On Identifying a Massive Number of Distributions

Sara Shahi, Daniela Tuninetti and Natasha Devroye
University of Illinois at Chicago, Chicago IL 60607, USA.
Email: sshahi7, danielat, devroye @uic.edu

Abstract—THIS PAPER IS ELIGIBLE FOR THE STUDENT PAPER AWARD. Finding the underlying probability distributions of a set of observed sequences under the constraint that each sequence is generated i.i.d by a distinct distribution is considered. The number of distributions, and hence the number of observed sequences, are let to grow with the observation blocklength $n$. Asymptotically matching upper and lower bounds on the probability of error are derived.

I. INTRODUCTION

Hypothesis testing is a classical problem in statistics where one is given a random observation vector and one seeks to identify the distribution from a given set of distributions that generated it. Pioneering work in classical hypothesis testing include the proof of the optimality of likelihood ratio tests under certain criteria in the Neyman-Pearson Theorem [1]. Derivation of error exponents of different error types and their trade-offs for binary and M-ary hypothesis testing in [2] and [3] and the analysis of sequential hypothesis testing in [4].

The classical identification problem, which includes hypothesis testing as a special case, is consist of a finite number of distinct sources, each generating a sequence of i.i.d samples. The problem is to find the underlying distribution of each sample sequence, given the constraint that each sequence is generated by a distinct distribution. With this constraint the number of hypothesis is exponential in the number of distributions. If one neglects the fact that the sequences are generated by distinct distributions, the problem boils down to multiple M-ary hypothesis testing problems. This approach is suboptimal as it fails to exploit some of the (possibly useful) constraints.

In [5], the authors study the the Logarithmically Asymptotically Optimal (LAO) Testing of identification problem for a finite number of distributions. In particular, they study the identification of only two different objects in detail and find the reliability matrix, which consist of the error exponents of all error types. Their optimality criterion is to find the largest error exponent for a set of error types for given values of the other error types error exponent. The same problem with a different optimality criterion was also studied in [6], where multiple, finite, sequences were matched to the source distributions. More specifically, they proposed a test for a generalized Neyman-Pearson-like optimality criterion to minimize the rejection probability given that all other error probabilities decay exponentially with a pre-specified slope.

In here, we assume $A$ sequences of length $n$ are generated i.i.d according to $A$ distinct distributions; in particular random vectors $X_i^n \sim P_{X_i}$, $i \in [1 : A]$, for some unknown permutation $\sigma$ of the distributions. The goal is to reliably identify the permutation $\sigma$ with vanishing error probability as $n \to \infty$ from an observation of $[X_1^n, \ldots, X_A^n]$. This problem has close ties with de-anonymization of anonymized data [6]. A different motivation is the identification of users using only channel output sequences, without the use of pilot / explicit identification signals [7]. In both scenarios, the problem’s difficulty increases with the number of users. In addition, in modeling the systems with a massive number of users (such as the Internet of Things), it may be reasonable to assume that the number of users grow with the transmission blocklength [7], [8], and that the user’s identities must be distinguished from the received data. As the result, it is useful to understand exactly how the number of distributions affects the system performance, in particular for the case that the cardinality of the distributions grows with the blocklength. Notice that in this scenario, the number of hypothesis, would be doubly exponential in blocklength and the analysis of the optimal decoder becomes much harder than the classical (with constant number of distributions) identification problems.

Contributions. In this paper, we consider the identification problem for the case that the number of distributions grow with the observation blocklength $n$ as motivated by the massive user identification problem in the Internet of Things paradigm. The key novel element in this work consist of analyzing and reducing the complexity of the optimal maximum likelihood decoder, with double exponential number of hypothesis, using a graph theoretic result. In particular, we find

1) Find matching upper and lower bounds on the probability of error. This result specifies the relation between the growth rate of the number of distributions and the pairwise distance of the distributions for reliable identification.

2) We show that the probability that more than two distributions are incorrectly identified is dominated by the probability of the event that only two distributions are incorrectly identified.

3) We show that the arithmetic mean of the cycle gains (where we define the cycle gain as the product of the edge weights within the cycle) in a graph can be upper bounded by a function of the sum of the squares of the edge weights. This may be of independent interest.

