Remarks on the singular set of suitable weak solutions to the 3D Navier-Stokes equations

Wei Ren∗, Yanqing Wang† and Gang Wu‡

Abstract

In this paper, let $\mathcal{S}$ denote the possible interior singular set of suitable weak solutions of the 3D Navier-Stokes equations. We improve the known upper box-counting dimension of this set from $360/277 \approx 1.300$ in [24] to $975/758 \approx 1.286$. It is also shown that $\Lambda(\mathcal{S}, r(\log(e/r))^{\sigma}) = 0(0 \leq \sigma < 27/113)$, which extends the previous corresponding results concerning the improvement of the classical Caffarelli-Kohn-Nirenberg theorem by a logarithmic factor in Choe and Lewis [3, J. Funct. Anal., 175: 348-369, 2000] and in Choe and Yang et al. [4, Comm. Math. Phys, 336: 171-198, 2015]. The proof is inspired by a new $\varepsilon$-regularity criterion proved by Guevara and Phuc in [7, Calc. Var. 56:68, 2017].

MSC(2000): 35B65, 35D30, 76D05

Keywords: Navier-Stokes equations; suitable weak solutions; box-counting dimension; generalized Hausdorff dimension

1 Introduction

We consider the following incompressible Navier-Stokes equations in three-dimensional space

$$\begin{cases}
    u_t - \Delta u + u \cdot \nabla u + \nabla \Pi = 0, \quad \text{div} \ u = 0, \\
    u|_{t=0} = u_0,
\end{cases}
$$

where $u$ stands for the flow velocity field, the scalar function $\Pi$ represents the pressure. The initial velocity $u_0$ satisfies $\text{div} \ u_0 = 0$.

In a series of papers, Scheffer in [19, 21] proposed a program to estimate the size of the potential space-time singular set $\mathcal{S}$ of (suitable) weak solutions obeying the local energy inequality to the Navier-Stokes system and proved that the Hausdorff dimension of this set of the 3D Navier-Stokes equations is at most $5/3$. A point is said to be a regular point of the suitable weak solution $u$ provided one has the $L^\infty$ bound of $u$ in some neighborhood of this point. The remaining points are called singular points. In this direction, the celebrated

∗College of Mathematics and Systems Science, Hebei University, Baoding, Hebei 071000, P. R. China Email: renwei4321@163.com
†Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, Henan 450002, P. R. China Email: wangyanqing20056@gmail.com
‡School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, P. R. China Email: wugangmaths@gmail.com
The Caffarelli-Kohn-Nirenberg theorem in [1] about the 3D Navier-Stokes system is that one-dimensional Hausdorff measure of $S$ is zero, which is deduced from the following $\varepsilon$-regularity criterion: there is an absolute constant $\varepsilon$ such that, if
\[ \limsup_{r \to 0} r^{-\frac{1}{2}} \| \nabla u \|_{L^2_t L^2_x(Q(r))} \leq \varepsilon, \] (1.2)
then $(x, t)$ is a regular point, where $Q(r) := B(r) \times (t - r^2, t)$ and $B(r)$ denotes the ball of center $x$ and radius $r$. From that time on, much effort has been devoted to the extension of the Caffarelli-Kohn-Nirenberg theorem and the $\varepsilon$-regularity criteria were presented in several works (see, e.g., [2–5, 7–15, 18, 22–24]).

Recently, in view of Bernoulli (total) pressure $\frac{1}{2} |u|^2 + \Pi$ as a signed distribution belonging to certain fractional Sobolev space of negative order in local energy inequality, Guevara and Phuc in [7] proved the following $\varepsilon$-regularity criterion: if
\[ \mu^{-\frac{3}{2}} \left( \left\| \frac{1}{2} |u|^2 \right\|_{L^{3/2}_t L^{3/2}_x(Q(\mu))} + \| \Pi \|_{L^{3/2}_t L^{3/2}_x(Q(\mu))} \right) < \varepsilon, \] (1.3)
where $(p, q)$ satisfying
\[ 2/p + 3/q = 7/2 \quad \text{with} \quad 1 \leq p \leq 2, \] (1.4)
then $(x, t)$ is a regular point. An especially interesting case of (1.3) is $p = q = 10/7$, which improves the following classical one shown in [11, 12] via blow-up procedure
\[ \mu^{-\frac{3}{2}} \left( \left\| \frac{1}{2} |u|^2 \right\|_{L^{3/2}_t L^{3/2}_x(Q(\mu))} + \| \Pi \|_{L^{3/2}_t L^{3/2}_x(Q(\mu))} \right) < \varepsilon. \] (1.5)

For the pair $(p, q)$ meeting with (1.4), we would like to mention an $\varepsilon$-regularity criterion in terms of Bernoulli pressure obtained in [13]
\[ \limsup_{\mu \to 0} \mu^{-\frac{3}{2}} \left( \left\| \frac{1}{2} |u|^2 + \Pi \right\|_{L^{3/2}_t L^{3/2}_x(Q(\mu))} \right) < \varepsilon. \]

One objective of this paper is to give an improvement of the known fractal upper box dimension of $S$ via (1.3). The relationship between Hausdorff dimension and the upper box dimension is that the first one is less than second one (see e.g. [6]). The definition of box dimension is via lower box dimension and upper box dimension. In what follows, box dimension and fractal dimension mean the upper box dimension. Before we state our theorem, we recall previous related results. With the help of (1.5), Robinson and Sadowski [14] proved that the upper box dimension of $S$ is at most 5/3. Shortly afterwards, Kukavica [9] showed that the box dimension of the singular set is less than or equal to $135/82 (\approx 1.646)$ and proposed a question whether this dimension of the singular set is at most 1. It was shown that the parabolic fractal dimension of the singular set is less than or equal to $45/29 (\approx 1.552)$ by Kukavica and Pei in [10]. Very recently, Koh and Yang [8] proved that the fractal upper box dimension of $S$ is bounded by $95/63 (\approx 1.508)$. In light of the arguments in [8] and some delicate estimates, the authors in [24] refined the upper box dimension to $360/277 (\approx 1.300)$.

Our first result in this paper is the following theorem:

**Theorem 1.1.** The (upper) box dimension of $S$ is at most $975/758 (\approx 1.286)$.

**Remark 1.1.** This improves the previous box dimension of $S$ obtained in [8, 10, 14, 24].
By contradiction arguments as in [24], Theorem 1.1 turns out to be the consequence of the following theorem.

**Theorem 1.2.** Suppose that the pair \((u, \Pi)\) is a suitable weak solution to (1.1). Then, for any \(\gamma < \frac{865}{2274}\), \((x, t)\) is a regular point provided there exist a sufficiently small universal positive constant \(\varepsilon_1\) and \(0 < r < 1\) such that

\[
\int \int_{Q(r)} |\nabla u|^2 + |u|^{10/3} + |\Pi - \Pi_{B(r)}|^{5/3} + |\nabla \Pi|^{5/4} \, dx \, ds \leq r^{5/3 - \gamma \varepsilon_1}.
\]  

(1.6)

The notations used here can be found at the end of this section.

**Remark 1.2.** Theorem 1.2 is an improvement of corresponding results proved in [10, 24].

**Remark 1.3.** Theorem 1.2 has been inspired by the new \(\varepsilon\)-regularity criterion (1.3). The main method in proving the above result is the one utilized in [8]. Furthermore, motivated by [24], we utilize the quantities \(\|\nabla \Pi\|_{L_{5/4}^{5/4}}\), \(\|\nabla u\|_{L_{5/3}^{2}}\) bounded by the initial energy as widely as possible. To apply (1.3), we need establish some decay estimates adapted to it, see Lemmas 2.1 and 2.2, which play an important role in the proof.

It is known that the Hausdorff dimension of the possible singular set of the suitable weak solution of 5D stationary Navier-Stokes equations is also at most 1 (see eg. [18, 23] and references therein). Therefore, a natural question is whether the box dimension of the singular set to the 5D stationary Navier-Stokes equations is at most one. Indeed, following the path of [8, 24], one could prove that the (upper) box dimension of the set of possible singular sets of suitable weak solutions to this system is at most \(15/13 (\approx 1.154)\). To this end, one just utilizes an analogue of \(\varepsilon\)-regularity criterion (1.5), since the \(\varepsilon\)-regularity criterion (1.3) for time-independent equations yields the same result. We leave this for the interested readers.

