BEYOND THE “PRINCIPLE OF LOCAL SYMMETRY”:
DERIVATION OF A GENERAL CRACK PROPAGATION LAW

Jennifer Hodgdon
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, NY 14853
James P. Sethna†
Laboratory of Applied Physics, Technical University of Denmark, DK-2800 Lyngby, DENMARK, and NORDITA, DK-2100 Copenhagen Ø, DENMARK

(January 30, 1992)

Submitted to the International Journal of Solids and Structures

Abstract

We derive a general crack propagation law for slow brittle cracking, in two and three dimensions, using symmetry, gauge invariance, and gradient expansions. Our derivation provides explicit justification for the “principle of local symmetry,” which has been used extensively to describe two dimensional crack growth, but goes beyond that principle to describe three dimensional crack phenomena as well. We also find that there are new materials properties needed to describe the growth of general cracks in three dimensions, besides the fracture toughness and elastic constants previously used to describe cracking.

I. Introduction

There are many aspects of the problem of crack growth that have received a
lot of attention recently. For instance, there has been much interest in dynamic fracture\textsuperscript{1} and the accompanying crack bifurcation\textsuperscript{2,3} and other instabilities.\textsuperscript{4} The transition between failure due to percolation of a network of many small cracks and failure due to a single dominating crack has also been explored,\textsuperscript{5} as well as the transition between brittle and ductile cracking.\textsuperscript{6} Pattern formation in multiple cracking\textsuperscript{7} has also been of interest. In light of all the interest in these rather complex phenomena of fracture, it is somewhat surprising to find that little is known about the growth laws for even slow-growing, simple (not multiple) three dimensional cracks, though there has been some work done on calculating the paths of cracks in two\textsuperscript{8–11} and three\textsuperscript{12} dimensions, and many measurements and calculations of the crack velocity for simple two-dimensional geometries.\textsuperscript{13–16}

The problem of finding a growth law for cracks would seem to be of fundamental interest; so, in this paper, we apply the standard tools of theoretical physics—gradient expansions, symmetry, and gauge symmetry—to find the most general possible growth law for a three dimensional crack growing slowly in a homogeneous, isotropic medium. Since it is possible to make precise numerical computations of the elastic fields for arbitrary three-dimensional geometries, with today’s computers, in a matter of hours, we do not consider the related problem of finding the stress state of the material containing the crack, but consider it to be completely known. We also compare the crack growth law we
derive here to previously derived and measured properties of cracks in two and three dimensions. In a second paper, we will discuss the detailed behavior of cracks growing under this law, using linear stability analysis as well as numerical simulations; in a third paper we will discuss experiments designed to measure material-dependent parameters appearing in the crack growth law.

II. Simplifications

We begin by simplifying the problem of crack propagation using length and time scale considerations. First, we smooth our crack problem over the length scale \( \ell_s \) which characterizes the size of inhomogeneities and anisotropies in the material containing the crack. (For example, in a glass, \( \ell_s \) is a few atomic sizes; in a polycrystal, it is the grain size; in concrete, it is the size or distance between the pebbles it contains.) Although for a single crystal, \( \ell_s \) is as large as the body containing the crack, for many situations of practical relevance, \( \ell_s \) is much smaller than the size of the body. In those cases, we can smooth the crack problem over \( \ell_s \) without losing much information, making the crack a smooth surface, and the material containing the crack continuous, homogeneous, and isotropic.

A second length scale in crack propagation problems arises because every material has some stress above which it fails to have linear elastic properties (e.g. begin to flow, with plastic or viscous behavior; emit dislocations; break bonds; or
have a martensitic transformation). For some materials, this stress is very low, and there is no linear elastic regime at all. For others, linear elasticity is valid except very near the crack tip, where the stress is much higher than in the bulk of the body. For these materials, there is a length scale \( \ell_{nl} \) which characterizes the size of the non-linear process zone around the crack tip; \( \ell_{nl} \) can range from a few angstroms in glass to tens of centimeters in concrete. In this analysis, we consider only materials for which \( \ell_{nl} \) is small compared to the length of the crack and the size of the body, so that the bulk of the material can be considered linear elastic. This work, then, describes materials usually considered linear and brittle, as well as materials exhibiting viscoelasticity, plasticity, and martensitic transformation toughening, as long as the length scale for these behaviors is sufficiently small. In principle, the non-linear properties of such materials could be included in a later version of this work to extend its applicability to smaller length scales.

A third length scale relevant to crack propagation is associated with the degree of translational invariance along the crack front. For many crack systems studied in the past, every plane perpendicular to the crack front is equivalent, which means that the problems can be considered two dimensional. On the other hand, most practical crack problems are not two dimensional, but instead have crack front curvature or stresses which vary along the crack front. If this is the case, then there is a length scale, which we call the dimensional crossover
length $\ell_{dc}$, above which the problem is three dimensional; $\ell_{dc}$ is either one of the geometric lengths associated with the crack geometry (such as the radius of curvature of the crack front), or is associated with the stress gradient: $\ell_{dc} \approx \frac{\sigma}{\nabla\sigma}$.

For this work, we assume that $\ell_{dc}$ is large, though in general not as large as the size of the body containing the crack, and we expand in powers of quantities which are inversely proportional to $\ell_{dc}$ (i.e. gradients).

Finally, we also simplify the crack propagation problem by considering cracks which are growing slowly enough that inertial and relativistic (relative to the sound, not light, velocity) effects are unimportant. Some of these effects could be included in future work, but the present analysis suffices for cracks which arrest after growing a certain distance, such as when a wedge is driven into a crack; cracks which grow at a constant speed, such as under constant displacement loading; and cracks which may eventually speed up, but which are currently growing slowly, as in the cases of fatigue cracks, sub-critical cracking, and the first stages of growth under constant force loading.

