Quantum Naked Singularities in 2d Dilaton Gravity

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Abstract

Roughly speaking, naked singularities are singularities that may be seen by timelike observers. The Cosmic Censorship conjecture forbids their existence by stating that a reasonable system of energy will not, under reasonable conditions, collapse into a naked singularity. There are however many (classical) counter-examples to this conjecture in the literature. We propose a defense of the conjecture through the quantum theory. We will show that the Hawking effect and the accompanying back reaction, when consistently applied to naked singularities in two dimensional models of dilaton gravity with matter and a cosmological constant, prevent their formation by causing them to explode or to emit radiation catastrophically. This contrasts with black holes which radiate slowly. If this phenomenon is reproduced in the four dimensional world, the radiation accompanying

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the formation of naked singularities should have observable consequences.
I. Introduction

Two dimensional models of gravity seem to mimic the behavior of their four dimensional counterparts. They have therefore long been considered excellent theoretical laboratories for the examination of the various conceptual problems posed by Hawking’s original work on black hole evaporation.¹ One important advantage of two dimensions is in the fact that the quantum stress energy tensor of a conformal field propagating in a curved two dimensional background can, in most cases, be calculated exactly from a knowledge of the trace anomaly. Thus, for example, Christensen and Fulling² were able to obtain the exact stress tensor of evaporation of a two dimensional Schwarzschild black hole, neglecting the back reaction of the spacetime, and their stress tensor exhibited most of the physical properties expected of the full four dimensional theory.

More recent interest in two dimensional models has been spurred by Witten’s discovery of an exact conformal field theory describing dilaton gravity in two dimensions.³ Witten’s model also arises from the compactification on an eight torus of ten dimensional dilaton gravity originating in heterotic string theory to lowest order in world-sheet perturbation theory,⁴ and as the effective action describing the radial modes of extreme dilatonic black holes in four dimensions.⁵ It possesses a rich structure with timelike and spacelike singularities and relatively simple dynamics, while reproducing most of the key features of four dimensional Einstein gravity. The model has been exploited, therefore, to revisit some of the fundamental problems raised by Hawking’s early work. As a consequence, quantum black holes in two dimensions have received, and are receiving, a good deal of attention.⁶,⁷,⁸,⁹

An important model of dynamical formation and evaporation of a black hole in two dimensional dilaton gravity was proposed by Callan, Giddings, Harvey and Strominger (CGHS).⁶ The authors coupled conformal matter degrees of freedom to the original Witten model and then examined an incoming shock wave of the matter fields, finding that it can be expected to radiate away all its energy before the black hole is actually formed. There were difficulties with this conclusion, however, because the back reaction of the spacetime was neglected as a first approximation and later could not be adequately taken into account. The dilaton coupling was also seen to be too large at the turn around point, further belying the original one loop calculation. Since this proposal, various generalizations and improvements have been made to the CGHS model⁷,⁸,⁹ and the evaporation of a black hole via particle production of massless conformal matter and including the back
reaction of the geometry of spacetime can be traced numerically and, in some models, analytically.

Naked singularities,\textsuperscript{10} on the other hand, have been for the most part ignored. Roughly speaking naked singularities are singularities that may be seen by physically allowed observers and are known to be formed under fairly generic conditions in four dimensions\textsuperscript{11}. Nevertheless, they have so far been considered intuitively undesirable and this has produced a conjecture forbidding their existence, the so-called cosmic censorship conjecture.\textsuperscript{12} The conjecture simply states that a physically “reasonable” system of energy will not, under “reasonable” initial conditions, evolve into a naked singularity. The original intent of the conjecture was of course to prohibit the formation of naked singularities in classical general relativity so that, for example, a non-rotating system would radiate away all multipole moments sufficiently rapidly that the final state would be a Schwarzschild black hole. However, there are many counter examples in the literature where naked singularities do indeed form from classically reasonable initial conditions.

The existence of naked singularities is still difficult to understand physically, and several suggestions have been made to maintain the viability of the cosmic censorship hypothesis. One is merely to reiterate that it holds classically but needs some fine tuning to describe the set of physically reasonable set of states from which a collapse may begin. The fine tuning would be such as to exclude all known examples of the formation of naked singularities. A second way to preserve the conjecture is to argue from the point of view of string theory. String theory is said to solve the short distance problems of general relativity by providing a fundamental length scale proportional to the inverse square root of the string tension. The expectation is therefore that the high energy behavior of string theory (in weak coupling) would forbid any observations of processes associated with high field gradients over short distances. Yet a third possibility, which we argue for here, is that the Hawking radiation and the accompanying back reaction will be so huge as to prevent a naked singularity from forming.

In ref.\textsuperscript{[13]} we examined the classical formation and quantum evaporation of naked singularities within the context of the CGHS shock-wave model. Our preliminary work seemed to indicate that naked singularities “explode” (as opposed to black holes that are expected to “evaporate”) as soon as they are formed. What we mean by this is that, in the regime in which the dilaton coupling is small, asymptotically timelike singularities were observed to give up all their energy in a burst upon forming, which contrasts with
the expectation of relatively slow evaporation from the black hole (except possibly in its final stage). Subsequent analysis of other models seems to verify this conclusion, if the naked singularities actually form classically. It seems possible, as we will show below, that initial conditions leading to the formation of a naked singularity are quantum mechanically inconsistent. Therefore naked singularities will not form. Thus, what we will describe below is evidence in favor of a cosmic censorship, one with its origins in the quantum theory.

The purpose of this article is to put together some of the key results, making a coherent case for the preservation of the cosmic censorship hypothesis by the quantum theory. We will consider two dimensional models that are described by an action of the form

$$S = \int d^2x \sqrt{-g} \left[ e^{-2\phi} (-R + 4(\nabla \phi)^2 + \Lambda) - \frac{1}{2} \sum_i (\nabla f_i)^2 + 4\mu^2 e^{-2\phi} U(f_i) \right],$$

(1.1)

where $U(f_i)$ is a consistent potential ascribed to the scalar fields. Consistency will mean the preservation of conformal invariance; this must be checked in the limit of weak dilaton coupling.\(^7\) We will show that the models classically admit both timelike and spacelike singularities, their behavior at infinity depending on the sign of the cosmological constant. They will then be analyzed in the semi-classical approximation. We first consider, in section II, the original CGHS model ($U(f_i) = 0$), but with a negative cosmological constant. A classical naked singularity appears, instead of a black hole, and we will argue that the naked singularity is unstable quantum mechanically, giving up all its energy in a burst so that an asymptotic observer sees only a burst of thermal radiation. In this model, the naked singularity is not “formed” in the sense of being in the future of the infalling matter. Instead, the matter shock wave and the singularity appear simultaneously in the spacetime.

To preserve scale and diffeomorphism invariance in the conformal gauge, the kinetic terms and the potential terms should get renormalized so that the full theory becomes a conformal field theory.\(^7\) In section III we consider what constraints must be imposed on the potential term, $U(f_i)$, so that the resulting theory is indeed a consistently quantizable conformal field theory. We will derive a simple consistency condition and present a solution. We will see that the sine-Gordon potential is a particular case of this solution and will go on to analyze the singularities induced by sine-Gordon solitons\(^{14}\) in section IV. The latter singularities are interesting even on the classical level. They are
neither purely spacelike nor are they purely timelike, being “hybrids” in the sense that they are timelike in some regions and spacelike in others: we will show that the nature of the Hawking radiation depends crucially on the asymptotic behavior of the singularities. Asymptotically spacelike singularities will “evaporate” while asymptotically timelike singularities, if they form, prefer to rapidly radiate their energy away at early times, or to “explode”. In section V we will examine a particularly interesting singularity consisting of two timelike pieces that are joined smoothly in the distant future by a spacelike line. This is an interesting object because the singularity is formed in the future of the infalling matter, and thus it represents a possible collapse scenario. However, we will hold that what seem to be natural initial conditions (no incoming flux of energy across $\mathcal{I}^-$ except for the soliton) leads to a violation of the weak energy conditions if quantum effects are accounted for. On the other hand requiring that the weak energy condition is maintained in the semi-classical theory implies that the incoming soliton is necessarily accompanied by an incoming flux of Hawking energy. The effect on the history of the collapse of this change in initial conditions is examined in section VI which deals with the back reaction of the spacetime in the context of the full quantum theory. In this section we examine in greater detail two of the naked singularities described before, viz., the shock wave induced naked singularity of section II and the singularity formed in the future of the incoming soliton of the previous section. In the first case, that of the shock wave, the full quantum theory admits no positive mass singularities, spacelike or timelike, if the Hawking boundary conditions are imposed. Thus, in this model, quantum effects do prevent the formation of a naked singularity. In the second case, the two sets of boundary conditions described above are separately examined. When we require that there is no incoming Hawking radiation but permit a violation of the positive energy conditions, the naked singularity still exists but has changed its character and location. If we demand that the semi-classical theory has no violation of the positive energy conditions we are forced to allow incoming Hawking radiation and the singularity no longer forms. We conclude in section VII with a brief discussion of the implications of these results. In what follows our conventions are those of Weinberg\textsuperscript{15}

**II. Shock wave induced naked singularity**

Consider the action

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\[ S = \int d^2x \sqrt{-g} \left[ e^{-2\phi} \left( -R + 4(\nabla \phi)^2 + \Lambda \right) - \frac{1}{2} \sum_i (\nabla f_i)^2 \right] \]  

where \( \Lambda \) is the cosmological constant, \( f_i(x) \) are \( N \) conformally coupled matter fields and \( \phi \) is the dilaton. It is the potential free version of (1.1).

