THE STRUCTURE OF SURFACES AND THREEFOLDS MAPPING TO THE MODULI STACK OF CANONICALLY POLARIZED VARIETIES

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ABSTRACT. Generalizing the well-known Shafarevich hyperbolicity conjecture, it has been conjectured by Viehweg that a quasi-projective manifold that admits a generically finite morphism to the moduli stack of canonically polarized varieties is necessarily of log general type. Given a quasi-projective threefold $Y^o$ that admits a non-constant map to the moduli stack, we employ extension properties of logarithmic pluri-forms to establish a strong relationship between the moduli map and the minimal model program of $Y^o$: in all relevant cases the minimal model program leads to a fiber space whose fibration factors the moduli map. A much refined affirmative answer to Viehweg’s conjecture for families over threefolds follows as a corollary. For families over surfaces, the moduli map can be often be described quite explicitly. Slightly weaker results are obtained for families of varieties with trivial, or more generally semi-ample canonical bundle.

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1. INTRODUCTION AND MAIN RESULTS

1.A. Introduction. Let $Y^o$ be a quasi-projective manifold that admits a morphism $\mu : Y^o \to \mathcal{M}$ to the moduli stack of canonically polarized varieties. Generalizing the classical Shafarevich hyperbolicity conjecture [Sha63], Viehweg conjectured in [Vie01 6.3] that $Y^o$
is necessarily of log general type if \( \mu \) is generically finite. Equivalently, if \( f^o : X^o \to Y^o \) is a smooth family of canonically polarized varieties, then \( Y^o \) is of log general type if the variation of \( f^o \) is maximal, i.e., \( \text{Var}(f^o) = \dim Y^o \). We refer to \( \text{[KK08]} \) for the relevant notions, for detailed references, and for a brief history of the problem, but see also \( \text{[KS06]} \).

Viehweg’s conjecture was confirmed for 2-dimensional manifolds \( Y^o \) in \( \text{[KK08]} \) using explicit surface geometry. Here, we employ recent extension theorems for logarithmic forms to study families over threefolds. If \( \dim Y^o \leq 3 \), we establish a strong relation-ship between the moduli map \( \mu \) and the logarithmic minimal model program of \( Y^o \): in all relevant cases, any logarithmic minimal model program necessarily terminates with a fiber space whose fibration factors the moduli map. This allows us to prove a much refined version of Viehweg’s conjecture for families over surfaces and threefolds, and give a positive answer to the conjecture even for families of varieties with only semi-ample canonical bundle. If \( Y^o \) is a surface we recover the results of \( \text{[KK08]} \) in a more sophisticated manner. In fact, going far beyond those results we give a complete geometric description of the moduli map in those cases when the variation cannot be maximal.

The proof of our main result is rather conceptual and independent of the argumentation of \( \text{[KK08]} \) which essentially relied on combinatorial arguments for curve arrangements on surfaces and on Keel-McKernan’s solution to the Miyanishi conjecture in dimension 2, \( \text{[KMc99]} \). Many of the techniques introduced here generalize well to higher dimensions, most others at least conjecturally.

Throughout the present paper we work over the field of complex numbers.

1.B. Main results. The main results of the present paper are summarized in the following theorems which describe the geometry of families over threefolds under increasingly strong hypothesis.

**Theorem 1.1** (Viehweg conjecture for families over threefolds). Let \( f^o : X^o \to Y^o \) be a smooth projective family of varieties with semi-ample canonical bundle, over a quasi-projective manifold \( Y^o \) of dimension \( \dim Y^o \leq 3 \). If \( f^o \) has maximal variation, then \( Y^o \) is of log general type. In other words,

\[
\text{Var}(f^o) = \dim Y^o \Rightarrow \kappa(Y^o) = \dim Y^o.
\]

**Remark 1.1.1.** The definition of Kodaira dimension \( \kappa(Y^o) \) for quasi-projective manifolds is recalled in Notation \( \text{2.3} \) below.

For families of canonically polarized varieties, we can say much more. The following much stronger theorem gives an explicit geometric explanation of Theorem 1.1

**Theorem 1.2** (Relationship between the moduli map and the MMP). Let \( f^o : X^o \to Y^o \) be a smooth projective family of canonically polarized varieties, over a quasi-projective manifold \( Y^o \) of dimension \( \dim Y^o \leq 3 \). Let \( Y \) be a smooth compactification of \( Y^o \) such that \( D := Y \setminus Y^o \) is a divisor with simple normal crossings.

Then any run of the minimal model program of the pair \( (Y, D) \) will terminate in a Kodaira or Mori fiber space whose fibration factors the moduli map birationally.

**Remark 1.2.1.** If \( \kappa(Y^o) = 0 \) in the setup of Theorem 1.2 then any run of the minimal model program will terminate in a Kodaira fiber space that maps to a single point. Since this map to a point factors the moduli map birationally, Theorem 1.2 asserts that the family \( f^o \) is necessarily isotrivial if \( \kappa(Y^o) = 0 \).

**Remark 1.2.2.** Neither the compactification \( Y \) nor the minimal model program discussed in Theorem 1.2 is unique. When running the minimal model program, one often needs to choose the extremal ray that is to be contracted.
In the setup of Theorem 1.2, if \( \kappa(Y^\circ) \geq 0 \), then the minimal model program terminates in a Kodaira fiber space whose base has dimension \( \kappa(Y^\circ) \). The following refined version of Viehweg’s conjecture is therefore an immediate corollary of Theorem 1.2.

**Corollary 1.3** (Refined Viehweg conjecture for families over threefolds cf. [KK08, 1.6]). Let \( f^\circ : X^\circ \to Y^\circ \) be a smooth projective family of canonically polarized varieties, over a quasi-projective manifold \( Y^\circ \) of dimension \( \dim Y^\circ \leq 3 \). Then either

1. \( \kappa(Y^\circ) = -\infty \) and \( \text{Var}(f^\circ) < \dim Y^\circ \), or
2. \( \kappa(Y^\circ) \geq 0 \) and \( \text{Var}(f^\circ) \leq \kappa(Y^\circ) \).

As a further application of Theorem 1.2, we describe the family \( f^\circ : X^\circ \to Y^\circ \) explicitly if the base manifold \( Y^\circ \) is a surface and the variation is not maximal.

**Theorem 1.4** (Description of the family in case of \( \text{Var}(f^\circ) = 1 \)). Let \( f^\circ : X^\circ \to Y^\circ \) be a smooth projective family of canonically polarized varieties, over a quasi-projective manifold \( Y^\circ \) of dimension \( \dim Y^\circ = 2 \). If \( \kappa(Y^\circ) < 2 \) and \( \text{Var}(f^\circ) = 1 \), then one of the following holds.

1. \( \kappa(Y^\circ) = 1 \), and there exists an open set \( U \subseteq Y^\circ \) and a Cartesian diagram of one of the following two types,

\begin{align*}
\begin{array}{ccc}
U & \xrightarrow{\gamma} & U \\
\text{étale} & & \text{étale} \\
\text{étale} & & \text{étale} \\
V & \xrightarrow{\pi} & V \\
\text{elliptic fibration} & & \text{elliptic fibration} \\
\text{étale} & & \text{étale} \\
\end{array}
\end{align*}

such that \( f^\circ_U : X^\circ \times_U U \to U \) is the pull-back of a family over \( V \).

2. \( \kappa(Y^\circ) = -\infty \), and there exists an open set \( U \subseteq Y^\circ \) of the form \( U = V \times \mathbb{A}^1 \) such that \( X^\circ \mid_U \) is the pull-back of a family over \( V \).

In order to complete the description of families with non-maximal variation over two-dimensional bases we include the following well-known statement. However, we would like to point out that this is a much easier statement and follows by simple abstract arguments, cf. the proof of Lemma 7.4.

**Theorem 1.5** (Description of the family in case of \( \text{Var}(f^\circ) = 0 \)). Let \( f^\circ : X^\circ \to Y^\circ \) be a smooth projective family of canonically polarized varieties, over a quasi-projective manifold \( Y^\circ \). If \( \text{Var}(f^\circ) = 0 \), then there exists an open set \( U \subseteq Y^\circ \) such that \( X^\circ \mid_U \) is isotrivial and further exists a finite étale cover \( \tilde{U} \to U \) such that \( f^\circ_{\tilde{U}} : X^\circ \times_U \tilde{U} \to \tilde{U} \) is trivial.

1.C. **Outline of proof, outline of paper.** The main results of this paper are shown in Sections 8–10 where we consider the cases \( \kappa(Y^\circ) = -\infty \), \( \kappa(Y^\circ) = 0 \) and \( \kappa(Y^\circ) > 0 \) separately, the most difficult case being when \( \kappa(Y^\circ) = 0 \). To keep the proofs readable, we have chosen to present many of the more technical results separately in the preparatory Sections 2–7. These may be of some independent interest. The reader who is primarily interested in a broad outline of the argument will likely want to take the technicalities on faith and move directly to Sections 8–10 on the first reading.

Section 11 introduces notation used in the remainder of the present paper. In Section 5, we discuss certain classes of singularities that appear in the minimal model program and
recall the Bogomolov vanishing result for log canonical threefolds. The standard construction of the global index-one cover for good minimal models of Kodaira dimension zero is recalled and summarized in Section 4.

Viehweg and Zuo have shown that the base of a family of positive variation often carries an invertible sheaf of pluri-differentials whose Kodaira-Iitaka dimension is at least the variation of the family. These Viehweg-Zuo sheaves, which are central to our argumentation, are introduced and discussed in Section 5. The existence of a Viehweg-Zuo sheaf of positive Kodaira-Iitaka dimension has strong consequences for the geometry if the underlying space. These are discussed in Section 6. We end the preparatory part of the paper with Section 7 where we discuss how families \( f : X \to Y \) over a fibered base \( \pi : Y \to C \) that are isotrivial over \( \pi \)-fibers often come from a family over \( C \), at least after passing to an étale cover.

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PART I. TECHNIQUES

2. NOTATION AND CONVENTIONS

2.A. Reflexive tensor operations. When dealing with sheaves that are not necessarily locally free, we frequently use square brackets to indicate taking the reflexive hull.

**Notation 2.1 (Reflexive tensor product).** Let \( Z \) be a normal variety and \( \mathcal{A} \) a coherent sheaf of \( \mathcal{O}_Z \)-modules. Given a number \( n \in \mathbb{N} \), set \( \mathcal{A}^{[n]} := (\mathcal{A}^{\otimes n})^{**} \). If \( \mathcal{A} \) is reflexive of rank one, we say that \( \mathcal{A} \) is \( \mathbb{Q} \)-Cartier if there exists a number \( n \) such that \( \mathcal{A}^{[n]} \) is invertible.

We will later discuss the Kodaira dimension of singular pairs and the Kodaira-Iitaka dimension of reflexive sheaves on normal spaces. Since this is perhaps not quite standard, we recall the definition here.

