - M'T' + 1 = 0.
\]

The remnant mass can now be obtained by substituting eq.(15) in eq.(16). This yields

\[
M'_{\text{rem}} = \frac{1}{T'} \left\{ a_2T'^4 + a_1T'^2 + 1 \right\}
\]

\[
\Rightarrow M_{\text{rem}} = \frac{M_p}{8\pi} \frac{\sqrt{6a_2}}{9a_2 \left\{ -a_1 + \sqrt{a_1^2 + 12a_2} \right\}^{\frac{3}{2}}} \left[ -a_1^2 + a_1 \sqrt{a_1^2 + 12a_2} + 12a_2 \right].
\]

Reassuringly the above result reduces to the result in \( a_2 \to 0 \) limit [34]

\[
M_{\text{rem}} = \frac{M_p\sqrt{a_1}}{4\pi}.
\]

Now for \( a_1 \to 0 \), \( a_2 \neq 0 \), the remnant mass is given by

\[
M_{\text{rem}} = \frac{M_p}{6\pi} (3a_2)^{\frac{3}{2}}.
\]

Also for \( a_1 \to 0 \), the mass-temperature relation (16) reads

\[
T'^4 - \frac{M'}{a_2}T' + \frac{1}{a_2} = 0.
\]

The solution of this bi-quadratic equation in \( T' \) yields

\[
T' = \left[ \frac{M'^2}{16a_2^2} + \sqrt{\frac{M'^4}{256a_2^4} - \frac{1}{27a_2^2}} \right]^{\frac{3}{2}} + \left[ \frac{M'^2}{16a_2^2} - \sqrt{\frac{M'^4}{256a_2^4} - \frac{1}{27a_2^2}} \right]^{\frac{3}{2}}.
\]

The above relation readily implies the existence of a critical mass below which the temperature will be a complex quantity

\[
M_{\text{cr}} = \frac{M_p (3a_2)^{\frac{3}{2}}}{6\pi}.
\]
This demonstrates that the remnant and critical masses are equal.

At this point, we would like to make a comment. It can be observed from the above analysis that analytical expressions for the remnant and critical masses can be obtained even if one retains terms of order of \((\Delta p)^8\) in the momentum uncertainty. This is because it leads to an equation of the form \(aT^8 + bT^6 + cT^4 + dT^2 + e = 0\) when the condition \(C = 0\) is imposed. This equation can be solved analytically to obtain the remnant mass. If we keep terms beyond this order in momentum uncertainty, analytical expressions for the remnant and critical masses can not be obtained.

The black hole entropy from the first law of black hole thermodynamics is given by

\[
S = \int c^2 \frac{dM}{T} = \int C \frac{dT}{T} = \int k_B \left[ \frac{\left( M_p c^2 \right)^2}{k_B T} + a_1 + 3a_2 \left( \frac{k_B T}{M_p c^2} \right)^2 \right] \frac{dT}{T} = \frac{k_B}{8\pi} \left[ \frac{1}{2T^2} + a_1 \log T' + \frac{3}{2} a_2 T'^2 \right].
\]

Eq.(16) yields up to \(O(a_1^2, a_2^2, a_1 a_2)\) to obtain an expression for the temperature \(T\) in terms of mass \(M\).

\[
T' = \frac{1}{M'} + \frac{a_1}{M'^3} + \frac{a_2}{M'^5} + \frac{2a_1^2}{M'^7} + \frac{6a_1 a_2}{M'^9} + \frac{4a_2^2}{M'^{11}}.
\]

Now the entropy expression in terms of the mass \(M\) of the black hole, we need to consider eq.(16) to obtain an expression for the temperature \(T\) in terms of mass \(M\). Eq.(16) yields up to \(O(a_1^2, a_2^2, a_1 a_2)\)

\[
\frac{S}{k_B} = \frac{1}{8\pi} \left[ \frac{1}{2} M^2 - a_1 - a_1 \log M' + \frac{a_1^2}{2M'^2} + \frac{a_2}{2M'^4} + \frac{a_1 a_2}{M'^6} + \frac{a_2^2}{2M'^8} \right] + O(a_1^2 a_2, a_1 a_2^2, a_1^3, a_2^3)
\]

