SU(2) × SU(2)-invariant Scattering Matrix

of the Hubbard Model

Fabian H.L. Eßler

Physikalisches Institut der Universität Bonn
Nussallee 12, 53115 Bonn

and

Vladimir E. Korepin

Institute for Theoretical Physics
State University of New York at Stony Brook
Stony Brook, NY 11794-3840

ABSTRACT

We consider the one-dimensional half-filled Hubbard model. We show that the excitation spectrum is given by the scattering states of four elementary excitations, which form the fundamental representation of SU(2) × SU(2). We determine the exact two-particle Scattering matrix, which a solution of the Yang-Baxter equation and reflects the SO(4) symmetry of the model. The results for repulsive and attractive Hubbard model are related by an interchange of spin and charge degrees of freedom.
1. Introduction

Strongly correlated electronic systems are currently under intense study in relation with the phenomenon of high-$T_c$ superconductivity. The two-dimensional Hubbard model is believed to be the most promising candidate for an electronic theory of superconductivity. It is believed to share important features with its one-dimensional analog\(^\text{[1]}\). An important problem in understanding the dynamics of the Hubbard model is the separation of spin and charge in one\(^\text{[2,3]}\) and two dimensions\(^\text{[4]}\). A particularly interesting feature of the Hubbard model is that in one dimension it can be solved exactly by means of the Bethe Ansatz\(^\text{[5]}\). Thus it is possible to obtain a complete and unambiguous picture of the dynamics in one dimension. The main purpose of this paper is to use the Bethe Ansatz solution to demonstrate that the one-dimensional Hubbard model is a factorizable scattering theory of four quasiparticles. These elementary excitations transform in the fundamental representation of $SU(2) \times SU(2)$.

The Hubbard hamiltonian is given by the following expression

$$H = -\sum_{j=1}^{L} \sum_{\sigma=\uparrow,\downarrow} \left( c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j,\sigma} \right) + 4U \sum_{j=1}^{L} (n_{j,\uparrow} - \frac{1}{2})(n_{j,\downarrow} - \frac{1}{2}). \quad (1.1)$$

Here $c_{j,\sigma}^\dagger$ are canonical fermionic creation operators on the lattice, $j$ labels the sites of a chain of even length $L$, $\sigma$ labels the spin degrees of freedom, $U$ is the coupling constant, and $n_{j,\sigma} = c_{j,\sigma}^\dagger c_{j,\sigma}$ is the number operator for spin $\sigma$ on site $j$. The hamiltonian (1.1) is invariant under a $SO(4)=SU(2) \times SU(2)/\mathbb{Z}_2$ algebra\(^\text{[6–10]}\). The two $SU(2)$ algebras are the spin $SU(2)$ generated by $S = \sum_{j=1}^{L} c_{j,\uparrow}^\dagger c_{j,\downarrow}$, $S^\dagger = \sum_{j=1}^{L} c_{j,\downarrow}^\dagger c_{j,\uparrow}$, $S^z = \sum_{j=1}^{L} \frac{1}{2}(n_{j,\uparrow} - n_{j,\downarrow})$ \quad (1.2) spin

and the $\eta$-$SU(2)$ algebra

$$\eta = \sum_{j=1}^{L} (-1)^j c_{j,\uparrow}^\dagger c_{j,\downarrow}, \quad \eta^\dagger = \sum_{j=1}^{L} (-1)^j c_{j,\downarrow}^\dagger c_{j,\uparrow}, \quad \eta^z = \frac{1}{2} \sum_{j=1}^{L} (n_{j,\uparrow} + n_{j,\downarrow} - 1). \quad (1.3) \text{eta}$$

All 6 generators commute with the hamiltonian (1.1). Clearly the operator $S^z + \eta^z$ has only integer eigenvalues (as $L$ is even) and all half-odd integer representations of $SU(2) \times SU(2)$ are projected out. Thus the symmetry algebra is $SO(4)$ as asserted above.

The nested Bethe Ansatz for the Hubbard model\(^\text{[5]}\) provides eigenstates of the hamiltonian that are parametrized by sets of so-called spectral parameters $k_j$ and $\Lambda_\alpha$. These parameters are subject to the Lieb-Wu equations

$$e^{ik_jL} = \prod_{\alpha=1}^{M} \frac{\sin(k_j) - \Lambda_\alpha + iU}{\sin(k_j) - \Lambda_\alpha - iU}, \quad j = 1, \ldots, M + N \quad (1.4) \text{liebwu}$$

$$\prod_{i=1}^{M+N} \frac{\sin(k_i) - \Lambda_\beta + iU}{\sin(k_i) - \Lambda_\beta - iU} = -\prod_{\alpha=1}^{M} \frac{\Lambda_\alpha - \Lambda_\beta + 2iU}{\Lambda_\alpha - \Lambda_\beta - 2iU}, \quad \beta = 1, \ldots, M.$$
These equations are the basis for the determination of ground state and excitation spectrum in the thermodynamic limit.

There has been substantial previous work on the excitation spectrum of the repulsive, half-filled Hubbard model. A.A. Ovchinnikov calculated the spin-triplet excitation \cite{11}, which was then re-examined by T.C. Choy and W. Young in \cite{12}. In his series of excellent papers \cite{2,3,13} F. Woynarovich gave a very detailed analysis of both spin and charge excitations. The attractive case was studied by B. Sutherland, who investigated the Bethe Ansatz structure of the ground state, and by F. Woynarovich, who determined ground state and excitation spectrum from the known results for the repulsive case by using discrete symmetries of the hamiltonian \cite{14}. This method, while providing a nice picture of the relations between the Bethe Ansätze for repulsion and attraction has the drawback of making assumptions about transformation properties of spectral parameters under the map connecting the $U > 0$ and $U < 0$ cases. In section 2 we present a direct Bethe Ansatz analysis of the ground state and excitation spectrum in the two-particle sector of the attractive Hubbard model. Our direct computation as expected confirms Woynarovich’s results for the dispersion relations \cite{14}. The main purpose of this section is to determine the so-called “shift-functions” \cite{15,16}, which are needed to evaluate the two-particle S-matrix. In section 3 we give a quasiparticle interpretation of the excitation spectrum. All excited states are scattering states of only four quasiparticles (or elementary excitations), which form the fundamental representation of $SU(2) \times SU(2)$. Two of these elementary excitations carry spin but no charge, and two carry charge but no spin. This reflects the separation of spin and charge degrees of freedom in the one-dimensional Hubbard model. In section 4 we evaluate the exact S-matrix of the attractive Hubbard model. It is a solution of the Yang-Baxter equation, which implies the two-particle reducibility of the $N$-body S-matrix. In section 5 we give the results for the repulsive case, and in section 6 we show that attractive and repulsive Hubbard model are related by an interchange of spin-and charge degrees of freedom. This relation holds both on the level of the quasiparticle spectrum and on the level of the S-matrix. Finally section 7 is devoted to a summary and discussion of our results. The appendix deals with the problem of showing that all excitations in the $N$-particle sector can be interpreted as scattering states of the four elementary excitations.

2. Attractive Hubbard Model

In this section we construct the ground state and all excitations in the two-particle sector for the attractive case $U < 0$. Starting point are the Lieb-Wu equations (1.4). The solutions of (1.4) can be split into three different types of subsets (strings), which are \cite{17}:

1. a single real momentum $k_i$
2. $m \Lambda_\alpha$’s combine into a string-type configuration (‘$\Lambda$-strings’); this includes the case $m = 1$, which is just a single real $\Lambda_\alpha$
(3) $2m$ $k_i$’s and $m$ $\Lambda_\alpha$’s combine into a different string-type configuration (‘$k$-$\Lambda$-strings’)

For a $\Lambda$-string of length $m$ the rapidities involved are

$$\Lambda_{\alpha}^{m,j} = \Lambda_{\alpha}^{m} + i(m + 1 - 2j)|U|, \quad \Lambda_{\alpha}^{m} \text{ real} \quad j = 1, 2, \ldots, m. \quad \text{(2.1) lambdastr}$$

The $k$’s and $\Lambda$’s involved in a $k$-$\Lambda$-string are

$$k_{\alpha}^1 = \sin^{-1}(\Lambda_{\alpha}^{m} + im|U|), \quad k_{\alpha}^2 = \sin^{-1}(\Lambda_{\alpha}^{m} + i(m - 2)|U|), \quad k_{\alpha}^3 = \pi - k_{\alpha}^2,$$

$$k_{\alpha}^4 = \sin^{-1}(\Lambda_{\alpha}^{m} + i(m - 4)|U|), \quad k_{\alpha}^5 = \pi - k_{\alpha}^4,$$

$$\ldots$$

$$k_{\alpha}^{2m-2} = \sin^{-1}(\Lambda_{\alpha}^{m} - i(m - 2)|U|), \quad k_{\alpha}^{2m-1} = \pi - k_{\alpha}^{2m-2}$$

$$k_{\alpha}^{2m} = \sin^{-1}(\Lambda_{\alpha}^{m} - im|U|). \quad \text{(2.2) klstr1}$$

and

$$\Lambda_{\alpha}^{m,j} = \Lambda_{\alpha}^{m} + i(m + 1 - 2j)|U|, \quad \Lambda_{\alpha}^{m} \text{ real} \quad j = 1, 2, \ldots, m. \quad \text{(2.3) klstr2}$$

Equations (2.1) - (2.3) are valid up to exponential corrections of order $\mathcal{O}(\exp(-\delta L))$, where $\delta$ is real and positive (as long as the real parts of the spectral parameters are much smaller than $L$). Note that the $k$-$\Lambda$-strings for the attractive Hubbard model are different from the repulsive case treated by M. Takahashi in [17].

We now consider a solution of (1.4) with $M_n$ strings of type (2.1) of length $n$, $M'_n$ $k$-$\Lambda$-strings (2.2), (2.3) of length $n$, and a total number of electrons $N_e$. Inserting the prescription (2.1) - (2.3) into the Lieb-Wu equations (1.4) and taking the logarithm we arrive at

$$k_j L = 2\pi I_j + \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M_n} \theta \left( \frac{\sin k_j - \Lambda_{\alpha}^{n}}{n|U|} \right) + \sum_{n=1}^{\infty} \sum_{\beta=1}^{M'_n} \theta \left( \frac{\sin k_j - \Lambda_{\beta}^{n}}{n|U|} \right), \quad \text{(4.4) pbc}$$

$$\sum_{j=1}^{N_e-2M'} \theta \left( \frac{\Lambda_{\alpha}^{n} - \sin k_j}{n|U|} \right) = 2\pi J_{\alpha}^{n} + \sum_{m=1}^{\infty} \sum_{\beta=1}^{M'_n} \theta \left( \frac{\Lambda_{\alpha}^{n} - \Lambda_{\beta}^{m}}{|U|} \right), \quad \text{(4.5) pbc1b}$$

$$L \left[ \sin^{-1} \left( \frac{\Lambda_{\alpha}^{n} + in|U|}{|U|} \right) + \sin^{-1} \left( \frac{-\Lambda_{\alpha}^{n} - in|U|}{|U|} \right) \right]$$

$$= 2\pi J_{\alpha}^{n} + \sum_{j=1}^{N_e-2M'} \theta \left( \frac{\Lambda_{\alpha}^{n} - \sin k_j}{n|U|} \right) + \sum_{m=1}^{\infty} \sum_{\beta=1}^{M'_n} \theta \left( \frac{\Lambda_{\alpha}^{n} - \Lambda_{\beta}^{m}}{|U|} \right), \quad \text{(4.6) pbc2}$$

where $L = 2 \times \text{odd}$ is the even length of the lattice, $I_j$, $J_{\alpha}^{n}$, and $J_{\alpha}^{n}$ are integer or half-odd integer numbers, $M' = \sum_{n=1}^{\infty} nM'_n$, $\theta(x) = 2 \arctan(x)$, and

$$\theta_{nm}(x) = \begin{cases} 
\theta \left( \frac{x}{n-m} \right) + 2 \theta \left( \frac{x}{n-m+2} \right) + \ldots + 2 \theta \left( \frac{x}{n+m} \right) + \theta \left( \frac{x}{m-n} \right) & \text{if } n \neq m \\
2 \theta \left( \frac{x}{2} \right) + 2 \theta \left( \frac{x}{4} \right) + \ldots + 2 \theta \left( \frac{x}{2n} \right) + \theta \left( \frac{x}{2} \right) & \text{if } n = m.
\end{cases} \quad \text{(4.7) theta}$$
The integer (half-odd integer) numbers in (2.4)-(2.6) have the ranges

\[
|J_\alpha^n| \leq \frac{1}{2}(N_e - 2M' - \sum_{m=1}^{\infty} t_{nm}M_m - 1),
\]

