MULTIPLY UNION FAMILIES IN \( \mathbb{N}^n \)

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ABSTRACT. Let \( A \subset \mathbb{N}^n \) be an \( r \)-wise \( s \)-union family, that is, a family of sequences with \( n \) components of non-negative integers such that for any \( r \) sequences in \( A \) the total sum of the maximum of each component in those sequences is at most \( s \). We determine the maximum size of \( A \) and its unique extremal configuration provided (i) \( n \) is sufficiently large for fixed \( r \) and \( s \), or (ii) \( n = r + 1 \).

1. Introduction

Let \( \mathbb{N} := \{0, 1, 2, \ldots\} \) denote the set of non-negative integers, and let \( [n] := \{1, 2, \ldots, n\} \). Intersecting families in \( 2^{[n]} \) or \( \{0, 1\}^n \) are one of the main objects in extremal set theory. The equivalent dual form of an intersecting family is a union family, which is the subject of this paper. In [3] Frankl and Tokushige proposed to consider such problems not only in \( \{0, 1\}^n \) but also in \( q^n \). They determined the maximum size of 2-wise \( s \)-union families (i) in \( q^n \) for \( n > n_0(q, s) \), and (ii) in \( \mathbb{N}^3 \) for all \( s \) (the definitions will be given shortly). In this paper we extend their results and determine the maximum size and structure of \( r \)-wise \( s \)-union families in \( \mathbb{N}^n \) for the following two cases: (i) \( n \geq n_0(r, s) \), and (ii) \( n = r + 1 \). Much research has been done for the case of families in \( \{0, 1\}^n \), and there are many challenging open problems. The interested reader is referred to [2, 3, 4, 8, 9].

For a vector \( x \in \mathbb{R}^n \), we write \( x_i \) or \( (x)_i \) for the \( i \)th component, so \( x = (x_1, x_2, \ldots, x_n) \). Define the weight of \( a \in \mathbb{N}^n \) by

\[ |a| := \sum_{i=1}^{n} a_i. \]

For a finite number of vectors \( a, b, \ldots, z \in \mathbb{N}^n \) define the join \( a \lor b \lor \cdots \lor z \) by

\[ (a \lor b \lor \cdots \lor z)_i := \max\{a_i, b_i, \ldots, z_i\}, \]

and we say that \( A \subset \mathbb{N}^n \) is \( r \)-wise \( s \)-union if

\[ |a_1 \lor a_2 \lor \cdots \lor a_r| \leq s \text{ for all } a_1, a_2, \ldots, a_r \in A. \]

In this paper we address the following problem.

**Problem.** For given \( n, r \) and \( s \), determine the maximum size \( |A| \) of \( r \)-wise \( s \)-union families \( A \subset \mathbb{N}^n \).
To describe candidates $A$ that give the maximum size to the above problem, we need some more definitions. Let us introduce a partial order $\prec$ in $\mathbb{R}^n$. For $a, b \in \mathbb{R}^n$ we let $a \prec b$ iff $a_i \leq b_i$ for all $1 \leq i \leq n$. Then we define a down set for $a \in \mathbb{N}^n$ by

$$
D(a) := \{c \in \mathbb{N}^n : c \prec a\},
$$

and for $A \subset \mathbb{N}^n$ let

$$
D(A) := \bigcup_{a \in A} D(a).
$$

We also introduce $S(a, d)$, which can be viewed as a part of sphere centered at $a \in \mathbb{N}^n$ with radius $d \in \mathbb{N}$, defined by

$$
S(a, d) := \{a + \epsilon \in \mathbb{N}^n : \epsilon \in \mathbb{N}^n, |\epsilon| = d\}.
$$

We say that $a \in \mathbb{N}^n$ is a balanced partition, if all $a_i$’s are as close to each other as possible, more precisely, $|a_i - a_j| \leq 1$ for all $i, j$. Let $1 := (1, 1, \ldots, 1) \in \mathbb{N}^n$.

