Real Lax spectrum implies spectral stability

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Abstract
We consider the dynamical stability of periodic and quasiperiodic stationary solutions of integrable equations with $2 \times 2$ Lax pairs. We construct the eigenfunctions and hence the Floquet discriminant for such Lax pairs. The boundedness of the eigenfunctions determines the Lax spectrum. We use the squared eigenfunction connection between the Lax spectrum and the stability spectrum to show that the subset of the real line that gives rise to stable eigenvalues is contained in the Lax spectrum. For non-self-adjoint members of the AKNS hierarchy admitting a common reduction, the real line is always part of the Lax spectrum and it maps to stable eigenvalues of the stability problem. We demonstrate our methods work for a variety of examples, both in and not in the AKNS hierarchy.

KEYWORDS
dynamical systems, solitons and integrable systems

1 | INTRODUCTION

A surprisingly large number of equations of physical significance are integrable and possess a Lax pair. An important feature of equations with a Lax pair is the Lax spectrum: the set of all Lax parameter values for which the solution of the Lax pair is bounded. For our purposes, this is important for determining the stability of solutions of a given integrable equation. Until recently, the Lax spectrum has only been determined explicitly for decaying potentials on the whole line or for self-adjoint problems with periodic coefficients. For such problems, the Floquet discriminant is a useful tool for numerically computing and giving a qualitative description of the Lax spectrum, but it is not used generally to get an explicit description of the Lax spectrum.
spectrum. A full description of the Lax spectrum can allow one to prove the stability of solutions to integrable equations with respect to certain classes of perturbations.

In Section 2, we define the type of stability we study for general nonlinear evolution equations. In Section 3, we restrict to a special class of evolution equations, those in the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy \cite{AKNS74}. We construct a function defined by an integral, whose zero level set determines the Lax spectrum for members of the AKNS hierarchy. Using this, we show that the Lax spectrum for members of the AKNS hierarchy admitting a common reduction contains a subset of the real line that maps to stable elements of the stability problem under the squared-eigenfunction connection. For non-self-adjoint members of the AKNS hierarchy with this reduction, this subset is the whole real line. In Section 4, we first show how these results apply to some example equations in the AKNS hierarchy. We finish Section 4 by showing how the results in Section 3 also apply to an equation that is not in the AKNS hierarchy. This leads to Section 5 in which we generalize the results of Section 3 to integrable equations possessing a $2 \times 2$ Lax pair that are not in the AKNS hierarchy. In Section 6, we apply the results of Section 5 to a number of examples. At the end of Section 6, we demonstrate how similar results are obtained for equations that do not fit into the framework of Section 5, suggesting that the work here may be more general than what we present here. For the interested reader, we construct the Floquet discriminant for these problems in the Appendix.

2 STABILITY SETUP

We consider the nonlinear evolution equation

$$u_t = \mathcal{N}(u, u_x, ..., u_{N_x}),$$

(1)

where $u(x, t)$ is a real- or complex-valued function (possibly vector valued) and $\mathcal{N}$ is a nonlinear function of $u$ and $N$ of its spatial derivatives. We assume that (1) is written in a frame in which there exists a nontrivial stationary solution, $u(x, t) = \bar{u}(x)$. In order to study the stability of the solution $\bar{u}$, we linearize about it by letting $u(x, t) = \bar{u}(x) + \epsilon v(x, t) + \mathcal{O}(\epsilon^2)$. Truncating at $\mathcal{O}(\epsilon)$ yields

$$v_t = \mathcal{L}(\bar{u}, \bar{u}_x, ..., \bar{u}_{N_x})v,$$

(2)

where $\mathcal{L}$ is a linear function. Since $\mathcal{L}(\bar{u}, \bar{u}_x, ..., \bar{u}_{N_x})$ is independent of $t$, we may separate variables with

$$v(x, t) = \tilde{v}(x)e^{\lambda t},$$

(3)

yielding the spectral problem for $\lambda$:

$$\lambda \tilde{v} = \mathcal{L}(\bar{u}, \bar{u}_x, ..., \bar{u}_{N_x})\tilde{v}.$$  

(4)

Definition 1. The stability spectrum is the set

$$\sigma_{\mathcal{L}} = \{\lambda \in \mathbb{C} : \tilde{v} \in S_{\mathcal{L}}\},$$

(5)
where \( S_L \) is a function space that depends on the perturbations of interest. In essence, we choose \( S_L \) to be the least restrictive space that works in our framework.

We are interested in Equations (1) that are Hamiltonian. The stability spectrum for such equations possesses a quadrafold symmetry: if \( \lambda \in \sigma_L \), then \( -\lambda, \lambda^*, -\lambda^* \in \sigma_L \) (here and throughout * represents the complex conjugate). There are many different types of stability. In this paper, whenever we refer to stability we are using the following definition of spectral stability.

**Definition 2.** A solution \( u(x, t) = \bar{u}(x) \) of (1) is stable if and only the stability spectrum of the corresponding operator \( \mathcal{L} = \mathcal{L}(\bar{u}, \bar{u}_x, ..., \bar{u}_{Nx}) \) is a subset of the imaginary axis, \( \sigma_L \subset i\mathbb{R} \).

Determining the stability spectrum is usually not a straightforward task. As we focus on integrable equations, significant progress can be made. We start with a special class of integrable equations known to be in the AKNS hierarchy.

### 3 | THE AKNS HIERARCHY  

#### 3.1 | The Lax pair, the Lax spectrum, and the squared-eigenfunction connection

We define an integrable equation to be an evolution equation of the form (1) that possesses a Lax pair. A Lax pair is a pair of ordinary differential equations (ODEs) of the form \( \phi_x = X\phi, \phi_t = T\phi \) where \( X \) and \( T \) are operators acting on a function \( \phi \). The integrable equation (1) is obtained by requiring that \( \partial_t \phi_x = \partial_x \phi_t \) holds.

The AKNS hierarchy is a special class of integrable equations containing a variety of physically important nonlinear evolution equations. The Lax pair for members of the AKNS hierarchy is

\[
\phi_x(x, t; \zeta) = \begin{pmatrix} -i\zeta & q(x, t) \\ r(x, t) & i\zeta \end{pmatrix} \phi(x, t; \zeta) = X\phi, \tag{6a}
\]

\[
\phi_t(x, t; \zeta) = \begin{pmatrix} A(x, t; \zeta) & B(x, t; \zeta) \\ C(x, t; \zeta) & -A(x, t; \zeta) \end{pmatrix} \phi(x, t; \zeta) = T\phi. \tag{6b}
\]

Here \( \zeta \in \mathbb{C} \) is called the Lax parameter, assumed to be independent of \( x \) and \( t \), and \( r, q, A, B, \) and \( C \) are complex-valued functions chosen such that the compatibility of mixed derivatives, \( \partial_t \phi_x = \partial_x \phi_t \) holds if and only if (1) holds. The compatibility condition defines the evolution equations

\[
q_t = B_x + 2i\zeta B + 2Aq, \tag{7a}
\]

\[
r_t = C_x - 2i\zeta C - 2Ar, \tag{7b}
\]

as well as the condition

\[
A_x = QC - RB. \tag{8}
\]
We are interested in studying the stability of stationary and periodic or quasiperiodic solutions of members of the AKNS hierarchy with

\[ r = \kappa q^*, \quad (9) \]

where \( \kappa = \pm 1 \). We make the following assumption.

**Assumption 1.** The functions \( q \) and \( r \) are related by (9), are \( t \)-independent, and have the form

\[ q(x) = e^{i\theta(x)}Q(x), \quad \text{and} \quad r(x) = \kappa e^{-i\theta(x)}Q(x), \quad (10) \]

where \( Q \) and \( \theta \) are real-valued functions and \( Q \geq 0 \) is \( P \)-periodic.

With this assumption, (7) gives

\[ B_x = -2i\zeta B - 2Aq, \quad (11a) \]
\[ C_x = 2i\zeta C + 2Ar. \quad (11b) \]

**Remark 1.** Many evolution equations are found by assuming \( A, B, \) and \( C \) can be written as a \( \zeta \)-power series. In such cases, a recursion operator is used to find \( A, B, \) and \( C \) (Ref. 18, chapter 2). Using the recursion operator, \( A \) is a function of products of \( q_{nx}^j r_{nx}^j \), where \( n, j \in \mathbb{N} \). Similarly, \( B \) is a function of products of the form \( q_{nx}^j r_{mx}^j \) and \( C \) is a function of products of the form \( q_{nx}^j r_{mx}^{j+1} \). Therefore, when Assumption 1 holds, \( A, B, \) and \( C \) are \( t \)-independent, bounded for \( x \in \mathbb{R} \), including \( \infty \), and \( A \) is \( P \)-periodic. In what follows we assume each of these statements are true.

**Definition 3.** The Lax spectrum is the set

\[ \sigma_L = \{ \zeta \in \mathbb{C} : \phi \in S_L \}, \quad (12) \]

where \( S_L \) is a function space that depends on the perturbations of interest. It turns out that \( S_L \) is related to the space \( S_L^\perp \).

