Abstract
In this paper, we construct two types of 7R closed single loop linkages by combining different factorizations of a general (non-vertical) Darboux motion. These factorizations are obtained by extensions of a factorization algorithm for a generic rational motion. The first type of 7R linkages has several one-dimensional configuration components and one of them corresponds to the Darboux motion. The other type is a 7R linkage with two degrees of freedom and without one-dimensional component. The Darboux motion is a curve in an irreducible two dimensional configuration component.

Keywords: 7R linkage, Darboux motion, factorization

1 Introduction
In 1881, G. Darboux determined all one-parametric spatial motions with only planar curves as point paths [1]. These properties determine a unique motion which is nowadays named after Darboux. It is the composition of a planar elliptic motion with a suitably oscillating translation perpendicular to that plane. If the elliptic motion degenerates to a rotation, the Darboux motion is called vertical, if this is not the case, we call the Darboux motion non-vertical or general. In both cases, the generic point paths are ellipses. The Darboux motion is a special Schoenfliess motion and we call the direction perpendicular to the planar elliptic motion component the axis direction. These and other properties are well known in literature, see for example [2] or [3, Chapter 9, §3].

In this article, we construct 7R linkages with one degree of freedom such that one link during (one component of) the motion performs a general Darboux motion. These results complement recently presented linkages that generate vertical Darboux motions [4]. The beginnings of this research can be found in [5] where we investigated spatial linkages with a straight line trajectory. The linkages we present in this article are novel. Unlike the linkages of [4], we generate a general (non-vertical) Darboux motion and our linkages are free of prismatic or cylindrical joints. One of our linkages has already be found in [5]. Our description here is more general and detailed.

Our linkage construction is based on the factorization of rational motions which has been first introduced in [6]. In the dual quaternion model of rigid body displacements, the Darboux motion is parameterized by a cubic motion polynomial. In general, a cubic motion can be generated by 6R linkages [7–9] but specialties of the Darboux motion prevent application of the factorization algorithm for generic motion polynomials. Therefore, we resort to new factorization techniques for non-generic motion polynomials [10–12], i.e., we multiply the motion polynomial with a real polynomial which divides the primal part of the motion polynomial. The new motion polynomial can be factored in different ways which allows us to obtain new closed 7R linkages by combining different factorizations. A unified algorithm for factorization of generic and non-generic motion polynomials is introduced in [11]. Here, we just mention the basic idea of this algorithm: By factoring out linear motion polynomial factors (from the left and from the right) we create a generic motion polynomial that can be factored by the original algorithm of [6].

We continue this text with a quick introduction to the dual quaternion model of SE(3), the group of rigid body displacements, and to motion polynomials and their factorization (Section 2). In Section 3 we describe the Darboux motion in the dual quaternion model, present different factorizations, and combine them to form 7R linkages.

2 Preliminaries
The investigations in this article are based on the dual quaternion model of the group SE(3) of rigid body displacements. We assume familiarity with dual quaternions and their relation to kinematics but we provide a quick introduction of concepts that cannot be found in standard textbooks. For
more information on basics of dual quaternions and kinematics we refer to [13–15]. A short presentation of the topic with close relations to this article can also be found in [16].

2.1 Dual quaternions

A dual quaternion is an object of the shape

\[ h = h_0 + h_1 i + h_2 j + h_3 k + \varepsilon(h_4 + h_5 i + h_6 j + h_7 k) \]  

(1)

with real numbers \( h_0, \ldots, h_7 \). The associative but non-commutative multiplication of dual quaternions is defined via the relations

\[ i^2 = j^2 = k^2 = ijk = -1, \quad \varepsilon^2 = 0, \]

\[ \varepsilon i = i \varepsilon, \quad \varepsilon j = j \varepsilon, \quad \varepsilon k = k \varepsilon. \]

