Dual formulation of spin network evolution

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Abstract

We illustrate the relationship between spin networks and their dual representation by labelled triangulations of space in 2+1 and 3+1 dimensions. We apply this to the recent proposal for causal evolution of spin networks. The result is labelled spatial triangulations evolving with transition amplitudes given by labelled spacetime simplices. The formalism is very similar to simplicial gravity, however, the triangulations represent combinatorics and not an approximation to the spatial manifold. The distinction between future and past nodes which can be ordered in causal sets also exists here. Spacelike and timelike slices can be defined and the foliation is allowed to vary. We clarify the choice of the two rules in the causal spin network evolution, and the assumption of trivalent spin networks for 2+1 spacetime dimensions and four-valent for 3+1. As a direct application, the problem of the exponential growth of the causal model is remedied. The result is a clear and more rigid graphical understanding of evolution of combinatorial spin networks, on which further work can be based.

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1 Introduction

A model for the causal evolution of spin networks in quantum gravity has recently been proposed by Markopoulou and Smolin \cite{1}. This proposal advocates exploiting the discreteness of space suggested by the appearance of spin networks in canonical gravity (where their diffeomorphism classes have been shown to provide a basis for the kinematical state space \cite{2,3,4,5,6,7,8}) in order to address the issue of time evolution. The model lies between the loop formulation of canonical gravity and the causal set approach \cite{9} since it assigns quantum amplitudes to special classes of causal sets which consist of spin networks representing quantum states of the gravitational field joined together by labelled null edges. The dynamics is specified by a choice of functions of the labellings of \(d+1\) dimensional simplices which represent elementary future light cones of events in these discrete spacetimes.

More specifically, given an initial spin network \(\Gamma_1\), there are two rules for the amplitudes of the transition from \(\Gamma_1\) to the next spin network \(\Gamma_2\). These rules do two things. First, they allow all possible moves from the spin network at one time instant to the later one, which is the creation of new edges (rule 1) or the recoupling of existing ones (rule 2), thus making use of the combinatorial nature of the spin networks. Second, the rules attempt to set up a causal structure in the resulting “spacetime network” – the combination of the spin networks that make up the spatial slices and the null edges (the lightcones) that connect them.

Given the two spin networks \(\Gamma_1\) and \(\Gamma_2\), the amplitude for the transition from \(\Gamma_1\) to \(\Gamma_2\), \(A_{\Gamma_1 \rightarrow \Gamma_2}\), can be constructed by applying these two rules. Let \(\mathcal{G}\) be the collection of all causal spacetime networks consistently built by the alternation of the two rules which has \(\Gamma_1\) as the zeroth spin network and \(\Gamma_f\) as the final one, and \(L_\mathcal{G}\) the number of such spin networks \(\Gamma_I\) in \(\mathcal{G}\). Then the transition amplitude from \(\Gamma_i\) to \(\Gamma_f\) is

\[
A_{\Gamma_i \rightarrow \Gamma_f} = \sum_{\mathcal{G}} L_{\mathcal{G}}^{-1} \prod_{I=0}^{L_{\mathcal{G}}-1} A_{\Gamma_I \rightarrow \Gamma_{I+1}}
\]

(1)

where the sum is over all allowed labellings and the amplitude is defined alternatively in terms of the two rules (see \cite{1} for more details).

\footnote{For the present purposes a causal set is a set of points which has the causal properties that may be assigned to sets of points in Minkowskian spacetime: to each pair either one is to the future of the other, or they are causally unrelated.}
This model was not derived from the classical theory, rather it is an attempt at constructing a transition amplitude between spin network states that is consistent with some discrete microscopic form of causality. Further, it displayed certain features of theories with critical behaviour, indicating that a renormalization group approach may be applicable (as it has been discussed in [10]).

