Regular graphs are antimagic

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Abstract

In this note we prove - with a slight modification of an argument of Cranston et al. [2] - that $k$-regular graphs are antimagic for $k \geq 2$.

1 Introduction

Throughout the note graphs are assumed to be simple. Given an undirected graph $G = (V, E)$ and a subset of edges $F \subseteq E$, $F(v)$ denotes the set of edges in $F$ incident to node $v \in V$, and $d_F(v) := |F(v)|$ is the degree of $v$ in $F$. A labeling is an injective function $f : E \to \{1, 2, \ldots, |E|\}$. Given a labeling $f$ and a subset of edges $F$, let $f(F) = \sum_{e \in F} f(e)$. A labeling is antimagic if $f(E(u)) \neq f(E(v))$ for any pair of different nodes $u, v \in V$.

A graph is said to be antimagic if it admits an antimagic labeling.

Hartsfield and Ringel conjectured [4] that all connected graphs on at least 3 nodes are antimagic. The conjecture has been verified for several classes of graphs (see e.g. [3]), but is widely open in general. In [2] Cranston et al. proved that every $k$-regular graph is antimagic if $k \geq 3$ is odd. Note that 1-regular graphs are trivially not antimagic. We have observed that a slight modification of their argument also works for even regular graphs, hence we prove the following.

Theorem 1. For $k \geq 2$, every $k$-regular graph is antimagic.

It is worth mentioning the following conjecture of Liang [5]. Let $G = (S, T; E)$ be a bipartite graph. A path $P = \{uw, vu\}$ of length 2 with $u, w \in S$ is called an S-link.

Conjecture 2. Let $G = (S, T; E)$ be a bipartite graph such that each node in $S$ has degree at most 4 and each node in $T$ has degree at most 3. Then $G$ has a matching $M$ and a family $\mathcal{P}$ of node-disjoint S-links such that every node $v \in T$ of degree 3 is incident to an edge in $M \cup (\bigcup_{P \in \mathcal{P}} P)$.

Liang showed that if the conjecture holds then it implies that every 4-regular graph is antimagic. The starting point of our investigations was proving Conjecture 2. As Theorem 1 provides a more general result, we leave the proof of Conjecture 2 for a forthcoming paper [1].

2 Proof of Theorem 1

A trail in a graph $G = (V, E)$ is an alternating sequence of nodes and edges $v_0, e_1, v_1, \ldots, e_t, v_t$ such that $e_i$ is an edge connecting $v_{i-1}$ and $v_i$ for $i = 1, 2, \ldots, t$, and the edges are all distinct (but there might be repetitions among the nodes). The trail is open if $v_0 \neq v_t$, and closed otherwise. The length of a trail is the number of edges in it. A closed trail containing every edge of the graph is called an Eulerian trail. It is well known that a graph has an Eulerian trail if and only if it is connected and every node has even degree.

Lemma 3. Given a connected graph $G = (V, E)$, let $T = \{v \in V : d_E(v) \text{ is odd}\}$. If $T \neq \emptyset$, then $E$ can be partitioned into $|T|/2$ open trails.

Proof. Note that $|T|$ is even. Arrange the nodes of $T$ into pairs in an arbitrary manner and add a new edge between the members of every pair. Take an Eulerian trail of the resulting graph and delete the new edges to get the $|T|/2$ open trails.

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The main advantage of Lemma 3 is that the edge set of the graph can be partitioned into open trails such that at most one trail starts at every node of \( V \). Indeed, there is a trail starting at \( v \) if and only if \( v \) has odd degree in \( G \). This is how we help the Lemma of [2].

**Corollary 4 (Helpful Lemma of [2]).** Given a bipartite graph \( G = (U, V; E) \) with no isolated nodes in \( U, V \) can be partitioned into subsets \( E_1, T_1, T_2, \ldots, T_l \) such that \( d_{E_i}(u) = 1 \) for every \( u \in U, T_i \) is an open trail for every \( i = 1, 2, \ldots, l \), and the endpoints of \( T_i \) and \( T_j \) are different for every \( i \neq j \).

**Proof.** Take an arbitrary \( E' \subseteq E \) with the property \( d_{E'}(u) = 1 \) for every \( u \in U \). A component of \( G - E' \) containing more than one node is called nontrivial. If there exists a nontrivial component of \( G - E' \) that only contains even degree nodes then let \( uw_1 \in E - E' \) be an edge in this component with \( u \in U \) and \( w_1 \in W \), and let \( uw_2 \in E' \). Replace \( uw_2 \) with \( uw_1 \) in \( E' \). After this modification, the component of \( G - E' \) that contains \( u \) has an odd degree node, namely \( w_1 \). Iterate this step until every nontrivial component of \( G - E' \) has some odd degree nodes. Let \( E'' = E' \) and apply Lemma 3 to get the decomposition of \( E - E'' \) into open trails. \( \square \)

In what follows we prove that regular graphs are antimagic: for sake of completeness we include the odd regular case, too. We emphasize the differences from the proof appearing in [2].

