HIGHER COHERENCE AND A GENERALIZATION OF
HIGHER CATEGORIFIED ALGEBRAIC STRUCTURES.

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Abstract. This article reflects (to some degree) a talk I gave in Hiroshima, and another in Himeji, in November 2016. I am grateful to the organizers of the symposia for the invitations.

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1. Introduction

1.1. Discovery or recognition of the right kind of algebraic structure is often important in the development of mathematical subjects. Starting perhaps with “group” in Galois theory, a list of examples would easily grow long. One might wish to systematically find and treat algebraic structures, especially for the use of higher categorical ideas, since more various structures are available with higher categorical dimensionality. Importance of higher category theory comes for example, from the necessity of it for both analysis and construction of topological field theories (TFT), as has become clear from the proof and generalization of Baez and Dolan’s cobordism hypothesis [1] by Lurie and Hopkins [10]. Higher category theory also seems promising since categorification (in the sense of Crane [5, 4], about which the most influential pioneer may have been Grothendieck) has been a useful method for finding important new structures.

In this article, we shall give a overview of the relatively non-technical aspects of our work [11], in the most fundamental part of which, a concrete understanding of higher categorical coherence leads to systematic views on some (quite general) kinds of algebraic structure. (On the other hand, we shall be able to touch on only a few of more topics which seem next fundamental, and are also treated in our work, building on what was mentioned. We refer the reader to [11] for those topics on which we could not touch in this article.) A central role will be played by a certain process resulting from an inductivity in the structure of higher coherence (Section 6 below),
which turns out to produce (for good consequences) more general kinds of structure than the process of categorification (or further, the process of lax categorification) does. The subject seems fundamental so the consequences of further research in the direction of our work seem likely to provide useful new viewpoints and methods not simply in algebra or geometry (e.g., through algebro-geometric methodology), but also in various subjects where algebraic methods are useful, such as mathematics, science, engineering and so on.

1.2. In order to proceed, we need a few reminders and notations.

1.3. The notion of coloured operad/multicategory is a generalization of the notion of symmetric monoidal category in the sense that a symmetric monoidal category $A$ has underlying multicategory, which we shall denote by $\Theta A$, from which $A$ can be recovered. An object of $\Theta A$ is an object of $A$, and, for a finite set $S$, an $S$-ary multimap $x = (x_s)_{s \in S} \to y$ in $\Theta A$ is a map $\otimes_s x_s \to y$ in $A$, where $\otimes$ denotes the monoidal multiplication operation on $A$. The structure of $A$ is recovered from the structure of $\Theta A$ since we have a universal $S$-ary multimap $x \to \otimes_s x_s$ in $\Theta A$ for every family $x$ of objects of $A$ indexed by $S$.

An important role of a multicategory is the role of controlling algebras over it. For a multicategory $U$, a $U$-algebra in a symmetric monoidal category $A$ is a functor $U \to \Theta A$ of multicategories. Thus, a multicategory is analogous to an algebraic theory in the sense of Lawvere [9], which controls algebraic structures of a specific kind in a more restricted context.

Many kinds of algebraic structure (in a symmetric monoidal category) are indeed controlled by an uncoloured operad:

- **Commutative:** by “Com”, the terminal multicategory
- **Associative:** by “$E_1$”, the “associative” operad
- **Bare object (no structure):** by “Init”, the initial uncoloured operad

and so on. With colours, more various structures can be controlled.

1.4. For a uncoloured operad $U$, there is a notion of $U$-monoidal category, which categorifies the notion of $U$-algebra in the sense that a $U$-monoidal category is morally a “$U$-algebra” in categories. Namely, a $U$-monoidal structure is an analogue on a category, of the structure of a $U$-algebra. By letting $U$ vary, we obtain symmetric, associative, braided [8], and other notions of monoidal category.

For a coloured multicategory $U$, a similar notion of $U$-monoidal category is also not difficult to define. This is an important example for us, of a categorified kind of structure.

1.5. There are in fact, different kinds of multicategory, among which, symmetric (discussed above), planar and braided [7] are quite commonly worked with. Before getting back to the subject, let us give a description of where those kinds come from, which will motivate development of a general framework.

