Growth of slip surfaces in 3D conical slopes

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Manuscript submitted to the International Journal for Numerical and Analytical Methods in Geomechanics on 27.01.2021
Revised and resubmitted on 13.04.2021

Abstract
Out-of-plane curvature of real submarine slopes imposes limitations on applicability of existing planar criteria for catastrophic growth of slip surfaces. In this paper, the growth of an initially weakened zone in three-dimensional (3D) convex and concave slopes is investigated using the process zone approach. The geometry of the problem is presented in a curvilinear coordinate system for which the governing equations for the three-dimensional slip surface growth are derived. Solution of these equations for an axisymmetric problem is obtained both analytically and numerically (using a finite differences scheme) and benchmarked against Coupled Eulerian-Lagrangian finite element simulations. It is shown that the application of the planar slope solution to conical slopes constitutes an overestimation of the slope’s stability. The closed form criteria for an unstable 3D slip surface growth in both convex and concave slopes are proposed.

Notations: $\tau_{g,cr}$, Critical gravitational shear stress; $\alpha_{cr}$, Critical vertical acceleration; $\gamma_p$, Engineering plastic shear strain; $\alpha_3$, Fitting parameter for the nonlinear criterion in planar geometry; $q_j$, Mises stress; $\hat{\lambda}$, Normalized area of the PSZ; $\hat{l}_{0,r}, \hat{l}_{0,\theta}$. Normalized dimensions of the PSZ for planar geometry; $\tilde{l}_{0,r}, \tilde{l}_{0,\theta}$, Normalized lengths of the mid axes of PSZ; $\Delta \tau_{g,cr}$, Normalized shear and normal stress differences; $F_{p,r}$, $F_{r,c}$, $\overline{F}$, Normalized shear stress ratios calculated for planar and convex slopes and their normalized difference; $\delta P$, Plastic displacement; $\delta \hat{P}$, Residual plastic displacement; $\gamma^n P$, The engineering plastic shear strain required to cause full softening of the weak layer; $\sigma_{ij}, \sigma_{g,ij}, \Delta \sigma_{ij}$, The ij-th component of the curvilinear acting, gravitational and ‘net’ stress tensors; $\psi_i$, The i-th component of the curvilinear velocity vector; $\hat{l}_{0,\lambda}, \hat{l}_{0,\mu}$, Characteristic lengths in the downslope and circumferential directions; $\hat{l}_{u,r}, \hat{l}_{u,\theta}$, Characteristic lengths of the planar slope problem; $l_{0,r}, l_{0,\theta}$, lengths of the mid axes of the PSZ in the downslope and circumferential directions; $\hat{\xi}, \hat{\delta}$, Normalized coordinates; $\alpha, \alpha_{cr}$, The length and critical length of the process zone; $A_{t}, A_{t+d}, A_{g}, A_{g+d} A_{s}$, Curvilinear element facet’s areas; $\hat{\delta}^P$, Accumulative equivalent plastic shear strain; $\delta\hat{P}$, equivalent plastic shear strain rate; $\hat{\delta}^P$, The accumulative equivalent plastic strain required to cause full softening of the weak layer; $\xi, \delta, \mu$, Incremental plastic strain rate tensor and its i-th component; $\eta$, Non-dimensional parameter used in governing equation; $E$, Young’s modulus; $E'$, Plane-strain modulus; $G$, Gravitational acceleration; $G_{as}$, Shear modulus of the sliding and weak layers, respectively; $h$, Thickness of sediment layer; $h_{as}$, FEM and FD element size; $J_2$, Second invariant of the deviatoric stress tensor; $L$, Downslope length of the slope; $m$, Fitting parameter for the nonlinear criterion of convex slopes; $N_{1j}, N_{2j}$, Number of grid points in the downslope and circumferential directions; $R(\xi, \eta)$, horizontal distance on the xy plane from the Cartesian z axis; $\tau_{g,0}$, Shear stress ratio and critical shear stress ratio; $R_0$, Crest/toe radius; $r_{c,max}, r_{c,min}$, Critical shear stress ratios of the convex, 2D planar and 1D/Axisymmetric problems, respectively; $s$, Thickness of the weak layer; $S_p$, Sensitivity of the weak layer; $u_e, \tilde{u}_e, \Delta u_e$, The i-th component of the curvilinear current, gravitational and ‘net’ displacement vectors; $x, y, z$, Cartesian coordinates; $\Delta \xi, \Delta \delta$, FD grid spacing in the downslope and circumferential directions; $\alpha$, Slope inclination; $\beta$, Parameter relating elastic and plastic shear strains between peak and residual strength; $\gamma'$, Submerged soil unit weight; $\lambda, \lambda_{crit}$, $\Delta \lambda$, Aspect ratio of the PSZ, aspect ratio at which the problem can be considered axisymmetric and the limiting value of $\lambda$ for infinitely large areas of the PSZ; $\gamma$, Poisson ratio; $\lambda_{cr}, \lambda_{as}$, Circumferential extent of FD calculation; $\delta_{cr}$, Characteristic angle; $\phi_0$, $\phi_{init}$, Initial section angle of the PSZ, initial angle at which the solution can be considered axisymmetric and the limiting value of $\phi_{init}$ for infinitely large areas of the PSZ; $\rho$, Bulk density of the soil; $\sigma_{g}(\xi, \eta)$, Gravitational normal and shear stresses of infinite planar slope; $\sigma_{g,P}, \sigma_{g,L}$, The ij-th component of the curvilinear stress tensor and gravitational curvilinear stress tensor; $\tau_{p, cr}$, Peak and residual strengths; $\tau_p$, Yield shear stress; $\xi, \delta, \eta$, Curvilinear coordinates

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and validated by fitting numerical results for various sizes and aspect ratios of the initially pre-softened zone.

KEYWORDS
3D failure criteria, large deformation analysis, slip surface growth, submarine landslides

1 | INTRODUCTION

Submarine landslides are responsible for both offshore and onshore damages and loss of life\(^1\)–\(^4\) and represent one of the largest natural hazards that exist. The failure of submarine slopes is often attributed to the presence of weak layers interbedded within the sediment mass.\(^5\)–\(^7\) These weak layers often exhibit strain softening behavior, resulting in progressive growth of slip surfaces followed by catastrophic slope failure. The process of the slip surface growth is initiated when a portion of the weak layer reaches peak strength due to increase in shear stresses or reduction of shear capacity (e.g., following increase in pore water pressures). As long as the equilibrium of the sliding layer can be re-established, the slip surface growth is stable. After reaching a certain critical length, this growth becomes unstable (catastrophic,\(^8\)) eventually causing active and passive failures at the top and bottom of the slope, respectively, bringing it to a global failure. During subsequent landslide evolution, the moving mass may either run-out or plough into the intact ground at the bottom\(^9\) and produce recurring upslope retrogressive failure at the top of the slope.\(^10,11\) These post-failure phenomena explain the devastating dimensions of the underwater mass movements and their extent has been quantified by simplified analytical solutions,\(^9,12\) limit analysis\(^3,13\) and numerically.\(^14,15\)

The process of the slip surface growth has been studied extensively in a 2D plane-strain set up for either planar geometries\(^16\)–\(^19\) or nonlinear slope profiles.\(^20,21\) Recent extension to a 3D scenario\(^22\) was limited to planar slope, not accounting for its possible in- and out-of-plane curvature. Real life bathymetries, however, consist of more complex geometries, including convexities, concavities, ridges and shoulders (Figure 1), affecting the stability of the slopes. In order to overcome the limitations of the 2D approach to slip surface growth, a number of strategies have been applied in practical applications.\(^23\) For example, the slopes can be subdivided into facets where they can be considered as two dimensional.\(^24,25\) These simplifications, however, do not allow assessing the possible effects of the out-of-plane curvature on the slope stability. In this paper, the influence of the out-of-plane curvature on the stability of submarine slopes is explored both analytically and numerically, leading to a closed form criterion for a 3D slip surface growth in convex and concave slopes for various sizes and aspect ratios of the initially pre-softened zone (PSZ).

2 | METHODOLOGY

Although convex and concave slopes are different in their geomorphological nature, mathematical descriptions of their geometry are rather similar. Consequently, for the reason of brevity, the following derivations and formulation of the catastrophic slip surface growth criterion are carried out for convex geometry and are later extended to the concave case.

2.1 | Problem formulation

The problem considered below is of a homogenous isotropic sediment layer of thickness \(h\) overlying a convex conical rigid base inclined by angle \(\alpha\) (Figure 2). The cone is truncated at the top by a horizontal plane, resulting in a circular cross-section of radius \(R_0\) denoted as the “crest.” The sediment layer is assumed to behave elastically, characterized by the Young’s Modulus (\(E\)) and Poisson ratio (\(\nu\)) with submerged specific weight \(\gamma’\). A thin weak layer is assumed at the base of the sediment layer. The weak layer behavior is elastic-plastic with softening, with shear strength undergoing linear strength degradation with increasing plastic deformations, from the peak (\(\tau_p\)) to the residual value (\(\tau_r\)) at the residual plastic displacement (\(\delta^p_r\)). It is assumed that a portion of the weak layer has already softened to its residual strength. Geometry of this portion is formed by the section of angle \(\theta_0\) bounded by two circular lines located at distances of \(\xi_0\) and \(\xi_0 + l_0^\xi\) from the crest (Figure 2), where \(l_0^\xi\) and \(\xi_0\) are the downslope length of the softened zone and the downslope distance from the crest to its top edge, respectively.
2.1.1 Equilibrium equations in a local curvilinear coordinate system

A right-hand orthogonal curvilinear coordinate system is introduced to describe the problem geometry as depicted in Figure 2. Its axes $\xi$, $\theta$, and $\eta$ correspond to the downslope distance from the crest parallel to the slope surface, the angle of rotation around the center of revolution (aligned with the Cartesian $z$ axis) and the depth orthogonal to the slope surface, respectively. The $\xi$ and $\eta$ axes are inclined to the Cartesian $xy$ plane by angles $\alpha$ and $\pi/2-\alpha$, respectively, and the coordinate transformation is given by

\[
\begin{align*}
x &= R(\xi, \eta) \cdot \cos \theta \\
y &= R(\xi, \eta) \cdot \sin \theta \\
z &= -R_0 \tan \alpha - \xi \sin \alpha + \eta \cos \alpha
\end{align*}
\]
where \( R(\xi, \eta) \) is the horizontal distance on the \( xy \) plane from the Cartesian \( z \) axis to any given point on the convex slope expressed by

\[
R(\xi, \eta) = R_0 + \xi \cos \alpha + \eta \sin \alpha
\]  

The above transformation between the two coordinate systems allows deriving the base vectors in the curvilinear coordinate system and hence obtaining the strain and stress tensors as well as the equilibrium equations by rigorously applying the principles of tensor analysis of continuum mechanics.

