I. INTRODUCTION

Decoherence is the most important obstacle in quantum computing. It causes a quantum computer to lose its quantum properties destroying its performance advantages over a classical computer. Therefore, in order to maintain quantum coherence in any computing system, it is important to remove the unwanted entanglement with its noisy environment. The unavoidable interaction between the open quantum computing system and its environment corrupts the information stored in the system and causes computational errors that may lead to wrong outputs. In general, environments may be very complex systems characterized by many uncontrollable degrees of freedom. The principal task of quantum error correction (QEC, [1, 2]) is to tackle this decoherence problem. For a comprehensive introduction to quantum error correction, we refer to the work presented in [3]. In summary, there exists two strategies to defend quantum coherence of a processing against environmental noise. The first strategy is that of quantum error correcting codes (QECC) [4, 5] where, in analogy to classical information theory, quantum information is stabilized by using redundant encoding and measurements. This is also known as an active strategy. The second strategy is known as noiseless quantum coding (also known as error avoiding quantum coding or decoherence free subspaces (DFSs)) [6–10]. This is a passive strategy where quantum information is stabilized by exploiting symmetry properties of the environment-induced noise for suitable redundant encoding.

The formal mathematical description of the qubit-environment interaction is usually given in terms of quantum channels. When noise errors act independently on each qubit, we talk about memoryless (noisy quantum) channels and independent error models. Instead, when noise errors do not affect qubits independently but correlations between errors on different qubits must be taken into consideration, we talk about memory (noisy quantum) channels and correlated error models. Correlations between errors may be considered either temporally over each use of a single channel, or spatially between uses of many parallel channels. QECC were developed under the assumption of i.i.d. (identically and independently distributed) errors. Recent studies on the performance of quantum codes for memory channels appear in [11, 12]. In Ref. [13], the performance of some codes (CSS codes and n-qubit repetition code) for spatially-correlated errors were studied and characterized by means of the lowest order temporal expansion of the fidelity of the density operator representing the quantum register after a single application of error correction.

In this Letter, we study the performance of simple quantum error correcting codes in the presence of correlated (classical-like) noise error models characterized by a correlation strength. Specifically, we consider bit flip (phase flip) noisy quantum memory channels and use repetition and noiseless quantum codes. We characterize the performance of such error correcting codes by means of the entanglement fidelity [14] as function of the error probability and degree of memory. Finally, comparing the entanglement fidelities of repetition codes and noiseless quantum codes, we find a threshold $\mu^*$ ($p$) for the correlation strength that allows to select the code with better performance.

The layout of this Letter is as follows. In Section II, the algorithmic structure of a basic quantum error correcting code and a brief description of independent and correlated error models are presented. In Section III, we introduce a simple error model in the presence of correlated errors. Specifically, we consider bit flip (or phase flip) noisy quantum
II. QUANTUM ERROR CORRECTION AND ERROR MODELS

In this Section, we present the algorithmic structure of a basic quantum error correcting code and briefly describe independent and correlated error models in quantum computation.

A. Algorithmic Structure of Quantum Error Correction

Characterizing the Error Model. We may deal with error models where errors occur on single qubits independently (independent error models). However, we may also consider models in which errors do not affect qubits independently. In this case we have to take into account any correlation between errors on different qubits (correlated error model). In any case, the error model is completely described by the Kraus operators \{A_j\} defining the quantum error operation \Lambda. The trace-preserving superoperator \Lambda is described as follows,

\[ \rho \overset{\text{def}}{=} |\psi\rangle \langle \psi| \xrightarrow{\Lambda} \Lambda(\rho) \overset{\text{def}}{=} \sum_j A_j |\psi\rangle \langle \psi| A_j^\dagger. \]  

Introducing Redundancy by Choosing the Encoding. The basic idea behind the redundancy is that even when errors corrupt some of the qubits in a codeword \(|\psi_{\text{enc}}\rangle\) \[\text{[1]},\] the remaining qubits contain enough information so that the logical qubit \(|\psi\rangle\), representing the quantum information to be transmitted through the quantum noisy communication channel \Lambda, can be recovered. In quantum error correction, encoding is implemented via a unitary operator \(U_{\text{enc}}\) that acts on the state we wish to encode, tensored with an ancilla of some fixed number of qubits in some specified initial state. The goal is to choose the encoding operation \(U_{\text{enc}}\) in such a way that the behavior of these transformed errors allows us to find a recovery operation \(R\) that gives back \(|\psi\rangle \langle \psi| \otimes \rho_{\text{noise}}\). The encoding operation is described as follows,

\[ |\psi\rangle \overset{\text{tensor product}}{\rightarrow} |\psi\rangle \otimes |00...0\rangle \overset{U_{\text{enc}}}{\rightarrow} U_{\text{enc}} (|\psi\rangle \otimes |00...0\rangle) \overset{\text{def}}{=} |\psi_{\text{enc}}\rangle. \] 

In other words, we consider the tensor product between the logical qubit \(|\psi\rangle\) and the ancilla qubit \(|00...0\rangle\), and then we encode \(|\psi\rangle \otimes |00...0\rangle\) via a unitary operator \(U_{\text{enc}}\), obtaining \(U_{\text{enc}} (|\psi\rangle \otimes |00...0\rangle) \overset{\text{def}}{=} |\psi_{\text{enc}}\rangle\).

Finding a Procedure for Error Recovery. The encoding operation \(U_{\text{enc}}\) can be seen as a way of transforming the encoded errors \(A'_j\) so that their action on the codeword states \(|\psi_{\text{enc}}\rangle\) is recoverable. Before encoding, the quantum error operation \(\Lambda'\) is defined as in \[\text{[1]}\]. After encoding, the new quantum error operation \(\Lambda'\) is defined as follows,

\[ \rho_{\text{enc}} \overset{\text{def}}{=} |\psi_{\text{enc}}\rangle \langle \psi_{\text{enc}}| \xrightarrow{\Lambda'} \Lambda'(\rho_{\text{enc}}) \overset{\text{def}}{=} \sum_j A'_j |\psi_{\text{enc}}\rangle \langle \psi_{\text{enc}}| A'_j^\dagger. \]

The noise operators \(A'_j\) act on the codeword \(|\psi_{\text{enc}}\rangle\) which lives in a larger dimensional space than that of the original quantum state \(|\psi\rangle\). The noise operators \(A_j\) act on \(|\psi\rangle\). Notice that in general,

\[ \text{Tr}_{\text{enc}} \left[ \sum_k U_{\text{enc}}^\dagger \left( \sum_j A'_j U_{\text{enc}} |\psi\rangle \langle 00...0| \langle 00...0| U_{\text{enc}}^\dagger A'_j^\dagger \right) U_{\text{enc}} \right] \neq |\psi\rangle \langle \psi|. \] 

Therefore, if we encode the quantum information \(|\psi\rangle\), subject it to the noise \(A'_j\) and decode using the inverse of the encoding operation, \(U_{\text{enc}}^\dagger = U_{\text{enc}}^{-1} \overset{\text{def}}{=} U_{\text{dec}}\), we will not always recover the original state \(|\psi\rangle\). To recover \(|\psi\rangle\) we need to
introduce an error recovery operation $\mathcal{R}$ that has the effect of undoing enough of the noise $A_{j}^\dagger$ on the codeword state $|\psi_{\text{enc}}\rangle$ so that after decoding and tracing out we are left with $|\psi\rangle$ [2],

$$\text{Tr}_{\text{enc}} \left[ \sum_{k} \mathcal{R}_{k} U_{\text{enc}}^\dagger \left( \sum_{j} A_{j}^\dagger U_{\text{enc}} |\psi\rangle \langle 00...0 | \right) \langle \psi | U_{\text{enc}} A_{j}^\dagger \right] U_{\text{enc}} \mathcal{R}_{k}^\dagger = |\psi\rangle \langle \psi|.$$  \hfill (5)

The design of a quantum error correcting code can be reduced to finding a unitary encoding operator $U_{\text{enc}}$ and a recovery operation $\mathcal{R}$ so that, given an error model corresponding to a specified set of error operators $A_{j}$, equation (4) is valid. The action of the recovery operator $\mathcal{R}$ may be interpreted as pushing all the noise into the ancilla $|00...0\rangle$ so that the errors are eliminated when the ancilla is traced out.