(2.8) ineq

\[
|J'_\alpha^n| \leq \frac{1}{2}(L - N_e + 2M' - \sum_{m=1}^{\infty} t_{nm}M'_m - 1), \quad -\frac{L-1}{2} \leq I_j \leq \frac{L-1}{2},
\]

where \( t_{mn} = 2\min\{m, n\} - \delta_{mn} \). Each set of “integers” \( \{I_j\}, \{J_\alpha^n\}, \{J'_\alpha^n\} \) is in one-to-one correspondence with a set of spectral parameters, which in turn unambiguously specifies one eigenstate of the Hamiltonian (1.1). Energy and momentum of a solution of the system (2.4)-(2.6) are found to be

\[
E = -2 \sum_{l=1}^{N_e-2M'} \cos(k_l) - 4 \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M'_n} \text{Re} \left\{ \sqrt{1 - (\Lambda^n_{\alpha} + i|U|n)^2} \right\} - 2UN + UL,
\]

(2.9) E

\[
P = \sum_{j=1}^{N_e-2M'} k_j + \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M'_n} (2\text{Re} \{\arcsin(\Lambda^n_{\alpha} + i|U|n)\} + \pi(n-1)) .
\]

(2.10) P

Equations (2.4) - (2.10) can now be used to determine the ground state energy and to construct excitations. The ground state can be found by taking the zero temperature limit of the Thermodynamic Bethe Ansatz based on (2.4)-(2.9), which can be constructed analogously to the repulsive case[17] (see the Appendix of this paper). An important element in our construction of excitations over the ground state is a theorem proved in [18], which states that

(i) All eigenstates of the Hamiltonian (1.1) with finite spectral parameters, \( N_\downarrow < N_\uparrow \), and \( N_e \leq L \) are lowest weight states of the \( SO(4) \) algebra (1.2),(1.3). These states are called regular Bethe states.

(ii) The set of states obtained by acting with the \( SO(4) \) raising-operators on the regular Bethe states is complete and forms a basis of the electronic Hilbert space.

This theorem implies that excitations constructed by means of the Bethe Ansatz are all lowest weight states of the \( SO(4) \) algebra. Furthermore all \( SO(4) \)-descendants of these excitations must be taken into account in order to obtain a complete set of excited states.

**Ground State**

The ground state for the half-filled band is characterized by choosing \( M'_1 = \frac{L}{2}, M_n = 0 \ \forall n, M'_n = 0 \ \forall n \geq 2 \), and filling all \( \frac{L}{2} \) vacancies given by (2.8) for the integers \( J'_\alpha^1 \) symmetrically between \(-\frac{L-2}{4}\) and \( \frac{L-2}{4} \). The Lieb-Wu equations (2.4)-(2.6) for the ground state are

\[
2L \text{Re} \{\arcsin(\Lambda_\alpha + i|U|)\} = 2\pi J'_\alpha^1 + \sum_{\beta=1}^{L \over 2} \theta \left( \frac{\Lambda_\alpha - \Lambda_\beta}{2|U|} \right), \quad \alpha = 1, \ldots, \frac{L}{2}.
\]

(2.11) GSEQ
Subtracting equations (2.11) for consecutive $\alpha$’s and using $J_{\alpha+1}^1 - J_{\alpha}^1 = 1$ we obtain an equation for the finite-interval density

$$\rho_0(\Lambda_\alpha) = \frac{1}{L(\Lambda_{\alpha+1} - \Lambda_\alpha)}.$$  \hfill (2.12) \rho

In the thermodynamic limit this equation turns into an integral equation for the ground state density $\rho_0(\Lambda)$ (which is defined as the limit of (2.12))

$$\rho_0(\Lambda) = \frac{1}{\pi} \text{Re} \left\{ \frac{1}{\sqrt{1 - (\Lambda + i|U|)^2}} \right\} - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\Lambda' \frac{4|U|}{4U^2 + (\Lambda - \Lambda')^2} \rho_0(\Lambda') ,$$  \hfill (2.13) GSIE

The ground state energy per site can be evaluated by using (2.9)

$$\frac{E_{GS}}{L} = |U| - 4 \int_{-\infty}^{\infty} d\Lambda \text{ Re} \left\{ \sqrt{1 - (\Lambda + i|U|)^2} \right\} \rho_0(\Lambda)$$

$$= -|U| - \int_{-\infty}^{\infty} d\omega \frac{\exp(-|\omega U|)}{\omega \cosh(\omega U)} J_0(\omega)J_1(\omega) .$$  \hfill (2.14) GSE

Next we will determine excitations over the ground state (2.13) by means of the so-called “shift-function” method, which is explained in detail in [16] and [15].
Charge-Triplet Excitation

The lowest weight state of the charge triplet \( \eta^z = -1 \) is obtained by choosing \( M_1' = \frac{L}{2} - 1 \). The allowed range (2.8) of the half-odd integers \( J_0' \) is \( [-\frac{L}{4}, \frac{L}{4}] \), so that there are \( \frac{L}{2} + 1 \) vacancies and thus two “holes”. We denote the spectral parameters corresponding to these holes by \( \Lambda^h_1 \) and \( \Lambda^h_2 \). The Lieb-Wu equations read

\[
2L \, \text{Re} \left\{ \arcsin(\tilde{\Lambda}_\alpha + i|U|) \right\} = 2\pi J_0'^{\prime} + \sum_{\beta=1}^{\frac{L}{2}+1} \theta(\frac{\tilde{\Lambda}_\alpha - \tilde{\Lambda}_\beta}{2|U|}) - \sum_{j=1}^{2} \theta(\frac{\tilde{\Lambda}_\alpha - \Lambda^h_j}{2|U|}) \tag{2.15} \]

Here the tilde indicates that the spectral parameters and the integers have changed as compared to their ground state distributions. Our convention is \( \tilde{J}_\alpha' - J_\alpha' = \frac{1}{2} \) for \( \alpha = 1 \ldots \frac{L}{2} \).

As compared to the ground state there is one more vacancy in the excited state and thus one more \( \tilde{J}_\alpha' \), which in our convention is the minimal allowed half-odd integer \( -\frac{L}{4} \). We denote the corresponding spectral parameter by \( \Lambda_{\min} \) and note that in the thermodynamic limit \( \Lambda_{\min} \to -\infty \). Subtracting (2.15) from (2.11) and using that \( \tilde{\Lambda}_\alpha - \Lambda_\alpha = O(\frac{1}{L}) \) we find

\[
2L \, \text{Re} \left\{ \frac{1}{\sqrt{1 - (\Lambda_\alpha + i|U|)^2}} \right\} \left( \tilde{\Lambda}_\alpha - \Lambda_\alpha \right) = \pi + \theta(\frac{\tilde{\Lambda}_\alpha - \Lambda_{\min}}{2|U|}) - \sum_{j=1}^{2} \theta(\frac{\tilde{\Lambda}_\alpha - \Lambda^h_j}{2|U|}) \\
+ \sum_{\beta=1}^{\frac{L}{2}} \frac{4|U|}{4U^2 + (\Lambda_\alpha - \Lambda_\beta)^2} \left( \tilde{\Lambda}_\alpha - \Lambda_\alpha - (\tilde{\Lambda}_\beta - \Lambda_\beta) \right) \tag{2.16} \]

We now define the “shift” function \( F_{CT} \) for the charge-triplet according to

\[
F_{CT}(\Lambda_\alpha) = \frac{\tilde{\Lambda}_\alpha - \Lambda_\alpha}{\Lambda_{\alpha+1} - \Lambda_\alpha} \tag{2.17} \]

and take the thermodynamic limit of (2.16)

\[
\frac{F_{CT}(\Lambda)}{\rho(\Lambda)} \left( 2 \, \text{Re} \left\{ \frac{1}{\sqrt{1 - (\Lambda + i|U|)^2}} \right\} - \int_{-\infty}^{\infty} d\lambda' \, \frac{4|U|}{4\lambda'^2 + (\Lambda - \lambda')^2} \rho(\lambda') \right) \\
= 2\pi - \int_{-\infty}^{\infty} d\lambda' \, \frac{4|U|}{4\lambda'^2 + (\Lambda - \lambda')^2} F_{CT}(\lambda') - \sum_{j=1}^{2} \theta(\frac{\Lambda - \Lambda^h_j}{2|U|}) \tag{2.18} \]

To order \( O(L^{-1}) \) \( \rho(\Lambda) \) is the same as \( \rho_0(\Lambda) \), so that we can use (2.13) in (2.18), obtaining an integral equation for \( F_{CT} \)

\[1\]The results for energy, momentum, and scattering phase are of course independent of our choice of convention.
\[
F_{CT}(\Lambda) = 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\Lambda' \frac{4|U|}{4U^2 + (\Lambda - \Lambda')^2} F_{CT}(\Lambda') - \frac{1}{2\pi} \sum_{j=1}^{2} \theta\left(\frac{\Lambda - \Lambda_h^j}{2|U|}\right).
\]  
(2.19) F2

Equation (2.19) has the solution
\[
F_{CT}(\Lambda) = 1 - \frac{1}{2\pi} \left(\gamma\left(\frac{\Lambda - \Lambda_h^1}{2|U|}\right) + \gamma\left(\frac{\Lambda - \ Lambda_h^2}{2|U|}\right)\right),
\]  
(2.20) F4

where \(\gamma(\mu)\) is defined as
\[
\gamma(\mu) = \int_{-\infty}^{\infty} \frac{d\omega}{i\omega} \frac{\exp(i\omega\mu)}{1 + \exp(|\omega|)} = i \ln \left(\Gamma\left(1 + \frac{i\mu}{2}\right)\Gamma(1 - \frac{i\mu}{2})\Gamma(1 + \frac{i\mu}{2})\Gamma(1 - \frac{i\mu}{2})\right).
\]  
(2.21) fadtakh

The excitation energy follows from (2.9)
\[
E_{CT} = 4 \int_{-\infty}^{\infty} d\Lambda \text{Re}\left\{ \frac{\Lambda + i|U|}{\sqrt{1 - \Lambda^2}} \right\} F_{CT}(\Lambda)
+ 4 \sum_{j=1}^{2} \left[ \text{Re}\left\{ \frac{\sqrt{1 - (\Lambda_h^j + i|U|)^2}}{1 - (\Lambda_h^j + i|U|)^2} |U| \right\} - |U| \right]
= \epsilon_{cw}(\Lambda_h^1) + \epsilon_{cw}(\Lambda_h^2),
\]  
(2.22) ECT

where
\[
\epsilon_{cw}(\lambda) = 2 \int_{0}^{\infty} d\omega \frac{J_1(\omega) \cos(\omega\lambda)}{\cosh(\omega U)}.
\]  
(2.23) es

The momentum of the excitation is computed by means of (2.10)
\[
P_{CT} = \int_{-\infty}^{\infty} d\Lambda 2\text{Re}\left\{ \frac{1}{\sqrt{1 - (\Lambda + i|U|)^2}} \right\} F_{CT}(\Lambda)
- \sum_{j=1}^{2} \text{Re}\left\{ \arcsin(\Lambda_h^j + i|U|) \right\} - \pi
= p_{cw}^h(\Lambda_h^1) + p_{cw}^h(\Lambda_h^2),
\]  
(2.24) PCT

where
\[
p_{cw}^h(\lambda) = - \int_{0}^{\infty} d\omega \frac{J_0(\omega) \sin(\omega\lambda)}{\omega \cosh(\omega U)} - \frac{\pi}{2} \leq p_{cw}^h(\lambda) \leq \frac{\pi}{2}.
\]  
(2.25) ps

The quantum numbers of this type of excitation are (due to the lowest weight theorem) \(S = 0, \eta^z = -1, \eta = 1\), and the state is lowest weight of \(SO(4)\). The complete multiplet is obtained by acting with \(\eta^\dagger\) and \((\eta^\dagger)^2\), which leaves the energy but not the momentum invariant \((\eta^\dagger\) does not commute with the momentum operator).

