Adding open string modes to the gauge/gravity correspondence

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ABSTRACT

We review some recent results on the extension of the gauge/gravity correspondence to include matter in the fundamental representation by adding D-branes to the supergravity backgrounds. Working in the quenched approximation, in which the D-branes are considered as probes, we show how to compute the meson spectrum for a general case of brane intersections which are dual to supersymmetric gauge theories with matter supermultiplets in several dimensions.
1 Introduction

The gauge/gravity correspondence [1, 2] provides a closed string description, based on classical supergravity, of the dynamics of gauge theories at large 't Hooft coupling. In its first formulation, this duality was applied to gauge theories in which all fields are in the adjoint representation. Obviously, a natural extension of the correspondence would be the inclusion of matter (quark) fields in the fundamental representation. This is equivalent to adding open string degrees of freedom to the supergravity side of the correspondence and can be achieved by adding D-branes to the supergravity background. If the number of added D-branes is small as compared to the number of branes that created the background, we can neglect their backreaction on the geometry and treat the extra branes as probes. The fluctuations of these probes correspond to degrees of freedom of open strings connecting the brane probe and those that generated the background [3]. On the field theory side these open strings are identified with fundamental matter multiplets of dynamical quarks whose masses are proportional to the distance between the two types of branes.

In Ref. [4] it was proposed to use this setup to add flavor to some supergravity duals. Indeed, let us consider the intersection of two branes of different dimensionalities. In the decoupling limit one sends the string scale $l_s$ to zero keeping the gauge coupling of the lower dimensional brane fixed. It is straightforward to see that the gauge coupling of the higher dimensional brane vanishes in this limit and, therefore, the corresponding gauge theory decouples and the gauge group of the higher dimensional brane becomes the flavor symmetry of the effective theory at the intersection. In Ref. [5] this approach was applied to the case of the intersection of D3- and D7-branes which share three common spatial directions (see also Ref. [6]). If the number of D3-branes is large and the number of D7-branes is small one can employ the probe approximation and identify the fluctuation modes of the D7-brane probe in the $\text{AdS}_5 \times S^5$ geometry with the mesons (or, more rigorously, bound states) of the gauge theory. Remarkably, the mesonic mass spectrum can be computed analytically in this case. Different flavor branes and their spectra for several backgrounds have been considered in the recent literature (see Refs. [7]-[24]).

In this paper we will first review the analysis of the fluctuation modes of a brane probe following mostly the approach of Ref. [25]. First of all, we shall determine a general class of BPS intersections by imposing a no-force condition. Then, we will particularize our formalism to study the fluctuation of transverse scalars of the probe in the background of a stack of Dp-branes. In general, the differential equations satisfied by the fluctuations, and the corresponding mass spectra, must be solved by numerical methods. However, when the background geometry is $\text{AdS}_5 \times S^5$ we will show that these equations can be solved analytically and the mass spectra can be given in closed form. This exactly solvable cases correspond to the intersections of D3-branes with D7-, D5- and D3-brane probes, whose induced worldvolume is of the form $\text{AdS}_{d+2} \times S^d$ for $d = 3, 2, 1$. As argued in Ref. [3], the AdS/CFT correspondence acts twice in these systems and, apart from the holographic description of the four dimensional field theory on the boundary of $\text{AdS}_5$, the fluctuations of the probe are conjectured to be dual to the physics confined to the boundary of $\text{AdS}_{d+2} \subset \text{AdS}_5$. Indeed, the D5- and D3-brane probes in $\text{AdS}_5 \times S^5$ are dual to $\mathcal{N} = 4, d = 4$ super Yang-Mills theories with codimension one or two defects and fundamental hypermultiplets.
localized at the defect. Finally, we will also briefly review the addition of supersymmetric D-brane probes to the $AdS_5 \times T^{1,1}$ background, which is dual to an $\mathcal{N} = 1, d = 4$ superconformal field theory.