The model with a positive cosmological constant, \( \Lambda = +4\lambda^2 \) was initially analyzed by CGHS and shown to be unstable against gravitational collapse, admitting black hole solutions produced by incoming \( f \) shock waves. It was then shown that quantum effects appeared to cause the \( f \)-wave to radiate away its energy before the formation of the horizon, so that it would seem that black hole states are excluded from the quantum spectrum. CGHS simultaneously pointed out several problems (some of which have been mentioned in the introduction) associated with this naive conclusion and attempts were subsequently made not simply to eliminate these obstacles but also to construct some exactly solvable models of black hole evaporation.\(^8,9,16\)

We will now see that the model with a negative cosmological constant, \( \Lambda = -4\lambda^2 \) yields a naked singularity when the spacetime is perturbed by an incoming \( f \) shock wave. As usual, the metric equations of motion,

\[ 0 = \mathcal{T}_{\mu\nu} = e^{-2\phi} \left[ 2\nabla_\mu \nabla_\nu \phi - \frac{1}{2} e^{2\phi} \sum_i \nabla_\mu f_i \nabla_\nu f_i ight. \\
+ g_{\mu\nu} \left( -2\nabla^2 \phi + 2(\nabla \phi)^2 + 2\lambda^2 + \frac{1}{4} e^{2\phi} \sum_i (\nabla f_i)^2 \right) \right] \]

where \( \nabla_\mu \) is the covariant derivative compatible with \( g_{\mu\nu} \), form a set of constraints on the allowable solutions of the field equations, viz.,

\[ 4\nabla^2 \phi - 4(\nabla \phi)^2 - R - 4\lambda^2 = 0 \]
\[ -\nabla^2 f_i = 0. \]

The system is best analyzed in the conformal gauge, \( g_{\mu\nu} = e^{2\rho} \eta_{\mu\nu} \), where \( \eta_{\mu\nu} \) is the Lorentz metric in two dimensions. We will also use light-cone coordinates, \( x^\pm = x^0 \pm x^1 \) where the constraint equations become

\[ \mathcal{T}_{++} = e^{-2\phi} \left[ 2\partial_+ \partial_+ \phi - 4\partial_+ \phi \partial_+ \rho \right] - \frac{1}{2} \sum_i \partial_+ f_i \partial_+ f_i = 0 \]
\[ \mathcal{T}_{--} = e^{-2\phi} \left[ 2\partial_- \partial_- \phi - 4\partial_- \phi \partial_- \rho \right] - \frac{1}{2} \sum_i \partial_- f_i \partial_- f_i = 0 \]
\[ \mathcal{T}_{+-} = e^{-2\phi} \left[ -2\partial_+ \partial_- \phi + 4\partial_+ \phi \partial_- \phi - \lambda^2 e^{2\rho} \right] = 0 \]
and the equations of motion are

\[
- 4 \partial_+ \partial_\phi + 4 \partial_+ \phi \partial_- \phi + 2 \partial_+ \partial_- \rho - \lambda^2 e^{2 \rho} = 0 \\
+ \partial_+ \partial_- f_i = 0.
\] (2.5)

Combining the first equation in (2.5) with the last in (2.4) shows that \( \rho(x) \) is the same as \( \phi(x) \) up to a harmonic function \( h(x) \). Because the conformal gauge does not fix the conformal subgroup of diffeomorphisms, a choice of \( h(x) \) is essentially a choice of coordinate system. If we work in the Kruskal gauge, with \( h(x) = 0 \), then the solution to the equations of motion, satisfying all the constraints (with \( f_i(x) = 0 \)) is unique up to an additive constant,

\[
e^{-2\phi} = e^{-2\rho} = \sigma = \lambda^2 x^+ x^- + \frac{M}{\lambda}.
\] (2.6)

We show below that it represents a naked singularity with Bondi mass \( M \) which we take to be greater than zero. When \( M \) is zero the spacetime is flat and the dilaton is linear in the spatial coordinate so that this solution is generally referred to as the linear dilaton vacuum. On the other hand, when \( M \neq 0 \), the curvature is given by

\[
R = + 2e^{-3\rho} \nabla^2 e^\rho - 2e^{-4\rho} (\nabla e^\rho)^2 \\
= + 4 \left[ \partial_+ \partial_- \sigma - \frac{\partial_+ \sigma \partial_- \sigma}{\sigma} \right] \\
= + \frac{4\lambda M}{\sigma}.
\] (2.7)

It is singular at \( \sigma(x) = 0 \) and the singularity is timelike. Its Penrose diagram is shown in figure I, in which the section of spacetime with \( \sigma > 0 \) is labeled I and II.

This spacetime admits a Killing vector \( \vec{\xi} \) given by

\[
(\xi^+, \xi^-) = \lambda (x^+, -x^-)
\] (2.8)

which is spacelike in region I of the diagram and timelike in region II. The ADM mass itself has no meaning, as spatial infinity is cut off by the singularity, and in such cases one seeks to define the mass, not at \( i^0 \) but along \( I^+ \) as follows. The existence of the Killing
vector implies the existence of a conserved current

\[ j_\mu = T_\mu \nu \xi^\nu. \]  

(2.9)

Let \( t_{\mu \nu} \) be a linearization of \( T_{\mu \nu} \) about the dilaton vacuum so that, to first order, \( j_\mu = t_{\mu \nu} \xi^\nu \) and consider the solution of the dilaton field that is asymptotic to the vacuum:

\[
\begin{align*}
\phi(x) &= \phi^{(0)}(x) + \delta \phi(x) \\
\phi^{(0)} &= -\frac{1}{2} \ln(\lambda^2 x^+ x^-)
\end{align*}
\]  

(2.10)

so that the current in (2.9) takes the form

\[
\begin{align*}
(j_+ + j_-) &= 2\lambda \partial_+ \left( e^{-2\phi^{(0)}} \left[ \delta \phi + x^+ \partial_+ \delta \phi + x^- \partial_- \delta \phi \right] \right) \\
(j_+ - j_-) &= 2\lambda \partial_- \left( e^{-2\phi^{(0)}} \left[ \delta \phi + x^+ \partial_+ \delta \phi + x^- \partial_- \delta \phi \right] \right).
\end{align*}
\]  

(2.11)

Conservation of this current, \( \nabla \cdot j = 0 \), implies the existence of two charges,

\[
\begin{align*}
Q^+ &= \int_{I_L^+} dx^- j_- \\
Q^- &= \int_{I_R^+} dx^+ j_+
\end{align*}
\]  

(2.12)

which are constant along \( x^+ \) and \( x^- \) respectively. The current densities are total derivatives and their integrals can be measured as surface terms on \( I^\pm \). For example,

\[
Q^- = 2\lambda \left( e^{-2\phi^{(0)}} \left[ \delta \phi + x^+ \partial_+ \delta \phi + x^- \partial_- \delta \phi \right] \right)_{I_R^+} = + M > 0.
\]  

(2.13)

is the Bondi mass on \( I_R^+ \) and similarly for \( I_L^+ \).

To introduce dynamical matter fields into the problem, consider a shock wave of incoming matter travelling in the \( x^- \) direction with strength \( a \) at constant \( x^+ = x_0^+ \)

\[
\frac{1}{2} \partial_+ f \partial_+ f = a \delta(x^+ - x_0^+).
\]  

(2.14)

The equations of motion are solved by

\[
\sigma = \lambda^2 x^+ x^- - a(x^+ - x_0^+) \Theta(x^+ - x_0^+),
\]  

(2.15)

which is the linear dilaton vacuum when \( x^+ < x_0^+ \), and the naked singularity with mass \( M = ax_0^+ \lambda \) when \( x^+ > x_0^+ \). The resulting metric and curvature are identical to those
in (2.6) and (2.7), with a shift in retarded time, \(x^− \rightarrow x^− - a/\lambda^2\). The mass of the singularity on \(I^+_R\) is now \(a\lambda x^+_0\) and the Penrose diagram for the spacetime is given in figure II. We should note that this solution does not actually describe the formation of a naked singularity as the latter appears together with the incoming shock wave and not in its future, contrasting with the scenario for the formation of a black hole.

As in any two dimensional system, Wald’s axioms and the trace anomaly can be used to study semi-classical quantum effects in this model. In two dimensions the conservation equations

\[
\nabla_\mu \langle T_{\mu\nu} \rangle = 0 \tag{2.16}
\]

and the trace anomaly

\[
\langle T_{\mu}^\mu \rangle = -\alpha R \tag{2.17}
\]

together completely determine the stress tensor of the matter fields up to a boundary condition dependent term. The trace anomaly, (2.17), gives the one loop correction to the stress energy tensor and \(\alpha\) is a positive spin dependent constant (it is \(1/24\pi\) for scalar fields). Integrating (2.16), using (2.17) then gives

\[
\langle T_{++} \rangle = -\int \frac{dx^-}{\sigma} \partial_+ (\sigma \langle T_+^- \rangle) + A(x^+) \\
\langle T_{--} \rangle = -\int \frac{dx^+}{\sigma} \partial_- (\sigma \langle T_+^- \rangle) + B(x^-) \tag{2.18}
\]

where \(A(x^+)\) and \(B(x^-)\) are the boundary condition dependent terms mentioned before and which we fix by the following physical requirements: (a) there should be no incoming radiation and (b) the stress tensor should vanish exactly in the linear dilaton vacuum (for \(x^+ < x^+_0\)). These conditions imply that

\[
A(x^+) = \frac{1}{2x^+2}, \quad B(x^-) = \frac{1}{2x^-2} \tag{2.19}
\]

In a coordinate system in which the metric is asymptotically flat, defined by

\[
x^+ = \frac{1}{\lambda} e^{\lambda \sigma^+} \\
x^- = \frac{1}{\lambda} e^{\lambda \sigma^-} + \frac{a}{\lambda^2} \tag{2.20}
\]
we find that \( \langle T_{\mu\nu} \rangle \) has the following behavior

\[
\begin{align*}
\langle T_{++}^{(\sigma)} \rangle & \to 0, \\
\langle T_{+-}^{(\sigma)} \rangle & \to 0 \\
\langle T_{--}^{(\sigma)} \rangle & = \frac{\alpha \lambda^2}{2} \left[ 1 - \frac{1}{(1 + (a/\lambda)e^{-\lambda \sigma})^2} \right].
\end{align*}
\]

on \( I^+_R \) and all components of the tensor vanish on \( I^+_L \).

\( \langle T_{--}^{(\sigma)} \rangle \) is the flux across \( I^+_R \) and its behavior is remarkably different from that calculated by CGHS for the black hole. It approaches a maximum, \( \alpha \lambda^2/2 \) in the far past of \( I^+_R \) as \( \sigma^- \to a/\lambda^2 \) and decreases smoothly to zero as \( i^+ \) is approached, i.e., as \( x^- \to \infty \). The total energy radiated is, of course, the integrated flux, \( \int_{-\infty}^{\sigma^-} d\sigma^- \langle T_{--}^{(\sigma)} \rangle \), but this is infinite because the flux approaches a steady state in the infinite past, as opposed to black hole evaporation in the same model where the flux is shown to increase steadily from zero on \( I^+_R \) to a steady state independent of the mass of the hole as the horizon of the hole is approached. Infinite energy cannot be radiated from a system with a finite amount of energy and the paradox expresses the fact that, beyond a certain point, one must account for the back reaction of the spacetime. It is safe, however, to say that this is at least an indication that naked singularities, if they form, tend to radiate cataclysmically (“explode”), as opposed to black holes that prefer to radiate slowly or “evaporate.”

The back reaction of the spacetime becomes important when the dilaton coupling gets large, that is the size of the dilaton coupling indicates the extent to which the effects of the spacetime itself on the matter system should be taken into account. However, the value of the dilaton field at the point at which the singularity bursts into radiation

\[
e^\phi = \frac{1}{\sqrt{a x^+_0}}
\]

depends on the inverse square root of its Bondi mass and consequently is small when the classical singularity is very massive. The one loop approximation above, ignoring the back reaction, therefore seems valid in this limit. However, as the mass decreases due to the evaporation, the dilaton coupling will grow and the above expressions will break down.

