Asymptotic formula for sum-free sets in abelian groups

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Abstract

Let $A$ be a subset of a finite abelian group $G$. We say that $A$ is sum-free if there is no solution of the equation $x + y = z$, with $x, y, z$ belonging to the set $A$. Let $SF(G)$ denote the set of all sum-free subsets of $G$ and $\sigma(G)$ denotes the number $n^{-1}(\log_2 |SF(G)|)$. In this article we shall improve the error term in asymptotic formula of $\sigma(G)$ obtained in [GR05]. The methods used are a slight refinement of methods of [GR05].

Let $G$ be a finite abelian group of order $n$. A subset $A$ of $G$ is said to be sum-free if there is no solution of the equation $x + y = z$, with $x, y, z$ belonging to the set $A$. Let $SF(G)$ denote the set of all sum-free subsets of $G$. This article is motivated by the question of studying the cardinality of the set $SF(G)$.

Definition:

(I) Let $\mu(G)$ denotes the density of a largest sum-free subset of $G$, so that any such subset has size $\mu(G)n$.

(II) Given a set $B \subset G$ we say that $(x, y, z) \in B^3$ is a Schur triple of the set $B$ if $x + y = z$.

Observing that all subsets of a sum-free set are sum-free we have the obvious inequality

$$|SF(G)| \geq 2^{\mu(G)n} \quad (1)$$

Let the symbol $\sigma(G)$ denotes the number $n^{-1}(\log_2 |SF(G)|)$. Then from (1) it follows trivially that $\sigma(G) \geq \mu(G)$.

In this article we improve the results of Ben Green and Imre Ruzsa [GR05] and prove the following two results. The theorem II follows immediately from theorem I and a result from [GR05], namely theorem 6. The methods used to prove theorem I are a slight refinements of methods in [GR05].
**Theorem 1.** Let $G$ be a finite abelian group of order $n$. Then we have the following asymptotic formula

$$\sigma(G) = \mu(G) + O\left(\frac{1}{(\ln n)^{1/27}}\right).$$

**Theorem 2.** There exist an absolute positive constant $\delta_0$ such that if $F \subset G$ as at-most $\delta n^2$ Schur triples, where $\delta \leq \delta_0$. Then

$$|F| \leq (\mu(G) + C\delta^{1/3})n$$

where $C$ is an absolute positive constant.

Earlier Ben Green and Ruzsa [GR05] proved the following:

**Theorem 3.** ([GR05], Theorem 1.8.) Let $G$ be a finite abelian group of order $n$. Then we have the following asymptotic formula

$$\sigma(G) = \mu(G) + O\left(\frac{1}{(\ln n)^{1/45}}\right).$$

**Theorem 4.** ([GR05], Proposition 2.2) Let $G$ be an abelian group, and suppose that $F \subset G$ has at-most $\delta n^2$ Schur triples. Then

$$|F| \leq (\mu(G) + 2^{20}\delta^{1/5})n$$

The following theorem is also proven in [GR05].

**Theorem 5.** ([GR05], Corollary 4.3.) Let $G$ be an abelian group, and suppose that $F \subset G$ has at-most $\delta n^2$ Schur triples. Then

$$|F| \leq (\max\left(1, \frac{1}{3}, \mu(G) + 3\delta^{1/3}\right))n$$

The theorem 2 follows immediately from theorem 5 in the case $\mu(G) \geq \frac{1}{3}$. In the case $\mu(G) < \frac{1}{3}$, the theorem 2 again follows immediately from theorem 5 in the case $\delta$ is not very “small”. In the case $\delta$ is small we require Lemma 10 where an estimate is done differently than in [GR05]. For the rest of results we require to prove theorem 2, the methods used are completely identical as in [GR05], but the results used are not identical.

For proving theorem 1 we use the following result from [GR05].

