On $R + \alpha R^2$ Loop Quantum Cosmology

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Abstract

Working in Einstein frame we introduce, in order to avoid singularities, holonomy corrections to the $f(R) = R + \alpha R^2$ model. We perform a detailed analytical and numerical study when holonomy corrections are taken into account in both Jordan and Einstein frames obtaining, in Jordan frame, a dynamics which differs qualitatively, at early times, from the one of the original model. More precisely, when holonomy corrections are taken into account the universe is not singular, starting at early times in the contracting phase and bouncing to enter in the expanding one where, as in the original model, it inflates. This dynamics is completely different from the one obtained in the original $R + \alpha R^2$ model, where the universe is singular at early times and never bounces. Moreover, we show that these holonomy corrections may lead to better predictions for the inflationary phase as compared with current observations.

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I. INTRODUCTION

Two kind of quantum geometric corrections come from the discrete nature of space-time assumed in Loop Quantum Cosmology (LQC): inverse volume corrections [1] and holonomy corrections (see for instance [2]). These last effects provide a big bounce that avoids singularities like the Big Bang and Big Rip (see for example [3]).

On the other side, it is well-known that, in general, F(R) gravity does not avoid singularities, except of particular non-singular cases where $R^2$ term plays an important role as it was demonstrated in [4]. In order to avoid them, one could introduce holonomy corrections in $f(R)$ gravity. The extension of Loop Quantum Gravity (LQG) to $f(R)$ gravity has been recently developed in [5, 6], where holonomy corrections are introduced in Einstein frame (EF), because in that frame the gravitational part of the Hamiltonian is linear in the scalar curvature and the matter part is given by a scalar field.

This extension simplifies very much when one consider the flat Friedmann-Lemaître-Robertson-Walker (FLRW) geometry. In that case, in order to take into account geometric effects, one has to replace the Ashtekar connection by a suitable sinus function (see for instance [7]) obtaining the holonomy corrected Friedmann equation in EF. Finally, from the holonomy corrected Friedmann equation in EF and through the relation between the corresponding variables in both frames, one obtains the holonomy corrected $f(R)$ theory in the Jordan frame (JF).

Our main objective is to apply, for the flat FLRW geometry, holonomy correction to the $f(R) = R + \alpha R^2$ model (also called $R^2$ gravity) and study its dynamics. To do this, first of all we perform a detailed analysis of $R^2$ gravity without corrections. When holonomy corrections in the model are taken into account one obtains a very complicated dynamical equation in the JF. Fortunately, dynamical equations simplify very much in EF, (in fact the dynamics is given by the well-known holonomy corrected Friedmann equation in LQC plus the Klein-Gordon equation in flat FLRW geometry) which allows us to perform a very deep analytical and numerical analysis, whose results can be translated to the JF. Our conclusion is that when holonomy corrections are taken into account the universe starts at the critical point ($H = 0, \dot{H} = 0$) (the Hubble parameter and its derivative vanish) and makes small oscillations around the critical point before entering the contracting phase, which it leaves bouncing (see [8] for a review of bounce cosmology), and enters the expanding phase where, as in the classical model, it reaches an inflationary stage which it leaves at late times and comes back to the critical point, once again, in an oscillating way.
The paper is organized as follows:

In Section II, we review \( f(R) \) gravity in Jordan and Einstein frames. In Section III, we introduce holonomy corrections to \( f(R) \) gravity. The idea is very simple: working in EF, \( f(R) \) gravity is formulated as Einstein gravity plus a scalar field. Then, the idea, as in standard LQC for the flat FLRW geometry, is to replace the Ashtekar connection by a suitable sinus function. Section IV is devoted to the study of \( R^2 \) gravity without holonomy corrections. After performing the change of variable \( p^2 = H \) where \( H \) is the Hubble parameter, the obtained dynamical equation can be understood as the dynamics of a particle under the action of a quadratic potential with dissipation. This system is very simple and the phase portrait can be drawn with all the details. In Section V, we analyze the model with holonomy corrections. We start working in EF due to the simplicity of equations and, once we have studied the dynamics in EF, we obtain the dynamics in JF from the formulae that relate both frames. Moreover, we obtain in EF the corrected expressions of the slow-roll parameters and the values of the spectral index for scalar perturbations and the ratio of tensor to scalar perturbations, showing that holonomy corrections help to match correctly the theoretical results obtained from \( R^2 \) gravity with current observations. Section VI is devoted to discuss a possible unification of inflation and current cosmic acceleration in the framework of Loop Quantum \( f(R) \) theories. We will show that when one consider the current suggested models for such unification this extension and/or its analytical study is, in general, unworkable. The only model we have been able to deal with is \( R^2 \) plus an small cosmological constant. For such a model, we have performed a detailed analytical study and the results are shown at the end of the work.

II. CLASSICAL DYNAMICAL EQUATIONS IN DIFFERENT FRAMES

In this Section we review the relations between Jordan and Einstein frames in \( f(R) \) gravity for the flat FLRW geometry.

The Lagrangian in JF for the flat FLRW geometry is given by \( \mathcal{L}_{JF} = \frac{a^3}{2} f(R) \), where the scalar curvature is \( R = 6\dot{H} + 12H^2 \) being \( H = \frac{\dot{a}}{a} \) the Hubble parameter, and the corresponding modified Friedmann equation in \( f(R) \) gravity can be obtained from Ostrogradskiis construction \[7\] giving as a result

\[
6f_{RR}(R)\dot{R}H + (6H^2 - R)f_R(R) + f(R) = 0,
\] (2.1)
where $f_R(R) \equiv \frac{\partial f(R)}{\partial R}$. Taking the derivative of equation (2.1) with respect to time and using the relation $R = 6(\dot{H} + 2H^2)$ one obtains the equivalent equation

$$f_{RR}(R)(\ddot{R} - \dot{R}H) + f_{RRR}(R)\dot{R}^2 + 2f_R(R) \left( \frac{R}{2} - 2H^2 \right) = 0. \tag{2.2}$$

To work in the Einstein frame (EF), one has to perform the change of variables [9]

$$\tilde{a} = \sqrt{f_R(R)} a; \quad d\tilde{t} = \sqrt{f_R(R)} dt. \tag{2.3}$$

Then, in that frame the Lagrangian density, for flat FLRW geometries, is

$$L_{EF} = \tilde{a}^3 \left( \frac{1}{2} \tilde{R} + \frac{1}{2} (\tilde{\phi}')^2 - V(\tilde{\phi}) \right) \iff L_{EF} = -3(\tilde{a}')^2 \tilde{a} + \tilde{a}^3 \left( \frac{1}{2} (\tilde{\phi}')^2 - V(\tilde{\phi}) \right), \tag{2.4}$$

where $'$ means the derivative with respect the time $\tilde{t}$. Here, $\tilde{a}$ and $\tilde{\phi}$ have to be considered as independent variables, and of course, $\tilde{R} = 6\tilde{H}' + 12\tilde{H}^2$.

