Classification of Steadily Rotating Spiral Waves for the Kinematic Model∗

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Abstract

Spiral waves arise in many biological, chemical, and physiological systems. The kinematical model can be used to describe the motion of the spiral arms approximated as curves in the plane. For this model, there appeared some results in the literature. However, these results all are based upon some simplification on the model or prior phenomenological assumptions on the solutions. In this paper, we use really full kinematic model to classify a generic kind of steadily rotating spiral waves, i.e., with positive (or negative) curvature. In fact, using our results (Theorem 8), we can answer the following questions: Is there any steadily rotating spiral wave for a given weakly excitable medium? If yes, what kind of information we can know about these spiral waves? e.g., the tip’s curvature, the tip’s tangential velocity, and the rotating frequency. Comparing our results with previous ones in the literature, there are some differences between them. There are only solutions with monotonous curvatures via simplified model but full model admits solutions with any given oscillating number of the curvatures.

Key Words: kinematic model, spiral waves, excitable media

AMS subject classifications. 37N25, 47N60, 93A30, 47N20, 65L05, 65G20

1 Introduction

Rotating spiral waves appear in many biological, physiological, and chemical systems. For example, spiral waves arise in cardiac arrhythmias [20, 30, 32, 46, 61], egg fertilization [12], the Belousov-Zhabotinsky (BZ) chemical reaction [4, 25],...
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31, 37, 49, 77, catalysis [10, 54], and aggregation of slime mold amoeba [1, 13, 66, 69]. These waves have been studied extensively from experimental [4, 25, 37, 47, 49, 52, 65, 71, 77], numerical [25, 26, 27, 39, 45, 74, 78], and analytical [5, 12, 21, 22, 24, 28, 36, 62, 63, 64, 68, 75] aspects.

Often, such systems exhibit the so-called “excitable” property [43, 76]. A spiral wave represents an excited moving spatial region with two spiral-like, thin phase-transition layers (or interfaces): wave front and wave back. These layers separate two different phase regions, excited and refractory (or recovery), and intersect at a tip (see Figure 12.5 in [30] or Figures 13, 17, and 22 in [40]).

Usually, the appearance of spiral waves is not desirable. For instance, heart attacks are caused by an abrupt shift from rhythmic pumping to spasmodic convulsion of the heart. Normally, with each heartbeat, an electric wavefront propagates across the interconnected muscle fibers, causing them to contract. However, because of some abnormality in the tissue, this wave can become stuck and start rotating as a spiral wave [58, 59]. Furthermore, if spiral waves occur in the cortex, they may lead to epileptic seizures. On the retina or the visual cortex, they may cause hallucinations [8, 9].

Spiral waves can be created by breaking propagating waves [25, 60, 65]. In [60], a circular wave is broken mechanically by ejecting a gentle blast of air into a small section of the wave. The process by which a broken wave evolves toward a rotating spiral wave has been addressed in [11, 18, 19, 41, 44, 76]. As pointed out in [19], along the interface separating the excited and the refractory regions, there exists a point where the normal velocity of the interface changes sign. This induces a twisting action on the interface motion, and as a result, the interface starts wrapping around some center of rotation, to form a spiral structure. Eventually, in an appropriate parameter range, a state of steady rotation is achieved, during which the spiral tip traces a circular trajectory at constant angular velocity, and the curvatures of the spiral-shaped interfaces keep constant in time. Such steadily rotating spiral waves have been observed in many experiments [48, 49, 71].

Another way to initiate spiral waves is by applying a spatially graded perturbation to a medium in the recovery phase after excitation [72, 73]. Besides steadily rotating spiral waves, non-steady forms of rotation, known as “meander” [73], are also confirmed by experiments [25, 60, 65] and numerical simulations [3, 25, 26, 39, 70]. When the controlled parameters are changed within some suitable range [25, 60, 65], or due to the anisotropic influences of the media [61], or the effect of electric field [50, 51], a transition from the steadily rotating spiral waves to meandering ones may occur. To study the above mentioned wave patterns, many models have been proposed, which we can categorize here as “phase field” type and “sharp interface” type models.

For the phase field type of models, well-known examples are the reaction-diffusion equations (for example, the FitzHugh-Nagumo model for the action potential wave in neurons [30]), the Oregonator model for the BZ reaction [16, 67], the bidomain model [30], and the chemotactic models [33, 69]. The bidomain

1We borrow these two terms from material science.
model is a mixture of elliptic and reaction-diffusion type of equations. It is used to describe the electrical activities of cardiac tissue. The chemotactic models describe the oriented movement of cells in response to the concentration gradient of chemical substances in their environment, such as the aggregation phenomena of slime mold amoeba.

Using the reaction-diffusion equation, the existence of steadily rotating spiral waves near a homogeneous steady state has been proved formally by Hagan [22] and later rigorously by Scheel [64]. Other more complicated wave patterns, e.g., meandering and hypermeander were also explored by [2, 3, 62, 63, 74].

For the sharp interface type of models, some typical examples are the free boundary models [17, 20, 24, 34, 36, 55, 56, 68] and the kinematic models [14, 15, 43, 44, 45, 76]. Other models are given in [68, 75].

If the two phase transition layers, as mentioned above, are very thin, we can view them as two plane curves meeting at the tip. Then, the free boundary model consists of an interface equation, a simplified phase field equation, and a dispersion relation with some boundary conditions. The interface equation (13) or (38) in [68] is derived from the so-called eikonal-curvature equation (11) there and governs the motion of a wave front; the simplified phase field equation (37) in [68] is derived from the original phase field model and affects the velocity appearing in the interface equation; and the dispersion relation (see section 3.3 in [68]) reflects the interaction between spiral waves (refractory tail effect).

In weakly excitable media, spiral waves rotate around a large circle, the excited region becomes rather narrow, and the spiral waves are sparse. In this case, we need to consider the wave front only. Thus, the spiral wave can be viewed as a single curve, which performs a motion in the plane regardless of the interaction between the spiral waves. That is, the dispersion relation can be ignored.

The kinematic model is then formulated in terms of a motion of a single curve with free end; see section 2 for the details. The validity of this kinematic model has been verified, since various wave patterns appearing in the experiments can also be produced with this approach [44].

Finally, we should point out that many main governing equations of the sharp interface models can be derived from the phase field models by taking various types of singular limits, e.g., the small parameters may represent the width of the thin transition layers. Therefore, it is natural to expect that there should be a close relation between the solutions of these two type of models. Indeed, it has been found that there is a good agreement for temporal periods of rotating spiral waves, when comparing the reaction-diffusion approach with the kinematic one [45].

In this paper, we use the really “full” kinematic model to prove the existence of steadily rotating spiral waves. In fact, using our results (Theorem 8), we can answer the following questions: Is there any steadily rotating spiral wave for a given weakly excitable medium (i.e., \( V_0, D \) are given below). If yes, what kind of informations we can know about these spiral waves? e.g., the tip’s curvature (\( \kappa_0 \) below), the tip’s tangential velocity (\( \Gamma \) below), and the rotating frequency (\( \omega \) below). In the literature, there appeared some results about kinematic model.
However, these results all are based upon some simplifications on the model or prior assumptions on the solutions. For examples, [14], [15], [21], and [44] considered only simplified model (see the Remarks in Section 2); [21], [24], [44], and [75] made a prior assumption on the solution, i.e., tip’s tangential velocity $G$ is 0, which was proven to be a very special case of steadily rotating spiral waves in [14], [21] is valid only for nonrotating ones, i.e., angular frequency $\omega$ is 0 (see the explanation after (16) below); [24] considered solutions with positive curvature $\kappa$. In this paper, we also consider solutions with unchangeable curvature’s sign (i.e., with positive or negative curvature). However, this condition is generic. Note that the formula (3.11) in [75] is $\omega(0; 0, l(0))$ in Theorem 8, i.e., a special case of our results with zero tip’s tangential velocity ($G = 0$) and monotonous curvature ($i = 0$). Note also that steadily rotating spiral waves with positive curvatures must have monotonous curvatures via simplified model (see [21]); however, using full model in this paper we have confirmed the existence of oscillating ones, i.e., the case “$i \neq 0$” in Theorem 8.

In section 2, we recall the kinematic model following Mikhailov et al. [44] and a new version by [14], [15]. In section 3, we state and then prove the main theorem of this paper. Numerical results are presented in section 4.

2 The Kinematic Model

The kinematic model, formulated in terms of a motion of curves, was first proposed for describing excitation waves in a cardiac muscle by Wiener and Rosenblueth [70]. They assumed that a plane curve moves in a normal direction with a constant velocity. They showed that such a curve, rotating around an obstacle, forms a spiral (which represents an involute of this obstacle) and approaches an Archimedian spiral far from it.

Since there occurs a singularity in the curvature of this spiral curve near the tip [70], and also by the subsequent analysis of wave propagation based on an excitable reaction-diffusion equation, it has been found that the normal velocity should be relevant to the local curvature $\kappa$. For example, the so-called eikonal-curvature relation or mean-curvature flow type equation (see (2) below) has been shown to be correct when linear approximation is considered.

