On colliding ultrarelativistic nuclei on a transverse lattice

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Abstract

We argue that the classical evolution of small x modes in the collision of two ultrarelativistic nuclei is described on a transverse lattice by the Kogut–Susskind Hamiltonian in 2+1-dimensions coupled to an adjoint scalar field. The initial conditions for the evolution are provided by the non–Abelian Weizsäcker–Williams fields which constitute the classical parton distributions in each of the nuclei. We outline how lattice techniques developed for real time simulations of field theories in thermal equilibrium can be used to study non–perturbatively, thermalization and classical gluon radiation in ultrarelativistic nuclear collisions.
1 Introduction

It is of considerable theoretical and experimental interest to understand the collisions of nuclei at ultrarelativistic energies and the putative evolution of the hot and dense matter created in these collisions into a thermalized, deconfined state of matter called a quark gluon plasma. The theoretical challenge is to understand the dynamics of the formation of this matter from QCD, and its properties, while the experimental challenge is to detect evidence that such a plasma was indeed formed [1].

The space–time evolution of the nuclei after the collision and the magnitudes and relevance of various proposed signatures of this hot and dense matter depend sensitively on the initial conditions for the evolution, namely, the parton distributions in each of the nuclei prior to the collision. In the conventional perturbative QCD approach to the problem, observables from the collision may be computed by convolving the parton distributions of each nucleus, determined from deep inelastic scattering experiments, with the elementary parton–parton scattering cross sections. The cross sections thereby obtained are often incorporated either in a multiple scattering formalism [2] or in a classical cascade approach to obtain the space–time evolution [3]. While the above approach provides a reasonable description of large transverse momentum processes at large $x$, it is not sufficient as we go to small $x$ or alternatively, towards central rapidities [4]. This is because at small $x$ partons in one nucleus may “see” more than one parton in the direction of the incoming nucleus resulting in a breakdown of the above described convolution of distributions. What is needed therefore to describe the collision of the “wee” nuclear partons is, roughly put, products of amplitudes as opposed to products of probabilities.

A model describing the small $x$ modes in large nuclei was formulated by McLerran and Venugopalan [5]. The model contains one dimensionful parameter, $\chi(y, Q^2)$, which is the total color charge squared per unit area integrated from the rapidity $y$ of interest to the beam rapidity. Since it is the only scale in the problem, the coupling
constant runs as a function of this scale. One therefore has weak coupling in the
limits where the color charge $\chi$ is large; either for $A >> 1$ or $s \rightarrow \infty$. It was argued
there that the classical background fields in this model are non–Abelian Weizsäcker–
Williams fields. Exact analytical expressions for these fields have been obtained
recently [3, 6]. Further, it has been shown explicitly that $\chi$ obeys renormalization
group equations in $y$ and $Q^2$ which reduce to the well known DGLAP and BFKL
equations [14] in the appropriate limits [3, 14].

The model was applied to the problem of nuclear collisions by Kovner, McLerran
and Weigert, who formulated the problem as the collision of Weizsäcker–Williams
fields [8]. The classical background fields after the collision then correspond to solu-
tions of the Yang–Mills equations in the presence of static, random sources of color
charge on the light cone. The classical background field for the two nuclei after the
collision was found and perturbative solutions obtained for modes with transverse mo-
menta $k_t >> \alpha_s \sqrt{\chi}$. After averaging over the Gaussian random light cone sources,
the energy and number distributions of physical gluons were computed. Further,
the classical gluon radiation from these perturbative modes was studied by these au-
thors and later in greater detail by several others [4, 10, 11]. While the perturbative
approach is very relevant and useful, it is still essential to consider the full non–
perturbative approach for the following reasons. Firstly, the classical gluon radiation
computed perturbatively is infrared singular and has to be cut-off at some scale. It
was argued in Ref. [8, 10] that a natural scale where the distributions are cut-off
is given by $k_t \sim \alpha_s \sqrt{\chi}$. However, since quantitative differences can be large, it is
important to perform a full calculation. Secondly, the non–perturbative approach is
crucial to study the possible thermalization of the system and the relevant time scales
for thermalization. This in turn has several ramifications for computations of various
signatures of the quark gluon plasma. For instance, if thermalization does occur, then
as proposed by Bjorken [16] hydrodynamic evolution of the system is reasonable. In
that event, our approach would provide the initial temperature and velocity profiles
necessary for such an evolution (see [17] and references therein).

