ESTIMATES FOR SUMS OF EIGENFUNCTIONS OF ELLIPTIC PSEUDO-DIFFERENTIAL OPERATORS ON COMPACT LIE GROUPS

DUVÁN CARDONA, JULIO DELGADO, AND MICHAEL RUZHANSKY

Abstract. We extend the estimates proved by Donnelly and Fefferman and by Lebeau and Robbiano for sums of eigenfunctions of the Laplacian (on a compact manifold) to estimates for sums of eigenfunctions of any positive and elliptic pseudo-differential operator of positive order on a compact Lie group. Our criteria are imposed in terms of the positivity of the corresponding matrix-valued symbol of the operator. As an application of these inequalities in the control theory, we obtain the null-controllability for diffusion models for elliptic pseudo-differential operators on compact Lie groups.

Contents

1. Introduction 2
1.1. Outline 2
1.2. Main result 2
1.3. Structure of the work 4
2. Preliminaries 5
2.1. Pseudo-differential operators on compact Lie groups 5
2.2. Global and local classes of pseudo-differential operators on the torus 14
2.3. Null-controllability of diffusion problems on Hilbert spaces 15
3. Donnelly-Fefferman inequalities on compact Lie groups 17
3.1. Sketch of the proof of Proposition 3.3 20
3.2. Computing the inverse of \( A(x, t, D, \partial_t) \) 21
3.3. \( L^2 \)-theory for the operator \( (-\partial_t^2 + L_G)A(x, t, D, \partial_t)^{-1} \) 24
3.4. Proof of Proposition 3.3 26
3.5. Applications to control theory: Null-controllability for diffusion models 35
4. Appendix: Construction of the cut-off function \( \psi \) 36
References 37

2020 Mathematics Subject Classification. 42B20, 42B37.
Key words and phrases. Pseudo-differential operator, Null-controllability, Fractional diffusion model, Microlocal Analysis, Spectral Inequality.

The authors are supported by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations and by the Methusalem programme of the Ghent University Special Research Fund (BOF) (Grant number 01M01021). Julio Delgado is also supported by Vic. Inv Universidad del Valle. Grant No. CI-7329, MathAmSud and Minciencias-Colombia under the project MATH-AMSUD 21-MATH-03. Michael Ruzhansky is also supported by EPSRC grant EP/R003025/2.
1. **Introduction**

1.1. **Outline.** Let \((M, g)\) be a compact \(C^\infty\)-Riemannian manifold. In the late 1980s H. Donnelly and C. Fefferman in their celebrated *Inventiones*’ paper [14] proved the doubling property

\[
\sup_{B(2R)} |\phi| \leq e^{C_1 \lambda + C_2} \sup_{B(R)} |\phi| \tag{1.1}
\]

for any eigenfunction of the Laplacian \(\Delta_g\) on \(M\), that is, \(-\Delta_g \phi = \lambda^2 \phi\), where \(B(2R)\) and \(B(R)\) represent concentric balls (associated to the geodesic distance) where the constants \(C_1\) and \(C_2\) are independent of \(R > 0\), and depending only on \(M\). The estimate in (1.1) remains valid for sums of eigenfunctions of \(\Delta_g\). In this work we extend such an estimate for sums of eigenfunctions of any positive elliptic pseudo-differential operator \(A\) when \(M\) is a compact Lie group. Even, we consider the general case where \(A\) has positive real order and belongs to the global \((\rho, \delta)\)-Hörmander classes. If this inequality holds in the complete range \(0 \leq \delta < \rho \leq 1\) was an open problem prior to this work. The fact of considering the setting of compact Lie groups is justified since on general compact manifolds the principal symbol of a pseudo-differential operator is invariantly defined only if \(0 \leq \delta < \rho \leq 1\) and \(\rho \geq 1 - \delta\), see Hörmander [27]. Here, we introduce a new approach, different from the one via Carleman estimates as developed by Donnelly and Fefferman in [14]. A reason to introduce a new approach comes from the lack of Carleman estimates in the case of non-local operators. To do this, looking for criteria on the operator \(A\) that allow the validity of the doubling property in (1.1) for the sums of its eigenfunctions, we connect this problem with the representation theory of a compact Lie group \(G\).

In order to present our main Theorem 1.1 let us introduce the required notation.

1.2. **Main result.** Indeed, by writing the elliptic operator \(A : C^\infty(G) \to C^\infty(G)\) in the convolution form

\[
Af(x) = \int_G R_A(x, xy^{-1})f(y)dy, \ f \in C^\infty_0(G), \tag{1.2}
\]

where the distribution \(R_A \in C^\infty(G, \mathcal{D}'(G))\) is associated via the Schwartz kernel theorem, one can associate a global symbol \(\sigma_A : G \times \widehat{G} \to \bigcup_{\ell \in \mathbb{N}} \mathbb{C}^{\ell \times \ell}\) to \(A\). Here, \(\widehat{G}\) denotes the unitary dual of \(G\), formed by the set of all continuous, unitary and irreducible representations \(\xi : G \to \text{Hom}(\mathbb{C}^\ell)\) of \(G\). Recall that \(d_\xi := \ell\) is usually called the dimension of the representation \(\xi\). Then, the global symbol \(\sigma_A\) of \(A\) is defined by the group Fourier transform of the distribution \(K_A(x, \cdot)\), which is given by

\[
\sigma_A(x, \xi) = \int_G R_A(x, z)\xi(y)^*dy, \ [\xi] \in \widehat{G}. \tag{1.3}
\]

Then, the Fourier inversion formula allows the Fourier representation of the operator \(A\) as follows

\[
Af(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}[\xi(x)\sigma_A(x, \xi)\hat{f}(\xi)], \ f \in C^\infty(G), \tag{1.4}
\]
with \( \hat{f}(\xi) = \int f(y)\xi(y)^*dy \) denoting the Fourier transform of a test function \( f \) at the representation \( \xi \). Above \( dy \) denotes the Haar measure on \( G \). This quantisation was consistently developed in [44], and we recall some of its relevant properties.

The Hörmander classes of pseudo-differential operators \( \Psi^m_{\rho,\delta}(G) \) can be characterised in terms of the global matrix-valued symbols \( \sigma_A \) obtained in the construction above. That \( A \in \Psi^m_{\rho,\delta}(G) \) means that in any local coordinate system the operator has the form (by identifying local coordinates systems in \( G \) with the corresponding Euclidean open subsets)

\[
A\phi(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \theta} \sigma(x, \theta)\hat{\phi}(\theta)d\theta, \quad \phi \in C_0^\infty(\mathbb{R}^n),
\]

where \( \hat{\phi} \) denotes the Euclidean Fourier transform of \( \phi \) and where the symbol \( \sigma \) associated to each chart satisfies growing estimates of \((\rho, \delta)\)-type, that is

\[
|\partial_x^\alpha \partial_\theta^\beta \sigma(x, \theta)| \leq C_{\alpha, \beta}(1 + |\theta|)^{m-\rho|\alpha|+\delta|\beta|}
\] (1.5)

uniformly on compact subsets of the chart. Indeed, there are required certain relations between \( \rho \) and \( \delta \), to have the classes \( \Psi^m_{\rho,\delta}(G) \) invariant under changes of coordinates, namely that

\[
0 \leq \delta < \rho \leq 1, \quad \rho \geq 1 - \delta.
\] (1.6)

Under this assumptions it was proved in [45], that \( A \in \Psi^m_{\rho,\delta}(G) \) if and only if its matrix-valued symbol \( \sigma_A \) satisfies the symbol estimates

\[
\|\partial_x^\alpha \partial_\theta^\beta \sigma_A(x, \xi)\|_{\text{End}(C^\infty)} \leq C_{\alpha, \beta}(\xi)^{m-\rho|\alpha|+\delta|\beta|}, \quad (x, \xi) \in G \times \hat{G}.
\] (1.7)

The weight \( \langle \xi \rangle := (1 + \lambda_{[\xi]} \frac{1}{2})^\frac{1}{2} \) is defined in terms of the spectrum \( \{\lambda_{[\xi]}\}_{[\xi] \in \hat{G}} \) of the positive Laplacian \( L_G \). Observe that the unitary dual of \( G \) is a discrete set and the difference operators \( \mathbb{D}^{\alpha} \) in (1.7) play the role of “derivatives” acting on functions/distributions defined on the unitary dual \( \hat{G} \).

An important feature of the description above for the Hörmander classes of pseudo-differential operators \( \Psi^m_{\rho,\delta}(G) \), \( m \in \mathbb{R} \), \( 0 \leq \delta < \rho \leq 1 \), and \( \rho \geq 1 - \delta \) is that still, when \( \rho < 1 - \delta \), one can define the classes

\[
\Psi^m_{\rho,\delta}(G \times \hat{G}) := \{ A : C^\infty(G) \to C^\infty(G) : \sigma_A \text{ satisfies (1.7)} \}
\] (1.8)

allowing a well-defined class of pseudo-differential operators in the complete range \( 0 \leq \delta < \rho \leq 1 \). The classes in (1.8) become effective in handling certain classes of operators, for example resolvent operators for vector fields on a compact Lie group \( G \) which belong to the class \( \Psi^0_{0,0}(G \times \hat{G}) \) or in parametrices of Hörmander sub-Laplacians that have symbols in the class \( \Psi^{-1}_{\frac{3}{2},0}(G \times \hat{G}) \). See [45] for this and for other examples of the appearance of different symbol classes as parametrices for hypoelliptic operators which cannot be handled by the standard theory in view of the restriction in (1.6).

The following Donnelly-Fefferman type inequality for elliptic pseudo-differential operators on a compact Lie group is the main theorem of this work.

**Theorem 1.1.** Let \( 0 \leq \delta < \rho \leq 1 \). Let \( A \in \Psi^m_{\rho,\delta}(G \times \hat{G}) \) be a positive elliptic pseudo-differential operator of order \( m > 0 \). Assume that \( \sigma_A(x, \xi) \geq 0 \) for all \( (x, [\xi]) \in G \times \hat{G} \).
Let \((e_j, \lambda_j^m)\), \(\lambda_j \geq 0\), be the corresponding spectral data of \(A\), determined by the eigenvalue problem \(Ae_j = \lambda_j^m e_j\) with the eigenfunctions \(e_j\) being \(L^2\)-normalised. Then the following spectral estimates are valid:

- For any non-empty open subset \(\omega \subset G\), we have
  \[
  \|\kappa\|_{L^2(G)} \leq C_1 e^{C_2 \lambda} \|\kappa\|_{L^2(\omega)}, \quad \kappa \in \text{span}\{e_j : \lambda_j \leq \lambda\},
  \]
  with \(C_1 = C_1(\omega)\) and \(C_2 = C_2(\omega)\) depending on \(\omega\), but not on \(\kappa\). \hfill (1.9)
- For any \(R > 0\) let \(B(x, R)\) be a ball defined by the geodesic distance, of radius \(R > 0\) and centred at \(x\). Then,
  \[
  \sup_{B(x, 2R)} |\kappa| \leq e^{C_1' \lambda} + C_2' \sup_{B(x, R)} |\kappa|, \quad \kappa \in \text{span}\{e_j : \lambda_j \leq \lambda\},
  \]
  with \(C_1' = C_1'(R)\) and \(C_2' = C_2'(R)\) depending only on the radius \(R > 0\) but not on \(\kappa\). \hfill (1.10)

**Remark 1.2.** In Subsection 3.5 we give an application of this result to the control theory. More precisely, we use Theorem 1.1 to prove that the heat equation

\[
\begin{cases}
u_t(x, t) + A^\gamma u(x, t) = g(x, t) \cdot 1_\omega(x), \quad (x, t) \in G \times (0, T), \\
u(0, x) = u_0,
\end{cases}
\]

associated to the fractional diffusion operator \(A^\gamma\) is null-controllable at any time \(T > 0\) provided that \(\gamma > 1/m\). The condition \(\gamma > 1/m\) is sharp if one considers the case of the powers \(A = L^{m/2}_T\) on the torus \(G = \mathbb{T}\), see Miller [40]. For the terminology and for the basic aspects related to the control theory we refer the reader to Subsection 2.3 and for the null-controllability result for the model (1.11) see Theorem 3.10 of Subsection 3.5.

**Remark 1.3.** The analysis of growth estimates for eigenfunctions of the Laplacian and of other elliptic differential operators is still a problem of wide interest. In particular for its relation with the geometric analysis of nodal sets. For classic references on the subject we refer the reader to Sunada [48], Atiyah, Donnelly and Singer [1], Borel and Garland [2], Jerison and Lebeau [28], Donnelly and Fefferman [13, 14, 15, 16], Donnelly and Garofalo [17, 18], and Lin [36]. As for recent works on the subject we refer the reader to Apraiz, Escauriaza, Wang, and Zhang [3], Blair and Sogge [6], Cavalletti and Farinelli [9], Enciso and Peralta-Salas [21], Georgiev [25], Kenig, Zhu, and Zhuge [29], Logunov [30, 31], Logunov, Malinnikova, Nadirashvili, and Nazarov [32], Tian and Yang [50] and Toth and Zelditch [51] just to mention a few. About the applications of spectral inequalities to the control theory we refer to Benabdallah and Naso [4], Fu, Lü, and Zhang [24], Lebeau and Robbiano [34], Lebeau and Zuazua [35], J.-L. Lions, [37], Micu and Zuazua [39], Miller [40, 41], Rousseau and Lebeau [42], Cardona [7], Rousseau and Robbiano [43], and the extensive list of references therein.

1.3. **Structure of the work.** In Section 2 we survey the rather extensive analytical backgrounds about the theory of pseudo-differential operators on compact Lie groups with the calculus based on the matrix-valued quantisation and on compact manifolds (with the notion of a symbol in the \((\rho, \delta)\)-class defined by local coordinate systems) in Section 2. We do a particular emphasis that these two points of view agree when
\( \rho \geq 1 - \delta \) and when \( 0 \leq \delta < \rho \leq 1 \). We then use the global theory of pseudo-differential operators and the matrix-valued quantisation to prove in Section 3 the Donnelly-Fefferman/Lebeau-Robbiano spectral inequalities in Theorem 1.1. Finally, our application to the control theory of diffusion problems on compact Lie groups is addressed in Theorem 3.10 of Subsection 3.5.

2. Preliminaries

In this section, we present the preliminaries about the theory of pseudo-differential operators on compact Lie groups as well as the matrix-valued quantisation. For our further applications, we recall some results about the control theory of heat equations on Hilbert spaces. The following standard notation will be employed during this work.

- For two Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), we denote by \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) the family of bounded and linear operators \( T : \mathcal{H}_1 \to \mathcal{H}_2 \).
- The spectrum of a densely defined linear operator \( A : \text{Dom}(A) \subset \mathcal{H}_1 \to \mathcal{H}_1 \) will be denoted by \( \sigma(A) \) and its resolvent set by \( \text{Resolv}(A) := \mathbb{C} \setminus \sigma(A) \).
- We write \( A \preceq B \) if \( A \leq cB \) where \( c > 0 \) does not depend on \( A \) and \( B \). If \( A \preceq B \) and \( B \preceq A \) we write \( A \asymp B \).
- \( G \) is a compact Lie group and \( L_G \) denotes its corresponding Laplace-Beltrami operator.

2.1. Pseudo-differential operators on compact Lie groups. To define pseudo-differential operators, the main tool is the Fourier transform. On compact Lie groups the Fourier transform is defined in terms of the representations of a group. Only irreducible and unitary representations are needed to have the Fourier inversion formula. We define these objects as follows.

2.1.1. The Fourier analysis of a compact Lie group. The \( L^p \)-spaces \( L^p(G) = L^p(G, dx) \) will be associated with the Haar measure \( dx \). The Hilbert space \( L^2(G) \) will be endowed with the inner product \( (f, g) = \int_G f(x)g(x)dx \). We will see that the spectral decomposition of \( L^2(G) \) can be done in terms of the entries of unitary representations on a compact Lie group \( G \).

A continuous and unitary representation of \( G \) on \( \mathbb{C}^\ell \) is any continuous mapping \( \xi \in \text{Hom}(G, U(\ell)) \), where \( U(\ell) \) is the Lie group of unitary matrices of order \( \ell \times \ell \). The integer number \( \ell = \text{dim} \xi \) is called the dimension of the representation \( \xi \) since it is the dimension of the representation space \( \mathbb{C}^\ell \).

A subspace \( W \subset \mathbb{C}^{d\xi} \) is called \( \xi \)-invariant if for any \( x \in G \), \( \xi(x)(W) \subset W \), where \( \xi(x)(W) := \{ \xi(x)v : v \in W \} \). The representation \( \xi \) is irreducible if its only invariant subspaces are \( W = \emptyset \) and \( W = \mathbb{C}^{d\xi} \), the trivial ones. On the other hand, any unitary representation \( \xi \) is a direct sum of unitary irreducible representations. We denote it by \( \xi = \xi_1 \otimes \cdots \otimes \xi_j \), with \( \xi_i \) being irreducible representations on factors \( \mathbb{C}^{d\xi_i} \) that decompose the representation space \( \mathbb{C}^{d\xi} = \mathbb{C}^{d\xi_1} \otimes \cdots \otimes \mathbb{C}^{d\xi_j} \).

