The Hertz contact in chain elastic collisions

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A theoretical analysis about the influence of the Hertz elastic contact on a three body chain collision is presented. In spite of the elastic character of the collision, the final velocity of each particle depends on the particular interaction between them. A system involving two elastic spheres falling together one in top of the other under the action of gravity, and colliding with an horizontal hard wall is studied in detail. The effect of the Hertz contact interaction can be easily put in evidence for some particular situations.

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I. INTRODUCTION

When two balls are dropped together, both vertically aligned and hitting the ground at the same time, an interesting rebound effect is obtained. If the smaller ball is above, it may rise surprisingly high into the air. This effect may still be enhanced if one drops a chain of balls, ordered by their relative weight, the lightest ball at the top.

This system was already discussed in pedagogical articles more than 30 years ago, assuming perfect and independent elastic collisions between each pair of balls [1–5]. It was considered an attractive and ideal system to study elastic collisions. However, if the balls are dropped together, all collisions will happen at the same time. Because they are not instantaneous, the final velocity distribution of the balls will be dependent on the particular interactions between them, even in those cases where the interactions are elastic, i.e., with no loss of energy. In their beautiful article [2], the class of W. G. Harter considered at some point non-instantaneous collisions. To study them, they introduced a simple quasi-linear model, which could be valid if we think not of spheres but of uni-dimensional springs. The dynamical differential equations were solved using an analog computer program. However, they concluded that the quasi-linear model did not agree with their experimental results, and they proposed a phenomenological non linear model. The interaction field was then obtained directly from empirical data, by dropping a ball onto a painted flat metal surface from varying heights. In this way the interaction could be plotted as a function of the spot size, and eventually as a function of the depression of the ball. Although the strategy to obtain the non linear interaction was remarkable, the discussion that followed was not sufficiently developed. In fact, from the theoretical point of view and as a first approximation, the analysis of the non-instantaneous multiple collisions effect should take into account the interaction between two elastic spheres obeying Hooke’s law, which is always valid for sufficiently small deformations. This problem is known in the literature as the Hertz contact [6].

In this article, a system involving multiple collisions between two different elastic spheres, one on top of the other, with the same initial velocity, hitting a planar hard horizontal wall is considered. In Section II, the results obtained for instantaneous elastic collisions are briefly reviewed. Next, the Hertz contact is introduced in the mathematical model of the system. At the end, the results of this theoretical analysis are presented. The main differences between independent and non-instantaneous collisions are outlined and discussed.

II. INDEPENDENT COLLISIONS

Consider a system with two elastic spheres, one on top of the other, falling with the same velocity \( v \), colliding with the ground, as represented in Fig. 1.

![Fig. 1. Two different elastic spheres falling with the same velocity colliding with the ground.](image)

Assume a very small gap between the spheres, such that all collisions can be treated individually. First, the lower sphere hits the rigid wall and returns with the same velocity \( v \). Next, the spheres collide elastically. The final velocities \( v_1 \) and \( v_2 \) (of lower and upper sphere respectively) can be calculated through momentum and energy conservation principle, and are independent of the particular interaction. If \( m = m_2/m_1 \) is the spheres mass ratio, we can write:
\[ u_1 = \frac{1-3m}{1+m} v \quad (1) \]
\[ u_2 = \frac{3-2m}{1+m} v, \quad (2) \]

where the positive sign corresponds to the upward orientation. The upper sphere may rebound with a velocity up to three times greater than its initial one, for a vanishingly small mass ratio. That is, if its mass is much smaller than the mass of the lower sphere.

In fact, the rebound may be enhanced if we add to this system extra spheres, all falling with the same initial velocity. To study this effect, consider first the elastic collision between two spheres with a mass ratio \( m \) but with different initial velocities. The upper sphere still moves with a downward velocity \( v \). The lower sphere has now previously collide with another sphere and moves upward with a velocity \( av \) (\( a > 1 \)). The final velocities are now given by:

\[ u_1 = \frac{a-(a+2)m}{1+m} v \quad (3) \]
\[ u_2 = \frac{1+2a-m}{1+m} v. \quad (4) \]

Naturally, equations (1) and (2) are recovered if \( a = 1 \). For a system with \( n \) spheres, the upper one may achieve a final velocity of \((2^n-1)v\), if all mass ratios are vanishingly small.

