Abstract

We give a purely scheme theoretic construction of the filtration by ramification groups of the Galois group of a covering. The valuation need not be discrete but the normalizations are required to be locally of complete intersection.

For a Galois extension of a complete discrete valuation field with not necessarily perfect residue field, the filtration by ramification groups on the Galois group is defined in a joint article [1] with Ahmed Abbes. Although the definition in [1] is based on rigid geometry, it is later observed that the use of rigid geometry can be avoided and the conventional language of schemes suffices ([2]). In this article, we reformulate the construction in [1] in the language of schemes. As a byproduct, we give a generalization for ramified finite Galois coverings of normal noetherian schemes and valuations not necessarily discrete.

All the ideas are present in [1], possibly in different formulation. As in [1], the main ingredients in the definition of ramification groups are the followings: First, we interpret a subgroup as a quotient of the fiber functor with a cocartesian property, Proposition [1.4.2]. Thus, the definition of ramification groups is a consequence of a construction of quotients of the fiber functor, indexed by elements of the rational value group of valuation.

The required quotients of the fiber functor are constructed as the sets of connected components of geometric fibers of dilatations defined by an immersion of the covering to a smooth scheme over the base scheme. Here a crucial ingredient is the reduced fiber theorem of Bosch-L"utkebohmert-Raynaud [2] recalled in Theorem [1.2.5]. This specializes to the finiteness theorem of Grauert-Remmert in the classical case where the base is a discrete valuation ring. A variant of the filtration is defined using the underlying sets of geometric fibers of quasi-finite schemes without using the sets of connected components.

To prove basic properties of ramification groups stated in Theorem [3.3.1] including the rationality of breaks, semi-continuity etc., a key ingredient is a generalization due to Temkin [11] of the semi-stable reduction theorem of curves recalled in Theorem [1.3.5].

Let $X$ be a normal noetherian scheme and $U \subset X$ be a dense open subscheme. The Zariski-Riemann space $\tilde{X}$ is defined as the inverse limit of proper schemes $X'_{\text{proper}}$ over $X$ such that $U' = U \times X X' \to U$ is an isomorphism. Points of $\tilde{X}$ on the boundary $\tilde{X} - U$ correspond bijectively to the inverse limits of the images of the closed points by the liftings of the morphisms $T = \text{Spec } A \to X$ for valuation rings $A \subseteq K = k(t)$ for points $t \in U$ such that $T \times X U$ consists of the single points $t$.

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Let $W \to U$ be a finite étale connected Galois covering of Galois group $G$. We will construct in Theorem 3.3.1 filtrations $(G_T^\gamma)$ and $(G_T^{\gamma+})$ on $G$ by ramification groups for a morphism $T \to X$ as above indexed by the positive part $(0, \infty)_{\Gamma_0} \subset \Gamma_0 = \Gamma \otimes \mathbb{Q}$ for the value group $\Gamma = K^*/A^*$. To complete the definition, we need to assume that for every intermediate covering $V \to U$, the normalization $Y$ of $X$ in $V$ is locally of complete intersection over $X$ to assure the cocartesian property in Proposition 1.4.2. The required cocartesian property Proposition 3.1.2 is then a consequence of a lifting property in commutative algebra recalled in Proposition 1.1.3.

The definition depends on $X$ not only on $W \to U$. In other words, for a normal noetherian scheme $X'$ over $X$ as above, the filtrations $(G_T^\gamma)$ and $(G_T^{\gamma+})$ defined in $X$ and those in $X'$ may be different. This arises from the fact that the formation of the normalization $Y$ need not commute with base change $X' \to X$. To obtain a definition depending only on $W \to U$, one would need to take inverse limit with respect to $X'$. This requires that the normalizations over $T$ be locally of complete intersection.

By Proposition 1.4.2, the definition of the filtrations $(G_T^\gamma)$ and $(G_T^{\gamma+})$ are reduced to the construction of surjections $F_T^\infty \to F_T^\gamma$ and $F_T^\infty \to F_T^{\gamma+}$ for a fiber functor $F_T^\infty$. To define them, for each intermediate covering $V \to U$, we take an embedding $Y \to Q$ of the normalization to a smooth scheme over $X$. Further taking a ramified covering and a blow-up $X'$, we find an effective Cartier divisor $R' \subset X'$ and a lifting $T' \to X'$ of $T \to X$ such that the valuation $v'(R')$ of $R'$ is $\gamma$ for each $\gamma \in \Gamma_0$. Then, we define a dilatation $Q^{(R')}$ over $X'$ to be the normalization of an open subscheme $Q^{(\gamma)}$ of the blow-up of the base change $Q' = Q \times_X X'$ at the closed subscheme $Y \times_X R' \subset Q \times_X X'$. To obtain a construction independent of the choice of $X'$, we apply the reduced fiber theorem of Bosch-Lütkebohmert-Raynaud for $Q^{(R')} \to X'$ to be flat and to have reduced geometric fibers.

Now the desired functor $F_T^\gamma(Y/X)$ is defined as the set of connected components of the geometric fiber of $Q^{(\gamma)} \to X'$ at the image of the closed point by $T' \to X'$. Example 2.1.11 and Remark 1.1.2 imply that we recover the construction in [1] in the classical case where $X = T$ is the spectrum of a complete discrete valuation ring. Its variant $F_T^{\gamma+}(Y/X)$ is defined more simply as the geometric fiber of the inverse image $Y' \times_{Q^{(\gamma)}} Q^{(R')}$ with respect to the morphism $Y' \times_X X' \to Q^{(R')}$ lifting the original immersion $Y \to Q$. The fact that the construction is independent of the choice of immersion $Y \to Q$ is based on a homotopy invariance of dilatations proved in Proposition 2.1.5.

To study the behavior of the functors $F_T^\gamma$ and $F_T^{\gamma+}$ thus defined for variable $\gamma$, we use a semi-stable curve $C$ over $X$ defined by $st = f$ for a non-zero divisor $f$ on $X$ defining an effective Cartier divisor $D \subset X$ such that $D \cap U = \emptyset$ as a parameter space for $\gamma$. Let $\bar{D} \subset C$ denote the effective Cartier divisor defined by $t$. Then, for $\gamma \in [0, v(D)]_{\Gamma_0}$, there is a lifting $T' \to C$ of $T \to X$ such that $v'(\bar{D}) = \gamma$. Using this together with a local description Proposition 1.3.3 of Cartier divisors on a semi-stable curve over a normal noetherian scheme and a combination of the reduced fiber theorem and the semi-stable reduction theorem over a general base scheme, we derive basic properties of $F_T^\gamma$ and $F_T^{\gamma+}$ in Proposition 3.1.8 to prove Theorem 3.2.6 and Theorem 3.3.1.

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Convention

In this article, we assume that for a noetherian scheme $X$, the normalization of a scheme of finite type over $X$ remains to be of finite type over $X$. This property is satisfied if $X$ is of finite type over a field, $\mathbb{Z}$ or a complete discrete valuation ring, for example.

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1 Preliminaries

1.1 Connected components

Definition 1.1.1 ([4, Définition (6.8.1)]). Let $f : X \to S$ be a flat morphism locally of finite presentation of schemes. We say that $f$ is reduced if for every geometric point $s$ of $S$, the geometric fiber $X_s$ is reduced.

In SGA 1 Exposé X Definition 1.1, reduced morphism is called separable morphism.

We study the sets of connected components of geometric fibers of a flat and reduced morphism of finite type. Let $S$ be a scheme and $s$ and $t$ be geometric points of $S$. Let $S_s$ denote the strict localization. A specialization $s \leftarrow t$ of geometric points means a morphism $S_s \leftarrow \eta$ over $S$.

Assume that $S$ is noetherian. Let $X \to S$ be a flat and reduced morphism of finite type and let $s \leftarrow t$ be a specialization of geometric points of $S$. We define the cospecialization mapping

\begin{equation}
\pi_0(X_s) \to \pi_0(X_t)
\end{equation}

as follows. By replacing $S$ by the closure of the image of $t$, we may assume that $S$ is integral and that $t$ is above the generic point $\eta$ of $S$. By replacing $S$ further by a quasi-finite scheme over $S$ such that the function field is a finite extension of $\kappa(\eta)$ in $\kappa(t)$, we may assume that the canonical mapping $\pi_0(X_t) \to \pi_0(X_\eta)$ is a bijection. Let $U \subset S$ be a dense open subset such that the canonical mapping $\pi_0(X_\eta) \to \pi_0(X_U)$ is a bijection. Then,
by [7 Corollaire (18.9.11)], the canonical mapping \( \pi_0(X_U) \to \pi_0(X) \) is also a bijection. Thus, we define the cospecialization mapping \( \Xi \) to be the composition

\[
\pi_0(X_s) \to \pi_0(X) \xrightarrow{\Xi} \pi_0(X_\eta) \xrightarrow{\Xi} \pi_0(X_t).
\]

We say that the sets of connected components of geometric fibers of \( X \to S \) are locally constant if for every specialization \( s \leftarrow t \) of geometric points of \( S \), the cospecialization mapping \( \pi_0(X_s) \to \pi_0(X_t) \) is a bijection. By [7 Théorème (9.7.7)] and by noetherian induction, there exists a finite stratification \( S = \coprod_i S_i \) by locally closed subschemes such that the sets of connected components of geometric fibers of the base change \( X \times_S S_i \to S_i \) are locally constant for every \( i \). We call this fact that the sets of connected components of geometric fibers of \( X \to S \) are constructible.

**Remark 1.1.2.** Let \( S = \text{Spec} \, \mathcal{O}_K \) for a discrete valuation ring \( \mathcal{O}_K \) and \( X = \text{Spec} \, A \) be an affine scheme of finite type over \( S \). Let \( \bar{s} \to S \) be the geometric closed point. Let \( X = \text{Spf} \, \hat{A} \) be the formal completion along the closed fiber and \( X_{\bar{K}} = \text{Spf} \, \hat{A} \otimes_{\mathcal{O}_K} \bar{K} \) be the associated affinoid variety over an algebraic closure \( \bar{K} \) of the fraction field \( K \) of \( \mathcal{O}_K \). If \( X \) is flat and reduced over \( S \), the cospecialization mapping \( \pi_0(X_{\bar{s}}) \to \pi_0(X_{\bar{K}}) \) is a bijection.

Let \( Y \to S \) be another flat and reduced morphism of finite type and let \( f: X \to Y \) be a morphism over \( S \). The cospecialization mappings \( \Xi \) form a commutative diagram

\[
\begin{array}{ccc}
\pi_0(X_s) & \to & \pi_0(X_t) \\
\downarrow & & \downarrow \\
\pi_0(Y_s) & \to & \pi_0(Y_t).
\end{array}
\]

(1.2)

**Lemma 1.1.3.** Let \( f: X \to Y \) be a morphism of schemes of finite type over a noetherian scheme \( S \). Assume that \( X \) is étale over \( S \) and that \( Y \) is flat and reduced over \( S \). Let \( A \) denote the subset of \( X \) consisting of the images of geometric points \( x \) of \( X \) satisfying the following conditions: Let \( s \) be the geometric point of \( S \) defined as the image of \( x \) and let \( C \subset Y_s \) be the connected component of the fiber containing the image of \( x \). Then, \( f^{-1}(C) \subset X_s \) consists of a single point \( x \).

Then \( A \) is closed.

**Proof.** By the constructibility of connected components of geometric fibers of \( Y \), the subset \( A \subset X \) is constructible. For a specialization \( s \leftarrow t \) of geometric points of \( S \), the upper horizontal arrow in the commutative diagram

\[
\begin{array}{ccc}
X_s & \to & X_t \\
\downarrow & & \downarrow \\
\pi_0(Y_s) & \to & \pi_0(Y_t)
\end{array}
\]

is an injection since \( X \to S \) is étale. Hence \( A \) is closed under specialization and is closed.

We have specialization mappings going the other way for proper morphisms. Let \( X \) be a proper scheme over \( S \). Let \( s \leftarrow t \) be a specialization of geometric points of \( S \). Then, the inclusion \( X_s \to X \times_S S_{(s)} \) induces a bijection \( \pi_0(X_s) \to \pi_0(X \times_S S_{(s)}) \) by [3 IV
Proposition (2.1)]. Its composition with the mapping \( \pi_0(X_t) \to \pi_0(X \times_S S(s)) \) induced by the morphism \( X_t \to X \times_S S(s) \) defines the specialization mapping
\[
(1.3) \quad \pi_0(X_s) \leftarrow \pi_0(X_t).
\]
For a morphism \( X \to Y \) of proper schemes over \( S \), the specialization mappings make a commutative diagram
\[
\begin{array}{ccc}
\pi_0(X_s) & \leftarrow & \pi_0(X_t) \\
\downarrow & & \downarrow \\
\pi_0(Y_s) & \leftarrow & \pi_0(Y_t).
\end{array}
\]

**Lemma 1.1.4.** Let \( f : X \to Y \) be a finite unramified morphism of schemes. Let \( B \) denote the subset of \( X \) consisting of the images of geometric points \( x \) of \( X \) satisfying the following conditions: For the geometric point \( y \) of \( Y \) defined as the image of \( x \), the fiber \( X \times_Y y \) consists of a single point \( x \).

Then, \( B \) is open.

**Proof.** The complement \( X \setminus B \) equals the image of the complement \( X \times_Y X \setminus X \) of the diagonal by a projection. Since \( X \to Y \) is unramified, the complement \( X \times_Y X \setminus X \subset X \times_Y X \) is closed. Since the projection \( X \times_Y X \to X \) is finite, the image \( X \setminus B \) is closed. \( \square \)

**Proposition 1.1.5.** Let
\[
\begin{array}{ccc}
Z' & \longrightarrow & X' \\
\downarrow & \square & \downarrow f \\
Z & \longrightarrow & X
\end{array}
\]
be a cartesian diagram of noetherian schemes. Assume that \( X \) is normal, the horizontal arrows are closed immersion, the right vertical arrow is quasi-finite and the left vertical arrow is finite. Assume further that there exists a dense open subscheme \( U \subset X \) such that \( U' = U \times_X X' \to U \) is faithfully flat and that \( U' \subset X' \) is also dense.

1. Let \( C \subset Z \) be an irreducible closed subset and let \( C' \subset f^{-1}(C) \) be an irreducible component. Then, \( C' \to C \) is surjective.

2. Let \( C \subset Z \) be a connected closed subset and let \( C' \subset f^{-1}(C) \) be a connected component. Then, \( C' \to C \) is surjective.

**Proof.** 1. By shrinking \( U \) if necessary, we may assume that \( U' \to U \) is finite. By Zariski’s main theorem, there exists a scheme \( \tilde{X}' \) finite over \( X \) containing \( X' \) as an open subscheme. By replacing \( \tilde{X}' \) by the closure of \( U' \), we may assume that \( U' \) is dense in \( \tilde{X}' \). Since \( U' \) is closed in \( \tilde{X}' \times_X U \), we have \( \tilde{X}' \times_X U = U' \). Since \( Z' = (\tilde{X}' \times_X Z) \cap X' \) is closed and open in \( \tilde{X}' \times_X Z \), by replacing \( X' \) by \( \tilde{X}' \), we may assume that \( f \) is finite.

Since \( f \) is a closed mapping, it suffices to show that the generic point \( z \) of \( C \) is the image of the generic point \( z' \) of \( C' \). Let \( x' \) be a point of \( C' \). Replacing \( X \) by an affine neighborhood of \( x = f(x') \in C \), we may assume \( X = \text{Spec} \ A \) and \( X' = \text{Spec} \ B \) are affine. Then, the assumption implies that \( A \to B \) is an injection and \( B \) is finite over \( A \). Since \( x \) is a point of the closure \( C = \{ z \} \), the assertion follows from [9, Chap. V, Section 2.4, Theorem 3].
2. Let $C_1 \subset C$ be an irreducible component such that $C_1 \cap f(C')$ is not empty. Then, there exists an irreducible component $C'_1$ of $f^{-1}(C_1) \subset f^{-1}(C)$ such that $C'_1 \cap C'$ is not empty. By 1, we have $C_1 = f(C'_1)$. Since $C'$ is a connected component of $f^{-1}(C)$ and $C'_1 \cap C' \neq \emptyset$, we have $C'_1 \subset C'$ and hence $C_1 = f(C'_1) \subset f(C')$. Thus, the complement $C - f(C')$ is the union of irreducible components of $C$ not meeting $f(C')$ and is closed. Since $f(C') \subset C$ is also closed and is non-empty, we have $C = f(C')$. \qed

**Corollary 1.1.6.** Let

$$
\begin{array}{c}
\begin{array}{c}
Z' \longrightarrow X' \longleftarrow Y_1' \longleftarrow \quad Y' \\
\downarrow \quad \square \quad f \quad \square \quad f_1 \quad \downarrow \quad f' \\
Z \longrightarrow X \longleftarrow Y_1 \longleftarrow \quad Y
\end{array}
\end{array}
$$

be a commutative diagram of noetherian schemes such that the left square is cartesian and satisfies the conditions in Proposition 1.1.5. Assume that $Y_1 \subset X$ is a closed subscheme, that the middle square is cartesian and that the four arrows in the right square are finite. Assume that there exists a dense open subscheme $V_1 \subset Y_1$ such that $V = V_1 \times Y_1 Y \subset Y$ is also dense and that $g|_{V'}: V \rightarrow V_1$ and $g'|_{V'}: V' = V \times_Y Y' \rightarrow V'_1 = V_1 \times Y_1 Y'_1$ are isomorphisms.

1. For any irreducible (resp. connected) component $C$ of $Y$, we have $f^{-1}(g(C)) = g'(f^{-1}(C))$. Consequently, we have $f^{-1}(g(Y)) = g'(Y')$.

2. Suppose that the mapping $Z \times_Y Y \rightarrow \pi_0(Y)$ is a bijection. Then, the diagram

$$
\begin{array}{c}
\begin{array}{c}
Z' \cap Y'_1 \longleftarrow \quad Z' \times_Y Y' \\
\downarrow \quad \downarrow \\
Z \cap Y_1 \longleftarrow \quad Z \times_Y Y
\end{array}
\end{array}
$$

of underlying sets induces a surjection $Z' \times_Y Y' \rightarrow (Z' \cap Y'_1) \times_{Z \cap Y_1} (Z \times_Y Y)$ of sets. If $Z \times_Y Y \rightarrow Z \cap Y_1$ is surjective, then $Z' \times_Y Y' \rightarrow Z' \cap Y'_1$ is also surjective. Further if $Y' \rightarrow Y$ is surjective, then $Z' \cap Y'_1 \rightarrow Z \cap Y_1$ is also surjective and the diagram (1.7) is a cocartesian diagram of underlying sets.

3. The diagram

$$
\begin{array}{c}
\begin{array}{c}
\pi_0(Z') \longleftarrow \quad Z' \cap Y'_1 \\
\downarrow \quad \downarrow \\
\pi_0(Z) \longleftarrow \quad Z \cap Y_1
\end{array}
\end{array}
$$

of sets induces a surjection $Z' \cap Y'_1 \rightarrow \pi_0(Z') \times_{\pi_0(Z)} (Z \cap Y_1)$ of sets. If $Z \cap Y_1 \rightarrow \pi_0(Z)$ is surjective, then $Z' \cap Y'_1 \rightarrow \pi_0(Z')$ is also surjective. Further if $Z' \cap Y'_1 \rightarrow Z \cap Y_1$ is surjective, the diagram (1.8) is a cocartesian diagram of sets.

**Proof.** 1. Let $C \subset Y$ be an irreducible component. The inclusion $f^{-1}(g(C)) \supset g'(f^{-1}(C))$ is clear. We show the other inclusion. Since $V$ is dense in $Y$, the intersection $C \cap V$ and hence its image $g(C) \cap V_1$ are not empty. Let $C'$ be an irreducible component of $f^{-1}(g(C)) \subset Y'_1$. Since $Y'_1 \rightarrow Y_1$ is finite and $g(C) \subset Y_1$ is an irreducible closed subset, we have $g(C) = f(C')$ by Proposition 1.1.5. Since $f(C' \cap V'_1) = f(C') \cap V_1 = g(C) \cap V_1$ is not empty, $C' \cap V'_1 = g'(g^{-1}(C' \cap V'_1))$ is also non-empty and hence is dense in $C'$. 


Since \( g'^{-1}(C' \cap V'_1) = g'^{-1}(C') \cap V' \subseteq f'^{-1}(C) \) and since \( g': Y' \to X' \) is proper, we have \( C' \subseteq g'(f'^{-1}(C)) \).

Since a connected component of \( Y \) and \( Y \) itself are unions of irreducible components of \( Y \), the remaining assertions follow from the assertion for irreducible components.

2. Let \( z' \in Z', Y \) and \( y \in Z \times X \) be points satisfying \( f'(z') = g(y) \) in \( Z, Y \). Let \( C \subseteq Y \) be the unique connected component containing \( y \). Since \( z' \in f'^{-1}(g(C)) = g'(f'^{-1}(C)) \) by 1, there exists a point \( y' \in Z \times X \), \( f'^{-1}(C) \subseteq Z \times X \) such that \( z' = g'(y') \). Since \( f'(y') \in Z \times X \) is a unique point contained in \( C \subseteq \pi_0(Y) \), we have \( y = f'(y') \). Thus, \( (z', y) \in (Z, Z \times X, Z \times Y) \) is the image of \( y' \in Z \times X \). \( Y' \).

If \( Z \times X \to Z \cap Y \) is surjective, then \( (Z' \cap Y'_1) \times_{Z \cap Y_1} (Z \times X, Y) \to Z' \cap Y'_1 \) is surjective and hence the first assertion implies the surjectivity of \( Z \times X, Y' \to Z' \cap Y'_1 \).

