Entangled inputs cannot make imperfect quantum channels perfect

F. G. S. L. Brandão1, J. Eisert2,3, M. Horodecki4, and D. Yang5

1 Departamento de Física, Universidade Federal de Minas Gerais, Belo Horizonte 30123-970, Brazil
2 Institute of Physics and Astronomy, University of Potsdam, 14476 Potsdam, Germany
3 Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany
4 Institute of Theoretical Physics and Astrophysics, University of Gdańsk, 80-952 Gdańsk, Poland and
5 Laboratory for Quantum Information, China Jiliang University, Hangzhou, Zhejiang 310018, China

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Entangled inputs can enhance the capacity of quantum channels, this being one of the consequences of the
celebrated result showing the non-additivity of several quantities relevant for quantum information science. In
this work, we answer the converse question (whether entangled inputs can ever render noisy quantum channels
have maximum capacity) to the negative: No sophisticated entangled input of any quantum channel can ever
enhance the capacity to the maximum possible value; a result that holds true for all channels both for the classical
as well as the quantum capacity. This result can hence be seen as a bound as to how “non-additive quantum
information can be”. As a main result, we find first practical and remarkably simple computable single-shot
bounds to capacities, related to entanglement measures. As examples, we discuss the qubit amplitude damping
and identify the first meaningful bound for its classical capacity.

How much information can one transmit reliably through a
quantum channel such as a tele-communication fiber? This
basic question is, despite much progress in recent years [1-5], still surprisingly wide open. Some suitable encoding and
decoding is necessary, needless to say, but the optimal achievable rates can still not be expressed in a computable closed
form. For classical information, the hope that the single-shot capacity would be sufficient to arrive at that goal was cor-
rborated by many examples of channels for which this is in fact true [2]. Alas, it was finally found to be unjustified with
the celebrated result [1] on the non-additivity of several quantities that are in the center of interest in quantum information
science [3-5]. In particular, entangled inputs help and do in-
crease the classical information capacity. This result showed
that the question of finding capacities of quantum channels is
more complicated than what one might have anticipated. In
the case of quantum information transmission, a similar situ-
ation has been known to be true already for a long time: in
general one must regularize the single-shot expression, given
by the coherent information, in order to attain the quantum
capacity [6].

To contribute to fix the coordinate system of channel
capacities, this insight begs for a resolution of the follow-
ing question: To what extent can entanglement help then? Is
the mentioned result rather an academic observation, mani-
festing itself in small violations of additivity in high physical
dimensions? An interesting question in this context is the fol-
lowing: Can suitably entangled inputs render noisy quantum
channels take their maximum possible capacity or make them
even perfect? This would be the other extreme, where the
non-additivity serves as a resource to overcome the noisiness
of channels.

In this work, we answer this question to the negative: For
all quantum channels, no matter how elaborate the entangled
coding over many uses of the channel might be, one can never
achieve the maximum possible capacity if this is not already
true on the single-shot level. This observation holds true both
for the classical as well as the quantum capacity.

We show this by introducing new upper bounds to these
capacities which can be evaluated on the single-shot level and
are computable, which constitute a main result of this work. We
connect questions of capacities to those of entanglement
measures of systems and their environments. These bounds
are useful in their own right, which will be shown by means
of an example of amplitude-damping channel.

Notation and setting. We start our discussion by fixing the
notation and clarifying some basic concepts that will be used
later on. We consider general quantum channels of arbitrary
finite dimension, \( T : S(\mathbb{C}^d) \rightarrow S(\mathbb{C}^d) \), modeling any gen-
eral noisy quantum evolution. \( T \) is hence an arbitrary trace-

 preserving completely positive map. Such a channel can al-
ways be written in terms of a Stinespring dilation as

\[
T(\rho) = \text{tr}_E(U \rho U^\dagger),
\]

labeling the input by \( A \), associated with the Hilbert space \( \mathbb{C}^d \),
the output by \( B \) and the environment by \( E \), equipped with Hilbert spaces \( \mathbb{C}^d \) and \( \mathbb{C}^d_{\text{env}} \), respectively. \( U \) is an isometry mapping the input on \( A \) onto a quantum state on \( B \) and \( E \).

