LARGE SCALE GEOMETRY OF 4-DIMENSIONAL
CLOSED NONPOSITIVELY CURVED
REAL ANALYTIC MANIFOLDS

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ABSTRACT. We study the asymptotic cones of the universal covering spaces of
closed 4-dimensional nonpositively curved real analytic manifolds. We show
that the existence of nonstandard components in the Tits boundary, discovered
by Christoph Hummel and Victor Schroeder [HS98], depends only on the quasi-
isometry type of the fundamental group.

1. INTRODUCTION

The concept of the asymptotic cone was introduced by van den Dries and
Wilkie [vdDW84] and by Gromov [Gro93]. It has been used by several authors
to study the large scale geometry of certain spaces and to distinguish the quasi-
isometry type of these spaces. Kleiner and Leeb [KL97b] used the asymptotic cone
to prove the Margulis conjecture regarding the quasi-isometry rigidity of symmetric
spaces of noncompact type. Kapovich and Leeb [KL95, KL97a] also used the
asymptotic cone to study the quasi-isometry types of Haken 3-manifolds.

Nonpositively curved closed 4-dimensional real analytic manifolds form an in-
teresting class of manifolds. Many examples of these manifolds have been con-
structed by Schroeder [Sch88, Sch89a] and later by Abresch and Schroeder [AS96].
The structure of flats in these manifolds was studied by Schroeder [Sch89b]. The
Tits alternative of their fundamental groups was studied by Xiangdong Xie [Xie04].

Given a Hadamard space $X$, the Tits boundary $\partial_T X = (X(\infty), Td)$ is a metric
space which reflects part of the asymptotic geometry of the space. The ideal
boundary of a Hadamard space in general is not a quasi-isometry invariant. Croke
and Kleiner gave the first example of such phenomenon. They constructed a pair
of compact piecewise Euclidean 2-complexes with nonpositive curvature which
are homeomorphic but whose universal covers have non-homeomorphic geometric
boundaries [CK00]. This is in contrast with the case of Gromov hyperbolic

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spaces, where quasi-isometric spaces have homeomorphic geometric boundaries. Buyalo [Buy98] and Croke and Kleiner [CK02] independently also showed that the Tits boundary is not equivariantly preserved under a quasi-isometry.

By investigating the Tits boundary of graph manifolds, Croke and Kleiner [CK02] discovered nontrivial connected components of the Tits boundary which are not subsets of unions of ideal boundaries of flats. These components are intervals of length less than $\pi$. Hummel and Schroeder [HS98] discovered similar components, which they called nonstandard components, in the Tits boundary of certain 4-dimensional nonpositively curved real analytic manifolds. It follows from [Sch89b] that the existence of these nonstandard components depends on the fundamental group of the 4-dimensional real analytic manifold.

In this paper we analyze, using some ideas of Kapovich and Leeb [KL95], the asymptotic cones of the universal cover of these manifolds, and show that the existence of nonstandard components depends only on the quasi-isometry type of the fundamental group of the manifold.

**Theorem 1.1.** Let $X_1$ and $X_2$ be the universal covers of two closed 4-dimensional nonpositively curved real analytic manifolds. If $X_1$ and $X_2$ are quasi-isometric, then $\partial_T X_1$ contains a nonstandard component if and only if $\partial_T X_2$ does.

If the Tits boundary of the universal covering space $X$ of a closed irreducible 4-dimensional nonpositively curved real analytic manifold does not contain any nonstandard components, we show in Section 4 that any asymptotic cone of $X$ is tree-graded in the sense of Druţu and Sapir [DS] (see Definition 1.10 in [DS]) with respect to a collection $\mathcal{F}$ of subsets, obtained as $\omega$-lim of higher rank submanifolds in $X$. The following corollary follows immediately from Theorem 8.5 of Druţu and Sapir [DS].

**Corollary 1.2.** Let $X$ be the universal cover of a closed irreducible 4-dimensional nonpositively curved real analytic manifold $M$. If $\partial_T X$ does not contain any nonstandard components, then the fundamental group $\pi_1(M)$ is relatively hyperbolic with respect to the fundamental groups of the maximal higher rank submanifolds in $M$.

The paper is organized as follows. Section 1 is an introduction. In Section 2 we recall the properties of 4-dimensional real analytic manifolds which we need through the rest of the paper. In Section 3 we show that for certain 4-dimensional real analytic manifolds all triangles are thin relative to a maximal higher rank submanifold. We use that to describe all flats in the asymptotic cone of these manifold.
manifolds which will be done in Section 4. In Section 5 we give the proof of Theorem 1.1.

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2. Background

Let $X^n$ be an $n$-dimensional Hadamard manifold, by that we mean a complete simply connected Riemannian manifold with nonpositive sectional curvature. By Cartan-Hadamard Theorem $X^n$ is diffeomorphic to $\mathbb{R}^n$. In fact the $\exp_p$ map at any point $p \in X$ is a diffeomorphism. For basic facts about Hadamard space, we refer the reader to [BGS85].

