On Farkas Lemma and Dimensional Rigidity of bar Frameworks *

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Abstract

We present a new semidefinite Farkas lemma involving a side constraint on the rank. This lemma is then used to present a new proof of a recent characterization, by Connelly and Gortler [7], of dimensional rigidity of bar frameworks.

1 Introduction

The celebrated Farkas lemma is at the core of optimization theory. It underpins duality theory of linear programming, and its semidefinite version plays a key role in strong duality results of semidefinite programming. As an example of theorems of the alternative, Farkas lemma establishes the infeasibility of a given linear matrix inequality by exhibiting a solution for another linear matrix inequality. In this paper, we present a new semidefinite Farkas lemma (Theorem 2.2 below) involving a side constraint on the rank. This Farkas lemma is then used to provide a new proof of a recent characterization, by Connelly and Gortler [7], of dimensional rigidity of bar frameworks.

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A bar framework in $\mathbb{R}^r$, denoted by $(G, p)$, is a simple connected undirected graph $G = (V, E)$ whose nodes are points $p^1, \ldots, p^n$ in $\mathbb{R}^r$; and whose edges are line segments, each joining a pair of these points. We say that $(G, p)$ is $r$-dimensional if the points $p^1, \ldots, p^n$ affinely span $\mathbb{R}^r$.

Let $(G, p)$ and $(G, p')$ be two $r$-dimensional and $s$-dimensional frameworks in $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively. Then $(G, p')$ is equivalent to $(G, p)$ if:

$$||p'^i - p'^j||^2 = ||p^i - p^j||^2$$

for each $\{i, j\} \in E(G)$, \hspace{1cm} (1)

where $||.||$ denotes the Euclidean norm and $E(G)$ denotes the edge set of $G$. Moreover, $(G, p')$ is said to be affinely equivalent to $(G, p)$ if $(G, p')$ is equivalent to $(G, p)$ and $p'^i = Ap^i + b$ for all $i = 1, \ldots, n$, where $A$ is an $r \times r$ matrix and $b$ is a vector in $\mathbb{R}^r$. Finally, two $r$-dimensional frameworks $(G, p)$ and $(G, p')$ in $\mathbb{R}^r$ are congruent if:

$$||p'^i - p'^j||^2 = ||p^i - p^j||^2$$

for all $i, j = 1, \ldots, n$. \hspace{1cm} (2)

An $r$-dimensional framework $(G, p)$ is said to be dimensionally rigid if no $s$-dimensional framework $(G, p')$, for any $s \geq r + 1$, is equivalent to $(G, p)$. On the other hand, if every $s$-dimensional framework $(G, p')$, for any $s$, that is equivalent to $(G, p)$ is in fact congruent to $(G, p)$, then framework $(G, p)$ is said to be universally rigid. It turns out that dimensional rigidity and universal rigidity are closely related.

**Theorem 1.1** (Alfakih [1]). Let $(G, p)$ be an $r$-dimensional bar framework on $n$ vertices in $\mathbb{R}^r$, for some $r \leq n - 2$. Then $(G, p)$ is universally rigid if and only if the following two conditions hold:

1. $(G, p)$ is dimensionally rigid.

2. There does not exist an $r$-dimensional framework $(G, p')$ in $\mathbb{R}^r$ that is affinely equivalent, but not congruent, to $(G, p)$.

The notion of a stress matrix plays a key role in the study of universal and dimensional rigidities. An equilibrium stress (or simply a stress) of $(G, p)$ is a real-valued function $\omega$ on $E(G)$ such that:

$$\sum_{j : \{i, j\} \in E(G)} \omega_{ij}(p^i - p^j) = 0$$

for all $i = 1, \ldots, n$. \hspace{1cm} (3)

Let $E(\overline{G})$ denote the edge set of graph $\overline{G}$, the complement graph of $G$, i.e.,

$$E(\overline{G}) = \{\{i, j\} : i \neq j, \{i, j\} \notin E(G)\},$$

$$2$$
and let $\omega = (\omega_{ij})$ be a stress of $(G, p)$. Then the $n \times n$ symmetric matrix $\Omega$ where
\[
\Omega_{ij} = \begin{cases} 
-\omega_{ij} & \text{if } \{i, j\} \in E(G), \\
0 & \text{if } \{i, j\} \in E(G), \\
\sum_{k: \{i,k\} \in E(G)} \omega_{ik} & \text{if } i = j,
\end{cases}
\]
is called the stress matrix associated with $\omega$, or a stress matrix of $(G, p)$.

The following result provides a sufficient condition for the dimensional rigidity of a given framework.

**Theorem 1.2** (Alfakih [1]). Let $(G, p)$ be an $r$-dimensional bar framework on $n$ vertices in $\mathbb{R}^r$, for some $r \leq n - 2$. Then $(G, p)$ is dimensionally rigid if it admits a positive semidefinite stress matrix $\Omega$ of rank $n - r - 1$.

Unfortunately, the sufficient condition in Theorem 1.2 is not necessary as was shown by Example 3.1 in [1] (see also Figure 1). Recently, Connelly and Gortler [7] bridged the gap between necessary and sufficient conditions for dimensional rigidity. Theorem 3.4 below is a refined version of their main result in [7] concerning dimensional rigidity.

The remainder of the paper is organized as follows. In Section 2 we review basic results on the facial structure of the semidefinite cone and we present our new Farkas lemma. The proof of this lemma is based on the Borwein-Wolkowicz facial reduction algorithm [4, 5]. In Section 3 we review basic results concerning the dimensional rigidity of bar frameworks, and we use our new Farkas lemma to present a proof of the Connelly-Gortler characterization of dimensional rigidity in [7]. Finally, numerical examples are presented in Section 4 to illustrate the results of the paper.

**1.1 Notation**

For the convenience of the reader, we collect here the notation used throughout the paper. $I_n$ denotes the identity matrix of order $n$. $0$ denote the zero vector or matrix of appropriate dimension. We denote by $e$ the vector of all 1’s in $\mathbb{R}^n$, and by $e^i$ we denote the $i$th standard unit vector in $\mathbb{R}^n$. For $i < j$, we let
\[
F^{ij} = (e^i - e^j)(e^i - e^j)^T.
\]
$||.||$ denotes the Euclidean norm. $S^n$ denotes the space of $n \times n$ symmetric real matrices. The set of $n \times n$ symmetric real positive semidefinite (positive definite) matrices is denoted by $S^n_+$ ($S^n_{++}$). We sometimes also use $A \succeq 0 (A \succ 0)$ to mean that $A$ is symmetric positive semidefinite (positive definite). We denote the relative
Figure 1: A 2-dimensional universally rigid bar framework in the plane, where the set of missing edges is \( E(G) = \{\{1, 4\}, \{3, 5\}\} \). It admits a positive semidefinite stress matrix of rank 1 but not of rank 2. The edge \( \{2, 4\} \) is drawn as an arc to make edges \( \{2, 3\} \) and \( \{3, 4\} \) visible.

interior of a set \( S \) in \( S^n \) by relint(\( S \)). For a matrix \( A \), \( \mathcal{N}(A) \) and \( \mathcal{R}(A) \) denote, respectively, the null space and the column space (or the range) of \( A \). The trace of \( A \) is denoted by \( \text{tr} (A) \). \( E(G) \) denotes the edge set of a simple graph \( G \), while \( E(G) \) denotes the edge set of the complement graph of \( G \), i.e., \( E(G) = \{\{i, j\} : i \neq j, \{i, j\} \notin E(G)\} \).

