Symplectic Slice for Subgroup Actions

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Abstract
Given a symplectic manifold \((M, \omega)\) endowed with a proper Hamiltonian action of a Lie group \(G\), we consider the action induced by a Lie subgroup \(H\) of \(G\). We propose a construction for two compatible Witt-Artin decompositions of the tangent space of \(M\), one relative to the \(G\)-action and one relative to the \(H\)-action. In particular, we provide an explicit relation between the respective symplectic slices.

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1. Introduction
Stability properties and bifurcations of relative equilibria can be determined by a method developed by Krupa [7], which states that the dynamics of an equivariant vector field, in a neighbourhood of a group orbit, is entirely governed by
the dynamics transverse to that group orbit by using the so-called slice coordinates, introduced by Palais [14]. While Krupa proved this result for compact Lie group actions, Fiedler et al. [4] extended it to proper Lie group actions. The Hamiltonian analogue has been studied by Mielke [10] and Guillemin and Sternberg [6], as well as Roberts and Sousa Dias [17] and Roberts, Wulff and Lamb [18]. By “dynamics transverse to the group orbit”, we mean that the vector field in question can be split into two parts: one part is defined along the group orbit and the other part belongs to a choice of normal subspace. In Hamiltonian systems, the flow of a Hamiltonian vector field with a fixed initial condition is confined to a level set of the momentum map, reflecting the conservation of momentum. The choice of normal space is therefore more restrictive than for general dynamical systems. Before giving its explicit form, we introduce some terminologies.

Let $G$ be a Lie group with Lie algebra $g$. There are natural actions of $G$ on $g$ and on its dual Lie algebra $g^*$, namely the adjoint action $\text{Ad} : (g, x) \in G \times g \mapsto \text{Ad}_g x \in g$ and the coadjoint action $\text{Ad}^* : (g, \lambda) \in G \times g^* \mapsto \text{Ad}^*_g \lambda \in g^*$. The respective infinitesimal actions are given by $\text{ad} : (x, y) \in g \times g \mapsto \text{ad}_x y = [x, y] \in g$ and $\text{ad}^* : (x, \lambda) \in g \times g^* \mapsto \text{ad}^*_x \lambda \in g^*$.

Definition 1.1. Assume that $G$ acts properly and canonically on a symplectic manifold $(M, \omega)$. In addition we assume that this group action is Hamiltonian and that the associated momentum map $\Phi_G : M \to g^*$ is equivariant with respect to the coadjoint action. Then the quadruple $(M, \omega, G, \Phi_G)$ is called a Hamiltonian $G$-manifold.

We fix a Hamiltonian $G$-manifold $(M, \omega, G, \Phi_G)$ and a point $m \in M$ with momentum $\mu = \Phi_G(m)$. The corresponding stabilizers are denoted by $G_m$ and $G_\mu$, and their respective Lie algebras by $g_m$ and $g_\mu$. Let $G \cdot m = \{g \cdot m \mid g \in G\}$ be the $G$-orbit of some point $m \in M$ and denote by $g \cdot m$ its tangent space at $m$. Elements of $g \cdot m$ are vectors of the form $x_M(m) := \frac{d}{dt} \bigg|_{t=0} \exp(tx) \cdot m$, where $x \in g$ and $\exp : g \to G$ is the group exponential. A symplectic slice $N_1$ at $m$ is a $G_m$-invariant subspace of $(T_m M, \omega(m))$ defined by

$$N_1 := \ker(\text{D}\Phi_G(m)) / g_\mu \cdot m,$$

(1.1)
where $D\Phi_G(m)$ is the differential of $\Phi_G$ at $m$. It is endowed with a symplectic structure $\omega_{N_1}$ coming from $\omega(m)$, and a linear Hamiltonian action of $G_m$ that makes it a Hamiltonian $G_m$-space. This subspace is of particular interest for the study of stability, persistence and bifurcations of relative equilibria (cf. Patrick et al. [15], Lerman and Singer [8] and Ortega and Ratiu [12], as well as Montaldi and Rodriguez-Olmos [11]). The construction of a symplectic slice is based on a Witt-Artin decomposition (relative to the $G$-action) of $T_mM$ i.e a decomposition into four $G_m$-invariant subspaces:

$$T_mM = T_0 \oplus T_1 \oplus N_0 \oplus N_1$$

(1.2)

with respect to which the skew-symmetric matrix associated to $\omega(m)$ has a specific normal form. The part $T_0 \oplus T_1 = g \cdot m$ corresponds to the directions tangent to $G \cdot m$ whereas the part $N_0 \oplus N_1$ is a choice of normal space such that $T_0 \oplus N_1 = \ker(D\Phi_G(m))$. This decomposition was first introduced by Witt [19] for symmetric bilinear forms. The construction involves choices, namely the subspaces $T_1, N_0$ and $N_1$.

Let $H \subset G$ be a Lie subgroup with Lie algebra $\mathfrak{h}$ and inclusion map $i_\mathfrak{h} : \mathfrak{h} \hookrightarrow \mathfrak{g}$. The dual map $i_\mathfrak{h}^* : \mathfrak{g}^* \rightarrow \mathfrak{h}^*$ is given by $i_\mathfrak{h}^*(\lambda) = \lambda|_\mathfrak{h}$, which is the restriction of the linear form $\lambda$ to the subalgebra $\mathfrak{h}$. Note that by definition, the projection $i_\mathfrak{h}^* : \mathfrak{g}^* \rightarrow \mathfrak{h}^*$ is $H$-equivariant. As the action of $H$ on $M$ is still Hamiltonian, it admits a momentum map $\Phi_H : M \rightarrow \mathfrak{h}^*$ given by $\Phi_H = i_\mathfrak{h}^* \circ \Phi_G$. Therefore $(M, \omega, H, \Phi_H)$ is a Hamiltonian $H$-manifold and we call $\Phi_H$ the induced momentum map. In this case, we can also consider a Witt-Artin decomposition of $T_mM$ relative to the $H$-action:

$$T_mM = \bar{T}_0 \oplus \bar{T}_1 \oplus \bar{N}_0 \oplus \bar{N}_1.$$  

(1.3)

In particular, the $H_m$-invariant subspace $\bar{N}_1$ is a symplectic slice for the $H$-action. It is chosen such that

$$\bar{N}_1 := \ker(D\Phi_H(m))/\mathfrak{h}_\alpha \cdot m,$$

(1.4)

where $\alpha := \Phi_H(m)$ is the restriction of the linear form $\mu \in \mathfrak{g}^*$ to $\mathfrak{h}$. In general two arbitrary decompositions (1.2) and (1.3) cannot be compared.
In the study of explicit symmetry breaking phenomena, Hamiltonian equations are perturbed in a way that the symmetry group $G$ breaks into one of its subgroup $H$ (cf. Fontaine [5]). The stability properties of the perturbed system rely on a symplectic slice relative to the $H$-action on $M$, which is “bigger” than a slice relative to the $G$-action. This leads us to find explicit relations between $N_1$ and $\tilde{N}_1$. It has been implicitly used in Grabsi, Montaldi and Ortega that if $G$ is a torus and $H$ is a subtorus, both acting freely on $M$, a symplectic slice $\tilde{N}_1$ at $m$ can be chosen of the form