The relation between both frames is given through the relations

$$\tilde{\phi} = \sqrt{3} \ln(f_R(R)); \quad V(\tilde{\phi}) = \frac{R f_R(R) - f(R)}{2 f_R^2(R)}, \tag{2.5}$$

and a simple calculation shows that the Friedmann equation in the EF, i.e. $\tilde{H}^2 = \frac{1}{3} \tilde{\rho}$, obtained from the Hamiltonian constrain

$$\mathcal{H}_{EF} \equiv \tilde{a}' \frac{\partial L_{EF}}{\partial \tilde{a}'} + \tilde{\phi}' \frac{\partial L_{EF}}{\partial \tilde{\phi}'} - L_{EF} = -3(\tilde{a}')^2 \tilde{a} + \tilde{a}^3 \left( \frac{1}{2} (\tilde{\phi}')^2 + V(\tilde{\phi}) \right) = 0, \tag{2.6}$$

where $\tilde{\rho} \equiv \frac{1}{2} (\tilde{\phi}')^2 + V(\tilde{\phi})$, is equivalent to equation (2.1). However, the Friedmann equation in EF, $\tilde{H}^2 = \frac{1}{3} \tilde{\rho}$, is a constrain instead of a dynamical equation. The dynamics is given by the conservation equation $\tilde{\rho}' = -3\tilde{H}(\tilde{\phi}')^2$ or the Raychauduri one $\tilde{H}' = -\frac{1}{2} (\tilde{\phi}')^2$ which are equivalent to equation (2.2).

Note that combining, in EF, the conservation and Friedmann equation one obtains

$$(\tilde{\rho}')^2 = 3\tilde{\rho}(\tilde{\phi}')^2, \tag{2.7}$$

and coming back to the JF this equation is a second order differential equation in $R$ (it only contains $R$, $\dot{R}$ and $\ddot{R}$) which is equivalent to equations (2.1) and (2.2).

Finally, we show the following relations between both frames, which will be important when we extend LQC to $R^2$ gravity:

$$H = \sqrt{f_R(R)} \left( \tilde{H} - \frac{1}{\sqrt{6}} \tilde{\phi}' \right); \quad R = f_R(R) \left( \tilde{R} + (\tilde{\phi}')^2 + \sqrt{6} \frac{\partial V(\tilde{\phi})}{\partial \tilde{\phi}} \right). \tag{2.8}$$
III. $f(R)$ LOOP QUANTUM COSMOLOGY

The idea to extend Loop Quantum Cosmology to $f(R)$ theories ($f(R)$ LQC) has been recently developed in [5, 6]. For a flat FLRW geometry the idea is very simple and goes as follows: Working in EF we can see that $\tilde{a}'$ and the scale factor $\tilde{a}$ are canonically conjugated variables with Poisson bracket $\{\tilde{a}', \tilde{a}\} = \frac{1}{3}$. Then, to obtain LQC one has to introduce the square root of the minimum eigenvalue of the area operator in LQG, namely $\lambda = \sqrt{\frac{\sqrt{3}}{2}}\gamma$ (where $\gamma$ is the Barbero-Immirzi parameter), and make the replacement (see [10] for a status report on LQC)

$$\tilde{a}' \to \tilde{a} \frac{\sin(\gamma \lambda \tilde{H})}{\lambda \gamma},$$

in the Hamiltonian (2.6), while keeping on the Poisson bracket $\{\tilde{a}', \tilde{a}\} = \frac{1}{3}$.

Finally, from the Hamilton equation $(\tilde{a}^3)' = \{\tilde{a}^3, H_{Ef,LQC}\}$ and the Hamiltonian constrain $H_{Ef,LQC} = 0$ (being $H_{Ef,LQC}$ the new Hamiltonian obtained from (2.6) after the replacement (3.1)), one obtains the corresponding holonomy corrected version of the classical Friedmann equation, that is,

$$\tilde{H}^2 = \frac{1}{3} \bar{\rho} \left(1 - \frac{\bar{\rho}}{\bar{\rho}_c}\right),$$

where $\bar{\rho}_c \equiv \frac{3}{\sqrt{\gamma}}$ is the so-called critical density in the EF.

As has been discussed in detail in [11] this equation depicts an ellipse in the plane $(\tilde{H}, \bar{\rho})$, and the dynamics along this curve is very simple: For a non-phantom field the universe moves clockwise from the contracting to the expanding phase starting and ending at the critical point $(0, 0)$ and bouncing only once at $(0, \bar{\rho}_c)$.

Finally, note that in the JF, the holonomy corrected Friedmann equation acquires the complicated form

$$6f_{RR}(R)\dot{R}H + (6H^2 - R)f_{R}(R) + f(R) = -\frac{\left(\frac{3}{2}f_{RR}(R)\dot{R}^2 + (Rf_{R}(R) - f(R))f_{R}(R)\right)^2}{2f_{R}^4(R)\bar{\rho}_c}. (3.3)$$

IV. $R^2$ GRAVITY

In this Section we study with all the details the classical model $f(R) = R + \alpha R^2$, with $\alpha > 0$. This model contains a quadratic correction to the scalar curvature and is a modified version of
the Starobinsky model \[12\], where the author considered quantum vacuum effects due to massless fields conformally coupled with gravity. Note that such (eternal) trace-anomaly driven inflation was proposed earlier in ref \[13\].

