BRAIDED HOPF ALGEBRAS AND GAUGE TRANSFORMATIONS

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Abstract. We study infinitesimal gauge transformations of an equivariant noncommutative principal bundle as a braided Lie algebra of derivations. For this, we analyze general $K$-braided Hopf and Lie algebras, for $K$ a (quasi)triangular Hopf algebra of symmetries, and study their representations as braided derivations. We then study Drinfeld twist deformations of braided Hopf algebras and of Lie algebras of infinitesimal gauge transformations. We give examples coming from deformations of abelian and Jordanian type. In particular we explicitly describe the braided Lie algebra of gauge transformations of the instanton bundle and of the orthogonal bundle on the quantum sphere $S^4_\theta$.

Contents

1. Introduction 2
2. Basic definitions and notations 4
   2.1. Hopf algebra modules and comodules 4
   2.2. Relative modules and $K$-equivariance 6
3. Quasitriangular structures and associated braidings 6
   3.1. Quasitriangular Hopf algebras 6
   3.2. Braided Hopf algebras 8
   3.3. Dual structures 12
4. Twisting braided Hopf algebras 13
   4.1. $K$-modules of linear maps 15
   4.2. Quasitriangular bialgebras 16
   4.3. Braided bialgebras 18
   4.4. Dual pairings of braided bialgebras and their twisting 21
5. Braided Lie algebras and their twisting 23
   5.1. Braided Lie algebras 23
   5.2. Braided Lie algebras of braided derivations 25
   5.3. Actions of braided Lie algebras and universal enveloping algebras 29
   5.4. Twisting braided Lie algebras 30
   5.5. Twisting braided derivations 33
6. Gauge group of Hopf–Galois extensions 35
   6.1. Hopf–Galois extensions 36
   6.2. Gauge group 36
1. Introduction

In this paper we study different notions of gauge group and infinitesimal gauge transformations of noncommutative principal bundles (Hopf–Galois extensions). This is a key ingredient for a differential geometry study of the theory of connections and of their moduli spaces. The group of gauge transformations was considered in [11] (see also [12] and [13]). A general feature of these works is that the group there defined is bigger than one would expect. In [6] we studied the gauge group of noncommutative bundles with commutative base space algebra and noncommutative Hopf algebra as structure group. In particular, we proved that noncommutative principal bundles arising via twist deformation of the structure Hopf algebra of commutative principal bundles have gauge group isomorphic to the gauge group of the initial classical bundle.

On one hand in the present paper we improve those results and study the case of principal bundles with noncommutative base space algebra. For any Hopf–Galois extension $B = A^\text{coH} \subseteq A$, with total space algebra $A$, base space algebra $B$ and structure group Hopf algebra $H$, we define the gauge group $\text{Aut}_B(A)$ of right $H$-comodule algebra morphisms that restrict to the identity on the subalgebra $B$. This definition encompasses the above mentioned Drinfeld twist deformations case where the gauge group remains classical. However, for arbitrary Hopf–Galois extensions it usually gives very small groups because in general there are few algebra maps on a noncommutative space. For example on Galois objects (Hopf–Galois extensions with $B$ the ground field) the gauge group is just made of characters.

On the other hand we study a second definition of gauge group for Hopf–Galois extensions with structure Hopf algebra $H$ and equivariant under a triangular Hopf algebra $K$. As expected, we show that in this case the natural notion of Lie algebra of gauge transformations is that internal to the representation category of the triangular Hopf algebra $K$. A Lie algebra object in that category is a $K$-module with a braided antisymmetric bracket compatible with the $K$-action, that is, it is a $K$-braided Lie algebra. Similarly, the gauge group is a $K$-braided Hopf algebra. The framework of braided Hopf algebras as Hopf algebras internal in the representation category of a quasitriangular Hopf algebra has been proposed in [29] [30]. A notion of braided gauge group was also considered in [20], albeit in a different braided context.

The advantage of considering braided Hopf and Lie algebras is clear when noncommutative principal bundles, with now noncommutative base space, are obtained via deformation quantization of classical ones. In this braided approach the (infinite dimensional) vector space underlying infinitesimal gauge transformations is the same as that of the undeformed bundle. This follows from the general categorical framework mentioned before, which is well adapted to Drinfeld twist deformations. It is also illustrated in the key

7. Braided Hopf algebra of gauge transformations
   7.1. Quantum principal bundle over quantum homogeneous space
   8. Braided Hopf algebra gauge symmetry from twist deformation
      8.1. Gauge transformations of twisted principal bundles
Appendix A. Proof of Proposition 4.14
References
examples of the instanton bundle and the orthogonal bundle on the 4-sphere $S^4$. In this latter both the base space and the structure group are noncommutative. These results (for the instanton bundle) suggest a correspondence with those on noncommutative instanton moduli spaces whose dimensions survives the $\theta$-deformation see for example [28] (see also [9, 10]).

Another relevant application of this braided gauge Lie algebra approach is for noncommutative coset spaces –not necessarily related to twist deformations– whose total space $A$ is a triangular Hopf algebra. The braided Lie algebra of infinitesimal gauge transformations is the subalgebra of vertical vector fields of the braided Lie algebra of vector fields. This latter is generated (over the base space algebra) by the right-invariant vector fields defining the bicovariant differential calculus à la Woronowicz on $A$. This relationship between differential calculi and braided infinitesimal gauge transformations supports their relevance for a theory of connections.

Infinitesimal gauge transformations similar to those in the present paper have appeared in the mathematical physics literature. We just mention the recent paper [15] where only trivial principal bundles are considered using formal deformation quantization with $\star$-products. A further independent argument in favour of a theory of noncommutative gauge groups that, like the braided one we develop, does not drastically depart from the classical one comes from the Seiberg-Witten map between commutative and noncommutative gauge theories [35]. This map establishes a one-to-one correspondence between the corresponding gauge transformations (yet of a different kind from the braided gauge transformations we consider) and hence points to noncommutative gauge equivalence classes that are a deformation of the classical ones.

The $K$-braided Lie algebra of infinitesimal gauge transformations we consider is an example of Lie algebra in the representation category of $K$-modules. The first part of the paper provides a self contained introduction to the theory of braided Hopf algebras, these are Hopf algebras in the representation category of a quasitriangular Hopf algebra $K$. In Section 4 we twist with a Drinfeld 2-cocycle $F \in K \otimes K$ the quasitriangular Hopf algebra $K$ to the quasitriangular Hopf algebra $K_F$ and recall the monoidal equivalence between $K$-modules and $K_F$-modules. $K$-braided Hopf algebras are twisted to $K_F$-braided Hopf algebras. In Section 5 we consider $K$ triangular, we discuss braided Lie algebras, that were pioneered by [22] as generalized Lie algebras. We twist these braided Lie algebras and their universal enveloping algebras, which are braided Hopf algebras. This leads to isomorphisms (denoted $D$) of $K_F$-braided Lie algebras that we study in detail in particular in the category of relative $K$-modules $B$-bimodules, that is, when there is a $K$-module algebra $B$ with a $K$-compatible action on the braided Lie algebra. Indeed, infinitesimal gauge transformations of a Hopf–Galois extension $B = A^{\text{co} H} \subseteq A$ form a $K$-module and a compatible $B$-module: they are a $K$-braided Poisson algebra.

In Section 6 the gauge group of a Hopf–Galois extension $B = A^{\text{co} H} \subseteq A$ as $H$-equivariant algebra endomorphisms of $A$ is considered, this does not require braided geometry notions. On the contrary, Section 7 applies the braided geometry of the initial sections to study the braided Lie algebra of infinitesimal gauge transformations of $K$-equivariant Hopf–Galois extensions $B = A^{\text{co} H} \subseteq A$. As discussed at the beginning of this introduction, this notion is natural in the $K$-representation category, and it is supported by the relationship between infinitesimal gauge transformations and differential calculi. The last section applies the Drinfeld twist deformation machinery to provide $K_F$-braided gauge Lie algebras from $K$-braided gauge Lie algebras. Thus gauge transformations of
twisted principal bundles arising from commutative principal bundles are presented. The
theory is first illustrated for a trivial principal bundle. Then the two main examples are
those of the instanton and orthogonal bundles on the $S^4_\theta$-spheres ($\mathcal{O}(S^4_\theta) \subseteq \mathcal{O}(S^4)$) and
$\mathcal{O}(S^4_\theta) \subseteq \mathcal{O}(SO_\theta(5,\mathbb{R}))$ respectively) where the braided Lie algebra of the gauge group
is explicitly presented. Two further examples considering Jordanian twist deformations
(rather than abelian $\theta$-deformations via actions of tori) are also considered. While all
these examples are on the ground field $\mathbb{C}$ a last example considers the general case in the
context of formal deformation quantization.

2. Basic definitions and notations

On this preliminary section we collect basic definitions and fix notation. All linear spaces
are $k$-modules, where $k$ is a commutative field with unit $1_k$, or the ring of formal power
series in a variable $h$ over a field. Much of what follows can be generalised to $k$ a
commutative unital ring. We denote the tensor product over $k$ by $\otimes$. For $k$-modules
$V, W$, we denote by $\tau$ the flip operator $\tau : V \otimes W \rightarrow W \otimes V$, $v \otimes w \mapsto w \otimes v$.

All algebras (coalgebras) are over $k$ and assumed to be unital and associative (counital
and coassociative). Morphisms of algebras (coalgebras) are unital (counital). The product
in an algebra $A$ is denoted $m_A : A \otimes A \rightarrow A$, and the unit map $\eta_A : k \rightarrow A$, with
$1_A := \eta_A(1_k)$ the unit element. The counit and coproduct of a coalgebra $C$ are denoted
$\varepsilon_C : C \rightarrow k$ and $\Delta_C : C \rightarrow C \otimes C$ respectively. We use the standard Sweedler notation
for the coproduct: $\Delta_C(c) = c_{(1)} \otimes c_{(2)}$ (with sum understood) and for its iterations,
$\Delta^n_C = (\text{id} \otimes \Delta_C) \circ \Delta^{n-1}_C : c \mapsto c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(n+1)}$, for all $c \in C$. Moreover, for a
Hopf algebra $H$, we denote $S_H : H \rightarrow H$ its antipode. We denote by $*$ the convolution
product in the dual k-module $C^* := \text{Hom}(C, k)$, $(f * g)(c) := f(c_{(1)}) g(c_{(2)})$, for all $f, g \in C^*$,
$c \in C$. For a coalgebra $C$, we denote $C^{cop}$ the co-opposite coalgebra: $C^{cop}$ is the coalgebra
structure on the k-module $C$ with comultiplication $\Delta_{C^{cop}} := \tau \circ \Delta_C$ and counit $\varepsilon_{C^{cop}} := \varepsilon_C$.
If $C$ is a bialgebra, such is $C^{cop}$ with multiplication and unit the same of $C$.

2.1. Hopf algebra modules and comodules. Given a bialgebra (or Hopf algebra) $H$,
a left $H$-module is a $k$-module $V$ with a left $H$-action: a $k$-module map $\triangleright_V : H \otimes V \rightarrow V$,
$h \otimes v \mapsto h \triangleright_V v$, such that

$$(hk) \triangleright_V v = h \triangleright_V (k \triangleright_V v), \quad 1 \triangleright_V v = v, \quad (2.1)$$

for all $v \in V$, $h, k \in H$. Given two $H$-modules $(V, \triangleright_V)$, $(W, \triangleright_W)$, a map $\psi : V \rightarrow W$ such that
$\psi(h \triangleright_V v) = h \triangleright_W (\psi(v))$, for all $h \in H$, $v \in V$, is called a morphism of
left $H$-modules. This condition will also be referred to as $H$-equivariance. The tensor
product (as $k$-modules) $V \otimes W$ is an $H$-module with action

$$\triangleright_{V \otimes W} : H \otimes V \otimes W \rightarrow V \otimes W, \quad h \otimes v \otimes w \mapsto (h_{(1)} \triangleright_V v) \otimes (h_{(2)} \triangleright_W w). \quad (2.2)$$

We denote by $H \mathcal{M}$ the category of left $H$-modules. It is a monoidal category, whose unit
object is $k$ with (trivial) action given by the counit map $\triangleright_k = \varepsilon_H : H \simeq H \otimes k \rightarrow k$.
For $H$ a Hopf algebra, the linear space $\text{Hom}(V, W)$ of $k$-linear maps $\psi : V \rightarrow W$ is a left
$H$-module with action

$$\triangleright_{\text{Hom}(V, W)} : H \otimes \text{Hom}(V, W) \rightarrow \text{Hom}(V, W)$$

$$h \otimes \psi \mapsto h \triangleright_{\text{Hom}(V, W)} \psi : v \mapsto h_{(1)} \triangleright_W \psi(\theta v) (h_{(2)}) \triangleright_V v. \quad (2.3)$$

This action is trivial on the subspace $\text{Hom}_{H\mathcal{M}}(V, W)$ consisting of $H$-module morphisms:
$h \triangleright_{\text{Hom}(V, W)} \psi = \varepsilon(h) \psi$. 

4
An algebra $A$ which is a left $H$-module, $(A, \triangleright A)$, is called a left $H$-module algebra if its multiplication $m_A : A \otimes A \to A$ and unit $\eta_A : k \to A$ are morphisms of $H$-modules:

$$h \triangleright_A (ab) = (h(1) \triangleright_A a)(h(2) \triangleright_A b), \quad h \triangleright_A 1 = \varepsilon(h)1$$

(2.4)

for all $a, b \in A, h \in H$. We denote by $_{H}A$ the category of left $H$-module algebras, with morphisms in it being $H$-module morphisms which are also algebra maps. The algebra $(\text{Hom}(V, V), \circ)$ of $k$-linear maps from an $H$-module $V$ to itself, with multiplication given by map composition and action as in (2.3), is a left $H$-module algebra. A module $V$ that is a left $H$-module and a left $A$-module, with $A$-action denoted $\cdot_V$, is called $(H, A)$-relative Hopf module if for all $h \in H, a \in A$, and $v \in V$,

$$h \triangleright_V (a \cdot_V v) = (h(1) \triangleright_V a) \cdot_V (h(2) \triangleright_V v).$$

(2.5)

A morphism of $(H, A)$-relative Hopf modules is a morphism of $H$-modules which is also a morphism of left $A$-modules.

Furthermore, a left $H$-module coalgebra is a coalgebra $C$ which is a left $H$-module and its coproduct and counit are morphisms of $H$-modules:

$$(h \triangleright_C c)_{(1)} \otimes (h \triangleright_C c)_{(2)} = (h(1) \triangleright_C c_{(1)}) \otimes (h(2) \triangleright_C c_{(2)}), \quad \varepsilon(h \triangleright_C c) = \varepsilon(h)\varepsilon(c)$$

(2.6)

for all $c \in C, h \in H$. We denote by $_{H}C$ the category of left $H$-module coalgebras; morphisms in $_{H}C$ are $H$-module maps which are also coalgebra maps.

Analogous definitions (and notations) are given for a right $H$-module $(V, \triangleleft_V)$, and corresponding categories of right $H$-module algebras $A_H$ and coalgebras $C_H$. Recall that if $H$ is a Hopf algebra, any left $H$-module $(V, \triangleright_V)$ can be made into a right $H$-module $(V, \triangleleft_V)$ with right action $\triangleleft_V : V \otimes H \to V, v \otimes h \mapsto v \triangleleft_V h := S(h) \triangleright_V v$. Moreover if $(V, \triangleright_V) \in A_H$, then $(V, \triangleleft_V) \in A_{H^{\text{cop}}}$, with $H^{\text{cop}}$ the co-opposite Hopf algebra.

Dually, a right $H$-comodule is a $k$-module $V$ with a $k$-linear map $\delta^V : V \to V \otimes H$ (a right $H$-coaction) such that

$$(\text{id} \otimes \Delta) \circ \delta^V = (\delta^V \otimes \text{id}) \circ \delta^V, \quad (\text{id} \otimes \varepsilon) \circ \delta^V = \text{id}. \quad (2.7)$$

In Sweedler notation we write $\delta^V : V \to V \otimes H$, $v \mapsto v_{(0)} \otimes v_{(1)}$ (with sum understood), and the right $H$-comodule properties (2.7) read

$$v_{(0)} \otimes (v_{(1)})_{(1)} \otimes (v_{(1)})_{(2)} = (v_{(0)})_{(0)} \otimes (v_{(0)})_{(1)} \otimes v_{(1)} =: v_{(0)} \otimes v_{(1)} \otimes v_{(2)}, \quad v_{(0)} \varepsilon(v_{(1)}) = v$$

for all $v \in V$. A morphism between $H$-comodules $V, W$ is a $k$-linear map $\psi : V \to W$ which is an $H$-comodule map, that is $\delta^W \circ \psi = (\psi \otimes \text{id}) \circ \delta^V$, or $\psi(v)_{(0)} \otimes \psi(v)_{(1)} = \psi(v_{(0)}) \otimes v_{(1)}$ for all $v \in V$. Also, the tensor product (as $k$-modules) $V \otimes W$ is an $H$-comodule with the right $H$-coaction

$$\delta^V \otimes W : V \otimes W \to V \otimes W \otimes H, \quad v \otimes w \mapsto v_{(0)} \otimes w_{(0)} \otimes v_{(1)} w_{(1)}. \quad (2.8)$$

We denote by $\mathcal{M}^H$ the category of right $H$-comodules; it is a monoidal category. The unit object is $k$ with coaction $\delta^k$ given by the unit map $\eta_H : k \to k \otimes H \simeq H$.

A right $H$-comodule algebra is an algebra $A$ which is a right $H$-comodule and has multiplication and unit which are morphisms of $H$-comodules. This is equivalent to requiring the coaction $\delta^A : A \to A \otimes H$ to be a morphism of unital algebras (where $A \otimes H$ has the usual tensor product algebra structure): for all $a, a' \in A$,

$$\delta^A(a a') = \delta^A(a) \delta^A(a'), \quad \delta^A(1_A) = 1_A \otimes 1_H.$$

We denote by $\mathcal{A}^H$ the category of right $H$-comodule algebras with morphisms just $H$-comodule maps which are also algebra maps.
Finally, a right $H$-comodule coalgebra is a coalgebra $C$ which is a right $H$-comodule and has coproduct and counit which are morphisms of $H$-comodules, that is,

$$(c(1))_{(0)} \otimes (c(2))_{(0)} \otimes (c(1))_{(1)}(c(2))_{(1)} = (c(0))_{(1)} \otimes (c(0))_{(2)} \otimes c(1), \quad \varepsilon(c(0))c(1) = \varepsilon(c)1_H$$  \hspace{3cm} (2.9)

for each $c \in C$. We denote by $C^H$ the category of right $H$-comodule coalgebras; its morphisms are $H$-comodule maps which are also coalgebra maps. Analogous notions and notation can be introduced for left comodules (algebras and coalgebras).

2.2. Relative modules and $K$-equivariance. Let $H$ be a bialgebra and let $A \in A^H$. An $(H,A)$-relative Hopf module $V$ is a right $H$-comodule with a left $A$-action $\triangleright_V$ which is a morphism of $H$-comodules: for all $a \in A$ and $v \in V$,

$$(a \triangleright_V v)_{(0)} \otimes (a \triangleright_V v)_{(1)} = a_{(0)} \triangleright_V v_{(0)} \otimes a_{(1)}v_{(1)}.$$  \hspace{3cm} (2.10)

A morphism of $(H,A)$-relative Hopf modules is a morphism of right $H$-comodules which is also a morphism of left $A$-modules. We denote by $A\mathcal{M}^H$ the category of $(H,A)$-relative Hopf modules. In a similar way one defines the categories of relative Hopf modules $\mathcal{M}_A^H$ for $A$ acting on the right, and $A\mathcal{A}_A^H$ for right and left $A$ compatible actions.

For two Hopf algebras $K$ and $H$, a $K$-equivariant $H$-comodule $V$ is an $(H,K)$-relative Hopf module with $K \in A^H$ with a trivial coaction $K \rightarrow H \otimes K, k \mapsto k \otimes 1_H$. The left $K$-action $\triangleright_V: K \otimes V \rightarrow V$ and the right $H$-coaction $\delta^V : V \rightarrow V \otimes H$ on $V$ thus satisfy $\delta \circ \triangleright_V = (\triangleright_V \otimes \text{id})$ o $\triangleright_V$ that is,

$$(k \triangleright_V v)_{(0)} \otimes (k \triangleright_V v)_{(1)} = k \triangleright_V v_{(0)} \otimes v_{(1)},$$  \hspace{3cm} (2.11)

for all $k \in K, v \in V$.

We denote by $K\mathcal{M}^H$ the category of $K$-equivariant $H$-comodules. It is a monoidal category: the tensor product of $V,W \in K\mathcal{M}^H$ is the object $V \otimes W \in K\mathcal{M}^H$ with tensor $K$-module structure in (2.2) and $H$-coaction in (2.8). We also consider the category $K\mathcal{A}^H$ of $K$-equivariant $H$-comodule algebras, where objects and morphisms are in $K\mathcal{A}^H$ if they are in $A^H$, $K\mathcal{A}$ and $K\mathcal{M}^H$.

3. Quasitriangular structures and associated braiding

3.1. Quasitriangular Hopf algebras.

We recall some basic definitions and properties of quasitriangular Hopf algebras referring to [25, Ch. 8] or [31, Ch. 2] for more details and proofs. This class of algebras is characterized by being cocommutative up to conjugation by a suitable element.

Definition 3.1. A bialgebra $K$, with coproduct $\Delta$, is called quasitriangular if there exists an element $R := R_0 \otimes R_0 \in K \otimes K$ (with an implicit sum) which is invertible: $\exists R \in K \otimes K$, such that $R \overline{R} = \overline{R} R = 1 \otimes 1$; $\Delta$ is quasi-cocommutative with respect to $R$:

$$\Delta^{\text{cop}}(k) = R\Delta(k)\overline{R}$$  \hspace{3cm} (3.1)

for each $k \in K$, with $\Delta^{\text{cop}} := \tau \circ \Delta$ and $\tau$ the flip; and such that

$$(\Delta \otimes \text{id}) R = R_{13} R_{23}, \quad \text{and} \quad (\text{id} \otimes \Delta) R = R_{13} R_{12}.$$  \hspace{3cm} (3.2)

Here $R_{12} = R_0 \otimes R_0 \otimes 1$ and similarly for $R_{13}$ and $R_{23}$. Then conditions (3.2) become

$$R_0(1) \otimes R_0(2) \otimes R_0 = R_0(1) \otimes R_0(2) \otimes R_0 \otimes R_0,$$

$$(3.3) \quad R_0(1) \otimes R_0(2) \otimes R_0 = R_0(1) \otimes R_0(2) \otimes R_0 \otimes R_0.$$  

The element $R$ is called a universal $R$-matrix of $K$. A quasitriangular bialgebra $(K,R)$ is triangular if $\overline{R} = R_{21}$, with $R_{21} = \tau(R) = R_0 \otimes R_0$.  


The $R$-matrix of a quasitriangular bialgebra $(K, R)$ is unital,

\[(\varepsilon \otimes \text{id})R = 1 = (\text{id} \otimes \varepsilon)R\]  

(3.4)

with $\varepsilon$ the counit of $K$, and satisfies the quantum Yang–Baxter equation

\[R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},\]

or in components,

\[R^a R^\beta \otimes R^\gamma \otimes R_\beta R_\gamma = R^\beta R^\gamma \otimes R^a R_\alpha \otimes R_\alpha R_\beta.\]  

(3.5)

By the defining properties for a $R$-matrix it follows that if $R$ is a $R$-matrix of $K$, then $\overline{R}_{21}$ is also a $R$-matrix of $K$. Moreover, $K$ is cocommutative, $\Delta^{\text{cop}} = \Delta$, if and only if $R = 1 \otimes 1$ is a $R$-matrix of $K$.

