Two-party Bell inequalities derived from combinatorics via triangular elimination

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Abstract. We establish a relation between the two-party Bell inequalities for two-valued measurements and a high-dimensional convex polytope called the cut polytope in polyhedral combinatorics. Using this relation, we propose a method, triangular elimination, to derive tight Bell inequalities from facets of the cut polytope. This method gives two hundred million inequivalent tight Bell inequalities from currently known results on the cut polytope. In addition, this method gives general formulas which represent families of infinitely many Bell inequalities. These results can be used to examine general properties of Bell inequalities.

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1. Introduction

Bell inequalities have been intensively studied in quantum theory \cite{1,2}, and it is known that they can be obtained from the structure of certain convex polytopes \cite{3,4,5}. Bell inequalities are not the only example of the use of convex polytopes in quantum theory. In a pioneering paper, McRae and Davidson \cite{6} used the theory of convex polytopes to obtain inequalities bounding the range of possible solutions to some problems in quantum mechanics. Their method is summarized as follows: First they prove that the possible solutions form a convex polytope and obtain the set of vertices of the polytope. Then they obtain a minimum set of inequalities that describe the polytope using a convex hull algorithm. Interestingly, one of the polytopes McRae and Davidson considered coincides with the correlation polytope Pitowsky introduced in \cite{3}, in connection with Bell inequalities. This polytope arises in many fields under different names, and a comprehensive source for results on this polytope and the related cut polytope (described later) is the book by Deza and Laurent \cite{7}.

In this paper we consider the results of correlation experiments between two parties, where one party has \( m_A \) choices of possible two-valued measurements and the other party has \( m_B \) choices. The relevant polytope can be described as follows. The results of a series of such correlation experiments are represented as a vector of \( m_A + m_B + m_A m_B \) probabilities. In classical mechanics, the set of vectors which are possible results of a correlation experiment forms an \( (m_A + m_B + m_A m_B) \)-dimensional convex polytope which is a projection of the correlation polytope onto the complete bipartite graph \( K_{m_A, m_B} \).

A Bell inequality is nothing but a linear inequality satisfied by all the points in such a polytope. A tight Bell inequality is a Bell inequality which cannot be represented as a positive weighted sum of other Bell inequalities and defines a facet of the polytope. Two examples of these facet defining inequalities are the nonnegativity inequality and the Clauser-Horne-Shimony-Holt (CHSH) inequality \cite{8}.

By considering these polytopes, Bell's original inequality \cite{9}, the CHSH and many other known Bell inequalities can be understood in a unified manner. Fine's necessary and sufficient conditions \cite{10} for \( m_A = m_B = 2 \) can be seen as the complete inequality representation of the correlation polytope of the complete bipartite graph \( K_{2,2} \). Pitowsky and Svozil \cite{11} and Collins and Gisin \cite{12} apply convex hull algorithms to obtain a complete list of tight Bell inequalities in other experimental settings. As a result, we know the complete list of Bell inequalities in the cases \( m_A = 2 \) \cite{12}, \((m_A, m_B) = (3, 3)\) \cite{11} and \((m_A, m_B) = (3, 4)\) \cite{12}. Several software packages for convex hull computation such as cdd \cite{13} and lrs \cite{14} are readily available. It is unlikely, however, that there exists a compact representation of the complete set of Bell inequalities in arbitrarily large settings. This follows from the fact that testing whether a vector of correlations lies in the correlation polytope of the bipartite graph \( K_{m_A, m_B} \) is NP-complete \cite{15}. Therefore it is natural to look for families of Bell inequalities, especially those that are facet producing. In this direction, Collins and Gisin \cite{12} give a family \( I_{m,m22} \) of Bell inequalities in the case \( m_A = m_B = m \) for general \( m \).
addition there are several extensions \[16, 17, 12\] of the CHSH inequality for multi-valued measurements.

In the field of polyhedral combinatorics a polytope isomorphic to the correlation polytope, called the cut polytope, has been studied in great detail \[7\]. The correlation and cut polytopes are isomorphic via a linear mapping \[18\] and so the inequalities representing them correspond one-to-one. This relationship enables us to apply results for the cut polytope to the study of Bell inequalities. Related to this, Pironio \[19\] uses lifting, which is a common approach in combinatorial optimization, to generate tight Bell inequalities for a larger system from those for a smaller system. Since the mathematical description of the facet structure of cut polytopes is simpler than that for correlation polytopes, the former are preferred in polyhedral combinatorics. Large classes of facets for the cut polytope \(\text{CUT}^\square_n\) of the complete graph \(K_n\) are known for general \(n\) \[7\], and a complete or conjectured complete list of all facets for \(\text{CUT}^\square_n\) is known for \(n \leq 9\) \[20\]. We make use of these results in this paper.

The cut polytope of the complete graph has been the most extensively studied. However, the case we are interested in corresponds to the correlation polytope of the complete bipartite graph \(K_{m_A, m_B}\), which maps to the cut polytope of the complete tripartite graph \(K_{1, m_A, m_B}\). To overcome this gap, we introduce a method called triangular elimination to convert an inequality valid for \(\text{CUT}^\square_n\) to another inequality valid for \(\text{CUT}^\square(K_{1, m_A, m_B})\), which is then converted to a Bell inequality via the isomorphism. The CHSH inequality and some of the other previously known inequalities can be explained in this manner. More importantly, triangular elimination converts a facet inequality of \(\text{CUT}^\square_n\) to a facet inequality of \(\text{CUT}^\square(K_{1, m_A, m_B})\), which corresponds to a tight Bell inequality.

A complete list of facets of \(\text{CUT}^\square_n\) for \(n \leq 7\) and a conjectured complete list for \(n = 8, 9\) are known. We apply triangular elimination to these facets to obtain 201,374,783 tight Bell inequalities. On the other hand, several formulas which represent many different inequalities valid for \(\text{CUT}^\square_n\) are known. We apply triangular elimination to these formulas to obtain new families of Bell inequalities. We discuss their properties such as tightness and inclusion of the CHSH inequality.

The rest of this paper is organized as follows. In Section 2 we introduce triangular elimination to derive tight Bell inequalities from facets of the cut polytope of the complete graph, and show its properties. We also give a computational result on the number of Bell inequalities obtained by triangular elimination. In Section 3 we apply triangular elimination to some of the known classes of facets of the cut polytope of the complete graph to obtain general formulas representing many Bell inequalities. Section 4 concludes the paper by giving the relation of our result to some of the open problems posed in 2.
2. Triangular elimination

2.1. Bell inequalities and facets of cut polytopes

Consider a system composed of subsystems A (Alice) and B (Bob). Suppose that on both subsystems, one of $m_A$ observables for Alice and one of $m_B$ observables for Bob are measured. For each observable, the outcome is one of two values (in the rest of the paper, we label the outcomes as 0 or 1). The experiment is repeated a large number of times. The result of such a correlation experiment consists of the probability distribution of the $m_A m_B$ joint measurements by both parties. Throughout this paper, we represent the experimental result as a vector $q$ in $m_A + m_B + m_A m_B$ dimensional space in the following manner: $q_{A_i}$, $q_{B_j}$ and $q_{A_iB_j}$ correspond to the probabilities $\Pr[A_i = 1]$, $\Pr[B_j = 1]$ and $\Pr[A_i = 1 \land B_j = 1]$ respectively.

In classical mechanics, the result of a correlation experiment must correspond to a probability distribution over all classical configurations, where a classical configuration is an assignment of the outcomes \{0, 1\} to each of the $m_A + m_B$ observables. The experimental result has a local hidden variable model if and only if a given experimental result can be interpreted as a result of such a classical correlation experiment.