We should note here that Wiener and Rosenblueth considered only pinned spirals that rotate around an obstacle. However, subsequent numerical simulations and experiments with various excitable media have revealed that the same media can also support spiral waves free of any obstacles [65, 72].

Later, Mikhailov et al. [44] proposed the following standard kinematic model:
\[
\frac{\partial \kappa}{\partial t} + \frac{\partial \kappa}{\partial s} \left( \int_{0}^{s} \kappa V d\xi + G \right) + \kappa^2 V + \frac{\partial^2 V}{\partial s^2} = 0, \quad 0 \leq s < \infty, \quad 0 \leq t < \infty, \quad (1)
\]
\[
V = V_0 - D\kappa, \quad 0 \leq s < \infty, \quad 0 \leq t < \infty, \quad (2)
\]
\[
G = G_0 - r\kappa_0, \quad 0 \leq t < \infty, \quad (3)
\]
\[
\frac{d\kappa_0}{dt} = -G \frac{\partial \kappa}{\partial s} \bigg|_{s=0}, \quad 0 < t < \infty, \quad \text{and} \quad (4)
\]
\[
\lim_{s \to \infty} \kappa(s, t) = 0, \quad 0 < t < \infty, \quad (5)
\]

where \( \kappa = \kappa(s, t) \) is the curvature of a plane curve depending on time \( t \) and the arc length \( s \) measured from the free tip, \( V = V(s, t) \) is the normal velocity, \( G = G(t) \) is the tangential velocity of the tip, \( \kappa_0 = \kappa_0(t) := \lim_{s \to 0} \kappa(s, t) \), and \( V_0, D, G_0, r \) are parameters determined from the media.

Here, (1) is a general equation satisfied by a plane curve of any point at which the curve moves in the normal direction (see (4.41) in \[40\] or (2.7) in \[44\]); (2) is the eikonal-curvature relation, which relates the normal velocity of the curve to the local curvature (see (4.10) in \[40\] or (1.1) in \[44\]); (3) is the tangential velocity of the tip\(^2\) and (4) and (5) are boundary conditions (see (2.10) in \[44\]).

We explain (3) as follows. The curvature near the tip, hence \( \kappa_0 \), affects the normal velocity near the tip by (2). Then, by (3.27) from \[40\], the normal velocity near the tip affects the width of the excited region near the tip. Moreover, it has been observed that the width of the excited region near the tip influences the tangential velocity of the tip \[50, 51\]. Hence \( \kappa_0 \) is related to \( G \).

Notice that (2) and (3) are good linear approximations of the perturbation expansions only when the curvature is small. Hence “\( \kappa \ll \nu_0 \)” in (2) (or \( \nu \approx \nu_0 \)) is required.

**Remarks** Elkin \textit{et al.} \[14\] used perturbations techniques and Fredholm alternative theorem to obtain more general equations compared to (3) and (4) here (see (28), (32), (36), and (37) in \[14\]). In fact, it has been found that (3) is only a special case of (32) in \[14\] (see the context below (37) in \[14\]) and (4) is not correct. Later Elkin \textit{et al.} \[15\] solved the above new model under some simplification, i.e., replacing \( V \) with \( V_0 \) in the term “\( \int_{0}^{s} \kappa V d\xi \)” in (1) (see (4) in \[15\]).

In this paper, we consider steadily rotating spiral waves (so \( \frac{\partial \kappa}{\partial t} = 0 \)). Let \( G \) be a constant and let \( \omega \) be the momentary rotational angular velocity of the tip. Then, using (2) and integrating the both sides of (1) give rise to:

\[
\kappa(s) \left( \int_{0}^{s} k(\xi)[V_0 - Dk(\xi)]d\xi + G \right) - Dk'(s) = \omega.
\]
Thus it follows that
\[
\kappa'(0) = \frac{G}{D}\kappa(0) - \frac{\omega}{D} = \frac{G}{D}\kappa_0 - \frac{\omega}{D},
\]
(7)
(see also (6) in [15]). After differentiating (6) once and replacing the integral term, we obtain:
\[
-D\kappa'' + \kappa^2(V_0 - D\kappa) + \frac{\kappa'}{\kappa}(\omega + D\kappa') = 0,
\]
(8)
provided \(\kappa \neq 0\). Without loss of generality, we assume \(\kappa > 0\) hereafter, since if \(\kappa < 0\), then (7) and (8) are invariant by replacing \(\kappa, \omega, V_0\) with \(-\kappa, -\omega, -V_0\), respectively. Let \(l(s) := \ln(\kappa(s))\). Then (5), (7), and (8) give rise to:
\[
\frac{d^2l}{ds^2} = -g(l(s)) + \frac{\omega}{D} \frac{dl}{ds} e^{-l(s)},
\]
(9)
\[
l(0) = \ln\kappa_0,
\]
(10)
\[
l'(0) = -\frac{\omega}{D\kappa_0} + \frac{G}{D},
\]
(11)
\[
\lim_{s \to \infty} l(s) = -\infty,
\]
(12)
where \(g(l) := e^{2l} - \frac{V_0}{D}e^l\). The above equations (9), (10), (11) are also equivalent to the following ones:
\[
l'(s) = v(s)
\]
(13)
\[
v'(s) = -g(l(s)) + \frac{\omega}{D} e^{-l(s)} v(s)
\]
(14)
\[
v(0) = -\frac{\omega}{D} e^{-l(0)} + \frac{G}{D}.
\]
(15)
Furthermore, if \(v(s) \neq 0\), then the solution \(v(s)\) in (13), (14) can be viewed as a function of \(l\) by the inverse function theorem and satisfies the following equation:
\[
\frac{dv}{dl} = -\frac{g(l)}{v} + \frac{\omega}{D} e^{-l}.
\]
(16)
Note that in [24], Ishimura et al. considered the above problem (9)-(12) with \(G = 0\) and \(l'(0) = 0\) (see (13) in [24]). Therefore, the results in [24] are valid only for the case “\(\omega = 0\)”. Now for any excitable media with given parameters \(V_0 > 0, D > 0\) which are fixed throughout this paper, we want to find suitable \(G, \omega, \text{and } l(0)\) (i.e., \(\kappa_0\)) such that (12), (13), (14), and (15) has global existence solutions. Let \(E(l, v) := \frac{1}{2}v^2 + \frac{1}{2}e^{2l} - \frac{V_0}{D}e^l\).
3 Behavior of the Solutions

**Lemma 1** For any \( \omega < 0 \), there are no global existence solutions of (13), (14) satisfying \( \lim_{s \to -\infty} l(s) = -\infty \).

**Proof.** Since \( \omega < 0 \), given any solution of (13), (14), say, \((l(s; \omega), v(s; \omega))\) (dropping out the dependence on initial data here for simplicity), \(E(l(\cdot; \omega), v(\cdot; \omega))\) is a decreasing function by (11) of [1]. Note also that \(E(l, v) \geq E(\ln \frac{V}{l}, 0) = -\frac{1}{2}(\frac{V}{l})^2\). Thus \(E(l(\cdot; \omega), v(\cdot; \omega))\) is bounded. If \(l(\cdot; \omega)\) exists globally with \(\lim_{s \to -\infty} l(s; \omega) = -\infty\), then we have \(\lim_{s \to -\infty} se^{2l(s)} = \frac{\omega}{2\sigma}\) by LEMMA 3 of [13]. This is impossible, since \(\lim_{s \to -\infty} se^{2l(s)} \geq 0 > \frac{\omega}{2\sigma}\). This completes the proof. \(\blacksquare\)

By above lemma and THEOREM 2 of [13], we only need to consider \( \omega > 0 \) now.

**Lemma 2** Given any global existence solution of (13), (14), \((l(s; \omega), v(s; \omega))\), if \(v(0; \omega) > 0\), then there is some \(\sigma > 0\) such that \(v(\sigma; \omega) = 0\) (no matter the sign of \(\omega\)).