**Theorem 6.** ([GR05], Proposition 2.1’) Let $G$ be an abelian group of cardinality $n$, where $n$ is sufficiently large. Then there is a family $F$ of subsets of $G$ with the following properties

(I) $\log_2 |F| \leq n(\ln n)^{-1/18}$;

(II) Every $A \in SF(G)$ is contained in some $F \in F$;

(III) If $F \in F$ then $F$ has at-most $n^2(\ln n)^{-1/9}$ Schur triples.

The theorem 1 follows immediately from theorem 6 and theorem 2. We shall reproduce the proof given in [GR05]. If $n$ is sufficiently large as required by theorem 6 then associated to each $A \in SF(G)$ there is an $F \in F$ for which $A \subset F$. For a given $F$, the number of $A$ which can arise in this way is at most $2^{|F|}$. Thus we have the bound

$$|SF(G)| \leq \sum_{F \in \mathcal{F}} 2^{|F|} \leq |F| \max_{F \in \mathcal{F}} |F|$$
Hence it follows that
\[ \sigma(G) \leq \mu(G) + C \left( \frac{1}{(\ln n)^{1/27}} + \frac{1}{\ln n^{1/18}} \right). \]  
(5)

But from the (1) we have the inequality \( \sigma(G) \geq \mu(G) \). Hence the theorem 1 follows.

In order to prove theorem 2 we shall require the value of \( \mu(G) \), which is now known for all finite abelian groups. In order to explain the results we need the following definition.

**Definition:** Suppose that \( G \) is a finite abelian group of order \( n \). If \( n \) is divisible by any prime \( p \equiv 2 \pmod{3} \) then we say that \( G \) is type I. We say that \( G \) is type \( I(p) \) if it is type I and if \( p \) is the least prime factor of \( n \) of the form \( 3l + 2 \). If \( n \) is not divisible by any prime \( p \equiv 2 \pmod{3} \), but \( 3 \mid n \), then we say that \( G \) is type II. Otherwise \( G \) is said to be type III. That is the group \( G \) is said to be of type III if and only all the divisors of \( n \) are congruent to 1 modulo 3.

The following theorem is due to P. H. Diananda and H. P. Yap [DY 69] for type I and type II groups and due to Green and Ruzsa [GR 05] for type III groups.

**Theorem 7.** ([GR 05], Theorem 1.5.) Let \( G \) be a finite abelian group of order \( n \). Then the following holds.

(I) If \( G \) is of type \( I(p) \) then \( \mu(G) = \frac{1}{3} + \frac{1}{3p} \).

(II) If \( G \) is of type II then \( \mu(G) = \frac{1}{3} \).

(III) If \( G \) is of type III then \( \mu(G) = \frac{1}{3} - \frac{1}{3m} \), where \( m \) is the exponent of \( G \).

1 proof of theorem 2

In case the group \( G \) is not of type III it follows from theorem 7 that \( \mu(G) \geq \frac{1}{3} \) and hence the theorem 2 follows immediate using theorem 5. Therefore we are required to prove theorem 2 for type III groups only.

For the rest of this article \( G \) will be a finite abelian group of type III and \( m \) shall denote the exponent of \( G \). The following proposition is an immediate corollary of theorem 7 and theorem 5.

**Proposition 8.** Let \( G \) be an abelian group of type III. Let the order of \( G \) be \( n \) and the exponent of \( G \) be \( m \). If \( F \subset G \) as at-most \( \delta n^2 \) Schur triples then

(I) \( |F| \leq (\mu(G) + \frac{1}{3m} + 3\delta^{1/3})n \).

(II) In the case \( \delta^{1/3}m \geq 1 \) then \( |F| \leq (\mu(G) + 4\delta^{1/3})n \), that is the theorem 2 holds in this case.

Therefore to prove the theorem 2 we are left with the following case.

