A Laplace operator and harmonics on the quantum complex vector space

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Abstract

The aim of this paper is to study the $q$-Laplace operator and $q$-harmonic polynomials on the quantum complex vector space generated elements $z_i, w_i, i = 1, 2, \ldots, n$, on which the quantum group $GL_q(n)$ (or $U_q(n)$) acts. The $q$-harmonic polynomials are defined as solutions of the equation $\Delta q p = 0$, where $p$ is a polynomial in $z_i, w_i, i = 1, 2, \ldots, n$, and the $q$-Laplace operator $\Delta_q$ is determined in terms of $q$-derivatives. The $q$-Laplace operator $\Delta_q$ commutes with the action of $GL_q(n)$. The projector $H_{m,m'}: A_{m,m'} \rightarrow H_{m,m'}$ is constructed, where $A_{m,m'}$ and $H_{m,m'}$ are the spaces of homogeneous (of degree $m$ in $z_i$ and of degree $m'$ in $w_i$) polynomials and homogeneous $q$-harmonic polynomials, respectively. By using these projectors, a $q$-analogue of the classical zonal spherical and associated spherical harmonics are constructed. They constitute an orthogonal basis of $H_{m,m'}$. A $q$-analogue of separation of variables is given. The quantum algebra $U_q(gl_n)$, acting on $H_{m,m'}$, determines an irreducible representation of $U_q(gl_n)$. This action is explicitly constructed. The results of the paper lead to the dual pair $(U_q(sl_2), U_q(gl_n))$ of quantum algebras.

I. INTRODUCTION

Laplace operators, harmonic polynomials, and related separations of variables of the classical analysis are of a great importance for mathematical and theoretical physics. They are closely related to the rotation groups $SO(n)$ (if we deal with Euclidean space) and to the unitary groups $U(n)$ (if we deal with the complex vector space) (see, for example, Ref. 1, Chaps. 10 and 11). In this paper we are interested in a $q$-analogue of Laplace operators, harmonic polynomials, and related separations of variables on complex spaces.

Harmonic polynomials on the $n$-dimensional complex vector space are defined by the equation $\Delta p = 0$, where $\Delta$ is the Laplace operator $\sum_{i=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_i} p$ and $p$ belongs to the space $R$ of polynomials in $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$ on the complex space $\mathbb{C}^n$. The space $\mathcal{H}$ of all harmonic polynomials on $\mathbb{C}^n$ decomposes as a direct sum of the subspaces $\mathcal{H}_{m,m'}$ of homogeneous harmonic polynomials of degree $m$ in $z_1, \ldots, z_n$ and of degree $m'$ in $\bar{z}_1, \ldots, \bar{z}_n$: $\mathcal{H} = \bigoplus_{m,m'}^{m=0} \mathcal{H}_{m,m'}$. The Laplace operator $\Delta$ on $\mathbb{C}^n$ commutes with the natural action of the unitary group $U(n)$ on the space $\mathbb{C}^n$. This means that the subspaces $\mathcal{H}_{m,m'}$ are invariant with respect to $U(n)$. The irreducible representation $T_{m,m'}$ of the group $U(n)$ with highest weight $(m, 0, \ldots, 0, -m')$ is realized on $\mathcal{H}_{m,m'}$.

The equation $\Delta p = 0$ permits solutions in separated variables on the space $\mathcal{H}_{m,m'}$. In other words, there exist different coordinate systems (spherical, polyspherical, etc.) on $\mathbb{C}^n$ and for each of them it is possible to find the corresponding basis of the space of solutions of the equation $\Delta p = 0$ consisting of products of functions depending on

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separated variables (see Ref. 2 for the general theory of separation of variables). To different coordinate systems there correspond different separations of variables. From the other side, to different coordinate systems there correspond different chains of subgroups of the group $U(n)$ (see Ref. 1, Chap. 11, for details of this correspondence). The bases of the space $H_{m,m'}$ in separated variables consist of products of Jacobi polynomials multiplied by $r^{m+m'}$ (different sets of Jacobi polynomials for different separations of variables), where $r$ is the radius. These polynomials (considered only on the unit sphere $S_{\mathbb{C}}^{-1}$ in $\mathbb{C}^n$) are matrix elements of the class 1 (with respect to the subgroup $U(n-1)$) irreducible representations $T_{m,m'}$ of $U(n)$ belonging to zero column (see Ref. 1, Chap. 11).

Many new directions of mathematical physics are related to quantum group and noncommutative geometry. It is natural to generalize the above-described theory to noncommutative spaces. Such generalizations can be of a great importance for further development of some branches of mathematical and theoretical physics related to noncommutative geometry.

The aim of this paper is to construct a $q$-deformation of the above-described classical theory. In the $q$-deformed case, instead of $\mathbb{C}^n$ we take the quantum complex vector space. It is defined by the associative algebra $\mathcal{A}$ generated by the elements $z_1, \ldots, z_n, w_1, \ldots, w_n$ satisfying a certain natural defining relations. The elements $z_1, \ldots, z_n$ play a role of Cartesian coordinates of $\mathbb{C}^n$ and $w_1, \ldots, w_n$ play a role of $\bar{z}_1, \ldots, \bar{z}_n$.

The $q$-Laplace operator $\Delta_q$ on $\mathcal{A}$ is defined in terms of $q$-derivatives (see formula (17) below). The quantum group $U_q(n)$ play a role of the unitary group $U(n)$ in the $q$-deformed case. It will be convenient for us to use the quantum algebra (that is, the quantized universal enveloping algebra) $U_q(\mathfrak{gl}_n)$ instead of the quantum group $U_q(n)$. The $q$-harmonic polynomials on the quantum complex vector space are defined as elements $p$ of the algebra $\mathcal{A}$ (that is, polynomials in $z_1, \ldots, z_n, w_1, \ldots, w_n$) for which $\Delta_q p = 0$. By using the quantum algebra $U_q(\mathfrak{gl}_n)$ we construct for $q$-harmonic polynomials a theory similar to the theory for classical harmonic polynomials. We construct the projector $H_{m,m'}: \mathcal{A}_{m,m'} \to H_{m,m'}$, where $\mathcal{A}_{m,m'}$ and $H_{m,m'}$ are the subspaces of homogeneous (of degree $m$ in $z_1, \ldots, z_n$ and of degree $m'$ in $w_1, \ldots, w_n$) polynomials in $\mathcal{A}$ and in the space $\mathcal{H}$ of all $q$-harmonic polynomials from $\mathcal{A}$, respectively. Using these projectors we can make different calculations in $H_{m,m'}$. In this way, zonal spherical and associated spherical polynomials can be calculated. The associated spherical polynomials of $H_{m,m'}$ constitute an orthogonal basis of this space. Here we obtain a $q$-analogue of the spherical separation of coordinates. We show that the natural action of the algebra $U_q(\mathfrak{gl}_n)$ on the quantum complex vector space realizes on the space $H_{m,m'}$ the irreducible representation of this algebra with highest weight $(m,0,\ldots,0,-m')$. Note that restrictions of zonal spherical and associated spherical polynomials from $H_{m,m'}$ to the quantum sphere in the quantum complex vector space coincide with matrix elements of irreducible representations $T_{m,m'}$ of the quantum group $U_q(n)$ corresponding to zero column (the latter matrix elements were calculated in Ref. 3; see also Ref. 4). Some our formulas coincide with formulas of Ref. 3. However, no Laplace operator and no $q$-harmonic polynomials are presented in Ref. 3.

Note that this paper is an extension of the results of our previous paper (see Ref. 5) (where we studied $q$-Laplace operator and $q$-harmonic polynomials on the quantum real vector space) to the case of quantum complex vector space. It is well known that in the classical case, the theory of Laplace operators and harmonic polynomials on $\mathbb{R}^{2n}$ can be reduced to the corresponding theory for the real space $\mathbb{R}^n$ (see Ref. 1, Chap.
It is not the case for the quantum spaces. The reason is that the quantum complex vector space cannot be obtained from the quantum real vector space in the same way as in the classical case.

Everywhere below we suppose that \( q \) is not a root of unity. Under considering a scalar product on the spaces \( \mathcal{A} \) and \( \mathcal{H} \) we assume that \( q \) is a positive real number. By \([a], a \in \mathbb{C}\), we denote the so called \( q \)-number defined as

\[
[a] = \frac{q^a - q^{-a}}{q - q^{-1}}.
\]

II. THE QUANTUM ALGEBRA \( U_q(\mathfrak{g}_n) \) AND THE QUANTUM VECTOR SPACE

The Drinfeld–Jimbo quantum algebra \( U_q(\mathfrak{g}_n) \) is generated by the elements \( k_i^{\pm 1} \equiv q^{h_i/2}, k_i^{-1/2} \equiv q^{-h_i/2}, i = 1, 2, \ldots, n \), and \( e_j, f_j, j = 1, 2, \ldots, n - 1 \), satisfying the relations

\[
k_i^{k_j} - k_j^{k_i} = 1, \quad k_ik_j = k_jk_i, \quad k_ie_jk_i^{-1} = q^{a_{ij}}e_j, \quad k_if_jk_i^{-1} = q^{-a_{ij}}f_j,
\]

\[
[e_i, f_j] = e_if_j - f_je_i = \delta_{ij} \frac{k_ik_{i+1}^{-1} - k_{i+1}^{-1}k_i}{q - q^{-1}},
\]

\[
e_i^2e_{i+1} - (q + q^{-1})e_ie_{i+1}e_i + e_{i+1}e_i^2 = 0,
\]

\[
f_i^2f_{i+1} - (q + q^{-1})f_if_{i+1}f_i + f_{i+1}f_i^2 = 0,
\]

\[
[e_i, e_j] = [f_i, f_j] = 0, \quad |i - j| > 1,
\]

where \( a_{ii} = 1, a_{i,i-1} = a_{i-1,i} = -1 \) and \( a_{ij} = 0 \) otherwise (see, for example, Ref. 6, Chap. 6).

The algebra \( U_q(\mathfrak{g}_n) \) is a Hopf algebra, and the Hopf algebra operations (comultiplication \( \Delta \), counit \( \varepsilon \) and antipode \( S \)) are given by the formulas

\[
\Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \Delta(e_i) = e_i \otimes k_i^{-1/2}k_{i+1}^{1/2} + k_i^{1/2}k_{i+1}^{-1/2} \otimes e_i,
\]

\[
\Delta(f_i) = f_i \otimes k_i^{-1/2}k_{i+1}^{1/2} + k_i^{1/2}k_{i+1}^{-1/2} \otimes f_i, \quad \varepsilon(k_i) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0,
\]

\[
S(k_i) = k_i^{-1}, \quad S(e_i) = -q^{-1}e_i, \quad S(f_i) = -qf_i.
\]

The group \( GL(n, \mathbb{C}) \) and its Lie algebra \( \mathfrak{g}(n, \mathbb{C}) \) act linearly on the \( n \)-dimensional complex vector space. Similarly, the quantum group \( GL_q(n, \mathbb{C}) \) and the algebra \( U_q(\mathfrak{g}_n) \) acts on the quantum (noncommutative) analogue of the complex vector space. This quantum space is determined by the algebra of polynomials \( \mathcal{A} \equiv \mathbb{C}_q[z_1, \ldots, z_n, w_1, \ldots, w_n] \) (see Ref. 7). This algebra is the associative algebra generated by elements \( z_1, z_2, \ldots, z_n, w_1, w_2, \ldots, w_n \) satisfying the defining relations

\[
z_iz_j = qz_jz_i, \quad w_iw_j = q^{-1}w_jw_i, \quad i < j, \tag{1}
\]

\[
w_jz_i = qz_iw_j, \quad i \neq j, \quad i, j = 1, 2, \ldots, n, \tag{2}
\]

\[
w_kw_k = z_kw_k + (1 - q^2) \sum_{s=1}^{k-1} z_sw_s. \tag{3}
\]

The elements \( w_1, \ldots, w_n \) play a role of \( \pi_1, \ldots, \pi_n \) in the classical analysis.
A *-operation can be defined on the algebra \( \mathcal{A} \) turning it into a *-algebra. This *-operation is uniquely determined by the relations \( z_i^* = w_i, w_i^* = z_i \), \( i = 1, 2, \ldots, n \). The compact quantum group \( U_q(g_l) \) acts on this *-algebra.

Note that the relations (3) are equivalent to the following ones:

\[
z_kw_k = w_kz_k - (1 - q^2) \sum_{s=1}^{k-1} q^{2(k-s-1)} w_s z_s.
\] (4)

The set of all monomials

\[
z_1^{r_1} z_2^{r_2} \cdots z_n^{r_n} w_1^{s_1} w_2^{s_2} \cdots w_n^{s_n}, \quad r_j, s_j = 0, 1, 2, \ldots,
\] (5)

form a basis of the algebra \( \mathcal{A} \) (see Ref. 8). The set

\[
w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n}, \quad r_j, s_j = 0, 1, 2, \ldots,
\] (6)

also form a basis of this algebra.

The vector space of the algebra \( \mathcal{A} \) can be represented as a direct sum of the vector subspaces \( \mathcal{A}_{m,m'} \) consisting of homogeneous polynomials of homogeneity degree \( m \) in \( z_1, z_2, \ldots, z_n \) and of homogeneity degree \( m' \) in \( w_1, w_2, \ldots, w_n \), \( m, m' = 0, 1, 2, \ldots \):

\[
\mathcal{A} = \bigoplus_{m=0}^{\infty} \bigoplus_{m'=0}^{\infty} \mathcal{A}_{m,m'}.
\] (7)

We have the linear space isomorphism

\[
\mathcal{A} \simeq \mathcal{A}_z \otimes \mathcal{A}_w,
\]

where the associative algebra \( \mathcal{A}_z \) (the associative algebra \( \mathcal{A}_w \)) is a subalgebra of \( \mathcal{A} \) coinciding with \( \bigoplus_{m=0}^{\infty} \mathcal{A}_{m,0} \) (respectively, with \( \bigoplus_{m'=0}^{\infty} \mathcal{A}_{0,m'} \)).

