Solutions to a class of forced drift-diffusion equations with applications to the magneto-geostrophic equations

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Abstract. We prove the global existence of classical solutions to a class of forced drift-diffusion equations with $L^2$ initial data and divergence free drift velocity $\{u^\nu\}_{\nu \geq 0} \subset L_t^\infty BMO_x^{-1}$, and we obtain strong convergence of solutions as the viscosity $\nu$ vanishes. We then apply our results to a family of active scalar equations which includes the three dimensional magneto-geostrophic $\{MG^\nu\}_{\nu \geq 0}$ equation that has been proposed by Moffatt in the context of magnetoostrophic turbulence in the Earth’s fluid core. We prove the existence of a compact global attractor $\{A^\nu\}_{\nu \geq 0}$ in $L^2(T^3)$ for the $MG^\nu$ equations including the critical equation where $\nu = 0$. Furthermore, we obtain the upper semicontinuity of the global attractor as $\nu$ vanishes.

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1. Introduction

Our motivation for addressing the limiting behaviour of a class of drift-diffusion equations comes from a model proposed by Moffatt and Loper [32], Moffatt [34] for magnetostrophic turbulence in the Earth’s fluid core. This model is derived from the full three dimensional magnetohydrodynamic (MHD) equations in the context of a rapidly rotating, densely stratified, electrically conducting fluid. For discussions about the MHD equations in geophysical contexts see, for example, [1], [4], [16], [27], [33], [35]. Following the notation of Moffatt and Loper [32], we write the equations in terms of dimensionless variables. The orders of magnitude of the resulting non-dimensional parameters are motivated by the physical postulates of their model:

$$R_0(\partial_t u + u \cdot \nabla u) + e_3 \times u = -\nabla P + e_2 \cdot \nabla b + R_m \cdot \nabla b + \theta e_3 + \nu \Delta u,$$  \hspace{1cm} (1.1)

$$R_m[\partial_t b + u \cdot \nabla b - b \cdot \nabla u] = e_2 \cdot \nabla u + \Delta b,$$  \hspace{1cm} (1.2)

$$\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta + S,$$  \hspace{1cm} (1.3)

$$\nabla \cdot u = 0, \nabla \cdot b = 0.$$  \hspace{1cm} (1.4)

The unknowns are $u(x,t)$ the velocity, $b(x,t)$ the magnetic field (both vector valued) and $\theta(x,t)$ the scalar (temperature field of the fluid). $P$ is the sum of the fluid and magnetic pressures, and the Cartesian unit vectors are given by $e_1, e_2$ and $e_3$. The physical forces governing this system are the Coriolis force, the Lorentz force and gravity acting via buoyancy, while the equation for the temperature is driven by a smooth function $S(x)$ that represents the external forcing of the MHD system.

The non-dimensional parameters in (1.1)-(1.4) are $R_0$ the Rossby number, $R_m$ the magnetic Reynolds number, $\nu$ a (non-dimensional) viscosity and $\kappa$ a (non-dimensional) thermal diffusivity. Moffatt and Loper argue that for the geophysical context they are modelling, all these parameters are...
small, with $\nu$ and $\kappa$ being extremely small. We note that the ratio of the Coriolis to Lorentz forces in their model is of order 1, so for notational simplicity we have set this parameter, denoted by $N^2$ in (32), equal to 1. Hence in (1.1)-(1.3) we have a system derived from an important physical problem that is very rich in small parameters. The mathematical properties of this system under various settings of some of the parameters to zero, or in the vanishing limits have been addressed in a sequence of different articles, [18], [20], [21], [22], [23], [24], [25]. Although the physically relevant boundary for a model of the Earth’s fluid core is a spherical annulus, for mathematical tractability these studies have considered the system on the periodic domain $T^3$, with all fields being mean free, a condition which is preserved by the equations. In this present paper we study the forced system (1.1)-(1.4) under these boundary conditions.

The system as investigated by Moffatt and Loper neglects the terms multiplied by $R_0$ and $R_{m}$ in comparison with the remaining terms. Essentially this means that the evolution equations (1.1) and (1.2) for the coupled velocity and magnetic vectors take a simplified “quasi-static” form. A linear relationship is then established between the vector fields and the scalar temperature $\theta$. The sole remaining nonlinearity in the system occurs in the evolution equation for $\theta$ given by (1.3). This equation is then an active scalar equation where the constitutive law that relates the vector $u$ and the scalar $\theta$ is obtained from the reduced linear system

$$e_3 \times u = -\nabla P + e_2 \cdot \nabla b + \theta e_3 + \nu \Delta u, \quad (1.5)$$
$$0 = e_2 \cdot \nabla u + \Delta b, \quad (1.6)$$
$$\nabla \cdot u = 0, \nabla \cdot b = 0. \quad (1.7)$$

This system encodes the vestiges of the physics in the problem, namely the Coriolis force, the Lorentz force and gravity. Vector manipulations of (1.5)-(1.7) give the expression

$$\{[\nu \Delta^2 - (e_2 \cdot \nabla)^2]u - \nu \Delta^2 - (e_2 \cdot \nabla)^2\} = -\nu \Delta^2 - (e_2 \cdot \nabla)^2\} \nabla \times (e_3 \times \nabla \theta) + (e_3 \cdot \nabla)\Delta(e_3 \times \nabla \theta).$$

(1.8)

We study the forced active scalar equation

$$\begin{cases}
\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta + S, \\
u = M^\nu[\theta], \theta(x, 0) = \theta_0(x)
\end{cases}$$

(1.9)

via an examination of the Fourier multiplier symbol of the operator $M^\nu$ obtained from (1.8). This active scalar equation is called the magnetogeostrophic (MG) equation. We also refer to (1.9) as the MG$^\nu$ equation when $\nu > 0$, and to the case when $\nu = 0$ as the MG$^0$ equation. In Section 7, we write the explicit expression for the Fourier multiplier symbol of $M^\nu$ obtained from (1.8). We observed that the limit $\nu \to 0$ is a highly singular limit: in particular, when $\nu > 0$ the operator $M^\nu$ is smoothing of degree 2, however when $\nu = 0$ the operator $M^0$ is singular of degree $-1$. The goal of the current article is to examine the convergence of solutions to (1.9) in the limit as the viscosity $\nu$ goes to zero.

Details of the singular behaviour of the Fourier multiplier symbols for the operator $M^0$ in certain regions of Fourier space are given in [23]. More general issues concerning the ill-posedness and well-posedness of the unforced MG$^0$ equation can be found in [20], [23], [24]. In particular, it is to be noted that the MG$^0$ with $\kappa > 0$ is the so-called critical MG equation in the sense of the delicate balance between the nonlinear term and the dissipative term. Various critical active scalar equations such as surface quasi-geostrophic equation (SQG) have received considerable attention in the past decade because of the challenging nature of this delicate balance, [3], [13], [14], [15], [17], [19], [28], [29]. On the other hand, as we discussed in [22], the MG$^\nu$ equation with $\nu > 0$, where the symbol $\hat{M}^\nu$ decays like $k^{-2}$, is a case where the dissipative term dominates the nonlinear term. In [22] it is shown that in the case of the MG$^\nu$ equation with $\nu > 0$ and $\kappa > 0$, even for singular initial data, the global solution is instantaneously $C^\infty$-smoothed and satisfied classically for all $t > 0$. With this dichotomy in mind, we seek to determine the long time behaviour of the forced critical MG$^0$ equation through the “vanishing viscosity” limit of the MG$^\nu$ equation. Friedlander and Vicol [23] studied a class of unforced active scalar equation of the type of associated with the singular operator $M^0$. They did this
in the context of a general class of drift diffusion equations where the divergence free drift velocity lies in $L^\infty BMO_x^{-1}$, which class includes the MG\textsuperscript{0} equation. We follow their approach using De Giorgi techniques to obtain global well-posedness results for a class of forced drift diffusion equations. More specifically, we study the following active scalar equation in $\nu > 0$, solutions of the system (1.10) for $t \geq \tau$, where $\theta_0$ is the initial condition and $S = S(x)$ is a given smooth function that represents the forcing of the system. \{T_{ij}^\nu\}_{\nu \geq 0}$ is a sequence of operators which satisfy:

(1.11) $\partial_t \partial_j T_{ij}^\nu f = 0$ for any smooth functions $f$.

(1.12) $T_{ij}^\nu : L^\infty(\mathbb{T}^d) \to BMO(\mathbb{T}^d)$ are bounded for all $\nu \geq 0$.

(1.13) There exists a constant $C_\ast > 0$ independent of $\nu$, such that for all $1 \leq i,j \leq d$, $\sup_{\nu \in (0,1]} \sup_{\{k \in \mathbb{Z}^d\}} |\tilde{T}_{ij}^\nu(k)| \leq C_\ast$;

\[
\sup_{\{k \in \mathbb{Z}^d\}} |\tilde{T}_{ij}^0(k)| \leq C_\ast,
\]

where $T_{ij}^0 = T_{ij}^\nu |_{\nu = 0}$.

(1.14) For each $1 \leq i,j \leq d$,

\[
\lim_{\nu \to 0} \sum_{k \in \mathbb{Z}^d} |\tilde{T}_{ij}^\nu(k) - \tilde{T}_{ij}^0(k)|^2 \hat{g}(k)^2 = 0
\]

for all $g \in L^2$.

The main results that we prove for the forced problem are stated in the following theorems:

**Theorem 1.1** (Existence of smooth solutions). Let $\theta_0 \in L^2$, $S \in C^\infty$ with $\|S\|_{L^\infty} < \infty$ and $\kappa > 0$ be given, and assume that \{T_{ij}^\nu\}_{\nu \geq 0} satisfy conditions (1.11)-(1.14). There exists a classical solution $\theta^\nu(x,t) \in C^\infty((0,\infty) \times \mathbb{T}^d)$ of (1.10), evolving from $\theta_0$ for all $\nu \geq 0$.

**Theorem 1.2** (Convergence of solutions as $\nu \to 0$). Let $\theta_0 \in L^2$, $S \in C^\infty$ with $\|S\|_{L^\infty} < \infty$ and $\kappa > 0$ be given, and assume that \{T_{ij}^\nu\}_{\nu \geq 0} satisfy conditions (1.11)-(1.14). If $\theta^\nu$, $\theta$ are $C^\infty$ smooth classical solutions of the system (1.10) for $\nu > 0$ and $\nu = 0$ respectively with initial data $\theta_0$, then given $\tau > 0$, for all $s \geq 0$, we have

\[
\lim_{\nu \to 0} \|\theta^\nu - \theta(t)\|_{H^s} = 0,
\]

whenever $t \geq \tau$.

**Theorem 1.3** (Existence of a global attractor for the MG equation). Assume $S \in L^p$ with $p > 3$. Then the system (1.9) with $\nu = 0$ possesses a compact global attractor $A$ in $L^2(\mathbb{T}^d)$, namely

\[
A = \{\theta_0 : \theta_0 = \theta(0) \text{ for some bounded complete "vanishing viscosity" solution } \theta(t)\}\text{.}
\]

For any bounded set $B \subset L^2(\mathbb{T}^d)$, and for any $\varepsilon, T > 0$, there exists $t_0$ such that for any $t_1 > t_0$, every "vanishing viscosity" solution $\theta(t)$ with $\theta(0) \in B$ satisfies

\[
\|\theta(t) - x(t)\|_{L^2} < \varepsilon, \forall t \in [t_1, t_1 + T]
\]

for some complete trajectory $x(t)$ on the global attractor ($x(t) \in A, \forall t \in (-\infty, \infty)$).

Furthermore, for $\nu \in [0,1]$, there exists a compact global attractor $A^\nu \subset L^2$ for (1.9) such that $A^0 = A$ and $A^\nu$ is upper semicontinuous at $\nu = 0$, which means that

\[
\sup_{\phi \in A^\nu} \inf_{\psi \in A} \|\phi - \psi\|_{L^2} \to 0 \text{ as } \nu \to 0.
\]
Our paper is organised as follows. In Section 2, we give some preliminaries and notations which will be used in later sections. In Section 3, we state and prove Theorem 1.1, namely the existence of a smooth solution to (1.10). In Section 4, we obtain a uniform $H^s$-bound on smooth solutions to (1.10) which will be used in proving Theorem 1.2. In Section 5, we give the proof of Theorem 1.2. In Section 6, we show there is no dissipation of energy in the limit as $\nu \to 0$. In Section 7 we show that the MG$\nu$ equation with $\nu \geq 0$ satisfies the general conditions formulated for the active scalar equation in (1.10)-(1.14). Hence Theorem 1.2 can be applied to prove convergence as $\nu \to 0$ of solutions of the subcritical MG$\nu$ equation to solutions of the critical MG$0$ equation. We introduce the concept of a “vanishing viscosity” weak solution of the MG$0$ equation and we prove that the forced critical MG$0$ equation possesses a compact global attractor in $L^2(\mathbb{T}^d)$ satisfying (1.16).

2. Preliminaries and notations

In this section, we give some preliminaries and notations which are useful in later sections.

Throughout this paper, we shall denote $L^p([0, \infty); L^q(\mathbb{R}^d))$ by $L^p_t L^q_x$ for $1 \leq p, q \leq \infty$, and similarly for $L^p_t BMO_x$. Also, $L^p_t I_x(I \times B) = L^p(I; L^p(B))$ for any $I \subset \mathbb{R}$ and $B \subset \mathbb{R}^d$.