**Proof of Theorem 1.** Note that it suffices to prove the theorem for connected regular graphs. Let \( G = (V, E) \) be a connected \( k \)-regular graph and let \( v^* \in V \) be an arbitrary node. Denote the set of nodes at distance exactly \( i \) from \( v^* \) by \( V_i \) and let \( q \) denote the largest distance from \( v^* \). We denote the edge-set of \( G[V_i] \) by \( E_i \). Apply Corollary 3 to the induced bipartite graph \( G[V_{i-1}, V_i] \) with \( U = V_i \) to get \( E_i^* \) and the trail decomposition of \( G[V_{i-1}, V_i] - E_i^* \) for every \( i = 1, \ldots, q \). The edge set of \( G[V_{i-1}, V_i] - E_i^* \) is denoted by \( E_i^\ast \).

Now we define the antimagic labeling \( f \) of \( G \) as follows. We reserve the \( |E_q| \) smallest labels for labeling \( E_q \), the next \( |E_{q-1}| \) smallest labels for labeling \( E_{q-1}^\ast \), the next \( |E_{q-2}| \) smallest labels for labeling \( E_{q-2} \), the next \( |E_{q-3}| \) smallest labels for labeling \( E_{q-3} \), etc. There is an important difference here between our approach and that of [2] as we switched the order of labeling \( E_i^\ast \) and \( E_i \), and we don’t yet define the labels, we only reserve the intervals to label the edge sets. Next we prove a claim that tells us how to label the edges in \( E_i^\ast \).

**Claim 5.** Assume that we have to label the edges of \( E_i^\ast \) from interval \( s, s + 1, \ldots, \ell \) (where \( |E_i| = \ell - s + 1 \)), and that we are given a trail decomposition of \( E_i^\ast \) into open trails. We can label \( E_i^\ast \) so that successive labels (in a trail) incident to a node \( v_i \in V_i \) have sum at most \( s + \ell \), and successive labels (in a trail) incident to a node \( v_{i-1} \in V_{i-1} \) have sum at least \( s + \ell \).

**Proof.** Our proof of this claim is essentially the same as the proof in [2]: we merely restate it for self-containedness. Let \( T \) be the trail decomposition of \( E_i^\ast \) into open trails. Take an arbitrary trail \( T = u_0, v_1, u_2, \ldots, v_t, u_t \) of length \( t \) from \( T \) and consider the following two cases (see Figure 1 for an illustration).

- **Case A:** If \( u_0 \in V_{i-1} \) then label \( e_1, \ldots, e_t \) by \( s, \ell, s + 1, \ell - 1, \ldots \) in this order. In this case the sum of 2 successive labels is \( s + \ell \) at a node in \( V_i \), and it is \( s + \ell + 1 \) at a node in \( V_{i-1} \).

- **Case B:** If \( u_0 \in V_i \) then label \( e_1, \ldots, e_t \) by \( \ell, s, s - 1, \ell + 1, \ldots \) in this order. In this case the sum of 2 successive labels is \( s + \ell - 1 \) at a node in \( V_i \), and it is \( s + \ell \) at a node in \( V_{i-1} \).

We prove by induction on \( |T| \). The proof is finished by the following cases.

1. If \( T \) contains a trail of even length, then let \( T \) be such a trail (and again \( t \) denotes the length of \( T \)). If the endpoints of \( T \) fall in \( V_{i-1} \) then apply Case A. On the other hand, if the endpoints of \( T \) fall in \( V_i \) then apply Case B. In both cases we use \( \frac{\ell}{2} \) labels from the lower end of the interval, and \( \frac{\ell}{2} \) labels from the upper end, therefore we can label the edges of the trails in \( T - T \) from the (remaining) interval \( s + \frac{\ell}{2}, s + \frac{\ell}{2} + 1, \ldots, \ell - 1 \), so that the lower bound \( s + \ell/2 + \ell/2 = s + \ell \) holds for the sum of 2 successive labels at every \( v_{i-1} \in V_{i-1} \), and the same upper bound holds at each node \( v_i \in V_i \).

2. Every trail in \( T \) has odd length. If \( T \) contains only one trail then label it using either of the two cases above and we are done. Otherwise let \( T_1 \) and \( T_2 \) be two trails from \( T \), and let \( t_i \) be the length of \( T_i \), for both \( i = 1, 2 \). Label first the edges of \( T_1 \) using Case A (starting at the endpoint of \( T_1 \) that lies in \( V_{i-1} \)). Note that the remaining labels form the interval \( s + \frac{\ell+1}{2}, \ldots, \ell - \frac{\ell-1}{2} \). Next label the edges of \( T_2 \) using Case B (starting at the endpoint of \( T_2 \) that lies in \( V_i \)). Note that the sum of successive labels in the trail \( T_2 \) becomes \( s + \frac{\ell+1}{2} + \ell - \frac{\ell-1}{2} - 1 = s + \ell \) at a node in \( V_i \), and it is \( s + \ell/2 + (\ell - \ell/2 - 1) = s + \ell + 1 \) at a node in \( V_{i-1} \), which is fine for us. Finally, the remaining labels form the interval \( s + \ell/2 + \ell/2, \ldots, \ell - \ell/2 - \ell/2 \), therefore we can label the edges of the trails in \( T - (T_1, T_2) \) from the remaining interval so that the lower bound \( s + \ell/2 + \ell/2 + \ell - \ell/2 - \ell/2 = s + \ell \) holds for the sum of 2 successive labels at every node of \( V_{i-1} \), and the same upper bound holds at every node of \( V_i \).

