Interpolation and cubature approximations and analysis for a class of wideband integrals on the sphere

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Abstract We propose, analyze, and implement interpolatory approximations and Filon-type cubature for efficient and accurate evaluation of a class of wideband generalized Fourier integrals on the sphere. The analysis includes derivation of (i) optimal order Sobolev norm error estimates for an explicit discrete Fourier transform type interpolatory approximation of spherical functions; and (ii) a wavenumber explicit error estimate of the order $O(\kappa^{-\ell}N^{-r_\ell})$, for $\ell = 0, 1, 2$, where $\kappa$ is the wavenumber, $2N^2$ is the number of interpolation/cubature points on the sphere and $r_\ell$ depends on the smoothness of the integrand. Consequently, the cubature is robust for wideband (from very low to very high) frequencies and very efficient for highly-oscillatory integrals because the quality of the high-order approximation (with respect to quadrature points) is further improved as the wavenumber increases. This property is a marked advantage compared to standard cubature that require at least ten points per wavelength per dimension and methods for which asymptotic convergence is known only with respect to the wavenumber subject to stable of computation of quadrature weights. Numerical results in this article demonstrate the optimal order accuracy of the interpolatory approximations and the wideband cubature.

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1 Introduction

The generalized Fourier integral operator $I^g_\kappa$, for a fixed phase function $g$ and wavenumber $\kappa \in (0, \infty)$, has the representation [4, 34]

$$I^g_\kappa f = \int_{\Omega} f(x) \exp(i \kappa g(x)) \, dx.$$  \hspace{1cm} (1.1)

Such integrals occur in various applications [4, 6, 13, 14, 32–34] including simulation of waves scattered by the surface of an obstacle when an incident wave (say, a plane wave $\exp(i \kappa \mathbf{x} \cdot \hat{d})$) strikes the obstacle from direction $\hat{d}$. In wave propagation models, for $\mathbf{x} \in \Omega$, the phase $g(x)$ depends on the incident direction $\hat{d}$ and the observed point $y \in \Omega$. Several fast algorithms and applications [4, 6, 13, 32, 33] require accurate and efficient evaluation of such integrals for large number of observed points and also for wideband frequencies (that is, for $\kappa$ in (1.1) taking a wide range of values from very small to very large).

Standard discretization techniques to evaluate such large numbers of generalized Fourier integrals is prohibitive, in particular for high-frequency wavenumber $\kappa$ and for $\Omega \subset \mathbb{R}^3$. This is mainly because standard quadrature rules require at least ten points per wavelength per dimension, even for low-order accuracy. This article is motivated by the paper of Domínguez et al. [9] on evaluating (1.1) in one dimension (with $\Omega = [-1, 1]$ and $g(s) = s$ on $\Omega$) and of Ganesh and Hawkins [13] on a fast algorithm to simulate high-frequency exterior acoustic scattering in three dimensions.

In addition to the classical asymptotic methods [4, 34], there is a large literature on efficient numerical evaluation of (1.1), see [17–20, 24, 26] and references therein. Such quadrature rules can be classified as Filon-type, Levin-type, or numerical steepest descent methods. A recent effort to avoid stability issues in such methods, for a one dimensional domain $\Omega$, is the shifted GMRES method [25] that requires solution of first order differential equations to evaluate $I^g_\kappa f$. Quadrature approximations for multivariate oscillatory integrals based on the three types of methods have also been investigated in the literature, see for example [18, 20, 24] and references therein. The main aim in this literature is to prove asymptotic convergence with respect to the wavenumber $\kappa$ and address stability issues associated with computing weights for large $\kappa$. Such asymptotic wavenumber convergence based results are not applicable for multivariate wideband oscillatory integrals. As an application of the efficient interpolatory approximation and robust analysis developed in this article, we introduce an $N$-point Filon-type cubature for a class of multivariate wideband oscillatory integral and prove robust wavenumber explicit spectrally accurate error bounds of the order $O(\kappa^{-\ell} N^{-r_\ell})$, for $\ell = 0, 1, 2$. Our fast evaluation algorithm to compute the wideband integrals does not require analytic continuation [17, 18] or derivative information [19, 20, 24].
We refer to the recent work [25, 26] for details of the need to develop efficient methods to compute (1.1) even with \( \Omega = [-1, 1] \) and discussions regarding the type of approach one needs to adopt depending on the properties of the phase function \( g \). In particular, Filon-type methods are efficient if the phase function \( g \) does not have stationary points. For wave propagation applications, whether the phase function \( g \) has stationary points or not depends on the observation points. For solving the full obstacle scattering wave propagation models, observation points should be considered in all regions of the scatterer. Consequently, various types of techniques are required to evaluate (1.1) depending, for example, on whether the observation point is near or away from the shadow boundary and whether the corresponding phase function has stationary [4, 34] or steepness [13] points and also whether the density function \( f \) has singular points in \( \Omega \).

For a boundary integral equation reformulation of the wave propagation model exterior to a convex scatterer, efficient approaches to evaluate \( I_k^g \) for stationary, steepness, and singular point cases are discussed in [13, 21] (and references therein), respectively, for two and three dimensional cases. In such applications, \( \Omega \) in (1.1) is the boundary of the convex obstacle, which can be diffeomorphically mapped onto \( S^{n-1} \), the unit sphere in \( \mathbb{R}^n \), \( n = 2, 3 \). Such a global mapping property was used in [13, 21] to transplant the wave propagation model to a boundary integral equation on \( S^{n-1} \), \( n = 2, 3 \) and linear combinations of appropriate polynomial basis functions are used to approximate the unknown surface current.

The focus of the paper by Domínguez et al. [9] is to evaluate \( I_k^g \) for a two dimensional scattering model [21] for the case when \( g \) does not have stationary points and \( f \) does not have singular points in \( \Omega \). (In a recent work [8], an extension of these techniques is developed to approximate integrals with stationary points by using graded meshes.) This leads to the requirement of evaluating (1.1) with \( \Omega = [-1, 1] \) and \( g(s) = s, s \in [-1, 1] \). We note that the integral considered in [9] is equivalent to (1.1) with \( \Omega = S^1 \) and \( g(x) = x \cdot \hat{d} \), \( x \in S^1 \) when the incident wave direction is assumed, without loss of generality (due to rotational symmetry of \( S^1 \)), to be \( \hat{d} = [0, 1]^T \).

The main aim of this article is to carry out the three dimensional counterpart of the method and analysis in [9], by taking \( \Omega = S^2 \) and \( g(x) = x \cdot \hat{d} \), \( x \in S^2 \) in (1.1) with \( \hat{d} = [0, 0, 1]^T \).

For the sphere case, even basic tools that are used to accomplish an efficient method and analysis in [9] is missing in the literature. A major contribution in this article is to first develop such tools and analysis, and hence derive an efficient Filon-type cubature and associated error estimates to evaluate the class of wideband integrals on the sphere defined by

\[
I_{k} f = \int_{S^2} f(x) \exp(ikx \cdot \hat{d}) \, dx, \quad \hat{d} = [0, 0, 1]^T, \quad k \in (0, \infty) .
\]  

The Filon-type quadrature rules (and wavenumber explicit analysis) to evaluate the generalized Fourier integral is based on the idea of replacing the density function \( f \) by an interpolatory approximation \( Q_N f \in X_N \) that can be efficiently computed. In addition for wavenumber explicit analysis, associated interpolatory approximation errors in Sobolev norms are required [9]. The choice of the finite dimensional space \( X_N \), associated basis functions, interpolation points and simple representation
of $Q_N f$ are crucial for efficient and stable approximations of the generalized Fourier integrals and analysis.

For example, in [9], with $X_N$ being the space of all polynomials of degree at most $N$ on $[-1, 1]$,

$$\tilde{I}_\kappa f = \int_{-1}^{1} f(x) \exp(i\kappa x) \, dx \approx \tilde{I}_{\kappa,N} f := \int_{-1}^{1} (Q_N f)(x) \exp(i\kappa x) \, dx, \quad (1.3)$$

where $Q_N f$ interpolates $f$ at $N + 1$ (quadrature) points in $[-1, 1]$ that are Chebyshev points $t_{j,N} = \cos(j\pi/N)$, $j = 0, \ldots, N$. In addition, $Q_N f$ can be explicitly expressed using the discrete Fast Fourier Transform (DFFT) type linear combination of the Chebyshev polynomial basis functions $T_n$, $n = 0, \ldots, N$:

$$Q_N f = \sum_{n=0}^{N} \langle f, T_n \rangle_N T_n, \quad \langle f, T_n \rangle_N = \frac{2}{N} \sum_{j=0}^{N} \cos(jn\pi/N) f(t_{j,N}), \quad (1.4)$$

where $\sum''$ means the first and last terms in the sum are to be halved. In (1.4), $\langle \cdot, \cdot \rangle_N$ is the discrete inner product approximation to the weighted $L^2$ inner product

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x) \frac{dx}{\sqrt{1-x^2}} = \int_{0}^{\pi} f(\cos \theta)g(\cos \theta) \, d\theta$$

with a crucial property that

$$\langle \psi_1, \psi_2 \rangle_N = \langle \psi_1, \psi_2 \rangle, \quad \psi_1, \psi_2 \in X_N. \quad (1.5)$$

The interpolation operator with explicit DFFT-type representation and quadrature property in (1.4) and (1.5) does not require solving any matrix equation. Such an operator is known as the DFFT-type matrix-free interpolation operator [15]. The error analysis in [9] crucially depends on the classical error estimates of approximating $2\pi$-periodic functions in the Sobolev space $H^\mu(-\pi, \pi)$, for $\mu = 0, 1$: With $\psi_c(\theta) = \psi(\cos \theta)$, for $0 \leq \mu \leq \nu, \nu > 1/2$, there is a constant $C_{v,\mu}$ such that [9]

$$\| f_c - (Q_N f)_c \|_{H^\mu(-\pi, \pi)} \leq C_{v,\mu} N^{\mu - v} \| f_c \|_{H^\nu(-\pi, \pi)}, \quad \mu = 0, 1. \quad (1.6)$$

For the sphere case, with $X_N$ being the $(N + 1)^2$ dimensional space of all spherical polynomials of degree at most $N \geq 3$, it is impossible to construct a DFFT-type matrix-free interpolation operator [28]. This seminal work of Sloan resulted in addressing several fundamental questions related to polynomial interpolation and numerical integration on the sphere, see [2, 16, 29] and extensive references therein.

Using a new class of finite dimensional spaces $X_N$ of spherical functions (some of which are not polynomials), Ganesh and Mhaskar [15] constructed matrix-free interpolation operators on the sphere. However, the analysis in [15] does not contain the Sobolev norm error estimate of the DFFT-type interpolation operators. In particular, for interpolation operator considered in this article we are not aware of error analysis in any norm. Deriving the Sobolev error estimates similar to (1.6) is one of the main contributions of this article.

This paper is organized as follows. In the next section using a finite dimensional space introduced in [12, 15] we prove that the interpolation problem onto the space is well defined for any set of $N + 1$ points in the latitudinal (elevation) angles in
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[0, \pi] (that include the poles) provided that 2N equally spaced points are chosen in the longitudinal (azimuth) angle interval \((-\pi, \pi]\). In Section 3 we study Sobolev-like Fourier spaces that orthogonally decompose the standard Sobolev spaces on the sphere and establish properties of functions in such spaces. In Section 4 we prove optimal order convergence, similar to (1.6), of a DFFT-type matrix-free interpolation operator induced by the Gauss–Lobatto latitudinal quadrature points. (Major part of proofs in Section 4 are deferred to Appendix.) In Section 5 we introduce a Filon-type cubature to evaluate the wideband integrals (1.2) and prove that the associated error is of the order \(O(\kappa^{-\ell} N^{-r_\ell})\), for \(\ell = 0, 1, 2\) where \(r_\ell\) depends on the smoothness of the density function \(f\). Numerical results in Section 6 demonstrate the robust theoretical analysis and efficiency of evaluating the wideband integrals for several low to high frequencies.

2 Interpolation on the sphere

In this section we introduce a class of interpolatory approximations to continuous functions on \(\mathbb{S}^2\). Following the standard convention, we identify functions on the sphere in terms of functions defined using the spherical polar coordinates,\n
\[ F(\theta, \phi) := (F^\circ \circ p)(\theta, \phi), \] \hspace{1cm} (2.1)\n
where we use the natural spherical polar coordinates parameterization

\[ \hat{x} = p(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^T, \quad \hat{x} \in \mathbb{S}^2. \] \hspace{1cm} (2.2)\n
Throughout this article, any function defined on the unit sphere \(\mathbb{S}^2\) will be superscripted with “\(^{\circ}\)”.

Using (2.2), we obtain the following periodic and reflectional symmetric properties of \(F\) in (2.1)

\[ F(\theta, \phi) = F(\theta + 2\pi, \phi) = F(\theta, \phi + 2\pi), \quad F(-\theta, \phi + \pi) = F(\theta, \phi). \] \hspace{1cm} (2.3)\n
Let \(\mathcal{C}(\mathbb{S}^2)\) be the space of all continuous functions on \(\mathbb{S}^2\) and define

\[ \mathcal{C} := \{F : \mathbb{R}^2 \to \mathbb{C} \mid F \text{ satisfies (2.1) for some } F^\circ \in \mathcal{C}(\mathbb{S}^2)\}. \] \hspace{1cm} (2.4)\n
Observe that if \(F \in \mathcal{C}\) then

\[ F(0, \cdot), \quad F(\pi, \cdot) \quad \text{are constants}. \] \hspace{1cm} (2.5)\n
Conversely, if \(F : \mathbb{R}^2 \to \mathbb{C}\) is continuous and satisfies (2.3) and (2.5) then there exists a unique \(F^\circ \in \mathcal{C}(\mathbb{S}^2)\) such that \(F = F^\circ \circ p\). Hence, \(\mathcal{C} \cong \mathcal{C}(\mathbb{S}^2)\).