Bell inequalities are valid linear inequalities for every experimental result which has a local hidden variable model. Specifically using the above formulation, we represent a Bell inequality in the form

$$\sum_{1 \leq i \leq m_A} b_{A_i} q_{A_i} + \sum_{1 \leq j \leq m_B} b_{B_j} q_{B_j} + \sum_{1 \leq i \leq m_A, 1 \leq j \leq m_B} b_{A_iB_j} q_{A_iB_j} \leq b_0,$$

for suitably chosen constants $b_x$.

For example, Clauser, Horn, Shimony and Holt [8] have shown that the following CHSH inequality is a valid Bell inequality:

$$-q_{A_1} - q_{B_1} + q_{A_1B_1} + q_{A_1B_2} + q_{A_2B_1} - q_{A_2B_2} \leq 0.$$

In general, the set of all experimental results with a local hidden variable model forms a convex polytope with extreme points corresponding to the classical configurations. If the results of the experiment are in the above form, the polytope is called a correlation polytope, a name introduced by Pitowsky [21]. (Such polyhedra have been discovered and rediscovered several times, see for instance Deza and Laurent [7].) From such a viewpoint, Bell inequalities can be considered as the boundary, or face inequalities, of that polytope. Since every polytope is the intersection of finitely many half spaces represented by linear inequalities, every Bell inequality can be represented by a convex combination of finitely many extremal inequalities. Such extremal inequalities are called tight Bell inequalities. Non-extremal inequalities are called redundant.

In polytopal theory, the maximal extremal faces of a polytope are called facets. Therefore, tight Bell inequalities are facet inequalities of the polytope formed by experimental results with a local hidden variable model. Note that for a given linear inequality $b^T q \leq b_0$ and $d$ dimensional polytope, the face represented by the inequality
is a facet of that polytope if and only if the dimension of the convex hull of the extreme points for which the equality holds is \( d - 1 \).

2.1.1. Cut polytope of complete tripartite graph  We introduce a simple representation of an experimental setting as a graph. Consider a graph which consists of vertices corresponding to observables \( A_i \) or \( B_j \) and edges corresponding to joint measurements between \( A_i \) and \( B_j \). In addition, to represent probabilities which are the results of single (not joint) measurements, we introduce a vertex \( X \) (which represents the trace operation of the other party) and edges between \( X \) and \( A_i \) for every \( 1 \leq i \leq m_A \), and between \( X \) and \( B_j \) for every \( 1 \leq j \leq m_B \). This graph is a complete tripartite graph since there exist edges between each party of vertices (observables) \( \{X\} \), \( \{A_i\} \) and \( \{B_j\} \). Using this graph, we can conveniently represent either the result probabilities or the coefficients of a Bell inequality as edge labels. We denote this graph by \( K_{1,m_A,m_B} \).

In polyhedral combinatorics, a polytope affinely isomorphic to the correlation polytope has been well studied. Specifically, if we consider the probabilities \( x_{A_iB_j} = \Pr[A_i \neq B_j] \) instead of \( q_{A_iB_j} = \Pr[A_i = 1 \wedge B_j = 1] \) for each edge, the probabilities form a polytope called the cut polytope. Thus, the cut polytope is another formulation of the polytope formed by Bell inequalities.

A cut in a graph is an assignment of \( \{0, 1\} \) to each vertex, 1 to an edge between vertices with different values assigned, and 0 to an edge between vertices with the same values assigned. In the above formulation, each cut corresponds to a classical configuration. Note that since the 0, 1 exchange of all values of vertices yields the same edge cut, we can without loss of generality assume that the vertex \( X \) is always assigned the label 0.

Let the cut vector \( \delta'(S') \in \mathbb{R}^{\{X_{A_i}\} \cup \{X_{B_j}\} \cup \{A_iB_j\}} \) for some cut \( S' \) be \( \delta'_{uv}(S') = 1 \) if vertices \( u \) and \( v \) are assigned different values, and 0 if assigned the same values. Then, the convex combination of all the cut vectors \( \text{CUT}^\square(K_{1,m_A,m_B}) = \{x = \sum_{S': \text{cut}} \lambda_{S'} \delta'(S') \mid \sum_{S': \text{cut}} \lambda_{S'} = 1 \text{ and } \lambda_{S'} \geq 0 \} \) is called the cut polytope of the complete tripartite graph. The cut polytope has full dimension. Therefore, \( \dim(\text{CUT}^\square(K_{1,m_A,m_B})) = m_A + m_B + m_A m_B \).

In this formulation, a tight Bell inequality \( b^T q \leq b_0 \) corresponds to a facet inequality \( a^T x \leq a_0 \) of the cut polytope. The affine isomorphisms between them are:

\[
\begin{align*}
    x_{XA_i} &= q_{A_i}, \\
    x_{XB_j} &= q_{B_j}, \\
    x_{A_iB_j} &= q_{A_i} + q_{B_j} - 2q_{A_iB_j},
\end{align*}
\]

\[
\begin{align*}
    q_{A_i} &= x_{XA_i}, \\
    q_{B_j} &= x_{XB_j}, \\
    q_{A_iB_j} &= \frac{1}{2}(x_{XA_i} + x_{XB_j} - x_{A_iB_j}).
\end{align*}
\]

Actually, because cut polytopes are symmetric under the switching operation (explained in Section 2.4) we can assume that the right hand side of a facet inequality of the cut polytope is always 0. This means that a given Bell inequality is tight if and only if for the corresponding facet inequality \( a^T x \leq 0 \) of the cut polytope, there exist \( m_A + m_B + m_A m_B - 1 \) linearly independent cut vectors \( \delta'(S') \) for which \( a^T \delta'(S') = 0 \).

For example, there exists a facet inequality \( -x_{A_1B_1} - x_{A_1B_2} - x_{A_2B_1} + x_{A_2B_2} \leq 0 \) for
A consequence of the above affine isomorphisms is that any theorem concerning facets of the cut polytope can be immediately translated to give a corresponding theorem for tight Bell inequalities. Recently, Collins and Gisin [12] gave the following conjecture about the tightness of Bell inequalities: if a Bell inequality $b^T q \leq b_0$ is tight in a given setting $m_A, m_B$, then for each $m_A' \geq m_A$ and $m_B' \geq m_B$, the inequality $b'^T q' \leq b_0$ is also tight. Here $b'$ is the vector $b'_{uv} = b_{uv}$ if the vertices (observables) $u, v$ appear in $b$ and is zero otherwise. They gave empirical evidence for this conjecture based on numerical experiments. In fact, a special case of the zero-lifting theorem by De Simone [22] gives a proof of their conjecture.

2. Triangular elimination

2.1. Cut polytope of complete graph

In the previous section we saw that the problem of enumerating tight Bell inequalities is equivalent to that of enumerating facet inequalities of the cut polytope of a corresponding complete tripartite graph. The properties of facet inequalities of the cut polytope of the complete graph $K_n$ are well studied and there are rich results. For example, several general classes of facet inequalities with relatively simple representations are known.

For $n \leq 7$ the complete list of facets is known [23], and for $n = 8, 9$ a conjectured complete list is known [21, 20]. In addition, the symmetry of the polytope is also well-understood. We show how to apply such results to our complete tripartite graph case.