The classical information capacity, or short classical capac-
ity, of a quantum channel is the rate at which one can reliably
send classical information. It is related to the Holevo–\( \chi \) [10]
or the single-shot classical capacity of that channel,

\[
\chi(T) = \max \left( S \left( \sum_j p_j T(\rho_j) \right) - \sum_j p_j S(T(\rho_j)) \right),
\]

where the maximum is taken over probability distributions.
and states, as the asymptotic regularization

\[ C(T) = \lim_{n \to \infty} \sup_{\rho} \frac{\chi(T^\otimes n)}{n}. \] (3)

The trivial maximum value that the capacity can possibly take is given by the maximum output entropy of the quantum channel,

\[ C(T) \leq \max_\rho S(T(\rho)) = S_{\text{max}}(T). \] (4)

We will say that whenever this bound is saturated, so when \( C(T) = S_{\text{max}}(T) \), that the channel has maximum capacity, giving rise to the maximum that is trivially possible. Of course, this notion includes the situation of a perfect quantum channel that has maximum output entropy of \( C(T) = S_{\text{max}}(T) = \log_2(d) \).

The quantum capacity of a quantum channel, in turn, is related to the rate at which one can reliably send quantum information through a quantum channel. Writing

\[ Q_1(T) = \max_\Psi \left( S(\omega_B) - S(\omega_E) \right), \] (5)
calculated in the state \( \omega = |\psi\rangle \langle \phi|, \Psi = |\psi\rangle \langle \psi| \), as a state on \( R, B, E, \) and where \( |\phi\rangle := U|\psi\rangle \), with \( U \) being again the isometry of \( T \), mapping \( A \) to \( B \) and \( E \), and \( |\psi\rangle \) being a state vector on \( R \) and \( A \). The quantum capacity is then

\[ Q(T) = \lim_{n \to \infty} \frac{Q_1(T^\otimes n)}{n}, \] (6)
again referred to as maximum if \( Q = S_{\text{max}}(T) \).

**Main result.** We can now formulate the main result.

**Observation 1** (Entanglement cannot enhance classical capacity of noisy quantum channels to its maximum value). Every quantum channel that is noisy (in the sense that the single-shot classical capacity is not the maximum output entropy) cannot be made having maximum capacity under the help of any sophisticated entangled input.

So if there is a gap to the maximum possible single-shot capacity, this gap will be preserved in the asymptotic limit, independent of \( n \): No entangled input can overcome this limitation. The single-shot classical capacity may be non-additive, as has been shown in Ref. [1]. Yet, entanglement can only help to some extent, and can, in particular, not make any imperfect channel perfect.

**Upper bounds for classical capacities.** In order to show this result and the equivalent one for the quantum capacity, we make use of upper bounds to channel capacities, starting with the classical capacity. The bounds forming the tools of the argument will be provided by quantities that capture the entanglement between a system and its environment in a dilation of the channel. We first show what properties a general quantity \( M : S(C^d \otimes C^{d_{env}}) \to \mathbb{R}^+ \), defined on bipartite quantum systems, should have. In order to be entirely clear, we will always give the tensor factors with respect to which an entanglement measure will be taken. For example, \( M(\sigma, A : B) \) would be the quantity evaluated for \( \sigma \) with respect to the split \( A : B \). Two properties will be important:

1. \( M \) has the property that

\[ E_F(\sigma, A : B) \geq \sum_{j=1}^n M(\sigma_{A_j,B_j} : A_j : B_j) \] (7)

for every bipartite state \( \sigma \) defined on \( n \) copies of a \( d \times e \)-dimensional quantum system, labelled \( B_1, \ldots, B_n \) and \( A_1, \ldots, A_n \), \( \sigma_{A_j,B_j} \) denoting the respective reduction.

