Klein-Gordon Transformation sans Extraneous Insertions: the Isomorphic Classical Complement to a Quantum System

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Abstract

The historical Klein-Gordon transformation of complex-valued first-order in time Schrödinger equations iterates these in a naively straightforward way which changes them into complex-valued second-order in time equations that have a plethora of extraneous solutions—the transformation is an operator-calculus analogue of the squaring of both sides of an algebraic equation. The real and imaginary parts of a Schrödinger equation, however, are well known to be precisely the dynamical equation pair of the real-valued classical Hamiltonian functional which is numerically equal to the expectation value of that Schrödinger equation’s Hermitian Hamiltonian operator. The purely real-valued second-order in time Euler-Lagrange equation of the corresponding classical Lagrangian functional is also isomorphic to that Schrödinger equation, and for symmetric Hamiltonians has exactly the same formal appearance as the corresponding naive complex-valued Klein-Gordon equation, but none of the latter’s extraneous solutions. These quantum Schrödinger-equation isomorphisms to classical Euler-Lagrange equations are the technical manifestation of a key theoretical aspect of the principle of complementarity, one which is elegantly illustrated by the isomorphic free-photon wave-function complement to the vector potential of source-free classical electrodynamics.

Introduction

It is well known that every Schrödinger equation follows from the purely real-valued classical Hamiltonian functional which is numerically equal to the quantum wave-vector expectation value of its Hermitian quantum Hamiltonian operator [1, 2]. If one then further proceeds to obtain the purely real-valued classical Lagrangian functional which corresponds to that particular classical Hamiltonian functional, its ensuing purely real-valued second-order in time classical Euler-Lagrange equation is naturally isomorphic to the original complex-valued first-order in time quantum Schrödinger equation. In that way the goal of Klein and Gordon [3] is universally achieved without unintentionally injecting extraneous solutions. In fact, when the Hermitian quantum Hamiltonian operator happens to be symmetric (and therefore real-valued), the ensuing purely real-valued second-order in time classical Euler-Lagrange equation has the very same formal appearance as the problematic complex-valued equation that Klein and Gordon obtained by naive operator squaring on both sides of the Schrödinger equation [3], a process whose algebraic analogue is notorious for the unintended injection of extraneous roots.

The detailed description below of the general class of quantum Schrödinger to classical Euler-Lagrange equation isomorphisms is the technical rendition of a key theoretical aspect of the principle of complementarity [4], one which is elegantly illustrated by the isomorphic free-photon wave-function complement to the vector potential of source-free classical electrodynamics [5, 6].

Isomorphic Euler-Lagrange complements to Schrödinger equations

As is well known, the Schrödinger equation,

\[ i\hbar \dot{\psi} = \hat{H}\psi \]  

(1a)

can be split into real and imaginary parts [2] by explicitly thus splitting both its complex-valued quantum wave vector \( \psi \),

\[ \psi = (2\hbar)^{-\frac{1}{2}}(\phi + i\pi) \]

(1b)

where \( \phi \) \( \equiv \) \( (\hbar/2)^{\frac{1}{2}}(\psi + \psi^*) \) and \( \pi \) \( \equiv \) \(-i(\hbar/2)^{\frac{1}{2}}(\psi - \psi^*) \).

and its Hermitian quantum Hamiltonian operator \( \hat{H} \),

\[ \hat{H} = \hbar(\hat{\Omega}_S + i\hat{\Omega}_A) \]

(1c)

\[ \text{where } \hat{\Omega}_S \equiv (\hat{H} + \hat{H}^*)/(2\hbar) = (\hat{H} + \hat{H}^T)/(2\hbar) \]

\[ \text{and } \hat{\Omega}_A \equiv -i(\hat{H} - \hat{H}^*)(2\hbar) = -i(\hat{H} - \hat{H}^T)/(2\hbar) \]

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from which we see that \( \hat{\Omega}_S \) is real-valued and symmetric, while \( \hat{\Omega}_A \) is real-valued and antisymmetric.

