Remarks on contact structures and vector fields on isolated complete intersection singularities

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Abstract

Let \((X, 0)\) be an isolated complete intersection complex singularity \((X\) can also be smooth at 0). Let \(K\) be its link, \(X\) its canonical contact structure and \(\mathcal{D}_X\) the complex vector bundle associated to \(X\). We prove that the bundle \(\mathcal{D}_X\) is trivial if and only if the Milnor number of \(X\) satisfies
\[
\mu(X, 0) \equiv (-1)^{n-1} \text{ modulo } (n - 1)!.
\]
This follows from a general theorem stating that the complex orthogonal complement of a vector field in \(X\) with an isolated singularity at 0 is trivial iff the GSV-index of \(v\) is a multiple of \((n - 1)!\). We have also an application to foliation theory: a holomorphic foliation \(F\) in a ball \(\mathbb{B}_r\) around the origin in \(\mathbb{C}^3\) with an isolated singularity at 0, admits a \(C^\infty\) normal section (away from 0) iff its multiplicity (or local index) is even, and this happens iff its normal bundle in \(\mathbb{B}_r \setminus \{0\}\) is topologically trivial.

0 Introduction

Let \(X \subset \mathbb{C}^N\) be an affine complex analytic variety of dimension \(n > 1\), with an isolated complete intersection singularity at 0. It is well known that the diffeomorphism type of its link \(K = X \cap S_\varepsilon\) does not depend on the choices of the embedding of \(X\) in \(\mathbb{C}^N\) nor on the sphere \(S_\varepsilon\), provided this is small enough. Moreover, according to \([19]\) one has a canonical contact structure \(\mathcal{X}\) on \(K\), which is again independent of the embedding of \(X\) in \(\mathbb{C}^N\) and the choice of the sphere, up to contact-isomorphism. We refer to \(\mathcal{X}\) as the canonical contact structure on \(K\). Our interest in looking at this contact structure comes from \([4]\), where the authors use it to make interesting applications to the theory of surface singularities.

This contact structure corresponds to the complex sub-bundle \(\mathcal{D}_X\) of the tangent bundle \(T_K\) whose fiber at each point is the complex orthogonal complement in \(T(X \setminus \{0\})\) of the unit outwards normal vector field \(\tau\) of \(K\) in \(X\).

For instance, if \(n = 2\) then one has a nowhere-vanishing holomorphic 2-form \(\Omega\) around 0 in \(X\), which determines an \(Sp(1)\)-structure on the complex bundle \(T(X - \{0\})\) (see \([10]\)). If, as before, we denote by \(\tau\) the unit outwards normal vector field of \(K\) in \(X\), then the bundle \(\mathcal{D}_X\) is the trivial 1-dimensional complex bundle spanned by the vector field \(j \cdot \tau\), obtained by multiplying the vector \(\tau(x)\) by the quaternion \(j\) at each point of \(K\). The vector field \(i \cdot \tau\) is, up to scaling, the Reeb vector field of the canonical contact structure \(\mathcal{X}\).

In this work we give a necessary and sufficient condition for \(\mathcal{D}_X\) to be trivial when \(n > 2:\)

**Theorem 1.** The complex bundle \(\mathcal{D}_X\) that defines the canonical contact structure on \(K\) is \(C^\infty\) trivial as a complex vector bundle iff the Milnor number \(\mu(X, 0)\) of the germ \((X, 0)\) satisfies:
\[
\mu(X, 0) \equiv (-1)^{n-1} \text{ mod } (n - 1)!.
\]

\(^*\)Supported by CONACYT and DGAPA-UNAM, Mexico
So, for instance, for the quadric \( X = \{ z_0^2 + \cdots + z_n^2 = 0 \} \) the bundle \( D_X \) is trivial if and only if \( n = 2 \) or \( n \) is an odd number.

Theorem 1 is a consequence of Theorem 2 below, which is proved via classical obstruction theory. We recall that the GSV-index of a vector field \( v \) on \( X \), singular only at \( 0 \), equals the Poincaré-Hopf index of a continuous extension of \( v \) to a Milnor fibre of \( X \) at \( 0 \) (see [5, 3]).