In the absence of inertia forces, formulating the linear momentum equations for an elementary volume (Figure 3) in the curvilinear coordinates results in the following equilibrium equations in the \( \xi, \theta, \eta \) directions, respectively

\[
\frac{\partial \sigma_{\xi \xi}}{\partial \xi} + \frac{\partial \sigma_{\eta \xi}}{\partial \eta} + \frac{(\sigma_{\xi \xi} - \sigma_{\theta \theta}) \cos \alpha + \sigma_{\eta \xi} \sin \alpha + \frac{\partial \sigma_{\theta \xi}}{\partial \theta}}{R(\xi, \eta)} = -\gamma' \sin \alpha
\]  

\[
\frac{\partial \sigma_{\xi \theta}}{\partial \xi} + \frac{\partial \sigma_{\eta \theta}}{\partial \eta} + \frac{2 (\sigma_{\xi \theta} \cos \alpha + \sigma_{\eta \theta} \sin \alpha) + \frac{\partial \sigma_{\theta \theta}}{\partial \theta}}{R(\xi, \eta)} = 0
\]  

\[
\frac{\partial \sigma_{\eta \eta}}{\partial \eta} + \frac{\partial \sigma_{\xi \eta}}{\partial \xi} + \frac{(\sigma_{\eta \eta} - \sigma_{\theta \theta}) \sin \alpha + \sigma_{\xi \eta} \cos \alpha + \frac{\partial \sigma_{\theta \eta}}{\partial \theta}}{R(\xi, \eta)} = \gamma' \cos \alpha
\]  

The complete derivation of the above equations is provided in the supplementary material.

While the analytical closed form solution for the complex gravitational stress field is difficult to obtain, the above equilibrium equations exhibit a rather useful feature, which can be potentially utilized for future simplifications. The third term on the left-hand-side of all three equations diminishes with increasing horizontal distance from the axis of revolution (i.e. large \( R(\xi, \eta) \)) provided there are no sharp stress gradients between adjacent soil elements. In the gravitational case, where no localizations of shear stresses and strains yet exist, the solution of Equations (3) and (5) approaches asymptotically that...
of the following equations:

\[
\frac{\partial \sigma_{\xi \xi}}{\partial \xi} + \frac{\partial \sigma_{\eta \xi}}{\partial \eta} = -\gamma' \sin \alpha \tag{6}
\]

\[
\frac{\partial \sigma_{\xi \eta}}{\partial \xi} + \frac{\partial \sigma_{\eta \eta}}{\partial \eta} = \gamma' \cos \alpha \tag{7}
\]

Note that in the gravitational case Equation (4) becomes redundant as the problem is symmetric in the circumferential direction.

Equations (6) (7) are identical to the equilibrium equations associated with two-dimensional infinite planar slopes (denoted as IPS hereafter), in plane strain conditions, where the \(\xi\) and \(\eta\) coordinates are interchangeable with the slope-parallel and slope-perpendicular planar coordinates, respectively. Moving away from the crest, it is reasonable to neglect any boundary effects since the depth to the failure surface in submarine slopes is normally much smaller compared to their length. In the context of the above equations, the derivatives of all stresses with respect to \(\xi\) can be neglected, so that Equations (6) and (7) are further reduced to

\[
\frac{\partial \sigma_{\eta \xi}}{\partial \eta} = -\gamma' \sin \alpha \tag{8}
\]

\[
\frac{\partial \sigma_{\eta \eta}}{\partial \eta} = \gamma' \cos \alpha \tag{9}
\]

In the absence of tractions at the slope surface, Equations (8) and (9) can be solved analytically:

\[
\sigma_{\eta \xi} = -\int_{-\eta}^{0} \gamma' \sin \alpha \cdot d\eta = -\gamma' \eta \sin \alpha = \tau_g (\eta) \tag{10}
\]

\[
\sigma_{\eta \eta} = \int_{-\eta}^{0} \gamma' \cos \alpha \cdot d\eta = \gamma' \eta \cos \alpha = \sigma_g (\eta) \tag{11}
\]

It appears that the asymptotic solution for the curvilinear shear and normal stresses (\(\sigma_{\eta \xi}\) and \(\sigma_{\eta \eta}\)) is identical to the gravitational slope-parallel shear stress and slope-perpendicular normal stress of IPS, denoted here as \(\tau_g (\eta)\) and \(\sigma_g (\eta)\), respectively. Note that for the chosen coordinate system \(\eta < 0\) everywhere in the slope which results in positive shear stresses and negative normal stresses (in compression).

### 2.1.2 Extent of application of the infinite slope stress field to convex slopes

Limits on the applicability of the known analytical solution for the IPS stress field (Equations (10) and (11)) to the curvilinear case can be defined as a criterion for the minimum distance from the crest at which the solutions of the equilibrium equations (3) and (5) degenerate to those of IPS. This criterion is formulated using the finite element (FEM) approach, where a homogenous elastic layer is subjected to the gravity load prior to any softening in the weak layer. The simulations are focused on the influence of the two most important factors of the solution: (a) the horizontal distance from the center of revolution, \(R(\xi, \eta)\), expressed via the downslope distance from the crest, \(\xi\), and crest radius, \(R_0\) and (b) the soil stiffness. An implicit static model was generated using the commercial code ABAQUS. Full integration eight-node brick elements of size \(h_{el}\) (in the \(\xi-\eta\) plane) are used in the simulations. The problem is considered symmetrical about the Cartesian \(z\) axis and hence displacements are restrained in the circumferential direction. The rigid base is simulated by restraining the displacements of the base nodes in all directions. The nodes at the center of revolution are permitted to displace only in the Cartesian \(z\) direction, while the far end downslope is fully constrained. Once the model is assembled, gravity is ramped using a smooth step. The base case geometry and soil parameters of the model are listed in Table 1.
A parametric study (Figure 4) was conducted to examine the effects of the mesh size, soil stiffness and slope geometry. After each analysis is successfully completed, the Cartesian stresses are extracted from the calculation output and converted into the curvilinear stresses (namely $\sigma_{\eta\xi}$ and $\sigma_{\eta\eta}$) using the curvilinear stress transformation (Equations 31–36 in the supplementary material). To prove convergence of the gravitational stress field to the IPS stresses, normalized stress differences are defined as follows:

$$\Delta \bar{\sigma}_{\eta\xi} = \left| \tau_g - \sigma_{\eta\xi} \right| / \tau_g$$  \hspace{1cm} (12)
A comparison between the normalized stresses in the curvilinear configuration and the normalized infinite slope stresses along the sediment layer for the case of $R_0/h = 1$. (A) Normalized shear stresses. (B) Normalized normal stresses

\[
\Delta \sigma_{\eta \eta} = \frac{\sigma_g - \sigma_{\eta \eta}}{\sigma_g}
\]

As the normalized stress differences approach zero, the gravitational stress field approaches the IPS stresses, with a normalized stress difference threshold of 5% adopted to establish convergence of the two solutions. First, the sensitivity of the solution to the choice of element size is investigated for the base case by varying $h_{el}$. The results are plotted as the normalized shear and normal stress differences, calculated at the base of the slope (i.e. $\eta = -h$) against the normalized distance from the crest, $\xi/h$. The dependency of the results on the mesh size is depicted in Figure 4(A,B), indicating very small variation with element size smaller than 0.5 m, justifying its use in subsequent analyses. Next, the effect of the soil stiffness on the solution is studied by varying the Young’s modulus ($E$) of the soil within a typical range of 1–100 MPa. The results, illustrated in Figure 4(C,D), seem almost unaffected by the choice of the soil stiffness, especially within the range of 10–100 MPa. Finally, Figure 4(E,F) investigate the significance of the crest’s radius and its impact on the results. The trend shows that as the crest radius becomes larger, the two solutions converge at smaller distances from the crest. This is in agreement with Equations (3)-(5) degenerating into the IPS equations with increasing horizontal distance from the center of revolution, $R(\xi, \eta)$. A closer inspection of Figure 4(E) suggests that for the distances larger than $\xi/h = 5$ from the crest, the IPS solution becomes applicable to all practically meaningful convex slope geometries.

To ascertain the convergence of the gravitational stress state to the IPS stress field throughout the entire sediment layer, the calculated normalized stresses in the layer are presented in Figure 5. The continuous lines represent curvilinear shear (A) and normal (B) stress distribution at different depths along the sediment layer for the base case. The stresses are normalized by the shear and normal stresses evaluated at the bottom of the sediment layer. To confirm convergence, the stress distributions ought to asymptotically approach the normalized IPS stresses calculated at the corresponding depths (represented by the dashed lines). Consistent with the results in Figure 4, for $\xi > 5h$ the normalized curvilinear shear stresses in Figure 5 converge to the equivalent normalized IPS stresses throughout the entire thickness of the sediment layer. It could, therefore, be concluded that for the consequent analysis of the catastrophic growth of slip surfaces, the IPS gravitational stress field can be applied to convex slopes at distances greater than $5h$ (i.e., five sliding layer thicknesses) from the crest.

2.1.3 | Formulation of the governing equation for a 3D slip surface growth

Once the initially weakened zone has softened to its residual strength, the stress field at its vicinity is disturbed, causing loading at the bottom and unloading at the upper portions of the slope. This loading-unloading process may cause progressive softening of the weak layer outside the PSZ leading to slip surface growth downslope as well as upslope. In order to establish the conditions at which the slip surfaces would propagate catastrophically and ultimately lead to failure of the slope, the process zone approach is applied. In the process zone approach, the equilibrium of the sliding layer in the
downslope direction is formulated in terms of deviation from the gravitational stress and displacement fields that exists in the slope prior to softening of the weakened zone. That is, the 'net' stresses \( \Delta \sigma_{ij} \) and displacements \( \Delta u_i \) can be expressed as \( \Delta \sigma_{ij} = \sigma_{ij} - \sigma_{g,ij} \) and \( \Delta u_i = u_i - u_{g,i} \), where \( \sigma_{ij} \) and \( \sigma_{g,ij} \) are the acting and gravitational stress components, respectively and \( u_i, u_{g,i} \) are the current and gravitational displacements, respectively. The weak layer has three distinctive zones: (a) the PSZ, (b) the process zone, flanking the PSZ, with soil experiencing strain softening from peak to residual strength and (c) the elastic shearing zone, where the 'net' displacements are purely elastic and the stresses decay in the far field to the gravitational stresses, i.e. the 'net' stresses approach zero. The stresses and displacements are averaged over the entire thickness of the sliding layer, whereas the shear stress at the bottom of the sliding layer is in equilibrium with the shear stress applied to the underlying weak layer. This approach was applied to two-dimensional and three-dimensional planar slopes by Zhang et al.,\textsuperscript{21,22,28} and the criteria derived were found to be in good agreement with large-deformations finite element simulations.