2. \( M \) is faithful. That is, \( M(\rho, A : B) > 0 \) for bipartite states \( \rho \) on \( A \) and \( B \) if and only if \( \rho \) is entangled with respect to this split.

Here, \( E_F \) denotes the entanglement of formation [3]. As it turns out, for any quantity satisfying Property 1, the following bound holds true:

**Observation 2** (Upper bound for the classical capacity). For any quantum channel \( T \) and any quantity \( M \) that satisfies the condition 1, we find the single-shot upper bound

\[ C(T) \leq \max_\rho \left( S(T(\rho)) - M(U \rho U^\dagger, B : E) \right). \] (8)

The argument leading to this bound is remarkably simple: Starting from Eq. (2), and defining \( \sigma = U^{\otimes n} \rho(U^{\otimes n})^\dagger \) with reductions \( \sigma_{B_1,\ldots,B_n} = \text{tr}_{E_1,\ldots,E_n}(\sigma) \), \( B \) being formed by \( B_1, \ldots, B_n \) and \( E \) by \( E_1, \ldots, E_n \), we find, using the MSW-correspondence [9],

\[ \chi(T^\otimes n) = \max_\rho \left( S(T^\otimes n(\rho)) - E_F(U^{\otimes n} \rho(U^{\otimes n})^\dagger, B : E) \right) \]

\[ = \max_\rho \left( S(\sigma_{B_1,\ldots,B_n}) - E_F(\sigma, B : E) \right) \]

\[ \leq \max_\rho \left( \sum_{j=1}^n S(\sigma_{B_j}) - E_F(\sigma, B : E) \right) \] (9)

using subadditivity, and hence, using Property 1,

\[ \chi(T^\otimes n) \leq \max_\rho \sum_{j=1}^n \left( S(\sigma_{B_j}) - M(\sigma_{B_j,E_j}, B_j : E_j) \right) \]

\[ \leq n \max_\rho \left( S(T(\rho)) - M(U \rho U^\dagger, B : E) \right), \] (10)

which is the above single-shot bound of Observation 2.

This bound is to be compared with the MSW expression [9] for the Holevo-\( \chi \) itself,

\[ \chi(T) = \max_\rho (S(T(\rho)) - E_F(U \rho U^\dagger, B : E)). \] (11)

This is very similar, except that now the entanglement of formation takes the role of the quantity \( M \). This indeed leads also to the conclusion of Observation 1 for the classical capacity: \( C(T) \) achieves the maximum upper bound \( \max_\rho S(T(\rho)) \) if and only if \( \chi(T) \) achieves it. This is because \( \chi \) achieves it if and only if

\[ E_F(U \rho U^\dagger, B : E) = 0 \] (12)

for the maximizing \( \rho \) in \( \max_\rho S(T(\rho)) \), which means that \( U \rho U^\dagger \) has to be separable. Now, if \( M \) is also faithful, i.e.,...
it satisfies Property 2, then we can see that also $C$ achieves $\max_p S(T(\rho))$ iff the optimal $U_p U^\dagger$ is separable \[\text{(II)},\] which proves Observation \[\text{[1]}\]. Below we shall provide a list of quantities, most of them satisfying both of the postulates.

*Identifying candidates for suitable entanglement measures.* This result, needless to say, leaves the question of finding entanglement measures exhibiting the above properties 1. and 2. i.e. we need at least one such measure to prove the claim. Moreover, any computable measure satisfying 1. will give rise to a useful bound for capacity.

