FREDHOLM MODULES ON P.C.F. SELF-SIMILAR FRACTALS
AND THEIR CONFORMAL GEOMETRY.

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Abstract. The aim of the present work is to show how, using the differential calculus associated to Dirichlet forms, it is possible to construct Fredholm modules on post critically finite fractals by regular harmonic structures \((D, r)\). The modules are \((d_{\mathcal{S}}, \infty)\)-summable, the summability exponent \(d_{\mathcal{S}}\) coinciding with the spectral dimension of the generalized laplacian operator associated with \((D, r)\). The characteristic tools of the noncommutative infinitesimal calculus allow to define a \(d_{\mathcal{S}}\)-energy functional which is shown to be a self-similar conformal invariant.

1. Introduction and description of the results.

The construction of Fredholm modules \((F, \mathcal{H})\) on compact topological spaces \(K\) is a generalization of the theory of elliptic differential operators on compact manifolds. In its odd form, one requires that the elements \(f\) of the algebra of continuous functions \(C(K)\) are represented as bounded operators \(\pi(f)\) on a Hilbert space \(\mathcal{H}\) on which it is considered a distinguished self-adjoint operator \(F\) of square one \(F^2 = 1\), the symmetry, in such a way that the commutators \([F, \pi(f)]\) are compact operators.

This notion, introduced by M. Atiyah [At], A.S. Mishchenko [Mis], Brown-Douglas-Fillmore [BDF], and G. Kasparov [Kas], lies at the core of the theory on noncommutative differential geometry created by A. Connes [C2], where the operator \(df := i[F, \pi(f)]\) is the operator theoretical substitute for the differential of \(f\). In its simplest example, \(F\) is the Hilbert transform acting on the space of square integrable functions on the circle. In Atiyah’s motivating case, \(\mathcal{H}\) is the module of square integrable sections of a smooth vector bundle \(\xi\) over a smooth manifold, the continuous functions acting in the natural way, while \(F\) arises from the parametrix of an elliptic pseudo-differential operator of order 0 on \(\xi\).

In the present work, we construct Fredholm modules on a class of self-similar fractal spaces, known as post critically finite (shortened as p.c.f. since now on). Self-similarity refers to the fact that such a space can be reconstructed as finite union of homeomorphic pieces of itself. The p.c.f. property, on the other hand, translates or generalizes mathematically, a property of finite ramification and it is for this reason that, in general, these spaces fail to be manifolds modelled on open Euclidean sets, so that the usual Leibniz-Newton infinitesimal calculus is no more available.

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These spaces have been largely investigated from the point of view of potential and spectral analysis (Dirichlet forms, Laplacians, heat kernels, Green functions, eigenvalues distribution) and probability theory (construction and analysis of diffusive Markov processes) (see for example [Ba], [FS], [Ki], [Ku]). Spaces of this class, including, for example, Koch’s curve, Sierpinski’s gasket, Hata’s tree-like set and Lindstrøm’s snowflake, exhibit singular behaviors when compared, from the above points of view, to differentiable riemannian manifolds. For example: i) their energy measures are, in general, singular with respect to any self-similar volume measure (see [BST], [Hi]); ii) in the strong symmetric case, they support localized eigenfunctions (see [FS], [Ki]); iii) in the so called arithmetic or lattice case, the integrated density of states is discontinuous (see [FS], [Ki]). It is worth to mention that the study of these exotic behaviors of fractals spaces was suggested and motivated for application to condensed matter physics (see [L], [RT]).

The first constructions of Fredholm modules over subsets of nonintegral Hausdorff dimension were given by A. Connes in [C2 IV.3] for quasi-circles embedded in the plane and Cantor subsets of the real line, while D. Guido - T. Isola considered in [GI1,2,3] more general subsets of \( \mathbb{R}^n \).

Our construction of Fredholm modules on p.c.f. fractals is based on the notion of regular harmonic structure introduced by J. Kigami [Ki] and on the differential calculus associated to Dirichlet forms we developed in [CS].

To any fixed harmonic structure on a p.c.f fractal one can associate its self-similar Dirichlet form \( (E,F) \). This is a lower semi-continuous quadratic form, defined on a uniformly dense subalgebra \( F \) of \( C(K) \) and satisfying the characteristic contraction property

\[
E[a \wedge 1] \leq E[a]
\]

which generalizes the Dirichlet integral of an Euclidean space.

Dirichlet forms can be canonically represented as graphs semi-norms

\[
E[a] = ||\partial a||^2_H
\]

of an essentially unique derivation \( \partial : F \to H \) [CS]. This is a map, taking values in a Hilbert space \( H \) which is a module over the algebra \( C(K) \), and satisfying the Leibnitz rule:

\[
\partial(ab) = (\partial a) \cdot b + a \cdot (\partial b).
\]

It is by the derivation \( \partial \) that the Dirichlet form \( E \) defines, in a natural way, a differential calculus on the fractal \( K \).

To quantize this calculus, we then define the Fredholm module \( (F,H) \) by the symmetry \( F \) corresponding to the subspace \( \text{Im} \partial \subset H \): \( F \) acts as the identity on the range \( \text{Im}\partial \) of the derivation and specularly on its orthogonal complement \( (\text{Im}\partial)^\perp \).

It is worth to recall that the Dirichlet form \( E \) is a quadratic form closable on the Lebesgue space \( L^2(K,\mu) \) with respect to a large set of positive Radon measures \( \mu \) (see [Ki]). By classical results of Dirichlet form theory (see [BD], [FOT]), the closure of \( E \) with respect to such a measure \( \mu \) is then the quadratic form of a positive, self-adjoint operator \( \Delta_\mu \) on \( L^2(K,\mu) \) which generates a Markovian semigroup \( e^{-t\Delta_\mu} \) whose heat kernel \( p(t, x, y) \) gives the transition probabilities of a Markovian diffusion process on \( K \).

The above definition for \( (F,H) \) is inspired by a result of Connes-Sullivan [Co IV.4] concerning a canonical construction of a Fredholm module on a even dimensional manifold \( V \). Their construction makes use of a fixed riemannian metric on \( V \) but the resulting Fredholm module directly determines the underlying conformal structure.
of $V$. This is explicitly seen by a formula reproducing the fundamental conformal invariant, namely the $\dim V$-homogeneous Dirichlet integral $\int_V |\nabla u|^{\dim V}$ through a suitable summation procedure known as the Dixmier trace.

In our setting, by analyzing the speed of vanishing of the sequences of the eigenvalues of the commutators $[F, \pi(f)]$, through the use of the Schatten’s classes of compact operators and other interpolation ideals, we are able to construct, still through the Dixmier trace, a new, densely defined, strongly local, convex energy functional $\mathcal{E}_C$ on $C(K)$. Its homogeneity exponent $d_S$ equals the spectral dimension of the heat semigroup generated by the generalized Laplacian associated to the closure of the Dirichlet form with respect to a natural self-similar measure on $K$, called by J. Kigami and M. Lapidus [KL] the Riemannian volume measure $V$. This is explicitly seen by a formula reproducing the fundamental conformal structure.

