A NOTE ON WEAK CONVERGENCE OF SINGULAR INTEGRALS IN METRIC SPACES

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Abstract. We prove that in any metric space $(X, d)$ the singular integral operators

$$T^k_{\mu, \varepsilon}(f)(x) = \int_{X \setminus B(x, \varepsilon)} k(x, y) f(y) d\mu(y).$$

converge weakly in some dense subspaces of $L^2(\mu)$ under minimal regularity assumptions for the measures and the kernels.

1. Introduction

A Radon measure on a metric space $(X, d)$ has $s$-growth if there exists some constant $c_\mu$ such that $\mu(B(x, r)) \leq c_\mu r^s$ for all $x \in X, r > 0$.

We say that $k(\cdot, \cdot) : X \times X \setminus \{(x, y) \in X \times X : x = y\} \to \mathbb{R}$ is an $s$-dimensional kernel if there exists a constant $c > 0$ such that for all $x, y \in X, x \neq y$:

$$|k(x, y)| \leq c d(x, y)^{-s}.$$

The kernel $k$ is antisymmetric if $k(x, y) = -k(y, x)$ for all distinct $x, y \in X$.

Given a positive Radon measure $\nu$ on $X$ and an $s$-dimensional kernel $k$, we define

$$T^k \nu(x) := \int k(x, y) d\nu(y), \quad x \in X \setminus \text{spt} \nu.$$

This integral may not converge when $x \in \text{spt} \nu$. For this reason, we consider the following $\varepsilon$-truncated operators $T^k_\varepsilon, \varepsilon > 0$:

$$T^k_\varepsilon \nu(x) := \int_{d(x,y) > \varepsilon} k(x, y) d\nu(y), \quad x \in X.$$

Given a fixed positive Radon measure $\mu$ on $X$ and $f \in L^1_{\text{loc}}(\mu)$, we write

$$T^k_\mu f(x) := T^k(f \mu)(x), \quad x \in X \setminus \text{spt}(f \mu),$$

and

$$T^k_{\mu, \varepsilon} f(x) := T^k_\varepsilon(f \mu)(x).$$

Concerning the limit properties of the operators $T^k_{\mu, \varepsilon}$ one can ask if the limit, the so called principal value of $T$,

$$\lim_{\varepsilon \to 0} T^k_{\mu, \varepsilon}(f)(x),$$

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exists $\mu$ almost everywhere. When $\mu$ is the Lebesgue measure in $\mathbb{R}^d$, and $k$ is a standard Calderón-Zygmund kernel, due to cancellations and the density of smooth functions in $L^1$, the principal values exist almost everywhere for $L^1$-functions. For more general measures, the question is more complicated. Let $n$ be an integer, $0 < n < d$, and consider the coordinate Riesz kernels

$$R^n_i(x) = \frac{x_i}{|x|^{n+1}}$$

for $i = 1, \ldots, d$.

Tolsa proved in [T] that if $E \subset \mathbb{R}^d$ has finite $n$-dimensional Hausdorff measure $\mathcal{H}^n$ the principal values

$$\lim_{\varepsilon \to 0} \int_{E \setminus B(x,\varepsilon)} \frac{x_i - y_i}{|x - y|^{m+1}} d\mathcal{H}^n(y)$$

exist $\mathcal{H}^n$ almost everywhere in $E$ if and only if the set $E$ is $n$-rectifiable i.e. if there exist $n$-dimensional Lipschitz surfaces $M_i$, $i \in \mathbb{N}$, such that

$$\mathcal{H}^n(E \setminus \bigcup_{i=1}^{\infty} M_i) = 0.$$ 

Mattila and Preiss had obtained the same result earlier, in [MP] under some stronger assumptions for the set $E$. It becomes obvious that the existence of principal values is deeply related to the geometry of the set $E$.