III. Relevant variables

Although the knowledge of length scales from the previous section has simplified our problem to a nearly two dimensional, isotropic, homogeneous, linear-elastic, continuous bulk medium with a smooth crack, at first glance it appears that there are still many variables which could influence the propagation of the
crack; for example, the load on the surface of the body, type of material, temperature, ambient atmosphere, and the stress and fracture history. However, we are concerned here with the propagation of a crack given the elastic fields in the body, not the precise conditions that produced those fields. Also, many variables, such as stress history and temperature, can be included implicitly in the materials constants, which we also assume are known. This means that for this work, the variables of relevance are the elastic fields near the crack tip, materials constants, and the current configuration of the crack.

Now, it is well known that for cracks in linear elastic media, the stress field near the tip of the crack—which is the only area we expect to influence crack growth—obeys a \( \sim \frac{1}{\sqrt{r}} \) power law in the distance \( r \) from the crack tip in the plane. There are three modes of cracking (see figure 1a, b, and c); each has a characteristic known angular dependence, which can be written\(^{17}\)

\[
\sigma_{ij}(r, \theta) = \frac{K_\alpha}{\sqrt{2\pi r}} f_{ij}^\alpha(\theta),
\]

where \( \theta \) is the angle made with \( \hat{n} \), the direction of cracking (see figure 1d), and the three \( K_\alpha \) are the mode I, II, and III Stress Intensity Factors (SIFs), numbers characterizing the strength of the stress singularity for each cracking mode. There are also similar expressions for the displacement field near the crack (see reference 17). From equation 1, we can see that the relevant information from the elastic field for a two-dimensional crack propagation problem, instead of being given by a displacement vector at every point in the plane, is reduced
to just three numbers, the SIFs. In weakly three dimensional problems, each non-equivalent plane has three SIFs which characterize the stress, so that the relevant information from the elastic field is given by three numbers at each point on the crack front, instead of a displacement vector at every point in the body.

Now, the configuration of the crack could be specified by giving equations for the two surfaces of the crack, but it can also be reduced to a smaller amount of information. This comes about because part of the information on the crack geometry is contained in the elastic fields, and the part of the crack surface far from the crack tip cannot influence the growth of the crack beyond affecting the elastic fields. So, the most relevant information about the crack geometry, aside from the elastic field, is given by the crack front curve, $\vec{x}(\lambda)$, and the vector $\hat{n}(\lambda)$ giving the current direction of crack growth, where $\lambda$ parameterizes the crack front curve. In figure 1d we show the three unit vectors associated with this description: $\hat{t}(\lambda) \equiv \frac{\partial \vec{x}}{\partial \lambda}$, the tangent to the curve, with $s$ the arc length; $\hat{n}(\lambda)$, the direction of crack growth, perpendicular to $\hat{t}(\lambda)$ (so that $\hat{t}(\lambda)$ and $\hat{n}(\lambda)$ define the crack plane at point $\lambda$), and $\hat{b}(\lambda) \equiv \hat{t}(\lambda) \times \hat{n}(\lambda)$, the normal to the local crack plane. Note that since the material containing the crack is isotropic, the coordinate system defined by these unit vectors is the only one physically relevant to crack growth, and all other quantities (such as the SIFs)
Fig. 1 (a) A Mode I crack (primary stress is $b-b$). (b) A Mode II crack (primary stress is $b-n$). (c) A Mode III crack (primary stress is $b-t$). The local stress field around a crack can always be decomposed into a linear sum of the three modes; the decomposition can be found from the modes’ different symmetry properties (see appendix B). (d) The vectors associated with a point on the crack front: $\hat{t}$ is the tangent to the crack front curve; $\hat{n}$, perpendicular to $\hat{t}$ and in the crack plane, is the direction of crack growth; $\hat{b} \equiv \hat{t} \times \hat{n}$ is the normal to the crack plane.
are understood to be defined in this coordinate system (see figure 1). Also, in
two dimensional cracks, \(\hat{t}, \hat{n},\) and \(\hat{b}\) are all constant along the crack front.

IV. The crack growth law

Now we are ready to derive a crack growth law in our relevant variables:
materials constants, the SIFs \(K_\alpha(\lambda),\) and the unit vectors \(\hat{n}(\lambda), \hat{t}(\lambda),\) and \(\hat{b}(\lambda).\)
That is, we are ready to derive an expression for the time evolution of the crack
front curve \(\vec{x}(\lambda)\) as an expansion in \(\frac{\partial}{\partial s} \equiv (\frac{\partial s}{\partial \lambda})^{-1} \frac{\partial}{\partial \lambda},\) the gradient along the
crack curve, of the relevant variables. (In two dimensions, \(\frac{\partial}{\partial s} = 0.)\) Noting that
the crack surface is smooth for all time, we know that the time derivative of \(\vec{x}\)
must lie in the crack plane, so we can write

\[
\frac{\partial \vec{x}(\lambda, t)}{\partial t} = v(\lambda, t)\hat{n}(\lambda, t) + w(\lambda, t)\hat{t}(\lambda, t), \tag{2}
\]

where \(v\) is the crack velocity, and \(w\) is a non-physical function which can be
chosen freely to determine how \(\lambda\) parameterizes the physical crack surface. That
is, a particular choice of \(w\) determines a gauge for \(\lambda\) (see appendix A); in two
dimensions, we take \(w = 0.\)

To determine the growth of the physical crack, then, we need to find \(v(\lambda, t)\)
and \(\hat{n}(\lambda, t)\) in equation 2, in terms of the relevant variables. Noting that \(\hat{n}(\lambda, 0)\)
is given, this means that to find \(\hat{n}(\lambda, t),\) we need to find the time derivative \(\frac{\partial \hat{n}}{\partial t}.\)
Now, since \(\hat{n}\) is a unit vector, its time derivative must be perpendicular to itself;
also, by definition $\hat{n}$ is perpendicular to $\hat{t} \equiv \frac{\partial \vec{x}}{\partial s}$. This gives us a constraint on the equation of motion for $\hat{n}$, obtained by setting $\frac{\partial}{\partial t}(\hat{n} \cdot \hat{t}) = 0$:

$$\frac{\partial \hat{n}}{\partial t} \cdot \hat{t} = -\frac{\partial v}{\partial s} - w \frac{\partial \hat{t}}{\partial s} \cdot \hat{n}. \quad (3)$$

Note that in two dimensions the right hand side of this equation vanishes, since $\frac{\partial}{\partial s} = 0$.