**Notation 2.2 (Kodaira-Iitaka dimension of a sheaf).** Let \( Z \) be a normal projective variety and \( \mathcal{A} \) a reflexive sheaf of rank one on \( Z \). If \( h^0(Z, \mathcal{A}^{[n]}) = 0 \) for all \( n \in \mathbb{N} \), then we say that \( \mathcal{A} \) has Kodaira-Iitaka dimension \( \kappa(\mathcal{A}) := -\infty \). Otherwise, set

\[
M := \{ n \in \mathbb{N} \mid h^0(Z, \mathcal{A}^{[n]}) > 0 \},
\]

recall that the restriction of \( \mathcal{A} \) to the smooth locus of \( Z \) is locally free and consider the natural rational mapping

\[
\phi_n : Z \dashrightarrow \mathbb{P}(H^0(Z, \mathcal{A}^{[n]}))^* \quad \text{for each} \ n \in M.
\]

The Kodaira-Iitaka dimension of \( \mathcal{A} \) is then defined as

\[
\kappa(\mathcal{A}) := \max_{n \in M} \left( \dim \phi_n(Z) \right).
\]

**Notation 2.3 (Kodaira dimension of a quasi-projective variety).** If \( Z^\circ \) is a quasi-projective manifold and \( Z \) a smooth compactification such that \( \Delta := Z \setminus Z^\circ \) is a divisor with at most simple normal crossings, define the Kodaira dimension of \( Z^\circ \) as \( \kappa(Z^\circ) := \kappa(\mathcal{O}_Z(K_Z + \Delta)) \). Recall the standard fact that this number is independent of the choice of the compactification.
2.B. **Logarithmic pairs.** The following fundamental definitions of logarithmic geometry will be used in the sequel.

**Definition 2.4 (Logarithmic pair).** A logarithmic pair \((Z, \Delta)\) consists of a normal variety \(Z\) and a reduced, but not necessarily irreducible Weil divisor \(\Delta \subset Z\). A morphism of logarithmic pairs, written as \(\gamma : (\tilde{Z}, \tilde{\Delta}) \to (Z, \Delta)\), is a morphism \(\gamma : \tilde{Z} \to Z\) such that \(\gamma^{-1}(\Delta) = \tilde{\Delta}\) set-theoretically.

**Definition 2.5 (Snc pairs).** Let \((Z, \Delta)\) be a logarithmic pair, and \(z \in Z\) a point. We say that \((Z, \Delta)\) is snc at \(z\), if there exists a Zariski-open neighborhood \(U\) of \(z\) such that \(U\) is smooth and such that \(\Delta \cap U\) has only simple normal crossings. The pair \((Z, \Delta)\) is snc if it is snc at all points.

Given a logarithmic pair \((Z, \Delta)\), let \((Z, \Delta)_{\text{reg}}\) be the maximal open set of \(Z\) where \((Z, \Delta)\) is snc, and let \((Z, \Delta)_{\text{sing}}\) be its complement, with the induced reduced subscheme structure.

**Definition 2.6 (Log resolution).** A log resolution of \((Z, \Delta)\) is a birational morphism of pairs \(\pi : (\tilde{Z}, \tilde{\Delta}) \to (Z, \Delta)\) such that the \(\pi\)-exceptional set \(\text{Exc}(\pi)\) is of pure codimension one, such that \((\tilde{Z}, \text{supp}(\tilde{\Delta} + \text{Exc}(\pi)))\) is snc, and such that \(\pi\) is isomorphic along \((Z, \Delta)_{\text{reg}}\).

If \((Z, \Delta)\) is a logarithmic pair, a log resolution is known to exist, cf. [Kol07].

2.C. **Minimal model program.** We will use the definitions and apply the techniques of the minimal model program frequently, sometimes without explicit references. On these occasions the reader is referred to [KM98] for background and details.

In particular, we will use the fact that the minimal model program asserts the existence of extremal contractions [KM98, 3.7, 3.31] on non-minimal varieties. These extremal contractions come in three different kinds: divisorial, small, and of fiber type. The first gives a birational morphism that contracts a divisor, the second leads to a flip [KM98, 2.8], and the third gives a fiber space. Recall that a fiber space \(\pi : Y \to Z\) is called proper if the general fiber \(F\) is of dimension \(0 < \dim F < \dim Y\). We will call an extremal contraction of fiber type non-trivial if the resulting fiber space is proper. Finally, recall that extremal contractions of divisorial or fiber type have relative Picard number one [KM98, 3.36].

Further note that since we are working in dimension at most 3, we do not need to appeal to the recent phenomenal advances in the Minimal Model Program by Hacon-McKernan and Birkar-Cascini-Hacon-McKernan [Cor07, BCHM06]. However, these results give us reasonable hope that the methods here may extend to all dimensions.

3. **Singularities of the Minimal Model Program**

3.A. **Dlt singularities of index one.** If \((Z, \Delta)\) is an snc pair of dimension \(\dim Z \leq 3\), the minimal model program yields a birational map to a pair \((Z_\lambda, \Delta_\lambda)\), where \(Z_\lambda\) is Q-factorial and \((Z_\lambda, \Delta_\lambda)\) is dlt —see [KM98, 2.37] for the definition of dlt. We remark for later use that dlt pairs of index one are snc in codimension two.

**Lemma 3.1.** Let \((Z, \Delta)\) be a dlt pair of index one, i.e., a pair where \(K_Z + \Delta\) is Cartier. Then

\[
\text{codim}_Z ((Z, \Delta)_{\text{sing}} \cap \Delta) \geq 3.
\]

**Remark 3.1.2.** It is important to note that \((Z, \Delta)\) has simple normal crossings away from \((Z, \Delta)_{\text{sing}}\), whereas having only normal crossings would give a much weaker result. This,
for example, implies that the components of $\Delta$ are smooth in codimension 1 which is not true for a boundary with only normal crossings, cf. [KM98] 2.38.

Proof. We will prove the statement by induction on the dimension.

Start of induction. First assume that $\dim Z = 2$. Then by definition of dlt singularities, [KM98] 2.37, there exists a finite subset $T \subset Z$ such that $(Z, \Delta)_\text{sing} \subseteq T$ and such that $Z$ is log terminal at the points of $T$, i.e., the discrepancy of any divisor $E$ that lies over $T$ is $a(E, Z, \Delta) > -1$. But since $K_Z + \Delta$ is Cartier, this number must be an integer, so $a(E, Z, \Delta) \geq 0$. This shows that $(Z, \Delta)$ is canonical at the points of $T$. Therefore it follows by [KM98] 4.5 that $T \cap \Delta = \emptyset$. In particular, (3.1.1) holds.

Inductive step. Now let $Z$ be of arbitrary dimension, and let $H \subseteq Z$ be a general hyperplane section. Set $\Delta_H := \Delta \cap H$. Since a Cartier divisor being smooth at a point implies that the ambient space is also smooth at that point, it follows, that for any $z \in H$, the pair $(H, \Delta_H)$ is snc at $z$ if and only if $(Z, \Delta)$ is snc at $z$. In other words, $(H, \Delta_H)_\text{sing} = (Z, \Delta)_\text{sing} \cap H$ and

$$\text{codim}_H((H, \Delta_H)_\text{sing} \cap \Delta_H) = \text{codim}_Z((Z, \Delta)_\text{sing} \cap \Delta).$$

Notice further that $(H, \Delta_H)$ is dlt of index one. The claim thus follows by induction. □

3.B. Dlc singularities. Given an snc pair of Kodaira dimension zero, the minimal model program terminates at a dlt pair $(Z, \Delta)$ where $\Delta$ is $\mathbb{Q}$-Cartier and $K_Z + \Delta$ is torsion. Much of the argumentation in Section 3 is based on the following observation:

If $\Delta \neq \emptyset$, and $\epsilon \in \mathbb{Q}^+$ sufficiently small, then $(Z, (1 - \epsilon)\Delta)$ is a dlt pair of Kodaira dimension $-\infty$. Therefore it admits at least one further extremal contraction.

Using the thinned down boundary to push the minimal model program further, we end with a logarithmic pair $(Z', \Delta')$ that might no longer be dlt, but still has manageable singularities.

Definition 3.2 (Dlc singularities). A logarithmic pair $(Z', \Delta')$ is called dlc if $(Z', \Delta')$ is log canonical, $\Delta'$ is $\mathbb{Q}$-Cartier and for any sufficiently small positive number $\epsilon \in \mathbb{Q}^+$, the pair $(Z', (1 - \epsilon)\Delta')$ is dlt.

Dlc singularities are of interest to us because sheaves of reflexive differentials on dlc surface pairs enjoy good pull-back properties, cf. Theorem 5.3 below. For future reference, we recall the relation between dlc and several other notions of singularity.

Remark 3.3 (Relationship with other singularity classes). By definition, a dlc pair $(Z, \Delta)$ is boundary-lc in the sense of [GKK08] Def. 3.6. If $\dim Z = 2$ this implies that $(Z, \Delta)$ is finitely dominated by analytic snc pairs [GKK08 Lem. 3.9].

3.C. Bogomolov-Sommese vanishing on singular spaces. If $(Z, \Delta)$ is an snc pair, the well-known Bogomolov-Sommese vanishing theorem asserts that for any number $1 \leq p \leq \dim Z$, any invertible subsheaf $\mathcal{E} \subseteq \Omega^p_Z(\log \Delta)$ has Kodaira-Iitaka dimension at most $p$. See [EV92] Sect. 6] for a thorough discussion. Much of the argumentation in this paper is based centrally on the fact that similar results also hold for reflexive sheaves of differentials on pairs with dlc, or more generally log canonical singularities.

The formulation of the general result we expect to be true is the following.

Conjecture 3.4 (Bogomolov-Sommese vanishing for log canonical varieties). Let $(Z, \Delta)$ be a logarithmic pair and assume that $(Z, \Delta)$ is log canonical. Let $\mathcal{A} \subseteq \Omega^p_Z(\log \Delta)$ be any reflexive subsheaf of rank one. If $\mathcal{A}$ is $\mathbb{Q}$-Cartier, then $\kappa(\mathcal{A}) \leq p$. 
At this time Conjecture 3.4 has been verified with the additional assumption $\dim Z \leq 3$ in [GKK08]:

**Theorem 3.5** (Bogomolov-Sommese vanishing for log canonical threefolds, [GKK08 Thm. 1.4]). Let $(Z, \Delta)$ be a logarithmic pair of dimension $\dim Z \leq 3$ and assume $(Z, \Delta)$ is log canonical. Let $\mathcal{O} \subseteq \Omega^p_Z (\log \Delta)$ be any reflexive subsheaf of rank one. If $\mathcal{O}$ is $\mathbb{Q}$-Cartier, then $\kappa(\mathcal{O}) \leq p$. □

4. Global index-one covers for varieties of Kodaira dimension zero

In this section, we consider good minimal models of pairs with Kodaira dimension 0. We briefly recall the main properties of the global index-one cover, as described in [KM98 2.50–53]. To prove (4.1.3) assume for the remainder of the proof that $K \sim_\mathbb{Q} (\log \Delta)$. Then immediately yields the claim.

**Theorem 4.1.** Let $(Z, \Delta)$ be a logarithmic pair. Assume that the log canonical divisor $K_Z + \Delta$ is torsion (in particular, it is $\mathbb{Q}$-Cartier), i.e., assume that there exists a number $m \in \mathbb{N}^+$ such that $m \cdot (K_Z + \Delta) \equiv 0$. Then there also exists a morphism of pairs $\eta: (Z', \Delta') \to (Z, \Delta)$, called the index-one cover, with the following properties.

1. The morphism $\eta$ is finite. It is étale wherever $Z$ is smooth. In particular, $\eta$ is étale in codimension one.
2. $K_{Z'} + \Delta'$ is Cartier and $\mathcal{O}_{Z'} (K_{Z'} + \Delta') \simeq \mathcal{O}_{Z'}$.
3. If $(Z, \Delta)$ is dlt, then $(Z', \Delta')$ is dlt as well. If furthermore $z' \in Z'$ is a point such that $(Z', \Delta')$ is snc at $z'$, then $(Z', \Delta')$ is canonical at $z'$.