Assuming $L^2(\mu)$-boundedness for the operators $T^k_\mu$ one could have expected that more could be deduced about the structure of $\mu$ and the existence of principal values, but this is a hard and, in a large extent, open problem. Dating from 1991 the David-Semmes conjecture, see [DS], asks if the $L^2(\mu)$-boundedness of the operators associated with the $n$-dimensional Riesz kernels suffices to imply $n$-uniform rectifiability, which can be thought as a quantitative version of rectifiability. In the very recent deep work [NToV], Nazarov, Tolsa and Volberg resolved the conjecture in the codimension 1 case, that is for $n = d - 1$. Mattila, Melnikov and Verdera in [MMV], using a special symmetrization property of the Cauchy kernel, had earlier proved the conjecture in the case of 1-dimensional Riesz kernels. For all other dimensions and for other kernels few things are known. In fact, there are several examples of kernels whose boundedness does not imply rectifiability, see [C], [D] and [H]. For some recent positive results involving other kernels see [CMPT].

Let $\mu$ be a finite Radon measure and let $k$ be an antisymmetric kernel in a complete metric space $(X, d)$ where the Vitali covering theorem holds for $\mu$ and the family of closed balls defined by $d$. Mattila and Verdera in [MV] showed that in this case the $L^2(\mu)$-boundedness of the operators $T^k_\mu$ forces them to converge weakly in $L^2(\mu)$. This means that there exists a bounded linear operator $T^k_\mu : L^2(\mu) \to L^2(\mu)$ such that for all $f, g \in L^2(\mu),$

$$\lim_{\varepsilon \to 0} \int T^k_\mu,\varepsilon (f)(x) g(x) d\mu(x) = \int T^k_\mu (f)(x) g(x) d\mu(x).$$

Furthermore notions of weak convergence have been recently used by Nazarov, Tolsa and Volberg in [NToV].
Motivated by these developments it is natural to ask if limits of this type might exist if we remove the very strong $L^2$-boundedness assumption. We prove that the operators $T_{\mu,\varepsilon}^k$ converge weakly in dense subspaces of $L^2(\mu)$ under minimal assumptions for the measures and the kernels in general metric spaces. Denote by $X_B$ the space of all finite linear combinations of characteristic functions of balls in $X$,

$$X_B = \left\{ \sum_{i=1}^n a_i \chi_{B(z_i,r_i)} : n \in \mathbb{N}, a_i \in \mathbb{R}, z_i \in X, r_i > 0 \right\}.$$ 

Whenever Vitali’s covering theorem holds for the closed balls in $(X,d)$ the space $X_B$ is dense in $L^2(\mu)$. When $X = \mathbb{R}^d$ Vitali’s covering theorem holds for any Radon measure $\mu$ and the closed balls defined by various metrics (including the standard $d_p$ metrics for $1 \leq p \leq \infty$) as a consequence of Besicovitch’s covering theorem, see [M, Theorem 2.8]. Furthermore Vitali’s covering theorem holds for any metric space $(X,d)$ whenever $\mu$ is doubling, that is when there exists some constant $C$ such that for all balls $B$, $\mu(2B) \leq C\mu(B)$, see [F, Section 2.8].

**Theorem 1.1.** Let $\mu$ be a finite Radon measure with $s$-growth and $k$ an antisymmetric $s$-dimensional kernel on a metric space $(X,d)$. If the Vitali Covering theorem holds for the closed balls in $(X,d)$ then there exists subsets $X_B' \subset X_B$ which are dense in $L^2(\mu)$ and the weak limits

$$\lim_{\varepsilon \to 0} \int T_{\mu,\varepsilon}^k f(x) g(x) d\mu(x)$$

exist for all $f, g \in X_B'$.

Until now Theorem 1.1 was only known for measures with $(d-1)$-growth in $\mathbb{R}^d$ under some smoothness assumptions for the kernels, see [CM]. We thus extend the result from [CM] to measures with $s$-growth for arbitrary $s$ in metric spaces where Vitali’s covering theorem holds for the family of closed balls without requiring any smoothness for the kernels. Our proof follows a completely different strategy using an “exponential growth” lemma for probability measures on intervals and is self contained (unlike the proof from [CM] which depends on several $L^2(\nu)$ to $L^2(\mu)$ boundedness results for separated measures $\nu$ and $\mu$).