For example

\[ (1 - \varepsilon i)(j + \varepsilon k) = j + \varepsilon k - \varepsilon ij - \varepsilon^2 ik = j + \varepsilon k - \varepsilon k - 0 = j. \]

Conjugate dual quaternion and dual quaternion norm are

\[ \overline{h} = h_0 - h_1 i - h_2 j - h_3 k + \varepsilon(h_4 - h_5 i - h_6 j - h_7 k), \]

\[ ||h|| = h \overline{h}. \]

The primal part of \( h \) is \( p = h_0 + h_1 i + h_2 j + h_3 k \) and the dual part is \( d = h_4 + h_5 i + h_6 j + h_7 k \). Primal and dual part are (ordinary) quaternions. The set of quaternions is denoted by \( \mathbb{H} \), the set of dual quaternions by \( \mathbb{DH} := \mathbb{D} \otimes \mathbb{R} \mathbb{H} \). Writing \( h = p + \varepsilon d \) with \( p, d \in \mathbb{H} \), we have \( ||h|| = \overline{p} \overline{p} + \varepsilon(\overline{p}q + q \overline{p}) \).

Note that both \( \overline{p} \overline{p} \) and \( \overline{p}q + q \overline{p} \) are real numbers and the norm itself is a dual number.

If the dual quaternion \( h = p + \varepsilon d \) has a real norm \( (\overline{p}q + q \overline{p} = 0) \), its action on a point \([x_0, x_1, x_2, x_3]\) of real projective three-space is defined via

\[ x \mapsto y = \frac{px \overline{p} + p \overline{q} - q \overline{p}}{p \overline{p}} \]  

(2)

where \( x = x_0 + x_1 i + x_2 j + x_3 k \), \( y = y_0 + y_1 i + y_2 j + y_3 k \) and the image point has projective coordinates \([y_0, y_1, y_2, y_3]\).

The map (2) describes a rigid body displacement and the composition of such displacements corresponds to dual quaternion multiplication. Note that \( h \) and \( \lambda h \) with \( \lambda \in \mathbb{R} \setminus \{0\} \) describe the same map.

The dual quaternion \( h \) as in (1) describes a rotation or translation, if and only if \( h_4 = 0 \). Among these, translations are characterized by \( h_1 = h_2 = h_3 = 0 \). We speak accordingly of rotation or translation quaternions. In the former case, the revolute axis has Plücker coordinates \([h_1, h_2, h_3, -h_5, -h_6, -h_7]\). In particular, the axis direction is \((h_1, h_2, h_3)\).

2.2 Motion polynomials

Now we study polynomials

\[ C = c_n t^n + \cdots + c_1 t + c_0 \]  

(3)

in the indeterminate \( t \) and with dual quaternion coefficients \( c_0, \ldots, c_0 \in \mathbb{DH} \). The set of these quaternions is denoted by \( \mathbb{DH}[t] \). Multiplication of polynomials in \( \mathbb{DH}[t] \) is defined by the convention that indeterminate \( t \) and coefficients commute. The conjugate polynomial \( \overline{C} \) is the polynomial with conjugate coefficients \( \overline{c_0}, \ldots, \overline{c_0} \).

A polynomial \( C \in \mathbb{DH}[t] \) is called a motion polynomial, if its leading coefficient \( c_n \) is invertible (has non-zero primal part) and the so-called norm polynomial \( QC \) has real coefficients. In this case, it parameterizes a rational motion via the map (2). Note that \( C \) and \( QC \) parameterize the same rational motion if \( Q \) is a real polynomial without zeros. This observation will be important in the next section.

The linear polynomial \( C = t - h \) with \( h \) as in (1) is a motion polynomial if and only if \( h_4 = h_1 h_5 + h_2 h_6 + h_3 h_7 = 0 \). Assuming \( (h_1, h_2, h_3) \neq (0, 0, 0) \), the displacements described by \( C \) for varying \( t \) are rotations about a fixed axis but with different revolute angle. The axis’ Plücker coordinates are \([h_1, h_2, h_3, -h_5, -h_6, -h_7]\) and \((h_1, h_2, h_3)\) is a direction vector of the axis.