$$\tilde{N}_1 = N_1 \oplus X_m$$

for some subspace $X_m \subset T_m M$ isomorphic to $\mathfrak{g}/\mathfrak{h} \times (\mathfrak{g}/\mathfrak{h})^*$. We generalize this observation for non-abelian connected Lie groups and non-free actions but with the assumption:

$$G_m \text{ acts on } H \text{ by conjugation.} \quad (A)$$

Given (A), we construct a Witt-Artin decomposition (1.2) at $m$ (relative to the $G$-action) and a Witt-Artin decomposition (1.3) at $m$ (relative to the $H$-action) that are compatible in the sense that the symplectic slice $\tilde{N}_1$ for $H$ can be expressed in terms of the symplectic slice $N_1$ for $G$. Explicitly

$$\tilde{N}_1 = N_1 \oplus X_m \oplus \mathfrak{s}(G, H, \mu) \cdot m$$

for some subspace $X_m \subset T_1 \oplus N_0$ symplectomorphic to a canonical cotangent bundle $\mathfrak{b} \times \mathfrak{b}^*$, and where $\mathfrak{s}(G, H, \mu) \cdot m$ is some symplectic vector subspace of $T_1$ (cf. Theorem 4.3 and Theorem 4.5). When $G$ is a torus and $H$ is a subtorus, both acting freely on $M$, we recover (1.5). In this case $\mathfrak{b}$ is isomorphic to $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{s}(G, H, \mu) \cdot m$ is trivial (cf. Example 1). In Perlmutter et al. [16] the subspace $\mathfrak{s}(G, H, \mu) \cdot m$ arises as a symplectic slice at $\mu$ for the $H$-action on the coadjoint orbit $G \cdot \mu$. We show in Proposition 4.1 that our construction coincides with theirs. In our construction $\mathfrak{s}(G, H, \mu)$ is defined as a subspace of the “symplectic orthogonal”

$$\mathfrak{h}^\perp_{\mu} := \{ x \in \mathfrak{g} \mid x_M(m) \in (\mathfrak{h} \cdot m)^{\omega(m)} \}$$

which is present in the context of geometric quantization (cf. Duval and al. [3]). The construction holds only if we assume that $\mathfrak{h}^\perp_{\mu}$ is $G_m$-invariant, which is
automatic if (A) is satisfied.

The paper is organized as follows. In §2 we explain how to make a specific choice of subspace $T_1$ in (1.2) in order to be able to find a compatible decomposition (1.3). In §3, the Symplectic Tube Theorem is stated. This yields a specific normal form for the momentum map (cf. Theorem 3.2). In §4 the construction introduced in §2 is used to show that a symplectic slice $\tilde{N}_1$ can be chosen of the form (1.6) (cf. Theorem 4.3). The symplectic form and the momentum map on it are specified (cf. Theorem 4.5 and Proposition 4.6). The last section discusses the construction of the other subspaces arising in (1.2) and (1.3).

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2. Witt-Artin decomposition

We use the following notations: if $(V, \omega)$ is a symplectic vector space and $W \subset V$ is a subspace, the symplectic orthogonal $W^{\omega}$ of $W$ in $V$ is the set of vectors $v \in V$ such that $\omega(v, w) = 0$ for every $w \in W$. Furthermore, if $W, U$ are two subspaces such that $U \subset W \subset V$, then $U^{\perp_W} = W$ is a direct sum.

Let $m \in M$ with momentum $\mu = \Phi_G(m)$ and assume that $(A)$ is satisfied. In particular $G_m$ acts on the stabilizer subalgebras $\mathfrak{h}_m$ and $\mathfrak{h}_\mu$ by mean of the adjoint action. We split the Lie algebra $\mathfrak{g}$ into three parts

$$\mathfrak{g} = \mathfrak{g}_m \oplus \mathfrak{m} \oplus \mathfrak{n}. \quad (2.8)$$

for some $G_m$-invariant subspaces $\mathfrak{m}$ and $\mathfrak{n}$, chosen as described below. Since the $G$-action on $M$ is proper, the stabilizer $G_m$ is compact. The Lie subalgebra $\mathfrak{g}_m$ can thus be decomposed into a direct sum of $G_m$-invariant subspaces $\mathfrak{g}_m = \mathfrak{h}_m \oplus \mathfrak{h}_m^{\perp_{\mathfrak{h}_m}}$. Note that $\mathfrak{h}_m$ is $G_m$-invariant by assumption $(A)$. Similarly $\mathfrak{h}_\mu = \mathfrak{h}_m \oplus \mathfrak{p}$ for some $G_m$-invariant complement $\mathfrak{p}$. Then

$$\mathfrak{g}_m + \mathfrak{h}_\mu = \mathfrak{h}_m \oplus \mathfrak{h}_m^{\perp_{\mathfrak{h}_m}} \oplus \mathfrak{p}. \quad (2.9)$$

Since $\mathfrak{g}_m + \mathfrak{h}_\mu \subset \mathfrak{g}_\mu$, we can choose a $G_m$-invariant complement $\mathfrak{b} := (\mathfrak{g}_m + \mathfrak{h}_\mu)^{\perp_{\mathfrak{g}_\mu}}$ so that

$$\mathfrak{g}_\mu = \mathfrak{h}_m \oplus \mathfrak{p} \oplus \mathfrak{h}_m^{\perp_{\mathfrak{h}_m}} \oplus \mathfrak{b} = \mathfrak{h}_m \oplus \mathfrak{h}_m^{\perp_{\mathfrak{h}_m}} \oplus \mathfrak{p} \oplus \mathfrak{b}. \quad (2.9)$$

In particular the $G_m$-invariant subspace $\mathfrak{m}$ in $(2.8)$ is chosen of the form $\mathfrak{m} := \mathfrak{p} \oplus \mathfrak{b}$. By $(2.9)$ it satisfies $\mathfrak{g}_\mu = \mathfrak{g}_m \oplus \mathfrak{m}$.

To define the subspace $\mathfrak{n}$ in $(2.8)$ we introduce the “symplectic orthogonal”:

$$\mathfrak{h}^{\perp_{\mu}} := \{ x \in \mathfrak{g} \mid x_M(m) \in (\mathfrak{h} \cdot m)^{\omega(m)} \}. \quad (2.10)$$

Note that, by assumption $(A)$, this subspace is $G_m$-invariant. The following alternative characterizations of $\mathfrak{h}^{\perp_{\mu}}$ are straightforward:
Proposition 2.1. The conditions below are equivalent:

(i) \( x \in h^{⊥μ} \)

(ii) \( \langle μ, [x, η] \rangle = 0 \) for every \( η \in h \)

(iii) \( \text{ad}^*_μ \in h^0 \) where \( h^0 := \{ λ \in g^* \mid \lambda \big|_h = 0 \} \) is the annihilator of \( h \) in \( g^* \)

Consider the projection \( α := μ \big|_h \in h^* \) and observe that \( h_α \subset h^{⊥μ} \). Indeed let \( x \in h_α, η \in h \), and note that

\[
\langle \text{ad}^*_x μ, η \rangle = \langle μ, [x, η] \rangle = \langle α, [x, η] \rangle = \langle \text{ad}^*_α μ, η \rangle = 0.
\] (2.11)

