Quasinormal modes of black holes. The improved semianalytic approach.

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We have extended the semianalytic technique of Iyer and Will for computing the complex quasinormal frequencies of black holes, $\omega$, by constructing the Padé approximants of the (formal) series for $\omega^2$. It is shown that for the (so far best documented) quasinormal frequencies of the Schwarzschild and Reissner-Nordström black holes the Padé transforms $P_{6}^{6}$ and $P_{7}^{6}$ are, within the domain of applicability, always in excellent agreement with the numerical results. We argue that the method may serve as the black box with the “potential” $Q(x)$ as an input and the accurate quasinormal modes as the output. The generalizations and modifications of the method are briefly discussed as well as the preliminary results for other classes of the black holes.

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I. INTRODUCTION

The physical black holes are not isolated systems, they interact in a variety of ways with their environment and changing their surrounding they change themselves. Especially interesting in this regard are the perturbations that can satisfactorily be described within the linear approximation. On general grounds one expects that the late-time behavior is dominated by the oscillations that are characteristic to a given black hole and independent of the initial cause of the perturbation. That means that the gravitational wave emitted by the perturbed black hole will carry the imprints of its characteristics on the unique set of complex numbers, $\omega$, simultaneously describing the rates of damping and the frequency of the oscillations. Indeed, numerical analysis of the evolution of the black holes formed in a gravitational collapse or in the collision of black holes indicates that each of them approaches such a ringdown phase. These quasinormal oscillations are expected to be crucial both in the black holes detection and, when discovered, in studying their properties. It is natural that the quasinormal oscillations of the black holes have been area of intense study for the last 40 years.
The quasinormal modes considered in this paper are the solutions of the second-order differential equation

$$\frac{d^2}{dx^2} \psi(x) + Q(x) \psi(x) = 0,$$  

(1)

where $-Q(x)$ is a potential function, which is assumed to be constant as $|x| \to \infty$ (the limits may be different) and to possess the maximum at some finite $x_0$, subjected to the particular set of the boundary conditions. The function $\psi(x)$ is the radial part of the free oscillations (with the assumed time-dependence of the form $e^{-i\omega t}$) which is purely “outgoing” as $|x| \to \infty$. Here we follow convention proposed in Ref. [1] and understand the term “outgoing” as “moving away from the potential barrier”. For a perturbation of a given spin weight, $s$, the quasinormal modes are labeled by the multipole number, $l$, and the overtone number $n$.

The quasinormal modes have been studied both numerically and analytically for the various perturbations of the black hole backgrounds. Especially interesting are the analytic or semianalytic methods allowing quick and accurate calculations for a wide range of black holes. Currently there are a few popular approaches to the quasinormal frequencies problem, each having its own merits. A high reputation of the continued fraction method is due to its great accuracy and possibility to calculate high overtones [2, 3]. In its original form it has been employed in the three term recurrence relation, however, the more complicated cases can also be addressed by reducing them to three term recurrence by Gauss elimination [4]. The Hill-determinant method proposed in Ref. [5] is in a sense complementary to the Leaver approach and allows for calculations of the high overtones by searching of the stable zeros of the high-order polynomials. Nollert [6, 7] studied the quasinormal modes via Laplace transform and analyzed the problem of overtones. The quasinormal modes (of the Schwarzschild black hole) have been calculated by Zaslavskii by reducing the problem to the well-known quantum anharmonic oscillator [8]. The competitive approaches include various incarnations of the phase integral method [9–11] and the modifications of the WKB approximation [1, 12]. The Iyer-Will method [9], which belongs to the latter class, and its generalization to the sixth order [13] gained great and well-deserved popularity. (See for example Refs. [14–19] and the references cited therein).

Typically, depending on the character of the problem, one is torn between the need for a high accuracy (also for overtones $n \gg l$) and the generality of the approach, allowing analysis of various potentials, even at the expense of some inherent limitations. Practically, these limitations may not be so serious as for the astrophysical black holes the least damped modes are most significant and simultaneously easiest to calculate. In our opinion the WKB method is a best choice due to
its generality and flexibility and may serve, with necessary modifications, as the black box with the “potential” $Q(x)$ as an input and the quasinormal modes as the output. In this paper we shall propose a modification of the Iyer-Will method \cite{12,13}. The modification is twofold: First we generalize their approach by extending calculations beyond sixth-order WKB and subsequently employ the powerful technique of the Padé transforms. Restricting to the so far best documented quasinormal modes of the four dimensional Schwarzschild and Reissner-Nordstr"om black holes we show that the approximation works very well. Typically, the deviations of the real and imaginary part of the complex frequencies from the accurate numerical results are smaller than those obtained within the framework of competing approaches. Moreover, for the low-lying modes the accuracy is comparable with or even better than the phase integral method in the optimal order.

The paper is organized as follows. In Sec. II we shall briefly introduce the method. In Sec. III we will discuss the results obtained for the various perturbations in the Schwarzschild and Reissner-Nordstr"om black holes and make a detailed comparison with the accurate numerical calculations. Finally, in Sec. IV we shall briefly discuss the possible extensions and modifications of the method as well as our preliminary results for other classes of black holes.

II. THE METHOD

As is well known the modification of the WKB approach proposed by Iyer and Will \cite{12,20,21} allows for configurations with closely lying classical turning points. The idea is to match simultaneously exterior WKB solutions across the two turning points. The differential equation (1) in the interior region is first simplified by expanding the potential into the Taylor series up to terms of order six and subsequently solved (approximately) in terms of the parabolic cylinder functions. The asymptotic approximation to the interior solution is used to match the third-order WKB solutions. This approach has been used to calculate the approximate quasinormal frequencies of the Schwarzschild \cite{22}, Reissner-Nordstr"om \cite{23} and Kerr \cite{24} black holes. A more profound comparison shows that the approximate results deviate (at worst) from the accurate numerical ones by a few percents. A natural question arises: Is it possible to modify the Iyer-Will approximation and get better results? Below we shall show that the answer to this question is affirmative.