The celebrated Caffarelli-Kohn-Nirenberg theorem for the three-dimensional time-dependent Navier-Stokes system can be written as \(\Lambda(S, r) = 0\), for the details of notation, see Sections 2. Some authors improve the Caffarelli-Kohn-Nirenberg theorem by a logarithmic factor, see, for example, [2–5]. Particularly, in [3], Choe and Lewis introduced the generalized Hausdorff measure \(\Lambda(S, r(\log(e/r))^\sigma)\) and proved that \(\Lambda(S, r(\log(e/r))^\sigma) = 0 (0 \leq \sigma < 3/44)\). By deriving a new local energy inequality in the absence of pressure, Choe and Yang in [4] studied the regularity of suitable weak solutions of the magnetohydrodynamic equations in dimension three and proved that \(\Lambda(S, r(\log(e/r))^\sigma) = 0\), where \(S\) denotes the potential interior singular set of suitable weak solutions for this system and \(\sigma\) is bounded by \(1/6\). The reader is referred to the recent work [2, 4] for the boundary case. The second goal of this paper is to improve the bound of \(\sigma\) mentioned above. Precisely, we have the following fact.

**Theorem 1.3.** Let \(S\) stand for the set of all the potential interior singular set of suitable weak solutions to (1.1) and \(0 \leq \sigma < 27/113\). There holds

\[
\Lambda(S, r(\log(e/r))^\sigma) = 0.
\]

**Remark 1.4.** Theorem 1.3 is an improvement of the known corresponding results in [3, 4].

**Remark 1.5.** A combination of arguments presents here and the \(\varepsilon\)-regularity criterion (1.5) implies that, \(\Lambda(S, r(\log(e/r))^\sigma) = 0 (0 \leq \sigma < 5/22)\). The \(\varepsilon\)-regularity criterion (1.3)
combined with the proof of Theorem 1.3 yields that $\Lambda(S, r(\log(e/r))^\sigma) = 0$ (0 $\leq \sigma < 28/127$). Naturally, it may be helpful to utilize the one with $p = 10/3$ below

$$
\mu^{-(5-2p)}\left(\left\|f\right\|^2_{L^{p/2}_x L^{p/2}_t(Q(\mu))} + \|\Pi\|_{L^{p/3}_x L^{p/2}_t(Q(\mu))}\right) < \varepsilon. \quad (1.7)
$$

However, one needs $J_q(\rho)$ with $q = 2$ in the proof, which contradicts (4.11). Based on this, for any $\kappa > 0$, we will apply (1.7) with $p = 10/3 - \kappa$. This allows us to obtain the desired result.

The remainder of this paper is divided into three sections. In Section 2, we present the definitions of upper box-counting dimension and generalized Hausdorff measure. Then, we recall the definition of suitable weak solutions to the Navier-Stokes equations and list some crucial bounds for the scaling invariant quantities. The third section is devoted to the box-counting dimension of the possible singular set of suitable weak solutions. Section 4 is concerned with generalized Hausdorff dimension of the potential singular set in the Navier-Stokes system.

**Notations:** Throughout this paper, the classical Sobolev norm $\|\cdot\|_{H^s}$ is defined as

$$
\left\|f\right\|^2_{H^s} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi, \quad s \in \mathbb{R}.
$$

We denote by $\check{H}^s$ homogenous Sobolev spaces with the norm $\|f\|_{\check{H}^s}^2 = \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi$. Denote by $L^q_2(\Omega)$ the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^q(\Omega)^n$, where $C_{0,\sigma}^\infty(\Omega) = \{u \in C_{0}^\infty(\Omega)^n; \text{div } u = 0\}$. The classical Sobolev space $W^{1,2}(\Omega)$ is equipped with the norm $\|f\|_{W^{1,2}(\Omega)} = \|f\|_{L^2(\Omega)} + \|\nabla f\|_{L^2(\Omega)}$. For $q \in [1, \infty]$, the notation $L^q(0, T; X)$ stands for the set of measurable functions on the interval $(0, T)$ with values in $X$ and $\|f(t, \cdot)\|_X$ belongs to $L^q(0, T)$. For simplicity, we write

$$
\|f\|_{L^q_{t,-\epsilon}(Q(\mu))} := \|f\|_{L^q(t-\delta^2, t; L^q(B(\mu)))} \quad \text{and} \quad \|f\|_{L^q(\epsilon Q(\mu))} := \|f\|_{L^q(\epsilon Q(\mu))}.
$$

Denote the average of $f$ on the set $\Omega$ by $\overline{f}_\Omega$. For convenience, $\overline{f}_r$ represents $\overline{f}_{B(r)}$. $K$ stands for the standard normalized fundamental solution of Laplace equation in $\mathbb{R}^n$ with $n \geq 2$. $|\Omega|$ represents the Lebesgue measure of the set $\Omega$. We will use the summation convention on repeated indices. $C$ is an absolute constant which may be different from line to line unless otherwise stated in this paper.

## 2 Preliminaries

First, we begin with the definitions of the (upper) box-counting dimension of a set and the generalized Hausdorff measure below, respectively.

**Definition 2.1.** The (upper) box-counting dimension of a set $X$ is usually defined as

$$
d_{box}(X) = \limsup_{\epsilon \to 0} \frac{\log N(X, \epsilon)}{-\log \epsilon},
$$

where $N(X, \epsilon)$ is the minimum number of balls of radius $\epsilon$ required to cover $X$.

**Definition 2.2 (cf. [3]).** Let $h$ be an increasing continuous function on $(0, 1]$ with $\lim_{r \to 0} h(r) = 0$ and $h(1) = 1$. For fixed parameter $\delta > 0$ and set $E \subset \mathbb{R}^3 \times \mathbb{R}$, we denote by $D(\delta)$ the family of all coverings $\{Q(x_i, r_i; \delta)\}$ of $E$ with $0 < r_i \leq \delta$. We denote

$$
\Psi_\delta(E, h) = \inf_{D(\delta)} \sum_i h(r_i)
$$

and define the generalized parabolic Hausdorff measure as
\[ \Lambda(E, h) = \lim_{\delta \to 0} \Psi_\delta(E, h). \]

Second, we recall the definition of suitable weak solutions to the Navier-Stokes equations (1.1).

**Definition 2.3.** A pair \( (u, \Pi) \) is called a suitable weak solution to the Navier-Stokes equations (1.1) provided the following conditions are satisfied,

1. \( u \in L^\infty(-T, 0; L^2(\mathbb{R}^3)) \cap L^2(-T, 0; H^1(\mathbb{R}^3)), \) \( \Pi \in L^{3/2}(-T, 0; L^{3/2}(\mathbb{R}^3)) \);
2. \( (u, \Pi) \) solves (1.1) in \( \mathbb{R}^3 \times (-T, 0) \) in the sense of distributions;
3. \( (u, \Pi) \) satisfies the following inequality, for a.e. \( t \in [-T, 0] \),
\[
\int_{\mathbb{R}^3} |u(x,t)|^2 \phi(x,t)dx + 2 \int_{-T}^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi dx ds \leq \int_{-T}^t \int_{\mathbb{R}^3} |u|^2 (\partial_s \phi + \Delta \phi) dx ds + \int_{-T}^t \int_{\mathbb{R}^3} u \cdot \nabla \phi (|u|^2 + 2\Pi) dx ds, \tag{2.1}
\]
where non-negative function \( \phi(x,s) \in C_0^\infty (\mathbb{R}^3 \times (-T, 0)) \).

