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Parallel Wavelet Schemes for Images
How to make the wavelet transform friendly to parallel architectures

Abstract In this paper, we introduce several new schemes for calculation of discrete wavelet transforms of images. These schemes reduce the number of steps and, as a consequence, allow to reduce the number of synchronizations on parallel architectures. As an additional useful property, the proposed schemes can reduce also the number of arithmetic operations. The schemes are primarily demonstrated on CDF 5/3 and CDF 9/7 wavelets employed in JPEG 2000 image compression standard. However, the presented method is general and it can be applied on any wavelet transform. As a result, our scheme requires only two memory barriers for 2-D CDF 5/3 transform compared to four barriers in the original separable form or three barriers in the non-separable scheme recently published. Our reasoning is supported by exhaustive experiments on high-end graphics cards.

Keywords Discrete wavelet transforms, Image processing, Parallel architectures

1 Introduction

The two-dimensional discrete wavelet transform (DWT) is a signal-processing transform suitable as a basis for sophisticated compression algorithms. For example, JPEG 2000 is an image coding system based on such compression technique. This paper focuses on the Cohen-Daubechies-Feauveau (CDF) 5/3 and 9/7 wavelets [2], which are often used for image compression. However, the methods are general and they are not limited to any specific type of transform. Of course, plenty of other applications are built over the discrete wavelet transform.

The one-dimensional discrete wavelet transform has undergone a gradual development in the last few decades. Probably the most important advance is the discovery of a factoring algorithm [3] referred to as the lifting scheme. In this context, the discrete wavelet transform or two band subband filtering can be represented by a polyphase matrix. The lifting scheme algorithm decomposes any wavelet transform with finite filters into a finite sequence of lifting steps, while reducing the number of arithmetic operations. The decomposition corresponds to a factorization of the polyphase matrix filters into elementary matrices. The resulting coefficients of 1-D transform are formed in two subbands. The subbands correspond to low-pass (L) and high-pass (H) filtered subsampled variants of the original signal.

In case of two-dimensional transform [15], one level of the transform can be realized using the separable decomposition scheme. In this scheme, the coefficients are evaluated by successive horizontal and vertical 1-D filtering, resulting in four disjoint groups (LL, HL, LH, and HH subbands). A naive algorithm of 2-D transform computation directly follows the horizontal and vertical filtering loops. As a consequence, the number of elementary polyphase matrices is doubled.

Unfortunately, this separable computation does not reflect the requirements of the parallel architectures where the scheme will need twice as many synchronizations. Such synchronizations often form a bottleneck of the overall calculation. State-of-the-art algorithms fuse the horizontal and vertical loops into a single one, which results in the single-loop approach. However, the number of the elementary polyphase matrices and thus the number of memory barriers remain unaffected.

To solve the outlined issue, we propose several novel spatial lifting structures computing the 2-D discrete wavelet transform with reduced number of memory barriers. These lifting structures are presented in the order in which they were gradually derived. The presented work is accompanied by exhaustive performance experiments.

A typical representative of parallel architectures is the graphics processing unit (GPU) capable of executing a general-purpose program. Actually, this is the architecture used to evaluate the performance of algorithms.
presented in this paper. We have employed OpenCL language for writing underlying implementations. These were then subject of performance measurements on significant graphics cards of two biggest vendors.

The rest of the paper is organized as follows. Section Related Work presents the theory in the necessary level of detail. This theory includes the lifting scheme basics and the spatial lifting structures recently proposed. Subsequent Section Proposed Schemes derives the new spatial lifting structures. Additionally, Section Improvements presents a simple trick proposed in order to reduce the number of arithmetic operations. Section Evaluation and Section Performance offer a thorough performance evaluation. Finally, Section Conclusions summarizes the paper.

2 Related Work

In this paper, we employ the well-known z-transform notation for the description of FIR filters. The transfer function of the FIR filter \( h_k \) is a Laurent polynomial defined as

\[
H(z) = \sum_{k=k_0}^{k_1-1} h_k z^{-k},
\]

where \( k_0 \) denotes the smallest and \( k_1 - 1 \) denotes the largest integer number \( k \) for which \( h_k \) is non-zero. The degree of a Laurent polynomial \( H(z) \) is defined as \( |H(z)| = k_1 - k_0 - 1 \). Analogously, the transfer function of the two-dimensional FIR filter \( h_{m,n,k} \) is a bivariate Laurent polynomial defined as

\[
H(z_m,z_n) = \sum_{k_m=k_0,m=k_0}^{k_1-1} \sum_{k_n=k_0,n=k_0}^{k_1-1} h_{m,k_n} z_m^{-k_m} z_n^{-k},
\]

where \( m \) refers to the horizontal axis and \( n \) to the vertical one. Moreover, to keep consistency with other papers, the \( H^*(z_m,z_n) = H(z_m,z_n) \) denotes a filter transposed to the \( H(z_m,z_n) \). For simplicity, we have made a small abuse of notation. Instead of the full notation \( H(z_m,z_n) \), we only use a shortened labeling, such as \( H \). Finally, we work with \( 2 \times 2 \) and \( 4 \times 4 \) matrices of Laurent polynomials. These are usually referred to as the polyphase matrices. The \( 2 \times 2 \) matrices refer to the 1-D systems, whereas the \( 4 \times 4 \) to the 2-D ones. For simplicity, we only use a shortened labeling for matrices as well. The superscript \( T \) denotes the vector or matrix transposition.

