Hamiltonian cycles in 7-tough \((P_3 \cup 2P_1)\)-free graphs

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Abstract

The toughness of a noncomplete graph \(G\) is the maximum real number \(t\) such that the ratio of \(|S|\) to the number of components of \(G - S\) is at least \(t\) for every cutset \(S\) of \(G\), and the toughness of a complete graph is defined to be \(\infty\). Determining the toughness for a given graph is NP-hard. Chvátal’s toughness conjecture, stating that there exists a constant \(t_0\) such that every graph with toughness at least \(t_0\) is hamiltonian, is still open for general graphs. A graph is called \((P_3 \cup 2P_1)\)-free if it does not contain any induced subgraph isomorphic to \(P_3 \cup 2P_1\), the disjoint union of \(P_3\) and two isolated vertices. In this paper, we confirm Chvátal’s toughness conjecture for \((P_3 \cup 2P_1)\)-free graphs by showing that every 7-tough \((P_3 \cup 2P_1)\)-free graph on at least three vertices is hamiltonian.

Keywords: Toughness; Hamiltonian cycle; \((P_3 \cup 2P_1)\)-free graph

1 Introduction

All graphs considered in this paper are undirected and simple. Let \(G\) be a graph. The vertex set and edge set of \(G\) are denoted by \(V(G)\) and \(E(G)\), respectively. For a vertex \(v \in V(G)\), the set of neighbors of \(v\) in \(G\) is denoted by \(N_G(v)\). Let \(H \subseteq G\) be a subgraph of \(G\), \(x \in V(G)\) and \(S \subseteq V(G)\). Define \(N_H(x) = N_G(x) \cap V(H)\), \(N_G(S) = \cup_{x \in S} N_G(x) \setminus S\) and \(N_H(S) = N_G(S) \cap V(H)\). We use \(G[S]\) to denote the subgraph of \(G\) induced by \(S\) and \(G - S\) to denote the subgraph \(G[V(G) \setminus S]\). For simplicity, \(G - \{x\}\) is written as \(G - x\). If \(uv \in E(G)\) is an edge, then we write \(u \sim v\) and \(u \nabla v\) otherwise. Let \(V_1, V_2 \subseteq V(G)\) be two disjoint vertex sets. Then the set of edges with one end in \(V_1\) and the other in \(V_2\) is denoted by \(E_G(V_1, V_2)\). The subscript \(G\) will be omitted in all the notation above if no confusion may arise.

A \textit{hamiltonian path} (resp. \textit{cycle}) in a graph \(G\) is a path (resp. cycle) which contains all the vertices of \(V(G)\). A graph is called \textit{hamiltonian connected} if there is a hamiltonian path between any two distinct vertices. Let \(c(G)\) denote the number of components in a graph \(G\). Chvátal \cite{9} defined a noncomplete graph \(G\) to be \(t\)-\textit{tough} if \(|S| \geq t \cdot c(G - S)\) for every
subset $S \subseteq V(G)$ with $c(G - S) \geq 2$, i.e., $S$ is a cutset of $G$. The toughness $\tau(G)$ of $G$ is the maximum real number $t$ for which $G$ is $t$-tough or is $\infty$ if $G$ is complete. A cutset $S$ is called a tough set in $G$ if $|S| = \tau(G) \cdot c(G - S)$. Let $S$ be a cutset of $G$. A component of $G - S$ is said to be trivial if it contains only one vertex, otherwise we call it nontrivial. Since every cycle is 1-tough, every hamiltonian graph is 1-tough. However, as noted by Chvátal, the converse only holds for graphs with at most six vertices. Figure 1 given by Chvátal is an example of 1-tough nonhamiltonian graph with seven vertices. Chvátal conjectured that large enough toughness in a graph $G$ would guarantee that $G$ is hamiltonian. In [9], he proposed the following conjecture.

**Conjecture 1.1.** (Chvátal’s toughness conjecture) There exists a constant $t_0$ such that every $t_0$-tough graph on at least three vertices is hamiltonian.

![Figure 1: 1-tough nonhamiltonian graph with seven vertices](image)

In 2000, Bauer, Broersma and Veldman [4] showed that for every $\epsilon > 0$, there exists a $\left(\frac{9}{4} - \epsilon\right)$-tough graph with no hamiltonian path. It follows that if Chvátal’s toughness conjecture is true, then $t_0 \geq \frac{9}{4}$. Chvátal’s toughness conjecture is still open. However, it is true for a number of well-studied classes of graphs including planar graphs [21], interval graphs [14], claw-free chordal graphs [2], planar chordal graphs [5], cocomparability graphs [10], split graphs [15], spider graphs [13], chordal graphs [8] [12], $k$-trees ($k \geq 2$) [7], $R$-free graphs for $R \in \{P_4, K_1 \cup P_3, 2K_1 \cup K_2\}$ [16], $2K_2$-free graphs [6] [19] [17], $(P_2 \cup P_3)$-free graphs [20], and $(P_2 \cup 3P_1)$-free graphs [11].