When we substitute Eqs. (1b) and (1c) into the Schrödinger Eq. (1a), we obtain,

\[
i\dot{\phi} - \dot{\pi} = \hat{\Omega}_S \phi - \hat{\Omega}_A \pi + i(\hat{\Omega}_S \pi + \hat{\Omega}_A \phi),
\]

which yields two purely real-valued first-order in time equations that together are equivalent to the Schrödinger Eq. (1a),

\[
\dot{\phi} = \hat{\Omega}_S \pi + \hat{\Omega}_A \phi, \quad \dot{\pi} = -\hat{\Omega}_S \phi + \hat{\Omega}_A \pi.
\] (2b)

It turns out that Eq. (2b) can also be obtained as the pair of dynamical equations of motion that follow from the real-valued classical Hamiltonian functional which is numerically equal to the wave-vector expectation value of the Hermitian quantum Hamiltonian operator [2],

\[
H_{\text{class}}^{\hat{\Omega}_S,\hat{\Omega}_A}[\phi, \pi] \overset{\text{def}}{=} \langle \psi* | \hat{\mathcal{H}} \psi \rangle = \frac{1}{2i}((\phi - i\pi), (\hat{\Omega}_S + i\hat{\Omega}_A)(\phi + i\pi)) = \frac{1}{4}[(\phi, \hat{\Omega}_S \phi) + (\pi, \hat{\Omega}_S \pi) + 2(\pi, \hat{\Omega}_A \phi)] = \frac{1}{4}[(\phi, \hat{\Omega}_S \phi) + (\pi, \hat{\Omega}_S \pi) - 2(\phi, \hat{\Omega}_A \pi)],
\] (3a)

where we have made use of three identities which follow from the facts that \( \hat{\Omega}_S \) is symmetric and \( \hat{\Omega}_A \) is antisymmetric. These identities are,

\[
(\phi, \hat{\Omega}_S \pi) = (\pi, \hat{\Omega}_S \phi), \quad (\phi, \hat{\Omega}_A \phi) = 0 = (\pi, \hat{\Omega}_A \pi), \quad (\pi, \hat{\Omega}_A \phi) = -(\phi, \hat{\Omega}_A \pi).
\]

We now proceed to obtain the pair of Hamilton’s classical dynamical equations of motion which follow from the classical Hamiltonian functional \( H_{\text{class}}^{\hat{\Omega}_S,\hat{\Omega}_A}[\phi, \pi] \) of Eq. (3a),

\[
\dot{\phi} = \delta H_{\text{class}}^{\hat{\Omega}_S,\hat{\Omega}_A}[\phi, \pi]/\delta\phi = \hat{\Omega}_S \pi + \hat{\Omega}_A \phi, \\
\dot{\pi} = -\delta H_{\text{class}}^{\hat{\Omega}_S,\hat{\Omega}_A}[\phi, \pi]/\delta\phi = -\hat{\Omega}_S \phi + \hat{\Omega}_A \pi.
\] (3b)

These equations have come out to be exactly the same as those of Eq. (2b), which are, in turn, equivalent to the Schrödinger Eq. (1a).

Now to convert the Eq. (2b) pair of first-order in time equations for \( \phi \) and \( \pi \) into an Euler-Lagrange style second-order in time equation for \( \phi \) alone requires that we solve for \( \pi \) in terms of \( \phi \) and \( \dot{\phi} \). If the real-valued symmetric operator \( \hat{\Omega}_S \) has the inverse \( \hat{\Omega}_S^{-1} \), we note from the first equality of Eq. (2b) that this is readily done, with the result,

\[
\pi = \hat{\Omega}_S^{-1}(\dot{\phi} - \hat{\Omega}_A \phi),
\] (4a)

which we can then substitute into the second equality of Eq. (2b), producing the second-order in time equation for \( \phi \) alone,

\[
\hat{\Omega}_S^{-1}(\ddot{\phi} - \hat{\Omega}_A \dot{\phi}) = -\hat{\Omega}_S \phi + \hat{\Omega}_A \hat{\Omega}_S^{-1}(\dot{\phi} - \hat{\Omega}_A \phi),
\] (4b)

which is more neatly rewritten as,

\[
\hat{\Omega}_S^{-1} \ddot{\phi} - (\hat{\Omega}_S^{-1} \hat{\Omega}_A + \hat{\Omega}_A \hat{\Omega}_S^{-1})\dot{\phi} + (\hat{\Omega}_S + \hat{\Omega}_A \hat{\Omega}_S^{-1} \hat{\Omega}_A)\phi = 0,
\] (4c)

an Euler-Lagrange style purely real-valued second-order in time equation for \( \phi \) alone. We take note that the sequential steps carried out in Eqs. (4a), (4b), and (4c) above are all reversible. In particular, it is apparent that Eq. (4c) is equivalent to Eq. (4b), and that Eq. (4a) is equivalent to the first equality of Eq. (2b). Furthermore, putting Eq. (4a) into Eq. (4b), or equally as well into Eq. (4c), produces the second equality of Eq. (2b). Therefore, not only does Eq. (2b) imply Eqs. (4a) and (4c), Eqs. (4a) and (4c) also imply Eq. (2b), which is equivalent to the Schrödinger Eq. (1a).