A generic motion polynomial \( C \) of degree \( n \) admits \( n! \) factorizations of the shape

\[ C = (t - h_1) \cdots (t - h_n) \]  

(4)

with rotation quaternions \( h_1, \ldots, h_n \). The precise statement and a factorization algorithm can be found in [6]. Here is suffices to say that a Darboux motion is not generic and the general theory of [6] gives no information about existence or non-existence of factorizations.

Equation (4) admits an important kinematic interpretation: Each factorization corresponds to a one-parametric motion of an open chain of \( n \) revolute joints whose end-effector follows the motion parameterized by \( C \). The revolute axes are determined by the rotation quaternions \( h_1, \ldots, h_n \). Different factorizations may be combined to form closed loop linkages with this coupler motion.

In the remainder of this text we need auxiliary results related to the factorization of motion polynomials. Their proofs can be found in [6]:

**Lemma 1 (Polynomial division).** If \( C \) and \( D \) are polynomials in \( \mathbb{DH}[t] \) and the leading coefficient of \( D \) is \( 1 \), there exist polynomials \( Q, R \in \mathbb{DH}[t] \) such that \( C = QD + R \) and \( \deg R < \deg D \).

Note that the computation of quotient \( Q \) and remainder \( R \) is possible by straightforward polynomial long division in a non-commutative setting.
Lemma 2 (Zeros and right factors). The dual quaternion $h$ is a zero of the polynomial $C \in \mathbb{DH}[t]$ if and only if there exists a polynomial $Q \in \mathbb{DH}[t]$ such that $C = Q(t-h)$.

The statement of Lemma 2 is less trivial than the corresponding statement for real polynomials. In particular, with $C$ as in (3), the value of $C(h)$ (and hence also the term “zero of $C$”) is defined as $c_0 t^0 + \cdots + c_1 t + c_0$. It is not admissible to simply plug $h$ in a factorized representation of the shape (4). With $C = (t-k)(t-h)$ and $h, k \in \mathbb{DH}[t]$ we have, for example, $C = t^2 - (h+k)t + kh$ and thus

$$C(h) = -kh + kh = 0 \quad \text{but} \quad C(k) = kh + kh \neq 0.$$  

Lemma 2 asserts that the “usual” relation between zeros linear factors holds at least for right factors.

3 Factorizations of the Darboux motion

Following [3, Equation (3.4)], the general Darboux motion is given by the parameteric equations

\[
X = x \cos \varphi - y \sin \varphi  \\
Y = x \sin \varphi + y \cos \varphi + a \sin \varphi  \\
Z = z + b \sin \varphi + c(1 - \cos \varphi).
\]

Here, $\varphi \in [-\pi, \pi]$ is the motion parameter, $a$, $b$, $c$ are real constants and $x, y, z$ resp. $X, Y, Z$ are coordinates in the moving and the fixed frame. For $a = 0$, the motion is a vertical Darboux motion. In this paper we exclude this case and assume $a \neq 0$.

Using the conversion formulas between matrix and dual quaternion representation of rigid body displacements (see for example [15]), we find the polynomial parameterization

\[
C_0 = (k + ce)^3 + (1 - \varepsilon(ia - c - bk - b)) t + 1
\]

for the Darboux motion. Left-dividing this motion polynomial by the leading coefficient $k + ce$ (this amounts to a coordinate change in the fixed frame), we arrive at

\[
C = t^3 - (k - \varepsilon(ia - bk)) t + (1 - \varepsilon(b + a - ck)) t - k + ce.
\]

Our further investigations will be based on this parametric representation of the Darboux motion.