Let $P_n$ denote a path with $n$ vertices. A graph is $(P_3 \cup 2P_1)$-free if it does not contain any induced subgraph isomorphic to $P_3 \cup 2P_1$, the disjoint union of $P_3$ and two isolated vertices. In this paper, we study the hamiltonicity of $(P_3 \cup 2P_1)$-free graphs under a given toughness condition and obtain the following result.

**Theorem 1.2.** Every 7-tough $(P_3 \cup 2P_1)$-free graph on at least three vertices is hamiltonian.

The remainder of this paper is organized as below. In next section, we introduce some notation and preliminaries. In Section 3, we prove Theorem 1.2.
2 Preliminaries

We start this section with some definitions and notation. For two integers \( p \) and \( q \), we let \([p, q] = \{i \in \mathbb{Z} : p \leq i \leq q\}\). A star-matching in a graph is a set of vertex-disjoint copies of stars. The vertices of degrees at least 2 in a star-matching are called the centers of the star-matching. In particular, if every star in a star-matching is isomorphic to \( K_{1, t} \), where \( t \geq 1 \) is an integer, we call the star-matching a \( K_{1, t} \)-matching. For a star-matching \( M \), we denote by \( V(M) \) the set of vertices covered by \( M \). And if \( x, y \in V(M) \) and \( xy \in E(M) \), we say \( x \) is a partner of \( y \).

Let \( C \) be an oriented cycle. For \( x \in V(C) \), denote the immediate successor of \( x \) on \( C \) by \( x^+ \). For \( u, v \in V(C) \), \( u \overset{C}{\rightarrow} v \) denotes the segment of \( C \) starting with \( u \), following \( C \) in the orientation, and ending at \( v \). Likewise, \( u \overset{C}{\leftarrow} v \) is the opposite segment of \( C \) with ends \( u \) and \( v \).

We assume all cycles in consideration afterwards are oriented. Let \( P \) be a path, \( u, v \in V(P) \) be two vertices in \( P \), the segment of \( P \) with ends \( u \) and \( v \) is denoted by \( uPv \). Let \( uPv \) and \( xQy \) be two disjoint paths. If \( v \) is adjacent to \( x \), we write \( uPvxQy \) as the concatenation of \( P \) and \( Q \) through the edge \( vx \).

We will need the following Lemmas in our proof.

**Lemma 2.1.** ([1], Theorem 2.10) Let \( G \) be a bipartite graph with partite sets \( X \) and \( Y \), and \( f \) be a function from \( X \) to the set of positive integers. If for every \( S \subseteq X \), \( |N_G(S)| \geq \sum_{v \in S} f(v) \), then \( G \) has a subgraph \( H \) such that \( X \subseteq V(H) \), \( d_H(v) = f(v) \) for every \( v \in X \), and \( d_H(u) = 1 \) for every \( u \in Y \cap V(H) \).

**Lemma 2.2.** ([3]) Let \( t > 0 \) be a real number and \( G \) be a \( t \)-tough graph on \( n \geq 3 \) vertices with \( \delta(G) > \frac{n}{t+1} - 1 \). Then \( G \) is hamiltonian.

**Lemma 2.3.** ([18]) Let \( t > 0 \) be a real number and \( G \) be a \( t \)-tough graph on \( n \geq 3 \) vertices. If the degree sum of any two nonadjacent vertices of \( G \) is greater than \( \frac{2n}{t+1} + t - 2 \), then \( G \) is hamiltonian.

**Lemma 2.4.** ([18]) Let \( t \geq 1 \) be a real number, \( G \) be a \( t \)-tough graph on \( n \geq 3 \) vertices and \( C \) be a nonhamiltonian cycle of \( G \). If \( x \in V(G) \setminus V(C) \) satisfies that \( d_C(x) > \frac{n}{t+1} - 1 \), then \( G \) has a cycle \( C' \) such that \( V(C') = V(C) \cup \{x\} \).

**Lemma 2.5.** Let \( G \) be a more than 1-tough \((P_3 \cup P_1)\)-free graph on \( n \geq 3 \) vertices. Then \( G \) is hamiltonian connected.