A Hopf algebra which is quasitriangular as a bialgebra is called a quasitriangular Hopf algebra. Its antipode is such that

\[(S \otimes \text{id})(R) = \overline{R}; \quad (\text{id} \otimes S)(R) = R; \quad (S \otimes S)(R) = R.\]  

(3.6)

and is invertible. Indeed, consider the invertible element $u_R := S(R_a) R^a$ with inverse $u_R = R_a S^a(R^a)$. Then, $S^a(k) = u_R k u_R$, for all $k \in K$. This relation and the fact that $S(u_R) u_R = u_R S(u_R)$ is central, yields the following expression for the inverse of $S$:

\[S^{-1}(k) = u_R S(k) u_R, \quad \forall k \in K.\]  

(3.7)

When $(K, R)$ is a quasitriangular bialgebra, the monoidal category $K\mathcal{M}$ of left $K$-modules is braided monoidal: if $(V, \triangleright_V)$, $(W, \triangleright_W)$ are two left $K$-modules, then the $K$-modules $V \otimes W$ and $W \otimes V$, with actions as in (2.2), are isomorphic via the map

\[\Psi_{V,W}^R : V \otimes W \rightarrow W \otimes V, \quad v \otimes w \mapsto (R_a \triangleright_W w) \otimes (R^a \triangleright_V v).\]  

(3.8)

Remark 3.2. In order to show that the braiding $\Psi_{V,W}^R$ is an isomorphism of left $K$-modules, only the condition $\overline{R} \Delta^{\text{cop}} = \Delta \overline{R}$ in (3.1) in Definition 3.1 is needed. Thus, since both $R$ and $\overline{R}_{21}$ are $R$-matrices for $K$, one could use $\overline{R}$ instead of $R_{21}$ in the definition of the braiding.

Additionally, the category of left $K$-module algebras $K\mathcal{A}$ is monoidal as well:

**Proposition 3.3.** Let $(K, R)$ be a quasitriangular bialgebra. Let $(A, \triangleright_A)$, $(C, \triangleright_C)$ be left $K$-module algebras, then the $K$-module $A \otimes C$ with tensor product action as in (2.2), is a left $K$-module algebra when endowed with the product

\[(a \otimes c) \bullet (a' \otimes c') := a \ \Psi_{C,A}^R (c \otimes a') c' = a (R_a \triangleright_A a') \otimes (R^a \triangleright_C c) c'.\]  

(3.9)

Moreover, if $\phi : A \rightarrow E$ and $\psi : C \rightarrow F$ are morphisms of $K$-module algebras, then so is the map $\phi \otimes \psi : A \otimes C \rightarrow E \otimes F$, $a \otimes c \mapsto \phi(a) \otimes \psi(c)$, where $A \otimes C$ and $E \otimes F$ are both endowed with the $\bullet$-products.

The $K$-module algebra $(A \otimes C, \bullet)$ is the braided tensor product algebra of $A$ and $C$. We denote it by $A \boxtimes C$ and write $a \boxtimes c \in A \boxtimes C$ for $a \in A$, $c \in C$, and similarly by $\phi \boxtimes \psi := \phi \otimes \psi : A \boxtimes C \rightarrow E \boxtimes F$ the tensor product of two morphisms. We denote by $(K,R)\mathcal{A}$, $\mathcal{B}$ or simply $(K,\mathcal{A},\mathcal{B})$ the monoidal category of left $K$-module algebras with respect to the quasitriangular structure $R$.  

3.2. Braided Hopf algebras.

We can next introduce braided Hopf algebras (see [31, §9.4.2] or [25, §10.3.3]):

**Definition 3.4.** Let $(K, R)$ be a quasitriangular Hopf algebra. Let $(L, \triangleright_L)$ be both a $K$-module algebra $(L, m_L, \eta_L, \triangleright_L)$ and a $K$-module coalgebra $(L, \Delta_L, \varepsilon_L, \triangleright_L)$. Then $L$ is a braided bialgebra associated with $(K, R)$ if it is a bialgebra in the braided monoidal category $(K, M, \otimes, \Psi^R)$ of $K$-modules:

$$
\varepsilon_L \circ m_L = m_k \circ (\varepsilon_L \otimes \varepsilon_L), \quad \Delta_L \circ m_L = m_{L \otimes L} \circ (\Delta_L \otimes \Delta_L).
$$

(3.10)

That is, $\varepsilon_L : L \to k$ and $\Delta_L : L \to L \otimes L$ are algebra maps with respect to the product $m_L$ in $L$ and the product $m_{L \otimes L} = (m_L \otimes m_L) \circ (\id_L \otimes \Psi^R_{L,L} \otimes \id_L)$ in $L \otimes L$ (as in [29]). We frequently term $L$ a $K$-braided bialgebra omitting to mention the choice of quasitriangular structure on $K$.

A braided biagebra $L$ is a braided Hopf algebra if there is a $K$-module map $S_L : L \to L$ which is the convolution inverse of the identity $\id_L : L \to L$:

$$
m_L \circ (\id_L \otimes S_L) \circ \Delta_L = \eta_L \circ \varepsilon_L = m_L \circ (S_L \otimes \id_L) \circ \Delta_L.
$$

(3.11)

Such a map is called a (braided) antipode.

For later use we recall that the antipode $S_L : L \to L$ of a braided Hopf algebra $L$ is a braided anti-module map and a braided anti-coalgebra map:

$$
S_L \circ m_L = m_L \circ \Psi^R_{L,L} \circ (S_L \otimes S_L), \quad \Delta_L \circ S_L = (S_L \otimes S_L) \circ \Psi^R_{L,L} \circ \Delta_L.
$$

(3.12)

**Example 3.5.** For any Hopf algebra $K$, the underlying algebra $(K, m_K, \eta_K)$ becomes a left $K$-module algebra with the left adjoint action

$$
\triangleright_{\text{ad}} : K \otimes K \to K, \quad h \otimes k \mapsto h \triangleright_{\text{ad}} k := h_{(1)} k S(h_{(2)}).
$$

(3.13)

In general, the coproduct $\Delta : K \to K \otimes K$ is not a morphism of $K$-modules, for $K \otimes K$ carrying the tensor product action as in (2.2). However, if $K$ is quasitriangular with $R$-matrix $R = R^a \otimes R_a$, then the ‘braided’ coproduct

$$
\Delta(k) := k_{(1)} S(R_a) \otimes (R^a \triangleright_{\text{ad}} k_{(2)}) = k_{(1)} R_\beta S(R_a) \otimes R^a k_{(2)} R^\beta
$$

(3.14)

is now a morphism between the $K$-modules $K$ and $K \otimes K$. This braided coproduct is coassociative and unital; together with the counit $\varepsilon_K$ it defines a coalgebra structure on $K$. We use the notation $\Delta(k) = k_{(1)} \otimes k_{(2)}$ to distinguish it from the coproduct in $K$.

The compatibility with the adjoint $K$-action implies that $(K, \Delta, \varepsilon_K, \triangleright_{\text{ad}})$ is a $K$-module coalgebra in $K\mathcal{C}$. Moreover, the map

$$
\widehat{S} : K \to K, \quad k \mapsto \widehat{S}(k) := R_\alpha S(R^a \triangleright_{\text{ad}} k) = R_\alpha \bar{u}_R S(k) R^\beta
$$

(3.15)

is a left $K$-module morphism and

$$
\widehat{K} := (K, m_K, \eta_K, \Delta, \varepsilon_K, \widehat{S}, \triangleright_{\text{ad}})
$$

is a braided Hopf algebra associated with the quasitriangular Hopf algebra $(K, R)$. 

In the category of $K$-modules an action of a $K$-module algebra $L$ on a $K$-module $M$ is an action $\triangleright_M : L \otimes M \to M$ of the algebra $L$ on $M$ which is $K$-equivariant:

$$
k \triangleright_M (\ell \triangleright_M m) = (k_{(1)} \triangleright_L \ell) \triangleright_M (k_{(2)} \triangleright_M m).
$$

(3.16)

Braided Hopf algebras are symmetry objects within the category of $K$-modules.
Lemma 3.6. Let \((K,R)\) be a quasitriangular Hopf algebra and \(L\) a braided bialgebra associated with \(K\) acting on \(K\)-modules \(M\) and \(N\) as in (3.16). There is an action
\[
\triangleright_{M\otimes N} : L \otimes M \otimes N \to M \otimes N
\]
on the tensor product \(M \otimes N\) given by
\[
\ell \triangleright_{M\otimes N} (m \otimes n) = \ell \triangleright_M (R_{\alpha} \triangleright_M m) \otimes (R^a \triangleright_L \ell(2)) \triangleright_N n.
\]
Proof. We show that this is an action: \((\ell \ell') \triangleright_{M\otimes N} (m \otimes n) = \ell \triangleright_{M\otimes N} \ell' \triangleright_{M\otimes N} (m \otimes n)\). For ease of notations in the proof we drop subscripts.
\[
\ell \triangleright (\ell' \triangleright (m \otimes n)) = \ell \triangleright (\ell'(1) \triangleright (R_{\alpha} \triangleright m) \otimes (R^a \triangleright \ell'(2)) \triangleright n)
\]
\[
= \ell(\ell(1) \triangleright (R_{\beta} \triangleright (\ell'(1) \triangleright (R_{\alpha} \triangleright m)) \otimes (R^a \triangleright \ell(2)))) \triangleright n
\]
\[
= (\ell(1)R_{\beta(1)} \triangleright \ell'(1)) \triangleright (R_{\beta(2)}R_{\alpha} \triangleright m) \otimes ((R^a \triangleright \ell(2))(R^a \triangleright \ell'(2))) \triangleright n
\]
where we have used condition (3.16) on \(M\) for the third equality and the quasitriangular condition (3.3) for the last one. On the other hand
\[
(\ell \ell') \triangleright (m \otimes n) = (\ell(1)R_{\beta} \triangleright \ell'(1)) \triangleright (R_{\alpha} \triangleright m) \otimes ((R^a \triangleright \ell(2))(R^a \triangleright \ell'(2))) \triangleright n
\]
where again we have used the quasitriangular condition for the last equality.
Next, we show \(K\)-equivariance of \(\triangleright_{M\otimes N}\). For all \(k \in K, \ell \in L, m \in M, n \in N, k \triangleright (\ell \triangleright (m \otimes n)) = k \triangleright (\ell(1) \triangleright (R_{\alpha} \triangleright m) \otimes (R^a \triangleright \ell(2))) \triangleright n\)
\[
= k_{(1)} \triangleright (\ell(1) \triangleright (R_{\alpha} \triangleright m)) \otimes k_{(2)} \triangleright ((R^a \triangleright \ell(2))) \triangleright n
\]
\[
= (k_{(1)} \triangleright \ell(1)) \triangleright (k_{(2)} \triangleright (R_{\alpha} \triangleright m)) \otimes (k_{(3)} \triangleright (R^a \triangleright \ell(2))) \triangleright (k_{(4)} \triangleright n)
\]
\[
= (k_{(1)} \triangleright \ell) \triangleright (k_{(2)} \triangleright m \otimes (k_{(3)} \triangleright n))
\]
In the second line we used that \(L\) is a \(K\)-module algebra, in the third that the action of \(L\) on \(M\) and on \(N\) is \(K\)-equivariant. Then quasi-cocommutativity yields the result. \(\square\)

An action of a braided bialgebra \(L\) on a \(K\)-module algebra \(A\) is a \(K\)-equivariant action
\[
\triangleright_A : L \otimes A \to A \quad (\text{cf. } (3.16))
\]
which satisfies the condition
\[
\ell \triangleright_A (ac) = (\ell(1) \triangleright_A (R_{\alpha} \triangleright_A a)) ((R^a \triangleright_L \ell(2)) \triangleright_A c),
\]
for all \(a, c \in A\).

An important example of the above construction is given by the adjoint action of a braided Hopf algebra \(L\) on itself.

Proposition 3.7. The \(k\)-linear map
\[
\triangleright_{\text{adj}} : L \otimes L \to L, \quad \ell \otimes \ell' \mapsto \ell \triangleright_{\text{adj}} \ell' := \ell(1)(R_{\alpha} \triangleright_L \ell') R^a \triangleright_L S_L(\ell(2)),
\]
is a \(K\)-equivariant action of \(L\) on the \(K\)-module \(L\):
\[
(\ell \ell') \triangleright_{\text{adj}} x = \ell \triangleright_{\text{adj}} (\ell' \triangleright_{\text{adj}} x)
\]
for all \( k \in K \) and \( \ell, \ell', x \in L \).

**Proof.** To lighten notation we drop all subscripts. The right hand side reads

\[
\ell \triangleright (\ell' \triangleright x) = \ell_1 (R_\beta \triangleright (R_\alpha \triangleright x)) R^\alpha \triangleright S(\ell_2) \]

were we used the quasitriangular condition in (3.19). The left hand side reads

\[
(\ell' \triangleright x) (\ell \triangleright y) = \ell_1 (R_\beta \triangleright (R_\alpha \triangleright x)) (R_\alpha \triangleright y) R^\beta \triangleright S(\ell_2, (3.19)),
\]

where we used the braided algebra map property of the coproduct (3.10), the braided antialgebra map property of the antipode (3.12) and again the quasitriangular condition in (3.13). The left hand side equals the right hand side using the Yang-Baxter equation and the fact that the antipode is \( K \)-equivariant. We are left to show \( K \)-equivariance: \( k \triangleright (\ell \triangleright x) = k \triangleright ((\ell_1 (R_\alpha \triangleright x)) R^\alpha \triangleright S(\ell_2)) \)

In the third line we used quasi-cocommutativity, in the fourth line the \( K \)-equivariance of the antipode and that \( L \) is a \( K \)-module coalgebra. \( \square \)

**Proposition 3.8.** The adjoint action \( \triangleright_{ad_r} \) in (3.18) is an action of \( L \) on the \( K \)-module algebra \( L \)

\[
\ell \triangleright_{ad_r} (x y) = (\ell_1 (R_\alpha \triangleright y)) R^\alpha \triangleright S(\ell_2) = (\ell_1 (R_\alpha \triangleright x)) (R_\beta \triangleright y) R^\beta R^\alpha \triangleright S(\ell_2) \tag{3.19}
\]

for all \( \ell, x, y \in L \); it is henceforth called braided adjoint action. It satisfies the Jacobi identity (it is a braided action with respect to the nonassociative product (3.13)),

\[
\ell \triangleright_{ad_r} (x \triangleright_{ad_r} y) = (\ell_1 \triangleright_{ad_r} (R_\alpha \triangleright x)) \triangleright_{ad_r} ((R^\alpha \triangleright \ell_2) \triangleright_{ad_r} y) \tag{3.20}
\]

for all \( \ell, x, y \in L \).

**Proof.** We again omit all subscripts. We start with the braided algebra map property (3.19) (cf. [20, Ex. 2.7], [30, App.]). The left hand side reads:

\[
\ell \triangleright (xy) = \ell_1 (R_\alpha \triangleright (xy)) R^\alpha \triangleright S(\ell_2) = \ell_1 (R_\alpha \triangleright x) (R_\beta \triangleright y) R^\beta R^\alpha \triangleright S(\ell_2) \tag{3.19}.
\]

It equals the right hand side:

\[
(\ell_1 \triangleright (R_\alpha \triangleright x)) ((R^\alpha \triangleright \ell_2) \triangleright y) = \ell_1 (R_\beta R_\gamma \triangleright x) (R^\beta \triangleright S(\ell_2)) (R^\alpha \triangleright \ell_3) (R_\gamma \triangleright y) (R^\gamma \triangleright S(\ell_4)) \tag{3.20}
\]

where in the second and third lines we used the quasitriangular condition in (3.3).
The proof of property (3.20) follows recalling that $\triangleright_{ad_H}$ is an action, indeed
\[
(\ell_1 \triangleright (R_\alpha \triangleright x)) \triangleright (R^\alpha \triangleright \ell_2) \triangleright y = ((\ell_1 \triangleright (R_\alpha \triangleright x)) (R^\beta \triangleright S(\ell_2)) \triangleright \ell_3) \triangleright y
\]
\[
= (\ell_1(R_\beta R_\alpha \triangleright x) (R^\beta \triangleright S(\ell_2)) R^\alpha \triangleright \ell_3) \triangleright y
\]
\[
= (\ell_1(R_\beta \triangleright x) R^\beta \triangleright (S(\ell_2)\ell_3)) \triangleright y
\]
\[
= (\ell_1 \triangleright y) \triangleright y
\]
where for the third equality we used again the equality $(\Delta \otimes \text{id})R = R_{13}R_{12}$. $\square$

Lemma 3.9. The map
\[
\mathcal{L} : (L, \triangleright_L) \rightarrow (\text{Hom}(L, L), \triangleright_{\text{Hom}(L, L)})
\]
\[
\ell \mapsto \mathcal{L}_\ell, \quad \mathcal{L}_\ell(x) := \ell \triangleright_{ad_L} x = \ell_1(R_\alpha \triangleright_L x) R^\alpha \triangleright_L S_L(\ell_2), \quad x \in L,
\]
is a morphism of $K$-modules.

Proof. We show that $\mathcal{L}_{k \triangleright_L \ell} = k \triangleright_{\text{Hom}(L, L)} \mathcal{L}_\ell$, for each $k \in K$, $\ell \in L$. Let $x \in L$, then
\[
\mathcal{L}_{k \triangleright_L \ell}(x) = (k \triangleright_1(R_\alpha \triangleright x) R^\alpha \triangleright S_L((k \triangleright_2) \triangleright_2)
\]
\[
= (k_1 \triangleright_1 (R_\alpha \triangleright S(k_{k_2}) \triangleright x)) (R^\alpha \triangleright S_L(\ell_2))
\]
\[
= (k_1 \triangleright_1 (R_\alpha S(k_{k_2}) \triangleright x)) (k_3(R^\alpha \triangleright S_L(\ell_2)))
\]
\[
= (k_1 \triangleright_1 (R_\alpha S(k_{k_3}) \triangleright x)) (R^\alpha(R_2 \triangleright S_L(\ell_2))),
\]
where for the last equality we used that $S_L$ is a morphism of $H$-modules. On the other hand, recalling the definition of $\triangleright_{\text{Hom}(L, L)}$ from (2.3),
\[
(k \triangleright \mathcal{L}_\ell)(x) = k_1 \triangleright \mathcal{L}_\ell(S(k_{k_2}) \triangleright x)
\]
\[
= k_1 \triangleright ((\ell_1(R_\alpha \triangleright S(k_{k_2}) \triangleright x)) \triangleright_2 (R^\alpha \triangleright S_L(\ell_2)))
\]
\[
= (k_1 \triangleright_1 (k_{k_2}R_\alpha S(k_{k_3}) \triangleright x)) (k_3(R^\alpha \triangleright S_L(\ell_2)))
\]
\[
= (k_1 \triangleright_1 (R_\alpha k_{k_3}S(k_{k_3}) \triangleright x)) (R^\alpha k_{k_2} \triangleright S_L(\ell_2)),
\]
using the quasi-cocommutativity of $K$ for the last equality. Thus, the equality holds. $\square$

Proposition 3.10. Let $(K, \triangleright_{ad})$ be the braided Hopf algebra associated with the quasitriangular Hopf algebra $(K, R)$ as is Example 3.2. Then, the braided adjoint action (3.13) coincides with the adjoint action (3.14): $\triangleright_{ad_H} = \triangleright_{ad}$. That is for all $h, k \in K$,
\[
h_1(R_\alpha \triangleright_{ad} k)(R^\alpha \triangleright_{ad} S(h_{21})) = h_{11}(R_\alpha \triangleright k) S(h_{21}).
\]

Proof. From the explicit expressions of the braided coproduct and antipode in $K$, we can rewrite the braided adjoint action in the left hand side as
\[
h_1(R_\alpha \triangleright_{ad} k) = h_{11}(R_\alpha S(R_\alpha) R_\alpha k S(R_\alpha R_\beta) R^\alpha R^\beta \bar{u}_R S(R^\mu h_{21} R^\nu) \bar{R}^\gamma S(R^\alpha_{21}).
\]
By using the quasi-triangularity of $K$ in (3.3), and then the properties (3.1) relating $R$ to $\bar{R}$ via the antipode, this is written as
\[
h_{11}(R_\alpha S(R_\alpha) R_\alpha k S(R_\alpha R_\beta) R^\alpha R^\beta \bar{u}_R S(R^\mu h_{21} R^\nu) \bar{R}^\gamma S(R^\beta R^\delta) = h_{11}(R_\alpha R_\beta S(R^\mu R_\gamma) \bar{R}_\gamma k R_\beta u_R R^\gamma \bar{u}_R S(R^\gamma) S(h_{21}) R^\mu \bar{R}^\gamma R^\beta.
\]
For the last equality we used
\[
R_\gamma \otimes u_R R^\gamma \bar{u}_R \otimes \bar{R}^\gamma = R_\gamma \otimes S(\bar{R}^\gamma) S(R_\alpha) \otimes R^\gamma.
\]
which is obtained from the expression (3.27) for the inverse of the antipode. Next the quantum Yang-Baxter equation (on the indices $\nu, \mu, \sigma$) gives

$$h_{(1)}R_{\nu}R_{\sigma}\overline{R}_{\beta}kR_{\beta}S(R_{\nu}R_{\sigma})S(h_{(2)})R_{\nu}R_{\sigma}\overline{R}_{\beta}R_{\beta}$$

and we finally obtain

$$h_{(1)}R_{\nu}R_{\sigma}\overline{R}_{\beta}kR_{\beta}S(R_{\nu})S(h_{(2)})R_{\nu}R_{\sigma}\overline{R}_{\beta}R_{\beta} = h_{(1)}kR_{\beta}S(R_{\nu})S(h_{(2)})R_{\nu}R_{\beta}$$

$$= h_{(1)}kS(R_{\nu}\overline{R}_{\beta})S(h_{(2)})R_{\nu}R_{\beta}$$

$$= h_{(1)}kS(h_{(2)})$$

thus establishing that $h \triangleright_{\text{ad}} k = h \triangleright_{\text{ad}} k$. \hfill \Box

### 3.3. Dual structures.

For later use we briefly recall the dual notion of coquasitriangular Hopf algebra and that of associated braided Hopf algebra, that was used in [3].

**Definition 3.11.** A bialgebra $H$ is called **coquasitriangular** (or dual quasitriangular) if it is endowed with a linear form $R : H \otimes H \to k$ such that:

1. $R$ is invertible for the convolution product in $H \otimes H$, with inverse denoted $\overline{R}$,
2. $R \circ (m \otimes \text{id}) = R_{13} \ast R_{23}$ and $R \circ (\text{id} \otimes m) = R_{13} \ast R_{12}$,
3. $R_{12} = R \otimes \varepsilon : H \otimes H \otimes H \to k$ and similarly for $R_{13}$ and $R_{23}$.

The linear form $R$ is called a **universal $R$-form** of $H$. A coquasitriangular bialgebra $(H, R)$ is called **cotriangular** if $R = \overline{R}_{21}$.

As for the quasitriangular case, tensor products of comodule algebras are comodule algebras and tensor products of comodule algebra maps are again comodule algebra maps:

**Proposition 3.12.** Let $(H, R)$ be a coquasitriangular bialgebra. Let $(A, \delta^A, (C, \delta^C) \in \mathcal{A}^H$ be right $H$-comodule algebras. Then the $H$-comodule $A \otimes C$, with tensor product coaction as in (2.8), is a right $H$-comodule algebra when endowed with the product

$$(a \otimes c)(a' \otimes c') := aa'_{(0)} \otimes c_{(0)}c' R(c_{(1)} \otimes a'_{(1)}). \tag{3.22}$$

Moreover, when $\phi : A \to E$ and $\psi : C \to F$ are morphisms of $H$-comodule algebras, then so is the map $\phi \otimes \psi : A \otimes C \to E \otimes F$, $a \otimes c \mapsto \phi(a) \otimes \psi(c)$, where $A \otimes C$ and $E \otimes F$ are endowed with the $\bullet$-products in (3.22).