For this model, the classical equation (2.1) becomes

\[
12\alpha H \dot{R} + 6H^2 + 12\alpha RH^2 - \alpha R^2 = 0 \iff H^2 = -12\alpha \left(3\dot{H}H^2 + H\ddot{H} - \frac{1}{2}\dot{H}^2\right), \tag{4.1}
\]

which coincides, when the parameter $\beta$ vanishes, with the dynamical equation studied in \[12\]

\[
H^2 = -12\alpha \left(3\dot{H}H^2 + H\ddot{H} - \frac{1}{2}\dot{H}^2\right) + \beta H^4, \quad \text{where} \quad \beta > 0. \tag{4.2}
\]

It is very simple to show that equation (4.1) leads to an inflationary epoch \[14, 15\]. Effectively, when the slow-roll initial condition $|\dot{H}| \ll H^2$ is fulfilled, equation (4.1) becomes $\ddot{H} = -3\dot{H}H - \frac{H}{12\alpha}$, which has the following particular solution in the expanding phase ($H > 0$)

\[
\dot{H}(t) = -\frac{1}{36\alpha} \implies H(t) = \frac{t_1 - t}{36\alpha} \implies a(t) = a(t_1)e^{-\frac{18\alpha H^2}{t_1}} \quad \text{for} \quad t < t_1. \tag{4.3}
\]

If $t_i$ and $t_f$ are the beginning and the end of inflation ($t_i < t_f < t_1$), then one will have

\[
a(t_f) = a(t_i)e^{18\alpha (H(t_i) - H^2(t_f))} \approx a(t_i)e^{18\alpha H^2(t_i)}, \tag{4.4}
\]

and the 60 e-folds needed to solve the flatness and horizon problems will be obtained when $\alpha H^2(t_i)$ is approximately 3.3.

Unfortunately, $R^2$ gravity contains singularities at early times, that is, all solutions have divergent scalar curvature at early times. To show that, one has to perform the change of variables $p^2(t) = H(t) > 0$ \[16\] (in this model the universe doesn’t bounce), then equation (4.1), which is not well-defined at singular value $H = 0$, becomes the following well-defined equation

\[
\frac{d}{dt} \left(\frac{\dot{p}^2}{2} + W(p)\right) = -3p^2\dot{p}^2, \tag{4.5}
\]

where $W(p) = \frac{p^2}{48\alpha}$.

We can see that the system (4.5) is dissipative. To understand its dynamics, we can imagine a ”particle” rolling down along the parabola $W(p)$ losing energy and oscillating, at late times, around $p = 0$. As a consequence, when time goes back the ”particle” gains energy and finally $|p| \to \infty$
$(H \to \infty)$, i.e., all the solutions are singular at early times. One also can check this fact as follows: We write equation (4.5) as

$$\ddot{p} + \frac{p}{24\alpha} = -3p^2\dot{p},$$

and look for, at early times, solutions of the form $p(t) = \frac{C}{(t-\bar{t})^r}$, where $C$ and $r$ are parameters. Inserting this expression in (4.6) and retaining the leading terms when $t \gtrsim \bar{t}$, one obtains the equation:

$$r(r + 1)C \frac{1}{(t - \bar{t})^{r+2}} = 3rC^3 \frac{1}{(t - \bar{t})^{3r+1}},$$

which has singular solutions at $t = \bar{t}$ of the form $p(t) = \sqrt{\frac{1}{2(t-\bar{t})}}$.

**Remark IV.1.** In the contracting phase we can perform the change of variable $p^2(t) = -H(t) > 0$, obtaining the system

$$\frac{d}{dt} \left( \frac{p^2}{2} + W(p) \right) = 3p^2\dot{p}^2,$$

where $W(p) = \frac{p^2}{18\alpha}$. We can see that in the contracting phase the system is anti-dissipative (the universe gains energy), in this case the universe starts oscillating around the bottom of the potential leaving it gradually and becomes singular at late times.

Equation (4.6) is also useful to obtain the inflationary period and the dynamics at late times. Effectively, when initially one has $\ddot{p} \cong 0$ equation (4.6) becomes $p\dot{p} = -\frac{1}{72\alpha}$, whose inflationary solution is once again

$$H(t) = p^2(t) = \frac{t_1 - t}{36\alpha}.$$  

On the other hand, to obtain the dynamics at late times we follow the same method used in chaotic inflation for a quadratic potential (see page 240 of [17]). Performing the change of variable

$$\dot{p}(t) = \sqrt{2f(t)} \cos(\theta(t)), \quad p(t) = \sqrt{48\alpha f(t) \sin(\theta(t))},$$

and inserting these expressions in equations (4.5) and (4.6) one gets the system

$$\begin{cases} 
\dot{f} = -18\alpha f^3(1 - \cos(4\theta)) \\
\dot{\theta} = \frac{1}{\sqrt{24\alpha}} + 144\alpha f^2 \sin^3(\theta) \cos(\theta).
\end{cases}$$

(4.11)
Since \( p \) goes to zero at late times, we can disregard the second term in the right hand side in the second equation of (4.11), obtaining \( \dot{\theta} = \frac{1}{\sqrt{24\alpha}} \), whose solution is \( \theta(t) = \frac{t}{\sqrt{24\alpha}} + \omega \), \( \omega \) being a constant of integration. Inserting this approximate solution in the first equation of (4.11), we obtain a solvable equation whose solution is given by

\[
\sqrt{\frac{1}{36\alpha t}} \left( 1 - \frac{\sin \left( \frac{2t}{\sqrt{6\alpha}} + 4\omega \right)}{\frac{2t}{\sqrt{6\alpha}}} \right) \approx \sqrt{\frac{1}{36\alpha t}} \left( 1 + \frac{\sin \left( \frac{2t}{\sqrt{6\alpha}} + 4\omega \right)}{\frac{2t}{\sqrt{6\alpha}}} \right). \tag{4.12}
\]

and thus the Hubble parameter reads

\[
H(t) \approx \frac{4}{3t} \left( 1 + \frac{\sin \left( \frac{2t}{\sqrt{6\alpha}} + 4\omega \right)}{\frac{2t}{\sqrt{6\alpha}}} \right) \sin^2 \left( \frac{t}{\sqrt{24\alpha}} + \omega \right). \tag{4.13}
\]

Now, choosing \( \omega = \pi/2 \) one obtains the well-known result \([14, 18, 19]\)

\[
H(t) \approx \frac{4}{3t} \left( 1 + \frac{\sin \left( \frac{2t}{\sqrt{6\alpha}} \right)}{\frac{2t}{\sqrt{6\alpha}}} \right) \cos^2 \left( \frac{t}{\sqrt{24\alpha}} \right), \tag{4.14}
\]

and after integrating by parts one gets as Starobinsky in \([12]\)

\[
a(t) \approx t^{2/3} \left( 1 + \frac{2 \sin \left( \frac{t}{\sqrt{6\alpha}} \right)}{\frac{t}{\sqrt{6\alpha}}} \right) \approx t^{2/3}. \tag{4.15}
\]