**Theorem 2.** Let \( v \) be a nowhere-zero, continuous vector field on a neighbourhood of \( M \) in \( W \). Then the complex orthogonal complement of \( v \) in \( T(X \setminus \{0\}) \) is a \( C^\infty \) trivial complex bundle if and only if the GSV-index of \( v \) is a multiple of \( (n-1)! \).

Theorem 2 was announced in [17], with an outline of its proof. Here we give a complete, self-contained proof of this result. This is also very much indebted to [11], where similar arguments are used in relation with framed cobordism.

We work always in the category of topological spaces and continuous maps, so our proofs of theorems 1 and 2 actually discuss topological triviality of the vector bundles in question. But everything becomes automatically \( C^\infty \) because every continuous map between smooth manifolds can be approximated by a smooth map.

A holomorphic vector field \( v \) on \( X \), singular only at \( 0 \) defines a 1-dimensional holomorphic foliation on \( X^* = X \setminus \{0\} \). In various situations one is naturally lead to considering the normal bundle of the foliation, and sections of it. Theorem 2 above implies the following:

**Corollary 1.** Assume \( X \) has complex dimension 3. Let \( F \) be a holomorphic, locally free, 1-dimensional foliation on \( X \), singular only at \( 0 \), and let \( v(F) \) be its normal bundle in \( X^* \). Let \( v \) be a vector field on \( X \) tangent to \( F \) and singular only at \( 0 \). The following conditions are equivalent:

i) The GSV-index of \( v \) at \( 0 \) is even.

ii) The bundle \( v(F) \) admits a nowhere-zero \( C^\infty \) section.

iii) The bundle \( v(F) \) is \( C^\infty \) trivial.

When \( X \) is smooth at \( 0 \), the condition of \( F \) being locally free is always satisfied, so it can be dropped. For \( X \) smooth, the GSV-index is the usual local Poincaré-Hopf index of \( v \).

1 **On compact parallelizable manifolds**

Let \( W \) be a \( 2n \)-dimensional, \( n \geq 1 \), compact, connected manifold with non-empty boundary \( M \) and trivial tangent bundle \( TW \). Let \( F_0 \) be a trivialization of \( TW \) and use this to define a complex structure on \( TW \) and an isomorphism \( TW \cong W \times \mathbb{C}^n \). Let \( v \) be a nowhere-zero, continuous vector field on a neighbourhood of \( M \) in \( W \). It is well-known that \( v \) can be extended to the interior of \( W \) with isolated singularities, and the total number of singularities, counted with their corresponding local Poincaré-Hopf indices, is independent of the choice of the extension of \( v \) to the interior of \( W \).

This number is the total Poincaré-Hopf index \( \text{Ind}_{PH}(v;W) \) of \( v \) in \( W \).

**Lemma 1.1** If \( \text{Ind}_{PH}(v;W) \) is a multiple of \( (n-1)! \), then \( v \) can be completed to a continuous trivialization of the complex vector bundle \( TW|_M \). That is, there exist \( (n-1) \) continuous sections \( \alpha_2, \ldots, \alpha_n \) of \( TW|_M \), such that the set \( \{ v, \alpha_2, \ldots, \alpha_n \} \) defines a trivialization of \( TW|_M \).

**Proof.** Since \( W \) is parallelizable and has non-empty boundary, there is an immersion \( I : W \rightarrow \mathbb{R}^{2n} \), by the immersion theorem of Hirsch-Poenaru [15]. Thus one has an induced (Gauss-type) continuous map \( M \xrightarrow{\psi_v} S^{2n-1} \), defined by

\[
\psi_v(x) = \frac{DI(v(x))}{|DI(v(x))|},
\]
where \( D \) is the derivative.

By obstruction theory (see [18]), \( v \) can be extended to all of \( W \) minus one point, say \( x_0 \), around which \( I \) can be assumed to be an embedding. Hence the topological degree of \( \psi_v \) equals \( \text{Ind}_{PH}(v; W) \). In particular one has that some other vector field \( v' \) on a neighborhood of \( M \) in \( W \) is homotopic to \( v \) (through never-vanishing vector fields) if and only if \( \text{Ind}_{PH}(v'; W) = \text{Ind}_{PH}(v; W) \).

