The local structure of Lie bialgebroids

Zhang-Ju Liu ∗
Laboratory of Pure and Applied Mathematics
Peking University
Beijing, 100871, China
email: liuzj@pku.edu.cn

Ping Xu †
Department of Mathematics
Pennsylvania State University
University Park, PA 16802, USA
email: ping@math.psu.edu

Dedicated to Professor Qian Min on the occasion of his 75th birthday

Abstract

We study the local structure of Lie bialgebroids at regular points. In particular, we classify all transitive Lie bialgebroids. In special cases, they are connected to classical dynamical r-matrices and matched pairs induced by Poisson group actions.

1 Introduction

Lie bialgebroids were introduced by Mackenzie and Xu in [1] as the infinitesimal versions of Poisson groupoids. Lie algebroids are, in a certain sense, generalized tangent bundles, while Lie bialgebroids can be considered as generalizations of both Poisson structures and Lie bialgebras. It is therefore tempting to classify the local structure of Lie bialgebroids. It is well known that at generic points (called regular points) a Lie algebroid $\mathcal{A}$ is locally isomorphic to the following “standard” one:

$$\mathcal{A} = D \oplus (M \times g),$$

(1)

where $D \subseteq TM$ is an integrable distribution, which is the image of the anchor, and $M \times g$ is a bundle of Lie algebras, namely the isotropy Lie algebras. Here one should think of $g$ as a vector space equipped with a family of Lie algebra structures parameterized by points on the base manifold $M$, denoted by $\{g_x | x \in M\}$. The Lie brackets are constant along leaves of $D$. I.e., for any $A, B \in g$ considered as constant sections of $\mathcal{A}$, $[A, B] \in C^\infty(M)^D \otimes g$, where $C^\infty(M)^D$ stands for leafwise constant functions on $M$. The anchor of $\mathcal{A}$ is the projection $\mathcal{A} \to D$. The bracket between sections of $\mathcal{A}$ is given by

$$[X + A, Y + B] = [X, Y] + (X(B) - Y(A) + [A, B]), \quad \forall X, Y \in \Gamma(D), \ A, B \in C^\infty(M, g).$$

(2)

∗Research partially supported by NSF of China and the Research Project of “Nonlinear Science”.
†Research partially supported by NSF grant DMS00-72171.

1Throughout the paper, we adapt the convention that elements in $g$ (or $g^*$) are automatically considered as constant sections on $\mathcal{A}$ (or $\mathcal{A}^*$) unless otherwise specified.
In particular, $\mathcal{A}$ is a transitive Lie bialgebroid if $D = TM$.

Consider the extended vector bundle
\[ \tilde{\mathcal{A}} := TM \oplus (M \times \mathfrak{g}) \supseteq D \oplus (M \times \mathfrak{g}) = \mathcal{A}. \]  

The bracket on $\Gamma(\mathcal{A})$ naturally extends to a bracket on sections of $\tilde{\mathcal{A}}$. This is done simply by applying the same formula: Equation (3) to $X + A$, $Y + B \in \mathfrak{X}(M) \oplus C^\infty(M, \mathfrak{g})$. Using the graded Leibniz rule, one may extend this bracket to sections of $\wedge^\cdot \tilde{\mathcal{A}}$ as well, which extends the ordinary Schouten brackets on $\Gamma(\wedge^\cdot \mathcal{A})$. However, $\tilde{\mathcal{A}}$ is in general not a Lie algebroid any more. For example, for $A, B \in \mathfrak{g}$, $X \in \mathfrak{X}(M)$, we have
\[ [X, [A, B]] + [B, [X, A]] + [A, [B, X]] = L_X [A, B], \]  
which may not be zero unless $[A, B]$ is a constant. This extended bundle $\tilde{\mathcal{A}}$ and the brackets on $\Gamma(\wedge^\cdot \tilde{\mathcal{A}})$ will be useful later in the paper in order to describe the Lie algebroid structure on $\mathcal{A}^*$.

For any section $\Lambda \in \Gamma(\wedge^2 \tilde{\mathcal{A}})$, by $[\cdot, \cdot]_\Lambda$ we denote the bracket on $\Gamma(\tilde{\mathcal{A}}^*)$ defined by
\[ [\phi, \psi]_\Lambda = L_{\Lambda^{\#} \phi} \psi - L_{\Lambda^{\#} \psi} \phi - d[\Lambda(\phi, \psi)] \quad \forall \phi, \psi \in \Gamma(\tilde{\mathcal{A}}^*). \]

The purpose of this paper is to classify all possible Lie bialgebroids $(\mathcal{A}, \mathcal{A}^*)$ for which $\mathcal{A}$ is of the standard form (4). We find that a Lie bialgebroid structure in this case can be totally characterized by a pair $(\Lambda, \delta)$, where $\Lambda$ is a section of $\wedge^2 \tilde{\mathcal{A}}$, and $\delta$ is a bundle map $M \times \mathfrak{g} \longrightarrow M \times \wedge^2 \mathfrak{g}$. Such a pair defines the Lie algebroid structure on $\mathcal{A}^*$. For transitive Lie bialgebroids, one can furthermore describe such a Lie bialgebroid using a quadruple $(\pi, \theta, \tau, \delta)$ satisfying certain equations (see Theorem 3.2), where $(\pi, \theta, \tau)$ are components of $\Lambda$. In this way, one establishes a one-to-one correspondence between transitive Lie bialgebroids and equivalence classes of certain quadruples.

As applications, we recover a construction, due to Lu [6], of a Lie algebroid arising from a matched pair induced by a Poisson group action. Another special case is connected to dynamical $r$-matrices coupled with Poisson manifolds, which is a natural generalization of the classical dynamical $r$-matrices of Felder [3].

Acknowledgments. We are grateful to Vladimir Drinfel’d for raising the local structure question. We also wish to thank Kirill Mackenzie for useful discussions and Alan Weinstein for numerous suggestions and comments. In addition to the funding sources mentioned in the first footnote, the second author would also like to thank Peking University for hospitality while part of this project was being done.