For $r, s, n, d \in \mathbb{N}$ with $0 \leq d \leq \left\lfloor \frac{n}{s} \right\rfloor$ and $a \in \mathbb{N}^n$ with $|a| = s - rd$ let us define a family $K$ by

$$
K = K(r, n, a, d) := \bigcup_{i=0}^{\left\lfloor \frac{n}{s} \right\rfloor} D(S(a + i1, d - ui)),
$$

where $u = n - r + 1$. This is the candidate family. Intuitively $K$ is a union of balls, and the corresponding centers and radii are chosen so that $K$ is $r$-wise $s$-union as we will see in Claim 3 in the next section.

**Conjecture.** Let $r \geq 2$ and $s$ be positive integers. If $A \subset \mathbb{N}^n$ is $r$-wise $s$-union, then

$$
|A| \leq \max_{0 \leq d \leq \left\lfloor \frac{n}{s} \right\rfloor} |K(r, n, a, d)|,
$$

where $a \in \mathbb{N}^n$ is a balanced partition with $|a| = s - rd$. Moreover if equality holds, then $A = K(r, n, a, d)$ for some $0 \leq d \leq \left\lfloor \frac{n}{s} \right\rfloor$.

We first verify the conjecture when $n$ is sufficiently large for fixed $r, s$. Let $e_i$ be the $i$-th standard base of $\mathbb{R}^n$, that is, $(e_i)_j = \delta_{ij}$. Let $\bar{e}_0 = 0$, and $\bar{e}_i = \sum_{j=1}^{i} e_j$ for $1 \leq i \leq n$, e.g., $\bar{e}_n = 1$.

**Theorem 1.** Let $r \geq 2$ and $s$ be fixed positive integers. Write $s = dr + p$ where $d$ and $p$ are non-negative integers with $0 \leq p < r$. Then there exists an $n_0(r, s)$ such that if $n > n_0(r, s)$ and $A \subset \mathbb{N}^n$ is $r$-wise $s$-union, then

$$
|A| \leq |D(S(\bar{e}_p, d))|.
$$

Moreover if equality holds, then $A$ is isomorphic to $D(S(\bar{e}_p, d)) = K(r, n, \bar{e}_p, d)$.

We mention that the case $A \subset \{0, 1\}^n$ of Conjecture is posed in [2] and partially solved in [4], and the case $r = 2$ of Theorem [1] is proved in [6] in a slightly stronger form. We also notice that if $A \subset \{0, 1\}^n$ is 2-wise $(2d + p)$-union, then the Katona’s $t$-intersection theorem [7] states that $|A| \leq |D(S(\bar{e}_p, d) \cap \{0, 1\})|$ for all $n \geq s$.

Next we show that the conjecture is true if $n = r + 1$. We also verify the conjecture on general $n$ if $A$ satisfies some additional properties described below.
Let $A \subset \mathbb{N}^n$ be $r$-wise $s$-union. For $1 \leq i \leq n$ let

$$m_i := \max\{x_i : x \in A\}.$$  \hspace{1cm} (2)

If $n - r$ divides $|m| - s$, then we define

$$d := \frac{|m| - s}{n - r} \geq 0,$$  \hspace{1cm} (3)

and for $1 \leq i \leq n$ let

$$a_i := m_i - d,$$  \hspace{1cm} (4)

and we assume that $a_i \geq 0$. In this case we have $|a| = s - rd$. Since $|a| \geq 0$ it follows that $d \leq \lfloor \frac{s}{r} \rfloor$. For $1 \leq i \leq n$ define $P_i \in \mathbb{N}^n$ by

$$P_i := a + de_i,$$  \hspace{1cm} (5)

where $e_i$ denotes the $i$th standard base, for example, $P_2 = (a_1, a_2 + d, a_3, \ldots, a_n)$.

**Theorem 2.** Let $A \subset \mathbb{N}^n$ be $r$-wise $s$-union. Assume that the sequences $P_i$ are well-defined and

$$\{P_1, \ldots, P_n\} \subset A.$$  \hspace{1cm} (6)

Then it follows that

$$|A| \leq \max_{0 \leq d' \leq \lfloor \frac{s}{r} \rfloor} |K(r, n, a', d')|,$$

where $a' \in \mathbb{N}^n$ is a balanced partition with $|a'| = s - rd'$. Moreover if equality holds, then $A = K(r, n, a', d')$ for some $0 \leq d' \leq \lfloor \frac{s}{r} \rfloor$.