Now assume that \( \text{Ind}_{PH}(v; W) \) is a multiple of \( (n - 1)! \), i.e., \( \text{Ind}_{PH}(v; W) = t(n - 1)! \) for some integer \( t \). Notice that one has on \( W \) vector fields with all possible Poincaré-Hopf total indices and never-zero on \( M \). Let \( v_0 \) be such a vector field with index \( t \).

We recall (see [2]) that the homotopy group \( \pi_{2n-1}(U(n)) \) of the unitary group \( U(n) \) is isomorphic to \( \mathbb{Z} \), and it has a canonical generator that maps to the generator 1 of \( \mathbb{Z} \). Let \( \xi: \mathbb{S}^{2n-1} \to U(n) \) represent this generator and define a map

\[
\phi_{v_0}: M \to U(n),
\]

by the composition \( \phi_{v_0} = \xi \circ \psi_v \). Now, following [3] [10], twist the trivialization \( \mathcal{F}_v \) on the boundary \( M \) using the map \( \phi_{v_0} \); we get a new trivialization \( \mathcal{F} \) of \( TW|_M \). This means that at each point \( x \in M \) we change the basis of \( T_x W \) given by \( \mathcal{F}_v \) into its image by the linear map \( \phi_{v_0}(x) \in U(n) \). We claim that \( \mathcal{F} \) has \( v \) as one of its \( n \) sections, up to homotopy; this will complete the proof of the lemma.

To prove the above claim notice first that one has a map \( \psi_{v_0}: \mathbb{S}^{2n-1} \to U(n) \) defined similarly to \( \psi_v \) but now using \( v_0 \). The previous discussion implies that \( \psi_{v_0} \) has degree \( t \).

There is a fibration

\[
U(n-1) \hookrightarrow U(n) \longrightarrow \mathbb{S}^{2n-1},
\]

and an associated long exact homotopy sequence,

\[
\cdots \to \pi_{2n-1}(U(n)) \xrightarrow{p_*} \pi_{2n-1}(\mathbb{S}^{2n-1}) \longrightarrow \pi_{2n-2}(U(n-1)) \longrightarrow \pi_{2n-2}(U(n)) \longrightarrow \cdots \tag{1.2}
\]

We know that \( \pi_{2n-1}(\mathbb{S}^{2n-1}) \cong \mathbb{Z} \). Bott’s calculations in [2] tell us that:

i) \( \pi_{2n-1}(U(n)) \cong \mathbb{Z} \),

ii) \( \pi_{2n-2}(U(n-1)) \cong \mathbb{Z}/(n-1)! \),

iii) \( \pi_{2n-2}(U(n)) \cong 0 \) and \( p_* \) is multiplication by \( (n - 1)! \).

Thus each section of \( \mathcal{F} \) has index \( \lfloor t \cdot (n - 1)! \rfloor \) and the result follows. \( \square \)

2 On highly connected manifolds

Now we assume that the manifold \( W^{2n} \) of §1 has the homotopy type of a bouquet of \( n \)-spheres, \( n > 1 \). In this case one has the converse of [11].

**Lemma 2.1** Let \( v \) be a continuous section of \( TW|_M \) which can be completed to a trivialization of the complex bundle \( TW|_M \); i.e., there exist continuous sections \( \alpha_2, \ldots, \alpha_n \) of \( TW|_M \) such that the set \( \mathcal{F} = \{ v, \alpha_2, \ldots, \alpha_n \} \) defines a trivialization of \( TW|_M \) as a complex vector bundle. Then \( \text{Ind}_{PH}(v; W) \) is a multiple of \( (n - 1)! \).

**Proof.** We equip \( W \) with a triangulation compatible with the boundary \( M \), and we refer to \( \mathcal{F} \) as a complex framing on \( M \), meaning by this a trivialization of the complex bundle \( TW|_M \). We try to extend \( \mathcal{F} \) to the interior of \( W \) using the usual “stepwise” process: first to the 0-skeleton, then the 1-skeleton and so on, as far as we can.

According to [18], the obstructions to extending \( \mathcal{F} \) as a complex framing over the interior of \( W \) are elements in the relative cohomology \( H^*(W, M; \mathbb{Z}) \). In fact these obstructions are all cocycles
that represent, by definition, the Chern classes of \( W \) relative to the framing \( F \) on \( M = \partial W \). Thus they live in the even-dimensional relative cohomology of \( (W, M) \).