Recall that if $k$ is the $(d-1)$-dimensional Riesz kernel in $\mathbb{R}^d$ and $\mu$ has $(d-1)$-growth and is $(d-1)$ purely unrectifiable, that is $\mu(E) = 0$ for all $(d-1)$-rectifiable sets $E$, the principal values diverge $\mu$ almost everywhere and the weak convergence in $L^2(\mu)$ fails. On the other hand it is of interest that weak convergence in the sense of Theorem 1.1 holds as it holds for any $s$-dimensional antisymmetric kernel and any finite measure with $s$-growth.

**2. Proof of Theorem 1.1**

We first prove the following lemma about exponential growth of probability measures on compact intervals. It is motivated by a similar result proved in [SUZ]. Here
Leb stands for the Lebesgue measure on the real line and $|I|$ denotes the length of an interval $I \subset \mathbb{R}$.

**Lemma 2.1.** For every integer $\lambda > 2$ the following holds. Let $\nu$ be a probability Borel measure on a compact interval $\Delta \subset \mathbb{R}$. Then for every interval $I \subset \Delta$ there exists a subset $I'(\lambda) \subset I$ such that $\text{Leb}(I'(\lambda)) > |I|(1 - 3(\lambda^{-1} + \lambda^{-2} + \ldots))$ and for every $t \in I'(\lambda)$,

$$\nu([t - \lambda^{3n}, t + \lambda^{3n}]) < \lambda^{-3n}$$

for all integers $n \geq 1$.

**Proof.** Let us partition the interval $I$ into $\lambda^2$ subintervals $J$ of length $|I|\lambda^{-2}$. Let $B_1$ be the family of all intervals $J$ from this partition for which $\nu(J) < \lambda^{-1}$. Obviously, there are at most $\lambda$ intervals in $B_1$. Thus

$$\#B_1 > \lambda^2 - \lambda = \lambda^2 \left(1 - \frac{\lambda}{\lambda^2}\right)$$

and

$$\text{Leb} \left( \bigcup \{J : J \in B_1\} \right) \geq |I| \left(1 - \frac{\lambda}{\lambda^2}\right) = |I| \left(1 - \frac{1}{\lambda}\right).$$

Next, each interval in $B_1$ is divided into $\lambda^2$ subintervals with disjoint interiors and of length $|I|\lambda^{-4}$, and we remove those subintervals for which $\nu(J) \geq \lambda^{-2}$. Denoting by $B_2$ the family of remaining intervals, we see that

$$\#B_2 \geq (\lambda^2)^2 \left(1 - \frac{\lambda}{\lambda^2}\right) - \lambda^2 = (\lambda^2)^2 \left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2}\right)$$

and

$$\text{Leb} \left( \bigcup \{J : J \in B_2\} \right) \geq |I| \left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2}\right).$$

Proceeding inductively, we partition the interval $I$ into disjoint intervals of length $|I|\lambda^{-2n}$. Next, we define in the same way the family $B_n$. It is formed by the intervals $J$ of this partition of $n$th generation, which are contained in some interval of the family $B_{n-1}$ and for which $\nu(J) < \lambda^{-n}$. Then

$$\text{Leb} \left( \bigcup \{J : J \in B_n\} \right) \geq \left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2} - \cdots - \frac{1}{\lambda^n}\right) |I|.$$ 

