SOME ABSTRACT PROPERTIES OF SEMIGROUPS APPEARING IN SUPERCONFORMAL THEORIES

Steven Duplij ∗†‡

Physics Department, University of Kaiserslautern, Postfach 3049, D-67653 KAIERSLAUTERN, Germany

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Abstract

A new type of semigroups which appears while dealing with $N = 1$ superconformal symmetry in superstring theories is considered. The ideal series having unusual abstract properties is constructed. Various idealisers are introduced and studied. The ideal quasicharacter is defined. Green’s relations are found and their connection with the ideal quasicharacter is established.

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∗Alexander von Humboldt Fellow
†On leave of absence from Theory Division, Nuclear Physics Laboratory, Kharkov State University, Kharkov 310077, Ukraine
‡E-mail: duplij@physik.uni-kl.de
1 Introduction

Mathematical objects with new properties often appear from concrete physical considerations and models. The discovery of supersymmetry \[3, 5, 36\] gave many new mathematical features, but its influence on the general abstract properties of the theory, in spite of the fact that among principal objects there were noninvertible ones and zero divisors \[15\], needs to be emphasized. The latter led to the conclusion that the abstract ground of supersymmetric theory should have semigroup nature \[8\]. It was also realised that the noninvertible transformations and semigroups appearing in that way have many new nontrivial properties \[7, 9\]. In particular, it would be interesting to work out the general abstract structure of the $N = 1$ superconformal semigroup, which is important in the consistent construction of the superstring unified theories \[16, 11\]. In this paper we provide a consideration of the superconformal semigroups from the abstract-algebraic point of view and present their abstract properties without proofs which will appear elsewhere.

2 Preliminaries

The semigroup of $N = 1$ superconformal transformations of $C^{1,1}$ complex superspace with the coordinates $(z, \theta)$ valued in the Grassmann algebra \[4\], where $z \in C^{1,0}$ and $\theta \in C^{0,1}$, is isomorphic to the semigroup $S$ of the even $C^{1,0} \to C^{1,0}$ and odd $C^{1,0} \to C^{0,1}$ functions satisfying some multiplication law (for details see \[7, 9\]). The even part of the law

$$s_3 = s_1 \ast s_2, \quad s_i \in S,$$

in terms of the even functions $g(z)$ can be presented as

$$g_3(z) = [g_1(\tilde{z}) + h_1(\tilde{z})] \cdot g_2(z),$$

where $\tilde{z}$ is some shifting and $h_1(z)$ is some even nilpotent function of second degree, i.e. $h_1^2(z) \equiv h_1(z) \cdot h_1(z) = 0$. We stress that, because of the shifting $z \to \tilde{z}$ and the second term in the brackets \[2\], $S$ differs from the semigroups
of functions with point by point multiplication \[6\] as well, as from the semi-
groups of functions \[23, 24\]. This leads to new unusual abstract properties of
\( S \) considered below. Further we note that to study this properties it is suffi-
cient to know the formal expression (2) only. This parametrisation of \( N = 1 \)
superconformal transformations was given in \[5, 9\] (where one can also find
the exact formulas and the concrete background). For other considerations
we refer to \[1, 26, 27, 14\].

Here we do not consider the physical interpretations of \( g(z) \) (see \[1, 5\])
and stress only that \( g(z) \) controls invertibility of the superconformal trans-
formations \[8\]. Therefore, the index of \( g(z) \) which is defined by

\[
\text{ind} g(z) \overset{\text{def}}{=} \{ n \in \mathbb{Z} \mid g^n(z) = 0, g^{n-1}(z) \neq 0 \} \tag{3}
\]

plays a crucial part in the following. We mention here that in (2) and (3) the
multiplication is a point by point one in the Grassmann algebra \( \mathbb{R} \) (for clarity
sometimes we use a point for it), but the star in (1) denotes the semigroup
multiplication.

So the semigroup \( S \) can be divided into two disjoint parts \( S = G \cup T \),
\( G \cap T = \emptyset \), where

\[
G \overset{\text{def}}{=} \{ s \in S | \text{ind} g(z) = \infty \}, \tag{4}
\]

\[
T \overset{\text{def}}{=} \{ s \in S | \text{ind} g(z) < \infty \}. \tag{5}
\]

Here \( G \) is a group corresponding to the invertible transformations. From the
multiplication law (2) it follows that \( T \) is a two-sided ideal. The unity element
\( e \in S \) has \( g(z) = 1, h(z) = 0 \), and the zero element has \( g(z) = 0, h(z) = 0 \)
(for other details see \[8, 4\]). From (3) and the relation \( \text{ind} h(z) = 2 \) it follows
that \( T \) is a nilsemigroup \[20, 12, 10, 33\], i.e. \( \forall t \in T \ \exists n \in \mathbb{Z}, t^n = z \) (here
the multiplication in the power expression is implied as the semigroup one
(1)). So every element from \( T \) is nilpotent without bound on its index and
of finite order, but every element from \( G \) is of infinite order.

