Iterated Tabu Search Algorithm for Packing Unequal Circles in a Circle

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Abstract

This paper presents an Iterated Tabu Search algorithm (denoted by ITS-PUCC) for solving the problem of Packing Unequal Circles in a Circle. The algorithm exploits the continuous and combinatorial nature of the unequal circles packing problem. It uses a continuous local optimization method to generate locally optimal packings. Meanwhile, it builds a neighborhood structure on the set of local minimum via two appropriate perturbation moves and integrates two combinatorial optimization methods, Tabu Search and Iterated Local Search, to systematically search for good local minima. Computational experiments on two sets of widely-used test instances prove its effectiveness and efficiency. For the first set of 46 instances coming from the famous circle packing contest and the second set of 24 instances widely used in the literature, the algorithm is able to discover respectively 14 and 16 better solutions than the previous best-known records.

Key words: Packing, Circle packing, Global optimization, Tabu search, Iterated local search

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1. Introduction

Given \( n \) circles and a container of predetermined shape, the circle packing problem is concerned with a dense packing solution, which can pack all the circles into the smallest container without overlap. The circle packing problem is a well-known challenge in discrete and computational geometry, and it arises in various real-world applications in the field of packing, cutting, container loading, communication networks and facility layout (Castillo et al., 2008). In the field of global optimization, the circle packing problem is a natural and challenging test bed for evaluating various global optimization methods.

This paper focuses on solving a classic circle packing problem, the Packing Unequal Circles in a Circle (PUCC) problem. As indicated in previous papers (Addis et al., 2008b; Grosso et al., 2010; Hifi and M’Hallah, 2009), the PUCC problem has an interesting and important characteristic that it has a both continuous and combinatorial nature. It has continuous nature because the position of each circle is chosen in \( \mathbb{R}^2 \). The combinatorial nature is due to the following two facts: (1) A packing pattern is composed of \( n \) circles, and shifting a circle to a different place would produce a new packing pattern; (2) The circles have different radiuses, and swapping the positions of two different circles may result in a new packing pattern.

In this paper, we pay special attention to the continuous and combinatorial characteristic of the PUCC problem. We propose an algorithm which integrates two kinds of optimization techniques: A continuous local optimization procedure which minimizes overlaps between circles and produces locally optimal packing patterns, and an Iterated Tabu Search (ITS) procedure which exploits the combinatorial nature of the problem and in-
telligently uses two appropriate perturbation moves to search for globally optimal packing patterns.

The proposed algorithm is assessed on two sets of widely used test instances, showing its effectiveness and efficiency. For the first set of 46 instances coming from the famous circle packing contest, the algorithm is able to discover 14 better solutions than the previous best-known records. For the second set of 24 instances widely used in the literature, the algorithm can improve 16 best-known solutions in a reasonable time.

The rest of this paper is organized as follows. Section 2 briefly reviews the most related literature. Section 3 formulates the PUCC problem. Section 4 describes the details of the proposed algorithm. Section 5 assesses the performance of the algorithm through extensive computational experiments. Section 6 analyzes some key ingredients of the algorithm to understand the source of its performance. Finally, Section 7 concludes this paper and proposes some suggestions for future work.

2. Related Literature

Over the last few decades, the circle packing problem has received considerable attention in the literature. The simplest and most widely studied cases are the packing of equal circles in a square or in a circle. Though researchers have spent significant effort on the two problems, only a few packings (up to tens of circles) have been proved to be optimal by purely analytical methods and computer-aided proving methods (Szabó et al., 2007; Graham et al., 1998). A second category of research aims at finding the best possible packings without optimality proofs. Following this spirit, various heuristic approaches have been proposed, including: Billiard simulation
(Graham et al., 1998), minimization of energy function (Nurmela and Östergård, 1997), nonlinear programming approaches (Birgin et al., 2005, 2010), Population Basin Hopping method (Addis et al., 2008a; Grosso et al., 2010), formulation space search heuristic algorithm (López and Beasley, 2011), quasi-physical global optimization method (Huang and Ye, 2011), greedy vacancy search method (Huang and Ye, 2010) and so on. With these approaches, best-known packings for up to thousands of circles have been found, which are reported and continuously updated on the Packomania website (Specht, 2013).

There are also a number of papers devoted to the unequal circle packing problem. Most previous papers on the unequal circle packing problem can be classified into two categories: Constructive approaches and global optimization approaches. The constructive approaches build a packing by successively placing a circle into the container. These approaches usually include two important components: A placement heuristic, which determines several candidate positions for a new circle in the container, and a tree search strategy, which controls the tree search process and avoids exhaustive enumeration of the solution space. The widely used placement heuristics include the principle of Best Local Position (BLP) (Hifi and M’Hallah, 2004, 2007, 2008; Akeb and Hifi, 2010) and the Maximum Hole Degree (MHD) rule (Huang et al., 2005, 2006; Lü and Huang, 2008; Akeb et al., 2009). The tree search strategies include the self look-ahead search strategy (Huang et al., 2005, 2006), Pruned-Enriched-Rosenbluth Method (PERM) (Lü and Huang, 2008), beam search algorithm (Akeb et al., 2009) and the hybrid beam search looking-ahead algorithm (Akeb and Hifi, 2010).