We will show that the assumption $(3)$ is satisfied when $n = r + 1$, see Corollary 3 in the last section.

Notation: For $a, b \in \mathbb{N}^n$ we define $a \setminus b \in \mathbb{N}^n$ by $(a \vee b) - b$, in other words, $(a \setminus b)_i := \max\{a_i - b_i, 0\}$. The support of $a$ is defined by $\text{supp}(a) := \{j : a_j > 0\}$.

2. Proof of Theorem 1 — the case when $n$ is large

Let $r, s$ be given, and let $s = dr + p$, $0 \leq p < r$. We consider the situation $n \to \infty$ for fixed $r, s, d$, and $p$.

**Claim 1.** $|\mathcal{D}(\mathcal{S}(\tilde{e}_p, d))| = \sum_{j=0}^{p} \binom{p}{j} \binom{n-j+d}{d} = (2^p / d!)n^d + O(n^{d-1})$.

**Proof.** By definition we have

$$\mathcal{D}(\mathcal{S}(\tilde{e}_p, d)) = \{x + y \in \mathbb{N}^n : |x| \leq d, y \prec \tilde{e}_p\}.$$  

We rewrite the RHS by classifying vectors according to their supports. For $I \subset [p]$ let $\tilde{e}_p|_I$ be the restriction of $\tilde{e}_p$ to $I$, that is, $(\tilde{e}_p|_I)_i$ is 1 if $i \in I$ and 0 otherwise, and let

$$R(I) := \{\tilde{e}_p|_I + z : \text{supp}(z) \subset I \cup ([n] \setminus [p]), |z| \leq d\}.$$  

Then we have $\mathcal{D}(\mathcal{S}(\tilde{e}_p, d)) = \bigcup_{I \subset [p]} R(I)$. For each $I \in \binom{[p]}{i}$ the number of $z$ in $R(I)$ equals the number of nonnegative integer solutions of $z_1 + z_2 + \cdots + z_{i+(n-p)} \leq d$. Thus it follows that $|R(I)| = \binom{n-(p-i)+d}{d}$, and

$$|\mathcal{D}(\mathcal{S}(\tilde{e}_p, d))| = \sum_{i=0}^{p} \binom{p}{i} \binom{n-(p-i)+d}{d} = \sum_{j=0}^{p} \binom{p}{j} \binom{n-j+d}{d}.$$
The RHS is further rewritten using \(\binom{n-j+d}{d} = \frac{n^d}{d!} + O(n^{d-1})\) and \(\sum_{j=0}^{p} \binom{p}{j} = 2^p\), as needed.

Let \(A \subset \mathbb{N}^n\) be \(r\)-wise \(s\)-union with maximal size. So \(A\) is a down set. We will show that \(|A| \leq |\mathcal{D}(S(\mathbb{e}_p, d))|\).

First suppose that there is a \(t\) with \(2 \leq t \leq r\) such that \(A\) is \(t\)-wise \((dt + p)\)-union, but not \((t - 1)\)-wise \((d(t - 1) + p)\)-union. In this case, by the latter condition, there are \(b_1, \ldots, b_{t-1} \in A\) such that \(|b| \geq d(t - 1) + p + 1\), where \(b = b_1 \lor \cdots \lor b_{t-1}\). Then, by the former condition, for every \(a \in A\) it follows that \(|a \lor b| \leq dt + p\), so \(|a \setminus b| \leq d - 1\). This gives us

\[
A \subset \{x + y \in \mathbb{N}^n : |x| \leq d - 1, \ y \prec b\}. 
\]

There are \(\binom{n+(d-1)}{d}\) choices for \(x\) satisfying \(|x| \leq d - 1\). On the other hand, the number of \(y\) with \(y \prec b\) is independent of \(n\) (so it is a constant depending on \(r\) and \(s\) only). In fact \(|b| \leq (t-1)s < rs\), and there are less than \(2^s\) choices for \(y\). Thus we get \(|A| < \binom{n+(d-1)}{d}2^s = O(n^{d-1})\) and we are done.

Next we suppose that \(A\) is \(t\)-wise \((dt + p)\)-union for all \(1 \leq t \leq r\).