**Proof.** Suppose not, i.e., \((l(s; \omega), v(s; \omega))\) always stays in the upper half plane: \(v > 0\). Note that the upper half plane is split into three parts: \(R_1 := \{l, v : v > 0 \text{ and } g(l) + \frac{\omega}{D} e^{-l} v > 0\}\), \(R_2 := \{l, v : v > 0 \text{ and } -g(l) + \frac{\omega}{D} e^{-l} v = 0\}\), and \(R_3 := \{l, v : v > 0 \text{ and } -g(l) + \frac{\omega}{D} e^{-l} v < 0\}\). If \((l(s; \omega), v(s; \omega))\) always stays in \(R_1\), then both \(l(\cdot; \omega), v(\cdot; \omega)\) increase as \(s \to -\infty\) by (13), (14), and so both \(\lim_{s \to -\infty} l(s; \omega)\) and \(\mu := \lim_{s \to -\infty} v(s; \omega) > 0\) exist. Thus clearly, we have \(\lim_{s \to -\infty} l(s; \omega) = \infty\). Since \(v'(s; \omega) > 0\) and \(\lim_{s \to -\infty} -g(l(s; \omega)) = -\infty\) now, we have \(\mu = \infty\) by (14). On the other hand, it follows from (14) that:

\[
v(s; \omega) + \frac{\omega}{D} e^{-l(s; \omega)} - (v(0; \omega) + \frac{\omega}{D} e^{-l(0; \omega)}) = -\int_0^s g(l(t; \omega))dt.
\]  

(17)

Taking limit as \(s \to -\infty\) on the both side of (17), we have

\[
\lim_{s \to -\infty} \left[ v(s; \omega) + \frac{\omega}{D} e^{-l(s; \omega)} - (v(0; \omega) + \frac{\omega}{D} e^{-l(0; \omega)}) \right] = \infty = -\int_0^\infty g(l(t; \omega)) dt.
\]  

(18)

However, \(-\int_0^\infty g(l(t; \omega)) dt = -\int_0^\delta g(l(t; \omega)) dt - \int_0^\infty g(l(t; \omega)) dt\), where \(l(\delta; \omega) = \ln \frac{V}{l}\). Since \(g(l) \geq 0\) as \(l \geq \ln \frac{V}{l}\) and \(l(t; \omega) \not\to \infty\) as \(t \not\to \infty\), we have \(-\int_0^\infty g(l(t; \omega)) dt < 0\) and so

\[
-\int_0^\infty g(l(t; \omega)) dt < -\int_0^\delta g(l(t; \omega)) dt < \infty.
\]  

(19)

Thus (18) contradicts (19). Now the only possible case is that \((l(s; \omega), v(s; \omega))\) goes through the curve \(R_2\), enters the region \(R_3\), and then stays there for ever.
Hence \( l(\cdot; \omega) \) and \( v(\cdot; \omega) \) are increasing and decreasing functions respectively now by (13). Clearly \( \mu := \lim_{s \to \infty} v(s; \omega) \in [0, \infty) \), since \( v(s; \omega) > 0 \). Note that the “Poincare-Bendixson theorem” implies \( \lim_{s \to \infty} l(s; \omega) = \infty \); otherwise, \((l(s; \omega), l'(s; \omega))\) will stay in a bounded region and so \( \lim_{s \to \infty} l(s; \omega) = \ln \frac{V}{\mu} \) and \( \lim_{s \to \infty} v(s; \omega) = 0 \), since \((\ln \frac{V}{\mu}, 0)\) is the only possible \( \omega^-\) limit point of \((l(s; \omega), l'(s; \omega))\). A contradiction then occurs, since \( \lim_{s \to \infty} l(s; \omega) \neq \ln \frac{V}{\mu} \) by (13).

Now it follows from “\( \lim_{s \to \infty} l(s; \omega) = \infty \)” and (14) that \( \lim_{s \to \infty} v'(s; \omega) = -\infty \) which contradicts “\( v(s; \omega) > 0, \forall s \geq 0 \).”

**Lemma 3** For \( \omega > 0 \), given any solution of (13), \((l(s; \omega), v(s; \omega))\), with maximal existence interval \([0, \Gamma), \) if \( E(l(s^\#; \omega), v(s^\#; \omega)) \geq 0 \) for some \( s^\# \in [0, \Gamma) \), then \( \Gamma < \infty \) and \( \lim_{s \to \infty} l(s; \omega) = \lim_{s \to \infty} v(s; \omega) = -\infty \), i.e., blows up in finite \( s \).

**Proof.** Assume the contrary that \( \Gamma = \infty \). Then by Lemma 2 \((l(s; \omega), v(s; \omega))\) will enter the lower half plane after a short time, say, \( \sigma \). Since \( E(l(s; \omega), v(s; \omega)) \) is increasing and \( E(l(s^\#; \omega), v(s^\#; \omega)) \geq 0 \) for some \( s^\# \in [0, \Gamma) \), we have \( E(l(s; \omega), v(s; \omega)) > 0 \) \( \forall s \in (s^\#, \infty) \). It then follows that \( v(s; \omega) < 0, \forall s \in (\max\{\sigma, s^\#\}, \infty) \), since the level curve \( E(l, v) = 0 \) is a barrier preventing \((l(s; \omega), v(s; \omega))\) from touching the \( l^-\)axis. Clearly, both \( \mu_1 := \lim_{s \to \infty} l(s; \omega) < \infty \) and \( \mu_2 := \lim_{s \to \infty} v(s; \omega) \leq 0 \) exists. Note that \( \mu_2 \neq 0 \); otherwise, again by the “Poincare-Bendixson theorem” we have \( \mu_1 = -\infty \) and so \( \lim_{s \to \infty} E(l(s; \omega), v(s; \omega)) = 0 \) which is impossible. Now it follows from “\( \mu_2 \in [-\infty, 0) \)” that \( \mu_1 = -\infty \). By (14), we then have \( \lim_{s \to \infty} v'(s; \omega) = -\infty \) and so \( \mu_2 = -\infty \). Moreover, we also have \( \lim_{s \to \infty} v''(s; \omega) = -\infty \) by differentiating both sides of (14) and taking limit.

Multiplying both sides of (14) by \( e^{l(s; \omega)} \) and integrating them, we then have

\[
\frac{\omega s}{D} + \int_{l(0; \omega)}^{l(s; \omega)} e^l (v(0; \omega) + \omega e^{-l(0; \omega)}) \int_0^s e^{l(p; \omega) - \Delta} dp = -\int_0^s e^{l(p; \omega) - \Delta} dp \int_0^p g(l(t; \omega)) dt dp.
\]

(20)

Now we want to show that \( e^{l(p; \omega)} = o\left(\frac{1}{p^3}\right) \) as \( p \to \infty \). Utilizing l’Hôpital’s rule, we have

\[
\lim_{p \to \infty} e^{l(p; \omega)} = \lim_{p \to \infty} \frac{p^3}{D} e^{-l(p; \omega)} = \lim_{p \to \infty} \frac{3p^2}{p^3} e^{-l(p; \omega)} = \lim_{p \to \infty} \frac{6p}{p^3} e^{-l(p; \omega)} - \frac{6p}{p^3} e^{-l(p; \omega)} = 0.
\]

(21)

Since \( g(l) \geq g(\ln \frac{V}{\mu}) \), we have \(-\int_0^{\infty} e^{l(p; \omega)} \int_0^p g(l(t; \omega)) dt dp \leq -g(\ln \frac{V}{\mu}) \int_0^{\infty} e^{l(p; \omega)} p dp < \infty \) by (21). Also, \( \lim_{s \to \infty} \int_{l(0; \omega)}^{l(s; \omega)} e^l dl = \int_{l(0; \omega)}^{\infty} e^l dl \in (-\infty, \infty) \)

and \( \int_0^{\infty} e^{l(p; \omega)} dp \in \)
(0, ∞) by \cite{21} again. By taking limit as $s \to \infty$ on the both sides of \cite{20}, a contradictions occurs. Therefore $\Gamma < \infty$ and so the proof is finished. \hfill \blacksquare

**Lemma 4** Given any $\omega \in (0, \frac{2\sqrt{2}}{D})$, there exists a unique $l^* = l^*(\omega) \in (\ln \frac{V_0}{D}, \ln \frac{2\sqrt{V_0}}{D})$ such that \cite{13, 14} has a global existence solution $(\tilde{l}(s;\omega), \tilde{v}(s;\omega))$ defined on $(-\infty, \infty)$ with $(\tilde{l}(0;\omega), \tilde{v}(0;\omega)) = (l^*, 0)$, $\tilde{l}(s;\omega) \searrow -\infty$, $\tilde{v}(s;\omega) \to 0$ as $s \to \infty$, and $\tilde{v}(s;\omega) < 0$ for all $s > 0$. Moreover, $(\tilde{l}(s;\omega), \tilde{v}(s;\omega))$ rotates counterclockwise around $(\ln \frac{V_0}{D}, 0)$ and tends to it as $s$ decreases to $-\infty$.

**Proof.** By THEOREM 3, LEMMA 4, and LEMMA 5 in [1], we just know there exists an interval $[\sup A, \inf B] \subset (\ln \frac{V_0}{D}, \ln \frac{2\sqrt{V_0}}{D})$ such that the solution of \cite{13, 14} with initial data $(l_0, 0)$, where $l_0 \in [\sup A, \inf B]$, has the properties mentioned in this lemma, denoted by $(\tilde{l}(s;\omega, l_0), \tilde{v}(s;\omega, l_0))$. Note that by Lemma \cite{3} we have $[\sup A, \inf B] \subset (\ln \frac{V_0}{D}, \ln \frac{2\sqrt{V_0}}{D})$. Now we want to prove $\sup A = \inf B$.

