PARAMETRIC SOLUTIONS TO THE REGULATOR-CONJUGATE MATRIX EQUATIONS

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ABSTRACT. The problem of solving regulator-conjugate matrix equations is considered in this paper. The regulator-conjugate matrix equations are a class of nonhomogeneous equations. Utilizing several complex matrix operations and the concepts of controllability-like and observability-like matrices, a special solution to this problem is constructed, which includes solving an ordinary algebraic matrix. Combined with our recent results on Sylvester-conjugate matrix equations, complete solutions to regulator-conjugate matrix equations can be obtained by superposition principle. The correctness and effectiveness are verified by a numerical example.

1. Introduction. Solving matrix equations such as the Lyapunov matrix equations, the Riccati matrix equations and the Sylvester matrix equations is always continued hot research topic in the past years. Actually, they are widely used in areas of science and engineering computation, such as control theory, transport theory, signal processing, neural network, stochastic filtering and statistics, and so on.

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At the same time, complex matrix equations have attracted much attention from mathematics fields. For instance, in [4], the necessary and sufficient condition of the existing of solution to complex matrix equations $X - AXB = C$ and $X - AXB = C$ are given by using the tool of real representation, and the problem is converted into solving real matrix equations. The complex matrix equation with the form $AX + BY = XF$ is investigated in [6], where explicit closed-form solutions are provided in terms of controllability and observability matrices with a significant degree of freedom. In addition, solutions to the matrix equation $XF - AX = C$ have also been deeply investigated ([9], [3], [2] and so on).

The so-called regulator-conjugate matrix equation is the complex matrix equation $AX - XF = BY + R$, where $A, B, F, R$ are known complex matrices with compatible dimensions and $X, Y$ are unknown matrices to be determined. When $R = 0$, it is obvious that this equation becomes Sylvester-conjugate equation mentioned above. In another aspect, if its coefficient matrices happen to be real ones, the regulator-conjugate matrix equation degenerates into regulation matrix equation, which is essential in output regulation problem of linear systems and eigenstructure assignment problem of linear second-order systems [13]. Furthermore, let $R = 0$, it becomes generalized Sylvester matrix equation, on which one can find a large amount of existing results, for example, [10], [7], [12] and references therein.

The work of this paper is an extension of the recent result in [8], where explicit parametric solutions to complex matrix equation $AX + BY = XF$ are given in a finite series form or in terms of a so-called controllability-like matrix and an observability-like matrix. In this paper, we dedicate to giving the solution to the regulator-conjugate matrix equation, i.e., nonhomogeneous Sylvester-conjugate matrix equation. The remaining of this paper is organized as the follows. Preliminaries are given in the next section. Section 3 presents the main results of this paper. An illustrative numerical example is provided in Section 4 to verify the effectiveness of the proposed method. Finally, we conclude this paper in Section 5.

Throughout this paper, for an arbitrary real $a$ we use $[a]$ to denote the integer part, that is, $a = [a] + p$ with $0 \leq p < 1$. We use $\lambda(A)$ to denote the set of eigenvalues of $A$.

2. Preliminaries.

2.1. A real representation of a complex matrix. Firstly, we define the following two matrices:

$$P_j = \begin{bmatrix} I_j & 0 \\ 0 & -I_j \end{bmatrix}, \quad Q_j = \begin{bmatrix} 0 & I_j \\ -I_j & 0 \end{bmatrix},$$

where $I_j$ is the $j \times j$ identity matrix. Let $A \in \mathbb{C}^{m \times n}$, then $A$ can be uniquely written as $A = A_1 + A_2i$, $A_1, A_2 \in \mathbb{R}^{m \times n}$, $i = \sqrt{-1}$. Real representation $\sigma$ is defined as

$$A_\sigma = \begin{bmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{bmatrix} \in \mathbb{R}^{2m \times 2n},$$

and $A_\sigma$ is called as the real representation matrix of matrix $A$. It should be noticed that the real representation [1] is different from the natural real representation

$$A \rightarrow \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix}.$$
The real representation \([1]\) is firstly proposed in \([4]\) for solving the complex matrix equation \(X - AXB = C\), and the following properties are pointed:

**Lemma 2.1.** *(The properties of the real representation)*

1. If \(A, B \in \mathbb{C}^{m \times n}, a \in \mathbb{R}\), then
   \[
   \begin{align*}
   (A + B)_{\sigma} &= A_{\sigma} + B_{\sigma}, \\
   (aA)_{\sigma} &= aA_{\sigma}, \\
   P_m A_{\sigma} P_n &= \overline{A}_{\sigma}.
   \end{align*}
   \]

2. If \(A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{n \times r}\), then
   \[
   (AB)_{\sigma} = A_{\sigma} P_n B_{\sigma} = A_{\sigma} (\overline{B})_\sigma P_r = P_m \overline{A}_{\sigma} B_{\sigma}.
   \]

3. If \(A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{n \times r}, C \in \mathbb{C}^{r \times q}\), then
   \[
   (ABC)_{\sigma} = A_{\sigma} B_{\sigma} C_{\sigma}.
   \]

4. If \(A \in \mathbb{C}^{n \times n}\), then \(A\) is nonsingular if and only if \(A_{\sigma}\) is nonsingular.

5. If \(A \in \mathbb{C}^{n \times n}\), then
   \[
   A^{2k} = (AA)^k_{\sigma} P_n = P_m \overline{A}_{\sigma} B_{\sigma}.
   \]

6. If \(A \in \mathbb{C}^{m \times n}\), then
   \[
   Q_m A_{\sigma} Q_n = A_{\sigma}.
   \]

2.2. **Operations on conjugate matrices.** For a complex matrix \(C\) and a positive integer number \(k\), we define \(C^{\ast k} = \overline{C}^{\ast (k-1)}\) with \(C^{\ast 0} = C\). With this definition, it is obvious that
   \[
   C^{\ast k} = \begin{cases} 
   C, & \text{for even } k \\
   \overline{C}, & \text{for odd } k
   \end{cases}
   \]

On such an operation, for arbitrary two positive integers \(k\) and \(l\), one can obtain
   \[
   (C^{\ast k})^{\ast l} = C^{\ast (k+l)}.
   \]

In the following, we introduce two operations on complex matrices.

**Definition 2.2.** \([8]\) For \(A \in \mathbb{C}^{n \times n}\), and \(k \in \mathbb{Z}\), the operation \(A^{\leftarrow k}\) and \(A^{\rightarrow k}\) are respectively defined as
   \[
   A^{\rightarrow k} = (A\overline{A})^{\left\lfloor \frac{k}{2} \right\rfloor} A^{k-2 \left\lfloor \frac{k}{2} \right\rfloor}, \\
   A^{\leftarrow k} = A^{k-2 \left\lfloor \frac{k}{2} \right\rfloor} (A\overline{A})^{\left\lfloor \frac{k}{2} \right\rfloor}.
   \]

According to this definition, it is obvious that
   \[
   \begin{align*}
   A^{\rightarrow 0} &= A; \\
   A^{\rightarrow 1} &= A; \\
   A^{\rightarrow 2} &= A\overline{A}, A^{\leftarrow 2} = \overline{A}\overline{A}; \\
   A^{\rightarrow 3} &= A^{\left\lfloor \frac{3}{2} \right\rfloor}, A^{\leftarrow 3} = A^{\left\lfloor \frac{3}{2} \right\rfloor}; \\
   A^{-1} &= \overline{A}^{-1}.
   \end{align*}
   \]

On the above defined two operations, there are many interesting properties from \([8]\).

