A formula of solutions for some Volterra integral equations

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Abstract. In this paper, we use the new modification of the Adomian decomposition method (NMADM) to formulate a solution of some weakly singular Volterra integral equations. In some cases, Taylor’s series is applied to simplify the formula. Some examples are given to illustrate the applicability and effectiveness of this method.

1. Introduction
In recent decades, several techniques have been proposed to solve Volterra integral equations [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], for example, the combined Laplace transform together with Adomian-decomposition method [11, 12], the optimal homotopy asymptotic method [13], the new modification of Adomian decomposition method (NMADM) [14, 15]. In this paper, we apply the NMADM to formulate the solutions of the weakly singular Volterra integral equation

\[ u(x) = \sum_{i=0}^{N} x^{m+r} + \int_{0}^{x} \frac{t^{\mu}}{x^{\mu}} u(t) dt, \]  

where \( \mu \) is a constant and \( m, r, q \) are integers such that \( 0 \leq r < q \). Several examples show the effectiveness of the formula.

2. Methods and Results
In this section, we applied the new modification of Adomian decomposition method to a weakly singular Volterra integral equation which is expressed as below:

\[ u(x) = x^{m+r} + \int_{0}^{x} \frac{t^{\mu}}{x^{\mu}} u(t) dt, \]  

where \( m, r, q \) are integers such that \( 0 < r \leq q \).

Applying the Adomian decomposition method, we assume that

\[ u(x) = \sum_{k=0}^{\infty} u_k(x). \]  

Applying the NMADM, we rewrite equation (2) as

\[ u(x) = \sum_{i=0}^{\infty} \alpha_i x^{m+r+i} + \sum_{i=0}^{\infty} \alpha_i x^{m+r+i} + x^{m+r} + \int_{0}^{x} \frac{t^{\mu}}{x^{\mu}} u(t) dt, \]
where \( \alpha_i \) and \( x^{m+\frac{\mu}{q}+i} \), \( i = 0, 1, 2, \ldots, N \) are accelerating components of the parameter and selective functions, respectively.

Applying the Adomian decomposition method, \( u(x) \) can be expressed as the series:

\[
u(x) = \sum_{k=0}^{\infty} u_k(x),\]

where the components \( u_k(x) \) will be determined recursively.

Applying the Adomian decomposition method, we develop the modified recursive relation

\[
u_0(x) = \sum_{i=0}^{\infty} \alpha_i x^{m+\frac{\mu}{q}+i},
\]

\[
u_1(x) = -\sum_{i=0}^{\infty} \alpha_i x^{m+\frac{\mu}{q}+i} + x^{m+\frac{\mu}{q}} + \int_0^x \frac{t^\mu}{x^\mu} u_0(t) dt,
\]

\[
u_{k+1}(x) = \int_0^x \frac{t^\mu}{x^\mu} u_k(t) dt, \quad k = 1, 2, 3, \ldots
\]

where \( \alpha_i \)'s are constant to be determined.

By assumption of the NMADM, we set

\[
u_1(x) = 0,
\]

which implies that

\[-\sum_{i=0}^{\infty} \alpha_i x^{m+\frac{\mu}{q}+i} + x^{m+\frac{\mu}{q}} + \int_0^x \frac{t^\mu}{x^\mu} u_0(t) dt = 0. \tag{4}\]

Substituting (4) into (4), we have

\[-\sum_{i=0}^{N} \alpha_i x^{r_i} + a_n x^n + \int_0^x \frac{t^\mu}{x^\mu} \sum_{i=0}^{\infty} \alpha_i x^{m+\frac{\mu}{q}+i} dt = 0.
\]

Therefore,

\[\alpha_0 = 1,
\]

and when \( k > 0 \)

\[\alpha_k = \frac{\Gamma(m + \mu + \frac{\mu}{q} + 1)}{\Gamma(m + \mu + \frac{\mu}{q} + k + 1)}.
\]

So the solution of this problem is

\[
u(x) = x^{m+\frac{\mu}{q}} + \frac{\Gamma(m + \mu + \frac{\mu}{q} + 1)}{x^\mu} \left[ \sum_{i=0}^{\infty} \frac{x^{i+\frac{\mu}{q}}}{\Gamma(i + \frac{\mu}{q} + 1)} - \sum_{i=0}^{m+\mu} \frac{x^{i+\frac{\mu}{q}}}{\Gamma(i + \frac{\mu}{q} + 1)} \right]. \tag{5}\]

When \( \mu + \frac{\mu}{q} \) is an integer, we get

\[
u(x) = x^{m+\frac{\mu}{q}} + \frac{(m + \mu + \frac{\mu}{q})!}{x^\mu} \left[ e^x - \sum_{i=0}^{m+\mu + \frac{\mu}{q}} \frac{x^i}{i!} \right]. \tag{6}\]
as \( \sum_{i=0}^{\infty} \frac{x^i}{i!} \) is a Taylor’s series expansion of \( e^x \). Therefore, the solution of an equation

\[
u(x) = x^m + \int_0^x \frac{t^\mu}{x^\mu} u(t) dt.
\]
is

\[
u(x) = x^m + \frac{(m + \mu)!}{x^\mu} \left[ e^x - \sum_{i=0}^{m + \mu} \frac{x^i}{i!} \right], \tag{7}
\]
where \( m \geq \mu \).