We finally remark that our construction allows to associate to each harmonic structure a topological invariant of $K$, called by J. Kigami and M. Lapidus [KL] the generalized conformal structure on $K$.

2. Laplacian and Dirichlet Forms on P.C.F. Self-Similar Sets

In this section we will briefly recall, for reader’s convenience, the main definitions and properties of the objects we will investigate. See [Ki] for details.

**Definition 2.1. (Self-similar structures)** Let $K$ be a compact metrizable topological space and let $\mathcal{S} := \{1, 2, \ldots, N\}$ for a fixed integer $N$ greater than one. For each $i \in \mathcal{S}$, let us denote by $F_i$ a fixed continuous injection of $K$ into itself. Then, $(K, \mathcal{S}, \{F_i : i \in \mathcal{S}\})$ is called a self-similar structure if there exists a continuous surjection $\pi : \Sigma \to K$ such that $F_i \circ \pi = \pi \circ \sigma_i$ for every $i \in \mathcal{S}$, where $\Sigma := \mathcal{S}^\infty$ is the one-sided shift space and $\sigma_i : \Sigma \to \Sigma$ denotes the injection $\sigma_i(w_1 w_2 w_3 \ldots) := iw_1 w_2 w_3 \ldots$ for each $w_1 w_2 w_3 \ldots \in \Sigma$.

Notice that if $(K, \mathcal{S}, \{F_i : i \in \mathcal{S}\})$ is a self-similar structure, then $K$ is self-similar in the sense that

\begin{equation}
K = \bigcup_{i \in \mathcal{S}} F_i(K).
\end{equation}

It is customary to denote by $W_m := \mathcal{S}^m$ the set of words of length $m \in \mathbb{N}$, composed using the letters of the alphabet $\mathcal{S}$, with the understanding that $W_0 := \emptyset$, setting also $W_* := \bigcup_{m \in \mathbb{N}} W_m$ for the whole vocabulary. Each word $w = w_1 \ldots w_m \in W_m$ defines a continuous injection $F_w : K \to K$ by $F_w := F_{w_1} \circ \cdots \circ F_{w_m}$, whose image $F_w(K)$ is denoted by $K_w$.

**Definition 2.2. (Post critically finite fractals)** Let $(K, \mathcal{S}, \{F_i : i \in \mathcal{S}\})$ be a self-similar structure. The critical set $\mathcal{C} \subset \Sigma$ and the post critical set $\mathcal{P} \subset \Sigma$ are defined by

$$
\mathcal{C} := \pi^{-1} \left( \bigcup_{i \neq j} K_i \cap K_j \right) \quad \text{and} \quad \mathcal{P} := \bigcup_{n \geq 1} \sigma^n(\mathcal{C}),
$$
where $\sigma : \Sigma \to \Sigma$ is the shift map defined by $\sigma(w_1w_2\ldots) := w_2w_3\ldots$. A self-similar structure is called post critically finite (p.c.f. for short) provided $\mathcal{P}$ is a finite set. One sets also $V_0 := \pi(\mathcal{P})$, considered as the boundary of $K$, and

$$V_m := \bigcup_{w \in W_m} F_w(V_0) \quad \text{and} \quad V_* := \bigcup_{m \in \mathbb{N}} V_m.$$ 

It is easy to see that $V_m \subset V_{m+1}$ and that $V_*$ is dense in $K$.

For a finite set $V$, equipped with the standard counting measure, denote by $l(V)$ the space of scalar functions on $V$ with its scalar product $(u|v) := \sum_{p \in V} u(p)v(p)$ and by $\mathcal{L}(V)$ the collection of the Laplacian operators on $V$, i.e. the generators $L$ of the conservative, symmetric Markovian semigroups $e^{-tL}$ on $l(V)$. These are, essentially, symmetric, positive definite matrices $L = \{L_{u,v} : u,v \in V\}$ such that $L_{u,v} \leq 0$ for $u \neq v$ and $\sum_{u \in V} L_{u,v} = 0$ for all $v \in V$.

For $L \in \mathcal{L}(V)$ let $\mathcal{E}_L$ the associated Dirichlet form on $l(V)$: $\mathcal{E}_L(u,v) = (L(u|v))$.

For a fixed self-similar structure $(K,\mathcal{S},\{F_i : i \in \mathcal{S}\})$ on $K$, a Laplacian $D \in \mathcal{L}(V_0)$ and a vector $r := (r_1,\ldots,r_N)$, where $r_i > 0$ for $i \in \mathcal{S}$, define for each $m \geq 0$ the quadratic form

$$\mathcal{E}^{(m)}[u] := \sum_{w \in W_m} \frac{1}{r_w} \mathcal{E}_D[u \circ F_w], \quad u \in l(V_m)$$

where $r_w := r_{w_1}\ldots r_{w_m}$ for $w = w_1\ldots w_m \in W_m$. It is easy to see that there exists $H_m \in \mathcal{L}(V)$ such that $\mathcal{E}^{(m)}[u] = (H_mu|u)$.

We now introduce the main object of analysis on fractals.

**Definition 2.3.** (Harmonic structures) $(D,r)$ is said to be a harmonic structure on $(K,\mathcal{S},\{F_i : i \in \mathcal{S}\})$ if for all $m \geq 0$ and for any $u \in l(V_m)$ one has

$$\mathcal{E}^{(m)}[u] = \min\{\mathcal{E}^{(m+1)}[v] : v \in l(V_{m+1}) , \ v|_{V_m} = u\}.$$ 

It is known that (2.3) holds for all $m \geq 0$ if and only if it holds for $m = 0$.

**Definition 2.4.** If $(D,r)$ is a harmonic structure on $(K,\mathcal{S},\{F_i : i \in \mathcal{S}\})$, define

$$\mathcal{F} := \{u \in l(V_\ast) : \lim_{m \to \infty} \mathcal{E}^{(m)}[u|_{V_m}] < \infty\}, \quad \mathcal{F}_0 := \{u \in \mathcal{F} : u|_{V_0} = 0\}$$

and

$$\mathcal{E}[u] := \lim_{m \to \infty} \mathcal{E}^{(m)}[u|_{V_m}] \quad \text{for} \quad u \in \mathcal{F}.$$ 

Since the quadratic form $(\mathcal{E},\mathcal{F})$ is defined in a self-similar way, it naturally satisfies the following self-similarity.