**Case:** The group \( G \) is an abelian group of type III, order \( n \) and exponent \( m \). The subset \( F \subset G \) has at most \( \delta n^2 \) Schur triples and \( \delta^{1/3}m < 1 \).
Let $\gamma$ be a character of $G$ and $q$ denotes the order of $\gamma$. Given such $\gamma$ we define $H_j = \gamma^{-1}(e^{2\pi ij/q})$. We also denote the set $H_0 = \ker(\gamma)$ by just $H$. Notice that $H$ is a subgroup of $G$ and $H_j$ are cosets of $H$. The cardinality of the coset $|H_j| = |H| = \frac{n}{q}$. The indices is to be considered as residues modulo $q$, reflecting the isomorphism $G/H \cong \mathbb{Z}/q\mathbb{Z}$.

For any set $F \subset G$ we also define $F_j = F \cap H_j$ and $\alpha_j = |F_j|/|H_j|$. 

**Proposition 9.** Let $G$ be a finite abelian group of order $n$. Let $F$ be a subset of $G$ having at most $\delta n^2$ Schur triples where $\delta \geq 0$. Let $\gamma$ be any character of $G$ and $q$ be its order. Also let $F_i$ and $\alpha_i$ be as defined above. Then the following holds.

(I) If $x$ belongs to $F_i$ and $y$ belongs to $F_j$ then $x + y$ belongs to $H_{i+j}$.

(II) The number of Schur triples $\{x, y, z\}$ of the set $A$ with $x$ belongs to $F_i$, $y$ belongs to $F_j$ and $z$ belongs to $F_{j+l}$ is at least $|F_i|(|F_j| + |F_{j+l}| - |H|)$. In other words there are at least $\alpha_i(\alpha_j + \alpha_{j+l} - 1)(\frac{n}{q})^2$ Schur triples $\{x, y, z\}$ of the set $F$ with $x$ belongs to the set $F_i$.

(III) Given any $t \in \mathbb{Z}/q\mathbb{Z}$ such that $\alpha_i > 0$, it follows that for any $j \in \mathbb{Z}/q\mathbb{Z}$ the inequality

$$\alpha_j + \alpha_{j+l} \leq 1 + \frac{\delta q^2}{\alpha_i}$$

holds.

(IV) Given any $t \in \mathbb{R}$ we define the set $L(t) \subset \mathbb{Z}/q\mathbb{Z}$ as follows. The set

$$L(t) = \{i \in \mathbb{Z}/q\mathbb{Z} : \alpha_i + \alpha_{2i} \geq 1 + t\}.$$

Then it follows that

$$\sum_{i \in L(t)} \alpha_i \leq \frac{\delta q^2}{t}$$

**Proof.**

(I) This follows immediately from the fact that $\gamma$ is an homomorphism.

(II) In the case $|F_i|(|F_j| + |F_{j+l}| - |H|) \leq 0$, there is nothing to prove. Hence we can assume that the set $F_i \neq \phi$. Then for any $x$ which belongs to the set $F_i$, the sets $x + F_j \subset H_{j+i}$. Since the set $F_{j+i}$ is also a subset of $H_{j+l}$ and $|F_j| + |F_{j+l}| - |H| > 0$, it follows that

$$|(x + F_j) \cap F_{j+l}| = |F_j| + |F_{j+l}| - |(x + F_j) \cup F_{j+l}| \geq |F_j| + |F_{j+l}| - |H|.$$

Now for any $z$ belonging to the set $(x + F_j) \cap F_{j+l}$ there exist $y$ belonging to $F_j$ such that $x + y = z$. hence the claim follows.

(III) From II there are at least $\alpha_i(\alpha_j + \alpha_{j+l} - 1)(\frac{n}{q})^2$ Schur triples of the set $F$. hence the claim follows by the assumed upper bound on the number of Schur triples of the set $F$.

