Checking the transverse Ward-Takahashi relation at one loop order in 4-dimensions

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Abstract. Some time ago Takahashi derived so called transverse relations relating Green’s functions of different orders to complement the well-known Ward-Green-Takahashi identities of gauge theories by considering wedge rather than inner products. These transverse relations have the potential to determine the full fermion-boson vertex in terms of the renormalization functions of the fermion propagator. He & Yu have given an indicative proof at one-loop level in 4-dimensions. However, their construct involves the 4th rank Levi-Civita tensor defined only unambiguously in 4-dimensions exactly where the loop integrals diverge. Consequently, here we explicitly check the proposed transverse Ward-Takahashi relation holds at one loop order in $d$-dimensions, with $d = 4 + \varepsilon$.

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1. Introduction

The Ward-Green-Takahashi identities [1, 2, 3] play an important role in the study of gauge theories and particularly in the implementation of consistent non-perturbative truncations of the corresponding Schwinger-Dyson equations [4, 5]. The Ward-Green-Takahashi identities involve contractions of vertices with external momenta and relate these to Green’s functions with a lesser number of external legs. The best known of these relates the 3 point vector vertex coupling a fermion-antifermion pair to the gauge boson, $\Gamma^\mu_V(p_1, p_2)$, to the difference of fermion propagators, $S_F^{-1}(p_1)$ and $S_F^{-1}(p_2)$, so that with $q = p_1 - p_2$

$$q_\mu \Gamma^\mu_V(p_1, p_2) = S_F^{-1}(p_1) - S_F^{-1}(p_2).$$

Such projections constrain the so called longitudinal component of the vertices, while leaving their transverse parts unrestricted.

While the Ward-Green-Takahashi identity follows from the divergence of the vector vertex, Takahashi made plausible the existence of new relations that follow from the curl (or wedge product) of the vertex [6]. These are referred to as transverse Ward-Takahashi relations and have the potential to restrict the transverse vertices from gauge symmetry alone. Kondo [7] rederived these relations for the 3-point functions in coordinate space using the path integral formulation. Subsequently, He, Khanna and Takahashi [8] using canonical field theory cast these relations in momentum space. However, He [9] then showed that an essential part of the Fourier transform was overlooked and that the correct relation in QED (in the simpler massless fermion) case is:

$$iq^\mu \Gamma^\nu_V - iq^\nu \Gamma^\mu_V = S_F^{-1}(p_1)\sigma^{\mu\nu} + \sigma^{\mu\nu}S_F^{-1}(p_2) - \frac{1}{2}(p_1 + p_2)_\lambda \left\{\sigma^{\mu\nu}, \Gamma^\lambda_V\right\}$$

$$+ \int \frac{d^4k}{(2\pi)^4} k_\lambda \left\{\sigma^{\mu\nu}, \tilde{\Gamma}^\lambda_V\right\},$$

where $\sigma^{\mu\nu} = i \left[\gamma^\mu, \gamma^\nu\right]/2$, $\Gamma^\mu_V = \Gamma^\mu_V(p_1, p_2)$ and $\tilde{\Gamma}^\lambda_V = \tilde{\Gamma}^\lambda_V(p_1, p_2; k)$. In contrast to what is written in the paper by He [9], it is the anticommutator $\left\{\sigma^{\mu\nu}, \gamma^\lambda\right\}$ that is used in the derivation of Kondo [7]; the identification of $\left\{\sigma^{\mu\nu}, \gamma^\lambda\right\}$ with $-2\varepsilon^{\lambda\mu\rho\nu}\gamma_\rho \gamma_5$ is only valid in exactly 4 space-time dimensions and should not be used when the requisite integrals contain divergences. It is this requirement that motivates the analysis presented here.

**Figure 1.** Fermion-vector boson vertex and its momenta
The last two terms of (2) are given by the momentum transform of

$$\lim_{x' \to x} -\frac{i}{2} \left( \partial_{x}^{\lambda} \right) \langle 0 | T \bar{\psi}(x') \left\{ \sigma^{\mu\nu}, \gamma^{\lambda} \right\} U_{P}(x', x) \psi(x) \bar{\psi}(x_{1}) \bar{\psi}(x_{2}) | 0 \rangle$$

(3)

with

$$U_{P}(x', x) = \mathcal{P} \exp \left( -ig \int_{x}^{x'} dy \rho A_{\mu}(y) \right) ,$$

(4)

where \( \psi \) and \( A \) are the fermion and gauge fields respectively. \( \mathcal{P} \) means the integral is path ordered, in fact a Wilson line integral. This implicitly defines the non-local vector vertex in our anticommutator \( \{ \sigma^{\mu\nu}, \tilde{\Gamma}^{\lambda} \} \) of (2), related to the axial one \( \tilde{\Gamma}_{A}(p_{1}, p_{2}; k) \) in He [9]. Being non-local these involve integrals in momentum space some of which cannot be represented diagrammatically in terms of Feynman graphs. An example is the last term of (2), which for later we denote by \( W^{\mu\nu} \). With a textbook factor of \( Z_{1} \) renormalising the vector vertices and \( Z_{2}^{-1} \) renormalising the fermion propagator, multiplicative renormalisability of the transverse Ward-Takahashi relation, (2), is ensured by the same condition \( Z_{1} = Z_{2} \) as required by the renormalisation of the longitudinal Ward-Green-Takahashi identity, (1).

To see how the Ward-Takahashi identity and the transverse relation constrain the fermion-boson vector vertex, first let us recall the very first Ward identity, which is the \( q \to 0 \) limit of (1), viz.

$$\Gamma_{V}^{\mu}(p, p) = \frac{\partial S_{F}^{-1}(p)}{\partial p_{\mu}} .$$

(5)

Let us separate the vertex into longitudinal and transverse components defined by

$$\Gamma_{V}^{\mu} = \Gamma_{L}^{\mu} + \Gamma_{T}^{\mu} , \quad \text{with} \quad q_{\mu} \Gamma_{T}^{\mu} = 0 .$$

(6)

We can arrange for \( \Gamma_{L}^{\mu} \) alone to satisfy the original Ward-Green-Takahashi identity, (1), and for \( \Gamma_{T}^{\mu} \) alone to contribute to the left hand side of the transverse Ward-Takahashi relation, (2), by writing

$$\Gamma_{L}^{\mu}(p_{1}, p_{2}) = \frac{q^{\mu}}{q^{2}} \left( S_{F}^{-1}(p_{1}) - S_{F}^{-1}(p_{2}) \right) , \quad \text{and} \quad \Gamma_{T}^{\mu} = \left( g^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q^{2}} \right) \Gamma_{V}^{\mu} .$$

(7)

This separation seemingly makes the longitudinal and transverse components unrelated. However, each component of (7) has a kinematic singularity at \( q^{2} = 0 \). Consequently, the Ward identity in (5) requires an inter-relation between \( \Gamma_{L}^{\mu} \) and \( \Gamma_{T}^{\mu} \). An alternative separation is to abandon (7) and ensure that each component is free of kinematic singularities. Then one can require that the longitudinal part alone not only satisfies (1), but its \( q \to 0 \) limit too, viz. (5). This is the Ball-Chiu construction [1]. With this definition of the longitudinal part, we have, writing the massless inverse fermion propagator as \( S_{F}(p) = \alpha(p^{2}) \rho \):

$$\Gamma_{L}^{\mu}(p_{1}, p_{2}) = \frac{1}{2} \left( \alpha(p_{1}^{2}) + \alpha(p_{2}^{2}) \right) \gamma^{\mu} + \frac{1}{2} \frac{\alpha(p_{1}^{2}) - \alpha(p_{2}^{2})}{p_{1}^{2} - p_{2}^{2}} (p_{1} + p_{2})(p_{1} + p_{2})^{\mu} .$$

(8)