Of course, \( \sigma_L \) depends on which norm is used. Although finding the Lax spectrum is an interesting and important problem in its own right, we are primarily interested in using the Lax spectrum to determine the stability of stationary solutions to integrable equations. The connection between the Lax spectrum and stability is through the so-called squared-eigenfunction connection. The squared-eigenfunction connection (sometimes referred to as quadratic eigenfunctions) gives a connection involving quadratic combinations of the eigenfunctions of (6) between eigenfunctions of the stability problem (4) and eigenfunctions of the Lax problem (6). In order to connect the Lax spectrum with the stability spectrum, we make the following observation (originally used in Ref.). When \( A, B, \) and \( C \) are \( t \)-independent (see Remark 1), (6b) may be solved by separation of variables. Equating

\[ \phi(x, t) = e^{\Omega t} \varphi(x), \quad (13) \]
where $\Omega$ is complex valued and $\varphi(x)$ is a complex vector-valued function, (6b) becomes a $2 \times 2$ eigenvalue equation for $\Omega$:

$$\Omega \varphi = T \varphi.$$  \hfill (14)

Using the expression (6b) for $T$,

$$\Omega^2 = A^2 + BC.$$  \hfill (15)

The following lemma establishes that $\Omega$ is not a function of $x$, in addition to the obvious fact that it is not a function of $t$.

**Lemma 1.** Under Assumption 1, $\Omega$ is independent of $x$ and $t$.

**Proof.** Independence of $t$ is immediate since $A$, $B$, and $C$ are independent of $t$ for stationary solutions (Remark 1). Multiplying (11a) by $C$ and (11b) by $B$ and adding the resulting equations yields

$$0 = \partial_x (BC) + 2A(qC - rB).$$  \hfill (16)

Using (8),

$$0 = \partial_x (BC + A^2) = \partial_x (\Omega^2),$$  \hfill (17)

so $\Omega^2$ is independent of $x$. \hfill $\blacksquare$

Thus $\Omega = \Omega(\zeta)$ is a function only of $\zeta$ and the solution parameters. The connection between the Lax spectrum and the stability spectrum is through $\Omega$. Comparing the exponential component of quadratic combinations of the two components of $\psi$ with $\nu$ from (3) yields

$$\lambda = 2\Omega(\zeta).$$  \hfill (18)

The squared-eigenfunction connection is known to be complete on the whole line,\textsuperscript{18} but this has not yet been shown in other settings. We have been able to show it is complete in every example that we have studied in depth,\textsuperscript{4,6,8,10,13} In such cases, (18) gives the entire stability spectrum if the set $\sigma_L$ is known. In other words, the map $\Omega : \sigma_L \mapsto \sigma_\nu$ is surjective. This gives us a connection between the function spaces defining the stability and the Lax spectrum: both are defined by the spatial boundedness of the eigenfunctions in question. The next step for studying stability is to find the Lax spectrum. Before doing so, we provide an example of how the steps in this section work for a well-known equation.

**Example 1** (The nonlinear Schrödinger (NLS) equation). The nonlinear Schrödinger (NLS) equation is

$$i\Psi_t + \frac{1}{2}\Psi_{xx} - \kappa \Psi |\Psi|^2 = 0,$$  \hfill (19)
where $\Psi(x, t)$ is a complex-valued function and $\kappa = -1$ and $\kappa = 1$ correspond to the focusing and defocusing equations, respectively. The Lax pair for the NLS equation\(^\text{19}\) is given by (6) with

$$ q = \Psi, \quad r = \kappa \Psi^*, \quad A = -i \xi^2 - i \kappa |\Psi|^2/2, \quad B = \xi \Psi + i \Psi_x/2, \quad C = \xi \kappa \Psi^* - i \kappa \Psi^*_x/2. \quad (20a) $$

Note that $A$, $B$, and $C$ satisfy the properties in Remark 1. Equating $\Psi(x, t) = e^{-i \omega t} \tilde{\psi}(x)$, where $\omega \in \mathbb{R}$ is constant, we obtain the NLS equation in a frame rotating with constant phase speed $\omega$,

$$ i \psi_t + \omega \psi + \frac{1}{2} \psi_{xx} - \kappa \psi |\psi|^2 = 0. \quad (21) $$

Equation (21) can be obtained from the compatibility of the new $t$-equation,

$$ \phi_t = \begin{pmatrix} -i \xi^2 - i \kappa |\psi|^2/2 + i \omega/2 & \xi \psi + i \psi_x/2 \\ \kappa \xi \psi^* - i \psi_x^*/2 & i \xi^2 + i \kappa |\psi|^2/2 - i \omega/2 \end{pmatrix} \phi, \quad (22) $$

and the $x$ equation (6a), which is unchanged.

The (quasi)periodic stationary solutions of (21) are called the elliptic solutions. The stability of the elliptic solutions of the defocusing and focusing NLS equation was studied in Refs. 4, 9, and 10, respectively. To do so, we linearize (21) about a stationary solution $\tilde{\psi}(x)$ by letting $\psi(x, t) = \tilde{\psi}(x) + \epsilon u(x, t) + \mathcal{O}(\epsilon^2)$. This results in

$$ U_t = \begin{pmatrix} u \\ \kappa u^* \end{pmatrix}_t = \begin{pmatrix} i \xi^2 - 2i \kappa |\tilde{\psi}|^2 + i \omega \\ -i \xi^2 - 2i \kappa |\tilde{\psi}|^2 - i \omega \end{pmatrix} \begin{pmatrix} u \\ \kappa u^* \end{pmatrix} = \mathcal{L}_{\text{NLS}} U. \quad (23) $$

Since $\mathcal{L}_{\text{NLS}}$ does not depend explicitly on $t$, we separate variables with

$$ U(x, t) = \begin{pmatrix} u(x, t) \\ \kappa u^*(x, t) \end{pmatrix} = e^{\lambda t} \begin{pmatrix} u(x) \\ \kappa u^*(x) \end{pmatrix} = e^{\lambda t} V(x), \quad (24) $$

resulting in the spectral problem

$$ \lambda V = \mathcal{L}_{\text{NLS}} V. \quad (25) $$

The squared-eigenfunction connection for the NLS equation\(^\text{4}\) gives

$$ U(x, t) = \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix}, \quad (26) $$

where $\phi = (\phi_1, \phi_2)^T$ is an eigenfunction of (22). Using (13), the eigenfunctions of $\mathcal{L}_{\text{NLS}}$ are given by

$$ U(x, t) = e^{\lambda t} V(x) = \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix} = e^{2 \Omega t} \begin{pmatrix} \varphi_1^2(x) \\ \varphi_2^2(x) \end{pmatrix}, \quad (27) $$
hence $\lambda = 2\Omega(\zeta)$ (18). The map $\Omega : \sigma_L \to \sigma_{\text{c-NLS}}$ is shown to be surjective in Refs. 4 and 10.

### 3.2 Finding the Lax spectrum

We begin by introducing the isospectral transformation

$$\Phi(x, t) = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} e^{-i\theta/2} \phi_1 \\ e^{i\theta/2} \phi_2 \end{pmatrix},$$

(28)

by which the Lax pair (6) becomes

$$\Phi_x = \begin{pmatrix} \alpha \\ \kappa Q(x) \alpha \end{pmatrix} \Phi,$$

(29a)

$$\Phi_t = \begin{pmatrix} A e^{-i\theta} & e^{-i\theta} B \\ e^{i\theta} C & -A \end{pmatrix} \Phi = \begin{pmatrix} A & \hat{B} \\ \hat{C} & -A \end{pmatrix} \Phi,$$

(29b)

where

$$\alpha = -i\zeta - i\theta x/2, \quad \hat{B} = e^{-i\theta} B, \quad \hat{C} = e^{i\theta} C.$$  

(30)

This form is helpful since, by Remark 1, $\hat{B}$ and $\hat{C}$ are $P$-periodical along with $A$. The compatibility conditions (8) and (11) become

$$A_x = \hat{Q}\hat{C} - \kappa\hat{Q}\hat{B}, \quad \hat{B}_x = 2(\alpha\hat{B} - AQ), \quad \hat{C}_x = -2(\alpha\hat{C} - \kappa AQ).$$

(31)

To find $\sigma_L$, we find the eigenfunctions of (6) using a technique first used in Ref. 4. We note that the eigenfunctions can be found using other techniques, see, for example, Refs. 20–22. From (29b), we see that the eigenfunctions are of the form

$$\Phi(x, t) = e^{\Omega t} y_1(x) \begin{pmatrix} -\hat{B}(x) \\ A(x) - \Omega \end{pmatrix}, \quad \text{or} \quad \Phi(x, t) = e^{\Omega t} y_2(x) \begin{pmatrix} A(x) + \Omega \\ \hat{C}(x) \end{pmatrix}.$$  

(32)

Here the scalar functions $y_1(x)$ and $y_2(x)$ are determined by the requirement that $\Phi(x, t)$ not only solves (29b), but also (29a), since (29 a-b) have a common set of eigenfunctions. We will use the first equation of (32), but both representations can be helpful (see Section 5.2 for more on this). Substitution in (29a) gives

$$-\hat{B}y_1' - \hat{B}_x y_1 = (-\alpha\hat{B} + Q(A - \Omega))y_1, \quad (A - \Omega)y_1' + A_x y_1 = (-\kappa Q\hat{B} - \alpha(A - \Omega))y_1,$$

(33)
so that different (but equivalent) representations for $y_1(x)$ are obtained from the first or second equation of (6a):

$$y_1 = \hat{y}_1 \exp \left( \int \frac{\alpha B - Q(A - \Omega) - B_x}{B} x \right), \quad y_1 = \hat{y}_1 \exp \left( - \int \frac{\kappa Q B + \alpha (A - \Omega) + A_x}{A - \Omega} \right),$$

(34)

where $\hat{y}_1$ and $\hat{y}_1$ are constants of integration. At this point, we define $S_L$, the function space that defines $\sigma_L$. Physically, the eigenfunctions $\phi$ should be bounded for $\mathbb{R} = \mathbb{R} \cup \{ \infty \}$. Therefore, we define $S_L$ to be the space where $\phi$ are bounded for $x \in \mathbb{R}$. Since $A$ and $B$ are bounded for $x \in \mathbb{R}$ (Remark 1), $y_1(x)$ must be bounded for $x \in \mathbb{R}$ in order for $\phi(x, t)$ to be an eigenfunction. We use the second expression of (34). Similar work is done for the first expression of (34) in Section 5.2.

