Gauge/string duality for QCD conformal operators

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Abstract

Renormalization group evolution of QCD composite light-cone operators, built from two and more quark and gluon fields, is responsible for the logarithmic scaling violations in diverse physical observables. We analyze spectra of anomalous dimensions of these operators at large conformal spins at weak and strong coupling with the emphasis on the emergence of a dual string picture. The multi-particle spectrum at weak coupling has a hidden symmetry due to integrability of the underlying dilatation operator which drives the evolution. In perturbative regime, we demonstrate the equivalence of the one-loop cusp anomaly to the disk partition function in two-dimensional Yang-Mills theory which admits a string representation. The strong coupling regime for anomalous dimensions is discussed within the two pictures addressed recently, — minimal surfaces of open strings and rotating long closed strings in AdS background. In the latter case we find that the integrability implies the presence of extra degrees of freedom – the string junctions. We demonstrate how the analysis of their equations of motion naturally agrees with the spectrum found at weak coupling.

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1 Introduction

The formalism of path-ordered exponentials, or Wilson loops, is an indispensable tool in QCD. It allows one to formulate complicated QCD dynamics in terms of gauge invariant degrees of freedom and express correlation functions as a sum over random walks, e.g.,

\[
\langle 0 | J_\mu(x) J_\nu(0) | 0 \rangle = \sum_C \text{e}^{-mL[C]} \phi_{\mu\nu}[C] \langle 0 | \tr P \exp \left( i \oint_C dx \mu A^\mu(x) \right) | 0 \rangle ,
\]

(1.1)

where \( J_\mu(x) = \overline{\Psi}(x) \gamma_\mu \Psi(x) \) is the electromagnetic current of a quark with mass \( m \). \( L[C] \) is the length of a closed path \( C = C[0,x] \) that passes through the points \( x \) and \( 0 \). \( \phi_{\mu\nu}[C] \) is a geometrical phase, the so-called Polyakov spin factor, that takes into account the variation of the quark spin upon parallel transport along the path \( C \). To evaluate (1.1), one has to calculate the (nonperturbative) expectation value of the Wilson loop for an arbitrary path \( C \) and perform resummation in the right-hand side of (1.1). Both tasks are extremely difficult and can not be performed in full at the current stage. Recently, a significant progress has been achieved in understanding the strong coupling dynamics of supersymmetric gauge theories [2, 3] based on the gauge/string correspondence [4, 5, 6]. One of the goals of the present paper is to establish a relation between certain QCD observables and their counterparts in string theory.

There exists a special class of QCD observables, for which the sum over paths in the right-hand side of (1.1) can be performed exactly. As a relevant physical example, let us consider a propagation of an energetic quark through a cloud of soft gluons. In the limit when its energy goes to infinity, the quark behaves as a point-like charged particle that moves along a straight line and interacts with soft gluons. This means that the sum over all paths in (1.1) is dominated in that case by a saddle point describing a propagation of a quark along its classical path. The Wilson loop corresponding to this path has the meaning of the eikonal phase acquired by the quark field upon interaction with gluons. In this way, the Wilson loop encodes universal features of soft radiation in QCD. Let us point out two important QCD observables, in which similar semiclassical regime occurs: the Isgur-Wise heavy-meson form factor, \( \xi(\theta) \), and parton distributions in a hadron, \( f(x) \), at the edge of the phase space, \( x \to 1 \). As we will demonstrate below, both observables are given by an expectation value of a Wilson loop with the integration contour \( C \) fixed by the kinematics of the process. A unique feature of the contour \( C \) is that it contains a few cusps at points in Minkowski space-time where the interaction with a large momentum has occurred in the underlying process.

In this way, Wilson loops with cusps, being fundamental objects in gauge theories, have a direct relevance for QCD phenomenology. Their calculation in the strong coupling (nonperturbative) regime is one of the prominent problems in gauge theories. In the present paper, we make use
of a recent progress in understanding the strong coupling behavior of the $\mathcal{N} = 4$ supersymmetric (SUSY) Yang-Mills (YM) theory to get some insights into properties of Wilson loops in QCD. Our analysis relies on the gauge/string duality between $\mathcal{N} = 4$ SUSY gauge theory and a string theory on $\text{AdS}_5 \times \text{S}^5$ background.\footnote{Note that recently there were several studies which aimed on the derivation of strong coupling results for high-energy QCD observables, most notably Refs.~\cite{7,8,9,10,11,12,13}.} A natural question arises: what is in common between QCD and $\mathcal{N} = 4$ SUSY Yang-Mills theory? The two theories have quite different dynamics at large distances, while at short distances they have many features in common. For instance, anomalous dimensions of twist-two operators, contributing to high-energy QCD processes, have a similar form in two theories including their behavior at large Lorentz spin. Having this relation in mind, we will study expressions for resummed anomalous dimensions in the $\mathcal{N} = 4$ SUSY Yang-Mills theory.\footnote{Since the anomalous dimensions originate from short distances, we find it appropriate to refer to them in the strong coupling regime as resummed anomalous dimensions rather than nonperturbative ones.}

We concentrate on two observations relevant to our present discussion. Recently, it was proposed that anomalous dimensions of twist-two composite operators with large Lorentz spin $J$ are equal in the strong coupling limit to the “Energy $- \text{Spin}$” of a folded closed string rotating in $\text{AdS}_5$ and having the shape of a long rigid rod (mimicking the adjoint QCD string of glue with heavy quarks at the folding points) \cite{7}

$$\gamma_J(\alpha_s) = E - J = 2\sqrt{\frac{\alpha_s N_c}{\pi}} \ln J + \mathcal{O}(J^0). \quad (1.2)$$

Another observation comes from the calculation of the Wilson loop in the strong coupling regime via the minimal surface, $A_{\text{min}}$, swept by an open string which goes into the fifth AdS dimension and whose ends trace its contour in Minkowski space \cite{2}. This picture naturally embeds the color flux tubes between the color sources, albeit penetrating into an extra Liouville dimension \cite{14} as compared to the conventional four-dimensional setup. From the QCD perspective, one is mostly interested in calculating Wilson loops with cusps. Such contours were considered in a number of studies \cite{8,9,10}. For the $\Pi$-shaped Wilson loop with two cusps (see Eq. (2.14) below) the result reads \cite{9,10}

$$W(v \cdot n \xi \mu) = \exp(iA_{\text{min}}), \quad A_{\text{min}} = i\sqrt{\frac{\alpha_s N_c}{\pi}} \ln^2((v \cdot n \xi \mu)). \quad (1.3)$$

Comparing (1.2) and (1.3), one notices that $\gamma_J$ and $A_{\text{min}}$ depend on the coupling constant in the same manner. The coincidence is not accidental, of course. Identifying $\sqrt{\alpha_s}$-factor as the leading term in the expression for the cusp anomalous dimension

$$\Gamma_{\text{cusp}}(\alpha_s) = \sqrt{\frac{\alpha_s N_c}{\pi}} + \mathcal{O}((\sqrt{\alpha_s})^0), \quad (1.4)$$
one can show that Eqs. (1.2) and (1.3) hold in a conformal gauge theory for arbitrary coupling constant $\alpha_s$. Eq. (1.4) defines the asymptotic behavior of $\Gamma_{\text{cusp}}(\alpha_s)$ in the $\mathcal{N} = 4$ SUSY YM theory in the strong coupling regime.

In the present paper, we will extend these results to composite QCD operators of higher twist, built from an arbitrary number of fields and having autonomous renormalization scale evolution. Such operators are known in QCD as multi-particle conformal operators. We determine the spectrum of their anomalous dimensions at large Lorentz spins, $J \gg 1$, both in weak and strong coupling regimes. At weak coupling, the spectrum has a hidden symmetry due to integrability of the dilatation operator in the underlying Yang-Mills theory. At strong coupling, the two different pictures, — the rotating folded long string and the minimal surface swept by an open string with ends attached to the cusp, — result into the same asymptotic expression for the anomalous dimension of conformal operators. We argue that integrability at weak coupling implies the presence of extra stringy degrees of freedom at strong coupling, — the string junctions, — and elucidate the relation between the anomalous dimensions of multi-particle conformal operators at strong coupling and solutions to the classical equations of motion for the string junctions.

To sew together the expressions for the anomalous dimensions at weak and strong coupling, one needs the stringy description of the weak coupling regime in Yang-Mills theory. One approach to the derivation of such stringy picture, based on the hidden integrability of evolution equations for the light-cone operators, has been developed in [13]. It relies on the identification of the underlying Yang-Mills dilatation operator as the Hamiltonian of $SL(2, \mathbb{R})$ Heisenberg spin chain. The $SL(2, \mathbb{R})$ group naturally appears in this context as a subgroup of the four-dimensional conformal group acting on the light-cone. Due to complete integrability of the spin chain model, the spectrum of the anomalous dimensions of multi-particle light-cone operators can be found exactly in terms of the Riemann surfaces whose genus is related to the number of the particles involved. As a consequence, the twist expansion on the light-cone was shown to correspond to the summation over the genera of the corresponding Riemann surfaces.

Our consequent presentation is organized as follows. In section 2 we review the relation of certain QCD observables to expectation values of Wilson lines and elucidate the physical meaning of the cusp anomaly. We also give there results for the two-loop cusp anomalous dimensions in supersymmetric theories. In section 3 we turn to multi-particle operators and show how conformal symmetry in gauge theory simplifies the problem of finding the spectrum of their anomalous dimensions. We demonstrate the way the integrability of the evolution equations arises through the identification of the underlying dilatation operator with the Hamiltonian of a Heisenberg spin chain. In the subsequent section we address the stringy interpretation of the gauge theory results. Our analysis suggests that the cusp anomaly at weak coupling is described by a string which is
different from the Nambu-Goto string. To identify this string we show that the cusp anomalous
dimension to one-loop order is equal to the transition amplitude for a test particle on the $SL(2, \mathbb{R})$
group manifold, which in its turn is given by the partition function of two-dimensional Yang-Mills
theory on a disk which admits a stringy representation. Next, in section 5 we discuss the strong
coupling computation within the open and closed string theory context for multi-particle operators
extending earlier results. Section 6 contains concluding remarks. In the appendix, we calculate the
contribution of vacuum polarization to the two-loop cusp anomaly in dimensional regularization
and dimensional reduction schemes.

2 Wilson loops as QCD observables

As was emphasized in the introduction, there are several QCD observables directly related to
expectation values of Wilson lines.

2.1 Isgur-Wise form factor

The Isgur-Wise form factor $\xi(\theta)$ describes the electromagnetic transition of a heavy meson $|M(v)\rangle$
with mass $m$ and momentum $p_\mu \equiv mv_\mu$, built from a heavy quark and a light component, to the
same meson with momentum $p'_\mu \equiv mv'_\mu$ (with $v_\mu^2, v'_\mu^2 = 1$) \[16\]

$$
\langle M(v')|\overline{\Psi}(0)\gamma_\mu\Psi(0)|M(v)\rangle = \xi(\theta)(v + v')_\mu.
$$

(2.1)

In the heavy-quark limit, $m \to \infty$, it depends only on the product of velocities $v' \cdot v = \cosh \theta$, or
equivalently on the angle $\theta$ between them in Minkowski space-time. The operator $\Psi(0)$ annihilates
the heavy quark inside the meson $|M(v)\rangle$. For $m \to \infty$, the heavy quark behaves as a classical
particle with the velocity $v_\mu$ interacting with the light component of the meson through its eikonal
current, $J_{\mu, eik}(x) = \int_{-\infty}^{0} d\tau v_\mu t^a \delta(4)(x - v\tau)$ with $t^a$ being the quark color charge. This allows one
to replace

$$
\begin{align*}
\Psi(x) &\to e^{-im(v \cdot x)} b_v \Phi_v[x; -\infty], & \Phi_v[x; -\infty] &\equiv P \exp \left( i \int_{-\infty}^{0} d\tau v \cdot A(x + v\tau) \right),
\end{align*}
$$

(2.2)

where $\Phi_v[x; -\infty]$ is the eikonal phase of a heavy quark in the fundamental representation of the
$SU(N_c)$ and $b_v$ amputates this quark inside the heavy meson. Applying similar transformation to
the quark field in the final state meson $|M(v')\rangle$, one obtains the following expression for the form
factor \[17\]

$$
\xi(\theta) = \langle \tilde{M}(v')|\Phi_v'[\infty; 0]\Phi_v[0; -\infty]M(v)\rangle \equiv \langle P \exp \left( i \int_{-}\int_{-\infty}^{0} dx_\mu A^\mu(x) \right) \rangle,
$$

(2.3)
with $|\tilde{M}(v)\rangle = b_v|M(v)\rangle$ standing for the light component of the meson with the amputated heavy quark. Here the net effect of nonperturbative interaction with the light component of the heavy meson is accumulated only via the Wilson line\footnote{Representing the “brown-muck” of the heavy meson by a wave function $|M_\ell\rangle = \int d^3k \phi(k) a^\dagger_k|0\rangle$, the transition of the light cloud from the initial to the final state can be described by an exact light-quark propagator in an external gluon field. In this manner the Isgur-Wise form factor will be rewritten as a correlation function of Wilson loop along the contour formed by the straight-line trajectories of heavy quarks and the phase of the light spectator quark in the world-line expression of its propagator \cite{13}.} evaluated along the contour consisting of two rays that run along the meson velocities $v_\mu$ and $v'_\mu$. It is important to notice that the contour has a cusp at the point 0, in which the interaction with the external probe has occurred.

The Isgur-Wise form factor $\xi(\theta)$ is a nonperturbative observable in QCD. It depends on hadronic, long-distance scales as well as on the ultraviolet cut-off $\mu \sim m$, which sets up the maximal energy of soft gluons. Although $\xi(\theta)$ cannot be calculated at present in QCD from the first principles, its dependence on $\mu$ can be found from the renormalization group equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \Gamma_{\text{cusp}}(\theta; \alpha_s)\right) \xi(\theta) = 0,$$

(2.4)

where $\alpha_s = g^2/(4\pi)$ is the QCD coupling constant and $\Gamma_{\text{cusp}}(\theta; \alpha_s)$ is the cusp anomalous dimension \cite{15}. To the lowest order in $\alpha_s$

$$\Gamma_{\text{cusp}}(\theta; \alpha_s) = \frac{\alpha_s C_F}{\pi} (\theta \coth \theta - 1) + O(\alpha_s^2),$$

(2.5)

where $C_F = (N_c^2 - 1)/(2N_c)$ is the Casimir operator of the $SU(N_c)$ group in the fundamental representation. The two-loop correction to (2.5) has been calculated in \cite{19} and its dependence on $\theta$ is more involved.

Eq. (2.4) follows from renormalization properties of the Wilson line in the right-hand side of (2.3). It acquires the anomalous dimension due to the presence of a cusp on the integration contour. The cusp anomalous dimension $\Gamma_{\text{cusp}}(\theta; \alpha_s)$ determines universal features of soft-gluon radiation and is known as the QCD bremsstrahlung function. As such, it is a positive definite function of the cusp angle (for real Minkowski angle $\theta$) at arbitrary value of the coupling constant $\Gamma_{\text{cusp}}(\theta; \alpha_s) \geq 0$.

(2.6)

To see this we recall that at the cusp point the heavy quark suddenly changes its velocity from $v_\mu$ to $v'_\mu$ and, due to instantaneous acceleration, it starts to emit soft (virtual and real) gluons with momentum $k < \mu$ with a cut-off $\mu \sim m$. Denoting the eikonal phase of the heavy quark as $\Phi \equiv \Phi_{+}(\infty; 0)\Phi_{-}(0; -\infty)$ and using its unitarity, one calculates the total probability for the heavy quark to undergo the scattering (the Bjorken sum rule) as

$$1 = \langle \tilde{M}(v)|\Phi^\dagger \Phi|\tilde{M}(v)\rangle = |\xi(\theta)|^2 + \sum X \left|\langle \tilde{M}_X(v')(\Phi)|\tilde{M}(v)\rangle\right|^2,$$

(2.7)
where in the right-hand side we inserted the decomposition of the unity operator over the physical hadronic states and separated the contribution of the ground state meson, \( |\tilde{M}(v)\rangle \), from excited states \( |\tilde{M}_X(v)\rangle \). The Wilson line \( (2.3) \) defines the probability of the elastic transition, \(|\xi|^2 \sim \exp(-w)\) with \( w = 2\int d(k/k) \Gamma_{\text{cusp}}(\theta; \alpha_s(k)) \). For \( \theta \neq 0 \), depending on the sign of \( \Gamma_{\text{cusp}}(\theta; \alpha_s) \), it either vanishes or goes to infinity for \( \mu \to \infty \). In order to preserve the unitarity condition \(|\xi|^2 \leq 1\) that follows from \((2.7)\), one has to require that \( \xi \to 0 \) for \( \mu \to \infty \) leading to \((2.6)\). At \( \theta = 0 \) the cusp vanishes, that is the heavy meson stays intact, the sum in \((2.7)\) equals zero and \( \xi(\theta = 0) = 1 \). This implies that the cusp anomalous dimension vanishes for \( \theta \to 0 \).

### 2.2 Parton distributions at \( x \to 1 \)

Our second example is provided by deeply inelastic scattering of a hadron \( H(p) \) with momentum \( p_\mu \) off a virtual photon \( \gamma^*(q) \) with momentum \(-q_\mu^2 = Q^2 \gg p^2\) in the exclusive limit \( x_{\text{Bj}} = Q^2/(2p \cdot q) \to 1 \), i.e. when the invariant mass of the final state system becomes small \((q+p)^2 \ll Q^2\).

In the scaling limit, \( Q^2 \to \infty \), the cross section of the process is expressed in terms of the twist-two quark distribution function \([20]\), see also Ref. \([21]\),

\[
f(x) = \int_{-\infty}^{\infty} d\xi \frac{e^{-ix\xi}}{2\pi} \langle H(p) | \bar{\Psi}(\xi n) \Gamma \Phi_n[\xi; 0] \bar{\Psi}(0) | H(p) \rangle ,
\]

\((2.8)\)
describing the probability to find a quark inside the hadron with the fraction \( x \) of its momentum \( p \).

The Wilson line stretched in between the quark fields makes the bilocal operator gauge invariant. It goes along the light-like direction \( n_\mu = (q_\mu + p_\mu x_{\text{Bj}})/(p \cdot q) \), so that \( n^2 = 0 \) and \( n \cdot p = 1 \).

The matrix \( \Gamma = \gamma \) in \((2.8)\) serves to select the quark states with opposite helicities. For our purposes, we will not specify \( \Gamma \) and treat it as a free parameter. The Mellin moments of the distribution function \((2.8)\) are related to the matrix elements of local twist-two operators

\[
\int_0^1 dx \ x^J f(x; \mu^2) = \langle H(p) | \bar{\Psi}(0) \Gamma (i n \cdot D)D^J \bar{\Psi}(0) | H(p) \rangle \equiv \langle O_f^J(\mu^2) \rangle ,
\]

\((2.9)\)
where \( D_\mu = \partial_\mu - i A_\mu \) is a covariant derivative. Their dependence on the ultraviolet cut-off \( \mu \) is described by an evolution equation, whose solution reads in terms of the anomalous dimensions

\[
\langle O_f^J(\mu^2) \rangle = (\mu/\mu_0)^{\gamma_f^J(\alpha_s)} \langle O_f^J(\mu_0^2) \rangle ,
\]

\((2.10)\)
where we assumed for simplicity that the coupling constant does not run, \( \beta = 0 \). The anomalous dimension of the twist-two operators, \( \gamma_f^J(\alpha_s) \), depends on the choice of the matrix \( \Gamma \). In particular, in case when \( \Gamma \) selects the same helicities of the quark fields in \((2.9)\), \( \Gamma = (1 + \gamma_5)\gamma \gamma_\perp \), the anomalous dimension is

\[
\gamma_f(\alpha_s) = \frac{\alpha_s}{\pi} C_F \left( 2\psi(J + 2) + 2\gamma_E - 3/2 \right) + \mathcal{O}(\alpha_s^2) ,
\]

\((2.11)\)
where \( \psi(J) = d \ln \Gamma(J)/dJ \) is the Euler psi-function and \( \gamma_E \) is the Euler constant. For other choices of \( \Gamma \), the anomalous dimensions have extra (rational in \( J \)) terms in addition to the \( \psi \)-function, see, e.g., [22]. As we will argue below, Eq. (2.11) has a hidden symmetry which is responsible for integrability of evolution equations for three-quark (baryonic) composite operators. Going over to the \( \mathcal{N} = 4 \) SUSY Yang-Mills theory, one finds that the same expression (2.11) defines (up to redefinition of the color factor \( C_F \rightarrow N_c \)) the anomalous dimensions of multiplicatively renormalizable twist-two operators [23].