We first consider a finite dimensional subspace \(\mathcal{X}_N\) of \(\mathcal{C}\), depending on a parameter \(N\), in which we shall seek interpolatory approximations. Such a space has been introduced and studied in [12, 15]. We refer to details in [12, 15] for arriving at the space by restricting the space of all bivariate trigonometric polynomials of degree at \(N\) to those functions that satisfy the essential spherical function properties (2.3) and (2.5).
For $M \in \mathbb{N}$, we consider the finite dimensional spaces
\[
\mathbb{D}^e_M := \text{span}\{ \cos n\theta : n = 0, \ldots, M \} = \text{span}\{ \cos^n \theta : n = 0, \ldots, M \},
\]
\[
\mathbb{D}^o_M := \text{span}\{ \sin \theta \cos n\theta : n = 0, \ldots, M \} = \text{span}\{ \sin \theta \cos^n \theta : n = 0, \ldots, M \}.
\]
(2.6)

Throughout this article $N \in \mathbb{N}$ with $N \geq 2$. For $p \in \mathbb{D}^e_N$ there exist $q \in \mathbb{D}^e_{N-2}$ and $r \in \mathbb{D}^o_1$ such that
\[
p(\theta) = r(\theta) + \sin^2 \theta q(\theta),
\]
Hence if $p \in \mathbb{D}^e_N$, we obtain
\[
p(0) = p(\pi) = 0 \iff p(\theta) = \sin^2 \theta q(\theta), \quad \text{with } q \in \mathbb{D}^e_{N-2}.
\]

Following [12, 15], we consider the finite dimensional subspace $\mathcal{X}_N \subset C$:
\[
\mathcal{X}_N := \left\{ p_0(\theta) + \sum_{-N < m \leq N - 1 \text{ even } m \neq 0} \sin^2 \theta p_m(\theta) \exp(im\phi) + \sum_{-N < m \leq N - 1 \text{ odd } m} p_m(\theta) \exp(im\phi) \right\}
\]
\[
\times p_0 \in \mathbb{D}^e_N, \quad p_{2\ell} \in \mathbb{D}^o_{N-2}, \quad \ell \neq 0, \quad p_{2\ell+1} \in \mathbb{D}^o_N \}
\]
(2.7)

It is easy to see that the dimension of $\mathcal{X}_N$ is $2N^2 - 2N + 2$.

Let $0 = \xi_0 < \xi_1 < \cdots < \xi_N = \pi$ with arbitrarily chosen set of $N - 1$ latitudinal angles in $(0, \pi)$. In this article, we choose $2N$ equally spaced azimuthal angles $\phi_k = k\pi/N, k = -N + 1, \ldots, N$. These sets induce a discrete set of $2N^2 - 2N + 2$ points on the unit sphere, by noticing from (2.2) that $p(\xi_0, \phi_k)$ and $p(\xi_N, \phi_k)$ are respectively the north and south pole, for $k = -N + 1, \ldots, N$.

Using these discrete sets of coordinates, we introduce the interpolation problem: For any $F \in C$, find $Q_N F \in \mathcal{X}_N$ such that
\[
( Q_N F ) \left( \xi_j, \frac{k\pi}{N} \right) = F \left( \xi_j, \frac{k\pi}{N} \right), \quad j = 0, \ldots, N, \quad k = -N + 1, \ldots, N. \quad (2.9)
\]

For any $F \in C$, since $F(0, \cdot), F(\pi, \cdot)$ are constants, (2.9) is equivalent to only $2N^2 - 2N + 2$ interpolation conditions. Next we constructively show that the interpolation with the arbitrary $N - 1$ elevation angles in $(0, \pi)$ is well-posed. Such a construction will also play a key role in developing an efficient approximation of the wideband integrals (1.2).

**Proposition 1** For any $F \in C$, the interpolation problem (2.9) has a unique solution.

**Proof** It is sufficient to show that such an interpolant can be always constructed. Set
\[
F^j_k := F \left( \xi_j, \frac{k\pi}{N} \right), \quad j = 0, \ldots, N, \quad k = -N + 1, \ldots, N,
\]
and define for \( j = 0, \ldots, N \)
\[
f_m^j := \frac{1}{2N} \sum_{-N < k \leq N} F_k^j \exp \left( -\frac{2\pi mk i}{2N} \right), \quad m = -N + 1, \ldots, N. \tag{2.10}
\]
The vector \( \mathbf{f}^j := (f_m^j)_{m=-N+1}^{N} \subset \mathbb{C}^{2N} \) is indeed (a symmetric variant of) the result of applying the inverse finite Fourier transform to the vector \( \mathbf{F}^j := (F_k^j)_{k=-N+1}^{N} \subset \mathbb{C}^{2N} \). Hence, \( \mathbf{F}^j \) can be recovered from \( \mathbf{f}^j \) via
\[
F_k^j = \sum_{-N < m \leq N} f_m^j \exp \left( \frac{2\pi mk i}{2N} \right), \quad k = -N + 1, \ldots, N. \tag{2.11}
\]
Using (2.3), we derive
\[
F_k^0 = F \left( 0, \frac{k\pi}{N} \right) = F(0, 0), \quad F_k^N = F \left( \pi, \frac{k\pi}{N} \right) = F(\pi, 0), \quad k = -N + 1, \ldots, N,
\]
and consequently
\[
f_m^0 = f_m^N = 0, \quad \text{for all } m \neq 0, \quad f_0^0 = F(0, \cdot), \quad f_0^N = F(\pi, \cdot). \tag{2.12}
\]

The interpolation problem: Find \( p_{2\ell} \in \mathbb{D}_N^\theta \) such that
\[
p_{2\ell}(\xi_j) = f_{2\ell}^j, \quad j = 0, \ldots, N,
\]
is equivalent, with the change of variables \( x = \cos \theta \), to a uniquely solvable polynomial interpolation problem on \([-1, 1]\). Moreover, (2.12) implies that for \( \ell \neq 0 \),
\[
p_{2\ell}(0) = p_{2\ell}(\pi) = 0. \tag{2.13}
\]

Similarly, for the odd coefficients, we solve the well-posed interpolation problem: Find \( \tilde{p}_{2\ell+1} \in \mathbb{D}_N^\theta \) such that
\[
\tilde{p}_{2\ell+1}(\xi_j) = f_{2\ell+1}^j, \quad j = 1, \ldots, N - 1,
\]
and set \( p_{2\ell+1} := \sin(\cdot) \tilde{p}_{2\ell+1} \in \mathbb{D}_{N-2}^0 \). Hence we obtain
\[
p_{2\ell+1} \in \mathbb{D}_{N-2}^0, \quad p_{2\ell+1}(\xi_j) = f_{2\ell+1}^j, \quad j = 0, \ldots, N.
\]
If we define
\[
F_N(\theta, \phi) := \sum_{-N < m \leq N} p_m(\theta) \exp(im\phi) \in X_N,
\]
using (2.11), we obtain
\[
F_N \left( \xi_j, \frac{k\pi}{N} \right) = \sum_{-N < m \leq N} p_m(\xi_j) \exp \left( \frac{2\pi mk i}{2N} \right) = \sum_{-N < m \leq N} f_m^j \exp \left( \frac{2\pi mk i}{2N} \right)
\]
\[
= F_k^j = F \left( \xi_j, \frac{k\pi}{N} \right).
\]
That is, \( Q_N F = F_N \) is the unique solution of the interpolation problem (2.9). \( \square \)
Note that for the well-posed interpolation problem (2.9), a priori, we are free to choose the \( N - 1 \) nodes at the elevation angle. The quality of the interpolant depends crucially on the choice of the latitudinal interpolation points \( 0 = \xi_0 < \xi_1 < \cdots < \xi_N = \pi \). In particular, for developing efficient Filon-type cubature to evaluate the wideband integrals (1.2), we require an interpolation operator \( Q_N \) on the sphere with optimal order convergence properties, similar to (1.6), in the Sobolev \( \mathcal{H}^s \) norm for \( s = 0, 1 \). (Definitions of these norms are presented in next section.)

That is, we require the Lebesgue constant of the interpolation operator \( Q_N \) to be uniformly bounded in \( \mathcal{H}^0 \) norm and grow only by \( O(N) \) in the \( \mathcal{H}^1 \) norm. In addition for efficient evaluation, similar to (1.4), it is ideal to construct \( Q_N F \) with a DFFT-type explicit representation.

The simple choice of \( \xi_j, j = 0, \ldots, N \) being equally spaced points in \([0, \pi]\) was considered in [12, 15] and the Lebesgue constant of the associated interpolation operator \( Q^u_N \) on the sphere in the uniform norm (that is, the standard norm in \( C \)) was proved in detail to be \( O(\log^2 N) \). The work in [15] in addition includes developing a grid-specific discrete inner product in \( \mathcal{X}_N \) and an associated orthonormal basis and hence a DFFT-type representation of \( Q^u_N \), see [15, Theorem 2.3]. While the space \( \mathcal{X}_N \) in (2.7) is same as that in [12] (with index \( m \) satisfying \(-N < m \leq N\)) the finite dimensional subspace of \( C \) in [15] is slightly different (with index \( m \) satisfying \(|m| \leq N\)). Results in [15] hold with \(|m| \leq N \) replaced with \(-N < m \leq N \).

A DFFT-type matrix-free interpolation operator \( Q^g^l_N \) on the sphere with the \( N + 1 \) latitudinal points based on the Gauss–Lobatto points was also introduced in [15, Theorem 2.2]: Let \( \{\eta_j\}_{j=0}^N \) be the nodes of the Gauss–Lobatto quadrature rule in \([-1, 1]\). That is, \( \eta_0 = -1, \eta_N = 1 \) and \( \eta_j \) for \( j = 1, \ldots, N - 1 \) are the roots of \( P'_N \), where \( P_N \) is the Legendre polynomial of degree \( N \) [2]. The \( N + 1 \) latitudinal Gauss–Lobatto points are

\[
\xi_j = \theta_j := \arccos(\eta_{N-j}), \quad j = 0, \ldots, N. \tag{2.14}
\]

Thus \( Q^g^l_N : C \to \mathcal{X}_N \) is defined, for \( F \in C \), as the solution of

\[
\left( Q^g^l_N F \right) \left( \theta_j, \frac{k\pi}{N} \right) = F \left( \theta_j, \frac{k\pi}{N} \right), \quad j = 0, \ldots, N, \quad k = -N + 1, \ldots, N. \tag{2.15}
\]

For a DFFT-type matrix-free analytical representation of \( Q^g^l_N \), we refer to [15, Equation (2.23)]. In this article we use \( Q^g^l_N \) for developing an efficient cubature rule for the wideband integrals.

Unlike the \( Q^u_N \) case, convergence properties of \( Q^g^l_N \) have not been established in any norm [15]. One of the key contributions in this article is to solve this open problem. In Section 4 we will prove optimal order convergence properties of \( Q^g^l_N \) in the Sobolev \( \mathcal{H}^s \) norm for \( s = 0, 1 \), after establishing various results in the next section. The convergence properties will be used in Section 5 to prove robust error estimates for a new class of cubature rules on the sphere, with respect to the wavenumber and quadrature points.
3 Sobolev and Sobolev-like Fourier coefficient spaces

The standard Sobolev spaces on the sphere $H^s(S^2)$, can be introduced in several, but equivalent, ways [1, 23]. Corresponding to $L^2(S^2)$, the space of all square integrable functions on the sphere, we define the equivalent space

$$H^0 = \left\{ F : \mathbb{R}^2 \to \mathbb{C} \left| F \text{ satisfies } (2.3)-(2.5), \quad \|F\|_{H^0} < \infty \right. \right\},$$

with

$$\|F\|_{H^0}^2 := \int_0^{2\pi} \int_0^\pi |F(\theta, \phi)|^2 \sin \theta \, d\theta \, d\phi.$$  \hspace{1cm} (3.1)

For $s > 0$, we similarly define equivalent spaces

$$H^s := \left\{ F \in H^0 \mid F = F^o \circ p, \text{ for some } F^o \in H^s(S^2) \right\}.$$  

In this article, $s = 0, 1, 2$ cases are important for our analysis. For $s = 1, 2$, respectively, the Sobolev space $H^s$ is equipped with the norm

$$\|F\|_{H^1}^2 := \frac{1}{4} \|F\|_{H^0}^2 + \|\nabla_{S^2} F\|_{H^0 \times H^0}^2,$$  \hspace{1cm} (3.2)

$$\|F\|_{H^2}^2 := \|\Delta_{S^2} F\|_{H^0}^2 + \frac{1}{2} \|\nabla_{S^2} F\|_{H^0 \times H^0}^2 + \frac{1}{16} \|F\|_{H^0}^2.$$  \hspace{1cm} (3.3)

3.1 Sobolev-like Fourier coefficient spaces

The standard orthonormal basis for $H^0 \cong L^2(S^2)$ is \{$Y_{m,n}$ | $n = 0, 1, 2, \ldots, |m| \leq n$\}, where

$$Y_{n,m}(\theta, \phi) := (-1)^{(m+|m|)/2} Q_n^m(\theta) e_m(\phi), \quad m = -n, \ldots, n, \quad n = 0, 1, \ldots,$$  \hspace{1cm} (3.4)

are spherical harmonics [2, 23], polynomials of degree $n$ on the sphere, with

$$Q_n^m(\theta) = Q_n^{-m}(\theta) := \left( \frac{2n+1 \ (n-m)!}{2 \ (n+m)!} \right)^{1/2} P_n^{|m|}(\cos \theta),$$  \hspace{1cm} (3.5)

and

$$e_m(\phi) := \frac{1}{\sqrt{2\pi}} \exp(i m \phi), \quad m \in \mathbb{Z},$$  \hspace{1cm} (3.6)

where $P_n^m$ are the associated Legendre functions [2, 23]. We observe that for any $F \in H^s$,

$$F = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \hat{F}_{n,m} Y_{n,m}^m, \quad \hat{F}_{n,m} := \int_0^{2\pi} \int_0^\pi F(\theta, \phi) Y_{n,m}^m(\theta, \phi) \sin \theta \, d\theta \, d\phi,$$  \hspace{1cm} (3.7)

with the series converging in $H^s$.

