GUIDING NONLINEAR FORCE-FREE MODELING USING CORONAL OBSERVATIONS: FIRST RESULTS USING A QUASI-GRAD–RUBIN SCHEME

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Received 2012 February 21; accepted 2012 July 1; published 2012 August 24

ABSTRACT

At present, many models of the coronal magnetic field rely on photospheric vector magnetograms, but these data have been shown to be problematic as the sole boundary information for nonlinear force-free field extrapolations. Magnetic fields in the corona manifest themselves in high-energy images (X-rays and EUV) in the shapes of coronal loops, providing an additional constraint that is not at present used as constraints in the computational domain, directly influencing the evolution of the model. This is in part due to the mathematical complications of incorporating such input into numerical models. Projection effects, confusion due to overlapping loops (the coronal plasma is optically thin), and the limited number of usable loops further complicate the use of information from coronal images. We develop and test a new algorithm to use images of coronal loops in the modeling of the solar coronal magnetic field. We first fit projected field lines with those of constant-α force-free fields to approximate the three-dimensional distribution of currents in the corona along a sparse set of trajectories. We then apply a Grad–Rubin-like iterative technique, which uses these trajectories as volume constraints on the values of α, to obtain a volume-filling nonlinear force-free model of the magnetic field, modifying a code and method presented by Wheatland. We thoroughly test the technique on known analytical and solar-like model magnetic fields previously used for comparing different extrapolation techniques and compare the results with those obtained by currently available methods relying only on the photospheric data. We conclude that we have developed a functioning method of modeling the coronal magnetic field by combining the line-of-sight component of the photospheric magnetic field with information from coronal images. Whereas we focus on the use of coronal loop information in combination with line-of-sight magnetograms, the method is readily extended to incorporate vector-magnetic data over any part of the photospheric boundary.

Key words: magnetic fields – plasmas – Sun: coronal – Sun: magnetic topology – Sun: UV radiation

Online-only material: color figures

1. INTRODUCTION

The ability to build adequate models of the coronal magnetic field is extremely important for understanding the physics of the solar corona. The corona is believed to be generally in a force-free (or at least low-β) state (Gary 2001). Destabilization of this state may lead to eruptions, with contributing factors including topological properties of the field, such as the existence of null points and excessive magnetic twist (Canfield et al. 1999). The amount of energy released in eruptions cannot exceed the amount of free magnetic energy at the time of destabilization. Moreover, as the coronal field generally evolves in such a way that its total helicity only changes due to helicity flux across the photosphere and into the heliosphere (Berger & Field 1984), the assessment of helicity at one point in time, such as prior to a coronal mass ejection, might be beneficial for studies of the evolution of the corona and heliosphere. Modeling of coronal heating is frequently performed as one-dimensional hydrodynamic (or static) models following magnetic field lines using values of magnetic field along these field lines as important input (e.g., Lundquist et al. 2008). So this modeling would also benefit from better models for the magnetic field.

The general problem of constructing a force-free magnetic field (hereafter FFF) to model the coronal field is formulated as follows (Nakagawa et al. 1971). The objective is to find a magnetic field \( \mathbf{B} \) which satisfies the divergence-free condition

\[
\nabla \cdot \mathbf{B} = 0
\]

and the force-free equation

\[
\nabla \times \mathbf{B} = \alpha \mathbf{B}, \quad (2)
\]

where \( \alpha \) is a proportionality constant between the magnetic field and magnetic current.4 Equations (1) and (2) must be solved for \( \mathbf{B} \) and \( \alpha \) in a volume domain \( V \) subject to boundary conditions \( \mathbf{B}\rvert_\partial V \) (or \( \mathbf{B} \cdot \hat{n}\rvert_\partial V \) and \( \alpha\rvert_\partial V \)). The problem is not in general linear and the solution is hence called a “nonlinear force-free field,” hereafter NLFFF. Particular cases include a linear force-free field (hereafter LFFF) that solves the system assuming \( \alpha(r) = \text{const} \), or a potential field (where \( \alpha(r) = 0 \)) which we refer to in the text as \( \mathbf{B}_P \).

Many difficulties arise when solving the problem of constructing an NLFFF. The underlying reasons for these difficulties are physical, mathematical, and computational. Physically, the full vector magnetic field at the lower boundary \( z = 0 \) is currently obtained only in the photosphere, where plasma forces are significant. That is to say, Equation (2) is not appropriate at the lower boundary level (Gary 2001). Also, the component of \( \mathbf{B} \) transverse to the line of sight at the photosphere is subject to an intrinsic 180° ambiguity, and measurements of boundary data at the top and side boundaries of the computational domain are not available at this time (see Demoulin et al. 1997, for an extensive discussion of these issues). Typically, assumptions are made about the side bound-

4 The parameter \( \alpha \) has topological meaning associated with the amount of twist in the field, e.g., Gold & Hoyle (1960).
aries, e.g., a field matching a potential source surface model (Schrijver & De Rosa 2003) is assumed, and there are various methods to resolve the azimuthal ambiguity (Metcalfe et al. 2006). Mathematically, the system is nonlinear and at the present stage the uniqueness, and even the existence, of a solution in general for a given boundary conditions are not proven. Finally, there are computational difficulties that have to do with the high instrumental uncertainty in the measurements of the transverse horizontal component of the photospheric magnetic field and the small spatial scale of current changes, possibly below the instrumental resolution, in the lower boundary. This uncertainty has more impact than it might seem at first sight because \( \mathbf{B} \cdot \nabla \alpha = 0 \), implying \( \alpha = \text{constant along magnetic field lines} \).\(^3\) Hence, field lines must connect points with the same \( \alpha \) on positive and negative polarities at the lower boundary so the boundaries must have equal amounts of incoming and outgoing magnetic flux for each value of \( \alpha \). Noise in \( \alpha \) at the lower boundary and limits to the field of view prevent this condition from being satisfied, and the problem is in general ill-posed (Aly 1989). Techniques exist for “pre-processing” of the boundary data to attempt to mitigate this problem (e.g., Wiegelmann et al. 2006, 2008) and for formulating a well-posed problem given the uncertainties in the measurements (Amari & Aly 2010; Wheatland & Régnier 2009; Wheatland & Leka 2011).

The existing methods to address the difficulties outlined above do not appear to be developed to a level such that photospheric vector magnetograms may be used to reliably model the coronal field. Different methods for solving the NLFFF problem, and even various implementations of the same method, applied to the same photospheric data, and even the same method applied to different polarities of the same data, frequently yield results inconsistent with each other and with the coronal features (Schrijver et al. 2006, 2008; Metcalfe et al. 2008; DeRosa et al. 2009). Such methods are, for example, vertical integration (Nakagawa 1974; Wu et al. 1990), magnetofrictional relaxation (e.g., Mikic & McClymont 1994; van Ballegooijen 2004; Valori et al. 2005), optimization (e.g., Wiegelmann 2004), and the Grad–Rubin method (e.g., Sakurai 1981; Amari et al. 1997, 1999; Wheatland 2007).

Some of the problems outlined above might be alleviated if vector magnetograms were available, routinely, for chromospheric heights (at level where the atmosphere is close to being force-free). However, at present these data are not commonly available. In the absence of these, we propose using coronal loops as an additional source of information to improve modeling.

Coronal loops, observed in X-ray and EUV images, are believed to follow lines of the magnetic field, and therefore they should be of help for magnetic extrapolations. Unlike vector magnetograms, this information originates in the force-free corona, where Equation (2) is appropriate. Field lines spread apart with height and so do bundles of coronal loops (though the field generally expands with height, individual loops are found to have nearly constant diameter with height, see Klimchuk 2000). Consequently, the structure of the magnetic field in the corona should be less fine than at the photospheric level so it might in principle be better resolved by currently available instruments. Observed loops also give an idea about the overall connectivity of the coronal field, which might otherwise be easily distorted by even minor noise present in photospheric vector magnetograms and therefore in \( \alpha \), as discussed above.