Let $\{\hat{\phi}_j\}_{j \in \mathbb{Z}}$ be a standard dyadic decomposition of the frequency space $\mathbb{R}^d$, with the Fourier support of the Schwartz function $\hat{\phi}_j$ being $\{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, and where $\hat{\phi}_j(\xi) = 1$ on $\mathbb{R}^d/\{0\}$. We define $\Delta_j f = \hat{\phi}_j * f$ for all Schwartz functions $f$. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$ the homogeneous Besov norm for $B^s_{p,q}$ is classically defined as

$$\|f\|_{B^s_{p,q}} = \|2^{js} \|\Delta_j f\|_{L^p} \|_{\ell^q(\mathbb{Z})}.$$  

We also recall the Chemin-Lerner space-time Besov space $\tilde{L}^r(I; B^s_{p,q})$, with norm given by

$$\|f\|_{\tilde{L}^r(I; B^s_{p,q})} = \left\|2^{js} \left( \int_I \|\Delta_j f(\cdot, t)\|_{L^r} dt \right)^{\frac{1}{r}} \right\|_{\ell^q(\mathbb{Z})},$$  

where $s \in \mathbb{R}$, $1 \leq r, p, q \leq \infty$ and $I$ is a time interval.

We will make use of the following well-known embedding theorems in later sections (refer to [2] for more details).

**Gagliardo-Nirenberg-Sobolev inequality:** Assume that $1 \leq p \leq d$. There exists a constant $C > 0$ depending only on $p$ and $d$, such that

$$\|f\|_{L^{d^p/(d-p)}} \leq C \|Df\|_{L^p},$$  

for all $f \in C^1_0(\mathbb{R}^d)$.

**Gagliardo-Nirenberg interpolation inequality:** Fix $1 \leq q, r \leq \infty$ and $m \in \mathbb{N}$. Suppose that $m, p, \gamma, j$ satisfy

$$\frac{1}{p} = \frac{j}{d} + \left(\frac{1}{r} - \frac{m}{d}\right)\gamma + \frac{1-\gamma}{q}, \quad \frac{j}{m} \leq \gamma \leq 1,$$

where $d$ is the dimension, then

$$\|D^j f\|_{L^r} \leq C \|D^m f\|_{L^r} \|f\|_{L^q}^{\frac{1-\gamma}{\gamma}},$$  

where $C > 0$ is a constant which depends only on $m, d, j, q, r, \gamma$.

**Gagliardo-Nirenberg interpolation inequality for homogeneous Sobolev space:** Let $q, r \in (1, \infty]$ and $\sigma, s \in (0, \infty)$ with $\sigma < s$. There exists a positive dimensional constant $C$ such that

$$\|f\|_{W^\sigma_q} \leq C \|f\|_{L^q}^{\gamma} \|f\|_{W^s_q}^{1-\gamma},$$  

with $\frac{1}{p} = \frac{2}{q} + \frac{1-2\sigma}{r}$ and $\gamma = 1 - \frac{2}{s}$.


**Besov Embedding Theorem:** Let \( s \in \mathbb{R} \). If \( 1 \leq p_1 \leq p_2 \leq \infty \) and \( 1 \leq r_1 \leq r_2 \leq \infty \), then
\[
\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-\frac{d}{p_1}+\frac{1}{2}}_{p_2,r_2}. \tag{2.4}
\]

### 3. Existence of smooth solutions

In this section, we prove the existence of smooth solutions to the forced non-linear problem \([1.10]\). It can be stated as follows.

**Theorem 3.1.** Let \( \theta_0 \in L^2 \), \( S \in C^\infty \) with \( ||S||_{L^\infty} < \infty \) and \( \kappa > 0 \) be given, and assume that \( \{T^\nu_{ij}\}_{\nu \geq 0} \) satisfy conditions \((1.11)-(1.14)\). There exists a classical solution \( \theta^\nu(x,t) \in C^\infty((0, \infty) \times \mathbb{T}^d) \) of \([1.10]\), evolving from \( \theta_0 \) for all \( \nu \geq 0 \).

Theorem 3.1 can be proved by the similar method as given in \([23]\) with modification for the presence of a forcing term \( S \) in \([1.10]\). Following the proof given in \([23]\), we first consider the linear problem:
\[
\partial_t \theta + (v \cdot \nabla) \theta = \kappa \Delta \theta + S, \tag{3.1}
\]
where the velocity vector \( v(t,x) = (v_1(t,x), \cdots, v_d(t,x)) \in L^2((0, \infty) \times \mathbb{R}^d) \) is given, and \((t,x) \in [0, \infty) \times \mathbb{T}^d \). Additionally, let \( v \) satisfy
\[
\partial_j v_j(t,x) = 0 \tag{3.2}
\]
in the sense of distributions. We express \( v_j \) as
\[
v_j(t,x) = \partial_j \partial_i V_{ij}(t,x) \text{ in } [0, \infty) \times \mathbb{R}^d, \tag{3.3}
\]
and we denoted \( V_{ij} = (-\Delta)^{-1} \partial_i v_j \). The matrix \( \{V_{ij}\}_{i,j=1}^d \) is given, and satisfies
\[
V_{ij} \in L^\infty((0, \infty); L^2(\mathbb{R}^d) \cap L^2((0, \infty); H^1(\mathbb{R}^d))) \tag{3.4}
\]
for all \( i,j \in \{1, \ldots, d\} \).

We first prove the following proposition for the existence of smooth solutions to \([3.1]-[3.3]\).

**Proposition 3.2.** Given \( \theta_0 \in L^2 \) and \( S \in C^\infty \) with \( ||S||_{L^\infty} < \infty \), and assume that \( \{V_{ij}\} \) satisfies \([3.4]\). Let
\[
\theta \in L^\infty([0, \infty); L^2(\mathbb{R}^d)) \cap L^2((0, \infty); H^1(\mathbb{R}^d)) \tag{3.5}
\]
be a global weak solution of the initial value problem associated to \([3.1]-[3.3]\). If additionally we have \( V_{ij} \in L^\infty([t_0, \infty); BMO(\mathbb{R}^d)) \) for all \( i,j \in \{1, \ldots, d\} \) and some \( t_0 > 0 \), then there exists \( \alpha > 0 \) such that \( \theta \in C^\alpha([t_0, \infty), \mathbb{T}^d) \).

**Remark 3.3.** Note that for divergence-free \( v \in L^2_{L^2} \), the existence of a weak solution \( \theta \) to \([5.1]-[5.3]\) evolving from \( \theta_0 \in L^2 \) is well-known (for example, see \([36]\) where the more general case \( v \in L^1_{loc} \) is discussed, also \([3]\) and references therein).

**Remark 3.4.** We further point out that, the condition on the forcing term \( S \) on the right side of \([3.1]\) can be relaxed to \( S \in L^r_x \) with \( r > 1 + \frac{d}{2} \) (see for example \([23]\) and \([31]\)).

**Proof of Proposition 3.2.** In view of Theorem 2.1 in \([23]\), we prove Proposition 3.2 in the following steps. Throughout the proof, we assume \( \kappa \equiv 1 \) for simplicity.

**Step 1:** A weak solution is bounded for positive time. In other words, there exists a positive constant \( C \) such that for all \( t > 0 \),
\[
\|\theta(\cdot, t)\|_{L^\infty} \leq C \left( \frac{\|\theta_0\|_{L^2}}{t^\frac{d}{4}} + \|S\|_{L^\infty}^\frac{d}{4} \|\theta_0\|_{L^2} \right). \tag{3.6}
\]
Proof. It follows by the similar method in proving Lemma 2.3 as in [8]. We notice that, condition (3.5) and the divergence-free condition on \( v \) imply that for each \( h > 0, \theta \) satisfies
\[
\partial_t (\theta - h)_+ + (v \cdot \nabla)(\theta - h)_+ \leq \Delta (\theta - h)_+ + S.
\] (3.7)

We multiply (3.7) by \( (\theta - h)_+ \) and use the divergence-free condition on \( v \) to obtain
\[
\int_{\mathbb{R}^d} |(\theta(\cdot, t_2) - h)_+|^2 + 2 \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |\nabla (\theta - h)|^2 \leq \int_{\mathbb{R}^d} |(\theta(\cdot, t_1) - h)_+|^2 + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |S(\theta - h)_+|
\] (3.8)
for all \( h > 0 \) and \( 0 < t_1 < t_2 < \infty \). Next, we apply De Giorgi iteration method based on (3.8). First we fix \( t_0 > 0 \) and define
\[
c_n = \sup_{t \geq t_0} \int_{\mathbb{R}^d} |\theta_n|^2 + 2 \int_{t_0}^{\infty} \int_{\mathbb{R}^d} |\nabla \theta_n|^2,
\]
where \( \theta_n = (\theta(\cdot, t) - h_n)_+ \), \( t_n = t_0 - \frac{t_n}{H} \), \( h_n = H - \frac{H}{n} \) and \( H \) to be chosen later. Then we have
\[
c_n \leq \frac{2^{n+1}}{t_0} \int_{t_{n-1}}^{\infty} \int_{\mathbb{R}^d} \theta_n^2 \cdot \chi_{\{\theta_n > 0\}} + \int_{t_{n-1}}^{\infty} \int_{\mathbb{R}^d} |S\theta_n|,
\] (3.9)
where \( \chi \) is the characteristic function and
\[
\chi_{\{\theta_n > 0\}} \leq \frac{2^n}{H^2} \theta_{n-1}.
\]
For the first term on the right side of (3.9), using Gagliardo-Nirenberg inequality (2.2), it can be estimated as follows.
\[
\frac{2^{n+1}}{t_0} \int_{t_{n-1}}^{\infty} \int_{\mathbb{R}^d} \theta_n^2 \cdot \chi_{\{\theta_n > 0\}} \leq \frac{2^{n+1}}{t_0} \left( \frac{2^n}{H} \right)^{-2} \int_{t_{n-1}}^{\infty} \int_{\mathbb{R}^d} \theta_n^p \leq C \frac{2^{n(p-1)+1}}{t_0 H^{p-2}} \int_{t_{n-1}}^{\infty} \left( \int_{\mathbb{R}^d} \theta_n^{-1} \right)^{\frac{p(1-\gamma)}{2}} \left( \int_{\mathbb{R}^d} \nabla \theta_n^{-1} \right)^{\frac{p\gamma}{2}}
\]
\[
\leq C \frac{2^{n(p-1)+1}}{H^{p-2}} \frac{\gamma}{2} C_{n-1}^\frac{\gamma}{2},
\]
where \( \gamma = \frac{2}{p}, p = 2(1 + \frac{2}{d}) \) and \( C > 0 \) is a dimensional constant independent of \( n \). Similarly, the second term on the right side of (3.9) is bounded by
\[
\int_{t_{n-1}}^{\infty} \int_{\mathbb{R}^d} |S\theta_n| \leq \|S\|_{L^\infty} \int_{t_{n-1}}^{\infty} \int_{\mathbb{R}^d} |\theta_n| \cdot \chi_{\{\theta_n > 0\}}^{-1}
\]
\[
\leq \|S\|_{L^\infty} \left( \frac{2^n}{H} \right)^{-p} \int_{t_{n-1}}^{\infty} \int_{\mathbb{R}^d} \theta_n^{-p} \leq C \|S\|_{L^\infty} \left( \frac{2^n}{H} \right)^{-p} \int_{t_{n-1}}^{\infty} \left( \int_{\mathbb{R}^d} \theta_n^{-1} \right)^{\frac{p(1-\gamma)}{2}} \left( \int_{\mathbb{R}^d} \nabla \theta_n^{-1} \right)^{\frac{p\gamma}{2}}
\]
\[
\leq C \|S\|_{L^\infty} \frac{2^{n(p-1)}}{H^{p-2}} C_{n-1}^{-\frac{\gamma}{2}}.
\]
Hence we conclude from (3.9) that
\[
c_n \leq C \frac{2^{n(p-1)+1}}{H^{p-2}} C_{n-1}^{-\frac{\gamma}{2}} + C \|S\|_{L^\infty} \frac{2^{n(p-1)}}{H^{p-2}} C_{n-1}^{-\frac{\gamma}{2}}.
\] (3.10)
By choosing a large enough constant \( H \), the nonlinear iteration inequality (3.10) implies that \( c_n \) converges to 0 as \( n \to \infty \). Hence \( \theta(x, t_0) \leq H \). Applying the same procedure to \( -\theta \) concludes the proof of (3.6).
Step 2: Next, we show that $\theta$ satisfies the first energy inequality, namely for any $0 < r < R$ and $h \in \mathbb{R}$, we have

$$
\|(\theta - h)\|_{L^\infty_t L^2_x(Q_r)} + \|\nabla(\theta - h)\|_{L^2_t L^\infty_x(Q_r)} \\
\leq \frac{CR}{(R - r)^2} \|(\theta - h)\|_{L^\infty_t L^2_x(Q_R)} \|\nabla(\theta - h)\|_{L^2_t L^\infty_x(Q_R)} + C R^y \|S\|_{L^\infty} \left\{ |\theta > h\} \cap Q_R \right\}^{1 - \frac{1}{2y}} \\
+ \frac{CR^{y+1}}{(R - r)} \|S\|_{L^\infty} \left\{ |\theta > h\} \cap L^2_x(Q_R) \right\}^{1 - \frac{1}{2y}}
$$

(3.11)

where $y = \frac{2(d + 2)}{d - 2} + 1$, $C = (d, \|V_{ij}\|_{L^\infty BMO_d}) > 0$ is a positive constant, and we have denoted $Q_r = [t_0 - \rho, t_0] \times B_r(x_0)$ for $\rho > 0$ and an arbitrary $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^d$. Notice that by (3.10), the right side of (3.11) is finite.

