Weyl-Dirac theory predictions on galactic scales

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Abstract

We consider the Weyl-Dirac theory within the framework of the weak field approximation and show that the resulting gravitational potential differs from that of Newtonian by a repulsive correction term increasing with distance. The scale of the correction term appears to be determined by the time variation rate of the gravitational coupling. It is shown that if the time variation rate of gravitational coupling is adopted from observational bounds, the theory can explain the rotation curves of typical spiral galaxies without resorting to dark matter. To check the consistency of our theoretical model with observation we use Likelihood analysis to find the best-fit values for the free parameters. The mean value for the most important free parameter, $\beta \times 10^{14}(1/\text{yr})$, using the Top-Hat and Gaussian priors are $6.38_{-2.46}^{+6.18}$ and $5.72_{-1.18}^{+1.23}_{-2.09}$, respectively. Although the interval for which $\beta$ is defined is wide, our results show that the goodness of the fit is, by and large, not sensitive to this quantity. The intergalactic effects and gravitational lensing of clusters of galaxies are estimated and seem to be consistent with observational data.

1 Introduction

It has long been known that standard gravitational theories cannot correctly predict the dynamics of large astronomical systems. The rotation curves of spiral galaxies and gravitational lensing are among the well known examples \cite{1, 2}. The rotation velocities usually tend to a constant or slightly rising values as a function of distance from the center of galaxy. This is in sharp contrast to the inverse square force law which implies a decline in velocity. The gravitational lensing of clusters of galaxies is the other observation that appears to be in conflict with the standard gravitational theories on extra galactic scales \cite{2}. These observations are usually explained by postulating the existence of dark matter \cite{3, 2}. Since there is no established observation of dark matter, the belief that standard gravitational theories should be modified on large length scales has been gaining momentum in recent years. Many attempts have emerged in this regard recently \cite{4}-\cite{14}.

The purpose of the present paper is to undertake an investigation on the predictions of the dynamics of large astronomical systems in the Weyl-Dirac theory \cite{15} which, in effect, is the Weyl theory \cite{15} modified by Dirac later on. Dirac made use of the theory to provide a framework to explain his large number hypothesis. This modified theory can be considered as a scalar-tensor theory of gravity and leads to a time varying gravitational coupling $G$ in accordance with the Dirac hypothesis. In comparison with Weyl theory, this modified theory is simpler and in agreement with the general theory of relativity. As we shall show below, its implications on the dynamics of astronomical objects is significant and therefore well worth studying.

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The organization of this paper is as follows: in the next section we briefly discuss the ingredients of the Weyl geometry followed by the introduction of the Weyl-Dirac model in section 3. In section 4 we present the spherically symmetric solutions of the model for a point-like mass source [16]. In section 5 we generalize the solutions to galaxies considered as spherically symmetric extended mass sources. The consistency of our model with observational rotation curves of typical spiral galaxies using Likelihood statistics will be checked and the best-fit values for free parameters of the model are determined in section 6. Sections 7 and 8 are devoted to the intergalactic effects and the gravitational lensing. Concluding remarks are presented in section 9.

In what follows we shall use units in which $\hbar = c = G = 1$. However, in order to interpret physical results we turn to conventional units from time to time. The signature of the metric is taken as $(-,+,+,+)$ and the convention for the sign of the curvature tensor is that of Misner, Thorne and Wheeler [17]. For simplicity, the scalar field indices are taken as their covariant derivatives.

## 2 Weyl geometry

Weyl geometry is a natural generalization of the Riemannian geometry [15]. In Weyl geometry the length of a vector is assumed to change as well as its direction in a parallel displacement. This means that if a vector has length $l$ at a point with coordinates $x^\mu$, then

$$\delta l = (k_\mu \delta x^\mu)l, \quad (1)$$

after a parallel displacement $\delta x^\mu$. The covariant vectors $k_\mu$ are considered as the field quantities and may be called the Weyl meson fields. By equation (1), the total displacement around a small closed loop is

$$\delta l = l F_{\mu \nu} \delta S^{\mu \nu}, \quad (2)$$

where

$$F_{\mu \nu} = k_{\nu, \mu} - k_{\mu, \nu}, \quad (3)$$

is called length curvature [18] and $\delta S^{\mu \nu}$ is the element of the area enclosed by the loop. It follows from definition (1) that the comparison of elements of length at two different points which are not separated by an infinitesimal distance can be made only with respect to a path joining the points and obviously, different paths lead to different results. The comparison then is possible if one arbitrarily defines the standards of length at each space-time point. Alternatively, the change in length $l$ can be parameterized by means of an arbitrary function $\Omega(x^\mu)$ such that

$$l' = \Omega(x^\mu) l, \quad (4)$$

and

$$k'_\mu = k_\mu + (\ln[\Omega(x^\mu)])_{, \mu}. \quad (5)$$

Such a transformation does not change the tensor field $F_{\mu \nu}$ defined in equation (3). One should note that the tensor field $F_{\mu \nu}$ satisfies

$$\nabla_\tau F_{\mu \nu} + \nabla_\nu F_{\tau \mu} + \nabla_\mu F_{\nu \tau} = 0. \quad (6)$$

Equations (3) and (6) are similar to Maxwell equations relating the vector potential to the electromagnetic tensor. It is therefore natural to interpret $k_\mu$ and $F_{\mu \nu}$ in just the same manner and transformations (5) as gauge transformations [19]. Hence the meson fields $k_\mu$ may be recognized as photons. However on the basis of the axiomatic formulation of spacetime theory presented by Ehlers, Pirani and Schild [20], this interpretation is unacceptable. It seems that the length curvature $F_{\mu \nu}$ must be related to a phenomena which has not been observed in nature. It has been shown that Weyl
mesons do not interact with leptons or quarks and other vector mesons in minimal form [21]-[23]. They only interact with gravitons and scalar mesons, that is, the Higgs fields. The question then naturally arises as to whether Weyl mesons may account for at least part of the dark matter in our universe. Some attempts on this subject can be found which consider the global structure of the universe [24]-[27]. In this paper we concentrate on the possibility of the Weyl-Dirac theory predicting dark matter effects on the dynamics of astronomical objects.