The superconformal transformations corresponding to \( G \) were studied
earlier in \[1, 4, 28\]. Therefore we concentrate our attention on the ideal
\( T \), which gives the evidence of some unusual abstract properties of such
parametrised superconformal semigroup \( S \).
3 Ideal series

To classify the elements from the ideal part $T$ we take $n$-th power of the equation (2) in the Grassmann algebra and, using the relation $\text{ind} h(z) = 2$, obtain

$$g_3^n(z) = \left[g_1^n(\tilde{z}) + n \cdot g_1^{n-1}(\tilde{z}) \cdot h_1(\tilde{z})\right] \cdot g_2^n(z). \quad (6)$$

We see that the natural classification can be done by means of the index of $g(z)$ (see (3)). Let us define the following sets

$$\Delta_n \overset{\text{def}}{=} \{s \in S|\text{ind } g(z) = n\}. \quad (7)$$

$$I_n \overset{\text{def}}{=} \bigcup_{k \leq n} \Delta_k. \quad (8)$$

Then we notice that $T$ is a disjoint union of the sets $\Delta_n$, because $T = \bigcup_n \Delta_n$, $\Delta_n \cap \Delta_{n-1} = \emptyset$. From (8) it follows that $I_{n-1} \subset I_n$ and $I_n \setminus I_{n-1} = \Delta_n$. Therefore we obtain the following infinite chain of the sets $I_n$

$$z \subset I_1 \subset I_2 \subset \ldots \subset I_n \subset \ldots \subset T. \quad (9)$$

To understand the meaning of $I_n$ we use (8) and obtain

$$S * I_n \subseteq I_n, \quad (10)$$

$$I_n * S \subseteq I_{n+1}, \quad (11)$$

$$S * I_n * S \subseteq I_{n+1}. \quad (12)$$

From these relations we can easily observe that the sets $I_n$ are left ideals of the semigroup $S$, but not right ideals, because of (11). Moreover, the appearance of $n+1$ in the right side of (11) and (12) is very unusual, and so these strange sets $I_n$ is natural to call "jumping ideals". Therefore $I_{n-1} \triangleleft l I_n$ and the chain (9) is a left and "jumping" ideal series. Then $I_n$ are quasiideals [32, 4] since they satisfy $S * I_n \cap I_n * S \subseteq I_n$. Simultaneously, the sets $I_n$ are biideals, because $I_n * S * I_n \subseteq I_n$ [3, 19]. It is exciting that in our case the regularity is not necessary for coincidence quasiideals and biideals in superconformal semigroup (as distinct from [19]). Because of the inclusion $I_n \triangleleft U \Rightarrow I_n \triangleleft S$, $\forall U \subset S$ the semigroup $S$ is a filial semigroup [17]. The indices in (9) form a well ordered set for which $n$ is an ordinal. Because of
In the chain (9) can be called a left ascending ideal series of $S$. From (11) and (12) we derive

$$S * I_n \cup I_n * S \subseteq I_{n+1},$$

(13)

This condition is opposite for the chain (11) to be an ascending annihilator series of $S$ [13, 30]. So we call it an ascending antiannihilator series of $S$.

The multiplication law for the sets $I_n$ and $\Delta_n$ is

$$
\begin{align*}
I_n * I_{n+k} & \subseteq I_{n+1}, \\
I_{n+k-1} * I_n & \subseteq I_n, \\
\Delta_n * \Delta_{n+k} & \subseteq I_{n+1}, \\
\Delta_{n+k-1} * \Delta_n & \subseteq I_n, \\
I_n * \Delta_{n+k} & \subseteq I_{n+1}, \\
I_{n+k-1} * \Delta_n & \subseteq I_n, \\
\Delta_n * I_{n+k} & \subseteq I_{n+1}, \\
\Delta_{n+k-1} * I_n & \subseteq I_n, \\
I_n * G & \subseteq I_{n+1}, \\
G * I_n & \subseteq I_n, \\
\Delta_n * G & \subseteq I_{n+1}, \\
G * \Delta_n & \subseteq \Delta_n,
\end{align*}
$$

(14)

where $k > 0$. It follows that the set $I_n$ is a subsemigroup of $S$, because from (14) we have $I_n * I_n \subseteq I_n$ but the set $\Delta_n$ is not a subsemigroup, since $\Delta_n * \Delta_n \subseteq I_n$. This is a consequence of the fact that our semigroup is defined over the Grassmann algebra [4] which contains nilpotents and zero divisors, and the latter fact should be taken into account properly [13].