2 Facial Reduction and Farkas Lemma

The proof of Theorem 2.2 below relies on the Borwein-Wolkowicz facial reduction algorithm \[1, 5\]. Thus we start this section by reviewing definitions and basic facts concerning the facial structure of the positive semidefinite cone \( S^n_+ \). For other applications of facial reduction see \[8, 9, 6\].

2.1 Facial Structure of \( S^n_+ \)

A subset \( K \in S^n \), the space of \( n \times n \) symmetric real matrices, is a \textit{cone} if for each \( X \in K \) and each \( \lambda \geq 0 \) we have \( \lambda X \in K \). Let \( K \) be a convex cone in \( S^n \). A subset \( F \subseteq K \) is a \textit{face} of \( K \) if for every \( X, Y \in K \) such that \( (X + Y) \in F \), it follows that \( X \in F \) and \( Y \in F \). A face \( F \) of convex cone \( K \) is said to be \textit{exposed} if there exists an \( A \in S^n \) such that \( F = \{X \in K : \text{tr} (AX) = 0\} \). A convex cone \( K \) is \textit{facially exposed} if every face \( F \) of \( K \) is exposed. Let \( S \) be a subset of a convex cone \( K \),
then the intersection of all faces of $K$ containing $S$ is called the minimal face of $S$, denoted by face($S$). It is easy to show that face($S$) is indeed a face of $K$. Moreover, if $S = \{A\}$, we write face($A$) instead of face($\{A\}$).

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It easy to see that $S^n_+$, the set of $n \times n$ symmetric positive semidefinite real matrices, is a closed convex cone. The faces of $S^n_+$ are well known to be in a one-to-one correspondence with the subspaces of $\mathbb{R}^n$ \cite{2 3 11}. In fact, $F$ is a face of $S^n_+$ if and only if

$$F = \{X \in S^n_+ : \mathcal{L} \subseteq \mathcal{N}(X)\},$$

for some subspace $\mathcal{L}$ of $\mathbb{R}^n$, where $\mathcal{N}(X)$ denotes the null space of $X$. Moreover,

$$\text{relint}(F) = \{X \in S^n_+ : \mathcal{L} = \mathcal{N}(X)\}.$$  \hspace{1cm} (7)

Thus, faces of $S^n_+$ are uniquely characterized by their relative interior. Hence, we have the following theorem.

**Theorem 2.1** \cite{2 3 11}. Let $A \in S^n_+$ of rank $r$ and let $A = [W \ U] \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W^T \\ U^T \end{bmatrix}$ be the spectral decomposition of $A$, where $\Lambda$ is the $r \times r$ diagonal matrix consisting of the positive eigenvalues of $A$. Then

$$\text{face}(A) = \{X \in S^n_+ : XU = 0\},$$  \hspace{1cm} (8)

$$= \{X \in S^n_+ : X = WYW^T \text{ for some } Y \in S^r_+\}. \hspace{1cm} (9)$$

Note that $A$ belongs to relint(face($A$)). Let $\mathcal{R}(A)$ denote the column space of $A$. Then for any $B$ in $S^n_+$ such that $\mathcal{R}(B) \subset \mathcal{R}(A)$, it follows that face($B$) $\subset$ face($A$). Hence, if $X \in$ face($A$), then rank $X \leq$ rank $A$. Moreover,

if $\mathcal{R}(B) = \mathcal{R}(A)$, then face($B$) = face($A$).

**Remark 2.1.** Observe that face($A$) in Theorem 2.1 has dimension $r(r+1)/2$. More precisely, face($A$) is isomorphic to $S^r_+$. Thus the faces of $S^n_+$ are isomorphic to smaller dimensional positive semidefinite cones. Furthermore, it is easy to see that $S^n_+ = \text{face}(I_n)$.

The following lemma can be used to provide a characterization of dimensional rigidity.

**Lemma 2.1.** Let $A^1, \ldots , A^m$ be given $n \times n$ symmetric matrices and let $b$ be a given nonzero vector in $\mathbb{R}^m$. Further, let $\mathcal{F} = \{X \in S^n_+ : \text{tr} (XA^i) = b_i \text{ for } i = 1, \ldots , m\}$. Assume that $X^* \in \mathcal{F}$ such that rank $X^* = r$. Then there does not exist an $X \in \mathcal{F}$ such that rank $X \geq r + 1$ if and only if $\mathcal{F} \subseteq \text{face}(X^*)$.  \hspace{1cm} (5)
Proof. Assume that $\mathcal{F} \subseteq \text{face}(X^*)$. Then rank $X \leq \text{rank } (X^*) = r$ for all $X \in \mathcal{F}$ since $\mathcal{R}(X) \subseteq \mathcal{R}(X^*)$.

To prove the other direction assume that rank $(X^*) = r = \max \{\text{rank } X : X \in \mathcal{F}\}$. Let $X'$ be any matrix in $\mathcal{F}$ and let $X = \alpha X^* + (1-\alpha)X'$ for some $\alpha : 0 < \alpha < 1$. Then $X \in \mathcal{F}$ since $\mathcal{F}$ is convex. Furthermore, $\mathcal{N}(X) = \mathcal{N}(X^*) \cap \mathcal{N}(X')$. Thus, $\mathcal{N}(X) \subseteq \mathcal{N}(X')$ and $\mathcal{N}(X) \subseteq \mathcal{N}(X^*)$. Hence, rank $X \geq r$. But, $X \in \mathcal{F}$. Thus, rank $X = r$. Consequently, $\mathcal{N}(X) = \mathcal{N}(X^*)$. Therefore, $\mathcal{N}(X^*) \subseteq \mathcal{N}(X')$. Hence, $X' \subseteq \text{face}(X^*)$ and thus $\mathcal{F} \subset \text{face}(X^*)$.

\[\Box\]

Remark 2.2. In fact, it follows from Lemma 2.1 that there does not exist an $X \in \mathcal{F}$ such that rank $X = r + 1$ if and only if $\text{face}(\mathcal{F}) = \text{face}(X^*)$. This follows since $X^* \in \mathcal{F}$ implies that $\text{face}(X^*) \subseteq \text{face}(\mathcal{F})$. On the other hand, $\mathcal{F} \subseteq \text{face}(X^*)$ implies that $\text{face}(\mathcal{F}) \subseteq \text{face}(X^*)$.

The following lemma plays a key role in this paper.

Lemma 2.2. Let $A^1, \ldots, A^m$ be given $n \times n$ symmetric matrices and let $b = (b_i)$ be a given nonzero vector in $\mathbb{R}^m$. Let

$$\mathcal{F} = \{X \in S^n : \text{tr } (XA^i) = b_i \text{ for } i = 1, \ldots, m\}.$$ 

Further, let $X^* \in \mathcal{F}$ and let $\mathcal{U}_j$ be a matrix with full column rank. If the following two conditions hold:

1. $\mathcal{F} \subset \text{face}(\mathcal{U}_j \mathcal{U}_j^T)$,

2. There exists $\Omega^j = \sum_{i=1}^m x^j_i A^i$ such that $\mathcal{U}_j^T \Omega^j \mathcal{U}_j \geq 0, \neq 0$ and $\text{tr } (\Omega^j X^*) \leq 0$.