II. NOTATION

Capital letters represent random variables that take on lower case letter values in calligraphic letter alphabets. For a set of finite alphabet $\mathcal{X}$, we use $\mathcal{P}_X$ to denote the set of all possible
More specifically, we are interested in finding a permutation $\pi$ and a sample $x_1, \ldots, x_n$ where $|S_n| = n!$, to denote the set of all possible permutations of a set of $n$ elements. For a permutation $\sigma \in S_n$, $\sigma$ denotes the $i$-th element of the permutation. $|x_i|$ is used to denote the remainder of $x$ divided by $i$. The indicator function of event $A$ is denoted by $\mathbb{1}(A)$. We use the notation $a_n = e^{nb}$ when $\lim_{n \to \infty} \frac{\log a_n}{n} = b$.

$K_k(a_1, \ldots, a_{\binom{k}{2}})$ is the complete graph with $k$ nodes with edge index $i \in [1 : \binom{k}{2}]$ and edge weights $a_i, i \in [1 : \binom{k}{2}]$. We may drop the edge argument and simply write $K_k$ when the edge specification is not needed. A cycle $c$ of length $r$ in $K_k$ may be interchangeably defined by a vector of vertices as $c^{(v)} = [v_1, \ldots, v_r]$ or by a set of edges $c^{(e)} = \{a_1, \ldots, a_r\}$ where $a_i$ is the edge between $(v_i, v_{i+1})$ to $v_i \in [1 : r-1]$ and $a_r$ is that between $(v_r, v_1)$. With this notation, $c^{(v)}(i)$ is then used to indicate the $i$-th vertex of the cycle. $C_k^{(r)}$ is used to denote the set of all cycles of length $r$ in the complete graph $K_k(a_1, \ldots, a_{\binom{k}{2}})$. The cycle gain, denoted by $G(c)$, for cycle $c = \{a_1, \ldots, a_r\} \in C_k^{(r)}$ is the product of the edge weights within the cycle. The edge index $\sigma$ is given by $X \sim i.i.d$.

### III. Problem Formulation

Let $P := \{P_1, \ldots, P_A\} \subset \mathcal{P}_X$ consist of $A$ distinct distributions and also let $\Sigma$ be uniformly distributed over $S_A$, the set of permutations of $A$ elements. In addition, assume that we have $A$ independent random vectors $\{X_1^n, X_2^n, \ldots, X_A^n\}$ of length $n$ each. For $\sigma$, a realization of $\Sigma$, assign the distribution $P_{\sigma|}\text{ to the random vector } X_1^n, \forall i \in [1 : A]$. After observing a sample $x_1^n, \ldots, x_A^n$ of the random vector $X_1^n, \ldots, X_A^n$, we would like to identify $P_{\sigma}, \forall i \in [1 : A]$. More specifically, we are interested in finding a permutation $\hat{\Sigma} : X_1^n \mapsto S_A$ to indicate that $X_1^n \overset{i.i.d.}\sim P_{\hat{\sigma}}, \forall i \in [1 : A]$. Let $\hat{\Sigma} = \hat{\sigma}(X_1^n)$.

The average probability of error for the set of distributions $P$ is given by

$$P_e(n) = \mathbb{P}\left[\hat{\Sigma} \neq \Sigma \right]$$

$$= \frac{1}{|A|} \sum_{\sigma \in S_A} \mathbb{P}\left[\hat{\Sigma} \neq \sigma | X_1^n \overset{i.i.d.}\sim P_\sigma, \forall i \in [1 : A]\right]$$

$$= \mathbb{P}\left[\hat{\Sigma} \neq [1 : A] | H(1, \ldots, A)\right],$$

where $H(1, \ldots, A) := \{X_1^n \overset{i.i.d.}\sim P_i, \forall i \in [1 : A]\}$.

We say that a set of distributions $P$ are identifiable if $\lim_{n \to \infty} P_e(n) \to 0$.

**Theorem 1.** A sequence of distributions $P = \{P_1, \ldots, P_A\}$ are identifiable iff

$$\lim_{n \to \infty} \sum_{1 \leq i \leq j \leq A_n} e^{-2nB(P, P_j)} = 0,$$

where $B(P, Q) = B(Q, P) = -\log \mathbb{P}_P \left[\left(\frac{Q}{P}\right)^{1/2}\right]$ is the Bhattacharyya distance between the distributions $P$ and $Q$.