We can define an action of the algebra \( U_q(g_l) \) on the vector space \( \mathcal{A} \). To determine this action we give the action of \( U_q(g_l) \) on \( z_j \) and \( w_j \) by the formulas\(^8\)

\[
k_i \triangleright z_j = q^{\delta_{ij}} z_j, \quad e_i \triangleright z_j = \delta_{j,i+1} z_{j-1}, \quad f_i \triangleright z_j = \delta_{j,i} z_{j+1},
\] (8)

\[
k_i \triangleright w_j = q^{-\delta_{ij}} w_j, \quad e_i \triangleright w_j = -\delta_{j,i} q^{-1} w_{j+1}, \quad f_i \triangleright w_j = -\delta_{j,i+1} q w_{j-1}.
\] (9)

and extend it to \( \mathcal{A} \) by using the comultiplication, that is, by means of the relation

\[
X \triangleright (p_1 p_2) = \sum (X(1) \triangleright p_1)(X(2) \triangleright p_2),
\]

where \( \Delta(X) = \sum X(1) \otimes X(2) \) (in the Sweedler notation), and linearity.

This action of the algebra \( U_q(g_l) \) on the vector space \( \mathcal{A} \) determines a representation of \( U_q(g_l) \) on this space (we denote it by \( L \)). Evidently, the subspaces \( \mathcal{A}_{m,m'} \) are invariant with respect to this action. Therefore, \( L \) determines representations of \( U_q(g_l) \) on these subspaces, which are denoted by \( L_{m,m'} \). We have \( L = \bigoplus_{m,m'=0}^{\infty} L_{m,m'} \).

3. OPERATORS ON THE ALGEBRA \( \mathcal{A} \)

In order to introduce the \( q \)-Laplace operator on \( \mathcal{A} \) and to study \( q \)-harmonic polynomials we need some operators on the linear space of the algebra \( \mathcal{A} \). By \( \gamma_i \) and \( \bar{\gamma}_i \) we denote the linear operators acting on monomials as

\[
\gamma_i (z_1^{r_1} z_2^{r_2} \cdots z_n^{r_n} w_1^{s_1} w_2^{s_2} \cdots w_n^{s_n}) = q^{r_i} z_1^{r_1} z_2^{r_2} \cdots z_n^{r_n} w_1^{s_1} w_2^{s_2} \cdots w_n^{s_n},
\]

\[
\bar{\gamma}_i (z_1^{r_1} z_2^{r_2} \cdots z_n^{r_n} w_1^{s_1} w_2^{s_2} \cdots w_n^{s_n}) = z_1^{r_1} z_2^{r_2} \cdots z_n^{r_n} w_1^{s_1} w_2^{s_2} \cdots w_n^{s_n}.
\]

\[\begin{array}{c}
\gamma_i (z_1^{r_1} z_2^{r_2} \cdots z_n^{r_n} w_1^{s_1} w_2^{s_2} \cdots w_n^{s_n}) = \\
\bar{\gamma}_i (z_1^{r_1} z_2^{r_2} \cdots z_n^{r_n} w_1^{s_1} w_2^{s_2} \cdots w_n^{s_n}) =
\end{array}\]
\[ \bar{\gamma}_i(w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n}) = q^{r_i} w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n}. \]

Definition of the operators $\gamma_i^{-1}$ and $\bar{\gamma}_i^{-1}$ is obvious.

By $\hat{z}_i$ and $\bar{z}_i$ we denote the linear operators of multiplication by the element $z_i$:

\[ \hat{z}_i(z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n} w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n}) = z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n} w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n}, \]
\[ \bar{z}_i(z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n} w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n}) = z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n} z_i w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n}. \]

The corresponding linear operators $\hat{w}_i$ and $\bar{w}_i$ are defined as

\[ \hat{w}_i(w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n}) = w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n}, \]
\[ \bar{w}_i(w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n}) = w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n} w_i z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n}. \]

We define on $\mathcal{A}$ the $q$-differentiations $\partial_i$ and $\bar{\partial}_i$. The linear operators $\partial_i$ act as $\partial_i p = 0$ on monomials $p$ of the form (5) not containing $z_i$ and as

\[ \partial_i = \hat{z}_i^{-1} \frac{\gamma_i - \gamma_i^{-1}}{q - q^{-1}} \]  

(10)

on monomials containing $z_i$. The $q$-differentiations $\bar{\partial}_i$ are linear operators acting as $\bar{\partial}_i p = 0$ on monomials $p$ of the form (6) not containing $w_i$ and as

\[ \bar{\partial}_i = \bar{w}_i^{-1} \frac{\gamma_i - \gamma_i^{-1}}{q - q^{-1}} \]  

(11)

on monomials containing $w_i$.

The action formulas (8) and (9) mean that the multiplication operators $\hat{z}_j$, $j = 1, 2, \ldots, n$, and $\bar{w}_j$, $j = 1, 2, \ldots, n$, constitute tensor operators transforming under the vector representation and under the contragredient to the vector representation, respectively.

The actions (8) and (9) of $U_q(\mathfrak{gl}_n)$ on $z_j$ and $w_j$ determines its action on the operators $\partial_j$ and $\bar{\partial}_j$:

\[ k_i \triangledown \partial_j = q^{-\delta_{ji}} \partial_j, \quad e_i \triangledown \partial_j = -\delta_{ji} q^{-1} \partial_{j+1}, \quad f_i \triangledown \partial_j = -\delta_{ji} q q_{j+1} \partial_{j-1}, \]
\[ k_i \triangledown \bar{\partial}_j = q^{\delta_{ji}} \bar{\partial}_j, \quad e_i \triangledown \bar{\partial}_j = \delta_{ji} q^{-1} \bar{\partial}_{j+1}, \quad f_i \triangledown \bar{\partial}_j = \delta_{ji} q q_{j+1} \bar{\partial}_{j-1}. \]

That is, the set $\bar{\partial}_j$, $j = 1, 2, \ldots, n$, (respectively, the set $\partial_j$, $j = 1, 2, \ldots, n$) is a tensor operator transforming under vector (respectively, contragredient to vector) representation.

The operators $\partial_i$, $\bar{\partial}_i$, $\hat{z}_i$, $\bar{z}_i$ satisfy the relations, which will be presented by means of the quantum $R$-matrix $R$ for the quantum algebra $U_q(\mathfrak{gl}_n)$ (see, for example, Refs. 6, section 8.1, and 7 for definition of the $R$-matrix). Let $\mathbf{R} = P R$, where the matrix $P$ permutes the spaces in the tensor product of two spaces on which $R$-matrix acts. Then

\[ R_{ik}^{\bar{j}j} = q^{\delta_{ki}} \delta_{ij} \delta_{jk} + (q - q^{-1}) \delta_{ik} \delta_{jl} \theta(j - i), \]

where $\theta(k) = 1$ if $k > 0$ and $\theta(k) = 0$ if $k \leq 0$. Its inverse matrix is

\[ \left( R^{-1} \right)^{\bar{j}j}_{ik} = q^{-\delta_{ji}} \delta_{ij} \delta_{jk} - (q - q^{-1}) \delta_{ik} \delta_{jl} \theta(i - j). \]

We also need the matrix $\Phi_{ik}^{\bar{j}j} = R_{ik}^{\bar{j}j} Q^{\delta_{i-j}}$, which satisfy the relations

\[ \sum_{j,l} \Phi_{ij}^{ul} (R^{-1})^{ji}_{kl} = \sum_{j,l} (R^{-1})^{ul}_{pj} \Phi_{lj}^{ji} = \delta_{up} \delta_{ik}, \]
\[ \sum_l \Phi_{lk}^{li} = \delta_{ik} q^{2(n-i)+1}, \quad \sum_k \Phi_{lk}^{jk} = \delta_{jl} q^{2l-1}. \]

The relations (1)–(3) rewritten for operators \( \hat{z}_i \) and \( \hat{w}_i \) can be presented as

\[ \hat{z}_i \hat{z}_j = q^{-1} R_{ij}^{kl} \hat{z}_k \hat{z}_l, \quad \hat{w}_i \hat{w}_j = q^{-1} R_{ij}^{kl} \hat{w}_k \hat{w}_l, \quad \hat{w}_i \hat{z}_j = q (R^{-1})^{ik} \hat{z}_k \hat{w}_l, \]

We also have the relations

\[ \partial_i \partial_j = q^{-1} R_{ij}^{kl} \partial_k \partial_l, \quad \partial_i \partial_j = q^{-1} R_{ij}^{kl} \bar{\partial}_k \bar{\partial}_l, \quad \partial_i \bar{\partial}_j = q^{-1} \Phi_{ij}^{kl} \bar{\partial}_k \partial_l, \]

\[ \partial_i \hat{w}_j = q (R^{-1})_{ij}^{kl} \hat{w}_k \partial_l, \quad \partial_i \hat{z}_j = q \Phi_{ij}^{kl} \hat{z}_k \bar{\partial}_l, \quad \partial_i \bar{\partial}_j = \gamma^1 \delta_{ij} + (R^\pm)^{ij} \hat{z}_k \partial_l, \]

which can be represented in the form

\[ \partial_i \partial_j = q^{-1} \partial_j \partial_i, \quad \partial_i \bar{\partial}_j = q \bar{\partial}_j \bar{\partial}_i, \quad i < j, \]

\[ \partial_i \partial_j = q \partial_j \partial_i, \quad \partial_i \bar{\partial}_j = \partial_i \bar{\partial}_j + (1 - q^2) \sum_{k>i} \partial_k \bar{\partial}_k, \]

\[ \partial_i \partial_k = \partial_i \partial_k + (1 - q^{-2}) \sum_{k>i} q^{2(k-i)} \partial_k \partial_k. \]

\[ \partial_i \hat{w}_i = \hat{w}_i \partial_i, \quad \partial_i \hat{w}_i - q \hat{w}_i \partial_i = (1 - q^2) \hat{w}_i \partial_i, \quad \partial_j \hat{w}_i = q \hat{w}_i \partial_i, \quad i < j, \]

\[ \partial_i \hat{z}_i = \hat{z}_i \partial_i, \quad \partial_i \hat{z}_i - q^{-1} \hat{z}_i \partial_i = (1 - q^{-2}) \hat{z}_i \partial_i, \quad \partial_j \hat{z}_i = q^{-1} \hat{z}_i \partial_j, \quad i < j, \]

\[ \partial_i \hat{z}_j = q \hat{z}_j \partial_i, \quad \partial_i \hat{w}_j = \hat{w}_j \partial_i, \quad \partial_i \hat{w}_i = q \hat{w}_i \partial_i, \quad i < j, \]

\[ \partial_i \hat{z}_i = q \hat{z}_i \partial_i + (q - q^{-1}) \sum_{k<i} \hat{z}_k \partial_k + \gamma^{-1} = q^{-1} \hat{z}_i \partial_i - (q - q^{-1}) \sum_{k<i} \hat{z}_k \partial_k + \gamma, \]

\[ \partial_i \hat{w}_i = q \hat{w}_i \partial_i + (q - q^{-1}) \sum_{k<i} \hat{w}_k \partial_k + \gamma^{-1} = q^{-1} \hat{w}_i \partial_i - (q - q^{-1}) \sum_{k<i} \hat{w}_k \partial_k + \gamma, \]

where \( \gamma = \gamma_1 \gamma_2 \cdots \gamma_n \) and \( \bar{\gamma} = \bar{\gamma}_1 \bar{\gamma}_2 \cdots \bar{\gamma}_n \). From last two lines, we obtain

\[ \sum_{k=1}^n \hat{z}_k \partial_k = \{ \gamma \} \equiv \frac{\gamma - \gamma^{-1}}{q - q^{-1}}, \quad \sum_{k=1}^n \hat{w}_k \partial_k = \{ \bar{\gamma} \} \equiv \frac{\bar{\gamma} - \bar{\gamma}^{-1}}{q - q^{-1}}. \]

We also have the relations

\[ \gamma \hat{z}_i = q \hat{z}_i \gamma, \quad \gamma \hat{w}_i = \hat{w}_i \gamma, \quad \bar{\gamma} \hat{z}_i = \hat{z}_i \bar{\gamma}, \quad \bar{\gamma} \hat{w}_i = q \hat{w}_i \bar{\gamma}, \]

\[ \gamma \partial_i = q^{-1} \partial_i \gamma, \quad \bar{\gamma} \partial_i = \partial_i \bar{\gamma}, \quad \bar{\gamma} \partial_i = q^{-1} \partial_i \bar{\gamma}. \]

Note that

\[ \gamma p = q^m p, \quad \bar{\gamma} p = q^{m'} p, \quad p \in A_{m,m'}. \quad (14) \]

To compare these relations with known from literature, we introduce the operators \( \partial_i' = \gamma \partial_i, \quad \bar{\partial}_i' = \gamma^{-1} \bar{\partial}_i \). Then the operators \( \hat{z}_i, \hat{w}_i, \partial_i', \bar{\partial}_i', i = 1, \ldots, n \), satisfy the relations from Ref. 9 which are known to be covariant with respect to \( U_q(gl_n) \).

Note that the above elements \( \hat{z}_1, \ldots, \hat{z}_n, \partial_1', \ldots, \partial_n' \) generate the \( q \)-Weyl algebra, that is, they satisfy the relations

\[ \hat{z}_i \hat{z}_j = q \hat{z}_j \hat{z}_i, \quad \partial_i' \partial_j' = q^{-1} \partial_j' \partial_i', \quad i < j, \quad \partial_i' \hat{z}_j = q \hat{z}_j \partial_i', \quad i \neq j, \]

\[ \hat{z}_i \hat{z}_j = q \hat{z}_j \hat{z}_i, \quad \partial_i' \partial_j' = q^{-1} \partial_j' \partial_i', \quad i < j, \quad \partial_i' \hat{z}_j = q \hat{z}_j \partial_i', \quad i \neq j. \]
\[ \partial_j' \hat{z}_i - q^2 \hat{z}_i \partial_j' = 1 + (q^2 - 1) \sum_{j > i} \hat{z}_j \partial_j' \]

(the definition of the q-Weyl algebra see, for example, in Ref. 6, Chap. 12). Similarly, the elements \( \hat{w}_1, \ldots, \hat{w}_n, \partial_1', \ldots, \partial_n' \) generate the \( q^{-1} \)-Weyl algebra.

The operators \( D := \sum_{k=1}^n \hat{z}_k \partial_k \) and \( \tilde{D} := \sum_{k=1}^n \hat{w}_k \partial_k \) are called the q-Euler operators. The formula (7) gives the decomposition of \( \mathcal{A} \) into a direct sum of eigenspaces of the operators \( D \) and \( \tilde{D} \).