**Computing the Entanglement Fidelity.** In [14], Schumacher introduced the concept of entanglement fidelity as a useful indicator of the efficiency of quantum error correcting codes. The entanglement fidelity is defined for a mixed state $\rho = \sum_{i} p_{i} \rho_{i} = \text{tr}_{\mathcal{H}_{R}} |\psi\rangle \langle \psi|$ in terms of a purification $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}_{R}$ to a reference system $\mathcal{H}_{R}$. The purification $|\psi\rangle$ encodes all of the information in $\rho$. Entanglement fidelity is a measure of how well the channel $\Lambda$ preserves the entanglement of the state $\mathcal{H}$ with its reference system $\mathcal{H}_{R}$. The entanglement fidelity is defined as follows [14],

$$\mathcal{F}(\rho, \Lambda) \overset{\text{def}}{=} \langle \psi | (\Lambda \otimes I_{\mathcal{H}_{R}}) (|\psi\rangle \langle \psi|) |\psi\rangle,$$  \hfill (6)

where $|\psi\rangle$ is any purification of $\rho$, $I_{\mathcal{H}_{R}}$ is the identity map on $\mathcal{M}(\mathcal{H}_{R})$ and $\Lambda \otimes I_{\mathcal{H}_{R}}$ is the evolution operator extended to the space $\mathcal{H} \otimes \mathcal{H}_{R}$, space on which $\rho$ has been purified. If the quantum operation $\Lambda$ is written in terms of its Kraus operator elements $\{A_{k}\}$ as, $\Lambda(\rho) = \sum_{k} A_{k} \rho A_{k}^\dagger$, then it can be shown that [10],

$$\mathcal{F}(\rho, \Lambda) = \sum_{k} \text{tr} (A_{k} \rho) \text{tr} \left( A_{k}^\dagger \rho \right) = \sum_{k} |\text{tr} (\rho A_{k})|^2.$$  \hfill (7)

This expression for the entanglement fidelity is very useful for explicit calculations. Finally, assuming that

$$\Lambda: \mathcal{M}(\mathcal{H}) \ni \rho \mapsto \Lambda(\rho) = \sum_{k} A_{k} \rho A_{k}^\dagger \in \mathcal{M}(\mathcal{H}), \text{dim}_{\mathcal{H}} = N$$  \hfill (8)

and choosing a purification described by a maximally entangled unit vector $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ for the mixed state $\rho = \frac{1}{\text{dim}_{\mathcal{H}}} I_{\mathcal{H}}$, we obtain

$$\mathcal{F} \left( \frac{1}{N} I_{\mathcal{H}}, \Lambda \right) = \frac{1}{N^2} \sum_{k} |\text{tr} A_{k}|^2.$$  \hfill (9)

The expression in (9) represents the entanglement fidelity when no error correction is performed on the noisy channel $\Lambda$ in (8). In this Letter, we will follow the general theoretical framework describing requirements for quantum error correcting codes presented in [4].

**B. Error Models**

To introduce noise models, we assume a quantum channel (CPTP map) $\Lambda$ on $n$-uses to be expressible by means of the following Kraus decomposition,

$$\Lambda^{(n)}(\rho) = \sum_{i_1, ..., i_n} p(i_n, i_{n-1}, ..., i_1) (A_{i_n} \otimes ... \otimes A_{i_1}) \rho (A_{i_n} \otimes ... \otimes A_{i_1})^\dagger.$$  \hfill (10)

If the probability $p(i_n, i_{n-1}, ..., i_1)$ is factorized in the product of $n$-independent probabilities, $p(i_n, i_{n-1}, ..., i_1) = \prod_{i=1}^{n} p(i_i)$, we are in the presence of a memoryless channel and $\Lambda^{(n)}(\rho) = \Lambda^{\otimes n}(\rho) \overset{\text{def}}{=} \Lambda^{\otimes ... \otimes \Lambda}$. On the contrary if $p(i_n, i_{n-1}, ..., i_1)$ is not separable in the product of of $n$-independent probabilities, then $\Lambda$ is a memory channel with $\Lambda^{(n)}(\rho) \neq \Lambda^{\otimes n}(\rho)$. For instance, a very important class of quantum memory channels is described by the Markovian correlated noise channels of length $n$,

$$\Lambda^{(n)}(\rho) \overset{\text{def}}{=} \sum_{i_1, ..., i_n} p(i_n|i_{n-1}) p(i_{n-1}|i_{n-2}) ... p(i_2|i_1) p(i_1) (A_{i_n} \otimes ... \otimes A_{i_1}) \rho (A_{i_n} \otimes ... \otimes A_{i_1})^\dagger,$$  \hfill (11)

with $p(i_l|i_{l-1}) \overset{\text{def}}{=} (1 - \mu) p_{i_l} + \mu \delta_{i_l, i_{l-1}}, \forall l = 1, ..., n$. The correlation parameter $\mu$ describes the degree of memory of the channel considered.
III. REPETITION CODES FOR CORRELATED BIT FLIP

In this Section, we introduce a simple error model in the presence of correlated errors. Specifically, we consider bit flip (or phase flip) noisy quantum memory channels and QEC is performed via odd and even repetition codes (RC) [17]. Although the error models considered are classical in nature, from this preliminary work we hope to gain useful insights for extending error correction techniques to quantum correlated error models. The performance of quantum error correcting codes is quantified by means of the entanglement fidelity $F^{(n)}_{RC} (\mu, p)$ as function of the error probability $p$ and degree of memory $\mu$.

A. CASE, $n_{odd} = 3$

Consider $n$ qubits and correlated errors in a bit flip quantum channel,

$$\Lambda^{(n)}(\rho) \overset{\text{def}}{=} \sum_{i_1, \ldots, i_n=0}^{1} p_{i_n|i_{n-1}}p_{i_{n-1}|i_{n-2}} \cdots p_{i_2|i_1}p_{i_1} (A_{i_n} \otimes \cdots \otimes A_{i_1}) \rho (A_{i_n} \otimes \cdots \otimes A_{i_1})^\dagger,$$  \hfill (12)

where $A_0 \overset{\text{def}}{=} I, A_1 \overset{\text{def}}{=} X$ are Pauli operators. Furthermore,

$$p_{i_k|i_j} = (1-\mu)p_{i_k} + \mu \delta_{i_k, i_j}, \quad p_{i_k=0} = 1 - p, p_{i_k=1} = p,$$  \hfill (13)

with,

$$\sum_{i_1, \ldots, i_n=0}^{1} p_{i_n|i_{n-1}}p_{i_{n-1}|i_{n-2}} \cdots p_{i_2|i_1}p_{i_1} = 1.$$  \hfill (14)

To simplify our notation, we may assume that $A_{i_n} \otimes \cdots \otimes A_{i_1} = A_{i_n} \ldots A_{i_1}$.

Error Operators. In the simplest example, consider three qubits $(n=3)$ and correlated errors in a bit flip channel,

$$\Lambda^{(3)}(\rho) \overset{\text{def}}{=} \sum_{i_1, i_2, i_3=0}^{1} p_{i_3|i_2}p_{i_2|i_1}p_{i_1} \left[ A_{i_3}A_{i_2}A_{i_1}\rho A_{i_1}^\dagger A_{i_2}^\dagger A_{i_3}^\dagger \right], \text{ with } \sum_{i_1, i_2, i_3=0}^{1} p_{i_3|i_2}p_{i_2|i_1}p_{i_1} = 1.$$  \hfill (15)

Substituting (13) in (15), it follows that the error superoperator $\mathcal{A}$ associated to channel (15) is defined in terms of the following error operators,

$$\mathcal{A} \leftrightarrow \{ A_0', \ldots, A_7' \} \text{ with } \Lambda^{(3)}(\rho) \overset{\text{def}}{=} \sum_{k=0}^{7} A_k' \rho A_k'^\dagger \text{ and, } \sum_{k=0}^{7} A_k'^\dagger A_k' = I_{8 \times 8}.$$  \hfill (16)

In an explicit way, the error operators $\{ A_0', \ldots, A_7' \}$ are given by,

$$A_0' = \sqrt{p_0^{(3)}} I \otimes I \otimes I, \quad A_1' = \sqrt{p_1^{(3)}} X \otimes I \otimes I, \quad A_2' = \sqrt{p_2^{(3)}} I \otimes X \otimes I,$$

$$A_3' = \sqrt{p_3^{(3)}} I \otimes I \otimes X, \quad A_4' = \sqrt{p_4^{(3)}} X \otimes X \otimes I, \quad A_5' = \sqrt{p_5^{(3)}} I \otimes I \otimes X,$$

$$A_6' = \sqrt{p_6^{(3)}} I \otimes X \otimes X, \quad A_7' = \sqrt{p_7^{(3)}} X \otimes X \otimes X,$$  \hfill (17)

where the coefficients $\hat{p}_k^{(3)}$ for $k = 1, \ldots, 7$ are given by,

$$\hat{p}_0^{(3)} = p_{00}^2p_0, \quad \hat{p}_1^{(3)} = p_{00}p_{10}p_0, \quad \hat{p}_2^{(3)} = p_{01}p_{10}p_0, \quad \hat{p}_3^{(3)} = p_{00}p_{01}p_1,$$

$$\hat{p}_4^{(3)} = p_{10}p_{11}p_0, \quad \hat{p}_5^{(3)} = p_{01}p_{10}p_1, \quad \hat{p}_6^{(3)} = p_{01}p_{11}p_1, \quad \hat{p}_7^{(3)} = p_{11}^2p_1,$$  \hfill (18)
with,
\[ p_0 = (1 - p), \quad p_1 = p, \quad p_{00} = ((1 - \mu)(1 - p) + \mu), \]
\[ p_{01} = (1 - \mu)(1 - p), \quad p_{10} = (1 - \mu)p, \quad p_{11} = ((1 - \mu)p + \mu). \]  