**Proof.** As it is obvious from the result of Theorem 1, for the case that $A_n = 1$, the sequence of distributions of $P$ are always identifiable and the probability of error in the identification problem decays to zero as the blocklength $n$ goes to infinity. The interesting aspect of Theorem 1 is in fact in the regime that $A_n$ increases exponentially with the blocklength.

To prove Theorem 1 we provide upper and lower bounds on the probability of error in the following subsections.

**A. Upper bound on the probability of error**

We use the optimal Maximum Likelihood (ML) decoder which minimizes the average probability of error, given by

$$\hat{\sigma}(x_1^n, \ldots, x_A^n) =: \mathbb{arg\ max}_{\Sigma \in \mathcal{S}_A} \sum_{i=1}^{A_n} \log \left(P_{\sigma_i}(x_i^n)\right),$$

where $P_{\sigma_i}(x_i^n) = \prod_{i=1}^{n} P_{\sigma_i}(x_{i,n})$. The average probability of error associated with the ML decoder can also be written as

$$P_e(n) = \mathbb{P}\left[\hat{\Sigma} \neq [1 : A_n] | H(1, \ldots, A_n)\right]$$

$$= \mathbb{P}\left[\bigcup_{\Sigma \neq [1 : A_n]} \hat{\Sigma} = \hat{\sigma} | H(1, \ldots, A_n)\right]$$

$$= \mathbb{P}\left[\bigcup_{r=2}^{A_n} \bigcup_{\sigma_{1:t} = [1 : r]} \hat{\Sigma} = \hat{\sigma} | H(1, \ldots, A_n)\right]$$

$$= \mathbb{P}\left[\bigcup_{r=2}^{A_n} \bigcup_{\sigma_{1:t} = [1 : r]} \sum_{i=1}^{A_n} \log \left(\frac{P_{\sigma_i}(X_i^n)}{P_{\hat{\sigma}_i}(X_i^n)}\right) \geq 0 | H(1, \ldots, A_n)\right]$$

where $\log \frac{P_{\sigma_i}(X_i^n)}{P_{\hat{\sigma}_i}(X_i^n)} := \sum_{i=1}^{n} \log \frac{P_{\sigma_i}(X_i)}{P_{\hat{\sigma}_i}(X_i)}$ and where $\mathbb{P}\left[\sum_{i=1}^{A_n} \mathbb{1}(\hat{\sigma}_i \neq \sigma_i) = r\right]$. Consider the two sequences $[i_1, \ldots, i_r]$ and $[\sigma_1, \ldots, \sigma_r]$ for which we have

$$\sum_{i=1}^{A_n} \mathbb{1}(\hat{\sigma}_i \neq \sigma_i) = r.$$
\( \tilde{r}_l \) for \( \tilde{r}_l < r \) such that \( \sum_{i=1}^{L} \tilde{r}_l = r \), then those cycles and their corresponding sequences are already taken into account in the (union of) set of \( \tilde{r}_l \) error events.

As an example, assume \( A_n = 4 \) and consider the error event
\[
\log P_2 \left( X_2^n \right) + \log P_1 \left( X_1^n \right) + \log P_4 \left( X_4^n \right) + \log P_3 \left( X_3^n \right) \geq 0,
\]
which corresponds to the (error) event of choosing \( \left[ \tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4 \right] = 2, 1, 4, 3 \) over \( \left[ 1, 2, 3, 4 \right] \) with \( r = 4 \) errors. In the graph representation, this gives two cycles of length 2 each, which correspond to
\[
\log P_2 \left( X_2^n \right) + \log P_1 \left( X_1^n \right) \geq 0 \cap \log P_4 \left( X_4^n \right) + \log P_3 \left( X_3^n \right) \geq 0,
\]
and are already accounted for in the events \( \left\{ \left[ \tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4 \right] = 2, 1, 3, 4 \right\} \cup \left\{ \left[ \tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4 \right] = 1, 2, 4, 3 \right\} \) with \( r = 2 \).