We start with the presentation of some basic informations concerning the strategy adopted in this paper. The method is largely due to Iyer and Will but with some necessary generalizations and modifications. Following Ref. 12 let us consider the two-turning-point problem with the turning
points located at, say, $x_1$ and $x_2$ ($x_1 < x_2$). We can distinguish three regions: The interior region (II) between the turning points and the regions I and III outside the points $x_1$ and $x_2$, respectively. In the exterior regions the (asymptotic) solution is given by the standard WKB solution constructed to the required order. Substituting

$$y(x) \sim \exp(S/\varepsilon),$$

where

$$S(x) = \sum_{k=0}^{\infty} \varepsilon^k S_k(x)$$

to

$$\varepsilon^2 \frac{d^2}{dx^2} \psi(x) + Q(x)\psi(x) = 0,$$

and collecting the terms with the like powers of $\varepsilon$ one obtains a chain of equations of ascending complexity. The expansion parameter $\varepsilon$, which helps to keep the order control of the complicated terms should be set to 1 at the end of the calculations. The first two solutions, $S_0$ and $S_1$, define the standard WKB approximation. On the other hand, in the region II, the strategy is different. First we expand $Q(x)$ into the Taylor series about the maximum of $-Q(x)$ (located at $x_0$) to the required order and making use of the substitutions

$$k = \frac{1}{2} Q''_0, \quad z_0^2 = -2Q_0/Q''_0,$$

and

$$s_n = \frac{2Q^{(n)}_0}{n!Q''_0}, \quad (n = 3, ...),$$

we rewrite Eq. (4) in the following form

$$\varepsilon^2 \frac{d^2}{dz^2} \psi + k(-z_0^2 + z^2 + \sum_{n=3}^{2N} s_n z^n) \psi = 0,$$

where $z = x - x_0$. With the aid of

$$t = (4k)^{1/4} e^{-i\pi/4} z^{1/2}/\varepsilon^{1/2},$$

$$\bar{s}_n = \frac{1}{4} s_n (4k)^{(2-n)/4} e^{n\pi/4}$$

and

$$\nu + \frac{1}{2} = -ik^{1/2}z_0^2/2\varepsilon - \sum_{n=2}^{N} \varepsilon^{n-1} \Lambda_n$$

(10)
this equation can be further transformed

\[ \frac{d^2 \psi}{dt^2} + \left\{ \nu + \frac{1}{2} - \frac{1}{4} t^2 - \sum_{n=1}^{N-1} \left[ \varepsilon^{2n-1/2} s_{2n+1} t^{2n+1} + \varepsilon^n \left( s_{2n+2} t^{2n+2} - \Lambda_{n+1} \right) \right] \right\} \psi = 0. \quad (11) \]

In the limit \( \varepsilon \to 0 \) the general solution of the equation (11) is the linear combination of the parabolic cylinder functions

\[ a D_\nu(t) + b D_{-\nu-1}(-it). \quad (12) \]

In a general case we will look for a solution of the form

\[ \psi(t) = f(t) D_\nu(g(t)), \quad (13) \]

where \( f(t) \) and \( g(t) \) are two functions that have to be determined. Substituting (13) into (11) and eliminating the term with the first derivative of the parabolic cylinder function one gets \( f = -(dg/dt)^{-1/2} \). As a result one obtains the differential equation for the function \( g(t) \) which can be solved assuming

\[ g(t) = t + \sum_{n=0}^{\infty} \varepsilon^{n/2} A_n(t), \quad (14) \]

where

\[ A_n(t) = \sum_{j=0}^{n} \alpha_{n}^{j} t^{n+1-2j}, \quad (n + 1 - 2j \geq 0). \quad (15) \]

It should be noted that by solving the resulting system of algebraic equations one obtains both the coefficients \( \alpha \) and \( \Lambda \). The whole procedure is algorithmic and can easily be implemented in any of the existing computer algebra system. Now, if we substitute \( \Lambda_i \) into (10) we will obtain the equation that relates \( \omega \), derivatives (up to \( 2N \)) of the potential at \( x_0 \) and \( \nu \).

Thus far the analysis has been carried out in the interior region. In the exterior regions the solutions are constructed within the framework of \( N \)th WKB method starting with physical optics approximation as the lowest order approximation. The asymptotic matching is to be performed outside the turning points, where \( |t| \to \infty \). The two different asymptotic representation of the parabolic cylinder function valid in a wedge \( |\arg t| < 3/4 \) and \( 1/4 \pi < \arg t < 5/4 \pi \), respectively, which are used to represent the function \( \psi \)

\[ \psi \sim \left( \frac{dg}{dt} \right)^{-1/2} \left[ AD_\nu(g) + BD_{-\nu-1}(ig) \right], \quad (16) \]

where \( A \) and \( B \) are constants, are matched with the “right” and “left” WKB approximants. The analysis of the structure of the asymptotic form of the parabolic cylinder function in the wedge
\[ \frac{iQ_0}{\sqrt{2Q_0'}} = \sum_{k=2}^{N} \Lambda_k = n + \frac{1}{2}, \]  

where each \( \Lambda_k \) is a function of the derivatives of \( Q(x) \) of ascending complexity. The first two terms \( \Lambda_2 \) and \( \Lambda_3 \) has been constructed in Ref. [12]. This result has been extended to the sixth order WKB by Konoplya in Ref. [13]. It should be noted that the third order result has been reconstructed using the phase integral [25] by Gal’tsov and Matiukhin in a very interesting development in Ref. [11].

In Refs. [12, 13] the terms \( \Lambda_k \) in Eq. (17) were summed. On the other hand we do not know in advance if adding the next term will improve or worsen the quality of the approximation. Similarly, we do not know if there is some optimal truncation. Consequently, it may be that summation of the \( \Lambda_k \) terms is not the best strategy.

In this paper we report on the extension of the results of Iyer and Will and of Konoplya to the 13th order WKB approximation, i.e., in addition to \( \Lambda_2 \) and \( \Lambda_3 \) calculated in Ref. [12] and \( \Lambda_4, \Lambda_5 \) and \( \Lambda_6 \) presented in Ref. [13] (in a few cases the typographical errors have been detected) we have calculated all \( \Lambda_n \) up to \( n = 13 \). The novelty of our method consists in construction, for any given \( l \) and \( n \), the Padé approximants. As have been observed by Bender and Orszag [21] “... Padé approximants often work quite well, even beyond their proven range of applicability...” and we believe that making use of this two powerful techniques will improve the quality of the results. In the next section we shall explicitly demonstrate that this is indeed the case.