First, we introduce the cut polytope of complete graph. The graph is denoted by $K_n$, has $n$ vertices, and has an edge between each pair of vertices. As before, a cut is an assignment of $\{0, 1\}$ to each vertex, and an edge is labeled by 1 if the endpoints of the edge are labeled differently or 0 if labeled the same. The cut vectors $\delta(S)$ of the complete graph are defined in the same manner as before. The set of all convex combinations of cut vectors $\text{CUT}^\square(K_n) = \{ x = \sum_{S:\text{cut}} \lambda_S \delta(S) \mid \sum_{S:\text{cut}} \lambda_S = 1 \text{ and } \lambda_S \geq 0 \}$ is called the cut polytope of the complete graph. $\text{CUT}^\square(K_n)$ is also written as $\text{CUT}_{n}^\square$.

In contrast to the complete tripartite graph, the space on which the cut polytope of the complete graph exists has elements corresponding to probabilities of joint measurement by the same party. Because of the restrictions of quantum mechanics, such joint measurements are prohibited. Therefore, if we want to generate tight Bell inequalities from the known facet inequalities of the cut polytope of the complete graph, we must transform the inequalities to eliminate joint measurement terms. In polyhedral terms, $\text{CUT}^\square(K_{1,m_A,m_B})$ is a projection of $\text{CUT}^\square(K_n)$ onto a lower dimensional space.

2.2. Definition of triangular elimination

A well known method for projecting a polytope is called Fourier-Motzkin elimination. This is essentially the summation of two facet inequalities to cancel out the target term. For example, it is well known that the triangle inequality $x_{uv} - x_{uw} - x_{wv} \leq 0$, for any three vertices $u, v, w$, is valid for the
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The triangular elimination is defined as follows:

For a given valid inequality for triangular elimination operation vertices, is almost always guaranteed to yield non-redundant inequalities. We call the Fourier-Motzkin elimination by triangle inequalities which introduce new triangle inequality introduced a new vertex

This means that the inequality with eliminated terms is redundant. The CHSH inequality is essentially the pentagonal inequality had no term with subscript labeled

In general, the result of Fourier-Motzkin elimination is not necessarily a facet. For example, it is known that the pentagonal inequality

is a facet inequality of $\text{CUT}^\square(K_5)$. If we eliminate joint measurement terms $x_{A_1A_2}$ and $x_{B_1B_2}$ by adding triangle inequalities $x_{A_1B_2} - x_{A_1A_2} - x_{A_2B_2} \leq 0$ and $x_{A_2B_1} - x_{A_1B_2} - x_{A_2B_2} \leq 0$, the result is $x_{X,A_1} + x_{X,A_2} - x_{X,B_1} - x_{X,B_2} - x_{A_1B_1} - 3x_{A_2B_2} \leq 0$. Therefore, this inequality is a valid inequality for $\text{CUT}^\square(K_{1,3,3})$. However, the inequality is a summation of four valid triangle inequalities for $\text{CUT}^\square(K_{1,3,3})$, namely $x_{X,A_1} - x_{X,B_1} - x_{A_1B_1} \leq 0$, $x_{X,A_2} - x_{X,B_2} - x_{A_2B_2} \leq 0$, $x_{X,A_2} - x_{X,B_2} - x_{A_2B_2} \leq 0$ and $-x_{X,A_2} + x_{X,B_2} - x_{A_2B_2} \leq 0$. This means that the inequality with eliminated terms is redundant.

Fourier-Motzkin elimination often produces large numbers of redundant inequalities, causing the algorithm to be computationally intractable when iterated many times. Therefore, it is important to find situations where the new inequalities found are guaranteed to be tight.

The difference between the two examples is that in the CHSH case, the second triangle inequality introduced a new vertex $B_2$ where “new” means that the first triangle inequality had no term with subscript labeled $B_2$. Generalizing this operation, we will show that Fourier-Motzkin elimination by triangle inequalities which introduce new vertices, is almost always guaranteed to yield non-redundant inequalities. We call the operation triangular elimination.

**Definition 2.1 (triangular elimination)** For a given valid inequality for $\text{CUT}^\square(K_{1+n_A+n_B})$

\[
\sum_{1 \leq i \leq n_A} a_{X,A_i}x_{X,A_i} + \sum_{1 \leq j \leq n_B} a_{X,B_j}x_{X,B_j} + \sum_{1 \leq i \leq n_A, 1 \leq j \leq n_B} a_{A,B_i}x_{A_i,B_j} \\
+ \sum_{1 \leq i < i' \leq n_A} a_{A,A_i,A_i'} + \sum_{1 \leq j < j' \leq n_B} a_{B,B_j,B_j'} \leq a_0, \tag{3}
\]

the triangular elimination is defined as follows:

\[
\sum_{1 \leq i \leq n_A} a_{X,A_i}x_{X,A_i} + \sum_{1 \leq j \leq n_B} a_{X,B_j}x_{X,B_j} + \sum_{1 \leq i \leq n_A, 1 \leq j \leq n_B} a_{A,B_i}x_{A_i,B_j}
\]
This is an inequality for \( \text{CUT}_0 \) Two-party Bell inequalities via triangular elimination or \( - \) original inequality, which is a triangle inequality. The terms elimination (see figure 2) of the pentagonal inequality (2). This inequality has two new nodes \( A \) one subsystem and are not allowed. Therefore, we eliminate these terms by adding two inequality a \( \text{CUT} \) inequality becomes the \( I \) terms of the vector \( x \) elimination gives another proof of the tightness of the inequality. The triangular elimination of a facet is a facet.

In this subsection, we show the main theorem of this paper: under a very mild condition, the triangular elimination of a facet is a facet.

2.3. Triangular elimination and facet

In this subsection, we show the main theorem of this paper: under a very mild condition, the triangular elimination of a facet is a facet.

Theorem 2.1 The triangular elimination of a facet inequality \( \mathbf{a}^T \mathbf{x} \leq 0 \) of \( \text{CUT}^\Box(K_1,m_A,m_B) \) is facet inducing for \( \text{CUT}^\Box(K_1,m_A,m_B) \) except for the cases that the inequality \( \mathbf{a}^T \mathbf{x} \leq 0 \) is a triangle inequality labelled as either \( -x_{A_1A_2} - x_{A_1A_2} - x_{A_2A_2} \leq 0 \) or \( -x_{A_1A_2} - x_{A_1A_2} - x_{A_2A_2} \leq 0 \).

For example, as we saw, the CHSH inequality is the triangular elimination of Bell’s original inequality, which is a triangle inequality. The \( I_{3322} \) inequality, found by Pitowsky.

![Figure 2](image-url)
and Svozil [11] and Collins and Gisin [12], is the triangular elimination of a pentagonal inequality.

Proof.

Let \( r_F \) be the set of cut vectors on the hyperplane \( \alpha^T x' = 0 \): \( r_F = \{ \delta'(S') \mid \alpha^T \delta' = 0, S' : \text{cut} \} \) for \( \text{CUT}^\square(K_{1,m_A,m_B}) \). We prove the theorem by exhibiting a linearly independent subset of these cut vectors with cardinality \( m_A + m_B + m_A m_B - 1 \).

In the following proof, we consider a simple case of \( n_B = 1 \). We consider the other case later. In addition, we assume that \( a_{A_i,A_j'} \leq 0 \) for all eliminated terms. For the other cases, the proof is similar.

By the above restriction, \( m_A + m_B + m_A m_B - 1 = (n_A^2 + 3n_A)/2 \).

A sketch of proof is as follows: first, we restrict \( r_F \) and decompose the whole space of \( \text{CUT}^\square(K_{1,m_A,m_B}) \) into two subspaces. For each subspace, we can pick a set of cut vectors which are linearly independent in that subspace. Next, we show that these sets of cut vectors are linearly independent in the whole space.