\(^3\) This follows from Equation (2) by taking the divergence of both sides of the equation.

Even if techniques of processing vector magnetograms are developed to the point that NLFFF models are generally reliable, coronal loops as an additional constraint might be of great benefit, for example, for studies of energy release in solar flares. Vector magnetograms undergo relatively minor changes during even major flares (e.g., Wang et al. 2012, found only a fractional change in the transverse component of the field in a small patch of the active region during a large X-class flare). In contrast, the changes in the connectivity of the coronal magnetic field can be large scale and dramatic even in smaller flares. As the connectivity of the magnetic field manifests itself in the shapes of coronal loops, the latter provide a powerful guide for tracking sudden changes in the field.

Making use of coronal loops is, however, a non-trivial task. The plasma is optically thin, and what is observed by instruments is the integrated emission of all the plasma along the line of sight. Extracting individual loops from bundles of overlapping loops is a non-trivial image processing task, with the possible exception of isolated loops far away from the core of the regions. Some progress, though, has been made in this direction (e.g., Aschwanden et al. 2008). Another difficulty is that all currently existing instruments, with the exception of STEREO satellites (Kaiser 2005), only observe the Sun in one projection, so the three-dimensional structure of the loops is not immediately obvious.

Coronal loops have been used to indirectly guide magnetic extrapolations in many studies. The common procedure in these studies was as follows. In the first step, a model of magnetic field was formulated, with a set of parameters defining this model. Second, a series of such models was computed with various values of these parameters. Third, the best-fitting parameter was selected by matching coronal or chromospheric images with lines of the model fields. The matching was usually performed for many observed features simultaneously, and the result was a value of the parameters which yielded an overall best-matching model. The parameters could be, for example, value of \( \alpha \) in an LFFF (e.g., Nakagawa et al. 1973; Aurass et al. 1999; Burnette et al. 2004), values of \( \alpha \), and a constant \( Q \) determining an extra term in Green’s function in an LFFF (Wu & Wang 1985), or a stretching parameter in a non-force-free field (Gary & Alexander 1999). The matching was performed visually or numerically and field lines to compare observables against were drawn from a pre-computed set or an additional step was made to find the best-matching loops.

Recently, substantial progress has been made in determining properties of the coronal magnetic field along individual loops. Wiegelmann & Neukirch (2002) fitted stereocorically derived loop trajectories with lines of an LFFF. Carcedo et al. (2003) and Lim et al. (2007) fitted plane-of-sky projections of coronal loops with lines of an LFFF. Wiegelmann & Neukirch (2002) and Lim et al. (2007) noticed that the best-fitting value of \( \alpha \) varies for individual loops within an active region. Malanushenko et al. (2009) showed that \( \alpha \) values obtained this way statistically correlate with \( \alpha \) values for an NLFFF model and developed a semi-automatic algorithm for such fits. Progress also has been made in extracting individual features from coronal images (see Aschwanden et al. 2008 for an overview). Numerous studies (e.g., Aschwanden et al. 2009) have demonstrated good results on triangulating loop trajectories using STEREO data.

This progress in evaluating the properties of individual loops allows the use of coronal data in a new way, principally different from the previous studies. That is, loop data may be used as an
input to directly constraint the model, rather than as a control variable to adjust model parameters.

In this paper, we develop and test such a method of constructing an NLFFF using data derived from coronal loops as direct constraints. We also draw attention to the value of the new data (trajectories and α values for individual loops) for magnetic modeling in general. These methods might in principle be of use in areas of plasma physics other than coronal studies. It might, for example, be desirable in laboratory plasma studies to estimate what kind of a force-free field would have a required topology and magnitude of currents.

Two approaches for the use of loop data in magnetic modeling are as follows. Reconstructed three-dimensional loop trajectories and α values along them may be determined approximately using the scheme from Malanushenko et al. (2009), hereafter the MLM09 fit. This provides information at least about the three-dimensional trajectories of some field lines and α in the corona along these field lines. Stereoscopically derived data offer another possibility; the inferred three-dimensional loop trajectories could be used in conjunction with values of the vector magnetic field at the loop footpoints. Vector magnetograms are of course prone to the problems outlined above. However, in the case of using loops, the field values only need to be accessed at a sparse set of locations in the lower boundary. The vector components of the magnetic field at the top layer of the chromosphere, if given in at least a few patches in an active region (provided a chromospheric vector magnetogram is available for this particular region), and provided that these patches contain footpoints of the stereoscopically determined loops trajectories, could be used with these trajectories as in the first approach, but with more accurate results.

The paper is organized as follows. In Section 2, we describe the quasi-Grad–Rubin scheme, enabling us to make use of coronal loops with and without vector magnetograms. In Section 3, we discuss various inputs. Section 4 describes the general scheme of a set of tests of the method and figures of merit obtained. The results of the tests are presented in detail in Section 5. Section 6 discusses the results, evaluating how successful the scheme is and its value for modeling of the coronal field.

2. DESCRIPTION OF THE QUASI-GRAD–RUBIN METHOD

Suppose there is a domain Ω with boundary ∂Ω and the following are given:

1. \( \mathbf{B} \cdot \mathbf{n}|_{\partial \Omega} \) (where \( \mathbf{n} \) is the normal to \( \partial \Omega \));
2. a set of trajectories \( \{ \mathcal{L}_i \}_{i=1}^N \) in Ω along which the force-free parameter values \( \{ \alpha_i \}_{i=1}^N \) are known (and are constant along each individual trajectory).

The objective is to find the field \( \mathbf{B} \) that solves Equation (2) and matches the boundary conditions (1) and the volume constraints (2).

The procedure is iterative and is similar to a Grad–Rubin iteration (Grad & Rubin 1958). It starts with potential field \( \mathbf{B}^{(0)} = \mathbf{B}_p \) as an initial guess for the field and an initial guess \( \alpha^{(0)} \) for the force-free parameter, which at each point in the domain is set equal to the closest point in the volume for which an \( \alpha \) value is known. Then on every iteration the updated cubes \( \mathbf{B}^{(n)} \) and \( \alpha^{(n)} \) are obtained as follows:

1. Impose the volume constraints by setting \( \alpha^{(n-1)} = \alpha_i \) along the trajectories \( \mathcal{L}_i \).

2. Calculate updated field values \( \mathbf{B}^{(n)} \) from

\[
\nabla \times \mathbf{B}^{(n)} = \alpha^{(n-1)} \mathbf{B}^{(n-1)}
\]

subject to the prescribed boundary conditions. This equation is solved using a vector potential \( \mathbf{A}^{(n)} \) such that \( \mathbf{B}^{(n)} = \nabla \times \mathbf{A}^{(n)} \), so the divergence-free condition is satisfied to truncation error.

3. Calculate an updated set of values for the force-free parameter \( \alpha^{(n)} \): for every point in \( \mathcal{V} \), assign \( \alpha^{(n)} = (\alpha^{(n-1)} \alpha^{(n-1)} \text{averaged along the field line in } \mathbf{B}^{(n)} \text{ that passes through that point} \). If a field line leaves the domain through any boundary but the lower one, the value of \( \alpha \) is set to zero along it (in common with the Wheatland 2007; GR scheme). This ensures that no currents go off to infinity so that the fields’ energy remains finite.

4. Repeat (1)–(3) until \( \mathbf{B}^{(n)} \approx \mathbf{B}^{(n-1)} \) and \( \alpha^{(n)} \approx \alpha^{(n-1)} \) to within a tolerance and therefore \( \nabla \times \mathbf{B}^{(n)} \approx \alpha^{(n)} \mathbf{B}^{(n)} \).