For any $t \in I$ let $J_n = J_n(t)$ be the interval of the $n$th partition such that $t \in J_n$. Thus, for every $t \in \bigcap_{n=1}^{\infty} \bigcup_{J \in B_n} J$, we have that $J_n(t) \in B_n$. Consequently, for all $t \in \bigcap_{n=1}^{\infty} \bigcup_{J \in B_n} J$, it holds that $\nu(J_n(t)) < \lambda^{-n}$ for all $n \geq 1$. Let now

$$C_n = \{t \in I : [t - |I|\lambda^{-3n}, t + |I|\lambda^{-3n}] \subset J_n(t)\}.$$ 

It is easy to see that $\text{Leb}(C_n^c) < 2|I|\lambda^{-n}$, and, therefore,

$$\text{Leb} \left( \bigcap_{n=1}^{\infty} C_n \right) > |I| \left(1 - 2\left(\frac{1}{\lambda} + \frac{1}{\lambda^2} + \cdots\right)\right).$$
Finally, setting
\[ I' := \left( \bigcap_{n=1}^{\infty} C_n \right) \cap \left( \bigcup_{i=1}^{\infty} \bigcup_{j \in B_i} J \right) \]
completes the proof. □

Proof of Theorem 1.1  We can assume that \( \mu(X) \leq 1 \). We define finite Borel measures on the unit interval for all \( z \in \text{spt} \mu \) by
\[ \mu_z(F) = \mu\{x \in X : d(x, z) \in F\}, \quad F \subset [0, 1]. \]
Let \( A_z = \bigcup_{\lambda > 2} I'_{\lambda}(\lambda) \) where \( I'_{\lambda}(\lambda) \) are the sets we obtain after we apply Lemma 2.1 to the measures \( \mu_z \). Then Lemma 2.1 implies that \( \mu_z(A_z) = \mu_z([0, 1]) \). Let \( G_z = \{ r \in (0, 1) : r \in A_z \} \) and
\[ X'_B = \left\{ \sum_{i=1}^{n} a_i \chi_{B(z_i, r_i)} : n \in \mathbb{N}, \quad a_i \in \mathbb{R}, \quad z_i \in \text{spt} \mu, \quad r_i \in G_z \right\}. \]
Then \( X'_B \) is dense in \( L^2(\mu) \).

Let \( f, g \in X'_B \) such that
\[ f = \sum_{i}^{n} a_i \chi_{B_i} \quad \text{and} \quad g = \sum_{j}^{m} b_j \chi_{S_j}, \]
where \( a_i, b_j \in \mathbb{R} \) and \( B_i, S_j \) are closed balls. Then for \( 0 < \delta < \varepsilon \),
\[ \int T_{\mu, \varepsilon}^{k} f(x) g(x) d\mu(x) - \int T_{\mu, \delta}^{k} f(x) g(x) d\mu(x) = \sum_{j=1}^{m} \sum_{i=1}^{n} a_i b_j \int_{S_j \setminus B_i} \int_{\delta < d(x, y) < \varepsilon} k(x, y) d\mu(y) d\mu(x). \]
Furthermore,
\[ \left| \int_{S_j \setminus B_i} \int_{\delta < d(x, y) < \varepsilon} k(x, y) d\mu(y) d\mu(x) \right| \]
\[ \leq \left| \int_{B_i \cap S_j} \int_{B_i \cap S_j} k(x, y) d\mu(y) d\mu(x) \right| + \left| \int_{S_j \setminus B_i} \int_{B_i \cap S_j} k(x, y) d\mu(y) d\mu(x) \right| \]
\[ + \left| \int_{S_j \setminus B_i} \int_{B_i \setminus S_j} k(x, y) d\mu(y) d\mu(x) \right| + \left| \int_{S_j \setminus B_i} \int_{B_i \setminus S_j} k(x, y) d\mu(y) d\mu(x) \right| \]
\[ \leq \int_{B_i} \int_{B_i^c} |k(x, y)| d\mu(y) d\mu(x) + 2 \int_{S_j} \int_{S_j^c} |k(x, y)| d\mu(y) d\mu(x). \]
The last inequality follows because by antisymmetry and Fubini’s theorem
\[ \int_{B_i \cap S_j} \int_{\delta < d(x,y) < \varepsilon} k(x,y) d\mu(y) d\mu(x) = 0. \]