To bound the exponential growth, we consider the real part of the exponential. Therefore we need the indefinite integral

$$I = \text{RE} \int \left( \frac{\kappa Q B}{A - \Omega} + \alpha + \frac{d}{dx} \log(A - \Omega) \right) dx,$$

(35)

to be bounded for $x \in \mathbb{R}$. By Remark 1 the integrand in $I$ is $P$-periodic, so it suffices to examine the average over one period. Therefore we need

$$J = \text{Re} \left\langle \alpha + \frac{\kappa Q B}{A - \Omega} + \frac{d}{dx} \log(A - \Omega) \right\rangle = 0,$$

(36)

where $\langle \cdot \rangle = \frac{1}{P} \int_0^P \cdot x$ is the average over a period. Since $A - \Omega$ is $P$-periodic, the logarithmic derivative has no contribution to $J$. Therefore $\zeta \in \sigma_L$ if and only if

$$J = \text{Re} \left\langle \alpha + \frac{\kappa Q B}{A - \Omega} \right\rangle = 0.$$

(37)

Remark 2. The condition (37) is a necessary and sufficient condition for $\zeta \in \sigma_L$. In general, this condition is nontrivial to work with. However, it does allow one to characterize large subsets of the Lax spectrum with relative ease. In particular, the large $\zeta$ asymptotic analysis of (37) is tractable, in that we can find an asymptotic approximation to the unbounded components of the Lax spectrum, in analogy to the asymptotic analysis used to find the essential spectrum for decayingsolutions on the real line.\textsuperscript{23} Below we examine (37) for $\zeta \in \mathbb{R}$.

When $\zeta \in \mathbb{R}$, $\alpha \in \mathbb{R}$ so it also has no contribution to $J$. Therefore when $\zeta \in \mathbb{R}$, we need

$$\text{Re} \left\langle \frac{\kappa Q B}{A - \Omega} \right\rangle = 0.$$

(38)

When using the recursion operator, $A(\mathbb{R}) \subset \mathbb{R}$ since $A(\zeta)$ is defined by a power series in $\zeta$ with imaginary coefficients (see Remark 1). We have the following lemma for $\zeta \in \mathbb{R}$.

**Lemma 2.** Consider a member of the AKNS hierarchy with $r = \kappa q^*$, where $\kappa = \pm 1$, and $q_t = r_t = 0$. Then $C = \kappa B^*$ when $\zeta \in \mathbb{R}$ and $A(\zeta) \in \mathbb{R}$, where $A$, $B$, and $C$ are defined in (6b).
Proof. We first establish that \( \hat{C} = \kappa \hat{B}^* \) (30) from which it follows that \( C = \kappa B^* \). From (31),
\[
0 = \text{Re}(A_x) = Q \text{Re}(C - \kappa \hat{B}),
\]
and
\[
0 = \text{Re}(\hat{C}_x - \kappa \hat{B}_x) = \text{Re}[-2\alpha(\hat{C} + \kappa \hat{B}) + 4\kappa QA] = \text{Re}[-2\alpha(\hat{C} + \kappa \hat{B})],
\]
since \( \kappa QA \in i\mathbb{R} \) when \( \zeta \in \mathbb{R} \), by assumption. If \( \alpha(\zeta) \neq 0 \), IM(\( \hat{C} \)) = \( -\kappa \) \( \text{IM}(\hat{B}) \) and (39) implies \( \hat{C} = \kappa \hat{B}^* \). If \( \alpha = 0 \), we have
\[
\partial_x(\hat{C} + \kappa \hat{B}) = \hat{C}_x + \kappa \hat{B}_x = 0. \tag{41}
\]
Since IM(\( \hat{C} + \kappa \hat{B} \)) = 0 at \( x \) with \( \alpha \neq 0 \), it must be zero for all \( x \). Therefore \( \hat{C} = \kappa \hat{B}^* \) for all \( x \). □

We can now identify part of the Lax spectrum with imaginary (stable) elements of the stability spectrum.

**Theorem 1.** Consider a member of the AKNS hierarchy (6) satisfying Assumption 1 and Remark 1. Assume further that \( A, \hat{B}, \) and \( \hat{C} \) from (29) are \( P \)-periodic. Let \( Q = \{ \zeta \in \mathbb{R} : A(\zeta) \in i\mathbb{R} \} \), \( S = \{ \zeta \in \mathbb{R} : \Omega(\zeta) \in i\mathbb{R} \} \) (\( S \) for stable), and \( U = \{ \zeta \in \mathbb{R} : \Omega(\zeta) \in \mathbb{R} \} \) (\( U \) for unstable).

When \( \kappa = -1 \), \( Q \subset S \). When \( \kappa = 1 \), \( Q \subset S \cup U \). Also, \( Q \cap S \subset \sigma_L \), and if \( \Omega \) is shown to be surjective, \( \Omega(Q \cap S) \subset \sigma_L \cap i\mathbb{R} \) for \( \kappa = \pm 1 \). In other words, all real \( \zeta \) for which \( \Omega(\zeta), A(\zeta) \in i\mathbb{R} \) are part of the Lax spectrum and map to stable elements of the stability spectrum.

Proof. Let \( \zeta \in Q \). By Lemma 2
\[
\Omega^2(\zeta) = A^2 + \kappa |B|^2 \in \mathbb{R}, \tag{42}
\]
so \( Q \subset S \cup U \). When \( \kappa = -1 \), \( \Omega^2(\zeta) < 0 \) so \( Q \subset S \). At this point, we are not assuming that \( \zeta \in \sigma_L \).

We show that this is the case by showing that \( \tilde{I}_1 = 0 \) (38). If \( \Omega(\zeta) \in i\mathbb{R} \), then
\[
\text{Re} \left( \frac{QB}{A - \Omega} \right) = \frac{1}{2} \left( \frac{QB}{A - \Omega} + \frac{QB^*}{A^* - \Omega^*} \right) \quad \text{when } \kappa \neq 0 \tag{43}
\]
\[
= \frac{1}{2} \left( \frac{QB - \kappa \hat{C}}{A - \Omega} \right) = -\frac{A_x}{2(A - \Omega)} = -\frac{1}{2} \frac{d}{dx} \log(A - \Omega).
\]
Since \( A \) is assumed to be periodic, \( \text{Re} \tilde{I}_1 = 0 \), establishing that \( \zeta \in \sigma_L \) and that \( Q \cap S \subset \sigma_L \). By definition, \( \Omega(Q \cap S) \subset i\mathbb{R} \cap \sigma_L \) when \( \Omega \) is surjective onto \( \sigma_L \). □

**Remark 3.** When using the recursion operator, \( A(\mathbb{R}) \subset i\mathbb{R} \) (see statement just before Lemma 2). Therefore when \( \kappa = -1 \), Theorem 1 implies that \( \mathbb{R} \subset \sigma_L \) and \( \Omega(\mathbb{R}) \subset i\mathbb{R} \) for all members of the AKNS hierarchy satisfying Assumption 1 and where Remark 1 applies.

When \( \kappa = 1 \), the spectral problem (6) is self-adjoint so \( \sigma_L \subset \mathbb{R} \). Therefore the assumption that \( \zeta \in \mathbb{R} \) is not actually an assumption for such problems. Theorem 1 establishes that all \( \{ \zeta \in \mathbb{R} : \quad \text{...} \} \)
FIGURE 1  The real versus imaginary part of the Lax and stability spectrum (left and right of each panel, respectively) for an elliptic solution of (A) the defocusing NLS equation and (B) the focusing NLS equation. The real component of $\sigma_L$ and its image under $\Omega$ is colored blue. The rest of $\sigma_L$ is in black. The Lax spectrum is computed analytically using (38) and the stability spectrum is the image under $\Omega$

$\Omega(\zeta) \in i\mathbb{R} \subset \sigma_L$. If one can establish that $\{\zeta \in \mathbb{R} : \Omega(\zeta) \in \mathbb{R} \setminus \{0\}\} \not\subset \sigma_L$ for a specific problem, then one has shown that the underlying solution is stable.