**Proof.** Properties (4.1.1) and (4.1.2) follow directly from the construction, cf. [KM98 2.50–53]. To prove (4.1.3) assume for the remainder of the proof that $(Z, \Delta)$ dlt. We need to show that $(Z', \Delta')$ is dlt as well. Observe that if $z' \in Z'$ is a point such that $(Z', \Delta')$ is snc at $\eta(z')$, then $(Z', \Delta')$ is snc at $z'$. The definition of dlt, together with the fact that discrepancies only increase under finite morphisms, [KM98 5.20], then immediately yields the claim.

Finally, if $z' \in Z'$ is any point where $(Z', \Delta')$ is not snc, then the discrepancy of any divisor $E$ that lies over $z'$ is $\alpha(E, Z', \Delta') > -1$. But since $K_{Z'} + \Delta'$ is Cartier, this number must be an integer, so $\alpha(E, Z', \Delta') \geq 0$. It follows that the pair $(Z', \Delta')$ is canonical at $z'$, hence (4.1.3) is shown. □

**Corollary 4.2.** Under the conditions of Proposition 4.1 if $\gamma: (\tilde{Z}, \tilde{\Delta}) \to (Z', \Delta')$ is any log resolution, then $\kappa(K_{\tilde{Z}} + \tilde{\Delta}) = 0$.

**Proof.** Since $(Z', \Delta')$ is canonical wherever it is not snc, the definition of canonical singularities, [KM98 2.26, 2.34], implies that $K_{\tilde{Z}} + \tilde{\Delta}$ is represented by an effective, $\gamma$-exceptional divisor. □

5. Viehweg-Zuo sheaves

5.1. Definition of Viehweg-Zuo sheaves. In the setup of Theorem 3.4 and in a few other cases, Viehweg and Zuo have shown in [VZ02 Thm. 1.4] that there exists a number $n \gg 0$ and an invertible sheaf $\mathcal{O} \subseteq \text{Sym}^n \Omega^1_Y (\log D)$ whose Kodaira-Iitaka dimension is at least the variation of $f^*\mathcal{O}$, i.e., $\kappa(\mathcal{O}) \geq \text{Var}(f^*)$. The existence of this sheaf is a cornerstone of our argumentation.

For technical reasons, it turns out to be more convenient to view $\mathcal{O}$ as a subsheaf of the tensor product, via the injection $\text{Sym}^n \Omega^1_Y (\log D) \hookrightarrow (\Omega^1_Y (\log D))^\otimes n$. It is also advantageous to extend studying these sheaves on singular varieties and then it is natural to
allow rank one reflexive sheaves instead of restricting to line bundles. These considerations give rise to the following definition.

**Definition 5.1** (Viehweg-Zuo sheaf). Let \((Z, \Delta)\) be a logarithmic pair. A reflexive sheaf \(\mathcal{A}\) of rank 1 is called a Viehweg-Zuo sheaf if there exists a number \(n \in \mathbb{N}\) and an embedding \(\mathcal{A} \subseteq (\Omega^1_Z(\log \Delta))^n\).

5.B. Pushing forward and pulling back. We often need to compare Viehweg-Zuo sheaves on different birational models of a pair. The following elementary statement shows that the push-forward of a Viehweg-Zuo sheaf under a birational map of pairs is often again a Viehweg-Zuo sheaf.

**Lemma 5.2** (Push forward of Viehweg-Zuo sheaves). Let \((Z, \Delta)\) be a logarithmic pair and assume that there exists a Viehweg-Zuo sheaf \(\mathcal{A}\) of rank 1 is called a Viehweg-Zuo sheaf if there exists a number \(n \in \mathbb{N}\) and an embedding \(\mathcal{A} \subseteq (\Omega^1_Z(\log \Delta))^n\).

Let \(\lambda : Z \to Z'\) be a birational map whose inverse does not contract any divisor, \(Z'\) is normal and \(\Delta'\) is the (necessarily reduced) cycle-theoretic image of \(\Delta\), then there exists a Viehweg-Zuo sheaf \(\mathcal{A}' \subseteq (\Omega^1_{Z'}(\log \Delta'))^n\) of Kodaira-Iitaka dimension \(\kappa(\mathcal{A}') \geq \kappa(\mathcal{A})\).

**Proof.** The assumption that \(\lambda^{-1}\) does not contract any divisors and the normality of \(Z'\) guarantee that \(\lambda^{-1} : Z' \to Z\) is a well-defined embedding over an open subset \(U \subseteq Z'\) whose complement has codimension \(\text{codim}_{Z'}(Z' \setminus U) \geq 2\), cf. Zariski’s main theorem \([Har77, V 5.2]\). In particular, \(\Delta'_U = (\lambda^{-1} |_U)^{-1}(\Delta)\). Let \(\iota : U \to Z'\) denote the inclusion and set \(\mathcal{A}' := \iota_*((\lambda^{-1} |_U)^*\mathcal{A})\). We obtain an inclusion of sheaves, \(\mathcal{A}' \subseteq (\Omega^1_{Z'}(\log \Delta'))^n\). By construction, we have that \(h^0(Z', \mathcal{A}'^m) \geq h^0(Z, \mathcal{A}^m)\) for all \(m > 0\), hence \(\kappa(\mathcal{A}') \geq \kappa(\mathcal{A})\).

If \(Z\) is a singular space with desingularization \(\pi : \tilde{Z} \to Z\), it follows almost by definition that any differential \(\sigma \in H^0(Z, \Omega^1_Z)\) pulls back to a differential \(\pi^*\sigma \in H^0(\tilde{Z}, \Omega^1_{\tilde{Z}})\), cf. \([Har77, II Prop.8.11]\). However, if \(\sigma\) is a reflexive differential, i.e., if \(\sigma \in H^0(Z, \Omega^1_Z)\), it is not clear— and generally false— that \(\pi^*\sigma\) can be interpreted as a differential on \(\tilde{Z}\). Likewise, if \((\tilde{Z}, \tilde{\Delta})\) is a logarithmic pair with log resolution \(\pi : (\tilde{Z}, \tilde{\Delta}) \to (Z, \Delta)\) and \(\mathcal{A} \subseteq (\Omega^1_Z(\log \Delta))^n\) is a Viehweg-Zuo sheaf, it is generally not possible to interpret the reflexive pull-back \(\pi^*(\mathcal{A})\) as a Viehweg-Zuo sheaf on \((\tilde{Z}, \tilde{\Delta})\). However, if the pair \((Z, \Delta)\) is log canonical, the extension theorems for differential forms studied in \([GKK08]\) show that an interpretation of \(\pi^*(\mathcal{A})\) as a Viehweg-Zuo sheaf often exists. The following theorem is an immediate consequence of Remark 5.3 and \([GKK08, Thm. 8.1]\). It summarizes the results of \([GKK08]\) that are relevant for our line of argumentation.

**Theorem 5.3** (Extension of Viehweg-Zuo sheaves. \([GKK08, Thm. 8.1]\)). Let \((Z, \Delta)\) be a dlc pair of dimension \(\text{dim} \ Z \leq 2\), and assume that there exists a Viehweg-Zuo sheaf \(\mathcal{A}\) with inclusion \(\iota : \mathcal{A} \to (\Omega^1_Z(\log \Delta))^n\). If \(\pi : (\tilde{Z}, \tilde{\Delta}) \to (Z, \Delta)\) is a log resolution, and \(E := \text{largest reduced divisor contained in } \pi^{-1}(\Delta) \cup \text{Exc}(\pi)\),

then there exists an invertible Viehweg-Zuo sheaf \(\mathcal{C} \subseteq (\Omega^1_Z(\log E))^m\) with the following property. For an arbitrary \(m \in \mathbb{N}\), the inclusion pulls back to give a sheaf morphism that factors through \(\mathcal{C}^\otimes m\),

\[
\iota^m : \pi^*(\mathcal{A}^m) \to \mathcal{C}^\otimes m \subseteq (\Omega^1_Z(\log E))^m \cdot m.
\]

In particular, \(\kappa(\mathcal{C}) \geq \kappa(\mathcal{A})\).
5.C. The reduction lemma. Like regular differentials, logarithmic differentials come with a normal bundle, and the corresponding restriction sequences, cf. [Ev92 2.3], [KK08 Lem. 2.13] and the references there. Since Viehweg-Zuo sheaves live in tensor products of the sheaf of differentials, this does not immediately translate into a sequence for a given Viehweg-Zuo sheaf. This makes the following lemma useful in the sequel.

Lemma 5.4 (Reduction lemma). Let Z be a reduced irreducible variety, E, F, G, H locally free sheaves, and A a rank one torsion-free sheaf on Z. Assume that there exists a short exact sequence

\[(5.4.1) \quad 0 \to F \to E \to G \to 0.\]

Then

\[(5.4.2) \quad \text{If there exists an inclusion } A \hookrightarrow E, \text{ then either } A \hookrightarrow F \text{ or } A \hookrightarrow G.\]

\[(5.4.3) \quad \text{If for some } m \in \mathbb{N}, \text{ there exists an inclusion } A \hookrightarrow H \otimes E^{\otimes m}, \text{ then there exists a } p \in \mathbb{N}, 0 \leq p \leq m \text{ such that } A \hookrightarrow H \otimes F^{\otimes p} \otimes G^{\otimes m-p}.\]

\[(5.4.4) \quad \text{If for some } m \in \mathbb{N}, \text{ there exists an inclusion } A \hookrightarrow E^{\otimes m}, \text{ and } F \simeq O_Z (\text{respectively } G \simeq O_Z), \text{ then there exists a } p \in \mathbb{N}, 0 \leq p \leq m \text{ such that } A \hookrightarrow F^{\otimes p} (\text{respectively } A \hookrightarrow F^{\otimes p}).\]

Proof. Suppose \(A \hookrightarrow E\) and let \(H = \ker[A \to G] \subseteq A\). If \(A \to G\) is injective at the general point of \(Z\), then \(H\) is a torsion sheaf and hence zero, so \(A \hookrightarrow G\). Since \(\text{rk } A = 1\), if \(A \to G\) is not injective at the general point, then it is zero. However, then \(A/\ker[A \to G] \subseteq G\) is a torsion sheaf and hence zero, so \(A \hookrightarrow F\). This proves (5.4.2). Taking \(H = O_Z\) it is easy to see that (5.4.4) is a special case of (5.4.3). To prove (5.4.3), we use induction.

Start of induction. If \(m = 1\), assertion (5.4.3) follows from applying (5.4.2) to the short exact sequence obtained by tensoring (5.4.1) with \(H\),

\[0 \to H \otimes F \to H \otimes E \to H \otimes G \to 0.\]

Note that if \(m = 1\), then either \(p = 0\) or \(m - p = 0\).

Induction step. Now assume that the statement is true for all numbers \(m' < m\). Consider the short exact sequence obtained by tensoring (5.4.1) with \(H \otimes E^{\otimes (m-1)}\),

\[0 \to H \otimes F \otimes E^{\otimes (m-1)} \to H \otimes E^{\otimes m} \to H \otimes G \otimes E^{\otimes (m-1)} \to 0.\]

Applying (5.4.2) for this short exact sequence yields that either \(A \hookrightarrow (H \otimes F) \otimes E^{\otimes (m-1)}\) or \(A \hookrightarrow (H \otimes G) \otimes E^{\otimes (m-1)}\). Setting \(H' := H \otimes F\) or \(H' := H \otimes G\), respectively, and applying the induction hypothesis to the sequence

\[0 \to H' \otimes F \otimes E^{\otimes (m-2)} \to H' \otimes E^{\otimes (m-1)} \to H' \otimes G \otimes E^{\otimes (m-2)} \to 0,\]

we obtain a number \(p \in \mathbb{N}, 0 \leq p \leq m - 1\) such that either \(A \hookrightarrow (H' \otimes F) \otimes E^{\otimes p} \otimes G^{\otimes m-1-p}\) or \(A \hookrightarrow (H' \otimes G) \otimes F^{\otimes p} \otimes G^{\otimes m-1-p}\). This proves (5.4.3). \(\square\)

6. Viehweg-Zuo sheaves on minimal models

The existence of a Viehweg-Zuo sheaf of positive Kodaira-Iitaka dimension clearly has consequences for the geometry of the underlying space. In case the underlying space is the end product of the minimal model program, we summarize the two most important consequences below, when \(\kappa = -\infty\) and \(\kappa = 0\).
6.A. The Picard-number of minimal models with non-positive Kodaira dimension.