2 Regular Lie bialgebroids

The aim of this section is to give a general description of a Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$ at a regular point. As mentioned in Introduction, any Lie algebroid $\mathcal{A}$ is always locally isomorphic to the “standard” one (3), so we will assume that $\mathcal{A}$ is isomorphic to this form throughout the paper. Indeed we say that a Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$ is regular if $\mathcal{A}$ is of the standard form (3). It is clear that $D^*$ is isomorphic to the quotient bundle $T^* M / D^\perp$. By $pr$, we denote the projection $T^* M \longrightarrow D^*$. By abuse of notation, we use the same symbol to denote its extension
\[ pr : T^* M \oplus (M \times \mathfrak{g}^*) \longrightarrow \mathcal{A}^* (\cong D^* \oplus (M \times \mathfrak{g}^*)). \]

For any $\alpha \in \Omega^1(M)$ (or more generally $\alpha \in \Gamma(T^* M \oplus (M \times \mathfrak{g}^*))$, by $\tilde{\alpha}$ we denote its corresponding section in $D^*$ (or $\mathcal{A}^*$). Hence
\[ d_{\mathcal{A}} f = \tilde{d} f, \quad \forall f \in C^\infty(M). \]

Even though we are mainly interested in the local structure, our results hold for more general situations as well. Indeed we only need to require that the first leafwise de-Rham cohomology group $H^1_{\mathcal{A}}(M)$ vanishes.

The main theorem of this section is the following:
Therefore, by definition \( \rho \), according to \( \rho(K,e) = \theta(K,e) \), we denote, respectively, the Lie bracket and the anchor of the dual Lie algebroid \( A^* \). Define \( K \in \Gamma(\wedge^2 A^* \oplus D) \) and \( \theta \in \mathfrak{X}(M) \) by

\[
K(\alpha + \xi, \beta + \eta) = \langle \rho_* \xi, \beta \rangle - \langle \rho_* \eta, \alpha \rangle; \quad \theta(\alpha, \xi) = -\langle \rho_* \xi, \alpha \rangle, \quad \forall \alpha, \beta \in \Omega^1(M), \xi, \eta \in \mathfrak{g}^*. \tag{8}
\]

Note that in general \( K \) is not a section of \( \wedge^2 A \). This is because the image of \( \rho_* \) is not necessarily contained in \( D \). By \( \theta^\#: \mathfrak{g}^* \to M \times \mathfrak{g} \), we denote the induced bundle map \( T^* M \to M \times \mathfrak{g} \):

\[
< \theta^\#(\alpha), \xi > = \theta(\alpha, \xi). \quad \tag{10}
\]

It is obvious to see that

\[
K^\#|_{T^* M} = \theta^\#: T^* M \to \mathfrak{g}; \quad \tag{11}
\]

\[
K^\#|_{M \times \mathfrak{g}^*} = \rho_* |_{M \times \mathfrak{g}^*} : M \times \mathfrak{g}^* \to TM. \tag{12}
\]

Let \( \pi \in \Gamma(\wedge^2 TM) \) be the Poisson structure on \( M \) induced from the Lie bialgebroid \( A^* \).

**Proposition 2.2.** Under the same hypothesis as in Theorem 2.1, we have

1. \( \pi \in \Gamma(\wedge^2 D) \);
2. \( \pi^\# = \rho_* |_{D \circ pr} \);
3. \( \forall \alpha, \beta \in \Omega^1(M), [\check{\alpha}, \check{\beta}]_* = [\alpha, \beta]_\pi \);
4. \( \forall f \in C^\infty(M), d_* f = -\pi^\#(df) - \theta^\#(df) \);
5. \( \forall f \in C^\infty(M), \xi \in \mathfrak{g}^*, [\check{\alpha}_A f, \xi]_* = -\check{\alpha}_A(\rho_*(\xi)f) + ad^*_{\rho_*(\xi)} (df) \xi \);
6. \( \forall \xi \in \mathfrak{g}^*, X \in \Gamma(D), [\rho_*(\xi), X] \in \Gamma(D) \);
7. \( \forall x \in M, \theta^\#(D_x^\perp) \subseteq Z(\mathfrak{g}_x) \), where \( Z(\mathfrak{g}_x) \) is the center of the Lie algebra \( \mathfrak{g}_x \).
8. \( \forall e \in \Gamma(A), [\check{K}, e] \in \Gamma(\wedge^2 A) \);
9. \( K \in \Gamma(\wedge^2 A) \) if \( \theta^\#(D_x^\perp) = 0 \), or \( \text{Im}(\rho_*) \subseteq D \).

**Proof.** By definition \( \pi \), \( \pi^\# = -\rho \circ \rho_* \), which implies that \( \pi^\#(T^* M) \subseteq D \). Thus, it follows that \( \pi \in \Gamma(\wedge^2 D) \). On the other hand, we also have \( \pi^\# = \rho_* \rho^\# \), thus \( \pi^\# = \rho_* |_{D \circ pr} \). This proves (2). Therefore

\[
\pi = \pi^\# + \rho_* |_{M \times \mathfrak{g}^*}, \tag{13}
\]
from which (4) follows immediately. (3) follows because $\rho^*$ is a Lie algebroid morphism from $T^*M$ to $\mathcal{A}^*\{\mathfrak{g}^*_0\}$, where $T^*M$ is equipped with the standard Lie algebroid structure induced from the Poisson tensor $\pi$.

As for (5), we have

$$\begin{align*}
[d_A f, \xi]_s &= -L_{d_A f} \xi \\
&= d_A \langle \theta^* (df), \xi \rangle + i_{\theta^* (df)} d_A \xi \\
&= -d_A (\rho_*(\xi)) f + ad^*_{\theta^* (df)} \xi.
\end{align*}$$

Here we have used the fact that $d_A \xi \in \Gamma(\wedge^2 (M \times \mathfrak{g}^*))$. Thus (5) is proved.

Given any $f \in C^\infty (M)^D$, we have $df \in \Gamma(D^1)$, and therefore $d_A f = 0$ according to Equation (1). By (5), we obtain that $d_A (\rho_*(\xi)) f = 0$ and $ad^*_{\theta^* (df)} \xi = 0$.

It is obvious that $ad^*_{\theta^* (df)} \xi = 0$ is equivalent to (7). Now $d_A (\rho_*(\xi)) f = 0$ is equivalent to $d(\rho_*(\xi)) f \in \Gamma(D^1)$, which implies that $X(\rho_*(\xi)) f = 0$ for any $X \in \Gamma(D)$. The latter is equivalent to $[X, \rho_*(\xi)](f) = 0$, which implies that $[X, \rho_*(\xi)] \in \Gamma(D)$ since $f$ is an arbitrary function in $C^\infty (M)^D$. Thus (6) is proved.