The discrete wavelet transform has undergone a gradual development [16] in the last few decades. First, S. Mallat [15] demonstrated the multi-scale wavelet decomposition computed with a pyramidal algorithm based on convolutions with quadrature mirror filters. In detail, the discrete wavelet transform splits the input signal \( x \) into to components \( L \) and \( H \), each subsampled at a factor of 2. Both these components can be computed by the discrete convolution with two FIR filters \( G_0(z) \) and \( G_1(z) \) followed by the subsampling. However, such computation is usually not the fastest computational scheme. The transform can also be represented by the polyphase matrix [19]. Using this representation, the input signal is initially split into the \( L \) and \( H \) components. No calculation is performed so far. After such splitting, the DWT

\[
y = M x.
\]

is described by the \( 2 \times 2 \) matrix \( M \) mapping the initial components

\[
x = \begin{bmatrix} L & H \end{bmatrix}^T
\]

onto the resulting ones

\[
y = \begin{bmatrix} L & H \end{bmatrix}^T.
\]

The polyphase matrix is initially assembled as a polynomial matrix

\[
M = \begin{bmatrix} G_{1,o} & G_{1,e} \\ G_{0,o} & G_{0,e} \end{bmatrix},
\]

where subscript \( e \) refers to the even coefficients, whereas \( o \) refers to the odd coefficients.

As a next step, W. Sweldens [3, 20] showed how any discrete wavelet transform can be decomposed into a sequence of simple filtering steps. These steps are referred to as the lifting steps and the scheme is known as the lifting scheme. The lifting scheme reduces the number of arithmetic operations up to 50\%. The lifting steps occur in \( K \) pairs. The first step is referred to as the predict and the second one to as the update. It may happen that the very first step of the lifting scheme is missing and the sequence of steps starts with the update step. Usually, the very last step has a different form compared to all the others. This one is then called the scaling step.

\[
M = \begin{bmatrix} \zeta & 0 \\ 0 & 1/\zeta \end{bmatrix} \prod_{k=K}^{0} \begin{bmatrix} 1 & U^{(k)} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & P^{(k)} \\ 0 & 1 \end{bmatrix},
\]

where \( \zeta \) is a non-zero scaling factor, \( P^{(k)} \) is the \( k \)-th predict convolution operator, and \( U^{(k)} \) is the \( k \)-th update convolution operator. In this paper, we focus on a single pair of the lifting steps. We thus omit the \((k)\) superscript. We also omit the scaling step, as applying this step is trivial.

In parallel environments [13], the processing of a single or several adjacent signal samples is mapped to independent processing units, commonly referred to as the threads. To avoid race conditions (the behavior where the output is dependent on the sequence or timing of other threads), the threads must use some type of synchronization method. In this paper, we will consider the use of memory barriers. When we return to the lifting scheme, these barriers are usually required before each of the individual lifting steps. However, a certain form of
This can be classified as a separable scheme. As the convolution is the linear operator, horizontal and vertical
transfers are defined like follows. Let the two-dimensional counterparts of the operators
are defined as

\[ x = \begin{bmatrix} \text{LL} & \text{HL} & \text{LH} & \text{HH} \end{bmatrix}^T \]  

onto the final ones

\[ y = \begin{bmatrix} \text{LL} & \text{HL} & \text{LH} & \text{HH} \end{bmatrix}^T \]  

Similarly to the 1-D case, this can be written as

\[ y = N_{P,U} \cdot x, \]

where \( P, U \) are 1-D predict and update convolution operators. Please notice the included initial barrier. We will further label this scheme as Polyphase.

To define the 2-D polyphase matrices, we must first migrate the predict and update operators into two dimensions. Coupled together with filter transposition defined above, the two-dimensional counterparts of the operators are defined like follows.

\[ \begin{bmatrix} P^* & U^* \end{bmatrix} = \begin{bmatrix} P(z_m) & U(z_n) \\ U(z_m) & P(z_n) \end{bmatrix} \]  

Roughly speaking, the \( P, U \) denote the filters oriented along the horizontal axes, whereas the \( P^*, U^* \) denote the filters oriented along the vertical one.