Thus Eqs. (4a) and (4c) above are equivalent to the Schrödinger Eq. (1a). This observation, however, is an infelicitous one as it stands because Eq. (4a) refers to \( \pi \), whereas the Schrödinger Eq. (1a) is exclusively
concerned with $\psi$. However, combining Eq. (4a) with the two Eq. (1b) definitions of $\pi$ and $\phi$ in terms of $\psi$ and $\psi^*$ in the form,

$$-i(h/2)^{\frac{1}{2}}(\psi - \psi^*) = \tilde{\Omega}_S^{-1}(\dot{\phi} - \tilde{\Omega}_A \phi), \quad (h/2)^{\frac{1}{2}}(\psi + \psi^*) = \phi, \quad (4d)$$

does enable the integration of Eq. (4a) into a formula for $\psi$ alone in terms of $\phi$ and $\dot{\phi}$. This is achieved by linearly combining the two equalities of Eq. (4d) so as to eliminate $\psi^*$ and give $\psi$ the coefficient of unity,

$$\psi = (2h)^{-\frac{1}{2}}[(1 - i\tilde{\Omega}_S^{-1}\tilde{\Omega}_A)\phi + i\tilde{\Omega}_S^{-1}\dot{\phi}]. \quad (4e)$$

It is therefore clear as a matter of mathematics that Eq. (4e) will, together with the Euler-Lagrange style Eq. (4c), permit derivation of the Schrödinger Eq. (1a). That fact can also be directly verified by making use of the "left factorization identities",

$$(\tilde{\Omega}_S + \tilde{\Omega}_A\tilde{\Omega}_S^{-1}\tilde{\Omega}_A) = (\tilde{\Omega}_S + i\tilde{\Omega}_A)(1 - i\tilde{\Omega}_S^{-1}\tilde{\Omega}_A) \quad \text{and} \quad (1 + i\tilde{\Omega}_A\tilde{\Omega}_S^{-1}) = (\tilde{\Omega}_S + i\tilde{\Omega}_A)\tilde{\Omega}_S^{-1},$$

after differentiating both sides of Eq. (4e) with respect to time and then inserting Eq. (4c) into the right-hand side of the result of that differentiation.

Of course the steps which we have displayed up to this point also make it apparent that Eq. (4e) and the Euler-Lagrange style Eq. (4c) both follow from the Schrödinger Eq. (1a) and the definitions and properties of $\phi$, $\tilde{\Omega}_S$ and $\tilde{\Omega}_A$ that are set out in Eqs. (1b) and (1c). Therefore, if the real-valued symmetric operator $\tilde{\Omega}_S$ indeed has the inverse $\tilde{\Omega}_S^{-1}$, then the relation of the Euler-Lagrange style Eq. (4c) to the Schrödinger Eq. (1a) is an isomorphic one via Eq. (4e).

Eq. (4e) is, of course, readily inverted after the extraction of its real and imaginary parts, with the result,

$$\phi = (h/2)^{\frac{1}{2}}(\psi + \psi^*), \quad (4f)$$

$$\dot{\phi} = -i(h/2)^{\frac{1}{2}}[(\tilde{\Omega}_S + i\tilde{\Omega}_A)\psi - (\tilde{\Omega}_S - i\tilde{\Omega}_A)\psi^*].$$

Of course the first equality of Eq. (4f) is nothing more than the definition of $\phi$ which is given in Eq. (1b), while the second equality of Eq. (4f) simply follows from straightforward application of the Schrödinger Eq. (1a) to the first equality.

In fact if we then proceed to likewise apply the Schrödinger Eq. (1a) to the second equality of Eq. (4f), and then use Eq. (4e) to eliminate $\psi$ and $\psi^*$ from the right hand side of what results, we find that (after a great many cancellations of multi-factor terms) the Euler-Lagrange style Eq. (4c) emerges.

In a nutshell, there are clearly many different ways (of varying algebraic complexity) to establish the isomorphism of the Euler-Lagrange style Eq. (4c) to the Schrödinger Eq. (1a) via the relationship expressed by Eq. (4e) (or by its inverse Eq. (4f)).

