Inequalities Associated to a Sequence of Dyadic Martingales

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Abstract: In this article, we establish some inequalities associated to a sequence of dyadic martingales. These inequalities are sub-Gaussian type estimates. We derive the inequalities for a regular sequence of dyadic martingales and also for a tail sequence.

Keywords: Dyadic Martingales, Square Function, Tail Square Function

Introduction

We first discuss the meaning of the word ‘martingale’. Originally martingale meant a strategy for betting in which you double your bet every time you lose. Let us consider a game in which the gambler wins his stake if a coin comes up heads and loses it if the coin comes up tails. The strategy is that the gambler doubles his bet every time he loses and continues the process, so that the first win would recover all previous losses plus win a profit equal to the original stake. This process of betting can be represented by a sequence of functions which is an example of dyadic martingale. Now we give the definition of dyadic martingales. For this let $\mathcal{D}_n$ denote the family of dyadic subintervals of the unit interval $[0, 1)$ of the form $\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$ where $n = 0, 1, 2 \ldots$ and $j = 0, 1, \ldots, 2^n-1$.

Definition 1.1 (Dyadic Martingale) (Bañuelos and Moore, 1999)

A dyadic martingale is a sequence of integrable functions, $\{f_n\}_{n=0}^{\infty}$ from $[0, 1) \to \mathbb{R}$ such that:

(i) For every $n, f_n$ is $\mathcal{F}_n$-measurable where $\mathcal{F}_n$ is the $\sigma$-algebra generated by dyadic intervals of the form $\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right), j \in \{0, 1, 2, \ldots, 2^n-1\}$

(ii) And the following conditional expectation condition for all $n \geq 0$ holds:

$$\mathbb{E}\left(f_{n+1} \mid \mathcal{F}_n\right) = f_n,$$

where, $\mathbb{E}\left(f_{n+1} \mid \mathcal{F}_n\right) = \frac{1}{|Q_n|}\int_{Q_n} f_{n+1}(y)dy$, for $Q_n \in \mathcal{D}_n$ and $x \in Q_n$.

A most general type of example of dyadic martingale is given by: Let $f \in L^1[0, 1)$ and $Q_n$ be a dyadic interval of length $\frac{1}{2^n}$ on $[0, 1)$. Define $f_n(x) = \frac{1}{|Q_n|}\int_{Q_n} f(y)dy$, $x \in Q_n$ where $|Q_n|$ is length of $Q_n$. Then $\{f_n\}_{n=1}^{\infty}$ is a dyadic martingale on $[0, 1)$. We now prove that the functions so defined are a dyadic martingale.

For this, we note that $\mathcal{F}_n = \{[0, 1), \phi, \mathcal{F}_n, [0, 1/2), [1/2, 1)\}$ and so on. We have $f_n(x) = \frac{1}{|Q_n|}\int_{Q_n} f(y)dy$, $x \in Q_n$ and this is the average of $f$ on $Q_n$.

Consequently, $f_n$ is constant on each of these $n$th generation dyadic intervals $Q_n = \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$ where $n = 0, 1, 2 \ldots$ and $j = 0, 1, \ldots, 2^n-1$. Thus for all $\lambda \in \mathbb{R}$, the set $\{x \in [0, 1) : f_n(x) > \lambda\}$ belongs to $\mathcal{F}_n$. Hence for each $n, f_n$ is $\mathcal{F}_n$-measurable. This shows that the first condition is satisfied. Next, we show that the expectation condition is also satisfied. Here:

$$\mathbb{E}\left(f_{n+1} \mid \mathcal{F}_n\right) = \frac{1}{|Q_n|}\int_{Q_n} f_{n+1}(y)dy$$

where, $x \in Q_n$ and $|Q_n| = \frac{1}{2^n}$. Let $Q_{n+1,1}$ and $Q_{n+1,2}$ be the $(n + 1)$th generation dyadic intervals such that $Q_n = Q_{n+1,1} \cup Q_{n+1,2}$. Using that fact that $f_{n+1}$ is constant on $Q_{n+1,1}$ and $Q_{n+1,2}$, we have:
B(x, r)

The proof of this Lemma using a different approach. Our proof is more analytic than the original probabilistic approach. We will use this Lemma in the proof of our inequalities.

Preliminaries
We first fix some notations, give some definitions which will be used in the course of the proof.