Another constraint on $\frac{\partial \hat{n}}{\partial t}$, and one on the crack velocity $v$ are obtained from symmetry. We consider symmetry operations, centered at some point on the crack front, which leave the unit vectors at that point fixed and reflect or rotate the material, preserving the physical properties that $\hat{t}$ is the tangent to the crack front curve and $\hat{n}$ is the direction of cracking at that point. (This is equivalent to leaving the material fixed and transforming the coordinates.) There are two such independent operations: (a) $180^\circ$ rotation about $\hat{n}$, and (b) reflection in the $n-t$ plane (the crack plane) (see figure 2). Under both of these operations, the physical law for $v$ must remain unchanged; that for $\frac{\partial \hat{n}}{\partial t} \cdot \hat{b}$ must change sign under both operations (see figure 2).

Let us now examine the case of a crack in two dimensions, where the only non-vector quantities we can form are combinations of the SIFs and material constants. Under symmetry operation (a), the material constants, $K_I$, and $K_{III}$ remain the same, while $K_{II}$ changes sign; under operation (b), both $K_{II}$ and $K_{III}$ change sign, and everything else remains the same (see appendix B). Since
Fig. 2 Transformations (a) rotation about \( \hat{n} \) and (b) reflection in the \( n-t \) plane on (c) the untransformed crack. To make a physical crack growth law, the crack velocity must be unchanged in both cases (if it changed sign, the crack would heal); \( \frac{\partial \hat{n}(\lambda,t)}{\partial t} \cdot \hat{b} \) must change sign in both cases, since the direction of \( \hat{b} \) is unaffected by the transformation, while the \emph{physical} normal to the crack plane shifts to the opposite direction.
the three SIFs have different transformation properties, and only $K_{II}$ transforms like $\frac{\partial \hat{n}}{\partial t} \cdot \hat{b}$, we see that the two-dimensional crack growth law must have the form:

$$\frac{\partial \vec{x}}{\partial t} = v \hat{n} \quad (4)$$

$$\frac{\partial \hat{n}}{\partial t} = -fK_{II} \hat{b},$$

where both $v$ and $f$ are functions of materials constants, $K_1$, $K_{II}^2$, and $K_{III}^2$. (The dependence of $f$ and $v$ on $K_{II}$ and $K_{III}$ must be quadratic to insure invariance under both symmetry operations.) The minus sign makes $f > 0$ correspond to the observed direction of crack growth under mode II loading; this is discussed in the next section.

In three dimensions, the gradient $\frac{\partial}{\partial s}$ is not strictly zero, and there are therefore non-vector quantities besides the SIFs and materials constants which can be formed from the relevant variables. Up to first order in $\frac{\partial}{\partial s}$, these are listed, along with their transformation properties, in table 1 of appendix B.

From the transformation properties of these quantities, and from the discussion in appendix A on gauge invariance, we can see that the physical crack growth law, to first order in $\frac{\partial}{\partial s}$, has the general three-dimensional form:

$$\frac{\partial \vec{x}}{\partial t} = v \hat{n} + w \hat{t}$$

$$\frac{\partial \hat{n}}{\partial t} = -\left[ \frac{\partial v}{\partial s} + w \frac{\partial \hat{t}}{\partial s} \cdot \hat{n} \right] \hat{t} +$$

$$\left[ -fK_{II} + g_1K_{III} \frac{\partial K_1}{\partial s} + g_{II}K_{II}K_{III} \frac{\partial K_{II}}{\partial s} + g_{III} \frac{\partial K_{III}}{\partial s} + h_{tb} \frac{\partial \hat{t}}{\partial s} \cdot \hat{b} + h_{nt}K_{II} \frac{\partial \hat{n}}{\partial s} \cdot \hat{t} + (h_{nb}K_{II}K_{III} + w) \frac{\partial \hat{n}}{\partial s} \cdot \hat{b} \right] \hat{b}, \quad (5)$$
where the $f$, $g_\alpha$, and $h_{ij}$ are functions of materials constants, $K_1$, $K_{II}^2$, and $K_{III}^2$; and $v$ is a function of materials constants, $K_1$, $K_{II}^2$, $K_{III}^2$, $(K_{II}K_{III}\frac{\partial K_1}{\partial s})$, $(K_{III}\frac{\partial K_{II}}{\partial s})$, $(K_{II}\frac{\partial K_{III}}{\partial s})$, $\dot{b}$, $(\frac{\partial n}{\partial s} \cdot \dot{t})$, and $(K_{III}\frac{\partial n}{\partial s} \cdot \dot{b})$, since these are all the quantities up to first order in $\frac{\partial}{\partial s}$ which are invariant under both symmetry operations.

V. The undetermined functions in the crack growth laws

Now, in equations 4 and 5 we have general forms for the crack growth law for two and three dimensional cracks. These equations contain many unspecified functions, which must be determined from considerations other than the symmetry considerations we used to find the general forms. Perhaps the most important of these functions is the crack growth velocity, $v$, a function of materials constants, $K_1$, $K_{II}^2$, and $K_{III}^2$ in two dimensions. This function has been measured for mode I cracks\textsuperscript{13} and usually has the form shown in figure 3a, with zero velocity for $K_1$ below some value $K_{1c}$, which depends on the material, and a monotonically increasing velocity above $K_{1c}$. This schematic form for the velocity has also been found in a theoretical calculation for a viscoelastic system.\textsuperscript{16} However, both in theory\textsuperscript{18–21} and in very clean experimental systems,\textsuperscript{19–21} the crack velocity has the form in figure 3b, with a negative velocity (crack healing) when $K_1 < K_{1c}$. This means that the crack velocity is a continuous function which passes through zero at $K_{1c}$, so that for SIFs near $K_{1c}$, (i.e. small crack
velocities), we can expand \( v \) as \( v(K_I) \approx v_0 \frac{K_I - K_{Ic}}{K_{Ic}} \), with \( v_0 \) a material dependent constant. For modes II and III cracks, as well as mixed mode cracks, since the elastic energy released per unit area of crack surface (the “energy release rate”) is proportional to \( (K_I^2 + K_{II}^2 + \frac{K_{III}^2}{(1-\nu)}) \), where \( \nu \) is Poisson’s ratio, we expect that a crack velocity function valid for all modes of cracking can be expanded as:

\[
v(K_I, K_{II}, K_{III}) \approx v_0 \frac{K - K_c}{K_c},
\]

where \( K \equiv (K_I^2 + K_{II}^2 + \frac{K_{III}^2}{(1-\nu)})^{\frac{1}{2}} \). Also, note that for fatigue cracking, where our growth laws must still hold (on time scales long compared to the load cycle), the crack velocity generally does not go to zero sharply at \( K_c \), but has a more gradual turn-on behavior (see figure 3a).