This sequence is similar to an existing Grad–Rubin method of solution of the NLFFF problem (Wheatland 2007; Wheatland & Régnier 2009). The only difference between these two schemes is how the updated \( \alpha^{(n)} \) cube is calculated in Step 3. In both schemes, at each point in the domain a field line is traced in \( \mathbf{B}^{(n)} \).

In the original Grad–Rubin code, positive or negative polarity is chosen at the lower boundary; and \( \alpha^{(n)} \) at each point in the volume is set to the value at the boundary point in the chosen polarity where it is crossed by the field line. The only exception is the boundary itself for all points in the chosen polarity where \( \alpha \) is kept constant. In the quasi-Grad–Rubin method, \( \alpha^{(n)} \) at each point is assigned the average of \( \alpha \) from the previous iteration along this field line. The only exception is the volume constraint paths where \( \alpha \) keeps the volume constraint value.

Let us consider some particular simple cases. First of all, it is clear that if the initial guess for \( \mathbf{B} \) and \( \alpha \) already satisfies \( \nabla \times \mathbf{B} = \alpha \mathbf{B} \), then the scheme keeps it unchanged. If the field or \( \alpha \) differs from a solution to Equation (2) even at one point, the field changes—though if it is only one point that is different, convergence is achieved in a single iteration. It is also clear that if \( \alpha = 0 \) everywhere, currents do not appear; since the scheme averages \( \alpha \) at every iteration it is incapable of introducing \( |\alpha| \geq \max(|\alpha|) \). These cases make sense: if the answer is close to the correct answer, convergence is achieved rapidly, and if no currents are specified to start with, the scheme does not change the input potential field.

But what happens in an intermediate situation: currents are known on some, but not all flux tubes in the domain? Can a solution be reached at all? If several solutions are plausible given the constraints, which solution (of any) is achieved? There are proofs of existence and uniqueness of solution for the force-free problem, achievable by Grad–Rubin iteration, if \( \alpha \) is sufficiently small in some sense (see Bineau 1972). However, the range of \( \alpha \) is not clearly defined and it is unclear whether solar-like fields are within this range. If they are outside of this range, that does not mean the original Grad–Rubin iteration necessarily fails. It is unclear if similar proofs exist for the proposed quasi-Grad–Rubin scheme. In the absence of this, we consider a process of numerical experimentation to test the scheme. We attempt also to determine how many field line trajectories \( \mathcal{L}_i \) are sufficient to enable convergence. In Section 5, we review some of our experiments.
solutions obtained using $\alpha$ values at the opposite polarities are consistent with each other to a tolerance. Since QGR does not use the lower boundary in the same way, we discard the switching between cycles and modify the way the $\alpha$ values in the volume are calculated (see previous section). Also, instead of using $B_z$ at $z = 0$ and a two-dimensional array of $\alpha$ values at $z = 0$ the modified code uses $B_z$ at $z = 0$ and two three-dimensional arrays: $\alpha_i$ along the trajectories and an initial guess everywhere else and a “mask,” i.e., another three-dimensional array with entries either unity along $L_i$ or zero everywhere else. These are the two principal changes to the CFit code. The calculation of the field on a given iteration, i.e., the solution of $\nabla \times \mathbf{B}^{n+1} = \alpha^{(n)}(\mathbf{B}^{n})$ is unchanged. The code uses a vector potential for this step and hence the field satisfies $\nabla \cdot \mathbf{B}^{n+1} = 0$.

4. DESCRIPTION OF METRICS FOR QGR SOLUTIONS

In this paper, we try to recover several known force-free fields, namely, those from Schrijver et al. (2006) and Schrijver et al. (2008). The iterations are initialized with the same potential fields used in the original studies. We also try to construct an NLFFF based on a dipole magnetogram and two specified loop trajectories (with a reference field which is not known).

We estimate the quality of the reconstruction and the relative force-freeness of the known solutions using metrics of which most have become standard in NLFFF modeling. These are as follows.

1. $E/E_P$ versus $E_{ref}/E_P$: energy in the reconstructed field versus energy of the reference field (for a perfect reconstruction the two would be equal).
2. $H/|\mathbf{B}_P|$ versus $H/|\mathbf{B}_{ref}|$: relative helicity in the reconstructed field versus that of the reference field (for a perfect reconstruction the two would be equal).
3. $\cos \theta = (\sum |\sin \theta||\mathbf{J}|)/(|\sum |\mathbf{J}|)$ versus $\cos \theta_{ref}$, where $|\sin \theta| = |\mathbf{J} \cdot \mathbf{B}|/|\mathbf{J}|B$: the total current-weighted sine of the angle between $\mathbf{B}$ and $\mathbf{J}$ (for a perfectly force-free field this is zero).
4. Metrics of similarity between $\mathbf{B}$ and $\mathbf{B}_{ref}$, normalized to equal unity if $\mathbf{B} = \mathbf{B}_{ref}$:
   
   \begin{enumerate}
   \item $C_{CS} = \frac{1}{N} \sum (\mathbf{B} \cdot \mathbf{B}_{ref}/(|\mathbf{B}|\mathbf{B}_{ref}))$ (where $N$ is the number of points in the domain): the average cosine of the angle between $\mathbf{B}$ and $\mathbf{B}_{ref}$;
   \item $C_{vec} = (\mathbf{B} \cdot \mathbf{B}_{ref})/(\sum |\mathbf{B}|\mathbf{B}_{ref})$: same as previous but with increased weight in regions of stronger field;
   \item $E_m = 1 - E_m$, where $E_m = \frac{1}{N} \sum (|\mathbf{B} - \mathbf{B}_{ref}|)/|\mathbf{B}_{ref}|$: the average relative difference between $\mathbf{B}$ and $\mathbf{B}_{ref}$;
   \item $E_n = 1 - E_n$, where $E_n = |\mathbf{B} - \mathbf{B}_{ref}|/|\mathbf{B}_{ref}|$: same as previous but with increased weight in regions of stronger field.
   \end{enumerate}

We omit metrics for how well $\nabla \cdot \mathbf{B}$ is satisfied, because the method (in common with Wheatland & Régnier 2009), uses a vector potential to calculate the field and hence achieves a divergence-free state to truncation error (Press et al. 1992).

5. SAMPLE APPLICATIONS OF QGR

5.1. QGR Solution for a Dipole Field

The first test case is a simple dipole field aligned in the E–W direction with the north half of both poles having negative twist and the south half of the poles having matching

### Table 1

| Table 1 | Possible Combinations of Different Inputs to QGR, in Addition to $\mathbf{B} \cdot \mathbf{a}_{3V}$ |
|---------|--------------------------------------------------------------------------------------------------|
| Values of $\alpha$ along Loop Trajectories | Known | Unknown |
| None | $L_a$ | ... |
| From the field lines of the model field ("ideal" input) | $L_b$ | $L_{IIb}$ |
| From the MLM09 approximation derived from two-dimensional projections of these field lines (realistic coronal input) | $L_c$ | $L_{IIc}$ |

Notes. Note that $L_a$ has the same input as the original GR scheme but the algorithm is different, so QGR with $L_a$ input is not equivalent to GR.

3. DIFFERENT TYPES OF THE INPUT DATA FOR QUASI-GRAD–RUBIN SCHEME

The quasi-Grad–Rubin numerical scheme (hereafter “QGR,” in contrast to “GR” for Grad–Rubin algorithm) could in principle be used with $\alpha$ constrained at any set of locations including the lower boundary. Hence, it may be used with vector magnetograms, setting $\alpha_i$ at the lower boundary to the vector magnetogram derived value,

$$\alpha|_{z=0} = \frac{1}{B_z} \left( \frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y} \right)|_{z=0}.$$

(4)

Vector magnetograms may also be used with or without the loop trajectories. We identify three different kinds of inputs for QGR: $\alpha$ along loop trajectories, $\alpha$ along loop trajectories and at the lower boundary, and $\alpha$ at the lower boundary only. Traditional schemes are designed to use $\alpha$ input at the lower boundary only and extrapolate $\alpha$ values into the volume.