3 The model

Consider the Weyl-Dirac action functional [28]

\[ S[\phi, k_\mu] = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + *R\phi^2 + \alpha \phi_{*\mu} \phi^{*\mu} \right\}, \]  

where \( \alpha \) is a dimensionless constant and \( \phi^* \) is the co-covariant derivative of the scalar field \( \phi \) and is defined as

\[ \phi_{*\mu} = \phi_{\mu} + k_\mu \phi, \]  

where \( *R \) is the modified curvature scalar given by

\[ *R = R + 6k^\mu_{\mu} - 6k_\mu k^\mu, \]  

with \( R \) being the scalar curvature. The original Weyl action contains the term \((*R)^2\) instead of the term \( *R\phi^2 \). Dirac proposed the term \( *R\phi^2 \), thereby avoiding the great complication of the Weyl action. Using definitions (9) and (10) one arrives at

\[
S[\phi, k_\mu] = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + R\phi^2 + \alpha \phi_{\mu} \phi^{\mu} + (\alpha - 6)\phi^2 k_\mu k^\mu + 2(\alpha - 6)\phi k^\mu \phi_{\mu} \\
+ 6(\phi^2 k^{\mu};_{\mu}) \right\},
\]  

where the term \( 6(\phi^2 k^{\mu};_{\mu}) \) is a total differential and can be ignored.

Action (11) is invariant under local unit transformations of units, that is under general conformal transformations. Here, local unit transformations are taken to be composed of gauge transformations defined in equations (2) and (3) with

\[ g_{\mu\nu} \rightarrow \Omega^2(x_\mu) g_{\mu\nu} , \]  

\[ \phi \rightarrow \Omega^{-1}(x_\mu) \phi . \]  

In order to incorporate the matter fields, we add the corresponding action \( S_m \) to (11) and consider \( S_m \) as being built out of matter fields in the usual manner. Variation of \( S[\phi, k_\mu] + S_m \) with respect to \( g_{\mu\nu}, \phi \) and \( k_\mu \) respectively gives

\[ G_{\mu\nu} = \phi^{-2} \{ T_{\mu\nu} + E_{\mu\nu} + \Theta_{\mu\nu} + \Sigma_{\mu\nu} \} , \]  

\[ \alpha \phi^{\mu}_{\mu} - R\phi - (\alpha - 6) \left\{ \phi k_\mu k^\mu - \phi k^{\mu}_{\mu} \right\} - \psi = 0 , \]  

\[ -(F^{\nu\mu})_{\nu\mu} + (\alpha - 6)(\phi^2 k^\nu + \phi \phi^\nu) + J^\nu = 0 . \]  

In Weyl’s geometry, a localized quantity \( \psi \) is called a co-tensor if under local unit transformation, it transforms as \( \psi' = \Omega^n(x_\mu) \psi \). Then \( \psi \) is said to be of power \( n \). Also, a co-covariant derivative is a modified covariant derivative and is a co-tensor. For a scalar field \( S \) it is defined as

\[ S_{*\mu} = S_{\mu} - nk_\mu S . \]  

Here \( \phi \) is a scalar field of power \( n = -1 \).
Here $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor and

$$ T_{\mu\nu} = \left( -2 + \frac{\alpha}{2} \right) g_{\mu\nu}\phi\phi\phi^\alpha + (2 - \alpha)\phi\phi\phi^\alpha - 2g_{\mu\nu}\phi(\phi^\alpha)_{;\alpha} + 2\phi\phi_{;\mu\nu}, \quad (17) $$

$$ E_{\mu\nu} = \frac{1}{4} F_{\alpha\beta} F_{\alpha\beta} g_{\mu\nu} - F_{\mu\alpha} F_{\nu\beta} g^{\alpha\beta}, \quad (18) $$

$$ \Theta_{\mu\nu} = (\alpha - 6) \left\{ \phi^2 \left[ -k_{\mu} k_{\nu} + \frac{1}{2} g_{\mu\nu} k^\alpha k_\alpha \right] - \phi \left( k_{\mu} \phi_{\nu} + k_{\nu} \phi_{\mu} - k^{\alpha} \phi^\alpha g_{\mu\nu} \right) \right\}, \quad (19) $$

where $\Sigma_{\mu\nu}$ is the stress tensor of the matter fields defined as

$$ \Sigma_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}, \quad (20) $$

and $J^\mu$ denotes the current density vector

$$ J^\mu = \frac{\delta S_m}{\delta k^\mu}, \quad (21) $$

with $\psi$ given by

$$ \psi = \frac{\delta S_m}{\delta \phi}. \quad (22) $$

It should be noted that equations (14)-(16) are not independent.