Another interesting limit to this system, for a certain distribution of mass ratios, concerns the case for which all lower spheres stop, and only the upper one rises into the air. In this situation, all initial energy was transmitted to only one sphere. In the simple system composed of only two spheres, the lower one has null final velocity for \( m = 1/3 \), whereas the upper sphere rebounds to reach twice its initial velocity.

III. NON INSTANTANEOUS COLLISIONS

However, if the spheres fall together, the assumption of independent collisions may no longer be accurate. For the system represented in Fig. 1, both collisions - of the lower sphere with the ground and of the two spheres - will happen at the same time. To obtain the final velocities, after the spheres separation, it is important to know the particular interaction between them. In the following subsections, one presents the simplest possible interaction: it will be assumed that all solids respect Hooke’s law of elasticity, valid for sufficiently small deformations. The dynamical equations for this model will then be established and solved.

A. The Hertz contact

The theory of the elastic contact between solids was first studied by H. Hertz, and it can be followed in detail in Ref. [6]. For two isotropic and homogeneous elastic spheres, with radius of curvature respectively of \( R_1 \) and \( R_2 \) (see Fig. 2), the elastic potential energy \( E_p \) depends on the combined deformation \( h \) as:

\[ E_p = \frac{2}{5} er^2 h^5, \quad (5) \]

where the reduced radius \( r = R_1 R_2/(R_1 + R_2) \) and the reduced elastic constant \( e \) is dependent on the Young moduli \( E_1, E_2 \) and on the Poisson coefficients \( \sigma_1, \sigma_2 \). It is given by:

\[ e = \frac{4}{3} \left( \frac{1-\sigma_1^2}{E_1} + \frac{1-\sigma_2^2}{E_2} \right)^{-1}. \quad (6) \]

The Young modulus increases with the solid rigidity, whereas the Poisson coefficient is typically slightly smaller than \( 1/2 \).

![FIG. 2. Elastic contact between two spheres.](attachment:image.png)

The elastic energy (5) is valid for a static deformation. Nevertheless, it can be considered a good approximation for the situation studied in this article if the velocity \( v \) is much smaller than the sound velocities of the solids involved. It is interesting to notice that this approximation fits well the experimental potential energy obtained by the class of W. G. Harter [2], in the first regime of small ball depressions. For large depressions, the energy increased with a greater power of \( h \), which is in agreement with the fact that Hooke’s elastic regime only take into account the first terms of the elastic energy power expansion on the deformation.

It is instructive to calculate the maximum height of deformation and the contact time of two colliding spheres. Suppose \( \nu \) is the relative velocity of the spheres before the collision, and \( \mu = m_1 m_2/(m_1 + m_2) \) their reduced mass. Then, the energy conservation principle reads:

\[ \frac{1}{2} \mu \left( \frac{dh}{dt} \right)^2 + \frac{2}{5} er^2 h^5 = \frac{1}{2} \mu \nu^2. \quad (7) \]

The maximum height of deformation occurs when the relative velocity is zero and can be written:

\[ h_M = \left( \frac{5\mu}{4er^2} \right)^{\frac{2}{5}} \nu^{\frac{2}{5}}. \quad (8) \]

The collision time corresponds to the time in which the deformation goes from 0 to \( h_M \) and back to 0 again. It is given by:
\[ \tau = 2 \left( \frac{25 \mu^2}{16 \epsilon^2 r \nu} \right)^{\frac{1}{2}} \int_{0}^{1} \frac{dx}{\sqrt{1 - x^2}} \approx 3.21 \left( \frac{\mu^2}{\epsilon^2 r \nu} \right)^{\frac{1}{2}}. \]  

These are the relevant length and time for elastic collision problems, and they will be useful to write down dimensionless dynamical equations.