**Proof.** Let \( u, v \in V(G) \) be any two distinct vertices. We find in \( G \) a longest \( uv \)-path \( P \). We may assume that \( P \) is not a hamiltonian path of \( G \). Let \( D \) be a component of \( G - V(P) \),
\[ W = N_P(V(D)) = \{x_0, x_1, \ldots, x_\ell\}. \] Then \( \ell \geq 2 \) by the toughness of \( G \). Assume for each \( i \in [1, \ell], x_{i-1} \) is in between \( u \) and \( x_i \) on \( P \), where note \( x_0 \) could be the same as \( u \) and \( x_\ell \) could be the same as \( v \). Let \( h = \ell \) if \( x_\ell \neq v \), and \( h = \ell - 1 \) if \( x_\ell = v \), and let \( W^+ = \{ x_i^+ : i \in [0, h] \} \) be the set of the immediate successors of vertices \( x_i \) on \( P \) along the direction from \( u \) to \( v \).

For any \( i \in [0, \ell - 1] \), we claim that \( x_i x_{i+1} \notin E(P) \). For otherwise, let \( x_i', x_{i+1}' \in V(D) \) such that \( x_i' x_i, x_{i+1}' x_{i+1} \in E(G) \), and let \( P' \) be an \( x_i' x_{i+1}' \)-path in \( D \). Then \( uPx_ix_i'P'x_{i+1}'x_{i+1}Pv \) is a longer \( uv \)-path than \( P \).

For any two distinct \( i, j \in [0, h] \), we claim that \( x_i^+ x_j^+ \notin E(G) \). For otherwise, let \( x_i', x_j' \in V(D) \) such that \( x_i' x_i, x_j' x_j \in E(G) \), and let \( P' \) be an \( x_i' x_j' \)-path in \( D \). Assume, by symmetry that \( i < j \). Then \( uPx_ix_i'P'x_j'x_jPx_i^+x_j^+Pv \) is a longer \( uv \)-path than \( P \).

For each \( i \in [0, \ell - 1] \), let \( L_i = x_ip_{x_{i+1}} - \{x_i, x_{i+1}\} \). Since \( x_i x_{i+1} \notin E(P) \), each \( L_i \) is a path with at least one vertex. Since \( G \) is \((P_3 \cup P_1)\)-free, each component of \( G - W - V(D) \) is a complete graph. For any two distinct \( i, j \in [0, h] \), as \( x_i^+ x_j^+ \notin E(G) \), \( L_i \) and \( L_j \) are contained in distinct components of \( G - W \). Those components containing \( L_i \)'s together with \( D \) gives \( c(G - W) \geq \ell + 1 \geq 3 \). Thus \( \tau(G) \leq \frac{|W|}{c(G - W)} \leq 1 \), giving a contradiction to \( \tau(G) > 1 \).

Let \( G \) be a graph, \( S \) be a cutset of \( G \), and \( D_1, D_2, \ldots, D_\ell \) be all the components of \( G - S \). For any \( i, j \in [1, \ell], \) if there exists \( S_i \subseteq N_G(V(D_i)) \cap S \) such that (i) \( |S_i| = 2s \) for some integer \( s \geq 1 \), (ii) if \( |V(D_i)| \geq 2 \), then \( S_i \) can be partitioned into \( S_{i1} \) and \( S_{i2} \) and \( N_{D_i}(S_i) \) can be partitioned into \( W_{i1} \) and \( W_{i2} \) with \( |S_{i1}| = |S_{i2}| = s \) such that \( S_{i1} \subseteq N_G(W_{i1}) \cap S \) and \( S_{i2} \subseteq N_G(W_{i2}) \cap S \); and (iii) \( S_i \cap S_j = \emptyset \) if \( i \neq j \), then we say \( G \) has a generalized \( K_{1,2s^*} \)-matching with centers as components of \( G - S \), and call vertices in \( S \) the partners of \( D_i \) from \( S \).

**Lemma 2.6.** Let \( G \) be a \( t \)-tough graph on \( n \) vertices for some \( t \geq 2 \), \( S \) be a cutset in \( G \), and let \( s = \lceil t/2 \rceil \). Then \( G \) has a generalized \( K_{1,2s^*} \)-matching with centers as components of \( G - S \).

**Proof.** Let \( D_1, D_2, \ldots, D_\ell \) be all the components of \( G - S \). For each \( D_i \), as \( \tau(G) \geq 2 \), \( |N_G(V(D_i)) \cap S| \geq 2t \). Furthermore, if \( |V(D_i)| \geq 2 \), then \( |N_{D_i}(S)| \geq 2 \). Otherwise, if \( |N_{D_i}(S)| = 1 \), then the neighbor of \( S \) in \( D_i \) would be a cutvertex in \( G \).