In the not infrequently occurring special circumstance that the Hermitian Hamiltonian operator $\tilde{H}$ is symmetric (and therefore real-valued), the real-valued antisymmetric operator $\tilde{\Omega}_A$ vanishes identically, which drastically simplifies the purely real-valued Euler-Lagrange style Eq. (4c). That equation thereupon becomes,

$$\ddot{\phi} + (\tilde{\Omega}_S)^2\phi = 0 \quad \text{or} \quad \ddot{\phi} + (\tilde{H}/h)^2\phi = 0, \quad (4g)$$

Eq. (4g) has the very same formal appearance as the equation for the complex-valued quantum wave vector $\psi$ which was proposed by Klein and Gordon [3]. In Eq. (4g), however, $\phi$ is (within a real-valued constant only the real part of $\psi$, as is completely apparent from Eq. (1b). This purely real-valued property of $\phi$ is obviously a critical aspect of the isomorphic relationship of the Schrödinger Eq. (1a) to Eq. (4c), which in this particular circumstance has reduced to Eq. (4g). In short, the complex-valued version of Eq. (4g) proposed by Klein and Gordon, wherein the real-valued $\phi$ is replaced by the complex-valued $\psi$, obviously destroys the isomorphic relationship of Eq. (4g) to the Schrödinger Eq. (1a) by doubling the number of its effective degrees of freedom relative to those of the Schrödinger equation, which obviously injects a plethora of extraneous solutions.
Taking up a loose end now which hasn’t yet been discussed, in case that \( \hat{\Omega}_S \) doesn’t actually have an inverse, then for a physically sensible system we would nevertheless expect the eigenvalue spectrum of both \( \hat{H} \) and \( \hat{\Omega}_S \) to be bounded below. So for a physically sensible system there ought to exist a nonnegative energy constant \( E_0 \) such that both \( \hat{H} + E_0 \) and \( \hat{\Omega}_S + (E_0/\hbar) \) do have inverses. Since the constants \( E_0 \) and \( (E_0/\hbar) \) commute with all operators, the thus modified quantum model will differ only almost trivially from the original one: one merely needs to subtract the constant value \( E_0 \) from all energy eigenvalues once the calculation is completed, while as well multiplying any time-evolved complex-valued quantum wave vector \( \psi(t; t_0) \) by the simple phase factor \( \exp[i(E_0/\hbar)(t - t_0)] \).

Finally, we wish to formally complete this discussion by explicitly passing from from the classical Hamiltonian functional \( H^{\text{class}}_{\Omega_S; \Omega_A} [\phi, \pi] \) of Eq. (3a) to its corresponding classical Lagrangian functional \( L^{\text{class}}_{\Omega_S; \Omega_A} [\phi, \dot{\phi}] \), and then obtain from the latter the formally resulting purely real-valued second-order in time Euler-Lagrange classical dynamical equation of motion for \( \phi \) alone. Since that Euler-Lagrange classical dynamical equation of motion must necessarily be consistent with the corresponding classical Hamilton pair of first-order in time dynamical equations of motion that are given by Eq. (3b), an equation pair which is identical to Eq. (2b), which is in turn a transcription of the real and imaginary parts of the Schrödinger Eq. (1a), there is simply no way that that Euler-Lagrange equation of motion could differ from the Euler-Lagrange style Eq. (4c), which is indeed isomorphic to both the Schrödinger Eq. (1a) as well as to the Eq. (2b) transcription of its real and imaginary parts. In fact, Eq. (4c) was specifically developed to be equivalent to the second equality of Eq. (2b) by way of the first equality of Eq. (2b).

Although there is thus no question that Eq. (4c) is the correct Euler-Lagrange classical equation of motion that is isomorphic to the Schrödinger Eq. (1a), it is still of technical interest to explicitly work out the classical Lagrangian functional \( L^{\text{class}}_{\Omega_S; \Omega_A} [\phi, \dot{\phi}] \) which is in fact complementary to the Eq. (1a) quantum Schrödinger system.

Passage from the Eq. (3a) classical Hamiltonian functional of \( \phi \) and \( \pi \) to its corresponding classical Lagrangian functional of \( \phi \) and \( \dot{\phi} \) of course requires that we solve for \( \pi \) in terms of \( \phi \) and \( \dot{\phi} \). But exactly that solution is given by Eq. (4a) above. What remains to obtain the classical Lagrangian functional \( L^{\text{class}}_{\Omega_S; \Omega_A} [\phi, \dot{\phi}] \) is largely algebra, albeit rather burdensome in its amount,