**III. Consistent Potentials**
It would be nice to see if the general features of the Hawking radiation from both spacelike and timelike singularities are reproduced in other models of two dimensional gravity. These models must be consistent generalizations of the CGHS model, in the sense that consistency means maintaining scale and diffeomorphism invariance as explained in the introduction. This leads to a consideration of what form the full quantum theory might take.

To incorporate quantum effects to lowest order in studying the back reaction, one must include the contribution of the conformal anomaly (which arises because the measure on the space of matter fields is non-invariant) in the effective action as we have done above. However, a consistent quantization of the theory in the conformal gauge requires that both the kinetic and potential terms get renormalized in such a manner that the theory becomes a conformal field theory. The requirements of scale and diffeomorphism invariance have been checked for the CGHS model by de Alwis and by Bilal and Calan.\textsuperscript{7}

If \( f \)-field potentials are included as in (1.1), their analysis must be modified and will yield a restriction on the allowed functionals \( U(\tilde{f}) \). Below we derive the condition that makes for a consistent theory. Our arguments will closely follow those of de Alwis in ref.[7]

In general terms, we can expect that the full quantum action, including matter but not including ghosts takes the form (in terms of some fiducial metric \( g = e^{2\rho} \hat{g} \))

\[
S[X, \hat{g}] = \int d^2x \sqrt{\hat{g}} \left[ -\frac{1}{2} \hat{g}^{\mu\nu} G_{ab} \nabla_\mu X^a \nabla_\nu X^b - \hat{R} \Phi + T(X) \right] \tag{3.1}
\]

where \( T \) is the tachyon potential and \( X^a \) is the \( N+2 \) dimensional vector

\[
X^a = \begin{bmatrix} \phi \\ \rho \\ f_1 \\ \vdots \\ \vdots \\ f_N \end{bmatrix}
\tag{3.2}
\]

including the \( N \) matter fields. The action in (1.1) is to be viewed as the weak coupling limit of (3.1), i.e., in the limit \( e^{2\phi} << 1 \) and including the anomaly term, (3.1) should
look like

\[
S[\phi, \rho, f_i] = \int d^2x \sqrt{g} \left[ e^{-2\phi} \left( 4(\nabla^2 \phi) - 4\nabla_\phi \cdot \nabla_\rho \right) - \alpha (\nabla_\rho)^2 \right. \\
- \hat{R} \left( e^{-2\phi} - \alpha \rho \right) + \Lambda e^{2(\rho - \phi)} \\
- \frac{1}{2} \sum_i (\nabla f_i)^2 + 4\mu^2 e^{2(\rho - \phi)} U(\vec{f}) \right]
\]

Comparing (3.1) and (3.3), we find that the low energy limit is given by

\[
G_{ab} = \begin{bmatrix}
-8e^{-2\phi} & 4e^{-2\phi} & 0 & \ldots & 0 \\
4e^{-2\phi} & 2\alpha & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & \ldots \\
0 & 0 & 0 & 1 & \ldots \\
\vdots & \ldots & \ldots & \ldots & \ldots \\
\vdots & \ldots & \ldots & \ldots & 1
\end{bmatrix}
\]

\[
\Phi = + \alpha \rho - e^{-2\phi}
\]

\[
T(X) = \Lambda e^{2(\rho - \phi)} + 4\mu^2 e^{2(\rho - \phi)} U(\vec{f}).
\]

Now consider a generalization of the above field space metric, but with the appropriate limit

\[
ds^2 = -8e^{-2\phi}(1+h_1(\phi))d\phi^2 + 8e^{-2\phi}(1+h_2(\phi))d\phi d\rho + 2\alpha(1+h_3(\phi))d\rho^2 + \sum_i df_i^2
\]

where \(h_i(\phi)\) are \(O(e^{2\phi})\) functionals of the dilaton, \(\phi\). We will restrict ourselves to the case \(h_3(\phi) = 0\), which leads to an exactly solvable model. Then the metric can be brought to a flat form by the following field redefinitions:

\[
\Omega = \sqrt{2|\alpha|} \left[ \rho - \frac{1}{\alpha} e^{-2\phi} + \frac{2}{\alpha} \int e^{-2\phi} h_2(\phi) d\phi \right]
\]

\[
\chi = \int P(\phi) d\phi
\]

where

\[
P(\phi) = \frac{e^{-2\phi}}{\sqrt{|\alpha|}} \left[ 8\alpha e^{2\phi}(1 + h_1) + 8(1 + h_2)^2 \right]^{\frac{1}{2}}
\]
which give for the metric in (3.5)

\[ ds^2 = \mp d\chi^2 \pm d\Omega^2 + \sum_i df_i^2. \quad (3.8) \]

where the upper sign is to be used for \( \alpha > 0 \) and the lower sign for \( \alpha < 0 \). In the weak field limit,

\[ \Omega \sim \sqrt{2|\alpha|} \left( \rho - \frac{1}{\alpha} e^{-2\phi} + \ldots \right) \]
\[ \chi \sim \frac{2}{|\alpha|} \left( \frac{\alpha}{2} \phi - \frac{1}{2} e^{-2\phi} + \ldots \right) \quad (3.9) \]

Let us now consider the beta-function equations for the action in (3.1)

\[ 0 = \beta_{ab}^G = -R_{ab} + 2\nabla_a \nabla_b \Phi - \nabla_a T \nabla_b T + \ldots \]
\[ 0 = \beta^\Phi = -R + 4G^{ab} \nabla_a \Phi \nabla_b \Phi - 4\nabla^2 \Phi + \frac{N - 24}{3} + \ldots \quad (3.10) \]
\[ 0 = \beta^T = -2\nabla^2 T + 4G^{ab} \nabla_a \Phi \nabla_b T - 4T + \ldots \]

It is easy to see, using (3.8), that the first of the above equations implies that \( \Phi \) is a linear function of the fields, and to have the appropriate limit it must be uniquely given by

\[ \Phi = \pm \sqrt{\frac{|\alpha|}{2}} \Omega \quad (3.11) \]

The second beta function equation gives trivially \( \alpha = (24 - N)/6 \), and the third implies that

\[ \pm \partial^2 \chi T = \partial^2 T \pm \sqrt{2|\alpha|} \partial_{\Omega} T - \sum_i \partial^2 \chi T^i T - 2T \quad = 0 \quad (3.12) \]

where we have retained only first order terms in the tachyon potential. Consider, first, that \( T(X) \) depends only on the fields \( (\chi, \Omega) \). Obviously, a solution that will reduce to the appropriate one in the limit of weak coupling is

\[ T(\chi, \Omega) = \Gamma e^{\sqrt{|\alpha|} (\Omega + \chi)} \sim \Gamma e^{2(\rho - \phi)} \quad (3.13) \]

where \( \Gamma \) is some constant.
Including the $f$ fields, we can now obtain a condition for what constitutes a consistent potential, coupled as in (1.1) to the metric. Thus, if $T$ is of the form

$$T(X) \sim e^{\sqrt{\frac{2}{|\alpha|} (\Omega \mp \chi)}} U(\vec{f})$$

(3.14)

i.e., if $T(\chi, \Omega, f_i)$ has the same coupling to the $\rho$ and $\phi$ fields as the cosmological term, then (3.12) can be satisfied only if $U(\vec{f})$ is harmonic,

$$\sum_i \partial^2_i U(\vec{f}) = 0$$

(3.15)

The general solution can therefore be expressed as arbitrary sums over functions of the form

$$U_k(\vec{f}) = e^{i\vec{\omega}_k \cdot \vec{f}}$$

(3.16)

where the $\vec{\omega}_k$ satisfy $\omega^2_k = 0$. For example, if one restricts the number, $N$, of fields, $f_i$, to be even, the potential

$$U(\vec{f}) = \prod_{i=1}^{N/2} \cos f_i \prod_{j=N/2+1}^N \cosh f_j$$

(3.17)

satisfies the condition (3.15). Now (3.12) is linear in $T$ (to this approximation), so

$$T(X) = \Lambda e^{2(\rho - \phi)} + 4\mu^2 e^{2(\rho - \phi)} \left( \prod_{i=1}^{N/2} \cos f_i \prod_{j=N/2+1}^N \cosh f_j - 1 \right)$$

(3.18)

is a consistent potential.

**IV. Singularities from sine-Gordon Solitons**

Before going on to examine (3.1) in detail, we note that the sine-Gordon theory is a special case of (3.18). This follows directly by taking $f_i = 0$ for $i \geq 2$. Therefore, here we will examine the singularities induced by sine-Gordon solitons in the Witten model. They are qualitatively different from the singularity discussed in section II being neither purely timelike nor purely spacelike but a combination of the two. Thus they combine both black holes and naked singularities, but the Hawking evaporation of the solitons
will depend only on their asymptotic properties. Consider the action in (1.1) with the sine-Gordon potential,

$$U(f) = (\cos f - 1),$$  \hspace{1cm} (4.1)$$

setting $f = f_1 \neq 0$ and $f_i = 0$ for all $i \geq 2$ in (3.18). As in section II, we will treat the action with a positive cosmological constant, $\Lambda = +4\lambda^2$, and with a negative cosmological constant, $\Lambda = -4\lambda^2$, separately. When $\Lambda = 4\lambda^2$, the classical constraints and equations of motion can be written in light cone coordinates as follows

$$0 = T_{++} = e^{-2\phi} \left[ -4\partial_+ \rho \partial_+ \phi + 2\partial_+^2 \phi \right] - \frac{1}{2}(\partial_+ f)^2$$

$$0 = T_{--} = e^{-2\phi} \left[ -4\partial_- \rho \partial_- \phi + 2\partial_-^2 \phi \right] - \frac{1}{2}(\partial_- f)^2$$

$$0 = T_{+-} = e^{-2\phi} \left[ -\partial_+ \partial_- \phi + 4\partial_+ \phi \partial_- \phi + \frac{\Lambda}{4} e^{2\rho} + \mu^2 e^{2\rho}(\cos f - 1) \right] - 4\partial_+ \partial_- \phi + 4\partial_+ \phi \partial_- \phi + 2\partial_+ \partial_- \rho + e^{2\rho} \left[ \frac{\Lambda}{4} + \mu^2(\cos f - 1) \right] + \partial_+ \partial_- f + \mu^2 e^{2(\rho - \phi)} \sin f = 0$$

with the solution in conformal gauge

$$f_{\text{kink}} = 4 \tan^{-1} e^{(\Delta - \Delta_0)}$$

$$\sigma = a + bx^+ + cx^- - \frac{\Lambda}{4} x^+ x^- - 2 \ln \cosh(\Delta - \Delta_0)$$  \hspace{1cm} (4.3)$$

in terms of $\Delta = \gamma_+ x^+ + \gamma_- x^-$, where

$$\gamma_\pm = \pm \mu \sqrt{\frac{1 \pm v}{1 \mp v}},$$  \hspace{1cm} (4.4)$$

$v$ is the velocity of the soliton, $f(x, t) = f(x + vt)$, $\Delta = \Delta_0$ is its center which we take without loss of generality to be greater than or equal to zero, and $a$, $b$ and $c$ are arbitrary constants. There is also the antikink solution

$$f_{\text{antikink}} = 4 \arctan e^{-(\Delta - \Delta_0)}$$

$$\sigma = -\frac{\Lambda}{4} x^+ x^- - 2 \ln \cosh(\Delta - \Delta_0)$$  \hspace{1cm} (4.5)$$

which may be analyzed along the same lines. To fix the constants $a$, $b$ and $c$ we require that the metric reduces to the linear dilaton vacuum in the absence of the incoming
soliton, i.e., in the limit as the soliton stress energy, $T_{\mu\nu}^f \to 0$, $\Delta_0 \to 0$. This gives for the solution in (4.3)

$$\sigma = -\frac{\Lambda}{4} x^+ x^- - 2 \ln \cosh(\Delta - \Delta_0). \quad (4.6)$$