From the last two relations of (14) and (12) we can obtain

$$G * \Delta_n \subseteq I_{n+1},$$

(15)

i.e. some of the elements from $\Delta_n$ are conjugated by the subgroup $G$ with the elements of the next set $\Delta_{n+1}$. By analogy with [34, 22, 21] we define $G$-normal subsets $A, B \subseteq S$ as follows $g^{-1} * A * g \subseteq B, \ g \in G$. Then from (15) we make a conclusion that any two sets $\Delta_n$ contain $G$-normal elements and one can reach any $\Delta_n$ using the subgroup action only. Further general abstract properties of such elements can be found in [22, 24].
4 Idealisers

The left (right, two-sided) idealiser $I_l(U)$ ($I_r(U)$, $I(U)$) of the subset $U \subseteq S$ can be defined as the largest subsemigroup of $S$ within which $U$ is a left (right, two-sided) ideal, i.e.

\[ I_l(U) \overset{def}{=} \{ s \subseteq S \mid s \ast U \subseteq U \} , \tag{16} \]
\[ I_r(U) \overset{def}{=} \{ s \subseteq S \mid U \ast s \subseteq U \} , \tag{17} \]
\[ I(U) \overset{def}{=} \{ s \subseteq S \mid s \ast U \subseteq U, U \ast s \subseteq U \} . \tag{18} \]

For the set $I(U)$ the set $U$ is really a subsemigroup, because $U \ast s \subseteq U$, $s \ast U \subseteq U$, $t \ast U \subseteq U \Rightarrow U \ast s \ast t \subseteq U \ast t \subseteq U$, $s \ast t \ast U \subseteq s \ast U \subseteq U$. Also, if $V$ is a subsemigroup of $U$ and $V \triangleleft U$, then $\forall v \in V \Rightarrow v \ast U \subseteq U, U \ast v \subseteq U \Rightarrow v \in I(U)$. Thus $V \subseteq I(U)$.

Let us consider the idealisers of the various introduced subsets of $S$. First the left idealiser for $I_n$ is $S$, as is follows directly from (10), i.e.

\[ I_l(I_n) = S. \tag{19} \]

From the last relation in (14) we find

\[ I_l(\Delta_n) = G. \tag{20} \]

For the right idealisers of $I_n$ the situation is more complicated. Using (11) we divide $S$ into two disjoint parts $S = S^I_n \cup S^\Delta_n$, where $S^I_n \cap S^\Delta_n = \emptyset$, and they satisfy the relations

\[ I_n \ast S^I_n \subseteq I_n , \tag{21} \]
\[ I_n \ast S^\Delta_n \subseteq \Delta_{n+1} . \tag{22} \]

By definition (17) $S^I_n$ is the right idealiser for $I_n$, i.e.

\[ I_r(I_n) = S^I_n . \tag{23} \]

Obviously, that $I_n \subset S^I_n$, since $I_n \ast I_n \subset I_n$. Therefore $S^I_n = I_n \cup S^H_n$. From (3) it follows that for the elements from $S^H_n$ the second term in the brackets should disappear, therefore we find

\[ S^H_n = \{ s \in T \setminus I_n \mid g_1^{n-1}(\tilde{z}) \cdot g_2^n(z) = 0, h_1(\tilde{z}) \cdot g_2^n(z) = 0 \} . \tag{24} \]
Then the ”jumping” set $S^\Delta_n$ from (22) is equal to $S^\Delta_n = (S \setminus I_n) \setminus S^{II}_n$.

Another way to vanish the second term in (6) is the consideration of the special superconformal transformations (they are called Ann-transformations in [9]) for which the relation $g^{n-1}(z) \cdot h(z) = 0$ is valid (see (4) and (5)). Let us divide $I_n$ in two disjoint parts $I_n = I^A_n \cup I^{\neq A}_n$, where $I^A_n \overset{def}{=} \{ s \in I_n | g^{n-1}(z) \cdot h(z) = 0 \}$. It was shown in [9] that Ann-property is preserved from the right only, and so we obtain $I^A_n * S \subseteq I^A_n$, which means that $I^A_n$ is a right ideal in $S$, then

$$I_r \left(I^A_n\right) = S. \quad (25)$$

For the sets $\Delta^A_n = I^A_n \setminus I^A_{n-1}$ we find $\Delta^A_n * G \subseteq \Delta^A_n$, therefore

$$I_r \left(\Delta^A_n\right) = G. \quad (26)$$

We note here that by means of the right group action we can reach a set $I_n$ with any large $n$, because the relation $\Delta^{\neq A}_n * G \subseteq \Delta^{\neq A}_{n+1}$ (see also (15)).