Two unitary representations \( \xi, \eta \in \text{Hom}(G, U(d\xi)) \) are equivalent if there exists a linear mapping \( F : \mathbb{C}^{d\xi} \to \mathbb{C}^{d\eta} \) such that for any \( x \in G \), \( F\xi(x) = \eta(x)F \). The mapping \( F \) is called an intertwining operator between \( \xi \) and \( \eta \). The set of all the intertwining operators between \( \xi \) and \( \eta \) is denoted by \( \text{Hom}(\xi, \eta) \). In view of
the 1905’s Schur lemma, if \( \xi \in \text{Hom}(G, U(d_\xi)) \) is irreducible, then \( \text{Hom}(\xi, \xi) = \mathbb{C} I_{d_\xi} \) is formed by scalar multiples of the identity matrix \( I_{d_\xi} \) of order \( d_\xi \).

The relation \( \sim \) on the set of unitary representations \( \text{Rep}(G) \) defined by: \( \xi \sim \eta \) if and only if \( \xi \) and \( \eta \) are equivalent representations, is an equivalence relation. The quotient

\[
\hat{G} := \text{Rep}(G)/\sim
\]

is called the unitary dual of \( G \). It encodes all the Fourier analysis on the group. Indeed, if \( \xi \in \text{Rep}(G) \), the Fourier transform \( \mathcal{F}_G \) associates to any \( f \in C^\infty(G) \) a matrix-valued function \( \mathcal{F}_G f \) defined on \( \text{Rep}(G) \) as follows

\[
(\mathcal{F}_G f)(\xi) \equiv \hat{f}(\xi) = \int_G f(x) \xi(x)^* dx, \ \xi \in \text{Rep}(G).
\]

The discrete Schwartz space \( \mathcal{S}(\hat{G}) := \mathcal{F}_G(C^\infty(G)) \) is the image of the Fourier transform on the class of smooth functions. This operator admits a unitary extension from \( L^2(G) \) into \( \ell^2(\hat{G}) \), with

\[
\ell^2(\hat{G}) = \{ \phi : \forall [\xi] \in \hat{G}, \phi(\xi) \in \mathbb{C}^{d_\xi \times d_\xi} \text{ and } \|\phi\|_{\ell^2(\hat{G})} := \left( \sum_{[\xi] \in \hat{G}} d_\xi \|\phi(\xi)\|_{\text{HS}}^2 \right)^{1/2} < \infty \}.
\]

The norm \( \|\phi(\xi)\|_{\text{HS}} \) is the standard Hilbert-Schmidt norm of matrices. The Fourier inversion formula takes the form

\[
f(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[\xi(x) \hat{f}(\xi)], \ f \in L^1(G), \ (2.1)
\]

where the summation is understood in the sense that from any equivalence class \( [\xi] \) we choose one (any) a unitary representation. The sum is independent of such choice.

### 2.1.2. The quantisation formula

Let \( A : C^\infty(G) \to C^\infty(G) \) be a continuous linear operator with respect to the standard Fréchet structure on \( C^\infty(G) \). The Schwartz kernel theorem associates to \( A \) a kernel \( K_A \in (C^\infty(G), \mathcal{D}'(G)) \) such that

\[
Af(x) = \int_G K_A(x, y)f(y)dy, \ f \in C^\infty(G).
\]

The distribution defined via \( R_A(x, xy^{-1}) := K_A(x, y) \) that provides the convolution identity

\[
Af(x) = \int_G R_A(x, xy^{-1})f(y)dy, \ f \in C^\infty(G),
\]

is called the right-convolution kernel of \( A \). By the Schwartz kernel theorem, one can associate a global symbol \( \sigma_A : G \times \hat{G} \to \bigcup_{\ell \in \mathbb{N}} \mathbb{C}^{\ell \times \ell} \) to \( A \). Indeed, in view of the identity \( Af(x) = (f \ast R_A(x, \cdot))(x) \), and after taking the Fourier transform with respect to \( x \in G \), we get

\[
\hat{A}f(\xi) = \hat{R}_A(x, \xi)\hat{f}(\xi).
\]
Then, the Fourier inversion formula gives the following representation of the operator $A$ in terms of the Fourier transform,

$$Af(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \Tr[\xi(x)\widehat{R}_A(x,\xi)f(\xi)], \ f \in C^\infty(G).$$

(2.2)

In view of the identity (2.2), from any equivalence class $[\xi] \in \widehat{G}$, we can choose one and only one irreducible unitary representation $\xi_0 \in [\xi]$, such that the matrix-valued function

$$\sigma_A(x,[\xi]) \equiv \sigma_A(x,\xi_0) := \widehat{R}_A(x,\xi_0), \ (x,[\xi]) \in G \times \widehat{G},$$

(2.3)

such that

$$Af(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \Tr[\xi_0(x)\sigma_A(x,[\xi])\widehat{f}(\xi_0)], \ f \in C^\infty(G).$$

(2.4)

The representation in (2.4) is independent of the choice of the representation $\xi_0$ from any equivalent class $[\xi] \in \widehat{G}$. This is a consequence of the Fourier inversion formula.

In the following quantisation theorem we observe that the distribution $\sigma_A$ in (2.4) is unique and can be written in terms of the operator $A$, see Theorems 10.4.4 and 10.4.6 of [44, Pages 552-553].

Theorem 2.1. Let $A : C^\infty(G) \to C^\infty(G)$ be a continuous linear operator. The following statements are equivalent.

- The distribution $\sigma_A(x,[\xi]) : G \times \widehat{G} \to \cup_{\ell \in \mathbb{N}} \mathbb{C}^{\ell \times \ell}$ satisfies the quantisation formula

$$\forall f \in C^\infty(G), \forall x \in G, \ Af(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \Tr[\xi(x)\sigma_A(x,[\xi])\widehat{f}(\xi)].$$

(2.5)

- $\forall (x,[\xi]) \in G \times \widehat{G}, \ \sigma_A(x,\xi) = \widehat{R}_A(x,\xi)$.

- $\forall (x,[\xi]), \sigma_A(x,\xi) = (\xi(x)^*A\xi(x), \ where \ A\xi(x) := (A\xi_{ij}(x))_{i,j=1}^d$.

Remark 2.2. In view of the quantisation formulae (2.4) and (2.5), a symbol $\sigma_A$ can be considered as a mapping defined on $G \times \widehat{G}$ or as a mapping defined on $G \times \text{Rep}(G)$ by identifying all the values $\sigma_A(x,\xi) = \sigma_A(x,\xi') = \sigma(x,[\xi])$ when $\xi', \xi \in [\xi]$.

Example 2.3 (The symbol of a Borel function of the Laplacian). Let $\mathbb{X} = \{X_1, \cdots, X_n\}$ be an orthonormal basis of the Lie algebra $\mathfrak{g}$. The positive Laplacian on $G$ is the second order differential operator

$$\mathcal{L}_G = -\sum_{j=1}^n X_j^2.$$  

(2.6)

The operator $\mathcal{L}_G$ is independent of the choice of the orthonormal basis $\mathbb{X}$ of $\mathfrak{g}$. The $L^2$-spectrum of $\mathcal{L}_G$ is a discrete set that can be enumerated in terms of the unitary dual $\widehat{G}$,

$$\text{Spect}(\mathcal{L}_G) = \{\lambda_{[\xi]} : [\xi] \in \widehat{G}\}.$$ 

(2.7)
For a Borel function \( f : \mathbb{R}_0^+ \to \mathbb{C} \), the right-convolution kernel \( R_{f(L_G)} \) of the operator \( f(L_G) \) (defined by the spectral calculus) is determined by the identity
\[
f(L_G)\phi(x) = \phi * R_{f(L_G)}(x), \quad x \in G.
\] (2.8)
This kernel satisfies the identity
\[
\hat{R}_{f(L_G)}([\xi]) = f(\lambda|\xi|)I_{d_k}.
\] (2.9)
Then the matrix-valued symbol of \( f(L_G) \) can be determined e.g. using Theorem 2.1 as follows
\[
\sigma_{f(L_G)}(x, \xi) = \hat{R}_{f(L_G)}([\xi]).
\] (2.10)
Since the operator \( f(L_G) \) is central the symbol \( \sigma_{f(L_G)}(\xi) = \sigma_{f(L_G)}(x, \xi) \) does not depend of the spatial variable \( x \in G \). Of particular interest for us will be the Japanese bracket function
\[
\langle t \rangle := (1 + |t|)^{\frac{1}{2}}, \quad t \in \mathbb{R}.
\] (2.11)
In particular the symbol of the operator \( \langle L_G \rangle \) is given by
\[
\sigma_{\langle L_G \rangle}([\xi]) := \langle \xi \rangle I_{d_k}, \quad \langle \xi \rangle := \langle \lambda_\xi \rangle.
\] (2.12)

2.1.3. Hörmander classes of pseudo-differential operators on compact Lie groups. In this section we denote for any linear mapping \( T \) on \( \mathbb{C}^f \) by \( \|T\|_{\text{op}} \) the standard operator norm
\[
\|T\|_{\text{op}} = \|T\|_{\text{End}(\mathbb{C}^f)} := \sup_{v \neq 0} \|Tv\|_{\mathbb{C}} / \|v\|_{\mathbb{C}},
\]
where \( \| \cdot \|_{\mathbb{C}} \) is the Euclidean norm.

For introducing the Hörmander classes on compact Lie groups we have to measure the growth of derivatives of symbols in the group variable, for this we use vector fields \( X \in T(G) \). To derive symbols with respect to the discrete variable \([\xi] \in \hat{G}\) we use difference operators. Before introducing the Hörmander classes on compact Lie groups we have to define these differential/difference operators.

So, if \( \{X_1, \ldots, X_j\} \) is an arbitrary family of left-invariant vector fields, we will denote by
\[
X_\alpha^\alpha := X_{1,x}^{\alpha_1} \cdots X_{n,x}^{\alpha_n}
\]
an arbitrary canonical differential operator of order \( m = |\alpha| \). Also, we have to take derivatives with respect to the “discrete” frequency variable \( \xi \in \text{Rep}(G) \). To do this, we will use the notion of difference operators. Indeed, the frequency variable in the symbol \( \sigma_A(x, [\xi]) \) of a continuous and linear operator \( A \) on \( C^\infty(G) \) is discrete. This is since \( \hat{G} \) is a discrete space.

If \( \xi_1, \xi_2, \ldots, \xi_k \), are fixed irreducible and unitary representation of \( G \), which do not necessarily belong to the same equivalence class, then each coefficient of the matrix
\[
\xi_\ell(g) - I_{d_\xi_\ell} = [\xi_\ell(g)_{ij} - \delta_{ij}]_{i,j=1}^{d_\xi_\ell}, \quad g \in G, \ 1 \leq \ell \leq k,
\] (2.13)
that is each function \( q_\ell^\delta_{ij}(g) := \xi_\ell(g)_{ij} - \delta_{ij}, \ g \in G \), defines a difference operator
\[
\Box_{\xi_\ell,i,j} := \mathcal{F}_G(\xi_\ell(g)_{ij} - \delta_{ij})\mathcal{F}_G^{-1}.
\] (2.14)
We can fix \( k \geq \dim(G) \) of these representations in such a way that the corresponding family of difference operators is admissible, that is,
\[
\text{rank}\{\nabla q_\ell^\delta_{ij}(e) : 1 \leq \ell \leq k\} = \dim(G).
\]
To define higher order difference operators of this kind, let us fix a unitary irreducible representation $\xi_\ell$. Since the representation is fixed we omit the index $\ell$ of the representations $\xi_\ell$ in the notation that will follow. Then, for any given multi-index $\alpha \in \mathbb{N}_0^{d_\ell}$, with $|\alpha| = \sum_{i,j=1}^{d_\ell} \alpha_{i,j}$, we write

$$D_\alpha := D_{1,1}^{\alpha_{1,1}} \cdots D_{d_\ell,d_\ell}^{\alpha_{d_\ell,d_\ell}}$$

for a difference operator of order $m = |\alpha|$. Now, we are ready for introducing the global Hörmander classes on compact Lie groups.

**Definition 2.4** (Global $(\rho, \delta)$-Hörmander classes in the whole range $0 \leq \delta, \rho \leq 1$). We say that $\sigma \in S^m_{\rho,\delta}(G \times \hat{G})$ if the following symbol inequalities

$$\|X_\rho D^\alpha \sigma(x,\xi)\|_{\text{op}} \leq C_{\alpha,\beta} (\xi)^{m-\rho|\gamma|+\delta|\beta|},$$

are satisfied for all $\beta$ and $\gamma$ multi-indices and for all $(x, [\xi]) \in G \times \hat{G}$, where $\langle \xi \rangle$ denotes the Japanese bracket function at $\lambda_\xi$ defined in (2.12).

The class $\Psi^m_{\rho,\delta}(G \times \hat{G}) \equiv \text{Op}(S^m_{\rho,\delta}(G \times \hat{G}))$ is defined by those continuous and linear operators on $C^\infty(G)$ such that $\sigma_A \in S^m_{\rho,\delta}(G \times \hat{G})$.

In the next theorem we describe some fundamental properties of the global Hörmander classes of pseudo-differential operators ([44]).

**Theorem 2.5.** Let $\rho, \delta \in [0, 1]$ be such that $0 \leq \delta \leq \rho \leq 1$, $\rho \neq 1$. Then $\Psi^\infty_{\rho,\delta}(G) := \bigcup_{m \in \mathbb{R}} \Psi^m_{\rho,\delta}(G)$ is an algebra of operators stable under compositions and adjoints, that is:

- the mapping $A \mapsto A^* : \Psi^m_{\rho,\delta}(G) \to \Psi^m_{\rho,\delta}(G)$ is a continuous linear mapping between Fréchet spaces.
- The mapping $(A_1, A_2) \mapsto A_1 \circ A_2 : \Psi^m_{\rho,\delta}(G) \times \Psi^m_{\rho,\delta}(G) \to \Psi^{m_1+m_2}(G)$ is a continuous bilinear mapping between Fréchet spaces.

Moreover, any operator in the class $\Psi^0_{\rho,\delta}(G)$ admits a bounded extension from $L^2(G)$ to $L^2(G)$.

**Remark 2.6.** The $L^2$-boundedness result in Theorem 2.5 is the global version of the Calderón-Vaillancourt theorem for compact Lie groups. Moreover, if $A \in \Psi^0_{\rho,\delta}(G)$ is such that $0 \leq \delta \leq \rho \leq 1$, $\rho \neq 1$, then

$$\|A\|_{\mathscr{B}(L^2)} \lesssim \sup \{C_{\alpha,\beta} : |\alpha| + |\beta| \leq \ell\},$$

where

$$C_{\alpha,\beta} := \sup_{(x,[\xi]) \in G \times \hat{G}} \langle \xi \rangle^{\rho|\alpha|-\delta|\beta|} \|\partial_x^\beta D^\alpha \sigma_A(x, \xi)\|_{\text{op}}$$

and $\ell \in \mathbb{N}_0$ is large enough.

2.1.4. **Hörmander classes of pseudo-differential operators on compact manifolds.** Next, we shall present the basics related to the classes of pseudo-differential operators on a compact manifold without boundary (closed manifold) by using charts, see Hörmander [27] and M. Taylor [49].
Definition 2.7 (Symbol classes on open sets). Let $U \subset \mathbb{R}^n$ be an open subset. The symbol $a \in C^\infty(U \times \mathbb{R}^n, \mathbb{C})$ belongs to the Hörmander class $S^m_{\rho,\delta}(U \times \mathbb{R}^n)$, with $0 \leq \rho, \delta \leq 1$, and $m \in \mathbb{R}$, if for every compact subset $K \subset U$ and for all $\alpha, \beta \in \mathbb{N}_0^n$, the inequalities
\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta,K}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|},
\] hold true uniformly in $(x, \xi) \in K \times \mathbb{R}^n$.

Remark 2.8 (Pseudo-differential operators on Euclidean open subsets). A continuous linear operator $A : C^\infty_0(U) \to C^\infty_0(U)$ (with respect to the standard Fréchet structure of $C^\infty_0(U)$ and of $C^\infty(U)$, respectively) is a pseudo-differential operator of order $m$, and of $(\rho, \delta)$-type, if there exists a symbol $a = a(x, \xi)$ in the class $S^m_{\rho,\delta}(U \times \mathbb{R}^n)$ such that
\[
\forall f \in C^\infty_0(U), \forall x \in \mathbb{R}^n, \quad Af(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi}a(x, \xi)(\mathcal{F}_{\mathbb{R}^n}f)(\xi)d\xi,
\]
where
\[
(\mathcal{F}_{\mathbb{R}^n}f)(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi}f(x)dx
\]
is the Euclidean Fourier transform of $f$ at $\xi \in \mathbb{R}^n$. We denote the family of pseudo-differential operators with symbols in the class $S^m_{\rho,\delta}(U \times \mathbb{R}^n)$ by $\Psi^m_{\rho,\delta}(U)$.

Now, let us extend the definition of the Hörmander classes on Euclidean topological subspaces to closed manifolds as follows.

Remark 2.9 (Pseudo-differential operators on compact manifolds). Given a closed manifold $M$, a continuous linear operator $A : C^\infty(M) \to C^\infty(M)$ is a pseudo-differential operator of order $m$, and of $(\rho, \delta)$-type, when
\[
\rho \geq 1 - \delta \quad \text{and} \quad 0 \leq \delta < \rho \leq 1,
\]
if for every coordinate patch $\omega : M_\omega \subset M \to U_\omega \subset \mathbb{R}^n$, and for every $\phi, \psi \in C^\infty_0(U_\omega)$, the operator
\[
T \psi := \psi(\omega^{-1})^*A\omega^*(\phi u), \quad u \in C^\infty_0(U_\omega),
\]
is a pseudo-differential operator with symbol $a_T \in S^m_{\rho,\delta}(U_\omega \times \mathbb{R}^n)$. Here, $\omega^*$ and $(\omega^{-1})^*$ are the pullbacks associated to the mappings $\omega$ and $\omega^{-1}$, respectively. All the operators $A$ with this property determines the family $A \in \Psi^m_{\rho,\delta}(M)$.