\[
L^{\text{class}}_{\Omega_S; \Omega_A} [\phi, \dot{\phi}] = (\dot{\phi} - H^{\text{class}}_{\Omega_S; \Omega_A} [\phi, \pi] |_{\pi=\hat{\Omega}_S^{-1}(\dot{\phi} - \hat{\Omega}_A \dot{\phi})}) = (\dot{\phi}, \hat{\Omega}_S^{-1}(\dot{\phi} - \hat{\Omega}_A \dot{\phi})) - \frac{1}{2}(\dot{\phi}, \hat{\Omega}_S \dot{\phi}) - \frac{1}{2}(\hat{\Omega}_S^{-1}(\dot{\phi} - \hat{\Omega}_A \dot{\phi}), (\dot{\phi} - \hat{\Omega}_A \dot{\phi})) + (\phi, \hat{\Omega}_A \hat{\Omega}_S^{-1}(\dot{\phi} - \hat{\Omega}_A \dot{\phi}))
\]

\[
= \frac{1}{2}[ (\dot{\phi}, \hat{\Omega}_S^{-1}(\dot{\phi} - \hat{\Omega}_A \dot{\phi})) - 2(\dot{\phi}, \hat{\Omega}_S \dot{\phi}) - (\phi, (\hat{\Omega}_S + \hat{\Omega}_A \hat{\Omega}_S^{-1} \hat{\Omega}_A) \dot{\phi})]
\]

\[
= \frac{1}{2}[ (\dot{\phi}, \hat{\Omega}_S^{-1}(\dot{\phi} - \hat{\Omega}_A \dot{\phi})) + 2(\phi, \hat{\Omega}_A \hat{\Omega}_S^{-1}(\dot{\phi} - \hat{\Omega}_A \dot{\phi})) - (\phi, (\hat{\Omega}_S + \hat{\Omega}_A \hat{\Omega}_S^{-1} \hat{\Omega}_A) \dot{\phi})]
\]

where multiply repeated use has been made of the symmetric nature of \( \hat{\Omega}_S^{-1} \) and the antisymmetric nature of \( \hat{\Omega}_A \).

We can now functionally differentiate \( L^{\text{class}}_{\Omega_S; \Omega_A} [\phi, \dot{\phi}] \) with respect to both \( \dot{\phi} \) and \( \phi \),

\[
\delta L^{\text{class}}_{\Omega_S; \Omega_A} [\phi, \dot{\phi}]/\delta \dot{\phi} = \hat{\Omega}_S^{-1} \dot{\phi} - \hat{\Omega}_A \dot{\phi},
\]

\[
\delta L^{\text{class}}_{\Omega_S; \Omega_A} [\phi, \dot{\phi}]/\delta \phi = \hat{\Omega}_A \hat{\Omega}_S^{-1} \dot{\phi} - (\hat{\Omega}_S + \hat{\Omega}_A \hat{\Omega}_S^{-1} \hat{\Omega}_A) \phi.
\]

Therefore, the Euler-Lagrange equation,

\[
d(\delta L^{\text{class}}_{\Omega_S; \Omega_A} [\phi, \dot{\phi}]/\delta \dot{\phi})/dt = \delta L^{\text{class}}_{\Omega_S; \Omega_A} [\phi, \dot{\phi}]/\delta \phi,
\]

works out to,

\[
\hat{\Omega}_S^{-1} \dot{\phi} - (\hat{\Omega}_S^{-1} \hat{\Omega}_A + \hat{\Omega}_A \hat{\Omega}_S^{-1}) \dot{\phi} + (\hat{\Omega}_S + \hat{\Omega}_A \hat{\Omega}_S^{-1} \hat{\Omega}_A) \phi = 0,
\]

namely exactly the same as Eq. (4c), which is definitely expected.
In the circumstance that $\hat{H}$ happens to be symmetric (and therefore real-valued), $\hat{\Omega}_A$ vanishes identically and $L_{\Omega_S,\Omega_A}^{\text{class}}[\phi, \dot{\phi}]$ greatly simplifies to become,

$$L_{\Omega_S}^{\text{class}}[\phi, \dot{\phi}] = \frac{i}{2}[(\phi, \hat{\Omega}_S^{-1}\dot{\phi}) - (\phi, \hat{\Omega}_S\dot{\phi})], \quad (5e)$$

whose correspondingly simplified real-valued Euler-Lagrange equation is of course given by Eq. (4g).