The equilibrium of the sliding layer in the downslope direction is derived using the 'net' stresses and considering an elementary volume of the convex slope (Figure 6):

\[
(\Delta \sigma_{\xi \xi} + d\Delta \sigma_{\xi \xi}) \cdot A_\xi - \Delta \sigma_{\eta \xi} \cdot A_\xi + d\xi + \left( (\Delta \sigma_{\theta \theta} + d\Delta \sigma_{\theta \theta}) \cdot A_\theta + \Delta \sigma_{\theta \xi} \cdot A_{\theta +d\theta} \right) \cdot \sin \frac{\theta}{2} \cdot \cos \alpha + \\
+ \left( (\Delta \sigma_{\theta \xi} + d\Delta \sigma_{\theta \xi}) \cdot A_\theta - \Delta \sigma_{\theta \xi} \cdot A_{\theta +d\theta} \right) \cos \frac{\theta}{2} - \Delta \sigma_{\eta \xi} \cdot A_\eta = 0
\]

(14)

where \( A_\xi, A_{\xi +d\xi}, A_\theta, A_{\theta +d\theta} \) and \( A_\eta \) are the areas of element facets shown in Figure 6.

Because the gravitational stresses are in equilibrium with the gravity induced driving force, this driving force does not appear in the 'net' equation (14). Revisiting Equation (2), one may notice that for large distances from the crest and mild slope inclinations, \( \xi \cos \alpha >> \eta \sin \alpha \) and therefore \( R(\xi, \eta) \approx R(\xi) = R_0 + \xi \cos \alpha \). Using the latter expression, neglecting second order terms and using small angle approximation for \( \alpha \), Equation (14) can be rewritten as

\[
\frac{d\Delta \sigma_{\xi \xi}}{d\xi} - \frac{\Delta \sigma_{\eta \xi}}{h} = \frac{(\Delta \sigma_{\xi \xi} - \Delta \sigma_{\theta \theta}) \cos \alpha}{R(\xi)} + \frac{1}{R(\xi)} \frac{d\Delta \sigma_{\eta \xi}}{d\theta} = 0
\]

(15)

The first, second and fourth terms in the above equation correspond to the gradient of the 'net' normal stress \( \Delta \sigma_{\xi \xi} \), the average gradient of 'net' shear stress \( \Delta \sigma_{\eta \xi} \), over the depth of the sliding layer, and the gradient of the intrinsic 'net' shear stress acting on the side facets of the elements. Those components are fundamental, and analogous to the ones found in the formulation of planar slopes.\textsuperscript{22} The third term in Equation (15) reflects the annular ring effect that exists in solids of revolution, in which the nominal value of the circumferential and radial stresses are present in the partial differential equation of equilibrium. In the context of convex slopes, the forces resulting from this term act in the direction of motion (as can be inferred from Figure 6) and therefore require additional shear resistance at the bottom of the sliding layer to maintain stability. Nevertheless, at those portions of the slope which can be approximated by the IPS stress field, the curvature of the slope decreases with increasing distance from the crest and the contribution of this term becomes negligible. In contrast, the fourth term cannot be neglected, as the stress gradient may become significant during the growth of the shear surface. This shear component is responsible for transferring loads between adjacent elements in the circumferential direction and therefore neglecting it would result in the slip surface growing solely in the downslope
direction. Note that the general structure of Equation (15) is very similar to the continua formulation in Equation (3), with some differences that can be mainly attributed to the averaging of the stresses across the element and to the different sign convention. A more elaborate explanation for these differences is provided in the supplementary material.

The solution of the governing equation relies on the following assumptions: (a) the averaging of the displacements across the thickness of the sliding layer implies that \( \frac{d \Delta u_\xi}{d \eta} = 0 \); (b) the 'net' circumferential displacements are negligible compared to the downslope ones; i.e. \( \Delta u_\theta \ll \Delta u_\xi \); (c) the out-of-plane strain (\( \Delta \varepsilon_{\theta \theta} \)) is assumed zero and (d) prior to the global slope failure, the slope-perpendicular stresses in the sliding layer remain constant, i.e. \( \Delta \sigma_{\eta \eta} = 0 \). The applicability of the assumptions is discussed in Appendix I.

The relation between the 'net' stresses and displacements of the sliding layer may be established using the linear elastic stress-strain relationship (Equation 49 in the supplementary material). Following these assumptions and omitting the third term in Equation (15), the governing equation can be written with respect to the 'net' downslope displacement (see supplementary material for derivations):

\[
\frac{d^2 \hat{u}}{d \hat{\xi}^2} + \left( \frac{a}{a + \hat{\xi}} \right)^2 \frac{d^2 \hat{u}}{d \hat{\theta}^2} = \frac{\Delta \sigma_{\eta \xi}}{\tau_p - \tau_r}
\]  

(16)

where \( \hat{\xi} = \xi / l_{u,\xi} \); \( \hat{\theta} = \theta / \theta_u \); \( \theta_u = l_{u,\theta} / R_0 \); \( \hat{u} = \Delta u_\xi / \delta_p \); \( a = R_0 / l_{u,\xi} \cos \alpha \); the characteristic lengths are given by

\[
l_{u,\xi} = \sqrt{\frac{E' h \delta_p}{\tau_p - \tau_r}}
\]

(17)

\[
l_{u,\theta} = \sqrt{\frac{G h \delta_p^p}{\tau_p - \tau_r}}
\]

(18)

\( E' \) is the plane strain modulus \((= E / (1 - \nu^2)) \) and \( G \) is the shear modulus of the sliding layer. Note that for undrained conditions \( E' = 4G \) and, therefore, \( l_{u,\xi} = 2l_{u,\theta} \).

The 'net' shear stress at the bottom of the sliding layer (right hand side of Equation (16)) is obtained by satisfying the stress and displacement compatibility conditions between the different zones of the weak layer. For the case of IPS in plane strain conditions,\(^{28}\) derived the normalized 'net' stress distribution as function of the 'net' displacements in the three regions for linear strength degradation

\[
\frac{\Delta \sigma_{\eta \xi}}{\tau_p - \tau_r} = \Delta r \leftrightarrow \left\{ \begin{array}{ll}
\hat{u} / (1 - \beta^2) & \hat{u} \leq (1 - \beta^2) (1 - r) \\
(1 - r) / \beta^2 (1 - \beta^2) (1 - r) & < \hat{u} \leq 1 - r + r \beta^2 \\
1 - r + r \beta^2 & < \hat{u}
\end{array} \right.
\]

Elastic shearing

(19)

\[
\beta^2 = 1 - \frac{s (\tau_p - \tau_r)}{G_w \delta_p^p}
\]

(20)

where \( G_w \) is the shear modulus of the weak layer and the shear stress ratio is introduced by

\[
r = \frac{\tau_g (-h) - \tau_r}{\tau_p - \tau_r}
\]

(21)

and \( \tau_g (-h) = \gamma' h \sin \alpha \) is the gravitational shear stress at the bottom of the sediment layer.

To complete the formulation of the boundary value problem of the slip surface growth it is required to satisfy stress and displacement compatibility conditions at the boundaries of the three different zones in the weak layer: (a) in the far field, the 'net' stresses and displacements vanish; (b) at the boundary between the elastic shearing zone and the process zone, the plastic deformation is zero and the peak strength is mobilized and (c) at the boundary between the process zone and the fully softened zone the shear stress reaches the residual value.
Zhang et al.\(^{22}\) showed that Equation (19), which was originally derived for a 1D growth of the slip surface, also applies to the case where two dimensional slip surface growth is considered in an IPS, for the reason that the compatibility requirements of stresses and displacements between the different zones are independent of the geometry and direction of propagation. Moreover, the gravitational shear stress at the bottom of the sediment layer remains unchanged in both problems and is equal to \(\gamma' h \sin \alpha\). In the same manner, for convex geometry, the same compatibility requirements hold and as was previously shown, the gravitational shear stress at the bottom of the sediment layer is equal to that of IPS at distances greater than \(5h\) from the crest. Therefore, Equation (19) may be used as the right hand side of the governing Equation (16) in those regions of the slope where the stress state corresponds to that of IPS. The process of propagation of shear bands near the crest is out of the scope of this paper and should be treated separately.

The governing Equation (16) is a nonlinear elliptical partial differential equation, and will be solved numerically using a finite difference (FD) scheme and later benchmarked against finite element simulations.

### 2.2 Solution of the governing equation using the FD approach

#### 2.2.1 The FD scheme

The problem's domain is divided into \(N_\xi\) by \(N_\theta\) grid points spaced \(\Delta_\xi\) and \(\Delta_\theta\) apart along the downslope and circumferential axes, respectively (Figure 7). The governing equation is approximated using the second order central finite difference scheme. The downslope grid index is denoted as \(i\) whereas the circumferential index as \(j\). The governing equation therefore reads

\[
\frac{\dot{u}_i^{j+1} - 2\dot{u}_i^{j} + \dot{u}_i^{j-1}}{\Delta_\xi^2} + \left(\frac{a}{a + \Delta_\xi (i - 1)}\right)^2 \frac{\dot{u}_i^{j+1} - 2\dot{u}_i^{j} + \dot{u}_i^{j-1}}{\Delta_\theta^2} = \Delta r \cdot \dot{\psi}_i^j
\]

$$\Delta r \cdot \dot{\psi}_i^j = \begin{cases} 
\frac{\dot{u}_i^j}{(1 - \beta^2)} & \dot{u}_i^j \leq (1 - \beta^2) (1 - r) \quad \text{Elastic} \\
(1 - r - \dot{u}_i^j) / \beta^2 & (1 - \beta^2) (1 - r) < \dot{u}_i^j \leq 1 - r + r \beta^2 \quad \text{Process zone} \\
- r & 1 - r + r \beta^2 < \dot{u}_i^j \quad \text{Fully softened}
\end{cases} \tag{23}$$

where \(\Delta_\xi = \Delta_\xi / l_{u,\xi}\) and \(\Delta_\theta = \Delta_\theta / \theta_u\).

The formulated FD scheme requires prior knowledge of the state of deformation; that is, whether the specific grid point mechanical behavior is elastic, at yielding or has softened to the residual strength. To this end, each grid point is first assigned with a mode attribute, which stores its current state: “elastic,” “process zone” or “fully softened.” Since the boundaries of the process zone are a priori unknown, all grid points which are not associated with the initial PSZ are
initially assigned as “elastic.” In each iteration, the grid points’ displacements are obtained and compared with those in Equation (23), after which the mode attribute is updated.

2.2.2 Boundary conditions

The governing equation is symmetric about the downslope axis of the PSZ. Hence, from the Neumann boundary conditions it follows that \( \hat{u}_i^2 = \hat{u}_i^0 \) \( \forall i \). The Dirichlet boundary conditions at the downslope far end and at the circumferential far end are set to zero, i.e. \( \hat{u}_{N_i} = \hat{u}_{N_j} = 0 \) \( \forall i, j \). At the crest, the change in downslope normal stress is prescribed zero, which results in \( \hat{u}_j^2 = \hat{u}_j^0 \) \( \forall j \). The indices \( i, j = 0 \) refer to ghost nodes extending out of the model boundaries, which are used only to prescribe the boundary conditions. The model boundaries \( L \) and \( \theta_{max} \) are adopted sufficiently large not to affect the solution.