The transverse component is then only constrained to satisfy the condition: \( \Gamma_{T}^{\mu}(p, p) = 0 \), which the separation of (7) does not. However, the transverse Ward-Takahashi
This follows directly from the identity relation, which is what we do here. Consequently, it is critical to check how to evaluate the transverse Ward-Takahashi relation now involves both $\Gamma_\mu^T$ and $\Gamma_\mu^L$ as well. We can illustrate the power of the transverse relation by considering the $q \to 0$ limit of this equation. We can then deduce to all orders in perturbation theory, and genuinely non-perturbatively, the constraint $W^\mu\nu(p + q, p)$ places on the transverse vertex. From \cite{10} we have:

$$W^\mu\nu(p_1, p_2) \equiv \int \frac{d^4k}{(2\pi)^4} k_\lambda \left\{ \sigma^{\mu\nu}, \bar{\Gamma}_V^\lambda(p_1, p_2; k) \right\}$$

$$= \left\{ - S^{-1}_F(p_1)\sigma^{\mu\nu} + \sigma^{\mu\nu} S^{-1}_F(p_2) - \frac{1}{2}(p_1 + p_2)_\lambda \left\{ \sigma^{\mu\nu}, \bar{\Gamma}_V^\lambda(p_1, p_2) \right\} \right.$$  

$$- iq^\mu \Gamma^\nu_V(p_1, p_2) + iq^\nu \Gamma^\mu_V(p_1, p_2) \right\}. \quad (9)$$

When $q \to 0$ then $W^\mu\nu(p_1 = p_2 = p) = 0$. The general transverse vertex in the massless fermion case involves 4 vectors orthogonal to $q^\mu$, the basis vectors $T^\mu_i$, which are listed in Ref. \cite{17}, so that

$$\Gamma^T_\mu(p + q, p) = \sum_{i=2,3,6,8} \tau_i((p + q)^2, p^2, q^2) T^\mu_i(p, q), \quad (10)$$

where the coefficients $\tau_i$ are themselves free of kinematic singularities and are functions of the three relevant invariants, $p^2$, $p^2$ and $q^2$. $T^\mu_2$ and $T^\mu_3$ are quadratic in $q$, while $T^\mu_6$ and $T^\mu_8$ are both linear in particular

$$T^\mu_8 = i\gamma^\mu p^\mu p^\nu \sigma_{\nu\lambda} + p^\lambda p^\nu - p^\nu p^\lambda. \quad (11)$$

The coefficients $\tau_i$ are all symmetric under $p_1 = p + q \leftrightarrow p_2 = p$, except for $\tau_6$, which is antisymmetric and so vanishes when $q \to 0$. We then have non-perturbatively to first order in the boson momentum $q$:

$$W^\mu\nu(p_1 = p + q, p_2 = p) = -i [(p^\mu q^\nu - p^\nu q^\mu) \not{p} + q \cdot p (p^\mu \gamma^\nu - p^\nu \gamma^\mu)] \alpha'(p^2)$$

$$+ 2i [(p^\mu q^\nu - p^\nu q^\mu) \not{p} - q \cdot p (p^\mu \gamma^\nu - p^\nu \gamma^\mu)]$$

$$+ p^2 (q^\mu \gamma^\nu - q^\nu \gamma^\mu) \tau_8(p, p), \quad (12)$$

where we expect $\alpha'(p^2)$ to result from terms like $(\alpha(p_1^2) - \alpha(p_2^2)) / (p_2^2 - p_2^2)$ as in \cite{8}. Such constraints restrict the form of the transverse coefficient $\tau_8$, for example, and have the potential to play a critical role in constructing consistent non-perturbative Feynman rules. Consequently, it is critical to check how to evaluate the transverse Ward-Takahashi relation, which is what we do here.

It is trivial to check that the transverse Ward-Takahashi relation holds at tree level. This follows directly from the identity

$$iq^\mu \gamma^\nu - iq^\nu \gamma^\mu = \not{p}_1 \sigma^{\mu\nu} + \sigma^{\mu\nu} \not{p}_2 - \frac{1}{2}(p_1 + p_2)_\lambda \left\{ \sigma^{\mu\nu}, \gamma^\lambda \right\}. \quad (13)$$

when the non-local term does not appear. Note that at tree level we are in 4-dimensions so we would be permitted to use the identity $\left\{ \sigma^{\mu\nu}, \gamma^\lambda \right\} = -2\varepsilon^{\lambda\mu\nu\rho} \gamma_\rho \gamma_5$. He and Yu \cite{10} have sketched a proof of the relation at one loop order. The purpose of this paper
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is to check this in detail in \( d \)-dimensions, where \( d = 4 + \epsilon \). For notational brevity we introduce the following form of the Chisholm identity:

\[
- i \left\{ \sigma^{\mu \nu}, \gamma^\lambda \right\} = \gamma^\lambda \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \gamma^\lambda .
\] (14)

There has been discussion in the literature \([11, 7]\) about the role of the Adler-Bell-Jackiw anomaly \([12, 13, 14]\) in these relations. It has been shown \([15, 16]\) that this plays no role in the axial-vector equivalent of (2). We will consider the transverse Ward-Takahashi relation in the massless fermion case for simplicity. We show that this only holds if appropriate care is taken of the integrals divergent in 4-dimensions. This allows us to complete the proof to one loop order outlined by He and Yu \([10]\).

2. Perturbative Derivation of the Transverse Ward-Takahashi Relation

Following the formal derivation of the transverse Ward-Takahashi relation in Ref. \([8]\), He and Yu \([10]\) give a proof of how the relation should hold at one loop order by considering the relevant integrands, but without evaluating any of the integrals. Here we will investigate the relation in greater detail. First in this section we will reconsider the proof by deducing the integrands in \( d \)-dimensions, where \( d = 4 + \epsilon \). To manipulate the divergent integrals integrals in \( d \)-dimensions we must clearly use the original object \( \{ \sigma^{\mu \nu}, \gamma^\lambda \} \), since once again its identification with \( -2\epsilon^\rho_{\lambda \mu \nu} \gamma^\rho \gamma_5 \) is valid only in 4-dimensions. We will then confirm this result in the next section by explicit evaluation of each of the contributing integrals.

To derive the transverse Ward-Takahashi relation at one-loop order, we begin with the tree-level relation of (13):

\[
i q^\mu \gamma^\nu - i q^\nu \gamma^\mu = \not{q}_1 \sigma^{\mu \nu} + \sigma^{\mu \nu} \not{q}_2 - \frac{1}{2} [ ( \not{p}_1 + \not{p}_2 ) \sigma^{\mu \nu} + \sigma^{\mu \nu} ( \not{p}_1 + \not{p}_2 ) ] .
\] (15)

It is obvious that this relation is trivially satisfied at tree-level, hence we need only concentrate on the one-loop corrections to the relation. To simplify the answer we introduce the following shorthand for the integration:

\[
\int \frac{d^d k}{(2\pi)^d} \equiv g^2 \int \frac{d^d k}{(2\pi)^d} \frac{-i}{k^2 a^2 b^2} ,
\] (16)
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\[ \begin{align*}
\begin{array}{c}
\begin{array}{c}
\mathbf{p} \\
\end{array}
\end{array}
& =
\begin{array}{c}
\begin{array}{c}
\mathbf{p} \\
\end{array}
\end{array}
+ \begin{array}{c}
\begin{array}{c}
\mathbf{p} - k \\
\end{array}
\end{array}
+ \cdots
\end{align*}\
\end{align*}\
\]

\textbf{Figure 3.} One loop correction to the fermion propagator

where \( a = p_1 - k \) and \( b = p_2 - k \) as shown in figure 2. We use the standard Feynman rules for QED with, for example,

\[ \Delta_{\alpha\beta} (k) = -i \frac{k \alpha k \beta}{k^2} \left( g_{\alpha\beta} + (\xi - 1) \frac{k \alpha k \beta}{k^2} \right) \] (17)

for the bare gauge boson propagator, where \( \xi \) is the usual covariant gauge parameter.