**Charge-Singlet Excitation**

The charge singlet is obtained by choosing \(M_1' = L_2 + 2\) and \(M_2' = 1\). The allowed range of vacancies for the integers \(J_{\alpha}^n\) is \([-\frac{L^2}{4}, \frac{L^2}{4}]\), so that there are again two holes with
corresponding spectral parameters $\Lambda_1^h, \Lambda_2^h$ in the distribution. The integer $J_{\alpha}^2$ can take the value 0 only. We denote the spectral parameter corresponding to the $k$-$\Lambda$-string of length 2 by $\kappa$. The Lieb-Wu equations (1.4) for this type of excitation are

$$2L \Re \left\{ \arcsin(\tilde{\Lambda}_\alpha + i|U|) \right\} = 2\pi J_{\alpha}^1 + \sum_{\beta=1}^{4} \theta(\frac{\tilde{\Lambda}_\alpha - \tilde{\Lambda}_\beta}{2|U|}) - \sum_{j=1}^{2} \theta(\frac{\tilde{\Lambda}_\alpha - \Lambda_j^h}{2|U|}) + \theta_{12}(\frac{\tilde{\Lambda}_\alpha - \kappa}{|U|}) \tag{2.26}$$

$$2L \Re \left\{ \arcsin(\kappa + 2i|U|) \right\} = \sum_{\beta=1}^{4} \theta_{21}(\frac{\kappa - \tilde{\Lambda}_\beta}{|U|}) - \sum_{j=1}^{2} \theta_{21}(\frac{\kappa - \Lambda_j^h}{|U|}) .$$

In the thermodynamic limit they turn into integral equations for an $F$-function and a density $\rho$ defined like in (2.17) and (2.12)

$$F_{CS}(\Lambda) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\Lambda' \frac{4|U|}{4U^2 + (\Lambda - \Lambda')^2} F_{CS}(\Lambda') + \theta_{12}(\frac{\Lambda - \kappa}{|U|}) - \sum_{j=1}^{2} \theta(\frac{\Lambda - \Lambda_j^h}{2|U|}) \tag{2.27}$$

$$2 \Re \left\{ \arcsin(\kappa + 2i|U|) \right\} = \int_{-\infty}^{\infty} d\Lambda \rho(\Lambda) \theta_{21}(\frac{\kappa - \Lambda}{|U|}) - \frac{1}{L} \sum_{j=1}^{2} \theta_{21}(\frac{\kappa - \Lambda_j^h}{|U|}) .$$

The second equation in (2.27) determines $\kappa$ as a function of $\Lambda_j^h$. The zeroth order part of this equation is fulfilled identically, and the $O(L^{-1})$ part leads to the requirement

$$\kappa = \frac{\Lambda_1^h + \Lambda_2^h}{2} . \tag{2.28}$$

The equation for $F_{CS}$ can then be solved by Fourier transforming

$$F_{CS}(\Lambda) = -\frac{1}{2\pi} \left( \gamma \left( \frac{\Lambda - \Lambda_1^h}{2|U|} \right) + \gamma \left( \frac{\Lambda - \Lambda_2^h}{2|U|} \right) \right) + \frac{1}{2\pi} \theta(\frac{\Lambda - \Lambda_1^h + \Lambda_2^h}{2|U|}) . \tag{2.29}$$

where $\gamma(\mu)$ is defined in (2.21). Energy and momentum can be determined by using (2.9) and (2.10). We find

$$E_{CS} = \epsilon_{cw}(\Lambda_1^h) + \epsilon_{cw}(\Lambda_2^h) , \tag{2.30}$$

$$P_{CS} = p_{cw}^h(\Lambda_1^h) + p_{cw}^h(\Lambda_2^h) ,$$

where $\epsilon_{cw}(\lambda)$ and $p_{cw}^h(\Lambda)$ are given by (2.23) and (2.25), and where $p_{cw}^h(\lambda) = \pi + p_{cw}^h(\lambda)$. Thus we see that this excitation has the same energy as the charge triplet excitation considered above, whereas the momentum differs by $\pi$. Charge and spin quantum numbers are $\eta = 0 = S$. 

Spin-Charge Scattering States
We now choose $M'_1 = \frac{L}{2} - 1$ and $M_e = N_e - 2M' = 1$. $M_e$ is the number of “elementary” $k_j$’s in a solution of (2.4)-(2.6). The number of vacancies (2.8) for the integers $J^1_{\alpha}$ is $\frac{L}{2}$, so that there is one hole with corresponding spectral parameter $\Lambda^h$ in the distribution. The spectral parameter of the one real $k$ is denoted by $k$. The Lieb-Wu equations read

\[
2L \text{Re} \left\{ \arcsin(\tilde{\Lambda}_\alpha + i|U|) \right\} = 2\pi \tilde{J}^h_{\alpha} + \sum_{\beta=1}^{\frac{L}{2}} \theta(\frac{\tilde{\Lambda}_\alpha - \tilde{\Lambda}_\beta}{2|U|}) - \theta(\frac{\tilde{\Lambda}_\alpha - \Lambda^h}{2|U|}) + \theta(\frac{\tilde{\Lambda}_\alpha - \sin(k)}{|U|}),
\]

\[
L_k = 2\pi \tilde{I} + \sum_{\beta=1}^{\frac{L}{2}} \theta(\frac{\sin(k) - \tilde{\Lambda}_\beta}{|U|}) - \theta(\frac{\sin(k) - \Lambda^h}{|U|}).
\]

The second equation in (2.31) determines $k$ as a function of the integer $\tilde{I}$, which has range $[-\frac{L-1}{2}, \frac{L+1}{2}]$. In the thermodynamic limit the first part of (2.31) turns into an integral equation for an $F$-function

\[
2\pi F_{\eta_s}(\Lambda) = -\int_{-\infty}^{\infty} d\Lambda' \frac{4|U|}{4U^2 + (\Lambda - \Lambda')^2} F_{\eta_s}(\Lambda') + \theta(\frac{\Lambda - \sin(k)}{|U|}) - \theta(\frac{\Lambda - \Lambda^h}{2|U|}),
\]

which has the solution

\[
F_{\eta_s}(\Lambda) = \frac{1}{\pi} \text{arctan} \left( \exp\left(\pi \frac{\Lambda - \sin(k)}{2|U|}\right) \right) - \frac{1}{4} - \frac{1}{2\pi} \gamma \left( \frac{\Lambda - \Lambda^h}{2|U|} \right).
\]

The excitation energy of the charge-spin scattering state is given by

\[
E_{\eta_s} = -2\cos(k) + 4 \int_{-\infty}^{\infty} d\Lambda \text{Re} \left\{ \theta(\frac{\Lambda + i|U|}{\sqrt{1-(\Lambda + i|U|)^2}}) \right\} + 4\text{Re} \left\{ \sqrt{1-(\Lambda^h + i|U|)^2} \right\} - 2|U|
\]

\[
= \epsilon_{cw}(\Lambda^h) + \epsilon_{sw}(k),
\]

where $\epsilon_{cw}(\lambda)$ is given by (2.23) and

\[
\epsilon_{sw}(k) = 2|U| - 2\cos(k) + 2 \int_0^{\infty} d\omega \frac{J_1(\omega)\cos(\omega \sin(k))}{\cosh(\omega U)} \exp(-|\omega U|).
\]

The momentum is found to be

\[
P_{\eta_s} = p_{cw}^h(\Lambda^h) + p_{sw}(k),
\]

where $p_{cw}^h(\lambda)$ is given by (2.25) and

\[
p_{sw}(k) = k - \int_0^{\infty} d\omega \frac{J_0(\omega)\sin(\omega \sin(k))}{\cosh(\omega U)} \exp(-|\omega U|).
\]
The quantum numbers of this excitation are \(-S^z = \frac{1}{2} = S\) and \(-\eta^z = \frac{1}{2} = \eta\) and the state is thus the lowest weight state of an \(SO(4)\) multiplet of dimension 4. The other states of the multiplet are obtained by acting with \(S^\dagger\), \(\eta^\dagger\), and both.

**Spin-Singlet Excitation**

The spin-singlet excitation is characterized by taking \(M'_1 = \frac{L}{2} - 1\), \(M_e = 2\), and \(M_1 = 1\). The total number of electrons is \(N_e = L\), so that there is no charge associated with this excitation. The Lieb-Wu equations are

\[
2L \Re \left\{ \arcsin(\tilde{\Lambda}_\alpha + i|U|) \right\} = 2\pi J^1_\alpha + \sum_{\beta=1}^{K-1} \theta(\frac{\tilde{\Lambda}_\alpha - \tilde{\Lambda}_\beta}{2|U|}) + \sum_{j=1}^{2} \theta(\frac{\tilde{\Lambda}_\alpha - \sin(k_j)}{|U|}),
\]

\[
L k_j = 2\pi I_j + \sum_{\beta=1}^{K-1} \theta(\frac{\sin(k_j) - \tilde{\Lambda}_\beta}{|U|}) + \theta(\frac{\sin(k_j) - \kappa}{|U|})
\]

(2.38)

Here \(\kappa\) is the spectral parameter corresponding to \(M_1 = 1\), and \(k_1, k_2\) are the momenta of the real \(k\)'s. There are \(\frac{L}{2} - 1\) vacancies for the half-odd integers \(J^1_\alpha\), so that there are no holes in the distribution. The integer \(J^1_\alpha\) is found to be 0 by (2.8), which implies that \(\kappa = \frac{\sin(k_1) + \sin(k_2)}{2}\) by the third equation of (2.38). The first equation in (2.38) once again turns into an integral equation for an \(F\)-function in the thermodynamic limit

\[
F_{SS}(\Lambda) = 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\Lambda' \frac{4|U|}{4U^2 + (\Lambda - \Lambda')^2} F_{SS}(\Lambda') + \frac{1}{2\pi} \sum_{j=1}^{2} \theta(\frac{\Lambda - \sin(k_j)}{|U|})
\]

\[
\Rightarrow F_{SS}(\Lambda) = \frac{1}{\pi} \sum_{j=1}^{2} \arctan \left( \frac{\exp(\frac{\Lambda - \sin(k_j)}{2|U|})}{\exp(-\frac{\Lambda - \sin(k_j)}{2|U|})} \right)
\]

(2.39)

Energy and momentum are given by

\[
E_{SS} = \epsilon_{sw}(k_1) + \epsilon_{sw}(k_2),
\]

\[
P_{SS} = p_{sw}(k_1) + p_{sw}(k_2),
\]

(2.40)

where \(\epsilon_{sw}(k)\) and \(p_{sw}(k)\) are given by (2.35) and (2.37). The quantum numbers are \(S = 0 = \eta\), so that the excitation is a singlet of \(SO(4)\).