Given a symmetric multicategory $U$, we obtain the relevant notion, that of $U$-graded multicategory (where grading is in a sense generalizing that e.g., of
a ring), from the notion of $\mathcal{U}$-algebra through a certain general process, which (as well as the output of which) we call theorization inspired by Lawvere’s notion. Examples include:

\[
\begin{align*}
E_1\text{-graded} &= \text{planar} \\
E_2\text{-graded} &= \text{braided} \\
\text{Init-graded multicategory} &= \text{category}
\end{align*}
\]

We consider a symmetric multicategory as ungraded. The notion of graded multicategory (similarly to that of symmetric multicategory) comes in a form enriched in a symmetric monoidal category. It turns out that a $\mathcal{U}$-graded multicategory enriched in sets, is an equivalent datum to a symmetric multicategory lying over $\mathcal{U}$:

\[
\text{Multicat}_\mathcal{U}(\text{Set}) = \text{Multicat}(\text{Set})/\mathcal{U}.
\]

(We may also replace Set with the groupoid-enriched category Gpd of groupoids. Namely, we have

\[
\text{Multicat}_\mathcal{U}(\text{Gpd}) = \text{Multicat}(\text{Gpd})/\mathcal{U},
\]

which is rather natural in the case where $\mathcal{U}$ is enriched in Gpd, e.g., for $\mathcal{U} = E_2$.)

The process of theorization is quite general and can also start from various other kinds of structure than $\mathcal{U}$-algebra. In general, the theorized structure turns out to be a more general kind than a categorified structure (and a lax categorified structure). For example, the notion of $\mathcal{U}$-graded multicategory is a common generalization of the notions of multicategory and of (possibly lax/oplax) $\mathcal{U}$-monoidal category.

**Example 1.1.** Consider a category $\mathcal{C}$ as a multicategory having only unary multimaps. Then, over $\mathcal{C}$, the notions we have are as follows.

**Algebra:** Functor on $\mathcal{C}$ (“left $\mathcal{C}$-module”).

**Categorification:** Functor $\mathcal{C} \to \text{Cat}$.

**Theorization:** In the case enriched in Set, it is category $\mathcal{X}$ equipped with a functor $\mathcal{X} \to \mathcal{C}$, among which, categorifications (and their lax morphisms) correspond to fibrations

\[
\begin{tikzcd}
\mathcal{X}^{\text{op}} \arrow{d} \\
\mathcal{C}^{\text{op}}
\end{tikzcd}
\]

(and not necessarily Cartesian functors over $\mathcal{C}^{\text{op}}$).

1.6. It seems to be a reasonable guess that the notion of $\mathcal{U}$-graded multicategory had been known before our work even in the form enriched in a symmetric monoidal category, and if it had in the enriched form, then it would probably have been a folklore that the notion more generally gets enriched in a $(\mathcal{U} \otimes E_1)$-monoidal category, namely, a $\mathcal{U}$-monoidal object in the 2-category of associative monoidal categories, or equivalently, associative monoidal object in the 2-category of $\mathcal{U}$-monoidal categories.

The idea of theorization leads to an explanation of this enrichment through a natural generalization. Indeed, the notion of $\mathcal{U}$-graded multicategory can
be enriched “along” a “2-dimensional” algebraic structure of a certain kind. The kind of structure is what we call \( \mathcal{U} \)-graded 2-theory, and is a natural theorization of the notion of \( \mathcal{U} \)-graded multicategory. (We refer to a multi-category also as a 1-theory.) This enrichment exists by a general nature of theorization, but turns out to specialize to the mentioned probable folklore.

We would like to take a glance at examples in the case \( \mathcal{U} = \text{Init} \). Note that a 2-category is a categorified Init-graded 1-theory, which therefore, is an instance of an Init-graded 2-theory by what we have remarked above. We shall denote a 2-category by \( \Theta C \) when we consider the former as an Init-graded 2-theory.

For an associative monoidal category \( \mathcal{A} \), let \( B \mathcal{A} \) denote its “categorical deloop”. Thus, \( B \mathcal{A} \) is a 2-category with a base object \( * \), which is the only object of \( B \mathcal{A} \), equipped with an equivalence \( \text{End}_{B \mathcal{A}}(*) \simeq \mathcal{A} \) of associative monoidal categories.