It is easy to see that (6) implies that $[K, X] \in \Gamma(\wedge^2 \mathcal{A})$, $\forall X \in \Gamma(D)$. Next we need to show that $[K, A] \in \Gamma(\wedge^2 \mathcal{A})$ for any $A \in \mathfrak{g}$ considered as a constant section of $\mathcal{A}$. For this purpose, let us choose a local coordinate system $(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+l})$ such that $D$ is spanned by $\{\frac{\partial}{\partial x_{k+j}} | j = 1, \ldots, l\}$. Then clearly $dx^i \in \Gamma(D^1)$, $i = 1, \ldots, k$. Let $A_i(x) = -\theta^* (dx^i)$. According to (7), we know that $A_i(x) \in Z(\mathfrak{g}_x)$, for any $x$ and $i = 1, \ldots, k$. Thus, locally we can write $K = K_1 + K_2$, where $K_1 = \sum_{i=1}^k A_i \wedge \frac{\partial}{\partial x_i}$ and $K_2 = \sum_{j=1}^l A_{k+j} \wedge \frac{\partial}{\partial x_{k+j}}$. It thus follows that $[K, A] = [K_1 + K_2, A] = [K_2, A] \in \Gamma(\wedge^2 \mathcal{A})$. Then (8) follows immediately.

Finally, we see, by definition, that $K \in \Gamma(\wedge^2 \mathcal{A})$ iff $\theta^* (D^2) = 0$ or $Im(\rho_*(\mathfrak{m} \times \mathfrak{g}^*)) \subseteq D$. The latter is equivalent to $Im(\rho_*) \subseteq D$ due to (1) and Equation (13). This completes the proof of the proposition.

**Corollary 2.3.** We have the following

1. for any $X, Y \in \Gamma(D)$ and $A \in \mathfrak{g}$, we have

$$[K, [X, Y]] = [[K, X], Y] + [X, [K, Y]];$$

$$[K, [X, A]] = [[K, X], A] + [X, [K, A]].$$

2. For $A, B \in \mathfrak{g}$, if $L_{\rho_*(e)} [A, B] = 0$ for any $e \in \Gamma(\mathcal{A}^*|_x)$, then

$$[K, [A, B]]_x = [[K, A], B]|_x + [A, [K, B]]|_x.$$  

**Proof.** Equation (14) is obvious. For Equation (15), we write $K = K_1 + K_2$ as in the proof of Proposition 2.2. From the fact that $A_i(x) \in Z(\mathfrak{g}_x)$, $i = 1, \ldots, k$, we have

$$[K_1, [X, A]] = [[K_1, X], A] + [X, [K_1, A]].$$

On the other hand, from the Jacobi identity of the Schouten bracket on $\Gamma(\wedge^2 \mathcal{A})$, it follows that

$$[K_2, [X, A]] = [[K_2, X], A] + [X, [K_2, A]].$$

Equation (15) thus follows.
Finally, we note that

\[ [K_2, [A, B]] = [[K_2, A], B] + [A, [K_2, B]], \]

while

\[ ([K_1, [A, B]] - [[K_1, A], B] - [A, [K_1, B]]) = \sum_{i=1}^{k} A_i \wedge L_{\frac{\partial}{\partial x_i}} [A, B]. \]

For any \( \xi \in \mathfrak{g}^* \), it is simple to see that \( \xi \mapsto (\sum_{i=1}^{k} A_i \wedge L_{\frac{\partial}{\partial x_i}} [A, B]) = L_{K^*_{1\pi}(\xi)} [A, B] \). On the other hand, \( \rho_*(\xi) = \pi^#(\xi) + K^*_{1\pi}(\xi) + K^*_{2\pi}(\xi) \). Since \( \pi^#(\xi), K^*_{2\pi}(\xi) \in \Gamma(D) \), hence \( L_{\rho_*(\xi)} [A, B] = L_{K^*_{2\pi}(\xi)} [A, B] \). The conclusion (2) thus follows immediately. \( \square \)

The following result describes the bracket of mixed terms in \( \Gamma(A^*) \).

**Proposition 2.4.** For any \( \alpha \in \Omega^1(M) \) and \( \xi \in C^\infty(M, \mathfrak{g}^*) \), we have

\[ [\tilde{\alpha}, \tilde{\xi}]_* = [\alpha, \xi]_{\pi + K}. \]  

**Proof.** First, let \( f \in C^\infty(M) \) be an arbitrary function, and \( \xi \in \mathfrak{g}^* \) an arbitrary element. Then according to Proposition 2.2 (5), together with Equations (11, 12), we have \( \tilde{\alpha}_f = [\tilde{\alpha}, \xi]_{\pi + K} \). On the other hand, it is easy to see that \( [\tilde{\alpha}, \xi]_{\pi} = 0 \). Thus, we have \( [\tilde{\alpha}_f, \xi]_* = [\tilde{\alpha}_f, \xi]_{\pi + K} \). Now Equation (17) follows immediately from the anchor properties of a Lie algebroid. \( \square \)

The last step is to analyze the bracket between elements in \( C^\infty(M, \mathfrak{g}^*) \). Note that \( \forall \xi, \eta \in C^\infty(M, \mathfrak{g}^*) \) and \( \forall f \in C^\infty(M) \), by Equation (12), one has

\[ [\xi, f \eta]_* = [\xi, f] [\eta, \xi]_K = f ([\xi, \eta]_*, -[\xi, \eta]_K). \]

This implies that \( [\xi, \eta]_* - [\xi, \eta]_K \) depends on \( \xi, \eta \) algebraically. On the other hand, \( [\xi, \eta]_* - [\xi, \eta]_K \) can be split into two parts:

\[ [\xi, \eta]_* - [\xi, \eta]_K = \Omega(\xi, \eta) + [\xi, \eta]_K, \]  

where \( \Omega(\xi, \eta) \in \Gamma(D^*) \) and \( [\xi, \eta]_K \in C^\infty(M, \mathfrak{g}^*) \). Now \( [,/]_K \) defines a fiberwise bracket on the bundle \( M \times \mathfrak{g}^* \), while \( \Omega(\xi, \eta) \) corresponds to a \( \mathfrak{g}^* \)-valued one-form on the bundle \( D \), i.e., \( \Omega \in \Gamma(D^*) \otimes (\wedge^2 \mathfrak{g}) \). Let \( \Omega^# : D \to M \times (\wedge^2 \mathfrak{g}) \) be its induced bundle map, and \( \delta^* : M \times \mathfrak{g} \to M \times (\wedge^2 \mathfrak{g}) \) the fiberwise cobracket corresponding to \( [,/]_K \).

**Lemma 2.5.**

\[ d_* = [\pi + K, \cdot] - \Omega^# + \delta^* \]  

**Proof.** By Propositions 2.2, 2.4 and the fact that \( [\alpha, \beta]_K = [\xi, \eta]_\pi = 0 \), we have

\[ [\tilde{\alpha} + \xi, \tilde{\beta} + \eta]_* = [\alpha + \xi, \beta + \eta]_{\pi + K} + \Omega(\xi, \eta) + [\xi, \eta]_K, \quad \forall \alpha, \beta \in \Omega^1(M), \xi, \eta \in C^\infty(M, \mathfrak{g}^*), \]

which implies Equation (19) immediately.