First, let the subset \( r'_F \) of \( r_F \) be those cuts such that, for any \( 1 \leq i < i' \leq n_A \), two vertices \( A_i \) and \( B_{A_i,A_i'} \) are assigned the same value. Then, consider the intersection of the space spanned by \( \delta'(S') \in r'_F \) and the subspace

\[
W = \left\{ (x_{X_{A_i}, x_{X_{B_j}, x_{A_i,B_j}, x_{A_i,B_i,A_i'}}})_{1 \leq i < i' \leq n_A, 1 \leq j \leq n_B} \right\}.
\]

From the definition of \( r'_F \), \( \delta'_{A_i',B_{A_i,A_i'}}(S') = 0 \). Therefore,

\[
\alpha^T \delta'(S') = \sum_{1 \leq i \leq n_A} a_{X_{A_i}} \delta_{X_{A_i}}(S') + \sum_{1 \leq j \leq n_B} a_{X_{B_j}} \delta_{X_{B_j}}(S') + \sum_{1 \leq i < i' \leq n_A} a_{A_i,A_i'} \delta_{A_i,A_i'}(S') = 0.
\]

This means that the intersection of space spanned by \( \delta'(S') \in r'_F \) and \( W \) is equivalent to the space spanned by the cut vectors \( r_f = \{ \delta(S) \mid \alpha^T \delta = 0, S : \text{cut} \} \) of \( \text{CUT}^\square(K_{n_A + n_B}) \). Therefore, from the assumption that the inequality \( \alpha^T x \leq 0 \) is facet supporting, we can pick \((n_A^2 + 3n_A)/2\) linearly independent cut vectors and transform the cut vectors of \( \text{CUT}^\square(K_{n_A + n_B}) \) into corresponding cut vectors of \( \text{CUT}^\square(K_{m_A,m_B}) \). Let this set of linearly independent cut vectors be \( D_0 \).

The remaining subspace of \( \text{CUT}^\square(K_{m_A,m_B}) \) is

\[
V = \bigoplus_{i < i'} V_{A_i,A_i'} = \bigoplus_{i < i'} \left\{ (x_{X_{B_{A_i,A_i'}}, x_{A_i,B_i,A_i'}, x_{A_i,B_i,A_i'}})_{i'' \neq i, i'} \right\}.
\]

for each eliminated term \( A_i A_i' \), \( 1 \leq i < i' \leq n_A \).

Instead of \( V \), we consider the space

\[
V' = \bigoplus_{i < i'} V_{A_i,A_i'} = \bigoplus_{i < i'} \left\{ (x_{X_{B_{A_i,A_i'}}, x_{A_i,B_i,A_i'}, x_{A_i,B_i,A_i'}})_{i'' \neq i, i'} \right\}
\]

where

\[
x_{\alpha_{A_i,A_i',A_i''}} = \begin{cases} 
\frac{1}{2}(x_{A_i',B_{A_i,A_i'}} - x_{A_i,B_{A_i,A_i'}} - x_{A_i',B_{A_i',A_i'}} + 3x_{A_i',B_{A_i',A_i'}}) (i'' < i') \\
\frac{1}{2}(x_{A_i',B_{A_i,A_i'}} - x_{A_i',B_{A_i',A_i'}} - x_{A_i,B_{A_i',A_i'}} - x_{A_i',B_{A_i',A_i'}}) (i'' < i')
\end{cases}
\]

for each eliminated term \( A_i A_i' \), \( 1 \leq i < i' \leq n_A \).


in the following. Since the transform $V$ to $V'$ is linear, the linear independence of vectors in $V$ is equivalent to that in $V'$.

Then, we consider the subset $r''_{F,A_iA'_i}$ of $r_F$ for each $A_iA'_i$ restricted as follows: $A_i$ must be assigned 0 and both $B'_{A_iA'_i}$ and $A'_i$ must be assigned 1. For other terms $A_iA'_iA''_iA'''_i (1 \leq i'' < i''' \leq n_A)$, vertices $A''_iA'''_i$ and $B'_{A_iA'_iA''_iA'''_i}$ must be assigned the same value. From that restriction, the equations

$$
\delta'_{XB'_{A_iA'_i}}(S''') - \delta'_{A_iA'_i}(S''') = -\delta_{X,A_i}(S), \\
\delta'_{A_iA'_i}(S''') = 1, \\
\delta'_{A_iA'_i}(S''') = -\delta_{A_iA'_i}(S)
$$

hold for $\delta'(S''') \in r''_{F,A_iA'_i}$. This means that the intersection of the space spanned by $\delta'(S''')$ and the subspace $V'_{A_iA'_i}$ is equivalent to that of the space spanned by $\delta(S) \in r_f$ and the subspace

$$
U_{A_iA'_i} = \left\{ \left( x_{XA_i}^1, 1, x_{A_iA'_i} \right)^T | x_{A_iA'_i} \neq 0 \right\}.
$$

Now, because $r_f$ is on the hyperplane $a^T x = 0$, the above intersection has dimension $n_A$ or $n_A - 1$. However, from the condition on the inequality $a^T x \leq 0$, the space spanned by $r_f$ is not parallel to $U_{A_iA'_i}$. Therefore, the dimension is $n_A$ and we can extract $n_A$ cut vectors which are linearly independent in the subspace $V'$ using the cut vectors from $r_f$. Let this set of cut vectors be $D_{A_iA'_i}$.

Finally, we show that $D_0 \cup \bigcup_{1 \leq i < i' \leq n_A} D_{A_iA'_i}$ is a linearly independent set of cut vectors. Suppose that the linear combination

$$
\sum_{\delta'(S'') \in D_0} \kappa_{S''} \delta'^T(S'') + \sum_{1 \leq i < i' \leq n_A} \sum_{\delta'(S'') \in D_{A_iA'_i}} \lambda_{S''}^{A_iA'_i} \delta'^T(S'') = 0
$$

holds. Consider the subspace $V'_{A_iA'_i}$ of the above linear combination. From the construction, for $D_0$ and $D_{A_iA'_i}$, the elements of cut vectors in that subspace are all zero. Therefore, for the linear combination to hold, it must be that

$$
\sum_{\delta'(S'') \in D_{A_iA'_i}} \lambda_{S''}^{A_iA'_i} \delta'^T(S'') = 0.
$$

However, the linear independence of $D_{A_iA'_i}$ means that the coefficients are all zero. By repeating this argument, we can conclude that the coefficient $\lambda_{S''}^{A_iA'_i}$ must be zero. So, from the linear independence of $D_0$, the coefficients $\kappa_{S''}$ are also zero. This completes the proof for the case $n_B = 1$.

Now we describe the outline of the proof for general case. The idea of the proof is to perform triangular elimination in two steps: eliminate the edges $A_iA'_i$ for $1 \leq i < i' \leq n_A$ in one step and then the edges $B_jB'_j$ for $1 \leq j < j' \leq n_B$ in the other. To do this, we need the notion of the cut polytope $\text{CUT}(G) \subseteq \mathbb{R}^E$ of a general graph $G = (V, E)$, which is obtained from the cut polytope of the complete graph on node set $V$ by removing the coordinates corresponding to the edges missing in $E$. In particular we consider the cut polytopes of the following two intermediate graphs: the graph $G_1(n_A, n_B)$ obtained from $K_{1,n_A,m_B}$ by adding edges $B_jB'_j$ and $B_jB'_{A_iA'_i}$, and the graph $G_2(n_A, n_B)$ obtained from $K_{1,m_A,m_B}$ by adding edges $B_jB'_{A_iA'_i}$. 
Two-party Bell inequalities via triangular elimination

The next lemma is a basic fact from polytope theory (see Lemma 26.5.2 (ii) in [7]; though the statement there restricts \( G \) to be a complete graph, that restriction is not necessary).