Now, taking the trace of equation (14), using definitions (17)-(19) and comparing the result with equations (15) and (16), one finds

$$ \Sigma_{\mu\mu} + (\alpha - 6) \left[ \phi^2 k^\mu + \phi^\mu \phi^\mu \right]_{;\mu} + \phi \psi = 0, \quad (23) $$

where from (16) we have

$$ (\alpha - 6) \left[ \phi^2 k^\mu + \phi^\mu \phi^\mu \right]_{;\mu} + J^\mu_{;\mu} = 0. \quad (24) $$

Taking the divergence of (14) and using (24) leads to

$$ \Sigma^{\nu}_{;\mu} - \frac{\phi_{;\nu}}{\phi} \Sigma^\mu = T^\nu_{\mu} - \left( k^\nu + \frac{\phi_{;\nu}}{\phi} \right) J^\mu_{;\mu}. \quad (25) $$

One may assign to the non-gravitational part of the fields a stress tensor defined as

$$ \tau_{\mu\nu} = T_{\mu\nu} + E_{\mu\nu} + \Theta_{\mu\nu}. \quad (26) $$

It may be proved [30], using equation (24) that

$$ \tau^{\nu}_{;\mu} - \frac{\phi_{;\nu}}{\phi} \tau^\mu_{;\mu} = T^\mu_{\nu} - \left( k^\nu + \frac{\phi_{;\nu}}{\phi} \right) J^\mu_{;\mu}. \quad (27) $$

From equations (25) and (27) we find the following relation for the total stress tensor

$$ (\Sigma^{\mu\nu} + \tau^{\mu\nu})_{;\mu} - \frac{\phi_{;\nu}}{\phi} (\Sigma^\mu_{\mu} + \tau^\mu_{\mu}) = 0. \quad (28) $$

It is worth noting that equation (25) may be used to derive the equation of motion for a test particle [29, 30].

To progress further, let us take the matter content in our model as having consisted of identical particles with rest mass $m$ and the Weyl charge $q$ in the form of dust, so that
where \( u^\mu \) is the 4-velocity and the mass density \( \rho \) is given by

\[
\rho = m \rho_n,
\]

with \( \rho_n \) being the particle density. Making use of the conservation of the number of particles, we find using equation (25), the equation of motion

\[
\frac{du^\mu}{ds} + \left\{ \frac{\mu}{\nu \lambda} \right\} u^\nu u^\lambda - \frac{\phi^\nu}{\phi}(g^\mu \nu - u^\mu u^\nu) = \frac{q}{m} u_\nu F^{\nu \mu},
\]

where \( \left\{ \frac{\mu}{\nu \lambda} \right\} \) denotes the Riemann Christoffel symbol. In the left hand side of equation (31) the term containing \( \frac{\phi^\nu}{\phi} \) may be taken as a variable mass term for the particle. In such a case we have \[30\]

\[
m = m_0 \phi,
\]

where \( m_0 \) is a constant.

### 4 Field of a mass source

Let us study the field of a mass source at rest at the origin and assume that it is in a current-free region. In such a case we can consider the vacuum solutions of the field equations, that is we take \( \Sigma_{\mu \nu} = 0 \) and \( J_\mu = 0 \). Now consider the spherically symmetric line-element

\[
ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\Omega^2,
\]

where \( \nu \) and \( \lambda \) are functions of \( t \) and \( r \). Neglecting the factor \( \alpha - 6 \) which can be shown to be small \[30, 10\] in equation (16), we find, using the above metric

\[
k_{0, r} = \frac{\gamma(t)e^{(\nu + \lambda)}}{r^2},
\]

where \( \gamma(t) \) is an arbitrary function of time \( t \) resulting from the integration. Using relation (34) in equation (14), one finds

\[
e^{-\lambda}\left( -\frac{\nu_r}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{1}{\phi} e^{-\nu}\left( \phi_t \lambda_t + 3\phi^2 \phi_t \right) - \frac{\gamma^2}{r^4 \phi^2},
\]

\[
e^{-\lambda}\left( -\frac{\nu_r}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{1}{\phi} e^{-\nu}\left( \phi_t \nu_t - \phi^2 \phi_t \nu_t - \phi_t \nu_t \right) - \frac{\gamma^2}{r^4 \phi^2},
\]

\[
-\frac{\nu_r}{r} = \frac{\phi_t \nu_t}{\phi}.
\]

The form of the left hand side of equations (36) and (37) suggest that

\[
e^\nu = f(t)e^{-\lambda}.
\]

Using equations (37) and (38) we see that equations (35) and (36) are identical provided \( f(t) \) satisfies

\[
\frac{\dot{f}t}{f} = 2\frac{\phi_t \nu_t}{\phi_t} - 4\frac{\phi_t}{\phi}.
\]

The last expression gives
\[ f = \frac{\phi^2}{\beta^2 \phi^4}, \]  
(40)

where \( \beta \) is a constant parameter. We are now in a position to obtain the generalization of the results derived in [30], that is

\[ e^\nu = \frac{\phi^2}{\beta^2 \phi^4} e^{-\lambda}, \]
(41)

with

\[ e^{-\lambda} = \frac{1}{2} - \frac{m}{\phi r} + \frac{q^2}{\phi^2 r^2} + \Delta, \]
(42)

\[ \Delta = \left[ \left( \frac{1}{2} - \frac{m}{\phi r} + \frac{q^2}{\phi^2 r^2} \right)^2 + \beta^2 \phi^2 r^2 \right]^{1/2}, \]
(43)

where we have assumed \( \gamma(t) = \frac{q}{\phi} \). The parameters \( m \) and \( q \) are constant and may be interpreted as mass and charge of the source.