The following theorem will be used later to show that a given pair is a Mori-Fano fiber space. This will turn out to be a crucial step in the proof of our main results.

**Theorem 6.1.** Let $(Z, \Delta)$ be a log canonical logarithmic pair where $Z$ is a projective \( \mathbb{Q} \)-factorial variety of dimension at most 3. Assume that the following holds:

1. There exists a Viehweg-Zuo sheaf \( \mathcal{A} \subseteq (\Omega^1_Z(\log \Delta))^[[n]] \) of positive Kodaira-Iitaka dimension, and
2. the anti log canonical divisor \(-(K_Z + \Delta)\) is nef.

Then the Picard number of minimal models with non-positive Kodaira-Iitaka dimension.

**Proof.** We argue by contradiction and assume that \( \rho(Z) = 1 \). Let \( C \subseteq Z \) be a general complete intersection curve. Since \( C \) is general, it avoids the singular locus \((Z, \Delta)_{\text{sing}}\). By (6.1.2), the restriction \( \Omega^1_Z(\log \Delta)|_C \) is a vector bundle of non-positive degree,

\[
\deg \Omega^1_Z(\log \Delta)|_C = (K_Z + \Delta) \cdot C \leq 0.
\]

We claim that the restriction \( \Omega^1_Z(\log \Delta)|_C \) is not anti-nef, i.e., that the dual vector bundle \( \mathcal{F}_Z(- \log \Delta)|_C \) is not nef. Equivalently, we claim that \( \Omega^1_Z(\log \Delta)|_C \) admits an invertible subsheaf of positive degree. Indeed, if \( \Omega^1_Z(\log \Delta)|_C \) was anti-nef, then none of its products \((\Omega^1_Z(\log \Delta)|_C)^[[n]]\) could contain a subsheaf of positive degree. However, since \( C \) is general, the restriction of the Viehweg-Zuo sheaf to \( C \) is a locally free subsheaf \( \mathcal{A}|_C \subseteq (\Omega^1_Z(\log \Delta)|_C)^[[n]] \) of positive Kodaira-Iitaka dimension, and hence of positive degree. This proves the claim.

As a consequence of the claim and of Equation (6.1.3), we obtain that \( \Omega^1_Z(\log \Delta) \) is not semi-stable and if \( \mathcal{B} \subseteq \Omega^1_Z(\log \Delta) \) denotes the maximal destabilizing subsheaf, then its slope \( \mu(\mathcal{B}) \) is positive. The assumption that \( \rho(Z) = 1 \) and the \( \mathbb{Q} \)-factoriality of \( Z \) then guarantees that \( \det \mathcal{B} \) is a \( \mathbb{Q} \)-Cartier and \( \mathbb{Q} \)-ample sheaf of \( p \)-forms. Notice that by its choice the rank of \( \mathcal{B} \) has to be strictly less than the rank of \( \Omega^1_Z(\log \Delta) \), hence \( p < \dim Z \). However, this leads to a contradiction. Because \( \mathcal{B} \) is \( \mathbb{Q} \)-ample, it follows that \( \kappa(\det \mathcal{B}) = \dim Z \) violating the Bogomolov-Sommese Vanishing Theorem \([5,3] \).

In the case when \( Z \) is a surface, this theorem immediately gives a criterion to guarantee that Viehweg-Zuo sheaves of positive Kodaira-Iitaka dimension cannot exist.

**Corollary 6.2.** Let \((Z, \Delta)\) be a projective, logarithmic dlt surface pair where \(-(K_Z + \Delta)\) is \( \mathbb{Q} \)-ample. If \( \mathcal{A} \) is any Viehweg-Zuo sheaf on \( Z \), then its Kodaira-Iitaka dimension is non-positive, i.e. \( \kappa(\mathcal{A}) \leq 0 \).

**Proof.** First recall from [KM98, Prop. 4.11] that \( Z \) is \( \mathbb{Q} \)-Cartier. The minimal model program then yields a morphism \( \lambda : (Z, \Delta) \to (Z_\lambda, \Delta_\lambda) \) to a \( \mathbb{Q} \)-Cartier model that does not admit any divisorial contractions. Note that the minimal model program for surfaces does not involve flips. Let \( \mathcal{A}_\lambda \) be the associated Viehweg-Zuo sheaf on \( Z_\lambda \), as given by Lemma \([5,2] \). It suffices to show that \( \kappa(\mathcal{A}_\lambda) \leq 0 \).

To this end, observe that \(- (K_{Z_\lambda} + \Delta_\lambda) \) is still \( \mathbb{Q} \)-ample. Theorem \([6,1] \) and the Cone Theorem \([KM98, 3.7] \) imply that there are at least two distinct contractions of fiber type, say \( \pi_1 : Z_\lambda \to C_1 \) and \( \pi_2 : Z_\lambda \to C_2 \). If \( F \) is a general fiber of \( \pi_1 \), then \( F \cong \mathbb{P}^1 \), the fiber \( F \) is entirely contained inside the snc locus of \( (Z_\lambda, \Delta_\lambda) \), and \( F \) intersects the boundary divisor \( \Delta_\lambda \) transversely in no more than one point. It follows from standard short exact sequences, \([KK08, \text{Lem. 2.13}] \), that

\[
\Omega^1_{Z_\lambda}(\log \Delta_\lambda)|_F \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a) \quad \text{with} \quad a \leq 0.
\]
In particular, $\Omega^1_Z(\log \Delta_\lambda) \mid F$ is anti-nef, and $\mathcal{A}_\lambda \mid F$ is necessarily trivial. But the same holds for the restriction of $\mathcal{A}_\lambda$ to general fibers of $\pi_2$. It follows that $\kappa(\mathcal{A}_\lambda) \leq 0$, as claimed. \hfill \square

6.B. Viehweg-Zuo sheaves on good minimal models for varieties of logarithmic Kodaira dimension zero. If $(Z, \Delta)$ is a good minimal model of Kodaira dimension zero, the existence of a Viehweg-Zuo sheaf of positive Kodaira-Iitaka dimension implies that $Z$ is uniruled. This is shown next.

**Theorem 6.3.** Let $(Z, \Delta)$ be a logarithmic pair where $Z$ is projective. Assume that the following holds:

1. There exists a Viehweg-Zuo sheaf $\mathcal{A} \subseteq (\Omega^1_Z(\log \Delta))^n$ of positive Kodaira-Iitaka dimension.
2. The log canonical divisor $K_Z + \Delta$ is $\mathbb{Q}$-Cartier and numerically trivial.

Then $Z$ is uniruled.

**Proof.** We argue by contradiction and assume that $Z$ is not uniruled. If $\pi : (\tilde{Z}, \tilde{\Delta}) \to (Z, \Delta)$ is any log resolution, this is equivalent to assuming that $K_{\tilde{Z}}$ is pseudo-effective, 

[BDPP04] cor. 0.3, see also [Laz04] sect. 11.4.C. Again by [BDPP04] thm. 0.2, this is in turn equivalent to the assumption that $K_{\tilde{Z}} \cdot \tilde{C} \geq 0$ for all moving curves $\tilde{C} \subset \tilde{Z}$.

As a first step, we will show that the assumption implies that the (Weil) divisor $\Delta$ is zero. To this end, choose a polarization of $Z$ and consider a general complete intersection curve $C \subset Z$. Because $C$ is a complete intersection curve, it intersects the support of the effective divisor $\Delta$ non-trivially if the support is not empty. By general choice, the curve $C$ is contained in the snc locus of $(Z, \Delta)$ and avoids the indeterminacy locus of $\pi^{-1}$. Its preimage $\tilde{C} := \pi^{-1}(C)$ is then a moving curve in $\tilde{Z}$ which intersects $\tilde{\Delta}$ positively if and only if the Weil divisor $\Delta$ is not zero. But

$$0 = (K_{\tilde{Z}} + \tilde{\Delta}) \cdot \tilde{C} = (K_Z + \Delta) \cdot \tilde{C} = K_{\tilde{Z}} \cdot \tilde{C} + \Delta \cdot \tilde{C},$$

so $\Delta \cdot \tilde{C} = 0$, and then $\Delta = 0$ as claimed. Combined with Assumption (6.3.2), this implies that the canonical divisor $K_{\tilde{Z}}$ is itself numerically trivial. The restrictions $\Omega^1_{Z|C}$ and $\mathcal{F}_{Z|C}$ are locally free sheaves of degree zero, and so is the product $(\Omega^1_{Z|C})^\otimes n$. On the other hand, the restriction $\mathcal{A} \mid C \subseteq (\Omega^1_{Z|C})^\otimes n$ has positive degree. In particular, $(\Omega^1_{Z|C})^\otimes n$ is not semi-stable. Since products of semi-stable vector bundles are again semi-stable, [HL97] Cor. 3.2.10, this implies that $\Omega^1_{Z|C}$ and $\mathcal{F}_{Z|C}$ are likewise not semi-stable. In particular, the maximal destabilizing subsheaf of $\mathcal{F}_{Z|C}$ is semi-stable and of positive degree, hence ample. In this setup, a variant [KST107] Cor. 5 of Miyaoka’s uniruledness criterion [Miy87] Cor. 8.6] applies to give the uniruledness of $Z$, contrary to our assumption. For more details on this criterion see the survey [KS06]. \hfill \square

As a corollary, we obtain a criterion to guarantee that the boundary is not empty. This will allow to apply the ideas described in Section 3.B above.

**Corollary 6.4.** In the setup of Theorem 6.2 if $(Z, \Delta)$ is dlc, then the boundary divisor $\Delta$ is not empty.
Proof. We argue by contradiction and assume $\Delta = \emptyset$. By the definition of dlc, the pair $(Z, \emptyset)$ is then dlt. Let $\eta : (Z', \emptyset) \to (Z, \emptyset)$ be the index-one-cover discussed in Proposition 4.1. Since $\eta$ is finite and étale in codimension one, there obviously exists an injection

$$\eta^* : (\Omega^1_{Z'})^n \subseteq (\Omega^1_Z)^n.$$

An application of Theorem 6.3, using the sheaf $\eta^* : (\mathcal{A}) \subseteq (\mathcal{A})$ as a Viehweg-Zuo sheaf on $(Z', \emptyset)$ then shows that $Z'$ is uniruled. If $\tilde{Z} \to Z'$ is a resolution, then $Z'$ is likewise uniruled. But Corollary 4.2 would then assert that $\kappa(K_{\tilde{Z}}) = 0$, in contradiction to uniruledness. \hfill \Box

7. UNWINDING FAMILIES

We will consider projective families $g : Y \to T$ where the base $T$ itself admits a fibration $\tilde{T} : T \to B$ such that $g$ is isotrivial defined over $B$. We will, however, show in this section that in some situations the family $g$ does become a pull-back after a suitable base change. Most results in this section are probably known to experts. We included full statements and proofs for the reader’s convenience, for lack of a suitable reference.

We use the following notation for fibered products that appear in our setup.