Since the solution $\tilde{v}(s;\omega, l_0) \neq 0$, for all $s \in (0, \infty)$, $\tilde{v}(s;\omega, l_0)$ can be viewed as a function of $l$ by the inverse function theorem, i.e., $\tilde{v}(s;\omega, l_0) = \tilde{v}(l;\omega, l_0)$, for $l \in [-\infty, l_0]$ and so satisfies \cite{20}. For any two solutions $\tilde{v}(l;\omega, l_0), \tilde{v}(l;\omega, l_0)$, denoted by $\tilde{v}_1(l)$ and $\tilde{v}_2(l)$ respectively for simplicity, by \cite{20} we have

$$
\frac{d(\tilde{v}_1 - \tilde{v}_2)}{dl} = \frac{g(l)}{\tilde{v}_1 \tilde{v}_2} (\tilde{v}_1 - \tilde{v}_2). \tag{22}
$$

Thus $(\tilde{v}_1 - \tilde{v}_2)(l) = (\tilde{v}_1 - \tilde{v}_2)(\ln \frac{V_0}{D}) \exp(-\int_{\ln \frac{V_0}{D}}^{\ln \frac{2\sqrt{V_0}}{D}} \frac{g(l)}{\tilde{v}_1 \tilde{v}_2(l)} dl)$ for $l < \ln \frac{V_0}{D}$. Suppose $(\tilde{v}_1 - \tilde{v}_2)(\ln \frac{V_0}{D}) \neq 0$. Then since $\tilde{v}_1, \tilde{v}_2 < 0$ and $g(l) < 0$ for $l < \ln \frac{V_0}{D}$, we have $\lim_{l \to -\infty} (\tilde{v}_1 - \tilde{v}_2)(l) \neq 0$. This is impossible, since $\lim_{s \to -\infty} \tilde{v}(s;\omega, l_0) = \lim_{s \to -\infty} \tilde{v}(s;\omega, l_0) = 0$. By uniqueness, we have $l_{01} = l_{02}$ and so $\sup A = \inf B$.

Let $l^*(\omega) := \sup A = \inf B$ and $(\bar{l}(s;\omega), \bar{v}(s;\omega)) := (\tilde{l}(s;\omega, l^*(\omega)), \tilde{v}(s;\omega, l^*(\omega)))$.

Finally, since the $\alpha$–limit set of $(\bar{l}(s;\omega), \bar{v}(s;\omega))$ consists of only one point $(\ln \frac{V_0}{D}, 0)$ which is a unstable focus, $(\bar{l}(s;\omega), \bar{v}(s;\omega))$ is also defined on $(-\infty, 0]$ and $\lim_{s \to -\infty} (\bar{l}(s;\omega), \bar{v}(s;\omega)) = (\ln \frac{V_0}{D}, 0)$. \hfill \blacksquare

Given any solution of \cite{13, 14}, say, $(l(s;\omega), v(s;\omega))$, $E(l(\cdot;\omega), v(\cdot;\omega))$ is an increasing function, since $\omega > 0$. Also, by LEMMA 5 in \cite{21}, $(\bar{l}(s;\omega), \bar{v}(s;\omega))$ always stays inside the curve: $E(l, v) = 0$ in the $l – v$ phase plane.

**Proposition 5** Suppose $\omega \in (0, \frac{2\sqrt{2}}{D})$. For any global existence solution of \cite{13, 14}, its orbit curve in the $l – v$ plane coincides with that of $(\bar{l}(s;\omega), \bar{v}(s;\omega))$, i.e., they both lie on the same solution curve in the $l – v$ plane.

**Proof.** Let $(l(s;\omega), v(s;\omega))$ be a global existence solution of \cite{13, 14}, i.e., existence interval contains $[0, \infty)$, (also dropping out the dependence on initial data here for simplicity). Suppose $(l(s;\omega), v(s;\omega))$ stays in a bounded region in the $l – v$
plane. Since the equilibrium \((\ln a, 0)\) is an unstable focus, the only possible \(\omega\)-limit set of \((l(s_1; \omega), v(s_1; \omega))\) consists of periodic orbits by the “Poincaré-Bendixson theorem”. However, by the “Bendixson criterion” (see [1]), there is no any periodic solution of \((13), (14)\). Thus our assumption is wrong. Therefore, any global existence solution is unbounded.

Note that by Lemma 3 \((l(s; \omega), v(s; \omega))\) always stays inside the curve: \(E(l, v) = 0\) in the \(l - v\) phase plane. Since \((l(s_2; \omega), v(s_2; \omega))\) is unbounded, we can choose some \(s_1\) large enough such that \((l(s_1; \omega), v(s_1; \omega))\) is located in the lower “left tail” region: \(\{(l, v) \in R^2 | E(l, v) < 0, v < 0, \text{ and } l < l_L\}, \) for some small enough \(l_L\) with \(E(l(s_1; \omega), v(s_1; \omega)) > E(l, 0)\) by the structure of level curve of energy \(E(l, v)\) with negative value, i.e., \(E(l, v) = c < 0\). Suppose \(v(s_2; \omega) = 0\) for some \(s_2 > s_1\). Then by the unboundedness of \((l(s; \omega), v(s; \omega))\), the direction of flow on the upper half plane, and the increasingness of \(E(l; \omega), v(\cdot; \omega))\), there exists some \(s_3 > s_2\) such that \(v(s_3; \omega) = 0\) and \(l(s_3; \omega) \in \{l^*, \ln \frac{2V_0}{l^*}\}\). Note that any solution curve which intersects the \(l\)-axis at \((l_R, 0)\) with \(l_R \in \{l^*, \ln \frac{2V_0}{l^*}\}\) must touch the level curve \(E(l, v) = 0\) and so blows up in finite time by Lemma 6 since \(l^* = \inf B\) (see [24] for the definition of set \(B\)) and the uniqueness theorem of ordinary differential equation. Thus \((l(s; \omega), v(s; \omega))\) blows up in finite time which leads to a contradiction. Therefore our assumption is wrong. We conclude that \(v(s; \omega) < 0\) for all \(s \geq s_1\) which leads to the following facts: \(\lim_{s \to -\infty} l(s; \omega) = -\infty\) and \(\lim_{s \to -\infty} v(s; \omega) = 0\).

Now using similar argument as in Lemma 4, the solution curve of \((l(s; \omega), v(s; \omega))\) must coincide with that of \((\tilde{l}(s; \omega), \tilde{v}(s; \omega))\) in the \(l - v\) plane.

Let \(l^*_{1R} = l^*(\omega), l^*_{2R} = l^*(\omega), \ldots, l^*_{iR} = l^*(\omega), \ldots < \ln \frac{V_0}{l^*}\) be the sequence of intersection points of \((\tilde{l}(s; \omega), \tilde{v}(s; \omega))\) with the \(l\)-axis greater than and less than \(\ln \frac{V_0}{l^*}\) respectively as \(s \to -\infty\). Note that \(l^*_{iR} > l^*_{jR}\) and \(l^*_{iL} < l^*_{jL}\) as \(i < j\).

**Lemma 6**  (i). \(l^*_{iR}(\cdot)\) and \(l^*_{iL}(\cdot)\) are decreasing and increasing functions respectively on \((0, \frac{2V_0}{D})\).

(ii). \(l^*_{iR}(\cdot)\) and \(l^*_{iL}(\cdot)\) are continuous functions on \((0, \frac{2V_0}{D})\).

(iii). \(\lim_{\omega \to 0^+} l^*_{iR}(\omega) = \ln \frac{2V_0}{D}\) and \(\lim_{\omega \to 0^+} l^*_{iL}(\omega) = -\infty, \) \(i = 1, 2, 3, \ldots\)

**Proof.** Since \(\tilde{v}(s_1; \omega_1) \neq 0, \tilde{v}(s_2; \omega_2) \neq 0, \) for all \(s \in (0, \infty), \tilde{v}(s; \omega_1), \tilde{v}(s; \omega_2)\) can be viewed as a function of \(l\) by the inverse function theorem, i.e., \(\tilde{v}(s; \omega_1) = \tilde{v}(l; \omega_1), \tilde{v}(s; \omega_2) = \tilde{v}(l; \omega_2)\) for \(l \in [-\infty, l^*_{iR}(\omega_1)], l \in [-\infty, l^*_{iR}(\omega_2)]\) respectively. Note that for \(\omega_1, \omega_2\), the corresponding solutions \(v_1(l) := v(l; \omega_1), v_2(l) := v(l; \omega_2)\) of \((13), (14)\) have the following relation:

\((v_1 - v_2)(l) = \ldots\)
(v_1 - v_2)(\tau) \exp(\int_\tau^t \frac{g(t)}{v_1(t)v_2(t)} dt) + (\frac{\omega_1 - \omega_2}{D}) \exp(\int_\tau^t \frac{g(t)}{v_1(t)v_2(t)} dt) \int_\tau^t \exp(-t - \int_\tau^t \frac{g(\rho)}{v_1(\rho)v_2(\rho)} d\rho) dt,

\text{(23)}

whenever both } v_1, v_2 \text{ exist on } [\tau, l] \text{ or } [l, \tau]. \text{ Now apply } 23 \text{ to } \tilde{v}_1(l) := \tilde{v}(l; \omega_1), \tilde{v}_2(l) := \tilde{v}(l; \omega_2), \text{ for } \omega_1, \omega_2 \in (0, \frac{\sqrt{D}}{2}) \text{ with } \omega_1 < \omega_2.

First we consider } l^*_R(\cdot).