**Proposition 2.6.** There exists a function \( \tau \in C^\infty(M, \wedge^2 \mathfrak{g}) \) and a fiberwise bracket \( [,/]_\tau \) on the bundle \( M \times \mathfrak{g}^* \) constant along leaves of \( D \) such that

\[ [\xi, \eta]_* = [\xi, \eta]_{\tau + [\xi, \eta]_\tau}, \quad \forall \xi, \eta \in C^\infty(M, \mathfrak{g}^*). \]
Thus, from Equation (18), we get

This concludes the proof of the proposition.

Moreover it is simple to see, from Equation (22), that

and Equations (14, 19), it follows that

\( \tau, A \)

is a cobracket which induces a fiberwise bracket \( \cdot \). Notice that in Proposition 2.6, \( \delta_\Lambda = \delta + [\cdot] \) is fixed. This completes the proof of the theorem.

Proof. For any \( X, Y \in \Gamma(D) \), from the compatibility condition \( d_*[X, Y] = [d_*X, Y] + [X, d_*Y] \) and Equations (14, 19), it follows that

\[
\Omega^\# [X, Y] = [\Omega^\# X, Y] + [X, \Omega^\# Y].
\]

This is equivalent to that, as a \( \wedge^2 g \)-valued one-form, \( \Omega \) is \( \Omega^D \)-closed. By assumption, there exists a function \( \tau \in C^\infty(M, \wedge^2 g) \) such that \( \Omega = \Omega^D \tau \). Hence \( \Omega^\#(X) = L_X \tau, \ \forall X \in \Gamma(D) \).

For any \( X \in \Gamma(D), A \in g \), the compatibility condition \( d_*[X, A] = [d_*X, A] + [X, d_*A] \), together with Equations (13, 14), implies that \( -[\Omega^\#(X), A] + [X, \delta^\# A] = 0 \). On the other hand, we have

\[
\begin{align*}
[\Omega^\#(X), A] - [X, \delta^\# A] &= [L_X \tau, A] - L_X \delta^\# A \\
&= L_X ([\tau, A] - \delta^\# A).
\end{align*}
\]

It thus follows that \( [\tau, A] - \delta^\# A \) is constant along leaves of \( D \). Let \( \delta A = \delta^\# A - [\tau, A] \). Thus, the map

\[
\delta = \delta^\# - [\tau, \cdot] : g \rightarrow C^\infty(M, \wedge^2 g)^D
\]  
(22)

is a cobracket which induces a fiberwise bracket \( [\cdot, \cdot]_\delta \) on \( M \times g^* \) constant along leaves of \( D \). Moreover it is simple to see, from Equation (22), that

\[
[\xi, \eta] = [\xi, \eta] + ad_{d^\tau \xi} \eta - ad_{d^\tau \eta} \xi, \quad \forall \xi, \eta \in C^\infty(M, g^*).
\]

Thus, from Equation (18), we get

\[
\begin{align*}
[\xi, \eta]_* &= [\xi, \eta]_K + [\xi, \eta] + ad_{d^\tau \xi} \eta - ad_{d^\tau \eta} \xi + (d_D \tau)(\xi, \eta) \\
&= [\xi, \eta]_K + [\xi, \eta] + [\xi, \eta]_\tau \\
&= [\xi, \eta]_{K+\tau} + [\xi, \eta]^\circ.
\end{align*}
\]

This concludes the proof of the proposition.

Proof of Theorem 2.7: According to Proposition 2.2 (3), Proposition 2.3, and Proposition 2.4, we conclude that for any \( \alpha, \beta \in \Gamma(D^*), \xi, \eta \in C^\infty(M, g^*) \),

\[
[\alpha \wedge \xi, \beta + \eta]_* = [\alpha \wedge \xi, \beta + \eta]_A + [\xi, \eta]^\circ,
\]

where \( A = \pi + K + \tau \). It thus follows that \( d_*[\Lambda, \cdot] + \delta \), where \( \delta : M \times g \rightarrow M \times \wedge^2 g \) is the cobracket dual to \( [\cdot, \cdot]^\circ \). Notice that in Proposition 2.4, \( \tau \) and \( \delta \) are not unique. They can differ by an element \( r_0 \in C^\infty(M, \wedge^2 g)^D \): \( \tau_1 = \tau + r_0 \) and \( \delta_1 = \delta - [r_0, \cdot] \). However, \( \delta^\# = \delta + [\tau, \cdot] \) is always fixed. This completes the proof of the theorem.

Theorem 2.7. Under the same hypothesis as in Theorem 2.4, we have

1. \( A \in \Gamma(\wedge^2 A) \) iff \( \text{Im}(\rho_*|_A) \subseteq D \).

2. For any \( x \in M \), the cobracket \( \delta_x : g_x \rightarrow \wedge^2 g_x \) is a Lie algebra 1-cocycle if \( L_{\rho_*|_x}(A, B) = 0 \), \( \forall A, B \in g \), \( e \in \Gamma(A^*|_x) \). In particular, \( \delta_x \) is always a 1-cocycle when \( \text{Im}(\rho_*|_x) \subseteq D_x \).
Proof. $K$ is the only term in $\Lambda$ which is not necessarily a section in $\wedge^2 A$. It is thus clear that $\Lambda \in \Gamma(\wedge^2 A)$ iff $\text{Im}(\rho_e) \subseteq D$ according to Proposition 2.2 (9).

For any $A, B \in \mathfrak{g}$, by the compatibility condition $d_\ast[A, B] = [d_\ast A, B] + [A, d_\ast B]$ and Equation (19), we have

$$\delta^\ast[A, B] - ([\delta^\ast A, B] + [A, \delta^\ast B]) = -[K, [A, B]] + [[K, A], B] + [A, [K, B]].$$

It thus follows from Corollary 2.3 (2) that $\delta_x^\ast$ is a cocycle if $L_{\rho_{\xi}\cdot(\xi)}[A, B] = 0$, $\forall \xi \in \mathfrak{g}^*$. Finally, note that $\delta_x$ being a cocycle is equivalent to $\delta_x^\ast$ being a cocycle since their difference is a coboundary $[\tau, \cdot]$. Therefore, the theorem is proved. \hfill \square

Definition 2.8. A regular Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$, where $\mathcal{A}$ is of the standard form: $\mathcal{A} = D \oplus (M \times \mathfrak{g})$, is called decomposable if $\text{Im}(\rho_e) \subseteq D$.