These analytic results are supported numerically in figure 1:

An important remark is in order: Note that in the model of \([12]\) one obtains the same equation (4.5) but with the potential \( W(p) = \frac{p^2}{48\alpha} - \frac{3p^6}{144\alpha} \). In this case the potential has an stable minimum at \( p = 0 \) and two unstable maximums at \( p = \pm \beta^{-1/4} \), which corresponds to the unstable de Sitter solution \( H = \beta^{-1/2} \). From the shape of this potential one deduces that there are only two unstable non-singular solutions (the ones that start at the de Sitter points and end at the bottom of the potential), and two that only are singular at late time (the ones that start at the de Sitter points and ends at \( |p| = \infty \)), all the other solutions are singular at early times. At late times, there are two kinds of solutions: the ones that have enough energy to overpass the wedge of the potential and become singular at late times, and
Figure 1: Phase portrait for $\alpha = 0.1$. The universe comes from a singularity at early times, when time goes forward it enters in the attractor inflationary phase, leaving it at early times when the universe starts to oscillate around $(0,0)$ without bouncing. In the first figure we have taken values of $H$ up to 6 to show clearly the inflationary stage, and the second one the values of $H$ are up to 1 to show, in more detail, the oscillatory phase. It’s clear from the pictures that orbits are unbounded, coming from $\infty$ at early times.

others with less energy that fall down against the wedge of the potential without clearing it due to the dissipation and, approaching to $p = 0$ with the same oscillatory behavior as in the $R + \alpha R^2$ model (see figure 2 for the shape of potentials, and figure 3 for the phase portrait of the Starobinsky model).

Note that what is really important in the Starobinsky model at late times, is the oscillatory behavior of the scale factor rather than its amplitude, because at late times the period of oscillation of the scale factor is much shorter than the Hubble time, meaning that for a few oscillations the amplitude of the scale factor can be considered constant. This behavior can be thought of as oscillations of a decaying field called *scalaron* [12] that creates light conformally coupled particles, which finally thermalize yielding a hot Friedmann universe that matches with the Standard Model.

V. LOOP QUANTUM $R^2$ GRAVITY

We start this section showing that there exists a wide range of values of $\alpha$ and $\tilde{\rho}_c$ for which the $R^2$ LQC model does not have any singularity. First at all, from the holonomy corrected Friedmann
Figure 2: Shape of the potentials $W(p)$ for $\alpha = 0.01$ and $\beta = 1$. The first picture corresponds to the potential given by $R^2$ gravity and the second one to the potential given by the model suggested in [12]. The dynamics is very easy: one can imagine a "particle" moving under the action of the potential $W$ and losing energy. For the first potential particles come at early times from $|p| = \infty$ and ends at late time at $p = 0$ in an oscillating way. For the second potential there are two unstable de Sitter solutions at $p = \pm (1/\beta)^{-1/4}$, so the "particle" could start at early times at these points and fall down into the wedge of the potential ending, at late time, at $p = 0$ in an oscillatory way. These are the only non-singular solutions. All the other orbits are singular at early and/or late times.

Using the Raychaudhuri equation in LQC, \( \tilde{H}' = -\frac{1}{2} (\tilde{\phi}')^2 \left( 1 - \frac{2\tilde{\phi}}{\tilde{\rho}_c} \right) \), one deduces that

\[ |\tilde{H}'| \leq \frac{1}{2} (\tilde{\phi}')^2 \leq \tilde{\rho}_c \implies |\tilde{R}| \leq 7\tilde{\rho}_c. \quad (5.4) \]

Moreover, the potential (5.2) satisfies

\[ \frac{\partial V(\tilde{\phi})}{\partial \tilde{\phi}} = \frac{1}{f_R(R)} \sqrt{\frac{V(\tilde{\phi})}{3\alpha}}, \quad (5.5) \]
Figure 3: Phase portrait for $\alpha = 0.01$ and $\beta = 1$ of the Starobinsky model. The unique non-singular solutions are the de Sitter one which correspond to the saddle point $(1, 0)$, which is the unique critical point of the system (painted brown), and the black curve that starts at the critical point and ends oscillating at $(0, 0)$.

which means (see the second equation of (2.8))

$$R = f_R(R) \left( \tilde{R} + (\tilde{\phi}')^2 \right) + \sqrt{\frac{2V(\tilde{\phi})}{\alpha}},$$

(5.6)

and thus,

$$R = \frac{1}{1 - 2\alpha(\tilde{R} + (\tilde{\phi}')^2)} \left( \tilde{R} + (\tilde{\phi}')^2 + \sqrt{\frac{2V(\tilde{\phi})}{\alpha}} \right).$$

(5.7)

From the bound $1 - 2\alpha(\tilde{R} + (\tilde{\phi}')^2) \geq 1 - 18\alpha\tilde{\rho}_c$, one easily deduces

$$|R| \leq \frac{1}{1 - 18\alpha\tilde{\rho}_c} \left( 18\tilde{\rho}_c + \sqrt{\frac{2\rho_c}{\alpha}} \right),$$

(5.8)

which is always bounded provided we choose $\alpha < \frac{1}{18\rho_c}$.

Finally, since $|R|$ is bounded, from the first equation of (2.8) one deduces that $|H|$ is bounded, and consequently $|\dot{H}| = \frac{1}{6} |R - 12H^2|$ is bounded, meaning that $R^2$ gravity in LQC has no singularities.

In fact, as we will see, in any case there are singularities when one takes into account holonomy corrections. However, when $8\alpha\tilde{\rho}_c > 1$ the scalar curvature $R$ can achieve very large values. To show that, we have to perform a detailed analysis in EF.
A. \( R^2 \) LQC in Einstein frame

To perform a deeper analysis of the model we will work in EF, where the dynamical equations are simpler than in the JF one. In fact, when \( f(R) = R + \alpha R^2 \) equation (3.3) becomes

\[
12\alpha \dot{R}H + 6H^2(1 + 2\alpha R) - \alpha R^2 = -\frac{\alpha^2[2\alpha(R^3 + 3\dot{R}^2) + R^2]}{2(1 + 2\alpha R)^4 \tilde{\rho}_c}. \quad (5.9)
\]

On the other hand, in EF, the field \( \tilde{\phi} \) satisfies the equation

\[
\tilde{\phi}'' + 3\tilde{H}\tilde{\phi}' + \frac{\partial V(\tilde{\phi})}{\partial \tilde{\phi}} = 0, \quad (5.10)
\]

where the potential \( \tilde{\phi} \) is given by (5.2).