Remark 2.10 (The principal symbol of a pseudo-differential operator). The symbol defined by localisations of a pseudo-differential operator $A$ is unique as an element in the quotient $\Psi^m_{\rho,\delta}(G)/\Psi^m_{\rho,\delta}(G)$. We call to this class the principal symbol of $A$. We denote it by
\[
a_m(x, \xi), \quad (x, \xi) \in T^*M.
\] The symbol is a well-defined section of the co-tangent bundle $T^*M$ if and only if
\[
\rho \geq 1 - \delta \quad \text{and} \quad 0 \leq \delta < \rho \leq 1.
\]
Then, the main feature of the principal symbol of a pseudo-differential operator is that it remains invariant under changes of coordinates.

In the next result, we summarise some fundamental properties of the calculus of pseudo-differential operators as defined by Hörmander ([27]).
Theorem 2.11. Let \( 0 \leq \delta < \rho \leq 1 \), be such that \( \rho \geq 1 - \delta \). Then \( \Psi^\infty_{\rho,\delta}(M) := \bigcup_{m \in \mathbb{R}} \Psi^m_{\rho,\delta}(M) \) is an algebra of operators stable under compositions and adjoints, that is:

- the mapping \( A \mapsto A^* : \Psi^m_{\rho,\delta}(M) \to \Psi^m_{\rho,\delta}(M) \) is a continuous linear mapping between Fréchet spaces.

- The mapping \((A_1, A_2) \mapsto A_1 \circ A_2 : \Psi^m_{\rho,\delta}(M) \times \Psi^m_{\rho,\delta}(M) \to \Psi^{m_1+m_2}_{\rho,\delta}(M) \) is a continuous bilinear mapping between Fréchet spaces.

Moreover, any operator in the class \( \Psi^0_{\rho,\delta}(M) \) admits a bounded extension from \( L^2(M) \) to \( L^2(G) \).

Remark 2.12. The \( L^2 \)-continuity statement of Theorem 2.11 is the microlocalised version of the celebrated Calderón-Vaillancourt theorem, see the classical reference [1]. Also, if \( A \in \Psi^0_{\rho,\delta}(\mathbb{R}^n) \) is such that \( 0 \leq \delta \leq \rho \leq 1 \), \( \rho \neq 1 \), then one has the estimate from above for the \( L^2 \)-operator norm of \( A \)

\[
\|A\|_{\mathcal{B}(L^2)} \lesssim \sup_{\alpha, \beta} C_{\alpha, \beta},
\]

where

\[
C_{\alpha, \beta} := \sup_{(x, \xi) \in \mathbb{R}^{2n}} (1 + |\xi|)^{\rho|\alpha|-\delta|\beta|} |\partial_\xi^\beta \partial_x^\alpha a(x, \xi)|.
\]

Remark 2.13. If \( U \subset M \) is an open subset and the dimension of \( M \) is \( n \), for all \( f \in C^\infty_0(U) \), by microlocalising \( A \in \Psi^0_{\rho,\delta}(M) \) when \( 0 \leq \delta < \rho \leq 1 \), and \( \rho \geq 1 - \delta \), one has

\[
\|Af\|_{L^2(U)} \lesssim \sup_{\alpha, \beta} C_{\alpha, \beta, U} \|f\|_{L^2(G)},
\]

where

\[
C_{\alpha, \beta, U} := \sup_{(x, \xi) \in U \times \mathbb{R}^n} (1 + |\xi|)^{\rho|\alpha|-\delta|\beta|} |\partial_\xi^\beta \partial_x^\alpha a(x, \xi)|,
\]

by making the identification of \( U \) with an open subset of \( \mathbb{R}^n \).

Remark 2.14 (Elliptic pseudo-differential operators). A pseudo-differential operator \( A \in \Psi^m_{\rho,\delta}(M) \) is elliptic of order \( m \), if in any local coordinate system \( U \), there exists \( R = R_U > 0 \), such that the symbol \( a = a_U \) of \( A \) associated to \( U \) satisfies uniformly on any compact subset \( K \subset U \) the growth estimate

\[
C_1 (1 + |\xi|)^m \leq |a(x, \xi)| \leq C_2 (1 + |\xi|)^m, |\xi| \geq R,
\]

uniformly in \((x, \xi) \in K \times \mathbb{R}^n\). One of the main aspects of the spectral theory of elliptic pseudo-differential operators is that their spectra are purely discrete sets ([27]).

Remark 2.15. The global Hörmander classes on compact Lie groups can be used to describe the Hörmander classes defined by local coordinate systems. We present the corresponding statement as follows.

Theorem 2.16 (Equivalence of classes in the range \( 0 \leq \delta < \rho \leq 1 \), \( \rho \geq 1 - \delta \) [44, 45]). Let \( A : C^\infty(G) \to \mathcal{D}'(G) \) be a continuous linear operator and let \( 0 \leq \delta < \rho \leq 1 \), with \( \rho \geq 1 - \delta \). Then, \( A \in \Psi^m_{\rho,\delta}(G) \) if and only if \( A \in \Psi^m_{\rho,\delta}(G \times \hat{G}) \).
2.1.5. Complex powers of an elliptic pseudo-differential operator on a compact Lie group. By using the Dunford-Riesz functional calculus, to any sector $\Lambda \subset \mathbb{C}$, of the complex plane we will associate a class class of elliptic operators $\Psi^m_{\rho,\delta}(G \times \hat{G}; \Lambda)$ as developed by the third author and J. Wirth in [46]. There, one extended for any $0 \leq \delta < \rho \leq 1$ the global functional calculus on compact manifolds due to Shubin [47] under the restrictions $0 \leq \delta < \rho \leq 1$ and $0 \leq \delta < \rho \leq 1$.

In practice, $\Lambda$ will be any angle with the vertex at some complex number $z_0 \in \mathbb{C}$. Now we will introduce the definition of parameter-elliptic global symbols as in [46].

**Definition 2.17 (Parameter-ellipticity with respect to a sector $\Lambda$).** Let

$$a = a(x, [\theta]) : G \times \hat{G} \to \bigcup_{|\xi| \in \hat{G}} \mathbb{C}^{d_x \times d_\xi}$$

be a matrix-valued symbol. For $m \in \mathbb{R}^+$, we say that $a$ is parameter-elliptic with respect to $\Lambda$, if the following conditions are satisfied:

- $\forall \lambda \in \Lambda$, $a(x, [\theta]) - \lambda I_{d_\theta} \in \text{GL}(d_\theta, \mathbb{C})$ is an invertible matrix.
- The symbol inequality

$$\| (a(x, [\theta]) - \lambda I_{d_\theta})^{-1} \|_{\text{op}} \leq C (1 + \langle \theta \rangle + |\lambda|^\frac{1}{m})^{-m},$$

holds uniformly in $x \in G$, for all $[\theta] \in \hat{G}$ and all $\lambda \in \Lambda$. In the case where $\Lambda = \{0\}$ is the trivial singleton, we just will say that the symbol $a(x, [\theta])$ is elliptic.

The following shows that we can use the notion of parameter-ellipticity in the construction of parametrices for the resolvent of an operator. The theorems below were proved in [46] and their corollaries are their immediate consequences. The proof of Lemma 2.21 below can be found in [8, Section 7.2].

**Theorem 2.18.** Let $m > 0$, and let $0 \leq \delta < \rho \leq 1$. Let $a(x, [\theta])$ be a parameter elliptic symbol with respect to a sector $\Lambda$. Then there exists a parameter-dependent parametrix of the resolvent $A - \lambda I$, with matrix-valued symbol $a^{-\#}(x, \theta, \lambda)$ satisfying the estimates

$$\sup_{\lambda \in \Lambda} \sup_{x,(\theta,\lambda) \in G \times \hat{G}} \| (|\lambda|^\frac{1}{m} + \langle \theta \rangle)^m (\langle \theta \rangle)^{\rho(|\alpha|-\delta|\beta|) + \lambda} \partial_x^k \partial_\theta^\alpha a^{-\#}(x, \theta, \lambda) \|_{\text{op}} < \infty,$$

for all $\alpha, \beta \in \mathbb{N}_0^n$ and $k \in \mathbb{N}_0$.

As a consequence of the previous theorem, we have an efficient classification of the symbol of the resolvent of an operator in the global Hörmander classes, see [46].

**Corollary 2.19.** Let $m > 0$, and let $a \in S^m_{\rho,\delta}(G \times \hat{G})$ where $0 \leq \delta < \rho \leq 1$. Let us assume that $\Lambda$ is a subset of the $L^2$-resolvent set of $A$, $\text{Resolv}(A) := \mathbb{C} \setminus \text{Spec}(A)$. Then $A - \lambda I$ is invertible on $\mathcal{D}'(G)$ and the symbol of the resolvent operator $\mathcal{R}_\Lambda := (A - \lambda I)^{-1}$, $\mathcal{R}_\Lambda(x, \xi)$ belongs to $S^{-m}_{\rho,\delta}(G \times \hat{G})$.

**Remark 2.20 (Functional calculus of global pseudo-differential operators).** Let $a \in S^m_{\rho,\delta}(G \times \hat{G})$ be a parameter elliptic symbol of order $m > 0$ with respect to the sector...
$\Lambda \subset \mathbb{C}$. For $A = \text{Op}(a)$, we will define the operator $F(A)$ by the (Dunford-Riesz) complex functional calculus

$$F(A) = -\frac{1}{2\pi i} \oint_{\partial \Lambda_\varepsilon} F(z)(A - zI)^{-1}dz,$$  \hspace{1cm} (2.22)

where

(A1): $\Lambda_\varepsilon := \Lambda \cup \{z : |z| \leq \varepsilon\}$, $\varepsilon > 0$, and $\Gamma = \partial \Lambda_\varepsilon \subset \text{Resolv}(A)$ is a positively oriented path in the complex plane $\mathbb{C}$.

(A2): $F$ is an holomorphic function in $\mathbb{C} \setminus \Lambda_\varepsilon$, and continuous on its closure.

(A3): We will assume decay of $F$ along $\partial \Lambda_\varepsilon$ in order that the operator (2.22) will be densely defined on $C^\infty(G)$ in the strong sense of the topology on $L^2(G)$.

Now, we will compute the matrix-valued symbols for operators defined by this complex functional calculus.

**Lemma 2.21.** Let $a \in S^m_{\rho,\delta}(G \times \hat{G})$ be a parameter elliptic symbol of order $m > 0$ with respect to the sector $\Lambda \subset \mathbb{C}$. Let $F(A) : C^\infty(G) \rightarrow \mathcal{D}'(G)$ be the operator defined by the analytical functional calculus as in (2.22). Under the assumptions (A1), (A2), and (A3) of Remark 2.20, the matrix-valued symbol of $F(A)$, $\sigma_{F(A)}(x, \xi)$ is given by,

$$\sigma_{F(A)}(x, \xi) = -\frac{1}{2\pi i} \oint_{\partial \Lambda_\varepsilon} F(z)\hat{R}_z(x, \xi)dz,$$

where $\hat{R}_z = (A - zI)^{-1}$ denotes the resolvent of $A$, and $\hat{R}_z \in S^{-m}_{\rho,\delta}(G \times \hat{G})$ is its symbol.

The decay assumption on $F$ will be clarified in the following theorem saying that the global calculus of pseudo-differential operators is stable under the action of the global complex functional calculus.

**Theorem 2.22.** Let $m > 0$, and let $0 \leq \delta < \rho \leq 1$. Let $a \in S^m_{\rho,\delta}(G \times \hat{G})$ be a parameter elliptic symbol with respect to $\Lambda$. Let us assume that $F$ satisfies the estimate $|F(\lambda)| \leq C|\lambda|^s$ uniformly in $\lambda$, for some $s < 0$. Then the symbol of $F(A)$, $\sigma_{F(A)} \in S^m_{\rho,\delta}(G \times \hat{G})$ admits an asymptotic expansion of the form

$$\sigma_{F(A)}(x, \xi) \sim \sum_{N=0}^{\infty} \sigma_{B_N}(x, \xi), \ (x, [\xi]) \in G \times \hat{G},$$  \hspace{1cm} (2.23)

where $\sigma_{B_N} \in S^{m_s-(\rho-\delta)N}_{\rho,\delta}(G \times \hat{G})$ and

$$\sigma_{B_0}(x, \xi) = -\frac{1}{2\pi i} \oint_{\partial \Lambda_\varepsilon} F(z)(a(x, \xi) - z)^{-1}dz \in S^{m_0}_{\rho,\delta}(G \times \hat{G}).$$

Moreover,

$$\sigma_{F(A)}(x, \xi) \equiv -\frac{1}{2\pi i} \oint_{\partial \Lambda_\varepsilon} F(z)a^{-\#}(x, \xi, \lambda)dz \mod S^{-\infty}(G \times \hat{G}),$$

where $a^{-\#}(x, \xi, \lambda)$ is the symbol of the parametrix to $A - \lambda I$, in Corollary 2.18.
Now, we present the construction of the complex powers \( A^z, z \in \mathbb{C} \).

**Corollary 2.23** (Complex powers of elliptic operators on compact Lie groups). Let \( \varepsilon > 0 \). Let \( A \in \Psi^m_{\rho,\delta}(G \times \hat{G}) \) be a parameter-elliptic pseudo-differential operator of order \( m > 0 \) with respect to a sector \( \Lambda \) of the complex plane \( \mathbb{C} \). Let \( \Lambda_\varepsilon := \Lambda \cup \{ z : |z| \leq \varepsilon \} \), \( \varepsilon > 0 \), and assume that \( \Gamma = \partial \Lambda_\varepsilon \subset \text{Resolv}(A) \) is a positively oriented curve in the complex plane \( \mathbb{C} \). Then, the mapping

\[
\varepsilon \in \mathbb{C} \mapsto A^\varepsilon := -\frac{1}{2\pi i} \int_{\partial \Lambda_\varepsilon} \lambda^\varepsilon (A - \lambda I)^{-1} d\lambda \in \Psi^{\text{Re}(\varepsilon)m}_{\rho,\delta}(G).
\tag{2.24}
\]

An immediate consequence of Theorem 2.23 is the construction of inverses for positive pseudo-differential operators. We record it in the following way.

**Corollary 2.24** (Inverse of positive pseudo-differential operators). Let \( A \in \Psi^m_{\rho,\delta}(G \times \hat{G}) \) be a positive elliptic pseudo-differential operator of order \( m > 0 \). Define \( A^z \) via the contour integral (2.24) where \( \Lambda \) is an acute angle centred at the origin 0, with its interior containing the interval \((-\infty, 0]\). Let \( E_0 := \text{Ker}(A) \) and let \( E_0' \) be its orthogonal complement in \( L^2(G) \). Then,

- \( A^z(E_0) = \{ 0 \} \), and \( A^z(E_0') \subset E_0' \).
- For any \( z, w \in \mathbb{C} \), \( A^{z+w} = A^z A^w \).
- If \( P_0 : L^2(G) \to E_0 \) is the orthogonal projection on \( E_0 \), then \( A^0 = I - P_0 \), and \( A^{-1} \) is the inverse of the operator \( A \) restricted to \( E_0' \).

### 2.2. Global and local classes of pseudo-differential operators on the torus.

Let us consider the torus \( \mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n, \mathbb{T} \cong \mathbb{S}^1 \). Different from the case of an arbitrary compact Lie group, here the local and the global Hörmander classes agree for all \( (\rho,\delta) \in [0,1]^2 \) such that \( 0 \leq \delta \leq 1, 0 < \rho \leq 1 \). We will present the required preliminaries in order to give the statement of this equivalence.

We will use the standard notation for this family of periodic pseudo-differential operators taken from [44].

**Definition 2.25** (Discrete Schwartz space). The Schwartz space \( S(\mathbb{Z}^n) \) on the lattice \( \mathbb{Z}^n \) is defined by the discrete functions \( \phi : \mathbb{Z}^n \to \mathbb{C} \) verifying the inequality

\[
\forall M \in \mathbb{R}, \exists C_M > 0, |\phi(\xi)| \leq C_M (1 + |\xi|)^M.
\tag{2.25}
\]

**Definition 2.26** (The Fourier transform on \( \mathbb{T}^n \)). The toroidal Fourier transform is defined for any test function \( f \in C^\infty(\mathbb{T}^n) \) by

\[
\hat{f}(\xi) := \int_{\mathbb{T}^n} e^{-i2\pi \langle x, \xi \rangle} f(x) dx, \ \xi \in \mathbb{Z}^n.
\]

Here, \( dx \) stands for the normalised Haar measure on the torus.