2.3 Finite element modeling of progressive failure

2.3.1 Model setup

The process of the slip surface growth induces large shear strains in the weak layer that cannot be accommodated within the standard Lagrangian FEM formulation due to significant distortion of the elements leading to loss of accuracy. The Coupled Eulerian Lagrangian (CEL) technique, built within the ABAQUS computing environment, allows modeling extreme large deformations and has been proven in the past to be suitable for modeling similar problems.\(^{15,29-32}\) This technique uses an Eulerian mesh, fixed through the course of the analysis, which is filled initially with different materials proportionally to the element volume fraction. The analysis is comprised of Lagrangian steps, where the elements are allowed to deform, followed by Eulerian re-meshing and mapping of the material points’ state variables back onto the undeformed Eulerian domain. The method is based on a dynamic explicit scheme using 8-node Eulerian brick elements with reduced integration. The relationships between curvilinear and Cartesian coordinates (Equation (1)) were applied to determine the nodal positions used to construct the FEM mesh. The weak layer is modeled using a row of elements of finite thickness, \( s \), positioned at the bottom of the model overlain by the elastic layer and void elements (Figure 8). The void elements are required to ensure that no material is lost from the analysis.

2.3.2 User defined constitutive laws

Following the conceptual physical model, three different materials are defined in the analysis; (a) elastic material for the sliding layer, (b) elastic perfectly plastic material in the PSZ and (c) an elastic-plastic material with strain softening for the elements within the weak layer outside the PSZ. The elastic behavior is assumed similar for all the materials and the residual strength of the process zone is set equal to the strength of the PSZ. The initial configuration of the materials in the Eulerian domain is presented in Figure 8. In the CEL technique, the initial assignment of material properties of an element may change through the course of the analysis, as the materials are allowed to “flow” freely through the Eulerian domain subject to the applied loads and boundary conditions. To overcome this problem, a user defined subroutine was written to ensure that each material behavior is restricted to the corresponding area in the model. The subroutine uses the von-Mises failure criterion to describe the deviatoric stress strain behavior of the user defined material. The Mises stress

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**Figure 8** Conceptual model vs. finite element representation
is defined as follows

\[ q_y = \sqrt{3J_2} \]  \hspace{1cm} (24)

where \( J_2 \) is the second invariant of the deviatoric stress tensor. To describe the deviatoric response, the Mises criterion was matched to the Tresca failure criterion in plane strain conditions so that

\[ q_y = \sqrt{3 \cdot \tau_y} \]  \hspace{1cm} (25)

The yield shear stress \( \tau_y \) of the weak layer is expressed as function of the plastic shear strains by

\[ \tau_y = \tau_y (\gamma_P) = \frac{s}{\delta_P} \]  \hspace{1cm} (26)

where \( s \) is the thickness of the weak layer, \( \gamma_P \) and \( \delta_P \) are the engineering plastic shear strain in the weak layer and plastic displacement of the sliding layer, respectively. The plastic shear strain in the weak layer and plastic displacement of the sliding layer required to cause full softening of the weak layer are given by

\[ \gamma_P = \frac{\delta_P}{s} \]  \hspace{1cm} (27)

Scaling the required shear strain to cause the material to soften using the thickness of the shear surface eliminates mesh dependency of the solution as shown by Stoecklin et al. \( ^{15} \).

The use of the Mises stress requires the use of its work conjugated strain measure, the accumulative equivalent plastic strain, obtained by integration of the equivalent plastic strain rates

\[ \bar{\varepsilon}^p = \int_0^t \dot{\varepsilon}^p dt \]  \hspace{1cm} (28)

\[ \bar{\varepsilon}^p = \sqrt{\frac{2}{3}} \varepsilon^p : \dot{\varepsilon}^p = \sqrt{\frac{2}{3}} \varepsilon^p \varepsilon^p = \frac{\dot{\varepsilon}^p}{\sqrt{3}} \]  \hspace{1cm} (29)

where \( \varepsilon^p \) is the plastic strain rate tensor and \( \dot{\varepsilon}^p \) is the shear strain rate. The integration results in

\[ \bar{\varepsilon}^p = \int_0^t \bar{\varepsilon}^p dt = \int_0^t \frac{\dot{\varepsilon}^p}{\sqrt{3}} dt = \frac{\gamma_P}{\sqrt{3}} \]  \hspace{1cm} (30)

Considering the above definitions, the deviatoric stress strain behavior of the user defined material can be expressed by

\[ q_y = \tau_p \sqrt{3} \cdot \begin{cases} \frac{1}{S_t} & \text{PSZ} \\ \max \left[ 1 - \frac{S_t - 1}{S_t \cdot \varepsilon^p \cdot \bar{\varepsilon}^p} ; \frac{1}{S_t} \right] & \text{Process Zone} \\ \infty & \text{Sliding layer} \end{cases} \]  \hspace{1cm} (31)

\[ \varepsilon^p = \frac{\gamma_P}{\sqrt{3}} = \frac{\delta_P}{s \sqrt{3}} \]  \hspace{1cm} (32)

where \( S_t = \tau_p / \tau_r \) is the sensitivity parameter and \( \varepsilon^p \) is the equivalent plastic strain required to cause full softening of the material.
The material behavior is controlled by assigning a predefined field variable to the nodes in each one of the distinguished zones. This field variable remains constant at the nodes and thereby assigns the appropriate material behavior (Equation (31)) in each element and consequently into each integration point.

2.3.3 Loading sequence and boundary conditions

The FEM allows obtaining the critical value of gravity \( a_{cr} \) that will cause instability of the sliding layer. For this purpose, the gravity level is increased until the sliding layer cannot further maintain stability and the kinetic energy experiences a rapid increase. The criterion of Zhang et al.,28 is used as a first estimate for \( a_{cr} \). The critical shear stress ratio \( r_{cr} \) is obtained by Equation (21) from which the critical gravitational shear stress can be calculated. The critical gravity is then evaluated using the definition of the critical gravitational shear stress

\[
\tau_{g,cr} = \alpha_{cr} \cdot \rho h \sin \alpha
\]  

where \( \rho \) is the bulk density of the soil.

The analysis consists of two steps; in the first step, the gravitational acceleration is ramped up to the value of 90% of the Zhang et al.,28 critical acceleration. In the consecutive step, the gravity level increases incrementally by 0.5% of the estimated \( a_{cr} \) followed by 10 seconds 'quiet time' in order to assure stability is maintained at each incremental increase in acceleration. In order to find the true critical acceleration, this process continues until the acceleration magnitude reaches about 110% of the Zhang et al.,28 critical acceleration.

Zero displacement boundary conditions cannot be imposed as the Eulerian mesh is stationary during the analysis. Instead, a zero velocity boundary condition is applied. The imposed boundary conditions include: (a) zero circumferential velocities at the circumferential boundary planes, i.e. \( v_\theta (\theta = 0) = v_\theta (\theta = \theta_{max}) = 0 \); (b) The rigid base is simulated by restraining the velocities of the base nodes in all directions; (c) the velocity of the nodes at the center of revolution are set zero aside for the component in the Cartesian z direction and (d) the far downslope end is fully constrained.

3 BENCHMARK OF THE AXISYMMETRIC CASE

The axisymmetric problem serves as a benchmark for the proposed methodology in order to ascertain the main assumptions used in formulating the governing equation. The axisymmetric case results in propagation only in the downslope direction allowing easier comparison between all models. Although axisymmetric conditions are not realistic since they require failure of a nearly full circular section, they serve as the limiting case where \( l_{0,\theta} > l_{0,\xi} \) (see Figure 2). Axial symmetry requires that the downslope displacements are identical across circumferential planes and hence the governing Equation (16) reduces to

\[
\frac{d^2 \hat{u}}{d\xi^2} = \Delta r^* \]

Equation (34) can be combined with Equations (19) and solved analytically. The solution to this equation is similar to the one obtained by Zhang et al.,28 for IPS and can be presented in terms of the critical shear stress ratio \( r_{cr} \) and the length of the process zone at the onset of catastrophic failure \( \omega_{cr} \):

\[
\frac{\tau_{g,cr} - \tau_r}{\tau_p - \tau_r} = \frac{2l_{u,x}}{l_{0,x} + 2l_{u,x}}
\]

\[
\frac{\omega_{cr}}{l_{u,x}} = \beta \arcsin \beta
\]

where \( l_{0,x} \) and \( l_{u,x} \) are the downslope length of the PSZ and the downslope characteristic length of the planar slope problem; \( \beta \) and \( l_{u,\xi} = l_{u,x} \) are given by Equations (20) and (17), respectively.
TABLE 2  Base case model parameters for benchmark analysis

| Parameter                                      | Symbol | Value |
|-----------------------------------------------|--------|-------|
| Sediment thickness (m)                        | $h$    | 10.0  |
| Slope's downslope length (m)                  | $L$    | 1000.0|
| Crest radius (m)                              | $R_0$  | 50.0  |
| Downslope distance to initial PSZ (m)         | $\xi_0$ | 500.0 |
| Downslope length of the PSZ (m)               | $l_{0,\xi}$ | 90    |
| Slope inclination (degrees)                   | $\alpha$ | 5     |
| Submerged specific weight (kN/m$^3$)          | $\gamma'$ | 6.0   |
| Young's modulus of both layers (kPa)          | $E$    | 1495  |
| Shear modulus of the weak layer (kPa)         | $G_w$  | 500   |
| Poisson ratio of both layers                  | $\nu$  | 0.495 |
| Peak shear strength (kPa)                     | $\tau_p$ | 10.0  |
| Sensitivity                                   | $S_t$  | 3, 5, 8|
| Plastic displacement to residual (m)          | $\delta p_r$ | 0.05, 0.1, 0.2, 0.5 |
| FE mesh size/FD grid size (m)                 | $h_{el, \Delta \xi}$ | 0.5   |
| Finite differences total error threshold      |       | $10^{-5}$ |

FIGURE 9  Comparison between the benchmark models results

In the context of the convex geometry, Equations (35) and (36) can be re-written using the curvilinear terms

$$ r_{cr} = \frac{2l_{u,\xi}}{l_{0,\xi} + 2l_{u,\xi}} $$

(37)

$$ \frac{\omega_{cr}}{l_{u,\xi}} = \beta \arcsin \beta $$

(38)

The model parameters used in this study for both the FD and FEM calculations are listed in Table 2. The propagation of slip surfaces (confined within the weak layer) is explored by varying the weak layer’s strength properties, namely (a) the ratio between peak and residual strengths ($S_t$) and (b) the residual plastic displacement ($\delta p_r$). For all cases, the peak strength is held constant while the residual plastic displacement and the sensitivity (hence the residual strength) are varied. In the FEM simulation, the PSZ is placed sufficiently far from the crest, ensuring that stable propagation of the slip surface occurs only in the areas where the infinite slope stress field applies.

