Effective theories of gauge-Higgs unification models in warped spacetime

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Abstract

We derive four-dimensional (4D) effective theories of the gauge-Higgs unification models in the warped spacetime. The effective action can be expressed in a simple form by neglecting subleading corrections to the wave functions. We have shown in our previous works that some Higgs couplings to the other fields are suppressed by factors that depend on $\bar{\theta}_H$ from the values in the standard model. Here $\bar{\theta}_H$ is the Wilson line phase along the fifth dimension, which characterizes the electroweak symmetry breaking. The effective action derived here explicitly shows a nonlinear structure of the Higgs sector, which clarifies the origins of those suppression factors.

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1 Introduction

In spite of many successes of the standard model, it is not believed to be the final theory because of some theoretical problems it has. One of the problems is an instability of the Higgs boson mass against radiative corrections. It has been argued that the Higgs mass suffers from quadratically divergent radiative corrections, which requires unnatural fine tuning of parameters in the theory unless it is protected by some symmetry. Supersymmetry is one of the most promising candidate of such symmetry. The supersymmetric models generally predict a light Higgs boson. In the minimal supersymmetric standard model, for example, the upper bound for the Higgs boson mass is about 130 GeV [1], which is relatively close to the experimental lower bound 114 GeV [2]. In recent years many alternative scenarios for the Higgs sector have been proposed, such as the little Higgs model [3], the Higgsless model [4], and so on. Among them the gauge-Higgs unification (GHU) scenario predicts various interesting properties in the Higgs couplings to the gauge and the fermion fields.

In the GHU scenario the Higgs field is unified with the gauge fields within the framework of higher dimensional gauge theory [5]-[19]. The extra-dimensional components of the gauge potentials play a role of the Higgs scalar fields in four-dimensional (4D) effective theory. The electroweak symmetry can be broken dynamically by non-Abelian Aharonov-Bohm phases (Wilson line phases) when the extra-dimensional space is nonsimply-connected [6]-[10]. This Hosotani mechanism also provides a finite mass to the 4D Higgs scalar by quantum dynamics. The Higgs mass is protected against the large radiative corrections by the higher dimensional gauge symmetry [11].

The simplest setup for the GHU scenario is five-dimensional (5D) gauge theory whose fifth dimension is compactified on an orbifold $S^1/Z_2$ [7], which naturally realizes chiral fermions in low energies. In the case that the 5D spacetime is flat, it has been shown that the Higgs mass is too small to satisfy the experimental lower bound unless the Wilson line phase $\tilde{\theta}_H$ is dynamically determined to a tiny value.\footnote{This is a generic feature of the GHU models in the flat spacetime. See Ref. [12], for example.} Besides, we have shown in Ref. [18] that trilinear couplings among the W and the Z bosons substantially deviate from the standard model values if $\tilde{\theta}_H = O(1)$, which is inconsistent with the experiments. The warped Randall-Sundrum spacetime [20] ameliorates these problems. The Higgs mass is enhanced by a factor $k\pi R \approx 35$ compared to the case of the flat spacetime [15] where $e^{k\pi R}$ is the warp factor that is used to explain a large hierarchy between the electroweak
scale and the Planck scale. The couplings among the \(W\) and the \(Z\) bosons are in good agreement with those in the standard model even in the case that \(\bar{\theta}_H = \mathcal{O}(1)\) [17, 18]. Therefore we consider the 5D GHU models in the warped spacetime whose fifth dimension is compactified on \(S^1/Z_2\) in this paper.

The conventional method to analyze models with an extra dimension is the so-called Kaluza-Klein (KK) expansion analysis. The procedure is as follows. Firstly we expand the 5D fields into infinite 4D KK modes by means of some complete sets of functions of the extra-dimensional coordinate \(y\). The 4D modes become mass-eigenstates if we choose the complete sets properly. Such functions of \(y\) are called the mode functions. The boundary conditions at both orbifold boundaries determine the mass spectrum and the mode functions. Substituting the mode-expanded expressions of the 5D fields into the 5D action and perform the \(y\)-integral, we obtain the 4D action. By means of this KK analysis we investigated the GHU models in the warped spacetime in our previous papers [16, 17, 18]. We have found that the Higgs couplings to the other fields deviate from their counterparts in the standard model. Namely the former are suppressed by factors that depend on \(\bar{\theta}_H\). This indicates that the Higgs sector has a nonlinear structure in the low-energy effective theory.

An alternative approach to analyze the GHU models is the so-called holographic approach proposed by Ref. [21], which is inspired by the AdS/CFT correspondence [22]. In this approach the field values on one boundary are treated as independent degrees of freedom from the bulk, and the 4D action is obtained by integrating out the latter. This approach is powerful for certain purposes, for example, the calculation of the Higgs potential or of the electroweak oblique parameters [23]. Such quantities can be estimated without explicitly summing contributions from the infinite KK modes [13, 24, 25]. Recently the authors of Ref. [26] derived the 4D effective actions of the 5D gauge theories including the GHU models in the holographic approach. They showed how to incorporate the 4D scalars coming from the fifth components of the 5D gauge fields into the 4D action. This work clarifies the symmetry structure of the effective action. The effective action derived in the holographic approach is expressed in the momentum space and in terms of the boundary values of the 5D fields, which are not mass-eigenstates. Thus the evaluation of the cubic (quartic) couplings requires calculations of the three- (four-) point functions. One of the main purposes of this paper is to understand the suppression factors for the Higgs couplings directly from the nonlinear structure of the Higgs sector. The KK analysis is suitable for
this purpose because the derived 4D action is expressed in terms of the mass-eigenstates and the coupling constants are directly read off from it. So we apply the latter approach to derive the effective action in this paper.

In the warped spacetime the mode functions are written in terms of the Bessel functions and the mass eigenvalues cannot be expressed in analytic forms. For the light modes below the KK mass scale $m_{KK}$, however, they have simple approximate expressions that are analytic functions of $\bar{\theta}_H$ [16, 17, 18]. Correction terms to them are suppressed by $O(\pi^2 m_f^2 / m_{KK}^2)$ where $m_f$ are masses of the light modes. This indicates that we can obtain a simple form of the effective action by neglecting the subleading contributions of $O(\pi^2 m_f^2 / m_{KK}^2)$. Besides, it is convenient to gauge away the background of the gauge fields $A_{y}^{bg}$ for the KK analysis. Then the background information is collected as the $\bar{\theta}_H$-dependent boundary conditions at one orbifold boundary. The nonlinear $\bar{\theta}_H$-dependences of the spectrum and the couplings are well seen in this approach. Since $\bar{\theta}_H$ corresponds to a vacuum expectation value (VEV) of the Higgs field, it is natural to promote it to the dynamical field in the above procedure. Namely we can see the nonlinear structure of the Higgs couplings manifestly by taking a gauge in which the fifth components of the dynamical gauge fields $A_y$ are zero.\footnote{This gauge is also useful to incorporate the 4D scalars coming from $A_y$ into the effective action in the holographic approach [26].} Interestingly the resultant effective actions have similar forms to those of 4D models in which the Higgs bosons are provided as the pseudo Nambu-Goldstone bosons just like the models proposed in Ref. [3]. This is consistent with the holographic interpretation.

The paper is organized as follows. In the next section, we explain our method in detail to derive the effective theory of the GHU models in the warped spacetime. Then we give some comments on the resultant effective action. In Sec. 3, we apply our method to specific models and investigate them from the viewpoint of the effective theory. Especially we see the nonlinear structure of the Higgs sector explicitly. Sec. 4 is devoted to the summary. The notations used in this paper are collected in Appendix A. A complemental calculation to Sec. 3.2.3 is shown in Appendix B.

## 2 Derivation of 4D effective theory

### 2.1 Setup

We consider a gauge theory with a gauge group $G$ in the warped 5D spacetime. The fifth dimension is compactified on an orbifold $S^1/Z_2$ with a radius $R$. We use, throughout the
paper, $M, N, \cdots = 0, 1, 2, 3, 4$ for the 5D curved indices, $A, B, \cdots = 0, 1, 2, 3, 4$ for the 5D flat indices in tetrads, and $\mu, \nu, \cdots = 0, 1, 2, 3$ for 4D indices. The background metric is given by

$$ds^2 = G_{MN}dx^M dx^N = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2,$$

(2.1)

where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1, 1)$, $\sigma(y) = \sigma(y + 2\pi R)$, and $\sigma(y) \equiv k|y|$ for $|y| \leq \pi R$. The cosmological constant in the bulk 5D spacetime is given by $\Lambda = -k^2$. The points $(x^\mu, -y)$ and $(x^\mu, y + 2\pi R)$ are identified with $(x^\mu, y)$. Thus the spacetime is equivalent to the interval in the fifth dimension with two boundaries at $y = 0$ and $y = \pi R$, which we refer to as the UV brane and the IR brane, respectively.

The 5D theory contains gauge fields $A_M$ and a matter fermion field $\Psi$. The former is decomposed as

$$A_M = A^I_M T^I,$$

(2.2)

where $T^I$ are the generators of $G$, which are normalized as

$$\text{tr}(T^I T^J) = \frac{1}{2} \delta^{IJ}.$$  

(2.3)

The 5D action is

$$S_5 = \int d^5 x \sqrt{-G} \left[ -\frac{1}{2} \text{tr} \left( F_{MN} F^{MN} \right) + i \bar{\Psi} \Gamma^N D_N \Psi - i M \varepsilon \bar{\Psi} \Psi \right],$$

(2.4)

where $G \equiv \text{det}(G_{MN})$ and $\Gamma^N \equiv e^A_N \Gamma^A$. The integral region for $y$ is $[0, \pi R]$. The 5D $\gamma$-matrices $\Gamma^A$ are related to the 4D ones $\gamma^\mu$ by $\Gamma^\mu = \gamma^\mu$ and $\Gamma^4 = \gamma_5$ which is the 4D chiral operator. Note that $\Gamma^0$ is anti-Hermitian in our notation. (See (A.3) in Appendix A.) Since the operator $\bar{\Psi} \Psi$ is anti-Hermitian and $Z_2$-odd, we need the factor $i$ and the periodic sign function $\varepsilon(y) = \sigma'(y)/k$ satisfying $\varepsilon(y) = \pm 1$ in (2.4). The field strength and the covariant derivative are defined by

$$F_{MN} \equiv \partial_M A_N - \partial_N A_M - ig_A [A_M, A_N],$$

$$D_M \Psi \equiv \left( \partial_M - \frac{1}{4} \omega^A_M \Gamma_{AB} - ig_A A_M \right) \Psi,$$

(2.5)

where $g_A$ is the 5D gauge coupling, and $\Gamma^{AB} \equiv \frac{1}{2} [\Gamma^A, \Gamma^B]$. The spin connection 1-form $\omega^{AB} = \omega^A_M dx^M$ determined from the metric (2.1) is

$$\omega^\nu = -\frac{d\sigma}{dy} e^{-\sigma} dx^\nu,$$

other components $= 0.$

(2.6)

As the background geometry preserves 4D Poincaré invariance, the curved 4D indices are not discriminated from the flat ones.
The orbifold $Z_2$-parity transformations around $y_0 = 0$ and $y_x = \pi R$, which preserve the orbifold structure, are written as

$$
\begin{pmatrix}
A_\mu \\
A_y
\end{pmatrix}
(x, y_j - y) = P_j \begin{pmatrix}
A_\mu \\
-A_y
\end{pmatrix}(x, y_j + y)P_j^{-1},
$$

$$
\Psi(x, y_j - y) = n_j P_j \gamma_5 \Psi(x, y_j + y),
$$

(2.7)

where $n_j = \pm 1$ ($j = 0, \pi$). The constant matrices $P_j$ belong to the group $G$ and satisfy $P_j^2 = 1$. The gauge group $G$ is generically broken to a subgroup $H$ due to the above orbifold conditions. The unbroken gauge group $H$ depends on the choice of $P_j$ ($j = 0, \pi$). The generators $T^I$ are then classified into two parts, i.e., $T^a$ and $\bar{T}^\dot{a}$, which are the generators of $H$ and $G/H$, respectively. In this paper we take the same $Z_2$-parity assignment at both boundaries, for simplicity. This assumption can be easily relaxed in the following derivation of the 4D effective theory.

The fermion field $\Psi$ belongs to an irreducible representation of the full gauge group $G$. It can be decomposed into components each of which belongs to an irreducible representation of $H$. For simplicity, we assume that $\Psi$ is decomposed into two such components, i.e.,

$$
\Psi = \begin{pmatrix} q \\ Q \end{pmatrix},
$$

(2.8)

where $q$ and $Q$ belong to irreducible representations of $H$ and have opposite $Z_2$-parities. In the case of $G = SU(3)$ and $H = SU(2) \times U(1)$, for example, an $SU(3)$ triplet $\Psi$ is decomposed into a doublet $q$ and a singlet $Q$ for the unbroken $SU(2)$. Extension to the cases that $\Psi$ is decomposed into more than two components is straightforward.

The $Z_2$-parities for the fields are collected in Table I. The fields which are even (odd) under the orbifold parity at $y = y_j$ obey the Neumann (Dirichlet) boundary conditions there if there are no additional dynamics on the boundary $y = y_j$. 