Constructing a model of causal spin network evolution turned out to be a mixture of combinatorial spin networks, causal sets and critical behaviour models. Presumably, progress on all of these features should be made before satisfactory results come out. In this paper, we take up the first aspect, the combinatorial structure of the spin networks as it was originally suggested by Penrose [11] (which may be interpreted as implying a fundamentally discrete space (and spacetime)). Indeed, the present application is part of work in progress where the appearance of spin networks in canonical gravity is regarded as evidence for discrete structure of space(time) and is viewed from the perspective of category theory which we believe is the relevant mathematical formalism. An immediate result is the duality between spin networks and triangulations, which we overview here and then apply it to the causal scheme of spin networks evolution, thus clarifying the evolution rules. We also describe how this viewpoint allows a simple remedy for the exponential growth in the causal model.

2 Spin networks and triangulations

In the causal evolution of spin networks [1] a spin network is regarded as a combinatorial labelled graph with nodes and edges labelled according to the rules satisfied by spin networks [12, 11]. The edges are labelled by representations of SU(2) and the nodes by intertwiners, which are distinct ways of extracting the identity representation from the products of representations of the incident edges. The spin networks are defined only by their combinatorics and no embedding in a spatial manifold is assumed.

A more rigid and reliable formalism, very useful when one needs to rely heavily on the combinatorics of spin networks, can be obtained if they are treated as the 1-skeletons of $n$-simplices triangulating the $n$-dimensional spatial manifold. This correspondence may be found, for example, in topological quantum field theory literature [13], and in observations on the relationship between spin networks and simplicial gravity [14]. A basic difference is that we treat triangulations exactly like their dual spin networks, namely
Figure 1: The 2-simplex triangulating 2-dimensional space and its dual trivalent spin network. The edges of the triangle are labelled by spins and its face by an intertwiner (omitted in an SU(2) spin network).

as combinatorial constructions. This is in contrast to their role in topological quantum field theory and simplicial gravity where the triangulation is an approximation to the spatial manifold. Further, this correspondence has not been explored in spin network canonical gravity since the emphasis has been on spin networks embedded in the three-dimensional space (spin network states are the basis for cylindrical functions, the elements of the Hilbert space of kinematical states) rather than regarding the appearance of spin networks as evidence for discrete, and possibly combinatoric, structure.

The relevant spatial triangulations and spacetime simplices are different in 2+1 and 3+1 dimensions. We describe the two sets of spatial triangulations in the next two subsections and in section 3 we describe their spacetime evolution.

2.1 2-dimensional space

In 2+1 spacetime dimensions, the simplex triangulating the 2-dimensional space is a triangle. One triangle corresponds to one node of the spin network, and its sides are labelled by the same colors as this dual spin network (figure 1). This implies that the spin networks employed in the description of 2-dimensional space are restricted to be trivalent. In general, the faces of the triangles are labelled by intertwiners. In the case of SU(2) spin networks they are trivial and hence the label does not appear.

This provides a triangulation of 2-dimensional space that corresponds to its kinematical description by a trivalent spin network (figure 2). Note that a closed spin network corresponds to closed space (figure 3).

A n-valent node in 2-dimensional space would be dual to a n-gon. This can be divided into triangles by the same method that an n-valent node can be split into trivalent ones by introducing virtual edges.
Figure 2: A 2-dimensional spatial manifold can be described by a (labelled) triangulation (left), or its dual trivalent spin network (right).

Figure 3: An example of a triangulated closed surface. It gives rise to a closed spin network.
Figure 4: Part of a four-valent spin network, five nodes, corresponds to five tetrahedra of a triangulation of 3-dimensional space.

Figure 5: The labelling of a tetrahedron used in a triangulation of 3-dimensional space. Its faces are labelled by spins and the tetrahedron itself by an intertwiner.

### 2.2 3-dimensional space

The 3-dimensional space is triangulated by tetrahedra. Again the dual spin network is the 1-skeleton of the tetrahedra, so that there is one node for each tetrahedron and one spin network edge puncturing each of its faces (figure 4). This means that we now allow only *four-valent* spin networks. The tetrahedra are labelled by intertwiners and their faces by spins (figure 5).

It is most intriguing that one can now propose a straightforward correspondence between spins puncturing faces of tetrahedra and the area of those faces, or the intertwiners labelling the tetrahedra and their volume, like the standard spin network results on area and volume [15]. We will investigate this and the relationship to simplicial gravity and Regge calculus it suggests in future work.