In this paper, we outline how one may perform real–time simulations of the full, non–perturbative evolution of classical non–Abelian Weizsäcker–Williams fields. Such a simulation is possible since the fields are classical. Similar real time simulations of classical fields have been performed in the context of sphaleron induced baryon number violation [12] and chirality violating transitions in hot gauge theories [13].

In brief, the idea is as follows. We write down the lattice Hamiltonian which describes the evolution of these classical gauge fields. It turns out to be the Kogut–Susskind Hamiltonian in 2+1–dimensions coupled to an adjoint scalar field. The lattice equations of motion for the fields are thereby determined straightforwardly. The initial conditions for the evolution are provided by the Weizsäcker–Williams fields for the nuclei before the collisions. Interestingly, the dependence on the static light cone sources does not enter through the Hamiltonian but instead from the initial conditions. Also, to reiterate, our results have to be averaged over by the above mentioned Gaussian measure.

The paper is organized as follows. In the following section we briefly discuss the problem of initial conditions as formulated by Kovner, McLerran and Weigert and their perturbative solution. In section 3, we derive the expression for the lattice Hamiltonian. The lattice equations of motion and initial conditions are discussed in section 4. We conclude with a discussion of observables that can be computed on the lattice and comment on checks that can be performed on such computations.

2 The classical background field of two nuclei on the light cone

In the work of McLerran and Venugopalan [5], the classical gluon field at small \( x \) for a nucleus in the infinite frame is obtained by solving the Yang–Mills equations
in the presence of a static source of color charge $\rho^a(r_t, \eta)$ on the light cone. Exact solutions for the classical field as functions of $\rho^a(r_t, \eta)$ were found by Jalilian–Marian et al. [6] and independently by Kovchegov [7]. Distribution functions are computed by averaging over products of the classical fields over a Gaussian measure in $\rho$ with the variance $\mu^2(\eta, Q^2)$. Here $\mu^2$ is the color charge squared per unit area per unit rapidity resolved at a scale $Q^2$ by an external probe. It is related to $\chi$ by the expression $\chi(\eta, Q^2) = \int_{\eta}^{\infty} \mu^2(\eta', Q^2)$.

The above picture of gluon fields in a nucleus at small $x$ was extended to describe nuclear collisions by Kovner, McLerran and Weigert [8]. We shall review and discuss their paper below. The classical background field of two nuclei is described by the Yang–Mills equations in the presence of two light cone sources— one on each light cone. We have then

$$D_\mu F^{\mu\nu} = J^\nu,$$  \hspace{1cm} (1)

where

$$J^{\nu, a}(r_t) = \delta^{\nu+} g\rho^2_+(r_t) \delta(x^-) + \delta^{\nu-} g\rho^a_-(r_t) \delta(x^+) .$$  \hspace{1cm} (2)

Gluon distributions are simply related to the Fourier transform $A^a_i(k_t)$ of the solution to the above equation by $< A^a_i(k_t) A^a_i(k_t) > \rho$. The averaging over the classical charge distributions is defined by

$$\langle O \rangle_\rho = \int d\rho_+ d\rho_- O(\rho_+ , \rho_-)$$

$$\times \exp \left( - \int d^2 r_t \frac{\text{Tr}[\rho^2_+(r_t) + \rho^2_-(r_t)]}{2\mu^2} \right) .$$  \hspace{1cm} (3)

The averaging over the color charge distributions is performed independently for each nucleus with equal Gaussian weight $\mu^2$.

The observant reader will notice that we have omitted the rapidity dependence of the charge distributions in the equations immediately above. We will justify
this omission in our discussion of the lattice Hamiltonian. We note that the rapidity dependence of the charge distribution is also absent in Ref. [8].