\[
= \frac{S_{BH}}{k_B} - \frac{a_1}{16\pi} \log \left( \frac{S_{BH}}{k_B} \right) - \frac{a_1}{16\pi} \log(16\pi) - \frac{a_1^2}{8\pi} + \frac{a_1 a_2}{(16\pi)^2} \left( \frac{S_{BH}}{k_B} \right)^{-1}
\]

\[
+ \frac{2a_1 a_2}{(16\pi)^3} \left( \frac{S_{BH}}{k_B} \right)^{-2} + \frac{a_2}{(16\pi)^4} \left( \frac{S_{BH}}{k_B} \right)^{-3} + O(a_1^2 a_2, a_1 a_2^2, a_1^3, a_2^3)
\]

where \(\frac{S_{BH}}{k_B} = \frac{4\pi G^2 M^2}{M_p^2}\) is the semi-classical Bekenstein-Hawking entropy for the Schwarzschild black hole. In terms of the black hole horizon area \(A = 4\pi r_s^2 = 16\pi G^2 M^2 / c^3 = 4l_p^2 S_{BH} / k_B\), the above entropy expression can be written as

\[
\frac{S}{k_B} = \frac{A}{4l_p^2} - \frac{a_1}{16\pi} \log \left( \frac{A}{4l_p^2} \right) - \frac{a_1}{16\pi} \log(16\pi) - \frac{a_1}{8\pi} + \frac{a_1^2}{(16\pi)^2} \left( \frac{A}{4l_p^2} \right)^{-1}
\]

\[
+ \frac{2a_1 a_2}{(16\pi)^3} \left( \frac{A}{4l_p^2} \right)^{-2} + \frac{a_2}{(16\pi)^4} \left( \frac{A}{4l_p^2} \right)^{-3} + O(a_1^2 a_2, a_1 a_2^2, a_1^3, a_2^3).
\]

This completes our discussion of the effect of the GUP on the thermodynamic properties of the Schwarzschild black hole. In Figure 1, we present the plot of the entropy of the black hole vs the horizon area for the GUP case and compare it with the standard case.
2.1 Energy output as a function of time

Due to radiation of the black hole, the mass of the black hole reduces while its temperature keeps on increasing. If one assumes that the energy loss is dominated by photons, then one can apply the Stefan-Boltzmann law to estimate the energy radiated as a function of time

\[
\frac{dM}{dt} = -\sigma A T_H^4
\]  

(27)

where \( \sigma \) is the Stefan-Boltzmann constant. In terms of Schwarzschild black hole mass \( M \) with the horizon area \( A = 4\pi r_s^2 = \frac{16\pi G^2 M^2}{c^4} \), above equation implies

\[
\frac{d}{dt} \left(\frac{8\pi M}{M_p}\right) = -\sigma \frac{2G^2 c^4 M_p^5}{k_B^4} \left(\frac{8\pi M}{M_p}\right)^2 T'^4
\]  

(28)

where we have used \( T' = \frac{k_B T_H}{M_p c^2} \).

We now write the above equation taking into account the effect of the GUP. Thus considering the mass-temperature relation (24), the radiation rate takes the following form

\[
\frac{dx}{dt} = -\frac{1}{t_{ch} x^2} \left[ \frac{1}{x} + \frac{a_1}{x^3} + \frac{a_2}{x^5} + \frac{2a_1^2}{x^3} + \frac{6a_1 a_2}{x^7} + \frac{4a_2^2}{x^9} \right]^4
\]  

(29)

where we have set \( x = \frac{8\pi M}{M_p} \) and the characteristic time \( t_{ch} \) is being defined as \( t_{ch} = \frac{k_B^4}{2\pi M_p c^4 G^2} \). If \( x_i \) refers to the initial mass at time \( t = 0 \), the solution of the above equation yields the mass-time relation. Upto \( \mathcal{O}(a_1, a_2) \), we have

\[
x = \left[ -\frac{3t}{t_{ch}} x_i^3 + 12a_1 x_i + \frac{12a_2}{x_i} + 12a_1 \left( x_i^3 - \frac{3t}{t_{ch}} \right)^\frac{1}{3} - 12a_2 \left( x_i^3 - \frac{3t}{t_{ch}} \right)^{-\frac{1}{3}} \right]^\frac{3}{5}
\]  

(30)

where

\[
\frac{t}{t_{ch}} = \frac{x_i^3}{3} - 4a_1 x_i + \frac{4a_2}{x_i}
\]  

(31)
In Figures 2 and 3 we have plotted the mass of the black hole as a function of time and the radiation rate as a function of time.