Proof. We follow the method for proving Lemma 2.6 in [23] and the only difference here comes from the extra forcing term $S$. Fix $h > 0$ and let $0 < r < R$ be such that $\frac{t_0}{2} - R^2 > 0$. Define $\eta(x, t) \in C_0^\infty((0, \infty) \times \mathbb{R}^d)$ to be a smooth cutoff function such that

- $0 \leq \eta \leq 1$ in $(0, \infty) \times \mathbb{R}^d$;
- $\eta \equiv 1$ in $Q_r(x_0, t_0)$ and $\eta \equiv 0$ in $\partial (Q^c_R(x_0, t_0) \cap \{t, x\} : t \leq t_0)$;
- $|\nabla \eta| \leq \frac{C}{R - r}$, $|\nabla \nabla \eta| \leq \frac{C}{(R - r)^2}$, $|\partial t \eta| \leq \frac{C}{(R - r)^2}$ in $Q_R(x_0, t_0)/Q_r(x_0, t_0)$,

for some positive constant $C$. Define $t_1 = t_0 - R^2$ and let $t_2 \in [t_0 - R^2, t_0]$ be arbitrary. Multiply (3.11) by $(\theta - h)_+ \eta^2$ and then integrate on $[t_1, t_2] \times \mathbb{R}^d$ to obtain

$$
\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \partial_t ((\theta - h)_+^2) \eta^2 dx dt - 2 \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \partial_j ((\theta - h)_+^2) \eta^2 dx dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \partial_i V_{ij} \partial_j ((\theta - h)_+^2) \eta^2 dx dt \\
= \int_{t_1}^{t_2} \int_{\mathbb{R}^d} (\theta - h)_+ \eta^2 S dx dt.
$$

(3.12)

From the estimates as shown in [23], it follows from (3.12) that

$$
\|(\theta - h)_+\|_{L^\infty_t L^2_x(Q_r)} + \|\nabla(\theta - h)_+\|_{L^2_t L^\infty_x(Q_r)} \\
\leq \frac{CR}{(R - r)^2} \|(\theta - h)_+\|_{L^\infty_t L^2_x(Q_R)} \|\nabla(\theta - h)_+\|_{L^2_t L^\infty_x(Q_R)} + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} (\theta - h)_+ \eta^2 S dx dt
$$

(3.13)

Using the Gagliardo-Nirenberg-Sobolev inequality [24] for $(\theta - h)_+ \in H^1_0(\mathbb{R}^d)$, the second term on the right side of (3.13) can be bounded by

$$
\left| \int_{t_1}^{t_2} \int_{\mathbb{R}^d} (\theta - h)_+ \eta^2 S dx dt \right| \leq \|S\|_{L^\infty} \left( \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |\eta(\theta - h)_+|^2 \right)^{\frac{d - 2}{2d}} \{|\theta > h\} \cap Q_R \right\}^{\frac{1}{2} + \frac{d - 2}{2d}} \\
\leq C \|S\|_{L^\infty} \left( \int_{Q_R} |\nabla(\eta(\theta - h)_+)|^2 \right)^{\frac{1}{2}} \{|\theta > h\} \cap Q_R \right\}^{\frac{1}{2} + \frac{d - 2}{2d}} \\
\leq C \|S\|_{L^\infty} \left( \int_{Q_R} |\nabla(\theta - h)_+|^2 \right)^{\frac{1}{2}} \{|\theta > h\} \cap Q_R \right\}^{\frac{1}{2} - \frac{d - 2}{2d} \cdot R^2 + \frac{1}{2}} \\
+ C \|S\|_{L^\infty} \left( \int_{Q_R} |(\theta - h)_+|^2 \right)^{\frac{1}{2}} \{|\theta > h\} \cap Q_R \right\}^{\frac{1}{2} - \frac{d - 2}{2d} \cdot R^2 + \frac{1}{2}} \\
\leq C \|S\|_{L^\infty} \left( \int_{Q_R} |\nabla(\theta - h)_+|^2 \right)^{\frac{1}{2}} \{|\theta > h\} \cap Q_R \right\}^{\frac{1}{2} - \frac{d - 2}{2d} \cdot R^2 + \frac{1}{2}} \\
+ C \|S\|_{L^\infty} \left( \int_{Q_R} |(\theta - h)_+|^2 \right)^{\frac{1}{2}} \{|\theta > h\} \cap Q_R \right\}^{\frac{1}{2} - \frac{d - 2}{2d} \cdot R^2 + \frac{1}{2}},
By letting \( H_b \) in Step 2, we have

\[
\text{Following the proof of Lemma 2.10 in [23], using (3.15) and the first energy inequality (3.11) as proved in Step 2, we have}
\]

\[
\sup_{Q_{r_0}} \theta \leq h_0 + C \left( \frac{|\{ \theta > h_0 \} \cap Q_{r_0}|}{r_0} \right)^{\frac{1}{2}} \left( \sup_{Q_{r_0}} \theta - h_0 \right)
\]

\[
(3.14)
\]

for some positive constant \( C = C(d, \Vert V_j \Vert_{L^\infty BMO_x}) \).

**Proof.** To facilitate the proof, we first introduce the following notations:

- \( A(h, r) = \{ \theta > h \} \cap Q_r \),
- \( a(h, r) = |A(h, r)| \),
- \( b(h, r) = \Vert (\theta - h) + \Vert_{L_{x,t}^2(Q_r)}^2 \),
- \( M(r) = \sup_{Q_r} \theta \),
- \( m(r) = \inf_{Q_r} \theta \).

Let \( 0 < r < R \) and \( 0 < h < H \). By the definitions of \( a \) and \( b \), we have

\[
a(H, r) \leq \frac{b(h, r)}{(H - h)^2}.
\]

Following the proof of Lemma 2.10 in [23], using (3.15) and the first energy inequality (3.11) as proved in Step 2, we have

\[
b(h, r) \leq Ca(h, r)^{\frac{1}{2}} \frac{R}{(R - r)^2} b(h, R)^{1 - \frac{1}{\alpha \nu}} \Vert (\theta - h) + \Vert_{L_{x,t}^2(Q_R)}^{\frac{2}{\alpha \nu}} + C(\theta - h) \Vert_{L_{x,t}^2(Q_R)}^{\frac{1}{2}} b(h, R)^{1 - \frac{1}{\alpha \nu}} (H - h)^{\frac{2}{\alpha \nu}}
\]

\[
+ C(a(h, r)^{\frac{1}{2}} \Vert S \Vert_{L^\infty} \frac{R^\nu}{(H - h)^2} b(h, R)^{1 - \frac{1}{\alpha \nu}} (H - h)^{\frac{2}{\alpha \nu}} + C(a(h, r)^{\frac{1}{2}} \Vert S \Vert_{L^\infty} \frac{R^{\nu + 1}}{(H - h)^{\frac{\nu + 1}{\nu + 2}} (H - h)^{\frac{2}{\alpha \nu}}}
\]

\[
(3.16)
\]

By combining (3.15) and (3.16), we obtain

\[
b(h, r) \leq \frac{CR}{(H - h)^{\frac{\nu + 1}{\nu + 2}} (R - r)^2} b(h, R)^{1 + \frac{1}{\alpha \nu}} \Vert (\theta - h) + \Vert_{L_{x,t}^2(Q_R)}^{\frac{2}{\alpha \nu}} + C(\theta - h) \Vert_{L_{x,t}^2(Q_R)}^{\frac{1}{2}} b(h, R)^{1 - \frac{1}{\alpha \nu}} (H - h)^{\frac{2}{\alpha \nu}}
\]

\[
+ C(a(h, r)^{\frac{1}{2}} \Vert S \Vert_{L^\infty} \frac{R^\nu}{(H - h)^{\frac{\nu + 1}{\nu + 2}}} b(h, R)^{1 - \frac{1}{\alpha \nu}} (H - h)^{\frac{2}{\alpha \nu}} + C(a(h, r)^{\frac{1}{2}} \Vert S \Vert_{L^\infty} \frac{R^{\nu + 1}}{(H - h)^{\frac{\nu + 1}{\nu + 2}} (H - h)^{\frac{2}{\alpha \nu}}}
\]

\[
(3.17)
\]

We now apply the De Giorgi iteration method based on (3.17). Let \( r_n = \frac{r_0}{2^n} + \frac{r_0}{2^{n+1}} \), \( \gamma_n = \gamma_{\infty} - \frac{(\gamma_{\infty} - \gamma_0)}{2^n} \), and \( \gamma_n = b(h_n, r_{n+1}) \), for all \( n \geq 0 \), where \( r_0 \) and \( \gamma_0 \) are as given and \( \gamma_{\infty} > 0 \) is to be chosen later. By letting \( H = h_{n+1} \), \( h = h_n \), \( r = r_{n+2} \), and \( R = r_{n+1} \) in (3.17), we have

\[
b_{n+1} \leq \frac{C(M(r_0) - h_0)^{\frac{1}{2}}}{{(\gamma_{\infty} - h_0)^{\frac{1}{2} + 2}}} + \frac{C\gamma_0^\nu \Vert S \Vert_{L^\infty}^2}{{(\gamma_{\infty} - h_0)^{\frac{1}{2} + 2}}} + \frac{C\gamma_0^{\frac{\nu - 1}{\nu} \Vert S \Vert_{L^\infty}}}{{(\gamma_{\infty} - h_0)^{\frac{2}{\nu + 2} + 1}}} 2^{\nu + 1} b_{n+1}^{\frac{1}{\alpha - 1}}
\]

\[
(3.18)
\]

We let \( B = 2^{4 + 2(d+2)} \) and choose \( \gamma_{\infty} \) large enough so that

\[
\left( \frac{C(M(r_0) - h_0)^{\frac{1}{2}}}{{(\gamma_{\infty} - h_0)^{\frac{1}{2} + 2}}} + \frac{C\gamma_0^\nu \Vert S \Vert_{L^\infty}^2}{{(\gamma_{\infty} - h_0)^{\frac{1}{2} + 2}}} + \frac{C\gamma_0^{\frac{\nu - 1}{\nu} \Vert S \Vert_{L^\infty}}}{{(\gamma_{\infty} - h_0)^{\frac{2}{\nu + 2} + 1}} \right) \leq 1 \]

\[
(3.19)
\]
then by induction, we obtain from (3.18) that \( b_n \leq \frac{b_0}{B^n} \) for all \( n \in \mathbb{N} \). The rest follows from the argument given in [23] and we omit the details here. \( \square \)

**Step 4:** We have the following *second energy inequality* in controlling the possible growth of level sets of the solution: fix an arbitrary \( x_0 \in \mathbb{R}^d \), let \( h \in \mathbb{R} \), \( 0 < r < R \), and \( 0 < t_1 < t_2 \). Then we have

\[
\|(\cdot,t_2) - h\|_{L^2(B_r)}^2 \leq \|(\cdot,t_1) - h\|_{L^2(B_r)}^2 + \frac{C_0 R^d (t_2 - t_1)}{(R - r)^2} \|(\cdot,t_2) - h\|_{L^\infty((t_1,t_2) \times B_R)}^2,
\]

where \( C_0 = C_0(d, \|V_{ij}\|_{L^\infty BMO_x}, \|S\|_{L^\infty}) \) is a sufficiently large positive constant. Notice that by (3.6), the right side of (3.19) is finite.

**Proof.** Similar to the first energy inequality, we follow the method for proving Lemma 2.11 in [23] and the only difference here comes from the forcing term \( S \). Given \( h, r, R, t_1, t_2 \), we define \( \eta \in C_0^\infty(\mathbb{R}^d) \) to be a smooth cutoff function such that

\[ \cdot 0 \leq \eta \leq 1 \text{ in } \mathbb{R}^d; \]
\[ \cdot \eta \equiv 1 \text{ on } B_r \text{ and } \eta \equiv 0 \text{ on } B_r^c; \]
\[ \cdot |\nabla \eta(x)| \leq \min \left\{ \frac{C}{R - r}, \frac{CR}{(R - r)^2} \right\} \|(\cdot,t_2) - h\|_{L^\infty((t_1,t_2) \times B_R)} \]

for some positive constant \( C \). Multiply (3.1) by \( (\cdot,t_1) - h \) and integrate on \([t_1, t_2] \times \mathbb{R}^d\), it follows from the estimates given in [23] that

\[ \|(\cdot,t_2) - h\|_{L^2(B_r)}^2 \leq \|(\cdot,t_1) - h\|_{L^2(B_r)}^2 + \frac{C R^d (t_2 - t_1)}{(R - r)^2} \|(\cdot,t_2) - h\|_{L^\infty((t_1,t_2) \times B_R)}^2 \]

\[ + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} (\cdot,t_2) - h, \eta^2 S \, dx \, dt, \]

where \( C = C(d, \|V_{ij}\|_{L^\infty BMO_x}) \) is a positive constant. Using the Gagliardo-Nirenberg-Sobolev inequality [24] for \( \eta \in C_0^\infty(\mathbb{R}^d) \), we bound the far right side of the above as follows.

\[ \left| \int_{t_1}^{t_2} \int_{\mathbb{R}^d} (\cdot,t_2) - h, \eta^2 S \, dx \, dt \right| \leq \|(\cdot,t_2) - h\|_{L^\infty((t_1,t_2) \times B_R)} \|S\|_{L^\infty} \left( \int_{B_R} |\eta|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{2d}} |B_R|^{\frac{1}{2} - \frac{d}{2d}} \]

\[ \leq CR^{d-1} \|(\cdot,t_2) - h\|_{L^\infty((t_1,t_2) \times B_R)} \|S\|_{L^\infty} \left( \int_{B_R} |\nabla \eta|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{2d}} \]

\[ \leq CR^{d-1} \|(\cdot,t_2) - h\|_{L^\infty((t_1,t_2) \times B_R)} \|S\|_{L^\infty} \left( \int_{B_R} |\nabla \eta|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{2d}} \]

\[ \leq \frac{C R^d}{(R - r)^2} \left( \int_{B_R} |\nabla \eta|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{2d}} \|S\|_{L^\infty} \left( \int_{B_R} |\nabla \eta|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{2d}} \]

By using (3.21) on (3.20), we can choose \( C_0 = C_0(d, \|V_{ij}\|_{L^\infty BMO_x}, \|S\|_{L^\infty}) > 0 \) sufficiently large enough so that (3.19) holds. \( \square \)

**Step 5:** Using the second energy inequality (3.19), we can bound \(|\{\cdot,t_2 \geq H\} \cap B_R|/|B_R|\) whenever \(|\{\cdot,t_1 \geq H\} \cap B_r|/|B_r| \leq \frac{1}{2}\). Fix \( \kappa_0 = (\frac{1}{2})^{\frac{1}{2}} \), let \( n_0 \geq 2 \) be the least integer such that \( \frac{2^{n_0}}{2^{n_0} - 2} \leq \frac{1}{2} \), and let \( \delta_0 = \frac{(1 - \kappa_0)^2}{12C_0 \kappa_0} \) where \( C_0 \) is the constant from (3.19). For \( t_1, R > 0 \), if

\[ |\{\cdot,t_1 \geq H\} \cap B_R| \leq \frac{1}{2} |B_R|, \]

then for all \( t_2 \in [t_1, t_1 + \delta_0 R^2] \) we have

\[ |\{\cdot,t_2 \geq H\} \cap B_R| \leq \frac{7}{8} |B_R|, \]
where we define $r = r_0 R$, $M = \sup_{(t_1, t_1 + \delta_0 R^2) \times B_R} \theta$, $m = \inf_{(t_1, t_1 + \delta_0 R^2) \times B_R} \theta$, $h = \frac{(M + m)}{2}$ and $H = M - \frac{(M - m)}{2n_0}$.