**Spin-Triplet Excitation**

The last type of excitation in the two-particle sector is the spin-triplet excitation. The quantum numbers of the lowest weight state are \(\eta = 0, S^z = -1, S = 1\). It is obtained
by choosing \( M'_1 = \frac{L}{2} - 1 \) and \( M_e = 2 \), which leads to a total number of electrons \( N_e = \frac{L}{2} \).

There are \( \frac{L}{2} - 1 \) vacancies for the half-odd integers \( J'_{\alpha} \), so that there are no holes in the distribution. The Lieb-Wu equations are

\[
2L \text{Re} \left\{ \arcsin(\tilde{\Lambda}_\alpha + i|U|) \right\} = 2\pi \tilde{J}^l_\alpha + \sum_{\beta=1}^{\frac{L}{2}-1} \theta(\frac{\tilde{\Lambda}_\alpha - \tilde{\Lambda}_\beta}{|U|}) + \sum_{j=1}^{2} \theta(\frac{\tilde{\Lambda}_\alpha - \sin(k_j)}{|U|}),
\]

(2.41) BAEST

\[
L k_j = 2\pi \tilde{I}_j + \sum_{\beta=1}^{\frac{L}{2}-1} \theta(\frac{\sin(k_j) - \tilde{\Lambda}_\beta}{|U|}).
\]

The \( F \)-function is found to be identical to \( F_{SS}(\Lambda) \), which implies that energy and momentum of the spin-triplet and spin-singlet are the same.

3. Quasiparticle Interpretation for \( U < 0 \)

All of the excited states constructed above have a natural interpretation as scattering states of four elementary excitations ("quasiparticles"), which form the fundamental representation of \( SU(2)_{\text{spin}} \times SU(2)_{\eta} \), i.e. they are grouped into doublets of the spin and \( \eta \)-SU(2) respectively. The chargeless spin-carriers ("spin-waves") have the dispersion

\[
p_{sw}(k) = k - \int_0^\infty d\omega \frac{J_0(\omega)\sin(\omega \sin(k))e^{-\omega|U|}}{\cosh(\omega U)}
\]

\[
\epsilon_{sw}(k) = 2|U| - 2 \cos(k) + 2 \int_0^\infty d\omega \frac{J_1(\omega)\cos(\omega \sin(k))e^{-\omega|U|}}{\cosh(\omega U)}.
\]

Their quantum numbers are \( \eta = 0, S = \frac{1}{2}, S^z = \pm \frac{1}{2} \), which corresponds to the representation \((\frac{1}{2}, 0)\) of \( SU(2)_{\text{spin}} \times SU(2)_{\eta} \).

The spinless charge-carriers ("charge-waves") (one particle, one hole) have the dispersions\[14\]

\[
p_{cw}^p(\lambda) = \pi - \int_0^\infty d\omega \frac{J_0(\omega)\sin(\omega \lambda)}{\cosh(\omega U)} = \pi + p_{cw}^h(\lambda),
\]

\[
\epsilon_{cw}(\lambda) = 2 \int_0^\infty d\omega \frac{J_1(\omega)\cos(\omega \lambda)}{\cosh(\omega U)},
\]

(3.2) aeps

Their quantum numbers are \( S = 0, \eta = \frac{1}{2}, \eta^z = \pm \frac{1}{2} \) and they form the \((0, \frac{1}{2})\) representation of \( SU(2)_{\text{spin}} \times SU(2)_{\eta} \).

Like in the case of the spin-\( \frac{1}{2} \) Heisenberg antiferromagnet the quasiparticles are “confined”, i.e. they do not exist as one-particle excitations. This is closely related to the fact that the symmetry of the Hubbard hamiltonian is \( SO(4) \) and all half-odd integer representations of \( SU(2) \times SU(2) \) (and thus the fundamental representation formed by the four elementary
excitations) are not present. The two-particle scattering states of the four quasiparticles are easily seen to reproduce exactly the twelve excitations of section 2.

(i) Scattering of two spin-waves

The dispersion relations for the scattering states of two spin-waves follow from (3.1) to be 
\[ E = \epsilon_{sw}(k_1) + \epsilon_{sw}(k_2), \quad P = p_{sw}(k_1) + p_{sw}(k_2), \]
where \( k_1 \) and \( k_2 \) are the rapidities of the spin-waves. The \( SO(4) \)-representation is simply \((\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) = (1, 0) \oplus (0, 0)\). By inspection we see that \((1, 0)\) is identical to the spin-triplet excitation of section 2, and \((0, 0)\) to the spin-singlet.

(ii) Scattering of two charge-waves

Scattering of two charge-waves leads to the \( SO(4) \)-representation \((0, \frac{1}{2}) \otimes (0, \frac{1}{2}) = (0, 1) \oplus (0, 0)\). The energy of all four states follows from (3.2) to be 
\[ E = \epsilon_{cw}(\Lambda_h^1) + \epsilon_{cw}(\Lambda_h^2), \quad P = p_{cw}^h(\Lambda_h^1) + p_{cw}^h(\Lambda_h^2), \]
where \( \Lambda_h^1 \) and \( \Lambda_h^2 \) are the rapidities of the charge-waves. The lowest weight state \( \eta_z = -\frac{1}{2} \) contains two holes and has momentum \( P = p_{cw}^h(\Lambda_h^1) + p_{cw}^h(\Lambda_h^2) \). It is identical to the charge-triplet state constructed in section 2. The other two states of the triplet have momenta \( P_{\eta_z=0} = p_{cw}^h(\Lambda_h^1) + p_{cw}^h(\Lambda_h^2) \) and \( P_{\eta_z=1} = p_{cw}^h(\Lambda_h^1) + p_{cw}^h(\Lambda_h^2) \) respectively, which is also in agreement with the results of section 2. The \((0, 0)\) state is seen to be identical to the charge-singlet constructed above.

(iii) Scattering of spin-waves and charge-waves

The relevant \( SO(4) \)-representation for this process is \((\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})\). The lowest weight state of the multiplet has quantum numbers \( \eta = \frac{1}{2} = S, \quad S^z = -\frac{1}{2} = \eta^z \) and a dispersion \( E = \epsilon_{cw}(\Lambda^h) + \epsilon_{sw}(k), \quad P = p_{cw}^h(\Lambda^h) + p_{sw}(k) \), where \( k \) and \( \Lambda^h \) denote the rapidities of the spin-wave and charge-wave respectively. This state is clearly identical to the spin-charge scattering state of section 2. The other states of the multiplet are identical to the ones found in section 2 as well.

A similar analysis can be carried out also in the sectors with four, six, etc. particles. All excitations have a natural interpretation in terms of scattering states of the four quasiparticles. The general proof of this statement is given in the appendix.

4. Scattering Matrix

In quantum mechanical scattering theory the scattering matrix can be extracted from the asymptotics of the wave-function of the scattering state\(^{[22]}\). The boundary conditions of the quantum mechanical problem are free. This is in contrast to the periodic boundary conditions imposed in the Bethe Ansatz solution. In \([23]\) it was shown how to modify quantum mechanical scattering theory to accommodate for this fact, and a general method for extracting the exact \( S \)-matrix from the Bethe Ansatz solution was given. In \([20,21,24,25]\)

\(^{[2]}\)This is completely analogous with the spin-\( \frac{1}{2} \) Heisenberg model, which is \( O(3) \)-symmetric for even chains, whereas the elementary excitations form the fundamental representation of \( SU(2) \)\(^{[20,21]}\).
this method was applied to evaluate the exact S-matrix for Heisenberg models. Here we will apply this method to the case of the nested Bethe Ansatz and the Hubbard model. The S-matrix for parity-eigenstates in one dimension is simply a phase factor. The two-particle scattering-phase is equal to the phase obtained by moving particle 2 through the whole interval (one-dimensional box) in the presence of particle 1 minus the one-particle phase-shift, which is equal to the phase picked up by moving particle 2 through the box when 1 is absent. The two-particle phases can be computed from the Lieb-Wu equations (1.4) by using our results for the two-particle excitations above. However there are no one-particle excitations present and thus it is impossible to evaluate the one-particle phase-shifts by means of the Bethe Ansatz. However it is possible to evaluate the relative phases between two-particle excitations by means of the Bethe Ansatz. Thus it is possible to evaluate the exact S-matrix up to an overall constant factor, which is the difference between a suitably chosen reference phase, and the true one-particle phase shift. In other words the method of [23] allows the evaluation of the logarithmic derivative of the S-matrix. In general this leads to the subtle issue of determining the overall phase of the S-matrix[26], which for models like the spin-$\frac{1}{2}$ Heisenberg antiferromagnet cannot be resolved in the framework of the Bethe Ansatz. For the case of the Hubbard model we are in much more fortunate position: In the limit $U \to \infty$ the charge sector of the repulsive Hubbard model reduces to scattering of free fermions. This can be seen by directly evaluating the many-body wave-functions in this limit[13]. Thus the S-matrix (in the specific sector) must reduce the the S-matrix for free fermions (which is 1) in the limit $U \to \infty$, which allows us to fix the constant. In the $U \to \infty$ limit of the attractive Hubbard model the charge-triplet must reduce to a scattering state of hard-core bosons with S-matrix $-1$.

The complete S-matrix for the Hubbard model is $16 \times 16$ dimensional and blockdiagonal. It breaks up into 4 blocks (due to conservation of spin and charge quantum numbers), describing scattering of spin-waves on spin-waves, spin-waves on charge-waves, charge-waves on spin-waves, and charge-waves on charge-waves respectively

$$S = \begin{pmatrix} S_{ss}(\mu_1) & 0 & 0 & 0 \\ 0 & S_{s\eta}(\mu_2) & 0 & 0 \\ 0 & 0 & S_{\eta s}(\mu_3) & 0 \\ 0 & 0 & 0 & S_{cc}(\mu_4) \end{pmatrix}.$$  \hspace{1cm} (4.1) SM

**Charge-Charge Scattering**

The four parity eigenstates are the charge-singlet and -triplet states. As the scattering phase is the same for all members of the triplet we will only consider the lowest weight state. The general form for the S-matrix of quasiparticles in the fundamental representation of $SU(2)$ is

$$S_{abcd}(\lambda_1, \lambda_2) = S_1(\lambda_1, \lambda_2)\delta_{ac}\delta_{bd} + S_2(\lambda_1, \lambda_2)\delta_{ad}\delta_{bc},$$  \hspace{1cm} (4.2) Sgen
where \(a, b, c, d\) label the colliding quasiparticles, and \(\lambda_1, \lambda_2\) are the corresponding spectral parameters. The phase factors for singlet and triplet are

\[
S_t = S_1 + S_2, \quad S_s = S_1 - S_2.
\]

We will now use the method of [23] to determine \(S_t\) and \(S_s\). We start by considering the charge-singlet excitation. The phase for moving particle 2 through the ring in presence of particle 1 is

\[
\delta_{CS}^{21}(\Lambda^h_1, \Lambda^h_2) = -2L \text{ Re} \left\{ \arcsin(\Lambda^h_2 + i|U|) \right\} + \sum_{\beta=1}^{4} \theta\left(\frac{\Lambda^h_2 - \Lambda^h_\beta}{2|U|}\right) - \theta\left(\frac{\Lambda^h_2 - \Lambda^h_1}{2|U|}\right) + \theta_{12}\left(\frac{\Lambda^h_2 - \Lambda^h_1}{2|U|}\right).
\]

In order to obtain the true phase shift we now ought to subtract the phase \(\delta_{\text{charge}}(\Lambda^h_2)\) obtained by moving particle 2 through the ring in the absence of particle 1 (i.e. the one-particle phase-shift). Due to confinement of the quasiparticles it is however impossible to compute \(\delta_{\text{charge}}\). Instead we will subtract a reference phase \(\delta_0(\Lambda^h_2)\), which is chosen such that the phase \(\delta_{CS}^{21}(\Lambda^h_1, \Lambda^h_2) - \delta_0(\Lambda^h_2)\) is a function of \(\Lambda^h_2 - \Lambda^h_1\) only. The difference \(\delta_{\text{charge}} - \delta_0\) leads to an overall phase-factor for the S-matrix (4.1), which will be fixed below. The reference phase is

\[
\delta_0(\Lambda^h_2) = -2L \text{ Re} \left\{ \arcsin(\Lambda^h_2 + i|U|) \right\} + \sum_{\beta=1}^{4} \theta\left(\frac{\Lambda^h_2 - \Lambda^h_\beta}{2|U|}\right),
\]

which leads to the following phase-shift for the charge-singlet

\[
\delta_{CS}(\Lambda^h_1, \Lambda^h_2) = \delta_{CS}^{21}(\Lambda^h_1, \Lambda^h_2) - \delta_0(\Lambda^h_2)
\]

\[
= - \int_{-\infty}^{\infty} d\Lambda' \frac{4|U|}{4U^2 + (\Lambda - \Lambda')^2} F_{CS}(\Lambda') + \theta_{12}\left(\frac{\Lambda^h_2 - \Lambda^h_1}{2|U|}\right) - \theta\left(\frac{\Lambda^h_2 - \Lambda^h_1}{2|U|}\right)
\]

\[
= 2\pi F_{CS}(\Lambda^h_2) = -\gamma(\mu_4) + \theta(\mu_4),
\]

where \(\mu_4 = \frac{\Lambda^h_2 - \Lambda^h_1}{2|U|} > 0\), \(\theta(x) = 2\arctan(x)\), and \(\gamma(\mu)\) is defined in (2.21).