As follows from (2.9), the asymptotics of the distribution function for \( x \to 1 \) is related to the contribution of twist-two operators of large Lorentz spins \( J \sim 1/(1-x) \gg 1 \). One finds from (2.11) that the anomalous dimension scales in this limit as

\[
\gamma_J(\alpha_s) = \frac{\alpha_s}{\pi} C_F \left\{ \ln(J+2) + \gamma_E - 3/4 - \frac{1}{2(J+2)} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n} (J+2)^{-2n} \right\} + \ldots,
\]

where \( B_n \)'s are the Bernoulli numbers, \( B_2 = 1/6 \), \( B_4 = -1/30 \), ... , and the ellipsis stands for higher order terms in \( \alpha_s \). It turns out that the leading scaling behavior \( \gamma_J(\alpha_s) \sim \ln J \) is a universal property of the anomalous dimensions of the twist-two operators (2.9) for arbitrary \( \Gamma \). It holds to all orders in \( \alpha_s \) and is intrinsically related to the cusp anomaly of the Wilson loops. The reason for this is that analyzing deeply inelastic scattering for \( x \to 1 \) one encounters the same physical phenomenon as in the case of the Isgur-Wise form factor, i.e. the struck quark carries almost the whole momentum of the hadron and, therefore, it interacts with other partons by exchanging soft gluons. In these circumstances, in complete analogy to the previous case, Eq. (2.2), the quark field can be approximated by an eikonal phase evaluated along the classical path in the direction of its velocity \( p_\mu = mv_\mu \),

\[
f(x) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{i(1-x)\xi} \langle \tilde{H}(p)|W_{\Pi}(v \cdot n \xi \mu - i0)|\tilde{H}(p)\rangle,
\]

where \( |\tilde{H}(p)\rangle \) is the state of the target hadron with amputated energetic quark and the causal \(-i0\) prescription ensures the correct spectral property, \( f(x) = 0 \) for \( x > 1 \). The \( n \)-shaped Wilson line in Eq. (2.13) consists out of two rays and one segment: a link from \(-\infty\) to 0 along the velocity of the incoming quark, next along the light-cone direction \( n_\mu \) to the point \( \xi n_\mu \) and, then, along \(-v_\mu\) from 0 to \( \infty \),

\[
W_{\Pi}(v \cdot n \xi \mu) = \Phi_{v}[\xi; -\infty] \Phi_{v}[\xi; 0] \Phi_{v}[0; -\infty].
\]

Substituting (2.13) into (2.9), one finds the following relation between the matrix elements of local composite operators at large spin \( J \) and the Wilson loop expectation value [15]

\[
\langle O_j^\Gamma(\mu^2) \rangle = \langle \tilde{H}(p)|W_{\Pi}(-iJ)|\tilde{H}(p)\rangle \equiv \langle P \exp \left(i \int_{\Pi} dx_\mu A^\mu(x) \right) \rangle.
\]
Here, the large Lorentz spin of the local operator defines the length of the light-cone segment:

\[ v \cdot n \xi \mu \rightarrow -iJ. \]  (2.16)

We would like to stress that Eq. (2.15) holds only for \( J \gg 1 \).

According to Eq. (2.15), the \( \mu \)-dependence of the twist-two operators follows from the renormalization of the Wilson line (2.14). The latter has two cusps located at the points 0 and \( \xi n_\mu \). In distinction with the previous case, one of the segments attached to the cusps lies on the light-cone, \( n^2 = 0 \), and the corresponding cusp angle is infinite, \( \theta \sim \frac{1}{2} \ln[(v \cdot n)^2/n^2] \rightarrow \infty \). In this limit, the cusp anomalous dimension scales to all orders in \( \alpha_s \) as [19]

\[ \Gamma_{\text{cusp}}(\theta; \alpha_s) = \theta \Gamma_{\text{cusp}}(\alpha_s) + O(\theta^0). \]  (2.17)

Here \( \Gamma_{\text{cusp}}(\alpha_s) \) is a universal anomalous dimension independent on \( \theta \). At weak coupling, it has the following form in QCD

\[ \Gamma_{\text{cusp}}(\alpha_s) = \frac{\alpha_s}{\pi} C_F + \left( \frac{\alpha_s}{\pi} \right)^2 C_F \left\{ N_c \left( \frac{67}{36} - \frac{\pi^2}{12} \right) - n_f \frac{5}{18} \right\} + O(\alpha_s^2), \]  (2.18)

where \( n_f \) is the number of quark flavors. This expression was obtained within the dimensional regularization scheme (DREG) by using the \( \overline{\text{MS}} \)-subtraction procedure, \( \alpha_s \equiv \alpha_{\overline{\text{MS}}} \).

The divergence of the anomalous dimension (2.17) for \( \theta \rightarrow \infty \) indicates that the Wilson line with a light-like segment satisfies an evolution equation different from (2.4). The modified equation looks like [15]

\[ \left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + 2\Gamma_{\text{cusp}}(\alpha_s) \ln[i(v \cdot n) \xi \mu] + \Gamma(\alpha_s) \right) \langle W_{\Pi}(v \cdot n \xi \mu) \rangle = 0. \]  (2.19)

Here the factor of 2 stems from the presence of two cusps on the \( n \)-shaped line contour and \( \Gamma(\alpha_s) \) is a process-dependent anomalous dimension. The explicit dependence of the anomalous dimensions on the renormalization scale \( \mu \) implies the absence of the multiplicative renormalizability of the light-like Wilson line. Combining together Eqs. (2.19) and (2.15), we obtain the renormalization group equation for local composite operators \( \langle O_f^J(\mu^2) \rangle \) at large \( J \). Matching its solution into (2.10), we find the asymptotic behavior of the anomalous dimensions of the twist-two quark operators for \( J \rightarrow \infty \)

\[ \gamma_f^{(qq)}(\alpha_s) = 2\Gamma_{\text{cusp}}(\alpha_s) \ln J + O(J^0). \]  (2.20)

Repeating a similar analysis for the twist-two gluon operators, one can show that their matrix elements satisfy (2.15) with the Wilson line defined in the adjoint representation. Therefore, their anomalous dimension satisfies (2.20) upon replacing \( C_F \rightarrow N_c \) leading to [15]

\[ \gamma_f^{(gg)}(\alpha_s) = \frac{N_c}{C_F} \gamma_f^{(qq)}(\alpha_s) + O(J^0). \]  (2.21)
In general, the quark and gluon operators mix with each other. However, at large $J$ the mixing occurs through the exchange of a soft quark with momentum $\sim 1/J$. Its contribution to the corresponding anomalous dimensions is suppressed by a power of $1/J$ leading to

$$\gamma_f^{(qg)}(\alpha_s) = \mathcal{O}(1/J), \quad \gamma_f^{(gq)}(\alpha_s) = \mathcal{O}(1/J).$$  \hfill (2.22)

We would like to stress that the relations (2.20)–(2.22) are valid to all orders in $\alpha_s$. Remarkably enough, they hold both in QCD and its supersymmetric extensions. In the latter case, the mixing matrix has a bigger size due to the presence of additional scalar fields. Nevertheless, this matrix remains diagonal at large $J$. Since the fields in supersymmetric YM theories belong to the adjoint representation, the diagonal matrix elements are the same

$$\gamma_f^{(ab)} = 2\delta_{ab} \Gamma_{\text{cusp}}(\alpha_s) \ln J + \mathcal{O}(J^0).$$  \hfill (2.23)

with $a, b = (q, g, s)$. However, one can not use for $\Gamma_{\text{cusp}}(\alpha_s)$ the two-loop expression (2.18) at $C_F = N_c$, because it was obtained within the dimensional regularization scheme which breaks supersymmetry and also lacks the contribution of possible scalars.

### 2.3 Cusp anomaly in supersymmetric theories

To calculate $\Gamma_{\text{cusp}}(\alpha_s)$ in a supersymmetric Yang-Mills theory, we use the regularization by dimensional reduction (DRED) [24, 25, 26]. In analogy to compactification, one gets this scheme by dimensionally reducing the four-dimensional theory down to $d = 4 - 2\varepsilon < 4$ dimensions. In comparison with the DREG scheme, the Lagrangian involves now the so-called epsilon-scalars generated by $2\varepsilon$ components of the four-dimensional gauge field. To two-loop accuracy, these scalar fields contribute to the Wilson loop by modifying the self-energy of a gluon at the level of $\mathcal{O}(\varepsilon)$ corrections. The calculation of the corresponding Feynman diagram is straightforward and details can be found in the Appendix A. It leads to the two-loop correction $-\left(\alpha_s^{\text{DR}}/\pi\right)^2 C_F N_c/12$ to the right-hand side of (2.18), resulting into

$$\Gamma_{\text{cusp}} = \frac{\alpha_s^{\text{DR}}}{\pi} N_c + \left(\frac{\alpha_s^{\text{DR}}}{\pi}\right)^2 N_c \left\{ N_c \left( \frac{16}{9} - \frac{\pi^2}{12} \right) - n_f \frac{5}{18} - n_s \frac{1}{9} \right\} + \mathcal{O}\left(\left(\alpha_s^{\text{DR}}\right)^3\right),$$  \hfill (2.24)

where $\alpha_s = \alpha_s^{\text{DR}}$ is the coupling constant in the DRED scheme with modified minimal subtractions. Here, in comparison with (2.18), we added the contribution of $n_s = n_{\tilde{g}} N_c$ real scalars and set $C_F = N_c$ since in a supersymmetric gauge theory fields belong to the adjoint representation of the $SU(N_c)$. Notice that one can obtain the same expression (2.24) by expanding (2.18) in powers of the coupling constant in the dimensional reduction scheme, $\alpha_s^{\text{DR}}$, which is related to the coupling...
constant in the dimensional regularization scheme, $\alpha_s^{\overline{\text{MS}}}$, by a finite renormalization \[27, 28\]
\[
\alpha_s^{\overline{\text{MS}}} = \alpha_s^{\overline{\text{DR}}} \left( 1 - \frac{N_c \alpha_s^{\overline{\text{DR}}}}{12} + \mathcal{O} \left( (\alpha_s^{\overline{\text{DR}}})^2 \right) \right). \tag{2.25}
\]

Eqs. (2.18) and (2.24) define the cusp anomalous dimensions in two different renormalization schemes, based on dimensional regularization and dimensional reduction, respectively. It is the latter scheme that does not break supersymmetry.

Using (2.20) and (2.24) we obtain the asymptotic behavior of the twist-two anomalous dimensions in various supersymmetric theories:

- In the $\mathcal{N} = 1$ YM theory, one has one Majorana fermion in the adjoint representation, $n_f = N_c$, and no scalars $n_s = 0$

\[
\Gamma_{\text{cusp}}^{\mathcal{N}=1} (\alpha_s) = \frac{\alpha_s N_c}{\pi} + \left( \frac{\alpha_s N_c}{\pi} \right)^2 \left( \frac{3}{2} - \frac{\pi^2}{12} \right) + \mathcal{O}(\alpha_s^3). \tag{2.26}
\]

- In the $\mathcal{N} = 2$ YM theory, one has two Majorana fermions in the adjoint representation, $n_f = 2N_c$, and two real scalar fields in the adjoint representation, $n_s = 2N_c$

\[
\Gamma_{\text{cusp}}^{\mathcal{N}=2} (\alpha_s) = \frac{\alpha_s N_c}{\pi} + \left( \frac{\alpha_s N_c}{\pi} \right)^2 \left( 1 - \frac{\pi^2}{12} \right) + \mathcal{O}(\alpha_s^3). \tag{2.27}
\]

- In the $\mathcal{N} = 4$ YM theory, one has four Majorana fermions in the adjoint representation, $n_f = 4N_c$, and six real scalar fields in the adjoint representation, $n_s = 6N_c$

\[
\Gamma_{\text{cusp}}^{\mathcal{N}=4} (\alpha_s) = \frac{\alpha_s N_c}{\pi} + \left( \frac{\alpha_s N_c}{\pi} \right)^2 \left( -\frac{\pi^2}{12} \right) + \mathcal{O}(\alpha_s^3). \tag{2.28}
\]

Notice that the two-loop correction to $\Gamma_{\text{cusp}} (\alpha_s)$ is positive in all cases except the $\mathcal{N} = 4$ theory. It becomes smaller as one goes from QCD to the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theory and, then, it becomes negative at $\mathcal{N} = 4$. We recall that $\Gamma_{\text{cusp}} (\alpha_s) > 0$ for arbitrary $\alpha_s$, Eq. (2.6).

Together with Eqs. (2.20)–(2.22), the expressions (2.26)–(2.28) establish the large-$J$ asymptotics of the anomalous dimensions of the twist-two operators in supersymmetric theories. Namely, the matrix of anomalous dimension is diagonal at large $J$ with the entries on the main diagonal equal to $2\Gamma_{\text{cusp}} (\alpha_s) \ln J$. Eq. (2.26) agrees with the results of explicit two-loop calculations in Ref. 29. Eq. (2.28) is in disagreement with the results of Ref. 23.\(^6\) Eq. (2.27) is a prediction since a two-loop calculation in that case has not been performed yet.

\(^5\)This simple “rule of substitutions” can be understood by noticing that $\Gamma_{\text{cusp}}$ governs the scale dependence of a physical observable, the Isgur-Wise form factor, and, therefore, it is renormalization scheme invariant.

\(^6\)Eq. (2.28) were in agreement with the results of Ref. 23 if one would assume that their result is given in the $\overline{\text{MS}}$ scheme and transforms the coupling constant to the $\overline{\text{DR}}$ scheme via (2.20).
3 Conformal operators and integrable spin chains

So far we have discussed two-particle composite operators. Let us now generalize our consideration to operators involving many fields. Such operators are of great phenomenological interest as their matrix elements define, e.g., baryon distribution amplitudes \[30\] and higher-twist corrections in various high-energy processes \[31\]. To start with, we define first a framework which solves partially the expected complications in the mixing problem in this case.

3.1 Two-particle conformal operators

The local twist-two operators \( \mathcal{O}_J \) can be obtained from the expansion of a nonlocal light-cone operator, cf. Eq. (2.8),

\[
\text{tr} \left\{ X_1(0) \Phi_n[0, \xi] X_2(\xi n) \Phi_{-n}[\xi, 0] \right\} = \sum_{J=0}^{\infty} \frac{(-i\xi)^J}{J!} \mathcal{O}_J(0).
\]

Here \( X_i(n\xi) \equiv X_i^a(n\xi)t^a \) denotes a general primary operator in the gauge theory defined in the adjoint representation of the \( SU(N_c) \) group, “living” on the light-cone \( n^2 = 0 \) and having definite quantum numbers with respect to transformations of the conformal group (see Eq. (3.3) below). The latter condition implies that \( X_i^a \) can be identified as the quasi-partonic operator \[22\], that is a scalar field, or a specific component of the quark field and the gluon strength tensor.

Two Wilson lines in (3.1) run along the light-cone in opposite directions between the points 0 and \( \xi \) and ensure the gauge invariance of the operator. The local twist-two operators \( \mathcal{O}_J \) were introduced in the previous section. Let us reinstate their definition again,

\[
\mathcal{O}_J(\xi) = \text{tr} \left\{ X_1(n\xi) \left( i n \cdot D \right)^J X_2(n\xi) \right\} = \text{tr} \left\{ X_1(\xi) \left( i \partial \right)^J X_2(\xi) \right\},
\]

where in distinction with the previous case, Eq. (2.9), the covariant derivative is defined in the adjoint representation, \( D_\mu X = \partial_\mu - ig[A_\mu, X] \). Here in the second relation we have chosen the gauge \( n \cdot A(x) = 0 \) and simplified the notation for the argument of the fields. The twist-two operators defined in this way are not renormalized multiplicatively and mix with operators containing total derivatives \( (i\partial)^l \mathcal{O}_{J-l}(\xi) \) with \( 1 \leq l \leq J \). Although the mixing can be neglected for forward matrix elements like (2.8), it survives in case one considers matrix elements with different momenta in the initial and final state, or when they are a part of multi-parton operators. In conformal theories, one can construct linear combinations of such operators, the so-called conformal operators \[32, 33, 34, 35, 36, 37\], in such a way that they are renormalized autonomously to all orders in the coupling constant.\(^7\) The mixing between these operators is protected by the \( SO(4, 2) \) conformal

---

\(^7\)In gauge theories with broken conformal symmetry, like QCD, this holds only to the lowest order in \( \alpha_s \).
symmetry of the gauge theory, more precisely, by its collinear \( SL(2, \mathbb{R}) \sim SU(1, 1) \) subgroup, which acts on the primary fields “living” on the light-cone as follows

\[
\xi \rightarrow \frac{a\xi + b}{c\xi + d}, \quad X_i(\xi) \rightarrow (c\xi + d)^{-2j_i} X_i\left(\frac{a\xi + b}{c\xi + d}\right),
\]

(3.3)

with \( a, \ldots, d \) real such that \( ad - bc = 1 \). Here \( j_i = (s_i + l_i)/2 \) is the conformal spin of the field \( X_i(\xi) \) equal to one half of the sum of its canonical dimension, \( 1 \), and projection of the spin onto the light-cone, \( \Sigma_{-+} X_i(\xi) = s_i X_i(\xi) \). By definition, the conformal operators \( \hat{O}_J(\xi) \) are composite operators built from the primary fields and satisfying (3.3) with \( j_i \) replaced by \( J + j_1 + j_2 \). It is easy to see that the operators (3.2) do not obey the latter condition.

To simplify the analysis, one chooses the axial gauge \( n \cdot A(x) = 0 \). Then, the operators in the left-hand side of (3.1) are given by the product of two primary fields. It is transformed under (3.3) in accordance with the direct product of two \( SL(2, \mathbb{R}) \) representations\(^8\) labelled by the spins \( j_1 \) and \( j_2 \). Decomposing this product into the sum of irreducible components, \([j_1] \otimes [j_2] = \sum_{J \geq 0} [J + j_1 + j_2]\), one can identify the spin-\( J \) component as defining the conformal operator \( \hat{O}_J(\xi) \). It has the following form

\[
\hat{O}_J(\xi) = \left. i^J (\partial_2 + \partial_1)^J P_j^{(\nu_1, \nu_2)} \left( \frac{\partial_2 - \partial_1}{\partial_2 + \partial_1} \right) \text{tr} \{X_1(\xi_1)X_2(\xi_2)\} \right|_{\xi_1 = \xi_2 = \xi},
\]

(3.4)

where \( \partial_a = \partial/\partial\xi_a \), \( \nu_a = 2j_a - 1 \) and \( P_j^{(\nu_1, \nu_2)} \) are the Jacobi polynomials. To restore gauge invariance, one has to substitute \( \partial X(\xi) = (n \cdot \mathcal{D}) X(\xi) \). Going back to (3.1), one obtains the operator product expansion for a nonlocal light-cone operator over the conformal operators, see, e.g., Refs. 38, 39, 40,

\[
\text{tr} \left\{X_1(0) \Phi_n[0, \xi] X_2(\xi n) \Phi_{-n}[\xi, 0]\right\} = \sum_{J = 0}^{\infty} C_J(\nu_1, \nu_2) \frac{(-i\xi)^J}{J!} \int_0^1 du u^{J + \nu_1} (1 - u)^{J + \nu_2} \hat{O}_J(u\xi),
\]

(3.5)

where

\[
C_J(\nu_1, \nu_2) = (2J + \nu_1 + \nu_2 + 1) \frac{\Gamma(J + 1) \Gamma(J + \nu_1 + \nu_2 + 1)}{\Gamma(J + \nu_1 + 1) \Gamma(J + \nu_2 + 1)}. \tag{3.6}
\]

The Lorentz operators \( \mathcal{O}_J \) can be re-expressed in terms of the conformal operators \( \hat{O}_J \) via

\[
\mathcal{O}_J = \sum_{j = 0}^J c_j^J(\nu_1, \nu_2) i^{J-j} (\partial_2 + \partial_1)^{J-j} \hat{O}_j,
\]

(3.7)

with the expansion coefficients \( c_j^J(\nu_1, \nu_2) = C_j(\nu_1, \nu_2) \int_0^1 du (1 - u)^{\nu_1 + \nu_2 + J} P_j^{(\nu_1, \nu_2)} (2u - 1) \), which can be calculated in terms of the hypergeometric function \( \, _3F_2 \). As was already mentioned, the

Beyond the leading order \( \hat{O}_J \)'s start to mix with \( (i\partial)^l \hat{O}_{J-l} \). The corresponding mixing anomalous dimensions are determined solely by the conformal anomalies, see the last paper of Ref. 36, Eqs. (51) and (113) for explicit expressions.