If we write the full vertex to one-loop order as

\[ \Gamma^\mu_V (p_1, p_2) = \gamma^\mu + \Lambda^\mu_\nu (p_1, p_2) \] (18)

we can read off from figure 2 that

\[ \Lambda^\mu_\nu (p_1, p_2) = \int \frac{dk}{k} \gamma^\alpha \gamma^\mu b \gamma^\alpha \] (19)

in the Feynman gauge, when \( \xi = 1 \). We see that the left-hand side of (2) can be obtained by sandwiching (15) between \( \gamma^\alpha a \) and \( b \gamma^\alpha \), then integrating over the measure \( \int \frac{dk}{k} \), so that:

\[ iq^\mu \Lambda^\nu_\lambda (p_1, p_2) - iq^\nu \Lambda^\mu_\nu (p_1, p_2) = \int \frac{dk}{k} \gamma^\alpha \delta (p_1 + p_2) \sigma^\mu + \sigma^\nu p_2) b \gamma^\alpha \]

\[ - \frac{1}{2} \int \frac{dk}{k} \gamma^\alpha \delta [(p_1 + p_2) \sigma^\mu + \sigma^\nu (p_1 + p_2)] \gamma^\alpha \] (20)

The second term on the right-hand side is the \( -\frac{1}{2} (p_1 + p_2) \lambda \left\{ \sigma^\mu, \Lambda^\nu_\lambda \right\} \) piece of the original relation and is one of the terms we wish to identify.

At the one-loop level we can write the fermion self energy parts of figure 3 as

\[ S_F (p_1) = \phi_1 - \Sigma_\nu (p_1) \], \hspace{1cm} \text{where} \hspace{1cm} \Sigma_\nu (p_1) = \int \frac{dk}{k} \gamma^\alpha \left( \phi \partial^2 \right) \gamma^\alpha \]

\[ S_F (p_2) = \phi_2 - \Sigma_\nu (p_2) \], \hspace{1cm} \text{where} \hspace{1cm} \Sigma_\nu (p_2) = \int \frac{dk}{k} \gamma^\alpha \left( a^2 \phi \right) \gamma^\alpha \] (21)

In order to extract such terms from (20), the unidentified term on the right hand side of this equation is re-expressed using the replacement:

\[ \int \frac{dk}{k} \gamma^\alpha \delta (\phi_1 \sigma^\mu + \sigma^\nu \phi_2) b \gamma^\alpha = \int \frac{dk}{k} \gamma^\alpha \left( -\sigma^\mu a^2 \phi - \phi \partial^2 \sigma^\nu \right) \gamma^\alpha \]

\[ + \int \frac{dk}{k} \gamma^\alpha \delta (\phi \sigma^\mu + \sigma^\nu \phi) b \gamma^\alpha \]

\[ + \int \frac{dk}{k} \gamma^\alpha \delta (2 \phi \sigma^\mu + 2 \sigma^\nu \phi) b \gamma^\alpha \] (22)
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To the first line on the right-hand side of (22) we commute the $\sigma$-matrices to the outside of the $\gamma^\alpha$ by using $\sigma^{\mu\nu}\gamma^\alpha \rightarrow \gamma^\alpha\sigma^{\mu\nu} - 2i\eta^{\mu\nu}\gamma^\alpha + 2i\eta^{\alpha\mu}\gamma^\nu$. This gives:

$$i q^\mu \Lambda^\nu_{(2)} (p_1, p_2) - i q^\nu \Lambda^\mu_{(2)} (p_1, p_2) = - \Sigma_{(2)}(p_1) \sigma^{\mu\nu} - \sigma^{\mu\nu} \Sigma_{(2)}(p_2)$$

$$- \frac{1}{2} (p_1 + p_2)_\lambda \left\{ \sigma^{\mu\nu}, \Lambda^\lambda_{V(2)} \right\}$$

$$+ \int \frac{d\vec{k}}{2\pi^d} k^\alpha (\gamma^\alpha \sigma^{\mu\nu} + \sigma^{\mu\nu} \gamma^\alpha)$$

$$+ 2 \int \frac{d\vec{k}}{2\pi^d} \left\{ \gamma^\alpha \sigma^{\mu\nu} + \sigma^{\mu\nu} \gamma^\alpha \right\} k^\alpha$$

$$- \left( (a^2 \gamma^\alpha + gb^2) \sigma^{\mu\nu} + \sigma^{\mu\nu} (a^2 \gamma^\alpha + gb^2) \right) \right\} .$$

(23)

To be able to express this in terms of the Wilson line integral in the transverse Ward-Takahashi relation, we need to introduce a factor of $\gamma^\alpha$ and $\gamma_\alpha$ either side of the last term in (23). On rearranging this becomes:

$$i q^\mu \Lambda^\nu_{(2)} (p_1, p_2) - i q^\nu \Lambda^\mu_{(2)} (p_1, p_2) = - \Sigma_{(2)}(p_1) \sigma^{\mu\nu} - \sigma^{\mu\nu} \Sigma_{(2)}(p_2)$$

$$- \frac{1}{2} (p_1 + p_2)_\lambda \left\{ \sigma^{\mu\nu}, \Lambda^\lambda_{V(2)} \right\}$$

$$+ \int \frac{d\vec{k}}{2\pi^d} k^\alpha (\gamma^\alpha \sigma^{\mu\nu} + \sigma^{\mu\nu} \gamma^\alpha)$$

$$- \int \frac{d\vec{k}}{2\pi^d} \left( \gamma^\alpha \sigma^{\mu\nu} + \sigma^{\mu\nu} \gamma^\alpha \right) (a^2 \gamma^\alpha + gb^2) \right\} \right\} .$$

(24)

Putting back in the tree-level result, together with the identification of last 3 lines with the Wilson-line integration over non-local contributions allows us to write the transverse Ward-Takahashi relation, derived for the massless case of QED in $d$-dimensions in the Feynman gauge, as

$$i q^\mu \Gamma^\nu_{V}(p_1, p_2) - i q^\nu \Gamma^\mu_{V}(p_1, p_2) = S^{-1}_F(p_1) \sigma^{\mu\nu} + \sigma^{\mu\nu} S^{-1}_F(p_2) - \frac{1}{2} (p_1 + p_2)_\lambda \left\{ \sigma^{\mu\nu}, \Gamma^\lambda_{V} \right\}$$

$$+ \int \frac{d^d k}{(2\pi)^d} k^\lambda \left\{ \sigma^{\mu\nu}, \tilde{\Gamma}^\lambda_{V} \right\} .$$

(25)

which is indeed what is found by He, though written with explicitly $d$-dimensional objects. It is straightforward to check that this relation would hold in any covariant gauge.
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3. The One Loop Integrals

Since the integrals are not finite and so the manipulations more subtle, we now proceed to explicit computation of the terms in the transverse Ward-Takahashi relation to one loop order. The calculation will be divided into the four parts, of which it was originally composed, plus the fifth correction piece. The following sections deal with the terms in (25) in turn

- section 3.1 \( i q^\mu \Gamma_\nu^V - i q^\nu \Gamma_\mu^V \)
- section 3.2 \( S_F^{-1}(p_1) \sigma^{\mu\nu} + \sigma^{\nu\mu} S_F^{-1}(p_2) \)
- section 3.3 \( \frac{1}{2}(p_1 + p_2)_\lambda \left\{ \sigma^{\mu\nu}, \Gamma_\lambda^V \right\} \)
- section 3.4 \( \int d^4 k \ k_\lambda \left\{ \sigma^{\mu\nu}, \tilde{\Gamma}_\lambda^V \right\} \)

The \( \mathcal{O}(\alpha) \) correction to each of these we denote by \( (i\alpha P_i^{\mu\nu}/4\pi) \) with \( i = 1, \ldots, 4 \). Then (25) if true would become simply

\[
P_1^{\mu\nu} = [P_2 + P_3 + P_4]^{\mu\nu} .
\] (26)

This is what we set out to prove. The calculation is performed in full for an arbitrary covariant gauge, specified as usual by \( \xi \), and in dimension \( d \). We will be particularly concerned with the results with \( d = 4 + \epsilon \) when \( \epsilon \to 0 \).