Let us show that the above relations for the operators \( \partial_i, \tilde{\partial}_i, \hat{z}_i, \hat{w}_i \) determine uniquely the formulas (10) and (11) for \( \partial_i, \tilde{\partial}_i \). We use the action formulas \( \partial_1 = \tilde{\partial}_1 = 0 \), take into account that \( \hat{z}_i, \hat{w}_i \) act as the operators of left multiplication on the basis elements \( (5) \) and \( (6) \), respectively, and \( \gamma, \tilde{\gamma} \) are gradation operators on \( \mathcal{A} \) (see (14)). By means of commutation relations between \( \partial_i \) and \( \hat{w}_i \), it is easy to obtain that \( \partial_i \hat{w}_1^r \hat{w}_2^s \cdots \hat{w}_n^s = 0 \).

To calculate \( \partial_i(z_i^r w_1^s w_2^s \cdots w_n^s) \) with \( r_i > 0 \), we use the relation

\[ \partial_i \hat{z}_i = q \hat{z}_i \partial_i + (q - q^{-1}) \sum_{k > i} \hat{z}_k \partial_k + \gamma^{-1}. \]

It gives \( \partial_i(z_i^r w_1^s w_2^s \cdots w_n^s) = [r_i] \hat{z}_i^{r_i-1} w_1^s w_2^s \cdots w_n^s \). Finally, we have the action formula

\[
\begin{align*}
\partial_i(z_1^r \cdots z_i^r \cdots z_n^r w_1^s \cdots w_n^s) &= q^{r_i(r_i+1) + \cdots + r_n} \partial_i(z_1^{r_1} \cdots z_i^{r_i-1} \cdots z_{i-1}^{r_{i-1}} \cdots z_n^{r_n} w_1^s \cdots w_n^s) \\
&= q^{r_i(r_i+1) + \cdots + r_n} \sum z_i^{r_i-1} \partial_i(z_i^r w_1^s \cdots w_n^s) \\
&= q^{r_i(r_i+1) + \cdots + r_n} [r_i] \hat{z}_i^{r_i-1} w_1^s \cdots w_n^s,
\end{align*}
\]

which exactly coincides with the action (10). The action formula for \( \tilde{\partial}_i \) is recovered in a similar way.

The action of the algebra \( U_q(gl_n) \) on \( \mathcal{A} \equiv \mathcal{A}_z \otimes \mathcal{A}_w \), defined by formulas (8) and (9), can be determined in terms of the operators \( \partial_i \) and \( \tilde{\partial}_i \). We first note that the action of \( U_q(gl_n) \) on \( \mathcal{A}_z \) is given by the operators

\[ L(k_i) = \gamma_i, \quad L(e_i) = q^{-1/2}(\gamma_i \gamma_{i+1})^{1/2} \hat{z}_i \partial_{i+1}, \quad L(f_i) = q^{1/2}(\gamma_i \gamma_{i+1})^{-1/2} \hat{z}_{i+1} \partial_i \]

and on \( \mathcal{A}_w \) by the operators

\[ L(k_i) = \tilde{\gamma}_i^{-1}, \quad L(e_i) = -q^{3/2}(\gamma_i \gamma_{i+1})^{-1/2} \hat{w}_{i+1} \tilde{\partial}_i, \quad L(f_i) = -q^{3/2}(\gamma_i \gamma_{i+1})^{-1/2} \hat{w}_i \tilde{\partial}_{i+1}. \]

Taking into account the comultiplication on \( U_q(gl_n) \) the action of \( U_q(gl_n) \) on the linear space \( \mathcal{A} \equiv \mathcal{A}_z \otimes \mathcal{A}_w \) can be written as \( L(k_i) = \gamma_i \otimes \tilde{\gamma}_i^{-1} \) and

\[
\begin{align*}
L(e_i) &= q^{-1/2}(\gamma_i \gamma_{i+1})^{1/2} \hat{z}_i \partial_{i+1} \otimes (\tilde{\gamma}_i \tilde{\gamma}_{i+1})^{1/2} - q^{-3/2}(\gamma_i \gamma_{i+1})^{1/2} \otimes (\tilde{\gamma}_i \tilde{\gamma}_{i+1})^{1/2} \hat{w}_{i+1} \partial_i, \\
L(f_i) &= q^{1/2}(\gamma_i \gamma_{i+1})^{-1/2} \hat{z}_{i+1} \partial_i \otimes (\tilde{\gamma}_i \tilde{\gamma}_{i+1})^{-1/2} - q^{3/2}(\gamma_i \gamma_{i+1})^{-1/2} \otimes (\tilde{\gamma}_i \tilde{\gamma}_{i+1})^{-1/2} \hat{w}_i \tilde{\partial}_{i+1}.
\end{align*}
\]

IV. SQUARED q-RADIUS AND q-LAPLACE OPERATOR

The element

\[ Q = \sum_{i=1}^n z_i w_i = \sum_{i=1}^n q^{2(n-i)} w_i z_i \in \mathcal{A}_{1,1} \] (15)
of the algebra \( \mathcal{A} \) is called the \textit{squared q-radius} on the quantum complex vector space. It is an important element in \( \mathcal{A} \). One can check by a direct computation that \( Q \) is \textit{invariant with respect to the representation} \( L_{1,1} \) (and hence with respect to the representation \( L \)), that is, \( L(k_{i}^{\pm 1})Q = Q \), \( L(e_{j})Q = 0 \) and \( L(f_{j})Q = 0 \). Similarly, the element \( Q^{k} \in \mathcal{A}_{k,k} \) is invariant with respect to the representation \( L_{k,k} \).

The squared \( q \)-radius \( Q \) belongs to the center of the algebra \( \mathcal{A} \), that is, \( Qz_{i} = z_{i}Q \), \( Qw_{i} = w_{i}Q \), \( i = 1, 2, \ldots, n \). We shall also use the elements

\[
Q_{j} = \sum_{i=1}^{j} z_{i}w_{i} = \sum_{i=1}^{j} q^{2(j-i)} w_{i}z_{i},
\]

which are squared \( q \)-radiuses for the subalgebras \( C_{q}[z_{1}, w_{1}, \ldots, z_{j}, w_{j}] \). They satisfy the relations

\[
Q_{j}Q_{i} = Q_{i}Q_{j}, \quad z_{i}w_{i} = Q_{i} - Q_{i-1}, \quad w_{i}z_{i} = Q_{i} - q^{-2}Q_{i-1},
\]

\[
z_{i}Q_{j} = q^{-2}Q_{j}z_{i}, \quad w_{i}Q_{j} = q^{2}Q_{j}w_{i} \quad \text{for} \quad i > j,
\]

\[
z_{i}Q_{j} = Q_{j}z_{i}, \quad w_{i}Q_{j} = Q_{j}w_{i} \quad \text{for} \quad i \leq j.
\]

It can be checked\(^8\) by direct computation that

\[
z_{i}^{k} w_{i}^{k} = Q_{i}^{k} \left( Q_{i-1}/Q_{i}; q^{-2} \right)_{k}, \quad w_{i}^{k} z_{i}^{k} = Q_{i}^{k} \left( q^{2} Q_{i-1}/Q_{i}; q^{2} \right)_{k},
\]

where

\[(a; q)_{s} = (1-a)(1-aq) \cdots (1-aq^{s-1}).\]

We consider on \( \mathcal{A} \) the operator

\[
\Delta_{q} = \partial_{1} \bar{\partial}_{1} + \partial_{2} \bar{\partial}_{2} + \cdots + \partial_{n} \bar{\partial}_{n} = \sum_{i=1}^{n} q^{2(i-1)} \bar{\partial}_{i} \partial_{i},
\]

which is called the \( q \)-\textit{Laplace operator} on the quantum complex vector space. Since \( \gamma \Delta_{q} = q^{-1} \Delta_{q} \gamma \) and \( \bar{\gamma} \Delta_{q} = q^{-1} \Delta_{q} \bar{\gamma} \), then \( \Delta_{q} : \mathcal{A}_{m,m'} \to \mathcal{A}_{m-1,m'-1} \).

To the element (15) there corresponds the operator \( \hat{Q} \) on \( \mathcal{A} \) defined as

\[
\hat{Q} = \hat{z}_{1} \hat{w}_{1} + \hat{z}_{2} \hat{w}_{2} + \cdots + \hat{z}_{n} \hat{w}_{n}.
\]

\textit{Proposition 1}: The operators \( \Delta_{q} \) and \( \hat{Q} \) satisfy the relations

\[
\Delta_{q} \hat{Q}^{k} - \hat{Q} \Delta_{q} = q^{-1} \hat{Q}^{k-1} [k \{ q^{k+n-1} \gamma \bar{\gamma} \}],
\]

\[
\Delta_{q} (Q^{k}) = q^{-1} Q^{k-1} [k][k + n - 1],
\]

where

\[
\{a\} = \frac{a - a^{-1}}{q - q^{-1}}
\]

and \([r] \equiv \{ q^{r} \}\) is a \textit{q-number}.

\textit{Proof}: First we prove the relation \( [\Delta_{q}, \hat{Q}] = q^{-1} \{ q^{n} \gamma \bar{\gamma} \} \). Using relations of section III we derive

\[
\Delta_{q} \hat{Q} = \sum_{k,l} \partial_{k} \bar{\partial}_{k} \hat{z}_{l} \hat{w}_{l} = \sum_{k,l,i,j} \partial_{k} \left( q^{-1} \Phi_{i k}^{j} \hat{z}_{i} \bar{\partial}_{j} \right) \hat{w}_{l}
\]

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that exists nonzero element \( p = q \).

Thus, \( \Delta \{ \) attain (18) by induction if to use the relation

Using explicit expressions for matrix elements of \( R \) we have

Using explicit expressions for matrix elements of \( R \) and \( R^{-1} \)

Thus, \( [\Delta_q, \hat{Q}] = q^{n-1} \gamma^{-1} \{q^n \gamma\} + q^{2n-1} \{ \gamma \} \hat{\gamma} = q^{n-1} \{q^n \gamma \} \). Now, it is easy to obtain (18) by induction if to use the relation \( \{q^n \gamma\} \hat{Q} = \hat{Q} \{q^{r+2} \gamma\} \) and the explicit expression for \( \{a\} \). Acting by both sides of (18) on 1 we obtain (19).

**Proposition 2:** The operators \( \Delta_q \) and \( \hat{Q} \) commute with the action of the algebra \( U_q(\text{gl}_n) \) on \( \mathcal{A} \), that is, with all operators of the representation \( L \) of \( U_q(\text{gl}_n) \).

**Proof:** It follows from (12) and (13) that \( k_i \triangleright \Delta_q = \Delta_q, \ e_j \triangleright \Delta_q = 0 \) and \( f_j \triangleright \Delta_q = 0 \). Now using the comultiplication for \( k_i \), \( e_j \) and \( f_j \), we obtain the proposition for the \( q \)-Laplace operator. For \( \hat{Q} \) the proposition is proved similarly.

V. q-HARMONIC POLYNOMIALS

A polynomial \( p \in \mathcal{A} \) is called \( q \)-harmonic if \( \Delta_q p = 0 \). The linear subspace of \( \mathcal{A} \) consisting of all \( q \)-harmonic polynomials is denoted by \( \mathcal{H} \). Let

**Proposition 3:** The space \( \mathcal{A}_{m,m'} \) can be represented as the direct sum

**Proof:** First we prove that \( \mathcal{H}_{m,m'} \cap Q \mathcal{A}_{m-1,m'-1} = \{0\} \). If it is not true, then there exists nonzero element \( p \in \mathcal{H}_{m,m'} \cap Q \mathcal{A}_{m-1,m'-1} \). Let \( k \) be a maximal integral such that \( p = Q^k p' \) with some nonzero polynomial \( p' \). Then it follows from \( \Delta_q(p) = 0 \) and (18) that

\[
0 = \Delta_q(Q^k p') = Q^k \Delta_q(p') + Q^{k-1} q^{n-1}[k][k + n - 1 + m + m' - 2k]p'.
\]
Since \( q^n[k][k + n - 1 + m + m' - 2k] \neq 0 \), then \( p' \) can be divided by \( Q \). This is a contradiction. Thus, \( H_{m,m'} \cap Q \mathcal{A}_{m-1,m'-1} = \{0\} \). Using this fact and the equality \( \ker \Delta_q = H_{m,m'} \), where \( \Delta_q \) is considered only on \( \mathcal{A}_{m,m'} \), we obtain the chain of inequalities

\[
\dim \mathcal{A}_{m,m'} - \dim \ker \Delta_q \geq \dim Q \mathcal{A}_{m-1,m'-1} = \dim \mathcal{A}_{m-1,m'-1} \geq \dim \ker \Delta_q.
\]

The last inequality follows from the fact that \( \Delta_q : \mathcal{A}_{m,m'} \to \mathcal{A}_{m-1,m'-1} \). Now we take into account the relation \( \dim \ker \Delta_q + \dim \ker \Delta_q = \dim \mathcal{A}_{m,m'} \). Thus, in fact, the above inequalities are exact equalities, and \( \mathcal{A}_{m,m'} = H_{m,m'} \oplus Q \mathcal{A}_{m-1,m'-1} \). Proposition is proved.

**Remark:** If \( n = 1 \), then \( \mathcal{H} \) consists of all polynomials in commuting elements \( z_1 \) and \( w_1 \). In this case, the space \( \mathcal{H} \) of \( q \)-harmonic polynomials has a basis consisting of the polynomials

\[
1, \ z_1^k, \ w_1^k, \ k = 1, 2, \ldots.
\]

The decomposition (20) has also the following consequences:

**Corollary 1:** If \( p \in H_{m,m'} \), then \( p \) cannot be represented as \( p = Q^k p', k \neq 0 \), with some polynomial \( p' \).