Definition 2.1
For a dyadic martingale, \( \{f_n\}_{n=0}^{\infty} \), we define:

(i) The increments: \( d_k = f_k - f_{k-1} \). So \( f_n(x) = \sum_{k=0}^{n} d_k(x) + f_0(x) \)
(ii) The quadratic characteristics or square function: \( S_n^2 f(x) = \sum_{k=0}^{n} d_k^2(x) \)
(iii) the limit function: \( S^2 f(x) = \lim_{n \to \infty} S_n^2 f(x) = \sum_{k=0}^{\infty} d_k^2(x) \)
(iv) the tail square function:

\[ S_n^2 f(x) = \left( S_n^2 f(x) \right)^2 = \sum_{k=n}^{\infty} d_k^2(x) \]

The martingale square function is a local version of variance and can also be understood as a discrete counterpart of the area function in Harmonic Analysis. From the definition, we note that for any \( x, y \in \mathbb{Q}_n \), we have \( S_n^2 f(x) = S_n^2 f(y) \). But the martingale tail square function, \( S_n^2 f(x) \), may not be equal to \( S_n^2 f(y) \). For more about martingales (Neveu and Speed, 1975).

Definition 2.2 (Hardy-Littlewood Maximal Function)
Let \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \). Then Hardy-Littlewood Maximal function associated to \( f \), denoted by \( Mf \), is defined as:

\[ Mf(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy, \]
where, \( B(x, r) \) is the ball with center at \( x \) and radius \( r \).

Proof of the Main Results
We first prove a Lemma. This Lemma is also known as Rubin’s Lemma (Pipher, 1993). The proof of this Lemma can also be found in (Chang et al., 1985). Here we give a proof of the Lemma using a different approach. Our proof is more analytic than the original probabilistic approach. We will use this Lemma in the proof of our inequalities.
Lemma 3

For a dyadic martingale \( \{f_n\} \), with \( f_0 = 0 \):

\[
\int_0^1 \exp \left( f_n(x) - \frac{1}{2} S_n^2 f(x) \right) dx \leq 1.
\]

Proof of Lemma 3

We claim that:

\[
g(n) = \int_0^1 \exp \left( \sum_{k=0}^n d_k(x) - \frac{1}{2} \sum_{k=0}^n d_k^2(x) \right) dx
\]

is a decreasing function of \( n \). Let \( Q_0 \) be an arbitrary \( n \)th generation dyadic interval. We have \( \sum_{k=0}^n d_k(x) = f_n(x) \) and \( f_n(x) \) is constant on \( Q_0 \). Using this we have:

\[
g(n+1) = \sum_{j=0}^{2n} \int_{Q_j} \exp \left( \sum_{k=0}^{n+1} d_k(x) - \frac{1}{2} \sum_{k=0}^{n+1} d_k^2(x) \right) dx
\]

\[
= \sum_{j=0}^{2n} \int_{Q_j} \exp \left( \sum_{k=0}^{n} d_k(x) - \frac{1}{2} \sum_{k=0}^{n} d_k^2(x) \right) \exp \left( d_{n+1}(x) - \frac{1}{2} d_{n+1}^2(x) \right) dx
\]

\[
= \sum_{j=0}^{2n} \int_{Q_j} \exp \left( \sum_{k=0}^{n} d_k(x) - \frac{1}{2} \sum_{k=0}^{n} d_k^2(x) \right) \left[ \int_{Q_j} \exp \left( d_{n+1}(x) - \frac{1}{2} d_{n+1}^2(x) \right) dx \right]
\]

Let \( Q_{(n+1)j} \) and \( Q_{(n+1)j}^c \) be the dyadic subintervals of \( Q_0 \). Suppose \( d_{n+1} \) takes the value \( \alpha \) on \( Q_{(n+1)j} \). Then by the expectation condition, \( d_{n+1} \) takes the value \( -\alpha \) on \( Q_{(n+1)j}^c \). This gives:

\[
\int_{Q_j} \exp \left( d_{n+1}(x) - \frac{1}{2} d_{n+1}^2(x) \right) dx
\]

\[
= \int_{Q_{(n+1)j}} \exp \left( \alpha - \frac{1}{2} \alpha^2 \right) dx + \int_{Q_{(n+1)j}^c} \exp \left( -\alpha - \frac{1}{2} \alpha^2 \right) dx
\]

\[
= \left[ \exp \left( \alpha - \frac{1}{2} \alpha^2 \right) + \exp \left( -\alpha - \frac{1}{2} \alpha^2 \right) \right] \frac{1}{2^{n+1}}
\]

\[
= 2 \exp \left( \frac{\alpha^2}{2} \right) \cos \alpha \frac{1}{2^{n+1}}
\]

Now, using the elementary fact that \( \cosh x \leq e^x \), we have:

\[
g(n+1) \leq \frac{1}{2^{n+1}} \sum_{j=0}^{2n} \left[ \exp \left( \sum_{k=0}^{n} d_k(x) - \frac{1}{2} \sum_{k=0}^{n} d_k^2(x) \right) \right] + 2 \exp \left( \frac{\alpha^2}{2} \right) \cos \alpha
\]

\[
= \frac{1}{2^{n+1}} \sum_{j=0}^{2n} \left[ \exp \left( \sum_{k=0}^{n} d_k(x) - \frac{1}{2} \sum_{k=0}^{n} d_k^2(x) \right) \right] + 2 \exp \left( \frac{\alpha^2}{2} \right) \cos \alpha
\]

\[
g(n) \leq \frac{1}{2^n} \int_0^1 \exp \left( f_n(x) - \frac{1}{2} S_n^2 f(x) \right) dx \leq 1.
\]