Therefore it is enough to show that for any “good” ball \( B = B(z,r) \) with \( z \in \text{spt} \mu \) and \( r \in G_z \)
\[ \lim_{\varepsilon \to 0} \int_{B^c} \int_{\delta < d(x,y) < \varepsilon} |k(x,y)| d\mu(y) d\mu(x) = 0, \]
which will follow by the monotone convergence theorem if we show that
\[ (2.1) \int_B \int_{B^c} |k(x,y)| d\mu(y) d\mu(x) < \infty. \]

Since \( B = B(z,r) \) and \( r \in G_z \) Lemma 2.1 implies that \( \mu(\partial B) = 0 \) hence it is enough to show that
\[ \int_{B^o} \int_{B^c} |k(x,y)| d\mu(y) d\mu(x) < \infty \]
where \( B^o \) stands for the interior of \( B \). For any \( x \in B^o \) let \( n(x) > 0 \) such that
\[ 2^{n(x)} d(x, \partial B) = 3 \]
and \( N(x) = \text{integer part of } n(x) + 1 \). Therefore, since \( \text{diam}(B) \leq 1 \),
\[ B(x, 2) \setminus B \subset \bigcup_{i=1}^{N(x)} B(x, 2^i d(x, \partial B)) \setminus B(x, 2^{i-1} d(x, \partial B)). \]

Hence for all \( x \in B^o \)
\[ \int_{B(x, 2) \setminus B} |k(x,y)| d\mu(y) \leq \int_{B(x, 2) \setminus B} d(x,y)^{-s} d\mu(y) \]
\[ = \sum_{i=1}^{N(x)} \mu(B(x, 2^i d(x, \partial B)) \setminus B(x, 2^{i-1} d(x, \partial B))) \int_{B(x, 2^i d(x, \partial B))} d(x,y)^{-s} d\mu(y) \]
\[ \leq \sum_{i=1}^{N(x)} \mu(B(x, 2^i d(x, \partial B)) (2^{i-1} d(x, \partial B))^{-s} d\mu(y) \]
\[ \lesssim N(x) \lesssim |\log d(x, \partial B)|, \]
and
\[ \int_{B^o} |k(x,y)| d\mu(y) \lesssim \int_{B(x, 2)^c} d(x,y)^{-s} d\mu(y) + |\log d(x, \partial B)| \]
\[ \lesssim 1 + |\log d(x, \partial B)|. \]
Since \( r \in G_z \) there exists some \( \lambda \in \mathbb{N} \) such that \( r \in I_z^\prime (\lambda) \). We write,

\[
\int_{B(z,r)^o} |\log d(x,\partial B)|d\mu(x) = \int_{B(z,r-\lambda^{-3})^o} |\log d(x,\partial B)|d\mu(x) \\
+ \sum_{n=1}^{\infty} \int_{\{x:r-\lambda^{-3n} \leq d(z,x) < r-\lambda^{-3(n+1)}\}} |\log d(x,\partial B)|d\mu(x)
\]

Notice that by Lemma 2.1

\[
\mu(\{x: r - \lambda^{-3n} \leq d(z,x) < r - \lambda^{-3(n+1)}\}) = \mu_z(\{r - \lambda^{-3n}, r - \lambda^{-3(n+1)}\}) \\
\leq \mu_z(\{r - \lambda^{-3n}, r + \lambda^{-3n}\}) \leq \lambda^{-n}.
\]

Therefore,

\[
\int_{B(z,r)^o} |\log d(x,\partial B)|d\mu(x) \lesssim 3 \log(\lambda)(r - \lambda^{-3})^s + \sum_{i=1}^{n} \lambda^{-n} |\log(\lambda^{-3(n+1)})| < \infty
\]

and this completes the proof of Theorem 1.1. \(\square\)

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