(IV) For any fixed $i \in L(t)$, taking $j = l = i$ in II, we get there are at least $\alpha_i t(\frac{n}{q})^2$ Schur triples $\{x, y, z\}$ of the set $F$ with $x$ belonging to the set $F_i$. Now for given any two $i_1, i_2 \in L(t)$ such that $i_1 \neq i_2$, the sets $F_{i_1}$ and $F_{i_2}$ have no element in common. Therefore there are at least $t \sum_{i \in L(t)} \alpha_i$ Schur triples of the set $F$. Hence the claim follows.

\[\square\]
Since the order of any character of an abelian group $G$ divides the order of group and $G$ is of type III, the order $q$ of any character $\gamma$ of $G$ is odd and congruent to 1 modulo 3. Therefore $q = 6k + 1$ for some $k \in \mathbb{N}$. Let $I, H, M, T \subset \mathbb{Z}/q\mathbb{Z}$ denotes the image of natural projection of the intervals $\{k + 1, k + 2, \cdots, 5k - 1, 5k\}, \{k + 1, k + 2, \cdots, 2k - 1, 2k\}, \{2k + 1, 2k + 2, \cdots, 4k - 1, 4k\}, \{4k + 1, 4k + 2, \cdots, 5k - 1, 5k\} \subset \mathbb{Z}$ to $\mathbb{Z}/q\mathbb{Z}$. Then the set $I$ is divided into $2k$ disjoint pairs of the form $(i, 2i)$ where $i$ belongs to the set $H \cup T$.

**Lemma 10.** Let $G$ be a finite abelian group of type III and order $n$. Suppose that $F \subset G$ has at-most $\delta n^2$ Schur triples. Let $\gamma$ be a character of $G$. Let the order of $\gamma$ be equal to $q = 6k + 1$. Then the following inequality holds.

$$\sum_{i=k+1}^{5k} \alpha_i \leq 2k + 2\delta^{1/2}q^{3/2}$$

**Proof.** The set $I = \{k + 1, k + 2, \cdots, 5k\}$ is divided into $2k$ disjoint pairs of the form $(i, 2i)$ where $i$ belongs to the set $H \cup T$. Therefore it follows that

$$\sum_{i=k+1}^{5k} \alpha_i = \sum_{i \in H \cup T} (\alpha_i + \alpha_{2i})$$

Given a $t > 0$ we divide the set $H \cup T$ into two disjoint sets as follows. We define the set

$$S = \{i \in H \cup T : \alpha_i + \alpha_{2i} \leq 1 + t\}$$

and

$$L = \{i \in H \cup T : \alpha_i + \alpha_{2i} > 1 + t\}.$$ 

Therefore the sets $S$ and $L$ are disjoint and the set $H \cup T = S \cup L$. Therefore it follows that

$$\sum_{i \in H \cup T} (\alpha_i + \alpha_{2i}) = \sum_{i \in S} (\alpha_i + \alpha_{2i}) + \sum_{i \in L} (\alpha_i + \alpha_{2i})$$

From proposition [6](#) we have the following inequality

$$\sum_{i \in L} \alpha_i \leq \frac{\delta q^2}{t}$$

Since for any $l \in \mathbb{Z}/q\mathbb{Z}$, the inequality $\alpha_l \leq 1$ holds trivially. It follows that

$$\sum_{i \in L} (\alpha_i + \alpha_{2i}) \leq |L| + \frac{\delta q^2}{t}.$$ 

(11)

Also the following inequality

$$\sum_{i \in S} (\alpha_i + \alpha_{2i}) \leq |S| + |S|t$$

holds just by the definition of the set $S$. Therefore from [6], [10], [12], [11] it follows that

$$\sum_{i=k+1}^{5k} \alpha_i \leq |L| + \frac{\delta q^2}{t} + |S| + |S|t \leq 2k + qt + \frac{\delta q^2}{t}$$

(13)

Now choosing $t = (\delta q)^{1/2}$ the lemma follows. \qed
Remark: The sum appearing in last Lemma was estimated as $2k + \delta q^2$ in [GR05]. There the estimate $\alpha_i + \alpha_{2i} \leq (\delta)^{1/2}q$ is used to estimate the right hand side of (9).