For $s \geq 0$, we introduce a decomposition of $H^s$ as an orthogonal direct sum. Such a decomposition, roughly speaking, comes from interchanging the order in which
the series in (3.7) is summed. In order to simplify the exposition, we start first with $F \in \mathcal{H}^0$. Hence

$$F(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \widehat{F}_{n,m} Y_n^m(\theta, \phi) = \sum_{m=-\infty}^{\infty} \left( \sum_{n=|m|}^{\infty} \widehat{F}_{n,m} Q_n^m(\theta) \right) e_m(\phi)$$

$$=: \sum_{m=-\infty}^{\infty} (\mathcal{F}_m F)(\theta) e_m(\phi). \quad (3.8)$$

The function $F(\theta, \cdot)$ is almost everywhere a function in $L^2(0, 2\pi)$ and as such it can be expanded in its Fourier series. By matching the Fourier coefficients of the same order, we conclude that $\mathcal{F}_m F$ admits the integral expression

$$\mathcal{F}_m F = \int_0^{2\pi} F(\cdot, \phi) e_m(\phi) \, d\phi = \int_0^{2\pi} F(\cdot, \phi) e_{-m}(\phi) \, d\phi. \quad (3.9)$$

Furthermore,

$$\|F\|_{\mathcal{H}^0}^2 = \int_0^{\pi} \left[ \int_0^{2\pi} |F(\theta, \phi)|^2 \, d\phi \right] \sin \theta \, d\theta = \sum_{m=-\infty}^{\infty} \|\mathcal{F}_m F\|_{L^2_{\sin}}^2 \quad (3.10)$$

with

$$\|f\|_{L^2_{\sin}} := \left[ \int_0^{\pi} |f(\theta)|^2 \sin \theta \, d\theta \right]^{1/2}.$$ 

Then, if $L^2_{\sin}$ is the corresponding $L^2$ weighted space, we obtain

$$\mathcal{H}^0 = \bigoplus_{m=-\infty}^{\infty} \{ f \otimes e_m \mid f \in L^2_{\sin}, f(\cdot + 2\pi) = f, f(-\cdot) = (-1)^m f \}, \quad (3.11)$$

where

$$(f \otimes e_m)(\theta, \cdot) := f(\theta)e_m(\phi) = \frac{1}{\sqrt{2\pi}} f(\theta) \exp(\imath m\phi).$$

Observe that the sum above is actually orthogonal.

We extend this decomposition to $\mathcal{H}^s$ by introducing new spaces, one for each Fourier mode. Let

$$W^s_m := \left\{ f : \mathbb{R} \to \mathbb{C} \mid f \otimes e_m \in \mathcal{H}^s \right\}, \quad (3.12)$$

endowed with the image norm

$$\|f\|_{W^s_m} := \|f \otimes e_m\|_{\mathcal{H}^s}. \quad (3.13)$$

Clearly, $W^s_m = W^s_{-m}$. We note that the definition of $W^s_m$ imposes naturally some periodicity and parity conditions on its elements, namely

$$f \in W^s_m \implies f(\cdot + 2\pi) = f, \quad f(-\cdot) = (-1)^m f \quad (3.14)$$

which are simply reflections on those satisfied by the elements of $\mathcal{H}^s$ (see (2.3)).
For \( s \geq 0 \), \( \{ (n + 1/2)^{-s} Y_n^m \mid n = 0, 1, 2, \ldots, |m| \leq n \} \) is an orthonormal basis for \( \mathcal{H}^s \) and the norm in \( \mathcal{H}^s \) can also be defined using the Fourier coefficient in (3.7) as

\[
\|F\|^2_{\mathcal{H}^s} := \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( n + \frac{1}{2} \right)^{2s} |\hat{F}_{n,m}|^2. \tag{3.15}
\]

The norms defined in (3.1), (3.2), and (3.3) coincide with that in (3.15), respectively for \( s = 0, 1, 2 \). It is easy to see that, for each \( m \in \mathbb{Z} \), \( W^s_m \) is a Hilbert space and a complete orthonormal set in \( W^s_m \) is \( \{ (n + 1/2)^{-s} Q_n^m : n \geq |m| \} \). Hence,

\[
f = \sum_{n=|m|}^{\infty} \hat{f}_m(n) Q_n^m, \quad \|f\|_{W^s_m} = \left( \sum_{n=|m|}^{\infty} \left( n + \frac{1}{2} \right)^{2s} |\hat{f}_m(n)|^2 \right)^{1/2} \tag{3.16}
\]

with

\[
\hat{f}_m(n) := \int_0^\pi f(\theta) Q_n^m(\theta) \sin \theta \, d\theta = \int_0^{2\pi} \int_0^\pi (f \otimes e_m)(\theta, \phi) Y_n^m(\theta, \phi) \, d\theta \, d\phi = (f \otimes e_m)_{n,m}.
\]

Clearly, \( \mathcal{F}_m : \mathcal{H}^s \rightarrow W^s_m \) is continuous and onto. Moreover, using (3.8)–(3.10),

\[
F = \sum_{m=-\infty}^{\infty} (\mathcal{F}_m F) \otimes e_m, \quad \|F\|^2_{\mathcal{H}^s} = \sum_{m=-\infty}^{\infty} \|\mathcal{F}_m F\|^2_{W^s_m}, \tag{3.17}
\]

with the first series converging in \( \mathcal{H}^s \). That is, we obtain an orthogonal direct sum decomposition of \( \mathcal{H}^s \):

\[
\mathcal{H}^s = \bigoplus_{m=-\infty}^{\infty} \{ f \otimes e_m \mid f \in W^s_m \}.
\]

### 3.2 Properties of Fourier coefficient spaces

We derive some properties of \( W^s_m \) that we will use in this article. Proofs of the results in this section are left as exercises to the reader. For further study of such spaces, we refer to [10] where a closely related family of spaces, namely that obtained by making the change of variables \( x = \cos \theta \), are studied.

It is easy to see that for \( r > s \) the injection \( W^r_m \subset W^s_m \) is compact. Moreover, because of (3.16),

\[
\|f\|_{W^s_m} \leq \left( |m| + \frac{1}{2} \right)^{s-r} \|f\|_{W^r_m}, \quad \forall r \geq s. \tag{3.18}
\]

Equations (3.17) and (3.18) prove that for \( r \geq s \)

\[
\sum_{m=-\infty}^{\infty} \left( |m| + \frac{1}{2} \right)^{2(r-s)} \|\mathcal{F}_m F\|^2_{W^r_m} \leq \|F\|^2_{\mathcal{H}^r}. \tag{3.19}
\]
Hence,

\[
\| F - \sum_{-N < m \leq N} (\mathcal{F}_m F) \otimes e_m \|_{\mathcal{H}^s} = \left[ \left( \sum_{m=N+1}^{\infty} + \sum_{m=-\infty}^{-N} \| \mathcal{F}_m F \|_{W^s_m}^2 \right) \right]^{1/2} \\
\leq \left( N + \frac{1}{2} \right)^{s-r} \left[ \sum_{|m| \geq N} \left( |m| + \frac{1}{2} \right)^{2(r-s)} \| \mathcal{F}_m F \|_{W^s_m}^2 \right]^{1/2} \\
\leq \left( N + \frac{1}{2} \right)^{s-r} \| F \|_{\mathcal{H}^s} \tag{3.20}
\]

for all \( r \geq s \).

**Proposition 2** For all \( m \in \mathbb{Z} \),

\[
\| f \|_{W^1_m}^2 = \frac{1}{4} \int_0^\pi |f(\theta)|^2 \sin \theta \, d\theta + m^2 \int_0^\pi |f(\theta)|^2 \frac{d\theta}{\sin \theta} \\
+ \int_0^\pi |f'(\theta)|^2 \sin \theta \, d\theta. \tag{3.21}
\]

Moreover, if \( |m| \geq 2 \),

\[
\| f \|_{W^2_m}^2 = \frac{1}{16} \int_0^\pi |f(\theta)|^2 \sin \theta \, d\theta + \frac{5m^2}{2} \int_0^\pi |f(\theta)|^2 \frac{d\theta}{\sin^2 \theta} \\
+ (m^4 - 4m^2) \int_0^\pi |f'(\theta)|^2 \frac{d\theta}{\sin^3 \theta} + \frac{1}{2} \int_0^\pi |f'(\theta)|^2 \sin \theta \, d\theta \\
+ (1 + 2m^2) \int_0^\pi |f''(\theta)|^2 \frac{d\theta}{\sin \theta} + \int_0^\pi |f''(\theta)|^2 \sin \theta \, d\theta. \tag{3.22}
\]

The norms (3.21) and (3.22) contain three and six integral terms respectively. We can, however, define equivalent norms with fewer terms. Let

\[
\| f \|_{Z^1_m}^2 := m^2 \int_0^\pi |f(\theta)|^2 \frac{d\theta}{\sin \theta} + \int_0^\pi |f'(\theta)|^2 \sin \theta \, d\theta \tag{3.23}
\]

\[
\| f \|_{Z^2_m}^2 := m^4 \int_0^\pi |f(\theta)|^2 \frac{d\theta}{\sin^3 \theta} + m^2 \int_0^\pi |f'(\theta)|^2 \frac{d\theta}{\sin \theta} \\
+ \int_0^\pi |f''(\theta)|^2 \sin \theta \, d\theta. \tag{3.24}
\]

The following result shows that, for \( i = 1, 2 \), the spaces \( W^i_m \) and \( Z^i_m \) are equivalent for all \( |m| \geq i \) with equivalence constants independent of \( m \).
Corollary 1 For all $m \in \mathbb{Z}$ with $|m| \geq 1$,
\[ \|f\|_{W^1_0} \leq \|f\|_{W^1_m} \leq \frac{\sqrt{5}}{2} \|f\|_{Z^1_m} \leq \frac{\sqrt{5}}{2} \|f\|_{W^1_m}. \] (3.25)
Moreover, for all $m \in \mathbb{Z}$ with $|m| \geq 2$,
\[ \frac{1}{\sqrt{3}} \|f\|_{W^2_m} \leq \|f\|_{Z^2_m} \leq \sqrt{6} \|f\|_{W^2_m}. \] (3.26)

Remark 1 Notice that if $f \in W^1_m = Z^1_m$ for $m \neq 0$, then $f^2$ and its derivative are in $L^1(0, \pi)$. Thus, from the Sobolev embedding theorem $|f|^2$ is continuous, and so is $f$. Moreover, $f$ vanishes at the end points $\{0, \pi\}$. This regularity property need not be true for functions in $W^1_0$. For instance, $|\log |\sin \theta||^{1/2} \in W^1_0$ but this function is obviously not continuous.

We complete this section by studying the regularity of $W_m^s$ from a classical Sobolev point of view. Let us introduce the $2\pi$-periodic Sobolev spaces
\[ H^r_\# := \left\{ f \in H^r_{\text{loc}}(\mathbb{R}) \left| f = f(\cdot + 2\pi) \right. \right\} \] (3.27)
endowed with the norm
\[ \|f\|^2_{H^r_\#} := |\widehat{f}(0)|^2 + \sum_{m \neq 0} |m|^{2r} |\widehat{f}(m)|^2, \quad \widehat{f}(m) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \exp(-im\theta) \, d\theta. \] (3.28)
For $r = 0$, $\|\cdot\|_{H^0_\#}$ is simply the $L^2(0, 2\pi)$ norm, and for non-negative integer values of $r$, an equivalent norm is given by
\[ \left[ \int_0^{2\pi} |f(\theta)|^2 \, d\theta + \int_0^{2\pi} |f^{(r)}(\theta)|^2 \, d\theta \right]^{1/2}. \] (3.29)

Proposition 3 For all $r > 0$ there exists $C_r > 0$ independent of $m$ and $f$ such that
\[ \|f\|_{H^r_\#} \leq C_r \|f\|_{W^{r+1/2}_m}, \quad \forall f \in W^{r+1/2}_m. \] (3.30)
Further,
\[ \|f\|_{W^0_m} \leq \|f\|_{H^0_\#}, \] (3.31)
\[ \|f\|_{W^1_m} \leq C(1 + |m|) \|f\|_{H^1_\#}, \] (3.32)
with $C$ independent of $f$ and $m$.

Proposition 4 There exists $C > 0$ such that for all $f \in W^1_0$
\[ \int_0^\pi |f(\theta)|^2 \, d\theta \leq C \left[ \int_0^\pi |f(\theta)|^2 \sin \theta \, d\theta \right. \]
\[ + \left. \left( \int_0^\pi |f(\theta)|^2 \sin \theta \, d\theta \right)^{1/2} \left( \int_0^\pi |f'(\theta)|^2 \sin \theta \, d\theta \right)^{1/2} \right]. \]
Proof Recall that if $\Omega$ is open and $\Gamma \subset \Omega$ is a Lipschitz curve, then \cite[Theorem 1.6.6]{5}
\[ \|\gamma_{\Gamma} u\|_{L^2(\Gamma)}^2 \leq C\Omega \|u\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)}. \]
The result follows by taking $\Omega = S^2$, $\Gamma$ being the maximum circle in $S^2$ and $u = f \circ e_0$.

\[ \square \]

4 Optimal order convergence analysis

In this section, we develop several results and hence prove the following optimal order convergence property of the interpolatory operator $Q^g_{N}$ on the sphere.

**Theorem 1** Let $s = 0, 1$. For all $r > s + 3/2$ there exists $C_r > 0$ such that for all $F \in H^r$ the following optimal order estimate holds:
\[ (1 - s) \|Q^g_{N} F - F\|_{H^0} + s N^{-1} \|Q^g_{N} F - F\|_{H^1} \leq C_r N^{-r} \|F\|_{H^r}. \] (4.1)

We demonstrate the optimal order convergence property with several numerical experiments in Section 6.

The discrete spaces and the functional frame where the analysis will be carried out have been already introduced in previous sections. The proof of Theorem 1 will rely on using certain semi-discrete operators $\rho^m_N$ and two interpolation operators $q^s_N$ and $q^0_N$ in the polar angle $\theta$. These operators facilitate a representation of the interpolation error as a sum of that introduced in the approximation of the central $2N$ Fourier coefficients of $F$ plus the error arising from ignoring the tail of the Fourier series (3.17).