We will construct a bipartite graph \( H \) in the following steps. For each \( D_i \) with \( |V(D_i)| \geq 2 \), let \( N_{D_i}(S) = W_i \). Let \( W_i^1 \cup W_i^2 \) be a partition of \( W_i \) such that both of them are nonempty. We contract all vertices in \( W_i^1 \) into a single vertex \( u_i \) and all vertices in \( W_i^2 \) into a single vertex \( v_i \). Then \( G[D_i] \) is changed into an edge \( u_iv_i \). The edges between \( D_i \) and \( S \) are now
between \(u_iv_i\) and \(S\). For each \(D_i\) with \(|V(D_i)| = 1\), let \(V(D_i) = \{w_i\}\). Split \(w_i\) into two vertices \(u_i\) and \(v_i\), and distribute the edges in \(G\) incident with \(w_i\) between \(u_i\) and \(v_i\).

Let \(T = \{u_i, v_i : 1 \leq i \leq \ell\}\) be the collection of vertices corresponding to \(D_i's\) and let \(H = H[S, T]\) be a bipartite graph with vertex set \(V(H) = S \cup T\) and edge set \(E(H) = E(S, \cup_{i=1}^{\ell} V(D_i))\). To complete the proof of Lemma 2.6 it is sufficient to prove that \(H\) has a \(K_{1, 2s}\)-matching saturating \(T\). Suppose not, by Lemma 2.1 there exists a nonempty subset \(T^* \subseteq T\) such that

\[
|N_H(T^*)| < s|T^*|.
\]

(1)

As \(T^*\) corresponds to at least \(|T^*|/2\) components of \(G - S\), (1) implies that

\[
c(G - N_H(T^*)) \geq \frac{|T^*|}{2} = \frac{s|T^*|}{2s} > \frac{1}{2s}|N_H(T^*)|,
\]

giving that \(\tau(G) < 2s \leq t\), a contradiction to the toughness of \(G\). This completes the proof of Lemma 2.6. \(\square\)

3 Proof of Theorem 1.2

Proof. We may assume \(G\) is not a complete graph. Since \(G\) is 7-tough and noncomplete, \(G\) is 14-connected and \(\delta(G) \geq 14\). By Lemma 2.2, we may further assume \(\delta(G) \leq \frac{n}{8} - 1\). It follows that

\[
n \geq 8\delta(G) + 8 \geq 120.
\]

(2)

Furthermore, by Lemma 2.3 we may assume that there exist two nonadjacent vertices in \(G\) with degree sum at most \(\frac{n}{4} + 5\). Let \(u, v \in V(G)\) be two nonadjacent vertices such that \(d(u) + d(v) \leq \frac{n}{4} + 5\). Let

\[
S_{uv} = N(u) \cup N(v).
\]

Then \(|V(G) \setminus (S_{uv} \cup \{u, v\})| \geq n - \frac{n}{4} - 5 - 2 \geq 1\). It follows that \(c(G - S_{uv}) \geq 3\). The following claim is obvious by the \((P_3 \cup 2P_1)\)-freeness of \(G\).

Claim 1. Each component of \(G - S_{uv}\) is a complete graph.

Let

\[
S_u = \{x \in S_{uv} : N_G(x) \cap (V(G) \setminus S_{uv}) = \{u\}\},
\]

\[
S_v = \{x \in S_{uv} \setminus S_u : N_G(x) \cap (V(G) \setminus (S_{uv} \setminus S_u)) = \{v\}\},
\]

\[
S = S_{uv} \setminus (S_u \cup S_v).
\]

In such a construction, \(u\) and \(v\) belong to distinct components of \(G - S\). Then we have the following claim by the construction of \(S\) and \((P_3 \cup 2P_1)\)-freeness of \(G\).
Claim 2. (i) \( c(G - S) = c(G - S_{uv}) \geq 3 \) and each component of \( G - S \) is a complete graph;

(ii) for any vertex \( x \in S, x \) is adjacent to vertices from at least two components of \( G - S \);

(iii) if \( c(G - S) \geq 4 \), then for each vertex \( x \in S, x \) is not adjacent to all vertices of at most one component of \( G - S \).

Case 1 For any component \( D \) of \( G - S_{uv}, |V(G) \setminus (S_{uv} \cup V(D))| > \frac{n}{8} - 1 \).