Figure 9 shows the FEM and FD calculations results in comparison to the analytical criteria (values provided in Table 6 in Appendix II: Discussion of the limitations of the FD and CEL approaches). The black continuous lines represent $r_{cr}$ and
based on the analytical solutions (37) and (38). The circles and triangles represent the results of the CEL simulations and the FD calculations, respectively. The left figure shows the prediction of \( r_{cr} \) at which catastrophic failure takes place. As expected, the FD scheme provides a good estimate as it solves the governing equation directly. The CEL results show a remarkable match to the analytical criterion despite the fact that in the FEM analysis, the deformations of the sliding layer are not averaged and the vertical stresses are free to change. The figure on the right-hand-side shows the predictions of \( \omega_{cr} \). While the FD calculations fit the analytical solution reasonably well, CEL results deviate more, which has also been observed by Zhang et al.\(^{28}\) Limitations of both approaches are discussed in Appendix II: Discussion of the limitations of the FD and CEL approaches.

To summarize, both the FD scheme and CEL framework can be considered reliable for the prediction of the initiation of the unstable slip surface growth in convex slopes. Furthermore, they show that averaging the stresses and strains along the sliding layer and neglecting the changes in slope perpendicular stresses (\( \Delta \sigma_{\eta\eta} \)) are valid assumptions. This justify their use in the subsequent 3D propagation analysis, for which an analytical criterion does not exist.

### 4 | A THREE-DIMENSIONAL SLIP SURFACE GROWTH

As previously shown, the load under which catastrophic growth of the slip surface takes place in the 1D planar and axisymmetric problems is closely related to the initial dimensions of the PSZ, namely its initial length in the downslope direction. The 2D planar problem\(^{22}\) and hence the 3D convex problem require the second dimension, which is the out-of-plane length of the PSZ. This dimension, in the context of the current problem, is denoted as \( l_{0,\phi} \) (Figure 2) and can be expressed by

\[
 l_{0,\phi} = \left[ R_0 + (\xi_0 + l_{0,\xi} / 2) \cos \alpha \right] \cdot \theta_0
\]

where \( \theta_0 \) is the PSZ section angle in radians. With the intention of establishing the criterion for the catastrophic 3D growth of the slip surface, the normalized aspect ratio and area of the PSZ, respectively, are defined by

\[
 \lambda = \bar{l}_{0,\phi} / \bar{l}_{0,\xi} = 2l_{0,\phi} / l_{0,\xi}
\]

\[
 \bar{A} = \bar{l}_{0,\phi} \cdot \bar{l}_{0,\xi}
\]

where the normalized dimensions are given by \( \bar{l}_{0,\phi} = l_{0,\phi} / l_{u,\phi} \) and \( \bar{l}_{0,\xi} = l_{0,\xi} / l_{u,\xi} \). Note that a very large aspect ratio \( \lambda \) indicates a very wide PSZ which ultimately reflects axisymmetry conditions. On the other hand a very small aspect ratio indicates a very narrow PSZ. The implication of the aspect ratio and its effect on the derived criterion are discussed below.

#### 4.1 | The bilinear analytical solution of planar geometry

In the axisymmetric problem, the solution of the governing equation is predominantly affected by the changes in the normal slope parallel force, while the change in slope parallel shear forces, resulting from the stress component \( \Delta \sigma_{\theta\xi} \), are negligible. However, for very narrow PSZ (\( \lambda << 1 \)), the change in lateral shear forces across the element prevail and the governing Equation (16) can be approximated by

\[
 \left( \frac{a}{a + \xi} \right)^2 \frac{d^2 \bar{u}_x}{d\xi^2} = \frac{\Delta \sigma_{\eta\xi}}{\tau_p - \tau_r}
\]

In the case of planar geometry, the squared term in front of the second derivative is equal to unity since the solution in planar slopes is independent of the position of the PSZ along the downslope axis. This allowed Zhang et al.,\(^{22}\) to obtain an analytical solution for the equation resulting the following criterion

\[
 r_{cr} = \frac{2l_{u,y}}{l_{0,y} + 2l_{u,y}}
\]

where \( l_{0,y} \) is length of the PSZ perpendicular to the direction of motion and \( l_{u,y} \) is the transverse characteristic length (\( = l_{u,\phi} \)) of the planar slope geometry. Both Equations (35) and (42) serve as bounds on \( r_{cr} \) in the planar case and were
Curvilinear shear stress distribution ($\sigma_{\theta\theta}$) in the weak layer at the onset of catastrophic failure for various aspect ratios with $\hat{A} = 50$ and $a = 1$.

**TABLE 3** Parameters used for the FD analysis of 3D slip surface growth

| Parameter                                      | Symbol | Value |
|------------------------------------------------|--------|-------|
| Sediment thickness (m)                        | $h$    | 10.0  |
| Downslope distance to PSZ (m)                 | $\xi_0$| 50.0  |
| Slope inclination (degrees)                   | $\alpha$| 5     |
| Submerged specific weight (kN/m$^3$)          | $\gamma'$| 6.0  |
| Young’s modulus of both layers (kPa)          | $E$    | 1495  |
| Shear modulus of the weak layer (kPa)         | $G_w$  | 500   |
| Poisson ratio of both layers                  | $\nu$  | 0.495 |
| Peak shear strength (kPa)                     | $\tau_p$| 10.0  |
| Sensitivity                                   | $S_t$  | 5     |
| Plastic displacement to residual (m)          | $\delta^p$| 0.25 |
| Downslope characteristic length (m)           | $l_{u,\xi}$| 25  |
| Circumferential characteristic length (m)     | $l_{u,\theta}$| 12.5 |
| Finite differences total error threshold      |       | $10^{-5}$ |

summarized as one bilinear criterion

\[
\frac{2}{\hat{l}_{0,x}} \left( \frac{1}{r_{cr}} - 1 \right) = \begin{cases} \\
1 & \hat{l}_{0,y}/\hat{l}_{0,x} > 1 \\
\hat{l}_{0,y}/\hat{l}_{0,x} & \hat{l}_{0,y}/\hat{l}_{0,x} \leq 1 
\end{cases}
\] (43)

With $\hat{l}_{0,y}$ and $\hat{l}_{0,x}$ being the normalized dimensions (by the corresponding characteristic lengths $l_{u,y}$ and $l_{u,x}$) of the planar PSZ.

Equation (41) could not be solved analytically for the convex geometry with the prescribed boundary conditions.

Figure 10 illustrates the process of the three-dimensional slip surface growth for curvilinear geometry. The solution was obtained using the FD scheme for a given normalized area and different aspect ratio values (using the model parameters provided in Table 3). The horizontal and vertical axes represent the normalized distance and normalized angle measured from the mid axes of the PSZ. Note that the far end extent is not identical in all models for the purpose of increasing...
computational accuracy and efficiency. Furthermore, because of the different aspect ratios, the initial downslope length of the PSZ $l_0,\xi$ is different in each case. This results in a different intercept value with the vertical axis, $(-\xi_0 - l_0,\xi/2)/l_0,\xi$. The white rectangles indicate the PSZ where the shear stress $\sigma_{\eta,\xi}$ in the weak layer corresponds to the residual strength, $\tau_r$. The black contour represents the border between the process zone and the elastic shearing zone where the peak strength ($\tau_p$) is mobilized. It is evident that for small aspect ratios, the border between both zones does not align with a single value of the angle $\theta$ but varies over the downslope axis. From the black contour outwards the stresses reduce to the gravitational stresses which are different for each case considering that catastrophic failure takes place under different shear stress ratios depending on the aspect ratio, as will be shown in a later stage. As the aspect ratio increases, the change in slope parallel forces becomes dominant and the slip surface grows almost solely downslope, similar to the axisymmetric case. This can be observed at the bottom figure in Figure 10, where the propagation is very limited in the circumferential direction.

4.2 Nonlinear planar slope criterion

Application of the bilinear criterion (43) for unstable growth of slip surfaces in planar slopes is limited to the two extreme cases, in which the PSZ aspect ratio is either very large or very small. For intermediate values of the aspect ratio this analytical criterion will result in too conservative estimates for $r_cr$. Zhang et al.\textsuperscript{22} have shown that the solution of the governing equation of a planar slope is symmetric about the aspect ratio of unity. By solving numerically the governing equation and curve fitting of the results, they derived a nonlinear criterion for the catastrophic growth of the slip surface in 2D planar geometry

$$\frac{2}{l_{0,x}} \left( \frac{1}{r_{cr}} - 1 \right) = \frac{l_{0,y}/l_{0,x}}{1+(l_{0,y}/l_{0,x})^{2a}} \frac{l_{0,x}}{l_{0,y}} \left[ 1+\left(\frac{l_{0,y}}{l_{0,x}}\right)^{2a} \right]^{1/a_3}$$

$$a_3 = 0.31 + 0.21 \ln (\tilde{A})$$

$a_3$ is a fitting parameter, numerically correlated to the normalized area of the PSZ $\tilde{A} = \tilde{l}_{0,y} \cdot \tilde{l}_{0,x}$

Note that their suggested functional (44) is symmetric about aspect ratio of unity and fulfills the limiting values of Equation (43).

The solution of the convex problem, however, is non-symmetrical about aspect ratio of unity due to the squared term before the second term of the governing Equation (16), repeated for convenience below

$$\frac{d^2 \Delta u_\xi}{d \xi^2} + \left( \frac{a}{a+\xi} \right)^2 \frac{d^2 \Delta u_\xi}{d \xi^2} = \frac{\Delta \sigma_{\eta,\xi}}{\tau_p - \tau_r} a = \frac{R_0}{l_{u,\xi} \cos \alpha}$$

The term $\left( \frac{a}{a+\xi} \right)^2$ depends on the radius of the crest $R_0$ and weakly on the slope inclination, as for most submarine slopes $\cos \alpha \approx 1$. Moreover it depends on the normalized downslope coordinate, which implicitly encapsulates the initial position of the PSZ along the downslope axis ($\xi_0$). Both terms $R_0$ and $\xi_0$ are directly linked to the curvature of the PSZ which is determined by the distance from the center of revolution, $R(\xi)$. The further away from center of revolution the initial PSZ is positioned, the milder becomes the curvature and the results should become closer to those of a planar slope. This implies that the nonlinear relationship curve for convex geometry will depend on both parameters $a$ and $\xi_0$.

Formulating a criterion that will not only depend on the normalized area and aspect ratio of the PSZ but also on the location of the initially PSZ and the crest radius may yield a cumbersome and somewhat impractical expression. A simpler criterion can be established by considering some practical simplifications which will be elaborated hereafter. Notwithstanding, one may solve Equation (16) using the outlined FD numerical scheme in order to predict the failure load for specific geometry and soil properties.