**Lemma 2.3.** If \(A \in \mathbb{C}^{n \times n}\) is invertible, then the following relations hold for \(k \in \mathbb{Z}\)
   \[
   A^{\rightarrow -k} = A^{-1} \overline{A}^{\left\lfloor \frac{k}{2} \right\rfloor}, A^{\leftarrow -k} = A^{-1} \overline{A}^{\left\lfloor \frac{k}{2} \right\rfloor}.
   \]
Lemma 2.4. For $A \in \mathbb{C}^{n \times n}$, $k, l \in \mathbb{Z}$, the following relations hold:

1. $\overrightarrow{A}^k = \overrightarrow{A}^k; \overrightarrow{A}^k = \overrightarrow{A}^k$.
2. $\overrightarrow{A}^0 = I; \overrightarrow{A}^{2l+1} = A(\overrightarrow{A})^l; \overrightarrow{A}^{2l} = (A\overrightarrow{A})^l; \overrightarrow{A}^{2l} = (\overrightarrow{A}A)^l$.
3. For odd $k$, $A^k = \overrightarrow{A}^k$; for even $k$, $A^k = \overrightarrow{A}^k$.
4. $\overrightarrow{A}^{k} = A_{k+1}; \overrightarrow{A}^{k} = A_{k+1};$
5. $(\overrightarrow{A}^k)^* \overrightarrow{A}^l = A^{k-l}; \overrightarrow{A}^l (\overrightarrow{A}^k)^* = A^{k-l};$
6. $(\overrightarrow{A}^{2l+1})^k = A^{k(2l+1)}; (\overrightarrow{A}^{2l+1})^k = A^{k(2l+1)}$.

In the above items, when $k$ or $l$ is negative, the matrix $A$ is required to be invertible.

By using Lemma 2.4 for an invertible square complex matrix $A$ and an integer $l$, one can obtain the following interesting results:

$$(\overrightarrow{A}^l)^* \overrightarrow{A}^{-l} = I; \overrightarrow{A}^{-l} (\overrightarrow{A}^l)^* = I.$$ 

At the end of this section, we provide a lemma on real representations of the operations in Definition 2.2. On its proof, one can refer [8].

Lemma 2.5. Given $M \in \mathbb{C}^{m \times n}$, $Z \in \mathbb{C}^{n \times p}$, $F \in \mathbb{C}^{p \times p}$, and $P_m$ is given by (2.1), then for integer $k \geq 0$ the following relation holds:

$$(MZ^*k F^{\overrightarrow{k}})_{\sigma} = P_{m+1} (M^*k)_{\sigma} Z_{\sigma} F^T_{\sigma}.$$  (3)

2.3. Other notations. For matrices $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times r}$, $C \in \mathbb{C}^{p \times n}$, we have the following notations associated with these matrices:

$\overrightarrow{\text{Ctr}}_t (A, B) = \begin{bmatrix} B & AB & \cdots & A^{t-1} & B^{t-1} \end{bmatrix},$

$\overrightarrow{\text{Ctr}} (A, B) = \overrightarrow{\text{Ctr}}_n (A, B),$

$\overrightarrow{\text{Obs}}_t (A, C) = \begin{bmatrix} C \\
CA \\
\cdots \\
C^{*t-1}A^{t-1} \end{bmatrix},$

$\overrightarrow{\text{Obs}} (A, C) = \overrightarrow{\text{Obs}}_n (A, C).$

Obviously, if $A$, $B$, and $C$ are all real, then matrices $\overrightarrow{\text{Ctr}} (A, B)$ and $\overrightarrow{\text{Obs}} (A, C)$ become the well-known controllability and observability matrices, respectively. Due to such a reason, for the sake of convenience the matrices $\overrightarrow{\text{Ctr}} (A, B)$ and $\overrightarrow{\text{Obs}} (A, C)$ will be referred to as controllability-like matrix of $(A, B)$ with index $t$, and observability-like matrix of $(A, C)$ with index $t$, respectively; $\text{Ctr} (A, B)$ and $\text{Obs} (A, C)$ will be called as controllability-like matrix of $(A, B)$ and observability matrix of $(A, C)$, respectively. In addition, for a matrix set $\mathcal{D} = \{ D_t \in \mathbb{C}^{r \times r}, t = 0, 1, \cdots, \varphi \}$, denote

$$S_t (\mathcal{D}) = \begin{bmatrix}
D_1 & D_2 & D_3 & D_4 & \cdots & D_t \\
D_2 & D_3 & D_4 & \cdots & D_t \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
D_{t-1} & \cdots & \cdots & \cdots & D_t \\
D_t^{*t} & D_t^{*t-1} & \cdots & \cdots & \cdots 
\end{bmatrix}$$  (4)
and define matrix function on complex variable \( s \)

\[
D(s) = \sum_{i=0}^{t} P_{r}^{i+1} (D_{r}^{*})_{i} s^{i}.
\]  

