A multi-parameter Hardy type inequality

Eskil Rydhe*

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Abstract

This note contains two simple observations. First, by the weak factorization of product $H^1$ (Ferguson–Lacey, Lacey–Terwilleger), we obtain a multi-parameter analogue of Hardy’s inequality. Second, as a dual statement, the Fourier transform of an essentially bounded function belongs to a certain product BMO-Sobolev space.

We let $H^p_A(\mathbb{C}_+)$ denote the standard Hardy space of analytic functions on the complex upper half-plane $\mathbb{C}_+$. Following the work of Hardy and Littlewood [6], Hille and Tamarkin [7] proved the following: There exists $C > 0$ such that, whenever $f \in H^1_A(\mathbb{C}_+)$,

$$\int_0^{\infty} \frac{|\hat{f}(\xi)|}{\xi} d\xi \leq C \|f\|_{H^1_A(\mathbb{C}_+)}.$$ (1)

We henceforth follow the standard convention that $C$ denotes a finite positive constant which is independent of $f$, but whose value may change from one occurrence to the next.

The proof of (1) relies on the factorization $H^1_A(\mathbb{C}_+) = H^2_A(\mathbb{C}_+) \cdot H^2_A(\mathbb{C}_+)$, essentially due to F. Riesz [10], which transforms (1) into an inequality for convolutions of functions supported on a half-line.

For the argument indicated above to work, it would have been sufficient to have the weak factorization $H^1_A(\mathbb{C}_+) = H^2_A(\mathbb{C}_+) \hat{\otimes} H^2_A(\mathbb{C}_+)$. In general, if $X$ is a space of analytic functions, consider the linear space

$$\left\{ \sum_{k=1}^N f_k g_k \mid f_k, g_k \in X \right\},$$
equipped with the norm

$$\|F\|_{X \hat{\otimes} X} = \inf \left\{ \sum_{k=1}^N \|f_k\|_X \|g_k\|_X \mid f_k, g_k \in X, F = \sum_{k=1}^N f_k g_k \right\}.$$*eskil.rydhe@math.lu.se, Centre for Mathematical Sciences, Lund University, Sweden. This work was supported by the Knut and Alice Wallenberg foundation, scholarship KAW 2016.0442.
The metric completion of this space is denoted \( X \hat{\otimes} X \).

Throughout this note, we consider a fixed positive integer \( d \). We denote by \( H^p_A \) the space of analytic functions \( f : (\mathbb{C}^+)^d \to \mathbb{C} \) such that

\[
\|f\|_{H^p_A} := \sup_{y > 0} \left( \int_{x \in \mathbb{R}^d} |f(x + iy)|^p \, dx \right)^{1/p} < \infty.
\]

The notation \( y > 0 \), where \( y \in \mathbb{R}^d \), means that \( y_j > 0 \) for each \( j \in \{1, \ldots, d\} \).

By means of non-tangential boundary values, \( H^p_A \) can be isometrically identified with a closed subspace of \( L^p = L^p(\mathbb{R}^d) \). This is used without further mention below.

Given a function \( \sigma : \{1, \ldots, d\} \to \{\pm 1\} \), we define the unitary transformation \( R_\sigma : (x_j) \mapsto (\sigma_j x_j) \) on \( \mathbb{R}^d \), and the partial reflection operator \( R_\sigma : f \mapsto f \circ R_\sigma \), where \( f : \mathbb{R}^d \to \mathbb{C} \) is a function. For each \( R_\sigma \) we define \( H^p_\sigma = R_\sigma H^p_A \). The multi-harmonic Hardy space \( H^p \) is the direct sum of all distinct \( H^p_\sigma \), c.f. Gundy and Stein [5]. The reader is cautioned not to confuse the multi-harmonic Hardy space \( H^p \) with the (mono-)harmonic Hardy space \( H^p(\mathbb{R}^d \times \mathbb{R}^+) \), e.g. C. Fefferman and Stein [2].

A function \( f \in L^2 \) belongs to \( H^1_A \) precisely when \( \hat{f} \) has support on \((0, \infty)^d \). Therefore \( H^2 = L^2 \). The orthogonal projection \( P_\sigma \) from \( H^2 \) onto \( H^2_\sigma \) is the Fourier multiplier induced by the indicator function of the orthant \( R_\sigma^d := R_\sigma(0, \infty)^d \). These Fourier multipliers can be used to characterize \( H^1 \) as

\[
H^1 = \{ f \in L^1 \mid P_\sigma f \in L^1 \text{ for each } \sigma \}.
\]

While the inclusion \( H^1_A \hookrightarrow L^1 \) is isometric, the inclusion \( H^1 \hookrightarrow L^1 \) is not even bounded below.

