Bounds on the Entropy of Patterns of I.I.D. Sequences

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Abstract—Bounds on the entropy of patterns of sequences generated by independently identically distributed (i.i.d.) sources are derived. A pattern is a sequence of indices that contains all consecutive integer indices in increasing order of first occurrence. If the alphabet of a source that generated a sequence is unknown, the inevitable cost of coding the unknown alphabet symbols can be exploited to create the pattern of the sequence. This pattern can in turn be compressed by itself. The bounds derived here are functions of the i.i.d. source entropy, alphabet size, and letter probabilities. It is shown that for large alphabets, the pattern entropy must decrease from the i.i.d. one. The decrease is in many cases more significant than the universal coding redundancy bounds derived in prior works. The pattern entropy is confined between two bounds that depend on the arrangement of the letter probabilities in the probability space. For very large alphabets whose size may be greater than the coded pattern length, all low probability letters are packed into one symbol. The pattern entropy is upper and lower bounded in terms of the i.i.d. entropy of the new packed alphabet. Correction terms, which are usually negligible, are provided for both upper and lower bounds.

I. INTRODUCTION

Several recent works (see, e.g., [1], [3]-[4], [6], [8], [9]) have considered universal compression for patterns of independently identically distributed (i.i.d.) sequences. The pattern of a sequence is a sequence of pointers that point to the actual alphabet letters, where the alphabet letters are assigned indices in order of first occurrence. For example, the pattern of the sequence “lossless” is “12331433”. A pattern sequence thus contains all positive integers from 1 up to a maximum value in increasing order of first occurrence, and is also independent of the alphabet of the actual data. Universal compression of patterns is interesting in applications that attempt to compress sequences generated by an initially unknown alphabet, such as a document in an unknown language. Utilization of the necessary coding of the unknown symbols can take place by ordering the symbols in their order of occurrence in the sequence, and then separately compressing the alphabet independent pattern of the sequence.

To the best of our knowledge, universal compression of patterns was first considered in [1], where it was proposed to compress sequences from a large known alphabet in which not all symbols are expected to occur by separating the representation of the occurring alphabet symbols from the pattern and compressing each separately. The paper considered compression of individual sequences. Later, patterns were rediscovered (and named) in a series of papers [3], [4] (and references therein) that considered universal compression for unknown alphabets and thoroughly studied the redundancy in universal coding of individual pattern sequences. These papers demonstrated that the individual sequence redundancy of patterns must decrease in universal compression compared to the redundancy obtained for simple universal compression of i.i.d. sequences. Furthermore, unlike the i.i.d. case, it was shown that the redundancy in universal pattern compression vanishes even if the alphabet is infinite. (This is, of course, related to the fact that we loose some information by coding the pattern instead of the actual sequence.) The universal average case was then studied in [6], [8], [9], where redundancy bounds for average case universal compression of patterns were derived.

The universal description length of patterns, however, consists of the pattern entropy and the redundancy of universally coding the pattern. While most of the emphasis in prior work was on the latter, it is clear that a pattern is a data processing over the actual sequence, and thus its entropy (the first term) must decrease. Furthermore, in [9] (see also [6]), we derived sequential codes for compressing patterns and bounded their description length. It was shown that for sufficiently large alphabets this description length was significantly smaller than the i.i.d. source entropy. This points out to the fact that not only is there an entropy decrease in patterns, but for large alphabets, this decrease is much more significant than the increase in description length due to the universal redundancy. Hence, to have better understanding even of universal compression of patterns, it is essential also to study the behavior of the pattern entropy. Pattern entropy is also important in learning applications. Consider all the new faces that a newborn sees. The newborn can identify these faces with the first time each was seen. There is no difference if it sees nurse A or nurse B (and never sees the other), as long as it is a nurse. The entropy of patterns can thus model the uncertainty of such learning processes. The exponent of the entropy gives an approximate count of the typical patterns one is likely to observe for the given source distribution. If the uncertainty goes to 0, we are likely to observe only one pattern.