2 BPS Intersections

Let us consider a ten-dimensional background corresponding to a $p_1$-brane, whose metric can be written as:

$$ds^2 = \left[ \frac{r^2}{R^2} \right]^{\gamma_1} dx_{1:p_1}^2 + \left[ \frac{R^2}{r^2} \right]^{\gamma_2} d\vec{y} \cdot d\vec{y},$$

(1)

where $dx_{1:p_1}^2$ denotes the $(p_1 + 1)$-dimensional Minkowski metric, $R$, $\gamma_1$ and $\gamma_2$ are constants, $\vec{y} = (y^1, \ldots, y^{9-p_1})$ and $r^2 = \vec{y} \cdot \vec{y}$. We will also assume that the dilaton in such background can be written as:

$$e^{-\phi(r)} = \left[ \frac{R^2}{r^2} \right]^{\gamma_3},$$

(2)

where $\gamma_3$ is a new constant. Let us now add to this background a probe $p_2$-brane sharing $d$ common spatial directions with the $p_1$-brane. The corresponding orthogonal intersection will be denoted as $(d|p_1 \perp p_2)$ and is depicted in figure 1. We will assume that the probe is extended along the directions $(x^1, \ldots, x^d, y^1, \ldots, y^{p_2-d})$ and we will denote by $\vec{z}$ the set of $y$ coordinates transverse to the probe. Notice that $|\vec{z}|$ represents the separation of the branes along the directions transverse to both background and probe branes. Actually, we shall consider first a static configuration in which the probe is located at a constant value of $|\vec{z}|$, namely at $|\vec{z}| = L$. The induced metric on the probe worldvolume for such a static configuration will be denoted by:

$$ds_I^2 = G_{ab} d\xi^a d\xi^b,$$

(3)
with $\xi^a$ being a set of worldvolume coordinates. In what follows we shall take these coordinates as the common cartesian coordinates $x^0, \ldots, x^d$, together with the spherical coordinates of the $y^1, \ldots, y^{p_2-d}$ hyperplane. Actually, assuming that $p_2 - d \geq 2$, we shall represent the line element of this hyperplane as:

$$\left(dy^1\right)^2 + \cdots + \left(dy^{p_2-d}\right)^2 = d\rho^2 + \rho^2 d\Omega^2_{p_2-d-1} ,$$

where $d\Omega^2_{p_2-d-1}$ is the line element of a unit $(p_2 - d - 1)$-sphere. It is now straightforward to verify that the induced metric $ds_I^2$ can be written as:

$$ds_I^2 = \left[\frac{\rho^2 + L^2}{R^2}\right]^{\gamma_1} dx_{1,d}^2 + \left[\frac{R^2}{\rho^2 + L^2}\right]^{\gamma_2} \left( d\rho^2 + \rho^2 d\Omega^2_{p_2-d-1} \right) .$$

The energy density $\mathcal{H}$ of the probe is determined by its Dirac-Born-Infeld action. For static configurations as those we are considering here, $\mathcal{H} = e^{-\phi} \sqrt{-\det \mathcal{G}}$, where $\mathcal{G}$ is the induced metric (5). Actually, one can easily prove by using eqs. (2) and (5) that

$$\mathcal{H} = \left[\frac{\rho^2 + L^2}{R^2}\right]^{\frac{1}{2}(d+1) - \frac{1}{2}(p_2-d) - \gamma_3} \rho^{p_2-d-1} \sqrt{\det \tilde{g}} ,$$

where $\tilde{g}$ is the metric of the unit $(p_2 - d - 1)$-sphere. We are interested in studying BPS intersections, in which the branes do not exert any force among each other. In our case this no-force condition requires that $\mathcal{H}$ be independent of the distance $L$ between the branes which, in view of the right-hand side of eq. (6), is only possible if the number $d$ of common dimensions is related to the total dimensionality $p_2$ of the probe as:

$$d = \frac{\gamma_2}{\gamma_1 + \gamma_2} p_2 + \frac{2\gamma_3 - \gamma_1}{\gamma_1 + \gamma_2} .$$

### 3 Dp-brane background

Let us now particularize our analysis to the case in which the background geometry is generated by a stack of $N$ parallel Dp-branes. In the string frame, the near-horizon supergravity solution corresponding to such a stack has the form displayed in eqs. (1) and (2), with $p_1 = p, R$ given by

$$R^{7-p} = 2^{5-p} \pi^{\frac{5-p}{2}} \Gamma\left(\frac{7-p}{2}\right) g_s N (\alpha')^{\frac{7-p}{2}} ,$$

and with the following values for the exponents $\gamma_i$:

$$\gamma_1 = \gamma_2 = \frac{7-p}{4} , \quad \gamma_3 = \frac{(7-p)(p-3)}{8} .$$