The curvature singularity is at $\sigma(x) = 0$. When $\Lambda > 0$ (4.5) represents a positive energy singularity combining a white hole a timelike singularity and a black hole, all smoothly joined along the soliton center (by a white hole we mean a spacelike naked singularity). On the contrary, when $\Lambda < 0$ it represents two naked singularities smoothly joined at the soliton center. For positive cosmological constant, using the (timelike) Killing vector $\xi^\mu = (x^+, x^-)$ where

$$x^+ = x^+ + \frac{2\gamma^-}{\lambda^2}, \quad x^- = x^- + \frac{2\gamma^+}{\lambda^2} \quad (4.7)$$

and a linearization of $T_{\mu\nu}$ about the dilaton vacuum, the conserved charge or the Bondi mass of the singularity is found to be

$$M_R = 2\lambda \left( \ln 2 - \frac{2\mu^2}{\lambda^2} + \Delta_0 \right). \quad (4.8)$$

on $I_R^+$ and

$$M_L = 2\lambda \left( \ln 2 - \frac{2\mu^2}{\lambda^2} - \Delta_0 \right) \quad (4.9)$$

on $I_L^+$. Without any loss of generality we will assume that $\Delta_0 > 0$ throughout, in which case the soliton never actually enters the left region. The Kruskal diagram is displayed in figure III. The soliton is seen to emerge from the merging of a white hole, and a timelike singularity at $(x^+ = 0, x^- = \Delta_0/\gamma_-)$, the white hole extending from $(x^+ = 2\gamma_-/\lambda^2, x^- = -\infty)$ on $I_R^-$ to $(x^+ = 0, x^- = \Delta_0/\gamma_-)$ where it smoothly turns into a timelike line proceeding to $x^- = 0, x^+ = \Delta_0/\gamma_+$ at which point the soliton is reabsorbed. Here the singularity once again turns spacelike smoothly and reaches $I_R^+$ at $(x^+ = \infty, x^- = -2\gamma_+/\lambda^2)$. The singularities are asymptotically spacelike and the metric approaches

$$\sigma \to -\lambda^2 x^+ x^- + 2\Delta_0 - \frac{4\mu^2}{\lambda^2} + 2 \ln 2 \quad (4.10)$$

near $I_R^+$. Although the singularities in figure III have been drawn for a specific value of the parameters, their qualitative behavior does not differ if both the left and right Bondi masses are positive.
As $\Delta_0$ approaches zero (figure IV) the left and right singularities merge forming, in the limit, a white hole that extends from $x^+ = -2\gamma_-/\lambda^2$ on $\mathcal{I}_R^-$ to $x^- = 2\gamma_+/\lambda^2$ on $\mathcal{I}_L^-$ and a black hole that stretches from $x^- = -2\gamma_+ / \lambda^2$ on $\mathcal{I}_R^+$ to $x^+ = 2\gamma_- / \lambda^2$ on $\mathcal{I}_L^+$, intersecting at the origin. The length of the timelike naked singularity has shrunk to zero and the masses measured on both null infinities are now the same. Even if they are zero ($2\mu^2 = \lambda^2 \ln 2$) soliton energy and momentum is present throughout the spacetime.

The spacetime with negative cosmological constant ($\Lambda = -4\lambda^2$)

$$\sigma = \lambda^2 x^+ x^- - 2 \ln \cosh(\Delta - \Delta_0) \quad (4.11)$$

is shown in figure V. All singularities are timelike as they approach $\mathcal{I}^\pm$. In the top region the timelike singularity approaches $\mathcal{I}_R^+$ at $(x^+ = \infty, x^- = 2\gamma_+ / \lambda^2)$ and approaches $\mathcal{I}_L^+$ at $(x^+ = -2\gamma_- / \lambda^2, x^- = \infty)$. The two timelike sections merge at a white hole in the region where the soliton center enters the spacetime. The latter emerges at $(x^+ = \Delta_0 / \gamma_+, x^- = 0)$ and travels to $i^0$. The Bondi mass depends on whether the asymptotic observer is located on $\mathcal{I}_R^+$ or $\mathcal{I}_L^+$,

$$M_R = 2\lambda \left( \frac{2\mu^2}{\lambda^2} + \Delta_0 + \ln 2 \right) \quad (4.12)$$

and

$$M_L = 2\lambda \left( \frac{2\mu^2}{\lambda^2} - \Delta_0 + \ln 2 \right),$$

the qualitative behavior again being independent of the parameter values chosen if both masses are positive. The masses are equal when $\Delta_0 = 0$. In this limit one has two naked singularities, the first extending from $x^- = -2\gamma_+ / \lambda^2$ on $\mathcal{I}_L^-$ to $x^- = 2\gamma_+ / \lambda^2$ on $\mathcal{I}_R^+$ and the other from $x^+ = 2\gamma_- / \lambda^2$ on $\mathcal{I}_R^-$ to $x^+ = -2\gamma_- / \lambda^2$ on $\mathcal{I}_L^+$, intersecting at the origin (figure VI).

Though our emphasis here is on the quantum behavior of (asymptotically) timelike singularities, this model also provides asymptotically spacelike singularities and it is instructive to compare the quantum behaviors of the two. We shall therefore examine the Hawking radiation both for positive and negative cosmological constant. For positive cosmological constant, the classical stress tensor of the $f^-$ soliton (right hand quadrant of figure III) is exponentially vanishing on $\mathcal{I}^+$ so that Hawking radiation is the dominant effect there. Again, the only possible one loop contribution to the trace of the stress
energy tensor must be of the form $-\alpha R$, $R$ being the only available geometric invariant. Thus we expect the quantum correction to take the form

$$T^{(q)\mu}_{\mu} = -4\sigma T^q_{++} = -\alpha R = -4\alpha \sigma \partial_+ \partial_- \ln \sigma$$

(4.13)

for some (positive) dimensionless constant $\alpha$. The two conservation equations can once again be integrated and the full stress tensor expressed in terms of two arbitrary functions $A(x^+)$ and $B(x^-)$ as before in (2.18). A consistent solution should admit no incoming radiation on $\mathcal{I}_R^+$ other than any matter fields that might be present and vanish in the absence of the soliton, that is, in the linear dilaton vacuum. The stress tensor satisfying these conditions is

$$\langle T_{++} \rangle = T^f_{++} + T^q_{++} = T^f_{++} - \alpha \left( \frac{\partial^2 \sigma}{\sigma} - \frac{1}{2} \left[ \frac{\partial_+ \sigma}{\sigma} \right]^2 \right) - \frac{\alpha}{2x'^2}$$

$$\langle T_{--} \rangle = T^f_{--} + T^q_{--} = T^f_{--} - \alpha \left( \frac{\partial^2 \sigma}{\sigma} - \frac{1}{2} \left[ \frac{\partial_- \sigma}{\sigma} \right]^2 \right) - \frac{\alpha}{2x^-^2}$$

(4.14)

$$\langle T_{+-} \rangle = T^f_{+-} + \alpha \partial_+ \partial_- \ln \sigma$$

where $x'^+ = x^+ + 2\gamma_-/\lambda^2$. It is most convenient to analyze the above expressions in the coordinate system in which the metric is manifestly asymptotically flat. Define, therefore the coordinates $\sigma^\pm = t \pm x$ by

$$x^+ = \frac{1}{\lambda} e^{\lambda \sigma^+} - \frac{2\gamma_-}{\lambda^2}$$

$$x^- = -\frac{1}{\lambda} e^{-\lambda \sigma^-} - \frac{2\gamma_+}{\lambda^2}$$

(4.15)

Thus $\sigma^+ \to \infty$ corresponds to the lightlike surface $x^+ = \infty$ and $\sigma^- \to \infty$ to the lightlike surface $x^- = -\infty$, while $\sigma^- \to \infty$ corresponds to the lightlike surface $x^- = -2\gamma_-/\lambda^2$.

Transforming the expressions in (4.14) to the new system, we find that

$$\langle T^{(\sigma)}_{++} \rangle \to 0, \quad \langle T^{(\sigma)}_{+-} \rangle \to 0$$

(4.16)

and

$$\langle T^{(\sigma)}_{--} \rangle \to \frac{\alpha \lambda^2}{2} \left[ 1 - \frac{1}{\left( 1 + \frac{2\gamma_+}{\lambda^2} e^{\lambda \sigma^-} \right)^2} \right].$$

(4.17)

Notice that the form of the solution is precisely the same as the solution for the collapsing shock wave. $\langle T^{(\sigma)}_{--} \rangle$ is the outgoing flux at $\mathcal{I}_R^+$. It grows smoothly from zero at $x^- = -\infty$ to
a maximum of $\alpha\lambda^2/2$ at $x^- = -2\gamma_+/\lambda^2$ on $I^+_R$ and depends on the soliton mass parameter but not on the mass of the singularity itself. This would seem to be a general feature of the Hawking evaporation from asymptotically spacelike singularities in these two dimensional models with collapsing matter, having been shown to be true for the radiation from a black hole formed by an incoming shock wave in the CGHS model. The integrated flux along $I^+_R$ is the total energy lost by the incoming soliton, but as the flux rapidly approaches its maximum value of $\alpha\lambda^2/2$, the integrated flux shows an infinite loss of energy if the integral is performed up to the future horizon at $x^- = -2\gamma_+/\lambda^2$. However, as CGHS pointed out, this is a consequence of having neglected the back reaction and must not be taken seriously. Instead, one can try to estimate the retarded time, $x^-_\tau$ at which the integrated Hawking radiation is equal to the mass of the singularity, $M = 4\lambda(\Delta_0 - 2\mu^2/\lambda^2 + \ln 2)$. We find

$$\int_{-\infty}^{\sigma_\tau} d\sigma^- \langle T_{\sigma^-}\rangle = \frac{\alpha\lambda}{2} \left[ 1 - \frac{1}{1 + \frac{2\gamma_+}{\lambda} e^{\lambda\sigma^-}} \right] + \ln \left( 1 + \frac{2\gamma_+}{\lambda} e^{\lambda\sigma^-} \right) = M \quad (4.18)$$