**Proposition 2.5.** [Ki] (Self-similar quadratic form) Let $(D,r)$ be a harmonic structure on $(K,\mathcal{S},\{F_i : i \in \mathcal{S}\})$. Then $u \in \mathcal{F}$ if and only if $u \circ F_i \in \mathcal{F}$ for all $i \in \mathcal{S}$ and in that case

$$\mathcal{E}[u] = \sum_{i \in \mathcal{S}} \frac{1}{r_i} \mathcal{E}[u \circ F_i], \quad u \in \mathcal{F}.$$ 

To construct Dirichlet forms on $K$ we need to fix measures on $K$. Here is a natural class one may consider.
Proposition 2.6. [Ki] (Self-similar measure) For a fixed vector of weights \((\mu_1, \ldots, \mu_N)\) with \(\mu_i > 0\) for \(i \in S\) and \(\sum_{i \in S} \mu_i = 1\), there exists a unique Borel measure \(\mu\) on \(K\) such that

\[
\int_K f \, d\mu = \sum_{i \in S} \mu_i \int_K f \circ F_i \, d\mu, \quad f \in C(K).
\]

\(\mu\) is called the self-similar measure with weights \((\mu_1, \ldots, \mu_N)\).

Theorem 2.7. [Ki] (Dirichlet forms and generalized laplacians) Let \((D, r)\) be a harmonic structure on a p.c.f. self-similar structure \((K, S, \{F_i : i \in S\})\), \(\mu\) the self-similar measure on \(K\) with weights \((\mu_1, \ldots, \mu_N)\) and assume that \(\mu_i r_i < 1\) for all \(i \in S\).

Then \(\mathcal{F}\) is naturally embedded in \(L^2(K, \mu)\), \((\mathcal{E}, \mathcal{F})\) and \((\mathcal{E}, \mathcal{F}_0)\) are regular, local Dirichlet form on \(K\) and their associated nonnegative self-adjoint operators \(H_N\) and \(H_D\) have compact resolvent.

Definition 2.8. (Eigenvalues distribution) Assuming the same hypotheses as in the previous theorem, define, for \(\ast = N, D\), the eigenspace corresponding to \(\lambda \in \mathbb{R}\) as

\[
E_\ast(\lambda) = \{u \in D(H_\ast) : H_\ast u = \lambda u\}.
\]

If the multiplicity \(\dim E_\ast(\lambda)\) is not zero then \(\lambda\) is said to be an \(\ast\)-eigenvalue and a non zero \(u \in E_\ast(\lambda)\) is said to be a \(\ast\)-eigenfunction belonging to the \(\ast\)-eigenvalue \(\lambda\). The collection \(\text{Sp}(H_\ast)\) of all the eigenvalues of \(H_\ast\) is called the spectrum of \(H_\ast\).

As \(H_\ast\) is unbounded with compact resolvent, \(\text{Sp}(H_\ast)\) is unbounded and discrete, consisting of isolated eigenvalues of finite multiplicity only.

The function

\[
\rho_\ast(\cdot, \mu) : \mathbb{R} \rightarrow \mathbb{N} \quad \rho_\ast(x, \mu) := \sum_{\lambda \leq x} \dim E_\ast(\lambda)
\]

is called the eigenvalues counting function of \(H_\ast\). As \(H_\ast\) is nonnegative and unbounded \(\rho_\ast(x, \mu) = 0\) if \(x < 0\) and \(\lim_{x \to +\infty} \rho_\ast(x, \mu) = +\infty\).

The following is the fractal analogue of the famous Weyl’s asymptotic formula for the eigenvalue distribution of the laplacian on a compact riemannian manifold.

Theorem 2.9. [Ki] Let \((D, r)\) be a harmonic structure on a p.c.f. self-similar structure \((K, S, \{F_i : i \in S\})\), \(\mu\) the self-similar measure on \(K\) with weights \((\mu_1, \ldots, \mu_N)\) and assume that \(\mu_i r_i < 1\) for all \(i \in S\).

Let \(d_\mathcal{S} = d_\mathcal{S}(\mu)\) be the unique positive real number satisfying

\[
\sum_{i \in S} \gamma_i^{d_\mathcal{S}} = 1,
\]

where \(\gamma_i := \sqrt{\mu_i} \sqrt{d_\mathcal{S}}\) for \(i \in S\). \(d_\mathcal{S}\) is called the spectral exponent of \((\mathcal{E}, \mathcal{F}, \mu)\). Then

\[
0 < \liminf_{x \to +\infty} \frac{\rho_\ast(x, \mu)}{x^{d_\mathcal{S}/2}} \leq \limsup_{x \to +\infty} \frac{\rho_\ast(x, \mu)}{x^{d_\mathcal{S}/2}} < +\infty,
\]

the \(\liminf\) and the \(\limsup\) are the same for \(\ast = N\) and \(\ast = D\). In the non-lattice case, where \(\sum_{i \in S} \zeta \log \gamma_i\) is dense subgroup of \(\mathbb{R}\), defining

\[
R(x) := \rho_D(x, \mu) - \sum_{i \in S} \rho_D(\gamma_i^2 x, \mu) \quad U(t) := e^{-td_\mathcal{S}} R(e^{2t})
\]
we have
\[
\lim_{x \to +\infty} \frac{\rho_*(x, \mu)}{x^{d_s/2}} = \left( -\sum_{i \in S} \gamma_i^{d_s} \log \gamma_i^{d_s} \right)^{-1} d_s \int_{R} U(t) dt.
\]

Comparing the above result with the Weyl’s classical one for compact Riemannian manifold, one is led to define

**Definition 2.10.** Let \((D, r)\) be a harmonic structure on a p.c.f. self-similar structure \((K, \mathcal{S}, \{F_i : i \in S\})\), \(\mu\) the self-similar measure on \(K\) with weights \((\mu_1, \ldots, \mu_N)\) and assume that \(\mu_i r_i < 1\) for all \(i \in S\). The spectral volume \(\text{vol}(K, \mu)\) is then defined by
\[
\text{vol}(K, \mu) := \left( -\sum_{i \in S} \gamma_i^{d_s} \log \gamma_i^{d_s} \right)^{-1} d_s \int_{R} U(t) dt.
\]

On compact Riemannian manifolds, the Connes’ trace formula [C1] allows to reconstruct the Riemannian measure through the knowledge of suitable eigenvalues distributions. This is done by the Dixmier trace \(\text{Tr}_\omega\), a trace functional on the space of compact operators on a Hilbert space, depending on the choice of certain ultrafilters on \(\mathbb{R}_+\). This functional is singular in the sense that it vanishes on the ideal of trace-class operators. The following generalization of the Connes’ trace formula has been proved on fractals by J. Kigami and M. Lapidus.

**Theorem 2.11.** [KL] Let \((D, r)\) be a harmonic structure on a p.c.f. self-similar structure \((K, \mathcal{S}, \{F_i : i \in S\})\), \(\mu\) the self-similar measure on \(K\) with weights \((\mu_1, \ldots, \mu_N)\) and assume that \(\mu_i r_i < 1\) for all \(i \in S\). Then there exists a unique positive Borel measure \(\nu_{\mu}\) on \(K\) such that
\[
\int_K f \, d\nu_{\mu} = \text{Tr}_\omega \left( f \circ H^{-d_s/2}_D \right),
\]
where the symbol \(f\) denotes both a continuous function on \(K\) as the associated multiplication operator on \(L^2(K, \mu)\). Moreover the total mass of \(\nu_{\mu}\) equals the spectral volume of \(K\): \(\nu_{\mu}(K) = \text{vol}(K, \mu)\).