\(^8\)These are unitary representations of the \( SL(2, \mathbb{R}) \) group of the discrete series.
conformal operators evolve autonomously under renormalization and their anomalous dimensions have a universal scaling behavior at large $J$, Eqs. (2.20) and (1.2).

Let us consider the forward matrix element of the both sides of (3.5) for large light-cone separations, $\xi \gg 1$. In this limit, typical wavelength of gluons exchanged between two fields $X_1(0)$ and $X_2(\xi n)$ scales as $\xi$ and the eikonal approximation (2.2) is justified. This allows us to replace the quantum operators $X_i(\xi)$ in the left-hand side of (3.5) by Wilson lines in the adjoint representations of the $SU(N_c)$. Then, the left-hand side of (3.5) can be approximated as $\langle p|\tilde{W}_\Pi(\xi)|p\rangle e^{-i\xi(p\cdot n)}$, where the Wilson line in the adjoint representation, $\tilde{W}_\Pi(\xi)$, evaluated along the same $\Pi$-like contour as in (2.14). In the right-hand side of (3.5), one can neglect the contribution of operators with total derivatives since their forward matrix elements vanish. In addition, at large $\xi$ the sum is dominated by the contribution of operators with large spin $J$. In this way, $\langle p|\hat{O}_J(u\xi)|p\rangle \sim \langle p|O_J(0)|p\rangle \sim (p\cdot n)^J \mu^{-2\Gamma_{cusp}(\alpha_s) \ln J}$ and

$$\langle p|\tilde{W}_\Pi(\xi)|p\rangle e^{-i\xi(p\cdot n)} \sim \sum_J \frac{(-i\xi)^J}{J!} (p\cdot n)^J \mu^{-2\Gamma_{cusp}(\alpha_s) \ln[(p\cdot n)\mu]}.$$  (3.8)

For $\xi \gg 1$ the sum in the right-hand side of this relation receives the dominant contribution from $J \sim -i\xi(p\cdot n)$ leading to

$$\langle p|\tilde{W}_\Pi(\xi)|p\rangle \sim \langle p|\hat{O}_J(0)|p\rangle \bigg|_{J=-i\xi(p\cdot n)} \sim \mu^{-2\Gamma_{cusp}(\alpha_s) \ln[-i\xi(p\cdot n)]}.$$  (3.9)

This relation establishes a correspondence between the Wilson line in the adjoint representation and the matrix element of the conformal operator analytically continued to large complex values of the spin $J$. Notice that in the multi-color limit, $N_c \to \infty$, one has $\langle \tilde{W}_\Pi(\xi) \rangle = \langle W_\Pi(\xi) \rangle^2$ with $W_\Pi(\xi)$ defined in the fundamental representation.

### 3.2 Multi-particle conformal operators

Let us generalize the above analysis to conformal operators built from three and more primary fields. As before, we construct a nonlocal operator containing $N$ primary (quasi-partonic) fields on the light-cone and expand it in powers of light-cone separations

$$\text{tr} \left\{ X_1(0)\Phi_n[0,\xi_2]X_2(\xi_2)\ldots X_N(\xi_N)\Phi_{-n}[\xi_N,0] \right\} = \sum_{j_2\ldots j_N \geq 0} \frac{(-i\xi_2)^{j_2}}{j_2!} \ldots \frac{(-i\xi_N)^{j_N}}{j_N!} \mathcal{O}_{j_2\ldots j_N}(0).$$  (3.10)

Here the Wilson lines run between two adjacent fields along the light-cone to ensure the gauge invariance. The local composite operators have the form

$$\mathcal{O}_{j_2\ldots j_N}(\xi) = \text{tr} \left\{ X_1(\xi) (in\cdot \mathcal{D})^{j_2} X_2(\xi) \ldots (in\cdot \mathcal{D})^{j_N} X_N(\xi) \right\},$$  (3.11)
where, as before, the covariant derivative is defined in the adjoint representation. The evolution of these operators under renormalization group transformations is much more complicated as compared with the previous case due to larger number of particles involved and complicated color flow. The latter can be simplified by going over to the multi-color limit \( N_c \to \infty \). In this limit, in the axial gauge \( n \cdot A(x) = 0 \), the planar Feynman diagrams contributing to the left-hand side of (3.10) have the topology of a cylinder\(^9\) as shown in Fig. 1.

In the multi-color limit, the operators (3.11) mix under renormalization among themselves and with operators containing total derivatives. The mixing occurs between the operators with the same total conformal spin \( J + \sum_{k=1}^{N} j_{X_k} \) with \( J = j_2 + \ldots + j_N \) and their number grows rapidly with \( N \), even for the forward matrix elements. In the latter case, the dimension of the mixing matrix scales as \( \sim J^{N-2} \) for large \( J \). At \( N = 2 \) this matrix has a single element for arbitrary \( J \). For the number of particles \( N \geq 3 \) the problem of constructing multiplicatively renormalizable operators \( \hat{O}_J \) is reduced to diagonalization of the mixing matrix whose size grows with \( J \). As we will argue below, the same problem is equivalent to solving a Schrödinger equation for a Hamiltonian of a Heisenberg spin chain model. Before doing this, let us analyze the picture from the point of view of the Wilson line formalism.

### 3.2.1 Wilson line approach

Following the Wilson line approach [15], one can obtain the scaling behavior of the anomalous dimensions of conformal operators at large spin \( J \). To this end, we examine Eq. (3.10) for large

\(^9\)For \( N = 3 \) the result is exact for arbitrary \( N_c \).
light-cone separations \(\xi_2 \sim \xi_3 \sim \ldots \sim \xi_N \gg 1\). As in the previous case, the nonlocal light-cone operator in the left-hand side of (3.10) is dominated by the contribution of soft gluons with the wavelength \(\sim \xi_k\). This allows one to apply the eikonal approximation and replace the quantum fields by eikonal phases, the Wilson lines in the adjoint representation evaluated along rays that run along momenta of particles and terminate at the light-cone. Assuming for simplicity that all particles have the same momentum \(p_\mu\) we apply the eikonal approximation

\[
X_i(\xi) \to e^{-i\xi(p-n)} \Phi^\dagger_p[-\infty,\xi] b_i(p) \Phi_p[-\infty,\xi],
\]

(3.12)

where \(b_i(p) = b_i^a(p) t^a\) is the annihilation operator of the \(i\)-th particle and \(\Phi^\dagger_p[-\infty,\xi] = \Phi_{-p}[\xi,\infty]\). Then, the matrix element of the operator entering the left-hand side of (3.10) between the vacuum and the \(N\)-particle state \(\langle b_1^\dagger(p) b_2^\dagger(p) \ldots b_N^\dagger(p) \rangle |0\rangle\) is given in the multi-color limit by (see Fig. 1)

\[
\langle \text{tr} W_\Pi[\xi_1,\xi_2] \rangle \langle \text{tr} W_\Pi[\xi_2,\xi_3] \rangle \ldots \langle \text{tr} W_\Pi[\xi_N,\xi_1] \rangle e^{-i(p-n)(\xi_2 + \ldots + \xi_N)},
\]

(3.13)

with \(\xi_1 = 0\) and \(|\xi_{k+1} - \xi_k| \sim \xi \gg 1\). (In arriving at this relation we applied the “vacuum dominance” property, \(\langle \text{tr} W_1 \text{tr} W_2 \rangle = \langle \text{tr} W_1 \rangle \langle \text{tr} W_2 \rangle + O(1/N_c^2)\).) Here \(W_\Pi[\xi_2,\xi_3]\) is the Wilson line in the fundamental representation evaluated along the \(n\)-like contour that runs along the momentum \(p_\mu\) from \(-\infty\) to the point \(\xi_2 n_\mu\), then along the light-cone to \(\xi_3 n_\mu\) and returns to infinity along \(-p_\mu\). Due to the presence of two cusps on the integration contour, \(\langle \text{tr} W_\Pi[\xi_1,\xi_2] \rangle\) acquires the cusp anomalous dimension \(\sim \mu^{-\Gamma_cusp(\alpha_s) \ln[-i\xi(p-n)]}\). In this way, one finds from (3.13) that the left-hand side of (3.10) scales as

\[
\sim \mu^{-N\Gamma_cusp(\alpha_s) \ln[-i\xi(p-n)]} e^{-i(p-n)(\xi_2 + \ldots + \xi_N)}.
\]

(3.14)

Let us now examine the scaling behavior of the right-hand side of (3.10). As before, for large light-cone separations, \(\xi \gg 1\), the sum is dominated by the contribution of terms with \(j_2 \sim \ldots \sim j_N \sim (-i\xi)\), or equivalently \(J = \sum_k j_k \sim -i\xi N\). The corresponding composite operator \(O_{j_2 \ldots j_N}(0)\) is renormalized multiplicatively and has the anomalous dimension

\[
\gamma_J^{(\text{max})} = N\Gamma_cusp(\alpha_s) \ln J.
\]

(3.15)

We recall that this result was obtained in the multi-color limit \(N_c \to \infty\) for \(J \gg 1\).

Anomalous dimensions of the \(N\)-particle conformal operators are defined as eigenvalues of the mixing matrix. As we argue in the next section, they can be parameterized by the set of \(N-2\) nonnegative integers \(\ell = (\ell_1, \ldots, \ell_{N-2})\) such that \(0 \leq \ell_1 \leq \ldots \leq \ell_{N-2} \leq J\). Their total number

10Here we used the relation between the eikonal phases defined in the fundamental (\(\Phi\)) and adjoint (\(\tilde{\Phi}\)) representations, \(t^\dagger[\tilde{\Phi}]_{ab} = \Phi^\dagger t^a \Phi\).
equals the size of the matrix and grows at large $J$ as $J^{N-2}$. For given spin $J$, the possible values of the anomalous dimensions occupy the band

$$\gamma_{J}^{(\text{min})} \leq \gamma_{J}(\ell_{1}, \cdots, \ell_{N-2}) \leq \gamma_{J}^{(\text{max})}. \quad (3.16)$$

Equation (3.15) sets up the upper bound in the spectrum of the anomalous dimensions.

We recall that (3.15) was obtained from the analysis of the nonlocal operator in the left-hand side of (3.10) at large light-cone separations between the fields. To establish the lower bound in (3.16), one has to relax the latter condition by allowing two or more fields to be closely located on the light-cone. It is convenient to choose the axial gauge $n \cdot A(x) = 0$ and consider $\text{tr} \{X_{1}(0)X_{2}(\xi_{2}) \cdots X_{N}(\xi_{N})\}$ in the region $\xi_{2} \ll 1$ and $|\xi_{k+1} - \xi_{k}| \sim \xi \gg 1$ with $k \geq 2$. The fields $X_{1}(0)$ and $X_{2}(\xi_{2})$ are separated along the light-cone by a (relatively) short distance. They interact with each other by exchanging particles with short wavelengths, thus invalidating the eikonal approximation. At the same time, the interaction of these two fields with the remaining fields still occurs through soft gluon exchanges. Since the soft gluons with the wavelength $\sim \xi$ can not resolve the fields $X_{1}(0)$ and $X_{2}(\xi_{2})$, they couple to their total color charge. This means that one can replace the bilocal operator $X_{1}(0)X_{2}(\xi_{2})$ by its expansion over the two-particle conformal operators $\hat{O}_{j}(0)$ and apply the eikonal approximation to $\hat{O}_{j}(\xi_{2})$ and remaining fields $X_{3}(\xi_{3}), \ldots, X_{N}(\xi_{N})$ afterwards. Repeating the analysis one arrives at the same expression as (3.13) with the only difference that the factor $\langle \text{tr} W_{\Pi}[\xi_{1}, \xi_{2}] \rangle$ is missing. As a consequence, the anomalous dimension of the nonlocal light-cone operator is given at large $J$ by (3.15) with $N$ replaced by $N - 2$. In other words, the coefficient in front of $\Gamma_{\text{cusp}}(\alpha_{s}) \ln J$ in Eq. (3.15) counts the number of fields separated along the light-cone by large distances $\xi \sim iJ$.

This suggests that the minimal anomalous dimension in (3.16) corresponds to the configuration when all fields in the left-hand side of (3.10) are grouped into two clusters on the light-cone located at the points 0 and $\xi \sim iJ$, respectively, leading to

$$\gamma_{J}^{(\text{min})} = 2\Gamma_{\text{cusp}}(\alpha_{s}) \ln J, \quad (3.17)$$

in agreement with the results of Refs. [41, 42, 43, 44, 45, 46]. In this case, the right-hand side of (3.10) receives contribution from the whole tower of spin-$J$ conformal operators with the anomalous dimensions satisfying (3.16). Going over to the limit $\mu \to \infty$, one finds that the right-hand side of (3.10) receives dominant contribution from the operators with minimal anomalous dimension given by (3.17).

As a function of large spin $J$, the anomalous dimensions $\gamma_{J}(\ell_{1}, \cdots, \ell_{N-2})$ form the family of (non-intersecting) trajectories labelled by the integers $\ell_{1}, \cdots, \ell_{N-2}$. Since the width of the band (3.16) scales as $\ln J$, while the total number of anomalous dimensions $\gamma_{J}(\ell_{1}, \cdots, \ell_{N-2})$ grows as a
power of \( J \), one obtains that for \( J \gg 1 \) the distribution of the trajectories inside the band can be described by a continuous function whose explicit form depends on the coupling constant. Going over from the weak to the strong coupling one finds that the trajectories do not intercept as \( \alpha_s \) increases although their distribution inside the band \((3.16)\) is modified.

### 3.2.2 Hamiltonian approach to evolution equations

The above analysis relies on the properties of Wilson loops. Let us now develop a “dual” picture based on the properties of conformal operators. In the multi-color limit, the conformal operators are given by linear combinations of composite operators \((3.11)\) including the operators with total derivatives. As was already discussed above, to construct \( N \)-particle conformal operators, one has to decompose the tensor product of \( N \) representations of the \( SL(2, \mathbb{R}) \) group labelled by conformal spins, \( j_k \equiv j_{X_k} \), of the fields \( X_i(\xi) \) and project out the nonlocal operator \((3.10)\) onto the spin-\( J \) representation

\[
[j_1] \otimes [j_2] \otimes \ldots \otimes [j_N] = \sum_{j \geq 0} [J + j_1 + j_2 + \ldots + j_N]. \tag{3.18}
\]

Subsequently applying the rule for the sum of two \( SL(2, \mathbb{R}) \) spins, \([j_1] \otimes [j_2] = \sum_{j_{12} \geq 0} [j_1 + j_2 + j_{12}]\), one finds that the spin-\( J \) representation has a nontrivial multiplicity \( n_J = (J + N - 2)!/[J!(N - 2)!] \). It is uniquely specified by the “external” conformal spins \( j_1, \ldots, j_N \) and “internal” spins \( 0 \leq j_{12} \leq j_{13} \leq \ldots \leq j_{12\ldots N-1} \leq J \) with \( j_{12\ldots k} \) defining the total spin in the \((12\ldots k)\)-channel. Each irreducible spin-\( J \) component gives rise to the following local composite operator

\[
\hat{O}_J^{(j)}(\xi) = P_j^{(j)}(i \partial_1, \ldots, i \partial_N) \text{tr}\left\{ X_1(\xi_1)X_2(\xi_2)\ldots X_N(\xi_N) \right\} \bigg|_{\xi_1=\ldots=\xi_N=\xi}, \tag{3.19}
\]

where \( \{j\} \equiv (j_{12}, \ldots, j_{12\ldots N-1}) \) and \( P_j^{(j)}(x_1, \ldots, x_N) \) is a homogeneous polynomial of degree \( N \) in momentum fractions \( x_k \). This polynomial is the highest weight vector of the spin-\( J \) \( SL(2, \mathbb{R}) \) representation of the discrete series. It satisfies the system of differential equations

\[
\left( \bar{L}_1 + \ldots + \bar{L}_k \right)^2 P_j^{(j)} = J_{12\ldots-k} (J_{12\ldots-k} - 1) P_j^{(j)}, \quad (k = 2, \ldots, N),
\]

\[
(L_1^+ + \ldots + L_N^+) P_j^{(j)} = 0, \quad (3.20)
\]

with \( J_{12\ldots-k} = j_1 + \ldots + j_k + j_{12\ldots-k} \) and \( j_{12\ldots N} \equiv J \). Here \( \bar{L}_k = (L_k^+, L_k^-, L_k^0) \) are the \( SL(2, \mathbb{R}) \) generators in the “momentum” representation

\[
L_k^- = -x_k, \quad L_k^+ = 2j_k \partial x_k + x_k \partial^2 x_k, \quad L_k^0 = j_k + x_k \partial x_k, \quad (3.21)
\]

with \( \bar{L}^2 = (L^+L^- + L^-L^+)/2 + L_0^2 \). At \( N = 2 \) the solution to \((3.20)\) is given by Jacobi polynomials (see Eq. \((3.4)\)). For higher \( N \), it can be constructed iteratively as a product of the \( N = 2 \) solutions.
The operators (3.19) are transformed under the $SL(2,\mathbb{R})$ conformal transformations as primary fields with the same conformal spin $J + \sum_k j_k$. However, they do not have autonomous evolution and mix under renormalization with each other. To construct multiplicative conformal operators, one has to diagonalize the corresponding $n_J \times n_J$ mixing matrix. Its eigenstates define the coefficients $c_{\ell,j}$ of the expansion of the conformal operators over the basis (3.19) and the corresponding eigenvalues provide their anomalous dimensions

$$\hat{O}_{J,\ell}(\xi) = \sum_{\{j\}} c_{\ell,j} \hat{O}_j(\xi) \equiv P_{J,\ell}(i\partial_1, \ldots, i\partial_N) \text{tr} \{X_1(\xi_1)X_2(\xi_2)\ldots X_N(\xi_N)\} \bigg|_{\xi_1=\ldots=\xi_N=\xi}, \quad (3.22)$$

where the subscript $\ell = (\ell_1, \ldots, \ell_{N-2})$ with $0 \leq \ell_1 \leq \ldots \leq \ell_{N-2} \leq J$ enumerates different conformal operators, or equivalently homogenous polynomials $P_{J,\ell}(x_1, \ldots, x_N) = \sum_{\{j\}} c_{\ell,j} P_j^{(j)}(x_1, \ldots, x_N)$.

Thus, in distinction with the $N = 2$ case, the conformal symmetry alone does not allow one to construct multiplicatively renormalizable conformal operators for $N \geq 3$. Nevertheless, it reduces the problem to diagonalizing the mixing matrix of dimension $n_J$. This matrix has a number of remarkable properties. To begin with, we notice that the homogenous polynomials entering (3.19) are orthogonal to each other with respect to the scalar product

$$\langle J, \{j\} | J', \{j'\} \rangle = \int_0^1 [d^N x] x_1^{2j_1-1} \ldots x_N^{2j_N-1} P_j^{(j)}(x_1, \ldots, x_N) P_{j'}^{(j')}(x_1, \ldots, x_N) \sim \delta_{J,J'} \delta_{jj'}, \quad (3.23)$$

where $[d^N x] = dx_1 \ldots dx_N \delta(1-\sum_k x_k)$ and integration goes over $0 \leq x_k \leq 1$. This follows from the fact that the $SL(2,\mathbb{R})$ generators (3.21) are self-adjoint operators on the vector space endowed with the scalar product (3.23). Then, the mixing matrix can be interpreted as a Hamiltonian acting on the Hilbert space (3.23), $\langle J, \{j\} | \mathcal{H}_N | J', \{j'\} \rangle$. Denoting $|J, \ell \rangle \equiv P_{J,\ell}(x_1, \ldots, x_N)$, one can determine the anomalous dimensions of the conformal operators $\gamma_j(\ell)$ as solutions to the $N$-particle Schrödinger equation

$$\mathcal{H}_N |J, \ell \rangle = \gamma_j(\ell) |J, \ell \rangle \quad (3.24)$$

under additional (highest weight) condition for its eigenstates

$$L^+_\text{tot} |J, \ell \rangle = 0, \quad L^0_\text{tot} |J, \ell \rangle = (J + \sum_{k=1}^N j_k) |J, \ell \rangle, \quad (3.25)$$

with $L_\text{tot} \equiv \sum_{k=1}^N \hat{L}_k$ being the total conformal spin. The Hamiltonian $\mathcal{H}_N$ commutes with the total $SL(2,\mathbb{R})$ spin, $[\mathcal{L}_\text{tot}, \mathcal{H}_N] = 0$ and is a self-adjoint operator on the Hilbert space (3.23). This ensures that the anomalous dimensions $\gamma_j(\ell)$ take real values.