We denote the ideal boundary $X(\infty)$ of $X$ equipped with the cone topology by $\partial_\infty X$, and denote the ideal boundary with the Tits metric by $\partial T X$. For any two different points $p, q \in X$, $\overrightarrow{pq}$, $\overrightarrow{pq}$ denote respectively the geodesic segment connecting $p$ to $q$, and the geodesic ray starting at $p$ and passing through $q$. By $\overrightarrow{pq}(\infty)$ we denote the limit point in $X(\infty)$ of the ray $\overrightarrow{pq}$.

Throughout the paper, with the exception of Proposition 3.3, $X$ will denote the universal cover of a closed irreducible 4-dimensional nonpositively curved real analytic manifold and we denote by $\Gamma$ its fundamental group.

Given a unit tangent vector $v$, the rank of $v$ is the dimension of the vector space of parallel Jacobi fields along the unique geodesic $c_v$ with $c_v'(0) = v$. If the isometry group of the Hadamard manifold $X$ satisfies the duality condition, for example if $X$ covers a manifold of finite volume, then parallel Jacobi fields can be integrated to produce flat strips, see Chapter IV in [Bal95] for details.

For any unit vector $v$, we write $v(\pm \infty) \in X(\infty)$ to denote the end points of the unique geodesic $c_v$. The parallel set of $c_v$ is denoted by $P_v$. This set consists of the union of all geodesics which are parallel to $c_v$. By analyticity, $P_v$ is a complete totally geodesic submanifold without boundary. Moreover $P_v$ splits isometrically as $Q \times \mathbb{R}$, where the $\mathbb{R}$-factor corresponds to the geodesic $c_v$.

A connected submanifold $F$ is said to be a higher rank submanifold of $X$ if it is totally geodesic submanifold with the property that every geodesic $c$ in $F$ has a parallel $c'$ in $F$ such that $c \neq c'$. We say $F$ is a maximal higher rank submanifold if it is not properly contained in any other higher rank submanifold. Given a unit vector $v$, $P_v$ is a higher rank submanifold unless rank$(v) = 1$. Schroeder [Sch89b]
gave a complete description of the higher rank submanifolds in $X$. If $X$ is irreducible, then there are exactly three types, $\mathbb{R}^2$, $\mathbb{R}^3$, and $Q \times \mathbb{R}$, where $Q$ is a 2-dimensional visibility manifold. Moreover all the maximal higher rank submanifolds are closed under the action of the fundamental group, i.e., for any maximal higher rank submanifold $F$, $\Gamma_F = \{ \phi \in \Gamma \mid \phi F = F \}$ operates with compact quotient on $F$.

We denote by $\mathcal{V}$ the set of all maximal higher rank submanifolds and by $\mathcal{W}$ the set of all maximal higher rank submanifolds of the form $Q \times \mathbb{R}$, where $Q$ is a 2-dimensional visibility manifold.

Schroeder [Sch89b] proved that modulo $\Gamma$ there are only finite number of maximal higher rank submanifolds. He also gave a description of the possible intersections of the maximal higher rank submanifolds. We now recall them. Given two different maximal higher rank submanifolds $F_1$ and $F_2$.

1. If $F_1 \approx \mathbb{R}^3$, then $F_1 \cap F_2 = \emptyset$.
2. If $F_1 \approx \mathbb{R}^2$ and $F_1 \cap F_2 \neq \emptyset$, then $F_2 \approx \mathbb{R}^2$ and $F_1 \cap F_2$ is a single point.
3. If $F_1 \in \mathcal{W}$ and $F_1 \cap F_2 \neq \emptyset$, then $F_2 \in \mathcal{W}$ and $F_1 \cap F_2$ is a 2-flat.

For any maximal higher rank submanifold $F \in \mathcal{V}$, $\partial_T F$ is a connected subset of $\partial_T X$. If $F \notin \mathcal{W}$, then $\partial_T F$ is isometric either to $S^1$ or $S^2$. If $F \in \mathcal{W}$, then $\partial_T F$ is a graph with two vertices and uncountable number of edges, where the length of each edge is $\pi$. The union of any two edges is a closed geodesic in $\partial_T F$ and it is the ideal boundary of a 2-flat in $F$. The two vertices are called singular points and they are precisely the end points of any singular geodesic, i.e., a geodesic of the form $\{q\} \times \mathbb{R} \subset Q \times \mathbb{R} = F$.

We now describe the possible intersections of the ideal boundaries of the maximal higher rank submanifolds. The only possible intersection is when $F_1, F_2 \in \mathcal{W}$ and $F_1 \cap F_2 = K$, where $K$ is a 2-flat. In such case $\partial_T F_1 \cap \partial_T F_2 = \partial_T K = S^1$.