It has been proved in [KL], that for certain classes of fractals, \(\nu_{\mu}\) is the self-similar measure on \(K\) with weights \(\nu_i = \gamma_i^{d_s}\).

3. **Fredholm Modules associated to Harmonic Structures on p.c.f. Fractals.**

In this section we consider a fixed harmonic structure \((D, r)\) on a p.c.f. self-similar structure \((K, \mathcal{S}, \{F_i : i \in S\})\).

Choosing a self-similar measure \(\mu\) on \(K\) with weights \((\mu_1, \ldots, \mu_N)\) such that \(\mu_i r_i < 1\) for all \(i \in S\), we can consider, by Theorem 2.7, the Dirichlet form \((\mathcal{E}, \mathcal{F})\) associated to \((D, r)\) on \(L^2(K, \mu)\).

Applying the general theory developed in [CS], it is possible to consider a differential calculus on the fractal \(K\), associated to the Dirichlet form \((\mathcal{E}, \mathcal{F})\). In other words:

**Proposition 3.1.** [CS] There exists an essentially unique derivation \(\partial : \mathcal{B} \to \mathcal{H}\), defined on the Dirichlet algebra \(\mathcal{B} = C(K) \cap \mathcal{F}\) with values in a real Hilbert module \(\mathcal{H}\), which is a differential square root of the Dirichlet form in the precise sense that
\[
\mathcal{E}[u] = \|\partial u\|_{\mathcal{H}}^2 \quad u \in \mathcal{B}.
\]
By this, we mean that $\mathcal{H}$ is a Hilbert space on which the algebra $C(K)$ acts continuously in such a way that the Leibniz rule holds true:

$$\partial(ab) = (\partial a)b + a(\partial b) \quad a, b \in \mathcal{B}.$$  

In turn, the self-adjoint operator $H_N$ associated to $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ appears as a generalized laplacian

$$H_N = \partial^*_\mu \circ \partial_\mu$$

where $\partial_\mu$ denotes the closure of $(\partial, \mathcal{B})$ in $L^2(K, \mu)$. A corresponding result holds true for $H_D$.

When the harmonic structure is regular, i.e. $r_i < 1$ for $i \in S$, then the Dirichlet algebra $\mathcal{B}$ coincide with the domain $\mathcal{F}$ of the Dirichlet form, which, in this case, is already a sub-algebra of $C(K)$ [Ki 3.3].

To recover the information potentially carried by the derivation, we consider its associated phase.

**Definition 3.2.** Let us consider a fixed harmonic structure $(D, r)$ on a p.c.f. self-similar structure $(K, S, \{F_i : i \in S\})$. Let $\partial : \mathcal{B} \rightarrow \mathcal{H}$ be the associated derivation, defined on the Dirichlet algebra $\mathcal{B} = C(K) \cap \mathcal{F}$ with values in the symmetric Hilbert module $\mathcal{H}$. Let $P \in \text{Proj}(\mathcal{H})$ the projection onto the closure $\text{Im} \partial$ of the range of the derivation

$$P\mathcal{H} = \text{Im} \partial$$

and $F = P - P^\perp : \mathcal{H} \rightarrow \mathcal{H}$ the associated phase or symmetry.

The following result shows that the phase operator associated to a regular harmonic structure on p.c.f. fractals is an elliptic operator on $K$ in the sense of M. Atiyah [At].

**Theorem 3.3. (Fredholm module structure on fractals)** Let $(D, r)$ be a fixed regular harmonic structure on a p.c.f. self-similar structure $(K, S, \{F_i : i \in S\})$. Then $(F, \mathcal{H})$ is a Fredholm module over $C(K)$ in the sense of [At] and a densely 2-summable Fredholm module over $C(K)$ in the sense of [C IV 1.1 Definition 8].

**Proof.** Clearly $F^* = F$, $F^2 = I$. Let us start to prove that $[F, a]$ is Hilbert-Schmidt for all real valued $a \in \mathcal{F}$. Since

$$[P, a] = PaP^\perp - P^\perp aP$$

and $a$ is real valued, we have

$$||[P, a]||^2 = ||PaP^\perp||^2 + ||P^\perp aP||^2$$

so that

$$||[F, a]||^2_{L^2} = 4||[P, a]||^2_{L^2} = 8||P^\perp aP||^2_{L^2}.$$  

Using the Leibnitz rule for the derivation $\partial$, the fact that $P \circ \partial = \partial$ and $P^\perp \circ \partial = 0$, we have, for all $b \in \mathcal{F}$,

$$P^\perp aP(\partial b) = P^\perp (a\partial b) = P^\perp (\partial(ab) - (\partial a)b) = -P^\perp ((\partial a)b)$$

so that

$$||P^\perp aP(\partial b)|| = ||P^\perp ((\partial a)b)|| \leq ||(\partial a)b||.$$
Let us choose a self-similar Borel measure $\mu$ on $K$ with weights $(\mu_1, \ldots, \mu_N)$ such that $\mu_i < 1$ for $i \in S$. By [Ki Theorem 3.4.7], $(\mathcal{E}, \mathcal{F}_0)$ has discrete spectrum \{0 < \lambda_1 \leq \lambda_2 < \ldots\} in $L^2(K, \mu)$. Denoting by $a_1, a_2, \ldots$ the corresponding eigenfunctions, we have that the vectors $\xi_k := \lambda_k^{-1/2}(\partial a_k)$, $k \geq 1$, form an orthonormal complete system in $PH$. Then by (3.7) and (3.9)

\[(3.10)\]

\[
||[F, a]\|_2^2 = 8||P^2aP||_2^2 = 8 \sum_{k=1}^{\infty} \lambda_k^{-1}||P^2aP(\partial a_k)||_H^2 \leq 8 \sum_{k=1}^{\infty} \lambda_k^{-1}||\partial a_k||_H^2
\]

\[
= 8 \sum_{k=1}^{\infty} \lambda_k^{-1} \int_K a_k^2 d\Gamma(a) = 8 \int_K (\sum_{k=1}^{\infty} \lambda_k^{-1} a_k^2) d\Gamma(a)
\]

\[
= 8 \int_K g d\Gamma(a)
\]

where $g$ is the restriction on the diagonal of $K \times K$ of the Green function $G(x, y) = \sum_{k=1}^{\infty} \lambda_k^{-1} a_k(x)a_k(y)$, kernel of the compact operator $H_D^{-1}$ on $L^2(K, \mu)$ (see [Ki 3.6]) and $\Gamma(a)$ is the energy measure of $a \in \mathcal{F}$ defined by the Dirichlet form [FOT]

\[
\int_K b \, d\Gamma(a) := (\partial a (\partial a)b) = \mathcal{E}(a|ab) - \frac{1}{2} \mathcal{E}(b|a^2), \quad b \in \mathcal{B} = C(K) \cap \mathcal{F}.
\]

Since the harmonic structure is regular, $G$ is continuous on $K \times K$ ([Ki Proposition 3.5.5]) and we have

\[
||[F, a]\|_2^2 \leq 8 \left( \sup_{x \in K} g(x) \right) \mathcal{E}[a] < +\infty
\]

for all $a \in \mathcal{F} \subset C(K)$. Since $\mathcal{F}$ is uniformly dense in $C(K)$, $[F, a]$ is norm continuous with respect to $a \in C(K)$ and the space of compact operators is norm closed, we have that $[F, a]$ is compact for all $a \in C(K)$.