The above solutions are valid for any gauge function \( \phi(t) \). We choose to work in the Einstein gauge by taking \( ds^2 \to d\bar{s}^2 = \phi^2 ds^2, \phi \to \phi = 1 \). In this case one gets

\[ ds^2 = -e^{-\lambda}dT^2 + e^{2\beta T} \left( e^\lambda dr^2 + r^2 d\Omega^2 \right), \]
(44)

where

\[ dT = \frac{\phi^2}{\beta \phi} dt = \frac{d\phi}{\beta \phi}, \]
(45)

which immediately yields

\[ \phi = \phi_0 e^{\beta T}. \]
(46)

It should be noticed that the factor \( \phi^{-2} \) associated with \( \Sigma_{\mu \nu} \) in (14) can be considered as the gravitational coupling \( G \) in natural units \((h = c = 1)\). Hence we may consider

\[ G \equiv \phi^{-2}. \]
(47)

Using (46), the last expression leads to

\[ G = G_0 e^{-2\beta T}, \]
(48)

where \( G_0 = \phi_0^{-2} \) may be interpreted as the Newtonian coupling constant. Therefore, the exponential factor \( e^{-2\beta T} \) shows the time evolution of \( G \) where \( T \) is the elapsed time. From (48) we find

\[ \frac{G_T}{G} = -2\beta. \]
(49)

The empirical measurements on the time evolution of \( G \) shows that [32]

\[ \frac{G_T}{G} = (4 \pm 9) \times 10^{-13} \text{ yr}^{-1}. \]
(50)

This sets the following restriction on \( \beta \)

\[ -6.5 \times 10^{-13} \leq \beta \leq 2.5 \times 10^{-13} \text{ yr}^{-1}. \]
(51)
5 Rotational velocity of galaxies

In this section we apply the results obtained above to the rotational velocity of typical spiral galaxies. In order to obtain the gravitational potential $U$ of the point mass source $m$ considered in section 4, we first note from (32) that in the Einstein gauge we have $m = m_0$. Now let us consider the weak field limit

$$g_{00} \cong -(1 + 2U).$$

From (44) and (42) in the conventional units, this leads to

$$U \approx -\frac{m_0 G}{r} + 4\left(\frac{m_0 G}{c^2}\right)^2 \frac{\beta^2}{r} + 2\left(\frac{m_0 G}{c^2}\right) \beta^2 + \left(\frac{m_0 G}{c^2}\right) \beta^2 r + \frac{\beta^2 r^2}{2}. \quad (53)$$

where we have neglected the term $\frac{1}{r^2}$ for large $r$. As it is clearly seen, equation (53) differs from the Newtonian gravitational potential by a number of correction terms. The presence of the variable gravitational coupling $G$ may equivalently account for this discrepancy, in accordance with reference [35]. One should note that parameter $\beta$ appears in all the correction terms. Taking into account the possible bound on $\beta$ which is given in (51), the correction terms are too small even if we consider $m_0$ to be of the order of a typical galactic mass. However, the last term, $\frac{\beta^2 r^2}{2}$, may result in significant effects at galactic scales. Therefore, the first and second terms of (53) may have important effects on galactic and larger scales. Hence we consider the gravitational potential of a point mass source as follows

$$U(r) \approx -\frac{m_0 G}{r} + \frac{\beta^2 r^2}{2}. \quad (54)$$

This gravitational potential differs from the Newtonian one by a repulsive correction term.

In order to study the consequences of the gravitational potential (54) on galactic objects let us generalize the gravitational potential $U$ for an extended object taken as a galaxy. To this end one may treat the exterior of a galaxy as a vacuum solution and its interior as filled with matter and discuss the junction conditions along the surface where the two solutions are to be joined. Alternatively, one may treat the whole space as filled with matter with an appropriate fall off condition on the density. In fact, we adopt the second strategy in a weak field gravitational approximation. This strategy has also been considered in [10, 11, 12, 13, 14]. We assume that a galaxy contains galactic objects such as solar systems which, on galactic scales, would be seen as point like sources and the gravitational potential for each individual galactic object obeys the form $U$ given by equation (54) when $m_0 = M_\odot$, where $M_\odot$ denotes the solar mass. We shall consider the matter density of these objects satisfying an appropriate fall off condition on galactic scales. In a weak gravitational field approximation the gravitational potential of the galaxy as a function of $r$, that is, the distance from the center of the galaxy, is given by

$$V(r) = \int \frac{4\pi \rho M(r')}{M_\odot} U(|r - r'|) r'^2 dr'. \quad (55)$$

Here $\rho M(r')$ is the mass density of the galaxy. We shall consider a galaxy core model for the mass distribution by assuming a spherically symmetric galaxy with a core density $\rho_c$ within a core radius $r < r_c$. It should be noted that in order to more accurately describe the core behavior, one may consider the bulge or thin disc effects according to the observational distribution of the luminous matter of a typical galaxy. However, the model we consider here yields a reasonable description of the velocity rotation curves without taking the above mentioned effects into account.