Furthermore \( g_μ \subset h^{⊥μ} \) since \( g_μ = \{ x \in g \mid \text{ad}^*_x μ = 0 \} \). We conclude that the inclusion \( g_μ + h_α \subset h^{⊥μ} \) holds. Knowing that \( g_μ \cap h_α = h_μ \), we choose a \( G_m \)-invariant complement \( a \) such that

\[
g_μ + h_α = \underbrace{h_μ \oplus h_μ^{⊥g}_m \oplus b \oplus a}_{g_μ} = \underbrace{h_μ \oplus a \oplus h_μ^{⊥g}_m \oplus b}_{h_α}.
\] (2.12)

Choose \( s(G, H, μ) \) to be a \( G_m \)-invariant complement to \( g_μ + h_α \) in \( h^{⊥μ} \). We can thus express (2.10) as a direct sum of \( G_m \)-invariant subspaces

\[
h^{⊥μ} = g_μ \oplus a \oplus s(G, H, μ).
\] (2.13)

In particular,

\[
q := a \oplus s(G, H, μ)
\] (2.14)

is a \( G_m \)-invariant complement to \( g_μ \) in \( h^{⊥μ} \). Finally, choosing a \( G_m \)-invariant complement \( (h^{⊥μ})^{⊥g}_n \) of \( h^{⊥μ} \) in \( g \) yields the decomposition

\[
g = \underbrace{h_μ \oplus h_μ^{⊥g}_m \oplus b \oplus q \oplus (h^{⊥μ})^{⊥g}_n}_{g_μ}.
\] (2.15)
By (2.9) and (2.15), the $G_m$-invariant subspaces $m$ and $n$ of (2.8) are

$$m := p \oplus b \quad \text{and} \quad n = q \oplus (h^\perp)^{\perp}$$

(2.16)

**Note.** The subspace $b \subset g_\mu$ is not a Lie subalgebra in general. However if $\mu$ is a regular value of $\Phi_G$, then $G_\mu$ is a maximal torus of $G$. In this case $K = G_m H_\mu$ is a subgroup of $G_\mu$ since $G_m H_\mu = H_\mu G_m$. It is a Lie subgroup by closedness of $G_m$ and $H_\mu$ and its Lie algebra is $\mathfrak{t} = \mathfrak{g}_m + \mathfrak{h}_\mu$. Since $\mathfrak{g}_\mu$ is abelian $\mathfrak{t}$ is trivially an ideal of $\mathfrak{g}_\mu$ making $b$ isomorphic to the Lie algebra $\mathfrak{g}_\mu / \mathfrak{t}$. If in addition $H_m = 1$, then $K = H_\mu \rtimes G_m$, as $G_m$ is normal in $G_\mu$.

The next theorem is a standard result whose proof can be found in Ortega and Ratiu [13] or Bates and Cushman [2].

**Theorem 2.2 (Witt-Artin decomposition).** Let $(M, \omega, G, \Phi_G)$ be a Hamiltonian $G$-manifold and let $m \in M$ with momentum $\mu = \Phi_G(m)$. Fix a splitting of $\mathfrak{g}$ into $G_m$-invariant subspaces as in (2.8). Then the tangent space $T_m M$ at $m$ decomposes as:

$$T_m M = T_0 \oplus T_1 \oplus N_0 \oplus N_1,$$

(2.17)

where the subspaces $T_0, T_1, N_0, N_1$ are constructed as follows:

(i) $T_0 := \ker(\mathcal{D}\Phi_G(m)) \cap \mathfrak{g} \cdot m = \mathfrak{g}_\mu \cdot m$.

(ii) $T_1 := n \cdot m$ which is a symplectic vector subspace of $(T_m M, \omega(m))$.

(iii) $N_1$ is a choice of $G_m$-invariant complement to $T_0$ in $\ker(\mathcal{D}\Phi_G(m))$. It is a symplectic subspace of $(T_m M, \omega(m))$ with symplectic form $\omega_{N_1} := \omega(m)|_{N_1}$. This subspace is called a symplectic slice. The linear action of $G_m$ on $N_1$ is Hamiltonian with momentum map $\Phi_{N_1} : N_1 \to \mathfrak{g}_m^*$ given by $\langle \Phi_{N_1}(\nu), x \rangle = \frac{1}{2} \omega(m)(x M(m), \nu)$ for $\nu \in N_1$ and $x \in \mathfrak{g}_m$.

(iv) $N_0$ is a $G_m$-invariant Lagrangian complement to $T_0$ in the symplectic orthogonal $(T_1 \oplus N_1)^{\omega(m)}$. There is an isomorphism $f : N_0 \to \mathfrak{m}^*$ given by $\langle f(w), y \rangle = \omega(m)(y_M(m), w)$ for every $w \in N_0$ and $y \in \mathfrak{m}$.
Furthermore, the subspaces $T_1, N_1$ and $T_0 \oplus N_0$ are mutually symplectically orthogonal.

**Note.** In (ii) of Theorem 2.2, the symplectic form $\omega(m)$ restricted to $T_1$ coincides with the Kostant-Kirillov-Souriau symplectic form. Besides the symplectic form $\omega(m)$ restricted to $T_0 \oplus N_0$ takes the form

$$\omega(m)(x_M(m) + w, x'_M(m) + w') = \langle f(w'), x \rangle - \langle f(w), x' \rangle$$

for $x, x' \in m$, $w, w' \in N_0$ and $f : N_0 \to m^*$ as in (iv). Indeed since $y_M(m) = 0$ for all $y \in g_m$, the elements of $T_0$ are of the form $x_M(m)$ with $x \in m$. Let $x, x' \in m$ and $w, w' \in N_0$. As both $T_0$ and $N_0$ are Lagrangian in $T_0 \oplus N_0$,

$$\omega(m)(x_M(m) + w, x'_M(m) + w') = \omega(m)(x_M(m), w) + \omega(x'_M(m), w)$$

which is equal to $\langle f(w'), x \rangle - \langle f(w), x' \rangle$ by definition of $f$.

Applying Theorem 2.2 to $(M, \omega, G, \Phi_G)$ with the subspaces in (2.8) taken as in (2.16), we get a decomposition

$$T_mM = T_0 \oplus T_1 \oplus N_0 \oplus N_1$$

with $T_0 = (g_m \oplus p \oplus b) \cdot m$ and $T_1 = (q \oplus (h^{1/2})^{1/2}) \cdot m$. Note that we have some freedom in the choice of $G_m$-invariant normal subspaces $N_0$ and $N_1$. As we did previously we set $\alpha := \Phi_H(m) = \mu|_b$ and we define $\tilde{T}_0 = h_\alpha \cdot m$. We shall give a specific choice of subspaces $\tilde{T}_1, \tilde{N}_0, \tilde{N}_1$ such that the tangent space of $M$ at $m$ decomposes as

$$T_mM = \tilde{T}_0 \oplus \tilde{T}_1 \oplus \tilde{N}_0 \oplus \tilde{N}_1$$

which is compatible with both, the decomposition (2.18) and the construction of Theorem 2.2 applied to $(M, \omega, H, \Phi_H)$. 