The case \(t = 1\) gives us \(|a| \leq d + p\) for every \(a \in A\). If \(p = 0\), then this means that \(A \subset \mathcal{D}(S(\mathbb{0}, d))\), which finishes the proof for this case. So, from now on, we assume that \(1 \leq p < r\). We will see that there is a \(u\) with \(u \geq 1\) such that there exist \(b_1, \ldots, b_u \in A\) satisfying

\[
|b| = u(d + 1),
\]

where \(b := b_1 \lor \cdots \lor b_u\). In fact we have (8) for \(u = 1\), if otherwise \(A \subset \mathcal{D}(S(\mathbb{0}, d))\). On the other hand, setting \(t = p + 1 \leq r\) in \(9\), we see that \(A\) is \((p + 1)\)-wise \(((p + 1)(d + 1) - 1)\)-union, and (8) fails if \(u = p + 1\). So we choose maximal \(u\) with \(1 \leq u \leq p\) satisfying (9), and fix \(b = b_1 \lor \cdots \lor b_u\). By this maximality, for every \(a \in A\), it follows that \(|a \lor b| \leq (u + 1)(d + 1) - 1\), and

\[
|a \setminus b| = |a \lor b| - |b| \leq d.
\]

Using (9) we have \(A \subset \bigcup_{i=0}^{d} A_i\), where

\[
A_i := \{x + y \in A : |x| = i, \ y \prec b\}.
\]

Then we have \(|A_i| \leq \binom{n+i}{i}2^{|b|}\). Noting that \(|b| \leq u(d + 1) < r(d + 1) = O(1)\) it follows \(\sum_{i=0}^{d-1} |A_i| = O(n^{d-1})\). So the size of \(A_d\) is essential.

We naturally identify \(a \in A\) with a subset of \([n] \times \{1, \ldots, d + p\}\). Formally let

\[
\phi(a) := \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq a_i\},
\]

for example, if \(a = (1, 0, 2)\), then \(\phi(a) = \{(1, 1), (3, 1), (3, 2)\}\). Define \(m = m(d)\) to be \(r + 1\) if \(d = 1\) and \(dr\) if \(d \geq 2\). We say that \(b' \prec b\) is rich if there exist \(m\) vectors \(c_1, \ldots, c_m\) of weight \(d\) such that \(b' \lor c_j \in A\) for every \(j\), and the \(m + 1\) subsets \(\phi(c_1), \ldots, \phi(c_m), \phi(b)\) are pairwise disjoint. In this case \(b'' \lor c_j \in A\) for all \(b'' \prec b'\) because \(A\) is a down set. This means that richness is hereditary, namely, if \(b'\) is rich and \(b'' \prec b'\), then \(b''\) is rich as well. Informally, \(b'\) is rich if it can be extended to
a (|b'| + d)-element subset of A in m ways disjointly outside b. We are comparing our family A with the reference family \( \mathcal{D}(\mathcal{S}(\tilde{e}_p), d) \), and we define \( \tilde{b} \) which plays the role of \( \tilde{e}_p \) in our family, namely, let us define

\[
\tilde{b} := \bigvee \{b' \prec b : b' \text{ is rich}\}.
\]