Before the nuclei collide \((t < 0)\), a solution of the equations of motion is

\[
A^\pm = 0, \\
A^i = \theta(x^-)\theta(-x^+)\alpha_1^i(r_t) + \theta(x^+)\theta(-x^-)\alpha_2^i(r_t),
\]

(4)

where \[9\]

\[
\alpha_{1,2}^i(r_t) = \frac{1}{ig} \left( Pe^{-ig \int_{+\eta_{proj}} \frac{d\eta'}{\sqrt{\perp}} \rho_{\pm}(q',r_t)} \right)^{\dagger} \nabla^i \left( Pe^{-ig \int_{+\eta_{proj}} \frac{d\eta'}{\sqrt{\perp}} \rho_{\pm}(q',r_t)} \right).
\]

(5)

Above, \(\eta = \eta_{proj} - \log(x^-/x_{proj}^-)\) is the rapidity of the nucleus moving along the positive light cone with the gluon field \(\alpha_1^i\) and \(\eta = -\eta_{proj} + \log(x_{proj}^+/x^+)\) is the rapidity of the nucleus moving along the negative light cone with the gluon field \(\alpha_2^i\). It is expected that at central rapidities (or \(x << 1\)) the source density varies slowly as a function of rapidity and \(\alpha^i \equiv \alpha^i(r_t)\). The above expression suggests that for \(t < 0\) the solution is simply the sum of two disconnected pure gauges.

For \(t > 0\) the solution is no longer pure gauge. Working in the Schwinger gauge

\[
x^+A^- - x^-A^+ = 0,
\]

(6)

the authors of Ref. [8] found that with the ansatz

\[
A^\pm = \pm x^\pm \alpha(\tau, r_t), \\
A^i = \alpha^i_\perp(\tau, r_t),
\]

(7)

where \(\tau = \sqrt{2x^+x^-}\), Eq. [8] could be written in the simpler form
\[
\frac{1}{\tau^3} \partial_\tau \tau^3 \partial_\tau \alpha + [D_i, [D^i, \alpha]] = 0, \\
\frac{1}{\tau} [D_i, \partial_\tau \alpha^i_\perp] + ig\tau [\alpha, \partial_\tau \alpha] = 0, \\
\frac{1}{\tau} \partial_\tau \tau \partial_\tau \alpha^i_\perp - ig\tau^2 [\alpha, [D^i, \alpha]] - [D^j, F^i_j] = 0.
\] (8)

The initial conditions for the fields \(\alpha(\tau, r_t)\) and \(\alpha^i_\perp\) are given in terms of the fields for each of the nuclei at \(t < 0\). We have

\[
\alpha^i_\perp|_{\tau=0} = \alpha^i_1 + \alpha^i_2, \\
\alpha|_{\tau=0} = \frac{ig}{2} [\alpha^i_1, \alpha^i_2].
\] (9)

Further, since the equations are very singular at \(\tau = 0\), the only condition on the derivatives of the fields that would lead to regular solutions are \(\partial_\tau \alpha|_{\tau=0}, \partial_\tau \alpha^i_\perp|_{\tau=0} = 0\).

In Ref. [8], solutions were found in the perturbative limit by expanding the initial conditions and the fields in powers \(\rho\) or equivalently, in powers of \(\alpha_S \mu/k_t\). Performing a gauge transformation of the above fields (such that the new fields satisfy the Coulomb gauge condition), at late times \(\tau >> \alpha_S \mu\), the new fields can be expanded as Fourier series with coefficients \(a_{1,2}^b(k_t)\) and their complex conjugates. The multiplicity distribution of classical gluons can then be written as

\[
\frac{dN}{dyd^2k_t} = \frac{1}{(2\pi)^3} \sum_{i,b} |a_{i,2}^b(k_t)|^2.
\] (10)

Detailed expressions for classical gluon radiation in the perturbative limit were obtained in Refs. [8, 9, 10]. However, all these expressions are infrared singular and have to be regulated by an infrared cutoff. One advantage of solving the Yang–Mills equations to all orders in \(\alpha_S \mu/k_t\) is that it will likely provide a self–consistent, infrared safe result for the multiplicity of classical gluon radiation in ultrarelativistic nuclear collisions.
3 Derivation of lattice Hamiltonian

While the Yang–Mills equations discussed above can be solved perturbatively in the limit $\alpha_S k_t << \mu$, it is unlikely that a simple analytical solution exists for Eq. 1 in general. The classical solutions have to be determined numerically for $t > 0$. The straightforward procedure would be to discretize Eq. 1 but it will be more convenient for our purposes to construct the lattice Hamiltonian and obtain the lattice equations of motion from Hamilton’s equations.