The $H$-comodule algebra $(A \otimes C, \bullet)$ is the **braided tensor product algebra** of $A$ and $C$. We denote it by $A \boxtimes C$, and write $a \boxtimes c \in A \boxtimes C$ for $a \in A$, $c \in C$ and by $\phi \boxtimes \psi := \phi \otimes \psi : A \boxtimes C \to E \boxtimes F$ the tensor product of two morphisms. With the tensor product $\boxtimes$ the category $(\mathcal{A}^H, \boxtimes)$ of $H$-comodule algebras is a monoidal category.

The definition of a braided Hopf algebra associated with a coquasitriangular Hopf algebra $(H, R)$ is the same as that in Definition 3.11 for the quasitriangular case, with the braided monoidal category of comodules for $(H, R)$ replacing that of modules for $(K, R)$.

In particular for any Hopf algebra $H$, the data $(H, \Delta_H, \varepsilon_H, \text{Ad})$ is an $H$-comodule coalgebra, with the right adjoint coaction $\text{Ad} : H \to H \otimes H$, $h \mapsto h_{(2)} \otimes S(h_{(1)})h_{(3)}$. When $(H, R)$ is coquasitriangular $\text{Ad}$ is an algebra map for the ‘braided’ product in $H$ given by

$$h \triangleright k := h_{(2)}k_{(2)}R(S(h_{(1)})h_{(3)} \otimes S(k_{(1)})), \quad h, k \in H. \tag{3.23}$$
Then $\mathcal{H} := (H, \cdot, \eta_H, \Delta_H, \varepsilon_H, \text{Ad})$ is a braided bialgebra associated with $(H, R)$. Furthermore $\mathcal{H}$ is a braided Hopf algebra with antipode $S$ defined by

$$S(h) := S(h_{(2)})R(S^2(h_{(1)})S(h_{(1)}) \otimes h_{(4)}), \quad h \in H. \quad (3.24)$$

4. Twisting braided Hopf algebras

We recall some basic definitions and properties of the theory of Drinfel’d twists [17, 18], see also [19].

**Definition 4.1.** Let $K$ be a bialgebra (or Hopf algebra). A twist for $K$ is an invertible element $F \in K \otimes K$ which is unital, $(\varepsilon \otimes \text{id})(F) = 1 = (\text{id} \otimes \varepsilon)(F)$, and satisfies

$$(F \otimes 1)[(\Delta \otimes \text{id})(F)] = (1 \otimes F)[(\text{id} \otimes \Delta)(F)], \quad (4.1)$$

referred to as the twist condition.

If $\bar{F}$ denotes the inverse of the twist, $F \bar{F} = 1 \otimes 1 = \bar{F}F$, the condition (4.1) can be written equivalently as

$$[(\Delta \otimes \text{id})(\bar{F})](F \otimes 1) = [(\text{id} \otimes \Delta)(\bar{F})](1 \otimes F). \quad (4.2)$$

We use the notation $F = F^\alpha \otimes F_\alpha$ and $\bar{F} = \bar{F}^\beta \otimes \bar{F}_\alpha$ with an implicit summation understood, for the twist and its inverse. The identities (4.1), (4.2), are then rewritten respectively as

$$F^\beta F_\alpha(1) \otimes F^\beta F_\alpha(2) \otimes F_\alpha = F^\alpha \otimes F^\beta F_\alpha(1) \otimes F^\beta F_\alpha(2) \quad (4.3)$$

$$\bar{F}^\beta \bar{F}_\alpha(1) \otimes \bar{F}^\beta \bar{F}_\alpha(2) \otimes \bar{F}_\alpha = \bar{F}^\alpha \otimes \bar{F}^\beta \bar{F}_\alpha(1) \otimes \bar{F}^\beta \bar{F}_\alpha(2) \bar{F}_\beta. \quad (4.4)$$

**Remark 4.2.** When $K$ is not finite dimensional over $k$, the twist $F$ may not necessarily be an element of $K \otimes K$, but rather it belongs to a topological completion of the tensor product algebra. In the examples of this paper we avoid this problem either by diagonalizing $F$ or by considering representations of $F$ involving only a finite number of addends in $F = F^\alpha \otimes F_\alpha$.

**Example 4.3.** The $R$-matrix $R$ of a quasitriangular bialgebra $K$ is a twist for $K$. Condition (4.1) follows from the quasitriangular condition (3.3) and the quantum Yang-Baxter equation: $(R \otimes 1)[(\Delta \otimes \text{id})R] = R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} = (1 \otimes R)[(\text{id} \otimes \Delta)]. \quad \blacksquare$

When $K$ has a twist, $K$ can be endowed with a second bialgebra structure which is obtained by deforming its coproduct and leaving its counit and multiplication unchanged:

**Proposition 4.4.** Let $F = F^\alpha \otimes F_\alpha$ be a twist on a bialgebra $(K, m, \eta, \Delta, \varepsilon)$. Then the algebra $(K, m, \eta, \Delta_F, \varepsilon)$ is a braided Hopf algebra with coproduct

$$\Delta_F(k) := F \Delta(k) \bar{F} = F^\alpha k(1) \bar{F}^\beta \otimes F_\alpha k(2) \bar{F}_\beta, \quad k \in K \quad (4.5)$$

and counit $\varepsilon$ is a bialgebra. If in addition $K$ is a Hopf algebra, then the twisted bialgebra $K_F := (K, m, \eta, \Delta_F, \varepsilon)$ is a Hopf algebra with antipode

$$S_F(k) := u_F S(k) \bar{u}_F,$$

where $u_F$ is the invertible element $u_F := F^\alpha S(F_\alpha)$ and $\bar{u}_F = S(\bar{F}^\alpha) \bar{F}_\alpha$ its inverse.

We use the notation $\Delta_F(k) := k(1) \otimes k(2)$ for the coproduct in $K_F$ to distinguish it from the original coproduct $\Delta(k) = k(1) \otimes k(2)$ in $K$.

Let us recall that the inverse $\bar{F}$ of the twist $F$ is a twist for the twisted bialgebra $K_F$, with the twist condition for $\bar{F} \in K_F \otimes K_F$ coinciding with (4.1). Clearly, using $\bar{F}$ to deform the coproduct in $K_F$ leads back to the initial bialgebra $(K_F)_F = K$. 


For later use, we next recall some results which are dual to those related to deformations of bialgebras via 2-cocycles \[16\], and which were addressed in \[9 \S 4\].

Let \( K \) be a bialgebra (or Hopf algebra) with a twist \( F \in K \otimes K \) and \( K_F \) the resulting twisted bialgebra, as above. Any \( K \)-module \( V \) with left action \( \triangleright_V \colon K \otimes V \to V \), is also a \( K_F \)-module with the same linear map \( \triangleright_V \), now thought as a map \( \triangleright_V \colon K_F \otimes V \to V \). Indeed, the module conditions in (2.1) only involve the algebra structure of \( K \), and the twisted bialgebra \( K_F \) coincides with \( K \) as algebra. When thinking of \( V \) as a \( K_F \)-module we denote it by \( V_F \), with action \( \triangleright_{V_F} \). Moreover, any \( K \)-module morphism \( \psi : V \to W \) can be thought as a morphism \( \psi_F : V_F \to W_F \) since, as stated, the action of \( K \) on any \( V,W \) coincides with the action of \( K_F \) on \( V_F,W_F \). This amounts to say that there is an equivalence of category \( \Gamma : K\mathcal{M} \to K_F\mathcal{M} \) with the functor \( \Gamma \) just the identity both on objects and on morphisms: \( \Gamma(V) := V_F \) and \( \Gamma(\psi) := \psi_F \). The use of \( \overline{F} \) as a twist for \( K_F \), which ‘twists back’ \( K_F \) to \( K \), inverts the construction going from \( K_F\mathcal{M} \to K\mathcal{M} \).

With the monoidal structure one needs the deformed coproduct (1.3). Explicitly, for \( K_F \) modules \( (V_F, \triangleright_{V_F}) \) and \( (W_F, \triangleright_{W_F}) \), the tensor product action of \( K_F \) on \( V_F \otimes_F W_F \simeq V \otimes W \) (as linear spaces) is given by

\[
\triangleright_{V_F \otimes_F W_F} : K_F \otimes (V_F \otimes F W_F) \to V_F \otimes F W_F,
\]

\[
k \otimes v \otimes F w \mapsto (k_{[1]} \triangleright_{V_F} v) \otimes F (k_{[2]} \triangleright_{W_F} w).
\]

We denote by \((K_F \mathcal{M}, \otimes_F)\) the monoidal category of left \( K_F \)-modules. We have then:

**Proposition 4.5.** The functor \( \Gamma : K\mathcal{M} \to K_F\mathcal{M} \) together with the natural isomorphism \( \varphi : \otimes_F \circ (\Gamma \times \Gamma) \Rightarrow \Gamma \circ \otimes \) given by the isomorphism of \( K_F \)-modules

\[
\varphi_{V,W} : V_F \otimes_F W_F \to (V \otimes W)_F
\]

\[
v \otimes_F w \mapsto (\overline{F} \triangleright_{V_F} v) \otimes_F (\overline{F} \triangleright_{W_F} w),
\]

is an equivalence of monoidal categories \((K\mathcal{M}, \otimes)\) and \((K_F\mathcal{M}, \otimes_F)\).

As a consequence of the twist condition (1.3) the isomorphisms \( \varphi_{-, -} \) satisfy

\[
\varphi_{V,W \otimes Z} \circ (\text{id}_{V_F} \otimes_F \varphi_{W,Z}) = \varphi_{V \otimes_F W, Z} \circ (\varphi_{V,W} \otimes_F \text{id}_{Z_F})
\]

for every \( K_F \)-modules \( V_F, W_F, Z_F \).

There is also an equivalence between the category \( K_A \) of left \( K \)-module algebras and the corresponding category \( K_F \mathcal{A} \) of \( K_F \)-module algebras

\[
\Gamma : K_A \to K_F \mathcal{A}, \quad (A, m_A, \eta_A, \triangleright_A) \mapsto (A_F, m_{A_F}, \eta_{A_F}, \triangleright_{A_F}).
\]

It is no longer the identity on objects. Given \( A \in K_A \) with multiplication \( m_A \) and unit \( \eta_A \), in order for the action \( \triangleright_{A_F} \) to be an algebra map one has to define a new product on the \( K \)-module \( A_F = \Gamma(A) \). The new algebra structure \( m_{A_F}, \eta_{A_F} \) on \( A_F \in K_F \mathcal{A} \) is defined by using the natural isomorphisms \( \varphi \) in (1.7):

\[
m_{A_F} := \Gamma(m_A) \circ \varphi_{A,A}, \quad \eta_{A_F} := \Gamma(\eta_A)
\]

Explicitly, the unit is unchanged, while the product is deformed to

\[
m_{A_F} : A_F \otimes_F A_F \to A_F
\]

\[
a \otimes_F a' \mapsto a \rtimes_F a' := (\overline{F} \triangleright_{A_F} a) (\overline{F} \triangleright_{A} a')
\]

(4.10)

The functor \( \Gamma \) is the identity on morphism: for any algebra map \( \psi : A \to A' \) one shows that \( \Gamma(\psi) = \psi_F : A_F \to A'_F \) is an algebra map for the deformed products.
Similarly to above, there is also an equivalence between the category $K\mathcal{C}$ of left $K$-module coalgebras and that of $K_\mathcal{F}$-module coalgebras
\[
\begin{align*}
\Gamma : K\mathcal{C} & \to K_\mathcal{F}\mathcal{C}, \quad (C, \Delta_C, \varepsilon_C, D_C) \mapsto (C_\mathcal{F}, \Delta_{C_\mathcal{F}}, \varepsilon_{C_\mathcal{F}}, D_{C_\mathcal{F}}). 
\end{align*}
\] (4.11)
Again, $\Gamma$ acts as the identity on morphisms, but not on objects. Each $K$-module coalgebra $C$ with costructures $\Delta_C, \varepsilon_C$ is mapped to the $K_\mathcal{F}$-module coalgebra $C_\mathcal{F} = \Gamma(C)$ with costructures $\Delta_{C_\mathcal{F}}, \varepsilon_{C_\mathcal{F}}$ defined by
\[
\Delta_{C_\mathcal{F}} := \varphi_{C_\mathcal{F},C}^{-1} \circ \Gamma(\Delta_C), \quad \varepsilon_{C_\mathcal{F}} := \Gamma(\varepsilon_C).
\] Explicitly, while the counit does not change, the deformed coproduct is
\[
\Delta_{C_\mathcal{F}} : C_\mathcal{F} \to C_\mathcal{F} \otimes_{\mathcal{F}} C_\mathcal{F}, \quad c \mapsto c_{[1]} \otimes c_{[2]} := (\mathcal{F}^a \triangleright_C c_{(1)}) \otimes (\mathcal{F}_a \triangleright_C c_{(2)}).
\] (4.12)

4.1. $K$-modules of linear maps. We study twist deformations of linear maps of $K$-modules for later applications in §5.3 to braided derivations.

Consider the $K$-module algebra $(\text{Hom}(V, V), \circ)$ of linear maps of a $K$-module $V$, with action $\triangleright_{\text{Hom}(V, V)}$ as in (2.3). On the one hand, we obtain a $K_\mathcal{F}$-module algebra $(\text{Hom}_\mathcal{F}(V, V), \circ_\mathcal{F})$ out of $(\text{Hom}(V, V), \circ)$ by changing the multiplication $\circ$ as in (4.10):
\[
\psi \circ_\mathcal{F} \phi = (\mathcal{F}^a \triangleright_{\text{Hom}(V, V)} \psi) \circ (\mathcal{F}_a \triangleright_{\text{Hom}(V, V)} \phi).
\] (4.13)
The $K_\mathcal{F}$-action $\triangleright_{\text{Hom}_\mathcal{F}(V, V)}$ coincides with $\triangleright_{\text{Hom}(V, V)}$ as linear map. On the other hand, there is the $K_\mathcal{F}$-module algebra $(\text{Hom}(V_\mathcal{F}, V_\mathcal{F}), \circ)$ of linear maps of the $K_\mathcal{F}$-module $V_\mathcal{F}$ with action
\[
\triangleright_{\text{Hom}(V_\mathcal{F}, V_\mathcal{F})} : K_\mathcal{F} \otimes \text{Hom}(V_\mathcal{F}, V_\mathcal{F}) \to \text{Hom}(V_\mathcal{F}, V_\mathcal{F})
\]
\[
k \otimes \psi \mapsto k \triangleright_{\text{Hom}(V_\mathcal{F}, V_\mathcal{F})} \psi : a \mapsto k_{[1]} \triangleright_{V_\mathcal{F}} \psi(S_F(k_{[2]}) \triangleright_{V_\mathcal{F}} a).
\] (4.14)

These two module algebras are isomorphic (cf. [4, Th. 4.7]):

**Proposition 4.6.** The $K_\mathcal{F}$-module algebras $(\text{Hom}_\mathcal{F}(V, V), \circ_\mathcal{F})$ and $(\text{Hom}(V_\mathcal{F}, V_\mathcal{F}), \circ)$ are isomorphic via the map
\[
\mathcal{D} : \text{Hom}_\mathcal{F}(V, V) \to \text{Hom}(V_\mathcal{F}, V_\mathcal{F})
\]
\[
\psi \mapsto \mathcal{D}(\psi) : v \mapsto (\mathcal{F}^a \triangleright_{\text{Hom}_\mathcal{F}(V, V)} \psi)(\mathcal{F}_a \triangleright_{V_\mathcal{F}} v)
\] (4.15)
with inverse
\[
\mathcal{D}^{-1} : \text{Hom}(V_\mathcal{F}, V_\mathcal{F}) \to \text{Hom}_\mathcal{F}(V, V)
\]
\[
\psi \mapsto \mathcal{D}^{-1}(\psi) : v \mapsto (\mathcal{F}^a \triangleright_{\text{Hom}(V_\mathcal{F}, V_\mathcal{F})} \psi)(\mathcal{F}_a \triangleright_{V_\mathcal{F}} v)
\] (4.16)

**Proof.** We first observe that from the twist condition and the definition of the action on endomorphisms, we can also write
\[
\mathcal{D}(\psi)(v) = (\mathcal{F}^a \triangleright_{V_\mathcal{F}} \psi(S_F(\mathcal{F}_a)u_F \triangleright_{V_\mathcal{F}} v);
\]
\[
\mathcal{D}^{-1}(\psi)(v) = (\mathcal{F}^a \triangleright_{V_\mathcal{F}} \psi(S_F(\mathcal{F}_a)u_F \triangleright_{V_\mathcal{F}} v).
\]
Then it is easy to see that the two maps are one the inverse of the other (also recalling that the actions $\triangleright_{V_\mathcal{F}}$ and $\triangleright_{V_\mathcal{F}}$ coincides as linear maps). We show the $K_\mathcal{F}$-equivariance $k \triangleright_{\text{Hom}(V_\mathcal{F}, V_\mathcal{F})} \mathcal{D}(\psi) = \mathcal{D}(k \triangleright_{\text{Hom}_\mathcal{F}(V, V)} \psi)$:
\[
(k \triangleright \mathcal{D}(\psi))(v) = k_{[1]} \triangleright \mathcal{D}(\psi)(S_F(k_{[2]}) \triangleright v)
\]
\[
= \mathcal{F}^a k_{[1]} \mathcal{F}^\beta \triangleright \mathcal{D}(\psi)(u_F S_F(k_{[2]} F^\beta)u_F \triangleright v)
\]
Next we show that \( \mathcal{D}(\psi \circ \phi) = \mathcal{D}(\psi) \circ \mathcal{D}(\phi) \). For that we compute

\[
(\psi \circ \phi)(v) = ((F^\alpha \triangleright \psi) \circ (F^\alpha \triangleright \phi))(v)
\]

\[
= (F^\alpha \triangleright \psi)(F^\alpha_{(2)} \triangleright \phi)(S(F^\alpha_{(2)} \triangleright v)
\]

\[
= F^\alpha_{(2)} \triangleright \psi(S(F^\alpha_{(2)} \triangleright v)
\]

\[
= F^\alpha_{(1)} \triangleright \psi(S(F^\alpha_{(2)} \triangleright v)
\]

\[
= F^\alpha \triangleright (v \triangleright (S(F^\alpha \triangleright v))
\]

\[
= \mathcal{D}(v \triangleright \psi)(v).
\]

thus showing that \( \mathcal{D} \) is an algebra morphism. \( \square \)

4.2. **Quasitriangular bialgebras.** We consider the case when the twist comes from a quasitriangular bialgebra \((K, R)\). The twisted bialgebra \( K_F \) is still quasitriangular. Moreover, the isomorphisms \( \varphi \) in (4.7) for module algebras are algebra maps (Proposition 4.7); for this we need the following result.

**Proposition 4.7.** Let \((K, R)\) be a quasitriangular bialgebra and \( F \in K \otimes K \) a twist for \( K \). Then the following identity holds in \( K^{\otimes 4} \):

\[
[(\Delta \otimes \Delta)F](1 \otimes R_{21} \otimes 1)[(id \otimes \tau \otimes id)(\Delta \otimes \Delta)F]
\]

\[
= (F \otimes F)(1 \otimes R_{F21} \otimes 1)\left[(id \otimes \tau \otimes id)(F \otimes F)\right], \quad (4.17)
\]

where \( R_{F21} = FR_{21} F_{21} \). That is,

\[
F^\gamma_{(1)} F^\delta_{(1)} \otimes F^\gamma_{(2)} R^\beta_{(2)} F_{(1)}^\alpha \otimes_F F_{(2)}^\alpha \otimes_F R^\delta_{(2)} F_{(1)}^\beta \otimes_F F_{(2)}^\alpha F_{(2)}^\beta \otimes_F F_{(2)}^\gamma \otimes_F F_{(2)}^\delta.
\]

\[
= F^\delta F^\gamma F^\beta R_{F}^\alpha F_{(1)}^\alpha \otimes_F F_{(2)}^\delta R_{F}^\gamma F_{(2)}^\gamma \otimes_F F_{(2)}^\delta F_{(2)}^\gamma \otimes_F F_{(2)}^\delta.
\]

**Proof.** Firstly we observe that the twist condition (4.1) for \( F \) gives

\[
[(\Delta \otimes \Delta)F] = (F \otimes 1 \otimes 1)[(1 \otimes (id \otimes \Delta)F)[(id \otimes \Delta^2)F]
\]

and, similarly, the twist condition (4.2) written for the inverse \( \overline{F} \) of \( F \) gives

\[
[(\Delta \otimes \Delta)F] = [(id \otimes \Delta^2)\overline{F}][(1 \otimes (id \otimes \Delta)F)(F \otimes 1 \otimes 1).\]

Then, identity (4.18) is equivalent to

\[
[(1 \otimes (id \otimes \Delta)F)[(id \otimes \Delta^2)F](1 \otimes R_{21} \otimes 1)(id \otimes \tau \otimes id)[[(id \otimes \Delta^2)F]1 \otimes (id \otimes \Delta)F] = (1 \otimes 1 \otimes \overline{F})(1 \otimes FR_{21} F_{21} \otimes 1)\left[(id \otimes \tau \otimes id)(1 \otimes 1 \otimes F)\right].
\]

16
We multiply both sides by \((1 \otimes 1 \otimes F)\) and use again the twist condition \((4.1)\) in the left hand side to simplify \((1 \otimes F \otimes 1)\) and obtain
\[
[(1 \otimes (\Delta \otimes \mathrm{id})F)][(\mathrm{id} \otimes \Delta^2)F](1 \otimes R_{21} \otimes 1)(\mathrm{id} \otimes \tau \otimes \mathrm{id})[[\mathrm{id} \otimes \Delta^2]F][1 \otimes (\mathrm{id} \otimes \Delta)F]
= (1 \otimes R_{21} F_{21} \otimes 1)([\mathrm{id} \otimes \tau \otimes \mathrm{id})(1 \otimes 1 \otimes F)).
\]
The quasitriangular condition \(\Delta^{cop} = R\Delta R\) gives
\[
(R_{21} \otimes 1)(\tau \otimes \mathrm{id})\Delta^2 = (R_{21}\Delta^{cop} \otimes \mathrm{id})\Delta = \Delta^2(R_{21} \otimes 1),
\]
which we can use to simplify the left hand side to
\[
[(1 \otimes (\Delta \otimes \mathrm{id})F)(1 \otimes R_{21} \otimes 1)(\mathrm{id} \otimes \tau \otimes \mathrm{id})][1 \otimes (\mathrm{id} \otimes \Delta)F]
\]
and further to
\[
(1 \otimes R_{21} \otimes 1)([1 \otimes (\Delta^{cop} \otimes \mathrm{id})F](\mathrm{id} \otimes \tau \otimes \mathrm{id})[1 \otimes (\mathrm{id} \otimes \Delta)F]
= (1 \otimes F_{21} \otimes 1)(\mathrm{id} \otimes \tau \otimes \mathrm{id})(1 \otimes 1 \otimes F).
\]

The identity \([(\Delta \otimes \mathrm{id})F][\mathrm{id} \otimes \Delta]F = (\mathrm{F} \otimes 1)(\otimes F)\) follows from the twist condition \((4.1)\), hence the identity \((4.18)\) in the Proposition is proven. \(\Box\)

As mentioned, the twisted bialgebra \(K_F\) is quasitriangular as well.