Notice that Lemma 10 holds for any character $\gamma$ of a group $G$ of type $III$. We would like to show that given $F \subset G$ having at most $\delta n^2$ Schur triples and also assuming that $(\delta)^{1/3}m < 1$ where $m$ is the exponent of $G$, there is a character $\gamma$ such that $\alpha_i \leq C(\delta q)^{1/2}i \in \{0, 1, 2, \cdots k\} \cup \{5k + 1, 5k + 2, \cdots, 6k\}$ where $C$ is an absolute positive constant, $q$ is the order of $\gamma$ and $k = \frac{1}{3\delta^{1/2}}$. To be able to do this we recall the concept of special direction as defined in [GR05]. The method of proof of this part is completely identical as in [GR05], though the results are not.

Given any set $B \subset G$, and a character $\gamma$ of $G$ we define $\hat{B}(\gamma) = \sum_{b \in B} \gamma(b)$. Given a set $B \subset G$ fix a character $\gamma_s$ such that $\text{Re} \hat{B}(\gamma)$ is minimal. We follow the terminology in [GR05] and call $\gamma_s$ to be the special direction of the set $B$.

The following Lemma is proven in [GR05], but we shall reproduce the proof here for the sake of completeness.

**Lemma 11.** ([GR05], Lemma 7.1, Lemma 7.3. (iv)) Let $G$ be an abelian group of type $III$. Let $F \subset G$ has at most $\delta n^2$ Schur triples. Let $\gamma_s$ be a special direction of the set $F$. Let $\alpha$ denotes the number $|F|/|G|$. Then the following holds.

(I) $\text{Re} \hat{F}(\gamma_s) \leq \left(\frac{\delta}{\alpha(1-\alpha)} - \frac{\alpha^2}{\alpha(1-\alpha)}\right)n$.

(II) In case $\delta \leq \eta/5$, then either $|F| \leq (\mu(G))n$ or the following inequality holds.

\[
q^{-1} \sum_{j=0}^{q-1} \alpha_j \cos \left(\frac{2\pi j}{q}\right) + \frac{\mu(\mathbb{Z}/q\mathbb{Z})^2}{1 - \mu(\mathbb{Z}/q\mathbb{Z})} < 6\delta. \tag{14}
\]

**Proof.** (I) The number of Schur triples in the set $F$ is exactly $n^{-1} \sum_{\gamma} (\hat{F}(\gamma))^2 \hat{F}(\gamma)$. This follows after the straightforward calculation, using the fact that

\[
\sum_{\gamma} \gamma(b) = 0 \iff b \neq 0,
\]  

and is equal to $n$ if $b = 0$ where $0$ here denotes the identity element of the group $G$. Therefore using the assumed upper bound on the number of Schur triples in the set $F$ it follows that

\[
n^{-1} \sum_{\gamma} (\hat{F}(\gamma))^2 \hat{F}(\gamma) = n^{-1}\sum_{\gamma \neq 1} (\hat{F}(\gamma))^2 \hat{F}(\gamma) + n^{-1}(\hat{F}(1))^2 \hat{F}(1) \leq \delta n^2,
\]

Where $\gamma = 1$ is the trivial character of the group $G$. Since $n^{-1}(\hat{F}(1))^2 \hat{F}(1) = (\alpha)^3n^2$, it follows that

\[
\text{Re} \hat{F}(\gamma_s) \sum_{\gamma \neq 1} (\hat{F}(\gamma))^2 \leq n^{-1}\sum_{\gamma \neq 1} (\hat{F}(\gamma))^2 \hat{F}(\gamma) \leq (\delta - \alpha^3)n^2.
\]