We should note that the role of $\mathcal{A}$ and $\mathcal{A}^*$ is not symmetric here. In other words, that $(\mathcal{A}, \mathcal{A}^*)$ is decomposable does not necessarily mean that $(\mathcal{A}^*, \mathcal{A})$ is decomposable. In fact $(\mathcal{A}^*, \mathcal{A})$ may even not be regular.

The following immediately follows from Proposition 2.2 (9).

Corollary 2.9. Given a regular Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$, if $\mathfrak{g}_x, \forall x \in M$, are center free (e.g., semisimple), then $(\mathcal{A}, \mathcal{A}^*)$ is decomposable.

In particular, if $\mathcal{A} = M \times \mathfrak{g}$ is a bundle of Lie algebras and $\mathfrak{g}_x, \forall x \in M$, are center free, $\mathcal{A}^* = M \times \mathfrak{g}^*$ is also a Lie algebra bundle such that $(\mathfrak{g}_x, \mathfrak{g}^*_x), \forall x \in M$, are all Lie bialgebras.

The following result is obvious.

Proposition 2.10. If $(\mathcal{A}, \mathcal{A}^*)$ is a decomposable Lie bialgebroid as in Definition 2.8 and $L$ is any leaf of $D$, then $(\mathcal{A}|_L, \mathcal{A}^*|_L)$ is a transitive Lie bialgebroid.

In other words, one can reduce the study of a decomposable Lie bialgebroid to the study of transitive ones, which is the main topic of the next section.

3 Transitive Lie bialgebroids

This section is devoted to the study of local structures of transitive Lie bialgebroids, which is a special case of regular ones. In this case, we have $\mathcal{A} = TM \oplus (M \times \mathfrak{g})$ and the fiberwise brackets on $M \times \mathfrak{g} \to M$ are constant. We also assume, throughout the section, that $H^1(M) = \{0\}$. Theorem 2.1 implies the following:

Theorem 3.1. Let $\mathcal{A} = TM \oplus (M \times \mathfrak{g})$ be a transitive Lie algebroid. Assume that $H^1(M) = \{0\}$. Then there is a one-to-one correspondence between Lie bialgebroids $(\mathcal{A}, \mathcal{A}^*)$ and equivalence classes $[(\Lambda, \delta)]$, where $\Lambda \in \Gamma(\wedge^2 \mathcal{A})$, $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is a Lie algebra 1-cocycle satisfying the property that

$$[\delta \Lambda + \frac{1}{2}[\Lambda, \Lambda], e] + \delta^2 e = 0, \quad \forall e \in \Gamma(\mathcal{A}),$$

and the equivalence stands for the gauge equivalence: $\Lambda \to \Lambda + r_0$, $\delta \to \delta - [r_0, \cdot]$ for some $r_0 \in \Lambda^2 \mathfrak{g}$.

Proof. Assume that $(\mathcal{A}, \mathcal{A}^*)$ is a transitive Lie bialgebroid. According to Theorem 2.1, we know that

$$d_\ast = [\Lambda, \cdot] + \delta$$

(24)
for $\Lambda \in \Gamma(\wedge^2\mathcal{A})$ and $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$. Moreover, by Theorem 2.7, $\delta$ is a Lie algebra 1-cocycle. It is simple to check that

$$d^2_\ast = [\delta \Lambda + \frac{1}{2}[\Lambda, \Lambda], \cdot] + \delta^2.$$  

(25)

Thus Equation (23) follows.

Conversely, given a pair $(\Lambda, \delta)$ satisfying Equation (23), let $d_* : \Gamma(\wedge^*\mathcal{A}) \to \Gamma(\wedge^{*+1}\mathcal{A})$ be as in Equation (24). Then $d_*$ defines a Lie algebroid on $\mathcal{A}^*$ if $d_2^\ast = 0$, which is equivalent to: $d_2^\ast f = 0$ and $d_2^\ast e = 0$ for any $f \in C^\infty(M)$ and $e \in \Gamma(\mathcal{A})$. Now we easily see that $\forall f \in C^\infty(M), e \in \Gamma(\mathcal{A})$,

$$d_2^\ast(fe) = fd_2^\ast e + (d_2^\ast f) \wedge e.$$ 

It thus follows that $d_2^\ast f = 0$ whenever the rank of $\mathcal{A}$ is greater than 2. If the rank of $\mathcal{A}$ is less than or equal to 2, by Equation (23), we have $d_2^\ast f = [\delta \Lambda + \frac{1}{2}[\Lambda, \Lambda], f] = 0$ since $\delta \Lambda + \frac{1}{2}[\Lambda, \Lambda]$ vanishes automatically. Finally, it is clear that the compatibility condition is satisfied automatically. $\square$

Now $\Lambda$ can be split into three parts: $\Lambda = \pi + K + \tau$, where $\pi \in \Gamma(\wedge^2TM), K \in \Gamma(\mathfrak{g} \wedge TM)$ and $\tau \in \Gamma(\wedge^3(M, \wedge^2\mathfrak{g})).$ Our next task is to spell out the meaning of Equation (23) in terms of these data. Let us explain some notations that will be needed below. Recall the element $\tau \in \wedge^3\mathfrak{g}$ defined by Equation (1), and the corresponding bundle map $\theta^\# : T^*M \to M \times \mathfrak{g}$. Note that $\delta$ extends naturally to a map, denoted by the same symbol, from $\mathcal{X}(M) \otimes \mathfrak{g}$ to $\mathcal{X}(M) \otimes (\wedge^2\mathfrak{g})$. If $\theta = \sum X_i \otimes A_i \in \mathcal{X}(M) \otimes \mathfrak{g}$, write

$$\delta \theta = \sum X_i \otimes \delta A_i, \quad [\tau, \theta] = \sum X_i \otimes [\tau, A_i]$$  

(26)

$$[\theta, \theta] = \sum [X_i, X_j] \otimes (A_i \wedge A_j), \quad \theta \wedge \theta = \sum X_i \wedge X_j \otimes [A_i, A_j].$$  

(27)

Note that $\theta^\# d\tau \in C^\infty(M, \wedge^2\mathfrak{g} \otimes \mathfrak{g})$. By $\text{Alt}(\theta^\# d\tau) \in C^\infty(M, \wedge^3\mathfrak{g})$, we denote its total antisymmetrization:

$$<\text{Alt}(\theta^\# d\tau), \xi \wedge \eta \wedge \zeta> = <(\theta^\# d\tau)(\xi, \eta), \zeta> + \text{c.p.}, \quad \forall \xi, \eta, \zeta \in \mathfrak{g}^*.$$  

(28)

\textit{From the above discussion, a transitive Lie bialgebroid is then determined by a quadruple $(\pi, \theta, \tau, \delta)$. We say two quadruples $(\pi_i, \theta_i, \tau_i, \delta_i), i = 1, 2,$ are \textit{equivalent} if $\pi_1 = \pi_2, \theta_1 = \theta_2,$ and $\tau_1 = \tau_2 + r_0$ and $\delta_1 = \delta_2 - [r_0, \cdot]$ for some $r_0 \in \wedge^2\mathfrak{g}$.