Then

$$\mathcal{F} \subset \text{face}(\mathcal{U}_{j+1} \mathcal{U}_{j+1}^T) \subset \text{face}(\mathcal{U}_j \mathcal{U}_j^T),$$

(10)

where $\mathcal{W}_j$ is a full column rank matrix such that $\mathcal{R}(\mathcal{W}_j) = \mathcal{N}(\mathcal{U}_j \mathcal{U}_j^T)$ and $\mathcal{U}_{j+1} = \mathcal{U}_j \mathcal{W}_j$.

Proof. $\mathcal{F} \subset \text{face}(\mathcal{U}_j \mathcal{U}_j^T)$ implies that $\mathcal{F} = \{X = \mathcal{U}_j Y \mathcal{U}_j^T : Y \succeq 0, \text{tr } (XA^i) = b_i \text{ for } i = 1, \ldots, m\}$. Then for every $X \in \mathcal{F}$ we have

$$\text{tr } (X \Omega^j) = \sum_{i=1}^m x^j_i \text{ tr } (\mathcal{U}_j Y \mathcal{U}_j^T A^i) = \sum_{i=1}^m x^j_i b_i = \sum_{i=1}^m x^j_i \text{ tr } (X^* A^i) = \text{ tr } (\Omega^j X^*) \leq 0.$$ 

But $\text{tr } (X \Omega^j) = \text{ tr } (\mathcal{U}_j^T \Omega^j \mathcal{U}_j Y)$. Therefore, $\mathcal{U}_j^T \Omega^j \mathcal{U}_j Y = 0$ since both $Y \succeq 0$ and $\mathcal{U}_j^T \Omega^j \mathcal{U}_j \succeq 0$. Hence, $Y = \mathcal{W}_j Y_j \mathcal{W}_j^T$ for some $Y_j \succeq 0$. Hence, $\mathcal{F} = \{X = \mathcal{U}_{j+1} Y_j \mathcal{U}_{j+1}^T : Y_j \succeq 0, \text{tr } (X^* A^i) = b_i \text{ for } i = 1, \ldots, m\}$; i.e., $\mathcal{F} \subset \text{face}(\mathcal{U}_{j+1} \mathcal{U}_{j+1})$. The result follows since $\mathcal{R}(\mathcal{U}_{j+1}) \subset \mathcal{R}(\mathcal{U}_j)$.

\[\Box\]
Remark 2.3. In Lemma 2.2, suppose that $U_j$ is $n \times s$. Thus, $\text{face}(U_jU_j^T)$ is isomorphic to $S_s^+$. Now if rank $(U_j^T\Omega U_j) = \delta$, then $W_j$ is $s \times (s - \delta)$ and hence, $U_{j+1}$ is $n \times (s - \delta)$ with full column rank. Consequently, $\text{face}(U_{j+1}U_{j+1}^T)$ is isomorphic to $S_s^{(s-\delta)}$. Therefore, the higher the rank of $(U_j^T\Omega U_j)$ is, the larger the difference between the dimension of $\text{face}(U_jU_j^T)$ and the dimension of $\text{face}(U_{j+1}U_{j+2}^T)$ will be.

2.2 A New Farkas Lemma

The following semidefinite Farkas lemma is well known. It is used to establish strong duality for semidefinite programming under Slater condition (see e.g [10]). It will also be used repeatedly in our proofs.

Lemma 2.3. Let $A^1, \ldots, A^m$ be given $n \times n$ symmetric matrices and let $b = (b_i)$ be a given nonzero vector in $\mathbb{R}^m$. Further, let

$$\mathcal{F} = \{ X \in S_n^+ : \text{tr}(XA^i) = b_i \text{ for } i = 1, \ldots, m \}.$$

Assume that there exists an $X^* \in \mathcal{F}$. Then exactly one of the following two statements holds:

1. There exists an $X \in \mathcal{F}$ such that $\text{rank } X \geq r + 1$.

2. There exist nonzero matrices $\Omega_0, \Omega_1, \ldots, \Omega_k$, for some $k \leq s - r$, such that:
   
   - (a) $\Omega^j = \sum_{i=1}^m x_i^j A^i (j = 0, 1, \ldots, k)$ for some scalars $x_i^j$,
   - (b) $U_j^T \Omega U_j \succeq 0$ for $j = 0, 1, \ldots, k$,
   - (c) $\text{tr}(X^* \Omega^j) \leq 0$ for $j = 0, 1, \ldots, k$.

Now are ready to state and prove our new semidefinite Farkas lemma.

Theorem 2.2. Let $A^1, \ldots, A^m$ be given $n \times n$ symmetric matrices and let $b = (b_i)$ be a given nonzero vector in $\mathbb{R}^m$. Let

$$\mathcal{F} = \{ X \in S_n^+ : \text{tr}(XA^i) = b_i \text{ for } i = 1, \ldots, m \}.$$

Let $U_0$ be a given $n \times s$ matrix with full column rank, and assume that $\mathcal{F} \subset \text{face}(U_0U_0^T)$. Let $X^* = U_0Y^*U_0^T$ be a matrix in $\mathcal{F}$ such that rank $X^* = r$, $r \leq s - 1$. Then exactly one of the following two statements holds.

1. There exists an $X$ in $\mathcal{F}$ such that rank $X \geq r + 1$.

2. There exist nonzero matrices $\Omega^0, \Omega^1, \ldots, \Omega^k$, for some $k \leq s - r$, such that:
   
   - (a) $\Omega^j = \sum_{i=1}^m x_i^j A^i (j = 0, 1, \ldots, k)$ for some scalars $x_i^j$,
   - (b) $U_j^T \Omega U_j \succeq 0$ for $j = 0, 1, \ldots, k$,
   - (c) $\text{tr}(X^* \Omega^j) \leq 0$ for $j = 0, 1, \ldots, k$,
(d) \( \text{rank } (U_0^T \Omega^0 U_0) + \text{rank } (U_1^T \Omega^1 U_1) + \cdots + \text{rank } (U_k^T \Omega^k U_k) = s - r, \)

where \( U_1, \ldots, U_{k+1} \), and \( W_0, W_1, \ldots, W_k \) are full column rank matrices defined as follows: For \( i = 0, 1, \ldots, k \), \( \mathcal{R}(W_i) = \mathcal{N}(U_i^T \Omega^1 U_i) \) and \( U_{i+1} = U_i W_i \).