If both \( Z \times X \to Z \cap Y_1 \) and \( Y' \to Y \) are surjective, then \( Z' \times X, Y' = Z \times X \) is also surjective and hence by the commutative diagram (1.17), the mapping \( Z' \cap Y'_1 \to Z \cap Y_1 \) is a surjection. This implies that the diagram (1.17) with \( Z' \times_X, Y' \) replaced by \( (Z' \cap Y'_1) \times_{Z \cap Y_1} (Z \times X, Y) \) is a cocartesian diagram of underlying sets. Hence the surjectivity of \( Z' \times X, Y' \to Z' \cap Y'_1 \times_{Z \cap Y_1} (Z \times X, Y) \) implies that the diagram (1.17) is a cocartesian diagram of underlying sets.

3. Let \( C' \subseteq Z' \) be a connected component and \( z \in Z \cap Y_1 \) be a point such that the connected component \( C \subseteq Z \) satisfying \( f(C') \subseteq C \) contains \( z \). Since \( f(C') = C \) by Proposition 1.1.5.2, the intersection \( C' \cap f^{-1}(z) \subseteq Z' \cap Y'_1 \) is not empty. Hence \( (C', z) \in \pi_0(Z') \times_{\pi_0(Z)} (Z \cap Y_1) \) is in the image of \( C' \cap f^{-1}(z) \subseteq Z' \cap Y'_1 \).

The remaining assertions are proved similarly as in 2. \( \square \)

### 1.2 Flat and reduced morphisms

Let \( k \geq 0 \) be an integer. Recall that a noetherian scheme \( X \) satisfies the condition \( (R_k) \) if for every point \( x \in X \) of \( \dim O_{X,x} \leq k \), the local ring \( O_{X,x} \) is regular [7 Définition (5.8.2)]. Recall also that a noetherian scheme \( X \) satisfies the condition \( (S_k) \) if for every point \( x \in X \), we have \( \text{prof} O_{X,x} \geq \min(k, \dim O_{X,x}) \) [7 Définition (5.7.2)].

**Proposition 1.2.1.** Let \( f: X \to S \) be a flat morphism of finite type of noetherian schemes and \( k \geq 0 \) be an integer. We define a function \( k: S \to \mathbb{N} \) by \( k(s) = \max(k - \dim O_{S,s}, 0) \).

1. If \( S \) satisfies the condition \( (R_k) \) and if the fiber \( X_s = X \times S s \) satisfies \( (R_{k(s)}) \) for every \( s \in S \), then \( X \) satisfies the condition \( (R_k) \).

2. If \( X \) satisfies the condition \( (R_k) \) and if \( f: X \to S \) is faithfully flat, then \( S \) satisfies the condition \( (R_k) \).

**Proof.** 1. Assume \( \dim O_{X,x} \leq k \) and set \( s = f(x) \). Then, we have \( \dim O_{S,s} \leq \dim O_{X,x} \leq k \) and \( \dim O_{S,s} = \dim O_{X,x} \leq \dim O_{S,s} \leq k(s) \) by [7 Proposition (5.7.2)]. Hence \( O_{S,s} \) and \( O_{X,s} \) are regular by the assumption. Thus \( O_{X,x} \) is regular by [7 Chap. 0tv Proposition (17.3.3) (ii)]

2. It follows from [7 Proposition (6.5.3) (i)]. \( \square \)

**Proposition 1.2.2.** Let \( f: X \to S \) be a flat morphism of finite type of noetherian schemes and \( k \geq 0 \) be an integer. Let the function \( k: S \to \mathbb{N} \) be as in Proposition 1.2.1.

1. If \( S \) satisfies the condition \( (S_k) \) and if the fiber \( X_s \) satisfies \( (S_{k(s)}) \) for every \( s \in S \), then \( X \) satisfies the condition \( (S_k) \).

2. If \( X \) satisfies the condition \( (S_k) \) and if \( f: X \to S \) is faithfully flat, then \( S \) satisfies the condition \( (S_k) \).
3. If $X$ satisfies the condition $(S_k)$ and if $S$ is of Cohen-Macaulay, then the fiber $X_s$ satisfies $(S_{k(s)})$ for every $s \in S$.

Proof. 1. Let $x \in X$ and $s = f(x)$. Then, we have $\text{prof} \mathcal{O}_{S,s} \geq \inf(k, \dim \mathcal{O}_{S,s})$ and $\text{prof} \mathcal{O}_{X,x} \geq \inf(k(s), \dim \mathcal{O}_{X,x})$ by the assumption. By $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{X_s} - \dim \mathcal{O}_{S,s}$ [7, Proposition (5.7.2)], we have $\inf(k, \dim \mathcal{O}_{S,s}) + \inf(k(s), \dim \mathcal{O}_{X,x}) = \inf(k, \dim \mathcal{O}_{X_s})$. Hence it follows from $\text{prof} \mathcal{O}_{X_s} = \text{prof} \mathcal{O}_{S,s} + \text{prof} \mathcal{O}_{X_s}$ [7, Proposition (6.3.1)].

2. It follows from [7, Proposition (6.4.1) (i)].

3. Let $x \in X$ and $s = f(x)$. Then by the assumption, we have $\text{prof} \mathcal{O}_{X,x} \geq \inf(k, \dim \mathcal{O}_{X,x})$ and $\text{prof} \mathcal{O}_{S,s} = \dim \mathcal{O}_{S,s}$. By $\text{prof} \mathcal{O}_{X,x} = \text{prof} \mathcal{O}_{X_s} - \text{prof} \mathcal{O}_{S,s} \geq 0$ [7, Proposition (6.3.1)] and $\dim \mathcal{O}_{X,s,x} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{S,s}$ [7, Proposition (5.7.2)] we have $\text{prof} \mathcal{O}_{X,s,x} \geq \inf(k - \dim \mathcal{O}_{S,s}, \dim \mathcal{O}_{X,x}) \geq k(s)$ and the assertion follows. □

Corollary 1.2.3. Let $f : X \to S$ be a flat morphism of finite type of noetherian schemes and let $U \subset X$ be the largest open subset smooth over $S$.

1. Assume that the fiber $X_s$ is reduced for every $s \in S$. Assume further that $S$ is normal and that for the generic point $s$ of each irreducible component, $X_s$ is normal. Then $X$ is normal.

2. For $s \in S$ and a geometric point $\bar{s}$ above $s$, we consider the following conditions:

   (1) The geometric fiber $X_{\bar{s}}$ is reduced.

   (2) $U_s$ is dense in $X_s$.

Then, we have (1) $\Rightarrow$ (2). Conversely, if $X$ is normal and $S$ is regular of dimension $\leq 1$, then we have (2) $\Rightarrow$ (1).

Proof. 1. By Serre’s criterion [7, Théorème (5.8.6)], $S$ satisfies $(R_2)$ and $(S_1)$. By [7, Proposition (5.8.5)], every fiber $X_s$ satisfies $(R_1)$ and $(S_0)$. Further if $s$ is the generic point of an irreducible component, the fiber $X_s$ satisfies $(R_2)$ and $(S_1)$. Since the function $k(s)$ for $k = 2$ satisfies $k(s) \leq 1$ unless $s$ is the generic point an irreducible component and $k(s) = 2$ for such point, the scheme $X$ satisfies the conditions $(R_2)$ and $(S_1)$ by Propositions [1.2.2.1 and 1.2.2.1]. Thus the assertion follows by [7, Théorème (5.8.6)].

2. (1) $\Rightarrow$ (2): Since $X_{\bar{s}}$ is reduced, there exists a dense open subset $V \subset X_{\bar{s}}$ smooth over $\bar{s}$. Since $f$ is flat, the image of $V$ in $X_s$ is a subset of $U_s$.

(2) $\Rightarrow$ (1): Since $X$ satisfies $(S_0)$ and $S$ is Cohen-Macaulay of dimension $\leq 1$, the fiber $X_s$ satisfies $(S_1)$ by Proposition [1.2.2.1]. Hence the geometric fiber $X_{\bar{s}}$ also satisfies $(S_1)$ by [7, Proposition (6.7.7)]. By (2), $X_{\bar{s}}$ satisfies $(R_0)$. Hence the assertion follow from [7, Proposition (5.8.5)]. □

Lemma 1.2.4. Let $S$ be a noetherian scheme and let $f : Y \to X$ be a quasi-finite morphism of schemes of finite type over $S$. Assume that $X$ is smooth over $S$ and that $Y$ is flat and reduced over $S$. Assume that there exist dense open subschemes $U \subset S$ and $U \times_S X \subset W \subset X$ such that $Y \times_X W \to W$ is étale and that for every point $s \in S$, the inverse image $f_s^{-1}(W_s) \subset Y_s = Y \times_S s$ of $W_s = W \times_S s \subset X_s = X \times_S s$ by $f_s : Y_s \to X_s$ is dense. Then, $Y \to X$ is étale.

Proof. If $S$ is regular, the assumption that $Y \times_X W \to W$ is étale and Corollary [1.2.3] implies that the quasi-finite morphism $Y \to X$ of normal noetherian schemes is étale in codimension $\leq 1$. Since $X$ is regular, the assertion follows from the purity theorem of Zariski-Nagata.
Since $X$ and $Y$ are flat over $S$, it suffices to show that for every point $s \in S$, the morphism $Y_s = Y \times_S s \to X_s$ is étale. Let $S' \to S$ be the normalization of the blow-up at the closure of $s \in S$. Then, there exists a point $s' \in S'$ above $s \in S$ such that the local ring $\mathcal{O}_{S', s'}$ is a discrete valuation ring. Since the assumption is preserved by the base change $\text{Spec} \mathcal{O}_{S', s'} \to S$, the morphism $Y_{s'} = Y \times_S s' \to X_{s'} = X \times_S s'$ is étale. Hence $Y_s \to X_s$ is also étale as required. \hfill \Box

The following statement is a combination of the reduced fiber theorem and the flattening theorem.

**Theorem 1.2.5** ([2] Theorem 2.1', [3] Théorème (5.2.2)). Let $S$ be a noetherian scheme and $U \subset S$ be a schematically dense open subscheme. Let $X$ be a scheme of finite type over $S$ such that $X_U = X \times_S U$ is schematically dense in $X$ and that $X_U \to U$ is flat and reduced. Then there exists a commutative diagram

\[
\begin{array}{ccc}
X & \leftarrow & X' \\
\downarrow & & \downarrow \\
S & \leftarrow & S'
\end{array}
\]

of schemes satisfying the following conditions:

(i) The morphism $S' \to S$ is the composition of a blow-up $S^* \to S$ with center supported in $S - U$ and a faithfully flat morphism $S' \to S^*$ of finite type such that $U' = S' \times_S U \to U$ is étale.

(ii) The morphism $X' \to S'$ is flat and reduced. The induced morphism $X' \to X \times_S S'$ is finite and its restriction $X' \times_{S'} U' \to X \times_S U'$ is an isomorphism.

If $X_U \to U$ is smooth and if $S'$ is normal, then $X'$ is the normalization of $X \times_S S'$ by Corollary 1.2.3.

For the morphism $S' \to S$ satisfying the condition (i) in Theorem 1.2.5 we have a following variant of the valuative criterion.

**Lemma 1.2.6.** Let $S$ be a scheme and $U$ be a dense open subscheme. Let $S_1 \to S$ be a proper morphism such that $U_1 = U \times_S S_1 \to U$ is an isomorphism and let $S' \to S$ be a quasi-finite faithfully flat morphism. Let $t \in U$, let $A \subset K = k(t)$ be a valuation ring and $T = \text{Spec} A \to S$ be a morphism extending $t \to U$. Then, there exist $t' \in U' = U \times_S S'$ above $t$, a valuation ring $A' \subset K' = k(t')$ such that $A = A' \cap K$ and a commutative diagram

\[
\begin{array}{ccc}
T' & \longrightarrow & S' \\
\downarrow & & \downarrow \\
T & \longrightarrow & S
\end{array}
\]

for $T' = \text{Spec} A'$. Further, if $t = T \times_S U$, then we have $t' = T' \times_{S'} U'$.

**Proof.** Since $S_1 \to S$ is proper and $U_1 \to U$ is an isomorphism, the morphism $T \to S$ is uniquely lifted to $T \to S_1$ by the valuative criterion of properness. Let $x_1 \in T \times_{S_1} S'$ be a closed point and let $t' \in t \times_{S_1} S'$ be a point above $t$ such that $x_1$ is contained in the closure $T_1 = \overline{\{t'\}} \subset T \times_{S_1} S'$ with the reduced scheme structure. Let $A' \subset k(t')$ be a valuation ring dominating the local ring $\mathcal{O}_{T_1, x_1}$. Then, we have a commutative diagram (1.10) for $T' = \text{Spec} A'$.

Since $t'$ is the unique point of $t \times_T T'$, the equality $t = T \times_S U$ implies $t' = T' \times_{S'} U'$. \hfill \Box
1.3 Semi-stable curves

Let $S$ be a scheme. Recall that a flat separated scheme $X$ of finite presentation over $S$ is a semi-stable curve, if every geometric fiber is purely of dimension $1$ and has at most nodes as singularities.

**Example 1.3.1.** Let $S$ be a scheme and $D \subset S$ be an effective Cartier divisor. Let $C' \to \mathbb{A}^{1}_{S}$ be the blow-up at $D \subset S \subset \mathbb{A}^{1}_{S}$ regarded as a closed subscheme by the 0-section. Then, the complement $C_{D} \subset C'$ of the proper transform of the 0-section is a semi-stable curve over $S$ and is smooth over the complement $U = S - D$. The exceptional divisor $\tilde{D} \subset C_{D}$ is an effective Cartier divisor satisfying $0 \leq \tilde{D} \leq D \times S C_{D}$. The difference $D \times S C_{D} - \tilde{D}$ equals the proper transform of $\mathbb{A}^{1}_{D}$.

If $S = \text{Spec } A$ is affine, $\mathbb{A}^{1}_{S} = \text{Spec } A[t]$ and if $D$ is defined by a non-zero divisor $f \in A$, we have $C_{D} = \text{Spec } A[s,t]/(st - f)$ and $D \subset C_{D}$ is defined by $t$.

**Lemma 1.3.2.** Let $S$ be a scheme and let $U \subset S$ be a schematically dense open subscheme. Let $C$ be a separated flat scheme of finite presentation over $S$ such that the base change $C_{U} = C \times_{S} U$ is a smooth curve over $U$. Then, the following conditions are equivalent:

1. $C$ is a semi-stable curve over $S$.
2. Etale locally on $C$ and on $S$, there exist an effective Cartier divisor $D \subset S$ such that $D \cap U$ is empty and an étale morphism $C \to C_{D}$ over $S$ to the semi-stable curve $C_{D}$ defined in Example 1.3.1.

**Proof.** This is a special case of [4, Corollaire 1.3.2].

Let $S$ be a normal noetherian scheme and $j: U = S - D \to S$ be the open immersion of the complement of an effective Cartier divisor $D$. Let $i: D \to S$ be the closed immersion and let $\pi_{D}: \tilde{D} \to D$ denote the normalization. Then, the valuations at the generic points of irreducible components of $D$ define an exact sequence $0 \to \mathbb{G}_{m,S} \to j_{*}\mathbb{G}_{m,U} \to i_{*}\pi_{D*}\mathbb{Z}_{D}$ of étale sheaves on $S$.

Let $f: C = C_{D} \to S$ be the semi-stable curve over $S$ defined in Example 1.3.1. Let $\tilde{j}: U_{C} = C \times_{S} U \to C$ denote the open immersion and let $\tilde{i}: D_{C} = C \times_{S} D \to C$ denote the closed immersion. Let $A \subset C$ be the exceptional divisor and $B = D_{C} - A$ be the proper transform of $\mathbb{A}^{1}_{D}$. Let $a: A \to C$ and $b: B \to C$ and $e: E = A \cap B \to C$ denote the closed immersions. Then, the Cartier divisors $A, B, D_{C} \subset C$ defines a commutative diagram

\[
\begin{array}{c}
f^{*}i_{*}\mathbb{Z} \\
\downarrow \\
f^{*}(j_{*}\mathbb{G}_{m,U}/\mathbb{G}_{m,S}) \longrightarrow \tilde{j}_{*}\mathbb{G}_{m,U_{C}/\mathbb{G}_{m,C}}
\end{array}
\]  

(1.11)

of étale sheaves on $C$.

**Proposition 1.3.3.** Let $S$ be a normal noetherian scheme and $D \subset S$ be an effective Cartier divisor. Let $f: C = C_{D} \to S$ be the semi-stable curve defined in Example 1.3.1. Then, the diagram (1.11) induces an exact sequence

\[
0 \longrightarrow f^{*}i_{*}\mathbb{Z} \longrightarrow f^{*}(j_{*}\mathbb{G}_{m,U}/\mathbb{G}_{m,S}) \oplus (a_{*}\mathbb{Z} \oplus b_{*}\mathbb{Z}) \longrightarrow \tilde{j}_{*}\mathbb{G}_{m,U_{C}/\mathbb{G}_{m,C}} \longrightarrow 0
\]  

(1.12)

of étale sheaves on $D_{C}$.
Proof. Let \( z \) be a geometric point of \( C \) and we show the exactness of the stalks of (1.12) at \( z \). Replacing \( S \) by the strict localization at the image \( x \) of \( z \), we may assume that \( S \) is strict local and that \( x \) is the closed point. For \( t \in S = S(\bar{z}) \), the Milnor fiber \( C(\bar{z}) \times_S t \) at \( t \) of the strict localization \( C(\bar{z}) \) at \( z \) is geometrically connected by [7 Théorème (18.9.7)]. Further, if \( z \in E \) and if \( t \in D \), the fiber at \( t \) of \( C(\bar{z}) = E(z) \) has 2 geometrically connected components.

First, we consider the case where \( C \) is smooth over \( S \) at \( z \). Then, since the Milnor fiber \( C(\bar{z}), t \) is connected, the canonical morphism \( f^*i_*\mathcal{Z}_D \to i_{C^*}\mathcal{Z}_D \) is an isomorphism. Hence the stalk of the lower horizontal arrow (1.11) at \( z \) is an injection. Further this is a surjection by flat descent.

We assume that \( C \to S \) is not smooth at \( z \). Let \( \tilde{D} \) be a Cartier divisor of \( C(\bar{z}) \) supported on \( D_{C(\bar{z})} = C(\bar{z}) \times_S D \). Then similarly as above, there exists a Cartier divisor \( D_1 \) on \( S \) supported on \( D \) such that \( D_0 = \tilde{D} - f^*D_1 \) is supported on the inverse image of \( A \). Define a \( \mathcal{Z} \)-valued function \( n \) on \( y \in E(\bar{z}) = D \) as the intersection number of \( D_0 \) with the fiber \( B \times_S y \). We show that the function \( n \) is constant. By adding some multiple of \( A \) to \( \tilde{D} \) if necessary, we may assume that \( D_0 \) is an effective Cartier divisor of \( C \) supported on \( A \). Since \( B \) is flat over \( D \), the pull-back \( D_0 \times_C B \) is an effective Cartier divisor of \( B \) finite flat over \( D \) by [7 Proposition (15.1.16) c)⇒b)]. Hence the function \( n \) is constant. Thus we have \( \tilde{D} = f^*D_1 + n \cdot A \) and the exactness of the stalks of (1.12) at \( z \) follows. \( \square \)

**Corollary 1.3.4.** Let \( S \) be a normal noetherian scheme and \( C \to S \) be a semi-stable curve. Let \( x \in S \) be a point and \( z \in C \times_S x \) be a singular point of the fiber. Assume that \( z \) is contained in the intersection of two irreducible components \( C_1 \) and \( C_2 \) of \( C \times_S x \). Let \( s_1 \colon S \to C \) and \( s_2 \colon S \to C \) be sections meeting with the smooth parts of \( C_1 \) and \( C_2 \) respectively.

Let \( U \subset S \) be a dense open subscheme such that \( C_U = C \times_S U \) is smooth over \( U \) and let \( \tilde{D} \subset C \) be an effective Cartier divisor such that \( \tilde{D} \cap C_U \) is empty. Define effective Cartier divisors \( D_1 = s_1^*\tilde{D} \) and \( D_2 = s_2^*\tilde{D} \) of \( S \) as the pull-back of \( \tilde{D} \).

Then, on a neighborhood of \( x \), we have either \( D_1 \leq D_2 \) or \( D_2 \leq D_1 \). Suppose we have \( D_1 \leq D_2 \) on a neighborhood of \( x \). Then, we have \( D_1 \times_S C \leq \tilde{D} \leq D_2 \times_S C \) on a neighborhood of \( z \).

Proof. In the notation of the proof of Proposition 1.3.3 we have \( \tilde{D} = f^*D_1 + nA \) for an integer \( n \) on an étale neighborhood of \( z \). Hence the assertion follows. \( \square \)

We recall a combination of a strong version of the semi-stable reduction theorem for curves over a general base scheme with the flattening theorem.

**Theorem 1.3.5 ([11 Theorem 2.3.3], [8 Théorème (5.2.2)])**. Let \( S \) be a noetherian scheme and \( U \subset S \) be a schematically dense open subscheme. Let \( C \to S \) be a separated morphism of finite type such that \( C \times_S U \to U \) is a smooth curve and that \( C \times_S U \subset U \) is schematically dense. Then, there exists a commutative diagram

\[
\begin{array}{ccc}
C' & & C \\
\downarrow & & \downarrow \\
S' & & S
\end{array}
\]

of schemes satisfying the following conditions:
(i) The morphism \( S' \to S \) is the composition of a proper modification \( S_1 \to S \) such that \( U_1 = U \times_S S_1 \to U \) is an isomorphism and a faithfully flat morphism \( S' \to S_1 \) such that \( U' = U \times_S S' \to U_1 \) is étale and \( U' \subset S' \) is schematically dense.