Figure 2: The mass of the black hole vs time. The mass is in units of Planck mass and the time is in units of characteristic time. Here thin line (lower curve) represents GUP case (considering both $a_2$ and $a_1$), the dashed line (middle curve) represents GUP case (considering only $a_1$) and thick line (upper curve) represents the standard case.

Figure 3: The Radiation Rate of the black hole vs time. The rate is in units of Planck mass per characteristic time and the time is in units of characteristic time. Here thin line (upper curve) represents GUP case (considering both $a_2$ and $a_1$), the dashed line (middle curve) represents GUP case (considering only $a_1$) and thick line (lower curve) represents the standard case.

3 Thermodynamics of Reissner-Nordstrøm black hole

In this section, we consider the Reissner-Nordstrøm (RN) black hole of mass $M$ and charge $Q$.

In this case, near the horizon of the black hole, the position uncertainty of a particle will be of the order of the RN radius of the black hole

$$\Delta x = \epsilon r_h$$

$$r_h = \frac{Gr_0}{c^2}$$

$$r_0 = M + \sqrt{M^2 - Q^2}$$

(32)

where $r_h$ is the radius of the horizon of the RN black hole. Substituting the value of $\Delta p$ and $\Delta x$ from eq.(3) and eq.(32), the GUP (5) can be rewritten as

$$r_0 = \frac{\hbar c^3}{2\epsilon Gk_B T} \left[ 1 + a_1 \left( \frac{l_p \Delta p}{\hbar} \right)^2 + a_2 \left( \frac{l_p \Delta p}{\hbar} \right)^4 \right].$$

(33)

Once again, in the absence of correction due to GUP, eq.(33) reduces to

$$r_0 = \frac{M^2 c^2}{2\epsilon k_B T}.$$  

(34)

Comparing the above relation with the semi-classical Hawking temperature $T = \frac{M^2 c^2 (Mr_0 - Q^2)}{2\pi k_B r_0^3}$, yields the value of $\epsilon$ to be

$$\epsilon = \frac{\pi r_0}{(Mr_0 - Q^2)}.$$  

(35)
This finally fixes the form of the mass-charge-temperature relation (33) to be
\[ \frac{r_0^2}{(r_0 - M)} = \frac{M_p}{2\pi} \left[ \frac{M_p c^2}{k_B T} + a_1 \frac{k_B T}{M_p c^2} + a_2 \left( \frac{k_B T}{M_p c^2} \right)^3 \right] \quad (36) \]
where the identity
\[ \frac{r_0}{(Mr_0 - Q^2)} = \frac{1}{(r_0 - M)} \quad (37) \]
has been used.

The heat capacity of the black hole can now be calculated using relation (9) and eq.(36):
\[ C = \frac{k_B(r_0 - M)^3}{2\pi r_0^2(2r_0 - 3M)} \left[ - \left( \frac{M_p c^2}{k_B T} \right)^2 + a_1 + 3a_2 \left( \frac{k_B T}{M_p c^2} \right)^2 \right]. \quad (38) \]

To express the heat capacity in terms of the mass, once again we make use of the relation (12) to recast eq.(36) in the form
\[ a_2 T'^4 + a_1 T'^2 - \frac{g(r_0)}{M_p} T' + 1 = 0 \quad (39) \]
where
\[ g(r_0) = \frac{2\pi r_0^2}{(r_0 - M)}. \]

Now to find out the temperature where the radiation process stops, we set C = 0. Eq.(38) therefore yields
\[ 3a_2 T'^4 + a_1 T'^2 - 1 = 0 \quad (40) \]
from which solution of \( T' \) reads
\[ T' = \frac{1}{\sqrt{6a_2}} \left\{ -a_1 + \sqrt{a_1^2 + 12a_2} \right\}^{\frac{1}{2}} \quad (41) \]
where the positive sign before the square root has been taken to reproduce the result corresponding to the limit \( a_2 \to 0 \) [34].