3.1. Part 1

The first part of the calculation is the \( (i q^\mu \Gamma_\nu^V - i q^\nu \Gamma_\mu^V) \) piece. This involves the full vertex \( \Gamma_\mu^V(p_1, p_2) \), as calculated for arbitrary covariant gauge by Kızılersü et al [17]. Once we know all of the relevant standard integrals, which are collected in the Appendix, the calculation is straightforward. We begin by writing the vertex to one-loop order as in (18)

\[
\Gamma_\mu^V(p_1, p_2) = \gamma^\mu + \Lambda_\mu^{(2)}(p_1, p_2) .
\] (27)

Defining the momentum flow from figure 2, we then have, using the standard Feynman rules, the vertex correction given by:

\[
\Lambda_\mu(p_1, p_2) = -\frac{i\alpha}{4\pi^3} \int d^4 \omega \frac{\gamma^\alpha (q_1 - q_2) \gamma^\mu (q_2 - q_1) \gamma^\beta}{\omega^2 (p_1 - \omega)^2 (p_2 - \omega)^2} \left( g_{\alpha\beta} + (\xi - 1) \frac{\omega_\alpha \omega_\beta}{\omega^2} \right) ,
\] (28)

where we have suppressed the \(+ i\epsilon\) prescribed to define the propagators. If we decompose this into its constituent tensor integrals,

\[
\Lambda_\mu(p_1, p_2) = -\frac{i\alpha}{4\pi^3} \left\{ \gamma^\alpha (p_1^\gamma p_2^\nu) \gamma_\alpha J^{(0)}(p_1, p_2) - \gamma^\alpha (p_1^\mu p_2^\nu) \gamma_\alpha J^{(1)}_\nu(p_1, p_2) \right.
\]

\[
+ \left[ (-\gamma^\nu p_1^\gamma p_2^\lambda - \gamma^\mu p_2^\nu p_2^\lambda) J^{(2)}(p_1, p_2) \right.
\]

\[
- \gamma^\mu B^{(0)}(p_1, p_2) + \left[ \gamma^\nu p_1^\gamma p_2^\lambda \right] J^{(2)}_{\nu\lambda}(p_1, p_2) \right\} .
\] (29)
Reducing the tensor integrals to scalar integrals, and collecting terms together with common Dirac structures gives

\[ \Gamma^\mu_\nu (p_1, p_2) = \gamma^\mu + \frac{\alpha}{4\pi} \{ \bar{\psi}_2 p_2^\nu [2J_A - 2J_C + (\xi - 1) I_D p_1^2] \]

\[ + \bar{\psi}_1 p_1^\mu [2J_B - 2J_E + (\xi - 1) I_D p_2^2] + \bar{\psi}_1 p_2^\nu [-2J_0 + 2J_A + 2J_B - 2J_D] \]

\[ + (\xi - 1) \left( -\frac{1}{2} J_0 - \frac{1}{2} I_C p_2^2 - I_D p_1 \cdot p_2 + \frac{1}{2} I_E p_1^2 + J_A \right) \]

\[ + \bar{\psi}_2 p_1^\mu [-2J_D + (\xi - 1) (I_C p_2^2 - J_A)] + \bar{\psi} \mu p_1 \left[ J_0 - J_A - J_B \right] \]

\[ + (\xi - 1) \left( \frac{1}{4} J_0 + \frac{1}{4} I_C p_2^2 + \frac{1}{2} I_D p_1 \cdot p_2 + \frac{1}{4} I_E p_1^2 - \frac{1}{2} J_A - \frac{1}{2} J_B \right) \]

\[ + \bar{\psi}_2 p_2^\mu \left[ -J_A p_2^2 - J_B p_1^2 + \frac{1}{2} J_C p_2^2 + J_D p_1 \cdot p_2 + \frac{1}{2} J_E p_1^2 + \frac{1}{2} \left( 1 + \frac{3\xi}{4} \right) K_0 \]

\[ + (\xi - 1) \left( -\frac{1}{2} J_A p_2^2 - \frac{1}{2} J_B p_1^2 + \frac{1}{2} K_0 \right) \} . \]

where each of the functions \( J_A, \ldots, J_E, K_0 \) depend on \( p_1, p_2 \) and are given in Appendix Eqs (A.16, A.17) as well as (32, 33, 34) below. Collecting the terms together with common Lorentz and Dirac structure at one loop we obtain:

\[ P_1^{\mu\nu} = \bar{\psi}_1 (p_2^\mu p_2^\nu - p_1^\mu p_1^\nu) \left[ (-I_E p_1^2 - I_D p_2^2 + J_B) (\xi - 1) - 2(J_B - J_D - J_E) \right] \]

\[ + \bar{\psi}_2 (p_2^\mu p_2^\nu - p_1^\mu p_1^\nu) [2(J_0 - 2J_A - J_B + J_C + J_D) \]

\[ + \frac{1}{2} (\xi - 1) \left( -2I_D p_1^2 + I_E p_1^2 - I_C p_2^2 + J_0 - 2J_B + 2I_D p_1 \cdot p_2 \right) \]

\[ + \bar{\psi}_1 \bar{\psi}_2 (\gamma^\nu p_2^\mu + \gamma^\mu p_2^\nu - \gamma^\mu p_1^\nu - \gamma^\nu p_1^\mu) [J_0 - J_A - J_B \]

\[ + \frac{1}{4} (\xi - 1) \left( I_E p_1^2 + I_C p_2^2 + J_0 - 2J_A - 2J_B + 2I_D p_1 \cdot p_2 \right) \]

\[ + (\gamma^\nu p_2^\mu + \gamma^\mu p_2^\nu - \gamma^\mu p_1^\nu - \gamma^\nu p_1^\mu) \left[ \frac{1}{2} \left( 2J_B p_1^2 - J_E p_1^2 + 2J_A p_2^2 - J_C p_2^2 \right. \right. \]

\[ \left. - K_0 - 2(2J_0 - 2J_A - 2J_B + J_D)(p_1 \cdot p_2 + 3) \right) \]

\[ - \frac{1}{2} (\xi - 1) \left( -J_B p_1^2 - J_A p_2^2 + 2I_D (p_1 \cdot p_2)^2 \right. \]

\[ + K_0 + \left( I_E p_1^2 + I_C p_2^2 + J_0 - 2J_A - 2J_B \right) (p_1 \cdot p_2) \} . \]

Except where explicitly stated otherwise, \( K_0, \ldots, J_E \) are functions of \( p_1, p_2 \) as given in the Appendix (A.16, A.17) as well as (32, 33, 34) below. This answer, (31), can then be substituted into the first part of the transverse Ward-Takahashi relation, (2, 25), i.e. in (20).
3.2. Part 2