**(a) The entanglement measure $G$:** Define as in Ref. \[\text{[7]}\]
\[
C_-(\rho, B : E) = S(\rho_B) - \inf_{P_{i=0}^{k-1}} \sum_{i=0}^{k-1} q_i S(\text{tr}_E((1 \otimes P_i) \rho (1 \otimes P_i)^\dagger)),
\]
where the infimum is performed over all Kraus operators $P_0, \ldots, P_{k-1}$ acting in $B$ only, satisfying
\[
\sum_{i=0}^{k-1} P_i = 1,
\]
and $q_i =$ tr$(1 \otimes P_i) \rho (1 \otimes P_i)^\dagger$. This is a computable single-shot quantity. We denote the convex hull of this function with $G$,
\[
G(\rho, B : E) = \min_j \sum_{j} p_j C_-(\rho_j, B : E),
\]
where $\rho = \sum_j p_j \rho_j$, and which is an “entanglement measure” in its own right (it is at least a monotone under one-way LOCC). We claim that this function has the right properties.

*Observation 3 (Bounding capacities in terms of classical correlations).* The quantity $G$ has the properties 1 and 2.

In fact, the validity of Property 2 is easily shown: Every separable state will have a convex combination in terms of products, for each of which $C_-$ will vanish. In turn, if a state is entangled, then there must in any convex combination be at least an entangled and hence correlated term, which will be detected by $C_-$. To show Property 1, we can make use of a result of Ref. \[\text{[12]}\]: For a pure tripartite state $\rho$ shared by $A$, $B$, and $C$, a duality relation gives rise to
\[
S(\rho_A) = E_F(\rho_{A,B}, A : B) + C_-(\rho_{A,C}, A : C).
\]

We now use the steps of Ref. \[\text{[12]}\] iteratively. For a mixed four-party state $\rho$ on $A$, $B$, $C$, and $D$, the optimal decomposition for $E_F(\rho, AB : CD)$ in terms of pure states being $\{p_j, \rho_j\}$, for each $\rho_j$ we have $S(\rho_{j,A,B}) = E_F(\rho_j_{A,B}, AB : C) + C_-(\rho_{j,A,B}, D, AB : D) \geq E_F(\rho_j, A, C : A) + C_-(\rho_{j,B,D}, B : D)$. Hence
\[
E_F(\rho, AB : CD) = \sum_j p_j S(\rho_j, A : B) \geq \sum_j p_j (E_F(\rho_j, A, C : A) + C_-(\rho_{j,B,D}, B : D)) \geq E_F(\rho_{A,C}, A : C) + G(\rho_{B,D}, B : D),
\]
arriving at Property 1. This gives rise to a computable bound. Explicitly, it reads
\[
C(T) \leq \max_{\rho} \left( S(T(\rho)) - \sum_j p_j C_-(U \rho_j U^\dagger, B : E) \right),
\]
with $\rho = \sum_j p_j \rho_j$, as a single maximization. A lower bound to this is
\[
C(T) \leq \max_{\rho} S(T(\rho)) - \min_{\rho} C_-(U \rho U^\dagger, B : E),
\]
which is usually less tight, but much simpler to compute.

**(b) Variants of the relative entropy of entanglement:** The measure proposed in Ref. \[\text{[13]}\] is superadditive and not larger than the entanglement of formation, implying Property 1. It is also shown to be faithful in Ref. \[\text{[13]}\], which is Property 2.

**(c) Squashed entanglement:** The squashed entanglement $E_{sq}$ is also known to be superadditive and is bounded from above by the entanglement of formation, so qualifies as a bound for the same reason. It is not easily computable, however, as it is based on a construction involving a state extension the dimension of which is not bounded. However a lower bound to squashed entanglement was provided in Ref. \[\text{[16]}\]:

\[
E_{sq}(U \rho U^\dagger, B : E) \geq \frac{1}{4 \ln(2) d_{env}} \left( \min_{\sigma} ||U \rho U^\dagger - \sigma||_1 \right)^2,
\]

in terms of the trace-norm distance to the set of separable quantum states $\sigma$ with respect to the split $B : E$.

**(d) Distillable entanglement:** A not efficiently computable but in instances practical bound is provided by the LOCC or PPT distillable entanglement with respect to $B : E$. (Note that either version of distillable entanglement does not satisfy property 2.)