(5)

3. Main results. In this section, we consider the regulator-conjugate matrix equation

\[
AX + BY = XF + R
\]

(6)

where \( A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times r}, F \in \mathbb{C}^{p \times p} \) and \( R \in \mathbb{C}^{n \times p} \) are given matrices, \( X \in \mathbb{C}^{n \times p}, Y \in \mathbb{C}^{r \times p} \) are unknown matrices to be determined.

When \( R = 0 \), this equation degenerates into the following form

\[
AX + BY = XF
\]

(7)

This equation is well known as Sylvester-conjugate matrix equation and has been widely investigated. There are many research results in the existing literatures. Here, we cite a previous research conclusion on this equation.

On the solution to the Sylvester-conjugate matrix equation (7), we have the following result ([8]).

**Lemma 3.1.** Given matrices \( A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times r} \) and \( F \in \mathbb{C}^{p \times p} \), suppose that \( \lambda(AA) \cap \lambda(FF) = \phi \). If there are a matrix set \( D = \{ D_{i} \in \mathbb{C}^{r \times r}, i = 0, 1, \cdots, t \} \) satisfying

\[
BD_{0} + ABD_{1} + AABD_{2} + \cdots + A^{\top} (BD_{t})^{*} t = 0
\]

(8)

then the matrices \( X \) and \( Y \) given by

\[
\begin{aligned}
X &= \text{Cir}_{t}(A,B)S_{t}(D)\overline{\text{Obs}_{t+1}(F,Z)} \\
Y &= [D_{0} \quad D_{1} \quad \cdots \quad D_{t}] \overline{\text{Obs}_{t+1}(F,Z)}
\end{aligned}
\]

(9)

with \( Z \in \mathbb{C}^{r \times p} \) being an arbitrarily chosen free parameter matrix, satisfy the matrix equation (7). Further, define \( D(s) \) as in (5). Then all the solutions to the Sylvester-conjugate matrix equation (7) can be parameterized by (9) if

\[
\det D(s) \neq 0, \text{ for any } s \in \lambda(F_{r})
\]

(10)

In the follows, we will present a proposition which is obvious and its proof is omitted.

**Proposition 1.** If \((X^{s},Y^{s})\) is a special solution to the regulator-conjugate matrix equation (6) and \((X^{*},Y^{*})\) is the general solution to the Sylvester-conjugate matrix equation (7), then the general solution to the regulator-conjugate matrix equation (6) is given by

\[
X = X^{s} + X^{*}, Y = Y^{s} + Y^{*}
\]

The above result can be called as superposition principle in linear regulator-conjugate matrix equation. According to this principle, to solve regulator-conjugate equation (6), one need solve the corresponding homogeneous equation and obtain a special solution of the equation itself. Up to this point, the only thing we should do is to find a special solution to matrix equation (6). The following theorem gives us a feasible approach.
Theorem 3.2. Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times r}$, $F \in \mathbb{C}^{p\times p}$ and $R \in \mathbb{C}^{n\times p}$ be known. If there exist scalars $t$ and $q$, a matrix set $\mathcal{D} = \{D_i^t \in \mathbb{C}^{r\times q}, i = 0, 1, \cdots, t\}$ and two matrices $V \in \mathbb{C}^{n\times n}, W \in \mathbb{C}^{p\times p}$ such that
\[
BD_0^t + ABD_1^t + A^2BD_2^t + \cdots + A^t(BD_t^t)^{s} = V \tag{11}
\]
and
\[
VW = R \tag{12}
\]
then a special solution to the regulator matrix equation (6) can be given by
\[
\begin{align*}
X^* &= \text{Crt}_t(A, B)S_t(\mathcal{D}) \text{Obs}_t(F, W) \\
Y^* &= \begin{bmatrix} D_0^t & D_1^t & \cdots & D_t^t \end{bmatrix} \text{Obs}_{t+1}(F, W) 
\end{align*} \tag{13}
\]