(ii) The morphism \( C' \to S' \) is a semi-stable curve and the morphism \( C' \to C \times_S S' \) is a proper modification such that \( C' \times_{S'} U' \to C \times_S U' \) is an isomorphism.

**Corollary 1.3.6.** Let \( S \) be a noetherian scheme and \( U \subset S \) be a schematically dense open subscheme. Let \( C \to S \) be a separated morphism of finite type such that \( C_U = C \times_S U \to U \) is a smooth curve and that \( C_U \subset C \) is schematically dense. Let \( X \to C \) be a separated morphism of finite type such that \( X_U = X \times_S U \subset X \) is schematically dense and that \( X_U \to C_U \) is flat and reduced. Then, there exists a commutative diagram

\[
\begin{array}{ccc}
X & \leftarrow & X' \\
\downarrow & & \downarrow \\
C & \leftarrow & C' \\
\downarrow & & \downarrow \\
S & \leftarrow & S'
\end{array}
\]

of schemes satisfying the following conditions:

(i) The morphism \( S' \to S \) is the composition of a proper modification \( S_1 \to S \) such that \( U_1 = U \times_S S_1 \to U \) is an isomorphism and a faithfully flat morphism \( S' \to S_1 \) such that \( U' = U \times_S S' \to U_1 \) is étale and \( U' \subset S' \) is schematically dense.

(ii) The morphism \( C' \to S' \) is a semi-stable curve and the morphism \( C' \to C \times_S S' \) is the composition of a proper modification \( C'_0 \to C \times_S S' \) such that \( C'_0 \times_{S'} U' \to C \times_S U' \) is an isomorphism, a faithfully flat morphism \( C'_1 \to C'_0 \) such that \( C'_1 \times_{S'} U' \to C'_0 \times_{S'} U' \) is étale and of a proper modification \( C' \to C'_1 \) such that \( C' \times_{S'} U' \to C'_1 \times_{S'} U' \) is an isomorphism.

(iii) The morphism \( X' \to C' \) is flat and reduced, the morphism \( X' \to X \times_C C' \) is finite and \( X' \times_{C'} U' \to X \times_C C' \times_{S'} U' \) is an isomorphism.

**Proof.** By the reduced fiber theorem Theorem 1.2.5 applied to \( X \to C \), there exists a commutative diagram

\[
\begin{array}{ccc}
X & \leftarrow & X_1 \\
\downarrow & & \downarrow \\
C & \leftarrow & C_1 
\end{array}
\]

satisfying the conditions (i) and (ii) loc. cit. Since \( C_1 \times_S U \to C \times_S U \) is étale and \( C_1 \times_S U \subset C_1 \) is schematically dense, by the combination Theorem 1.3.5 of the stable reduction theorem and the flattening theorem, there exists a commutative diagram

\[
\begin{array}{ccc}
C_1 & \leftarrow & C_2 \\
\downarrow & & \downarrow \\
S & \leftarrow & S_2 
\end{array}
\]

satisfying the conditions (i) and (ii) loc. cit.
By the flattening theorem [8, Théorème (5.2.2)] applied to $S_2 \to S$, there exists a commutative diagram

$$
\begin{array}{ccc}
S_2 & \to & S' \\
\downarrow & & \downarrow \\
S & \to & S_1
\end{array}
$$

satisfying the condition (i). We show that $C' = C_2 \times_{S_2} S'$ and $X' = X_1 \times_{C_1} C'$ satisfy the required conditions. The base change $C' \to S'$ of a semi-stable curve $C_2 \to S_2$ is a semi-stable curve. Since $C_1 \to C$ is obtained by applying Theorem 1.3.5, the composition $C' = C_2 \times_{S_2} S' \to C'_1 = C_1 \times_S S' \to C \times_S S'$ satisfies the condition in (ii). Finally, the base change $X' \to C'$ of a flat and reduced morphism $X_1 \to C_1$ is flat and reduced. Since $X' \to C'$ is obtained by applying Theorem 1.2.5 the morphism $X' \to X \times_C C'$ satisfies the condition (iii). \hfill \Box

1.4 Subgroups and fiber functor

For a finite group $G$, let (Finite $G$-sets) denote the category of finite sets with left $G$-action.

**Definition 1.4.1.** We say that a category $C$ is a finite Galois category if there exist a finite group $G$ and an equivalence of categories $F: C \to$ (Finite $G$-sets). If $F: C \to$ (Finite $G$-sets) is an equivalence of categories, we say that $G$ is the Galois group of the finite Galois category $C$ and call the functor $F$ itself or the composition $C \to$ (Finite sets) with the forgetful functor also denoted by $F$ a fiber functor of $C$.

We say that a morphism $F \to F'$ of functors $F, F': C \to$ (Finite sets) is a surjection if $F(X) \to F'(X)$ is a surjection for every object $X$ of $C$. For a subgroup $H \subset G$ and for a fiber functor $F: C \to$ (Finite $G$-sets), let $F_H$ denote the functor $C \to$ (Finite sets) defined by $F_H(X) = H \setminus F(X)$. The canonical morphism $F \to F_H$ is a surjection.

Surjections $F \to F_H$ are characterized as follows.

**Proposition 1.4.2** (cf. [1, Proposition 2.1]). Let $C$ be a finite Galois category of Galois group $G$ and $F: C \to$ (Finite sets) be a fiber functor. Let $F': C \to$ (Finite sets) be another functor and $F \to F'$ be a surjection of functors. Then, the following conditions are equivalent:

1. For every surjection $X \to Y$ in $C$, the diagram

$$
\begin{array}{ccc}
F(X) & \longrightarrow & F'(X) \\
\downarrow & & \downarrow \\
F(Y) & \longrightarrow & F'(Y)
\end{array}
$$

is a cocartesian diagram of finite sets. For every pair of objects $X$ and $Y$ of $C$, the morphism $F'(X) \amalg F'(Y) \to F'(X \amalg Y)$ is a bijection.

2. There exists a subgroup $H \subset G$ such that $F \to F'$ induces an isomorphism $F_H \to F'$.

**Proof.** (1)$\Rightarrow$(2): We may assume $C =$ (Finite $G$-sets) and $F$ is the forgetful functor. For $X = G$, the mapping $F(G) = G \to F'(G)$ is a surjection of finite sets. Define an equivalence relation $\sim$ on $G$ by requiring that $G/\sim \to F'(G)$ to be a bijection and set $H = \{x \in G \mid x \sim e\}$. Then, since the group $G$ acts on the object $G$ of $C$ by the right
action, the relation \( x \sim y \) is equivalent to \( xy^{-1} \in H \). Since \( \sim \) is an equivalence relation, the transitivity implies that \( H \) is stable under the multiplication, the reflexivity implies \( e \in H \) and the symmetry implies that \( H \) is stable under the inverse. Hence \( H \) is a subgroup and the surjection \( F(G) = G \to F'(G) \) induces a bijection \( H \backslash G \to F'(G) \).

Let \( X \) be an object of \( C = \text{Finite } \mathcal{G} \text{-sets} \) and regard \( G \times X \) as a \( G \)-set by the left action on \( G \). Then, since the functor \( F' \) preserves the disjoint union, we have a canonical isomorphism \( F'(G \times X) \to F'(G) \times X \to (H \backslash G) \times X \). Further, the cocartesian diagram \([1.13]\) for the surjection \( G \times X \to X \) in \( C \) defined by the action of \( G \) is given by

\[
\begin{array}{ccc}
G \times X & \longrightarrow & (H \backslash G) \times X \\
\downarrow & & \downarrow \\
X & \longrightarrow & F'(X).
\end{array}
\]

(1.14)

Thus we obtain a bijection \( H \backslash X \to F'(X) \).

The other implication (2) \( \Rightarrow \) (1) is clear. \( \square \)

**Corollary 1.4.3.** Let the notation be as in Proposition 1.4.2 and let \( G' \) be a quotient group. Let \( C' \subset C \) be the full subcategory consisting of objects \( X \) such that \( F(X) \) are \( G' \)-sets. Then the subgroup \( H' \subset G' \) defined by the surjection \( F|_{C'} \to F'|_{C'} \) of the restrictions of the functors equals the image of \( H \subset G \) in \( G' \).

**Proof.** If a \( G \)-set \( X \) is a \( G' \)-set, the quotient \( H \backslash X \) is \( H' \backslash X \). \( \square \)

**Corollary 1.4.4.** Let \( C \) be a finite Galois category of Galois group \( G \) and \( F: C \to \text{Finite } \mathcal{G} \text{-sets} \) be a fiber functor. Let \( G' \to G \) be a morphism of groups and let \( F \) also denote the functor \( C \to \text{Finite } G' \text{-sets} \) defined as the composition defined by \( G' \to G \). Let \( F': C \to \text{Finite } G' \text{-sets} \) be another functor and \( F \to F' \) be a surjection of functors such that the composition with the forgetful functor satisfies the condition (1) in Proposition 1.4.2.

Let \( H \subset G \) be the subgroup satisfying the condition (2) and \( G'_1 \subset G \) be the image of \( G' \to G \). Then, the functor \( F' \) induces a functor \( C \to \text{Finite } G'_1 \text{-sets} \) and \( G'_1 \subset G \) is a subgroup of the normalizer \( N_G(H) \) of \( H \).

**Proof.** For an object \( X \) of \( C \), \( F(X) \) regarded as a \( G' \)-set is a \( G'_1 \)-set. Since \( F(X) \to F'(X) \) is a surjection of \( G' \)-sets, \( F'(X) \) is also a \( G'_1 \)-set. Since the left action of \( G'_1 \subset G \) on the \( G \)-set \( F(G) = G \) induces an action on \( F'(G) = H \backslash G \), the subgroup \( H \) is normalized by \( G'_1 \). \( \square \)

## 2 Dilatations

### 2.1 Functoriality of dilatations

Let \( X \) be a noetherian scheme and we consider morphisms

\[
(2.1) \quad D \longrightarrow X \longleftarrow Q \longleftarrow Y
\]

of separated schemes of finite type over \( X \) satisfying the following condition:

(i) \( D \subset X \), \( D_Y = D \times_X Y \subset Y \) and \( D_Q = D \times_X Q \subset Q \) are effective Cartier divisors and \( Y \to Q \) is a closed immersion.
In later subsections, we will further assume the following condition:

(ii) $X$ is normal and $Q$ is smooth over $X$.

We give examples of constructions of $Q$ for a given $Y$ over $X$.

**Example 2.1.1.** Assume that $X$ and $Y$ are separated schemes of finite type over a noetherian scheme $S$.

1. Assume $S = \text{Spec} \, A$ and $Y = \text{Spec} \, B$ are affine. Then, taking a surjection $A[T_1, \ldots, T_n] \to B$, we obtain a closed immersion $Y \to Q = A^n \times_S X$.

2. Assume that $Y$ is smooth over $S$. Then, $Q = Y \times_S X \to X$ is smooth and the canonical morphism $Y \to Q = Y \times_S X$ is a closed immersion.

3. Assume that $\pi: Y \to X$ is finite flat and define a vector bundle $Q$ over $X$ by the symmetric $O_X$-algebra $S^{\bullet} \pi_* O_Y$. Then the canonical surjection $S^{\bullet} \pi_* O_Y \to \pi_* O_Y$ defines a closed immersion $Y \to Q$.

For morphisms (2.1) satisfying the condition (i) above, we construct a commutative diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & Q^{(D)} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Q^{(D)} \\
\end{array}
\]

of schemes over $X$ as follows. Let $\mathcal{I}_D \subset O_X$ and $\mathcal{I}_Y \subset O_Q$ be the ideal sheaves defining the closed subschemes $D \subset X$ and $Y \subset Q$. Let $Q' \to Q$ be the blow-up at $D_Y = D \times_X Y \subset Q$ and define the dilatation $Q^{(D)}$ at $Y \to Q$ and $D$ to be the largest open subset of $Q'$ where $\mathcal{I}_D O_{Q'} \supset \mathcal{I}_Y O_{Q'}$. Since $D_Y$ is a divisor of $Y'$, by the functoriality of blow-up, the immersion $Y \to Q$ is uniquely lifted to a closed immersion $Y \to Q^{(D)}$. Let $Y'$ and $Q^{(D)}$ be the normalization of $Y$ and $Q^{(D)}$ and let $Y \to Q^{(D)}$ be the morphism induced by the morphism $Y \to Q^{(D)}$. If there is a risk of confusion, we let $Q^{(D)}$ and $Q^{(D)}$ also be denoted by $Q^{(D,y)}$ and $Q^{(D,y)}$ to make $Y$ explicit.

Locally, if $Q = \text{Spec} \, A$ and $Y = \text{Spec} \, A/I$ are affine and if $D \subset X$ is defined by a non-zero divisor $f$, we have

\[
Q^{(D)} = \text{Spec} \, A[I/f]
\]

for the subring $A[I/f] \subset A[1/f]$ and the immersion $Y \to Q^{(D)}$ is defined by the isomorphism $A[I/f]/(I/f)A[I/f] \to A/I$.

**Example 2.1.2.** Let $X$ be a noetherian scheme and $D \subset X$ be an effective Cartier divisor.

1. Let $Q$ be a smooth separated scheme over $X$ and $s: X \to Q$ be a section. Let $Y = s(X) \subset Q$ be the closed subscheme. Then, $Q^{(D)}$ is smooth over $X$. If $X$ is normal, the canonical morphism $Q^{(D)} \to Q^{(D)}$ is an isomorphism.

2. Assume that $X$ is normal. Let $Q$ be a smooth curve over $X$ and let $s_1, \ldots, s_n: X \to Q$ be sections. Define a closed subscheme $Y \subset Q$ as the sum $\sum_{i=1}^n s_i(X)$ of the sections regarded as effective Cartier divisors of $Q$. Assume that $D \subset \sum_{i=1}^n \bar{s}_i(X)$ for $i = 1, \ldots, n-1$. Then $Q^{(nD)} \to Q$ is smooth and $Y \times_{Q^{(nD)}} Q^{(nD)} \subset Q^{(nD)}$ is the sum $\sum_{i=1}^n \bar{s}_i(X)$ of the sections $\bar{s}_i: X \to Q^{(nD)}$ lifting $s_i: X \to Q$.

In fact, we may assume that $X = \text{Spec} \, A$ is affine and, locally on $Q$, take an étale morphism $Q \to A^n_X$. Then, we may assume that $Q = A^n_X = \text{Spec} \, A[T]$ and $Y$ is defined by $P = \prod_{i=1}^n (T - a_i)$ for $a_i \in A$. We may further assume that $D$ is defined by a non-zero
divisor $a \in A$ dividing $a_1, \ldots, a_n$. Then, we have $Q^{[nD]} = \text{Spec} A[T][P/a^n]$ and $T' = T/a$ satisfies $\prod_{i=1}^{n}(T' - a_i/a) = P/a^n$ in $A[T][1/a]$. Hence we have $Q^{(nD)} = \text{Spec} A[T']$ and this equals $Q^{[D,n(X)]}$ and is smooth over $X$. The section $Y \to Q^{[nD]}$ is defined by $P/a^n = 0$ and hence $Y \times_{Q^{[nD]}} Q^{(nD)} \subset Q^{(nD)}$ is defined by $A[T']/\prod_{i=1}^{n}(T' - a_i/a)$.

We study the base change $Q^{[D]} \times_X D$.

**Lemma 2.1.3.** 1. The canonical morphism $Q^{[D]} \to Q$ induces

$$Q^{[D]} \times_X D = Q^{[D]} \times_Q D_Y \to D_Y.$$  

2. If $Y \to Q$ is a regular immersion and if $T_YQ$ and $T_DX$ denote the normal bundles, we have a canonical isomorphism

$$T_YQ(-D_Y) \times_Y D_Y = (T_YQ \times_Y D_Y) \otimes (T_DX \times_D D_Y)^{\otimes -1} \to Q^{[D]} \times_X D.$$  

The isomorphism (2.5) depends only on the restriction $D_Y \to Q$ and not on $Y \to Q$ itself.

3. Assume that $Q$ is smooth over $X$ and $X = Y \to Q$ is a section. Let $T(\mathcal{Q}/X)$ denote the relative tangent bundle defined by the symmetric $\mathcal{O}_Q$-algebra $S_{\mathcal{O}_D}^\bullet (\mathcal{O}_Q)^{\otimes 1}/\mathcal{O}_X$. Then, we have a canonical isomorphism

$$T(\mathcal{Q}/X)(-D) \times_Q D = (T(\mathcal{Q}/X) \times_Q D) \otimes T_DX^{\otimes -1} \to Q^{[D]} \times_X D.$$  

The isomorphism (2.6) depends only on the restriction $D \to Q$ and not on the section $X \to Q$ itself.

**Proof.** 1. Since $\mathcal{I}_D\mathcal{O}_{Q[D]} \supset \mathcal{I}_Y\mathcal{O}_{Q[D]}$ on $Q^{[D]}$ by the definition of $Q^{[D]}$, we have $Q^{[D]} \times_X D = Q^{[D]} \times_Q D_Y$. Hence, we obtain a morphism $Q^{[D]} \times_X D \to D_Y$.

2. Assume that $Y \to Q$ is a regular immersion. Then, $D_Y \to Q$ is also a regular immersion and the normal bundle $T_{D_Y}Q$ fits in an exact sequence $0 \to T_{D_Y}DQ \to T_{D_Y}Q \to T_DX \times_D D_Y \to 0$ depending only on $D \to X$ and $D_Y \to Q$ and not on $Y \to Q$. Let $Q' \to Q$ be the blow-up at $D_Y \subset Q$. Then, the exceptional divisor $Q' \times_Q D_Y$ is canonically identified with the projective space bundle $P(T_{D_Y}Q)$ over $D_Y$. Its open subset $Q^{[D]} \times_Q D_Y$ is identified as in (2.5) since $T_{D_Y}DQ = T_YQ \times_Y D_Y$.

3. Since the normal bundle $T_XQ$ is canonically identified with the restriction $T(\mathcal{Q}/X) \times_Q X$ of the relative tangent bundle, the assertion follows from 2.

We give a sufficient condition for the morphism $\bar{Y} \to Q^{(D)}$ to be an immersion.

**Lemma 2.1.4.** Assume that $X$ and $Y = D_Y$ are normal and let $\pi: \bar{Y} \to Y$ be the normalization. Assume that $\bar{Y} \to X$ is étale and that $\pi_*\mathcal{O}_{\bar{Y}}/\mathcal{O}_Y$ is an $\mathcal{O}_{D_Y}$-module. Then, the finite morphism $\bar{Y} \to Q^{(2D)}$ is a closed immersion.

**Proof.** Since the assertion is étale local on $Y$, we may assume that $Y \to X$ is finite and that the étale covering $\bar{Y} \to X$ is split. We may further assume that $X, Y$ and $Q$ are affine and that $D$ is defined by a non-zero divisor $f$ on $X$. Let $Y = \text{Spec} A, \bar{Y} = \text{Spec} \bar{A}, Q = \text{Spec} B, Q^{[2D]} = \text{Spec} B^{(2D)}, Q^{(2D)} = \text{Spec} B^{(2D)}$ for $A = B/I, B^{(2D)} = B[I/f^2] \subset B[1/f]$ and the normalization $B^{(2D)}$ of $B^{(2D)}$. Since $\bar{Y} \to X$ is a split étale covering, it suffices to show that for every idempotent $e \in \bar{A}$, there exists a lifting $\bar{e} \in B^{(2D)}$.

Since $\bar{A}/A$ is annihilated by $f$, the product $fe = g$ is an element of $A$. Let $\bar{g} \in B$ be a lifting of $g$. Since $e^2 = e$, the element $h = \bar{g}^2 - f\bar{g} \in B$ is contained in $I$ and hence
\( h/f^2 \in B[1/f] \) is an element of \( B^{[2D]} \). Thus \( \tilde{e} = \tilde{g}/f \in B[1/f] \) is a root of the polynomial \( T^2 - T - h/f^2 \in B^{[2D]}[T] \) and is an element of \( B^{(2D)} \). Since \( \tilde{e} \) is a lifting of \( e \), the assertion follows.

We study the functoriality of the construction. We consider a commutative diagram

\[
\begin{array}{ccc}
D \times_X X' \subset D' \subset X' & \leftarrow & Q' \leftarrow Y' \\
\downarrow & & \downarrow & & \downarrow \\
D & \subset & X & \leftarrow & Q & \leftarrow Y
\end{array}
\]

(2.7)

of schemes such that the both lines satisfy the condition (i) on the diagram (2.1). Then, by the functoriality of dilatations and normalizations, we obtain a commutative diagram

\[
\begin{array}{ccc}
Y' & \longrightarrow & Q^D' \leftarrow Q^{(D')} \leftarrow Y' \\
\downarrow & & \downarrow & & \downarrow \\
Y & \longrightarrow & Q^D \leftarrow Q^{(D)} \leftarrow \bar{Y}.
\end{array}
\]

(2.8)

The diagram (2.8) induces a morphism

\[
Q^D' \times_{Q^D} Y' \rightarrow Q^D \times_{Q^D} Y.
\]

(2.9)

Let \( \bar{x} \) be a geometric point of \( D \) and \( \bar{x}' \) be a geometric point of \( D \times_X X' \) above \( \bar{x} \). Then the diagram (2.8) also induces a mapping

\[
\pi_0(Q^D'_{\bar{x}'}) \rightarrow \pi_0(Q^D_{\bar{x}})
\]

(2.10)

of the sets of connected components of the geometric fibers.

First we study the dependence on \( Q \).

**Proposition 2.1.5.** Suppose \( X = X', Y = Y' \) and \( D = D' \) and let \( \bar{x} \) be a geometric point of \( D \).