**Remark 2.27** (The Fourier inversion formula). The Fourier inversion formula is given by the representation of any function \( f \in L^1(\mathbb{T}^n) \) in its Fourier series

\[
f(x) = \sum_{\xi \in \mathbb{Z}^n} e^{i2\pi \langle x, \xi \rangle} \hat{f}(\xi), \ x \in \mathbb{T}^n.
\]
Definition 2.28 (Hörmander classes on the torus). The toroidal Hörmander class $S_{\rho,\delta}^{m}(\mathbb{T}^{n} \times \mathbb{R}^{n})$, $0 \leq \rho, \delta \leq 1$, are defined by those functions $a = a(x, \xi)$ which are smooth in $(x, \xi) \in \mathbb{T}^{n} \times \mathbb{R}^{n}$ and which satisfy the inequalities

$$|\partial_{\xi}^{\alpha} a(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}. \quad (2.26)$$

Remark 2.29. Note that symbols in $S_{\rho,\delta}^{m}(\mathbb{T}^{n} \times \mathbb{R}^{n})$ are symbols in $S_{\rho,\delta}^{m}(\mathbb{R}^{n} \times \mathbb{R}^{n})$ (see [44]) of order $m$ which are 1-periodic in $x$. Then, if $a \in S_{\rho,\delta}^{m}(\mathbb{T}^{n} \times \mathbb{R}^{n})$, the corresponding pseudo-differential operator is defined by the quantisation formula

$$a(X, D_x)f(x) = \int_{\mathbb{T}^{n}} \int_{\mathbb{R}^{n}} e^{i2\pi(x-y,\xi)} a(x, \xi) f(y) d\xi dy. \quad (2.27)$$

Definition 2.30 (Hörmander classes on the torus II). The class $S_{\rho,\delta}^{m}(\mathbb{T}^{n} \times \mathbb{Z}^{n})$, $0 \leq \rho, \delta \leq 1$, consists of those functions $a(x, \xi)$ which are smooth in $x \in \mathbb{T}^{n}$, for all $\xi \in \mathbb{Z}^{n}$ and which satisfy the symbol inequalities

$$\forall \alpha, \beta \in \mathbb{N}^{n}, \exists C_{\alpha,\beta} > 0, \ |\Delta_{\xi}^{\alpha} a(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}. \quad (2.28)$$

The operator $\Delta$ is the standard difference operator defined in $\mathbb{Z}^{n}$, [44]. In this case for any $\alpha \in \mathbb{N}_{0}$, $\Delta^{\alpha} = D^{\alpha}$. The toroidal operator with symbol $a$ is defined as

$$a(x, D) f(x) = \sum_{\xi \in \mathbb{Z}^{n}} e^{i2\pi(x,\xi)} a(x, \xi) \hat{f}(\xi), \ f \in C^\infty(\mathbb{T}^{n}). \quad (2.29)$$

Remark 2.31. We denote the corresponding toroidal class of operators associated with toroidal symbols in $S_{\rho,\delta}^{m}(\mathbb{T}^{n} \times \mathbb{Z}^{n})$ (resp. $S_{\rho,\delta}^{m}(\mathbb{T}^{n} \times \mathbb{R}^{n})$) by $\Psi_{\rho,\delta}^{m}(\mathbb{T}^{n} \times \mathbb{Z}^{n})$, (resp. $\Psi_{\rho,\delta}^{m}(\mathbb{T}^{n} \times \mathbb{R}^{n})$).

There exists a process allowing the interpolation of the second argument of the symbols defined on $\mathbb{T}^{n} \times \mathbb{Z}^{n}$ in a smooth way to get a smooth symbol defined on $\mathbb{T}^{n} \times \mathbb{R}^{n}$. It leads to the following toroidal equivalence-of-classes-theorem.

Theorem 2.32. Let $(\rho, \delta) \in [0, 1]^{2}$ be such that $0 \leq \delta \leq 1$, $0 < \rho \leq 1$. Then the symbol $a \in S_{\rho,\delta}^{m}(\mathbb{T}^{n} \times \mathbb{Z}^{n})$ if and only if there exists an Euclidean symbol $a' \in S_{\rho,\delta}^{m}(\mathbb{T}^{n} \times \mathbb{R}^{n})$ such that $a = a'|_{\mathbb{T}^{n} \times \mathbb{Z}^{n}}$. Moreover, we have

$$\Psi_{\rho,\delta}^{m}(\mathbb{T}^{n} \times \mathbb{Z}^{n}) = \Psi_{\rho,\delta}^{m}(\mathbb{T}^{n} \times \mathbb{R}^{n}). \quad (2.30)$$

Moreover, any $A \in \Psi^{0}_{\rho,\delta}(\mathbb{T}^{n} \times \mathbb{Z}^{n})$ is bounded on $L^{2}(\mathbb{T}^{n})$, and

$$\|A\|_{B(L^{2})} \lesssim \sup_{|\alpha|+|\beta| \leq [n/2]+1} \sup_{(x,\xi)} |\Delta_{\xi}^{\alpha} a(x, \xi)|. \quad (2.31)$$

Proof. The proof of (2.30) can be found in [44]. The proof of the $L^{2}$-estimate in (2.31) can be found in [12]. \qed

2.3. Null-controllability of diffusion problems on Hilbert spaces. This section is dedicated to presenting the functional analysis related to the control theory of fractional problems for self-adjoint linear operators on Hilbert spaces, we will follow Miller [40]. We use the following notation/fact:

- the norm of a Hilbert space $H$ will be denoted by $\| \cdot \|$ without using subscript.

In general $H_{1}, H_{2},$ etc. denote Hilbert spaces. In what follows, any Hilbert space will be identified with its topological dual in the canonical way.
Remark 2.33 (Observation operator and Control operator). Let $H$ be a separable Hilbert space and let $A: \operatorname{Dom}(A) \subset H \to H$ be a positive self-adjoint operator with dense domain $\operatorname{Dom}(A) \subset H$. Consider $H_1$ the Hilbert space obtained by choosing on the domain $\operatorname{Dom}(A)$ the graph norm. We extend $\{e^{-tA} : t > 0\}$ to a semigroup on the dual space $H_1^*$. Let $S$ be an observation operator from $H$ to a Hilbert space of inputs $U$, and let us consider the control operator $B = S^* \in \mathcal{B}(U, H_1^*)$.

Assume the following properties on $S$ and $B$.

**Assumption 2.34.** Let $S \in \mathcal{B}(H, U)$ be an observation operator and let us consider its adjoint (the control operator) $B = S^* \in \mathcal{B}(U, H_1^*)$. Assume that, for some $T > 0$ (and hence, for any $T > 0$), the following estimates hold.

- There exists $K_T > 0$, such that
  \[
  \forall v_0 \in \operatorname{Dom}(A), \quad \int_0^T \|S e^{-tA} v_0\|^2 \leq K_T \|v_0\|^2. \tag{2.32}
  \]

- We have
  \[
  \forall u \in L^2_{\text{loc}}(\mathbb{R}, U), \quad \int_0^T \|e^{-tA} Bu(t)\|^2 dt \leq K_T \int_0^T \|u(t)\|^2 dt. \tag{2.33}
  \]

The assumptions (2.32) and (2.33) identify the necessary hypotheses for the existence and uniqueness of the solution of the model

\[
\phi_t + A\phi = Bu, \phi(0) = \phi_0 \in H, u \in L^2_{\text{loc}}(\mathbb{R}, U). \tag{2.34}
\]

We summarise this in the following result.

**Proposition 2.35.** Under the hypothesis (2.32) and (2.33), for any input $u \in L^2_{\text{loc}}(\mathbb{R}, U)$, there exists a unique solution $u \in C(\mathbb{R}^+_0, U)$ to (2.34) such that

\[
\phi(t) = e^{-tA} \phi_0 + \int_0^T e^{(s-t)A} Bu(s) ds. \tag{2.35}
\]

We precise the notion of null-controllability in the following definition.

**Definition 2.36.** The model (2.34) is null-controllable in time $T > 0$, if for any initial state $\phi_0 \in H$, there exists an input function $u \in L^2_{\text{loc}}(\mathbb{R}^+_0, H)$ such that its solution (2.35) satisfies $\phi(T) = 0$.

**Remark 2.37.** Let us consider the adjoint model to (2.34) without the source term, that is

\[
v_t + Av = 0. \tag{2.36}
\]

Since $B = S^*$ the null-controllability of (2.34) is equivalent to the following observability inequality: there exists $C_T > 0$, such that

\[
\forall v_0 \in H, \quad \|e^{-T A} v_0\| \leq C_T \|S e^{-t A} v_0\|_{L^2((0, T), U)}. \tag{2.37}
\]
The smallest constant $C_T > 0$ is called the cost of controllability in time $T > 0$. Note that by the duality argument, the cost of controllability in time $T > 0$, is the smallest constant $C_T > 0$ satisfying that
\[ \forall \phi_0 \in H, \exists u \text{ in Definition (2.36) such that } \| u \|_{L^2((0,T),U)} \leq C_T \| \phi_0 \|. \tag{2.38} \]

**Remark 2.38 (Spectral inequalities and null-controllability).** Theorem 2.39 says that a spectral inequality for the power $A^\gamma$ which is defined by the functional calculus of the operator $A$, is a sufficient condition for the null-controllability of the model (2.34). Next, we give the precise statement. Here, $E^T_\lambda : = E^T(0,\lambda) : H \to H$ denotes an arbitrary projection of the spectral measure $\{ E^T_\lambda \}_{\lambda > 0}$ associated to a positive and densely defined linear operator $T : H \to H$.

**Theorem 2.39 (Miller [40], 2006).** Assume that for some $\gamma \in (0,1)$, the fractional operator $A^\gamma$ satisfies the spectral inequality
\[ \forall \lambda > 0, \forall u \in E^\gamma(A^\gamma(H), \exists d_1, d_2 > 0, \| v \| \leq d_1 e^{d_2 \lambda} \| Sv \|. \] \tag{2.39} \]
Then, the problem (2.34) is null-controllable in time $T > 0$. Moreover, the controllability cost $C_T$ over short times $T$, satisfies the inequality
\[ \forall \beta > \frac{\gamma}{1 - \gamma}, \exists C_1, C_2, \forall T \in (0,1), C_T \leq C_1 e^{C_2 T^{-\beta}}. \tag{2.40} \]

3. **Donnelly-Fefferman inequalities on compact Lie groups**

In this section we prove our main Theorem 1.1. We employ the following notation.
- We denote by $\omega \neq \emptyset$ an open non-empty subset in $G$, and a generic compact subset in $M$ will be denoted by $\tilde{K}$.
- For any $T > 0$, let us consider the space-time manifold $G_T : = G \times [0,T]$, and the Sobolev space $H^s(G_T)$ of order $s \in \mathbb{N}$, is defined by the norm
\[ \| f \|^2_{H^s(G_T)} = \sum_{0 \leq j \leq s} \int_0^T \int_G \left[ |\partial_t^j f(x,t)|^2 + |(1 + \mathcal{L}_G)^{\frac{j}{2}} f(x,t)|^2 \right]dxdt < \infty, \tag{3.1} \]
where $\mathcal{L}_G$ is the positive Laplace-Beltrami operator on $G$.

**Remark 3.1 (A topological construction).** For our further analysis we will make a topological construction. We do it by the following steps.
- Step 1. Let us consider the cylinder $G_T = G \times [0,T]$. See Figure 1.
- Step 2. We fix a parameter $\varepsilon > 0$ and we extend the cylinder $G_T$ in the time-variable $t$ until obtaining a new cylinder $G_{T,\varepsilon} = G \times [-T-\varepsilon, T+\varepsilon]$, and we identify its lateral boundaries $G \times \{ -T-\varepsilon \} \sim G \times \{ T+\varepsilon \}$. See Figure 2.
- Step 3. After the identification $G \times \{ -T-\varepsilon \} \sim G \times \{ T+\varepsilon \}$ the lateral boundaries of the manifold $G_{T,\varepsilon} = G \times [-T-\varepsilon, T+\varepsilon]$ can be glued until obtaining the Lie group $G \times \mathbb{T}(T,\varepsilon)$ where $\mathbb{T}(T,\varepsilon)$ is the flat torus
\[ \mathbb{T}(T,\varepsilon) : = \mathbb{R}/2(T+\varepsilon) \mathbb{Z} \cong \{ -(T+\varepsilon), T+\varepsilon \}. \]
The manifold $G \times \mathbb{T}(T,\varepsilon)$ can be seen as a “torus” where any transversal section is a copy of $G$. See Figure 3.
Remark 3.2 (The operator $-\partial_t^2 + A^2_{\frac{m}{2}}$ on $G \times \mathbb{T}(T, \varepsilon)$). Let $A \in \Psi^m_{\rho,\delta}(G \times \hat{G})$ be a positive elliptic matrix-valued pseudo-differential operator of order $m > 0$. The operator $-\partial_t^2$ became, up to a constant, the (positive) Laplace-Beltrami type operator on $\mathbb{T}(T, \varepsilon)$.

Note that $-\partial_t^2 \in \Psi^2_{0,0}(\mathbb{T}(T, \varepsilon))$ is a positive and elliptic differential operator of second order on $\mathbb{T}(T, \varepsilon)$. Indeed, let us consider the orthogonal basis of $L^2(\mathbb{T}(T, \varepsilon))$ formed by the exponential functions $t \in \mathbb{T}(T, \varepsilon) \mapsto \tilde{e}_k^\varepsilon(t) = \exp \left( \frac{2\pi i k t}{2(T+\varepsilon)} \right)$, $k \in \mathbb{Z}$, \hspace{1cm}(3.2)

and let us consider the $L^2$-normalised system of $2(T+\varepsilon)$-periodic eigenfunctions $e_k^\varepsilon := \tilde{e}_k^\varepsilon / \sqrt{2(T+\varepsilon)}$, of the Laplacian $-\partial_t^2$. The global symbol of $-\partial_t^2$ is given by

$$\sigma_{-\partial_t^2}(k) = \frac{4\pi^2 k^2}{4(T+\varepsilon)^2} = \left( \frac{\pi k}{T+\varepsilon} \right)^2, \quad k \in \mathbb{Z}.$$ \hspace{1cm}(3.3)

The ellipticity of $-\partial_t^2$ follows from the following inequality

$$\exists C_1, C_2 > 0, \forall k \in \mathbb{Z}, \quad C_1 |k|^2 \leq |\sigma_{-\partial_t^2}(k)| \leq C_2 |k|^2.$$ \hspace{1cm}(3.3)

Note that the constants $C_1$ and $C_2$ are independent of $\varepsilon > 0$ if $\varepsilon \in (0, 1)$. Note that $A(x, t, D, \partial_t) = -\partial_t^2 + A^2_{\frac{m}{2}} \in \Psi^2_{0,0}(G \times \mathbb{T}(T, \varepsilon))$ is also a positive and elliptic pseudo-differential operator on the Lie group $G \times \mathbb{T}(T, \varepsilon)$. Note that $G_T$ can be viewed as an open sub-manifold of the Lie group $G \times \mathbb{T}(T, \varepsilon)$.

Figure 1. Step 1: To consider the space-time manifold $G_T = G \times [0, T]$.

Our analysis starts with the spectral inequality in Proposition 3.3 below that corresponds to the spectral inequality (1.9) in Theorem 1.1 in the case where $A$ satisfies the lower bound $A \geq cI$, for some $c > 0$. Then, the proof of (1.9) in Theorem 1.1 will be deduced from this particular situation.
Figure 2. We extend the cylinder $G_T$ in the time-variable $t$ until obtaining a new cylinder $G_{T,\varepsilon} = G \times [-T - \varepsilon, T + \varepsilon]$, and we identify its lateral boundaries $G \times \{-T - \varepsilon\} \sim G \times \{T + \varepsilon\}$.

Figure 3. We have constructed the Lie group $G \times \mathbb{T}(T, \varepsilon)$ where $\mathbb{T}(T, \varepsilon)$ is the flat torus $\mathbb{T}(T, \varepsilon) := \mathbb{R}/2(T + \varepsilon)\mathbb{Z} \cong -(T + \varepsilon), (T + \varepsilon]$. This closed manifold is a compact Lie group.

Proposition 3.3. Let $A \in \Psi^m_{\rho,0}(G \times \hat{G})$ be a positive elliptic pseudo-differential operator of order $m > 0$. Assume that $\sigma_A(x, \xi) \geq 0$ for all $(x, [\xi]) \in G \times \hat{G}$. Moreover, assume that for some $c > 0$, $A \geq cI$ in $L^2(G)$, and that $\sigma_A(x, [\xi]) \geq cI_{d_\xi}$ on every representation space.

Then, for any non-empty open subset $\omega \subset M$, any $a_j \in \mathbb{R}$, and all $\lambda > 0$, the following spectral inequality holds

$$\left( \sum_{\lambda_j \leq \lambda} a_j^2 \right)^{\frac{1}{2}} \leq C_1 e^{C_2 \lambda} \left\| \sum_{\lambda_j \leq \lambda} a_j e_j(x) \right\|_{L^2(\omega)}, \quad (3.4)$$

where $C_1 > 0$ and $C_2 > 0$ may depend on $\omega$ but not on $a_j$, $\lambda > 0$ or on the eigenfunctions $e_j$.

We postpone the proof of Proposition 3.3 for a moment. Indeed, for our further analysis we require the following interpolation inequality. It is formulated in the case of the compact Lie groups but it is still valid in the case of a compact Riemannian manifold $(M, g)$, see e.g. [35].

Lemma 3.4. Let us consider the operator $L(x, t, D, \partial_t) = -\partial^2_t + \mathcal{L}_G \in \Psi^2_{1,0}(G_T)$ with $G_T = G \times (0, T)$. Let $\omega$ be a non-empty open subset in $M.$
Then, for any $T > 0$ and all $\alpha \in (0, T/2)$, there exists $\delta \in (0, 1)$ such that
\[
\| \phi \|_{H^1(G \times (a, T - a))} \leq C \| \phi \|_{H^1(G_T)}^\delta \left( \| L(x, t, D, \partial_t) \phi \|_{L^2(G \times (0, T))} + \| \partial_t \phi \|_{L^2(\omega)} \right)^{1-\delta},
\]
for all $\phi \in H^2(G \times (0, T))$ such that $\phi = 0$ in $G \times \{0\}$.

We refer the reader to Rousseau and Lebeau [42] for extensions of this result even, for second-order elliptic operators with Lipchitz coefficients. In the following section we describe the approach that we introduce for the proof of Proposition 3.3.