In the light of the natural scaling property of the time-dependent Navier-Stokes equations, we introduce the following dimensionless quantities:

\[
E(\mu) = \mu^{-1} \|u\|_{L^\infty,2(Q(\mu))}^2, \quad E_s(\mu) = \mu^{-1} \|\nabla u\|_{L^2(\mu)}^2, \\
E_p(\mu) = \mu^{p-5} \|u\|_{L^p(\mu)}^p, \quad P_{5/4}(\mu) = \mu^{-5/4} \|\nabla \Pi\|_{L^{5/4}(\mu)}^{5/4}, \\
P_{10/7}(\mu) = \mu^{-15/7} \left\| \Pi - \Pi_B(\mu) \right\|_{L^{10/7}(\mu)}^{10/7}, \quad P_{5/3}(\mu) = \mu^{-5/3} \left\| \Pi - \Pi_B(\mu) \right\|_{L^{5/3}(\mu)}^{5/3}, \\
J_q(\mu) = \mu^{2q-5} \|\nabla u\|_{L^q(\mu)}^q.
\]

As said before, we need to establish some decay estimates of scaling invariant quantities to consist with (1.3) for \( p = q = 10/7 \). The first estimate is partially motivated by [24, Lemma 2.1, p.222]. We refer the reader to [1, 2, 11, 12, 23] for different versions.

**Lemma 2.1.** For \( 0 < \mu \leq \frac{1}{2} \rho, \frac{7}{2} \leq b \leq 6 \) and \( 3p/5 \leq q \leq 2(p \geq 1, q \geq 1) \), there is an absolute constant \( C \) independent of \( \mu \) and \( \rho \), such that
\[
E_{20/7}(\mu) \leq C \left( \frac{\rho}{\mu} \right)^{10/7} E_{7(6-2b)}^{7b-20} (\rho) E_{7(6-2q)}^{36} (\rho) + C \left( \mu / \rho \right)^{20/7} E^{10/7} (\rho), \tag{2.2}
\]
\[
E_p(\mu) \leq C \left[ \left( \frac{\rho}{\mu} \right) \frac{p+10-5q}{2} E_{7(6-2b)}^{7b-20} (\rho) J_q(\rho) + C \left( \frac{\rho}{\mu} \right)^p E^{p/2} (\rho) \right]. \tag{2.3}
\]

**Proof.** By utilizing the Hölder inequality twice and the Poincaré-Sobolev inequality, for any \( 20/7 < b \leq 6 \), we infer that
\[
\int_{B(\mu)} |u - \bar{u}_B(\rho)|^{20/7} dx \leq C \left( \int_{B(\mu)} |u - \bar{u}_B(\rho)|^2 dx \right)^{7b-20} \left( \int_{B(\mu)} |u - \bar{u}_B(\rho)|^b dx \right)^{\frac{6}{7b-20}}.
\]
\[ \leq C\mu^{\frac{2(6-b)}{3q}} \left( \int_{B(\rho)} |u|^2 \right)^{\frac{7b-20}{7(b-2)}} \left( \int_{B(\rho)} |\nabla u|^2 \right)^{\frac{3b}{7(b-2)}}. \]

According to the triangle inequality and the last inequality, we see that
\[ \int_{B(\mu)} |u|^{20/7} \leq C \int_{B(\mu)} |u - \bar{u}_{B(\rho)}|^{20/7} + C \int_{B(\mu)} |\bar{u}_{\rho}|^{20/7} \]
\[ \leq C\mu^{\frac{2(6-b)}{3q}} \left( \int_{B(\rho)} |u|^2 \right)^{\frac{7b-20}{7(b-2)}} \left( \int_{B(\rho)} |\nabla u|^2 \right)^{\frac{3b}{7(b-2)}} + \frac{\mu^3 C}{\rho^{\frac{3q}{2}}} \left( \int_{B(\rho)} |u|^2 \right)^{10/7}. \]

Integrating this inequality in time on \((t - \mu^2, t)\) and utilizing the Hölder inequality, for any \(b \geq 7/2\), we get
\[ \int \int_{Q(\mu)} |u|^{20/7} \leq C\mu^{\frac{2}{7}} \left( \sup_{t - \mu^2 \leq s \leq t} \int_{B(\rho)} |u|^2 \right)^{\frac{7b-20}{7(b-2)}} \left( \int \int_{Q(\rho)} |\nabla u|^2 \right)^{\frac{3b}{7(b-2)}} + C\mu^{\frac{5}{7}} \left( \sup_{t - \mu^2 \leq s \leq t} \int_{B(\rho)} |u|^2 \right)^{10/7}, \]

which yields that
\[ E_{20/7}(\mu) \leq C \left( \frac{\mu}{\rho} \right)^{10/7} E_{\frac{7b-20}{7(b-2)}}(\rho) E_{\frac{3b}{7(b-2)}}(\rho) + C \left( \frac{\mu}{\rho} \right)^{20/7} E^{10/7}(\rho). \]

Let us now move to the proof of (2.3). Thanks to the Hölder inequality and the Poincaré-Sobolev inequality, for any \(3p/5 \leq q \leq 2\), we know that
\[ \int_{B(\mu)} |u - \bar{u}_{B(\rho)}|^p \leq \left( \int_{B(\mu)} |u - \bar{u}_{B(\rho)}|^2 \right)^{\frac{p-q}{2}} \left( \int_{B(\mu)} |u - \bar{u}_{B(\rho)}|^\frac{2q}{p+q} \right)^{\frac{2+q-p}{2}} \]
\[ \leq C\mu^{\frac{5a-3b}{2}} \left( \int_{B(\rho)} |u|^2 \right)^{\frac{p-q}{2}} \left( \int_{B(\rho)} |\nabla u|^q \right). \]

Taking advantage of the triangle inequality, the Hölder inequality and the Poincaré-Sobolev inequality, for any \(3p/5 \leq q \leq 2\), we know that
\[ \int_{B(\mu)} |u|^p \leq C \int_{B(\mu)} |u - \bar{u}_{B(\rho)}|^p \leq C \int_{B(\mu)} |u - \bar{u}_{B(\rho)}|^p \]
\[ \leq C\mu^{\frac{5a-3b}{2}} \left( \int_{B(\rho)} |u|^2 \right)^{\frac{p-q}{2}} \left( \int_{B(\rho)} |\nabla u|^q \right) + \frac{\mu^3 C}{\rho^{\frac{3q}{2}}} \left( \int_{B(\rho)} |u|^2 \right)^{p/2}. \]

Integrating this inequality in time on \((-\mu^2, 0)\) and using the Hölder inequality, we obtain
\[ \int \int_{Q(\mu)} |u|^p \leq C\mu^{\frac{5a-3b}{2}} \left( \sup_{-\mu^2 \leq s \leq 0} \int_{B(\rho)} |u|^2 \right)^{\frac{p-q}{2}} \left( \int \int_{Q(\rho)} |\nabla u|^q \right)^{\frac{p}{2}}, \]

\[ + C\mu^{\frac{5}{3q}} \left( \sup_{-\mu^2 \leq s \leq 0} \int_{B(\rho)} |u|^2 \right)^{p/2}. \]
which in turn implies that
\[ E_p(r) \leq C \left[ \left( \frac{\rho}{r} \right)^{\frac{n+10-5q}{2}} E^{(p-q)}(\rho)J_q(\rho) + C \left( \frac{\rho}{r} \right)^p E^{p/2}(\rho) \right]. \]

This achieves the proof of this lemma.

In the spirit of [17, Lemma 2.1, p.222], we can make full use of the interior estimate of harmonic function to establish the following decay estimate of pressure \( \Pi \) in terms of \( \nabla \Pi \) in equations enables us to apply this lemma in the proof of Theorem [1.2] and Theorem [1.3].

**Lemma 2.2.** For \( 0 < \mu \leq \frac{1}{8} \rho \), there exists an absolute constant \( C \) independent of \( \mu \) and \( \rho \) such that
\[
\begin{align*}
P_{10/7}(\mu) &\leq C \left( \frac{\rho}{\mu} \right)^{15/7} E_{20/7}(\rho) + C \left( \frac{\mu}{\rho} \right)^{16/7} P_{10/7}(\rho), \\
P_{p/2}(\mu) &\leq C \left[ \left( \frac{\rho}{\mu} \right)^{5-p} E^{(p-q)}(\rho)J_q(\rho) + \left( \frac{\mu}{\rho} \right)^{3p-4} P_{p/2}(\rho) \right],
\end{align*}
\]
where \( p \) and \( q \) are defined in Lemma [2.1].