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3. Symplectic Tube Theorem

In this section we introduce a fundamental result to study both the local dynamics and the local geometry of Hamiltonian $G$-manifolds. It provides a local model for a $G$-invariant open neighbourhood of a $G$-orbit in $M$. Take a point $m \in M$ with momentum $\mu = \Phi_G(m)$ and define the splittings as in (2.8). Let $N_1$ be a symplectic slice at $m$. Since $N_1$ is $G_m$-invariant, there is an action of $G_m$ on the product $G \times m^* \times N_1$ given by

$$k \cdot (g, \rho, \nu) = (g k^{-1}, \text{Ad}_{k^{-1}}^* \rho, k \cdot \nu).$$

(3.20)

This action is free and proper by freeness and properness of the action on the $G$-factor. The orbit space $Y$ is thus a smooth manifold whose points are equivalence classes of the form $[(g, \rho, \nu)]$. The group $G$ acts smoothly and properly on $Y$, by left multiplication on the $G$-factor. Let $m^*_0 \subset m^*$ and $(N_1)_0 \subset N_1$ be $G_m$-invariant neighbourhoods of zero in $m^*$ and $N_1$, respectively. Then

$$Y_0 := G \times_{G_m} (m^*_0 \times (N_1)_0)$$

(3.21)

is a neighbourhood of the zero section in $Y$. It comes with a symplectic structure $\omega_{Y_0}$ if it is chosen sufficiently small (cf. [13] Proposition 7.2.2). Define the Chu map $\Psi : M \to Z^2(\mathfrak{g})$ associated to the $G$-action by

$$\Psi(m)(x, y) := \omega(m)(x_M(m), y_M(m)).$$

(3.22)

Note that $\Psi(m)(x, y) = -\langle \mu, [x, y] \rangle$, and thus $\Psi(m)$ coincides with the Kostant-Kirillov-Souriau symplectic form on the coadjoint orbit $G \cdot \mu$ whenever $x, y \in \mathfrak{n}$.

The next theorem is the well-known Symplectic Tube Theorem. It was obtained by MARLE [9] and generalized by GUILLÉMIN AND STERNBERG [6], BATES AND LERMAN [1], ORTEGA AND RATIU [13]. A proof is available in [13] (cf. Theorem 7.4.1).
Theorem 3.1 (Symplectic Tube Theorem). Let \((M, \omega, G, \Phi_G)\) be a Hamiltonian \(G\)-manifold. Let \(m \in M\) with momentum \(\mu = \Phi_G(m)\). If the neighbourhood \(Y_0\) defined in (3.21) is sufficiently small, it admits a symplectic structure \(\omega_{Y_0}\). In this case, there exists a \(G\)-invariant neighbourhood \(U \subset M\) of \(m\) and a \(G\)-equivariant symplectomorphism
\[
\varphi : (Y_0, \omega_{Y_0}) \to (U, \omega\big|_U)
\]
such that \(\varphi([e, 0, 0]) = m\).

We call the triplet \((\varphi, Y_0, U)\) a symplectic \(G\)-tube at \(m\) and we also say that \((Y_0, \omega_{Y_0})\) is a symplectic local model for \((U, \omega\big|_U)\). Besides the momentum map \(\Phi_G : M \to g^*\) can be expressed in terms of the slice coordinates:

\[
\Phi_G([g, \rho, \nu]) = \text{Ad}_{g^{-1}}(\Phi_G(m) + \rho + \Phi_N(\nu)). \tag{3.23}
\]

If \(G\) is connected then \(\Phi_G\) coincides with \(\Phi_G\big|_U\) when pulled back along \(\varphi^{-1}\).

4. Compatible symplectic slices

In this section we explain how to choose the symplectic slice \((\tilde{N}_1, \omega_{\tilde{N}_1})\) at \(m\) arising in (2.19). Explicitly
\[
\tilde{N}_1 = \mathfrak{s}(G, H, \mu) \cdot m \oplus X_m \oplus N_1, \tag{4.24}
\]
where \( s(G, H, \mu) \) is a \( G_m \)-invariant complement to \( g_\mu + h_\alpha \) in \( h^{\perp_\mu} \), and \( X_m \subset T_m M \) is some subspace symplectomorphic to \( b \times b^* \) with the canonical symplectic form. We show in Lemma 4.4 that \( s(G, H, \mu) \cdot m \) is a symplectic subspace of \( (T_m M, \omega(m)) \). This subspace depends on the choice of group, subgroup, and on the momentum \( \mu \). However the space \( b = (g_m + h_\mu)^{\perp_\mu} \) also depends on the dynamics of the \( G \)-action on \( M \) as it involves the stabilizer subalgebra \( g_m \). The next proposition is a geometric description of \( s(G, H, \mu) \cdot m \).

**Proposition 4.1.** The subspace \( s(G, H, \mu) \cdot m \) is identified with a symplectic slice at \( \mu \) for the \( H \)-action on the coadjoint orbit \( G \cdot \mu \).

**Proof —** The subgroup \( H \) acts on the coadjoint orbit \( G \cdot \mu \) by left multiplication. Since the momentum map for the standard \( G \)-action on \( G \cdot \mu \) is just the inclusion \( G \cdot \mu \hookrightarrow g^* \), the momentum map \( \Phi : G \cdot \mu \rightarrow h^* \) for the \( H \)-action is given by \( \Phi(\text{Ad}_{g^{-1}}^* \mu) = i_h^*(\text{Ad}_{g^{-1}}^* \mu) \). The kernel of its differential is \( \ker(D\Phi(\mu)) = (a \oplus s(G, H, \mu)) \cdot \mu \). Indeed, a straightforward calculation shows that

\[
x_{\mu^*}(\mu) \in \ker(D\Phi(\mu)) \iff \text{ad}_{\mu}^* \mu \in h^0.
\]

By Proposition 2.1, \( x \in h^{\perp_\mu} \). Because of the identification \( g^* = n^0 \oplus T_{\mu}(G \cdot \mu) \) and (2.13), we must have \( x \in a \oplus s(G, H, \mu) \). The momentum of \( \mu \) is \( \Phi(\mu) = i_h^*(\mu) = \alpha \) and thus a symplectic slice for the \( H \)-action on \( G \cdot \mu \) is a complement to \( h_\alpha \cdot \mu \) in \( \ker(D\Phi(\mu)) \). By construction, this complement is \( s(G, H, \mu) \cdot \mu \) which can be identified with \( s(G, H, \mu) \cdot m \) as \( s(G, H, \mu) \) has trivial intersection with \( g_m \) and \( g_\mu \).

**Proposition 4.2.** Let \((M, \omega, G, \Phi_G)\) be a Hamiltonian \( G \)-manifold with \( G \) connected and let \( \Phi_H : M \rightarrow h^* \) be the induced momentum map. Then

\[
\ker(D\Phi_H(m)) = \ker(D\Phi_G(m)) \oplus \mathcal{M},
\]

where \( \mathcal{M} \subset T_m M \) is isomorphic to \( q \cdot m \times b^* \) as defined in (2.16).
Proof — It is clear from the definitions that there is an inclusion of subspaces
\[ \ker (D\Phi_G(m)) \subset \ker (D\Phi_H(m)). \] (4.25)

Let \((\varphi, G \times_G m, (m_0^* \times (N_1)_0), U)\) be a symplectic \(G\)-tube at \(m\). Linearising \(\varphi^{-1}\) at \(m\) yields a linear symplectomorphism
\[ T_m \varphi^{-1} : T_0 \oplus T_1 \oplus N_0 \oplus N_1 \rightarrow T_{\varphi^{-1}(m)} (G \times_G m (m^* \times N_1)). \]