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11 One can obtain the same expression by subsequently applying the fusion rules to the product of $N$ primary fields.
In perturbation theory, the Hamiltonian $H_N$ can be obtained from explicit calculation of Feynman diagrams in the multi-color limit $N_c \to \infty$ \cite{22}. To the lowest order in $\alpha_s$, due to cylinder-like topology of the planar diagrams, the interaction occurs only between nearest neighbors (see Fig. 1)

$$
H_N = \frac{\alpha_s N_c}{\pi} \left( H_{12} + H_{23} + \ldots + H_{N1} \right) + \mathcal{O} \left( (\alpha_s N_c)^2 \right),
$$

(3.26)

where the two-particle Hamiltonian $H_{k,k+1}$ acts on the “coordinates” $x_k$ and $x_{k+1}$ only.\textsuperscript{12} Conformal invariance implies that $H_{k,k+1}$ depends on the sum of two $SL(2, \mathbb{R})$ spins

$$
(J_k + L_{k+1})^2 = J_{k,k+1}(J_{k,k+1} - 1) = -x_1 x_2 (\partial_1 - \partial_2)^2 + 2(j_2 x_1 - j_1 x_2)(\partial_1 - \partial_2) + (j_1 + j_2)(j_1 + j_2 - 1),
$$

(3.27)

where $\partial_k = \partial/\partial x_k$. The general expression for $H_{k,k+1}$ looks like

$$
H_{k,k+1} = \psi(J_{k,k+1}) + \ldots,
$$

(3.28)

where $\psi(x) = d\ln \Gamma(x)/dx$ is the Euler psi-function. Here ellipses denote additional, rational in $J_{k,k+1}$ terms which are subleading for $J_{k,k+1} \gg 1$. In distinction with the first term in the right-hand side of (3.28), they depend on the type of particles involved.

The two-particle Hamiltonian (3.28) has a universal form for large spins $J_{k,k+1} \gg 1$

$$
H_{k,k+1} = \ln J_{k,k+1} + \mathcal{O} \left( (J_{k,k+1})^0 \right).
$$

(3.29)

At $N = 2$ the Hamiltonian (3.26) equals $H_{N=2} = 2(\alpha_s N_c/\pi)H_{12}$ and its eigenvalues, $\gamma_{N=2}(J) \sim 2(\alpha_s N_c/\pi) \ln J$ for $J \gg 1$, define the anomalous dimension of the two-particle conformal operator at weak coupling, Eq. (2.20). To find the spectrum of anomalous dimension for $N \geq 3$, one has to solve the Schrödinger equation (3.24) for the $N$-particle Hamiltonian (3.26). It turns out the quantum-mechanical system with the Hamiltonian (3.24) possesses a hidden symmetry and is intrinsically related to Heisenberg spin magnets.

We would like to emphasize that the equivalence between evolution equations and dynamical Hamiltonian systems is a rather general phenomenon in Yang-Mills theories. In particular, similar integrable structures appear in the Regge asymptotics \cite{17} with the evolution “time” being the logarithm of the energy scale $t_{\text{Regge}} = \ln s$ and in the low-energy behavior of the $N = 2$ effective action with the “time” $t_{\text{SYM}} = \ln \Lambda_{\text{QCD}}$ \cite{48}.\textsuperscript{12}

\textsuperscript{12}We recall that $x_k$ has the meaning of the momentum fraction carried by the particle described by the field $X_k(\xi_k)$.
3.2.3 Heisenberg spin chains

Let us substitute (3.28) into (3.26) and define the following Hamiltonian

$$\mathcal{H}_{N}^{\text{quan}} = \frac{\alpha_{s} N c}{\pi} \sum_{k=1}^{N} \psi(J_{k,k+1}),$$  \hspace{1cm} (3.30)

with $J_{N,N+1} \equiv J_{N,1}$. Notice that, in general, it differs from the exact QCD Hamiltonian (3.26) by terms that vanish for $J_{k,k+1} \gg 1$. Therefore one should expect that in spite of the fact that the fine structure of the energy spectrum of two Hamiltonians may be different, their asymptotic behavior for $J \gg 1$ is the same. Exact calculations at $N = 3$ confirm such expectations [41, 42, 43, 44, 45, 46]. The main advantage of dealing with (3.30) is that the Hamiltonian (3.30) possesses a set of integrals of motion, $q = (q_2, \ldots, q_N)$,

$$[\mathcal{H}_{N}^{\text{quan}}, q_n] = [q_n, q_m] = 0,$$  \hspace{1cm} (3.31)

and, as a consequence, the Schrödinger equation for $\mathcal{H}_{N}^{\text{quan}}$ turns out to be completely integrable [41, 42, 43, 44, 45, 46]. The Hamiltonian (3.30) is well-known in the theory of integrable models. It has been constructed in [49, 50, 51, 52] as a generalization of the celebrated spin-1/2 XXX Heisenberg magnet to higher spin representations of the $SU(2)$ and $SL(2,\mathbb{R})$ groups.

Eq. (3.30) defines periodic Heisenberg spin chain model of length $N$ and spins being the $SL(2,\mathbb{R})$ generators. The value of the spin at the $k$-th site is given by the conformal spin of the corresponding primary field. Such identification allows one to solve the spectral problem for the Hamiltonian (3.30) exactly by the Quantum Inverse Scattering Method [49, 50, 51, 52, 53, 54]. In particular, using the Lax operator for the XXX Heisenberg spin magnet

$$\mathbb{L}_k(u) = \begin{pmatrix} u + iL_k^0 & iL_k^+ \\ iL_k^- & u - iL_k^0 \end{pmatrix},$$  \hspace{1cm} (3.32)

with $L_k^0$ and $L_k^\pm$ being the $SL(2,\mathbb{R})$ generators (3.21), one can obtain the explicit form of the integrals of motion $q_2, \ldots, q_N$ from the expansion of the transfer matrix $t_N(u)$ in powers of the spectral parameter $u$

$$t_N(u) = \text{tr} [\mathbb{L}_1(u)\mathbb{L}_2(u) \ldots \mathbb{L}_N(u)] = 2u^N + q_2 u^{N-2} + \ldots + q_N,$$  \hspace{1cm} (3.33)

with $q_2 = -J_{\text{tot}}^2 + \sum_{k=1}^{N} j_k (j_k - 1)$ depending on the total spin of the system $J$. Due to complete integrability of the model, the energy spectrum is uniquely specified by their eigenvalues, $E_N = E_N(q_2, \ldots, q_N)$. Applying the Bethe Ansatz [49, 50, 51, 52, 53, 54], one can calculate explicitly both the energy spectrum and the corresponding eigenfunctions. We recall that the former defines the anomalous dimensions of conformal operators, while the latter determine the polynomials $P_{J,\ell}(x_1, \ldots, x_N)$ entering (3.22).
For our purposes, we are interested in finding the large $J$ asymptotic behavior of the energy spectrum. Assuming for simplicity that the particles have the same $SL(2,\mathbb{R})$ spin, $j_1 = \ldots = j_N \equiv j$ one obtains the following asymptotic expression for the energy $[55, 42, 44]$

$$
\gamma_J(\ell) = \frac{\alpha_s N_c}{\pi} \left\{ \ln 2 + \Re \left[ \sum_{k=1}^{N} \psi(j + i\lambda_k) + \gamma_E \right] + O(1/J^2N) \right\},
$$

(3.34)

where $\lambda_k$ are roots of the transfer matrix (3.33), $t_N(u) = 2\prod_{k}(u - \lambda_k)$. According to (3.33), the quantum numbers $q_k$ are given by symmetric polynomials of degree $k$ in $\lambda_1, \ldots, \lambda_N$. For $J \gg 1$ one finds that the roots are real and they scale differently with $J$ in the upper and lower part of the spectrum. In the upper part of the spectrum, all roots scale as $\lambda_k \sim J$ with $k = 1, \ldots, N$ leading to $q_k = O(J^{k})$

$$
\gamma_J^{(\text{max})} = \frac{\alpha_s N_c}{\pi} \ln |\lambda_1 \lambda_2 \ldots \lambda_N| = \frac{\alpha_s N_c}{\pi} \ln q_N = N \frac{\alpha_s N_c}{\pi} \ln J.
$$

(3.35)

In the lower part of the spectrum, $\lambda_1 \sim -\lambda_2 \sim O(J)$ and $\lambda_k \sim J^0$ for $k \geq 3$ (notice that $\sum_k \lambda_k = 0$ due to absence of the $u^{N-1}$ term in the right-hand side of (3.33)). This leads to $q_k \sim J^2$ and

$$
\gamma_J^{(\text{min})} = \frac{\alpha_s N_c}{\pi} \ln |\lambda_1 \lambda_2| = 2 \frac{\alpha_s N_c}{\pi} \ln J.
$$

(3.36)

We observe a perfect agreement of these expressions with Eqs. (3.15) and (3.17) obtained within the Wilson line approach.

In the Wilson line approach, the asymptotic behaviour of the anomalous dimensions, Eqs. (3.15) and (3.17) is tied to the scaling scaling behaviour of the two-particle spins $J_{k,k+1}$ while in the Hamiltonian approach Eqs. (3.35) and (3.36) follow from similar behaviour of the roots $\lambda_k$ of the transfer matrix (3.33). Let us demonstrate that $\lambda_k \sim J_{k,k+1}$ at large $J$.

By the definition, $J_{k,k+1}$ is the sum of two $SL(2,\mathbb{R})$ spins. Its eigenvalues satisfy $j_k + j_{k+1} \leq J_{k,k+1} \leq J$, where $J$ is the total spin of $N$ particles. Since $[H_N, J_{k,k+1}] \neq 0$, one can not assign a definite value of $J_{k,k+1}$ to the eigenstates of the Hamiltonian $H_N$. Nevertheless, for $J \to \infty$ the system of $N$ particles approaches a quasiclassical regime in which quantum fluctuations are frozen and the spins $J_{k,k+1}$ can be treated as classical variables. To see this one notices that for quantum-mechanical systems defined by the Hamiltonian (3.30) the “effective” Planck constant equals unity, $\hbar = 1$, and the energy scale is defined by the total spin $J$. This suggests that for $J \gg \hbar$ one can solve the Schrödinger equation (3.24) by the WKB methods [55].

In the WKB approach, one looks for the solution to (3.24) in the form

$$
|J,\ell\rangle \equiv P_{J,\ell}(x_1, \ldots, x_N) = \exp \left( \frac{i}{\hbar} S_0(\vec{x}) + iS_1(\vec{x}) + O(\hbar) \right),
$$

(3.37)

where $\vec{x} = (x_1, \ldots, x_N)$. To find the functions $S_0(\vec{x})$, $S_1(\vec{x})$, ..., one requires that the wave function (3.37) has to be an eigenstate of the transfer matrix (3.33), or equivalently diagonalize...
simultaneously the integrals of motion \( q_2, \ldots, q_N \). In this way, one obtains that the leading term \( S_0(\vec{x}) \) satisfies the Hamilton-Jacobi equations in the underlying classical system, while subleading terms can be expressed in terms of \( S_0(\vec{x}) \). To go over to the classical limit, one applies the operator of the two-particle spin defined in (3.27) to the WKB wave function (3.37)

\[
\hbar^2 (\vec{L}_k + \vec{L}_{k+1})^2 e^{iS_0/\hbar} = \left\{ x_k x_{k+1} (p_k - p_{k+1})^2 + \mathcal{O}(\hbar) \right\} e^{iS_0/\hbar},
\]

where we have restored in the left-hand side the dependence on the Planck constant and \( p_k = -i\hbar \partial_{x_k} S_0(\vec{x}) \) is a classical momentum of the \( k \)-th particle. In a similar manner, the \( SL(2, \mathbb{R}) \) spin operators (3.21) can be replaced by classical functions on the phase space of \( N \) particles

\[
L_{k, cl}^- = -x_k, \quad L_{k, cl}^+ = -x_k p_k^2, \quad L_{k, cl}^0 = ix_k p_k.
\]

Notice that the dependence on a single-particle spin disappears since \( j_k = \mathcal{O}(\hbar) \). One verifies that, in agreement with (3.38),

\[
(\vec{L}_{k, cl} + \vec{L}_{k+1, cl})^2 = (J_{k, k+1}^{cl})^2 = x_k x_{k+1} (p_k - p_{k+1})^2.
\]

Here \( J_{k, k+1}^{cl} \) defines the classical limit of the two-particle spin \( J_{k, k+1} \). Then, replacing \( J_{k, k+1} \to \hbar J_{k, k+1}^{cl} \) in (3.30), one expands the Hamiltonian \( H^{\text{quan}}_N \) in powers of \( \hbar \) and identifies the leading term of the expansion as the Hamiltonian of the classical model

\[
H^{cl}_N = \frac{\alpha_s N_c}{2\pi} \sum_{k=1}^{N} \ln \left( J_{k, k+1}^{cl} \right)^2.
\]

Remarkably enough, this Hamiltonian inherits integrability properties of the quantum model. It contains a set of integrals of motion \( q_k^{cl} \) (\( k = 2, \ldots, N \))

\[
\{ \mathcal{H}^{cl}_N, q_k^{cl} \} = \{ q_n^{cl}, q_k^{cl} \} = 0,
\]

with the Poisson bracket defined as \( \{ f, g \} = \partial_x f \partial_p g - \partial_p f \partial_x g \). To obtain their explicit form, one replaces the \( SL(2, \mathbb{R}) \) spin operators in (3.32) by their classical counterparts (3.39) and substitutes the resulting expression for the Lax operator into (3.38). This leads to

\[
q_n^{cl} = \sum_{1 \leq j_1 < \cdots < j_n \leq N} x_{j_1} \ldots x_{j_{n-1}} x_{j_n} (p_{j_1} - p_{j_2}) \ldots (p_{j_{n-1}} - p_{j_n}) (p_{j_n} - p_{j_1}),
\]

with \( n = 2, \ldots, N \). One observes that \( (q_n^{cl})^2 = \prod_{k=1}^{N} (J_{k, k+1}^{cl})^2 \) and, therefore, the classical Hamiltonian (3.41) can be written as

\[
\mathcal{H}^{cl}_N = \frac{\alpha_s N_c}{\pi} \ln q_N^{cl}.
\]

By construction, \( \mathcal{H}^{cl}_N \) defines a classical limit of the Heisenberg \( SL(2, \mathbb{R}) \) spin magnet model. Evaluating \( \mathcal{H}^{cl}_N \) along the orbits of classical motion of \( N \) particles, one obtains the energy spectrum
of the quantum magnet to the leading order of the WKB expansion, or equivalently the large-
J behavior of the anomalous dimensions \( \gamma_J(\ell) \) of conformal operators, Eq. (3.24). According
to (3.44), this behavior is controlled by the large-J scaling of the “highest” integral of motion
\( q_N^\dagger \). In the WKB approach, the spectrum of \( q_N \) is obtained by imposing the Bohr-Sommerfeld
quantization conditions on the periodic orbits of classical motion of \( N \) particles [55]. As was shown
in Ref. [42, 43, 44, 45], the WKB spectrum of anomalous dimensions derived in this manner is in
agreement with the exact expressions (3.35) and (3.36).

Finally, let us establish the relation between the roots \( \lambda_k \) of the transfer matrix and the two-
particle spins \( J_{k,k+1}^{cl} \). Substituting \( t_N(u) = 2\prod_k(u - \lambda_k) \) into (3.33) one obtains that the roots
parameterize the eigenvalues of the integrals of motion \( q_n \). At large \( J \), replacing \( q_n \) by their
classical counterparts defined in (3.43), one finds that \( q_n^{cl} \) are given by symmetric polynomials
in \( \lambda_1, \ldots, \lambda_N \) of degree \( n \) (with \( n = 2, \ldots, N \)). Therefore, the large-J behavior of the momenta
\( p_k - p_{k+1} \) and the roots \( \lambda_k \) are in one-to-one correspondence with each other. In particular, in the
upper part of the spectrum, Eq. (3.35), one gets from \( \lambda_k \sim J \) that \( p_k - p_{k+1} \sim J \), or equivalently
\( J_{k,k+1}^{cl} \sim J \) for \( k = 1, \ldots, N \). In similar manner, in the lower part of the spectrum, Eq. (3.36),
\( \lambda_k \sim J^0 \) leads to \( p_k - p_{k+1} \sim J^0 \) and \( J_{k,k+1}^{cl} \sim J^0 \). We recall that the \( x_k \)-variables entering (3.40)
have the meaning of momentum fractions carried by \( N \) particles described by quantum fields
\( X_k(\xi_k) \). Then, conjugated to them the \( p_k \)-variables are the light-cone coordinates of the same
fields, \( p_k \equiv \xi_k \). Thus, in the upper and the lower part of the spectrum one has \( \xi_k - \xi_{k+1} \sim J \)
and \( \xi_k - \xi_{k+1} \sim J^0 \), respectively. These properties are in agreement with the results obtained in
section 3.2.1 within the Wilson line approach.

We notice that the anomalous dimensions of \( N \)-particle conformal operators, Eq. (3.35) and
(3.36), were obtained using the lowest-order expression for the QCD evolution kernels whereas
Eqs. (3.15) and (3.17) hold to all orders in the coupling constant. The above analysis suggests
that at large \( J \) the higher order corrections modify the classical Hamiltonian (3.41) and (3.44) in
the following way

\[
\mathcal{H}_{N}^{cl} = \frac{1}{2} \Gamma_{\text{cusp}}(\alpha_s) \sum_{k=1}^{N} \ln (J_{k,k+1}^{cl})^2 = \Gamma_{\text{cusp}}(\alpha_s) \ln q_N^{cl},
\]

with \( J_{k,k+1}^{cl} \) and \( q_N^{cl} \) given by the same expressions as before, Eqs. (3.40) and (3.43), respectively.

4 Cusp anomaly at weak coupling

Let us revisit the computation of the cusp anomalous dimension to the lowest order of perturbation
theory aiming on an analogy with the stringy computation of Wilson loops within the AdS/CFT
framework. As we will see momentarily, the cusp anomaly in the weak coupling regime can be
interpreted as a \textit{quantum} transition amplitude for a test particle propagating in the radial time $\ln r$ and the angular coordinate $\theta$. This should be compared with the strong coupling calculation \cite{8, 9, 10}, in which the same quantity is given by a \textit{classical} action function for a particle propagating on the same phase space.

### 4.1 Cusp anomaly in perturbation theory

To the lowest order in the coupling constant, the Wilson line expectation value is given by

$$ W = \langle P \exp \left( i \oint_C A^\mu \right) \rangle = 1 + \frac{ig^2}{2} t^a t^a \int_C dx_\mu \int_C dy_\nu \, D^{\mu\nu} (x-y) + \mathcal{O}(g^4), $$

(4.1)

where $D^{\mu\nu}(x-y)\delta^{ab} = \langle 0 | T A^a_\mu(x) A^b_\nu(y) | 0 \rangle$ is a gluon propagator and $t^a t^a = N_c$ is the Casimir operator in the adjoint representation of the $SU(N_c)$. To calculate the cusp anomaly we choose the integration contour $C$, see Fig. 2 (left), to be the same as for the Isgur-Wise form factor, Eq. (2.3). In this way, we obtain

$$ W(v \cdot v') = 1 - \frac{\alpha_s N_c}{\pi} \left( w(v \cdot v') - w(1) \right) + \mathcal{O}(\alpha_s^2), $$

(4.2)

where $v_\mu$ and $v'_\mu$ are tangents to the integration contour in the vicinity of the cusp, $v^2 = v'^2 = 1$, $v \cdot v' = \cosh \theta$, $w(1) = w(v \cdot v) = w(v' \cdot v')$ and

$$ w(v \cdot v') = \int_{-\infty}^0 ds \int_0^\infty dt \frac{v \cdot v'}{(vs - vt')^2}, $$

(4.3)

with $s$ and $t$ being proper times. Going over to higher orders in $\alpha_s$, one takes into account that the Wilson loop possesses the property of non-abelian exponentiation \cite{56}

$$ \ln W = \sum_k \left( \frac{\alpha_s N_c}{\pi} \right)^k w_k, $$

(4.4)

where the weights $w_k$ receive contribution from diagrams to the $k$-th order in $\alpha_s$ with maximally nonabelian color structure. In our case, the exponentiation property states that $w_1 = w(v \cdot v') - w(1)$.