In order to determine \(S_t\) we have to consider the charge triplet excitation. The phase-shift for moving particle 2 through the ring in presence of particle 1 is

\[
\delta_{CT}^{21}(\Lambda^h_1, \Lambda^h_2) = -2L \text{ Re} \left\{ \arcsin(\Lambda^h_2 + i|U|) \right\} + \sum_{\beta=1}^{4} \theta\left(\frac{\Lambda^h_2 - \Lambda^h_\beta}{2|U|}\right) - \theta\left(\frac{\Lambda^h_2 - \Lambda^h_1}{2|U|}\right) + \theta\left(\frac{\Lambda^h_2 - \Lambda^h_{\text{min}}}{2|U|}\right) + \pi.
\]
The extra $\pi$ is due to the fact that the $J^q_A$ are half-odd integers for the charge-triplet but integers for the charge-singlet and the ground state, which will be used for reference purposes. The phase-shift is found to be

$$
\delta_{CT}(\Lambda^h, \Lambda^s) = \delta_{CT}^{\Lambda^s}(\Lambda^h_1, \Lambda^h_2) - \delta_0(\Lambda^h_2) = 2\pi \ F_{CT}(\Lambda^h_2) \mod 2\pi = \pi - \gamma(\mu_4) \mod 2\pi , \ \mu_4 = \frac{\Lambda^h_2 - \Lambda^h_1}{2|U|} > 0 \quad (4.8) \ \text{dct}
$$

Using (4.8) and (4.6) we can now determine the phase-factors $S_s$ and $S_t$

$$
S_s = \exp(i\delta_{CS}) = -\frac{\mu_4 - i}{\mu_4 + i} \frac{\Gamma(1+\frac{i\mu_4}{2})\Gamma(1-\frac{i\mu_4}{2})}{\Gamma(1-\frac{i\mu_4}{2})\Gamma(1+\frac{i\mu_4}{2})} \quad (4.9) \ \text{scc}
$$

Thus the final result for the $4 \times 4$ block of the complete S-matrix describing scattering of Charge-waves on Charge-waves is found to be

$$
S_{cc}(\Lambda^h_1, \Lambda^h_2) = -\frac{\Gamma(1+\frac{i\mu_4}{2})\Gamma(1-\frac{i\mu_4}{2})}{\Gamma(1-\frac{i\mu_4}{2})\Gamma(1+\frac{i\mu_4}{2})} \left( \frac{\mu_4 + i}{\mu_4 + i} I + \frac{i}{\mu_4 + i} \Pi \right) . \quad (4.10) \ \text{Scc}
$$

where $I$ and $\Pi$ are the $4 \times 4$ identity and permutation matrices respectively. This S-matrix is as a function of spectral parameter (but not of momentum) identical to the well-known S-matrix for the spin-$\frac{1}{2}$ Heisenberg antiferromagnet$^{[20,21]}$ and the $SU(2)_1$ Wess-Zumino-Witten-Novikov (WZWN) model$^{[27]}$ (this S-matrix also describes scattering of particles in the $SU(2)$ Gross-Neveu model$^{[19]}$, and spinons in the Kondo model$^{[28,29]}$).

**Scattering of Charge on Spin and Spin on Charge**

The relevant excitation for these processes is the spin-charge scattering state of section 2. We first consider the charge-wave to be the active scatterer. The phase for moving the charge-wave with spectral parameter $\Lambda^h$ around the ring in presence of the spin-wave with spectral parameter $k$ is

$$
\delta_{\eta_s}^{21}(\Lambda^h, k) = -2L \ \text{Re} \left\{ \arcsin(\Lambda^h + i|U|) \right\} + \sum_{\beta=1}^{L} \theta(\frac{\Lambda^h - \Lambda^s_\beta}{2|U|}) + \theta(\frac{\Lambda^h - \sin(k)}{|U|}) . \quad (4.11) \ \text{d12es}
$$

The reference phase is once again given by (4.5), which yields the following phase-shift for charge-spin scattering

$$
\delta_{\eta_s}(\Lambda^h, k) = \delta_{\eta_s}^{21}(\Lambda^h, k) - \delta_0(\Lambda^h) = 2\pi \ F_{\eta_s}(\Lambda^h_2) \mod 2\pi
$$

$$
= 2 \ \arctan(\exp(\pi\mu_3)) - \frac{\pi}{2} , \ \mu_3 = \frac{\Lambda^h - \sin(k)}{2|U|} > 0 \quad (4.12) \ \text{des}
$$
The corresponding $4 \times 4$ block of the S-matrix is
\[ S_{\eta s}(\Lambda^h, k) = -i \frac{1 + i \exp(\pi \mu_3)}{1 - i \exp(\pi \mu_3)} I. \]  
\[(4.13)\]

Now we consider the spin-wave as the active scatterer. The total two-particle phase shift is
\[ \delta_{s\eta}^{21}(\Lambda^h, k) = Lk - \sum_{\beta=1}^{L} \theta\left(\frac{\sin(k) - \tilde{\Lambda}_\beta}{|U|}\right) + \theta\left(\frac{\sin(k) - \Lambda^h}{|U|}\right). \]  
\[(4.14)\]

To get the phase-shift $\delta_{s\eta}$ we ought to subtract the one-particle phase-shift for a spin-wave $\delta_{\text{spin}}(k)$, which is unknown due to confinement. By the same method as above we can, however, determine the phase-shift up to a constant. The reference phase is $\tilde{\delta}_0(k) = Lk - \sum_{\beta=1}^{L} \theta\left(\frac{\sin(k) - \Lambda^\beta}{|U|}\right)$, which yields the following result for $\delta_{s\eta}$
\[ \delta_{s\eta}(\Lambda^h, k) = \delta_{s\eta}^{21}(\Lambda^h, k) - \tilde{\delta}_0(k) \]
\[ = \int_{-\infty}^{\infty} d\Lambda' \frac{2|U|}{U^2 + (\sin(k) - \Lambda')^2} F_{\eta s}(\Lambda') + \theta\left(\frac{\sin(k) - \Lambda^h}{|U|}\right) \]  
\[(4.15)\]

This is a priori correct up to a constant, which we will now show to be zero. We know that $\delta_{qs}(\sin(x), x) = \delta_{s\eta}(\sin(x), x)$, as in this case the quasiparticles are at rest relative to each other. Inspection of (4.15) and (4.12) now shows that the constant must vanish. As a result we see that it does not matter whether spin or charge is taken to be active. The form of the S-matrix (4.13) is the same, only the definition of $\mu$ changes.

**Spin-Spin Scattering**

In the spin-spin sector the S-matrix is once again of the form (4.2). We first determine the phase-shift for the triplet, as the $I_j$’s are integers both for the spin triplet and the spin-charge scattering state we used to determine the reference phase in the spin-sector. The phase for moving particle 2 through the ring in presence of particle 1 is
\[ \delta_{ST}^{21}(k_1, k_2) = Lk_2 - \sum_{\alpha=1}^{L} \theta\left(\frac{\sin(k_2) - \tilde{\Lambda}_\alpha}{|U|}\right). \]  
\[(4.16)\]

Subtracting the one-particle phase-shift for spin-waves $\tilde{\delta}_0(k_2)$ we arrive at the following expression for the triplet scattering phase
\[ \delta_{ST}(k_1, k_2) = \gamma(\mu_1) \mod 2\pi, \quad \mu_1 = \frac{\sin(k_2) - \sin(k_1)}{2|U|} > 0. \]  
\[(4.17)\]
The spin-singlet phase-shift is found to be

$$
\delta_{SS}(k_1, k_2) = \gamma(\mu_1) - 2 \arctan(\mu_1) + \pi \mod 2\pi,
$$

(4.18) \text{s}\text{s}

which implies the following form for the S-matrix in the spin-spin sector

$$
S_{ss}(\sin(k_1^h), \sin(k_2^h)) = \frac{\Gamma(1 - \frac{i\mu_1}{2})\Gamma(1 + \frac{i\mu_1}{2})}{\Gamma(1 + \frac{i\mu_1}{2})\Gamma(1 - \frac{i\mu_1}{2})} \left( \frac{\mu_1}{\mu_1 - i} I - \frac{i}{\mu_1 - i} \Pi \right).
$$

(4.19) \text{S}\text{s}\text{s}

Last but not least we have to fix the overall factor of the S-matrix by considering the $U \to -\infty$ limit. In this limit the wave-function for the charge-triplet state reduces to the one for hard-core bosons. Therefore the charge-triplet phase-shift has to become $\pi$ in this limit (the S-matrix for hard-core bosons is $-1$, the one for free fermions $1$). This fixes the constant factor. We see that our expression for the charge-triplet phase-shift (4.8) has the correct asymptotic behaviour, so that our constant factor is already the correct one. This completes our computation of the exact S-matrix for the attractive Hubbard model.

The analytic structure of the S-matrix is related to the existence of bound states of the elementary excitations. The physical strips for spin-waves and charge-waves are defined by $|\text{Im}(\sin(k))| \leq 2|U|$ and $|\text{Im}(\Lambda^h)| \leq |U|$ respectively. Using these ranges in the expressions for the S-matrices $S_{ss}$, $S_{ns}$ and $S_{cc}$ we find poles at $\sin(k_2) - \sin(k_1) = 2i|U|$ in the spin-spin sector, at $\Lambda^h - \sin(k) = \pm i|U|^3$ in the charge-spin sector, and at $\Lambda^h_2 - \Lambda^h_1 = -2i|U|$ in the charge-charge sector. The pole in the charge-charge sector is at the boundary of the physical strip and corresponds to zero total energy and momentum. Thus it does not correspond to a physical bound state. The analysis in the spin-spin and charge-spin sectors is more difficult. For large values of $|U|$ one can show that the poles either lie on the unphysical sheet (and thus lead to anti-bound states, which due to non-normalizability drop out of the physical Hilbert space), or lead to excitations with fixed momentum (and thus cannot be interpreted as particles) and higher energies than the scattering states. We conclude that there are no bound states and the set of four elementary excitations is complete.