Now, we saw in the previous section that the crack velocity in three dimensions can also depend on gradient quantities, besides the SIFs; the dependence of the crack velocity on these quantities has not been measured, to our knowledge. However, there is no reason to suppose that the dependence on these quantities has special behavior (e. g. zero-crossing or very strong dependence on SIFs) near \( K_c \), the value of the SIF where the crack velocity becomes positive. So, for small velocities, where \( K \approx K_c \), we can approximate the dependence of the crack velocity on these quantities by a constant function (i. e. its value at \( K_c \)).
Fig. 3  (a) Form of the crack velocity $v$ as a function of stress intensity factor $K$ in ordinary crack experiments, where the crack velocity is zero below a critical value $K_c$ of the stress intensity factor, and then has a sharp turn-on. The dotted line shows the behavior under fatigue, where the crack velocity increases more gradually. (b) Form of the crack velocity in very clean experiments, where crack healing can take place.
Similarly, we expect that the seven functions \( f, g_\alpha, \) and \( h_{ij} \) in equations 4 and 5, which are allowed by symmetry to be functions of materials constants, \( K_I, K_{II}^2, \) and \( K_{III}^2, \) can be approximated as constants when the velocity is small. This means that to find the material-specific form of the crack growth law, for small velocities, it is a reasonable approximation to measure only the linearized dependence of the crack velocity on \( K, \) and the constant parts of \( f, g_\alpha, \) and \( h_{ij}. \)

VI. Predictions of the crack growth laws

Let us now examine the two dimensional crack growth law, equation 4:

\[
\frac{\partial \vec{x}}{\partial t} = v \hat{n} \\
\frac{\partial \hat{n}}{\partial t} = -f K_{II} \hat{b}.
\]  

When \( K_{II} = 0, \) this equation says that the crack grows in a straight line (since \( \frac{\partial \hat{n}}{\partial t} = 0 \)), in agreement with the “principle of local symmetry”\(^{23}\) generally used to predict crack growth in two dimensions. However, the principle of local symmetry also says that \( K_{II} = 0 \) is maintained at all times by the propagating crack—in effect, that the crack curves in such a way as to keep \( K_{II} = 0. \) Our law, in contrast, says that it is only a non-zero \( K_{II} \) which can make the crack curve, but that (with \( f > 0 \)) the crack curves in such a way as to make \( K_{II} \) smaller (see figure 4).

Now, we can resolve the differences between the principle of local symmetry and our crack propagation law by writing the crack velocity \( v \) as the time
Fig. 4 Schematic of observed crack growth in mode II, where the crack curves so as to reduce the mode II stress, leaving only mode I stress. (a) Our picture, where the crack curves gradually to the direction where $K_{II} = 0$, on a length scale of $\frac{2u}{fK_i}$. (b) The traditional picture, where there is a sharp kink to the direction where $K_{II} = 0$. Note that in the $f \to \infty$ limit, the two pictures agree.
derivative of the crack length $\frac{\partial a}{\partial t}$, writing $\frac{\partial}{\partial t} = \frac{\partial a}{\partial t} \frac{\partial}{\partial a}$, and by writing $\hat{n}$ and $\hat{b}$ in terms of the angle $\theta$ that $\hat{n}$ makes with the $x$-axis. With these changes, equation 4 becomes:

$$\frac{\partial \vec{x}}{\partial a} = (\cos \theta, \sin \theta)$$

$$\frac{\partial \theta}{\partial a} = -\left(\frac{f}{v}\right) K_{II}.$$  

(7)

In principle, $f$, $v$, and $K_{II}$ are functions of $\vec{x}$ and $\theta$. However, in the case of a small amount of growth at the end of a long crack, we expect $f$ and $v$ to be nearly constant as the crack grows, since the SIFs only change by a small amount during the growth (see appendix C.) Also, when $\theta$ differs from the angle that makes $K_{II} = 0$ by only a small amount $\Delta \theta(x)$, we can approximate $K_{II}$ as

$$K_{II}(x) = K_{I}(0) \frac{\Delta \theta(x)}{2} \left[ 1 + O\left(\frac{x}{a}\right) \right],$$

(8)

where $a$ is the length of the original long crack (see appendix C). Using equation 8 in equation 7, we see that

$$\frac{\partial \Delta \theta}{\partial a} = -\left(\frac{f K_{I}(0)}{2v}\right) \Delta \theta,$$  

(9)

which we can immediately solve, taking $f$ and $v$ constant, to find that

$$\Delta \theta(a) = e^{-\frac{fK_{I}(0)a}{2v}}.$$  

(10)

That is, if we start a crack with a small deviation from the direction predicted by the principle of local symmetry, then our crack propagation law says that the deviation decays with a characteristic distance of $\frac{2v}{fK_{I}}$. This length scale must
be very small, about the size of the non-linear process zone or the smoothing length, because these microscopic lengths are the only length scales that appear in two-dimensional crack problems (see section II). In the limit that the length scale is zero, or $f \to \infty$, we can now see that our crack propagation law for two dimensions agrees with the principle of local symmetry.