In this circumstance that $\hat{H}$ is symmetric, $\hat{H} = \hbar\hat{\Omega}_S$ from Eq. (1c), and the Schrödinger Eq. (1a) is simply written as,

$$\dot{\psi} = -i\hat{\Omega}_S\psi \quad \text{or} \quad \dot{\psi}^* = i\hat{\Omega}_S\psi^*, \quad (6a)$$

from which we can see Eq. (4f) simplifies to,

$$\phi = (\hbar/2)^\frac{1}{2}(\psi + \psi^*), \quad \dot{\phi} = -i(\hbar/2)^\frac{1}{2}\hat{\Omega}_S(\psi - \psi^*), \quad (6b)$$

whose inverse is also much simpler than Eq. (4e),

$$\psi = (2\hbar)^{-\frac{1}{2}}(\phi + i\hat{\Omega}_S^{-1}\dot{\phi}), \quad (6c)$$

and whose Euler-Lagrange Eq. (4g) follows from application of the Schrödinger Eq. (6a) to the second equality of Eq. (6b),

$$\ddot{\phi} + (\hat{\Omega}_S)^2\phi = 0 \quad \text{or} \quad \ddot{\phi} + (\hat{H}/\hbar)^2\phi = 0, \quad (6d)$$

Since source-free classical electrodynamics in radiation gauge obeys the classical wave equation, which has the form of the simplified Euler-Lagrange Eq. (6d), we can immediately deduce from it the applicable real-valued symmetric operator $\hat{\Omega}_S$, and therefore the nature of its complementary isomorphic Schrödinger equation, which, as pointed out in the first sentence of the foregoing paragraph, has the Hamiltonian operator $\hat{H} = \hbar\hat{\Omega}_S$.

Is is instructive and lamentable to note that notwithstanding the work of Klein and Gordon with quantum-mechanics related equations of the form of Eq. (6d), the isomorphic quantum complement to source-free classical electrodynamics which we are about to discuss was a completely closed book for those theorists because the equations of classical electrodynamics are, of course, strictly real-valued, whereas Klein and Gordon insisted on considering Eq. (6d) forms only in conjunction with strictly complex-valued solutions [3]. In this we have a worthwhile reminder that physics confronts beleaguered theorists with a multitude of subtle ways to wander astray. Still, it can reasonably be contended that Klein and Gordon ought to have regarded the surfeit of extraneous solutions which they soon encountered as a stern warning to backtrack and rethink.

The isomorphic quantum complement to source-free classical electrodynamics

Source-free classical electrodynamics in radiation gauge has vanishing scalar potential, while its vector potential is transverse and satisfies the classical wave equation [7], i.e.,

$$A^0 = 0, \quad \nabla \cdot \mathbf{A} = 0, \quad \ddot{\mathbf{A}} - c^2 \nabla^2 \mathbf{A} = \mathbf{0}. \quad (7a)$$

The classical wave equation certainly has the special simplified Euler-Lagrange form set out in Eq. (6d) with $\Omega_S = c(-\nabla^2)^\frac{1}{2}$ or $\hat{H} = \hbar c(-\nabla^2)^\frac{1}{2}$. Now in quantum mechanical configuration representation the three-momentum operator $\hat{\mathbf{p}}$ for a particle is equal to $-i\hbar \nabla$, so that the quantum Hamiltonian operator $\hat{\mathcal{H}} = \hbar c(-\nabla^2)^\frac{1}{2}$ which we have just deduced from the classical wave equation corresponds in particle three-momentum quantum operator terms to $\hat{\mathcal{H}} = c\hat{\mathbf{p}}$, which is the relativistic quantum Hamiltonian operator of a zero rest-mass free particle that has the three-momentum quantum operator $\hat{\mathbf{p}}$. In other words, it appears that the quantum complement of classical source-free electrodynamics is the Schrödinger equation of the free photon, which, of course has zero rest mass.

Further information about the complex-valued photon wave function ought to be available from Eq. (6c) above. However, because the source-free classical electrodynamics potential $\mathbf{A}$ is a vector field, it would
naturally be expected that Eq. (6c) would be slightly modified to assume the complex-valued vector wave function form,

\[
\Psi = (2\hbar)^{-\frac{1}{4}}(\Phi + (i/c)(-\nabla^2)^{-\frac{1}{4}}\Phi).
\] (7b)

We obviously hope to identify the real-valued vector field \(\Phi\) as just the radiation gauge classical vector potential \(A\) itself. However, if we require the dimension of \(\Psi\) to be that of a quantum mechanical wave function, the identification of \(\Phi\) as \(A\) itself isn't dimensionally feasible. That minor dimensional impasse is readily resolved, however, by inserting dimensionally needed bits and pieces of already known physical factors that actually pertain to this classical electrodynamic system, namely by letting,

\[
\Phi \overset{\text{def}}{=} e^{-\frac{i}{\hbar}}(-\nabla^2)^{\frac{1}{4}}A,
\] (7c)

so that Eq. (7b) becomes,

\[
\Psi = (2\hbar c)^{-\frac{1}{4}}((-\nabla^2)^{\frac{1}{4}} + (i/c)(-\nabla^2)^{-\frac{1}{4}}\dot{A}),
\] (7d)

and \(\Psi\) is readily verified to have the correct dimension for a quantum mechanical wave function.