Since \( G - S_{uv} \) has at least one component \( D^* \) which does not contain \( u \) or \( v \), by the assumption of Case 1, we know that \( G - S_{uv} \) has at least one component other than \( D^* \) not containing \( u \) or \( v \). It follows that \( c(G - S_{uv}) \geq 4 \). Then \( c(G - S) \geq 4 \). Furthermore, by Claim 2(iii), the following claim holds.

Claim 3. For each vertex \( x \in S \), we have \( d_{G-S}(x) > \frac{n}{8} - 1 \).

In the following, we will construct a cycle that covers all the vertices of \( G - S \) firstly, then insert the remaining vertices of \( S \) into the cycle by applying Lemma 2.4 repeatedly. Since \( G \) is 7-tough and so is 2-tough, applying Lemma 2.6 with \( t = 2 \), there exist distinct vertices \( x_1, y_1, x_2, y_2, \ldots, x_t, y_t \) in \( S \) such that \( x_i, y_i \) are partners of \( D_i \). As each \( D_i \) is a complete graph, there exists a hamiltonian path \( P_i \) in \( D_i \) such that the two ends \( a_i, b_i \) of \( P_i \) are adjacent to \( x_i \) and \( y_i \), respectively. Note that \( a_i = b_i \) if \( |V(D_i)| = 1 \). Since \( y_1 \) is not adjacent to at most one component of \( G - S \) by Claim 2(iii), without loss of generality, we may assume that \( y_1 \sim a_2 \). Recursively we can assume that \( y_i \sim a_{i+1} \) for each \( i \in [1, t - 2] \). If both of \( y_{t-1} \sim a_t \) and \( x_1 \sim b_t \) hold, then we get a cycle \( C = x_1 a_1 P_1 b_1 y_1 a_2 P_2 b_2 y_2 \ldots y_{t-1} a_t P_t b_t x_1 \) that contains all vertices of \( \bigcup_{i=1}^{t} V(D_i) \).

Thus, \( y_{t-1} \sim a_t \) or \( x_1 \sim b_t \). By Claim 2(iii), we have that \( y_{t-1} \sim a_1 \) or \( x_1 \sim b_{t-1} \). In either case, we can find a cycle \( C' \) such that \( \bigcup_{i=1}^{t-1} V(D_i) \subseteq V(C') \). We will extend \( C' \) to a larger cycle \( C \) containing also vertices of \( D_t \) in the following and give a claim first.

Claim 4. (i) For any two vertices \( u \in N_{C'}(x_i) \) and \( w \in N_{C'}(y_t), uw \notin E(C') \).

(ii) \( x_i \) and \( y_t \) can not be adjacent to all vertices of any nontrivial component of \( G - S \).

As a consequence, \( |V(D_i)| = 1 \) for each \( i \in [3, t] \).

Proof. (i) Otherwise \( C = (C' \setminus \{uw\}) \cup \{ux_i, y_tw\} \cup P_t \) is a desired cycle containing all vertices of \( G - S \).

(ii) Let \( D \) be a nontrivial component of \( G - S \) such that both of \( x_i \) and \( y_t \) are adjacent to all vertices of \( D \). By the construction of \( C' \) and \( D \) being a complete graph, we know that there exist two adjacent vertices \( u, w \in V(C') \cap V(D) \) such that \( uw \in E(C') \), contradicting (i). So
for each component $D$ of $G - S$ such that $x_t$ and $y_t$ are both adjacent to all vertices of $D$, it holds that $|V(D)| = 1$. By Claim [2]iii, for any vertex $x \in S$, each of $x_t$ and $y_t$ is not adjacent to all vertices of at most one component of $G - S$. Since $|V(D_1)| \geq |V(D_2)| \geq \ldots \geq |V(D_t)|$, it follows that $|V(D_i)| = 1$ for each $i \in [3, t]$.

As $G$ is 7-tough and $|S| \leq \frac{n}{4} + 5$, it follows that

$$|V(D_1)| + |V(D_2)| \geq n - |S| - \left(\frac{|S|}{7} - 2\right) = \frac{5n}{7} - 4 > 81$$

by (2). By the assumption of Case 1, we know that

$$|V(D_2)| > \frac{n}{8} - 1 \geq \frac{|S|}{7} - 2 > \frac{5n}{56} > 10$$

also by (2).