\textbf{Case 1. } l^*_R(\omega_1) < l^*_R(\omega_2).

Then there is a small } \varepsilon > 0 \text{ such that } \tilde{v}_1(l), \tilde{v}_2(l) \text{ are defined on } (-\infty, l^*_R(\omega_1) - \varepsilon] \text{ with } \tilde{v}_1(l^*_R(\omega_1) - \varepsilon)) > \tilde{v}_2(l^*_R(\omega_1) - \varepsilon)). \text{ Taking } \tau = l^*_R(\omega_1) - \varepsilon \text{ and } l \leq \tau, \text{ by } 23 \text{ we obtain:}

\[(\tilde{v}_1 - \tilde{v}_2)(l) \geq (\tilde{v}_1 - \tilde{v}_2)(\tau) \exp(\int_\tau^l \frac{g(t)}{v_1(t)v_2(t)} dt).

\text{Since } \lim_{l \to -\infty} (\tilde{v}_1 - \tilde{v}_2)(\tau) \exp(\int_\tau^l \frac{g(t)}{v_1(t)v_2(t)} dt) > 0, \text{ we have } \lim_{l \to -\infty} (\tilde{v}_1 - \tilde{v}_2)(l) > 0.

\text{A contradiction occurs.}

\textbf{Case 2. } l^*_R(\omega_1) = l^*_R(\omega_2) \text{ and there is } \varepsilon > 0 \text{ such that } \tilde{v}_1(l^*_R(\omega_1) - \varepsilon)) \geq \tilde{v}_2(l^*_R(\omega_1) - \varepsilon)).

Then taking } \tau = l^*_R(\omega_1) - \varepsilon \text{ and } l \leq \tau, \text{ by } 23 \text{ we obtain:}

\[(\tilde{v}_1 - \tilde{v}_2)(l) \geq (\frac{\omega_1 - \omega_2}{D}) \exp(\int_\tau^l \frac{g(t)}{v_1(t)v_2(t)} dt) \int_\tau^l \exp(-t - \int_\tau^t \frac{g(\rho)}{v_1(\rho)v_2(\rho)} d\rho) dt.

\text{Since } \lim_{l \to -\infty} \exp(\int_\tau^l \frac{g(t)}{v_1(t)v_2(t)} dt) \int_\tau^l \exp(-t - \int_\tau^t \frac{g(\rho)}{v_1(\rho)v_2(\rho)} d\rho) dt > 0, \text{ we also have a contradiction. Hence the only possible case is } l^*_R(\omega_1) = l^*_R(\omega_2) \text{ and } \tilde{v}_1(l) < \tilde{v}_2(l) \text{ for } l \in (-\infty, l^*_R(\omega_1)) \text{ other than } l^*_R(\omega_1) > l^*_R(\omega_2). \text{ Taking any } \tau \in [\ln \frac{1}{2}, l^*_R(\omega_1)) \text{ and } l \leq l^*_R(\omega_1), \text{ by } 23 \text{ we have}

\[(\tilde{v}_1 - \tilde{v}_2)(l) < (\frac{\omega_1 - \omega_2}{D}) \exp(\int_\tau^l \frac{g(t)}{v_1(t)v_2(t)} dt) \int_\tau^l \exp(-t - \int_\tau^t \frac{g(\rho)}{v_1(\rho)v_2(\rho)} d\rho) dt < 0.

\text{(24)}

Clearly, the right hand side of } 24 \text{ will not tend to } 0, \text{ as } l \to l^*_R(\omega_1)^+. \text{ This contradicts to } \lim_{l \to l^*_R(\omega_1)^-} (\tilde{v}_1 - \tilde{v}_2)(l) = 0, \text{ since } l^*_R(\omega_1) = l^*_R(\omega_2).

By above discussions, we conclude that } l^*_R(\omega_1) > l^*_R(\omega_2), \text{ if } \omega_1 < \omega_2.

Now we consider } l^*_L(\cdot). \text{ Given any } s_{L_1} < 0 \text{ such that the solution curves of } (\tilde{l}(s; \omega_1), \tilde{v}(s; \omega_1)) \text{ and } (\tilde{l}(s; \omega_2), \tilde{v}(s; \omega_2)) \text{ stay in the upper half plane when } s \in [s_{L_1}, 0), \text{ both } \tilde{v}(s; \omega_1) \text{ and } \tilde{v}(s; \omega_2) \text{ then can also be viewed as functions of } l, \tilde{v}_1(\cdot), \tilde{v}_2(\cdot) \text{ as before and so } 23 \text{ is also valid. Due to } l^*_R(\omega_1) > l^*_R(\omega_2), \text{ we can take } \tau = l^*_R(\omega_2) - \delta \text{ for some small } \delta > 0 \text{ such that both } \tilde{v}_1(\cdot) \text{ and } \tilde{v}_2(\cdot) \text{ are defined on } (\max\{l^*_L(\omega_1), l^*_L(\omega_2)\}, \tau] \text{ with } \tilde{v}_1(\tau) > \tilde{v}_2(\tau). \text{ By } 23 \text{ we have}

\[(\tilde{v}_1 - \tilde{v}_2)(l) > (\frac{\omega_1 - \omega_2}{D}) \exp(\int_\tau^l \frac{g(t)}{v_1(t)v_2(t)} dt) \int_\tau^l \exp(-t - \int_\tau^t \frac{g(\rho)}{v_1(\rho)v_2(\rho)} d\rho) dt > 0,

\text{(25)}
for \( l \in (\max\{ \lambda_{1L}^*(\omega_1), \lambda_{1L}^*(\omega_2) \}, \tau) \). Since the right hand side of \( \tau \) will not tend to 0 as \( l \rightarrow \max\{ \lambda_{1L}^*(\omega_1), \lambda_{1L}^*(\omega_2) \}^+ \), we conclude that \( \lambda_{1L}^*(\omega_1) < \lambda_{1L}^*(\omega_2) \), if \( \omega_1 < \omega_2 \).

Arguing in the same way as above, we can derive \( \lambda_{1R}^*(\omega_1) > \lambda_{1R}^*(\omega_2) \) from "\( \lambda_{i-1L}^*(\omega_1) < \lambda_{i-1L}^*(\omega_2) \)" and then \( \lambda_{1L}^*(\omega_1) < \lambda_{1L}^*(\omega_2) \) from "\( \lambda_{1R}^*(\omega_1) > \lambda_{1R}^*(\omega_2) \)" for \( i = 2, 3, 4, \ldots \) This completes (i).

Now we want to prove (ii). First consider \( \lambda_{1R}^*(\omega) \). Note that by (i) and Lemma \( \lambda_{10} \) \( \lambda_{1R}^*(\omega) \) exists for any given \( \omega \in (0, \ln \frac{2V_1}{D}) \) and \( l_0 \in (\ln \frac{\omega}{D}, \ln \frac{2V_1}{D}) \).

Suppose \( l_0 = \ln \frac{2V_1}{D} \), i.e., the initial data and parameter pair \((\lambda_{1R}^*(\omega), 0), \omega) \) tends to \((\ln \frac{2V_1}{D}, 0), \omega) \). Then by the "continuous dependence on the initial data and parameter" theorem of ordinary differential equation and Lemma \( \lambda_{10} \) a contradiction occurs, since \( E(\ln \frac{2V_1}{D}, 0) = 0 \) and \( E(l(\omega), v(\omega)) \) is increasing. Therefore we have \( l_0 \in (\ln \frac{\omega}{D}, \ln \frac{2V_1}{D}) \). Also by the "continuous dependence" property, the solution of \( (13), (14) \) with initial data and parameter pair \((l_0, 0), \omega) \) must stay in the region \( \{ (l, v) \in R^2 \mid E(l, v) < 0 \} \) and \( v < 0 \). Then by Lemma 1 in [1] this solution is global existence one. Thus by Lemma 4 and Proposition 5 we have \( l_0 = \lambda_{1R}^*(\omega_0) \) and so \( \lim_{\omega \rightarrow \omega_0} \lambda_{1R}^*(\omega) = \lambda_{1R}^*(\omega_0) \). Hence

\[ \lambda^*(\omega) = \lambda_{1R}^*(\omega) \text{ is a continuous function on } (0, \ln \frac{2V_1}{D}). \]

For \( \lambda_{1L}^*(\omega), i = 1, 2, 3, \ldots \) and \( \lambda_{1R}^*(\omega), i = 2, 3, \ldots \), by the continuity of \( \lambda_{1R}^*(\omega) \) and the "continuous dependence" property of o.d.e. when choosing long enough existence intervals \([s_L, 0], [s_L, 0] \) such that the orbit curve of \( (\tilde{l}(s; \omega), \tilde{v}(s; \omega)) \) intersects the \( l \)-axis \( 2i \) and \( 2i - 1 \) times respectively, both \( \lambda_{1L}^*(\omega) \) and \( \lambda_{1R}^*(\omega) \) will stay close to \( \lambda_{1L}^*(\omega_0) \) and \( \lambda_{1R}^*(\omega_0) \) respectively as \( \omega \) is close to \( \omega_0 \). Therefore \( \lambda_{1L}^*(\omega) \) and \( \lambda_{1R}^*(\omega) \) are continuous functions.