**Notation 7.1.** Let $T$ be a scheme, $Y$ and $Z$ schemes over $T$ and $h : Y \to Z$ a $T$-morphism. If $t \in T$ is any point, let $Y_t$ and $Z_t$ denote the fibers of $Y$ and $Z$ over $t$. Furthermore, let $h_t$ denote the restriction of $h$ to $Y_t$. More generally, for any $T$-scheme $\tilde{T}$, let

$$h_{\tilde{T}} : Y \times_T \tilde{T} \to Z \times_T \tilde{T}$$

denote the pull-back of $h$ to $\tilde{T}$. The situation is summarized in the following commutative diagram.

$$\begin{array}{ccc}
Y & \xrightarrow{h_{\tilde{T}}} & Z \\
\downarrow h & & \downarrow h \\
T & \to & T
\end{array}$$

The setup of the current section is then formulated as follows.

**Assumption 7.2.** Throughout the present section, consider a sequence of morphisms between algebraic varieties,

$$\begin{array}{ccc}
Y & \xrightarrow{g} & T & \xrightarrow{\varphi} & B, \\
\text{smooth, projective} & & \text{smooth, rel. dim.} = 1 & \text{quasi-projective}
\end{array}$$

where $g$ is a smooth projective family and $\varphi$ is smooth quasi-projective of relative dimension 1. Assume further that for all $b \in B$, there exists a smooth variety $F_b$ such that for all $t \in T_b$, there exists an isomorphism $Y_t \simeq F_b$.

7.A. Relative isomorphisms of families over the same base. To start, recall the well-known fact that an isotrivial family of varieties of general type over a curve becomes trivial after passing to an étale cover of the base. As we are not aware of an adequate reference, we include a proof here.

**Lemma 7.3.** Let $b \in B$ and assume that $\text{Aut}(F_b)$ is finite. Then the natural morphism $
 \iota : I = \text{Isom}_g(Y_b, T_b \times F_b) \to T_b$ is finite and étale. Furthermore, pull-back to $I$ yields an isomorphism of $I$-schemes $Y_I \simeq I \times F_b$. 

Proof. Consider the $T_b$-scheme

$$H := \Hilb_{T_b}(Y_b \times_{T_b} (T_b \times F_b)) \simeq \Hilb_{T_b}(Y_b \times F_b).$$

By Assumption 7.2, $H_t \simeq \Hilb(F_b \times F_b)$ for all $t \in T_b$. Similarly, $I_t \simeq \Aut(F_b)$ hence $I$ is one-dimensional, $\text{length}(I_t)$ is constant on $T_b$ and $I \to T_b$ is dominant. Since $I$ is open in $H$, the closure of $I$ in $H$, denoted by $H^I$, consists of a union of components of $H$. Therefore $H^I$ is also one-dimensional and since $H^I \to T$ is dominant, it is quasi-finite.

Recall that $H \to T_b$ is projective, so $H^I \to T_b$ is also projective, hence finite. Since $H \to T_b$ is flat, $\text{length}(H^I_t)$ is constant. Furthermore, $I \subseteq H^I$ is open, so $H^I_t = I_t$ and hence $\text{length}(H^I_t) = \text{length}(I_t)$ for a general $t \in T_b$. However, we observed above that $\text{length}(I_t)$ is also constant, so we must have that $\text{length}(H^I_t) = \text{length}(I_t)$ for all $t \in T_b$, and since $I \subseteq H^I$, this means that $I = H^I$ and $\tau : I \to T_b$ is finite and unramified, hence étale.

In order to prove the global triviality of $Y_t$, consider $\text{Isom}_I(Y_t, I \times F_b)$. Recall that taking $\Hilb$ and $\text{Isom}$ commutes with base change, and so we obtain an isomorphism

$$\text{Isom}_I(Y_t, I \times F_b) \simeq I \times_{T_b} \text{Isom}_{T_b}(Y_b, T_b \times F_b) \simeq I \times T_b I.$$

This scheme admits a natural section over $I$, namely its diagonal, which induces an $I$-isomorphism between $Y_t$ and $I \times F_b$. \hfill \Box

The preceding Lemma 7.3 can be used to compare two families whose associated moduli maps agree. In our setup any two such families become globally isomorphic after base change.

**Lemma 7.4.** In addition to Assumption [22] suppose that there exists another projective morphism, $Z \to T$, with the following property: for any $b \in B$ and any $t \in T_b$, we have $Y_t \simeq Z_t \simeq F_b$. Then

(7.4.1) there exists a surjective morphism $\tau : \tilde{T} \to T$ such that the pull-back families of $Y$ and $Z$ to $\tilde{T}$ are isomorphic as $\tilde{T}$-schemes, i.e., we have a commutative diagram as follows:

$$
\begin{array}{ccc}
Y_T & \xrightarrow{\tau^{-\text{isom.}}} & Z_{\tilde{T}} \\
\downarrow & & \downarrow \\
T & \xrightarrow{\tau} & T
\end{array}
$$

Furthermore, if for all $b \in B$, the group $\Aut(F_b)$ is finite, then $\tilde{T}$ can be chosen such that the following holds. Let $\tilde{T}' \subseteq \tilde{T}$ be any irreducible component. Then

(7.4.2) $\tau$ is quasi-finite,

(7.4.3) the image set $\tau(\tilde{T}')$ is a union of $\varnothing$-fibers, and

(7.4.4) if $\tilde{T}'$ dominates $B$, then there exists an open subset $B^o \subseteq (\varnothing \circ \tau)(\tilde{T}')$ such that $\tau_{|B^o}$ is finite and étale over $B^o$. More precisely, if we set $T^\circ := \varnothing^{-1}(B^o)$ and $\tilde{T}^\circ := \tau^{-1}(T^\circ) \cap \tilde{T}'$, then the restriction $\tau_{|\tilde{T}^\circ} : \tilde{T}^\circ \to T^\circ$ is finite and étale.

**Remark 7.4.5.** In Lemma 7.4 we do not claim that $\tilde{T}$ is irreducible or connected.
Corollary 7.5. connected fibers. Let \( \tilde{T} := \text{Isom}_T(Y, Z) \) and let \( \tau : \tilde{T} \to T \) be the natural morphism. Again, taking \( \text{Isom} \) commutes with base change, and we have an isomorphism \( \tilde{T} \times_T \tilde{T} \cong \text{Isom}_{\tilde{T}}(Y_{\tilde{T}}, Z_{\tilde{T}}) \). Similarly, for all \( b \in B \), and for all \( t \in T_b \), there is a natural one-to-one correspondence between \( \tilde{T}_t \) and \( \text{Aut}(F_b) \). In particular, we obtain that \( \tau \) is surjective. As before, observe that \( \tilde{T} \times_T \tilde{T} \) admits a natural section, the diagonal. This shows (7.4.1). If for all \( b \in B \), \( \text{Aut}(F_b) \) is finite, then the restriction of \( \tau \) to any \( \varphi \)-fiber, \( \tau_b : \tilde{T}_b \to T_b \) is finite étale by Lemma 7.3. This shows (7.4.2) and (7.4.3). Furthermore, it implies that if \( \tilde{T}' \subseteq \tilde{T} \) is a component that dominates \( B \), neither the ramification locus of \( \tau \mid_{\tilde{T}'} \) nor the locus where \( \tau \mid_{\tilde{T}'} \) is not finite dominates \( B \).

Let \( \widehat{B} \subseteq T \) be a multisection of \( \varphi : T \to B \), i.e., a closed subvariety that dominates \( B \) and is of equal dimension. In particular, the morphism \( \varphi \mid_{\widehat{B}} : \widehat{B} \to B \) is quasi-finite. The scheme \( \text{Isom}_{\widehat{B}}(Y, Y) \) is quasi-finite and quasi-projective over \( \widehat{B} \), hence over \( B \) as well. Then there exists an open subset \( B^\circ \subseteq B \) where \( \text{length}(\text{Isom}_{\widehat{B}}(Y, Y))_b \) is constant for \( b \in B \). It is easy to see that (7.4.4) holds for \( B^\circ \).

7.B. Families where \( \varphi \) has a section. In addition to Assumption 7.2, assume that the morphism \( \varphi \) admits a section \( \sigma : B \to T \). Using \( \sigma : B \to T \), define \( Y_B := Y \times_T B \) and let \( Z := Y_B \times_B T \) be the pull-back of \( Y_B \) to \( T \). With these definitions, Lemma 7.4 applies to the families \( Y \to T \) and \( Z \to T \). As a corollary, we will show below that in this situation \( \tilde{T} \) contains a component \( \tilde{T}' \) such that the pull-back family \( Y_{\tilde{T}'} \) comes from \( B \). Better still, the restriction \( \tau \mid_{\tilde{T}'} \) is “relatively étale” in the sense that \( \tau \mid_{\tilde{T}'} \) is étale and that \( \varphi \mid_{\tilde{T}'} \) has connected fibers.

Corollary 7.5. Under the conditions of Lemma 7.4 assume that \( \varphi \) admits a section \( \sigma : B \to T \), and that \( Z = Y_B \times_B T \). Then there exists an irreducible component \( \tilde{T}' \subseteq \tilde{T} \) such that

\begin{align*}
&\text{(7.5.1)} \quad \tilde{T}' \text{ surjects onto } B, \text{ and} \\
&\text{(7.5.2)} \quad \text{the restricted morphism } \tilde{\varphi} := \varphi \mid_{\tilde{T}'} : \tilde{T}' \to B \text{ has connected fibers.}
\end{align*}

Proof. It is clear from the construction that \( Y_B \cong Z_B \). This isomorphism corresponds to a morphism \( \tilde{\sigma} : B \to \text{Isom}_T(Y, Z) = \tilde{T} \). Let \( \tilde{T}' \subseteq \tilde{T} \) be an irreducible component that contains the image of \( \tilde{\sigma} \). Observe that \( \tilde{\sigma} \) is a section of \( \tilde{\varphi} : \tilde{T}' \to B \) and that the existence of a section guarantees that \( \tilde{\varphi} \) is surjective and its fibers are connected. 

One particular setup where a section is known to exist is when \( T \) is a birationally ruled surface over \( B \). The following will become important later.

Corollary 7.6. In addition to Assumption 7.2, suppose that \( B \) is a smooth curve and that the general \( \varphi \)-fiber is isomorphic to \( \mathbb{P}^1 \), \( \mathbb{A}^1 \) or \( (\mathbb{A}^1)^* = \mathbb{A}^1 \setminus \{0\} \). Then there exist non-empty Zariski open sets \( B^\circ \subseteq B \), \( T^\circ := (\varphi^{-1}(B^\circ)) \) and a commutative diagram

\[
\begin{array}{ccc}
\tilde{T}^\circ & \xrightarrow{\tau} & T^\circ \\
\text{conn. fibers} & \searrow & \varphi \\
& B^\circ & \nwarrow
\end{array}
\]

such that

\begin{align*}
&\text{(7.6.1) the fibers of } \varphi \circ \tau \text{ are again isomorphic to } \mathbb{P}^1, \mathbb{A}^1 \text{ or } (\mathbb{A}^1)^*, \text{ respectively, and}
\end{align*}
(7.6.2) the pull-back family $Y_{\tilde{\tau}^o}$ comes from $B^o$, i.e., there exists a projective family $Z \to B^o$ and a $\tilde{T}^o$-isomorphism

$$Y_{\tilde{\tau}^o} \simeq Z_{\tilde{\tau}^o}.$$ 

Remark 7.6.3. If the general $\varphi$-fiber is isomorphic to $\mathbb{P}^1$ or $\mathbb{A}^1$, the morphism $\tau$ is necessarily an isomorphism. Shrinking $B^o$ further, if necessary, $\varphi : T^o \to B^o$ will then even be a trivial $\mathbb{P}^1$- or $\mathbb{A}^1$-bundle, respectively.