It is useful here to note that the many integrals appearing in evaluation of the loop graphs are expressible in terms of a set of basis functions $J_0$, $L$, $L'$, $C$ and $S$ given by:

$$J_0 = \frac{2}{\Delta} \left[ S_p \left( \frac{p_1^2 - p_1 \cdot p_2 + \Delta}{p_1^2} \right) - S_p \left( \frac{p_1^2 - p_1 \cdot p_2 - \Delta}{p_1^2} \right) \right]$$

$$+ \frac{1}{2} \ln \left( \frac{p_1 \cdot p_2 - \Delta}{p_1 \cdot p_2 + \Delta} \right) \ln \left( \frac{(p_2 - p_1)^2}{p_1^2} \right)$$

with $\Delta = \sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}$, and

$$L = \ln \left( -\frac{p_1^2}{\mu^2} \right); \quad L' = \ln \left( -\frac{p_2^2}{\mu^2} \right); \quad S = \frac{1}{2} \ln \left( -\frac{(p_1 - p_2)^2}{\mu^2} \right)$$

$$C = -\frac{2}{\varepsilon} - \gamma - \ln(\pi) .$$

In terms of these the $K$-scalar integrals which are about to appear are given simply by:

$$K_0(p_1, p_2) = 2(C + 2 - 2S)$$

$$K_0(0, p_2) = 2(C + 2 - L')$$

$$K_0(p_1, 0) = 2(C + 2 - L) .$$

With these definitions the $(S_F^{-1}(p_1)\sigma^{\mu\nu} + \sigma^{\mu\nu} S_F^{-1}(p_2))$ piece of (25) requires the evaluation of the inverse fermion propagator to one-loop. We note that the inverse propagator can be written as $S_F^{-1}(p_1) = p_1 - \Sigma_{(2)}(p_1)$, where

$$\Sigma_{(2)}(p_1) = -\frac{i\alpha}{4\pi^3} \int d^4k \gamma^\alpha \left( \frac{1}{k^2 - (p_1 - k)^2} \right) \left( g_{\alpha\beta} + \left( \xi - 1 \right) \frac{k_\alpha k_\beta}{k^2} \right)$$

$$= -\frac{i\alpha}{4\pi^3} (-2 - \varepsilon) \left[ p_1 K^{(0)}(p_1, 0) - \gamma^\lambda K^{(1)}_\lambda(p_1, 0) \right]$$

$$-\frac{i\alpha}{4\pi^3} (\xi - 1) \left[ \gamma^\lambda p_1 \gamma^\nu J^{(2)}_{\lambda\nu}(p_1, 0) + \gamma^\lambda K^{(1)}_\lambda(p_1, 0) \right] .$$

Once the tensor integrals have been substituted, the inverse propagator is

$$S_F^{-1}(p_1) = p_1 \left( 1 + \frac{\alpha\xi}{4\pi} (C + 1 - L) \right) ,$$

with a similar expression for $S_F^{-1}(p_2)$ with the replacement $p_1 \to p_2$ in (36). From these we deduce that

$$P_2^{\mu\nu} = p_1 \left( \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \right) \frac{\xi}{4} (K_0(p_1, 0) - 2)$$

$$+ (p_2 (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) + 4\gamma^\mu p_2^\nu - 4\gamma^\nu p_2^\mu) \frac{\xi}{4} (K_0(0, p_2) - 2) .$$

Substituting for these in the transverse Ward-Takahashi relation, (22-25), will give us the second piece.
3.3. Part 3

The third part we wish to compute involves the anticommutator of $\sigma^{\mu\nu}$ with the vector vertex with momentum routing as defined in figure 2. In the work of He [9] the anticommutator is replaced by its 4-dimensional identity, hence the appearance of the axial vector therein. We choose not to follow this path for obvious reasons — the integrals must be evaluated in $d = 4 + \varepsilon$ dimensions.

Thus we begin with

$$
- \frac{1}{2} \left( p_1 + p_2 \right)_{\lambda} \{ \sigma^{\mu\nu}, \Gamma^\lambda_V \} = \frac{i \alpha}{24 \pi^3} \int d^4 k \frac{\gamma^\alpha \left( \not{p}_1 - \not{k} \right) \left\{ \sigma^{\mu\nu}, \gamma^\lambda \right\} \left( \not{p}_2 - \not{k} \right) \gamma^\beta}{k^2 \left( p_1 - k \right)^2 \left( p_2 - k \right)^2} \times \left( g_\alpha^\beta + (\xi - 1) \frac{k_\alpha k_\beta}{k^2} \right).
$$

(38)

After extensive use of Dirac algebra identities and noting the definition of $r^{\lambda\mu\nu}$ in (14), we deduce

$$
- \frac{1}{2} \left( p_1 + p_2 \right)_{\lambda} \{ \sigma^{\mu\nu}, \Gamma^\lambda_V \} = - \frac{i}{2} \left( p_1 + p_2 \right)_{\lambda} r^{\lambda\mu\nu} \\
+ \frac{\alpha}{4 \pi^3} \left( p_1 + p_2 \right)_{\lambda} \left\{ \left( \gamma^\beta r^{\lambda\mu\nu} \gamma^\delta \right) \left( p_{1\delta} p_{2\beta} J^{(0)}(p_1, p_2) - p_{1\delta} J^{(1)}_{\beta}(p_1, p_2) - p_{2\beta} J^{(1)}_{\delta}(p_1, p_2) \right) \right. \\
- \frac{1}{2} \gamma^\alpha \gamma^\beta r^{\lambda\mu\nu} \gamma^\delta \gamma_\alpha J^{(2)}_{\beta\delta}(p_1, p_2) \\
+ (\xi - 1) \left[ \frac{1}{2} \gamma^\beta \not{p}_1 r^{\lambda\mu\nu} \left( J^{(1)}_{\beta}(p_1, p_2) - \not{p}_2 \gamma^\delta I^{(2)}_{\beta\delta}(p_1, p_2) \right) \\
+ \frac{1}{2} r^{\lambda\mu\nu} \left( \not{p}_2 \gamma^\beta J^{(1)}_{\beta}(p_1, p_2) - K^{(0)}(p_1, p_2) \right) \right].
$$

(39)
Checking the transverse Ward-Takahashi relation at one loop order

Substituting the forms of the integrals from the Appendix, we again collect the terms according to their Lorentz and Dirac structures