**Proposition 4.8.** If \((K,R)\) is a quasitriangular bialgebra and \(F \in K \otimes K\) is a twist for \(K\), then the twisted bialgebra \(K_F\) with twisted coproduct \(\Delta_F\) (see Proposition 4.4) is quasitriangular with \(R\)-matrix
\[
R_F := F_{21} R F = F_{\alpha} R^{\beta} F^{\gamma} \otimes F_{\alpha} R^{\beta} F^{\gamma}
\]
and inverse \(R_F := F R F_{21} = F^{\alpha} R^{\beta} F^{\gamma} \otimes F^{\alpha} R^{\beta} F^{\gamma}\). If \((K,R)\) is triangular, so is \((K_F,R_F)\). Also, if \(K\) is a quasitriangular Hopf algebra, then so is \(K_F\) with twisted antipode \(S_F\). \(\Box\)

The category \(K_F\mathcal{M}\) of left \(K_F\)-modules is a braided monoidal category with braiding
\[
\Psi_{V,W}^{R_F} : V \otimes W \longrightarrow W \otimes V, \quad v \otimes w \longmapsto (R_{F_{\alpha} \triangleright W} w) \otimes (R_{F^{\alpha} \triangleright V} v),
\]
for any pair of left \(K_F\)-modules \((V, \triangleright W), (W, \triangleright V)\).

**Remark 4.9.** The matrix \(R_F\) has been used in the Proposition 4.7. By inspection one sees that the identity \((4.17)\) can be written as:
\[
1 \otimes \Psi_{K_F,K_F}^{R_F} \otimes 1 = (F \otimes F)[(\Delta \otimes \Delta)F][1 \otimes \Psi_{K_F,K_F}^{R_F} \otimes 1)(\Delta \otimes \Delta)F](F \otimes F).
\]
Here \(K\) (respectively \(K_F\)) is seen as a left \(K\)-module (respectively left \(K_F\)-module).

For quasitriangular bialgebras we compare the monoidal categories of module algebras \((K,A,\boxtimes)\) and \((K_F,A,\boxtimes_F)\). Proposition 4.7 and Proposition 4.8 imply they are equivalent.
Proposition 4.10. Let \((K, R)\) be a quasitriangular bialgebra and \(F \in K \otimes K\) a twist. There is an equivalence of monoidal categories between \((K, \mathcal{A}, \otimes)\) and \((K_r, \mathcal{A}_F, \otimes_F)\) given by the functor \(\Gamma: K \mathcal{A} \to K_r \mathcal{A}\) in (4.9) and the isomorphisms in \(K_r \mathcal{A}\)

\[ \varphi_{A,C} : A_F \boxtimes_F C_F \longrightarrow (A \boxtimes C)_F, \quad a \boxtimes_F c \longmapsto (F^1 \triangleright_A a) \boxtimes (F^2 \triangleright_C c), \]

with \(A_F \boxtimes_F C_F\) the braided tensor product of the algebras \(A_F\) and \(C_F\), and \((A \boxtimes C)_F\) the image via \(\Gamma\) of the braided tensor product of the algebras \(A\) and \(C\).

Proof. This is really a corollary of Proposition (4.7). We need to show that the isomorphisms \(\varphi_{A,C}\) in \(K_r \mathcal{M}\), or equivalently their inverses

\[ \varphi^{-1}_{A,C} : (A \boxtimes C)_F \longrightarrow A_F \boxtimes_F C_F, \quad a \boxtimes c \longmapsto (F^1 \triangleright_A a) \boxtimes (F^2 \triangleright_C c), \]

are algebra maps:

\[ m_{A_F \boxtimes_F C_F} \circ (\varphi^{-1}_{A,C} \otimes \varphi^{-1}_{A,C}) = \varphi_{A,C} \circ m_{(A \boxtimes C)_F}. \]  

To lighten notations, we omit the subscripts in the actions. The product \(m_{A_F \boxtimes_F C_F}\) in the l.h.s. is the product in the braided tensor product algebra \(A_F \boxtimes_F C_F\), defined as in (3.9), but now with respect to the \(R\)-matrix \(R_F\):

\[ m_{A_F \boxtimes_F C_F} ((a \boxtimes c) \otimes (a' \boxtimes c')) = a \triangleright_R (R_{F,a} \triangleright a') \boxtimes_R (R_{F,a} \triangleright c) \triangleright_R c', \]

\[ = (F^3 \triangleright a)(F^3 \triangleright_R a'(R_{F,a} \triangleright c)(F^3 \triangleright_R c)). \]

The product \(m_{(A \boxtimes C)_F}\) in the r.h.s. is the deformation (as in (4.10)) of the product \(\boxtimes\) in the tensor product algebra \(A \boxtimes C\) in (3.9):

\[ m_{(A \boxtimes C)_F} ((a \boxtimes c) \otimes (a' \boxtimes c')) = (F^1 \triangleright (a \boxtimes c)) \boxtimes (F^1 \triangleright (a' \boxtimes c')) \]

\[ = (F^1 \triangleright a) \boxtimes (F^1 \triangleright (a' \boxtimes c')) \]

\[ = (F^1 \triangleright a)(R_{F,a} \triangleright a') \boxtimes (R_{F,a} \triangleright c)(F^1 \triangleright_R c'). \]

Then, the l.h.s. of (4.22) on the generic element \((a \boxtimes a') \otimes (c \boxtimes c')\) gives

\[ m_{A_F \boxtimes_F C_F}(\varphi^{-1}_{A,C}(a \boxtimes c) \otimes \varphi^{-1}_{A,C}(a' \boxtimes c')) = (F^3 \triangleright a)(F^3 \triangleright_R a'(R_{F,a} \triangleright c)(F^3 \triangleright_R c)). \]

Similarly, the r.h.s. of equation (4.22) gives

\[ \varphi_{A,C} \circ m_{(A \boxtimes C)_F}((a \boxtimes a') \otimes (c \boxtimes c')) = (F^1 \triangleright (a \boxtimes a')) \boxtimes (F^1 \triangleright (c \boxtimes c')). \]

Equation (4.22) then follows from (4.18) in Proposition 4.7.

4.3. Braided bialgebras. Next consider a braided bialgebra \((L, m_L, \eta_L, \Delta_L, \varepsilon_L, >_L)\), associated with the quasitriangular bialgebra \((K, R)\).

On the one hand, \((L, m_L, \eta_L, >_L)\) is a \(K\)-module algebra that we can deform using the functor in (4.9) into the \(K_r\)-module algebra \((L_F, m_{L_F}, \eta_{L_F}, >_{L_F})\); this differs from \(L\) only for the product given by \(\xi \triangleright \xi' = (F^0 \triangleright \xi)(F^0 \triangleright \xi')\), for all \(\xi, \xi' \in L_F\), as in (4.10).

On the other hand, \((L, \Delta_L, \varepsilon_L, >_L)\) is a \(K\)-module coalgebra that we can deform using the functor in (4.11) so to obtain the \(K_r\)-module coalgebra \((L_{F_r}, \Delta_{L_F}, \varepsilon_{L_F}, >_{L_F})\) with twisted coproduct \(\Delta_{L_F}(\xi) = \xi_{[1]} \otimes \xi_{[2]} := (F^0 \triangleright \xi_{[1]})(F^0 \triangleright \xi_{[2]})\) for each \(\xi \in L_F\), as in (4.12).

We have the following result (cf. [4] Prop. 4.7 for a dual result):
Proposition 4.11. The twist deformation of \((L, m_L, \eta_L, \Delta_L)\) as a \(K\)-module algebra and of \((L, \Delta_L, \varepsilon_L, \triangleright_L)\) as a \(K\)-module coalgebra is a braided bialgebra associated with the twisted Hopf algebra \(K_F\). That is, \((L, m_L, \eta_L, \Delta_L, \varepsilon_L, \triangleright_L)\) is a bialgebra in the braided monoidal category \((\mathcal{K}, \mathcal{M}, \otimes_F, \Psi^R_F)\) of \(K_F\)-modules. If \(L\) is a braided Hopf algebra, then \(L_F\) is a braided Hopf algebra, with antipode \(S_{L_F} = \Gamma(S_L)\) (equal to \(S_L\) as a linear map).

Proof. We have to show that \(\Delta_{L_F} : L_F \to L_F \otimes_F L_F\) and \(\varepsilon_{L_F} : L_F \to \mathbf{k}\) are algebra maps:
\[
\varepsilon_{L_F} \circ m_{L_F} = m_{\mathbf{k}} \circ (\varepsilon_{L_F} \otimes \varepsilon_{L_F}), \quad \Delta_{L_F} \circ m_{L_F} = m_{L_F \otimes_F L_F} \circ (\Delta_{L_F} \otimes_F \Delta_{L_F}),
\]
(4.23)
for the product in \(L_F\), \(m_{L_F}(\xi \otimes \zeta) = (F^\alpha \triangleright \xi)(F^\alpha \triangleright \zeta)\), with \(\xi, \zeta \in L_F\) and the product \(m_{L_F \otimes_F L_F} = (m_{L_F} \otimes m_{L_F}) \circ (id_{L_F} \otimes_F \Psi^R_{L_F \otimes L_F} \otimes_F id_{L_F})\) in the braided tensor product algebra \(L_F \otimes_F L_F\). This latter is given from (3.9) by
\[
m_{L_F \otimes_F L_F}((\xi \otimes_F \zeta) \otimes (\xi' \otimes_F \zeta')) = (F^\beta \triangleright \xi)(F^\beta \triangleright \zeta) \otimes_F (F^\gamma \triangleright \xi')(F^\gamma \triangleright \zeta').
\]

The identity involving the counit follows from the algebra property of \(\varepsilon_L\), by using the module coalgebra axiom (2.6) and the fact that \(F\) is unital.

For the second identity, with coproduct \(\Delta_{L_F} : \xi \mapsto \xi_{(1)} \otimes \xi_{(2)} := (F^\alpha \triangleright \xi_{(1)}) \otimes (F^\alpha \triangleright \xi_{(2)})\), we compute the l.h.s. by using the module coalgebra axiom (2.6) and the property of the coproduct \(\Delta_L\) to be a braided algebra map: on the generic element \(\xi \otimes \zeta\) one has
\[
l.h.s. = \Delta_{L_F} \circ m_{L_F}(\xi \otimes \zeta)
\]
\[
= \Delta_{L_F} \left( (F^\alpha \triangleright \xi)(F^\alpha \triangleright \zeta) \right)
\]
\[
= F^\gamma \triangleright ((F^\alpha \triangleright \xi_{(1)})R_{\beta} \triangleright (F^\alpha \triangleright \zeta_{(1)}) \otimes F^\gamma \triangleright (R^\beta \triangleright (F^\alpha \triangleright \xi_{(2)})F^\alpha \triangleright \zeta_{(2)})
\]
\[
= F^\gamma \triangleright ((F^\alpha \triangleright \xi_{(1)})R_{\beta} F^\alpha_{(1)} \triangleright \zeta_{(1)}) \otimes F^\gamma \triangleright ((R^\beta F^\alpha_{(2)} \triangleright \xi_{(2)}(F^\alpha_{(2)} \triangleright \zeta_{(2)}))
\]
\[
= (F^\gamma_{(1)}F^\gamma \triangleright \xi_{(1)})(F^\gamma_{(2)}R_{\beta} F^\alpha_{(1)} \triangleright \zeta_{(1)}) \otimes (F^\gamma_{(1)}R_{\beta} F^\alpha_{(2)} \triangleright \xi_{(2)})(F^\gamma_{(2)} F^\alpha_{(2)} \triangleright \zeta_{(2)})
\]

Similarly, we compute the r.h.s.:
\[
r.h.s. = m_{L_F \otimes_F L_F} \circ (\Delta_{L_F} \otimes_F \Delta_{L_F})(\xi \otimes \zeta)
\]
\[
= (F^\alpha \triangleright \xi_{(1)})(F^\beta R_{\alpha} \triangleright \zeta_{(1)} \otimes F^\alpha \triangleright \xi_{(2)})(F^\beta \triangleright \zeta_{(2)})
\]
\[
= (F^\gamma F^\gamma \triangleright \xi_{(1)})(F^\beta R_{\alpha} F^\delta \triangleright \zeta_{(1)} \otimes (F^\gamma R_{\alpha} F^\gamma \triangleright \xi_{(2)})(F^\gamma F^\delta \triangleright \zeta_{(2)})
\]

The two expressions coincide from (1.13) in Proposition 4.7.

If \(L\) is a Hopf algebra with antipode \(S_L\), which is a \(K\)-module map, its image under \(\Gamma\), \(\xi \mapsto S_{L_F}(\xi) = S_L(\xi)\), is a \(K_F\)-module map. We have only to show that \(S_{L_F}\) satisfies the antipode conditions (3.11) for the twisted bialgebra \(L_F\). For any element \(\xi \in L_F\) one has
\[
m_{L_F} \circ (id_{L_F} \otimes S_{L_F}) \circ \Delta_{L_F}(\xi) = F^\gamma \triangleright (F^\beta \triangleright \xi_{(1)}) \otimes F^\alpha \triangleright (S_L(F^\beta \triangleright \xi_{(2)}))
\]
\[
= (F^\gamma F^\gamma \triangleright \xi_{(1)})(F^\beta F^\beta \triangleright \xi_{(2)}) = \varepsilon_{L_F}(\xi)1_{L_F}.
\]

Analogously one shows that \(m_{L_F} \circ (S_{L_F} \otimes id_{L_F}) \circ \Delta_{L_F} = \eta_{L_F} \circ \varepsilon_{L_F}\). \(\square\)

Example 4.12. Let \(K := (K, m_K, \eta_K, \Delta_K, \varepsilon_K, S, \triangleright_{ad})\) be the braided Hopf algebra associated with the quasitriangular Hopf algebra \((K, R)\), as in Example 3.5.

If \(F\) is a twist for \(K\), the data
\[
K_F := (K, m_K, \eta_K, \Delta_F, \varepsilon_F, S_F, \triangleright_{ad})
\]
(4.24)
is a braided Hopf algebra associated with $K_F$. From the construction above, $K_F$ has the same unit as $K_F$ while its product is the deformed one as in (4.10):

$$m_{K_F}(h \otimes k) := h \ast k = (F^\alpha \triangleright h)(F_\alpha \triangleright k)$$

Its coproduct is the deformed one as in (4.12):

$$\Delta_F(k) := F^\alpha \triangleright_{ad} (k(1)S(R_\beta)) \otimes (F_\alpha R^\beta) \triangleright_{ad} k(2).$$

The left adjoint action of $K_F$ on $K_F$ is the initial (unchanged) left adjoint action,

$$\triangleright_{ad}: K_F \otimes K_F \rightarrow K_F, \quad h \otimes k \mapsto h \triangleright_{ad} k := h(1)kS(h(2)).$$

Finally, the antipode $S_F$ of $K_F$ coincides with that of $K_F$, $S_F: k \mapsto R_\alpha S(R^\alpha \triangleright_{ad} k)$, as a linear map.

**Example 4.13.** Rather than first considering the braided Hopf algebra $K_F$ and then twisting it to $K_F$, one could start with the quasitriangular Hopf algebra $K_F$ and associate to it the braided Hopf algebra

$$K_F := (K_F, m_{K_F}, \eta_{K_F}, \Delta_F, \varepsilon_{K_F}, S_F, \triangleright_{adr}),$$

again constructed as in Example 3.5. This consists of the algebra $(K_F, m_{K_F}, \eta_{K_F})$ endowed with the same counit as $K_F$ and with the braided coproduct

$$\Delta_F(k) := k(1)_1 \otimes k(2) = F^\alpha k(1)F^\beta \otimes F_\alpha h(2)F_\beta$$

is the coproduct (4.5) of $K_F$. Here the left adjoint action is

$$\triangleright_{adr}: K_F \otimes K_F \rightarrow K_F, \quad h \otimes k \mapsto h \triangleright_{adr} k := h(1)_1kS(h(2)),$$

with $S_F$ the antipode of $K_F$: $S_F(k) = u_F S(k)u_F = F^\alpha S(F_\alpha)S(k)S(F^\delta)F_\beta$. On elements,

$$h \otimes k \mapsto h \triangleright_{adr} k := F^\alpha h(1)F^\beta F\gamma S(F_\gamma)S(F_\alpha h(2)F_\beta)S(F^\delta)F_\delta.$$

With these structures $K_F$ is a $K_F$ module algebra and module coalgebra, moreover $\Delta_F$ is an algebra map (with respect to the braided tensor product). The antipode is given by

$$S_F: K_F \rightarrow K_F, \quad k \mapsto S_F(k) := R_{F_\alpha}S(R_{F^\alpha} \triangleright_{adr} k)$$

and is a left $K_F$-module morphism.

The following Proposition is dual to Theorem 4.9 in [6]; its proof is in Appendix A.

**Proposition 4.14.** The two braided Hopf algebras $K_F$ and $K_F$ associated with $K_F$ are isomorphic via the map (cf. (4.15))

$$D: K_F \rightarrow K_F, \quad k \mapsto D(k) := (F^\alpha \triangleright_{ad} k)F_\alpha = F^\alpha kS(F_\alpha)u_F$$

with inverse

$$D^{-1}: K_F \rightarrow K_F, \quad k \mapsto D^{-1}(k) = (F_\alpha \triangleright_{adr} k)F_\alpha = F^\alpha kS(F_\alpha)u_F.$$
4.4. Dual pairings of braided bialgebras and their twisting.

**Definition 4.15.** A dual pairing of two braided bialgebras \((L, m_L, η_L, ∆_L, ε_L, ▷_L)\) and \((N, m_N, η_N, ∆_N, ε_N, ▷_N)\) associated with the quasitriangular bialgebra \((K, R)\) is a bilinear map \(⟨·, ·⟩ : L \otimes N → k\) such that

\[
⟨ξ, xy⟩ = ⟨Δ_L ξ, x ⊗ y⟩ := ⟨ξ(1), R_α ▷ N x⟩(R^α ▷ L ξ(2), y)
\]

\[
⟨ξη, x⟩ = ⟨ξ ⊗ η, ∆_N x⟩ := ⟨ξ, R_α ▷ N x(1)⟩(R^α ▷ L η, x(2))
\]

(4.30)

and

\[
⟨ξ, 1_N⟩ = ε_L(ξ), \quad ⟨1_L, x⟩ = ε_N(x)
\]

(4.31)

for all \(ξ, η ∈ L\) and \(x, y ∈ N\). Moreover \(⟨·, ·⟩ : L \otimes N → k\) is a morphism of \(K\)-modules:

\[
⟨k_1 ▷ L ξ, k_2 ▷ N x⟩ = ε(k)⟨ξ, x⟩
\]

(4.32)

for each \(k ∈ K\), \(ξ ∈ L\) and \(x ∈ N\). We say that \(L\) and \(N\) are dually paired if they admit a dual pairing.

**Lemma 4.16.** When \(K\) is a Hopf algebra, the condition for the pairing \(⟨·, ·⟩\) to be a morphism of \(K\)-modules can be given equivalently as

\[
⟨k ▷ L ξ, x⟩ = ⟨ξ, S(k) ▷ N x⟩, \quad ∀ ξ ∈ L, x ∈ N,
\]

(4.33)

or, with \(S^{-1}\) the inverse of the antipode of \(K\),

\[
⟨ξ, k ▷ N x⟩ = ⟨S^{-1}(k) ▷ L ξ, x⟩, \quad ∀ ξ ∈ L, x ∈ N.
\]

(4.34)

**Proof.** If (4.32) holds, then for all \(ξ ∈ L, x ∈ N\) we have

\[
⟨k ▷ L ξ, x⟩ = ⟨k(1) ▷ L ξ, ε(k(2)) ▷ N x⟩ = ⟨k(1) ▷ L ξ, k(2) ▷ N (S(k(3)) ▷ N x))⟩ = ⟨ξ, S(k) ▷ N x⟩.
\]

Conversely, if (4.33) holds, then

\[
⟨k_1 ▷ L ξ, k_2 ▷ N x⟩ = ⟨ξ, S(k(1)) ▷ N (k_1(2) ▷ N x))⟩ = ⟨ξ, (S(k(1))k(2)) ▷ N x⟩ = ε(k)⟨ξ, x⟩.
\]

Being the antipode of a quasitriangular Hopf algebra invertible (see (3.7)), identity (4.33) gives (4.34), when replacing \(k\) by \(S^{-1}(k)\).

**Lemma 4.17.** If \(⟨·, ·⟩ : L \otimes N → k\) is a dual pairing between two braided Hopf algebras \(L\) and \(N\) associated with the quasitriangular Hopf algebra \((K, R)\), then

\[
⟨S_L(ξ), x⟩ = ⟨ξ, S_N(x)⟩, \quad ∀ ξ ∈ L, x ∈ N.
\]

(4.35)

**Proof.** We omit the subscripts \(L\) and \(N\). Using the pairing axioms (4.30), we compute

\[
⟨S(ξ), x⟩ = ⟨S(ξ(1)), R_α ▷ x(1)⟩ε(R^α ▷ ξ(2))ε(x(2))
\]

\[
= ⟨S(ξ(1)), R_α ▷ x(1)⟩(R^α ▷ ξ(2), x(2)S(x(3))
\]

\[
= ⟨S(ξ(1)), R_α ▷ x(1)⟩(R^α ▷ ξ(2), (R^β ▷ ξ(3)))
\]

\[
= ⟨S(ξ(1)), R_α ▷ x(1)⟩(Rα ▷ ξ(2), x(2))⊥ (R^α ▷ ξ(2), x(2))
\]

\[
= ⟨S(ξ(1)), R_α ▷ x(1)⟩(R^α ▷ ξ(2), x(2))\langle R^α ▷ ξ(2), S(x(3))⟩
\]

where we used the module coalgebra axioms (2.6) in the last equality. Next, we use the quasitriangularity condition \((Δ \otimes id)R = R_{13}R_{23}\) and get

\[
⟨S(ξ), x⟩ = ⟨S(ξ(1)), R_α R_β ▷ x(1)⟩(R^α ▷ ξ(2), x(2))\langle R^α ▷ ξ(2), S(x(3))⟩.
\]

On the other hand, with analogous computations we have

\[
⟨ξ, S(x)⟩ = ε(ξ(1))ε(R_γ ▷ x(1))\langle R^γ ▷ ξ(2), S(x(3))⟩
\]

\[
= ⟨S(ξ(1)), R_γ ▷ x(1)⟩(R^γ ▷ ξ(2), x(2))\langle R^γ ▷ ξ(2), S(x(3))⟩
\]

\[
= ⟨S(ξ(1)), R_γ ▷ (R^γ ▷ x(1))⟩\langle R^γ ▷ ξ(2), R_γ ▷ x(1)⟩\langle R^γ ▷ ξ(2), S(x(3))⟩.
\]

\[
= ⟨S(ξ(1)), R_α ▷ (R_γ ▷ x(1))⟩\langle R^γ ▷ ξ(2), x(2))\langle R^γ ▷ ξ(2), S(x(3))⟩
\]

\[
= ⟨S(ξ(1)), R_α ▷ (R_γ ▷ x(1))⟩\langle R^γ ▷ ξ(2), x(2))\langle R^γ ▷ ξ(2), S(x(3))⟩.
\]
Finally, the quasitriangular condition (id ⊗ Δ)R = R_{12}R_{13} gives
\langle ξ, S(x) \rangle = \langle S(ξ(1)), R_α R_γ (1) ▷ x(1) \rangle \langle R^α ▷ ξ(2), R_γ (2) ▷ x(2) \rangle \langle R^γ ▷ ξ(3), S(x(3)) \rangle.