(19)

**Encoding and Decoding Operators.** Consider a repetition code that encodes 1 logical qubit into 3-physical qubits. We have,
\[ |0\rangle \longrightarrow |0\rangle \otimes |00\rangle = |000\rangle \text{ def } |0_L\rangle, \quad |1\rangle \longrightarrow |1\rangle \otimes |00\rangle = |100\rangle \quad U_{\text{CNOT}}^{12} \otimes I^3 |110\rangle \quad U_{\text{CNOT}}^{12} \otimes I^3 |111\rangle \text{ def } |1_L\rangle. \]

(20)

The operator \( U_{\text{CNOT}}^{ij} \) is the CNOT gate from qubit \( i \) to \( j \) defined as,
\[ U_{\text{CNOT}}^{ij} \text{ def } \frac{1}{2} \left[(I^i + Z^i) \otimes I^j + (I^i - Z^i) \otimes X^j\right]. \]

(21)

Finally, the encoding operator \( U_{\text{enc}} \) such that \( U_{\text{enc}} |000\rangle = |000\rangle \) and \( U_{\text{enc}} |100\rangle = |111\rangle \) is defined as,
\[ U_{\text{enc}} \text{ def } (U_{\text{CNOT}}^{13} \otimes I^2) \circ (U_{\text{CNOT}}^{12} \otimes I^3). \]

(22)

**Recovery Operators.** The set of error operators satisfying the detectability condition \[17\], \( P_C A_p P_C = \lambda A_p P_C \), where \( P_C = |0_L\rangle \langle 0_L| + |1_L\rangle \langle 1_L| \) is the projector on the code subspace \( C = \text{Span} \{ |0_L\rangle, |1_L\rangle \} \) is given by,
\[ A_{\text{detectable}} = \{ A_0, A_1, A_2, A_3, A_4, A_5, A_6 \} \subseteq A. \]

(23)

The only non-detectable error is \( A_5 \). Furthermore, since all the detectable errors are invertible, the set of correctable errors is such that \( A_{\text{correctable}} \subseteq A_{\text{detectable}} \) is detectable. It follows then that,
\[ A_{\text{correctable}} = \{ A_0, A_1, A_2, A_3 \} \subseteq A_{\text{detectable}} \subseteq A. \]

(24)

The action of the correctable error operators \( A_{\text{correctable}} \) on the codewords \( |0_L\rangle \) and \( |1_L\rangle \) is given by,
\[ |0_L\rangle \rightarrow A_0' |0_L\rangle = \sqrt{p_0^{(3)}} |000\rangle, \quad A_1' |0_L\rangle = \sqrt{p_1^{(3)}} |100\rangle, \quad A_2' |0_L\rangle = \sqrt{p_2^{(3)}} |010\rangle, \quad A_3' |0_L\rangle = \sqrt{p_3^{(3)}} |001\rangle \]
\[ |1_L\rangle \rightarrow A_0' |1_L\rangle = \sqrt{p_0^{(3)}} |111\rangle, \quad A_1' |1_L\rangle = \sqrt{p_1^{(3)}} |101\rangle, \quad A_2' |1_L\rangle = \sqrt{p_2^{(3)}} |110\rangle, \quad A_3' |1_L\rangle = \sqrt{p_3^{(3)}} |100\rangle. \]

(25)

The two four-dimensional orthogonal subspaces \( \mathcal{V}^{0L} \) and \( \mathcal{V}^{1L} \) of \( \mathcal{H}_2^3 \) generated by the action of \( A_{\text{correctable}} \) on \( |0_L\rangle \) and \( |1_L\rangle \) are given by,
\[ \mathcal{V}^{0L} = \text{Span} \{ |v_1^{0L}\rangle = |000\rangle, |v_2^{0L}\rangle = |100\rangle, |v_3^{0L}\rangle = |010\rangle, |v_4^{0L}\rangle = |001\rangle \}, \]

(26)

and,
\[ \mathcal{V}^{1L} = \text{Span} \{ |v_1^{1L}\rangle = |111\rangle, |v_2^{1L}\rangle = |011\rangle, |v_3^{1L}\rangle = |101\rangle, |v_4^{1L}\rangle = |110\rangle \}, \]

(27)

respectively. Notice that \( \mathcal{V}^{0L} \oplus \mathcal{V}^{1L} = \mathcal{H}_2^3 \). The recovery superoperator \( R \leftrightarrow \{ R_i \} \) with \( l = 1,...,4 \) is defined as \[4\],
\[ R_l \text{ def } \sum_{i=0}^{1} |v_i^l\rangle \langle v_i^l|, \]

(28)

where the unitary operator \( V_i \) is such that \( V_i |v_i^l\rangle = |i_L\rangle \) for \( i \in \{0, 1\} \). Substituting (20) and (27) into (28), it follows that the four recovery operators \( \{ R_1, R_2, R_3, R_4 \} \) are given by,
\[ R_1 = |0_L\rangle \langle 0_L| + |1_L\rangle \langle 1_L|, \quad R_2 = |0_L\rangle \langle 100| + |1_L\rangle \langle 011|, \]
\[ R_3 = |0_L\rangle \langle 010| + |1_L\rangle \langle 101|, \quad R_4 = |0_L\rangle \langle 001| + |1_L\rangle \langle 110|. \]  

(29)
Using simple algebra, it turns out that the $8 \times 8$ matrix representation $[R_l]$ with $l = 1, \ldots, 4$ of the recovery operators is given by,

$$[R_1] = E_{11} + E_{88}, \quad [R_2] = E_{12} + E_{87}, \quad [R_3] = E_{13} + E_{86}, \quad [R_4] = E_{14} + E_{85},$$

(30)

where $E_{ij}$ is the $8 \times 8$ matrix where the only non-vanishing element is the one located in the $ij$-position and it equals 1. It follows that $\mathcal{R} \leftrightarrow \{R_l\}$ is indeed a trace preserving quantum operation since,

$$\sum_{l=1}^{4} R_l^\dagger R_l = I_{8 \times 8}. \quad (31)$$

Considering this recovery operation $\mathcal{R}$, the map $\Lambda^{(3)}(\rho)$ in (10) becomes,

$$\Lambda^{(3)}_{\text{recover}}(\rho) \equiv (\mathcal{R} \circ \Lambda^{(3)})(\rho) \overset{\text{def}}{=} \sum_{k=0}^{7} \sum_{l=1}^{4} (R_l A_k^\dagger) \rho (R_l A_k^\dagger)\dagger. \quad (32)$$

**Entanglement Fidelity.** We want to describe the action of $\mathcal{R} \circ \Lambda^{(3)}$ restricted to the code subspace $\mathcal{C}$. Therefore, we compute the $2 \times 2$ matrix representation $[R_l A_k^\dagger]_\mathcal{C}$ of each $R_l A_k^\dagger$ with $l = 1, \ldots, 4$ and $k = 0, \ldots, 7$ where,

$$[R_l A_k^\dagger]_\mathcal{C} \overset{\text{def}}{=} \begin{pmatrix} 0_L |R_l A_k^\dagger 0_L\rangle & \langle 0_L |R_l A_k^\dagger 1_L\rangle \\ \langle 1_L |R_l A_k^\dagger 0_L\rangle & \langle 1_L |R_l A_k^\dagger 1_L\rangle \end{pmatrix}. \quad (33)$$

Substituting (25) and (29) into (33), it turns out that the only matrices $[R_l A_k^\dagger]_\mathcal{C}$ with non-vanishing trace are given by,

$$[R_1 A_0^\dagger]_\mathcal{C} = \sqrt{p_0^{(3)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [R_2 A_1^\dagger]_\mathcal{C} = \sqrt{p_1^{(3)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$[R_3 A_2^\dagger]_\mathcal{C} = \sqrt{p_2^{(3)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [R_4 A_3^\dagger]_\mathcal{C} = \sqrt{p_3^{(3)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (34)$$

Therefore, the entanglement fidelity $\mathcal{F}^{(3)}_{\mathcal{RC}}(\mu, p)$ defined as,

$$\mathcal{F}^{(3)}_{\mathcal{RC}}(\mu, p) \overset{\text{def}}{=} \mathcal{F}^{(3)} \left( \frac{1}{2} I_{2 \times 2}, \mathcal{R} \circ \Lambda^{(3)} \right) = \frac{1}{(2)^2} \sum_{k=0}^{7} \sum_{l=1}^{4} \left| \text{tr} \left( [R_l A_k^\dagger]_\mathcal{C} \right) \right|^2, \quad (35)$$

results,

$$\mathcal{F}^{(3)}_{\mathcal{RC}}(\mu, p) = p_0^{(3)} + p_1^{(3)} + p_2^{(3)} + p_3^{(3)}. \quad (36)$$

The expression for $\mathcal{F}^{(3)}_{\mathcal{RC}}(\mu, p)$ in (35) represents the entanglement fidelity quantifying the performance of the error correction scheme provided by the repetition code here considered. The quantum operation $\mathcal{R} \circ \Lambda^{(3)}$ appearing in (35) is defined in equation (32) and the recovery operators $R_l$ are explicitly given in (29). The action of $R_l A_k^\dagger$ in (30) is restricted to the code space $\mathcal{C}$ defined in (20).