For \(x + y \in g_m \oplus m\) and \(z \in n\) we have
\[ T_m \varphi^{-1} \cdot ((x + y)_M(m) + z_M(m) + w + \nu) = T_{(e,0,0)} \rho \cdot (x + y + z, f(w), \nu) \]
where \(\rho : G \times m^* \times N_1 \rightarrow G \times_G m (m^* \times N_1)\) is the orbit map. By definition, the subspace \(\ker (D\Phi_H(m))\) consists of the elements
\[ ((x + y)_M(m) + z_M(m) + w + \nu) \in T_0 \oplus T_1 \oplus N_0 \oplus N_1 \]
satisfying \(D(\Phi_H \circ \varphi \circ \rho)(m) \cdot (x + y + z, f(w), \nu) = 0\). Equivalently
\[
0 = \frac{d}{dt} \bigg|_{t=0} \Phi_H \circ \varphi \left( (\exp(t(x + y + z)), tf(w), tv) \right) \\
= \frac{d}{dt} \bigg|_{t=0} i^*_h \left( \text{Ad}_{\exp(-t(x+y+z))}^* (\mu + tf(w) + \Phi_{N_1}(tv)) \right) \\
= i^*_h (-\text{ad}_z^* \mu + f(w))
\]

where the normal form for the momentum map is given by Theorem 3.2. Then \(-\text{ad}_z^* \mu + f(w) \in h^\circ\) since the kernel of \(i^*_h\) is equal to \(h^\circ\). We conclude that
\[ \ker (D\Phi_H(m)) = \ker (D\Phi_G(m)) \oplus \mathcal{M} \] where
\[ \mathcal{M} := \{ z_M(m) + w \in T_1 \oplus N_0 \mid -\text{ad}_z^* \mu + f(w) \in h^\circ \}. \] (4.26)

It remains to show that \(\mathcal{M}\) is isomorphic to \(q \cdot m \times b^*\). By construction
\[ T_1 = n \cdot m = (q \oplus (h^\perp)^\perp) \cdot m \]
and $N_0$ is isomorphic to $m^* = p^* \oplus b^*$. An element $z_M(m) + w \in \mathcal{M}$ can thus be written uniquely as $u_M(m) + v_M(m) + w$ for some unique elements $u \in q$ and $v \in (h^{+\nu})^{+\theta}$. In addition, we set $f(w) = \pi + \beta$ for $\pi \in p^*$ and $\beta \in b^*$. By definition of $\mathcal{M}$ the following relation holds:

$$\langle -\text{ad}_{u+v}^* \mu + \pi + \beta, \eta \rangle = 0 \quad \text{for every} \quad \eta \in h. \quad (4.27)$$

From the decomposition

$$g^*_\mu = h^*_\mu \oplus (h_{m^e}^*)^* \oplus b^*, \quad (4.28)$$

we see that $\langle \beta, \eta \rangle = 0$ for every $\eta \in h$ since $g^*_\mu \cap h = h_\mu$ on which $\beta$ vanishes. In addition, $\langle -\text{ad}_{u+v}^* \mu, \eta \rangle = \langle -\text{ad}_{v}^* \mu, \eta \rangle$ for every $\eta \in h$ as $u \in q \subseteq h^{+\nu}$. Hence (4.27) reduces to

$$\langle -\text{ad}_{v}^* \mu + \pi, \eta \rangle = 0 \quad \text{for every} \quad \eta \in h. \quad (4.28)$$

In particular, if $\eta \in h_\mu$, we are left with $\langle \pi, \eta \rangle = 0$ and thus $\pi = 0$. Since $\langle -\text{ad}_{v}^* \mu, \eta \rangle = 0$ for every $\eta \in h$, this implies that $v \in h^{+\nu} \cap (h^{+\nu})^{+\theta} = \{0\}$. Therefore the element $z_M(m) + w$ we started with is such that $z = u \in q$ and $f(w) = \beta \in b^*$.

Conversely, it is straightforward to check from the argument above that an element $z_M(m) + w \in q \cdot m \oplus N_0$ such that $f(w) = \beta \in b^*$ satisfies $-\text{ad}_{v}^* \mu + \beta \in h^\circ$. We showed that

$$\mathcal{M} = \{u_M(m) + w \in q \cdot m \oplus N_0 \mid f(w) \in b^*\}. \quad (4.29)$$

The isomorphism is $F : u_M(m) + w \in \mathcal{M} \mapsto (u_M(m), f(w)) \in q \cdot m \times b^*$. \hfill \blacksquare

Because of the assumption arising in Theorems 3.2 and 4.2 we assume from now on that $G$ is a connected Lie group.

**Theorem 4.3 (Compatible Symplectic Slice).** Given (2.18), a symplectic slice $\widetilde{N}_1$ at $m$ relative to the $H$-action can be chosen of the form

$$\widetilde{N}_1 = s(G, H, \mu) \cdot m \oplus X_m \oplus N_1, \quad (4.29)$$

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where $X_m = b \cdot m \oplus Y_m$ with $Y_m \subset N_0$ isomorphic to $b^*$.

Proof — Let a Witt-Artin decomposition of $M$ as in (2.18). Then by (2.9)

$$
\ker (D\Phi_G(m)) = g_\mu \cdot m \oplus N_1 \\
= (h_\mu \oplus h_m^\perp \oplus b) \cdot m \oplus N_1
$$

(4.30)

By Proposition 4.2, there is a subspace $Y_m \subset N_0$ isomorphic to $b^*$ such that

$$
\ker (D\Phi_H(m)) = \ker (D\Phi_G(m)) \oplus q \cdot m \oplus Y_m \\
= h_\mu \cdot m \oplus b \cdot m \oplus N_1 \oplus q \cdot m \oplus Y_m \quad \text{from (4.30)}.
$$

In (2.12) and (2.14) we obtained $h_\alpha = h_\mu \oplus a$ and $q = a \oplus s(G, H, \mu)$. Therefore

$$
h_\mu \cdot m \oplus q \cdot m = h_\alpha \cdot m \oplus s(G, H, \mu) \cdot m.
$$

Setting $X_m = b \cdot m \oplus Y_m$, we conclude that

$$
\ker (D\Phi_H(m)) = h_\alpha \cdot m \oplus s(G, H, \mu) \cdot m \oplus X_m \oplus N_1.
$$

(4.31)

A symplectic slice $\widetilde{N}_1$ at $m$ for the $H$-action must satisfy

$$
\ker (D\Phi_H(m)) = h_\alpha \cdot m \oplus \widetilde{N}_1.
$$

Hence we choose $\widetilde{N}_1 = s(G, H, \mu) \cdot m \oplus X_m \oplus N_1$. 