We are now ready to state the main theorem of this section.}

\textbf{Theorem 3.2.} Under the same hypothesis as in Theorem 2.7, there is a one-one correspondence between transitive Lie bialgebroids $(\mathcal{A}, \mathcal{A}^*)$ and equivalence classes of quadruples $(\pi, \theta, \tau, \delta)$ satisfying the following properties:

1. $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is a Lie algebra 1-cocycle;
2. $\pi \in \Gamma(\wedge^2TM)$ is a Poisson tensor;
3. $\theta^\# : (T^*M, \pi) \to \mathfrak{g}$ is Lie algebroid morphism;
4. $\delta \theta + \frac{1}{2}[\theta, \theta] = [\tau, \theta] - \pi^\# (d\tau)$;
5. $\delta \tau + \frac{1}{2}[\tau, \tau] + \text{Alt}(\theta^\# d\tau) \in \wedge^3\mathfrak{g}$ is constant on $M$;
6. $\delta^2 + [\delta \tau + \frac{1}{2}[\tau, \tau] + \text{Alt}(\theta^\# d\tau), \cdot] = 0$, as a map from $\mathfrak{g}$ to $\wedge^3\mathfrak{g}$. 

Proof. Each fiber of $\mathcal{A}$ is a vector space direct sum $T_m M \oplus \mathfrak{g}$, and therefore admits a natural bigrading: elements in $T_m M$ have the degree $(1, 0)$ while elements in the second component $\mathfrak{g}$ have the degree $(0, 1)$. This also induces a bigrading on the fibers of exterior powers of $\mathcal{A}$ and consequently on their sections. It is simple to see that $[\pi, \pi]$ is of degree $(3, 0)$, $[\pi, K]$ is of degree $(2, 1)$, $[\pi, \tau]$ is of degree $(1, 2)$, and $[\tau, \tau]$ is of degree $(0, 3)$. On the other hand, $[K, \tau]$ consists of elements of degree $(1, 2)$ and of $(0, 3)$, and $[K, K]$ consists of elements of degree $(1, 2)$ and of $(2, 1)$. For any $S \in \Gamma(\wedge^3 \mathcal{A})$, let $S = \sum_{0 \leq i, j \leq 3} S^{(i,j)}$ be its decomposition with respect to this bigrading. The following lemma can be easily verified by a direct computation.

**Lemma 3.3.** With the above notations,

1. as a $\wedge^2 \mathfrak{g}$-valued vector field on $M$, we have $[K, K]^{(1,2)} = [\theta, \theta]$;
2. As a $\mathfrak{g}$-valued bivector field on $M$, $[K, K]^{(2,1)} = 2 \theta \wedge \theta$;
3. $\delta K = \delta \theta$; $[K, \tau]^{(1,2)} = -[\tau, \theta]$;
4. $[K, \tau]^{(0,3)} = \text{Alt}(\theta^# d\tau)$;
5. $[\pi, K] = d\pi \theta$.

Write $T = \delta \Lambda + \frac{1}{2}[\Lambda, \Lambda] \in \Gamma(\wedge^3 \mathcal{A})$.

A direct computation, using Lemma 3.3, yields that

\[
T^{(3,0)} = \frac{1}{2} [\pi, \pi];
\]
\[
T^{(2,1)} = [\pi, K] + \frac{1}{2} [K, K]^{(2,1)} = \theta \wedge \theta + d_\pi \theta;
\]
\[
T^{(1,2)} = \delta K + [K, \tau]^{(1,2)} + [\pi, \tau] + \frac{1}{2} [K, K]^{(1,2)} = \delta \theta - [\tau, \theta] + \frac{1}{2} [\theta, \theta] + \pi^# (d\tau);
\]
\[
T^{(0,3)} = \delta \tau + \frac{1}{2} [\tau, \tau] + [K, \tau]^{(0,3)} = \delta \tau + \frac{1}{2} [\tau, \tau] + \text{Alt}(\theta^# d\tau).
\]

For any $X \in \mathfrak{X}(M)$, according to Equation (25), we have $d^2 X = L_X T$. Hence $d^2 X = 0$, $\forall X \in \mathfrak{X}(M)$, is equivalent to that $T^{(i,3-i)} = 0$, $i = 1, \cdots, 3$, and $T^{(0,3)}$ is a constant function. The latter is equivalent to Conditions (2)-(5).

For any $A \in C^\infty(M, \mathfrak{g})$, it is easy to see that $(d^2 A)^{(3,0)}$, $(d^2 A)^{(2,1)}$, and $(d^2 A)^{(1,2)}$ all vanish by Conditions (2)-(5). And the only nontrivial term remaining is

\[
(d^2 A)^{(0,3)} = \delta^2 A + [\delta \tau + \frac{1}{2} [\tau, \tau] + \text{Alt}(\theta^# d\tau), A].
\]

Therefore, we see that $d^2 e = 0$, $\forall e \in \Gamma(\mathcal{A})$ iff Conditions (2)-(6) hold. The conclusion thus follows by Theorem 3.1. 

\[\square\]
4 Applications

As applications, in this section, we will consider two special cases of transitive Lie bialgebroids. They are connected, respectively, to the Lie algebroids arising from Poisson group actions \( \mathcal{A}_1 \) and dynamical \( r \)-matrices \( \mathcal{A}_2 \).

Let \( (\mathcal{A}, \mathcal{A}^*) \) be a transitive Lie bialgebroid corresponding to a quadruple \( (\pi, \theta, \tau, \delta) \) as in Theorem 3.2. From Proposition 2.1, it is simple to see that the subbundle \( M \times g^* \) of \( \mathcal{A}^* = T^*M \oplus (M \times g^*) \) is a Lie subalgebroid iff \( dr = 0 \), or \( \tau \in \wedge^2 g \) is a constant.