**Proof.** We choose the usual smooth cut-off function \( \phi \in C_0^\infty(B(\rho/2)) \) such that \( \phi \equiv 1 \) on \( B(\frac{3}{8}\rho) \) with \( 0 \leq \phi \leq 1 \) and \( |\nabla \phi| \leq C \rho^{-1}, |\nabla^2 \phi| \leq C \rho^{-2} \).

It follows from divergence free condition that
\[ \partial_i \partial_i (\Pi \phi) = -\phi \partial_i \partial_j [U_{i,j}] + 2\partial_j \phi \partial_i \Pi + \Pi \partial_i \partial_i \phi, \]
where \( U_{i,j} = (u_j - \overline{u}_j)/(u_i - \overline{u}_i)/(r^2/2) \).

For any \( x \in B(\frac{3}{8}\rho) \), we derive from integrations by parts that
\[
\Pi(x) = K \ast \{-\phi \partial_i \partial_j [U_{i,j}] + 2\partial_j \phi \partial_i \Pi + \Pi \partial_i \partial_i \phi\}
= -\partial_i \partial_j K \ast (\phi[U_{i,j}])
+ 2\partial_j K \ast (\partial_j \phi[U_{i,j}]) - K \ast (\partial_i \partial_j \phi[U_{i,j}])
+ 2\partial_i K \ast (\partial_i \phi \Pi) - K \ast (\partial_i \partial_i \phi \Pi)
=: P_1(x) + P_2(x) + P_3(x),
\]
where \( K \) represents the standard normalized fundamental solution of Laplace equation.

Thanks to \( \phi(x) = 1 \ (x \in B(\rho/4)) \), we know that
\[ \Delta (P_2(x) + P_3(x)) = 0. \]

In the light of the interior estimate of harmonic function and Hölder’s inequality, we see that, for every \( x_0 \in B(\rho/8), \)
\[
|\nabla (P_2 + P_3)(x_0)| \leq \frac{C}{\rho^4} \|(P_2 + P_3)\|_{L^1(B_{x_0}(\rho/8))}
\leq \frac{C}{\rho^4} \|(P_2 + P_3)\|_{L^1(B(\rho/4))}
\leq \frac{C}{\rho^{p=6}/p} \|(P_2 + P_3)\|_{L^p(B(\rho/4))},
\]

7
which in turn implies
\[
\| \nabla (P_2 + P_3) \|_{L^\infty(B(\rho/8))}^{p/2} \leq C \rho^{- (p+6)/2} \| (P_2 + P_3) \|_{L^{p/2}(B(\rho/4))}^{10/7},
\]
This combined with the mean value theorem yields that, for any \( \mu \leq \frac{1}{8} \rho, \)
\[
\| (P_2 + P_3) - (P_2 + P_3)_{B(\mu)} \|_{L^{p/2}(B(\mu))}^{p/2} \leq C \mu^2 \| (P_2 + P_3) - (P_2 + P_3)_{B(\mu)} \|_{L^\infty(B(\mu))}^{p/2} \\
\leq C \mu^{(p+6)/2} \| \nabla (P_2 + P_3) \|_{L^\infty(B(\rho/8))}^{p/2} \\
\leq C \left( \frac{\mu}{\rho} \right)^{(p+6)/2} \| (P_2 + P_3) \|_{L^{p/2}(B(\rho/4))}^{p/2}.
\]
Note that \((P_2 + P_3) - (P_2 + P_3)_{B(\rho/4)}\) is also a harmonic function on \(B(\rho/4),\) hence, there holds
\[
\| (P_2 + P_3) - (P_2 + P_3)_{B(\mu)} \|_{L^{p/2}(B(\mu))}^{p/2} \\
\leq C \left( \frac{\mu}{\rho} \right)^{(p+6)/2} \| (P_2 + P_3) - (P_2 + P_3)_{B(\rho/4)} \|_{L^{p/2}(B(\rho/4))}^{p/2}.
\]
By the triangle inequality, we deduce that
\[
\| (P_2 + P_3) - (P_2 + P_3)_{B(\rho/4)} \|_{L^{p/2}(B(\rho/4))} \\
\leq \| \Pi - \Pi_{B(\rho/4)} \|_{L^{p/2}(B(\rho/4))} + \| P_1 - P_{B(\rho/4)} \|_{L^{p/2}(B(\rho/4))} \\
\leq C \| \Pi - \Pi_{B(\rho)} \|_{L^{p/2}(B(\rho/4))} + C \| P_1 \|_{L^{p/2}(B(\rho/4))},
\]
which tells us that
\[
\| (P_2 + P_3) - (P_2 + P_3)_{B(\mu)} \|_{L^{p/2}(B(\mu))}^{p/2} \\
\leq C \left( \frac{\mu}{\rho} \right)^{(p+6)/2} \left( \| \Pi - \Pi_{B(\rho)} \|_{L^{p/2}(B(\rho/4))} + \| P_1 \|_{L^{p/2}(B(\rho/4))} \right). \tag{2.9}
\]
The classical Calderón-Zygmund theorem ensures that
\[
\int_{B(\rho/4)} |P_1(x)|^{p/2} dx \leq C \int_{B(\rho/2)} |u - \bar{u}_{B(\rho/2)}|^{p} dx, \tag{2.10}
\]
from which it follows that, for any \( \mu \leq \frac{1}{8} \rho, \)
\[
\int_{B(\mu)} |P_1(x)|^{p/2} dx \leq C \int_{B(\rho/2)} |u - \bar{u}_{B(\rho/2)}|^{p} dx. \tag{2.11}
\]
Employing time integration on \((t - \mu^2, t)\) and the triangle inequality, we conclude using (2.9)-(2.11) that
\[
\int_{Q(\mu)} |\Pi - \Pi_{B(\mu)}|^{p/2} dx \forall s \\
\leq \int_{Q(\mu)} |P_1 - \Pi_{B(\mu)}|^{p/2} dx \forall s + \int_{Q(\mu)} |P_2 + P_3 - (P_2 + P_3)_{B(\mu)}|^{p/2} dx \forall s \\
\leq C \int_{Q(\mu)} |P_1|^{p/2} dx \forall s + C \left( \frac{\mu}{\rho} \right)^{(p+6)/2} \left( \| \Pi - \Pi_{B(\rho)} \|_{L^{p/2}(B(\rho/4))} + \| P_1 \|_{L^{p/2}(B(\rho/4))} \right) \\
\leq C \int_{Q(\rho/2)} |u|^{p} dx \forall s + C \left( \frac{\mu}{\rho} \right)^{(p+6)/2} \| \Pi - \Pi_{B(\rho)} \|_{L^{p/2}(B(\rho))}, \tag{2.12}
\]
which means (2.7). A slight modified the above the proof of the latter inequality together with (2.4) gives (2.7). The proof of this lemma is completed. \qed
3 Proof of Theorem 1.2

The main part of this sections is the proof of Theorem 1.2. The method follows closely the recent developments in [8, 24]. The main ingredient is to apply (1.3) and decay-type estimates established in Section 2.