Proof. By virtue of (11), directly computation gives
\[
AX^* = A\text{Crt}_t(A, B)S_t(\mathcal{D}) \text{Obs}_t(F, W) \\
= \begin{bmatrix} AB & A^2B & \cdots & A^{t-1}B^{t-1} & A^tB^t \end{bmatrix} S_t(\mathcal{D}) \text{Obs}_t(F, W) \\
= \begin{bmatrix} \sum_{i=1}^{t} A^i(BD_i^t)^{s} & \cdots & \sum_{i=1}^{t-1} A^i(BD_i^t)^{s} & ABD_t^t \end{bmatrix} \text{Obs}_t(F, W) + VW \\
= \begin{bmatrix} -BD_0^t & \sum_{i=1}^{t-1} A^i(BD_i^t)^{s} & \cdots & ABD_t^t \end{bmatrix} \text{Obs}_{t+1}(F, W) + VW
\]

Similarly, we have
\[
\begin{align*}
\overline{X^*}F &= \text{Crt}_t(A, B)S_t(\mathcal{D}) \text{Obs}_t(F, W)F \\
&= \text{Crt}_t(A, B) \begin{bmatrix} 0 & S_t(\mathcal{D}) \end{bmatrix} \text{Obs}_{t+1}(F, W) \\
&= \begin{bmatrix} \sum_{i=0}^{t-1} A^i(BD_i^t)^{s} & \cdots & BD_t^t \end{bmatrix} \text{Obs}_{t+1}(F, W)
\end{align*}
\]

Therefore, using (12), we have
\[
AX^* - \overline{X^*}F = \begin{bmatrix} -BD_0^t & BD_1^t & \cdots & BD_t^t \end{bmatrix} \text{Obs}_{t+1}(F, W) + VW \\
= -B \begin{bmatrix} D_0^t & D_1^t & \cdots & D_t^t \end{bmatrix} \text{Obs}_{t+1}(F, W) + R \\
= -BY^* + R
\]

This shows that the matrix pair $(X^*, Y^*)$ given in (13) satisfies the regulator equation (6). \hfill \Box

Remark 1. Here, we give a simple introduction on how to solve equations (11) and (12). Without loss of generality, we can firstly give a matrix $W$, and solve equation (12) to obtain matrix $V$. Next, by substituting the obtained matrix $V$ into (11), equation (11) becomes a common linear algebraic equation in the augmented form of
\[
\begin{bmatrix} B & AB & \cdots & A^tB^t \end{bmatrix} \begin{bmatrix} (D_0^t)^T & (D_1^t)^T & \cdots & (D_t^t)^T \end{bmatrix}^T = V
\]

which can be easily solved by any method provided in linear algebra textbook.

Based on the above results, we can summarize the complete solutions to the regulator-conjugate matrix equation in the following.
Theorem 3.3. Given matrices $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times r}$, $F \in \mathbb{C}^{p \times p}$, and $R \in \mathbb{C}^{n \times p}$, suppose that $\lambda(A^T A) \cap \lambda(F F^T) = \emptyset$. If there exist two scalars $t$ and $q$, a matrix set $D^p = \{D_i \in \mathbb{C}^{r \times q}, i = 0, 1, \ldots, t\}$ and two matrices $V \in \mathbb{C}^{n \times q}$, $W \in \mathbb{C}^{q \times p}$ satisfying (11) and (12), matrix set $D = \{D_i \in \mathbb{C}^{r \times r}, i = 0, 1, \ldots, t\}$ satisfying (8), then the matrices $X$ and $Y$ given by

$$
X = \text{Ctri}(A, B) \left( S_t(D) \text{Obs}_t(F, Z) + S_t(D^p) \text{Obs}_t(F, W) \right)
$$

$$
Y = \left[ \begin{array}{cccc}
D_0 & D_1 & \cdots & D_t \\
\text{Obs}_{t+1}(F, Z) & \text{Obs}_{t+1}(F, W)
\end{array} \right]
$$