Moreover, the Dp-brane solution is endowed with a Ramond-Ramond $(p+1)$-form potential, whose expression is not relevant in what follows. Applying eq. (7) to this background, we get the following relation between $d$ and $p_2$:

$$d = \frac{p_2 + p - 4}{2} .$$
Let us now consider the case in which the probe brane is another D-brane. As the brane of the background and the probe should live in the same type II theory, \( p_2 - p \) should be even. Since \( d \leq p \), we are left with the following three possibilities \( p_2 = p, p + 2, p + 4 \), for which eq. (10) gives \( d = p - 2, p - 1, p \) respectively. Thus, we get the following well-known[26] set of orthogonal BPS intersections of D-branes:

\[ \langle p| Dp \perp D(p + 4) \rangle, \quad \langle p - 1| Dp \perp D(p + 2) \rangle, \quad \langle p - 2| Dp \perp Dp \rangle. \]  (11)

Let us now study the fluctuations around the \( |\vec{z}| = L \) static embedding just discussed. Without loss of generality we can take \( z^1 = L \), \( z^m = 0 \) \( (m > 1) \) as the unperturbed configuration and consider a fluctuation of the type:

\[ z^1 = L + \chi^1, \quad z^m = \chi^m \quad (m > 1), \]  (12)

where the \( \chi \)'s are small. The dynamics of the fluctuations is determined by the Dirac-Born-Infeld lagrangian which, for the fluctuations of the transverse scalars we study in this section, reduces to \( \mathcal{L} = -e^{-\phi}\sqrt{-\det g} \), where \( g \) is the induced metric on the worldvolume for an embedding of the probe as given by eq. (12). By expanding this lagrangian and keeping up to second order terms, one can prove that:

\[ \mathcal{L} = \frac{1}{2} \rho^{p_2-d-1} \sqrt{\det g} \left[ \frac{R^2}{\rho^2 + L^2} \right]^{\frac{7-d}{4}} g^{ab} \partial_a \chi^m \partial_b \chi^m, \]  (13)

where \( g^{ab} \) is the (inverse of the) metric (5). The equations of motion derived from \( \mathcal{L} \) are:

\[ \partial_a \left[ \rho^{p_2-d-1} \sqrt{\det g} \left( \frac{R^2}{\rho^2 + L^2} \right)^{\frac{7-d}{4}} g^{ab} \partial_b \chi \right] = 0, \]  (14)

where we have dropped the index \( m \) in the \( \chi \)'s. Using the explicit form of the metric elements \( g^{ab} \), eq. (14) can be written as the following differential equation:

\[ \frac{R^7-p}{(\rho^2 + L^2)^{\frac{7-d}{2}}} \partial^\mu \partial_\mu \chi + \frac{1}{\rho^{p_2-d-1}} \partial_\rho (\rho^{p_2-d-1} \partial_\rho \chi) + \frac{1}{\rho^2} \nabla^i \nabla_i \chi = 0, \]  (15)

where the index \( \mu \) corresponds to the directions \( x^\mu = (t, x^1, \ldots, x^d) \) and \( \nabla_i \) is the covariant derivative on the \( (p_2 - d - 1) \)-sphere. To solve this equation, let us separate variables as:

\[ \chi = \xi(\rho) e^{ikx} Y^l(S^{p_2-d-1}), \]  (16)

where the product \( kx \) is performed with the flat Minkowski metric and \( Y^l(S^{p_2-d-1}) \) are scalar spherical harmonics on the \( (p_2 - d - 1) \)-dimensional sphere, which satisfy:

\[ \nabla^i \nabla_i Y^l(S^{p_2-d-1}) = -(l + p_2 - d - 2) Y^l(S^{p_2-d-1}). \]  (17)

If we redefine the variables as:

\[ \varrho = \frac{\rho}{L}, \quad \tilde{M}^2 = -R^7-p L^{p-5} k^2, \]  (18)

the differential equation (15) becomes:

\[ \partial_\rho \left( \varrho^{p_2-d-1} \partial_\rho \varrho \right) + \left[ \tilde{M}^2 \frac{\varrho^{p_2-d-1}}{(1 + \varrho^2)^{\frac{7-d}{2}}} - l(l + p_2 - d - 2) \varrho^{p_2-d-3} \right] \xi = 0. \]  (19)
4  \( AdS_5 \times S^5 \) background

When the background is generated by a stack of D3-branes, the background geometry is \( AdS_5 \times S^5 \) and the list (11) of BPS intersections reduces to:

\[
(3|D3 \perp D7) , \quad (2|D3 \perp D5) , \quad (1|D3 \perp D3) .
\]