Having now discussed the relevant triangulation of space, in the next section we describe the corresponding picture of causal time evolution.
3 Spacetime evolution and causality

We shall discuss this mainly in 2+1 dimensions, as the 3+1 case involves 4-dimensional spacetime simplices which are rather hard to visualize. At the level of the present discussion, the observations in 3+1 are completely analogous to those of 2+1, and thus none of the important insights are lost with this simplification.

3.1 2+1 evolution

In 2+1 spacetime dimensions, evolution will be a transition from a triangulation of the 2-dimensional space at a given time instant to a later one. The way to do this is by placing spacetime tetrahedra, whose faces are the triangles described in 2.1, on top of the initial triangulation. (These are spacetime tetrahedra and unrelated to the spatial ones dual to four-valent spin networks in 3+1 dimensions).

The tetrahedra can then be regarded as a triangulation of spacetime. An initial slice is the spin network drawn on the faces of the initial spatial triangulation. A time step consists of tetrahedra being placed on this triangulation, their bottom faces being the triangles of the initial triangulation. On their top faces, the new triangles, we can again draw the dual spin network. This is the new spatial slice (figure 6). (Whether there is a change of all the nodes by such a placing of a tetrahedron in each time step is discussed in section 4.)

Note that we define a time step to last the entire “height” of a spacetime simplex. That is, in this model spatial slices do not cut through the simplices. One can also construct discrete theories where this happens, and may be necessary to consider such cases. However, this is not something we shall investigate here (see, for example, the approach of J. Zapata in [16]).

In the spin network model of [1], causality was built in by introducing a distinction between future nodes, produced by intersections of lightcones of past nodes, and those past nodes. Here this translates to the distinction between future and past faces of the spacetime tetrahedron. As it is placed on the spatial triangulation, the bottom faces correspond to its past nodes and the top to the future nodes. Then the labels on future faces and edges depend only on the past faces and edges of the same tetrahedron. Thus, there is a natural partial ordering of the faces and we can again arrange them into causal sets.

The model is local in the sense that if at time $t_1$ the four triangles of
Figure 6: Evolution of a spin network in 2+1 spacetime.

Figure 7: Local evolution.

Figure 7 are causally unconnected, i.e. an observer in the middle triangle is not communicating with the three adjacent ones, in the next step \( t_2 \) all four triangles in the region bounded by spins \( a, ..., f \), but no more, will be in touch with each other.

Further, having seen what a spatial slice looks like and what is the causal ordering of the nodes, we can now give a definition of a spacelike slice: a spacelike slice should not combine future and past nodes of the same spacetime simplex, i.e. a spacelike slice should not wrap around a spacetime simplex. In contrast, timelike surfaces will be those that do contain both future and past nodes of the same spacetime simplex. It is important to note that this allows a varying “foliation” of spacetime, subject to the slices that follow the given initial surface being spacelike.

Let us now write down conditions on the transition amplitudes from an initial to a final spin network that may be inferred from this formulation. The spacetime tetrahedron is the amplitude for the possible transitions between past and future nodes and can be evaluated using an appropriate type of 6j-symbol. It has 4 faces, hence 4 dual nodes, and the various ways it can be placed on the past triangulation correspond to either 2 past nodes.
going to 2 new ones, or 1 past node to 3 new and vice versa (fig. 8). The 2–2 case is a tetrahedron standing on one edge and gives rise to recoupling of edges. There are two 2–2 possibilities, fig. 9 which we take to have the same weight. A tetrahedron with one face down will be the process where one node splits to a triangle of three new ones (1–3), while the inverted version of this will be the opposite process (3–1) (fig. 9).

It is clear that these are all the possibilities that can arise when tetrahedral amplitudes are placed on a trivalent spin network. Note that the result is the same as Rules 1 and 2 of the causal evolution of spin networks. Rule 1 produced three new nodes/edges for each past node, thus it is the 1–3 move. Rule 2 allowed for all possible recouplings of the new spin network, hence it is the 2–2 case. We can now see that the choice of those rules is justified, as they are simply the full set of possibilities for evolution of a combinatoric trivalent spin network. We can also see that the 3–1 case was omitted in [1]. In section 4 we show that this omission is responsible for the problematic exponential growth of the spin network in that case.