**Table I:** The $Z_2$-parities of the gauge and fermion fields. We take the same parity assignment at both boundaries.

| Field | $A^a_\mu$ | $A^\dot{a}_\mu$ | $A^a_y$ | $A^\dot{a}_y$ | $q_L$ | $Q_L$ | $q_R$ | $Q_R$ |
|-------|----------|---------------|--------|----------|--------|--------|--------|--------|
|       | +        | $-$           | $-$    | $+$       | $+$    | $-$    | $+$    | $+$    |
2.2 Gauging away of $A_y$

We can always gauge $A_y^a$ away without changing the boundary conditions listed in Table I. Namely $A_y^a$ are not physical degrees of freedom. On the other hand, $\hat{A}_y^\alpha$ cannot be completely gauged away because corresponding gauge transformation mixes components with different $Z_2$-parities. This means that $\hat{A}_y^\alpha$ contain physical modes. However we can formally perform the following gauge transformation to remove the whole $A_y$ from the bulk.

$$\hat{A}_M = \Omega A_M \Omega^{-1} - \frac{i}{g_A} (\partial_M \Omega) \Omega^{-1},$$

(2.9)

where

$$\Omega(x, y) \equiv \mathcal{P} \exp \left\{ ig_A \int_0^{-\pi R} dy' A_y(x, y') \right\}.$$  

(2.10)

The symbol $\mathcal{P}$ stands for the path ordering operator from left to right. This transformation must be discontinuous at $y = 0$ to maintain the $Z_2$-parities of the fields. If we parametrize $\Omega$ as

$$\Omega(x, y) = \exp \left\{ i\varphi^a(x, y) T^a \right\} \exp \left\{ i\theta^\alpha(x, y) T^\alpha \right\},$$

(2.11)

the transformation parameters $\theta^\alpha(x, y)$ are odd under the $Z_2$-parity at both boundaries while $\varphi^a(x, y)$ are even. Since $\Omega(x, y)$ satisfies that

$$\Omega(x, \pi R) = 1,$$

(2.12)

i.e., $\varphi^a(x, \pi R) = \theta^\alpha(x, \pi R) = 0$, all the transformation parameters $(\varphi^a, \theta^\alpha)$ are continuous at the IR brane. On the other hand, the parameters $\theta^\alpha(x, y)$ become discontinuous at the UV brane, i.e.,

$$\left[ \theta^\alpha(x, y) \right]_{y=\epsilon}^{y=-\epsilon} = 2\theta^\alpha(x, \epsilon) \neq 0,$$

(2.13)

where $\epsilon$ is a positive infinitesimal. These gaps $\theta^\alpha(x, \epsilon)$ at $y = 0$, which are equal to the Wilson line phases by definition, are the physical degrees of freedom, and are identified with the Higgs fields as we will see below. The parameters $\varphi^a(x, y)$ are continuous also at the UV brane, and can be absorbed by a gauge transformation for the residual gauge symmetry $H$.

Now the action (2.4) becomes

$$S_5 = S_5^{\text{gauge}} + S_5^{\text{fermi}},$$

$$S_5^{\text{gauge}} = \int d^5x \left[ -\frac{1}{2} \text{tr} \left\{ \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + 2e^{-2\sigma} \left( \partial_\mu \tilde{A}^\mu \partial_\nu \tilde{A}^\nu \right) \right\} \right],$$

$$S_5^{\text{fermi}} = \int d^5x \, i \left\{ \tilde{\Psi} \left( e^\sigma \gamma^\mu \tilde{D}_\mu + \gamma_5 \partial_y - M \tilde{\varepsilon} \right) \tilde{\Psi} \right\},$$

(2.14)
where

\[ \tilde{\Psi} = \begin{pmatrix} \tilde{q} \\ \tilde{Q} \end{pmatrix} \equiv e^{-2\sigma} \Omega \Psi, \]

\[ \tilde{D}_\mu \tilde{\Psi} \equiv \left( \partial_\mu - ig_A \tilde{A}_\mu \right) \tilde{\Psi}. \]  

(2.15)

The boundary conditions for \( (\tilde{A}_\mu, \tilde{\Psi}) \) at \( y = \pi R \) are the same as those of \( (A_\mu, e^{-2\sigma(y)}\Psi) \) because \( \Omega(x, \pi R) = 1 \). Namely they obey the conditions:

\[
\partial_y \tilde{A}_\mu^a = 0, \quad \tilde{A}_\mu^a = 0, \\
(\partial_y + M)\tilde{q}_L = 0, \quad \tilde{Q}_L = 0, \\
\tilde{q}_R = 0, \quad (\partial_y - M)\tilde{Q}_R = 0, 
\]  

(2.16)

(2.17)

at \( y = \pi R \). On the other hand, the boundary conditions at \( y = 0 \) are neither Neumann nor Dirichlet conditions any more due to the appearance of \( \theta^a(x, \epsilon) \). In the rest of this paper we take the integration region for \( y \) as \( [\epsilon, \pi R] \) in order to avoid the discontinuity at \( y = 0 \). Then \( \sigma(y) = ky \) and \( \epsilon(y) = 1 \) in the following.

### 2.3 Gauge sector

The equations of motion for the gauge fields \( \tilde{A}_\mu^I \) are obtained from (2.14) as

\[
\left[ D_\nu \tilde{F}^{\mu\nu} \right]^I - \partial_y \left( e^{-2\sigma} \partial_y \tilde{A}_\mu^I \right) + e^\sigma g_A \tilde{\Psi} \gamma^\mu T^I \tilde{\Psi} = 0.
\]  

(2.18)

The symbol \([\cdots]^I\) denotes the \( I \)-component in the decomposition by the generators \( T^I \), i.e.,

\[
[C]^I \equiv 2\text{tr}(T^I C) = C^I,
\]  

(2.19)

for some matrix \( C = C^J T^J \). From the 4D point of view, the second term in (2.18) corresponds to the mass term. We focus on the modes that are much lighter than the KK mass scale \( m_{KK} \equiv k\pi/(e^{k\pi R} - 1) \) in the following. They can be regarded as massless modes in the first approximation. Thus we impose the following “massless condition” to extract the zero-modes from the 5D fields.

\[
\partial_y \left( e^{-2\sigma} \partial_y \tilde{A}_\mu^I \right) = 0.
\]  

(2.20)
Solving this with the conditions (2.16), the \( y \)-dependences of \((\tilde{A}_a^\mu, \tilde{A}^{\hat{a}}_\mu)\) are completely determined as
\[
\tilde{A}_a^\mu(x, y) = \tilde{A}_a^\mu(x, \epsilon),
\tilde{A}^{\hat{a}}_\mu(x, y) = \frac{e^{2k\pi R} - e^{2\sigma(y)}}{e^{2k\pi R} - 1} \tilde{A}^{\hat{a}}_\mu(x, \epsilon).
\] (2.21)

Substituting these into \( S_{\text{gauge}}^5 \) in (2.14) and performing the \( y \)-integral, we obtain the 4D action.
\[
S_{\text{gauge}}^4 \simeq \int d^4 x \left\{ -\frac{\pi R}{2} \text{tr} \left( \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \right) - ke^{-2k\pi R} \tilde{A}^{\hat{a}}_\mu \tilde{A}^{\hat{a}\mu} \right\}_{y=\epsilon}. \tag{2.22}
\]

Here we have neglected corrections suppressed by a factor of \((k\pi R)^{-1}\), which is about a few percent when \(e^{k\pi R} = \mathcal{O}(10^{15})\). Note that each component of \( \tilde{A}_a^\mu(x, \epsilon) \) is expressed by \( A_a^\mu(x, \epsilon) \), \( \varphi^a(x, \epsilon) \) and \( \theta^{\hat{a}}(x, \epsilon) \) since \( A^{\hat{a}}_\mu(x, 0) = 0 \). As mentioned in the previous subsection, the \( \varphi^a(x, \epsilon) \)-dependence can be removed by a 4D gauge transformation for \( \mathbb{H} \). In the following the 4D fields \( \tilde{A}_I^\mu(x, \epsilon) \) are understood as the fields after this gauge transformation. Then they are expressed in terms of \( A_a^\mu(x, \epsilon) \) and \( \theta^{\hat{a}}(x, \epsilon) \). For example,
\[
\tilde{A}^{\hat{a}}_\mu(x, \epsilon) = \left[ \Omega_0 A_\mu(x, \epsilon) \Omega^{-1}_0 - \frac{i}{g_A} \partial_\mu \Omega_0 \Omega^{-1}_0 \right]^{\hat{a}} = \frac{1}{g_A} \left\{ \partial_\mu \theta^{\hat{a}}(x, \epsilon) + g_A C_{abc} A_b^\mu(x, \epsilon) \theta^c(x, \epsilon) + \cdots \right\}, \tag{2.23}
\]

where \( C_{IJK} \) are the structure constants of \( \mathbb{G} \) defined in (A.5), and
\[
\Omega_0(x) \equiv \exp \left\{ i \theta^{\hat{a}}(x, \epsilon) T^{\hat{a}} \right\}. \tag{2.24}
\]

The ellipsis in the second line of (2.23) denotes higher order terms for \( \theta^{\hat{a}}(x, \epsilon) \). Thus the second term in (2.22) corresponds to the kinetic terms for \( \theta^{\hat{a}}(x, \epsilon) \). On the other hand, the \( \theta^{\hat{a}} \)-dependence of the first term in (2.22) is cancelled, \( i.e. \),
\[
\text{tr} \left( \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \right)_{y=\epsilon} = \text{tr} \left( F_{\mu\nu} F^{\mu\nu} \right)_{y=\epsilon} = \frac{1}{2} \left( F_{\mu
u} F^{\mu\nu} \right)_{y=\epsilon}, \tag{2.25}
\]

Therefore we redefine the fields as
\[
A_a^\mu(x) \equiv \sqrt{\pi R} A_a^\mu(x, \epsilon),
H^{\hat{a}}(x) \equiv \frac{\sqrt{2k}e^{-k\pi R}}{g_A} \theta^{\hat{a}}(x, \epsilon), \tag{2.26}
\]
so that the fields are canonically normalized. The resultant 4D action is
\[
S_{\text{gauge}}^4 \simeq \int d^4 x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \tilde{D}^{(4)}_{\mu} H^{\hat{a}} \tilde{D}^{(4)\mu} H^{\hat{a}} \right\}. \tag{2.27}
\]
where
\[ F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \bar{g}_A C_{abc} A_\mu^b A_\nu^c, \]
\[ \tilde{D}^{(4)}_\mu H^{\bar{a}} \equiv \frac{\sqrt{2ke^{-k\pi R}}}{\sqrt{\pi R}} \left[ \Omega_0(A_\mu T^a)\Omega_0^{-1} - \frac{i}{\bar{g}_A} \partial_\mu \Omega_0 \Omega_0^{-1} \right]^{\bar{a}} \]
\[ = \partial_\mu H^{\bar{a}} + \bar{g}_A C_{abc} A_\mu^b H^{\bar{c}} + \mathcal{O}(H^2). \] (2.28)

Here a dimensionless constant \( \bar{g}_A \equiv g_A/\sqrt{\pi R} \) is the 4D gauge coupling, and \( \Omega_0 \) is expressed in terms of \( H^{\bar{a}} \) as
\[ \Omega_0(x) = \exp \left\{ \frac{i}{f_H} H^{\bar{a}}(x)T^{\bar{a}} \right\}, \]
\[ f_H \equiv \frac{\sqrt{2ke^{-k\pi R}}}{g_A}. \] (2.29)

### 2.4 Fermion sector

The equation of motion for the fermion field \( \tilde{\Psi} \) is
\[ \left\{ \gamma^\mu \tilde{D}_\mu + e^{-\sigma} (\gamma_5 \partial_y - M) \right\} \tilde{\Psi} = 0. \] (2.30)

Multiplying the differential operator: \( \left\{ \gamma^\nu \tilde{D}_\nu + e^{-\sigma} (\gamma_5 \partial_y + M) \right\} \) from the left, we obtain
\[ \tilde{D}_\mu \tilde{D}^\mu \tilde{\Psi} + e^{-2\sigma} \left\{ \partial_y^2 - k (\partial_y - \gamma_5 M) - M^2 \right\} \tilde{\Psi} = 0. \] (2.31)

The second term corresponds to the mass term from the 4D point of view. Then the "massless conditions" are written as
\[ \left\{ \partial_y^2 - k (\partial_y - \gamma_5 M) - M^2 \right\} \left( \frac{\tilde{q}}{\tilde{Q}} \right) = 0. \] (2.32)