**Definition 2.1.** A connected component of $\partial_T X$ is called **standard** if it contains a boundary point of a flat and **nonstandard** if it is not a single point and not standard.

The main result of Hummel and Schroeder in [HS98] was to show that $\partial_T X$ contains nonstandard components precisely when there exist two maximal higher rank submanifolds $F_1, F_2 \in \mathcal{W}$ which intersect in a 2-flat, and that all the nonstandard components are intervals of length less than $\pi$. 
3. Fat Triangles

Throughout this section, with the exception of Proposition 3.3, $X$ will denote the universal cover of a closed irreducible 4-dimensional nonpositively curved real analytic manifold which has no nonstandard component in $\partial_T X$.

If $X$ contains any higher rank submanifold, then it is not Gromov $\delta$-hyperbolic space for any $\delta \geq 0$. Easy examples of triangles which are not $\delta$-thin are triangles which lie in a flat in a higher rank submanifold. The goal of this section is to show that all “fat” triangles have to lie near a maximal higher rank submanifold, see Theorem 3.2. We begin with the following definition.

**Definition 3.1.** Let $\delta > 0$ and let $F$ be a maximal higher rank submanifold in $X$, a triangle $\Delta(p, q, r)$ is called $\delta$-thin relative to $F$, if every side is contained in a $\delta$-neighborhood of the union of the other two sides and $F$. This definition resembles the definition of thin triangles relative to flats used by Hruska in [Hru04].

**Theorem 3.2.** There exists some constant $\delta > 0$ such that any triangle in $X$ is $\delta$-thin relative to some maximal higher rank submanifold.

Most of the proofs in this section depends on the following observation.

**Proposition 3.3.** Let $X^n$ be an $n$-dimensional Hadamard manifold, not necessarily analytic. Let $p \in X$ and let $x_n$ be a sequence of geodesic segments such that $x_n, y_n$ converge respectively to $\xi_x, \xi_y \in X(\infty)$. If $d(p, x_ny_n)$ goes to infinity then $Td(\xi_x, \xi_y) \leq \pi$.

**Proof.** The proof is easy and resembles the proof of Lemma 2.1 in [Bal82]. Assume that the proposition is not true. If $Td(\xi_x, \xi_y) > \pi$ then there exists a geodesic $\sigma$ in $X$ such that $\sigma(\infty) = \xi_x$ and $\sigma(-\infty) = \xi_y$. Notice that for any two points $p, q \in X$ the metrics $\angle_p$ and $\angle_q$ define the same topology on $\overline{X} = X \cup \partial_\infty X$. So without loss of generality and to simplify the notation, we assume that $p = \sigma(0)$.

Let $\sigma_n$ be the complete geodesic extending $x_n, y_n$ parameterized such that $\sigma_n(0)$ is the closest point to $\sigma(0)$. It is easy to see that for large values of $n$, $x_n$ and $y_n$ have to be on opposite sides of $\sigma_n(0)$. Therefore $d(\sigma_n(0), p)$ goes to infinity. Fix $R > 0$ and let $q_n$ be the point on $p\sigma_n(0)$ such that $d(p, q_n) = R$. By passing to a subsequence we assume that $q_n$ converges to $q$ and that the rays $\overline{q_nx_n}$ and $\overline{q ny_n}$ converge respectively to two rays starting at $q$ and asymptotic to $p\xi_x$ and $p\xi_y$. In fact these two rays form a complete geodesic, see [Bal82] for details. Therefore $\sigma$ is contained in a flat strip of width $R$. Since $R$ is arbitrary number, the geodesic $\sigma$
There exists a constant \( \pi \) such that if \( F \) is a maximal higher rank submanifold, then either \( d(p, F(p)) > \pi \) or \( d(q, F(q)) > \pi \). This contradicts the assumption that \( F \) is a maximal higher rank submanifold. \( \square \)

For any convex subset \( K \) of a Hadamard manifold \( X \), we denote by \( \pi_K \) the projection map from \( X \) to \( K \).

**Proposition 3.4.** There exists a constant \( \delta > 0 \) which depends only on the analytic 4-manifold \( X \), such that for any maximal higher rank submanifold \( F \), any \( q \in F \), and any \( p \notin F \), \( d(\pi_F(p), \overline{pq}) \leq \delta \). In particular the triangle \( \Delta(p, q, \pi_F(p)) \) is \( \delta \)-thin.

