Article

Contribution of Using Hadamard Fractional Integral Operator via Mellin Integral Transform for Solving Certain Fractional Kinetic Matrix Equations

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Abstract: Recently, the importance of fractional differential equations in the field of applied science has gained more attention not only in mathematics but also in electrodynamics, control systems, economic, physics, geophysics and hydrodynamics. Among the many fractional differential equations are kinetic equations. Fractional-order kinetic Equations (FOKEs) are a unifying tool for the description of load vector behavior in disorderly media. In this article, we employ the Hadamard fractional integral operator via Mellin integral transform to establish the generalization of some fractional-order kinetic equations including extended \((k, \tau)\)-Gauss hypergeometric matrix functions. Solutions to certain fractional-order kinetic matrix Equations (FOKMEs) involving extended \((k, \tau)\)-Gauss hypergeometric matrix functions are also introduced. Moreover, several special cases of our main results are archived.

Keywords: Hadamard fractional integral; Mellin transform; fractional-order kinetic equations; extended \((k, \tau)\)-Gauss hypergeometric matrix functions

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1. Introduction

The theory of fractional calculus [1] appears in numerous problems of applied mathematics, statistics, physical phenomena, approximation theory hydrodynamics and many other applications. There are various definitions of fractional calculus. The Hadamard fractional integral operator is one fractional integral operator involved in those definitions, and it was first proposed by Hadamard in 1892 [2]. Later on, expansion formulas in terms of integer-order derivatives for the Hadamard fractional integral and derivative were proposed by Pooseh et al. [3]. Boutiara et al. investigated a boundary value problem for the nonlinear Caputo–Hadamard fractional differential equation with a Hadamard fractional integral in [4]. Moreover, Gong et al. [5] listed the two main differences between the Hadamard fractional derivative and the Riemann–Liouville fractional derivative. Furthermore, Li Ma and Changpin Li [6] discussed the Hadamard fractional operators in three aspects and their reciprocal properties. Recently, Zhou et al. [7] obtained the generalized proportional Hadamard fractional integral operators.

On the contrary, the fractional differential equations have appeared in many applications in modern analysis, economic models, nuclear physics and engineering [8,9]. The kinetic equations particularly describe the continuity of the substance motion. The kinetic (reaction-type) equations have a prime importance as mathematical tools widely used in describing several astrophysical and physical phenomena [10,11]. The production and destruction of nuclei in chemical (thermonuclear) reactions can be described by reaction-type
Reactions characterized by a time-dependent quantity $N = N(t)$ can be formally represented by the following Cauchy problem [11]:

$$\frac{dN}{dt} = -\delta(N) + p(N), \quad N(0) = N_0,$$

(1)

where $\delta$ and $p$ are the destruction rate and the production rate of $N$, respectively, and $N_0$ is the initial data. Haubold and Mathai [11] studied a special case of this Cauchy problem, given by

$$\frac{dN}{dt} = -\theta N, \quad \theta \in \mathbb{R}^+, \quad N(0) = N_0.$$

(2)

Equation (2) is known as the standard kinetic equation. An alternative form of Equation (2) can be obtained as

$$N(t) - N_0 = -\theta t D_t^{-1}N(t), \quad \theta, t \in \mathbb{R}^+, \quad \nu \in \mathbb{R}^+.$$

(3)

where $D_t^{-1}$ is the standard integral operator. Haubold and Mathai [11] have introduced a fractional generalization of the standard kinetic Equation (3) as

$$N(t) - N_0 = -\theta^\nu t D_t^{-\nu} N(t), \quad \theta, t \in \mathbb{R}^+, \quad \nu \in \mathbb{R}^+.$$

(4)

The solution of the fractional kinetic Equation (4) is of the form

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(vn+1)}(\theta t)^{vn}.$$

(5)

At present, one very interesting application of special functions in allied sciences is solving generalized fractional kinetic equations by using various integral transforms; for instance, in recent papers, fractional kinetic equations involving generalized k-Bessel function via Sumudu transform were presented by Agarwal et al. [12]. Suthar et al. [13] introduced a solution of fractional kinetic equations associated with the $(p,q)$-Mathieu-type series. The application of Laplace transform to a fractional kinetic equation pertaining to the generalized Galue-type Struve function was studied by Habenom et al. in [14]. Furthermore, Samraiz et al. established weighted $(k,s)$-Riemann–Liouville fractional integral operators and a solution to a fractional kinetic equation in [15]. Very recently, Ahmed et al. [16] introduced a solution to a fractional kinetic equation for a Hadamard-type fractional integral via Mellin transform.

On the other hand, there has been much recent activity about extending applications from the fractional differential equations into the matrix framework of fractional differential equations by many researchers; for example, Garrappa and Popolizio discussed the use of matrix functions for fractional partial differential equations in [17]. Solutions of certain matrix fractional differential equations were considered by Kiliçman and Ahmad in [18], while the Cauchy problem for matrix factorizations of the Helmholtz equation in a three-dimensional bounded domain were archived in [19] by Juraev et al. Moreover, certain fractional matrix equations of some special matrix functions were recently studied by Abdalla and others in [20,21]. Furthermore, many modern problems can be modeled mathematically by matrix fractional differential equations (MFDEs), such as the population of fractional oscillators, electromagnetic waves, the fractional Lorenz system and the nonlinear oscillation of an earthquake.

Motivated by the above ideas, in our present study, we introduce a generalization of the kinetic fractional equations in terms of an extended $(k, \tau)$-Gauss hypergeometric matrix.
function. The solutions of a generalized fractional kinetic equation using the Hadamard fractional integral operator via Mellin integral transform are derived and established as an application of the extended \((k, \tau)\)-Gauss hypergeometric matrix functions. The results archived in this article generalize many results available in the literature and are capable of generating several applications in the theory of matrix fractional differential equations.