Note that the primal part of $C$ is $(t^2 + 1)(t - k)$. The presence of a real factor violates the assumptions of [6, Theorem 1] (existence of factorizations) and prevents application of Algorithm 1 (computation of factorizations) of that paper. Nonetheless, we will see below that $C$ admits infinitely many factorizations with three linear motion polynomials. Moreover, suitable polynomial multiples of $C$ admit factorizations as well and parameterize the same Darboux motion. This we will use for our construction of 7R Darboux linkages.

In [5], the authors gave two types of factorizations for the motion $C$ in Equation (6). The first type is with three linear motion polynomials (FI), the second type (FII) is with five linear motion polynomials whose product does not equal $C$ but $C$ times a real quadratic polynomial $P$. This makes an algebraic difference but does not change the kinematics: Both $C$ and $PC$ parameterize the same motion. Moreover, two successive linear factors in FII are identical. Hence, it gives rise to an open 4R chain which can generate the Darboux motion. In combination with the 3R chain obtained from FI we obtain a 7R linkage.

In this section, we recall the construction of [5] and we give two further factorizations (FIII and FIV) of (6). Similar to FII, these factorizations also use five linear motion polynomials but give only four axes because two successive linear motion polynomials are same. Combining it with the already known 3R chain, we obtain further 7R Darboux linkages.

3.1 First factorization FII

We want to find linear motion polynomials $Q_1, Q_2, Q_3$ such that $C = Q_1 Q_2 Q_3$. By Lemma 2, a necessary and sufficient condition for $Q_3 = t - q_3, q_3 \in \mathbb{DH}$, to be a right factor of $C$ is $C(q_3) = 0$. A straightforward calculation shows that $q_3$ is necessarily of the shape

\[
q_3 = k + \varepsilon(vi + wj), \quad v, w \in \mathbb{R}.
\]

Next, we consider the quotient polynomial $Q = P + \varepsilon D$, defined by $C = QQ_3$. Its primal part is $P = t^2 + 1$. Thus, the motion parameterized by $Q$ is a curvilinear translation. Since $Q$ is quadratic and $P$ has no real zero, the trajectories are congruent ellipses. This motion admits a factorization $Q = Q_1 Q_2$ of the required shape only if it is a circular translation (see [16] for a detailed proof of this statement). This puts a condition on $q_3$, namely

\[
v = -a^{-1}bc, \quad w = -(2a)^{-1}(a^2 + b^2 + c^2). \quad (7)
\]

If (7) is satisfied, the remainder $Q$ is a circular translation and can be factored in infinitely many ways as $Q = Q_1 Q_2$ with linear rotation polynomials $Q_1, Q_2$. Each pair of these factorizations correspond to a parallelogram linkage that generates the circular translation. One example of a factorization of $C$ is

\[
Q_1 = t + E - \frac{bc}{a} \varepsilon + \frac{a^2 + c^2 - b^2}{2a} \varepsilon - b \varepsilon k,
\]

\[
Q_2 = t - E,
\]

\[
Q_3 = t - k + \frac{bc}{a} \varepsilon + \frac{a^2 + b^2 - c^2}{2a} \varepsilon k
\]

where

\[
E = \frac{2ac}{a^2 + b^2 + c^2} \varepsilon i + \frac{2ab}{a^2 + b^2 + c^2} \varepsilon j + \frac{a^2 - b^2 - c^2}{a^2 + b^2 + c^2} \varepsilon k
\]

In particular, we have shown
Proposition 3. The general Darboux motion parameterized by (6) can be decomposed into a rotation about an axis parallel to the Darboux motion’s axis direction and a circular translation in a plane orthogonal to the vector quaternion $D$ of (9).

Combining two of the infinitely many factorizations with three linear factors gives a parallelogram linkage to whose coupler a dangling link (corresponding to the right factor $Q_3$ which is the same for all factorizations) is attached. This 6R linkage has two trivial degrees of freedom and is not particularly interesting.