**Example 1** Consider the following weakly singular Volterra integral equation:

\[
u(x) = 1 + \int_0^x \frac{t}{x} u(t) dt.
\]

We can see that

\[ m = 0, r = 1, \mu = 1. \]

From (7), the exact solution is

\[
u(x) = x^0 + \frac{(0 + 1)!}{x^1} \left[ e^x - \sum_{i=0}^{0+1} \frac{x^i}{i!} \right] = 1 + \frac{1}{x} \left[ e^x - 1 - x \right] = \frac{e^x}{x} - \frac{1}{x}.
\]

**Example 2** Consider the following weakly singular Volterra integral equation:

\[
u(x) = x^{0.5} + \int_0^x \frac{t^{0.5}}{x^{0.5}} u(t) dt.
\]

We can see that

\[ m = 0, \quad \frac{r}{q} = 0.5, \mu = 0.5. \]

From (7), the exact solution is

\[
u(x) = x^{0.5} + \frac{(0.5 + 0.5)!}{x^{0.5}} \left[ e^x - \sum_{i=0}^{0.5+0.5} \frac{x^i}{i!} \right] = \frac{e^x}{x^{0.5}} + \frac{1}{x^{0.5}} \left[ e^x - 1 - x \right] = \frac{e^x}{\sqrt{x}} - \frac{1}{\sqrt{x}}.
\]

The following proposition shows that the formula is also effective when the given function is a linear combination of such functions. The proof of this proposition is obvious.

**Proposition 1** If \( u(x) = f_1(x) \) and \( u(x) = f_2(x) \) are solutions of equations

\[
u(x) = g_1(x) + \int_0^x \frac{t^\mu}{x^\mu} u(t) dt
\]
and
\[ u(x) = g_2(x) + \int_0^x \frac{t^\mu}{x^\mu} u(t) dt, \]
respectively. Then \( u(x) = c_1 f_1(x) + c_2 f_2(x) \) is a solution of an equation
\[ u(x) = c_1 g_1(x) + c_2 g_2(x) + \int_0^x \frac{t^\mu}{x^\mu} u(t) dt, \]
where \( c_1 \) and \( c_2 \) are constants.

**Example 3** Consider the following weakly singular Volterra integral equation:
\[ u(x) = x^2 + 4 + \int_0^x \frac{t^2}{x^2} u(t) dt \]
(8)

From (7), the solutions of
\[ u(x) = x^2 + \int_0^x \frac{t^2}{x^2} u(t) dt \]
and
\[ u(x) = 4 + \int_0^x \frac{t^2}{x^2} u(t) dt \]
are
\[ u(x) = x^2 + \frac{24e^x}{x^2} - \frac{24}{x^2} - 12 - 4x - x^2 \]
and
\[ u(x) = 4 + \frac{8e^x}{x^2} - \frac{8}{x^2} - 8x - 4, \]
respectively. Therefore, the solution of equation (8) is
\[ u(x) = \frac{32e^x}{x^2} - \frac{32}{x^2} - \frac{32}{x} - 4x - 12. \]

The formula can be applied for some initial value problem differential equations.

**Example 4** Consider the following IVP:
\[ \frac{du}{dx} = 1 + 2x + u(x), \, u(0) = 0. \]
(9)

In operator form (9) becomes
\[ Lu = 1 + 2x + u(x), \]
(10)
where \( L(.) = \frac{d}{dx}(.). \) Then inverse of \( L \) is \( L^{-1}(.) = \int_0^x u(t) dt \). Applying \( L^{-1} \) to both sides of (10) we find that
\[ u(x) = x + x^2 + \int_0^x u(t) dt \]
(11)
which is of the form that its solution can be found by using formula (6). Applying such the formula, we get
\[ u(x) = 3e^x - 2x - 3. \]
Example 5 Consider the following weakly singular Volterra integral equation:

\[
u(x) = x^{0.5} + \int_{0}^{x} \frac{t}{x} u(t) dt.
\]

We can see that 

\[m = 0, r = 1, q = 2 \text{ and } \mu = 1.\]

From (6), the analytical solution of the above equation is

\[
u(x) = x^{0.5} + \Gamma(0 + 1 + 0.5 + 1) \left[ \sum_{i=0}^{\infty} \frac{x^{i+0.5}}{i+1} - \sum_{i=0}^{0+1} \frac{x^{i+0.5}}{i+0.5 + 1} \right]
\]

\[= \sqrt{x} + \frac{\Gamma(\frac{5}{2})}{x} \left[ \sum_{i=0}^{\infty} \frac{x^{i+0.5}}{\Gamma(i+0.5 + 1)} - \frac{\sqrt{x}}{\Gamma(\frac{3}{2})} \right].
\]

3. Discussion

We first apply the NMADM [14] for finding a solution of weakly singular Volterra integral equations for the case when the given function is of the form \(x^{m+\frac{r}{q}}\) where \(m, r\) and \(q\) are integers such that \(0 \leq r < q\). Proposition 1 and Example 3,4 show that it is also effective when the given function is a linear combination of functions of this form. Shown by Example 2, the solution is closed-form when \(m + \mu + \frac{r}{q}\) is an integer. Example 5 shows that our formula can be used to find the analytical solution for any given function that can be represented by the finite sum of \(x^i\) where \(i \geq \mu\).

4. Conclusions

This work is aimed at finding the solution formula of the weakly singular Volterra integral equation:

\[
u(x) = x^{m+\frac{r}{q}} + \int_{0}^{x} \frac{t^\mu}{x^\mu} u(t) dt.
\]

The solution obtained is

\[
u(x) = x^{m+\frac{r}{q}} + \frac{\Gamma(m + \mu + \frac{r}{q} + 1)}{x^\mu} \left[ \sum_{i=0}^{\infty} \frac{x^{i+\frac{r}{q}}}{\Gamma(i+\frac{r}{q} + 1)} - \sum_{i=0}^{m+\mu} \frac{x^{i+\frac{r}{q}}}{\Gamma(i+\frac{r}{q} + 1)} \right].
\]

The examples show that this formula is very effective.

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