\( \square \)
This concludes the proof of $E$.

Although the proof of (ii) can be found in [2], we also present it here to make the paper self-contained. The proof is very similar to the even case. So assume that $k$ is odd. In this case $p(v)$ is the sum of an even number of labels. We pair up these labels using the trail decomposition of $E_i$ to get the bounds needed.

1. Take a node $v_i \in V_i$. Note that $f(e) < s$ for every $e \in E(v_i) - E_i'$. Let $t = d_{E_i'}(v_i)$.

   (a) If $t$ is even then $\sum_{e \in E_i' \cap E(v_i)} f(e) \leq \frac{k-2}{2} (s+\ell) + \ell$ by Claim 5, giving $p(v_i) \leq \frac{k-2}{2} (s+\ell) + (k-1-t)s \leq \frac{k-2}{2} (s+\ell) + \ell$.

   (b) If $t$ is odd then $\sum_{e \in E_i' \cap E(v_i)} f(e) \leq \frac{k-1}{2} (s+\ell) + \ell$ by Claim 5, giving $p(v_i) \leq \frac{k-1}{2} (s+\ell) + \ell + (k-1-t)s \leq \frac{k-1}{2} (s+\ell) + \ell$.

2. Now take a node $v_i-1 \in V_{i-1}$. Note that $f(e) > \ell$ for every $e \in E(v_i-1) - E_i'$. Let again $t = d_{E_i'}(v_i)$.

   (a) If $t$ is even then $\sum_{e \in E_i' \cap E(v_i-1)} f(e) \geq \frac{k-2}{2} (s+\ell) + s$ by Claim 5, giving $p(v_i) \geq \frac{k-2}{2} (s+\ell) + (k-1-t)\ell \geq \frac{k-2}{2} (s+\ell) + s$.

   (b) If $t$ is odd then $\sum_{e \in E_i' \cap E(v_i-1)} f(e) \geq \frac{k-1}{2} (s+\ell) + s$ by Claim 5, giving $p(v_i) \geq \frac{k-1}{2} (s+\ell) + s + (k-1-t)\ell \geq \frac{k-1}{2} (s+\ell) + s$.

This concludes the proof of (ii).
(b) If \( t \) is odd then \( \sum_{e \in E'(\cap E(v_i))} f(e) \leq \frac{t-1}{2}(s+\ell)+\ell+(k-1-t)s \leq \frac{k-1}{2}(s+\ell) \).

2. Now take a node \( v_{i-1} \in V_{i-1} \). Note that \( f(e) > \ell \) for every \( e \in E(v_{i-1}) - E'_i \). Let again \( t = d_{E'_i}(v_{i-1}) \).

(a) If \( t \) is even then \( \sum_{e \in E'_i \cap E(v_i)} f(e) \geq \frac{t}{2}(s+\ell) \) by Claim 5, giving \( p(v_i) \leq \frac{t-1}{2}(s+\ell)+\ell \).

(b) If \( t \) is odd then \( \sum_{e \in E'_i \cap E(v_i)} f(e) \geq \frac{t-1}{2}(s+\ell)+s \) by Claim 5, giving \( p(v_{i-1}) \geq \frac{t-1}{2}(s+\ell)+s+(k-1-t)\ell \geq \frac{k-1}{2}(s+\ell) \).

This concludes the proof of (ii), and we are done.

The assignment of the labels implies \( f(\sigma(v_i)) < s \) and \( f(\sigma(v_{i-1})) > \ell \) for \( v_i \in V_i \) and \( v_{i-1} \in V_{i-1} \). Claim 6 yields \( f(E(v_i)) < f(E(v_{i-1})) \), finishing the proof of Theorem 1.

**Remark 7.** Observe that the proof of Theorem 1 does not really use the regularity of the graph, it merely relies on the fact that the degree of a node \( v_i \in V_i \) is not smaller than that of a node \( v_j \in V_j \) where \( i < j \). Hence the following result immediately follows.

**Theorem 8.** Assume that a connected graph \( G = (V,E) \) \(|V| \geq 3\) has a node \( v^* \in V \) of maximum degree such that \( d_E(v_i) \geq d_E(v_j) \) whenever \( v_i \in V_i, v_j \in V_j \) and \( i < j \), where \( V_\ell \) denotes the set of nodes at distance exactly \( \ell \) from \( v^* \). Then \( G \) is antimagic.

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