Following the Mallat’s scheme, the predict and update lifting steps are applied in both directions sequentially. This can be classified as a separable scheme. As the convolution is the linear operator, horizontal and vertical steps can be arbitrary interleaved. We will consider the baseline formulation of this scheme as follows. The predict steps are always preceding the update ones. Such separable scheme can be formally described by

\[ y = S_U^H \cdot S_U^H \cdot T_P^H \cdot T_P^H \cdot x, \]

where the individual matrices are defined as follows. Let us mention a short comment on the matrix notation used. For example, the matrix \( T_P^H \) is parameterized by the \( P \) polynomial. Further in the text, the same matrix appears parameterized by different polynomials, which is completely valid. As it can be expected, the matrix \( T_P^H \) definition is not repeated for such case. For better understanding, the corresponding signal-processing block diagram is shown in Fig. 1. For the CDF 5/3 wavelet, these steps are also graphically illustrated in Fig. 2. Note that for readers not familiar with such signal-processing notations, a relationship of the block and data-flow diagrams is explained in Appendix.

\[ T_P^H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ P & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & P & 1 \end{bmatrix} \]
Please note the barriers in between each of the lifting steps. In total, four barriers are required for each pair of the original 1-D lifting steps. We will further label this scheme as Sweldens.

Contemporary approaches on parallel architectures most commonly reflects this separable Sweldens scheme. Exceptionally, the Convolution scheme is employed. Considering the independent horizontal and vertical filtering steps, several different strategies of 2-D DWT implementation can be used. These strategies can be divided into three groups – row–column, block-based, and pipelined methods. The row–column methods process all of the horizontal filtering steps prior to the vertical ones. The row–column method applied on the entire 2-D image was used for instance in [1; 4–7; 21; 22]. In some papers, the transition between the horizontal and vertical stage is accompanied with data transposition. The pipelined methods was used, e.g., in [13] and [14]. These methods uses moving window for the vertical part of the transform. However, the horizontal and vertical part remain separated. The block-based methods were used, e.g., in [17] and [1]. The transform is tiled into blocks, in which the horizontal and vertical processing still remain separated. However, between these parts, the data remain loaded in the local memory (making them faster accessible).

Going back to the Polyphase scheme, the polyphase matrix

\[
N_{P,U} = \begin{bmatrix}
V^*V & V^*U & U^*V & U^*U \\
V^*P & V^* & U^*P & U^* \\
P^*V & P^*U & V & U \\
P^*P & P^* & P & 1
\end{bmatrix}
\]  

(17)

can expressed using the auxiliary polynomial \( V = PU + 1 \). The matrix can be obtained as the product of individual matrices of the Sweldens scheme. In this scheme, it is no longer possible to distinguish the vertical and horizontal filtering. Only an initial barrier is required for this scheme. Unfortunately, the number of arithmetic operations has grown in proportion to the square of filter sizes. For the CDF 5/3 wavelet, these operations are illustrated in Fig. 4. The corresponding generic signal-processing diagram is shown in Fig. 3.

Recently, Iwahashi et al. [8–10] presented the non-separable lifting scheme, consisting of three spatial lifting steps. As in the previous case, it is not possible to distinguish the vertical and horizontal filtering. The three steps can be described as follows. Initially, a 2-D lifting step leading to the computation of the HH coefficient is performed. This step corresponds to a spatial predict convolution operator. This is followed by parallel computation of the HL and LH coefficients, using the original 1-D predict and update filters. In the third step, the LL coefficient is computed using another 2-D filter. The last step can be understood as a spatial update operator. In the matrix notation, the scheme can be defined as

\[
y = S_U \circ R_P \circ T_P \circ x,
\]  

(18)
Fig. 5 Block diagram of the Iwahashi scheme. The dashed vertical lines indicate barriers.

Fig. 6 2-D dataflow diagram, CDF 5/3 wavelet, Iwahashi lifting scheme. The solid box corresponds to the output coefficients.

where the individual matrices are defined as follows. For the CDF 5/3 wavelet, the individual steps are illustrated in Fig. 6. The signal-processing diagram is shown in Fig. 5.

\[
T^f_P = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
PP^* & P^* & P & 1
\end{bmatrix}
\]  \hspace{1cm} (19)

\[
R^f_{P,U} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & U^* \\
P^* & 0 & 1 & U \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  \hspace{1cm} (20)

\[
S^f_U = \begin{bmatrix}
1 & U & U^* & -UU^* \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  \hspace{1cm} (21)

Three barriers are required in between these steps. As for the Polyphase scheme, the number of arithmetic operations increased proportionally to the square of filter sizes. However, the total number of operations is significantly lower. We will further label this scheme as Iwahashi.

When we compare the separable Sweldens and non-separable Iwahashi schemes, some findings becomes obvious at first glance. The number operations tends to be considerably smaller for the separable case. On the other hand, the number of memory barriers in the non-separable scheme was reduced to 75% (from four to three barriers). The Polyphase scheme stands apart from these two schemes. It needs only an initial memory barrier. Unfortunately, the number of arithmetic operations is unreasonably large. This is caused by the number of non-zero elements in the corresponding polyphase matrix as well as by the degree of the longest filter \(V\). For clarification, the product of a Laurent polynomial of degree \(|P(z)|\) and a Laurent polynomial of degree \(|U(z)|\) is a Laurent polynomial of degree \(|P(z)| + |U(z)|\). Finally, the Convolution scheme employing four 2-D filters is even worse in terms of the operations. Anyway, only an initial memory is required here as well. Detailed quantitative comparison is provided in Section Evaluation.