**Remark 3.4.** The proof given above shows that the regularity of a function $a \in \mathcal{B} = \mathcal{F} \cap C(K)$, can be detected using the energy form $\mathcal{E}$ and an auxiliary reference measure $\mu$ with respect to which $\mathcal{E}$ has discrete spectrum (in this respect see [Ki Theorem 3.4.6, Corollary 3.4.7]). The effectiveness of the upper bound on the Hilbert-Schmidt norm of the commutator $[F, a]$ depends on the integrability of the diagonal part of the Green function (of $\mathcal{E}$ with respect to $\mu$) with respect to the energy measure $\Gamma(a)$. The same proof thus provides a method for constructing Fredholm modules even for non regular harmonic structures. In these situations, one no more has $\mathcal{F} \subseteq C(K)$ but may uses the core of harmonic functions associated to the harmonic structure

$\mathcal{H}_s =: \{a \in \mathcal{F} : a$ is an $m$–harmonic function for some $m \geq 0\} \subset \mathcal{B}$.

In particular see [Ki 3.2] and the proof of [Ki Theorem 3.4.6].

We are now interested to investigate finer summability properties of the commutators $[F, a]$. The following lemma contains an estimate we will need below. It is essentially [Ki Lemma 5.3.5].

**Lemma 3.5.** Let $(D, r)$ be a fixed regular harmonic structure on a p.c.f. self-similar structure $(K, S, \{F_i : i \in S\})$ and let $\mu$ be a self-similar Borel measure on $K$ with

\[
\sum_{i \in S} \mathcal{E}[F_i] < +\infty.
\]
weights \((\mu_1, \ldots, \mu_N)\) such that \(r_i \mu_i < 1\) for \(i \in S\). Denote by \(d_S\) the spectral exponent of \((E, F, \mu)\). Then the potential operators

\[
G_p := \int_0^\infty t^{\frac{p}{2} - 1} e^{-tH_D} dt, \quad d_S < p \leq 2
\]

are compact in \(L^2(K, \mu)\) and their integral kernels \(g_p\) are positive continuous functions satisfying, for some \(c_1 > 0\),

\[
g_p(x, y) \leq c_1 \left( \frac{1}{\lambda_1} + \frac{2}{p - d_S} \right).
\]

**Proof.** By the Spectral Theorem \(G_p = \Gamma(\frac{p}{2}) H_D^{-\frac{p}{2}}\), so that the compactness follows from the discreteness of the spectrum of the laplacian. Let \(p_D(t, x, y)\) be the kernel of the heat semigroup \(e^{-tH_D}\) so that

\[
g_p(x, y) = \int_0^\infty t^{\frac{p}{2} - 1} p_D(t, x, y) dt.
\]

By [Ki Lemma 5.3.5] there exists \(c_1 > 0\) such that

\[
p_D(t, x, y) \leq \begin{cases} 
  c_1 t^{-\frac{d_0}{2}} & t \in (0, 1] \\
  c_1 e^{-(t-1)\lambda_1} & t \geq 1,
\end{cases}
\]

(in fact \(c_1 = \|e^{-H_D}\|_{L^1 \to L^\infty}\)) from which we get

\[
g_p(x, y) \leq c_1 \left\{ \int_1^\infty t^{\frac{p}{2} - 1} \cdot t^{-\frac{d_0}{2}} dt + \int_1^\infty t^{\frac{p}{2} - 1} \cdot e^{-(t-1)\lambda_1} dt \right\} \leq c_1 \left( \frac{1}{\lambda_1} + \frac{2}{p - d_S} \right).
\]

**Theorem 3.6. (Commutators and Shatten classes)** Let \((D, r)\) be a fixed regular harmonic structure on a p.c.f. self-similar structure \((K, S, \{F_i : i \in S\})\) and let \(\mu\) be a self-similar Borel measure on \(K\) with weights \((\mu_1, \ldots, \mu_N)\) such that \(r_i \mu_i < 1\) for all \(i \in S\).

Then \((F, \mathcal{H})\) is a densely \(p\)-summable Fredholm module over \(C(K)\) for all \(d_S < p \leq 2\). In particular

\[
\text{Trace} \left( \left[ [F, a] \right]^p \right) \leq c_2(p) \frac{\mu}{2} \cdot \left( E[a] \right)^{\frac{\mu}{2}} \cdot \left[ \text{Trace} \left( H_D^{-\frac{\mu}{2}} \right) \right]^{1 - \frac{\mu}{2}},
\]

where \(c_2(p) := 16c_1 (\frac{\lambda_1}{\lambda_1} + \frac{2}{p - d_S}) \Gamma \left( \frac{p}{2} \right)^{-1}\), for all \(a \in F\) (\(c_1\) being the constant appearing in Lemma 3.5).

**Proof.** Let us fix a \(a \in F\) real valued and denote by \(\{\mu_k(T) : k \geq 0\}\) the non vanishing singular values of a compact operator \(T\) arranged in decreasing order and repeated according to their multiplicity. Recall that \(\mu_k(T) = \mu_k(T^*)\). Setting \(S := [P, a]\) and \(T := P a P\), from (3.5) we get

\[
|S| = |T^*| + |T|
\]

and then

\[
\mu_{n+m}(S) = \mu_{n+m}(|T^*| + |T|) \leq \mu_n(T^*) + \mu_m(T) = \mu_n(T) + \mu_m(T)
\]

\[
\mu_k(S) \leq 2\mu_k(T) \leq 2\mu_k(T), \quad k \geq 0
\]
and finally

\[
\text{Trace (} |S|^p \text{) } = \sum_{k=0}^{\infty} \mu_k (|S|^p) = \sum_{k=0}^{\infty} \mu_k (S)^p \\
\leq 2^p \sum_{k=0}^{\infty} \mu_k (T)^p = 2^p \text{ Trace (} |T|^p \text{)}.
\]

(3.20)