Solving these equations with the boundary conditions in (2.17), the \( y \)-dependence of the 5D fermion fields are completely determined as follows.
\[ \tilde{q}_L(x, y) = e^{-c\sigma(y)} \tilde{q}_L(x, \epsilon), \]
\[ \tilde{Q}_L(x, y) = e^{-c\sigma(y)} \left( \frac{e^{(k+2M)\pi R} - e^{(1+2c)\sigma(y)}}{e^{(k+2M)\pi R} - 1} \right) \tilde{Q}_L(x, \epsilon), \]
\[ \tilde{q}_R(x, y) = e^{c\sigma(y)} \left( \frac{e^{(k-2M)\pi R} - e^{(1-2c)\sigma(y)}}{e^{(k-2M)\pi R} - 1} \right) \tilde{q}_R(x, \epsilon), \]
\[ \tilde{Q}_R(x, y) = e^{c\sigma(y)} \tilde{Q}_R(x, \epsilon), \] (2.33)
where \( c \equiv M/k \). Each component of \( \tilde{\Psi}(x, \epsilon) \) is expressed by \( q_L(x, \epsilon), Q_R(x, \epsilon), \varphi^a(x, \epsilon) \) and \( \theta^\hat{a}(x, \epsilon) \) since \( Q_L(x, 0) = q_R(x, 0) = 0 \). As mentioned in the previous subsection, \( \varphi^a \) can be removed by the 4D gauge transformation for \( \mathbb{H} \). So \( \tilde{\Psi}(x, \epsilon) \) is understood as the 4D field after this transformation. Namely,

\[
\begin{align*}
\begin{pmatrix} \tilde{q}_L \\ \tilde{Q}_L \end{pmatrix}(x, \epsilon) &= \Omega_0 \begin{pmatrix} q_L \\ 0 \end{pmatrix}(x, \epsilon), \\
\begin{pmatrix} \tilde{q}_R \\ \tilde{Q}_R \end{pmatrix}(x, \epsilon) &= \Omega_0 \begin{pmatrix} 0 \\ Q_R \end{pmatrix}(x, \epsilon).
\end{align*}
\]

Making use of the following equations followed from (2.33),

\[
(\partial_y + M)\tilde{q}_L = (\partial_y - M)\tilde{Q}_R = 0,
\]

the fermionic part of the 5D action \( S_5^{\text{fermi}} \) in (2.14) can be rewritten as

\[
S_5^{\text{fermi}} = \int d^5 x \left\{ e^\sigma \tilde{\Psi} \gamma^\mu \tilde{D}_\mu \tilde{\Psi} + \partial_y \left( \tilde{q}_L \tilde{q}_R - \tilde{Q}_R \tilde{Q}_L \right) \right\}
\]

\[
= \int d^4 x \left\{ e^\sigma \tilde{\Psi} \gamma^\mu \tilde{D}_\mu \tilde{\Psi} + \int d^4 y \left( \tilde{q}_L \tilde{q}_R - \tilde{Q}_R \tilde{Q}_L \right) \right\}_{y = \epsilon}.
\]

The surface terms on the IR brane vanish due to the boundary conditions (2.17).

In the following we assume that \(|c| \geq 1/2\).

### 2.4.1 case of \( c \geq 1/2 \)

The solution (2.33) is approximated as

\[
\begin{align*}
\tilde{q}_L(x, y) &= e^{-\sigma(y)}\tilde{q}_L(x, \epsilon), \quad \tilde{Q}_L(x, y) \simeq e^{-\sigma(y)}\tilde{Q}_L(x, \epsilon), \\
\tilde{q}_R(x, y) &\simeq e^{(1-c)\sigma(y)}\tilde{q}_R(x, \epsilon), \quad \tilde{Q}_R(x, y) = e^{\sigma(y)}\tilde{Q}_R(x, \epsilon).
\end{align*}
\]

Substituting these into (2.36) and performing the \( y \)-integral, we obtain the 4D action,

\[
S_4^{\text{fermi}} \simeq \int d^4 x \left\{ \frac{1}{2M - k} \tilde{\Psi}_L \gamma^\mu \tilde{D}_\mu \tilde{\Psi} + \frac{e^{(k+2M)\pi R}}{k+2M} \left( -e^{-k\pi R} \right) \begin{pmatrix} 0 \\ Q_R \end{pmatrix} \right\}_y \]

\[
= \int d^4 x \left\{ \tilde{q}_L \tilde{q}_R + \tilde{Q}_R \tilde{Q}_L \right\}_y.
\]

Since \( e^{(1-c)\sigma(y)} \ll e^{\sigma(y)} \) in most part of the bulk, the contribution of \( \tilde{q}_R \) in the \( y \)-integral in (2.36) is negligible. We have dropped corrections suppressed by a factor of \((k \pi R)^{-1}\) coming from the integral of terms involving \( \tilde{A}_\mu^\hat{a} \) in \( \tilde{D}_\mu \).
We can see from (2.34) that the $\theta^a$-dependence of the first term in (2.38) is cancelled, i.e.,

\[
\left( \tilde{\Psi}_L \gamma^\mu \hat{D}_\mu \tilde{\Psi}_L \right)_{y=\epsilon} = \left( \bar{q}_L, 0 \right) \gamma^\mu \hat{D}_\mu \left( q_L \right)_{y=\epsilon},
\]

where

\[
\hat{D}_\mu \equiv \partial_\mu - igA_\mu.
\]

On the other hand, such cancellation does not occur for the right-handed components because the contribution of $\tilde{q}_R$ in the kinetic term is negligible.

Here we decompose $\Omega_0$ into four parts so that the relation: $\tilde{\Psi}(x, \epsilon) = \Omega_0 \Psi(x, \epsilon)$ is rewritten as

\[
\left( \bar{q}(x, \epsilon) \hspace{1cm} \bar{Q}(x, \epsilon) \right) = \left( \Omega_0^{qq} \hspace{1cm} \Omega_0^{qQ} \hspace{1cm} \Omega_0^{Qq} \hspace{1cm} \Omega_0^{QQ} \right) \left( q(x, \epsilon) \hspace{1cm} Q(x, \epsilon) \right). \tag{2.41}
\]

Then (2.38) becomes

\[
S^\text{fermi}_4 \simeq \int d^4 x \left\{ \frac{1}{2M - k} (\bar{q}_L, 0) \gamma^\mu \hat{D}_\mu \left( q_L \right) + \frac{e^{(k+2M)\pi R}}{k + 2M} (0, \bar{Q}_R (\Omega_0^{QQ})^\dagger) \gamma^\mu \hat{D}_\mu \left( 0 \right) \right. \\
- \bar{q}_L (\Omega_0^{qq})^\dagger \Omega_0^{qQ} Q_R + \bar{Q}_R (\Omega_0^{QQ})^\dagger \Omega_0^{Qq} q_L \left\} y=\epsilon. \tag{2.42}
\]

To normalize the fields canonically, we redefine them as

\[
\psi_L(x) \equiv \frac{q_L(x, \epsilon)}{\sqrt{2M - k}}, \tag{2.43}
\]

\[
\chi_R(x) \equiv i \left( \frac{e^{(k+2M)\pi R}}{k + 2M} \right)^{1/2} \Omega_0^{QQ} Q_R(x, \epsilon) \tag{2.44}
\]

Then the above action is rewritten as

\[
S^\text{fermi}_4 \simeq \int d^4 x \left\{ \bar{\psi}_L \gamma^\mu \hat{D}_\mu^{(4)} \psi_L + \bar{\chi}_R \gamma^\mu \hat{D}_\mu^{(4)} \chi_R \\
- i \left( \frac{4M^2 - k^2}{e^{(2M+k)\pi R}} \right)^{1/2} \left( \bar{\psi}_L (\Omega_0^{QQ})^\dagger \chi_R - \bar{\chi}_R (\Omega_0^{Qq})^\dagger \psi_L \right) \right\}. \tag{2.45}
\]

Here we have used the relation followed from the unitarity condition of $\Omega_0$:

\[
(\Omega_0^{qq})^\dagger \Omega_0^{qQ} = -(\Omega_0^{Qq})^\dagger \Omega_0^{QQ}. \tag{2.46}
\]
The covariant derivatives $\mathcal{D}_\mu^{(4)}$ and $\tilde{\mathcal{D}}_\mu^{(4)}$ are defined by

$$
\mathcal{D}_\mu^{(4)} \psi_L \equiv \left( \partial_\mu - ig_A A_\mu T^a \right) \psi_L,
$$

$$
\tilde{\mathcal{D}}_\mu^{(4)} \chi_R \equiv \left( \partial_\mu - ig_A A_\mu T^a \right) \chi_R,
$$

where

$$
\tilde{A}_\mu^a \equiv \left[ \Omega_0 (A_\mu T^a) \Omega_0^{-1} - i \frac{\bar{g}_A}{g_A} \partial_\mu \Omega_0 \Omega_0^{-1} \right]^a.
$$

(2.48)

2.4.2 case of $c \leq -1/2$

Now the solution (2.33) is approximated as

$$
\tilde{q}_L(x, y) = e^{-c\sigma(y)}\tilde{q}_L(x, \epsilon), \quad \tilde{q}_R(x, y) \simeq e^{c\sigma(y)}\tilde{q}_R(x, \epsilon),
$$

$$
\tilde{Q}_L(x, y) \simeq e^{(1+c)\sigma(y)}\tilde{Q}_L(x, \epsilon), \quad \tilde{Q}_R(x, y) = e^{c\sigma(y)}\tilde{Q}_R(x, \epsilon).
$$

(2.49)

Then $S_4^{\text{fermi}}$ is calculated as

$$
S_4^{\text{fermi}} \simeq \int d^4x \begin{pmatrix} e^{(k-2M)\pi R} & 0 \\ k - 2M & 0 \end{pmatrix} \gamma_\mu \tilde{D}_\mu \left( \begin{array}{c} \tilde{q}_L \\ 0 \end{array} \right) + \frac{1}{k + 2M} \tilde{\Psi}_R \gamma_\mu \tilde{D}_\mu \tilde{\Psi}_R
$$

$$
- \tilde{q}_L \tilde{q}_R + \tilde{Q}_R \tilde{Q}_L \bigg|_{y=\epsilon}.
$$

(2.50)

Since $e^{-c\sigma(y)} \gg e^{(1+c)\sigma(y)}$ in most part of the bulk, the contribution of $\tilde{Q}_L$ in the kinetic term is negligible. We have again dropped terms suppressed by a factor of $(k\pi R)^{-1}$.

The $\theta^a$-dependence of the second term in (2.50) is cancelled, i.e.,

$$
\left\{ \tilde{\Psi}_R \gamma_\mu \tilde{D}_\mu \tilde{\Psi}_R \right\}_{y=\epsilon} = \left\{ (0, \tilde{Q}_R) \gamma_\mu \tilde{D}_\mu \left( \begin{array}{c} 0 \\ Q_R \end{array} \right) \right\}_{y=\epsilon},
$$

(2.51)

while such cancellation does not occur for the left-handed components. Thus (2.50) becomes

$$
S_4^{\text{fermi}} \simeq \int d^4x \begin{pmatrix} e^{(k-2M)\pi R} & 0 \\ k - 2M & 0 \end{pmatrix} \gamma_\mu \tilde{D}_\mu \left( \begin{array}{c} \Omega^{qq}_0 q_L \\ 0 \end{array} \right) + \frac{1}{k + 2M} (0, Q_R) \gamma_\mu \tilde{D}_\mu \left( \begin{array}{c} 0 \\ Q_R \end{array} \right)
$$

$$
- \tilde{q}_L (\Omega^{qq}_0)^\dagger \Omega^{QQ}_0 Q_R + \tilde{Q}_R (\Omega^{QQ}_0)^\dagger \Omega^{QQ}_0 q_L \bigg|_{y=\epsilon}.
$$

(2.52)

To normalize the fields canonically, we redefine them as

$$
\chi_L(x) \equiv i \left( \frac{e^{(k-2M)\pi R}}{k - 2M} \right)^{1/2} \Omega^{qq}_0 q_L(x, \epsilon)
$$

$$
= \frac{i e^{-\frac{3\pi R}{k - 2M}}}{\sqrt{k - 2M}} q_L(x, \pi R),
$$

(2.53)

$$
\psi_R(x) \equiv \frac{Q_R(x, \epsilon)}{\sqrt{k + 2M}}.
$$

(2.54)
Then the above action is rewritten as

\[
S^{\text{fermi}}_4 \simeq \int d^4 x \ i \left\{ \overline{\chi}_L \gamma^\mu \tilde{D}^{(4)}_\mu \chi_L + \overline{\psi}_R \gamma^\mu D^{(4)}_\mu \psi_R \right.
- i \left( \frac{k^2 - 4M^2}{e(k-2M)\pi R} \right)^{1/2} \left( \chi_L \Omega^Q_0 \psi_R - \overline{\psi}_R (\Omega^Q_0)^\dagger \chi_L \right) \left. \right\}.
\]

(2.55)

Here we have used the Hermitian conjugate of (2.46),

\[
(\Omega^Q_0)^\dagger \Omega^Q_0 = -(\Omega^Q_0)^\dagger \Omega^Q_0.
\]

(2.56)

### 2.5 comments

#### 2.5.1 Review of 4D action

Let us review the derived 4D effective action. It consists of the Higgs fields \( H^{\hat{a}}(x) \) (or the Wilson line phase \( \theta^{\hat{a}}(x, \epsilon) \)) and the 4D boundary values of the 5D fields at the UV brane. Note that the gauge fields \( \tilde{A}_a^a_{\mu} \) that appear in the covariant derivatives \( \tilde{D}^{(4)}_\mu \chi_{R,L} \) are dressed by the Higgs fields. Such “dressed gauge fields” \( \tilde{A}_a^a_{\mu} \) and the covariant derivatives of the Higgs fields \( \tilde{D}^{(4)}_\mu H^{\hat{a}} \) can be read off from the gauged Maurer Cartan 1-form \( \alpha_\mu \) as follows. (See (2.28) and (2.48).)

\[
i\alpha_\mu \equiv \Omega_0 \left( \bar{g}_A A_\mu^a T^a \right) \Omega_0^{-1} - i \partial_\mu \Omega_0 \Omega_0^{-1}
= \bar{g}_A \tilde{A}_\mu^a T^a + \frac{1}{f_H} \tilde{D}^{(4)}_\mu H^{\hat{a}} T^{\hat{a}}.
\]

(2.57)

Here \( \Omega_0 \) is an element of \( \mathbb{G}/\mathbb{H} \) parametrized by the Higgs fields as (2.29). Using these ingredients we can construct the effective action \( S_4 = S_4^{\text{gauge}} + S_4^{\text{fermi}} \) by the formulae (2.27) and (2.45) (or (2.55)).

