Coherent States for the Deformed Algebras

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Abstract

We provide a unified approach for finding the coherent states of various deformed algebras, including quadratic, Higgs and q-deformed algebras, which are relevant for many physical problems. For the non-compact cases, coherent states, which are the eigenstates of the respective annihilation operators, are constructed by finding the canonical conjugates of these operators. We give a general procedure to map these deformed algebras to appropriate Lie algebras. Generalized coherent states, in the Perelomov sense, follow from this construction.

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Recently, deformed Lie algebras have attracted considerable attention in the context of various physical and mathematical problems [1]. The quadratic algebra was discovered by Sklyanin [2,3], in the context of statistical physics and field theory. The well-known Higgs algebra, a cubic algebra, was manifest in the study of the dynamical symmetries of the quantum oscillator and the Coulomb problem in a space of constant curvature [4,5]. Other examples of deformation of Lie algebras appear in the description of the degeneracy structures and as dynamical symmetries of many conventional quantum mechanical problems, like singular oscillators and Hartmann potential [6,7]. They have also appeared in interacting models of Calogero-Sutherland type [8,9]. The presence of ambiguities in the definition of the generators of the Lie algebras, responsible for the degeneracy in these problems, have led many authors to the deformed Lie algebras. The celebrated quantum groups is another example of deformation, originating from the physical problems of the spin-chains and lower dimensional integrable models [10].

In this communication, we present a unified approach for finding the coherent states (CS) of these deformed algebras. Coherent states, needless to say, occupy a very special place in physics, having relevance to many problems of physical interest [11,12]. Hence, apart from its intrinsic interest, the method of construction presented here, will greatly facilitate the physical applications of these algebras. For the non-compact cases, the construction of the CS, which are the eigenstates of the lowering operators, takes place in two steps. First, we find the canonical conjugates of these operators. The CS, corresponding to the deformed algebras are then obtained by the action of the exponential of the respective conjugate operators on the vacuum [13,16,17]; this is in complete parallel to the harmonic oscillator case. Another CS, which in a sense to be made precise in the text, is dual to the first one, naturally follows from the above construction. We also provide a mapping between the deformed algebras and their undeformed counterparts. This connection is then utilized to find the CS in the Perelomov sense [14]. Apart from obtaining the known CS of the $SU(1,1)$ algebra, we construct the CS for the quadratic, cubic and the quantum group cases. Although our
method is general, we will confine ourselves here to finding the CS of the deformed $SU(1, 1)$ and $SU(2)$ algebras.

The CS in our construction will be characterized by the eigenvalues of the Casimir operator. This operator for the deformed algebra, written in the Cartan-Weyl basis,

$$[H, E_{\pm}] = \pm E_{\pm} \quad , \quad \quad [E_+, E_-] = f(H) \quad , \quad \quad (1)$$

can be written in the form [17],

$$C = E_- E_+ + g(H) \quad ,$$
$$= E_+ E_- + g(H - 1) \quad . \quad \quad (2)$$

Here, $f(H) = g(H) - g(H - 1)$, $g(H)$ can be determined up to the addition of a constant. The eigenstates are characterized by the values of the Casimir operator and the Cartan subalgebra $H$. We make use of these relations to construct the canonical conjugate of the lowering operator of the $SU(1, 1)$ algebra and then write down the corresponding coherent states. This is done for the purpose of comparing with the known results in the literature and to illustrate our method. This approach is then extended to the deformed algebras a straightforward way. In what follows, the relationships derived between various operators are valid only on suitable Hilbert spaces.

For the $SU(1, 1)$ algebra,

$$[K_+, K_-] = -2K_0 \quad , \quad [K_0, K_{\pm}] = \pm K_{\pm} \quad , \quad \quad (3)$$

one finds, $f(K_0) = -2K_0$ and $g(K_0) = -K_0(K_0 + 1)$. The quadratic Casimir operator is given by $C = K_- K_+ + g(K_0) = K_- K_+ - K_0(K_0 + 1)$. $\tilde{K}_+$, the canonical conjugate of $K_-$, satisfying

$$[K_-, \tilde{K}_+] = 1 \quad , \quad \quad (4)$$

can be written in the form,
\[ \tilde{K}_+ = K_+ F(C, K_0) \quad . \] (5)

Eq.(4) then yields,

\[ F(C, K_0) K_- K_+ - F(C, K_0 - 1) K_+ K_- = 1 \quad ; \] (6)

making use of the Casimir operator relation given earlier, one can solve for \( F(C, K_0) \) in the form,

\[ F(C, K_0) = \frac{K_0 + \alpha}{C + K_0(K_0 + 1)} \quad . \] (7)

The constant, arbitrary, parameter \( \alpha \) in F can be determined by demanding that Eq.(4) is valid in the entire Hilbert space. For the purpose of clarity, we illustrate this point, with the one oscillator realization of the \( SU(1,1) \) generators.