**B. Dynamical equations**

For the system represented in Fig. 1, the total potential energy is given by the sum:

\[ E_p = \frac{2}{5} e_{01} r_{01}^2 h_{01}^2 + \frac{2}{5} e_{12} r_{12}^2 h_{12}^2, \]  

where the first term takes into account the interaction between the first sphere and the rigid ground, with infinite Young modulus and radius of curvature, and the second term is the interaction between both spheres. The deformations depend on the positions \( x_1 \) and \( x_2 \) of the centers of the spheres (the ground defining the reference position):

\[ h_{01} = H(R_1 - x_1), \]  
\[ h_{12} = H((R_1 + R_2) - (x_2 - x_1)). \]

The function \( H(x) = x \) if \( x > 0 \) and \( H(x) = 0 \) otherwise. The equations of motion are:

\[ m_1 \ddot{x}_1 = -\partial E_p / \partial x_1, \quad (i = 1, 2), \]

will simply be written:

\[ m_1 \ddot{x}_1 = e_{01} r_{01}^2 h_{01}^2 - e_{12} r_{12}^2 h_{12}^2, \]
\[ m_2 \ddot{x}_2 = e_{12} r_{12}^2 h_{12}^2. \]  

If one introduces natural units, \( L = (m_1 \nu^2 / e_{01} r_{01}^2) \) and \( T = L / \nu \), which are associated with the maximum height of depression and the time of contact between the ground and the first sphere, it is possible to define dimensionless variables. The equations of motion may be written in a simpler form:

\[ \ddot{x}_1 = h_{01}^2 - k h_{12}^2, \]  
\[ m \ddot{x}_2 = k h_{12}^2. \]

They only depend on two parameters,

\[ m = m_2 / m_1, \]
\[ k = e_{12} r_{12}^2 / e_{01} r_{01}^2. \]

Note that \( k \) is defined as a ratio of the reduced elastic constants and radius. If the infinite rigidity of the planar wall is taken into account, we have \( e_{01} > e_{12} \) and \( r_{01} > r_{12} \). For this system, one concludes that \( k < 1 \).

**C. Numerical resolution**

The set of differential dynamical equations (15) and (16) was solved numerically using a simple and standard Euler method, for different values of the parameters. Fig. 3 shows several solutions found for the positions \( x_1 \) and \( x_2 \) as functions of time, starting when the two spheres initially in contact hit the ground with the same velocity \( v \), until they separate with constant final velocities \( u_1 \) and \( u_2 \). The axis have arbitrary units. The solutions were calculated for spheres with the same Young modulus and density, but four different mass ratios \( m = 0.01, 0.5, 1, 5 \) (wich were obtained choosing appropriate sphere radii).

Surprisingly, for a very small mass ratio, the spheres have almost the same velocity \( u_1 \approx u_2 \approx v \) after the rebound. This situation contrasts with the results obtained considering independent collisions, for which the maximum velocity gain would be expected. In fact, if the mass ratio \( m \) is negligible, it is possible to see from Eq. 16 that the deformation \( h_{12} \) will also be very small, exerting almost no influence in the motion of the lower sphere (see Eq. 15). The latter sphere rebounds with the wall and returns with velocity \( u_1 \approx v \). Since \( h_{12} \approx R_2 = 0 \), it may be concluded that \( R_1 \approx x_2 - x_1 \). Then \( \dot{x}_1 \approx \dot{x}_2 \), which means that both spheres stick together during the collision as if they were one whole body [7].

It is also interesting to notice that for mass ratios greater than one (in the case of the figure with \( m = 5 \)) the three body collision gets more complex, with the bottom sphere making several rebounds between the ground and the other sphere.

It is important to mention that the final velocities do not depend on the amplitude of the spheres’ deformations
during the collisions. If both spheres were more rigid, but keeping the same ratio \( k \), the deformations could be very small, and still the final velocities would be equal to the ones represented in Fig. 3.

In Fig. 4, the bold lines represent the final velocity distributions \( u_1 \) and \( u_2 \) as functions of \( m < 1 \), for different values of \( E_2/E_1 = 0.1, 0.5, 1, 10 \). The thin lines show the results of equations (1) and (2), considering independent collisions.