2. Basic Notions

In this section, we recall some definitions and terminologies which will be used to prove the main results. Throughout our present work, let \(\mathbb{N}, \mathbb{R}^+, \mathbb{Z}_0^+\), and \(\mathbb{C}\) be the sets of positive integers, positive real numbers, non-positive integers and complex numbers, respectively, and let \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\) and \(\mathbb{R}^+_0 := \mathbb{R}^+ \cup \{0\}\). In addition, let \(\mathbb{C}^{m \times m}\) be the vector space of all the square matrices of order \(m \in \mathbb{N}\) whose entries are in \(\mathbb{C}\). Furthermore, let \(I\) and \(0\) denote the identity and zero matrices corresponding to a square matrix of any order, respectively.

If \(V \in \mathbb{C}^{m \times m}\), the spectrum \(\sigma(V)\) is the set of all eigenvalues of \(V\), which is denoted by \([22]\)

\[
\mu(V) = \max \{\text{Re}(z) : z \in \sigma(V)\} \quad \text{and} \quad \bar{\mu}(V) = \min \{\text{Re}(z) : z \in \sigma(V)\},
\]

where \(\mu(V)\) refers to the spectral abscissa of \(T\) and for which \(\bar{\mu}(V) = -\mu(-V)\). A matrix \(V\) is said to be a positive stable matrix if and only if \(\bar{\mu}(V) > 0\).

If \(V\) is a positive stable matrix in \(\mathbb{C}^{m \times m}\) and \(k \in \mathbb{R}^+\), then the \(k\)-gamma matrix function \(\Gamma_k(V)\) is well-defined as follows \([23]\):

\[
\Gamma_k(V) = \int_0^\infty u^{V-l} e^{-\frac{uk}{\lambda}} \, du, \quad u^{V-l} := \exp \left( (V-I) \ln u \right).
\]

If \(V\) is a matrix in \(\mathbb{C}^{m \times m}\) such that \(V + \ell k I\) is an invertible matrix for every \(\ell \in \mathbb{N}_0\) and \(k \in \mathbb{R}^+\), then \(\Gamma_k(V)\) is invertible, its inverse is \(\Gamma_k^{-1}(V)\), and the \(k\)-Pochhammer matrix symbol is defined by \([24]\)

\[
(V)_{\ell,k} = V(V + kI) \cdots (V + (\ell - 1)kI) = \Gamma_k(V + kI) \Gamma_k^{-1}(V) \quad (\ell \in \mathbb{N}_0, k \in \mathbb{R}^+).
\]

**Remark 1.** For \(k = 1\), (2.2) and (2.3) will reduce to the gamma matrix function \(\Gamma(V)\), the inverse gamma matrix function \(\Gamma^{-1}(V)\) and the Pochhammer matrix symbol \((V)_n\), respectively \([25, 26]\).

For a non-singular matrix \(S\) in \(\mathbb{C}^{s \times s}\) we define \(S^\nu\) for an arbitrary real number \(\nu\) by

\[
S^\nu = \exp(\nu \log S),
\]

where the logarithm is the principal matrix logarithm \([17]\).

In general, it is not true that \((S^\nu)^\mu = (S^\mu)^\nu\) for real \(\nu\) and \(\mu\), although for symmetric positive definite matrices this identity does hold because the eigenvalues are real and positive.

If \(Y = S^\nu\), does it follow that \(S = Y^\frac{1}{\nu}\)? Clearly, the answer is no in general because, for example, \(Y = S^2\) does not imply \(S = Y^{1/2}\). Using the matrix unwinding function, it can be shown that \((S^\nu)^\frac{1}{\nu} = S\) for \(\nu \in [-1, 1]\). Hence, the function \(G(S) = S^\frac{1}{\nu}\) is the inverse function of \(F(S) = S^\nu\) for \(\nu \in [-1, 1]\) \([17, 27]\).

Let \(V\) be a positive stable matrix in \(\mathbb{C}^{s \times s}\); an extension of the \(k\)-gamma function of a matrix argument (8) is then defined in \([23]\) as follows:

\[
\Gamma_k^\nu(V) = \int_0^\infty u^{V-l} e^{\left( -\frac{uk}{\lambda} \frac{\theta^k}{\ln^k \lambda} \right)} \, du \quad (\theta \in \mathbb{R}^+_0, \ k \in \mathbb{R}^+).
\]
Assume that $D, E, F, G$ and $H$ are positive stable matrices in $\mathbb{C}^{m \times m}$, such that $G + \ell I$ and $H + \ell I$ are invertible for all $\ell \in \mathbb{N}_0, \rho \in \mathbb{R}_0^+$ and $k, \tau \in \mathbb{R}^+$. Then, for $|\xi| < 1$, the extended $(k, \tau)$-Gauss hypergeometric matrix function is defined in [28] by

$$
\begin{align*}
3W_2^{(k, \tau \rho)}(\xi) := & \ 3W_2^{(k, \tau \rho)} \left[ \begin{array}{c}
(D; k, \rho), (E, k), (F, k) \\
(G, k), (H, k)
\end{array} ; \xi \right] \\
:= & \ \Gamma_k^{-1}(E) \Gamma_k(G) \Gamma_k^{-1}(F) \Gamma_k(H) \\
& \times \sum_{s=0}^{\infty} (D; \rho)_s k^{-1}(G + ktsI) \Gamma_k(E + ktsI) \\
& \times \Gamma_k^{-1}(H + ktsI) \Gamma_k(F + ktsI) \frac{\xi^s}{s!},
\end{align*}
$$

where $\Gamma_k^{-1}(H)$ and $\Gamma_k^{-1}(G)$ are the inverse gamma matrix functions of $\Gamma_k(H)$ and $\Gamma_k(G)$, respectively, and $(D; \rho)_s$ is the generalized $k$-Pochhammer matrix symbols defined as

$$
(D; \rho)_s k = \begin{cases} \\
\Gamma_k^p(D + sI) \Gamma_k^{-1}(D), & (\mu(D) > 0, \rho, k \in \mathbb{R}^+, s \in \mathbb{N}) \\
(D)_s, & (p = 0, \ k \in \mathbb{R}^+, s \in \mathbb{N}) \\
I, & (s = 0, \ p = 0, \ k = 1)
\end{cases}
$$

**Remark 2.** The following are some of the special cases of the extended $(k, \tau)$-Gauss hypergeometric matrix functions $3W_2^{(k, \tau \rho)}$ given by Equation (11).