The technique of the Padé approximants has been used in Ref. [24] in the calculations of the gravitational perturbations of the Kerr black hole and in Ref. [20] but in a different context. Let us assume, for simplicity, that the function \( Q(x) \) is of the form

\[ Q(x) = \omega^2 - V(x), \]
where $V(x)$ does not contain $\omega$. Now, we approximate the right hand side of the equation

$$\omega^2 = V(x) - i \left( n + \frac{1}{2} \right) \sqrt{2Q''_0} \varepsilon - i \sqrt{2Q''_0} \sum_{i=2}^{N} \varepsilon^i \Lambda_j$$

(treated as a polynomial of $\varepsilon$) by $P^6_6$ and $P^6_7$. Note that the functions $\Lambda_k$ depend on $n$. We have chosen to work with $P^6_6$ and $P^6_7$ although all table of approximants $P^\tilde{m}\tilde{n}$ satisfying $\tilde{m} + \tilde{n} + 1 \leq 14$ can be constructed. We have not attempted to analyze the optimal (from a point of view of calculational effectiveness) approximants and prefer to work with $\omega^2$.

As is well known the methods of calculating frequencies of the quasinormal modes based on the WKB approximation break down when the overtone number $n$ exceeds the angular harmonic index, $l$, so one expects the reasonable results only for $n \leq l$, or $n$ slightly bigger than $l$. On the other hand the results are progressively better with increasing $l$.

The formulas describing the higher-order $\Lambda_n$ are rather long and complicated and the calculation is time-consuming. Indeed, the number of terms and their complexity quickly grows with $n$ as can be seen in Tab. I. Fortunately the calculations of the general terms have to be executed only once, and to avoid proliferation of extremely long formulas we do not present them here. All $\Lambda_n$ ($n = 2, ..., 13$) stored in various formats can be obtained from the first author upon request. On the other hand, the calculation of the quasinormal modes for a given potential function is quite fast.

Although the $\Lambda_j$ are complicated products of the various powers of the derivatives of the function $Q$ evaluated at the maximum they can easily be calculated numerically. Indeed, for each $j$ the maximal derivative of $Q$ is $2j$ and the length of each term in $\Lambda_j$ is $L^{(2j-1)/2}$ (where $Q^{(k)}$ - the $k$th derivative of $Q$ has the length $L^{-k}$) and the calculations reduce to simple multiplications. The derivatives of $Q$ with respect to the general tortoise coordinate are easily programmable and the determination of the radial coordinate of the maximum of the potential requires only elementary numerics.

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1 The Iyer-Will formulas can be reproduced in a few second on a typical budget laptop. On the other hand calculations of the general terms up to $\Lambda_{13}$ can take a few hours

2 The results are stored in Mathematica, Maple and Maxima syntax.
III. THE NORMAL MODES OF BLACK HOLES

A. Schwarzschild black hole

Here we give only basic informations necessary to calculate the quasinormal modes. The odd-parity perturbation of the Schwarzschild black hole are governed by the equation

$$\frac{d^2 \psi}{dx_*^2} + \left[ \omega^2 - \left( 1 - \frac{2}{x} \right) \left( \frac{\lambda_1}{x^2} + \frac{2\beta}{x^3} \right) \right] \psi,$$

(20)

where $x_*$ is the Regge-Wheeler coordinate, $\lambda_1 = l(l+1)$ and $\beta = 1, 0, -3$ for the scalar, vector and gravitational perturbations, respectively. For the gravitational perturbations we shall also calculate the quasinormal modes for the even-parity (Zerilli) potential

$$\frac{d^2 \psi}{dx_*^2} + \omega^2 \psi - \frac{2\Delta}{x^5(x_2x + 3)^2} \left[ \lambda_2^2(x_2 + 1)x^2 + 3x_2x + 9 \right] \psi,$$

(21)

where $\Delta = x^2 - 2x$ and $\lambda_2 = (l - 1)(l + 2)/2$. The normal mode frequencies should be the same for the both potentials and this can be used for the consistency check.

Now we shall analyze the accuracy of the WKB method with the Padé transform and contrast the thus obtained results with the numerical calculations of Fröman et al. presented in Ref. [9].

The frequencies of the odd-parity modes of the scalar, electromagnetic and gravitational are given in Tables II, III and IV, respectively. The results of the calculations of the gravitational even-parity modes are presented in Table V. Although it is quite possible that the order of the Padé approximants is not always optimal, we have decided to focus our attention only on $P_6^6$ and $P_7^6$. Such a detailed comparison is possible only if the results of the numerical calculations are known in advance. Typically, for a new $Q(x)$, such results are absent and we have to choose the calculational strategy (to certain extent) blindly.\(^3\) The Padé and numerical results are compared with the analogous result calculated within the framework of the very popular sixth-order “pure” WKB method. The appropriate formulas have been constructed previously by Konoplya. Our results have been obtained as a by-product of the calculations of the $\Lambda_{13}$, and, when applied to the perturbations of the Schwarzschild black hole they extend the calculations of Ref. [13].

The general features of the approximation based on the WKB method are shared by Padé approximants: the quality of the approximation increases with the increase of the harmonic index, $l$, and deteriorates with the the overtone number $n$. The best results are obtained for $n \leq l$ though they remain reasonable if the overtone number slightly excesses $l$. On the other hand, the quality

\(^3\) Our codes are written in such a way that one can calculate any element of the table of the Padé approximants.
of the approximation within the region of validity is really superb and the quasinormal frequencies are close to the accurate numerical results. Generally, as expected, they are also more accurate than those calculated using sixth-order WKB. Moreover, the Padé approximants $P^6_6$ and $P^6_7$ give $\omega$ that are comparable with or even better than the phase-integral method in the optimal order. Of course, in order to construct better approximants one has to take into account the influence of the remaining turning points.