We first considered pattern entropy in [7], where we bounded the range of values within which the entropy of a pattern can be, depending on the specific distribution, as a function of the i.i.d. entropy. We showed that for larger alphabets, the pattern entropy must decrease with respect to (w.r.t.) the i.i.d. one. However, the results were limited to distributions that contain only letters with sufficiently large letter probability. An upper bound that extends the results for
unbounded distributions was derived in [10]. Subsequently to our initial paper [7], pattern entropy was independently studied with different approaches and from a different view of the problem in [2] and [5], where the focus has been on limiting results for the entropy rate of patterns.

In this paper, we continue and generalize the results in [7] and [10]. We derive general upper and lower bounds for the entropy of patterns generated by large and very large alphabets. The bounds are presented as functions of a related i.i.d. entropy, the alphabet size, and the alphabet letter probabilities. The related i.i.d. entropy is that of the i.i.d. source if no letters with very low probabilities exist. Otherwise, all the probabilities smaller than a threshold are packed into one symbol, and the i.i.d. entropy is that of the new alphabet. Since the detailed proofs of most of the bounds require lengthy rigorous analysis, we only include road maps of the proofs in this paper. The complete proofs are presented in [11].

The technique used to derive the bounds in this paper relies on partitioning the probability space into a grid of points. Between each two points, we obtain a bin. For a typical i.i.d. sequence of the source, each permutation of the sequence letters that only permutates among letters in the same bins, has almost the same probability as the typical sequence, and results in the same pattern. Such permutations exchange all occurrences of one letter by all occurrences of another. The probability of the pattern increases from that of the i.i.d. sequence by the number of such permutations. This, in turn, yields a decrease in the pattern entropy. This idea is used directly to derive some of the bounds, and is extended to include low probabilities to derive the more general bounds. To derive a general upper bound, we propose a low-complexity sequential (non-universal) code for compressing patterns, which achieves the bound. The algorithm is, again, based on the idea of bins. The use of bins is not easy because the grids that determine the bins need to be wisely designed to efficiently utilize the probability space. In particular, the grid points are taken in increasing spacing. The reason is that for large probabilities, the decrease in probability assigned to a typical sequence is slower as we shift away from the true letter probability.

The outline of the paper is as follows. In Section II, we define the notation. Section III reviews initial simple, easy to derive, bounds on the entropy, and motivates the remainder of the paper. Then, in Section IV, we derive the upper and lower bounds for pattern entropy of i.i.d. sources with sufficiently large probabilities, and show the range of values that the pattern entropy can take in this case, depending on the actual source distribution. Finally, Section V contains the derivations of more general upper and lower bounds, that do not require a condition on the letter probabilities.

II. NOTATION AND DEFINITIONS

Let \( x^n \triangleq (x_1, x_2, \ldots, x_n) \) be a sequence of \( n \) symbols over an alphabet \( \Sigma \) of size \( k \). The parameter \( \theta \triangleq (\theta_1, \theta_2, \ldots, \theta_k) \) contains the probabilities of the alphabet letters. Since the order of these probabilities does not affect the pattern, we assume, without loss of generality, that \( \theta_1 \leq \theta_2 \leq \cdots \leq \theta_k \), and that \( \Sigma = \{i, 1 \leq i \leq k\} \). In general, boldface letters will denote vectors, whose components will be denoted by their indices. Capital letters will denote random variables.

The pattern of \( x^n \) will be denoted by \( \psi^n \triangleq \Psi(x^n) \). Different sequences have the same pattern. For example, for the sequences \( x^n = "\text{lossless}" \), \( x^n = "\text{sold sell}" \), and \( x^n = "\text{76887288}" \), the pattern is \( \Psi(x^n) = "\text{12331433}" \). Therefore, for given \( \Sigma \) and \( \theta \), the probability of a pattern \( \psi^n \) induced by an i.i.d. underlying probability is given by

\[
P_\theta(\psi^n) = \sum_{y^n: \Psi(y^n) = \psi^n} P_\theta(y^n). \tag{1}
\]