(14)

with $Z \in \mathbb{C}^{r \times p}$ being an arbitrarily chosen free parameter matrix, satisfy the matrix equation (6). Further, define $D(s)$ as in (5). Then all the solutions to the regulator-conjugate matrix equation (6) can be parameterized by (14) if

$$
\det D(s) \neq 0, \text{ for any } s \in \lambda(F^\sigma).
$$

For ease of use, we will present a detailed computing process for solving the regulator-conjugate matrix equation.

Algorithm 1. (Parametric solving algorithm of regulator-conjugate matrix equation)

1. Judge whether conditions $\lambda(A^T A) \cap \lambda(F F^T) = \emptyset$ and (15) are satisfied. If so, go to the next step; otherwise, exit.
2. Solve linear complex matrix equations (11) and (12) to obtain a group of matrices $D_i, i = 0, 1, 2, \ldots, t$ and two matrices $V \in \mathbb{C}^{n \times q}$, $W \in \mathbb{C}^{q \times p}$.
3. According to (13), compute a special solution $X^s$ and $Y^s$.
4. Solve linear complex matrix equation (8) to obtain a group of matrices $D_i, i = 0, 1, 2, \ldots, t$.
5. According to (9), compute the parametric general solutions to (7) as $X^* = X^s + X^*$ and $Y^* = Y^s + Y^*$.
6. The general solutions to (6) are given by $X = X^s + X^*$, $Y = Y^s + Y^*$.

4. An illustrative example.

Example 1. Consider regulator-conjugate matrix equation (6) with the following parameters

$$
A = \begin{bmatrix}
1 & -2 - i & -1 + i \\
0 & i & 0 \\
0 & -1 & 1 - i
\end{bmatrix}, \quad F = \begin{bmatrix}
2i & i \\
1 & -1 + i
\end{bmatrix},
$$

$$
B = \begin{bmatrix}
-1 + i & 1 \\
0 & i \\
-1 & 1 - 2i
\end{bmatrix}, \quad R = \begin{bmatrix}
1 & i \\
i & 1 \\
0 & 1 - i
\end{bmatrix}
$$

The corresponding homogeneous equation of the considered equation has been dealt with in [8].

For this matrix equation, $n = 3$. We choose $t = 2$. A direct calculation gives

$$
\begin{bmatrix}
B & A^T B & AAB
\end{bmatrix}
= \begin{bmatrix}
-1 + i & 1 & -2 - 2i & -3 + i & -2 + 4i & -6 + 3i \\
0 & i & 0 & 1 & 0 & i \\
-i & 1 - 2i & 1 + i & 3 + 2i & -2i & -5i
\end{bmatrix}.
$$
Let 

\[ W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

According to (12), we have 

\[ V = R = \begin{bmatrix} 1 & i \\ i & 1 \\ 0 & 1 - i \end{bmatrix} \]

By this, solving the matrix equation (11) generates 

\[ D_{s0} = \begin{bmatrix} -2 - 2i & 1 + 6i \\ 1 & 0 \end{bmatrix}, \quad D_{s1} = \begin{bmatrix} \frac{3}{2} + \frac{1}{2}i & -2 - 2i \\ 0 & 0 \end{bmatrix}, \quad D_{s2} = \begin{bmatrix} 1 & -2 \\ 0 & -i \end{bmatrix}. \]

Furthermore, we can compute that 

\[ \overrightarrow{\text{Ctr}}_2(A, B) = \begin{bmatrix} B & AB \end{bmatrix}, \quad S_2(D^*) = \begin{bmatrix} D_1 & D_2 \\ D_2 & 0 \end{bmatrix}. \]

\[ \overleftarrow{\text{Obs}}_2(F, W) = \begin{bmatrix} W \\ WF \end{bmatrix}, \quad \overleftarrow{\text{Obs}}_3(F, W) = \begin{bmatrix} W \\ WF \\ WFF \end{bmatrix}. \]