Proof of Theorem 1.2 From (1.6), we choose \(2\rho < 1\) such that \(\rho^g < 1/2\), where \(\beta\) will be determined later and

\[
\int \int_{Q(2\rho)} |\nabla u|^2 + |u|^{10/3} + |\Pi - \Pi_{B(2\rho)}|^{5/3} + |\nabla \Pi|^{5/4} dxds \leq (2\rho)^{5/3-\gamma} \varepsilon_1. \tag{3.1}
\]

First, one can derive

\[
E(\rho) \leq C\varepsilon_1^{3/5} \rho^{-\frac{7\gamma}{5}}, \quad (\gamma \leq 5/12), \tag{3.2}
\]

from (3.1) via the local energy inequality (2.1), which is proved in [24]. Here we omit the details, see [24, Proof of theorem 1.2, p.1768-1769] for details. Second, iterating (2.6) in Lemma 2.2, we see that

\[
P_{10/7}(\theta^N\mu) \leq C \sum_{k=1}^N \theta^{-\frac{15}{4} + \frac{16(k-1)}{7}} E_{20/7}(\theta^{N-k}\mu) + C\theta^{16N/7} P_{10/7}(\mu). \tag{3.3}
\]

With the help of the Poincaré-Sobolev inequality and Hölder’s inequality, we get

\[
\|\Pi - \Pi_{B(\mu)}\|_{10/7}^{10/7} \leq \|\Pi - \Pi_{B(\mu)}\|_{L_{5/4,15/7}(Q(\mu))}^{5/7} \|\Pi - \Pi_{B(\mu)}\|_{L_{5/3,15/14}(Q(\mu))}^{5/7} \leq C\mu^{5/7} \|\nabla \Pi\|_{L_{5/4}(Q(\mu))}^{5/7} \|\Pi - \Pi_{B(\mu)}\|_{L_{5/3}(Q(\mu))}^{1/2}. \tag{3.4}
\]

Dividing both sides of the last inequality by \(\mu^{15/7}\), we arrive at

\[
P_{10/7}(\mu) \leq CP_{5/4}(\mu) P_{5/3}(\mu).
\]

We substitute the above inequality into (3.3) to obtain that

\[
P_{10/7}(\theta^N\mu) \leq C \sum_{k=1}^N \theta^{-\frac{15}{4} + \frac{16(k-1)}{7}} E_{20/7}(\theta^{N-k}\mu) + C\theta^{16N/7} P_{5/4}(\mu) P_{5/3}(\mu). \tag{3.5}
\]

To proceed further, we set \(r = \rho^\alpha = \theta^N\mu, \theta = \rho^\beta, r_i = \mu = \theta^{-i} r = \rho^{-i\beta}(1 \leq i \leq N)\), where \(\alpha\) and \(\beta\) are determined by \(\gamma\). Their precise selection will be given in the end. Hence, we derive from (3.5) that

\[
P_{10/7}(r) + E_{20/7}(r) \leq C \sum_{k=1}^N \theta^{-\frac{15}{4} + \frac{16(k-1)}{7}} E_{20/7}(r_k) + C\theta^{16N/7} P_{5/4}(r_N) P_{5/3}(r_N) \tag{3.6}
\]

where we have used the fact that \(E_{20/7}(u, r) \leq C\theta^{-\frac{15}{4}} E_{20/7}(u, \theta^{-1} r)\). Our aim below is to resort to (1.3) to complete the proof, that is, there exists a constant \(r > 0\) such that
$P_{10/7}(r) + E_{20/7}(r) < \varepsilon_0$. To this end, we adopt (2.2) with $b = 7/2$ in Lemma 2.1 (3.2) and (3.1) to obtain

$$E_{20/7}(r_k) \leq C\left(\frac{\rho}{r_k}\right)^{20/7} E^{3/7}(\rho) E_\star(\rho) + C\left(\frac{r_k}{\rho}\right)^{20/7} E^{10/7}(\rho)$$

$$\leq C\varepsilon_1^{6/7} \left( \rho^{\frac{44}{21} - \frac{10\alpha}{7} - \frac{2\alpha}{7} + \frac{3\beta}{7}} + \rho^{\frac{3\beta}{7} + \frac{2\alpha}{7} - \frac{2\alpha}{7}} \right).$$

Substituting the last inequality into $I$ produces that

$$I \leq C\varepsilon_1^{6/7} \sum_{k=1}^N \left( \rho^{\frac{3\beta}{7} + \frac{2\alpha}{7} - \frac{2\alpha}{7}} + \rho^{\frac{3\beta}{7} + \frac{2\alpha}{7} - \frac{2\alpha}{7}} \right).$$

To minimise the righthand side of this inequality, we choose

$$\alpha = \frac{7}{30} \left( \frac{26\beta}{7} + 104 \frac{21}{5} - 2\gamma + 4N\beta \right)$$

(3.7)

to conclude that, for sufficiently large $N$,

$$I \leq C\varepsilon_1^{6/7} \left( \rho^{-\frac{3\beta}{7} + \frac{2\alpha}{7} - \frac{2\alpha}{7}} + \rho^{-\frac{3\beta}{7} + \frac{2\alpha}{7} - \frac{2\alpha}{7}} \right)$$

(3.8)

To bound $II$, we will temporarily assume that $r_N \leq \rho$, namely

$$\rho^{\alpha - N\beta} \leq \rho.$$ (3.9)

Combining (3.1) and (3.7), we see that

$$II \leq C\rho^{\frac{16N\beta}{21}} r_N^{-\frac{20}{7}} \left( \int_{Q(2\rho)} |\nabla \Pi|^{5/4} dx \right)^{4/7} \left( \int_{Q(r_N)} |\Pi - \Pi_{2\rho}|^{5/3} dx \right)^{3/7}$$

$$\leq C\rho^{\frac{16N\beta}{21}} r_N^{-\frac{20}{7}} \left( \int_{Q(2\rho)} |\nabla \Pi|^{5/4} dx \right)^{4/7} \left( \int_{Q(2\rho)} |\Pi - \Pi_{2\rho}|^{5/3} dx \right)^{3/7}$$

(3.10)

In order to conclude that $I + II \leq C\varepsilon_1^{6/7} \leq \varepsilon_0$, we need $-\frac{4\beta}{21} + \frac{1}{9} - \frac{118\gamma}{105} - \frac{4N\beta}{21} \geq 0$ and $\frac{74N\beta}{21} + \frac{1}{63} - \frac{13\gamma}{15} - \frac{26\beta}{21} \geq 0$. In addition, it follows from (3.9) that $\alpha - N\beta - 1 \geq 0$. Hence, we sum up all the restrictions of $\gamma$ below

$$\gamma \leq \min \left\{ \frac{5(28 - 12N\beta - 123\beta)}{354}, \frac{5(1 + 222N\beta - 78\beta)}{273}, \frac{5(7 - 39N\beta + 39\beta)}{21}, \frac{5}{12} \right\}.$$ (3.11)

Maximising this bound on $\gamma$ with respect to $N\beta$, we obtain $N\beta = 135/1516$. Furthermore, it follows (3.11) from that

$$\beta = \frac{135}{1516N} \leq \frac{118}{205} \left( \frac{865}{2274} - \gamma \right).$$

Hence, choosing $\beta$ sufficiently small by selecting $N$ sufficiently large, we can have any $\gamma < 865/2274$. Then, we pick $\alpha = \frac{7}{30} \left( \frac{26\beta}{7} - \frac{2\alpha}{5} + \frac{38821}{7959} \right)$. In this stage, from (3.6), (3.8) and (3.10), we get

$$P_{10/7}(r) + E_{20/7}(r) \leq C\varepsilon_1^{6/7} \leq \varepsilon_0,$$

with $r = \rho^\alpha$. By (1.3), we know that $(x, t)$ is a regular point in this stage. This completes the proof of Theorem 1.2.
4 Proof of Theorem 1.3

In the spirit of [2, 3], we begin with some technical lemmas for the proof of Theorem 1.3. These lemmas are parallel to the one of [3]. It is worth remarking that the proof of Lemma 4.2 is slightly different from the ones in [2–5]. In what follows, we set \( m(r) = (\Gamma(r))^\sigma = (\log(e/r))^\sigma \), where \( \sigma \in (0, 1) \) will be determined later.