As we have already explained, due to the holonomy effects in EF the universe starts in the contracting phase with zero energy, and the energy density increases as far as it catches up with the critical value \( \tilde{\rho}_c \), where the universe bounces and enters in the expanding phase.

Performing the change of variable \( \sqrt{\frac{2}{3}}\tilde{\phi} = \ln \tilde{\psi} \), i.e. \( \tilde{\psi} = f_R(R) = 1 + 2\alpha R \) (essentially \( \tilde{\psi} \) is like \( R \)), one gets

\[
\tilde{\psi}''\tilde{\psi} - (\tilde{\psi}')^2 + 3\tilde{H}\tilde{\psi}'\tilde{\psi} + \frac{1}{6\alpha} \left( \tilde{\psi} - 1 \right) = 0. \quad (5.11)
\]

From equation (5.11) one can show that the orbits in the plane \( (\tilde{\psi}, \tilde{\psi}') \), are symmetric with respect the axis \( \tilde{\psi}' = 0 \) in the expanding and contracting phase, because equation (5.11) remains invariant after performing the replacement \( \tilde{t} \rightarrow -\tilde{t} \) and \( \tilde{H} \rightarrow -\tilde{H} \). To be more precise, consider in the plane \( (\tilde{\psi}, \tilde{\psi}') \), a trajectory (a solution of (5.11)) \( \sigma_1(t) = (\tilde{\psi}(t), \tilde{\psi}'(t)) \) in the contracting \( \tilde{H} < 0 \) (resp. expanding \( \tilde{H} > 0 \)) phase. Then, \( \sigma_2(t) = (\tilde{\psi}(-t), -\tilde{\psi}'(-t)) \) is a trajectory in the expanding \( \tilde{H} > 0 \) (resp. contracting \( \tilde{H} < 0 \)) phase.

The energy density, using the new variables, is given by

\[
\tilde{\rho} = \frac{3}{4\tilde{\psi}^2} \left( (\tilde{\psi}')^2 + \frac{1}{6\alpha} (\tilde{\psi} - 1)^2 \right), \quad (5.12)
\]

which means that \( \tilde{H} \) vanishes at the point \( (\tilde{\psi}, \tilde{\psi}') = (1, 0) \) and over the curve \( \tilde{\rho} = \tilde{\rho}_c \), with equation

\[
\frac{(\tilde{\psi}')^2}{4\tilde{\rho}_c} + \frac{(\tilde{\psi} - \frac{1-8\alpha\tilde{\rho}_c}{8\alpha\tilde{\rho}_c})^2}{3(1-8\alpha\tilde{\rho}_c)} = 1, \quad (5.13)
\]
which produces an ellipse for $1 - 8\alpha \tilde{\rho}_c > 0$, an hyperbola for $1 - 8\alpha \tilde{\rho}_c < 0$ and a parabola for $1 - 8\alpha \tilde{\rho}_c = 0$. Note also that $(1, 0)$ is the unique critical point corresponding to $\tilde{\rho} = 0$, which means that all the orbits start and end at this point (the universe starts and ends at this point), and in the curve (5.13) the universe in EF bounces, because it corresponds to $\tilde{\rho} = \tilde{\rho}_c$.

From the previous analysis we can conclude that the dynamics, working in EF, goes as follows: the universe starts in the contracting phase $\tilde{H} < 0$ oscillating around the unique critical point $(1, 0)$ and increasing the amplitude of oscillations, then it reaches the curve $\tilde{\rho} = \tilde{\rho}_c$ where it bounces and enters in the expanding phase $\tilde{H} > 0$ coming back once again to $(1, 0)$ in an oscillatory way (our analytical study is supported numerically in figure 4).

![Figure 4](imageURL) Figure 4: In the first picture we have the phase space portrait of an orbit in EF for the case $1 - 8\alpha \tilde{\rho}_c > 0$ ($\alpha = 0.1$ and $\tilde{\rho}_c = 1$). The universe starts, in the contracting phase $\tilde{H} < 0$, oscillating around $(1, 0)$ (red curve) and arriving to the ellipse defined by equation (5.13) (blue curve), where the universe bounces entering in the expanding phase $\tilde{H} > 0$ and coming back to $(1, 0)$ oscillating (black curve). In second picture we draw an orbit in EF for the case $1 - 8\alpha \tilde{\rho}_c < 0$ ($\alpha = 0.1$ and $\tilde{\rho}_c = 15$). The dynamics is similar, the only difference is that now the blue curve is an hyperbola. At the top of the picture we have inserted and increased in size the oscillatory behavior around the critical point $(1, 0)$.

Two important remarks are in order:

1. Strictly speaking, the phase portrait in the plane $(\tilde{\psi}, \tilde{\psi}')$ shows the dynamics of two dynamical systems, because equation (5.11) defines two different differential equations, one with $\tilde{H} > 0$ and the other one with $\tilde{H} < 0$. Then, since we have two different autonomous dynamical
systems, at each point of the plane \((\tilde{\psi}, \tilde{\psi}')\) two different orbits, one with \(\tilde{H} > 0\) and the other one with \(\tilde{H} < 0\), cross.

2. It is important to realize that the system does not contain singularities because all the orbits start and end at the critical point \((1, 0)\). In the case \(1 - 8\alpha \tilde{\rho}_e > 0\), the variables \(\tilde{\psi}\) and \(\tilde{\psi}'\) move inside an ellipse (a compact domain) meaning that, in this case, all the quantities are bounded. Effectively, inside the ellipse the quantities \(\tilde{H}, \tilde{R}, \tilde{\phi}, \tilde{\phi}'\) and \(\tilde{\psi} = 1 + 2\alpha R\) are bounded. Consequently, it follows from (2.8) that \(H\) is bounded. On the other hand, in the case \(1 - 8\alpha \tilde{\rho}_e < 0\) the variables \(\tilde{\psi}\) and \(\tilde{\psi}'\) move inside an unbounded region delimited by an hyperbola, meaning that there are orbits where \(\tilde{\psi}\), and consequently the scalar curvature \(R\), achieve very large values, which never happens in the other case.