### 3.1. Sketch of the proof of Proposition 3.3.

As it was proved by Jerison and Lebeau [28], the interpolation inequality (3.5) can be used to prove the inequality (3.4) for the Laplacian $\Delta_G$ (or even for the Laplacian $-\Delta_g$ on an arbitrary compact Riemannian manifold $(M, g)$). From the point of view of the pseudo-differential calculus the operators
\[
L(x, t, D, \partial_t) = -\partial_t^2 + \mathcal{L}_G \quad \text{and} \quad \mathcal{A}(x, t, D, \partial_t) = -\partial_t^2 + A^\frac{2}{m}
\]
are similar. They are elliptic pseudo-differential operators of order 2. So, in order to prove (3.4), we will construct a suitable function $\phi \in H^2$ in terms of the eigenfunctions $e_j$, were the index $j$ is such that $\lambda_j \leq \lambda$, and from the inequality in (3.5), we will deduce an inequality of the form
\[
\| \phi \|_{H^1(G \times (a, T - a))} \leq C \| \phi \|_{H^1(G_T)}^\delta \left( \| \mathcal{A}(x, t, D, \partial_t) \phi \|_{L^2(G \times (0, T))} + \| \partial_t \phi \|_{L^2(\omega)} \right)^{1-\delta},
\]
and by following the strategy by Jerison and Lebeau in [28] from the inequality in (3.6) we will deduce the inequality (3.4).

Now, observe that the interpolations inequalities (3.5) and (3.6) are essentially similar. Indeed, they differ just by the $L^2$-norms
\[
\| L(x, t, D, \partial_t) \phi \|_{L^2(G \times (0, T))} \quad \text{and} \quad \| \mathcal{A}(x, t, D, \partial_t) \phi \|_{L^2(G \times (0, T))}.
\]
Informally, if we were able to compute the inverse of the operator $\mathcal{A}(x, t, D, \partial_t)$, then observe that the composition
\[
L(x, t, D, \partial_t) \mathcal{A}(x, t, D, \partial_t)^{-1}
\]
would be a pseudo-differential operator of order zero. Then, an application of a suitable “Calderon-Vaillancourt” theorem would provide the $L^2$-boundedness of the operator $L(x, t, D, \partial_t) \mathcal{A}(x, t, D, \partial_t)^{-1}$. All this informal argument is constructed under the pseudo-differential philosophy, indeed, the $L^2$-boundedness of such an operator would give the following estimate
\[
\| L(x, t, D, \partial_t) \phi \|_{L^2} = \| L(x, t, D, \partial_t) \mathcal{A}(x, t, D, \partial_t)^{-1} \mathcal{A}(x, t, D, \partial_t) \phi \|_{L^2} 
\lessapprox \| \mathcal{A}(x, t, D, \partial_t) \phi \|_{L^2}.
\]
Then, if all this works, from (3.5) we could obtain (3.6). So, the main difficulty in computing the inverse of the operator $\mathcal{A}(x, t, D, \partial_t) = -\partial_t^2 + A^\frac{2}{m}$ is that it is acting on distributions defined in the cylinder $G_T = G \times [0, T]$. This is a compact manifold whose lateral boundaries are $G \times \{0\}$ and $G \times \{T\}$, and the global pseudo-differential calculus in [44] does not allow the construction of parametrics and inverses on this kind of manifold. However, inspired a little bit by the topological constructions by Donaldson in [19], we will fix a parameter $\varepsilon > 0$ and we will embed the cylinder
\[
G_T = G \times [0, T]
\]
into the manifold

$$G \times \mathbb{T}(T, \varepsilon)$$

that was constructed in Remark 3.1, (see Figures 1, 2 and 3). Our strategy will be to construct the inverse/parametrix of the operator $A(x, t, D, \partial_t) = -\partial_t^2 + A^\frac{z}{\varepsilon}$ on the compact Lie group $G \times \mathbb{T}(T, \varepsilon)$, and then by using the calculus in [44] and the parameter-ellipticity notion developed in [46] we will prove an inequality of the form (3.6). The dependence of the parameter $\varepsilon$ will be eliminated forcing it to go to zero and showing that the auxiliary inequality that we get is stable under this limit procedure. Moreover, the inequality that we obtain will be given for the manifold $G \times [-T, T]$ but a symmetry property in the auxiliary function $\phi$ (that we construct later) will give us the required interpolation inequality on the interval $[0, T]$. To do this, we force the function $\phi = \phi(x, t)$ to be odd with respect to the time variable, that is $\phi(x, t) = -\phi(x, -t)$. Summarising, the proof of Proposition 3.3 will be constructed by following three steps:

- Step 1: To compute the inverse of the operator $A(x, t, D, \partial_t)$. We do this in Subsection 3.2.
- Step 2: To establish the $L^2$-theory for the operator $(-\partial_t^2 + \mathcal{L}_G)A(x, t, D, \partial_t)^{-1}$. This will be done in Subsection 3.3.
- Step 3: To use the Steps 1 and 2 in the proof of Proposition 3.3. This will be presented in Subsection 3.4.

3.2. Computing the inverse of $A(x, t, D, \partial_t)$. According to the hypothesis in Proposition 3.3, let us consider the operator $A \geq cI$. We will analyse the invertibility of the operator

$$A(x, t, D, \partial_t) = -\partial_t^2 + A^\frac{z}{\varepsilon} : H^2(G \times \mathbb{T}(T, \varepsilon)) \to L^2(G \times \mathbb{T}(T, \varepsilon)).$$

Note that we have embedded the manifold $G_T$ (with lateral boundary $\partial G_T = (G \times \{0\}) \cup (G \times \{T\})$) on the compact Lie group $G \times \mathbb{T}(T, \varepsilon)$. In the next lemma we prove that $A(x, t, D, \partial_t)$ is parameter elliptic with respect to a suitable angle of the complex plane.

**Lemma 3.5.** Let us consider the operator $A \geq cI$ of Proposition 3.3. Then, it is parameter-elliptic with respect to the sector

$$\Lambda = \{\lambda + i\lambda' : |\lambda'| \leq -\lambda, \lambda \leq 0\}.$$  \hfill (3.7)

**Proof.** In order to prove that $A$ is parameter elliptic with respect to the sector in (3.7), see Figure 4, we have to prove that the matrix-valued symbol $a(x, \theta) = \sigma_A(x, [\theta])$ of $A$ satisfies the following estimate

$$\|(a(x, [\theta]) - zI_{d_t})^{-1}\|_{op} \leq C(1 + \langle \theta \rangle + |z|^{\frac{1}{m}})^{-m},$$

uniformly in $x \in G$, for all $[\theta] \in \hat{G}$ and all $z \in \Lambda$. Note that $A$ is elliptic and then, we have the estimate

$$\|a(x, [\theta])\|_{op} \geq C\langle \theta \rangle^m, \quad [\theta] \in \hat{G}. \hfill (3.8)$$

In some basis of the representation space, in view of the positivity of its symbol $a(x, [\theta]) \geq 0$, we can write it in diagonal form, that is

$$a(x, [\theta]) \equiv a(x, \theta) = \text{diag}[\lambda_{11}(x, \theta), \cdots, \lambda_{d_d d_\theta}(x, \theta)], \quad \lambda_{ii}(x, \theta) \geq 0, \quad 1 \leq i \leq d_\theta. \quad (3.9)$$
Then, using that the order of \( A \) is \( m \) and (3.8) we have that
\[
\|a(x, [\theta])\|_{\text{op}} = \sup_{1 \leq i \leq d_{\theta}} \lambda_{ii}(x, \theta) \asymp \langle \theta \rangle^m. \tag{3.10}
\]
Note that the positivity hypothesis \( a = \sigma_A(x, [\theta]) \geq cI_{d_{\theta}} \) in Proposition 3.3, implies the invertibility of the matrix-valued symbol \( a = a(x, [\theta]) \) in any representation space. Then, the symbol
\[
a(x, [\theta])^{-1}
\]
is elliptic of order \(-m\). In particular it satisfies the inequality
\[
\forall [\theta] \in \widehat{G}, \quad \|a(x, [\theta])^{-1}\|_{\text{op}} = \sup_{1 \leq i \leq d_{\theta}} \lambda_{ii}(x, \theta)^{-1} \asymp \langle \theta \rangle^{-m}.
\]
Then we have that
\[
\inf_{1 \leq i \leq d_{\theta}} \lambda_{ii}(x, \theta) \asymp \langle \theta \rangle^m. \tag{3.11}
\]
Note that (3.10) and (3.10) imply that for any \( z = \lambda + i\lambda' \in \Lambda \) we have
\[
\|a(x, [\theta]) - zI_{d_{\xi}}\|^{-1}_{\text{op}} \leq \sup_{1 \leq i \leq d_{\theta}} |\lambda_{ii}(x, \theta) - z|^{-1} = \sup_{1 \leq i \leq d_{\theta}} |\lambda_{ii}(x, \theta) - \lambda - i\lambda'|^{-1}
\]
\[
\lesssim \sup_{1 \leq i \leq d_{\theta}} (\lambda_{ii}(x, \theta) - \lambda + |\lambda'|)^{-1}
\]
\[
\lesssim (\langle \theta \rangle^m - \lambda + |\lambda'|)^{-1}
\]
\[
\asymp (\langle \theta \rangle + (-\lambda + |\lambda'|)^{-m}^{-m}
\]
\[
\asymp (\langle \theta \rangle + |z|^m)^{-m},
\]
as desired. The proof of Lemma 3.5 is complete.

\[\square\]

Remark 3.6 (Complex powers of \( A(x, t, D, \partial_t) \)). Let us consider the sector in (3.7), see Figure 4. Let \( c > 0 \) be the lower bound in the condition on the operator \( A \geq cI \) in Proposition 3.3. If
\[
0 < \varepsilon < \frac{c^2}{1000},
\]
consider the complex sector \( \Lambda_{\varepsilon} = \Lambda \cup \{z \in \mathbb{C} : |z| \leq \varepsilon\} \) in Figure 5 below.

Note that any \( z \in \Lambda \) belongs to the resolvent of \( A(x, t, D, \partial_t) \). Indeed, the lower bound \( A \geq cI, c > 0 \), implies that the spectrum of \( A \) is contained in the infinite interval \([c, \infty)\). Moreover, the spectral mapping theorem implies that the spectrum
of $A_{\frac{2}{n}}$ is contained in $[c^2, \infty)$. Since $-\partial_t^2$ is a positive operator on $L^2(\mathbb{T}(T, \varepsilon))$ and $\lambda = 0$ belongs to its spectrum, we have that

$$A(x, t, D, \partial_t) = -\partial_t^2 + A_{\frac{2}{n}}$$

is positive on $L^2(G \times \mathbb{T}(T, \varepsilon))$ with its spectrum contained in $[c^2, \infty)$. This analysis proves the inclusion $\Lambda_{\varepsilon} \subset \text{Resolv}(A(x, t, D, \partial_t))$.

In view of Lemma 3.5 and of Theorem 2.23 we have that

$$z \in \mathbb{C} \mapsto G^z := A(x, t, D, \partial_t)^z, \quad (3.12)$$

is a holomorphic family of pseudo-differential operators, that maps any $z \in \mathbb{C}$ into the class $\Psi^{2\Re(z)}_{\rho, \delta}(G \times \mathbb{T}(T, \varepsilon))$, where

$$G^z f(x) = -\frac{1}{2\pi i} \int_{\partial\Lambda_{\varepsilon}} \lambda^{-1}(A(x, t, D, \partial_t) - \lambda I)^{-1} f(x) d\lambda, \quad f \in C^\infty(G \times \mathbb{T}(T, \varepsilon)). \quad (3.13)$$

In particular, with $z = -1$, we have the inverse $G^{-1}$,

$$G^{-1} f(x) = -\frac{1}{2\pi i} \int_{\partial\Lambda_{\varepsilon}} \lambda^{-1}(A(x, t, D, \partial_t) - \lambda I)^{-1} f(x) d\lambda, \quad f \in C^\infty(G \times \mathbb{T}(T, \varepsilon)), \quad (3.14)$$

of $A(x, t, D, \partial_t)$ on the orthogonal complement of its kernel. This is, if $P_0$ is the orthogonal projection on the subspace $\text{Ker}(A(x, t, D, \partial_t))$, $G^{-1} G = I - P_0$, see Corollary 2.24. In view of the lower bound $A \geq cI$, we deduce that $P_0$ is the null operator.

Moreover, we have the following property.

**Proposition 3.7.** Let $0 < \varepsilon < 1$, and let us consider the operator norm

$$B_{\varepsilon} = \|A(t, x, D, \partial_t)^{-1}\|_{\mathcal{B}(L^2(G \times \mathbb{T}(T, \varepsilon)), H^2(G \times \mathbb{T}(T, \varepsilon)))}.$$ 

Then

$$B := \sup_{0 < \varepsilon < 1} B_{\varepsilon} \leq 1 + 1/c, \quad (3.15)$$

where $c > 0$ in the constant is the positivity condition $A \geq cI$ of Proposition 3.3.

**Proof.** Let us consider the orthogonal basis of $L^2(\mathbb{T}(T, \varepsilon))$ formed by the exponential functions

$$t \in \mathbb{T}(T, \varepsilon) \mapsto \tilde{e}^\varepsilon_k(t) = \exp \left( \frac{2\pi i t k}{2(T + \varepsilon)} \right), \quad k \in \mathbb{Z}, \quad (3.16)$$

Figure 5. The new sector $\Lambda_{\varepsilon} = \Lambda \cup \{z \in \mathbb{C} : |z| \leq \varepsilon\}$. 
and let us consider the $L^2$-normalised system of $2(T + \varepsilon)$-periodic eigenfunctions
\[ e_k^\varepsilon := \frac{\tilde{e}_k^\varepsilon}{\sqrt{2(T + \varepsilon)}}, \]
of the Laplacian $-\partial_t^2$. The corresponding eigenvalues of $-\partial_t^2$ are given by
\[ \mu_{k,\varepsilon} = \frac{4\pi^2 k^2}{4(T + \varepsilon)^2} = \left( \frac{\pi k}{T + \varepsilon} \right)^2, \quad k \in \mathbb{Z}. \]
Since $\{e_k^\varepsilon \otimes e_j\}$ is a basis for $L^2(G \times \mathbb{T}(T, \varepsilon))$ the spectrum of the operator $-\partial_t^2 + A_{\hat{\pi}}$ is determined by the sequence
\[ \mu_{k,\varepsilon} + \lambda_j^2 = \left( \frac{\pi k}{T + \varepsilon} \right)^2 + \lambda_j^2, \quad k \in \mathbb{Z}, \quad j \in \mathbb{N}_0. \]
Since $A \geq cI$, we have the eigenvalue inequality $\lambda_k \geq c$, and then for any $f \in L^2(G \times \mathbb{T}(T, \varepsilon))$ we have that
\[ \|A(t, x, D, \partial_t)^{-1} f\|_{H^2}^2 = \left\| \sum_{k,j} \left( \mu_{k,\varepsilon} + \lambda_j^2 \right)^{-1} (f, e_k^\varepsilon \otimes e_j) e_k^\varepsilon \otimes e_j \right\|_{H^2}^2 \]
\[ = \sum_{k,j} (1 + \mu_{k,\varepsilon} + \lambda_j^2)^2 (\mu_{k,\varepsilon} + \lambda_j^2)^{-2} |(f, e_k^\varepsilon \otimes e_j)|^2 \]
\[ \leq \left( \frac{1}{c} + 1 \right)^2 \sum_{k,j} |(f, e_k^\varepsilon \otimes e_j)|^2 = \left( \frac{1}{c} + 1 \right)^2 \|f\|_{L^2}^2. \]
From the previous analysis we deduce that $B \leq 1 + \frac{1}{c}$ as desired. \qed

3.3. $L^2$-theory for the operator $(-\partial_t^2 + \mathcal{L}_G)A(x, t, D, \partial_t)^{-1}$. The global Calderón-Vaillancourt theorem is a sharp $L^2$-estimate for pseudo-differential operators. We will apply it in the proof of our spectral inequality. Indeed, let us consider the operator $A \geq cI$ of Proposition 3.3. Observe that
\[ (-\partial_t^2 + \mathcal{L}_G) \in \Psi^2_{1,0}(G \times \mathbb{T}(T, \varepsilon) \times \hat{G} \times 2(T + \varepsilon)\mathbb{Z}), \]
belongs to the Hörmander class of order 2. Also, we have that
\[ A(x, t, D, \partial_t) = -\partial_t^2 + A_{\hat{\pi}} \in \Psi^2_{1,\delta}(G \times \mathbb{T}(T, \varepsilon) \times \hat{G} \times 2(T + \varepsilon)\mathbb{Z}), \]
and for the inverse of $A(x, t, D, \partial_t)$ we have
\[ A(x, t, D, \partial_t)^{-1} \in \Psi^{-2}_{1,\delta}(G \times \mathbb{T}(T, \varepsilon) \times \hat{G} \times 2(T + \varepsilon)\mathbb{Z}). \]
The pseudo-differential calculus implies that
\[ F(x, t, D, \partial_t) := (-\partial_t^2 + \mathcal{L}_G)A(x, t, D, \partial_t)^{-1} \in \Psi^0_{1,\delta}(G \times \mathbb{T}(T, \varepsilon) \times \hat{G} \times 2(T + \varepsilon)\mathbb{Z}). \]
The global Calderón-Vaillancourt theorem implies that $F(x, t, D, \partial_t)$ is a bounded operator on $L^2(G \times \mathbb{T}(T, \varepsilon) \times \hat{G} \times 2(T + \varepsilon)\mathbb{Z})$. Note that the global quantisation allows writing the operator $F(x, t, D, \partial_t)$ as follows
\[ F(x, t, D, \partial_t)u(x, t) = \sum_{k \in \mathbb{Z}^n} \sum_{|\xi| \in \hat{G}} d\xi \text{Tr}[(\xi \otimes e^{\frac{ik(x + \varepsilon)}{2(T + \varepsilon)}})(x, t)\sigma(x, t, \xi, k)\hat{u}(\xi, k)], \quad (3.17) \]
where, for any \( u \in C^\infty_0(G \times \mathbb{T}(T, \varepsilon)) \), the Fourier transform of \( u \) at

\[
(\xi, k) \in \hat{G} \times \hat{\mathbb{T}}(T, \varepsilon) \cong \hat{G} \times 2(T + \varepsilon)\mathbb{Z},
\]

is defined by

\[
\hat{u}(\xi, k) = \int_{\mathbb{T}(T, \varepsilon)} \int_{\hat{G}} e^{-i\pi(t,k)} \xi(x)^* u(x, t) dx dt, \quad k \in \mathbb{Z}, \quad [\xi] \in \hat{G}.
\]  