From the classical wave equation, namely the third equality of Eq. (7a), we have that \(\dot{A} = c^2(-\nabla^2)A\). Using this, we obtain from Eq. (7d) (after performing some algebra) that,

\[
\Psi = -ic(-\nabla^2)^{\frac{1}{4}}\Psi,
\] (7e)

which is consistent with the Schrödinger equation for the free photon because the free-photon Hamiltonian operator is \(\hat{H} = \hbar c(-\nabla^2)^{\frac{1}{2}} = c\hat{p}\).

In addition to this deduction of the Schrödinger equation for the free photon, we can use the second equality of Eq. (7a), namely that \(\nabla \cdot A = 0\), to deduce from Eq. (7d) that,

\[
\nabla \cdot \Psi = 0.
\] (7f)

Therefore the free-photon complex-valued vector wave function is transverse, which implies that the free photon has transverse polarization (i.e., transverse “spin”).

Finally, it will be feasible to invert Eq. (7d), obtaining the classical \(A\) and \(\dot{A}\) in terms of the quantum \(\Psi\) and \(\Psi^*\), from which the classical wave equation can be obtained as a consequence of the Schrödinger equation, e.g., Eq. (7a). This is the complement of what was established when the effective Schrödinger Eq. (7e) was obtained through use on Eq. (7d) of the classical wave equation. That the quantum and classical equations in fact imply each other is, of course, the heart of the complementary isomorphism.

Adding Eq. (7d) to its complex conjugate, and subtracting its complex conjugate from it permits us to obtain its inversion, which is,

\[
A = ((\hbar c)/2)^{\frac{1}{4}}((-\nabla^2)^{\frac{1}{4}}(\Psi + \Psi^*),
\] (7g)

\[
\dot{A} = -ic((\hbar c)/2)^{\frac{1}{4}}((-\nabla^2)^{\frac{1}{4}}(\Psi - \Psi^*).
\]

Note that, as is usual is these cases, the second equality of Eq. (7g) follows from the first by application to it of the Schrödinger equation, e.g., \(\dot{A}\) can be obtained through application of Eq. (7e) to \(\Psi\) and \(\Psi^*\) on the right-hand side of the first equality of Eq. (7g).

It is now furthermore readily seen from Eqs. (7g) and (7e) that application of the Schrödinger equation to the second equality of Eq. (7g) proves to be equivalent to the classical wave equation for \(A\), i.e., to the third equality of Eq. (7a). In addition, application of Eq. (7f) to the first equality of Eq. (7g) shows that \(\nabla \cdot A = 0\), i.e., it demonstrates the second equality of Eq. (7a). Therefore the quantum Eqs. (7e) and (7f) for \(\Psi\) imply the two classical Eq. (7a) equalities for \(A\). Those implications are, of course, the needed second part of establishing the complementary isomorphism of the classical equations for \(A\) to the quantum equations for \(\Psi\), as is pointed out in the paragraph which precedes the one immediately above.

A last fact which is of interest is that it is feasible swap out of Eqs. (7d) and (7g) the source-free radiation gauge electromagnetic potential \(A\) described by the three equalities of Eq. (7a) in favor of the source-free electric and magnetic fields. Those source-free electric and magnetic fields are governed by the four well-known source-free classical Maxwell field equations,

\[
\dot{\mathbf{B}} = -c\nabla \times \mathbf{E}, \quad \dot{\mathbf{E}} = c\nabla \times \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = 0.
\] (8a)
It turns out that these four source-free classical Maxwell equations not only imply the free-photon effective Schrödinger Eq. (7e) and the free-photon transverse-spin condition of Eq. (7f), they are as well in turn implied by those free-photon quantum requirements. Thus we also have a situation of perfect complementary isomorphism of the physical behavior of the quantum free photon to the complete set of source-free classical Maxwell equations. Faraday and Maxwell were therefore the first physicists to practice quantum mechanics—indeed on the ultra-relativistic free photon and its peculiar transverse spin at that.