It is straightforward to perform the integration in (4.3). For our purposes, however, we change the integration variables to $r_\max = \max(t, -s)$ and $r_\min = \min(t, -s)$ and apply the identity

$$ \frac{v \cdot v'}{(vs - vt')^2} = -\frac{1}{r_\max^2} \left\{ \sum_{n=1}^\infty (-1)^n \binom{n}{r_\min} \frac{(n+1)^2}{n(n+2)} U_n(v \cdot v') + \frac{1}{4} \right\}, $$

(4.5)

where $U_n(\cosh \theta) = \sinh((n+1)\theta)/\sinh \theta$ are Chebyshev polynomials of the second kind \cite{57}. Its substitution into (4.3) leads to

$$ w(v \cdot v') = -2 \left\{ \sum_{n=1}^\infty (-1)^n \frac{n+1}{n(n+2)} U_n(\cosh \theta) + \frac{1}{4} \right\} \int_{r_\min}^{r_\max} \frac{dr_\max}{r_\max} = \theta \coth \theta \ln(\mu r_\max), $$

(4.6)
where $r_{\text{min}} \sim 1/\mu$ and $r_{\text{max}}$ are ultraviolet and infrared cut-offs, respectively. Combining together (4.2) and (4.6), one verifies that the Wilson line satisfies the evolution equation (2.4).

Eqs. (4.5) and (4.6) have a simple interpretation within the radial quantization approach [58, 59]. In this formalism, one performs a quantization procedure using four-dimensional (Euclidean) polar coordinates $r^2 = x^2_\mu$ and $v_\mu = x_\mu/r$. This allows one to separate the dynamics in the radial and angular coordinates by decomposing the propagators of fields over partial waves defined as eigenstates of the operator of the angular momentum

$$L^2 = \frac{1}{8} \epsilon^{\mu\nu\lambda} l_{\mu\nu}, \quad l_{\mu\nu} = i (x_\mu \partial_\nu - x_\nu \partial_\mu),$$

which has the meaning of the Laplace-Beltrami operator on a sphere $v_\mu^2 = 1$. Then, in the radial quantization the lowest order contribution to the cusp anomaly, Eq. (4.6), takes the factorized form [59]

$$w(v \cdot v') = -\frac{1}{2} \left( \langle -v' | 1/L^2 | v \rangle + 1 \right) \int d \ln r,$$

where $|v\rangle$ denotes a point on the (hyper)sphere $SO(3,1)/SO(3)$ defined by the unit vector $v_\mu$ and additional minus sign inside $\langle -v' |$ takes into account that two tangents have opposite orientation at the cusp. Thus at the weak coupling, the cusp anomalous dimension is given by

$$\Gamma_{\text{cusp}}(\theta; \alpha_s) = -\frac{\alpha_s N_c}{2\pi} \left( \langle -v' | 1/L^2 | v \rangle - \langle v | 1/L^2 | v \rangle \right) + \mathcal{O}(\alpha_s^2).$$

Let us demonstrate that the matrix elements entering this expression coincide with the propagator of a test particle on the time-like hyperboloid $v_\mu^2 = v_0^2 - v_1^2 - v_2^2 - v_3^2 = 1$.

### 4.2 Particle propagation on a sphere

To avoid complications due to the infinite volume of the (noncompact) $SL(2,\mathbb{R})$ group manifold, we will calculate the propagator of a particle on a unit sphere $S^3$ and perform an analytic continuation from Euclidean to Minkowski signatures afterwards.
The transition amplitude for the particle on the $S^3$ sphere to go from the point $v_\mu$ to $-v'_\mu$ is equal to the sum over all paths $P$ connecting these two points, Fig. 3

$$G[v, -v'] = \sum_{P \in S^3} e^{-iA[P]},$$  \hspace{1cm} (4.10)

where summation goes over connected paths $P$ of the length $A[P]$ on the $S^3$ sphere. With an arbitrary point on the sphere, $v_\mu \in S^3$, one can associate an element of the $SU(2)$ group

$$g_v = v_0 + i \sum_{a=1}^{3} v_a \sigma_a, \quad \text{tr} [g_{v_1}^{-1} g_{v_2}] = 2(v_1 \cdot v_2),$$  \hspace{1cm} (4.11)

with $\sigma_a$ being Pauli matrices. Eq. (4.10) defines a quantum dynamics of a particle on the $SU(2)$ group manifold. Introducing local (angular) coordinates on the sphere $X^a = (\rho, \tau, \phi)$

$$v_0 = \cos \rho \cos \tau, \quad v_1 = \sin \rho \sin \phi, \quad v_2 = \sin \rho \cos \phi, \quad v_3 = \cos \rho \sin \tau,$$  \hspace{1cm} (4.12)

one obtains the metric on this manifold as

$$ds^2 = \frac{1}{2} \text{tr} [g_v^{-1} dg_v]^2 = -d\rho^2 - \sin^2 \rho d\phi^2 - \cos^2 \rho d\tau^2 \equiv G_{MN}(X)dX^M dX^N.$$

The action of a particle has the meaning of the length of the path on this manifold

$$A[P] = \int_0^1 d\sigma \sqrt{-G_{MN}(X)\partial_\sigma X^M \partial_\sigma X^N} = \int_0^1 d\sigma \left\{ \frac{e(\sigma)}{4} - \frac{1}{e(\sigma)} G_{MN}(X)\partial_\sigma X^M \partial_\sigma X^N \right\},$$  \hspace{1cm} (4.14)

where $\sigma$ is a local coordinate on the trajectory. The two expressions coincide upon extremizing with respect to the einbein field $e(\sigma)$. 

Figure 3: Transition amplitude on a sphere between two points $v_\mu$ and $-v'_\mu$ with $\theta_2 - \theta_1 = \pi - \theta$ and $\tau_2 - \tau_1 = \tau$ (left). Classical trajectories which saturate the amplitude are multiple windings around the sphere over principles circles, e.g., $\ell = 2$ trajectories (right), separated from each other for illustration purposes only.
Following, for instance, [60], one can express this path integral as an integral over the proper time \( \tau = \int_0^1 d\sigma e(\sigma) \) of the matrix element of the transition operator

\[
G[v, -v'] = -i \int_0^\infty d\tau \langle -v' | e^{i\tau \mathcal{H}} | v \rangle = \langle -v' | 1/\mathcal{H} | v \rangle .
\] (4.15)

Here the Hamiltonian \( \mathcal{H} = \mathbf{L}^2 \) is the Laplace-Beltrami operator on the \( S^3 \). It defines a quantum-mechanical model (symmetric top) with the \( SU(2) \) dynamical symmetry. Since the Hamiltonian coincides with the Casimir operator of this group, its eigenstates correspond to unitary irreducible \( SU(2) \) representations of spin \( j = 0, \frac{1}{2}, 1, \ldots \)

\[
\mathcal{H} Y^j_{m_1 m_2}(g) = (j + 1) Y^j_{m_1 m_2}(g) , \quad Y^j_{m_1 m_2}(g) = (2j + 1)^{1/2} d^j_{m_1 m_2}(g) ,
\] (4.16)

where \( d^j_{m_1 m_2}(g) = \langle j, m_1 | d^j(g) | j, m_2 \rangle \), the matrix elements of the spin-\( j \) group representation \(-j \leq m_1, m_2 \leq j\), are the well-known Wigner functions [61]. The transition matrix element can be expanded over characters of the \( SU(2) \) representation

\[
\langle -v' | e^{i\tau \mathcal{H}} | v \rangle = \sum_{j=0,\frac{1}{2},1,\ldots} (2j + 1) \text{tr} [d^j(g^{-1}_v d^j(g_v))] e^{ij(j+1)}
\]

\[
= \sum_{j=0,\frac{1}{2},1,\ldots} (2j + 1) \chi_j[g^{-1}_v g_v] e^{ij(j+1)} .
\] (4.17)

Here, the \( SU(2) \) character is defined as [61]

\[
\chi_j[g_v] = \sum_{m=-j}^j d^j_{mm}(g_v) = \frac{\sin(2j + 1)\Theta/2}{\sin \Theta/2} ,
\] (4.18)

where the Euler angle is defined as \( \cos(\Theta/2) = \text{tr} g_v/2 \) with \( g_v \) given by (4.11) leading to \( \Theta = 2(\pi - \theta) \). Substituting (4.18) into (4.17) and making use of the second relation in (4.11), one calculates the matrix element entering (4.15) as

\[
\langle -v' | e^{i\tau L^2} | v \rangle = \sum_{j=0,\frac{1}{2},1,\ldots} (2j + 1) e^{ij(j+1)} \frac{\sin((2j + 1)\tilde{\theta})}{\sin \tilde{\theta}} ,
\] (4.19)

where \( \tilde{\theta} = \pi - \theta \) is the angle between the vectors \( v_\mu \) and \( -v'_\mu \). Substituting (4.19) into (4.15), one notices that the \( j = 0 \) term in (4.19) leads to a divergent contribution upon integration in the right-hand side of (4.15). It does not depend, however, on the cusp angle \( \theta \) and cancels in the difference \( G[v, -v'] - G[v, v] \) leading to

\[
F_{\text{cusp}}(\theta; \alpha_s) = -\frac{\alpha_s N_c}{2\pi} \left( G[v, -v'] - G[v, v] \right) = \frac{\alpha_s N_c}{\pi} (\theta \cot \theta - 1) ,
\] (4.20)

where \( \theta \) is the Euclidean cusp angle \( \langle v \cdot v' \rangle = \cos \theta \). Going over to Minkowski space, we continue \( \theta \to i\theta \) and reproduce the correct expression for the cusp anomalous dimension, Eq. (2.5), upon identification of the Casimirs \( C_F = N_c \).
The following comments are in order. The sum in (4.19) can be expressed, via the Poisson summation formula, as a derivative of Jacobi theta functions, namely,

\[
\langle -v'|q^L^2|v \rangle = \frac{1}{2q^{1/4}\sin \theta} \frac{\partial}{\partial \tilde{\theta}} \left[ \theta_2(\tilde{\theta}/\pi, q) + \theta_3(\tilde{\theta}/\pi, q) \right],
\]

where \( q = e^{i\tau} \) and \( \tilde{\theta} = \pi - \theta \). One can rewrite the same expression as

\[
\langle -v'|e^{i\tau}L^2|v \rangle = -\frac{1}{\pi \sin(\pi \tau)} \frac{\partial}{\partial \tilde{\theta}} \sum_{\ell=-\infty}^{\infty} e^{-i(\tilde{\theta}+2\pi\ell)^2/\tau-i\tau/4}.
\]

Eq. (4.21) has a remarkably simple physical meaning. As was shown in [62, 63, 64, 65], see also Refs. [66, 67], the semiclassical expression for the transition amplitude on the \( SU(2) \) group manifold coincides with the exact solution, Eq. (4.21), i.e. the path integral collapses from a sum over all paths to a sum over classical paths. In the semiclassical approach, the right-hand side of (4.21) comes about as a sum over classical trajectories “dressed” by quadratic fluctuations. The classical trajectories are geodesics and run along the principal circle on the unit sphere between the two points, \( v_\mu \) and \( -v'_\mu \), and wrap around this circle \( \ell \)-times in the (anti-)clockwise direction depending on the sign of \( \ell \). The trajectories fall into two homotopy classes depending on the \( \ell \)-parity. Denoting \( \theta(\sigma) \) the angular variable on this circle, one can parameterize classical trajectories as \( \theta(\sigma) = (\tilde{\theta} + 2\pi\ell)\sigma/\tau \) with \( 0 \leq \sigma \leq \tau \). The metric (4.13) on the classical trajectories equals

\[
ds^2 = G_{MN}dX^M dX^N = -d\theta^2
\]

and the classical action (4.14) is given in the gauge \( e(\sigma) = \text{const} \) by

\[
A_{cl}[P] = \int_0^\tau d\sigma \left( \frac{1}{4} + \dot{\theta}(\sigma)^2 \right) = \frac{\tau}{4} + \frac{1}{\tau}(\tilde{\theta} + 2\pi\ell)^2.
\]

Coming back to the original sum (4.10), we conclude that the exact expression for the transition amplitude (4.21), and as a consequence the cusp anomalous dimension (4.20), is given by the sum over classical trajectories. This property of the path integral is a manifestation of the Duistermaat-Heckman “localization” phenomenon in quantum dynamical systems on Lie groups [68].

The derivation of (4.19) was based on the identification of the unit sphere \( S^3 \) as the \( SU(2) \) group manifold. Going over from Euclidean to Minkowski kinematics, one has to substitute the sphere \( S^3 \) by the time-like hyperboloid \( H_3^+ \), or equivalently the 3-dimensional Lobachevsky space \( \text{AdS}_3 \). The appropriate group manifold is provided by the \( SO(3,1)/SO(3) \) coset. In distinction with the previous case, the dynamical symmetry group is noncompact and we have to deal with quantum mechanics on the space of constant negative curvature. The analysis goes along the
same line as above with the only difference that the $SU(2)$ representations of spin-$j$ have to be substituted by the unitary, continuous representations of the $SO(3,1)$ group:

- fundamental series: $j = -1/2 + i\nu/2$ with $-\infty < \nu < \infty$
- complementary series: $-1 < j < 0$

It turns out that the resulting expression can be obtained from (4.19) through analytical continuation in spin $j$. To see this, one applies the Barnes-Mellin transformation to rewrite the sum over half-integer $j$ in (4.19) as a contour integral over the complex spin that runs parallel to the imaginary axis to the right from $j = 0$

$$G[v,-v'] - G[v,v] = -\int_{\delta-i\infty}^{\delta+i\infty} \frac{dj}{2\pi i} \frac{\pi}{\sin(2\pi j)} \frac{(2j+1)^2}{j(j+1)} \left\{ \frac{\sin((2j+1)\theta)}{(2j+1)\sin\theta} - 1 \right\}. \quad (4.24)$$

with $0 < \delta < 1$. One verifies that moving the integration contour to the right and picking up the residues at half-integer $j$ one reproduces the known expression for the $SU(2)$ propagator, Eq. (4.20). Let us now move the integration contour in (4.24) to the left parallel to the imaginary axis until it reaches $\Re j = -1/2$. Since the integrand has a pole at $j = 0$, the deformed contour will contain an additional addendum that encircles the segment $-1/2 < j < 0$ on the real axis. Changing the integration variable as $j = -1/2 + i\nu/2$ and going over to Minkowski kinematics, $\theta \to i\theta$, one finds

$$G[v,-v'] - G[v,v] = \left( \int_{-\infty}^{\infty} + \frac{1}{2} \int_{\gamma} \right) \frac{d\nu \nu}{2(1+\nu^2)} \frac{\sin(\nu \theta)}{\sinh(\pi \nu) \sinh \theta}. \quad (4.25)$$

Here in the second integral we made use of the symmetry of the integrand under $j \to -1 - j$ and extended the integration to the contour $\gamma$, which encircles the segment $[-i,i]$ in the anticlockwise direction. The two integrals in the right-hand side of (4.25) describe the contribution of the fundamental and complimentary series, correspondingly. Since the integrand in (4.25) is an odd function of $\nu$, the former integral vanishes, while the latter is given by the residue at the poles $\nu = \pm i$, or equivalently $j = 0, -1$. The resulting expression for the propagator (4.25) coincides with (4.20) upon replacing $\theta \to i\theta$. Similar to the previous case, one can expand the cusp anomalous dimension as the sum over classical trajectories on the hyperplane $v_0^2 - v_1^2 = 1$ defined by the time-like vectors $v_\mu$ and $v'_\mu$ in the AdS space $v_0^2 - v_1^2 - v_2^2 - v_3^2 = 1$. Since the trajectories do not “penetrate” into transverse $(n_2,n_3)$-directions, the sum will be the same if one changes the metric of the AdS$_3$ space from Euclidean to Lorentzian signature, $v_3 \to iv_3$. It is this version of the AdS$_3$ space that one encounters in the strong-coupling calculation of the cusp anomaly [7, 9, 10].
4.3 Analytic structure of cusp anomaly

Examining (4.20) one faces a paradox. By definition, \( \cos \theta = (v \cdot v') \), the cusp angle is defined up to \( \theta \to \theta + 2\pi n \) with \( n \) an arbitrary integer. At the same, (4.20) is not invariant under this transformation indicating that \( \Gamma_{\text{cusp}}(\theta; \alpha_s) \) is a multivalued function of the cusp angle. To understand the origin of this non-analyticity we observe that the sum in (4.19) diverges at \( \theta \to \pi \), or equivalently \( v'_\mu = -v_\mu \). This divergence has a simple meaning in terms of the sum over random paths on the \( S^3 \)-sphere, Eq. (4.10). For \( 0 \leq \theta < \pi \) the minimal length path connecting the points \( v_\mu \) and \( v'_\mu \) is unique. For \( \theta = \pi \) the points \( v_\mu \) and \( v'_\mu \) are opposite poles on the sphere, the length of the minimal path equals \( \pi \) and the number of such paths is infinite. The same singularity has a clear meaning in QCD in context of the heavy quark form factor. We recall that \( v_\mu \) and \( v'_\mu \) are velocities of the heavy quark before and after interaction with space-like external momentum \( q \), \(-q^2 \equiv Q^2 = -m^2(v - v')^2 = 2m^2[(v \cdot v') - 1] > 0 \). Re-expressing the cusp anomalous dimension (2.5) as a function of \( x \equiv 4m^2/Q^2 = -1/\sin^2(\theta/2) \),

\[
\Gamma_{\text{cusp}}(\theta; \alpha_s) = -\frac{\alpha_s N_c}{\pi} \frac{1}{\sqrt{1+x}} \left\{ \sqrt{1+x} - \left(1 + \frac{x}{2}\right) \ln \frac{\sqrt{1+x}+1}{\sqrt{1+x}-1} \right\},
\]

(4.26)
one finds that the singularity at \( x = -1 \), i.e. in the non-physical point \( Q^2 = -4m^2 \), corresponding to \( \theta = \pi \). The heavy quark form factor, analytically continued from \( Q^2 > 0 \) to \( Q^2 < 0 \), describes the threshold creation of a pair of heavy quarks. For \( Q^2 > 4m^2 \), i.e. above the threshold, the cusp anomalous dimension acquires an imaginary part.

4.4 Cusp anomalous dimension and 2D gauge theory

In the previous section, we identified the one-loop cusp anomaly with a transition amplitude for a test particle on the \( SL(2, \mathbb{R}) \) group manifold. As a next step, we will express this amplitude in terms of a disk partition function in a two-dimensional gauge theory and use the stringy representation of the latter. We will demonstrate that the emerging stringy description of the cusp anomaly at weak coupling is different from the one dictated by the Nambu-Goto string.

In two dimensions, the Yang-Mills theory does not contain transverse gauge degrees of freedom and, therefore, it can be reduced to a quantum-mechanical model. Its partition function on an arbitrary Euclidean two-dimensional manifold \( \Sigma \) of genus \( G \) with the metric tensor \( g_{\mu\nu} \) can be calculated through the heat kernel expansion \[69\]

\[
Z[g^2A] = \int \mathcal{D}A e^{-\frac{\alpha_s N_c}{4\pi} \int_\Sigma d^2x \sqrt{\text{det} g_{\mu\nu}} \text{tr} F^2} = \sum_R (\dim R)^2 e^{-g^2 A C_2(R)/2},
\]

(4.27)
where \( F \equiv F_{01}^a[A] t^a \) is the only nontrivial component of the strength tensor \( F_{\mu\nu}^a[A] \) with generators in the fundamental representation normalized as \( \text{tr}(t^a t^b) = \delta^{ab}/2 \). \( A \) is the area of the target
manifold Σ. The sum in the right-hand side of (4.27) runs over the unitary representations \( R \) of the gauge group of dimension \( \dim R \) and quadratic Casimir \( C_2(R) \). As in section 4.2, we will consider the \( SU(2) \) gauge group and perform analytical continuation to the \( SL(2, \mathbb{R}) \) group afterwards. In that case, \( C_2(R) = j(j + 1) \) and \( \dim R = (2j + 1) \) with \( j \) being non-negative (half)integer. Then, the sum in (4.27) has a striking resemblance with a similar sum defining the transition amplitude (4.17).