To analyze the quality of the Padé approximants in more details let us define deviation of the real part of the frequency

$$\Delta^{(r)}(\omega_k) = \frac{\Re(\omega_k) - \Re(\omega_{num})}{\Re(\omega_{num})} 100\%$$

and similarly the deviation of its imaginary part

$$\Delta^{(i)}(\omega_k) = \frac{\Im(\omega_k) - \Im(\omega_{num})}{\Im(\omega_{num})} 100\%,$$

where $\omega_k$ is the approximate complex frequency of the quasinormal mode and $\omega_{num}$ is its accurate numerical value. First, let us consider the odd-parity scalar modes. Inspection of Table II shows that both $P^6_6$ and $P^6_7$ are very accurate and except for the lowest mode ($l = 0, n = 0$) it is always superior to the sixth-order WKB. Indeed, $\Delta^{(r)}$ calculated for $P^6_6$ and $P^6_7$ approximants of the lowest mode is $7.9 \times 10^{-1}\%$ and $9.8 \times 10^{-2}\%$, respectively, whereas within the framework of WKB approximation one has $1.2 \times 10^{-2}\%$. (The deviation of $P^5_6$ is slightly smaller). On the other hand, deviation of the imaginary part of $P^6_6$ and $P^6_7$ for that mode is $-8.6 \times 10^{-3}\%$ and $2.8 \times 10^{-2}\%$, respectively. This may be compared with deviation calculated within the framework of the sixth-order WKB, which is $3.9\%$. For the lowest modes ($l \geq 1$) and overtones the Padé approximants are at least two order of magnitude better that the WKB. (For $l = 3, n = 0$ both $P^6_6$ and $P^6_7$ are exactly the same as the numerically calculated frequencies). The imaginary part of the Padé approximants to the quasinormal modes is always far better than those calculated using WKB.

The Padé approximants for the vector and gravitational odd-parity modes are amazingly accurate as can be seen from Table III and IV. Now, let us analyze the gravitational even-parity modes. The results of the calculations are displayed in Table V. In Table VI the deviations $\Delta^{(r)}$ and $\Delta^{(i)}$ are presented for all calculated modes. The accuracy of the Padé approximants is excellent and ranges from $5.2 \times 10^{-1}\%$ for ($l = 2, n = 3$) to $10^{-8}\%$ for ($l = 4, n = 1$). This can be contrasted with the WKB results, which deviates by $3.9\%$ from the exact value for ($l = 2, n = 3$) and $1.6\%$ for ($l = 3, n = 5$).
TABLE II: The quasinormal modes of the odd-parity scalar perturbations of the Schwarzschild black hole. Numerical results are taken from Ref. [9]. The WKB results have been obtained previously in [13]. Here we have calculated them once again and retained more digits.

| l  | n | Numerical value of \( \omega \) | Regge-Wheeler potential \( P^6 \) | Regge-Wheeler potential \( P^7 \) | Sixth order WKB |
|----|---|---------------------|---------------------|---------------------|---------------------|
| 0  | 0 | 0.1104543-0.1048943 i | 0.111328786-0.104885261 i | 0.11034643-0.104924037 i | 0.110467018-0.100816251 i |
|    | 1 | 0.0861169-0.3480524 i | 0.086972660-0.347994964 i | 0.087477758-0.347875503 i | 0.089029100-0.344528856 i |
| 1  | 0 | 0.2929361-0.0970660 i | 0.292936154-0.097660269 i | 0.292936143-0.097659978 i | 0.292909644-0.0977616179 i |
|    | 1 | 0.26444865-0.30625759 i | 0.264449307-0.306258587 i | 0.264449319-0.306257973 i | 0.264471051-0.306518241 i |
| 2  | 0 | 0.229539335-0.54073425 i | 0.229227121-0.54073930 i | 0.229227121-0.54073886 i | 0.231014233-0.54265498 i |
|    | 1 | 0.203258386-0.788297823 i | 0.202925543-0.78885062 i | 0.202747941-0.788724995 i | 0.222094287-0.795168920 i |
|    | 2 | 0.174773568-0.525187599 i | 0.174483574-0.524812107 i | 0.174388313-0.52548762 i | 0.173988785-0.530056622 i |
| 3  | 0 | 0.430544054-0.508558402 i | 0.430453707-0.508556964 i | 0.430453707-0.508556573 i | 0.430385788-0.508698985 i |
|    | 1 | 0.393863063-0.738096585 i | 0.393853836-0.738081384 i | 0.393855027-0.738098967 i | 0.393206798-0.739885188 i |

TABLE III: The quasinormal modes of the odd-parity electromagnetic perturbations of the Schwarzschild black hole. Numerical results are taken from Ref. [9]. The WKB results have been obtained previously in [13]. Here we have calculated them once again and retained more digits.

| l  | n | Numerical value of \( \omega \) | Regge-Wheeler potential \( P^6 \) | Regge-Wheeler potential \( P^7 \) | Sixth order WKB |
|----|---|---------------------|---------------------|---------------------|---------------------|
| 1  | 0 | 0.248263272-0.092487709 i | 0.248263238-0.0924876590 i | 0.248263273-0.092487749 i | 0.248191418-0.0926370275 i |
|    | 1 | 0.21451542-0.293667646 i | 0.214496169-0.293664604 i | 0.214503194-0.293668427 i | 0.214252900-0.294118148 i |
| 2  | 0 | 0.174773568-0.525187599 i | 0.174483574-0.524812107 i | 0.174388313-0.52548762 i | 0.173988785-0.530056622 i |
|    | 1 | 0.141676999-0.771908924 i | 0.144481853-0.772141225 i | 0.147368451-0.772137961 i | 0.159002152-0.796775458 i |
| 2  | 0 | 0.457595512-0.095004426 i | 0.457595511-0.0950044260 i | 0.457595512-0.0950044257 i | 0.457593408-0.0950111516 i |
|    | 1 | 0.436542386-0.290710143 i | 0.436542317-0.290710154 i | 0.436542359-0.290710162 i | 0.436533960-0.290729026 i |
|    | 2 | 0.401186734-0.501587346 i | 0.401189409-0.501584248 i | 0.401185685-0.501588318 i | 0.400990598-0.501728128 i |
| 3  | 0 | 0.362595032-0.730198514 i | 0.362585402-0.730166653 i | 0.362589034-0.730120508 i | 0.361158598-0.732452614 i |
|    | 1 | 0.656898670-0.0956162179 i | 0.656898670-0.0956162179 i | 0.656898666-0.0956171153 i | 0.656898466-0.0956171153 i |
|    | 2 | 0.613832926-0.492066258 i | 0.613832147-0.492066236 i | 0.613832001-0.492066269 i | 0.613787638-0.492063925 i |
| 3  | 0 | 0.577918506-0.70633083 i | 0.577918507-0.706329774 i | 0.577918194-0.706330849 i | 0.577481153-0.706497836 i |