The global optimization approaches formulate the unequal circle packing problem as a mathematical programming problem, then the task becomes to
find the global minimum of a mathematical model. These kind of approaches include the quasi-physical quasi-human algorithm by Wang et al. (2002), the Tabu Search and Simulated Annealing hybrid approach (Zhang and Deng, 2005), the Population Basin Hopping algorithm (Addis et al., 2008b; Grosso et al., 2007, 2010), the GP-TS algorithm by Huang et al. (2012a), the Iterated Local Search algorithm by Huang et al. (2012b), the Formulation Search Space algorithm by López and Beasley (2012) and the Iterated Tabu Search algorithm by Fu et al. (2013) for the circular open dimension problem.

For the circle packing problem, there also exist many important literature not mentioned here. Interested readers are referred to the review articles by Castillo et al. (2008) and Hifi and M’Hallah (2009), the book by Szabó et al. (2007) and the Packomania website (Specht, 2013).

3. Problem Formulation

Given $n$ disks, each having radius $r_i$ ($i = 1, 2, \cdots, n$), the PUCC problem consists in finding a dense packing solution, which can pack all $n$ disks into the smallest circular container of radius $R$ without overlap. We designate the container center as the origin of the cartesian coordinate system and locate disk $i$ ($i = 1, 2, \cdots, n$) by the coordinate position of its center $(x_i, y_i)$. The PUCC problem can be formulated as:

\[
\begin{align*}
\text{minimize} & \quad R, \quad \text{s.t.:} \\
\sqrt{x_i^2 + y_i^2} + r_i & \leq R \quad (1) \\
\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} & \geq r_i + r_j \quad (2)
\end{align*}
\]

where $i, j = 1, 2, \cdots, n; i \neq j$. Eq.(1) ensures that each disk is completely in the container and Eq.(2) guarantees that no overlap exists between any two
disks. Note that, this problem can also be formulated in other ways, see for example Birgin et al. (2005) and Grosso et al. (2010).

A packing solution is described by two variables: The radius of the container $R$ and the packing pattern denoted by the positions of all $n$ disks $X = (x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$. The infeasibility of a packing can be caused by two kinds of overlaps: Overlaps between two disks and overlaps between a disk and the exterior of the container. We define the overlapping depth between disks $i$ and $j$ ($i, j = 1, 2, \cdots, n; i \neq j$) as:

$$o_{ij} = \max\left\{0, r_i + r_j - \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}\right\}.$$  \hspace{1cm} (3)

and the overlapping depth between disk $i$ ($i = 1, 2, \cdots, n$) and the exterior of the container as:

$$o_{0i} = \max\left\{0, \sqrt{x_i^2 + y_i^2 + r_i - R}\right\}.$$  \hspace{1cm} (4)

Adding all squares of overlapping depth together, we get a penalty function measuring overlaps of a packing

$$E(X, R) = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} o_{ij}^2.$$  \hspace{1cm} (5)

Thus, a packing $(X, R)$ is feasible (non-overlapping) if and only if $E(X, R) = 0$.

Sometimes, we fix the radius of the container at a constant value $R$ and the penalty function becomes

$$E_R(X) = E(X, R).$$  \hspace{1cm} (6)

Note that, finding a packing pattern $X$ with $E_R(X) = 0$ corresponds to solving the following circle-packing decision problem (Birgin et al., 2005):
Given a circular container with fixed radius $R$, find out a feasible pattern $X$ which can pack all the circles into the container without overlap.

Our original PUCC problem aims to find the smallest container of radius $R^*$ and a corresponding non-overlapping packing pattern $X$. In practice, the PUCC problem can be solved as a serial of circle-packing decision problems with descending $R$ (Huang and Ye, 2010). The main steps are as follows:

1. Let $\bar{R}$ be an upper bound of $R^*$. Initialize $\bar{R}$ with a relatively large number such that all circles can be easily packed into the container of radius $\bar{R}$ without overlap.
2. Set $R \leftarrow \bar{R}$ and launch an algorithm to find a feasible $X$ with $E_R(X) = 0$ (i.e., to solve the corresponding circle-packing decision problem).
3. Tighten the packing $(X, R)$, i.e, to minimize $R$ while keeping $X$ basically unchanged. This step can be achieved using various approaches, like the simple bisection method described in Huang and Ye (2011), the simple penalty method described in Huang and Ye (2010), the standard local optimization solver SNOPT adopted by Addis et al. (2008b) and the more sophisticated augmented Lagrangian method (Andreani et al., 2007; Birgin and Sobral, 2008; Birgin and Martínez, 2009). After this step, we can usually obtain a better (at least not worse) packing $(X', R')$.
4. Set $\bar{R} \leftarrow R'$ and go to step 2. The loop of steps 2-4 is ended until a certain termination criterion (like time limit) is satisfied.

In the rest of this paper, we will first introduce an Iterated Tabu Search algorithm to solve the circle-packing decision problem, and then use it to search for dense packing solutions for the PUCC problem in the computational experiments section.
4. Iterated Tabu Search Algorithm

This section describes the Iterated Tabu Search (ITS) procedure for solving the circle-packing decision problem. As indicated in Section 3, this problem can be transformed to an unconstrained global optimization problem:

$$\text{minimize } E_R(X).$$

(7)

This subproblem is very difficult because there exist enormous local minima in the solution space. Grosso et al. (2010) have shown that, even for the equal circle packing problem, the number of local minima tends to increase very quickly with the number of circles. For the more complex unequal circle packing problem, it is very possible that the number of local minima will be significantly larger.