Now, let us move to consideration of equation 5,

$$\frac{\partial \bar{x}}{\partial t} = v \hat{n} + w \hat{t}$$

$$\frac{\partial \hat{n}}{\partial t} = - \left[ \frac{\partial v}{\partial s} + w \frac{\partial \hat{t}}{\partial s} \cdot \hat{n} \right] \hat{t} + \left[ -f K_{II} + g_1 K_{III} \frac{\partial K_I}{\partial s} + g_{11} K_{II} K_{III} \frac{\partial K_{II}}{\partial s} + g_{III} \frac{\partial K_{III}}{\partial s} + h_{tb} \frac{\partial \hat{t}}{\partial s} \cdot \hat{b} + h_{nt} K_{II} \frac{\partial \hat{n}}{\partial s} \cdot \hat{t} + \left( h_{nb} K_{II} K_{III} + w \right) \frac{\partial \hat{n}}{\partial s} \cdot \hat{b} \right] \hat{b},$$

our crack propagation law for three dimensions. First, note that the principle of local symmetry term, $-f K_{II}$, appears in this law, just as in two dimensions, and we do not expect $f$ to be different here from its value in two dimensions (that is, $\frac{f}{v K_I}$ is the inverse of a microscopic length). In contrast to $f$, the other functions appearing in equation 5 do not contain any length scales—both $\frac{g_\alpha}{v K_I}$ and $\frac{h_{ij}}{v K_I}$ are dimensionless. The length scales in these terms come from the gradient $\frac{\partial}{\partial s}$; if the dimensionless forms of $g$ and $h$ are of order 1, these terms act over the length scales of the gradients. As noted in the previous section, we do not know $g_\alpha$ or $h_{ij}$ from symmetry principles; simple experiments to measure the sign and magnitude of these material-dependent functions, and their physical interpretations, will be discussed in a separate paper.
Our crack growth law, since it contains terms besides the principle of local symmetry term, predicts three-dimensional behavior beyond the scope of the principle of local symmetry; one example of this is the so-called "factory roof" structure seen in mode III cracks. We will explore this and other predictions of our crack propagation law more fully in a separate paper, and also discuss the stability of straight cracks to wavy perturbations.

VII. Conclusions

We have seen that from symmetry principles, we can derive a crack growth law for both two and three dimensional geometries which agrees with the principle of local symmetry in the limit that the microscopic length scales in the crack problem are truly zero. Our law also predicts behavior seen in three dimensions that is not predicted by the principle of local symmetry; more detailed predictions of our crack growth law in three dimensions will be explored in a separate paper. The laws we derived, equations 4 and 5, contain several functions, such as the crack velocity, which are not determined by symmetry alone and must be measured in controlled cracking experiments or atomic simulations; we will discuss experiments designed to measure the sign and order of magnitude of these functions, and their physical interpretations, in a third paper.

By using symmetry principles, separation of length scales, gauge invariance, and gradient expansions, then, we have derived effective, macroscopic equations
governing the growth of cracks in three dimensions. We have coarse-grained
the problem so that microscopic details—such as atomic bond breaking, crys-
talline grain morphology, deformation near the crack tip in response to strain,
and surface effects—are on such small length scales that they cannot affect the
macroscopic crack growth. Understanding the microscopic origins of our effec-
tive growth equations, and describing crack growth on very small length scales,
where our crack growth law is not valid, will demand calculations that include
these microscopic details.

Acknowledgments

We acknowledge the support of (US) DOE Grant #DE-FG02-88-ER45364; JH
was funded by an NSF Graduate Fellowship and a DOE National Needs
in Materials Physics fellowship. We would like to thank P. Wash Wawrzynek,
Dave Potyondy, and Tony Ingraffea for introducing us to the problem of crack
propagation, for many discussions about the subject, and for allowing us to use
their two-dimensional finite element analysis code (FRANC); Jim Krumhansl
for his continuing support and critical evaluation of this research; and Andy
Ruina, Robb Thomson, and Jim Rice for their comments on earlier versions of
this work. JPS would like to thank Stephen Langer for introducing him to the
gauge invariance of curves. We would also like to thank the Technical University
of Denmark and NORDITA for support and hospitality.
Appendix A: Gauge symmetry and cracks

There are many cases where the natural mathematical description of a problem introduces fictitious degrees of freedom with no physical relevance. The most well-known example is in electromagnetism, where all physical quantities are unchanged when the gradient of an arbitrary function $\chi(\vec{r})$ is added to the vector potential $\vec{A}$. The transformation $\vec{A} \rightarrow \vec{A} + \nabla \chi$ is an example of a “gauge transformation;” the invariance of physical quantities under such a transformation is called “gauge invariance.” The strong and weak forces of particle physics also have gauge transformations associated with them. In general relativity, the choice of coordinate system for space-time is arbitrary; this gauge invariance can be used to derive momentum and energy conservation. Another use of gauge symmetry is in site-disorder spin glasses, where a gauge transformation is used to show that the certain forms of disorder do not result in spin glass properties, but in ferromagnetic behavior.

The term “gauge” is particularly appropriate for the gauge symmetry of our problem, where the parameterization $\lambda$ of the crack front curve $\vec{x}(\lambda, t)$ is arbitrary: how one “gauges” (measures) the points along the curve cannot affect the growth of the crack. There are two different types of gauge symmetry for cracks. The first type is the freedom to change the parameterization at any one time, which we call the “one-time gauge symmetry.” The second is the freedom to choose how the parameterization at some time is related, through
the growth equation, to the parameterization at a later time; we call this the "time-dependent gauge symmetry." Crack growth laws must satisfy both gauge symmetries; that is, neither the one-time nor the time-dependent gauge transformation can change the physical crack growth equation.

A crack growth equation that satisfies the one-time gauge symmetry must not have any direct dependence on the value of $\lambda$ at a point on the crack front curve, but only depend on physical quantities evaluated at that value of $\lambda$. Also, derivatives along the crack front cannot enter the growth equation as $\frac{\partial}{\partial \lambda}$, but must instead be in terms of the arc length $s$, because $\frac{\partial}{\partial s}$ is gauge invariant. Since we have written our crack growth law in terms of $\frac{\partial}{\partial s}$, and have not included explicit $\lambda$-dependence, it does satisfy the one-time gauge symmetry.