**Lemma 2.2** Let \( G \) be a graph and \( G' \) be a subgraph of \( G \). If \( \mathbf{a}^T \mathbf{x} \leq 0 \) is facet inducing for \( \text{CUT}^\square(G) \) and \( a_e = 0 \) for all edges \( e \) belonging to \( G \) but not to \( G' \), then \( \mathbf{a}^T \mathbf{x} \leq 0 \) is facet inducing also for \( \text{CUT}^\square(G') \).

The inequality after the first step of triangular elimination is as follows:

\[
\sum_{1 \leq i \leq n_A} a_{XA_i} x_{XA_i} + \sum_{1 \leq j \leq n_B} a_{XB_j} x_{XB_j} + \sum_{1 \leq i \leq n_A, 1 \leq j \leq n_B} a_{A_iB_j} x_{A_iB_j} \\
+ \sum_{1 \leq i < i' \leq n_A} (a_{A_iA_{i'}} x_{A_iA_{i'}} - |a_{A_iA_{i'}} x_{A_iA_{i'}}|) + \sum_{1 \leq j < j' \leq n_B} a_{B_jB_{j'}} x_{B_jB_{j'}} \leq a_0. \tag{5}
\]

For the case \( n_B = 1 \), the inequality \( (5) \) is exactly the same as \( (4) \). We proved above for the case \( n_B = 1 \) that the inequality \( (5) \) is facet inducing for \( \text{CUT}^\square(K_{1,n_A,m_B}) \). Except for when the original inequality is a triangle inequality, we can extend this argument to prove that the inequality \( (5) \) is facet inducing also for \( \text{CUT}^\square(G_1(n_A, n_B)) \). This can be generalized for the case \( n_B > 1 \): the inequality \( (5) \) is facet inducing for \( \text{CUT}^\square(G_1(n_A, n_B)) \). Then we can repeat a similar argument to prove the final inequality \( (4) \) is facet inducing for \( \text{CUT}^\square(G_2(n_A, n_B)) \). Since \( G_2(n_A, n_B) \) is a supergraph of the desired graph \( K_{1,m_A,m_B} \), the inequality \( (4) \) is facet inducing also for \( \text{CUT}^\square(K_{1,m_A,m_B}) \) from Lemma 2.2.

2.4. Triangular elimination and symmetry

Many Bell inequalities are equivalent to each other due to the arbitrariness in the labelling of the party, observable and value identifiers. This corresponds to symmetries of the underlying polytope. We consider ways of representing nonequivalent Bell inequalities in this section.

The nonequivalence of Bell inequalities can be translated into two questions about facet inequalities \( f \) and \( f' \) of a given cut polytope of a complete graph, and their triangular eliminations \( F \) and \( F' \), respectively:

(i) does the equivalence of \( f \) and \( f' \) imply the equivalence of \( F \) and \( F' \)?
(ii) does the equivalence of \( F \) and \( F' \) imply the equivalence of \( f \) and \( f' \)?

The answers are both affirmative if we define equivalence appropriately, so equivalence before triangular elimination is logically equivalent to equivalence after triangular elimination. This means that, for example, to enumerate the nonequivalent Bell inequalities, we need only enumerate the facet inequalities of the cut polytope of the complete graph up to symmetry by party, observable and value exchange.

In \( \text{CUT}^\square(K_{1,m_A,m_B}) \), the relabelling of all vertices of Alice to that of Bob and vice versa corresponds to a party exchange. On the other hand, the local relabelling of some vertices of Alice (or Bob) corresponds to an observable exchange. Thus by the
observable exchange of Alice represented by the permutation \( \sigma \) over \( \{A_1, \ldots, A_{m_A}\} \), an inequality \( a^T x \leq a_0 \) is transformed into \( a'^T x \leq a_0 \) where \( a'_{\sigma(A_i)V} = a_{A_iV} \) for any vertex \( V \).

In addition, there is an operation which corresponds to a value exchange of some observables, called a switching in the theory of cut polytopes. By the switching corresponding to the value exchange of an Alice’s observable \( A_i \), an inequality \( a^T x \leq a_0 \) is transformed into \( a'^T x \leq a_0 - \sum_V a_{A_iV} \) where \( a'_{A_iV} = -a_{A_iV} \), and \( a'_{A_iV} = a_{A_iV} \) for any \( i \neq i_0 \) and any vertex \( V \neq A_i \) (definitions for Bob’s exchange are similar).

It is well known, and easily shown, that by repeated application of the switching operation we may reduce the right hand side of any facet inequality to zero.

Let \( n_A \leq n_B \) and \( n = 1 + n_A + n_B \). Let \( f \) and \( f' \) be facets of \( \text{CUT}^\square_n \) where the \( n \) nodes of \( K_n \) is labelled by \( V = \{A_1, \ldots, A_{n_A}, B_1, \ldots, B_{n_B}, X\} \). The two facets \( f \) and \( f' \) are said to be equivalent and denoted \( f \sim f' \) if \( f \) can be transformed to \( f' \) by applying zero or more of the following operations: (1) (only applicable in the case \( A_i = B_j \)) swapping labels of nodes \( A_i \) and \( B_j \) for all \( 1 \leq i \leq n_A \), (2) relabelling the nodes within \( A_1, \ldots, A_{n_A} \), (3) relabelling the nodes within \( B_j, \ldots, B_{n_B} \), and (4) switching.\(^\dagger\)

Two facets \( F \) and \( F' \) of \( \text{CUT}^\square(K_{1,m_A,m_B}) \) are said to be equivalent and denoted \( F \sim F' \) if \( F \) can be transformed to \( F' \) by applying permutation which fixes node \( X \), switching, or both. This notion of equivalence of facets of \( \text{CUT}^\square(K_{1,m_A,m_B}) \) corresponds to equivalence of tight Bell inequalities up to party, observable and value exchange.

**Theorem 2.3** Let the triangular elimination of facet inequalities \( f \) and \( f' \) be \( F \) and \( F' \), respectively. Then, \( f \sim f' \iff F \sim F' \).

Proof. A sketch of the proof is as follows. Since the permutation and switching operations are commutative, it is sufficient to prove the proposition under each operation separately. Because the \( \Rightarrow \) direction is straightforward for both permutation and switching, we concentrate on the proof of \( \Leftarrow \) direction.

First, consider switching. Suppose \( F \) is obtained from a switching of \( F' \). The switching could involve either (i) a new observable introduced by the triangular elimination, or (ii) an observable which had a joint measurement term eliminated. Since a switching of type (i) has no effect on \( f \) and \( f' \), we need only consider type (ii). We can view the triangular elimination of the term \( A_iA_{i'} \) as addition of triangle inequality \( x_{A_iA_{i'}} - x_{A_iB_{i'}A_{i'}} - x_{A_{i'}B_{i'}A_{i'}} \leq 0 \) or its switching equivalent inequality \( -x_{A_iA_{i'}} + x_{A_iB_{i'}A_{i'}} + x_{A_{i'}B_{i'}A_{i'}} \leq 0 \) according to the sign of the coefficient \( a_{A_iA_{i'}} \). Thus, if \( F \) is switching of \( F' \) of vertices \( A_i \) and \( B_{i'}A_{i'} \), then \( f \) is switching of \( f' \) of \( A_i \).