When we consider the linearity of the convolution and the dependencies in between the individual lifting steps, several gaps can be inferred in the schemes described above. Recombining the operations into a new form could lead to the removal of unnecessary barriers. Actually, exactly this idea is investigated in the following section, in which several novel 2-D schemes are proposed.

Since this work is based on our previous work in \[11\] and \[12\], it should be explained what the difference between this work and \[11; 12\] is. In \[11\], we have presented a block-based method employing a scheme foregoing the schemes proposed in this paper. Moreover, in \[12\], we present a particular implementation of one of the schemes presented here, designed for CDF 5/3 wavelet. Unlike both of these works, the schemes presented in this paper are defined by general predict and update operators.

3 Proposed Schemes

In this section, we reassemble the polyphase matrices known so far in order to obtain the schemes suitable for parallel architectures. All of the schemes discussed here are general and they can be used for any discrete wavelet transform.

When we take a detailed look at the original 1-D lifting scheme, a certain pattern can be identified in the predict and update steps. Particularly, the predicts transmit data from \(L\) into \(H\) samples, whereas the updates transmit data from \(H\) into \(L\). The transmission can be viewed from two perspectives – the data flows out from a source component (similarly to an explosion); or the data flows in into a destination component (an implosion). As it can be expected, the Sweldens scheme exactly follows this pattern, since this scheme is a mere extension of 1-D lifting into two dimensions. Roles of source and destination samples properly turns during four lifting steps (horizontal and vertical, predict and update). This procedure can be also seen as a data transmission in direction from \(LL\) into \(HH\) component (using 1-D predicts), and a transmission from \(HH\) into \(LL\) one (using updates). The
HL and LH components are not relevant in this view. The situation is clearly visible in Fig. 1. In contrast to this, the Iwahashi scheme has a different structure. The leading step transmit data into HH component (using predicts), while the trailing one transmit them into LL one (updates). However, no exclusive source components can be identified in this case. The remaining step in the middle is not relevant. See the block diagram in Fig. 5. Regarding to the perspectives outlined above, the Iwahashi scheme can be classified as an implosive one. However, this is not the only three-step possibility (two-steps scheme is discussed below in the text). Similar scheme can be formulated using data explosions instead of the implosions. Particularly, the LL component spreads the data into its neighborhood during the predict step, whereas the data flows out from the HH component in the update step. No exclusive destination components can be identified here as well. Again, the step in the middle is not relevant. For further purposes, we will label this newly proposed scheme as Explosive. The steps for the CDF 5/3 wavelet are also illustrated in Fig. 8. The block diagram is shown in Fig. 7. Formally, the scheme can be defined as

\[ y = S_E^T R_{E,U}^T T_P^E x, \]  

where the individual matrices follows. Three barriers are required, as in the case of the Iwahashi scheme.
Table 1  CDF 5/3 wavelet. Shapes of spatial lifting steps for selected schemes. The step in the middle raised from the combination of the original predict and update steps. Illustrative purpose only.

where the $S_U$ and $T_P$ are defined as follows. For the CDF 5/3 wavelet, the scheme is graphically illustrated in Fig. 9. Moreover, the hypothetical signal-processing diagram is shown in Fig. 10.

$$T_P = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & P & P^* & 1
\end{bmatrix}$$  \hspace{1cm} (27)

$$S_U = \begin{bmatrix}
1 & U & U^* & UU^* \\
0 & 1 & 0 & U^* \\
0 & 0 & 1 & U \\
0 & 0 & 0 & 1
\end{bmatrix}$$  \hspace{1cm} (28)

The total number of operations remained the same as for the Iwahashi scheme. However, the number of the explicit barriers has been reduced to only two. This is a crucial contribution of our work. Further in the paper, we will label this scheme as Monolithic. One can easily verify the correctness of the proposed scheme by comparing the product $S_UT_P$ to the matrix $N_{P,U}$ of the Polyphase scheme.

A comparison of the shapes for selected schemes can be found in Table 1. Regarding the Polyphase scheme, no spatial predict nor update step can be identified in its calculation.

Two simple observations can be made from the scheme presented so far. The Sweldens scheme requires the lowest number of operations. In contrast to this, the non-separable scheme proposed above requires the lowest number of memory barriers. Combining these two observations together, new schemes can be formed. This possibility is investigated below.

4 Improvements

Additionally, we have made another observation. The operation composed as a product of monomials with the exponent of $z_n$ and $z_n$ being equal to zero (i.e. scalars) never touch the coefficients belonging to the surrounding threads. As the convolution is the linear operation, this monomial can be detached from the original operator and subsequently calculated using the Sweldens scheme. This scheme has a minimal number of arithmetic operations. The rest of the original polynomial shall be computed using different scheme, according to suitability for a particular platform.