Proposition 4.18. Let \langle -, - \rangle : L ⊗ N → k be a dual pairing between two braided bialgebras (L, m_L, η_N, Δ_L, ε_L, ▷_L) and (N, m_N, η_N, Δ_N, ε_N, ▷_N) associated with the quasitriangular bialgebra (K, R). Then the braided bialgebras (L_F, m_{L_F}, η_{L_F}, Δ_{L_F}, ε_{L_F}, ▷_{L_F}) and (N_F, m_{N_F}, η_{N_F}, Δ_{N_F}, ε_{N_F}, ▷_{N_F}) associated with the quasitriangular bialgebra (K_F, R_F) (as given in Proposition 4.17) are dually paired via
\langle -, - \rangle_F : L_F ⊗ N_F → k,
\xi ⊗_N x → \langle ξ, x \rangle_F := \langle F^\xi ▷ L, F_α ▷ N, x \rangle

Proof. Firstly we show that this map \langle -, - \rangle_F is a K_F-module map. With the coproduct for K_F given in (4.35), we have
\langle k_{[1]} ▷ L, k_{[2]} ▷ N, x \rangle_F = \langle F^α k_{(1)} ▷ L, F_α k_{(2)} ▷ N, x \rangle_F = \langle k_{(1)} ▷ L, k_{(2)} ▷ N, x \rangle_F
= ε(k) \langle F^β ▷ L, F_β ▷ N, x \rangle_F = ε(k) \langle ξ, x \rangle_F

where we have used that the pairing \langle -, - \rangle : L ⊗ N → k is a K-module map. Next, being F unital, the compatibility properties (4.31) with the counits are easily verified. We show the first of properties (4.30), the other one can be established similarly. To lighten the notation we omit the subscripts. We have
\langle ξ, x ∗ y \rangle_F = \langle F^α ▷ ξ, F_α ▷ ((F^β ▷ x)(F_β ▷ y)) \rangle
= \langle F^α ▷ ξ, (F_α ▷ F^β ▷ x)(F_α ▷ F_β ▷ y) \rangle
= \langle F^α ▷ ξ, F_α ▷ (F^β ▷ x)(F_α ▷ F_β ▷ y) \rangle
= \langle F^α ▷ ξ, F_α ▷ (F^β ▷ x)(F_α ▷ F_β ▷ y) \rangle
= \langle F^α ▷ ξ, F_α ▷ (F^β ▷ x)(F_α ▷ F_β ▷ y) \rangle

where for the last two identities we have used the twist condition (4.4), followed by the module coalgebra condition (2.6). This can be further simplified by using the quasitriangular condition (3.1) of the coproduct of K and then the property (3.32) for the pairing to be a K-module map:
\langle F^α ▷ ξ, F^β ▷ x, F_α ▷ (F^γ ▷ R^F ▷ ξ, F_α ▷ y) \rangle
= \langle F^α ▷ ξ, R_γ F_β ▷ x, F_α ▷ y \rangle

By using again the twist condition and the coproduct in L_F, we have
\langle F^\gamma ▷ ξ, F_β ▷ x, R_γ F_μ ▷ y \rangle \langle F^\gamma ▷ R^F ▷ ξ, F_α ▷ y \rangle
= \langle F^3 ▷ ξ, R_γ F_β ▷ x, F^\gamma ▷ R^F ▷ ξ, F_α ▷ y \rangle
= \langle F^3 ▷ ξ, F_β ▷ R_γ F_μ ▷ x, F^\gamma ▷ R^F ▷ ξ, F_α ▷ y \rangle
= \langle F^3 ▷ ξ, F_β ▷ R_γ F_μ ▷ x, F^\gamma ▷ R^F ▷ ξ, F_α ▷ y \rangle

where in the last but one equality we have used once again the quasitriangular condition (3.1) in the Definition 3.1 of the coproduct and in the last one the definition of R_F. Then, the twist condition and the K-module map property (3.32) of the pairing yields
\langle F^\gamma ▷ ξ, F_β ▷ x, R_γ F_μ ▷ y \rangle \langle F^\gamma ▷ R^F ▷ ξ, F_α ▷ y \rangle


\[
\begin{align*}
= (F^3 \triangleright \xi_{[1]}, F_\beta R_\gamma \triangleright x) (F^\gamma R_\alpha \triangleright \xi_{[2]}, F_\alpha \triangleright y) \\
= (\xi_{[1]}, R_{F_\alpha} \triangleright x) F (R_\alpha \triangleright \xi_{[2]}, y) F,
\end{align*}
\]
thus concluding the proof.

5. BRAIDED LIE ALGEBRAS AND THEIR TWISTING

Further insights in the use of braided Hopf algebras as algebras of symmetries come from the study of associated braided Lie algebras: Lie algebras in the braided monoidal category of \( K \)-modules, with \((K, R)\) triangular.

5.1. Braided Lie algebras.

Definition 5.1. A **braided Lie algebra** associated with a triangular Hopf algebra \((K, R)\), or simply a \(K\)-braided Lie algebra, is a \(K\)-module \(g\) with a bilinear map (a nonassociative non-unital multiplication)

\[
[\ , \ ] : g \otimes g \rightarrow g
\]
that satisfies the conditions:

(i) \(K\)-equivariance:

\[
k \triangleright [u, v] = [k_{(1)} \triangleright u, k_{(2)} \triangleright v],
\]

(ii) braided antisymmetry:

\[
[u, v] = - [R_\alpha \triangleright v, R^\alpha \triangleright u],
\]

(iii) braided Jacobi identity:

\[
[u, [v, w]] = [[u, v], w] + [R_\alpha \triangleright v, [R^\alpha \triangleright u, w]],
\]
for all \(u, v, w \in g, k \in K\). A \(K\)-braided Lie algebra morphism \(\varphi : g \rightarrow g'\) is a morphism of the \(K\)-modules \(g\) and \(g'\) such that \(\varphi \circ [\ , \ ] = [\ , \ ]' \circ (\varphi \otimes \varphi)\), for \([\ , \ ]\) and \([\ , \ ]'\) the brackets of \(g\) and \(g'\) respectively.

We add the subscript \([\ , \ ]_R\) to the bracket when we need to emphasize the \(R\)-matrix structure we are using.

Lemma 5.2. Let \((K, R)\) be a triangular Hopf algebra. A \(K\)-module algebra \(A\) is a \(K\)-braided Lie algebra with bracket

\[
[\ , \ ] : A \otimes A \rightarrow A, \quad a \otimes b \mapsto [a, b] = ab - (R_\alpha \triangleright b) (R^\alpha \triangleright a).
\]

(5.1)

Proof. The \(K\)-equivariance follows from the \(K\)-module algebra property (2.4) of \(A\) and quasi-cocommutativity of \(K\):

\[
k \triangleright [a, b] = (k_{(1)} \triangleright a)(k_{(2)} \triangleright b) - (k_{(1)} R_\alpha \triangleright b) (k_{(2)} R^\alpha \triangleright a)
\]

\[
= (k_{(1)} \triangleright a)(k_{(2)} \triangleright b) - (R_\alpha k_{(2)} \triangleright b)(R^\alpha k_{(1)} \triangleright a)
\]

\[
= [k_{(1)} \triangleright a, k_{(2)} \triangleright b].
\]

The braided antisymmetry follows from the triangularity of \(K\):

\[
[R_\alpha \triangleright b, R^\alpha \triangleright a] = (R_\alpha \triangleright b)(R^\alpha \triangleright a) - (R_\alpha R^\alpha \triangleright a) (R^\beta R_\alpha \triangleright b)
\]

\[
= (R_\alpha \triangleright b)(R^\alpha \triangleright a) - ab = -[a, b].
\]
We are left to prove Jacobi identity. On the one hand, using \( K \)-equivariance of \([\, , \,] \) just shown, and condition \((\text{id} \otimes \Delta) R = R_{13} R_{12} \) together with quasi-co-commutativity, we have

\[
[a, [b, c]] = a[b, c] - (R_\alpha \triangleright [b, c])(R^\alpha \triangleright a)
= a[b, c] - [R_{\alpha(1)} \triangleright b, R_{\alpha(2)} \triangleright c](R^\alpha \triangleright a)
= abc - a(R_\beta \triangleright c)(R^\beta \triangleright b) - (R_\alpha \triangleright b)(R_{\alpha(2)} \triangleright c)(R^\alpha \triangleright a)
+ (R_\beta R_{\alpha(2)} \triangleright c)(R^\beta R_{\alpha(1)} \triangleright b)(R^\alpha \triangleright a)
= abc - a(R_\beta \triangleright c)(R^\beta \triangleright b) - (R_\alpha \triangleright b)(R_{\alpha(2)} \triangleright c)(R^\alpha \triangleright a)
+ (R_{\alpha(1)} R_\beta \triangleright c)(R_{\alpha(2)} R^\beta \triangleright b)(R^\alpha \triangleright a)
= abc - a(R_\beta \triangleright c)(R^\beta \triangleright b) - (R_\gamma \triangleright b)(R_\rho \triangleright c)(R^\alpha R^\gamma \triangleright a)
+ (R_\alpha R_\beta \triangleright c)(R_\rho R^\beta \triangleright b)(R^\gamma R^\alpha \triangleright a).
\]

On the other hand, using also \((\Delta \otimes \text{id}) R = R_{13} R_{23}\), analogous computations give

\[
[[a, b], c] + [R_\alpha \triangleright b, [R^\alpha \triangleright a, c]]
= [a, b]c - (R_\beta \triangleright c)(R^\beta \triangleright [a, b]) + (R_\alpha \triangleright b)[R^\alpha \triangleright a, c] - (R_\beta \triangleright [R^\alpha \triangleright a, c])(R^\beta R_\alpha \triangleright b)
= [a, b]c - (R_\beta \triangleright c)(R^\beta \triangleright [a, b]) + (R_\alpha \triangleright b)[R^\alpha \triangleright a, c]
- [R_{\beta(1)} R^\alpha \triangleright a, R_{\beta(2)} \triangleright c](R^\beta R_\alpha \triangleright b)
= [a, b]c - (R_\gamma R_\beta \triangleright c)(R^\gamma \triangleright a, R^\beta \triangleright b) + (R_\alpha \triangleright b)[R^\alpha \triangleright a, c]
- [R_\gamma R^\alpha \triangleright a, R_\beta \triangleright c](R^\beta R^\gamma R_\alpha \triangleright b)
= [a, b]c - (R_\gamma R_\beta \triangleright c)(R^\gamma \triangleright a, R^\beta \triangleright b) + (R_\alpha \triangleright b)[R^\alpha \triangleright a, c]
- [a, R_\beta \triangleright c](R^\beta \triangleright b) + (R_\alpha \triangleright b)[R^\alpha \triangleright a, c]
= abc - a(R_\beta \triangleright c)(R^\beta \triangleright b) - (R_\alpha \triangleright b)(R_{\alpha(2)} \triangleright c)(R^\alpha \triangleright a)(R^\beta \triangleright b)
+ (R_\gamma R_\beta \triangleright c)(R_{\gamma} R^\beta \triangleright b)(R^\gamma R^\alpha \triangleright a)
- (R_\alpha \triangleright b)(R_{\beta} \triangleright c)(R^\beta R^\alpha \triangleright a) - a(R_\beta \triangleright c)(R^\beta \triangleright b) + (R_\alpha R_\beta \triangleright c)(R^\alpha \triangleright a)(R^\beta \triangleright b)
\]

where for the fourth equality we have used the triangularity of \( R \). This expression equals that for \([a, [b, c]]\) found above.

\( \square \)

**Remark 5.3.** Let \((K, R)\) be a triangular Hopf algebra. A \( K \)-braided Poisson algebra \((P, \cdot, [\, , \,])\) is a \( K \)-module algebra \((P, \cdot)\), and a \( K \)-braided Lie algebra \((P, [\, , \,])\) such that the bracket is a braided derivation of the multiplication: for all \( p, r, s \in P \),

\[
[p, r \cdot s] = [p, r] \cdot s + R_\alpha \triangleright r \cdot [R^\alpha \triangleright p, s].
\] (5.2)

In particular, any \( K \)-braided algebra \((A, \cdot)\) is canonically a \( K \)-braided Poisson algebra \((A, \cdot, [\, , \,])\) with the bracket defined in \([5,1]\).

An enveloping algebra of \( \mathfrak{g} \) is the datum \((A, \alpha)\) of a \( K \)-module algebra \( A \), with braided Lie algebra structure as in Lemma \([5,2]\) and a braided Lie algebra morphism \( \alpha : \mathfrak{g} \to A \), that is a \( K \)-equivariant map such that \( \alpha([u, v]) = [\alpha(u), \alpha(v)] \), for all \( u, v \in \mathfrak{g} \).

The universal enveloping algebra \( \mathcal{U}(\mathfrak{g}, \iota) \) of \( \mathfrak{g} \) is characterised by the following universal property: given an enveloping algebra \((A, \alpha)\) there is a unique lift of \( \alpha \) to a \( K \)-equivariant algebra map \( \hat{\alpha} : \mathcal{U}(\mathfrak{g}) \to A \), such that \( \hat{\alpha} \circ \iota = \alpha : \mathfrak{g} \to A \).
Existence of \((\mathcal{U}(g), \iota)\) follows from considering the usual tensor algebra \(T(g)\), canonically a \(K\)-module algebra, and the quotient \(\mathcal{U}(g) := T(g)/I\) where \(I\) is the algebra ideal generated by \(u \otimes v - R_a \triangleright v \otimes R^a \triangleright u - [u, v]\). The quotient is a \(K\)-module algebra since the ideal \(I\) is preserved by the \(K\)-action due to the quasi-cocommutativity of \(K\). The \(K\)-equivariant map \(\hat{\iota} : g \rightarrow T(g), u \mapsto \hat{\iota}(u) = u\) induces the braided Lie algebra morphism \(\iota : g \rightarrow \mathcal{U}(g)\) on the quotient. As in the classical case, the universality of \((\mathcal{U}(g), \iota)\) follows from that of \(T(g)\) and uniqueness (up to isomorphism) from the universal property.

**Proposition 5.4.** The universal enveloping algebra \(\mathcal{U}(g)\) of a braided Lie algebra \(g\) is a braided Hopf algebra associated with the triangular Hopf algebra \((K, R)\).

**Proof.** We first show that the tensor algebra \(T(g)\) is a braided Hopf algebra. We define the coproduct \(\Delta(u) = u \boxtimes 1 + 1 \boxtimes u\) for degree one elements \(u \in g\) and extend it to all \(T(g)\) by requiring it to be a unital algebra map between \(T(g)\) and the braided tensor algebra \(T(g) \boxtimes T(g)\) (cf. Proposition 3.3):

\[
\Delta(\zeta \otimes \xi) = (\zeta(1) \otimes R_\alpha \triangleright \xi(1)) \boxtimes (R^\alpha \triangleright \zeta(2) \otimes \xi(2))
\]

for all \(\zeta, \xi \in T(g)\). Coassociativity of \(\Delta\) on degree one elements is trivial, it is then easy to see that if it holds for elements \(\zeta\) and \(\xi\) in \(T(g)\) it holds also for their product. The counit \(\varepsilon : T(g) \rightarrow k\) is the unital algebra map defined by \(\varepsilon(u) = 0\) for all \(u \in g\). Then \((T(g), \Delta, \varepsilon)\) is a \(K\)-module coalgebra as in (2.6). Indeed, the \(K\)-module coalgebra property \(\Delta(k \triangleright \zeta) = k(1) \triangleright \zeta(1) \boxtimes k(2) \triangleright \zeta(2)\) holds on degree one elements and, using quasi-cocommutativity of \(K\), it is seen to hold on the product \(\zeta \otimes \xi\) if it holds on the single elements \(\zeta, \xi \in T(g)\). The condition on the counit is trivially satisfied. Since \(\Delta\) has been defined as an algebra map \(T(g) \rightarrow T(g) \boxtimes T(g)\) we have that \(T(g)\) is a braided bialgebra associated with \((K, R)\). It is a braided Hopf algebra with antipode the \(K\)-equivariant map \(S(u) = -u\) for \(u \in g\), extended to \(T(g)\) as braided anti-algebra map.

Finally, the quotient \(\mathcal{U}(g) = T(g)/I\) is a braided Hopf algebra since the ideal \(I = \langle u \otimes v - R_a \triangleright v \otimes R^a \triangleright u - [u, v]\rangle\) is a braided Hopf ideal. Indeed, as already mentioned \(I\) is a \(K\)-module; it is also a coideal since

\[
\Delta(u \otimes v - R_a \triangleright v \otimes R^a \triangleright v - [u, v]) \in I \boxtimes T(g) + T(g) \boxtimes I
\]

for all \(u, v \in g\), as can be shown by using triangularity of the universal \(R\) matrix. \(\Box\)

**Remark 5.5.** Over a field of characteristic zero a braided Lie algebra \(g\) is the \(K\)-submodule in \(\mathcal{U}(g)\) of primitive elements: \(\text{Prim}(\mathcal{U}(g)) = g\). More in general, a connected braided cocommutative Hopf algebra \(L\) is the universal enveloping algebra of the braided Lie algebra \(g = \text{Prim}(L)\) of its primitive elements, see [23].

### 5.2. Braided Lie algebras of braided derivations

We now study derivations of \(K\)-module algebras.

Let \(A\) be a \(K\)-module algebra and \((\text{Hom}(A, A), \triangleright_{\text{Hom}(A,A)})\) the \(K\)-module of linear maps from \(A\) to \(A\) as in (2.3). We denote

\[
\text{Der}(A) := \{\psi \in \text{Hom}(A, A) \mid \psi(aa') = \psi(a)a' + (R_a \triangleright a)(R^a \triangleright_{\text{Hom}(A,A)} \psi)(a')\}\quad (5.3)
\]

the \(K\)-module of braided derivations of \(A\). We add the superscript \(R\) and write \(\text{Der}^R(A)\) when we wish to emphasize the role of the braiding. Braided derivations are a \(K\)-submodule of \(\text{Hom}(A, A)\):
Lemma 5.6. For \((K, R)\) quasi-triangular, \(\text{Der}(A)\) is a \(K\)-module with action given by the restriction of \(\triangleright_{\text{Hom}(A, A)}\):

\[
\triangleright_{\text{Der}(A)}: K \otimes \text{Der}(A) \to \text{Der}(A)
\]

\[
k \otimes \psi \mapsto k \triangleright_{\text{Der}(A)} \psi : a \mapsto k(1) \triangleright \psi(S(k(2)) \triangleright a).
\]

(5.4)

Proof. We only need to show that the restriction of \(\triangleright_{\text{Hom}(A, A)}\) to \(\text{Der}(A)\) has image in \(\text{Der}(A)\), that is, \(k \triangleright_{\text{Hom}(A, A)} \psi\) is a braided derivation if \(\psi\) is such, for each \(k \in K\).

\[
(k \triangleright \psi)(ab) = k(1) \triangleright \psi(S(k(2)) \triangleright (ab))
\]

\[
= k(1) \triangleright \psi((S(k(3)) \triangleright a)(S(k(2)) \triangleright b))
\]

\[
= k(1) \triangleright \left(\psi(S(k(3)) \triangleright a)(S(k(2)) \triangleright b) + (R_a S(k(3)) \triangleright a) (R^a \triangleright \psi)(S(k(2)) \triangleright b)\right)
\]

\[
= k(1) \triangleright \psi(S(k(4)) \triangleright a) (k(2) S(k(3)) \triangleright b)
\]

\[
+ (k(1)R_a S(k(4)) \triangleright a) k(2) \triangleright \left((R^a \triangleright \psi)(S(k(3)) \triangleright b)\right)
\]

using that \(\psi\) is a braided derivation and \(A\) is a module algebra. Then, the first term in the sum simplifies to \((k \triangleright \psi)(a)b\), while for the second term we have

\[
(k_1R_a S(k(4)) \triangleright a) k_2 \triangleright \left((R^a \triangleright \psi)(S(k(3)) \triangleright b)\right)
\]

\[
= (k_1R_a S(k(4)) \triangleright a) (k_2 R^a(1)) \triangleright \psi(S(k(3))R^a(2)) \triangleright b)
\]

\[
= (k_1R_a R_\beta S(k(4)) \triangleright a) (k_2 R^a) \triangleright \psi(S(k(3))R^\beta) \triangleright b)
\]

\[
= (R_a k(2)S(R_\beta k(3)) \triangleright a) (R^a k(1)) \triangleright \psi(S(R^\beta k(4)) \triangleright b)
\]

\[
= (R_a R_\beta \triangleright a) (R^a k(1)) \triangleright \psi(S(R^\beta k(4)) \triangleright b)
\]

using quasitriangularity for the second equality, \((\text{id} \otimes S)\overline{R} = R\) for the third one, and quasi-cocommutativity twice. Finally, again by quasitriangularity this is rewritten as

\[
(R_a \triangleright a) \left((R^a k(1)) \triangleright \psi(S(R^a k(2)) \triangleright b) = (R_a \triangleright a) \left((R^a k) \triangleright \psi\right)(b)
\]

thus showing that \(k \triangleright \psi\) is a braided derivation:

\[
(k \triangleright \psi)(ab) = (k \triangleright \psi)(a)b + (R_a \triangleright a) (R^a \triangleright (k \triangleright \psi))(b).
\]

This ends the proof. \(\square\)

As in Lemma 5.2 when the Hopf algebra \(K\) is triangular, associated with the \(K\)-module algebra \((\text{Hom}(A, A), \circ)\) there is the braided Lie algebra \((\text{Hom}(A, A), [ , ]\)) where \([ , ]\) is the braided commutator as in (5.1).

For \(a \in A\) we define

\[
\ell_a \in \text{Hom}(A, A) , \quad \ell_a : A \to A , \quad a' \mapsto \ell_a(a') = aa' .
\]

(5.5)

Using the \(K\)-module structure (2.3) of \(\text{Hom}(A, A)\), one shows that \(\ell : A \to \text{Hom}(A, A)\), \(a \mapsto \ell_a\), is \(K\)-equivariant, \(k \triangleright \ell_a = \ell_{ka}\) for all \(k \in K\), \(a \in A\). It follows that a linear map \(\psi \in \text{Hom}(A, A)\) is a \((K, R)\)-braided derivation, \(\psi(ab) = \psi(a)b + (R_a \triangleright a) (R^a \triangleright \psi)(b)\), if and only if \(\psi(\ell_a(b)) = \ell_{\psi(a)} (b) + (\ell_{R_a \circ a} \circ (R^a \triangleright \psi))(b)\), that is if and only if

\[
[\psi, \ell_a] = \ell_{\psi(a)}
\]

(5.6)

for all \(a \in A\).
Proposition 5.7. Let \((K,R)\) be triangular. For each \(K\)-module algebra \(A\), the \(K\)-module of braided derivations with

\[
\begin{align*}
[\ , \ ] : \text{Der}(A) \otimes \text{Der}(A) & \to \text{Der}(A) \\
\psi \otimes \lambda & \mapsto [\psi, \lambda] := \psi \circ \lambda - (R_\alpha \triangleright_{\text{Der}(A)} \lambda) \circ (R^\alpha \triangleright_{\text{Der}(A)} \psi)
\end{align*}
\tag{5.7}
\]

is a braided Lie subalgebra of \(\text{Hom}(A,A)\).