5. Repulsive Hubbard Model

The analysis of the Lieb-Wu equations (1.4) for the repulsive case is similar to the attractive case above. The analog of the logarithmic equations (2.4) - (2.6) was derived by M. Takahashi in [17]. The main difference is the form of the $k$-$\Lambda$-strings: in the repulsive case $k^1_\alpha = \pi - \arcsin(\Lambda^m_\alpha + i mU)$, $k^{2m}_\alpha = \pi - \arcsin(\Lambda^m_\alpha - i mU)$, whereas all other $k^j_\alpha$’s are the same as in (2.2)(see [17]). Equations (2.5) and (2.6) are the same for the repulsive case if we take $|U| \to U$. The analog of equation (2.4) for the case of repulsion is obtained

\footnote{The sign depends on whether we consider the spin-wave or charge-wave to be the active scatterer.}
by taking $|U| \to -U$ in (2.4). The boundaries of integers (2.8) are the same for $U > 0$ and $U < 0^4$. Energy and momentum are

$$E = -2 \sum_{l=1}^{N_e-2M'} \cos(k_l) + 4 \sum_{n=1}^{\infty} \sum_{a=1}^{M'_n} \text{Re} \left\{ \sqrt{1 - (\Lambda^*_n + i|U|n)^2} \right\} - 2UN_e + UL , \quad (5.1)$$

$$P = \sum_{j=1}^{N_e-2M'} k_j - \sum_{n=1}^{\infty} \sum_{a=1}^{M'_n} (2\text{Re} \{ \text{arcsin}(\Lambda^*_n + i|U|n) \} + \pi(n - 1)) . \quad (5.2)$$

The half-filled repulsive ground state is characterized by taking $N_e = L$, $M_1 = \frac{L}{2}$, and then filling both the integers $J^1_\alpha$ and the half-odd integers $I^1_j$ symmetrically around zero. It is described by a set of two coupled integral equations and was found by Lieb and Wu in [5]. The two filled Fermi seas of the $I^1_j$'s and the $J^1_\alpha$'s correspond to spin and charge degrees of freedom respectively. The excitation spectrum over this ground state has been previously studied by many authors\cite{[11,3,13,2,30,12,31]}, although the Lie-algebraic structure of the excitation spectrum was not fully revealed. Like in the attractive case there are 12 excitations in the two-particle sector. Unlike for the attractive case now two $F$-functions occur as the ground state consists of two Fermi seas. In order to explain our notation we will discuss the charge-triplet excitation in some detail and then quote the results for the other excitations. The Lieb-Wu equations for the repulsive ground state are

$$L k_l = 2\pi I_l - \sum_{a=1}^{\frac{L}{2}} \theta \left( \frac{\sin(k_l) - \Lambda^*_a}{U} \right) ,$$

$$\sum_{l=1}^{L} \theta \left( \frac{\Lambda^*_a - \sin(k_l)}{U} \right) = 2\pi J^1_\alpha - \sum_{\beta=1}^{\frac{L}{2}} \theta \left( \frac{\Lambda^*_a - \Lambda^*_\beta}{2U} \right) , \quad (5.3)$$

where $I_l = l - \frac{L+1}{2}$, $l = 1 \ldots L$, and $J^1_\alpha = \alpha - \frac{L+2}{4}$, $\alpha = 1 \ldots \frac{L}{2}$. In the thermodynamic limit (5.3) turn into the well-know set of coupled integral equations\cite{[5]}

$$\rho(k) = \frac{1}{2\pi} + \frac{1}{2\pi} \cos(k) \int_{-\infty}^{\infty} d\Lambda \frac{2U}{U^2 + (\sin(k) - \Lambda)^2} \sigma(\Lambda) , \quad (5.4)$$

$$\sigma(\Lambda) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dk}{U^2 + (\sin(k) - \Lambda)^2} \rho(k) - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\Lambda' \frac{4U}{4U^2 + (\Lambda - \Lambda')^2} \sigma(\Lambda') .$$

The ground state energy per site for the repulsive model is\cite{[5]}

$$E_{GS}^r(U > 0) = -U - \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \exp(-|\omega|U) \cosh(\omega U) J_0(\omega)J_1(\omega) . \quad (5.5)$$

$^4$This is important for the validity of the completeness proof of [18] for both attraction and repulsion.

$^5$For a collection of reprints see [32].
The charge-triplet excitation over the ground state is described by the following set of equations \((N_e = L - 2, M_1 = \frac{L - 2}{2})\)

\[
L \tilde{k}_l = 2\pi \tilde{l}_l - \sum_{\alpha=1}^{L-1} \theta \left( \frac{\sin(\tilde{k}_l) - \tilde{\Lambda}_\alpha}{U} \right)
\]

\[
\sum_{l=1}^{L} \theta \left( \frac{\tilde{\Lambda}_\alpha - \sin(\tilde{k}_l)}{U} \right) - \sum_{j=1}^{2} \theta \left( \frac{\tilde{\Lambda}_\alpha - \sin(k^h_j)}{U} \right) = 2\pi \tilde{j}_\alpha - \sum_{\beta=1}^{L-2} \theta \left( \frac{\tilde{\Lambda}_\alpha - \tilde{\Lambda}_\beta}{2U} \right).
\] (5.6) rBAECT

There are two holes corresponding to spectral parameters \(\sin(k^h_1), \sin(k^h_2)\) in the distribution of the integers \(\tilde{l}_j\). The half-odd integers \(\tilde{j}_\alpha^1\) are distributed symmetrically between \(-\frac{L-4}{4}\) and \(\frac{L-4}{4}\). Our convention is \(\tilde{l}_j - I_j = \frac{1}{2}\) and \(\tilde{j}_\alpha^1 - J_\alpha^1 = \frac{1}{2}\), where \(I_j\) and \(J_\alpha^1\) are the ground state distributions of “integers”. Subtracting (5.3) from (5.6) and then taking the thermodynamic limit we can derive coupled integral equations for the \(F\)-functions \(F^s(\Lambda_\alpha) = \frac{\tilde{\Lambda}_\alpha - \tilde{\Lambda}_\beta}{\tilde{\Lambda}_\alpha + 1 - \tilde{\Lambda}_\beta}\) and \(F^c(k^h_j) = \frac{\tilde{k}^h_{j+1} - k^h_j}{\tilde{k}^h_{j+1} + 1 - k^h_j}\)

\[
F^c_{CT}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \frac{2U}{U^2 + (\sin(k) - \Lambda)^2} F^s_{CT}(\Lambda)
\]

\[
F^s_{CT}(\Lambda) = -1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda' \frac{4U}{4U^2 + (\sin(k) - \Lambda')^2} F^s_{CT}(\Lambda') + \frac{1}{2\pi} \sum_{j=1}^{2} \theta \left( \frac{\Lambda - \sin(k^h_j)}{U} \right).
\] (5.7) rFCT

These equations can once again be solved by Fourier transforming

\[
F^s_{CT}(\Lambda) = -1 + \frac{1}{\pi} \sum_{j=1}^{2} \arctan(\exp(\pi \frac{\Lambda - \sin(k^h_j)}{2U}))
\]

\[
F^c_{CT}(k) = -\frac{1}{2} + \frac{1}{2\pi} \sum_{j=1}^{2} \gamma \left( \frac{\sin(k) - \sin(k^h_j)}{2U} \right),
\] (5.8) rFCT2

where we have used (2.21). Energy and momentum of the excitation are given by

\[
E_{CT}(k^h_1, k^h_2) = 4U + 2 \sum_{j=1}^{2} \cos(k^h_j) + 2 \int_{-\pi}^{\pi} dk \sin(k) F^c_{CT}(k) = \epsilon_h(k^h_1) + \epsilon_h(k^h_2),
\] (5.9) rEPCT

\[
P_{CT}(k^h_1, k^h_2) = -2 \sum_{j=1}^{2} k^h_j + \int_{-\pi}^{\pi} dk F^c_{CT}(k) = p_h(k^h_1) + p_h(k^h_2),
\]

where

\[
p_h(k) = \frac{\pi}{2} - k - \int_{0}^{\infty} \frac{d\omega}{\omega} J_0(\omega) \frac{\sin(\omega \sin(k)) e^{-|U|}}{\cosh(\omega U)}
\]

\[
\epsilon_h(k) = 2U + 2 \cos(k) + 2 \int_{0}^{\infty} \frac{d\omega}{\omega} J_1(\omega) \frac{\cos(\omega \sin(k)) e^{-|U|}}{\cosh(\omega U)}.
\] (5.10) repc
Here $\epsilon_h(k)$ and $p_h(k)$ are energy and momentum of a quasiparticle carrying charge $+e$ and no spin. This quasiparticle, which transforms in the $(0, \frac{1}{2})$-representation of $SU(2)_{spin} \times SU(2)_{eta}$, is called holon. The charge-triplet state is a scattering state of two holons with rapidities $\sin(k_1^h)$ and $\sin(k_2^h)$.

The charge-singlet is characterized by $N_c = L$, $M_1 = \frac{L}{2} - 1$, $M'^1 = 1$. It is a two-parameter excitation with rapidities $\sin(k_1^h)$ and $\sin(k_2^h)$ and leads to the following set of $F$-functions

\[
F_{CS}^s(\Lambda) = -1 + \frac{1}{\pi} \sum_{j=1}^{2} \arctan(\exp(\pi \frac{\Lambda - \sin(k_j^h)}{2U})) \quad \text{(5.11)}
\]

\[
F_{CS}^c(k) = -\frac{1}{\pi} \arctan\left(\frac{\sin(k) - \frac{\sin(k_1^h) + \sin(k_2^h)}{2}}{U}\right) + \frac{1}{2\pi} \sum_{j=1}^{2} \gamma \left(\frac{\sin(k) - \sin(k_j^h)}{2U}\right),
\]

Energy and momentum are found to be \[^{[3]}

\[
E_{CS}(k_1^h, k_2^h) = \epsilon_h(k_1^h) + \epsilon_{ah}(k_2^h),
\]

\[
P_{CS}(k_1^h, k_2^h) = p_h(k_1^h) + p_{ah}(k_2^h),
\]

where $\epsilon_{ah}(k) \equiv \epsilon_h(k)$ and $p_{ah}(k) = -\pi + p_h(k)$ are energy and momentum of the other member of the $(0, \frac{1}{2})$-doublet. It is connected to the holon by action of $\eta^i$ and is called antiholon. The charge-singlet is thus the scattering state of one holon and one antiholon.