The probability of \( \Psi(x^n) \) can be expressed as in \( \Psi \) by summing over all sequences that have the same pattern with a fixed parameter vector. However, we can also express it by fixing the actual sequence and summing over all permutations of occurring symbols of the parameter vector

\[
P_\theta[\Psi(x^n)] = \sum_{\sigma} P_\theta(\sigma)(x^n), \tag{2}
\]

where the summation is over all permutation vectors \( \sigma \) that differ among each other in the index of the probability parameter assigned to at least one occurring letter, and \( \theta(\sigma_i) \) denotes the \( i \)th component of the permuted vector \( \theta \), permuted according to \( \sigma \). For example, if \( \theta = (0.7, 0.1, 0.2) \) and \( \sigma = (3, 1, 2) \), then \( \theta(\sigma) = (0.2, 0.7, 0.1) \) and \( \theta(\sigma_2) = \theta = 0.7 \).

The entropy rate of an i.i.d. source will be denoted by \( H_\theta(X) \). The sequence entropy for an i.i.d. source is \( H_\theta(x^n) = nH_\theta(X) \). The pattern sequence entropy of order \( n \) of a source \( \theta \) is defined as

\[
H_\theta(\Psi^n) \triangleq - \sum_{\psi^n} P_\theta(\psi^n) \log P_\theta(\psi^n). \tag{3}
\]

As described in Section II, we will grid the probability space in order to derive the bounds. Letters whose probabilities lie in the same bin between two adjacent grid points will be grouped together. We will use two different grids, as defined below, to derive the bounds. For an arbitrarily small \( \varepsilon > 0 \), let \( \eta \triangleq (\eta_0, \eta_1, \eta_2, \ldots, \eta_b, \ldots, \eta_B) \) be a grid of \( B + 1 \) points, where \( \eta_0 = 0, \eta_1 = 1/n^{1+\varepsilon} \), and let

\[
\tau_b \triangleq \sum_{j=1}^b \frac{2(j - 1)}{n^{1+2\varepsilon}} = \frac{b^2}{n^{1+2\varepsilon}}. \tag{4}
\]

Then,

\[
\eta_b = \tau_{b+\lfloor n^{1/2}\rfloor}; \quad \forall b \geq 2, \tag{5}
\]

i.e., \( \eta_2 = 1/n^{1-\varepsilon} \), and so on. Clearly, there are \( B \) nonzero grid points, where \( B \) is the rounded down integer of \( \sqrt{n^{1+2\varepsilon} - n^{3\varepsilon}/2 + 2} \).

We will use \( k_0 \) to denote the number of letters \( \theta_i \in (\eta_b, \eta_{b+1}) \). In particular, \( k_0 \) will denote the number of letters in \( \Sigma \) with probability not greater than \( 1/n^{1+\varepsilon} \), \( k_1 \) the number of letters with probabilities in \( (1/n^{1+\varepsilon}, 1/n^{1-\varepsilon}] \), and \( k_{01} \) their sum. Let \( \varphi_b \) be the total probability of letters in bin \( b \) of grid \( \eta \). Of particular importance will be \( \varphi_0, \varphi_1, \)
defined w.r.t. bins 0, 1, respectively, and \( \varphi_0 + \varphi_1 \). We use \( L_0 \) and \( L_b \) for the mean number of total letters, and \( L_b = \sum_{\theta_i \in (\eta_b, \eta_{b+1}]} [1 - (1 - \theta_i)^n] \). It is easy to see that
\[
k_b - \sum_{\theta_i \in (\eta_b, \eta_{b+1}]} e^{-n\theta_i} \leq L_b \leq k_b - \sum_{\theta_i \in (\eta_b, \eta_{b+1}]} e^{-n(\theta_i - \theta_i^*)},
\]
where in the upper bound summation only \( \theta_i \leq 3/5 \) are included. In particular, for bin 0,
\[
n\varphi_0 - \left( \frac{n}{2} \right) k_0 \geq \varphi_0 - \left( \frac{n}{2} \right) k_0 \theta_0^2 \leq L_0 \leq n\varphi_0 - \left( \frac{n}{2} \right) k_0 \sum_{i=1}^k \theta_i^2 + \left( \frac{n}{3} \right) k_0 \sum_{i=1}^k \theta_i^3.
\]
(7)