Thus both $D_1$ and $D_2$ are nontrivial components of $G - S$. By Claim [4]ii, we assume, without loss of generality, that $x_t$ is adjacent to all vertices of $D_1$ and $y_t$ is adjacent to all vertices of $D_2$. If $y_{t-1} \sim a_1$, then

$$C = x_t b_1 P_1 a_1 y_{t-1} a_{t-1} y_{t-2} a_{t-2} \ldots y_2 a_3 y_2 b_2 P_2 a_2 y_t a_t x_t$$

is a cycle containing all vertices of $\cup_{i=1}^t V(D_i)$. If $x_1 \sim b_{t-1}$, then

$$C = x_t b_1 P_1 a_1 x_1 a_{t-1} y_{t-2} a_{t-2} \ldots y_2 a_3 y_2 b_2 P_2 a_2 y_t a_t x_t$$

is a cycle containing all vertices of $\cup_{i=1}^t V(D_i)$. Note that $a_i = b_i$ for each $i \in [3, t]$ by Claim [4]ii. By Claim [3] for each vertex $x \in S \setminus V(C)$, $d_C(x) > \frac{n}{8} - 1$. Applying Lemma 2.4 recursively on vertices of $S$ that are out of the current cycle containing $V(C)$, we get a hamiltonian cycle of $G$.

**Case 2** There exists a component $D_0$ of $G - S_{uv}$ such that $|V(G) \setminus (S_{uv} \cup V(D_0))| \leq \frac{n}{8} - 1$.

Let the components of $G - S$ be $D_1, D_2, \ldots, D_t$ with $|V(D_1)| \geq |V(D_2)| \geq \ldots \geq |V(D_t)|$ for some integer $t \geq 3$. In this case, $|V(D_1)| \geq |V(D_0)| \geq \frac{5n}{8} - 4$ since $|S_{uv}| \leq \frac{n}{4} + 5$. Let $S_1 \subseteq S$ be a largest subset of $S$ such that $|N_{D_1}(S_1)| < 2|S_1|$, $S_2 = S \setminus S_1$ and $S^* = N_{D_1}(S_1) \cup S_2$. By the choice of $S_1$, it can be seen that for any subset $T \subseteq S_2$, $|N_{D_1}(T) \setminus N_{D_1}(S_1)| \geq 2|T|$. By Lemma 2.1 there exists a $K_{1, 2}$-matching $M$ between $S_2$ and $V(D_1) \setminus N_{D_1}(S_1)$ with centers as vertices of $S_2$. Let $Q = V(M) \setminus S_2$ and $D^*_1 = D_1 - N_{D_1}(S_1) - Q$.

As $|V(D_1)| \geq \frac{5n}{8} - 4$, $|S_{uv}| \leq \frac{n}{4} + 5$, and $|N_{D_1}(S_1)| < 2|S_1|$, it follows that

$$|V(D^*_1)| > \frac{5n}{8} - 4 - 2 \left(\frac{n}{4} + 5\right) = \frac{n}{8} - 14 \geq 1,$$

where the last inequality above follows by (2). Thus $G - S^*$ has a component containing $D^*_1$. 

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Claim 5. If there are disjoint paths $Q_1, \ldots, Q_k$ for some integer $k \geq 1$ in $G$ such that (i) vertices of $G - S^*$ - $V(D_1)$ are contained as internal vertices of one or more $Q_i$’s, and (ii) each $Q_i$ has its two endvertices from $S^*$ and those two are the only vertices of $S^*$ that are contained in $Q_i$, then $G$ has a hamiltonian cycle.

Proof. Let $i \in [1, k]$ and $x_i, y_i$ be the endvertices of $Q_i$, where note $x_i, y_i \in S^*$. If $x_i, y_i \in N_{D_1}(S_1)$, then let $P_i = Q_i$. If $x_i \in N_{D_1}(S_1), y_i \in S_2$, then let $y'_i$ be a partner of $y_i$ and $P_i = x_iQ_iy_iy'_i$. If $x_i \in S_2, y_i \in N_{D_1}(S_1)$, then let $x'_i$ be a partner of $x_i$ and $P_i = x'_ix_iQ_iy_i$. If $x_i, y_i \in S_2$, then let $x'_i$ and $y'_i$ be a partner of $x_i$ and $y_i$, respectively, and $P_i = x'_ix_iQ_iy_iy'_i$. Then the endvertices of $P_i$ belong to $V(D_1)$ and all those paths are pairwise disjoint. Furthermore, for each $x \in S_2 \backslash (\cup_{i=1}^k V(P_i))$, the partners of $x$ belong to $V(D_1)$ and are not contained in any $P_i$. Since $D_1$ is a complete graph, there exists a hamiltonian cycle $C$ in $D_1$ such that the two endvertices of each $P_i$ are consecutive on $C$ and the two partners of each $x \in S_2 \backslash (\cup_{i=1}^k V(P_i))$ are also consecutive on $C$. Now for each edge $wz \in E(C)$, if $w$ and $z$ are endvertices of some $P_i$, we replace $wz$ by $P_i$; if $w$ and $z$ are the two partners of some $x \in S_2 \backslash (\cup_{i=1}^k V(P_i))$, we replace $wz$ by $wzx$. After doing this replacement for all such edges $wz$ of $C$, we have obtained a hamiltonian cycle of $G$. \hfill \square

Subcase 2.1 $c(G - S^*) \geq 3$.