3.2 Second factorization FII

Now we construct a further factorization FII of the general Darboux motion. We do, however, alter the parameterization (6) and multiply it with the quadratic polynomial $P := t^2 + 1$. This changes the motion polynomial but not the parameterized motion and gives additional freedom to find factorizations. In contrast to FII, we have more than three linear factors.

We begin by setting the right factor to $Q_4 := t - k$ and then use polynomial division (Lemma 1) to compute

$$C_1 = t^2 + \epsilon \langle ai - bk \rangle t + 1 + cek,$$

such that $C = C_1Q_4$. The polynomial $PC_1$ can be written as the product of four linear motion polynomials. Again, there exist infinitely many factorizations. In order to keep the number of joints small, we choose one with identical middle factors, i.e., $PC_1 = Q_1Q_2Q_3Q_5$, where

$$Q_1 = t + i + \frac{a + c}{2} - \epsilon j - \frac{b}{2} ck,$$

$$Q_6 = t - i,$$

$$Q_5 = t + i + \frac{a - c}{2} - \epsilon j - \frac{b}{2} ck.$$

The multiplicity of the middle factor $Q_6$ allows us to make a 3R chain for the motion parameterized by $PC_1$. Together with $C = Q_1Q_2Q_3$ and $Q_4$, we can get a 7R linkage. It can be seen that the axes of $Q_1$, $Q_2$ are parallel, as are the axes of $Q_3$, $Q_4$, $Q_5$, $Q_6$, $Q_7$.

For a concrete example, a Gröbner basis computation reveals that the configuration set of this 7R linkage is one dimensional [17]. Note that its configuration curve consists of different components and only one corresponds to the Darboux motion parameterized by $C$. We also want to mention that the construction of the second factorization FII leaves many degrees of freedom. We just presented one factorization of this type.

3.3 Third factorization FIII

In order to find a third factorization FIII that is suitable for linkage construction, we proceed in a similar fashion to case FII, multiplying the motion polynomial $C$ of Equation (6) with $P = t^2 + 1$. But we do so prior to splitting off a right factor. The polynomial $PC$ has infinitely many factorizations into products of five linear motion polynomials, one of them being of the shape $CP = Q_1Q_2Q_3Q_4^2$.

We start by finding a suitable factor $Q_4$. Because $Q_4^2$ is a factor of $(CP)(CP) = P^2$, we have the necessary constraint $Q_4^2Q_4 = P$ but still have infinitely many choices. We take

$$Q_4 = t - k - xi\varepsilon - yj\varepsilon$$

with $x, y \in \mathbb{R}$. After division of $PC$ by $(Q_4)^2$ (Lemma 1), we are left with the cubic motion polynomial $C_2$ defined by

$$C_2 = t^3 + (k + \epsilon (ai - bk + 2ai + 2aj))t^2 + (1 + \epsilon (ai + c) + 2ai - 2aj + bj)t + k - c\varepsilon.$$

This new cubic motion polynomial $C_2$ parameterizes again a Darboux motion but the parameterization is not in the standard form (6). Nonetheless, there exist again infinitely many factorizations similar to (8). For example, we have $C_2 = Q_2Q_6Q_5$, where

$$Q_2 = t + k + \frac{(b^2 - ab - 2bcy - c^2)}{2(T^2 + 4x^2)} + \frac{(ab^2 - ac^2 + 2b^2y + 4bcx - 2c^2 + a + y)}{2(T^2 + 4x^2)},$$

$$Q_6 = \frac{2(ac - 2bx + cy)}{b^2 + c^2 + T^2 + 4x^2} + \frac{2(ab + 2by + 2cx)}{b^2 + c^2 + T^2 + 4x^2} + \frac{Sx + bcT + 4(ay + x^2 + y^2)}{T^2 + 4x^2} + \frac{(S + 4x^2)at + 4(ay(a(3y + a) + 2y^2) - abcx)}{2(aT^2 + 4x^2)} - bke,$$

and we abbreviated $S = a^2 - b^2 - c^2$ and $T = a + 2y$.