Remarkably, it turns out that in this case the differential equation (19) for the fluctuations of the transverse scalars can be solved analytically in terms of hypergeometric functions and the spectrum of values of \( \bar{M} \) can be found exactly. To prove this fact, let us introduce the quantity \( \lambda \), related to the rescaled mass \( \bar{M} \) as:

\[
\bar{M}^2 = 4\lambda(\lambda + 1) .
\]

Then, the solution of (19) for \( p = 3 \) that is regular when \( \varrho \to 0 \) is:

\[
\xi(\varrho) = \varrho^l (\varrho^2 + 1)^{-\lambda} F(-\lambda, -\lambda + l - 1 + \frac{p_2 - d}{2}; l + \frac{p_2 - d}{2}; -\varrho^2) .
\]

We also want that \( \xi \) vanishes when \( \varrho \to \infty \). A way to ensure this is by imposing that

\[
-\lambda + l - 1 + \frac{p_2 - d}{2} = -n , \quad n = 0, 1, 2, \cdots .
\]

When the quantization condition (23) is imposed, the series defining the hypergeometric function in (22) truncates, and the highest power of \( \varrho \) is \( (\varrho^2)^n \). As a consequence \( \xi \) vanishes as \( \varrho^{-(l+p_2-d-2)} \) when \( \varrho \to \infty \). Moreover, notice that the quantization condition (23) of the values of \( \lambda \) implies that the allowed values of \( \bar{M}^2 \) are:

\[
\bar{M}^2 = 4 \left( n + l - 1 + \frac{p_2 - d}{2} \right) \left( n + l + \frac{p_2 - d}{2} \right) .
\]

Notice that, for the three cases in (20), \( p_2 = 2d + 1 \) for \( d = 3, 2, 1 \). By using this relation between \( p_2 \) and \( d \), one can rewrite the mass spectra (24) of scalar fluctuations for the intersections (20) as:

\[
M = \frac{2L}{R^2} \sqrt{\left( n + l + \frac{d - 1}{2} \right) \left( n + l + \frac{d + 1}{2} \right)} ,
\]

where \( M^2 = -k^2 \) and we have taken into account that, in this case, \( \bar{M}^2 = -R^4 L^{-2} k^2 \) (see eq. (18)). Moreover, in this \( AdS_5 \times S^5 \) background, the induced metric on the probe (5) can be written as:

\[
ds_2^2 = \frac{\rho^2 + L^2}{R^2} \, ds_{1,d}^2 + \frac{R^2}{\rho^2 + L^2} \, d\rho^2 + R^2 \frac{\rho^2}{\rho^2 + L^2} \, d\Omega_d^2 .
\]

It is clear by inspecting (26) that, in the conformal case \( L = 0 \), the metric \( ds_2^2 \) reduces to that of a product space of the form \( AdS_{d+2} \times S^d \). Interestingly, this same form of the metric
is achieved for \( L \neq 0 \) in the ultraviolet limit \( \rho \to \infty \). This \( \rho \to \infty \) limit is simply the high energy regime of the theory, where the mass of the quarks, which are proportional to the brane separation \( L \), can be ignored and the theory becomes conformal. Therefore, the \( \rho \to \infty \) behaviour of the fluctuations should provide us information about the conformal dimension \( \Delta \) of the corresponding dual operators. Indeed, in the context of the AdS/CFT correspondence in \( d + 1 \) dimensions, it is well known that, if the fields are canonically normalized, the normalizable modes behave at infinity as \( \rho^{-\Delta} \), whereas the non-normalizable ones should behave as \( \rho^{\Delta-d-1} \). In the case in which the modes are not canonically normalized the behaviours of both types of modes are of the form \( \rho^{-\Delta+\gamma} \) and \( \rho^{\Delta-d-1+\gamma} \) for some \( \gamma \). In our case the fluctuations are given in terms of a hypergeometric function in eq. (22). Taking into account that for large \( \rho \) the hypergeometric function behaves as:

\[
F(a_1, a_2; b; -\rho^2) \approx c_1 \rho^{-2a_1} + c_2 \rho^{-2a_2}, \quad (\rho \to \infty),
\]

one immediately gets that the associated conformal dimension is:

\[
\Delta = \frac{d+1}{2} + a_2 - a_1.
\]