Since \( \{\xi_k := \lambda_k^{-1/2} \partial a_k : k \geq 1\} \) is a complete orthonormal family in the Hilbert space \( PH \) and, by assumption, \( p \leq 2 \), we can use inequality [S Remark 1 page 17] and (3.9) to get

\[
\text{Trace (} |T|^p \text{) } = \sum_{k=0}^{\infty} \mu_k (T)^p \leq \sum_{k=0}^{\infty} \| T \xi_k \|^p = \sum_{k=0}^{\infty} \left( \| T \xi_k \|^2 \right)^{p/2}
\]

(3.21)

\[
= \sum_{k=0}^{\infty} \left( \| \lambda_k^{-1/2} (\partial a_k \xi_k) \|^2 \right)^{p/2}
\]

\[
= \sum_{k=0}^{\infty} \left( \int_K \lambda_k^{-1} a_k^2 d\Gamma(a) \right)^{p/2}.
\]

By Hölder’s inequality in the spaces \( l^q(\mathbb{N}) \), with conjugate exponents \( 2/p \) and \( 2/(2-p) \), we obtain

\[
\text{Trace (} |T|^p \text{) } \leq \sum_{k=0}^{\infty} \left( \int_K \lambda_k^{-1} a_k^2 d\Gamma(a) \right)^{p/2}
\]

(3.22)

\[
= \sum_{k=0}^{\infty} \lambda_k^{-\frac{p(2-p)}{2}} \left( \int_K \lambda_k^{-\frac{p}{2}} a_k^2 d\Gamma(a) \right)^{p/2}
\]

\[
= \left( \sum_{k=0}^{\infty} \lambda_k^{-\frac{p}{2}} \right)^{1-\frac{q}{2}} \left( \int_K \sum_{k=0}^{\infty} \lambda_k^{-\frac{p}{2}} a_k^2 d\Gamma(a) \right)^{p/2}
\]

\[
= \left[ \text{Trace (} H^{-\frac{p}{2}} D^{-\frac{p}{2}} \text{) } \right]^{1-\frac{q}{2}} \left( \int_K \sum_{k=0}^{\infty} \lambda_k^{-\frac{p}{2}} a_k^2 d\Gamma(a) \right)^{p/2}.
\]

Since \( G_p = \Gamma(p/2) H^{-\frac{p}{2}} \), we have \( \sum_{k=0}^{\infty} \lambda_k^{-\frac{p}{2}} a_k^2 (x) = \Gamma(p/2)^{-1} g_p (x, x) \). From Lemma 3.5 and (3.22) we have

\[
\text{Trace (} |T|^p \text{) } \leq \Gamma(p/2)^{-\frac{p}{2}} \left[ \text{Trace (} H^{-\frac{p}{2}} D^{-\frac{p}{2}} \text{) } \right] \left( \int_K g_p (x, x) \Gamma(a)(dx) \right)^{p/2}
\]

(3.23)

\[
\leq c_1^p \left( \frac{1}{\lambda_1} + \frac{2}{p-d_s} \right)^{\frac{p}{2}} \Gamma(p/2)^{-\frac{p}{2}} (\mathcal{E}[a])^{p/2} \left[ \text{Trace (} H^{-\frac{p}{2}} D^{-\frac{p}{2}} \text{) } \right]^{1-\frac{q}{2}}.
\]

Noticing that \( [F, a] = 2S \), we finally obtain from (3.20) and (3.23)

\[
\text{Trace (} [F, a]^p \text{) } \leq 4^p c_1^p \left( \frac{1}{\lambda_1} + \frac{2}{p-d_s} \right)^{\frac{p}{2}} \Gamma(p/2)^{-\frac{p}{2}} (\mathcal{E}[a])^{p/2} \left[ \text{Trace (} H^{-\frac{p}{2}} D^{-\frac{p}{2}} \text{) } \right]^{1-\frac{q}{2}}
\]

(3.24)

\[
= c_2 (p)^{\frac{p}{2}} \cdot (\mathcal{E}[a])^{p/2} \left[ \text{Trace (} H^{-\frac{p}{2}} D^{-\frac{p}{2}} \text{) } \right]^{1-\frac{q}{2}}.
\]
In order to proceed further, we need the following intermediate result.

**Lemma 3.7.** Let \( u \in L^1_{\text{loc}}([1, +\infty)) \) be a positive, locally integrable function such that \( u \in L^s([1, +\infty)) \) for \( s \in (1, 2) \) and

\[
(3.25) \quad \int_1^\infty u(t)^s \, dt \leq \frac{c}{s-1} \quad s \in (1, 2)
\]

for some constant \( c > 0 \). Then there exists a constant \( c' > 0 \) such that

\[
(3.26) \quad \int_1^x u(t) \, dt \leq c' \ln x \quad x \in (1, +\infty).
\]

**Proof.** By Hölder inequality and for \( x \geq 1, s \in (1, 2] \), we have

\[
\int_1^x u(t) \, dt \leq \left( \int_1^x u(t)^s \, dt \right)^{1/s} \cdot (x-1)^{1-1/s} \leq \left( \frac{c}{s-1} \right)^{1/s} \cdot (x-1)^{1-1/s}.
\]

Setting \( h(s) := \frac{1}{s} \ln \frac{c}{s-1} + \frac{1}{s-1} \ln(x-1) \) we have \( \int_1^x u(t) \, dt \leq e^{h(s)} \). Evaluating \( h(s) \) at its critical point, where \( \ln(x-1) = \frac{s}{s-1} + \ln \frac{c}{s-1} \), we get \( h(s) = 1 + \ln \frac{c}{s-1} \) and

\[
\int_1^x u(t) \, dt \leq \frac{ec}{s-1}.
\]

As \( s \in (1, 2] \), we have \( \ln(x-1) = \frac{s}{s-1} + \ln \frac{c}{s-1} \geq \ln ec + \frac{1}{s-1} \geq \ln ec^2 \), which implies \( x \geq 1 + ce^2 \) and finally

\[
\int_1^x u(t) \, dt \leq \frac{ec}{s-1} \leq ec \ln \frac{x-1}{ec} \leq c' \ln x
\]

for all \( x \geq 1 + ce^2 \) and \( c' := oec \) where \( \alpha > 1 \) is such that \( ec \geq (\alpha - 1)^{(\alpha-1)/\alpha} \). \( \square \)

We can now prove the finest summability properties for the quantum derivative of functions with finite energy on fractals.

**Theorem 3.8.** *(Commutator and Interpolation ideals)* Let \((D, \mathbf{r})\) be a fixed regular harmonic structure on a p.c.f. self-similar structure \((K, \mathcal{S}, \{F_i : i \in \mathcal{S}\})\) and let \( \mu \) be a self-similar Borel measure on \( K \) with weights \((\mu_1, \ldots, \mu_N)\) such that \( r_i \mu_i < 1 \) for \( i \in \mathcal{S} \).