The points \( b \geq 1 \) in grid \( \xi = (\xi_0, \xi_1, \ldots, \xi_B) \) are defined as in (2), but where \( -\varepsilon \) replaced \( 2\varepsilon \), and also \( \xi_0 = 0 \). Here, we will use \( \kappa_b, b \geq 1 \), to denote the number of letters whose probabilities are in the three adjacent bins surrounding \( b \), i.e., \( \theta_i \in (\xi_{b-1}, \xi_{b+1}] \), with the exception of \( \kappa_1 \) which will only count the letters with probabilities in \( (\xi_1, \xi_2] \).

Using the definitions above, we can now define two i.i.d. entropy expressions, where some of the low probability symbols are packed into one symbol,
\[
H_\theta^{(0)} (X) \triangleq -\varphi_0 \log \varphi_0 - \sum_{i=k_0+1}^k \theta_i \log \theta_i,
\]
(8)
\[
H_\theta^{(0,1)} (X) \triangleq -\sum_{b=0}^k \varphi_b \log \varphi_b - \sum_{i=k_0+1}^k \theta_i \log \theta_i.
\]
(9)

### III. BACKGROUND AND SIMPLE BOUNDS

It is clear that the pattern entropy satisfies the following bounds:

**Theorem 1:** If \( k \leq n \),
\[
nH_\theta (X) - \log (k! \leq H_\theta (\Psi^n) \leq nH_\theta (X).
\]
(10)

Otherwise,
\[
nH_\theta (X) - \log \frac{k!}{(k-n)!} \leq H_\theta (\Psi^n) \leq nH_\theta (X).
\]
(11)

The upper bound is trivial, and the lower bounds are proved in [11]. For \( k = o(n) \), the simple bound in (10) already points the fact that if the i.i.d. entropy rate of the source is not vanishing, the entropy rate of patterns is equal to the i.i.d. one. However, it is clear that for many sources the bounds above are not tight.

In [9], we derived a universal sequential algorithm for coding patterns. The bound on its description length provides a bound on the pattern entropy. In particular, if \( k > e^{19/18 \cdot n^{1/3}} \), where \( k \) is the number of alphabet letters that occur in \( x^n \) with probability at least \( (1 - \varepsilon) \), it was shown that the pattern entropy must decrease from the i.i.d. one, and
\[
H_\theta (\Psi^n) \leq nH_\theta (X) - (1 - \varepsilon) \frac{3}{2} k \log \frac{k}{e^{19/18 n^{1/3}}}. 
\]
(12)

For sources with very high entropy, for example, \( \theta_i = n^{-\alpha}, \forall i \), for some constant \( \alpha \geq 1 \), the bound increases with \( n \) and becomes loose. The derivation in [9] can thus be used to replace the first term in the bound by \( n \log n \). However, this still yields a very loose bound on the entropy.

### IV. BOUNDS FOR SMALL AND LARGE ALPHABETS

We now consider sources in which \( \theta_1 > 1/n^{1-\varepsilon} \), i.e., \( k_0 = 0 \). We present an upper bound and a lower bound for this case, and discuss the range of values the entropy can take, where for sufficiently large \( k \), it must decrease from the i.i.d. one.

**A. An Upper Bound**

The following theorem upper bounds the pattern entropy.

**Theorem 2:** Let \( \theta_i > 1/n^{1-\varepsilon}, \forall i, 1 \leq i \leq k \). Then,
\[
H_\theta (\Psi^n) \leq nH_\theta (X) - (1 - \varepsilon) \sum_{b=2}^B \log (k_b!).
\]
(13)

To prove Theorem 2, we lower bound the probability of patterns generated only from typical sequences \( x^n \) by the sum of probabilities of all typical sequences that have this pattern. Using (2) and this idea, \( P_0 [\Psi (x^n)] \) is lower bounded by the partial sum of permutations \( \sigma \) of \( \theta \), that contains only permutations for which for every \( i \) and every \( b \), \( \theta_i \in (\eta_b, \eta_{b+1}] \Rightarrow \theta (\sigma_i) \in (\eta_b, \eta_{b+1}] \). For all such permutations and a typical sequence \( x^n \), the probability assigned to \( x^n \) decreases at most negligibly w.r.t. the actual probability of \( x^n \). Hence, for a typical \( x^n \),
\[
\log P_0 [\Psi (x^n)] \geq \log P_0 (x^n) + \log M_\theta - o(k),
\]
where \( M_\theta \) is the number of such permutations \( \sigma \). Computing \( M_\theta \) and accounting for the probability of non-typical sequences yields the bound of (13).