#### 2.5.2 Analogy to PNG Higgs models and holographic interpretation

Obviously the above construction is nothing but the nonlinear realization of \( \mathbb{G}/\mathbb{H} \). This suggests an analogy between 5D GHU model in the warped spacetime and 4D model where the Higgs bosons are realized as the pseudo Nambu-Goldstone (PNG) bosons just like the models in Ref. [3]. This analogy has been discussed in detail in Ref. [24]. The Higgs fields \( H^{\hat{a}} \) and the constant \( f_H \) correspond to the PNG bosons and their decay constant, respectively. The large gauge invariance of \( S_4 \) (or the periodicity of \( H^{\hat{a}} \)) is manifest since
\( H^\hat{a} \) are the coordinates on the compact manifold \( \mathbb{G}/\mathbb{H} \). If we formally set \( A^a_\mu \) and \( \psi_{L,R} \) to zero, the effective action \( S_4 \) is invariant under the following nonlinear \( \mathbb{G} \)-transformations.

\[
H^\hat{a} \rightarrow H'^\hat{a}, \quad \chi_{R,L} \rightarrow h\chi_{R,L},
\]

(2.58)

where \( H'^\hat{a} \) and \( h \in \mathbb{H} \) are defined as

\[
\xi \Omega_0^{-1}(H^\hat{a}) = \Omega_0^{-1}(H'^\hat{a})h,
\]

(2.59)

for an arbitrary group element \( \xi \in \mathbb{G} \). Thus \( h \) depends not only on \( \xi \) but also on \( H^\hat{a} \).

Under the above transformations, it follows from (2.47) and (2.57) that

\[
\tilde{A}^a_\mu T^a \rightarrow h(\tilde{A}^a_\mu T^a)h^{-1} - \frac{i}{g_A} \partial_\mu hh^{-1},
\]

\[
\tilde{D}^{(4)}_\mu H^\hat{a} T^a \rightarrow h(\tilde{D}^{(4)}_\mu H^\hat{a} T^a)h^{-1},
\]

\[
\tilde{D}^{(4)}_\mu \chi_{R,L} \rightarrow h\tilde{D}^{(4)}_\mu \chi_{R,L},
\]

(2.60)

Here \( \tilde{A}^a_\mu \) are purely made of \( H^\hat{a} \). The invariance under these transformations ensures the masslessness of \( H^\hat{a} \). Namely they can be identified as exact NG bosons in this case. Now we turn on \( A^a_\mu \) and \( \psi_{L,R} \) in the effective action. Then the above \( \mathbb{G} \) invariance is broken to \( \mathbb{H} \) explicitly so that the NG bosons \( H^\hat{a} \) become PNG bosons.\(^4\) Recall that both \( \chi_{R,L} \) and \( H^\hat{a} \) correspond to zero-modes of \( \Psi \) and \( A^a_\mu \) localized near the IR brane in the KK analysis. On the other hand \( \psi_{L,R} \) and \( A^a_\mu \) correspond to zero-modes of \( \Psi \) and \( A^a_\mu \) that are localized near the UV brane and spread over the bulk, respectively. Hence only the modes localized near the IR brane respect the \( \mathbb{G} \) symmetry which is realized nonlinearly.

The above features are consistent with the holographic interpretation [21, 24, 25, 26], which is based on a conjecture that 5D theories on the warped spacetime are dual to 4D theories with a strongly interacting sector. In this interpretation, the 5D bulk corresponds to some strongly coupled conformal sector in a 4D theory, and the UV and the IR branes correspond to the UV cutoff scale \( \Lambda_{UV} \) and the spontaneous breakdown of the conformal symmetry at \( \Lambda_{IR} \), respectively. Due to the conformal symmetry breaking, there appears a mass gap in the theory and the CFT spectrum is discretized. Namely bound states appear, which are identified with the modes localized near the IR brane in the 5D picture. There is a massless bound state \( \chi_R \) or \( \chi_L \) among such modes depending on the parameter \( c = M/k \), which corresponds to the dimension of a CFT operator relevant to \( \chi_{R,L} \). The gauge

\(^4\)The Higgs fields \( H^a \) acquire nonzero masses at quantum level.
symmetry $G$ in the 5D theory is identified with a global symmetry in the conformal sector of the 4D theory, and is spontaneously broken to the subgroup $\mathbb{H}$ at $\Lambda_{\text{IR}}$. The unbroken symmetry $\mathbb{H}$ is gauged by $A^a_\mu$ that are external to the conformal sector. The elementary fields $A^a_\mu$ and $\psi_{L,R}$, which are coupled to the conformal sector at $\Lambda_{\text{UV}}$, are provided by the boundary values of the 5D fields $A^a_\mu$ and $\Psi$ at the UV brane. This is consistent with (2.26) and (2.43) (or (2.54)). In this holographic interpretation, the facts mentioned in the previous paragraph are understood as follows. If we turn off the external fields $A^a_\mu$ and $\psi_{L,R}$, the global symmetry $G$ becomes exact and thus the NG modes $H^a$ associated with $G/\mathbb{H}$ are massless. The effective action consists of the bound states $H^a$ and $\chi_{R,L}$, and has an invariance under the nonlinear $G$-transformation. The gauge connections $\tilde{A}^a_\mu$ in $\mathcal{D}^{(4)}_{\mu} \chi_{R,L}$ are purely made of the NG bosons $H^a$, i.e., the CFT bound states. After including the external fields $A^a_\mu$ and $\psi_{L,R}$, $G$ is broken to $\mathbb{H}$ explicitly so that $H^a$ become PNG bosons. The connection $\tilde{A}^a_\mu$ are now the mixing states between the elementary states $A^a_\mu$ and the CFT bound states.

The symmetry structure mentioned above can also be seen in Ref. [26] where the 4D effective action is derived in the holographic procedure. The holographic procedure is useful to calculate the effective potential of the Higgs fields or the electroweak oblique parameters [13, 24, 25]. On the other hand, our 4D action is suitable to see the whole structure of the nonlinear Higgs couplings among the light modes as we will see in the next section.

### 2.5.3 Validity of approximations

In our derivation of the effective action $S_4$, we have taken two approximations. Firstly we have neglected masses of the light modes that appear in $S_4$ in determining their $y$-dependences. (See (2.20) and (2.32).) Secondly we have dropped terms suppressed by a factor of $(k\pi R)^{-1}$ coming from the $y$-integral of terms involving $\tilde{A}^a_\mu$. To be precise, the light modes in $S_4$ get nonzero masses by the Higgs mechanism when $H^a$ have nontrivial VEV. The typical scale of such masses is characterized by the $W$ boson mass $m_W$. On the other hand, the cutoff scale of the effective theory is given by the KK mass scale $m_{\text{KK}}$. In fact corrections to the expressions (2.27) and (2.45) (or (2.55)) are estimated to be of or less

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5Precisely speaking, the UV boundary values of the 5D fields contain contributions from the KK modes which we have neglected. However they are exponentially suppressed because the KK modes are localized near the IR brane in the warped spacetime.

6The fermion masses are smaller than $m_W$ for $|c| > 1/2$. 

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than \( \mathcal{O}(\pi^2 m_W^2/m_{KK}^2) \), which is \( \mathcal{O}(1/k\pi R) \) as we will see in the end of Sec. 3.1. Therefore the second approximation is consistent with the first one. In the derivation of \( S_{4}^{\text{fermi}} \) we have not considered the case that \( |c| < 1/2 \) because we do not obtain a simple form of the effective action by our method in such a case. This is related to the fact that the fermion mass becomes larger than \( m_W \) when \( |c| < 1/2 \) and the error mentioned above increases.

### 3 Specific models

In this section we consider specific models and derive their effective actions by the method proposed in the previous section.

#### 3.1 SU(3) model

Here we consider the \( SU(3) \) model investigated in Ref. [16]. The \( SU(3) \) gauge field \( A_M \) is decomposed as

\[
A_M = A_M^I \frac{\lambda^I}{2},
\]

where \( \lambda^I \) \((I = 1, 2, \cdots, 8)\) are the Gell-Mann matrices. As a matter field we introduce a fermion field \( \Psi \) that is an \( SU(3) \) triplet. We choose \( P_j \) and \( \eta_j \) \((j = 0, \pi)\) in (2.7) as

\[
P_0 = P_\pi = \begin{pmatrix}
-1 \\
-1 \\
1
\end{pmatrix},
\eta_0 = \eta_\pi = +1,
\]

in the fundamental representation. Then \( \mathcal{G} = SU(3) \) is broken to \( \mathcal{H} = SU(2) \times U(1) \). The unbroken and the broken generators \( T^a \) and \( \bar{T}^a \) are

\[
T^a = \frac{\lambda^a}{2}, \quad (a = 1, 2, 3, 8)
\]
\[
\bar{T}^a = \frac{\lambda^{\bar{a}}}{2}, \quad (\bar{a} = 4, 5, 6, 7)
\]

The \( SU(3) \)-triplet \( \Psi \) is decomposed into

\[
\Psi = \begin{pmatrix}
q \\
Q
\end{pmatrix},
\]

where \( q \) and \( Q \) are a doublet and a singlet under the unbroken \( SU(2) \), respectively.
3.1.1 4D Effective action

There appear four real scalar fields $H^\hat{a}$ ($\hat{a} = 4, 5, 6, 7$) in low energies. They form an $SU(2)$-doublet and play a role of the Higgs doublet in the standard model. They do not have a potential at the classical level due to the 5D gauge invariance. So VEV of $H^\hat{a}$ is determined by the quantum effects, which is not discussed in this paper. Once $H^\hat{a}$ have a nonvanishing VEV, $SU(2) \times U(1)$ is broken to the $U(1)_{\text{EM}}$ subgroup. Making use of the $SU(2) \times U(1)$ symmetry of the effective action, such nonvanishing VEV can always be aligned to the $T^6$-direction. Then, after the breaking of $SU(2) \times U(1)$, the Higgs field $H^\hat{6}$ is expanded as

$$ H^\hat{6} = f_H \bar{\theta}_H + \tilde{H}, $$

(3.5)

where the first and the second terms denote VEV and the fluctuation around it, respectively. In this notation, $\bar{\theta}_H$ becomes the VEV of the Wilson line phase, and

$$ \langle \Omega_0 \rangle = \exp \left\{ i \frac{\bar{\theta}_H x^6}{2} \right\} = \begin{pmatrix} 1 & i \bar{s}_H \\ i \bar{c}_H & \bar{c}_H \end{pmatrix}, $$

(3.6)

where $\bar{c}_H \equiv \cos \frac{1}{2} \bar{\theta}_H$ and $\bar{s}_H \equiv \sin \frac{1}{2} \bar{\theta}_H$. The other scalars $H^\hat{\alpha}$ ($\hat{\alpha} = 4, 5, 7$) are eaten by the gauge bosons for $SU(2) \times U(1)/U(1)_{\text{EM}}$ and thus are unphysical. In fact, we can move to the unitary gauge in which $H^\hat{\alpha} = 0$ ($\hat{\alpha} = 4, 5, 7$) after the breaking of $SU(2) \times U(1)$. Thus we focus on $H^\hat{6}$ among the four real scalars and see an explicit form of the effective action.