The main rationale behind the ITS procedure is as follows: (1) Each local minimum of $E_R(X)$ corresponds to a packing pattern of $n$ disks in the container. (2) If we perturb the current local minimizer $X$ by swapping the positions of two different disks (or shifting the position of one disk) and then call the LBFGS procedure to minimize $E_R(X)$, we can obtain a new local minimizer. (3) By systematically using the two perturbation moves, swap and shift, we can obtain a set of neighboring local minima from the current local minima. Furthermore, we can build a neighborhood structure on the set of local minima of $E_R(X)$. (4) Since there is a neighborhood structure, some Stochastic Local Search methods (Hoos and Stützle, 2005), such as Tabu Search (Glover and Laguna, 1998) and Iterated Local Search (Lourenço et al., 2003), can be employed to search for good local minima.

The outline of the ITS procedure is given in Algorithm 1. The procedure performs searches on the set of local minima of $E_R(X)$ and follows an
Iterated Local Search schema. In Algorithm 1 we run the ITS procedure in a multi-start fashion. At each run, the algorithm starts from a randomly generated local minimum (steps 2-3). It goes through the SwapTabuSearch procedure (step 4) and reaches a swap-optimal local minimum (which will be defined in the next section). Then the search explores the solution space by repeatedly escaping from local optima traps (step 6) and moving to another local optimum (step 7). This process is repeated until the best-found solution has not been improved during the last PerturbDepth iterations.

**Input:** Radiuses of \( n \) disks, radius of the container \( R \)

**Output:** A feasible packing pattern \( X \) of \( n \) disks in the container

1. while not time out do
   2. \( X \leftarrow \) Randomly scatter \( n \) disks into the container ;
   3. \( X \leftarrow \) Minimize \( E_R(X) \) using LBFGS ;
   4. \( X \leftarrow \) SwapTabuSearch(\( X \)) ; /* local search */
   repeat
   5. \( Y \leftarrow \) ShiftPerturb (\( X \)) ; /* perturb */
   6. \( Y \leftarrow \) SwapTabuSearch(\( Y \)) ; /* local search */
   7. if \( E_R(Y) \leq E_R(X) \) then
      8. \( X \leftarrow Y; \)
   end
   until the best-found solution has not been improved in the last PerturbDepth iterations;
12. end

**Algorithm 1:** The ITS procedure
4.1. The SwapTabuSearch Procedure

In the SwapTabuSearch procedure, we build a neighborhood structure on the set of local minima of $E_R(X)$. A swap move performed on a packing pattern $X$ is defined as swapping the positions of two disks with different radiiues and then locally minimizing $E_R(X)$. For two local minima of $E_R(X)$, $X$ and $X'$, we say $X'$ is a neighbor of $X$ if and only if $X'$ can be reached by performing a swap move on $X$. The neighborhood of $X$ is a set containing all the neighbors of $X$. We use $E_R(X)$ as the evaluation function, and a local minimum $X$ of $E_R(X)$ is called a swap-optimal local minimum if it has better solution quality than all its neighbors (or it cannot be improved via any swap move).

Totally, there are $n \times (n - 1)$ possible swap moves for a packing pattern with $n$ disks. However, for efficiency purposes, a restricted neighborhood is used in this paper. We first sort the disks in a nondecreasing order w.r.t. their radius values, such that for disks $i$ and $j$, $r_i \leq r_j$ if $i < j$. A swap move can only be performed on a pair of disks with neighboring radius values. That is to say, disk $i$ can only exchange positions with disks $i - 1$ and $i + 1$. Then, there are in total $n - 1$ swap moves and a local minimum $X$ has at most $n - 1$ different neighbors.

The SwapTabuSearch procedure follows a Tabu Search strategy. At the beginning of the search, the tabu list is empty and all swap moves are admissible. At each step, the algorithm chooses a best admissible move which leads to the best nontabu solution. The aspiration criterion is used such that a tabu move can be selected if it generates a solution that is better than the best-found solution. Once a move is selected, it is declared tabu for the next TabuTenure steps. The procedure is repeated until the best-found solution has not been improved with the last TabuDepth steps. The sketch
of the SwapTabuSearch procedure is presented in Algorithm 2.

| Algorithm 2: The SwapTabuSearch procedure |
|------------------------------------------|
| **Input:** An initial local minimum of $E_R(X)$ |
| **Output:** A swap-optimal minimum of $E_R(X)$ |
| 1 repeat |
| 2 choose a best candidate swap move $mv$; |
| 3 perform move $mv$; |
| 4 declare move $mv$ tabu for $TabuTenure$ iterations; |
| 5 until The best-found solution has not been improved in the last $TabuDepth$ iterations; |
| 6 return the best-found solution $X^*$; |