As the result, in order to avoid double counting, in calculating the value of (4) for each \( r \) we only consider the sets of sequences which produce a single cycle of length \( r \). Hence, we can upper bound the probability of error in (4) as (where we drop the conditioning for ease of notation)
\[
P_e \left( G \right) = \sum_{r=2}^{A_n} \sum_{c \in \mathcal{C}_A} \mathbb{P} \left[ \sum_{i=1}^{r} \log \frac{P_c \left( \left( i+1 \right) \right)}{P_0 \left( i \right)} \left( X_2^n \right) ight] \geq 0 \]
\[
\leq \sum_{r=2}^{A_n} \sum_{c \in \mathcal{C}_A} \mathbb{E} \left[ \exp \left\{ -n \inf_{c} \log \mathbb{E} \left[ \prod_{i=1}^{r} \frac{P_c \left( \left( i+1 \right) \right)}{P_0 \left( i \right)} \left( X_2^n \right) \right] \right\} \right],
\]
\[
= \exp \left\{ -n \inf_{c} \log \mathbb{E} \left[ \prod_{i=1}^{r} \frac{P_c \left( \left( i+1 \right) \right)}{P_0 \left( i \right)} \left( X_2^n \right) \right] \right\},
\]
In (5), we have also defined \( e^{-nB(P_0, P_c)} \) to be the edge weight between vertices \( (i, j) \) in the complete graph \( K_{A_n} \). Hence, \( G(c) = e^{-n \sum_{i=1}^{A_n} B(P_c(i), P_0(i))} \) is the gain of cycle \( c \).

The fact that we used \( t = 1/2 \) in (7) instead of finding the exact optimizing \( t \), comes from the fact that \( t = 1/2 \) is the optimal choice for \( r = 2 \) and as we will see later, the rest of the error events are dominated by the set 2 incorrect distributions. This can be seen as follows for \( X_1^n \sim P_1 \) \( X_2^n \sim P_2 \)
\[
P \left[ \log P_2 \left( X_2^n \right) + \log P_1 \left( X_1^n \right) \right] \geq 0
\]
\[
= \sum_{r=2}^{A_n} \mathbb{P} \left\{ \exp \left\{ -nD \left( \tilde{P}_1 \parallel P_1 \right) - nD \left( \tilde{P}_2 \parallel P_2 \right) \right\} \right\},
\]
\[
\leq e^{-nD \left( \tilde{P}_1 \parallel P_1 \right) - nD \left( \tilde{P}_2 \parallel P_2 \right)} = e^{-2nB(P_1, P_2)},
\]
where \( \tilde{P} \) in the first equality in (8), by using the Lagrangian method, can be shown to be equal to \( \tilde{P}(x) = \sqrt{P_1(x)P_2(x)} \) and subsequently the second inequality in (8) is proved.

In order to further simplify the expression in (5), we use the following graph theoretic Lemma, the proof of which is given in the Appendix.

**Lemma 1.** In a complete graph \( K_k \left( a_1, \ldots, a_n \right) \) and for the set of cycles of length \( r, \mathcal{C}_A^r \) \( = \left\{ c_1, \ldots, c_n \right\} \) we have
\[
\frac{1}{N_{r,k}} \left( G(c_1) + \ldots + G(c_n) \right) \leq \left( \frac{a_1^2 + \ldots + a_n^2}{n_k} \right)^{r/2}
\]
where \( N_{r,k}, n_k \) are the number of cycles of length \( r \) and the number of edges in the complete graph \( K_k \), respectively.

By Lemma 1 and (5) we can write
\[
P_e \left( G \right) \leq \sum_{r=2}^{A_n} \sum_{c \in \mathcal{C}_A} G(c)
\]
\[
\leq \sum_{r=2}^{A_n} \frac{N_{r,A_n}}{n_k} \left( a_1^2 + \ldots + a_n^2 \right)^{r/2}
\]
\[
\leq \sum_{r=2}^{A_n} \left( \frac{1}{n_k} \sum_{1 \leq i < j \leq A_n} e^{-2nB(P_1, P_2)} \right)^{r/2}
\]
\[
\leq \left( \frac{16 \sum_{1 \leq i < j \leq A_n} e^{-2nB(P_1, P_2)}}{1 - \sqrt{16 \sum_{1 \leq i < j \leq A_n} e^{-2nB(P_1, P_2)}}} \right)^{r/2}
\]
where (9) is by Fact 1 (see Appendix) and
\[
\frac{N_{r,A_n}}{\left( n_k \right)^{r/2}} = \left( \frac{A_n}{r} \right)^{r/2} \leq 4^r.
\]
The upper bound on the probability of error in (10) goes to zero if
\[
\lim_{n \to \infty} \sum_{1 \leq i < j \leq A_n} e^{-2nB(P_1, P_2)} = 0.
\]
As a result of Lemma 1, it can be seen from (9) that the sum of probabilities that \( r \geq 3 \) distributions are incorrectly identified is dominated by the probability that only \( r = 2 \) distributions are incorrectly identified. This shows that the most probable error event is indeed the error events with two wrong distributions.
B. Lower bound on the probability of error