**Proof.** By using (3.19), the proof of (3.22) follows by the same argument given by the proof of Lemma 2.12 in [23] and we omit the details here. □

**Step 6:** Applying Step 1 to Step 5, the proof of Proposition 3.2 now follows by showing that there exists $\beta \in (0, 1)$ independent of $R$, such that

$$osc(Q_1) \leq \beta osc(Q_2),$$

where $Q_1 = [t_1, t_1 + \delta_0 r^2] \times B_r$, $Q_2 = [t_1, t_1 + \delta_0 R^2] \times B_R$ and $osc(Q) = \sup_Q \theta - \inf_Q \theta$. Here $\delta_0, n_0, M, m, h, R$ are defined as in Step 5, and we recall that $t_1 > 0$ and $R > 0$ are arbitrary. We refer the reader to (23, pp. 293–294) for details in proving (3.23). The estimate (3.23) implies the Hölder regularity of the solution (the Hölder exponent $\alpha \in (0, 1)$ may be calculated explicitly from $\beta$) which finishes the proof of Proposition 3.2. □

**Proof of Theorem 3.1.** We now focus back on the non-linear problem (1.10) and give the proof of Theorem 3.1. To begin with, we notice that given $\theta_0 \in L^2$ and $\nu \geq 0$, there exists a global in time Leray-Hopf weak solution $\theta^\nu$ of (1.10) evolving from $\theta_0$ (a proof for it can be found in (23)). Using the same method as in proving (3.6), we have $\theta^\nu \in L_{t,x}^\infty$ and it follows from the Caldern-Zygmund theory of singular integrals that $T^\nu_i \theta = : V^\nu_{ij} \in L^\infty([t_0, \infty); BMO(\mathbb{R}^d))$, for any $t_0 > 0$ and $\nu \geq 0$, where $i, j \in \{1, ..., d\}$. Therefore, we may treat (1.10) as a linear evolution equation (see also [3, 15, 23]), where the divergence-free velocity field $u$ is given, and $u \in L^2((0, \infty); L^2(\mathbb{R}^d)) \cap L^\infty((t_0, \infty); BMO^{-1}(\mathbb{R}^d))$, for any $t_0 > 0$. This is precisely the setting of Proposition 3.2 for the linear evolution equation and it can be applied to the nonlinear problem (1.10) to give Hölder regularity of the solution. Finally, since Hölder regularity is sub-critical for the natural scaling of (1.10), one may bootstrap to prove that the solution is in a higher regularity class. We refer to [23] for further details and conclude the proof of Theorem 3.1. □

### 4. Uniform bounds on smooth solutions

We have the following uniform $H^s$-bound on smooth solutions to (1.10) which will be used in proving Theorem 1.2.

**Theorem 4.1.** Assume that the hypotheses and notations of Theorem 3.1 are in force. Then given $0 < t_1 < t_2$ and $s \geq 0$, there exists a positive constant $C(C_*, t_1, t_2, s, d, \kappa, S, \|\theta_0\|_{L^2}) > 0$ independent of $\nu$ such that

$$\sup_{t \in [t_1, t_2]} \|\theta^\nu(\cdot, t)\|_{H^s} + \int_{t_1}^{t_2} \|\theta^\nu(\cdot, t)\|_{H^{s+1}(\mathbb{R}^d)}^2 dt \leq C(C_*, t_1, t_2, s, d, \kappa, S, \|\theta_0\|_{L^2}),$$

where $C_*$ is the constant as stated in condition (1.13).

**Proof of Theorem 4.1.** Fix $I = [t_1, t_2]$ for some $0 < t_1 < t_2$. By Theorem 3.1, there exists $\alpha \in (0, 1)$ such that for each $\nu \geq 0$,

$$\theta^\nu \in L^\infty(I; L^2(\mathbb{R}^d) \cap L^2(I; \dot{H}^1(\mathbb{R}^d)) \cap L^\infty(I; C^\alpha(\mathbb{R}^d))).$$

Depending on the value of $\alpha$, we consider the following 2 cases:

**Case 1:** $\alpha \in (0, \frac{1}{2})$. The proof is based on the one given in [25] and we just need to take extra care of the forcing term $S$. It is given in the following steps:

**Step (i):** Assume further that

$$\theta^\nu \in L^2(I; \dot{B}_{p,2}^{1}(\mathbb{R}^d))$$

(4.3)
for some $p \geq 2$, then we have

$$\theta^\nu \in \tilde{L}^2(I; \tilde{B}^{1}_{q, r}(\mathbb{R}^d))$$

(4.4)

for all $1 \leq r \leq \infty$, and for all $q \in (p, m_{\alpha}p)$, where $m_{\alpha} = \frac{1 - \alpha}{2\alpha} > 1$.

**Proof.** We apply $\Delta_j$ to (1.10), multiply by $\Delta_j \theta |\Delta_j \theta|^{q-2}$, integrate over $\mathbb{R}^d$, and use Proposition 29.1 in [30] (also refer to [5]) to obtain, for $j \in \mathbb{Z}$,

$$\frac{1}{q} \frac{d}{dt} \|\Delta_j \theta^\nu\|_{L^q}^q + C 2^j \|\Delta_j \theta^\nu\|_{L^q}^q \leq \left| \int \Delta_j (u^\nu \cdot \nabla) \Delta_j \theta^\nu \Delta_j \theta^\nu |\Delta_j \theta^\nu|^{q-2} + \int \Delta_j (S) \Delta_j \theta^\nu |\Delta_j \theta^\nu|^{q-2} \right|,$$

(4.5)

where $C = C(d, q) > 0$ is a positive constant independent of $\nu$. Using Hölder inequality, the second term on the right side of (4.6) is bounded by $\|\Delta_j \theta^\nu\|_{L^q}^q \|\Delta_j (S)\|_{L^q}$. Hence applying the estimates given in the proof of Lemma 2 in [23] pp. 259–261, we obtain

$$\frac{d}{dt} \|\Delta_j \theta^\nu\|_{L^q} + C 2^j \|\Delta_j \theta^\nu\|_{L^q} \leq C \|\theta^\nu\|_{C_{\alpha}} \sum_{k \leq j} 2^{k(1-\frac{q}{2} - \alpha(1-\frac{q}{2}))} \|\Delta_k \theta^\nu\|_{L^p} \frac{2^j}{2^{j-1}}$$

$$+ C \|\theta^\nu\|_{C_{\alpha}} \sum_{|k-j| \leq 2} 2^{k(1-\frac{q}{2} - \alpha(1-\frac{q}{2}))} \|\Delta_k \theta^\nu\|_{L^p} \frac{2^j}{2^{j-1}}$$

$$+ C \|\theta^\nu\|_{C_{\alpha}} \sum_{k \geq j-1} 2^{k(1-\frac{q}{2} - \alpha(1-\frac{q}{2}))} \|\Delta_k \theta^\nu\|_{L^p} \frac{2^j}{2^{j-1}}$$

(4.6)

where $C = C(C_*) > 0$ is a positive constant independent of $\nu$ and $C_*$ is defined in condition (1.13). Applying Grönwall’s inequality on (1.6),

$$\|\Delta_j \theta^\nu(t)\|_{L^q} \leq e^{-c 2^j(t-t_1)} \|\Delta_j \theta^\nu(t_1)\|_{L^q}$$

$$+ C \|\theta^\nu\|_{L^\infty(I; C_{\alpha})} 2^{j(1-\alpha)} \sum_{k \leq j} 2^{k(1-\frac{q}{2} - \alpha(1-\frac{q}{2}))} \Theta_{j,k}(t)$$

$$+ C \|\theta^\nu\|_{L^\infty(I; C_{\alpha})} 2^{j(2-\frac{q}{2} - \alpha(1-\frac{q}{2}))} \sum_{|k-j| \leq 2} \Theta_{j,k}(t)$$

$$+ C \|\theta^\nu\|_{L^\infty(I; C_{\alpha})} 2^j \sum_{k \geq j-1} 2^{k(1-\frac{q}{2} - \alpha(1-\frac{q}{2}))} \Theta_{j,k}(t)$$

$$+ \|\Delta_j S\|_{L^q} \int_{t_1}^t e^{-c(t-\tau)2^j} d\tau,$$

(4.7)

where

$$\Theta_{j,k}(t) = \int_{t_0}^t e^{-c(t-\tau)2^j} (2^k \|\Delta_k \theta^\nu(s)\|_{L^p}) d\tau.$$

We take the $L^2(I)$ norm of (4.7) and apply the estimates given in [23] pp. 260–261 to obtain

$$\|\Delta_j \theta^\nu(t)\|_{L^2(I; L^q)} \leq C \|\theta^\nu(t_1)\|_{L^q} \|\theta^\nu(t_1)\|_{C_{\alpha}} 2^{-j(1-\frac{q}{2})} \min\{2^{-j}, |I|^{\frac{1}{2}}\}$$

$$+ C \|\theta^\nu\|_{L^\infty(I; C_{\alpha})}\|\theta^\nu\|_{L^q(I; H^1_{x} L^p)} |I| \frac{2^j}{2^{j-1}} 2^{j(2-\frac{q}{2} - \alpha(1-\frac{q}{2}))} \min\{C2^{-2j}, |I|\}$$

$$+ C \|\Delta_j S\|_{L^q} |I| \frac{1}{2} \min\{C2^{-\gamma}, |I|\}.$$

(4.8)
Multiply the above on both sides by $2^j$ and take an $\ell^r(\mathbb{Z})$-norm,

$$
\|\theta^\nu\|_{L^2(I;\dot{B}^{1}_{q,r})} \leq C \|\theta^\nu(t_1)\|_{L^q}^{1-\frac{2\nu}{q}} \|\theta^\nu(t_1)\|_{C^{\alpha}}^{\frac{2\nu}{q}} \min\{2^{-j}\alpha(1-\frac{2\nu}{q})\} \|\theta^\nu(t_1)\|_{\ell^r(\mathbb{Z})}
$$

$$
+ C \|\theta^\nu\|_{L^\infty(I;C^{\alpha})} \|\theta^\nu\|_{L^2(I;\dot{B}^{1}_{p,r})} |I|^\frac{2\nu}{q} \min\{2^{-j}\alpha(1-\frac{2\nu}{q})\} \|\theta^\nu(t_1)\|_{\ell^r(\mathbb{Z})}
$$

$$
+ C \|\theta^\nu\|_{L^2(I;\dot{B}^{1}_{q,r})} |I|^\frac{2\nu}{q} \min\{2^{-j}\alpha(1-\frac{2\nu}{q})\} \|\theta^\nu(t_1)\|_{\ell^r(\mathbb{Z})}.
$$

(4.9)

Since $q \in (p, m_\alpha p)$, the two $\ell^r$ norms on the right side of the above estimate are finite for any $1 \leq r \leq \infty$. On the other hand, the last term on the right side of (4.9) can be bounded by $\|S\|_{L^\infty}|I|^\frac{2\nu}{q}$. Hence we have $\theta^\nu \in \dot{L}^2(I;\dot{B}^{1}_{q,r})$ which finishes the proof of (4.4).