**Corollary 2:** The space \( \mathcal{A}_{m,m'} \) decomposes into the direct sum

\[
\mathcal{A}_{m,m'} = \bigoplus_{j=0}^{\min(m,m')} Q^j H_{m-j,m'-j}.
\]

**Corollary 3:** For dimension of the space of \( q \)-harmonic polynomials \( H_{m,m'} \) we have the formula

\[
\dim H_{m,m'} = \frac{(m + n - 2)!(m' + n - 2)!(m + m' + n - 1)!}{(n - 1)!(n - 2)!m!m'}.
\]

**Corollary 4:** The space of \( q \)-harmonic polynomials \( \mathcal{H} \) can be represented in the form of a direct sum

\[
\mathcal{H} = \bigoplus_{m=0}^{\infty} \bigoplus_{m'=0}^{\infty} H_{m,m'}.
\]

Corollary 1 is a direct consequence of formula (20). Corollary 2 easily follows from repeated application of (20). Corollary 3 is proved in the same way as in the classical case (see, for example, Ref. 1, Chap. 10). For this we note that

\[
\dim \mathcal{A}_{m,m'} = \frac{(n + m - 1)!(n + m' - 1)!}{(n - 1)!m!m'}.
\]

Hence, for \( \dim H_{m,m'} = \dim \mathcal{A}_{m,m'} - \dim \mathcal{A}_{m-1,m'-1} \) we obtain the expression stated in the corollary. In order to prove Corollary 4 we note that

\[
\mathcal{H} = \bigoplus_{m \geq 0} \bigoplus_{m' \geq 0} \bigoplus_{j=0}^{p} Q^j H_{m-j,m'-j} = \bigoplus_{m \geq 0} \bigoplus_{m' \geq 0} \left( Q^j H_{m-j,m'-j} \right)_{j=1}^{p},
\]

where \( p = \min(m,m') \). Now Corollary 4 follows from here and Corollary 1.

**Theorem 1:** The linear space isomorphism \( \mathcal{A} \simeq \mathbb{C}[Q] \otimes \mathcal{H} \) is true, where \( \mathbb{C}[Q] \) is the space of all polynomials in \( Q \).
This theorem follows from Corollary 2.

The decomposition $\mathcal{A} \simeq \mathbb{C}[Q] \otimes \mathcal{H}$ is a $q$-analogue of the theorem on separation of variables for Lie groups in an abstract form. It follows from this decomposition that

$$\mathcal{A} \simeq \mathbb{C}[Q] \otimes \mathcal{H} \simeq \mathbb{C}[Q] \otimes \bigoplus_{m \geq 0} \bigoplus_{m' \geq 0} \mathcal{H}_{m,m'} = \bigoplus_{m \geq 0} \bigoplus_{m' \geq 0} (\mathbb{C}[Q] \otimes \mathcal{H}_{m,m'}). \quad (23)$$

Since the subspaces $\mathcal{A}_{m,m'}$ are invariant with respect to the action of the algebra $U_q(gl_n)$, it follows from Proposition 2 for $\Delta_q$ that the subspace $\mathcal{H}_{m,m'}$ is invariant with respect to the representation $L_{m,m'}$ of $U_q(gl_n)$. We denote the restriction of this representation to $\mathcal{H}_{m,m'}$ by $T_{m,m'}$. It follows from Proposition 2 for $Q$ and from (22) that

$$L_{m,m'} = \bigoplus_{j=0}^{\min(m,m')} T_{m-j,m'-j}. \quad (24)$$

Proposition 4: The representations $T_{m-j,m'-j}$ of $U_q(gl_n)$ in (24) are irreducible with highest weights $(m-j,0,\cdots,0,-m'+j)$, respectively.

Proof: Let us show that the representation $L_{m,0} = T_{m,0}$ in the space of holomorphic polynomials $\mathcal{A}_{m,0}$ is irreducible with highest weight $(m,0,\cdots,0)$. In fact, a direct calculation shows that the monomials $z_1^{m_1} \cdots z_n^{m_n}$, $m_1 + \cdots + m_n = m$, are weight vectors of this representation. The highest weight vector coincides with $z_1^m$. Therefore, the irreducible representation with highest weight $(m,0,\cdots,0)$ is a subrepresentation of $L_{m,0} = T_{m,0}$. Since their dimensions coincide, $L_{m,0} = T_{m,0}$ is an irreducible representation with highest weight $(m,0,\cdots,0)$. It can be proved in the same way that the representation $L_{0,m'} = T_{0,m'}$ in the space of polynomials $\mathcal{A}_{0,m'}$ is irreducible with highest weight $(0,0,\cdots,0,-m')$.

Now we can prove the proposition by the induction. Assume that the proposition is true for the representations $T_{m-1-j,m'-1-j}$ which are contained in the decomposition

$$L_{m-1,m'-1} = \bigoplus_{j=0}^{\min(m-1,m'-1)} T_{m-1-j,m'-1-j}. \quad (25)$$

Note that since $\mathcal{A}_{m,m'} = \mathcal{H}_{m,m'} \oplus QA_{m-1,m'-1}$, then $L_{m-1,m'-1}$ is a subrepresentation in $L_{m,m'}$ and

$$\dim \mathcal{A}_{m-1,m'-1} = \dim L_{m-1,m'-1} = \sum_{j=0}^{\min(m-1,m'-1)} \dim T_{m-1-j,m'-1-j}.$$ 

The space $\mathcal{A}_{m,m'}$ contains the highest weight vector $z_1^{m_1}w_1^{m_1'}$ which is of the weight $(m,0,\cdots,0,-m')$. Therefore, $L_{m,m'}$ contains an irreducible representation $\hat{T}_{m,m'}$ of $U_q(gl_n)$ with highest weight $(m,0,\cdots,0,-m')$. This irreducible representation is absent in the decomposition (25). Hence, $\hat{T}_{m,m'}$ is a subrepresentation in $T_{m,m'}$. By the formula for dimensions of irreducible representations of $U_q(gl_n)$ and by Corollary 3 we have $\dim \hat{T}_{m,m'} = \dim \mathcal{H}_{m,m'}$. Therefore, $\hat{T}_{m,m'}$ is equivalent to $T_{m,m'}$. Proposition is proved.

Thus, we proved that the action of the algebra $U_q(gl_n)$ on the space $\mathcal{A}$ realizes the irreducible representations $T_{m,m'}$ on the subspaces $\mathcal{H}_{m,m'}$ of homogeneous $q$-harmonic polynomials, respectively.
We denote by \( \mathcal{A}^{U_q(\mathfrak{g}_n)} \) the space of elements of \( \mathcal{A} \) consisting of invariant elements with respect to the action of \( U_q(\mathfrak{g}_n) \).

**Proposition 5:** We have \( \mathcal{A}^{U_q(\mathfrak{g}_n)} = \mathbb{C}[Q] \) and

\[
\mathcal{A}^{U_q(\mathfrak{g}_{n-1})} \simeq \bigoplus_{k,l} \mathbb{C}[Q_{n-1}] z_{n}^{k} w_{n}^{l} \simeq \bigoplus_{k,l} \mathbb{C}[Q] z_{n}^{k} w_{n}^{l}.
\]

**Proof:** The formula (23) leads to the decomposition of the representation \( L \) on \( \mathcal{A} \) into irreducible subrepresentations of \( U_q(\mathfrak{g}_n) \) (the representation multiple to the irreducible representation \( T_{m,m'} \) is realized on \( \mathbb{C}[Q] \otimes H_{m,m'} \)). Since the trivial representation of \( U_q(\mathfrak{g}_n) \) is realized only on \( H_{0,0} \), then \( \mathcal{A}^{U_q(\mathfrak{g}_n)} \) coincides with \( \mathbb{C}[Q] \otimes H_{0,0} \equiv \mathbb{C}[Q] \simeq \mathbb{C}[Q] \).

In order to prove the second equality we note that for \( U_q(\mathfrak{g}_{n-1}) \)-module \( \mathcal{A} \) we have

\[
\mathcal{A} = \mathbb{C}[z_1, w_1, \ldots, z_n, w_n] = \bigoplus_{k,l} \mathbb{C}[z_1, w_1, \ldots, z_{n-1}, w_{n-1}] z_{n}^{k} w_{n}^{l}.
\]

The action of the subalgebra \( U_q(\mathfrak{g}_{n-1}) \) on monomials \( z_{n}^{k} w_{n}^{l} \) is trivial. Moreover, \( \mathbb{C}[z_1, w_1, \ldots, z_n, w_n]^{U_q(\mathfrak{g}_{n-1})} = \mathbb{C}[Q_{n-1}] \), where \( Q_{n-1} = z_1 w_1 + \cdots + z_{n-1} w_{n-1} \).

Since \( Q = Q_{n-1} + z_n w_n \), we have \( \mathcal{A}^{U_q(\mathfrak{g}_{n})} \simeq \bigoplus_{k,l} \mathbb{C}[Q_{n-1}] z_{n}^{k} w_{n}^{l} \simeq \bigoplus_{k,l} \mathbb{C}[Q] z_{n}^{k} w_{n}^{l} \).

Proposition is proved.

**VI. THE DUAL PAIR \((U_q(\mathfrak{sl}_2), U_q(\mathfrak{g}_n))\)**

The formulas

\[
ke = q^2 ek, \quad kf = q^{-2} fk, \quad ef - fe = \frac{k - k^{-1}}{q - q^{-1}} \tag{26}
\]

determine the quantum algebra \( U_q(\mathfrak{sl}_2) \) generated by the elements \( k, k^{-1}, e, f \). Let \( \mathcal{L}(\mathcal{A}) \) be the space of linear operators on the algebra \( \mathcal{A} \). It is directly verified by means of formula (18) that the operators

\[
\omega(k) = q^n \gamma \tilde{\gamma}, \quad \omega(e) = q^{-n+1} \tilde{Q}, \quad \omega(f) = -\Delta_q \tag{27}
\]

satisfy relations (26). This means that the algebra homomorphism \( \omega : U_q(\mathfrak{sl}_2) \to \mathcal{L}(\mathcal{A}) \) uniquely determined by formulas (27) is a representation of \( U_q(\mathfrak{sl}_2) \).

Since the operators \( \omega(k), \omega(e), \omega(f) \) commute with the operators \( L(X), X \in U_q(\mathfrak{g}_n) \) we can introduce the representation \( \omega \times L \) of the algebra \( U_q(\mathfrak{sl}_2) \times U_q(\mathfrak{g}_n) \) on \( \mathcal{A} \), where \( L \) is the above defined natural action of \( U_q(\mathfrak{g}_n) \) on \( \mathcal{A} \). This representation is reducible.

Let us decompose it into irreducible constituents.

By (23), we have \( \mathcal{A} = \bigoplus_{m,m' \geq 0} (\mathbb{C}[Q] \otimes H_{m,m'}) \). The subspaces \( \mathbb{C}[Q] \otimes H_{m,m'} \) are invariant under \( U_q(\mathfrak{sl}_2) \times U_q(\mathfrak{g}_n) \), since the space \( \mathbb{C}[Q] \) is elementwise invariant under \( U_q(\mathfrak{g}_n) \), and for \( f \in \mathbb{C}[Q] \) and \( h_{m,m'} \in H_{m,m'} \) we have

\[
\tilde{Q}(f(Q) \otimes h_{m,m'}) = Qf(Q) \otimes h_{m,m'}, \tag{28}
\]

\[
\Delta_q(Q^r \otimes h_{m,m'}) = q^{r-1}[r] (r + m + m' + n - 1)! Q^{r-1} \otimes h_{m,m'}, \tag{29}
\]

\[
\gamma \tilde{\gamma}(Q^r \otimes h_{m,m'}) = q^{2r+m+m'} (Q^r \otimes h_{m,m'}). \tag{30}
\]

These formulas show that \( U_q(\mathfrak{sl}_2) \) acts on \( \mathbb{C}[Q] \) and \( U_q(\mathfrak{g}_n) \) acts on \( H_{m,m'} \). However, this action of \( U_q(\mathfrak{sl}_2) \) depends on the component \( H_{m,m'} \). Taking the basis

\[
|r| := q^{-r(n-1)}(r + m + m' + n - 1)!^{-1} Q^r, \quad r = 0, 1, 2, \ldots,
\]
in the space \( \mathbb{C}[Q] \), we find from (28)–(30) that
\[
\omega(k)|r\rangle = q^{2r+m+m'+n}|r\rangle, \quad \omega(f)|r\rangle = -|r|/r - 1,
\]
\[
\omega(e)|r\rangle = |r+m+m'+n|/r + 1).
\]
Comparing this representation with the known irreducible representations of \( U_q(sl_2) \) (see, for example, Ref. 11) we derive that the irreducible representation of \( U_q(sl_2) \) of the discrete series with lowest weight \( m + m' + n \) is realized on the component \( \mathbb{C}[Q] \otimes \mathcal{H}_{m,m'} \). We denote this representation of \( U_q(sl_2) \) by \( D_{m+m'+n} \).

Thus, we have derived that on the subspace \( \mathbb{C}[Q] \otimes \mathcal{H}_{m,m'} \subset A \) the irreducible representation \( D_{m+m'+n} \times T_{m,m'} \) of the algebra \( U_q(sl_2) \times U_q(gl_n) \) acts. This means that for the reducible representation \( \omega \otimes L \) we have the following decomposition into irreducible components:
\[
\omega \otimes L = \bigoplus_{m,m'} D_{m+m'+n} \times T_{m,m'},
\]
that is, each irreducible representation of \( U_q(gl_n) \) in this decomposition determines uniquely the corresponding irreducible representation of \( U_q(sl_2) \) and vice versa. This means that \( U_q(sl_2) \) and \( U_q(gl_n) \) constitute a dual pair under the action on \( A \).

VII. RESTRICTION OF \( q \)-HARMONIC POLYNOMIALS ONTO THE QUANTUM SPHERE

The associative algebra \( \mathcal{F}(S_{q,n-1}^C) \) generated by the elements \( z_1, \cdots, z_n, w_1, \cdots, w_n \) satisfying the relations (1)–(3) and the relation
\[
z_1w_1 + z_1w_1 + \cdots + z_nw_n = 1
\]
is called the algebra of functions on the quantum sphere \( S_{q,n-1}^C \) (see Refs. 6, Chap. 11, and 7). It is clear that the following canonical algebra isomorphism has place:
\[
\mathcal{F}(S_{q,n-1}^C) \simeq A/I,
\]
where \( I \) is the two-sided ideal of \( A \) generated by the element \( Q - 1 = \sum_i z_iw_i - 1 \). We denote by \( \tau \) the canonical algebra homomorphism
\[
\tau : A \rightarrow A/I \simeq \mathcal{F}(S_{q,n-1}^C).
\]

This homomorphism is called the restriction of polynomials of \( A \) onto the quantum sphere \( S_{q,n-1} \).