4.1 Fourier analysis

Recalling the definition of $F_m$ in (3.8), the aim of this part is to investigate properties of the semi-discrete maps
\[ (\rho^m_N F)(\theta) := \sum_{\ell = -\infty}^{\infty} (F_{m + 2\ell N} F)(\theta), \quad m = -N + 1, \ldots, N. \] (4.2)

The following bound will be used repeatedly in this article: For $r > 1$ and for all $|m| \leq N$
\[ \sum_{\ell \neq 0} |m + 2\ell N|^{-r} \leq C_r N^{-r}, \quad C_r := \sum_{\ell = 1}^{\infty} \frac{1}{\ell^r} < \infty. \] (4.3)

**Lemma 1** For all $r > 1$ there exists $C_r > 0$ such that for any $F \in H^r$
\[ \sum_{m = -\infty}^{\infty} \sum_{n = |m|} \left| \hat{F}_{n,m} Y^m_n \right| = \frac{1}{\sqrt{2\pi}} \sum_{m = -\infty}^{\infty} \sum_{n = |m|} \left| \hat{F}_{n,m} Q^m_n \right| \leq C_r \|F\|_{H^r}. \]
Moreover,

\[ F(0, \cdot) = (\mathcal{F}_0 F)(0), \quad F(\pi, \cdot) = (\mathcal{F}_0 F)(\pi), \]

and therefore \( F \in \mathcal{C} \).

**Proof** Since \( F \in \mathcal{H}^{1+\varepsilon} \), for some \( \varepsilon > 0 \), using (3.4),
\[
\sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} |\hat{F}_{n,m} Y_n^m(\theta, \phi)| \leq \frac{1}{\sqrt{2\pi}} \left( \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} |\hat{F}_{n,m}|^2 \left( n + \frac{1}{2} \right)^{2+2\varepsilon} \right)^{1/2} \\
\times \left( \sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right)^{-2-2\varepsilon} \sum_{m=-n}^{n} |Q_n^m(\theta)|^2 \right)^{1/2} \\
= \frac{1}{\sqrt{2\pi}} \left\{ \left[ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} |\hat{F}_{n,m}|^2 \left( n + \frac{1}{2} \right)^{2+2\varepsilon} \right]^{1/2} \\
\times \left[ \sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right)^{-1-2\varepsilon} \right]^{1/2} \right\} \\
\leq C_\varepsilon \|F\|_{\mathcal{H}^{1+\varepsilon}}.
\]

In the penultimate step, we have used the addition theorem [23, Theorem 2.4.5]
\[
\sum_{m=-n}^{n} |Q_n^m(\theta)|^2 = n + \frac{1}{2}, \quad \forall \theta.
\]

In particular, \( F \) is continuous. Moreover, for \( m \neq 0 \), \( Q_n^m(0) = Q_n^m(\pi) = 0 \) for all \( n \). Hence using (3.8),
\[
F(0, \phi) = \sum_{n=0}^{\infty} \hat{F}_{n,0} Q_n^0(0) = (\mathcal{F}_0 F)(0), \quad \forall \phi.
\]

Analogously, it can be shown that \( F(\pi, \cdot) = (\mathcal{F}_0 F)(\pi) \). Thus \( F \in \mathcal{C} \). \( \square \)

**Lemma 2** For all \( |m| \leq N \) and \( r > 1 \), \( \rho_N^m \in C[0, \pi] \) and
\[
(\rho_N^m F)(0) = (\rho_N^m F)(\pi) = 0, \quad \text{if } m \neq 0, \quad (4.4a) \\
(\rho_N^0 F)(0) = (\mathcal{F}_0 F)(0) = F(0, \phi), \quad (\rho_N^0 F)(\pi) = (\mathcal{F}_0 F)(\pi) = F(\pi, \phi), \quad \forall \phi \in \mathbb{R}. \quad (4.4b)
\]

**Proof** Follows from the definition of \( \mathcal{F}_m \) in (3.8) and Lemma 1. \( \square \)
Given a general interpolation operator \( Q_N : \mathcal{C} \to \mathcal{X}_N \) fulfilling the conditions stated in (2.9), we consider the associated one dimensional interpolants: For \( f \in \mathcal{C}[0, \pi] \) find \( q_N^c f \in \mathbb{D}_N^c \) and \( q_N^o f \in \mathbb{D}_N^o - 2 \) such that
\[
q_N^c f(\xi_i) = f(\xi_i), \quad i = 0, \ldots, N, \quad (4.5a)
\]
\[
q_N^o f(\xi_j) = f(\xi_j), \quad j = 1, \ldots, N - 1. \quad (4.5b)
\]

With the help of these interpolants we are able to write \( Q_N F \) in terms of the Fourier coefficients of \( F \).

**Lemma 3** For all \( F \in \mathcal{H}_r \) with \( r > 1 \)
\[
( Q_N F)(\theta, \phi) = \sum_{-N+1 \leq m \leq N} (q_N^e \rho_N^m F)(\theta) e_m(\phi) + \sum_{-N+1 \leq m \leq N} (q_N^o \rho_N^m F)(\theta) e_m(\phi).
\]

**(4.6)**

**Proof** Note that, because of (4.4), for even \( m \neq 0 \), \( (q_N^e \rho_N^m)(0) = (q_N^e \rho_N^m)(\pi) = 0 \) which proves that the element of the right hand side of (4.6) is contained in \( \mathcal{X}_N \). We then just have to prove that the right hand side of (4.6) satisfies the interpolation conditions (2.9).

Since \( F \in \mathcal{H}_r \), with \( r > 1 \), Lemma 1 shows that the Laplace series of \( F \) converges absolutely. Hence the following manipulations are fully justified. For \( k = -N + 1, \ldots, N \),
\[
F \left( \theta, \frac{k\pi}{2N} \right) = \sum_{m=-\infty}^{\infty} (F_m F)(\theta) e_m \left( \frac{k\pi}{2N} \right)
\]
\[
= \sum_{-N+1 \leq m \leq N} \sum_{\ell=-\infty}^{\infty} (F_{m+2\ell N} F)(\theta) e_{m+2\ell N} \left( \frac{k\pi}{2N} \right)
\]
\[
= \sum_{-N+1 \leq m \leq N} \left( \sum_{\ell=-\infty}^{\infty} (F_{m+2\ell N} F)(\theta) \right) e_m \left( \frac{k\pi}{2N} \right)
\]
\[
= \sum_{-N+1 \leq m \leq N} (\rho_N^m F)(\theta) e_m \left( \frac{k\pi}{2N} \right).
\]

Therefore, for \( j = 0, \ldots, N \) and \( k = -N + 1, \ldots, N \),
\[
F \left( \xi_j, \frac{k\pi}{2N} \right) = \sum_{-N+1 \leq m \leq N} (\rho_N^m F)(\xi_j) e_m \left( \frac{k\pi}{2N} \right)
\]
\[
= \sum_{-N+1 \leq m \leq N} (q_N^e \rho_N^m F)(\xi_j) e_m \left( \frac{k\pi}{2N} \right)
\]
\[
+ \sum_{-N+1 \leq m \leq N} (q_N^o \rho_N^m F)(\xi_j) e_m \left( \frac{k\pi}{2N} \right).
\]

Thus we obtain the representation (4.6) \( \square \)
Denote by
\[ q^e_{N,gl} : C[0, \pi] \to \mathbb{D}^e_N, \quad \text{and} \quad q^o_{N,gl} : C[0, \pi] \to \mathbb{D}^o_{N-2} \]
respectively the interpolant projections defined in (4.5) with nodes \( \{\theta_j\}_j^{N-1} \), the Gauss–Lobatto points. Then using Lemma 3,
\[ Q^g_{N,gl} F(\theta, \phi) = \sum_{-N+1 \leq m \leq N \atop \text{even } m} (q^e_{N,gl} \rho^m_N F)(\theta) e_m(\phi) + \sum_{-N+1 \leq m \leq N \atop \text{odd } m} (q^o_{N,gl} \rho^m_N F)(\theta) e_m(\phi). \]
(4.8)

For any \( m \in \mathbb{Z} \), we introduce a simpler notation
\[ q^m_{N,gl} := \begin{cases} q^e_{N,gl}, & \text{if } m \text{ is even}, \\ q^o_{N,gl}, & \text{if } m \text{ is odd}. \end{cases} \] (4.9)

We recall the spaces \( Z^i_m \), for \( i = 1, 2 \), induced respectively by the norms (3.23) and (3.24) and that for \( |m| \geq i \), these are equivalent to the spaces \( W^i_m \) in (3.12) with respective norms given by (3.21) and (3.22). The next two results give properties of \( q^m_{N,gl} \) when applied to elements in these spaces. Derivation of these results require new technical tools and hence we postpone the proofs of the following two results to Appendix.

**Lemma 4** There exists \( C > 0 \) such that for all \( f \in Z^1_m \) with \( m \neq 0 \) and \( N \geq 2 \)
\[ \|q^m_{N,gl} f\|_{L^2_{\text{sin}}} \leq C \left[ \|f\|_{L^2_{\text{sin}}} + \frac{1}{N} \|f\|_{Z^1_m} \right]. \] (4.10)

Besides, for all \( f \in Z^2_m \) with \( m \geq 2 \)
\[ \|q^m_{N,gl} f\|_{Z^1_m} \leq C \left[ \|f\|_{Z^1_m} + \frac{1}{N} \|f\|_{Z^2_m} \right], \] (4.11)

where \( C \) is independent of \( m \) and \( N \).

**Proof** See Lemma 9 and Corollary 2 in Appendix. \( \square \)

**Proposition 5** Let \( s = 0, 1 \). Then, for \( s = 0 \) and \( r > 1 \) and for \( s = 1 \) and \( r \geq 2 \), there exists \( C_r > 0 \) such that for any \( N \geq 2 \) and \( f \in W^r_m \)
\[ \|q^m_{N,gl} f - f\|_{W^r_m} \leq C_r N^{s-r} \|f\|_{W^r_m}. \]

Moreover, for \( m \neq 0 \) and \( s = 0 \) the result holds also for \( r = 1 \).

**Proof** See Propositions 6 and 7 in Appendix. \( \square \)

The main result of this section is stated and proven below.
Theorem 2 Let $s = 0, 1$. For $r > 3/2 + s$, there exists $C_r > 0$ such that for all $N \geq 2$ and $|m| \leq N$,

$$
\|q_N^m \rho_N^m F - \mathcal{F}_m F\|_{W^{s}_m} \leq C_r N^{-r} \left[ \sum_{\ell = -\infty}^{\infty} \|\mathcal{F}_m + 2\ell N F\|_{W^{s+2}_{m+2\ell N}}^2 \right]^{1/2} 
\leq C_r N^{s-r} \|F\|_{H^r}.
$$

(4.12)

Proof We have

$$
\|q_N^m \rho_N^m F - \mathcal{F}_m F\|_{W^{0}_m} = \|q_N^m (\rho_N^m F - \mathcal{F}_m F)\|_{L^2_{\text{sin}}} + \|q_N^m \mathcal{F}_m F - \mathcal{F}_m F\|_{L^2_{\text{sin}}}
=: \|E_1\|_{L^2_{\text{sin}}} + \|E_2\|_{L^2_{\text{sin}}}.
$$

For the first term in the bound, we obtain

$$
\|E_1\|_{W^{0}_m}^2 \leq \left[ \sum_{\ell \neq 0} |m + 2\ell N|^{2-2r} \right] \left[ \sum_{\ell \neq 0} |m + 2\ell N|^{2r-2} \|q_N^m \mathcal{F}_m + 2\ell N F\|_{L^1_{m+2\ell N}}^2 \right]
\leq C_r N^{2-2r} \left[ \sum_{\ell \neq 0} |m + 2\ell N|^{2r-2} \right]
\times \left( \|\mathcal{F}_m + 2\ell N F\|_{L^2_{\text{sin}}}^2 + N^{-2} \|\mathcal{F}_m + 2\ell N F\|_{Z^1_{m+2\ell N}}^2 \right)
\leq C_r N^{2-2r} \left[ \sum_{\ell \neq 0} |m + 2\ell N|^{2r-2} \right]
\times (|m + 2\ell N|^{-2} + N^{-2}) \|\mathcal{F}_m + 2\ell N F\|_{W^{1}_{m+2\ell N}}^2
\leq 2C_r N^{-2r} \left[ \sum_{\ell \neq 0} \|\mathcal{F}_m + 2\ell N F\|_{W^{1}_{m+2\ell N}}^2 \right],
$$

(4.13)

where we have applied successively (4.3), (4.10) of Lemma 4, Corollary 1 and (3.18).

On the other hand, Proposition 5 yields

$$
\|E_2\|_{W^{0}_m} \leq C' N^{-r} \|\mathcal{F}_m F\|_{W^{r}_m}.
$$

(4.14)

The result for the case $s = 0$ follows now from (4.13) and (4.14).
For \( s = 1 \), we proceed as before, but using now (4.11) of Lemma 4:

\[
\| E_1 \|_{W_1^m}^2 \leq \left[ \sum_{\ell \neq 0} |m + 2\ell N|^{4-2r} \right] \left[ \sum_{\ell \neq 0} |m + 2\ell N|^{2r-4} \| q_N^m \mathcal{F}_{m+2\ell N} F \|_{W_1^m}^2 \right] \\
\leq C_r N^{4-2r} \left[ \sum_{\ell \neq 0} |m + 2\ell N|^{2r-4} \right] \left( \| \mathcal{F}_{m+2\ell N} F \|_{W_2^{m+2\ell N}}^2 + N^{-2} \| \mathcal{F}_{m+2\ell N} F \|_{W_2^{m+2\ell N}}^2 \right) \\
\leq C_r N^{4-2r} \left[ \sum_{\ell \neq 0} |m + 2\ell N|^{2r-4} \right] \left( |m + 2\ell N|^{-2} + 6N^{-2} \right) \| \mathcal{F}_{m+2\ell N} F \|_{W_2^{m+2\ell N}}^2 \\
\leq 7C_r N^{2-2r} \left[ \sum_{\ell \neq 0} \| \mathcal{F}_{m+2\ell N} F \|_{W_2^{m+2\ell N}}^2 \right]. \tag{4.15}
\]

Finally, Proposition 5 yields

\[
\| E_2 \|_{W_1^m} \leq C_r N^{1-r} \| \mathcal{F}_m F \|_{W_2^m}.
\]

\[\square\]

4.2 Proof of Theorem 1

For \( s = 0, 1 \), using (4.8) and (3.17) we obtain

\[
\| \mathcal{Q}_N^{gl} F - F \|_{H^s}^2 = \sum_{-N < m \leq N} \| q_N^m \mathcal{Q}_N^{gl} \rho_N^m F - \mathcal{F}_m F \|_{W_2^m}^2 \\
+ \left( \sum_{m = -\infty}^{-N} + \sum_{m = N+1}^{\infty} \right) \| \mathcal{F}_m F \|_{W_2^m}^2 \\
=: D_1 + D_2.
\]

The last term can be estimated using (3.20)

\[
D_2 \leq N^{2s-2r} \| F \|_{H^r}^2.
\]
On the other hand, and with the help of Theorem 2, we deduce that
\[ D_1 \leq C_r N^{2s-2r} \sum_{N+1 \leq m \leq N} \sum_{\ell = -\infty}^{\infty} \|F_{m+2\ell N}F\|_{W^r_{m+2\ell N}}^2 \]
\[ \leq C_r N^{2s-2r} \sum_{m = -\infty}^{\infty} \|F_mF\|_{W^r_m}^2 = C_r N^{2s-2r} \|F\|_{\mathcal{H}^r}^2. \]

\[ \square \]

5 Cubature and analysis for wideband integrals

In this section, using the tools developed in previous sections, we propose efficient approximations to the class of wideband integrals on the sphere defined in (1.2) and analyze the error in such cubature approximations.