The ground states defined by \( K_- | 0 > = \frac{1}{2} a^2 | 0 > = 0 \), are, \( | 0 > \) and \( | 1 > = a^\dagger | 0 > \), in terms of the oscillator Fock space. Making use of the results,

\[ K_0 | 0 > = \frac{1}{4} (2a^\dagger a + 1) | 0 > = \frac{1}{4} | 0 > \quad , \] (8)

and

\[ C | 0 > = \frac{3}{16} | 0 > \quad , \] (9)

we find that, \([K_-, \tilde{K}_+] | 0 > = K_- \tilde{K}_+ | 0 >\), yields \( \alpha = \frac{3}{4} \). Similarly, for the other case

\[ [K_-, \tilde{K}_+] | 1 > = | 1 > \quad , \] (10)

leads to \( \alpha = \frac{1}{4} \). Hence, there are two disjoint sectors characterized by the \( \alpha \) values \( \frac{3}{4} \) and \( \frac{1}{4} \), respectively. These results match identically with the earlier known ones \([13]\), once we rewrite \( F \) as,

\[ F(C, K_0) = \frac{K_0 + \alpha}{C + K_0(K_0 + 1)} \quad , \] (11)

\[ = \frac{K_0 + \alpha}{K_- K_+} \quad . \] (12)
The unnormalized coherent state $|\beta\rangle$, which is the annihilation operator eigenstate, i.e.,
$K_-|\beta\rangle = \beta |\beta\rangle$, is given in the vacuum sector by

$$|\beta\rangle = e^{\beta\tilde{K}^+} |0\rangle .$$

(13)

Analogous construction holds in the other sector, where $\alpha = \frac{1}{4}$. These states, provide a
realization of the Cat states \[18\], and play a prominent role in the quantum measurement
theory. As has been noticed earlier \[15\], $[K_-, \tilde{K}^+] = 1$, also yields,

$$[\tilde{K}^+\dagger, K^+] = 1 .$$

(14)

From this, one can find the eigenstate of $\tilde{K}^+\dagger$ operator, in the form,

$$|\gamma\rangle = e^{\gamma K^+} |0\rangle .$$

(15)

This CS, after proper normalization is the well-known Yuen state \[19\]. Our construction
can be easily generalized to various other realizations of the $SU(1, 1)$ algebra.

We now extend the above procedure to the quadratic algebra. As has been mentioned
earlier, this algebra has relevance to statistical physics and field theory; a simpler version
appears in quantum mechanical problems:

$$[N_0, N_\pm] = \pm N_\pm , \quad [N_+, N_-] = 2N_0 + aN_0^2 .$$

(16)

In this case, $f_1(N_0) = 2N_0 + aN_0^2 = g_1(N_0) - g_1(N_0 - 1)$, where,

$$g_1(N_0) = N_0(N_0 + 1) + \frac{a}{3}N_0(N_0 + 1)(N_0 + \frac{1}{2}) .$$

(17)

Representation theory of the quadratic algebra has been studied in the literature \[17\]; it
shows a rich structure depending on the values of ‘a’. In the non-compact case, i.e, for the
values of ‘a’ such that the unitary irreducible representations (UIREP) are either bounded
below or above, we can construct the canonical conjugate $\tilde{N}_+$ of $N_-$ such that $[N_-, \tilde{N}_+] = 1$.
It is given by $\tilde{N}_+ = N_+ F_1(C, N_0)$, with
\[ F_1(C, N_0) = \frac{N_0 + \delta}{C - N_0(N_0 + 1) - \frac{2}{3}N_0(N_0 + 1)(N_0 + \frac{1}{2})} \cdot \]  

(18)

As can be easily seen, in the case of the finite dimensional UIREP, \( \tilde{N}_+ \) is not well defined since \( F_1(C, N_0) \) diverges on the highest state. As mentioned earlier, the values of \( \delta \) can be fixed by demanding that the relation, \([N_-, \tilde{N}_+] = 1\), holds in the vacuum sector \(|v >_i\) where, \(|v >_i\) are annihilated by \( N_- \). This gives \( N_-\tilde{N}_+ |v >_i|v >_i\), which leads to \((N_0+\delta)|v >_i|v >_i\),

the value of the Casimir operator, \( C = N_-N_+ + g_1(N_0) \), can be easily calculated. Hence, the unnormalized coherent state \(|\mu >_i\); \( N_- |\mu >_i = \mu |\mu >_i\), is given by \( e^{\mu \tilde{N}_+} |v >_i\).

The other coherent state originating from \([\tilde{N}_+^\dagger, N_+] = 1\) is given by \(|\nu >_i = e^{\nu N_+} |v >_i\).

This can be recognized as the (unnormalized) CS in the Perelomov sense. Depending on the UIREP being infinite or finite dimensional, this quadratic algebra can also be mapped into \( SU(1, 1) \) and \( SU(2) \) algebras, respectively; leaving aside the commutators not affected by this mapping, one gets,

\[ [N_+, \tilde{N}_-] = -2bN_0 ; \]  

(19)

where \( b = 1 \) corresponds to the \( SU(1, 1) \) and \( b = -1 \) gives the \( SU(2) \) algebra. Explicitly,

\[ \tilde{N}_- = N_-G_1(C, N_0) , \]  

(20)

and

\[ G_1(C, N_0) = \frac{(N_0^2 - N_0)b + \epsilon}{C - g_1(N_0 - 1)} , \]  

(21)

\( \epsilon \) being an arbitrary constant. One can immediately construct CS in the Perelomov sense as \( U |v >_i\), where \( U = e^{\xi N_+ - \xi^* N_-} \). For the compact case, the CS are analogous to the spin and atomic coherent states \([20, 21]\). We would like to point out that, earlier the generators of the deformed algebra have been written in terms of the undeformed ones \([17]\). However, in our approach the undeformed \( SU(1, 1) \) and \( SU(2) \) generators are constructed from the deformed generators.