B. Reissner-Nordström black hole

Technically speaking, the equations governing the perturbations of the Reissner-Nordström black hole are only slightly more complicated than the analogous equations for the Schwarzschild black hole. The main difference lies in the fact that for the charged black holes the are no pure electromagnetic or pure gravitational oscillations. The (odd-parity) normal oscillations are governed
by the system of the differential equations

\[ \frac{d^2 \psi_j(-)}{dx^2} + \omega^2 \psi_j(-) - \frac{\Delta}{r^5} \left[ \lambda_1 r - p_k + \frac{4q^2}{x} \right] \psi_j(-), \quad (j \neq k) \tag{24} \]

where \( q = |e|/M \) with \( e \) being the electric charge, \( \Delta = x^2 - 2x + q^2 \), \( i, j = 1, 2 \) and

\[ p_1 = 3 + (9 + 8\lambda_2 q^2)^{1/2}, \tag{25} \]

\[ p_2 = 3 - (9 + 8\lambda_2 q^2)^{1/2}. \tag{26} \]

The tortoise coordinate \( x_* \) is defined in the standard way. Note that for uncharged black hole \( q = 0 \) the functions \( \psi_1(-) \) and \( \psi_2(-) \) describe electromagnetic and gravitational perturbations, respectively. The last term in Eq. (24) defines the minus potential, i.e., \( (-V_j(-)\psi_j(-)) \). The potential of the even parity modes is given by

\[ V_j^{}(+) = V_j^{}(-) + 2p_i \frac{d}{dx_*} \left( \frac{\Delta_1}{2\lambda_2 r^2 + p_i} \right) \tag{27} \]

and the perturbation equations are given by (24) with the last term in the right hand side replaced by \( -V_j^{}(+)\psi_j^{}(+) \) and \( \psi_j(-) \) by \( \psi_j^{}(+) \).

The quasinormal modes of the Reissner-Nordström black hole have been studied by a number of authors [4, 23, 26–30]. Here we restrict ourselves to the odd-parity electromagnetic and gravitational modes as the extension to the even-parity modes does not present any problems and the results are of the same quality. The Padé approximants of the quasinormal frequencies are compared with the accurate numerical calculations of Andersson [26] for the nonextreme black holes and with the results of Onozawa et al. [28] for the extreme ones. The results are displayed in Tables VII–IX. A more detailed comparison of the Padé approximants of the gravitational modes with the accurate numerical results is given in Tables XI and XIX. It is seen that the Padé approximants are very close to the numerical \( \omega \) and the accuracy of the results does not depend on \( q \). Similarly, for the extreme configuration the Padé approximants give excellent agreement with the numerical results presented in Ref. [28].

**IV. CONCLUDING REMARKS**

We have used the the thirteenth order WKB method and the Padé transforms to calculate the quasinormal modes of the Schwarzschild and Reissner-Nordström black holes and demonstrated that our results are very close to the accurate numerical calculations. The method can be modified
TABLE IV: The quasinormal modes of the odd-parity gravitational perturbations of the Schwarzschild black hole. Numerical results are taken from Ref. [9]. The WKB results have been obtained previously in [13]. Here we have calculated them once again and retained more digits.

| $l$ | $n$ | Numerical value of $\omega$ | Regge-Wheeler potential $P^6_6$ | Regge-Wheeler potential $P^7_7$ | Sixth order WKB |
|-----|-----|-------------------------------|-----------------------------|-----------------------------|-------------------|
| 2   | 0   | 0.373671684-0.088962315 i     | 0.373672999-0.088966626 i    | 0.373619357-0.0888909796 i   | 0.37369357-0.088909796 i |
| 1   |     | 0.346710997-0.273914875 i     | 0.346735243-0.273772106 i    | 0.346296571-0.273479774 i    | 0.346735243-0.273772106 i |
| 2   |     | 0.301053455-0.478276983 i     | 0.301756472-0.476569695 i    | 0.298519956-0.477560223 i    | 0.301756472-0.476569695 i |
| 3   |     | 0.251504962-0.705148202 i     | 0.250329362-0.705148202 i    | 0.24715477-0.709595458 i     | 0.250329362-0.705148202 i |
| 2   | 0   | 0.346710997-0.273914875 i     | 0.346735243-0.273772106 i    | 0.346296571-0.273479774 i    | 0.346735243-0.273772106 i |
| 1   |     | 0.301053455-0.478276983 i     | 0.301756472-0.476569695 i    | 0.298519956-0.477560223 i    | 0.301756472-0.476569695 i |
| 2   |     | 0.251504962-0.705148202 i     | 0.250329362-0.705148202 i    | 0.24715477-0.709595458 i     | 0.250329362-0.705148202 i |
| 3   |     | 0.205411205-1.15245777 i      | 0.205411205-1.15245777 i     | 0.2024715477-0.709595458 i   | 0.205411205-1.15245777 i |
| 4   | 0   | 0.809178378-0.094163961 i     | 0.809178378-0.094163961 i    | 0.809178378-0.094163961 i    | 0.809178378-0.094163961 i |
| 1   |     | 0.796631532-0.284334349 i     | 0.796631532-0.284334349 i    | 0.796631532-0.284334349 i    | 0.796631532-0.284334349 i |
| 2   |     | 0.77209545-0.479998196 i      | 0.77209545-0.479998196 i     | 0.77209545-0.479998196 i     | 0.77209545-0.479998196 i |
| 3   |     | 0.73963675-0.639234319 i      | 0.73963675-0.639234319 i     | 0.73963675-0.639234319 i     | 0.73963675-0.639234319 i |
| 4   | 0   | 0.701515509-0.898238972 i     | 0.701515429-0.898239961 i    | 0.701515717-0.898238277 i    | 0.701515509-0.898238972 i |

To allow for more complicated potentials, with the function $V$ depending on $\omega$. Our general formulas can be applied straightforwardly to any black hole potential given by Eq. (18) and the only limit of the calculations is their scale. For example, the quasinormal modes of the $d$-dimensional Schwarzschild-Tangherlini black holes [31], the asymptotically (anti)-de Sitter [32–34] black holes and many others can be calculated without necessity to change the codes. It should be emphasized that some potentials may not be strictly positive and their complexity grows with the dimension.