The important issue is to find which amplitudes should be assigned to the spacetime tetrahedra to give us the new labels as functions of the old ones, satisfying our requirements of local and causal evolution. This remains unresolved. However, the construction itself provides some guidance. First, we are using combinatorial spin networks, hence we are looking for $6j$-type
amplitudes, i.e. combinatorial expressions for the new labels. Second, the spacetime simplices are directed because we distinguish between their future and past faces. Consequently, the amplitudes we are looking for will have less symmetry than the Regge-Ponzano $6j$-symbols.

Let us then name the 1–3 amplitude $J(mnp;ijk)$, $m,n,p$ being the new labels and $i,j,k$ the old ones. We are then allowed to permute each of the two sets $m,n,p$ and $i,j,k$ but not to mix them. Similarly we can call the 3–1 amplitude $J'(mnp;ijk)$. In the 2–2 case (see fig. 9), we have recoupling that sends an old edge $n$ into a new one $m$, keeping the outer labels $i,j,k,l$ fixed. Presumably, in this case we can only permute $i,j,k,l$, hence let us name this amplitude $P(m;ijkl;n)$ and the other 2–2 case $P'(m;ijkl;n)$.

A site in a given spatial triangulation is where one spacetime tetrahedron is placed and it may contain one, two, or three triangles. If it only has one, then the amplitude $J'$ applies, if two then it’s either $P$ or $P'$, and if it has three triangles we need $J$. Selecting the right amplitude is the same as selecting the site, hence we do not need to count both. As a result, the amplitude to go from a given spin network $\Gamma_0$ to the next one $\Gamma_1$ is given by

$$A_{\Gamma_0 \rightarrow \Gamma_1} = \prod_{\text{sites in the triangulation dual to } \Gamma_0} A(\text{site})$$

where $A(\text{site})$ can be $J$, $J'$, $P$, or $P'$, according to the type of site.

If, between an initial and a final spin network $\Gamma_i$ and $\Gamma_f$, or equivalently the boundaries of a given spacetime triangulation, a foliation of spacelike slices contains a finite number $N$ of spin networks, then the amplitude from
Figure 10: The 4-simplex, the spacetime transition amplitude for 3-dimensional spatial triangulations.

\[ A_{\Gamma_i \rightarrow \Gamma_f \mid \text{fixed triangulation}} = \prod_{I=0}^{I=N-1} A_{\Gamma_{I} \rightarrow \Gamma_{I+1}}. \] (3)

And if we further allow the spacetime triangulation to vary then the overall amplitude is given by

\[ A_{\Gamma_i \rightarrow \Gamma_f} = \sum_{\text{possible spacetime triangulations}} \prod_{I=0}^{I=N-1} A_{\Gamma_{I} \rightarrow \Gamma_{I+1}}. \] (4)

Implementing causality now involves finding the correct choices of these amplitudes, which at this stage is an unsolved problem.

### 3.2 3+1 evolution

In 3+1 dimensions one repeats the above construction for the three-dimensional spatial manifold triangulated by tetrahedra (four-valent spin networks). The spacetime is now triangulated by the 4-simplex of figure 10. It is made up of 10 faces labelled by spins and 5 tetrahedra labelled by intertwiners. Placing 4-simplices on top of the spatial tetrahedra again gives us a distinction between past and future nodes. Since it has 5 tetrahedra, and hence 5 dual nodes which can be separated to past and future ones, different placements of the 4-simplex represent the transition amplitudes 1–4 of 1 past node to 4 future, 2–3, two past nodes to 3 future, and vice versa (the issue of the appropriate 15j symbols that respect causality also applies here). These are the Pachner moves, shown in figure 11.