Example 2 (The NLS equation, continued). We begin by demonstrating the statement of Remark 2. The condition (37) can be studied asymptotically for large $\zeta$. For both the defocusing and the focusing NLS equations, $\Omega \sim i\zeta^2$ for large $\zeta$. Since $A(\mathbb{R}) \subset i\mathbb{R}$, $A - \Omega \in i\mathbb{R}$ if $\zeta \in \mathbb{R}$.

Further, $\alpha(\mathbb{R}) \in i\mathbb{R}$ and $xQB \sim xQ^2\zeta \in \mathbb{R}$ for real $\zeta$. Therefore for large real $\zeta$,

$$\alpha + \frac{xQB}{A - \Omega} \in i\mathbb{R} \quad (44)$$

implying that $\zeta \in \sigma_L$. This gives two unbounded components of $\sigma_L$: one as $\zeta \to \infty$ and another as $\zeta \to -\infty$. Next we demonstrate that these two are the only unbounded components of $\sigma_L$. For large $\zeta$, the $x$-equation of the Lax pair (6a) becomes diagonal, giving

$$\phi \sim \begin{pmatrix} e^{-i\zeta x} \\ e^{i\zeta x} \end{pmatrix}.$$  

The only large $\zeta$ which leaves $\phi$ bounded for all $x \in \mathbb{R}$ is $\zeta \in \mathbb{R}$. Therefore the two unbounded components asymptotic to the real line as $\zeta \to \pm \infty$ found above are the only two unbounded components of the Lax spectrum.

The defocusing NLS equation has $q = r^*$ so the Lax pair is self-adjoint and $\sigma_L \subset \mathbb{R}$. In Ref. 4, the authors establish that $\{\zeta \in \mathbb{R} : \Omega(\zeta) \in \mathbb{R} \setminus \{0\}\} \not\subset \sigma_L$ by working with (38) directly. Since $\sigma_L = \{\zeta \in \mathbb{C} : \Omega(\zeta) \in i\mathbb{R}\}$ and $\Omega : \sigma_L \to \sigma_{\text{NLS}}$ is surjective, the elliptic solutions are stable (see Figure 1A).

The situation for the elliptic solutions of the focusing NLS equation is more complicated since the Lax pair is not self-adjoint and $\sigma_L$ is not a subset of the real line. To find the rest of $\sigma_L$, we work with (36) by computing the integral in terms of elliptic functions and working with the result, but it may also be found using the Floquet Discriminant (see the Appendix). It was originally shown in Ref. 9 that $\mathbb{R} \subset \sigma_L$ by working with the complicated expression for (36) directly. In Ref. 10, we establish that the set $\mathcal{E} = \mathbb{R} \cup \{\zeta \in \mathbb{C} : \Omega(\zeta) = 0\}$ are the only elements in $\sigma_L$ that map to $i\mathbb{R} \subset \sigma_L$ under $\Omega$. Everything else in the spectrum maps to unstable modes, hence the solutions are, in general, unstable. Special classes of perturbations exist, the subharmonic perturbations, for which only members of $\mathcal{E}$ are excited. Some of the elliptic solutions are stable with respect to this class of
perturbations. This demonstrates the power of what is established here: as we know what maps to stable elements of the stability spectrum, we can find the class of perturbations for which the solution is stable. See Figure 1B for an example.

4 | AKNS EXAMPLES

In this section, we apply the results of Section 3 to three integrable equations with connections to the AKNS hierarchy. The first equation is in the AKNS hierarchy with the reduction (9); Theorem 1 applies directly. Some results are new and others confirm known results. The second equation is in the AKNS hierarchy but does not have the reduction (9). Theorem 1 does not apply immediately, but similar conclusions are made. The third equation is in the AKNS hierarchy and (9) holds. However, a change of variables is required to make the problem physical that takes it out of the AKNS hierarchy. Still, we find results identical to Theorem 1. This leads us to search for a generalization of Theorem 1 in Section 5.

4.1 | The modified Korteweg-de Vries equation

The modified Korteweg-de Vries (mKdV) equation is given by

\[ u_t - 6 \kappa u^2 u_x + u_{xxx} = 0, \quad (46) \]

where \( u \) is a real-valued function and \( \kappa = -1 \) and \( \kappa = 1 \) correspond to the focusing and defocusing cases, respectively. Equation (46) is a member of the AKNS hierarchy (Section 3) with the Lax pair

\[
A = -4i \xi^3 - 2i \xi q r, \quad B = 4 \xi^2 q + 2q^2 r + 2i \xi q_x - q_{xx}, \quad C = 4 \xi^2 r + 2q^2 r - 2i \xi r_x - r_{xx}, \quad (47)
\]

and \( r = \kappa q = xu \). Letting \((y, \tau) = (x - ct, t)\), where \( c \in \mathbb{R} \) is constant, gives mKdV in the traveling frame,

\[ u_{\tau} - cu_y - 6ku^2u_y + u_{yyy} = 0. \quad (48) \]

The Lax pair for (48) changes accordingly:

\[ \phi_y = X \phi, \quad \phi_\tau = (T + cX) \phi. \quad (49) \]

The elliptic solutions of mKdV are found upon equating \( u_\tau = 0 \). The Lax pair can be found by using the recursion operator as mentioned in Remark 1, so the remarks there holds and Assumption 1 applies.

Theorem 1 applies with \( Q = \mathbb{R} \) since \( A(\mathbb{R}) \subset i\mathbb{R} \). When \( \kappa = -1 \), \( \mathbb{R} \subset \sigma_L \) and \( \Omega(\mathbb{R}) \subset i\mathbb{R} \cap \sigma_L \). This result is new: it can be used as a first step for studying the stability of the elliptic solutions of the focusing mKdV equation. Solutions are stable with respect to perturbations that excite only real elements of the Lax spectrum. We do not yet know what class of perturbations this corresponds to. When \( \kappa = 1 \), \( S = \{ \zeta \in \mathbb{R} : \Omega(\zeta) \in i\mathbb{R} \} \subset \sigma_L \), \( \Omega(S) \subset i\mathbb{R} \cap \sigma_L \), and \( \Omega(S) \subset \sigma_L \). Since \( \mathbb{R} \subset \sigma_L \), one must establish that \( \{ \zeta \in \mathbb{R} : \Omega(\zeta) \notin i\mathbb{R} \setminus \{0\} \} \not\subset \sigma_L \) to establish stability.
This is proven in Ref. 8 to show that the elliptic solutions for the defocusing mKdV equation are stable.

### 4.2 The PT-symmetric reverse space nonlocal NLS equation

The PT-symmetric reverse space nonlocal NLS equation is given by Ref. 24

\[ i\Psi_t(x,t) + \frac{1}{2}\Psi_{xx}(x,t) - \kappa\Psi(x,t)^2\Psi^*(-x,t) = 0, \quad (50) \]

where \( \kappa = \pm 1 \). This equation is a member of the AKNS hierarchy (Section 3) with the Lax pair\(^4\)

\[ A(x,t) = -i\zeta^2 - iq r/2, \quad B = \zeta q + iq_x/2, \quad C = \zeta r - ir_x/2, \quad (51) \]

and \( r(x,t) = \kappa q^*(-x,t) = \kappa \Psi^*(-x,t) \in \mathbb{C} \). Solutions of \((19)\) with

\[ \Psi(x,t) = \Psi(-x,t), \quad (52) \]

are also solutions of \((50)\). Thus all even solutions of \((19)\) examined in Ref. 4 for \( \kappa = 1 \) and in Ref. 10 for \( \kappa = -1 \) are solutions to \((50)\). These solutions, and other periodic and quasi-periodic solutions of \((50)\), were first reported in Ref. 25. Almost every solution found in Ref. 25 is even in \( x \), except for one that is odd. For the even solutions, the Lax spectrum remains unchanged since \( \Psi(x,t) = \Psi(-x,t) \): hence the stability results in Refs. 4 and 10 hold for these solutions.