$i$ When $k = 1$, Equation (11) reduces to the following the extended $\tau$-Gauss hypergeometric matrix function [29]:

$$
\begin{align*}
3W_2^{(\tau \rho)}(\xi) := & \ 3W_2^{(\tau \rho)} \left[ \begin{array}{c}
(D; \rho), (E, \tau), (F) \\
(G, \tau)
\end{array} ; \xi \right] \\
:= & \Gamma^{-1}(E) \Gamma(G) \Gamma^{-1}(F) \Gamma(H) \\
& \times \sum_{s=0}^{\infty} (D; \rho)_s \Gamma^{-1}(G + \tau sI) \Gamma(E + \tau sI) \\
& \times \Gamma^{-1}(H + \tau sI) \Gamma(F + \tau sI) \frac{\xi^s}{s!}, \ |\xi| < 1,
\end{align*}
$$

where $D, E, F, G$ and $H$ are positive stable matrices in $\mathbb{C}^{m \times m}$, such that $G + \ell I$ and $H + \ell I$ are invertible for all $\ell \in \mathbb{N}_0, \rho \in \mathbb{R}_0^+$ and $\tau \in \mathbb{R}^+$.

$ii$ When $\tau = 1$ in Equation (11), using some properties of $k$-Pochhammer matrix symbols, we obtain the extended $k$-Gauss hypergeometric matrix function [29]:

$$
\begin{align*}
3W_2^{(k \rho)}(\xi) := & \ 3W_2^{(k \rho)} \left[ \begin{array}{c}
(D; k, \rho), (E, k), (F, k) \\
(G, k), (H, k)
\end{array} ; \xi \right] \\
:= & \sum_{s=0}^{\infty} (D; \rho)_s k(E)_{s,k} (F)_{s,k} [(G)_{s,k}]^{-1} [(H)_{s,k}]^{-1} \frac{\xi^s}{s!} \ |\xi| < 1,
\end{align*}
$$

where $D, E, F, G$ and $H$ are positive stable matrices in $\mathbb{C}^{m \times m}$, such that $G + \ell I$ and $H + \ell I$ are invertible for all $\ell \in \mathbb{N}_0, \rho \in \mathbb{R}_0^+$ and $k \in \mathbb{R}^+$. 


iii When \( F = H \), Equation (11) reduces to the extended \((k, \tau)\)-Wright hypergeometric matrix function \( \mathbf{2R}_1^{(k, \tau)}(\xi) \) defined by

\[
\mathbf{2R}_1^{(k, \tau)}(\xi) := \mathbf{2R}_1^{(r)}((D,k\rho),(E,k);(G,k);\xi)r
\]

\[
:= \Gamma^{-1}_k(E)\Gamma_k(G)\sum_{s=0}^{\infty}(D;r)_s\Gamma^{-1}_k(G+k\tau sI)\Gamma_k(E+k\tau sI)\xi^s s! |\xi| < 1,
\]

where \( D, E \) and \( G \) are positive stable matrices in \( \mathbb{C}^{m \times m} \), such that \( E + \ell I \) and \( G + \ell I \) are invertible for all \( \ell \in \mathbb{N}_0, \rho \in \mathbb{R}_0^+ \) and \( \tau \in \mathbb{R}^+ \).

iv If we set \( \rho = 0 \) and \( F = H \), then Equation (11) reduces to the \((k, \tau)\)–Gauss hypergeometric matrix function \( \mathbf{2R}_1^{(k, \tau)}(\xi) \) given by

\[
\mathbf{2R}_1^{(k, \tau)}(\xi) := \mathbf{2R}_1^{(r)}((D,k),(E,k);(G,k);\xi)
\]

\[
:= \Gamma^{-1}_k(E)\Gamma_k(G)\sum_{s=0}^{\infty}(D;\tau)_s\Gamma^{-1}_k(G+k\tau sI)\Gamma_k(E+k\tau sI)\xi^s s! |\xi| < 1,
\]

where \( D, E \) and \( G \) are positive stable matrices in \( \mathbb{C}^{m \times m} \), such that \( E + \ell I \) and \( G + \ell I \) are invertible for all \( \ell \in \mathbb{N}_0, k, \tau \in \mathbb{R}^+ \).

v When \( \tau = 1 \) in Equation (16), using some properties of \( k \)-Pochhammer matrix symbols, we obtain the \( k \)-hypergeometric matrix function [30]:

\[
\mathbf{H}^k(D, E; G; \xi) = \sum_{s=0}^{\infty}(D,s\xi)_s(F)_s\xi^{-1} s! |\xi| < 1,
\]

where \( k \in \mathbb{R}^+ \) and \( D, E \) and \( G \) are positive stable matrices in \( \mathbb{C}^{m \times m} \) such that \( G + \ell I \) is invertible for all \( \ell \in \mathbb{N}_0 \).

vi When \( k = 1 \), Equation (16) reduces to the following \( \tau \)-Wright hypergeometric matrix function [21,31]:

\[
\mathbf{2R}_1^{(\tau)}(D, E; G; \xi) := \Gamma^{-1}((E)\Gamma(G)\sum_{s=0}^{\infty}(D;\tau)_s\Gamma^{-1}(G+\tau sI)\Gamma(E+\tau sI)\xi^s s! |\xi| < 1,
\]

where \( \tau \in \mathbb{R}^+ \) and \( D, E \) and \( G \) are positive stable matrices in \( \mathbb{C}^{m \times m} \) such that \( G + \ell I \) is invertible for all \( \ell \in \mathbb{N}_0 \).

vii If we set \( \tau = 1 \) and \( k = 1 \), Equation (16) will yield the hypergeometric matrix function defined in [26].

viii Furthermore, taking \( D, E \) and \( G \in \mathbb{C}^{1 \times 1} \) in (2.12), we obtain the generalized hypergeometric function [32].