*Example: The amplitude qubit damping channel.* To find any non-trivial bound for the capacity of the amplitude damping channel has been an open problem for some time \[\text{[17]}\]. The methods proposed here give rise to such bounds. The Kraus
operators of \( T(\rho) = K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger \) are given by
\[
K_0 = \sqrt{p}(0)\langle 1 |, \quad K_1 = |0\rangle\langle 0 | + \sqrt{1 - p}|1\rangle\langle 1 | \quad (21)
\]
for \( p \in [0, 1] \). The isometry \( U \) of this qubit channel maps
\[
|0\rangle \rightarrow |0\rangle , \quad |1\rangle \rightarrow \sqrt{p}(0, 1) + \sqrt{1 - p}|1, 0 \rangle . \quad (22)
\]
To bound the correlation measure \( C_\ast(U \rho U^\dagger, B : E) \), any choice for \( k \) and for \( P_0, \ldots, P_{k-1} \) giving rise to a positive operator valued measure amounts to a valid bound. This gives rise to the bound depicted in Fig. 2 \[19\] for \( p \in [0, 1/2] \). Note that it is significantly tighter than the trivial bound \( C(T) \leq S_{\text{max}}(T) \), which here takes the value 1. It is easy to see that for \( p \in [0, 1/2] \) there always exists an input diagonal in the computational basis that yields an output \( \text{diag}(1, 1/2) \) with unit entropy. For \( p = 0 \) the channel becomes the perfect channel with \( C(T) = 1 \). (The entanglement assisted classical information capacity \[15\] is also a crude upper bound, but yields values even larger than 1 for \( p \in [0, 1/2] \). We have hence established a first non-trivial bound for the amplitude damping channel. Needless to say, the same techniques can be applied to any finite-dimensional quantum channel.

Quantum capacity. Indeed, an argument very similar to the above one for the classical capacity of a quantum channel holds true also for the quantum capacity. We arrive at the following conclusion (for details, see EPAPS \[22\]). Again, entanglement can help to a certain degree, but never uplift channels to the maximum possible value.

Observation 4 (Entanglement cannot enhance the quantum capacity to its maximum value). For every quantum channel for which the single-shot quantum capacity is not yet already given by the trivial upper bound \( S_{\text{max}}(T) \), the same will hold true for the quantum capacity.

Summary and outlook. In this work, we have investigated the converse question to the additivity problem: How much can entanglement help enhance capacities of quantum channels. In the focus of interest was the question whether entanglement can ever enhance the capacity to its trivial maximum if a single invocation does not yet reach that. We affirmatively answer that question to the negative, including the quantum and classical capacity. In doing so, we have established practical computable upper bounds to capacities, relating them to entanglement measures and rendering bounds and witnesses to the latter quantities useful to assess capacities. There is though an interesting challenge: all the quantities from our list exhibit a sort of monogamy, i.e., for states which are highly sharable they have to be small, implying that the bounds may become loose. An example is a channel whose Stinespring dilation gives rise to a \( d \)-dimensional anti-symmetric space. A normalized projector onto this subspace is \( d \)-sharable, which means that for large \( d \) all our bounds would tend to \( \log d \), while a direct approach of Ref. \[21\] shows that the capacity is bounded by a constant independent of \( d \). An open question is therefore how to find a quantity satisfying our Property 1, but that would not necessarily drop for sharable states. It is the hope that the present work triggers further work on how “small” violations of additivity really are in practice and what role entanglement plays after all in quantum communication.

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\[18\] With \( \rho^i = (\mathbb{1} \otimes P_i)\rho(\mathbb{1} \otimes P_i)^\dagger/q_i \), one finds
\[
C_\ast(\rho, B : E) = \sum_{i=0}^{k-1} q_i S(\rho_B^{i\dagger} || \rho_B^i ). \quad (23)
\]
Still, the quantity $C_{\psi}$ is not convex.