4.3 The convex slope criterion

4.3.1 The implication of slope curvature on the failure load

The curvature of the slope around the area of the PSZ is determined by the crest radius $R_0$ (encapsulated in the non-dimensional parameter $a$) and by the distance from the crest to the upper boundary of the PSZ, $\xi_0$. The study of the
The non-linear criterion results for different positions of the initial PSZ with constant initial normalized area of $\hat{A} = 50$ and $a = 1$.

Effect of slope curvature on the slope stability is therefore divided into two parts. The influence of $\xi_0$ on the solution is illustrated in Figure 11, which presents numerical results obtained for a given area of the PSZ $\hat{A} = 50$ and a constant crest radius $R_0$ corresponding to $a = 1$. In the calculations, the normalized distances from the crest along the downslope axis, $\xi_0/h$, was varied. The results of the planar slope (marked as grey markers in the figure) were obtained by solving numerically the governing equation of planar slopes outlined in Zhang et al. using a modified FD scheme. The criterion of Zhang et al. i.e. Equation (44), was not used in this case as it involves averaging of a large dataset and may, for the sake of comparison, bias the results. According to Figure 11, for increasing values of $\xi_0/h$ (i.e. decreasing curvature) the non-linear criterion for convex slope reverts to that of planar slope as expected. Using the planar criterion for convex slopes would be, an attractive option if it gave a conservative estimate for $r_{cr}$. Unfortunately, this is not the case, because the planar slope criterion in Figure 11 gives higher critical shear stress ratios than those obtained from the convex slope calculations.

To be able to apply the IPS stress field to the convex geometry, the downslope position of the initially softened zone, $\xi_0$, must be at a distance greater than $5h$ as previously discussed. Strictly speaking, the downslope position must be even greater than this value so that even at the stage where the slip surface has grown to its critical length, the stress field outside the elastic shearing zone will still correspond to that of IPS. This condition is met when $\xi_0 \approx 5h + \omega_{cr}$.

Figure 11 shows that decreasing values of $\xi_0/h$ (greater curvature) are associated with lower predictions for the failure load and therefore adopting $\xi_0 = 5h$ will constitute a conservative bound for the criterion. This makes the criterion independent of $\omega_{cr}$.

After constraining one geometry parameter by adopting $\xi_0/h = 5$, the influence of the parameter $a$ on the stability of the slope can be assessed by conducting a parametric study. As the governing equation suggests, this parameter will mainly affect the results where the change in lateral shear forces becomes dominant (i.e. in small aspect ratios) and on the other hand will have slight effect if any in the case of large PSZ aspect ratio.

Figure 12 presents the results obtained for various areas of the PSZ. The parameter $a$ was varied across four orders of magnitude, implying very large increase in the crest’s radius from about 0.25 m to 2500 m. The black horizontal line in the figures represent the axisymmetric solution (37), to which all results adhere regardless of the area size. For better illustration, the purple and grey dashed lines (corresponding to $a = 100$ and $a = 0.01$, respectively) in each of the figures represent the difference in percentage from the solution obtained for $a = 1.0$. As expected, the major difference appears at very small aspect ratios and relaxes towards larger values. The maximum difference obtained was about 19.8% for $\hat{A} = 400$ and $a = 100$ with the smaller critical shear stress ratio corresponding to $a = 1.0$. This is in agreement with the former conclusion that greater curvature, or alternatively shorter distances from the center of revolution, correspond to lower critical shear stress ratios. With aspect ratios value larger than 1.0 the results differed mostly less than 5% suggesting that the value of $a$ does not play a major role in this aspect ratio range. It can therefore be concluded that by formulating a single criterion using the smallest curvature possible will yield the most conservative estimate. However, the true dimensions of the slopes under discussion should also be considered. In order to satisfy both requirements, it is suggested that the criterion shall be formulated using $a = 1.0$, under which $R_0 \approx l_{u,\xi}$.
FIGURE 12  Results of nonlinear convex slope criterion for various areas of the PSZ and curvature determined by the parameter ‘α’

4.3.2  Formulation of the nonlinear criterion

Figure 12 shows that while all results converge categorically to the axisymmetric solution regardless of the size of the PSZ area, it appears that the solution converges at smaller aspect ratios the larger the normalized area becomes. This implies that the fitting functional should involve a truncated function with the truncation aspect ratio, denoted as $\lambda_L$, depending on the normalized area such that

$$2 \left( \frac{1 - \lambda}{\lambda} \right) = f(\lambda, \lambda_L(\hat{A}))$$

Three trial truncated functions were considered to be fitted to the results of the FD analysis:

a. a truncated hyperbolic function

$$f(\lambda, \lambda_L) = \frac{\lambda}{(1 + \lambda) \alpha_H}$$

$$\alpha_H = (\lambda_L - 1) / \lambda_L$$

b. A truncated logarithmic function

$$f(\lambda, \lambda_L) = \lambda - \alpha_L \cdot \lambda [\ln (1 + \lambda)]^R$$

$$\alpha_L = \frac{\lambda_L - 1}{\lambda_L [\ln (1 + \lambda_L)]^R}$$

$$R = \frac{(1 + \lambda_L) \ln (1 + \lambda_L)}{\lambda_L (\lambda_L - 1)}$$
and (c) a truncated elliptic function of the form:

\[ f(\lambda, \lambda_L) = \sqrt{1 - \left(\frac{\lambda - \lambda_L}{\lambda_L}\right)^2} \]  \hspace{1cm} (47)

All three functions are truncated such that they satisfy the condition \( f(\lambda \geq \lambda_L) = 1 \).

The results from the numerical analysis were obtained for 24 different values of normalized area ranging from 1–3000, each for which 12–18 aspect ratios were considered. The range of the aspect ratios for each normalized area was determined individually as the maximum aspect ratio (corresponding to \( \theta_0 = 2\pi \)) decreases with increasing area of the PSZ.

Figure 13 depicts the curve fitting of the three functions for the same normalized areas presented in Figure 12. For each value of the normalized area, the three functions were fitted to the results using the least squares method by modifying the free parameter \( \lambda_L \). The logarithmic function seems to be the least fitting function as it is unable to provide the required curvature at moderate aspect ratios. For the normalized area values in the range of 20–100 the hyperbolic function shows the best fit however for smaller values, e.g. \( \hat{A} = 10 \) the fitting becomes very poor as the truncation point extends to \( \lambda_L = 127 \) (top left figure in Figure 13). The elliptic function shows a rather good fit in the entire range of normalized areas and especially at larger areas where both the logarithmic and hyperbolic functions under predict the truncation point. Another advantage of the elliptic function is its ability to provide a safe estimate in the lower aspect ratio range. The magnified area in the top right in Figure 13 shows the difference between the functions in the lower aspect ratio range. Both logarithmic and hyperbolic functions satisfy \( \frac{\partial f}{\partial \lambda}(\lambda = 0) = 1 \) and therefore yield similar results to the bi-linear planar slope criterion (43), which is non-conservative for convex geometries. The elliptic function on the other hand satisfies \( \frac{\partial f}{\partial \lambda}(\lambda = 0) \to \infty \).
FIGURE 14  Results and curve fitting of the nonlinear convex slope criterion (left) and the resulting relationship between the minimal section angle of the PSZ required to consider the problem axisymmetric and the normalized area of the PSZ (right)

which results in the curve passing slightly over and on top of the numerical results and therefore provides a safer estimation in comparison to the other functions for all values of $\hat{A}$.

In planar slope geometry, the two dimensions of the PSZ are independent and their length unrestricted so that for each value of $\hat{A}$ the aspect ratio may vary in the range of $0 < \lambda < \infty$. In the convex case, on the other hand, some geometrical constraints apply. Considering $R_0$ and $\xi_0$ restricted each to $l_{u,\xi}$ and $2h$, respectively, and $0 < \theta_0 \leq 2\pi$ the following limiting cases impose geometrical constraints: $\hat{A} \to \infty$ entails $l_{0,\xi} \to \infty$ and therefore $\lambda \to \frac{l_{u,\xi}}{2l_{u,\theta}} \theta_0 \cos \alpha = \theta_0 \cos \alpha$. In this case $\lambda_L(\hat{A})$ should adhere to $\lambda_L(\hat{A} \to \infty) = \lambda_{0,\infty} \cos \alpha$ where $\lambda_{L,\infty}$ and $\theta_{0,\infty}$ represent the limiting aspect ratio and initial section angle of the PSZ, respectively, at which the solution can be considered axisymmetric for infinitely large values of $\hat{A}$. The limiting case $\hat{A} \to 0$ does not impose strict mathematical constraints as either $l_{0,\xi} \to 0 (\lambda \to \infty)$ or $\theta_0 \to 0 (\lambda \to 0)$ satisfy this condition. The latter is not physical because axisymmetric conditions require $\theta_0 >> 0$. As a result the condition $\lambda_L(\hat{A} \to 0) \to \infty$ must be satisfied.

The numerically obtained results describing the relationship between $\lambda_L$ and $\hat{A}$ are plotted in Figure 14 (left) using the gray circular markers. A simple hyperbolic function was chosen to describe the relationship between $\lambda_L$ and $\hat{A}$

$$\lambda_L = \lambda_{L,\infty} + \frac{m}{\hat{A}} \quad (48)$$

Note that this function satisfies the geometrical constraints discussed above.

The parameter $\lambda_{L,\infty}$ is read out from the result as the limiting aspect ratio at $\hat{A} \to \infty$ and $m$ is obtained by curve fitting. The best fitting curve is shown using the solid black line in Figure 14 (left). Considering the asymptotic value of the curve and fitting the results using the least square method to the function in Equation (48) results in the following constants

$$\lambda_{L,\infty} = 4.4$$
$$m = 124.5 \quad (49)$$

This relationship provides the threshold value for the aspect ratio at which the problem can be considered axisymmetric or alternatively as an infinite planar slope. The convex slope criterion can then be summarized using Equations (47) and (49) by

$$\frac{2}{\theta_{0,\xi}} \left( \frac{1}{r_{cr}} - 1 \right) = \begin{cases} 1 - \left( \frac{\lambda - \lambda_L}{\lambda_L} \right)^2 & \lambda < \lambda_L \\ 1 & \lambda \geq \lambda_L \end{cases}$$

$$\lambda_L = 4.4 + \frac{124.5}{\hat{A}}$$

$$\lambda_L = 4.4 + \frac{124.5}{\hat{A}}$$

$\lambda_L = 4.4 + \frac{124.5}{\hat{A}}$
Another way to interpret the relationship between $\lambda_L$ and $\hat{A}$, and perhaps more intuitive, is by writing the section angle of the PSZ in terms of the aspect ratio using Equations (39) and (40)

$$\theta_{0,L}(\hat{A}) = \frac{\hat{A} \cdot \lambda_L(\hat{A}) / \cos \alpha}{\hat{A} + 2(a + \xi_0) \sqrt{\hat{A} \cdot \lambda_L(\hat{A})}}$$

(51)

where $\xi_0 = \xi_0/l_u \xi$

This relationship determines the minimal section angle of the PSZ, $\theta_{0,L}$, at which the convex slope problem can be simplified into the axisymmetric solution (right figure in Figure 14). The limits of this function therefore determine the minimal section angle below which the problem cannot be considered axisymmetric and the maximal section angle above which the problem can be considered axisymmetric

$$\lim_{\hat{A} \to \infty} \theta_{0,L}(\hat{A}) = \theta_{0,\infty} = \lambda_{L,\infty} / \cos \alpha$$

$$\lim_{\hat{A} \to 0} \theta_{0,L}(\hat{A}) = \theta_{0,\min} = \frac{\sqrt{m}}{2 (a + \xi_0) \cos \alpha}$$

(52)

Considering $m$ and $\lambda_{L,\infty}$ provided in (49) and the previous assumptions $a = 1$ and $\xi_0 = 5h$, the problem cannot be considered axisymmetric for any size of PSZ area if its section angle is smaller than $0.59\pi \approx 107^\circ$ and on the other hand can always be considered axisymmetric if the PSZ section angle exceeds $\theta_{0,\infty} = 1.4\pi \approx 253^\circ$.