Lemma 4.4. The subspace $s(G, H, \mu) \cdot m = \{x_M(m) \mid x \in s(G, H, \mu)\}$ is a symplectic vector subspace of $(T_mM, \omega(m))$. The restriction of $\omega(m)$ on $s(G, H, \mu) \cdot m$ coincides with the Kostant-Kirillov-Souriau symplectic form.
Proof — Using (2.14), the complement to \( \mathfrak{g}_\mu \) in \( \mathfrak{g} \) defined in (2.16) reads

\[
\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{s}(G, H, \mu) \oplus (\mathfrak{h}^\perp)^g.
\] (4.32)

To show that \( \mathfrak{s}(G, H, \mu) \cdot m \) is symplectic, we use that \( \mathfrak{n} \cdot m = \{ z_M(m) \mid z \in \mathfrak{n} \} \) is a symplectic vector subspace of \((T_mM, \omega(m))\). The restriction of \( \omega(m) \) on \( \mathfrak{n} \cdot m \) is non-degenerate and takes the form

\[
\Psi(m)(x, y) = -\langle \mu, [x, y] \rangle.
\]

Therefore \( \omega(m) \) restricted to \( \mathfrak{n} \cdot m \) coincides with the Kostant-Kirillov-Souriau symplectic form. Let us show that it is also non-degenerate when restricted to \( \mathfrak{s}(G, H, \mu) \cdot m \). Assume \( x \in \mathfrak{s}(G, H, \mu) \) is such that \( \Psi(m)(x, y) = 0 \) for every \( y \in \mathfrak{s}(G, H, \mu) \). To show non-degeneracy we must show that \( x_M(m) = 0 \). By (4.32), any \( z \in \mathfrak{n} \) can be written uniquely as \( z = u + y + v \) with \( u \in \mathfrak{a}, y \in \mathfrak{s}(G, H, \mu) \) and \( v \in (\mathfrak{h}^\perp)^g \). This yields

\[
\Psi(m)(x, z) = \Psi(m)(x, u) + \Psi(m)(x, v)
\] (4.33)

as the term \( \Psi(m)(x, y) \) vanishes by assumption. Note that

\[
\Psi(m)(x, u) = -\langle \mu, [x, u] \rangle = 0
\]

since \( x \in \mathfrak{h}^\perp \) by (2.13) and \( u \in \mathfrak{a} \subset \mathfrak{h} \) by (2.12). Moreover the last term of (4.33) vanishes. To see this we construct a Witt-Artin decomposition at \( m \) relative to the \( H \)-action:

\[
T_mM = \tilde{T}_0 \oplus \tilde{T}_1 \oplus \tilde{N}_0 \oplus \tilde{N}_1
\] (4.34)

with \( \tilde{N}_1 \) as in Theorem 4.3. Recall that

\[
\ker(D\Phi_H(m)) = \tilde{T}_0 \oplus \tilde{N}_1.
\]

Furthermore since \( \ker(D\Phi_H(m)) = (\mathfrak{h} \cdot m)^{\omega(m)} \), we can write

\[
\mathfrak{h}^\perp = \{ x \in \mathfrak{g} \mid x_M(m) \in \tilde{T}_0 \oplus \tilde{N}_1 \}.
\] (4.35)
There are two possibilities:

(i) If $v \in h$ then $v_M(m) \in \tilde{T}_1$ since $v \in (h^\perp)^{\perp_\phi}$. The subspaces $\tilde{T}_1$ and $\tilde{N}_1$ are symplectically orthogonal. Hence $\Psi(m)(x, v) = 0$.

(ii) Otherwise $v_M(m) \in \tilde{N}_0$. Indeed since $v \in (h^\perp)^{\perp_\mu} \perp g$, it cannot belong to $\tilde{T}_0 \oplus \tilde{N}_1$ by (4.35). Since $x_M(m) \in s(G, H, \mu) \cdot m \subset \tilde{N}_1$ and $\tilde{T}_0 \oplus \tilde{N}_0$ and $\tilde{N}_1$ are symplectically orthogonal, we conclude that $\Psi(m)(x, v) = 0$.

Therefore (4.33) reduces to $\Psi(m)(x, z) = 0$ for every $z \in n$. Since $n \cdot m$ is symplectic we get $x_M(m) = 0$ and we are done.

\begin{theorem}
With respect to the splitting of Theorem 4.3, the symplectic form $\omega_{\tilde{N}_1}$ reads $\Psi(m) \oplus \omega_{X_m} \oplus \omega_{N_1}$ with $\Psi(m)$ as in Lemma 4.4 and

$$\omega_{X_m}(b_M(m) + w, b'_M(m) + w') = \langle f(w'), b \rangle - \langle f(w), b' \rangle$$

for every $b, b' \in b$, $w, w' \in Y_m$, and $f$ as in Theorem 2.2 (iv).
\end{theorem}

\begin{proof}
By Lemma 4.4, the symplectic form on $s(G, H, \mu) \cdot m$ is given by $\Psi(m)$. Denote by $\omega_{X_m}$ the restriction of $\omega(m)$ to $X_m$. It coincides with the pullback of the canonical symplectic form on $b \times b^*$ along the isomorphism

$$b_M(m) + w \in X_m = b \cdot m \oplus Y_m \mapsto (b, f(w)) \in b \times b^*.$$

Therefore $\omega_{X_m}(b_M(m) + w, b'_M(m) + w') = \langle f(w'), b \rangle - \langle f(w), b' \rangle$ for all $b, b' \in b$ and $w, w' \in Y_m$. This yields the decomposition $\omega_{\tilde{N}_1}(m) = \Psi(m) \oplus \omega_{X_m}(m) \oplus \omega_{N_1}$ as stated.
\end{proof}

\begin{proposition}
With respect to the splitting of Theorem 4.3, the momentum map $\Phi_{\tilde{N}_1} : \tilde{N}_1 \to h^*_m$ associated to the linear Hamiltonian $H_m$-action on $\tilde{N}_1$ decomposes as

$$\langle \Phi_{\tilde{N}_1}(\nu), \eta \rangle = -\frac{1}{2} \langle (ad_x^\ast)^2 \mu, \eta \rangle + \langle -ad_x^\ast f(w), \eta \rangle + \frac{1}{2} \omega_{N_1}(\eta_{N_1}(\nu), \nu)$$

\end{proposition}
for every $\eta \in h_m$, where $\bar{\nu} = x_M(m) + (b_M(m) + w) + \nu \in \bar{N}_1$ with $x \in s(G, H, \mu), b \in b, w \in Y_m$ and $\nu \in N_1$.

Proof — By linearity of the Hamiltonian $H_m$-action on $\bar{N}_1$, the momentum map $\Phi_{\bar{N}_1}$ takes the form

$$\langle \Phi_{\bar{N}_1}(\bar{\nu}), \eta \rangle = \frac{1}{2} \omega_{\bar{N}_1} \left( \eta_{\bar{N}_1}(\bar{\nu}), \bar{\nu} \right)$$

(4.36)

for all $\bar{\nu} \in \bar{N}_1$ and $\eta \in h_m$. With respect to the decomposition of $\bar{N}_1$ in Theorem 4.3, we write

$$\bar{\nu} = x_M(m) + (b_M(m) + w) + \nu \in \bar{N}_1$$

where $x \in s(G, H, \mu), b \in b, w \in Y_m$ and $\nu \in N_1$. For $\eta \in h_m$ we get

$$\eta_{\bar{N}_1}(x_M(m)) = \frac{d}{dt} \Big|_{t=0} \exp(t\eta) \cdot x_M(m)$$

$$= \frac{d}{dt} \Big|_{t=0} \left( \text{Ad}_{\exp(t\eta)} x \right)_M(m)$$

$$= [\eta, x]_M(m).$$

Similarly $\eta_{\bar{N}_1}(b_M(m)) = [\eta, b]_M(m)$. By Theorem 4.5, the symplectic form on $\bar{N}_1$ decomposes as $\omega_{\bar{N}_1}(m) = \Psi(m) \oplus \omega_{X_m}(m) \oplus \omega_{N_1}$ and then (4.36) reads

$$\frac{1}{2} \omega_{\bar{N}_1} \left( \eta_{\bar{N}_1}(\bar{\nu}), \bar{\nu} \right) = \frac{1}{2} \Psi(m)([\eta, x], x)$$

$$+ \frac{1}{2} \omega_{X_m}(m) \left( [\eta, b]_M(m) + \eta_{\bar{N}_1}(w), b_M(m) + w \right)$$

$$+ \frac{1}{2} \omega_{N_1}(\eta_{N_1}(\nu), \nu).$$

By definition the second term of the above is $\frac{1}{2} \left( \langle f(w), [\eta, b] \rangle - \langle f(\eta_{\bar{N}_1}(w)), b \rangle \right)$.