### 4.2. The ShiftPerturb Procedure

In the ShiftPerturb procedure, the algorithm escapes a local optimum by a series of shift moves. A shift move performed on a packing pattern $X$ is defined as shifting the position of a randomly chosen disk to a random place in the container and then locally minimizing $E_R(X)$. The number of times the shift move is performed is controlled by a parameter $PerturbStrength$. As pointed out in previous research (Lourenço et al., 2003), the perturbation strength is very important for Iterated Local Search. If it is too weak, the local search may undo the perturbation and the search will be confined in a small area of the solution space. On the contrary, if the perturbation is too strong, the Iterated Local Search will behave like random restart, leading to poor performance. After preliminary computational tests, we choose the value of $PerturbStrength$ to be a random integer from $[1, n/8]$. 
5. Performance Assessment

In this section, we assess the performance of the proposed algorithm through computational experiments on two sets of widely-used test instances. We also compare the results of our algorithm with some state-of-the-art algorithms in the literature.

5.1. Experimental Protocol

The algorithm is programmed in C++ and compiled using GNU G++. All computational experiments are carried out on a personal computer with 4Gb memory and a 2.8GHz AMD Phenom II X6 1055T CPU. Table 1 gives the settings of the four important parameters of the algorithm. Note that all the computational results are obtained without special tuning of the parameters, i.e., all the parameters used in the algorithm are fixed for all the tested instances.

5.2. Test Instances

Two sets of test problems are considered, in total constituting 70 instances. The first set comes from the famous circle packing contest (see [http://www.recmath.org/contest/CirclePacking/index.php](http://www.recmath.org/contest/CirclePacking/index.php)). This contest started on October 2005 and ended on January 2006. During this period, the participants were invited to propose densest packing solutions to
pack \( n(n = 5, 6, \ldots, 50) \) circles, each having radius \( r_i = i (i = 1, 2, \ldots, n) \) into the smallest containing circle without overlap. 155 groups from 32 countries took part in the contest and submitted a total of 27490 tentative solutions. After the contest, these results were further improved respectively by Müller et al. (2009), Eckard Specht (Specht, 2013), Zhanhua Fu et al. (Specht, 2013). Currently, all the best-known records are published and continuously updated on the Packomania website.

The second set of instances consists of 24 problem instances first presented by Huang et al. (2005). These instances are frequently used in the literature by many authors, see for example Huang et al. (2006); Akeb et al. (2009); Akeb and Hifi (2010). The size of these instances ranges from \( n = 10 \) to 60. A detailed description of these instances can be found in Huang et al. (2005).

5.3. Computational results on the circle packing contest instances

For the circle packing contest instances, researchers usually pay more attention on the solution quality. Especially during the contest, people mostly focus on finding better solutions than the best-known records and rarely consider the computational resource used. After researchers have solved these instances using various approaches and large amount of computational resource, this set of instances now becomes a challenging benchmark to test the discovery capability (Grosso et al., 2007) of a new algorithm. Therefore, our first experiment concentrates on searching for high-quality solutions. For each run of each instance, we usually set the time limit to 24 hours, run the algorithm multiple times and record the best-found solutions.

Table 2 gives the computational results. Column 1 lists the best-known records on the Packomania website. Columns 2-4 respectively report the
solution difference between some top reference results and the best-known records. These include: the best results found by Addis et al. (2008b) (who is the champion of the circle packing contest) using PBH algorithm, the best records obtained by all the participants in the contest, the best results found by Müller et al. (2009) using Simulated Annealing (SA) algorithm. Column 5 gives the solution difference between our results and the best-known records. The results indicated in bold are better than the best-known ones. Table 2 omits the results for $n = 5, 6, \ldots, 20$, because our results and all the reference results are the same on these instances. Note that, our program generates solutions with a maximum error on the distances of $10^{-9}$. We have sent all the improved results to Eckard Specht. Using his own local optimization solver, he has processed our results to a high precision ($10^{-28}$) and published them on the Packomania website.

Table 3 summarizes the comparison of our results with the reference results. The rows better, equal and worse respectively denote the number of instances for which the proposed algorithm gets solutions that are better, equal and worse than each reference result. Table 3 shows that the proposed algorithm is able to discover a number of better solutions than the previous best reference results, demonstrating its efficacy in finding high-quality solutions. In fact, we also tested the proposed algorithm on the larger instances of $n = 51, 52, \ldots, 100$. Some preliminary experiments show that the algorithm can improve almost all previous best-known results. Interested readers can refer to the Packomania website.

All the reference algorithms in Table 2 concentrate on finding high-quality solutions and do not reveal their computational statistics. In order to further evaluate the proposed algorithm in terms of search efficiency, we conduct additional experiments to compare the proposed algorithm with two
Table 2: Comparison of solution quality on the circle packing contest instances