The spin-charge scattering state is obtained by taking $N_c = L - 1$, $M_1 = \frac{L}{2} - 1$ and filling the seas of integers $J_j$ and $J'^1_\alpha$. There is one hole in both distributions with corresponding spectral parameters $\Lambda^h$ and $\sin(k^h)$. The $F$-functions for this state are of the form

\[
F_{\eta s}^s(\Lambda) = \frac{1}{\pi} \arctan(\exp(\pi \frac{\Lambda - \sin(k^h)}{2U})) - \frac{1}{4} - \frac{1}{2\pi} \gamma \left(\frac{\Lambda - \Lambda^h}{2U}\right),
\]

\[
F_{\eta s}^c(k) = \frac{1}{4} + \frac{1}{\pi} \arctan(\exp(\pi \frac{\sin(k) - \Lambda^h}{2U})) + \frac{1}{2\pi} \gamma \left(\frac{\sin(k) - \sin(k^h)}{2U}\right).
\]

The dispersion relation is again of quasiparticle form

\[
E_{\eta s}(k^h, \Lambda^h) = \epsilon_h(k^h) + \epsilon_s(\Lambda^h),
\]

\[
P_{\eta s}(k^h, \Lambda^h) = p_h(k^h) + p_s(\Lambda^h),
\]

where $\epsilon_s(\Lambda)$ and $p_s(\Lambda)$ are energy and momentum a quasiparticle in the $(\frac{1}{2}, 0)$-representation of $SU(2)_{spin} \times SU(2)_{eta}$, which is called spinon

\[
p_s(\Lambda) = \frac{\pi}{2} - \int_0^{\infty} d\omega \frac{J_0(\omega)\sin(\omega\lambda)}{\omega \cosh(\omega U)},
\]

\[
\epsilon_s(\Lambda) = 2\int_0^{\infty} d\omega \frac{J_1(\omega)\cos(\omega\lambda)}{\omega \cosh(\omega U)}.
\]
Spin-triplet and singlet states are scattering states of two spinons. The lowest weight state of the triplet is obtained from the Bethe Ansatz by taking $N_e = L$, $M_1 = \frac{L}{2} - 1$. There are two holes with spectral parameters $\Lambda^h_1, \Lambda^h_2$ in the distribution of the $\Lambda_\alpha$. The $F$-functions are found to be

$$F^s_{ST}(\Lambda) = \frac{1}{2} - \frac{1}{2\pi} \sum_{j=1}^{2} \gamma \left( \frac{\Lambda - \Lambda^h_j}{2U} \right),$$

$$F^c_{ST}(k) = \frac{1}{\pi} \sum_{j=1}^{2} \arctan(\exp(\pi \sin(k) - \Lambda^h_j)).$$

(5.16) rFST

The spin-singlet state is characterized by $N_e = L$, $M_1 = \frac{L}{2} - 2$, $M_2 = 1$. Again there are two holes, and the $F$-functions are

$$F^s_{SS}(\Lambda) = -\frac{1}{2\pi} \sum_{j=1}^{2} \gamma \left( \frac{\Lambda - \Lambda^h_j}{2U} \right) + \frac{1}{\pi} \arctan(\frac{\Lambda - \frac{\Lambda^h_1 + \Lambda^h_2}{2}}{U}),$$

$$F^c_{SS}(k) \equiv F^c_{ST}(k).$$

(5.17) rFSS

Energy for both triplet and singlet states are of quasiparticle form

$$E(\Lambda^h_1, \Lambda^h_2) = \epsilon_s(\Lambda^h_1) + \epsilon_s(\Lambda^h_2),$$

$$P(\Lambda^h_1, \Lambda^h_2) = p_s(\Lambda^h_1) + p_s(\Lambda^h_2),$$

(5.18) rEPspin

where $\epsilon_s(\lambda)$ and $p_s(\lambda)$ are given by (5.15). The two-particle S-matrix for the repulsive case can be computed by the same method as for the attractive case. The phase-shifts are found to be

$$\delta_{CT} = 2\pi F^c_{CT}(k^h_2) + \pi = \ln \left( \frac{\Gamma(\frac{i+\mu_2}{2})\Gamma(1-i\frac{\mu_2}{2})}{\Gamma(\frac{i-\mu_4}{2})\Gamma(1+i\frac{\mu_4}{2})} \right), \mu_4 = \frac{\sin(k^h_2) - \sin(k^h_1)}{2U} > 0,$$

$$\delta_{CS} = 2\pi F^c_{CS}(k^h_1) + \pi = 2 \arctan(\mu_4) + \ln \left( \frac{\Gamma(\frac{i+\mu_4}{2})\Gamma(1-i\frac{\mu_4}{2})}{\Gamma(1+i\frac{\mu_4}{2})\Gamma(1+i\frac{\mu_4}{2})} \right),$$

$$\delta_{ST} = 2\pi F^s_{ST}(\Lambda^h_2) = \pi - \ln \left( \frac{\Gamma(\frac{i-\mu_3}{2})\Gamma(1-i\frac{\mu_3}{2})}{\Gamma(\frac{i+\mu_3}{2})\Gamma(1+i\frac{\mu_3}{2})} \right), \mu_1 = \frac{\Lambda^h_2 - \Lambda^h_1}{2U} > 0,$$

$$\delta_{SS} = 2\pi F^s_{SS}(\Lambda^h_2) = 2 \arctan(\mu_1) - \ln \left( \frac{\Gamma(\frac{i-\mu_3}{2})\Gamma(1-i\frac{\mu_3}{2})}{\Gamma(\frac{i+\mu_3}{2})\Gamma(1+i\frac{\mu_3}{2})} \right),$$

$$\delta_{\eta_s} = 2\pi F^c_{\eta_s}(k^h) + \pi = 2 \arctan(\exp(\pi \mu_3)) - \frac{\pi}{2}, \mu_3 = \frac{\sin(k^h) - \Lambda^h}{2U} > 0,$$

$$\delta_{\eta_s} = 2\pi F^s_{\eta_s}(\Lambda^h) = 2 \arctan(\exp(\pi \mu_2)) - \frac{\pi}{2}, \mu_2 = \frac{\Lambda^h - \sin(k^h)}{2U} > 0.$$  

(5.19) rphase1

(5.20) rphase2

(5.21) rphase3

Here we have fixed the overall constant by considering the charge-triplet in the $U \to \infty$ limit, in which the wave functions reduce to the ones for free fermions. Thus the S-matrix
must reduce to 1 in this limit. The complete S-matrix is then again of form (4.1), where now
\[ S_{ss}(\Lambda^h_1, \Lambda^h_2) = -\frac{\Gamma{(1+im_1/2)}\Gamma{(1-im_1/2)}}{\Gamma{(1+im_2/2)}\Gamma{(1-im_2/2)}} \left( \frac{\mu_1}{\mu_1 + i} + \frac{i}{\mu_1 + i} \right), \quad (5.22) \]
\[ S_{sn}(\Lambda^h, k) = -i \frac{1 + \exp(\pi \mu_2)}{1 - \exp(\pi \mu_2)} I, \quad (5.23) \]
\[ S_{\eta s}(\Lambda^h, k) = -i \frac{1 + \exp(\pi \mu_3)}{1 - \exp(\pi \mu_3)} I, \quad (5.24) \]
\[ S_{cc}(k^h_1, k^h_2) = \frac{\Gamma{(1-im_4/2)}\Gamma{(1+im_4/2)}}{\Gamma{(1+im_4/2)}\Gamma{(1-im_4/2)}} \left( \frac{\mu_4}{\mu_4 - i} + \frac{i}{\mu_4 - i} \right). \quad (5.25) \]

The analytic structure of the repulsive S-matrix can easily be inferred from the structure of the attractive one. There are again no physical poles of the S-matrix in the complex plane.

6. Repulsion versus Attraction

Attractive and repulsive quasiparticle spectra are related by an interchange of spin and charge degrees of freedom. To see this we first make the substitution \( k \rightarrow \pi - k \) in the attractive case. This renaming of spectral parameters has of course no physical consequences. The spinon energy (5.15) for \( U > 0 \) is the same as the energy (3.2) of the charge waves in the attractive case. The same holds for the holon energy (5.10) and the spin-wave energy (3.1). On the level of the quasiparticle momenta these equivalences hold only up to constants. The spinon momentum is related to the momentum of the charge waves for \( U < 0 \) like \( p_s(\lambda) = p_{cw}^h(\lambda) + \frac{\pi}{2} = p_{cw}^p(\lambda) - \frac{\pi}{2} \). The spin-wave momentum \( p_{sw}(k) \) is almost the same as the holon/antiholon momenta for \( U > 0 \) \( p_{cw}(k) = p_h(k) + \frac{\pi}{2} = p_{ah}(k) - \frac{\pi}{2} \). These differences in the quasiparticle momenta are due to the fact that the transformation \( c_{j,\uparrow} \leftrightarrow (-1)^j c_{j,\uparrow} \) that interchanges spin and charge degrees of freedom does not commute with the momentum operator but changes the total momentum by \( \pi \). This can also be easily seen on the level of the two \( SU(2) \) algebras: the spin-\( SU(2) \), which commutes with total momentum, is transformed into the \( \eta- SU(2) \), which changes momentum by \( \pi \). The S-matrices for attraction and repulsion after interchange of spin and charge labels are seen to be identical as functions of the uniformizing parameter \( \mu \).

7. Discussion

In this paper we derived the quasiparticle interpretation for the one-dimensional Hubbard model and evaluated the exact S-matrix describing the scattering of these quasiparticles. The classification of elementary excitations into spinons (which carry spin but no charge)
and holons/antiholons (which carry charge but no spin) clearly reflects spin-and charge separation in the one-dimensional Hubbard model\(^6\). The S-matrix is a $SU(2) \times SU(2)$-invariant solution of the Yang-Baxter equation, which establishes the one-dimensional Hubbard model as a factorizable $SO(4)$-scattering theory. Our S-matrix is very similar to the S-matrix for the $SU(2) \times SU(2)$ principal chiral model with Wess-Zumino term of level 1\(^{33}\). The charge-charge and charge-spin sectors are also identical to the S-matrix for electron-electron and electron-kink scattering in the “exactly screened case” of the Kondo problem\(^{34}\). Apart from its relevance in the field-theory limit the S-matrix is an important tool in the study of transport properties\(^{35,36}\), and plays a crucial role in the general theory of integrable models, where S-matrix and R-matrix are often proportional. In [37] a R-matrix for an embedding of the Hubbard hamiltonian in a commuting family of transfer matrices was derived. This R-matrix, which belongs to a fundamental model (where L-operator and R-matrix are essentially identical\(^{15}\)), is different from our S-matrix. It is quite natural to search for a different embedding of the Hubbard hamiltonian, using an R-matrix proportional to our S-matrix. The L-operator for such a construction would have to be different from the R-matrix.

Acknowledgements:

It is a pleasure to thank Prof. C.N. Yang and Dr. E. Melzer for stimulating discussions. This work was supported in part by the National Science Foundation under grant 9309888 and by NATO 9.15.02 RG 901098 Special Panel on Chaos Order and Patterns ‘Functional Integral Methods in Statistical Mechanics and Correlations’.

\(^6\)The analogous statement for spin-waves and charge-waves in the attractive case is of course also true.
I. Appendix: 2N-Quasiparticle Sector

In this appendix we use the Thermodynamic Bethe Ansatz (TBA)\cite{38} to prove the validity of the quasiparticle interpretation (obtained by analyzing the two-particle sector) in the 2N-quasiparticle sector. We will discuss the $U < 0$ case explicitly, the repulsive case can be treated in an analogous manner. The TBA-part of the analysis in the attractive case discussed here is very similar to the one for repulsion, which is presented in great detail in \cite{17}. We refer to that paper for explanations and notations and merely give the results here. Starting point for the TBA are the Bethe equations (2.4) - (2.6). In the thermodynamic limit these equations turn into coupled integral equations for the particle-densities $\rho$, $\sigma_n$, $\sigma'_n$ and hole-densities $\bar{\rho}$, $\bar{\sigma}_n$, $\bar{\sigma}'_n$ ($n = 1, \ldots \infty$) describing the distribution of elementary $k$'s, $\Lambda$-strings of length $n$, and $k - \Lambda$-strings of length $n$ respectively

\[
\rho(k) + \bar{\rho}(k) = \frac{1}{2\pi} - \frac{\cos(k)}{2\pi} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \frac{2n|U|}{n^2U^2 + (\sin k - \Lambda)^2}(\sigma_n(\Lambda) + \sigma'_n(\Lambda))
\]

\[
\bar{\sigma}_n(\Lambda) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \frac{2n|U|}{n^2U^2 + (\Lambda - \sin k)^2}\rho(k) - \sum_{m=1}^{\infty} A_{nm} * \sigma_m \bigg|_{\Lambda}
\]

\[
\bar{\sigma}'_n(\Lambda) = \frac{1}{\pi} \text{Re} \left\{ \frac{1}{\sqrt{1 - (\Lambda + in|U|)^2}} - \int_{-\pi}^{\pi} dk \frac{2n|U|\rho(k)}{2\pi n^2U^2 + (\sin(k) - \Lambda)^2} - \sum_{m=1}^{\infty} A_{nm} * \sigma'_m \bigg|_{\Lambda} \right\}
\]

where

\[
A_{nm} * f \bigg|_{\lambda} = \delta_{nm} f(\lambda) + \frac{1}{2\pi} \frac{d}{d\lambda} \int_{-\infty}^{\infty} d\lambda' \theta_{n,m}(\frac{\lambda - \lambda'}{|U|}) f(\lambda')
\]