We start from the QCD action (without dynamical quarks) which is given by

$$S_{\text{QCD}} = \int d^4x \sqrt{-g} \left\{ \frac{1}{4} g^{\mu\lambda} g^{\nu\sigma} F_{\mu\nu} F_{\lambda\sigma} - j^\mu A_\mu \right\},$$

(11)

where $g = \det|g_{\mu\nu}|$. In the forward light cone ($t > 0$) it is convenient to work with the $\tau, \eta, \vec{r}_t$ co–ordinates where $\tau = \sqrt{2x^+x^-}$ is the proper time, $\eta = \frac{1}{2} \log(x^+/x^-)$ is the space–time rapidity and $\vec{r}_t = (x, y)$ are the two transverse Euclidean co–ordinates. In these co–ordinates, the metric is diagonal with $g^{\tau\tau} = -g^{xx} = -g^{yy} = 1$ and $g^{\eta\eta} = -1/\tau^2$.

After a little algebra, the Hamiltonian can be written as

$$H = \int d\eta d\vec{r}_t \tau \left\{ \frac{1}{2} p^\eta p^\eta + \frac{1}{2\tau^2} p^\tau p^\tau + \frac{1}{2\tau^2} F_{\eta r} F_{\eta r} + \frac{1}{4} F_{xy} F_{xy} + j^\eta A_\eta + j^\tau A_\tau \right\}.$$  

(12)

Here we have adopted the gauge condition of Eq. 3, which is equivalent to requiring $A^\tau = 0$. Also, $p^\eta = \frac{1}{\tau} \partial_\eta A_\eta$ and $p^\tau = \tau \partial_\tau A_\tau$ are the conjugate momenta.

Consider the field strength $F_{\eta r}$ in the above Hamiltonian. If we assume approximate boost invariance, or

$$A_r(\tau, \eta, \vec{r}_t) \approx A_r(\tau, \vec{r}_t); \ A_\eta(\tau, \eta, \vec{r}_t) \approx \Phi(\tau, \vec{r}_t),$$

(13)

we obtain

$$F_{\eta r}^a = -D_r \Phi^a,$$

(14)
where $D_r = \partial_r - igA_r$ is the covariant derivative. Further, if we express $j^{\mu r}$ in terms of the $j^\pm$ defined in Eq. 3 we obtain the result that $j^{\mu r} = 0$ for $\tau > 0$. Finally, since

$$\Phi = \tau^2 \alpha(\tau, \vec{r}_i); \ A_r = \alpha^r_\perp(\tau, \vec{r}_i),$$

(15)

we can perform the integration over the space–time rapidity to re–write the Hamiltonian in Eq. 12 as

$$H = \int d\vec{r}_t \tau \eta \left\{ \frac{1}{2} \left( \frac{\partial \Phi^r}{\partial \tau} \right)^2 + \frac{1}{4} F^a_{xy} F^a_{xy} + \frac{1}{2\tau^2} \left( D_\beta \left[ \tau^2 \alpha \right] \right)^2 \right\}.$$  

(16)

Here the index $\beta = (\tau, \vec{r}_i)$. We have thus succeeded in expressing the Hamiltonian in Eq. 12 as the Yang–Mills Hamiltonian in 2+1–dimensions coupled to an adjoint scalar. The discrete version of the above Hamiltonian is well known and is the Kogut–Susskind Hamiltonian [19] in 2+1–dimensions coupled to an adjoint scalar field. We shall now restrict the discussion to an SU(2) Yang–Mills theory, in order to keep notation simple. Generalization to an arbitrary SU(N) gauge group is straightforward.

The Kogut-Susskind analogue of Eq. 16 is

$$H_L = \frac{1}{2\tau} \sum_{l=\{j,\hat{n}\}} E_l^a E_l^a + \tau \sum_{\sigma} \left( 1 - \frac{1}{2} \text{Tr} U_{\sigma} \right) + \frac{1}{4\tau} \sum_{j,\hat{n}} \text{Tr} \left( \Phi_j - U_{j,\hat{n}} \Phi_{j+\hat{n}} U_{j,\hat{n}}^\dagger \right)^2 + \frac{\tau}{4} \sum_j \text{Tr} p_j^2,$$

(17)

where $E_l$ are generators of right covariant derivatives on the group and $U_{j,\hat{n}}$ is a component of the usual SU(2) matrices corresponding to a link from the site $j$ in the direction $\hat{n}$. The first two terms correspond to the contributions to the Hamiltonian from the chromoelectric and chromomagnetic field strengths respectively. In the last equation $\Phi \equiv \Phi^a \sigma^a$ is the adjoint scalar field with its conjugate momentum $p \equiv p^a \sigma^a$.