Before presenting the proof of Theorem 2.2, we outline the key idea and intuition behind it. By Remark 2.2, Statement 1 of Theorem 2.2 does not hold if and only if \( \text{face}(\mathcal{F}) = \text{face}(X^*) \). Borwein and Wolkowicz [4, 5] presented a facial reduction algorithm for finding \( \text{face}(\mathcal{F}) \). At each step of this algorithm, a smaller dimensional face of \( S_n^+ \) containing \( \text{face}(\mathcal{F}) \) is found. Thus, this algorithm will find matrices \( U_1, \ldots, U_{k+1} \) such that

\[
\text{face}(\mathcal{F}) = \text{face}(U_{k+1} U_{k+1}^T) \subset \cdots \subset \text{face}(U_1 U_1^T) \subset \text{face}(U_0 U_0^T),
\]

where \( \mathcal{R}(U_{k+1}) \subset \cdots \subset \mathcal{R}(U_1) \subset \mathcal{R}(U_0) \). Hence, Statement 1 in Theorem 2.2 does not hold if and only if \( \text{face}(U_{k+1} U_{k+1}^T) = \text{face}(X^*) \) if and only if \( \mathcal{R}(U_{k+1}) = \mathcal{R}(X^*) \).

**Proof.** First, we prove that if Statement 1 does not hold, then Statement 2 holds. Therefore, assume that there does not exist an \( X \in \mathcal{F} \) such that rank \( X \geq r + 1 \), i.e., assume that \( \text{face}(\mathcal{F}) = \text{face}(X^*) \). Then, there does not exist an \( s \times s \) matrix \( Y \succ 0 \) such that \( \text{tr}(U_0^T A U_0) = b_i \) for \( i = 1, \ldots, m \). Thus by Lemma 2.3 there exists \( \Omega^0 = \sum_{i=1}^m x_i^0 A^i \) such that \( U_0^T \Omega^0 U_0 \succeq 0, \neq 0 \) and \( \text{tr}(X^* \Omega^0) \leq 0 \). If rank \( (U_0^T \Omega^0 U_0) = s - r \), then we are done and \( k = 0 \) in the theorem. Therefore assume that rank \( (U_0^T \Omega^0 U_0) = s - r - \delta_1 \), where \( \delta_1 \geq 1 \), and let \( W_0 \) be a full column rank matrix such that \( \mathcal{R}(W_0) = \mathcal{N}(U_0 \Omega^0 U_0) \). Since \( \mathcal{F} \subset \text{face}(U_0 U_0^T) \), it follows from Lemma 2.2 that

\[
\mathcal{F} \subset \text{face}(U_1 U_1^T) \subset \text{face}(U_0 U_0^T),
\]

where \( U_1 = U_0 W_0 \) is \( n \times (r + \delta_1) \) with full column rank. Moreover, since \( \text{face}(\mathcal{F}) = \text{face}(X^*) \neq \text{face}(U_1 U_1^T) \), there exist \( Y_1 \succ 0 \) such that \( \text{tr}(U_1^T A U_1 Y_1) = b_i \) for all \( i = 1, \ldots, m \). Thus, by Lemma 2.3 there exists \( \Omega^1 = \sum_{i=1}^m x_i^1 A^i \) such that \( U_1^T \Omega^1 U_1 \succeq 0, \neq 0 \) and \( \text{tr}(X^* \Omega^1) \leq 0 \). If rank \( U_1^T \Omega^1 U_1 = \delta_1 \), then rank \( (U_0^T \Omega^0 U_0) + \text{rank } (U_1^T \Omega^1 U_1) = s - r \) and we are done and \( k = 1 \) in the theorem. Therefore, assume that rank \( (U_1^T \Omega^1 U_1) = \delta_1 - \delta_2 \), where \( \delta_1 - 1 \geq \delta_2 \geq 1 \), and let \( W_1 \) be a full column rank matrix such that \( \mathcal{R}(W_1) = \mathcal{N}(U_1 \Omega^1 U_1) \). Since \( \mathcal{F} \subset \text{face}(U_1 U_1^T) \), it follows from Lemma 2.2 that

\[
\mathcal{F} \subset \text{face}(U_2 U_2^T) \subset \text{face}(U_1 U_1^T) \subset \text{face}(U_0 U_0^T),
\]

where \( U_2 = U_1 W_1 \) is \( n \times (r + \delta_2) \) with full column rank.

Observe that at each step, a lower dimensional face containing \( \mathcal{F} \) is obtained. Thus after at most \( s - r \) steps, we must arrive at the case where rank \( (U_k \Omega^k U_k) = \delta_k \) and hence Statement 2 holds.
Second, we prove that if Statement 2 holds, then Statement 1 does not hold. Therefore, assume that \( k = 0 \) in the theorem, i.e., there exists \( \Omega^0 = \sum_{i=1}^{m} x_i^0 A^i \) such that \( \mathcal{U}_k^T \Omega^0 \mathcal{U}_0 \succeq 0 \), \( \text{rank}(\mathcal{U}_0^T \Omega^0 \mathcal{U}_0) = s - r \) and \( \text{tr}(X^* \Omega^0) \leq 0 \). Since \( \mathcal{F} \subset \text{face}(\mathcal{U}_0 \mathcal{U}_0^T) \), it follows from Lemma 2.2 that
\[
\mathcal{F} \subset \text{face}(\mathcal{U}_1 \mathcal{U}_1^T),
\]
where \( \mathcal{U}_1 = \mathcal{U}_0 \mathcal{W}_0 \) is \( n \times r \) with full column rank. Hence, \( \text{face}(X^*) = \text{face}(\mathcal{U}_1 \mathcal{U}_1^T) \) and thus Statement 1 does not hold.

Now assume that \( k = 1 \) in the theorem, i.e., there exist \( \Omega^0 = \sum_{i=1}^{m} x_i^0 A^i \), \( \text{tr}(\Omega^0 X^*) \leq 0 \) and \( \Omega^1 = \sum_{i=1}^{m} x_i^1 A^i \), \( \text{tr}(\Omega^1 X^*) \leq 0 \) such that \( \mathcal{U}_0^T \Omega^0 \mathcal{U}_0 \succeq 0, \neq 0 \) and \( \mathcal{U}_1^T \Omega^1 \mathcal{U}_1 \geq 0, \neq 0 \) where \( \text{rank}(\Omega^0) = n - r - \delta_1 \) and \( \text{rank}(\mathcal{U}_0^T \Omega^0 \mathcal{U}_0) = \delta_1 \). Let \( \mathcal{W}_0 \) and \( \mathcal{W}_1 \) be full column rank matrices such that \( \mathcal{R}(\mathcal{W}_0) = \mathcal{N}(\mathcal{U}_0 \Omega^0 \mathcal{U}_0) \) and \( \mathcal{R}(\mathcal{W}_1) = \mathcal{N}(\mathcal{U}_1 \Omega^1 \mathcal{U}_1) \). Then it follows from Lemma 2.2 that \( \mathcal{F} \subset \text{face}(\mathcal{U}_1 \mathcal{U}_1^T) \) where \( \mathcal{U}_1 = \mathcal{U}_0 \mathcal{W}_0 \) is \( n \times (r + \delta_1) \) with full column rank. Applying Lemma 2.2 again we have that \( \mathcal{F} \subset \text{face}(\mathcal{U}_2 \mathcal{U}_2^T) \) where \( \mathcal{U}_2 = \mathcal{U}_1 \mathcal{W}_1 \) is \( n \times r \) with full column rank. Hence, \( \text{face}(X^*) = \text{face}(\mathcal{U}_2 \mathcal{U}_2^T) \) and thus Statement 1 does not hold.