\begin{align*}
\mathbf{P}_3^{\mu \nu} &= \psi_1 \left[ (E_1 p_1^2 - I_D p_2^2 - J_B) (\xi - 1) (p_2^\nu p_1^\mu - p_1^\nu p_2^\mu) - 2(J_B + J_D + J_E) (p_2^\mu p_1^\nu - p_1^\nu p_2^\mu) \right] \\
&+ \psi_1 \psi_2 (\gamma^\nu p_2^\mu - \gamma^\mu p_2^\nu) [J_0 + J_A - J_B - 2J_C + 2J_D] \\
&+ \frac{1}{4} (\xi - 1) \left( 4I_D p_1^2 + I_E p_1^2 - 3I_C p_2^2 + J_0 + 2J_A - 2J_B + 2I_D p_1 \cdot p_2 \right) \\
&+ \psi_2 (p_2^\mu p_1^\nu - p_1^\nu p_2^\mu) \left[ 2(J_0 - J_B - J_C + J_D) \\
&+ \frac{1}{4} (\xi - 1) \left( 2I_D p_1^2 + I_E p_1^2 - I_C p_2^2 + J_0 - 2J_B + 2I_D p_1 \cdot p_2 \right) \right] \\
&+ \psi_1 \psi_2 (\gamma^\nu p_1^\mu - \gamma^\mu p_1^\nu) [J_0 - J_A + J_B + 2J_D - 2J_E] \\
&+ \frac{1}{4} (\xi - 1) \left( -3I_E p_1^2 + I_C p_2^2 + 4I_D p_1^2 + J_0 - 2J_A + 2J_B + 2I_D p_1 \cdot p_2 \right) \\
&+ \frac{1}{2} \psi_2 (\gamma^\nu \gamma^\nu - \gamma^\mu \gamma^\mu) \left[ \frac{1}{2} \left( -2J_0 p_1^2 + 2J_A p_1^2 - 4J_D p_1^2 - 3J_E p_1^2 + 2J_A p_2 - 2J_C p_2^2 \right) \\
&+ J_0 p_1^2 - 2J_A p_1^2 - 2J_A p_2^2 + 2K_0 - 2 \left( I_D p_1^2 - I_C p_2^2 + 2J_A \right) p_1 \cdot p_2 \right] \\
&+ \frac{1}{2} \psi_1 (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \left[ \frac{1}{2} \left( 2J_B p_1^2 - J_E p_1^2 - 2J_B p_2^2 + 2J_B p_2^2 + 3J_C p_2^2 - 4J_D p_2^2 - K_0 \right) \\
&+ 2(2J_B + J_D - 2J_E) p_1 \cdot p_2 - 5) + \frac{1}{4} (1 - \xi) \left( -3I_C p_1^2 + 4I_D p_1^2 p_2^2 + I_E p_1^2 p_2^2 \right) \\
&+ J_0 p_2^2 - 2J_B p_2^2 - 2J_B p_1^2 + 2K_0 - 2 \left( -2I_E p_1^2 + I_D p_2^2 + 2J_B \right) p_1 \cdot p_2 \right) \\
&+ (\gamma^\mu p_2^\nu - \gamma^\nu p_2^\mu) \left[ \frac{1}{2} \left( -2J_B p_1^2 - 4J_D p_1^2 + 3J_E p_1^2 + 2J_A p_2^2 - J_C p_2^2 - K_0 \right) \\
&+(4J_0 + 4J_A - 4J_B - 8J_C + 6J_D) p_1 \cdot p_2 - 5) \\
&+ \frac{1}{2} (1 - \xi) \left( -2I_E p_1^4 + 2I_D p_2^2 p_1^2 + J_B p_1^2 - J_A p_2^2 - 2I_D (p_1 \cdot p_2)^2 + K_0 \right) \\
&- \left( 4I_D p_1^2 + I_E p_1^2 - 3I_C p_2^2 + J_0 + 2J_A - 2J_B \right) p_1 \cdot p_2 \right] \\
&+ (\gamma^\mu p_1^\nu - \gamma^\nu p_1^\mu) \left[ \frac{1}{2} \left( 2J_B p_1^2 - J_E p_1^2 - 4J_0 p_2^2 + 2J_A p_2^2 + 4J_B p_2^2 + 3J_C p_2^2 - 4J_D p_2^2 \right) \\
&- K_0 - 2(2J_0 - 2J_A - 2J_B + J_D) p_1 \cdot p_2 - 5) + \frac{1}{2} (1 - \xi) \left( -I_C p_1^4 + 2I_D p_1^2 p_2^2 \right) \\
&+ I_E p_2^2 p_1^2 + J_0 p_2^2 - J_A p_2^2 - 2J_B p_2^2 - J_B p_1^2 + 2I_D (p_1 \cdot p_2)^2 + K_0 \right) \\
&+ \left( I_E p_1^2 + I_C p_2^2 + 2I_D p_2^2 + J_0 - 2J_A - 2J_B \right) p_1 \cdot p_2 \right] \ .
\end{align*}
The second set of contributions to the Wilson-line at one-loop is:

\[
\int \frac{d^4k}{(2\pi)^4} k_\lambda \left\{ \sigma^{\mu\nu}, \bar{\Gamma}^\lambda_V \right\}
\]

\[
= - \frac{i\alpha}{4\pi^3} \int d^4k \ k_\lambda \frac{\gamma^\alpha (p_1 - k') \left\{ \sigma^{\mu\nu}, \bar{\Gamma}^\lambda_V \right\} (p_2 - k') \gamma^\beta}{k^2 (p_1 - k)^2 (p_2 - k)^2} \left( g_{\alpha\beta} + (\xi - 1) \frac{k_\alpha k_\beta}{k^2} \right) \]

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Collecting these together and again ordering the terms according to their Lorentz and Dirac structure, we have:

\[
P_{4\mu\nu} = (\not{p}_1 \gamma^\nu p_1^\mu - \not{p}_2 \gamma^\mu p_2^\nu) \left[ 2(J_B + J_D - J_E) + (-I_E p_1^2 + I_D p_2^2 + J_B) (\xi - 1) \right]
\]

\[+ (\not{p}_1 \gamma^\nu p_1^\mu - \not{p}_2 \gamma^\mu p_2^\nu) \left[ 2(J_A - J_C + J_D) + (I_D p_1^2 - I_C p_2^2 + J_A) (\xi - 1) \right]
\]

\[+ \not{p}_2 \left[ -2I_D (\xi - 1) (p_2^\mu p_1^\nu - p_1^\mu p_2^\nu) p_1^2 - 4(J_A - J_C) (p_2^\mu p_1^\nu - p_1^\mu p_2^\nu) \right]
\]

\[+ \not{p}_1 \left[ 4J_D (p_2^\mu p_1^\nu - p_1^\mu p_2^\nu) + 2 (J_B - I_E p_1^2) (\xi - 1) (p_2^\mu p_1^\nu - p_1^\mu p_2^\nu) \right]
\]

\[+ (\gamma^\nu p_2^\mu - \gamma^\mu p_2^\nu) \left[ -2J_D p_1^2 + 2J_C p_2^2 - K_0 + 4(J_A - J_C + J_D) p_1 \cdot p_2 \right.
\]

\[+ \frac{1}{2} (1 - \xi) \left[ -2I_E p_1^4 + 2I_D p_2^2 p_1^2 + J_E p_1^2 - 3J_C p_2^2 + 2K_0
\]

\[-2 \left( 2I_D p_1^2 - 2I_C p_2^2 + 2J_A + J_D \right) p_1 \cdot p_2 - 3 \right] + 2 \right]
\]

\[- \frac{1}{2} \not{p}_1 (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \left[ J_E p_1^2 - 2J_A p_2^2 + J_C p_2^2 + 2J_D p_1 \cdot p_2 - 4
\right.
\]

\[+ \frac{1}{2} (1 - \xi) \left[ -2I_C p_1^4 + 2I_E p_2^2 p_1^2 + 2J_A p_2^2 - 4J_D p_2^2 + K_0 - 4J_E p_1 \cdot p_2 - 4 \right] - 1 \right]
\]

\[+ \frac{1}{2} \not{p}_2 (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \left[ 2J_B p_1^2 - J_E p_1^2 - J_C p_2^2 - 2J_D p_1 \cdot p_2
\right.
\]

\[+ \frac{1}{4} (\xi - 1) \left[ -3I_E p_1^4 + J_C p_2^2 p_1^2 + 4I_D p_2^2 p_1^2 + J_0 p_1^2 - 4J_D p_1^2 + J_E p_1^2 - 3J_C p_2^2
\]

\[+3K_0 - K_0(p_1, 0) - 2 \left( I_D p_1^2 - 2I_C p_2^2 + 2J_A + 2J_C + J_D \right) p_1 \cdot p_2 - 5 \right] + 1 \right]
\]

\[+ (\gamma^\mu p_1^\nu - \gamma^\nu p_1^\mu) \left[ 4J_A p_2^2 - 2J_C p_2^2 - 2J_D p_2^2 - K_0 + 4(J_B - J_E) p_1 \cdot p_2 + 4
\right.
\]

\[+ \frac{1}{2} (\xi - 1) \left[ -I_C p_1^4 + 2I_D p_2^2 p_1^2 + J_E p_1^2 p_2^2 + J_0 p_2^2 - 4J_D p_2^2 + K_0
\]

\[-K_0(p_1, 0) + \left( 2I_D p_2^2 - 4J_E \right) p_1 \cdot p_2 - 2 \right] + 4 \right]
\]

\[+ \frac{1}{2} \not{p}_1 (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \left[ \frac{1}{2} (4 - K_0(p_1, 0)) (\xi - 1) - 2 \right]
\]

\[+ \frac{1}{2} \not{p}_2 (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \left[ \frac{1}{2} (4 - K_0(0, p_2)) (\xi - 1) + 2 \right]
\]

\[+ (\gamma^\mu p_1^\nu - \gamma^\nu p_1^\mu) \left[ K_0(p_1, 0) + \xi - 5 \right]
\]

\[+ (\gamma^\nu p_2^\mu - \gamma^\mu p_2^\nu) \left[ 1 + \xi K_0(0, p_2) \right] \]
Checking the transverse Ward-Takahashi relation at one loop order

4. Summing terms gives the result

We now collect together the terms from (31, 37, 40, 44) to form

\[ Q^{\mu \nu} = [P_1 - P_2 - P_3 - P_4]^{\mu \nu}. \]  

(45)

The fact that each term is individually antisymmetric in \( \mu \) and \( \nu \) assists the checking. Rather than report every term of this somewhat tedious exercise, we illustrate that the answer for \( Q^{\mu \nu} \) is indeed zero by considering two structures. First those proportional to \( C \) of (33), which are individually singular in \( 1/\varepsilon \), then we will consider those proportional to unity.