The time-dependent gauge symmetry is slightly more complicated. An equation for $\frac{\partial \vec{x}(\lambda,t)}{\partial t}$, which is how we have chosen to write the crack growth equation, tells us how a point with parameter value $\lambda$ evolves in time. This means that the time evolution of the parameterization is implicit in our formulation, and the time-dependent gauge transformation changes the growth equation (unlike the one-time gauge transformation). This change happens in a well-defined way: if we have some crack growth law in terms of a parameter $\lambda$,

$$\frac{\partial \vec{x}(\lambda,t)}{\partial t} = A\hat{n} + B\hat{t}$$

$$\frac{\partial \hat{n}(\lambda,t)}{\partial t} = C\hat{t} + D\hat{b},$$

(11)
where the right hand sides are implicit functions of \( \lambda \), when we introduce a time-dependent gauge transformation to a new parameter \( \mu(\lambda, t) \), then the crack growth law becomes

\[
\frac{\partial \vec{x}(\mu, t)}{\partial t} = A\hat{n} + B\hat{t} + \frac{\partial \vec{x}}{\partial \mu} \frac{\partial \mu}{\partial t}
\]
\[
\frac{\partial \hat{n}(\mu, t)}{\partial t} = C\hat{t} + D\hat{b} + \frac{\partial \hat{n}}{\partial \mu} \frac{\partial \mu}{\partial t}
\]  

(12)

with the right hand sides now implicit functions of \( \mu \). Writing \( \frac{\partial}{\partial \mu} = \frac{\partial s}{\partial \mu} \frac{\partial}{\partial s} \) (with \( s \) the arc length), defining a new function \( w \equiv \frac{\partial s}{\partial \mu} \frac{\partial}{\partial s} \), and using the definition of \( \hat{t} \equiv \frac{\partial \vec{x}}{\partial s} \), we can write this as

\[
\frac{\partial \vec{x}(\mu, t)}{\partial t} = A\hat{n} + (B + w)\hat{t}
\]
\[
\frac{\partial \hat{n}(\mu, t)}{\partial t} = C\hat{t} + D\hat{b} + w \frac{\partial \hat{n}}{\partial s}
\]  

(13)

\[
= (C + w \frac{\partial \hat{n}}{\partial s} \cdot \hat{t})\hat{t} + (D + w \frac{\partial \hat{n}}{\partial s} \cdot \hat{b})\hat{b}.
\]

There are three particular time-dependent gauges that we have found to be of special use in our study of crack growth. First, there is the “reference gauge,” where \( B + w \) in equation 13 is zero. In this case, curves of constant parameter value \( \lambda_n \) are the integral curves of \( \hat{n} \), and the growth equation can be written:

\[
\frac{\partial \vec{x}(\lambda_n, t)}{\partial t} = v\hat{n}
\]
\[
\frac{\partial \hat{n}(\lambda_n, t)}{\partial t} = -\frac{\partial v}{\partial s} \hat{t} + D\hat{b}.
\]  

(14)

Here the \( \frac{\partial v}{\partial s} \hat{t} \) term comes from the requirement that \( \hat{n} \cdot \hat{t} = 0 \) be preserved at all times; the function \( D \) is free, as far as gauge symmetry is concerned. This is the only possible form of the growth law when \( \frac{\partial \vec{x}}{\partial t} \) is along \( \hat{n} \), and we use this as our
reference gauge for discussing other time-dependent gauges. In fact, comparing equation 14 to equations 11 and 13, we see that the growth law in a general time-dependent gauge can be written as:

\[
\frac{\partial \vec{x}(\mu, t)}{\partial t} = v \hat{n} + w \hat{t} = \frac{\partial \hat{n}(\mu, t)}{\partial t} = \left( -\frac{\partial v}{\partial s} + w \frac{\partial \hat{n}}{\partial s} \cdot \hat{t} \right) \hat{t} + (D + w \frac{\partial \hat{n}}{\partial s} \cdot \hat{b}) \hat{b},
\]

(15)

with \( w = \frac{\partial s}{\partial \mu} \frac{\partial n}{\partial t} \) as above. Note that the function \( w \) characterizes the time-dependent gauge used in the crack growth law, while \( D \) and \( v \) are the physical functions describing the crack growth.

Another particular time-dependent gauge of interest is the “arc length gauge,” where the parameter \( \lambda \) is always equal to the arc length \( s \) along the crack front curve, measured from some starting point (such as the edge of the body). Since the arc length along the crack front is \( s(\lambda) = \int_{\lambda_0}^{\lambda} \left| \frac{\partial \vec{x}}{\partial \lambda} \right| d\lambda \), if we begin in the arc length gauge (by making a one-time gauge transformation), then we can remain in that gauge by choosing

\[
w(\lambda_s) = \int_{\lambda_0}^{\lambda_s} v(\lambda') \left| \frac{\partial \hat{n}}{\partial \lambda'} \right| \frac{\partial \hat{n}}{\partial \lambda'} \cdot \hat{t}(\lambda') d\lambda'
\]

(16)
in equation 15. Although arc length is physically the most natural parameterization for a curve, the arc length gauge is not usually very convenient, as \( w(\lambda_s) \) is a non-local function—as the crack grows, if the arc length of a section near some point \( \lambda_1 \) stretches (or shrinks), \( \lambda_s \) must shift upwards (downwards) for all points with \( \lambda_s > \lambda_1 \).
A third time-dependent gauge, the “z gauge,” is useful for cracks which have crack fronts which always point nearly along some axis, which we take to be the z-axis. In this case, it is natural to use a gauge where the parameter λ of the crack front is the z-coordinate (as long as the crack front $\vec{x}(\lambda_z)$ is a single-valued function). To achieve this, the crack growth equation for $\vec{x}$ must have zero z-component; that is, we must choose

$$w(\lambda_z) = -v(\lambda_z)\frac{\hat{n}_z}{t_z}. \quad (17)$$

Similar ideas can be used to make special gauges for cracks which are nearly circular, parameterizing with the angle $\theta$ from the x-axis, for instance, and for other common crack geometries.