In this case, each component of $G - S^*$ is a complete graph by the freeness of $P_3 \cup 2P_1$. Let $R_1, R_2, \ldots, R_\ell$ be all the components of $G - S^*$ with $R_i \neq D_1 - N_{D_1}(S_1)$ for each $i \in [1, \ell]$. Applying Lemma 2.6 with $t = 2$, $G$ has a generalized $K_{1,2}$-matching with centers as $R_1, R_2, \ldots, R_\ell$. We let $x_i$ and $y_i$ be the partners of $R_i$ from $S^*$, and let $a_i, b_i \in V(R_i)$ such that $a_ix_i, b_iy_i \in E(G)$, where note that $a_i = b_i$ if $|V(R_i)| = 1$. Moreover, there is a hamiltonian path $P'_i$ from $a_i$ to $b_i$ in $R_i$. Then we are done by Claim 5 by letting $Q_i = x_ia_iP'_ib_iy_i$ for each $i \in [1, \ell]$.

Subcase 2.2 $c(G - S^*) = 2$.

Let $D^*_2$ be the other component of $G - S^*$ other than the component containing $D^*_1$. As $c(G - S) \geq 3$ and $D^*_1$ is a subgraph of the original component $D_1$ of $G - S$, it follows that $|V(D^*_2)| \geq 2$. Note that $D^*_2$ is $(P_3 \cup P_1)$-free by the $(P_3 \cup 2P_1)$-freeness of $G$. Since $G$ is 14-connected and so is 2-connected, there exist two distinct vertices $a_0, b_0 \in V(D^*_2)$ and distinct vertices $x_0, y_0 \in S^*$ such that $a_0x_0, b_0y_0 \in E(G)$. If $D^*_2$ is hamiltonian connected, let $P$ be a hamiltonian path in $D^*_2$ between $a_0$ and $b_0$. Define $Q_1 = x_0a_0Pb_0y_0$, then we apply Claim 5 to find a hamiltonian cycle of $G$.

Thus assume that $D^*_2$ is not hamiltonian connected. Applying Lemma 2.5, we conclude that $\tau(D^*_2) \leq 1$. Let $W$ be a minimum tough set of $D^*_2$. Each component of $D^*_2 - W$ is a complete graph as $D^*_2$ is $(P_3 \cup P_1)$-free. Moreover, each vertex $x \in W$ is adjacent to at least
two components of $D^*_2 - W$ by $W$ being a tough set of $D^*_2$.

**Subcase 2.2.1** \(c(D^*_2 - W) = 2\).

Then \(1 \leq |W| \leq 2\). Let $F_1$ and $F_2$ be the two components of $D^*_2 - W$. Since $G$ is 7-tough, $G_1 = G - W$ is 6-tough and so is 2-tough. We can find in $G_1$ a generalized $K_{1,2}$-matching with centers as $F_1$ and $F_2$. If $|W| = 1$ or $W$ is a clique, then as each $F_i$ is a complete graph and each $F_i$ has two partners from $S^*$ such that these partners have at least two neighbors from $F_i$ in $G$ if $|V(F_i)| \geq 2$, we can find a hamiltonian $ab$-path $P$ of $D^*_2$ such that $a \sim x$ and $b \sim y$ for distinct $x, y \in S^*$. Now letting $Q_1 = xaPby$, we can find a hamiltonian cycle of $G$ by Claim 5.

Thus we assume $W = \{w_1, w_2\}$ and $w_1 \not\sim w_2$. Let $x_i, y_i$ be two partners of $F_i$ from $S$ and $a_i, b_i \in V(F_i)$ such that $a_i, x_i, b_i, y_i \in E(G)$ for each $i \in [1, 2]$. Assume first that $|V(D^*_2)| \leq 7$. As $G$ is 7-tough and noncomplete, $\delta(G) \geq 14$. Thus we can find distinct vertices $x_3, y_3, x_4, y_4 \in S \setminus \{x_1, x_2, y_1, y_2\}$ such that $w_1 \sim x_3, y_3$ and $w_2 \sim x_4, y_4$. We find in $F_1$ a hamiltonian $a_1b_1$-path $P'_1$, in $F_2$ a hamiltonian $a_2b_2$-path $P'_2$. Let $Q_1 = x_1a_1P'_1b_1y_1$, $Q_2 = x_2a_2P'_2b_2y_2$, $Q_3 = x_3w_1y_3$ and $Q_4 = x_4w_2y_4$. Then we apply Claim 5 to find a hamiltonian cycle of $G$.