If the $\alpha$ values, wherever set, are approximate, this will introduce uncertainties. To properly test QGR, we try to recover several known fields and use both approximated trajectories and those drawn from the known field. We note that there may be applications of QGR even if it would not work with approximate data, for example, for problems of the following kind: constructing a magnetic field with twist prescribed along certain trajectories. It could also be used with stereoscopically triangulated loops and currents derived from the photospheric vector magnetograms, or $\alpha$ values and trajectories obtained by other means.

In this paper, we test various possible inputs and compare the results with the reference fields. Table 1 outlines the degrees of freedom available for such tests. We refer to various inputs and schemes using this chart, e.g., $L_{IIb}$ is QGR applied to volume constraints alone, drawn from the reference field.

The QGR scheme was implemented by modifying an existing GR code (which we refer to as CFit version 1.3), described in Wheatland & Régnier (2009). It is a “self-consistent” scheme, in that it chooses a polarity, finds a force-free solution using boundary data from that polarity, hence obtaining values of $\alpha$ everywhere in the volume including the other polarity at the boundary; then the $\alpha$ map at the other polarity is updated with the weighted average of the values obtained from this new solution and the values that existed before this solution was found. It then repeats the cycle using the updated $\alpha$ map from the other polarity. The cycles are continued until the two
positive twist. This model could be viewed as a simple representation of an emerged untwisted flux rope whose footpoints became distorted in such a way that the field at one (leading) polarity has been inclined more than the field at the second (following) one, perhaps due to subsurface flows. Such a difference in inclinations is observed for solar active regions (Howard 1991).

To construct the field, we calculate two constant-$\alpha$ fields confined to half-spaces (Chiu & Hilton 1977) with equal and opposite twist ($\alpha_0 = \pm 1.5\pi/L$, where $L$ is the size of the domain). We draw one field line for $\mathcal{L}_i$ from each of these and use these field lines (which imitate coronal loops) as volume constraints. The initial guess for $\alpha$ is $\pm\alpha_0$ in the S and N halves, respectively. The fields, field lines, and the locations of the constraints are shown in Figure 1.

QGR is found to converge to a nonlinear force-free field, as shown in Figure 2. Achievement of the force-free state is shown by the distribution of $\sin(\mathbf{B}, \mathbf{J})$ peaking at zero. A second smaller peak at one is a contribution from the current-free regions of the field, and is due to $\mathbf{J}$ in these regions being exclusively numeric noise. To support this, the shaded histogram in Figure 2 shows the distribution of the same quantity evaluated for a potential field. The nonlinearity of the field, that is, the presence of different $\alpha$ values on different field lines, is illustrated by a distribution of $\alpha$ on a horizontal slice. These additional diagrams are shown in Figures 2(a)–(c).

The field lines of the solution, interestingly, are found to follow different trajectories than the two constraining field lines. In fact, the solution, having $\alpha$ close to the two imposed constraints in the two halves, appears much closer to a potential field than the highly twisted constant-$\alpha$ fields we drew the loops from! This is shown in Figure 2(d) which illustrates the constraining loops and two field lines of the solution initiated at the midpoints of these loops.

The fact that the NLFFF solution with currents similar in magnitude as in the Chiu & Hilton (1977) constant-$\alpha$ field appears to have much less twisted field lines is due to at least two effects. First, as open field lines are required to carry no currents in the NLFFF model constructed by a QGR (in common with the Wheatland 2007, implementation of GR), the volume containing currents has to be contained in the computational domain, while the Chiu & Hilton (1977) fields have currents in the entire half-space. As the volume containing currents in the NLFFF is smaller, the current density has to be stronger for the field lines to have similar shape. Second, currents which run in opposing directions close to each other might counteract each other in influencing the shape of field lines.

To illustrate these effects we perform a simple numeric experiment. We repeat the computation but set $\alpha_i = \pm f\alpha_0$ on the constraining paths for several values of $f$. The results are shown in Figure 3, first column. We also repeat the experiment for a box of smaller size (cropped on the sides and the top) but with otherwise identical setup (Figure 3, second column). Finally, we repeat the computation with the same lower boundary but different volume constraints: a field line in the lower half of the domain is drawn from a constant-$\alpha$ field with the same sign of $\alpha$ as the upper half but smaller in magnitude, with $\alpha = -\alpha_0/4 = -5\pi/8L$ (Figure 3, third column). For the original setup, the best match between the constraining loops (red dashed curves) and field lines of the solution (black solid curves) initiated at the midpoints of these loops is achieved with $f \approx 10$–12, while if the same (or at least similar) field lines are required to exist in a field with currents confined to a smaller domain, the best match between the loops and the solution is achieved with $f \approx 6$–10. (No steady solution is found for $f = 14$ for the setup in the first and third columns; in this case the QGR solution continues to oscillate. The solutions for $f = 14$ in the second column and $f = 12$ in the third columns exhibits oscillations but of small magnitude. Later we discuss these oscillations which may result from the input being inconsistent with an NLFFF solution and describe a procedure to damp them. In this section, however, our point is to illustrate the significance of the scale factor $f$.)

The scale factor $f$ cannot be evaluated a priori, but it may be estimated using observables, i.e., coronal loops. This can be done by minimizing the difference between the shapes of coronal loops and field lines of different solutions corresponding to different values of $f$. In the next two sections, we demonstrate that scaling factors obtained this way are indeed a proportionality constant between $\alpha$ on lines from an NLFFF and $\alpha$ of the approximation of these field lines by lines of constant-$\alpha$ fields of the type constructed by Chiu & Hilton (1977).
Figure 2. Solution for the dipole test case. The QGR iteration converges as shown in panel (a), which displays the change in the free energy of the field consecutive iterations. The constructed field $B$ is force-free, as shown in panel (b) by a histogram of $|\sin(B, J)|$ for $B_p$ (shaded curve) and $B$ (solid curve). The peak at unity is due to the current-free regions in the field, as explained in the previous section. The initial field $B_p$ is current-free and $B$ retains current-free regions (in particular where field lines leave the domain through side or top boundary). The field $B$ is nonlinear, as seen in panel (c), which shows a horizontal slice of $\alpha$ at a height of 6 pixels in the box (out of 64). In this panel, black corresponds to negative $\alpha$ and white to positive $\alpha$. The field lines of $B$, however, do not match the constraints! Field lines of $B$ are shown in panel (d) as solid black while the constraints are dashed red lines. Field lines are initiated at midpoints, rather than footpoints, of the loops.

(A color version of this figure is available in the online journal.)

5.2. QGR Solutions for Low & Lou Fields

In this section, we try to reconstruct the test field from Schrijver et al. (2006) (the “reference field” further in this section) using QGR. This test case is a member of family of analytic NLFFFs introduced by Low & Lou (1990).