Since using (15) it follows that $\sum_{\gamma \neq 1} (\hat{F}(\gamma))^2 = \alpha(1 - \alpha^2)n^2$, the claim follows.
(II) We have $Re\tilde{F}(\gamma_s) = |H| \sum_j \alpha_j \cos(\frac{2\pi j}{q})$. Therefore in the case $|F| \geq \mu(G)$, then from (I) it follows that

$$q^{-1} \sum_{j=0}^{q-1} \alpha_j \cos(\frac{2\pi j}{q}) \leq \frac{\delta}{\alpha(1-\alpha)} - \frac{\alpha^2}{\alpha(1-\alpha)} \quad (16)$$

$$q^{-1} \sum_{j=0}^{q-1} \alpha_j \cos(\frac{2\pi j}{q}) + \frac{\mu(G)^2}{1-\mu(G)} \leq \frac{\delta}{\alpha(1-\alpha)} \quad (17)$$

Since from theorem 7 that $\mu(G) \geq \mu(\mathbb{Z}/q\mathbb{Z})$ it follows that

$$\frac{\mu(G)^2}{1-\mu(G)} \geq (\mu(\mathbb{Z}/q\mathbb{Z}))^2 \frac{1-\mu(\mathbb{Z}/q\mathbb{Z})}{1-\mu(G)}.$$ 

The claim follows using this and the fact that $\mu(G) \geq \frac{1}{q}$, which implies that $\frac{\delta}{\alpha(1-\alpha)} \leq 6\delta$. \qed

**Proposition 12.** Let $G$ be an abelian group of type III. Let $n$ and $m$ denotes the order and exponent of $G$ respectively. Let $F \subset G$ has at most $\delta n^2$ Schur triples and $\delta^{1/3} m \leq 1$. Let $|F| \geq \mu(G)n$. Let $\gamma_s$ be a special direction of the set $F$ and $q$ be the order of $\gamma_s$. Let $q = 6k + 1$ and $\alpha_i$ be as defined above. There exist an positive absolute constants $q_0$ and $\delta_0$ such that if $q \geq q_0$ and $\delta \leq \delta_0$, then the following holds

$$\alpha_i \leq c(\delta q)^{1/2} \text{ for all } i \in \{0, 1, \ldots, k-1, k\} \cup \{5k + 1, 5k + 2, \ldots, 6k - 1\}, \quad (18)$$

where $c$ is an positive absolute constant.

**Proof.** If $F \subset G$ be the set as given, then $-F \subset G$ is also a set which satisfies the same hypothesis as required in the statement of proposition. It is also the case that $|F_j| = |(-F)_{-j}|$. Therefore to prove the proposition it is sufficient to show that

$$\alpha_i \leq c(\delta q)^{1/2} \text{ for all } i \in \{0, 1, \ldots, k-1, k\}$$

for some positive absolute constant $c$.

Let $S = \sum_{j=0}^{q-1} \alpha_j \cos(\frac{2\pi j}{q}) + \frac{\mu(\mathbb{Z}/q\mathbb{Z})^2}{1-\mu(\mathbb{Z}/q\mathbb{Z})}$. Then from Lemma 11 we have that

$$S \leq 6\delta. \quad (19)$$

Now let for some $l \in \{0, 1, \ldots, k-1, k\}$, $\alpha_l > c(\delta q)^{1/2}$ (where $c$ is a positive number which we shall choose later), then we shall show that this violates (14), provided $q$ and $c$ are sufficiently large and $\delta$ is sufficiently small. For this we shall find the lower bound of $M = \sum_{j=0}^{q-1} \alpha_j \cos(\frac{2\pi j}{q})$.