Preliminary calculations and first comparisons with the known results look promising. Indeed, our calculations of the fundamental tensor gravitational quasinormal modes of the higher dimensional Schwarzschild-Tangherlini black holes ($5 \leq d \leq 11$) give exactly the same results as those presented in Refs. [35, 36]. In the $d = 11$-dimensional case for $2 \leq l \leq 11$ and the lowest overtones the Padé approximants yield good results. For example, for $l = 2$ and $n = 0, 1, 2$ the Padé approximants $P^6_6$ (rounded to four decimal places) give, respectively, $\omega = 4.3920 - 1.0577 i$, $\omega = 3.3393 - 3.0283 i$ and $\omega = 1.8026 - 3.6527 i$, which is identical ($l = 0, n = 0$) or close to the frequencies calculated by Rostworowski [35] and the accuracy of our calculations rapidly grows with $l$. The behavior of the low-lying modes may be contrasted with the WKB approximation. For example for $l = 2$ one has $\omega = 4.4007 - 1.0601 i$, $\omega = 3.1165 - 3.7864 i$ and $\omega = 0.5276 - 5.6632 i$. The real part of the frequency of the mode $l = 2$ and $n = 2$ is wrong at any order of the WKB.
TABLE V: The quasinormal modes of the even-parity gravitational perturbations of the Schwarzschild black hole. Numerical results are taken from Ref. 9.

| $l$ | $n$ | Numerical value of $\omega$ | Zerilli potential $P^0_6$ | Zerilli potential $P^0_7$ | Sixth order WKB |
|-----|-----|-----------------------------|--------------------------|--------------------------|-----------------|
| 2   | 0   | 0.373671684-0.088962351 i   | 0.373671632-0.088962365 i| 0.373671627-0.088962347 i| 0.373707325-0.0889233432 i |
| 2   | 1   | 0.346710997-0.273914875 i   | 0.346710995-0.273913980 i| 0.346710995-0.273910070 i| 0.346715435-0.273875995 i |
| 2   | 2   | 0.301053455-0.478276983 i   | 0.301080694-0.478292024 i| 0.301133004-0.478206943 i| 0.300049631-0.478829264 i |
| 3   | 0   | 0.599443288-0.092703048 i   | 0.599443288-0.092703048 i| 0.599443373-0.0927027654 i| 0.582643803-0.281298113 i |
| 3   | 1   | 0.582643803-0.281298113 i   | 0.582643798-0.281298116 i| 0.582643797-0.281298115 i| 0.582642128-0.281292695 i |
| 3   | 2   | 0.551685901-0.479092751 i   | 0.551685635-0.479091963 i| 0.551685718-0.479091911 i| 0.551595711-0.479055726 i |
| 3   | 3   | 0.511961911-0.690337096 i   | 0.511964029-0.690332484 i| 0.511965550-0.690331591 i| 0.511107845-0.690490597 i |
| 3   | 4   | 0.470174006-0.915649393 i   | 0.470171037-0.915616065 i| 0.470157447-0.915641228 i| 0.466882696-0.917994938 i |
| 3   | 5   | 0.431364791-1.15215136 i    | 0.431254524-1.15207234 i| 0.431276665-1.15224185 i| 0.424371238-1.16253344 i |
| 4   | 0   | 0.890178378-0.094163961 i   | 0.890178378-0.094163961 i| 0.890178349-0.094160473 i| 0.796631532-0.284334349 i |
| 4   | 1   | 0.796631532-0.284334349 i   | 0.796631532-0.284334349 i| 0.796631532-0.284334349 i| 0.796631252-0.284334186 i |
| 4   | 2   | 0.772709543-0.479908175 i   | 0.772709544-0.479908170 i| 0.772695376-0.479899772 i| 0.739836754-0.683924275 i |
| 4   | 3   | 0.739836754-0.683924275 i   | 0.739836652-0.683924275 i| 0.739836564-0.683924275 i| 0.739665134-0.683901916 i |
| 4   | 4   | 0.701515509-0.898238972 i   | 0.701515239-0.898238200 i| 0.701515431-0.898238235 i| 0.700641016-0.898461166 i |

TABLE VI: Deviations of the real and imaginary part of the quasinormal frequencies of the even-parity gravitational perturbations for the Schwarzschild black hole from the accurate numerical results. The Padé approximation always gives better results that the 6-th order WKB.