From (55) the gravitational potential can be obtained as follows

$$V(r) = -\frac{GM(r)}{r} + \frac{\beta^2 M}{2M_\odot} r^2 + \frac{\beta^2 I_{cm}}{2M_\odot}. \quad (56)$$

In equation (56), $M$ is the galaxy mass.
\[ M = M_* + M_{HI} + M_{DB}, \] 
with \( M_* \), \( M_{HI} \) and \( M_{DB} \) respectively denoting the visible mass, the neutral hydrogen mass and the possible dark baryon mass and gas with \( I_{cm} \) being the moment of inertia about the center of mass and

\[ \mathcal{M}(r) = 4\pi \int_0^r dr' r'^2 \rho_c(r'), \] 

is the mass inside the luminous core of the galaxy described by a ball of radius \( r = r_c \). Inside the ball the dynamics is described by Newtonian theory and the effect of the last two terms in (56) is negligible. Moreover outside the core the effect of the second term becomes more important. A simple model of \( \mathcal{M}(r) \) is given by [14]

\[ \mathcal{M}(r) = M \left( \frac{r}{r + r_c} \right)^{3\gamma}, \] 

where

\[ \gamma = \begin{cases} 1 & \text{HSB galaxies} \\ 2 & \text{LSB and dwarf galaxies}. \end{cases} \] 

Well outside the core radius, namely for \( r \gg r_c \), from (59) we have

\[ \mathcal{M}(r) \longrightarrow M. \] 

Also, well inside the core radius the density \( \rho(r) \) has a constant value for HSB galaxies and \( \rho(r) \propto r^3 \) for LSB and dwarf galaxies. From (56) the square of the rotational velocity of a solar object about the center of a galaxy can be obtained as

\[ v^2(r) \approx \frac{\mathcal{M}(r)G}{r} + \frac{M}{M_\odot} \beta^2 r^2. \] 

6 Observational constraint on the model parameters

To find the consistency of this model with observational data, we compare the predictions of our model with data directly obtained from observations and find the best fitting parameters. Since one cannot expect the theory to exactly explain observational data, we give confidence intervals for the free parameters of the model using likelihood analysis.

To begin with, using equation (59), we rewrite equation (62) as follows

\[ v^2(r) = \frac{\alpha}{r} \left( \frac{r}{r + r_c} \right)^{3\gamma} + \alpha \beta^2 r^2. \] 

where \( \alpha = M/M_\odot \). In order to compare the theoretical results with the observational data, we must compute the rotational velocities given by equation (63). For this purpose, we redefine free parameters of the model as

\[ \alpha = 10^{10} \times \bar{\alpha} \] 
\[ \beta = 10^{-14} \times \bar{\beta} \left( \frac{1}{yr} \right) \]

Let us compute the quality of the fitting through the least squared fitting quantity \( \chi^2 \) defined by

\[ \chi^2(\bar{\alpha}, \bar{\beta}, r_c) = \sum_i \frac{(v_{\text{obs}}(r_i) - v_{\text{th}}(r_i; \bar{\alpha}, \bar{\beta}, r_c))^2}{\sigma_i^2}, \]
where $\sigma_i$ is the observational uncertainty in the rotational velocity. To constrain the parameters of model, we use the Likelihood statistical analysis

$$
\mathcal{L}(\bar{\alpha}, \bar{\beta}, r_c) = \mathcal{N} e^{-\chi^2(\bar{\alpha}, \bar{\beta}, r_c)/2},
$$

(67)

where $\mathcal{N}$ is a normalization factor. In the presence of every nuisance parameters, Likelihood function should be marginalized (integrated out). Using equation (66) we can find the best fit-values of the model parameters as the values that minimize $\chi^2(\bar{\alpha}, \bar{\beta}, r_c)$. Table I shows different priors on the model parameters used in the likelihood analysis.

**Table I: Priors on the parameter space, used in the likelihood analysis.**

| Parameters | prior |
|------------|-------|
| $\bar{\alpha}$ | $[0 - 15]$ | Top-Hat |
| $\bar{\beta}$ | $[0 - 20]$ | Top-Hat |
| $\beta$ | $10.0 \pm 0.5$ | Gaussian |
| $r_c$ | $[0 - 7]$ kpc | Top-Hat |
| $\gamma$ | 1 or 2 | Depends to the kinds of Galaxy |

The best-fit values for the parameters of the model at $1\sigma(68.3\%)$ and $2\sigma(95.4\%)$ confidence intervals with corresponding reduced $\chi^2 = \chi^2/N$ ($N$ is number of degrees of freedom) for various galaxies are presented in Table III. Table IV shows the best-fit values, using the Gaussian prior for $\bar{\beta}$. To infer the Gaussian prior, we rely on the determination of $\bar{\beta}$ from empirical measurement of the time evolution of $G$ mentioned in section 4. Figure 1 compares the fitting rotational velocity curves derived by using the best fitting parameters of some galaxies with observational data points.

Since a proper theory should have a constant $\beta$, we infer a mean value with its variance at $1\sigma$ and $2\sigma$ confidence level for $\bar{\beta}$ from the best fit-values reported in Table III and IV as

$$
\bar{\beta} = 6.38^{+2.44}_{-3.46} \pm 6.18
$$

and

$$
\bar{\beta} = 5.72^{+1.22}_{-1.18} \pm 2.90
$$

The fist value corresponds to the Top-Hat prior for $\bar{\beta}$ and the second is related to the Likelihood analysis with Gaussian prior. As mentioned before, due to some systematic and random errors in the observational data, one cannot expect the Likelihood analysis to give a unique value for $\beta$ for our theoretical model so that one is inclined to find a confidence interval. On the other hand we should point out that in spite of many attempts in previous models [33], we have found a mostly confined interval for $\beta$ which is in agreement with pervious predictions such as empirical measurement of the time evolution of $G$. In addition, Weyl-Dirac theory gives a better prediction for rotational velocity curves than Newtonian theory.