Since the linear map $f$ is $H_m$-equivariant,

$$\langle f(\eta_{\bar{N}_1}(w)), b \rangle = \langle -\text{ad}^*_f w, b \rangle = -\langle f(w), [\eta, b] \rangle.$$
Finally
\[
\Psi(m)([\eta, x], x) = -\langle \mu, [\eta, x] \rangle \\
= -\langle \text{ad}^*_x \mu, [x, \eta] \rangle \\
= -\langle (\text{ad}^*_x)^2 \mu, \eta \rangle.
\]

We thus obtain
\[
\langle \Phi_{\tilde{N}_1}(\tilde{\nu}), \eta \rangle = -\frac{1}{2} \langle (\text{ad}^*_x)^2 \mu, \eta \rangle + \langle -\text{ad}^*_x f(w), \eta \rangle + \frac{1}{2} \omega_{N_1}(\eta_{N_1}(\nu), \nu).
\]

Example 1 (Abelian groups). Let \((M, \omega, G, \Phi_G)\) be a Hamiltonian \(G\)-manifold where \(G\) is abelian and let \(H\) be a subgroup of \(G\). For simplicity we assume that this action is free i.e. all the stabilizers \(G_m\) are trivial. If \(m \in M\) has momentum \(\mu = \Phi_G(m)\), then \(g_\mu = g\) and \(h_\alpha = h_\mu = h\). In particular \(g_\mu + h_\alpha = g\). Since \(G\) is abelian \(h^{+\mu} = g\), and thus \(s(G, H, \mu) = 0\) as it is the orthogonal complement of \(g_\mu + h_\alpha\) in \(h^{+\mu}\). On the other hand \(b = h^{+\nu}\) is isomorphic to \(g/h\). Theorem 4.3 implies that
\[
\tilde{N}_1 = N_1 \oplus X_m
\]
where \(X_m\) is isomorphic to \(g/h \times (g/h)^*\).

Example 2. Let \((M, \omega, G, \Phi_G)\) where \(G = SO(3)\) is the group of rotations in \(\mathbb{R}^3\). Assume that this action is free. Let \(H = SO(2)\) be the subgroup of rotations about the axis defined by a vector \(x \in \mathbb{R}^3\). The Lie algebra \(g\) is the space of \(3 \times 3\) skew-symmetric matrices. It is identified with \(\mathbb{R}^3\) and so is its dual \(g^*\) by using the standard dot product. Let \(m \in M\) be a point with momentum \(\Phi_G(m) = \mu \in g^*\) where \(\mu := \chi \in \mathbb{R}^3\). Similarly an element \(y \in g\) is identified with \(y := y \in \mathbb{R}^3\). Clearly \(g_{\mu} := \text{span}(\chi) \subset \mathbb{R}^3\) and \(h := \text{span}(x) \subset \mathbb{R}^3\). Since \(\text{ad}^*_y \mu := \chi \times y \in \mathbb{R}^3\) the symplectic orthogonal \(h^{+\mu}\) is the subspace of \(\mathbb{R}^3\) defined by
\[
h^{+\mu} := \{y \in \mathbb{R}^3 \mid (\chi \times x) \cdot y = 0\}.
\]
There are three cases to be considered: when $\chi$ and $x$ are not collinear, when they are collinear, and when $\chi = 0$.

(i) If $\chi$ and $x$ are not collinear then there are no elements in $H$ fixing $\chi$. Therefore $h_\mu = 0$. As $H$ is abelian, $h_\alpha = h$ and thus $g_\mu + h_\alpha \defeq \text{span}(\chi, x)$. Furthermore

$$h_{-\mu} \defeq \text{span}(\chi, x)$$

is a two-dimensional plane. We conclude that $s(G, H, \mu) = 0$. The other subspace of interest is $b = (g_m + h_\mu)_{-\mu}$. In this case, as $g_m + h_\mu = 0$, we deduce that $b = g_\mu$. By applying Theorem 4.3, the symplectic slice for the $H$-action is given by

$$\tilde{N}_1 = N_1 \oplus X_m, \quad (4.39)$$

where $X_m$ is isomorphic to $g_\mu \times g_\mu^*$.

(ii) When $\chi$ and $x$ are collinear, all the elements of $H$ fix $\chi \in \mathbb{R}^3$. Consequently $h_\mu = h = g_\mu$ and

$$h_{-\mu} = g := \mathbb{R}^3.$$ In this case $s(G, H, \mu) = n$ which is the orthogonal complement to $g_\mu$ in $g$. In $\mathbb{R}^3$, it is identified with the plane through the origin that is orthogonal to $\chi$. It is also naturally isomorphic to the tangent space at $\chi$ of the 2-sphere of radius $\|\chi\|$ which corresponds to the coadjoint orbit $G \cdot \mu$. However, as $h_\mu = h = g_\mu$, we find that $b = 0$. Therefore,

$$\tilde{N}_1 = N_1 \oplus s(G, H, \mu) \cdot m \quad (4.40)$$

where $s(G, H, \mu) \cdot m = n \cdot m$.

(iii) When $\mu := \chi = 0$ we have $g_\mu = g$ and $h_\mu = h$. As $H$ is abelian, $h_\alpha = h$ and we find $g_\mu + h_\alpha = g := \mathbb{R}^3$. This implies that $s(G, H, \mu) = 0$. The subspace $b$ is just $h_{-\mu} \cong g/h$ which is the orthogonal complement to $h$ in $g$. Therefore

$$\tilde{N}_1 = N_1 \oplus X_m, \quad (4.41)$$

where $X_m$ is isomorphic to $g/h \times (g/h)^*$. 20
5. The case of the other subspaces

As observed in the previous example, a compatible symplectic slice $\tilde{N}_1$ at $m$ depends on the choice of group $G$, subgroup $H$, momentum $\mu$, and whether $G_m$ is trivial or not. In this section we look at what happens for the other members of the four-fold decomposition relative to the $H$-action:

$$T_m M = \tilde{T}_0 \oplus \tilde{T}_1 \oplus \tilde{N}_0 \oplus \tilde{N}_1. \quad (5.42)$$

According to our choice of splittings, we set

$$\tilde{T}_0 = h \cdot m = p \cdot m \oplus a \cdot m \quad \text{and} \quad \tilde{N}_0 \simeq p^* \oplus a^*.$$ 

Recall that $n$ was defined such that $g = g_\mu \oplus n$. Similarly we define $\bar{n}$ such that $h = h_\alpha \oplus \bar{n}$. Hence $\bar{T}_1 = \bar{n} \cdot m$ and the symplectic slice $\bar{N}_1$ is as in Theorem 4.3.