The Mellin transform of a suitable integrable function \( \mathcal{G}(t) \) is defined [33], as usual, by

\[
\mathcal{G}(\delta) = \mathcal{M}\{\mathcal{G}(t) : t \to \delta\} = \int_0^{\infty} t^{\delta-1} \mathcal{G}(t) \, dt \quad (\delta \in \mathbb{C}),
\]

provided that the improper integral in Equation (19) exists. Furthermore, the inverse Mellin transform is

\[
\mathcal{G}(t) = \mathcal{M}^{-1}\{\mathcal{G}(\delta) : \delta \to t\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-\delta} \mathcal{G}(\delta) \, d\delta \quad (c = \text{Re} \delta).
\]
If there is any matrix $V$ in $\mathbb{C}^{m \times m}$, the Mellin transform becomes
\[ \mathcal{M}\{f(t); V\} := F(s) = \int_{0}^{\infty} f(t) t^{V-1} dt. \]  

We conclude with the following example to illustrate this definition.

**Example 1.** The Mellin transform of
\[ \mathcal{M}\left\{ \frac{1}{1+t}; V \right\} = \Gamma(V)\Gamma(I-V). \]

Furthermore, the Mellin convolution of two functions $\theta(t)$ and $\phi(t)$ is defined as
\[ (\theta * \phi)(t) = \int_{0}^{t} \theta\left(\frac{t}{x}\right) \phi(x) \frac{dx}{x}. \]  

**Lemma 1** ([28]). For a matrix $R \in \mathbb{C}^{m \times m}$, $\sigma \in \mathbb{R}_{0}^{+}$, and $k, \delta \in \mathbb{R}^{+}$, we have
\[ \mathcal{M}\{\Gamma_{k}^{\sigma}(R) ; \delta\} = \Gamma_{k}(\delta I) \Gamma_{k}(R + \delta I) \quad (\bar{\mu}(R + \delta I) > 0 \text{ when } k = 1), \]
where $\Gamma_{k}^{\sigma}(R)$ is the extended $k$-gamma of a matrix argument defined in (11).

**Theorem 1** ([28]). The Mellin transform of the matrix function $3W_{2}(k,\tau,p)(\xi)$ is given by
\[ \mathcal{M}\left\{ 3W_{2}(k,\tau,p) \left(\begin{array}{c} (D,k;\rho), (E,k), (F,k) \\ (G,k), (H,k) \end{array} \right ; \xi \right\} : \rho \rightarrow \varepsilon \}
\]
\[ = \Gamma_{k}(\varepsilon) (D)_{\varepsilon,k} 3W_{2}(k,\tau,p) \left(\begin{array}{c} (D + \varepsilon I,k;\rho), (E,k), (F,k) \\ (G,k), (H,k) \end{array} \right ; \xi \right\}, \]
where $\Re(\varepsilon) > 0$ and $\bar{\mu}(D + \varepsilon I) > 0$ when $\rho = 0$ and $k = 1$.

**Definition 1** ([33]). Let $\Re(\gamma) > 0$. The left-sided and the right-sided Hadamard fractional integrals of order $\gamma \in \mathbb{C}$ are defined, respectively, as
\[ (H^{\gamma}_{\tau}f)(t) = \frac{1}{\Gamma(\gamma)} \int_{0}^{t} \left(\log \frac{t}{\tau}\right)^{\gamma-1} f(\tau) d\tau, \quad t > 0, \]
and
\[ (H^{\gamma}_{\tau}f)(t) = \frac{1}{\Gamma(\gamma)} \int_{t}^{\infty} \left(\log \frac{\tau}{t}\right)^{\gamma-1} f(\tau) d\tau, \quad t > 0. \]

**Lemma 2** ([33]). If $\Re(\gamma) > 0$, $\tau \in \mathbb{C}$, and the Mellin transform $\mathcal{M}(f)(\tau)$ exists for a function $f$, the following holds true:
\[ \mathcal{M}(H^{\gamma}_{\tau}f)(\tau) = (-\tau)^{-\gamma}(\mathcal{M}f)(\tau), \quad \Re(\tau) < 0, \]
and
\[ \mathcal{M}(H^{\gamma}_{\tau}f)(\tau) = (\tau)^{-\gamma}(\mathcal{M}f)(\tau), \quad \Re(\tau) > 0. \]

**Theorem 2** ([34]). For $t \in [0,\xi]$,
\[ \mathcal{M}[f(t)](\tau) = \mathcal{F}(\tau) = \int_{0}^{\xi} \xi^{-\tau} t^{\gamma-1} f(t) dt. \]
and

\[ f(t) = \mathcal{M}^{-1}[\mathcal{F}(\tau)](t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\mathcal{F}(\tau)}{\tau} d\tau. \]

3. Statement of Results

In this section, we study fractional kinetic equations with Hadamard fractional integrals involving generalized extended \((k, \tau)-Gauss\) hypergeometric matrix functions.

**Theorem 3.** Let \(D_{\mu}, E_{\mu}, F_{\mu}, G_{\mu}, H_{\mu}\) and \(C\) be positive stable matrices in \(\mathbb{C}^{m \times m}\), such that \(G_{\mu} + \ell I\) and \(H_{\mu} + \ell I\) are invertible for all \(\mu \in \mathbb{N}\), \(\ell \in \mathbb{N}_0\), \(\alpha, \rho \in \mathbb{R}_0^+\), and \(d, k, \tau, \zeta \in \mathbb{R}^+\). Then, for \(\gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-,\) and \(t \in [0, \zeta]\) the generalized fractional kinetic equation