Again, the use of 4-simplices and Pachner moves has the same effect as applying Rules 1 and 2 to four-valent spin networks, as was proposed in [1]. There, no justification for the choice of four-valent spin networks for
3+1 dimensions was made, but this is now simply the spin network dual to a triangulation of the 3-dimensional space. (Again nodes of higher valence than four corresponding to polyhedra can be broken down to fourvalent ones by reducing the polyhedron to tetrahedra.)

Thus it becomes clear how the features introduced in [1] of a discrete spacetime described by a local, causal evolution of combinatorial spin networks relate to a triangulation of space and spacetime. In the next subsection we discuss further how formulating this problem in terms of triangulations helps to clarify certain aspects of causal evolution.

### 3.3 Space and spacetime

The existing proposals for evolving spin networks often come under the names of spacetime or 4-dimensional formalisms. An issue here is whether one should extend spin networks to 4 dimensions and thus have a “4-dimensional spin network” or restrict spin networks to represent space only and obtain the 4-dimensional theory in some other way. A further confusion arises because of the similarities between 3-dimensional space and 2+1 Euclidean gravity.

The 4-dimensional models involving spin networks in the literature—apart from the spacetime network of Markopoulou and Smolin being considered here—are those of Reisenberger and Rovelli [17] and Baez [18] which feature a spacetime construction using surfaces.\(^3\) They can be seen as branching

\(^3\)There is also the simplicial model for euclidean general relativity of Reisenberger
surfaces connecting the initial and final spin networks, or, conversely, spin networks can be regarded as slicings of the labelled spacetime surfaces ("spin foams"). Presumably, some of the features of these models are truly essential and possibly common in the different models, while others serve mostly as visual aids or in calculational strategies. The duality of spin networks to triangulations can help a great deal in elucidating the essential features.

We have seen that starting from an initial spin network $\Gamma_i$ dual to a triangulation, the next one is obtained by local changes of the first, whose amplitudes can be represented by spacetime simplices (this is illustrated in more detail in the next section). Now these simplices clearly are amplitudes and not spin networks themselves. No extra non-spatial edges are introduced by the spacetime simplex. That is, all edges one can draw on a spacetime simplex belong to some spatial slice. Therefore, the “internet” or null edges of $[1]$ are not really there. They duplicate labels that were already in $\Gamma_i$. To see this let us consider figure 12. Two nodes in the initial spin network $\Gamma_i$ go to six nodes in the next one, $\Gamma_f$. The null edges used to write the transition amplitude $A_{\Gamma_i \to \Gamma_f}$ are in fact labelled with the same spins as the spatial edges they come from (this is dictated by the causality requirement). Therefore, they duplicate existing labels. In the dual triangulation on the right of fig. 12 all spin labels are spatial, i.e. they belong to the some spatial slice.

The null edges have one more important role, to keep track of causality, namely determining the dependence of new labels on old. In triangulations this is done automatically by identifying the top and bottom faces (tetrahedra in 3+1) of a spacetime simplex to be the future and past nodes. Also, timelike surfaces can still be defined, as explained earlier in section 3.

Keeping in mind that $(n - 1)$-dimensional space is an $n$-valent spin network dual to spatial $(n - 1)$-simplices, and that this spin network evolves with amplitudes given by $n$-simplices (which are not themselves associated with dual spin networks), spacetime built this way may be thought of as made up of all evolution amplitudes in equations (3)–(4).

[19], and in topological quantum field theory the work in 4 dimensions of Crane [13, 20], Ooguri [21] and others. In particular, Reisenberger has already pointed out how the duality between spin networks and triangulations clarifies a relationship between the causal evolution rules and topological quantum field theory [22].
Figure 12: The null edges duplicate labels of the initial spin network. In the
dual triangulation all labels are spacelike.

4 Controlling the growth of the spin network

The evolution model described in [1] is the extreme case where every node
of the spin network is a spacetime event, that is, new edges are created at
every time step and for every node. One may object that in this case the
spin network evolves too fast. However, this can be easily controlled, as
we will discuss in this section. Again, for clarity, we discuss the 2+1 case.
There is no significant difference in 3+1.