To make the correspondence exact, one introduces the amplitude of the two-dimensional Yang-Mills theory on a disk, with radial coordinate \( x^0, 0 \leq x^0 \leq T \), and angular \( x^1, 0 \leq x^1 \leq L \), of area \( A = LT/2 \), and a holonomy at its boundary \( C = \partial \Sigma \),

\[
U = P \exp \left( i \oint_C dx \cdot A(x) \right).
\]

The partition function of the disk is \[69, 70\]

\[
\mathcal{Z}[U; g^2 A] = \int \mathcal{D}A_\mu \delta \left( P e^{i \int_C dx \cdot A(x)} , U \right) e^{-\frac{1}{g^2} \int_C d^2x \sqrt{\det g_{\mu\nu}} \text{tr} F^2} = \sum_j (2j + 1) \chi_j[U] e^{-g^2 A (j + 1)/2},
\]

(4.28)

where and \( \chi_j[U] \) is its characters for the spin-\( j \) representation of the gauge group. The path integral representation is due to Ref. \[71\], where the conjugation invariant delta function\(^{13}\) is defined by a group Fourier transform \( \delta(U, U') = \sum_R \chi_R[U] \chi_R[U'] \). The disk transition amplitude (4.28) is used to build the ones for arbitrary manifolds of genus \( G \) by a gluing procedure \[70\]. Eq. (4.28) reduces to the partition function via (4.27) by \( \mathcal{Z} = \int dU \mathcal{Z}[U] \). Thus,

\[
\mathcal{Z}[U; g^2 A = 2\tau] = \langle -v' | e^{-\tau H} | v \rangle,
\]

c.f. Eq. (4.17), where \( U = g_{v'} g_v \) and \( \text{tr} [U(\tilde{\theta})] = 2 \cos \tilde{\theta} \) with \( \tilde{\theta} = \pi - \theta \). Thus, the one-loop cusp anomalous dimension (4.20) is given by the integral of the wave functional in two-dimensional Yang-Mills theory on the disc with respect to its area

\[
\Gamma_{\text{cusp}}(\theta; \alpha_s) = -\frac{\alpha_s N_c}{2\pi} \int_0^\infty d\tau \left( \mathcal{Z}[U; 2\tau] - \mathcal{Z}[1; 2\tau] \right).
\]

(4.30)

As we have already mentioned, the transition amplitude of the particle on the \( SU(2) \) group manifold has two different representations, Eqs. (4.19) and (4.21). The first one coincides with (4.29). While the second one is related to the saturation of the partition function on a Riemann surface by a sum over classical saddle points in the path integral \[72\] (see also \[73\]), — a consequence of the Duistermaat-Heckman localization. It can be rephrased in physical terms as instanton mechanism of confinement in two-dimensional Yang-Mills theory \[74\]. The instantons

\(^{13}\) It equates the eigenvalues of two unitary matrices.
under consideration are solutions to the Yang-Mills equations of motion on the two-dimensional disk with the boundary conditions set by the holonomy \( U[A(x^0 = T, x^1)] = 2 \cos \tilde{\theta} \). The classical configurations in the \( A^0(x^0, x^1) = 0 \) gauge correspond to straight paths connecting the initial and final points and read in the topological charge-\( \ell \) sector \( A^1_\ell(x^0, x^1) = x^0(\sigma_3/2)(\tilde{\theta} + 2\pi \ell)/A \). The action evaluated on these instanton solutions reads

\[
S[A_\ell] = 2(\tilde{\theta} + 2\pi \ell)^2/(g^2 A),
\]

and the weight factor \( w_\ell = \tilde{\theta} + 2\pi \ell \) in \( w_\ell \exp(-S_\ell) \). These properties are in a perfect agreement with our findings in section 4.2, Eq. (4.21).

It is well-known that the two-dimensional \( SU(N) \) Yang-Mills theory is a string theory. Its partition function is given by the sum of maps from two-dimensional worldsheet to two-dimensional target manifold \( \Sigma \). The explicit relation between the partition functions in two theories looks as follows

\[
\ln Z[g^2 A] = Z_{str} \left[ g_s = 1/N, \alpha' = 1/(\pi g^2 N) \right] \tag{4.31}
\]

with \( N = 2 \). Combining together Eqs. (4.30) and (4.31), we conclude that the one-loop cusp anomalous dimension admits the same stringy representation.

Eq. (4.31) is a counterpart of the relation between the transition amplitude of firstly quantized particle and partition function of secondly quantized field theory. Namely, it relates firstly quantized string with two-dimensional Yang-Mills theory. According to (4.31), the partition function in the latter theory is equal to the transition amplitude in the string theory with the boundary conditions specified by the holonomy \( U(\tilde{\theta}) \) on the disk boundary. Moreover, one can establish a one-to-one correspondence between the gauge invariant states in the Hilbert space of two-dimensional Yang-Mills theory, \( \text{tr} \left[ U^n(\tilde{\theta}) \right] \) with \( n = 1, 2, \ldots \), and stringy excitations (the oscillatory modes). The relation between the two looks as follows

\[
\text{tr} \left[ U^n(\tilde{\theta}) \right] \to (\alpha_n + \bar{\alpha}_{-n})|\tilde{\theta}\rangle, \tag{4.32}
\]

where the \( \alpha_n \) and \( \bar{\alpha}_n \) satisfy the commutation relations \( [\alpha_n, \alpha_m] = [\bar{\alpha}_n, \bar{\alpha}_m] = n \delta_{n+m,0} \) and \( [\alpha_n, \bar{\alpha}_m] = 0 \) and \( |\tilde{\theta}\rangle \) is the stringy coherent state

\[
|\tilde{\theta}\rangle = \exp \left( \sum_{n=1}^{\infty} \frac{e^{in\tilde{\theta}} \alpha_n}{n} \right) \exp \left( \sum_{n=1}^{\infty} \frac{e^{-in\tilde{\theta}} \bar{\alpha}_n}{n} \right) |0\rangle.
\]

The holonomy \( U(\tilde{\theta}) \) is defined in the fundamental representation of the \( SU(2) \) group.\(^{14}\)

\(^{14}\)The closed string picture for \( SU(N_c) \) two-dimensional YM theory is valid perturbatively only for \( N_c \to \infty \). However, the stringy representation exists for arbitrary \( N_c \).
We recall that $U(\hat{\theta}) = g^{-1} g_{v}$ with $g_{v}$ given in (4.11) so that $\text{tr} \left[ U^{n}(\hat{\theta}) \right] = 2 \cos(n \hat{\theta})$. Notice that $\text{tr} \left[ U^{n} \right]$ and the characters $\chi_{j}[U]$ provide two linear independent bases on the space of invariant functions on the $SU(2)$ group and, therefore, they are related to each other by a linear transformation [79], a form of the Frobenius character formula [80]. This allows one to rewrite Eq. (4.29) in terms of $\text{tr} \left[ U^{n} \right]$ and make use of (4.32) in order to re-express the transition amplitude for a particle on a cylinder, deduced by gluing the opposite arcs of the boundary holonomies of the disk amplitude [70],

$$Z[U_{1},U_{2};2\tau] = \int d\Omega \ Z[\Omega U_{1} \Omega^{\dagger} U_{2};2\tau], \quad (4.33)$$

in terms of a string amplitude

$$\mathcal{Z}[U(\theta),U(0);2\tau] = \langle -v'| e^{-\tau \mathcal{H}} |v \rangle = \langle \hat{\theta}| e^{-\tau \mathcal{H}_{\text{str}}} |\hat{\theta} = 0 \rangle, \quad (4.34)$$

where $|\hat{\theta} = 0 \rangle$ corresponds to a trivial holonomy $U(0) = 1$, thus resulting merely to a disk amplitude $\mathcal{Z}[U(\hat{\theta}),U(0);2\tau] = \mathcal{Z}[U(\hat{\theta});2\tau]$. The group theory Hamiltonian $C_{2}(R)$ is not diagonal in the basis spanned by string states and involves splitting and joining of strings. The string Hamiltonian is defined for the $SU(2)$ group as [81]

$$\mathcal{H}_{\text{str}} = 2(L_{0} + \bar{L}_{0}) + \frac{1}{2}(L_{0} - \bar{L}_{0})^{2} + (V + \bar{V}), \quad (4.35)$$

where the Virasoro generator and interaction vertex operator read

$$L_{0} = \frac{1}{2} \sum_{n} \alpha_{-n} \alpha_{n}, \quad V = \frac{1}{2} \sum_{n,m>0} (\alpha_{-n-m} \alpha_{n} \alpha_{m} + \alpha_{-n} \alpha_{-m} \alpha_{n+m}),$$

with $\bar{L}_{0}$ and $\bar{V}$ given by similar expressions. Substituting (4.34) into (4.30) one could obtain the representation for one-loop cusp anomalous dimension in terms of string propagator

$$\Gamma_{\text{cusp}}(\theta;\alpha_{s}) = -\frac{\alpha_{s} N_{c}}{2\pi} \left[ \langle \hat{\theta}|1/\mathcal{H}_{\text{str}}|0 \rangle - (0)|1/\mathcal{H}_{\text{str}}|0 \rangle \right]. \quad (4.36)$$

We would like to stress that the string corresponding to the one-loop cusp anomaly is not of the Nambu-Goto type [14]. At the same time, the cusp anomaly at strong coupling is described by the Nambu-Goto string on the AdS background whose tension scales as $(\alpha_{s} N_{c})^{1/2}$ (see next section). Going over from weak to strong coupling regime one expects to find the transition from the former string to the latter. The mechanism governing such transition remains unclear. One of possible scenarios was proposed in Ref. [82]. It is based on the identification of the two-dimensional Yang-Mills theory (4.27), rewritten as

$$-\frac{1}{g^{2}} \text{tr} F^{2} \rightarrow \text{tr} \left[ 2F \phi + g^{2} \phi^{2} \right],$$

as a topological string (for $g^{2} = 0$) perturbed by a rigidity term [60] [83]. The Nambu-Goto action was conjectured to arise through the dimensional transmutation mechanism at strong coupling.
It is worth mentioning on the relation of Yang-Mills theory with AdS background. Its appearance can be understood by noticing that after analytical continuation of the cusp anomaly from the Euclidean to Minkowski kinematics, one has to deal with a two-dimensional Yang-Mills theory with the $SL(2, \mathbb{R})$ gauge group. It is well-known \cite{84} that at $g^2 = 0$ such theory is equivalent to topological Jackiw-Teitelboim gravity with the action

$$S = \int d^2x \sqrt{\det g_{\mu\nu}} \left( R(g) - \Lambda \right) \eta, \quad (4.37)$$

where $\eta$ is the dilaton field and $\Lambda$ is the cosmological constant. Solutions to the classical equations of motion give rise to the AdS$_2$ gravity coupled to the dilaton.

5 Cusp anomaly at strong coupling

The main goal of our previous discussion of the cusp anomaly at weak coupling was to emphasize its quantum nature as a transition amplitude for a particle on the AdS$_3$ space. In this section, we will argue that at strong coupling the cusp anomaly is given by Hamilton-Jacobi action function corresponding to a classical mechanical system defined on the same space.

According to the AdS/CFT correspondence \cite{5, 6}, the strong coupling regime in gauge theories is related to the supergravity limit of a string theory on the AdS$_5 \times$S$^5$ background. In the present discussion we are interested in operators with large angular momentum $J$ where the conventional (supergravity field)/(Yang-Mills operator) correspondence is not applicable, and one has to solve the string theory semiclassically \cite{7}. For the light-cone observables discussed here, like quark distribution functions \cite{2, 3} and light-like Wilson loops \cite{2, 3}, the full conformal QCD group $SO(4, 2)$ is effectively reduced to its collinear subgroup $SU(1, 1)$. It is only the latter which acts non-trivially on the field operators “living” on the light-cone. The group $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ is an isometry of the AdS$_3$. Therefore, applying the gauge/string correspondence, instead of the full AdS$_5$ space it will be enough to consider only its AdS$_3$ subspace.

Let us remind a few elementary facts about anti-de Sitter space. The AdS$_3$ space with the Lorentzian signature is a hypersurface embedded in flat $\mathbb{R}^{2,2}$

$$X_0^2 - X_1^2 - X_2^2 + X_3^2 = R^2. \quad (5.1)$$

We set its radius to be $R = 1$ for simplicity. The $SU(1, 1)$ group structure becomes manifest via the following parametrization

$$g = \begin{pmatrix} X_0 + iX_3 & X_1 - iX_2 \\ X_1 + iX_2 & X_0 - iX_3 \end{pmatrix} = \sum_{a=0}^3 X_a \tau_a, \quad (5.2)$$
with \( \tau_a = \{ 1, \sigma_1, \sigma_2, i\sigma_3 \} \). To parametrize the hypersurface (5.1), we will use two different sets of coordinates \[85\]. In the global \((\rho, \tau, \phi)\)-coordinates the AdS space looks like

\[
X_0 + iX_3 = \cosh \rho \ e^{i\tau}, \quad X_1 + iX_2 = \sinh \rho \ e^{i\phi}.
\]

and in local Poincaré \((u, x, t)\)-coordinates

\[
X_0 + X_1 = u, \quad X_2 = ux, \quad X_3 = ut.
\]

The transformation from one set of coordinates to the other is achieved via the map

\[
\begin{align*}
    u &= \cosh \rho \ \cos \tau + \sinh \rho \ \cos \phi, \\
    x &= \frac{\sinh \rho \ \sin \phi}{\cosh \rho \ \cos \tau + \sinh \rho \ \cos \phi}, \\
    t &= \frac{\cosh \rho \ \sin \tau}{\cosh \rho \ \cos \tau + \sinh \rho \ \cos \phi}.
\end{align*}
\]

For the discussion which follows we introduce the Rindler coordinates on the conformally flat part \(\mathbb{R}^{1,1}\) of AdS\(_3\) in Poincaré parametrization

\[
t = r \cosh \theta, \quad x = r \sinh \theta.
\]

The metric on the AdS space in the global coordinates reads, see e.g., \[85\],

\[
ds^2 = \frac{1}{2} \text{tr} \left( g^{-1}dg \right)^2 = -\cosh^2 \rho \ d\tau^2 + \sinh^2 \rho \ d\phi^2 + d\rho^2
\]

and in the Poincaré-Rindler coordinates

\[
ds^2 = \frac{du^2}{u^2} + u^2 \left( dx^2 - dt^2 \right) = \frac{du^2}{u^2} + u^2 \left( -dr^2 + r^2 d\theta^2 \right).
\]

Here, the two sets of coordinates have different physical interpretation. In Eq. (5.7), \(0 \leq \rho < \infty\) defines the radial coordinate on the AdS space, \(0 \leq \phi < 2\pi\) is the azimuthal angle and \(-\infty < \tau < \infty\) sets up the AdS time. Notice that the latter is different from the time variable in (5.8). In Eq. (5.8), \(0 \leq u^2 < \infty\) is the Liouville coordinate, \(t\) and \(x\) are the time and the spacial coordinates, respectively, on the hyperplane in Minkowski space to which the contour entering the definition of the Wilson loop (2.3) (see also Fig. 2) belongs to. In the polar coordinates, choosing \(r = 0\) at the cusp, one identifies \(\theta\) in (5.5) as the cusp angle. Since the relation (5.5) between two sets of the coordinates is nonlinear, the same trajectory of a test particle on the AdS space looks differently in the \((\rho, \phi, \tau)\) and \((r, \theta, u)\) coordinates.

Let us now turn to the analysis of the cusp anomaly in the strong coupling regime. As was mentioned in section 2 there are two apparently different approaches to calculate the cusp
anomalous dimension within the gauge/string correspondence. One of them relies on the relation between the large-$J$ behavior of the anomalous dimensions of twist-two composite operators and cusp anomaly, Eq. (2.20). In this way, following [7], one can calculate $\Gamma_{\text{cusp}}(\alpha_s)$ as the energy minus spin of a folded closed string rapidly rotating in the AdS space in the $(\rho, \phi, \tau)$ coordinates, Eq. (1.2). In the second approach [8, 9, 10], one calculates $\Gamma_{\text{cusp}}(\alpha_s)$ from the minimal surface of the worldsheet of an open string propagating in the AdS space in the $(r, \theta, u)$ coordinates with the ends sliding along two rays at the boundary $u = \infty$ with the cusp angle $\theta \to \infty$. In both approaches, the result for the cusp anomaly is expressed in terms of solutions to classical equations of motion. The main difference between the two cases is the form of the classical Hamiltonian and underlying picture of classical motion. In this section, we will demonstrate the equivalence between the two approaches.

5.1 Open string and Wilson loop

To begin with, we recall the calculation of a Wilson loop with a cusp. The Nambu-Goto action for a string propagating in the $\text{AdS}_3$ target space looks like

$$S = 2 \frac{R^2}{2\pi \alpha'} \int d^2 \sigma \sqrt{-\det |G_{MN} \partial_a X^M \partial_b X^N|}, \quad (5.9)$$

where $R^2/\alpha' = \sqrt{g^2 N_c}$ and $G_{MN}$ is the metric tensor on the $\text{AdS}_3$ space, $ds^2 = G_{MN} dX^M dX^N$, Eqs. (5.7) and (5.8). The Wilson loop is defined by a classical configuration that minimizes this action, $W \sim \exp(iS_{\text{min}})$. The additional factor 2 in the right-hand side of (5.9) takes into account that the Wilson loop is taken in the adjoint representation of the $SU(N_c)$ and, in multi-color limit

Another difference is that the calculation of Ref. [7] can be performed only in the AdS$_3$ space with Lorentzian signature, whereas in the case of the Wilson loop, Refs. [8, 9, 10], the result can be reproduced by an analytical continuation from Euclidean space.
$N_c \to \infty$, it is just the square of the same loop in the fundamental representation. Also, the minus sign under the square-root in (5.9) ensures that $S$ is real in Minkowski signature.

The main contribution to the cusp anomaly comes from the vicinity of the cusp, see Fig.4. In the Poincaré-Rindler $(r, \theta, u)$ coordinates the minimal surface can be written as

$$u = \frac{f(\theta)}{r}. \quad (5.10)$$

Choosing $(\sigma_1, \sigma_2) = (\theta, r)$ as local coordinates on the worldsheet, one finds the induced metric as

$$G_{MN} \partial_a X^M \partial_b X^N = \begin{pmatrix} (\dot{f}/f)^2 + f^2 & -\dot{f}/(fr) \\ -\dot{f}/(fr) & (1 - f^2)/r^2 \end{pmatrix}, \quad (5.11)$$

where $\dot{f} = \partial f(\theta)/\partial \theta$. Then, the action becomes

$$S_{\text{min}} = 2\sqrt{\alpha_s N_c / \pi} \int dr \int d\theta \sqrt{\dot{f}^2 - f^2 + f^4} \equiv i\Gamma_{\text{cusp}}(\theta, \alpha_s) \ln \frac{r_{\text{max}}}{r_{\text{min}}}, \quad (5.12)$$

where the cusp anomalous dimension is given by

$$\Gamma_{\text{cusp}}(\theta; \alpha_s) = \frac{2\sqrt{\alpha_s N_c / \pi}}{\pi} \int_0^\theta \frac{d\theta}{\sqrt{-\dot{f}^2 + f^2 - f^4}}. \quad (5.13)$$

This allows us to interpret $\Gamma_{\text{cusp}}(\theta, \alpha_s)$ as a classical action of a particle with the Lagrangian $\mathcal{L}[f] = (-\dot{f}^2 + f^2 - f^4)^{1/2}$, where $f(\theta)$ and $\theta$ play the role of the coordinate and the time, respectively. The energy $E$ and momentum $P(\theta)$ of the particle take the form

$$E = \dot{f} \frac{\partial \mathcal{L}[f]}{\partial \dot{f}} - \mathcal{L} = \frac{-f^2 + f^4}{\sqrt{-\dot{f}^2 + f^2 - f^4}}, \quad P = \frac{\partial \mathcal{L}[f]}{\partial \dot{f}} = -\frac{\dot{f}}{\sqrt{-\dot{f}^2 + f^2 - f^4}}. \quad (5.14)$$

Being the integral of motion, $E = \text{const}$, the energy determines the solutions to the classical equations of motion. The action calculated along classical trajectory satisfies the Hamilton-Jacobi equation and is given by

$$\Gamma_{\text{cusp}}(\theta; \alpha_s) = \frac{2\sqrt{\alpha_s N_c / \pi}}{\pi} \int_0^\theta d\theta \left( P(\theta) \dot{f}(\theta) - E \right). \quad (5.15)$$

Since the minimal surface ends at the boundary of the AdS space, $u = \infty$, the classical trajectories have to satisfy the boundary condition $f(0) = \infty$. As was shown in Refs.9,10, the asymptotic behavior of the cusp anomaly at large $\theta$ is governed by the contribution of classical solutions with the energy $E = -1/2$. In this case, the classical trajectory starts at infinity, $f(0) = \infty$, and approaches $f(\theta) \to 1/\sqrt{2}$ as $\theta \to \infty$. Since $P(\theta) \sim \dot{f}(\theta)$ vanishes in this limit, $\dot{f} = 0$, one finds from (5.15)

$$\Gamma_{\text{cusp}}(\theta; \alpha_s) = \frac{2\sqrt{\alpha_s N_c / \pi}}{\pi} \theta, \quad (5.16)$$

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in agreement with (2.17) and (1.4). The corresponding minimal surface can be translated into the
global coordinates by noticing that \( f^2(\theta) = X_3^2 - X_2^2 \rightarrow 1/2 \). Then, one gets from the equation
for AdS embedding (5.1),

\[
X_3^2 - X_2^2 = 1/2, \quad X_0^2 - X_1^2 = 1/2, \tag{5.17}
\]

which is the result of Ref. [9].