In particular, for \( F \in \mathcal{H}^r \) with \( r > 1 \) and \( f = \mathcal{F}^0 \) in (1.2), the wideband integrals (with wavenumber \( \kappa \in (0, \infty) \) and incident direction \( \widehat{d} = [0, 0, 1]^T \)) can be written as
\[
I_\kappa(F) := \int_0^{2\pi} \int_0^{\pi} F(\theta, \phi) \exp(i \kappa \cos \theta) \sin \theta \, d\theta \, d\phi
= \int_{\mathbb{S}^2} F^0(x) \exp(i \kappa x \cdot \widehat{d}) \, dx. \quad (5.1)
\]

We are interested in robust cubature rules that give accurate results for small, large and very large values of the wavenumber \( \kappa \).

Following the ideas of Filon rules, or those more general product integration rules (see for instance [22]), we propose
\[
I_\kappa(F) \approx \int_0^{2\pi} \int_0^{\pi} (Q_{N}^g F)(\theta, \phi) \exp(i \kappa \cos \theta) \sin \theta \, d\theta \, d\phi =: I_{\kappa,N}^g(F). \quad (5.2)
\]

Using (3.9) and Lemma 3, we obtain for \( F \in \mathcal{H}^r \) with \( r > 1 \),
\[
I_\kappa(F) = \sqrt{2\pi} \int_0^{\pi} (\mathcal{F}_0 F)(\theta) \exp(i \kappa \cos \theta) \sin \theta \, d\theta, \quad (5.3)
I_{\kappa,N}^g(F) = \sqrt{2\pi} \int_0^{\pi} (\mathcal{F}_0 Q_N F)(\theta) \exp(i \kappa \cos \theta) \sin \theta \, d\theta
= \sqrt{2\pi} \int_0^{\pi} (q_{N,g}^c \rho_N^0 F)(\theta) \exp(i \kappa \cos \theta) \sin \theta \, d\theta. \quad (5.4)
\]

Consequently,
\[
|I_\kappa(F) - I_{\kappa,N}^g(F)| \leq \sqrt{2\pi} \left| \int_0^{\pi} (\mathcal{F}_0 F - q_{N,g}^c \rho_N^0 F)(\theta) \exp(i \kappa \cos \theta) \sin \theta \, d\theta \right|.
\]

Before entering into the analysis of such approximations, we observe that computation of \( q_{N,g}^c \rho_N^0 F \) does not require evaluations of either \( \mathcal{F}_{2\ell N} F \) for all \( \ell \) or \( Q_N^g F \). This is a substantial computational advantage even compared to using the matrix-free FFT.
type representation of $Q_{N}^{gl} F$. An efficient algorithm to compute the cubature, based on the proof of Proposition 1 is as follows.

Algorithm
1. Set
   $$f_{0}^{0} = F(0, 0), \quad f_{0}^{N} = F(\pi, 0),$$
   $$f_{0}^{j} := \frac{1}{2N} \sum_{-N+1 \leq k \leq N} F\left(\xi_{j}, \frac{k\pi}{N}\right), \quad j = 1, \ldots, N - 1.$$  
2. Solve the one dimensional interpolation problem at Gauss–Lobatto based nodes:
   $$\mathbb{D}_{N}^{e} \ni p_{N} \quad \text{s.t.} \quad p_{N}(\xi_{j}) = f_{0}^{j}, \quad j = 0, \ldots, N.$$  
   (Observe that $p_{N} = \frac{1}{\sqrt{2\pi}} q_{N, gl}^{e} F_{0}$.)
3. Compute
   $$2\pi \int_{0}^{\pi} p_{N}(\theta) \exp(i\kappa \cos \theta) \sin \theta \, d\theta. \quad (5.5)$$

Now we require a robust approach for evaluating (5.5). After performing the change of variables $\theta = \arccos x$ we have to evaluate
   $$\int_{-1}^{1} p_{N}(\arccos x) \exp(i\kappa x) \, dx.$$  
Since $p_{N}(\arccos \cdot)$ is now a polynomial, there are some accurate, robust, and fast methods in the literature for evaluating this integral, see [9] and references therein. In our numerical implementation, we use the approach developed in [9] for evaluating the one dimensional integrals (5.5).

Crucial to our wavenumber explicit error analysis of the cubature rule is Theorem 2 which is essentially the main result for proving optimal order error bounds for the interpolatory approximation $Q_{N}^{gl} F$ in the $\mathcal{H}^0$ and $\mathcal{H}^1$ norms. In addition, for the error analysis we require additional convergence estimates in stronger norms, defined (3.27)–(3.29).

Lemma 5 Let $g \in W_{0}^{r}$ and $g_{N} \in \mathbb{D}_{N}^{e}$. Then for all $s \geq 0$ and $r > s + 1/2$ there exists $C_{r} > 0$ so that
   $$\|g - g_{N}\|_{\mathcal{H}^{s}_{\#}} \leq C_{r}\left[N^{s-r+1/2}\|g\|_{W_{0}^{r}} + N^{s+1/2}\|g - g_{N}\|_{W_{0}^{r}} + N^{s-1/2}\|g - g_{N}\|_{W_{0}^{r}}\right]. \quad (5.6)$$

Proof We consider the following $H_{\#}^{s}$ projection on $\mathbb{D}_{N}^{e}$:
   $$s_{N} g(\theta) := \sum_{m=0}^{N} \tilde{g}(m) \cos m\theta, \quad \tilde{g}(m) := \begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) \, d\theta, & \text{if } m = 0, \\ \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos m\theta \, d\theta, & \text{if } m > 0. \end{cases}$$
Hence for even \( g \), as it is in our case, it is easy to prove from (3.28) (see also [27]) that

\[
\| s_0 g - g \|_{H^0_\kappa} \leq N^{-r} \| g \|_{H^r_\kappa}. 
\]

The other result we need in this proof is the well-known and easy to prove inverse inequality

\[
\| g_N \|_{H^s_\kappa} \leq N^{s-r} \| g_N \|_{H^r_\kappa}, \quad \forall g_N \in \text{span} \{ \exp(i m \cdot) : |m| \leq N \}, \quad s \geq r.
\]

Using (5.7) and (5.8), we obtain

\[
\| g - g_N \|_{H^r_\kappa} \leq \| g - s_0 g \|_{H^r_\kappa} + \| s_0 g - g_N \|_{H^r_\kappa} \\
\leq N^{s-r+1/2} \| g \|_{H^r_\kappa}^{-1/2} + N^s \| s_0 g - g_N \|_{H^r_\kappa} \\
\leq N^{s-r+1/2} \| g \|_{H^r_\kappa}^{-1/2} + N^s \| g - g_N \|_{H^r_\kappa} \\
\leq 2N^{s-r+1/2} \| g \|_{H^r_\kappa}^{-1/2} + N^s \| g - g_N \|_{H^r_\kappa} \\
\leq C_r N^{s-r+1/2} \| g \|_{H^r_\kappa} + \sqrt{2}N^s \| g - g_N \|_{L^2(0,\pi)}. 
\]

where in the last step we have used (3.30). We next apply Proposition 4 to the last term and obtain

\[
\| g - g_N \|_{L^2(0,\pi)} \leq C \left[ \| g - g_N \|_{L^2_{\text{sin}}}^2 + \| g - g_N \|_{L^2_{\text{sin}}} \| (g - g_N)' \|_{L^2_{\text{sin}}} \right]^{1/2} \\
\leq C \left( 1 + N^{1/2} \right) \| g - g_N \|_{L^2_{\text{sin}}} + N^{-1/2} \| (g - g_N)' \|_{L^2_{\text{sin}}}.
\]

The result (5.6) now follows by applying Proposition 2 to the right hand side above.

Now we are ready to prove the wavenumber explicit error bounds for the cubature approximations \( I_{\kappa,N}^{\text{gl}}(F) \) of the wideband integrals \( I_{\kappa}(F) \). Such error bounds provide robust error estimates for convergence with respect to the number of cubature points and the wavenumber.

**Theorem 3** For \( \ell = 0, 1 \) and \( r > 3/2 \) or \( \ell = 2 \) and \( r > 4 \) there exists \( C_r \) such that for all \( F \in \mathcal{H}'_\kappa \),

\[
| I_{\kappa}(F) - I_{\kappa,N}^{\text{gl}}(F) | \leq C_r \kappa^{-\ell} N^{\eta(\ell) - r} \| F \|_{\mathcal{H}'_\kappa},
\]

where

\[
\eta(\ell) := \begin{cases} 
0, & \text{if } \ell = 0, \\
3/2, & \text{if } \ell = 1, \\
4, & \text{if } \ell = 2.
\end{cases}
\]

**Proof** Let

\[
r_N := \sqrt{2\pi} \left( q_{N,gl}^e \rho_0^N F - F_0 F \right).
\]
For $F$ smooth enough (precise information about the regularity requirements on $F$ will be given below), $r_N'(0) = r_N''(\pi) = 0$ and therefore the function

$$\varphi_N(\theta) := \begin{cases} \frac{r_N'(\theta)}{\sin \theta}, & \text{if } \theta \in (0, \pi), \\ r_N''(0), & \text{if } \theta = 0, \\ -r_N''(\pi), & \text{if } \theta = \pi, \end{cases}$$

is continuous. Hence, using integration by parts we derive

$$|I_\kappa(F) - I_{\kappa,N}(F)| = \left| \int_0^\pi r_N(\theta) \exp(i\kappa \cos \theta) \sin \theta \, d\theta \right| \quad (5.11)$$

Thus (5.10) holds for $\ell = 0$. 

$$\|r_N\|_{L_0^2} = \|r_N\|_{W_{0}^2} \leq C_r N^{-r} \|F\|_{H^r}.$$
For $\ell = 1, 2$, we need to bound $\|r_N\|_{H^s_{\rho}}$ for different values of $s$. Applying (5.6), with $g = F_0 F$ and $g_N = q_N^\rho F_0 F$ and (4.12) of Theorem 2 we obtain for $r > s + 1/2$

$$\|r_N\|_{H^s_{\rho}} \leq C_r \left[ N^{s-r+1/2}\|F_0 F\|_{W^{s}_{0}} + N^{s+1/2}\|r_N\|_{W^{s}_{0}} + N^{s-1/2}\|r_N\|_{W^{s}_{1}} \right]$$

$$\leq C_r \left[ N^{s-r+1/2}\|F_0 F\|_{W^{s}_{r}} + N^{s-r+1/2} \left[ \sum_{\ell = -\infty}^{\infty} \|F_{m+2\ell} N F\|_{W^{s}_{r}}^{2m+2s} / N \right]^{1/2} \right]$$

$$\leq C_r \sqrt{2} N^{s-r+1/2} \|F\|_{H^r}. \quad (5.17)$$

Applying (5.17) first in (5.15) and second in (5.16) with $s = 3, s = 0$ and $s = s' < r - 1/2$, we obtain the result (5.10) for $\ell = 1, 2$. \qed

6 Numerical experiments

In the section, we demonstrate the algorithms and analysis developed in this article for interpolatory approximations of functions on the sphere with various smoothness properties and efficient evaluation of the wideband integrals in (5.1) induced by such functions. For a fixed observation point $x^* \in S^2$ and a smoothness parameter $s$, we consider $F_s^*: S^2 \to \mathbb{R}$, defined by

$$F_s^*(x) = \|x - x^*\|^s = [(x_1 - x^*_1)^2 + (x_2 - x^*_2)^2 + (x_3 - x^*_3)^2]^{s/2}, \quad x \in S^2,$$

(6.1)

where $x = [x_1, x_2, x_3]^T$ and $x^* = [x^*_1, x^*_2, x^*_3]^T$.

Clearly, for $s < 0$, $F_s^*$ is a discontinuous spherical function; for $s \geq 0$ and even, $F_s^*$ is a spherical polynomial; and for $s \geq 1$ and odd $F_s^* \in H^r(S^2)$, for all $r < s + 1$. For numerical experiments, we chose $x^* = [2/3, 1/3, 2/3]^T$; $s = 1, 3, 5$ and various wideband wavenumbers $\kappa = 10^m$ with $m = -2, -1, 0, 1, 2, 3, 4, 5$.