From the solution (22) one gets that \( a_1 = -\lambda \) and \( a_2 = -\lambda + l + \frac{d-1}{2} \). By applying eq. (28) to this case, we get the following value for the dimension of the operator associated to the scalar fluctuations:

\[
\Delta = l + d.
\]

The field theory dual of the three intersections (20) is well-known. Indeed, the \( (3|D3 \perp D7) \) intersection is the case extensively studied in Ref. [5] and corresponds in the UV to an \( AdS_5 \times S^3 \subset AdS_5 \times S^5 \) embedding. In this case the D7-brane is a flavor brane for the \( \mathcal{N} = 4 \) gauge theory coupled to an \( \mathcal{N} = 2 \) fundamental hypermultiplet and the fluctuation modes of the probe can be identified with the mesons of the theory. It turns out that the mass spectra of all the Born-Infeld modes (and not only those reviewed here that correspond to the transverse scalars) can be computed analytically[5]. This full set of fluctuation modes can be accommodated in multiplets, and the mass spectra display the expected degeneracy. The dual operators in the gauge theory side can be matched with the fluctuations by looking at the UV dimensions and at the R-charge quantum numbers. Generically, the dual fields are bilinear in the fundamental fields and contain the powers of the adjoint fields needed to construct the appropriate representation of the R-charge symmetry.

In the conformal limit \( L = 0 \) the \( (2|D3 \perp D5) \) intersection represents an \( AdS_4 \times S^2 \) defect in \( AdS_5 \times S^5 \). The codimension one dual defect CFT has has been studied in detail in Ref. [27]. It corresponds to \( \mathcal{N} = 4, d = 4 \) super Yang-Mills theory coupled to \( \mathcal{N} = 4, d = 3 \) fundamental hypermultiplets localized at the defect. In Ref. [27] the action of the model was constructed and a precise dictionary between operators of field theory and fluctuation modes of the probe was obtained (see also Refs. [28, 29]). The meson spectrum for this defect field theory when the separation of the branes is non-vanishing has been obtained, for the full set of fluctuation modes, in Ref. [25]. The corresponding mesonic mass levels can be obtained analytically and are compatible with the supermultiplet assignments made in Ref. [27].
The $(1|D3 \perp D3)$ intersection corresponds, in the conformal limit, to an $AdS_3 \times S^1$ defect in $AdS_5 \times S^5$ which is codimension two in the gauge theory directions[30]. This system can be regarded as the dual of two $\mathcal{N} = 4$ four-dimensional theories coupled to each other through a bifundamental hypermultiplet living on the two-dimensional defect. If a non-zero mass is given to the hypermultiplet, a mass gap is introduced in the theory and the mass spectrum of all modes can be obtained in closed form[25].

5 Approximate methods for the Dp-brane background

When $p \neq 3$ the differential equation (19) cannot be solved analytically and, thus, we have to make use of other techniques in order to obtain the fluctuation spectrum. One of these techniques is converting eq. (19) into a Schrödinger equation of the type:

$$\partial_y^2 \psi - V(y) \psi = 0,$$

where $V$ is some potential. The change of variables needed to transform eq. (19) into (30) is:

$$e^y = \varrho \,, \quad \psi = \varrho^{\frac{p_2-d-2}{2}} \xi .$$

Notice that in this change of variables $\varrho \to \infty$ corresponds to $y \to \infty$, while the point $\varrho = 0$ is mapped into $y = -\infty$. Moreover, the resulting potential $V(y)$ takes the form:

$$V(y) = \left( l - 1 + \frac{p_2-d}{2} \right)^2 - \bar{M}^2 \frac{e^{2y}}{(e^{2y} + 1)^{\frac{7-p}{2}}} .$$

In these new variables, the problem of finding the mass spectrum can be rephrased as that of finding the values of $\bar{M}$ such that a zero-energy level for the potential (32) exists. By inspecting the form of variables $\varrho \to \infty$ corresponds to $y \to \infty$, while the point $\varrho = 0$ is mapped into $y = -\infty$. Moreover, the resulting potential $V(y)$ takes the form:

$$V(y) = \left( l - 1 + \frac{p_2-d}{2} \right)^2 - \bar{M}^2 \frac{e^{2y}}{(e^{2y} + 1)^{\frac{7-p}{2}}} .$$