\[ N(t)I - N_0 \tau^{\nu-1} \prod_{\mu=1}^{n} \mathcal{W}_2^{(k_{\mu}, \gamma_{\mu}, \rho_{\mu})} \left[ (D_{\mu}, k_{\mu}; \rho_{\mu}), (E_{\mu}, k_{\mu}), (F_{\mu}, k_{\mu}), (G_{\mu}, k_{\mu}), (H_{\mu}, k_{\mu}) \right] ; \tau^\gamma \right] = -C^T H^T I^T N(t), \quad (25) \]

is solvable. The solution to Equation (25) is given by the formula

\[
N(t)I = N_0 \tau^{\nu-1} \log t \prod_{\mu=1}^{n} \Gamma^{-1}_{k_{\mu}}(E_{\mu}) \Gamma_{k_{\mu}}(G_{\mu}) \Gamma^{-1}_{k_{\mu}}(F_{\mu}) \Gamma_{k_{\mu}}(H_{\mu}) \times
\]

\[
\times \sum_{s=0}^{\infty} (D_{\mu}; \rho)_{s,k_{\mu}} \Gamma^{-1}_{k_{\mu}}(G_{\mu} + k_{\mu} \tau s I) \Gamma_{k_{\mu}}(E_{\mu} + k_{\mu} \tau s I) \times
\]

\[
\times \Gamma_{k_{\mu}}^{-1}(H_{\mu} + k_{\mu} \tau s I) \Gamma_{k_{\mu}}(F_{\mu} + k_{\mu} \tau s I) \left( \frac{d^{\mu} \gamma_{\mu} s}{s!} \right)^{\mu} \times
\]

\[
\times \sum_{r=0}^{\infty} \sum_{\ell=0}^{\infty} \left[ -\left( \log t \right)^{\gamma} \left[ \log t^{-\gamma \mathbb{N}^+} \right]^{\ell} \Gamma[1 - (\gamma r + \ell + 2)], \right. \quad (26)
\]

where \(3\mathcal{W}_2^{(k_{\mu}, \gamma_{\mu}, \rho_{\mu})}\) is generalized of (11).

**Proof.** Recall that the Mellin transform for the Hadamard fractional integral is

\[ \mathcal{M} \left[ H^T I^T N(t) \right](z) = z^{-\gamma} \mathcal{N}(z), \]

where \(\mathcal{N}(z)\) is the Mellin transform of \(N(t)\).

Applying the Mellin transform to Equation (25) gives

\[ \mathcal{N}(z) \left[ I + z^{-\gamma} \mathcal{C} \right] = N_0 \prod_{\mu=1}^{n} \Gamma^{-1}_{k_{\mu}}(E_{\mu}) \Gamma_{k_{\mu}}(G_{\mu}) \Gamma^{-1}_{k_{\mu}}(F_{\mu}) \Gamma_{k_{\mu}}(H_{\mu}) \times
\]

\[
\times \sum_{s=0}^{\infty} (D_{\mu}; \rho)_{s,k_{\mu}} \Gamma^{-1}_{k_{\mu}}(G_{\mu} + k_{\mu} \tau s I) \Gamma_{k_{\mu}}(E_{\mu} + k_{\mu} \tau s I) \times
\]

\[
\times \Gamma_{k_{\mu}}^{-1}(H_{\mu} + k_{\mu} \tau s I) \Gamma_{k_{\mu}}(F_{\mu} + k_{\mu} \tau s I) \left( \frac{d^{\mu} \gamma_{\mu} s}{s!} \right)^{\mu} \mathcal{M} \left[ \gamma^{\mu s + \alpha - 1} \right](z). \]

However, for \(t \in [0, \zeta]\),

\[ \mathcal{M} \left[ \gamma^{\mu s + \alpha - 1} \right](z) = \frac{\gamma^{\mu s + \alpha - 1}}{z + \gamma \mathbb{N}^+ + \alpha - 1}, \quad z \in \mathbb{C}. \]
Hence,
\[
\mathcal{N}(z) = N_0 \prod_{\mu=1}^{n} \Gamma_{k_{\mu}}^{-1}(E_{\mu}) \Gamma_{k_{\mu}}(G_{\mu}) \Gamma_{k_{\mu}}^{-1}(F_{\mu}) \Gamma_{k_{\mu}}(H_{\mu}) \times \\
\times \sum_{s=0}^{\infty} (D_{\mu}; \rho)_{s,k_{\mu}} \Gamma_{k_{\mu}}^{-1}(G_{\mu} + k_{\mu} \tau s I) \Gamma_{k_{\mu}}(E_{\mu} + k_{\mu} \tau s I) \times \\
\times \Gamma_{k_{\mu}}^{-1}(H_{\mu} + k_{\mu} \tau s I) \Gamma_{k_{\mu}}(F_{\mu} + k_{\mu} \tau s I) \left( \frac{d^{\gamma}}{s!} \right)^{\mu} \frac{\xi^{\gamma s + \alpha - 1}}{z + \gamma \mu s + \alpha - 1} \times \\
\times \sum_{r=0}^{\infty} (-1)^r C^{\gamma r} z^{-\gamma r}
\]
which can be written as
\[
\mathcal{N}(z) = N_0 \prod_{\mu=1}^{n} \Gamma_{k_{\mu}}^{-1}(E_{\mu}) \Gamma_{k_{\mu}}(G_{\mu}) \Gamma_{k_{\mu}}^{-1}(F_{\mu}) \Gamma_{k_{\mu}}(H_{\mu}) \times \\
\times \sum_{s=0}^{\infty} (D_{\mu}; \rho)_{s,k_{\mu}} \Gamma_{k_{\mu}}^{-1}(G_{\mu} + k_{\mu} \tau s I) \Gamma_{k_{\mu}}(E_{\mu} + k_{\mu} \tau s I) \times \\
\times \Gamma_{k_{\mu}}^{-1}(H_{\mu} + k_{\mu} \tau s I) \Gamma_{k_{\mu}}(F_{\mu} + k_{\mu} \tau s I) \left( \frac{d^{\gamma}}{s!} \right)^{\mu} \xi^{\gamma s + \alpha - 1} \times \\
\times \sum_{r=0}^{\infty} (-1)^r C^{\gamma r} \frac{z^{-\gamma r}}{z + \gamma \mu s + \alpha - 1}.
\] 