Finally, we should comment on a key assumption in the above derivation, namely, the boost invariance of the fields. This invariance results in Eq. 16 thereby allowing us to restrict ourselves to a transverse lattice alone. To clarify the issue we are compelled to make a few historical remarks. As we mentioned earlier, the authors of Ref. 8
found a solution which was explicitly boost invariant. However, this result was a consequence of the original assumption of McLerran and Venugopalan that the color charge density factorizes, \( \rho^a(r_t, \eta) \rightarrow \rho^a(r_t) \delta(x^-) \). It was noticed in Ref. [20], this factorized form for the charge density results in infrared singular correlation functions which diverge as the square of the lattice size. This problem was resolved in Ref. [6] where the authors realized that a rapidity dependent charge density \( \rho^a(r_t, \eta) \) would give infrared safe solutions. This might be interpreted as implying that the boost invariance assumption of Ref. [8] should be given up as well.

Fortunately, this is not necessary. In principle, the rapidity dependence of the color charge density can be arbitrarily weak since that is sufficient to obtain infrared safe correlation functions. In Ref. [21], an explicit model was constructed for the color charge distribution in the fragmentation region. It was shown there that for \( \eta < \eta_{proj} \) the color charge distribution had a very weak dependence on \( \eta \). We should note too that it was shown recently by Gyulassy and McLerran [9] that the initial conditions in Eq. (8) are unaffected by the smearing in rapidity.

## 4 Lattice equations of motion and initial conditions

Lattice equations of motion follow directly from \( H_L \) of Eq. (17). For any dynamical variable \( v \) with no explicit time dependence \( \dot{v} = \{H_L, v\} \), where \( \dot{v} \) is the derivative with respect to \( \tau \), and \( \{\} \) denote Poisson brackets. We take \( E_l, U_l, p_j, \) and \( A_j \) as independent dynamical variables, whose only nonvanishing Poisson brackets are

\[
\{p^a_i, \Phi^b_j\} = \delta_{ij} \delta_{ab}; \quad \{E^a_i, U_m\} = -i \delta_{lm} U_l \sigma^a; \quad \{E^a_i, E^b_m\} = 2 \delta_{lm} \epsilon_{abc} E^c_l
\]

(no summing of repeated indices). The equations of motion are consistent with a set of local constraints (Gauss’ laws). These are

\[
C^a_j \equiv \sum_{\bar{n}} \left[ -\frac{1}{2} E^b_{j, \bar{n}} \text{Tr} \left( \sigma^a U_{j, \bar{n}} \sigma^b U_{j, \bar{n}}^\dagger \right) - E^a_{j-\bar{n}, \bar{n}} \right] - 2 \epsilon_{abc} p^b_j A^c_j = 0. \tag{18}
\]
The initial conditions for the fields and momenta in Eq. 3 can be discretized as follows. To every lattice site \( j \) we assign two SU(2) matrices, \( V_{1,j} \) and \( V_{2,j} \). Each of these two defines a lattice gauge field configuration (corresponding to the two pure gauges discussed in section 2) whose link variables are

\[
U_{j,\hat{n}}^q \equiv V_{q,j} V_{q,j+\hat{n}}^\dagger,
\]

where \( q = 1, 2 \) labels the two nuclei. An initial condition for the link matrices, having the correct formal continuum limit, is then

\[
U_{j,\hat{n}}|_{\tau=0} = \left. \frac{\sum_q U_{j,\hat{n}}^q - I}{\left( \frac{1}{2} \text{Tr}(\sum_q U_{j,\hat{n}}^q - I)(\sum_q U_{j,\hat{n}}^q - I) \right)^{1/2}} \right|_{\tau=0},
\]

(19)

where \( I \) is the identity matrix. Introducing \( \alpha_q \) through \( \exp(-ia\alpha_q) = U^q \), one can easily verify that for smooth initial field configurations

\[
1 - \frac{1}{2} U_\Box \to \frac{a^4}{2} B^\nu B^\nu
\]

in the limit of vanishing lattice spacing \( a \). Here \( B^\nu \) is the \( \nu \)th color component of the SU(2) magnetic field corresponding to the gauge potential \( \alpha_1 + \alpha_2 \). Thus the initial plaquette term of Eq. 17 has the correct formal continuum limit.