1. Assume that \( Q \) and \( Q' \) are smooth over \( X \). Then, the square

\[
\begin{array}{ccc}
Q^D & \leftarrow & Q^{(D)} \\
\downarrow & & \downarrow \\
Q^D & \leftarrow & Q^{(D)}
\end{array}
\]

(2.11)

is cartesian. The induced morphism \( Q^D \times_{Q^D} Y \rightarrow Q^{(D)} \times_{Q^D} Y \) \text{(2.9)} is an isomorphism over \( Y \) and the induced mapping \( \pi_0(Q^D') \rightarrow \pi_0(Q^D) \) \text{(2.10)} is a bijection.

2. Assume that \( Q' \rightarrow Q \) is smooth and let \( T = T(Q'/Q) \) denote the relative tangent bundle of \( Q' \) over \( Q \). Then \( Q^D \rightarrow Q^{(D)} \) is also smooth and there exists a cartesian diagram

\[
\begin{array}{ccc}
T(-D) \times_{Q'} D_Y & \leftarrow & Q^D \times_X D \\
\downarrow & & \downarrow \\
D_Y & \leftarrow & Q^{(D)} \times_X D.
\end{array}
\]

(2.12)
Proof. 2. First, we show the case where \( Q' \to Q \) admits a section \( Q \to Q' \) extending \( Y \to Q' \). The section \( Q \to Q' \) defines a section \( Q^{[D]} \to Q'^{ \times}Q^{[D]} \). Define \( (Q' \times_Q Q^{[D]})^{[D]}_{Q^{[D]}}, Q^{[D]} \) to be the dilatation of \( Q' \times_Q Q^{[D]} \) for the section \( Q^{[D]} \to Q' \times_Q Q^{[D]} \) and a divisor \( D_{Q^{[D]}} = D \times_Q Q^{[D]} \) over \( Q^{[D]} \). We show that the canonical morphism \( Q^{[D]} \to Q' \times_Q Q^{[D]} \) induces an isomorphism

\[
(2.13) \quad Q^{[D]} \to (Q' \times_Q Q^{[D]})^{[D]}_{Q^{[D]}}, Q^{[D]}].
\]

Since the question is étale local on \( Q' \), we may assume that \( Q' = \mathbf{A}^n_q \) and the section \( Q \to Q' \) is the 0-section. Further, we may assume that \( Q = \text{Spec} \ A \) and \( Y = \text{Spec} \ A/I \) are affine and that \( D \subset X \) is defined by a non-zero divisor \( f \) on \( X \). We set \( A' = A[T_1, \ldots, T_n] \) and \( Q' = \text{Spec} \ A' \). The 0-section \( Q \to Q' \) is defined by the ideal \( J = (T_1, \ldots, T_n) \subset A' \). We have \( Q^{[D]} = \text{Spec} \ A[I/f] \) and \( Q^{[D]} = \text{Spec} \ A'[I'/f'] \) for \( I' = IA' + J \). Since \( A'[I'/f'] = A[I/f][T_1/f, \ldots, T_n/f] \) as a subring of \( A'[1/f] \), we obtain an isomorphism \((2.13)\).

By the isomorphism \((2.13)\) and Example 2.1.1, the morphism \( Q^{[D]} \to Q^{[D]} \) is smooth. Further by Lemma 2.1.3, we obtain a cartesian diagram \((2.12)\), depending only on \( D \to X \), \( D_Y \to Q \) and \( D_Y \to Q' \) but not on the choice of section \( Q \to Q' \) extending \( Y \to Q' \).

We prove the general case. Since \( Q' \to Q \) has a section on \( Y \subset Q \), locally on \( Q' \), there exist a closed subscheme \( Q_1 \subset Q' \) étale over \( Q \) such that \( Y' \to Q' \) is induced by \( Y' \to Q \). For the smoothness of \( Q^{[D]} \to Q^{[D]} \), since the assertion is étale local, we may assume that \( Q_1 = Q \) is a section. Hence the smoothness \( Q^{[D]} \to Q^{[D]} \) follows. Further since the cartesian diagram \((2.12)\) defined étale locally is independent of the choice of section, we obtain \((2.12)\) for \( Q' \) by patching.

1. First, we show the case where \( Q' \to Q \) is smooth. Then by 2, \( Q^{[D]} \to Q^{[D]} \) is also smooth and the fibered product \( Q^{[D]} \times_{Q^{[D]}} Q^{[D]} \) is normal. Hence the square \((2.11)\) is cartesian and the morphism \((2.9)\) is an isomorphism. By the cartesian squares \((2.11)\) and \((2.12)\), \( Q^{[D]} \) is a vector bundle over \( Q^{[D]} \). Hence \((2.10)\) is a bijection.

We show the general case. A morphism \( f : Q' \to Q \) is decomposed as the composition of the projection \( \text{pr}_2 : Q' \times_Q Q \to Q \) and a section of the projection \( \text{pr}_1 : Q' \times_Q Q \to Q' \). Hence, the cartesian squares \((2.11)\) and the bijections \((2.10)\) for the projections imply those for \( f \) respectively. The cartesian square \((2.11)\) for \( f \) implies an isomorphism \((2.9)\) for \( f \).

Corollary 2.1.6. Assume that \( Q \) and \( Q' \) are smooth over \( X \). Then, the morphism \( Q^{[D]} \times_{Q^{[D]}} Y' \to Q^{[D]} \times_{Q^{[D]}}, Y \) \((2.9)\) is independent of \( Q' \to Q \). Let \( \bar{x} \) be a geometric point of \( D \) and \( \bar{x}' \) be a geometric point of \( D' \) above \( \bar{x} \). Then the mapping \( \pi_0(Q^{[D]}_{\bar{x}'}) \to \pi_0(Q^{[D]}_{\bar{x}'}) \) \((2.10)\) is independent of morphism \( Q' \to Q \).

Proof. Decompose a morphism \( Q' \to Q \) as \( Q' \to Q' \times_X Q \to Q \). Then the isomorphism \((2.9)\) and the bijection \((2.10)\) for \( Q' \to Q' \times_X Q \) are the inverses of those for the projection \( Q' \times_X Q \to Q' \). Hence the assertion follows.

By the canonical isomorphism \((2.9)\), the finite scheme \( Y \times_{Q^{[D]}} Q^{[D]} \) over \( Y \) is independent of \( Q \). We let it denoted by \( Y^{(D)} \).
Lemma 2.1.7. Suppose that the squares

\[
\begin{array}{ccc}
D' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
D & \longrightarrow & X,
\end{array}
\quad
\begin{array}{ccc}
Q' & \longleftarrow & Y' \\
\downarrow & & \downarrow \\
Q & \longleftarrow & Y
\end{array}
\]

are cartesian.

1. The morphism \(Q^{[D']} \to Q^{[D]} \times_Q Q'\) is a closed immersion and \(Q^{[D']} \to Q^{(D)} \times_Q Q'\) is finite. Consequently, the morphism \(Q^{(D')} \times_Q Y' \to Q^{(D)} \times_Q Y\) is finite if \(Y' \to Y\) is finite. Further, if \(Q\) and \(Q'\) are normal, then \(Q^{(D')}\) equals the normalization of \(Q^{(D)} \times_Q Q'\) in \(Q' = D' \times_X Q'\).

2. If \(Q' \to Q\) is flat, the square

\[
\begin{array}{ccc}
Q^{[D']} & \longrightarrow & Q' \\
\downarrow & & \downarrow \\
Q^{[D]} & \longrightarrow & Q
\end{array}
\]

is cartesian.

Proof. Since the assertion is local on a neighborhood of \(Y' \subset Q'\), we may assume that \(Q = \text{Spec } A, Y = \text{Spec } A/I, Q' = \text{Spec } A'\) and \(Y' = \text{Spec } A'/IA'\) are affine and that \(D\) is defined by a non-zero divisor \(f\) on \(X\). Then, we have \(Q^{[D]} = \text{Spec } A[I/f]\) and \(Q^{[D']} = \text{Spec } A'[IA'/f]\).

1. Since \(A[I/f] \otimes_A A' \to A'[IA'/f]\) is a surjection, the morphism \(Q^{[D']} \to Q^{[D]} \times_Q Q'\) is a closed immersion. The remaining assertions follow from this immediately.

2. If \(A \to A'\) is flat, the injection \(A[I/f] \to A[1/f]\) induces an injection \(A' \otimes_A A[I/f] \to A' \otimes_A A[1/f] = A'[1/f]\). Hence the surjection \(A' \otimes_A A[I/f] \to A'[IA'/f]\) is an isomorphism. \(\square\)

The construction of \(Q^{(D)}\) commutes with base change if \(Q^{(D)} \to X\) is flat and reduced.

Lemma 2.1.8. Suppose that the diagram (2.7) is cartesian and \(D' = D \times_X X'\). Assume that one of the following conditions is satisfied:

(i) \(X'\) is normal, \(Q \to X\) is smooth and \(Q^{(D)} \to X\) is flat and reduced.

(ii) \(X' \to X\) is smooth.

Then the square

\[
\begin{array}{ccc}
Q^{(D)} & \longleftarrow & Q^{(D')} \\
\downarrow & & \downarrow \\
X & \longleftarrow & X'
\end{array}
\]

(2.14)

is cartesian.

Proof. By Lemma 2.1.7, \(Q^{(D')}\) is the normalization of \(Q^{(D)} \times_X X'\). If the condition (i) is satisfied, then \(Q^{(D)} \times_X X'\) is normal by Corollary 1.2.3.1. If \(X' \to X\) is smooth, then \(Q^{(D)} \times_X X'\) is smooth over \(Q^{(D)}\) and is normal. Hence the square (2.14) is cartesian in both cases. \(\square\)

We study the dependence on \(D\) and show that the canonical morphism contracts the closed fiber.
Lemma 2.1.9. Suppose $X = X', Y = Y'$ and $Q = Q'$ and that $D_1 = D' - D$ is an effective Cartier divisor of $X$. Then, the morphism $Q^{[D]} \to Q^{[D]}$ (resp. $Q^{[D']}$ to $Q^{(D)}$) induces a morphism $Q^{[D']} \times_Q D_{1,Y} \to D_{1,Y} \subset Y \subset Q^{[D]}$ (resp. $Q^{(D')} \times_Q D_{1,Y} \to Q^{(D)} \times_Q D_{1,Y} \subset Q^{(D)}$).

Proof. We consider the immersion $Y \to Q^{[D]}$ lifting $Y \to Q$. Then, the morphism $Q^{[D']} \to Q^{[D]}$ induces an isomorphism $Q^{[D']} \to (Q^{[D]})^{[D_1]}$ to the dilatation $(Q^{[D]})^{[D_1]}$ of $Q^{[D]}$ for $Y \to Q^{[D]}$ and $D_1 \subset X$. Hence the morphism (2.3) defines a morphism $Q^{[D']} \times_Q D_{1,Y} \to D_{1,Y}$. The assertion for $Q^{(D')}$ follows from this.

\[\square\]

2.2 Dilatations and complete intersection

We give a condition for the right square in (2.7) to be cartesian.

Lemma 2.2.1. Let $S$ be a noetherian scheme and let $Q \to P$ be a quasi-finite morphism of smooth schemes of finite type over $S$. If $Q \to P$ is flat on dense open subschemes, then $Q \to P$ is flat and locally of complete intersection of relative virtual dimension $0$.

Proof. Let $U \subset P$ and $V \subset Q$ be dense open subschemes such that $V \to U$ is flat. Then the relative dimension of $V \to S$ is the same as that of $U \to S$. Hence, we may assume that the relative dimensions of $P \to S$ and $Q \to S$ are the same integer $n$.

The morphism $Q \to P$ is the composition of the graph $Q \to Q \times_S P$ and the projection $Q \times_S P \to P$. For every point $x \in P$, the fiber $Q \times_P x \to Q \times_S x$ is a regular immersion of codimension $n$. Hence by [7, Proposition (15.1.16) c)⇒b)] applied to the immersion $Q \to Q \times_S P$ over $P$, the immersion $Q \to Q \times_S P$ is also a regular immersion of codimension $n$ and $Q \to P$ is flat.

Lemma 2.2.2. Let $S$ be a noetherian scheme and let $Y \to X$ be a morphism of schemes of finite type over $S$.

1. Suppose that there exists a cartesian diagram

\[
\begin{array}{ccc}
Q & \xleftarrow{\square} & Y \\
\downarrow & & \downarrow \\
P & \xleftarrow{} & X
\end{array}
\]

(2.15)

of schemes of finite type over $S$ satisfying the following conditions:

$P$ and $Q$ are smooth over $S$ and $Q \to P$ is quasi-finite and is flat on dense open subschemes. The horizontal arrows are closed immersions.

Then $Y \to X$ is quasi-finite, flat and locally of complete intersection of relative virtual dimension $0$.

2. Conversely, suppose that $Y \to X$ is finite (resp. quasi-finite) and locally of complete intersection of relative virtual dimension $0$. Then $Y \to X$ is flat and, locally on $X$ (resp. locally on $X$ and on $Y$), there exists a cartesian diagram (2.15) satisfying the following conditions:

$P$ and $Q$ are smooth of the same relative dimension over $S$ and $Q \to P$ is quasi-finite and flat. The horizontal arrows are closed immersions.
Proof. 1. By Lemma 2.2.1, the quasi-finite morphism \( Q \to P \) is flat and locally of complete intersection. Hence \( Y \to X \) is also quasi-finite, flat and locally of complete intersection of relative virtual dimension 0.

2. Since the assertion is local, we may assume that \( S, X \) and \( Y \) are affine. Take a closed immersion \( Q_1 = \mathbb{A}^m_X \to Y \). Since the immersion \( Y \to Q_1 \) is a regular immersion of codimension \( m \) and since \( Y \to X \) is finite (resp. quasi-finite), after shrinking \( X \) (resp. \( Q \) and \( Y \)), we may assume that the ideal defining \( Y \subset Q_1 \) is generated by \( m \) sections \( f_1, \ldots, f_m \) of \( \mathcal{O}_{Q_1} \). Also take a closed immersion \( P_1 = \mathbb{A}^m_S \to X \) and an open subscheme \( Q \subset \mathbb{A}^m_{P_1} \) to obtain a cartesian diagram

\[
\begin{array}{ccc}
Q & \leftarrow & Q_1 \\
\downarrow & & \downarrow \\
P_1 & \leftarrow & X.
\end{array}
\]

(2.16)

Taking sections \( \tilde{f}_1, \ldots, \tilde{f}_m \) of \( \mathcal{O}_Q \) lifting \( f_1, \ldots, f_m \) after shrinking \( Q \) if necessary, define a morphism \( Q \to P = \mathbb{A}^m_{P_1} \). Then, we obtain a cartesian diagram

\[
\begin{array}{ccc}
Q & \leftarrow & Q_1 \\
\downarrow & & \downarrow \\
P & \leftarrow & \mathbb{A}^m_X \\
\end{array}
\]

where the lower right horizontal arrow \( \mathbb{A}^m_X \to X \) is the 0-section.

The schemes \( P = \mathbb{A}^{n+m}_S \) and \( Q \subset \mathbb{A}^{n+m}_S \) are smooth over \( S \). Since \( Y \to X \) is quasi-finite, after replacing \( Q \) by a neighborhood of \( Y \) if necessary, the morphism \( Q \to P \) is quasi-finite. Since \( Q \) and \( P \) are smooth of the same relative dimension over \( S \), the morphism \( Q \to P \) is flat on dense open subschemes. By Lemma 2.2.1, the quasi-finite morphism \( Q \to P \) is flat and hence \( Y \to X \) is also flat.

We give examples of construction of the diagram (2.15).

Example 2.2.3. Assume that \( X \) and \( Y \) are schemes of finite type over a noetherian scheme \( S \).

1. Assume \( X = \text{Spec} A \) and \( Y = \text{Spec} B \) are affine. Let \( A[T_1, \ldots, T_n]/(f_1, \ldots, f_n) \to B \) be an isomorphism and define a morphism \( Q = \mathbb{A}^n_X = \text{Spec} A[T_1, \ldots, T_n] \to P = \mathbb{A}^n_X \) by \( f_1, \ldots, f_n \). Then, we obtain a cartesian diagram (2.15) by defining the section \( X \to P = \mathbb{A}^n_X \) to be the 0-section.

2. Assume that \( X \) and \( Y \) are smooth over a noetherian scheme \( S \). Then, we obtain a cartesian diagram

\[
\begin{array}{ccc}
Y & \rightarrow & Q = Y \times_S X \\
\downarrow & & \downarrow \\
X & \rightarrow & P = X \times_S X.
\end{array}
\]

Assume that \( Q^{(D)} \to P^{(D)} \) is étale on a neighborhood of \( Q^{(D)} \times_X D \). Let \( \bar{x} \) be a geometric point of \( D \) and let \( 0_{\bar{x}} \) denote the geometric point above the origin of the vector space \( P^{(D)}_{\bar{x}} \) over \( \bar{x} \). Then, since \( Q^{(D)}_{\bar{x}} \to P^{(D)}_{\bar{x}} \) is finite étale, we have an action of the fundamental group \( \pi_1(P^{(D)}_{\bar{x}}, 0_{\bar{x}}) \) on \( Y^{(D)}_{\bar{x}} = Q^{(D)}_{\bar{x}} \times_{P^{(D)}_{\bar{x}}} 0_{\bar{x}} \). The action on \( Y^{(D)}_{\bar{x}} \) is compatible
with the canonical mapping $\pi(D) \to \pi_0(Q(D))$ with respect to the trivial action on $\pi_0(Q(D))$ and is transitive on the inverse image of each element of $\pi_0(Q(D))$.

Since $Q(D) \to P(D) \times_P Q$ is an isomorphism by Lemma 2.1.7, for a geometric point $\bar{y}$ of $Y_\bar{s}$ and for the geometric point $0\bar{y}$ of $Q(D)$ above $P(D)$, we have canonical isomorphisms $Q(D) = Q(D) \times_Q \bar{y} \to P(D)$ and $\pi_1(Q(D), 0\bar{y}) \to \pi_1(P(D), 0\bar{y})$. The action of $\pi_1(P(D), 0\bar{y})$ on $Y(D)$ is compatible with the action of $\pi_1(Q(D), 0\bar{y})$ on $Y(D) \times_Y \bar{y}$. For a morphism $Q' \to Q$, the canonical morphism $\pi_1(Q(D), 0\bar{y}) \to \pi_1(Q(D), 0\bar{y})$ is compatible with the actions on $Y(D) \times_Y \bar{y}$.

We study the relation between the étaleness of $Q(D) \to P(D)$ and the annihilator of $O_Y(D) \otimes_O Y(X/Q)$.

**Lemma 2.2.4.** Let

\[
\begin{array}{ccc}
Q & \longrightarrow & Y \\
\downarrow & & \downarrow \\
P & \longleftarrow & X
\end{array}
\]

be a cartesian diagram of separated schemes of finite type over $X$. Assume that $P$ and $Q$ are smooth over $X$ and that the vertical arrows are quasi-finite and flat.

Assume that there exists an effective Cartier divisor $D_1 \subset D = D_1 + D_0$ of $X$ such that $O_Y(D) \otimes_O Y(X/Q)\Omega^1_{Y/X}$ is annihilated by $I_{D_1} \subset O_X$ and that we have an equality $D_0 = D$ of underlying sets. Then, there exists an open neighborhood $W \subset Q(D)$ of $Q(D) \times_X D$ such that $Q(D) \to P(D)$ is étale on $W = (Q(D) \times_X D)$.

**Proof.** It suffices to show that each irreducible component $Z \subset Q(D)$ of the inverse image of the support of $\Omega^1_{Q/P}$ is either a subset of $Q(D) \times_X D$ or does not meet $Q(D) \times_X D$, since $Q(D) \to Q$ is an isomorphism on the complement of the inverse images of $D$. Assume that $Z$ is not a subset of $Q(D) \times_X D$ but does meet $Q(D) \times_X D$ and regard $Z$ as an integral closed subscheme of $Q(D)$. Then, $D \times_X Z \subset Z$ is a non-empty effective Cartier divisor.

Since the assertion is étale local on $Y$, we may assume that $Y \to X$ is faithfully flat and finite. Let $T_0 \subset Z \times_Y Y(D)$ be the closure of the complement $Z \times_X Y(D) \subset D \times_X (Z \times_Y Y(D))$ and $T$ be its normalization. Then, since $Y \to X$ is finite surjective, $T \to Z$ is also finite surjective. Hence $D_T = D \times_X T \subset T$ is a non-empty effective Cartier divisor.

By the assumption that $O_{Y(D)} \otimes_O Y(X/Q)\Omega^1_{Y/X}$ is annihilated by $I_{D_1} \subset O_X$, the $O_T$-module $O_T \otimes_O Y(X/Q)\Omega^1_{Y/X}$ is annihilated by $I_{D_1} \cdot O_T$. Since $D_T$ is a scheme over $Q(D) \times_X D$, we have an isomorphism $O_{D_T} \otimes_O Q_{Q/P} \Omega^1_{Q/P} \to O_{D_T} \otimes_O Y(X/Q)\Omega^1_{Y/X}$ by Lemma 2.1.3.1. Thus $O_{D_T} \otimes_O Q_{Q/P} \Omega^1_{Q/P}$ is also annihilated by $I_{D_1} \cdot O_{D_T}$. Since $D = D_1 + D_0$, this means an inclusion $I_{D_1} \cdot O_{D_T} \otimes_Q Q_{Q/P} \Omega^1_{Q/P} \subset I_{D_0} \cdot I_{D_1} \cdot O_T \otimes_Q Q_{Q/P} \Omega^1_{Q/P}$. By Nakayama’s lemma, we have $I_{D_1} \cdot O_T \otimes_Q Q_{Q/P} \Omega^1_{Q/P} = 0$ on a neighborhood of $D_0 \times_X T$.