1. Inflation in Einstein frame

The slow-roll parameters in EF are given by (see for example [20])

\[
\tilde{\epsilon} \equiv -\frac{\tilde{H}'}{\tilde{H}^2} \quad \text{and} \quad \tilde{\eta} \equiv \tilde{\epsilon} - \tilde{\delta} = 2\tilde{\epsilon} - \frac{\tilde{\epsilon}'}{2\tilde{H}\tilde{\epsilon}},
\]

(5.14)

where \(\tilde{\delta} = \frac{\tilde{\phi}''}{\tilde{H}\tilde{\phi}'}.\)

Slow-roll dynamics requires \((\tilde{\phi}')^2 \ll V(\tilde{\phi})\) and \(\tilde{\phi}'' \ll \tilde{H}\tilde{\phi}'.\) Then, in the slow-roll phase the dynamical equations read

\[
\tilde{H}^2 = \frac{V(\tilde{\phi})}{3} \left(1 - \frac{V(\tilde{\phi})}{\tilde{\rho}_e}\right) \quad \text{and} \quad 3\tilde{H}\tilde{\phi}' + \frac{\partial V(\tilde{\phi})}{\partial \tilde{\phi}} = 0,
\]

(5.15)

and thus, in this phase, the slow-roll parameters are approximately

\[
\tilde{\epsilon} \simeq \frac{1}{2} \left(1 - \frac{V(\tilde{\phi})}{\tilde{\rho}_e}\right) \left(\frac{1 - 2V(\tilde{\phi})}{V(\tilde{\phi})}\right) \quad \text{and} \quad \tilde{\eta} \simeq \frac{1}{V(\tilde{\phi})} \frac{\partial^2 V(\tilde{\phi})}{\partial \tilde{\phi}^2} \left(1 - \frac{V(\tilde{\phi})}{\tilde{\rho}_e}\right).
\]

(5.16)

For the potential given by \(R^2\) gravity, i.e. for (5.2), slow-roll conditions \(|\tilde{\epsilon}| \ll 1\) and \(|\tilde{\eta} \ll 1|\) are only satisfied for large positive values of the field. In that case, equation (5.16) becomes

\[
\tilde{\epsilon} \simeq \frac{4}{3} \frac{e^{-\sqrt{\frac{4\tilde{\phi}}{8\alpha \tilde{\rho}_e}}} \left(1 - \frac{(1 - e^{-\sqrt{\frac{4\tilde{\phi}}{8\alpha \tilde{\rho}_e}})^2}{4\alpha \tilde{\rho}_e}\right)}{(1 - e^{-\sqrt{\frac{4\tilde{\phi}}{8\alpha \tilde{\rho}_e}}}^4)} \left(1 - \frac{(1 - e^{-\sqrt{\frac{4\tilde{\phi}}{8\alpha \tilde{\rho}_e}})^2}{8\alpha \tilde{\rho}_e}\right)^2.
\]

(5.17)
\[ \eta \cong \frac{4}{3} \frac{e^{-\sqrt{3}\phi} - e^{-\sqrt{2}\phi}}{(1 - e^{-\sqrt{3}\phi})^2} \left( \frac{1}{1 - (1-e^{-\sqrt{3}\phi})^2} \right). \] (5.18)

To calculate inflation ends, the values of the slow-roll parameters must be of the order 1, which happens, for positive values of the field \( \tilde{\phi} \), when it satisfies the equation

\[ \frac{e^{-\sqrt{3}\tilde{\phi}}}{(1 - e^{-\sqrt{3}\tilde{\phi}})^2} \cong \frac{\sqrt{3}}{2}, \] (5.19)

whose solution is

\[ \tilde{\phi}_{\text{end}} = -\sqrt{\frac{3}{2}} \ln \left( \frac{1 + \sqrt{3} - \sqrt{2\sqrt{3} + 1}}{\sqrt{3}} \right) > 0. \] (5.20)

And to calculate the number of e-folds that the scale factor increases during the period of inflation

\[ \tilde{N} \equiv \int_{\tilde{\phi}_{\text{ini}}}^{\tilde{\phi}_{\text{end}}} \tilde{H} d\tilde{t} = \int_{\tilde{\phi}_{\text{ini}}}^{\tilde{\phi}_{\text{end}}} \frac{H}{\dot{\tilde{\phi}}} d\tilde{\phi}, \] (5.21)

we have to use the slow roll equations (5.15) obtaining

\[ \tilde{N} \cong \int_{\tilde{\phi}_{\text{ini}}}^{\tilde{\phi}_{\text{end}}} V(\tilde{\phi}) \left( 1 - \frac{V(\tilde{\phi})}{\rho_c} \right) d\tilde{\phi} \] (5.22)

In the case of our potential (5.2), the final number of e-folds is approximately

\[ \tilde{N} \cong \frac{3}{4} e^{\sqrt{3} \tilde{\phi}_{\text{ini}}}. \] (5.23)

On the other hand, for a given value of \( \tilde{N} \) the slow-roll parameters are:

\[ \tilde{\epsilon} \cong \frac{3}{4N^2} \left( 1 - \frac{1}{8\alpha \rho_c} \right)^2 \quad \text{and} \quad \tilde{\eta} \cong -\frac{1}{N} \left( \frac{1}{1 - \frac{1}{8\alpha \rho_c}} \right). \] (5.24)

With these values, the spectral index of scalar perturbations, namely \( \tilde{n}_s \), and the ratio of tensor to scalar perturbations, namely \( \tilde{r} \), are approximately

\[ \tilde{n}_s \cong 1 - 6\tilde{\eta} + 2\tilde{\eta} \cong 1 - \frac{2}{N} \left( \frac{1}{1 - \frac{1}{8\alpha \rho_c}} \right), \quad \tilde{r} \cong 16\tilde{\epsilon} \cong \frac{12}{N^2} \left( \frac{1}{1 - \frac{1}{8\alpha \rho_c}} \right)^2. \] (5.25)
which coincide, when holonomy corrections are disregarded, i.e. when \( \tilde{\rho}_c \to \infty \), with the values obtained in [21].

A very important remark is in order: The latest Planck data gives for the spectral index the approximate value \( \tilde{n}_s = 0.9603 \pm 0.0073 \). If one disregards the loop corrections, to achieve the value 0.96 one has to take \( \tilde{N} = 50 \) e-folds, which does not give enough inflation to solve the flatness and horizon problems. However, if one takes into account holonomy corrections, for the values \( 8 \alpha \tilde{\rho}_c \cong 6 \) and \( \tilde{N} = 60 \) (the minimum number of e-folds required to solve the horizon and flatness problems) one obtains the desired result. Moreover, for these same values one obtains \( \tilde{r} = 0.0031 \), which satisfies the current bound \( \tilde{r} < 0.11 \).