Let \( Q_{11} \) and \( Q_{12} \) be the dyadic subintervals of \( Q_0 \). Assume that \( d_1 \) takes value \( \theta \) on \( Q_{11} \) so that it takes value \( -\theta \) on \( Q_{12} \):

\[
g(1) = \int_0^1 \exp \left( d_1(x) - \frac{1}{2} d_1^2(x) \right) dx
\]

\[
= \int_{Q_{11}} \exp \left( \theta - \frac{1}{2} \theta^2 \right) dx + \int_{Q_{12}} \exp \left( -\theta - \frac{1}{2} \theta^2 \right) dx
\]

\[
= \exp \left( \theta - \frac{1}{2} \theta^2 \right) \frac{1}{2} + \exp \left( -\theta - \frac{1}{2} \theta^2 \right) \frac{1}{2}
\]

\[
= \exp \left( \frac{-1}{2} \theta^2 \right) \frac{e^\theta + e^{-\theta}}{2}
\]

\[
= \exp \left( \frac{-1}{2} \theta^2 \right) \cosh \theta
\]

\[
\leq \exp \left( \frac{-1}{2} \theta^2 \right) \exp \left( \frac{1}{2} \theta^2 \right)
\]

\[
= 1.
\]

Since \( g(n) \) is decreasing and \( g(1) \leq 1 \) we conclude:

\[
\int_0^1 \exp \left( \sum_{k=0}^n d_k(x) - \frac{1}{2} \sum_{k=0}^n d_k^2(x) \right) dx \leq 1.
\]

Hence:

\[
\int_0^1 \exp \left( f_n(x) - \frac{1}{2} S_n^2 f(x) \right) dx \leq 1.
\]

This completes the proof.

Remark 4

Note that if we rescale the sequence \( \{f_n\} \) by \( \lambda \), then Lemma 3 gives:

\[
\int_0^1 \exp \left( \lambda f_n(x) - \frac{1}{2} \lambda^2 S_n^2 f(x) \right) dx \leq 1.
\]

This shows that this lemma is an inhomogeneous type inequality. We won't need this fact in the sequel.
Proof of Inequality 1

Fix \( n \). Let \( \lambda > 0 \), \( \gamma > 0 \). Then for every \( m \leq n \):

\[
f_m(x) = \frac{1}{|Q_n|} \int_{Q_n} f_m(y) dy, \quad x \in Q_n, \quad |Q_n| = \frac{1}{2^n}.
\]

Fix \( x \): Then \( \sup_{y \in Q_n} |f_m(x)| \leq M |f_m(x)| \) where \( M_{f_m} \) is the Hardy-Littlewood maximal function of \( f_m \). Then using Jensen’s inequality we have:

\[
\exp(\gamma |f_m(x)|) = \exp\left(\gamma \int_{Q_n} f_m(y) dy \left( \frac{y}{|Q_n|} \right) \right)
\]
\[
\leq \frac{1}{|Q_n|} \int_{Q_n} \exp(\gamma |f_m(y)|) dy
\]
\[
\leq M \left( e^{\|f_m\|_{\gamma}} \right)(x).
\]

Using the Hardy-Littlewood maximal estimate, we have:

\[
\left| \left\{ x \in [0,1) : \sup_{y \in Q_n} |f_m(x)| > \lambda \right\} \right|
\]
\[
= \left| \left\{ x \in [0,1) : \sup_{y \in Q_n} e^{\gamma |f_m(y)|} > e^{\frac{\lambda}{\gamma}} \right\} \right|
\]
\[
\leq \left| \left\{ x \in [0,1) : M \left( e^{\|f_m\|_{\gamma}} \right)(x) > e^{\frac{\lambda}{\gamma}} \right\} \right|
\]
\[
\leq \frac{3}{e^\gamma} \int_{[0,1]} \exp(\gamma |f_m(y)|) dy
\]
\[
\leq \frac{3}{e^\gamma} \exp\left( \frac{\gamma^2}{2} \|S_f \|_{\gamma}^2 \right) \int_{[0,1]} \exp\left( \gamma |f_m(y)| - \frac{\gamma^2}{2} S_f(y) \right) dy.
\]