From now on, we will assume that \( M \times g^* \) is a Lie subalgebroid. Thus one may simply assume that \( \tau = 0 \) via a gauge transformation. By Theorem 3.2 (6), we have \( \delta^2 = 0 \). This implies that \( g^* \) is a Lie algebra such that \( (g, g^*) \) is indeed a Lie bialgebra. Theorem 3.2 (4) is equivalent to that the linear map \( - (\theta^\#)^* : g^* \to X(M) \) obtained by taking the opposite of the dual of \( \theta^\# \) is a Lie algebra morphism. Hence, it defines a right action of \( g^* \) on \( M \). Theorem 3.2 (3) is equivalent to saying that this is a Poisson action (Proposition 5.4 in [10]). In conclusion, when \( \tau = 0 \), the conditions in Theorem 3.2 reduce to the statement that \( (g, g^*) \) is a Lie bialgebra and \( M \) is a Poisson \( g^* \)-manifold. In this case, the Lie algebroid structure on \( \mathcal{A}^* = T^*M \oplus (M \times g^*) \) can be described more explicitly. First, we note that the subbundle \( T^*M \) is also a Lie subalgebroid of \( \mathcal{A}^* \).

In other words, both summands of \( \mathcal{A}^* \) are Lie subalgebroids. Therefore, in order to describe the bracket on \( \Gamma(\mathcal{A}^*) \), it suffices to consider the bracket between the mixed terms. For this purpose, consider the maps:

\[
\phi : T^*M \to CDO(M \times g^*), \quad \phi(\alpha)(\xi) = L_{\pi^\#\alpha}\xi + ad_{\theta^\#\alpha}\xi \quad \text{and}
\]

\[
\psi : M \times g^* \to CDO(T^*M), \quad \psi(\xi)(\alpha) = -<\xi, d(\theta^\#\alpha)> - i_{(\theta^\#\alpha)\xi}d\alpha,
\]

\( \forall \alpha \in \Omega^1(M) \) and \( \xi \in C^\infty(M, g^*) \). Here \( CDO \) stands for covariant differential operators \( [7] \), and \( <\cdot, \cdot> \) means the pairing between \( g^* \) and \( g \). Now a simple computation, using Proposition 2.1, yields that

\[
[a, \xi]_\pi = [\alpha, \xi]_{\pi + K} = L_{\pi^\#\alpha}\xi + L_{K^\#\alpha}\xi - L_{K^\#\xi}\alpha - d< K^\#\alpha, \xi >
\]

\[
L_{\pi^\#\alpha}\xi + ad_{\theta^\#(\alpha)}\xi - <\theta^\#(\alpha), d\xi > + i_{(\theta^\#\alpha)\xi}d\alpha + d< \xi, \theta^\#(\alpha) > = \phi(\alpha)(\xi) - \psi(\xi)(\alpha).
\]

As a consequence, we conclude that (1). both \( \phi \) and \( \psi \) are Lie algebroid representations, (2). \( (T^*M, M \times g^*) \) is a matched pair, and (3). \( \mathcal{A}^* \) is isomorphic to the corresponding Lie algebroid \( T^*M \bowtie (M \times g^*) \) \( \mathcal{A}_2 \). This Lie algebroid was studied in detail by Lu in [6].

Now we can summarize the discussion above in the following:

**Theorem 4.1.** A quadruple \( (\pi, \theta, 0, \delta) \) as in Theorem 3.4 satisfies the conditions in Theorem 3.2 if \& only if \( \pi \) is a Poisson tensor, \( \delta \) defines a Lie bialgebra, and \( \theta \) induces a right \( g^* \)-Poisson action. In this case, the dual Lie algebroid \( \mathcal{A}^* \) is isomorphic to the matched pair \( T^*M \bowtie (M \times g^*) \) as mentioned above.

In other words, under the bijection of Theorem 3.3, the classes of quadruples \( (\pi, \theta, 0, \delta) \) correspond to those transitive Lie bialgebroids \( (\mathcal{A}, \mathcal{A}^*) \) where \( \mathcal{A}^* \) is the double of the matched pairs \( (T^*M, M \times g^*) \) of a Poisson group action.

**Remark 4.2.** More generally, instead of \( \tau = 0 \), one may consider the situation where Conditions
\(\delta\theta + \frac{1}{2}[\theta, \theta] = 0; \quad (29)\)
\([\tau, \theta] - \pi^#(d\tau) = 0; \quad (30)\)
\(\delta^2 = 0; \quad \text{and} \quad (31)\)
\(\delta \tau + \frac{1}{2}[\tau, \tau] + \text{Alt}(\theta^#d\tau) = 0. \quad (32)\)

Equations (29, 31) imply that \(\delta\) defines a Lie bialgebra and \(\theta\) induces a right \(g^*\)-Poisson action, and therefore one can form a Lie algebroid \(T^*M \bowtie (M \times g^*)\) as in Theorem 4.1. It is simple to see that Equations (30, 32) mean that \(\tau \in C^\infty(M, \wedge^2 g)\), considered as a section of \(\wedge^2 A\), is a Hamiltonian operator, i.e., its graph \(\Gamma_\tau\) is a Dirac structure of \(A \oplus A^*\) \([4]\). In this case \(A^*\) is isomorphic to the corresponding Lie algebroid \(\Gamma_\tau\). This is a generalization of the situation studied in \([5]\). Note that Equation (30) is equivalent to that
\[X_f \tau = \text{ad}_{\theta^#(df)} \tau, \quad \forall f \in C^\infty(M), \quad (33)\]
which implies that \(\tau\) is completely determined, on a symplectic leaf, by its value at a particular point of the leaf. It would be interesting to investigate what kind of equation \(\tau\) satisfies when being viewed as a function on the leaf space (compare with Theorem 3.14 in \([2]\)).

Another special case of Theorem 3.2 is when the quadruple \((\pi, \theta, \tau, \delta)\) is equivalent to \((\pi, \theta, \tau, 0)\). In this case, the corresponding Lie bialgebroid must be a coboundary Lie bialgebroid with \(\Lambda = \pi + K + \tau\).

**Theorem 4.3.** There is a one-one correspondence between coboundary Lie bialgebroids \((A, A^*)\), where \(A = TM \oplus (M \times g)\), and triples \((\pi, \theta, \tau)\) satisfying the properties:

1. \(\pi \in \Gamma(\wedge^2 TM)\) is a Poisson tensor;
2. \(\theta^#: (T^*M, \pi) \to g\) is Lie algebroid morphism;
3. \(\frac{1}{2}[\theta, \theta] = [\tau, \theta] - \pi^#(d\tau)\);
4. \(\text{Alt}(\theta^#d\tau) + \frac{1}{2}[\tau, \tau] \in (\wedge^3 g)^S\) (i.e., it is constant on \(M\) as well as \(\text{ad}\)-invariant).