Note that the plots in figure 1 give the impression as if the velocity curves tend to bend slightly upwards. However, this is not the case since from the gravitational potential (56) obtained for an extended object such as a galaxy, we may infer that there is a limiting radius for the formation.
of galaxies. In fact, this gravitational potential differs from the Newtonian one by two repulsive correction terms. For distances near the center of a galaxy the effect of the first term is important. However at larger distances the first term falls off and the potential is dominated by the other two terms. There is a limiting distance $r$ for which we have $V(r) = 0$ and far from it the force becomes repulsive and the solar orbit becomes unstable. Therefore, we see that the Weyl-Dirac theory predicts a limiting radius for a galaxy.

The observational Tully-Fisher relation [34] implies that $v_{\text{out}}^4 \sim G_0 M$ where $v_{\text{out}}$ is the observed velocity at the outermost observed radial position and $M$ is the galaxy mass. Milgrom’s phenomenological law [7] or MOND model predict the Tully-Fisher relation by assuming that the mass to luminosity ratio, $M/L$, is constant across all galaxies. In Milgrom’s phenomenological model we have

$$v_{\text{out}}^4 = G M a_0,$$  \hfill (68) 

at sufficiently low acceleration $a \ll a_0$ with $a_0 \sim 10^{-8}$ cm/s. Relation (68) does depend on the magnitude of the acceleration $a_0$, but not on the radial distance $r$ and the rotation velocity is constant out to an infinite range. In contrast, our model does depend on the radius $r$ and the time variation rate of the gravitational coupling. Despite the beauty of Milgrom’s law in explaining flat rotation curves of galaxies it seems problematic to embed the theory within a comprehensive relativistic theory of gravity. Therefore it is not clear that the theory is as successful for explaining gravitational lensing of clusters and other curved space time effects. In fact strong gravitational lensing indicates a larger mass concentration at cluster centers than accounted for by the present form of Milgrom’s theory [43].

The properties of observational rotational velocity data points for some famous galaxies used in our model are summarized in Table II. In order to make a comparison of our results for galaxy masses with other approaches we also list the mass obtained by Milgrom phenomenological MOND model and the Moffat relativistic MSTG model.

| Galaxy      | Surface Brightness | $M(10^{10}M_\odot)$ MOND | $M(10^{10}M_\odot)$ MSTG | ref. |
|-------------|--------------------|--------------------------|---------------------------|------|
| NGC 1560    | LSB (Dwarf)        | 0.59±0.05                | 0.79±0.05                 | 36   |
| NGC 3109    | LSB (Dwarf)        | 0.62±0.04                | 0.78±0.04                 | 37   |
| NGC 55      | LSB (Dwarf)        | 0.91±0.07                | 1.17±0.07                 | 38   |
| UGC 2259    | HSB (Dwarf)        | 0.55±0.02                | 0.77±0.02                 | 40   |
| NGC 5585    | HSB (Dwarf)        | 0.9±0.06                 | 1.17±0.07                 | 42   |
| NGC 247     | HSB                | 1.46±0.14                | 2.27±0.17                 | 39   |
| IC 2574     | HSB                | -                        | -                         | 41   |
| Milkyway    | HSB                | 10.60±0.37               | 9.12±0.28                 | 41   |

Column six shows the references used to extract the observational data. Column three shows the values $M$ in $10^{10}M_\odot$ which are used to fit the velocity curves in our model. In columns four and five we present the galaxy masses obtained from the MOND model and the MSTG model [14]. Comparison of the results in column 3 with that of columns 4 and 5 shows that the masses obtained by the Weyl-Dirac theory are within the range obtained from the MOND and MSTG models.

7 Intergalactic effects

The galaxy M31 and our own galaxy (the Galaxy) are dynamically the only members of the so-called Local Group. Astronomical observations show that the center of M31 and the Galaxy approach each
other with a speed of 119 m/s [44]. A possible explanation of this phenomena was given by Kahn and Woltjer [45]. They showed that the effect requires the reduce mass of M31 and the Galaxy to be 100 times larger. This discrepancy is usually accounted for by the incorporation of dark matter. Recently the effect has been explained by the nonsymmetric gravitational theory [12]. Here we attempt to explain the effect by using the Weyl-Dirac theory without resorting to the dark matter hypothesis.

We first note that in the absence of intergalactic matter, M31 and the Galaxy form a double galaxy. In this case, using (56), the equation of motion for the system becomes

\[ \ddot{r} + \frac{M_{LG} G}{r^2} + \frac{M_{LG} \beta^2}{M_\odot} r = 0 \]  

(69)

where \( M_{LG} \) is the total mass of M31 and the Galaxy and \( r \) is the relative distance between them. Because the distance between M31 and the Galaxy is about 700 kpc, the term \( \frac{M_{LG} G}{r^2} \) in (69) is negligible and can be ignored. In this case we have

\[ \ddot{r} + \frac{M_{LG} \beta^2}{M_\odot} r = 0. \]  

(70)

This equation is equivalent to the equation of motion derived by Kahn and Woltjer by assuming the intergalactic dark matter spreading homogeneously through the Local Group with mass density \( \rho \equiv \frac{3 M_{LG}}{4 \pi G M_\odot} \beta^2 \). Therefore, it appears that the Weyl-Dirac theory may help us to make a better understanding of the intergalactic effects without the exotic dark matter hypothesis.