Then \((F, \mathcal{H})\) is a densely \((d_S, \infty)\)-summable Fredholm module over \( C(K) \):

\[
(3.27) \quad [F, a] \in \mathcal{L}^{(d_S, \infty)}(\mathcal{H}) \quad a \in \mathcal{F}
\]

where \( \mathcal{L}^{(d_S, \infty)}(\mathcal{H}) \) is the interpolation ideal defined, for instance, in [C2 Chapter IV].

**Proof.** By the upper bound (2.11) on the eigenvalue counting function, there exists a constant \( c_3 > 0 \) such that

\[
(3.28) \quad \rho_+(x, \mu) \leq c_3 x^{d_S/2}, \quad x \geq \lambda_1.
\]

As \( k \leq \rho_+(\lambda_k, \mu) \leq c_3 \lambda_k^{d_S/2} \), we have \( c_3^{-2/d_S} \lambda_k^{2/d_S} \leq \lambda_k \) and also

\[
(3.29) \quad \text{Trace} \left( H_{D^{m/2}}^{-m/2} \right) = \sum_{k=1}^{\infty} \lambda_k^{-m/2} \leq c_3^{-m/2} \frac{d_S}{p - d_S}.
\]

As \( p - d_S < 1 \), we have, for the constant \( c_2(p) \) in (3.14) the bound

\[
(3.30) \quad c_2(p) \leq 16 c_1 \left( \frac{1}{\lambda_1} + 2 \right) \Gamma(p/2)^{-1} \frac{1}{p - d_S}.
\]
Combining (3.16), (3.29) and (3.30), we then have

\[ \text{Trace} \left( |[F, a]|^p \right) \leq \left[ 16c_1 \left( \frac{1}{\lambda_1} + 2 \right) \Gamma(p/2)^{-1} \right]^\frac{p}{2} c_3 \sum_{k} \frac{d_s^{1 - \frac{p}{2}}}{d_s} \frac{1}{p - d_s} \]

so that for a suitable \( c > 0 \) independent on \( p \in (1, 2] \)

\[ \sum_{k=1}^{\infty} \mu_k(T)^p \leq c \frac{d_s}{p - d_s}, \quad d_s < p \leq 2, \]

where now \( T := [F, a] \). Setting \( s := p/d_s \) and \( u(t) := \mu_{[t]}(T)^{d_s} \) for \( t \geq 1 \), we have \( s < 1, 2 \) and the thesis follows applying the previous lemma:

\[ \sup_{N \geq 2} \frac{1}{\ln N} \sum_{k=1}^{N} \mu_k(T)^{d_s} < +\infty \]

so that \( [F, a] \in L^{d_s, \infty}([H]) \) for all \( a \in \mathcal{F} \) as promised. \( \square \)

Our final goal in this section is to provide a bound similar to (3.16) in Theorem 3.6 but now involving Dixmier traces.

**Theorem 3.9. (Dixmier trace summability)** Let \((D, r)\) be a fixed regular harmonic structure on a p.c.f. self-similar structure \((K, S, \{ F_i : i \in S \})\), let \( \mu \) be a self-similar Borel measure on \( K \) with weights \((\mu_1, \ldots, \mu_N)\) such that \( \tau_i \mu_i < 1 \) for \( \tau \in S \) and \((F, H)\) the associated densely \((d_s, \infty)\)-summable Fredholm module over \( C(K) \).

Then, for any Dixmier trace \( \tau_\omega \), the following upper bound holds true:

\[ \tau_\omega \left( |[F, a]|^{d_s} \right) \leq c_2(d_s) d_s^{-\frac{d_s}{2}} \cdot \left( \mathcal{E}[a] \right)^{-\frac{d_s}{2}} \cdot \left( \tau_\omega \left( H_D^{-\frac{d_s}{2}} \right) \right)^{1 - \frac{d_s}{2}}, \quad \forall a \in \mathcal{F} \]

where \( c_2(d_s) := 32c_1 \Gamma(d_s/2)^{-1} \).

**Proof.** By Theorem 3.8, \( \tau_\omega \left( |[F, a]|^{d_s} \right) \) is finite for all Dixmier functionals \( \omega \) on \( L^\infty(\mathbb{R}_+) \) and, by [CPS Lemma 5.1], we have the identity

\[ d_s \tau_\omega \left( |[F, a]|^{d_s} \right) = \omega - \lim_{r \to \infty} \frac{1}{r} \tau \left( |[F, a]|^{d_s} \right) \]

where \( \omega := \omega \circ L \) is the Dixmier functional on \( L^\infty(\mathbb{R}) \) corresponding to \( \omega \) through the map \( L : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}_+) \) given by \( Lf := f \circ \log \).

By Lemma 3.7 applied to the bound (3.29), we have that \( \tau_\omega \left( H_D^{-\frac{d_s}{2}} \right) \) is finite for all Dixmier functionals \( \omega \) on \( L^\infty(\mathbb{R}_+) \) so that, again by [CPS Lemma 5.1], we have the identity

\[ d_s \tau_\omega \left( H_D^{-\frac{d_s}{2}} \right) = \omega - \lim_{r \to \infty} \frac{1}{r} \tau_\omega \left( H_D^{-\frac{d_s}{2}} \right) \]

The desired bound (3.34) then follows by (3.16) in Theorem 3.6. \( \square \)

The previous result naturally suggests the consideration of a new energy functional which should be a conformal invariant in the sense of Alain Connes [C2].

**Definition 3.10.** The functional \( \Phi_{d_s}^\omega : \mathcal{F}_0 \to [0, +\infty) \)

\[ \Phi_{d_s}^\omega(a) := \tau_\omega \left( |[F, a]|^{d_s} \right) \]

will be referred to as the \( d_s \)-energy functional of the harmonic structure \((D, r)\).
Corollary 3.11. For all $a \in \mathcal{F}$ we have

$$
(3.38) \quad \sum_{i=1}^{N} \Phi^d_{\omega}(a \circ F_i) \leq c_2(d_S) \frac{d_S}{d_S} \cdot \left( \mathcal{E}[a] \right)^{\frac{d_S}{d_S}} \cdot \left[ \tau_\omega \left( H_D \frac{d_S}{d_S} \right) \right]^{1-\frac{d_S}{d_S}}.
$$

Proof. Setting $c := c_2(d_S) \frac{d_S}{d_S} \cdot \left[ \tau_\omega \left( H_D \frac{d_S}{d_S} \right) \right]^{1-\frac{d_S}{d_S}}$ and, for $a \in \mathcal{F}$, applying (3.34) to $a \circ F_i \in \mathcal{F}$, we have

$$
(3.39) \quad \Phi^d_{\omega}(a \circ F_i) \leq c \cdot \left( \mathcal{E}[a \circ F_i] \right)^{\frac{d_S}{d_S}}.
$$

By Hölder inequality we then have

$$
\sum_{i=1}^{N} \Phi^d_{\omega}(a \circ F_i) \leq c \cdot \sum_{i=1}^{N} \left( \mathcal{E}[a \circ F_i] \right)^{\frac{d_S}{d_S}}
$$

$$
= c \cdot \sum_{i=1}^{N} r_i^{\frac{d_S}{d_S} \left( r_i^{\frac{d_S}{d_S}} \mathcal{E}[a \circ F_i] \right)^{\frac{d_S}{d_S}}}
$$

$$
\leq c \cdot \sum_{i=1}^{N} r_i^{\frac{d_S}{d_S} \left( r_i^{\frac{d_S}{d_S}} \mathcal{E}[a \circ F_i] \right)^{\frac{d_S}{d_S}}}
$$

$$
= c \cdot \left( \mathcal{E}[a] \right)^{\frac{d_S}{d_S}}.
$$

The previous result suggests that the $d_S$-energy functional may be conformal, as we now prove that it is indeed, by means of the uniqueness result of [CS].