**Appendix B: Symmetry operations on the relevant variables**

In this appendix, we examine the two symmetry operations (a) 180° rotation of the material about $\hat{n}$ at some point $\lambda_0$, while keeping the coordinates fixed, and (b) reflection of the material in the $n-t$ plane (the crack plane) at $\lambda_0$, while keeping the coordinates fixed (see figure 2), as applied to the relevant variables for crack growth and their derivatives. Note that only the signs, and not the magnitudes, of the variables and derivatives can change under these two operations. Also note that the signs of the SIFs are defined in terms of the three unit vectors, and that the gradient operator $\frac{\partial}{\partial s}$ along the crack front curve goes in the direction of $\hat{t}$. 

Now, $K_I$ transforms like the $\sigma_{bb}$ stress component, $K_{II}$ like $\sigma_{nb}$, and $K_{III}$ like $\sigma_{bt}$ (see figure 1). Symmetry operation (a) takes material at $(x_n, x_b, x_t)$, in terms of the coordinates with axes $\hat{n}, \hat{b},$ and $\hat{t}$ and origin at $\lambda_0$, to $(x_n, -x_b, -x_t)$; symmetry operation (b) takes material at $(x_n, x_b, x_t)$ to $(x_n, -x_b, x_t)$. Therefore, $K_I \rightarrow K_I$ under both (a) and (b), $K_{II} \rightarrow -K_{II}$ under both, and $K_{III} \rightarrow K_{III}$ under (a) and $-K_{III}$ under (b).

Now let us consider a case where one of the SIFs, before transformation, is greater, in absolute value, for $x_t > 0$ than for $x_t < 0$, so that $\frac{\partial|K|}{\partial s} > 0$. Then, under transformation (a), the material at $x_t > 0$ with the greater $|K|$ moves to $x_t < 0$, so that $\frac{\partial|K|}{\partial s} < 0$; transformation (b) leaves $x_t$ unchanged, so that $\frac{\partial|K|}{\partial s} > 0$. This can be combined with the transformation properties of the SIFs themselves to give the transformation properties of the gradients of the SIFs; the results are in table 1.

We also need to consider the transformation properties of gradients of the unit vectors—quantities of the form $\frac{\partial \hat{a}}{\partial s} \cdot \hat{b}$, where $a$ and $b$ belong to $\{n, b, t\}$. Noting that

$$\frac{\partial(\hat{a} \cdot \hat{b})}{\partial s} = 0 = \frac{\partial \hat{a}}{\partial s} \cdot \hat{b} + \frac{\partial \hat{b}}{\partial s} \cdot \hat{a}$$

(18)

for all $a$ and $b$, since $\hat{n}$, $\hat{b}$, and $\hat{t}$ are mutually orthogonal unit vectors, we can see that there are only three independent quantities to consider, which we take to be $\frac{\partial \hat{n}}{\partial s} \cdot \hat{t}$, $\frac{\partial \hat{n}}{\partial s} \cdot \hat{b}$, and $\frac{\partial \hat{t}}{\partial s} \cdot \hat{b}$. From figure 5, we can see that the transformation properties of these three quantities are as shown in table 1.
| Quantity | (a) $180^\circ \hat{n}$ rotation | (b) $n-t$ reflection |
|----------|---------------------------------|---------------------|
| $K_I$    | $+$                             | $+$                 |
| $K_{II}$ | $-$                             | $-$                 |
| $K_{III}$| $+$                             | $-$                 |
| $\frac{\partial K_I}{\partial s}$ | $-$                             | $+$                 |
| $\frac{\partial K_{II}}{\partial s}$ | $+$                             | $-$                 |
| $\frac{\partial K_{III}}{\partial s}$ | $-$                             | $-$                 |
| $\frac{\partial \hat{n}}{\partial s} \cdot \hat{t}$ | $+$                             | $+$                 |
| $\frac{\partial \hat{n}}{\partial s} \cdot \hat{b}$ | $+$                             | $-$                 |
| $\frac{\partial i}{\partial s} \cdot \hat{b}$ | $-$                             | $-$                 |

**Table 1** Transformation properties of relevant variables and their derivatives under two symmetry operations; see figure 5 and text. Note that products of these quantities transform as the product of the transformation properties (e.g. $(K_{II}K_{III})$ is $-$ under (a) and $+$ under (b)).

**Appendix C: Approximation of $K_{II}$ in two dimensions**

In this appendix, we derive equation 8,

$$K_{II}(x) = K_{II}(0) - \frac{1}{2} \theta(x) K_1(0) - \left( \frac{2}{\pi} \right)^{\frac{1}{2}} T \int_0^x \frac{\theta(x')}{(x-x')^{\frac{1}{2}}} dx', \quad (8)$$

which gives $K_{II}$ after the crack has grown a distance $x$ from the end of a long crack of length $a$, in terms of the deviation $\Delta \theta(x)$ of $\theta$ (the angle that $\hat{n}$ makes with the $x$-axis) from the angle that makes $K_{II} = 0$. First, we can use the results of Cotterell and Rice\textsuperscript{11} to find that as a function of the $x$-coordinate of $\bar{x}$, measured from the end of the original long crack,