Proof. We only need to show that the image of \([\ , \ ]\) is in \(\text{Der}(A)\). Let \(\psi, \lambda \in \text{Der}(A)\) and \(a \in A\). Using the characterization (5.6) of braided derivations and the Jacobi identity in \(\text{Hom}(A,A)\), we have

\[
\begin{align*}
[[\psi, \lambda], \ell_a] &= [\psi, [\lambda, \ell_a]] - [R_\alpha \triangleright \lambda, [R^\alpha \triangleright \psi, \ell_a]] \\
&= [\psi, \ell_{\lambda(a)}] - [R_\alpha \triangleright \lambda, \ell_{(R^\alpha \triangleright \psi)(a)}] \\
&= \ell_{\psi(\lambda(a))} - \ell_{(R_\alpha \triangleright \lambda)(R^\alpha \triangleright \psi)(a)} \\
&= \ell_{\psi, \lambda(a)}
\end{align*}
\]

and thus \([\psi, \lambda]\) is a braided derivation. \(\Box\)

Recall that the \(K\)-module algebra \(A\) is \textit{quasi-commutative} (or braided commutative) when

\[
a \cdot a' = (R_\alpha \triangleright a')(R^\alpha \triangleright a),
\tag{5.8}
\]

for all \(a, a' \in A\). In this case the braided Lie algebra \(\text{Der}(A)\) is also a left \(A\)-submodule of \(\text{Hom}(A,A)\) by defining

\[
(a\psi)(c) := a \psi(c)
\tag{5.9}
\]

for \(\psi \in \text{Hom}(A,A), a, c \in A\), that is, \(a\psi := \ell_a \circ \psi\). Indeed, for \(a \in A, \psi \in \text{Der}(A)\) one has:

\[
\begin{align*}
\psi(cc') &= \psi(c)c' + a(R_\alpha \triangleright c)(R^\alpha \triangleright \psi)(c') = \psi(c)c' + (R_\beta R_\alpha \triangleright c)(R^\beta \triangleright a)(R^\alpha \triangleright \psi)(c') \\
&= \psi(c)c' + (R_\beta \triangleright c)(R^\alpha_{(1)} \triangleright a)(R^\alpha_{(2)} \triangleright \psi)(c') = \psi(c)c' + (R_\beta \triangleright a)(R^\alpha \triangleright \psi)(c')
\end{align*}
\]

The compatibility with the \(K\)-action, as in (2.3), shows that \(\text{Der}(A)\) is a \((K,A)\)-relative Hopf module. The compatibility of the \(A\)-module and the braided Lie algebra structures follows from \((\text{Hom}(A,A),\circ,[\ , \ ])\) being a braided Poisson algebra (cf. Remark 5.3).

Proposition 5.8. Let \((K,R)\) be triangular and let \(A\) be a quasi-commutative \(K\)-module algebra. The Lie bracket of \(\text{Der}(A)\) is a derivation of the \(A\)-module \(\text{Der}(A)\), that is, it satisfies, for all \(a, a' \in A, \psi, \psi' \in \text{Der}(A)\),

\[
[[\psi, a'], \psi'] = \psi(a')\psi' + R_\alpha \triangleright a'[R^\alpha \triangleright \psi, \psi']
\]

and more generally

\[
[[a\psi, a'\psi'] = a\psi(a')\psi' + a(R_\alpha \triangleright a')[R^\alpha \triangleright \psi, \psi'] \\
+ R_\gamma R_\alpha \triangleright a'(R_\beta R_\gamma \triangleright \psi')(R^\beta R^\gamma \triangleright a)R^\alpha R^\gamma \triangleright \psi
\tag{5.10}
\]

Remark 5.9. The knowledge of the bracket on a \(K\)-submodule \(X\) of \(\text{Der}(A)\) (not necessarily a braided Lie subalgebra) that generates \(\text{Der}(A)\) as a left \(A\)-module is therefore sufficient to determine via (5.10) the \(K\)-braided Lie algebra \(\text{Der}(A)\).
Proof. Let \((P, \cdot, [\ , \ ])\) be a \(K\)-braided Poisson algebra as in Remark 5.3. From (5.2) and the braided antisymmetry of the bracket we have, \([p \cdot q, r] = p \cdot [q, r] + [p, R_\alpha \triangleright r] \cdot R^\alpha \triangleright q\), for all \(p, q, r \in P\). Using this and (5.2) again we obtain more generally
\[
[p \cdot q, p' \cdot q'] = p \cdot [q, p'] \cdot q' + [p, R_\alpha \triangleright p'] \cdot R^\alpha \triangleright q \cdot q' + R_\beta R_\alpha \triangleright p' \cdot R^\beta \triangleright p \cdot [R^\alpha \triangleright q, q']
\]
\[
+ R_\beta R_\alpha \triangleright p' \cdot [R^\beta \triangleright p, R_\gamma \triangleright q] \cdot R^\gamma R^\alpha \triangleright q .
\] (5.11)

We now consider the case \((P, \cdot, [\ , \ ]) = (\text{Hom}(A, A), \circ, [\ , \ ]))\). When \(A\) is quasi-commutative the left action of \(\text{Der}(A)\) is \(a \psi = \ell_a \circ \psi\) for all \(a \in A\), \(\psi \in \text{Der}(A)\), so that \([\psi, \ell_a'] \circ \psi' = \ell_{\psi(a')} \circ \psi' = \psi(a')\psi'\). Quasi-commutativity of \(A\) in the form \([\ell_a, \ell_{a'}] = 0\) for all \(a, a' \in A\) implies \([\ell_a, \ell_{R_\alpha \triangleright a}] \otimes R^\alpha = 0\). Then, one obtains equation (5.10) setting \(p = \ell_a\), \(p' = \ell_{a'}\), \(q = \psi\), \(q' = \psi'\) in (5.11).

\[\square\]

Example 5.10. Braided Lie algebra of a bicovariant differential calculus on a cotriangular Hopf algebra. Let \((A, R)\) be a cotriangular Hopf algebra with dual triangular Hopf algebra \((U, \mathcal{R})\), with pairing \(\langle \ , \rangle : U \otimes A \to k\) such that \(R(a \otimes a') = \langle \mathcal{R}, a \otimes a' \rangle := \langle \mathcal{R}^a, a \rangle \langle \mathcal{R}_a, a' \rangle\), for all \(a, a' \in A\). The opposite Hopf algebra \(U^\text{op}\) is triangular with universal matrix \(\mathcal{R}\). The algebra \(A\) is a \(U^\text{op} \otimes U\)-module via \(\triangleright : U^\text{op} \otimes U \otimes A \to A\), \(\langle \zeta \otimes \xi \rangle \triangleright a = \langle \zeta, a_{(1)} \rangle \langle \xi, a_{(2)} \rangle\). Furthermore, \(U^\text{op} \otimes U\) is triangular with \(\mathfrak{R} = (\text{id} \otimes \text{flip} \otimes \text{id})(\mathcal{R} \otimes \mathcal{R})\).

It is not difficult to see that the property \(m_{\text{op}} = R \ast m \ast R\) of the cotriangular Hopf algebra \((A, R)\) is the quasi-commutativity property (5.8) of \(A\) with respect to the triangular structure \(\mathfrak{R}\) of \(U^\text{op} \otimes U\). In this case the canonical notion of braided derivations of \(A\) is given by equation (5.3) where the braiding is the one associated with \(\mathfrak{R}\); we denote this braided Lie algebra by \(\text{Der}^3(A)\). Hence \(\text{Der}^3(A)\) is a \(U^\text{op} \otimes U\)-braided Lie algebra and a \((U^\text{op} \otimes U, A)\)-relative Hopf module, the compatibility between these structures being as in Proposition 5.8.

As shown in [1, §3.3], if \(U\) separates the elements of \(A\), the braided derivations \(\text{Der}^3(A)\) define a bicovariant differential calculus on \(A\), the unique bicovariant differential calculus compatible with the cotriangular structure \(R\) studied in [21, §4.3]. The braided derivations \(\text{Der}^3(A)\) give the \(A\)-bicovariant bimodule dual to that of one-forms.

By the fundamental theorem of bicovariant bimodules \(\text{Der}^3(A) = A \otimes \text{Der}^3(A)_{\text{inv}}\), that is, braided derivations are freely generated as a left \(A\)-module by the right-invariant ones
\[
\text{Der}^3(A)_{\text{inv}} := \{u \in \text{Der}^3(A) : \Delta(u(a)) = u(a_{(1)}) \otimes a_{(2)}\} \quad \text{for all} \ a \in A .
\] (5.12)

We write \(\text{Der}^3(A)_{\text{inv}}\) rather than \(\text{Der}^3_{A^+}(A)\) to conform with Woronowicz’s notation [38]. Right-invariance implies that the \(U\)-action is trivial, \(\xi \triangleright u = \varepsilon(\xi)u\) for all \(\xi \in U\), \(u \in \text{Der}^3(A)_{\text{inv}}\). Indeed \(\xi \triangleright (u(a)) = u(a)_{(1)} \langle \xi, u(a_{(2)}) \rangle = u(a_{(1)}) \langle \xi, u(a_{(2)}) \rangle = u(a) \triangleright a\).

Also the \(U^\text{op}\)-action closes in \(\text{Der}^3(A)_{\text{inv}}\):
\[
(\xi \triangleright u)(a) = \xi(1) \triangleright (u(S^{-1}(\xi(2)) \triangleright a))
\]
\[
= \langle S^{-1}(\xi(2)), a_{(1)} \rangle \xi(1) \triangleright (u(a_{(2)}))
\]
\[
= \langle S^{-1}(\xi(2)), a_{(1)} \rangle \langle \xi(1), u(a_{(2)})_{(1)} \rangle u(a_{(2)})_{(2)}
\]
\[
= \langle S^{-1}(\xi(2)), a_{(1)} \rangle \langle \xi(1), u(a_{(2)}) \rangle a_{(3)}
\] (5.13)

for all \(\xi \in U^\text{op}\), \(u \in \text{Der}^3(A)_{\text{inv}}\), \(a \in A\), and
\[
\Delta(\xi \triangleright u)(a) = \langle S^{-1}(\xi(2)), a_{(1)} \rangle \langle \xi(1), u(a_{(2)}) \rangle a_{(3)} \otimes a_{(4)} = (\xi \triangleright u)(a_{(1)}) \otimes a_{(2)} .
\] (5.14)
This implies that $\text{Der}^{\text{op}}(A)_{\text{inv}}$ is a $U^{\text{op}} \otimes U$-submodule and a braided subalgebra of $\text{Der}^{\text{op}}(A)$. Since
\[
(\mathfrak{g}_a \triangleright a) \mathfrak{R}^a \triangleright u = (\mathfrak{R}_a \triangleright (\mathfrak{R}_\beta \triangleright a)) \mathfrak{R}_\beta^a \triangleright (\mathfrak{R}_\beta^a \triangleright u) = (\mathfrak{R}_a \triangleright a) \mathfrak{R}^a \triangleright u ,
\]
we further have
\[
\text{Der}^{\text{op}}(A)_{\text{inv}} = \text{Der}^{\text{op}}(A)_{\text{inv}} .
\quad (5.15)
\]
This is a $U^{\text{op}}$-braided Lie algebra isomorphism.

5.3. Actions of braided Lie algebras and universal enveloping algebras. An action of a $K$-braided Lie algebra $g$ on a $K$-module $W$ is a $K$-equivariant map
\[
\mathcal{L} : g \otimes W \to W , \quad u \otimes w \mapsto \mathcal{L}_u(w)
\]
such that $\mathcal{L}_{u,v} = \mathcal{L}_u \circ \mathcal{L}_v - \mathcal{L}_{\alpha \triangleright u} \circ \mathcal{L}_{\alpha \triangleright v}$ for all $u,v \in g$. Equivalently, an action such that $(\text{Hom}(W,W), \mathcal{L})$ is an enveloping algebra of $g$.

When $W = A$ is a $K$-module algebra, the action of $g$ is in addition required to satisfy the braided Leibniz rule:
\[
\mathcal{L}_u(ab) = \mathcal{L}_u(a)b + (R_a \triangleright a) \mathcal{L}_{R^a \triangleright u}(b) ,
\quad (5.16)
\]
for all $u \in g, a,b \in A$. Equivalently, $\mathcal{L} : g \to \text{Hom}(A,A)$ is required to have image in $\text{Der}(A)$. Hence it is a morphism $\mathcal{L} : g \to \text{Der}(A)$ of braided Lie algebras.

Proposition 5.11. Let $\mathcal{U}(g)$ be the universal enveloping algebra of a braided Lie algebra $g$ associated with $(K,R)$. Any action $\mathcal{L} : g \otimes A \to A$ of a braided Lie algebra $g$ on a $K$-module algebra $A$ lifts to an action $\mathcal{L} : \mathcal{U}(g) \otimes A \to A$ of the braided Hopf algebra $\mathcal{U}(g)$ on $A$.

Proof. By definition of universal enveloping algebra, the $K$-braided Lie algebra morphism $\mathcal{L} : (g,[ , ] ) \to (\text{Hom}(A,A),[ , ] )$ lifts to a unique $K$-equivariant algebra morphism $\hat{\mathcal{L}} : (\mathcal{U}(g),\cdot ) \to (\text{Hom}(A,A),\circ )$ that with slight abuse of notation we still denote $\mathcal{L} : \mathcal{U}(g) \to \text{Hom}(A,A)$.

We show that condition (5.17) holds; for all $\xi \in \mathcal{U}(g)$,
\[
\mathcal{L}_\xi(ab) = \mathcal{L}_\xi(\xi_{(1)}(R_\gamma \triangleright a)\mathcal{L}_{R^\gamma \triangleright \xi_{(2)}}(b))
\quad (5.17)
\]
(where we dropped the subscripts of the $K$-actions). This is indeed the case if $\xi \in g \subset \mathcal{U}(g)$. The equality is then proven inductively, by showing that if $\zeta,\xi \in \mathcal{U}(g)$ satisfy (5.17) then so does the product $\zeta \xi$. On one hand we have
\[
\mathcal{L}_{\zeta \xi}(ab) = \mathcal{L}_\zeta(\mathcal{L}_\xi(ab)) = \mathcal{L}_\zeta(\mathcal{L}_{\xi_{(1)}}(R_\gamma \triangleright a)\mathcal{L}_{R^\gamma \triangleright \xi_{(2)}}(b))
\]
\[
\mathcal{L}_{\zeta_{(1)}}(R_\alpha \triangleright \mathcal{L}_{\xi_{(1)}}(R_\gamma \triangleright a))\mathcal{L}_{R^\alpha \triangleright \xi_{(2)}}(b)
\]
\[
= \mathcal{L}_{\xi_{(1)}}(\mathcal{L}_{R^\alpha \triangleright \xi_{(1)}}(R_{\alpha(2)}R_\gamma \triangleright a))\mathcal{L}_{R^\gamma \triangleright \xi_{(2)}}(R^\alpha \triangleright \xi_{(2)}(b))
\]
\[
= \mathcal{L}_{\xi_{(1)}}(R_{\alpha \triangleright \xi_{(1)}}(R_{\alpha(2)}R_\gamma \triangleright a))\mathcal{L}_{R^\gamma \triangleright \xi_{(2)}}(R^\alpha \triangleright \xi_{(2)}(b))
\]
\[
= \mathcal{L}_{\xi_{(1)}}(R_{\alpha \triangleright \xi_{(1)}}(R_{\alpha(2)}R_\gamma \triangleright a))\mathcal{L}_{R^\alpha \triangleright \xi_{(2)}}(R^\alpha \triangleright \xi_{(2)}(b))
\]
On the other hand, since $\Delta(\xi \xi) = \Delta(\xi) \mathcal{L}(\xi) = \xi_{(1)}(R_\alpha \triangleright \xi_{(1)}) \otimes (R_{\gamma} \triangleright \xi_{(2)})\xi_{(2)}$, we have
\[
\mathcal{L}_{\xi_{(1)}}(R_\gamma \triangleright a)\mathcal{L}_{R^\gamma \triangleright \xi_{(2)}}(b) = \mathcal{L}_{\xi_{(1)}}(R_\alpha \triangleright \xi_{(1)})(R_\gamma \triangleright a)\mathcal{L}_{(R^\gamma \triangleright \xi_{(2)})}(R_{\gamma(2)} \triangleright \xi_{(2)})(b)
\]
\[
= \mathcal{L}_{\xi_{(1)}}(R_\alpha \triangleright \xi_{(1)})(R_\gamma \triangleright a)\mathcal{L}_{(R^\gamma \triangleright \xi_{(2)})}(R_{\gamma(2)} \triangleright \xi_{(2)})(b)
\]
\[
= \mathcal{L}_{\xi_{(1)}}(R_\alpha \triangleright \xi_{(1)})(R_\gamma \triangleright a)\mathcal{L}_{(R^\gamma \triangleright \xi_{(2)})}(R_{\gamma(2)} \triangleright \xi_{(2)})(b) .
\]
Comparing these two expressions we obtain $L_{\xi\zeta}(ab) = L_{(\xi\zeta)}(R_{\gamma} \triangleright a)L_{R_{\alpha} \triangleright (\xi\zeta)}(b)$. □

When $g = \text{Der}(A)$ with action $\text{Der}(A) \otimes A \to A$ given by the evaluation, we obtain:

**Corollary 5.12.** Let $A$ be a $K$-module algebra and $\text{Der}(A)$ the associated braided Lie algebra of derivations. The universal enveloping algebra $\mathcal{U}(\text{Der}(A))$ acts on $A$ as a $K$-braided Hopf algebra.

Thus, a $K$-module algebra $A$ is canonically a braided $\mathcal{U}(\text{Der}(A))$-module algebra with respect to the braided Hopf algebra $\mathcal{U}(\text{Der}(A))$. We notice that in this context primitive elements $\psi \in \text{Der}(A) \subseteq \mathcal{U}(\text{Der}(A))$ act on $A$ as braided derivations:

$$L_{\psi}(ab) = L_{\psi(1)}(R_{\alpha} \triangleright_A a)(R_{\alpha} \triangleright_{\text{Der}(A)} L_{\psi(2)})(b) = L_{\psi}(a)b + (R_{\alpha} \triangleright a)L_{R_{\alpha} \triangleright_{\text{Der}(A)} \psi}(b).$$

There is a canonical action of a braided Lie algebra $g$ on the $K$-module algebra $\mathcal{U}(g)$, the adjoint action.

**Proposition 5.13.** Given a braided Lie algebra $g$ associated with $(K,R)$, the braided adjoint action $(3.18)$ is an action of $g$ on the braided Hopf algebra $\mathcal{U}(g)$; equivalently, a morphism $g \to \text{Der}(\mathcal{U}(g))$ of braided Lie algebras.

**Proof.** Let $g$ be a braided Lie algebra and $\mathcal{U}(g)$ its universal enveloping algebra, a braided Hopf algebra for Proposition 5.4. From Lemma 3.9 there is a morphism of $K$-modules:

$$L : (\mathcal{U}(g), \triangleright_{\mathcal{U}(g)}) \to (\text{Hom}(\mathcal{U}(g), \mathcal{U}(g)), \triangleright_{\text{Hom}(\mathcal{U}(g), \mathcal{U}(g))})$$

$$\xi \mapsto L_{\xi}, \quad L_{\xi}(\zeta) := \xi \triangleright \zeta = \xi_{(1)}(R_{\alpha} \triangleright \zeta)R_{\alpha} \triangleright S_{\mathcal{U}(g)}(\xi_{(2)}), \quad \zeta \in \mathcal{U}(g).$$

This restricts to the $K$-module morphism

$$L : (g, \triangleright) \to (\text{Der}(\mathcal{U}(g)), \triangleright_{\text{Der}(\mathcal{U}(g))})$$

$$u \mapsto L_u, \quad L_u(\zeta) = u \triangleright_{\text{ad}_R} \zeta = u\zeta - (R_{\alpha} \triangleright \zeta)(R_{\alpha} \triangleright u), \quad \zeta \in \mathcal{U}(g).$$

Indeed, the map $L_u$ is a braided derivation: from $(3.19)$ the map $\triangleright_{\text{ad}_R}$ is an action, then

$$L_u(\xi\zeta) = u \triangleright_{\text{ad}_R} (\xi\zeta)$$

$$= (u(1) \triangleright_{\text{ad}_R} (R_{\alpha} \triangleright \xi))(R_{\alpha} \triangleright u(2) \triangleright_{\text{ad}_R} \xi)$$

$$= (u \triangleright_{\text{ad}_R} \xi)\zeta + (R_{\alpha} \triangleright \xi)(R_{\alpha} \triangleright u \triangleright_{\text{ad}_R} \xi)$$

$$= L_u(\xi)\zeta + (R_{\alpha} \triangleright \xi)(L_{R_{\alpha} \triangleright u} \triangleright_{\text{ad}_R} \xi)$$

$$= L_u(\xi)\zeta + (R_{\alpha} \triangleright \xi)(R_{\alpha} \triangleright L_u(\xi)).$$

where the last equality uses $K$-equivariance of $L$. □

### 5.4. Twisting braided Lie algebras.

We next consider twist deformations of braided Lie algebras. Let $g$ be a braided Lie algebra associated with a triangular Hopf algebra $(K,R)$ and $F$ be a twist for $K$. The $K_F$-module $g_F$ (this is just $g$ with $K$-action; see the construction after Proposition 4.3) inherits from $g$ a twisted bracket:

**Proposition 5.14.** The $K_F$-module $g_F$ with bilinear map

$$[ , ]_F = g_F \otimes g_F \to g_F, \quad u \otimes v \rightarrow [u,v]_F := [F_u \triangleright v, F_{\alpha} \triangleright v]. \tag{5.18}$$

is a braided Lie algebra associated with $(K_F,R_F)$.  

30
Proof. The $K_F$-equivariance is easily proven using the analogue property for $\langle \ , \ \rangle$ and recalling from \ref{K-F-eq} that $\Delta_K = K \Delta_K F$: for all $u, v \in g, k \in K_F$,

$$k \triangleright [u, v]_F = [k^{(1)} F^\alpha \triangleright u, k^{(2)} F^\alpha \triangleright v] = [F^\alpha_k^{(1)} \triangleright u, F^\alpha_k^{(2)} \triangleright v] = [k^{(1)} \triangleright u, k^{(2)} \triangleright v]_F.$$  

Similarly, the braided antisymmetry $[u, v]_F = -[R_F \beta \triangleright v, R_F \beta \triangleright u]_F$ follows from the braided antisymmetry of $\langle \ , \ \rangle$ and recalling that $R_F = F_21 R_F$.