According to (13), a special solution to regulator-conjugate equation can be given by 

\[ X^s = \overrightarrow{\text{Ctr}}_2(A, B) S_2(D^*) \overleftarrow{\text{Obs}}_2(F, W) \]

\[ = \begin{bmatrix} 1 + i & -1 + i \\ 1 & -1 \\ -3.5 - 0.5i & -4i \end{bmatrix} \]

\[ Y^s = \begin{bmatrix} D_0 & D_1 & D_2 \end{bmatrix} \overleftarrow{\text{Obs}}_3(F, W) \]

\[ = \begin{bmatrix} 3 & 0.5 + 2.5i \\ 2 + i & 1 - 2i \end{bmatrix} \]

It is easily verified that this special solution meets the considered matrix equation very well. Combined with the corresponding homogeneous solutions given in [8], the complete parametric solutions to the regulator-conjugate matrix equation can be obtained.

5. Conclusions. The problem of solving regulator-conjugate matrix equation is studied in this paper. The work in this paper is an extension of the previous discussion on Sylvester-conjugate matrix equation in [8], where all the solutions to the Sylvester-conjugate matrix equation are provided in the parametric and explicit form. Utilizing some algebraic technique, a special solution to regulator-conjugate matrix equation is found. The main computation includes controllability-like and observability-like matrices and solving an ordinary linear algebraic equation. By superposition principle, complete parametric solutions to the considered matrix equation can be obtained. Finally, a numerical example is presented to verify the effectiveness of the proposed approach.
REFERENCES

[1] P. Benner, J. R. Li and T. Penzl, Numerical solution of large scale Lyapunov equations, Riccati equations, and linear quadratic optimal control problems. *Numerical Linear Algebra with Applications*, 15 (2008), 755–777.

[2] J. Bevis, F. Hall and R. Hartwig, The matrix equation $AX - XB = C$ and its special cases, *SIAM Journal on Matrix Analysis and Applications*, 60 (2010), 95–111.

[3] Y. Hong and R. Horn, A canonical form for matrices under consimilarity. *Linear Algebra and its Applications*, 102 (1988), 143–168.

[4] T. Jiang and M. Wei, On solutions of the matrix equations $X - AXB = C$ and $X - AXB = C$. *Linear Algebra and its Applications*, 367 (2003), 225–233.

[5] X. Jiang and Y. Zhang, A smoothing-type algorithm for absolute value equations. *Journal of Industrial & Management Optimization*, 9 (2013), 789–798.

[6] A. Wu, G. Feng, G. Duan and W. Wu. Closed-form solutions to Sylvester-conjugate matrix equations. *Computers & Mathematics with Applications*, 60 (2010), 95–111.

[7] A. Wu and G. Duan, Solution to the generalised Sylvester matrix equation $AV + BW = EVF$. *IET Control Theory & Applications*, 1 (2007), 402–408.

[8] A. Wu, L. Lv, G. Duan and W. Liu, Parametric solutions to Sylvester-conjugate matrix equations. *Computers & Mathematics with Applications*, 62 (2011), 3317–3325.

[9] A. Wu, G. Duan and H. Yu, On solutions of the matrix equations $XF - AX = C$ and $XF - AX = C$. *Applied Mathematics and Computation*, 183 (2006), 932–941.

[10] C. Yang, J. Liu and Y. Liu. Solutions of the generalized Sylvester matrix equation and the application in eigenstructure assignment. *Asian Journal of Control*, 14 (2012), 1669–1675.

[11] K. F. C. Yiu, K. L. Mak and K. L. Teo, Airfoil design via optimal control theory. *Journal of Industrial & Management Optimization*, 1 (2005), 133–148.

[12] B. Zhou and G. Duan, A new solution to the generalized Sylvester matrix equation $AV-EVF= BW$. *Systems & Control Letters*, 55 (2006), 193–198.

[13] B. Zhou, G. Duan and Z. Li, A Stein matrix equation approach for computing coprime matrix fraction description. *IET Control Theory & Applications*, 3 (2009), 691–700.

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