Before going further, we set
\[
F(m) = \left\{ (x,t) \mid \limsup_{r \to 0} \frac{E_\ast(r)}{m(r)} \leq 1 \right\}.
\]

Lemma 4.1. Assume that \((x,t) \in F(m) \cap S\) and the pair \((p, q)\) is used in Lemma 2.1. Then, there exists a positive constant \(c_1\) and \(c_2\) independent of \((x,t)\) such that
\[
\limsup_{r \to 0} \frac{E(r)}{m^2(r)} \leq c_1,
\]
\[
\limsup_{r \to 0} \frac{P_{p/2}(r)}{m^{p-1}(r)} \leq c_2.
\]

**Proof.** The reader is referred to [3, Lemma 1, page 357] for the detail of (4.1). We outline the proof of (4.2). Let \( g(r) = \frac{P_{p/2}(r)}{m^{p-1}(r)} \), from (2.3) with \( q = 2 \), we see that
\[
g(\mu) \leq C \left[ \left( \frac{\mu}{\rho} \right)^{5-p} \left( \frac{E_\ast(\rho)}{m^2(\rho)} \right)^{\frac{p-2}{2}} \frac{E_\ast(\rho)}{m(\rho)} + \left( \frac{\mu}{\rho} \right)^{\frac{3p-4}{2}} g(\rho) \right]
\]
\[
\leq C \left[ \left( \frac{\mu}{\rho} \right)^{5-p} + \left( \frac{\mu}{\rho} \right)^{\frac{3p-4}{2}} g(\rho) \right],
\]
where we have utilized the hypothesis and (4.1). This together with the iteration method see, (e.g. [17]) allows us to obtain (4.2). \( \square \)

Lemma 4.2. Let \((x,t) \in F(m) \cap S\). Then, there exists a positive constant \(c_2\) independent of \((x,t)\) such that
\[
\liminf_{r \to 0} J_q(r)m(r)^\tau \geq c_3,
\]
where \( \tau = \frac{\mu^2 + (6-3q)p + 4q}{3p-4} \) and the pair \((p, q)\) is utilized in Lemma 2.1.

**Proof.** Assume that the statement fails, then, for any \( \eta > 0 \), there exists a singular point \((x,t)\) and a sequence \( r_n \to 0 \) such that
\[
J_q(r_n)m(r_n)^\tau < \eta.
\]
It follows from (2.3), Lemma 1.1 and (4.3) that, for \( \theta_n < 1/8 \),
\[
E_p(\theta_n r_n) + P_{p/2}(\theta_n r_n) \leq C \theta_n^p m^p(r_n) + C \theta_n^{-\frac{p+10-5q}{2}} m(r_n)^{p-q} J_q(r_n)
\]
\[
+ C \theta_n^{\frac{3p-4}{2}} m^{p-1}(r_n) + C \theta_n^{-(5-p)} m(r_n)^{p-q} J_q(r_n)
\]
\[
\leq C \theta_n^{\frac{3p-4}{2}} m^p(r_n) + C \theta_n^{-(5-p)} m(r_n)^{p-q} J_q(r_n)
\]
\[
\leq C m(r_n)^{-\frac{q(3p-4)}{6+p}} J_q(r_n)^{\frac{3p-4}{6+p}}
\]
\[
\leq c \eta^{\frac{3p-4}{6+p}},
\]
where \( \theta_n = m(r_n)^{-2q/(6+p)} J_q^{2/(6+p)} (r_n) \). Note that \( \theta_n \) goes to 0 as \( n \to \infty \) by (4.3). Let \( \rho_n = \theta_n r_n \) and \( \varepsilon_2 = c n^{\frac{3-4}{6+p}} \) such that \( \varepsilon_2 < \min\{1, \varepsilon_0^{10/7}/2\} \). For sufficiently large \( n \), we see that
\[
E_p(\theta_n r_n) + P_{p/2}(\theta_n r_n) \leq \varepsilon_2.
\]
This together with (1.14) implies that \( (x, t) \) is a regular point. Thus, we reach a contradiction and finish the proof.

**Corollary 4.3.** Suppose that \( (x, t) \in F(m) \cap S \) and the pair \( (p, q) \) is defined in Lemma 2.7, then, there exists a small constant \( c_4 \) such that
\[
\liminf_{r \to 0} J_q(r)m(r)^{\tau} \geq c_3/2,
\]
where \( J_q(r) = r^{2q-5} \int_{Q(r) \cap \{ |u(x, t)| > c_4 r^{-2} m(r) \}} |\nabla u|^{q} dx ds \) and \( c_3 \) is defined as in Lemma 4.2.

**Proof.** After a straightforward computation, we get
\[
J_q(r) - \bar{J}_q(r) = r^{2q-5} \int_{Q(r) \cap \{ |\nabla u(x, t)| \leq c_4 r^{-2} m(r) \}} |\nabla u|^{q} dx ds
\leq c_4 r^{2q-5} c_4 r^{-2q} m(r)^{-\tau} r^5 = c c_4 m(r)^{-\tau},
\]
which yields that
\[
\limsup_{r \to 0} m(r)^{\frac{27-5q}{5}} |J_q(r) - \bar{J}_q(r)| \leq cc_4.
\]
Combining (4.5) and Lemma 4.2 ensures that
\[
\liminf_{r \to 0} \bar{J}_q(r)m(r)^{\frac{27-5q}{5}} \geq \liminf_{r \to 0} J_q(r)m(r)^{\frac{27-5q}{5}} + \liminf_{r \to 0} [J_q(r) - J_q(r)] m(r)^{\frac{27-5q}{5}}
\geq c_3 - cc_4
\geq c_3/2,
\]
where \( c_4 = c_3/2c \). This concludes the proof of this lemma.

Now we are in a position to show Theorem 1.3.

**Proof of Theorem 1.3.** Let \( G_k \) denote the set of \( (x, t) \in F(m) \cap S \) such that
\[
c_2/4 \leq m(r)^{\tau} \bar{J}_q(r) \quad \text{and} \quad E_+ (r) \leq 2m(r),
\]
for \( 0 < r < \frac{1}{k} \). From Corollary 4.3, we know that \( F(m) \cap S = \bigcup_{k=1}^{\infty} G_k = \lim_{k \to \infty} G_k \). Let \( r_0 = 1/k \), then it follows from (4.6) that
\[
E_+(r_1) \leq c \ m(r_1)m(r_2)^{\tau} \bar{J}_q(r_2),
\]
for any \( 0 < r_1, r_2 < r_0 \).

We denote \( d^k(x, t) = \inf \{ |x - y| + |t - s|^{\frac{1}{2}} : (y, s) \in G_k \} \) and define the neighbourhood of \( G_k \) by \( L^k(r) = \{ (x, t) | d(x, t) < r \} \), \( \bar{L}^k(r) = L^k(r) \cap \bar{K}(r) \), where \( \bar{K}(r) = \{ (x, t) :
\[ |\nabla u(x, t)| > c_3 r^{-2} m(r) \frac{r^2}{t^2} \]. By the classical Vitali covering lemma, \( G_k \subset \mathcal{S} \) and (1.2), we know that there is a sequence of parabolic cylinders \( \{Q(x_i, t_i; r)\} \) such that

\[ G_k \subset \bigcup_i Q(x_i, t_i; 5r), \]

\[ (x_i, t_i) \in G_k, \]

\[ Q(x_m, t_m; r) \cap Q(x_n, t_n; r) = \emptyset, \ m \neq n, \]

\[ r \leq \varepsilon^{-1} \int_Q |\nabla u|^2 dxdt. \quad (4.8) \]

Moreover, we would like to point out that the radius \( r \) in \( \{Q(x_i, t_i; r)\} \) above is independent on the points \( (x_i, t_i) \), which can be examined by Vitali covering lemma. For this fact, see also [1, Proof of Theorem B, p.807] and [9, Proof of theorem 2.1 assuming theorem 2.2, p.2892].