Relatively recently, Lacey and Terwilleger [8] identified the weak product \( H^2_\sigma \hat{\otimes} H^2_A \) as \( H^1_A \). We mention also the significant contributions by Ferguson and Sadosky [4], and Ferguson and Lacey [3]. The Lacey–Terwilleger result allows us to extend (1) to the multi-harmonic setting.

**Theorem 1.** There exists a constant \( C = C_d \) such that

\[
\int_{\xi \in \mathbb{R}^d} \frac{|\hat{f}(\xi)|}{\prod_{j=1}^d |\xi_j|} \, d\xi \leq C \|f\|_{H^2}
\]

whenever \( f \in H^1 \).

**Proof.** Since the partial reflections \( R_\sigma \) commute with the Fourier transform, it suffices to consider \( f \in H^2_A \). The Lacey–Terwilleger factorization further reduces the proof to the case \( f = gh \), where \( g, h \in H^2_A \), and \( \|g\|_{L^2} \|h\|_{L^2} \leq C \|f\|_{H^1} \). In this situation,

\[
\hat{f}(\xi) = \int_{\eta \in Q_\xi} \hat{g}(\xi - \eta) \hat{h}(\eta) \, d\eta,
\]

where \( Q_\xi \) denotes the rectangle \( \{ \eta \in \mathbb{R}^d ; 0 < \eta < \xi \} \).
For $\xi \in \mathbb{R}^d$, let $\prod_{\xi} = \prod_{j=1}^{d} \xi_j$. By the triangle inequality, and an obvious change of variables,

$$\int_{\xi > 0} \frac{|\hat{f}(\xi)|}{\prod_{\xi}} \, d\xi \leq \int_{\xi, \eta > 0} \frac{|\hat{g}(\xi)| |\hat{h}(\eta)|}{\prod_{\xi+\eta}} \, d\eta \, d\xi.$$ 

By the factorization $|\hat{g}(\xi)||\hat{h}(\eta)| = (\prod_{\xi} / \prod_{\eta})^{1/4} |\hat{g}(\xi)| \cdot (\prod_{\eta} / \prod_{\xi})^{1/4} |\hat{h}(\eta)|$, and the Cauchy–Schwarz inequality, the above right-hand side is less than

$$\left( \int_{\eta, \xi > 0} \frac{\prod_{\xi}^{1/2} |\hat{g}(\xi)|^2}{\prod_{\eta}^{1/2} \prod_{\xi+\eta}} \, d\eta \, d\xi \right)^{1/2} \left( \int_{\eta, \xi > 0} \frac{\prod_{\eta}^{1/2} |\hat{h}(\eta)|^2}{\prod_{\xi}^{1/2} \prod_{\xi+\eta}} \, d\eta \, d\xi \right)^{1/2}.$$

Since the numerical value of the expression

$$\int_{\eta > 0} \frac{\prod_{\xi}^{1/2}}{\prod_{\eta}^{1/2} \prod_{\xi+\eta}} \, d\eta$$

does not depend on $\xi$, we obtain that

$$\int_{\xi > 0} \frac{|\hat{f}(\xi)|}{\prod_{\xi}} \, d\xi \leq C \|\hat{g}\|_{L^2} \|\hat{h}\|_{L^2} = C \|g\|_{L^2} \|h\|_{L^2} \leq C \|f\|_{H^1},$$

which completes the proof.

Consider the Schwartz class $S = S(\mathbb{R}^d)$, and the subclass $S_0$ consisting of $f \in S$ for which the Fourier transform vanishes on each coordinate face of co-dimension 1, i.e. $\hat{f}|_{P_k} \equiv 0$ for each $P_k = \{(\xi_j) \in \mathbb{R}^d \mid \xi_k = 0\}, 1 \leq k \leq d$. In the next lemma, we interpret this condition on $\hat{f}$ as a cancellation condition.

**Lemma 2.** Let $f \in S_0$, and define the function

$$F(x) = \int_{y < x} f(y) \, dy.$$ 

Then $F \in S$.

**Proof.** The main part of proving this statement is to show that $\lim_{|x| \to \infty} F(x) = 0$. Once we have this, since $f$ is Schwartz function, it is clear that the decay of $F$ and it’s derivatives is sufficiently fast.

Decompose $x \in \mathbb{R}^d$ as $(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$. We will prove that

$$\lim_{|x_d| \to \infty} F(x', x_d) = 0,$$

where the convergence is uniform with respect to $x'$. By a mere change of notation, we obtain similar statements whenever $x_d$ is replaced with another variable $x_j$. This implies the conclusion of the lemma.
Let $\varepsilon > 0$, and choose $R$ such that $\int_{|x_d| > R} |f(x)| \, dx < \varepsilon$. Clearly, $x_d < -R \implies |F(x)| < \varepsilon$.