**Proof.** Modulo the action of the fundamental group \( \Gamma \), there are only a finite number of maximal higher rank submanifolds. If the proposition is false, then there exists a maximal higher rank submanifold \( F \) and a sequence of triangles \( \Delta(p_n, q_n, \pi_F(p_n)) \) such that \( d(\pi_F(p_n), \overline{p_nq_n}) \geq n \). Since \( F \) is closed, we could assume that \( \pi_F(p_n) \) is contained in a compact subset of \( F \). By passing to a subsequence we assume that \( p_n, q_n, \pi_F(p_n) \) converge respectively to \( \xi_p \in \partial_{\infty}X \), \( \xi_q \in \partial_{\infty}F \), and \( c \in F \). By Proposition 3.3, \( \xi_p \in \partial_{\infty}F \), which is a contradiction. This finishes the proof. \( \square \)

**Lemma 3.5.** There exists a constant \( D_1 > 0 \) which depends only on the space \( X \) such that if \( F \) is a maximal higher rank submanifold and \( \overline{pq} \) is a geodesic segment, then either \( d(\pi_F(p), \pi_F(q)) \leq D_1 \) or \( d(\overline{pq}, F) \leq D_1 \).

**Proof.** Assume that the statement is false, then there is a maximal higher rank submanifold \( F \) such that for every \( n \in \mathbb{N} \) there exists a geodesic segment \( \overline{p_nq_n} \) such that \( d(\pi_F(p_n), \pi_F(q_n)) \geq n \) and \( d(\overline{p_nq_n}, F) \geq n \). Since \( F \) is closed, we could assume that \( \pi_F(p_n) \) is contained in a compact subset of \( F \). By passing to a subsequence we assume that \( \pi_F(p_n), \pi_F(q_n), p_n, \) and \( q_n \) converge respectively to \( c \in F \), \( \eta \in \partial_{\infty}F \), and \( \xi_p, \xi_q \in \partial_{\infty}X \). It is not hard to see that since \( F \) is a totally geodesic submanifold, \( \xi_p \notin \partial F \). Since \( d(c, \overline{\pi_F(q_n)q_n}) \geq n - 1 \) for large values of \( n \) and \( d(c, \overline{p_nq_n}) \geq n \), using Proposition 3.3 we see that \( \xi_p \) is path connected to \( \eta \in \partial_{\infty}F \), and therefore \( \xi_p \in \partial F \) which is a contradiction. This finishes the proof of the lemma. \( \square \)

**Corollary 3.6.** There exists a constant \( D_2 > 0 \) such that if \( F \) is a maximal higher rank submanifold and \( \overline{pq} \) is a geodesic segment such that \( d(\pi_F(p), \pi_F(q)) > D_2 \) then there are two points \( p', q' \in \overline{pq} \) such that \( d(p', \pi_F(p)) \leq D_2 \) and \( d(q', \pi_F(q)) \leq D_2 \). In particular \( \overline{pq} \) runs within \( D_2 \) distance from the path consisting of the geodesic segments \( \overline{p\pi_F(p)}, \pi_F(p)\pi_F(q), \) and \( \overline{\pi_F(q)q} \).
Proof. Let $D_2 = D_1 + \delta$, where $D_1$ is the constant in Lemma 3.5 and $\delta$ is the constant in Proposition 3.4. Let $r \in \overline{pq}$ be a point which is closest to $F$. By Lemma 3.5, $d(r, \pi_F(r)) \leq D_1$ and therefore the two geodesic segments $\overline{pr}$ and $\overline{p\pi_F(r)}$ are at most $D_1$ apart. By Proposition 3.4, there exists a point on $p\pi_F(r)$ which is $\delta$ close to $\pi_F(p)$. Therefore there is a point on $\overline{pr}$ which is $D_2$ close to $\pi_F(p)$. Similarly there exists a point on $\overline{rq}$ which is $D_2$ close to $\pi_F(q)$ and the corollary follows. \[\Box\]

Now we start the proof of the main result in this section.

Proof of Theorem 3.2. Assume that the statement is false, then there exists a sequence of triangles $\Delta(p_n, q_n, r_n)$ which are not $n$-thin relative to any maximal higher rank submanifold. In particular they are not $n$-thin triangles. Therefore there exists a point $c_n$ on $\overline{p_nq_n}$ such that $d(c_n, \overline{p_nr_n}) \geq n$ and $d(c_n, \overline{qnr_n}) \geq n$. Since $\Gamma$ acts cocompactly on $X$, we assume that $c_n$ are contained in a compact subset. By passing to a subsequence we could assume that $c_n$ converges to a point $c \in X$. Notice that $p_n$, $q_n$, and $r_n$ diverge to infinity. By passing further to a subsequence we could assume that $p_n$, $q_n$, and $r_n$ converge respectively to $\xi_p$, $\xi_q$, and $\xi_r$ in $\partial_\infty X$. By Proposition 3.3, $Td(\xi_p, \xi_r) \leq \pi$ and $Td(\xi_r, \xi_q) \leq \pi$ and therefore they belong to a connected component of the Tits boundary. Clearly $\xi_p \neq \xi_q$, therefore there exists a maximal higher rank submanifold $F$ such that $\xi_p, \xi_q, \xi_r \in \partial_T F$.