Since \( \mathcal{R}(\mathcal{U}_k) \subset \mathcal{R}(\mathcal{U}_{k-1}) \subset \cdots \subset \mathcal{R}(\mathcal{U}_1) \), after at most \( s - r \) steps we must have \( \text{rank}(\mathcal{U}_k^T \Omega^k \mathcal{U}_k) = \delta_k \). Thus \( \mathcal{U}_{k+1} = \mathcal{U}_k \mathcal{W}_k \) is \( n \times r \). Thus \( \text{face}(X^*) = \text{face}(\mathcal{U}_{k+1} \mathcal{U}_{k+1}^T) \), and thus Statement 1 does not hold.

\( \square \)

In Theorem 2.2 the assumption that \( \mathcal{F} \subset \text{face}(\mathcal{U}_0 \mathcal{U}_0) \) was made in order to make the application of Theorem 2.2 to the dimensional rigidity problem straightforward; i.e., this assumption was made for the purposes of the paper. Dropping this assumption is equivalent to setting \( \mathcal{U}_0 = I_n \), since \( S_n^c = \text{face}(I_n) \). The following lemma is a restatement of Theorem 2.2 without the aforementioned assumption.

**Lemma 2.4.** Let \( A^1, \ldots, A^m \) be given \( n \times n \) symmetric matrices and let \( b = (b_i) \) be a given nonzero vector in \( \mathbb{R}^m \). Further, let
\[
\mathcal{F} = \{ X \in S_n^c : \text{tr}(X A^i) = b_i \text{ for } i = 1, \ldots, m \}.
\]
Let \( X^* \) be a matrix in \( \mathcal{F} \) such that \( \text{rank}(X^*) = r \). Then exactly one of the following two statements holds:

1. There exists an \( X \) in \( \mathcal{F} \) such that \( \text{rank}(X) \geq r + 1 \).
2. There exist nonzero matrices \( \Omega^0, \Omega^1, \ldots, \Omega^k \), for some \( k \leq n - r \), such that:
   - (a) \( \Omega^j = \sum_{i=1}^{m} x_i^j A^i \) \( j = 0, 1, \ldots, k \) for some scalars \( x_i^j \),
   - (b) \( \Omega^0 \succeq 0, \mathcal{U}_0^T \Omega^0 \mathcal{U}_0 \succeq 0, \ldots, \mathcal{U}_k^T \Omega^k \mathcal{U}_k \succeq 0 \),
   - (c) \( \text{tr}(X^* \Omega^j) \leq 0 \) for \( j = 0, 1, \ldots, k \),
\[ \text{rank } \Omega^0 + \text{rank } (U_1^T \Omega^1 U_1) + \cdots + \text{rank } (U_k^T \Omega^k U_k) = n - r, \]

where \( U_1, \ldots, U_{k+1}, \) and \( W_0, W_1, \ldots, W_k \) are full column rank matrices defined as follows: For \( i = 0, 1, \ldots, k, \) \( \mathcal{R}(W_i) = \mathcal{N}(U_i^T \Omega U_i), \) and \( U_{i+1} = U_i W_i \) with \( U_0 = I_n. \)

## 3 Dimensional Rigidity of Bar Frameworks

In the section, we use Theorem 2.2 to prove a recent result, by Connelly and Gortler [7], concerning the dimensional rigidity of bar frameworks (or frameworks for short). To make the dimensional rigidity problem amenable to semidefinite programming methodology, we use Gram matrices to represent the configuration of a framework. We start by characterizing the set of all frameworks that are equivalent to a given framework \((G, p)\).

### 3.1 The Set of Equivalent Frameworks

Let \((G, p)\) be an \(r\)-dimensional framework on \(n\) vertices in \(\mathbb{R}^r\). The \(n \times r\) matrix

\[
P = \begin{bmatrix}
(p^1)^T \\
\vdots \\
(p^n)^T
\end{bmatrix}
\]

is called the configuration matrix of \((G, p)\). We will find it convenient to make the following assumption in the sequel. Recall that \(e\) denotes the vector of all 1’s in \(\mathbb{R}^n\).

**Assumption 3.1.** \(P^T e = 0\) for any configuration matrix \(P\), i.e., the origin coincides with the centroid of the points \(p^1, \ldots, p^n\).

In terms of the configuration matrix \(P\), the Gram matrix of \((G, p)\) is given by \(PP^T\). Note that \(\text{rank } (PP^T) = r\) since \((G, p)\) is \(r\)-dimensional, i.e., \(P\) has full column rank. Beside being positive semidefinite, Gram matrices of frameworks are invariant under orthogonal transformation. Moreover, by Assumption 3.1, Gram matrices of frameworks are also invariant under translations. Hence, congruent frameworks have the same Gram matrix. Thus, Gram matrices can be used to characterize all frameworks \((G, p')\) that are equivalent to \((G, p)\).

Let \(B'\) be the Gram matrix of framework \((G, p')\). Recall the definition of matrix \(F^{ij}\) in (5). Then

\[
\text{tr } (F^{ij} B') = B'_{ii} + B'_{jj} - 2B'_{ij} = ||p^i - p^j||^2.
\]

Thus, \((G, p')\) is equivalent to \((G, p)\) if and only if

\[
\text{tr } (F^{ij} B') = ||p^i - p^j||^2 \text{ for all } \{i, j\} \in E(G),
\]
The following theorem characterizes the set of all frameworks that are equivalent to \((G,p)\).

**Theorem 3.1.** Let \((G,p)\) be a given \(r\)-dimensional framework on \(n\) nodes in \(\mathbb{R}^r\), \(r \leq n - 2\), and let

\[
\mathcal{F} = \{B' \in S_n \cup : B'e = 0, \, \text{tr} \{F^{ij}B'\} = ||p^i - p^j||^2 \forall \{i,j\} \in E(G)\}. 
\tag{14}
\]

Then \((G,p')\) is an \(r'\)-dimensional framework that is equivalent to \((G,p)\) if and only if the Gram matrix of \((G,p')\) belongs to \(\mathcal{F}\), where \(r' = \text{rank } B'\).

The following Theorem is an immediate corollary of Theorem 3.1 and Lemma 2.1.

**Theorem 3.2.** Let \((G,p)\) be a given \(r\)-dimensional framework on \(n\) nodes in \(\mathbb{R}^r\), \(r \leq n - 2\), and let

\[
\mathcal{F} = \{B' \in S_n \cup : B'e = 0, \, \text{tr} \{F^{ij}B'\} = ||p^i - p^j||^2 \forall \{i,j\} \in E(G)\}. 
\tag{15}
\]

Then \((G,p)\) is dimensionally rigid if and only if \(\mathcal{F} \subset \text{face}(PP^T)\), where \(P\) is the configuration matrix of \((G,p)\).

Let \(V\) be an \(n \times (n - 1)\) matrix such that

\[
V^Te = 0 \quad \text{and } V^TV = I_{n-1}. \tag{16}
\]

Then, by Theorem 2.1, it follows that \(\mathcal{F}\) in (14) is a subset of \(\text{face}(VV^T)\). Thus Theorem 3.1 can be equivalently stated as follows.