**Claim 2.** \(|\tilde{b}| \leq p\).

**Proof.** Suppose the contrary. Then \(|\tilde{b}| > p \) and we can find rich \( b'_1, b'_2, \ldots, b'_{p+1} \) (with repetition if necessary) such that \(|b'_1 \lor \cdots \lor b'_{p+1}| \geq p + 1 \). Since richness is hereditary we may assume that \(|b'_1 \lor \cdots \lor b'_{p+1}| = p + 1 \). Let \( c_1^{(i)}, \ldots, c_m^{(i)} \) support the richness of \( b'_i \). By definition \( \phi(c_1^{(i)}), \ldots, \phi(c_m^{(i)}) \) and \( \phi(b) \) are pairwise disjoint. Let \( a_1 := b'_1 \lor c_{j_1}^{(1)} \in A \), say, \( j_1 = 1 \). Then choose \( a_2 := b'_2 \lor c_{j_2}^{(2)} \) so that \( \phi(c_{j_1}^{(1)}) \) and \( \phi(c_{j_2}^{(2)}) \) are disjoint. If \( i \leq p \), then having \( a_1, \ldots, a_i \), chosen, we only used \( id \) elements as \( \bigcup_{i=1}^{p} \phi(c_p^{(i)}) \), which intersect at most \( id \) of \( c_1^{(i+1)}, \ldots, c_m^{(i+1)} \). Then, since \( id \leq pd < rd \leq m \), we still have some \( c_{j_{i+1}}^{(i+1)} \), which is disjoint from any already chosen vectors. So we can continue this procedure until we get \( a_{p+1} := b'_{p+1} \lor c_{j_{p+1}}^{(p+1)} \in A \) such that all \( \phi(c_{j_1}^{(1)}), \ldots, \phi(c_{j_{p+1}}^{(p+1)}) \) and \( \phi(b) \) are disjoint. However, these vectors yield that

\[
|a_1 \lor \cdots \lor a_{p+1}| = |b'_1 \lor \cdots \lor b'_{p+1}| + |c_{j_1}^{(1)}| + \cdots + |c_{j_{p+1}}^{(p+1)}| = (p + 1) + (p + 1)d = (p + 1)(d + 1),
\]

which contradicts \( [\tilde{b}] \) at \( t = p + 1 \). □

If \( y \prec b \) is not rich, then

\[
\{\phi(x) : x + y \in A_d, |x| = d\}
\]

is a family of \( d \)-element subsets on \((d + p)n\) vertices, which has no \( m \) pairwise disjoint subsets (so the matching number is \( m - 1 \) or less). Thus, by the Erdős matching theorem \([\tilde{b}]\), the size of this family is \( O(n^{d-1}) \). There are at most \( 2^{\tilde{b}} = O(1) \) choices for non-rich \( y \prec b \), and we can conclude that the number of vectors in \( A_d \) coming from non-rich \( y \) is \( O(n^{d-1}) \). Then the remaining vectors in \( A_d \) come from rich \( y \prec \tilde{b} \), and the number of such vectors is at most \( 2^{\tilde{b}}(n^d/d) \). Note also that \( \sum_{i=0}^{d-1} |A_i| = O(n^{d-1}) \). Consequently we get

\[
|A| \leq 2^{\tilde{b}} \left( \frac{n + d}{d} \right) + O(n^{d-1}) = (2^{\tilde{b}}/d!) \ n^d + O(n^{d-1}).
\]

Recall that the reference family is of size \((2^p/d!)n^d + O(n^{d-1})\), and \(|\tilde{b}| \leq p \) from Claim 2. So we only need to deal with the case when \(|\tilde{b}| = p \) and there are exactly \( 2^p \) rich sets. In other words, \( b = \tilde{e}_p \) (by renaming coordinates if necessary) and every \( b' \prec \tilde{e}_p \) is rich. We show that \( A \subset \mathcal{D}(\mathcal{S}(\tilde{e}_p, d)) \). Suppose the contrary, then there is an \( a \in A \) such that \(|a'| \geq d + 1 \), where \( a' = a \setminus \tilde{e}_p \). Since \( A \) is a down set we may assume that \(|a'| = d + 1 \). Now \( \tilde{e}_p \) is rich and let \( c_1, \ldots, c_m \) be vectors assured by the richness. We remark that \( m - (d + 1) \geq r - 1 \). In fact if \( d = 1 \) then
Let \( m - (d + 1) = r - 1 \), and if \( d \geq 2 \) then \( m - (d + 1) = (r - 1)(d - 1) + r - 2 \geq r - 1 \). So we may assume that \( \phi(c_1), \ldots, \phi(c_{r-1}) \) are pairwise disjoint and disjoint to \( \phi(a) \) as well. Let \( a_i := \tilde{e}_p \lor c_i \in A \) for \( 1 \leq i \leq r - 1 \). Then we get

\[
|a \lor a_1 \lor \cdots \lor a_{r-1}| = |\tilde{e}_p \lor a'| + |c_1| + \cdots + |c_{r-1}|
= (p + d + 1) + (r - 1)d = dr + p + 1 = s + 1,
\]

which contradicts that \( A \) is \( r \)-wise \( s \)-union. This completes the proof of Theorem 1.