Since $Z$ is a subset of the inverse image of support of $\Omega^1_{Q/P}$, the annihilator ideal of $O_T \otimes_Q Q_{Q/P}$ is 0. This contradicts to that $D_0 \times_X T = D_T$ is non-empty. \qed

**Lemma 2.2.5.** Assume $X$ is normal and let

\[
\begin{array}{ccc}
Q & \longrightarrow & Y \\
\downarrow & & \downarrow \\
P & \longleftarrow & X
\end{array}
\]
be a cartesian diagram of separated schemes of finite type over $X$. Assume that $P$ and $Q$ are smooth over $X$ and that the vertical arrows are quasi-finite and flat.

Let $Y_0$ be a closed subscheme of $Y$ étale over $X$ satisfying an equality $D_{Y_0} = D_Y$ of underlying sets and let $I_0 \subset \mathcal{O}_D$ be the nilpotent ideal defining $D_{Y_0} \subset D_Y$. Let $n \geq 1$ be an integer satisfying $I_0^n = 0$ and let $D_0 \subset D$ be an effective Cartier divisor on $X$ satisfying $nD_0 \leq D$.

Assume that $Y^{(D)} = Y \times_{Q^{(D)}} Q^{(D)}$ is étale over $X$. Then $\mathcal{O}_{Y^{(D)}} \otimes \mathcal{O}_Y \Omega^1_{Y/X}$ is annihilated by the ideal $I_{D-D_0} \subset \mathcal{O}_X$ defining $D-D_0 \subset X$.

**Proof.** Let $I \subset I_0 \subset \mathcal{O}_Q$ and $I_D \subset I_{D_0} \subset \mathcal{O}_X$ be the ideals defining the closed subschemes $Y_0 \subset Y \subset Q$ and $D_0 \subset D \subset X$. Let $Y_0^{(n)} \subset Q$ denote the closed scheme defined by the ideal $I_0^n \subset \mathcal{O}_Q$. Let $Q^{[D_0,Y_0]} \to Q$ denote the dilatation for $Y_0 \to Q$ and $D_0$. We also define a dilatation $Q^{[nD_0,Y_0^{(n)}]} \to Q$ for $Y_0^{(n)} \to Q$ and $nD_0$.

Since $Y_0$ is étale over $X$, the scheme $Q^{[D_0,Y_0]}$ is smooth over $X$ by Example 2.1.2.1 and equals its normalization $Q^{(D_0,Y_0)}$. The canonical morphism $Q^{[D_0,Y_0]} \to Q^{[nD_0,Y_0^{(n)}]}$ is finite and induces an isomorphism $Q^{(D_0,Y_0)} \to Q^{[nD_0,Y_0^{(n)}]}$ on the normalizations.

By the assumptions $I_0^n = 0$ and $nD_0 \leq D$, we have $I_0^n \subset I + I_D \subset I + I_{nD_0}$. Hence we have a morphism $Q^{[nD_0]} \to Q^{[nD_0,Y_0^{(n)}]}$. Further by $nD_0 \leq D$, we obtain a morphism $Q^{(D)} \to Q^{(D_0,Y_0)}$ of normalizations.

The dilatation $P^{(D)}$ of $P$ for the section $X \to P$ and $D$ is smooth over $X$ by Example 2.1.2.1 and hence is equal to the normalization $P^{(D)}$. Since $Y^{(D)} \to X$ is étale and since the diagram

$$
\begin{array}{ccc}
Y^{(D)} & \longrightarrow & Q^{(D)} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Q^{[D]} \longrightarrow Q \\
\downarrow & & \downarrow \downarrow \\
X & \longrightarrow & P^{(D)} \longrightarrow P \\
\end{array}
$$

is cartesian by Lemma 2.1.7.2, the quasi-finite morphism $Q^{(D)} \to P^{(D)}$ of normal schemes is étale on a neighborhood $W \subset Q^{(D)}$ of $Y^{(D)}$ by [7 Théorème (18.10.16)].

The commutative diagram

$$
\begin{array}{ccc}
Q^{(D)} & \longrightarrow & Q^{(D_0,Y_0)} \longrightarrow Q \\
\downarrow & & \downarrow \\
P^{(D)} & \longrightarrow & P
\end{array}
$$

of schemes defines a commutative diagram

$$
\begin{array}{ccc}
\Omega^1_{Q^{(D)}/X} \longrightarrow \Omega^1_{Q^{(D_0,Y_0)}/X} & \longrightarrow & \Omega^1_{Q/X} \\
\Omega^1_{P^{(D)}/X} \longrightarrow \Omega^1_{P/X}
\end{array}
$$

of locally free $\mathcal{O}_W$-modules. Since $Q^{(D)} \to P^{(D)}$ is étale on $W$, the left vertical arrow is an isomorphism.
Since $X \to X$ and $Y_0 \to X$ are étale, the lower horizontal arrow (resp. the upper right horizontal arrow) induces an isomorphism $\mathcal{O}_W \otimes \Omega^1_{P/X} \to \mathcal{I}_D \cdot \mathcal{O}_W \otimes \Omega^1_{P(D)/X}$ (resp. $\mathcal{O}_W \otimes \Omega^1_{Q/X} \to \mathcal{I}_{D_0} \cdot \mathcal{O}_W \otimes \Omega^1_{Q(D_0,Y_0)/X}$). Hence $\mathcal{I}_{D-D_0} \cdot \mathcal{O}_W \otimes \Omega^1_{Q/X} = \mathcal{I}_D \cdot \mathcal{O}_W \otimes \Omega^1_{Q(D_0,Y_0)/X}$ is contained in the image of $\mathcal{O}_W \otimes \Omega^1_{P/X}$. Or equivalently, $\mathcal{O}_W \otimes \Omega^1_{Q/P}$ is annihilated by $\mathcal{I}_D - D_0$. Hence its pull-back $\mathcal{O}_{Y(D)} \otimes \mathcal{O}_Y \Omega^1_{Y/X}$ is also annihilated by $\mathcal{I}_D - D_0$.

\section{Ramification}

\subsection{Ramification of quasi-finite schemes}

Let $X$ be a normal noetherian scheme and $D$ be an effective Cartier divisor of $X$. Let $Y$ be a quasi-finite scheme over $X$ such that $D_Y = D \times_X Y \subset Y$ is a Cartier divisor.

Locally on $X$, there exists a smooth scheme $Q$ over $X$ and a closed immersion $Y \to Q$ over $X$. Then, by Proposition 2.1.5 and Corollary 2.1.6, the scheme $Y(D)$ over $Q$ defined as $Y \times Q(D)$ is canonically independent of $Q$. Hence a finite scheme $Y(D)$ over $Y$ is defined by patching. Similarly, for a geometric point $\bar{x}$ above a point $x \in D$, the set $\pi_0(Q(D)_{\bar{x}})$ of connected components of the geometric fiber is canonically independent of $Q$.

**Definition 3.1.1.** Let $X$ be a normal noetherian scheme and $D$ be an effective Cartier divisor of $X$. Let $Y$ be a quasi-finite scheme over $X$ such that $D_Y = D \times_X Y \subset Y$ is a Cartier divisor and let $\bar{Y}$ be the normalization of $Y$. Let $\bar{x}$ be a geometric point above a point $x \in D$.

By taking a closed immersion $Y \to Q$ to a smooth scheme $Q$ over $X$ defined on a neighborhood of $x$, we define finite sets $F^D_x(Y/X)$ and $F^{D+}_x(Y/X)$ by

\begin{equation}
F^D_x(Y/X) = \pi_0(Q(D)_{\bar{x}}), \quad F^{D+}_x(Y/X) = Y(D)_{\bar{x}}
\end{equation}

equipped with canonical mappings

\begin{equation}
\begin{array}{ccc}
\bar{Y} & \xrightarrow{\varphi^D_x} & F^D_x(Y/X) \\
\nearrow \varphi^D_x & & \downarrow \varphi^D_x \\
F^D_x(Y/X) & \rightarrow & Y_{\bar{x}}
\end{array}
\end{equation}

induced by the morphisms

\begin{equation}
\begin{array}{ccc}
\bar{Y} & \xrightarrow{Y(D)} & Y(D) \\
\downarrow & & \downarrow \\
Q(D) & \rightarrow & Q
\end{array}
\end{equation}

We consider a commutative diagram

\begin{equation}
\begin{array}{ccc}
Y' & \longrightarrow & X' \supset D' \supset D \times_X X' \longleftarrow \bar{x}' \\
\downarrow & & \downarrow & & \downarrow \\
Y & \longrightarrow & X \supset D \longleftarrow \bar{x}
\end{array}
\end{equation}
of noetherian schemes. We assume that $X'$ is normal, $D' \subset X'$ is an effective Cartier divisor, $Y'$ is quasi-finite over $X'$ and that $D'_Y, \subset Y'$ is an effective Cartier divisor. Then, the commutative diagram (2.3) induces a commutative diagram

$$
\begin{align*}
\bar{Y}'_{x'} & \xrightarrow{\varphi_{x'}^{D'}} F_{x'}^{D'}(Y'/X') \longrightarrow F_{x'}^{D'}(Y'/X') \longrightarrow \bar{Y}'<x'>
\
\bar{Y}_{x} & \xrightarrow{\varphi_x^{D'}} F_{x}^{D'}(Y/X) \longrightarrow F_{x}^{D'}(Y/X) \longrightarrow \bar{Y}_x.
\end{align*}
$$

(3.4)

For effective Cartier divisors $D$ and $D'$ of a scheme $X$ defined by the ideal sheaves $\mathcal{I}_D, \mathcal{I}_{D'} \subset \mathcal{O}_X$ and for $x \in D$, we write $D < D'$ at $x$ if we have a strict inclusion $\mathcal{I}_{D,x} \supset \mathcal{I}_{D',x}$. If $X = X', Y = Y', \bar{x} = \bar{x}'$ and if $D < D'$ at the image $x$ of $\bar{x}$ as Cartier divisors, further we have an arrow $F_{x}^{D'}(Y/X) \rightarrow F_{x}^{D}(Y/X)$ making the two triangles obtained by dividing the middle square commutative by Lemma 2.1.9.

**Proposition 3.1.2.** Assume that $Y \rightarrow X$ is quasi-finite, flat and locally of complete intersection and that the normalization $\bar{Y}$ of $Y$ is étale over $X$.

1. The arrows in

$$
\begin{align*}
\bar{Y}_{x} & \xrightarrow{\varphi_x^{D}} F_{x}^{D}(Y/X) \\
F_{x}^{D}(Y/X) & \xrightarrow{\varphi_x^{D}} \bar{Y}_x
\end{align*}
$$

(3.5)

are surjections.

2. Let $Y' \rightarrow Y$ be a surjective morphism locally of complete intersection of quasi-finite and flat schemes over $X$. Assume that the normalization $\bar{Y}'$ of $Y'$ is étale over $X$. Then, the diagram

$$
\begin{align*}
\bar{Y}'_{x} & \xrightarrow{\varphi_{x}^{D}} F_{x}^{D}(Y'/X) \longrightarrow F_{x}^{D}(Y'/X) \longrightarrow \bar{Y}'_x
\
\bar{Y}_{x} & \xrightarrow{\varphi_{x}^{D}} F_{x}^{D}(Y/X) \longrightarrow F_{x}^{D}(Y/X) \longrightarrow \bar{Y}_x
\end{align*}
$$

(3.6)

is a cocartesian diagram of surjections.

**Proof.** By replacing $X$ by the strict localization $X_{(\bar{x})}$, we may assume that $\bar{x} \rightarrow X$ is a closed immersion and that $Y \rightarrow X$ is finite.

1. By Lemma 2.2.2.2, we may assume that there exists smooth schemes $P$ and $Q$ over $X$ and a cartesian diagram

$$
\begin{align*}
Y & \longrightarrow Q \\
\downarrow & \phantom{\downarrow} \phantom{\downarrow} \\
X & \longrightarrow P
\end{align*}
$$

of schemes over $X$ such that the horizontal arrows are closed immersions and that the vertical arrows are quasi-finite and flat. We verify that the diagram

$$
\begin{align*}
Q^{(D)}_{x} & \longrightarrow Q^{(D)} \leftarrow Y^{(D)} \leftarrow \bar{Y} \\
\downarrow & \phantom{\downarrow} \phantom{\downarrow} \\
P^{(D)}_{x} & \longrightarrow P^{(D)} \leftarrow X \phantom{\downarrow} \phantom{\downarrow} X
\end{align*}
$$

(3.6)
satisfies the assumptions in Corollary 1.1.6. Since $P^{[D]} \to X$ is smooth, we have $P^{(D)} = P^{[D]}$. By Lemma 2.1.7.2, the diagram

\[
\begin{array}{ccc}
Q & \leftarrow & Q^{[D]} \leftarrow Y \\
\downarrow & & \downarrow \\
P & \leftarrow & P^{[D]} \leftarrow X
\end{array}
\]

is cartesian. Hence the middle square in (3.6) is also cartesian.

The diagram (3.6) satisfies the finiteness assumption in Corollary 1.1.6 by Lemma 2.1.7.1. Since $X = X_{(\bar{x})}$ is strictly local, the assumption that the canonical mapping $\bar{x} \to \pi_0(\bar{X})$ is a bijection is satisfied. Since $P^{(D)}$ is a vector space over $\bar{x}$ and is connected, the mapping $\bar{x} \to P^{(D)} \cap X \to \pi_0(P^{(D)})$ are bijections of sets consisting of single elements. We may assume that the finite étale morphism $\bar{Y} \to X$ is surjective since if otherwise the assertion is trivial. Hence by Corollary 1.1.6.2 (resp. 3), the mapping $\bar{Y} \bar{x} \to \bar{Y}^{(D)}\bar{x} = Y^{(D)}\bar{x}$ (resp. $F^{D+}(Y/X) = Y^{(D)}\bar{x} \to \pi_0(Q^{(D)}\bar{x}) = F^{D}(Y/X)$) is surjective.

Similarly, applying Corollary 1.1.6.2 to the diagram

\[
\begin{array}{ccc}
Y_{\bar{x}} & \longrightarrow & Y \leftarrow Y \leftarrow \bar{Y} \\
\downarrow & \square & \downarrow & \square & \downarrow & \square \\
\bar{x} & \longrightarrow & X \leftarrow X \longrightarrow X
\end{array}
\]

we see that $\bar{Y}_{\bar{x}} \to Y_{\bar{x}}$ is a surjection.

2. By Lemma 2.2.2.2, we may assume that there exists smooth schemes $Q$ and $Q'$ over $X$ and a cartesian diagram

\[
\begin{array}{ccc}
Y' & \longrightarrow & Q' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Q
\end{array}
\]

of schemes over $X$ such that the horizontal arrows are closed immersions and that the vertical arrows are quasi-finite and flat.

We verify that the diagram

\[
\begin{array}{ccc}
Q^{(D)}_{\bar{x}} & \longrightarrow & Q'^{(D)} \leftarrow Y'^{(D)} \leftarrow \bar{Y}' \\
\downarrow & \square & \downarrow & \square & \downarrow & \square \\
Q^{(D)}_{\bar{x}} & \longrightarrow & Q^{(D)} \leftarrow Y^{(D)} \leftarrow \bar{Y}
\end{array}
\]

satisfies the assumptions in Corollary 1.1.6. The middle square is cartesian by Lemma 2.1.7.2. The finiteness assumption in Corollary 1.1.6 is satisfied by Lemma 2.1.7.1. Since the finite étale covering $\bar{Y} \to X$ is split and $X$ is connected, the assumption that the canonical mapping $\bar{Y}_{\bar{x}} \to \pi_0(\bar{Y})$ is a bijection is satisfied. By 1, $\bar{Y}_{\bar{x}} \to Y^{(D)}_{\bar{x}} \to \pi_0(Q^{(D)}_{\bar{x}})$ are surjective. We may assume that $Y$ and $Y'$ are finite over $X$. Since $Y' \to Y$ is surjective, the morphism $\bar{Y}' \to \bar{Y}$ of finite étale schemes over $X$ is also surjective. Hence by Corollary 1.1.6.2 (resp. 3), the right square (resp. the middle square) of (3.5) is a cocartesian diagram of surjections.
Similarly, applying Corollary 3.1.6.2 to the diagram

\[
\begin{array}{ccc}
Y'_x & \longrightarrow & Y' \\
\downarrow & \downarrow & \downarrow \\
Y_x & \longrightarrow & Y
\end{array}
\]

we see that the big rectangle in (3.5) is a cocartesian diagram of surjections. \hfill \Box

**Corollary 3.1.3.** Assume that $Y \to X$ is locally of complete intersection. Let $P$ and $Q$ be smooth schemes over $X$ and let

\[
\begin{array}{ccc}
Y & \longrightarrow & Q \\
\downarrow & \downarrow & \downarrow \\
X & \longrightarrow & P
\end{array}
\]

be a cartesian diagram of schemes over $X$ such that the horizontal arrows are closed immersions and that the vertical arrows are quasi-finite and flat. Then, the mapping $\bar{Y}_x \to F_x^{D^+}(Y/X)$ is an injection on the inverse image of $y \in Y$ if and only if $Q^{(D)} \to P^{(D)}$ is étale on the inverse image of $y$ by $Y^{(D)} \to Y$.

**Proof.** Since the assertion is étale local, we may assume that $Y \to X$ and $Q \to P$ are finite and that $y$ is the unique point of the inverse image of $x$. Then, by Proposition 3.1.2.1, $\bar{Y}_x \to Y_x^{(D)} = F_x^{D^+}(Y/X) \subset Q_x^{(D)}$ is a bijection of finite sets. Hence $Q \to P$ is étale at $x$ by [7, Théorème (18.10.16)]. \hfill \Box

**Definition 3.1.4.** Let $X$ be a normal noetherian scheme and $U \subset X$ be a dense open subscheme. Let $Y$ be a quasi-finite scheme over $X$ such that $V = U \times_X Y \to U$ is étale. Let $D$ be an effective Cartier divisor of $X$ such that $U \cap D$ is empty and that $D_Y = D \times_X Y$ is an effective Cartier divisor.

1. For $x \in D$, we consider the following condition on $X, Y$ and $D$:
   (RF) There exist an open neighborhood $W$ of $x \in X$, a smooth scheme $Q$ over $W$ and a closed immersion $Y \times_X W \to Q$ such that the normalization $\bar{Y}$ of $Y$ is étale over $W$ and that the normalization $Q^{(D)}$ of the dilatation $Q^{[D]}$ is flat and reduced over $W$. If the condition (RF) is satisfied at every $x \in D$, we say that $Y$ over $X$ satisfies the condition (RF) for $D$.

2. Let $x \in D$ and assume that $Y$ over $X$ satisfies the condition (RF) for $D$ at $x$.
   Let $y$ be a point of $\bar{Y} \times_X x \subset \bar{Y} \times_X D$. We say that the ramification of $Y \to X$ is bounded by $D$ (resp. by $D^+$) at $y$, if the mapping $\varphi^D_x: \bar{Y}_x \to F_x^D(Y/X)$ (resp. $\varphi_x^{D^+}: \bar{Y}_x \to F_x^{D^+}(Y/X)$) is an injection on the inverse image of $y$.
   We say that the ramification of $Y \to X$ is bounded by $D$ (resp. by $D^+$) at $x$, if the mapping $\bar{\varphi}^D_x: \bar{Y}_x \to F_x^D(Y/X)$ (resp. $\bar{\varphi}^{D^+}_x: \bar{Y}_x \to F_x^{D^+}(Y/X)$) is an injection.
   If the ramification is bounded by $D$, it is bounded by $D^+$. We show that the condition (RF) is independent of the choice of $Q$.

**Lemma 3.1.5.** Let $X$ be a normal noetherian scheme and $U \subset X$ be a dense open subscheme. Let $Y$ be a quasi-finite scheme over $X$ such that $V = U \times_X Y \to U$ is étale. Let $D$ be an effective Cartier divisor of $X$ such that $U \cap D$ is empty and that $D_Y \subset Y$ is an effective Cartier divisor. Let $x \in D$. 27
1. Assume that \( Y \) over \( X \) satisfies (RF) for \( D \) at \( x \). Let \( W \subset X \) be an open neighborhood of \( x \), \( Q \) be a smooth scheme over \( W \) and \( Y \times_X W \to Q \) be a closed immersion. Then, there exists an open neighborhood \( W' \subset W \) of \( x \), such that \( (Q \times_W W')(D \times_X W') \to W' \) is flat and reduced.

2. Let \( X' \to X \) be a morphism of normal noetherian scheme such that \( U' = U \times_X X' \) is a dense open subscheme and that \( D'_Y = D_Y \times_X X' \subset Y' = Y \times_X X' \) is an effective Cartier divisor. Let \( x' \) be a point of \( D' = D \times_X X' \) above \( x \). We consider the following conditions:

   (1) \( Y \) over \( X \) satisfies (RF) for \( D \) at \( x \).
   (2) \( Y' \) over \( X' \) satisfies (RF) for \( D' \) at \( x' \).

We have \((1) \Rightarrow (2)\). Conversely, if \( X' \to X \) is smooth at \( x' \), we have \((2) \Rightarrow (1)\).

**Proof.**
1. Set \( D_W = D \times_X W \). After shrinking \( W \) if necessary, we may assume that there exist a smooth scheme \( Q_0 \) over \( W \) and a closed immersion \( Y \times_X W \to Q_0 \) such that \( Q_0(D_W) \to W \) is flat and reduced. Since \( Q(D_W) \leftarrow (Q \times_W Q_0)(D_W) \to Q_0(D_W) \) are smooth by Proposition 2.1.5, the assertion follows.

2. \((1) \Rightarrow (2)\): This follows from Lemma 2.1.8.