**B. A Lower Bound**

The next theorem shows a bound of similar nature to the bound of Theorem 2.

**Theorem 3:** Let \( \theta_i > 1/n^{1-\varepsilon}, \forall i, 1 \leq i \leq k \). Then,
\[
H_\theta (\Psi^n) \geq nH_\theta (X) - \sum_{b=1}^B \log (k_b!) - o(1).
\]
(15)

To prove Theorem 3, we first define a typical pattern \( \psi^n \) as one that is the pattern of at least one typical \( x^n \). The number of typical sequences \( x^n \) that have a given typical pattern is then upper bounded by the product of factorials that leads to the second term of the bound. It is then shown that the contribution of non-typical sequences to the probability of any typical pattern decays exponentially in \( n^{\varepsilon \sigma} \), where \( \alpha \) is some constant. It is necessary to show that even if a typical pattern is the pattern of very few typical sequences, the many non-typical sequences of this pattern still contribute negligibly to its probability. To show that, each set of non-typical sequences that have pattern \( \psi^n \) is shown to result from a permutation of a typical sequence, where the probability of such a non-typical permutation multiplied by a bound on the number of such permutations is still negligible w.r.t. the probability of the original typical sequence. Finally, a straightforward set of equations that breaks the pattern entropy computation into typical and non-typical sequences, yields the bound of (15).
C. Entropy Range

We now consider the overall range of values the pattern entropy can take, regardless of how the letter probabilities are lined up in the probability space. It is clear that the lower bound in (10) is tight for a uniform distribution for \( \theta_i > 1/n^{1-\varepsilon} \). The upper bound, however, is restricted by the minimum number of permutations that yield a typical sequence after permuting another typical sequence. For the simple bound in (10), only the identity permutation is counted. However, if the number of alphabet symbols is sufficiently large, there must be more than one such permutation, because more than one letter probability must fall within a single bin of \( \eta \). Letters with probabilities in the same bin in a typical \( x^n \) can be permuted among themselves to another sequence \( y^n \) that is typical, gives the same pattern, and has almost equal probability to \( x^n \).

Not to violate the condition \( \sum \theta_i = 1 \), most of the letter probabilities must be distributed in essentially \( O(n^{(1+\varepsilon)/3}) \) lower bins of \( \eta \). For sufficiently large alphabets, using the smallest possible number of such permutations, yields

**Theorem 4:** Let \( \theta_i > 1/n^{1-\varepsilon}, \forall i, 1 \leq i \leq k \), and let \( k \geq n^{(1+\varepsilon)/3} \). Then,

\[
nH \theta (X) - \log (k!) \leq H \theta (\Psi^n) \leq nH \theta (X) - (1-\varepsilon)3\frac{k \log k}{e^2/3^{1/3}}.
\]

Theorem 4 gives a range within which the pattern entropy must be, depending on the actual letter probabilities. Figure 1 shows the region of decrease in the pattern entropy w.r.t. the i.i.d. one. For large alphabets, the entropy must decrease essentially by at least \( 1.5 \log (k/n^{1/3}) \) bits per alphabet symbol.

V. BOUNDS FOR VERY LARGE ALPHABETS

We now consider a more general case, where there is no lower bound on the letter probabilities.

A. An Upper Bound

A general upper bound on \( H_\theta (\Psi^n) \) is derived through a sequential probability assignment code. A new symbol is assigned a joint probability of its index and its bin in the grid \( \eta \). We thus code the joint sequence \( (\psi^n, \beta^n) \), where \( \beta^n \) is the sequence of bin indices corresponding to \( x^n \). The average description length of this code upper bounds the joint entropy \( H_\theta (\Psi^n, B^n) \), which in turn upper bounds \( H_\theta (\Psi^n) \).