Next, consider the permutation corresponding to an observable exchange. Observe that for any vertex \( A_i (1 \leq i \leq n_A) \), triangular elimination does not change the number of terms \( A_iV \) with non-zero coefficient. In addition, it can be shown that for any facet inequality \( f \) of the cut polytope of the complete graph other than the triangle inequality,

\( \dagger \) The two facets \( f \) and \( f' \) are said to be equivalent and denoted \( f \sim f' \) if \( f \) can be transformed to \( f' \) by permutation and switching where the permutation \( \tau \) on \( V \) satisfies: (1) \( \tau(X) = X \) and (2) \( \tau \) either fixes two sets \( \{A_1, \ldots, A_{n_A}\} \) and \( \{B_1, \ldots, B_{n_B}\} \) setwise or (in the case \( n_A = n_B \)) swaps these two sets.
Table 1. The number of inequivalent facets of CUT$^\Box_n$ and the number of inequivalent tight Bell inequalities obtained as the triangular eliminations of the facets of CUT$^\Box_n$. Asterisk (*) indicates the value is a lower bound.

| n  | Facets of CUT$^\Box_n$ | Tight Bell ineqs. via triangular elimination |
|----|------------------------|---------------------------------------------|
| 3  | 1                      | 2                                           |
| 4  | 1                      | 2                                           |
| 5  | 2                      | 8                                           |
| 6  | 3                      | 22                                          |
| 7  | 11                     | 323                                         |
| 8  | 147*                   | 40,399*                                     |
| 9  | 164,506*               | 201,374,783*                                |

there is no vertex satisfying the following conditions: (a) there are exactly two terms $A_iV$, with non-zero coefficients, and (b) for those non-zero coefficients $a_{A_iW}$ and $a_{A_iU}$, $|a_{A_iW}| = |a_{A_iU}|$. This means that if $F \sim F'$, then the corresponding permutation $\sigma$ is always in the following form: for permutations $\tau_A$ over $\{A_1, \ldots, A_{n_A}\}$ and $\tau_B$ over $\{B_1, \ldots, B_{n_B}\}$, $\sigma(A_i) = \tau_A(A_i)$ and $\sigma(B_{A_iA_i'}) = B_{\tau_A(A_i)\tau_A(A_i')}$.

Therefore, $f$ and $f'$ are equivalent under the permutations $\tau_A$ and $\tau_B$.

2.5. Computational results

By Theorem 2.3, we can compute the number of the classes of facets of CUT$^\Box_n(K_{1,m_A,m_B})$ of the same type obtained by applying triangular elimination to non-triangular facets of CUT$^\Box_n$. We consulted De Simone, Deza and Laurent [25] for the H-representation of CUT$_7$, and the “conjectured complete description” of CUT$_8$ and the “description possibly complete” of CUT$_9$ in SMAPO [20]. The result is summarized in Table 1. For $n = 8$ and 9, the number is a lower bound since the known list of the facets of CUT$^\Box_n$ is not proved to be complete. A program to generate Bell inequalities from the list in [20] are available from an author’s webpage at [http://www-imai.is.s.u-tokyo.ac.jp/~tsuyoshi/bell/](http://www-imai.is.s.u-tokyo.ac.jp/~tsuyoshi/bell/) The list of the generated Bell inequalities for $n = 8$ is also available.

3. Families of Bell inequalities

While a large list of individual tight Bell inequalities is useful in some applications, a few formulas which give many different Bell inequalities for different values of parameters are easier to treat theoretically. The cut polytope of the complete graph has several classes of valid inequalities whose subclasses of facet-inducing inequalities are partially known (see [7] Chapters 27–30 for details). In this section, we apply triangular elimination to two typical examples of such classes to obtain two general formulas for Bell inequalities. In addition, we prove sufficient conditions for these formulas to give a tight Bell inequality.
In this section, terms of the left hand side of an inequality are arrayed in the format introduced by Collins and Gisin [12], each row corresponds to coefficients of each observable of party $A$ and each column corresponds to that of party $B$. Because of switching equivalence, we can assume that the right hand side of inequality are always zero. The example of the CHSH $-q_{A_1} - q_{B_1} + q_{A_1B_1} + q_{A_1B_2} + q_{A_2B_1} - q_{A_2B_2} \leq 0$ is arrayed in the form as follows:

$$
\begin{pmatrix}
  -1 & 0 \\
  1 & 1 \\
  0 & -1
\end{pmatrix} \leq 0.
$$

### 3.1. Bell inequalities derived from hypermetric inequalities

Hypermetric inequalities are a fundamental class of inequalities valid for the cut polytope of the complete graph. Here we derive a new family of Bell inequalities by applying triangular elimination to the hypermetric inequalities. A special case of this family, namely the triangular eliminated pure hypermetric inequality, contains four previously known Bell inequalities: the trivial inequalities like $q_{A_i} \leq 1$, the well known CHSH inequality found by Clauser, Horne, Shimony and Holt [8], the inequality named $I_{3322}$ by Collins and Gisin [12], originally found by Pitowsky and Svozil [11], and the $I_{3322}^2$ inequality by Collins and Gisin [12].

Let $s$ and $t$ be nonnegative integers and $b_{A_1}, \ldots, b_{A_s}, b_{B_1}, \ldots, b_{B_t}$ be integers. We define $b_X = 1 - \sum_{i=1}^s b_{A_i} - \sum_{j=1}^t b_{B_j}$. Then it is known that $\sum_{uv} b_u b_v x_{uv} \leq 0$, where the sum is taken over the $s+t+1$ edges of the complete graph on nodes $X, A_1, \ldots, A_s, B_1, \ldots, B_t$, is valid for $\text{CUT}_{s+t+1}$. This inequality is called the hypermetric inequality defined by the weight vector $b = (b_X, b_{A_1}, \ldots, b_{A_s}, b_{B_1}, \ldots, b_{B_t})$.

We apply triangular elimination to this hypermetric inequality. Let $s_+$ and $t_+$ be the number of positive entries of the form $b_{A_i}$ and of the form $b_{B_j}$, respectively. Without loss of generality, we assume that $b_{A_1}, \ldots, b_{A_{s_+}}, b_{B_1}, \ldots, b_{B_{t_+}} > 0$, and $b_{A_{s_++1}}, \ldots, b_{A_s}, b_{B_{t_++1}}, \ldots, b_{B_t} \leq 0$. By assigning $a_{uv} = b_u b_v$ in the formula (4), the Bell inequality obtained by triangular elimination is:

$$
\begin{align*}
\sum_{i=1}^{s_+} b_{A_i} \left( \frac{1 - b_{A_i}}{2} - \sum_{i' = s_+ + 1}^{i-1} b_{A_{i'}} \right) q_{A_i} + \sum_{i=s_+ + 1}^{s} b_{A_i} \left( \frac{1 - b_{A_i}}{2} - \sum_{i' = s_+ + 1}^{i-1} b_{A_{i'}} \right) q_{A_i} \\
+ \sum_{j=1}^{t_+} b_{B_j} \left( \frac{1 - b_{B_j}}{2} - \sum_{j' = 1}^{j-1} b_{B_{j'}} \right) q_{B_j} + \sum_{j=t_+ + 1}^{t} b_{B_j} \left( \frac{1 - b_{B_j}}{2} - \sum_{j' = t_+ + 1}^{j-1} b_{B_{j'}} \right) q_{B_j} \\
+ \sum_{j=1}^{t_+} \sum_{j' = t_+ + 1}^{t} b_{B_j} b_{B_{j'}} q_{A_j A_{j'}} + \sum_{i=1}^{s_+} \sum_{i' = s_+ + 1}^{i} b_{A_i} b_{A_{i'}} q_{A_i A_{i'}} - \sum_{i=1}^{s} \sum_{j=1}^{t} b_{A_i} b_{B_j} q_{A_i B_j} \\
- \sum_{1 \leq i < i' \leq s} b_{A_i} b_{A_{i'}} q_{A_i A_{i'}} + \sum_{1 \leq i < i' \leq s} \left| b_{A_i} b_{A_{i'}} \right| q_{A_i A_{i'}} \\
- \sum_{1 \leq j < j' \leq t} b_{B_j} b_{B_{j'}} q_{A_j A_{j'}} + \sum_{1 \leq j < j' \leq t} \left| b_{B_j} b_{B_{j'}} \right| q_{A_j A_{j'}} \leq 0.
\end{align*}
$$
Though the formula (6) represents a Bell inequality for any choice of weight vector $b$, this Bell inequality is not always tight. Many sufficient conditions for a hypermetric inequality to be facet-inducing are known in study of cut polytopes. By Theorem 2.1, these sufficient conditions give sufficient conditions for the Bell inequality (6) to be tight. The sufficient conditions stated in [7, Corollary 27.2.5] give the following theorem.