For our converse, we use the optimal ML decoder, and as a lower bound to the probability of error in (4), we only consider the set of error events with only two incorrect distributions, i.e., the set of events with \( r = 2 \). In this case we have

\[
P_{e}^{(2)} \geq \mathbb{P} \left\{ \log \frac{P_j}{P_i} (X^i) + \log \frac{P_i}{P_j} (X^j) \geq 0 \right\}
\]

\[
\geq \left( \sum_{1 \leq i,j \leq A_n} \mathbb{P} [\xi_{i,j}] \right)^2
\]

\[
= \left( \sum_{\{i,j\} \neq \{k,l\}} \mathbb{P} [\xi_{i,j}, \xi_{k,l}] \right)^2
\]

where (11) is by (9) and where

\[
\xi_{i,j} := \left\{ \log \frac{P_j}{P_i} (X^i) + \log \frac{P_i}{P_j} (X^j) \geq 0 \right\}
\]

If the number of distributions grows with the blocklength \( n \), we found matching upper and lower bounds on the probability of error. This result characterizes the relation between the number of distributions and the pairwise distance of the distributions for reliable identification.

IV. CONCLUSION

In this paper, we generalized the identification problem to the case that the number of distributions grows with the blocklength \( n \). We found matching upper and lower bounds on the probability of identification error. This result characterizes the relation between the number of distributions and the pairwise distance of the distributions for reliable identification.

APPENDIX

We first consider the case that \( r \) is an even number and then prove

\[
r(n_k)^{n-1} \left( G(c_1) + \ldots + G(c_{n_k}) \right) \leq \frac{N_{r,k}^n}{n_k} \left( a_1^2 + \ldots + a_{n_k}^2 \right)^{\frac{n}{2}}.
\]

We may drop the subscripts and use \( N := N_{r,k} \) and \( n := n_k \) in the following for notational ease. Our goal is to expand the right hand side (RHS) of (16) such that all elements have coefficient 1. Then, we parse these elements into \( N \) different groups (details will be provided later) such that using the AM-GM inequality (i.e., \( n \prod_{a_i=1}^n a_i^{\frac{1}{n}} \leq \sum_{a_i=1}^n a_i \)) on each group, we get one of the \( N \) terms on the LHS of (16). Before stating the rigorous proof, we provide an example of this strategy for the graph with \( k = 4 \) vertices shown in Fig. 1. In this example, we consider the Lemma for \( r = 4 \) cycles (for which we have \( N = 3 \)).

\[
\begin{align*}
8 \left( \sum_{1 \leq i,j \leq A_n} e^{-2nB(P_i,P_j)} \right)^{\frac{3}{2}} &+ \left( \sum_{1 \leq i,j \leq A_n} e^{-2nB(P_i,P_j)} \right)^{\frac{1}{2}} \\
&= \sqrt{\sum_{1 \leq i,j \leq A_n} e^{-2nB(P_i,P_j)}} \frac{8}{\sum_{1 \leq i,j \leq A_n} e^{-2nB(P_i,P_j)}}.
\end{align*}
\]

where (14) is by Lemma 1. As it can be seen from (15), if \( \lim_{n \to \infty} \sum_{1 \leq i,j \leq A_n} e^{-2nB(P_i,P_j)} = 0 \), the probability of error is bounded away from zero. As the result, we have

\[
\lim_{n \to \infty} \sum_{1 \leq i,j \leq A_n} e^{-2nB(P_i,P_j)} = 0.
\]
It can be easily seen that if we use the AM-GM inequality on $\Theta_1$, $\Theta_2$ and $\Theta_3$, we can get the lower bound equal to $24(\{a_1a_2a_3a_4\}), 24(\{a_1a_3a_4a_5\}) and 24(\{a_1a_2a_4a_5\})$, respectively where $rn^{-\frac{1}{2}} = 24$ and hence holds in this case. We propose to prove Lemma 14 for arbitrary $k$ and (even) $r \geq 2$. We propose the following scheme to group the elements on the RHS of (16) and then we prove that this grouping indeed leads to the claimed inequality in the Lemma.