We claim that $c$ belongs to $F$. If not, then the geodesic segments $\overline{p_nq_n}$ would converge to a geodesic $\sigma$ passing through $c$ and parallel to $F$. Let $\sigma'$ be a geodesic in $F$ which is parallel to $\sigma$. By the analyticity of $X$, $\sigma$ and $\sigma'$ are contained in a 2-flat. If $F \notin \mathcal{W}$, i.e. isometric to $\mathbb{R}^2$ or $\mathbb{R}^3$ then the parallel set $P_{\sigma'}$ of $\sigma'$ properly contains $F$ which contradicts the maximality of $F$. If $F \in \mathcal{W}$, then $\sigma'$ can not be a singular geodesics in $F$, otherwise the parallel set $P_{\sigma'}$ is 4-dimensional and $X$ would be reducible. So, we assume that $\sigma'$ is not a singular geodesic in $F$. In this case $P_{\sigma}$ is 3-dimensional and therefore has to be of the form $Q \times \mathbb{R}$ and therefore in $\mathcal{W}$. But by the assumption on $X$ that can not happen. Otherwise $\partial_T X$ would have a nonstandard component.

Let $p'_n = \pi_F(p_n)$, $q'_n = \pi_F(q_n)$, and $r'_n = \pi_F(r_n)$. Without loss of generality assume that $d(q'_n, r'_n) \geq d(p'_n, r'_n)$, we might need to pass to a subsequence to guarantee that for every $n$. It is easy to see that $d(c, p'_n)$ and $d(c, q'_n)$ go to infinity since otherwise $p_n$ respectively $q_n$ would not converge to a point in $\partial_T F$. This implies that $d(p'_n, q'_n)$ goes to infinity, and therefore $d(q'_n, r'_n)$ goes to infinity. We need to show that $d(r'_n, p'_n)$ goes to infinity as well. If not then by passing to a subsequence we could assume that $d(p'_n, r'_n) \leq C$, for some constant $C$.
By Corollary 3.6, there exist four points \( t_n, t'_n \in \overline{p_nq_n} \) and \( s_n, s'_n \in \overline{q_nr_n} \) such that \( d(t_n, p'_n), d(t'_n, q'_n), d(s_n, r'_n), \) and \( d(s'_n, q'_n) \) are smaller than or equal \( D_2 \). Therefore \( d(t_n, s_n) \leq 2D_2 + C \). Therefore the two geodesic segments \( \overline{q_ns_n} \) and \( \overline{q_n t_n} \) are at most \( 2D_2 + C \) apart. But this contradicts that \( d(c, q_n r_n) \) goes to infinity. Therefore \( d(p'_n, r'_n) \) goes to infinity as well. Again by Corollary 3.6, there are two points \( t_n, t'_n \in \overline{p_nq_n} \) such that \( d(t_n, t'_n) \) and \( d(t'_n, r'_n) \) are smaller than or equal \( D_2 \). Now it is easy to see that all the triangles \( \Delta(p_n, q_n, r_n) \) are \( D_2 \)-thin relative to \( F \), contradicting the choice of these triangles. This is finishes the proof of the theorem. \( \square \)

Remark 3.7. The proof of Theorem 3.2 shows that the “fat” part of any triangle is close to a maximal higher rank submanifold.

4. The Asymptotic cone of \( X \)

In this section we analyze the asymptotic cone of the Hadamard manifold \( X \). As in Section 3, we still assume that \( X \) is the universal cover of a closed irreducible 4-dimensional nonpositively curved real analytic manifold without nonstandard components in \( \partial_T X \).

We now recall the definition of the asymptotic cone of a metric space \((X, d)\). Fix a non-principle ultrafilter \( \omega \) on \( \mathbb{N} \), a sequence of base points \( x_n \in X \), and a sequence of rescalling factors \( \lambda_n \) such that \( \omega \)-\( \lim \lambda_n = \infty \). The based ultralimit of \((X, \lambda_n^{-1} \cdot d), x_n)\) decomposes generically into uncountable number of components. The asymptotic cone is defined to be the component of the ultralimit containing the base point. We refer the reader to [KL95, KL97b] for further details.

We recall two well known facts about the asymptotic cone which we will use. For any non-principle ultrafilter \( \omega \), \( \text{Cone}_\omega(\mathbb{R}^n) = \mathbb{R}^n \). If \( X \) is a quasi-homogeneous Gromov hyperbolic space with uncountable number of ideal boundary points, then \( \text{Cone}_\omega(X) \) is an \( \mathbb{R} \)-tree with uncountable branching.

If \( X \) is a Hadamard manifold, then \( \text{Cone}_\omega(X) \) is a Hadamard space. Any sequence of flats in \( X \) gives rise to a flat in the ultralimit of \((X, \lambda_n^{-1} \cdot d), x_n)\). If the distance between that flat and the base point in the asymptotic cone is finite, then the flat is in \( \text{Cone}_\omega(X) \). The goal of this section is to show that these are the only flats which appear in \( \text{Cone}_\omega(X) \).