We are left to prove the Jacobi identity: for all $u, v, w \in g_F$, we have

$$[u, [v, w]]_F = [F^\gamma \triangleright u, [F^\beta \triangleright _{\ell}(1), F^\alpha \triangleright v, F^\beta \triangleright _{\ell}(2) F^\alpha \triangleright w]]$$

$$= [[F^\beta \triangleright u, [F^\beta \triangleright _{\ell}(1), F^\alpha \triangleright v], F^\beta \triangleright _{\ell}(2) F^\alpha \triangleright w] + [R_\gamma, F^\beta \triangleright _{\ell}(1), F^\alpha \triangleright v, R_\gamma F^\beta \triangleright u, F^\beta \triangleright _{\ell}(2) F^\alpha \triangleright w]]$$

$$= [F^\beta \triangleright [F^\beta \triangleright u, F^\beta \triangleright _{\ell}(1), F^\alpha \triangleright v], F^\beta \triangleright _{\ell}(2) F^\alpha \triangleright w] + [R_\gamma, F^\beta \triangleright _{\ell}(1), F^\alpha \triangleright v, R_\gamma F^\beta \triangleright u, F^\beta \triangleright _{\ell}(2) F^\alpha \triangleright w]]$$

$$= [[u, v]_F, w]_F + [F^\beta R_{F, \gamma} \triangleright v, [R_{F, \beta(1)} F^\alpha \triangleright v, R_{F, \beta(2)} F^\alpha \triangleright w]]$$

$$= [[u, v]_F, w]_F + [F^\beta R_{F, \gamma} \triangleright v, [R_{F, \beta(1)} F^\alpha \triangleright v, R_{F, \beta(2)} F^\alpha \triangleright w]]$$

For the first two equalities we used the $K$-equivariance and Jacobi identity for $\langle \ , \ \rangle$, for the third the twist property \ref{K-F-eq}, for the forth the $K$-equivariance and the quasi-cocommutativity of $K$. Then $R_F F = F_21 R_F$ and the twist property end the proof. \qed

Example 5.15. When $g$ is a $K$-module algebra $A$ as in Lemma \ref{K-K-lemma} its braided commutator is twisted to

$$[a, b]_F = \langle F^\alpha \triangleright A a, (F^\alpha \triangleright A b) - (R_F a \triangleright A b) \rangle (F^\beta \triangleright A a)$$

$$= \langle F^\alpha \triangleright A a, (F^\alpha \triangleright A b) - (F^\beta R_F a \triangleright A b) \rangle (F^\alpha R_F \beta \triangleright A a)$$

Since $\triangleright_A$ and $\triangleright_{A, F}$ coincide, this shows that the braided Lie algebra $g_F$ is just the twisted $K_F$-module algebra $A_F$ with braided commutator defined as in \ref{K-F-eq} but using the universal $R$-matrix $K_F$ of $K_F$ and the product in $A_F$. \qed

Example 5.16. As a particular case of the previous example, consider the $K$-module algebra $(\text{Hom}(V, V), \circ)$ of linear maps of a $K$-module $V$ with action \ref{K-F-action}. The twist deformation of the braided Lie algebra $(\text{Hom}(V, V), \circ, [\ , \ ]_F)$ is the braided Lie algebra $(\text{Hom}_F(V, V), \circ_F, [\ , \ ]_F)$, that is the $K_F$-module algebra $(\text{Hom}_F(V, V), \circ_F)$, see \ref{K-F-eq}, with braided commutator

$$[\psi, \lambda]_F = \psi \circ_F \lambda - (R_F a \triangleright \text{Hom}(V, V) \lambda) \circ_F (R_F a \triangleright \text{Hom}(V, V) \psi).$$

We prove that it is isomorphic to the braided Lie algebra $(\text{Hom}(V^*_F, V^*_F), \circ, [\ , \ ]_{R_F})$ associated with the triangular Hopf algebra $(K_F, R_F)$. Indeed, the $K_F$-module algebra isomorphism $D : (\text{Hom}_F(V, V), \circ_F) \rightarrow (\text{Hom}(V^*_F, V^*_F), \circ)$ in Proposition \ref{K-F-prop} is also a braided Lie algebra isomorphism:

$$D([\psi, \lambda]_F) = D(\psi) \circ D(\lambda) - D(R_{F a} \triangleright \text{Hom}(V, V) \lambda) \circ D(R_{F a} \triangleright \text{Hom}(V, V) \psi)$$

$$= D(\psi) \circ D(\lambda) - R_{F a} \triangleright \text{Hom}(V^*_F, V^*_F) D(\lambda) \circ R_{F a} \triangleright \text{Hom}(V^*_F, V^*_F) D(\psi)$$

$$= [D(\psi), D(\lambda)]_{R_F}.$$
Here we used that \( D \) is an algebra map and a morphism of \( K_F \)-modules.

Furthermore, from the \( K \)-braided Poisson algebra structure \((\text{Hom}(V,V), \circ, [\ , \ ]_F)\), we have that \((\text{Hom}_F(V,V), \circ_F, [\ , \ ]_F)\) is a \( K_F \)-braided Poisson algebra. Indeed, the derivation property of the bracket \([\ , \ ]_F\) with respect to the multiplication \( \circ_F \) is proven exactly as in the proof of the Jacobi identity of Proposition 5.14 (just replace the second bracket \([\ , \ ]_F\) with the multiplication \( \circ_F \)). Then it follows that \( D : (\text{Hom}_F(V,V), \circ_F, [\ , \ ]_F) \rightarrow (\text{Hom}(V_F,V_F), \circ, [\ , \ ]_{R_F}) \) is a \( K_F \)-braided Poisson algebra isomorphism.

The braided Hopf algebra \( U(g)_F \) associated with \((K_F, R_F)\) is given, as in Proposition 5.11 by the twist deformation of the universal enveloping algebra \( U(g) \) of the braided Lie algebra \( g \). We now show that it coincides (up to isomorphism) with the universal enveloping algebra \( U(g_F) \) of the braided Lie algebra \( g_F \), given as in Proposition 5.14. In other words, the universal enveloping algebra construction commutes with twist deformation: \( g \rightarrow g_F \rightarrow U(g_F) \) equals \( g \rightarrow U(g) \rightarrow U(g)_F \).

The isomorphism \( U(g_F) \simeq U(g)_F \) is induced from the \( K_F \)-module isomorphism
\[
\varphi^{(2)} := \varphi_{g,g} : g_F \otimes_F g_F \rightarrow (g \otimes g)_F, \quad u \otimes_F v \mapsto F^1 \triangleright_F \alpha(u) \triangleright_F \beta(v),
\]
given as in (4.7), and its lifts \( \varphi^{(n)} \) to \( n \)-th tensor products \( \varphi^{(n)} : g_F^{\otimes n} \rightarrow (g^{\otimes n})_F \), for \( n \in \mathbb{N} \), defined recursively, for \( n \geq 2 \), by
\[
\varphi^{(n+1)} = \varphi_{g^{\otimes n}, g} \circ (\text{id}_{g_F} \otimes_F \varphi^{(n)}),
\]
with the maps \( \varphi_{g^{\otimes n}, g} \) again as in (4.7), and \( \varphi^{(0)} := \text{id}_{g_F}, \varphi^{(1)} := \text{id}_{g^{\otimes 2}} \). Explicitly,
\[
\varphi^{(n)}(u_1 \otimes_F \ldots u_{n-1} \otimes_F u_n) = \text{F}^{n+1} \triangleright_F u_1 \otimes_F \text{F}^{a_1} \triangleright_F \ldots \triangleright_F u_n \otimes_F \text{F}^{a_{n-1}} \triangleright_F (\text{F}^{a_{n-1}} \triangleright_F u_{n-1} \otimes_F \text{F}^{a_{n-2}} \triangleright_F \ldots \triangleright_F u_1),
\]
for all \( u_i \in g_F, i = 1, \ldots, n \). Property (4.8) of the isomorphisms \( \varphi_{-, -} \), generalized to \( n \)-th tensor products, gives

**Lemma 5.17.** For integers \( p, q \in \mathbb{N} \) with \( p + q = n \),
\[
\varphi^{(n)} = \varphi_{g^{\otimes p}, g^{\otimes q}} \circ (\varphi^{(p)} \otimes_F \varphi^{(q)}).
\]  

**Proof.** By induction: the identity holds for \( n \leq 2 \); assume it holds for \( n \), then
\[
\varphi^{(n+1)} = \varphi_{g^{\otimes n}, g} \circ (\text{id}_{g_F} \otimes_F \varphi^{(n)})
\]
\[
= \varphi_{g^{\otimes n}, g} \circ (\text{id}_{g_F} \otimes_F \varphi_{g^{\otimes p}, g^{\otimes q}} \circ (\varphi^{(p)} \otimes_F \varphi^{(q)}))
\]
\[
= \varphi_{g^{\otimes n}, g} \circ (\text{id}_{g_F} \otimes_F \varphi_{g^{\otimes p}, g^{\otimes q}} \circ (\text{id}_{g_F} \otimes_F \varphi^{(p)} \otimes_F \varphi^{(q)}))
\]
\[
= \varphi_{g^{\otimes n}, g} \circ (\varphi_{g^{\otimes p+1}, g^{\otimes q}} \circ (\varphi^{(p+1)} \otimes_F \varphi^{(q)}))
\]
where for the fourth equality we have used property (4.8) of the isomorphisms \( \varphi_{-, -} \). □

**Proposition 5.18.** The braided Hopf algebras \( U(g)_F \) and \( U(g_F) \) associated with the triangular Hopf algebra \((K_F, R_F)\) are isomorphic.

**Proof.** The isomorphisms \( \varphi^{(n)} : g_F^{\otimes n} \rightarrow (g^{\otimes n})_F \) induce the isomorphism
\[
\varphi^* := \oplus_{n \geq 0} \varphi^{(n)} : T(g_F) \rightarrow T(g)_F
\]
of (graded) \( K_F \)-modules. We show it is a braided Hopf algebra isomorphism.
Let us denote by $m_{\otimes F}$ the multiplication in the tensor algebra $T(\mathfrak{g})$, and by $(m_{\otimes})_F$ the one in $T(\mathfrak{g})_F$ given by $(m_{\otimes})_F(\tilde{\zeta}, \tilde{\xi}) := \mathcal{F}^a \triangleright \tilde{\zeta} \otimes \mathcal{F}_a \triangleright \tilde{\xi}$ on homogeneous elements $\tilde{\zeta} \in (\mathfrak{g}^{\otimes n})_F, \tilde{\xi} \in (\mathfrak{g}^{\otimes m})_F$, as in (5.19). Then for all $\zeta \in \mathfrak{g}^{\otimes n}_F, \xi \in \mathfrak{g}^{\otimes m}_F$,

$$(m_{\otimes})_F \circ (\varphi^{(n)} \times \varphi^{(m)})(\zeta, \xi) = (m_{\otimes})_F(\varphi^{(n)}(\zeta), \varphi^{(m)}(\xi))$$

$$= \mathcal{F}^a \triangleright \varphi^{(n)}(\zeta) \otimes \mathcal{F}_a \triangleright \varphi^{(m)}(\xi)$$

$$= \mathcal{F}^a \triangleright (\varphi^{(n)} \otimes \mathcal{F} \varphi^{(m)})(\zeta \otimes_F \xi)$$

$$= \varphi^{(n+m)} \circ m_{\otimes_F}(\zeta, \xi)$$

using (5.19) for the last equality. This implies $(m_{\otimes})_F \circ (\varphi^* \times \varphi^*) = \varphi^* \circ m_{\otimes_F}$ so that $\varphi^* : T(\mathfrak{g}) \to T(\mathfrak{g})_F$ is an algebra isomorphism, in fact a $K_F$-module algebra isomorphism.

We are left to show that $\varphi^*$ is a coalgebra morphism. Since the counits are zero (but on $k$), the condition $\varepsilon_{T(\mathfrak{g})} = \varepsilon_{T(\mathfrak{g})_F} \circ \varphi^*$ is trivially true. To show the compatibility with the coproducts, $(\varphi^* \otimes_F \varphi^*) \circ \Delta_{T(\mathfrak{g})} = \Delta_{T(\mathfrak{g})_F} \circ \varphi^*$, it suffices to check it on degree zero and one elements since both sides are braided algebra maps. Being $\varphi^{(0)}$ and $\varphi^{(1)}$ both identity maps, this reduces to show that $\Delta_{T(\mathfrak{g})} = \Delta_{T(\mathfrak{g})_F}$ on $u \in \mathfrak{g}_F$. But this is immediate. Indeed, $T(\mathfrak{g})_F$ has coproduct $\Delta_{T(\mathfrak{g})}$ given by $\Delta_{T(\mathfrak{g})}(u) = u \otimes_F 1 + 1 \otimes_F u$ for $u \in \mathfrak{g}_F$, (and extended as a braided algebra map, see Proposition 5.4). On the other hand, the coproduct on $T(\mathfrak{g})_F$ is the twist deformation as (4.12) of the coproduct of $T(\mathfrak{g})$,

$$\Delta_{T(\mathfrak{g})_F} := \varphi^{(1)}_{T(\mathfrak{g}), T(\mathfrak{g})} \circ \Gamma(\Delta_{T(\mathfrak{g})}) : \quad \tilde{\xi} \mapsto \tilde{\xi}^{(1)} \otimes_F \tilde{\xi}^{(0)}$$

which reduces to $\Delta_{T(\mathfrak{g})_F}(u) = u \otimes_F 1 + 1 \otimes_F u$ in degree one.

The braided Hopf algebra isomorphism $\varphi^*$ induces such an isomorphism $\mathcal{U}(\mathfrak{g}_F) \simeq \mathcal{U}(\mathfrak{g})_F$ on the quotients. Firstly, the ideal $T(\mathfrak{g}_F) \supseteq I_{\mathfrak{g}_F} = \langle u \otimes_F v - R_{\mathfrak{g}_F} \triangleright v \triangleright u \otimes_F \rangle$ is mapped by $\varphi^*$ to the ideal $T(\mathfrak{g})_F \supseteq (I_{\mathfrak{g}})_F = \langle u \otimes v - R_{\mathfrak{g}} \triangleright v \triangleright R_{\mathfrak{g}} \triangleright u - [u, v] \rangle$. For all $u, v \in \mathfrak{g}_F$, since $\varphi^{(1)} = \text{id}_{\mathfrak{g}_F}$ and using $\mathcal{F}_{(2)} R_{\mathfrak{g}} = R_{\mathcal{F}}$, we compute

$$\varphi^*(u \otimes_F v - R_{\mathfrak{g}_F} \triangleright v \triangleright u - [u, v])$$

$$= \mathcal{F}^\alpha \triangleright u \otimes \mathcal{F}_\alpha \triangleright v - \mathcal{F}^\gamma \triangleright \mathcal{F}_\beta \triangleright v \otimes \mathcal{F}_\alpha \triangleright u - [u, v]$$

$$= \mathcal{F}^\alpha \triangleright u \otimes \mathcal{F}_\alpha \triangleright v - R_{\mathfrak{g}} \triangleright \mathcal{F}_\beta \triangleright v \otimes R_{\mathfrak{g}} \triangleright u - \mathcal{F}^\gamma \triangleright u, \mathcal{F}_\alpha \triangleright v.$$
Theorem 5.19. The braided Lie algebras $(\text{Der}(A)_F, [\cdot, \cdot]_F)$ and $(\text{Der}(A_F), [\cdot, \cdot]_{R_F})$ are isomorphic via the map $D : \text{Der}(A)_F \to \text{Der}(A_F)$, the restriction of the isomorphism $\mathcal{D} : ((\text{Hom}(A, A)_F, \circ), [\cdot, \cdot]_F) \to ((\text{Hom}(A_F, A_F), \circ), [\cdot, \cdot]_{R_F})$.

Proof. Since $\text{Der}(A)_F \subset \text{Hom}(A, A)_F$ and $\text{Der}(A_F) \subset \text{Hom}(A_F, A_F)$ are braided Lie subalgebras, we just need to prove that $D : \text{Der}(A)_F \to \text{Der}(A_F)$ is a bijection between the sets $\text{Der}(A) = \text{Der}(A)_F$ of $(K, R)$-braided derivations of $A$ and $\text{Der}(A_F)$ of $(K_F, R_F)$-braided derivations of $A_F$. Recall that a linear map $\psi \in \text{Hom}(A, A)$ is a $(K, R)$-braided derivation if and only if (5.6) holds:

$$[\psi, \ell_a]_R = \ell_{\psi(a)}$$  \hspace{1cm} (5.20)

for all $a \in A$. This condition is linear in $\psi$ and $a$ and, since $A$ and $\text{Der}(A)$ are $K$-modules, it implies

$$[F^a \triangleright \psi, \ell_{F_a \triangleright a}]_R = \ell_{(F^a \triangleright \psi)(F_a \triangleright a)}$$  \hspace{1cm} (5.21)

for all $a \in A$. Similarly, applying $F$ to $F^{-1} \triangleright (\psi \otimes a)$ we see that (5.20) is equivalent to (5.21). In turn, from the $K$-equivariance of $\ell$ the latter is equivalent to

$$[\psi, \ell_a]_F := [F^a \triangleright \psi, F_a \triangleright \ell_a]_R = \ell_{D(\psi)(a)}.$$

Applying the isomorphism $\mathcal{D} : ((\text{Hom}(A, A)_F, \circ), [\cdot, \cdot]_F) \to ((\text{Hom}(A_F, A_F), \circ), [\cdot, \cdot]_{R_F})$ of $K_F$-braided Lie algebras shows that (5.20) is equivalent to

$$[D(\psi), D(\ell_a)]_{R_F} = D(\ell_{D(\psi)(a)}).$$  \hspace{1cm} (5.22)

Thus $\psi$ is a $(K, R)$-braided derivation if and only if (5.22) holds. The theorem is proven by showing that the latter is equivalent to $D(\psi)$ being a $(K_F, R_F)$-braided derivation. To this end we evaluate (5.22) on any $a' \in A$ and, using $K$-equivariance of $\ell$, we obtain

$$D(\psi)(D(\ell_a)(a')) - (R_{F_a} \triangleright D(\ell_a))(\ell_{K_F}(\psi)(a')) = D(\psi)(a) \triangleright a'.$$

that using also $K_F$-equivariance of $\mathcal{D}$ is equivalent to

$$D(\psi)(a \triangleright a') = D(\psi)(a) \triangleright a' + (R_{F_a} \triangleright a) \triangleright \ell_{D(\psi)(a')}. $$  \hspace{1cm} (5.23)

The implication (5.20) $\Rightarrow$ (5.23) shows that the bijection $D : \text{Hom}(A, A)_F \to \text{Hom}(A_F, A_F)$ restricts to the injection $D : \text{Der}(A)_F \to \text{Der}(A_F)$ (recall that $\text{Hom}(A, A) = \text{Hom}_F(A, A)$ as sets). Since any element of $\text{Hom}(A_F, A_F)$ can be written as $D(\psi)$ for some $\psi \in \text{Hom}(A, A)_F$, the implication (5.23) $\Rightarrow$ (5.20) shows that $D : \text{Der}(A)_F \to \text{Der}(A_F)$ is a surjection.

When $A$ is quasi-commutative, the $K$-braided Lie algebra $\text{Der}(A)$ has an $A$-module structure defined in (5.9) that is compatible with the Lie bracket of $\text{Der}(A)$ (cf. Proposition 5.8). This implies that also the $K_F$-braided Lie algebras $\text{Der}(A)_F$ and $\text{Der}(A_F)$ have compatible $A_F$-module structures and the isomorphism $\mathcal{D} : \text{Der}(A)_F \to \text{Der}(A_F)$ maps one into the other. The $(K, A)$-relative Hopf module $\text{Der}(A)$ with $A$-module structure (5.9) is twisted to the $(K_F, A_F)$-relative Hopf module $\text{Der}(A)_F$ with

$$a \triangleright_F \psi := (F^a \triangleright_A a)(F_a \triangleright_{\text{Der}(A)} \psi)$$  \hspace{1cm} (5.24)

for all $a \in A_F, \psi \in \text{Der}(A)_F$, that is $a \triangleright_F \psi = \ell_a \circ_F \psi$. The braided derivation property of the bracket with respect to this module structure reads, for all $\psi, \psi' \in \text{Der}(A)_F, a \in A_F$,

$$[\psi, a \triangleright_F \psi]_F = [\psi, a]_F \triangleright_F \psi + (R_{F_a} \triangleright a) \triangleright_F [R_{F_a} \triangleright \psi, \psi']_F$$  \hspace{1cm} (5.25)

where $[\psi, a]_F := (F^a \triangleright \psi)(F_a \triangleright a)$ in agreement with $[\psi, \ell_a]_F = [F^a \triangleright \psi, F_a \triangleright \ell_a] = \ell_{(F^a \triangleright \psi)(F_a \triangleright a)}$. We recall that $(\text{Hom}(A, A)_F, \circ, [\cdot, \cdot]_F)$ is a $K_F$-braided Poisson algebra (cf. Example 5.16). Then the proof of (5.25) is along the proof of Proposition 5.8.
where now we consider a $K_F$-braided Poisson algebra $(P_F, \cdot_F, [\cdot, \cdot]_F)$ and the $K$-equivariant map $\ell : A \to \text{Hom}(A, A)_F$, with $\ell_a$ defined in [5.13], is seen as a $K_F$-equivariant map $\ell : A_F \to \text{Hom}(A, A)_F$. Setting $p = \text{id}_{A_F}$, $q = \psi$, $p' = \ell_a$, $q' = \psi'$ in the $K_F$-braided Poisson algebra version of (5.11) we obtain
\[
[\psi, a \cdot_F \psi']_F = [\psi, \ell_a \circ_F \psi']_F = [\psi, \ell_a \circ_F \psi'] + (R_{F_{\alpha \circ a}} \triangleright a) \circ_F [R_{F_{\alpha}} \triangleright \psi, \psi']_F
\]
\[
= \ell_{\cdot_F \circ \psi}(F_{\alpha \circ a}) \circ_F \psi + (F_{R_{F_{\alpha \circ a}} \circ a} \circ_F [R_{F_{\alpha}} \triangleright \psi, \psi'])_F
\]
\[
= \ell_{\cdot_F \circ \psi}(F_{\alpha \circ a}) \circ_F \psi + (F_{R_{F_{\alpha}} \circ a} \circ_F [R_{F_{\alpha}} \triangleright \psi, \psi'])_F
\]
\[
= [\psi, a]_F \cdot_F \psi + (R_{F_{\alpha}} \circ a) \cdot_F [R_{F_{\alpha}} \triangleright \psi, \psi']_F .
\]

This establishes the braided derivation property [5.23]. More generally, setting $p = \ell_a$, $q = \psi$, $p' = \ell_a'$, $q' = \psi'$ in [5.11] we obtain
\[
[a \cdot_F \psi, a' \cdot_F \psi']_F = a \cdot_F [a', \psi]_F \cdot_F \psi' + a (R_{F_{\alpha}} \circ a') \cdot_F [R_{F_{\alpha}} \triangleright \psi, \psi']_F + R_{F_{\beta}} R_{F_{\alpha}} \circ a' \cdot_F [R_{F_{\alpha}} \circ a \circ F_{\beta} R_{F_{\gamma}} \triangleright \psi, R_{F_{\beta}} R_{F_{\alpha}} \circ a]_F \circ a' \cdot F R_{F_{\alpha}} \triangleright \psi .
\]

The above shows that we have a $(K_F, R_F)$-braided Lie algebra $\text{Der}(A)_F$ with compatible left $A_F$-module structure [5.24]. In addition we have also that $\text{Der}(A_F)$ is a $(K_F, R_F)$-braided Lie algebra with compatible $A_F$-module structure defined by
\[
(a \cdot \tilde{\psi})(a') := a \cdot a' \tilde{\psi}(a')
\]
for all $a, a' \in A_F, \tilde{\psi} \in \text{Der}(A_F)$, as in [5.19]. Indeed if $A$ is $K$-quasi-commutative then $A_F$ is $K_F$-quasi-commutative so that we can apply Proposition [5.8] to this case.

**Corollary 5.20.** If the $K$-module algebra $A$ is quasi-commutative the braided Lie algebra $\text{Der}(A)_F$ of Theorem [5.13] is also an isomorphism of the $A_F$-modules $\text{Der}(A)_F$ and $\text{Der}(A_F)$.

**Proof.** For any $a \in A_F$ we have $\ell_a : A \to A$ and $D(\ell_a) : A_F \to A_F$. Let $D(\psi)$ be an element of $\text{Der}(A_F)$, we show that $a \cdot \cdot D(\psi) = D(\ell_a) \circ D(\psi)$. For any $a' \in A_F$,
\[
D(\ell_a) \circ D(\psi)(a') = D(\ell_a)(D(\psi)(a')) = \ell_{\cdot_F \circ a}(F_{\alpha} \circ (D(\psi)(a'))) = a \cdot \cdot (D(\psi)(a'))
\]
\[
= (a \cdot \cdot D(\psi))(a') .
\]
We then have for any $\psi \in \text{Der}(A)$, $D(a \cdot \cdot \psi) = D(\ell_a \circ_F \psi) = D(\ell_a) \circ D(\psi) = a \cdot \cdot D(\psi).$

We denote $\tilde{\psi} = D(\psi)$ for the image of $\psi \in \text{Der}(A)_F$. When applying $D$ to (5.26), we obtain the equality,
\[
[a \cdot \cdot \tilde{\psi}, a' \cdot \cdot \tilde{\psi'}]_F = a \cdot a' \cdot \cdot \tilde{\psi}(a') \otimes \tilde{\psi}' + a \cdot (R_{F_{\alpha}} \circ a') \cdot \cdot [R_{F_{\alpha}} \triangleright \cdot \cdot \tilde{\psi}, \cdot \cdot \tilde{\psi'}]_F + R_{F_{\beta}} R_{F_{\alpha}} \circ a' \cdot \cdot [R_{F_{\alpha}} \circ a \circ F_{\beta} R_{F_{\gamma}} \triangleright \cdot \cdot \tilde{\psi}, R_{F_{\beta}} R_{F_{\alpha}} \circ a]_F \circ a' \cdot F R_{F_{\alpha}} \triangleright \cdot \cdot \tilde{\psi} .
\]

for all $\tilde{\psi}, \tilde{\psi}' \in \text{Der}(A_F)$. We used that $[\psi, a]_F = (F_{\alpha} \circ \psi)(F_{\alpha} \circ a) = D(\psi)(a)$.