To be more precise, if one disregards loop corrections, 60 e-folds are only achieved when \( 0.9666 \leq \tilde{n}_s \leq \tilde{n}_{s,\text{max}} = 0.9676 \), in fact for \( \tilde{n}_s = 0.9676 \) one obtains 61.72 e-folds, which means that, in this model without corrections, it is impossible for the universe to inflate more that 61.72 e-folds. However, including loop quantum effects one easily achieves a greater number of e-folds; for example, for \( \tilde{n}_s = 0.9676 \) one obtains 70 e-folds choosing \( 8 \alpha \tilde{\rho}_c \cong 8.46 \). To sum up, we have shown that loop quantum corrections could be essential to match correctly \( R^2 \) inflation with the current observational data.

\section*{B. \( R^2 \) LQC in Jordan frame}

To study the dynamics in the JF from the results obtained in the EF, we look for the points in the space \((\tilde{\psi}, \tilde{\psi}')\) where the universe could bounce in the JF, i.e., we look for the points where \( H = 0 \). Since, \( H = \sqrt{f_R(R)} \left( \tilde{H} - \frac{1}{\sqrt{6}} \tilde{\psi}' \right) \), one has to solve the equation \( \tilde{H}^2 = \frac{1}{4} (\tilde{\psi}')^2 \), which gives, for \( \tilde{\psi} > 1 \) the following curve

\[
\frac{(\tilde{\psi}')^2}{12(1-\sqrt{8\alpha \tilde{\rho}_c})} + \frac{(\tilde{\psi} - \frac{1-\sqrt{2\alpha \tilde{\rho}_c}}{1-\sqrt{8\alpha \tilde{\rho}_c}})^2}{2\alpha \tilde{\rho}_c \left(1-\sqrt{8\alpha \tilde{\rho}_c}\right)^2} = 1,
\]

which, as in EF, produces an ellipse for \( 1 - 8\alpha \tilde{\rho}_c > 0 \), an hyperbola for \( 1 - 8\alpha \tilde{\rho}_c < 0 \) and a parabola for \( 1 - 8\alpha \tilde{\rho}_c = 0 \). And, for \( 0 < \tilde{\psi} < 1 \) the curve is

\[
\frac{(\tilde{\psi}')^2}{12(1+\sqrt{8\alpha \tilde{\rho}_c})} + \frac{(\tilde{\psi} - \frac{1+\sqrt{2\alpha \tilde{\rho}_c}}{1+\sqrt{8\alpha \tilde{\rho}_c}})^2}{2\alpha \tilde{\rho}_c \left(1+\sqrt{8\alpha \tilde{\rho}_c}\right)^2} = 1,
\]

for \( 1 - 8\alpha \tilde{\rho}_c = 0 \).
which is always an ellipse. Then, when in EF the orbits in the plane \((\tilde{\psi}, \tilde{\psi}')\) reach those curves, the universe in the JF could bounce. To assure that it bounces the equation \(\tilde{H} = \frac{\tilde{\psi}'}{2\tilde{\psi}}\) must be satisfied.

Now we are ready to explain the dynamics in JF from the results already obtained in EF: In EF the dynamics starts in the contracting phase and ends in the expanding one at the critical point \((\tilde{\psi}, \tilde{\psi}') = (1, 0)\). From the relation between both frames

\[
H = \sqrt{\tilde{\psi}} \left( \tilde{H} - \frac{\tilde{\psi}'}{2\tilde{\psi}} \right), \quad \dot{H} = \frac{\tilde{\psi}'}{2} \left( \tilde{H} - \frac{\tilde{\psi}'}{2\tilde{\psi}} \right) + \tilde{\psi} \left( \tilde{H}' - \frac{1}{2} \left( \frac{\tilde{\psi}'}{\tilde{\psi}} \right) \right),
\]

(5.28)

which is obtained from the first equation of (2.8) and its derivative, one deduces that, in JF, the universe starts and ends at \((H = 0, \dot{H} = 0)\). Note that to calculate explicitly \(\dot{H}\) one has to use the Raychaudhuri equation \(\tilde{H}' = -\frac{3}{4} \left( 1 - \frac{3\tilde{\rho}}{\tilde{\rho}_c} \right) \left( \frac{\tilde{\psi}'}{\tilde{\psi}} \right)^2\) and the field equation (5.11). Moreover, since in EF the orbits of the system at early and late times oscillate around the point \((\tilde{\psi}, \tilde{\psi}') = (1, 0)\), crossing many times the curves \((5.26)\) and \((5.27)\) one can conclude that in JF the orbits of the system at early times oscillate around the point \((H, \dot{H}) = (0, 0)\) meaning that the universe makes small bounces many times, and when it leaves this oscillatory regime it enters in the contracting phase and bounces (in EF when the orbit reach the curve \((5.13)\)) to enter in the expanding phase, where the universe inflates and finally, at late times, it goes asymptotically to the critical point \((0, 0)\) in an oscillating way, that is, bouncing again many times.

Note that this behavior is completely different from the one obtained disregarding holonomy corrections where, in JF, as we have already seen in section IV, the universe never bounces and is singular at early times. Moreover, it is important to remark that the holonomy corrected equation \((5.9)\) is not singular at \(H = 0\), and thus the orbits can cross the axis \(H = 0\), which allows the universe to bounce. Of course, that does not happen in classical \(R^2\) gravity where the corresponding dynamical equation (eq. \((4.1)\)) is not defined at \(H = 0\).

Numerically, the dynamics in the plane \((H, \dot{H})\) is easily derived via \((5.28)\) from the one in EF, which is very simple as we have already shown. In figure 5, we have depicted in the plane \((H, \dot{H})\) the orbits depicted in figure 4.