For a small mass singularity, the retarded time is given by

$$x^-_\tau = -\frac{2\gamma_+}{\lambda^2} \left( 1 + \frac{\alpha\lambda}{M} \right) \quad (4.19)$$

which, when naively traced backwards, corresponds to the point

$$(x^+_\tau, x^-_\tau) = \left( \frac{\Delta_0}{\gamma_+} + \frac{2\gamma_-}{\lambda^2} \left[ 1 + \frac{\alpha\lambda}{M} \right], -\frac{2\gamma_+}{\lambda^2} \left[ 1 + \frac{\alpha\lambda}{M} \right] \right) \quad (4.20)$$

on the soliton trajectory. If $x^+_\tau < 0$, the soliton energy has evaporated earlier than the appearance of its center in the spacetime. An observer on $I^+_R$ sees a white hole which rapidly radiates away all its energy. This of course is true only if the mass of the singularity is small. On the other hand, if $x^+_\tau$ is greater than zero, the soliton does enter the spacetime evaporating eventually by $x^-_\tau$ in (4.19). How reliable is the estimate of its lifetime above? Assuming that the soliton center does enter the spacetime before evaporating completely, the dilaton coupling constant at the turn around point,

$$e^\phi = \frac{1}{\sqrt{2 \left( 1 + \frac{\alpha\lambda}{M} \right) \left[ \frac{M}{4\lambda} - \ln 2 - \frac{2\mu^2\alpha}{\lambda^2 M} \right]}} \quad (4.21)$$

is large for a small mass singularity and signals the breakdown of the one loop approximation if the back reaction is ignored.
On the other hand, if $M$ is large,

$$\int_{-\infty}^{\sigma_\tau} d\sigma^- \langle T^-(\sigma) \rangle \sim \frac{\alpha \lambda}{2} \left[ \ln \frac{2\gamma_+}{\lambda} + \lambda \sigma_\tau \right] = M$$

(4.22)

or

$$x^-_\tau = -\frac{2\gamma_+}{\lambda^2} \left[ 1 + e^{-2M/\alpha \lambda} \right]$$

(4.23)

which, when traced back corresponds to the point

$$(x^+_\tau, x^-_\tau) = \left( \frac{\Delta_0}{\gamma_+} + \frac{2\gamma_-}{\lambda^2} \left[ 1 + e^{-2M/\alpha \lambda} \right], -\frac{2\gamma_+}{\lambda^2} \left[ 1 + e^{-2M/\alpha \lambda} \right] \right)$$

(4.24)

on the soliton trajectory. The dilaton coupling at this point has the value

$$e^\phi = \frac{1}{\sqrt{2 \left( 1 + e^{-2M/\alpha \lambda} \right) \left( \frac{M}{4\lambda} - \ln 2 - \frac{2\mu^2}{\lambda} e^{-2M/\alpha \lambda} \right)}}$$

(4.25)

and is small in the limit of large $M$. The soliton evaporates completely by the time the observer has reached the event horizon and the black hole never forms. Moreover this is the limit in which the one loop approximation is a satisfactory indication of what may actually be happening. Similar conclusions can be drawn in the CGHS (shock wave) model. In the low mass limit that the dilaton coupling is large at the turn around point, being proportional only to $\sqrt{1/\alpha}$. However, in the large mass limit the dilaton coupling behaves as the inverse square root of the mass. Thus, there seems to be no essentially new feature in the evaporation of the soliton.

Next consider an observer in the left quadrant of figure III. As we have mentioned, because of our choice of $\Delta_0 > 0$ the soliton center never enters this region and this observer lives in a universe inhabited only by the tail of the soliton energy and the singularity described earlier. The appropriate boundary conditions on the Hawking stress tensor are (a) its vanishing in the absence of the soliton and (b) no flux across $\mathcal{I}^-$. It follows that the quantum contribution to the stress tensor is given by

$$T^q_{++} = -\alpha \left( \frac{\partial^2 \sigma}{\sigma} - \frac{1}{2} \left[ \frac{\partial_+ \sigma}{\sigma} \right]^2 \right) - \frac{\alpha}{2x^+}$$

$$T^q_{--} = -\alpha \left( \frac{\partial^2 \sigma}{\sigma} - \frac{1}{2} \left[ \frac{\partial_- \sigma}{\sigma} \right]^2 \right) - \frac{\alpha}{2x^-}$$

$$T^q_{+-} = \alpha \partial_+ \partial_- \ln \sigma$$

(4.26)
where \( x' = x - 2\gamma_+ / \lambda^2 \). Going to the system \( \sigma^\pm = t \pm x \) given by

\[
\begin{align*}
  x^+ &= -\frac{1}{\lambda} e^{-\lambda \sigma^+} + \frac{2\gamma_+}{\lambda^2} \\
  x^- &= \frac{1}{\lambda} e^{\lambda \sigma^-} + \frac{2\gamma_-}{\lambda^2}
\end{align*}
\]

(4.27)

in which the metric is manifestly flat at null infinity, \( (\sigma^+ \to \infty \text{ corresponds to the lightlike line } x^+ = 2\gamma_- / \lambda^2 \) and \( \sigma^- \to -\infty \) to the lightlike line \( x^+ = 2\gamma_+ / \lambda^2 \) while \( \sigma^+ \to -\infty \) and \( \sigma^- \to \infty \) correspond to the respective lightlike infinities) we obtain (on \( I^+_L \))

\[
\langle T^{(\sigma)}_{--} \rangle \to 0, \quad \langle T^{(\sigma)}_{+-} \rangle \to 0 \quad (4.28)
\]

and

\[
\langle T^{(\sigma)}_{++} \rangle = \frac{\alpha \lambda^2}{2} \left[ 1 - \frac{1}{\lambda^2 e^{2\lambda \sigma^+}} \right] \quad (4.29)
\]

In this part of the world, the radiation also grows steadily from zero in the infinite past to its maximum value (of \( \alpha \lambda^2 / 2 \)) in the far future of \( I^+_L \). The integrated flux is infinite and, as we have argued before, that is a signal that the back reaction is to be taken into account.

We turn now to solitons in 2d gravity with a negative cosmological constant. We consider the observer in the top quadrangle. In the absence of \( I^- \) the only reasonable boundary condition one may impose upon the stress tensor is that it vanishes in the absence of the soliton, a limit we have defined earlier. The quantum corrections to the stress tensor then take the form

\[
\begin{align*}
  T^q_{++} &= -\alpha \left( \frac{\partial^2 \sigma}{\sigma} - \frac{1}{2} \left[ \frac{\partial_+ \sigma}{\sigma} \right]^2 \right) - \frac{\alpha}{2x^+} \\
  T^q_{--} &= -\alpha \left( \frac{\partial^2 \sigma}{\sigma} - \frac{1}{2} \left[ \frac{\partial_- \sigma}{\sigma} \right]^2 \right) - \frac{\alpha}{2x^-} \\
  T^q_{+-} &= \alpha \partial_+ \partial_- \ln \sigma
\end{align*}
\]

(4.30)

Again, to analyze the tensors it is convenient to go to a system in which the metric is...
asymptotically flat: define the system $\sigma^\pm = t \pm x$ by

$$
\begin{align*}
  x^+ &= \frac{1}{\lambda} e^{\lambda \sigma^+} - \frac{2\gamma_-}{\lambda^2}, \\
  x^- &= \frac{1}{\lambda} e^{\lambda \sigma^-} + \frac{2\gamma_+}{\lambda^2}
\end{align*}
$$

Thus, $\sigma^- \to -\infty$ corresponds to the lightlike line $x^- = 2\gamma_+ / \lambda^2$ and $\sigma^+ \to -\infty$ to the lightlike line $x^+ = -2\gamma_- / \lambda^2$ while $\sigma^\pm \to \infty$ correspond to the respective lightlike infinities.

The fluxes across both $\mathcal{I}_L^+$ and $\mathcal{I}_R^+$ are now non-vanishing, each approaching a maximum of $\alpha \lambda^2 / 2$ at early times and decreasing steadily in the far future, as $t^0$ is approached. Thus, on $\mathcal{I}_R^+$, for instance, we find

$$
\langle T^{(\sigma)}_{++} \rangle \to 0, \quad \langle T^{(\sigma)}_{+-} \rangle \to 0 \quad (4.32)
$$

and

$$
\langle T^{(\sigma)}_{-\gamma} \rangle = \frac{\alpha \lambda^2}{2} \left[ 1 - \frac{1}{\left( 1 + \frac{2\gamma_+}{\lambda} e^{-\lambda \sigma^-} \right)^2} \right], \quad (4.33)
$$

and on $\mathcal{I}_L^+$

$$
\langle T^{(\sigma)}_{-\gamma} \rangle \to 0, \quad \langle T^{(\sigma)}_{++} \rangle \to 0 \quad (4.34)
$$

and

$$
\langle T^{(\sigma)}_{+\gamma} \rangle = \frac{\alpha \lambda^2}{2} \left[ 1 - \frac{1}{\left( 1 - \frac{2\gamma_-}{\lambda} e^{-\lambda \sigma^+} \right)^2} \right]. \quad (4.35)
$$

The tensors are again independent of the mass of the singularities but depend on the soliton mass parameter. The integrated flux over any interval is again infinite because the flux itself approaches a steady state at early times. Of course this is a consequence of having neglected the back reaction of the radiation on the spacetime geometry.

This picture is also similar to that developed in the shock wave model. Even if quantum gravity does permit the formation of naked singularities, they will evaporate catastrophically ("explode") due to the Hawking radiation. To justify this statement, one must check the validity of the one loop picture by considering the strength of the dilaton...
coupling constant at the point on the soliton center at which the singularities are expected to detonate. On $\mathcal{I}_R^+$ this is at the retarded time $x^- = 2\gamma_+ / \lambda^2$ which, when traced back corresponds to the point $(x^+_+, x^-_+) = (1 / \gamma_+ (\Delta_0 + 2\mu^2 / \lambda), 2\gamma_+ / \lambda^2)$ on the soliton center and gives for the dilaton coupling

$$e^\phi = \frac{1}{\sqrt{2 \left( \frac{2\mu^2}{\lambda^2} + \Delta_0 \right)}}. \quad (4.36)$$

This is indeed small when the Bondi mass, $M_R$, is large, i.e., the soliton mass parameter is large. On the other hand, on $\mathcal{I}_L^+$ this is at the advanced time $x^+ = -2\gamma_- / \lambda^2$ which, when traced back, corresponds to the point $(x^+_-, x^-_-) = (-2\gamma_- / \lambda^2, 1 / \gamma_- (\Delta_0 - 2\mu^2 / \lambda))$ on the soliton center and gives for the dilaton coupling

$$e^\phi = \frac{1}{\sqrt{2 \left( \frac{2\mu^2}{\lambda^2} - \Delta_0 \right)}} \quad (4.37)$$

which is again small when the Bondi mass, $M_L$, is large (or the soliton mass parameter is large).