$$K_{II}(x) = K_{II}(0) + \frac{1}{2} \theta(x) K_1(0) - \left( \frac{2}{\pi} \right)^{\frac{1}{2}} T \int_0^x \frac{\theta(x')}{(x-x')^{\frac{1}{2}}} dx', \quad (19)$$
Fig. 5  (a) Top view of the surface of a planar crack with a curved crack front, which gives a non-zero value for $\frac{\partial \hat{n}}{\partial s} \cdot \hat{t}$, with both $\frac{\partial \hat{n}}{\partial s} \cdot \hat{b}$ and $\frac{\partial \hat{t}}{\partial s} \cdot \hat{b}$ zero (since both $\hat{n}$ and $\hat{t}$ are always in the same plane, perpendicular to $\hat{b}$). Under rotation of the crack about $\hat{n}$ at a point, keeping the vectors fixed (symmetry operation a), the value of $\frac{\partial \hat{n}}{\partial s} \cdot \hat{t}$ stays the same; under reflection of the crack in the $n$-$t$ plane at a point, again keeping the vectors fixed (symmetry operation b), it is also invariant. (b) A non-planar crack with non-zero $\frac{\partial \hat{n}}{\partial s} \cdot \hat{b}$, where both $\frac{\partial \hat{n}}{\partial s} \cdot \hat{t}$ and $\frac{\partial \hat{t}}{\partial s} \cdot \hat{b}$ are zero (since $\hat{b}$ and $\hat{n}$ are always in the same plane, perpendicular to $\hat{t}$). Under symmetry operation a, $\frac{\partial \hat{n}}{\partial s} \cdot \hat{b}$ is invariant; it changes sign under symmetry operation b. (c) A non-planar crack with non-zero $\frac{\partial \hat{t}}{\partial s} \cdot \hat{b}$, where both $\frac{\partial \hat{n}}{\partial s} \cdot \hat{t}$ and $\frac{\partial \hat{n}}{\partial s} \cdot \hat{b}$ are zero (since $\hat{b}$ and $\hat{t}$ are always in the same plane, perpendicular to $\hat{n}$). $\frac{\partial \hat{t}}{\partial s} \cdot \hat{b}$ changes sign under symmetry operations a and b.
where $T$ is the non-singular tensile stress at the end of the crack. So, when the principle of local symmetry is satisfied, $\theta(x)$ has the value which makes $K_{II}(x) = 0$; if $\theta$ differs from this value by a small amount $\Delta \theta(x)$, and if we take $T = b \frac{K_{I}(0)}{\sqrt{a}}$, with the appropriate geometrical factor $b$, then we find that

$$K_{II}(x) = K_{I}(0) \frac{\Delta \theta(x)}{2} \left[ 1 - 4b \left( \frac{2x}{\pi a} \right)^{\frac{1}{2}} \int_{0}^{x} \frac{\Delta \theta(x')}{\Delta \theta(x)} \left( \frac{x}{x-x'} \right)^{\frac{1}{2}} \frac{dx'}{2x} \right].$$

(20)

Now, if $\frac{\Delta \theta(x')}{\Delta \theta(x)}$ were constant, then the integral on the right hand side would be equal to 1; small variations of $\Delta \theta(x)$ from a constant function leave the integral approximately 1. Noting that the integral is multiplied by $\left( \frac{x}{a} \right)^{1/2}$, which is small by assumption, we can therefore approximate $K_{II}$ as

$$K_{II}(x) = K_{I}(0) \frac{\Delta \theta(x)}{2} \left[ 1 + O \left( \frac{x}{a} \right) \right],$$

(8)

which is equation 8. We also note here that under the approximations of this appendix, and with the results of Cotterell and Rice, $K_{I}$ is approximately constant.

**References**

1. L. B. Freund, *Dynamic Fracture Mechanics*, Cambridge (1990).
2. A. Shukla, H. Nigam, and H. Zervas, Engr. Fracture Mech. 36, 429 (1990).
3. John P. Dempsey, Mao-Kuen Kuo, and Diane L. Bentley, Int. J. Solids Structures 22, 333 (1986).
4. Jay Fineberg, Steven P. Gross, M. Marder, and Harry L. Swinney, Phys. Rev. Lett. 67, 457 (1991).
5. W. A. Curtin and H. Scher, Phys. Rev. Lett. 67, 2457 (1991).
6. Kin S. Cheung and Sidney Yip, Phys. Rev. Lett. 65, 2804 (1990).
7. Paul Meakin, Science 252 (12 April), 226 (1991).
8. Anthony R. Ingraffea, Tulio N. Bittencourt, and Jose Luiz Antunes O. Sousa, “Automatic fracture propagation for 2D finite element models,” to appear in proc. of MECOM90: XI Congress Ibero-Americana Sobre Metodos Computacionales en Engenharia (1990).
9. Asher A. Rubinstein, Int. J. Fracture 47, 291 (1991).
10. Norman A. Fleck, John W. Hutchinson and Zhigang Suo, Int. J. Solids Structures 27, 1683 (1991).
11. B. Cotterell and J. R. Rice, Int. J. Fracture 16, 155 (1980).
12. J. B. Leblond, “Crack kinking and curving in three-dimensional elastic solids—application to the study of crack path stability in hydraulic fracturing,” preprint, presented at the International Conference on Mixed Mode Fracture and Fatigue, Vienna (1991).
13. D. R. Clarke and K. T. Faber, J. Phys. Chem. Solids 48, 1115 (1987).
14. J. M. Huntley, Proc. R. Soc. Lond. A 430, 525 (1990).
15. M. Marder, Phys. Rev. Lett. 66, 2484 (1991).
16. M. Barber, J. Donley, and J. S. Langer, Phys. Rev. A 40, 366 (1989).
17. Melvin F. Kanninen and Carl H. Popelar, Advanced Fracture Mechanics, Oxford (1985).
18. J. R. Rice, J. Mech. Phys. Solids 26, 61 (1978).
19. Brian R. Lawn, David H. Roach, and Robb M. Thomson, J. Mat. Sci. 22, 4036 (1987).
20. B. R. Lawn and S. Lathabai, Mat. Forum 11, 313 (1988).
21. Kai-Tak Wan, Nicholas Aimard, S. Lathabai, Roger G. Horn, and Brian R. Lawn, J. Mater. Res. 5, 172 (1990).
22. David Broek, Elementary Engineering Fracture Mechanics, fourth edition; Nijhoff Publishers, Dordrecht, The Netherlands (1986).
23. R. V. Gol’dstein and R. L. Salganik, Int. J. Fracture 10, 507 (1974).
24. François Hourlier, Hubert d’Hondt, Michel Truchon, and André Pineau, p. 228 in Multiaxial Fatigue, ASTM STP 853 (YEAR??).
25. Lewis H. Ryder, Quantum Field Theory (see chapter 3), Cambridge (1985).
26. K. Binder and A. P. Young, Rev. Mod. Phys. 58, 801 (see pages 846-7) (1986).
27. Stephen A. Langer and Raymond E. Goldstein, “Dissipative Dynamics of Closed Curves in Two Dimensions,” to be published; we also thank Karsten Jacobsen for pointing out the particular appropriateness of the term “gauge” for this problem (1991).