The probability that is assigned to the joint pattern and bin sequence is given by \( Q (\psi^n, \beta^n) = \prod_{j=1}^n Q (\psi_j, \beta_j | (\psi^{j-1}, \beta^{j-1}) \right) \). If \( \psi_j \) is an index that already occurred in the pattern \( \psi^{j-1} \), then

\[
Q (\psi_j, \beta_j | (\psi^{j-1}, \beta^{j-1})) = \rho \beta_j,
\]

where \( \rho_b \triangleq \varphi_b/k_b \) for \( b \geq 2 \), and \( \rho_0 \) and \( \rho_1 \) are values assigned to letters in the first two bins, that will be optimized later. Once an index occurred, it only occurs jointly with the same bin number that occurred with its first occurrence. If \( \psi_j \) is a new index, and its bin is \( \beta_j \), the pair is assigned probability

\[
Q (\psi_j, \beta_j | (\psi^{j-1}, \beta^{j-1})) = \varphi_b - c (\psi^{j-1}, \beta^{j-1}) \cdot \rho \beta_j,
\]

where \( c (\psi^{j-1}, \beta^{j-1}) \) is the number of distinct indices that jointly occurred with bin index \( \beta_j \) in \( (\psi^{j-1}, \beta^{j-1}) \) (e.g., if \( \psi^{j-1} = 1232345 \) and \( \beta^{j-1} = 1222242 \) then \( c (\psi^{j-1}, \beta^{j-1}) \) is 3 for \( \beta_j = 2 \), 1 for \( \beta_j = 1 \) and \( \beta_j = 4 \), and is 0, otherwise).

This probability assignment groups the probability of all the symbols in the same bin into one symbol. Then, each occurrence of a new symbol in bin \( b \), it codes a new index with the remaining group probability, extracting one count of the mean bin probability from the remaining probability in the bin. Each re-occurrence of an index assigns the index and its attached bin the mean bin probability of the respective bin. For bins \( b = 0, 1 \), the mean is replaced by \( \rho_0 \) and \( \rho_1 \), respectively.

Upper bounding the average description length of this code, optimizing \( \rho_0 \) and \( \rho_1 \) to minimize the bound, yields the following upper bound on the pattern entropy.

**Theorem 5:** The pattern entropy is upper bounded by

\[
H_\theta (\Psi^n) \leq nH_\theta^{(0,1)} (X) - \sum_{b=2}^B (1-\varepsilon) \log (k_b!)
+ (n\varphi_1 - L_1) \log \left( \min \{ k_1, n \} \right) + n\varphi_1 h_2 \left( \frac{L_1}{n\varphi_1} \right)
+ \left( \frac{n^2 \sum_{i=1}^k \theta_i^2}{2} \right) \log \left( \frac{2e \cdot \varphi_0 \cdot \min \{ k_0, n \}}{n \sum_{i=1}^k \theta_i^2} \right),
\]

where \( h_2 (\alpha) \triangleq -\alpha \log \alpha - (1-\alpha) \log (1-\alpha) \).

The bound consists of: the packed i.i.d. entropy with bins 0 and 1 as one symbol each (the first term), the pattern gain in first occurrences of any letter within the remaining bins (the second term), the loss in packing bin \( b = 1 \) (the next two terms), and the loss in packing bin \( b = 0 \) (the last term). The greatest contribution of the third and the fourth term can be shown to be \( (1-\varepsilon)n\varphi_1 \log n \), and that of the last term \( 0.5\varphi_0 n^{1-\varepsilon} \log (2en^{1+\varepsilon}) \), which is clearly negligible if \( H_\theta^{(0,1)} (X) \) is non-vanishing.
B. A Lower Bound