**Theorem 3.3.** Let \((G,p)\) be a given \(r\)-dimensional framework on \(n\) nodes in \(\mathbb{R}^r\), \(r \leq n - 2\), and let

\[
\mathcal{F} = \{B' = VYV^T : Y \in S_n \cup^-1, \, \text{tr} \{YV^TF^{ij}V\} = ||p^i - p^j||^2 \forall \{i,j\} \in E(G)\}. \tag{17}
\]

Then \((G,p')\) is an \(r'\)-dimensional framework that is equivalent to \((G,p)\) if and only if the Gram matrix of \((G,p')\) belongs to \(\mathcal{F}\), where \(r' = \text{rank } Y\).

### 3.2 Quasi-Stress Matrices

We saw earlier that stress matrices play an important role in the problem of dimensional rigidity. However, for the purposes of this paper, it will be convenient to introduce the notion of a quasi-stress matrix.

It is clear from the definition of a stress matrix in (4) that the columns of the matrix \([P \ e]\) belong to the null space of any stress matrix of \((G,p)\), where \(P\) is the
configuration matrix of \((G, p)\). Hence, the rank of a stress matrix of an \(r\)-dimensional framework on \(n\) vertices is \(\leq n - r - 1\).

An \(n \times n\) symmetric matrix \(\Omega\) is said to be a quasi-stress matrix of \((G, p)\) if it satisfies the following properties:

\[
\begin{align*}
(a) & \quad P^T \Omega P = 0, \\
(b) & \quad \Omega_{ij} = 0 \text{ for all } \{i, j\} \in E(G), \\
(c) & \quad \Omega e = 0.
\end{align*}
\]

It immediately follows that if a quasi-stress matrix \(\Omega\) is positive semidefinite, then \(\Omega\) is a stress matrix since in this case \(P^T \Omega P = 0\) implies that \(\Omega P = 0\). For later use we remark here that for any \(A \in \mathbb{S}^n\), \(Ae = 0\) if and only if \(A = \sum_{i<j} \omega_{ij} F_{ij}\) for some \(\omega_{ij}\)'s, where \(F_{ij}\) is as defined in (5). As a result, any quasi-stress matrix \(\Omega\) can be written as

\[
\Omega = \sum_{\{i,j\} \in E(G)} \omega_{ij} F_{ij} \text{ for some scalars } \omega_{ij}.
\]

### 3.3 Characterizing Dimensional Rigidity

A characterization of dimensional rigidity in terms of the minimal face of \(PP^T\) was given in Theorem 3.2. Another characterization can be obtained from Theorem 2.2. The following theorem is a refined version of Connelly and Gortler main result (Corollary 2 in [7]) concerning dimensional rigidity.

**Theorem 3.4.** Let \((G, p)\) be an \(r\)-dimensional framework on \(n\) vertices in \(\mathbb{R}^r\), \(r \leq n - 2\). Then \((G, p)\) is dimensionally rigid if and only if there exist nonzero quasi-stress matrices: \(\Omega^0, \Omega^1, \ldots, \Omega^k\), for \(k \leq n - r - 1\), such that:

1. \(\Omega^0 \succeq 0, \quad \mathcal{R}(\Omega^0) = \mathcal{N}(P^T e^T)\)
2. \(\mathcal{R}(\Omega^1) = \mathcal{N}(\rho_1^T \Omega^1), \ U_1 = [P^T \rho_1] \text{ for } i = 1, \ldots, k \text{ and } \rho_{i+1} = \rho_i \xi_i \text{ for all } i = 1, \ldots, k - 1.
3. \(\Omega^k \succeq 0, \quad \mathcal{R}(\Omega^k) = \mathcal{N}(\rho_k^T \Omega^k), \ U_k = [P^T \rho_k] \text{ for } i = 1, \ldots, k \text{ and } \rho_{i+1} = \rho_i \xi_i \text{ for all } i = 1, \ldots, k - 1.

\(\mathcal{N}\) and \(\rho\) denote the null space and null vector, respectively.
Proof. framework \((G,p)\) is dimensionally rigid if and only if \(\mathcal{F} \subset \text{face}(PP^T)\), i.e., if and only if there does not exist a \(B' \in \mathcal{F}\) such that rank \(B' \geq r + 1\), where \(\mathcal{F}\) is defined in (17). Note that \(\mathcal{F} \subset \text{face}(VV^T)\), where \(V\) is as defined in (10). Therefore, it follows from Theorem 2.2 that \((G,p)\) is dimensionally rigid if and only if there exist nonzero matrices \(\Omega^0, \ldots, \Omega^k\), for some \(k \leq n - 1 - r\), such that:

1. \(\Omega^l = \sum_{i,j \in E(G)} \omega^l_{ij} F^l_{ij} (l = 0, 1, \ldots, k)\) for some scalars \(\omega^l_{ij}\).
2. \(V^T \Omega^0 V \geq 0, U^T_1 \Omega^l U_l \geq 0\) for \(l = 1, \ldots, k\).
3. \(\text{tr}(PP^T \Omega^l) \leq 0\) for \(l = 0, 1, \ldots, k\).
4. \(\text{rank}(V^T \Omega^0 V) + \text{rank}(U^T_1 \Omega^1 U_1) + \cdots + \text{rank}(U^T_k \Omega^k U_k) = n - 1 - r\),

where \(U_1, \ldots, U_{k+1}\), and \(W_0, W_1, \ldots, W_k\) are full column rank matrices such that for \(i = 0, 1, \ldots, k\), we have \(\mathcal{R}(W_i) = \mathcal{N}(U^T_i \Omega^1 U_i)\), and \(U_{k+1} = U_k W_i, U_0 = V\).

Now it follows from the definition of \(V\) in (16) that

\[VV^T = I_n - \frac{ee^T}{n}.\]

Let \(\overline{\Omega^0} = V^T \Omega^0 V\), then \(\Omega^0 = V \overline{\Omega^0} V^T\) since \(\Omega^0 e = 0\). Therefore, \(\overline{\Omega^0} \succeq 0\) if and only if \(\Omega^0 \succeq 0\). Thus, \(\overline{\Omega^0}\) is a stress matrix of \((G,p)\) and hence \(\Omega^0 P = 0\). Moreover, \(\text{rank} \(\overline{\Omega^0} = \text{rank} \(V^T \Omega^0 V\).\)

Also, it is easy to see that if \(x \in \mathcal{N}(\Omega^0)\), then \(V^T x \in \mathcal{N}(V^T \Omega^0 V)\). Thus, by the definition of \(\rho_1\), we have that \(\mathcal{R}([P \ e \ \rho_1]) = \mathcal{N}(\Omega^0)\). Thus \(\mathcal{R}([V^T P \ V^T \rho_1]) = \mathcal{N}(V^T \Omega^0 V) = W_0\). Hence, \(U_1 = V W_0 = [P \ \rho_1]\) since \(P^T e = 0\) and \(\rho_1 e = 0\).

On the other hand, since

\[U^T_1 \Omega^1 U_1 = \begin{bmatrix} P^T \Omega^1 P & P^T \Omega^1 \rho_1 \\ \rho_1^T \Omega^1 P & \rho_1^T \Omega^1 \rho_1 \end{bmatrix} \succeq 0,
\]

and since \(\text{tr}(P^T \Omega^1 P) \leq 0\), it follows that

\(P^T \Omega^1 P = 0\) and hence \(P^T \Omega^1 \xi = 0\).