4.1. Equating C’s

We denote the singular term in \( Q^{\mu \nu} \) by \( Q_\varepsilon^{\mu \nu} \). Collecting these parts we have:

\[ Q_\varepsilon^{\mu \nu} \equiv \gamma^\mu (p_2^\nu - p_1^\nu) C + (\xi - 1) \gamma^\mu (p_2^\nu - p_1^\nu) C \quad (+P_1^{\mu \nu}) \]

\[ - \xi \left[ \frac{1}{2} (p_1 + p_2) \gamma^\mu \gamma^\nu + 2 \gamma^\mu p_2^\nu \right] C \quad (-P_2^{\mu \nu}) \]

\[ + \xi \left[ \frac{1}{2} (p_1 + p_2) \gamma^\mu \gamma^\nu + \gamma^\mu (p_1^\nu + p_2^\nu) \right] C \quad (-P_3^{\mu \nu}) \]

\[ + 2 \gamma^\mu (p_2^\nu - p_1^\nu) C + 2 \gamma^\mu (p_1^\nu - p_2^\nu) C \]

\[ + (\xi - 1) \left[ \frac{1}{2} (p_1 + p_2) \gamma^\mu \gamma^\nu + 2 \gamma^\mu p_2^\nu \right] C \]

\[ - (\xi - 1) \left[ \frac{1}{2} (p_1 + p_2) \gamma^\mu \gamma^\nu + 2 \gamma^\mu p_2^\nu \right] C \quad (-P_4^{\mu \nu}) \]

\[ - (\mu \leftrightarrow \nu) \]

\[ = 0 \quad (46) \]

The \( P_i^{\mu \nu} \) in brackets indicate from which part the terms originate.
4.2. Equating 1's

We next collect together the terms proportional to unity. We denote this collection by $Q^{(1)}_{\mu\nu}$:

$$Q^{(1)}_{\mu\nu} = \frac{1}{\Delta^2} \left\{ (2 - \xi) \left( -p_1^\mu p_2^\nu \right) \left[ p_2^2 \left( p_1^2 - p_1 \cdot p_2 \right) + p_1^2 \left( p_2^2 - p_1 \cdot p_2 \right) \right] 
+ 2 (\xi - 1) \gamma^\mu (p_2^\nu - p_1^\nu) \Delta^2 \right\} (+P_{1\mu\nu})$$

$\quad - \xi \left[ \frac{1}{2} (p_1 + p_2) \gamma^\mu \gamma^\nu + 2 \gamma^\mu p_2^\nu \right] \Delta^2 \right\} (-P_{2\mu\nu})$

$\quad + \frac{(4 + \xi)}{2} \left( p_1 + p_2 \right) \gamma^\mu \gamma^\nu \Delta^2 + 6 \gamma^\mu (p_1^\nu + p_2^\nu) \Delta^2$

$\quad - (2 - \xi) \left\{ \gamma^\mu p_1^\nu \left[ p_2^2 \left( p_1^2 + p_1 \cdot p_2 \right) + 2 \Delta^2 \right] - \gamma^\mu p_2^\nu \left( p_2^2 + p_1 \cdot p_2 \right) 
+ p_1^\mu p_2^\nu \left[ p_2^2 \left( p_1^2 + p_1 \cdot p_2 \right) - p_1^2 \left( p_2^2 + p_1 \cdot p_2 \right) \right] 
+ p_1^\mu p_2^\nu \left[ p_2^2 \left( p_1^2 + p_1 \cdot p_2 \right) - p_1^2 \left( p_2^2 + p_1 \cdot p_2 \right) \right] \right\} (-P_{3\mu\nu})$

$\quad - \left( 2 p_1^\mu \gamma^\nu + 4 \gamma^\mu p_2^\nu \right) \Delta^2 - \left( 2 p_2^\mu \gamma^\nu + 4 \gamma^\mu p_2^\nu \right) \Delta^2$

$\quad + (2 - \xi) \left\{ \gamma^\mu p_1^\nu \left[ p_2^2 \left( p_1^2 + p_1 \cdot p_2 \right) - \gamma^\mu p_1^\nu \left( p_2^2 + p_1 \cdot p_2 \right) 
+ p_1^\mu p_2^\nu \left( 2 p_2^2 p_1^2 - 2 p_1^2 p_1 \cdot p_2 \right) \right] 
- \left( \mu \leftrightarrow \nu \right) \right\}$

$$= 0 \quad . \quad (47)$$

4.3. The Result

We can similarly check for all the terms in $Q^{\mu\nu}$, i.e. for the combination of $P_{i\mu\nu}$ ($i = 1, 4$) of (45), that the result is indeed zero.

Thus in conclusion we have deduced that the transverse Ward-Takahashi identity to one loop order in $d$ close to 4 dimensions is indeed given by (25):
Checking the transverse Ward-Takahashi relation at one loop order

\begin{equation}
q^\mu \Gamma^\nu_V(p_1, p_2) - i q^\nu \Gamma^\mu_V(p_1, p_2) = \frac{1}{2}(p_1 + p_2) \lambda \left\{ \sigma^\mu \Gamma^\lambda_V \right\} 
+ \int \frac{d^d k}{(2\pi)^d} k\lambda \left\{ \sigma^\mu \tilde{\Gamma}^\lambda_V \right\}.
\end{equation}

This is a critical step in developing non-perturbative Feynman rules for the key fermion-boson interaction so central to Abelian and non-Abelian gauge theories in the strong coupling regime.

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Appendix A.

Note that in the text these functions will be assumed to depend upon \( p_1 \) and \( p_2 \) unless explicitly indicated.

\begin{align}
K^{(0)}(p_1, p_2) &= \int d^4 \omega \frac{1}{(p_1 - \omega)^2 (p_2 - \omega)^2} = \frac{i \pi^2}{2} K_0(p_1, p_2), \\
K^{(0)}(0, p_2) &= \int d^4 \omega \frac{1}{\omega^2 (p_2 - \omega)^2} = \frac{i \pi^2}{2} K_0(0, p_2), \\
K^{(0)}(p_1, 0) &= \int d^4 \omega \frac{1}{(p_1 - \omega)^2 \omega^2} = \frac{i \pi^2}{2} K_0(p_1, 0), \\
K^{(1)}_{\mu}(p_1, p_2) &= \int d^4 \omega \frac{\omega^\mu}{(p_1 - \omega)^2 (p_2 - \omega)^2} = \frac{i \pi^2}{4} (p_1 + p_2)_\mu K_0(p_1, p_2), \\
K^{(1)}_{\mu}(0, p_2) &= \int d^4 \omega \frac{\omega^\mu}{\omega^2 (p_2 - \omega)^2} = \frac{i \pi^2}{4} p_{2\mu} K_0(0, p_2), \\
K^{(1)}_{\mu}(p_1, 0) &= \int d^4 \omega \frac{\omega^\mu}{(p_1 - \omega)^2 \omega^2} = \frac{i \pi^2}{4} p_{1\mu} K_0(p_1, 0), \\
J^{(0)}(p_1, p_2) &= \int d^4 \omega \frac{1}{\omega^2 (p_1 - \omega)^2 (p_2 - \omega)^2} = \frac{i \pi^2}{2} J_0,
\end{align}
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\[ J^{(1)} (p_1, p_2) = \int d^4 \omega \frac{\omega_\mu}{\omega^2 (p_1 - \omega)^2 (p_2 - \omega)} = \frac{i\pi^2}{2} [p_{2\mu} J_A + p_{1\mu} J_B] , \quad (A.8) \]