Our first example is about a generalization of delooping for planar multicategories.

**Example 1.2.** For a planar multicategory \( \mathcal{M} \), there is a functorial construction of an Init-graded 2-theory \( B \mathcal{M} \), for which we have

\[
B \Theta \mathcal{A} \simeq \Theta B \mathcal{A}
\]

for an associative monoidal category \( \mathcal{A} \). A category enriched in \( \mathcal{M} \) is in fact an equivalent datum to an Init-graded 1-theory enriched “along” \( B \mathcal{M} \).

For the next example, for a nicely behaving associative monoidal category \( \mathcal{A} \), let \( \text{Alg}_1(\mathcal{A}) \) denote its “Morita” 2-category (due to Bénabou [3]) of associative algebras and bimodules.

**Example 1.3.** For a planar multicategory \( \mathcal{M} \), there is a functorial construction of an Init-graded 2-theory \( \text{Alg}_1(\mathcal{M}) \), for which we have

\[
\text{Alg}_1(\Theta \mathcal{A}) \simeq \Theta \text{Alg}_1(\mathcal{A})
\]

for \( \mathcal{A} \) as above. There is forgetful functor

\[
\text{Alg}_1(\mathcal{M}) \rightarrow B \mathcal{M}
\]

of Init-graded 2-theories, generalizing the forgetful lax functor

\[
\text{Alg}_1(\mathcal{A}) \rightarrow B \mathcal{A}.
\]

A category enriched in \( \mathcal{M} \), or equivalently, an Init-graded 1-theory enriched along \( B \mathcal{M} \) (Example 1.2), has a canonical lift to a category enriched along \( \text{Alg}_1(\mathcal{M}) \).

**Remark 1.4.** In the situation of Example 1.3 if \( \mathcal{M} \) is of the form \( \Theta \mathcal{A} \), then there is another lift of the enriched category resulting from the obvious section \( B \mathcal{A} \rightarrow \text{Alg}_1(\mathcal{A}) \) to the forgetful lax functor, which sends the base object of \( B \mathcal{A} \) to the unit algebra \( 1 \) in \( \mathcal{A} \). However, a multicategory does not in general have a unit object, namely, the nullary tensor product.

1.7. We shall discuss the role of theorization in algebra, and a method for actually theorizing various kinds of algebraic structures iteratively.
2. Higher theories

2.1. The notion of theorization results from the desire to control algebraic structures. For example, ask what may control graded multicategories. A traditional answer to this says that they are controlled by a multicategory, at least under some restriction. For example, for a multicategory $\mathcal{U}$, Baez and Dolan [2] constructs a multicategory $\mathcal{U}^+$, which they call the slice of $\mathcal{U}$, whose algebras in the category of sets are $\mathcal{U}$-graded multicategories with restricted colours:

$$\text{Alg}_{\mathcal{U}^+}(\text{Set}) = \text{Multicat}_{\text{Same colours as } \mathcal{U}^*(\text{Set})/\mathcal{U}}.$$ 

For example, in the case $\mathcal{U} = E_1$, the slice multicategory controls uncoloured planar operads.

While slicing was the main construction for Baez and Dolan, a much simpler structure can in fact control all $\mathcal{U}$-graded multicategories without a restriction on colours. Namely, they are controlled by a certain (ungraded) 2-theory $\Theta_\mathcal{U}$:

$$\text{Multicat}_\mathcal{U} = \text{Alg}_{\Theta_\mathcal{U}},$$

where the construction of $\Theta_\mathcal{U}$ is in a simple one step (of replacing each composition operation in $\mathcal{U}$ with the “bimodule” “corepresented” by it $[\Pi]$), which is essentially as simple as the construction of the underlying multicategory $\Theta_\mathcal{A}$ (recalled in Section 1) of a symmetric monoidal category $\mathcal{A}$.

The key for this simplicity (as well as the ability to treat colours naturally) is that a 2-theory is a 2-dimensional structure. It appears

natural to see 1-dimensional structure (such as multicategory) as algebra over a 2-dimensional structure.