V. Formation of a Naked Singularity

We now come to the last remaining case, the asymptotically timelike singularity in the lower quadrant of figure V. This case is particularly interesting in as much as it provides a bonafide model of the “formation” of a naked singularity. The singularity is formed in the future, an observer being obstructed from reaching future null infinity, and this contrasts with our previous examples in which the naked singularity was seen to form simultaneously with the emergence of the soliton center. The analysis of the Hawking radiation in this case requires somewhat different considerations from those laid out above.

The lower branch is made up of a spacelike piece in the far future joined smoothly to two timelike singularities on either end, being null in the neighborhood of $\mathcal{I}^-$, and intersecting $\mathcal{I}_R^-$ at $x^- = -\infty, x^+ = 2\gamma_- / \lambda^2$ and $\mathcal{I}_L^-$ at $x^+ = -\infty, x^- = -2\gamma_+ / \lambda^2$. Classical soliton energy entering the spacetime therefore produces a naked singularity in the future. Accompanying the production of the singularity is Hawking radiation, but we expect that the stress tensor of the evaporation is regular at all points in the spacetime.
except at the singularity, $\sigma = 0$. Na"ively it may seem that natural conditions are (a) there is no incoming flux of energy other than the soliton’s and (b) the Hawking radiation vanishes in the absence of the classical soliton stress energy, i.e., when $\mu = 0 = \Delta_0$. However, the tensor satisfying these conditions is not regular everywhere. This is because requiring the absence of incoming energy on $I^-$ (other than the soliton, condition (a)) implies that

$$A(x^+) = -\frac{1}{2x^+^{\gamma^2}}, \quad B(x^-) = -\frac{1}{2x^-^{\gamma^2}} \quad (5.1)$$

where

$$x^+ = x^+ - \frac{2\gamma_-}{\lambda^2}, \quad x^- = x^- + \frac{2\gamma_+}{\lambda^2}, \quad (5.2)$$

As the bracketed term on the right hand side of (4.30) is finite everywhere, no incoming flux on $I^-$ implies that $\langle T_{++} \rangle \to -\infty$ on the lightlike line $x^+ = 0$ and $\langle T_{--} \rangle \to -\infty$ on the lightlike line $x^- = 0$. This behavior is physically unacceptable on the grounds that there is nothing special about the spacetime along these lines and, furthermore, infinite negative fluxes to the left and right are in violation of the positive energy conditions.

On the other hand the tensor that is regular everywhere within the spacetime (except at the singularity, $\sigma = 0$) and vanishes in the absence of the soliton is given by

$$\langle T_{++} \rangle = T_{++}^f + T_{++}^q = T_{++}^f - \alpha \left( \frac{\partial_+^2 \sigma}{\sigma} - \frac{1}{2} \left[ \frac{\partial_+ \sigma}{\sigma} \right]^2 \right) - \frac{\alpha}{2x^+^{\gamma^2}}$$

$$\langle T_{--} \rangle = T_{--}^f + T_{--}^q = T_{--}^f - \alpha \left( \frac{\partial_-^2 \sigma}{\sigma} - \frac{1}{2} \left[ \frac{\partial_- \sigma}{\sigma} \right]^2 \right) - \frac{\alpha}{2x^-^{\gamma^2}} \quad (5.3)$$

$$\langle T_{+-} \rangle = T_{+-}^f + \alpha \partial_+ \partial_- \ln \sigma.$$  

In coordinates that are manifestly asymptotically flat

$$x^+ = -\frac{1}{\lambda} e^{-\lambda \sigma^+} + \frac{2\gamma_-}{\lambda^2} \quad (5.4)$$

the tensor approaches

$$\langle T^{(\sigma)}_{--} \rangle \to 0, \quad \langle T^{(\sigma)}_{++} \rangle \to 0,$$

$$\langle T^{(\sigma)}_{+-} \rangle \to \frac{\alpha \lambda^2}{2} \left[ 1 - \frac{1}{\left( 1 - \frac{2\gamma_- e^{\lambda \sigma^+}}{\lambda^2} \right)^2} \right] \quad (5.5)$$
on $\mathcal{I}^-_R$ and

$$
\langle T^{(\sigma)}_{++} \rangle \to 0, \quad \langle T^{(\sigma)}_{+-} \rangle \to 0
$$

$$
\langle T^{(\sigma)}_{--} \rangle \to \frac{\alpha \lambda^2}{2} \left[ 1 - \frac{1}{\left( 1 + \frac{2\gamma_+ e^{\lambda \sigma}}{\lambda^2} \right)^2} \right]
$$

(5.6)
on $\mathcal{I}^-_L$. Therefore consistency seems to require an incoming flux of radiation across past null infinity. This flux is seen to increase smoothly from zero on $i^-$ to the constant value $\alpha \lambda^2/2$ as the lightlike lines $x^- = -2\gamma_+/\lambda^2$ and $x^+ = 2\gamma_-/\lambda^2$ are approached. The total energy flowing into the spacetime is the integrated flux from $i^-$ to the point $x^+ = 2\gamma_-/\lambda^2$ on the right and $x^- = -2\gamma_+/\lambda^2$ on the left. This is obviously infinite and it is easy to see that this will occur whenever the singularity cuts off a portion of $\mathcal{I}^-$ which in turn is possible only because of the timelike part of the singularity.

Thus the following scenario seems to emerge. Ignoring quantum effects, the observer would enter spacetime at $i^-$ accompanied by highly localized soliton energy and doomed to finally crash into a singularity. Before doing so he would have to cross $x^+ = 2\gamma_-/\lambda^2$ or $x^- = -2\gamma_+/\lambda^2$ or both. Once he has crossed these lines he would be able to receive information from the singularity and would realize that the cosmic censorship hypothesis could not hold in his universe. As we have seen, however, quantum effects will play an important role. The stress tensor that is regular throughout the spacetime except at the singularity itself indicates that on entering the spacetime the observer encounters a flux of incoming energy across $\mathcal{I}^-_L$ and $\mathcal{I}^-_R$ and accompanying the soliton. Neglecting the back reaction, the flux appears to increase steadily from $i^-$ leading to an infinite total energy entering the spacetime before the singularity becomes actually visible.

VI. The Back Reaction

In as much as naked singularities are concerned, the previous case studies fall in two categories. In the first, matter enters the spacetime together with the formation of an asymptotically naked singularity. The shock wave induced naked singularity and the soliton induced singularity in the upper quadrant fall in this category. In the second, a soliton attempts to form an asymptotically timelike singularity in the future. In this section we will briefly examine the back reaction in two specific examples of these categories: the singularity formed by a shock wave (the first case studied) and the attempt to form a naked singularity by an incoming soliton (the last of the cases studied earlier).
Equation (3.1) represents the full quantum action including matter but not including ghosts and reduces, in the weak coupling limit, to the CGHS action (modified to include the anomaly term). After the field redefinitions in (3.6), the action may be written in terms of the fields $\chi$ and $\Omega$ as follows,

\[
S = \int d^2x \left[ -\partial_+ \chi \partial_- \chi + \partial_+ \Omega \partial_- \Omega + \sum_i \partial_+ f_i \partial_- f_i \\
+ \frac{1}{2} \left( \Lambda + 4\mu^2 U(\vec{f}) \right) e^{\sqrt{\frac{2}{|\alpha|}}(\Omega \mp \chi)} \right]
\]

(6.1)

plus a ghost action. From the above action follow the equations of motion for $\chi$ and $\Omega$ as well as the constraints

\[
\partial_+ \partial_- \Omega = \pm \partial_+ \partial_- \chi = \sqrt{\frac{2}{|\alpha|}} \left( \frac{\Lambda}{4} + \mu^2 U(\vec{f}) \right) e^{\sqrt{\frac{2}{|\alpha|}}(\Omega \mp \chi)}
\]

\[
\partial_+ \partial_- f_i = \mu^2 e^{\sqrt{\frac{2}{|\alpha|}}(\Omega \mp \chi)} \frac{\partial U(\vec{f})}{\partial f_i}
\]

\[
T_{\pm\pm} = \pm \frac{1}{2} \partial_+ \chi \partial_- \chi \pm \frac{1}{2} \partial_+ \Omega \partial_- \Omega + \frac{1}{2} \sum_i \partial_+ f_i \partial_- f_i + \sqrt{\frac{|\alpha|}{2}} \partial_\pm \Omega + t_{\pm\pm}
\]

(6.2)

where $t_{\pm\pm}$ is the ghost stress energy and $U(\vec{f})$ is the matter field potential, being zero for the shock wave, and

\[
U(\vec{f}) = \left( \prod_{i=1}^{N/2} \cos f_i \prod_{j=N/2+1}^{N} \cosh f_j - 1 \right)
\]

(6.3)

for the soliton solutions described in sections IV and V.

We are now in a position to derive the exact solutions to quantum dilaton gravity in two dimensions. Evidently we may choose $\chi(x^+, x^-) = \pm \Omega(x^+, x^-)$ and the solution which obeys the constraints $T_{\mu\nu} = 0$, with $t_{\pm\pm} = 0$, may be written down directly in comparison with the solutions given in the previous sections for the weak coupling limit. In all cases

\[
\Omega(x^+, x^-) = \pm \chi(x^+, x^-) = -\sqrt{\frac{2}{|\alpha|}} \Gamma(x^+, x^-)
\]

(6.4)
where
\[
\Gamma(x^+, x^-) = \begin{cases} 
-\frac{A}{4} x^+ x^- - a(x^+ - x_0^+) \Theta(x^+ - x_0^+) \\
-\frac{A}{4} x^+ x^- - 2 \ln \cosh(\Delta - \Delta_0)
\end{cases}
\] (6.5)

where the first is for the incoming shock wave and the second for the sine-Gordon solitons.