**Theorem 3.1** The Bell inequality (6) is tight if one of the following conditions is satisfied.

(i) For some $l > 1$, the integers $b_{A_1}, \ldots, b_{A_s}, b_{B_1}, \ldots, b_{B_t}$ and $b_X$ contain $l + 1$ entries equal to 1 and $l$ entries equal to $-1$, and the other entries (if any) are equal to 0.

(ii) At least 3 and at most $n - 3$ entries in $b_{A_1}, \ldots, b_{A_s}, b_{B_1}, \ldots, b_{B_t}$ and $b_X$ are positive, and all the other entries are equal to $-1$.

Now we consider some concrete cases when the formula (6) represents a tight Bell inequality. If we let $s + t = 2l$, $s \leq l$, $l > 1$, $b_{A_1} = \ldots = b_{A_s} = b_{B_1} = \ldots = b_{B_{l-s}} = 1$, and $b_{B_{l-s+1}} = \ldots = b_{B_t} = -1$, then $b_X = 1$ and by case (i) of Theorem 3.1, the Bell inequality (6) is tight. In this case, the Bell inequality (6) is in the following form.

\[
- \sum_{i=1}^{s} (i - 1) q_{A_i} - \sum_{j=1}^{l-s} \sum_{j'=l-s+1}^{t} q_{A_{ij}'} - \sum_{j=1}^{l-s} (j - 1) q_{B_j} - \sum_{j=l-s+1}^{t} (j - (l-s)) q_{B_j} \\
- \sum_{i=1}^{s} \sum_{j=1}^{l-s} q_{A_i, B_j} + \sum_{i=1}^{s} \sum_{j=l-s+1}^{t} q_{A_i, B_j} - \sum_{1 \leq i < i' \leq s} q_{A_i B_{i'}} + \sum_{1 \leq i < i' \leq s} q_{A_i B_{i'}} \\
- \sum_{1 \leq j < j' \leq l-s} q_{A_{ij}' B_j} - \sum_{l-s+1 \leq j < j' \leq t} q_{A_{ij}' B_j} + \sum_{j=1}^{l-s} \sum_{j'=l-s+1}^{t} q_{A_{ij}' B_{j'}} + \sum_{1 \leq j < j' \leq t} q_{A_{ij}' B_{j'}} \leq 0. \tag{7}
\]

Examples of tight Bell inequality in the form (7) are $I_{3322}^1$ and $I_{3422}^2$ inequalities [12].

In case of $l = 1$, Theorem 3.1 does not guarantee that the Bell inequality (7) is tight. However, in cases of $(l, s, t) = (1, 1, 1)$ and $(1, 1, 2)$, the Bell inequality (7) becomes trivial and CHSH inequalities, respectively, both of which are tight.

Letting $(l, s, t) = (2, 2, 2)$ in (7) gives:

\[- q_{A_1} - q_{B_1} - 2q_{B_2} + q_{A_1 B_1} + q_{A_1 B_2} + q_{A_2 B_1} + q_{A_2 B_2} - q_{A_1 B_{12}} + q_{A_1 B_{12}} - q_{A_{12} B_1} + q_{A_{12} B_2} \leq 0.\tag{8}\]

Following the notation in [12], we write the inequality (8) by arraying its coefficients:

\[
\begin{pmatrix}
(A_2) & (A_1) & (A_{12}) \\
(B_2) & -1 & 0 & 0 \\
(B_1) & -2 & 1 & 1 & 1 \\
(B_{12}) & -1 & 1 & 1 & -1 \\
(B_{12}') & 0 & 1 & -1 & 0
\end{pmatrix}
\leq 0.
\]

Now it is clear that the Bell inequality (8) is $I_{3322}^1$ inequality.
Letting \((l, s, t) = (2, 1, 3)\) in (7) gives:

\[
\begin{pmatrix}
(A_1) & 0 & 1 & 1 & -1 \\
(A'_{13}) & -1 & 0 & 1 & 1 \\
(A'_{12}) & -1 & 1 & 0 & 1 \\
(A'_{23}) & 0 & -1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
(B_2) & (B_3) & (B_1) \\
-1 & -2 & 0
\end{pmatrix} \leq 0. \tag{9}
\]

After exchanging the two values 1 and 0 of the observable \(A_1\), and doing the same to the two values of the observable \(B_3\), the Bell inequality (9) becomes:

\[
\begin{pmatrix}
(A_1) & -1 & -1 & 1 & 1 \\
(A'_{13}) & 0 & 0 & -1 & 1 \\
(A'_{12}) & -1 & 1 & 0 & 1 \\
(A'_{23}) & 1 & -1 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
(B_2) & (B_3) & (B_1) \\
0 & 1 & -1
\end{pmatrix} \leq 0,
\]

which is \(I^2_{3422}\) inequality \([12]\). This means that the Bell inequality (9) is equivalent to \(I^2_{3422}\) inequality.

3.2. Bell inequalities derived from pure clique-web inequalities

Clique-web inequalities \([7, \text{Chapter 29}]\) are generalization of hypermetric inequalities. One of the important subclasses of clique-web inequalities are the pure clique-web inequalities, which are always facet-inducing. Here we introduce an example of Bell inequalities derived from some pure clique-web inequalities.

For nonnegative integers \(s, t\) and \(r\) with \(s \geq t \geq 2\) and \(s - t = 2r\), we consider the pure clique-web inequality with parameters \(n = s + t + 1, p = s + 1, q = t\) and \(r\). After relabelling the \(n\) vertices of \(K_n\) by \(A_1, \ldots, A_s, X, B_1, \ldots, B_t\) in this order, the Bell inequality (4) corresponding to the clique-web inequality is:

\[
- \sum_{i=r+1}^{s-r} (i - r - 1)q_{A_i} - 2t \sum_{i=s-r+1}^{s} q_{A_i} - \sum_{j=1}^{t} (j - r)q_{B_j} + \sum_{i=1}^{s} \sum_{j=1}^{t} q_{A_i}q_{B_j} \\
+ \sum_{1 \leq i' \leq i, r+1 \leq j' \leq s-r} (-q_{A_i'}q_{B_{j'}}) + \sum_{1 \leq j < j' \leq t} (-q_{A_{j'}}q_{B_j} + q_{A_{j'}}q_{B_{j'}}) \leq 0. \tag{10}
\]

The next theorem is a direct consequence of Theorem 2.1.