5.2 Rotating closed string

Let us now recapitulate the consideration of a rotating folded closed string in the AdS
space around its center-of-mass following [7], see also [86]. In distinction with the previous case, one
chooses to work in the global \((\rho, \phi, \tau)\)-coordinates (5.7) and assumes that the center of the string
lies at \( \rho = 0 \). The string action is given by the same Nambu-Goto expression (5.9) but with
different boundary conditions. Choosing the gauge \( \sigma_1 = \tau \) and \( \sigma_2 = \rho \), one finds the induced
metric as

\[
G_{MN} \partial_a X^M \partial_b X^N = \left( \begin{array}{cc}
-cosh^2 \rho + \dot{\phi}^2 \sinh^2 \rho & 0 \\
0 & 1
\end{array} \right),
\tag{5.18}
\]

where \( \phi = \phi(\tau) \) is an azimuthal angle of a point on the string with the AdS time and radial
coordinates, \( \tau \) and \( \rho \), respectively, and \( \dot{\phi} \equiv \partial \phi / \partial \tau \) the corresponding angular velocity. As a
consequence, the action of the rotating stretched string looks like

\[
S = 4 \frac{R^2}{2\pi \alpha'} \int d\tau \int_0^{\rho_0} d\rho \sqrt{\cosh^2 \rho - \dot{\phi}^2(\tau) \sinh^2 \rho} \equiv \int d\tau L[\phi]. \tag{5.19}
\]

Here the additional factor 4 counts the number of segments of the folded string rotating around
\( \rho = 0 \) and the maximal radial coordinate \( \rho \leq \rho_0 \) is determined by

\[
\coth^2 \rho - \dot{\phi}^2(\tau) \geq 0. \tag{5.20}
\]

Eq. (5.19) defines a classical mechanical model of a rotating rod with the Lagrangian \( L[\phi] \). Its
energy and angular momentum are

\[
E = \phi \frac{\partial L[\phi]}{\partial \dot{\phi}} - L = -4 \sqrt{\frac{\alpha_s N_c}{\pi}} \int_0^{\rho_0} d\rho \frac{\cosh^2 \rho}{\sqrt{\cosh^2 \rho - \dot{\phi}^2 \sinh^2 \rho}}, \tag{5.21}
\]

\[
J = \frac{\partial L[\phi]}{\partial \phi} = -4 \sqrt{\frac{\alpha_s N_c}{\pi}} \int_0^{\rho_0} d\rho \frac{\dot{\phi} \sinh^2 \rho}{\sqrt{\cosh^2 \rho - \dot{\phi}^2 \sinh^2 \rho}}. \tag{5.22}
\]

Both quantities are integrals of motion so that the classical trajectories are specified by the values
of \( E \) and \( J \). The action (5.19) evaluated along the classical trajectory is given by\(^{16}\)

\[
S_{cl} = \int d\tau \left( J \dot{\phi} - E \right) = 2\gamma J(\alpha_s) \ln \frac{r_{max}}{r_{min}}, \tag{5.23}
\]

\(^{16}\)Note that the pair \((J, \phi)\) defines the action-angle variables for the system under consideration.
where $\tau_{\text{max/min}} = \ln r_{\text{max/min}}$ and
\[
\gamma_J(\alpha_s) = \frac{1}{2} \int_0^{2\pi} \frac{d\tau}{2\pi} \left( J \dot{\phi} - E \right) = \frac{1}{2}(-E + J\omega)
\]
with $\omega = \dot{\phi}$ being the angular velocity of the rod. The additional factor 2 in the right-hand side of (5.23) counts the number of end-points of the folded string. The anomalous dimension defined in this way is the coefficient in front of the AdS time in the expression for the action function. The latter is the solution to the Hamilton-Jacobi equations for the system (5.19). In the limit of long strings,
\[
\rho_0 = \frac{1}{2} \ln(1/\eta) \gg 1, \quad \omega = 1 + 2\eta,
\]
for $\eta \to 0$, one finds the energy and angular momentum of the folded string as
\[
E = 2\sqrt{\frac{\alpha_s N_c}{\pi}}(\eta^{-1} - \ln \eta), \quad J = 2\sqrt{\frac{\alpha_s N_c}{\pi}}(\eta^{-1} + \ln \eta).
\]
Substituting these relations into (5.24), one obtains
\[
\gamma_J(\alpha_s) = 2\sqrt{\frac{\alpha_s N_c}{\pi}} \ln J.
\]
This expression defines the anomalous dimension of the twist-two operators $\hat{O}_J(0)$ at strong coupling.

### 5.3 Multi-particle operators: minimal surfaces

As we have seen in the previous sections, the anomalous dimensions of the twist-two operators at strong coupling can be obtained using two different approaches based on the calculation of the Wilson loop with a cusp and the classical energy of a long string rotating on the AdS background. In this section we will generalize these results to $N$-particle conformal operators of higher twist.

We have demonstrated in section 3.2.1 that the anomalous dimensions of such operators at large spins $J$ occupy the band (3.10) whose boundaries, Eqs. (3.15) and (3.17), are defined by the cusp anomalous dimension. Since this result holds for arbitrary coupling constant, one makes use of (1.4) to replace $\Gamma_{\text{cusp}}(\alpha_s)$ by its asymptotic behavior at strong coupling. Remarkably enough, the same result can be obtained from the gauge/string duality.

Following the approach described in section 5.1, one has to construct the minimal surface on the AdS$_3$ target space whose boundary involves multiple cusps. The number of cusps, $k$, varies along the band. On the upper and lower boundary it equals $N$ and 2, respectively. At large $N_c$, the expectation value of the product of Wilson loops factorizes into the product of their expectation values (see Fig. 1)
\[
\langle \text{tr} W_{\Pi}[\xi_1,\xi_2] \ldots \text{tr} W_{\Pi}[\xi_k,\xi_1] \rangle = \langle \text{tr} W_{\Pi}[\xi_1,\xi_2] \rangle \ldots \langle \text{tr} W_{\Pi}[\xi_k,\xi_1] \rangle
\]
(5.28)
This implies that the area of the minimal surface corresponding to the Wilson loop with \( k \) cusps is given by the sum of \( k \) elementary areas derived in section 5.1

\[
\langle \text{tr} W_{\Pi}[\xi_j, \xi_{j+1}] \rangle \sim \exp \left( S(\theta_j) + S(\theta_{j+1}) \right) = \mu^{\Gamma_{\text{cusp}}(\theta_j; \alpha_s) + \Gamma_{\text{cusp}}(\theta_{j+1}; \alpha_s)} = \mu^{(\theta_j + \theta_{j+1})} \Gamma_{\text{cusp}}(\alpha_s). 
\]

(5.29)

Substituting this relation into (5.28), we calculate the total area of the minimal surface

\[
S(\theta_1, \ldots, \theta_k) = 2 \sum_{j=1}^{k} S(\theta_j) = 2 k \sum_{j=1}^{k} \theta_j \ln \mu \sim 2k \theta \Gamma_{\text{cusp}}(\alpha_s) \ln \mu 
\]

(5.30)

for \( \theta_1 \sim \ldots \sim \theta_k \sim \theta \gg 1 \). As before, the coefficient in front of the \( \ln \mu \) at \( k = 2 \) and \( k = N \) can be identified as the anomalous dimensions of the \( N \)-particle conformal operators, \( \gamma^\text{min}_J \) and \( \gamma^\text{max}_J \), respectively, for \( \theta \sim J \).

### 5.4 Multi-particle operators: revolving closed string

Let us turn to the picture of a rotating closed string. We remind that the anomalous dimension of the twist-two conformal operators, \( \gamma^\text{tw}_{J-2}(\alpha_s) \), at strong coupling is related to the energy of the rotating Nambu-Goto string evaluated on a classical configuration with the minimal energy for a given angular momentum \( J \gg 1 \). In the AdS background such configuration corresponds to a folded rotating long string. It is worth mentioning that the emerging picture is a generalization of the well-known hadronic string for the meson states from flat to curved background, see, e.g., [87]. Attempting to extend the Gubser-Klebanov-Polyakov approach to \( N \)-particle conformal operators, one immediately encounters the following difficulty. In distinction with the \( N = 2 \) case, the anomalous dimensions occupy the band (3.16). On the stringy side, this indicates the existence of additional stringy degrees of freedom. One expects that their total number should be \( N - 2 \) in accordance with the total number of integrals of motion \( q_2, \ldots, q_N \) in the Heisenberg spin chain (both in classical and quantum cases). The spectrum of the integrals of motion is specified by the total angular momentum \( J \) and the set of integers \( \ell_1, \ldots, \ell_{N-2} \).\(^{17}\) Going over to the strong coupling regime, the integrability properties of the evolution equations may be lost, but the analytical structure of the energy spectrum remains intact. In other words, for large \( J \) the anomalous dimensions of \( N \)-particle operators are parameterized by the same set of integers, \( \gamma_J = \gamma_J(\alpha_s; \ell_1, \ldots, \ell_{N-2}) \) although the explicit form of this dependence may be different at strong and weak coupling. As we will argue below, these additional degrees of freedom can be identified as string junctions.

Let us consider the simplest case of the \( N = 3 \) conformal operators. Similar to the \( N = 2 \) case, we expect to recover a folded closed string rotating on the AdS background. Its total angular

\(^{17}\)These integers appear in the Bohr-Sommerfeld quantization conditions imposed on the orbits of classical motion.
momentum equals the Lorentz spin of the conformal operator. Since the logarithmically enhanced contribution, $\sim \ln J$, to the energy of the string originates from the boundary region, the large $J$ asymptotics of the anomalous dimension depends on how many bits of the folded string approach the boundary (see Fig. 5). The fact that the anomalous dimensions on the upper boundary scale as $3 \ln J$ suggests that the corresponding stringy configuration consists of three long bits, which are interconnected at some point close to the center of the AdS. Similar $Y$-shaped configurations are well-known in QCD as describing baryonic string and following we will refer to the string vertex as the string junction. There is however a number of important differences.

Since the quarks in QCD belong to the fundamental representation of the $SU(N_c)$ group for $N_c = 3$, the color-singlet baryonic operators are built from $N_c$ quarks and the baryonic vertex in the corresponding $Y$-shaped hadronic string contains the same number of string bits. Within the AdS/CFT framework, at large $N_c$, similar baryonic vertices have been constructed in Ref. 89. In supersymmetric theories, fermions belong to the adjoint representation of the $SU(N_c)$, which allows one to construct color-singlet composite operators containing an arbitrary number $N \geq 2$ of fermions, see e.g. Eq. (3.11). This leads to important differences in the renormalization properties of such operators both at the weak and strong coupling. Namely, at large $N_c$, the interaction between fermions in the adjoint representation occurs only between nearest neighbours while in the fundamental representation, due to antisymmetry of the baryonic vertex under permutation of quarks, the interaction between any pair of quarks is allowed. In the string representation, one can effectively replace a Wilson line in the adjoint representation by a pair of Wilson lines in the fundamental representation running in opposite directions. Then, one can construct the $Y$-shaped baryonic vertex as shown in Fig. 5. In distinction with the previous case, this vertex contains six string bits, but, as before, we shall call it the string junction.

Figure 5: Baryon string with a junction.
The Y-shaped string on the AdS background is described by the action

\[
S = 2 \frac{R^2}{2\pi\alpha'} \sum_{k=1}^{N=3} \int d\sigma_1^{(k)} d\sigma_2^{(k)} \sqrt{\det G^{MN} \partial_a X_M^{(k)} \partial_b X_N^{(k)}},
\]

(5.31)

where the superscript \((k)\) enumerates three different “arms” and \(\sigma_a^{(k)}\) are local coordinates on the worldsheet of the \(k\)-th arm. Making use of the reparameterization invariance and assigning \(\sigma_2^{(k)} = 0\) and \(\sigma_2^{(k)} = \pi\) to the folding point and string junction, respectively, one can write the boundary condition along the string junction as \(X_M^{(1)}(\sigma_1^{(1)}, \pi) = X_M^{(2)}(\sigma_1^{(2)}, \pi) = X_M^{(3)}(\sigma_1^{(3)}, \pi)\). The string equations of motion corresponding to the action \((5.31)\) take the following form in the conformal gauge \([87, 88, 90]\)

\[
\partial_+ \partial_- X_M^{(k)} + (\partial_+ X_M^{(k)} \cdot \partial_- X_M^{(k)}) = 0, \quad T_{\pm}^{(k)} = (\partial_\pm X^{(k)} \cdot \partial_\pm X^{(k)}) = 0,
\]

(5.32)

where \((X \cdot Y) \equiv G^{MN}X_M Y_N\) is the scalar product on the AdS space and \(\partial_\pm \equiv \partial/\partial \sigma_\pm\) with \(\sigma_\pm = \sigma_1 \pm \sigma_2\), being light-cone coordinates on the world-sheet. Eqs. \((5.32)\) are invariant under reparametrization \(\sigma_\pm \rightarrow f(\sigma_\pm)\). To fix this ambiguity we identify the local coordinate on the world-sheet as the time coordinate on the AdS space \(\sigma_1 = \tau\). Then, the energy and the angular momentum of the string are given by

\[
E = 2 \frac{R^2}{2\pi\alpha'} \sum_{k=1}^{3} \int_0^{\beta_k(\tau)} d\sigma_2^{(k)} G_{\tau\tau}(X^{(k)})
\]

\[
J = 2 \frac{R^2}{2\pi\alpha'} \sum_{k=1}^{3} \int_0^{\beta_k(\tau)} d\sigma_2^{(k)} \phi_k G_{\phi\phi}(X^{(k)})
\]

(5.33)

where \(G_{\tau\tau} = \cosh^2 \rho_{(k)}(\tau)\) and \(G_{\phi\phi} = \sinh^2 \rho_{(k)}(\tau)\) define the AdS3 metric \((5.7)\), while \(\rho_{(k)}\) and \(\phi_{(k)}\) are the radial and angular AdS coordinates of the \(k\)-th arm. Notice that in this gauge the local parameters \(\sigma_2\) corresponding to each arm take the same value at the folding point, \(\sigma_2^{(k)} = 0\) and different values at the junction \(\sigma_2^{(k)} = \beta_k(\tau)\). To find the explicit form of \(\beta_k(\tau)\) one has to impose the junction conditions. In covariant form they take the form \([87]\)

\[
\sum_{k=1}^{3} \frac{dX_M^{(k)}(\tau, \beta_k(\tau))}{d\tau} = \sum_{k=1}^{3} \left( \partial_\tau + \dot{\beta}_k(\tau) \partial_{\sigma_2} \right) X_M^{(k)}(\tau, \sigma_2) \bigg|_{\sigma_2 = \beta_k(\tau)} = 0,
\]

(5.34)

together with

\[
X_M^{(\text{junction})}(\tau) \equiv X_M^{(1)}(\tau, \beta_1(\tau)) = X_M^{(2)}(\tau, \beta_2(\tau)) = X_M^{(3)}(\tau, \beta_3(\tau)).
\]

(5.35)

Solving the system of equations \((5.32), (5.34)\) and \((5.35)\), one can find the classical motion of the Y-shaped string on the AdS background and apply \((5.33)\) to calculate the corresponding energy and angular momentum.

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In the flat space, for hadronic QCD string, this analysis has been carried out in Refs. [87]. In that model, the \( \mathcal{Y} \)-shaped string describes the spectrum of baryons and the string junction plays the rôle of their additional degree of freedoms. It was found that the dependence of the mass of baryons, \( m = E \), on their angular momentum \( J \) takes the Regge form, \( J/m^2 = \alpha'/\kappa \) with \( \kappa \) depending on the classical dynamics of the junction and taking the values within the band \( 2 \leq \kappa \leq 3 \). The maximal value \( \kappa = 3 \) corresponds to the configuration when the string junction is at rest and three bits of the string have the form of the rods of the same length, rotating with the same angular velocity and forming the same angle \( 2\pi/3 \) against each other (recall the analogy with the interior minimal surface of three joint soap bubbles). The minimal value \( \kappa = 2 \), corresponds to the meson-like Regge trajectory, i.e. \( N = 2 \) in Eq. (5.31). In this case, the baryon has the diquark-quark structure, that is the end-points of two bits are located close to each other and to the string junction. As we will argue in a moment, similar picture emerges in the AdS geometry.

To start with, we notice that short strings rotating around the center of the AdS5 do not feel its curvature and, therefore, look the same as strings in a flat background. That is, the dependence of the energy, \( E \), and the angular momentum, \( J \), of the string on its angular velocity \( \omega \) is the same in two cases. The only difference is due to different representation of the fermions – the open hadronic string in QCD is replaced by a folded closed string in the supersymmetric case. In the long-string limit, as was first shown in Ref. [7], a long folded string rotating on the AdS background gives rise to the anomalous dimensions of local composite operators of large spin \( J \). According to (1.2), the anomalous dimension scales as \( \ln J \) with the prefactor depending on the number of string bits reaching the boundary (5.26). In particular, meson-like long folded string gives rises rise to the anomalous dimension of two-particle conformal operators, Eq. (5.27).

Generalizing this picture to the case of baryon-like folded strings, we consider the same \( \mathcal{Y} \)-shaped configuration as in the hadronic string with the only difference that each bit of the “fundamental” string is replaced by two bits of the folded string. One can verify that such configuration satisfies the classical equations of motion on the AdS background. Since the string junction vertex is at rest, the energy of the rotating folded \( \mathcal{Y} \)-shaped string is given by the sum of energies of three arms, i.e. we can choose the gauge \( \sigma_{1(k)} = \tau_{(k)} \) and \( \sigma_{2(k)} = \rho_{(k)} \). The same is true for the total angular momentum. Due to symmetry of the configuration, three arms have the same energy and the angular momentum which in their turn are equal to half of the energy and the angular momentum of the meson-like folded string discussed in section 5.2. As a consequence, the energy spectrum of the \( \mathcal{Y} \)-shaped baryonic string with the string junction at rest and mesonic string are related to each other as

\[
E_\mathcal{Y}(\omega) = \frac{3}{2} E_1(\omega), \quad J_\mathcal{Y}(\omega) = \frac{3}{2} J_1(\omega).
\]

For \( \omega \gg 1 \), in the limit of short string [7], (5.36) coincides with the known relation between Regge
trajectories of mesons and baryons described by the hadronic QCD string. For \( \omega \to 1 \), in the limit of long strings, one calculates the anomalous dimension of the \( N = 3 \) particle conformal operators as

\[
\gamma_{J}^{N=3} = E_{Y}(\omega) - J_{Y}(\omega) = \frac{3}{2} [E_{I}(\omega) - J_{I}(\omega)] = \frac{3}{2} \gamma_{J}^{k=2} = 3 \Gamma_{cusp}(\alpha_{s}) \ln J. \tag{5.37}
\]

We observe that this expression coincides with the upper bound in the spectrum of the anomalous dimensions of \( N = 3 \)-particle operators, Eq. (3.16). We would like to stress that (5.37) corresponds to the \( Y \)-shaped string with the string junction at rest. In general, the classical solutions to the string equations of motion are parameterized by the classical trajectory of the junction, \( X_{M} = X_{M}^{(junction)}(\tau) \). For given total angular momentum \( J \) the minimal classically allowed energy of the \( Y \)-shaped string depends on the junction trajectories. It is well-known that on the flat background and junction moving, the minimal energy of the string with the total angular momentum \( J = J_{Y} \) is smaller than the energy \( E_{Y}(\omega) \) defined in (5.36). Obviously, the same property holds for short strings on the AdS background. Going over to the limit of long strings, one expects that the energy levels do not collide and, as a consequence, the same hierarchy is preserved. In other words, the minimal energy of the \( Y \)-shaped long string with the string junction moving is smaller than the energy with the junction at rest. This implies that the anomalous dimensions of the corresponding \( N = 3 \)-particle conformal operators is smaller than the anomalous dimension \( \gamma_{J}^{N=3} \) defined in (5.37). Moreover, the minimal energy of the \( Y \)-shaped string for the given total angular momentum \( J \) corresponds to the diquark-quark configuration when the string junction is located near the folding point. In that case, the energy and the angular momentum of the string approaches the energy and the angular momentum of the mesonic string, \( E_{I}(\omega) \) and \( J_{I}(\omega) \), respectively, and, as a consequence, the anomalous dimension of the \( N = 3 \)-particle conformal operator coincides with the twist-two anomalous dimension, \( 2 \Gamma_{cusp}(\alpha_{s}) \ln J \). These properties are in a perfect agreement with the expression obtained before within the Wilson line approach Eq. (3.16).