5 | DISCUSSION

So far the investigation of the slip surface growth in convex geometry was manifested through rigorous examination of the curvature influence on the failure load. The criterion for the catastrophic slip surface growth was then correlated to the area and aspect ratio of the PSZ, i.e. Equation (50), by solving the governing equation using FD scheme. After derivation of the criterion, a comparison between the planar and convex slope criteria can be drawn in order to evaluate the difference between the two and most importantly, to shed light on the risks involved by applying the planar criterion to convex geometry. For the sake of comparison, two normalized variables are introduced:

$$F_c = r_{cr,c} / r_{cr,1D} = \frac{l_{0,\xi}/2 + 1}{l_{0,\xi}/2 \cdot n_c + 1}; \quad n_c = \min \left[ \sqrt{1 - \left( \frac{\lambda - \lambda_L}{\lambda_L} \right)^2}, 1 \right]$$

$$F_p = r_{cr,p} / r_{cr,1D} = \frac{l_{0,x}/2 + 1}{l_{0,x}/2 \cdot n_p + 1}; \quad n_p = \frac{\lambda}{(1 + \lambda^{2\alpha_3})^{1/\alpha_3}}$$

(53)

where the critical shear stress ratio of the convex problem, $r_{cr,c}$ is obtained from Equation (50); the critical shear ratio of the 2D planar problem, $r_{cr,p}$ is obtained from Equation (44) and the critical shear stress ratio of the axisymmetric/1D problem, denoted as $r_{cr,1D}$ is obtained from Equation (35). The parameters $\alpha_3$ and $\lambda_L$ are given in Equations (44) and (50), respectively.

Note that these ratios do not correspond to the actual safety factor but rather relate to the available safety margin depending on the residual strength (Equation (35)). Both ratios should approach unity as the convex/planar nonlinear solutions approach the axisymmetric/1D planar solutions, i.e. in the large aspect ratio range.
In order to assess the difference between the two, a normalized difference ($\bar{F}$) is introduced

$$\bar{F} = \frac{F_c - F_p}{F_c} \quad (54)$$

With this definition, $\bar{F} < 0$ indicates a higher shear stress ratio prediction by the planar criterion leading to non-conservative estimation of the failure load.

Figure 15 presents the comparison between both criteria for four different values of normalized area of the PSZ varying from 1–1000 and the parameters in Table 3. The black solid lines and the red dashed lines represent the distribution of $F_c$ and $F_p$, respectively, as function of the normalized aspect ratio. As $\lambda / \lambda_L$ approaches unity, $F_c$ approaches unity as well (marked as the solid grey lines) owing to the nonlinear criterion of the convex slope degenerating to the axisymmetric solution. The figure shows that both criteria yield very high safety margins in the low aspect ratio range and that this margin relaxes as the aspect ratio approaches $\lambda_L$. It follows that the 1D solution becomes overly conservative for small aspect ratios, where the 2D planar/3D convex criteria should be applied. The normalized difference between the two criteria $\bar{F}$ is depicted in the figure using the black dotted lines. According to the figure, $\bar{F}$ is almost exclusively negative, implying that the planar criterion almost always over-estimates the critical shear stress ratio of a convex slope with PSZ of similar size and dimensions. In fact, it is highly likely that in rare cases, when the convex slope criterion yields higher predictions for the critical shear stress ratio, it is only due to the numerical fitting of both criteria. The difference between the two seem to become larger the larger the area of the PSZ becomes and reach the value of 100%. However, it should be noted that the larger difference occurs normally in the low aspect ratio range, where failure is predominantly governed by growth of the slip surface in the circumferential direction, and reduces as the solution reverts to that of the axisymmetric/1D planar problems. The difference between the two criteria at $\lambda = \lambda_L$ originates from the fact that the nonlinear planar slope criterion approaches asymptotically the 1D planar solution and is not truncated as the convex slope criterion.
6 | EXTENSION TO CONCAVE SLOPES

It can be shown that the governing equations for the case of concave slopes (Figure 16) take the form of

\[
\frac{\partial \sigma_{\xi\xi}}{\partial \xi} + \frac{\partial \sigma_{\eta\xi}}{\partial \eta} + \frac{(\sigma_{\xi\xi} - \sigma_{\theta\theta}) \cos \alpha - \sigma_{\eta\xi} \sin \alpha + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}}{R(\xi, \eta)} = \gamma' \sin \alpha
\]

(55)

\[
\frac{\partial \sigma_{\xi\theta}}{\partial \xi} + \frac{\partial \sigma_{\eta\theta}}{\partial \eta} + \frac{2(\sigma_{\xi\theta} \cos \alpha - \sigma_{\eta\theta} \sin \alpha) + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}}{R(\xi, \eta)} = 0
\]

(56)

\[
\frac{\partial \sigma_{\eta\eta}}{\partial \eta} + \frac{\partial \sigma_{\xi\eta}}{\partial \xi} + \frac{(\sigma_{\theta\theta} - \sigma_{\eta\eta}) \sin \alpha + \sigma_{\xi\eta} \cos \alpha + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}}{R(\xi, \eta)} = \gamma' \cos \alpha
\]

(57)

where \(R_0\) is the radius of the toe as depicted in Figure 16 and \(R(\xi, \eta)\) is the horizontal distance on the xy plane from the Cartesian z axis to any given point on the concave slope expressed by

\[
R(\xi, \eta) = R_0 + \xi \cos \alpha - \eta \sin \alpha
\]

(58)

Following the procedure for a convex element, the equilibrium of the ‘net’ stresses can be formulated for a concave element as

\[
\frac{d\Delta \sigma_{\xi\xi}}{d\xi} - \frac{\Delta \sigma_{\eta\xi}}{h} + \frac{(\Delta \sigma_{\xi\xi} - \Delta \sigma_{\theta\theta}) \cos \alpha}{R(\xi)} + \frac{1}{R(\xi)} \frac{d\Delta \sigma_{\theta\theta}}{d\theta} = 0
\]

(59)

Similar to convex slopes, the equilibrium equations for concave slopes degenerate to those of IPS with increasing distance from the toe. It was verified that the same criterion of the minimum distance of \(5h\) from the crest required for the degeneration of the convex stress field to that of IPS could be adopted to concave slopes but with distance larger than \(5h\) from the toe instead of from the crest. Note that sign of the third term in Equation (59) differs from that in Equation (15). In contrast to convex geometry, the forces resulting from the annular ring effect in the concave case have a component acting against the direction of motion, thus providing additional stabilizing force to the shear resistance at the bottom of the sliding layer. As the curvature of the slope decreases with increasing distance from the toe, the contribution of this term becomes negligible and could also be omitted. Therefore, both governing equations (15) and (59) can be approximated by Equation (16), resulting in the same criterion (50) for both geometries. For the concave geometry, however, this criterion is slightly more conservative because of neglecting the small stabilizing forces. Closer to the toe, the lateral confinement provided by the surrounding soil has a larger stabilizing effect on the concave slope potentially preventing global failure, which explains why concave geometries are in general more favorable in terms of stability compared to convex geometries, as observed by, e.g., Zhang et al.,\textsuperscript{34} and Kelesoglu.\textsuperscript{35}
TABLE 4 Parameters for application example

| Parameter                                      | Symbol | Value |
|-----------------------------------------------|--------|-------|
| Sediment thickness (m)                        | \( h \) | 10.0  |
| Crest radius (m)                              | \( R_0 \) | 350.0 |
| Downslope distance to initial PSZ (m)         | \( \xi_0 \) | 200.0 |
| PSZ angle (degrees)                           | \( \delta_0 \) | 7     |
| Downslope length of the PSZ (m)               | \( l_{0,\xi} \) | 100.0 |
| Slope inclination (degrees)                   | \( \alpha \) | 5     |
| Submerged specific weight (kN/m³)              | \( \gamma' \) | 6.0   |
| Young's modulus of both layers (kPa)           | \( E \) | 1500  |
| Poisson ratio of both layers                   | \( \nu \) | 0.5   |
| Peak shear strength (kPa)                     | \( \tau_p \) | 10.0  |
| Residual shear strength (kPa)                  | \( \tau_r \) | 2.0   |
| Plastic displacement to residual (m)          | \( \delta_{p,r} \) | 0.25  |

7 | APPLICATION EXAMPLE

Consider a convex slope as depicted in Figure 2. The geometry and properties of the sediment layer and weak layer are given in Table 4.

Based on the suggested framework, the factor of safety of the slope against catastrophic failure can be calculated according to the following steps:

(a) Define the geometry of the pre-softened zone (Equation (39)).

(b) Use the geometry and soil parameters to calculate the characteristic lengths \( l_{0,\xi} \) and \( l_{0,\theta} \) using Equations (17) and (18).

(c) Calculate the normalized downslope \( \left( \hat{l}_{0,\xi} = l_{0,\xi}/l_{u,\xi} \right) \) and circumferential \( \left( \hat{l}_{0,\theta} = l_{0,\theta}/l_{u,\theta} \right) \) lengths of the PSZ as well as its initial normalized aspect ratio \( \hat{\lambda} = \left( l_{0,\theta}/l_{0,\xi} \right) \) and area \( \hat{A} = l_{0,\theta} \cdot \hat{l}_{0,\xi} \).

(d) Use the nonlinear criterion (Equation (50)) to estimate \( r_{cr} \).

(e) Use Equation (35) together with the obtained \( r_{cr} \) and peak and residual strengths to obtain the critical shear stress at the bottom of the sediment layer \( \tau_{g,cr} = \sigma_{0,cr}/\gamma \) for initiation of the unstable slip surface growth.

(f) Use Equation (10) to calculate the gravitational shear stress at the bottom of the sediment layer, \( \tau_g (-h) \).