The Lax pair is not self adjoint for \( \kappa = 1 \), so \( \sigma_L \) is not necessarily a subset of \( \mathbb{R} \). With the equation written in a uniformly rotating frame so that all components are time independent, Assumption 1 and Remark 1 apply. Upon assuming that \( q \) and \( r \) are \( P \)-periodic or quasiperiodic and \( \zeta \in \mathbb{R} \), we have that \( \zeta \in \sigma_L \) if

\[ \text{Re} \left\langle \frac{qC}{A - \Omega} \right\rangle = 0. \quad (53) \]

This condition is found just as \((38)\) was found and is an example of several equivalent conditions for the Lax spectrum derived in Section 5.2. For \( \zeta \in \mathbb{R} \),

\[ A^*(-x) = -A(x), \text{ and } C^*(-x) = \kappa B(x). \quad (54) \]

so that

\[ \Omega^2(\zeta) = A(x)^2 + B(x)C(x) = A(-x)^2 + B(-x)C(-x) \]

\[ = (A^2 + BC)^* = (\Omega^2(\zeta))^*, \quad (55) \]

and \( \Omega(\mathbb{R}) \subset \mathbb{R} \cup i\mathbb{R} \). If \( \zeta \in \mathbb{R} \) and \( \Omega(\zeta) \in i\mathbb{R} \),
\[ \text{Re} \int_0^P \left( \frac{q(x)C(x)}{A(x) - \Omega} \right)^x = \frac{1}{2} \left( \int_0^P \frac{q(x)C(x)}{A(x) - \Omega}^x - \int_0^{-P} \frac{q^*(-x)C^*(-x)}{A^*(-x) - \Omega^*}^x \right) \]

\[ = \frac{1}{2} \left( \int_0^P \frac{q(x)C(x)}{A(x) - \Omega}^x + \int_0^0 \frac{q^*(-x)C^*(-x)}{A(x) + \Omega}^x \right) \]

\[ = \frac{1}{2} \left( \int_0^P \frac{q(x)C(x) - q^*(-x)C^*(-x)}{A(x) - \Omega}^x \right) \]

\[ = \frac{1}{2} \int_0^P \frac{A_x}{A(x) - \Omega}^x = 0, \]

since \( A \) is periodic when \( q \) and \( r \) are. It follows from (53) that \( \{ \zeta \in \mathbb{R} : \Omega(\zeta) \in i\mathbb{R} \} \subset \sigma_L \) for \( \kappa = \pm 1 \).

4.3 | The sine- and sinh-Gordon equations

The sine-Gordon (s-G) equation in light-cone coordinates is given by

\[ u_{\xi \eta} = \sin u, \]  \hspace{1cm} (57)

where \( u(\xi, \eta) \) is real valued. Equation (57) is a member of the AKNS hierarchy (Section 3) with Lax pair

\[ A = \frac{i}{4\zeta} \cos(u), \quad B = \frac{i}{4\zeta} \sin(u), \quad C = \frac{i}{4\zeta} \sin(u). \]  \hspace{1cm} (58)

We write \((\xi, \eta)\) instead of \((x, t)\) to distinguish between the light-cone coordinates \((\xi, \eta)\) and the space-time coordinates \((x, t)\). Equation (57) is equivalent to the compatibility of mixed derivatives, \( \partial_\eta u_\xi = \partial_\xi u_\eta \), by requiring that \( r = -q = u_\xi / 2 \). Since \( r = -q \), (57) is not self-adjoint. A self-adjoint variant of the s-G equation is the sinh-Gordon (sh-G) equation,

\[ u_{\xi \eta} = \sinh u. \]  \hspace{1cm} (59)

Equation (59) is a member of the AKNS hierarchy (Section 3) with Lax pair

\[ A = \frac{i}{4\zeta} \cosh(u), \quad B = -\frac{i}{4\zeta} \sinh(u), \quad C = \frac{i}{4\zeta} \sinh(u), \]  \hspace{1cm} (60)

and is equivalent to the compatibility of mixed derivatives under the reduction \( r = q = u_\xi / 2 \).
The Lax pair for both the s-G and the sh-G equations can be found by using the recursion operator as mentioned in Remark 1, so the remarks there and Assumption 1 apply. Theorem 1 applies to the s-G equation when \( \kappa = -1 \) and to the sh-G equation when \( \kappa = 1 \). For \( \kappa = -1 \), \( \mathbb{R} \subset \sigma_L \) and \( \Omega(\mathbb{R}) \in \sigma_L \cap i\mathbb{R} \). For \( \kappa = 1 \), \( \mathbb{R} \cap \{ \xi \in \mathbb{R} : \Omega(\xi) \in i\mathbb{R} \} \subset \sigma_L \). However, since \( \eta \) is not a timelike variable, stability results mean little for this equation in these variables. Instead, we transform from light-cone to laboratory coordinates. To transform (57) from light-cone to laboratory coordinates, we let \((x, t) = (\eta + \xi, \eta - \xi)\) to obtain
\[
 u_{tt} - u_{xx} + \sin(u) = 0. \tag{61}
\]
The same coordinate transformation on (59) gives
\[
 u_{tt} - u_{xx} + \sinh(u) = 0. \tag{62}
\]
The Lax pair for both systems is
\[
 w_x = \frac{1}{2}(T + X)w = \hat{X}w, \quad w_t = \frac{1}{2}(T - X)w = \hat{T}w. \tag{63}
\]
Note that (61) and (62) are not members of the AKNS hierarchy. Nonetheless we show that statements very similar to those made in Theorem 1 hold for this equation. This gives us a bridge from the AKNS framework to generalizations.

We move the s-G equation (61) and the sh-G equation (62) to a traveling frame by letting \((z, \tau) = (x - Vt, t)\) for constant \( V \in \mathbb{R} \) and find
\[
 (V^2 - 1)u_{zz} - 2Vu_{zt} + u_{\tau\tau} + \sin(u) = 0, \tag{64}
\]
and
\[
 (V^2 - 1)u_{zz} - 2Vu_{zt} + u_{\tau\tau} + \sinh(u) = 0, \tag{65}
\]
respectively. The new Lax pair is given by
\[
 w_z = \hat{X}w, \quad w_\tau = (\hat{T} + V\hat{X})w = Tw. \tag{66}
\]
Periodic stationary solutions are found by letting \( u_\tau = 0 \) and assuming \( u \) is periodic in \( x \). Assumption 1 holds and the Lax pair is of the form mentioned in Remark 1. As the transformation to lab coordinates is isospectral, the spectrum \( \sigma_L \) does not change: \( \mathbb{R} \subset \sigma_L \) and \( \Omega(\mathbb{R}) \subset i\mathbb{R} \cap \sigma_L \) (the squared-eigenfunction connection gives \( \lambda = 2\Omega(\xi) \) here as well\(^7\)). Solutions are stable with respect to perturbations that excite only real elements of the Lax spectrum. These results are known and have been shown in Ref. 7 (whose results agree with Ref. 26 where the Lax spectrum is not used). For the sh-G equation, \( S = \{ \xi \in \mathbb{R} : \Omega(\xi) \in i\mathbb{R} \} \subset \sigma_L \), \( \Omega(S) \subset i\mathbb{R} \cap \sigma_L \), and \( \Omega(S) \subset \sigma_L \). As \( \mathbb{R} \subset \sigma_L \), one must establish that \( U^* = \{ \xi \in \mathbb{R} : \Omega(\xi) \in i\mathbb{R} \setminus \{0\} \subset \sigma_L \) to establish stability. This result is new and \( U^* \notin \sigma_L \) has not yet been shown.

The above results show that Theorem 1 may be applicable to some non-AKNS integrable equations. In the next section, we establish a theorem to this effect. However, before continuing on we continue to study the spectral problem for (61) and (62) for exposition purposes.
The Lax pair (63) defines a quadratic eigenvalue problem (QEP),

\[ Q(\zeta)w = (M\zeta^2 + N\zeta + K)w = 0. \] (67)

There are two choices for \( M, N, \) and \( K \):

\[
M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N_1 = \begin{pmatrix} -2i\partial_x & iq \\ -ir & 2i\partial_x \end{pmatrix}, \quad K_1 = \begin{pmatrix} ia & ib \\ ic & -ia \end{pmatrix},
\]

\[
M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} -2i\partial_x & iq \\ ir & -2i\partial_x \end{pmatrix}, \quad K_2 = \begin{pmatrix} ia & ib \\ -ic & ia \end{pmatrix},
\]

(68a)

(68b)

where

\[ a = \zeta A, \quad b = \zeta B, \quad c = \zeta C. \] (69)

Then \( \zeta \in \mathbb{C} \) is an eigenvalue of \( Q \) if and only if \( Q(\zeta)w = 0 \) for all bounded \( w \). A QEP is classified as self-adjoint if \( M, N, \) and \( K \) are self-adjoint. The eigenvalues for self-adjoint QEPs are either real or come in complex-conjugate pairs. If \( M_1, N_1, \) and \( K_1 \) are chosen, then \( Q(\lambda) \) is self-adjoint if \( r = q^*, a^* = -a, \) and \( c^* = b \), which is the case for the sh-G equation. If \( M_2, N_2, \) and \( K_2 \) are chosen, then \( Q(\lambda) \) is self-adjoint if \( r = -q^*, a^* = -a, \) and \( c^* = -b \), which is the case for the s-G equation. It follows that for either equation, the Lax spectrum consists of real or complex-conjugate spectral elements. This confirms what we know from the isospectral transform to light-cone coordinates. The whole real line is part of the Lax spectrum for the s-G equation and the Lax spectrum for the sh-G equation is a subset of the real line. To determine the subset of \( \sigma_L \) off the real line, one may use the integral condition (36) (used in Ref. 7 to find \( \sigma_L \)) or the Floquet discriminant (Section A).

5 | GENERALIZATION OF THE AKNS RESULTS

As seen from the AKNS examples, the real line of the Lax spectrum plays an important role in stability. For the non self-adjoint members of the AKNS hierarchy (those with \( \kappa = -1 \)), the real line is part of the Lax spectrum and maps to stable elements of the stability spectrum. For the self-adjoint members of the hierarchy (those with \( \kappa = 1 \)), the real line is a subset of the Lax spectrum. If one can establish that \( \{ \zeta \in \mathbb{R} : \Omega(\zeta) \in \mathbb{R} \setminus \{0\} \} \not\subseteq \sigma_L \), then the solution of interest is stable. The sine-Gordon and sinh-Gordon examples indicate that this trend holds even for integrable systems not in the AKNS hierarchy. In what follows, we extend the AKNS results to integrable systems that are not in the AKNS hierarchy.