Since
\[
\mathcal{M}^{-1} \left[ \frac{z^{-\gamma r}}{z + \gamma \mu s + \alpha - 1} \right](t) = \int_{0}^{\infty} t^{-\gamma} \frac{z^{-\gamma r}}{z + \gamma \mu s + \alpha - 1} dz \\
= \sum_{\ell=0}^{\infty} \left[ - (\gamma \mu s + \alpha - 1) \right]^\ell \int_{0}^{\infty} t^{-\gamma} z^{-(\gamma r + \ell + 2)} dz \\
= \sum_{\ell=0}^{\infty} \left[ - (\gamma \mu s + \alpha - 1) \right]^\ell [\log t]^{\gamma r + \ell + 1} \Gamma(1 - (\gamma r + \ell + 2)),
\]
then, taking the inverse Mellin transform on both sides of Equation (27) yields
\[
N(t) = N_0 \prod_{\mu=1}^{n} \Gamma_{k_{\mu}}^{-1}(E_{\mu}) \Gamma_{k_{\mu}}(G_{\mu}) \Gamma_{k_{\mu}}^{-1}(F_{\mu}) \Gamma_{k_{\mu}}(H_{\mu}) \times \\
\times \sum_{s=0}^{\infty} (D_{\mu}; \rho)_{s,k_{\mu}} \Gamma_{k_{\mu}}^{-1}(G_{\mu} + k_{\mu} \tau s I) \Gamma_{k_{\mu}}(E_{\mu} + k_{\mu} \tau s I) \times \\
\times \Gamma_{k_{\mu}}^{-1}(H_{\mu} + k_{\mu} \tau s I) \Gamma_{k_{\mu}}(F_{\mu} + k_{\mu} \tau s I) \left( \frac{d^{\gamma}}{s!} \right)^{\mu} \xi^{\gamma s + \alpha - 1} \times \\
\times \sum_{r=0}^{\infty} (-1)^r C^{\gamma r} \sum_{\ell=0}^{\infty} \left[ - (\gamma \mu s + \alpha - 1) \right]^\ell [\log t]^{\gamma r + \ell + 1} \Gamma(1 - (\gamma r + \ell + 2)),
\]
which is the targeted result of Equation (26). 

Several special cases can be introduced here.
Corollary 1. Let $D_{\mu}, E_{\mu}, F_{\mu}, G_{\mu}, H_{\mu}$ and $C$ be positive stable matrices in $\mathbb{C}^{m \times m}$, such that $G_{\mu} + \ell I$ and $H_{\mu} + \ell I$ are invertible for all $\mu \in \mathbb{N}, \ell \in \mathbb{N}_0, \rho \in \mathbb{R}^+_0$, and $d, k, \tau, \xi \in \mathbb{R}^+$. Then, for $\gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and $t \in [0, \xi]$ the generalized fractional kinetic equation is

$$N(t) - N_0 \prod_{\mu=1}^{n} 3\mathcal{W}_2^{(k_{\mu}, \tau_{\mu})} \left( (D_{\mu}, k_{\mu} ; \rho), (E_{\mu}, k_{\mu}), (F_{\mu}, k_{\mu}) \\ (G_{\mu}, k_{\mu}), (H_{\mu}, k_{\mu}) \right) ; d^r t^\gamma \right] = -C_H^{\gamma} I^\gamma_t N(t), \quad (28)$$

where $3\mathcal{W}_2^{(k_{\mu}, \tau_{\mu})}$ is generalized of (11). The solution to Equation (28) is given by the formula

$$N(t) = N_0 \log t \prod_{\mu=1}^{n} (D_{\mu} ; k_{\mu}) (E_{\mu} ; k_{\mu}) (F_{\mu} ; k_{\mu}) (G_{\mu} + k_{\mu} \tau_\mu I) (H_{\mu} + k_{\mu} \tau_\mu I) \times$$

$$\times \sum_{s=0}^{\infty} \left( D_{\mu} ; k_{\mu} \Gamma_{k_{\mu}}^{-1}(G_{\mu} + k_{\mu} \tau_\mu I) \right) (H_{\mu} + k_{\mu} \tau_\mu I) \times$$

$$\times \Gamma_{k_{\mu}}^{-1}(H_{\mu} + k_{\mu} \tau_\mu I) (F_{\mu} + k_{\mu} \tau_\mu I) \left( \frac{d^r \xi^{\frac{r}{s!}}}{s!} \right) \times$$

$$\times \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} \left[ \left( \log t^C \right)^r \left[ \log t^{-(\gamma + \mu)} \right] \Gamma[1 - (\gamma r + \ell + 2)] \right]. \quad (29)$$

Corollary 2. Let $D, E, F, G, H$ and $C$ be positive stable matrices in $\mathbb{C}^{m \times m}$, such that $G + \ell I$ and $H + \ell I$ are invertible for all $\ell \in \mathbb{N}_0, \alpha, \rho \in \mathbb{R}^+_0$, and $d, k, \tau, \xi \in \mathbb{R}^+$. Then, for $\gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and $t \in [0, \xi]$ the generalized fractional kinetic equation

$$N(t) = N_0 \log t \prod_{\mu=1}^{n} (D_{\mu} ; k_{\mu}) (E_{\mu} ; k_{\mu}) (F_{\mu} ; k_{\mu}) (G_{\mu} + k_{\mu} \tau_\mu I) (H_{\mu} + k_{\mu} \tau_\mu I) \times$$

$$\times \sum_{s=0}^{\infty} \left( D_{\mu} ; k_{\mu} \Gamma_{k_{\mu}}^{-1}(G_{\mu} + k_{\mu} \tau_\mu I) \right) (H_{\mu} + k_{\mu} \tau_\mu I) \times$$

$$\times \Gamma_{k_{\mu}}^{-1}(H_{\mu} + k_{\mu} \tau_\mu I) (F_{\mu} + k_{\mu} \tau_\mu I) \left( \frac{d^r \xi^{\frac{r}{s!}}}{s!} \right) \times$$

$$\times \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} \left[ \left( \log t^C \right)^r \left[ \log t^{-(\gamma + \alpha + 1)} \right] \Gamma[1 - (\gamma r + \ell + 2)] \right]. \quad (30)$$