Using Lemma 3 we have:

\[
\int_{[0,1]} \exp\left( \gamma |f_m(y)| - \frac{\gamma^2}{2} S_f(y) \right) dy
\]
\[
= \int_{\gamma < f_m(y)} \exp\left( \gamma f_m(y) - \frac{\gamma^2}{2} S_f(y) \right) dy
\]
\[
+ \int_{\gamma > f_m(y)} \exp\left( -\gamma f_m(y) - \frac{\gamma^2}{2} S_f(y) \right) dy
\]
\[
\leq \int_{[0,1]} \exp\left( \gamma f_m(y) - \frac{\gamma^2}{2} S_f(y) \right) dy
\]
\[
+ \int_{[0,1]} \exp\left( -\gamma f_m(y) - \frac{\gamma^2}{2} S_f(y) \right) dy
\]
\[
\leq 2.
\]

So:

\[
\left| \left\{ x \in [0,1) : \sup_{y \in Q_n} |f_m(x)| > \lambda \right\} \right| \leq \frac{6}{e^\gamma} \exp\left( \frac{\gamma^2}{2} \|S_f \|_{\gamma}^2 \right).
\]

Choose \( \gamma = \frac{\lambda}{\|S_f \|_{\gamma}} \). With this \( \gamma \), the above inequality becomes:

\[
\left| \left\{ x \in [0,1) : \sup_{y \in Q_n} |f_m(x)| > \lambda \right\} \right| \leq 6 \exp\left( \frac{-\lambda^2}{2\|S_f \|_{\gamma}^2} \right).
\]

Note that for the dyadic martingale \( \{f_m\} \):

\[
S_m^2 f(x) = \sum_{k=1}^{m} d_k^2(x) \rightarrow S^2 f(x) = \sum_{k=1}^{\infty} d_k^2(x).
\]

Consequently:

\[
\frac{-1}{2\|S_f \|_{\gamma}^2} \leq \frac{-1}{2\|S' \|_{\gamma}^2}.
\]

Recall the continuity property of Lebesgue measure: If \( \{E_n\} \) is a sequence of sets with \( E_n \subset E_{n+1} \) for all \( n \) and \( E = \bigcup_{n=1}^{\infty} E_n \), then \( |E| = \lim_{n \to \infty} |E_n| \). Using this we get:

\[
\left| \left\{ x \in [0,1) : \sup_{y \in Q_n} |f_m(x)| > \lambda \right\} \right| \leq 6 \exp\left( \frac{-\lambda^2}{2\|S' \|_{\gamma}^2} \right).
\]

This completes the proof of the first inequality.

Proof of Inequality 2

Fix \( n \). Define a sequence \( \{g_m\} \) as follows:

\[
g_m(x) = \begin{cases} 0, & \text{if } m \leq n; \\ f_m(x) - f_{m+1}(x), & \text{if } m > n. \end{cases}
\]  \( (1) \)

We first show that \( \{g_m\} \) is a dyadic martingale. Clearly for every \( m \), \( g_m \) is measurable with respect to the sigma algebra \( \mathcal{F}_m \): Let \( m > n \). Then using the fact that \( f_m \) is constant on the cube \( Q_m \) we have:

\[
E(g_{m+1} | \mathcal{F}_m)(x) = \frac{1}{|Q_m|} \int_{Q_m} \left[ f_{m+1}(x) - f_m(x) \right] dx
\]
\[
= \frac{1}{|Q_m|} \int_{Q_m} f_{m+1}(x) dx - \frac{1}{|Q_m|} \int_{Q_m} f_m(x) dx
\]
\[
= \frac{1}{|Q_m|} \int_{Q_m} f_{m+1}(x) dx - f_m(x)
\]
\[
= f_{m+1}(x) - f_m(x)
\]
\[
= g_m(x).
\]

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Thus we have $E(g_{m+1} \mid \mathcal{F}_n) = g_m$. This shows that $\{g_m\}$ is a martingale. Then applying the inequality 1 for this martingale, we get:

$$\left| \left\{ x \in [0,1) : \sup_{m \leq n} g_m(x) > \lambda \right\} \right| \leq 6\exp\left( -\frac{\lambda^2}{2\|S'_n\|} \right).$$

But, $g_m(x) = 0$ for $m \leq n$. Hence:

$$\left| \left\{ x \in [0,1) : \sup_{m \leq n} g_m(x) > \lambda \right\} \right| \leq 6\exp\left( -\frac{\lambda^2}{2\|S'_n\|} \right).$$

Again:

$$S^2 g(x) = \sum_{k=0}^{\infty} d_k(x) = \sum_{k=0}^{\infty} g_k(x) - g_k(x).$$

By the triangle inequality we have:

$$\sup_{m \leq n} \left| f_m(x) - f_n(x) \right| \leq \sup_{m \leq n} \left| f_m(x) - f_n(x) + f_n(x) - f_n(x) \right| = \left| f_m(x) - f_n(x) \right| + \sup_{m \leq n} \left| f_n(x) - f_m(x) \right|.$$
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Ethics

This is a mathematical research article. No ethical issues will arise after the publication of the article.

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