Substituting (18) and (19) into (36), we finally obtain

$$\mathcal{F}^{(3)}_{\mathcal{RC}}(\mu, p) = \mu^2 (2p^3 - 3p^2 + p) + \mu (-4p^3 + 6p^2 - 2p) + (2p^3 - 3p^2 + 1). \quad (37)$$

Notice that for a vanishing degree of memory $\mu$, the entanglement fidelity becomes,

$$\mathcal{F}^{(3)}_{\mathcal{RC}}(0, p) = \frac{1}{m} \binom{3}{m} p^m (1-p)^{3-m} = 2p^3 - 3p^2 + 1. \quad (38)$$

**Remarks on the coding for phase flip memory channels.** The code for the phase flip channel has the same characteristics as the code for the bit flip channel. These two channels are unitarily equivalent since there is a unitary operator, the Hadamard gate $H$, such that the action of one channel is the same as the other, provided the first channel is preceded by $H$ and followed by $H^\dagger$ [1],

$$\Lambda^{\text{phase}}(\rho) \overset{\text{def}}{=} (H \circ \Lambda^{\text{bit}} \circ H^\dagger)(\rho) = (1-p) \rho + p Z \rho Z, \quad (39)$$
where,

$$
\Lambda^{\text{bit}}(\rho) \equiv (1 - p) \rho + pX\rho X \text{ and, } H \equiv \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right).
$$

(40)

These operations may be trivially incorporated into the encoding and error-correction operations. The encoding for the phase flip channel is performed in two steps: i) first, we encode in three qubits exactly as for the bit flip channel; ii) second, we apply a Hadamard gate to each qubit,

$$
|0\rangle \xrightarrow{\text{tensor}} |000\rangle \xrightarrow{U_{\text{enc}}^{\text{bit}}} |0_L\rangle \equiv |++\rangle, \quad |1\rangle \xrightarrow{\text{tensor}} |100\rangle \xrightarrow{U_{\text{enc}}^{\text{bit}}} |1_L\rangle \equiv |--\rangle,
$$

where,

$$
\left( \begin{array}{c} |+\rangle \\ |\rangle \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \right).
$$

(42)

The unitary encoding operator $U_{\text{enc}}^{\text{phase}}$ is given by $U_{\text{enc}}^{\text{phase}} \equiv H^{\otimes 3} \circ U_{\text{enc}}^{\text{bit}}$ with $U_{\text{enc}}^{\text{bit}}$ defined in (22). Furthermore, in the phase flip code, the recovery operation is the Hadamard conjugated recovery operation from the bit flip code, $R_k^{\text{phase}} \equiv H^{\otimes 3} R_k^{\text{bit}} H^{\otimes 3}$.

B. CASE, $n_{\text{even}} = 4$

Here, we apply the even length repetition code with $n = 4$ to the correlated bit-flip.

**Error Operators.** In this case, the memory channel to consider is given by,

$$
\Lambda^{(4)}(\rho) \equiv \sum_{i_1, i_2, i_3, i_4 = 0}^{1} p_{i_1 i_3} p_{i_1 i_3} p_{i_2 i_4} p_{i_2 i_4} p_{i_1} \left[ A_{i_4} A_{i_3} A_{i_2} A_{i_1} \rho A_{i_4}^\dagger A_{i_3}^\dagger A_{i_2}^\dagger A_{i_1}^\dagger \right],
$$

(43)

where the error operators $\{A_k\}$ with $r = 1, 2, 3, 4$ act on 1 qubit quantum states. The error superoperator $A$ associated to channel $\mathbb{E}$ may be defined in terms of the following encoded error operators $\{A_k^{\text{enc}}\}$ with $k = 0, \ldots, 15$,

$$
A \longleftrightarrow \{A_0^{\text{enc}}, A_1^{\text{enc}}\} \text{ with } \Lambda^{(4)}(\rho) \equiv \sum_{k=0}^{15} A_k^{\text{enc}} \rho A_k^{\text{enc}} \text{ and, } \sum_{k=0}^{15} A_k^{\text{enc}} A_k^{\text{enc}} = I_{16 \times 16}.
$$

(44)

Omitting the symbol $\otimes^n$, the error operators $\{A_0^{\text{enc}}, A_1^{\text{enc}}\}$ are given by,

$$
A_0' = \sqrt{p_0^{(4)}} I_1^2 I_2^2 I_3^2 I_4^2, \quad A_1' = \sqrt{p_1^{(4)}} X_1^2 I_2^2 I_3^2 I_4^2, \quad A_2' = \sqrt{p_2^{(4)}} I_1^2 X_2^2 I_3^2 I_4^2, \quad A_3' = \sqrt{p_3^{(4)}} I_1^2 X_2^2 I_3^2 I_4^2, \quad A_4' = \sqrt{p_4^{(4)}} I_1^2 I_2^2 I_3^2 X_4^2,
$$

$$
A_5' = \sqrt{p_5^{(4)}} X_1^2 X_2^2 I_3^2 I_4^2, \quad A_6' = \sqrt{p_6^{(4)}} X_1^2 I_2^2 X_3^2 I_4^2, \quad A_7' = \sqrt{p_7^{(4)}} X_1^2 I_2^2 I_3^2 X_4^2, \quad A_8' = \sqrt{p_8^{(4)}} I_1^2 X_2^2 X_3^2 I_4^2, \quad A_9' = \sqrt{p_9^{(4)}} I_1^2 X_2^2 I_3^2 X_4^2,
$$

$$
A_{10}' = \sqrt{p_{10}^{(4)}} I_1^2 X_2^2 X_3^2 X_4^2, \quad A_{11}' = \sqrt{p_{11}^{(4)}} X_1^2 X_2^2 I_3^2 X_4^2, \quad A_{12}' = \sqrt{p_{12}^{(4)}} X_1^2 X_2^2 I_3^2 X_4^2, \quad A_{13}' = \sqrt{p_{13}^{(4)}} X_1^2 I_2^2 X_3^2 X_4^2, \quad A_{14}' = \sqrt{p_{14}^{(4)}} X_1^2 I_2^2 X_3^2 X_4^2, \quad A_{15}' = \sqrt{p_{15}^{(4)}} X_1^2 X_2^2 X_3^2 X_4^2.
$$

(45)

The coefficients $p_k^{(4)}$ for $k = 1, \ldots, 15$ are formally given by $p_k^{(4)} = p_{i_1 i_2} p_{i_3 i_4} p_{i_1 i_3} p_{i_2 i_4} p_{i_1} p_{i_2}$ where the $i_{4j}^{(4)} \in \{0, 1\}$ are determined by the relation $A_k^{\text{enc}} |0_L\rangle = |i_1^{(4)} i_2^{(4)} i_3^{(4)} i_4^{(4)}\rangle$. Following the line of reasoning used for the odd repetition codes and omitting technical details that will appear in Appendix A, the entanglement fidelity $F^{(4)}_{\text{RC}}(\mu, p)$ becomes,

$$
F^{(4)}_{\text{RC}}(\mu, p) = \mu^2 \left( 2p^3 - 3p^2 + p \right) + \mu \left( -4p^3 + 6p^2 - 2p \right) + \left( 2p^3 - 3p^2 + 1 \right).
$$

(46)

Notice that $F^{(4)}_{\text{RC}}(\mu, p) = F^{(3)}_{\text{RC}}(\mu, p)$ and, in absence of correlations,

$$
F^{(4)}_{\text{RC}}(0, p) = \sum_{m=0}^{1} \binom{4}{m} p^m (1-p)^{4-m} + \frac{1}{2} \binom{4}{2} p^2 (1-p)^2 = 2p^3 - 3p^2 + 1 = F^{(3)}_{\text{RC}}(0, p).
$$