Corollary 3. Let $D, E, F, G, H$ and $C$ be positive stable matrices in $\mathbb{C}^{m \times m}$, such that $G + \ell I$ and $H + \ell I$ are invertible for all $\ell \in \mathbb{N}_0, \rho \in \mathbb{R}^+_0$, and $d, k, \tau, \xi \in \mathbb{R}^+$. Then, for $\gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and $t \in [0, \xi]$ the generalized fractional kinetic equation

$$N(t) = N_0 \prod_{\mu=1}^{n} 3\mathcal{W}_2^{(k_{\mu}, \tau_{\mu})} \left( (D_{\mu}, k_{\mu} ; \rho), (E_{\mu}, k_{\mu}), (F_{\mu}, k_{\mu}) \\ (G, k), (H, k) \right) ; d^r t^\gamma \right] = -C_H^{\gamma} I^\gamma_t N(t), \quad (32)$$
with $\mathcal{W}_2^{(k,\tau,\rho)}$ as defined in (11), is solvable. The solution to Equation (32) is given by the formula

$$N(t) = N_0 \log(t) \Gamma_k^{-1}(E) \Gamma_k(G) \Gamma_k^{-1}(F) \Gamma_k(H) \times$$

$$\times \sum_{s=0}^{\infty} (D; \rho)_s \Gamma_k^{-1}(G + ktsI) \Gamma_k(E + ktsI) \times$$

$$\times \Gamma_k^{-1}(H + ktsI) \Gamma_k(F + ktsI) \left( \frac{e^s}{s!} \right) \times$$

$$\times \sum_{r=0}^{\infty} \sum_{t=0}^{\infty} \left[ - \left( \log t \right)^{\gamma} \right]^t \left( \log t^{-\gamma} \right) \Gamma[1 - (\gamma r + \ell + 2)].$$

Theorem 4. Let $D, E, F, G, H$ and $C$ be positive stable matrices in $\mathbb{C}^{m \times m}$, such that $G + \ell I$ and $H + \ell I$ are invertible for all $\ell \in \mathbb{N}_0$, $\rho \in \mathbb{R}_0^+$, and $k, \gamma, \xi \in \mathbb{R}_0^+$. Then, for $\gamma \in \mathbb{C} \setminus \mathbb{Z}^+_0$, and $t \in [0, \xi]$ the generalized fractional kinetic equation

$$N(\rho) I - N_0 \mathcal{W}_2^{(k,\tau,\rho)} \left[ (D, k; \rho), (E, k), (F, k) \right]_{z} = -C_H \int_{\rho}^{\tau} N(\rho),$$

with $\mathcal{W}_2^{(k,\tau,\rho)}$ as defined in (11), is solvable. The solution to Equation (34) is given by the formula

$$N(\rho) I = N_0 \Gamma_k^{-1}(E) \Gamma_k(G) \Gamma_k^{-1}(F) \Gamma_k(H) \Gamma_k^{-1}(D) \times$$

$$\times \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \Gamma_k^{-1}(G + ktsI) \Gamma_k(E + ktsI) \times$$

$$\times \Gamma_k^{-1}(H + ktsI) \Gamma_k(F + ktsI) \times$$

$$\times \Gamma_k^\rho(D + ktsI) + (\log \rho)^{\gamma-1} \left( \frac{e^s}{s!} \right).$$

Proof. Substituting $\mathcal{W}_2^{(k,\tau,\rho)}$ from (11) into Equation (34), yields

$$N(\rho) I - N_0 \sum_{s=0}^{\infty} \Lambda_{k,s} \Gamma_k^\rho(D + ktsI) = -C_H \int_{\rho}^{\tau} N(\rho),$$

where

$$\Lambda_k = \Gamma_k^{-1}(E) \Gamma_k(G) \Gamma_k^{-1}(F) \Gamma_k(H) \Gamma_k^{-1}(D),$$

$$\Lambda_{k,s} = \Gamma_k^{-1}(G + ktsI) \Gamma_k(E + ktsI) \Gamma_k^{-1}(H + ktsI) \Gamma_k(F + ktsI) \frac{e^s}{s!}. (37)$$

Let $\mathcal{N}(z) := \mathcal{M}[N(\rho); z]$ be the Mellin transform of $N(\rho)$. Then, applying the Mellin transform to Equation (36) gives

$$\mathcal{N}(z)[I + C^\gamma z^{-\gamma}] = N_0 \sum_{s=0}^{\infty} \Lambda_{k,s} \mathcal{M}[\Gamma_k^\rho(D + ktsI); z].$$

According to Lemma 1, we have

$$\mathcal{M}[\Gamma_k^\rho(D + ktsI); z] = \Gamma_k(zI) \Gamma_k(D + (kts + z)I).$$

Hence,

$$\mathcal{N}(z) I = N_0 \sum_{s=0}^{\infty} \Lambda_{k,s} \Gamma_k(zI) \Gamma_k\left(D + ktsI + \frac{z}{k}\right) \sum_{r=0}^{\infty} (-1)^r C^\gamma z^{-\gamma},$$
which can be rewritten as
\[ N(z)I = N_0 \Lambda_k \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} (-1)^r C^{s+r} \Lambda_k \Gamma_k(zI) \Gamma_k(D + (kts + z)I)z^{\gamma r}. \] (38)

Since
\[ M^{-1} [\Gamma_k(zI) \Gamma_k(D + (kts + z)I)z^{-\gamma r}] (\rho) = \Gamma^\rho_k(D + ktsI) * (\log \rho)^{\gamma r - 1} \]
\[ = \int_0^{\infty} \left( \frac{\log \rho}{x} \right)^{\gamma r - 1} \Gamma^\rho_k(D + ktsI) \frac{dx}{x}, \]
then, taking the inverse Mellin transform on both sides of Equation (38) yields
\[ N(z)I = N_0 \Lambda_k \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} (-1)^r C^{s+r} \Lambda_k \Gamma^\rho_k(D + ktsI) * (\log \rho)^{\gamma r - 1}, \]
which is the targeted result of Equation (44). \( \square \)

Next, we introduce some special cases.