By Lefschetz duality one has \( H^i(W; M) \cong H_{2n-i}(W) \), hence all these groups vanish, except for \( i = n, 2n \), since \( W \) is assumed to have the homotopy of a bouquet of \( n \)-spheres.

Let us assume first that \( n \) is odd. Since Chern classes live in even dimensions, in this case the only possible obstruction to extending \( F \) to the interior of \( W \) is the top relative Chern class \( c_n(W, F) \in H^{2n}(W, M; \mathbb{Z}) \). By definition, this class is the obstruction to extending to the interior of \( W \) one of the sections that define \( F \), that we can take to be \( v \). Hence \( F \) can be extended to all of \( W \) minus one point, say \( x_o \), and \( \text{Ind}_{\partial W}(v; W) \) can be regarded as being both, the local Poincaré-Hopf index at \( x_o \) of the extension of \( v \) to \( W \setminus \{x_o\} \), and also the Lefschetz dual \( c_n(W, F)[W, M] \in H_0(W) \) of the Chern class \( c_n(W, F) \), where \( [W, M] \) is the fundamental cycle of the pair.

Since \( F \) is already extended to a trivialization of \( T(W \setminus \{x_o\}) \), one has that \( c_n(W, F) \) can be identified, by excision, with the Chern class of a small disc \( D_\varepsilon \) in \( W \) centered at \( x_o \), relative to the framing \( F \) on \( \partial D_\varepsilon \). Then the exact sequence [1.1] implies that \( c_n(W, F) \) is a multiple of \((n-1)!\), proving the lemma when \( n \) is odd.

Consider now the case \( n \) is even, say \( n = 2m \), so \( W \) has real dimension \( 4m \). We need:

**Lemma 2.2** The framing \( F \) on \( M \) extends to a trivialization \( \hat{F} \) of the complex bundle \( T(W \setminus T) \), where \( T \) is a torus \( S^n \times S^{n-1} \) embedded in the interior \( \hat{W} \) of \( W \).

**Proof.** Now there is a first possibly non-zero obstruction \( g(F) \in H^n(W, M; \mathbb{Z}) \) for extending \( F \) to the interior of \( W \). Let \( S_\hat{F} \in H_n(W) \) be the lchets dual of \( g(F) \).

By hypothesis \( W \) is simply connected. Hence Lemma 6 in [13] implies that \( S_\hat{F} \) can be represented by an \( n \)-sphere \( S_\hat{F} \) embedded in \( W \). We claim:

i) that \( S_\hat{F} \) is actually embedded in \( \hat{W} \) with trivial normal bundle, so the boundary of a tubular neighbourhood of \( S_\hat{F} \) is homeomorphic to a torus \( S^n \times S^{n-1} \); and

ii) the framing \( F \) extends to a complex framing \( \hat{F} \) on all of \( W \setminus S_\hat{F} \).

These two claims obviously prove the lemma [22]. We prove (ii) first, and then use (ii) to prove that \( S_\hat{F} \) is embedded with trivial normal bundle. We need:

**Lemma 2.3** The cohomology groups \( H^i(W \setminus S, M) \) vanish for \( i = 0, 1, \cdots, n-2, n+2, \cdots, 2n-2 \); i.e., for \( i \neq n \pm 1, 2n-1, 2n \).

Maybe this lemma can be improved, but this is all we need.

**Proof.** We will prove that \( H^i(W \setminus S, M) \) and \( H^i(W, M) \) are isomorphic in the above dimensions. The lemma then follows from the fact that \( W \) is (homotopically) a bouquet of \( n \)-spheres.

Consider the exact cohomology sequences of the pairs \( (W, M) \) and \( (W \setminus S_\hat{F}, M) \). The “Five Lemma” implies that \( H^i(W, M) \) is isomorphic to \( H^i(W \setminus S_\hat{F}, M) \) whenever the groups \( H^j(W) \) and \( H^j(W \setminus S_\hat{F}) \) are isomorphic for \( j = i-1 \) and \( j = i \). Then the exact sequence of the pair \( (W, W \setminus S_\hat{F}) \), together with Alexander duality and the fact that \( S_\hat{F} \) is an \( n \)-sphere, imply the lemma. □

Let us return to the proof of (ii). Let \( N \) be an open tubular neighbourhood of \( S_\hat{F} \) in \( W \) and consider a triangulation \( T \) of the compact manifold \( W \setminus N \), compatible with its boundary \( M \cup N \). Denote by \( T^{(i)} \) the corresponding \( i \)th-skeleton.