**Definition 4.4.** For a Poisson manifold \((M, \pi)\) and a Lie algebra \(g\), assume that there exists a tensor \(\theta \in \mathfrak{X}(M) \otimes g\) such that \(\theta^#: (T^*M, \pi) \to g\) is a Lie algebroid morphism. A function \(\tau \in C^\infty(M, \wedge^2 g)\) is called a dynamical \(r\)-matrix coupled with the Poisson manifold \((M, \pi)\) via \(\theta\) if both conditions (3) and (4) in Theorem 4.3 are satisfied. Here \(\theta\) is called a coupling tensor, and the equation
\[\text{Alt}(\theta^#d\tau) + \frac{1}{2}[\tau, \tau] = \Omega \in (\wedge^3 g)^S, \quad (34)\]
is called the generalized DYBE coupled with \((\pi, \theta)\).

With this definition, Theorem 4.3 can be rephrased as follows.

**Theorem 4.5.** A function \(\tau \in C^\infty(M, \wedge^2 g)\) is a dynamical \(r\)-matrix coupled with a Poisson tensor \(\pi \in \Gamma(\wedge^2 TM)\) via \(\theta\) iff \(\Lambda := \pi + K + \tau\) defines a coboundary Lie bialgebroid structure for the Lie algebroid \(TM \oplus (M \times g)\).

Now we consider an example, which is in fact the main motivation for the above definition.
Example 4.6. Assume that $M = \mathfrak{h}^*$ where $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra. Then $M$ is a Poisson manifold with the Lie-Poisson structure $\pi$. In this case, $T^*M \cong \mathfrak{h}^* \times \mathfrak{h}$. Let $\theta^\# : T^*M \to \mathfrak{g}$ be the projection: $(\xi, v) \mapsto v, \langle \xi, v \rangle \in \mathfrak{h}^* \times \mathfrak{h}$. Clearly $\theta^\#$ is a Lie algebra morphism. Let us fix a basis of $\mathfrak{h}$, say $\{e_1, e_2, \ldots, e_k\}$, and let $\{\lambda_1, \ldots, \lambda_k\}$ be its corresponding coordinate system on $\mathfrak{h}^*$. Then we have $\theta = \sum \frac{\partial}{\partial \lambda_i} \otimes e_i$, where $\frac{\partial}{\partial \lambda_i}$, $1 \leq i \leq k$, are considered as constant vector fields on $\mathfrak{h}^*$. Clearly $[\theta, \theta] = 0$. Thus, for any $\tau \in C^\infty(\mathfrak{h}^*, \wedge^2 \mathfrak{g})$, Condition (3) in Theorem 4.3 (3) takes the form $[\tau, \theta] = \pi^\#(d\tau)$, which, according to Equation (33), is equivalent to that

$$ad_\xi \tau = X_{l_\xi}(\tau), \quad \forall \xi \in \mathfrak{h}. \quad (35)$$

Here $X_{l_\xi}$ denotes the Hamiltonian vector field of the linear function $l_\xi$ on $\mathfrak{h}^*$. Thus, If $H$ denotes a connected Lie group with Lie algebra $\mathfrak{h}$, Equation (33) is equivalent to that the map $\tau : \mathfrak{h}^* \to \wedge^2 \mathfrak{g}$ is $H$-equivariant. Condition (4) in Theorem 4.3 becomes

$$\text{Alt}(d\tau) + \frac{1}{2}[\tau, \tau] \in (\wedge^3 \mathfrak{g})^\theta.$$ 

In other words, $\tau$ is a classical dynamical $r$-matrix in the sense of Felder [3, 2]. We thus recover the main result in [3].

We end the paper with the following

Example 4.7. Let $(M, \pi)$ be any Poisson manifold and $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ with the standard generators $\{H, E_+, E_-\}$:

$$[H, E_+] = E_+, \quad [H, E_-] = -E_-, \quad [E_+, E_-] = 2H.$$ 

Take $\theta = X_f \otimes H$, where $f$ is a smooth function on $M$ and $X_f$ its Hamiltonian vector field. Clearly $\theta^\# : (T^*M, \pi) \to \mathfrak{g}$ is Lie algebra morphism because $d_\pi \theta + \theta \wedge \theta = 0$.

Let

$$\tau = e^f H \wedge E_+ + e^{-f} H \wedge E_- + E_+ \wedge E_-,$$

which can be considered as a twist of standard $r$-matrix on $\mathfrak{sl}(2, \mathbb{R})$. Obviously $[\theta, \theta] = 0$ and

$$[\tau, \theta] = \pi^\#(d\tau) = X_f \otimes (e^f H \wedge E_+ - e^{-f} H \wedge E_-)$$

so that the condition in Theorem 4.3 (3) holds. For Theorem 4.3 (4), note that $\text{Alt}(\theta^\# d\tau) = 0$ and $[\tau, \tau] = 3H \wedge E_+ \wedge E_- \in (\wedge^3 \mathfrak{g})^\theta$. Hence $\tau$ is indeed a dynamical $r$-matrix coupled with $\pi$ via $\theta$.

In particular, let $M = \mathbb{R}^2$ be equipped with the standard symplectic structure $\omega = dx \wedge dy$ and $f = ax - by$. Then we have

$$\tau(x, y) = e^{ax-by}H \wedge E_+ + e^{-(ax-by)}H \wedge E_- + E_+ \wedge E_-.$$

References

[1] Bangoura, M. and Kosmann-Schwarzbach, Y., Equation de Yang-Baxter dynamique classique et algebroides de Lie, C. R. Acad. Sci. Paris Serie I, 327 541- 546, (1998).

[2] Etingof, P. and Varchenko A., Geometry and classification of solutions of the classical dynamical Yang-Baxter equation, Comm. Math. Phys., 192 77-120, (1998).
[3] Felder, G., Conformal field theory and integrable systems associated to elliptic curves, *Proc. ICM Zürich*, Birkhäuser, Basel, (1994), 1247-1255.

[4] Liu, Z.-J., Weinstein, A., and Xu, P., Manin triples for Lie bialgebroids, *J. Differential Geom.* 45 (1997), 547–574.

[5] Liu, Z.-J., Xu, P., Dirac structures and dynamical $r$-matrices, *Ann. Inst. Fourier* 51 (2001), 831-859.

[6] Lu, J.-H., Poisson homogeneous spaces and Lie algebroids associated to Poisson actions, *Duke Math. J.* 86 (1997), 261–304.

[7] Mackenzie, K., Lie groupoids and Lie algebroids in differential geometry, London Mathematical Society Lecture Note Series, 124 Cambridge University Press, 1987.

[8] Mackenzie, K., Notions of double for Lie algebroids, preprint.

[9] Mackenzie, K. and Xu, P., Lie bialgebroids and Poisson groupoids, *Duke Math. J.* 18 (1994), 415-452.

[10] Xu, P., On Poisson groupoids, *Internat. J. Math.* 6 (1995), 101-124.