Thus we have $|V(D^*_2)| \geq 8$. As $G$ is 7-tough and noncomplete, it is 14-connected. Note also that $|S^*| \geq 2\tau(G) \geq 14$. By the connectivity, there are distinct $a_i \in V(D^*_2)$ and distinct $x_i \in S^*$ such that $a_i \sim x_i$ for each $i \in [1, 7]$. As $W$ is a tough set of $D^*_2$, we conclude that $\tau(D^*_2) = 1$ and so $D^*_2$ is 2-connected. Since each of $F_1$ and $F_2$ is a complete graph and $|W| = 2$, we can find a hamiltonian cycle $C$ of $D^*_2$ such that $a_i, a_j \in E(C)$ for some distinct $i, j \in [1, 7]$. Now letting $Q_1$ be obtained from $C$ by deleting $a_ia_j$ and adding $a_ix_i$ and $a_jx_j$, we can apply Claim 5 to find a hamiltonian cycle of $G$.

**Subcase 2.2.2** \(c(D^*_2 - W) \geq 3\).

Then $D^*_2$ contains a complete bipartite graph with bipartition as $W$ and $V(D^*_2) \setminus W$ by \((P_3 \cup P_t)\)-freeness of $D^*_2$. Since $G$ is 7-tough and so $G_1 = G - \{x_0, y_0\}$ is 6-tough, applying Lemma 2.6 with $t = 4$, $G_1$ has a generalized $K_{1,4}$-matching with centers as components of $D^*_2 - W$. If a component $D$ of $D^*_2 - W$ has at least two vertices, then $V(D)$ has two disjoint subsets such that vertices from each of them are adjacent in $G$ to two distinct vertices from $S^* \cup W$. Thus for at least $c(D^*_2 - W) - \frac{3}{2}|W|$ components $F_i$ of $D^*_2 - W$, we can find a generalized $K_{1,2}$-matching in $G_1 - W$ with them as centers.

Let $\mathcal{F}$ be the collection of all components of $D^*_2 - W$ and $\mathcal{F}^*$ be those that are centers of the generalized $K_{1,2}$-matching of $G_1 - W$. If $a_0, b_0 \in W$, let $\mathcal{F}_1 \subseteq \mathcal{F}^*$ such that $|\mathcal{F}_1| = |\mathcal{F}| - (|W| - 1)$. If $a_0 \in W$ and $b_0 \in V(D^*_2 - W)$, then let $\mathcal{F}_1 \subseteq \mathcal{F}^*$ such that $|\mathcal{F}_1| = |\mathcal{F}| - |W|$ and that the vertex $b_0$ is not contained in any component of $\mathcal{F}_1$ (requires at most
If \( a_0, b_0 \in V(D_2^* - W) \), then we let \( F_1 \subseteq F^* \) such that
\[
|F_1| = |F| - (|W| + 1)
\]
and none of the vertices \( a_0 \) and \( b_0 \) is contained in any component of \( F_1 \) (requires at most \( |F| - (|W| + 1) + 2 \) components from \( F^* \)). Note that such \( F_1 \) exists as at least \( c(D_2^* - W) - \frac{1}{2}|W| \) components of \( D_2^* - W \) have two partners from \( S \setminus \{x_0, y_0\} \), and
\[
c(D_2^* - W) - \frac{1}{2}|W| \geq |F| - (|W| - 1) = c(D_2^* - W) - |W| + 1.
\]

Let \( F_2 = F \setminus F_1 \). Since the induced subgraph between \( W \) and \( D_2^* - W \) is a complete bipartite graph, there is a hamiltonian path \( P \) between \( a_0 \) and \( b_0 \) containing all vertices from \( W \) and components in \( F_2 \). For each \( F_i \in F_1 \), let \( x_i, y_i \) be its partners from \( S^* \) and \( a_i, b_i \in V(F_i) \) such that \( a_ix_i, b_iy_i \in E(G) \). As each component of \( D_2^* - W \) is complete, there is a path \( P'_i \) between \( a_i \) and \( b_i \) containing all vertices of each \( F_i \in F_1 \). Now let \( Q_0 = x_0a_0P_0b_0y_0 \), and \( Q_i = x_ia_iP'_ib_iy_i \) for each \( F_i \in F_1 \). Applying Claim 5, we find a hamiltonian cycle of \( G \).

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