Now we start the process of trying to extend \( F \) from \( M \) to all of \( W \setminus N \), step by step. By the previous lemma, there are no obstructions up to dimension \((n-2)\), and \((n-1)\) is odd, so we can extend \( F \) to \( M \cup T^{(n-1)} \) because the obstructions appear in even dimensions.
Notice one has a commutative diagram:

\[
\begin{array}{ccc}
H^n(W, W \setminus S_F) & \xrightarrow{j^*} & H^n(W, M) & \xrightarrow{k^*} & H^n(W \setminus S_F, M) \\
\downarrow L & & \downarrow P & & \\
H_n(S_F) & \xrightarrow{i_*} & H_n(W) & & \\
\end{array}
\] (2.4)

where the horizontal arrows are induced by the inclusions, \(L, P\) denote Lefschetz and Poincaré duality and the first row is exact in the middle. This implies that the obstruction class \(\varrho(F)\) is in the image of \(j^*\), so it is mapped to 0 by \(k^*\), thus implying that there is no obstruction to extending \(F\) to a complex framing \(\hat{F}\) on the \(n\)-skeleton of the triangulation on \(W \setminus N\).

Now, \(n + 1\) is odd, so we can extend the framing over \(M \cup T^{n+1}\), and then the previous lemma grants we can continue the process up to dimension \(2n - 2\), and therefore to dimension \(2n - 1\) since this is odd.

We finally meet an obstruction in the top dimension \(2n\). Regarded in homology, this means the obstruction has support at isolated points, say \(q_1, \cdots, q_r\) which we can assume to be in the interior of \(W \setminus N\). We can now retract \(N\) to \(S_F\) smoothly, thus extending \(F\) to all of \(W \setminus (S_F \cup \{q_1, \cdots, q_r\})\). Finally, we can move these points in \(W\) by an isotopy so that they become contained in the sphere \(S_F\), and we arrive to statement (ii).

To prove the first claim we notice that, since \(W\) is parallelizable, Lemma 7 in [14] implies that \(S_F\) is embedded in \(\hat{W}\) with trivial normal bundle if and only if its self-intersection number \(S_F \cdot S_F\) is 0, that is if and only if

\[
\varrho(F) \cup \varrho(F) = 0.
\]

Let us prove that this happens. Recall we have the trivialization \(F_o\) of \(TW\) that we used to define a complex structure on \(TW\). Of course the Chern classes \(c_i(W, M; F_o)\) of \(W\) relative to the framing \(F_o\) on \(TW|_M\) are all zero.

In general one has that given two complex framings \(\beta, \beta'\) on the boundary \(M\), each defines relative Chern classes \(c_i(W; \beta), c_i(W; \beta')\) in \(H^{2i}(W, M)\) for all \(i \geq 0\) even, and in each dimension their difference is a contribution of the boundary

\[
H^{2i-1}(M) \xrightarrow{\delta^*} H^{2i}(W, M),
\]

given by the difference cocycle (see [18], also [14 23]).

In our case, since the classes \(c_i(W, M; F_o)\) are all zero, this implies that \(c_m(W, M; F)\) is a coboundary, i.e., \(\varrho(F) = \delta^*(a)\) for some \(a \in H^{2m-1}(M)\). Thus one has:

\[
\varrho(F) \cup \varrho(F) = \delta^*(a) \cup \varrho(F) = j^*\delta^*(a) \cup \varrho(F) = 0,
\]

where \(j^*: H^*(W, M) \to H^*(W)\) is induced by the inclusion, proving lemma 2.2 \(\square\)

Let us now complete the proof of lemma 2.1 when \(n\) is even, \(n = 2m\).

From the previous lemma we know that \(F\) is a complex framing that extends \(F\) to all of \(W\) minus the interior \(\text{Int} \widehat{T}\) of the solid torus \(\widehat{T} \cong S^n \times \mathbb{B}^n\), which is a tubular neighbourhood of the \(n\)-sphere \(S_F\) in \(W\). Set \(T = \partial \widehat{T}\).