The matrix $\Omega_0$ is calculated as

$$ \Omega_0 = \begin{pmatrix} 1 & i \bar{s}_H \\ i \bar{c}_H & \bar{c}_H \end{pmatrix} + \cdots, $$

(3.7)

where $c_H \equiv \cos \frac{1}{2} \theta_H(x)$, $s_H \equiv \sin \frac{1}{2} \theta_H(x)$, and

$$ \theta_H(x) \equiv \frac{H^\hat{6}}{f_H} = \bar{\theta}_H + \frac{\tilde{H}(x)}{f_H}. $$

(3.8)
The ellipsis in (3.7) denotes terms involving $H^a$ ($a = 4, 5, 7$). Then, from (2.57), the “dressed gauge fields” and the covariant derivatives of the Higgs fields are read off as

\[
\begin{align*}
\tilde{A}_\mu^1 + i\tilde{A}_\mu^2 &= c_\mu (A_\mu^1 + iA_\mu^2), \\
\tilde{A}_\mu^3 &= \frac{1 + c_\mu^2}{2} A_\mu^3 + \frac{\sqrt{3}}{2} s_H A_\mu^8, \\
\tilde{A}_\mu^8 &= \frac{\sqrt{3}}{2} s_H A_\mu^3 + \frac{3c_\mu^2 - 1}{2} A_\mu^8, \\
\tilde{D}^{(4)}_\mu (H^3 + iH^5) &= ig_A f_{HS} (A_\mu^1 + iA_\mu^2), \\
\tilde{D}^{(4)}_\mu (H^6 + iH^7) &= \partial_\mu \tilde{H} - ig_A f_{HS} c_H (A_\mu^3 - \sqrt{3} A_\mu^8).
\end{align*}
\] (3.9)

Substituting these into (2.27) and (2.45), we obtain the 4D effective action.

\[
\begin{align*}
S^\text{gauge}_4 &\sim \int d^4x \left\{-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} \partial_\mu \tilde{H} \partial_\mu \tilde{H} - \frac{g_A^2 f_{H}^2 s_H^2}{2} (A_\mu^1 A_\mu^1 + A_\mu^2 A_\mu^2) \\
&\quad \quad - \frac{g_A f_{HS} c_H^2}{2} (A_\mu^3 - \sqrt{3} A_\mu^8) (A_\mu^9 - \sqrt{3} A_\mu^8) + \cdots \right\}, \\
S^\text{fermion}_4 &\sim \int d^4x \left\{ \bar{\psi}_L \gamma_\mu \tilde{D}^{(4)}_\mu \psi_L + \bar{\chi}_R \gamma_\mu \tilde{D}^{(4)}_\mu \chi_R \\
&\quad \quad - i \left( \frac{4M^2 - k^2}{e^{(2M+k)\pi R}} \right)^{1/2} (\bar{\psi}_L \Omega_0^{Qq})^\dagger \chi_R - \bar{\chi}_R \Omega_0^{Qq} \psi_L + \cdots \right\}. \tag{3.10}
\end{align*}
\]

The field strengths are

\[
\begin{align*}
F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_A \epsilon_{abc} A_\mu^b A_\nu^c, \quad (a = 1, 2, 3) \\
F_{\mu\nu}^8 &= \partial_\mu A_\nu^8 - \partial_\nu A_\mu^8, \tag{3.11}
\end{align*}
\]

where $\epsilon_{abc}$ is the completely antisymmetric tensor of $SU(3)$, and the covariant derivatives and $\Omega_0^{Qq}$ are

\[
\begin{align*}
D^{(4)}_\mu \psi_L &= \left\{ \partial_\mu - i g_A \sum_{a=1}^3 A_\mu^a \sigma_a/2 - i \frac{g_A}{2\sqrt{3}} A_\mu^8 \right\} \psi_L, \\
\tilde{D}^{(4)}_\mu \chi_R &= \left( \partial_\mu + \frac{i g_A}{\sqrt{3}} A_\mu^8 \right) \chi_R \\
&= \left\{ \partial_\mu + i g_A \sqrt{3} \left( \frac{\sqrt{3}}{2} s_H^2 A_\mu^3 + \frac{3c_H^2 - 1}{2} A_\mu^8 \right) \right\} \chi_R, \tag{3.12}
\end{align*}
\]

Here we have assumed that $c > 1/2$. The fields $\psi_L$ and $\chi_R$ are a doublet and a singlet chiral spinors, whose components are denoted as

\[
\begin{align*}
\psi_L &= \begin{pmatrix} \psi_L^+ \\ \psi_L^e \end{pmatrix}, \quad \chi_R &= \chi_R^e. \tag{3.13}
\end{align*}
\]
3.1.2 Electroweak breaking phase

After the Higgs field gets nonvanishing VEV, the gauge fields \((A^1_\mu, A^2_\mu, A^3_\mu, A^8_\mu)\) are redefined to the mass eigenstates as

\[
W_\mu \equiv \frac{1}{\sqrt{2}} (A^1_\mu + i A^2_\mu), \\
\left( Z_\mu \right) \equiv \begin{pmatrix} \cos \theta_W & -\sin \theta_W \\ \sin \theta_W & \cos \theta_W \end{pmatrix} \left( A^3_\mu \right) = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \left( A^8_\mu \right). \tag{3.14}
\]

Here \(A^\gamma_\mu\) corresponds to the unbroken U(1) symmetry, which is identified as the photon. Thus the Weinberg angle \(\theta_W\) in this model is calculated as

\[
\sin \theta_W = \frac{\sqrt{3}}{2}, \tag{3.15}
\]

which is too large compared to the experimental value: \(\sin^2 \theta_W^{\exp} \approx 0.23\). Therefore the \(SU(3)\) model cannot be a realistic model.

The effective action after the breaking of \(SU(2) \times U(1)\) becomes

\[
S_{\text{EWB}}^{\text{gauge}} \simeq \int d^4x \left\{ -\frac{1}{4}\bar{W}^{\mu\nu} W^{\mu\nu} - \frac{1}{4} F^{\mu} Z^{\mu\nu} - \frac{1}{4} \bar{F}^{\mu} F^{\mu\nu} - \frac{1}{2} \partial_\mu \bar{H} \partial^\mu \bar{H} \\
i \left( \bar{g} \cos \theta_W \bar{F}^{\mu} Z^{\mu\nu} + \bar{e} \bar{F}^{\mu}_g \right) W^{\mu\nu} + \frac{\bar{g}^2}{2} \left\{ |W^{\mu\nu}|^2 - (W^{\mu\nu})^2 \right\} \\
- m^2_W W^{\mu\nu} - \frac{m^2_Z}{2} Z^{\mu\nu} + \cdots \right\},
\]

\[
S_{\text{EWB}}^{\text{fermi}} \simeq \int d^4 \left\{ \bar{\psi}_L \gamma^\mu D^{(4)}_\mu \psi_L + \bar{\psi}_R \gamma^\mu D^{(4)}_\mu \psi_R + \bar{\chi}_R \gamma^\mu \bar{D}^{(4)}_\mu \chi_R \\
- m_e \left( \bar{\psi}_L \psi_L + \bar{\chi}_R \psi_R \right) + \cdots \right\}, \tag{3.16}
\]

where the ellipses denote interaction terms with the Higgs field \(\bar{H}\). The other Higgs scalars \(H^{\hat{a}}\) (\(\hat{a} = 4, 5, 7\)) are set to zero in the unitary gauge. The field strengths and the mass parameters are defined as

\[
W_{\mu\nu} \equiv D^{(4)}_{\mu} W_{\nu} - D^{(4)}_{\nu} W_{\mu}, \\
D^{(4)}_{\mu} W_{\nu} \equiv \{ \partial_\mu + i g_A A^3_\mu \} W_{\nu} = \{ \partial_\mu + i g_A \cos \theta_W Z_\mu + i e A^8_\mu \} W_{\nu}, \\
F^{\mu}_{Z} \equiv \partial_\mu Z_\nu - \partial_\nu Z_\mu, \quad F^{\gamma}_{\mu\nu} \equiv \partial_\mu A^8_\nu - \partial_\nu A^8_\mu, \\
m^2_W \equiv \bar{g}^2 f_H^2 \sin^2 \bar{\theta}_H, \\
m^2_Z \equiv 4 g_A^2 f_H^2 e_H^2 c_H^2 = \bar{g}^2 f_H^2 \sin^2 \bar{\theta}_H, \\
m_e \equiv \left( \frac{4 M^2 - k^2}{e(2 M + k) \pi R} \right)^{1/2} \sin \frac{\bar{\theta}_H}{2}. \tag{3.17}
\]
Here $\bar{e} \equiv g_A \sin \theta_W$ is the $U(1)_{EM}$ gauge coupling.

The expressions of the mass parameters in (3.17) agree with those derived by the KK analysis. The corrections to them are estimated as $O(\pi^2 m_f^2 / m_{KK}^2)$ for $m_f^2$ ($f = W, Z, e$). (See Eq.(5.5) in Ref. [16].) For the $W$ boson, for example, this ratio becomes

$$\frac{\pi^2 m_W^2}{m_{KK}^2} \approx \frac{1}{k \pi R} \sin^2 \frac{\bar{\theta}_H}{2},$$

(3.18)

which means a few percents error for $e^{k \pi R} = O(10^{15})$ and $\bar{\theta}_H = O(1)$.

Note that the masses are not proportional to the “Higgs VEV” $\bar{\theta}_H$. This nonlinearity comes from the nonlinear structure of the Higgs couplings in (3.10). Furthermore the $\bar{\theta}_H$-dependence of $m_W$ and $m_Z$ are different. So the $\rho$ parameter depends on $\bar{\theta}_H$ and deviates from one for general values of $\bar{\theta}_H$.

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = \frac{1}{\cos^2 \frac{1}{2} \bar{\theta}_H}. \quad (3.19)$$

This is another problem of the $SU(3)$ model.

The interaction terms will be discussed in more realistic model considered in the next subsection.

### 3.2 $SO(5) \times U(1)_{B-L}$ model

Here we consider the $SO(5) \times U(1)_{B-L}$ model that is analyzed in our previous papers [17, 18]. We have the $SO(5)$ gauge field $A_M$ and the $U(1)_{B-L}$ gauge field $B_M$. The former is decomposed as

$$A_M = A_M^I T^I, \quad (3.20)$$

where $T^I$ ($I = 1, 2, \ldots, 10$) are the generators of $SO(5)$. The spinorial representation of $T^I$ is tabulated in (A.6) in Appendix A. As a matter field we introduce a fermion field $\Psi$ in the spinorial representation of $SO(5)$ (i.e., 4 of $SO(5)$). We choose $P_j$ and $\eta_j$ ($j = 0, \pi$) in (2.7) as

$$P_0 = P_\pi = \begin{pmatrix} 1_2 \\ -1_2 \end{pmatrix},$$

$$\eta_0 = \eta_\pi = -1. \quad (3.21)$$

Then $G = SO(5) \times U(1)_{B-L}$ is broken to $H = SO(4) \times U(1)_{B-L} \sim SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. The unbroken generators are the $SU(2)_L \times SU(2)_R$ generators ($T^a_L, T^a_R$).

---

7This type of model is first studied in Ref. [13].
(a_L, a_R = 1, 2, 3) and the U(1)_{B−L} generator \( Q_{B−L} \), while the broken ones are \( T^\hat{a} \) (\( \hat{a} = 1, 2, 3, 4 \)). The fermion \( \Psi \) is decomposed as

\[
\Psi = \begin{pmatrix} q \\ Q \end{pmatrix},
\]

(3.22)

where \( q \) and \( Q \) belong to \( (2, 1) \) and \( (1, 2) \) of \( SU(2)_L \times SU(2)_R \), respectively. The covariant derivative of \( \Psi \) is defined as

\[
D_M \Psi \equiv \left( \partial_M - \frac{1}{4} \omega_M^{AB} \Gamma_{AB} - ig_A A_M^I T^I - ig_B B_M Q_{B−L} \right) \Psi,
\]

(3.23)

where \( I = (a_L, a_R, \hat{a}) \).

In contrast to the previous \( SU(3) \) model, the unbroken gauge symmetry \( SU(2)_L \times SU(2)_R \times U(1)_{B−L} \) is too large to be identified as the electroweak symmetry. In Ref. [17], we add an additional dynamics on the UV brane in order to construct a realistic model. It breaks \( SU(2)_R \times U(1)_{B−L} \) to a subgroup \( U(1)_Y \) spontaneously at a relatively high energy scale \( M_R \). Then the following mass terms are induced on the UV brane below the scale \( M_R \).

\[
\mathcal{L}_{\text{mass}} = - \left\{ M_1^2 \left( A_{1\mu}^1 A_{1\mu}^1 + A_{2\mu}^2 A_{2\mu}^2 \right) + M_2^2 A_{3\mu}^3 A_{3\mu}^3 \right\} \delta(y),
\]

(3.24)

where \( M_1, M_2 = \mathcal{O}(M_R) \), and

\[
\begin{pmatrix} A_{1\mu}^{3R} \\ A_{\mu}^Y \end{pmatrix} \equiv \begin{pmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{pmatrix} \begin{pmatrix} A_{1\mu}^{3R} \\ B_\mu \end{pmatrix},
\]

\[
c_\phi \equiv \frac{g_A}{\sqrt{g_A^2 + g_B^2}}, \quad s_\phi \equiv \frac{g_B}{\sqrt{g_A^2 + g_B^2}}
\]

(3.25)

The constants \( g_A \) and \( g_B \) are the gauge couplings for \( SO(5) \) and \( U(1)_{B−L} \), respectively. The mass terms (3.24) effectively change the boundary conditions for \( (A_{1\mu}^{1R}, A_{2\mu}^{2R}, A_{3\mu}^{3R}) \) at the UV brane from the Neumann to the Dirichlet conditions. This does not cause an essential change of the derivation of the effective theory in Sec. 2. Recall that the solutions (2.21) and (2.33) are determined only by the boundary conditions at the IR brane. The only effect of \( \mathcal{L}_{\text{mass}} \) in (3.24) is to force the field values of \( (A_{1\mu}^{1R}, A_{2\mu}^{2R}, A_{3\mu}^{3R}) \) at the UV brane down to zero. Namely,

\[
\tilde{g}_A \left( A_{\mu}^{aL} T^{aL} + A_{\mu}^{aR} T^{aR} \right) + \frac{\tilde{g}_B}{2} Q_{B−L} B_\mu = \tilde{g}_A \left( A_{\mu}^{aL} T^{aL} + s_\phi A_{\mu}^Y T^{3R} \right) + \frac{\tilde{g}_B}{2} Q_{B−L} (c_\phi A_{\mu}^Y) = \tilde{g} A_{\mu}^{aL} T^{aL} + \tilde{g}' Q_Y A_{\mu}^Y,
\]

(3.26)

where \( \tilde{g} \equiv \tilde{g}_A, \tilde{g}' \equiv s_\phi \tilde{g}_A = c_\phi \tilde{g}_A \) are the 4D gauge couplings for \( SU(2)_L \) and \( U(1)_Y \) respectively, and \( Q_Y \equiv T^{3R} + Q_{B−L}/2 \) is the charge of \( U(1)_Y \). Thus we can obtain the 4D effective action of this model by setting the 4D gauge fields \( (A_{\mu}^{1R}, A_{\mu}^{2R}, A_{\mu}^{3R}) \) to zero at the last step of the procedure.
3.2.1 4D Effective action

There appear four real scalars $H^\hat{a}$ ($\hat{a} = 1, 2, 3, 4$) in low energies. They form an $SU(2)_L$-doublet and play a role of the Higgs doublet in the standard model. Once $H^\hat{a}$ have non-vanishing VEV, $SU(2)_L \times U(1)_Y$ is broken to the electromagnetic gauge group $U(1)_{EM}$. Making use of the $SU(2)_L \times U(1)_Y$ symmetry of the effective action, such VEV can always be aligned along the $T^4$-direction. Then the Higgs field $H^\hat{4}$ is expanded as

$$H^\hat{4} = \sqrt{2} f_H \bar{\theta}_H + \tilde{H}, \quad (3.27)$$

where the first and the second terms denote VEV and the fluctuation field, respectively.