Finally we prove (iii). Namely, the limit \( \xi := \lim_{\omega \rightarrow 0^+} \lambda_{1R}^*(\omega) \) exists with \( \xi \in (\ln \frac{\omega}{D}, \ln \frac{2V_1}{D}) \). Consider the solution of \( (13), (14) \) with initial value-parameter pair \((\xi, 0), 0) \) periodic with some period, say, \( T \in (0, \infty) \). Applying the "continuous dependence" property of o.d.e. when choosing existence interval \([0, T] \), solution \( (\tilde{l}, \tilde{v}) : (\tilde{l}(s; \omega), \tilde{v}(s; \omega)) := (l(s_i; \omega), v(s_i; \omega)) \) over the existence interval \([0, i T] \) where \( l(s_i; \omega), v(s_i; \omega) = (l_{1R}^*(\omega), 0) \) must intersect the "negative" \( l \)-axis (less than \( \ln \frac{\omega}{D} \)) at least \( i \) times as \( \omega \rightarrow 0^+ \). This is a contradiction. Thus \( \lim_{\omega \rightarrow 0^+} \lambda_{1R}^*(\omega) = \ln \frac{2V_1}{D} \).

By (i) again, we have \( \xi := \lim_{\omega \rightarrow 0^+} \lambda_{1L}^*(\omega) \) exists and \( \xi \in [\ln \frac{\omega}{D}, \ln \frac{2V_1}{D}] \). Suppose \( \xi \in (-\infty, \ln \frac{\omega}{D}) \). Now we apply the "continuous dependence" theorem of o.d.e. to the solution of \( (13), (14) \) with the initial-value parameter pair \((\xi, 0), 0) \). When we choose a long enough existence interval \([0, s_R] \) such that the \( l \) value at \( s_R \) of the above solution is less than \( \xi - 1 \), we have \( \lambda_{1L}^*(\omega) < \xi - 1 \) as \( \omega \) is close to 0 enough by \( \lim_{\omega \rightarrow 0^+} \lambda_{1R}^*(\omega) = \ln \frac{2V_1}{D} \) above and the "continuous dependence" property. This contradicts to \( \lambda_{1L}^*(\omega) = \xi \). Thus \( \lim_{\omega \rightarrow 0^+} \lambda_{1L}^*(\omega) = -\infty \) and then (iii) follows. □
Note that by Lemma 6 and its proof we can roughly say that given \( \omega_1 < \omega_2 \), \((l(s; \omega_1), \tilde{\nu}(s; \omega_1))\) traces out an orbit curve which is closer to \((\ln \frac{2\sqrt{v}}{D}, 0)\) than \((l(s; \omega_1), \tilde{\nu}(s; \omega_1))\) does as \( s \to \infty \), i.e., \((l(s; \omega_2), \tilde{\nu}(s; \omega_2))\) is “enclosed” by \((l(s; \omega_1), \tilde{\nu}(s; \omega_1))\).

**Lemma 7.** Given any \( l(0) \in (\infty, \ln \frac{2\sqrt{v}}{D}) \), \( G \), and \( i \in \{0, 1, 2, \ldots \} \), there is at most one \( \omega = \omega(G, i, l(0)) \) among \( (0, \frac{2\sqrt{v}}{D}) \) such that \( (13), (14), (15) \) has a global solution \((l(\cdot; \omega), \nu(\cdot; \omega)) \) on \([0, \infty) \) which has exact \( 2i \) intersection points with the \(-\)axis as \( s \) increases from \( 0 \) to \( \infty \).

**Proof.** Note that any global solution \((l(\cdot; \omega), \nu(\cdot; \omega)) \) of \((13), (14) \) has the same orbit curve as \((\tilde{l}(\cdot; \omega), \tilde{\nu}(\cdot; \omega)) \), i.e., \((l(s; \omega), \nu(s; \omega)) = (\tilde{l}(s + s_0; \omega), \tilde{\nu}(s + s_0; \omega)) \) for some \( s_0 \) by Proposition 5. Therefore without loss of generality, we consider \((l(\cdot; \omega), \nu(\cdot; \omega)) \) on \([0, \infty) \). Fix \( l = l(0) \) and \( G \) in \((13)\). Note that the initial value \((l(0), \nu(0)) \) is on the curve:

\[
v = I(l; \omega, G) := -\frac{\omega}{D} e^{-l} + \frac{G}{D}.
\]

On one hand \( v_0 := \nu(0) = I(l(0); \omega) \) is a decreasing function of \( \omega \) on \((0, \infty) \).

Let \((\tilde{l}_i(\cdot; \omega), \tilde{\nu}_i(\cdot; \omega))\) be the segment of \((\tilde{l}(\cdot; \omega), \tilde{\nu}(\cdot; \omega)) \)'s orbit curve connecting \( l_{i+1}(\cdot; \omega) \) and \( l_i(\cdot; \omega) \) in the lower \(-\)axis plane. Then on the other hand, given \( \omega_1 < \omega_2 \) and \( s_1, s_2 \) such that \( \tilde{l}_i(s_1; \omega_1) = \tilde{l}_i(s_2; \omega_2) = l(0) \), it can be easily seen from Lemma 6 (i) and its proof that \( \tilde{\nu}_i(s_1; \omega_1) < \tilde{\nu}_i(s_2; \omega_2) \), if both \( \tilde{\nu}_i(s_1; \omega_1), \tilde{\nu}_i(s_2; \omega_2) \) are nonpositive, i.e., increasing in \( \omega \). Since there is at most one intersection point for increasing and decreasing functions, this lemma then follows.

Now we state our main result:

**Theorem 8.** (i). Given any \( l(0) \in (\infty, \ln \frac{2\sqrt{v}}{D}) \) and \( i \in \{0, 1, 2, \ldots \} \), there is a corresponding \( G_i = G_i(l(0)) > 0 \) such that for any given \( G \in [0, G_i] \) there is a unique \( \omega = \omega(G; i, l(0)) \in (0, \frac{2\sqrt{v}}{D}) \) such that \((13), (14), (15) \) has a global existence solution \((l(\cdot; \omega), \nu(\cdot; \omega)) \) on \((\infty, \infty) \) which rotates clockwise around \((\ln \frac{2\sqrt{v}}{D}, 0) \), has \( 2i \) or \( 2i + 1 \) (exact \( 2i \) for \( G = 0 \) case) intersection points with the \(-\)axis as \( s \) increases from \( 0 \) to \( \infty \), and after the \( 2i \) or \( 2i + 1 \) intersections we have \( v(s; \omega) < 0, l(s; \omega) \to -\infty, v(s; \omega) \to -\infty \) as \( s \to \infty \). Moreover, given any \( G \in \cap_{i} [0, G_i(l(0))] \), \( \omega(G; \cdot, l(0)) \) is a decreasing function with the properties that \( \omega(G; i, l(0)) \to 0^- \) as \( i \to \infty \) for \( l(0) \in (\infty, \ln \frac{2\sqrt{v}}{D}) \setminus \{ \ln \frac{\sqrt{v}}{D} \} \), \( G \in \cap_{i} [0, G_i(l(0))] \) and \( \omega(0; i, l(0)) \to 0^+ \) as \( i \to \infty \).

(ii). For \( l(0) \in [\ln \frac{2\sqrt{v}}{D}, \infty) \) and any \( G \), we have that \( \omega = 0 \) is the only number such that \((13), (14), (15) \) has global existence solution \((l(\cdot; 0), \nu(\cdot; 0)) \). Moreover, if \( G > 0 \), then there is some \( \theta \in (0, \infty) \) such that \( v(s; 0) > 0 \).
For $s \in [0, \vartheta)$, $v(\vartheta; 0) = 0$, and $v(s; 0) < 0$ for $s \in (\vartheta, \infty)$ with the asymptotic behavior $l(s; 0) \to -\infty$, $v(s; 0) \to 0^-$ as $s \to \infty$; if $G \leq 0$, then $v(s; 0) < 0$ for $s \in (0, \infty)$ with $l(s; 0) \to -\infty$, $v(s; 0) \to 0^-$ as $0 \leq s \to \infty$.

(iii). If $G \in (-\infty, -V_0]$, $i \in \{0, 1, 2, \ldots\}$ has no global existence solutions. When $G \in (-V_0, 0)$, given any $l(0) \in \left(\ln\left[\frac{V_0 - \sqrt{V_0^2 - G^2}}{D}\right], \ln\left[\frac{V_0 + \sqrt{V_0^2 - G^2}}{D}\right]\right)$ and $i \in \{0, 1, 2, \ldots\}$, there is a unique $\omega = \omega(i, l(0); G) \in (0, \frac{2V}{D})$ such that $l(i, \omega), v(i, \omega)$ has a global existence solution $(l(\cdot; \omega), v(\cdot; \omega))$ satisfying all the properties described in (i) (but having exact $2i$ intersection points with the $l-$axis as $s$ increases from 0 to $\infty$ here); on the other hand, $l(i, \omega), v(i, \omega)$ has no global existence solutions whenever $l(0) \notin \left(\ln\left[\frac{V_0 - \sqrt{V_0^2 - G^2}}{D}\right], \ln\left[\frac{V_0 + \sqrt{V_0^2 - G^2}}{D}\right]\right)$ in this case.