(3.18)

For the toroidal variable, we have used the toroidal calculus, see Subsection 2.2 or [44] for details. From now, \( \Delta_k \) denotes the difference operator on a lattice. Note that the matrix-valued symbol \( \sigma_F(x, t, \xi, k) \) of the operator \( F(x, t, D, \partial_t) \) admits an asymptotic expansion of the form

\[
\sigma_F(x, t, \xi, k) \sim \sum_{j=0}^{\infty} \sigma_{m-j}(x, t, \xi, k), \quad (x, t, \xi, k) \in G \times \mathbb{T}(T, \varepsilon) \times \hat{G} \times 2(T + \varepsilon)\mathbb{Z},
\]

in the sense that

\[
\forall N \in \mathbb{N}, \quad \sigma_F(x, t, \xi, k) - \sum_{j=0}^{N} \sigma_{m-j}(x, t, \xi, k) \in S^{m-(N+1)(1-\delta)}(G \times \mathbb{T}(T, \varepsilon) \times \hat{G} \times 2(T + \varepsilon)\mathbb{Z}).
\]

Let \( a(x, \xi)^{\hat{\pi}} \) denote the matrix-valued symbol of \( A^{\hat{\pi}} \), (defined by the functional calculus of matrices). The matrix-valued component \( \sigma_F \) of higher order of the quotient operator

\[
F(x, t, D, \partial_t) := (-\partial_t^2 + \mathcal{L}_G)A(x, t, D, \partial_t)^{-1}
\]

is given by

\[
\sigma_F(x, t, \xi, k) = \sigma_F^\varepsilon(x, t, \xi, k) := \left( (\frac{\pi k}{T + \varepsilon})^2 + (\xi)^2 \right) \left( (\frac{\pi k}{T + \varepsilon})^2 I_{d_x} + a(x, [\xi])^{\hat{\pi}} \right)^{-1}.
\]  

(3.19)

Since the ellipticity of \( A^{\hat{\pi}} \), implies that

\[
C_1 \xi^2 \leq \|a(x, \xi)^{\hat{\pi}}\|_{op} \leq C_2 \xi^2, \quad [\xi] \in \hat{G},
\]

the symbol \( \sigma_F \) is elliptic of order zero, satisfying the inequality

\[
\tilde{C}_1 \leq \|\sigma_F(x, t, \xi, k)\| \leq \tilde{C}_2,
\]

with \( C_1 \) and \( C_2 \) independent of \( \varepsilon \in (0, 1) \). In view of the positivity hypothesis \( \sigma_A(x, [\xi]) \geq c I_{d_x} \) on every representation space, the family

\[
[0, 1] \mapsto \sigma_F(x, t, \xi, k) = \sigma_F^\varepsilon(x, t, \xi, k)
\]

is a smooth function from the unit interval \([0, 1]\) to the class

\[
S_{1,\delta}^0(G \times \mathbb{T}(T, \varepsilon) \times \hat{G} \times 2(T + \varepsilon)\mathbb{Z}),
\]

endowed with its natural Fréchet structure. As a consequence the supremum

\[
\sup_{\varepsilon \in [0, 1]} \sup_{(x, t, [\xi], k)} (1 + |k| + (\xi))^{\alpha + |\gamma| - |\delta|} \|\partial_x^\alpha D^\gamma \Delta_k^\varepsilon \sigma_F(x, t, \xi, k)\|_{op} < \infty,
\]

is bounded. This implies that \( \sigma_F \) satisfies inequalities of the type

\[
\|\partial_x^\alpha D^\gamma \Delta_k^\varepsilon \sigma_F(x, t, \xi, k)\|_{op} \leq C_{\alpha,\beta,\gamma}(1 + |k| + (\xi))^{-|\alpha| - |\gamma| + |\delta|},
\]
where the constants $C_{\alpha,\beta,\gamma}$ are independent of the parameter $\varepsilon \in [0,1]$. Since the Calderón-Vaillancourt estimates the $L^2$-boundeness of $F$ in terms of the constants $C_{\alpha,\beta,\gamma}$ and of $\tilde{C}_2$, (see Remark 2.13) that is, for any $u \in C_0^\infty(\mathbb{T}(T,\varepsilon) \times \tilde{U})$,

$$\|F(x,t,D,\partial_t)u\|_{L^2} \leq \left( \sup_{|\alpha|+|\beta|+|\gamma| \leq \ell} \{C_{\alpha,\beta,\gamma}, \tilde{C}_2\} \right) \|u\|_{L^2},$$

(3.21)

where $\ell \in \mathbb{N}$ is big enough. As a consequence of this discussion we have proved the following lemma.

**Lemma 3.8.** Let $0 < \varepsilon < 1$, and let us consider the operator norm

$$C_\varepsilon = \|(-\partial_x^2 + \mathcal{L}_G)A(t,x,D,\partial_t)^{-1}\|_{\mathcal{B}(L^2(G \times [T,\varepsilon]))}.$$  

Then

$$C := \sup_{0<\varepsilon<1} C_\varepsilon < \infty. \quad (3.22)$$

Moreover, there is $\ell \in \mathbb{N}_0$ large enough such that

$$C \lesssim \left( \sup_{|\alpha|+|\beta|+|\gamma| \leq \ell} \{C_{\alpha,\beta,\gamma}, \tilde{C}_2\} \right) \|u\|_{L^2}. \quad (3.23)$$

### 3.4. Proof of Proposition 3.3.

We shall reduce the proof of this proposition to an interpolation inequality. We explain this strategy as follows.

**Remark 3.9.** Let us consider a spectra parameter $\lambda > 0$. Let $\varphi \in \text{Im}(E_\lambda)$. Then, $\varphi$ can be written as linear combinations of the eigenfunctions $e_j$, where $\lambda_j \leq \lambda$, that is

$$\varphi(x) = \sum_{\lambda_j \leq \lambda} a_j e_j(x). \quad (3.24)$$

We note that for the proof of (3.4), is enough to show that

$$F(x,t) := \sum_{\lambda_j \leq \lambda} \frac{\sinh(\lambda_j t)}{\lambda_j} a_j e_j(x), \quad (x,t) \in G_T := G \times [0,T], \quad (3.25)$$

satisfies the interpolation inequality

$$\|F\|_{H^1(G \times [0,T])} \leq C \|F\|_{H^1(G_T)}^{\kappa} \|\varphi\|_{L^2(\omega)}^{1-\kappa}. \quad (3.26)$$

Indeed, by the Parseval theorem we have that

$$\|F\|_{H^1(G \times [0,T])}^2 \geq C \|F\|_{L^2(G \times [0,T])}^2$$

$$= \int_0^T \int_G \left| \sum_{\lambda_j \leq \lambda} \frac{\sinh(\lambda_j t)}{\lambda_j} a_j e_j(x) \right|^2 \, dx \, dt$$

$$= \sum_{\lambda_j \leq \lambda} |a_j|^2 \int_0^T \left| \frac{\sinh(\lambda_j t)}{\lambda_j} \right|^2 \, dt$$

$$\geq \sum_{\lambda_j \leq \lambda} |a_j|^2 \int_0^T t^2 \, dt$$
\[ C_\alpha \sum_{\lambda_j \leq \lambda} |a_j|^2. \]

Observing that
\[ \partial_t F(x, 0) = \sum_{\lambda_j \leq \lambda} a_j e_j(x), \tag{3.27} \]
and that
\[ \|F\|_{H^1(G_T)}^2 \lesssim e^{2T\lambda^2} \sum_{\lambda_j \leq \lambda} |a_j|^2 \]
we deduce the inequality
\[ C_\alpha \sum_{\lambda_j \leq \lambda} |a_j|^2 \lesssim \kappa T^{2(1-\kappa)} \sum_{\lambda_j \leq \lambda} a_j e_j(x) \|L^2(\omega)\|. \tag{3.28} \]

Hence
\[ \left( \sum_{\lambda_j \leq \lambda} |a_j|^2 \right)^{1-\kappa} \lesssim e^{\kappa T\lambda^2(1-\kappa)} \sum_{\lambda_j \leq \lambda} a_j e_j(x) \|L^2(\omega)\|^{2(1-\kappa)}, \]
and consequently
\[ \left( \sum_{\lambda_j \leq \lambda} |a_j|^2 \right)^{\frac{1}{2}} \lesssim e^{\kappa T\lambda^{\kappa/(1-\kappa)}(1-\kappa)} \sum_{\lambda_j \leq \lambda} a_j e_j(x) \|L^2(\omega)\|^{(1-\kappa)}, \]
proving (3.4). Note also that we can estimate \( e^{\kappa T\lambda^{\kappa/(1-\kappa)}(1-\kappa)} \lesssim e^{C_2 \kappa T\lambda/(1-\kappa)} \) for some \( C_2 > 0 \).

**Proof of Proposition 3.3.** In view of Remark 3.9 we proceed with the proof of the inequality (3.26). Note that, by normalising \( \kappa \) on \( L^2(G) \) we can assume without loss of generality that \( \|\kappa\|_{L^2(G)} = 1 \).

### 3.4.1. An auxiliary interpolation inequality on \([0, T + \varepsilon)\).
Let \( \varepsilon \in (0, 1) \) be a positive parameter whose conditions will be imposed later. Firstly, by replacing in Lemma 3.4 the open interval \( I_T := (0, T) \) by \( I_{T+\varepsilon} := (0, T + \varepsilon) \), and with \( \bar{G}_{T+\varepsilon} := G \times (0, T + \varepsilon) \), we shall make use of the following interpolation inequality:

\[
\text{For any } T > 0 \text{ and all } \alpha \in (0, T/2), \text{ there exists } \kappa \in (0, 1) \text{ such that } \]
\[
\|\phi\|_{H^1(G \times (\alpha, T-\alpha))} \leq C \|\phi\|_{H^1(G_{T+\varepsilon})} \left( \|(-\partial_t^2 + L_G)\phi\|_{L^2(G_{T+\varepsilon})} + \|\partial_t \phi(x, 0)\|_{L^2(\omega)} \right)^{1-\kappa} \tag{3.29}
\]

for all \( \phi \in H^2(G_{T+\varepsilon}) \) such that \( \phi = 0 \) in \( G \times \{0\} \).
3.4.2. Construction of a suitable function $\phi$: Let us apply (3.29) with $\phi$ defined as follows. Consider $\psi \in C^\infty(G \times [0, T + \varepsilon])$ satisfying that

$$\psi(t) := \begin{cases} C, & t \in [0, T], x \in G, \\ 0, & t \in [T + \frac{2\varepsilon}{T}, T + \varepsilon], x \in G, \end{cases}$$

(3.30)

where $0 < C \leq \varepsilon$. Assume that there exists $M_0 > 0$, independent of $\varepsilon$ such that

$$\|\psi^{(i)}\|_{L^\infty} \leq M_0,$$

(3.31)

for all $i \in \{1, 2, 3, 4\}$. We construct a function with this characteristics in Lemma 4.1. Then, by considering the function

$$F(x, t) := \sum_{\lambda_j \leq \lambda} \frac{\sinh(\lambda_j t)}{\lambda_j} a_j e_j(x), \quad (x, t) \in G_T := G \times [0, T],$$

(3.32)

and its extension to the set $I_{T+\varepsilon} = [0, T+\varepsilon]$ by the constant function equal to $F(x, T)$, that is

$$F(x, t) = \begin{cases} \sum_{\lambda_j \leq \lambda} \frac{\sinh(\lambda_j t)}{\lambda_j} a_j e_j(x), & (x, t) \in [0, T], x \in G, \\ F(x, T), & t \in [T, T + \varepsilon], x \in G, \end{cases}$$

(3.33)

we consider

$$\phi(x, t) := F(x, t)\psi(t), \quad (x, t) \in G_{T+\varepsilon}.$$ 

(3.34)

Note that $\phi(x, 0) = F(x, 0) = 0 = \phi(x, T + \varepsilon) = 0$. Note also that $\phi$ is an extension of $F$ from $G_T$ to $G_{T+\varepsilon}$.

Now, let us consider the odd extension of $\phi$ to the whole interval $[-(T + \varepsilon), T + \varepsilon]$, that is,

$$\phi(x, t) = -\phi(x, -t) = -\psi(-t)F(x, -t), \quad -(T + \varepsilon) \leq t \leq 0.$$ 

(3.35)

3.4.3. The norm $\|\partial_t \phi(x, 0)\|_{L^2(\omega)}$. Note that $\psi$ has been defined on $[0, T + \varepsilon]$ and it has been extended to $[-(T + \varepsilon), 0]$ using its odd extension, that is, the one defined via $\psi(-t) = -\psi(t)$, for $t \in [0, T + \varepsilon]$.

The Leibniz rule gives for any $t$ in a neighborhood of $t = 0$, the identity

$$\partial_t \phi(x, t) = \psi'(t)F(x, t) + \psi(t)\partial_t F(x, t).$$

By evaluation both sides of this identity at $t = 0$ we have

$$\partial_t \phi(x, 0) = \psi'(0)F(x, 0) + \psi(0)\partial_t F(x, 0) = \psi(0)\partial_t F(x, 0),$$

(3.36)

from which we have the identity

$$\|\partial_t \phi(x, 0)\|_{L^2(\omega)} = \|\psi(0)\| \times \|\partial_t F(x, 0)\|_{L^2(\omega)} = \|\psi(0)\| \times \|\mathcal{K}\|_{L^2(\omega)}.$$ 

(3.37)

3.4.4. Embedding of $G_T$ in a closed manifold $G \times \mathbb{T}(T, \varepsilon)$, $\mathbb{T}(T, \varepsilon) \cong S^1$: Now, we will proceed with a topological construction.

It is clear that in the variable $t \in [-T + \varepsilon, T + \varepsilon]$ the function $\phi$ can be extended in the periodic way to the whole line $\mathbb{R}$, or in other words, we can identify $\phi$ with a distribution on the dilated torus

$$\mathbb{T}(T, \varepsilon) = \mathbb{R}/(2(T + \varepsilon)\mathbb{Z}) = [-T + \varepsilon, T + \varepsilon],$$

(3.38)

where in the resulting manifold $[-T + \varepsilon, T + \varepsilon]$ we identify the endpoints $-(T + \varepsilon) \sim T + \varepsilon$. This construction allows the manifold $\mathbb{T}(T, \varepsilon)$ to be diffeomorphic to the circle.
S^1, and in consequence the function \( \phi \in \mathcal{D}'(G \times T(T, \varepsilon)) \) is smooth on the product space \( G \times T(T, \varepsilon) \) which is a compact manifold of \( C^\infty \)-class without boundary. In particular, we have the inclusion: \( \forall T, \varepsilon > 0, G_T \subset G \times T(T, \varepsilon) \).