There are, in fact, two ways to control growth. First, the problem can be
fixed by simply noticing that the model did not include the 3–1 case which
we encountered in its translation into triangulations, where three past nodes
evolve to only one future node (top right of figure 9). Evolution by placing a
spacetime simplex over each possible site at each time step (as in fig. 6) but
including the 3–1 move means that the number of nodes no longer increases
exponentially.

It may well be that this is all one needs to do. However, there is a second
intriguing option that arises here, and at present there is no reason to prefer
one of the two. This we call the non-maximal case, and its treatment would
be reminiscent of directed percolation. It produces the effect of multifingered
time.

Let us recall the example of the statistical model of directed percolation,
most easily seen in its 1+1 form [23]. The causal links (null edges between
nodes) are represented (in figure 13) as future pointing arrows. Each arrow
can be on or off and there can be a probability amplitude \( p \) associated with
the various possibilities. One can understand this model as information
flowing from the bottom to the top, with \( P \) being the probability that infor-
mation from the bottom succeeds in flowing through to the top. The critical
behaviour of this model may be seen by graphing \( P \) against the probability
Figure 13: (a) Percolation lattice with all arrows on. (b) A history in the percolation model. (c) A possible assignment of probabilities to the various cases: $2p_1 + p_2 + p_3 = 1$, $\sum q_i = 1$. (d) Probability to reach infinity against the probabilities $p$ assigned to the nodes.

Figure 14: Maximal and non-maximal evolution of the 1+1 spacetime network of \[1\].

$p$ assigned to each node (essentially the number of arrows that are on). In the extreme case where all arrows are on, $P$ is of course 1. As the number of arrows on decreases, $P$ remains close to 1, down to some critical value $p_c$. A short while after $p_c$ is crossed, information flow fails to make it to the top (fig. 13).

In our context of spin networks, a large number of arrows on corresponds to a large number of nodes being spacetime events, which could mean too much information flowing into a given region, or space expanding too fast. The interesting situation instead would be when the system operates close to its $p_c$, where the 1+1 spacetime network of \[1\] might look like that on the right of figure 14. In this case not all nodes are spacetime events at every time step. We call this case non–maximal, while the situation where all the nodes are spacetime events will be referred to as maximal.
In terms of triangulations, the non-maximal case is the situation where, in one time step, spacetime simplices are placed only over some spatial sites, thus only changing parts of the spin network, like multifingered time in the canonical theory. In figure 15 we have placed only two tetrahedra on the given triangulation of the 2-dimensional space. The result again is that the growth of the spin network is controlled. This case also slows down the build-up of spacetime by spacetime simplices, as it takes many more time steps to achieve a result similar to the maximal case.

5 Conclusions

We have illustrated the duality between spin networks and spatial triangulations, and used this relationship to clarify the recent causal spin network evolution model, particularly its assumptions of a connection between valence and dimensionality of space, and its two evolution rules. We discussed the interpretation of locality and causality in the triangulated version. This useful duality helped us to make important extensions to the causal spin network model. In particular, adding the missing 3–1 move, controlling the growth of the spin network, allowing the definition of spacelike slices and varying foliations, and admitting multifingered time. The important issue remains the choice of the amplitude functions that will be consistent with the causality requirement, although the triangulation formalism provides some indication as to which symmetries are the desired ones. We may note
that we have only worked with very basic features of the model, for example, we have not discussed q-deformed spin networks or considered the normalisation of the amplitudes we propose, which need to be addressed next.

Since we believe that spin networks in canonical gravity point towards a discrete spacetime theory, it will be useful to import results and techniques from simplicial gravity. Clearly, setting up the actual correspondence between spin networks and triangulations is the first step. The results of Barett and Foxon in Euclidean and Lorentzian Regge calculus [24], together with null-strut calculus [25], are likely to shed light on the correct amplitudes for a Lorentzian spacetime. We shall discuss this elsewhere.

Finally, we note that the dual picture makes the increased use of the graphical representation of the spin network combinatorics more reliable. For example, in the treatment of spin network evolution as a critical phenomenon by application of renormalisation group techniques, the groupings suggested by triangulations are more natural and justifiable than when using spin networks (fig. 16), although irregularity is still a major problem in higher dimensions.

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