| n   | Best-Known | Solution difference (i.e., this result - best-known) | PBH | Contest record | SA | ITS-PUCC |
|-----|------------|------------------------------------------------------|-----|----------------|----|----------|
| 21  | 62.5888709 | 0.00118149                                           | 0   | 0              | 0  | 0        |
| 22  | 66.7602862 | 0                                                    | 0   | 0              | 0  | 0        |
| 23  | 71.1994616 | 0                                                    | 0   | 0              | 0  | 0        |
| 24  | 75.7491425 |                        0.00356154                      | 0.00356154 | 0              | 0  | 0        |
| 25  | 80.2858644 | 0                                                    | 0   | 0              | 0  | 0        |
| 26  | 84.9895916 |                        0.11634365                      | 0.08646206 | 0              | 0  | -0.01174810 |
| 27  | 89.7509628 |                        0.07861113                      | 0.04121888 | 0              | 0  |          |
| 28  | 94.5258771 |                        0.17998508                      | 0.02410937 | 0.0006594      | 0  |          |
| 29  | 99.4831156 |                        0.02920634                      | 0.02920634 | 0              | 0  |          |
| 30  | 104.5403676 |                       0.20743552                      | 0.03819132 | 0.0008052      | 0  |          |
| 31  | 109.6820427 |                       0.08990423                      | 0.04121888 | 0.0004137      | 0  | -0.05280209 |
| 32  | 114.7981146 |                       0.06562367                      | 0.06562367 | 0.0411343      | 0  |          |
| 33  | 120.0658963 |                       0.15129751                      | 0.15129751 | 0.0001869      | 0  |          |
| 34  | 125.3669392 |                       0.24548366                      | 0.06656255 | 0              | 0  | 0.0761871 |
| 35  | 130.8490784 |                       0.31742394                      | 0.30725789 | 0.0685492      | 0  |          |
| 36  | 136.4921355 |                       0.04277728                      | 0.04277728 | 0.0001210      | -0.18421273 |
| 37  | 141.9243775 |                       0.32389631                      | 0.24254278 | 0.1189077      | -0.14870433 |
| 38  | 147.4521166 |                       0.53945928                      | 0.40557489 | 0.0047288      | 0.16654317 |
| 39  | 153.0070280 |                       0.30312533                      | 0.25459839 | 0.0793799      | -0.00009525 |
| 40  | 159.1797260 |                       0.39413040                      | 0.39926227 | 0.0026352      | -0.12653569 |
| 41  | 164.8870421 |                       0.40486751                      | 0.40486751 | 0.1498584      | 0  |          |
| 42  | 170.8953190 |                       0.03044253                      | 0.03044253 | 0.0000083      | -0.11479840 |
| 43  | 176.8257438 |                       0.41388066                      | 0.24859621 | 0.2256308      | -0.02750344 |
| 44  | 183.0428935 |                       0.32678190                      | 0.13277222 | 0.0593545      | -0.11226929 |
| 45  | 189.9513856 |                       0.48403531                      | 0.44030654 | 0.0077934      | -0.02405842 |
| 46  | 195.5263640 |                       0.38439932                      | 0.38439932 | 0.0000710      | -0.17039157 |
| 47  | 201.7279255 |                       0.50009381                      | 0.45768615 | 0              | -0.05939136 |
| 48  | 208.0901590 |                       0.54578742                      | 0.54578742 | 0              | -0.04513139 |
| 49  | 214.2905550 |                       0.36989651                      | 0.36989651 | 0.0033920      | -0.00983298 |
| 50  | 220.5640026 |                       0.52435233                      | 0.52435233 | 0.0350184      | 0.50085590 |

Table 3: Summary of comparison of solution quality on the 30 circle packing contest instances with \( n = 21, 22, \ldots, 50 \)

| n   | Better | Equal | Worse | Contest record | SA | Best-known |
|-----|--------|-------|-------|----------------|----|------------|
| 27  | 25     | 20    | 14    | 0              |     |            |
| 3   | 4      | 7     | 13    | 0              |     |            |
| 0   | 1      | 3     | 3     | 0              |     |            |

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recently published algorithms in a time-equalized basis. For each instance of \( n = 5, 6, \ldots, 32 \), we set the maximum time limit to 10000 seconds. We record the best-found solution and the elapsed time when it is first detected by the algorithm. To reduce the impact of randomness, each instance is independently solved for 10 times.

Table 4 gives the computational results. Columns 2-3, 4-5 respectively list the best-found solution and the needed computing time of TS/NP algorithm and FSS algorithm. Columns 2 and 3 are extracted from Al-Mudahka et al. (2010) where the algorithm ran on a computer with a Pentium IV, 2.66 Ghz CPU and 512Mb RAM. Columns 4 and 5 are extracted from López and Beasley (2012). Their experiments were done on a computer with a Intel(R) Core(TM) i5-2500 3.30 GHz CPU and 4.00 GB RAM. Columns 6-8 give the computational statistics of our algorithm, including the best-found solution, the number of hit times and the averaged computing time to detect the best-found solution.

Columns 6-8 show that, for all the 28 instances, the proposed algorithm can reach (or improve) the previous best-known records listed in Table 2 within the given time limit. Especially for \( n \leq 25 \), the algorithm can robustly detect the best-known records in a short time. When compared with the two reference algorithms, one observes that the proposed algorithm can usually find better solutions within the time limit. These results provide evidence of the search efficiency of ITS-PUCC algorithm.