To pass from the source-free radiation gauge potential of Eq. (7a) to the source-free electric and magnetic fields, we note that the first Eq. (7a) equality, namely $A^0 = 0$, implies that $\mathbf{E} = -\hat{\mathbf{A}}/c$, or $\mathbf{A} = -c\mathbf{E}$ . Also the fact that $\mathbf{B} = \nabla \times \mathbf{A}$, together with the second Eq. (7a) equality, namely $\nabla \cdot \mathbf{A} = 0$, implies that $\nabla \times \mathbf{B} = -\nabla^2 \mathbf{A}$, or $\mathbf{A} = (-\nabla^2)^{-1}(\nabla \times \mathbf{B})$. With these relations in hand, we reexpress Eq. (7d) in terms of the source-free $\mathbf{E}$ and $\mathbf{B}$ fields,

\[ \Psi = (2\hbar c)^{-\frac{1}{2}}((-\nabla^2)^{-\frac{1}{2}}(\nabla \times \mathbf{B}) - i(-\nabla^2)^{-\frac{3}{2}}\mathbf{E}), \]  

(8b)

From the particular Eq. (8a) source-free classical Maxwell equation $\nabla \cdot \mathbf{E} = 0$, Eq. (8b) immediately yields,

\[ \nabla \cdot \psi = 0, \]  

(8c)

which is exactly the same free-photon transverse spin requirement as we previously obtained in the form of Eq. (7f). Furthermore, since it can be straightforwardly shown that the four Eq. (8a) source-free Maxwell equations imply both of the relations $\mathbf{E} = c(\nabla \times \mathbf{B})$ and $\nabla \times \mathbf{B} = -c(-\nabla^2)\mathbf{E}$, we also obtain from Eq. (8b) (after performing some algebra) that,

\[ \Psi = -i c(-\nabla^2)^{\frac{3}{2}}\psi, \]  

(8d)

which is exactly the same effective Schrödinger equation for a free photon as we previously obtained in the form of Eq. (7e).

If we extract the real and imaginary parts of Eq. (8b), it turns out that we can readily solve those for the $\mathbf{B}$ and $\mathbf{E}$ fields, expressed, respectively, in terms of the real and imaginary parts of the complex-valued free-photon wave function $\Psi$. The result of this straightforward exercise is,

\[ \mathbf{B} = ((\hbar c)/2)^{\frac{1}{2}}(-\nabla^2)^{-\frac{1}{2}}(\nabla \times (\Psi + \psi^*)), \]  

(8e)

\[ \mathbf{E} = i((\hbar c)/2)^{\frac{1}{2}}(-\nabla^2)^{\frac{1}{2}}(\Psi - \psi^*), \]

which can be checked by insertion of this result into Eq. (8b), bearing in mind the Eq. (8c) requirement of transverse free-photon spin.

We see that the source-free Maxwell equation $\nabla \cdot \mathbf{B} = 0$ holds identically for the $\mathbf{B}$ of Eq. (8e), while the source-free Maxwell equation $\nabla \cdot \mathbf{E} = 0$ follows from the $\mathbf{E}$ of Eq. (8e) in conjunction with the Eq. (8c) requirement of transverse free-photon spin.

The source-free Maxwell equation $\hat{\mathbf{E}} = c(\nabla \times \mathbf{B})$ follows from application of the Eq. (8d) effective Schrödinger equation for the free photon to the $\mathbf{E}$ of Eq. (8e) plus from taking the curl of the $\mathbf{B}$ of Eq. (8e) while bearing in mind the Eq. (8c) requirement of transverse free-photon spin.

The source-free Maxwell equation $\hat{\mathbf{B}} = -c(\nabla \times \mathbf{E})$ follows from application of the Eq. (8d) effective Schrödinger equation for the free photon to the $\mathbf{B}$ of Eq. (8e) plus from taking the curl of the $\mathbf{E}$ of Eq. (8e).

We thus see that the quantum free-photon effective Schrödinger equation and transverse spin requirement both follow from the four classical source-free Maxwell equations, while those four classical source-free Maxwell equations conversely follow from the quantum free-photon effective Schrödinger equation and transverse spin requirement. Therefore we here also have an archetypal quantum-classical complementary isomorphism.

It is certainly fascinating that an isomorphic mapping of the mathematics which describes a quantum Schrödinger-equation physical system (which here is the complex-valued transverse-vector free-photon wave function) can be so perfectly suited to the description of a related classical complementary physical system (which here is the source-free classical electromagnetic field).
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