5.1 | Setup

In this section, we consider integrable equations (1) possessing a \( 2 \times 2 \) Lax pair of the form,

\[
\phi_x(x, t; \zeta) = \begin{pmatrix} \alpha(x, t; \zeta) & \beta(x, t; \zeta) \\ \gamma(x, t; \zeta) & -\alpha(x, t; \zeta) \end{pmatrix} \phi(x, t; \zeta) = X\phi,
\]

(70a)
where \( \alpha, \beta, \gamma, A, B, \) and \( C \) are complex-valued functions. As in our analysis of the AKNS hierarchy, we restrict our analysis to Lax pairs where the elements of \( P = \{ \alpha, \beta, \gamma, A, B, C \} \) are bounded for all \( x \in \mathbb{R} \) and are autonomous in \( t \). Since we are interested in studying stationary solutions, we assume again that \( \alpha_t = \beta_t = \gamma_t = 0 \). In the stationary frame, the compatibility of (70) defines the conditions

\[
A_x = \beta C - \gamma B, \\
B_x = 2(\alpha B - \beta A), \\
C_x = -2(\alpha C - \gamma A).
\]

The definition for the Lax spectrum (12) is unchanged.

5.2 Computing the Lax spectrum

With \( A, B, \) and \( C \) \( t \)-independent, (70b) may be solved by separation of variables resulting in once again (15). Here \( \Omega \) has the same properties as for the AKNS hierarchy and Lemma 1 holds with a nearly identical proof. We again consider the special case of the reduction

\[
\gamma = \kappa \beta^*, \quad \kappa = \pm 1.
\]

With such a reduction, we let

\[
\beta(x; \zeta) = \eta(x; \zeta)e^{i\vartheta(x; \zeta)}, \quad \gamma(x; \zeta) = \kappa \eta(x; \zeta)e^{-i\vartheta(x; \zeta)},
\]

where \( \eta \) and \( \vartheta \) are real-valued functions with \( \eta(x; \zeta) \geq 0 \). We also assume that \( \eta \) is a \( P \)-periodic function. Using the isospectral transformation (28), the Lax pair (70) becomes

\[
\Phi_x = \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\gamma} & -\hat{\alpha} \end{pmatrix} \Phi, \quad \Phi_t = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & -\hat{A} \end{pmatrix} \Phi,
\]

where

\[
\hat{\alpha} = \alpha - i\vartheta_x/2, \quad \hat{\beta} = \beta e^{-i\vartheta} = \eta, \quad \hat{\gamma} = \gamma e^{i\vartheta} = \kappa \eta, \\
\hat{A} = A, \quad \hat{B} = e^{-i\vartheta}B, \quad \hat{C} = e^{i\vartheta}C.
\]
The eigenfunctions here are identical to (32). Similar to what we did there, we get two ODEs for \( y_1 \), but also two for \( y_2 \). This gives two expressions for \( y_1 \) and two for \( y_2 \). In each case, the exponential term needs to be bounded for \( x \in \mathbb{R} \). Inspired by the results for the AKNS hierarchy, we assume that \( \hat{A} \), \( \hat{B} \), and \( \hat{C} \) are \( P \)-periodic. With these assumptions, we obtain eight boundedness conditions that define the Lax spectrum (four from \( y_1 \) and \( y_2 \), and four from rewriting those four using (71)):

\[
\begin{align*}
\text{Re} \left\langle \hat{\alpha} - \frac{\hat{\beta}(A - \Omega)}{\hat{B}} \right\rangle &= 0, \\
\text{Re} \left\langle \hat{\alpha} + \frac{\gamma \hat{B}}{A - \Omega} \right\rangle &= 0, \\
\text{Re} \left\langle \hat{\alpha} + \frac{\hat{\beta} \hat{C}}{A + \Omega} \right\rangle &= 0, \\
\text{Re} \left\langle \frac{\hat{\beta} \Omega}{\hat{B}} \right\rangle &= 0, \\
\text{Re} \left\langle \hat{\alpha} - \frac{\gamma(A + \Omega)}{\hat{C}} \right\rangle &= 0, \\
\text{Re} \left\langle \hat{\alpha} + \frac{\hat{\beta} \hat{C}}{A - \Omega} \right\rangle &= 0, \\
\text{Re} \left\langle \hat{\alpha} + \frac{\gamma \hat{B}}{A + \Omega} \right\rangle &= 0, \\
\text{Re} \left\langle \frac{\gamma \Omega}{\hat{C}} \right\rangle &= 0,
\end{align*}
\] (76)

If any of these conditions are satisfied for a particular \( \hat{\zeta} \in \mathbb{C} \), then \( \hat{\zeta} \in \sigma_L \). Some of these conditions are new and some have been used in Ref. 4, 6–10. This is the first time all are written down in full generality.

**Remark 4.** The conditions (76) are generalizations of the Lax condition (37) for members of the AKNS hierarchy. Each condition is necessary and sufficient for \( \zeta \in \sigma_L \). This condition is typically nontrivial to work with directly, but it does allow one to find large subsets of the Lax spectrum by considering large \( \zeta \) asymptotics. We note that the last condition in (76) implies \( \{ \zeta \in \mathbb{C} : \Omega(\zeta) = 0 \} \subset \sigma_L \).

Lemma 2 and Theorem 1 have immediate analogues here, whose proof is nearly identical.

**Theorem 2.** Consider an integrable equation (1) possessing the Lax pair (70) with the reduction (72). Assume that \( \hat{A} \), \( \hat{B} \), and \( \hat{C} \) from (75) are \( P \)-periodic. Let

\[
\begin{align*}
Q_- &= \{ \zeta \in \mathbb{C} : \alpha(\zeta), A(\zeta) \in i\mathbb{R} \text{ and } \beta = -\gamma^* \}, \\
Q_+ &= \{ \zeta \in \mathbb{C} : \alpha(\zeta), A(\zeta) \in i\mathbb{R} \text{ and } \beta = \gamma \}, \\
S &= \{ \zeta \in \mathbb{C} : \Omega(\zeta) \in i\mathbb{R} \}, \text{ and } \ U^* = \{ \zeta \in \mathbb{C} : \Omega(\zeta) \in \mathbb{R} \}.
\end{align*}
\] (77)

Then \( Q_- \subset S \), \( Q_+ \subset S \cup U^* \), \( Q_+ \cap S \subset \sigma_L \), and \( \Omega(Q_+ \cap S) \subset \sigma_L \cap i\mathbb{R} \) if \( \Omega \) is shown to be surjective.

In other words, assume \( \alpha(\zeta), A(\zeta) \in i\mathbb{R} \). Then \( \Omega(\zeta) \in i\mathbb{R} \) when \( \beta(\zeta) = -\gamma(\zeta)^* \), (akin to the \( \kappa = -1 \) case in Theorem 1), and those \( \zeta \) are in the Lax spectrum, \( \sigma_L \). Further, they map to stable elements of the stability spectrum. When \( \beta(\zeta) = \gamma(\zeta)^* \), (akin to the \( \kappa = 1 \) case in Theorem 1), \( \Omega(\zeta) \in i\mathbb{R} \cup \mathbb{R} \) and the \( \zeta \) such that \( \Omega(\zeta) \in i\mathbb{R} \) are in the Lax spectrum and map to stable elements of the stability spectrum.

The proof of this theorem is nearly identical to the proof of Theorem 1, so we omit it here. Before moving on to examples that demonstrate the applicability of Theorem 2, we remark that our results do address the intersection of \( \sigma_L \) with \( Q_\pm \cup U^* \). Indeed, establishing whether this is true is important for understanding the stability of the solution in question. We do not address this difficulty here.
6 | EXAMPLES

In this section, we provide examples for which Theorem 2 applies. Next, we provide examples for which the theorem does not apply directly, but for which similar conclusions can be drawn. We do this to demonstrate that these results can be generalized and are not necessarily limited to the special cases considered here.

6.1 Derivative nonlinear Schrödinger equation

The derivative NLS (dNLS) equation,

\[ iq_t = -q_{xx} + \kappa (|q|^2 q)_x, \quad \kappa = \pm 1, \]  

(78)

was first solved on the whole line using the Inverse Scattering Transform in Ref. 28. The Lax pair for (78) is given by (70) with Ref. 28

\[
\begin{align*}
\alpha &= -i\zeta^2, \quad \beta = q\zeta, \quad \gamma = r\zeta, \\
A &= -2i\zeta^4 - i\zeta^2rq, \quad B = 2\zeta^3q + i\zeta q_x + \zeta rq^2, \quad C = 2\zeta^3r - i\zeta r_x + \zeta r^2q,
\end{align*}
\]

(79)

where \( r = \kappa q^* \in \mathbb{C} \). Using \( q(x, t) \mapsto e^{-i\omega t} q(x, t) \) where \( \omega \) is a real constant, (78) becomes

\[ iq_t = -q_{xx} + i\kappa (|q|^2 q)_x - \omega q, \]

(80)

and \( A \mapsto A + i\omega / 2 \); otherwise (79) remains the same. Stationary solutions satisfy

\[ -q_{xx} + i\kappa (|q|^2 q)_x - \omega q = 0. \]

(81)

Quasi-periodic elliptic solutions to the stationary problem were found in Ref. 29.