**Corollary 4.** Let \( D, E, F, G, H \) and \( C \) be positive stable matrices in \( \mathbb{C}^{m \times m} \), such that \( G + \ell I \) and \( H + \ell I \) are invertible for all \( \ell \in \mathbb{N}_0, \rho \in \mathbb{R}^+ \), and \( \tau, \xi \in \mathbb{R}^+ \). Then, for \( \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^+ \), \( t \in [0, \xi] \) and \( 3W_2^{(\tau \rho)} \) are the extended \( \tau \)-Gauss hypergeometric matrix function defined by Equation (13), and the generalized fractional kinetic equation
\[ N(\rho)I - N_0 3W_2^{(\tau \rho)} \begin{bmatrix} (D, \rho), (E), (F) \cr (G), (H) \end{bmatrix} = -C_H \Gamma^\rho_H N(\rho), \] (39)
is solvable. The solution to Equation (34) is given by the formula
\[ N(\rho)I = N_0 \Gamma^{-1}(E) \Gamma(G) \Gamma^{-1}(F) \Gamma(H) \Gamma^{-1}(D) \times\]
\[ \times \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \Gamma^{-1}(G + \tau s I) \Gamma(E + \tau s I) \times\]
\[ \times \Gamma^{-1}(H + \tau s I) \Gamma(F + \tau s I) \times\]
\[ \times \Gamma^\rho(D + s I) * (\log \rho)^{\gamma r - 1} \left( \frac{\xi^s}{s!} \right). \] (40)

**Corollary 5.** Let \( D, E, F, G, H \) and \( C \) be positive stable matrices in \( \mathbb{C}^{m \times m} \), such that \( G + \ell I \) and \( H + \ell I \) are invertible for all \( \ell \in \mathbb{N}_0, \rho \in \mathbb{R}^+ \), and \( k, \xi \in \mathbb{R}^+ \). Then, for \( \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^+ \), \( t \in [0, \xi] \) and \( 3W_2^{(k \rho)} \) are the extended \( k \)-Gauss hypergeometric matrix function defined by Equation (14), and the generalized fractional kinetic equation
\[ N(\rho)I - N_0 3W_2^{(k \rho)} \begin{bmatrix} (D, k; \rho), (E, k), (F, k) \cr (G, k), (H, k) \end{bmatrix} = -C_H \Gamma^\rho_H N(\rho), \] (41)
is solvable. The solution to Equation (34) is given by the formula
The Gauss hypergeometric function is one of the special functions that arise frequently provided its solutions using the Hadamard fractional integral operator and Mellin integral transform. The obtained results are generalizations of many previously known results.

Furthermore, the current study may open the door for further investigations concerning the practical applications of matrix functions associated with fractional differential equations involving the extended fractional kinetic equation for all \( k, \tau, \xi \in \mathbb{R}^+ \)

\[
N(\rho)I = N_0 \Gamma_k^{-1}(E)\Gamma_k^{-1}(G)\Gamma_k^{-1}(F)\Gamma_k(H)\Gamma_k^{-1}(D) \times \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{\Gamma_k^{-1}(G + ksl)\Gamma_k(E + ksl)}{\Gamma_k^{-1}(H + ksl)\Gamma_k(F + ksl)} \times \Gamma_k^\rho(D + ksl) \times (\log \rho)^{\gamma - 1} \left( \frac{\rho}{s!} \right).
\]

**Corollary 6.** Let \( D, E, G, \) and \( C \) be positive stable matrices in \( \mathbb{C}^{m \times m} \), such that \( G + \ell I \) is invertible for all \( \ell \in \mathbb{N}_0, \rho \in \mathbb{R}^+ \), and \( k, \tau, \xi \in \mathbb{R}^+ \). Then, for \( \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^+ \), \( t \in [0, \xi] \) and \( 2^R_1(k, \tau, \rho) \) are the extended \((k, \tau)\)-Wright hypergeometric matrix function defined by Equation (15), and the generalized fractional kinetic equation

\[
N(\rho)I = N_0 2^R_1[(D, k; \rho), (E, k); (G, k); \xi] = C_H^\rho I G N(\rho),
\]

is solvable. The solution to Equation (34) is given by the formula

\[
N(\rho)I = N_0 2^R_1[(D, k; \rho), (E, k); (G, k); \xi] = C_H^\rho I G N(\rho),
\]

**Remark 3.** Similarly, using special cases in Remark 1, we can indicate other results similar to Theorems 3 and 4.

**4. Conclusions**

The motivation of the present work arises from the following:

(A) The Gauss hypergeometric function is one of the special functions that arise frequently in a wide variety of problems in theoretical physics, differential equations, statistics, engineering science and other sciences. In [10], the authors used hypergeometric-type functions to analyze the physical phenomenon of the Casimir effect and Bose–Einstein condensation for particular situations.

(B) Matrix generalizations of special functions have become important during the last few years. The study here depends on the papers on Gauss hypergeometric matrix functions and related matrix functions which were recently published [25].

(C) This work grew out of interesting applications of special functions in solving generalized fractional kinetic equations by using various integral transforms such as Sumudu transform [12], Laplace transform [13,14] and Mellin transform [16].

Depending on all the above, in this article, we introduced generalized kinetic fractional equations involving the extended \((k, \tau)\)-Gauss hypergeometric matrix function, and provided its solutions using the Hadamard fractional integral operator and Mellin integral transform. The obtained results are generalizations of many previously known results. Furthermore, the current study may open the door for further investigations concerning the practical applications of matrix functions associated with fractional differential equations [17,18].

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