![FIG. 4. Velocity distributions \( u_1 \) and \( u_2 \) as functions of \( m < 1 \). The thin lines correspond to the results obtained for independent collisions and they are independent of the ratio between the Young moduli. In bold, the velocities obtained considering the Hertz contact for \( E_2/E_1 = 0.1, 0.5, 1, 10 \).](image)

The independent collisions results are recovered for small values of the ratio \( E_2/E_1 \), and consequently for small \( k \). This is particularly true for the final velocity distribution of the lower sphere \( u_1 \). In fact, if \( k \to 0 \), it is possible to see from Eq. 15 that in a first moment, the motion of the lower sphere will depend mainly on its interaction with the wall — the deformation \( h_{12} \) between both spheres will have a negligible contribution when it is compared with the deformation \( h_{01} \) between the lower sphere and the wall. However, if the collision time between both spheres is large enough (as it is shown in Eq. 9, the collision time increases for small \( k \)), after the lower sphere-wall interaction, the terms containing the deformation \( h_{12} \) will be the only ones present in the equations of motion. Therefore, the collisions can be considered as almost independent and subsequential. Nevertheless, as the final velocity \( u_2 \) is concerned, the convergence does not occur for all values of \( m \): it is possible to see in Fig. 4 \((E_2/E_1 = 0.1)\) that the final speed \( u_2 \to v \) as the mass ratio \( m \to 0 \).

As \( E_2/E_1 \) (or \( k \)) increases, small changes can be seen in the final velocity distributions, and the independent collisions approximation is no longer applicable. These changes are limited because the infinite rigidity assumption requires \( k < 1 \). In fact, the velocity distributions for \( E_2/E_1 \to \infty \) is not very different from the one represented for \( E_2/E_1 = 10 \)

The left part of Fig. 5 represent the rebound maximum speed \( u_{2,m} \) as a function of \( k < 1 \). On the right, the mass ratio \( m_M \) corresponding to the maximum rebound speed is also plotted. \( m_M \) decreases rapidly to zero as \( k \to 0 \).

![FIG. 5. Maximum rebound speed \( u_{2,m} \) and corresponding mass ratio \( m_M \) as a function of \( k < 1 \) (according to the infinite rigidity of the wall assumption).](image)

The discontinuity on the slope observed for the final velocity \( u_1 \) in Fig. 4 \((E_2/E_1 = 0.1)\) corresponds to the critical value \( m_c \) for which the lower sphere stops completely. If \( m > m_c \), this sphere rebounds twice off the ground, before it reaches the final velocity represented in the figure. If the ratio between the Young moduli increases, the discontinuity changes first its position \((E_2/E_1 = 0.5)\), and eventually it disappears \((E_2/E_1 = 1 \) and larger values).

However, these discontinuities will happen again for larger values of \( m \), as the lower sphere rebounds more and more in between the ground and the upper sphere. This curious feature can be seen in Fig. 6, which represents the velocity distributions \( u_1 \) and \( u_2 \) as functions of \( 1/m < 1 \), for \( E_2/E_1 = 1 \). As \( 1/m \) approaches 0, i.e., when the lower sphere becomes much lighter than the top one, the number of collisions increases to infinity.

![FIG. 6. Velocity distributions \( u_1 \) and \( u_2 \) as functions of \( 1/m < 1 \), for \( E_2/E_1 = 1 \). These results correspond to the results obtained only considering non-instantaneous collisions.](image)

### IV. DISCUSSION AND CONCLUSIONS

In this article, the influence of the Hertz contact on multiple chain collisions was studied in detail. It was shown that the independent collisions approximation is no longer valid for small values of the mass ratio \( m \) and large values of the rigidity ratio \( k \).

To understand the results obtained here, it is important to consider the typical two-body times of collision (see Eq. 9). First, suppose \( m \) is of the order of unity. If...
is small, the two-body collision time between the lower sphere and the wall, $\tau_{01}$, is smaller than the two-body collision time between both spheres, $\tau_{12}$. As $k \to 0$, the first collision occurs almost instantaneously, and subsequently, the two spheres collide. Thus, the results obtained considering this limit approach the independent collisions results.

However, as $k$ increases, $\tau_{01}$ also increases. The collisions become more evolved, and it is difficult to treat them separately.

Despite its complexity, the final velocity distributions $u_1$ and $u_2$ approach each other for large $k$ and small $m$. In this limit ($m \to 0$), as it was proved through the analysis of the equations of motion, both spheres behave as if they belonged to the same body, changing their velocities together as they interact with the ground. At the end, both spheres rebound with equal final velocities $u_1 \approx u_2 \approx v$.

The results obtained with this model were summarized in Fig. 5, which clearly indicates the best parameters $m$ and $k$ to experimentally put in evidence the influence of the Hertz contact on chain elastic collisions.

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