(47)
Finally, it can also be shown that $F^{(6)}_{RC}(\mu, p) = F^{(5)}_{RC}(\mu, p)$ with,

$$F^{(5)}_{RC}(\mu, p) = \mu^4 (-6p^5 + 15p^4 - 12p^3 + 3p^2) + \mu^3 (24p^5 - 60p^4 + 52p^3 - 18p^2 + 2p) +$$

$$+ \mu^2 (-36p^5 + 90p^4 - 78p^3 + 27p^2 - 3p) + \mu (24p^5 - 60p^4 + 48p^3 - 12p^2) +$$

$$+ (-6p^5 + 15p^4 - 10p^3 + 1),$$

(48)

and $F^{(8)}_{RC}(\mu, p) = F^{(7)}_{RC}(\mu, p)$ with,

$$F^{(7)}_{RC}(\mu, p) = \mu^6 (20p^7 - 70p^6 + 90p^5 - 50p^4 + 10p^3) + \mu^5 (-120p^7 + 420p^6 - 564p^5 + 360p^4 - 108p^3 + 12p^2) +$$

$$+ \mu^4 (300p^7 - 1050p^6 + 1440p^5 - 975p^4 + 336p^3 - 54p^2 + 3p) +$$

$$+ \mu^3 (-400p^7 + 1400p^6 - 1920p^5 + 1300p^4 - 448p^3 + 72p^2 - 4p) +$$

$$+ \mu^2 (300p^7 - 1050p^6 + 1410p^5 - 900p^4 + 270p^3 - 30p^2) + \mu (-120p^7 + 420p^6 - 540p^5 + 300p^4 - 60p^3) +$$

$$+ (20p^7 - 70p^6 + 84p^5 - 35p^4 + 1).$$

(49)

In Figure 1, we plot $F^{(3)}_{RC}(\mu, p)$, $F^{(5)}_{RC}(\mu, p)$ and $F^{(7)}_{RC}(\mu, p)$ vs. $\mu$ for $p = 0.45$. From this plot, it is clear that the entanglement fidelity $F^{(n)}_{RC}(\mu, p)$ increases with increasing $n$ and decreases with the correlation parameter $\mu$.

### IV. DECOHERENCE FREE SUBSPACES FOR CORRELATED BIT FLIP

In this Section, we tackle our decoherence problem via the decoherence-free subspaces formalism. This is a passive quantum error correction method where the key idea is that of avoiding decoherence by encoding quantum information into special subspaces that are protected from the interaction with the environment by virtue of some specific dynamical symmetry. For a detailed review, we refer to [13].

#### A. CASE, $n_{odd} = 3$

Let us consider the correlated bit-flip noisy error model as defined in [15] and [16].

**Encoding and Decoding Operators.** Consider the following quantum code encoding 1 logical qubit into 3-physical qubits,

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{1}{\sqrt{2}} (|0\rangle_1 + |1\rangle_1) \otimes (|0\rangle_2 + |1\rangle_2) \otimes (|0\rangle_3 + |1\rangle_3) \equiv |+++angle,$$

$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{1}{\sqrt{2}} (|0\rangle_1 - |1\rangle_1) \otimes (|0\rangle_2 - |1\rangle_2) \otimes (|0\rangle_3 - |1\rangle_3) \equiv |--\rangle,$$

(50)

with $|++\rangle = |++\rangle + |++\rangle = (++|--|--|)$ and $|--\rangle = |--\rangle + |--\rangle = (++|--|--|) = 0$.

**Recovery Operators.** The set of error operators satisfying the detectability condition $P_C A_k^c P_C = \lambda A_k^c P_C$ where $P_C = |0_L\rangle \langle 0_L| + |1_L\rangle \langle 1_L|$ is the projector operator on the code subspace $C = \text{Span} \{ |0_L\rangle, |1_L\rangle \}$ is given by,

$$A_{\text{detectable}} = \{ A_0', A_4', A_5', A_6' \} \subseteq A.$$

(51)

Furthermore, since all the detectable errors are invertible, the set of correctable errors is such that $A_{\text{correctable}} \subseteq A_{\text{detectable}}$. It follows then that,

$$A_{\text{correctable}} = \{ A_0', A_4', A_5', A_6' \} = A_{\text{detectable}}.$$

(52)
The action of the correctable error operators $A_{\text{correctable}}$ on the codewords $|0_L\rangle$ and $|1_L\rangle$ is given by,

$$
|0_L\rangle \rightarrow A'_r |0_L\rangle = \sqrt{P_r^{(3)}} |0_L\rangle, \quad A'_r |1_L\rangle = \sqrt{P_r^{(3)}} |1_L\rangle,
$$

(53)

for $r = 0, 4, 5, 6$. From (53), it follows that $C = \text{Span} \{ |0_L\rangle , |1_L\rangle \}$ is a decoherence-free subspace for the correctable error operators in $A_{\text{correctable}}$. The two one-dimensional orthogonal subspaces $\mathcal{V}^{0L}$ and $\mathcal{V}^{1L}$ of $\mathcal{H}^2_2$ generated by the action of $A_{\text{correctable}}$ on $|0_L\rangle$ and $|1_L\rangle$ are given by,

$$
\mathcal{V}^{0L} = \text{Span} \{ |v^{0L}_1\rangle = |++\rangle \}, \quad (54)
$$

and,

$$
\mathcal{V}^{1L} = \text{Span} \{ |v^{1L}_1\rangle = |---\rangle \}, \quad (55)
$$

respectively. Notice that $\mathcal{V}^{0L} \oplus \mathcal{V}^{1L} \neq \mathcal{H}^2_2$. This means that the trace preserving recovery superoperator $R$ is defined in terms of one standard recovery operator $R_1$ and by the projector $R_\perp$ onto the orthogonal complement of $\bigoplus_{i=0}^{1} \mathcal{V}^i_L$, i.e. the part of the Hilbert space $\mathcal{H}^2_2$ which is not reached by acting on the code $C$ with the correctable error operators. In the case under consideration,

$$
\begin{align*}
R_1 & \equiv |++\rangle \langle ++| + |---\rangle \langle ---|, \quad R_\perp = \sum_{s=1}^{6} |r_s\rangle \langle r_s|,
\end{align*}
$$

(56)

where $\{|r_s\rangle\}$ is an orthonormal basis for $(\mathcal{V}^{0L} \oplus \mathcal{V}^{1L})^\perp$. A suitable basis $B_{(\mathcal{V}^{0L} \oplus \mathcal{V}^{1L})^\perp}$ is given by,

$$
B_{(\mathcal{V}^{0L} \oplus \mathcal{V}^{1L})^\perp} = \{ r_1 = |+-\rangle, \quad r_2 = |+-\rangle, \quad r_3 = |++\rangle, \quad r_4 = |--\rangle, \quad r_5 = |--\rangle, \quad r_6 = |+-\rangle \}.
$$

(57)

Therefore, $R \leftrightarrow \{ R_1, R_\perp \}$ is indeed a trace preserving quantum operation,

$$
R_l^\dagger R_1 + R_1^\dagger R_\perp = I_{8 \times 8}.
$$

(58)

Considering this recovery operation $R$ with $R_2 \equiv R_\perp$, the map $\Lambda^{(3)}(\rho)$ in (10) becomes,

$$
\Lambda^{(3)}_{\text{recover}}(\rho) \equiv \left( R \circ \Lambda^{(3)} \right)(\rho) \equiv \sum_{k=0}^{7} \sum_{l=1}^{2} (R_l A'_k) \rho (R_l A'_k)^\dagger,
$$

(59)

\textit{Entanglement Fidelity.} We want to describe the action of $R \circ \Lambda^{(3)}$ restricted to the code subspace $C$. Therefore, we compute the $2 \times 2$ matrix representation $[R_l A'_k]_C$ of each $R_l A'_k$ with $l = 1, 2$ and $k = 0, ..., 7$ where,

$$
[R_l A'_k]_C \equiv \begin{pmatrix}
0_L & R_l A'_k |0_L\rangle \\
1_L & R_l A'_k |1_L\rangle
\end{pmatrix}.
$$

(60)

Substituting (53) and (56) into (60), it turns out that the only matrices $[R_l A'_k]_C$ with non-vanishing trace are given by,

$$
[R_l A'_k]_C = \sqrt{p_r^{(3)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

(61)

for $r = 0, 4, 5, 6$. Therefore, the entanglement fidelity $F^{(3)}_{\text{DFS}}(\mu, p)$ defined as,

$$
\begin{align*}
F^{(3)}_{\text{DFS}}(\mu, p) & \equiv \mathcal{F}^{(3)} \left( \frac{1}{2} I_{2 \times 2}, \mathcal{R} \circ \Lambda^{(3)} \right) = \frac{1}{2} \sum_{k=0}^{7} \sum_{l=1}^{2} \text{tr} \left( [R_l A'_k]_C \right)^2, \quad (62)
\end{align*}
$$

results,

$$
F^{(3)}_{\text{DFS}}(\mu, p) = p_0^{(3)} + p_4^{(3)} + p_5^{(3)} + p_6^{(3)}.
$$

(63)

The expression for $F^{(3)}_{\text{DFS}}(\mu, p)$ in (62) represents the entanglement fidelity quantifying the performance of the error correction scheme provided by the noiseless code here considered. The quantum operation $R \circ \Lambda^{(3)}$ appearing in (62) is defined in equation (53) and the recovery operators $R_l$ are explicitly given in (56). The action of $R_l A'_k$ in (62) is restricted to the code space $C$ defined in (53).