3. The Polytope \( P \) and Proof of Theorem 2

Let \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \) with \( |a| = s - rd \) for some \( d \in \mathbb{N} \). We introduce a convex polytope \( P \subset \mathbb{R}^n \), which will play a key role in our proof. This polytope is defined by the following \( n + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-r+1} \) inequalities:

\[
x_i \geq 0 \quad \text{if} \quad 1 \leq i \leq n, \tag{10}
\]

\[
\sum_{i \in I} x_i \leq \sum_{i \in I} a_i + d \quad \text{if} \quad 1 \leq |I| \leq n - r + 1, \quad I \subset [n]. \tag{11}
\]

Namely,

\[
P := \{ x \in \mathbb{R}^n : x \text{ satisfies (10) and (11)} \}.
\]

Let \( L \) denote the integer lattice points in \( P \):

\[
L = L(r, n, a, d) := \{ x \in \mathbb{N}^n : x \in P \}.
\]

**Lemma 1.** The two sets \( K \) (defined by (1)) and \( L \) are the same, and \( r \)-wise \( s \)-union.

**Proof.** This lemma is a consequence of the following three claims.

**Claim 3.** The set \( K \) is \( r \)-wise \( s \)-union.

**Proof.** Let \( x_1, x_2, \ldots, x_r \in K \). We show that \( |x_1 \lor x_2 \lor \cdots \lor x_r| \leq s \). We may assume that \( x_j \in S(a + i_j 1, d - u i_j) \), where \( u = n - r + 1 \). We may also assume that \( i_1 \geq i_2 \geq \cdots \geq i_r \). Let \( b := a + i_1 1 \). Then, informally, \( |x \setminus b| = |x \lor b| - b \) counts the excess of \( x \) above \( b \), more precisely, it is \( \sum_{j \in [n]} \max \{0, x_j - b_j\} \). Thus we have

\[
|x_1 \lor x_2 \lor \cdots \lor x_r| \leq |b| + \sum_{j=1}^r |x_j \setminus b|
\]

\[
\leq |a| + ni_1 + \sum_{j=1}^r ((d - u i_j) - (i_1 - i_j))
\]

\[
= |a| + dr + (n - r)i_1 - \sum_{j=1}^r (u - 1)i_j
\]

\[
= s - (n - r) \sum_{j=2}^r i_j \leq s,
\]

as required. \( \square \)

**Claim 4.** \( K \subset L \).
Proof. Let $x \in K$. We show that $x \in L$, that is, $x$ satisfies (10) and (11). Since (10) is clear by definition of $K$, we show that (11). To this end we may assume that $x \in S(a + i, d - ui)$, where $u = n - r + 1$ and $i \leq \lceil \frac{d}{n} \rceil$. Let $I \subset [n]$ with $1 \leq |I| \leq u$. Then $i|I| \leq ui$. Thus it follows
\[
\sum_{j \in I} x_j \leq \sum_{j \in I} a_j + i|I| + (d - ui) \leq \sum_{j \in I} a_j + d,
\]
which confirms (11).

Claim 5. $K \supset L$.

Proof. Let $x \in L$. We show that $x \in K$, that is, there exists some $i'$ such that $0 \leq i' \leq \lfloor \frac{d}{n-r+1} \rfloor$ and
\[
|x \setminus (a + i'1)| \leq d - (n - r + 1)i'.
\]
We write $x$ as
\[
x = (a_1 + i_1, a_2 + i_2, \ldots, a_n + i_n),
\]
where we may assume that $d \geq i_1 \geq i_2 \geq \cdots \geq i_n$. We notice that some $i_j$ can be negative. Since $x \in L$ it follows from (11) (a part of the definition of $L$) that if $1 \leq |I| \leq n - r + 1$ and $I \subset [n]$, then
\[
\sum_{j \in I} i_j \leq d.
\]

Let $J := \{ j : x_j \geq a_j \}$ and we argue separately by the size of $|J|$.