5.4. Computational results on the NR instances

This section tests the proposed algorithm on the 24 NR instances. For each instance, we set the time limit to 10000 seconds, and record the best-found solution and the elapsed time when it is first detected by the algo-
Table 4: Comparison of search efficiency on the circle packing contest instances

| n   | TS/NP   | R   | time(s) | FSS   | R   | time(s) | ITS-PUCC | R   | #hits | time(s) |
|-----|---------|-----|---------|-------|-----|---------|----------|-----|-------|---------|
| 5   | 9.001398 | 426 | 9.00139775 | 461   | 9.00139774 | 10/10 | 1       |
| 6   | 11.05704 | 686 | 11.0570404 | 667   | 11.05704039 | 10/10 | 1       |
| 7   | 13.46211 | 1511| 13.46211068 | 721   | 13.46211067 | 10/10 | 1       |
| 8   | 16.22175 | 2551| 16.22174668 | 1028  | 16.22174667 | 10/10 | 1       |
| 9   | 19.39734 | 4051| 19.23193931 | 1404  | 19.2319393 | 10/10 | 1       |
| 10  | 22.34516 | 5760| 22.0019301 | 1438  | 22.0019301 | 10/10 | 1       |
| 11  | 24.96063 | 2094| 24.96063429 | 1820  | 24.96063428 | 10/10 | 1       |
| 12  | 28.67863 | 3548| 28.37138944 | 2299  | 28.37138943 | 10/10 | 1       |
| 13  | 32.00719 | 4054| 31.5456702 | 2905  | 31.5456701 | 10/10 | 1       |
| 14  | 35.41261 | 6146| 35.09564714 | 2970  | 35.09564714 | 10/10 | 2       |
| 15  | 39.00243 | 6996| 38.83799682 | 3904  | 38.8379955 | 10/10 | 1       |
| 16  | 42.92185 | 9684| 42.45811644 | 4917  | 42.45811643 | 10/10 | 6       |
| 17  | 46.77237 | 10168| 46.34518193 | 5264  | 46.29134211 | 10/10 | 24      |
| 18  | 50.65635 | 14312| 50.20889346 | 6224  | 50.11976262 | 10/10 | 23      |
| 19  | 55.02744 | 14925| 54.36009421 | 7349  | 54.24029359 | 10/10 | 39      |
| 20  | 59.04547 | 19825| 58.48047359 | 7517  | 58.40056747 | 10/10 | 82      |
| 21  | 63.49768 | 5923 | 63.00078332 | 8924  | 62.55877709 | 10/10 | 190     |
| 22  | 68.10291 | 6636 | 66.96471591 | 10762 | 66.76028624 | 10/10 | 127     |
| 23  | 72.70501 | 7209 | 71.69822657 | 13018 | 71.1994616 | 10/10 | 268     |
| 24  | 76.49105 | 8552 | 76.1231197 | 13004 | 75.74914258 | 10/10 | 704     |
| 25  | 81.56595 | 11409| 80.8168236 | 15569 | 80.28586443 | 10/10 | 633     |
| 26  | 86.43809 | 12062| 85.487438 | 18320 | 84.97819106 | 9/10  | 3538    |
| 27  | 91.15366 | 13657| 90.93173506 | 18544 | 89.75096268 | 7/10  | 5287    |
| 28  | 96.34813 | 14364| 95.6406414 | 21931 | 94.5258771 | 10/10 | 1568    |
| 29  | 101.7251 | 15185| 100.72003313 | 25455 | 99.4831156 | 7/10  | 2915    |
| 30  | 107.1161 | 20745 | 105.881772 | 25658 | 104.5403638 | 5/10  | 4538    |
| 31  | 111.8966 | 21424 | 111.077126 | 29973 | 109.6292407 | 2/10  | 8551    |
| 32  | 117.6701 | 22781 | 116.6122668 | 34445 | 114.7998147 | 4/10  | 3885    |
rithm. Each instance is solved for 10 times from different randomly generated starting points.

The computational results are presented in Table 5. Column 1 gives the instance name. Columns 2-3, 4-5, 6-7, 8-9, respectively present the best-found solution and the needed computing time of A1.5 Algorithm in Huang et al. (2006), Beam Search (BS) algorithm in Akeb et al. (2009), Algorithm 2 in Akeb et al. (2010) and GP-TS algorithm in Huang et al. (2012a). Columns 10-12 give the computational statistics of our algorithm, including the best-found solution, the number of hit times and the averaged computing time for detecting the best-found solution. In experiments, our program generates solutions with a maximum error on the distance of $10^{-9}$. However, in order to keep consistent with previous papers, we report in Table 5 the results with 4 significant digits after the decimal point.

Table 5 demonstrates that, for all the tested 24 instances, the proposed algorithm can find 16 better solutions than the best results found by the references algorithms (as indicated in bold in the table). For the other 8 instances, it can reach the best-known solutions efficiently and robustly. These results further provide evidence of the competitiveness of the proposed algorithm.