The Lax pair defines a QEP (67). There are two choices for \( M, N, \) and \( K \),

\[
\begin{align*}
M_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & N_1 &= \begin{pmatrix} 0 & iq \\ -ir & 0 \end{pmatrix}, & K_1 &= \begin{pmatrix} -i\partial_x & 0 \\ 0 & i\partial_x \end{pmatrix}, \\
M_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & N_2 &= \begin{pmatrix} 0 & iq \\ ir & 0 \end{pmatrix}, & K_2 &= \begin{pmatrix} -i\partial_x & 0 \\ 0 & -i\partial_x \end{pmatrix}.
\end{align*}
\]

(82)

If \( M_1, N_1, \) and \( K_1 \) are chosen, then \( Q(\lambda) \) is self-adjoint if \( r = q^* \). If \( M_2, N_2, \) and \( K_2 \) are chosen, then \( Q(\lambda) \) is self-adjoint if \( r = -q^* \). It follows that eigenvalues are real or come in complex-conjugate pairs for either choice of \( \kappa \).

Since Assumption 1 holds and Remark 1 applies here, Theorem 2 applies with different results depending on \( \kappa \) and \( \zeta \). We have \( \zeta \in Q_- \) when \( \zeta \in \mathbb{R} \) and \( \kappa = -1 \) or when \( \zeta \in i\mathbb{R} \) and \( \kappa = 1 \). Alternatively, \( \zeta \in Q_+ \) when \( \zeta \in \mathbb{R} \) and \( \kappa = 1 \) or when \( \zeta \in i\mathbb{R} \) and \( \kappa = -1 \). Defining \( \Omega = \{ \zeta \in \mathbb{C} : \Omega(\zeta) \in i\mathbb{R} \}, \mathbb{R} \cup (\Omega_1 \cap i\mathbb{R}) \subset \sigma_L \) and \( \Omega(\mathbb{R} \cup (\Omega_1 \cap i\mathbb{R})) \subset i\mathbb{R} \) for \( \kappa = -1 \) (see Figure 2A). If \( \kappa = 1, i\mathbb{R} \cup (\mathbb{R} \cap \Omega_1) \subset \sigma_L \) and \( \Omega(i\mathbb{R} \cup (\mathbb{R} \cap \Omega_1)) \subset i\mathbb{R} \) (see Figure 2B). To compute the spectrum of the
FIGURE 2  The real versus imaginary part of the Lax and stability spectrum (left and right, respectively) for a periodic solution of (A) the focusing dNLS equation and (B) the defocusing dNLS equation. The Lax spectrum is computed numerically using Ref. 30 and the stability spectrum is the image under $\Omega$. The Lax spectrum on the real and imaginary axes are colored blue and red, respectively. Their image under the map $\Omega$ is colored in the stability spectrum accordingly. The Lax spectrum of the real and imaginary axes and its image under $\Omega$ is black.

real or imaginary axes, one must examine the integral conditions (76) or construct the Floquet discriminant (the Appendix).

6.2  Vector and matrix nonlinear Schrödinger equations

The \textit{Manakov system} or two-component Vector NLS (VNLS) equation is given by

\begin{align*}
    i \frac{\partial q_1}{\partial t} + \frac{\partial^2 q_1}{\partial x^2} + 2 \left( |q_1|^2 + |q_2|^2 \right) q_1 &= 0, \\
    i \frac{\partial q_2}{\partial t} + \frac{\partial^2 q_2}{\partial x^2} + 2( |q_1|^2 + |q_2|^2 )q_2 &= 0,
\end{align*}

where \( q_1 \) and \( q_2 \) are complex-valued functions. The system (83) was shown to be integrable in Ref. 31. Its finite-genus solutions (including its elliptic solutions) were explicitly constructed in Ref. 32. Its Lax pair is

\begin{align*}
    v_x &= \begin{pmatrix} \alpha & \beta^\top \\ \gamma & \rho \end{pmatrix} v = Xv, \\
    v_t &= \begin{pmatrix} A & B^\top \\ C & D \end{pmatrix} v = T v,
\end{align*}

with

\begin{align*}
    \alpha &= -i\zeta, \quad \beta = q, \quad \gamma = -q^*, \quad \rho = i\zeta I_2, \\
    A &= -2i\zeta^2 + iq^\top q^*, \quad B = 2\zeta q + iq_x, \quad C = -2\zeta q^* + iq^*_x, \quad D = 2i\zeta^2 I_2 - iq^* q^\top,
\end{align*}

where \( q = (q_1, q_2)^\top \) and where \( I_n \) is the \( n \times n \) identity matrix. This example does not fit the results found in Section 3 or Section 5. However, we show that similar results are found for this system: \( Q = \{ \zeta \in \mathbb{R} : \Omega(\zeta) \in i\mathbb{R} \} \subset \sigma_L \) and \( \Omega(Q) \subset i\mathbb{R} \cap \sigma_L \), as has been established for other examples.
The compatibility conditions are

\[ A_x = \beta^T C - B^I \gamma, \quad (86a) \]

\[ B_x = 2\alpha B^T - 2A\beta^T, \quad (86b) \]

\[ C_x = 2\rho C + 2\gamma A, \quad (86c) \]

\[ D_x = \gamma B^T - C\beta^T. \quad (86d) \]

As before, \( \Omega \) is found by separation of variables and satisfies

\[
\begin{pmatrix}
A - \Omega & B^T \\
C & D - \Omega I_2
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix} = 0,
\]

for nontrivial eigenvectors \( \phi = (\phi_1, \phi_2)^T \). Note that \( \Omega \) does not have the form (15). Instead, \( \Omega \) satisfies

\[
0 = \det \left( A - \Omega \quad B^T \\
C \quad D - \Omega I_2
\right) = \begin{cases}
(A - \Omega) \det \left( (D - \Omega I_2) - CB^T/(A - \Omega) \right), & \Omega \notin \sigma(A), \\
\det(D - \Omega I_2) \left( (A - \Omega) - B^T(D - \Omega I_2)^{-1} C \right), & \Omega \notin \sigma(D),
\end{cases}
\]

(88)

where \( \sigma(L) \) represents the spectrum of \( L \). As usual, \( \Omega(\zeta) \) defines a Riemann surface. In the genus-one case,\(^{32} \) it is represented by

\[ f(\zeta, \Omega) = (\Omega + 2i\zeta^2)(\Omega - 2i\zeta^2)^2 + (2\lambda_2 \zeta + \lambda_3)(\Omega - 2i\zeta^2) + \mu_0 = 0, \quad (89a) \]

\[ \lambda_2 = -i(q^T \ddot{q}_x - \dot{q}_x^T \ddot{q}), \quad (89b) \]

\[ \lambda_3 = q_x^T \ddot{q}_x + (q^T \ddot{q})^2, \quad (89c) \]

\[ \mu_0 = i|q_{1,x} q_2 - q_{2,x} q_1|^2. \quad (89d) \]
Since
\[
\det(D - \Omega I_2) = (\Omega - 2i\zeta^2)(A + \Omega),
\]
we use the second expression in (88) only when \(\Omega = 2i\zeta^2\) or \(A + \Omega = 0\). But \(\Omega = 2i\zeta^2\) satisfies (89a) only if \(q_2\) and \(q_1\) are proportional, and \(\Omega = A\) satisfies (89a) only if \(|q_1|^2 + |q_2|^2\) is constant.

In the first case, (83) reduces to two uncoupled NLS equations, for which the spectrum is known. In the second case, (83) reduces to two uncoupled linear Schrödinger equations, for which the spectrum is known. Therefore, we assume that \(D - \Omega I_2\) is invertible.