For the first QGR test, we use field lines of the reference field as trajectories $L_i$ and take $\alpha_i$ values corresponding to the correct $\alpha$ values for the reference field. Physically, this could correspond to stereoscopically derived loops with chromospheric vector field known around their footpoints. These data are hard to obtain at present; we use them mainly to test QGR alone, that is, on the “ideal” data not contaminated by measurement errors. We use 113 randomly selected field lines $L_{i,ref}$ (out of a bigger sample, chosen to be closed field lines, i.e., with both footpoints on the lower boundary). We calculate $\alpha_{i,ref}$ numerically everywhere in $B_{ref}$ and evaluate it along these 113 field lines. We discard all but 27 of these, retaining the ones that may be well fitted using MLM09, to make this test consistent with the next, realistic test presented later in this section. We construct two solutions of the same size as the reference field ($64^3$ pixels), with and without the additional constraint of vector field data at the lower boundary (i.e., schemes I.b and II.b from Table 1). Figures of merit are calculated in the same subdomain as in Schrijver et al. (2006). This center subdomain is also used to estimate a best-matching scaling factor $f$.

In the second test, we determine $L_i$ and $\alpha_i$ based on a fit to the resulting MLM09 field derived from the normal field at the lower boundary as in Malanushenko et al. (2009). We calculate QGR solutions using these data with and without vector field data at the lower boundary (I.c and II.c from Table 1). This tests the applicability of QGR to realistically available coronal data. It is not obvious that using approximations to loop trajectories and approximate values of $\alpha$ along them is sufficient to create a field model at least as good as those derived from vector magnetogram data. To investigate this, we project the same 113 field lines as for the ideal data to the $z = 0$ plane to simulate the appearance of loops in the plane of the sky. We treat these two-dimensional projections as synthetic loops and use them to obtain $\alpha$ values $\alpha_{i,MLM09}$ along trajectories $L_{i,MLM09}$ using the MLM09 fit procedure. We discard loops which upon visual examination are poorly fitted, which leaves 27 loops that appear to have a near-perfect fit. The trajectories of these loops and their MLM09 approximations are shown in Figure 4.
Keeping in mind the results from Section 5.1, we calculate several solutions for II.c with seven different scaling factors $f$ applied to input $\alpha$ but with the calculations otherwise identical. Three of these solutions (corresponding to $f = 1.23, 1.69, \text{and } 2.15$) are shown in Figure 5, top row. For each of the solutions we estimate how closely the field lines match the synthetic two-dimensional loops used to construct the trajectories $L_i$. We calculate the average distance between the projected loops.
and the corresponding field lines of the solution. The result (as a function of $f$) for all seven solutions is shown in Figure 5 (bottom left plot). The solution for $f \approx 1.69$ is the closest match to the loops. The bottom right panel in the same figure is a scatter plot of $\alpha_{\text{fit}}$ and $\alpha_{\text{ref}}$ for individual loops. The fit appears to underestimate the value of $\alpha$. The underestimation factor is remarkably close to 1.69 (dashed line on the same plot). This suggests that such underestimation could be derived a posteriori from the observed loops.

The results of our tests (I.c and II.c with $f = 1.69$) are summarized in Table 2 and Figure 6. Figure 7 shows that convergence is achieved for all solutions. We conclude that QGR is able to reconstruct the reference field, including the shape of field lines, the structure of the currents, and the distribution of $\alpha$, remarkably well. The figures of merit show that the QGR reconstructions are at least as close (and typically closer) to the right answer as other methods. In particular, the smallest estimate for the free energy we obtain is closer to the correct answer than any of the estimates based on the vector field boundary values alone reported by Schrijver et al. (2006).

5.3. QGR Applied to a Solar-like Field

In this section, we investigate whether QGR is applicable to solar data. The Low & Lou (1990) family of fields has axial symmetry which is not in general observed in active regions, and both magnetic field and current vary unrealistically

\[ f = 1.23 \]
\[ f = 1.69 \]
\[ f = 2.15 \]

Figure 5. Top row: several QGR solutions for the reference field from Schrijver et al. (2006) for input II.c (see Table 1) corresponding to the same input but different scaling factor $f$ for $\alpha$. Red dashed lines show lines of $B_{\text{ref}}$ projected onto $z = 0$ and used as loops which were approximated by MLM09 to construct trajectories $L_i$. Black lines show corresponding lines of $B$ (initiated at midpoints of the lines of $B_{\text{ref}}$). Field lines in the core of the domain are the most affected by the choice of $f$. Bottom left: average distance between loops (projections of lines of $B_{\text{ref}}$) and corresponding lines of $B$ in projection onto the plane of the sky ($z = 0$). The difference is smallest for $f \approx 1.69$. This coefficient is remarkably close to the scaling coefficient between $\alpha_{\text{ref}}$ and $\alpha_{\text{fit}}$ from MLM09, shown at the lower right panel as a dashed line. Diamonds show $\alpha_{\text{ref}}$ of the individual field lines of $B_{\text{ref}}$ vs. $\alpha_{\text{fit}}$ of the MLM09 approximations of these field lines.

(A color version of this figure is available in the online journal.)
Figure 6. Reconstruction of Low & Lou field from Schrijver et al. (2008) using schemes II.b and II.c, i.e., QGR with ideal and realistic loop input, that is, reconstructed from two-dimensional loop projections (see Table 1). Panels (a)–(i): field lines, line-of-sight-integrated magnitude of current and horizontal slices of $\alpha$ for $B_{\text{ref}}$ and $B$ for II.b and II.c. Panel (j): field lines of $B_{\text{obs}}$ (all field lines are traced from the same starting points). Panels (h) and (l): line-of-sight-integrated volume constraints for II.b and II.c.

(A color version of this figure is available in the online journal.)
smoothly through the lower boundary by comparison with vector magnetogram data. Hence, we repeat the experiments from the previous section, but with choosing a more realistic solar-like field as $B_{\text{ref}}$. We use two particular NFFF solutions from Schrijver et al. (2008), who presented NFFF reconstructions of the coronal field for AR 10930 before and after a major flare using several extrapolation methods applied to Hinode vector magnetograms. They found that the extrapolations which best matched observed coronal features were GR solutions obtained with the Wheatland (2007) code, using $\alpha$ values from the positive polarity of the magnetograms (hereafter $Wh_{\alpha}^+_{\text{pp}}$). We use those as our reference field. These solutions also had the largest free energy of all extrapolations. Another advantage of these fields for our study is that they use the same boundary conditions and nearly the same numeric implementation as the QGR scheme. We emphasize that the objective is in this case not to create a realistic representation of coronal field but to test the new algorithm on known NFFFs that are expected to more closely resemble the coronal field overlying a solar active region.

**Table 2**

|                  | $C_{\text{vec}}$ | $C_{\text{CS}}$ | $1 - E_{\text{in}}$ | $1 - E_{\text{out}}$ | CWsin | $E/E_P$ | $H(B|B_P)$ |
|------------------|------------------|------------------|---------------------|----------------------|-------|---------|------------|
| Reference field  | 1.00             | 1.00             | 1.00                | 1.00                 | 0.01  | 1.24    | 1.00       |
| Quasi-Grad–Rubin with vector magnetograms |                 |                  |                     |                      |       |         |            |
| I.a              | 0.99             | 0.93             | 0.80                | 0.64                 | 0.03  | 1.27    | 0.67       |
| I.b              | 0.99             | 0.96             | 0.83                | 0.70                 | 0.02  | 1.19    | 0.81       |
| I.c              | 1.00             | 0.97             | 0.88                | 0.75                 | 0.02  | 1.23    | 0.91       |
| Quasi-Grad–Rubin with loop trajectories alone |                 |                  |                     |                      |       |         |            |
| II.b             | 0.99             | 0.96             | 0.81                | 0.68                 | 0.02  | 1.17    | 0.77       |
| II.c             | 0.99             | 0.97             | 0.86                | 0.78                 | 0.02  | 1.23    | 0.93       |
| Ranges reported in Schrijver et al. (2006) |                 |                  |                     |                      |       |         |            |
|                  | 0.94–1.00        | 0.54–0.91        | 0.48–0.92           | −2.2–0.66            | 0.03–0.57 | 0.82–1.14 | ...        |
| Potential field  | 0.86             | 0.87             | 0.50                | 0.44                 | ...   | 1.00    | 0.00       |

**Notes.** The values for I.c and II.c are reported for the optimal solution with $f = 1.69$. The values for $B_{\text{ref}}$ and $B_P$ are shown for comparison, and so are the ranges of values for different NFFF extrapolations reported in Schrijver et al. (2006). Relative helicity is stated in fractions of that of the reference field. For notation, refer to Table 1.
Figure 8. Values of CWsin and \(E/E_P\) in the center of the domain (the same region as used in Schrijver et al. 2008) for the QGR calculation for one of the solar-like fields demonstrating oscillatory behavior. Different stages of this cycle are discussed in the text. The values of \(E_{\text{ref}}\) and \(CW_{\text{sin ref}}\) are shown as dashed lines.