For $s = 1, 3, 5$, Theorem 1 indicates that the estimated order of convergence (EOC) for approximating $F_s$ by $Q_N^d F_s$ in the $H^0$ and $H^1$ norms are approximately $s + 1$ and $s$. Results in Tables 1, 2, and 3 substantiate the efficient approximations and analysis for spherical functions with various smoothness properties. The Sobolev norm errors were computed by approximating integrals in (3.1) and (3.2) using rectangular rules with over 160,000 quadrature points.

| $N$ | $\|Q_N^d F_1 - F_1\|_{H^0}$ | EOC($H^0$) | $\|Q_N^d F_1 - F_1\|_{H^1}$ | EOC($H^1$) |
|-----|-----------------------------|-------------|-----------------------------|-------------|
| 5   | 4.0077e-02                  | 3.3936e-01  | 1.0870                    | 1.0870      |
| 10  | 1.1338e-02                  | 1.8159      | 1.6166e-01                | 1.6166e-01  |
| 20  | 2.7244e-03                  | 2.0629      | 8.7935e-02                | 8.7935e-02  |
| 40  | 6.7527e-04                  | 2.0124      | 3.9452e-02                | 3.9452e-02  |
| 80  | 1.6554e-04                  | 2.0283      | 2.0598e-02                | 2.0598e-02  |
Table 2  $H^0$ and $H^1$ errors in approximation of $F_3$ by $Q_{N}^{gl} F_3$ and EOC

| $N$ | $\| Q_{N}^{gl} F_3 - F_3 \|_{H^0}$ | EOC($H^0$) | $\| Q_{N}^{gl} F_3 - F_3 \|_{H^1}$ | EOC($H^1$) |
|-----|-----------------|-------------|-----------------|-------------|
| 5   | 9.1450e-03      |             | 6.4589e-02      |             |
| 10  | 5.7752e-04      | 3.9850      | 6.8649e-03      | 3.2340      |
| 20  | 3.4725e-05      | 4.0558      | 9.7395e-04      | 2.8173      |
| 40  | 2.2636e-06      | 3.9393      | 1.0621e-04      | 3.1970      |
| 80  | 1.4386e-07      | 3.9758      | 1.4652e-05      | 2.8577      |

Table 3  $H^0$ and $H^1$ errors in approximation of $F_3$ by $Q_{N}^{gl} F_3$ and EOC

| $N$ | $\| Q_{N}^{gl} F_3 - F_3 \|_{H^0}$ | EOC($H^0$) | $\| Q_{N}^{gl} F_3 - F_3 \|_{H^1}$ | EOC($H^1$) |
|-----|-----------------|-------------|-----------------|-------------|
| 5   | 8.7351e-03      |             | 5.9412e-02      |             |
| 10  | 1.0756e-04      | 6.3436      | 1.2333e-03      | 5.5901      |
| 20  | 1.5172e-06      | 6.1476      | 4.0722e-05      | 4.9206      |
| 40  | 2.5432e-08      | 5.8986      | 1.1236e-06      | 5.1796      |
| 80  | 4.0461e-10      | 5.9740      | 3.9233e-08      | 4.8399      |

Table 4  Cubature errors $| I_\kappa(F_1) - I_{\kappa,N}^{gl}(F_1) |$ for $\kappa = 10^m$, $m = -2, -1, 0, 1, 2, 3, 4, 5$

| $N$ | $\kappa = 10^{-2}$ | $\kappa = 10^{-1}$ | $\kappa = 10^0$ | $\kappa = 10^1$ | $\kappa = 10^2$ | $\kappa = 10^3$ | $\kappa = 10^4$ | $\kappa = 10^5$ |
|-----|--------------------|--------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 5   | 2.0e-03            | 2.1e-03            | 3.2e-03         | 1.1e-02         | 7.3e-05         | 9.5e-07         | 7.7e-09         | 7.4e-11         |
| 10  | 1.3e-03            | 1.3e-03            | 3.6e-03         | 2.6e-05         | 1.8e-07         | 1.4e-09         | 1.3e-11         |                 |
| 20  | 3.7e-05            | 3.7e-05            | 1.3e-04         | 1.4e-05         | 1.4e-07         | 1.8e-09         | 1.9e-11         |                 |
| 40  | 1.6e-05            | 1.6e-05            | 1.7e-05         | 1.4e-05         | 8.1e-09         | 9.7e-11         | 9.7e-13         |                 |

The respective $| I_\kappa(F_1) |$ values are: 16.76; 16.73; 14.27; 1.054; 0.100; 0.0141; 0.00079; 0.00006

Table 5  Cubature errors $| I_\kappa(F_3) - I_{\kappa,N}^{gl}(F_3) |$ for $\kappa = 10^m$, $m = -2, -1, 0, 1, 2, 3, 4, 5$

| $N$ | $\kappa = 10^{-2}$ | $\kappa = 10^{-1}$ | $\kappa = 10^0$ | $\kappa = 10^1$ | $\kappa = 10^2$ | $\kappa = 10^3$ | $\kappa = 10^4$ | $\kappa = 10^5$ |
|-----|--------------------|--------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 5   | 1.2e-04            | 1.4e-04            | 3.9e-04         | 1.7e-03         | 1.5e-05         | 1.9e-07         | 1.6e-09         | 1.4e-11         |
| 10  | 3.0e-05            | 3.0e-05            | 3.0e-05         | 2.3e-04         | 1.7e-06         | 2.1e-08         | 1.7e-10         | 1.6e-12         |
| 20  | 1.7e-07            | 1.7e-07            | 2.0e-07         | 1.2e-06         | 3.1e-07         | 3.2e-09         | 4.0e-11         | 4.1e-13         |
| 40  | 2.3e-08            | 2.3e-08            | 2.8e-08         | 2.8e-08         | 2.1e-11         | 3.2e-13         | 3.3e-15         |                 |

The respective $| I_\kappa(F_3) |$ values are: 40.21; 40.16; 35.30; 3.555; 0.3683; 0.0396; 0.0036; 0.0003
Further, for $s = 1, 3, 5$, using Theorem 3, the error in approximating the wideband integrals by the new class of cubature, $|I_κ(F_s) - I_{κ,N}^{gl}(F_s)|$, is similar to that of $\|Q_{N}^{gl}F_s - F_s\|_{H^0}$ for all wideband wavenumbers and that the cubature error is much smaller for large wavenumbers. Results in Tables 4, 5, and 6 (with exact decaying values of $I_κ(F_s)$, for $s = 1, 3, 5$) demonstrate the very efficient approach developed in the article to approximate the wideband integrals and associated error analysis. For a fixed $N$ and for large $κ ≥ 100$, the error results in Tables 4–6 indicate $O(κ^{-2})$ decay of the cubature error. This is the case even for $F_1$ that does not satisfy the smoothness assumption in Theorem 3. In particular, for a fixed large $κ$, accurate solutions have been obtained even for small values $N$ and hence we do not observe consistent error improvement for certain increment in values of $N$ for the $F_1$ case. Consequently, similar to [9, Table 2] and as explained in [9, Experiment 2], for a fixed large $κ$, the estimate in Theorem 3 for $F_1$ is only an upper bound for the error.

Using the algorithm described in Section 5, in addition to obtaining high-order accuracy, the total MATLAB computation time taken to produce all results in Tables 4–6 was less than one-tenth of one second of a single core of a 3.4GHz INTEL CORE i7-2600 CPU. This is a marked computational advantage over (i) standard cubature approximations that require at least ten points per wavelength to compute such wideband integrals with low-order accuracy; and (ii) asymptotic based approximations and analysis that are applicable only for very large wavenumbers.

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Appendix: Proofs of Lemma 4 and Proposition 5

In this section we prove several auxiliary results, most concerned with convergence estimates of the interpolants $q_{m,N}^{m,gl}$, introduced in (4.7)–(4.9), in the spaces $W_{m}^{r}$, which are, up to our knowledge, new or sharper than that found in the literature (see for instance [3] or [11, §6.6] and references therein).

The first result is an inverse inequality for $D_{N}^{ε}$ and $D_{N}^{0}$ which holds in $W_{m}^{r}$, uniformly in $m$ and $N$. 

---

**Table 6** Cubature errors $|I_κ(F_5) - I_{κ,N}^{gl}(F_5)|$ for $κ = 10^m$, $m = -2, -1, 0, 1, 2, 3, 4, 5$

| $N$ | $κ = 10^{-2}$ | $κ = 10^{-1}$ | $κ = 10^0$ | $κ = 10^1$ | $κ = 10^2$ | $κ = 10^3$ | $κ = 10^4$ | $κ = 10^5$ |
|-----|---------------|---------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 5   | 3.1e-05       | 7.0e-05       | 1.5e-04     | 8.1e-04     | 7.2e-06     | 1.0e-07     | 7.0e-10     | 6.3e-12     |
| 10  | 1.7e-06       | 1.7e-06       | 1.8e-06     | 4.5e-05     | 4.2e-07     | 5.5e-09     | 4.3e-11     | 4.1e-13     |
| 20  | 1.4e-09       | 1.4e-09       | 2.4e-09     | 2.9e-08     | 1.9e-08     | 1.9e-10     | 2.3e-12     | 2.3e-14     |
| 40  | 9.0e-11       | 9.0e-11       | 9.0e-11     | 1.3e-10     | 2.7e-10     | 8.1e-13     | 3.2e-15     | 4.8e-18     |

The respective $|I_κ(F_5)|$ values are: 114.89; 114.77; 103.22; 12.389; 1.2664; 0.1282; 0.0126; 0.0013.
Lemma 6 There exists $C > 0$ such that for any $m \in \mathbb{Z}$, $g_N \in \mathbb{D}^e_N \cup \mathbb{D}^o_N$ with $g_N(0) = g_N(\pi) = 0$ if $m \neq 0$,
\[ \|g_N\|_{W^1_m} \leq C(1 + |m|)N\|g_N\|_{W^0_m}. \]

Proof We first prove the result for even $m$. Let $g_N \in \mathbb{D}^e_N$ and take $m' = 2$ if $g_N(0) = g_N(\pi) = 0$ and $m' = 0$ otherwise. Note that
\[ g_N \in \text{span}\{Q^m_n : n = m', \ldots, N\}, \]
this property is not valid for $m' > 2$. Since $\{Q^m_n\}_{n=m'}^{\infty}$ is an orthonormal basis of $L^2_{\sin} = W^0_m$, we obtain
\[ g_N = \sum_{n=m'}^{N} \left[ \int_0^\pi g_N(\theta)Q^m_n(\theta) \sin \theta \, d\theta \right]Q^m_n = \sum_{n=m'}^{N} (\widehat{g_N})_{n,m'}(n)Q^m_n. \]
Hence
\[ \|g_N\|^2_{W^1_m} = \sum_{n=m'}^{N} \left( n + \frac{1}{2} \right)^2 |(\widehat{g_N})_{n,m'}(n)|^2 \leq \left( N + \frac{1}{2} \right)^2 \sum_{n=m'}^{N} |(\widehat{g_N})_{n,m'}(n)|^2 \]
\[ = \left( N + \frac{1}{2} \right)^2 \|g_N\|^2_{W^0_m}. \]
This concludes the proof for $|m| = 0, 2$. For even $|m| > 2$, we apply (3.21) of Proposition 2 to derive
\[ \|g_N\|_{W^1_m} \leq \frac{|m|}{2} \|g_N\|_{W^2_m} \leq \frac{|m|}{2} \left( N + \frac{1}{2} \right) \|g_N\|_{L^2_{\sin}} = \frac{|m|}{2} \left( N + \frac{1}{2} \right) \|g_N\|_{W^0_m}. \]
The proof for odd $m$ is similar: Since $\mathbb{D}^o_N = \text{span}\{Q^1_n : n = 1, \ldots, N\}$, as above the result can be derived first for $m = 1$ and using again (3.21) of Proposition 2, the result follows for all odd values of $m$. $\square$

Let
\[ s^m_Nf := \sum_{|m| \leq n \leq N-1} \widehat{f}_m(n)Q^m_n, \quad \widehat{f}_m(n) = \int_0^\pi f(\theta)Q^m_n(\theta) \sin \theta \, d\theta, \quad (A.1) \]
with the convention that, for $|m| \geq N$, $s^m_Nf = 0$. Clearly $s^m_N$ is just the orthogonal projection onto span $\{Q^m_n : n = |m|, \ldots, N - 1\}$ in $W^s_{m}$ for all $s$. Moreover $s^{2\ell}_{N}f \in \mathbb{D}^e_{N-1} \subset \mathbb{D}^e_{N}$, $s^{2\ell+1}_{N}f \in \mathbb{D}^o_{N-2}$, and therefore
\[ q^e_Ns^{2\ell}_{N} = s^{2\ell}_{N}, \quad q^o_Ns^{2\ell+1}_{N} = s^{2\ell+1}_{N}. \quad (A.2) \]

Lemma 7 For all $r \geq s$ and $N \geq 2$,
\[ \|f - s^m_Nf\|_{W^r_m} \leq \left( N + \frac{1}{2} \right)^{s-r} \|f\|_{W^r_m}. \quad (A.3) \]
Moreover, if $m \neq 0$,
\[ s^m_Nf(0) = s^m_Nf(\pi) = 0, \quad (A.4) \]
and
\[
\max \left\{ |s_N^0 f(0) - f(0)|, |s_N^0 f(\pi) - f(\pi)| \right\} \leq \frac{N^{1-r}}{(2r - 2)^{1/2}} \| f \|_{W_0^r}, \tag{A.5}
\]
for all \( r > 1 \).

**Proof** Take \( N' = \max\{ N, |m| \} \). Then
\[
\| s_N^m f - f \|_{W_m^r}^2 = \sum_{n \geq N'} \left( n + \frac{1}{2} \right)^{2r} |\hat{f}_m(n)|^2 \leq \left( N' + \frac{1}{2} \right)^{2s-2r} \times \sum_{n \geq N'} \left( n + \frac{1}{2} \right)^{2r} |\hat{f}_m(n)|^2 \leq \left( N + \frac{1}{2} \right)^{2s-2r} \| f \|_{W_m^r}^2,
\]
which proves the first result. Using (3.5),
\[
Q_n^m(0) = Q_n^m(\pi) = 0, \quad \text{for } m \neq 0,
\]
and hence (A.4) follows.