Moreover, consider the function $g: x' \mapsto \int_{x_d \in \mathbb{R}} f(x', x_d) \, dx_d$.

It is clear that $\hat{g} (\xi') = \hat{f} (\xi', 0) = 0$, and so $g \equiv 0$. It follows that $\lim_{x_d \to \infty} F(x', x_d) = \int_{y' < x'} \int_{x_d \in \mathbb{R}} f(y', x_d) \, dx_d \, dy' = 0$.

It remains to prove that this convergence is uniform with respect to $x'$.

Partition $\mathbb{R}$ into finitely many intervals $\{ I^{(k_1)} \}$, in such a way that for each $k_1$

$$\int_{I^{(k_1)} \times \mathbb{R} \times [-R,R]} |f(x)| \, dx < \varepsilon.$$ 

Then proceed inductively to construct partitions $\{ I^{(k_2)} \}, \{ I^{(k_3)} \}, \ldots, \{ I^{(k_d-1)} \}$ of $\mathbb{R}$ into finitely many intervals, with the property that if $l \in \{1, 2, \ldots, d-1\}$, and $N_l$ denotes the number of intervals in the partition $\{ I^{(k_l)} \}$, then for each $(k_1, \ldots, k_l)$

$$\int_{I^{(k_1)} \times I^{(k_2)} \times \ldots \times I^{(k_l)} \times \mathbb{R} \times [-R,R]} |f(x)| \, dx < \frac{\varepsilon}{N_1 \cdots N_{l-1}}.$$ 

In each rectangle $Q^{(k_1, \ldots, k_{d-1})} := I^{(k_1)} \times \ldots \times I^{(k_{d-1})}$ choose a point $x^{(k_1, \ldots, k_{d-1})}$. By what we proved in the previous paragraph, we may choose $R^{(k_1, \ldots, k_{d-1})} \geq R$ such that $x_d > R^{(k_1, \ldots, k_{d-1})} \implies |F(x^{(k_1, \ldots, k_{d-1})}, x_d)| < \varepsilon$.

By our construction of $Q^{(k_1, \ldots, k_{d-1})}$, it holds that $x' \in Q^{(k_1, \ldots, k_{d-1})}$, $x_d > R^{(k_1, \ldots, k_{d-1})} \implies |F(x', x_d)| < (d+2)\varepsilon$.

Let $\tilde{R} = \max R^{(k_1, \ldots, k_{d-1})}$. It follows that $|x_d| > \tilde{R} \implies |F(x', x_d)| < (d+2)\varepsilon$.

Hence, $F(x', x_d) \to 0$, uniformly in $x'$, as $|x_d| \to \infty$. \hfill \Box

**Lemma 3.** The space $S_A = \{ f \in S \mid \text{spt} \hat{f} \subset (0, \infty)^d \}$ is dense in $H^1_A$. 


Proof. Given \( f \in H^1_A \), since \( H^1_A \) is isometrically embedded into \( L^1 \), it suffices to find a sequence of \( f_n \in S_A \) such that \( f_n \to f \) in \( L^1 \).

Choose an even function \( \varphi \in S \), where \( \hat{\varphi} \) has compact support, and \( \hat{\varphi}(0) = 1 \). Furthermore, consider its \( L^1 \)-normalized dilations, given by \( \varphi_\varepsilon(x) = \frac{1}{\varepsilon^d} \varphi \left( \frac{x}{\varepsilon} \right) \).

Since \( \varphi_\varepsilon \ast f \to f \) as \( \varepsilon \to 0 \), and \( (\varphi_\varepsilon \ast f) \) has compact support, it suffices to consider the case where \( \hat{f} \) has compact support.

Recall that \( \hat{\varphi}_\varepsilon \) vanishes outside \((0, \infty)^d\). Given \( \nu \in (0, \infty)^d \), we consider the modulation \( f_h(x) = e^{2\pi i h \cdot \nu} f(x) \). Since \( f_h \to f \) as \( h \to 0 \), we may assume that \( \text{spt} \hat{f} \subset (0, \infty)^d \).

Finally, let \( f_\varepsilon = \hat{\varphi}_\varepsilon f \). Since \( \hat{f}_\varepsilon = \varphi_\varepsilon \ast \hat{f} \) has support in \((0, \infty)^d\), provided that \( \varepsilon \) is sufficiently small, and \( \hat{\varphi}_\varepsilon \to 1 \) as \( \varepsilon \to 0 \), it follows that \( f_\varepsilon \to f \) as \( \varepsilon \to 0 \).