In more detail, we have split the original filters into two halves as $P = P_0 + P_1$, and $U = U_0 + U_1$, where

Fig. 11  2-D dataflow diagram, CDF 5/3 wavelet, common steps for all improved schemes.
Fig. 12 2-D dataflow diagram, CDF 5/3 wavelet, Iwahashi* scheme. The solid box corresponds to the output coefficients.

Fig. 13 2-D dataflow diagram, CDF 5/3 wavelet, Explosive* scheme. The solid box corresponds to the output coefficients.

Fig. 14 2-D dataflow diagram, CDF 5/3 wavelet, Monolithic* scheme. The solid box corresponds to the output coefficients.

Fig. 15 2-D dataflow diagram, CDF 5/3 wavelet, Polyphase* scheme. The solid box corresponds to the output.

\[ y = S_U^V S_U^H S_{U_1}^V | R_{P_1,U_1}^L | T_{P_1}^L | T_{P_0}^V T_{P_0}^H x, \]  
\[ y = S_U^V S_U^H S_{U_1}^E | R_{P_1,U_1}^E | T_{P_1}^E | T_{P_0}^V T_{P_0}^H x, \]

Also in this case, the number of barriers remains the same as for the original scheme. Analogously to the previous case, this scheme will be referred to as Explosive*. The dataflow diagram for the CDF 5/3 wavelet is shown in Fig. 13.

As a next step, consider a new construction based on the Monolithic scheme. The same trick can be utilized here as well. In the matrix notation, the newly composed scheme is composed as

\[ y = S_U^V S_U^H S_{U_1}^E | T_{P_0}^V T_{P_0}^H T_{P_1}^E x, \]  

where the individual matrices are defined above in the text. For the CDF 5/3 wavelet, this scheme is graphically illustrated in Fig. 14. We will label this scheme as Monolithic*.

The schemes described above are formed such a way that the first lifting step (comprising P₁, U₁) after the barrier access coefficients of the surrounding threads. The subsequent or preceding steps (comprising P₀, U₀) read only the local coefficients, which are not accessed by the other threads. Then, the whole sequence can be repeated. Of course, the calculation of transforms consisting of several pairs of lifting steps comprises several such connected schemes.

Finally, we have decided to remove the last explicit barrier, leaving only the initial one. The trick lies in the appropriate combination of the Sweldens and Polyphase schemes. This time, the non-separable parts are merged into a joint step N_{P_1,U_1}. This step is inherently preceded by a barrier. In case of an initial pair of lifting steps, the
barriers at the beginning of the computation is used for this purpose. In more detail, after the input data has been read by each computation unit, the calculations \( T_{P_0} T_{P_0}^H \) are immediately performed. At this point, the intermediate results can be appropriately shared. This is followed by the initial barrier. Regarding the transforms consisting of several such schemes, the barrier between the connecting schemes is gratefully exploited. In any case, the scheme is thus composed as

\[
y = S_{U_0}^V S_{U_0}^H N_{P_1, U_1} | T_{P_0}^V T_{P_0}^H x
\]  

including the discussed barrier. For the CDF 5/3 wavelet, the steps are illustrated in Fig. 15. We will label this scheme as Polyphase*. For the sake of clarity, we will now summarize the proposed schemes. By reversing the direction of filtering steps in the Iwahashi scheme, the new Explosive scheme was formed. As a next step, we have reassembled the polynomials of the original polyphase matrix into a new two-step form. In between the steps, a memory barrier have to be placed. This scheme is denoted as Monolithic. Moreover, we have reduced the number of arithmetic operation by splitting the polynomial into two parts. These newly formed polynomials are then employed in appropriate schemes. In this manner, the number of barriers remains unaffected, while the number of operations has been reduced. This simple trick has resulted in the Iwahashi*, Explosive*, Monolithic*, and Polyphase* schemes. Once again, we would like to emphasize that the schemes presented in this paper are general and they are not limited to any specific type of transform.

5 Evaluation

This section analyzes in detail various attributes of the schemes described in the previous sections. Namely, synchronization and memory demands for different wavelets are examined. We realize that such properties do not provide sufficient information on a performance in real environments. For this reason, we are interested in comparing the performance of the discussed schemes on real graphics cards in terms of memory bandwidth in the next section.

The evaluation is presented using the following three wavelets. The first wavelet we have employed is the CDF 5/3 wavelet. This one is used for a lossless compression in the JPEG 2000 compression standard. The lifting scheme is defined by

\[
\begin{bmatrix}
P(z) \\
U(z)
\end{bmatrix} = \begin{bmatrix} -1/2(1 + z^{-1}) \\
1/4(1 + z)
\end{bmatrix},
\]  

and the scaling factor \( \zeta = \sqrt{2} \).