5 Ideal quasicharacter

Let us define

$$\chi(s) \overset{def}{=} \{ n \in N | \ind g(z) = n \}. \quad (27)$$

Using (11) and (11) we obtain

$$\max \chi(s * t) = \begin{cases} \chi(t), & \chi(s) \geq \chi(t) \\ \chi(s) + 1, & \chi(s) < \chi(t) \end{cases}. \quad (28)$$

In particular,

$$\chi(g * s) = \chi(s), \quad (29)$$

$$\chi(s * g) = \chi(s) + 1.$$ 

From (28) it follows that $n_\delta = |\chi(s * t) - \chi(s) - \chi(t)|$ is bounded. This value $n_\delta$ shows how much the mapping $s \rightarrow \chi(s)$ differs from a homomorphism [18]. The limitedness of $n_\delta$ allows us to conclude that $\chi(s)$ is a quasicharacter [31] which can be called an ideal quasicharacter. The elements of $S$ having finite ideal quasicharacter are nilpotent and belong to the ideal $T,$
and \( \chi(g) = \infty, g \in G \). Another description of the ideal quasicharacter can be written as follows \( \chi(s) = n \iff s \in \Delta_n \). Since \( \Delta_n \cap \Delta_m = \emptyset, n \neq m \), we conclude that \( \chi(s) \) indeed disjoins the elements of \( S \), and the relation \( \pi \) defined as \( s \pi t \iff \chi(s) = \chi(t) \) is an equivalence relation in \( S \).

6 Green’s relations

In our notations the Green’s \( L \) and \( R \) relations are

\[
s \mathrel{L} t \iff \exists u, v \in S, \ u * s = t, \ v * t = s,
\]

\[
s \mathrel{R} t \iff \exists u, v \in S, \ s * u = t, \ t * v = s.
\]

(30)

Let us find \( L \) and \( R \) equivalent elements in the superconformal semigroup \( S \). Using (10) and (28) we find that \( s \mathrel{L} t \Rightarrow \chi(s) \leq \chi(t) \wedge \chi(t) \leq \chi(s) \Rightarrow \chi(s) = \chi(t) \). Therefore \( L = \pi \), and \( L \)-equivalent elements have the same ideal quasicharacter,

\[
s \mathrel{L} t \Rightarrow \chi(s) = \chi(t),
\]

(31)

and they belong to the same set \( \Delta_n \). By analogy from (11) for the \( R \)-equivalent elements we derive \( s \mathrel{R} t \Rightarrow \chi(s) \leq \chi(t) + 1 \wedge \chi(t) \leq \chi(s) + 1 \). Then the ideal quasicharacters of the \( R \)-equivalent elements can differ only by 1 or coincide, i.e.

\[
s \mathrel{R} t \Rightarrow |\chi(s) - \chi(t)| \leq 1.
\]

(32)

Since \( H = L \cap R \), the sets \( \Delta_n \) consist also of \( H \)-equivalent elements.

Consider the \( L \)-equivalent elements. Let \( s \neq t, s \neq z, t \neq z \). From (30) we derive that \( s = v * (u * s) = (v * u) * s = (v * u)^k * s \) for any \( k \in N \). If \( v \in T \lor u \in T \), then \( (v * u)^k \in T \), since \( T \) is an ideal in \( S \). Because of \( T \) is a nilsemigroup \( \exists n \in N \) such that \( (v * u)^n = z \). Through the arbitrariness of \( k \) we choose \( k = n \) and obtain \( s = (v * u)^n * s = z * s = z \) or \( s = t \), which contradicts the initial assumptions. The same is valid for other Green’s relations. Therefore \( v \in G \land u \in G \), i.e. nontrivial \( L \) and \( R \) equivalences can be constructed with regard to the invertible elements of \( S \) only. Then the principal left and right ideals generated by \( \forall t \in S \) and defined by \( L(t) \overset{\text{def}}{=} S * t \) and \( R(t) \overset{\text{def}}{=} t * S \), as a matter of fact are some analogies of the left and right cosets of \( G \) in \( S \) introduced in [23, 24].
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