**Proposition 6:** We have \( \tau \mathcal{H} \simeq \mathcal{F}(S_{q,n-1}^C) \). This means that \( \tau : \mathcal{H} \rightarrow \mathcal{F}(S_{q,n-1}^C) \) is a one-to-one mapping, that is, the restriction of a \( q \)-harmonic polynomial to the sphere \( S_{q,n-1}^C \) determines this polynomial uniquely.

**Proof:** By Theorem 1, we have \( \mathcal{F}(S_{q,n-1}^C) = \tau A = \tau(\mathbb{C}[Q] \otimes \mathcal{H}) = \tau \mathcal{H} \). Since \( Q \) is invariant with respect to the action of the algebra \( U_q(gl_n) \), then the ideal \( I \) is an invariant subspace under the action of \( U_q(gl_n) \) on \( A \). Therefore, an action of \( U_q(gl_n) \) on \( A/I \) is defined. This action coincides with the action in Ref. 8. The homomorphism \( \tau \) intertwines the action of \( U_q(gl_n) \) on \( A \) and on \( A/I \). Since \( \tau \mathcal{H}_{m,m'} \neq \{0\} \), then the action of \( U_q(gl_n) \) realizes the same irreducible representation on \( \mathcal{H}_{m,m'} \) and on \( \tau \mathcal{H}_{m,m'} \). This means that \( \dim \mathcal{H}_{m,m'} = \dim \tau \mathcal{H}_{m,m'} \), that is, the mapping \( \tau \) is one-to-one on \( \mathcal{H}_{m,m'} \). Therefore, it is one-to-one on \( \mathcal{H} \). Proposition is proved.
Proposition 6 allows us to determine a scalar product on \( \mathcal{H} \). For this, we use the invariant functional \( h \) on the quantum sphere defined in Ref. 8, section 4.2. This functional \( h \) is determined by introducing a linear gradation in \( \tau \mathcal{A} \): \( \tau \mathcal{A} = \sum_{\lambda \in \mathbb{Z}^n} (\tau \mathcal{A})^\lambda \), where \( (\tau \mathcal{A})^\lambda = \{ p \in \tau \mathcal{A} \mid p(tz, t^{-1}w) = t^\lambda p(z, w) \} \), \( t = (t_1, t_2, \ldots, t_n) \) are \( n \) indeterminates, and

\[
\text{tz} = (t_1z_1, \ldots, t_nz_n), \quad t^{-1}w = (t_1^{-1}w_1, \ldots, t_n^{-1}w_n), \quad t^\lambda = t_1^{\lambda_1} \cdots t_n^{\lambda_n}.
\]

The subalgebra \( (\tau \mathcal{A})^0 \) is spanned by the monomials \( z_1^{\mu_1} \cdots z_n^{\mu_n} w_1^{\nu_1} \cdots w_n^{\nu_n} \) (or by the monomials \( w_1^{\mu_1} \cdots w_n^{\mu_n} z_1^{\nu_1} \cdots z_n^{\nu_n} \)), \( \mu_i = 0, 1, 2, \ldots \). The functional \( h \) is defined as a linear mapping \( h : \tau \mathcal{A} \to \mathbb{C} \) such that \( h(p) = 0 \) if \( p \in (\tau \mathcal{A})^\lambda \), \( \lambda \neq 0 \), and

\[
h(w_1^{\mu_1} \cdots w_n^{\mu_n} z_1^{\nu_1} \cdots z_n^{\nu_n}) = \frac{(q^2; q^2)^{\mu_1} \cdots (q^2; q^2)^{\nu_n} (q^2; q^2)^{n-1}}{(q^2; q^2)^{\mu_1+\cdots+\mu_n+n-1}}.
\]

The following assertions are proved in Ref. 8:

(a) The subalgebra \( (\tau \mathcal{A})^0 \) is a commutative algebra generated by the elements \( Q_{n-1}, Q_{n-2}, \ldots, Q_1 \).

(b) The algebra \( (\tau \mathcal{A})^0 \) is isomorphic to the polynomial algebra in \( n-1 \) commuting indeterminates.

(c) For any polynomial \( p(z, w) = f(Q_1, \ldots, Q_{n-1}) \in (\tau \mathcal{A})^0 \) the value \( h(p) \) is expressed in term of Jackson integral:

\[
h(p) = \frac{(q^2; q^2)^{n-1}}{(1-q^2)^{n-1}} \int_0^1 \cdots \int_0^1 f(Q_1, \ldots, Q_{n-1}) d_q Q_1 \cdots d_q Q_{n-1}
\]

(the definition of Jackson integral see, for example, in Ref. 12, Chap. 1).

Now we can introduce a scalar product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{H} \):

\[
\langle p_1, p_2 \rangle = h((\tau p_1)(\tau p_2)^*),
\]

where \( a^* \) determines an element conjugate to \( a \in \mathcal{A} \) under action of the *-operation.

**Proposition 7:** We have \( \mathcal{H}_{m, m'} \perp \mathcal{H}_{r, r'} \) if \( (m, m') \neq (r, r') \).

Proof follows from the fact that \( (\tau p_1)(\tau p_2)^* \notin (\tau \mathcal{A})^0 \) if \( p_1 \in \mathcal{H}_{m, m'}, p_2 \in \mathcal{H}_{r, r'} \), and \( (m, m') \neq (r, r') \).

**VIII. THE PROJECTION** \( \mathcal{A}_{m, m'} \to \mathcal{H}_{m, m'} \)

Let us go back to the decomposition (20) and construct the projector

\[
\mathcal{H}_{m, m'} : \mathcal{A}_{m, m'} = \mathcal{H}_{m, m'} \oplus Q \mathcal{A}_{m-1, m'-1} \to \mathcal{H}_{m, m'}.
\]

We present this projector in the form

\[
\mathcal{H}_{m, m'} p = \sum_{k=0}^{\min(m, m')} \alpha_k Q^k \Delta^k p, \quad \alpha_k \in \mathbb{C}, \quad p \in \mathcal{A}_{m, m'}.
\]

We have to calculate values of the coefficients \( \alpha_k \). In order to do this, we act by the operator \( \Delta_\mu \) upon both parts of (32) and use the relation (18). Under this action, the left hand side vanishes. Equating the right hand side to 0, we derive a recurrence relation

\[
q^{m-1}[k][m + m' + n - k - 1] + \alpha_k + \alpha_{k-1} = 0
\]
for $\alpha_k$ which gives

$$\alpha_k = (-1)^k q^{-(n-1)k} \frac{(m+m'+n-k-2)!}{[k]!(m+m'+n-2)!}$$

(33)

where $[s]! = [s][s-1][s-2] \cdots [1]$ for $s \neq 0$ and $[0]! = 1$.

Note that the coefficients $\alpha_k$ are determined by the recurrence relation uniquely up to a constant. In (33) we have chosen this constant in such a way that $H_{m,m'}p = p$ for $p \in \mathcal{H}_{m,m'}$. This means that $\mathcal{H}_{m,m'} = \mathcal{H}_{m,m'}$.

**Proposition 8:** The operator $H_{m,m'}$ commutes with the action of $U_q(gl_n)$, that is, with the operators of the representation $L_{m,m'}$ of $U_q(gl_n)$.

**Proof:** This assertion follows from the fact that the operators $L_{m,m'}(X)$, $X \in U_q(gl_n)$, commute with $\hat{Q}$ and $\Delta_q$ (see Proposition 2). Proposition is proved.

A polynomial $\varphi$ of the space $\mathcal{H}_{m,m'}$ is called **zonal** if it is invariant with respect to operators $L_{m,m'}(X)$, $X \in U_q(gl_{n-1})$. We shall show below that zonal polynomials can be expressed in terms of the basic hypergeometric function $\varphi_1$ which is defined by the formula

$$\varphi_1(a, b; c; q, x) = \sum_{k=0}^{\infty} \frac{(a; q)_k(b; q)_k}{(c; q)_k(q; q)_k} x^k$$

(see Refs. 12 and 13 for properties of this function).

**Proposition 9:** (a) The subspace of zonal polynomials in $\mathcal{H}_{m,m'}$ is one-dimensional.

(b) Up to a constant, a zonal polynomial of $\mathcal{H}_{m,m'}$ is given by the formula

$$\varphi_{m,m'} = z_{m,m'} Q^m \sum_{s=0}^{m'} \frac{(q^{-2m'}; q^2)_s(q^{2(m+n-1)}; q^2)_s Q^s}{(q^{2(n-1)}; q^2)_s(q^2; q^2)_s} Q^{s-1} q^{2s}$$

(34)

if $m \geq m'$ and by the formula

$$\varphi_{m,m'} = Q^m \sum_{s=0}^{m'} \frac{(q^{-2m}; q^2)_s(q^{2(m'+n-1)}; q^2)_s Q^s}{(q^{2(n-1)}; q^2)_s(q^2; q^2)_s} Q^{s-1} q^{2s} w_{m'-m}^{m'}$$

(35)

if $m \leq m'$.

**Proof:** (a) As we have seen, the irreducible representation $T_{m,m'}$ of $U_q(gl_n)$ with highest weight $(m, 0, \cdots, 0, -m')$ is realized on $\mathcal{H}_{m,m'}$. It is known that this representation, under restriction to $U_q(gl_{n-1})$, contains trivial (one-dimensional) representation of this subalgebra with multiplicity 1. This proves the first assertion.

(b) We construct a zonal polynomial of $\mathcal{H}_{m,m'}$ by using the projection operator $H_{m,m'}$. In order to do this, we have to take a polynomial $p \in \mathcal{A}_{m,m'}$ invariant with respect to $U_q(gl_{n-1})$ and to act upon it by the operator $H_{m,m'}$. Since the projector $H_{m,m'}$ commutes with the action of $U_q(gl_{n-1})$, a polynomial obtained in this way is a zonal polynomial. Clearly, the polynomial $p = z_n^m w_n^{m'}$ belongs to $\mathcal{A}_{m,m'}$ and is invariant under the action of $U_q(gl_{n-1})$. In order to find an expression for $H_{m,m'}(z_n^m w_n^{m'})$ we first assume that $m \geq m'$.

Using the second expression for $\Delta_q$ in (17) and relation $\tilde{\partial}_n \hat{z}_n = \hat{z}_n \tilde{\partial}_n$ we have

$$\varphi_{m,m'} := H_{m,m'}(z_n^m w_n^{m'}) = \sum_{s=0}^{m'} \alpha_s \hat{Q}^s \Delta_q z_n^m w_n^{m'}$$

$$= \sum_{s=0}^{m'} \alpha_s \hat{Q}^s z_n^m w_n^{m'}$$

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where we used formula (II.6) from Appendix II in Ref. 12.

Taking into account the expression for the coefficients \( \alpha_s \) and using the formulas

\[
[s] = \frac{(q^2;q^2)_{s}(1-q)^{s}}{(q-q^{-1})^s q^{-s(s+1)/2}}, \quad [m] = \frac{(q^{-2m};q^2)_{s}}{(q-q^{-1})^{s} q^{ms-m(s-1)/2}}
\]

we obtain

\[
\varphi_{m,m'} = z_{n}^{m-m'} \sum_{s=0}^{m'} q^{2s} (q^2;q^2)_s (q^{-2m'};q^2)_s \frac{z^s}{z^m q^{s(s+1)/2}} w_{n}^{m'-s}.
\]

Using the first relation in (16) we obtain from (36) that

\[
\varphi_{m,m'} = Q^{m'} z_{n}^{m-m'} \sum_{s=0}^{m'} q^{2s} \frac{(q^2;q^2)_s (q^{-2m'};q^2)_s}{(q^2;q^2)_s (q^{-2(m+m'+n-2)};q^2)_s} Q^{-m'-s} w_{n}^{m'-s}.
\]

Since (see relation (II.4) from Appendix II in Ref. 12)

\[
(Q_{n-1}/Q; q^{-2})_{m'-s} = \sum_{\nu=0}^{m'-s} q^{2\nu} (q^{-2(m'-s)}; q^2)_\nu Q_{n-1}/Q^\nu,
\]

we have

\[
\varphi_{m,m'} = Q^{m'} z_{n}^{m-m'} \sum_{s=0}^{m'} \frac{q^{2s} (q^{-2m';q^2})_s(q^{-2m'};q^2)_s}{(q^2;q^2)_s (q^{-2(m+m'+n-2)};q^2)_s} \sum_{\nu=0}^{m'-s} q^{2\nu} (q^{-2(m'-s)}; q^2)_\nu Q_{n-1}/Q^\nu.
\]

Applying relation (I.7) and then relation (I.13) from Appendix I in Ref. 12 we find

\[
(q^{-2(m'-s)}; q^2)_\nu = (-1)^\nu q^{-2m'\nu} q^{\nu(\nu-1)} \frac{(q^{2m'-2\nu+2}; q^2)_\nu (q^{-2m'+2\nu}; q^2)_s}{(q^{-2m'}; q^2)_s}.
\]

Therefore, for the sum over \( s \) in (37) (which will be denoted by \( I_\nu \)) we obtain the expression

\[
I_\nu = (-1)^\nu q^{-2m'\nu} q^{\nu(\nu-1)} \frac{(q^{2m'-2\nu+2}; q^2)_\nu (q^{-2m'+2\nu}; q^2)_s}{(q^2;q^2)_s} \sum_{s=0}^{m'-\nu} \frac{(q^{-2m'+2\nu}; q^2)_s(q^{-2m'}; q^2)_s q^{2s}}{(q^2;q^2)_s (q^{-2(m+m'+n-2)};q^2)_s}.
\]

The sum over \( s \) here is the basic hypergeometric function

\[
\varphi_1(q^{-2m}; q^{-2m'+2\nu}; q^{2}; q^2) = \frac{q^{-2m'+2\nu} q^{-2m'} q^{\nu(\nu-1)} q^{-2m'+2\nu}}{(q^{-2(m+m'+n-2)}; q^2)_{m'-\nu}},
\]

where we used formula (II.6) from Appendix II in Ref. 12.