As the second wavelet, we have chosen the CDF 9/7 wavelet. In the JPEG 2000 standard, this wavelet is used as a basis for a lossy compression. The underlying scheme is given by

\[
\begin{align*}
P^{(0)}(z) &= \left[ \alpha(1 + z^{-1}) \right] \\
U^{(0)}(z) &= \left[ \beta(1 + z^{-1}) \right] \\
P^{(1)}(z) &= \left[ \gamma(1 + z^{-1}) \right] \\
U^{(1)}(z) &= \left[ \delta(1 + z^{-1}) \right],
\end{align*}
\]  

where the \( \alpha, \beta, \gamma, \delta, \) and the \( \zeta \) are defined in [3]. Both the CDF wavelets have predict and update convolution operators of degree 1 (two-tap symmetric filters).

The last wavelet included in the comparison is (4, 4) interpolating transform built from the interpolating Deslauriers–Dubuc [20], defined by

\[
\begin{align*}
P(z) &= \left[ 1/16(z + z^{-2}) - 9/16(1 + z^{-1}) \right] \\
U(z) &= \left[ 9/32(1 + z) - 1/32(z^{-1} + z^2) \right].
\end{align*}
\]  

Table 2: Number of operations and memory barriers examined for various wavelets.

| wavelet     | scheme     | barriers | operations |
|-------------|------------|----------|------------|
| CDF 5/3     | Sweldens   | 4        | 16         |
| CDF 5/3     | Iwahashi   | 4        | 16         |
| CDF 5/3     | Iwahashi*  | 3        | 18         |
| CDF 5/3     | Explosive  | 3        | 24         |
| CDF 5/3     | Explosive* | 3        | 18         |
| CDF 5/3     | Monolithic | 2        | 24         |
| CDF 5/3     | Monolithic*| 2        | 18         |
| CDF 5/3     | Polyphase  | 1        | 63         |
| CDF 5/3     | Polyphase* | 1        | 23         |
| CDF 5/3     | Convolution| 1        | 64         |
| CDF 9/7     | Sweldens   | 8        | 32         |
| CDF 9/7     | Iwahashi   | 6        | 48         |
| CDF 9/7     | Iwahashi*  | 6        | 36         |
| CDF 9/7     | Explosive  | 6        | 48         |
| CDF 9/7     | Explosive* | 6        | 36         |
| CDF 9/7     | Monolithic | 4        | 48         |
| CDF 9/7     | Monolithic*| 4        | 36         |
| CDF 9/7     | Polyphase  | 2        | 126        |
| CDF 9/7     | Polyphase* | 2        | 46         |
| CDF 9/7     | Convolution| 1        | 256        |
| DD 13/7     | Sweldens   | 4        | 32         |
| DD 13/7     | Iwahashi   | 3        | 64         |
| DD 13/7     | Iwahashi*  | 3        | 50         |
| DD 13/7     | Explosive  | 3        | 64         |
| DD 13/7     | Explosive* | 3        | 50         |
| DD 13/7     | Monolithic | 2        | 64         |
| DD 13/7     | Monolithic*| 2        | 50         |
| DD 13/7     | Polyphase  | 1        | 255        |
| DD 13/7     | Polyphase* | 1        | 203        |
| DD 13/7     | Convolution| 1        | 256        |
The first examined parameters include the number of arithmetic operations (the scaling steps were omitted) and the number of memory barriers. The schemes presented in this paper can be directly applied on the CDF 5/3 and DD 13/7 transforms, as these comprise only a single pair of lifting steps. The CDF 9/7 transform is computed by two such connected schemes. The comparison is shown in Table 2.

Several expectations can be made from the Table 2. On architectures based on serial computation, the schemes presented in this paper can be directly applied on the CDF 5/3 and DD 13/7 transforms, as these comprise only a single pair of lifting steps. The CDF 9/7 transform is computed by two such connected schemes. The comparison is shown in Table 2.

### Table 3

| scheme     | barriers | single | double |
|------------|----------|--------|--------|
| Sweldens   | 4        | 2      | 3      |
| Iwahashi   | 3        | 3      | (6) 4  |
| Iwahashi*  | 3        | 3      | 4      |
| Explosive  | 3        | 2      | 3      |
| Explosive* | 3        | 2      | 3      |
| Monolithic | 2        | 3      | 6      |
| Monolithic*| 2        | 3      | 6      |
| Polyphase  | 1        | 4      | (8) 4  |
| Polyphase* | 1        | 4      | (8) 4  |

#### Table 3

Number of memory barriers and local memory cells per quadruple required by the schemes discussed in this paper. Memory cells are given for a single buffering (two barriers) as well as a double buffering (only a single barrier). The numbers in parentheses are valid in the case of connecting schemes. Best features in bold.

### Table 4

| scheme     | write | read degree-1 | read degree-3 |
|------------|-------|---------------|---------------|
| Sweldens   | $1 + 4K$ | $8K$ | $24K$ |
| Iwahashi   | $2 + 4K$ | $10K$ | $42K$ |
| Iwahashi*  | $6K$   | $10K$ | $42K$ |
| Explosive  | $4K$   | $10K$ | $42K$ |
| Explosive* | $4K$   | $10K$ | $42K$ |
| Monolithic | $6K$   | $10K$ | $42K$ |
| Monolithic*| $6K$   | $10K$ | $42K$ |
| Polyphase  | $4K$   | $21K$ | $117K$ |
| Polyphase* | $4K$   | $12K$ | $117K$ |

#### Table 4

Number of local memory reads and writes for all schemes and wavelets under examination. The $K$ denotes the number of predict/update pairs. The degree-1 polynomials correspond to factorizations of CDF wavelets, whereas degree-3 to DD 13/7.