**Theorem 3.2** For any nonnegative integers \(s, t\) and \(r\) with \(s \geq t \geq 2\) and \(s - t = 2r\), the Bell inequality (10) is tight.
3.3. Inclusion relation

Collins and Gisin [12] pointed out that the following $I_{3322}$ inequality becomes the CHSH inequality if we fix two measurements $A_3$ and $B_1$ to a deterministic measurement whose result is always 0.

$$I_{3322}:egin{pmatrix} (A_1) & (A_2) & (A_3) \\ -1 & 0 & 0 \\ (B_1) & -2 & 1 & 1 & 1 \\ (B_2) & -1 & 1 & 1 & -1 \\ (B_3) & 0 & 1 & -1 & 0 \end{pmatrix} \leq 0,$$

$$\text{CHSH}:egin{pmatrix} (A_1) & (A_2) \\ -1 & 0 \\ (B_2) & -1 & 1 & 1 \\ (B_3) & 0 & 1 & -1 \end{pmatrix} \leq 0.$$

As stated in [12], this fact implies the CHSH inequality is irrelevant if the $I_{3322}$ inequality is given. In other words, if a quantum state satisfies the $I_{3322}$ inequality with every set of measurements, then it also satisfies the CHSH inequality with every set of measurements.

We generalize this argument and define inclusion relation between two Bell inequalities: A Bell inequality $a^Tq \leq 0$ includes another Bell inequality $b^Tq \leq 0$ if we can obtain the inequality $b^Tq \leq 0$ by fixing some measurements in the inequality $a^Tq \leq 0$ to deterministic ones.

We do not know whether all the Bell inequalities (except positive probability) include the CHSH inequality. However, we can prove that many Bell inequalities represented by (6) or (10) include the CHSH inequality.

**Theorem 3.3** If $b_{A_1} = b_{A_2} = 1$ and $b_{B_{t+1}} = -1$, then the Bell inequality represented by (6) contains the CHSH inequality.

**Proof.** The Bell inequality (6) contains $s + \frac{t}{2}$ observables of Alice and $t + \frac{s}{2}$ observables of Bob. By fixing all but 4 observables $A_1, A_2, B_{t+1}$ and $B'_{l_2}$ to the one whose value is always 0, we obtain the following CHSH inequality: $-q_{A_2} - q_{B_{t+1}} + q_{A_1B_{t+1}} + q_{A_2B_{t+1}} - q_{A_1B'_{l_2}} + q_{A_2B'_{l_2}} \leq 0$.

**Theorem 3.4** All the Bell inequalities in the form (10) include the CHSH inequality.

**Proof.** By fixing all but 4 observables $A_{r+1}, A_{r+2}, B_{r+1}$ and $B'_{r+1,r+2}$ to the one whose value is always 0, the Bell inequality (10) becomes the following CHSH inequality: $-q_{A_{r+2}} - q_{B_{r+1}} + q_{A_{r+1}B_{r+1}} + q_{A_{r+2}B_{r+1}} - q_{A_{r+1}B'_{r+1,r+2}} + q_{A_{r+2}B'_{r+1,r+2}} \leq 0$.

3.4. Relationship between $I_{mm22}$ and triangular eliminated Bell inequality

Collins and Gisin [12] proposed a family of tight Bell inequalities obtained by the extension of CHSH and $I_{3322}$ as $I_{mm22}$ family, and conjectured that $I_{mm22}$ is always facet supporting (they also confirmed that for $m \leq 7$, $I_{mm22}$ is actually facet supporting by
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computation). Therefore, whether their $I_{mm22}$ can be obtained by triangular elimination
of some facet class of CUT$\square (K_n)$ is an interesting question.

The $I_{mm22}$ family has the structure as follows:

\[
\begin{pmatrix}
-1 & 0 & \ldots & 0 & 0 & 0 \\
-(m-1) & 1 & 1 & \ldots & 1 & 1 \\
-(m-2) & 1 & 1 & \ldots & 1 & -1 \\
-(m-3) & 1 & 1 & \ldots & 1 & -1 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
-1 & 1 & 1 & -1 & 0 & \ldots & 0 \\
0 & 1 & -1 & 0 & 0 & \ldots & 0
\end{pmatrix} \leq 0.
\]

From its structure, it is straightforward that if $I_{mm22}$ can be obtained by triangular
elimination of some facet class of CUT$\square (K_n)$, then only $A_m$ and $B_m$ are new vertices
introduced by triangular elimination, since the other vertices have degree more than 2. For $m = 2, 3, 4$, the $I_{mm22}$ inequality is the triangular elimination of the triangle,
pentagon and Grishukhin inequality
\[
\sum_{1 \leq i < j \leq 4} x_{ij} + x_{56} + x_{57} - x_{67} - x_{16} - x_{36} - x_{27} - x_{47} - 2 \sum_{1 \leq i \leq 4} x_{i5} \leq 0,
\]
respectively. In general, $I_{mm22}$ inequality is the triangular elimination of a facet-inducing inequality of CUT$\square (K_n)$ and it is tight [26].

3.5. Known tight Bell inequalities other than the triangular elimination of CUT$\square (K_n)$

Since we have obtained a large number of tight Bell inequalities by triangular elimination
of CUT$\square (K_n)$, the next question is whether they are complete i.e., whether all families
and their equivalents form the whole set of facets of CUT$\square (K_1^{m_A}, m_B)$.

For the case $m_A = m_B = 3$, the answer is affirmative. Both Śliwa [27] and
Collins and Gisin [12] showed that there are only three kinds of inequivalent facets:
positive probabilities, CHSH and $I_{3322}$, corresponding to the triangle facet, the triangular
elimination of the triangle facet and the triangular elimination of the pentagonal facet
of CUT$\square (K_n)$, respectively.

On the other hand, in the case $m_A = 3$ and $m_B = 4$, the answer is negative.
Collins and Gisin enumerated all of the tight Bell inequalities and classified them into
6 families of equivalent inequalities [12]. While positive probabilities, CHSH, $I_{3322}$ and
$I_{3422}$ inequalities are either facets of CUT$\square (K_n)$ or their triangular eliminations, the
other two are not:

\[
I_{3422}^1 = \begin{pmatrix}
1 & 1 & -2 \\
1 & -1 & -1 & 1 \\
0 & -1 & 1 & 1 \\
0 & 1 & -1 & 1 \\
1 & -1 & -1 & -1
\end{pmatrix} \leq 2, \quad I_{3422}^3 = \begin{pmatrix}
1 & 0 & -1 \\
0 & -2 & 1 & 1 \\
0 & 0 & -1 & 1 \\
-1 & 1 & 1 \\
2 & -1 & -1 & -1
\end{pmatrix} \leq 2.
\]
4. Concluding remarks

We introduced triangular elimination to derive tight Bell inequalities from the facet inequalities of the cut polytope of the complete graph. Though it does not give the complete list of Bell inequalities, this method derives not only many individual tight Bell inequalities from individual known facet inequalities of cut polytope, but also several families of Bell inequalities. This gives a partial answer to the $N = K = 2$ case of the problem posed by Werner [2, Problem 1].

Gill poses the following problem in [2, Problem 26.B]: is there any Bell inequality that holds for all quantum states, other than the inequalities representing nonnegativity of probabilities? Theorems 3.3 and 3.4 give a partial answer to this problem. If a Bell inequality $a^T q \leq 0$ includes the CHSH inequality, then the Bell inequality $a^T q \leq 0$ is necessarily violated in any quantum states violating the CHSH inequality.

Further investigation of inclusion relation and families of Bell inequalities may be useful to understand the structure of Bell inequalities such as the answer to Gill’s problem.

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