If $|J| \leq n - r + 1$, then we may choose $i' = 0$. In fact,
\[
|x \setminus a| = \max\{0, i_1\} + \max\{0, i_2\} + \cdots + \max\{0, i_{n-r+1}\} = \max\left\{ \sum_{j \in I} i_j : I \subset [n-r+1] \right\} \leq d.
\]

If $|J| \geq n - r + 2$, then we may choose $i' = i_{n-r+2}$. In fact, by letting $i':= i_{n-r+2}$, we have
\[
|x \setminus (a + i'1)| = (i_1 - i') + (i_2 - i') + \cdots + (i_{n-r+1} - i') \leq d - (n - r + 1)i'.
\]

We need to check $0 \leq i' \leq \lfloor \frac{d}{n-r+1} \rfloor$. It follows from $|J| \geq n - r + 2$ that $i' \geq 0$. Also $d \geq i_1 \geq i_2 \geq \cdots \geq i_{n-r+2}$ and $i_1 + i_2 + \cdots + i_{n-r+1} \leq d$ yield $i' \leq \lfloor \frac{d}{n-r+1} \rfloor$.

This completes the proof of Lemma 1.

Let
\[
\sigma_k(a) := \sum_{K \in \binom{\{1, \ldots, n\}}{k}} \prod_{i \in K} a_i
\]
be the $k$th elementary symmetric polynomial of $a_1, \ldots, a_n$. 

Lemma 2. The size of $K(r, n, a, d)$ is given by
\[ |K(r, n, a, d)| = \sum_{j=0}^{n} \binom{d + j}{j} \sigma_{n-j}(a) \]
\[ + \sum_{i=1}^{\frac{n}{r}} \sum_{j=u+1}^{n} \left( \binom{d - ui + j}{j} - \binom{d - ui + u}{j} \right) \sigma_{n-j}(a + i1), \]
where $u = n - r + 1$. Moreover, for fixed $n, r, d$ and $|a|$, this size is maximized if and only if $a$ is a balanced partition.

Proof. For $J \subseteq [n]$ let $x|_J$ be the restriction of $x$ to $J$, that is, $(x|_J)_i$ is $x_i$ if $i \in J$ and 0 otherwise.

First we count the vectors in the base layer $\mathcal{D}(S(a, d))$. To this end we partition this set into $\bigsqcup_{J \subseteq [n]} A_0(J)$, where
\[ A_0(J) = \{ a|_J + e + b : \text{supp}(e) \subseteq J, |e| \leq d, \text{supp}(b) \subseteq [n] \setminus J, b_i \leq a_i \text{ for } i \not\in J \}. \]
The number of vectors $e$ with the above property is equal to the number of non-negative integer solutions of the inequality $x_1 + x_2 + \cdots + x_{|J|} \leq d$, which is $\binom{d + |J|}{|J|}$.
The number of vectors $b$ is clearly $\prod_{l \in [n] \setminus J} a_l$. Thus we get
\[ \sum_{J \subseteq [n]} |A_0(J)| = \sum_{J \subseteq [n]} \binom{d + |J|}{|J|} \prod_{l \in [n] \setminus J} a_l = \binom{d + j}{j} \sigma_{n-j}(a), \]
and $|\mathcal{D}(S(a, d))| = \sum_{j=0}^{n} \binom{d + j}{j} \sigma_{n-j}(a)$.