6 Performance

To evaluate the considered schemes, we have decided to use high-performance GPUs programmed using the OpenCL framework. In terms of the OpenCL, the schemes are computed using parallel tasks referred to as the kernels. One item from a collection of parallel executions of a kernel is referred to as the work-item or thread. The threads that execute on a single compute unit are grouped into so-called work-groups. The threads in the group execute the same kernel and share local memory. Each work-group can synchronize the threads via memory barriers. Work-groups cannot synchronize with each other. Considering the processing of images, we map overlapping (in order to properly compute the coefficients near tile boundaries) image tiles onto the work-groups. Moreover, each thread is responsible for a single quadrature of transform coefficients (LL, HL, LH, and HH). At the beginning of the computation, the input image is placed in...
the global memory. The tiles are then transferred into the local memory. After the scheme computation, the resulting coefficients are copied back into the global memory. Such strategy fulfills the definition of a single-loop data processing (therefore without unnecessary data transfers).

The evaluation was performed primarily on two high-end GPUs – AMD Radeon HD 6970 and AMD Radeon HD 5870. We now describe their technical parameters. The AMD Radeon HD 6970 GPU consists of 24 multiprocessors. Each of them has 1 SIMT (single instruction multiple thread) unit with 16 VLIW4 (very long instruction word with a length of 4) processors. Taken together, the GPU comprises 1 536 processors (384 VLIW4 processors) clocked at 880 MHz with a capability of executing 2 703 FLOPS (1 352 MAD FLOPS). The card is equipped with 1 GB GDDR5 memory at 1 375 MHz connected to the GPU with 256-bit bus. This gives memory throughput of 176 GB/s. The AMD Radeon HD 5870 GPU consists of 20 multiprocessors, where each of them has 1 SIMT unit with 16 VLIW5 processors. In total, the GPU consists of 1 600 processors (320 VLIW5 processors) clocked at 850 MHz with a capability of executing 2 720 FLOPS (1 360 MAD FLOPS). The card is equipped by 1 GB GDDR5 memory at 1 200 MHz, connected with 256-bit bus, with memory throughput of 154 GB/s. On both of the cards, variable length VLIW instructions are executed using blocks of 64 threads. In more detail, VLIW instructions can be categorized into several groups (load/store instructions, barrier instructions, control flow instructions and ALU instructions). To utilize whole processing capability, the VLIW instructions should be of maximal length. In other words, as much as possible blocks of independent instructions should be presented in a kernel.

Several possibilities raise during the implementations of the presented schemes. All of the schemes require several memory cells to interchange the intermediate coefficients. Considering the GPUs, these coefficients can be efficiently stored in the local memory. Unfortunately, it is not possible to rewrite these coefficients using a single memory barrier. As a consequence, two possibilities occur – double buffering using a single memory barrier, and single buffering using two of them. The double buffering increases the memory requirements while maintaining the number of synchronizations. Conversely, the single buffering introduces an addition barrier – separating reading and rewriting of the coefficients. For details, see Table 3. Moreover, another possibility lies in the method of input and output data delivery. For evaluation purposes, it is possible to completely omit the input and output of data. The transform is not limited by memory bandwidth in this case. For real scenarios, the data can be delivered using the global or texture memory. In our experiments, we chose the latter option.

In the following paragraphs, three fundamental experiments on the described GPUs are presented. The first experiment studies the performance of the baseline schemes mentioned in this paper. The second experiment examines the influence of the improvement proposed in Section Improvements. Finally, the third experiment measures the real performance with CDF 9/7 wavelet and texture memory.

In the first experiment, we have examined the performance of the baseline schemes (without improvements proposed in Section Improvements). The measurements were conducted on the AMD 6970 card with two different lifting scheme shapes (degree-1 and degree-3 operators). Only the transform performance was measured, without the influence of memory throughput. The presented results are the average of ten measurements. The results are shown in Fig. 16. One can easily observe a different behavior for short and long lifting operators. For the short operators, the reduction of the number of lifting steps clearly improves the performance. The situation actually corresponds directly to the number of memory barriers.
Conversely, in the case of the long operators, the situation is tilted in favor of the number of arithmetic operations. Note that the horizontal axes are in a logarithmic scale. The vertical axes express the transform throughput in GB/s (gigabytes per second).