The Cubic algebra, which is also popularly known as the Higgs algebra in the literature, manifested in the study of the degeneracy structure of eigenvalue problems in a curved space \([\mathbb{E}]\). The generators satisfy,
\[ [M_0, M_\pm] = \pm M_\pm \, , \, [M_+, M_-] = 2cM_0 + 4hM_0^3 \, , \] \tag{22}

where, \( f_2(M_0) = 2cM_0 + 4hM_0^3 = g_2(M_0) - g_2(M_0 - 1) \), and

\[ g_2(M_0) = cM_0(M_0 + 1) + hM_0^2(M_0 + 1)^2 \, . \] \tag{23}

Analysis of its representation theory yields a variety of UIREP’s, both finite and infinite dimensional, depending on the values of the parameters \( c \) and \( h \) \cite{22}. Physically, \( h \) represents the curvature of the manifold. In the non-compact case the canonical conjugate is given by,

\[ \tilde{M}_+ = M_+ \mathcal{F}_2(C, M_0) \, , \] \tag{24}

where,

\[ \mathcal{F}_2(C, M_0) = \frac{M_0 + \zeta}{C - cM_0(M_0 + 1) - hM_0^2(M_0 + 1)^2} \, . \] \tag{25}

As before, the annihilation operator eigenstate is given by

\[ | p >_{i} = e^{\rho \tilde{M}_+} | p >_i \, , \] \tag{26}

where, \( | p >_i \) are the states annihilated by \( M_- \). Like the previous cases, the dual algebra yields another coherent state. This algebra can also be mapped in to \( SU(1,1) \) and \( SU(2) \) algebras, as has been done for the quadratic case:

\[ [M_+, M_-] = -2dM_0 \, , \] \tag{27}

where, \( d = 1 \) and \( d = -1 \) correspond to the \( SU(1,1) \) and \( SU(2) \) algebras respectively. Here,

\[ \tilde{M}_- = M_- \mathcal{G}_2(C, M_0) \, , \] \tag{28}

where,

\[ \mathcal{G}_2(C, M_0) = \frac{(M_0^2 - M_0)d + \sigma}{C - g_2(M_0 - 1)} \, , \] \tag{29}

\( \sigma \) being a constant. The coherent state in the Perelomov sense is then \( U \mid v >_i \), where, \( U = e^{\phi M_+ - \phi^* M_-} \).
For the sake of completeness, we now extend the above construction to the quantum group case. The quantum deformed $SU(2)$ algebra is given by,

$$[D_0, D_{\pm}] = \pm D_{\pm}, \quad [D_+, D_-] = \frac{q^{D_0} - q^{-D_0}}{q - q^{-1}},$$

(30)

for which [17],

$$g_3(D_0) = \frac{q^{D_0 + \frac{1}{2}} + q^{-D_0 - \frac{1}{2}}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(q - q^{-1})}.$$

(31)

The canonical conjugate $\tilde{D}_+$, of, $D_-$, valid for the non-compact case, is

$$\tilde{D}_+ = D_+ F_3(C, D_0),$$

(32)

where,

$$F_3(C, D_0) = \frac{D_0 + \eta}{C - g_3(D_0)}.$$

(33)

One can then easily construct the coherent state like the previous examples. This algebra can also be mapped in to $SU(1,1)$ and $SU(2)$ algebras:

$$[D_+, \tilde{D}_-] = -2f D_0,$$

(34)

where, $f = 1$ and $f = -1$ gives the $SU(1,1)$ and $SU(2)$ algebras respectively (the other relations of the algebra not being affected in this mapping). Explicitly,

$$\tilde{D}_- = D_- G_3(C, D_0),$$

(35)

where,

$$G_3(C, D_0) = \frac{(D_0^2 - D_0)f + \omega}{C - g_3(D_0 - 1)}.$$

(36)

The coherent state in the Perelomov sense then follows naturally.

To conclude, we have found a general method for constructing the coherent states for various deformed algebras. Since the method is algebraic and relies on the group structure of Lie algebras, the precise nature of the non-classical behaviour of these CS can be
easily inferred from our construction. It will be of particular interest to see the role of the
deformation parameters in this behaviour. Since many of these algebras are related to
quantum mechanical problems with non-quadratic, non-linear Hamiltonians, quantum op-
tical problems involving 3-photon and four photon processes [23], spin-chains and various
other physical problems [24–26], a detailed study of the properties of the CS associated with
these non-linear and deformed algebras is of physical relevance.

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