For \( m = 0 \) we note first that
\[
Q_n^0(0) = \left( n + \frac{1}{2} \right)^{1/2} = (-1)^n Q_n^0(\pi).
\]

Then
\[
|s_N^0 f(0) - f(0)| \leq \sum_{n \geq N} |\hat{f}_0(n)| Q_n^0(0) = \sum_{n \geq N} \left( n + \frac{1}{2} \right)^{1/2} |\hat{f}_0(n)|
\]
\[
\leq \left[ \sum_{n \geq N} \left( n + \frac{1}{2} \right)^{-(2r-1)} \right]^{1/2} \left[ \sum_{n \geq N} \left( n + \frac{1}{2} \right)^{2r} |\hat{f}_0(n)|^2 \right]^{1/2}
\]
\[
\leq \frac{N^{1-r}}{(2r - 2)^{1/2}} \| f \|_{W_0^r}
\]
where, for \( \alpha > 1 \), we have applied
\[
\sum_{n \geq N} \left( n + \frac{1}{2} \right)^{-\alpha} \leq \int_{N-1/2}^{\infty} \left( x + \frac{1}{2} \right)^{-\alpha} dx \leq \frac{1}{\alpha - 1} N^{1-\alpha}.
\]
Proceeding in a similar way, we can bound \( |s_N^0 f(\pi) - f(\pi)| \) and hence we obtain (A.5). \( \square \)

Our next result gives us information about the distribution of the nodes \( \{ \theta_j \}_{j=0}^N \), the nodes of \( Q_{N}^{gl} \) in the elevation angle. Roughly speaking, this distribution turns out to be quasi-uniform.
Lemma 8 For \( N > 0 \) define
\[
\begin{align*}
a_j &= \frac{(j - 1/4)\pi}{N + 1/2}, & b_j &= \frac{(j + 1)\pi}{N + 1/2}, \quad j = 1, \ldots, \left\lceil \frac{N - 1}{2} \right\rceil, \\
a_{N/2} &= \pi - \frac{5\pi/8}{N + 1/2}, & b_{N/2} &= \frac{5\pi/8}{N + 1/2}, \text{ for even } N, \\
b_j &= \pi - a_{N-j}, & a_j &= \pi - b_{N-j}, \quad j = \left\lceil \frac{N+1}{2} \right\rceil, \ldots, N - 1.
\end{align*}
\]

Let \( \{\theta_j\}_{j=0}^N \) be as in (2.14). Then \( \theta_j \in [a_j, b_j], \quad j = 1, \ldots, N - 1. \)

Proof Note that for even \( N, \theta_{N/2} = \arccos \xi_{N/2} = \arccos 0 = \pi/2 \) and therefore there is nothing to prove in this case. On the other hand,
\[
\arccos \xi_{N-j+1} < \theta_j < \arccos \xi_{N-j} < \arccos \xi_{N-j+1},
\]
where \( \{\xi_j\}_{j=1}^N \) are the zeros of \( P_N \), the Legendre polynomial of degree \( N \). We can apply well-known results on the asymptotic distribution of these roots (see for instance [11, (3.1.2)–(3.1.3)]) to deduce that for \( 1 \leq j < N/2, \)
\[
a_j = \frac{(j - 1/4)\pi}{N + 1/2} < \arccos \xi_{N-j+1} < \theta_j < \arccos \xi_{N-j} < \frac{(j + 1)\pi}{N + 1/2} = b_j.
\]
(See also [30, Theorem 6.21.3] or [3, Theorem 2.4].) For \( j > N/2 \) the result follows from the first part of the proof and the symmetric disposition of the nodes.

Let us introduce the Gauss–Lobatto quadrature rule
\[
\sum_{j=0}^{N} \omega_j g(\eta_j) \approx \int_{-1}^{1} g(s) \, ds,
\]
which is exact for any polynomial of degree up to \( 2N - 1 \) [7, 22]. We point out that the weights of this formula satisfy (see for instance [11, Section 3.5], [3, Corollary 2.8])
\[
\omega_0 = \omega_N = \frac{2}{N(N + 1)}
\]
and
\[
0 < \omega_j \leq \frac{C}{N} \sin \theta_j, \quad j = 1, \ldots, N - 1 \quad (A.6)
\]
with \( C \) independent of \( N \).
Consequently, the quadrature rule
\[
\mathcal{L}_N f := \sum_{j=0}^{N} \omega_j f(\theta_j) \approx \int_{0}^{\pi} f(\theta) \sin \theta \, d\theta = \int_{-1}^{1} f(\arccos s) \, ds
\]
is exact for any \( p_N \in \mathbb{D}^e_{2N-1} \) (recall that \( \theta_j = \arccos \eta_{N-j} \)). Hence,
\[
\mathcal{L}_N(\|p_N\|^2) = \|p_N\|_{L^2_{\sin}}^2, \quad \forall p_N \in \mathbb{D}^e_{N-1} \cup \mathbb{D}^o_{N-2}.
\]
If \( p_N \in \mathbb{D}_N^e \cup \mathbb{D}^o_{N-1} \) (see (2.6)), the rule fails to be exact for \( |p_N|^2 \in \mathbb{D}^e_{2N} \). However, we have instead the bound

\[
\int_0^\pi |p_N(\theta)|^2 \sin \theta \, d\theta \leq C \sum_{n=0}^N \omega_n |p_N(\theta_n)|^2, \quad p_N \in \mathbb{D}_N^e \cup \mathbb{D}^o_{N-2} \tag{A.7}
\]

with \( C \) independent of \( N \) and \( p_N \). This bound, which is a consequence of a well known stability result for Gauss–Lobatto formulas (see [11, Theorem 3.8.2]), will play an essential role in what follows.

We are ready to prove the first inequality in Lemma 4. In fact below we prove a slightly more general result (see Corollary 1 which gives an equivalent norm for \( W^1_m \)).

**Lemma 9** There exists \( C > 0 \) such that for all \( N \geq 2 \) and for any \( f \in C^0[0, \pi] \) with \( f' \in L^2_{\sin} \)

\[
\|q^m_{N,\text{gl}} f\|_{L^2_{\sin}} \leq C \left[ \|f\|_{L^2_{\sin}} + \frac{1}{N} \|f'\|_{L^2_{\sin}} + \frac{1}{N} (|f(0)| + |f(\pi)|) \right],
\]

with \( C \) is independent of \( f \) and \( N \).

**Proof** Without loss of generality we assume that \( f \) is a real valued function. Recall that from the definition of \( q^m_{N,\text{gl}} \),

\[
q^2_{N,\text{gl}} f(\theta_j) = q^e_{N,\text{gl}} f(\theta_j) = f(\theta_j), \quad j = 0, \ldots, N, \\
q^{2\ell+1}_{N,\text{gl}} f(\theta_j) = q^o_{N,\text{gl}} f(\theta_j) = f(\theta_j), \quad j = 1, \ldots, N-1, \\
q^{2\ell+1}_{N,\text{gl}} f(0) = q^{2\ell+1}_{N,\text{gl}} f(\pi) = 0.
\]

Then, using (A.7) we obtain

\[
\int_0^\pi \left[q^m_{N,\text{gl}} f\right]^2(\theta) \sin \theta \, d\theta \leq C_m \sum_{n=0}^N \omega_n \left[q^m_{N,\text{gl}} f\right]^2(\theta_n) \leq C_m \sum_{n=0}^N \omega_n |f|^2(\theta_n),
\]

where \( C_m \) is independent of \( f \) and \( N \). Note that \( C_m = 1 \) for \( m \) odd (that is, for \( q^o_{N,\text{gl}} f \)) or for \( m \) even with \( q^e_{N,\text{gl}} f \in \mathbb{D}^e_{N-1} \). From Lemma 8 we have

\[
\theta_n \in [a_n, b_n], \quad \text{with} \quad \frac{\pi}{N} \leq \frac{5\pi}{4(N+1/2)} = (b_n - a_n) \leq \frac{2\pi}{N}, \quad n = 1, \ldots, N-1. \tag{A.8}
\]
Hence from (A.6) and (A.8) we conclude that

\[
\int_0^\pi \left[ q_m N, gl f \right]^2(\theta) \sin \theta \, d\theta \leq \frac{2C_m}{N(N + 1)} \left( f^2(0) + f^2(\pi) \right) \\
+ C \left[ \frac{1}{N} \sum_{n=1}^{N-1} f^2(\theta_n) \sin \theta_n \right] \\
\leq \frac{2C_m}{N(N + 1)} \left( f^2(0) + f^2(\pi) \right) \\
+ \frac{C}{\pi} \sum_{n=1}^{N-1} (b_n - a_n) f^2(\theta_n) \sin \theta_n. \quad (A.9)
\]

We point out that the above sum can be understood as a composite rectangular rule applied to \( f^2(\theta) \sin \theta \). Using the well-known bound (with \( c \in [a, b] \)),

\[
\left| \int_a^b g(\theta) \, d\theta - g(c)(b - a) \right| \leq (b - a) \int_a^b |g'(\theta)| \, d\theta,
\]

and noticing that

\[
\sum_{n=1}^{N-1} \int_{a_n}^{b_n} |g(\theta)| \, d\theta \leq 2 \int_0^\pi |g(\theta)| \, d\theta, \quad \forall g \in L^1(0, \pi),
\]

we deduce

\[
\sum_{n=1}^{N-1} (b_n - a_n) f^2(\theta_n) \sin \theta_n
\]

\[
= \sum_{n=1}^{N-1} \left[ (b_n - a_n) f^2(\theta_n) \sin \theta_n - \int_{a_n}^{b_n} f^2(\theta) \sin \theta \, d\theta \right] + \sum_{n=1}^{N-1} \int_{a_n}^{b_n} f^2(\theta) \sin \theta \, d\theta
\]

\[
\leq \sum_{n=1}^{N-1} (b_n - a_n) \int_{a_n}^{b_n} \left| f^2(\theta) \sin \theta \right| \, d\theta + 2 \int_0^\pi f^2(\theta) \sin \theta \, d\theta
\]

\[
\leq \frac{4\pi}{N} \left[ \int_0^\pi f^2(\theta) \cos \theta \, d\theta + \int_0^\pi 2|f(\theta)f'(\theta)| \sin \theta \, d\theta \right] + 2 \int_0^\pi f^2(\theta) \sin \theta \, d\theta
\]

\[
\leq \frac{8\pi}{N} \left[ \int_0^\pi f^2(\theta) \, d\theta + \left( \int_0^\pi f^2(\theta) \sin \theta \, d\theta \right)^{1/2} \left( \int_0^\pi |f'(\theta)|^2 \sin \theta \, d\theta \right)^{1/2} \right]
\]

\[
+ 2 \int_0^\pi f^2(\theta) \sin \theta \, d\theta
\]

\[
\leq C'' \int_0^\pi f^2(\theta) \sin \theta \, d\theta + \frac{C''}{N} \left( \int_0^\pi f^2(\theta) \sin \theta \, d\theta \right)^{1/2} \left( \int_0^\pi |f'(\theta)|^2 \sin \theta \, d\theta \right)^{1/2}
\]

where we have applied again (A.8) and used in the last step Proposition 4. Finally, the inequality

\[
2ab \leq N a^2 + N^{-1} b^2
\]
proves that
\[
\sum_{n=1}^{N-1} (b_n - a_n) f_n^2(\theta_n) \sin \theta_n \leq C \left[ \int_0^\pi f(\theta) \sin \theta \, d\theta + \frac{1}{N^2} \int_0^\pi |f'(\theta)|^2 \sin \theta \, d\theta \right].
\] (A.10)

Substituting (A.10) in (A.9), the desired result follows.

Remark 2 Inequalities such as those proved in Lemma 9 have been used in the literature for studying the convergence of polynomial interpolants in weighted Sobolev spaces (see for instance [11] or [3] and references therein).

Note that with the change of variables \( s = \cos \theta \), Lemma 4 implies that for \( f \in C^1[-1, 1] \) with \( f(\pm 1) = 0 \), and if \( p_N \) is the polynomial of degree \( N \) which interpolates \( f \) at Gauss–Lobatto nodes \( \{\eta_j\}_{j=0}^N \), we have the bound
\[
\int_{-1}^1 |p_N(s)|^2 \, ds \leq C \left[ \int_{-1}^1 |f(s)|^2 \, ds + \frac{1}{N^2} \int_{-1}^1 |f'(s)|^2 (1 - s^2) \, ds \right]
\]
with \( C \) independent of \( N \) and \( f \). This bound is an improvement of Theorem 4.1 in [3] where a similar result is proven but without the weight function \( (1 - s^2) \) in the second integral.

Next we prove Proposition 5 for the \( s = 0 \) case.

Proposition 6 For \( r > 1 \) there exists \( C_r > 0 \) so that
\[
\|q_{N, gl}^m f - f\|_{W^0_m} \leq C_r N^{-r} \|f\|_{W^r_m}.
\]
Moreover, the estimate above holds also for \( r = 1 \) if \( m \neq 0 \).

Proof Assume first that \( m \neq 0 \). We recall from Remark 1 that any \( f \in W^1_m \) is continuous and vanishes at \( \{0, \pi\} \). From (A.2), Lemma 9 (see also Lemma 4), Proposition 2, and (A.3) of Lemma 7 we deduce that for \( r \geq 1 \),
\[
\|q_{N, gl}^m f - f\|_{W^0_m} = \|q_{N, gl}^m (f - s^m_N f) - (f - s^m_N f)\|_{W^0_m}
\leq C \left[ \|f - s^m_N f\|_{W^0_m} + N^{-1} \|f - s^m_N f\|_{W^1_m} \right] \leq C N^{-r} \|f\|_{W^r_m}.
\]
If \( m = 0 \), we have instead
\[
\|q_{N, gl}^0 f - f\|_{W^0_0} \leq C \left[ \|f - s^0_N f\|_{W^0_0} + N^{-1} \|f - s^0_N f\|_{W^1_0} + |(f - s^0_N f)(0)| + |(f - s^0_N f)(\pi)| \right].
\]
Using (A.3) as before, and (A.5) for taking care of the pointwise errors at \( \{0, \pi\} \) (and here using the restriction \( r > 1 \) we conclude that
\[
\|q_{N, gl}^0 f - f\|_{W^0_0} \leq C N^{-r} \|f\|_{W^r_0}.
\]
To derive convergence estimates in $W_m^1$ we require additional technical results. To this end, it is convenient to introduce the norms
\[
\| f \|_{L^2_{\sin^k}} := \left[ \int_0^\pi |f(\theta)|^2 \sin^k \theta \, d\theta \right]^{1/2},
\]
and the associated spaces $L^2_{\sin^k}$. Below we will use these norms for $k = -3, -1$ and 2.