\((2) \Rightarrow (1)\): After shrinking \( X' \) if necessary, we may assume that \( X' \to X \) is smooth. Let \( W \) be an open neighborhood of \( x \), let \( Y \times_X W \to Q \) be a closed immersion to a smooth scheme \( Q \) over \( W \) and \( W' = W \times_X X' \). Then the morphism \( (Q \times_W W')(D' \times_X W') \to Q(D' \times_X W) \times_W W' \) is an isomorphism by Lemma 2.1.8. Hence the assertion follows.

**Lemma 3.1.6.** Let \( X \) be a normal noetherian scheme and \( U \subset X \) be a dense open subscheme. Let \( Y \) be a quasi-finite scheme over \( X \) such that \( V = U \times_X Y \to U \) is étale. Let \( D \subset D' \) be effective Cartier divisors of \( X \) such that \( U \cap D' \) is empty and that \( D'_Y \subset Y \) is an effective Cartier divisor. Let \( x \in D \) and assume that \( Y \) over \( X \) satisfies (RF) for \( D \) and \( D' \) at \( x \).

Let \( y \in Y \) be a point above \( x \). If the ramification of \( Y \) over \( X \) is bounded by \( D^+ \) at \( y \) and if \( D < D' \) at \( x \), then the ramification of \( Y \) over \( X \) is bounded by \( D' \) at \( y \).

**Proof.** It follows from Lemma 2.1.9.

**Lemma 3.1.7.** Let \( X \) be a normal noetherian scheme and \( U \subset X \) be a dense open subscheme. Let \( Y \) be a quasi-finite scheme over \( X \) such that \( V = U \times_X Y \to U \) is étale. Let \( D \) be an effective Cartier divisor of \( X \) such that \( U \cap D \) is empty and that \( D_Y \subset Y \) is an effective Cartier divisor. Assume that \( Y \) over \( X \) satisfies the condition (RF) for \( D \).

Let \( S \subset D_Y \) (resp. \( S^+ \subset D_Y \)) denote the subset consisting of points \( y \in D_Y \) where the ramification of \( Y \to X \) is bounded by \( D \) (resp. by \( D^+ \)).

1. We have \( S \subset S^+ \).
2. The subset \( S \subset D_Y \) is closed and the subset \( S^+ \subset D_Y \) is open.

**Proof.**
1. It follows from the commutative diagram (3.1).
2. By Lemma 1.1.3 applied to \( Y \to Q(D) \), we see that \( S \) is closed. Similarly, by Lemma 1.1.4 applied to \( Y \to Y(D) \) we see that \( S^+ \) is open.

**Proposition 3.1.8.** Let \( X \) be a normal noetherian scheme and \( U \subset X \) be a dense open subscheme. Let \( Y \) be a quasi-finite scheme over \( X \) such that \( V = U \times_X Y \to U \) is étale. Let \( D \) be an effective Cartier divisor of \( X \) such that \( U \cap D \) is empty and that \( D_Y \subset Y \) is an effective Cartier divisor.
Let $C$ be a semi-stable curve over $X$ such that $C_U = C \times_X U \to U$ is smooth. Let $x \in X$ be a point of $D$ and $z \in C$ be a singular point of the fiber $C_x$. Assume that there exist two irreducible components $C_1$ and $C_2$ of the fiber $C_x$ meeting at $z$ and let $\zeta_1$ and $\zeta_2$ be their generic points. Let $D_1 \subset D_2$ be effective Cartier divisors on $X$ and let $D \subset C$ be an effective Cartier divisor such that $D_1 < D_2$ at $x$ and that $\tilde{D} = D_1 \times_X C = D_{1,C}$ on a neighborhood of $\zeta_i$ for $i = 1, 2$.

Assume that $Y_C = Y \times_X C$ over $C$ satisfies the condition (RF) for $\tilde{D}$ at $z$.

1. $Y$ over $X$ satisfies the condition (RF) for $D_1$ and $D_2$ at $x$.
2. We have a commutative diagram

$$
(3.7) \quad \begin{array}{c}
F_{\tilde{D}_2}^{D_2}(Y/X) \quad F_{\tilde{D}}^{\tilde{D}}(Y_C/C) \quad F_{\tilde{D}_1}^{D_1}(Y/X) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
F_{\tilde{D}_2}(Y/X) \quad F_{\tilde{D}}(Y_C/C) \quad F_{\tilde{D}_1}(Y/X).
\end{array}
$$

3. The lower left horizontal arrow $F_{\tilde{D}_2}^{D_2}(Y/X) \to F_{\tilde{D}}^{\tilde{D}}(Y_C/C)$ in (3.7) is an injection. The upper right horizontal arrow $F_{\tilde{D}}^{\tilde{D}}(Y_C/C) \to F_{\tilde{D}_1}^{D_1}(Y/X)$ in (3.7) is an injection on the image of $Y_{\tilde{z}}$.

Proof. 1. Since $\zeta_1$ and $\zeta_2$ are contained in any open neighborhood of $z$, the scheme $Y_C$ over $C$ satisfies (RF) for $\tilde{D}$ at $\zeta_1$ and $\zeta_2$. Since $C \to X$ is smooth at $\zeta_1$ and $\zeta_2$, the scheme $Y$ over $X$ satisfies (RF) for $D_1$ and $D_2$ at $x$ by Lemma 2.1.8.

2. Let $D_{1,C}$ and $D_{2,C}$ be the pull-backs of $D_1$ and $D_2$ to $C$. Then, we have $D_{1,C} < \tilde{D} < D_{2,C}$ at $z$. Hence by (3.4) with the slant arrow added, we obtain a commutative diagram

$$
(3.8) \quad \begin{array}{c}
F_{\tilde{D}_2}^{D_2,C}(Y_C/C) \quad F_{\tilde{D}}^{\tilde{D}}(Y_C/C) \quad F_{\tilde{D}_1}^{D_1,C}(Y_C/C) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
F_{\tilde{D}_2}(Y_C/C) \quad F_{\tilde{D}}(Y_C/C) \quad F_{\tilde{D}_1}(Y_C/C).
\end{array}
$$

Since $Y$ over $X$ satisfies (RF) for $D_1$ and $D_2$ at $x$ by 1, the pull-back defines canonical isomorphisms from the left and right columns of (3.7) to those of (3.8) by Lemma 2.1.8. Thus we obtain (3.7).

3. By functoriality of cospecialization mappings, we obtain a commutative diagram

$$
(3.9) \quad \begin{array}{c}
F_{\tilde{D}_2}^{D_2,C}(Y_C/C) \quad F_{\tilde{D}_2}^{\tilde{D}}(Y_C/C) \quad F_{\tilde{D}_2}(Y/X) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
F_{\tilde{D}_2}(Y_C/C) \quad F_{\tilde{D}}(Y_C/C).
\end{array}
$$

By Lemma 2.1.8 and by $\tilde{D} = D_{2,C}$ at $\zeta_2$, the composition $F_{\tilde{D}_2}^{D_2}(Y/X) \to F_{\tilde{D}_2}^{\tilde{D}}(Y_C/C)$ is a bijection. Hence $F_{\tilde{D}_2}^{D_2}(Y/X) \to F_{\tilde{D}}^{\tilde{D}}(Y_C/C)$ is injective.

Since the second assertion is étale local on $X$, we may assume that $Y \to X$ is finite.
By functoriality of specialization mappings, we obtain a commutative diagram

\[
\begin{array}{ccc}
F^D_{\tilde{\mathcal{Z}}}(Y/C) & \xrightarrow{\text{sp.}} & F^D_{\tilde{\mathcal{Z}}}(Y/C) \\
\downarrow & & \downarrow \\
F^{D_1}(Y/X) & \xleftarrow{\text{sp.}} & F^{D_1}(Y/C) \\
\end{array}
\]

Since the composition \(F^D_{\tilde{\mathcal{Z}}}(Y/C) \to F^{D_1}(Y/C)\) is a bijection, the vertical arrow \(F^D_{\tilde{\mathcal{Z}}}(Y/C) \to F^{D_1}(Y/C)\) is an injection on the image of \(\bar{Y}_{\tilde{\mathcal{Z}}}\). Hence the assertion follows. \(\square\)

### 3.2 Ramification and valuations

For a valuation ring \(A \subset K\), let \(v: K^\times \to \Gamma = K^\times/A^\times\) denote the valuation.

**Definition 3.2.1.** Let \(X\) be a normal separated noetherian scheme, \(U \subset X\) be a dense open subscheme and \(A\) be a valuation ring. We say that a morphism \(T = \text{Spec} \ A \to X\) is \(U\)-external if \(T \times_X U\) consists of a single point \(t\).

For a morphism \(T = \text{Spec} \ A \to X\) and an effective Cartier divisor \(D \subset X\), let \(v(D) \in \Gamma\) denote the valuation \(v(f)\) of a non-zero divisor \(f\) defining \(D \subset X\) on a neighborhood of the image of \(T\).

Let \(\bar{X} = \lim X'\) be the inverse limit of proper schemes \(X' \to X\) such that \(U' = U \times_X X' \to U\) is an isomorphism. Then, points of \(\bar{X} \cap U\) correspond bijectively to the inverse limits of the images of the closed points by the liftings of \(U\)-external morphisms \(T \to X\) defined by valuation rings of the residue fields of points of \(U\) by [6, 5.4].

**Lemma 3.2.2.** Let \(X\) be a normal noetherian scheme, \(U \subset X\) be a dense open subscheme, \(t \in U\) be a point, \(A \subset K = k(t)\) be a valuation ring and \(T = \text{Spec} \ A \to X\) be a \(U\)-external morphism.

1. Let \(g \in \Gamma(U', \mathcal{O}_{U'}^\times)\) be an invertible function defined on an open neighborhood \(U' \subset U\) of \(t \in U\) such that \(v(g) = \gamma \geq 0\). Then, there exist a normal scheme \(X'\) of finite type over \(X\) such that \(U \times_X X' = U'\), that \(g\) is extended to a non-zero divisor on \(X'\) defining an effective Cartier divisor \(R' \subset X'\) and that \(U' = X' \cap R'\) is the complement of an effective Cartier divisor \(D' \subset X'\) and a \(U'\)-external morphism \(T \to X'\) lifting \(T \to X\) and \(v(R') = \gamma\).

2. Let \(K'\) be a finite separable extension of \(K = k(t)\) and \(A' \subset K'\) be a valuation ring such that \(A' \cap K = A\). Set \(T' = \text{Spec} \ A'\) and let \(\gamma > 0\) be a positive element of the value group \(\Gamma'\) of \(A'\). Then, there exist a commutative diagram

\[
\begin{array}{ccc}
U' & \longrightarrow & X' \leftarrow T' \\
\downarrow & & \downarrow \\
U & \longrightarrow & X \leftarrow T
\end{array}
\]

of schemes, a point \(t' \in U'\) above \(t\), an isomorphism \(K' \to k(t')\) over \(K\) and an effective Cartier divisor \(R'\) of \(X'\) satisfying the following conditions (i)-(iv):

(i) \(X'\) is a normal scheme of finite type over \(X\),
(ii) The left square is cartesian and $U'$ is a dense open subscheme of $X'$ étale over $U$.
(iii) $T' \to X'$ is a $U'$-external morphism extending $t' \to U'$.
(iv) $R' \cap U' = \emptyset$ and $v'(R') = \gamma$.

3. Let

\[
\begin{array}{ccc}
U' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
U & \longrightarrow & X
\end{array}
\]

be a commutative diagram, $t_1 \in U_1$ and $t' \in U'$ be points above $t \in U$ and $R_1 \subset X_1$ and $R' \subset X'$ be effective Cartier divisors satisfying the following conditions (i)-(iv):

(i) $X_1$ and $X'$ are normal noetherian schemes and $X_1 \to X$ is of finite type.
(ii) The left square and the left parallelogram are cartesian and $U_1 \to U$ is étale. The open subschemes $U_1 \subset X_1$ and $U' \subset X'$ are dense.
(iii) $T_1 = \text{Spec } A_1$ and $T' = \text{Spec } A'$ for valuation rings $A_1 \subseteq K_1 = k(t_1)$ and $A' \subseteq K' = k(t')$ satisfying $A_1 \cap K = A' \cap K = A$. The morphism $T_1 \to X_1$ is $U_1$-external and $T' \to X'$ is $U'$-external.
(iv) $R_1 \cap U_1$ and $R' \cap U'$ are empty and we have $v_1(R_1) \leq v'(R')$ in $\Gamma'_{\mathcal{Q}}$.

Then, there exist a commutative diagram

\[
\begin{array}{ccc}
U'_1 & \longrightarrow & X'_1 \\
\downarrow & & \downarrow \\
U' \times_U U_1 & \longrightarrow & X' \times_X X_1
\end{array}
\]

and $t'_1 \in U'_1$ above $t$ satisfying the following conditions (i)-(iv):

(i) $X'_1$ is a normal scheme of finite type over $X'$.
(ii) $U'_1$ is $U' \times_X X'_1$ and is a dense open subscheme of $X'_1$.
(iii) $T'_1 = \text{Spec } A'_1$ for a valuation ring $A'_1 \subset K'_1 = k(t'_1)$.
(iv) For the pull-backs $R'_1 = R_1 \times_{X_1} X'_1$ and $R'_2 = R' \times_{X'} X'_1$, we have $R'_1 \leq R'_2$.

Proof. 1. Let $Z$ and $Z'$ be closed subschemes such that $U = X - Z$ and $U' = X - Z'$. By replacing $X$ by the normalization of the blow-up at $Z$ and at $Z'$ and by the valuative criterion of properness, we may assume that $U = X - D$ and $U' = X - D'$ are the complements of effective Cartier divisors $D, D' \subset X$.

Let $x \in X$ be the image of the closed point of $T$ and let $W = \text{Spec } B \subset X$ be an open neighborhood of $x$ such that $D \cap W, D' \cap W$ are principal divisors defined by $f, f' \in B$. Then we have $U' \cap W = W - D' \cap W = \text{Spec } B[1/f']$. Set $g = h/f'' \in B[1/f']$. The function $g$ and hence also $h \in B$ are also invertible on $U' \cap W$. Set $\alpha = v(f), \alpha' = v(f') \in \Gamma$. Since $T \to X$ is $U$-external, we have $\Gamma + [1/\alpha] = \Gamma$. Hence after replacing $f$ by its power, we may assume that $\alpha' \leq \alpha$.

Let $W' \to W$ be the normalization of the blow-up at the ideals $(f'' h)$ and $(f, f')$. Since $f, f'$ and $h$ are invertible on $U' \cap W$, the morphism $W' \to W$ induces an isomorphism $U' \times_X W' \to U' \cap W$. Since $W' \to W$ is proper, the morphism $T \to W$ is uniquely lifted to $T \to W'$. Since the generic point $t \in T$ is the unique point of $U \times_X T \supset (U' \times_X W') \times_W T$, the morphism $T \to W'$ is $U'$-external.
Let $x' \in W'$ be the image of the closed point of $T$. Since the ideals $(f^m, h), (f, f') \subset \mathcal{O}_{W, x'}$ are principal ideals and since $v(h) \geq v(f^m)$ and $v(f') \geq v(f)$, there exists an open neighborhood $X'$ of $x' \in W$ such that $U' \subset X'$ where we have inclusions $(f^m) \supset (h)$ and $(f) \supset (f')$. Then, $g = h/f^m$ defines a Cartier divisor $R'$ on $X'$ satisfying $R' \cap U' = \emptyset$ and $v(R') = \gamma$. We also have an inclusion $U \times_X X' = X' \times U \subset X' \times X = U' \times_X X' = U'$. Since the other inclusion is obvious, we have $U' = U \times_X X'$.

2. We may take an étale scheme $U_1 \to U$ such that $t' = \text{Spec } K' = t \times_U U_1$ and a finite scheme $X_1 \to X$ containing $U_1$ as a dense open scheme. After shrinking $U_1$ if necessary, we may take an invertible function $g \in \Gamma(U_1, \mathcal{O}_{U_1}^{\times})$ such that $\gamma = v'(g)$. Since $T'$ is a localization of the normalization of $T \times_X X_1$, the morphism $t' \to U_1 \subset X_1$ is uniquely extended to $T' \to X_1$.

Then, by 1 applied to the open subschemes $U_1 \subset U \times_X X_1 \subset X_1$, to the morphism $T' \to X_1$ and to the invertible function $g \in \Gamma(U_1, \mathcal{O}_{U_1}^{\times})$, the assertion follows.

3. Let $T_{(\bar{x})}, T_{1,(\bar{x}_1)}$ and $T'_{(\bar{x}'_1)}$ denote the strict localizations. We take a point $\bar{t}_1 \in T'_{(\bar{x}'_1)} \times T_{(\bar{x})}$ above the generic point of $T'_{(\bar{x}'_1)}$. Then the normalization $\tilde{T}'_1$ of $T'_{(\bar{x}'_1)}$ in $\tilde{R}'_1 = \text{Spec } A_{\bar{x}'_1}^{\text{sh}}$ for a strictly local valuation ring $A_{\bar{x}'_1}^{\text{sh}}$. Let $t'_1 \in t' \times t_1 \subset T' \times_T T_1$ be the image of $\bar{t}_1$ and set $K'_1 = k(t'_1)$ and $A'_1 = A_{\bar{x}'_1}^{\text{sh}} \cap K'_1$. Let $T'_1 = \text{Spec } A'_1$ and $\bar{x}'_1$ be the geometric point of $T'_1$ defined by a geometric closed point of $\tilde{T}'_1$.

Let $X_0'$ be the normalization of $X' \times_X X_1$ in $U'_1 = U' \times_U U_1$. Define effective Cartier divisors of $X'_0$ by $R'_{01} = R_1 \times_{X_1} X_0'$ and $R'_{02} = R' \times_{X_1} X_0'$. Let $\tilde{X}_1' \to X_0'$ be the normalization of the blow-up at $R''_0 \cap R'_1 = R_0 \times_{X_1'} R'_1$ and define effective Cartier divisors of $\tilde{X}_1'$ by $\tilde{R}'_1 = R_1 \times_{X_1'} \tilde{X}_1'$ and $\tilde{R}'_2 = R' \times_{X_1'} \tilde{X}_1'$. Since $\tilde{X}_1' \to X' \times_X X_1$ is proper, the morphism $t'_1 \to t' \times t_1 \subset U' \times_U U_1$ is uniquely lifted to $T'_1 \to \tilde{T}'_1$ by the valuative criterion of properness.

Let $x'_1 \in \tilde{X}_1'$ be the image of the closed point of $T'_1$. The intersection $\tilde{R}'_1 \cap \tilde{R}'_2 \subset \tilde{X}_1'$ is the exceptional divisor and hence is an effective Cartier divisor. Since $v'_1(\tilde{R}'_1) \leq v'_1(\tilde{R}'_2)$, on an open neighborhood $X'_1 \subset \tilde{X}_1'$ of $x'_1$, we have $\tilde{R}'_1 \cap \tilde{R}'_2 = \tilde{R}'_1 \leq \tilde{R}'_2$ by Nakayama’s lemma.

Let $X$ be a normal noetherian scheme and $U \subset X$ be a dense open subscheme. Let $t \in U$ and $T = \text{Spec } A \to X$ be a $U$-external morphism defined by a valuation ring $A \subset K = k(t)$ of the residue field at a point $t \in U$. Let $\bar{x}$ and $\bar{t}$ be geometric points of $T$ supported on the closed point and on the generic point respectively. Recall that $T_{(\bar{x})}$ denotes the strict localization and that a specialization $\bar{x} \leftarrow \bar{t}$ is a morphism $T_{(\bar{x})} \leftarrow \bar{t}$ of schemes.

Let $A'$ be a valuation ring and $T' = \text{Spec } A' \to T$ be a faithfully flat morphism. We identify $\Gamma$ as a subgroup of the value group $\Gamma' \to T$ by the canonical injection $\Gamma \to \Gamma'$. Let $\bar{x}'$ and $\bar{t}'$ be geometric points of $T'$ above $\bar{x}$ and $\bar{t}$ respectively. We say that a specialization $\bar{x}' \leftarrow \bar{t}'$ is a lifting of $\bar{x} \leftarrow \bar{t}$ if the diagram

\[
\begin{array}{cccc}
\bar{x}' & \longrightarrow & T' & \leftarrow & \bar{t}' \\
\downarrow & & \downarrow & & \downarrow \\
\bar{x} & \longrightarrow & T & \leftarrow & \bar{t}
\end{array}
\]

is commutative.
We consider a commutative diagram

\[
\begin{array}{ccc}
X' & \xleftarrow{\alpha} & T' \\
\downarrow & & \downarrow \\
X & \xleftarrow{\gamma} & T
\end{array}
\]

(3.10)

of schemes equipped with an effective Cartier divisor $R' \subset X'$ and a lifting $\bar{x}' \leftarrow \bar{t}$ to $T'$ of the specialization $\bar{x} \leftarrow t$ satisfying the following conditions (i)-(iii):

(i) $X'$ is a normal noetherian scheme of finite type over $X$ such that $U' = U \times_X X' \subset X'$ is a dense open subscheme étale over $U$.

(ii) $T' = \text{Spec} A' \to X'$ is a $U'$-external morphism defined by a valuation ring $A' \subseteq K' = k(t')$ of the residue field at a point $t' \in U'$ above $t$ such that $A' \cap K = A$.

(iii) $R' \cap U' = \emptyset$ and $\nu'(R') = \gamma$ in the value group $\Gamma'$ of $A'$.

For elements $\alpha \leq \beta$ of a totally ordered group $\Gamma$, let $(\alpha, \beta)_\Gamma \subset \Gamma$ denote the subset \{\(\gamma \in \Gamma \mid \alpha < \gamma < \beta\)\}. Similarly, we define $(\alpha, \beta)_\Gamma, (\alpha, \infty)_\Gamma \subset \Gamma$ etc.