Therefore, for the function \( \varphi_{m,m'} \) we have the expression

\[
\varphi_{m,m'} = Q^{m'} z_{n}^{m-m'} \sum_{\nu=0}^{m' \nu} Q_{n-1}/Q^\nu (-1)^\nu q^{-2m'\nu} q^{\nu(\nu-1)} q^{-2m'+2\nu}
\]

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monic polynomials on the unitary (complex Euclidean) space $E^1$. If $m$ can be written in terms of these polynomials as

$$q^m \leq q^m$$

By formula (I.8) of Appendix I in Ref. 12 we have

$$q^{2m-2\nu+2}, q^2 = (q^{2m-2\nu+4}, q^2)_{\nu}(q^{2m-2n+4}, q^2)_{\nu}(q^{2m+2n-2}, q^2)_{m-\nu}.$$ 

And by formula (I.11) from Appendix I in Ref. 12 we obtain

$$\frac{(q^{2m-2n+4}, q^2)_{\nu}}{(q^{2m+2n-2}, q^2)_{m-\nu}} = q^{-2m\nu} \frac{(q^{2m-2n+4}, q^2)_{\nu}}{(q^{2m+2n-2}, q^2)_{m-\nu}}.$$ 

For this reason, we have

$$H_{m,m'}(z_{n}^m, w_{n}^{m'}) = \varphi_{m,m'} = q^{-2m\nu} \frac{(q^{2m-2n+4}, q^2)_{\nu}}{(q^{2m+2n-2}, q^2)_{m-\nu}} Q_{m'-m}^m \times$$

$$\frac{m'}{m} \sum_{n=0}^{m'} \frac{(q^{2m'}, q^2)_{\nu}(q^{2n+1}, q^2)_{\nu}, Q_{n-1}^m}{(q^{2n+1}, q^2)_{\nu}} q^{2n}$$

$$= \frac{(q^{2n-1}, q^2)_{m'} Q_{m'-m}^m}{(q^{2n+1}, q^2)_{m'}} 2\varphi_1(q^{-2m'}, q^{2n+1}, q^2)_{\nu} Q_{n-1}^m.$$

This proves the second assertion of the proposition for the case $m \geq m'$. The case $m < m'$ is proved in the same way. Proposition is proved.

The formula

$$P_k^{(n, \beta)}(x; q) = 2\varphi_1(q^{-k}, q^{\alpha+\beta+k+1}; q^2; q^k)$$

defines the so-called little $q$-Jacobi polynomials. The zonal polynomials from Proposition 9 can be written in terms of these polynomials as

$$\varphi_{m,m'} = Q^{m'} z_{n}^{m-m'} P_{m'}^{(n-2, m-m')} (Q_{n-1}^m, q^2)$$

if $m \geq m'$ and as

$$\varphi_{m,m'} = Q^m P_{m}^{(n-2, m-m')} (Q_{n-1}^m, q^2) w_{n}^{m'-m}$$

if $m \leq m'$. Restricting these polynomials onto the quantum sphere $S_{q,n-1}^C$ we obtain

$$\tau_{\varphi_{m,m'}} = z_{n}^{m-m'} P_{m'}^{(n-2, m-m')} (Q_{n-1}^m, q^2)$$

if $m \geq m'$ and as

$$\tau_{\varphi_{m,m'}} = P_{m}^{(n-2, m-m')} (Q_{n-1}^m, q^2) w_{n}^{m'-m}$$

if $m \leq m'$. These polynomials are called zonal spherical functions on the quantum sphere $S_{q,n-1}^C$ and were calculated in Ref. 8 (see also Refs. 3 and 4).

IX. $q$-ANALOGUE OF ASSOCIATED SPHERICAL HARMONICS WITH RESPECT TO $U_q(\mathfrak{gl}_{n-1})$

It is known (see Ref. 1, Chap. 11) that in the space of classical homogeneous harmonic polynomials on the unitary (complex Euclidean) space $E_n^C$ there exist different
orthonormal bases. They correspond to different separations of variables. Each separation of variables corresponds to a certain chain of subgroups of the unitary group \( U(n) \). A similar picture has place for the spaces \( \mathcal{H}_{m,m'} \) of homogeneous \( q \)-harmonic polynomials. We consider in this section a \( q \)-analogue of separation of variables corresponding to spherical coordinates on the sphere \( S^n_{q} \) (see Ref. 1, Chap. 11).

In the classical case, the tree method distinguishes different separations of variables. Different separations of variables are in a one-to-one correspondence with different chains of subgroups of \( U(n) \). The same tree method can be used for \( q \)-harmonic polynomials, but instead of chains of subgroups of \( U(n) \) we have to take the corresponding chains of subalgebras of the algebra \( U_q(gl_n) \). A certain orthogonal basis corresponds to such a chain of subalgebras.

The aim of this section is to construct an orthogonal basis of the space \( \mathcal{H}_{m,m'} \) of homogeneous \( q \)-harmonic polynomials which corresponds to the chain

\[
U_q(gl_n) \supset U_q(gl_{n-1}) \supset \cdots \supset U_q(gl_2) \supset U_q(gl_1).
\]

This basis is a \( q \)-analogue of the set of associated spherical harmonics on the complex vector space which are products of certain Jacobi polynomials (see, Ref. 1, Chap. 11).

The basis elements give solutions of the equation \( \Delta_q \psi = 0 \) in "separated coordinates".

So, we obtain a \( q \)-analogue of the classical separation of variables.

**Lemma 1:** Let \( f_k(z') \) and \( g_l(w') \) be homogeneous polynomials of degrees \( k' \) in \( z' = (z_1, z_2, \ldots, z_{n-1}) \) and of degrees \( l' \) in \( w' = (w_1, w_2, \ldots, w_{n-1}) \), respectively. Then for any nonnegative integers \( k \) and \( l \) we have

\[
\Delta_q(z_n^k w_n^l f_k(z') g_l(w')) = q^{l-k} z_n^{k'} w_n^{l'} \Delta_{q,n-1}(f_k(z') g_l(w'))
\]

or

\[
\Delta_q(z_n^k w_n^l f_k(z') g_l(w')) = q^{2(l-k)} q^{l-k} \sum_{i=1}^{n-1} \tilde{\Delta}_i \partial_i \partial_i f_k(z') g_l(w'),
\]

where \( \Delta_{n-1} = \sum_{i=1}^{n-1} q^{2l-i} \tilde{\Delta}_i \partial_i \) is the \( q \)-Laplace operator for the elements \( z' = (z_1, \ldots, z_{n-1}) \) and \( w' = (w_1, \ldots, w_{n-1}) \).

**Proof:** Using the relations for the operators from section III we derive

\[
\tilde{\Delta}_n \partial_n z_n^{k'} w_n^{l'} f_k(z') g_l(w') = q^{(l-k)' \partial_n \partial_n f_k(z') z_n^{k'} w_n^{l'} g_l(w')}
\]

\[
= q^{l-k}' \partial_n \partial_n f_k(z') z_n^{k-1} w_n^{l} g_l(w') = q^{l-k} \partial_n z_n^{k-1} f_k(z') w_n^{l} g_l(w')
\]

\[
= q^{l-k} \partial_n z_n^{k-1} f_k(z') w_n^{l} g_l(w') = q^{-2k'+l(k'+l)} \partial_n z_n^{k-1} f_k(z') g_l(w') w_n^{l}
\]

\[
= q^{-2k' + l} \partial_n z_n^{k-1} f_k(z') w_n^{l} g_l(w') = q^{-2k' + l} \partial_n z_n^{k-1} w_n^{l} f_k(z') g_l(w').
\]

Since \( \partial_i w_n = q \tilde{w}_i \partial_i + (1 - q^2) \tilde{w}_i \partial_i \), \( i < n \), and \( \partial_n (w_n^l f_k(z') g_l(w')) = 0 \), we have

\[
\tilde{\Delta}_i \partial_i z_n^{k'} w_n^{l'} f_k(z') g_l(w') = \tilde{\Delta}_i z_n^{k'} \partial_i w_n^{l'} f_k(z') g_l(w') = q^l \tilde{\Delta}_i z_n^{k'} \partial_i f_k(z') g_l(w').
\]

Using recurrently the relation \( \tilde{\Delta}_i z_n^{k'} = q^{-1} \tilde{z}_i \partial_i + (1 - q^2) q^{2(l-i)} \tilde{z}_i \partial_i \), we obtain

\[
\tilde{\Delta}_i \partial_i z_n^{k'} w_n^{l'} f_k(z') g_l(w') = q^{-k} \tilde{z}_n^{k'} w_n^{l'} \tilde{\Delta}_i \partial_i f_k(z') g_l(w')
\]

\[
+ q^{2(l-i)} (1 - q^2) q^{l+i} \tilde{z}_i \partial_i f_k(z') g_l(w').
\]

Thus, one has

\[
\Delta_{n-1}(z_n^{k'} w_n^{l'} f_k(z') g_l(w')) = q^{-k} \tilde{z}_n^{k'} w_n^{l'} \Delta_{n-1} f_k(z') g_l(w').
\]

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where the relation
\[ \sum_{i=1}^{n-1} \bar{z}_i \partial_i f_{k'}(z') g_{\nu'}(w') = \sum_{i=1}^{n} \bar{z}_i \partial_i f_{k'}(z') g_{\nu'}(w') = \{ \gamma \} f_{k'}(z') g_{\nu'}(w') = [k'] f_{k'}(z') g_{\nu'}(w') \]
has been used. From the above results and from the equality \( \Delta_{q} = q^{2(n-1)} \partial_{i} \partial_{n} + \Delta_{n-1} \), the lemma follows.

**Proposition 10:** Let \( s \) and \( s' \) be integers such that \( 0 \leq s \leq m \) and \( 0 \leq s' \leq m' \). Let \( h_{s,s'}(z',w') \) be a homogeneous harmonic polynomial of degree \( s \) in \( z' = (z_1, z_2, \cdots, z_{n-1}) \) and of degree \( s' \) in \( w' = (w_1, w_2, \cdots, w_{n-1}) \). Then for \( z_{n}^{m-s} u_{n}^{m-s'} h_{s,s'}(z',w') \in A_{m,m'} \) we have
\[ H_{m,m'}(z_{n}^{m-s} u_{n}^{m-s'} h_{s,s'}(z',w')) = z_{n}^{m-s-m'+s'} Q^{m'-s'} d_{s,s'}^{m'n'} h_{s,s'}(z',w'), \quad (39) \]
where \( m - s \geq m' - s' \),
\[ d_{s,s'}^{m'n'} = q^{-2(m-s)(m'-s')}(q^{-2m' - 2s - 2n + 4}; q^{2})_{m'-s'} \times 2\varphi_1(q^{-2(m'-s')}; q^{2(s+s'+n-1)}; q^{2}; q^{2} Q_{n-1}/Q), \]
and
\[ H_{m,m'}(z_{n}^{m-s} u_{n}^{m-s'} h_{s,s'}(z',w')) = Q^{m-s} a_{s,s'}^{m'n'} u_{n}^{m'-s'-s+m} h_{s,s'}(z',w'), \quad (40) \]
where \( m - s \leq m' - s' \),
\[ d_{s,s'}^{m'n'} = q^{-2(m-s)(m'-s')}(q^{-2m' - 2s - 2n + 4}; q^{2})_{m-s} \times 2\varphi_1(q^{-2(m-s)}; q^{2(m'+s+n-1)}; q^{2}; q^{2} Q_{n-1}/Q). \]

**Proof:** The proof of this proposition is similar to that of Proposition 6 and we shall omit details. Taking into account formula (32) for the projector \( H_{m,m'} \) and Lemma 1, we obtain
\[ H_{m,m'}(z_{n}^{m-s} u_{n}^{m-s'} h_{s,s'}(z',w')) = \sum_{k=0}^{\min(m,m')} \alpha_{k} Q^{k} \Delta_{q} z_{n}^{m-s} u_{n}^{m'-s'} h_{s,s'}(z',w') \]
\[ = \sum_{k=0}^{b} \alpha_{k} Q^{k} q^{2(n-1)k}(q^{s+s'})^{k} \frac{[m-s]![m'-s']!}{[m-s-k]![m'-s'-k]!} z_{n}^{m-s-k} u_{n}^{m'-s'-k} h_{s,s'}(z',w'), \]
where \( b = \min(m-s, m'-s') \). Let \( m - s \geq m' - s' \), then
\[ H_{m,m'}(z_{n}^{m-s} u_{n}^{m-s'} h_{s,s'}(z',w')) = z_{n}^{m-s-m'+s'} Q^{m'-s'} a_{s,s'}^{m'n'} h_{s,s'}(z',w'), \]
where
\[ d_{s,s'}^{m'n'} = \sum_{k=0}^{m'-s'} q^{k} \frac{(q^{-2(m-s)}; q^{2})_{k}(q^{-2(m'-s')}; q^{2})_{k}}{(2; q^{2})_{k}(q^{-2(m'+s+n-2)}; q^{2})_{k}} \sum_{\nu=0}^{d} q^{\nu} \frac{(q^{-2(m'-s'-k)}; q^{2})_{\nu}}{(q^{2}; q^{2})_{\nu}} Q^{\nu}. \]
Here $d = m' - s' - k$. Changing the order of summations in the last expression we have

$$d_{ss'}^{mn'} = \sum_{\nu=0}^{\sigma'} \left( Q_{n-1}/Q \right)^{2\nu} q^{2\nu} \sigma'^{-\nu} \sum_{k=0}^{\nu} \frac{q^{2k}}{(q^2; q^2)_k} \times \frac{(q^{-2(m-s)}; q^2)_k (q^{-2(m-s'-k)}; q^2)_\nu (q^{-2(m'-s')}; q^2)_k}{(q^2; q^2)_k (q^{2(m+m' + n + 2)}; q^2)_k},$$