3.4.5. **Proof of the interpolation inequality.** Now, we are ready for the proof of the interpolation inequality (3.26)

\[
\| F \|_{H^1(G \times (a, T-a))} \leq C_{s_0, s_{00}} \| F \|_{H^1(G_T)} \| \mathcal{K} \|_{L^2(\omega)}^{1/\kappa}.
\]

In the identity (3.42) below, we will prove that with \( \phi \) defined in (3.34), we have that

\[
\| (-\partial_t^2 + \mathcal{L}_G)\phi(x, t) \|_{L^2(G_{T+\varepsilon})} = 1/\sqrt{2} \| (-\partial_t^2 + \mathcal{L}_G)\phi(x, t) \|_{L^2(G \times T(T, \varepsilon))}.
\]

The positivity condition

\[
A \geq c I, \quad c > 0,
\]

and the spectral mapping theorem applied to \( A \) gives the lower bound

\[
A^{\frac{2}{m}} \geq c^{\frac{2}{m}} I, \quad c > 0,
\]

which gives the invertibility of the pseudo-differential operator

\[
\mathcal{A}(t, x, D, \partial_t) = -\partial_t^2 + A^{\frac{2}{m}} : H^2(G \times T(T, \varepsilon)) \to L^2(G \times T(T, \varepsilon)).
\]

Then the operator

\[
\mathcal{A}(t, x, D, \partial_t)^{-1} : L^2(G \times T(T, \varepsilon)) \to H^2(G \times T(T, \varepsilon))
\]

is bounded, see Lemma 3.7. Moreover,

\[
\| (-\partial_t^2 + \mathcal{L}_G)\phi(x, t) \|_{L^2(G \times T(T, \varepsilon))} \leq C \| (-\partial_t^2 + A^{\frac{2}{m}})\phi(x, t) \|_{L^2(G \times T(T, \varepsilon))},
\]

where the constant \( C \) is independent of \( \varepsilon > 0 \). Now, let us use the identity

\[
\| (-\partial_t^2 + \mathcal{L}_G)\phi(x, t) \|_{L^2(G \times T(T, \varepsilon))} = \| (-\partial_t^2 + \mathcal{L}_G)(-\partial_t^2 + A^{\frac{2}{m}})^{-1}(-\partial_t^2 + A^{\frac{2}{m}})\phi(x, t) \|_{L^2(G \times T(T, \varepsilon))}.
\]

From Lemma 3.8, the operator \((-\partial_t^2 + \mathcal{L}_G)(-\partial_t^2 + A^{\frac{2}{m}})^{-1}\) belongs to the class \( \Psi^0_{1, \delta} \) and the matrix-valued Calderón-Vaillancourt theorem gives its boundedness on \( L^2 \), with its operator norm bounded by a constant \( C > 0 \), independent of \( \varepsilon > 0 \). Consequently,

\[
\| (-\partial_t^2 + \mathcal{L}_G)(-\partial_t^2 + A^{\frac{2}{m}})^{-1}(-\partial_t^2 + A^{\frac{2}{m}})\phi(x, t) \|_{L^2(G \times T(T, \varepsilon))} \leq C \| (-\partial_t^2 + A^{\frac{2}{m}})\phi(x, t) \|_{L^2(G \times T(T, \varepsilon))}.
\]

In what follows we estimate the norm:

\[
Z_1 := \| (-\partial_t^2 + A^{\frac{2}{m}})\phi(x, t) \|_{L^2(G \times T(T, \varepsilon))},
\]

and let us keep in mind that the partial analysis above gives us the inequality

\[
\| \phi \|_{H^1(G \times (a, T-a))} \lesssim \| \phi \|_{H^1(G_{T+\varepsilon})} (Z_1 + |\psi(0)|\| \mathcal{K} \|_{L^2(\omega)})^{1-\kappa}, \quad (3.39)
\]

in view of the identity

\[
\psi(0)\mathcal{K} = \partial_t\phi(x, 0) = \psi(0)\partial_t F(x, 0) = \psi(0) \sum_{\lambda_j \leq \lambda} a_j e_j(x).
\]
Indeed, we recall that for every \( t \in [0, T] \), we have that

\[
\partial_t F(x, t) = \sum_{\lambda_j \leq \lambda} \sinh(\lambda_j t) a_j e_j(x),
\]

\[
\partial_t^2 F(x, t) = \sum_{\lambda_j \leq \lambda} \sinh(\lambda_j t) \lambda_j a_j e_j(x).
\]

### 3.4.6. Estimate of \( Z_1 \)

By the spectral properties of \( A \) we have

\[
A^2 F(x, t) = \sum_{\lambda_j \leq \lambda} \frac{\sinh(\lambda_j t)}{\lambda_j} a_j A^2 \phi(x_j)(x) = \sum_{\lambda_j \leq \lambda} \frac{\sinh(\lambda_j t)}{\lambda_j} a_j \lambda_j^2 e_j(x)
\]

\[
= \partial_t^2 F(x, t),
\]

for all \( t \in [0, T] \). Since \( \phi(x, t) = \psi(t) F(x, t) \) on \([0, T] \), and \( \psi \) is constant on \([0, T] \) we have that

\[
\forall x \in G, \forall t \in (0, T), (-\partial_t^2 + A^2) \phi(x, t) = 0. \tag{3.40}
\]

First, note that the following symmetry property is valid due to the identity \( \phi(x, t) = -\phi(x, -t) \),

\[
\|(\partial_t^2 + A^2) \phi(x, t)\|_{L^2(G \times T(\epsilon, T))}^2 = \|(-\partial_t^2 + A^2) \phi(x, t)\|_{L^2(G_{T+\epsilon})}^2. \tag{3.41}
\]

Moreover, once proved (3.41), we have in the particular case (where \( A = +L_G \)) of the Laplacian, the following inequality.

\[
\|(\partial_t^2 + \mathcal{L}_G) \phi(x, t)\|_{L^2(G \times T(\epsilon, T))}^2 = 2\|(-\partial_t^2 + L_G) \phi(x, t)\|_{L^2(G_{T+\epsilon})}^2. \tag{3.42}
\]

Indeed, for the proof of the identity of norms in (3.41) observe that

\[
\|(\partial_t^2 + A^2) \phi(x, t)\|_{L^2(G \times T(\epsilon, T))}^2
\]

\[
= \int_G \int_0^{T-\epsilon} \int_0 \int_0 (|\partial_t^2 + A^2) \phi(x, t)|^2 dt \, dx + \int_G \int_0^{T+\epsilon} (|\partial_t^2 + A^2) \phi(x, t)|^2 dt \, dx
\]

\[
= \int_G \int_0^{T-\epsilon} \int_0 \int_0 (|\partial_t^2 + A^2) \phi(x, -t)|^2 dt \, dx + \int_G \int_0^{T+\epsilon} (|\partial_t^2 + A^2) \phi(x, t)|^2 dt \, dx
\]

\[
= \int_G \int_0^{T-\epsilon} \int_0 \int_0 \int_0 (\phi_{tt}(x, -t) - A^2 \phi(x, -t)|^2 dt \, dx + \int_G \int_0^{T+\epsilon} (|\partial_t^2 + A^2) \phi(x, t)|^2 dt \, dx
\]

\[
= 2 \int_G \int_0^{T+\epsilon} (|\partial_t^2 + A^2) \phi(x, t)|^2 dt \, dx.
\]
Taking into account (3.40), the symmetry property \( \phi(x, t) = -\phi(x, -t) \), and the positivity of the operator \((-\partial_t^2 + A^{\frac{2}{n}})\) on \(L^2(G \times \mathbb{T}(T, \varepsilon))\) (that is, making use of the self-adjointness of \((-\partial_t^2 + A^{\frac{2}{n}}))\) imply that

\[
\|(-\partial_t^2 + A^{\frac{2}{n}})\phi(x, t)\|^2_{L^2(G(T + \varepsilon))} = \frac{1}{2}\|(-\partial_t^2 + A^{\frac{2}{n}})\phi(x, t)\|^2_{L^2(G \times \mathbb{T}(T, \varepsilon))} = \frac{1}{2}|(-\partial_t^2 + A^{\frac{2}{n}})\phi(x, t), (-\partial_t^2 + A^{\frac{2}{n}})\phi(x, t)|_{L^2(G \times \mathbb{T}(T, \varepsilon))}|
\]

\[
\leq \int \int_{G \setminus [T, T + \varepsilon]} |(-\partial_t^2 + A^{\frac{2}{n}})^2 \phi(x, t)| |\phi(x, t)| dxdt = \int \int_{G \setminus [T, T + \varepsilon]} |(-\partial_t^2 + A^{\frac{2}{n}})^2 \phi(x, t)| |\phi(x, t)| dxdt.
\]

Therefore, we have the estimate

\[
\|(-\partial_t^2 + A^{\frac{2}{n}})\phi(x, t)\|^2_{L^2(G(T + \varepsilon))} \leq \int \int_{G \setminus [T, T + \varepsilon]} |(-\partial_t^2 + A^{\frac{2}{n}})^2 \phi(x, t)| |\phi(x, t)| dxdt
\]

\[
\leq \int \int_{G \setminus [T, T + \varepsilon]} |\phi(x, t)| dxdt \times \|(-\partial_t^2 + A^{\frac{2}{n}})^2 \phi\|_{L^\infty}
\]

\[
= I \times II,
\]

where

\[
I = \int \int_{G \setminus [T, T + \varepsilon]} |\phi(x, t)| dx dt, \quad II = \|(-\partial_t^2 + A^{\frac{2}{n}})^2 \phi\|_{L^\infty(G \times [T, T + \varepsilon])}.
\]

Now, we will estimate each one of these norms.

3.4.7. Estimate for \(I\). Note that

\[
I \leq \text{Vol}(G) \times \varepsilon \times \|\phi\|_{L^\infty(G \times [T, T + \varepsilon])} = \text{Vol}(G) \times \varepsilon \|\psi(t)\|_{L^\infty[T, T + \varepsilon]} \|F(x, t)\|_{L^\infty}.
\]

Now, and for any \(t\) fixed, and for all \(s_0 \in \mathbb{N}\) observe that

\[
|F(x, t)| \leq \sup_{0 \leq s \leq s_0} \| (1 + A)^{\frac{2}{n}} F(:, t) \|_{L^\infty(G)}.
\]

(3.43)

In view of the Sobolev embedding theorem, any \(s_0 > n/2\) satisfies that

\[
\sup_{0 \leq s \leq s_0} \| (1 + A)^{\frac{2}{n}} F(:, t) \|_{L^\infty(G)} \leq \sup_{0 \leq s \leq s_0} \| (1 + A)^{\frac{n+s_0}{n}} F(:, t) \|_{L^2(G)} \leq \sup_{0 \leq s \leq s_0 + s_00} \| (1 + A)^{\frac{2}{n}} F(:, t) \|_{L^2(G)}.
\]
The spectral properties of the operator $(1 + A)^{\frac{\alpha}{2}}$ give the estimates

\[
\|(1 + A)^{\frac{\alpha}{2}} F(\cdot, t)\|_{L^2(G)}^2 = \left\| \sum_{\lambda_j \leq \lambda} \frac{\sinh(\lambda_j t)}{\lambda_j} (1 + \lambda_j^m)^{\frac{\alpha}{2m}} a_j e_j(x) \right\|_{L^2(G)}^2
\]

\[
= \sum_{\lambda_j \leq \lambda} \left| \frac{\sinh(\lambda_j t)}{\lambda_j} \right|^2 (1 + \lambda_j^m)^{\frac{\alpha}{2m}} |a_j|^2
\]

\[
\leq \sum_{\lambda_j \leq \lambda} \left| \frac{\sinh(\lambda_j t)}{\lambda_j} \right|^2 \lambda_j^{2s} |a_j|^2 \leq \sum_{\lambda_j \leq \lambda} e^{\lambda_j t} \lambda_j^{2s-2} |a_j|^2
\]

\[
\lesssim s_0, s_0 \lesssim C \sum_{\lambda_j \leq \lambda} |a_j|^2
\]

\[
= e^{C_0 T} \| \partial_t F(\cdot, 0) \|_{L^2(G)}^2,
\]

for some $C > 1$. So, we deduce the inequality

\[
\forall s \in [0, s_0], \forall s_0 > n/2, \quad \|(1 + A)^{\frac{\alpha}{2}} F\|_{L^\infty} \lesssim s_0, s_0 e^{C_0 T/2} \| \partial_t F(\cdot, 0) \|_{L^2(G)},
\]

(3.44)

as well as the Sobolev inequality

\[
\forall s_0 > n/2, \forall s \in [0, s_0 + s_0], \quad \|(1 + A)^{\frac{\alpha}{2}} F\|_{L^2(G)} \lesssim s_0, s_0 e^{C_0 T/2} \| \partial_t F(\cdot, 0) \|_{L^2(G)}.
\]

(3.45)

With $s_0 = 0$, we have that $\|F\|_{L^\infty} \lesssim s_0, s_0 e^{T\lambda/2} \| \partial_t F(\cdot, 0) \|_{L^2(G)}$. Putting all these estimates together we have the inequality:

\[
I \lesssim \text{Vol}(G) \times \varepsilon \|\psi(t)\|_{L^\infty[T, T+\varepsilon]} e^{C_0 T/2} \| \partial_t F(\cdot, 0) \|_{L^2(G)} = \text{Vol}(G) \times \varepsilon \|\psi(t)\|_{L^\infty[T, T+\varepsilon]} e^{C_0 T/2}.
\]

(3.46)

In the last line we have used the identity $\| \partial_t F(\cdot, 0) \|_{L^2(G)} = \| \mathcal{X} \|_{L^2(G)} = 1$. Summarising, we have the inequality

\[
I \leq C' \text{Vol}(G) \times \varepsilon \|\psi(t)\|_{L^\infty[T, T+\varepsilon]} e^{C_0 T/2},
\]

for some $C' > 0$ independent of $\varepsilon > 0$.

3.4.8. Estimating $II$: To estimate the second term, we start by observing the inequality

\[
II = \|(-\partial_t^2 + A^{\frac{\alpha}{2}})^2 \psi\|_{L^\infty(G \times [T, T+\varepsilon])} = \|(-\partial_t^2 + A^{\frac{\alpha}{2}})^2 [\psi(t) F(x, T)]\|_{L^\infty(G \times [T, T+\varepsilon])}.
\]

Since

\[
(-\partial_t^2 + A^{\frac{\alpha}{2}})^2 [\psi(t) F(x, T)] = (-\partial_t^2 + A^{\frac{\alpha}{2}})(-\partial_t^2 + A^{\frac{\alpha}{2}}) [\psi(t) F(x, T)]
\]

\[
= (-\partial_t^2 + A^{\frac{\alpha}{2}}) [-\psi \partial_t F(x, T) + \psi(t) A^{\frac{\alpha}{2}} F(x, T)]
\]

\[
= \psi^{(4)}(t) F(x, T) - 2 \psi \partial_t F(x, T) A^{\frac{\alpha}{2}} F(x, T)
\]

\[
+ \psi(t) A^{\frac{\alpha}{2}} (F(x, T)),
\]

for $s_0 \geq 4$, and with $s_0 > n/2$, the Sobolev inequality in (3.44) implies that

\[
\|(-\partial_t^2 + A^{\frac{\alpha}{2}})^2 [\psi(t) F(x, T)]\|_{L^\infty(G \times [T, T+\varepsilon])}
\]

\[
\leq \psi^{(4)} \|_{L^\infty[T, T+\varepsilon]} \|F(x, T)\|_{L^\infty}
\]
Now dividing both sides of this inequality by \( \| \psi \|_{L^\infty[T,T+\varepsilon]} \), we get

\[
\| \psi(t) \|_{L^\infty[T,T+\varepsilon]} = \left\| \frac{\psi(t) F(x,t)}{\psi(0)} \right\|_{H^1(G \times (\alpha,T-\alpha))}
\]

3.4.9. Estimate for \( Z_1 \). In view of the estimates for I and II above, we have that

\[
\|(-\partial_t^2 + E(x,D)^{\frac{2}{3}}) \phi(x,t) \|_{L^2(G_{T+\varepsilon})} \lesssim \text{Vol}(G) \times \| \psi(t) \|_{L^\infty[T,T+\varepsilon]} e^{T\lambda} \left( \| \psi(t) \|_{L^\infty[T,T+\varepsilon]} + 2\| \psi(t) \|_{L^\infty[T,T+\varepsilon]} + \| \psi \|_{L^\infty[T,T+\varepsilon]} \right)^{\frac{1}{2}} + \| \psi(0) \|_{L^2(\omega)}^{1-\kappa}.
\]

Now, dividing both sides of this inequality by \(|\phi(0)|\) and using that \( \psi(0) = \psi(T) = \psi(t), \ 0 \leq t \leq T \), we get

\[
\| F(x,t) \|_{H^1(G \times (\alpha,T-\alpha))} = \left\| \frac{\psi(t) F(x,t)}{\psi(0)} \right\|_{H^1(G \times (\alpha,T-\alpha))}
\]

3.4.10. Final Analysis. The estimates above for \( Z_1 \) lead to the following inequality in view of the interpolation inequality (3.39)
and taking the limit when $\varepsilon \to 0^+$ in both sides of this estimate we conclude the expected inequality,
\[
\|F\|_{H^1(G \times (\alpha, T-\alpha))} \leq C_{s_0, s_{00}} \|F\|_{H^1(G_T)}^{\frac{1}{2}} \|\mathcal{K}\|_{L^2(\omega)}^{\frac{1}{2}},
\]  
(3.47)
where we have used that $\psi(t)/\psi(0) = \psi(t)/\psi(T) = 1$, $0 \leq t \leq T$, the smoothness of $\psi$, the following identities (see proposition 4.1)
\[
\lim_{\varepsilon \to 0^+} \|\psi^{(4)}\|_{L^2(T, T+\varepsilon)} = \lim_{\varepsilon \to 0^+} \|\psi\|_{L^2(T, T+\varepsilon)} = 0,
\]
and the following facts
\[
\lim_{\varepsilon \to 0^+} \|\psi(t)/\psi(0)\|_{L^\infty(T, T+\varepsilon)} = 1,
\]  
(3.48)
and
\[
\lim_{\varepsilon \to 0^+} \|\psi(t)F(x, t)/\psi(0)\|_{H^1(G_{T+\varepsilon})} = \|F(x, t)\|_{H^1(G_T)}.
\]  
(3.49)
For the proof of (3.49) note that for $\tilde{\psi} := \psi(t)/\psi(0)$, and using that $F(x, t) = F(x, T)$ if $0 \leq t \leq T + \varepsilon$, we have
\[
\lim_{\varepsilon \to 0^+} \|\psi(t)F(x, t)/\psi(0)\|_{H^1(G_{T+\varepsilon})} = \lim_{\varepsilon \to 0^+} \sum_{j=0,1} \int_0^{T+\varepsilon} \|\partial_x^{(j)}(\tilde{\psi}(t)F(x, t))\|_{H^1(G)}^2 dt
\]  
\[
= \lim_{\varepsilon \to 0^+} \sum_{j=0,1} \int_0^T \|\partial_x^{(j)}(\tilde{\psi}(t)F(x, t))\|_{H^1(G)}^2 dt + \lim_{\varepsilon \to 0^+} \sum_{j=0,1} \int_0^{T+\varepsilon} \|\partial_x^{(j)}(\tilde{\psi}(t)F(x, t))\|_{H^1(G)}^2 dt
\]  
\[
= \sum_{j=0,1} \int_0^T \|\partial_x^{(j)}(F(x, t))\|_{H^1(G)}^2 dt + \lim_{\varepsilon \to 0^+} \sum_{j=0,1} \int_0^T \|\tilde{\psi}^{(j)}(t)F(x, T)\|_{H^1(G)}^2 dt
\]  
\[
= \|F(x, t)\|_{H^1(G_T)}^2,
\]
where we have used that when $t \to T$, $\tilde{\psi}^{(j)}(t) \to 0$. Now, from (3.47) we can follow the standard Lebeau-Robbiano argument that has been described at the beginning of the section to conclude the proof of the spectral inequality. Having proved (3.4), the proof of Proposition 3.3 is complete.