More precisely, (I.1) determine the hole-densities in terms of the particle densities. The particle densities (at finite temperature) are determined by minimizing the thermodynamic potential $\Omega = E - TS - \mu N$ ($\mu$ is the chemical potential\footnote{We will set $\mu = 2U$ as we are interested in the half-filled case.}, $T$ the temperature, $S$ the entropy, and $E$ the energy following from (2.9)). This leads to the following equations for the ratios
of densities $\zeta(k) = \frac{\rho(k)}{\rho(k)}$, $\eta_n(\Lambda) = \frac{\rho_n(\Lambda)}{\sigma_n(\Lambda)}$, and $\eta'_n(\Lambda) = \frac{\rho'_n(\Lambda)}{\sigma'_n(\Lambda)}$

$$\ln(\zeta(k)) = \frac{-2 \cos(k) - \mu}{T} + \sum_{n=1}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \frac{2n|U|}{n^2 U^2 + (\sin(k) - \Lambda)^2} \ln\left(1 + \frac{1}{\zeta(k)}\right),$$

$$\ln(1 + \eta_n(\Lambda)) = \sum_{n=1}^{\infty} A_{nm} * \ln\left(1 + \frac{1}{\eta_m}\right) + \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \frac{2n|U| \cos(k)}{} \ln\left(1 + \frac{1}{\zeta(k)}\right),$$

$$\ln(1 + \eta'_n(\Lambda)) = \frac{-4Re\left\{ \sqrt{1 - (\Lambda + \imath n|U|)^2} \right\}}{T} + \sum_{n=1}^{\infty} A_{nm} * \ln\left(1 + \frac{1}{\eta'_m}\right) + \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \frac{2n|U| \cos(k)}{} \ln\left(1 + \frac{1}{\zeta(k)}\right).$$

The ratios of densities are related to the dressed energies $\kappa(k)$ (for elementary $k$'s), $\epsilon_n(\Lambda)$ (for $\Lambda$-strings of length $n$) and $\epsilon'_n(\Lambda)$ (for $k - \Lambda$-strings of length $n$) via $\zeta(k) = \exp\left(\frac{\kappa(k)}{T}\right)$, $\eta_n(\Lambda) = \exp\left(\frac{\epsilon_n(\Lambda)}{T}\right)$, and $\eta'_n(\Lambda) = \exp\left(\frac{\epsilon'_n(\Lambda)}{T}\right)$. In the zero-temperature limit all dressed energies except $\epsilon'_1(\Lambda)$ are greater or equal to zero, whereas $\epsilon'_1(\Lambda) \leq 0$ on the whole real line. This implies that the attractive ground state is obtained by filling all vacancies for $k - \Lambda$-strings of length 1, which is how we constructed it in section 2. In the $T = 0$ limit it is also possible to completely solve the system (I.2) for the dressed energies. We find

$$
\epsilon'_1(\Lambda) = -2 \int_{0}^{\infty} \frac{d\omega}{\omega} \frac{J_1(\omega) \cos(\omega \Lambda)}{\cosh(\omega U)}, \\
\kappa(k) = 2|U| - 2 \cos(k) + 2 \int_{0}^{\infty} \frac{d\omega}{\omega} \frac{J_1(\omega) \cos(\omega \sin(k)) e^{-\omega|U|}}{\cosh(\omega U)}, \\
\epsilon'_n(\Lambda) = 0 \forall n \geq 2, \quad \epsilon_n(\Lambda) = 0 \forall n \geq 1.
$$

This is in perfect agreement with our analysis of section 2: making a hole at spectral parameter $\Lambda^h$ in the distribution of $\sigma'_1(\Lambda)$ costs energy $-\epsilon'_1(\Lambda^h)$. This obviously corresponds to a charge-wave, and we identify $\epsilon'_1 = -\epsilon_{cw}$. Introducing one elementary $k$ increases the energy by $\kappa(k)$, which is seen to be identical to the spin-wave energy $\epsilon_{sw}(k)$. The additional information we get out of (I.3) is that only these two dressed energies are nonvanishing. Thus the energy of any excited state is the sum over $-\epsilon'_1$'s and $\kappa$'s. A similar analysis can be carried out for the total momentum of excitations at $T = 0$. Using the expression

$$P = \frac{2\pi}{L} \left( \sum_{j=1}^{L} I_j - \sum_{n=1}^{M_2} \sum_{j_n=1}^{M_{2n}} J_n + \sum_{n=1}^{\infty} \sum_{\alpha=1}^{\infty} M_n \sum_{\alpha=1}^{\infty} \sum_{\alpha=1}^{\infty} J_n \right) + \pi \sum_{n=1}^{\infty} M_{2n}$$

for the total momentum it can be shown that only $k - \Lambda$-strings of length 1 and elementary $k$'s contribute dynamically ($k - \Lambda$-strings of lengths greater than 1 contribute $\pi(n - 1)$).
Thus we conclude that both energy and momentum of any excitation breaks up into sums over the quasiparticle energies and momenta. What remains to be shown is now that the quasiparticle interpretation (QPI) predicts the correct number of $SO(4)$-representations with spin $S$ and $\eta$-spin $\eta$ in the sector with $2N$ quasiparticles (recall that there always are an even number). The prediction of the QPI for the number of lowest-weight scattering states of $M_e$ spin-waves with spin $S$ (note that there is a constraint that $\frac{M_e}{2} - S$ is always an integer) is simply

$$\chi_1 = \left( \frac{M_e}{2} - S \right) - \left( \frac{M_e}{2} - S - 1 \right).$$

This number has to be multiplied by the number of lowest-weight scattering states of $2N-M_e$ charge-waves, which is

$$\chi_2 = \left( 2N - M_e \right) - \left( 2N - M_e \right).$$

Are these numbers reproduced by the Bethe Ansatz? The answer is of course “Yes”. Excitations over the ground state are constructed based on the allowed ranges of integers (2.8). By analyzing these equations we can determine the exact number of Bethe Ansatz excitations (i.e. lowest weight states of $SO(4)$ multiplets). We fix the number of holes in the $\Lambda^l$ distribution to be $2N-M_e$ and the number of elementary $k$’s to be $M_e$. The dispersion of such states is by the above arguments identical to the dispersion of a scattering state of $M_e$ spin-waves and $2N-M_e$ charge-waves. If we further fix the values $S$ of spin and $\eta$ of $\eta$-spin, the total number of Bethe states with these characteristics is given by

$$\sum_{M_1, M_2, \ldots} \prod_{n=1}^{\infty} \left( M_e - \sum_{m=1}^{\infty} t_{mn} M_m \right) \times \sum_{M'_1, M'_2, \ldots} \prod_{n=2}^{\infty} \left( 2N - M_e - \sum_{m=2}^{\infty} (t_{mn} - 2) M'_m \right).$$

These sums can simply be looked up in [18] or [40] and we find that the two factors exactly coincide with $\chi_1$ and $\chi_2$. This establishes the validity of the QPI in the $2N$-quasiparticle sector.

References

1. P.W. Anderson, Science 235 (1987) 1196.  
2. F. Woynarovich, J. Physics C16 (1983) 5293.  
3. F. Woynarovich, J. Physics C15 (1982) 85.
4. G. Baskaran, Z. Zou, P.W. Anderson, *Solid State Comm.* **63** (1987) 973.
5. E.H. Lieb, F.Y. Wu, *Phys. Rev. Lett.* **20** (1968) 1445.
6. O.J. Heilmann, E.H. Lieb, *Ann. New York Acad. Sci.* **172** (1971) 583.
7. C.N. Yang, *Phys. Rev. Lett.* **63** (1989) 2144.
8. C.N. Yang and S. Zhang, *Mod. Phys. Lett.* B**4** (1990) 759.
9. M. Pernici, *Europhys. Lett.* **12** (1990) 75.
10. I. Affleck in talk given at the Nato Advanced Study Institute on *Physics, Geometry and Topology*, Banff, August 1989.
11. A.A. Ovchinnikov, *Sov. Phys. JETP* **30** (1970) 1160.
12. T.C. Choy, W. Young, *J. Physics* C**15** (1982) 521.
13. F. Woynarovich, *J. Physics* C**15** (1982) 97.
14. F. Woynarovich, *J. Physics* C**16** (1983) 6593.
15. V.E. Korepin, G. Izergin and N.M. Bogoliubov, *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz*, Cambridge University Press, 1993
16. V.E. Korepin, G. Izergin and N.M. Bogoliubov, *Exactly Solvable Problems in Condensed Matter and Relativistic field Theory*, B.S. Shastry, S.S. Jha, V. Singh (eds.) Lecture Notes in Physics, v.242, Berlin: Springer Verlag, (1985), p.220.
17. M. Takahashi, *Prog. Theor. Phys.* **47** (1972) 69.
18. F.H.L. Eßler, V.E. Korepin, K. Schoutens, *Nucl. Phys.* B**372** (1992) 559, *Nucl. Phys.* B**384** (1992) 431, *Phys. Rev. Lett.* **67** (1991) 3848.
19. N. Andrei, J.H. Lowenstein, *Phys. Lett.* B**91**B (1980) 401.
20. L.D. Faddeev, L. Takhtajan, *J. Soviet Math.* **24** (1984) 241.
21. L.D. Faddeev, L. Takhtajan, *Phys. Lett.* **85A** (1981) 375.
22. L.D. Landau, E.M. Lifshitz, “*Quantum Mechanics*”, Pergamon Press 1975.
23. V.E. Korepin, *Theor. Mat. Phys.* **76** (1980) 165.
24. L. Takhtajan, *Phys. Lett.* **87A** (1982) 479.
25. A.N. Kirillov, N.Yu. Reshetikhin, *J. Physics* A**20** (1987) 1565.
26. T.R. Klassen, E. Melzer, *preprint ITP-SB-92-36*.
27. P.B. Wiegmann, *Phys. Lett.* **141**B (1984) 217.
28. N. Andrei, summer course on *Low-dimensional Quantum Field Theories for Condensed Matter Physicists*, Trieste 1992, unpublished.

These lecture notes, which were brought to our attention after completion of our paper, also discuss the computation of phase-shifts in the Hubbard model.
29. N. Andrei, *private communication*.
30. A. Klümper, A. Schadschneider, J. Zittartz, *Z.Phys.* B**78** (1990) 99.
31. N. Kawakami, A. Okiji, *Phys. Rev.* B**40** (1989) 7066.
32. *Exactly Solvable Models of Strongly Correlated Electrons*, eds V.E. Korepin, F.H.L. Eßler, World Scientific, Spring 1994
33. A.B. Zamolodchikov, Al.B. Zamolodchikov, *Nucl. Phys.* B**379** (1992) 602.
34. P. Fendley, *Kinks in the Kondo-problem, preprint BUHEP-93-10*.
35. J. Carmelo, A.A. Ovchinnikov, *J. Physics Cond. Mat.* 3 (1991) 757.
36. J. Carmelo, P. Horsch, A.A. Ovchinnikov, *Phys. Rev.* B46 (1992) 14728.
37. B.S. Shastry, *J. Stat. Phys.* 50 (1988) 57.
38. C.N. Yang, C.P. Yang, *J. Math. Phys.* 10 (1969) 1115.
39. L. Mezincescu, R.I. Nepomechie, *preprint UMTG-170*.
40. M. Takahashi, *Prog. Theor. Phys.* 46 (1971) 401.