8 Gravitational lensing

The gravitational lensing or deflection of light by massive bodies provides another way to test the relativistic effects of gravitational theories at large scales. The gravitational lensing gives the most accurate information about the mass of the core of galaxies [46] and the cosmic telescope effects of gravitational lenses have enabled us to study faint and distant galaxies which happen to be strongly magnified by galaxy clusters. Also, the statistics of gravitational lensing events can be used to estimate cosmological parameters [47]. Observations of strong lensing by clusters indicate a larger mass concentration at cluster centers than their visible matter [2]. This discrepancy is usually accounted for by the existence of dark matter. Here we describe this effect by the Weyl-Dirac theory.

The deflection angle of light rays passing through a gravitational potential \( V \) is given by

\[ \alpha = \frac{2}{c^2} \int_s^o \nabla_\perp V \, dl, \]  

(71)

where the integral is evaluated along the path traversed by light joining the source (s) to distant observer (o) and \( \nabla_\perp \) denotes the derivative in the direction perpendicular to the path. Let us now consider deflection the of light from a cluster of galaxies. At these scales we can safely use the point mass approximation (54). Clearly, to study microlensing effects for smaller scales, we may use gravitational potential (56). Using (54), we find the deflection angle

\[ \alpha = \frac{4GM}{c^2 b} \left( 1 + \frac{b^2 \beta^2 d_{os}}{2MG} \right), \]  

(72)

where \( b \) and \( d_{os} \) denote the impact parameter and the distance between the observer and source, respectively. Relation (72) shows a larger light deflection than is expected from standard gravitational theories. For example for a cluster with \( b \sim 4 \times 10^4 \) kpc , \( d_{os} = 10 \) Gpc and \( \beta \sim 0.74 \times 10^{-13} \) /yr we obtain

\[ \frac{b^2 \beta^2 d_{os}}{2M_\odot G} \sim 10. \]  

(73)

Therefore, the increase in the light deflection may be explained without the dark matter hypothesis.
9 Conclusions

We have considered the Weyl-Dirac theory and showed that the spherically symmetric solutions of the theory lead to a gravitational potential which differs from the Newtonian one due to the appearance of a repulsive correction term increasing with distance. The correction term appears to be determined by the time variation rate of the gravitational coupling. We saw that the correction term for an extended massive source such as a galaxy becomes more important, especially at galactic distances. Hence one may expect the theory to imply significant consequences for galaxies and on galactic scales.

We also performed well-known Likelihood analysis on the free parameters in our model to check the consistency of our predictions to that of the observational data. In doing so, we considered eight galaxies which are chosen from different types, namely the low and the high surface brightness galaxies. Using two different types of prior for model parameters, the best fit-values for free parameters at $1\sigma(68.3\%)$ and $2\sigma(95.4\%)$ confidence intervals have been determined. The mean value for $\beta$ using Top-Hat and Gaussian priors are $6.38^{+2.44+6.18}_{-3.46-6.71}$ and $5.72^{+1.22+2.90}_{-1.18-2.69}$, respectively. However we found a relatively wide interval for the best fit-values of $\beta$ for each galaxy, but this gives a small deviation in the goodness of the fitting curves with observation, in other words the goodness of the fit is almost insensitive to major fluctuations of $\beta$. Also Likelihood analysis lets us to obtain a much more smaller confidence interval for $\beta$ than the former extracted values which are also consistent with previous inferring. The fit to data seems to be acceptable in terms of the total mass $M$ of each galaxy and a best fit-value for $\beta$ without the exotic dark matter hypothesis at the expense of a rather wide interval range for $\beta$. This could be regarded as the main disadvantage of the model presented here. However, it should also be pointed out that the goodness of the fit is acceptable for the underlaying observational data set used in this analysis.

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Table III: Best fitting parameters at 1σ and 2σ confidence levels with corresponding $\chi^2_\nu$ for various galaxies.

| Galaxy   | $\alpha$       | $\beta$       | $r_c$ kPc | $\gamma$ | $\chi^2_\nu$ |
|----------|----------------|---------------|-----------|----------|---------------|
| NGC 1560 | $1.20^{+0.05}_{-0.20}$ | $6.95^{+1.25}_{-1.10}$ | $1.00^{+0.05}_{-0.10}$ | 2 | 3.26          |
|          | $1.20^{+0.10}_{-0.35}$  | $6.95^{+2.15}_{-0.55}$  | $1.00^{+0.10}_{-0.20}$  |           |
| NGC 3109 | $0.65^{+0.25}_{-0.10}$ | $9.40^{+1.00}_{-2.10}$ | $1.30^{+0.15}_{-0.10}$ | 2 | 0.99          |
|          | $0.65^{+0.50}_{-0.25}$ | $9.40^{+4.10}_{-3.50}$ | $1.30^{+0.30}_{-0.20}$ |           |
| NGC 55  | $3.35^{+0.30}_{-0.30}$ | $2.50^{+0.50}_{-0.55}$ | $1.60^{+0.05}_{-0.05}$ | 2 | 0.86          |
|          | $3.35^{+0.80}_{-0.65}$ | $2.50^{+4.10}_{-1.30}$ | $1.60^{+0.15}_{-0.15}$ |           |
| UGC 2259 | $0.95^{+0.15}_{-0.10}$ | $11.30^{+0.55}_{-2.30}$ | $0.45^{+0.05}_{-0.05}$ | 2 | 0.46          |
|          | $0.95^{+0.25}_{-0.20}$ | $11.30^{+1.10}_{-4.60}$ | $0.45^{+0.15}_{-0.10}$ |           |
| NGC 5585 | $2.90^{+0.25}_{-0.30}$ | $3.05^{+0.70}_{-0.20}$ | $1.25^{+0.05}_{-0.10}$ | 2 | 2.39          |
|          | $2.90^{+0.55}_{-0.55}$ | $3.05^{+1.25}_{-0.80}$ | $1.25^{+0.10}_{-0.15}$ |           |
| NGC 247 | $2.30^{+0.25}_{-0.25}$ | $6.35^{+0.50}_{-0.40}$ | $2.80^{+0.15}_{-0.15}$ | 1 | 1.05          |
|          | $2.30^{+0.35}_{-0.45}$ | $6.35^{+1.40}_{-0.90}$ | $2.80^{+0.30}_{-0.30}$ |           |
| IC 2574 | $1.05^{+0.45}_{-0.25}$ | $7.95^{+1.50}_{-1.50}$ | $5.35^{+0.65}_{-0.40}$ | 1 | 1.63          |
|          | $1.05^{+1.00}_{-0.45}$ | $7.95^{+3.30}_{-2.85}$ | $5.35^{+1.50}_{-0.95}$ |           |
| Milkyway | $14.32^{+0.20}_{-0.15}$ | $3.55^{+0.10}_{-0.15}$ | $1.00^{+0.05}_{-0.05}$ | 2 | 1.75          |
|          | $14.32^{+1.20}_{-1.05}$ | $3.55^{+0.35}_{-0.25}$ | $1.00^{+0.10}_{-0.10}$ |           |
Table IV: Best fitting parameters at $1\sigma$ and $2\sigma$ confidence levels with corresponding $\chi^2_\nu$ for various galaxies using the Gaussian prior for $\beta$ in the Likelihood analysis.