Thanks to the definition of \( L^k(r) \), we infer that

\[ L^k(r) \subset \bigcup_i Q(x_i, t_i; 6r), \]

which yields that

\[ \int_{L^k(r)} |\nabla u|^2 dxdt \leq \sum_i \int_{Q(x_i, t_i; 6r)} |\nabla u|^2 dxdt. \]

By (1.7), for \( 0 < r < r_0 \), we arrive at that

\[ \int_{L^k(r)} |\nabla u|^2 dxdt \leq C r^{2q-4} m(6r)m(r)^r \sum_i \int_{Q(x_i, t_i; r) \cap K(r)} |\nabla u|^q dxdt \]

\[ = C r^{2q-4} m(r)^{r+1} \int_{L^k(r)} |\nabla u|^q dxdt. \quad (4.9) \]

Define \( d^k_n(x, t) = \max\{d^k(x, t), \frac{r}{n}\} \) with \( n > k \). Multiplying (4.9) by \( r^{-1} \) and integrating the obtained inequality over \( (n^{-1}, r_0) \), we get

\[ \int_{L^k(r_0)} [\Gamma(d_n) - \Gamma(r_0)] |\nabla u|^2 dxdt = \int_{n^{-1}}^{r_0} \int_{L^k(r)} r^{-1} |\nabla u|^2 dxdsdr \]

\[ \leq C \int_{n^{-1}}^{r_0} r^{2q-5} \Gamma(r)^{(r+1)\sigma} \int_{L^k(r)} |\nabla u|^q dxdsdr, \quad (4.10) \]

where we used the definition of \( \Gamma(r) \).

Thanks to Tonelli’s theorem, we interchange the order of integration for the right-hand side of the inequality (4.10) to arrive at

\[ \int_{L^k(r_0)} [\Gamma(d_n) - \Gamma(r_0)] |\nabla u|^2 dxdt \]

\[ \leq C \int_{L^k(r_0)} |\nabla u|^q \int_{n^{-1}}^{r_0} \chi_{K(r) \cap L^k(r)}(x, t) r^{2q-5} \Gamma(r)^{(r+1)\sigma} dr dxds \]

\[ \leq C \int_{L^k(r_0)} |\nabla u|^q \int_{n^{-1}}^{r_0} \min\{\chi_{K(r)}(x, t), \chi_{L^k(r)}(x, t)\} r^{2q-5} \Gamma(r)^{(r+1)\sigma} dr dxds. \]
Due to the properties of $\Gamma(r)$ and the definition of $L^k(r)$, for $q < 2$, we find
\[ \int_{r_0}^r \chi_{L^k(r)}(x,t)r^{2q-5}\Gamma(r)^{(\tau+1)\sigma} \, dr \leq Cd^2n^{-q-4}\Gamma(d_n)^{(\tau+1)\sigma}. \tag{4.11} \]

For $(x,t) \in \tilde{K}(r)$, it is clear that
\[ r^{-1} \leq c_3^{-\frac{1}{2}}|\nabla u(x,t)|^{\frac{1}{2}}\Gamma(r)^{\frac{\tau\sigma}{2q}}, \tag{4.12} \]

In the light of $\lim_{r \to 0} r\Gamma(r) = 0$, it turns out that
\[ \Gamma(r) \leq \frac{C}{r}. \]

Consequently, we can obtain that
\[ r^{-1} \leq C|\nabla u(x,t)|^{\frac{1}{2}r^{-\frac{\tau\sigma}{2q}}}, \]

which in turn implies
\[ r^{-1} \leq C|\nabla u(x,t)|^{\frac{1}{2}(1-\delta)}, \]

where $\delta = \frac{\tau\sigma}{2q} \in (0,1)$. With the help of the properties of $\Gamma(r)$, we infer that
\[ \Gamma(r) \leq \Gamma(C|\nabla u(x,t)|^{\frac{1}{2}(1-\delta)} \leq C(\delta\Gamma(|\nabla u(x,t)|^{-\frac{1}{2}}). \tag{4.13} \]

Combining this and (4.12), we get the following result
\[ r^{-1} \leq C|\nabla u(x,t)|^{\frac{1}{2}\Gamma(|\nabla u(x,t)|^{-\frac{1}{2}})^{\delta}}. \tag{4.14} \]

From (4.13) and (4.14), for $3p/5 < q < 2$, we see that
\[ \int_{r_0}^r \chi_{\tilde{K}(r)}(x,t)r^{2q-5}\Gamma(r)^{(\tau+1)\sigma} \, dr \leq C\int_{r_0}^r \chi_{\tilde{K}(r)}(x,t)r^{2q-5}dr\Gamma(|\nabla u(x,t)|^{-\frac{1}{2}})^{(\tau+1)\sigma} \tag{4.15} \]
\[ \leq C|\nabla u(x,t)|^{2-q}\Gamma(|\nabla u(x,t)|^{-\frac{1}{2}})^{(\tau+1)\sigma} + \Gamma(|\nabla u(x,t)|^{-\frac{1}{2}})^{\frac{(2-q)\tau\sigma}{4}+(\tau+1)\sigma}. \]

It follows from (4.11) and (4.15) that
\[ \int \int_{L^k(r_0)} [\Gamma(d_n) - \Gamma(r_0)]|\nabla u|^2 \, dx \, ds \]
\[ \leq C \int \int_{L^k(r_0)} |\nabla u|^q \min\{d_n^{2q-4}\Gamma(d_n)^{\delta_1}, |\nabla u(x,t)|^{2-q}\Gamma(|\nabla u(x,t)|^{-\frac{1}{2}})^{\delta_2}\} \, dx \, ds, \tag{4.16} \]

where
\[ \delta_1 = (\tau + 1)\sigma \quad \text{and} \quad \delta_2 = \frac{(2-q)\tau\sigma}{q} + (\tau + 1)\sigma. \tag{4.17} \]

By $\sigma < 27/113$, we can choose $q$ sufficiently close to 2 and $p$ sufficiently close to $10/3$ to guarantee that
\[ \delta_2 = \frac{(2-q)\tau\sigma}{q} + (\tau + 1)\sigma < 1. \tag{4.18} \]
In case $|\nabla u| \geq d_n^{-2}$, we see that
\[
|\nabla u|^q \min\{d_n^{2q-4}\Gamma(d_n)^\delta_1, |\nabla u(x, t)|^{2-q}\Gamma(|\nabla u(x, t)|\frac{1}{t})^{\delta_2}\} \leq |\nabla u|^2 \Gamma(d_n)^{\delta_1} \leq |\nabla u|^2 \Gamma(d_n)^{\delta_2}.
\]
Otherwise, if $|\nabla u| < d_n^{-2}$, we get
\[
|\nabla u|^q \min\{d_n^{2q-4}\Gamma(d_n)^\delta_1, |\nabla u(x, t)|^{2-q}\Gamma(|\nabla u(x, t)|\frac{1}{t})^{\delta_2}\} \leq |\nabla u|^2 \Gamma(d_n)^{\delta_2}.
\]
So, no matter in which case, we always choose $r_0$ sufficiently small to get
\[
C \ |\nabla u|^2 \Gamma(d_n)^{\delta_2} \leq \frac{1}{4} |\nabla u|^2 \Gamma(d_n).
\]
This together with (4.16) implies that
\[
\iint_{L^k(r_0)} \Gamma(d_n)|\nabla u|^2 dxds \leq c(\sigma, q, r_0) < \infty.
\]
We deduce from monotone convergence theorem in the last inequality that
\[
\iint_{L^k(r_0)} \Gamma(d)|\nabla u|^2 dxds < \infty. \tag{4.19}
\]
Since $\tilde{K}(r) = \{(x, t) : |\nabla u(x, t)| > c_4 r^{-2} m(r)^{-\frac{1}{q}}\}$, by means of Chebyshev’s inequality, we infer that
\[
\iint_{Q(x, t; r) \cap \tilde{K}(r)} dxds \leq \frac{1}{(c_4 r^{-2} m(r)^{-\frac{1}{q}})^2} \iint_{Q(x, t; r) \cap \tilde{K}(r)} |\nabla u|^2 dxds.
\]
For $q < 2$, by the Hölder inequality and the last inequality, we have
\[
\iint_{Q(x, t; r) \cap \tilde{K}(r)} |\nabla u|^q dxds \leq \left( \iint_{Q(x, t; r) \cap \tilde{K}(r)} |\nabla u|^2 dxds \right)^{q/2} \left( \iint_{Q(x, t; r) \cap \tilde{K}(r)} dxds \right)^{(2-q)/2} \\
\leq C r^{4-2q} m(r)^{\frac{q(2-q)}{q}} \left( \iint_{Q(x, t; r) \cap \tilde{K}(r)} |\nabla u|^2 dxds \right). \tag{4.20}
\]
By virtue of the definition of $\Psi_\delta(E, h)$ and the above inequality, we derive from (4.16), (4.20) and (4.18) that, for every $k \geq 2$, $0 < r \leq r_0 \leq 1/2$,
\[
\Psi_{5r}(G_k, t\Gamma^\sigma(t)) \leq \sum_i (5r)\Gamma^\sigma(5r) \\
\leq \sum_i r m(r)^{1+\sigma} \tilde{J}_q(r) \\
\leq C t(1+\sigma) \frac{r^{2-q}}{q} \sum_i \left( \iint_{Q(x, t; r) \cap \tilde{K}(r)} |\nabla u|^2 dxds \right) \tag{4.21}
\]
\[
\leq C r^{\delta_2-1} \iint_{L^k(r_0)} \Gamma(d)|\nabla u|^2 dxds,
\]
which together with (4.19) implies that
\[
\Lambda(S \cap F(m), r\Gamma^\sigma) = 0. \tag{4.22}
\]
To complete the proof, we have to show that $\Lambda(\mathcal{S}\{F(m), r\Gamma(r)^\sigma\}) = 0$. Indeed, for $(x, t) \in \mathcal{S}\{F(m)$, we deduce from (1.2) and the definition of $F(m)$ that