\[ J^{(1)} (0, p_2) = \int d^4 \omega \frac{\omega_\mu}{\omega^4 (p_2 - \omega)^2} = \frac{i\pi^2 p_{2\mu}}{p_2^2} , \quad (A.9) \]

\[ J^{(1)} (p_1, 0) = \int d^4 \omega \frac{\omega_\mu}{\omega^4 (p_1 - \omega)^2} = \frac{i\pi^2 p_{1\mu}}{p_1^2} , \quad (A.10) \]

\[ J^{(2)}_{\mu\nu} (p_1, p_2) = \int d^4 \omega \frac{\omega_\mu \omega_\nu}{\omega^2 (p_1 - \omega)^2 (p_2 - \omega)^2} \]

\[ = \frac{i\pi^2}{2} \left\{ \frac{g_{\mu\nu}}{d} K_0 + \left( p_{2\mu} p_{2\nu} - g_{\mu\nu} \frac{p_2^2}{4} \right) J_C + \left( p_{1\mu} p_{1\nu} - g_{\mu\nu} \frac{p_1^2}{4} \right) J_E \right. \]

\[ + \left. \left( p_{1\mu} p_{2\nu} + p_{2\mu} p_{1\nu} - g_{\mu\nu} \frac{p_1 \cdot p_2}{2} \right) J_D \right\} , \quad (A.11) \]

\[ J^{(2)}_{\mu\nu} (0, p_2) = \int d^4 \omega \frac{\omega_\mu \omega_\nu}{\omega^4 (p_2 - \omega)^2} \]

\[ = \frac{i\pi^2}{2} \left( \frac{g_{\mu\nu}}{4} K_0 (0, p_2) + \frac{p_{2\mu} p_{2\nu}}{p_2^2} \right) , \quad (A.12) \]

\[ J^{(2)}_{\mu\nu} (p_1, 0) = \int d^4 \omega \frac{\omega_\mu \omega_\nu}{\omega^4 (p_1 - \omega)^2} \]

\[ = \frac{i\pi^2}{2} \left( \frac{g_{\mu\nu}}{4} K_0 (p_1, 0) + \frac{p_{1\mu} p_{1\nu}}{p_1^2} \right) , \quad (A.13) \]

\[ I^{(1)} (p_1, p_2) = \int d^4 \omega \frac{\omega_\mu}{\omega^4 (p_1 - \omega)^2 (p_2 - \omega)^2} \]

\[ = \frac{i\pi^2}{2} [p_{2\mu} I_A + p_{1\mu} I_B] , \quad (A.14) \]

\[ I^{(2)}_{\mu\nu} (p_1, p_2) = \int d^4 \omega \frac{\omega_\mu \omega_\nu}{\omega^4 (p_1 - \omega)^2 (p_2 - \omega)^2} \]

\[ = \frac{i\pi^2}{2} \left\{ \frac{g_{\mu\nu}}{4} J_0 + \left( p_{2\mu} p_{2\nu} - g_{\mu\nu} \frac{p_2^2}{4} \right) I_C + \left( p_{1\mu} p_{1\nu} - g_{\mu\nu} \frac{p_1^2}{4} \right) I_E \right. \]

\[ + \left. \left( p_{1\mu} p_{2\nu} + p_{2\mu} p_{1\nu} - g_{\mu\nu} \frac{p_1 \cdot p_2}{2} \right) I_D \right\} . \quad (A.15) \]

A.1. J-scalar functions

\[ J_A = \frac{1}{\Delta^2} \left\{ \frac{J_0}{2} \left( -p_1^2 (p_2^2 - p_1 \cdot p_2) \right) + p_1 \cdot p_2 L' - p_1^2 L - 2 \left( p_1 \cdot p_2 - p_1^2 \right) S \right\} \]

\[ J_B = \frac{1}{\Delta^2} \left\{ \frac{J_0}{2} \left( -p_2^2 (p_1^2 - p_1 \cdot p_2) \right) + p_1 \cdot p_2 L - p_2^2 L' - 2 \left( p_1 \cdot p_2 - p_2^2 \right) S \right\} \]

\[ J_C = \frac{1}{4 \Delta^2} \left\{ 2p_1^2 - 4p_1 \cdot p_2 S + 2p_1 \cdot p_2 L' + \left( 2p_1 \cdot p_2 p_2^2 - 3p_1^2 p_2^2 \right) J_A - p_1^4 J_B \right\} \]
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\[ J_D = \frac{1}{4\Delta^2} \left\{ 2p_1 \cdot p_2 p_2^2 J_A + 2p_1 \cdot p_2 p_1^2 J_B - 2p_1 \cdot p_2 + 2p_2^2 S - p_2^2 L' - p_1^2 p_2 J_A \\
+ 2p_1^2 S - p_1^2 L - p_1^2 p_2 J_B \right\} \]

\[ J_E = \frac{1}{4\Delta^2} \left\{ 2p_2^2 - 4p_1 \cdot p_2 S + 2p_1 \cdot p_2 L + \left( 2p_1 \cdot p_2 p_2^2 - 3p_1^2 p_2^2 \right) J_B - p_2^4 J_A \right\} \quad (A.16) \]

A.2. I-scalar functions

\[ I_A = \frac{1}{\Delta^2} \left\{ -\frac{p_1^2 - p_1 \cdot p_2}{2} J_0 + 2 \left( 1 - \frac{p_1 \cdot p_2}{p_2^2} \right) S - L' + \frac{p_1 \cdot p_2}{p_2^2} L \right\} \]

\[ I_B = \frac{1}{\Delta^2} \left\{ -\frac{p_2^2 - p_1 \cdot p_2}{2} J_0 + 2 \left( 1 - \frac{p_1 \cdot p_2}{p_1^2} \right) S - L + \frac{p_1 \cdot p_2}{p_1^2} L' \right\} \]

\[ I_C = \frac{1}{4\Delta^2} \left\{ 2p_1^2 J_0 - 4 \frac{p_1 \cdot p_2}{p_1^2} + \left( 2p_1 \cdot p_2 - 3p_1^2 \right) J_A - p_1^2 J_B - p_1^4 J_A \\
+ \left( 2p_1 \cdot p_2 p_1^2 - 3p_1 p_2^2 \right) I_A \right\} \]

\[ I_D = \frac{1}{4\Delta^2} \left\{ -2p_1 \cdot p_2 J_0 + 4 + \left( 2p_1 \cdot p_2 - p_2^2 \right) J_A + \left( 2p_1 \cdot p_2 - p_1^2 \right) J_B \\
+ \left( 2p_1 \cdot p_2 p_2^2 - p_1^2 p_2^2 \right) I_A + \left( 2p_1 \cdot p_2 p_1^2 - p_1^2 p_2^2 \right) I_B \right\} \]

\[ I_E = \frac{1}{4\Delta^2} \left\{ 2p_2^2 J_0 - 4 \frac{p_1 \cdot p_2}{p_1^2} + \left( 2p_1 \cdot p_2 - 3p_2^2 \right) J_B - p_2^4 J_A - p_2^4 I_A \\
+ \left( 2p_1 \cdot p_2 p_2^2 - 3p_1 p_2^2 \right) I_B \right\} \quad (A.17) \]
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