We recall that at \( N = 3 \) the spectrum of the anomalous dimensions, \( \gamma_{J}(\ell_{1}) \), is parameterized by integer \( 0 \leq \ell_{1} \leq J \). We have demonstrated that the two “extreme” classical trajectories of the junction, that is the junction at rest and rotating along the AdS boundary, are mapped into the upper and lower boundaries of the band, \( \ell_{1} = 0 \) and \( \ell_{1} = J \), respectively. We expect that similar correspondence exists for arbitrary \( 0 < \ell_{1} < J \).

Let us now consider the \( N \)-particle conformal operators. As was shown in section 3.2, the spectrum of their anomalous dimensions at weak coupling is parameterized by the set of \( N - 2 \) integers \( \ell_{i} \). Going over to the strong coupling limit, we expect that the same structure should be present. In other words, the string configuration describing such operators has to manifest the \( (N-2) \) additional degrees of freedom. At \( N = 3 \) such degree of freedom is provided by the string
junction. For $N \geq 4$ one can use the Y-shaped folded string as a building block to construct the classical string with an arbitrary number of bits. An example is shown in Fig. 6. Notice that the number of junctions for the string with $N$ folding points equals $N - 2$. It worth mentioning that similar configurations in hadronic QCD string describe exotic mesons and baryons [87].

A general analysis of such string configurations is rather involved, even on the flat background. Nevertheless, the asymptotic behavior of the anomalous $E - J \sim N \Gamma_{\text{cusp}} \ln J$ can be easily derived by considering the limiting case when all $N$ folding points approach the AdS boundary. As in the $N = 3$ case, the $(N - 2)$ junctions are at rest so that the energy and angular momentum of the string with $N$ arms receives an additive contribution from each arm. This result agrees with the upper bound in the spectrum of the $N$-particle conformal operator, Eq. (3.15).

A natural question arises about the possible physical interpretation of the string junction. If the string junction is a genuine physical degree of freedom of $N$-particle operators, it should be also found at weak coupling. As we have discussed in section 3.2.3, the anomalous dimensions at weak coupling are identified as the eigenvalues of the one-loop QCD dilatation operator which coincides at large Lorentz spin $J$ with spin chain Hamiltonian, Eq. (3.31). In the quasiclassical approach, the anomalous dimensions are calculated by imposing the Bohr-Sommerfeld quantization conditions on the orbits of classical motion of $N$ particles on the light-cone. The latter can be significantly simplified by going over from the original, light-cone $\xi$-coordinates to the separated $z$-coordinates. The Hamilton-Jacobi equations for the action function $S_0(z)$ take the following form in the separated coordinates

$$y^2 = t_N(z) - 4z^{2N}$$

where $y = 2z^N \sinh S_0^2(z)$ and $t_N(z) = 2z^N + q_2z^{N-2} + \ldots + q_N$ with $q_2 = -J^2$ at large $J$ and $q_k$ being the higher integrals of motion. This hyperelliptic curve of genus $N - 2$ is a “surface of equal energy” for a given set of the integrals of motion $q_k$ which define the coordinates on the moduli
space of the complex structures of the Riemann surface $\text{(5.38)}$. Quasiclassical calculation of the anomalous dimension at weak coupling, Eq. $\text{(3.34)}$, amounts to the quantization of the moduli space of these complex structures.

In particular, at $N = 3$ the Riemann surface $\text{(5.38)}$ corresponding to the baryonic operator has the topology of a torus. In that case, the collective degrees of freedom “live” on this surface. The Bohr-Sommerfeld quantization conditions allow one to find the quantized values of the integral of motion $q_3 = q_3(J, \ell_1)$ and calculate the corresponding anomalous dimension $\gamma_J(J, \ell_1)$. It turns out that the upper and the lower bounds in their spectrum, $\gamma_J(J, \ell_1 = 0)$ and $\gamma_J(J, \ell_1 = J)$, correspond to $q_3^2 = J^6/27$ and $q_3 = 0$, respectively. At these values of $q_3$ the Riemann surface $\text{(5.38)}$ becomes degenerate, that is one of the cycles shrinks into a point. From the point of view of classical mechanics this corresponds to freezing out the collective degrees of freedom. We observe that the same phenomenon occurs at the strong coupling. Namely, the string junction is at rest at the upper bound of the spectrum and at the di-quark center-of-mass at the lower bound of the spectrum. This suggests that the classical dynamics of the string junction is governed by yet another Riemann surface of the same genus. Indeed, it is known that the general solutions to the string equations of motion $\text{(5.32)}$ are parametrized by a hyperelliptic curve of higher genus $\text{[90]}$. The explicit form of this curve is fixed by the boundary conditions. In the case of the string with $N - 2$ junctions such conditions are given by Eqs. $\text{(5.34)}$. It would be interesting to compare $\text{(5.38)}$ with the curve emerging at strong coupling.

6 Concluding remarks

The present paper was devoted to studies of the anomalous dimensions of conformal operators at weak and strong coupling. We have demonstrated that in the both regimes the anomalous dimensions behave asymptotically as $\sim \Gamma_{\text{cusp}}(\alpha_s) \ln J$ at large Lorentz spin $J$ while the dependence of the cusp anomalous dimension on the coupling constant is different. At weak coupling, we calculated the first two terms of perturbative expansion of $\Gamma_{\text{cusp}}(\alpha_s)$ in a generic gauge theory involving scalars. While at large coupling we obtained its leading asymptotic behavior using classical limit of string propagating on the AdS background.

In perturbative regime, we have found the one-loop cusp anomaly corresponds to angular gluon propagation on a cylinder. This allows to establish a relation of the former to the quantum transition amplitude of a spherical top. Due to localization phenomena, it is saturated by classical trajectories, i.e. multiple windings of paths around the principle circles of a sphere. All of these properties are naturally incorporated into the two-dimensional gauge theory on a disk. There, the cusp anomaly is expressed as an integral of the partition function with a boundary holonomy.
with respect to the area of the disk. The well-known relation of Yang-Mills in two dimensions to a string theory, give us the opportunity to give a stringy representation of the cusp anomalous dimension at weak coupling.

At strong coupling, we extended the Gubser-Klebanov-Polyakov results for twist-two operators as a rapidly rotating closed string to multi-particle cases. The integrability of one-loop interaction kernel implies that the $N$-particle anomalous dimension is a function of $N$ parameters, — conserved charges. One expects that strong coupling will share similar property, — the anomalous dimension will keep this properties. In the stringy picture, these new degrees of freedom are encoded into the string junctions. The equation of motion for the latter are parametrized by a hyperelliptic curve, i.e. their classical dynamics of the string is driven by a Riemann surface. This will be discussed elsewhere.

To establish the relation between the two expressions for $\Gamma_{\text{cusp}}(\alpha_s)$ using the gauge/string correspondence, one has to provide the explicit mapping between the conformal operators in Yang-Mills theory and eigenstates of the stringy Hamiltonian on some background. Then, one can identify the anomalous dimensions of the conformal operators for arbitrary $\alpha_s$ as the energy of the corresponding stringy excitations. To go over from the strong coupling regime to arbitrary coupling constant in gauge theory, one needs to know the whole spectrum of the quantum string. At present this problem can not be solved in full due to lack of the quantization of the strings on AdS$_5 \times S^5$ background.

It is known that this difficulty can be avoided by considering the Penrose limit of the AdS$_5 \times S^5$ background. It is relevant to calculation of the anomalous dimension of local operators in the $\mathcal{N} = 4$ YM theory with large $R$-charge [91]. The string theory on this background is exactly solvable and, as a consequence, the spectrum of the stringy excitations can be found. In this case, the gauge/string correspondence looks as follows. There are six adjoint scalars $\Phi_i$ in $\mathcal{N} = 4$ theory and the $R$-symmetry rotates two of them, say $\Phi_1$ and $\Phi_2$. The stringy oscillator states are mapped into the so-called BMN operators constructed from the complex field $Z(x) = \Phi_1(x) + i\Phi_2(x)$. Namely, the operator $\text{tr} [Z^J(0)]$ with the large $R$-charge $J \gg 1$ is dual to the ground state of the string of the length $J$, $|0, J\rangle$, while the operator with two “impurities” is dual to excited oscillatory stringy state

$$ a_i^\dagger a_j^\dagger |0, J\rangle \leftrightarrow \sum_{l=0}^J e^{2i\pi ln/J} \text{tr} [\Phi_i^l \Phi_j^l Z^{J-l}] , \quad (6.1) $$

where $i, j = 1, \ldots, 6$. The exact spectrum of the string Hamiltonian in the pp-wave background gives rise to the anomalous dimensions of the BMN operators with large $R$-charge for arbitrary coupling constant. At strong coupling they coincide with expressions obtained in the quasiclassical approximation while at weak coupling they match the first few terms of perturbative expansion [92 93].
It turned out that the quantum string on the pp-background is intrinsically related to integrable spin chains. The latter appear when one examines the renormalization of the local operators $\text{tr} \left[ \Phi_{i_1}(0) \ldots \Phi_{i_k}(0) \right]$ at weak coupling in the multi-color limit. These operators mix under renormalization already at one-loop level and their eigenvalues can be found by diagonalizing the corresponding mixing matrix. As was found in, the one-loop mixing matrix coincides with the Hamiltonian of a completely integrable $SO(6)$ Heisenberg spin chain defined in an appropriate basis. The length of the spin chain is equal to $k$ and the spin operators belong to the fundamental representation of the $SO(6)$ group. The appearance of this group can be traced back to the fact that the same group is the isometry group of the $S^5$.

We observe a striking similarity between renormalization properties of such operators and conformal operators discussed above. In both cases, the one-loop mixing matrix gives rise to an integrable spin chain. The dynamical symmetry group of the spin chain, — the $SO(6)$ group for the local scalar operators and the $SL(2, \mathbb{R})$ group for the conformal operators, — is dictated by the isometry of the relevant part of the background, the $S^5$ and the AdS parts, respectively. In spite of the fact that two spin chains are different their energy spectrum can be obtained within the Bethe Ansatz in a similar manner by quantizing their spectral curves. For the $SL(2, \mathbb{R})$ magnet the spectral curve, Eq. (5.38), is hyperelliptic and its genus equals the number of fields involved. For the $SO(6)$ magnet the curve is more complicated and it can be reduced to hyperelliptic curve of the genus $J$ if one considers only its $SO(3)$ subgroup. Having these properties in mind, one may consider a more general case of renormalization of a local composite operator built from an arbitrary number of scalar fields and covariant derivatives acting along different light-cone directions, $(n_j \cdot D) \Phi_i(0)$ with $n_j^2 = 0$ and $i = 1, \ldots, 6$. One might expect that the corresponding one-loop mixing matrix is related to the spin chain with the symmetry group $SO(2, 4) \times SO(6)$, which is the isometry group of the $AdS_5 \times S^5$ background.

Going over to the strong coupling regime, one should ask about the fate of integrability of the mixing matrix. For the scalar, BMN like operators it has been suggested that integrability holds to higher loop orders. Would it be the case, the transition of the anomalous dimensions from weak to strong coupling regime would correspond to the flow in the space of integrable Hamiltonians with respect to the coupling constant $\alpha_s$. Moreover, had the same property be valid for conformal operators, it would allow one to calculate their anomalous dimensions at large spin $J$ for arbitrary $\alpha_s$. This question certainly deserves further studies.

We would like to stress that the origin of integrability of the one-loop mixing matrix remains obscure. A possible explanation could come from the stringy picture for the cusp anomalous dimension as weak coupling discussed in section. We have argued that the corresponding string picture emerges from the two-dimensional Yang-Mills theory which in its turn is equivalent to
topological theory at $g^2 = 0$ with the gauge group $SL(2, \mathbb{R})$. The latter theory is the limit of the $SL(2, \mathbb{R})$ Chern-Simons theory at the level $k \to \infty$. It is known that the correlation functions of the Wilson lines in the Chern-Simons theory exhibit integrable structure related to the XXZ Heisenberg spin chain with symmetry group $SL(2, \mathbb{R})$ and anisotropy $q = e^{2\pi i/(k-2)}$. In this way, for $k \to \infty$ one recovers the homogeneous XXX spin chain. The use of Chern-Simons approach for the calculation of the anomalous dimensions of the operators with arbitrary conformal spins and the corresponding stringy picture behind will be discussed elsewhere.

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A Cusp anomaly in dimensional reduction

The difference between two-loop expressions for the cusp anomalous dimension in the dimensional regularization and dimensional reduction schemes, Eqs. (2.18) and (2.24), respectively, is solely due to difference of the corresponding one-loop gluon polarization operators. In gauge theory it receives contribution from gauge bosons, $n_f$ fundamental fermions and $n_s$ scalars. In the momentum representation in two different schemes one obtains (with the Feynman gauge)

- **dimensional regularization (DREG):**

$$
\Pi_{\mu\nu}^{\text{DREG}}(q) = \frac{\alpha_s}{2\pi} \left( \frac{4\pi \mu^2}{-q^2} \right) \frac{\varepsilon}{\Gamma(\varepsilon) \Gamma(1-\varepsilon) \Gamma(2-\varepsilon)} \left( q^2 g_{\mu\nu}^{(d)} - q_\mu q_\nu \right) \left[ N_c (5 - 3\varepsilon) - 2n_f (1 - \varepsilon) - n_s / 2 \right].
$$

(A.1)

- **dimensional reduction (DRED):**

$$
\Pi_{\mu\nu}^{\text{DRED}}(q) = \frac{\alpha_s}{2\pi} \left( \frac{4\pi \mu^2}{-q^2} \right) \frac{\varepsilon}{\Gamma(\varepsilon) \Gamma(1-\varepsilon) \Gamma(2-\varepsilon)} \left( q^2 g_{\mu\nu}^{(d)} - q_\mu q_\nu \right) \left[ N_c (5 - 4\varepsilon) - 2n_f (1 - \varepsilon) - n_s / 2 \right] \times \left\{ q^2 \left( g_{\mu\nu}^{(4)} - g_{\mu\nu}^{(d)} \right) \left[ N_c - n_f + n_s / 2 \right] + \left( q^2 g_{\mu\nu}^{(4)} - q_\mu q_\nu \right) \left[ N_c (5 - 4\varepsilon) - 2n_f (1 - \varepsilon) - n_s / 2 \right] \right\},
$$

(A.2)

where $g_{\mu\nu}^{(4)}$ and $g_{\mu\nu}^{(d)}$ are metric tensors in the Minkowski space-time of dimension 4 and $d = 4 - 2\varepsilon$ with $\varepsilon > 0$, respectively, and $\alpha_s$ is a bare coupling constant. Note that in supersymmetric theories the $d$-dimensional Lorentz non-covariant part vanishes in the right-hand side of (A.2)

$$
N_c - n_f + n_s / 2 = 0,
$$

(A.3)
which is easy to very at $\mathcal{N} = 1$ ($n_f = N_c$, $n_s = 0$), $\mathcal{N} = 2$ ($n_f = 2N_c$, $n_s = 2N_c$) and $\mathcal{N} = 4$ ($n_f = 4N_c$, $n_s = 6N_c$).

The regularized polarization operators in the DREG and DRED schemes have the same residue at the pole in $\varepsilon$ and differ by a finite $O(\varepsilon^0)$-term due to contribution of $\varepsilon$-scalars in the DRED scheme. To renormalize the obtain expressions, we apply the modified minimal subtraction procedure. In this way, subtracting the ultraviolet pole from $\Pi_{\mu\nu}^{DRED}$, one defines the so-called $\overline{\text{DR}}$ renormalization scheme for the coupling constant. The counter-term in the $\overline{\text{DR}}$ scheme is given by

$$\text{div}_{\overline{\text{DR}}} \Pi_{\mu\nu}^{DRED} = \frac{\alpha_s^{\overline{\text{DR}}}}{12\pi} \left( q^2 g_{\mu\nu}^{(4)} - q_\mu q_\nu \right) \left( 5N_c - 2n_f - n_s/2 \right) \left( \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi \right). \quad (A.4)$$

One can define yet another renormalization scheme by adding a finite term to the right-hand side of (A.4)

$$\text{div}_{\text{MS}} \Pi_{\mu\nu}^{DRED} = \frac{\alpha_s^{\text{MS}}}{12\pi} \left( q^2 g_{\mu\nu}^{(4)} - q_\mu q_\nu \right) \left( 5N_c - 2n_f - n_s/2 \right) \left( \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi \right) - N_c. \quad (A.5)$$

In this scheme, the renormalized polarization operator $\Pi_R^{DRED} = \Pi^{DRED} - \text{div} \Pi^{DRED}$ coincides with the polarization operator (A.1) renormalized within the conventional dimensional regularization $\overline{\text{MS}}$ scheme, that is $\Pi_R^{DRED} = \Pi_{\text{MS}}^{DRED}$. That is the reason why one usually refers to (A.5) as the dimensional reduction $\overline{\text{MS}}$ scheme. The coupling constants in two schemes are related to each other through the scheme transformation

$$\alpha_s^{\text{MS}} = \alpha_s^{\overline{\text{DR}}} \left( 1 - \frac{N_c \alpha_s^{\overline{\text{DR}}}}{12 \pi} + O \left( \left( \alpha_s^{\overline{\text{DR}}} \right)^2 \right) \right). \quad (A.6)$$

The polarization operator modifies the gluon propagator by the term

$$D_{\mu\nu}(x) \rightarrow D_{\mu\nu}^{(1)}(x) = -i \int \frac{d^d q}{(2\pi)^d} e^{-iq \cdot x} \frac{\Pi_{\mu\nu, R}(q)}{q^4}, \quad (A.7)$$

with $\Pi_R \equiv \Pi - \text{div} \Pi$. Its substitution into (4.1) yields the following contribution to the Wilson loop evaluated along the contour shown in Fig. 2

$$W^{(1)} = -ig^2 \mu^4 a \int \frac{d^d q}{(2\pi)^d} v_\mu \Pi_{\mu\nu, R}(q) v'_\nu. \quad (A.8)$$

Since the velocity vectors are two-dimensional, $(g_{\mu\nu}^{(4)} - g_{\mu\nu}^{(d)}) v_\nu = 0$, so that the first term in Eq. (A.2) does not contribute. Calculating this integral in the $\overline{\text{DR}}$-scheme, one can determine the contribution to the two-loop cusp anomalous dimension (2.24), coming from $n_f$ fermions, $n_s$ scalars and part of the $N_c$ term. The remaining terms $\sim N_c$ originate from other two-loop Feynman diagrams. In the dimensional reduction, the only difference between $W^{(1)}$ evaluated
in the \(\overline{\text{MS}}\)- and \(\text{DR}\)-schemes comes from \((-N_c)\) term in the right-hand side of (A.5). A simple evaluation using the integral

\[
\int \frac{d^d q}{(4\pi)^d} \frac{v \cdot v'}{[-q^2 + \lambda^2]^m (q \cdot v)(q \cdot v')} = 2i \frac{\Gamma(m - d/2 + 1)}{\Gamma(m)} \frac{\theta \coth \theta}{\lambda^{2m-d+2}}, \tag{A.9}
\]

with \(\lambda^2\) being an infrared cut-off, gives

\[
W_{\text{DR}}^{\text{DRED}} - W_{\text{MS}}^{\text{DRED}} = \left(\frac{\alpha_s^{\text{DR}}}{\pi}\right)^2 \frac{C_F N_c}{12} \theta \coth \theta \ln \frac{\mu}{\lambda}. \tag{A.10}
\]

Notice that, by construction, \(W_{\text{MS}}^{\text{DRED}} = W_{\text{MS}}^{\text{DREG}}\) to two-loop level. At large \(\theta\), Eq. (A.10) is translated into similar relation between the cusp anomalous dimension in two schemes, Eqs. (2.18) and (2.24).

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