$$\limsup_{r \to 0} r^{-1} \int_{Q(x,t; r)} |\nabla u|^2 \, dx \, ds \geq \varepsilon$$

and

$$\limsup_{r \to 0} \frac{1}{r \Gamma(r)^\sigma} \int_{Q(x,t; r)} |\nabla u|^2 \, dx \, ds \geq 1.$$

Let $\delta > 0$, for each $(x, t) \in \mathcal{S}\{F(m)$, we can choose $Q(x, t; r)$ with $r < \delta$ such that

$$\int_{Q(x,t; r)} |\nabla u|^2 \, dx \, ds \geq \varepsilon r/2$$

and

$$\int_{Q(x,t; r)} |\nabla u|^2 \, dx \, ds \geq r \Gamma(r)^\sigma/2.$$}

From classical Vitali covering lemma, we know that there exists a disjoint subfamily $\{Q(x_i, t_i; r_i)\}$ such that

$$\mathcal{S}\{F(m) \subset \bigcup_i Q(x_i, t_i; 5r_i)$$

and

$$\bigg| \bigcup_i Q(x_i, t_i; r_i) \bigg| \leq C\delta^4 \sum_i r_i \leq C\delta^4 \int_{\bigcup_i Q(x_i, t_i; r_i)} |\nabla u|^2 \, dx \, ds \leq C\delta^4,$$

where $C$ is independent of $\delta$. In addition, we know that

$$\sum_i r_i \Gamma(r_i)^\sigma \leq 2 \sum_i \int_{Q(x_i, t_i; r_i)} |\nabla u|^2 \, dx \, ds = 2 \int_{\bigcup_i Q(x_i, t_i; r_i)} |\nabla u|^2 \, dx \, ds.$$

Note that $\delta$ is arbitrary. Therefore, it follows from absolutely continuity of the integral of $|\nabla u|^2$ that

$$\Lambda(\mathcal{S}\{F(m), r\Gamma(r)^\sigma\}) = 0,$$

Combined this and (4.22) implies $\Lambda(\mathcal{S}, r\Gamma(r)^\sigma) = 0$. This ends the proof of Theorem 1.3.

Acknowledgement

The authors would like to express their deepest gratitude to two anonymous kind referees and the editors for careful reading of our manuscript, the invaluable comments and suggestions which helped to improve the paper greatly. In particular, the proof of (4.21) was generously suggested by the referee. Wang was partially supported by the National Natural Science Foundation of China under grant No. 11601492. Wu was partially supported by the National Natural Science Foundation of China under grant No. 11771423 and No. 11671378.

References

[1] L. Caffarelli, R. Kohn and L. Nirenberg, Partial regularity of suitable weak solutions of Navier-Stokes equation, *Comm. Pure. Appl. Math.*, **35** (1982), 771–831.
[2] H. Choe, Boundary regularity of suitable weak solution for the Navier-Stokes equations. *J. Funct. Anal.* **268** (2015), 2171–2187.

[3] H. Choe and J. Lewis, On the singular set in the Navier-Stokes equations, *J. Funct. Anal.* **175** (2000) 348–369.

[4] H. Choe and M. Yang, Hausdorff measure of the singular set in the incompressible magnetohydrodynamic equations, *Comm. Math. Phys.* **336** (2015) 171–198.

[5] H. Choe and M. Yang, Hausdorff measure of boundary singular points in the magnetohydrodynamic equations. *J. Differential Equations.* **260** (2016), 3380–3396.

[6] K. Falconer, Fractal Geometry: Mathematical Foundations and Applications (New York: Wiley) 1990.

[7] C. Guevara and N. C. Phuc, Local energy bounds and $\varepsilon$-regularity criteria for the 3D Navier-Stokes system. *Calc. Var.* (2017) 56:68.

[8] Y. Koh and M. Yang, The Minkowski dimension of interior singular points in the incompressible Navier-Stokes equations. *J. Differential Equations.* **261** (2016), 3137–3148.

[9] I. Kukavica, The fractal dimension of the singular set for solutions of the Navier-Stokes system. *Nonlinearity.*, **22** (2009), 2889–2900.

[10] I. Kukavica and Y. Pei, An estimate on the parabolic fractal dimension of the singular set for solutions of the Navier-Stokes system. *Nonlinearity.*, **25** (2012), 2775–2783.

[11] O. Ladyzenskaja and G. Seregin, On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations, *J. Math. Fluid Mech.*, **1** (1999), 356–387.

[12] F. Lin, A new proof of the Caffarelli-Kohn-Nirenberg Theorem, *Comm. Pure Appl. Math.*, **51** (1998), 241–257.

[13] C. Miao and Y. Wang, Regularity conditions for the suitable weak solutions of the Navier-Stokes system from its rotation form. *Pacific J. Math.*, **288** (2017), 189–215.

[14] J. Robinson and W. Sadowski, Almost-everywhere uniqueness of Lagrangian trajectories for suitable weak solutions of the three-dimensional Navier-Stokes equations, *Nonlinearity.*, **22**, (2009) 2093–2099.

[15] I. Kukavica, On the Dimension of the Singular Set of Solutions to the Navier-Stokes Equations, *Comm. Math. Phys.*, **309** (2012), 497–506.

[16] G. Seregin, On smoothness of $L_{3,\infty}$-solutions to the Navier-Stokes equations up to boundary. *Math. Ann.*, **332** (2005), 219–238.

[17] G. Seregin, Estimates of suitable weak solutions to the Navier-Stokes equations in critical Morrey spaces. *J. Math. Sci.*, **143** (2007), 2961–2968.

[18] M. Struwe, On partial regularity results for the navier-stokes equations, *Comm. Pure. Appl. Math.*, **41** (1988), 437–458.

[19] V. Scheffer, Partial regularity of solutions to the Navier-Stokes equations, *Pacific J. Math.*, **66** (1976), 535–552.
[20] ______, Hausdorff measure and the Navier-Stokes equations, *Comm. Math. Phys.*, 55 (1977), 97–112.

[21] ______, The Navier-Stokes equations in space dimension four, *Comm. Math. Phys.*, 61 (1978), 41–68.

[22] A. Vasseur, A new proof of partial regularity of solutions to Navier-Stokes equations, *NoDEA Nonlinear Differential Equations Appl.*, 14 (2007), 753–785.

[23] Y. Wang and G. Wu, A unified proof on the partial regularity for suitable weak solutions of non-stationary and stationary Navier-Stokes equations, *J. Differential Equations.*, 256 (2014), 1224–1249.

[24] ______, On the box-counting dimension of potential singular set for suitable weak solutions to the 3D Navier-Stokes equations, *Nonlinearity.*, 30 (2017), 1762-1772.