Our analysis is similar to the analysis done by Kapovich and Leeb in [KL95]. And we will use some of their results. We will often mention the results without proof and refer the reader to their paper for the proofs.
Given a maximal higher rank submanifold \( F \in \mathcal{V} \), if \( F \approx \mathbb{R}^2, \mathbb{R}^3 \), or \( Q \times \mathbb{R} \), the sequence \( F_n = F \) has a limit \( \mathbb{R}^2, \mathbb{R}^3 \), or \( T \times \mathbb{R} \), where \( T \) is an \( \mathbb{R} \)-tree, in \( \text{Cone}_\omega(X) \).

The group \( \Gamma^* = \prod_n \Gamma \) acts by isometries on \( \text{Cone}_\omega(X) \). Recall that up to the action of \( \Gamma \) there is only a finite number of maximal higher rank submanifolds, we denote them by \( K_1, \ldots, K_r \). Any sequence \( F_n \) of maximal higher rank submanifolds in \( \mathcal{V} \) gives rise to a partition \( A_1 \sqcup \cdots \sqcup A_r \) of \( \mathbb{N} \) as follows: \( n \in A_s \) if \( F_n = \phi(K_s) \) for some isometry \( \phi \in \Gamma \). By basic properties of ultrafilters, there is exactly one subset \( A_s \in \omega \). Therefore the sequence \( F_n \) and the subsequence corresponding to \( A_s \) give rise to the same limit in \( \text{Cone}_\omega(X) \). Therefore without loss of generality we could assume that for a fixed \( K_s \), \( F_n = \phi_n(K_s) \), where \( \phi_n \in \Gamma \). Since \( \Gamma^* \) acts by isometries on \( \text{Cone}_\omega(X) \), it is easy to see that the limit of \( F_n \) is \( \phi^*(\omega\text{-lim}K_s) \) where \( \phi^* = (\phi_1, \phi_2, \ldots) \in \Gamma^* \).

We denote by \( \mathcal{F} \) the limits in \( \text{Cone}_\omega(X) \) of all sequences of maximal higher rank submanifolds of \( X \). The above discussion shows the following,

**Proposition 4.1.** Every element in \( \mathcal{F} \) is isometric to \( \mathbb{R}^2, \mathbb{R}^3 \), or \( T \times \mathbb{R} \), where \( T \) is an \( \mathbb{R} \)-tree.

**Definition 4.2.** Let \( X \) be a \( \text{CAT}(0) \) space. A triangle \( \Delta(p_1, p_2, p_3) \) in \( X \) is called an open triangle if \( p_1, p_2, p_3 \) are different points, and \( \overline{p_i p_j} \cap \overline{p_i p_k} = \{p_i\} \), where \( i, j, k \in \{1, 2, 3\} \) and \( i \neq j \neq k \).

The following lemma is a rephrase of Proposition 4.3 in [KL95].

**Lemma 4.3.** The asymptotic cone \( \text{Cone}_\omega(X) \) satisfies the following properties:
1. Every open triangle is contained in some \( F \in \mathcal{F} \).
2. Any two different elements \( F, F' \in \mathcal{F} \) have at most one point in common.

**Proof.** The proof of the first part is identical to the proof of the first part in Proposition 4.3 in [KL95], and the proof carries over to our setting.

Now we give the proof of the second part. Notice that every \( F \in \mathcal{F} \) is a convex subset of \( \text{Cone}_\omega(X) \). Assume that \( F \) and \( F' \) intersect in two different points \( x \) and \( y \). Therefore \( \overline{xy} \subset F \cap F' \). Let \( F_n \) be a sequence of maximal higher rank submanifolds such that \( \omega\text{-lim}F_n = F \). Let \( z \in F' \) be any point such that the triangle \( \Delta(x, y, z) \) is an open triangle. The goal is to show that \( z \in F \). Choose a sequence of triangles \( \Delta_n = \Delta(x_n, y_n, z_n) \) in \( X \) such that they converge to \( \Delta(x, y, z) \) in the asymptotic cone. We can choose \( x_n, y_n \in F_n \). Let \( z'_n = \pi_{F_n}(z_n) \) be the projection of \( z_n \) to \( F_n \).