**Grouping scheme:** For each cycle $c_i = \{a_{i1}, \ldots, a_{in_i}\}$, we need a group of elements, $\Theta_i$, from the RHS of (16). In this regard, we consider all possible subsets of the edges of cycle $c_i$ with $1 : \frac{r}{2}$ elements (e.g. $\{a_{i1}, a_{i2}, \ldots, a_{ip}, a_{i2}, \ldots, a_{ip}, \ldots, a_{in_i}\}$). For each one of these subsets, we find the respective elements from the RHS of (16) that is the multiplication of the elements in that subset. For example, for the subset $\{a_{i1}, a_{i3}, a_{i5}\}$, we consider the elements like $a_{i1}a_{i3}a_{i5}$ for all possible $a_{i1}, a_{i3}, a_{i5} > 0$ from the RHS of (16). However, note that we do not assign all such elements to cycle $c_i$ only. If there are $l$ cycles of length $r$ that all contain $\{a_{i1}, a_{i3}, a_{i5}\}$, we should assign $\frac{1}{l}$ of the elements like $a_{i1}a_{i3}a_{i5}$, $a_{i1}, a_{i3}, a_{i5} > 0$ to cycle $c_i$ (so that we can assign the same amount of elements to other cycles with similar edges).

We state some facts, which can be easily verified:

**Fact 1.** In a complete graph $K_k$, there are $N = N_{r,k} = \binom{k}{2}(r-1)!$ cycles of length $r$.

**Fact 2.** By expanding the RHS of (16) such that all elements have coefficient 1, we end up with $\left(\frac{N_r}{n}\right)n^{\frac{1}{r}}$ elements.

**Fact 3.** Expanding the RHS of (16) such that all elements have coefficient 1, and finding their product yields

$$(a_1 \times \ldots \times a_n)^{\frac{1}{n}} = n^{\frac{1}{r}} - 1,$$

**Fact 4.** In above grouping scheme each element on the RHS of (16) is summed in exactly one group. Hence, by symmetry and Fact 2, each group is the sum of $rn^{-\frac{1}{2}}$ elements.

Now, consider any two cycles $c_i^{(e)}(e) = \{a_{i1, \ldots, a_{in_i}}\}$, $c_j^{(e)} = \{a_{j1, \ldots, a_{jn_j}}\}$. Assume that using the above grouping scheme, we get the group of elements $\Theta_1, \Theta_2$, (where by fact 3 each one is the sum of $rn^{-\frac{1}{2}}$ elements). If we apply the AM-GM inequality on each of the two groups, we get

$$\Theta_1 \geq rn^{-\frac{1}{2}} \left(\frac{a_{1n_1} \times \ldots \times a_{nn_n}}{rn^{-\frac{1}{2}}}\right),$$

$$\Theta_2 \geq rn^{-\frac{1}{2}} \left(\frac{a_{2n_2} \times \ldots \times a_{2n_n}}{rn^{-\frac{1}{2}}}\right),$$

where $\prod_{t=1}^{r} a_{it}^{n_{it}}$ is the product of the elements in $\Theta_i$. By symmetry of the grouping scheme for different cycles, it is obvious that $\forall i \in [1 : r], n_{it} = n_{jt}$. Hence $n_{it} = n_{jt} = p$, $\forall i, j \in [1 : N]$, i.e., we have

$$\Theta_1 \geq rn^{-\frac{1}{2}} \left(\frac{a_{p1} \times \ldots \times a_{zp}}{rn^{-\frac{1}{2}}}\right).$$

By symmetry of the grouping scheme over the elements of each cycle, we also get that $n_{ik} = n_{ij} = q$, $\forall i, k, l \in [1 : r]$, i.e.