First, we explain the physical meaning of the above results and then prove this theorem.

3.1 Interpretation of Theorem $\S$

(i). Remember that $l(0) = \ln \kappa_0$, where $\kappa_0$ is the tip’s curvature and $G$ is the tangential velocity of the tip. Theorem $\S$ (i) tells us that given a weakly excitable medium ($V_0, D$ are fixed), if the tip’s curvature $\kappa_0$ is not too large, i.e., $\kappa_0 \in (0, \frac{2V}{D})$, there is a corresponding range for the tangential velocity of the tip so that for each $G$ in this range there is a unique rotating frequency $\omega(G; i, \ln \kappa_0) \in (0, \frac{2V}{D})$ such that this medium supports a normally propagating, steadily rotating, and growing (since $G \geq 0$) plane curve with angular frequency $\omega(G; i, \ln \kappa_0)$ and tip curvature $\kappa_0$, the curvature of which $\kappa(s) = e^{l(s)}$ changes its monotonicity $2i$ or $2i + 1$ times (exact $2i$ for $G = 0$ case) as arc length $s \to \infty$ and finally decreases to 0 in the Archimedean spiral’s way, i.e., $\lim_{s \to \infty} \kappa(s)^2s = \frac{\omega(G; i, \ln \kappa_0)}{2V_0}$ (see (14) in [22]). Especially, the case $"i = 0, G = 0"$ is just the one appearing in $\S$.

(ii). On the other hand, if the tip’s curvature $\kappa_0$ is too large, i.e., $\kappa_0 \in [\frac{2V}{D}, \infty)$, then the medium supports only “nonrotating” spiral waves ($\omega = 0$). Moreover, given any $G > 0$ ($G \leq 0$, respectively), there is a unique growing (contracting, respectively) spiral wave with tip’s curvature $\kappa_0$ and tangential velocity $G$, the curvature of which changes its monotonicity exactly once (0 time, respectively) and the asymptotic behavior of which is the same as in above (i) but the curvature $\kappa(s)$ decays to 0 in different order, not Archimedean one (see THEOREM 2 of [24]).

(iii) tells us that it is impossible for an excitable medium to support a spiral wave with too large contracting velocity of the tip, i.e., $G \in (-\infty, -V_0]$. For smaller contracting velocity, $G \in (-V_0, 0)$, there is a corresponding curvature’s range of the tip, $(\frac{V_0 - \sqrt{V_0^2 - G^2}}{D}, \frac{V_0 + \sqrt{V_0^2 - G^2}}{D})$, such that when the tip’s curvature $\kappa_0$ does not belong to this range, there is no any spiral wave but when given
and any \( \kappa_0 \) in this range, there is a unique \( \omega(i, \ln \kappa_0; G) \in (0, \frac{2V_0^2}{D}) \) such that this medium supports a spiral wave with tip’s contracting velocity \( G \), tip’s curvature \( \kappa_0 \), rotating frequency \( \omega(i, \ln \kappa_0; G) \), and the same properties as mentioned in (i) above (but the curvature changes its monotonicity exact 2\( i \) times).

Note that we also have similar explanations as above by replacing \( \kappa, \omega, V_0 \) with \(-\kappa, -\omega, -V_0\), respectively when \( \kappa < 0 \).

### 3.2 Proof of Theorem 8

**Proof.** As explained in the proof of Lemma 7, it suffices to consider \((\vec{l}(\cdot; \omega), \vec{v}(\cdot; \omega))\).

Letting \( \omega = \frac{2V_0^2}{D} \), (26) becomes

\[
v = I(l; \frac{2V_0^2}{D}, G) := -2\frac{V_0}{D} e^{-l} + \frac{G}{D}.
\] (27)

Plugging (27) into \( E(l, v) \), we obtain

\[
E(l, -2\frac{V_0}{D} e^{-l} + \frac{G}{D}) = \frac{1}{2} e^{-2l}[4\frac{V_0}{D}]^4 + e^{3l} - \frac{2V_0}{D} e^{3l} + (\frac{G}{D})^2 e^{2l} - 4\frac{G}{D} \frac{V_0}{D} e^{l}.
\] (28)

It can be easily derived that “\( x^4 - 2ax^3 + 4a^4 > 0 \), \( \forall x \in R, a \neq 0 \)” which implies \( E(l, -2\frac{V_0}{D} e^{-l}) > 0 \) for all \( l \in R \) (i.e., \( G = 0 \)). Hence given any \( l \), we have that \( I(l; \frac{2V_0^2}{D}, 0) \lhd \) the negative \( v \) value of curve “\( E(l, v) = 0 \)” , i.e., curve (27) lies below \( E(l, v) = 0 \) when \( G = 0 \). Then by continuity, there is a \( G(l) > 0 \) such that \( I(l; \frac{2V_0^2}{D}, G) \leq \) the negative \( v \) value of curve “\( E(l, v) = 0 \)” , whenever \( G \in (-\infty, G(l)] \). From above observation and Lemma 8, we have that if there is some \( \omega > 0 \) such that (13), (14), (15) has global solutions, then \( \omega \in (0, \frac{2V_0^2}{D}) \). Thus we have obtained the uniqueness property by Lemma 7. It is left for (i) to prove the “existence” of such \( \omega \). The strategy is to find suitable \( \omega \) such that there are intersections points between the curve (26) and some special segment of the orbit curve of \((\vec{l}(\cdot; \omega), \vec{v}(\cdot; \omega))\), which implies (13), (14), (15) has global solutions with the properties mentioned in this theorem. By Lemma 8, the ranges of \( l^*_i \) \( R(\cdot) \) and \( l^*_{0, L}(\cdot) \) over \((0, \frac{2V_0^2}{D})\) are open intervals, say, \((R_{i, 1}, \ln \frac{2V_0^2}{D})\) and \((-\infty, L_i)\) respectively, where \( l^*_{0, L}(\cdot) := -\infty \) and \( L_0 := -\infty \). For any \( i \in \{0, 1, 2, \ldots\} \) and \( l(0) \in (L_i, R_{i, 1}) \), \( \vec{v}_i(l(0); \omega) \) is well-defined for any \( \omega \in (0, \frac{2V_0^2}{D}) \), where \( \vec{v}_i(l; \omega) := \vec{v}(s; \omega) \) if there is some \( s \) such that \( \vec{l}(s; \omega) = l \) and \( \vec{l}(s; \omega), \vec{v}(s; \omega) \) is on the “lower” orbit curve connecting \( l^*_i \) \( R(\omega) \) and \( l^*_{0, L}(\omega) \). Thus given any \( l(0) \in (L_i, R_{i, 1}) \) and \( G \in (-\infty, G_i(l(0))) \), where \( G_i(\cdot) := \gamma(\cdot) \), there is some positive \( \omega_{P_i} \) close to \( \frac{2V_0^2}{D} \) such that \( \vec{v}_i(l(0); \omega_{P_i}) \) > the negative \( v \) value of “\( E(l(0), v) = 0 \)” \( \geq I(l(0); \omega_{P_i}, G) \) by Lemma 8 the continuity of \( I(l(0); \cdot, G) \), and the observation in the beginning of this proof; for \( l(0) = L_i \) or \( R_{i, 1} \), it still holds by Lemma 8 (i),(ii). Now consider \( l(0) \in (R_{i, 1}, \ln \frac{2V_0^2}{D}) \) or \( l(0) \in (-\infty, L_i) \). By definition, there is \( \omega_0 \in (0, \frac{2V_0^2}{D}) \) such that \( l^*_i R(\omega_0) = l(0) \) (or
\( l_{1L}(\omega_0) = l(0) \). We require now that \( I(l(0); \omega_0, G) \leq 0 \) which is equivalent to

\[
G \leq \omega_0 e^{-l(0)}.
\]

(29)