2.2. There are further technical advantages in introducing higher dimensional structures rather than staying in the world of multicategories. For example, the purpose of Baez and Dolan was to give a definition of an $n$-category, which took a few more steps from the slice construction. On the other hand, we can further theorize the notion of 2-theory iteratively with the same method (to be described in Section 6) and the resulting “higher theories” or more specifically, “$n$-theories”, already include $n$-categories, which are iterated categorifications. Thus, (iterative) theorization is a more direct route to $n$-categories than the slicing.

Dimensionality is the key here again: an $n$-category is naturally $n$-dimensional as is an $n$-theory, rather than 1-dimensional like a multicategory.

As we have seen in Section 1 theorization also helps with enriching notions.

2.3. One may wonder what examples of higher theories there are. Higher theories arise for example, through various general constructions from various inputs.

- “Delooping” construction $\mathbb{B}$, of an $(n+1)$-theory from an $n$-theory. For a symmetric monoidal category $\mathcal{A}$, which one may call a categorified “0-theory”, there is an equivalence

$$\mathbb{B}^n\Theta_\mathcal{A} \simeq \Theta^{n+1}B^n\mathcal{A}$$
of \((n+1)\)-theories, where the right hand side is the \((n+1)\)-theory which corresponds to the symmetric monoidal \((n+1)\)-category \(B^nA\), the \(n\)-fold deloop of \(A\).

- The “pull-back” of grading (Section 4), and “push-forward” constructions which respectively have a universal property as the left and the right adjoint.
- A generalization of the Day convolution \([6]\).

Many of these constructions raise theoretic order, so interests at least in multicategories, i.e., 1-theories, would naturally lead eventually to interests in all higher theories.

There are also some concrete constructions of higher theories in more specific situations.

3. Meaning of theorization

3.1. The effectiveness of use of multicategories results from the ability to express algebras of interest as functors of multicategories (whenever possible). This is analogous to a lesson learned in linear algebra: matrices and their multiplication become most clearly understandable when one regards matrices as linear maps between vector spaces.

Theorization is a method for finding for a given kind of algebraic structure, say “\(X\)-algebra”, a new kind of structure, say “\(X\)-theory”, such that structures similar (in a way) to \(X\)-algebras can be expressed as (“coloured”) morphisms of \(X\)-theories. (We can naturally include colours when an \(X\)-algebra may have colours as a multicategory does for example.) We consider structures expressible as coloured morphisms of \(X\)-theories as controlled by the source \(X\)-theory, and enriched “along” the target \(X\)-theory.

The relation of theorization with categorification is understood from this point of view. For us, relevance of categorification comes from the principle that

\[ \text{a given notion “\(X\)-algebra”, makes sense in a categorified form of \(X\)-algebra.} \]

E.g., for a multicategory \(\mathcal{U}\), there is a notion of \(\mathcal{U}\)-algebra in a \(\mathcal{U}\)-monoidal category \(\mathcal{A}\). Such a structure is equivalent to (or defined as) a lax \(\mathcal{U}\)-monoidal functor \(1^0_{\mathcal{U}} \to \mathcal{A}\), where \(1^0_{\mathcal{U}}\) denotes the unit \(\mathcal{U}\)-monoidal category. This is an enriched notion of \(\mathcal{U}\)-algebra, and in general, a theorized structure (\(\mathcal{U}\)-graded multicategory in this case) is a more general place where the original notion (the notion of \(\mathcal{U}\)-algebra here) can be enriched.

3.2. To be more specific, by theorizing a notion “\(X\)-algebra” to a notion “\(X\)-theory”, we want a consequence of the form “for a categorified form \(\mathcal{A}\) of \(X\)-algebra,

\[ \text{the datum of an \(X\)-algebra in } \mathcal{A} \text{ is equivalent to the datum of a “coloured” functor } 1^\text{Theory}_X \to \Theta \mathcal{A} \text{ of } X\text{-theories}, \]

where

- \(1^\text{Theory}_X\) denotes the terminal \(X\)-theory,
- \(\Theta \mathcal{A}\) denotes \(\mathcal{A}\) as an \(X\)-theory,
- we omit explanation of colouring, which is only technical.
By an X-algebra in \( \mathcal{A} \), we mean a suitably coloured lax functor \( 1^\text{Algebra}_X \to \mathcal{A} \) of X-algebras.