We know, essentially by definition, that

\[
c_n(W; F)[W, M] = \text{Ind}_{\text{PH}}(v; W),
\]

where \([W, M]\) is the fundamental cycle of the pair \((W, M)\), and

\[
c_n(W; F)[W, M] = c_n(\widehat{T}, \hat{F})(\widehat{T}, T)
\]

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because \( \hat{F} \) extends \( F \). Let us prove that the latter integer is a multiple of \((n-1)!\).

Let \( F_o \) be the trivialization of \( TW \) used to determine an almost-complex structure on \( W \). Since \( \pi_n(U(n)) = 0 \) (see [4]), \( n > 1 \), we can assume that \( F_o \) and \( \hat{F} \) coincide, up to homotopy, over a parallel \((S^n \times *)\) of \( T \), where \( * \) is a point in \( \partial B^3 \).

Recall that, following [3], we can think of the framing \( \hat{F}|_T \) as being obtained from \( F_o|_T \), twisting it by a map \( f : T \rightarrow U(n) \). This map is, at each point \( z \) of \( T \), the linear transformation that carries the base of \( Tw \) given by \( F_o(z) \) into the one given by \( \hat{F}(z) \). In the sequel, we identify the framing \( \hat{F}|_T \) with the map \( f \).

Notice that \( T \cong S^n \times S^{n-1} \) has a CW-decomposition as \( e^{0} \cup e^{n-1} \cup e^n \cup e^{2n} \), with \( S^n = e^0 \cup e^n \) and \( S^{n-1} = e^0 \cup e^{n-1} \).

Now define a complex framing \( \beta \) on \( T \) by

\[
\beta(x, y) = \hat{F}(x_o, y),
\]

where \( x_o \) is the projection of \( e^0 \) to \( S^n \) and we are thinking of \( \hat{F}|_T \) as being the map \( f \). Then \( \beta \) and \( \hat{F}|_T \) agree over \( e^{0} \cup e^{n-1} \cup e^n \), by construction, and therefore they differ by a map \( \partial e^{2n} \rightarrow U(n) \), i.e., by an element in \( \pi_{2n-1}(U(n)) \).

We claim that one has \( c_n(T; \beta)|_T = 0 \). For this, it is convenient to think of this integer as being the degree of \( \beta \), i.e., if we write \( \beta \) as the \( n \)-frame \( \{\xi_1, \ldots, \xi_n\} \), where the \( \xi_i \) are vector fields, then \( c_n(T; \beta)|_T = \text{Ind}_PH(b_1 ; T) \) for all \( i = 1, \ldots, n \). Take one of these vector fields, say \( b_1 \). Notice that the framing \( F_o \) provides an isomorphism \( TW \cong W \times \mathbb{C}^n \). Restricting this to \( T \) and projecting into \( \mathbb{C}^n \) one gets a well defined map \( \hat{\beta} = p \circ b_1 \),

\[
\begin{array}{c}
T \xrightarrow{b_1} T^\# \xrightarrow{\hat{\beta}} S^{2n-1}
\end{array}
\]

whose degree equals \( \text{Ind}_PH(b_1 ; T) \). To prove that this map has degree 0 it is enough to show that \( \hat{\beta} \) extends to a map \( T \rightarrow S^{2n-1} \) (see for instance [6]). This follows because \( \beta \) is constant on \( S^n \times \{\ast\} \) and \( \hat{\beta}|_{e^0 \cup e^{n-1}} \) is nullhomotopic because \( S^{2n-1} \) is \((2n-2)\)-connected.

Therefore we have that \( c_n(T; \beta)|_T = 0 \) and, by construction, \( \beta \) differs from \( \hat{F} \) on \( T \) by an element in \( \pi_{2n-1}(U(n)) \). The result now follows from the exact sequence [12] where \( p^* \) is multiplication by \((n-1)!\). □

3 Proof of the theorems

We now take the manifold \( W \) to be a Milnor fiber of \( X \). More precisely, we assume the \( n \)-dimensional complete intersection germ \((X, 0)\) is defined by a reduced function

\[
f = (f_1, \cdots, f_k) : (U, 0) \rightarrow (\mathbb{C}, 0),
\]

with a critical point at 0, where \( U \) is an open neighbourhood of 0 in \( \mathbb{C}^{n+k} \), and we let \( W = f^{-1}(t) \cap B_\varepsilon \), where \( B_\varepsilon \) is a ball around 0 \( \in \mathbb{C}^{n+k} \) of sufficiently small radius \( \varepsilon > 0 \), and \( t \) is a regular value with \( |t| \) sufficiently small with respect to \( \varepsilon \). We are assuming also that 0 is an isolated singularity in \( X \), which means it is an isolated critical point of \( f \) in \( X \).