Proof. Shrinking $B$, if necessary, we may assume that all $\varphi$-fibers are isomorphic to $\mathbb{P}^1$, $\mathbb{A}^1$ or $(\mathbb{A}^1)^*$, and hence that $T$ is smooth. Then it is always possible to find a relative smooth compactification of $T$, i.e. a smooth $B$-variety $\overline{T} \to B$ and a smooth divisor $D \subset T$ such that $\overline{T} \setminus D$ and $T$ are isomorphic $B$-schemes.

By Tsen’s theorem, [Sha94, p. 73], there exists a section $\sigma : B \to \overline{T}$. In fact, there exists a positive dimensional family of sections, so that we may assume without loss of generality that $\sigma(B)$ is not contained in $D$.

Let $B^o \subset B$ be the open subset such that for all $b \in B^o$, $\overline{T}_b \simeq \mathbb{P}^1$, $T_b$ is isomorphic to $\mathbb{P}^1$, $\mathbb{A}^1$ or $(\mathbb{A}^1)^*$, respectively, and $\sigma(b) \not\in D$. Using that any connected finite étale cover of $T_b$ is again isomorphic to $T_b$, and shrinking $B^o$ further, Corollary [7.5] yields the claim. □

Remark 7.7. Throughout the article we work over the field of complex numbers $\mathbb{C}$, thus we kept that assumption here as well. However, we would like to note that the results of this section work over an arbitrary algebraically closed base field $k$.

PART II. THE PROOFS OF THEOREMS 1.1, 1.2 AND 1.4

8. The case $\kappa(Y^o) = -\infty$

8.A. Setup. Let $f^o : X^o \to Y^o$ be a smooth projective family of varieties with semiample canonical bundle, over a quasi-projective manifold $Y^o$ of dimension $\dim Y^o \leq 3$ and logarithmic Kodaira dimension $\kappa(Y^o) = -\infty$.

Consider a smooth compactification $Y$ of $Y^o$ where $D := Y \setminus Y^o$ is a divisor with simple normal crossings. Let $\lambda : Y \dashrightarrow Y_\lambda$ be a sequence of extremal divisorial contractions and flips given by the minimal model program, and let $D_\lambda \subset Y_\lambda$ be the cycle-theoretic image of $D$. We may assume that $(Y_\lambda, D_\lambda)$ satisfies the following properties:

8.1 Properties of $(Y_\lambda, D_\lambda)$.

(8.1.1) The variety $Y_\lambda$ is $\mathbb{Q}$-factorial, and $(Y_\lambda, D_\lambda)$ is a logarithmic dlt pair.

(8.1.2) The pair $(Y_\lambda, D_\lambda)$ does not admit a divisorial or small extremal contraction.

(8.1.3) As $\kappa(Y^o) = -\infty$, either

- $\rho(Y_\lambda) = 1$ and $(Y_\lambda, D_\lambda)$ is $\mathbb{Q}$-Fano, or

- $\rho(Y_\lambda) > 1$ and $(Y_\lambda, D_\lambda)$ admits a non-trivial extremal contraction of fiber type.

8.B. Proof of Theorem 1.2. To prove Theorem [1.2] assume that $f^o$ is a family of canonically polarized varieties and that $f^o$ has positive variation, $\Var(f^o) > 0$. By [VZ02, Thm. 1.4] and Lemma [5.2] this implies that there exists a Viehweg-Zuo sheaf $\mathcal{A}_\lambda$ of positive Kodaira-Iitaka dimension, $\kappa(\mathcal{A}_\lambda) \geq \Var(f^o) > 0$ on $(Y_\lambda, D_\lambda)$. Since $(Y_\lambda, D_\lambda)$ is $\mathbb{Q}$-factorial and dlt, in particular log canonical, Theorem [6.1] implies that $\rho(Y_\lambda) > 1$. Therefore, by (8.1.3), there exists an extremal contraction of fiber type $\pi : Y_\lambda \to C$. Let $F \subset Y_\lambda$ be a general $\pi$-fiber, and $D_{\lambda,F} := D_\lambda|_F$ the restriction of the boundary divisor.
We will now push the family $f^\circ$ down to $F$, to the maximum extent possible. Since the inverse map $\lambda^{-1}$ does not contract any divisor, we may use $\lambda^{-1}$ to pull the family $f^\circ : X^\circ \to Y^\circ$ back to obtain a smooth family of canonically polarized varieties,

$$f_\lambda : X_\lambda \to Y_\lambda \setminus (D_\lambda \cup T), \quad \text{where codim}_{Y_\lambda} T \ge 2.$$ 

Let $f_{\lambda,F} := f_\lambda|_F$ be the restriction of this family to $F$. To prove Theorem 1.2 in our context, it suffices to show that the family $f_{\lambda,F}$ is isotrivial. This will be carried out next.

8.B.1. **Proof of Theorem 1.2 when $F$ is a curve.** If $F$ is a curve, it is entirely contained inside the snc locus of $(Y_\lambda, D_\lambda)$ and does not intersect $T$. Furthermore, it follows from the adjunction formula that $F \cong \mathbb{P}^1$ and that $D_{\lambda,F}$ contains no more than one point. In this situation, the isotriviality of $f_{\lambda,F}$ is well-known, [Kov00, 0.2] and [VZ01, Thm. 0.1]. This shows that the variation $\text{Var}(f^\circ)$ cannot be maximal and finishes the proof of Theorem 1.2.

8.B.2. **Proof of Theorem 1.2 when $F$ is a surface.** Again, we need to show that $f_{\lambda,F}$ is isotrivial. We argue by contradiction and assume that this is not the case. By general choice of $F$, the pair $(F, D_{\lambda,F})$ is again dlt and

$$\text{codim}_F T_F = \text{codim}_{Y_\lambda} T \ge 2, \quad \text{where} \ T_F := T \cap F.$$ 

We claim that there exists a Viehweg-Zuo sheaf $\mathcal{B}_\lambda$ on $(F, D_{\lambda,F})$ which is of positive Kodaira-Iitaka dimension. In fact, an embedded resolution of $D_{\lambda,F} \cup T_F \subseteq F$ provides an snc pair $(\tilde{F}, \tilde{D})$ and a proper morphism $\eta : \tilde{F} \to F$ such that $\eta(\tilde{D}) = D_{\lambda,F} \cup T_F$. The family $f_{\lambda,F}$ pulls back to a family on $\tilde{F} \setminus \tilde{D}$, and [VZ02, Thm. 1.4] asserts the existence of a Viehweg-Zuo sheaf $\mathcal{B}$ on $(\tilde{F}, \tilde{D})$ with $\kappa(\mathcal{B}) > 0$. The existence of a Viehweg-Zuo sheaf $\mathcal{B}_\lambda$ on $(F, D_{\lambda,F})$ with $\kappa(\mathcal{B}_\lambda) \ge \kappa(\mathcal{B}) > 0$ then follows from Lemma 5.2.

On the other hand, $-(K_F + D_{\lambda,F})$ is $\mathbb{Q}$-ample because $\pi$ is an extremal contraction of fiber type. Corollary 6.2 thus asserts that $\kappa(\mathcal{B}_\lambda) \le 0$, a contradiction. This finishes the proof of Theorem 1.2 in case $\kappa(Y^\circ) = -\infty$. \qed

8.C. **Proof of Theorem 1.4.** We maintain the notation and assumptions made in Section 8.B above and assume in addition that $Y$ is a surface. The minimal model map $\lambda$ is then a morphism. As we have seen in Section 8.B.1 the general fiber $F'$ of $\pi \circ \lambda$ is again a rational curve which intersects the boundary in at most one point and that then the restriction of the family $f^\circ$ to the fiber $F' \cap Y^\circ$ is necessarily isotrivial. The detailed descriptions of $Y^\circ$ and of the moduli map in case $\kappa(Y^\circ) = -\infty$ which are asserted in Theorem 1.4 then follow from Corollary 7.6 and Remark 7.6.3. This finishes the proof of Theorem 1.4 in case $\kappa(Y^\circ) = -\infty$. \qed

8.D. **Proof of Theorem 1.1.** To prove Theorem 1.1 we argue by contradiction and assume that both $\kappa(Y^\circ) = -\infty$ and that $\text{Var}(f^\circ) = \dim Y^\circ$. Lemma 5.2 and [VZ02, Thm. 1.4] then give the existence of a big Viehweg-Zuo sheaf $\mathcal{A}_\lambda$ on $(Y_\lambda, D_\lambda)$. The argumentation of Section 8.B applies verbatim and shows the existence of a proper fibration of $\pi : Y_\lambda \to C$ such that the induced family is isotrivial when restricted to the general $\pi$-fiber. That, however, contradicts the assumption that the variation is maximal. Theorem 1.1 is thus shown in the case $\kappa(Y^\circ) = -\infty$. \qed
9. The case $\kappa(Y^\circ) = 0$

9.A. Setup. Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of varieties with semiample canonical bundle, over a quasi-projective variety $Y^\circ$ of dimension $\dim Y^\circ \leq 3$ and logarithmic Kodaira dimension $\kappa(Y^\circ) = 0$. To prove Theorems 1.1 and 1.2, in this case, it suffices to show that $f^\circ$ is not of maximal variation, and even isotrivial if its fibers are canonically polarized. Since those families give rise to Viehweg-Zuo sheaves of positive Kodaira-Iitaka dimension by [VZ02, Thm. 1.4], Theorems 1.1 and 1.2 immediately follow from the following proposition.

**Proposition 9.1.** Let $(Z, \Delta)$ be a dlt logarithmic pair where $Z$ is a $\mathbb{Q}$-factorial variety of dimension $\dim Z \leq 3$ and $\kappa(K_Z + \Delta) = 0$. If $\mathcal{A}$ is any Viehweg-Zuo sheaf on $(Z, \Delta)$, then $\kappa(\mathcal{A}) \leq 0$.

Observe that once Theorem 1.2 holds, the assertion of Theorem 1.4 is vacuous in our case. Accordingly, we do not consider Theorem 1.4 here.

We show Proposition 9.1 in the remainder of the present Section. The proof proceeds by induction on $\dim Z$. If $\dim Z = 1$, the statement of Proposition 9.1 is obvious. We will therefore assume throughout the proof that $\dim Z > 1$, and that the following holds.

**Induction Hypothesis 9.2.** Proposition 9.1 is already shown for all pairs $(Z', \Delta')$ of dimension $\dim Z' < \dim Z$.

We argue by contradiction and assume the following.

**Assumption 9.3.** There exists a Viehweg-Zuo sheaf $\mathcal{A}$ of positive Kodaira-Iitaka dimension $\kappa(\mathcal{A}) > 0$.

We run the minimal model program and obtain a birational map $\lambda : Z \dashrightarrow Z_\lambda$, where $Z_\lambda$ is $\mathbb{Q}$-factorial. If $\Delta_\lambda$ is the cycle-theoretic image, the pair $(Z_\lambda, \Delta_\lambda)$ is dlt, and $K_{Z_\lambda} + \Delta_\lambda$ is semi-ample. Since $\kappa(K_{Z_\lambda} + \Delta_\lambda) = 0$, the divisor $K_{Z_\lambda} + \Delta_\lambda$ is $\mathbb{Q}$-torsion, i.e.,

(9.3.1) $\exists m \in \mathbb{N}$ such that $\mathcal{O}_{Z_\lambda}(m(K_{Z_\lambda} + \Delta_\lambda)) \cong \mathcal{O}_{Z_\lambda}$.