(41)

where $\sigma' = m' - s'$. Since

$$\frac{(q^{-(m-s'-k)}; q^2)_\nu}{(q^{-2(m-s'-k)}; q^2)_\nu} = q^{\nu(\nu-1)} (-q^{-2(m-s'-k)}) \nu \left( q^{2m'-2s'-2k} q^{-2\nu+2}; q^2 \right)_\nu$$

$$= (-1)^\nu q^{\nu(\nu-1)} q^{-2(m-s')\nu} \frac{(q^{2m'-2s'-2\nu+2}; q^2)_\nu (q^{-2(m'+s'+2\nu)}; q^2)_k}{(q^{2m'+2s'+2\nu}; q^2)_k},$$

for the sum over $k$ in (41) we have

$$\frac{(-1)^\nu q^{\nu(\nu-1)} (q^{-2\nu} q^{2m'-2s'+2}; q^2)_\nu}{q^{2(m-s')\nu}} \sum_{k=0}^{\nu} \frac{(q^{-2m'+2s'+2\nu}; q^2)_\nu (q^{-2(m-s)}; q^2)_k}{(q^2; q^2)_k (q^{2(m+m' + n + 2)}; q^2)_k} q^{2k}$$

$$= a_{\nu} (-1)^\nu q^{\nu(\nu-1)} q^{-2(m-s')\nu} 2^{\nu} \left( q^{-2(m-s)}; q^{-2(m-s)\nu} + 2\nu; q^2 \right)$$

$$= a_{\nu} (-1)^\nu q^{\nu(\nu-1)} q^{-2(m-s')\nu} \left( q^{-2m'-2s'-2\nu+4}; q^2 \right)_{m'-s'-\nu} q^{-2(m-s)}(m'-s')$$

$$= a_{\nu} (-1)^\nu q^{\nu(\nu-1)} q^{-2m'-2s'-2\nu+4}(q^{-2m'-2s'-2\nu+4}; q^2)_{m'-s'-\nu} q^{-2(m-s)}(m'-s'),$$

where $a_{\nu} = (q^{-2\nu} q^{2m'-2s'+2}; q^2)_\nu$. Since

$$a_{\nu} \equiv (q^{-2\nu} q^{2m'-2s'+2}; q^2)_\nu = (-1)^\nu q^{\nu(\nu-1)} q^{2(m'-s')\nu} (q^{-2m'+2\nu}; q^2)_\nu,$$

for $d_{ss'}^{mn'}$ we have the expression

$$d_{ss'}^{mn'} = e_{ss'}^{mn'} 2^{\nu} \left( q^{-2(m-s)}; q^{2(m+s'+n+1)}; q^2; q^2 Q_{n-1}/Q \right),$$

where

$$e_{ss'}^{mn'} = q^{-2(m-s)}(m'-s') \frac{(q^{-2m'-2s'-2\nu+4}; q^2)_{m'-s'}}{(q^{-2m'-2s'-2\nu+4}; q^2)_{m'-s'}} = \frac{(q^{2(s+n+1)}; q^2)_{m'-s'}}{(q^{2(m+m'+n+2)}; q^2)_{m'-s'}}.$$

In the case when $m - s \leq m' - s'$, the proof is similar and we omit it. Proposition is proved.

**Remark:** If $n = 2$, then polynomials $h_{s,s'}(z_1, w_1)$ in Proposition 10 are multiple to elements from (21), that is, we have $s = 0$ or $s' = 0$ or $s = s' = 0$.

The expressions for $d_{ss'}^{mn'}$ from Proposition 10 can be represented in terms of little $q$-Jacobi polynomials $P_k^{(\alpha, \beta)}(x; q)$ as

$$d_{ss'}^{mn'} = e_{ss'}^{mn'} P_{m'-s'}^{(s+s'+n-2, m-s-s')} \left( Q_{n-1}/Q \right),$$

if $m - s \geq m' - s'$ and as

$$d_{ss'}^{mn'} = e_{ss'}^{mn} P_{m-s}^{(s+s'+n-2, m-s-s')} \left( Q_{n-1}/Q \right),$$

if $m - s < m' - s'$.
if \( m - s \leq m' - s' \).

We denote the expression \( z_n^{m-s+m'+s'} P^{m-s} q^{m-m'} d_{ss'}^{m,m'} \) from (39) and the expression \( Q^{m-s} q^{m-m'} w_n^{m-s+m+s} \) from (40) by \( \iota_{s,s'}^{m,m'} \). Then

\[
H_{m,m'}(z_n^{m-s} u_n^{m'-s'} h_{s,s'}(z', w')) = \iota_{s,s'}^{n,m,m'} h_{s,s'}(z', w').
\]

Moreover, the space \( H_{m,m'} \) can be represented as the direct sum

\[
H_{m,m'} = \bigoplus_{s=0}^{m} \bigoplus_{s'=0}^{m'} \iota_{s,s'}^{n;m,m'} H_{s,s'}^{(n-1)},
\]

where \( H_{s,s'}^{(n-1)} \) are the corresponding spaces of homogeneous \( q \)-harmonic polynomials in \( z_i, w_i, i = 1, 2, \ldots, n - 1 \). To prove this, we note that the subspaces \( \iota_{s,s'}^{n;m,m'} H_{s,s'}^{(n-1)} \) pairwise do not intersect and \( \bigoplus_{s=0}^{m} \bigoplus_{s'=0}^{m'} \iota_{s,s'}^{n;m,m'} H_{s,s'}^{(n-1)} \subset H_{m,m'} \). Now the equality (43) follows from the fact that dimensions of the spaces on the right and on the left coincide.

To have a correspondence with the classical case, below we denote \( \iota_{s,s'}^{2;m,m'} \) (in this case \( s = 0 \) or \( s' = 0 \)) by \( \iota_{s,s'}^{2;m,m'} \) if \( s' = 0 \) and by \( \iota_{s,s'}^{2;m,m'} \) if \( s = 0 \).

Taking into account the orthogonality relation (7.3.3) in Ref. 12 for little \( q \)-Jacobi polynomials we obtain for the scalar product of \( \iota_{s,s'}^{n;m,m'} h_{s,s'}^{(n-1)} \) and \( \iota_{r,r'}^{n,m,m'} h_{r,r'}^{(n-1)} \), \( h_{r,r'}^{(n-1)} \in \mathcal{H}_{p,p'}^{(n-1)} \), the expression

\[
\langle \iota_{s,s'}^{n;m,m'} h_{s,s'}^{(n-1)}, \iota_{r,r'}^{n,m,m'} h_{r,r'}^{(n-1)} \rangle = \delta_{ss'} \delta_{rr'} (\epsilon_{ss'}^{m,m'})^{-2} b_{ss'}^{mm'} (h_{s,s'}^{(n-1)}, h_{r,r'}^{(n-1)})_{(n-1)},
\]

where \( \langle \cdot, \cdot \rangle_{(n-1)} \) is the scalar product in the space \( \mathcal{H}_{s,s'}^{(n-1)} \) and

\[
b_{ss'}^{mm'} = \frac{(1 - q^2(n+s-s')) q^{2(2m')-s'}(n+s-s') \left(q^2 q^2 m-q^2 q^2 m-s'\right)}{(1 - q^2(2m+n-1))(q^2(2n+s'-s') q^2 m-q^2 q^2 m-s').}
\]

Note that a calculation of this scalar product reduces to \( q \)-integration (see Refs. 3 and 4 on calculation of \( q \)-integrals of this type).

Now we apply the decomposition (43) to the subspaces \( \mathcal{H}_{s,s'}^{(n-1)} \) and obtain

\[
H_{m,m'} = \bigoplus_{s=0}^{m} \bigoplus_{s'=0}^{m'} \bigoplus_{r=0}^{m} \bigoplus_{r'=0}^{m'} \iota_{s,s'}^{n;m,m'} H_{s,s'}^{(n-2)} H_{r,r'}^{(n-2)},
\]

where \( H_{r,r'}^{(n-2)} \) are the subspaces of homogeneous \( q \)-harmonic polynomials in \( z_i, w_i, i = 1, 2, \ldots, n - 2 \). Continuing such decompositions we obtain the decomposition

\[
H_{m,m'} = \bigoplus_{m,m', m_1} \Xi_{m,m', m_1} (z, w),
\]

where the polynomials \( \Xi_{m,m', m_1} \) are given by the formula

\[
\Xi_{m,m', m_1} (z, w) = \iota_{m,m'}^{n-1, m-1} \iota_{m-1, m_1}^{n-2, m_1} \iota_{m_1, m_1}^{3, m, m'} \iota_{m_1, m_1}^{2, m_1} \iota_{m_1, m_1}^{1, m_1}, \tag{44}
\]

and the summation is over all sets of \( 2n - 3 \) integers \( m = (m_{n-1}, \ldots, m_3), \ m' = (m'_{n-1}, \ldots, m'_3), \) \( m_1 \) such that \( m_i \geq 0, m'_i \geq 0, i = 2, 3, \ldots, n - 1, \) \( m_2 \geq m_1 \geq -m'_2, \)

\[
m \geq m_{n-1} \geq m_{n-2} \geq \cdots \geq m_2, \quad m' \geq m'_{n-1} \geq m'_{n-2} \geq \cdots \geq m'_2.
\]
Here \( t_{m_{p-1},m_{p-1}}^{n,m_p,m_p'} \) and \( t_{m_1}^{2,m_2,m_2'} \) are determined by formulas given above and

\[
t^{1:m_1} = z_1^{m_1} \quad \text{for} \quad m_1 > 0, \quad t^{1:0} = 1, \quad t^{1:m_1} = w_1^{-m_1} \quad \text{for} \quad m_1 < 0.
\]

It is easy to show that the basis (44) is orthogonal with respect to the scalar product introduced above.

At \( q = 1 \), polynomials (44) turn into the basis elements of the spaces of homogeneous harmonic polynomials on \( \mathbb{C}^n \) in separated coordinates determined by formulas (2) of section 11.1.4 in Ref. 1. These classical homogeneous harmonic polynomials, restricted to the sphere \( S_{n-1}^c \), coincide with associated spherical functions from section 11.3 in Ref. 1. They are matrix elements of zero column of the corresponding irreducible representations of the group \( U(n) \).

The basis elements (44) give solutions of the equation \( \Delta p = 0 \) in \( \mathcal{H}_{m,m'} \). A representation of solutions in the form (44) can be considered as a \( q \)-analogue of the corresponding classical separation of variables.

In order to have an orthonormal basis in \( \mathcal{H}_{m,m'} \) we replace each \( t_{m_{n-1},m_n,m_n'}^{n-1,m_{n-1},m_{n-1}} \) in the expression (44) for \( \hat{\Xi}_{m,m',m_1}(z,w) \) by

\[
t_{m_{n-1},m_n,m_n'}^{n-1,m_{n-1},m_{n-1}} + e_{m,n,m,n'} \cdot (t_{m_{n-1},m_n,m_n'}^{n-1,m_{n-1},m_{n-1}}) - 1/2 t_{m_{n-1},m_n,m_n'}^{n-1,m_{n-1},m_{n-1}}.
\]

We denote the expression (44) with such the replacement by \( \hat{\Xi}_{m,m',m_1}(z,w) \). These polynomials constitute an orthonormal basis of \( \mathcal{H}_{m,m'} \).

It was shown above that the irreducible representation \( T_{m,m'} \) with highest weight \( \langle m,0,\cdots,0,-m' \rangle \) acts on the space \( \mathcal{H}_{m,m'} \). The following assertion is true.

**Proposition 11:** The operators \( T_{m,m'}(e_j), T_{m,m'}(f_j), T_{m,m'}(k_j) \), corresponding to the generating elements \( e_j, f_j, k_j \) of the algebra \( U_q(\mathfrak{gl}_n) \), act upon the basis elements \( \hat{\Xi}_{m,m',m_1} \equiv \{ m,m',m_1 \} \) as

\[
T_{m,m'}(e_j)|m,m',m_1\rangle = A(m,m')|m,m',m_1\rangle + B(m,m')|m,m',m_1\rangle,
\]

\[
T_{m,m'}(f_j)|m,m',m_1\rangle = A(m,m')^{-1}|m,m',m_1\rangle + B(m,m')^{-1}|m,m',m_1\rangle,
\]

\[
T_{m,m'}(k_j)|m,m',m_1\rangle = q^{m_j-m_{j-1}+m_{j-1}-m_j-1}|m,m',m_1\rangle,
\]

where

\[
A(m,m') = \left( \frac{[m_j-m_{j-1}][m_j+m_{j-1}+j-1][m_j+1-m_{j-1}+j-1]}{[m_{j-1}+m_j+j-2][m_{j-1}+1+m_j+j-1]} \right)^{1/2},
\]

\[
B(m,m') = \left( \frac{[m_j+m_{j-1}+1][m_j+m_{j-1}+j-2][m_{j-1}-m_{j-2}+1]}{[m_{j-1}+1+m_j+j-2][m_{j-1}+m_j+j-3]} \right)^{1/2},
\]

\( m_n \equiv m, m'_n \equiv m', m_j^{\pm 1} \) denotes the set of the numbers \( m_{j-1} \) with \( m_{j-1} \) replaced by \( m_{j-1} \pm 1 \), respectively.

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A proof of this proposition is awkward. Since it is similar to that of Theorem 1 in Ref. 5, we omit it.

X. q-ANALOGUE OF ASSOCIATED SPHERICAL HARMONICS WITH RESPECT TO
$U_q(gl_p) \times U_q(gl_{n-p})$

In section IX we found an orthogonal basis of the space $\mathcal{H}_{mm'}$ of homogeneous $q$-harmonic polynomials corresponding to the chain of subalgebras (38). In this section we shall find orthogonal bases of the same space corresponding to the reductions

$$U_q(gl_n) \supset U_q(gl_p) \times U_q(gl_{n-p}) \supset \cdots.$$ (45)

In the classical case (see Ref. 1, Chap. 11), further reductions can be made taking any chain of subgroups of the groups $U(p)$ and $U(n-p)$. In particular, the usual tree method (see Ref. 1, section 10.2) can be used to describe different chains of these groups corresponding to different orthogonal bases of $\mathcal{H}_{mm'}$. In our case, there are some difficulties with construction of orthogonal bases corresponding to any chain of subalgebras in (45). For this reason, we construct orthogonal bases corresponding to the case, when we take chains of the type (38) for the subalgebras $U_q(gl_p)$ and $U_q(gl_{n-p})$ in (45).