**Definition 3.2.3.** Let $X$ be a normal noetherian scheme and $U \subset X$ be a dense open subscheme. Let $t \in U$, $A \subset k(t)$ be a valuation ring of the residue field at $t$ and $T = \text{Spec} A \to X$ be a $U$-external morphism. Let $\gamma \in (0, \infty)_{\Gamma_Q}$ for $\Gamma_Q = \Gamma \otimes \mathbb{Q}$. Let $Y$ be a quasi-finite flat scheme over $X$ such that $V = Y \times_X U \to U$ is étale.

We define a commutative diagram

\[
\begin{array}{ccc}
F^\infty_T(Y/X) & \xrightarrow{\varphi^+_Y} & F^+_T(Y/X) \\
\varphi^+_Y \downarrow & & \downarrow \\
F^+_T(Y/X) & \xrightarrow{\varphi^+_T} & F^0+(Y/X)
\end{array}
\]

(3.11)

as the inverse limit of

\[
\begin{array}{ccc}
Y'_{x'} & \xrightarrow{\varphi^+_{x'}} & F^+_T(Y'/X') \\
\varphi^+_{x'} \downarrow & & \downarrow \\
F^+_T(Y'/X') & \xrightarrow{\varphi^+_{T}} & Y_{x'}
\end{array}
\]

(3.12)

for commutative diagrams (3.10) satisfying the conditions (i)-(iii).

We say that the ramification of $Y$ over $X$ at $T$ is bounded by $\gamma$ (resp. by $\gamma^+$) if $F^\infty_T(Y/X) \to F^+_T(Y/X)$ (resp. $F^+_T(Y/X) \to F^+_{T}(Y/X)$) is an injection.

By Lemma 3.2.2, the limit is a filtered limit.

**Lemma 3.2.4.** 1. There exist a commutative diagram (3.10) satisfying the conditions (i)-(iii), an effective Cartier divisor $R' \subset X'$ satisfying $R' \cap U' = \emptyset$ and $x' \in R'$ such that $Y'$ over $X'$ satisfies (RF) for $R'$ at the image $x' \in R'$ of the closed point of $T'$.

2. For $x' \in R' \subset X'$ satisfying the condition in 1, the canonical morphism from (3.11) to (3.12) is an isomorphism. The diagram (3.11) is a diagram of finite sets.

**Proof.** 1. By Lemma 3.2.2, after replacing $X$ by a normal scheme of finite type over $X$ if necessary, we may assume that there exist an effective Cartier divisor $R \subset X$ such
that \( v(R) = \gamma \) and a closed immersion \( Y \to Q \) over \( X \) to a smooth scheme \( Q \) over \( X \). Applying Theorem 1.2.5 to \( Q(R) \to X \) and taking the normalizations, we obtain a morphism \( X' \to X \) of finite type of normal noetherian schemes satisfying the following properties: The morphism \( X' \to X \) is the composition of a blow-up \( X^* \to X \) with center supported in \( X = U \) and a faithfully flat morphism \( X' \to X^* \) of finite type such that \( U' = X' \times_X U \to U \) is étale. The morphism \( Q'(R) \to X' \) is flat and reduced. Hence \( Y' \) over \( X' \) satisfies the condition (RF) for \( R' \). The morphism \( T \to X \) is lifted to \( T' \to X' \) by Lemma 1.2.6.

2. By 1 and Lemma 3.2.2 among commutative diagrams 3.10 those such that the base change \( Y' = Y \times_X X' \) over \( X' \) satisfies the condition (RF) for \( R' \) at \( x' \) are cofinal. Hence the assertion follows from Lemma 2.1.8. \( \square \)

We study functoriality of the construction of \( F_T^{\gamma'}(Y/X) \) and \( F_T^{\gamma'+}(Y/X) \). We consider a commutative diagram

\[
\begin{array}{ccccccccc}
Y' & \longrightarrow & X' & \longleftarrow & T' & \longleftarrow & \bar{x}' & \longleftarrow & \bar{t}' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y & \longrightarrow & X & \longleftarrow & T & \longleftarrow & \bar{x} & \longleftarrow & \bar{t},
\end{array}
\]

(3.13)

dense open subschemes \( U \subset X \) and \( U' \subset U \times_X X' \subset X' \) and \( \gamma \in (0, \infty)_{TQ} \) and \( \gamma' \in (0, \infty)_{T'Q} \), satisfying the following properties:

(i) \( X' \to X \) is a morphism of normal noetherian schemes.

(ii) \( T = \text{Spec} \ A \to X \) and \( T' = \text{Spec} \ A' \to X' \) are \( U \)-external and \( U' \)-external morphisms for valuation rings \( A \subset K = k(t) \) and \( A' \subset K' = k(t') \) of the residue fields at \( t \in U \) and \( t' \in U' \). The morphism \( T' \to T \) is faithfully flat.

(iii) \( Y \to X \) and \( Y' \to X' \) are quasi-finite morphisms such that \( Y \times_X U \to U \) and \( Y' \times_{X'} U' \to U' \) are étale.

(iv) \( \gamma' \leq \gamma' \).

(v) \( \bar{x}' \longleftarrow \bar{t}' \) is a lifting of \( \bar{x} \longleftarrow \bar{t} \).

**Lemma 3.2.5.** We keep the notation above.

1. We have a commutative diagram

\[
\begin{array}{ccccccccc}
F_T^{\infty}(Y'/X') & \longrightarrow & F_T^{\gamma'+}(Y'/X') & \longrightarrow & F_T^{\gamma'}(Y'/X') & \longrightarrow & F_T^{\infty}(Y'/X') & \longrightarrow & F_T^{\gamma'+}(Y'/X') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_T^{\infty}(Y/X) & \longrightarrow & F_T^{\gamma'+}(Y/X) & \longrightarrow & F_T^{\gamma'}(Y/X) & \longrightarrow & F_T^{\infty}(Y/X) & \longrightarrow & F_T^{\gamma'+}(Y/X)
\end{array}
\]

(3.14)

of finite sets. Further if \( \gamma < \gamma' \), we have an arrow

\[
F_T^{\gamma'}(Y'/X') \to F_T^{\gamma'+}(Y/X)
\]

making the two triangles obtained by dividing the middle square commutative.

2. If the left square in (3.13) is cartesian and if \( \gamma = \gamma' \), the vertical arrows in (3.14) are bijections.

**Proof.** By Lemma 3.2.41, we may assume that there exists an effective Cartier divisor \( R \subset X \) such that \( R \cap U = \emptyset \) and \( v(R) = \gamma \) and that \( Y \) over \( X \) satisfies the condition (RF) for \( R \). Further by Lemma 3.2.41 and Lemma 3.2.23, we may assume that there exists an
effective Cartier divisor \( R' \subset X' \) such that \( R' \cap U' = \emptyset, v'(R') = \gamma \) and \( R' \geq R \times_X X' \) and that \( Y' \) over \( X' \) satisfies the condition (RF) for \( R' \). Then, by Lemma 3.2.3.2, we may identify \( F_{T}^\gamma(Y/X) = F_{x}^{R}(Y/X), F_{T}^{\gamma+}(Y/X) = F_{x}^{R+}(Y/X) \) and \( F_{T}^\gamma(Y'/X) = F_{x}^{R}(Y'/X'), F_{T}^{\gamma+}(Y'/X') = F_{x}^{R+}(Y'/X') \).

1. The assertion now follows from the functoriality of dilatation (3.1).

2. In the notation above, we may further assume that \( R' = R \times_X X' \). Hence the assertion follows from Lemma 2.1.8.

Let \( T^h \) be the henselization at the closed point \( x \in T \) and let \( t^h \in T^h \) denote the generic point. Then, the absolute Galois group \( D_T = \text{Gal}(t^h/t) \) acts on the specialization \( \bar{x} \leftarrow \bar{t} \) of geometric points of \( T \). Hence the commutative diagram (3.1) admits a canonical action of \( D_T \).

**Theorem 3.2.6.** Let the notation be as in Definition 3.2.3. Then, there exist an element \( \beta_0 \in (0, \infty)_{\Gamma_Q} \) and finite pairs \((\alpha_i, \beta_i)_{i \in I} \) of elements of \( [0, \beta_0]_{\Gamma_Q} \) satisfying the following properties (i)-(iii):

(i) \( [0, \beta_0]_{\Gamma_Q} = \bigcup_{i \in I} [\alpha_i, \beta_i]_{\Gamma_Q} \).

(ii) For \( \gamma > \beta_0 \) (resp. \( \gamma \geq \beta_0 \)), \( F^\gamma_{T}(Y/X) \leftarrow F^\infty_{T}(Y/X) \) (resp. \( F_{T}^{\gamma+}(Y/X) \leftarrow F_{T}^{\infty}(Y/X) \)) is an injection.

(iii) Let \( i \in I \) and \( \gamma \in (\alpha_i, \beta_i)_{\Gamma_Q} \). Then, \( F^\gamma_{T}(Y/X) \leftarrow F^\beta_{T}(Y/X) \) is an injection and \( F_{T}^{\alpha_0+}(Y/X) \leftarrow F_{T}^{\gamma+}(Y/X) \) is an injection on the image of \( F_{T}^{\infty}(Y/X) \).

**Proof.** Since we may take base change, we may assume that \( Y \to X \) is finite and that the normalization \( \bar{Y} \to X \) is finite étale. Hence by Lemma 2.1.1, we may assume that there exists an effective Cartier divisor \( R \subset X \) such that \( R \cap U = \emptyset \) and \( \bar{Y} \to Y(R) \) is a closed immersion.

Set \( \beta_0 = v(R) \in \Gamma \). Then, by Lemma 3.2.3, after replacing \( X \) if necessary, we may assume that \( Y \) over \( X \) satisfies the condition (RF) for \( R \). Since \( \bar{Y} \to Y(R) \) is a closed immersion and \( F^\gamma_{T}(Y/X) \to \bar{Y}_x \) is a bijection, \( \bar{Y}_x = F^\gamma_{T}(Y/X) \to Y_i(R) = F^\gamma_{T+}(Y/X) \) is an injection. For \( \gamma > \beta_0 \), the composition \( F_{T}^{\beta_0+}(Y/X) \leftarrow F_{T}^{\gamma}(Y/X) \leftarrow F_{T}^{\gamma+}(Y/X) \) is an injection. Hence the condition (ii) is satisfied.

Let \( Q \) be a smooth scheme over \( X \) and let \( Y \to Q \) be a closed immersion. As in Example 1.3.1, we define a semi-stable curve \( C_R \to X \) by the effective Cartier divisor \( R \subset X \). Define an effective Cartier divisor \( \bar{R} \subset C_R \) to be the exceptional divisor. Applying Theorem 1.3.3 to \((Q \times_X C_R)[R] \to C_R \to X \) and taking the normalizations, we obtain a commutative diagram

\[
\begin{array}{ccc}
C_R & \overset{\phi}{\longleftarrow} & C' \\
\downarrow & & \downarrow \\
X & \overset{\phi}{\longleftarrow} & X'
\end{array}
\]

where \( Y_{C'} = Y \times_X C' \) over \( C' \) satisfies the condition (RF) for \( R' = \bar{R} \times_{C_R} C' \) and \( C' \to X' \) is a semi-stable curve.

By Lemma 1.2.6, there exist a finite extension \( K' \) of \( K \) and a valuation ring \( A' \) such that \( A = A' \cap K \) and that \( T \to X \) is lifted to \( T' = \text{Spec } A' \to X' \). Let \( x' \in X' \) denote the image of the closed point of \( T' \). Further, for \( \gamma \in [0, \beta_0]_{\Gamma_Q} \), after replacing \( K' \) by a finite extension if necessary, we may assume that \( \gamma \) is an element of \( [0, \beta_0]_{\Gamma'} \).

Let \( I_1 \) be the set of irreducible components of the fiber \( C' \times_{X'} x' \). For \( i \in I_1 \), let \( C_i \subset C' \times_{X'} x' \) denote the corresponding connected component. Let \( I_2 \) denote the set of
singular points of the fiber $C' \times_{X'} x'$. For $i \in I_2$, let $z_i \subset C' \times_{X'} x'$ denote the corresponding singular point. Set $I = I_1 \amalg I_2$.

Since the assertion is étale local on $X'$, we may assume that for each $i \in I_1$, there exists a section $s_i : X' \to C'$. For $i \in I_1$, set $\alpha_i = \beta_i = v'(s_i^* R') \in \Gamma^+$. Since $\alpha_i = v'(\tilde{R})$ for the composition $T' \to X' \to C' \to C_R$, we have $\alpha_i \in [0, \beta_0]_{\Gamma_Q}$. For $i \in I_2$, if $z_i$ is contained in two irreducible components $C_{i_1}$ and $C_{i_2}$ such that $\alpha_{i_1} \leq \alpha_{i_2} \in \Gamma^+$, we define $\alpha_i = \alpha_{i_1} \leq \beta_i = \alpha_{i_2} \in [0, \beta_0]_{\Gamma_Q}$. If $z_i$ is contained in a unique irreducible component $C_i$, we define $\alpha_i = \beta_i = \alpha_i \in [0, \beta_0]_{\Gamma_Q}$.

We show that the condition (i) is satisfied. Since $\alpha_i, \beta_i \in [0, \beta_0]_{\Gamma_Q}$, we have the inclusion $[0, \beta_0]_{\Gamma_Q} \supset \bigcup_{i \in I} [\alpha_i, \beta_i]_{\Gamma_Q}$. Let $\gamma$ be an element of $[0, \beta_0]_{\Gamma_Q}$. Then, we may assume $\gamma \in [0, \beta_0]_{\Gamma'}$. Then, since $T' \to X$ has a lifting to $T' \to C_R$ such that $v'(\tilde{R}) = \gamma$ and since $C' \to C_R \times_X X'$ is proper and birational, there exists a unique lifting $T' \to C'$ of $T' \to C_R$ by the valuative criterion. If the image of closed point by $T' \to C'$ is contained in the smooth part $C_i \cap C_{i_2}^{\text{sm}}$ of an irreducible component $C_i \subset C'$, then we have $\gamma = \alpha_i$. If the image of closed point by $T \to C'$ is the singular point $z_i \in C'_x$ for $i \in I_2$, then we have $\gamma \in [\alpha_i, \beta_i]_{\Gamma_Q}$ by Corollary 1.3.4. Thus, the condition (i) is also satisfied.

We show that the condition (iii) is satisfied. For $i \in I_1$ or $i \in I_2$ such that $\alpha_i = \beta_i$, there is nothing to prove. Assume that $i \in I_2$ and that $z_i$ is contained in two irreducible components $C_{i_1}$ and $C_{i_2}$ such that $\alpha_i = \alpha_{i_1} < \beta_i = \alpha_{i_2} \in \Gamma^+$ and set $\gamma \in (i, \beta_i)_{\Gamma_Q}$. Then, we may assume $\gamma \in (\alpha_i, \beta_i)_{\Gamma'}$. By Corollary 1.3.4 after replacing $T'$ by an extension if necessary, we may take a morphism $T' \to C'$ such that the image of the closed point $x' \in T'$ is $z_i$ and $v'(R') = \gamma$. Since $F_{T}(Y/X) = F_{T}(Y_{C'}/C')$ and $F_{T}^{0+}(Y/X) = F_{T}^{0+}(Y_{C'}/C')$ by Lemma 3.2.5 and Lemma 3.2.4, the assertion follows from Proposition 3.1.8.

We study some variants.

Let $X$ be a normal noetherian scheme, $U$ be a dense open subscheme and let $V \to U$ be a finite étale morphism. We consider a cartesian diagram

$$
\begin{array}{ccc}
Y' & \leftarrow & V \\
\downarrow & & \downarrow \\
X' & \leftarrow & U
\end{array}
$$

(15)

of schemes of finite type over $X$ satisfying the following conditions: The horizontal arrows are dense open immersions, $X'$ is normal, $X' \to X$ is a proper birational morphism inducing the identity on $U$ and $Y'$ is finite flat over $X'$.

Let $A \subset K = k(t)$ be a valuation ring of the residue field at a point $t \in U$ and $T = \text{Spec} A \to X$ be a $U$-external morphism. Let $x \in T$ denote the closed point and $\overline{x}$ be a geometric point above $x$. For $\gamma \in \Gamma_{Q > 0}$, we define

$$
F_{T}^{\infty}(V/U) \longrightarrow F_{T}^{0+}(V/U)
$$

(16)

$$
\begin{array}{ccc}
F_{T}(V/U) & \longrightarrow & F_{T}^{0+}(V/U) \\
\downarrow & & \downarrow \\
F_{T}(V/U) & \longrightarrow & F_{T}^{0+}(V/U)
\end{array}
$$

(17)

to be the inverse limit of

$$
\begin{array}{ccc}
F_{T}^{\infty}(Y'/X') & \longrightarrow & F_{T}^{0+}(Y'/X') \\
\downarrow & & \downarrow \\
F_{T}(Y'/X') & \longrightarrow & F_{T}^{0+}(Y'/X')
\end{array}
$$

(18)
Let \( T_V \) denote the normalization of \( T \) in \( V \times_X T \). For \( X' \) in (3.15), let \( X'_T \subset X' \) denote the reduced closed subscheme supported on the closure of \( t \in U \subset X' \) and let \( x' \in X'_T \) denote the image of the unique morphism \( T \to X' \) lifting \( T \to X \). Then, since \( A = \lim_{X' \to X} \mathcal{O}_{X'_T,x} \), we have \( F^0_T(V/U) = T_V \times_T \bar{x} \).

**Lemma 3.2.7.** Suppose that the normalization \( T_V \) of \( T \) in \( V \times_X T \) is finite and flat over \( T \). Then, there exists a finite and flat \( Y'' \to X' \) such that \( T_V = Y'' \times_{X'} T \). For such \( Y'' \to X' \), the diagram (3.16) is isomorphic to (3.17).

**Proof.** Since \( A = \lim_{X' \to X} \mathcal{O}_{X'_T,x} \) in the notation above, the existence of finite flat \( Y'' \to X' \) such that \( T_V = Y'' \times_{X'} T \) follows. By the flattening theorem [8, Théorème (5.2.2)], such \( Y'' \to X' \) are cofinal among commutative diagrams (3.15). Hence the assertion follows from Lemma 3.2.8.

For a normal noetherian scheme \( X \), a formal \( \mathbb{Q} \)-linear combination \( R = \sum_i r_i D_i \) with positive coefficients \( r_i \geq 0 \) of irreducible closed subsets \( D_i \) of codimension 1 is called an effective \( \mathbb{Q} \)-Cartier divisor if a non-zero multiple is an effective Cartier divisor. The union \( \bigcup_i D_i \) for \( r_i > 0 \) is called the support of \( R \). For an open subset \( U \subset X \), if \( U \) does not meet the support of \( R \), we write \( R \cap U = \emptyset \) by abuse of notation. For a \( U \)-external morphism \( T = \text{Spec} A \to X \), the valuation \( v(R) \) is defined as an element of \([0, \infty)_{\Gamma_\mathbb{Q}}\).

**Definition 3.2.8.** Let \( X \) be a normal noetherian scheme and \( U \subset X \) be a dense open subscheme. Let \( Y \) be a quasi-finite flat scheme over \( X \) such that \( V = Y \times_X U \to U \) is finite étale. Let \( R \) be an effective \( \mathbb{Q} \)-Cartier divisor of \( X \) such that \( U \cap R \) is empty and let \( x \in X \) be a point contained in the support of \( R \).

We say that the ramification of \( Y \) over \( X \) is bounded by \( R \) (resp. by \( R^+ \)) at \( x \), if for every \( U \)-external morphism \( T \to X \), the ramification of \( Y \to X \) is bounded by \( v(R) \) (resp. by \( v(R^+) \)) in the sense of Definition 3.2.8.

**Lemma 3.2.9.** Let the notation be as in Definition 3.2.8. Then, the following conditions (1), (1‘) and (2) are equivalent:

1. The ramification of \( Y \to X \) is bounded by \( R \) (resp. by \( R^+ \)) in the sense of Definition 3.2.8.

1‘. The condition in Definition 3.2.8 with \( T \) restricted to be a discrete valuation ring is satisfied.

2. For every morphism \( f : X' \to X \) of finite type of normal noetherian schemes such that \( U' = U \times_X X' \to U \) is étale, that \( R' = f^* R \) is an effective Cartier divisor and that \( Y' = Y \times_X X' \to X' \) satisfies the condition (RF) in Definition 3.1.4 for \( R' \), the ramification of \( Y' \to X' \) is bounded by \( R' \) (resp. by \( R'^+ \)) at every point of \( R' \) in the sense of Definition 3.1.4.

**Proof.** (1‘)⇒(2): Let \( X' \to X \) be as in (2) and \( x' \in X' \) be a point. Let \( X'_1 \to X' \) be the normalization of the blow-up at the closure of \( x' \). Then, the local ring \( A' = \mathcal{O}_{X'_1,x'_1} \) at the generic point \( x'_1 \) of an irreducible component of the inverse image of \( x' \) is a discrete valuation ring. The morphism \( T' = \text{Spec} A' \to X'_1 \to X' \) is \( U' \)-external and the image of the closed point is \( x' \).
For \( \gamma' = v'(R') \), by Lemma 3.2.42, the commutative diagram (3.12) is canonically identified with

\[
F^\infty_T(Y'/X') \xrightarrow{\varphi'_{\gamma'}} F^\infty_{T'}(Y'/X') \cong Y'_{x'}.
\]

Further, this commutative diagram is canonically identified with (3.11) for \( \gamma = v(R) \) by Lemma 3.2.5.2. Hence the assertion follows.

(2)\( \Rightarrow \) (1): Let \( T \to X \) be a \( U \)-external morphism and \( \gamma = v(R) \). Then by Lemma 3.2.42, the commutative diagram (3.11) is canonically identified with (3.12). Hence the assertion follows.