We claim that the two sequences \( (z_n)_{n \in \mathbb{N}} \) and \( (z'_n)_{n \in \mathbb{N}} \) represent the same point \( z \) in \( \text{Cone}_\omega(X) \), which implies that \( z \in \omega\text{-lim}F_n = F \). To see that, assume that
\((z'_n)_{n \in \mathbb{N}}\) converges to \(z' \in \text{Cone}_\omega(X)\) and \(z \neq z'\). Using Proposition 3.4, for every \(n \in \mathbb{N}\), we can choose two points \(s_n \in \overline{zx_n}\) and \(t_n \in \overline{z'y_n}\) within distance at most \(\delta\) from \(z'_n\), where \(\delta\) is the constant given in Proposition 3.4. Clearly, \((z'_n)_{n \in \mathbb{N}}, (s_n)_{n \in \mathbb{N}}\), and \((t_n)_{n \in \mathbb{N}}\) represent the same point \(z' \in \text{Cone}_\omega(X)\) which implies that \(\{z\} \nsubseteq z'z' \subset \overline{zx} \cap \overline{zy}\). This is a contradiction since we assumed that \(\Delta(x, y, z)\) is an open triangle.

Notice that the set of points in \(F'\) where the triangle \(\Delta(x, y, z)\) is open is an open dense subset of \(F'\), for the three different possibilities of \(F'\) given by Proposition 4.1. By a continuity argument we see that \(F' \subset F\). Similarly we show that \(F \subset F'\) which finishes the proof. \(\square\)

**Remark 4.4.** The proof of Lemma 4.3 rules out the possibility that the limit of a sequence of maximal higher rank submanifolds which are isometric to \(\mathbb{R}^2\) is contained in the limit of a sequence of maximal higher rank submanifolds of higher dimensions.

For every element \(F \in \mathcal{F}\), we denote by \(\pi_F: \text{Cone}_\omega(X) \rightarrow F\) the projection map which is well defined and distance non-increasing since \(F\) is a convex subset of \(\text{Cone}_\omega(X)\) which is CAT(0) space. Now we state Lemma 4.4 and Lemma 4.5 from [KL95] in our new setting where 2-flats are replaced by maximal higher rank submanifolds.

**Lemma 4.5.** Let \(\gamma: I \rightarrow \text{Cone}_\omega(X)\) be a curve in the complement of \(F\). Then \(\pi_F \circ \gamma\) is constant.

**Lemma 4.6.** Every embedded closed curve \(\gamma \subset \text{Cone}_\omega(X)\) is contained in some element \(F \in \mathcal{F}\).

The proofs of these two lemmas in [KL95] carry over to our new setting. The only ingredient used there was that the limits of 2-flats in the asymptotic cones of the universal covers of certain Haken manifolds, satisfy the two properties in Lemma 4.3. The proofs carry over to our new sitting after replacing 2-flats by maximal higher rank submanifolds and using Lemma 4.3.

Lemma 4.3 easily implies that the only flats in \(\text{Cone}_\omega(X)\) are the ones which are subsets of elements of \(\mathcal{F}\). While Lemma 4.6 shows that every embedded disk of dimension at least 2 is contained in an element of \(\mathcal{F}\).

5. Proof of the main theorem

In this section we give the proof of Theorem 1.1. We assume that \(\partial_T X_1\) has a nonstandard component and \(\partial_T X_2\) does not have any nonstandard components. If
\( f: X_1 \rightarrow X_2 \) is a quasi-isometry, then it induces a map \( \text{Cone}_\omega(f): \text{Cone}_\omega(X_1) \rightarrow \text{Cone}_\omega(X_2) \) which is bi-Lipschitz homeomorphism. Since \( \partial_T X_1 \) contains a nonstandard component, there exist two maximal higher rank submanifolds \( H_1 = Q_1 \times \mathbb{R} \) and \( H_2 = Q_2 \times \mathbb{R} \) which intersect in a 2-flat, which we denote by \( K \). In the asymptotic cone \( Q_1 \times \mathbb{R} \) and \( Q_2 \times \mathbb{R} \) give rise to two convex subsets \( W_1 = T_1 \times \mathbb{R} \) and \( W_2 = T_2 \times \mathbb{R} \), where \( T_1 \) and \( T_2 \) are real-trees. The real-factors of \( W_1 \) and \( W_2 \), which we call the singular directions are different. The 2-flat \( K \) gives rise to a 2-flat, which we denote by \( L \), in the asymptotic cone \( \text{Cone}_\omega X_1 \). Clearly \( W_1 \cap W_2 \supseteq L \). The goal of the following proposition is to show that the intersection of \( W_1 \) and \( W_2 \) is precisely \( L \).

**Proposition 5.1.** Using the above notation, \( W_1 \cap W_2 = L \).

**Proof.** Let \( z \in W_1 \cap W_2 \). Choose two sequences \( (x_n)_{n \in \mathbb{N}} \subset H_1 \) and \( (y_n)_{n \in \mathbb{N}} \subset H_2 \) which represent the point \( z \) in \( \text{Cone}_\omega(X_1) \). This implies that \( \omega \lim d(x_n, y_n) / \lambda_n = 0 \). Since \( H_1 \) and \( H_2 \) are orthogonal to each other at the intersection (see Lemma 3.4 in [Sch89b]), therefore \( \pi_{H_i}(y_n) \in K = H_1 \cap H_2 \). Since \( d(\pi_{H_1}(y_n), y_n) \leq d(x_n, y_n) \), it easy to see that \( (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}, \) and \( (\pi_{H_1}(y_n))_{n \in \mathbb{N}} \) represent the same point in \( \text{Cone}_\omega(X_1) \). This shows that \( z \in L \), which finishes the proof. \( \square \)

Since \( \partial_T X_2 \) does not contain any nonstandard components, Section 4 describe all the flats in \( \text{Cone}_\omega(X_2) \). We are finally ready to start the proof of the main theorem.