Because $C = Q_1Q_2Q_3$ and $PC = C_2Q_6^2 = Q_1Q_2Q_3Q_4^2$ parameterize the same motion, we can combine these two factorizations to form a 7R linkage where each rotation is defined by $Q_7$, $Q_6$, $Q_5$, $Q_4$, $Q_3$, $Q_2$, $Q_1$. It can be seen that the axes of $Q_1$, $Q_2$ are parallel, as are the axes of $Q_3$, $Q_1$, $Q_5$ and $Q_2$, $Q_6$.

As revealed by a Gröbner basis computation, the configuration space of a concrete numeric example of this 7R linkage is really a curve [17]. Thus, we have indeed constructed another 7R linkage whose coupler motion is a non-vertical Darboux motion. Note that the configuration curve contains several components, not all of them rational. One component corresponds to the rational curve parameterized by $C$.

In Figure 1, we present nine configurations (the first one and the last one are from same configuration) of this linkage in an orthographic projection parallel to $k$.
Figure 1: The first 7R linkage generates a non-vertical Darboux motion.
3.4 Fourth factorization FIV

Finally, we present an exceptional case of the 7R linkage obtained from the third factorization FIII. Among the infinitely many choices for \( a, b, c, x, y \) we choose the special case \( a = 1, b = 2, c = 0, x = 0, y = 0 \). We then have

\[
\begin{align*}
Q'_1 &= t + \frac{4}{5}j - \frac{3}{5}k - \frac{3}{2}je - 2ke, \\
Q'_2 &= t - \frac{4}{5}j + \frac{3}{5}k, \\
Q'_3 &= t - k + \frac{5}{2}je, \\
Q'_4 &= t - k, \\
Q'_5 &= t + k + \frac{5}{2}je, \\
Q'_6 &= t + \frac{4}{5}j - \frac{3}{5}k, \\
Q'_7 &= t - \frac{4}{5}j + \frac{3}{5}k - \frac{3}{2}je - 2ke.
\end{align*}
\]

We can combine this factorization to form a 7R linkage where each rotation is defined by \( Q'_1, Q'_2, Q'_3, Q'_4, Q'_5, Q'_6, Q'_7 \). It can be seen that the axes of \( Q'_1, Q'_2, Q'_3, Q'_6, Q'_7 \) are parallel, as are the axes of \( Q'_1, Q'_4, Q'_5 \). Since four neighboring axes are parallel, this linkage contains a planar 4-bar linkage. Furthermore, there are two pairs of parallel 3R subchains which give us two Sarrus linkages if we fix the remaining joint. Hence, this 7R linkage has at least two degrees of freedom.

A Gröbner basis computation confirms that the configuration space is indeed two-dimensional [17]. Moreover, a decomposition of the configuration variety shows that the linkage is not kinematotropic, which means that there are components of different dimension [18]: It consists of three irreducible 2-dimensional components. Two components are rational. We do not know whether the other two-dimensional component which contains the rational curve parameterized by \( C \) is rational or not.

In Figure 2, we present eleven configurations (the first one and the last one are the same) of this linkage in an orthographic projection parallel to the direction of \( \mathbf{k} \). Because the axes of \( Q'_1, Q'_2, Q'_6, \) and \( Q'_7 \) are parallel, the remaining axes are depicted as points throughout the motion. We can observe the parallelity of axes and possible singularities (intersections). The two configurations where the axes of \( Q'_5 \) coincide give us two rational configuration sets of dimension two: A planar 4R loop (the coupler motion is a circular translation) and a single rotation rational component. None of these two contains the Darboux motion.