The solutions are the same as in the CGHS theory except that now they are given in terms of \( \chi \) and \( \Omega \) which incorporate all the quantum effects. In the limit of small coupling, equation (3.9) implies that \( \chi = \Omega \sim -e^{-2\rho} \sim -e^{-2\phi} \). This is to be expected of course and it gives the classical solution admitting the singularities that have already been described, but the singularities appear in the strong coupling regime and the strong coupling expansion of (3.6) and (3.7) should have been used. Moreover, the quantum matter stress energy contains a boundary condition dependent piece (the contribution from the conformal anomaly has already been accounted for) so that, if the constraint \( T_{\mu\nu} = 0 \) is to be maintained, the ghost contribution must cancel this contribution and therefore cannot be zero. Therefore the solutions in (6.4) and (6.5) need to be modified to account for the boundary condition dependence of the Hawking radiation. For definiteness, we will take \( \alpha > 0 \ (N < 24) \) in what follows. Referring back to equation (3.6) we are reminded of the fact that the fields, \( \chi \) and \( \Omega \), are defined in terms of the dilaton and conformal factor but that this definition depends rather strongly on the way in which the field space metric, \( G_{ab} \), and the tachyon, \( T(X) \), are renormalized. There are many possibilities and we shall consider but one such here, taking
\[
h_1(\phi) = -\frac{3\alpha}{4} e^{2\phi}, \quad h_2(\phi) = -\alpha e^{2\phi}
\] (6.6)

which amounts to a renormalization of the field space metric but not the tachyon and has the advantage of making the square-rooted term in the definition of \( P(\phi) \) a perfect square. This model is similar to that of Russo, Susskind and Thorlacius, and it gives
\[
P(\phi) = 2\sqrt{2\alpha e^{-2\phi} \left[ 1 - \frac{\alpha}{2} e^{2\phi} \right]}
\]
\[
\chi = \int d\phi P(\phi) = -\sqrt{2\alpha} \left[ \phi + \frac{e^{-2\phi}}{\alpha} \right]
\]
\[
\Omega = \sqrt{2\alpha} \left[ \rho - 2\phi - \frac{e^{-2\phi}}{\alpha} \right] \] (6.7)

so that the solution \( \chi = \Omega \) implies that \( \rho = \phi \) and \( \Omega - \chi = \sqrt{2\alpha}(\rho - \phi) \).
Now there is a certain arbitrariness in the choice of the ghost stress energy, which is associated with the freedom to choose the boundary conditions. One possible way to fix this is to prescribe its form in such a way as to be consistent with the semi-classical Hawking radiation on $I^+$. This was seen to approach a constant at early retarded times and decay to zero in the far future, so that the integrated flux, the total energy leaving the spacetime from any point $x^-$ on $I^+_R$ to future timelike infinity, $i^+$, is then

$$\int_{\sigma^-}^{\infty} d\sigma^- \langle T^{(\sigma)} \rangle = \frac{\alpha \lambda}{2} \left[ \ln \left( 1 + \frac{a}{\lambda} e^{-\lambda\sigma^-} \right) + \frac{1}{1 + \frac{a}{\lambda} e^{\lambda\sigma^-}} \right] \quad (6.8)$$

in the double null coordinates of (2.20).

Requiring that the solution reproduce the Hawking boundary conditions means that the Bondi mass, measured on $I^+$, should decrease along $I^+_R$ at the same rate as the flux of outgoing Hawking radiation, either becoming infinitely negative by the time $i^+$ is approached if the initial mass is finite, or possessing an infinite initial mass if the final mass (on $i^+$) is finite. Moreover, to be consistent with the semi-classical calculation, the mass on $I^+_L$ should be constant, and the total energy along both null infinities must be conserved. The following choice for the ghost stress energy

$$t_{++} = \frac{\alpha}{2x^+}, \quad t_{--} = \frac{\alpha}{2x^-}, \quad (6.9)$$

where

$$x^-' = x^- - \frac{a}{\lambda^2} \Theta(x^+ - x^+_0) \quad (6.10)$$

satisfies all these conditions. It leads to the solution for $\chi(x^+, x^-)$ obeying the modified constraints

$$\Omega = \chi = -\sqrt{\frac{2}{\alpha}} \left( \lambda^2 x^+ x^- - a(x^+ - x^+_0) \Theta(x^+ - x^+_0) - \frac{\alpha}{2} \ln(\lambda^2 x^+ x^-) \right) \quad (6.11)$$

In order to compute the Bondi mass of the solution following the arguments of section II, we need a reference solution, $\Omega^{(0)}$, and an expansion about it of the form $\Omega = \Omega^{(0)} + \delta\Omega$. It is natural to take this reference to be the linear dilaton solution, which is the value of
$\Omega$ in the absence of the incoming shock wave,

$$\Omega^{(0)} = -\sqrt{\frac{2}{\alpha}} \left[ \lambda^2 x^+ x^- - \frac{\alpha}{2} \ln(\lambda^2 x^+ x^-) \right]$$  \hspace{1cm} (6.12)

Comparing it with the full solution in (6.11) we then get

$$\delta \Omega = -\sqrt{\frac{2}{\alpha}} \left[ \frac{M}{\lambda} - a x^+ - \frac{\alpha}{2} \ln(\frac{x^-}{x^-}) \right]$$  \hspace{1cm} (6.13)

whereupon, using the stress tensor in (6.2), the charge evolving the system along $I^+_R$ are seen to be given by

$$Q^- = \lambda \sqrt{\frac{\alpha}{2}} \left[ -\delta \Omega + x^+ \partial_+ \delta \Omega + x^- \partial_- \delta \Omega \right]_{I^+_R}$$

$$= \lambda \left[ \frac{M}{\lambda} + \frac{\alpha}{2} \ln(\frac{x^-}{x^-}) + \frac{\alpha}{2} \left( 1 - \frac{x^-}{x^-} \right) \right]$$  \hspace{1cm} (6.14)

First we remark that the constant $M$ is the Bondi mass measured by an observer on $i^+$. The charge is a constant on $I^+_L$, and infinite in the retarded past on $I^+_R$, at $x^- = a/\lambda^2$. This behavior is required to reproduce the semi-classical Hawking radiation given in section II: an evaporating singularity that gives up an infinite amount of energy to end up with a finite mass must have been infinitely massive to begin with. The total energy, including the integrated flux, is conserved. This is easy to see by rewriting the expression in (6.14) in terms of the coordinates $\sigma^\pm$,

$$Q^- = M(\sigma^-) = M + \frac{\alpha \lambda}{2} \left[ \ln \left( 1 + \frac{a}{\lambda} e^{-\lambda \sigma^-} \right) + \frac{1}{1 + \frac{a}{\lambda} e^{-\lambda \sigma^-}} \right]$$  \hspace{1cm} (6.15)

and one recognizes the last two terms on the right hand side as the integrated flux over the interval $\sigma^-$ to $\infty$ on $I^+_R$, given in (6.8). In fact, the Hawking flux across $I^+_R$ may be calculated as the negative of the rate of change of the mass in (6.15),

$$- \frac{dM(\sigma^-)}{d\sigma^-} = \frac{\alpha \lambda^2}{2} \left[ 1 - \frac{1}{(1 + \frac{a}{\lambda} e^{-\lambda \sigma^-})^2} \right]$$  \hspace{1cm} (6.16)

which is the proper way to think of the Hawking effect in the full theory.
When $M = 0$, (6.11) represents a shifted dilaton vacuum. When $M \neq 0$, a singularity is formed at $\chi = \chi_{\text{min}}$.

\[
\frac{\alpha}{2} \ln \left( \frac{2e}{\alpha} \right) = \lambda^2 x^+ x^- + \frac{M}{\lambda} - \frac{\alpha}{2} \ln(\lambda^2 x^+ x^-).
\]  

Clearly there are no solutions for $M > 0$ because the minimum of the sum of the first and last term in (6.17) is precisely equal to the left hand side. For $M < 0$ there are two real solutions. These are given by

\[
\lambda^2 x^+ x^- = -\frac{\alpha}{2} W(k, -e^{2M/\alpha - 1}).
\]

where $W(k, x)$ is Lambert’s function\(^{17}\) and $k = 0, -1$. The branch corresponding to $k = 0$ is the principal branch and is the one that is analytic at $x = 0$ ($M \to -\infty$).

As $M$ is the Bondi mass measured at future timelike infinity and on $\mathcal{I}^+_L$, the only singularities possible within the full quantum theory are spacelike and would have negative Bondi masses on $\mathcal{I}^+_L$. Measured on $\mathcal{I}^+_R$, however, they would begin with a positive mass in the past and rapidly give up energy over retarded time to turn eventually into spacelike negative energy singularities. The Penrose diagram is displayed in figure VII (where the physical regions in which $\sigma > 0$ are denoted by I and III). For positive $M$, quantum effects have prevented the singularity from forming. For negative $M$, quantum effects turn the naked singularity into a black hole in region III and a white hole in region I. (Even classically the singularity is a black hole for negative $M$, but of course negative $M$ is physically not allowed.)

The next case study we examine is that of section V, where the incoming soliton attempts to form an asymptotically naked singularity in the future. Two possible choices of the ghost contribution to the full stress energy were discussed. The first choice involved no incoming Hawking radiation and led to an negative infinite flux of energy on the future horizon in the semi-classical approximation. The second choice avoided this problem by allowing an influx of Hawking energy across past null infinity. We will now discuss the choices of ghost stress energy that correspond to each of these boundary conditions.

If there is no incoming Hawking flux at past null infinity, the appropriate choice for
the ghost stress energy is
\[ t_{\pm\pm} = \frac{\alpha}{2x_{\pm}^2} \] (6.19)
which leads to the following solution
\[ \Omega = \chi = -\sqrt{\frac{2}{\alpha}} \left( \chi^2 x^+ x^- - 2 \ln \cosh(\Delta - \Delta_0) - \frac{\alpha}{2} \ln(\lambda^2 x^+ x^-) \right). \] (6.20)

To see that this is indeed correct, consider the Bondi mass as measured by an observer on \( I^- \). For example, choosing \( \Omega^{(0)} \) (the vacuum) as before, in (6.12), on \( I^-_R \) \( (x^- \to -\infty) \) we have
\[ \delta \Omega = -\sqrt{\frac{2}{\alpha}} [2 \ln \cosh(\Delta - \Delta_0)] \]
\[ \sim 2\sqrt{\frac{2}{\alpha}} \left[ \gamma_+ x^+ + \gamma_- x^- - \Delta_0 - \ln 2 \right] \] (6.21)
so that the charge evolving the system along \( I^-_R \) is constant:
\[ Q^+ = \lambda \sqrt{\frac{2}{\alpha}} \left[ -\delta \Omega + x^+ \partial_+ \delta \Omega + x^- \partial_- \delta \Omega \right] |_{I^-_R} \]
\[ = 2\lambda \left[ \frac{2\mu^2}{\lambda^2} + \Delta_0 + \ln 2 \right] = M_R \] (6.22)
where
\[ x^+^' = x^+ - \frac{2\gamma_-}{\lambda^2}. \]
Likewise, on \( I^-_L \), the Bondi mass is calculated to be
\[ Q^- = \lambda \sqrt{\frac{2}{\alpha}} \left[ -\delta \Omega + x^+ \partial_+ \delta \Omega + x^-^' \partial_- \delta \Omega \right] |_{I^-_L} \]
\[ = 2\lambda \left[ \frac{2\mu^2}{\lambda^2} - \Delta_0 + \ln 2 \right] = M_L \] (6.23)
where
\[ x^-^' = x^- + \frac{2\gamma_+}{\lambda^2} \]
in the notation of sections IV and V.
The singularity occurs at
\[
\frac{\alpha}{2} \ln \left( \frac{2e}{\alpha} \right) = \lambda^2 x^+ x^- - 2 \ln \cosh(\Delta - \Delta_0) - \frac{\alpha}{2} \ln (\lambda^2 x^+ x^-). \tag{6.24}
\]
This curve is obtained quite easily in parametrized form (using \(\Delta\) as a parameter) and can be written in terms of the Lambert’s \(W\) function as follows
\[
x^- = \frac{1}{2\gamma_-} \left[ \Delta + \sqrt{\Delta^2 - 2\alpha \mu^2 \mathcal{W}[k, f(\Delta)]} \right]
\]
\[
x^+ = \frac{\Delta - \gamma_- x^-}{\gamma_+}
\]
\[
f(\Delta) = - \exp \left[ -1 - \frac{4}{\alpha} \ln \cosh(\Delta - \Delta_0) \right]
\]
for \(k = 0, -1\). The range of \(\Delta\) is \((-\infty, \infty)\) and the singularities are displayed in figure VIII. An observer entering spacetime in region III will see a naked singularity before inevitably crashing into it. Region II is unphysical. An observer in region I would emerge from a white hole.