**Lemma 10** For all $N \geq 2$ and for all $p_N \in \mathbb{D}^e_N \cup \mathbb{D}^o_{N-1}$,
\[
\| p_N \|_{L^2(0,\pi)} \leq \left[ \frac{1}{\sin \left( \frac{\pi}{2N+4} \right)} \right]^{1/2} \| p_N \|_{L^2_{\sin}}.
\]

**Proof** In this proof we will apply the fact that the quadrature rule
\[
\frac{\pi}{N+2} \sum_{j=0}^{N+1} f \left( \frac{(j+1/2)\pi}{N+2} \right) \approx \int_0^\pi f(\theta) \, d\theta
\]
is exact for any $f \in \mathbb{D}^e_{2N+2}$. Notice that, by hypothesis, $|p_N|^2 \in \mathbb{D}^e_{2N}$. Hence,
\[
\| p_N \|^2_{L^2(0,\pi)} = \frac{\pi}{N+2} \sum_{j=0}^{N+1} |p_N \left( \frac{(j+1/2)\pi}{N+2} \right)|^2
\]
\[
\leq \left[ \sin \left( \frac{\pi}{2N+4} \right) \right]^{-2} \frac{\pi}{N+2} \sum_{j=0}^{N+1} \left| (p_N \sin) \left( \frac{(j+1/2)\pi}{N+2} \right) \right|^2
\]
\[
= \left[ \sin \left( \frac{\pi}{2N+4} \right) \right]^{-2} \| p_N \|_{L^2_{\sin^2}}^2. \tag{A.11}
\]

We point out that, in the usual notation of the theory of interpolation of Hilbert spaces, we have
\[
L^2_{\sin} = \left[ L^2(0, \pi), L^2_{\sin^2} \right]^{1/2}
\]
(see for instance [31, Lemma 23.1]). Hence, with (A.11), we obtain the desired result:
\[
\| p_N \|^2_{L^2(0,\pi)} \leq \left[ \sin \left( \frac{\pi}{2N+4} \right) \right]^{-1} \| p_N \|_{L^2_{\sin}}^2, \quad \forall p_N \in \mathbb{D}^e_N \cup \mathbb{D}^o_{N-1}.
\]

**Lemma 11** There exists $C > 0$ so that for all $f \in W_m^2$ with $|m| \geq 2$,
\[
\| q_{N, gl}^m f \|_{L^2_{\sin^{-1}}} \leq C \left[ \| f \|_{L^2_{\sin^{-1}}}^2 + \frac{1}{N} \left( \| f \|_{L^2_{\sin^{-3}}}^2 + \| f \|_{L^2_{\sin^{-1}}}^2 \right) \right]. \tag{A.12}
\]

**Proof** First, consider the case of even $m$ so that $q_{N, gl}^m f = q_{N, gl}^e f$. Then, it is easy to check that
\[
(q_{N, gl}^e f)(\theta) / \sin \theta = q_{N, gl}^o \left( f / \sin \right)(\theta).
\]
(Note that in this case $f(0) = f(\pi) = 0$.) We can proceed as in the proof of Lemma 9, see (A.9) and (A.10), and derive

$$\|q_{N,gl}^0 f\|_{L^2_{sin^{-1}}}^2 = \|q_{N,gl}^0 (f/\sin)\|_{L^2_{sin}}^2 \leq C \sum_{n=1}^{N-1} (b_n - a_n) (f/\sin)^2 (\theta_n) \sin \theta_n$$

$$\leq C' \left[ \int_0^\pi f^2(\theta) \frac{d\theta}{\sin \theta} + \frac{1}{N^2} \int_0^\pi |(f/\sin)'(\theta)|^2 \sin \theta \, d\theta \right]$$

$$\leq 2C' \left[ \int_0^\pi f^2(\theta) \frac{d\theta}{\sin \theta} + \frac{1}{N^2} \int_0^\pi f^2(\theta) \frac{d\theta}{\sin^3 \theta} $$

$$+ \frac{1}{N^2} \int_0^\pi |f'(\theta)|^2 \frac{d\theta}{\sin \theta} \right].$$

This proves the result for the even $m$ case.

To prove the result for the odd $m$ case, observe that now $q_{N,gl}^0 f/\sin \in \mathbb{D}_{N-2}^e$. However, if we try to apply the same ideas as before, unlike the previous case, we have to take care also of the pointwise values at 0 and $\pi$. Indeed, following again (A.9) and (A.10), and noticing that $C_m = 1$ in (A.9), we derive

$$\|q_{N,gl}^0 f\|_{L^2_{sin^{-1}}}^2 = \|q_{N,gl}^0 f/\sin\|_{L^2_{sin}}^2$$

$$\leq \frac{2}{N(N+1)} \left( |(q_{N,gl}^0 f/\sin)(0)|^2 + |(q_{N,gl}^0 f/\sin)(\pi)|^2 \right)$$

$$+ C \sum_{n=1}^{N-1} (b_n - a_n) (f/\sin)^2 (\theta_n) \sin \theta_n$$

$$\leq \frac{2}{N(N+1)} \left( |(q_{N,gl}^0 f/\sin)(0)|^2 + |(q_{N,gl}^0 f/\sin)(\pi)|^2 \right)$$

$$+ 2C' \left[ \int_0^\pi f^2(\theta) \frac{d\theta}{\sin \theta} + \frac{1}{N^2} \int_0^\pi f^2(\theta) \frac{d\theta}{\sin^3 \theta}$$

$$+ \frac{1}{N^2} \int_0^\pi |f'(\theta)|^2 \frac{d\theta}{\sin \theta} \right].$$

(A.13)

It remains to bound

$$R_N(f) := \frac{2}{N(N+1)} \left( |(q_{N,gl}^0 f/\sin)(0)|^2 + |(q_{N,gl}^0 f/\sin)(\pi)|^2 \right)$$

$$\leq \frac{4}{N(N+1)} \|q_{N,gl}^0 f/\sin\|_{L^\infty(0,\pi)}^2.$$
for suitable coefficients $\alpha_{2n}$. Hence, by applying the Cauchy-Schwarz inequality we obtain

$$\|q_{N,gl}^0 f / \sin \|_{L^2(0,\pi)}^2 \leq \left[ \sum_{n=0}^{\lfloor (N-2)/2 \rfloor} |\alpha_{2n}|^2 \right] \leq \frac{2}{\pi} \left[ \frac{\pi |\alpha_0|^2}{2} + \sum_{n=1}^{\lfloor (N-2)/2 \rfloor} |\alpha_{2n}|^2 \frac{\pi}{2} \right] \times \frac{N}{2}$$

$$= \frac{N}{\pi} \|q_{N,gl}^0 f / \sin \|_{L^2(0,\pi)}^2.$$ 

Recall that $q_{N,gl}^0 f / \sin \in D^e_N - 2 \subset D^e_N - 1$. Then we can apply Lemma 10 to get

$$\|q_{N,gl}^0 f / \sin \|_{L^2(0,\pi)}^2 \leq \frac{N}{\pi \sin(\pi/(2N + 2))} \|q_{N,gl}^0 f / \sin \|_{L^2_{\sin}}^2,$$

and therefore

$$R_N(f) \leq \frac{4}{\pi (N + 1) \sin(\pi/(2N + 2))} \|q_{N,gl}^0 f / \sin \|_{L^2_{\sin}}^2 =: c(N) \|q_{N,gl}^0 f \|_{L^2_{\sin}}^2.$$ 

We point out that

$$0 < c(N) \leq c(2) = \frac{8}{3\pi} < 1, \quad \forall N \geq 2.$$ 

Using this bound in (A.13), we obtain

$$\|q_{N,gl}^0 f \|_{L^2_{\sin}}^2 \leq \frac{2C}{1 - c(2)} \left[ \int_0^{\pi} f^2(\theta) \frac{d\theta}{\sin \theta} \right. \right.$$

$$+ \frac{1}{N^2} \left( \int_0^{\pi} f^2(\theta) \frac{d\theta}{\sin^3 \theta} + \int_0^{\pi} |f'(\theta)|^2 \frac{d\theta}{\sin \theta} \right).$$

(A.14)

For odd $f$, that is for functions with $f(\theta) = -f(\pi - \theta)$, we can proceed similarly and show that a similar bound to (A.14) also holds in this case.

We have proven the result for the subspace of even and odd functions. To extend the result for arbitrary $f$ we note first that

$$f = f_e + f_o, \quad q_{N,gl}^0 f = q_{N,gl}^0 f_e + q_{N,gl}^0 f_o$$
where \( f_\varepsilon(\theta) := \frac{1}{2}(f(\theta) + f(\theta - \pi)) \) and \( f_o(\theta) = \frac{1}{2}(f(\theta) - f(\theta - \pi)) \) are even and odd parts of \( f \). We deduce that
\[
\|q^{0}_{N,gl} f\|_{L^{2}_{\sin^{-1}}}^2 = \|q^{0}_{N,gl} f_\varepsilon\|_{L^{2}_{\sin^{-1}}}^2 + \|q^{0}_{N,gl} f_o\|_{L^{2}_{\sin^{-1}}}^2
\leq C \left[ \|f_\varepsilon\|_{L^{2}_{\sin^{-1}}}^2 + \|f_o\|_{L^{2}_{\sin^{-1}}}^2 \right]
\times \frac{1}{N^2} \left( \|f_\varepsilon\|_{L^{2}_{\sin^{-3}}}^2 + \|f_o\|_{L^{2}_{\sin^{-3}}}^2 + \|f_o'\|_{L^{2}_{\sin^{-1}}}^2 + \|f_o'\|_{L^{2}_{\sin^{-1}}}^2 \right)
= C \left[ \|f\|_{L^{2}_{\sin^{-1}}}^2 + \frac{1}{N^2} \left( \|f\|_{L^{2}_{\sin^{-3}}}^2 + \|f'\|_{L^{2}_{\sin^{-1}}}^2 \right) \right].
\]
Thus the result \((A.12)\) follows. \(\Box\)

We are now ready to prove the second inequality in Lemma 4.

**Corollary 2** For all \( |m| \geq 2 \),
\[
\|q^{m}_{N,gl} f\|_{Z^{1}_{m}} \leq C \left[ \|f\|_{Z^{1}_{m}} + \frac{1}{N} \|f\|_{Z^{2}_{m}} \right].
\]

**Proof** Note that
\[
\|q^{m}_{N,gl} f\|_{Z^{1}_{m}} \leq m^2 \|q^{m}_{N,gl} f\|_{L^{2}_{\sin^{-1}}}^2 + 2 \|q^{m}_{N,gl} f - q^{m}_{N,gl} f\|_{L^{2}_{\sin^{-1}}}^2 + 2 \|q^{m}_{N,gl} f\|_{L^{2}_{\sin^{-1}}}^2 + 2 \|q^{m}_{N,gl} f\|_{L^{2}_{\sin^{-1}}}^2 =: I_1 + I_2 + I_3.
\]

We bound the first term using Lemma 11:
\[
I_1 \leq C m^2 \left[ \|f\|_{L^{2}_{\sin^{-1}}}^2 + \frac{1}{N^2} \left( \|f\|_{L^{2}_{\sin^{-3}}}^2 + \|f'\|_{L^{2}_{\sin^{-1}}}^2 \right) \right] \leq C \left[ \|f\|_{Z^{1}_{m}}^2 + \frac{1}{N^2} \|f\|_{Z^{2}_{m}}^2 \right].
\]

For the second term, using first Proposition 2 and combining the inverse inequalities stated in Lemma 6 with Lemma 7 and Proposition 6, we obtain
\[
I_2 \leq 2 \|q^{m}_{N,gl} f - q^{m}_{N,gl} f\|_{W^{2}_{1}}^2 \leq C N^2 \|q^{m}_{N,gl} f - q^{m}_{N,gl} f\|_{W^{0}_{m}}^2 \leq C' \|f\|_{Z^{1}_{m}}^2,
\]
where in the last step we have used \((3.25)\) of Corollary 1. We bound the last term as
\[
I_3 \leq 2 \|q^{m}_{N,gl} f\|_{W^{1}_{m}}^2 \leq 2 \|f\|_{W^{1}_{m}}^2 \leq \frac{5}{2} \|f\|_{Z^{1}_{m}}^2.
\]
Thus we obtain the desired result. \(\Box\)

Finally, we prove Proposition 5 for the \( s = 1 \) case.

**Proposition 7** For all \( r \geq 2 \) there exists \( C_r > 0 \) so that
\[
\|q^{r}_{N,gl} f - f\|_{W^{1}_{m}} \leq C_r N^{1-r} \|f\|_{W^{r}_{m}}.
\]
Proof. Note that
\[
\|q_N^m, gl (f - s_N f) - (f - s_N f)\|_{W^1_m} \leq \|q_N^m, gl (f - s_N f)\|_{W^1_m} + N^{1-\gamma} \|f\|_{W^r_m}
\]
where we have applied (A.2) and (A.3) of Lemma 7.

It remains to bound the first term. For |m| \leq 1, we make use of Lemma 6 and Proposition 6 to obtain,
\[
\|q_N^m, gl (f - s_N f)\|_{W^1_m} \leq CN \|q_N^m, gl (f - s_N f)\|_{W^0_m} \leq CN \left[ \|q_N^m, gl (f - s_N f)\|_{W^0_m} + \|f - s_N f\|_{W^0_m} \right] \leq C' N^{1-\gamma} \|f\|_{W^r_m}.
\]

For |m| \geq 2, this approach is not valid since the inverse inequalities of Lemma 6 contains m as a penalizing term. Thus, we follow a different approach and use Corollaries 1 and 2 to obtain
\[
\|q_N^m, gl (f - s_N f)\|_{W^1_m} \leq \frac{\sqrt{5}}{2} \|q_N^m, gl (f - s_N f)\|_{Z^1_m} \leq C \left( \|f - s_N f\|_{Z^1_m} + N^{-1} \|f - s_N f\|_{Z^1_m} \right) \leq C \left( \|f - s_N f\|_{W^1_m} + \sqrt{6} N^{-1} \|f - s_N f\|_{W^2_m} \right) \leq C' N^{1-\gamma} \|f\|_{W^r_m}.
\]
Hence the desired result follows. \(\square\)

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