The eigenfunctions of (84) are
\[
v(x, t) = e^{i\Omega t} y_1(x) \left( a - (D - \Omega I_2)^{-1} Ca \right),
\]
where \(a \in \mathbb{C}\) is an arbitrary scalar. The scalar function \(y_1(x)\) is determined by substitution in the \(x\) equation (84):
\[
y_1' = (\alpha - \beta^\dagger(D - \Omega I_2)^{-1} C) y_1,
\]
so that
\[
y_1 = \hat{y}_1 \exp \left( \int (\alpha - \beta^\dagger(D - \Omega I_2)^{-1} C) \chi \right),
\]
where \(\hat{y}_1\) is a constant. Thus, \(\zeta \in \sigma_L\) provided that
\[
\left| \text{Re} \int (\alpha - \beta^\dagger(D - \Omega I_2)^{-1} C) \chi \right| < \infty
\]
for all \(x \in \mathbb{R}\). For periodic potentials and \(\zeta \in \mathbb{R}\), this becomes
\[
\text{Re} \langle \beta^\dagger(D - \Omega I_2)^{-1} C \rangle = 0.
\]

For \(\zeta \in \mathbb{R}\),
\[
A^\dagger = -A, \quad C^\dagger = -B^\dagger, \quad D^\dagger = -D, \quad (\beta^\dagger)^\dagger = -\gamma,
\]
where \(F^\dagger = (F^*)^\dagger\) is the conjugate transpose of \(F\). It follows that \(T\) defined by (84) is skew-adjoint and \(\Omega \in i\mathbb{R}\). Further,
\[
(D - \Omega I_2)^\dagger = -(D - \Omega I_2).
\]
It follows that

\[
\text{Re} \beta^\top (D - \Omega I_2)^{-1} C = \frac{1}{2} \left[ \beta^\top (D - \Omega I_2)^{-1} C + C^\dagger ((D - \Omega I_2)^{-1})^\dagger (\beta^\top)^\dagger \right]
\]

\[
= \frac{1}{2} \left[ \text{Tr} (\beta^\top (D - \Omega I_2)^{-1} C - B^\top (D - \Omega I_2)^{-1} \gamma) \right]
\]

\[
= \frac{1}{2} \left[ \text{Tr} ((D - \Omega I_2)^{-1} C \beta^\top - (D - \Omega I_2)^{-1} \gamma B^\top) \right]
\]

\[
= \frac{1}{2} \left[ \text{Tr} ((D - \Omega I_2)^{-1} (C \beta^\top - \gamma B^\top)) \right]
\]

\[
= \frac{1}{2} \left[ \text{Tr} ((D - \Omega I_2)^{-1} (-D \chi)) \right]
\]

\[
= -\frac{1}{2} \frac{\partial_x \det(D - \Omega I_2)}{\det(D - \Omega I_2)}
\]

\[
= -\frac{1}{2} \partial_x \log \det(D - \Omega I_2),
\]

so that

\[
\text{Re} \left< \beta^\top (D - \Omega I_2)^{-1} C \right> = 0,
\]

(99)

and \(\Omega(\zeta) \in i\mathbb{R}\) for \(\zeta \in \mathbb{R}\). This can be verified using the method described in Ref. 30.

The work above can be generalized to the Matrix NLS (MNLS) equation,

\[
iU_t + U_{xx} - 2\kappa U U^* U = 0,
\]

(100)

where \(U\) is an \(\ell_1 \times \ell_2\) matrix for \(\ell_1, \ell_2 \in \mathbb{N}\), and \(\kappa = -1\) and \(\kappa = 1\) correspond to the focusing and defocusing cases, respectively. The Lax pair for the MNLS equation is given by

\[
\Psi_x = \begin{pmatrix} -i\zeta I_{\ell_1} & Q \\ R & i\zeta I_{\ell_2} \end{pmatrix} \Psi = X \Psi,
\]

(101a)

\[
\Psi_t = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Phi = T \Psi,
\]

(101b)

\[
A = -2i\zeta^2 I_{\ell_1} - iQR, \quad B = 2\zeta Q + iQ_x, \quad C = 2\zeta R - iR_x, \quad D = 2i\zeta^2 I_{\ell_2} + iRQ.
\]

(101c)

Here, \(Q\) and \(R\) are \(\ell_1 \times \ell_2\) and \(\ell_2 \times \ell_1\) complex-valued matrices, respectively. The \(x\) equation may be written as a spectral problem,

\[
\zeta \Psi = \begin{pmatrix} iI_{\ell_1} \partial_x & -iQ \\ iR & -iI_{\ell_2} \partial_x \end{pmatrix} \Psi = L \Psi.
\]

(102)
$L$ is self-adjoint if $R^* = Q$, hence $\sigma_L \subset \mathbb{R}$ if $R^* = Q$. The compatibility conditions are the same as \((86d)\). The steps above apply in a straightforward but cumbersome manner to establish that $Q = \{\zeta \in \mathbb{R} : \Omega(\zeta) \in \mathbb{i} \mathbb{R}\} \subset \sigma_L$ and $\Omega(Q) \subset \mathbb{i} \mathbb{R} \cap \sigma_L$. Details are omitted here for brevity.

7 CONCLUSION

The stability spectrum and the Lax spectrum for solutions of many integrable equations on the whole line have been characterized for some time. The same level of understanding for the periodic problem does not exist. One reason the whole line problem is more straightforward to study is the ability to do spatial asymptotics to find the essential spectrum that contains the unbounded components of the spectrum. In this work, we have given an asymptotic characterization of all unbounded components of the Lax spectrum for a number of integrable equations, using \((76)\) (see Remark 4).

We provided two theorems (Theorems 1 and 2) with easily verifiable assumptions that establish that real Lax spectra corresponds to stable modes of the linearization for a number of equations in and not in the AKNS hierarchy. We applied the theorems to a number of examples. The methods described in this paper can be applied to other equations not mentioned in Sections 4 and 6. Some other examples include: the KdV equation, Hirota’s equation (or the mixed generalized NLS-generalized mKdV equation)\(^{23,34}\), the Modified Thirring Model,\(^{36}\) the $O_4$ nonlinear $\sigma$-model,\(^{37}\) the complex reverse space-time nonlocal mKdV equation,\(^{38}\) and the reverse space-time nonlocal generalized sine-Gordon equations.\(^{38}\)

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APPENDIX A: THE FLOQUET DISCRIMINANT

A common tool for characterizing the Lax spectrum for periodic potentials is the Floquet discriminant.5,12,14,15 The Floquet discriminant is typically approximated numerically as the eigenfunctions of the $x$ equation are unknown for generic potentials. In our framework, we have explicit expressions for the eigenfunctions (32). As $\Omega(\zeta)$ is defined by its square, (15) defines two different values of $\Omega$ for every value of $\zeta$ for which $\Omega(\zeta) \neq 0$. Hence (32) defines two linearly independent solutions of (74) except for when $\Omega(\zeta) = 0$. When $\Omega(\zeta) = 0$, only one solution is generated by (32) and a second solution is found using the method of reduction of order. The solution found by reduction of order is algebraically unbounded so it is not an eigenfunction. For $\Omega(\zeta) \neq 0$, the two eigenfunctions of (74) are

$$\phi_{\pm}(x, t) = e^{\pm \Omega t} y_{\pm}(x) \begin{pmatrix} -\hat{B}(x) \\ \hat{A}(x) - \Omega_{\pm} \end{pmatrix}.$$  

We use one choice of (32) and one choice of $y_1$. The following computations proceed similarly for the other choices. A fundamental matrix solution (FMS) of the $x$-equation of (74) is

$$M(x) = \begin{pmatrix} -\hat{B}(x)y_+(x) & -\hat{B}(x)y_-(x) \\ (\hat{A}(x) - \Omega_+)y_+(x) & (\hat{A}(x) - \Omega_-)y_-(x) \end{pmatrix},$$  

where dependence on $t$ has been omitted. The FMS normalized to the identity is given by

$$\tilde{M}(x; x_0) = M^{-1}(x_0)M(x).$$  

To simplify notation, we define

$$I_{\pm}(x; \zeta) = -\int \left( \hat{\alpha} + \frac{\hat{\beta}_C}{\hat{A} - \Omega_{\pm}} \right) x.$$  

In this section, we use Assumption A.1 instead of assuming that $\alpha \in i\mathbb{R}$.

**Assumption A.1.** $\hat{\alpha}$ and $\hat{\beta}_C/(\hat{A} - \Omega)$ are periodic in $x$ with the same period $P$ as the solution.

Under Assumption A.1, each of the integrands in (76) is $P$-periodic, and we may use any of the representations for $I$. It follows that

$$I_{\pm}(x + P; \zeta) = I_{\pm}(x; \zeta) + I_{\pm}(P; \zeta).$$  

Then

$$y_{\pm}(x + P) = y_{\pm}(x)e^{I_{\pm}(x; \zeta)}e^{I_{\pm}(P; \zeta)} = y_{\pm}(x)\Gamma_{\pm}(P),$$
and

$$\tilde{M}(x + P; x_0) = M^{-1}(x_0)M(x + P) = M^{-1}(x_0)M(x)\Gamma(P) = \tilde{M}(x; x_0)\Gamma(P),$$  \hspace{1cm} (A.7)

where

$$\Gamma(P) = \begin{pmatrix} \Gamma_+(P) & 0 \\ 0 & \Gamma_-(P) \end{pmatrix}$$  \hspace{1cm} (A.8)

is the transfer matrix. In order for solutions to be bounded in space, it must be that the eigenvalues of the transfer matrix have unit modulus. Thus,

$$\text{Re} \left( I_{\pm}(P; \zeta) \right) = 0.$$  \hspace{1cm} (A.9)

If Assumption A.1 holds, this is equivalent to (76). The Floquet discriminant is defined by

$$\Delta(\zeta) = \text{tr}(\Gamma(P)) = \Gamma_+(P) + \Gamma_-(P),$$  \hspace{1cm} (A.10)

and

$$\sigma_L = \{ \zeta \in \mathbb{C} : \text{IM}(\Delta(\zeta)) = 0 \text{ and } |\Delta(\zeta)| \leq 2 \}.$$  \hspace{1cm} (A.11)

Both definition (A.11) and (76) require numerical computation or the use of special functions. We prefer working with (76) directly, but we present the Floquet discriminant because of its popularity.