**Step (ii):** Assume that $\theta^\nu$ satisfies (4.12), we then have

$$
\nabla \theta^\nu \in L^2(I;L^\infty(\mathbb{R}^d)).
$$

(4.10)

**Proof.** We follow the proof of Lemma 3 in [25] pp. 262–263. First, we note that $\dot{H}^1 = \dot{B}^{1}_{2,2}$, so we may apply Step (i) with $p = 2$ and obtain that $\theta \in L^2(I;\dot{B}^{1}_{2,2})$ for any $q \in (2, m_\alpha)$. Since $m_\alpha > 1$, we may bootstrap and apply Step (i) once more to obtain that $\theta \in L^2(I;\dot{B}^{1}_{\frac{d}{2},2})$ for all $q \in (2, m_\alpha^2)$. For any fixed $p > 2$, we have $m_\alpha > 1$ and hence $m_\alpha^k \to \infty$ as $k \to \infty$. By iterating Step (i) finitely many times, we obtain

$$
\theta^\nu \in \dot{L}^2(I;\dot{B}^{1}_{q,r}) \text{ for all } r \in [1,\infty].
$$

(4.11)

Fix $p$ large enough (which will be explicitly chosen later), and let $q = \frac{p(1+m_\alpha)}{2}$. From the estimate (4.8), for any $\varepsilon > 0$,

$$
2^{j(1+\varepsilon)} \|\Delta_j \theta^\nu(t)\|_{L^2(I;L^q)} \leq C \|\theta^\nu(t_1)\|_{L^q}^{1-\frac{2\nu}{q}} \|\theta^\nu(t_1)\|_{C^{\alpha}}^{\frac{2\nu}{q}} \min\{2^{j(1+\varepsilon)}|I|^{\frac{2\nu}{q}}\}
$$

$$
+ C \|\theta^\nu\|_{L^\infty(I;C^{\alpha})} \|\theta^\nu\|_{L^2(I;\dot{B}^{1}_{\frac{d}{2},2})} |I|^\frac{2\nu}{q} \min\{2^{j(1+\varepsilon)}|I|^{\frac{2\nu}{q}}\}
$$

$$
+ C \|\Delta_j S\|_{L^q} \|\theta^\nu\|_{\ell^r(\mathbb{Z})} \min\{2^{j(1+\varepsilon)}|I|\}.
$$

(4.12)

Choose

$$
\varepsilon = \frac{1}{2} \min\left\{\frac{\alpha^2}{2-3\alpha}, \frac{(1-2\alpha)(2-3\alpha)}{(1-\alpha)(2-3\alpha)}\right\},
$$

then $\varepsilon > 0$ for all $\alpha \in (0,\frac{1}{2})$. By taking the $\ell^r$ norm of (4.12) and using Besov embedding theorem (2.4), we have

$$
\theta^\nu \in \dot{L}^2(I;\dot{B}^{1+\varepsilon}_{q,1}) \subset L^2(I;\dot{B}^{1+\varepsilon}_{q,1}) \subset L^2(I;\dot{B}^{1+\varepsilon}_{\infty,1}) \subset L^2(I;\dot{B}^{1+\varepsilon-\frac{2d}{p+pm_\alpha}}_{\infty,1}).
$$

(4.13)

We pick $p > 2$ so that $\varepsilon - \frac{2d}{p+pm_\alpha} = 0$, then we obtain from (4.13) that

$$
\nabla \theta^\nu \in L^2(I;\dot{B}^{0}_{\infty,1}).
$$

Lastly, from condition (4.2), we have $\nabla \theta^\nu \in L^2(I;L^2 \cap \dot{B}^{0}_{\infty,1})$, hence (4.10) follows from the borderline Sobolev embedding theorem that $L^2 \cap \dot{B}^{0}_{\infty,1} \subset B^{0}_{\infty,1} \subset L^\infty$. □

Using the results obtained from Step (i) and Step (ii) as described above, we are ready to prove the bound (4.2) for $\alpha \in (0,\frac{1}{2}]$. First, by using (4.10), we have

$$
\int_{t_1}^{t_2} \|\nabla \theta^\nu(t)\|_{L^\infty} dt \leq C(C_* t_1, t_2, d, \kappa, S, \|\theta_0\|_{L^2}),
$$

(4.14)
where \( C_\ast, t_1, t_2, d, \kappa, S, \| \theta_0 \|_{L^2} > 0 \) is a positive constant independent of \( \nu \). Next, by the condition (1.2), \( u^\nu \) is divergence free and we have the a priori estimate derived from (1.10) that

\[
\frac{1}{2} \frac{d}{dt} \| \nabla \theta^\nu (\cdot, t) \|_{L^2} + \kappa \| \Delta \theta^\nu (\cdot, t) \|_{L^2} 
\leq \int \partial_k u^\nu_j \partial_k \theta^\nu \partial_j \theta^\nu + \int S \Delta \theta^\nu 
\leq C_\ast \| \Delta \theta^\nu (\cdot, t) \|_{L^\infty} \| \nabla \theta^\nu (\cdot, t) \|_{L^2} + \| \nabla \theta^\nu (\cdot, t) \|_{L^2} \| \nabla S (\cdot, t) \|_{L^2} 
\leq \kappa \frac{1}{2} \| \Delta \theta^\nu (\cdot, t) \|_{L^2}^2 + C(C_\ast, \kappa, S) \| \nabla \theta^\nu (\cdot, t) \|_{L^2}^2 \| \nabla \theta^\nu (\cdot, t) \|_{L^2}^2.
\]  

(4.15)

By absorbing the term \( \frac{\kappa}{2} \| \Delta \theta^\nu (\cdot, t) \|_{L^2}^2 \) on the left side of (4.15) and using the bound (4.14), we obtain, for all \( t \geq t_1 \),

\[
\| \theta^\nu (\cdot, t) \|_{H^1}^2 \leq \| \theta^\nu (\cdot, t_2) \|_{H^1}^2 e^{\int t_2^{t_1} C(C_\ast, \kappa, S) \| \nabla \theta^\nu (\cdot, t) \|_{L^2}^2 dt} \leq \| \theta^\nu (\cdot, t_2) \|_{H^1}^2 e^{C(C_\ast, t_1, t_2, d, \kappa, S, \| \theta_0 \|_{L^2})}.
\]  

(4.16)

Since \( \theta^\nu \in L^2([t_1, t_2]; H^1) \) and \( L^2 \) functions are finite a.e., \( \| \theta^\nu (\cdot, t_2) \|_{H^1} \) is finite for a.e. \( t_2 > 0 \), with bounds in terms of \( \| \theta_0 \|_{L^2} \) but independent of \( \nu \). Hence (4.16) implies \( \theta^\nu \in L^\infty([t_1, t_2]; H^1(\mathbb{R}^d)) \cap L^2([t_1, t_2]; H^2(\mathbb{R}^d)) \) with

\[
\sup_{t \in [t_1, t_2]} \| \theta^\nu (\cdot, t) \|_{H^1} + \int_{t_1}^{t_2} \| \theta^\nu (\cdot, t) \|_{H^2(\mathbb{R}^d)}^2 dt \leq C(C_\ast, t_1, t_2, d, \kappa, S, \| \theta_0 \|_{L^2}).
\]

By further taking derivatives of the equation (4.10) and repeating the above argument, (4.1) also holds for all \( s > 1 \) and we finish the proof of (4.1) for \( \alpha \in (0, \frac{1}{2}) \).

**Case 2:** \( \alpha \in (\frac{1}{2}, 1) \). We adopt the method given in [25] pp. 257–258. Similar to Case 1, the goal is to prove that (4.10) holds for \( \theta^\nu \). Once (4.10) is proved, same argument given in the previous case can then be applied which gives the bound (4.1). First, note that if \( \theta^\nu \) satisfies (1.2), then \( \theta^\nu \in L^\infty([0, \infty); B^p_{\alpha, \infty}) \), where \( \alpha_p = (1 - \frac{2}{p}) \alpha \) and \( p \in [2, \infty) \) is fixed and to be chosen later. Then, for \( j \in \mathbb{Z} \) fixed, we have

\[
\frac{1}{p} \frac{d}{dt} \| \Delta_j \theta^\nu \|_{L^p}^p + C 2^j \| \Delta_j \theta^\nu \|_{L^p}^p \leq \int | \Delta_j \theta^\nu |^{p-2} \Delta_j \theta^\nu | \Delta_j \theta^\nu | | u \cdot \nabla \theta^\nu | + \int | \Delta_j(S) \Delta_j \theta^\nu | | \Delta_j \theta^\nu |^{p-2}.
\]

(4.17)

Using the Bony paraproduct decomposition and the method given in [25], the first term on the right side of (4.17) can be bounded by \( C 2^{(2-2\alpha)p} \| \theta^\nu \|_{C^{\alpha_p}} \| \theta^\nu \|_{B^p_{\alpha, \infty}} \), where \( C > 0 \) is a constant which may depend on \( C_\ast \) as in condition (1.13) but independent of \( \nu \). On the other hand, the term \( \int | \Delta_j(S) \Delta_j \theta^\nu | | \Delta_j \theta^\nu |^{p-2} \) can be bounded by \( \| \Delta_j \theta^\nu \|_{L^p}^{p-1} \| \Delta_j S \|_{L^p} \). By applying the bounds on (4.17), using Gröwall inequality and the Besov embedding theorem (2.4), we obtain

\[
\theta^\nu \in L^\infty([t, \infty); \dot{B}^{2\alpha}_{p, \infty}(\mathbb{R}^d) \subset L^\infty([t, \infty); \dot{B}^{2\alpha-\frac{4\alpha}{p}+rac{4d}{p}}_{\infty, \infty}(\mathbb{R}^d))
\]

for all \( t \geq t_1 \). Choose \( p > \frac{4d}{2\alpha-1} \), then \( 2\alpha - \frac{4\alpha}{p} > 1 \). Since \( L^\infty \cap \dot{B}^{2\alpha-\frac{4\alpha}{p}+rac{4d}{p}}_{\infty, \infty} = C^{2\alpha-\frac{4\alpha}{p}+rac{4d}{p}} \), we conclude that (4.10) holds for \( \theta^\nu \) and we finish the proof of (4.1) for \( \alpha \in (\frac{1}{2}, 1) \).

5. **Proof of Theorem 1.2**

The proof can be divided into two cases:

**Case 1:** \( s = 0 \). Let \( \phi = \theta^\nu - \theta \), then \( \phi \) satisfies the following equation:

\[
\partial_t \phi + u^\nu \cdot \nabla \phi + (u^\nu - u) \cdot \nabla \theta = \kappa \Delta \phi.
\]

(5.1)
Multiply (5.1) by $\phi$ and integrate,

$$\frac{1}{2} \frac{d}{dt} \| \phi(\cdot, t) \|_{L^2}^2 + \frac{\kappa}{2} \| \nabla \phi(\cdot, t) \|_{L^2}^2 = - \int (u^\nu - u) \cdot \nabla \theta \cdot \phi(x, t) dx. \tag{5.2}$$

We estimate the right side of (5.2) as follows. For each $t > 0$,

$$\left| - \int (u^\nu - u) \cdot \nabla \theta \cdot \phi(x, t) dx \right| \leq \| (u^\nu - u)(\cdot, t) \|_{L^2} \| \phi(\cdot, t) \|_{L^2} \| \nabla \theta(\cdot, t) \|_{L^\infty} \leq \frac{\kappa}{4C_s^2} \| (u^\nu - u)(\cdot, t) \|_{L^2}^2 + \frac{4C_s^2}{\kappa} \| \phi(\cdot, t) \|_{L^2}^2 \| \nabla \theta(\cdot, t) \|_{L^2}^2, \tag{5.3}$$

where $C_s > 0$ is the constant as stated in condition (1.13). We focus on the term $\| (u^\nu - u)(\cdot, t) \|_{L^2}^2$ as in (5.3), and for simplicity we sometime drop the variable $t$. Using Plancherel Theorem, for each $j$,

$$\| (u_j - u_j)(\cdot, t) \|_{L^2}^2 = \sum_{k \in \mathbb{Z}^3} |(\hat{u}_j - u_j)(k)|^2 \leq \sum_{k \in \mathbb{Z}^3} |(\hat{T}_{ij}^\nu \theta^\nu - \hat{T}_{ij}^0 \theta)(k)|^2 \leq \sum_{k \in \mathbb{Z}^3} |\hat{T}_{ij}^\nu (k) - \hat{T}_{ij}^0 (k)|^2 |\nabla \phi(k)|^2 \leq \sum_{k \in \mathbb{Z}^3} C_s^2 \| \nabla \phi(k) \|_{L^2}^2 \| \nabla \theta(k) \|_{L^2}^2 \left( \hat{\theta}^\nu (k) - \hat{\theta}^0 (k) \right)^2 \leq I_1 + I_2. \tag{5.4}$$

Using the condition (1.13), the term $I_1$ can be estimated by

$$I_1 \leq C_s^2 \sum_{k \in \mathbb{Z}^3} |\nabla \phi(k)|^2 = C_s^2 \| \nabla \phi(\cdot, t) \|_{L^2}^2. \tag{5.5}$$

For the term $I_2$, by Theorem 3.1, $\theta$ is smooth and $\| \nabla \theta(\cdot, t) \|_{L^2} < \infty$. So the condition (1.4) can be applied and we have

$$\lim_{\nu \to 0} I_2 = \lim_{\nu \to 0} \sum_{k \in \mathbb{Z}^3, k \neq 0} |\hat{T}_{ij}^\nu (k) - \hat{T}_{ij}^0 (k)|^2 |\nabla \theta(k)|^2 = 0. \tag{5.6}$$

We apply (5.3) on (5.4) to obtain

$$\| (u^\nu - u)(\cdot, t) \|_{L^2}^2 \leq C_s^2 \| \nabla \phi(\cdot, t) \|_{L^2}^2 + I_2, \tag{5.7}$$

and hence using (5.7) on (5.3),

$$\left| - \int (u^\nu - u) \cdot \nabla \theta \cdot \phi(x, t) dx \right| \leq \frac{\kappa}{4} \| \nabla \phi(\cdot, t) \|_{L^2}^2 + \frac{\kappa}{4C_s^2} I_2 + \frac{4C_s^2}{\kappa} \| \phi(\cdot, t) \|_{L^2}^2 \| \nabla \theta(\cdot, t) \|_{L^\infty}^2. \tag{5.8}$$

Applying (5.8) on (5.2), using Grönwall’s inequality, taking $\nu \to 0$ and using (5.3), for all $t > 0$, we conclude that

$$\lim_{\nu \to 0} \| (\theta^\nu - \theta)(\cdot, t) \|_{H^s}^2 = \lim_{\nu \to 0} \| \phi(\cdot, t) \|_{L^2}^2 = 0. \tag{5.9}$$

**Case 2:** $s > 0$. We apply the Gagliardo-Nirenberg interpolation inequality for homogeneous Sobolev space (2.3) to obtain, for $s > 0$ and $t > 0$,

$$\| (\theta^\nu - \theta)(\cdot, t) \|_{H^s} \leq C \| (\theta^\nu - \theta)(\cdot, t) \|_{L^2}^\gamma \| (\theta^\nu - \theta)(\cdot, t) \|_{H^{s+1}}^{1-\gamma}, \tag{5.10}$$

where $\gamma \in (0, 1)$ depends on $s$, and $C > 0$ is a positive constant which depends on $d$ but is independent of $\nu$. Using the bounds (4.1) as in Theorem 4.1, the term $\| (\theta^\nu - \theta)(\cdot, t) \|_{H^{s+1}}^{1-\gamma}$ is bounded uniformly in $\nu$ for all $t \geq \tau$. Hence by taking $\nu \to 0$ and applying the $L^2$ convergence (5.9), we have

$$\lim_{\nu \to 0} \| (\theta^\nu - \theta)(\cdot, t) \|_{H^s} = 0 \text{ for } t \geq \tau,$$
which finishes the proof of Theorem 1.2.