The implication (1)\( \Rightarrow \) (1') is obvious. \( \square \)

**Proposition 3.2.10.** Let the notation be as in Definition 3.2.8 and assume that the ramification of \( Y \) over \( X \) is bounded by \( R^+ \). Assume that \( Y \) is locally of complete intersection over \( X \) and let

\[
\begin{array}{ccc}
Q & \leftarrow & Y \\
\downarrow & & \downarrow \\
P & \leftarrow & X
\end{array}
\]

be a cartesian diagram of schemes over \( X \) such that \( P \) and \( Q \) are smooth over \( X \), the vertical arrows are quasi-finite and flat and the horizontal arrows are closed immersions.

Let \( X' \) be a normal noetherian scheme over \( X \) such that \( R' = R \times_X X' \) is an effective Cartier divisor, that \( Y' = Y \times_X X' \) over \( X' \) satisfies the condition (RF) for \( R' \).

Then, the morphism \( Q^{(R')} \to P^{(R')} \) is étale on a neighborhood of \( Q^{(R')} \times_{X'} R' \).

**Proof.** First, we show that we may assume that there exist a closed subscheme \( Y_0' \subset Y' \) étale over \( X' \), an integer \( n \geq 1 \) and an effective Cartier divisor \( D_0' \subset R' \) satisfying the following conditions: We have an equality \( R_{Y_0}' = R_Y' \), of underlying sets. Let \( J_0' \subset O_{R_Y'} \) be the nilpotent ideal defining \( R_{Y_0}' \subset R_Y' \). Then, we have \( J_0'^n = 0 \) and \( (n + 1)D_0' = R' \).

Under the condition (RF), the formation of \( Q^{(R')} \to P^{(R')} \) commutes with base change by Lemma 2.1.8 and Example 2.1.2.1. Since \( Q^{(R')} \) and \( P^{(R')} \) are flat over \( X' \), the étaleness of \( Q^{(R')} \to P^{(R')} \) is checked fiberwise. Hence, we may take base change. Let \( x' \in R' \) be a point and let \( X'' \to X' \) be the normalization of the blow-up at the closure of \( x' \). Then, there exists a point \( x'' \in X'' \) above \( x' \) such that the local ring \( O_{X'',x''} \) is a discrete valuation ring. Hence, by replacing \( X' \) by \( \text{Spec} O_{X'',x''} \), we may assume that \( X' \) is the spectrum of a discrete valuation ring.

Then, we may assume that \( Y'' \subset Q' \) is a union of sections \( X' \to Q' \). There exists a disjoint union \( Y_0'' \subset Y'' \) of sections such that we have an equality \( R_{Y_0}'' = R_Y' \), of underlying sets. Let \( n \geq 1 \) be an integer satisfying \( J_0'^n = 0 \) in the notation above. After replacing \( X' \) by a ramified covering if necessary, there exists effective Cartier divisor \( D_0' \) of \( X' \) satisfying \( (n + 1)D_0' = R' \).

The finite morphism \( Y^{(R')} \to X' \) is étale by Corollary 3.1.3. Hence by the existence of \( Y_0', D_0' \) and \( n \) and by Lemma 2.2.3, the \( O_{Y^{(R')}} \)-module \( O_{Y^{(R')}} \otimes_{O_Y} \Omega^1_{Y/X} \) is annihilated by \( \mathcal{I}_{nD_0'} \). Hence by Lemma 2.2.4, there exists an open neighborhood \( W_1 \subset Q^{(R')} \) of \( Q^{(R')} \times_{X'} R' \) such that \( Q^{(R')} \to P^{(R')} \) is étale on \( W_1 \).
The morphism $Q^{(R)} \to P^{(R)}$ is étale also on a neighborhood $W_2$ of $Y^{(R)} \subset Q^{(R)}$. Since the vector bundle $P^{(R)} \times_X R' \to R'$ has irreducible fibers, $W_2 \subset Q^{(R)}$ is dense in the fiber of every point of $R'$ by Proposition 1.4.2. Hence the assertion follows from Lemma 3.2.4. □

3.3 Ramification groups

Theorem 3.3.1. Let $X$ be a connected normal noetherian scheme and $U \subset X$ be a dense open subscheme. Let $G$ be a finite group, $W \to U$ be a connected $G$-torsor and let $C$ be the category of finite étale schemes over $U$ trivialized by $W$. Assume that for every morphism $V_1 \to V_2$ of $C$, the morphism $Y_1 \to Y_2$ of normalizations of $X$ in $V_1$ and in $V_2$ is locally of complete intersection.

Let $t \in U$ and $T = \text{Spec} \ A \to X$ be a $U$-external morphism for a valuation ring $A \not\subseteq K = k(t)$. Let $\bar{x}$ (resp. $\bar{i}$) be a geometric point above the closed point $x$ (resp. the generic point $t$) of $T$ and $\bar{x} \leftarrow \bar{i}$ be a specialization. Fix a lifting of $\bar{x}$ to the normalization $T_W$ of $T$ in $W \times_X T$ and let $I_{\bar{x}} \subset G$ be the inertia group at the image of the lifting of $\bar{x}$ to the normalization $Y_W$ of $X$ in $W$ by $T_W \to Y_W$.

For an object $V$ of $C$, let $Y$ denote the normalization of $X$ in $V$ and consider the fiber functor sending $V$ to $F_T^\infty(Y/X)$.

1. There exist decreasing filtrations $G^\gamma_T \supset G^\gamma_T^+$ of $G$ indexed by $\gamma \in (0, \infty)_{\Gamma Q}$ such that, for every object $V$ of $C$, the canonical surjections $F_T^\infty(Y/X) \to F_T^{\gamma^+}(Y/X) \to F_T^\gamma(Y/X)$ induce bijections

$$G^\gamma_T \setminus F_T^\infty(Y/X) \to F_T^{\gamma^+}(Y/X), \quad G^\gamma_T \setminus F_T^\infty(Y/X) \to F_T^\gamma(Y/X).$$

For $I_{\bar{x}} = G^0_T$, the mapping

$$G^0_T \setminus F_T^\infty(Y/X) \to F_T^0(Y/X)$$

is a bijection.

2. There exists a finite increasing sequence $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n$ of elements of $[0, \infty)_{\Gamma Q}$ such that we have

$$G^\alpha_{i-1} = G^\gamma = G^\gamma_T^+ = G^\alpha_i \quad \text{for } \gamma \in (\alpha_{i-1}, \alpha_i)_{\Gamma Q}, 1 \leq i \leq n,$$

$$G^\alpha_n = G^\gamma = G^\gamma_T^+ = 1 \quad \text{for } \gamma \in (\alpha_n, \infty)_{\Gamma Q}.$$

3. Let $D_T \subset G$ be the decomposition group of $T$ in $W \times_X T$. Then, $D_T$ normalizes $G^\gamma$ and $G^\gamma_T^+$.

Proof. 1. By Proposition 3.1.2 and Lemma 3.2.4, 2, the diagram

$$
\begin{array}{cccc}
F_T^\infty(Y'/X) & \longrightarrow & F_T^{\gamma^+}(Y'/X) & \longrightarrow & F_T^\gamma(Y'/X) & \longrightarrow & F_T^{0^+}(Y'/X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_T^\infty(Y/X) & \longrightarrow & F_T^{\gamma^+}(Y/X) & \longrightarrow & F_T^\gamma(Y/X) & \longrightarrow & F_T^{0^+}(Y/X)
\end{array}
$$

is a cocartesian diagram of surjections. Further the functors $F_T^\gamma$ and $F_T^{\gamma^+}$ preserve disjoint unions. Hence by Proposition 1.4.2 we obtain filtrations $(G^\gamma_T)^\gamma$ and $(G^\gamma_T^+)^\gamma$ indexed by
$\gamma \in (0, \infty)_{r_{\mathbb{Q}}} r_{\mathbb{Q}}$ characterized by the bijections (3.19). For $\gamma = 0$, the bijection (3.19) follows from $F_{T}^{0+}(Y/X) = Y_{x}$.

2. Since $C$ has only finitely many connected objects and $F_{T}^{\infty}(Y/X) \to F_{T}^{r+}(Y/X) \to F_{T}^{r}(Y/X)$ are surjections, it follows from Theorem 3.2.6.

3. Since the surjections $F_{T}^{\infty}(Y/X) \to F_{T}^{r+}(Y/X) \to F_{T}^{r}(Y/X)$ are compatible with the actions of $D_{T} \subset G$, the subgroup $D_{T} \subset D_{x}$ normalizes $G^{\gamma}$ and $G^{\gamma+}$ by Corollary 1.4.4.

By the definition of the filtrations, the ramification of $Y/X$ at $T$ is bounded by $\gamma$ (resp. by $\gamma+$) if and only if the action of $G^{\gamma}$ (resp. of $G^{\gamma+}$) on $F_{T}^{\infty}(Y/X)$ is trivial. By Corollary 1.4.3, the filtrations $(G^{\gamma})$ and $(G^{\gamma+})$ are compatible with quotients. We have the following functoriality. Let

\[
\begin{array}{c}
X' \leftarrow T' \\
\downarrow \quad \downarrow \\
X \leftarrow T
\end{array}
\]

be a commutative diagram of schemes. Assume that $X' \to X$ is a morphism of normal connected noetherian schemes and let $U' \subset U \times_{X} X' \subset X'$ be a dense open subscheme. The horizontal arrows $T \to X$ and $T' \to X'$ are $U$-external and $U'$-external and the vertical arrow $T' \to T$ is faithfully flat. Let $W'$ be a connected $G'$-torsor over $U'$ for a finite group $G'$ and let $W' \to W$ be a morphism over $U' \to U$ compatible with a morphism $G' \to G$ of finite groups. Assume that $W' \to U'$ satisfies the complete intersection property as in Theorem 3.3.1 and let $(G^{'r})$ and $(G^{'r+})$ be the filtrations of $G'$ indexed by $\gamma' \in (0, \infty)_{r_{\mathbb{Q}}}$. Then, for $\gamma \in (0, \infty)_{r_{\mathbb{Q}}}$, the morphism $G' \to G$ induces

\begin{equation}
(3.22) \quad G^{'r} \to G^{\gamma}, \quad G^{'r+} \to G^{\gamma+}
\end{equation}

by the functoriality Lemma 3.2.5.1.

We consider a variant. Let $A \subseteq K$ be a valuation ring and $L$ be a finite Galois extension of $K$ of Galois group $G$. We define a filtration of $G$ by ramification groups under the following assumptions: For every intermediate extension $K \subset M \subset L$, the normalization $A_{M}$ of $A$ in $M$ is a valuation ring finite flat and of complete intersection over $A$. There exist an irreducible normal noetherian scheme $X$ such that $K$ is the residue field at the generic point $t$ and a morphism $T = \text{Spec} A \to X$ extending $t \to X$.

Let $T \to X$ be as above. Then by Lemma 3.2.2, there exist a dense open subscheme $U \subset X$, a normal scheme $X'$ of finite type over $X$ satisfying the following conditions: The morphism $U' = U \times_{X} X' \to U$ is an isomorphism. The morphism $T \to X$ is lifted to $T \to X'$. For every intermediate extension $M$, there exists a finite flat scheme $Y'M \to X'$ locally of complete intersection such that $U' \times_{X'} Y'M \to U'$ is finite étale and $T \times_{X} Y'M = \text{Spec} A_{M}$. Then applying Theorem 3.3.1 we obtain filtrations $(G'^{r})$ and $(G'^{r+})$ by normal subgroups of $G = D_{T}$ indexed by $(0, \infty)_{r_{\mathbb{Q}}}$.

In the rest of the article, we consider the case where $X = T = \text{Spec} O_{K}$ for a complete discrete valuation ring $O_{K}$. For a finite Galois extension of the fraction field $K$ of Galois group $G$, the decreasing filtrations $(G'^{r})_{r_{T} \geq 0}$ and $(G'^{r+})_{r_{T} \geq 0}$ by normal subgroups indexed by rational numbers are defined.

Let $L$ be a finite separable extension of degree $n$ of $K$ and $Y = \text{Spec} O_{L}$ for the integer ring $O_{L}$. We recall the classical case where $O_{L}$ is generated by one element over $O_{K}$, using the Herbrand function. Take a closed immersion $Y = \text{Spec} O_{L} \to Q = \mathbb{A}^{1}_{X}$.
Spec $\mathcal{O}_K[T]$, and let $P \in \mathcal{O}_K[T]$ be the monic polynomial such that we have an isomorphism $\mathcal{O}_K[T]/(P) \to \mathcal{O}_L$.

Let $K'$ be a finite separable extension containing the Galois closure of $L$ and $X' = \text{Spec } \mathcal{O}_{K'}$. Let $v' : K' \to \mathbb{Q} \cup \{\infty\}$ be the valuation extending the normalized valuation of $K$. Let $r > 0$ be a rational number in the image of $v'$ and let $R' \subset X'$ be the effective Cartier divisor such that $v'(R') = r$. Let $Q' \supset Y'$ be the base change of $Q \supset Y$ by $X' \to X$ and let $Q'^{(r)} = Q'^{(R')}$ denote the dilatation. We compute $Q'^{(r)}$ using the Herbrand function whose definition we briefly recall.

Decompose $P$ as $P = \prod_{i=1}^n(T - a_i)$ in $\mathcal{O}_{K'}[T]$ and set $b_i = a_i - a_n \in \mathcal{O}_{K'}$. Set $P(T_1 + a_n) = \prod_{i=1}^n(T_1 - b_i) = T_1^a + c_1T_1^{a-1} + \cdots + c_{n-1}T_1$ in $\mathcal{O}_{K'}[T_1]$. Changing the numbering if necessary, we assume that the valuations $v_i = v'(b_i) \in \mathbb{Q}$ are increasing in $i$. Note that the increasing sequence $s_0 = 0 \leq s_1 \leq \cdots \leq s_{n-1} < s_n = \infty$ is independent of the choice of $a_n$. The valuation $v'(c_n) = \sum_{k=1}^{n-1}s_k$ equals the valuation $v'(D_{L/K})$ of the different $D_{L/K}$. It is further equal to the length of the $\mathcal{O}_L$-module $\Omega_{L/K}^{1}$ divided by the ramification index $e_{L/K}$ by [10] Chap. III §7 Corollaire 2 à Proposition 11.

The largest piecewise linear convex continuous function $p : [0, n + 1] \to [0, v'(D_{L/K})]$ such that the graph is below the points $(0, 0)$ and $(k, \text{ord}_{K}c_k)$ for $k = 1, \ldots, n$ is defined by

$$p(x) = \sum_{i=1}^{k-1}s_i + s_k(x - k + 1)$$

on $[k-1, k]$ for $k = 1, \ldots, n - 1$. The graph of $p$ is the Newton polygon of the polynomial $P(T_1 + a_n)$. The Herbrand function $\varphi : [0, \infty) \to [0, \infty)$ is a piecewise linear concave continuous function defined by

$$\varphi(s) = \sum_{i=1}^{n-1}\min(s_i, s) + s.$$  

We have

$$\varphi(s) = \sum_{i=1}^{k-1}(n - i + 1) \cdot (s_i - s_{i-1}) + (n - k + 1) \cdot (s - s_{k-1})$$

on $[s_{k-1}, s_k]$ for $k = 1, \ldots, n$.

**Example 3.3.2.** Let $s \in (s_{k-1}, s_k) \mathbb{Q}$, $r = \varphi(s)$ and let $t$ be an element of a finite separable extension $K'$ of $K$ such that $\text{ord}_{K}t = s$. By (3.23) and Example 2.1.2 $Q'^{(r)}$ is obtained as an iterated dilatation defined inductively by $Q'_0 = Q'$,

$$Q'^{(r)} = \begin{cases} Q'^{(n-(i-1))-(s_i-s_{i-1})} & \text{for } 0 < i < k \\ Q'^{(n-k-(s-s_{k-1}))} & \text{for } i = k \end{cases}.$$  

Hence $Q'^{(r)} \to X'$ is smooth. Let $C \subset Q'^{(r)} \times_{X'} X'$ be the connected component meeting the section $s_n' : X' \to Q'^{(r)}$ lifting $s_n : X \to Q$ defined by $T = a_n$. Let $k$ be the smallest integer $k = 1, \ldots, n$ satisfying $s \leq s_k$. Then, $\text{Spec } \mathcal{O}_{K'}[T']$ for $T' = T_1/t$ is a neighborhood of $C \subset Q'^{(r)}$. Further on $\text{Spec } \mathcal{O}_{K'}[T']$, the closed subscheme $Y'^{(r)} \subset Q'^{(r)}$ is defined by $\prod_{i=k}^{n}(T' - b_i/t)$.
Consequently, the surjection $\bar{Y} = \{a_1, \ldots, a_n\} \to F^r_X(Y/X)$ (resp. $\to F^r_{X^+}(Y/X)$) is given by the equivalence relation $v'(a_i - a_j) \geq s$ (resp. $v'(a_i - a_j) > s$). In particular, $r_{L/K} = \varphi(s_{n-1}) = v'(D_{L/K}) + s_{n-1}$ is the unique rational number $r$ such that the ramification of $Y$ over $X$ is bounded by $r+$ but not by $r$.

We give a slightly simplified proof of the proposition below giving characterizations of unramified extensions and tamely ramified extensions.

**Lemma 3.3.3** ([10 Chap. III §7 Proposition 13], [1 Proposition A.3]). Let $L$ be a finite separable extension of a complete discrete valuation field $K$. Assume that $O_L$ is generated by one element over $O_K$ and let $r_{L/K} = \varphi(s_{n-1}) = v'(D_{L/K}) + s_{n-1}$ be as in Example 3.3.2.

1. The following conditions are equivalent:
   (1) $L$ is an unramified extension of $K$.
   (2) $r_{L/K} = 0$.
   (3) $r_{L/K} < 1$.

2. The following conditions are equivalent:
   (1) $L$ is a tamely ramified extension of $K$.
   (2) $r_{L/K} = 0$ or $1$.
   (3) $r_{L/K} \leq 1$.

**Proof.** By [1 Proposition A.3], we have $v'(D_{L/K}) \geq 1 - 1/e_{L/K}$ and the equality holds if and only if $e_{L/K}$ is tamely ramified. We have $s_{n-1} \geq 0$ and the equality holds if and only if $L$ is unramified. If $L$ is ramified, we have $s_{n-1} \geq 1/e_{L/K}$ and the equality holds if and only if $L$ is tamely ramified. The assertions follows from these. \[\square\]

**Proposition 3.3.4** ([1 Proposition 6.8]). Let $L$ be a finite separable extension of a complete discrete valuation field $K$.

1. The following conditions are equivalent:
   (1) $L$ is an unramified extension of $K$.
   (2) The ramification of $L$ over $K$ is bounded by $1^+$.

2. The following conditions are equivalent:
   (1) $L$ is a tamely ramified extension of $K$.
   (2) The ramification of $L$ over $K$ is bounded by $1^+$.

**Proof.** (1) $\Rightarrow$ (2): Since $O_L$ is generated by one element over $O_K$, this follows from Example 3.3.2 and Lemma 3.3.3.

(2) $\Rightarrow$ (1): 1. Let $L$ be a finite separable extension such that the ramification over $K$ is bounded by 1 and assume that $L$ was ramified over $K$.

Let $G$ be the Galois group of a Galois closure of $L$ over $K$ and let $1 \nsubsetneq I \subset G = \text{Gal}(L/K)$ be the wild inertia subgroup and the inertia subgroup. By replacing $K$ and $L$ by the subextensions corresponding to $I$ and to a maximal subgroup $H \nsubseteq I$, we may assume that $L$ is a cyclic extension of prime degree since $I$ is solvable.

Then, either the ramification index $e_{L/K}$ is 1 and the residue extension is a purely inseparable extension of degree $p$ or $L$ is totally ramified extension. Hence $O_L$ is generated by an element and the assertion follows from Example 3.3.2 and Lemma 3.3.3.

2. If the integer ring $O_L$ is generated by one element over $O_K$, the assertion follows from Example 3.3.2 and Lemma 3.3.3. We prove the general case by reducing to this case by contradiction.
Let \( L \) be a finite separable extension such that the ramification over \( K \) is bounded by \( 1^+ \) and assume that \( L \) was wildly ramified over \( K \).

Let \( G \) be the Galois group of a Galois closure of \( L \) over \( K \) and let \( 1 \subsetneq P \subset I \subset G = \text{Gal}(L/K) \) be the wild inertia subgroup and the inertia subgroup. By replacing \( K \) and \( L \) by the subextensions corresponding to \( I \) and to a maximal subgroup \( H \subsetneq P \), we may assume that \([L : K] = mp\) for an integer \( m \) prime to \( p \).

By construction, there exists a sequence \( K \subset K_0 \subset K_1 \subset \cdots \subset K_n = K' \) such that \( K_0 \) is an unramified extension of \( K \) and that \( K_i \) is an extension of \( K_{i-1} \) of degree \( p \) of ramification index 1 with inseparable residue field extension for each \( i = 1, \ldots, n \). Since \([LK_0 : K_0] = mp\), we have \( n > 0 \). By taking the smallest such \( n \), we may assume \([LK_{n-1} : K_{n-1}] = mp\).

Further by the functoriality (3.22), we may replace \( K \) and \( L \) by \( K_{n-1} \) and \( LK_{n-1} \). Hence, we may assume that \([K' : K] = p \) and \( K' \subset L \). Since \([K' : K] = p \), the integer ring \( \mathcal{O}_{K'} \) is generated by one element over \( \mathcal{O}_K \). Since \( K' \subset L \), the ramification of \( K' \) over \( K \) is bounded by \( 1^+ \). Hence \( K' \) is tamely ramified over \( K \). This contradicts to that the residue field extension of \( K' \) over \( K \) is inseparable.

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