**Proof of Theorem 1.1.** Assume that \( f: X_1 \rightarrow X_2 \) is a quasi-isometry. The induced map \( \text{Cone}_\omega(f): \text{Cone}_\omega(X_1) \rightarrow \text{Cone}_\omega(X_2) \) is a bi-Lipschitz homeomorphism. Assume that \( T_i \times \mathbb{R} \subset \text{Cone}_\omega(X_1) \), for \( i = 1, 2 \), where \( T_i \) is an real-tree such that \( T_1 \times \mathbb{R} \cap T_2 \times \mathbb{R} = \mathbb{R}^2 \) as mentioned above. By Lemma 4.6 the image of each flat in \( T_i \times \mathbb{R} \), for \( i = 1, 2 \), has to be contained inside some element \( F \in \mathcal{F} \). For any two 2-flats in \( T_i \times \mathbb{R} \), there exists a third flat which intersects both of them in half planes. That shows that the images of \( T_i \times \mathbb{R} \), for \( i = 1, 2 \), have to be contained in some \( F \in \mathcal{F} \). We need to show this is not possible for the three different types of \( F \), namely \( \mathbb{R}^2, \mathbb{R}^3 \), and \( T \times \mathbb{R} \).

**Case 1:** \( F = \mathbb{R}^2 \). This case is easy, a bi-Lipschitz embedding of any 2-flat in \( T_1 \times \mathbb{R} \) has to be onto \( F \), contradicting that \( \text{Cone}_\omega(f) \) is a bijection.

**Case 2:** \( F = \mathbb{R}^3 \). Since \( T_i = \text{Cone}_\omega(Q_i) \) where \( Q_i \) are visibility 2-manifolds (with cocompact action) and therefore hyperbolic, then \( T_i \) branches uncountably many
times at each point. Now it is not hard to see that an \( \mathbb{R} \)-tree which branches uncountably many times at each point cannot be bi-Lipschitz embedded in \( \mathbb{R}^3 \). One easy way to see that is to fix a base point \( p \) in the tree, and take a sequence of points \( p_i \) such that \( d(p, p_i) = 1 \) and \( d(p_i, p_j) = 2 \), for \( i \neq j \). But Cone\(_\omega(f)(p_i)\) is contained in a compact subset of \( \mathbb{R}^3 \) and therefore by passing to a subsequence it will converge in \( \mathbb{R}^3 \), which contradicts that Cone\(_\omega(f)\) is bi-Lipschitz homeomorphism.

**Case 3:** \( F = T \times \mathbb{R} \). First we recall Lemma 2.14 in [KL95] which states that if \( T \) is a metric tree then the image of any bi-Lipschitz embedding \( \pi: \mathbb{R}^2 \to T \times \mathbb{R} \), is a flat in \( T \times \mathbb{R} \). Notice that any two different flats in \( F = T \times \mathbb{R} \) either do not intersect, intersect in a line, a strip, or a half plane.

Denote by \( \alpha_1, \alpha_2 \) the unique geodesics in \( T_1, T_2 \) respectively which satisfies that \( \alpha_1 \times \mathbb{R} = \alpha_2 \times \mathbb{R} \) is the unique 2-flat which is the intersection of \( T_1 \times \mathbb{R} \) and \( T_2 \times \mathbb{R} \). Parameterize \( \alpha_1 \) and \( \alpha_2 \) such \( p = \alpha_1(0) = \alpha_2(0) \) is the unique intersection point. Choose a geodesic \( \beta_1 \) in \( T_1 \) such that, \( \beta_1 \) follows \( \alpha_1 \) until they reach the point \( p \) and then branches at \( p \). Choose a geodesic \( \beta_2 \) in \( T_2 \) such the intersection of \( \alpha_2 \) and \( \beta_2 \) is the unique point \( p \). Consider the two flats \( \beta_1 \times \mathbb{R} \subset T_1 \times \mathbb{R} \) and \( \beta_2 \times \mathbb{R} \subset T_2 \times \mathbb{R} \). It is not hard to see that these two planes intersect in a half line. But by Lemma 2.14 in [KL95] mentioned above, the image of these two planes in \( F \) are two flats. This is a contradiction since a half line is not bi-Lipschitz homeomorphic to either a line, a strip or a half plane. □

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