For both pre- and post-flare reference fields, we select random sets of field lines and evaluate \(\langle \alpha \rangle\) on each field line. These field lines are used as trajectories \(L_i\) in II.b setup and their projections onto \(z = 0\) plane are used as loops for II.c setup. We also use \(B_z\) at the lower boundary and start with the same initial \(B_P\) as the other schemes in Schrijver et al. (2008). We perform final tests on the full-sized domain but determine the scaling factor \(f\) on the domain downsampled by a factor of 0.5 (this is done to speed up computations and to allow the possibility that currents have structure finer than the grid size, which is likely to be the case for real data).

For both sets of loops (volume constraints for the downsampled domain were only imposed at a small fraction of pixels, at about 1.7% of the current-carrying volume and the bigger one covered about 6.6% of the current-carrying volume), the solution for the pre-flare configuration does not converge with the II.b inputs. Instead, it enters a remarkably stable oscillatory cycle with a period of \(\approx 50\) iterations. This cycle develops at \(\approx 200\) iterations, as shown in Figure 8. We ran the code for a few thousands iterations to verify that the cycle is indeed stable. As energy slowly increases, a sheared arcade forms similar to the one in \(B_{\text{ref}}\), but as the energy reaches its maximum and becomes most similar to \(B_{\text{ref}}\), the field experiences drastic changes. Some of the current-carrying field lines rapidly “escape” the domain via the \(y = -100\) boundary, what changes the \(\alpha\) values on these field lines (\(\alpha\) is set to zero), as explained in Section 2. In the stage of the cycle with the lowest energy most of the field lines from the core of the region connect to the \(y = -100\) boundary and so carry no currents. This may be a valid force-free solution, though it is not consistent with the volume constraints which require \(\alpha\) to be non-zero in some points in the volume. When the \(\alpha\) values from the volume constraints are reimposed again at each iteration, the currents gradually build up again and the cycle repeats. The escape of field lines does not represent a physical evolution of the field as the iterations are not related to any physically meaningful time-like variable. The same oscillatory behavior is found in numerous experiments with this particular test case. Schrijver et al. (2008) report that the pre-flare solution which we use as \(B_{\text{ref}}\) did not fully converge either; it kept oscillating.

Below we discuss factors possibly causing the oscillations and a way to damp them. These factors are (1) numerical noise in \(\alpha\) and therefore in the electric currents that appear even in current-free areas and (2) deviations of the input data from a force-free field (as we discuss below, variations in \(\alpha\) due to numerical uncertainties are \(|\alpha| \lesssim 0.005\) arcsec\(^{-1}\)). As we use the same numeric solver, the noise in our case is expected to be of similar nature and magnitude.

Figure 9. Left panel: image of horizontal slice of \(\alpha_{\text{ref}}\) in the pre-flare Wh + close to the lower boundary (the gray scale goes from \(-0.8\) to 0.8 arcsec\(^{-1}\)) and two profiles of \(\alpha_{\text{ref}}\) (solid line on both profiles) and \(\alpha_{\text{pot}}\) (dashed line on both profiles) in this slice. Top right panel: areas with significant currents. Bottom right panel: areas with no currents in both \(B_{\text{ref}}\) and \(B_P\). Variations in \(\alpha\) due to numerical uncertainties are \(|\alpha| \lesssim 0.005\) arcsec\(^{-1}\).
Figure 10. Histograms of $\alpha_{\text{ref}}$ evaluated numerically on closed field (black line) and open field regions (gray line) in the pre-flare Wh$^+$ field. No currents are allowed on the open field by the GR scheme, used to calculate $B_{\text{ref}}$. Hence, the gray line shows the numerical noise. The distribution of the noise, evaluated from this plot, has a half-width at half-maximum of $\approx 10^{-3}$ arcsec$^{-1}$.

$B_{\text{ref}}$ in this case is not exactly force-free even at full resolution. The damping that we consider allows the calculated field to have small variations in $\alpha$ along field lines as well as small deviations of the solution from the volume constraints.

The first factor is the influence of numerical noise when solving $\nabla \times B^{(n+1)} = \alpha^{(n)} B^{(n)}$ (Step 2 in the algorithm in Section 2), especially around sharp edges in $\alpha^{(n)}$, and with the artifacts introduced by the Fourier transforms around these edges. These effects introduce noise in the $\alpha^{(n+1)}$ values obtained in the next step. Figures 9 and 10 clarify the amount of such noise and its relative size to the signal. In areas of closed field $|\alpha_{\text{ref}}| \lesssim 0.8$ arcsec$^{-1}$ and in the areas with open field (and hence no currents) $|\alpha_{\text{ref}}| \lesssim 5 \times 10^{-3}$ arcsec$^{-1}$. The flux-weighted distribution of $\alpha$ evaluated numerically in the current-free region has half-width at half-maximum of $\approx 10^{-3}$ arcsec$^{-1}$. We chose arcseconds as the units of length to maintain consistency with the

Figure 11. Variation in $\alpha$ in the pre-flare Wh$^+$ field used as a solar-like test case. Left column: selected field lines of $B_{\text{ref}}$ (dashed red) and stream lines of $J_{\text{ref}}$ initiated at points along these field lines. Right column: profiles of $\alpha_{\text{ref}}$ along these field lines. These panels indicate both small-scale and large-scale variation in $\alpha_{\text{ref}}$ significantly above the noise threshold. The values of $\langle \alpha_{\text{ref}} \rangle$ and $\langle \alpha_{\text{ref}} \rangle \pm \sigma$ are shown as dashed and dotted lines, respectively.

(A color version of this figure is available in the online journal.)
study where this region was introduced (Schrijver et al. 2008), and arcseconds$^{-1}$ as units of $\alpha$ to maintain consistency with the studies where the fitting algorithm was verified (Malanushenko et al. 2009, 2011). These units may be converted to Mm and Mm$^{-1}$ as follows: the lower boundary is a square with the side 200$''$, which for this particular observation is $\approx 142$ Mm and the $\alpha$ values we report in this study are typically $|\alpha| \lesssim 0.5$ arcsec$^{-1} \approx 0.71$ Mm$^{-1}$.

The second factor is errors in the volume constraint data. In the case discussed in this section, $B_{\text{ref}}$ has significant non-zero magnetic forces from the perspective of QGR. As discussed in Section 2, for convergence QGR requires not only that the Lorentz force is small everywhere in the volume, but also that the integrals of the Lorentz force along the field lines are small. These conditions are not met for $B_{\text{ref}}$ as shown in Figure 11, $\alpha$ changes substantially along central field lines. This is due to the Wh$_{pp}^+$ solutions themselves not converging precisely during the GR iteration used to calculate them (as mentioned in Schrijver et al. 2008).