Let $\gamma_j$ denotes $\frac{(\alpha_j + \alpha_{j+l})}{2}$. Then we have

$$M = \frac{1}{q^2 \cos(\frac{\pi l}{q})} \sum_{j=0}^{q-1} \alpha_j \left( \cos(\frac{(2j+l)\pi}{q}) + \cos(\frac{(2j-l)\pi}{q}) \right).$$

That is we have

$$M = \frac{1}{q \cos(\frac{\pi l}{q})} \sum_{j=0}^{q-1} \gamma_j \cos(\frac{(2j+l)\pi}{q}) \quad (20)$$
Notice that \(\cos\left(\frac{\pi l}{q}\right)\) is not well defined if we consider \(l\) as an element of \(\mathbb{Z}/q\mathbb{Z}\). This is because the function \(\cos\left(\frac{\pi t}{q}\right)\) as a function of \(t\) is not periodic with period \(q\) but is periodic with period \(q^2\). But we have assumed that \(l \in \{0, 1, \ldots, k - 1, k\}\), therefore the above computation is valid.

Since \(d^{1/2} q^{3/2} \leq d^{1/2} m^{3/2} < 1\) is true by assumption, recalling Lemma 9 it follows that

\[
2\gamma_j = \alpha_j + \alpha_{j+l} \leq 1 + \frac{1}{c}d^{1/2} q^{3/2} \leq 1 + \frac{1}{c}, \text{ for any } j \in \mathbb{Z}/q\mathbb{Z} \quad (21)
\]

and

\[
\sum_j \gamma_j = \sum_j \alpha_j \geq \mu(G)n \geq 2k. \quad (22)
\]

The inequality (22) follows from the assumption that \(|F| \geq \mu(G)n\).

Let \(t_c\) denotes the number \(1+1/c\). Let \(E(c, q)\) denotes the minimum value of \(\sum_{j=0}^{q-1} \gamma_j \cos\left(\frac{(2j+l)\pi}{q}\right)\)
subject to the constraints that \(0 \leq \gamma_j \leq \frac{l}{2}\) and \(\sum_j \gamma_j \geq 2k\).

The function \(f: \mathbb{Z} \rightarrow \mathbb{R}\) given by \(f(x) = \cos\left(\frac{(q+x)\pi}{q}\right)\) is an even function with period \(2q\).

Also for \(0 \leq x \leq q\) we have the following

\[
f(0) < f(1) < f(2) < f(3) < \ldots < f(q-1) < f(q) \quad (23)
\]

Now to determine the minimum value of \(E(c, q)\), we should choose \(\gamma_j\) to be as large as we can when the function \(\cos\left(\frac{2j+l}{q}\right)\) takes the small value. Now we have the two cases to discuss, the one when \(l\) is even and when \(l\) is odd. Now the image of function \(g: \mathbb{Z}/q\mathbb{Z} \rightarrow \mathbb{R}\) given by \(g(j) = \cos\left(\frac{(2j+l)\pi}{q}\right)\) is equal to \(\{f(x): x \text{ is even}\}\) in case \(l\) is odd
and is equal to \(\{f(x): x \text{ is odd}\}\) in case \(l\) is even. From this it is also easy to observe that the number of \(j \in \mathbb{Z}/q\mathbb{Z}\) such that the function \(\cos\left(\frac{2j+l}{q}\right)\) is periodic is at most \(\frac{q+1}{2}\).

Now let \(-\frac{q-1}{2} \leq j \leq \frac{q-1}{2} - l\) so that \(-q \leq 2j+l \leq q\). Now in case \(l\) is odd then consider the case when \(\gamma_j = \frac{t_c}{2}\) if

\[
2j + l = q - \left[\frac{k}{t_c}\right], \ldots, q - 2, q, q + 1, \ldots, q + \left[\frac{k}{t_c} - \frac{1}{2}\right] \text{ and } \gamma_j = 0 \text{ otherwise}. \quad (24)
\]

The condition \(2\left[\frac{k}{t_c} - 1/2\right] + 1 \geq \frac{q+1}{2}\) ensures that in the above configuration for all possible negative values of \(\cos\left(\frac{(2j+l)\pi}{q}\right)\) the maximum possible weight \(\frac{t_c}{2}\) is chosen. This condition can be ensured if \(q \geq 11\) by choosing \(c \geq c_1\) where \(c_1\) is sufficiently large positive absolute constant.