Substituting (10) into (63), we finally obtain

$$
F^{(3)}_{\text{DFS}}(\mu, p) = \mu^2 \left( -4p^3 + 6p^2 - 2p \right) + \mu \left( 8p^3 - 12p^2 + 4p \right) + \left( -4p^3 + 6p^2 - 3p + 1 \right).
$$

(64)
B. CASE, $n_{\text{even}} = 4$

Let us now consider the correlated bit-flip noisy error model as defined in \cite{13}.

**Encoding and Decoding Operators.** Consider the following quantum code encoding 1 logical qubit into 4-physical qubits,

$$|0\rangle \rightarrow |0_L\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{2}} (|0\rangle_1 + |1\rangle_1) \otimes (|0\rangle_2 + |1\rangle_2) \otimes (|0\rangle_3 + |1\rangle_3) \otimes (|0\rangle_4 + |1\rangle_4) \equiv |+++\rangle,$$

$$|1\rangle \rightarrow |1_L\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{2}} (|0\rangle_1 - |1\rangle_1) \otimes (|0\rangle_2 - |1\rangle_2) \otimes (|0\rangle_3 - |1\rangle_3) \otimes (|0\rangle_4 + |1\rangle_4) \equiv |---\rangle, \quad (65)$$

with $\langle+++|+++\rangle = \langle---|---\rangle = 1$ and $\langle---|+++\rangle = \langle+++|---\rangle = 0$. Following the line of reasoning used for the odd case and omitting technical details that will appear in Appendix B, the entanglement fidelity $F_{\text{DFS}}^{(4)}(\mu, p)$ becomes,

$$F_{\text{DFS}}^{(4)}(\mu, p) = \mu^3 (-8p^4 + 16p^3 - 10p^2 + 2p) + \mu^2 (24p^4 - 48p^3 + 28p^2 - 4p) +$$

$$+ \mu (-24p^4 + 48p^3 - 30p^2 + 6p) + (8p^4 - 16p^3 + 12p^2 - 4p + 1). \quad (66)$$

In Figure 2, we plot $F_{\text{RC}}^{(4)}(\mu, p)$ and $F_{\text{DFS}}^{(4)}(\mu, p)$ for three values of the error probability $p = 0.45$, $p = 0.40$ and $p = 0.35$. Following the line of reasoning presented above, it can be shown that $F_{\text{DFS}}^{(5)}(\mu, p)$ and $F_{\text{DFS}}^{(6)}(\mu, p)$ are given by,

$$F_{\text{DFS}}^{(5)}(\mu, p) = \mu^4 (-16p^5 + 40p^4 - 36p^3 + 14p^2 - 2p) + \mu^3 (64p^5 - 160p^4 + 136p^3 - 44p^2 + 4p) +$$

$$+ \mu^2 (-96p^5 + 240p^4 - 204p^3 + 66p^2 - 6p) + \mu (64p^5 - 160p^4 + 144p^3 - 56p^2 + 8p) +$$

$$+ (16p^5 - 40p^4 + 40p^3 + 20p^2 - 5p + 1), \quad (67)$$

and,

$$F_{\text{DFS}}^{(6)}(\mu, p) = \mu^5 (-32p^6 + 97p^5 - 115p^4 + 67p^3 - 19p^2 + 2p) +$$

$$+ \mu^4 (160p^6 - 484p^5 + 546p^4 - 280p^3 + 62p^2 - 4p) +$$

$$+ \mu^3 (-320p^6 + 966p^5 - 1068p^4 + 519p^3 - 103p^2 + 6p) +$$

$$+ \mu^2 (320p^6 - 964p^5 + 1078p^4 - 546p^3 + 120p^2 - 8p) +$$

$$+ \mu (-160p^6 + 481p^5 - 561p^4 + 320p^3 - 90p^2 + 10p) +$$

$$+ (32p^6 - 96p^5 + 120p^4 - 80p^3 + 30p^2 - 6p + 1). \quad (68)$$

respectively. It turns out that $F_{\text{DFS}}^{(5)}(\mu, p) \leq F_{\text{DFS}}^{(3)}(\mu, p)$ and $F_{\text{DFS}}^{(6)}(\mu, p) \leq F_{\text{DFS}}^{(4)}(\mu, p)$ for $\mu \in [0, 1]$ and $p < 0.5$. Moreover, $F_{\text{DFS}}^{(4)}(\mu, p)$ is greater than $F_{\text{DFS}}^{(3)}(\mu, p)$ for $\mu \geq \mu_{\text{min}} \geq 0.2$. Therefore, in the high correlation regime where $\mu \rightarrow 1$, $F_{\text{DFS}}^{(4)}(\mu, p)$ achieves the highest value for arbitrary error probabilities $p$ less than 0.5.

V. FINAL REMARKS

Because of the results obtained in the previous Section, it follows that there must be a certain threshold value $\mu^* (p)$ that allows to select the better code between the repetition and the noiseless quantum code for our noisy quantum
memory channel. Considering the case with \( n = 4 \), we may obtain a curve \( \mu^* = \mu^* (p) \) defined in such a way that, \( \mathcal{F}^{(4)}_{\text{DFS}} (\mu^* (p), p) \) \( - \mathcal{F}^{(4)}_{\text{RC}} (\mu^* (p), p) = 0 \). For example, in Figure 2 we have plotted \( \mathcal{F}^{(4)}_{\text{RC}} (\mu, p) \) and \( \mathcal{F}^{(4)}_{\text{DFS}} (\mu, p) \) for few values of \( p \). From this plot, we see the emergence of threshold values \( \mu^* (0.45) \approx 0.34 \), \( \mu^* (0.40) \approx 0.45 \) and \( \mu^* (0.35) \approx 0.52 \) when the curves \( \mathcal{F}^{(4)}_{\text{RC}} (\mu, p) \) and \( \mathcal{F}^{(4)}_{\text{DFS}} (\mu, p) \) cross. This means that for \( p = 0.45 \) the noiseless quantum code outperforms the repetition code when \( \mu \geq \mu^* (0.45) \) \( \approx 0.34 \). Finally, in Figure 3 we plot the threshold curve \( \mu^* (p) \) for all permitted values of the error probability \( p \). In conclusion, we have shown in an explicit way that the repetition code (be it even or odd) works better than the noiseless quantum code in the low correlations regime. On the contrary, in the high correlation regime, the noiseless quantum codes work better. The proper quantities defining the correlation regimes are the threshold values \( \mu^* (p) \).

In conclusion, in this Letter we have analyzed the performance of simple quantum error correcting codes in the presence of correlated noise error models characterized by a correlation strength \( \mu \). Specifically, we have considered bit flip (phase flip) noisy quantum memory channels and used repetition and noiseless quantum codes. We have characterized the performance of the codes by means of the entanglement fidelity \( \mathcal{F} (\mu, p) \) as a function of the error probability \( p \) and degree of memory \( \mu \). We have shown in an explicit way that the entanglement fidelity \( \mathcal{F}^{(\text{RC})}_{\text{RC}} (\mu, p) \) equals \( \mathcal{F}^{(\text{RC})}_{\text{RC}} (\mu, p) \) and that \( \mathcal{F}^{(\text{RC})}_{\text{RC}} (\mu, p) \) increases with the length \( n \) of the code and decreases with the correlation parameter \( \mu \). Furthermore, we also used the decoherence free subspaces formalism and showed that the performance of such QECCs quantified in terms of the entanglement fidelity \( \mathcal{F}^{(\text{DFS})}_{\text{DFS}} (\mu, p) \) is better than the one of repetition codes in the high correlation regime where \( \mu \rightarrow 1 \). The noiseless quantum code with \( n = 4 \) performs better than the other (noiseless) codes considered in this work in the high correlation regime. Comparing the entanglement fidelities of repetition codes and noiseless quantum codes, we found a threshold \( \mu^* (p) \) for the correlation strength that allows to select the quantum code with better performance.

The above results suggest that it may be convenient to concatenate decoherence-free subspaces with standard quantum error correcting codes in order to achieve higher entanglement fidelity values in both low and high correlations regimes. This will be the object of future investigations.