Let \( G_i(l) := \omega_0 e^{-l} \). Therefore, given any \( G \in (-\infty, G_i(l(0))] \), there is also some \( \omega_P \) close to \( \omega_0 \) such that \( \tilde{v}_i(l_0(0); \omega_P) \geq I(l(0); \omega_P, G) \) by virtue of Lemma 6 (i), (ii). On the other hand, given any \( G \in [0, \infty) \) and \( l(0) \in (-\infty, \ln \frac{2\sqrt{v}}{D}) \), there exists some positive \( \omega_N \) close to 0 such that \( \tilde{v}_i(l(0); \omega_N) \) is well-defined and \( \tilde{v}_i(l(0); \omega_N) < I(l(0); \omega_N, G) \), since \( \lim_{\omega \to 0^+} \tilde{v}_i(l(0); \omega, G) = \frac{G}{\omega} \geq 0 \) and

\[
\lim_{\omega \to 0^+} \tilde{v}_i(l(0); \omega) = -\text{the negative } \nu \text{ value of "} E(l(0), v) = 0 \text{" by Lemma 6 (iii)}.
\]

and the continuous dependence theorem of o.d.e. Note that \( \tilde{v}_i(l(0); \cdot) \) is a continuous function on \( [\omega_N, \omega_P] \) by Lemma 6 (ii). Obviously \( I(l(0); \cdot, G) \) is also a continuous function. Therefore choosing \( l(0) \in (-\infty, \ln \frac{2\sqrt{v}}{D}) \) and then \( G \in [0, G_i(l(0))) \), there must be some \( \sigma_i \in (\omega_N, \omega_P) \subset (0, \frac{2\sqrt{v}}{D}) \) such that \( \tilde{v}_i(l(0); \sigma_i) = I(l(0); \sigma_i, G) \) by the intermediate value theorem of continuous functions. Let \( \omega(G; i, l(0)) := \sigma_i \). Hence “existence” is guaranteed. Given \( G \in \cap_i [0, G_i(l(0))] \), the decreasingness of \( \omega(G; \cdot, l(0)) \) as \( i \to \infty \) immediately follows from the explanation after Lemma 6 i.e., “\( \tilde{v}(s; \omega_2), \tilde{v}(s; \omega_1) \) is enclosed by \( \tilde{v}(s; \omega_1), \tilde{v}(s; \omega_1) \) as \( \omega_1 < \omega_2 \)” and the decreasingness of \( I(l(0); \cdot, 0) \). Now we prove \( \lim_{l \to \infty} \omega(G; i, l(0)) = 0 \) for \( l(0) \in (-\infty, \ln \frac{2\sqrt{v}}{D}) \setminus \{ \ln \frac{\sqrt{v}_2}{D} \} \), \( G \in \cap_i [0, G_i(l(0))] \)

and \( \lim_{l \to \infty} l_{1L}(i, l(0)) = 0 \). For the first part, assume the contrary that \( \lim_{l \to \infty} \omega(G; i, l(0)) = \omega_f(G; l(0)) > 0 \). Note that \( \lim_{l \to \infty} l_{1L}^*(\omega_f(G; l(0))) = \lim_{l \to \infty} l_{1L}(\omega_f(G; l(0))) = \ln \frac{\sqrt{v}_2}{D} \).

By \( l(0) \neq \ln \frac{\sqrt{v}_2}{D} \), there must be some \( i \) large enough such that \( l(0) \in \left( l_{i+1}^*, l_{i+1}^*(\omega_f(G; l(0))) \right) \subset (R_{i+1}, \ln \frac{2\sqrt{v}}{D}) \) or \( l(0) \in (-\infty, l_{i}^*(\omega_f(G; l(0)))) \subset (-\infty, L_i) \). Arguing as before, then there is some \( \omega \in (0, \omega_f(G; l(0))) \) such that \( \tilde{v}_i(l(0); \omega) < I(l(0); \omega, G) \). By uniqueness, this \( \omega \) must be \( \omega(G; i, l(0)) \) which leads to a contradiction, since it is impossible that \( \omega(G; i, l(0)) < \omega_f(G; l(0)) \). For the second part, note that \( \tilde{v}_i(l(0); \omega_f; \ln \frac{\sqrt{v}_2}{D}) = 0 \) as explained in the last of the proof of Lemma 6. Thus for \( i \) large enough we have \( \tilde{v}_i(l(0); \omega_f; \ln \frac{\sqrt{v}_2}{D}) > I(l(0); \omega_f; \ln \frac{\sqrt{v}_2}{D}) \), \( I(l(0); \omega_f; \ln \frac{\sqrt{v}_2}{D}) < 0 \). Then again by Lemma 6 (iii) and the intermediate value theorem, there must be some \( \omega \in (0, \omega_f; \ln \frac{\sqrt{v}_2}{D}) \) such that \( \tilde{v}_i(l(0); \omega) = I(l(0); \omega) \) which also leads to a contradiction. This completes (i).

Part (ii) follows from Lemma 6 and THEOREM 2 in [24] immediately.

Now we prove (iii). When \( G \in (-\infty, -V_0) \), the curve \( l(0; l, \omega, G) \) entirely lies outside the curve “\( E(l, v) = 0 \)” for any \( \omega > 0 \). Thus by Lemma 6 we have that \( l_{1L}(0; l, \omega, G) \) has no global existence solutions. When \( G \in (-V_0, 0) \), the parts of the curve \( l(0; l, \omega, G) \), where \( l(0) \notin (\ln \left[ \frac{V_0 - \sqrt{\omega^2 - G^2}}{D} \right], \ln \left[ \frac{V_0 + \sqrt{\omega^2 - G^2}}{D} \right] \) also lie outside the curve “\( E(l, v) = 0 \)” for any \( \omega > 0 \). As explained above, we only need to consider \( l(0) \in (\ln \left[ \frac{V_0 - \sqrt{\omega^2 - G^2}}{D} \right], \ln \left[ \frac{V_0 + \omega^2 - G^2}{D} \right] \) now. Note that the right-hand side of \( l_{1L}(0; l, \omega, G) \) is greater than 0 \( \forall l \), for any \( G < 0 \) and [24] is also valid. Arguing as in (i), we then have completed (iii).
Therefore we have finished the proof. ■

4 Numerical Results

The following formulas can be referred to [40, p. 45].

Let

\[ \theta_0(t) = \omega t + \theta_0(0) \]  \hspace{1cm} (30)

be the angle of the tip at time \( t \), where \( \omega \) is the rotating frequency determined by Theorem 8 and \( \theta_0(0) \) is the initial angle of the tip, which can be chosen arbitrary. Let also \( \theta(s, t) \) be the angle of the position away from the tip at a distance “arclength \( s \)” at time \( t \). Then

\[ \theta(s, t) = \theta_0(t) - \int_0^s e^{l(\rho)} d\rho, \]  \hspace{1cm} (31)

where \( l(s) \) is the function from Theorem 8.

The cartesian coordinates of the tip at time \( t \) are given by

\[
\begin{align*}
\frac{d}{dt}X_0(t) &= (De^{l(0)} - V_0) \sin \theta_0(t) - G \cos \theta_0(t) = (De^{l(0)} - V_0) \sin[\omega t + \theta_0(0)] - G \cos[\omega t + \theta_0(0)] \quad \text{and} \\
\frac{d}{dt}Y_0(t) &= \left(V_0 - De^{l(0)}\right) \cos \theta_0(t) - G \sin \theta_0(t) = \left(V_0 - De^{l(0)}\right) \cos[\omega t + \theta_0(0)] - G \sin[\omega t + \theta_0(0)],
\end{align*}
\]

(32)

where the initial condition \((X_0(0), Y_0(0))\) can be chosen arbitrarily. Then, the cartesian coordinates \((X(s, t), Y(s, t))\) of the position away from the tip at a distance “arclength \( s \)” at time \( t \) are

\[
\begin{align*}
X(s, t) &= \int_0^s \cos \theta(\xi, t) d\xi + X_0(t) = \int_0^s \cos[\omega t + \theta_0(0)] - \int_0^\xi e^{l(\rho)} d\rho d\xi + X_0(t) \\
Y(s, t) &= \int_0^s \sin \theta(\xi, t) d\xi + Y_0(t) = \int_0^s \sin[\omega t + \theta_0(0)] - \int_0^\xi e^{l(\rho)} d\rho d\xi + Y_0(t),
\end{align*}
\]

(33)

which is the same as (4.43) in [40, p. 45].

Now we explain how to determine the parameters \( D, V_0, \theta_0(0), l(0), G, \omega \) and then to draw the pictures of spiral waves in Theorem 8.

First, parameters \( D > 0, V_0 > 0, \theta_0(0) \) can be chosen arbitrarily.

In Theorem 8 (i), we choose \( l(0) \in (-\infty, \ln \frac{2V_0}{D}) \), \( i \in \{0, 1, 2, 3, \ldots\} \) first. Then we choose \( G \in [0, G_i(l(0))] \) (for example choose \( G = 0 \)). Finally, there is a unique \( \omega = \omega(G; i, l(0)) \in (0, \frac{2V_0}{D}) \) by Theorem 8 (i). Plug above parameters into (32) and (33) and then the corresponding spiral wave has the properties described in Theorem 8 (i).

For (ii), we choose \( l(0) \in [\ln \frac{2V_0}{D}, \infty) \), \( \omega = 0 \), and any \( G \). Plug above parameters into (32) and (33) and then the corresponding spiral wave has the properties described in Theorem 8 (ii).
For (iii), we first choose $G \in (-V_0, 0)$ and then $l(0) \in \left(\ln\left[\frac{V_0 - \sqrt{V_0^2 - G^2}}{B}\right], \ln\left[\frac{V_0 + \sqrt{V_0^2 - G^2}}{B}\right]\right)$ and any $i \in \{0, 1, 2, 3, \ldots\}$. Finally there is a unique $\omega(i, l(0); G) \in (0, \frac{2V_0}{B})$. Also, plugging above parameters into (32) and (33), the corresponding spiral wave then has the properties described in Theorem 8 (iii).

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