| $l$ | $n$ | $P^0_6$ | $P^0_7$ | $P^0_6$ | $P^0_7$ | Sixth order WKB | Sixth order WKB |
|-----|-----|---------|---------|---------|---------|-----------------|-----------------|
| 2   | 0   | 1.4 × 10^{-5} | 1.5 × 10^{-5} | 1.4 × 10^{-4} | 1.5 × 10^{-4} | 8.1 × 10^{-2} |
| 2   | 1   | 1.9 × 10^{-2} | 9.4 × 10^{-3} | 1.2 × 10^{-2} | 1.3 × 10^{-2} | 1.6 × 10^{-1} |
| 2   | 2   | 1.4 × 10^{-2} | 2.6 × 10^{-2} | 8.4 × 10^{-2} | 1.5 × 10^{-1} | 1.5 × 10^{-1} |
| 3   | 0   | 6.9 × 10^{-8} | 6.6 × 10^{-8} | 1.3 × 10^{-7} | 1.4 × 10^{-5} | 5.7 × 10^{-4} |
| 3   | 1   | 9.3 × 10^{-7} | 1.1 × 10^{-6} | 1.7 × 10^{-7} | 1.1 × 10^{-2} | 3.7 × 10^{-3} |
| 3   | 2   | 1.3 × 10^{-4} | 5.4 × 10^{-4} | 1.8 × 10^{-4} | 1.6 × 10^{-5} | 9.6 × 10^{-3} |
| 3   | 3   | 4.1 × 10^{-4} | 6.7 × 10^{-4} | 1.7 × 10^{-4} | 1.7 × 10^{-1} | 1.9 × 10^{-2} |
| 4   | 0   | 5.7 × 10^{-8} | 6.9 × 10^{-8} | 3.5 × 10^{-4} | 7.1 × 10^{-1} | 2.5 × 10^{-1} |
| 4   | 1   | 4.1 × 10^{-8} | 1.1 × 10^{-7} | 1.4 × 10^{-5} | 3.6 × 10^{-5} | 1.2 × 10^{-4} |
| 4   | 2   | 9.8 × 10^{-8} | 9.8 × 10^{-8} | 1.1 × 10^{-6} | 1.8 × 10^{-5} | 1.8 × 10^{-3} |
| 4   | 3   | 6.4 × 10^{-8} | 3.2 × 10^{-8} | 9.8 × 10^{-8} | 2.3 × 10^{-2} | 3.3 × 10^{-3} |
| 4   | 4   | 3.9 × 10^{-8} | 8.6 × 10^{-8} | 1.1 × 10^{-5} | 8.2 × 10^{-5} | 1.2 × 10^{-4} |
Specifically, within the sixth order WKB approximation, it is over three times smaller than the accurate numerical value calculated by Rostworowski. On the other hand, however, the approximation works better for larger $l$, as expected. This comparison indicates that one should be cautious with any approximation based on the WKB method even for the low overtones ($n = 2$) of the low-lying modes $l = 2$ and $l = 3$. On the other hand, our results demonstrate the usefulness of the Padé approximation and suggest that there is still a room for improvements and new ideas.
TABLE VIII: The quasinormal $l = 2$ modes of the odd-parity gravitational perturbations of the Reissner-Nordström black hole for several exemplary values of $q = |e|/M$. The numerical results are taken from Ref. [26].

| Method  | $q$ | $n = 0$ | $n = 1$ | $n = 2$ |
|---------|-----|---------|---------|---------|
| numerical | 0.1 | 0.37393238-0.088990533 i | 0.34968103-0.27399495 i | 0.3013348-0.47839381 i |
|         |     | 0.37393222-0.08898960 i | 0.34655495-0.27401278 i | 0.29943057-0.47869569 i |
|         |     | 0.37393371-0.08894903 i | 0.34700547-0.27385072 i | 0.30202310-0.47667901 i |
| $P^6_6$ | 0.2 | 0.37474443-0.089074786 i | 0.34786203-0.27423284 i | 0.30222215-0.47873633 i |
|         |     | 0.37474440-0.089083239 i | 0.34740014-0.27426511 i | 0.30027463-0.47911441 i |
|         |     | 0.37474583-0.089079319 i | 0.34785085-0.27408448 i | 0.30286554-0.47700100 i |
| numerical | 0.3 | 0.37619678-0.089212946 i | 0.34935008-0.27461849 i | 0.30384543-0.47927509 i |
|         |     | 0.37619695-0.089221471 i | 0.34892699-0.27467057 i | 0.30184885-0.47976635 i |
|         |     | 0.37619829-0.089217694 i | 0.34937482-0.27446442 i | 0.30441295-0.47751621 i |
| $P^6_6$ | 0.4 | 0.37843689-0.089398114 i | 0.35172753-0.27512438 i | 0.30642361-0.47994045 i |
|         |     | 0.37843726-0.089406811 i | 0.35131155-0.27519530 i | 0.30440527-0.48055153 i |
|         |     | 0.37843856-0.089403029 i | 0.35175035-0.27501248 i | 0.30688908-0.47817569 i |
| numerical | 0.5 | 0.38167715-0.089612379 i | 0.35521299-0.27568438 i | 0.31028277-0.48058204 i |
|         |     | 0.38167772-0.089621446 i | 0.35480784-0.27576354 i | 0.30831023-0.48125731 i |
|         |     | 0.38167903-0.089617226 i | 0.35522981-0.27552809 i | 0.31064235-0.47885652 i |
| $P^6_6$ | 0.6 | 0.38621729-0.089813675 i | 0.35977578-0.27620272 i | 0.31409566-0.48143936 i |
|         |     | 0.38621829-0.089823511 i | 0.35977587-0.27620272 i | 0.31409556-0.48143936 i |
| $P^7_7$ | 0.7 | 0.39249849-0.089904426 i | 0.36071729-0.27615005 i | 0.31588482-0.48087942 i |
|         |     | 0.39250093-0.089915592 i | 0.36681075-0.27606507 i | 0.32266295-0.48012476 i |
|         |     | 0.39250088-0.089906645 i | 0.36712177-0.27601444 i | 0.32427673-0.47886675 i |
| numerical | 0.8 | 0.40121719-0.08964323 i | 0.37690401-0.27494337 i | 0.33517151-0.47656389 i |
|         |     | 0.40122292-0.089647871 i | 0.3769266-0.27485827 i | 0.33521872-0.47580206 i |
|         |     | 0.40121726-0.089643579 i | 0.37690333-0.27482471 i | 0.33570503-0.47578542 i |
| $P^6_6$ | 0.9 | 0.4135705-0.08333012 i | 0.39045691-0.27001504 i | 0.34959435-0.46521762 i |
|         |     | 0.41357316-0.08335816 i | 0.3904903-0.26997448 i | 0.35028615-0.46480977 i |
|         |     | 0.41357020-0.08332922 i | 0.39059923-0.26997219 i | 0.35064801-0.46456627 i |
| numerical | 0.99 | 0.42929679-0.084265944 i | 0.40356088-0.25700759 i | 0.35441725-0.44384073 i |
|         |     | 0.42929725-0.084268033 i | 0.40355372-0.25700435 i | 0.35539583-0.44397284 i |

In our personal view the method advocated by Matiukhin and Gal'tsov [11] and its possible generalizations, although technically very hard, look promising. We also observe that the calculated functions $\Lambda_k$ can be applied in the analysis of the potential barrier tunneling. These problems are actively investigated and the results will be presented elsewhere. Finally, observe that in the calculations of the quasinormal modes one often has to choose between the generality and simplicity of the approach on the one hand and the great accuracy on the other, and, consequently, the quality...
Reissner-Nordström black holes are given in Table VII. The numerical results are taken from Ref. [28]. To conform with [28] we have rounded our results appropriately.