6. Algorithm Analysis

In this section, we turn our attention to analyzing the two most important ingredients of the proposed algorithm: the SwapTabuSearch procedure and the ShiftPerturb procedure.
Table 5: Comparison of search efficiency on the 24 NR instances

| Instance | A1.5  | BS  | Algorithm 2 | GP-TS | ITS-PUCC |
|----------|-------|-----|-------------|-------|----------|
|          | R     | time(s) | R     | time(s) | R     | time(s) | R     | time(s) | #hits | time(s) |
| NR10-1   | 99.89 | 1     | 99.89 | 19     | 99.88 | 1     | 99.88 | 1     | 9.988 | 10/10  |
| NR11-1   | 60.71 | 1     | 60.71 | 28     | 60.71 | 2     | 60.71 | 2     | 6.071 | 10/10  |
| NR12-1   | 65.30 | 6     | 65.47 | 151    | 65.03 | 248   | 65.02 | 483   | 65.02 | 10/10  |
| NR13-1   | 113.84| 2     | 114.29| 151    | 113.56 | 286  | 113.56 | 286  | 113.56 | 10/10  |
| NR14-1   | 38.97 | 25    | 38.94 | 509    | 38.91 | 4628  | 38.91 | 4628  | 38.91 | 10/10  |
| NR15-2   | 38.85 | 6     | 38.82 | 1179   | 38.82 | 23540 | 38.82 | 23540 | 38.82 | 10/10  |
| NR16-1   | 143.44| 71    | 143.76| 139    | 143.56 | 139  | 143.56 | 139  | 143.56 | 10/10  |
| NR16-2   | 128.29| 44    | 128.05| 28     | 128.05 | 28   | 128.05 | 28   | 128.05 | 10/10  |
| NR17-1   | 49.25 | 30    | 49.26 | 234    | 49.18 | 258   | 49.18 | 258   | 49.18 | 10/10  |
| NR18-1   | 197.40| 8     | 198.28| 18     | 198.28 | 18   | 198.28 | 18   | 198.28 | 10/10  |
| NR19-1   | 125.53| 39    | 125.63| 764    | 125.52 | 19945 | 125.52 | 19945 | 125.52 | 10/10  |
| NR20-1   | 122.21| 318   | 122.19| 351    | 122.03 | 15095 | 122.03 | 15095 | 122.03 | 10/10  |
| NR21-1   | 148.82| 683   | 149.13| 638    | 148.34 | 12080 | 148.34 | 12080 | 148.34 | 10/10  |
| NR23-1   | 175.47| 1229  | 175.40| 3072   | 174.94 | 58430 | 174.94 | 58430 | 174.94 | 10/10  |
| NR24-1   | 138.38| 2339  | 138.27| 510    | 138.05 | 37140 | 138.05 | 37140 | 138.05 | 10/10  |
| NR25-1   | 190.47| 4614  | 190.18| 1493   | 189.47 | 37053 | 189.47 | 37053 | 189.47 | 10/10  |
| NR26-1   | 246.75| 1019  | 247.54| 583    | 246.17 | 18620 | 246.17 | 18620 | 246.17 | 10/10  |
| NR26-2   | 303.38| 5164  | 303.21| 11240  | 302.58 | 190600| 302.58 | 190600| 302.58 | 10/10  |
| NR27-1   | 222.58| 4436  | 222.49| 3750   | 221.63 | 177000| 221.63 | 177000| 221.63 | 10/10  |
| NR30-1   | 178.66| 1365  | 178.01| 5045   | 177.64 | 160700| 177.64 | 160700| 177.64 | 10/10  |
| NR30-2   | 173.70| 1078  | 173.43| 9217   | 173.22 | 155050| 173.22 | 155050| 173.22 | 10/10  |
| NR40-1   | 357.00| 12109 | 357.06| 22140  | 355.65 | 158700| 355.65 | 158700| 355.65 | 10/10  |
| NR50-1   | 380.00| 9717  | 378.58| 21400  | 378.00 | 186000| 378.00 | 186000| 378.00 | 10/10  |
| NR60-1   | 522.93| 13256 | 521.27| 13975  | 519.84 | 116270| 519.84 | 116270| 519.84 | 10/10  |
6.1. Analysis of The SwapTabuSearch Procedure

The SwapTabuSearch procedure is a key component of the proposed algorithm, which enables the algorithm to intelligently examines the neighboring packing patterns through swap moves. In order to make sure the Tabu Search strategy makes a meaningful contribution, we conduct experiments to compare the Tabu Search strategy with a simple local search strategy called Steepest Descent (Hoos and Stützle, 2005).

For comparison, we use the same neighborhood structure as described in Section 4.1 and implement the Steepest Descent strategy as follows. At each iteration, the search examines each neighbor of the current solution and find out the best neighbor with the least objective value $E_R$. If the best neighbor $X'$ is better than the current solution $X$, i.e., $E_R(X') \leq E_R(X)$, then the search moves to $X'$ and proceeds to the next iteration; otherwise the search stops and declares reaching a local minimum.

A representative instance $NR15-2$ is chosen as our test bed. This instance is nontrivial. Though many previous papers have tested it, only few state-of-the-art algorithms, like Beam Search (Hifi and M’Hallah, 2008), PBH (Addis et al., 2008b; Grosso et al., 2010), SA (Müller et al., 2009) can obtain the optimal packing pattern. We set the radius of container $R$ to the best-known value, randomly generate initial $X$ and call both algorithms to minimize $E_R(X)$.

We run both algorithms 1000 times from different randomly generated starting points and record in Table 6 respectively the best-found solution (Column 2), the average solution quality (Column 3), the average number of search steps for each local search (Column 4) and the average elapsed time for each local search (Column 5). From Table 6 we observe that, the Tabu Search strategy shows clear advantage over Steepest Descent strategy.