(g) Calculate the factor of safety by \( FS = \tau_{g,cr}/\tau_g (-h) \)

The above calculations are provided in Table 5. The results show that given the above parameters and geometry, if the shear stress at the bottom of the sediment layer will increase (for example due to earthquake loading) from its gravitational value of 5.23 kPa to its critical value of 5.89 kPa, the slope will fail catastrophically. The overall factor of safety is therefore \( FS = \sigma_{0,cr}/\tau_g = 5.89/5.23 \approx 1.12 \).

A 3D CEL model was generated to simulate the example problem using the parameters listed in Table 4. The shear stress ratio was ramped gradually to a value of 0.4275 over a period of 20 seconds. This was achieved by increasing the gravity level and calculating the shear stress ratio using Equations (33) and (35). In the second step, the shear stress ratio was increased slowly similar to the loading sequence of the benchmark analysis to the value of 0.495. The kinetic energy of the model was monitored and the critical shear stress ratio was determined as the one corresponding to the moment of sudden and rapid increase in kinetic energy. The evolution of the kinetic energy with increasing shear stress ratio is provided in the supplementary material. The resulting critical shear stress ratio is 0.47 which is only 3% lower than the one obtained by the suggested analytical criterion.

8 | SUMMARY AND CONCLUSIONS

The paper proposes a framework for predicting catastrophic failure in 3D convex and concave conical slopes due to the growth of slip surfaces from an initially weakened zone. The suggested framework applies to the portions of the slopes
TABLE 5  Calculated parameters for evaluating the factor of safety of the example case

| Parameter                              | Symbol | Equation | Value |
|----------------------------------------|--------|----------|-------|
| Circumferential length of PSZ (m)      | $l_{0,\theta}$ | $\approx [R_0 + (\xi_0 + l_{0,\xi}/2) \cos \beta]_0$ | 73.18 |
| Plane-strain modulus (kPa)             | $E'$   | $E/(1 - \nu^2)$ | 2000.0|
| Shear modulus (kPa)                    | $G$    | $E'/4$    | 500.0 |
| Downslope characteristic length (m)    | $l_{u,\xi}$ | Equation (17) | 25.0  |
| Circumferential characteristic length (m) | $l_{u,\theta}$ | Equation (18) | 12.5  |
| Normalized downslope length            | $\hat{l}_{0,\xi}$ | $l_{0,\xi}/l_{u,\xi}$ | 4.0   |
| Normalized circumferential length      | $\hat{l}_{0,\theta}$ | $l_{0,\theta}/l_{u,\theta}$ | 5.85  |
| Normalized aspect ratio                | $\hat{\lambda}$ | $l_{0,\theta}/l_{0,\xi}$ | 1.46  |
| Normalized area                        | $\hat{A}$ | $l_{0,\xi} \cdot l_{0,\theta}$ | 23.42 |
| Critical shear stress ratio            | $r_{cr}$ | Equation (50) | 0.486 |
| Critical shear stress at the bottom of the sediment layer (kPa) | $\tau_{g,cr}$ | $\tau_r + r_{cr} \cdot (\tau_p - \tau_r)$ | 5.89  |
| Gravitational shear stress at the bottom of the sediment layer (kPa) | $\tau_g(-h)$ | $\gamma' h \sin \alpha$ | 5.23  |
| Factor of safety                       | $FS$   | $\tau_{g,cr}/\tau_{g}(-h)$ | 1.12  |

where the stress state approaches that of infinite planar slopes, which was shown to occur at distances greater than five thicknesses of the sliding layer from the slope crest (or toe) for the case of convex (or concave) slopes. The governing equation for the 3D propagation was derived in the local curvilinear coordinate system and solved analytically for the axisymmetric case in which the aspect ratio of the initially weakened zone approaches theoretically infinity. The finite differences scheme, formulated to solve the governing equation for intermediate values of the aspect ratio, was first benchmarked against large-deformations finite element simulations for the axisymmetric case. The results were shown to be in good agreement, validating the suggested framework for subsequent analysis of the 3D problem.

The finite differences scheme was then used to establish a nonlinear criterion to three-dimensional slip surface growth in convex geometries. The numerical investigation showed that increased convexity leads to decreased stability of the slope and as such suggests that using the planar slope criterion to predict failure of convex slopes is not conservative. In fact, it was shown that for small aspect ratios of the pre-softened zone geometry, the planar slope criterion may over-predict the critical shear stress ratio by as much as 100%. Considering the geometrical restrictions arising from the complex geometry, we derived the minimum section angle of the pre-softened zone for which the convex problem solution degenerates to the axisymmetric one.

Though the criterion derived in the paper was developed for convex geometries, it was shown to be valid for concave slopes and provide a safe estimate for the critical shear stress ratio. The suggested framework provides an easy to use criterion that for the first time could be applied to estimate the stability of true three-dimensional slopes subjected to

TABLE 6  Results of the benchmark models

| $\delta_p^b/s$ | $S_i$ | $l_{0,\xi}/l_{u,\xi}$ | $\beta$ | $\omega/l_{u,\xi}$ | $F_S$ | $r_{cr}$ |
|----------------|-------|------------------------|---------|---------------------|------|----------|
| $\delta_p^b/s$ |   |                       |         | Analytical          | CEL | FD       | Analytical          | CEL | FD       |
| 0.01           | 3    | 7.385                  | 0.931   | 1.114               | 1.046| 1.190    | 0.213               | 0.206| 0.212    |
| 0.02           | 3    | 5.222                  | 0.966   | 1.265               | 1.204| 1.306    | 0.277               | 0.273| 0.276    |
| 0.04           | 3    | 3.693                  | 0.983   | 1.364               | 1.231| 1.313    | 0.351               | 0.356| 0.349    |
| 0.1            | 5    | 8.090                  | 0.917   | 1.062               | 0.989| 1.124    | 0.198               | 0.189| 0.197    |
| 0.02           | 5    | 5.720                  | 0.959   | 1.232               | 1.096| 1.303    | 0.259               | 0.254| 0.258    |
| 0.04           | 5    | 4.045                  | 0.980   | 1.342               | 1.371| 1.393    | 0.331               | 0.334| 0.330    |
| 0.1            | 5    | 2.558                  | 0.992   | 1.432               | 1.499| 1.421    | 0.439               | 0.446| 0.438    |
| 0.1            | 8    | 8.461                  | 0.908   | 1.035               | 0.799| 1.081    | 0.191               | 0.181| 0.190    |
| 0.02           | 8    | 5.983                  | 0.955   | 1.214               | 1.197| 1.230    | 0.251               | 0.245| 0.249    |
| 0.04           | 8    | 4.230                  | 0.978   | 1.330               | 1.410| 1.340    | 0.321               | 0.321| 0.320    |
failure due to the slip surface growth. The simple analytical form of the criterion can be utilized in risk assessment of large submarine basins after incorporating it into GIS based platforms.

ACKNOWLEDGEMENTS
The authors would like to thank Marc Kohler (ETH Zurich, Switzerland) for valuable discussions on the topic. The work was supported by the Swiss National Science Foundation, SNF grant 200021_168998.

DATA AVAILABILITY STATEMENT
The data that supports the findings of this study are available in the supplementary material of this article.

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APPENDIX I: THE APPLICABILITY OF THE ASSUMPTIONS MADE FOR OBTAINING THE GOVERNING EQUATION

Averaging the displacements across the thickness of the sliding layer is a common simplification imposed in order to obtain an analytical solution. This assumption was proven to be reasonable for the prediction of the failure load by validation against FEM analysis for the 2D solution by 28 and in the current work in the benchmark analysis of the axisymmetric problem. The assumption where the displacements occur only in the downslope direction was first adopted by 22 when modeling the 2D slip surface growth in planar slopes. In order to ascertain the validity of this assumption to the current problem, a finite element model of the entire 3D problem of convex geometry was modeled. The model parameters used are listed in Table 3. The modeling technique is identical to that used for the axisymmetric problem, only using a much larger number of elements and different boundary conditions similar to those imposed on the FD grid for modeling the 3D slope surface growth.

Figure 17 presents the nodal velocity vectors at the onset of catastrophic failure on top of the accumulated equivalent plastic strain distribution in the weak layer. The figure indicates clearly that the majority of the velocity vectors are aligned
FIGURE 17  Velocity vectors and equivalent plastic strains distribution in the weak layer at the onset of catastrophic failure for \( \hat{A} = 10 \) and \( a = 100 \) and aspect ratio of unity

with the downslope direction thus justifying the assumption that the velocities (and hence displacements) in the circumferential direction are negligible compared to the downslope components.

The out-of-plane ‘net’ strain (\( \Delta \varepsilon_{\theta \theta} \)) in convex geometry is related to the ‘net’ displacements using the following relation (Equation 23 in the supplementary material):

\[
\Delta \varepsilon_{\theta \theta} = \frac{\Delta u_\xi \cos \alpha + \Delta u_\eta \sin \alpha + \frac{\partial \Delta u_\theta}{\partial \theta}}{R(\xi, \eta)}
\]

It is reasonable to assume that this ‘net’ strain component vanishes at the portions of the slope, where the initial stress field is equivalent to that of IPS for two reasons: (a) both terms \( \frac{\Delta u_\xi \cos \alpha}{R(\xi, \eta)} \) and \( \frac{\Delta u_\eta \sin \alpha}{R(\xi, \eta)} \) approach zero since \( \Delta u_\xi, \Delta u_\eta \ll R(\xi, \eta) \) and (b) after establishing that \( \Delta u_\theta = 0 \) through the entire domain it follows that \( \frac{\partial \Delta u_\theta}{\partial \theta} = 0 \).

APPENDIX II: DISCUSSION OF THE LIMITATIONS OF THE FD AND CEL APPROACHES

For the axisymmetric case, predictions of the length of the process zone \( \omega_{cr} \), as well as the prediction of the critical shear stress ratio \( r_{cr} \), in both FD and CEL approaches (see Table 6). Shortcoming of the FD scheme lies at evaluation of the solution only at the grid points. This gives rise to a source of error in the prediction of \( \omega_{cr} \) as the choice of grid space influences its value. In the studied problem, this may induce an error of approximately 2.5-5% depending on the value of \( l_{u_\xi} \). The CEL results, on the other hand, are sensitive for the output sampling time. During the catastrophic failure, the shear band propagates at a very high velocity, resulting in large propagation length increments between two adjacent time sampling points. In the current analysis, the results were sampled every 0.5 seconds, meaning twenty times in each acceleration increase. The prediction of the gravity level is therefore rather accurate since in twenty sampling time points the acceleration is constant. However, under the same gravity level and during the catastrophic propagation, the length of the process zone changes drastically. Another source of error in the FEM calculation is introduced by the reduced integration scheme. The integration is performed over a single integration point rather than over eight as required for full integration eight-node brick linear element. As a result, the equivalent plastic strain is constant within the element which influences the measured length of the process zone, just as in the FD scheme. Regardless of the above, the numerical CEL results show a rather good fit, in both trend and absolute values.