The other scalars $H^\hat{a}$ ($\hat{a} = 1, 2, 3$) are eaten by the gauge bosons and thus unphysical. In fact, we can move to the unitary gauge in which $H^\hat{a} = 0$ ($\hat{a} = 1, 2, 3$) after the breaking of $SU(2)_L \times U(1)_Y$. Thus we focus on $H^\hat{4}$ among the four real scalars and see an explicit form of the effective action. The matrix $\Omega_0$ is calculated as

$$\Omega_0 = \begin{pmatrix} c_H & i s_H \\ i s_H & c_H \end{pmatrix} \otimes 1_2 + \cdots, \quad (3.28)$$

where $c_H \equiv \cos \frac{1}{2} \theta_H(x)$, $s_H \equiv \sin \frac{1}{2} \theta_H(x)$, and

$$\theta_H(x) \equiv \frac{H^\hat{4}(x)}{\sqrt{2} f_H} = \bar{\theta}_H + \frac{\tilde{H}(x)}{\sqrt{2} f_H}. \quad (3.29)$$

The ellipses in (3.28) and in the following expressions denote terms involving $H^\hat{a}$ ($\hat{a} = 1, 2, 3$). From the gauged Maurer Cartan 1 form $\alpha_\mu$ defined in (2.57), we can read off the “dressed gauge fields” and the covariant derivatives of the Higgs fields as

$$\tilde{A}_{\mu}^{\hat{a}L} = c_H^2 A_{\mu}^{aL} + s_H^2 A_{\mu}^{aR} + \cdots,$$

$$\tilde{A}_{\mu}^{\hat{a}R} = s_H^2 A_{\mu}^{aL} + c_H^2 A_{\mu}^{aR} + \cdots,$$

$$\tilde{B}_\mu = B_\mu,$$

$$\tilde{D}_{\mu}^{(4)} H^\hat{a} = -\sqrt{2} g_A f_H s_H c_H (A_{\mu}^{aL} - A_{\mu}^{aR}) + \cdots, \quad (a_L = a_R = \hat{a} = 1, 2, 3)$$

$$\tilde{D}_{\mu}^{(4)} H^\hat{4} = \sqrt{2} f_H \partial_\mu \bar{\theta}_H = \partial_\mu \tilde{H} + \cdots. \quad (3.30)$$
Substituting these into (2.27) and (2.45), the following expressions are obtained.

\[ S^\text{gauge}_4 \simeq \int d^4x \left\{ -\frac{1}{4} \left( F_{\mu\nu}^a F^{a\mu\nu} + F_{\mu\nu}^r F^{r\mu\nu} + F_{\mu\nu}^B F^{B\mu\nu} \right) \right. \\
- \frac{1}{2} \partial_\mu \tilde{H} \partial^\mu \tilde{H} - \tilde{g}_A^2 \tilde{H} \tilde{c}_H^2 \left( A_{\mu}^{aL} - \tilde{A}_{\mu}^{aR} \right) \left( A_{\mu}^{aL\mu} - \tilde{A}_{\mu}^{aR\mu} \right) \left\} + \cdots, \right.

\[ S^\text{fermi}_4 \simeq \int d^4x i \left\{ \bar{\psi}_L \gamma^\mu \left( \partial_\mu - i \tilde{g} A_{\mu}^{aL} \frac{\sigma^a}{2} - \frac{\tilde{g}_B}{2} B_\mu Q_{B-L} \right) \psi_L \right.
\\+ \bar{\chi}_R \gamma^\mu \left( \partial_\mu - i \tilde{g} A_{\mu}^{aL} \left( c_H^2 A_{\mu}^{aL} + s_H^2 A_{\mu}^{aR} \right) \frac{\sigma^a}{2} - \frac{\tilde{g}_B}{2} B_\mu Q_{B-L} \right) \chi_R
\\- \left( \frac{4M^2 - k^2}{e(2M+k)\pi R} \right)^{1/2} s_H \left( \bar{\psi}_L \chi_R + \bar{\chi}_R \psi_L \right) \left\} + \cdots, \right. \tag{3.31}

where \( F_{\mu\nu}^a \) and \( F_{\mu\nu}^B \) are the field strengths of \( A_{\mu}^{aL,R} \) and \( B_\mu \), respectively. We have assumed that \( c > 1/2 \). By setting \( (A_{\mu}^{1L}, A_{\mu}^{2R}, A_{\mu}^{3R}) \) to zero in the above expressions, we obtain the 4D effective action.

\[ S^\text{gauge}_4 \simeq \int d^4x \left\{ -\frac{1}{4} \left( F_{\mu\nu}^a F^{a\mu\nu} + F_{\mu\nu}^Y F^{Y\mu\nu} \right) - \frac{1}{2} \partial_\mu \tilde{H} \partial^\mu \tilde{H} \right.
\\- \frac{\tilde{g}_A^2 \sin^2 \theta_H}{4} \left\{ A_{\mu}^{1L} A_{\mu}^{1L} + A_{\mu}^{2L} A_{\mu}^{2L} + (A_{\mu}^{3L} - s_\phi A_{\mu}^Y) (A_{\mu}^{3L} - s_\phi A_{\mu}^Y) \right\}, \\
\left. + \cdots, \right. \tag{3.32}

\[ S^\text{fermi}_4 \simeq \int d^4x i \left\{ \bar{\psi}_L \gamma^\mu D_\mu^{(4)} \psi_L + \bar{\chi}_R \gamma^\mu D_\mu^{(4)} \chi_R - \left( \frac{4M^2 - k^2}{e(2M+k)\pi R} \right)^{1/2} s_H \left( \bar{\psi}_L \chi_R + \bar{\chi}_R \psi_L \right) \right\}, \\
\left. + \cdots, \right. \tag{3.32}

where \( F_{\mu\nu}^Y \) is the field strength of \( A_{\mu}^Y \), and

\[ D_\mu^{(4)} \psi_L = \left\{ \partial_\mu - i \tilde{g} A_{\mu}^{aL} \frac{\sigma^a}{2} - i \tilde{g}' A_{\mu}^Y \frac{Q_{B-L}}{2} \right\} \psi_L, \]
\[ D_\mu^{(4)} \chi_R = \left\{ \partial_\mu - i \tilde{g} s_H^2 A_{\mu}^{aL} \frac{\sigma^a}{2} - i \tilde{g}' A_{\mu}^Y \left( c_H^2 \frac{\sigma^3}{2} + \frac{Q_{B-L}}{2} \right) \right\} \chi_R. \tag{3.33} \]

Note that \( Q_Y = Q_{B-L}/2 \) on \( \psi_L \) since \( T^{3R} = 0 \).

### 3.2.2 Electroweak breaking phase

After the Higgs field gets nonvanishing VEV, the mass eigenstates become

\[ W_\mu \equiv \frac{1}{\sqrt{2}} \left( A_{\mu}^{1L} + i A_{\mu}^{2L} \right), \]
\[ \begin{pmatrix} Z_\mu \\ A_{\mu}^Y \end{pmatrix} \equiv \begin{pmatrix} \cos \theta_W & -\sin \theta_W \\ \sin \theta_W & \cos \theta_W \end{pmatrix} \begin{pmatrix} A_{\mu}^{3L} \\ A_{\mu}^Y \end{pmatrix} = \frac{1}{\sqrt{1 + s_\phi^2}} \begin{pmatrix} 1 & -s_\phi \\ s_\phi & 1 \end{pmatrix} \begin{pmatrix} A_{\mu}^{3L} \\ A_{\mu}^Y \end{pmatrix}. \tag{3.34} \]
Thus the Weinberg angle $\theta_W$ is read off as

$$\tan \theta_W = s_\phi. \quad (3.35)$$

Then the effective action in the electroweak breaking phase becomes

$$S_{\text{gauge}}^{\text{EWB}} \simeq \int d^4x \left\{ -\frac{1}{2} W_{\mu\nu} W^{\mu\nu} - \frac{1}{4} F_{\mu\nu}^Z F^{Z\mu\nu} - \frac{1}{4} F_{\mu\nu}^{\gamma} F^{\gamma\mu\nu} - \frac{1}{2} \partial_\mu \tilde{H} \partial^\mu \tilde{H} 
+ i \left( \bar{g} \cos \theta_W F_{\mu\nu}^{Z} + \bar{e} F_{\mu\nu}^{\gamma} \right) W^{\mu\nu} + \frac{\bar{g}^2}{2} \left\{ |W_{\mu\nu}|^2 - (W_{\mu}^{\mu} W_{\nu})^2 \right\} 
- \frac{\bar{g}^2 f_\phi^2}{2} \sin^2 \theta_H W_{\mu}^{\mu} W_{\nu} - \frac{\bar{g}^2 f_\phi^2}{4 \cos^2 \theta_W} \sin^2 \theta_H Z_\mu Z^\mu \right\},$$

$$S_{\text{fermi}}^{\text{EWB}} \simeq \int d^4x \left\{ \bar{\psi}_L \gamma^\mu D_\mu^{(4)} \psi_L + \bar{\chi}_R \gamma^\mu \tilde{D}_\mu^{(4)} \chi_R 
- \left( \frac{4 M^2 - k^2}{e^{(2M+k)\pi R}} \right)^{1/2} \frac{\theta_H}{2} \left( \bar{\psi}_L \chi_R + \bar{\chi}_R \psi_L \right) \right\}, \quad (3.36)$$

where $\bar{g} \equiv g \sin \theta_W$ is the $U(1)_{\text{EM}}$ gauge coupling. We have taken the unitary gauge in which $\dot{H}^a = 0$ ($\dot{a} = 1, 2, 3$). The definitions of the field strengths are the same as in (3.17) but now

$$D_\mu^{(4)} W_\nu \equiv \left\{ \partial_\mu + i\bar{g} A_\mu^a \right\} W_\nu = \left\{ \partial_\mu + i\bar{g} \cos \theta_W Z_\mu + i\bar{e} A_\mu^a \right\} W_\nu. \quad (3.37)$$

From the couplings to the Higgs field, the mass parameters are read off as

$$m_W^2 \equiv \frac{\bar{g}^2 f_\phi^2 \sin^2 \theta_H}{2},$$

$$m_Z^2 \equiv \frac{\bar{g}^2 f_\phi^2 \sin^2 \theta_H}{4 \cos^2 \theta_W} = \frac{m_W^2}{\cos^2 \theta_W},$$

$$m_e \equiv \left( \frac{4 M^2 - k^2}{e^{(2M+k)\pi R}} \right)^{1/2} \frac{\theta_H}{2}. \quad (3.38)$$

These expressions agree with those derived by the KK analysis [17]. In contrast to the $SU(3)$ model, the $W$ and the $Z$ boson masses have the same $\theta_H$-dependence so that the $\rho$ parameter is now independent of $\theta_H$ and equals one, which is consistent with the experiments. This can be understood as a result of the custodial symmetry. Note that the Higgs fields $H^a$ ($\dot{a} = 1, 2, 3, 4$) form a doublet not only for $SU(2)_L$ but also for $SU(2)_R$. Thus the Higgs sector of $S_4^{\text{gauge}}$ in (3.32) has a global symmetry $SU(2)_L \times SU(2)_R$ if we set $s_\phi = 0$ (or $g_B = 0$). After $H^4$ gets VEV, this global symmetry is broken to its diagonal subgroup $SU(2)_D$. This custodial symmetry ensures $\rho = 1$. 