In these new variables, the problem of finding the mass spectrum can be rephrased as that of finding the values of $\bar{M}$ such that a zero-energy level for the potential (32) exists. By inspecting the form of $V(y)$ one readily concludes that, for $p < 5$, it has a unique minimum at $e^{2y} = \frac{2}{5-p}$. Notice that the classically allowed region in the Schrödinger equation (30) corresponds to the values of $y$ such that $V(y) \leq 0$. We would have a discrete spectrum if this region is of finite size or, equivalently, if the points $y = \pm \infty$ are not allowed classically. When $p < 5$ the second term in (32) vanishes at $y = \pm \infty$ and, thus, only the first term of $V$ (which is always non-negative) remains in this limit. This means that, indeed, there is a discrete spectrum of values of $\bar{M}$. A very useful (and in some cases accurate) tool to estimate this spectrum is the semiclassical WKB method, whose starting point is the WKB quantization rule:

$$(n + \frac{1}{2})\pi = \int_{y_1}^{y_2} dy \sqrt{-V(y)} , \quad n \geq 0 ,$$

where $n \in \mathbb{Z}$ and $y_1$, $y_2$ are the turning points of the potential ($V(y_1) = V(y_2) = 0$). The WKB method has been very successful [31, 32] in the calculation of the glueball spectrum in the context of the gauge/gravity correspondence. In this method one evaluates the right-hand side of eq. (33) by expanding it as a power series in $1/\bar{M}$ and keeping the leading and subleading terms of this expansion. One obtains in this way the expression of $\bar{M}$ as a function of the principal quantum number $n$ which is, in principle, reliable for large $n$,.
although in some cases it happens to give the exact result. The outcome [25] of this analysis is the following expression for the mass levels:

\[ M_{WKB}^{n,l} = 2\sqrt{\pi} \frac{L^2}{R^2} \frac{\Gamma\left(\frac{7-p}{4}\right)}{\Gamma\left(\frac{2-p}{4}\right)} \sqrt{(n+1) \left(n + \frac{7-p}{5-p} \left(l-1 + \frac{p_2-d}{2}\right)\right)} . \quad (34) \]

| \( (2|D2 \perp D6) \) with \( l = 0 \) | \( (4|D4 \perp D8) \) with \( l = 0 \) | \( (3|D4 \perp D6) \) with \( l = 0 \) |
|---|---|---|
| \( n \) | WKB | Numerical | \( n \) | WKB | Numerical | \( n \) | WKB | Numerical |
| 0 | 11.46 | 11.34 | 0 | 4.31 | 4.68 | 0 | 2.15 | 1.68 |
| 1 | 36.67 | 36.54 | 1 | 11.48 | 11.88 | 1 | 7.18 | 6.78 |
| 2 | 75.63 | 75.50 | 2 | 21.53 | 21.94 | 2 | 15.07 | 14.72 |
| 3 | 128.34 | 128.20 | 3 | 34.45 | 34.86 | 3 | 25.84 | 25.58 |
| 4 | 194.80 | 194.66 | 4 | 50.24 | 50.66 | 4 | 39.48 | 39.34 |
| 5 | 275.01 | 274.88 | 5 | 68.91 | 69.34 | 5 | 55.99 | 56.02 |

Table 1: Numerical and WKB mass levels for some \( (2|Dp \perp D(p+4)) \) and \( (2|Dp \perp D(p+2)) \) intersections.

The mass levels can also be computed numerically by means of the shooting technique. To apply this technique one first notices that those solutions of eq. (19) which are regular in the IR behave as \( \xi \sim q^l \) near \( q \approx 0 \), while for large \( q \) they should decrease as \( \xi \sim q^{-(l+p_2-d-2)} \). In the shooting technique one solves numerically the differential equation (19) for the fluctuations by imposing the regular behaviour \( \xi \sim q^l \) at \( q \approx 0 \) and then one scans the values of \( \bar{M} \) until the UV behaviour \( \xi \sim q^{-(l+p_2-d-2)} \) is fine tuned. This occurs only for a discrete set of values of \( \bar{M} \), which determines the mass spectrum we are looking for. The numerical values obtained for some intersections are given in table 1, where they are compared with the masses obtained with the WKB mass formula (34). Notice that, indeed, for large \( n \) the WKB estimate (34) is rather accurate.