4 Conclusions

Using factorization of motion polynomials in non-generic cases, we constructed several examples of new 7R Darboux linkages. The main idea is to find an irreducible real polynomial \( P \) and factor \( PC \) instead of \( C \). The obtained 7R linkages generate the general (non-vertical) Darboux motion and exhibit some interesting specialities like parallel axes or circular translations as relative motions between certain links. One can replace two R-joints by other R-joints which generate same circular translations.

Verifying the validity of the presented factorizations and corresponding linkages is trivial but, admittedly, we did not explain in detail how to actually find the factorizations. The factorization theory for non-generic motion polynomials, including theoretical results, algorithms and upper bounds for the number of factors, is currently being worked out. Its presentation is left to future publications.

Acknowledgements

This research was supported by the Austrian Science Fund (FWF): P 26607 (Algebraic Methods in Kinematics: Motion Factorisation and Bond Theory).

References

[1] Darboux G. Sur le déplacement d’une figure invariable. *Comptes rendus de l’Académie des sciences*, pp. 118–121, 1881.
[2] Blaschke W. Zur Kinematik. *Abh. Math. Sem. Univ. Hamburg*, 22(1):171–175, 1958.
[3] Bottema O. and Roth B. *Theoretical Kinematics*. Dover Publications, 1990.
[4] Lee C.C. and Hervé J.M. On the vertical Darboux motion. In J. Lenarcic and M.L. Husty, eds., *Latest Advances in Robot Kinematics*, pp. 99–106. Springer, 2012.
[5] Li Z., Schicho J., and Schröcker H.P. Spatial straight line linkages by factorization of motion polynomials. Submitted for publication, 2014.
[6] Hegedüs G., Schicho J., and Schröcker H.P. Factorization of rational curves in the Study quadric and revolute linkages. *Mech. Mach. Theory*, 69(1):142–152, 2013.
[7] Hegedüs G., Schicho J., and Schröcker H.P. Four-pose synthesis of angle-symmetric 6R linkages. *ASME J. Mechanisms Robotics*, 7(4), 2015. 1309.4959.
[8] Li Z. and Schicho J. Classification of angle-symmetric 6R linkages. *Mechanism and Machine Theory*, 70:372 – 379, 2013.
[9] Li Z. and Schicho J. Three types of parallel 6R linkages. In F. Thomas and A. Perez Gracia, eds., *Computational Kinematics*, volume 15 of *Mechanisms and
Figure 2: The second 7R linkage generates a non-vertical Darboux motion.
[10] Gallet M., Koutschan C., Li Z., Regensburger G., Schicho J., and Villamizar N. Planar linkages following a prescribed motion. Submitted for publication, 2015. arxiv.org/abs/1502.05623.

[11] Li Z., Schicho J., and Schröcker H.P. Factorization of motion polynomials, feb 2015. arxiv.org/abs/1502.07600.

[12] Li Z., Schicho J., and Schröcker H.P. Rational motions of minimal dual quaternion degree to a given trajectory. In preparation, 2015.

[13] McCarthy J.M. An Introduction to Theoretical Kinematics. MIT Press, Cambridge, Massachusetts, London, England, 1990.

[14] Selig J. Geometric Fundamentals of Robotics. Monographs in Computer Science. Springer, 2 edition, 2005.

[15] Husty M. and Schröcker H.P. Kinematics and algebraic geometry. In J.M. McCarthy, ed., 21st Century Kinematics. The 2012 NSF Workshop, pp. 85–123. Springer, 2012.

[16] Li Z., Rad T.D., Schicho J., and Schröcker H.P. Factorisation of rational motions: A survey with examples and applications. In: Proceedings of the 14th World Congress in Mechanism and Machine Science, Taipei, 2015.

[17] https://people.ricam.oeaw.ac.at/z.li/softwares/7Rdlinkages.html. https://people.ricam.oeaw.ac.at/z.li/softwares/7Rdlinkages.html

[18] Galletti C. and Fanghella P. Singleloop kinematotropic mechanisms. Mechanism and Machine Theory, 36:743–761, 2001.