As we have already discussed, regularity requires the vanishing of the initial transverse color electric fields. If follows from Eq. 15 that the adjoint scalar \( A \) must also vanish initially. On the other hand, values of \( p \), the conjugate momentum of \( A \), need not be zero at \( \tau = 0 \). In the continuum

\[
p_{\eta}|_{\tau=0} = 2\alpha.
\]

This initial condition can be written on the lattice as follows:

\[
p_{\hat{n}}|_{\tau=0} = -i \sum_{\hat{n}} \left( [U_{j,\hat{n}}^1, U_{j,\hat{n}}^2] + [U_{j-\hat{n},\hat{n}}^1, U_{j-\hat{n},\hat{n}}^2] \right).
\]

10
Note that the initial conditions as described satisfy the Gauss constraints 1.

Finally, we comment on numerical integration of the lattice equations of motion. A method of choice for energy-conserving Hamiltonian equations is the leapfrog algorithm, wherein, the fields are updated at every step of the integration while their conjugate momenta are kept fixed and vice versa. In our case, the Hamiltonian depends explicitly on time and the energy is not conserved. Nevertheless, only a minor modification of the algorithm is necessary in order to maintain the same time step accuracy as in the energy-conserving case. As explained in [22], the leapfrog algorithm has the useful property of respecting Gauss’ laws exactly (regardless of the time step) for any Hamiltonian which is a sum of potential and kinetic terms.

5 Outlook

We have outlined above a procedure to solve for the gauge field configurations produced in the collision of the Weizsäcker–Williams fields of two nuclei. These can then be used to compute a large variety of observables as a function of proper time $\tau$. An observable that can be computed directly in our Hamiltonian approach is the energy density (and correlations in the energy density). Also, by looking at the Fourier decomposition of the co–ordinate space correlators of the electric fields, one can determine the energy and number distributions of the modes as well as the energy dispersion relation for the soft modes. These can be directly related to the multiplicity and transverse energy of mini–jets (for a review see Ref. [23]). Some care however must be exercised in ensuring that the residual gauge freedom of the fields is fixed properly.

The field configurations generated immediately after the collision are completely

\footnote{The lattice initial conditions are not completely specified by requiring that they reduce to the corresponding conditions in the continuum upon taking the formal continuum limit. This arbitrariness can be removed by a consistent derivation of the initial conditions entirely within the lattice theory. Such derivation will be presented elsewhere [26].}
out of equilibrium. It is an interesting problem to study if and how the system approaches equilibrium. This can be done by studying whether the multiplicity saturates as a function of time. Alternatively, one can follow the example of the Duke group and study the behaviour of the Lyapunov exponents as a function of time [24, 25]. If the system does indeed thermalize, energy density/temperature and velocity profiles can be extracted for use as initial conditions in hydrodynamic simulations.

A useful check of the results of our proposed simulation (besides the usual technical ones) is to reproduce the results of Ref. [8, 9, 10] in the “Abelian limit” of large transverse momenta, $k_t >> \alpha S\mu$. The sensitivity of various observables to the lattice spacing and lattice size must also be carefully studied.

There are several open questions which have still to be resolved. Primarily, our simulation is completely classical–how accurate is the classical description? It would be interesting to study whether the Wilson renormalization group evolution for the scale $\chi(\eta, Q^2)$ can be implemented on the lattice. Another issue is our assumption of boost invariance–relaxing this assumption however is not conceptually difficult but may be numerically time consuming.

A more detailed discussion of the above and detailed numerical simulations will be presented at a later date [26].

Acknowledgments

One of us (R.V.) would like to thank Miklos Gyulassy, Larry McLerran and Berndt Müller for useful discussions. He would also like to thank the organizers of the “International conference on the physics and astrophysics of the quark gluon plasma” held in Jaipur, India, from March 17th–21st.

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