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3.2.3 Interaction terms

Now we discuss the interaction terms. Firstly we can immediately see from (3.36) that self-interactions of the gauge fields, such as the $WWZ$, $WWWW$ and $WWZZ$ couplings, are the same as those of the standard model. This is consistent with the results obtained by the KK analysis [17, 18].

Next we investigate the couplings among the gauge and the Higgs fields, i.e., the third line of $S_{\text{EWB}}$ in (3.36). By expanding the corresponding terms in terms of the fluctuation $\tilde{H}$ around VEV, we obtain

$$L^\text{int}_4 = -\frac{\bar{g}^2 f_H^2}{2} \sin^2 \left( \bar{\theta}_H + \frac{\tilde{H}}{\sqrt{2} f_H} \right) \mathcal{W}^\dagger_\mu \mathcal{W}^\mu + \ldots$$

$$= -\frac{\bar{g}^2 f_H^2}{2} \left\{ \sin^2 \bar{\theta}_H + 2 \sin \bar{\theta}_H \cos \bar{\theta}_H \cdot \frac{\tilde{H}}{\sqrt{2} f_H} + \cos 2 \bar{\theta}_H \cdot \frac{\tilde{H}^2}{2 f_H^2} + \mathcal{O}(\tilde{H}^3) \right\} \mathcal{W}^\dagger_\mu \mathcal{W}^\mu + \ldots$$

$$= -m_W^2 \mathcal{W}^\dagger_\mu \mathcal{W}^\mu - \lambda_{WWW} \mathcal{W}^\dagger_\mu \mathcal{W}^\mu \tilde{H} - \frac{\lambda_{WWW}^2}{4} \mathcal{W}^\dagger_\mu \mathcal{W}^\mu \tilde{H}^2 + \ldots. \quad (3.39)$$

where

$$\lambda_{WWW} = \bar{g} m_W \cos \bar{\theta}_H,$$

$$\lambda_{WWW}^2 = \bar{g}^2 \cos 2 \bar{\theta}_H. \quad (3.40)$$

The $WWW$ coupling $\lambda_{WWW}$ is consistent with the result obtained in the KK analysis [17, 18]. It is suppressed by a factor $\cos \bar{\theta}_H$ compared to the counterpart in the standard model. This suppression factor can be easily understood from a nonlinear structure in the Higgs sector of the effective action. Eq.(3.40) shows that the $WWW$ coupling is suppressed by a factor $\cos 2 \bar{\theta}_H$ compared to the standard model. This seems to contradict the result obtained in Ref. [18] where the suppression factor is estimated as $\left(1 - \frac{2}{3} \sin^2 \bar{\theta}_H\right)$. However we have to notice that the couplings calculated in Ref. [18] are the bare couplings $\lambda_{WWW}^2$ bare. In energies below the compactification scale $m_{\text{KK}}$, the massive KK modes are integrated out and induce additional contributions to some couplings among the light modes. In fact a diagram depicted in Fig. 1 also contributes to the $WWW$ couplings in the low-energy effective theory. As shown in the appendix B, this additional contribution to $\lambda_{WWW}^2$ bare is estimated as

$$\delta \lambda_{WWW}^2 \simeq -\frac{4}{3} \bar{g}^2 \sin^2 \bar{\theta}_H. \quad (3.41)$$
Thus the effective \( \lambda_{WWHH}^{\text{eff}} \) coupling becomes

\[
\lambda_{WWHH}^{\text{eff}} = \lambda_{WWHH}^{\text{bare}} + \delta \lambda_{WWHH}^{\text{eff}} \\
\simeq \bar{g}^2 (1 - 2 \sin^2 \theta_H) = \bar{g}^2 \cos 2\theta_H,
\]

which is consistent with (3.40). A situation is similar for the \( ZZH \) and the \( ZZHH \) couplings.

Finally we discuss the fermion sector. The effective action (3.36) reproduces the results of Ref. [17, 18] also for the gauge couplings of the fermions. From (3.33), the covariant derivative of \( \psi_L \) is written in the electroweak breaking phase as

\[
D^{(4)}_\mu \psi_L = \left\{ \partial_\mu - i \bar{g} - \sqrt{2} \left( W^\dagger_\mu \right) \right\} \\
W_\mu \\
- \frac{i \bar{g}}{\cos \theta_W} \left( \frac{\sigma_3}{2} - \sin^2 \theta_W Q_{\text{EM}} \right) Z_\mu - i e A_\mu^\gamma Q_{\text{EM}} \right\} \left( \begin{array}{c} \psi_L^e \\ \psi_L^c \end{array} \right),
\]

where \( Q_{\text{EM}} = T^3_L + Q_Y = T^3_L + T^3_R + Q_{B-L}/2 \) is the \( U(1)_{\text{EM}} \) charge operator. This agrees with that of the standard model, and is consistent with the experiments. On the other hand, the gauge couplings for \( \chi_R \) deviate from the standard model. The covariant derivative of \( \chi_R \) becomes

\[
\tilde{D}^{(4)}_\mu \chi_R = \left\{ \partial_\mu - i \bar{g} - \sqrt{2} s_W \right\} \\
W_\mu \\
- \frac{i \bar{g}}{\cos \theta_W} \left( \frac{s_W \sigma_3}{2} - \sin^2 \theta_W Q_{\text{EM}} \right) Z_\mu - i e A_\mu^\gamma Q_{\text{EM}} \right\} \left( \begin{array}{c} \chi_R^L \\ \chi_R^c \end{array} \right).
\]

The couplings to the \( W \) and the \( Z \) bosons substantially deviate from the standard model values for \( \tilde{\theta}_H = O(1) \). Especially the right-handed fermion \( \chi_R \) couples to the \( W \) boson. The situation is similar also in the case of \( c < -1/2 \). The gauge couplings for the left-handed
fermion $\chi_L$ deviate from the standard model values in that case. From the viewpoint of our effective action, these deviations stem from the fact that the gauge fields that couple to the modes localized near the IR brane are dressed by the Higgs field. Therefore the gauge couplings of the fermions inevitably deviate from the standard model values unless they are localized near the UV brane. This problem can be solved by introducing some additional chiral fermions on the orbifold boundaries and mixing them with the bulk fermion $\Psi$. This possibility will be investigated thoroughly in Ref. [19].

The suppression of the Yukawa couplings, which is discussed in the $SU(3)$ model in Ref. [16], can also be explained by the nonlinear structure of the Higgs sector in the effective action (3.10).

4 Summary

We have derived the 4D effective actions of the 5D gauge-Higgs unification models in the warped spacetime. The derivation is simple and straightforward. To extract light modes that appear in the effective theory from the 5D fields, we imposed the “massless conditions” (2.20) and (2.32). Although these modes can obtain nonzero masses after the Higgs fields have nontrivial VEV, our prescription is a good approximation because the their masses $m_f$ are much lighter than the KK excitation scale $m_{KK}$ in the warped spacetime. In the determination of the mode functions of the light modes in the KK analysis, the effects of $m_f$ are suppressed by $m_f^2/m_{KK}^2$ and subdominant. Imposing the “massless conditions” corresponds to just neglecting such subdominant effects. The nonzero masses $m_f$ are safely reproduced with a good approximation through the couplings to the Higgs fields. The “massless conditions” greatly simplifies the derivation of the effective theory because they determine the mode functions of the light modes without using the detailed information on the mass spectrum. Notice that only the boundary conditions at the IR brane are necessary to determine the mode functions. The role of those at the UV brane is to set the UV-boundary values of some 5D fields to zero and select 4D fields that are dynamical in the effective theory.

The nonlinear structure of the Higgs couplings can be explicitly seen by deriving the 4D action in the gauge where $A_y = 0$. The resultant 4D action is obtained by calculating the gauged Maurer Cartan 1-form $\alpha_\mu$. All the coupling constants can be read off from the formulae (2.27), (2.45) (or (2.55)). The self-interactions among the gauge fields are the
same as those in the standard model. The Higgs couplings to the $W$ and the $Z$ bosons are suppressed by factors that depend on the Wilson line phase $\bar{\theta}_H$ from the standard model. We can directly read off those suppression factors which we have obtained by somewhat complicated calculations in Ref. [16, 17, 18]. The problematic deviations of the fermionic gauge couplings are also manifest in our effective action. These are the advantage of our approach.

The gauge group $G$ is broken by the boundary conditions at both boundaries. From the viewpoint of the effective theory, the symmetries broken at the IR brane are realized nonlinearly while those at the UV brane are explicitly broken. In fact the effective action has the same structure as those of 4D models in which the Higgs fields are provided as the pseudo Nambu-Goldstone (PNG) bosons. These properties are consistent with the holographic interpretation [21, 24, 25, 26].

The Higgs field $\tilde{H}$ in this paper and the scalar zero-mode $H^{(0)}$ in Ref. [17] represent the same degree of freedom. However they have different forms of the interactions. The former has interaction terms at any order while the latter does not. In fact the latter has only up to quartic couplings since it comes from the fluctuation mode of the fifth component of the gauge potential $A_y$. Furthermore they have different couplings to the $W$ and the $Z$ bosons as we have seen in Sec. 3.2.3. The latter has nonvanishing $W^{(0)}W^{(n)}H^{(0)}$ couplings ($n \neq 0$) that induce the correction to the $WWHH$ coupling. This means that we cannot simply drop the KK excitation modes to obtain the effective action. On the other hand, $\tilde{H}$ has the correct value of the $WWHH$ coupling in the effective action. Therefore $\tilde{H}$ is regarded as a field obtained by the field redefinition of $H^{(0)}$:

$$\tilde{H} = H^{(0)} + \mathcal{O}(H^{(0)2}),$$

(4.1)

so that the $W^{(0)}W^{(n)}\tilde{H}$ coupling vanishes.

Finally we comment on the limits of validity for our approach. First we should emphasize that our method is valid only in the warped spacetime. In the flat limit ($k \pi R \to 0$), the KK mass scale $m_{KK}$ becomes closer to the electroweak scale $m_W$ unless $\bar{\theta}_H \ll 1$. (See Fig. 1 of Ref. [16] or Fig. 1 of Ref. [18].) Therefore we cannot neglect the KK modes like we did in this paper. They provide non-negligible effects to the low energy physics when they are integrated out. It is known that $\bar{\theta}_H$ must be tiny to avoid too light Higgs boson in the flat spacetime. When $\bar{\theta}_H \ll 1$, our method is applicable even in the flat case, but the nonlinear structure of the Higgs sector almost disappears and the effective theory
is reduced to the ordinary standard model. When the warp factor is large enough (i.e., 
\( k\pi R \gg 1 \)), the effects of the KK modes are negligible and our method is safely applied. 
In fact the corrections of the mass formulae in (3.17) and (3.38) are of order or less than 
\( O(1/k\pi R) \).

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A Notations

The metric convention of the 4D Minkowski space is taken as

\[
\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1),
\]  

(A.1)

and the Clifford algebra is given by

\[
\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}. 
\]  

(A.2)

An explicit representation of the \( \gamma \)-matrices is given by

\[
\Gamma^0 = \gamma^0 = \begin{pmatrix} i1_2 \\ i1_2 \end{pmatrix}, \quad \Gamma^j = \gamma^j = \begin{pmatrix} i\sigma_j \\ -i\sigma_j \end{pmatrix}, \quad \Gamma^4 = \gamma_5 = \begin{pmatrix} 1_2 \\ -1_2 \end{pmatrix}, 
\]  

(A.3)

where \( j = 1, 2, 3 \) and \( \sigma_j \) are the Pauli matrices.