Theorem 3.12. (Conformal invariance) The $d_S$-energy functional is a self similar conformal invariant

$$
(3.40) \quad \Phi^d_{\omega}(a) = \sum_{i=1}^{N} \Phi^d_{\omega}(a \circ F_i) \quad a \in \mathcal{F}.
$$

Proof. Let us consider the Hilbert space $\mathcal{H}^N = \bigoplus_{i=1}^{N} \mathcal{H}$ endowed with the action of $C(K)$ given by

$$
a \left( \bigoplus_{i=1}^{N} \xi_i \right) := \bigoplus_{i=1}^{N} (a \circ F_i) \xi_i, \quad a \in C(K), \quad \xi_i \in \mathcal{H}, \quad i = 1, \ldots, N,
$$

and the involution given by $J^N : \mathcal{H}^N \to \mathcal{H}^N$

$$
J^N \left( \bigoplus_{i=1}^{N} \xi_i \right) := \bigoplus_{i=1}^{N} J \xi_i, \quad \xi_i \in \mathcal{H}.
$$

It is easily verified that $(C(K), \mathcal{H}^N, J^N)$ is a symmetric Hilbert module over $C(K)$ and the map $\partial^N : \mathcal{F} \to \mathcal{H}^N$ given by

$$
\partial^N(a) := \bigoplus_{i=1}^{N} r_i^{-1/2} \partial(a \circ F_i)
$$
is a symmetric derivation such that

\[ \|\partial^N(a)\|^2_{\mathcal{H}^N} = \bigoplus_{i=1}^N r_i^{-1}\|\partial(a \circ F_i)\|_{\mathcal{H}}^2 = \bigoplus_{i=1}^N r_i^{-1} E[a \circ F_i] = E[a]. \]

In other words, \((\partial^N, \mathcal{H}^N, J^N)\) is a new symmetric derivation representing the Dirichlet form \((E, F)\), isomorphic to the older one \((\partial, \mathcal{H}, J)\) by [CS Theorem 8.3]. Since the corresponding Fredholm modules are unitarily isomorphic, the \(d_S\)-energy functional \(\Phi_{d_S}^F\) is unchanged if computed using the new structure.

\[ \square \]
REFERENCES

[At] M.F. Atiyah, Global theory of elliptic operators, *Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969)* (1970), 21–30 Univ. of Tokyo Press, Tokyo.

[Ba] M.T. Barlow, “Diffusions on fractals”, Lectures Notes in Mathematics 1690, Springer, 1998.

[BST] O. Ben-Bassat, R.S. Strichartz, A. Teplayev, What is not in the domain of the Laplacian on Sierpinski gasket type fractals, *J. Funct. Anal.* 166 (1999), 197–217.

[BeDe] A. Beurling and J. Deny, Dirichlet Spaces, *Proc. Nat. Acad. Sci.* 45 (1959), 208-215.

[BDF] L.G. Brown, R.G. Douglas, P.F. Fillmore, Extentions of C*-algebras and K-homology, *Ann. of Math.* 105 (1977), 265–324.

[CPS] A. Carey, J. Phillips, F. Sukochev, Spectral Flows and Dixmier Traces, *Advances in Anal.* 173 (2003), no. 1, 68–113.

[CS] F. Cipriani, J.-L. Sauvageot, Derivations as square roots of Dirichlet forms, *J. Funct. Anal.* 201 (2003), no. 1, 78–120.

[C1] A. Connes, The action functional in noncommutative geometry, *Comm. Math. Phys.* 117 (1998), 673-683.

[C2] A. Connes, “Noncommutative Geometry”, Academic Press, 1994.

[Dav] E.B. Davies, “Heat Kernels and Spectral Theory”, Cambridge University Press, 1989.

[Dix] J. Dixmier, “Les C*-algèbres et leurs représentations”, Gauthier–Villars, Paris, 1969.

[FS] M. Fukushima, T. Shima, On a spectral analysis for the Sierpinski gasket, *Potential Anal.* 1 (1992), 1–35.

[GI1] D. Guido, T. Isola, Fractals in noncommutative geometry, *Fields Inst. Commun.*, Amer. Math. Soc., Providence, RI, 30 (2001), 171–186.

[GI2] D. Guido, T. Isola, Dimensions and singular traces for spectral triples, with applications to fractals, *J. Funct. Anal.* 203 (2003), no. 2, 362–400.

[GI3] D. Guido, T. Isola, Dimensions and spectral triples for fractals in $\mathbb{R}^n$, in “Advances in operator algebras and mathematical physics”, Theta Ser. Adv. Math., 5, Theta, Bucharest, (2005), 89-108.

[Ku] S. Kusuoka, Dirichlet forms on fractals and products of random matrices, *Publ. Res. Inst. Math. Sci.* 25 (1989), 659-680.

[L] S.H. Liu, Fractals and their applications in condensed matter physics, *Solid State Physics* 39 (1986), 207–283.

[Hi] M. Hino, On singularity of energy measures on self-similar sets, *Probab. Th. Rel. Fields* 132 (2005), 265-290.

[Kas] G. Kasparov, Topological invariants of elliptic operators, I. K-homology, *Math. SSSR Izv.* 9 (1975), 751–792.

[Ki] J. Kigami, “Analysis on Fractals”, Cambridge Tracts in Mathematics vol. 143, Cambridge University Press, 2001.

[KL] J. Kigami, M. Lapidus, Self-Similarity of the volume measure for laplacians on p.c.f. self-similar fractals, *Comm. Math. Phys.* 217 (2001), 165-180.

[Mis] A.S. Mishchenko, Infinite-dimensional representations of discrete groups and higher signatures, *Math. SSSR Izv.* 8 (1974), 85–112.
[RT] R. Rammal, G. Toulouse, Random walks on fractal structures and percolation clusters, *J. Phys. Lett.* **44** (1983), L13–L22.

[S] B. Simon, “Trace ideals and their applications”, Lecture Note Series vol. **35**, Cambridge University Press, 1979.

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