We represent the set $(z, w) = (z_1, \cdots, z_n; w_1, \cdots, w_n)$ as $(y, t)$, where $y = (z_1, z_2, \cdots, z_p, w_1, w_2, \cdots, w_p)$ and $t = (z_{p+1}, \cdots, z_n, w_{p+1}, \cdots, w_n)$. Then the $q$-Laplace operator $\Delta_q$ can be written as

$$\Delta_q = \Delta_{(y)} + \Delta_{(t)},$$ (46)

where

$$\Delta_{(y)} = \partial_{t_1} \tilde{\partial}_1 + \cdots + \partial_{t_p} \tilde{\partial}_p, \quad \Delta_{(t)} = \partial_{p+1} \tilde{\partial}_{p+1} + \cdots + \partial_n \tilde{\partial}_n = \sum_{i=1}^{n-p} q^{2(i-1)} \tilde{\partial}_{i+p} \tilde{\partial}_{p+i}. $$ (47)

The operator $\Delta_q$ can be also represented as

$$\Delta_q = \Delta_{(y)} + q^{2p} \Delta_{(t)},$$

where

$$\hat{\Delta}_{(y)} = \tilde{\partial}_1 \partial_1 + q^2 \tilde{\partial}_2 \partial_2 + \cdots + q^{2(p-1)} \tilde{\partial}_p \partial_p.$$ (48)

We have

$$\Delta_{(y)} - \hat{\Delta}_{(y)} = (1 - q^{2p}) \Delta_{(t)}.$$ (49)

In order to find bases of $\mathcal{H}_{m,m'}$ corresponding to the reduction (45) we take non-negative numbers $r, r', s, s'$ such that

$$u := m - r - s = m' - r' - s' \geq 0.$$ (50)

We wish to find a harmonic projection of the polynomials

$$Q^h_{x,s',r'}(t)h_{r,r'}(y) \in \mathcal{A}_{mm'}, \quad h_{x,s'}(t) \in \tilde{\mathcal{H}}_{s,s'}, \quad h_{r,r'}(y) \in \mathcal{H}_{r,r'}^{(y)},$$

where $Q_y := z_1w_1 + \cdots + z_pw_p$, $\mathcal{H}_{r,r'}^{(y)}$ is the space of homogeneous $q$-harmonic polynomials in $y = (z_1, z_2, \cdots, z_p, w_1, w_2, \cdots, w_p)$, and $\tilde{\mathcal{H}}_{s,s'}^{(t)}$ is the space obtained in the following way. We take the space $\mathcal{H}_{s,s'}^{(n-p)}$ of homogeneous $q$-harmonic polynomials in $(z_1, \cdots, z_{n-p}, w_1, \cdots, w_{n-p})$ and, using the relations between $z_i$ and $w_j$, represent each
Lemma is proved.

of a linear combination of monomials \( z_1, \ldots, z_{n-p} \) stand before the elements \( w_1, \ldots, w_{n-p} \). Then we replace \( z_1, \ldots, z_{n-p}, w_1, \ldots, w_{n-p} \) by \( z_{p+1}, \ldots, z_n, w_{p+1}, \ldots, w_n \), respectively, in each of these polynomials. The space of these polynomials in \( z_{p+1}, \ldots, z_n, w_{p+1}, \ldots, w_n \) is denoted by \( \tilde{H}_{s,s'}^{(t)} \).

Lemma 2: Polynomials \( P \) of \( \tilde{H}_{s,s'}^{(t)} \) satisfy the conditions \( \partial_i P = 0, \bar{\partial}_i P = 0, i = 1, 2, \ldots, p \).

Proof: Fulfillment of the conditions \( \partial_i P = 0, i = 1, 2, \ldots, p \), follow from the construction of polynomials of the space \( \tilde{H}_{s,s'}^{(t)} \). In order to prove the fulfillment of the conditions \( \bar{\partial}_i P = 0, i = 1, 2, \ldots, p \), we note that according to formulas (8) and (9) the space \( \tilde{H}_{s,s'}^{(t)} \) is elementwise invariant with respect to the subalgebra \( U_q(\mathfrak{gl}_p) \). Moreover, this space is invariant and irreducible with respect to the subalgebra \( U_q(\mathfrak{gl}_{n-p}) \) acting on \( t \).

Now we rearrange elements \( z_{p+1}, \ldots, z_n, w_{p+1}, \ldots, w_n \) in each of polynomials of \( \tilde{H}_{s,s'}^{(t)} \) such that in each summand (monomial) elements \( w_{p+1}, \ldots, w_n \) stand before the elements \( z_{p+1}, \ldots, z_n \). We denote the space \( \tilde{H}_{s,s'}^{(t)} \) with this rearrangement in polynomials by \( \mathcal{L}_{s,s'}^{(t)} \). Because of elementwise invariance with respect to \( U_q(\mathfrak{gl}_p) \), the space \( \mathcal{L}_{s,s'}^{(t)} \) can be represented as a direct sum

\[
\mathcal{L}_{s,s'}^{(t)} = R_{s,s'} \oplus Q_y R_{s-1,s'} \oplus Q_y^2 R_{s-2,s'} \oplus \cdots, \tag{51}
\]

where \( R_{s-j,s'-j} \) denote the space of homogeneous polynomials in which \( w_{p+1}, \ldots, w_n \) stand before \( z_{p+1}, \ldots, z_n \). Due to formulas (8) and (9), the spaces \( R_{s-j,s'-j} \) are invariant with respect to \( U_q(\mathfrak{gl}_{n-p}) \). However, the representation of \( U_q(\mathfrak{gl}_{n-p}) \) on \( \mathcal{L}_{s,s'}^{(t)} \) is irreducible. Therefore, the decomposition (51) contains only one summand and \( \mathcal{L}_{s,s'}^{(t)} = R_{s,s'} \). It is clear that for elements of \( R_{s,s'} \) the conditions \( \bar{\partial}_i P = 0, i = 1, 2, \ldots, p \), are fulfilled. Lemma is proved.

Corollary 1: Elements \( P \) of the space \( \tilde{H}_{s,s'}^{(t)} \) satisfy the relation \( \Delta^{(t)} P = 0 \).

Corollary 2: Elements \( P \) of the space \( \tilde{H}_{s,s'}^{(t)} \) are \( q \)-harmonic, that is, \( \Delta q P = 0 \).

Corollary 1 follows from (47)–(49). Corollary 2 follows from Corollary 1 and formula (46).

Lemma 3: For polynomial \( h_{s,s'}(t) \in \tilde{H}_{s,s'}^{(t)} \) and arbitrary polynomial \( f(y) \) we have

\[
\Delta^{(y)} h_{s,s'}(t) f(y) = q^{s-s'} h_{s,s'}(t) \Delta^{(y)} f(y). \tag{52}
\]

Proof: We first prove the relations \( \partial_i h_{s,s'}(t) f(y) = q^i h_{s,s'}(t) \partial_i f(y), i = 1, \ldots, p. \) The polynomial \( h_{s,s'}(t) \) can be represented in the form of a linear combination of monomials \( z_{p+1}^{k_{p+1}} \ldots z_n^{k_n} w_{p+1}^{l_{p+1}} \ldots w_n^{l_n} \), where \( k_{p+1} + \cdots + k_n = s, l_{p+1} + \cdots + l_n = s' \). We have

\[
\partial_i z_{p+1}^{k_{p+1}} \ldots z_n^{k_n} w_{p+1}^{l_{p+1}} \ldots w_n^{l_n} f(y) = z_{p+1}^{k_{p+1}} \ldots z_n^{k_n} (\partial_i w_{p+1}^{l_{p+1}} \ldots w_n^{l_n} f(y)) = q^i z_{p+1}^{k_{p+1}} \ldots z_n^{k_n} w_{p+1}^{l_{p+1}} \ldots w_n^{l_n} (\partial_i f(y)),
\]

where the relation \( \partial_i f(y) = 0 \) and relations from section III were used. It proves our relations. We analogously prove the relations \( \bar{\partial}_i h_{s,s'}(t) f(y) = q^{-s} h_{s,s'}(t) \bar{\partial}_i f(y), i = 1, \ldots, p. \) In this case, it is useful to represent the polynomial \( h_{s,s'}(t) \) in the form of a linear combination of monomials \( w_{p+1}^{l_{p+1}} \ldots w_n^{l_n} z_{p+1}^{k_{p+1}} \ldots z_n^{k_n} \) (such representation is possible due to Lemma 2). Now the lemma follows from explicit formula for \( \Delta^{(y)} \). Lemma is proved.
Since $\Delta_{(t)} \left( Q^u_{h_s,s'}(t) h_{r,r'}(y) \right) = 0$, then using Lemma 3 and relation (18) with $n$ replaced by $p$ we have

$$
\Delta_q \left( Q^u_{h_s,s'}(t) h_{r,r'}(y) \right) = \hat{\Delta}_q \left( Q^u_{h_s,s'}(t) h_{r,r'}(y) \right) = q^n h_{s,s'}(t) \Delta_q Q^u_{h_{r,r'}}(y)
$$

$$
= q^n \frac{[p + u + r + r' - 1]}{[r + r' + p + u - 1]} Q^{u-1}_{h_{s,s'}}(t) h_{r,r'}(y),
$$

where $a = 2(s - s')u + s' - s$.

Now we may find a harmonic projection of the polynomials (50). Denoting this projection by $h^{(r',r';s,s')}_{m,m'}(z, w)$ we have

$$
h^{(r',r';s,s')}_{m,m'}(z, w) = \min (m, m') \sum_{k=0}^{\min (m, m')} \alpha_k Q^k \Delta_q \left( Q^u_{h_s,s'}(t) h_{r,r'}(y) \right)
$$

$$
= \sum_{k=0}^{\min (m, m')} \alpha_k Q^k q^{(-s'+r'-p-1)k} \frac{[u]! [r + r' + p + u - 1]}{[r + r' + p + u - k - 1]} Q^{u-k}_y h_{s,s'}(t) h_{r,r'}(y),
$$

where $\alpha_k$ is determined by formula (33). Denoting the expression in the parentheses by $t^{n,p;m,m'}_{r,r';s,s'}(Q_y, Q_t)$, we have

$$
h^{(r',r';s,s')}_{m,m'}(z, w) = H_{m,m'} \left( Q^m_{y} - s h_{s,s'}(t) h_{r,r'}(y) \right) = t^{n,p;m,m'}_{r,r';s,s'}(Q_y, Q_t) h_{s,s'}(t) h_{r,r'}(y).
$$

(52)

After some simple transformations, we obtain for $t^{n,p,m,m'}_{r,r';s,s'}(Q_y, Q_t)$ the expression

$$
t^{n,p,m,m'}_{r,r';s,s'}(Q_y, Q_t) = Q^u_y \sum_{k=0}^{\min (m, m')} \left( \frac{q^{-2u}; q^2_k}{(q^{-2(m+m'+n-2)}; q^2)^2_k} Q^k Q^q_{y} \right),
$$

where $\sigma = -2n - 2s' + 2 + 2p$. Taking into account the definition of the basis hypergeometric function $2\varphi$, we derive

$$
t^{n,p,m,m'}_{r,r';s,s'}(Q_y, Q_t) = Q^u_y 2\varphi_1(q^{-2u}, q^{-2(r+r'+p+u-1)}; q^{-2(m+m'+n-2)}; q^2, Q^{q-1}_y q^2).
$$

Applying the relation

$$
2\varphi_1(q^{-n}, b; c; q, z) = q^{-(n+1)/2}(z)\sum_{n=0}^{\infty} \left( \frac{b; q}{c; q} \right)_n q^{n-1} / b z
$$

(see, for example, formula (2) of Section 14.1.8 in Ref. 14) we reduce this expression to

$$
t^{n,p,m,m'}_{r,r';s,s'}(Q_y, Q_t) = (-q^n)^u q^{-(-u+1)u} \left( \frac{q^{-2(r+r'+p+u-1)}; q^2}{(q^{-2(m+m'+n-2)}; q^2)^2_u} Q^u_y \right)
$$

$$
\times 2\varphi_1(q^{-2u}, q^{2(m+m'+n-u-1)}; q^{2(r+r'+p)}; q^2, q^{-2s+2} Q_y / Q).
$$

Using the definition of the little $q$-Jacobi polynomials, we derive from here that

$$
t^{n,p,m,m'}_{r,r';s,s'}(Q_y, Q_t) = (-q^n)^u q^{-(-u+1)u} \left( \frac{q^{-2(r+r'+p+u-1)}; q^2}{(q^{-2(m+m'+n-2)}; q^2)^2_u} Q^u_y \right)
$$

$$
\times \sum_{p=1}^{2s+2} Q_y / Q^2.
$$

(53)
Thus, we proved that the projection $H_{m,m'} (Q_{y} \frac{m-r-s}{r,r'} h_{s,s'} (t) h_{r,r'} (y))$ is given by formula (52), where $t_{r,r';s,s'}^{n;m,m'}$ is determined by (53). The restriction $\tau_{m,m'}^{r,r';s,s'} (z, w)$ of this projection onto the quantum sphere $S_{q,n-1}$ is given by

$$\tau_{m,m'}^{r,r';s,s'} (z, w) = (\tau_{r,r';s,s'}^{n;p,m,m'} (Q_{y}) h_{s,s'} (t) h_{r,r'} (y),$$

where $(\tau_{r,r';s,s'}^{n;p,m,m'}) (Q_{y}) = c_{P}^{(r+r'+p-1,s+s'+n-p-1)} (Q_{y}/Q; q^{2})$ (c is the multiplier from the right hand side of (53)).

For the scalar product of polynomials of the form (52) we have

$$\langle h_{m,m'}^{r,r';s,s'}, h_{m,m'}^{r',r';s',s'} \rangle = 0 \text{ if } (r, r', s, s') \neq (r'', r''', s'', s''')$$

(since the spaces $H_{r,r'}^{(y)}$, and $H_{s,s'}^{(y)}$, and the spaces $\tilde{H}_{s,s'}^{(t)}$ and $\tilde{H}_{s,s'}^{(t)}$ are orthogonal). If $(r, r', s, s') = (r'', r''', s'', s''')$, then the norm of the polynomial (52) reduces to the orthogonality relation for $q$-Jacobi polynomials and to norms of $h_{s,s'} (t)$ and $h_{r,r'} (y)$.

In order to obtain a $q$-analogue of separation of variables in this case we have to take bases of the spaces $H_{r,r'}^{(y)}$ and $H_{s,s'}^{(t)}$ in separated coordinates (as it was made in section IX).

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