Hydrodynamic modeling of QGP expansion using an exact solution of Riemann problem

Zuzana Fecková1,2 and Boris Tomášik2,3

1 University of Pavol Jozef Safrík, Slobárova 2, 04001 Košice, Slovakia
2 Matej Bel University, Tajovského 40, 97401 Banská Bystrica, Slovakia
3 FNSPE, Czech Technical University in Prague, Břehová 7, 11519 Prague 1, Czech Republic
E-mail: zuzana.feckova1@student.upjs.sk, boris.tomasik@umb.sk

Abstract. Hydrodynamic modelling of quark-gluon plasma requires sophisticated numerical schemes that have low numerical viscosity and are able to cope with high gradients of energy density that may appear in initial conditions. We propose to use the Godunov method with an exact Riemann solver for ideal hydrodynamic modelling to meet these conditions. We present the results of numerical tests of the method, such as the sound wave propagation and the shock tube problem, which show both high precision and low numerical viscosity.

1. Introduction
Relativistic hydrodynamics has proved to be a useful tool for studying the expansion of the quark-gluon plasma created in heavy ion collisions, e.g. [1, 2]. In the present paper we will introduce a new numerical scheme for such modelling based on the exact solution of the Riemann problem with an arbitrary equation of state and a possible source term. It aims for high precision and the suitability to work with complicated initial conditions and to capture shocks that might arise from jets propagating in the medium [3, 4, 5].

2. Numerical method
Ideal relativistic hydrodynamic equations express the conservation of energy and momentum as well as conserved charges, e.g. the baryon number in heavy-ion collisions. In collisions at highest energies the net baryon density is practically zero, thus we only consider energy and momentum:

\[ \partial_\mu T^{\mu\nu}(0) = 0, \]

\[ T^{\mu\nu}(0) = (\epsilon + p)u^\mu u^\nu - pg^{\mu\nu}, \]

and the equation of state which completes the system of equations relates only the pressure and the energy density \( p = p(\epsilon) \).

The presented equations are solved numerically using a Godunov scheme with slope-limited linear reconstruction of states. The flows of conserved quantities are calculated at every cell boundary [6, 7] by exactly solving the Riemann problem for an arbitrary equation of state with the presence of tangential velocities [8, 9]. The Riemann problem is solved by reconstructing the corresponding wave pattern at the boundary considering possible shock and rarefaction waves and a contact discontinuity.
3. Numerical tests

3.1. Sound wave problem

We simulate a sound wave over one wavelength in the numerical grid with these initial conditions:

\[ p_{\text{init}}(x) = p_0 + \delta p \sin \frac{2\pi x}{\lambda}, \]
\[ v_{\text{init}}(x) = \frac{\delta p}{c_s(e_0 + p_0)} \sin \frac{2\pi x}{\lambda}, \]

with parameters \( p_0 = 10^3 \text{ fm}^{-4}, \delta p = 10^{-1} \text{ fm}^{-4} \). The variation of pressure is sufficiently small \( \delta p \ll p_0 \), we can thus consider the linearized analytic solution, which is a sound wave of velocity \( c_s \). To evaluate the precision of the scheme the numerical solution \( p \) is compared to the analytic one denoted \( p_s \) using \( L_1 \) norm [10]:

\[ L(p(N_{\text{cell}}), p_s) = \sum_{i=1}^{N_{\text{cell}}} |p(x_i; N_{\text{cell}}) - p_s(x_i)| \frac{\lambda}{N_{\text{cell}}}, \]

We have studied the dependence of \( L_1 \) norm on the number of cells in the numerical grid, which is shown in the left panel of Fig. 1. As expected, the precision improves with finer discretization and is comparable to other numerical schemes presented in [10].

![Graph](image)

**Figure 1.** Dependence of \( L_1 \) norm (left) and numerical viscosity \( \eta_{num} \) (right) on the number of cells in the numerical grid.

The \( L_1 \) norm can also be used to assess the numerical dissipation that is due to limited precision. The quark-gluon plasma is expected to have very low viscosity, therefore it is very important to control the numerical viscosity, evaluated as

\[ \eta_{num} = -\frac{3\lambda}{8\pi^2 c_s(e_0 + p_0)} \ln \left[ 1 - \frac{\pi}{2\lambda\delta p} L(p(N_{\text{cell}}), p_s) \right]. \]

We show the dependence of numerical viscosity \( \eta_{num} \) on the number of cells in the grid in the right panel of Fig. 1. Its values are adequately small, comparable again to other numerical schemes used in heavy ion collision modelling [10].

To compare the numerical dissipation with the possible values of physical dissipation a more suitable parameter is the ratio of viscosity to entropy density \( \eta/s \). We have estimated the value of this ratio for the numerical viscosity in our scheme and in Fig. 2 we are comparing its values with the estimated values for pion gas [11] and the well-known quantum limit \( \eta/s = 1/4\pi \) from AdS/CFT calculations [12].
3.2. Shock tube problem

To test the scheme’s capability to cope with large discontinuities in energy density and velocity as well as its ability to resolve the shock and rarefaction wave as precisely as possible we impose special initial conditions in the numerical grid. They consist of two constant states separated by a discontinuity. The imposed energy density and tangential velocity vary in the grid: in the left (right) half they are: $\epsilon_L = 1 \text{ GeV}, \epsilon_R = 20 \text{ GeV}, v^t_L = 1/3c \ (v^t_R = 1/2c)$. The initial normal velocity is $v^x = c/2$ over the whole grid. The analytic solution is such that with time the discontinuity is dissolved via waves: a rarefaction wave propagates into the region of higher energy density, a shock wave into the region of lower energy density and the two are connected by a contact discontinuity. In Fig. 3 we compare the analytic solution to the numerical solution with profiles of the energy density and the normal velocity in the grid after 100 time-steps. We conclude that the wave patterns are very well reproduced by the numerical solution.

Finally, in Fig. 4 we show the importance of linear reconstruction of states in the numerical grid. In the left panel, the profile of the tangential velocity after 100 time-steps when using a
Figure 4. Profile of tangential velocity using a piecewise constant distribution of variables (left) and using a linear reconstruction of states (right) in the numerical grid after 100 time-steps (our scheme in blue circles, analytic solution in black solid line, initial conditions in red dashed line).

4. Summary
We have built and tested an ideal relativistic hydrodynamic scheme based on exact solution of the Riemann problem in one spatial dimension with presence of tangential velocity. The sound wave propagation test shows a good precision and low numerical viscosity which will become important when introducing dissipation into the model. The shock tube problem reveals that the scheme is able to capture shock and rarefaction waves very well. We will extend this scheme to three dimensions and then apply it in the description of the flow in ultrarelativistic nuclear collisions.

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