We know from [13] (see also [12]) that \( X \) is parallelizable and it has the homotopy type of a bouquet of \( \mu = \mu(X, 0) \) spheres of dimension \( n \), for some integer \( \mu > 0 \) which is known as the Milnor number of \((X, 0)\). In fact one has more: the tangent bundle \( TW \) is naturally equipped with a complex structure, since \( W \) is defined by a holomorphic function, and one has:

\[
TW|_W \cong TW \oplus \nu W,
\]
as $C^\infty$ vector bundles, where $\nu W$ is the normal bundle of $W$. The bundle $\nu W$ is canonically trivialized by the gradient vector fields $(\nabla f_1, \cdots, \nabla f_k)$, where

$$\nabla f_i = \left( \frac{\partial f_i}{\partial z_1}, \cdots, \frac{\partial f_i}{\partial z_{n+k}} \right).$$

This implies that $TW$ is stably trivial as a complex vector bundle (see [10]), i.e., its tangent bundle plus a trivial bundle is trivial. Thus, by [10], it is actually trivial as a complex bundle, because $W$ is connected with non-empty boundary and therefore $H^{2n}(W) \cong 0$.

Thus we are in the situation envisaged in sections 1 and 2 above, and Theorem 2 follows from lemmas 1.1 and 2.1, together with the fact that the Thom isotopy theorem (see for instance [1]) allows us to identify the link $K = X \cap \partial B_c$ with the boundary of $W$ (see [14, 12]). □

Now take $v$ to be a radial, outwards-pointing vector field on $X$. Then its GSV-index equals its Poincaré-Hopf index in $W$, where we are identifying $K$ with the boundary of $W$. In this case $v$ is everywhere transversal to $\partial W$, pointing outwards. Hence, by the theorem of Poincaré-Hopf for manifolds with boundary, its index in $W$ is $\chi(W)$, the Euler-Poincaré characteristic. By [14, 7] (see also [12]), this equals $1 + (-1)^n \mu(X, 0)$, since $W$ has the homotopy type of a bouquet of $\mu(X, 0)$ spheres of dimension $n$. Hence Theorem 2 implies

$$1 + (-1)^n \mu(X, 0) \equiv 0 \mod (n - 1)!,$$

and we arrive to Theorem 1. □

4 Proof of Corollary 1 and an example

Now we prove Corollary 1. Notice first that $X$ is locally a cone over its link $K$ with vertex at 0, by [14]. Hence in statements (ii) and (iii) of Theorem 3 we can restrict the discussion to the link.

The equivalence between statements (i) and (iii) is immediate from Theorem 2, and it is obvious that (iii) implies (ii), so we only must prove that (ii) implies (iii). Let $\xi$ be a never-zero continuous section of the normal bundle $\nu(F)$. This spans a 1-dimensional continuous complex line sub-bundle $L$ of $\nu(F)$. The bundle $L$ is trivial iff $\nu(F)$ is trivial. But $K$ is 2-connected, by [14]. Hence every complex line bundle over $K$ is trivial.

To finish this note, we give an example. Let $X$ be a hypersurface in $\mathbb{C}^{2n}$ defined by some function $f : (\mathbb{C}^{2n}, 0) \rightarrow (\mathbb{C}, 0)$. Then the Hamiltonian vector field

$$v = \left( \frac{\partial f}{\partial z_2}, -\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_4}, -\frac{\partial f}{\partial z_3}, \cdots, \frac{\partial f}{\partial z_{2n}}, -\frac{\partial f}{\partial z_{2n-1}} \right)$$

is obviously tangent to $X^*$, since $df(v) \equiv 0$ everywhere. This implies also that $v$ is tangent to all fibers of $f$. Hence its GSV-index is 0. Thus, by Theorem 2, the normal bundle of the holomorphic foliation that $v$ spans on $X^*$ is topologically trivial.

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