The same choice of splittings allows us to construct a Witt-Artin decomposition relative to the $G$-action:

$$T_m M = T_0 \oplus T_1 \oplus N_0 \oplus N_1. \quad (5.43)$$

By (2.9) we get $T_0 = g_\mu \cdot m = p \cdot m \oplus b \cdot m$. Thus $T_0$ is contributing to two different parts of (5.42), namely $\tilde{T}_0$ and $\tilde{N}_1$. Furthermore $N_0$ is isomorphic to $m^*$, where $m$ is a complement of $g_m$ in $g_\mu$. By (2.16) there is an isomorphism $N_0 \simeq p^* \oplus b^*$. Hence $N_0$ contributes to $\tilde{N}_0$ and $\tilde{N}_1$.

Let us now specify $T_1$. Note that $h^+ \cdot m \cap h \cdot m = h_\alpha \cdot m$. Therefore $\bar{n} \subset (h^+)^+ \cap \bar{n} \cdot m$ and we denote by $\bar{r}$ an $H_m$-invariant complement in $(h^+)^+ \cap \bar{n} \cdot m$ so that $(h^+)^+ = \bar{n} \oplus \bar{r}$. By (4.32), $n$ reads

$$n = a \oplus s(G, H, \mu) \oplus \bar{n} \oplus \bar{r}, \quad (5.44)$$

which implies that $T_1 = n \cdot m = a \cdot m \oplus s(G, H, \mu) \cdot m \oplus \bar{n} \cdot m \oplus \bar{r} \cdot m$. We show in Lemma 5.1 below that there is an isomorphism

$$T_1 \simeq n \cdot m = a \cdot m \oplus s(G, H, \mu) \cdot m \oplus \bar{n} \cdot m \oplus a^*. \quad (5.45)$$

Hence $T_1$ contributes to every subspace appearing in (5.42). Finally the symplectic slice $N_1$ contributes to $\tilde{N}_1$ only.
Example 3. To illustrate the above discussion, we come back to Example 2. Note that, since $H$ is abelian, the subspace $\tilde{T}_1$ is trivial (cf. Figure 1).

\[ T_0 \oplus T_1 \oplus N_0 \oplus N_1 \]

\[ \tilde{T}_0 \oplus \tilde{N}_0 \oplus \tilde{N}_1 \]

*Figure 1*: Witt-Artin decompositions for $G$ (top line) and $H$ (bottom line).

As $G_\mu$ is a copy of $SO(2)$ in $SO(3)$, its Lie algebra $\mathfrak{g}_\mu := \text{span}(\chi)$ is a copy of $\mathbb{R}$ in $\mathbb{R}^3$. Therefore $T_0 = \mathfrak{g}_\mu \cdot m$ is identified with a copy of $\mathbb{R}$ in $\mathbb{R}^3$, and $T_1$ is the orthogonal complement that we identify with $\mathbb{R} \oplus \mathbb{R}$ where the first $\mathbb{R}$-factor corresponds to $\mathfrak{h} := \text{span}(x)$ when $\chi$ and $x$ are not collinear.

This is not the case if $\chi$ and $x$ are collinear. Indeed, we have seen in (4.40) that, in this case, $T_1 = \mathfrak{n} \cdot m$ is sent to $\tilde{N}_1$. Since the $G$-action on $M$ is free, the subspace $N_0$ is isomorphic to $\mathfrak{g}_\mu^* := \mathbb{R}^*$. Regarding the subgroup $H$, we have that $\tilde{T}_0 = \mathfrak{h} \cdot m$ which is identified with the copy of $\mathbb{R}$ corresponding to span$(x)$ in $\mathbb{R}^3$. Again by freeness of the $G$-action, $\tilde{N}_0$ is identified with $\mathfrak{h}^* := \mathbb{R}^*$. Figures 2 and 3 below show the relations between the two decompositions.

*Figure 2*: $\chi$ and $x$ are not collinear.

*Figure 3*: $\chi$ and $x$ are collinear.

The remaining case is when $\mu := \chi \in \mathbb{R}^3$ is equal to zero. Since $\mathfrak{g}_\mu = \mathfrak{g}$, the subspace $T_1$ is trivial and $T_0 = \mathfrak{g} \cdot m$ is identified with $\mathbb{R}^3$ that we can see as $\mathbb{R} \oplus \mathbb{R}^2$, where the first $\mathbb{R}$-factor is the copy of $\mathbb{R}$ corresponding to $\mathfrak{h} := \text{span}(x)$. The remaining $\mathbb{R}^2$-factor corresponds to $\mathfrak{b} = \mathfrak{h}^\perp$ (cf. (4.41)) and this copy is sent to $\tilde{N}_1$, as is its dual $(\mathbb{R}^2)^*$ (cf. Figure 4).
Lemma 5.1. The space $Z_m := a \cdot m \oplus r \cdot m$ is a symplectic vector subspace of $(T_m M, \omega(m))$ and is isomorphic to $a \cdot m \oplus a^*$. In particular, $f : r \cdot m \to a^*$ is an isomorphism (cf. Theorem 2.2 (iv)).

Proof — To show the first statement we use the same strategy as in the proof of Lemma 4.4, by using the fact that $n \cdot m$ is a symplectic vector subspace of $(T_m M, \omega(m))$ where

$$n = a \oplus r \oplus \mathfrak{s}(G, H, \mu) \oplus \tilde{n}. \tag{5.46}$$

Let $x_M(m) \in Z_m$ such that $\Psi(m)(x, y) = \omega(m)(x_M(m), y_M(m)) = 0$ for every $y_M(m) \in Z_m$. We will show that $\Psi(m)(x, z) = 0$ for every $z \in n$. Pick some $z \in n$ and write it as $z = y + u + v$ where $y \in a \oplus r$, $u \in \mathfrak{s}(G, H, \mu)$ and $v \in \tilde{n}$ by (5.46). Then

$$\Psi(m)(x, z) = \Psi(m)(x, y) + \Psi(m)(x, u) + \Psi(m)(x, v) = 0.$$

Indeed, the first term vanishes by hypothesis. The remaining terms vanish because $Z_m \subset \tilde{T}_0 \oplus \tilde{N}_0$ whereas $\tilde{n} \cdot m \subset \tilde{T}_1$ and $\mathfrak{s}(G, H, \mu) \cdot m \subset \tilde{N}_1$. Since those subspaces are symplectically orthogonal, $\Psi(m)(x, u) = 0$ as well as $\Psi(m)(x, v) = 0$. Using that $\omega(m)$ is non-degenerate on $n \cdot m$, we get $x_M(m) = 0$.

The second statement follows if we show that $a \cdot m$ is a Lagrangian subspace of $Z_m$, that is, $\omega(m)$ vanishes identically on $a \cdot m$. Let $x, y \in a$. In particular $x \in \mathfrak{h}^{-\mu}$ and $y \in \mathfrak{h}$ by construction of $a$. Therefore

$$\Psi(m)(x, y) = -\langle \mu, [x, y] \rangle = 0, \tag{5.47}$$

which shows that $a \cdot m$ is a Lagrangian subspace of $Z_m$. In particular, there is an isomorphism $Z_m \simeq a \cdot m \oplus (a \cdot m)^*$. Since $a$ has trivial intersection with $\mathfrak{g}_m$, we get that $a \cdot m \simeq a$. ■
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