Lemma 5.2 guarantees the existence of a Viehweg-Zuo sheaf $\mathcal{A}_\lambda$ on $(Z_\lambda, \Delta_\lambda)$ with $\kappa(\mathcal{A}_\lambda) > 0$. Raising $\mathcal{A}$ and $\mathcal{A}_\lambda$ to a suitable reflexive power, if necessary, we assume without loss of generality that $\mathcal{A}_\lambda$ is invertible and that $h^0(Z_\lambda, \mathcal{A}_\lambda) > 0$.

9.B. Outline of the proof. Since the proof of Proposition 9.1 is slightly more complicated than most other proofs here, we outline the main strategy for the convenience of the reader.

The main idea is to apply induction, using a component of the boundary divisor $\Delta_\lambda$. For that, we show in Section 9.C that $\mathcal{A}_\lambda$ is not trivial on the boundary, and that there exists a component $\Delta'_\lambda \subseteq \Delta_\lambda$ such that $\kappa(\mathcal{A}_\lambda|_{\Delta'_\lambda}) > 0$. Passing to the index-one cover, we will then in Section 9.D construct a Viehweg-Zuo sheaf of positive Kodaira-Iitaka dimension on the associated boundary component and verify that this component with its natural boundary satisfies all the requirements of Proposition 9.1. This clearly contradicts the Induction Hypothesis 9.3 and finishes the proof.

In order to find $\Delta'_\lambda$ we need to analyze the geometry of $Z_\lambda$ in more detail. For that, we will show in Section 9.C that the minimal model $Z_\lambda$ admits further contractions if one is willing to modify the coefficients of the boundary, compare the remarks in Section 3.B. A second application of the minimal model program then brings us to a dlc logarithmic pair $(Z_\mu, \Delta_\mu)$ that shares many of the good properties of $(Z_\lambda, \Delta_\lambda)$. In addition, it will turn out in Section 9.D that $Z_\lambda$ has the structure of a Mori fiber space. An analysis of the Viehweg-Zuo sheaf along the fibers will be essential.
9.C. Minimal models of \((Z_{\lambda}, (1 - \varepsilon)\Delta_{\lambda})\). As a first step in the program outlined in Section 9.B, we claim that the boundary \(\Delta_{\lambda}\) is not empty, \(\Delta_{\lambda} \neq \emptyset\). In fact, using (9.3.1) and the existence of the Viehweg-Zuo sheaf \(\mathcal{A}_{\lambda}\), this follows immediately from Corollary 6.4. In particular, (9.3.1) implies that \(K_{Z_{\lambda}} \equiv -\Delta_{\lambda}\) and it follows that for any rational number \(0 < \varepsilon < 1\),

\[
\kappa(K_{Z_{\lambda}} + (1 - \varepsilon)\Delta_{\lambda}) = \kappa(\varepsilon K_{Z_{\lambda}}) = \kappa(Z_{\lambda}) = -\infty.
\]

Now choose one \(\varepsilon\) and run the log minimal model program for the dlt pair \((Z_{\lambda}, (1 - \varepsilon)\Delta_{\lambda})\). This way one obtains a birational map \(\mu : Z_{\lambda} \dashrightarrow Z_{\mu}\). Let \(\Delta_{\mu}\) be the cycle-theoretic image of \(\Delta_{\lambda}\). The variety \(Z_{\mu}\) is \(\mathbb{Q}\)-factorial and the pair \((Z_{\mu}, (1 - \varepsilon)\Delta_{\mu})\) is then dlt.

Claim 9.4. The logarithmic pair \((Z_{\mu}, \Delta_{\mu})\) is dlc.

Proof. By (9.3.1) some positive multiples of \(K_{Z_{\lambda}}\) and \(-\Delta_{\lambda}\) are numerically equivalent. For any two rational numbers \(0 < \varepsilon', \varepsilon'' < 1\), the divisors \(K_{Z_{\lambda}} + (1 - \varepsilon')\Delta_{\lambda}\) and \(K_{Z_{\lambda}} + (1 - \varepsilon'')\Delta_{\lambda}\) are thus again numerically equivalent up to a positive rational multiple.

The birational map \(\mu\) is therefore a minimal model program for the pair \((Z_{\lambda}, (1 - \varepsilon)\Delta_{\lambda})\), independently of the number \(\varepsilon\) chosen in its construction. It follows that \((Z_{\mu}, (1 - \varepsilon')\Delta_{\mu})\) has dlt singularities for all \(0 < \varepsilon' < 1\), so \((Z_{\mu}, \Delta_{\mu})\) is indeed dlc. \(\square\)

9.D. The fiber space structure of \(Z_{\mu}\). Since the Kodaira-dimension of \((Z_{\lambda}, (1 - \varepsilon)\Delta_{\lambda})\) is negative by (9.3.2), either \(\rho(Z_{\mu}) = 1\), or \(\rho(Z_{\mu}) > 1\) and the pair \((Z_{\mu}, (1 - \varepsilon)\Delta_{\mu})\) admits an extremal contraction of fiber type. We apply Theorem 6.1 in order to show that the Picard number cannot be one.

Proposition 9.5. The Picard number of \(Z_{\mu}\) is not one. The pair \((Z_{\mu}, (1 - \varepsilon)\Delta_{\mu})\) therefore admits a non-trivial extremal contraction of fiber type, \(\pi : Z_{\mu} \to \tilde{W}\).

Proof. As the birational map \(\mu\) is a sequence of extremal divisorial contractions and flips, the inverse of \(\mu\) does not contract any divisors. This has two consequences. First, the divisor \(K_{Z_{\mu}} + \Delta_{\mu}\) is torsion, and \(- (K_{Z_{\mu}} + \Delta_{\mu})\) is nef. On the other hand, Lemma 5.2 applies and shows the existence of a Viehweg-Zuo sheaf \(\mathcal{A}_{\mu}\) of positive Kodaira-Iitaka dimension. Since we have seen in Claim 9.4 that \((Z_{\mu}, \Delta_{\mu})\) is dlc, in particular log canonical, and since we know that \(Z_{\mu}\) is \(\mathbb{Q}\)-factorial, Theorem 6.1 then gives that \(\rho(Z_{\mu}) > 1\), as desired. \(\square\)

Now let \(F \subset Z_{\mu}\) be a general fiber of \(\pi\), and set \(\Delta_F := \Delta_{\mu} \cap F\). Since normality is preserved when passing to general elements of base point free systems, \([\text{BS95, Thm. 1.7.1}]\), and since discrepancies only increase, the logarithmic pair \((F, \Delta_F)\) is again dlc.

Remark 9.6. The adjunction formula gives that \(K_F + \Delta_F\) is torsion. On the other hand, \(\pi\) is an extremal contraction so \(- (K_F + (1 - \varepsilon)\Delta_F)\) is \(\pi\)-ample. It follows that the boundary divisor of \(F\) cannot be empty, \(\Delta_F \neq \emptyset\). It is not clear to us whether in general \(F\) is necessarily \(\mathbb{Q}\)-factorial.

9.E. Non-triviality of \(\mathcal{A}_{\mu}|_{\Delta_{\mu}}\). As in Section 9.A, Lemma 5.2 guarantees the existence of a Viehweg-Zuo sheaf \(\mathcal{A}_{\mu}\) on \((Z_{\mu}, \Delta_{\mu})\) with \(\kappa(\mathcal{A}_{\mu}) \geq \kappa(\mathcal{A}_{\lambda}) > 0\). Again, passing to a suitable reflexive power, we can assume that \(\mathcal{A}_{\mu}\) is invertible and that \(h^0(Z_{\mu}, \mathcal{A}_{\mu}) > 0\).

Proposition 9.7. The restriction \(\mathcal{A}_{\mu}|_F\) has Kodaira-Iitaka dimension zero, \(\kappa(\mathcal{A}_{\mu}|_F) = 0\).

Proof. Consider the open set \(F^\circ := (F, \Delta_F)_{\text{reg}} \cap (Z_{\mu}, \Delta_{\mu})_{\text{reg}}\). The fiber \(F\) being general, it is clear that \(\text{codim}_F(F \setminus F^\circ) \geq 2\). On \(F^\circ\), the standard conormal sequence \([\text{KK08}]\).
Lem. 2.13] for logarithmic differentials then gives a short exact sequence of locally free sheaves, as follows,

\[ 0 \rightarrow \pi^* \left( \Omega^1_W \right)_{|F_0} \rightarrow \Omega^1_{Z_\mu} (\log \Delta_\mu)_{|F_0} \rightarrow \Omega^1_F (\log \Delta_F)_{|F_0} \rightarrow 0. \]

(9.7.1)

By the definition of a “Viehweg-Zuo sheaf”, there exists a number \( n \in \mathbb{N} \) and an embedding \( \mathcal{A}_\mu \rightarrow (\Omega^1_{Z_\mu} (\log \Delta_\mu)_{|F_0})^\otimes n \). The first term in (9.7.1) being trivial, Lemma 5.3 gives a number \( m \leq n \) and an injection

\[ \mathcal{A}_\mu \mid_{F_0} \hookrightarrow (\Omega^1_F (\log \Delta_F)_{|F_0})^\otimes m. \]

(9.7.2)

Recall that \( \mathcal{A}_\mu \) is invertible. Then by (9.7.2) we obtain an injection between the reflexive hulls \( \mathcal{A}_\mu \mid_{F} \hookrightarrow (\Omega^1_F (\log \Delta_F)) [m] \), i.e., we realize \( \mathcal{A}_\mu \mid_{F} \) as a Viehweg-Zuo sheaf on \((F, \Delta_F)\).

The log canonical divisor \( K_F + \Delta_F \) being torsion, Proposition 9.7 follows immediately if \( F \) is a curve. We will thus assume for the remainder of the proof that \( \dim F = 2 \).

It remains to show that the Viehweg-Zuo sheaf \( \mathcal{A}_\mu \mid_{F} \) on \((F, \Delta_F)\) has Kodaira-Iitaka dimension \( \kappa(\mathcal{A}_\mu \mid_{F}) \leq 0 \). The fact that \( \kappa(\mathcal{A}_\mu) > 0 \) will then imply that \( \kappa(\mathcal{A}_\mu \mid_{F}) = 0 \), as claimed. In order to do this, consider a log resolution \( \psi : (\widetilde{F}, \widetilde{\Delta}_F) \rightarrow (F, \Delta_F) \). Setting

\[ E := \text{maximal reduced divisor in } \psi^{-1}(\Delta_F) \cup \text{Exc}(\psi), \]

it follows immediately from the definition of dlc that \( \widetilde{K}_F + E \) is represented by the sum of a torsion divisor and an effective, \( \psi \)-exceptional divisor. In particular, \( \kappa(K_F + E) = 0 \), and Theorem 5.3 gives the existence of a Viehweg-Zuo sheaf \( \mathcal{E} \) on the snc pair \((\widetilde{F}, \widetilde{\Delta}_F)\) with \( \kappa(\mathcal{E}) \geq \kappa(\mathcal{A}_\mu \mid_{F}) \). However, this contradicts the Induction Hypothesis 9.2 which asserts that \( \kappa(\mathcal{E}) \leq 0 \).

**Corollary 9.8.** The restriction \( \mathcal{A}_\mu \mid_{F} \) is trivial, i.e., \( \mathcal{A}_\mu \mid_{F} \cong \mathcal{O}_F \).

**Proof.** Since \( \mathcal{A}_\mu \) is invertible and \( h^0(Z_\mu, \mathcal{A}_\mu) > 0 \), there exists an effective Cartier divisor \( D \) on \( Z_\mu \) with \( \mathcal{A}_\mu \cong \mathcal{O}_{Z_\mu}(D) \). Decompose \( D = D^h + D^v \), where \( D^h \) consists of those components that dominate \( W \), and \( D^v \) of those components that do not. We need to show that \( D^h = 0 \). Again, we argue by contradiction and assume that \( D^h \) is non-trivial.