**Proof of Theorem 1.1.** Let $\mu, c > 0$ be two positive parameters and assume that $0 < c < \mu^m$. Define
\[
\tilde{E}(x, D) = A + c.
\]
Note that $\tilde{E}(x, D) \geq cI$. Since $a(x, [\xi]) \geq 0$, the global symbol
\[
\tilde{E}(x, [\xi]) = a(x, [\xi]) + cI_{d_\xi},
\]
satisfies the positivity condition
\[
\forall [\xi] \in \hat{G}, \tilde{E}(x, [\xi]) \geq cI_{d_\xi}.
\]
On the other hand, if $\{\mu_j := \lambda_j^{m}, e_j\}$ are the corresponding spectral data
\[
A e_j = \lambda_j^m e_j, \lambda_j \geq 0,
\]

\[
\]
of $A$, then $\{\mu_j + c := \lambda_j^m + c, e_j\}$ are the corresponding spectral data

$$\tilde{E}(x, D)e_j = (\lambda_j^m + c)e_j, \; \lambda_j \geq 0$$

of the operator $\tilde{E}(x, D)$. Let $\lambda := (\mu^m + c)^\frac{m}{2}$. From Proposition 3.3 we deduce the spectral inequality

$$\left( \sum_{(\lambda_j^m + c)^\frac{m}{2} \leq \lambda} a_j^2 \right)^{\frac{1}{2}} \leq C_1 e^{C_2 \lambda} \left\| \sum_{(\lambda_j^m + c)^\frac{m}{2} \leq \lambda} a_j e_j(x) \right\|_{L^2(\omega)}.$$

(3.50)

Note that $(\lambda_j^m + c)^\frac{m}{2} \leq \lambda$ becomes equivalent to the inequality $\lambda_j \leq \mu$ and since $0 < c < \mu^m$, then $\lambda < 2\mu$. Thus, we have proved the spectral inequality

$$\left( \sum_{\lambda_j \leq \mu} a_j^2 \right)^{\frac{1}{2}} \leq C_1 e^{2C_2 \mu} \left\| \sum_{\lambda_j \leq \mu} a_j e_j(x) \right\|_{L^2(\omega)}.$$

(3.51)

In consequence the proof of (1.9) is complete. For the proof of (1.10) we can use (1.9) and the Sobolev embedding theorem. Indeed, let $R > 0$ and let us consider $s \in \mathbb{R}$ such that $s > n/2$. For the proof of (1.10) we can use (1.9) and the Sobolev embedding theorem. Indeed, let $R > 0$ and let us consider $s \in \mathbb{R}$ such that $s > n/2$. With $\omega = B(x, R)$ a ball of radius $R > 0$ we have that

$$\|x\|_{L^\infty(B(x,2R))} \leq \|x\|_{L^\infty(G)}.$$

(3.52)

Now, the Sobolev embedding theorem and the inequality in (1.9) imply that

$$\|x\|_{L^\infty(G)} \lesssim \|(1 + A)^{\frac{m}{2}} x\|_{L^2(G)} \lesssim (1 + \lambda)^s \|x\|_{L^2(G)}$$

$$\lesssim (1 + \lambda)^s C_{1,R} e^{C_{2,R} \lambda} \|x\|_{L^2(B(x,R))}.$$

By using (3.52) and (1.9) we conclude this analysis with the inequality

$$\|x\|_{L^\infty(B(x,2R))} \lesssim \|x\|_{L^\infty(G)} \leq e^{C_{2,R} + C_{1,R} \lambda} \|x\|_{L^2(B(x,2R))} \leq e^{C_{2,R}^* + C_{1,R} \lambda} \|x\|_{L^\infty(B(x,2R))},$$

for some $C_{1,R}^* > C_{1,R}$ and $C_{2,R}^* > C_{2,R}$. The proof of Theorem 1.1 is complete. □

3.5. Applications to control theory: Null-controllability for diffusion models. Now, we give a consequence of Theorem 1.1 which we present in the following way.

**Theorem 3.10.** Let $A$ be a positive and elliptic pseudo-differential operator of order $m > 0$ in the Hörmander class $\Psi_{\alpha,0}(G \times \hat{G})$ and let $u_0 \in L^2(G)$ be an initial datum.

Then, for any $\alpha > 1/m$, the fractional diffusion model

$$\begin{cases}
  u_t(x, t) + A^\alpha u(x, t) = g(x, t) \cdot 1_\omega(x), & (x, t) \in G \times (0, T), \\
  u(0, x) = u_0,
\end{cases}$$

(3.53)

is null-controllable at any time $T > 0$, that is, there exists an input function $g = g(x, t) \in L^2(G)$ such that for any $x \in G$, $u(x, T) = 0$. 

The spectral inequality in (3.4) allows us to make use of Theorem 2.39 with $A = A^\alpha$ and with $B = S = M_{1_{\omega}}$ being the multiplication operator by the characteristic function $1_{\omega}$. Note that $M_{1_{\omega}}$ is bounded on $H = L^2(G)$. Observe that $A^\gamma = A^{\alpha\gamma} = A^{\frac{1}{m}}$ satisfies (3.4) (that is, the inequality (2.39) holds) for $\alpha\gamma = 1/m$. Because $\gamma \in (0, 1)$ if and only if $\alpha > 1/m$, Theorem 2.39 guarantees that this inequality on the fractional order $\alpha$ is a sufficient condition in order that (3.53) will be null-controllable in time $T > 0$. The proof of Theorem 3.10 is complete. □

In the following result we analyse the controllability cost of the model (3.53) when the time is small.

**Corollary 3.11.** The controllability cost $C_T$ for the fractional heat equation (3.53) over short times $T \in (0, 1)$ satisfies

$$C_T \leq C_1 e^{C_2 T - \beta}, \quad (3.54)$$

where $\beta > 1/(\alpha m - 1)$.

**Proof.** For the proof, note that $A^\gamma = A^{\alpha\gamma} = A^{\frac{1}{m}}$, satisfies (3.4) for $\alpha\gamma = 1/m$. Then, from Theorem 2.39 we have the estimate $C_T \leq C_1 e^{C_2 T - \beta}$ for any $\beta > \gamma/(\gamma - 1) = 1/(\alpha m - 1)$. The proof of Corollary 3.11 is complete. □

4. **Appendix: Construction of the cut-off function $\psi$**

In this appendix we construct the regularising function $\psi$ used in the proof of Proposition 3.3. For any $\varepsilon \in (0, 1)$, let $a := 3\varepsilon/4$. We summarise the analysis above and some their straightforward consequences in the following lemma.

**Lemma 4.1.** The function $\psi$ as defined in (4.5) satisfies the following properties.

A. $0 < \psi(0) < \varepsilon$.

B. $\psi^{(i)}(T) = 0$, for $i \in \{1, 2, 3, 4\}$.

C. For $i \in \{1, 2, 3, 4\}$, $\psi^{(i)} \in C^\infty(0, T + \varepsilon)$, and there is a constant $M_0 > 0$, independent of $\varepsilon \in (0, 1)$, such that

$$\|\psi^{(i)}\|_{L^\infty} \leq M_0,$$

for all $i = 1, 2, 3, 4$.

**Proof.** We do this by the following steps.

Step 1. Define the function

$$E(t) = \begin{cases} 
   e^{-\frac{1}{a^2 - t^2}}(a^2 - t^2)^{10}, & t \in [0, a], \\
   0, & t \in [a, \varepsilon].
\end{cases} \quad (4.2)$$

Step 2. By straightforward computation one can show that for any $t \in [0, a]$,

1. $E_t(t) = 2t \exp(-1/(a^2 - t^2))((a^2 - t^2)^8(-10a^2 + 10t^2 - 1))$.

2. $E_{tt}(t) = -2 \exp(-1/(a^2 - t^2))(a^2 - t^2)^6(10a^6 + a^4(1 - 210t^2) + a^2(390t^4 - 38t^2) - 190t^6 + 37t^4 - 2t^2)$.
The analysis above shows that the function 
\[ s \text{ satisfies the required properties of the lemma.} \]

References
1. M. F. Atiyah, H. Donnelly, I. M. Singer, Geometry and analysis of Shimizu L-functions. Proc. Nat. Acad. Sci. U.S.A. 79(18), 5751, (1982).
2. A. Borel, H. Garland. Laplacian and the discrete spectrum of an arithmetic group. Amer. J. Math. 105(2), 309–335, (1983).
3. J. Apraiz, L. Escauriaza, G. Wang, C. Zhang. Observability inequalities and measurable sets. J. Eur. Math. Soc. 16(11), 2433–2475, (2014).
4. A. Benabdallah and M. G. Naso, Null controllability of a thermoelastic plate, Abstr. Appl. Anal. 7 (2002), 585–599.
5. U. Biccari, V. Hernández-Santamaría. Controllability of a one-dimensional fractional heat equation: theoretical and numerical aspects. IMA J. Math. Control Inf. 36(4), 1199-1235, (2019).
6. M. D. Blair, C. D. Sogge, On Kakeya-Nikodym averages, Lp-norms and lower bounds for nodal sets of eigenfunctions in higher dimensions. J. Eur. Math. Soc. 17(10), 2513–2543, (2015).
7. D. Cardona. Spectral inequalities for elliptic pseudo-differential operators on closed manifolds, preprint.
8. D. Cardona, M. Ruzhansky. Subelliptic pseudo-differential operators and Fourier integral operators on compact Lie groups. arXiv:2008.09651.
9. F. Cavalletti, S. Farinelli. Indeterminacy estimates and the size of nodal sets in singular spaces. Adv. Math. 389, Paper No. 107919, 38 pp. (2021).
10. L. Boutet de Monvel. Boundary problems for pseudo-differential operators. Acta Math., 126(1-2), 11–51, (1971).
11. A. P. Calderon, R. Vaillancourt. On the boundedness of pseudo-differential operators. J. Math. Soc. Jpn. 23(2), 374–378, (1971).
12. J. Delgado, M. Ruzhansky. Lp-bounds for pseudo-differential operators on compact Lie groups, J. Inst. Math. Jussieu, 18(3), 531–559, (2019).
13. H. Donnelly, C. Fefferman. $L^2$ cohomology of the Bergman metric. Proc. Nat. Acad. Sci. U.S.A. 80(10-i), 3136–3137, (1983).
14. H. Donnelly and C. Fefferman, Nodal sets of eigenfunctions on Riemannian manifolds, Invent. Math. 93, 161–183, (1988).
15. H. Donnelly, C. Fefferman. Nodal sets for eigenfunctions of the Laplacian on surfaces. J. Amer. Math. Soc. 3(2), 333–353, (1990).
16. H. Donnelly, C. Fefferman. Nodal domains and growth of harmonic functions on noncompact manifolds. J. Geom. Anal. 2(1), 79–93, (1992).
17. H. Donnelly, N. Garofalo. Riemannian manifolds whose Laplacians have purely continuous spectrum. Math. Ann. 293(1), 143–161, (1992).
18. H. Donnelly, N. Garofalo, Schrödinger operators on manifolds, essential self-adjointness, and absence of eigenvalues. J. Geom. Anal. 7(2), 241–257, (1997).
19. S. Donaldson. The geometry of 4-manifolds. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 43–54, Amer. Math. Soc., Providence, RI, 1987.
20. J. J. Duistermaat, J. A. C. Kolk. Lie groups. Universitext, Springer, Berlin Heidelberg, 2000.
21. A. Enciso, D. Peralta-Salas. Eigenfunctions with prescribed nodal sets. J. Differential Geom. 101(2), 197–211, (2015).
22. C. Fefferman, D. H. Phong. On positivity of pseudo-differential operators. Proc. Nat. Acad. Sci. U.S.A. 75(10), 4673–4674, (1978).
23. C. Fefferman, D. H. Phong. Symplectic geometry and positivity of pseudodifferential operators. Proc. Nat. Acad. Sci. U.S.A. 79(2), 710–713, (1982).
24. X. Fu, Q. Lü, X. Zhang. Carleman Estimates for Second Order Elliptic Operators and Applications, a Unified Approach. In: Carleman Estimates for Second Order Partial Differential Operators and Applications. Springer Briefs in Mathematics. Springer, Cham. (2019).
25. B. Georgiev. On the lower bound of the inner radius of nodal domains. J. Geom. Anal. 29(2), 1546–1554, (2019).
26. L. Hormander, (1963). Linear partial differential operators. Berlin: Springer.
27. L. Hörmander. The analysis of the linear partial differential operators, Vol. III-IV. Springer-Verlag, (1985).
28. D. Jerison, G. Lebeau. Nodal sets of sums of eigenfunctions. Harmonic analysis and partial differential equations (Chicago, IL, 1996), Chicago Lectures in Math, 223–239, (1999).
29. C. Kenig, J. Zhu, J. Zhuge. Doubling inequalities and nodal sets in periodic elliptic homogenization. Comm. Partial Differential Equations, 47(3), 549–584, (2022).
30. A. Logunov. Nodal sets of Laplace eigenfunctions: proof of Nadirashvili’s conjecture and of the lower bound in Yau’s conjecture. Ann. of Math. 1. 187(1), 241–202, (2018).
31. A. Logunov. Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure. Ann. of Math. 2. 187(1), 221–239, (2018).
32. A. Logunov, E. Malinnikova, N. Nadirashvili, F. Nazarov. The sharp upper bound for the area of the nodal sets of Dirichlet Laplace eigenfunctions. Geom. Funct. Anal. 31(5), 1219–1244, (2021).

33. M. Léautaud, Spectral inequalities for non-selfadjoint elliptic operators and application to the null-controllability of parabolic systems, J. Funct. Anal. 258, 2739–2778, (2010).

34. G. Lebeau, L. Robbiano, Contrôle exact de l’équation de la chaleur, Comm. Partial Diff. Equations., 20, 335–356, (1995).

35. G. Lebeau, E. Zuazua. Null-Controllability of a System of Linear Thermoelasticity. Arch. Rational Mech. Anal. 141(4), 297–329, (1998).

36. F. H. Lin, Nodal sets of solutions of elliptic and parabolic equations, Comm. Pure Appl. Math. 44, 287–308, (1991).

37. J. L. Lions. Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués: Perturbations. Tome I. Collection RMA, Masson, 1988.

38. Q. Lu A lower bound on local energy of partial sum of eigenfunctions for Laplace–Beltrami operators. ESAIM Control Optim. Calc. Var. 19, 255–273, (2013).

39. S. Micu and E. Zuazua. On the controllability of a fractional order parabolic equation. SIAM J. Control Optim., 44(6), 1950–1972, (2006).

40. L. Miller. On the controllability of anomalous diffusions generated by the fractional Laplacian. Math. Control Signals Systems 18(3), 260–271, (2006).

41. L. Miller, On the cost of fast controls for thermoelastic plates, Asymptot. Anal. 51, 93–100, (2007).

42. J. Le Rousseau, G. Lebeau. On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. Esaim: Cocv 18, 712–747, (2012).

43. J. Le Rousseau, L. Robbiano. Spectral inequality and resolvent estimate for the bi-Laplace operator. J. Eur. Math. Soc., 22, 1003–1094, (2020).

44. M. Ruzhansky, V. Turunen. Pseudo-differential Operators and Symmetries: Background Analysis and Advanced Topics Birkhäuser-Verlag, Basel, 2010.

45. M. Ruzhansky, V. Turunen, J. Wirth. Hörmander class of pseudo-differential operators on compact Lie groups and global hypoellipticity, J. Fourier Anal. Appl. 20 (2014), 476–499.

46. M. Ruzhansky, J. Wirth. Global functional calculus for on compact Lie groups, J. Funct. Anal. 267(1), 144–172, (2014).

47. M. A. Shubin. Pseudodifferential operators and spectral theory. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1987. Translated from the Russian by Stig I. Andersson.

48. T. Sunada. Trace formula and heat equation asymptotics for a nonpositively curved manifold. Amer. J. Math. 104(4), 795–812, (1982).

49. M. Taylor. Pseudodifferential Operators, Princeton Univ. Press, Princeton, N.J., (1981).

50. L. Tian, X. Yang. Measure upper bounds for nodal sets of eigenfunctions of the bi-harmonic operator. J. Lond. Math. Soc. (2) 105(3), 1936–1973, (2022).

51. Toth, John A.; Zelditch, Steve Nodal intersections and geometric control. J. Differential Geom. 117, no. 2, 345–393, (2021).

DUVÁN CARDONA:
DEPARTMENT OF MATHEMATICS: ANALYSIS, LOGIC AND DISCRETE MATHEMATICS
GHENT UNIVERSITY, BELGIUM
E-mail address duvanc306@gmail.com, duvan.cardonasanchez@ugent.be

JULIO DELGADO:
DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DEL VALLE
CALI-COLOMBIA
E-mail address delgado.julio@correounivalle.edu.co

MICHAEL RUZHANSKY:
