Superalgebra structure on differential forms of manifold

Kentaro Mikami Tadayoshi Mizutani

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1 Introduction

Typical examples of Lie superalgebra are the direct sum of exterior power of tangent bundle of a differential manifold with the Schouten bracket or the matrix algebra whose elements are divided into 4 parts, and the artificial bracket operation. The Schouten bracket in the first example is useful to describe a 2-vector field \( \pi \) to be a Poisson tensor by \([\pi, \pi] = 0\). In this article, we introduce a notion of Lie superalgebra structure on the direct sum of exterior power of cotangent bundle of a differential manifold with a natural bracket. Also we introduce an extension of the Lie superalgebra structure by the 1-vector fields.

1.1 Quick review of \( \mathbb{Z} \)-graded Lie superalgebras

First we recall the definition of Lie superalgebra or pre Lie superalgebra we call sometimes.

**Definition 1** (\( \mathbb{Z} \)-graded Lie superalgebra). Suppose a real vector space \( g \) is graded by \( \mathbb{Z} \) as \( g = \bigoplus_{j \in \mathbb{Z}} g_j \) and has a \( \mathbb{R} \)-bilinear operation \([\cdot, \cdot]\) satisfying

\[
\begin{align*}
[\cdot, \cdot] & \subset g_i + g_j, \\
[X, Y] + (-1)^{xy}[Y, X] & = 0 \quad \text{where } X \in g_x \text{ and } Y \in g_y, \\
(-1)^{xz}[[X, Y], Z] + (-1)^{xy}[[Y, Z], X] + (-1)^{yz}[[Z, X], Y] & = 0.
\end{align*}
\]

Then we call \( g \) a \( \mathbb{Z} \)-graded or (pre) Lie superalgebra.

**Remark 1.** Super Jacobi identity (1.3) above is equivalent to the one of the following.

\[
\begin{align*}
[X, [Y, Z]] & = [[X, Y], Z] + (-1)^{xy}[[Y, Z], X] + (-1)^{yz}[[Z, X], Y] \quad \text{(1.4)} \\
[[X, Y], Z] & = [X, [Y, Z]] + (-1)^{yz}[[X, Z], Y] \quad \text{(1.5)}
\end{align*}
\]

**Definition 2** (Sub superalgebra). For a general \( \mathbb{Z} \)-graded Lie superalgebra \( g = \bigoplus_i g_i \) if \( h_i \) is a subspace of \( g_i \) for each \( i \), and satisfy \([h_i, h_j] \subset h_{i+j}\), then \( \bigoplus_i h_i \) is a \( \mathbb{Z} \)-graded Lie superalgebra, we call a Lie sub-superalgebra of \( g \).

**Remark 2.** Let \( g = \bigoplus_i g_i \) be a \( \mathbb{Z} \)-graded Lie superalgebra. From definitions, we see some obvious properties.

1. Assume \( g_\ell \neq (0) \). Then \( \tilde{g}^\ell = \bigoplus_{i>\ell} g_i \) or \( \bar{g}^\ell = g_0 + \tilde{g}^\ell \) are \( \mathbb{Z} \)-graded Lie sub-superalgebras of \( g \). If \( \ell \) is even, they are Lie algebras.

2. Fix \( \ell \geq 0 \). \( \bigoplus_{i \geq \ell} g_i \) is a \( \mathbb{Z} \)-graded Lie sub-superalgebra of \( g \).

2' Fix \( \ell \leq 0 \). \( \bigoplus_{i \leq \ell} g_i \) is a \( \mathbb{Z} \)-graded Lie sub-superalgebra of \( g \).
2 Super brackets on differential forms

We introduce a new example of $\mathbb{Z}$-graded Lie superalgebra with a super bracket associated from the exterior differentiation. We have a typical $\mathbb{Z}$-graded Lie superalgebra of the direct sum of $j$-multivectors $\Lambda^j T(M)$ whose super grade (or weight) is $j-1$ for $j=0, \ldots, \dim M$, where $M$ is a manifold.

| $\Lambda^* T(M)$ | 1-vec | 2-vec | 3-vec | $\cdots$ |
|-------------------|-------|-------|-------|---------|
| grading           | $\cdots$ | $-3$  | $-2$  | $-1$   |
| $\Lambda^* T^*(M)$ | $\cdots$ | 2-form | 1-form | 0-form | $\cdots$ |

So, the super grade of differential $i$-forms $\Lambda^i T^*(M)$ of "$\mathbb{Z}$-graded Lie superalgebra" of the direct sum of differential $i$-forms $\Lambda^i T^*(M)$ are expected to be $-(i+1)$, and we define the super grade (weight) of differential forms as follows:

**Definition 3.** The grade (in super sense) or weight of $\alpha \in \Lambda^a T^*(M)$ is defined by $-(1+a)$. Sometimes we abbreviate the grade $-(1+a)$ of $\alpha \in \Lambda^a T^*(M)$ as $a'$.

**Definition 4.** We define a bi-linear map

\[
\llbracket \alpha, \beta \rrbracket = (-1)^a d(\alpha \wedge \beta) = \alpha \wedge d\beta - (-1)^{a'b'} \beta \wedge d\alpha
\]

for $\alpha \in \Lambda^a T^*(M)$ and $\beta \in \Lambda^{b'} T^*(M)$.

**Theorem 5.** The direct sum $\Lambda^a T^*(M) \oplus \cdots \oplus \Lambda^1 T^*(M) \oplus \Lambda^0 T^*(M)$ becomes a $\mathbb{Z}$-graded Lie superalgebra with the bracket (2.1) and the grading by Definition 3.

**Proof:** Suppose $\alpha \in \Lambda^a T^*(M) = \mathfrak{g}_{[a]}$ and $\beta \in \Lambda^{b'} T^*(M) = \mathfrak{g}_{[b']}$, where $a' = -(1+a)$ and $b' = -(1+b)$. Then $\llbracket \alpha, \beta \rrbracket \in \Lambda^{a'+b'} T^*(M) = \mathfrak{g}_{[a'+b']}$ because $-1 - (1+a+b) = a' + b'$.

\[
\llbracket \beta, \alpha \rrbracket \llbracket \beta, \alpha \rrbracket = \beta \wedge d\alpha - (-1)^{b'a'} \alpha \wedge \beta = (-1)^{a'+b'} \alpha \wedge d\beta - (-1)^{a'b'} \beta \wedge d\alpha = (-1)^{a'+b'} \llbracket \alpha, \beta \rrbracket
\]

\[
= (-1)^{a'+b'} (0 - (-1)^{a'(b'+c')} (\beta \wedge d\gamma - (-1)^{b'c'} \gamma \wedge d\beta) \wedge d\alpha)
\]

\[
= (-1)^{b'} \beta \wedge d\gamma \wedge d\alpha + (-1)^{b'} \gamma \wedge d\beta
\]

and so

\[
\mathfrak{g}_{a,b',\gamma} (-1)^{a'c'} \llbracket \alpha, \llbracket \beta, \gamma \rrbracket \rrbracket = 0.
\]

\[\square\]

**Remark 3.** We remember that in the superalgebra $\sum_{i\geq 0} \Lambda^i T(M)$, $[f, g]_{S} = 0$ because of $\mathfrak{g}_{-2} = (0)$. But the definition above says if $\alpha, \beta$ are 0-forms, namely functions $f, g$, then $[f, g] = d(fg)$, which is a 1-form.

**Remark 4.** It is a great surprise to the authors if this theorem was previously unknown.

Concerning to choice of bracket, we do not claim that (2.1) is unique. In fact, a common constant multiple of (2.1) is also super bracket.
Remark 5. A constant 1 satisfies $[1, 1] = 0$ and $\phi : A \mapsto [1, A]$ is a derivation of degree 1, and satisfies $\phi \circ \phi = 0$ like the Poisson cohomology story. In fact, this is the coboundary operation of de Rham cohomology theory.

Some values of super bracket are
\[
[1, \alpha] = (-1)^a [\alpha, 1] = d\alpha , \\
[1, \alpha \wedge \beta] = d(\alpha \wedge \beta) = (-1)^a [\alpha, \beta] , \\
[1, \alpha \wedge \beta] = d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^a \alpha \wedge d\beta = [1, \alpha] \wedge \beta + (-1)^a \alpha \wedge [1, \beta] .
\]

More generally
\[
[\gamma, \alpha \wedge \beta] = [\gamma, \alpha] \wedge \beta + (-1)^c(1+a')\alpha \wedge [\gamma, \beta] + (-1)^c (d\gamma) \wedge \alpha \wedge \beta ,
\]

because
\[
[\gamma, \alpha \wedge \beta] = (-1)^c d(\alpha \wedge \beta \wedge \gamma) = (-1)^c (d\alpha) \wedge \beta \wedge \gamma + \alpha \wedge (d\beta) \wedge \gamma + (-1)^a \alpha \wedge \beta \wedge (d\gamma)
\]
\[
= [\gamma, \alpha] \wedge \beta + (-1)^{a+ac} \alpha \wedge \gamma \wedge d\beta
\]
\[
= [\gamma, \alpha] \wedge \beta + (-1)^{a+ac} \alpha \wedge ([\gamma, \beta] - (-1)^c (d\gamma) \wedge \beta)
\]
\[
= [\gamma, \alpha] \wedge \beta + (-1)^{a+ac} \alpha \wedge [\gamma, \beta] + (-1)^{c+1} (d\gamma) \wedge \alpha \wedge \beta .
\]

Remark 6. We may give a new grading by the grade of $i$-form by $1 + i$ if we only deal with the direct sum of forms, but we expect to handle differential forms and multi-vector fields, too.

3 Homology groups of our Lie superalgebra

3.1 Quick review of homology groups of general Lie superalgebra

In a usual Lie algebra homology theory, $m$-th chain space is the exterior product $\Lambda^m g$ of $g$ and the boundary operator essentially comes from the operation $X \wedge Y \mapsto [X, Y]$.

Likewise, in the case of $\mathbb{Z}$-graded Lie superalgebras, ”exterior algebra” is defined as the quotient of the tensor algebra $\otimes^m g$ of $g$ by the two-sided ideal generated by

\[
X \otimes Y + (-1)^xy \otimes X \quad \text{where} \quad X \in g_x, Y \in g_y ,
\]

and we denote the equivalence class of $X \otimes Y$ by $X \triangle Y$. Since $X_{\text{odd}} \triangle Y_{\text{odd}} = Y_{\text{odd}} \triangle X_{\text{odd}}$ and $X_{\text{even}} \triangle Y_{\text{any}} = -Y_{\text{any}} \triangle X_{\text{even}}$ hold, $\triangle^m g_k$ has a symmetric property for odd $k$ and has a skew-symmetric property for even $k$ with respect to $\triangle$.

Definition 6. For a $\mathbb{Z}$-graded Lie superalgebra $g = \sum_i g_i$, $m$-th chain space is defined by $C_m = \underbrace{g \triangle \cdots \triangle g}_{m}$ and the boundary operator is given by

\[
\partial(A_1 \triangle A_2 \triangle \cdots \triangle A_m) = \sum_{i<j} (-1)^{i-1+a_i} \sum_{i<j} \sum_{j<s} a_s A_1 \triangle \cdots \hat{A}_i \cdots \triangle [A_i, A_j] \triangle \cdots \triangle A_m
\]

where $A_i \in g_{a_i}$. We see that $\partial \circ \partial = 0$ holds, and have the homology groups by

\[
H_m(g) = \ker(\partial : C_m \to C_{m-1}) / \partial(C_{m+1}) .
\]
**Definition 7** (Weight). For each non-zero element \( u \) in \( g_{i_1} \cdots \cdots g_{i_m} \), we define \( u \) has the weight \( i_1 + \cdots + i_m \). We define the subspace of \( C_m \) by

\[
C_{m,w} = \sum_{i_1 \leq \cdots \leq i_m, \sum s=1^m i_s = w} g_{i_1} \cdots \cdots g_{i_m} ,
\]

which is the space of common weight \( w \).

The following is known well.

**Proposition 3.1.** The weight \( w \) is preserved by \( \partial \), i.e., we have \( \partial(C_{m,w}) \subset C_{m-1,w} \). Thus, for a fixed \( w \), we have \( w \)-weighted homology groups

\[
H_{m,w}(g) = \ker(\partial : C_{m,w} \to C_{m-1,w})/\partial(C_{m+1,w}) .
\]

Since the boundary operator is defined by (3.2), we see

\[
\begin{align*}
(3.3) \quad \partial(A_1 \triangle \cdots \triangle A_p \land B_1 \triangle \cdots \triangle B_q) &= \partial(A_1 \triangle \cdots \triangle A_p) \land B_1 \triangle \cdots \triangle B_q \\
&+ (-1)^p A_1 \triangle \cdots \triangle A_p \land \partial(B_1 \triangle \cdots \triangle B_q) \\
&+ [A_1 \triangle \cdots \triangle A_p, B_1 \triangle \cdots \triangle B_q]_{sz}
\end{align*}
\]

where

\[
\begin{align*}
(3.4) \quad [A_1 \triangle \cdots \triangle A_p, B_1 \triangle \cdots \triangle B_q]_{sz} &= \sum (-1)^{i-1+a_i(\sum_{i<s} a_i + \sum_{s<j} b_j) \cdots A_i\cdot \cdots \triangle A_p \land B_1 \triangle \cdots [A_i, B_j] \cdots} \\
(3.5) &= \sum (-1)^{p+j+(\sum_{i<s} a_i + \sum_{s<j} b_j) b_j} \sum [A_i, B_j] \cdots \triangle A_p \land B_1 \triangle \cdots \hat{B}_j \cdots \\
(3.6) &= \sum (-1)^{i+a_i \sum_{s<s} a_i + \sum_{s<j} b_j} \sum [A_i, B_j] \cdots \triangle A_p \land B_1 \triangle \cdots \hat{B}_j \cdots .
\end{align*}
\]

This new bracket \([\cdot, \cdot]_{sz}\), which is similar to the Schouten bracket in some case, satisfies

\[
[A_1 \cdots A_p, B_1 \cdots B_q \triangle C_1 \cdots C_r]_{sz} = [A_1 \cdots A_p, B_1 \cdots B_q]_{sz} \triangle C_1 \cdots C_r \\
+ (-1)^{1+q(\sum A_i)(\sum b_j)} B_1 \cdots B_q \triangle [A_1 \cdots A_p, C_1 \cdots C_r]_{sz}
\]

and also

\[
[A_1 \cdots A_p \triangle B_1 \cdots B_q, C_1 \cdots C_r]_{sz} = (-1)^p A_1 \cdots A_p \triangle [B_1 \cdots B_q, C_1 \cdots C_r]_{sz} \\
+ (-1)^{1+q(\sum b_j)(\sum_{s<s} c_s)} [A_1 \cdots A_p, C_1 \cdots C_r]_{sz} \triangle B_1 \cdots B_q .
\]

Assume that \( A_1 = \cdots = A_p = A \in g_{odd} \) in (3.4). Then we have

\[
(3.7) \quad [\triangle^p A, B_1 \triangle \cdots \triangle B_q]_{sz} = p(-1)^p \triangle^{p-1} A \triangle \sum_j (-1)^{j+b_j} \sum_{s<j} b_s \sum_{j=1}^q [A, B_j] \triangle B_1 \cdots
\]

\[
(3.8) \quad = p \triangle^{p-1} (-A) \triangle [A, B_1 \triangle \cdots \triangle B_q]_{sz} .
\]

Furthermore, assume every \( B_j \) has degree even, then

\[
(3.9) \quad [A, B_1 \triangle \cdots \triangle B_q]_{sz} = \sum_j B_1 \triangle \cdots \triangle [A, B_j]_{sz} \triangle \cdots \triangle B_q .
\]

Contrary, assume every \( C_j \) has degree odd, then

\[
(3.10) \quad [A, C_1 \triangle \cdots \triangle C_r]_{sz} = \sum_j (-1)^{j+1} C_1 \triangle \cdots \triangle [A, C_j]_{sz} \triangle \cdots \triangle C_r
\]

\[
(3.11) \quad = \sum_j [A, C_j]_{sz} \triangle C_1 \triangle \cdots \triangle \hat{C}_j \triangle \cdots \triangle C_r ,
\]

Thus, for a fixed \( w \), we have \( w \)-weighted homology groups

\[
H_{m,w}(g) = \ker(\partial : C_{m,w} \to C_{m-1,w})/\partial(C_{m+1,w}) .
\]
and

\[ [A, B_1 \triangle \cdots \triangle B_q \triangle C_1 \triangle \cdots \triangle C_r]_{sz} = [A, B_1 \triangle \cdots \triangle B_q]_{sz} \triangle C_1 \triangle \cdots \triangle C_r \]

\[ + B_1 \triangle \cdots \triangle B_q \triangle [A, C_1 \triangle \cdots \triangle C_r]_{sz} . \]

In particular, the boundary operator is written by left action as follows:

\[ \partial(A_0 \triangle A_1 \triangle \cdots \triangle A_m) = -A_0 \triangle \partial(A_1 \triangle \cdots \triangle A_m) + A_0 \cdot (A_1 \triangle \cdots \triangle A_m) \]

(3.13)

where

\[ A_0 \cdot (A_1 \triangle \cdots \triangle A_m) = \sum_{i=1}^{m} (-1)^{a_0 \sum_{s \leq i} a_s} A_1 \triangle \cdots \cdots \triangle [A_0, A_i] \triangle \cdots \triangle A_m \]

(3.14)

\[ = - [A_0, A_1 \triangle \cdots \triangle A_m]_{sz} \]

(3.15)

for each homogeneous elements \( A_i \in g_a \).

In lower degree, the boundary operator is given as below:

\[ \partial(A \triangle B) = [A, B] \]

(3.16)

\[ \partial(A \triangle B \triangle C) = -A \triangle [B, C] + [A, B] \triangle C + (-1)^{ab} B \triangle [A, C] \]

(3.17)

for each homogeneous elements \( A \in g_a, B \in g_b, C \in g_c \).

3.2 Some works of (co)homology groups of \( \mathbb{Z} \)-graded Lie superalgebra with the Schouten bracket

We only refer to [2] and [3].

4 Homology groups of superalgebra of left invariant Lie groups

Assume a Lie group \( G \) acts on \( M \). Then we are able to discuss \( G \)-invariant theory of \( \mathbb{Z} \)-graded Lie superalgebra of tangent bundle or cotangent bundle. In particular case of \( M = G \), we already have studied of \( \mathbb{Z} \)-graded Lie superalgebra of Lie algebra, where let \( \xi_i \) be a basis of Lie algebra with bracket relation

\[ [\xi_i, \xi_j] = \sum_k c^k_{ij} \xi_k \]

(4.1)

with \( c^k_{ij} \) are constants of structure. Let \( z_i \) be the dual of \( \xi_j \). Since

\[ (d z_i)(\xi_j, \xi_k) = \xi_j(z_i, \xi_k) - \xi_k(z_i, \xi_j) - \langle z_i, [\xi_j, \xi_k] \rangle = -c^j_{ik} \]

\[ d z_i = \sum_{j,k} (d z_i)(\xi_j, \xi_k) z_j \otimes z_k = - \sum_{j,k} c^j_{jk} z_j \otimes z_k \]

\[ = - \frac{1}{2} \sum_{j,k} c^j_{jk} z_k \otimes z_k \]

we have

\[ d z_i = - \frac{1}{2} \sum_{j,k} c^j_{jk} z_j \wedge z_k = - \sum_{j<k} c^j_{jk} z_j \wedge z_k , \]

(4.2)
4.1 When $M = G$ and dim $G = 2$

The $w$-weighted $m$-th chain space $C_{m,w}$ is given by

$$\sum_{i_1,i_2,i_3} g^{i_1}_{i_2} g^{i_2}_{i_3} \Delta g^{i_3}_{i_3} \text{ where } i_1 + i_2 + i_3 = m \text{ and } i_1 + 2i_2 + 3i_3 = -w.$$  

We see that $m \leq -w$ and $3m \geq -w$, namely $Q(m) \leq 0$, where $Q(m) = (-w - m)(-w - 3m)$. $m$ runs from ceil((-w)/3) to $-w$. Since the dimension of $g_{-2}$ is 2 and $\Delta^2 g_{-2}$ is a skew-symmetric power, and so it vanishes if $i_2 > 2$. Thus, solving linear equations of $i_1,i_3$ for cases of $i_2 = 0,1,2$, we see that

$$C_{m,w} = \begin{cases} 
\Delta^{m-3K} g_{-1} \Delta^0 g_{-2} \Delta^K g_{-3} + \Delta^{w-3K-1} g_{-1} \Delta^2 g_{-2} \Delta^{K-1} g_{-3} & \text{if } -w-m = 2K \\
\Delta^{m-3L-2} g_{-1} \Delta^1 g_{-2} \Delta^L g_{-3} & \text{if } -w-m = 2L+1,
\end{cases}$$

in short

$$C_{m,w} = \begin{cases} 
\Delta^{w-3K} K + \Delta^{w-3K-1} \Delta z_1 \Delta z_2 \Delta^{K-1} V & \text{if } -w-m = 2K \\
\Delta^{w-3L-2} \Delta^1 z_1 \Delta L V & \text{if } -w-m = 2L+1.
\end{cases}$$

(4.3)

For lower weight $w$, the chain complexes are as below.

| $w$ | $C_{1,w}$ | $C_{2,w}$ | $C_{3,w}$ | $C_{4,w}$ | $C_{5,w}$ | $C_{6,w}$ |
|-----|------------|------------|------------|------------|------------|------------|
| $-1$ | $g_{-1}$   | 0          | 0          | 0          | 0          | 0          |
| $-2$ | $g_{-2}$   | $\Delta^2 g_{-1}$ | 0          | 0          | 0          | 0          |
| $-3$ | $g_{-3}$   | $g_{-2} \Delta g_{-1}$ | $\Delta^3 g_{-1}$ | 0          | 0          | 0          |
| $-4$ | 0          | $g_{-3} \Delta g_{-1} + \Delta^2 g_{-2}$ | $g_{-2} \Delta \Delta^2 g_{-1}$ | $\Delta^4 g_{-1}$ | 0          | 0          |
| $-5$ | 0          | $g_{-3} \Delta g_{-2}$ | $g_{-3} \Delta \Delta^2 g_{-1} + \Delta^2 g_{-2} \Delta g_{-1}$ | $g_{-2} \Delta \Delta^3 g_{-1}$ | $\Delta^5 g_{-1}$ | 0          |

The dimension of each chain space is directly given by

$$\dim C_{m,w} = \begin{cases} 
1 & \text{if } Q(m) = 0 \\
2 & \text{if } -w-m = 2K > 0 \text{ and } Q(m) < 0 \\
2 & \text{if } -w-m = 2L+1 \text{ and } Q(m) < 0 \\
0 & \text{otherwise}
\end{cases}$$

Non-abelian Lie algebra is unique and the structure equation is $[\xi_1, \xi_2] = \xi_1$ and so $dz_1 = -z_1 \wedge z_2$, $dz_2 = 0$. We put $z_1 \wedge z_2$ by $V$. Then $dV = 0$.

The multiplications of super bracket are shown in the table below.

| $1 \in g_{-1}$ $z_i \in g_{-2}$ $V = z_1 \wedge z_2 \in g_{-3}$ |
|-------------------------------|-------------------------------|-------------------------------|
| $1 \in g_{-1}$                | $z_i \in g_{-2}$             | $V = z_1 \wedge z_2 \in g_{-3}$ |
| $1 \in g_{-1}$                | $z_i \in g_{-2}$             | $V = z_1 \wedge z_2 \in g_{-3}$ |
| $z_i \in g_{-1}$              | $z_i \in g_{-2}$             | $V = z_1 \wedge z_2 \in g_{-3}$ |
| $z_i \in g_{-1} \wedge z_2$   | $z_i \in g_{-2} \wedge z_2$  | $V = z_1 \wedge z_2 \in g_{-3}$ |

We see that

(4.4) $\partial(\Delta^a 1 \Delta^c V) = 0$ because of $[1,1] = 0, [1,V] = [V,1] = 0$ and $[V,V] = 0$.

(4.5) $\partial(\Delta^a 1 \Delta z_1 \Delta z_2 \Delta^c V) = az_2 \Delta^{a-1} 1 \Delta^{c+1} V$
\( \partial(\Delta^a 1 \Delta z_1 \Delta^c V) = (-1)^a a \Delta^{a-1} 1 \Delta (z_1 \cdot 1) \Delta^c V = \begin{cases} (-1)^a a \Delta^{a-1} 1 \Delta^{c+1} V & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases} \)

because of

\[
\partial(\Delta^a 1 \Delta z_1 \Delta z_2 \Delta^c V) = \partial(z_1 \Delta z_2 \Delta^a 1 \Delta^c V) = -z_1 \Delta \partial(z_2 \Delta^a 1 \Delta^c V) + z_1 \cdot (\Delta z_2 \Delta^a 1 \Delta^c V)
\]
\[
\partial(z_2 \Delta^a 1 \Delta^c V) = -z_2 \Delta \partial(\Delta^a 1 \Delta^c V) + z_2 \cdot (\Delta^a 1 \Delta^c V) = 0
\]
\[
z_1 \cdot (z_2 \Delta^a 1 \Delta^c V) = (z_1 \cdot z_2) \Delta^a 1 \Delta^c V + a z_2 \Delta (z_1 \cdot 1) \Delta^{-1} a \Delta^{-1} 1 \Delta^c V + c_z \Delta^a 1 \Delta (z_1 \cdot 1) \Delta^{-1} V
\]
\[
0 + a z_2 \Delta (V) \Delta^{-1} a \Delta^{-1} 1 \Delta^c V + 0 = a z_2 \Delta^{-1} 1 \Delta^c V.
\]
\[
\partial(\Delta^a 1 \Delta z_i \Delta^c V) = (-1)^a \partial(z_i \Delta^a 1 \Delta^c V) = (-1)^a a \Delta^{-1} a \Delta^{-1} 1 \Delta (z_i \cdot 1) \Delta^c V.
\]

From (4.4) to (4.6), (4.3) shows generators of the boundary image of each chain space as follows:

\[
\begin{align*}
\partial C_{m,w} &= \begin{cases} 0 & \text{if } Q(m) = 0 \\ (-w - 3K - 1) z_2 \Delta^{-w-3K-2} 1 \Delta^K V & \text{if } Q(m) < 0 \text{ and } -w - m = 2K \\ (-1)^{-w-3L-2}(-w - 3L - 2) \Delta^{-w-3L-3} 1 \Delta^{L+1} V & \text{if } Q(m) < 0 \text{ and } -w - m = 2L + 1 \\ 0 & \text{otherwise} \end{cases} \\
\end{align*}
\]

and so

\[
\dim(\partial C_{m,w}) = \begin{cases} 0 & \text{if } Q(m) = 0 \text{ or } m = \lceil -w/3 \rceil \\ 1 & \text{if } Q(m) < 0 \text{ and } m > \lceil -w/3 \rceil \text{ or } m = \lceil -w/3 \rceil \\ 0 & \text{otherwise} \end{cases}
\]

In short, both ends of chain complex vanish by the boundary operator \( \partial \). Each chain space between both ends has rank 1. Thus, we have the following tables of space dimensions and rank of the boundary operator and the Betti numbers: When \(-w = 3\Omega + \varepsilon \) with \( \varepsilon = \pm 1 \) and we put \( \Omega_0 = \lceil -w/3 \rceil = \Omega + (\varepsilon + 1)/2 \).

| \( -w = 1 \) | \( -w = 2 \) | \( -w = 3\Omega \) |
|---|---|---|
| \( \dim \) | 1 | 2 |
| \( \text{rank} \) | 0 | 0 |
| \( \text{Betti} \) | 1 | 2 |

| \( -w = 3\Omega + \varepsilon > 3 \) |
|---|
| \( \Omega_0 \) | \( \Omega_0 + 1 \) |

| \( -w = 3\Omega + \varepsilon \) |
|---|
| \( \Omega_0 + 2 \) | \( 3\Omega + \varepsilon - 2 \) |

| \( \text{SPdim} \) | 1 | 2 | 2 | 2 | 2 | 1 |
| \( \text{rank} \) | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| \( \text{Betti} \) | 1 | 0 | 0 | \( \cdots \) | 0 | 1 | 1 |

| \( -w = 3\Omega + \varepsilon \) |
|---|
| \( \Omega_0 + 2 \) | \( 3\Omega + \varepsilon - 1 \) | \( 3\Omega + \varepsilon \) |

| \( \text{SPdim} \) | 2 | 2 | 2 | \( \cdots \) | 2 | 2 | 1 |
| \( \text{rank} \) | 1 | 1 | 1 | \( \cdots \) | 1 | 0 | 0 |
| \( \text{Betti} \) | 1 | 0 | 0 | \( \cdots \) | 0 | 1 | 1 |
4.2 When $M = G$ and $\dim G = 3$

4.2.1 Common issues

“General” multiplication table of super bracket is shown as follows:

|       | $1 \in \mathfrak{g}_{-1}$ | $z_i \in \mathfrak{g}_{-2}$ | $z_j \land z_k \in \mathfrak{g}_{-3}$ | $z_1 \land z_2 \land z_3 \in \mathfrak{g}_{-4}$ |
|-------|-----------------------------|-------------------------------|---------------------------------|----------------------------------|
| $1 \in \mathfrak{g}_{-1}$ | 0                           | $dz_i$                      | $d(z_j \land z_k)$               | 0                                |
| $z_p \in \mathfrak{g}_{-2}$ | $-dz_p$                  | $-d(z_p \land z_i)$           | 0                               | 0                                |
| $z_q \land z_r \in \mathfrak{g}_{-3}$ | $d(z_p \land z_q)$   | 0                             | 0                               | 0                                |
| $z_1 \land z_2 \land z_3 \in \mathfrak{g}_{-4}$ | 0                           | 0                             | 0                               | 0                                |

We denote the 2-form $z_i \land z_{i+1}$ by $w_{i+2}$, where indices are reduced by modulo 3. Put $z_1 \land z_2 \land z_3$ by $V$. Each chain space $C_{m, w}$ consists of spaces $\triangle^i \mathfrak{g}_{-1} \triangle^2 \mathfrak{g}_{-2} \triangle^3 \mathfrak{g}_{-3} \triangle^4 \mathfrak{g}_{-4}$ where $i_2 = 0, 1, 2, 3$ and $i_4 = 0, 1,$ and (4.9)

$$m = i_1 + i_2 + i_3 + i_4 \quad \text{and} \quad w = i_1 + 2i_2 + 3i_3 + 4i_4.$$  

From $i_1 + i_2 + i_3 = m - i_4$ and $i_1 + 2i_2 + 3i_3 = -w - 4i_4$, we see $m - i_4 \leq -w - 4i_4 \leq 3(m - i_4)$, i.e.,

$$\frac{-w - 1}{3} \leq m \leq -w.$$  

Solving the linear equations (4.9), we have

$$2i_1 = 3m + w - i_2 + i_4, \quad 2i_3 = -w - m - i_2 - 3i_4.$$  

Depending on the cases $-w - m = 2K$ or $-w - m = 2L + 1$, the above equations become

$$2i_1 = -2w - 6K - i_2 + i_4, \quad 2i_3 = 2K - i_2 - 3i_4$$  

$$2i_1 = -2w - 6L - 3 - i_2 + i_4, \quad 2i_3 = 2L + 1 - i_2 - 3i_4.$$  

When $-w = 3\Omega + \varepsilon$, $\varepsilon = 0, \pm 1$,

$$2i_1 = -2w - 6K - i_2 + i_4 = (2(3\Omega + \varepsilon) - 6K - i_2 + i_4), \quad 2i_3 = 2K - i_2 - 3i_4 \quad \text{if} \quad i_2 + i_4 \text{ is even},$$  

$$2i_1 = -2w - 6L - 3 - i_2 + i_4 = 2(3\Omega + \varepsilon) - 6L - 3 - i_2 + i_4, \quad 2i_3 = 2L + 1 - i_2 - 3i_4 \quad \text{if} \quad i_2 + i_4 \text{ is odd}.$$  

Thus,

$$(4.11) \quad C_{w, -w - 2K} = \sum_{i_2 + i_4 = \text{even}} \triangle^{3\Omega + \varepsilon - 3K - i_2/2 + i_4/2} \mathfrak{g}_{-1} \triangle^2 \mathfrak{g}_{-2} \triangle^{K - i_2/2 - 3i_4/2} \mathfrak{g}_{-3} \triangle^4 \mathfrak{g}_{-4}$$

$$= \triangle^{w - 3K} \mathfrak{g}_{-1} \triangle^{K} \mathfrak{g}_{-3} \triangle^4 \mathfrak{g}_{-4} + \triangle_{-w - 3K - 1} \mathfrak{g}_{-1} \triangle^2 \mathfrak{g}_{-2} \triangle^{K - 1} \mathfrak{g}_{-3} \triangle^0 \mathfrak{g}_{-4}$$

$$(4.12) \quad C_{w, -w - 2L - 1} = \sum_{i_2 + i_4 = \text{odd}} \triangle^{3\Omega + \varepsilon - 3L - 3 - i_2/2 + i_4/2} \mathfrak{g}_{-1} \triangle^2 \mathfrak{g}_{-2} \triangle^{L + 1/2 - i_2/2 - 3i_4/2} \mathfrak{g}_{-3} \triangle^4 \mathfrak{g}_{-4}$$

$$= \triangle^{w - 3L - 2} \mathfrak{g}_{-1} \triangle^L \mathfrak{g}_{-3} \triangle^0 \mathfrak{g}_{-4} + \triangle_{-w - 3L - 3} \mathfrak{g}_{-1} \triangle^3 \mathfrak{g}_{-2} \triangle^{L - 1} \mathfrak{g}_{-3} \triangle^0 \mathfrak{g}_{-4}$$

The expressions above imply the following dimension formula, where $\binom{p}{q}$ is $\frac{p!}{q!(p-q)!}$ only when $p \geq 0$, $q \geq 0$ and $p \geq q$. $\binom{p}{q}$ is 0 if $p < 0$ or $q < 0$ or $p < q$.

$$\dim C_{w, -w - 2K} = \left( -1 + 3\Omega + \varepsilon - 3K \right) \left( -1 + K \right)$$

$$= \left( 1 + 3\Omega + \varepsilon - 3K \right) \left( 3 - 1 + K \right)$$

$$= \left( 1 + 3\Omega + \varepsilon - 3K \right) \left( 3 - 1 + K \right)$$
\[ (4.13b) \quad + \left(1-1+3\Omega+\varepsilon-3K\right) \binom{1}{1} \binom{3-1+K-2}{K-2} + \left(1-1+3\Omega+\varepsilon-3K-1\right) \binom{3}{3} \binom{3-1+K-3}{K-3} \]

\[ (4.13c) \quad = \binom{3\Omega+\varepsilon-3K}{0} \binom{K+2}{2} + 3\binom{3\Omega+\varepsilon-3K-1}{0} \binom{K+1}{K-1} \]
\[ + 3\binom{3\Omega+\varepsilon-3K}{0} \binom{K-2}{2} + \binom{3\Omega+\varepsilon-3K-1}{0} \binom{K-1}{K-3} \]

\[ (4.13d) \quad = \begin{cases} \binom{K+2}{2} + 3\binom{K+1}{K-1} + 3\binom{K}{K-2} + \binom{K-1}{K-3} & \text{if } K \leq \Omega + \frac{\varepsilon-1}{3} \\
0 & \text{if } \varepsilon = 0 \text{ and } K = \Omega \end{cases} \]

\[ (4.13e) \quad \dim C_{w-2L-1,w} = \binom{1-1+3\Omega+\varepsilon-3L-2}{1} \binom{3}{3} \binom{3-1+L}{3-1} + \binom{1-1+3\Omega+\varepsilon-3L-3}{1} \binom{3}{3} \binom{3-1+L-1}{L-1} + \binom{1-1+3\Omega+\varepsilon-3L-2}{1} \binom{3-1+L-1}{L-1} + \binom{1-1+3\Omega+\varepsilon-3L-3}{1} \binom{3-1+L-2}{L-2} \]
\[ (4.13f) \quad = \binom{3\Omega+\varepsilon-3L-2}{0} \binom{L+2}{2} + \binom{3\Omega+\varepsilon-3L-3}{0} \binom{L+1}{L-1} + \binom{3\Omega+\varepsilon-3L-2}{0} \binom{L+1}{L-1} + \binom{3\Omega+\varepsilon-3L-3}{0} \binom{L+1}{L-1} \]
\[ (4.13g) \quad = 3\binom{3\Omega+\varepsilon-3L-2}{0} \left( \binom{L+2}{2} + \binom{L}{L-2} \right) + \left( \binom{3\Omega+\varepsilon-3L-3}{0} + \binom{3\Omega+\varepsilon-3L-1}{0} \right) \binom{L+1}{L-1} \]
\[ (4.13h) \quad = \begin{cases} 3\binom{L+2}{2} + 2\binom{L+1}{L-1} & \text{if } L \leq \Omega + \frac{\varepsilon-1}{3} \\
\binom{L+1}{L-1} + 2\binom{L+1}{L-1} & \text{if } \varepsilon = -1 \text{ and } L = \Omega - 1 \\
\binom{L+1}{L-1} & \text{if } \varepsilon = 1 \text{ and } L = \Omega \end{cases} \]

For lower weight \( w \), the chain complexes are as below.

| \( w \) | \( C_{1,w} \) | \( C_{2,w} \) | \( C_{3,w} \) | \( C_{4,w} \) | \( C_{5,w} \) | \( C_{6,w} \) |
|------|---------|---------|---------|---------|---------|---------|
| -1   | \( g_{-1} \) | 0       | 0       | 0       | 0       | 0       |
| -2   | \( g_{-2} \) | \( \triangle^2 g_{-1} \) | 0       | 0       | 0       | 0       |
| -3   | \( g_{-3} \) | \( g_{-3} \triangle g_{-2} \) | \( \triangle^3 g_{-1} \) | 0       | 0       | 0       |
| -4   | \( g_{-4} \) | \( g_{-1} \triangle g_{-3} + \triangle^2 g_{-2} \) | \( \triangle^2 g_{-1} \triangle g_{-2} \) | \( \triangle^4 g_{-1} \) | 0       | 0       |
| -5   | 0       | \( g_{-1} \triangle g_{-4} + g_{-2} \triangle g_{-3} \) | \( \triangle^2 g_{-1} \triangle g_{-3} + g_{-1} \triangle^2 g_{-2} \) | \( \triangle^3 g_{-1} \triangle g_{-2} \) | \( \triangle^5 g_{-1} \) | 0       |
| -6   | 0       | \( g_{-2} \triangle g_{-4} + \triangle^2 g_{-3} \) | (*)     | (**)    | \( \triangle^4 g_{-1} \triangle g_{-2} \) | \( \triangle^6 g_{-1} \) |

where \((*) = g_{-1} \triangle g_{-2} \triangle g_{-3} + \triangle^2 g_{-1} \triangle g_{-4} + \triangle^3 g_{-2}\) and \((**) = \triangle^3 g_{-1} \triangle g_{-3} + \triangle^2 g_{-1} \triangle^2 g_{-2}\).

The dimensions of those chain spaces are the following:

| \( w \) | \( \dim C_{1,w} \) | \( \dim C_{2,w} \) | \( \dim C_{3,w} \) | \( \dim C_{4,w} \) | \( \dim C_{5,w} \) | \( \dim C_{6,w} \) |
|------|---------|---------|---------|---------|---------|---------|
| -1   | 1       | 0       | 0       | 0       | 0       | 0       |
| -2   | 3       | 1       | 0       | 0       | 0       | 0       |
| -3   | 3       | 3       | 1       | 0       | 0       | 0       |
| -4   | 1       | 6       | 3       | 1       | 0       | 0       |
| -5   | 0       | 10      | 6       | 3       | 1       | 0       |
| -6   | 0       | 9       | 11      | 6       | 3       | 1       |

Before discussing individual cases depending on each Lie algebra structure, We now study of common behavior of the boundary operator. We use impolite notations here like \( 1^a \) instead of \( \triangle^a 1 \) or \( WC = w_1^a w_2^b w_3^c \) instead of
\[ \Delta^{c_j} w_1 \Delta^{c_2} w_2 \Delta^{c_3} w_3. \] And we abbreviate  \( Z^B = z_1^{b_1} \triangle z_2^{b_2} \triangle z_3^{b_3} \) by  \( Z^B = z_1^{b_1} z_2^{b_2} z_3^{b_3} \) where  \( b_i = 0, 1 \). Now a typical generator of the chain space is written as  \( 1^a \triangle Z^B \triangle W^C \triangle V^\ell \) or  \( 1^a Z^B W^C V^\ell \) where  \( \ell = 0, 1 \), and whose weight is  \( a + 2|B| + 3|C| + 4\ell \) and degree is  \( a + |B| + |C| + \ell \). Using properties of \( \partial \) here,

\[
\partial(1^a \triangle Z^B \triangle W^C) = \partial(1^a) \triangle Z^B \triangle W^C + (-1)^a \partial(Z^B) \triangle W^C + [1^a, Z^B \triangle W^C]_{sz}
\]

\[
= 0 + (-1)^a ((\partial Z^B) \triangle W^C + (-1)^{|B|} Z^B \triangle \partial W^C + [Z^B, W^C]_{sz})
\]

\[
+ [1^a, Z^B \triangle W^C]_{sz}
\]

\[
= (-1)^a (\partial Z^B) \triangle W^C + 0 + [1^a, Z^B \triangle W^C]_{sz}
\]

thus, we have

\[
(4.14a) \quad \partial(1^a \triangle Z^B \triangle W^C) = (-1)^a(\partial Z^B) \triangle W^C + [1^a, Z^B \triangle W^C]_{sz}
\]

\[
(4.14b) \quad [1^a, Z^B \triangle W^C]_{sz} = a(-1)^{a-1}[1^a, Z^B \triangle W^C]_{sz}
\]

\[
(4.14c) \quad = a(-1)^{a-1}([1^a, Z^B]_{sz} \triangle W^C + Z^B \triangle [1^a, W^C]_{sz})
\]

and

\[
(4.14d) \quad \partial(1^a \triangle Z^B \triangle W^C \triangle V^\ell) = \partial(1^a \triangle Z^B \triangle W^C) \triangle V^\ell.
\]

And also

\[
(4.14e) \quad [1, z_i \triangle z_j]_{sz} = [1, z_i]_{sz} \triangle z_j + z_i \triangle [1, z_j]_{sz},
\]

\[
(4.14f) \quad \partial(z_i \triangle z_j) = [z_i, z_j],
\]

\[
(4.14g) \quad [1, z_i \triangle z_2 \triangle z_3]_{sz} = [1, z_i]_{sz} \triangle z_2 \triangle z_3 + z_i \triangle [1, z_2]_{sz} \triangle z_3 + z_i \triangle z_2 \triangle [1, z_3]_{sz},
\]

\[
(4.14h) \quad \partial(z_i \triangle z_2 \triangle z_3) = \mathcal{G} [z_i, z_2] [z_i, z_3].
\]

Each subspace  \( \triangle^a g_{-1} \triangle^b g_{-2} \triangle^c g_{-3} \triangle^\ell g_{-4} \), which is a part of chain space, and is denoted by  \( 1^a Z^b W^c V^\ell \) in short. More precise expression by using basis is given as follows:

\[
1^a Z^b W^c V^\ell = \sum_{|B|=b,|C|=c} \lambda^a_{B,[C,\ell]} 1^a Z^B W^C V^\ell
\]

where  \( B = (b_1, b_2, b_3) \in \{0, 1\}^3, C = (c_1, c_2, c_3) \in \mathbb{N}^3, |B| = b_1 + b_2 + b_3, |C| = c_1 + c_2 + c_3, \ell = 0 \text{ or } 1 \), and  \( \lambda^a_{B,[C,\ell]} \) are scalars.

As described in (4.11) or (4.12), each chain space is given:

\[
C_{w-2K, w} = 1^{-w-3K} Z^0 W^K + 1^{-w-3K-1} Z^2 W^K-1 + 1^{-w-3K} Z^1 W^{K-2} V + 1^{-w-3K-1} Z^3 W^{K-3} V
\]

\[
C_{w-2L, w} = 1^{-w-3L-2} Z^1 W^L + 1^{-w-3L-3} Z^3 W^{L-1} + 1^{-w-3L-1} Z^0 W^{L-1} V + 1^{-w-3L-2} Z^2 W^{L-2} V.
\]

Thus, using the fact that  \( [1, w_i]_{sz} = u V \) for some  \( u \), and so  \( [1, w_i]_{sz} V = 0 \), the boundary image is written in general as follows:

When  \( -w - m = 2K \), then each element in  \( \partial C_{m, w} \) is given by

\[
X = + \sum \lambda^{[-w-3K,0]}_{[000],[C,K]} ([1^{-w-3K}, W^C]_{sz})
\]

\[
+ \sum \lambda^{[-w-3K-1,0]}_{[011],[C,K]-1} ((-1)^{-w-3K-1} (\partial(z_2 z_3)) W^C + [1^{-w-3K-1}, z_2 z_3 W^C]_{sz})
\]

\[
+ \sum \lambda^{[-w-3K-1,0]}_{[101],[C,K]-1} ((-1)^{-w-3K-1} (\partial(z_1 z_3)) W^C + [1^{-w-3K-1}, z_1 z_3 W^C]_{sz})
\]

\[
+ \sum \lambda^{[-w-3K-1,0]}_{[110],[C,K]-1} ((-1)^{-w-3K-1} (\partial(z_1 z_2)) W^C + [1^{-w-3K-1}, z_1 z_2 W^C]_{sz})
\]
from the original form

\[ X_{\text{org}} = \sum_i \lambda_{[w-3K,0]}^{[001],[C,K]_2} [1^{w-3K}, z_{i3}] W^C V \]

\[ + \sum_i \lambda_{[w-3K-1,0]}^{[011],[C,K]_2} [1^{w-3K-1}, z_{i3} W^C]_{z_{i3}} \]

\[ + \sum_i \lambda_{[w-3K-1,0]}^{[010],[C,K]_2} [1^{w-3K-1}, z_{i3}] W^C V \]

\[ + \sum_i \lambda_{[w-3K,1]}^{[010],[C,K]_2} [1^{w-3K}, z_{i3}] W^C V \]

\[ + \sum_i \lambda_{[w-3K-1,1]}^{[111],[C,K]_2} [1^{w-3K-1}, z_{i3} W^C]_{z_{i3}} \]

Similarly, when \(-w - m = 2L + 1\), then each element in \(\partial C_{m,w}\) is given by

\[ Y = \sum_i \lambda_{[w-3L-2,0]}^{[001],[C,L]_2} [1^{w-3L-2}, z_{i3}] W^C]_{z_{i3}} \]

\[ + \sum_i \lambda_{[w-3L-2,0]}^{[010],[C,L]_2} [1^{w-3L-2}, z_{i3}] W^C]_{z_{i3}} \]

\[ + \sum_i \lambda_{[w-3L-2,0]}^{[100],[C,L]_2} [1^{w-3L-2}, z_{i3}] W^C]_{z_{i3}} \]

\[ + \sum_i \lambda_{[w-3L-3,0]}^{[111],[C,L]_2} [1^{w-3L-3}, z_{i3}] W^C]_{z_{i3}} \]

\[ + \sum_i \lambda_{[w-3L-2,1]}^{[011],[C,L]_2} [1^{w-3L-2}, z_{i3}] W^C]_{z_{i3}} \]

\[ + \sum_i \lambda_{[w-3L-2,1]}^{[010],[C,L]_2} [1^{w-3L-2}, z_{i3}] W^C]_{z_{i3}} \]

\[ + \sum_i \lambda_{[w-3L-2,1]}^{[100],[C,L]_2} [1^{w-3L-2}, z_{i3}] W^C]_{z_{i3}} \]

Since both \(\partial\) and \([\cdot, \cdot]_{z_{i3}}\) depend on the Lie algebra structure, our discussion needs to be developed individually.

### 4.2.2 Case of SO(3):

Now we have

\[ dz_i = -2w_i \quad i = 1, 2, 3 \]

and the brackets of superalgebra are given as follows:

\[
\begin{align*}
[z_1, 1] & = 0 \quad [z_1, z_j] = -2w_j \quad [1, w_j] = 0 \quad [1, V] = 0 \\
[z_2, 1] & = 2w_1 \quad [z_2, z_j] = 0 \quad [z_2, w_j] = 0 \quad [z_2, V] = 0 \\
[w_1, 1] & = 0 \quad [w_1, z_j] = 0 \quad [w_1, w_j] = 0 \quad [w_1, V] = 0 \\
[V, 1] & = 0 \quad [V, z_j] = 0 \quad [V, w_j] = 0 \quad [V, V] = 0
\end{align*}
\]

\(V\) and \(w_i\) are central elements of this algebra. \((4.14a) \sim (4.14b)\) imply

\[ \partial(1^a Z^B W^C V^\ell) = a(-1)^{a-1}[1, Z^B]_{z_{i3}} W^C V^\ell \]

\[ [1, z_j]_{z_{i3}} = -2w_j \]
\[ [1, z_j \triangle z_k]_{xz} = (z_j \triangle (-2w_k) + z_k \triangle (-2w_j)) \]
\[ [1, z_1 \triangle z_2 \triangle z_3]_{xz} = [1, z_1] \triangle z_2 \triangle z_3 + z_1 \triangle [1, z_2] \triangle z_3 + z_1 \triangle z_2 \triangle [1, z_3] \]
\[ = z_1 \triangle z_2 \triangle (-2w_3) - z_1 \triangle z_3 \triangle (-2w_2) + z_2 \triangle z_3 \triangle (-2w_1) \]

We remember the decomposition of chain spaces by (4.11) and (4.12), and knowing the properties of the boundary operator which is defined by the Lie algebra structure, we first have direct sum decomposition of subspaces as below:

When \( m = -w - 2K \), a general element in \( \partial C_{m,w} \) is written as
\[
X = +2 \sum \lambda_{[-w-3K-1,0]} \bigl(-w-3K-1\bigr)(-II)^{-w-3K-2}(z_3 w_2 - z_2 w_3)W^C
+2 \sum \lambda_{[-w-3K-1,1]} \bigl(-w-3K-1\bigr)(-II)^{-w-3K-2}(z_3 w_1 - z_1 w_3)W^C
+2 \sum \lambda_{[-w-3K-1,1]} \bigl(-w-3K-1\bigr)(-II)^{-w-3K-2}(z_2 w_1 - z_1 w_2)W^C
-2 \sum \lambda_{[-w-3K,1]} \bigl(-w-3K\bigr)(-II)^{-w-3K-1}w_3 W^C V
-2 \sum \lambda_{[-w-3K,1]} \bigl(-w-3K\bigr)(-II)^{-w-3K-1}w_2 W^C V
-2 \sum \lambda_{[-w-3K,0]} \bigl(-w-3K\bigr)(-II)^{-w-3K-1}w_1 W^C V
-2 \sum \lambda_{[-w-3K,1]} \bigl(-w-3K\bigr)(-II)^{-w-3K-2}(z_3 z_1 w_1 - z_3 z_2 w_2 + z_1 z_2 w_3)W^C V
\]

And when \( m = -w - 2L - 1 \), a general element in \( \partial C_{m,w} \) is written as
\[
Y = -2 \sum \lambda_{[-w-3L-2,0]} \bigl(-w-3L-2\bigr)(-II)^{-w-3L-3}w_3 W^C
-2 \sum \lambda_{[-w-3L-2,0]} \bigl(-w-3L-2\bigr)(-II)^{-w-3L-3}w_2 W^C
-2 \sum \lambda_{[-w-3L-2,0]} \bigl(-w-3L-2\bigr)(-II)^{-w-3L-3}w_1 W^C
-2 \sum \lambda_{[-w-3L-3,0]} \bigl(-w-3L-3\bigr)(-II)^{-w-3L-4}(z_3 z_1 w_1 - z_3 z_2 w_2 + z_1 z_2 w_3)W^C
-2 \sum \lambda_{[-w-3L-2,1]} \bigl(-w-3L-2\bigr)(-II)^{-w-3L-3}(z_1 z_2 w_2 + z_1 z_2 w_3)W^C V
+2 \sum \lambda_{[-w-3L-2,1]} \bigl(-w-3L-2\bigr)(-II)^{-w-3L-3}(z_3 w_1 - z_1 w_3)W^C V
+2 \sum \lambda_{[-w-3L-2,1]} \bigl(-w-3L-2\bigr)(-II)^{-w-3L-3}(z_2 w_1 - z_1 w_2)W^C V
\]

Thus, when \( m = -w - 2K \)
\[
\text{dim} C_{m,w} = 3 \binom{-w-3K-2}{0} \binom{2+K-1}{K-1} + 3 \binom{-w-3K-1}{0} \binom{2+K-2}{K-2} + \binom{-w-3K-2}{0} \binom{2+K-3}{K-3} ,
\]
and when \( m = -w - 2L - 1 \)
\[
\text{dim} C_{m,w} = 3 \binom{-w-3L-3}{0} \binom{2+L}{L} + \binom{-w-3L-4}{0} \binom{2+L-1}{L-1} + 3 \binom{-w-3L-3}{0} \binom{2+L-2}{L-2} .
\]

4.2.3 \( SL(2, \mathbb{R}) \):

In the case of \( SL(2, \mathbb{R}) \), we have \( dz_i = 2w_i \), \( dz_i = -2w_i \quad i = 1, 2, 3 \) and the brackets of superalgebra are given as follows:
\[
[1, 1] = 0 \quad [1, z_j] = -2(1 - 2\delta_{1j}) w_j \quad [1, w_j] = 0 \quad [1, V] = 0
\]
\[
[z_i, 1] = 2(1 - 2\delta_{1i}) w_i \quad [z_i, z_j] = 0 \quad [z_i, w_j] = 0 \quad [z_i, V] = 0
\]
\[
[w_i, 1] = 0 \quad [w_i, z_j] = 0 \quad [w_i, w_j] = 0 \quad [w_i, V] = 0
\]
\[
[V, 1] = 0 \quad [V, z_j] = 0 \quad [V, w_j] = 0 \quad [V, V] = 0
\]
$V$ and $w_i$ are central elements of this algebra. We may say this algebra is isomorphic to the algebra associated with $SO(3, \mathbb{R})$.

4.2.4 dim[\mathfrak{g}, \mathfrak{g}] = 2:

In the case of dim[\mathfrak{g}, \mathfrak{g}] = 2, we may take a basis $[\xi_1, \xi_2] = 0$, $[\xi_2, \xi_3] = \kappa \xi_2$, $[\xi_3, \xi_1] = -\xi_1$ where $\kappa \neq 0$, and we have $dz_1 = 2w_2$, $dz_2 = -2\kappa w_1$, $dz_3 = 0$. The brackets of superalgebra are given as follows:

\begin{equation}
\partial(1^a Z^B W^C V^\ell) = (-1)^a (\partial Z^B) W^C V^\ell + a(-1)^{a-1}([1^a, Z^B]_{sz} W^C + Z^B [1^a, W^C]_{sz}) V^\ell
\end{equation}

because $w_1, w_2$ are central. From now on, we abbreviate $z_i \triangle z_j$ as $z_{ij}$ and $z_1 \triangle z_2 \triangle z_3$ as $z_{1,2,3}$. $\partial Z^B = 0$ if $|B| \leq 1$, $\partial(z_{ij}) = [z_i, z_j] = \begin{cases} -2(1 + \kappa)V & (i, j) = (1, 2), \\ 0 & \text{otherwise} \end{cases}$

$[1, z_{i,j}]_{sz} = [1, z_{i}]_{sz} \triangle z_j + z_i \triangle [1, z_j]_{sz}$ and $[1, z_{i,2,3}]_{sz} = 2z_2 \triangle z_3 \triangle w_2 + 2\kappa z_1 \triangle z_3 \triangle w_1$, and

$[1, W^C]_{sz} = c_3[1, w_3]_{sz} W^{c_1, c_2, c_3-1} = 2c_3(1 + \kappa)V \triangle W^{c_1, c_2, c_3-1} = 2(1 + \kappa)V \triangle \frac{c_3}{w_3} W^C$.

Using (4.15), we have an expression of $\partial$-image for each generator in (4.11) as follows:

\begin{enumerate}
\item \textbf{When $-w - m = 2K$},
\begin{align}
(4.18a) & \quad Kd = (-w - 3K)(-1)^{-w-3K-1} \sum \lambda_{001}^{[-w-3K, 0]} ([1, W^C]_{sz}) \\
(4.18b) & \quad + (-w - 3K - 1)(-1)^{-w-3K-2} \sum \lambda_{011}^{[-w-3K-1, 0]} (2\kappa z_3 w_1 W^C + z_2 z_3 [1, W^C]_{sz}) \\
(4.18c) & \quad + (-w - 3K - 1)(-1)^{-w-3K-2} \sum \lambda_{001}^{[-w-3K-1, 0]} (2\kappa z_2 w_2 W^C + z_1 z_3 [1, W^C]_{sz}) \\
(4.18d) & \quad + (-1)^{-w-3K-2} \sum \lambda_{101}^{[-w-3K-1, 0]} (1(1 + \kappa) V) W^C \\
& \quad + (-w - 3K - 1)(-2\kappa w_1 z_1 - 2w_2 z_2) W^C + z_1 z_2 [1, W^C]_{sz}) \\
(4.18e) & \quad + (-w - 3K)(-1)^{-w-3K-1} \sum \lambda_{100}^{[-w-3K-1, 0]} (2\kappa w_2 W^C) \\
(4.18f) & \quad + (-w - 3K)(-1)^{-w-3K-1} \sum \lambda_{111}^{[-w-3K-1, 1]} (2\kappa w_1 W^C) \\
(4.18g) & \quad + (-w - 3K - 1)(-1)^{-w-3K-2} \sum \lambda_{111}^{[-w-3K-1, 1]} ((2\kappa w_1 z_1 z_3 + 2w_2 z_2 z_3) W^C) \\
\end{align}

Assume $-w - m = 2K$ and $\kappa + 1 = 0$. Then $[1, W^C]_{sz} = 0$ and the expression above becomes simpler like

\begin{align}
Kdz = (-w - 3K - 1)(-1)^{-w-3K-2} \sum \lambda_{011}^{[-w-3K-1, 0]} (2\kappa z_3 w_1 W^C) \\
& \quad + (-w - 3K - 1)(-1)^{-w-3K-2} \sum \lambda_{011}^{[-w-3K-1, 0]} (2\kappa w_2 W^C) \\
& \quad + (-1)^{-w-3K-2} \sum \lambda_{101}^{[-w-3K-1, 0]} (2\kappa w_1 z_1 - 2w_2 z_2) W^C \\
& \quad + (-w - 3K)(-1)^{-w-3K-1} \sum \lambda_{100}^{[-w-3K-1, 0]} (2\kappa w_2 W^C) \\
& \quad + (-w - 3K)(-1)^{-w-3K-1} \sum \lambda_{111}^{[-w-3K-1, 1]} (2\kappa w_1 z_1 z_3 + 2w_2 z_2 z_3) W^C)
\end{align}

\end{enumerate}
If \(-w - 3K - 1 < 0\), then \(Kdz = 0\), i.e., \(\dim \mathbf{C}_{m,w} = 0\). If \(-w - 3K - 1 = 0\), then

\[
Kdz = (-w - 3K)(-1)^{-w-3K-1} \left( \sum \lambda_{\{00\},[C,K]} \left[ -w-3K,0 \right]_1 W_C \right) + \sum \lambda_{\{10\},[C,K-2]} \left[ -w-3K,1 \right]_2 W_C
\]
i.e., the space is a union of two subspaces of dimension \((2+K-2)/K-3\) and those intersection is \((2+K-3)/K-3\) dimensional. Thus \(\dim \mathbf{C}_{m,w} = 2\left(2+K-2\right)/K-3\).

If \(-w - 3K - 1 > 0\), then we apply the same discussion for the first two subspaces, and we get

\[
\dim \mathbf{C}_{m,w} = 2\left(2+K-1\right)/K-3 + 2\left(2+K-2\right)/K-3 + 2\left(2+K-3\right)/K-3 + 2\left(2+K-3\right)/K-3
\]
Assume \(-w - m = 2K\) and \(\kappa + 1 \neq 0\). We use the formula \((4.18a) \sim (4.18c)\).

If \(-w - 3K - 1 < 0\) then the source element is zero and \(Kd = 0\). If \(-w - 3K - 1 = 0\) then we have

\[
X = \sum \lambda_{\{00\},[C,K]} \left[ -w-3K,0 \right]_1 W_C + \sum \lambda_{\{10\},[C,K-2]} \left[ -w-3K,1 \right]_2 W_C
\]
and so the rank is \((2+K-1)/K-1\).

If \(-w - 3K - 1 > 0\) then the above shows they are divided into 4 linearly independent parts: The first consists of \((4.18a), (4.18c)\) and \((4.18f)\), which have the common factor \(V\), the second is \((4.18b)\) and \((4.18c)\) which have the common factor \(z_3\), the third part is \((4.18d)\) whose rank is \((2+K-1)/K-1\), the fourth part is \((4.18g)\) whose rank is \((2+K-3)/K-3\). Since

\[
(4.18a) + (4.18c) + (4.18f) = -(w - 3K)(-1)^{-w-3K-2} \sum \lambda_{\{00\},[C,K]} \left[ -w-3K,0 \right]_1 W_C + \sum \lambda_{\{10\},[C,K-2]} W_C
\]

and so the rank is \((2+K-1)/K-1\). We get

\[
(4.18b) + (4.18c) + (4.18f) = (w - 3K - 1)(-1)^{-w-3K-2} \sum \lambda_{\{01\},[C,K-1]} \left[ -w-3K,0 \right]_1 W_C + \sum \lambda_{\{10\},[C,K-2]} W_C
\]

the rank of the first part \((4.18a) + (4.18c) + (4.18f)\) is \((2+K-1)/K-1\).
generators: \((\kappa w_1 W^C - z_2(1 + \kappa)V_{\frac{\alpha}{\alpha}_2}W^C)\) and \((\omega_2 W^C + z_1(1 + \kappa)V_{\frac{\alpha}{\alpha}_2}W^C)\) where \(|C| = |C'| = K - 1\). Since \(1 + \kappa \neq 0\), if \(c_3 \neq 0\) or \(c_3' \neq 0\) then they are linearly independent, and there the rank is \(2(\frac{2+K}{K-1}) - (\frac{1+K}{K-1})\). Otherwise, we can recall the same old discussion and the rank is \(2(\frac{1+K}{K-1}) - (\frac{1+K}{K-2})\). Finally, \(\dim C_{m,w} = 4(\frac{2+K}{K-1}) - (\frac{1+K}{K-2}) + (\frac{2+K}{K-3})\) when \(-w = m = 2K\) and \(1 + \kappa \neq 0\).

We study when \(-w - m = 2L + 1\). Since \([1, z_i]_{sz} V = 0\), we have

\[
Ld = (-w - 3L - 2)(-1)^{-w-3L-3} \sum \lambda_{[001],[C,L]}^{[-w-3L-2,0]} (z_3[1,W^C]_{sz}) + \cdots
\]

4.2.4.1 Assume \(\kappa + 1 = 0\) Then \([1, W^C]_{sz} = 0\) and the expression becomes simpler.

\[
Ldz = 2(-w - 3L - 2)(-1)^{-w-3L-3} \sum \lambda_{[001],[C,L]}^{[-w-3L-2,0]} ((-\kappa w_1 W^C) + \sum \lambda_{[100],[C,L]}^{[-w-3L-2,0]} ((w_2 W^C)))
\]

4.2.4.2 We assume \(\kappa + 1 \neq 0\)

\[
Ld = (-w - 3L - 2)(-1)^{-w-3L-3} \sum \lambda_{[001],[C,L]}^{[-w-3L-2,0]} ([1,W^C]_{sz}) + \cdots
\]

\[
+ (\omega_2 W^C + z_1(1 + \kappa)V_{\frac{\alpha}{\alpha}_2}W^C)\) where \(|C| = |C'| = K - 1\). Since \(1 + \kappa \neq 0\), if \(c_3 \neq 0\) or \(c_3' \neq 0\) then they are linearly independent, and there the rank is \(2(\frac{2+K}{K-1}) - (\frac{1+K}{K-1})\). Otherwise, we can recall the same old discussion and the rank is \(2(\frac{1+K}{K-1}) - (\frac{1+K}{K-2})\). Finally, \(\dim C_{m,w} = 4(\frac{2+K}{K-1}) - (\frac{1+K}{K-2}) + (\frac{2+K}{K-3})\) when \(-w = m = 2K\) and \(1 + \kappa \neq 0\).

We study when \(-w - m = 2L + 1\). Since \([1, z_i]_{sz} V = 0\), we have

\[
Ld = (-w - 3L - 2)(-1)^{-w-3L-3} \sum \lambda_{[001],[C,L]}^{[-w-3L-2,0]} (z_3[1,W^C]_{sz}) + \cdots
\]

4.2.4.1 Assume \(\kappa + 1 = 0\) Then \([1, W^C]_{sz} = 0\) and the expression becomes simpler.

\[
Ldz = 2(-w - 3L - 2)(-1)^{-w-3L-3} \sum \lambda_{[001],[C,L]}^{[-w-3L-2,0]} ((-\kappa w_1 W^C) + \sum \lambda_{[100],[C,L]}^{[-w-3L-2,0]} ((w_2 W^C)))
\]

4.2.4.2 We assume \(\kappa + 1 \neq 0\)

\[
Ld = (-w - 3L - 2)(-1)^{-w-3L-3} \sum \lambda_{[001],[C,L]}^{[-w-3L-2,0]} ([1,W^C]_{sz}) + \cdots
\]
because

\[
(\sum \lambda_{[010],[C,L]}^{[w-3L-2,0]}((-2\kappa w_1 W^C + z_2 [1, W^C]_{sz}) + \sum \lambda_{[100],[C,L]}^{[w-3L-2,0]}((2w_2 W^C + z_1 [1, W^C]_{sz}))
\]
\[
+ \sum \lambda_{[110],[C,L]}^{[w-3L-3,0]}((-1)^{-w-3L-3}(2(1 + \kappa)z_3)W^C)
\]
\[
+ (-w - 3L - 3)(-1)^{-w-3L-4}((2\kappa w_1 z_1 z_3 + 2w_2 z_2 z_3)W^C + z_1 z_2 z_3 [1, W^C]_{sz})
\]
\[
+ (-w - 3L - 2)(-1)^{-w-3L-3} \sum \lambda_{[110],[C,L]}^{[w-3L-2,1]}((-2\kappa w_1 z_1 - 2w_2 z_2)W^C)V)
\]

If \(-w - 3L - 3 < 0\) then \(Ld = 0\). If \(-w - 3L - 3 = 0\) then

\[
Ld = z_3 \triangle (\sum \lambda_{[010],[C,L]}^{[w-3L-2,0]}([1, W^C]_{sz}) + \sum \lambda_{[101],[C,L]}^{[w-3L-2,1]}((2\kappa w_1 W^C)V) + \sum \lambda_{[110],[C,L]}^{[w-3L-2,1]}((-2w_2 W^C)V)
\]
\[
+ (\sum \lambda_{[010],[C,L]}^{[w-3L-3,0]}((-2\kappa w_1 W^C + z_2 [1, W^C]_{sz}) + \sum \lambda_{[101],[C,L]}^{[w-3L-3,0]}((2w_2 W^C + z_1 [1, W^C]_{sz}))
\]
\[
+ \sum \lambda_{[110],[C,L]}^{[w-3L-3,0]}((2(1 + \kappa)z_3)W^C)
\]
\[
+ \sum \lambda_{[110],[C,L]}^{[w-3L-2,1]}((-2\kappa w_1 z_1 - 2w_2 z_2)W^C)V)
\]

the first term gives \(2^{2L-1}_{L-1}\), the second term gives \(2^{2L}_{L-1} - 2\) and the third term gives \(2^{2L-2}_{L-2}\). Thus, the rank is \(2^{2L+1}_L + 2^{2L-2}_{L-2}\).

If \(-w - 3L - 3 > 0\) then the first term gives the rank \(2^{2L-1}_{L-1}\), the second term gives \(2^{2L}_{L-1} - 2\), the third term gives \(2^{2L-1}_{L-1}\), the fourth term gives \(2^{2L-2}_{L-2}\). Thus, the rank is

\[
2^{2L-1}_{L-1} + 2^{2L-1}_{L-1} + 2^{2L-1}_{L-1} + 2^{2L-1}_{L-1} = 2^{2L+1}_L + 2^{2L-2}_{L-2}
\]

4.2.5 \(\dim [g, g] = 1\) and \(\dim [g, g] \nsubseteq Z(g)\):

In the case of \(\dim [g, g] = 1\) and \(\dim [g, g] \nsubseteq Z(g)\), we may take a basis \([\xi_1, \xi_2] = -[\xi_2, \xi_1] = \xi_2\) and the others are 0. Now we have \(dz_1 = 0, dz_2 = -2w_3, dz_3 = 0\). The brackets of superalgebra are given as follows:

| 1 | 1 | 1 | 1 |
|---|---|---|---|
| z_1 | z_2 | z_3 | w_1 | w_2 | w_3 | V |
| 0 | 0 | 0 | -2w_3 | 0 | -2V | 0 | 0 |
| z_1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| z_2 | 2w_3 | 0 | 0 | 2V | 0 | 0 | 0 |
| z_3 | 0 | 0 | -2V | 0 | 0 | 0 | 0 |
| w_1 | -2V | 0 | 0 | 0 | 0 | 0 | 0 |
| w_2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| w_3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

\(\partial(1^a Z^B W^C V^\ell) = (-1)^a (\partial Z^B) W^C V^\ell + a (-1)^{a-1} ([1, Z^B]_{sz} W^C - [1, W^C]_{sz} Z^B) V^\ell\)

where

\[
\partial Z^{111} = -2z_1 \triangle V, \quad \partial Z^{110} = \partial Z^{101} = 0, \quad \partial Z^{011} = 2V,
\]
\[
[1, Z^B]_{sz} = 2(-1)^{1+b_2} \frac{b_2}{z_2} Z^B w_3
\]
\[
[1, W^C]_{sz} = (-2V) \frac{c_1}{w_1} W^C
\]

because

\[
[1, Z^B]_{sz} = b_1 [1, z_1]_{sz} z_2 z_3 + b_2 z_1 [1, z_2]_{sz} z_3 + b_3 z_1 z_2 [1, z_3]_{sz} = b_2 z_1 (-2w_3) z_3
\]

\[17\]
\[= b_2 (-1)^{1+b_1} z_1^{b_1} z_3^{b_3} w_3 = 2 (-1)^{1+b_1} b_2 z_2^3 Z^B w_3\]

\[\begin{align*}
[1, W^C]_{sz} &= c_1 [1, w_1]_{sz} w_1^{c_1-1} w_2^C w_3^c = [1, w_1]_{sz} c_1 W^C = (-2V) c_1 W^C
\end{align*}\]

Assume \(-w - m = 2K\). Then a general element in \(\partial C_{m,w}\) is given by

\[X = \sum \lambda_{[000],[C,K]}^{[-w-3K,0]} (-w-3K)(-1)^{-w-3K-1} [1, W^C]_{sz} \]

\[+ \sum \lambda_{[010],[C,K-1]}^{[-w-3K-1,0]} (2(-1)^{-w-3K-1} V W^C \]

\[+ (-w-3K-1)(-1)^{-w-3K-2} [2 z_3 w_3^C + z_2 z_3 [1, W^C]_{sz}] \]

\[+ \sum \lambda_{[110],[C,K-1]}^{[-w-3K-1,0]} \lambda_{[010],[C,K-1]}^{[-w-3K-1,0]} (z_1 z_3 [1, W^C]_{sz}) \]

\[+ \sum \lambda_{[110],[C,K-3]}^{[-w-3K-1,0]} (0 + (-w-3K-1)(-1)^{-w-3K-2} (2 z_1 z_3 w_3^C + 0)V) \]

\[= (-w-3K)(-1)^{-w-3K-1} \sum \lambda_{[000],[C,K]}^{[-w-3K,0]} [1, W^C]_{sz} \]

\[+ \sum \lambda_{[010],[C,K-1]}^{[-w-3K-1,0]} (2(-1)^{-w-3K-1} V W^C \]

\[+ (-w-3K-1)(-1)^{-w-3K-2} [2 z_3 w_3^C + z_2 z_3 [1, W^C]_{sz}] \]

\[+ (-w-3K-1)(-1)^{-w-3K-2} \sum \lambda_{[110],[C,K-1]}^{[-w-3K-1,0]} (z_1 z_3 [1, W^C]_{sz}) \]

\[+ \sum \lambda_{[110],[C,K-3]}^{[-w-3K-1,0]} ((2 z_1 z_3 w_3^C + 0)V) \]

\[= (+-3K)(-1)^{-w-3K-1} \sum \lambda_{[000],[C,K]}^{[-w-3K,0]} [1, W^C]_{sz} + \sum \lambda_{[010],[C,K-1]}^{[-w-3K-1,0]} (-2w_3^C)V) \]

\[+ \sum \lambda_{[010],[C,K-1]}^{[-w-3K-1,0]} (2(-1)^{-w-3K-1} V W^C \]

\[+ (-w-3K-1)(-1)^{-w-3K-2} [2 z_3 w_3^C + z_2 z_3 [1, W^C]_{sz}] \]

\[+ (-w-3K-1)(-1)^{-w-3K-2} \sum \lambda_{[110],[C,K-1]}^{[-w-3K-1,0]} (z_1 z_3 [1, W^C]_{sz}) \]

\[+ \sum \lambda_{[110],[C,K-3]}^{[-w-3K-1,0]} ((2 z_1 z_3 w_3^C + z_1 z_3 [1, W^C]_{sz}) \]

If \(-w - 3K - 1 < 0\), then \(X = 0\), i.e., the boundary operator is zero map.

If \(-w - 3K - 1 = 0\),

\[X = \sum \lambda_{[000],[C,K]}^{[-w-3K,0]} [1, W^C]_{sz} + \sum \lambda_{[010],[C,K-2]}^{[-w-3K-1,0]} (-2w_3^C)V) + \sum \lambda_{[010],[C,K-1]}^{[-w-3K-1,0]} 2V W^C \]

\[= 2V (\sum \lambda_{[000],[C,K]}^{[-w-3K,0]} [1, W^C]_{sz} + \sum \lambda_{[010],[C,K-2]}^{[-w-3K-1,0]} (-2w_3^C)V) + \sum \lambda_{[010],[C,K-1]}^{[-w-3K-1,0]} 2V W^C \]

\[= 2V (\sum \lambda_{[000],[C,K]}^{[-w-3K,0]} [1 + c_1] + (-1)^K \lambda_{[010],[C,K-2]}^{[-w-3K-1,0]} (-2w_3^C)V) + \sum \lambda_{[010],[C,K-1]}^{[-w-3K-1,0]} 2V W^C \]

and we have the set of generators:

\[-\lambda_{[000],[1+c_1,c_2,c_3,K]}^{[-w-3K,0]} (1 + c_1) + (-1)^K \lambda_{[010],[1+c_1,c_2,c_3,K-2]}^{[-w-3K-1,0]} (-2w_3^C)V) + \sum \lambda_{[010],[1+c_1,c_2,c_3,K-1]}^{[-w-3K-1,0]} 2V W^C \]
with $|C| = K - 1$, and the rank is $\binom{2+K-1}{K-1}$.

If $-w - 3K - 1 > 0$, we have linearly independent 4 blocks:

$$X_1 = (-w - 3K)(-1)^{-w-3K-1}(-2V)(\sum \lambda_{[w-3K,0]}(\frac{1}{w_1}W^C) - (-1)^K(\sum \lambda_{[w-3K,1]}(w_3W^C))$$

$$X_2 = \sum \lambda_{[w-3K-1,0]}(2(-1)^{-w-3K-1}VW^C + (-w - 3K - 1)(-1)^{-w-3K-2}(2z_3w_3W^C + z_2z_3[1, W^C]_{sz}))$$

$$X_3 = (+w - 3K - 1)(-1)^{-w-3K-2}z_1z_3(-2V)(\sum \lambda_{[w-3K-1,0]}(\frac{1}{w_1}W^C - (-1)^K(\sum \lambda_{[w-3K-1,1]}(w_3W^C))$$

we see that generators do not have common factors are

$$\lambda_{[w-3K,0]}(\frac{1}{w_1}W^C) - (-1)^K(\sum \lambda_{[w-3K,1]}w_3W^C)$$

$$\lambda_{[w-3K-1,0]}(1 + c_1) - (-1)^K(\sum \lambda_{[w-3K-1,1]}w_3W^C)$$

with $c_1 + c_2 + c_3 = K - 1$

Their freedom are $\binom{2+K-1}{K-1}$ and $\binom{2+K-2}{K-2}$ respectively. Thus, the rank is $3\binom{2+K-1}{K-1} + \binom{2+K-2}{K-2}$.

When $-w - m = 2L + 1$, a general element in $\partial C_{m,w}$ is given by

$$Y = \sum \lambda_{[w-3L-2,0]}((-w - 3L - 2)(-1)^{-w-3L-3}z_3[1, W^C]_{sz})$$

$$\sum \lambda_{[w-3L-2,0]}((-w - 3L - 2)(-1)^{-w-3L-3}(-(2w_3W^C + z_2[1, W^C]_{sz}))$$

$$\sum \lambda_{[w-3L-2,0]}((-w - 3L - 2)(-1)^{-w-3L-3}z_1[1, W^C]_{sz})$$

$$\sum \lambda_{[w-3L-2,0]}(-(2(-1)^{-w-3L-3}z_1VW^C)$$

$$\sum \lambda_{[w-3L-2,0]}((-w - 3L - 3)(-1)^{-w-3L-4}(2z_3z_3w_3W^C + z_1z_2z_3[1, W^C]_{sz})$$

By the same discussion when $-w - m = 2K$, we also get $\dim C_{m,w}$ for $-w - m = 2L + 1$ as follows:
Proposition 4.1. When \(-w - m = 2K\), then \(\dim \partial C_{m,w} = \begin{cases} 3\binom{2+K-1}{K-1} + \binom{2+K-2}{K-2} & \text{if } -w - 3K - 1 > 0 \\ \binom{2+K-1}{K-1} & \text{if } -w - 3K - 1 = 0 \\ 0 & \text{otherwise.} \end{cases}\)

When \(-w - m = 2L + 1\), then \(\dim \partial C_{m,w} = \begin{cases} 3\binom{2+L-1}{L-1} + \binom{2+L}{L} & \text{if } -w - 3L - 2 > 1 \\ 2\binom{2+L-1}{L-1} + \binom{2+L}{L} & \text{if } -w - 3L - 2 = 1 \\ 0 & \text{otherwise.} \end{cases}\)

4.2.6 \(\dim [g,g] = 1\) and \([g,g] \subset Z(g)\):

In the case of \(\dim [g,g] = 1\) and \([g,g] \subset Z(g)\), we may take a basis \([\xi_1, \xi_2] = -[\xi_2, \xi_1] = \xi_3\) and the others are 0. Now we have \(dz_1 = 0\), \(dz_2 = 0\), \(dz_3 = -2w_3\). The brackets of superalgebra are given as on the right (4.14a) implies

\[
\partial(1^n Z^B W C V^\ell) = a(-1)^{n-1}[1, Z^B]_{sz} W C V^\ell = a(-1)^{n-1} z_1 b_1 z_2 b_2 z_3 (-2w_3) W C V^\ell.
\]

\[
X = \sum \lambda_{[000],[C,K]}^{[-w-3K,0]}(0) + \sum \lambda_{[011],[C,K-1]}^{[-w-3K-1,0]}(-2(-w - 3K - 1)(-1)^{-w-3K-2}z_2 w_3 W C) + \sum \lambda_{[100],[C,K-1]}^{[-w-3K,1]}(0)
\]

\[
Y = \sum \lambda_{[000],[C,L]}^{[-w-3L,0]}(0) - \sum \lambda_{[011],[C,L-1]}^{[-w-3L-1,0]}(-2(-w - 3L - 2)(-1)^{-w-3L-2} z_2 w_3 W C) + \sum \lambda_{[100],[C,L-1]}^{[-w-3L,1]}(0)
\]

Thus, when \(-w - m = 2K\), \(\partial C_{m,w}\) is spanned by

\[
(-1)^{-w-3K-2} z_2 (-2w_3) W C \quad \text{and} \quad (-1)^{-w-3K-2} z_1 (-2w_3) W C \quad \text{with } |C| = K - 1,
\]

\[
(-1)^{-w-3K-1} (-2w_3) W C V \quad \text{with } |C| = K - 2,
\]

\[
(-1)^{-w-3K-2} z_1 z_2 (-2w_3) W C V \quad \text{with } |C| = K - 3.
\]

and \(\dim \partial C_{m,w} = (-w-3K-2) 2^{2K-1} K^{-1} + (-w-3K-2) 2^{2K-2} K^{-2} + (-w-3K-2) 2^{2K-3} K^{-3}\).

By the same argument, if \(-w - m = 2L + 1\), then

\[
\dim \partial C_{m,w} = (-w-3L-3) 2^{2L+1} L^{-1} + (-w-3L-4) 2^{2L} L^{-2} + (-w-3L-3) 2^{2L+2} L^{-3}.
\]
The $m$-th Betti number is given by $\dim C_{m,w} - \dim \partial C_{m,w} - \dim \partial C_{m+1,w}$, say $\text{Bet}_m$. Thus, if $-w - m = 2K$ then

$$\text{Bet}_m = \dim C_{m,w} - \left(\left(-\frac{w-3K-2}{0}\right)2^{(2+K-1)} + \left(-\frac{w-3K-1}{0}\right)2^{(2+K-2)} + \left(-\frac{w-3K-2}{0}\right)2^{(2+K-3)}\right)$$

$$- \left(\left(-\frac{w-3L-3}{0}\right)2^{(2+L-1)} + \left(-\frac{w-3L-1}{0}\right)2^{(2+L-2)} + \left(-\frac{w-3L-2}{0}\right)2^{(2+L-3)}\right)$$

and if $-w - m = 2L + 1$ then

$$\text{Bet}_m = \dim C_{m,w} - \left(\left(-\frac{w-3L-3}{0}\right)2^{(2+L-1)} + \left(-\frac{w-3L-1}{0}\right)2^{(2+L-2)} + \left(-\frac{w-3L-2}{0}\right)2^{(2+L-3)}\right)$$

4.2.7 Look at once

So far, for each chain complex of Lie superalgebra of a given 3-dimensional Lie algebra, we got all kernel dimensions, and so we can get Betti numbers. For instance, the following table shows the case of weight $-3,-5,-10$ chain spaces simultaneously. $d3: \ker$ means the list of kernel dimensions of the Lie algebra $\dim [g, g] = 3$, $d2y :$ means the Lie algebra $\dim [g, g] = 2$ and $\kappa + 1 = 0$, $d2n :$ means the Lie algebra $\dim [g, g] = 2$ and $\kappa + 1 \neq 0$, $d1n :$ means the Lie algebra $\dim [g, g] = 1$ and $[g, g] \notin Z(g)$, and $d1y :$ means the Lie algebra $\dim [g, g] = 1$ and $[g, g] \subset Z(g)$.

| $w = -3$ | 1 2 3 | $w = -5$ | 1 2 3 4 5 | $w = -10$ | 1 2 3 4 5 6 7 8 9 10 |
|----------|-------|----------|----------|----------|-------------------|
| $\text{Space dim}$ | 3 3 1 | $\text{Space dim}$ | 0 10 6 3 1 | $\text{Space dim}$ | 0 0 6 38 27 18 11 6 3 1 |
| $d3 : \ker$ | 3 0 1 | $d3 : \ker$ | 0 10 3 0 1 | $d3 : \ker$ | 0 0 6 32 11 7 4 3 0 1 |
| $\text{Bet}$ | 0 0 1 | $\text{Bet}$ | 0 7 0 0 1 | $\text{Bet}$ | 0 0 0 16 0 0 1 0 0 1 |
| $d2y : \ker$ | 3 1 1 | $d2y : \ker$ | 0 10 3 1 1 | $d2y : \ker$ | 0 0 6 33 12 8 5 3 1 1 |
| $\text{Bet}$ | 1 1 1 | $\text{Bet}$ | 0 7 1 1 1 | $\text{Bet}$ | 0 0 1 18 2 2 2 1 1 1 |
| $d2n : \ker$ | 3 1 1 | $d2n : \ker$ | 0 10 2 1 1 | $d2n : \ker$ | 0 0 6 32 11 7 4 2 1 1 |
| $\text{Bet}$ | 1 1 1 | $\text{Bet}$ | 0 6 0 1 1 | $\text{Bet}$ | 0 0 0 16 0 0 0 0 1 1 |
| $d1y : \ker$ | 3 2 1 | $d1y : \ker$ | 0 10 4 2 1 | $d1y : \ker$ | 0 0 6 35 16 11 7 4 2 1 |
| $\text{Bet}$ | 2 2 1 | $\text{Bet}$ | 0 8 3 2 1 | $\text{Bet}$ | 0 0 3 24 9 7 5 3 2 1 |
| $d1n : \ker$ | 3 2 1 | $d1n : \ker$ | 0 10 3 2 1 | $d1n : \ker$ | 0 0 6 32 12 8 5 3 2 1 |
| $\text{Bet}$ | 2 2 1 | $\text{Bet}$ | 0 7 2 2 1 | $\text{Bet}$ | 0 0 0 17 2 2 2 2 2 1 |

5 Long $\mathbb{Z}$-graded Lie superalgebra

5.1 Examples of $\mathbb{Z}$-graded trivially long Lie superalgebra

Let $M$ be an $n$-dimensional manifold. It is known that

$$\Lambda^{0}T(M) \oplus \Lambda^{1}T(M) \oplus \cdots \oplus \Lambda^{n}T(M) \quad \text{grade of } \Lambda^{i}T(M) = i - 1$$

is a Lie superalgebra with the Schouten bracket with grading of $\Lambda^{i}T(M) = i - 1$ and $[f, g]_{S} = 0$ for functions. In this article, we have shown that

$$\Lambda^{n}T^{*}(M) \oplus \cdots \oplus \Lambda^{1}T^{*}(M) \oplus \Lambda^{0}T^{*}(M) \quad \text{grade of } \Lambda^{j}T^{*}(M) = -j$$

becomes a Lie superalgebra with the bracket defined by (2.1). Now let

$$g = \Lambda^{n}T^{*}(M) \oplus \cdots \oplus \Lambda^{1}T^{*}(M) \oplus \Lambda^{0}T^{*}(M) \oplus \Lambda^{1}T(M) \oplus \cdots \oplus \Lambda^{n}T(M)$$
We easily get a “long trivial” $\mathbb{Z}$-graded Lie superalgebra as follows:

**Proposition 5.1.** Define the grading of (5.3) by $\mathfrak{g}_i = \Lambda^{-i}T^*(M)$ for $i < 0$, and $\mathfrak{g}_i = \Lambda^iT(M)$ for $i \geq 0$, i.e., $\Lambda^jT(M) = \mathfrak{g}_{-j}$ for $j > 0$, and $\Lambda^jT^*(M) = \mathfrak{g}_{-j}$ for $j \geq 0$.

On $\mathfrak{g}$, define a bracket by $[x, y]_{\text{new}} = 
\begin{cases}
[x, y]_S & x, y \text{ vectors} \\
[x, y] & x, y \text{ forms} \\
0 & \text{otherwise}
\end{cases}
$. Then $\mathfrak{g}$ becomes a trivial Lie superalgebra.

We restate the proposition above in abstract way as below:

**Proposition 5.2.** Let $\mathfrak{g} = \sum_i \mathfrak{g}_i$ and $\mathfrak{h} = \sum_j \mathfrak{h}_j$ be $\mathbb{Z}$-grade Lie superalgebras. Then we have a “trivial” superalgebra as follows: $\mathfrak{g}_i = \mathfrak{g}_i$ for $i < 0$ and $\mathfrak{h}_j = \mathfrak{h}_j$ for $j \geq 0$, and the bracket is defined by

$$
[x, y]_{\text{new}} = 
\begin{cases}
[x, y]_S & \text{if } x \in \mathfrak{g}_i, y \in \mathfrak{h}_j (i, j < 0) \\
[x, y] & \text{if } x \in \mathfrak{g}_i, y \in \mathfrak{h}_j (i, j \geq 0) \\
0 & \text{otherwise}
\end{cases}
$$

6 One step extended superalgebra

**Theorem 8.** Let $M$ be an $n$-dimensional manifold. As (5.2), we have a superalgebra $\mathfrak{h} = \Lambda^nT^*(M) \oplus \cdots \oplus \Lambda^0T^*(M)$. Take $\mathfrak{h} = \mathfrak{h} \oplus \Lambda T(M)$ with $\mathfrak{h}_0 = T(M)$.

On $\mathfrak{h}$, using Lie derivative $L_X$ of $X \in T(M)$, we define a bracket by

$$
[x, y]_{\text{new}} = [x, y]_S \text{ for 1-vectors } x, y,
$$

$$
[x, y]_{\text{new}} = [x, y] \text{ for forms } x, y,
$$

$$
[x, y]_{\text{new}} = L_x y \text{ for 1-vector } x \text{ and form } y
$$

$$
[y, x]_{\text{new}} = -L_x y \text{ for 1-vector } x \text{ and form } y.
$$

Then $\mathfrak{h}$ becomes a Lie superalgebra. $\mathfrak{h}$ is a (super subalgebra) or sub superalgebra of $\mathfrak{h}$ naturally.