Once we have a notion of X-theory satisfying this, it is reasonable to define algebra over an X-theory \( \mathcal{U} \) as a (coloured) functor on \( \mathcal{U} \), so we will have the picture

\[
\begin{array}{ccc}
\text{X-theory} & \ni & \mathcal{U} \\
\xrightarrow{\text{theorize}} & & \downarrow \text{control} \\
\text{X-algebra} & \xleftarrow{\text{forbid colours}} & \mathcal{U}\text{-algebra} \\
\end{array}
\]

We will obtain enriched notions of algebra, which are interrelated as follows.

\[
\begin{array}{ccc}
\text{Functor} & \xleftarrow{\text{forbid colours}} & \text{U-algebra} \\
\mathcal{U} \to \mathcal{V} & \xrightarrow{\mathcal{U}=\Theta^\text{Theory}_X} & \text{X-algebra} \\
\downarrow & \downarrow & \downarrow \\
\mathcal{V}=\Theta & \xleftarrow{\text{forbid colours}} & \text{U-graded \( (n+1) \)-theory} \\
\text{U-algebra in } \mathcal{A} & \xrightarrow{\mathcal{U}=\Theta^\text{Theory}_X} & \text{X-algebra} \\
\end{array}
\]

where

\( \mathcal{U}, \mathcal{V} : \text{X-theories} \)

\( \mathcal{A} : \text{categorified X-algebra.} \)

3.3. The way how we can actually theorize in the desired manner in many situations, will be discussed in Section 6.

4. Graded higher theories

4.1. Let us describe a large system to which iterative theorization of the notion of multicategory leads.

4.2. Given an \( n \)-theory \( \mathcal{U} \), there is a notion of \( \mathcal{U}\text{-graded } n\text{-theory} \), which theorizes the notion of \( \mathcal{U}\text{-algebra.} \)

The following theorem generalizes our method of controlling graded multicategories mentioned in Section 2. As before, it is easy to construct an \( (n+1)\)-theory \( \Theta \mathcal{U} \) “underlying” \( \mathcal{U} \).

**Theorem 4.1.** The following forms of datum are equivalent:

- a \( \mathcal{U}\text{-graded } n\text{-theory} \\
- a \( \Theta \mathcal{U}\text{-algebra.} \\

There is also a notion of \( \mathcal{U}\text{-graded } (n+1)\text{-theory} \), which naturally theorizes the notion of \( \mathcal{U}\text{-graded } n\text{-theory} \), and we have the following theorem.

**Theorem 4.2.** The following forms of datum are equivalent:

- a \( \mathcal{U}\text{-graded } (n+1)\text{-theory} \\
- a \( \Theta \mathcal{U}\text{-graded } (n+1)\text{-theory.} \\

By the general principle, a \( \mathcal{U}\text{-graded } (n+1)\text{-theory} \) is a natural place where the notion of \( \mathcal{U}\text{-graded } n\text{-theory} \) can be enriched. In the case \( n = 1 \), this specializes to the enrichment of the notion of \( \mathcal{U}\text{-graded} \) multicategory in a \( (\mathcal{U} \otimes E_1)\)-monoidal category, as mentioned in Section 1.
The above theorems lead to iterative theorization. Namely, they make it immediate to figure out the right notion of $U$-graded $m$-theory for $m \geq n+2$.

4.3. There are also graded “lower” theories.

Let $U$ be an $n$-theory. Then, for an integer $m$ such that $0 \leq m \leq n-1$, there is a notion of $U$-graded $m$-theory so that

the notion of $U$-graded $(m+1)$-theory theorizes the notion of $U$-graded $m$-theory,

and

the notion of $U$-algebra is equivalent to the notion of $U$-graded $(n-1)$-theory.

Moreover, there is a notion of $\ell$-theory graded by a $U$-graded $m$-theory. We obtain for example, equivalence of the following notions for every integer $m \geq 0$:

- $1^m_U$-graded $\ell$-theory
- $U$-graded $\ell$-theory

where $1^m_U$ denotes the terminal $U$-graded $m$-theory.