Finally, note that equation \((5.11)\) defines two different dynamical systems, which means that in the plane \((H, \dot{H})\), two different orbits, one with \(\tilde{H} > 0\) and the other one with \(\tilde{H} < 0\), cross at each point. Moreover, the invariance of the equation \((5.11)\) with respect to the the replacement \(\tilde{t} \rightarrow -\tilde{t}\) and \(\tilde{H} \rightarrow -\tilde{H}\), means that the phase portrait in the plane \((H, \dot{H})\) has a symmetry with respect the
In the first picture we have the phase space portrait of an orbit in JF for the case $1 - 8\alpha \tilde{\rho}_c > 0$ ($\alpha = 0.1$ and $\tilde{\rho}_c = 1$). The universe starts oscillating around $(0, 0)$ then enters in the contracting phase ($H < 0$) and bounces entering in the expanding phase $H > 0$ coming back to $(0, 0)$ oscillating. In the second picture we draw an orbit in JF for the case $1 - 8\alpha \tilde{\rho}_c < 0$ ($\alpha = 0.1$ and $\tilde{\rho}_c = 15$). The dynamics is similar to that described in the other picture, but there is enough inflation here in the expanding phase.

axis $H = 0$. More precisely, given a piece of an orbit with $\tilde{H} > 0$ (resp. $\tilde{H} < 0$) in EF, there is a symmetric piece with respect to the axis $H = 0$, of an orbit with $\tilde{H} < 0$ (resp. $\tilde{H} > 0$) in EF.

## VI. INFLATION AND DARK ENERGY IN $R^2$ LQC

Some time ago the unification of the early time inflation with late time Dark Energy (DE) in frames of modified gravity was proposed ([22]). Later, several improved models containing Dark Energy (DE) have been suggested to unify inflation with the current acceleration of the universe. In this work, the idea is to add to $R^2$ gravity a correction $g(R)$ given a model of the form $f(R) = R + \alpha R^2 + g(R)$ that takes into account the accelerated expansion of the universe and passes the Solar system tests. Two of the best regarded examples of these corrections are: $g(R) = \lambda (e^{-bR} - 1)$ being $\lambda$ and $b$ positive constants [23], and $g(R) = -m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1}$, where $n > 0$ and $c_1, c_2$ are dimensionless parameters [24].

The problem with this kind of models is that they lead to very complicated potentials in EF, complicating considerably their extension to LQC. Moreover, it is nearly impossible to perform a detailed
analytical study and it is not evident how to perform numerical computations. For this reason in order to deal with DE we will consider the simplest model: we will add a small cosmological constant to our model, i.e., we will consider the \( f(R) = R + \alpha R^2 - 2\Lambda \) model.

When one does not take into account holonomy corrections, the system after the change \( p = H^2 \) has the same form as (4.5) but with the potential \( W(p) = \frac{p^2}{48\alpha} + \frac{\Lambda}{144\alpha p^2} \). This potential satisfies \( V(0) = V(\infty) = \infty \) meaning that the dynamics can be restricted to positive values of \( p \). The potential only has a minimum at the point \( p = \left( \frac{\Lambda}{3} \right)^{1/4} \) (de Sitter solution), and thus at late times all the solutions go asymptotically to this point oscillating around it. Moreover, the inflationary solution given in (4.9) is also an attractor when the cosmological constant is taken into account. Finally, it is easy to show that the solutions are singular at early times. When a cosmological constant is considered, there are two kind of solutions: the ones that, as in \( R^2 \) gravity without cosmological constant, are given by \( p(t) = \sqrt{\frac{t}{2(t-\bar{t})}} \), and the other ones given by \( p(t) = \left( \frac{\Lambda}{36\alpha}(t - \bar{t}) \right)^{1/6} \), which vanish at \( t = \bar{t} \) but have divergent scalar curvature.

Incorporating the cosmological constant to the EF model we have obtained the following potential \( V(\tilde{\phi}) = \frac{1}{8\alpha} \left( 1 - e^{-\sqrt{\frac{3}{2}}\tilde{\phi}} \right)^2 + \Lambda e^{-\sqrt{\frac{3}{2}}\tilde{\phi}} \), which has a minimum at \( \tilde{\phi}_{\text{min}} = \sqrt{\frac{3}{2}} \ln(1 + 8\alpha \Lambda) \). That means that, at late times in the plane \( (\tilde{\phi}, \tilde{\phi}') \) of EF, all the solutions oscillate around \( \tilde{Q}_{\text{min}} \equiv (\tilde{\phi}_{\text{min}}, 0) \). When we introduce Loop Quantum effects in EF, the orbits will oscillate initially around \( \tilde{Q}_{\text{min}} \) in the contracting phase, i.e., \( \tilde{H} < 0 \). In fact, \( \tilde{Q}_{\text{min}} \) in the contracting phase corresponds to the anti de Sitter solution \( \tilde{H}_- = -\sqrt{\frac{V(\tilde{\phi}_{\text{min}})}{3} \left( 1 - \frac{V(\tilde{\phi}_{\text{min}})}{\rho_c} \right)} \), where \( V(\tilde{\phi}_{\text{min}}) = \frac{\Lambda}{8\alpha \Lambda + 1} \) is the minimum value of the potential. After leaving the anti de Sitter phase the orbits move into the contracting phase before bouncing and entering in the expanding one where the universe inflates, and finally oscillate asymptotically to the de Sitter solution \( \tilde{H}_+ \equiv -\tilde{H}_- \).

In JF, the dynamics is very similar: the universe starts oscillating around the anti de Sitter solution \( H_- = \sqrt{8\alpha \Lambda + 1} \tilde{H}_- \), after leaving the anti de Sitter phase moves in the contracting phase \( H < 0 \), which it leaves bouncing, enters the expanding phase where it inflates and finally, at late times, it oscillates around the de Sitter solution \( H_+ = \sqrt{8\alpha \Lambda + 1} \tilde{H}_+ \). This oscillatory behavior at late times is essential, because it excites the light fields coupled with gravity that will re-heat the universe \([12, 14, 18]\), yielding a hot universe that matches with the \( \Lambda \)CDM model.
VII. CONCLUSIONS

We have introduced holonomy corrections to $R^2$ gravity in order to avoid early time singularities that appear in this model. We have performed a detailed analytical and numerical analysis which shows that the new model is not singular due to the quantum geometric corrections (holonomy corrections) coming from the discrete nature of space-time assumed in LQC. The new model is more involved than the original one. For this reason, in order to understand the dynamics in JF, a previous analysis must be performed in EF, where the dynamical equations greatly simplify. This allows us to perform a detailed study of its dynamics, what is essential in order to have a global idea of the system in JF. From this analysis we conclude that, when quantum geometric corrections are taken into account, the universe evolves from the contracting phase to the expanding one through a big bounce, and when it enters in the expanding phase, as in the classical model, it inflates in such a way that, these holonomy corrections lead to theoretical predictions that match correctly with current observational data. Finally, to remark that it would be interesting to study different versions of F(R) gravity, for instance, with several power-law type terms in order to understand how such theories which normally do not support the inflation behave in LQC approach.

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