**Proof:** We only check super Jacobi identity by the properties

$$
L_X (\alpha \wedge \beta) = L_X (\alpha \wedge \beta + \alpha \wedge L_X (\beta))
$$

$$
L_{[X, Y]} \alpha = [L_X, L_Y] \alpha = L_X (L_Y \alpha) - L_Y (L_X \alpha).
$$

In fact, for $\alpha \in \Lambda^aT^*(M)$

$$
[[X, Y], \alpha]_{\text{new}} = L_X (L_Y \alpha) - L_Y (L_X \alpha) = L_X [Y, \alpha]_{\text{new}} - L_Y [X, \alpha]_{\text{new}}
$$

$$
= [X, [Y, \alpha]_{\text{new}}]_{\text{new}} - [Y, [X, \alpha]_{\text{new}}]_{\text{new}} = [X, [Y, \alpha]_{\text{new}}]_{\text{new}} + [[X, \alpha]_{\text{new}}, Y]_{\text{new}}
$$

$$
= [X, [Y, \alpha]_{\text{new}}]_{\text{new}} + (-1)^{b(a-1)}[[X, \alpha]_{\text{new}}, Y]_{\text{new}}
$$

$$
[X, [\alpha, \beta]_{\text{new}}]_{\text{new}} = [L_X ((-1)^a d(\alpha \wedge \beta)) = (-1)^a d(L_X (\alpha \wedge \beta)) = (-1)^a d(L_X (\alpha \wedge \beta + \alpha \wedge L_X (\beta)))
$$

$$
= [[L_X \alpha, \beta]_{\text{new}} + [\alpha, L_X \beta]_{\text{new}} = [[X, \alpha]_{\text{new}}, \beta]_{\text{new}} + (-1)^{b(a-1)}[[\alpha, X]_{\text{new}}, \beta]_{\text{new}}
$$

$\square$
6.1 General argument

Since the degree or weight of $X \in T(M)$ is 0, we have

$$C_{m,w} = \sum_{k=0}^{n} \triangle^k T(M) \triangle C_{m-k,w}$$

6.2 About low dimensional Lie algebras

In this case,

$$C_{m,w} = \sum_{k=0}^{n} \triangle^k g \triangle C_{m-k,w}$$

$$\sum_{m} (-1)^m \dim C_{m,w} = \sum_{m} (-1)^m \sum_{k=0}^{n} \dim C_{m-k,w} = \sum_{m,k} (-1)^k \binom{n}{k} (-1)^{m-k} \dim C_{m-k,w}$$

$$= \sum_{k} (-1)^k \binom{n}{k} \sum_{m} (-1)^{m-k} \dim C_{m-k,w} = 0$$

Thus, the Euler number is always 0.

6.3 Double weighted super homology theory on $\mathbb{R}^n$

Let $x_1, \ldots, x_n$ be a Cartesian coordinate of $\mathbb{R}^n$. Then vector fields and differential forms with only polynomial coefficients are written as follows:

$$X = \sum_{i=1}^{n} F_i(x) \frac{\partial}{\partial x_i} \quad \text{where } F_i(x) \text{ is a polynomial of } x = (x_1, \ldots, x_n)$$

$$\Omega = \sum_{A} G_A(x) dx^A \quad \text{where } G_A(x) \text{ is a polynomial of } x, dx^A = dx_1^{a_1} \wedge \cdots \wedge dx_n^{a_n} \quad \text{and}$$

$$A = (a_1, \ldots, a_n) \in \{0, 1\}^n, \text{ we sometimes denote } a_1 + \cdots + a_n \text{ by } |A|.$$ 

They provide Lie sub superalgebras. We divide polynomials by homogeneity and define the secondary degree or weight as follows.

Definition 9. Assume $X = \sum_{i=1}^{n} F_i(x) \frac{\partial}{\partial x_i} \neq 0$. The grading in super sense or the weight of $X$ is 0 by definition.

If for each $i$, $F_i(x) = 0$ or homogeneous with the same homogeneity $h$, i.e., $F_i(tx) = t^h F_i(x)$ for all $i$, then we define the secondary weight of $X$ is $h - 1$.

Non-zero $\Omega = \sum_{|A|=a} G_A(x) dx^A$ has the degree in super sense or the (primary) weight $-1 - a$ by the definition.

Now the secondary weight is defined by $h - 1$ when $G_A(tx) = t^h G_A(x)$ for all $A$ with $|A| = a$. Let $g_{<i,h>}$ be the space consisted by $(h+1)$-homogeneous $(-1-i)$-forms for $i < 0$, and let $g_{<0,h>}$ be the space consisted by $(h+1)$-homogeneous 1-vectors. $g_i = \sum_{h} g_{<i,h>}$ and $g_0 = \sum_{h} g_{<0,h>}$ hold.

We easily see that

Proposition 6.1. $[g_{<i,h>}, g_{<j,k>}]_{\text{new}} \subset g_{<i+j,h+k>}$ for $i, j \leq 0$ and $h, k \geq -1$. 

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Thus, we have doubly weighted chain complex

\[ C_{m,w,h} = \sum_{i=1}^{m} \{ g_{<w_i,h_i>} \mid \sum w_i = w \text{ and } \sum h_i = h \}. \]

**Theorem 10.** The Betti numbers are 0 when \( w - h \neq 0 \) for the double weighted homology groups of the above double weighted chain complex \( C_{m,w,h} \) of \( \mathbb{R}^n \).

**Proof:** Using the Euler vector field \( E = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \), we have

\[
\begin{align*}
[E, x^P \partial / \partial x_j] &= (|P| + |A|) x^P \partial / \partial x_j = SW(x^P \partial / \partial x_j) x^P \partial / \partial x_j \\
[E, x^P dx^A] &= (|P| - 1) x^P \partial / \partial x_j = PW(x^P \partial / \partial x_j) x^P \partial / \partial x_j
\end{align*}
\]

where \( PW(\alpha) \) and \( SW(\alpha) \) mean the primary and the secondary weight of \( \alpha \).

Since the double weight of \( E \) is \((0,0)\), the definition of the boundary operator implies

\[
\partial(E \triangle Y_1 \triangle \cdots \triangle Y_m) = -E \triangle \partial(Y_1 \triangle \cdots \triangle Y_m) + \sum_{i=1}^{m} Y_i \triangle \cdots \triangle [E, Y_i] \triangle \cdots \triangle Y_m
\]

When \( Y_i \in g_{<w_i,h_i>}, [E, Y_i] = (h_i + 1)Y_i \) because \( E \) is the Euler vector field, so we have

\[
\begin{align*}
\partial(E \triangle Y_1 \triangle \cdots \triangle Y_m) &= -E \triangle \partial(Y_1 \triangle \cdots \triangle Y_m) + \sum_{i=1}^{m} (h_i - w_i) Y_1 \triangle \cdots \triangle Y_m \\
&= -E \triangle \partial(Y_1 \triangle \cdots \triangle Y_m) + (h - w) Y_1 \triangle \cdots \triangle Y_m
\end{align*}
\]

for \( Y_1 \triangle \cdots \triangle Y_m \in C_{m,w,h} \). Thus, if \( h - w \neq 0 \), we see each \( m \)-cycle is exact and so the Betti number is 0. \( \square \)

## A Appendix

We have a new super bracket on forms in \([5]\) which is older version of this article. We first fix a candidate as \( L(a,b) d\alpha \wedge \beta + R(a,b) \alpha \wedge d\beta \). By super symmetry and super Jacobi identity we got \( L \) and \( R \) up to constant.

That is our super bracket. On the other hand, the Schouten bracket is a prototype of super brackets, and there is a way to understand it. We first recall the way in the next subsection, then in the final subsection, we explain our super bracket can be understood in a similar way.

### A.1 Schouten bracket

First we review one procedure to get the Schouten bracket. Let \( x, y \) be vector fields, and \( \alpha \) be a differential form on \( M \). Then we have the magic formula and a formula

\[
\begin{align*}
L_x \alpha &= (dt_x + t_x d) \alpha \\
t_{[x,y]} \alpha &= L_x t_y \alpha - t_y L_x \alpha, \quad \text{i.e.,} \quad t_y L_x = L_x t_y + t_{[y,x]}
\end{align*}
\]

From \((A.1)\), it is known
Lemma 1.

\[(A.3)\quad d \circ i_{x_1} \circ \cdots \circ i_{x_p} - (-1)^p i_{x_1} \circ \cdots \circ i_{x_p} \circ d = \sum_{j=1}^{p} (-1)^{j+1} i_{x_1} \circ \cdots \circ L_{x_j} \cdots \circ i_{x_p} \cdot \]

Symbolically, we denote them as \(i_{x_1} \circ \cdots \circ i_{x_p} = i_{x_1} \wedge \cdots \wedge i_{x_p} = X \) and \(\sum_{j=1}^{p} (-1)^{j+1} i_{x_1} \circ \cdots \circ L_{x_j} \cdots \circ i_{x_p} = \overline{X} \).

The relation \((A.3)\) is symbolically

\[(A.4)\quad [d, X] = d \cdot X - (-1)^p X \cdot d = \overline{X} \]

We use \((A.2)\) a lot and get

\[
i_{y} \overline{X} = \sum_{j=1}^{p} i_{x_1} \circ \cdots \circ L_{x_j} \circ \cdots \circ i_{x_p} = \sum_{j=1}^{p} i_{x_1} \circ \cdots \circ (L_{x_j} i_{y} + i_{[y,x_j]}) \circ \cdots \circ i_{x_p}
= (-1)^{p-1} \overline{X} \cdot i_{y} + L_{y} X \quad \text{where} \quad L_{y} X = \sum_{j=1}^{p} x_1 \wedge \cdots \wedge [y, x_j] \wedge \cdots \wedge x_p.
\]

\[
i_{y} i_{x_1} \overline{X} = (-1)^{p-1} \overline{X} \cdot i_{y} + L_{y} i_{x_1} = (-1)^{p-1} \overline{X} \cdot i_{y} + i_{x_1} L_{y} = (-1)^{2(p-1)} \overline{X} \cdot i_{y} + (-1)^{p-1} L_{x_1} \cdot i_{y} + i_{x_1} L_{y}.
\]

By induction on the degree of \(Y\), we get

\[
Y \cdot \overline{X} = (-1)^q(p-1) \overline{X} \cdot Y + [Y, X]_S, \quad \text{i.e.,} \quad [Y, [d, X]] = [Y, X]_S,
\]

where

\[
[Y, X]_S = \sum_{k=1}^{q} (-1)^{(p-1)(q-k)} y_1 \wedge \cdots \wedge L_{y_k} X \wedge \cdots \wedge y_q
= \sum_{k} (-1)^{j+k} [y_k, x_j] \cdot y_1 \wedge \cdots \wedge \hat{y}_k \wedge \cdots \wedge y_q \wedge x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_p.
\]

This \([Y, X]_S\) is the Schouten bracket of multi-vector fields \(Y\) and \(X\).

A.2 Super bracket on forms

We apply the explanation so far to reach our bracket on forms. Let us consider the \(\mathbb{Z}\)-graded vector space \(\mathfrak{h} = \sum_i \mathfrak{h}_i\) where \(\mathfrak{h}_i = \Lambda^i T^*(M)\). The composition is just the wedge product and it holds

\[
d \cdot A = (dA) + (-1)^a A \cdot d \quad \text{more precisely} \quad d \cdot (A \cdot \omega) = (dA) \cdot \omega + (-1)^a A \cdot (d \cdot \omega).
\]

Unlike the previous discussion, \([[d, A], B]] = [dA, B] = 0\) because \([A, B] = A \cdot B - (-1)^{ab} B \cdot A\) and so \([d, A] = dA\). So we try another bracket

Definition 11. \{A, B\} = A \cdot B - (-1)^{ab} B \cdot A and \{A, B\}_n = \{A, \{d, B\}\}.

Direct computation implies

\[(A.5)\quad \{A, B\}_n = \{A, \{d, B\}\} = 2A \cdot (dB) + 2(-1)^{a+b+ab} B \cdot (dA) + 4(-1)^{b} A \cdot B \cdot d
\]

Since \(d \cdot d = 0\), \(d \cdot \beta = d\beta + (-1)^b \beta \cdot d\) and \(d \beta \in \mathfrak{h}\), the algebra generated \(d\) and \(\mathfrak{h}\) is \(\mathfrak{h} \oplus \mathfrak{h} \cdot d\). Applying the projection from \(\mathfrak{h} \oplus \mathfrak{h} \cdot d\) onto \(\mathfrak{h}\) to \((A.5)\), we get and define a revised bracket as follows.
Definition 12.

\[(A.6) \quad \{A, B\}_r = A \cdot (dB) + (-1)^{a+b+ab} B \cdot (dA) = (-1)^a d(A \cdot B)\]

for each \(A \in \mathfrak{h}_a\) and \(B \in \mathfrak{h}_b\).

**Theorem 13.** The bracket \(\{\cdot, \cdot\}_r\) is a superalgebra bracket by the new grading \(\text{ndeg}(\mathfrak{h}_i) = i + 1\).

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