If, on the other hand, we do allow for incoming Hawking radiation on \(I^-\), the appropriate choice for the ghost stress energy is
\[
t_{\pm\pm} = \frac{\alpha}{2 x^{\pm 2}} \tag{6.26}
\]
which gives the following solution for \(\Omega\)
\[
\Omega = - \sqrt{\frac{2}{\alpha}} \left[ \lambda^2 x^+ x^- - 2 \ln \cosh(\Delta - \Delta_0) - \frac{\alpha}{2} \ln \lambda^2 x^+ x^- \right]. \tag{6.27}
\]
That this solution indeed represents incoming Hawking radiation on \(I^-\) is checked by calculating the Bondi mass of the spacetime there. One gets
\[
\delta \Omega = - \sqrt{\frac{2}{\alpha}} \left[ - 2 \ln \cosh(\Delta - \Delta_0) - \frac{\alpha}{2} \left( \ln \frac{x^+}{x^-} + \ln \frac{x^-}{x^+} \right) \right]. \tag{6.28}
\]
Thus, on \(I^-_R\) \((x^- \to -\infty)\),
\[
\delta \Omega \sim 2 \sqrt{\frac{2}{\alpha}} \left[ \gamma_+ x^+ + \gamma_- x^- - \Delta_0 - \ln 2 + \frac{\alpha}{4} \ln \frac{x^+}{x^-} \right] \tag{6.29}
\]
giving,

\[ Q^+ = \lambda \left[ \frac{M_R}{\lambda} + \frac{\alpha}{2} \ln \frac{x^+}{x^+'} + \frac{\alpha}{2} \left( 1 - \frac{x^+'}{x^+} \right) \right] \quad (6.30) \]

and, similarly, on \( \mathcal{I}_L^- \),

\[ Q^- = 2\lambda \left[ \frac{M_L}{\lambda} + \frac{\alpha}{2} \ln \frac{x^-}{x^-'} + \frac{\alpha}{2} \left( 1 - \frac{x^-'}{x^-} \right) \right] \quad (6.31) \]

These are, of course, the Bondi masses measured on the respective past null infinities.

Changing to asymptotically flat coordinates defined by

\[
\begin{align*}
    x^+ &= -\frac{1}{\lambda} e^{-\lambda \sigma^+} + \frac{2\gamma^-}{\lambda^2} \\
    x^- &= -\frac{1}{\lambda} e^{-\lambda \sigma^-} - \frac{2\gamma_+}{\lambda^2}
\end{align*}
\]  

one finds that

\[
\begin{align*}
    M_R(\sigma^+) &= M_R + \frac{\alpha \lambda}{2} \left[ \ln \left( 1 - \frac{2\gamma^-}{\lambda} e^{\lambda \sigma^+} \right) + \left( \frac{1}{1 - \frac{\lambda}{2\gamma^-} e^{-\lambda \sigma^+}} \right) \right] \\
    M_L(\sigma^-) &= M_L + \frac{\alpha \lambda}{2} \left[ \ln \left( 1 + \frac{2\gamma_+}{\lambda} e^{\lambda \sigma^-} \right) + \left( \frac{1}{1 + \frac{\lambda}{2\gamma_+} e^{-\lambda \sigma^-}} \right) \right]
\end{align*}
\]  

Therefore,

\[
\begin{align*}
    \frac{dM_R(\sigma^+)}{d\sigma^+} &= \langle T^{(\sigma)}_{++} \rangle \\
    \frac{dM_L(\sigma^-)}{d\sigma^-} &= \langle T^{(\sigma)}_{--} \rangle
\end{align*}
\]  

on past null infinity, where \( \langle T^{(\sigma)}_{\pm\pm} \rangle \) were given in (5.5) and (5.6).

We have not been able to obtain a closed form expression for the singularity curve in this case. Yet, one can analyze this curve in the asymptotic regions where, for example on \( \mathcal{I}_R^- \), its equation reads

\[
\frac{\alpha}{2} \ln \left( \frac{2\gamma}{\alpha} \right) = \lambda^2 x^+ x^- + \frac{M}{\lambda} - \frac{\alpha}{2} \ln(\lambda^2 x^+ x^-').
\]  

with

\[ x^- = x^- - \frac{2\gamma_+}{\lambda^2}. \]

As \( x^- \to -\infty \), this equation is similar to the corresponding one for the shock wave (equation 6.17) and the conclusions are the same. There is no solution for \( M > 0 \).
$M = 0$ is the (shifted) vacuum and, when $M < 0$, the singularity is (asymptotically) spacelike, intersecting $I_R$ at $x^+ = 0$. Obviously, a parallel conclusion can be drawn for the singularity on $I_L$ and a negative mass singularity would intersect $I_L$ at $x^- = 0$. It looks as if quantum effects prevent the singularity from forming if $M > 0$ and change it to a black hole if $M < 0$.

VII. Discussion

This paper represents an attempt to explore the consequences of a violation of the classical cosmic censorship conjecture. One may imagine that Einstein’s theory predicts that some reasonable physical systems will collapse to form naked singularities. It is generally, but not universally, believed that the breakdown due to the formation of a timelike (naked) singularity is more serious than the breakdown due to the formation of a spacelike (black hole) singularity. In either case one does expect quantum effects to play an important role. Since string theory is the only available theory that gives a consistent quantum theory of gravity, it represents the scenario in which problems involving gravitational collapse to a naked singularity should properly be explored. Our work represents an attempt to arrive at an understanding of the details of gravitational collapse on the classical and the quantum levels.

On the classical level, and in the case of the soliton induced singularities, the changing character of each singularity is fascinating. In the same spacetime the singularity may alternate and smoothly change its nature from being spacelike to timelike and back to spacelike. Quite possibly this type of behavior may be typical of the collapse of a chaotic distribution of matter. The induced radiation accompanying the collapse is in all cases very significant and implies that a major revision of the classical conclusions is necessary.

On the quantum level, we have analyzed the problem first semi-classically as this is the basis of our physical intuition. Here already we found indications of a weak cosmic censorship hypothesis. If a naked singularity forms, the total energy radiated will exceed the mass of the incoming physical field. In the case of a black hole the energy radiated will build up slowly but in the case of a naked singularity the radiated energy builds up rapidly. Thus, even at this stage one might conjecture that when the collapse of a physical system is predicted by Einstein’s theory to lead to a naked singularity, quantum effects would induce an explosive burst of radiation such that the back reaction would step in to prevent the singularity from actually forming. The quantum treatment including the
back reaction seems to verify this conjecture, if attention is restricted to positive (Bondi) mass singularities and boundary conditions that obey the weak energy conditions in the semi-classical approximation.

The tantalizing question is then: do phenomena such as these exist in nature and do they lead to observable astrophysical effects?

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Figure Captions:

**Figure I:** Penrose diagram of the naked singularity represented by (2.6)

**Figure II:** Penrose diagram of the naked singularity which appears simultaneously with an incoming $f-$ shock wave (equation 2.15).

**Figure III:** The Kruskal diagram for $\Lambda = 4\lambda^2 > 0$ and $\Delta_0 > 0$. Regions I & III are physical ($\sigma > 0$). The singularities are spacelike at null infinity and joined smoothly along the soliton center by a timelike singularity. The figure was drawn for the following parameter values: $\mu^2 = 1$, $\lambda^2 = 4$, $\gamma_+ = \sqrt{3}$, $\gamma_- = -1/\sqrt{3}$, $\Delta_0 = 0.1$ and $\alpha = 1/24\pi$.

**Figure IV:** The Kruskal diagram for $\Lambda = 4\lambda^2 > 0$ and $\Delta_0 = 0$. Regions I & III are physical ($\sigma > 0$). Both singularities are spacelike so that a black hole and a white hole are seen to intersect at the origin. The figure was drawn for the following parameter values: $\mu^2 = 1$, $\lambda^2 = 4$, $\gamma_+ = \sqrt{3}$, $\gamma_- = -1/\sqrt{3}$ and $\alpha = 1/24\pi$.

**Figure V:** The Kruskal diagram for $\Lambda = -4\lambda^2 < 0$ and $\Delta_0 > 0$. Regions II & IV are physical ($\sigma > 0$). The singularities are timelike at null infinity and joined smoothly at the soliton center. The figure was drawn for the following parameter values: $\mu^2 = 1$, $\lambda^2 = 0.25$, $\gamma_+ = \sqrt{3}$, $\gamma_- = -1/\sqrt{3}$, $\Delta_0 = 5.0$ and $\alpha = 1/24\pi$.

**Figure VI:** The Kruskal diagram for $\Lambda = -4\lambda^2 < 0$ and $\Delta_0 = 0$. Regions II & IV are physical ($\sigma > 0$). Two timelike singularities intersect at the origin. The figure was drawn for the following parameter values: $\mu^2 = 1$, $\lambda^2 = 0.25$, $\gamma_+ = \sqrt{3}$, $\gamma_- = -1/\sqrt{3}$ and $\alpha = 1/24\pi$.

**Figure VII:** The Penrose diagram for the negative mass (on $I^+_L$) singularity formed in the full quantum theory by an incoming $f-$ shock wave. Regions I and III are physical.

**Figure VIII:** The Kruskal diagram for the $k = 0$ branch of the singularity formed by an incoming soliton in the full quantum theory. The boundary conditions are no incoming radiation on $I^-$. The figure was drawn for the following parameter values: $\Delta_0 = 1$, $\lambda^2 = 1$, $\gamma_+ = \sqrt{3}$, $\gamma_- = -1/\sqrt{3}$ and $\alpha = 23/6$. Regions I and III are physical.
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Linear Dilaton Vacuum

$\mathcal{I}_L^+$ $\mathcal{I}_R^+$

$\mathcal{I}_L^-$ $\mathcal{I}_R^-$

$x = a/\lambda^2$

FIGURE II
FIGURE III
Path of Soliton Center

FIGURE IV
FIGURE VI
FIGURE VII