The generators of \( G \) are normalized as

\[
\text{tr}(T^IT^J) = \frac{1}{2}\delta^{IJ}, 
\]  

(A.4)

and the structure constants are defined as

\[
[T^I, T^J] = iC_{IJK}T^K. 
\]  

(A.5)

The spinorial representation of the \( SO(5) \) generators is given by

\[
T^{aL} \equiv \frac{1}{2} \begin{pmatrix} \sigma_{aL} \\ 0_2 \end{pmatrix}, \quad T^{aR} \equiv \frac{1}{2} \begin{pmatrix} 0_2 \\ \sigma_{aR} \end{pmatrix}, \quad T^{\hat{a}} \equiv \frac{i}{2\sqrt{2}} \begin{pmatrix} \sigma_{\hat{a}} \\ -\sigma_{\hat{a}}^\dagger \end{pmatrix},
\]  

(A.6)

where \( \sigma_{\hat{a}} \equiv (\vec{\sigma}, -i1_2) \). Here \( T^{aL,aR} (a_L, a_R = 1, 2, 3) \) and \( T^{\hat{a}} (\hat{a} = 1, 2, 3, 4) \) are the generators 
of \( SO(4) \sim SU(2)_L \times SU(2)_R \) and \( SO(5)/SO(4) \), respectively.
B Correction to the $WWHH$ coupling

Here we calculate the contribution of Fig. 1 to the quartic coupling $\lambda_{WWHH}^{2 \text{bare}}$ by the KK analysis. It is estimated as

$$\delta \lambda_{WWHH}^2 \simeq -4 \sum_{n=1}^{\infty} \frac{\lambda_{WWnH}^2}{m_W^{(n)^2}}, \quad (B.1)$$

where $m_W^{(n)}$ is a mass of the $n$-th KK excitation mode of the $W$ boson, and $\lambda_{WWnH}$ is a trilinear coupling appearing in the 4D Lagrangian as

$$\mathcal{L}_{4}^{\text{int}} = \sum_{n} \lambda_{WWnH} \left(W_{\mu}^{(0)\dagger}W^{(n)\mu} + W_{\mu}^{(n)\dagger}W^{(0)\mu}\right) H^{(0)} + \cdots. \quad (B.2)$$

Here $W_{\mu}^{(n)}$ and $H^{(0)}$ denote the $n$-th KK mode of the $W$ boson and the fluctuation zero-mode around the Higgs VEV, respectively. The numerical factor in (B.1) is a statistical factor of the Feynmann diagram in Fig. 1 and the factor $1/m_W^{(n)^2}$ comes from the propagator of $W_{\mu}^{(n)}$. Following the procedure of Ref. [17], the coupling $\lambda_{WWnH}$ is expressed as the following overlap integral of the mode functions.

$$\lambda_{WWnH} = \frac{g_A k}{2} \int_{1}^{z_{\pi}} \frac{dz}{z} \tilde{h}_{\varphi,0}^{\pm} \left\{ \tilde{h}_{A,0}^{\pm L} \partial_z \left( \tilde{h}_{A,n}^{\pm L} - \tilde{h}_{A,n}^{\pm R} \right) - \partial_z \tilde{h}_{A,n}^{\pm L} \left( \tilde{h}_{A,0}^{\pm L} - \tilde{h}_{A,0}^{\pm R} \right) \right. + \tilde{h}_{A,n}^{\pm R} \partial_z \left( \tilde{h}_{A,0}^{\pm R} - \tilde{h}_{A,0}^{\pm L} \right) - \partial_z \tilde{h}_{A,n}^{\pm R} \left( \tilde{h}_{A,0}^{\pm L} - \tilde{h}_{A,0}^{\pm R} \right) \right\}, \quad (B.3)$$

where $z_{\pi} \equiv e^{k\pi R}$ is the warp factor,

$$\tilde{h}_{\varphi,0}^{\pm}(z) = \sqrt{\frac{2}{k(z_{\pi}^2 - 1)}} z \quad (B.4)$$

is the mode function of the Higgs field $H^{(0)}$, and

$$\tilde{h}_{A,n}^{\pm L}(z) = C_{A,n}^{\pm L} z F_{1,0}(\lambda_n z, \lambda_n z_{\pi}),$$

$$\tilde{h}_{A,n}^{\pm R}(z) = C_{A,n}^{\pm R} z F_{1,0}(\lambda_n z, \lambda_n z_{\pi}),$$

$$\tilde{h}_{A,n}^{\pm}(z) = C_{A,n}^{\pm} z F_{1,1}(\lambda_n z, \lambda_n z_{\pi}) \quad (B.5)$$

are those of $W_{\mu}^{(n)}$. Here the functions $F_{\alpha,\beta}(u, v)$ are defined in terms of the Bessel functions as

$$F_{\alpha,\beta}(u, v) \equiv J_{\alpha}(u) Y_{\beta}(v) - Y_{\alpha}(u) J_{\beta}(v), \quad (B.6)$$

and $\lambda_n \equiv m_W^{(n)}/k$ is the mass eigenvalues determined by

$$F_{1,0} \left\{ \pi^2 \lambda_n^2 z_{\pi}^2 F_{0,0} F_{1,1} - 2 \sin^2 \tilde{\theta}_H \right\} = 0. \quad (B.7)$$
Here and henceforth, $F_{\alpha,\beta}$ without the argument denotes $F_{\alpha,\beta}(\lambda_n, \pi z)$. The coefficients ($C_{A,n}^{\pm L}, C_{A,n}^{\pm R}, C_{A,n}^{\pm}$) are given as follows.

**Case 1:** The eigenvalue $\lambda_n$ is determined by $F_{1,0} = 0$.

$$C_{A,n}^{\pm L} = (1 - \cos \bar{\theta}_H) \hat{C}_{n}^{(1)},$$
$$C_{A,n}^{\pm R} = -(1 + \cos \bar{\theta}_H) \hat{C}_{n}^{(1)},$$
$$C_{A,n}^{\pm} = 0,$$  \hspace{1cm} (B.8)

where

$$\hat{C}_{n}^{(1)} \equiv \frac{\sqrt{k}}{\sqrt{1 + \cos^2 \bar{\theta}_H}} \left\{ \frac{4}{\pi^2 \lambda_n^2} - F_{0,0}^2 \right\}^{-1/2}.$$  \hspace{1cm} (B.9)

**Case 2:** The eigenvalue $\lambda_n$ is determined by $\pi z F_{0,0} F_{1,1} = 2 \sin^2 \bar{\theta}_H$.

$$C_{A,n}^{\pm L} = (1 + \cos \bar{\theta}_H) \hat{C}_{n}^{(2)},$$
$$C_{A,n}^{\pm R} = (1 - \cos \bar{\theta}_H) \hat{C}_{n}^{(2)},$$
$$C_{A,n}^{\pm} = -\sqrt{2} \sin \bar{\theta}_H \frac{F_{1,0}}{F_{1,1}} \hat{C}_{n}^{(2)},$$  \hspace{1cm} (B.10)

where

$$\hat{C}_{n}^{(2)} \equiv \frac{\sqrt{k}}{\sqrt{1 + \cos^2 \bar{\theta}_H}} \left\{ \frac{4}{\pi^2 \lambda_n^2} + \frac{\pi^2 \lambda_n^2 z^2 \pi^2 F_{1,0}^2 F_{0,0}^2}{\sin^2 \bar{\theta}_H (1 + \cos^2 \bar{\theta}_H)} - \frac{2F_{1,0}^2}{1 + \cos^2 \bar{\theta}_H} - \frac{2F_{0,0}^2}{\sin^2 \bar{\theta}_H} \right\}^{-1/2}.$$  \hspace{1cm} (B.11)

The detailed derivation of the above expressions are provided in Ref. [18].

When the warp factor $z_\pi$ is large enough, the mode function of the lowest mode (i.e., the $W$ boson mode) is approximated as

$$\tilde{h}_{A,0}^{\pm L}(z) \simeq \frac{1 + \cos \bar{\theta}_H}{2\sqrt{\pi R}}, \quad \tilde{h}_{A,0}^{\pm R}(z) \simeq \frac{1 - \cos \bar{\theta}_H}{2\sqrt{\pi R}}, \quad \tilde{h}_{A,0}^{\pm}(z) \simeq -\frac{\sin \bar{\theta}_H}{\sqrt{2\pi R}} \left( 1 - \frac{z^2}{z_\pi^2} \right).$$  \hspace{1cm} (B.12)

Then the overlap integral (B.3) is simplified as

$$\lambda_{WW,nH}^{(1)} \simeq -\frac{4g_A \sqrt{k} \hat{C}_{n}^{(1)} \sin \bar{\theta}_H}{\lambda_n z^2 \sqrt{\pi R}} \left( \frac{4}{\pi \lambda_n^2} + F_{0,0} \right),$$
$$\lambda_{WW,nH}^{(2)} \simeq -\frac{4g_A \sqrt{k} \hat{C}_{n}^{(2)} \sin \bar{\theta}_H \cos \bar{\theta}_H}{z^3 \sqrt{\pi R}} \left\{ \left( 1 - \frac{8}{\lambda_n^2} \right) F_{1,0} + \frac{16}{\pi \lambda_n^3} + \frac{4F_{0,0}}{\lambda_n} \right\},$$  \hspace{1cm} (B.13)

where the superscript (1) or (2) denotes that $W_{\mu}^{(n)}$ belongs to Case 1 or Case 2, respectively.

We have used the formulae collected in the appendix C in Ref. [18].
The dominant contribution in (B.1) comes from the modes which satisfy $m_W^{(n)} \ll k$ (or $\lambda_n \ll 1$). Thus we focus on such modes in the following. Then $F_{\alpha,\beta}$ are approximated as
\[
F_{0,0} \simeq -\frac{2 \ln \lambda_n}{\pi} J_0(\lambda_n z_\pi), \quad F_{1,0} \simeq \frac{2}{\pi \lambda_n} J_0(\lambda_n z_\pi), \quad F_{1,1} \simeq \frac{2}{\pi \lambda_n} J_1(\lambda_n z_\pi). \quad (B.14)
\]
Making use of these approximations, we can obtain a simple expression for $\delta \lambda^2_{W\lambda H\lambda H}$.
In Case 1, the mass eigenvalues $\lambda_n$ satisfy $J_0(\lambda_n z_\pi) \simeq 0$ and
\[
\delta \lambda^{(1)2}_{W\lambda H\lambda H} \simeq -4 \sum_n \frac{\lambda^{(1)2}_{W\lambda H}}{m_W^{(n)2}} \lambda_n \simeq -4 \sum_n \frac{16 g_A^2 \sin^2 \theta_H}{(1 + \cos^2 \theta_H) \lambda_n^4 z_\pi^2 R} \left( \frac{4}{\pi \lambda_n^2} + F_{0,0} \right)^2 \left( \frac{4}{\pi^2 \lambda_n^2} - F_{0,0}^2 \right)^{-1}
\]
\[
\simeq -4 \bar{g}^2 \frac{\sin^2 \theta_H}{1 + \cos^2 \theta_H} \sum_n \left( \frac{2}{\lambda_n^2 z_\pi^2} \right)^6, \quad (B.15)
\]
where $\bar{g} \equiv g_A/\sqrt{\pi R}$.
In Case 2, the equation that determines $\lambda_n$ is approximated as
\[
J_0(\lambda_n z_\pi) J_1(\lambda_n z_\pi) \simeq \frac{\sin^2 \theta_H}{2 \lambda_n z_\pi \ln \lambda_n} \ll 1. \quad (B.16)
\]
Thus $\lambda_n$ satisfy
\[
J_0(\lambda_n z_\pi) \ll J_1(\lambda_n z_\pi) \ll O(1), \quad (n = 2m + 1)
\]
\[
J_1(\lambda_n z_\pi) \ll J_0(\lambda_n z_\pi) \ll O(1), \quad (n = 2m + 2) \quad (B.17)
\]
for $m = 0, 1, 2, \ldots$. In this case,
\[
\delta \lambda^{(2)2}_{W\lambda H\lambda H} \simeq -4 \sum_n \frac{\lambda^{(2)2}_{W\lambda H}}{m_W^{(n)2}} \lambda_n \simeq -4 \sum_n \frac{g_A^2 \sin^2 \theta_H \cos^2 \theta_H}{(1 + \cos^2 \theta_H) \lambda_n^2 z_\pi^2 R} \left\{ \left( 1 - \frac{8}{\lambda_n^2} \right) F_{1,0} + \frac{16}{\pi \lambda_n^2} + \frac{4 F_{0,0}}{\lambda_n} \right\}^2
\]
\[
\cdot \left\{ \frac{4}{\pi^2 \lambda_n^2} + \frac{\pi^2 \lambda_n^2 z_\pi^2 R^2}{\sin^2 \theta_H (1 + \cos^2 \theta_H)} \frac{F_{1,0}^2 F_{0,0}^2}{1 + \cos^2 \theta_H} - \frac{2 F_{1,0}^2}{\sin^2 \theta_H} - \frac{2 F_{0,0}^2}{\sin^2 \theta_H} \right\}^{-1/2}
\]
\[
\simeq -4 \bar{g}^2 \frac{\sin^2 \theta_H \cos^2 \theta_H}{1 + \cos^2 \theta_H} \sum_n \left( \frac{2}{\lambda_n^2 z_\pi^2} \right)^6 \left( 1 - J_0(\lambda_n z_\pi) \right)^2 \left( 1 - J_0(\lambda_n z_\pi) \right)^2
\]
\[
\cdot \left\{ 1 - \frac{2 J_0^2(\lambda_n z_\pi)}{1 + \cos^2 \theta_H} + \frac{J_0^2(\lambda_n z_\pi) \sin^2 \theta_H}{(1 + \cos^2 \theta_H) J_0^2(\lambda_n z_\pi)} \right\}^{-1}
\]
\[
\simeq -4 \bar{g}^2 \frac{\sin^2 \theta_H \cos^2 \theta_H}{1 + \cos^2 \theta_H} \sum_n \left( \frac{2}{\lambda_n^{2m+1} z_\pi^2} \right)^6. \quad (B.18)
\]
We have used (B.17) at the last step.

Therefore (B.1) is estimated as

\[
\delta \lambda_{WWHH}^2 = \delta \lambda_{WWHH}^{(1)2} + \delta \lambda_{WWHH}^{(2)2}
\]

\[
\simeq -4 \bar{\theta}_H \sin^2 \bar{\theta}_H \sum_n \left( \frac{2}{x_n} \right)^6 - 4 \bar{\theta}_H \cos^2 \bar{\theta}_H \sum_n \left( \frac{2}{x_n} \right)^6
\]

\[
= -4 \bar{\theta}_H \sin^2 \bar{\theta}_H \sum_n \left( \frac{2}{x_n} \right)^6
\]

\[
= -\frac{4}{3} \bar{\theta}_H \sin^2 \bar{\theta}_H,
\]

where \( x_n \) are zeros of \( J_0(x) \). In the last equality, we have used the formula

\[
\sum_n \left( \frac{2}{x_n} \right)^6 = \frac{1}{3}.
\]

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