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Table 6: Computational statistics of Tabu Search strategy and Steepest Descent strategy from 1000 randomly generated initial packings

| Search Strategy       | Best-found solution | Average solution quality | Search steps | Time (s) |
|-----------------------|----------------------|--------------------------|--------------|----------|
| Tabu Search           | 0.000000             | 0.000000                 | 1269         | 2        |
| Steepest Descent      | 0.092194             | 2.085742                 | 7            | 0        |

Each time, the Tabu Search strategy can find the global minimum from a randomly generated starting point, while the Steepest Descent strategy fails for all 1000 runs. In fact, we try to run the Steepest Descent strategy from 100000 randomly generated starting points, it still cannot find the global minimum.

The main reason for the difference is that, with the Steepest Descent strategy, the search is easily trapped in poor local minimum. As shown in Table 6, the average number of search step for each local search is only 7. However, with the Tabu Search strategy, the search can escape from low-quality local minimum trap and proceed to explore the neighboring area. Figure 1 shows a typical search trajectory of Tabu Search, compared with the search trajectory of Multistart Steepest Descent. In Figure 1, both algorithms start from the same initial solution, a packing pattern with $E_R = 8.77845794$. After 7 search steps, both of them encounter a local minimum with $E_R = 1.37297106$. At this time, the Steepest Descent strategy is trapped, the search has to proceed from a new randomly generated initial solution. However, with the Tabu Search strategy, the search is able to escape from the local minimum with $E_R = 1.3729106$, proceed to examine the neighboring area, and finally find the global minimum at the 362th search step.
These experiments reveal that, the Tabu Search strategy helps to perform an intensified examination around the incumbent packing pattern and makes possible discovering those hidden good solutions. The same experiments have been performed on several other instances, leading to similar observation.

6.2. Analysis of the ShiftPerturb Procedure

In order to verify the effectiveness of the ShiftPerturb procedure, we conduct experiments to compare the proposed ITS algorithm with a Multistart Tabu Search algorithm. In the Multistart Tabu Search algorithm, when the SwapTabuSearch procedure finishes, the search proceeds from a new randomly generated initial solution. The parameter setting of SwapTabuSearch is the same as listed in Table 1. We test the Multistart Tabu Search algorithm on the 28 circle packing instances with $5 \leq n \leq 32$. Each instance is solved for 10 times. The time limit for each run is also set to 10000 seconds.
The computational results show clear advantage of ITS algorithm over Multistart Tabu Search algorithm. For the instances of $5 \leq n \leq 24$, the Multistart Tabu Search algorithm can also detect the best-known records, but with lower success rates and relatively longer time. Nevertheless, for each instance of $25 \leq n \leq 32$, the Multistart Tabu Search algorithm fails to detect the best-known solution for all the 10 runs within the given time limit.

We conjecture the superior of ITS over Multistart Tabu Search may be explained from the following two aspects. First, the Iterated Local Search framework helps the search to perform a more intensified examination around the incumbent solution, making it possible to repeatedly discover better solutions. This is supported by our observations from computational experiments that, with the ITS algorithm, the search can usually generate a sequence of local minima with descending objective value. The final solution obtained by one run of ITS is usually much better than that found by the first run of \textit{SwapTabuSearch}. Second, the shift move in the \textit{ShiftPerturb} procedure is complementary to the swap move, enabling the search to reach some packing patterns which are hard to detect only through swap moves.

7. Conclusion and Future Work

In this paper, we have presented a heuristic global optimization algorithm for solving the unequal circle packing problem. The proposed algorithm uses a continuous local optimization method to generate locally optimal packings and integrates two combinatorial optimization methods, Tabu Search and Iterated Local Search, to systematically search for good
local minima. The efficiency and effectiveness of the algorithm have been demonstrated by computational experiments on two sets of widely used test instances. For the 46 challenging circle packing contest instances and the 24 widely-used NR instances, the algorithm can respectively improve 14 and 16 previous best-known records in a reasonable time.

There are two main directions for future research. On the one hand, the presented algorithm can be further improved by incorporating other advanced strategies. Possible improvements include the following: First, reduce the solution space by first ignoring several smaller disks and only looking for optimal packing pattern of the remaining larger disks. The smaller disks can be inserted into the holes after the larger disks have been placed into the container. This strategy was proposed in Addis et al. (2008b) and had proved to be very useful. Second, test other Stochastic Local Search methods, such as Simulated Annealing used in Müller et al. (2009), Variable Neighborhood Search (Hoos and Stützle, 2005) and so on. Third, the proposed algorithm is a single-solution based method. It is possible to strengthen the robustness of the algorithm by employing some population-based methods, like the Population Basin Hopping method proposed in Grosso et al. (2007).

On the other hand, the ideas behind the proposed algorithm can also be applied to other hard global optimization problems. Many real-world global optimization problems, such as the cluster optimization problem in computational chemistry (Ye et al., 2011) and the protein folding problem in computational biology (Huang et al., 2006), have the same characteristics as the unequal circle packing problem, i.e., they have both a continuous and combinatorial nature. For these kinds of problems, it is possible to build a neighborhood structure on the set of local minima via appropriate perturba-
tion moves, and then to employ some advanced combinatorial optimization methods to systematically search for good local minima.

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