The remnant mass can now be computed by substituting eq.(41) in eq.(39). This would then give
\[ \frac{g(r_0)}{M_p} = \frac{\sqrt{6a_2}}{9a_2 \left\{ -a_1 + \sqrt{a_1^2 + 12a_2} \right\}^{\frac{1}{2}}} \left[ -a_1^2 + a_1 \sqrt{a_1^2 + 12a_2} + 12a_2 + 12a_2 \right]. \quad (42) \]

Thus we finally obtain the following cubic equation for the remnant mass
\[ \left( \frac{2M_p Z}{\pi} \right) M_{rem}^3 - \left( \frac{M_p Z}{2\pi} \right)^2 M_{rem}^2 - \left( \frac{2M_p Z}{\pi} \right) Q^2 M_{rem} + Q^4 + \left( \frac{M_p Z}{2\pi} \right)^2 Q^2 = 0 \quad (43) \]
where
\[ Z = \frac{\sqrt{6a_2}}{9a_2 \left\{ -a_1 + \sqrt{a_1^2 + 12a_2} \right\}^{\frac{1}{2}}} \left[ -a_1^2 + a_1 \sqrt{a_1^2 + 12a_2} + 12a_2 + 12a_2 \right]. \quad (44) \]
Solution of eq. (43) gives us the exact expression of remnant mass for the RN black hole. This yields

$$M_{\text{rem}} = \frac{M_p Z}{24\pi} \left[ 1 + \left( \frac{M_p^2 Z^2}{4\pi^2} + 48Q^2 \right)^{1/3} + \frac{4\pi^2}{M_p^2 Z^2} \left( \frac{B}{2} \right)^{1/3} \right]$$

(45)

where

$$B = \frac{M_p^6 Z^6}{32\pi^6} - 18 \frac{M_p^4 Z^4}{\pi^4} Q^2 - 108 \frac{M_p^2 Z^2}{\pi^2} Q^4$$

$$+ \sqrt{\frac{27}{16}} \frac{M_p^{10} Z^{10}}{\pi^{10}} Q^2 - \frac{350}{27} \frac{M_p^8 Z^8}{\pi^8} Q^4 - 4968 \frac{M_p^6 Z^6}{\pi^6} Q^6 + 11664 \frac{M_p^4 Z^4}{\pi^4} Q^8.$$  

(46)

The above expression for the remnant mass reduces to the remnant mass for the Schwarzschild black hole in the $Q \to 0$ limit.

Finally, we proceed to compute the entropy of the RN black hole. To do that, we first obtain an expression of $T'(T)$ from eq. (39) in terms of the mass and the charge of the RN black hole.

This gives us $O(a_1^2, a_1a_2, a_2^2)$

$$T' = \frac{M_p}{g(r_0)} \left[ 1 + a_1 \left( \frac{M_p}{g(r_0)} \right)^2 + a_2 \left( \frac{M_p}{g(r_0)} \right)^4 + 2a_1^2 \left( \frac{M_p}{g(r_0)} \right)^4 + 6a_1a_2 \left( \frac{M_p}{g(r_0)} \right)^6 + 4a_2^2 \left( \frac{M_p}{g(r_0)} \right)^8 \right].$$

(47)

From this one now can calculate the entropy for the RN black hole using eq. (38) and eq. (47):

$$\frac{S}{k_B} = \frac{\pi r_h^2}{l_p^2} - \frac{a_1}{16\pi} \ln \left( \frac{\pi r_h^2}{l_p^2} \right) - \left( \frac{a_1 Q^2}{2^3 M_p^2 - 2^2 \pi^2} \right) \left( \frac{\pi r_h^2}{l_p^2} \right)^{-1} - \left( \frac{a_2}{2^3 \pi M_p^2 - 2^1 M_p^6} \right) \left( \frac{\pi r_h^2}{l_p^2} \right)^{-2}$$