Therefore, \(\Omega^1\) is a quasi-stress matrix of \((G,p)\). Moreover, since

\[\mathcal{R}(W_1) = \mathcal{N}(U^T_1 \Omega^1 U_1) = \mathcal{N}(\begin{bmatrix} 0 & 0 \\ 0 & P^T \Omega^1 \rho_1 \end{bmatrix}),\]

it follows that \(W_1 = \begin{bmatrix} I_\tau & 0 \\ 0 & \xi_1 \end{bmatrix}\). Hence, \(U_2 = U_1 W_1 = [P \ \rho_1 \xi_1] = [P \ \rho_2]\). The rest of the proof for \(\Omega^2, \ldots, \Omega^k\) proceeds in an analogous fashion to the proof for \(\Omega^1\). \(\square\)
We end this section with the following observation regarding the computation of $\Omega^1, \ldots, \Omega^k$. While the matrices \{\(F^{ij} : \{i, j\} \in E(G)\}\} are linearly independent, in the second and subsequent steps of the Borwein-Wolkowicz facial reduction algorithm, the matrices \(\{U_i^T F^{ij} U_i : \{i, j\} \in E(G)\}\) may become linearly dependent. Thus some of the distance constraints in the definition of \(F\), namely the constraints
\[
\text{tr} (Y_i U_i^T F^{ij} U_i) = ||p^i - p^j||^2 : \{i, j\} \in E(G)
\]
may become redundant. This can be used to our advantage. Since if the constraint corresponding to edge \(\{\hat{i}, \hat{j}\}\) is redundant, then \(\Omega^l_{\hat{i}, \hat{j}}\) is 0. For instance, in Example 4.2, the framework \((G, p)\) (see Figure 1) has a clique \(\{p^2, p^3, p^4\}\) where \(p^2, p^3\) and \(p^4\) are collinear. As a result,
\[
U_i^T F^{24} U_i = 4 U_i^T F^{23} U_i = 4 U_i^T F^{34} U_i.
\]
Hence, we may choose \(\Omega^1_{24} = \Omega^1_{34} = 0\).

4 Numerical Examples

To illustrate the Borwein-Wolkowicz facial reduction algorithm used in the proof of Theorem 2.2, and consequently in Theorem 3.4, we present two numerical examples.

Example 4.1. Consider the bar framework \((G, p)\) in Figure 2 given in [7]. Its configuration matrix and a corresponding positive semidefinite stress matrix \(\Omega^0\) are given by
\[
P = \begin{bmatrix}
-1 & -2 \\
-1 & 2 \\
1 & 2 \\
1 & -2 \\
-1 & 0 \\
1 & 0
\end{bmatrix} \quad \text{and} \quad \Omega^0 = \begin{bmatrix}
1 & 0 & 0 & -2 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
-2 & 0 & -2 & 0 & 0 \\
0 & -2 & 0 & 0 & 0
\end{bmatrix}.
\]
Thus \(\delta_1 = n - r - 1 - \text{rank} \Omega^0 = 1\). Then \(\rho_1\), the matrix whose columns form a basis of null space \(\begin{bmatrix}
\Omega^0 \\
P^T \\
e^T
\end{bmatrix}\), is given by
\[
\rho_1 = \begin{bmatrix}
-1 \\
1 \\
-1 \\
1 \\
0 \\
0
\end{bmatrix}.
\]
Hence, \(U_1 = [P \ \rho_1] = \begin{bmatrix}
-1 & -2 & -1 \\
-1 & 2 & 1 \\
1 & 2 & -1 \\
1 & -2 & 1 \\
-1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix} \).
It admits a positive semidefinite stress matrix of rank 2 but not of rank 3. The edges \{1,2\} and \{3,4\} are drawn as arcs to make edges \{1,5\}, \{2,5\}, \{3,6\} and \{4,6\} visible.

Thus

\[ \mathcal{F} = \{ B' : B' = U_1 Y_1 U_1^T : Y_1 \in \mathcal{S}_+^3, \text{tr} (U_1^T F_{ij} Y_1 U_1) = ||p^i - p^j||^2 \forall \{i,j\} \in E(G) \}, \]

where,

\[ U_1^T F_{12} U_1 = 4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \quad U_1^T F_{15} U_1 = 4 \begin{bmatrix} 4 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 4 \end{bmatrix}, \quad U_1^T F_{25} U_1 = 4 \begin{bmatrix} 4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 4 \end{bmatrix}, \quad U_1^T F_{34} U_1 = 4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 1 \end{bmatrix}, \quad U_1^T F_{36} U_1 = 4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad U_1^T F_{46} U_1 = 4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

Note that since \( U_1^T F_{12} U_1 = 4 \), \( U_1^T F_{15} U_1 = 4 \), \( U_1^T F_{25} U_1 = 4 \), \( U_1^T F_{34} U_1 = 4 \), and \( U_1^T F_{36} U_1 = 4 \), we only include the distance constraints for edges \{1,2\} and \{3,4\}. The distance constraints corresponding to edges \{1,5\}, \{2,5\}, \{3,6\} and \{4,6\} are redundant. Thus \( Y_1 = (y_{ij}) \) must satisfy
\[
\begin{aligned}
4 \ y_{22} + 4 \ y_{23} + y_{33} &= 4 \\
4 \ y_{22} - 4 \ y_{23} + y_{33} &= 4 \\
y_{11} - 2 \ y_{13} + y_{33} &= 1 \\
y_{11} + 2 \ y_{13} + y_{33} &= 1 \\
4 \ y_{11} &= 4.
\end{aligned}
\]

Hence, \(y_{11} = y_{22} = 1\), \(y_{13} = y_{23} = y_{33} = 0\), and \(y_{12} = \alpha\) is a free parameter. Thus \(Y_1\) is a function of \(\alpha\), and it is given by

\[
Y_1(\alpha) = \begin{bmatrix}
1 & \alpha & 0 \\
\alpha & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Thus, \(Y_1(\alpha)\) is positive semidefinite iff 

\[-1 \leq \alpha \leq 1.\]

Furthermore, rank \(Y_1(\alpha) \leq 2\). Thus \(F \cap \text{relint(face } (U_1U_1^T)) = \emptyset\); i.e., there does not exist \(Y_1 \succeq 0\) such that \(\text{tr}(Y_1U_1f_{ij}U_1^T) = ||p_i - p_j||^2\) for all \(\{i, j\} \in E(G)\).