[19] This is numerically evaluated by optimizing over mixed-state ensembles $\rho = \sum_j p_j \rho_j$ using global optimization both based on the routine fmincon in Matlab with randomly sampled initial conditions and simulated annealing.

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[22] See EPAPS.

I. EPAPS

In this supplementary material, we detail the proof of Observation 4 of the main text. We again label in the Stinespring dilation $T(\rho) = \text{tr}_E(U \rho U^\dagger)$ of the quantum channel $T$ the input by $A$, the output by $B$ and the environment by $E$, but also keep a system $R$ holding a purification of the input. We have to show that if $Q(T) = S_{\text{max}}(T)$, then also

$$Q_1(T) = S_{\text{max}}(T).$$

(24)

If $Q(T) = S_{\text{max}}(T)$ indeed holds true, then we find for the classical and quantum entanglement-assisted capacities $C_E$ and $Q_E$, respectively,

$$C_E(T) = 2Q_E(T) \geq 2Q(T) = 2S_{\text{max}}(T).$$

(25)

We also know that

$$C_E(T) = \max_{\Psi} I(\Psi, R : B),$$

(26)

where the maximum is taken over pure states $\Psi$. One way of expressing the right hand side, so the mutual information, in our notation of input $A$, output $B$, environment $E$, and purification $R$ is

$$C_E(T) = \max_{\rho} \{ S(\rho_R) + S(\rho_B) - S(\rho) \},$$

(27)

where $\rho$ is a mixed state that is shared between $B$ and $R$ that is obtained as $\rho = (\mathbb{1}_R \otimes T)(\Psi)$. From the subadditivity of the von-Neumann entropy, it follows that for the input state that achieves the maximum in Eq. (27),

$$2S(\rho_B) \geq S(\rho_R) + S(\rho_B) - S(\rho) \geq 2S_{\text{max}}(T).$$

(28)

Since at the same time, by definition $S(\rho_B) \leq S_{\text{max}}(T)$, we find that there exists an input pure state $\Psi = |\psi\rangle\langle\psi|$ shared between $R$, $B$, and $E$, with $|\psi\rangle = U|\phi\rangle$, $|\phi\rangle$ being shared between $R$ and $A$, such that

$$S(\rho_B) = S_{\text{max}}(T),$$

(29)

$$S(\rho_R) + S(\rho_B) - S(\rho) = 2S_{\text{max}}(T).$$

(30)

The latter equality also implies that

$$S(\rho_{BE}) = S(\rho_B) + S(\rho_E),$$

(31)

which in turn means $\rho_{BE} = \rho_B \otimes \rho_E$.

In the final step, it is the aim to construct an input to the channel that certifies that the single shot quantum capacity $Q_1(T) = S_{\text{max}}(T)$. Based on the above properties of the channel, this can easily be done. By virtue of the Schmidt decomposition, there exists a unitary $V$ supported on $R$ such that

$$V|\psi\rangle = \sum_{i,j} \alpha_i \beta_j |i\rangle_{R_1} |j\rangle_{R_2} |e_i\rangle_E,$$

(32)

where for convenience the tensor factors have been indicated. So the input

$$V|\phi\rangle = \sum_{i,j} \alpha_i \beta_j |i\rangle_{R_1} |j\rangle_{R_2} |d_{i,j}\rangle_A,$$

(33)

will give rise to the output as in Eq. (29). From this input to the original problem we can construct a new input which achieves the desired bound: Take

$$|\nu\rangle = \sum_{i} \alpha_i |i\rangle_{R_1} |0\rangle_{R_2} |d_{i,0}\rangle_A,$$

(34)

yielding the output

$$U|\nu\rangle = |\xi\rangle_{R_1,B} |0\rangle_{R_2} |e_0\rangle_E,$$

(35)

which is a tensor product between $R$ and $E$. This means that $Q_1(T) = \max_{\Psi} (S(\rho_R) - S(\rho)) = S_{\text{max}}(T)$, which is that was to be shown.