### 6. Dissipation of energy

In this section, we discuss about the dissipation energy to (1.10). For \( d = 2, 3 \), we prove that there is no dissipation of energy in the limit as \( \nu \to 0 \). More precisely, we have the following result:

**Theorem 6.1.** Let \( \theta^\nu \) be the \( C^\infty \) smooth classical solutions of the system (1.10) for \( \nu > 0 \) with initial data \( \theta_0 \in L^2 \), and assume that \( S \) be sufficiently smooth with \( S \in L^1 \cap L^\infty \). Then if \( d < 4 \), we have

\[
\lim_{\nu \to 0} \nu \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{\mathbb{R}^d} |\nabla \theta^\nu(x, s)|^2 dxds \right) = 0
\]  

(6.1)

**Proof.** Throughout the proof, we assume \( \kappa \equiv 1 \) for simplicity. Multiply \( \theta^\nu \) to (1.10) and integrate, for each \( T > 0 \), we have

\[
\int_0^T \int_{\mathbb{R}^d} |\nabla \theta^\nu(x, s)|^2 \leq \int_0^T |\theta_0|^2 + 2 \int_0^T \int_{\mathbb{R}^d} |S \theta^\nu(x, s)| dxds.
\]  

(6.2)

Using the \( L^\infty \) bound (3.6) for \( \theta^\nu \) and the assumption that \( d < 4 \), we bound the second term on the right side of (6.2) as follows.

\[
2 \int_0^T \int_{\mathbb{R}^d} |S \theta^\nu(x, s)| dxds \leq 2 \int_0^T C\|S\|_{L^1} \left( \frac{\|\theta_0\|_{L^2}}{s^{\frac{d}{2}}} + \|S\|_{L^\infty} \|\theta_0\|_{L^2} \right) ds
\]

\[
\leq 2C\|S\|_{L^1} \left( \|\theta_0\|_{L^2} T^{-\frac{d}{4}} + \|S\|_{L^\infty} \|\theta_0\|_{L^2} T \right).
\]  

(6.3)

Hence by applying (6.3) on (6.2), we conclude that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{\mathbb{R}^d} |\nabla \theta^\nu(x, s)|^2 dxds \leq \lim_{T \to \infty} \left[ \frac{1}{T} \|\theta_0\|_{L^2}^2 + 2C\|S\|_{L^1} \left( \|\theta_0\|_{L^2} T^{-\frac{d}{4}} + \|S\|_{L^\infty} \|\theta_0\|_{L^2} \right) \right]
\]

\[
= 2C\|S\|_{L^1} \|S\|_{L^\infty} \|\theta_0\|_{L^2},
\]

and therefore

\[
\lim_{\nu \to 0} \nu \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{\mathbb{R}^d} |\nabla \theta^\nu(x, s)|^2 dxds \right) \leq \lim_{\nu \to 0} \nu \cdot \left( 2C\|S\|_{L^1} \|S\|_{L^\infty} \|\theta_0\|_{L^2} \right) = 0,
\]

which finishes the proof of (6.1).

### 7. The MG equations and the existence of a global attractor

#### 7.1. The explicit MG equation

We now return to the magnetogeostricophic active scalar equation discussed in the introduction. Specifically, we are interested in the following active scalar equation in the domain \( \mathbb{T}^3 \times (0, \infty) = [0, 2\pi]^3 \times (0, \infty) \) (with periodic boundary conditions):

\[
\begin{align*}
\partial_t \theta^\nu + u^\nu \cdot \nabla \theta^\nu &= \kappa \Delta \theta^\nu + S, \\
u &= M^\nu[\theta^\nu], \quad \theta(x, 0) = \theta_0(x)
\end{align*}
\]  

(7.1)

via a Fourier multiplier operator \( M^\nu \) which relates \( u^\nu \) and \( \theta^\nu \). More precisely,

\[
u_j^\nu = M_j^\nu[\theta^\nu] = (M_j^\nu \hat{\theta}^\nu)^\wedge
\]
for \( j \in \{1, 2, 3\} \). The explicit expression for the components of \( \hat{M}^\nu \) as functions of the Fourier variable \( k = (k_1, k_2, k_3) \in \mathbb{Z}^3 \) are obtained from the constitutive law (1.9) to give
\[
\begin{align*}
\hat{M}^\nu_1(k) &= |k_2k_3|k^2 - k_1k_3(k_2^2 + \nu|k|^4)|D(k)^{-1}, \\
\hat{M}^\nu_2(k) &= -k_1k_3|k|^2 - k_2k_3(k_2^2 + \nu|k|^4)|D(k)^{-1}, \\
\hat{M}^\nu_3(k) &= [(k_1^2 + k_2^2)(k_2^2 + \nu|k|^4)]D(k)^{-1},
\end{align*}
\]
where
\[
D(k) = |k|^2k_3^2 + (k_2^2 + \nu|k|^4)^2.
\]
Here \( \kappa > 0 \) and \( \nu \geq 0 \) are some diffusive constants, \( \theta_\nu \) is the initial condition, and \( S = S(x) \) is a given smooth function that represents the forcing of the system. Furthermore, we restrict the system (7.1) to the function spaces where all functions (including the forcing \( S \) and initial data \( \theta_\nu \)) have zero mean with respect to \( x_3 \) (refer to section 4 of [23] for further discussion of this restriction). When \( \nu = 0 \), we write \( M^0 = M^\nu\big|_{\nu=0} \), \( \theta = \theta^\nu\big|_{\nu=0} \) and \( u = M^0[\theta] \). We also write
\[
u_j' = M^\nu_j[\theta] = \partial_\nu T^\nu_{ij},
\]
where we have denoted
\[
T^\nu_{ij} = -\partial_\nu((-\Delta)^{-1}M^\nu) \text{ for } \nu \geq 0.
\]
In order to show that conditions (1.11)-(1.14) are satisfied, we need the following lemmas for \( \{T^\nu_{ij}\}_{\nu \geq 0} \):

**Lemma 7.1.** Let \( T^\nu_{ij}, T^0_{ij} \) be as defined in (7.6)-(7.7) in terms of \( M^\nu \) and \( M^0 \). There are constants \( C_1, C_2 > 0 \) independent of \( \nu \) such that, for all \( 1 \leq i, j \leq 3 \),
\[
\begin{align*}
\sup_{\nu \in (0, 1)} \sup_{\{k \in \mathbb{Z}^3 : k \neq 0\}} \left| \hat{T}^\nu_{ij}(k) \right| &\leq \sup_{\nu \in (0, 1)} \sup_{\{k \in \mathbb{Z}^3 : k \neq 0\}} \frac{\left| \hat{M}^\nu(k) \right|}{|k|} \leq C_1, \\
\sup_{\{k \in \mathbb{Z}^3 : k \neq 0\}} \left| \hat{M}^0_{ij}(k) \right| &\leq \sup_{\{k \in \mathbb{Z}^3 : k \neq 0\}} \frac{\left| \hat{M}^0(k) \right|}{|k|} \leq C_2.
\end{align*}
\]

**Proof.** The bound (7.9) follows from the discussion in ([23], Section 4) and we omit the proof.

To show the bound (7.8), we only give the details for \( \hat{M}^\nu_1 \) since the cases for \( \hat{M}^\nu_2 \) and \( \hat{M}^\nu_3 \) are almost identical.

To prove (7.8), we fix \( \nu \in (0, 1] \) and consider the following cases:

**Case 1:** \( |k| > \nu^{-\frac{1}{2}} \). Then for each \( k \in \mathbb{Z}^3 / \{k = 0\} \),
\[
\left| \hat{M}^\nu_1(k) \right| = \frac{|k_2k_3|k^2 - k_1k_3(k_2^2 + \nu|k|^4)|}{|k||k|^2k_3^2 + (k_2^2 + \nu|k|^4)^2|}.
\]

Since \( k \neq 0 \), so \( |k| \geq |k_j| \geq 1 \) for \( j = 1, 2, 3 \), in particular \( |k|^{-1} < \nu^\frac{1}{2} \). Hence we obtain
\[
\left| \hat{M}^\nu_1(k) \right| \leq \frac{|k_2k_3|k^2}{|k||k|^2k_3^2} + \frac{|k_1k_3||k_2^2 + \nu|k|^4|}{|k^3k_3^2| + (k^2 + \nu|k|^4)^2} \leq \frac{1}{|k_3|} + \frac{1}{|k_3|} + \frac{1}{\nu|k|^2} \leq 1 + \frac{\nu}{\nu} = 3.
\]
Case 2: $|k| \leq \nu^{-\frac{1}{2}}$. Then for each $k \in \mathbb{Z}^3 \setminus \{ k = 0 \}$,
\[
\frac{\hat{M}_1'(k)}{|k|} \leq \frac{|k_2k_3||k|^2}{|k|^3k_3^2} + \frac{|k_1k_3|^2}{|k|^3k_3^2} + \frac{\nu|k_1k_3||k|^4}{|k|^3k_3^2}
\leq \frac{1}{|k_3|} + \frac{1}{|k_3|} + \frac{\nu|k|^2}{|k_3|}
\leq 2 + \nu \cdot (\nu^{-\frac{1}{2}})^2 = 3.
\]

Combining two cases, we have
\[
\sup_{\nu \in (0, 1) \setminus \{ k \in \mathbb{Z}^3 : k \neq 0 \}} \sup_{k \leq \nu} \frac{|\hat{M}_1'(k)|}{|k|} \leq 3,
\]
and hence (7.8) holds for some $C_1 > 0$ independent of $\nu$.

**Lemma 7.2.** For each $L > 0$,\n\[
\lim_{\nu \to 0} \sup_{k \in \mathbb{Z}^3 \setminus \{ k \neq 0 \}, |k| \leq L} \frac{|\hat{M}_1'(k) - \hat{M}_0'(k)|}{|k|} = 0 \tag{7.10}
\]

**Proof.** Again we only give the details for $\hat{M}_1'$. We fix $L > 0$, then for each $k \in \mathbb{Z}^3 \setminus \{ k = 0 \}$ with $|k| \leq L$, we have
\[
\frac{|\hat{M}_1'(k) - \hat{M}_0'(k)|}{|k|} \leq \frac{1}{|k|^3k_3^2} + \frac{|\nu|k_1k_3^2|k|^4}{|k|^3k_3^2} - \frac{2\nu^2k_1k_3|k|^6}{|k|^3k_3^2}.
\]

Hence
\[
\lim_{\nu \to 0} \sup_{k \in \mathbb{Z}^3 \setminus \{ k \neq 0 \}, |k| \leq L} \frac{|\hat{M}_1'(k) - \hat{M}_0'(k)|}{|k|} = 0.
\]

**Lemma 7.3.** Let $g$ be a function such that $\|\nabla g\|_{L^2} < \infty$. Then we have
\[
\lim_{\nu \to 0} \sum_{k \in \mathbb{Z}^3 : k \neq 0} \frac{|\hat{M}_1'(k) - \hat{M}_0'(k)|^2||\nabla g(k)|^2}{|k|^2} = 0 \tag{7.11}
\]

**Proof.** Fix $g$ with $\|\nabla g\|_{L^2} < \infty$ and let $\varepsilon > 0$ be given. Then $\sum_{k \in \mathbb{Z}^3} |\nabla g(k)|^2 < \infty$, so there exists $L = L(\varepsilon) > 0$ such that $\sum_{k \in \mathbb{Z}^3, |k| > L} |\nabla g(k)|^2 < \varepsilon$. Hence
\[
\sum_{k \in \mathbb{Z}^3 : k \neq 0} \frac{|\hat{M}_1'(k) - \hat{M}_0'(k)|^2||\nabla g(k)|^2}{|k|^2} \leq \sum_{k \in \mathbb{Z}^3 : k \neq 0, |k| \leq L} \frac{|\hat{M}_1'(k) - \hat{M}_0'(k)|^2||\nabla g(k)|^2}{|k|^2} + \sum_{k \in \mathbb{Z}^3 : k \neq 0, |k| > L} \frac{|\hat{M}_1'(k)|^2 + |\hat{M}_0'(k)|^2||\nabla g(k)|^2}{|k|^2}
\leq \left( \sup_{k \in \mathbb{Z}^3 : k \neq 0, |k| \leq L} \frac{|\hat{M}_1'(k) - \hat{M}_0'(k)|}{|k|} \right)^2 \|\nabla g\|_{L^2}^2 + (C_1 + C_2)\varepsilon. \tag{7.12}
\]
Using Lemma 7.2 and taking $\nu \to 0$ on (7.12),
\[
\lim_{\nu \to 0} \sum_{k \in \mathbb{Z}^{3}; k \neq 0} \frac{|\widehat{M_{\nu}}(k) - \widehat{M_{\theta}}(k)|^2 |\nabla g(k)|^2}{|k|^2} \leq (C_1 + C_2)\varepsilon.
\]
Since $\varepsilon > 0$ is arbitrary, (7.11) follows.