$$\Theta_1 \geq rn^{-\frac{1}{2}} \left(\frac{a_{q1} \times \ldots \times a_{qn}}{rn^{-\frac{1}{2}}}\right).$$

It can be seen from (17) and (18) that all the elements of all groups have the same power $n_i = p, \forall i \in [1 : N], l \in [1 : r]$. i.e.,

$$\Theta_1 \geq rn^{-\frac{1}{2}} \left(\frac{a_{p1} \times \ldots \times a_{pn}}{rn^{-\frac{1}{2}}}\right).$$

Since each element on the RHS of (16) is assigned to one and only one group and since $\prod_{t=1}^{r} a_{it}^{n_{it}} = \prod_{t=1}^{r} a_{jt}^{n_{jt}}$ is the product of the elements of each group $\Theta_i$, the product of all elements in $\Theta_1 + \ldots + \Theta_N$ (which is equal to product of the elements in the expanded version of the RHS of (16)) is $\prod_{i=1}^{N} \prod_{t=1}^{r} a_{it}^{n_{it}}$.

In addition, since each $a_i$ appears in exactly $\frac{N_r}{n}$ of the cycles, by Fact 3 and a double counting argument, we have

$$p \times \frac{N_r}{n} = \left(\frac{N_r}{n}\right) rn^{-\frac{1}{2}},$$

and hence $p = rn^{-\frac{1}{2}} - 1$. Hence, the lower bound of the AM-GM inequality on the $\Theta_1 + \ldots + \Theta_N$, will result in

$$rn^{-\frac{1}{2}} - 1 G(c_1) + \ldots + an^{-\frac{1}{2}} - 1 G(c_N),$$

and the Lemma is proved for even $r$.

For odd values of $r$, the problem that may arise by using the grouping strategy in its current form, is when $r < \frac{k}{2}$. In this case, some of the terms on the RHS of (16) may contain multiplication of $a_i$’s that are not present in any of the $G(c_i)$’s. To overcome this, take both sides to the power of $2m$ for the smallest $m$ such that $rn^{-\frac{1}{2}} - 1 > 2m$. Then the RHS of (16) is at most the multiplication of $rn$ different $a_i$’s and on the LHS the LHS of (16), there are $2rn$ cycles of length $r$ multiplied together. By our choice of $2m$, now, all possible combinations of $a_i$’s on the RHS are present in at least one cycle multiplication in the LHS. Hence, we can now continue the proof with the same strategy as even values of $r$ for the odd values of $r$.

**REFERENCES**

[1] J. Neyman and E. S. Pearson, “On the problem of the most efficient tests of statistical hypotheses,” Philosophical Transactions of the Royal Society of London. Series A. Containing Papers of a Mathematical or Physical Character, vol. 231, pp. 289–337, 1933.

[2] R. Blahut, “Hypothesis testing and information theory,” IEEE Transactions on Information Theory, vol. 20, no. 4, pp. 405–417, 1974.

[3] E. Tuncel, “Extensions of error exponent analysis in hypothesis testing,” in IEEE International Symposium on Information Theory (ISIT), 2005.

[4] A. Wald, “Sequential tests of statistical hypotheses,” The Annals of Mathematical Statistics, vol. 16, no. 2, pp. 117–186, 1945.

[5] R. Ahlswede and E. Haroutunian, “On logarithmically asymptotically optimal testing of hypotheses and identification,” in General Theory of Information Transfer and Combinatorics. Springer, 2006, pp. 553–571.

[6] J. Unnikrishnan, “Asymptotically optimal matching of multiple sequences,” IEEE Transactions on Information Theory, vol. 61, no. 1, pp. 452–468, 2015.

[7] S. Shahi, D. Tuninetti, and N. Devroye, “On the capacity of strong asynchronous multiple access channels with a large number of users,” in IEEE International Symposium on Information Theory (ISIT), July 2016, pp. 1486–1490.

[8] X. Chen, T.-Y. Chen, and D. Guo, “Capacity of gaussian many-access channels,” IEEE Transactions on Information Theory, vol. 63, no. 6, pp. 3516–3539, 2017.

[9] K. L. Chung and P. Erdos, “On the application of the borel-cantelli lemma,” Transactions of the American Mathematical Society, vol. 72, no. 1, pp. 179–186, 1952.