To account for these issues, we introduce two uncertainty thresholds: $\Delta \alpha_{\text{err}}$ and $\Delta \alpha_{\text{noise}}$. The first allows the solution to have $\alpha$ values slightly different from the $\alpha$ values imposed along loop trajectories $L_i$, but only at points satisfying $|\alpha^{(n-1)} - \alpha_i| \geq \Delta \alpha_{\text{err}}$.

1. Impose the volume constraints by setting $\alpha^{(n-1)} = \alpha_i$ along loop trajectories $L_i$, but only at points satisfying $|\alpha^{(n-1)} - \alpha_i| \geq \Delta \alpha_{\text{err}}$.

2. Calculate updated field values $B^{(n)}$ from Equation (3) subject to the prescribed boundary conditions. This equation is solved using a vector potential $A^{(n)}$ such that $B^{(n)} = \nabla \times A^{(n)}$, so the divergence-free condition is satisfied to truncation error.

3. Calculate an updated set of values for the force-free parameter $\alpha^{(n)}$: for every point in $V$, assign $\alpha^{(n)} = \langle \alpha^{(n-1)} \rangle$ averaged along the field line in $B^{(n)}$ that passes through that point, but only at points satisfying $|\alpha^{(n)} - \langle \alpha^{(n-1)} \rangle| \geq \Delta \alpha_{\text{noise}}$. Otherwise retain the value of $\alpha$ from the previous iteration. If a field line leaves the domain through any boundary but the lower one, the value of $\alpha$ is set to zero along it (in common with the Wheatland 2007 GR scheme). This ensures that no currents go off to infinity so that the fields’ energy remains finite.

4. Repeat (1)–(3) until $B^{(n)} \approx B^{(n-1)}$ and $\alpha^{(n)} \approx \alpha^{(n-1)}$ to within a tolerance and therefore $\nabla \times B^{(n)} \approx \alpha^{(n)} B^{(n)}$.

For the case of the Wh$_{pp}^+$ field we choose $\Delta \alpha_{\text{err}} = \Delta \alpha_{\text{noise}} \approx 5 \times 10^{-4}$ arcsec$^{-1}$. For comparison, significant $\alpha$ for the pre-flare $B_{\text{ref}}$ are $|\alpha_{\text{ref}}| \propto 0.8$ arcsec$^{-1}$, as shown in Figure 9. Two exceptions are the II.b pre-flare solution on the downsampled domain for fewer loops case and II.c pre-flare solution on

Figure 12. Energy $E/E_F$ at each iteration as demonstration of convergence of different schemes for the pre-flare and post-flare Wh$_{pp}^+$ field fields. These particular plots correspond to the downsampled datacube for the case with fewer loops.
the full-size domain for more loops case, for which the error threshold is increased to $5 \times 10^{-3}$ arcsec$^{-1}$ in order to damp oscillations. Figure 12 shows convergence plots for the II.b and II.c cases for the case with the fewer loops.

The correction factors $f$ are determined in the same way as in Section 5.2. We find a best-fitter $f = 4.0$ for both pre- and post-flare data. In both cases, this factor matches the coefficient between $\alpha_{\text{ref}}$ and $\alpha_{\text{fit}}$ (see Figures 13 and 14) for individual loops, which provides further evidence in favor of this method of estimating $f$.

The results for the pre- and post-flare reference fields are summarized in Tables 3 and 4 and Figures 15–17. In each case, the QGR reproduces the overall shape of field lines and the large-scale features of the current distribution. The reconstructed fields have $\geq 50\%$ of the free energy and $\geq 25\%$ of the relative helicity of the reference fields. The reconstructions using half-resolution data are slightly inferior to those using full-resolution data, based on the metrics in Tables 3 and 4, but the solutions still reproduce at least half of the free energy and the quarter of helicity of the reference field and the large-scale structure of currents.

6. DISCUSSION AND CONCLUSIONS

We demonstrate that coronal loop trajectories can be used to directly constrain the structure of a model magnetic field. While the observed loops do not cover all of the coronal volume, they provide information about the shape of the coronal field lines, which boundary data alone lack.

We propose a method that constructs nonlinear force-free fields using line-of-sight magnetograms and coronal loops observed in the plane-of-sky projection. This may mitigate the problems NLFFF schemes encounter with currents determined from vector magnetograms (Demoulin et al. 1997). The loops are first approximated by lines of constant-$\alpha$ fields, with different $\alpha$ values for each loop. This is done using an existing scheme developed by Malanushenko et al. (2009) which we refer to as the MLM09 fit in this paper. The approximate $\alpha$ values along the approximate loop trajectories are treated as volume constraints in a quasi-Grad–Rubin algorithm, using the code modified from Wheatland & Régnier (2009).

The method, which we refer to as the quasi-Grad–Rubin method (or QGR) is tested on several nonlinear force-free fields and the results demonstrate good performance of the method. While traditional extrapolations of coronal magnetic fields have been found to provide poor matches to coronal features observed in X-rays and EUV (DeRosa et al. 2009), the fields created by QGR are constructed with the effort to match observed coronal features and thus may provide a more realistic model of the actual coronal magnetic field.

The problem of constructing a nonlinear force-free field is typically viewed as a boundary value problem requiring an extrapolation of the field from the boundary to the volume of the corona. However, throughout this paper we purposely avoid referring to QGR as an extrapolation scheme, because it is not. It is a mixture of extrapolation of magnetic field and interpolation of electric currents. The reason we tend to view the step of filling the volume with $\alpha$ values as an interpolation-like procedure is as follows. At each iteration, $\alpha$ is averaged along lines of the field at the present iteration. If the solution has not yet converged, this field is different from the one on the previous iteration, so $\alpha$ is averaged across lines of the field of the previous iteration (see Section 2). Substantial differences in $\alpha$ values on such field lines result in extreme values getting “spread” through the volume and the values of $\alpha$ being “smoothed” along field lines. Smaller differences between the fields from two consecutive iterations should result in $\alpha$ smoothing out on shorter distances. So on each consecutive iteration $n$ the $\alpha$ values are smoothed across field lines to a distance which depends on the magnitude of

| Table 3 | Metrics for the Pre-flare Reference Field |
|---------|------------------------------------------|
| $C_{\text{vec}}$ | $C_{\text{CS}}$ | $1 - E_n$ | $1 - E_m$ | CWsin | $E/E_P$ | $H(B|B_0)$ |
| Half-resolution |
| Reference field | 1.00 | 1.00 | 1.00 | 1.00 | 0.35 | 1.31 | 1.00 |
| QGR with loop trajectories alone, fewer loops case | II.b | 0.98 | 0.98 | 0.83 | 0.84 | 0.37 | 1.16 | 0.62 |
| II.c | 0.97 | 0.97 | 0.79 | 0.80 | 0.32 | 1.20 | 0.36 |
| QGR with loop trajectories alone, more loops case | II.b | 0.98 | 0.98 | 0.82 | 0.83 | 0.34 | 1.23 | 0.60 |
| II.c | 0.97 | 0.97 | 0.77 | 0.76 | 0.30 | 1.27 | 0.21 |
| Potential field | 0.86 | 0.94 | 0.62 | 0.70 | ... | 1.00 | 0.00 |
| Full resolution |
| Reference field | 1.00 | 1.00 | 1.00 | 1.00 | 0.24 | 1.32 | 1.00 |
| QGR with loop trajectories alone, fewer loops case | II.b | 0.98 | 0.99 | 0.85 | 0.86 | 0.11 | 1.18 | 0.64 |
| II.c | 0.98 | 0.98 | 0.80 | 0.81 | 0.07 | 1.27 | 0.43 |
| QGR with loop trajectories alone, more loops case | II.b | 0.98 | 0.99 | 0.83 | 0.84 | 0.09 | 1.26 | 0.62 |
| II.c | 0.97 | 0.97 | 0.77 | 0.77 | 0.08 | 1.30 | 0.29 |
| Potential field | 0.86 | 0.94 | 0.62 | 0.70 | ... | 1.00 | 0.00 |