In view of Lemma 7.1–7.3, the sequence of operators $\{T_{ij}^\nu\}_{\nu \geq 0}$ given by (7.6)–(7.7) satisfy the conditions (1.11) and (1.13)-(1.14). Moreover, following the discussion given in [23] pp. 298–299, $\{T_{ij}^\nu\}_{\nu \geq 0}$ also satisfy (1.12). By Theorem 3.1 and Theorem 4.1, there exists classical solutions $\theta^n(x,t) \in C^\infty((0, \infty) \times \mathbb{R}^3)$ of (7.1)–(7.5), evolving from $\theta_0$ which satisfy the uniform bounds (4.1). The abstract Theorem 1.1 and Theorem 1.2 may therefore be applied to the MG equations in order to obtain the convergence of smooth solutions, and hence we have proven:

**Theorem 7.4.** Let $\theta_0 \in L^2$, $S \in C^\infty$ with $\|S\|_{L^\infty} < \infty$ and $\kappa > 0$ be given. There exists a classical solution $\theta^n(x,t) \in C^\infty((0, \infty) \times \mathbb{T}^3)$ of (7.1)–(7.5), evolving from $\theta_0$ for all $\nu \geq 0$.

**Theorem 7.5.** Let $\theta_0 \in L^2$, $\kappa > 0$ and $S = S(x) \in C^\infty$ be given with $\|S\|_{L^\infty} < \infty$. Then if $\theta^n, \theta$ are $C^\infty$ smooth classical solutions of (7.1)–(7.5) for $\nu > 0$ and $\nu = 0$ respectively with initial data $\theta_0$, then given $\tau > 0$, for all $s \geq 0$, we have
\[
\lim_{\nu \to 0} \|\theta^n - \theta(\cdot, t)\|_{H^s} = 0,
\]
whenever $t \geq \tau$.

Furthermore, since $d = 3 < 4$ for the case of MG equation, Theorem 6.1 can be applied to obtain:

**Theorem 7.6.** Let $\theta_0 \in L^2$, $\kappa > 0$ and $S = S(x) \in C^\infty$ be given, and assume $S \in L^1 \cap L^\infty$. If $\theta^n$ is the $C^\infty$ smooth classical solutions of (7.1)–(7.5) for $\nu > 0$ with initial data $\theta_0$, then we have
\[
\lim_{\nu \to 0} \nu \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{\mathbb{R}^d} |\nabla \theta^n(x,s)|^2 dx ds \right) = 0.
\]

### 7.2. The existence of a global attractor

The issue of the existence of a global attractor for active scalar equations is important in the general context of the long time averages of solutions to forced fluid equations (c.f., [11]). In particular there are recent results concerning the existence of a global attractor for the dynamics of the forced critical SQG equation [8], [12], [13]. In [8], Cheskidov and Dai prove that the forced critical SQG equation possesses a global attractor in $L^2(\mathbb{T}^3)$ provided the force $S$ is in $L^p(\mathbb{T}^3)$, $p > 2$. They use "classical" viscosity solutions and the abstract framework of evolutionary systems introduced by Cheskidov and Foias [6], [9]. We prove an analogous result for the forced three dimensional critical MG equation using the concept of a "vanishing viscosity" solution that arises naturally from the results of Theorem 7.5 concerning the convergence in the "vanishing viscosity" limit of solutions for the MG$^\nu$ and MG$^0$ equations.

We prove the existence of a compact global attractor in $L^2(\mathbb{T}^3)$ for the MG$^\nu$ equations (7.1)–(7.5) including the critical equation where $\nu = 0$, and we further obtain the upper semicontinuity of the global attractor as $\nu$ vanishes. First, we define a class of solutions to (1.10)–(1.12) as follows.

**Definition 7.7.** A weak solution to (7.1)–(7.5) with $\nu = 0$ is a function $\theta \in C_w([0,T];L^2(\mathbb{T}^3))$ with zero spatial mean that satisfies (7.1) in a distributional sense. That is, for any $\phi \in C_0^\infty(\mathbb{T}^3 \times (0,T))$,
\[
- \int_0^T \langle \theta, \phi_t \rangle dt - \int_0^T \langle u\theta, \nabla \phi \rangle dt + \kappa \int_0^T \langle \nabla \theta, \nabla \phi \rangle dt = \langle \theta_0, \phi(x,0) \rangle + \int_0^T \langle S, \phi \rangle dt,
\]
where $u = u_{\nu = 0}$ and $\langle \cdot, \cdot \rangle$ is the standard $L^2$-inner product. A weak solution $\theta(t)$ to (7.1) on $[0,T]$ with $\nu = 0$ is called a "vanishing viscosity" solution if there exist sequences $\nu_n \to 0$ and $\{\theta^{\nu_n}\}$ such that $\{\theta^{\nu_n}\}$ are smooth solutions to (7.1) as given by Theorem 3.1 and $\theta^{\nu_n} \to \theta$ in $C_w([0,T];L^2)$ as $\nu_n \to 0$. 

Remark 7.8. By Theorem 7.4 and Theorem 7.5, for any initial data \( \theta_0 \in L^2 \), there exists a “vanishing viscosity” solution \( \theta \) of \((1.10)\) on \([0, \infty)\) with \( \theta(0) = \theta_0 \).

We prove that the equation \((7.1)\) driven by a force \( S \) possesses a compact global attractor in \( L^2(T^3) \) which is upper semicontinuous at \( \nu = 0 \). More precisely, we have

Theorem 7.9. Assume \( S \in L^p \) with \( p > 3 \). Then the system \((7.1), (7.5)\) with \( \nu = 0 \) possesses a compact global attractor \( \mathcal{A} \) in \( L^2(T^3) \), namely

\[
\mathcal{A} = \{ \theta_0 : \theta_0 = \theta(0) \} \text{ for some bounded complete “vanishing viscosity” solution } \theta(t) \}.
\]

For any bounded set \( \mathcal{B} \subset L^2(T^3) \), and for any \( \varepsilon, T > 0 \), there exists \( t_0 \) such that for any \( t_1 > t_0 \), every “vanishing viscosity” solution \( \theta(t) \) with \( \theta(0) \in \mathcal{B} \) satisfies

\[
\| \theta(t) - x(t) \|_{L^2} < \varepsilon, \forall t \in [t_1, t_1 + T],
\]

for some complete trajectory \( x(t) \) on the global attractor \( (x(t) \in \mathcal{A}, \forall t \in (-\infty, \infty)) \).

Furthermore, for \( \nu \in [0, 1] \), there exists a compact global attractor \( \mathcal{A}^\nu \subset L^2 \) for \((7.1)\) such that \( \mathcal{A}^0 = \mathcal{A} \) and \( \mathcal{A}^\nu \) is upper semicontinuous at \( \nu = 0 \), which means that

\[
\sup_{\phi \in \mathcal{A}^\nu} \inf_{\psi \in \mathcal{A}} \| \phi - \psi \|_{L^2} \to 0 \text{ as } \nu \to 0. \tag{7.14}
\]

Before we give the proof of Theorem 7.9, we state and prove the following propositions.

Proposition 7.10. Let \( \nu > 0 \) and \( \theta_0 \in L^3(T^3) \cap L^2(T^3) \) be given. Let \( s \in [0, 1] \) and \( p \in (3, \infty) \). Assume that \( S \in L^q \) where \( q > p \). Then there exists \( T \in (0, 1) \) such that the classical solution \( \theta^\nu \) to \((7.1)-(7.5)\) as given by Theorem 3.1 satisfies

\[
\theta^\nu \in BC((0, T); L^d(T^3)),
\]

\[
t^{\frac{1}{2} - \frac{3}{p}} \theta^\nu \in C((0, T); W^{s,p}(T^3)). \tag{7.16}
\]

In particular, \( \theta^\nu(\cdot, t) \to \theta_0 \) in \( L^3 \) as \( t \to 0^+ \).

Proof. The proof follows by the similar argument given in the proof of Theorem 2.3 in \([22]\). We rewrite \((7.1)\) into the integral equation

\[
\theta^\nu(x, t) = G(t) \theta_0(x) + \int_0^t G(t - \tau) S(x) d\tau + B(\theta^\nu(x, t), \theta^\nu(x, t)), \tag{7.17}
\]

where \( G(t) \) is the convolution operator with kernel given in Fourier variables by \( \tilde{G}(\xi, t) = e^{-\kappa t |\xi|^2} \) and \( B(\cdot, \cdot) \) is the bilinear form

\[
B(\phi(x, t), \psi(x, t)) = -\int_0^t G(t - \tau) [\nabla \cdot (u(\phi(x, \tau)) \psi(x, \tau))] d\tau.
\]

To begin with, one can show that there exists a constant \( C_1 > 0 \) such that for any \( T \in (0, 1] \) and \( \phi, \psi \)

\[
\sup_{0 < t < T} t^{\frac{1}{2} - \frac{3}{p}} \| B(\phi(\cdot, t), \psi(\cdot, t)) \|_{L^p} \leq C_1 \left( \sup_{0 < t < T} t^{\frac{1}{2} - \frac{3}{p}} \| \phi(\cdot, t) \|_{L^p} \right) \left( \sup_{0 < t < T} t^{\frac{1}{2} - \frac{3}{p}} \| \psi(\cdot, t) \|_{L^p} \right). \tag{7.18}
\]

If we define

\[
X_T = \{ f \text{ measurable: } t^{\frac{1}{2} - \frac{3}{p}} f \in C((0, T); L^p) \}
\]

and \( \| f \|_{X_T} = \sup_{0 < t < T} t^{\frac{1}{2} - \frac{3}{p}} \| f \|_{L^p} \), then \( X_T \) is a Banach space and by \((7.15)\), we have that \( B(\cdot, \cdot) \) is a bounded bilinear form on \( X_T \). On the other hand, there exists a constant \( C_2 > 0 \) (refer to Lemma 5.1 in \([22]\) such that for \( \theta_0 \in L^2 \cap L^3 \),

\[
\| G(t) \theta_0 \|_{L^p} \leq C_2 t^{-\frac{1}{2} + \frac{3}{p}} \| \theta_0 \|_{L^3},
\]
Hence given \( \theta \) are uniformly bounded respectively in \( L^p \) while for the external force \( S \), for \( 0 < t \leq T \),
\[
\left\| \int_0^t G(t - \tau)S(x)d\tau \right\|_{L^p} \leq C_2 t^{1 - \frac{1}{2} \left( \frac{n}{2} - \frac{1}{2} \right)} \| S \|_{L^q} \leq C_2 t^{\frac{3}{2} - 1 + \frac{1}{2} p} \| S \|_{L^q}.
\]
Hence given \( \delta > 0 \), there exists \( T \in (0, 1) \) such that
\[
\left\| G(t)\theta_0(x) + \int_0^t G(t - \tau)S(x)d\tau \right\|_{L^p} \leq \sup_{0 \leq \tau < T} t^{\frac{3}{2} - \frac{1}{2} \delta} \| G(t)\theta_0(x) + \int_0^t G(t - \tau)S(x)d\tau \|_{L^p} \leq \delta,
\]
and therefore by Lemma 5.2 in \cite{22}, there exists \( \theta' \in X_T \) which satisfies (7.17) with
\[
\sup_{0 \leq \tau < T} t^{\frac{3}{2} - \frac{1}{2} \delta} \| \theta'(\cdot, t) \|_{L^p} \leq 2 \delta.
\]
To proceed further, we consider the Picard sequence defined by:
\[
\theta_1(x, t) = G(t)\theta_0(x),
\]
\[
\theta_{n+1}(x, t) = \theta_1(x, t) + \int_0^t G(t - \tau)S(x)d\tau + B(\theta_n(x, t), \theta_n(x, t)), \text{ for } n \in \mathbb{N}.
\]
Following the lines of \cite{22} Theorem 5.3, one can show that the sequences \( \{ \theta_n \}_{n \in \mathbb{N}} \) and \( \{ t^{\frac{3}{2} + \frac{1}{2} \frac{3}{2} \delta} \theta_n \}_{n \in \mathbb{N}} \) are uniformly bounded respectively in \( L^\infty((0, T); L^3) \) and \( L^\infty((0, T); \dot{W}^{s, p}) \). Hence there exists a subsequence of \( \{ \theta_n \}_{n \in \mathbb{N}} \) which converges towards some \( \tilde{\theta} \) weak-* in \( L^\infty((0, T); L^3) \) and consequently in \( D'((\mathbb{T}^3 \times (0, T)) \). On the other hand, we know that \( \theta_n \rightarrow \theta' \) in \( X_T \), which implies convergence in \( D'((\mathbb{T}^3 \times (0, T)) \) as well. Therefore, \( \theta' = \tilde{\theta} \in L^\infty((0, T); L^3) \). The time-continuity of \( \theta' \) follows by using the fact that \( \theta' \) belongs to \( X_T \) and it solves (7.17). By similar methods, we can prove that
\[
t^{\frac{3}{2} + \frac{1}{2} \frac{3}{2} \delta} \theta' \in C((0, T); \dot{W}^{s, p})
\]
and we omit further details to avoid redundancy. 