Let \( C_0 = C_0(\mathbb{R}^d) \) denote the space of continuous functions vanishing at infinity. Elements of \( H^1 \) satisfy a rather strong cancellation property, which we express in the next lemma.

**Lemma 4.** Let \( f \in H^1 \), and define the function

\[
F(x) = \int_{y < x} f(y) \, dy.
\]

Then \( F \in C_0 \).

**Remark 5.** The function \( F \) defined in Lemma 4 is of course smoother than just continuous. Our main interest lies in the fact that \( F \) vanishes at infinity.

**Proof.** By symmetry, it is enough to consider \( f \in H^1_A \). Also, if \( f_1, f_2 \in H^1 \), then \( \|F_1 - F_2\|_{L^\infty} \leq \|f_1 - f_2\|_{H^1} \). By Lemma 3 we may restrict attention to \( f \in S_A \). The result is now immediate from Lemma 2.

The map \( H^1 \ni f \mapsto F \in C_0 \) indicated by Lemma 4 is injective, and its left inverse is given by \( D = \prod_{j=1}^d \partial/\partial x_j \). This leads us to define the multi-harmonic Hardy–Sobolev space

\[
H^1_1 := \{ f \in C_0(\mathbb{R}^d) \mid Df \in H^1 \}.
\]

We equip \( H^1_1 \) with the norm

\[
\|f\|_{H^1_1} := \|Df\|_{H^1}.
\]

By the standard relation between derivatives and Fourier multipliers, we immediately obtain a corollary to Theorem 1.

**Corollary 6.** There exists a constant \( C = C_d \) such that

\[
\int_{\xi \in \mathbb{R}^d} |\hat{f}(\xi)| \, d\xi \leq C\|f\|_{H^1_1}
\]

whenever \( f \in H^1_1 \).
An alternative phrasing of the above result is that the Fourier transform \( F: H^1 \to L^1 \) bounded. Since \( F \) is self-adjoint with respect to the standard distributional pairing, \( F: L^\infty \to (H^1)^* \) also is bounded.

Since \( D: H^1 \to H^1 \) is a bijection, the same is true for \( D: (H^1)^* \to (H^1)^* \).

The space \( (H^1)^* \) is called multi-harmonic BMO, and has been characterized by Chang and R. Fefferman [1]: Consider the set \( D(R) \) of dyadic intervals \([k2^{-l},(k+1)2^{-l}]\), \( k, l \in \mathbb{Z} \), and the set \( D(R^d) \) of dyadic rectangles \( \prod_{j=1}^d I_j \), \( I_j \in D(R) \). Furthermore, let \( v \) be a Schwartz function with \( \text{spt} \hat{v} \subset \{ \xi \in \mathbb{R} | \tfrac{2}{3} \leq |\xi| \leq \tfrac{8}{3} \} \), and \( v_I(x) = \frac{1}{|I|^{1/2}} v \left( \frac{x-c_I}{|I|} \right) \), whenever \( I \) is a dyadic interval with centre \( c_I \). It is a celebrated result by Y. Meyer that \( v \) can be chosen in such a way that \( \{ v_I \}_{I \in D(R)} \) becomes an orthonormal basis for \( L^2(R) \), e.g. [9, p. 75 et seq.]. For one such \( v \), we define the multi-parameter wavelet \( \{ w_R \}_{R \in D(R^d)} \), where, for \( R = \prod_{j=1}^d I_j \), \( w_R(x) = \prod_{j=1}^d v_{I_j}(x_j) \). A tempered distribution \( f \) belongs to \( (H^1)^* \) if and only if there exists \( C > 0 \) such that

\[
\sum_{R \in D(R^d), R \subset \Omega} |\langle f, w_R \rangle|^2 \leq C^2 |\Omega|
\]

for all open sets \( \Omega \subset \mathbb{R}^d \). The smallest such \( C \) is called \( \| f \|_{\text{BMO}} \), and is comparable to \( \| f \|_{(H^1)^*} \).

The foregoing discussion motivates the definition of the multi-harmonic BMO–Sobolev space

\[
\text{BMO}_{-1} := \{ Df; f \in \text{BMO} \},
\]

with norm

\[
\| Df \|_{\text{BMO}_{-1}} := \| f \|_{\text{BMO}}.
\]

**Corollary 7.** The operator

\[
F: L^\infty \to \text{BMO}_{-1}
\]

is bounded.

Corollary 7 provides a natural end point analogue of the embedding \( F: L^p \to D^{1-\frac{2}{p}} L^p, \quad p > 2 \), considered by the author in [11].

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