Recall that \( \pi : Z_\mu \rightarrow W \) is a contraction of an extremal ray and that the relative Picard number \( \rho(Z_\mu/W) \) is therefore one. The divisor \( D^h \) is thus relatively ample, contradicting Proposition 9.7. \( \square \)

**Corollary 9.9.** There exists a component \( \Delta_{\lambda, 1} \subseteq \Delta_\lambda \) such that \( \kappa(\mathcal{A}_\lambda \mid_{\Delta_{\lambda, 1}}) > 0 \).

**Proof.** We have seen in Remark 9.6 that \( \Delta_F = \Delta_\mu \cap F \) is not empty. So, there exists a component \( \Delta_{\mu, 1} \subseteq \Delta_\mu \) that intersects all \( \pi \)-fibers. Let \( \Delta_{\lambda, 1} \subseteq \Delta_\lambda \) be its strict transform. Since the birational map \( \mu \) does not contract \( \Delta_{\lambda, 1} \), and since \( \mu^{-1} \) does not contract any divisors, \( \mu \) induces an isomorphism of open sets \( U_\lambda \subseteq Z_\lambda \) and \( U_\mu \subseteq Z_\mu \) such that \( \Delta_{\lambda, 1} := \Delta_{\lambda, 1} \cap U_\lambda \) and \( \Delta_{\mu, 1} := \Delta_{\mu, 1} \cap U_\mu \) are both non-empty.

For an arbitrary \( m \in \mathbb{N} \) we obtain a commutative diagram of linear maps,

\[
\begin{array}{ccc}
H^0(Z_\lambda, \mathcal{A}_\lambda^{\otimes m}) & \xrightarrow{\alpha_1 \text{ restr.}} & H^0(\Delta_{\lambda, 1}, \mathcal{A}_\lambda^{\otimes m}|_{\Delta_{\lambda, 1}}) \\
\mu_1 & & \mu_2 \\
H^0(Z_\mu, \mathcal{A}_\mu^{\otimes m}) & \xrightarrow{\beta_1 \text{ restr.}} & H^0(\Delta_{\mu, 1}, \mathcal{A}_\mu^{\otimes m}|_{\Delta_{\mu, 1}})
\end{array}
\]

\[ \xrightarrow{\beta_2 \text{ restr.}} H^0(\Delta_{\mu, 1}, \mathcal{A}_\mu^{\otimes m}|_{\Delta_{\mu, 1}}). \]
where the $\mu_i$, $i = 1, 2$ are the obvious push-forward morphisms coming from the construction of $\mathcal{A}_\mu$ in Lemma 5.2. Since $\mu_1$ and $\beta_2$ are clearly injective, Corollary 9.9 will follow once we show that $\beta_1$ is injective as well. Now, let $\sigma \in H^0(\mathcal{O}_Z, \mathcal{A}_\mu(\sigma))$ and assume that $\sigma$ is in the kernel of $\beta_1$. By choice of $\Delta_\mu, 1$, any general fiber $F$ intersects $\Delta_\mu, 1$ in at least one point. The triviality of $\mathcal{A}_\mu|_F$ asserted in Corollary 9.8 then implies that $\sigma$ vanishes along $F$. The fiber $F$ being general, we obtain that $\sigma = 0$ on all of $Z_\mu$. Corollary 9.9 follows.

9.F. **Existence of pluri-forms on the boundary.** Now consider the index-one-cover $\gamma : (Z_\lambda', \Delta_\lambda') \rightarrow (Z_\lambda, \Delta_\lambda)$, as described in Proposition 9.1. The pair $(Z_\lambda', \Delta_\lambda')$ is then dlt, the log canonical divisor is trivial, $\mathcal{O}_{Z_\lambda'}(K_{Z_\lambda'} + \Delta_\lambda') \cong \mathcal{O}_{Z_\lambda'}$, and the pull-back $\mathcal{A}_\lambda' := \gamma^* (\mathcal{A}_\lambda)$ is an invertible Viehweg-Zuo sheaf on $(Z_\lambda', \Delta_\lambda')$ with $\kappa(\mathcal{A}_\lambda') > 0$. Better still, if $\Delta_\lambda', 1 \subseteq \gamma^{-1}(\Delta_\lambda, 1)$ is any component, Corollary 9.9 immediately implies that $\kappa(\mathcal{A}_\lambda'|_{\Delta_\lambda', 1}) > 0$.

Now recall from Lemma 5.1 that $(Z_\lambda', \Delta_\lambda')$ is snc along the boundary away from a closed subset $W$ with $\text{codim}_Z(W \cap \Delta_\lambda') \geq 3$. The divisor $\Delta_\lambda'$ is therefore Cartier in codimension two and inversion of adjunction applies, cf. [KM98 Sect. 5.4]. Setting

$$\Delta_\lambda'' : = (\Delta_\lambda' - \Delta_\lambda', 1)|_{\Delta_\lambda', 1},$$

this yields the following:

1. (9.9.1) the subvariety $\Delta_\lambda', 1$ is normal [KM98 Cor. 5.52] and
2. (9.9.2) the pair $(\Delta_\lambda', 1, \Delta_\lambda'')$ is again logarithmic and dlt [KM98 Prop. 5.59].

**Observation 9.10.** It follows from the adjunction formula that the log canonical divisor $K_{\Delta_\lambda', 1} + \Delta_\lambda''$ is trivial.

**Proposition 9.11.** The pair $(\Delta_\lambda', 1, \Delta_\lambda'')$ admits a Viehweg-Zuo sheaf of positive Kodaira-Iitaka dimension.

**Proof.** Consider the standard conormal sequence for logarithmic differentials [KK08 Lem. 2.13] on the open subset $\Delta_\lambda'' := (\Delta_\lambda', 1, \Delta_\lambda')_{\text{reg}},$

$$\Omega_{\Delta_\lambda''}^1 ((\log \Delta_\lambda'')) \rightarrow \Omega_{\Delta_\lambda'}^1 ((\log \Delta_\lambda')) \rightarrow \Omega_{\Delta_\lambda'}^1 ((\log \Delta_\lambda'))|_{\Delta_\lambda'} \rightarrow \mathcal{O}_{\Delta_\lambda''} \rightarrow 0.$$

The last term in (9.11.1) being trivial, Lemma 5.4 gives a number $m \leq n$ and an injection

$$\mathcal{A}_\lambda'|_{\Delta_\lambda''} \hookrightarrow \left( \Omega_{\Delta_\lambda''}^1 ((\log \Delta_\lambda'')) \right)^{\otimes m}.$$

Using that $\mathcal{A}_\lambda'$ is invertible and that $\text{codim}_{\Delta_\lambda''} (W \cap \Delta_\lambda', 1) \geq 2$, we pass to reflexive hulls and realize $\mathcal{A}_\lambda'$ as a Viehweg-Zuo sheaf on $\Delta_\lambda, 1$,

$$\mathcal{A}_\lambda'|_{\Delta_\lambda'} \subseteq \left( \Omega_{\Delta_\lambda'}^1 ((\log \Delta_\lambda')) \right)^{[m]}.$$

9.G. **Completion of the proof.** Recall that we have seen that the pair $(\Delta_\lambda', 1, \Delta_\lambda'')$ is dlt, has trivial log canonical class and admits a Viehweg-Zuo sheaf of positive Kodaira-Iitaka dimension. Since $\text{dim} \Delta_\lambda', 1 \leq 2$, being dlt implies that the variety $\Delta_\lambda, 1$ is Q-factorial, cf. [KM98 Prop. 4.11]. This clearly contradicts the Induction Hypothesis 9.2. Assumption 9.3 is therefore absurd. This finishes the proof of Proposition 9.1. Consequently, Theorems 1.1 and 1.2 are shown in case $\kappa(Y^o) = 0$. □
10. The Case $\kappa(Y^o) > 0$

10.A. Setup. Let $f^o : X^o \to Y^o$ be a smooth projective family of varieties with semiample canonical bundle over a quasi-projective variety $Y^o$ of dimension $\dim Y^o \leq 3$ and logarithmic Kodaira dimension $\kappa(Y^o) > 0$.

Again, let $Y$ be a compactification of $Y^o$ where $D := Y \setminus Y^o$ is a divisor with simple normal crossings, and let $\lambda : (Y, D) \to (Y_\lambda, D_\lambda)$ be the map to a minimal model. The divisor $K_{Y_\lambda} + D_\lambda$ is then semi-ample by the log abundance theorem [KMMc04] and defines a map $\pi : Y_\lambda \to C$ with $\dim C = \kappa(Y^o)$.

10.B. Proof of Theorem 1.2. To prove Theorem 1.2, assume that $f^o$ is a family of canonically polarized manifolds. We may also assume without loss of generality that the family $f^o$ is not isotrivial and that $\kappa(Y^o) < \dim Y$. Blowing up $Y$ and pulling back the family, we obtain a diagram as follows,

If $\bar{F} \subset \bar{Y}$ is the general $\pi$-fiber, recall the standard fact that $\kappa(K_{\bar{F}} + \bar{D}|_{\bar{F}}) = 0$, cf. [Iit82, sect. 11.6]. We saw in Section 9 that then the family $\bar{f}^o$ must be isotrivial over $\bar{F}$. This shows that the fibration $\pi$ factors the moduli map birationally, and proves Theorem 1.2 in case $\kappa(Y^o) > 0$. $\square$

10.C. Proof of Theorem 1.4. It remains to prove Theorem 1.4 and give a detailed description of the moduli map if $Y$ is a surface.

To this end, we maintain the notation and assumptions made in Section 10.A above and assume in addition that $Y$ is a surface, that $\text{Var}(f^o) > 0$, and that $\kappa(Y^o) = 1$. As there are no flipping contractions in dimension two, $\lambda$ is a birational morphism, and $K_{Y_\lambda} + D_\lambda$ is trivial on the general $\pi$-fiber $F_\lambda \subset Y_\lambda$. In particular, one of the following holds:

- $F_\lambda$ is an elliptic curve and no component of $D_\lambda$ dominates $C$, or
- $F_\lambda$ is isomorphic to $\mathbb{P}^1$ and intersects $D_\lambda$ in exactly two points.

If the general fibers of $\pi$ are isomorphic to $(\mathbb{A}^1)^*$, Corollary 7.6 gives the statement of Theorem 1.4.

Otherwise, let $V \subset C$ be an open subset such that $\pi$ is a smooth elliptic fibration over $V$. Let $\bar{V} \subset Y_\lambda$ be a general hyperplane section. Restricting $V$ further if necessary we may assume that $\bar{V}$ is étale over $V$. Taking a base change to $V$, we obtain a section $\sigma : \bar{V} \to \bar{U} := U \times_V \bar{V}$. Finally, set $\bar{X} := X \times_U \bar{U}$, and $Z := \bar{V} \times_\sigma \bar{X}$. Shrinking $V$ further, if necessary, an application of Lemma 7.4 completes the proof. $\square$

10.D. Proof of Theorem 1.1. To prove Theorem 1.1, we argue by contradiction and assume that $0 < \kappa(Y^o) < \dim Y^o$ and that $\text{Var}(f^o) = \dim Y^o$. The argumentation of Section 10.B applies verbatim and shows the existence of a proper fibration of $\bar{\pi} : \bar{Y} \to C$ such that the family $f^o$ is isotrivial when restricted to the general $\bar{\pi}$-fiber. That, however, contradicts the assumption that the variation is maximal. Theorem 1.1 is thus shown in case $\kappa(Y^o) > 0$. $\square$
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