In the second experiment, we have examined the contribution of the improvements proposed in Section Improvements. The measurements were performed on both of the cards under the evaluation. This time we have focused on the degree-1 schemes only. As in the previous case, only the transform performance was measured using the average of ten measurements. The results are shown in Fig. 17. Again, the horizontal axes are in a logarithmic scale and the vertical ones express the pure transform throughput. As expected, the improvements slightly increase the transform performance. However, the order of the schemes still corresponds to the number of memory barriers. Several schemes perform even worse than the original separable Sweldens scheme – namely, the original Iwahashi and both Polyphase schemes. It is not surprising for the original Polyphase scheme, as this one exhibit quite a high number of operations and load instructions (see Table 4 and Table 2). In case of Polyphase* scheme, the decisive factor was the number of load instructions coupled with a high local memory footprint (see Table 5). A little surprising is the situation regarding the original Iwahashi scheme. In this case, the scheme contains a relatively high number of operations, wherein there is no additional advantage. For convenience, the values at the end of plots in Fig. 17 are listed in Table 5.

In the last experiment, we were interested in a real performance. This experiment was performed on both of the cards with CDF 9/7 wavelet. The input as well as output raster were supplied in the texture memory. This time, we show only the improved schemes as these always outperform the original ones. The results are shown in Table 5. The degree-1 schemes on AMD 6970 and AMD 5870. The performance of a transform code without the memory throughput is listed. Values given in GB/s at the end of plots in Fig. 17.

Table 5
| Scheme          | Throughput AMD 6970 | Throughput AMD 5870 |
|-----------------|---------------------|---------------------|
| Monolithic*     | 117.426             | 121.579             |
| Monolithic      | 109.865             | 105.407             |
| Explosive*      | 97.214              | 105.344             |
| Explosive       | 95.263              | 97.877              |
| Iwahashi*       | 89.748              | 92.288              |
| Sweldens        | 82.336              | 88.924              |
| Iwahashi        | 80.284              | 80.283              |
| Polyphase*      | 51.776              | 43.619              |
| Polyphase       | 32.593              | 27.462              |

Fig. 17 The schemes on AMD 6970 and AMD 5870. Evaluation with the degree-1 schemes. Only the performance of a transform code without the memory throughput was measured.
7 Conclusions

In this paper, we have proposed several non-separable lifting schemes for the calculation of the discrete wavelet transform. The proposed schemes produce exactly the same results as the commonly used separable lifting scheme. Using out schemes, the transform can be computed in a smaller number of steps. On parallel architectures, this property has resulted in a smaller number of synchronizations.

Namely, we have proposed two-step 2-D lifting scheme compatible to the commonly used four-step separable one. Unlike the separable scheme, the proposed scheme consists of spatial predict and update operators. Since, the number of the lifting steps was halved, our scheme reduces also the number of memory barriers, which form a major bottleneck on parallel architectures. In addition, we have proposed the three-step scheme reducing the memory access overhead. For a moment, let $K$ denote the number of predict-update pairs. In absolute numbers, the original separable scheme requires to write $1 + 4K$ coefficients per predict/update pair, whereas our three-step scheme requires $4K$ coefficients only. Additionally, the proposed two-step scheme requires three memory cells per thread, whereas the proposed three-step scheme requires two cells only (same as the separable scheme). Finally, we have proposed an improvement valid for all non-separable scheme, including the already known ones. This improvement significantly reduces the number of arithmetic operations. More specifically, the original non-separable schemes require $24$ arithmetic operations for CDF 5/3 wavelet. Whereas the improved variants require $18$ operations only for the same case. Even greater savings are achieved in the case of a non-factorized polyphase matrix (same as the convolution for the CDF 5/3 wavelet). In this case, the proposed improvement reduces the number of operations from 63 to 23. All the proposed schemes are general and can be used in conjunction with any discrete wavelet transform.

The proposed schemes were subjected to performance measurements. In experiments on the two selected high-end GPUs (AMD Radeon HD 6970 and 5870), the proposed schemes outperform all the others for short lifting filters. This includes the well known CDF 5/3 and CDF 9/7 wavelets, employed e.g. in JPEG 2000 compression standard.

Future work, we would like to do, consists of extensions to multi-dimensional systems, and extensions to another subband transforms.
In this paper, we work with $4 \times 4$ matrices of Laurent polynomials, usually referred to as the polyphase matrices, for example, this one:

$$T^H_P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ P & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & P \end{bmatrix}. \quad (36)$$

Since these matrices define a linear mapping from vectors of form $[LL \; HL \; LH \; HH]^T$ to vectors of the same form, we can simply illustrate this mapping by the block diagram in Fig. 19(a).

Moreover, the matrices are composed of elementary lifting operators like

$$P(z) = -1/2(1 + z^{-1}). \quad (37)$$

If we substitute such particular polynomials into the matrix, the mapping gets a specific shape, as illustrated by the dataflow diagram in Fig. 19(b). The solid arrows correspond to multiplication by $-1/2$ along with subsequent summation. The dotted arrows similarly correspond to multiplication by factor of 1, since the matrix $T^H_P$ contains ones on the main diagonal.

For reader’s convenience, we use two-dimensional diagrams to illustrate the schemes with CDF 5/3 wavelets. For the example above, such a diagram is shown in Fig. 19(c) whereas the elementary quadratures of coefficients are highlighted by solid and dotted boxes.

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