TABLE IX: The quasinormal modes of the odd-parity gravitational perturbations of the extreme (|e| = M) Reissner-Nordström black hole. The numerical results are taken from Ref. [28]. To conform with [28] we have rounded our results appropriately.

| Method | n = 0                      | n = 1                      | n = 2                      |
|--------|----------------------------|----------------------------|----------------------------|
| l = 2  | numerical                  | 0.43134-0.083460 i         | 0.40452-0.25498 i          | 0.35340-0.44137 i          |
|        | P₆                          | 0.43134-0.083462 i         | 0.40457-0.25498 i          | 0.35412-0.44167 i          |
|        | P₇                          | 0.43134-0.083460 i         | 0.40459-0.25499 i          | 0.35515-0.44191 i          |
| l = 3  | numerical                  | 0.70430-0.05973 i          | 0.68804-0.25992 i          | 0.65624-0.44007 i          |
|        | P₆                          | 0.70430-0.05973 i          | 0.68804-0.25992 i          | 0.65624-0.44007 i          |
|        | P₇                          | 0.70430-0.05973 i          | 0.68804-0.25992 i          | 0.65624-0.44007 i          |
| l = 4  | numerical                  | 0.96576-0.087001 i         | 0.95381-0.26212 i          | 0.93020-0.44064 i          |
|        | P₆                          | 0.96576-0.087001 i         | 0.95381-0.26212 i          | 0.93020-0.44064 i          |
|        | P₇                          | 0.96576-0.087001 i         | 0.95381-0.26212 i          | 0.93020-0.44064 i          |

TABLE X: Deviations of the real and imaginary part of the Padé approximants, P₆, from the accurate numerical results. The (l = 2) quasinormal frequencies of the odd-parity gravitational perturbations of the Reissner-Nordström black holes are given in Table VII.

| q     | Δᵣ (%) | Δᵢ (%) | Δᵣ (%) | Δᵢ (%) | Δᵣ (%) | Δᵢ (%) |
|-------|---------|---------|---------|---------|---------|---------|
| n = 0 | 0.0001  | 0.0000  | 0.0001  | 0.0000  | 0.0001  | 0.0000  |
| n = 1 | 0.0001  | 0.0000  | 0.0001  | 0.0000  | 0.0001  | 0.0000  |
| n = 2 | 0.0001  | 0.0000  | 0.0001  | 0.0000  | 0.0001  | 0.0000  |

of the approximation has been judged not only by comparison with the exact numerical results but also with the competing analytic or semianalytic approaches.

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TABLE XI: Deviations of the real and imaginary part of the Padé approximants, $P_l^6$, from the accurate numerical results. The ($l = 2$) quasinormal frequencies of the odd-parity gravitational perturbations of the Reissner-Nordström black holes are given in Table VIII.

| $q$   | $\Delta^{(r)} \, (\%)$ | $\Delta^{(i)} \, (\%)$ | $\Delta^{(r)} \, (\%)$ | $\Delta^{(i)} \, (\%)$ | $\Delta^{(r)} \, (\%)$ | $\Delta^{(i)} \, (\%)$ |
|-------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| 0.1   | $-3.6 \times 10^{-4}$  | $-4.9 \times 10^{-3}$  | $-7.1 \times 10^{-3}$  | $5.3 \times 10^{-2}$  | $-2.3 \times 10^{-1}$  | $3.6 \times 10^{-1}$  |
| 0.2   | $-3.7 \times 10^{-4}$  | $-5.1 \times 10^{-3}$  | $-7.1 \times 10^{-3}$  | $5.4 \times 10^{-2}$  | $-2.1 \times 10^{-1}$  | $3.6 \times 10^{-1}$  |
| 0.3   | $-4.1 \times 10^{-4}$  | $-5.3 \times 10^{-3}$  | $-7.1 \times 10^{-3}$  | $5.6 \times 10^{-2}$  | $-1.9 \times 10^{-1}$  | $3.7 \times 10^{-1}$  |
| 0.4   | $-4.4 \times 10^{-4}$  | $-5.5 \times 10^{-3}$  | $-6.5 \times 10^{-3}$  | $5.8 \times 10^{-2}$  | $-1.5 \times 10^{-1}$  | $3.7 \times 10^{-1}$  |
| 0.5   | $-4.9 \times 10^{-4}$  | $-5.4 \times 10^{-3}$  | $-4.7 \times 10^{-3}$  | $5.7 \times 10^{-2}$  | $-1.2 \times 10^{-1}$  | $3.6 \times 10^{-1}$  |
| 0.6   | $-5.3 \times 10^{-4}$  | $-4.6 \times 10^{-3}$  | $-1.3 \times 10^{-3}$  | $5.1 \times 10^{-2}$  | $-0.9 \times 10^{-2}$  | $3.2 \times 10^{-1}$  |
| 0.7   | $-4.1 \times 10^{-4}$  | $-2.5 \times 10^{-3}$  | $3.2 \times 10^{-3}$   | $3.6 \times 10^{-2}$  | $-1.2 \times 10^{-1}$  | $2.6 \times 10^{-1}$  |
| 0.8   | $-1.9 \times 10^{-5}$  | $-3.9 \times 10^{-4}$  | $-8.3 \times 10^{-3}$  | $4.3 \times 10^{-2}$  | $-1.6 \times 10^{-1}$  | $1.6 \times 10^{-1}$  |
| 0.9   | $7.3 \times 10^{-5}$   | $1.1 \times 10^{-4}$   | $-3.6 \times 10^{-2}$  | $1.6 \times 10^{-2}$  | $-3.1 \times 10^{-1}$  | $1.4 \times 10^{-1}$  |
| 0.99  | $1.1 \times 10^{-4}$   | $6.8 \times 10^{-4}$   | $-8.3 \times 10^{-3}$  | $3.9 \times 10^{-3}$  | $-4.2 \times 10^{-1}$  | $-8.9 \times 10^{-2}$ |
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