| Galaxy       | $\alpha$        | $\beta$        | $r_c$ kPc | $\gamma$ | $\chi^2_\nu$ |
|--------------|-----------------|----------------|-----------|----------|-------------|
| NGC 1560     | $1.20^{+0.10}_{-0.10}$ | $6.95^{+0.10}_{-0.10}$ | $1.00^{+0.05}_{-0.05}$ | 2        | 3.29        |
|              | $1.20^{+0.15}_{-0.15}$ | $6.95^{+0.65}_{-0.55}$ | $1.00^{+0.10}_{-0.10}$ |          |             |
| NGC 3109     | $1.00^{+0.10}_{-0.10}$ | $6.75^{+0.35}_{-0.45}$ | $1.50^{+0.05}_{-0.10}$ | 2        | 1.09        |
|              | $1.00^{+0.25}_{-0.20}$ | $6.75^{+1.35}_{-1.05}$ | $1.50^{+0.20}_{-0.15}$ |          |             |
| NGC 55       | $2.45^{+0.25}_{-0.15}$ | $4.05^{+0.25}_{-0.55}$ | $1.40^{+0.10}_{-0.05}$ | 2        | 1.93        |
|              | $2.45^{+0.60}_{-0.35}$ | $4.05^{+0.70}_{-0.95}$ | $1.40^{+0.15}_{-0.10}$ |          |             |
| UGC 2259     | $1.30^{+0.10}_{-0.05}$ | $6.70^{+0.80}_{-0.50}$ | $0.55^{+0.05}_{-0.05}$ | 2        | 0.79        |
|              | $1.30^{+0.20}_{-0.15}$ | $6.70^{+1.45}_{-1.30}$ | $0.55^{+0.10}_{-0.10}$ |          |             |
| NGC 5585     | $2.15^{+0.20}_{-0.05}$ | $4.60^{+0.15}_{-0.20}$ | $1.10^{+0.05}_{-0.05}$ | 2        | 3.29        |
|              | $2.15^{+0.45}_{-0.25}$ | $4.60^{+0.70}_{-1.00}$ | $1.10^{+0.10}_{-0.10}$ |          |             |
| NGC 247      | $2.30^{+0.15}_{-0.15}$ | $6.35^{+0.45}_{-0.40}$ | $2.80^{+0.10}_{-0.15}$ | 1        | 1.05        |
|              | $2.30^{+0.35}_{-0.35}$ | $6.35^{+1.05}_{-0.85}$ | $2.80^{+0.25}_{-0.25}$ |          |             |
| IC 2574      | $1.40^{+0.20}_{-0.20}$ | $6.60^{+0.65}_{-0.55}$ | $5.95^{+0.70}_{-0.65}$ | 1        | 1.65        |
|              | $1.40^{+0.35}_{-0.35}$ | $6.60^{+1.39}_{-1.15}$ | $5.95^{+0.70}_{-0.65}$ |          |             |
| Milkyway     | $13.45^{+1.05}_{-0.15}$ | $3.80^{+0.10}_{-0.35}$ | $0.95^{+0.05}_{-0.05}$ | 2        | 2.09        |
|              | $13.45^{+1.30}_{-0.80}$ | $3.80^{+0.29}_{-0.45}$ | $0.95^{+0.10}_{-0.10}$ |          |             |
Figure 1: The rotational velocity of various galaxies in terms of distance. The solid line corresponds to the theoretical prediction, equation (63), with the best fit values for the free parameters derived by using Top-Hat prior for $\bar{\beta}$. The theoretical fitting corresponding to the Gaussian prior for $\bar{\beta}$ has been indicated by the dashed-dot curves. The dashed curve shows the Newtonian-Keplerian prediction. Data from observations indicated by symbols are for the following galaxies: NGC 247, NGC 55, NGC 1560, NGC 3109, IC 2574, NGC 5585, UGC 2259 and Milkyway.
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