We also have the following energy equality which is important in obtaining an absorbing ball for (7.1) (see Remark 7.12 below).

**Proposition 7.11.** (The energy equality) Let \( \theta(t) \) be a “vanishing viscosity” solution of (7.1) on \([0, \infty)\) with \( \theta(0) \in L^2 \). Then \( \theta(t) \) satisfies the following energy equality:
\[
\frac{1}{2} \| \theta(t) \|^2_{L^2} + \kappa \int_{t_0}^t \| \nabla \theta(x, s) \|^2_{L^2} ds = \frac{1}{2} \| \theta(t_0) \|^2_{L^2} + \int_{t_0}^t \int_{\mathbb{T}^3} S \theta dx ds,
\]
for all \( 0 \leq t_0 \leq t \).

**Proof.** By Proposition 7.10 and Theorem 7.5, since \( \theta \) is a “vanishing viscosity” solution, we have
\[
\lim_{t \to 0^+} \| \theta(\cdot, t) \|_{L^2} = \| \theta(0) \|_{L^2}.
\]
It remains to show that the flux term \( \int_{t_0}^t \int_{\mathbb{T}^3} u \theta \cdot \nabla \theta dx ds \) equals zero; for which a proof can be found in \cite{8} and we omit the details. The proof uses techniques of Littlewood-Paley decomposition. It is analogous to the proof given in \cite{7} that the energy flux is zero for weak solutions of the three dimensional Euler equation that are smoother than Onsager critical. 

**Remark 7.12.** Based on the equality (7.19), we can see that every “vanishing viscosity” solution to (7.1) is strongly continuous in \( t \).

**Remark 7.13.** Moreover, in view of (7.19), there exists an absorbing ball \( \mathcal{Y} \) for (7.1) given by
\[
\mathcal{Y} = \{ \theta \in L^2 : \| \theta \|_{L^2} \leq R \},
\]
where \( R \) is any number larger than \( \kappa^{-1} \| S \|_{H^{-1}(\mathbb{T}^3)} \). Then for any bounded set \( \mathcal{B} \subset L^2 \), there exists a time \( t_0 \) such that
\[
\theta(t) \in \mathcal{Y}, \quad \forall t \geq t_0,
\]
for every “vanishing viscosity” solution \( \theta(t) \) with the initial data \( \theta(0) \in \mathcal{B} \).
We are ready to prove Theorem 7.9. We first denote the strong and weak distances on $L^2(\mathbb{T}^3)$ respectively by
\[
d_s(\phi, \psi) = \|\phi - \psi\|_{L^2}; \quad d_w(\phi, \psi) = \sum_{k \in \mathbb{Z}^3} \frac{1}{2|k|} \frac{|\hat{\phi}_k - \hat{\psi}_k|}{1 + |\hat{\phi}_k - \hat{\psi}_k|},
\]
where $\hat{\phi}_k$ and $\hat{\psi}_k$ are the Fourier coefficients of $\phi$ and $\psi$. Also, we let
\[\mathcal{T} = \{I : I = [T, \infty) \subset \mathbb{R}, \text{ or } I = (-\infty, \infty)\},\]
and for each $I \subset \mathcal{T}$, let $\mathcal{F}(I)$ denote the set of all $\mathcal{Y}$-valued functions on $I$ (here $\mathcal{Y}$ is the absorbing ball given by \((7.20)\) in Remark 7.12).

**Proof of Theorem 7.9.** We now define
\[\mathcal{E}(T, \infty) = \{\theta(\cdot) : \theta(\cdot) \text{ is a “vanishing viscosity” solution of } (1.10) \text{ on } [T, \infty) \text{ and } \theta \in \mathcal{Y} \text{ for all } t \in [T, \infty)\},\]
\[\mathcal{E}(-\infty, \infty) = \{\theta(\cdot) : \theta(\cdot) \text{ is a “vanishing viscosity” solution of } (1.10) \text{ on } (-\infty, \infty) \text{ and } \theta \in \mathcal{Y} \text{ for all } t \in (-\infty, \infty)\},\]
then $\mathcal{E}$ is an evolutionary system (see [3] and [8] for the definition), so by Theorem 4.5 in [8], there exists a weak global attractor $A_w$ to $\mathcal{E}$ with
\[A_w = \{\theta_0 : \theta_0 = \theta(0) \text{ for some } \theta \in \mathcal{E}((-\infty, \infty))\}. \quad (7.21)\]
Furthermore, by the Aubin-Lions Lemma (also refer to [11] for the case of Navier-Stokes equation), if $\theta_n(t)$ is any sequence of “vanishing viscosity” solutions of (1.10) such that $\theta_n(t) \in \mathcal{Y}$ for all $t \geq 0$, then there exists a subsequence $\theta_{n_k}$ of $\theta_n$ that converges in $C([t_0, T]; \mathcal{Y}_w)$ to some “vanishing viscosity” solution $\theta(t)$ (here $\mathcal{Y}_w$ refers to the metric space $(\mathcal{Y}, d_w)$). Using Proposition 7.10 and applying the arguments given in [8], $\mathcal{E}$ satisfies all the following properties:

A1. $\mathcal{E}([0, \infty))$ is a compact set in $C([0, \infty); \mathcal{Y}_w)$ ($\mathcal{Y}_w$ is endowed with the weak topology induced by $d_w$);

A2. for any $\epsilon > 0$, there exists $\delta > 0$ such that for every $\theta \in \mathcal{E}([0, \infty))$ and $t > 0$,
\[\|\theta(t)\|_{L^2} \leq \|\theta(t)\|_{L^2} + \epsilon,\]
for $t_0$ a.e. in $(t - \delta, t) \cap [0, \infty)$;

A3. if $\theta_n \in \mathcal{E}([0, \infty))$ and $\theta_n \rightarrow \theta$ in $\mathcal{E}([0, \infty))$ in $C([0, \infty); \mathcal{Y}_w)$ for some $T > 0$, then $\theta_n(t) \rightarrow \theta(t)$ strongly a.e. in $[0, T]$.

Therefore, together with Remark 7.11, Theorem 4.5 in [8] can then be applied again to our evolutionary system $\mathcal{E}$, which implies that

- the strong global attractor $A_s$ exists, it is strongly compact and $A := A_s = A_w$; and
- for any bounded set $B \subset L^2(\mathbb{T}^3)$, and for any $\epsilon, T > 0$, there exists $t_0$ such that for any $t_1 > t_0$, every “vanishing viscosity” solution $\theta(t)$
\[\|\theta(t) - x(t)\|_{L^2} < \epsilon, \forall t \in [t_1, t_1 + T],\]
for some complete trajectory $x(t)$ on the global attractor ($x(t) \in A, \forall t \in (-\infty, \infty)$).

Finally, to prove that \((7.11)\) holds, we define $\pi^\nu : L^2 \rightarrow L^2$ as the map $\pi^\nu \theta_0 = \theta^\nu$, where $\theta^\nu$ is the solution to \((7.1)-(7.5)\) given by Theorem 7.4. Then for each $\nu \in (0, 1]$, $\pi^\nu$ is a semigroup $\{\pi^\nu(t)\}_{t \geq 0}$ on $L^2$ and there exists a compact global attractor $A^\nu \subset L^2$ for $\pi^\nu$ given by
\[A^\nu = \{\theta_0 : \theta_0 = \theta^\nu(0)\}, \quad \text{where } \theta^\nu \text{ is a solution of } (7.1)-(7.5)\]
defined on $(-\infty, \infty)$ and $\theta^\nu \in \mathcal{Y}$ for all $t \in (-\infty, \infty)$.

Moreover, for $\nu = 0$, $A^0 = A$ is the global attractor for $\pi^0(\cdot)$ satisfying
- $\pi^0(t)A = A$ for all $t \in \mathbb{R}$;
- for any bounded set $B$, $\sup_{\phi \in \pi^0(t)B} \inf_{\psi \in A} \|\phi - \psi\|_{L^2} \rightarrow 0$ as $t \rightarrow 0$. 

According to [26], (7.14) holds if we can prove that:

L1. \(\pi_\nu(\cdot)\) has a global attractor \(\mathcal{A}_\nu\) for every \(\nu \in [0, 1]\);

L2. there is a compact subset \(\mathcal{K}\) of \(L^2\) such that \(\mathcal{A}_\nu \subset \mathcal{K}\) for every \(\nu \in [0, 1]\);

L3. for \(t > 0\), \(\pi_\nu(t)\theta_0\) is continuous in \(\nu\), uniformly for \(\theta_0\) in compact subsets of \(L^2\).

Hence it remains to prove L3. Let \(\mathcal{K}\) be a compact subset of \(L^2\) and fix \(\theta_0 \in \mathcal{K}\). Let \(\nu_1, \nu_2 \in [0, 1]\), then for each \(t > 0\), we have

\[
\frac{1}{2} \frac{d}{dt} \| (\theta^{\nu_1} - \theta^{\nu_2})(\cdot, t) \|_{L^2}^2 + \frac{\kappa}{2} \| \nabla (\theta^{\nu_1} - \theta^{\nu_2})(\cdot, t) \|_{L^2}^2 = - \int (u^{\nu_1} - u^{\nu_2}) \cdot \nabla \theta^{\nu_2} \cdot (\theta^{\nu_1} - \theta^{\nu_2})(x, t) dx, \tag{7.22}
\]

where \(\theta^{\nu_i}(0) = \theta_0\) for \(i = 1, 2\). We follow the argument given in the proof of Theorem 1.2 to obtain

\[
- \int (u^{\nu_1} - u^{\nu_2}) \cdot \nabla \theta^{\nu_2} \cdot (\theta^{\nu_1} - \theta^{\nu_2})(x, t) dx \leq \frac{\kappa}{4C^2} \| (u^{\nu_1} - u^{\nu_2})(\cdot, t) \|_{L^2}^2 + \frac{4C^2}{\kappa} \| (\theta^{\nu_1} - \theta^{\nu_2})(\cdot, t) \|_{L^2}^2 \| \nabla \theta^{\nu_2}(\cdot, t) \|_{L^\infty}^2. \tag{7.23}
\]

Using the bound (4.11) for \(\theta^{\nu_2}\), the second term on the right side of (7.23) is bounded by \(C \| (\theta^{\nu_1} - \theta^{\nu_2})(\cdot, t) \|_{L^2}^2\), for some constant \(C > 0\) independent of \(\theta_0\) but depends on the compact set \(\mathcal{K}\) and \(t\). On the other hand, the term \(\| (u^{\nu_1} - u^{\nu_2})(\cdot, t) \|_{L^2}^2\) can be bounded as follows.

\[
\| (u^{\nu_1} - u^{\nu_2})(\cdot, t) \|_{L^2}^2 = \sum_{k \in \mathbb{Z}^3} \| (\partial_i T_{ij}^{\nu_1} \theta^{\nu_1} - \partial_i T_{ij}^{\nu_2} \theta^{\nu_2})(k) \|^2 \leq \sum_{k \in \mathbb{Z}^3} \| T_{ij}^{\nu_1}(k) \|^2 \| \nabla (\theta^{\nu_1} - \theta^{\nu_2})(k) \|^2 + \sum_{k \in \mathbb{Z}^3} \| T_{ij}^{\nu_1}(k) - T_{ij}^{\nu_2}(k) \|^2 \| \nabla \theta^{\nu_2}(k) \|^2 \leq C^2 \| \nabla (\theta^{\nu_1} - \theta^{\nu_2})(\cdot, t) \|_{L^2}^2 + \sum_{k \in \mathbb{Z}^3} \| T_{ij}^{\nu_1}(k) - T_{ij}^{\nu_2}(k) \|^2 \| \nabla \theta^{\nu_2}(k) \|^2,
\]

where \(T_{ij}^{\nu_1}, T_{ij}^{\nu_2}\) are defined in (7.14). Using the bound (4.11) for \(\theta^{\nu_2}\) again and applying the similar argument given in the proof of Lemma 7.1 and Lemma 7.2, there exists constant \(C = C(\nu_1, \nu_2, \mathcal{K}, t) > 0\) such that

\[
\sum_{k \in \mathbb{Z}^3} \| T_{ij}^{\nu_1}(k) - T_{ij}^{\nu_2}(k) \|^2 \| \nabla \theta^{\nu_2}(k) \|^2 \leq C |\nu_1 - \nu_2|.
\]

Hence we conclude from (7.22) that

\[
\frac{1}{2} \frac{d}{dt} \| (\theta^{\nu_1} - \theta^{\nu_2})(\cdot, t) \|_{L^2}^2 \leq C \left[ |\nu_1 - \nu_2| + \| (\theta^{\nu_1} - \theta^{\nu_2})(\cdot, t) \|_{L^2}^2 \right],
\]

which further implies

\[
\| (\pi^{\nu_1}(t)\theta_0 - \pi^{\nu_2}(t)\theta_0) \|_{L^2}^2 = \| (\theta^{\nu_1} - \theta^{\nu_2})(\cdot, t) \|_{L^2}^2 \leq C |\nu_1 - \nu_2|.
\]

It proves the continuity of \(\pi_\nu\) in \(\nu\) uniformly for \(\theta_0\) in \(\mathcal{K}\), and thus L3 holds. This completes the proof of Theorem 7.9.

\[\square\]

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