5. More general higher theories: some examples

5.1. We can also theorize some structures which (unlike a multicategory) involve operations with multiple inputs and multiple outputs. More precisely, consider each symmetric monoidal category $B$ as controlling symmetric monoidal functors $B \to A$, where $A$ varies among symmetric monoidal categories. This “kind” of structure “controlled” by $B$, can be theorized iteratively if $B$ is generated in a certain nice manner. We refer to the resulting notion as the notion of $B$-graded $n$-theory.

Remark 5.1. In this terminology, a symmetric multicategory will be a “Fin-graded” 1-theory, where Fin denotes the category of finite sets made symmetric monoidal under the operations of disjoint union. Note the difference of this notion from a $\Theta\text{Fin}$-graded 1-theory in the sense of Section 1. In the terminology here, the term 1-theory does not translate as multicategory, which is the point of the generalization.

For example, the notion of coloured properad of Vallette [12] turns out to be equivalent to the notion of 1-theory graded by $\text{Cocorr}(\text{Fin})$, the core correspondence category (enriched in groupoids) on Fin. The notions of $\text{Cocorr}(\text{Fin})$-graded $n$-theory iteratively theorize the notion of coloured properad.

5.2. In another instance, we find structures of somewhat unexpected nature. Namely, for an infinity 1-category $C$, there is a “Bord$_1$-graded” 1-theory $Z_C$, where Bord$_1$ denotes the oriented 1-dimensional bordism category (enriched in groupoids) “controlling” 1-dimensional oriented (equivalently, framed) TFT’s, such that every “1-dimensional TFT” in $Z_C$, namely, functor

$$1^1_{\text{Bord}_1} \longrightarrow Z_C$$

of Bord$_1$-graded 1-theories, is of the form

$$1^1_{\text{Bord}_1} = Z_1 \xrightarrow{Z_x} Z_C$$
for an unique object
\[ x: 1 \rightarrow C \]
of \( C \).

This looks very different from field theories in the original, untheorized context, which, according to the cobordism hypothesis, are classified by dualizable objects of a symmetric monoidal infinity 1-category \([10]\). Yet the notions are parallel, so a unified treatment (as far as that goes) is at least natural.

6. Higher coherence and iterative theorization

6.1. The way how we can iteratively theorize kinds of structure discussed in the previous sections, is by using an inductivity embedded in the structure of the coherence for higher associativity.

To describe the idea of this inductivity, suppose that we have the following data.

\[ m: \text{a collection of operations wanting to be associative, operating as maps in an symmetric monoidal infinity 1-category } A. \text{ E.g., “m with a coherent associativity” may define an “X-algebra” enriched in A.} \]

\[ m': \text{collection of 2-isomorphisms/homotopies giving an associativity of m.} \]

Then we would wish to make \( m' \) coherent.

In the case where “X-algebra” (i.e., the kind of the structure being constructed in the described situation) means any of the kinds considered in the previous sections for theorization, there is a way to organize \( m' \) so we can see it as a collection of operations themselves in such a manner that associativity of those operations \( m' \) makes sense. (This is explicitly visible for instance, in the definition of a symmetric \( n \)-theory in \([11]\).)

Remark 6.1. It may not seem natural to try to see \( m' \) as operations since we required \( m' \) to consist of 2-isomorphisms rather than possibly non-invertible 2-morphisms. However, for \( m' \) in the suitably organized form as mentioned, we do not actually need to require invertibility since, without invertibility, we would still have the structure of an (op)lax X-algebra, with which we could be contented.

Moreover, we obtain equivalence of the following forms of datum:

- a coherence of the associativity \( m' \) for \( m \)
- a coherent associativity of \( m' \) as operations.

6.2. The described inductivity is relevant to theorization since many kinds of structure can be theorized using the idea that, if \( m \) gives an “X-algebra” structure, then its theorization, “X-theory”, will be a kind defined (in the form enriched in a symmetric monoidal category \( A \)) by operations \( m' \) considered in the “categorical deloop” \( BA \) (so 2-morphisms \( m' \) are maps in \( A = \text{End}_{BA}(*)) \).

Since the structure is inductive, we can iterate theorization, and get to a notion of \( n \)-theory for every integer \( n \geq 0 \), generalizing in particular, the notion of \( n \)-category. The definition of an \( n \)-theory can be written explicitly \([11]\).
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