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Appendix A: Repetition Codes, \( n \text{even} = 4 \)

Recovery Operators. The set of error operators satisfying the detectability condition \( P_{\mathcal{C}} A_k^l P_{\mathcal{C}} = \lambda A_k^l P_{\mathcal{C}} \) where \( P_{\mathcal{C}} = |0_L\rangle \langle 0_L| + |1_L\rangle \langle 1_L| \) is the projector operator on the code subspace \( \mathcal{C} = \text{Span} \{ |0_L\rangle, |1_L\rangle \} \) with \( |0_L\rangle \equiv |0000\rangle \) and \( |1_L\rangle \equiv |1111\rangle \) is given by \( \mathcal{A}_{\text{detectable}} = A_{\{1\}_{11}} \subseteq \mathcal{A} \). Furthermore, since all the detectable errors are invertible, the set of correctable errors is such that \( \mathcal{A}_{\text{correctable}} \subseteq \mathcal{A}_{\text{detectable}} \) is correctable. It follows then that,

\[
\mathcal{A}_{\text{correctable}} = \{ A'_0, A'_1, A'_2, A'_3, A'_4, A'_5, A'_6 \} \subseteq \mathcal{A}_{\text{correctable}} \subseteq \mathcal{A}.
\]  

(A1)

The action of the correctable error operators \( \mathcal{A}_{\text{correctable}} \) on the codewords \( |0_L\rangle \) and \( |1_L\rangle \) is given by,

\[
|0_L\rangle \rightarrow A'_0 |0_L\rangle = \sqrt{\frac{1}{p_0}} |0000\rangle, \quad A'_1 |0_L\rangle = \sqrt{\frac{1}{p_1}} |1000\rangle, \quad A'_2 |0_L\rangle = \sqrt{\frac{1}{p_2}} |0100\rangle, \quad A'_3 |0_L\rangle = \sqrt{\frac{1}{p_3}} |0010\rangle,
\]

\[
A'_4 |0_L\rangle = \sqrt{\frac{1}{p_4}} |0001\rangle, \quad A'_5 |0_L\rangle = \sqrt{\frac{1}{p_5}} |1100\rangle, \quad A'_6 |0_L\rangle = \sqrt{\frac{1}{p_6}} |1010\rangle, \quad A'_7 |0_L\rangle = \sqrt{\frac{1}{p_7}} |1001\rangle,
\]

and,

\[
|1_L\rangle \rightarrow A'_0 |1_L\rangle = \sqrt{\frac{1}{p_{0'}}} |1111\rangle, \quad A'_1 |1_L\rangle = \sqrt{\frac{1}{p_{1'}}} |0111\rangle, \quad A'_2 |1_L\rangle = \sqrt{\frac{1}{p_{2'}}} |0111\rangle, \quad A'_3 |1_L\rangle = \sqrt{\frac{1}{p_{3'}}} |0110\rangle, \quad A'_4 |1_L\rangle = \sqrt{\frac{1}{p_{4'}}} |1110\rangle,
\]

\[
A'_5 |1_L\rangle = \sqrt{\frac{1}{p_{5'}}} |1110\rangle, \quad A'_6 |1_L\rangle = \sqrt{\frac{1}{p_{6'}}} |0011\rangle, \quad A'_7 |1_L\rangle = \sqrt{\frac{1}{p_{7'}}} |0101\rangle, \quad A'_8 |1_L\rangle = \sqrt{\frac{1}{p_{8'}}} |0110\rangle.
\]  

(A2)
respectively. The two eight-dimensional orthogonal subspaces $\mathcal{V}^0_L$ and $\mathcal{V}^1_L$ of $\mathcal{H}_2^L$ generated by the action of $\mathcal{A}_{\text{correctable}}$ on $|0_L\rangle$ and $|1_L\rangle$ are given by,

$$\mathcal{V}^0_L = \text{Span} \left\{ |\psi^0_{0_L}\rangle = |0000\rangle, |\psi^0_{2_L}\rangle = |1000\rangle, |\psi^0_{3_L}\rangle = |0100\rangle, |\psi^0_{4_L}\rangle = |0010\rangle \right\},$$

(A4)

and,

$$\mathcal{V}^1_L = \text{Span} \left\{ |\psi^1_{0_L}\rangle = |1111\rangle, |\psi^1_{2_L}\rangle = |0111\rangle, |\psi^1_{3_L}\rangle = |1011\rangle, |\psi^1_{4_L}\rangle = |1101\rangle \right\},$$

(A5)

Notice that $\mathcal{V}^0_L \oplus \mathcal{V}^1_L = \mathcal{H}_2^L$. The recovery superoperator $\mathcal{R} \leftrightarrow \{R_l\}$ with $l = 1,.., 8$ is defined as,

$$R_l \overset{\text{def}}{=} V_l \sum_{i=0}^{1} |\psi^i_{l}\rangle \langle \psi^i_{l}|,$$

(A6)

where the unitary operator $V_l$ is such that $V_l |\psi^i_{l}\rangle = |i_L\rangle$ for $i \in \{0, 1\}$. Substituting (A4) and (A5) into (A6), it follows that the eight recovery operators $\{R_1, .., R_8\}$ are given by,

$$R_1 = |0_L\rangle \langle 0_L| + |1_L\rangle \langle 1_L|, \quad R_2 = |0_L\rangle \langle 1000| + |1_L\rangle \langle 0111|, \quad R_3 = |0_L\rangle \langle 0100| + |1_L\rangle \langle 1011|,$$

$$R_4 = |0_L\rangle \langle 0011| + |1_L\rangle \langle 1110|, \quad R_5 = |0_L\rangle \langle 0001| + |1_L\rangle \langle 1101|, \quad R_6 = |0_L\rangle \langle 1100| + |1_L\rangle \langle 0011|,$$

$$R_7 = |0_L\rangle \langle 1010| + |1_L\rangle \langle 0101|, \quad R_8 = |0_L\rangle \langle 0100| + |1_L\rangle \langle 0110|.$$  

(A7)

It can be shown that $\mathcal{R} \leftrightarrow \{R_l\}$ with $l = 1,.., 8$ is indeed a trace preserving quantum operation since,

$$\sum_{l=1}^{8} R_l^\dagger R_l = I_{16 \times 16}.$$  

(A8)

Considering this recovery operation $\mathcal{R}$, the map $\Lambda_{\text{recover}}^{(4)}(\rho)$ in (A3) becomes,

$$\Lambda_{\text{recover}}^{(4)}(\rho) \equiv (\mathcal{R} \circ \Lambda^{(4)})(\rho) \overset{\text{def}}{=} \sum_{k=0}^{15} \sum_{l=1}^{8} (R_l A_k^l) \rho (R_l A_k^l)^\dagger.$$  

(A9)

**Entanglement Fidelity.** We want to describe the action of $\mathcal{R} \circ \Lambda^{(4)}$ restricted to the code subspace $\mathcal{C}$. We simply compute the $2 \times 2$ matrix representation $[R_l A_k^l]_C$ of each $R_l A_k^l$ with $l = 1,.., 4$ and $k = 0,.., 7$ where,

$$[R_l A_k^l]_C \overset{\text{def}}{=} \begin{pmatrix} 0_L & R_l A_k^l 0_L \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} R_l A_k^l 1_L \end{pmatrix}.$$  

(A10)

Substituting (A2), (A3) and (A7) into (A10), it turns out that the only matrices $[R_l A_k^l]_C$ with non-vanishing trace are given by,

$$[R_1 A_0^0]_C = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad [R_2 A_1^0]_C = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad [R_3 A_2^0]_C = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad [R_4 A_3^0]_C = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 0 \end{pmatrix},$$

$$[R_5 A_4^0]_C = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad [R_6 A_5^0]_C = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad [R_7 A_6^0]_C = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 0 \end{pmatrix}.$$  

(A11)

Therefore, the entanglement fidelity $\mathcal{F}^{(4)}_{\mathcal{R}C}(\mu, p)$ defined as,

$$\mathcal{F}^{(4)}_{\mathcal{R}C}(\mu, p) \overset{\text{def}}{=} \mathcal{F}^{(4)}_{\mathcal{R}C} \left( \frac{1}{2} I_{2 \times 2}, \mathcal{R} \circ \Lambda^{(4)} \right) = \frac{1}{16} \sum_{k=0}^{15} \sum_{l=1}^{8} |\text{tr} \left( [R_l A_k^l]_C \right)|^2,$$

(A12)
results,
\[
F_{RC}^{(4)}(\mu, p) = \tilde{\rho}_1^{(4)} + \rho_1^{(4)} + \tilde{\rho}_2^{(4)} + \rho_2^{(4)} + \tilde{\rho}_3^{(4)} + \rho_3^{(4)} + \tilde{\rho}_4^{(4)} + \rho_4^{(4)} + \tilde{\rho}_5^{(4)} + \rho_5^{(4)} + \tilde{\rho}_7^{(4)}. \tag{A13}
\]
Substituting (19) into (A13), we finally obtain
\[
F_{RC}^{(4)}(\mu, p) = \mu^2 (2p^3 - 3p^2 + p) + \mu \left(-4p^3 + 6p^2 - 2p\right) + (2p^3 - 3p^2 + 1). \tag{A14}
\]
Notice that \( F_{RC}^{(4)}(\mu, p) = F_{RC}^{(3)}(\mu, p) \) and, in absence of correlations,
\[
F_{RC}^{(4)}(0, p) = \sum_{m=0}^{1} \left(\frac{4}{m}\right) p^m \left(1-p\right)^{4-m} + \frac{1}{2} \left(\frac{4}{2}\right) p^2 \left(1-p\right)^2 = 2p^3 - 3p^2 + 1 = F_{RC}^{(3)}(0, p). \tag{A15}
\]
Finally, following the same line of reasoning presented above, it can be shown that \( F_{RC}^{(6)}(\mu, p) = F_{RC}^{(5)}(\mu, p) \) and \( F_{RC}^{(8)}(\mu, p) = F_{RC}^{(7)}(\mu, p) \).