Next we count the vectors in the $i$th layer:
\[ \mathcal{D}(S(a + i1, d - ui)) \setminus \left( \bigcup_{j=0}^{i-1} \mathcal{D}(S(a + j1, d - uj)) \right). \]
For this we partition the above set into $\bigsqcup_{J \subseteq [n]} A_i(J)$, where
\[ A_i(J) = \{ (a + i1)|_J + e + b : \text{supp}(e) \subseteq J, d - u(i - 1) - |J| < |e| \leq d - ui, \text{supp}(b) \subseteq [n] \setminus J, b_i \leq a_i + i \text{ for } l \not\in J \}. \]
In this case we need $d - u(i - 1) < |J| + |e|$ because the vectors satisfying the opposite inequality are already counted in the lower layers $\bigsqcup_{J \subseteq [n]} A_j(J)$. We also notice that $d - u(i - 1) - |J| < d - ui$ implies that $|J| > u$. So $A_i(J) = \emptyset$ for $|J| \leq u$. Now we count the number of vectors $e$ in $A_i(J)$, or equivalently, the number of non-negative integer solutions of
\[ d - u(i - 1) - |J| < x_1 + x_2 + \cdots + x_{|J|} \leq d - ui. \]
This number is $\binom{d - ui + j}{j} - \binom{d - ui + u}{j}$, where $j = |J|$. On the other hand, the number of vectors $b$ in $A_i(J)$ is $\prod_{l \in [n] \setminus J} (a_l + i)$. Consequently we get
\[ \sum_{J \subseteq [n]} |A_i(J)| = \sum_{j=u+1}^{n} \left( \binom{d - ui + j}{j} - \binom{d - ui + u}{j} \right) \sigma_{n-j}(a + i1). \]
Summing this term over $1 \leq i \leq \lfloor \frac{n}{d} \rfloor$ we finally obtain the second term of the RHS of $|K|$ in the statement of this lemma. Then, for fixed $|a|$, the size of $K$ is maximized when $\sigma_{n-j}(a)$ and $\sigma_{n-j}(a + i1)$ are maximized. By the property of symmetric polynomials, this happens if and only if $a$ is a balanced partition, see e.g., Theorem 52 in section 2.22 of [3].

□

Proof of Theorem 2. Let $A \subset \mathbb{N}^n$ be an $r$-wise $s$-union with (3). For $I \subset [n]$ let

$$m_I := \max \left\{ \sum_{i \in I} x_i : x \in A \right\}.$$

Claim 6. If $I \subset [n]$ and $1 \leq |I| \leq n - r + 1$, then

$$m_I = \sum_{i \in I} a_i + d.$$

Proof. Choose $j \in I$. By (3) we have $P_j \in A$ and

$$m_I \geq \sum_{i \in I} (P_j)_i = \sum_{i \in I} a_i + d. \quad (12)$$

We need to show that this inequality is actually an equality. Let $[n] = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_r$ be a partition of $[n]$. Then it follows that

$$s \geq m_{I_1} + m_{I_2} + \cdots + m_{I_r} \geq \sum_{i \in [n]} a_i + rd = s,$$

where the first inequality follows from the $r$-wise $s$-union property of $A$, and the second inequality follows from (12). Since the left-most and the right-most sides are the same $s$, we see that all inequalities are equalities. This means that (12) is equality, as needed.

By this claim if $x \in A$ and $1 \leq |I| \leq n - r + 1$, then we have

$$\sum_{i \in I} x_i \leq m_I = \sum_{i \in I} a_i + d.$$

This means that $A \subset L$. Finally the theorem follows from Lemmas 1 and 2.

□

Corollary 3. If $n = r + 1$, then Conjecture is true.

Proof. Let $n = r + 1$ and let $A \subset \mathbb{N}^{r+1}$ be $r$-wise $s$-union with maximum size. Define $m$ by (3). Since $n - r = 1$ we can define $d$ by (3). Then define $a$ by (4). We need to verify $a_i \geq 0$ for all $i$. To this end we may assume that $m_1 \geq m_2 \geq \cdots \geq m_{r+1}$. Then $a_i \geq a_{r+1} = m_{r+1} - d$, so it suffices to show $m_{r+1} \geq d$. Since $A$ is $r$-wise $s$-union it follows that $m_1 + m_2 + \cdots + m_r \leq s$. This together with the definition of $d$ implies $d = |m| - s \leq m_{r+1}$, as needed. So we can properly define $P_i$ by (3).

Next we check that $x \in A$ satisfies (10) and (11). By definition we have $x_i \leq m_i = a_i + d$, so we have (10). Since $A$ is $r$-wise $s$-union, we have

$$(x_1 + x_2) + m_3 + \cdots + m_{r+1} \leq s,$$

or equivalently,

$$(x_1 + x_2) + (a_3 + d) + \cdots + (a_{r+1} + d) \leq s = |a| + rd.$$
Rearranging we get $x_1 + x_2 \leq a_1 + a_2 + d$, and we get the other cases similarly, so we obtain (11). Thus $A \subset L$ and the result follows from Lemmas 1 and 2. □

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