Now let \(\Omega_1 = (\omega_{ij})\). Since the distance constraints corresponding to edges \(\{1, 5\}, \{2, 5\}, \{3, 6\}\) and \(\{4, 6\}\) are redundant, we set \(\omega_15 = \omega_25 = \omega_36 = \omega_46 = 0\). Then \(P^T \Omega_1 P = 0\) and \(P^T \Omega_1 \rho_1 = 0\) imply that

\[
\begin{aligned}
\omega_{14} + \omega_{23} + \omega_{56} &= 0, \\
\omega_{14} - \omega_{23} &= 0, \\
\omega_{12} - \omega_{34} &= 0, \\
\omega_{12} + \omega_{34} &= 0.
\end{aligned}
\]

Thus, \(\omega_{12} = \omega_{34} = 0\) and \(\omega_{56} = -2\omega_{14} = -2\omega_{23}\). Also, \(\rho_1 \Omega_1 \rho_1\) is nonzero positive semidefinite if \(\omega_{12} + \omega_{34} + \omega_{14} + \omega_{23} > 0\). Therefore, set \(\omega_{14} = \omega_{23} = 1\) and \(\omega_{56} = -2\). Hence,

\[
\Omega_1 = \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -2 \\
0 & 0 & 0 & 0 & 2 & 2
\end{bmatrix}.
\]

Then

\[
U_1^T \Omega_1 U_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \rho_1^T \Omega_1 \rho_1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 8
\end{bmatrix} \succeq 0.
\]
Thus $\rho_1 \Omega^1 \rho_1 = 8$ is nonsingular. Moreover, $\mathcal{R}(\begin{bmatrix} I_2 \\ 0 \end{bmatrix}) = \mathcal{N}(U_1^T \Omega^1 U_1)$. Hence, $U_2 = U_1 \begin{bmatrix} I_2 \\ 0 \end{bmatrix} = [P \ \rho_1] \begin{bmatrix} I_2 \\ 0 \end{bmatrix} = P$. Therefore, $\mathcal{F} \subseteq \text{face}(P P^T)$ and rank $\Omega^0 + \text{rank } (U_1^T \Omega^1 U_1) = 2 + 1 = 3 = n - r - 1$.

**Example 4.2.** Consider the bar framework $(G, p)$ in Figure 7. Its configuration matrix and a corresponding positive semidefinite stress matrix $\Omega^0$ are given by

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 1 & 1 \\ 2 & 0 \\ -3 & -3 \end{bmatrix} \quad \text{and } \Omega^0 = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -2 & 1 & 0 \end{bmatrix}.$$ 

Thus $\delta_1 = n - r - 1 - \text{rank } \Omega^0 = 1$. Then $\rho_1$, the matrix whose columns form a basis of null space $\begin{bmatrix} \Omega^0 \\ P^T \\ e^T \end{bmatrix}$, is given by

$$\rho_1 = \begin{bmatrix} -4 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \quad \text{Hence, } U_1 = [P \ \rho_1] = \begin{bmatrix} 0 & 0 & -4 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \\ -3 & -3 & 1 \end{bmatrix}.$$ 

Thus

$\mathcal{F} = \{B' : B' = U_1 Y_1 U_1^T \in S_+^3, \text{tr } (U_1^T F^{ij} U_1 Y_1) = ||p^j - p^j||^2 \forall \{i, j\} \in E(G)\}$,

where, $U_1^T F^{24} U_1 = 4 \ U_1^T F^{23} U_1 = 4 \ U_1^T F^{34} U_1 = 4 \ U_1^T F^{12} U_1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $U_1^T F^{12} U_1 = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 1 & 5 \\ 1 & 5 & 25 \end{bmatrix}$, $U_1^T F^{15} U_1 = \begin{bmatrix} 9 & 9 & -15 \\ 9 & 9 & -15 \\ -15 & -15 & 25 \end{bmatrix}$, $U_1^T F^{25} U_1 = \begin{bmatrix} 25 & 15 & 0 \\ 15 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $U_1^T F^{45} U_1 = \begin{bmatrix} 25 & 15 & 0 \\ 15 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Note that since $U_1^T F^{24} U_1 = 4 \ U_1^T F^{23} U_1 = 4 \ U_1^T F^{34} U_1$, we only include the distance constraint for edge $\{2, 3\}$. The distance constraints corresponding to edges $\{2, 4\}$ and $\{3, 4\}$ are redundant. Thus $Y_1 = (y_{ij})$ must satisfy

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\[
\begin{align*}
y_{11} - 2y_{12} + y_{22} &= 2 \\
25y_{11} + 30y_{12} + 9y_{22} &= 34 \\
9y_{11} + 30y_{12} + 25y_{22} &= 34
\end{align*}
\]

and
\[
\begin{align*}
y_{11} + 18y_{12} + 9y_{22} - 30y_{13} - 30y_{23} + 25y_{33} &= 18 \\
y_{11} + 2y_{12} + y_{22} + 10y_{13} + 10y_{23} + 25y_{33} &= 2 \\
y_{22} + 20y_{23} + 25y_{33} &= 4
\end{align*}
\]

Hence, \( Y_1 \) is unique (recall that \((G,p)\) is universally rigid), and it is given by
\[
Y_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Furthermore, \( \text{rank } Y_1 = 2 \). Thus \( \mathcal{F} \cap \text{relint}(\text{face } (U_1U_1^T)) = \emptyset \).

Now let \( \Omega^1 = (\omega_{ij}) \). Since the distance constraints corresponding to edges \{2,4\} and \{3,4\} are redundant, we set \( \omega_{24} = \omega_{34} = 0 \). Then \( P^T\Omega^1P = 0 \) and \( P^T\Omega^1\rho_1 = 0 \) imply that
\[
\begin{align*}
\omega_{13} - 3\omega_{15} &= 0, \\
2\omega_{12} + \omega_{13} - 3\omega_{15} &= 0, \\
\omega_{13} + 9\omega_{15} + \omega_{23} + 9\omega_{25} + 25\omega_{45} &= 0, \\
\omega_{13} + 9\omega_{15} - \omega_{23} + 15\omega_{25} + 15\omega_{45} &= 0, \\
4\omega_{12} + \omega_{13} + 9\omega_{15} + \omega_{23} + 25\omega_{25} + 9\omega_{45} &= 0.
\end{align*}
\]

Moreover, we require \( \rho_1^T\Omega^1\rho_1 = 25(\omega_{12} + \omega_{13} + \omega_{15}) \) to be positive semidefinite. Hence, \( \omega_{12} = 0, \omega_{13} = 24, \omega_{15} = 8, \omega_{23} = 6, \) and \( \omega_{25} = \omega_{45} = -3 \); i.e.,
\[
\Omega^1 = \begin{bmatrix} 32 & 0 & -24 & 0 & -8 \\ 0 & 3 & -6 & 0 & 3 \\ -24 & -6 & 30 & 0 & 0 \\ 0 & 0 & 0 & -3 & 3 \\ -8 & 3 & 0 & 3 & 2 \end{bmatrix}.
\]

Then
\[
U_1^T\Omega^1U_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \rho_1^T\Omega^1\rho_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 800 \end{bmatrix} \succeq 0.
\]
Hence, $\rho_1\Omega^1\rho_1 = 800$ is nonsingular. Moreover, $\mathcal{R}(\begin{bmatrix} I_2 \\ 0 \end{bmatrix}) = \mathcal{N}(\mathcal{U}_1^T\Omega^1\mathcal{U}_1)$. Hence, 
\[ \mathcal{U}_2 = \mathcal{U}_1 \begin{bmatrix} I_2 \\ 0 \end{bmatrix} = [P \rho_1] \begin{bmatrix} I_2 \\ 0 \end{bmatrix} = P. \]
Therefore, $\mathcal{F} \subseteq \text{face}(PP^T)$ and rank $\Omega^0 + \text{rank } (\mathcal{U}_1^T\Omega^1\mathcal{U}_1) = 2+1 = 3 = n - r - 1$.

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