**Appendix B: Decoherence Free Subspaces, \( n_{even} = 4 \)**

**Recovery Operators.** The set of error operators satisfying the detectability condition \( P_C A'_r P_C = \lambda A'_r P_C \) where \( P_C = |0_L\rangle \langle 0_L| + |1_L\rangle \langle 1_L| \) is the projector operator on the code subspace \( C = \text{Span} \{ |0_L\rangle, |1_L\rangle \} \) is given by,
\[
A_{\text{detectable}} = \{ A'_0, A'_5, A'_6, A'_7, A'_8, A'_9, A'_{10}, A'_{15} \} \subseteq A. \tag{B1}
\]
Furthermore, since all the detectable errors are invertible, the set of correctable errors is such that \( A'_{\text{correctable}} \) is a subset of \( A_{\text{detectable}} \) is correctable. It follows then that,
\[
A_{\text{correctable}} = \{ A'_0, A'_5, A'_6, A'_7, A'_8, A'_9, A'_{10}, A'_{15} \} = A_{\text{detectable}}. \tag{B2}
\]
The action of the correctable error operators \( A_{\text{correctable}} \) on the codewords \( |0_L\rangle \) and \( |1_L\rangle \) is given by,
\[
|0_L\rangle \rightarrow A'_{0r} |0_L\rangle = \sqrt{\tilde{\rho}_r^{(4)}} |0_L\rangle, \ |1_L\rangle \rightarrow A'_{r} |1_L\rangle = \sqrt{\tilde{\rho}_r^{(4)}} |1_L\rangle, \tag{B3}
\]
for \( r = 0, 5, 6, 7, 8, 9, 10, 15 \). From (B3), it follows that \( C = \text{Span} \{ |0_L\rangle, |1_L\rangle \} \) is a decoherence-free subspace for the correctable error operators in \( A_{\text{correctable}} \). The two one-dimensional orthogonal subspaces \( V^{0L} \) and \( V^{1L} \) of \( H^4_2 \) generated by the action of \( A_{\text{correctable}} \) on \( |0_L\rangle \) and \( |1_L\rangle \) are given by,
\[
V^{0L} = \text{Span} \{ |v^{0L}_1\rangle = |+++\rangle \}, \tag{B4}
\]
and,
\[
V^{1L} = \text{Span} \{ |v^{1L}_1\rangle = |---\rangle \}, \tag{B5}
\]
respectively. Notice that \( V^{0L} \otimes V^{1L} \neq H^4_2 \). This means that the trace preserving recovery superoperator \( R \) is defined in terms of one standard recovery operator \( R_1 \) and by the projector \( R_\perp \) onto the orthogonal complement of \( \bigoplus_{i=0}^{14} V^{iL} \), i. e. the part of the Hilbert space \( H^4_2 \) which is not reached by acting on the code \( C \) with the correctable error operators. In the case under consideration,
\[
R_1 \overset{\text{def}}{=} |++\rangle \langle ++| + |---\rangle \langle ---|, \quad R_\perp = \sum_{s=1}^{14} |r_s\rangle \langle r_s|, \tag{B6}
\]
where \( \{ |r_s\rangle \} \) is an orthonormal basis for \( (V^{0L} \otimes V^{1L})^\perp \). It can be shown that \( R \leftrightarrow \{ R_1, R_\perp \} \) is a trace preserving quantum operation,
\[
R_1^\dagger R_1 + R_\perp^\dagger R_\perp = I_{16 \times 16}. \tag{B7}
\]
Considering this recovery operation $\mathcal{R}$ with $R_2 = R_\perp$, the map $\Lambda^{(4)} (\rho)$ in (3) becomes,

$$\Lambda^{(4)}_{\text{recover}} (\rho) \equiv \left( \mathcal{R} \circ \Lambda^{(4)} \right) (\rho) \equiv \sum_{k=0}^{15} \sum_{l=1}^{2} (R_l A'_k) \rho (R_l A'_k)^\dagger,$$

(B8)

**Entanglement Fidelity.** We want to describe the action of $\mathcal{R} \circ \Lambda^{(4)}$ restricted to the code subspace $\mathcal{C}$. Therefore, we compute the $2 \times 2$ matrix representation $[R_l A'_k]_C$ of each $R_l A'_k$ with $l = 1, 2$ and $k = 0, \ldots, 15$ and it turns out that the only matrices $[R_l A'_k]_C$ with non-vanishing trace are given by,

$$[R_l A'_k]_C = \sqrt{p^{(k)}_{l}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(B9)

for $r = 0, 5, 6, 7, 8, 9, 10, 15$. Therefore, the entanglement fidelity $F^{(4)}_{\text{DFS}} (\mu, p)$ defined as,

$$F^{(4)}_{\text{DFS}} (\mu, p) \equiv F^{(4)}_{\text{DFS}} \left( \frac{1}{2} I_{2 \times 2}; \mathcal{R} \circ \Lambda^{(4)} \right) = \frac{1}{(2)^2} \sum_{k=0}^{15} \sum_{l=1}^{2} \left| \text{tr} \left( [R_l A'_k]_C \right) \right|^2,$$

(B10)

is given by,

$$F^{(4)}_{\text{DFS}} (\mu, p) = \tilde{p}^{(4)}_0 + \tilde{p}^{(4)}_5 + \tilde{p}^{(4)}_6 + \tilde{p}^{(4)}_7 + \tilde{p}^{(4)}_8 + \tilde{p}^{(4)}_9 + \tilde{p}^{(4)}_{10} + \tilde{p}^{(4)}_{15}.$$  

(B11)

After some algebra, it follows that,

$$F^{(4)}_{\text{DFS}} (\mu, p) = \mu^3 \left( -8p^4 + 16p^3 - 10p^2 + 2p \right) + \mu^2 \left( 24p^4 - 48p^3 + 28p^2 - 4p \right) +$$

$$+ \mu \left( -24p^4 + 48p^3 - 30p^2 + 6p \right) + \left( 8p^4 - 16p^3 + 12p^2 - 4p + 1 \right).$$  

(B12)

Finally, following the same line of reasoning presented above, $F^{(5)}_{\text{DFS}} (\mu, p)$ and $F^{(6)}_{\text{DFS}} (\mu, p)$ can be computed as well.

[1] M. A. Nielsen and I. L. Chuang, "Quantum Computation and Information", Cambridge University Press (2000).
[2] P. Kaye, R. Laflamme and M. Mosca, "An Introduction to Quantum Computing", Oxford University Press (2007).
[3] D. Gottesman, "An Introduction to Quantum Error Correction and Fault-Tolerant Quantum Computation", arXiv:quant-ph/0004257 (2000).
[4] E. Knill and R. Laflamme, "Theory of quantum error-correcting codes", Phys. Rev. A55, 909 (1997).
[5] A. R. Calderbank et al., "Quantum Error Correction and Orthogonal Geometry", Phys. Rev. Lett. 78, 405 (1997).
[6] P. Zanardi and M. Rasetti, "Error Avoiding Quantum Codes", Mod. Phys. Lett. B11, 1085 (1997).
[7] P. Zanardi and M. Rasetti, "Noiseless Quantum Codes", Phys. Rev. Lett. 79, 3306 (1997).
[8] D. A. Lidar et al., "Decoherence-Free Subspaces for Quantum Computation", Phys. Rev. Lett. 81, 2594 (1998).
[9] D. A. Lidar et al., "Concatenating Decoherence-Free Subspaces with Quantum Error Correcting Codes", Phys. Rev. Lett. 82, 4556 (1999).
[10] D. Bacon et al., "Universal Fault-Tolerant Computation on Decoherence-Free Subspaces", Phys. Rev. Lett. 85, 1758 (2000).
[11] A. D’Arrigo et al., "Memory effects in a Markov chain dephasing channel", Int. J. Quantum Info. 6, 651 (2008).
[12] A. Shabani, "Correlated errors can lead to better performance of quantum codes", Phys. Rev. A77, 022323 (2008).
[13] J. P. Clemens et al., "Quantum error correction against correlated noise", Phys. Rev. A69, 062313 (2004).
[14] B. Schumacher, "Sending entanglement through noisy quantum channels", Phys. Rev. A54, 2614 (1996).
[15] D. A. Lidar and K. B. Whaley, "Decoherence-Free Subspaces and Subsystems", arXiv:quant-ph/0301032 (2003).
[16] M. A. Nielsen, "The entanglement fidelity and quantum error correction", arXiv: quant-ph/9906012 (1996).
[17] E. Knill et al., "Introduction to Quantum Error Correction", arXiv:quant-ph/020717 (2002).