Notes. The numbers for II.c solution are reported for the $f = 4.0$ solution. For notation, refer to Table 1.

| Table 4 | Metrics for the QGR Results for the Post-flare Reference Field |
|---------|------------------------------------------|
| $C_{\text{vec}}$ | $C_{\text{CS}}$ | $1 - E_n$ | $1 - E_m$ | CWsin | $E/E_P$ | $H(B|B_0)$ |
| Half-resolution |
| Reference field | 1.00 | 1.00 | 1.00 | 1.00 | 0.13 | 1.16 | 1.00 |
| QGR with loop trajectories alone | II.b | 0.99 | 0.99 | 0.86 | 0.87 | 0.10 | 1.07 | 0.48 |
| II.c | 0.99 | 0.99 | 0.88 | 0.87 | 0.07 | 1.13 | 0.63 |
| Potential field | 0.94 | 0.97 | 0.76 | 0.80 | ... | 1.00 | 0.00 |
| Full resolution |
| Reference field | 1.00 | 1.00 | 1.00 | 1.00 | 0.17 | 1.14 | 1.00 |
| QGR with loop trajectories alone | II.b | 0.99 | 0.99 | 0.89 | 0.88 | 0.13 | 1.09 | 0.50 |
| II.c | 0.99 | 0.99 | 0.89 | 0.88 | 0.10 | 1.14 | 0.69 |
| Potential field | 0.93 | 0.97 | 0.75 | 0.80 | ... | 1.00 | 0.00 |

Notes. The numbers for II.c are reported for the $f = 4.0$ solution. For notation, refer to Table 1.
the angle between $\mathbf{B}^{(n)}$ and $\mathbf{B}^{(n-1)} \times \mathbf{J}^{(n)}$, and therefore on the Lorentz force at the $n$th iteration. The process therefore results in a smooth distribution of $\alpha$ in the volume and decreasing the Lorentz forces implies smaller scale changes in this already smooth distribution. The described scheme cannot produce $\alpha$ bigger in magnitude than the volume constraints and it tends to produce a smooth transition of $\alpha$ between these constraints in such a way as to minimize Lorentz forces. This explains the interpolation-like nature of QGR with respect to $\alpha$. This scheme does not resolve fine structure of the “interpolated” variable (in this case, $\alpha$), as no interpolation scheme can, but it successfully approximates general trends, as expected from an interpolation scheme.

We also develop a way to deal with the uncertainties in the input data and the numeric noise. The uncertainties in the observables produce inconsistency with a force-free solution. As such uncertainties are expected, this is an important feature of the method. QGR in the form described in Section 5.3 allows the volume constraints not to be reimposed if the average $\alpha$ on a given field line which passes through a given constraint point is within a small prescribed amount of $\alpha_i$ of the constraint. This means that a magnetic field which is force-free but imperfectly matches given volume constraints for $\alpha$ would not be changed by the method. As with any numerical scheme, QGR is also prone to numerical noise. We are able to determine the range of this noise for a given problem. The method assumes that $\alpha$ below this noise level is numerically not distinguished from zero. It also assumes that average $\alpha$ along a given field line may vary within this noise range. It therefore does not replace $\alpha$ by the newly defined average along the field line if that average differs to less than the numerical noise threshold from the previously determined value.

We noted that fitting loops with lines of Chiu & Hilton (1977) constant-$\alpha$ fields results in the underestimation of $\alpha$ and verify

![Figure 13](image13.png)

**Figure 13.** Left panel: average distance between loops from the pre-flare Wh$_{pp}$ field and corresponding lines of $\mathbf{B}$ for the QGR solution in the same manner as in Figure 5. The factor $f = 4.0$, which yields the best-matching solution, is again close to scaling coefficient between $\alpha_{\text{fit}}$ from MLM09 underestimates $\alpha_{\text{ref}}$. Right panel: a scatter plot of $\alpha_{\text{ref}}$ and $\alpha_{\text{fit}}$ for individual loops and a line with the slope which equals to the best-matching $f$. Vertical error bars indicate the average variation of $\alpha$ along loops in the reference field.

![Figure 14](image14.png)

**Figure 14.** Same as Figure 12, but for the QGR solution for the post-flare reference field. The best-matching scaling factor $f$ is found to be the same as for the pre-flare field.
Figure 15. QGR solutions for the pre-flare WHF reference field in full resolution for fewer loops case using schemes II.b and II.c (QGR with ideal and realistic loop input, that is, reconstructed from two-dimensional loop projections—refer to Figure 1). Panels (a)—(i): field lines, line-of-sight-integrated magnitude of current and horizontal slice of $\alpha$ for $B_{\text{ref}}$ and $B$ for II.b and II.c. Panel (j): field lines of $B_P$ (all field lines are traced from the same starting points). Panels (h) and (l): line-of-sight-integrated volume constraints for II.b and II.c.

(A color version of this figure is available in the online journal.)
Figure 16. QGR solutions for the pre-flare $W_{Wh}$ reference field in half-resolution for more loops using schemes II.b and II.c (QGR with ideal and realistic loop input, that is, reconstructed from two-dimensional loop projections—refer to Figure 1). Panels (a)–(i): field lines, line-of-sight-integrated magnitude of current and horizontal slice of $\alpha$ for $B_{ref}$ and $B$ for II.b and II.c. Panel (j): field lines of $B_P$ (all field lines are traced from the same starting points). Panels (h) and (l): line-of-sight-integrated volume constraints for II.b and II.c.

(A color version of this figure is available in the online journal.)

that it is at least partly due to a difference in the size of the volumes which contain currents (finite in reference cases and half-space in the fields used for fitting, see Section 5.1). We determine that the underestimation coefficient is roughly the same for most loops and that this coefficient may be determined from observables (projected loops). Applying this determined coefficient leads to a good match between the reference field and the model, as demonstrated in Section 4. A rigorous proof of the nature of such a coefficient and its analytic evaluation are subjects of future studies.
We do not find substantial differences when reconstructing a reference field on a full size domain or on a downsampled domain. This might be due to the smoothness of fields created by the QGR due to its interpolation-like nature for the reasons discussed above. This is an important result, as the test case in Section 5.3 has structure of currents finer than the grid size in the downsampled test, which may also be the case when modeling coronal fields.
While developed for currents approximated from loops in EUV and X-ray images, QGR yields better results when currents are measured exactly, e.g., from vector magnetograms (II.b inputs in Sections 5.2 and 5.3). This gives hope that as vector magnetograms become more applicable for NLFFF modeling (at least in the cores of active regions), the performance on the II.b level could be achieved. This method could also benefit from the exact knowledge of the three-dimensional shape of the loops, e.g., drawn from STEREO and Solar Dynamics Observatory (SDO) satellites combined.

Overall, we find that the method developed in this paper is able to recover over half of the free energy and over a quarter of the helicity for the solar-like test case fields, which is more than was reported for previously tested methods (Metcalf et al. 2008; Schriever et al. 2008; DeRosa et al. 2009). The method recovers large-scale features of the field well, such as structure of currents, shape of field lines, and the connectivity of the field, but it fails to resolve fine structure. We nonetheless find that the large structure determines at least half of the free energy and a quarter of the relative helicity and therefore QGR may be used to provide estimates of these quantities.

This work was supported by AIA contract NNG04EA00C to the Lockheed Martin Advanced Technology Center through a grant to Montana State University, in collaboration with the University of Sydney. Hinode is a Japanese mission developed and launched by ISAS/JAXA, with NAOJ as domestic partner and NASA and STFC (UK) as international partners. It is operated by these agencies in cooperation with ESA and NSC (Norway).

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