To lower bound $H_0(\Psi^n)$, the contributions of large and small probabilities are separated. The former, of probabilities greater than $1/n^{1-\varepsilon}$, is bounded using derivation as in Theorem 4. The latter is bounded by a straightforward derivation. To separate the two, we define a random sequence $Z^n$, such that $Z_j = 0$ if $\theta_{x_j} \leq 1/n^{1-\varepsilon}$ and 1 otherwise. Using $Z^n$, $H_0(\Psi^n)$ can be expressed as

$$H_0(\Psi^n) = H_0(\Psi^n | Z^n) + H_0(Z^n) - H_0(Z^n | \Psi^n). \quad (20)$$

The first term of (20) can now be bounded by splitting a particular value $z^n$ of $Z^n$ into the elements for which $z_j = 1$ and those for which $z_j = 0$, and bounding $H_0(\Psi^n | z^n)$ separately for each of these sets. We use the relation $H_0(\Psi^n | z^n) = \sum_{j=1}^{n} H_0(\psi_j | \psi^{j-1}, z^n)$.

The third term of (20) complicates the analysis if there is no clear separation between small and large probabilities, i.e., there exists $\varepsilon$ values for which there are $k_2^a > 0$ letters with probabilities in $\{1/(2n^{1-\varepsilon}), 1/n^{1-\varepsilon}\}$ and $k_2^b > 0$ letters with probabilities in $\{1/n^{1-\varepsilon}, 3/(2n^{1-\varepsilon})\}$. A permutation between letters in the first bin and letters in the second may still result in a typical sequence. Hence, the separation must be a correction term. Applying all the above considerations yields the following lower bound:

Theorem 6: The pattern entropy is lower bounded by

$$H_0(\Psi^n) \geq n H_0^{(1)}(X) - \sum_{b=1}^{B} \log(B) + \sum_{i=1}^{k_0-1} \left[ n \theta_i - 1 + e^{-n \left( \theta_i + \frac{a^2}{n} \right)} \right] \log \frac{\varphi_{01}}{\theta_i} + (n \theta_{k_0}) \log \frac{\varphi_{01}}{\theta_{k_0}} + (\log e) \sum_{i=1}^{(L_0-1)} (L_0 - i) \frac{\theta_i}{\varphi_{01}} - \log \left( k_2^a + k_2^b \right) - o(1). \quad (21)$$

The first term in (21) is the i.i.d. entropy in which all letters with probability not greater than $1/n^{1-\varepsilon}$ are packed into one symbol. The second term is the decrease in entropy due to first occurrences of large probability letters. The next three terms are due to the contribution of low probability letters beyond that of the super-symbol that merges them. The first two of these represent the penalty in packing in repetition of these letters, where the third one is the penalty in first occurrence of such a letter. We note that the first two of these three terms can be separated into contributions of the first $k_0$ letters and the following $k_1$ letters, to obtain expressions that resemble the bound in (12). The sixth term of (21) is the correction term from separating small and large probability letters. Finally, the last term of $o(1)$ absorbs all the lower order terms of the bound. As in the upper bound in (13), all the terms beyond the first two and the first element of the third term, can be shown to contribute at most $O((n^{\varepsilon} \log n)$ for letters that result from bin 1 of $\eta$, and $o(n)$ for letters that result from bin 0 of $\eta$.

There are several other forms that the bound in (21) can be brought to. In particular, the second term, representing the decrease due to first occurrences of large probability letters may not be tight if the distribution is close to uniform, but symbols appear in very few separate adjacent bins formed by $\xi$. If this is the case, it may be beneficial to derive a bound on the large probabilities using similar methods to the bound derived on the low probabilities. In such a bound, the second term of (21) will be replaced by two terms that take the form of the third and fifth terms of (21), where the $n \theta_i$ leading element of the third term is omitted.

VI. SUMMARY AND CONCLUSIONS

We studied the entropy of patterns of i.i.d. sequences. We provided upper and lower bounds on this entropy as functions of a related i.i.d. source entropy, the alphabet size, the letter probabilities, and their arrangement in the probability space. The bounds provided a range of values the pattern entropy can take, and showed that in many cases it must decrease substantially from the original i.i.d. sequence entropy. It was shown that low probability symbols contribute mostly as a single super-symbol to the pattern entropy, where in particular, very low probability symbols contribute negligibly over the contribution of this super-symbol.

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