$$- \left( \frac{a_2 Q^4}{2^7 M_p^4} \right) \left( \frac{\pi r_h^2}{l_p^2} \right)^{-3} - \left( \frac{a_2 \pi Q^6}{2^8 M_p^6} \right) \left( \frac{\pi r_h^2}{l_p^2} \right)^{-4} - \left( \frac{1}{5} \frac{a_2 \pi^2 Q^8}{2^8 M_p^8} \right) \left( \frac{\pi r_h^2}{l_p^2} \right)^{-5} + O(a_1^2, a_1a_2, a_2^2)$$

$$= \frac{S_{BH}}{k_B} - \frac{a_1}{16\pi} \ln \left( \frac{S_{BH}}{k_B} \right) - \left( \frac{a_1 Q^2}{2^3 M_p^2 - 2^2 \pi^2} \right) \left( \frac{S_{BH}}{k_B} \right)^{-1} - \left( \frac{a_2}{2^3 \pi M_p^2 - 2^1 M_p^6} \right) \left( \frac{S_{BH}}{k_B} \right)^{-2}$$

$$- \left( \frac{a_2 Q^4}{2^7 M_p^4} \right) \left( \frac{S_{BH}}{k_B} \right)^{-3} - \left( \frac{a_2 \pi Q^6}{2^8 M_p^6} \right) \left( \frac{S_{BH}}{k_B} \right)^{-4} - \left( \frac{1}{5} \frac{a_2 \pi^2 Q^8}{2^8 M_p^8} \right) \left( \frac{S_{BH}}{k_B} \right)^{-5} + O(a_1^2, a_1a_2, a_2^2)$$

(48)

where $\frac{S_{BH}}{k_B} = \frac{\pi r_h^2}{l_p^2}$ is the semi-classical Bekenstein-Hawking entropy for the RN black hole. In terms of the area of the horizon $A = 4\pi r_h^2 = 4l_p^2 \frac{S_{BH}}{k_B}$, the above equation can be written as

$$\frac{S}{k_B} = \frac{A}{4l_p^2} - \frac{a_1}{16\pi} \ln \left( \frac{A}{4l_p^2} \right) - \left( \frac{a_1 Q^2}{2^3 M_p^2 - 2^2 \pi^2} \right) \left( \frac{A}{4l_p^2} \right)^{-1} - \left( \frac{a_2}{2^3 \pi M_p^2 - 2^1 M_p^6} \right) \left( \frac{A}{4l_p^2} \right)^{-2}$$

$$- \left( \frac{a_2 Q^4}{2^7 M_p^4} \right) \left( \frac{A}{4l_p^2} \right)^{-3} - \left( \frac{a_2 \pi Q^6}{2^8 M_p^6} \right) \left( \frac{A}{4l_p^2} \right)^{-4} - \left( \frac{1}{5} \frac{a_2 \pi^2 Q^8}{2^8 M_p^8} \right) \left( \frac{A}{4l_p^2} \right)^{-5} + O(a_1^2, a_1a_2, a_2^2)$$

(49)

which is the area theorem for the RN black hole with corrections from the GUP containing higher order terms in the momentum uncertainty.

We would like to conclude this section by mentioning that in [53], it has been pointed out that there is a part of the information (leaking out of the black hole due to Hawking radiation) related to non-thermal GUP correlations. This insight may be important to provide a solution for the well-known information loss paradox and is worth investigating in future.
4 Conclusions

In this paper, we have investigated the modifications of the various thermodynamic properties of Schwarzschild and Reissner-Nordström black holes using higher order momentum uncertainty terms in the GUP. We obtain the GUP modified mass-temperature relation. This then leads to the existence of a remnant mass thereby preventing the complete evaporation of the black hole. The expression for the remnant and critical masses have been obtained analytically. In this regard, we observe that analytical expressions for these masses can be obtained even if we keep terms of the order of $(\Delta p)^8$ in the momentum uncertainty. Beyond this it is no longer possible to obtain analytical expression for the critical and remnant masses. We also compute the mass and energy outputs as functions of time using the Stefan-Boltzman law. We observe that these expressions get modified from the standard case as well as the case where the simplest form of the GUP is used. The expression for the entropy exhibits the well known area theorem in terms of the horizon area in both cases upto leading order corrections from the GUP.

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