Several comments concerning the meson spectra just found are in order. First of all, the masses for the full set of fluctuations of the intersections (11) have been obtained, both numerically and semiclassically, in Ref. [25]. Moreover, we notice from (34) that, for large \( n \), the mass grows linearly with the excitation number \( n \) (i.e. \( n \sim n \) for \( n \to \infty \)). This is in contrast with the \( M \sim \sqrt{n} \) behaviour expected [33] in confining gauge theories. Notice also that the mass gap of the theory is just the coefficient relating \( \bar{M}^2 \) and \( M^2 = -k^2 \) in eq. (18), namely \( R^{p-7}L^{5-p} \) (see also eq. (34)). Let us express this quantity in terms of the quark mass \( m_q \) and the Yang-Mills coupling constant \( g_{YM} \), given by:

\[ m_q = \frac{L}{2\pi \alpha'}, \quad g_{YM}^2 = (2\pi)^{p-2} g_s (\alpha')^{\frac{p-3}{2}} . \quad (35) \]

After using the expression (8) of \( R \), one easily gets:

\[ R^{p-7}L^{5-p} = \frac{2^{p-2} \pi^{p+1}}{\Gamma\left(\frac{7-p}{2}\right)} \frac{(m_q)^{5-p}}{g_{YM}^2 N} . \quad (36) \]
It is interesting to rewrite this result in terms of the effective dimensionless coupling constant $g_{\text{eff}}(U)$ at the energy scale $U$, which is given by [35]:

$$g_{\text{eff}}^2(U) = g_{YM}^2 N U^{p-3}.$$  \hspace{1cm} (37)

It follows from (36) and (37) that, up to a numerical coefficient, the mass gap of the theory is:

$$M \sim \frac{m_q}{g_{\text{eff}}(m_q)},$$  \hspace{1cm} (38)

a result that seems to be universal and coincident with the UV/IR relation found in Ref. [36].

6 Reduced supersymmetry

In order to extend the results reviewed in the previous sections to more realistic scenarios one should consider the addition of branes to less supersymmetric backgrounds. Here we will mostly concentrate on reviewing the case in which the five-sphere of the $AdS_5 \times S^5$ geometry is substituted by an Einstein space $X_5$ with less isometries. If, in particular, $X_5$ is taken as the $T^{1,1}$ space we have the so-called Klebanov-Witten (KW) model[37], which is the background corresponding to having a stack of $N$ D3-branes at the tip of the conifold. The corresponding dual field theory is a four-dimensional $\mathcal{N} = 1$ superconformal field theory with gauge group $SU(N) \times SU(N)$ coupled to four chiral superfields $A_i, B_i$ ($i = 1, 2$) in the bifundamental representation. The ten-dimensional metric for the KW model has the form:

$$ds^2 = \frac{r^2}{L^2} dx_{1,3}^2 + \frac{L^2}{r^2} dr^2 + L^2 ds_{T^{1,1}}^2,$$  \hspace{1cm} (39)

where $L^4 = \frac{27}{4} \pi g s N \alpha' r^2$ and $ds_{T^{1,1}}^2$ is the metric of the $T^{1,1}$ space. This metric can be written[38] by using the fact that this space can be realized as the coset $(SU(2) \times SU(2))/U(1)$ and that it is a $U(1)$ bundle over $S^2 \times S^2$. Actually, if $(\theta_1, \phi_1)$ and $(\theta_2, \phi_2)$ are the standard coordinates of the $S^2$'s and if $\psi \in [0, 4\pi)$ parametrizes the $U(1)$ fiber, the metric $ds_{T^{1,1}}^2$ is:

$$ds_{T^{1,1}}^2 = \frac{1}{6} \sum_{i=1}^{2} (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + \frac{1}{9} (d\psi + \sum_{i=1}^{2} \cos \theta_i d\phi_i)^2.$$  \hspace{1cm} (40)

To determine the supersymmetric embeddings of the different D-brane probes in this background we do not have at our disposal the no-force argument employed above for the geometries created by D-branes in flat space. Instead we have to use kappa symmetry, which is based on the fact that there exists a matrix $\Gamma_\kappa$ such that, if $\epsilon$ is a Killing spinor of the background, only those embeddings obeying $\Gamma_\kappa \epsilon = \epsilon$ are supersymmetric. The matrix $\Gamma_\kappa$ depends on the embedding of the probe in the background and, therefore, if the Killing spinors of the latter are known, the kappa symmetry condition allows to determine such supersymmetric embeddings in a systematic way. This analysis has been carried out in detail in Ref. [16] and it will be summarized here. First of all, let us recall that the $T^{1,1}$ space is topologically $S^2 \times S^3$. Thus, there is the possibility of having D3-branes wrapping a three-cycle of $T^{1,1}$,
which allows for a rich zoology of BPS intersections. Moreover, the $T^{1,1}$ is a Sasaki-Einstein space and therefore, its six-dimensional cone $C(T^{1,1})$ with metric $dt^2 + r^2 ds_{T^{1,1}}^2$ is a Calabi-Yau manifold with complex dimension three. A very convenient set of complex coordinates of $C(T^{1,1})$ is the following:

$$z_i = \tan \left( \frac{\theta_i}{2} \right) e^{i\phi_i} \quad (i = 1, 2), \quad z_3 = r^3 \sin \theta_1 \sin \theta_2 e^{-i\psi} . \quad (41)$$

It turns out that there exists a family of three-cycles $C_3 \in T^{1,1}$ such that a D3-brane located at the center of the $AdS_5$ space and wrapping $C_3$ is $1/8$ supersymmetric. In terms of the complex coordinates (41) these cycles can be simply characterized as the intersection of the locus of the polynomial equation

$$z_1^{m_1} z_2^{m_2} = \text{constant} \quad (42)$$

in the cone $C(T^{1,1})$ with its $T^{1,1}$ base. To identify the field theory dual of the D3-branes wrapping the three-cycles (42) one must take into account that, according to the standard AdS/CFT arguments, the volume of the cycle determines the conformal dimension of the dual field theory operator. Moreover, in the KW theory there is a $U(1)$ baryon number symmetry under which the $A_i$ ($B_i$) fields have baryon number $+1$($-1$). On the gravity side of the AdS/CFT correspondence, the baryon number can be identified with the third homology class of the three-cycle $C_3$ over which the D3-brane is wrapped. By using these facts one can verify that these D3-branes are dual to dibaryonic operators built out from the $A_i$ and $B_i$ fields. For example[39], the D3-brane wrapping the three-cycle with $m_1 = 1$, $m_2 = 0$ is dual to an operator of the form $\det(A)$, whereas for arbitrary values of $m_1$ and $m_2$ it corresponds to an operator with higher baryon number. It is also interesting to point out that one can study[40] the fluctuations of the D3-branes, which correspond to mesonic excitations of the dibaryon operators.

One can also find supersymmetric embeddings of D5-branes which give rise to defect theories, similar to the $(2|D3 \perp D5)$ intersections in flat space studied above. In this case the D5-brane must be extended along two Minkowski coordinates, as well as in the holographic coordinate $r$, and wrap a two-cycle $C_2$ of $T^{1,1}$, which can be simply characterized by the equations:

$$\theta_2 = \theta_1 , \quad \phi_2 = 2\pi - \phi_1 , \quad (43)$$

with the coordinate $\psi$ being constant.

The flavor branes of the KW model are D7-branes filling the four spacetime Minkowski directions $x^\mu$ and extended along a certain submanifold $M_4$ of the conifold $C(T^{1,1})$. In terms of the complex coordinates (41) this submanifold $M_4$ can be described by the following polynomial equation

$$z_1^{m_1} z_2^{m_2} z_3^{m_3} = \text{constant} . \quad (44)$$

The meson spectrum for the flavor D7-branes embedded on the conifold geometry as in eq. (44) has been computed in Ref. [10] (see also Ref. [17]).

The same methodology reviewed here for the conifold has been applied in Refs. [41] and [42] to the metrics of the form $AdS_5 \times Y^{p,q}$ and $AdS_5 \times L^{a,b,c}$, where $Y^{p,q}$ and $L^{a,b,c}$ are the recently discovered[43] five dimensional Sasaki-Einstein spaces. Moreover, the kappa
symmetry condition $\Gamma_\epsilon = \epsilon$ has also been employed in Refs. [12] and [44] to determine the supersymmetric embeddings of D-branes in the confining Maldacena-Núñez[46] background (for a review on the extension of the gauge/gravity duality to confining theories, see Ref. [47]).

7 Concluding Remarks

In this paper we have reviewed the holographic description of mesons by using the fluctuation of brane probes in supergravity backgrounds. We have restricted ourselves to describe the simplest configurations with a large amount of supersymmetry. For models more suitable for the phenomenological description of QCD the interested reader should consult, specially, Refs. [8], [9] and [20]. Let us finish by mentioning that there have been different attempts to go beyond the probe approximation and to obtain a backreacted supergravity solution [45].

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