Therefore after doing a small calculation one may check that for \(c \geq c_1\) the following inequality

\[
E(c, q) \geq -t_c \frac{\sin \frac{2\pi [k - 1/2]}{q}}{2q \sin \pi/l \cos \pi l / q} - \frac{1}{q} \quad (25)
\]

holds. In case \(l\) is even and \(c \geq c_1\) then choosing \(\gamma_j = \frac{t_c}{2}\) if

\[
2j + l = q - \left[\frac{k}{t_c}\right], \ldots, q - 1, q + 1, \ldots, q + \left[\frac{k}{t_c}\right] \text{ and weights 0 otherwise}, \quad (26)
\]

we get that the following inequality

\[
E(c) \geq -t_c \frac{\sin \frac{2\pi [k] + 1}{q}}{2q \sin \pi/l \cos \pi l / q} - \frac{t_c}{q} \quad (27)
\]
holds. Using this we get

\[ S \geq -t_c \frac{\sin \frac{2\pi l}{q}}{2q \sin \frac{\pi}{q} \cos \frac{l}{q}} + \frac{\mu(\mathbb{Z}/q\mathbb{Z})^2}{1 - \mu(\mathbb{Z}/q\mathbb{Z})} \quad \text{when } l \text{ is even} \tag{28} \]

\[ S \geq t_c \frac{\sin \frac{2\pi (l - 1)}{q}}{2q \sin \frac{\pi}{q} \cos \frac{l}{q}} - \frac{1}{q} + \frac{\mu(\mathbb{Z}/q\mathbb{Z})^2}{1 - \mu(\mathbb{Z}/q\mathbb{Z})} \quad \text{when } l \text{ is odd}. \tag{29} \]

\[ S \geq -t_c \frac{\sin \frac{2\pi l}{q}}{2q \sin \frac{\pi}{q} \cos \frac{l}{q}} - \frac{1}{q} + \frac{\mu(\mathbb{Z}/q\mathbb{Z})^2}{1 - \mu(\mathbb{Z}/q\mathbb{Z})} \quad \text{when } l \text{ is odd}. \tag{30} \]

Now as \( q \to \infty \) right hand side of (28) as well as (29) converges to the

\[ -t_c \sin \frac{2\pi l}{3} \sin \frac{2\pi}{q} + \frac{1}{6} \]

Then let \( \eta = 2^{-20} \), then choosing \( c \geq c_2 \) and \( q \geq q_0 \) we get and noticing that \( l \leq \frac{q}{6} \) we get that

\[ S \geq -\frac{1}{2\pi} + \frac{1}{6} - \eta = 8\delta_0 \text{ say}. \tag{31} \]

The above quantity is strictly positive absolute constant. Then if \( \delta < \delta_0 \), this contradicts (19). Hence the Lemma follows. \[ \square \]

To complete the proof of theorem 2 we require the following result from [GR05].

**Lemma 13.** ([GR05], Proposition 7.2) Let \( G \) be an abelian group of type III and \( n \), \( m \) be its order and exponent respectively. Let \( F \subset G \) has at most \( \delta n^2 \) Schur triples, with \( \delta^{1/3} m < 1 \). Let \( q \) be the order of special direction such that \( q \leq q_0 \), where \( q_0 \) is a positive absolute constant as in Lemma 12. Also assume that \( \delta \leq \frac{\eta}{q^5} = \delta' \), where \( \eta = 2^{-50} \), then either \( |F| \leq \mu(G)n \) or \( \alpha_i \leq 64\delta^{1/3} q^{2/3} \).

Combining Lemma 10, Lemma 12 and Lemma 13 the theorem follows in the case \( \delta^{1/3} m < 1 \). In the case \( \delta^{1/3} m > 1 \) the theorem follows from proposition 8.

**References**

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