6. **Gauge group of Hopf–Galois extensions**

The notion of gauge group of a quantum principal bundle $A^{coH} \subset A$ has been studied in [11, 12] as the group of invertible $H$-equivariant unital maps $A \to A$. These are not required to be algebra maps and in the case of a commutative principal bundle the resulting gauge group is much bigger than the usual commutative one (see [6, Ex. 3.1]).

Here, along the lines of [6] we follow a different route to the gauge group. We first study the group of unital $H$-equivariant algebra maps (right $H$-comodule algebra maps)
from $A$ to $A$. Examples are provided that show when this notion is too restrictive and when it is not. In the next section we introduce $K$-equivariant Hopf–Galois extensions $A^{coH} \subset A$, with $K$ an external triangular Hopf algebra of symmetries of the extension. In that context we provide a more general definition of gauge symmetries as braided Hopf algebras.

6.1. Hopf–Galois extensions. We consider noncommutative principal bundles as Hopf–Galois extensions. These are $H$-comodule algebras $A$ with a canonically defined map $\chi : A \otimes_B A \to A \otimes H$ which is required to be invertible.

Definition 6.1. Let $H$ be a Hopf algebra and let $A \in A^H$ be an $H$-comodule algebra with coaction $\delta^A$. Consider the subalgebra $B := A^{coH} = \{b \in A \mid \delta^A(b) = b \otimes 1_H\} \subseteq A$ of coinvariant elements and let $A \otimes_B A$ be the corresponding balanced tensor product with multiplication $m : A \otimes_B A \to A$ induced from $m_A : A \otimes A \to A$. The extension $B \subseteq A$ is called an $H$-Hopf–Galois extension provided the map

$$\chi := (m \otimes \id) \circ (\id \otimes \delta^A) : A \otimes_B A \to A \otimes H, \quad a' \otimes_B a \mapsto a'_1 \otimes a_1(0) \otimes a_1(1),$$

the canonical map, is bijective.

Being $\chi$ left $A$-linear, its inverse, when it exists, is determined by the translation map:

$$\tau = \chi^{-1}|_{\otimes_B} : 1 \otimes H \cong H \to A \otimes_B A, \quad h \mapsto h^{<1>} \otimes h^{<2>}.$$ 

A Hopf–Galois extension is cleft if there is a convolution invertible morphism of $H$-comodules $j : H \to A$ (the cleaving map), where $H$ has coaction $\Delta$. This is equivalent to an isomorphism $A \cong B \otimes H$ of left $B$-modules and right $H$-comodules, where $B \otimes H$ is a left $B$-module via multiplication on the left and a right $H$-comodule via $\id \otimes \Delta$. When the cleaving map is an algebra map, the algebra $A$ is isomorphic to a smash product algebra $A \cong B^H \otimes H$.

The extension $B = A^{coH} \subset A$ is called $H$-principal comodule algebra if it is Hopf–Galois and $A$ is $H$-equivariantly projective as a left $B$-module, i.e., there exists a left $B$-module and right $H$-comodule morphism $s : A \to B \otimes A$ that is a section of the (restricted) product $m : B \otimes A \to A$. This is equivalent to requiring that the total space algebra $A$ is faithfully flat as a left $B$-module. Cleft Hopf–Galois extensions are always principal comodule algebras.

6.2. Gauge group. Let $\text{Hom}_{A^H}(A, A)$ be the $k$-module of $H$-comodule maps from $A$ to $A$ and let $\text{Hom}_{A^H}(A, A)$ be the subset of algebra maps from $A$ to $A$. The following theorem improves previous results in [6, Prop. 3.6] (where $A$ was quasi-commutative) and [23, Prop. 3.3] (where the faithfully flat condition was used), see also [34, Rem. 3.11].

Theorem 6.2. Let $B = A^{coH} \subseteq A$ be an $H$-Hopf–Galois extension. The set

$$\text{Aut}_B(A) := \{F \in \text{Hom}_{A^H}(A, A) \mid F_{|B} = \id\}$$

(6.2)

of right $H$-comodule algebra morphisms that restrict to the identity on the subalgebra $B$ is a group with respect to the composition of maps

$$F : G := G \circ F$$

for all $F, G \in \text{Aut}_B(A)$. For $F \in \text{Aut}_B(A)$ its inverse $F^{-1} \in \text{Aut}_B(A)$ is given by

$$F^{-1} := m \circ (\id \otimes m) \circ (\id \otimes F \otimes_B \id) \circ (\id \otimes \tau) \circ \delta^A : A \to A$$

$$a \mapsto a_1(0) F(a_1(1)) a_1(2).$$

(6.3)
Proof. The map $F^{-1}$ in (6.3) is well-defined because $F \otimes_B \text{id}$ is well-defined due to the right $B$-linearity of $F$. The theorem is proven by showing that $F^{-1}$ is the inverse of $F$. We first prove that $F^{-1} \circ F = \text{id}_A$. Indeed, being $F$ an $H$-equivariant of algebra map,

$$F^{-1}(F(a)) = F(a_0)F(a_0^{<1>} a_{(1)}^{<2>}) = F(a_0 a_0^{<1>}) a_{(1)}^{<2>} = F(1) a = a .$$

(6.4)

Here in the last but one equality we used that $\chi$ and its inverse are left $A$-linear so that $a_0(\tau(a_1)) = a_0(\chi^{-1}(1 \otimes a_1)) = \chi^{-1}(a_0 \otimes a_1) = a_0(\chi(1 \otimes a)) = 1 \otimes a$.

To prove that $F \circ F^{-1} = \text{id}_A$ we notice that the balanced tensor product map $F \otimes_B F : A \otimes_B A \to A \otimes_B A$ is well defined because $F$ is left and right $B$-linear. We show that

$$(F \otimes_B F) \circ \tau = \tau .$$

This is equivalent to $\chi \circ (F \otimes_B F) \circ \tau = 1_A \otimes \text{id}_H$; for all $h \in H$,

$$\chi(F(h^{<1>}) \otimes_B F(h^{<2>} )) = F(h^{<1>}) F(h^{<2>}(0)) \otimes (h^{<2>})_{(1)}$$

$$= F(h^{<1>}) F(h^{<2>}(0) \otimes h^{<2>}_{(1)})$$

$$= F(h^{<1>} h^{<2>}(0)) \otimes h^{<2>}_{(1)}$$

$$= (F \otimes \text{id}_H)(\chi(h^{<1>} \otimes_B h^{<2>}))$$

$$= 1_A \otimes h$$

where in the second equality we used $H$-equivariance of $F$, in the last that $\tau = \chi^{-1}|_{1 \otimes H}$.

Using that $F$ is an algebra map and the above property it is now easy to show that $F \circ F^{-1} = F^{-1} \circ F = \text{id}_A$; for all $a \in A$,

$$F(F^{-1}(a)) = F(a_0) F(a_0^{<1>} a_{(1)}^{<2>}) = F(a_0) F(F(a_0^{<1>})) F(a_{(1)}^{<2>})$$

$$= F(a_0) F(a_0^{<1>} a_{(1)}^{<2>}) = F^{-1}(F(a)) = a .$$

where in the last but one equality we used equation (6.4). \qed

In general this notion of gauge group is too restrictive. For instance, a Hopf algebra $H$ coacting on itself with its coproduct, can be seen as the Hopf–Galois extension of the ground field $k$. In this case the gauge group coincides with the group of characters of $H$. Indeed for $\chi : H \to k$ a character, the map $\alpha_\chi := (\chi \otimes \text{id}) \circ \Delta : H \to H$ is a right $H$-comodule algebra morphism. Vice versa, given a right $H$-comodule algebra morphism $\alpha$, the composition $\chi = \varepsilon \circ \alpha$ is an algebra map. These two constructions are one the inverse of the other. This equivalence holds also for locally compact quantum groups $[13]$. For example, the gauge group of the extension $C \subseteq O_q(SL(2))$, $q^2 \neq 1$, is just the multiplicative group $\mathbb{C}^\times$.

More in general, for a faithfully flat Galois object $A$ over $k$ (a Hopf–Galois extension $k = A^{coH} \subset A$) we have $\text{Aut}_B(A) = \text{Char}(L)$. This is the group of characters of the Hopf algebra $L := (A \otimes A)^{coH}$ of coinvariant elements of the $H$-comodule algebra $A \otimes A$ under the diagonal coaction $[33] \S 3.1$, $[23] \S 6.1$.

The gauge group $\text{Aut}_B(A)$ of an $H$-Hopf–Galois extension is in general very small because, being a group, it is a classical object associated to a noncommutative principal bundle. Nonetheless in the relevant case of deformation quantization this notion is adequate. We show that for twist deformations of Hopf–Galois extensions, it leads to a gauge group which is isomorphic to the classical gauge group of the undeformed principal bundle.
Example 6.3. For simplicity we begin with the Galois object \( k \subseteq H \) with \( H \)-coaction given by the coproduct. Let \( H \) be commutative with trivial cotriangular structure \( R = \varepsilon \otimes \varepsilon \). Let \( \gamma : H \otimes H \to k \) be a 2-cocycle on \( H \) with convolution inverse \( \hat{\gamma} \) (every twist on a Hopf algebra dual to \( H \) gives a 2-cocycle on \( H \) via \( \gamma(h \otimes h') = (F^\alpha, h) \langle F_{\alpha}, h' \rangle \), see e.g. [2 App. A]). As in [5 §4] and dually to the theory in Section 5, the Hopf algebra \((H, R = \varepsilon \otimes \varepsilon)\) is deformed to the cotriangular Hopf algebra \((H_\gamma, R_\gamma = \gamma_2 \ast \hat{\gamma})\). The total space algebra \((A = H, \cdot, \Delta)\) is deformed as an \( H \)-comodule algebra to \((A_\gamma = H_{\gamma \ast}, \cdot, \Delta)\), where \( h \cdot h' = h_{(1)}h_{(1)}'(\hat{\gamma}(h_{(2)} \otimes h'_{(2)})) \) for all \( h, h' \in H \). We thus obtain the Hopf–Galois extension \( k \subseteq H_{\gamma \ast} \). This is a cleft extension with cleaving map \( j = \text{id}_H : H_\gamma \to H_{\gamma \ast} \), but in general needs not be a trivial extension since \( H_\gamma \) and \( H_{\gamma \ast} \) are in general not isomorphic as \( H_{\gamma \ast} \)-comodule algebras. The algebra \( H_{\gamma \ast} \) is quasi-commutative: \( h \cdot h' = h'_{(1)} \cdot h_{(1)} R(h_{(2)} \otimes h'_{(2)}) \) for all \( h, h' \in H \).

The gauge group of this noncommutative cleft extension \( k \subseteq H_{\gamma \ast} \) is isomorphic to the gauge group of the trivial extension \( k \subseteq H \). Indeed an algebra map \( F : H \to A \) that is also a right \( H \)-comodule map satisfies \( \Delta(F(h)) = F(h_{(1)}) \otimes h_{(2)} \) and therefore
\[
F(h \cdot h') = F(h_{(1)}h'_{(1)})\hat{\gamma}(h_{(2)} \otimes h'_{(2)}) = F(h_{(1)})F(h'_{(1)})\hat{\gamma}(h_{(2)} \otimes h'_{(2)}) = F(h \cdot h')
\]
(6.5) for all \( h, h' \in H_{\gamma \ast} \). This shows that \( F \) is also an algebra map \( F : H_{\gamma \ast} \to H_{\ast} \) and an \( H_{\gamma \ast} \)-comodule algebra map (since the coproduct of \( H_\gamma \) is the same as that of \( H \)).

Example 6.5. Consider the Hopf–Galois extension \( B = A^{\text{co}H} \subseteq A \) with \( A = \mathcal{O}(P) \), \( B = P/G \), \( H = \mathcal{O}(G) \), of a principal \( G \)-bundle \( P \to P/G \) of affine varieties. Its twisted deformation \( B = A^{\gamma \text{co}H} \subseteq A_{\gamma} \) has gauge group isomorphic to the one of \( P \to P/G \).

7. Braided Hopf algebra of gauge transformations

We study the gauge symmetry of a \( K \)-equivariant Hopf–Galois extension, for \( K \) a triangular Hopf algebra. This is an \( H \)-Hopf Galois extension \( B = A^{\text{co}H} \subseteq A \) with \( A \) a \( K \)-equivariant \( H \)-comodule algebra, as defined in Section 2.2 and \( B \) a \( K \)-submodule (this
holds for example when $K$ is flat as $k$-module, cf. \cite[Prop. 3.12]{2}). In this context there is a $K$-braided Hopf algebra of (infinitesimal) gauge transformations that generalizes the notion of gauge group of the previous section.

According to the definition of gauge group $\text{Aut}_B(A)$ in \cite[(6.2)]{2}, infinitesimal gauge transformations of a Hopf–Galois extension $B = A^\text{coH} \subseteq A$ are $H$-comodule maps that are vertical derivations

$$\text{aut}_B(A) := \{ u \in \text{Hom}(A, A) \mid \delta(u(a)) = u(a_{(0)}) \otimes a_{(1)}, \quad u(aa') = u(a)a' + au(a'), \quad u(b) = 0, \text{ for all } a, a' \in A, b \in B \}.$$  

They form a Lie algebra with Lie bracket given by the commutator.

In the $K$-equivariant case we can require a braided derivation property.

**Definition 7.1.** Let $(K, R)$ be a triangular Hopf algebra. Infinitesimal gauge transformations of a $K$-equivariant Hopf–Galois extension $B = A^\text{coH} \subseteq A$ are $H$-comodule maps that are braided vertical derivations

$$\text{aut}^R_B(A) := \{ u \in \text{Hom}(A, A) \mid \delta(u(a)) = u(a_{(0)}) \otimes a_{(1)}, \quad u(aa') = u(a)a' + (R_a \triangleright a)(R^\ast \triangleright u)(a'), \quad u(b) = 0, \text{ for all } a, a' \in A, b \in B \}.$$  

In this context infinitesimal gauge transformations form a Lie algebra object in the category of $K$-modules. For a $(K, R)$-module algebra $A$, recall that $\text{Der}^R(A)$ as in \cite[(5.3)]{5, 3} denotes the braided Lie algebra of braided derivations of $A$.

**Proposition 7.2.** The linear space $\text{aut}^R_B(A)$ with bracket

$$[\ , \ ]_R : \text{aut}^R_B(A) \otimes \text{aut}^R_B(A) \to \text{aut}^R_B(A)$$

$$u \otimes u' \mapsto [u, u']_R := u \circ u' - R_a \triangleright u' \circ R^\ast \triangleright u,$$

for all $u, u' \in \text{aut}^R_B(A)$, is a $K$-braided Lie subalgebra of $\text{Der}^R(A)$.

**Proof.** We show that $\text{aut}^R_B(A)$ is the intersection of three $K$-modules. Firstly, the $k$-submodule $\text{Hom}_{A^H}(A, A) \subseteq \text{Hom}(A, A)$ of $H$-equivariant $k$-linear maps from $A$ to $A$ is a $K$-submodule of $\text{Hom}(A, A)$. Indeed commutativity of the $K$-action with the $H$-coaction on $A$ is equivalent to $H$-equivariance, $\delta(\mathcal{L}_a) = (\mathcal{L}_a \otimes \text{id}_A)\delta(a)$, of the map $\mathcal{L}_a : A \to A$, $\mathcal{L}_a(a) = k \triangleright a$ for all $k \in K$ and $a \in A$. Then for $u \in \text{Hom}_{A^H}(A, A)$ and from $(k \triangleright_{\mathcal{L}(a)} u)(a) = k_{(1)} \triangleright (u(S(k_{(2)}) \triangleright a))$ for all $a \in A$ (cf. \cite[(2.3)]{2}), we have $k \triangleright_{\mathcal{L}(a)} u = \mathcal{L}_{k_{(1)}} \circ u \circ \mathcal{L}_{S(k_{(2)})} \in \text{Hom}_{A^H}(A, A)$ since composition of $H$-equivariant maps.

Since $B$ is a $K$-module, the $k$-submodule $\text{Hom}_B(A, A) \subseteq \text{Hom}(A, A)$ of linear maps that annihilate $B$ (that is of vertical maps) is a $K$-submodule of $\text{Hom}(A, A)$. Hence $\text{aut}^R_B(A) = \text{Hom}_{A^H}(A, A) \cap \text{Der}^R(A) \cap \text{Hom}_B(A, A)$ is a $K$-submodule since intersection of $K$-submodules.

The bracket in \cite[(7.3)]{2} is just the restriction to $\text{aut}^R_B(A)$ of the one of $\text{Der}^R(A)$ defined in \cite[(5.7)]{5}. It closes in $\text{aut}^R_B(A)$: being the braided Lie bracket of $\text{Der}^R(A)$ a braided commutator, $H$-equivariance and verticality follow. \hfill $\square$

In this braided context infinitesimal gauge transformations of a $K$-equivariant Hopf–Galois extension $B = A^\text{coH} \subseteq A$ are thus encoded in the braided Lie algebra $\text{aut}^R_B(A)$,
or equivalently, in the braided Hopf algebra $\mathcal{U}(\text{aut}^R_B(A))$ associated with $\text{aut}^R_B(A)$ (cf. Corollary 5.12). We recover the Lie algebra (7.1) of (usual) vertical derivations when $K = k$ with $R = 1 \otimes 1$.

### 7.1. Quantum principal bundle over quantum homogeneous space

A quantum subgroup of a Hopf algebra $A$ is a Hopf algebra $H$ together with a surjective bialgebra (and thus Hopf algebra) homomorphism $\pi : A \to H$. Then $A$ is a right $H$-comodule algebra via the projection of the coproduct

$$\delta := (\text{id} \otimes \pi) \circ \Delta : A \to A \otimes H.$$ (7.4)

When $B = A^{coH} \subseteq A$ is a Hopf–Galois extension, we call $A$ a quantum principal bundle over the quantum homogeneous space $B$ (see e.g. [12, §5.1]).

As in Example 5.10 let $A$ be cotriangular with dual Hopf algebra $U$ that is triangular with $\mathcal{R} \in U \otimes U$ (here a topological tensor product is understood if $U$ is not finite dimensional over $k$). From the $U^{op} \otimes U$-action $\triangleright : U^{op} \otimes U \otimes A \to A$, $(\zeta \otimes \xi) \triangleright a = (\zeta, a_{(1)}) (\xi, a_{(2)})$ the right coaction of $A$ on itself $\Delta : A \to A \otimes A$ and the $U^{op}$-action commute, hence so do the right $H$-coaction $\delta : A \to A \otimes H$ and the $U^{op}$-action (this is in general not the case for the $U$-action). It follows that the quantum principal bundle $A$ over the quantum homogeneous space $B$ is a $K$-equivariant Hopf–Galois extension $B = A^{coH} \subseteq A$ with $(K, R) = (U^{op}, \mathcal{R})$. The associated $K$-braided Lie algebra of infinitesimal gauge transformations is $\text{aut}^R_B(A) \subseteq \text{Def}^R(A)$.

We study the relation between $\text{aut}^R_B(A)$ and the braided Lie algebra of braided derivations $\text{Der}^{\mathcal{R}}(A) = A \otimes \text{Der}^{\mathcal{R}}(A)_{\text{inv}}$ of the cotriangular Hopf algebra $A$ in Example 5.10. Let $(H, R_H)$ be cotriangular and the cotriangular structure on $A$ be given by that on $H$, that is, $R(a \otimes a') = R_H(\pi(a) \otimes \pi(a'))$ for all $a, a' \in A$. In terms of the triangular structure $(U_H, \mathcal{R}_H)$ dual to $(H, R_H)$ we have $U_H \subseteq U$ and $\mathcal{R}_H = \mathcal{R} \in U_H \otimes U_H \subseteq U \otimes U$. This implies $\mathcal{R}_H := \mathcal{R}_H = \mathcal{R}$ so that

$$\text{Der}^{\mathcal{R}}_H(A) = \text{Der}^R(A)$$ (7.5)

as linear spaces of braided derivations; the first is a $K_H = U_H^{op}$-braided Lie algebra, the second is a $K = U^{op}$-braided Lie algebra. Similarly, $\mathcal{R}_H = \mathcal{R}$, so that $\text{Der}^{\mathcal{R}}_H(A) = \text{Der}^{\mathcal{R}}(A)$ as linear spaces; the first is a $U_H^{op} \otimes U_H$-braided Lie algebra, the second is a $U^{op} \otimes U$-braided one. The linear subspace of $H$-equivariant derivations of $A$,

$$\text{Der}^{\mathcal{R}}_M(A) = \{u \in \text{Der}^{\mathcal{R}}_H(A) : \delta(u(a)) = u(a_{(0)} \otimes a_{(1)}) \text{ for all } a \in A\}$$

is a $U_H^{op} \otimes U_H$-braided Lie subalgebra of $\text{Der}^{\mathcal{R}}_H(A)$. Indeed the action of $U_H$ is trivial while the action of $U_H^{op}$ closes in $\text{Der}^{\mathcal{R}}_M(A)$, with proofs similar to those for $\text{Der}^{\mathcal{R}}_H(A)_{\text{inv}}$ in Example 5.10.

**Theorem 7.3.** Let $(A, R)$ be a cotriangular Hopf algebra with dual triangular Hopf algebra $(U, \mathcal{R})$. Let $B = A^{coH} \subseteq A$ be a quantum principal bundle over the quantum homogeneous space $B$, with Hopf algebra projection $\pi : A \to H$. Assume $(H, R_H)$ is cotriangular with $\mathcal{R} = R_H \otimes (\pi \otimes \pi)$.

Then the braided gauge transformations $\text{aut}^R_B(A)$ are the vertical braided vector fields in $B \otimes \text{Der}^{\mathcal{R}}_M(A)_{\text{inv}}$, where $\text{Der}^{\mathcal{R}}_M(A)_{\text{inv}}$ are the right-invariant vector fields defining the bicovariant differential calculus on $(A, R)$. This linear space isomorphism is a $U_H^{op}$-braided Lie algebra isomorphism, where $U_H \subseteq U$ is the triangular Hopf algebra dual to $H$. 

40
Proof. Recall from Example 5.10 that Der\(^{\mathfrak{R}}\)(A) = A \otimes \text{Der}\(^{\mathfrak{R}}\)(A)\text{inv}. Here Der\(^{\mathfrak{R}}\)(A)\text{inv} are the right-invariant vector fields for the A-coaction (and thus for the H-coaction); they are H-equivariant. Then,

\[
\text{Der}\(_{\mathfrak{M}^H}\)\text{inv}(A) = B \otimes \text{Der}\(_{\mathfrak{M}^H}\)(A)\text{inv}.
\]

Recall the linear space equalities Der\(^{\mathfrak{R}}\)(A) = Der\(^{\mathfrak{R}H}\)(A) and the analogous one in (7.5). From the proof of Proposition 7.2, Der\(_{\mathfrak{M}^H}\)(A) = Der\(_{\mathfrak{R}H}\)(A) \cap \text{Hom}_{\mathfrak{M}^H}(A, A) is a U\(^{op}\)-braided Lie algebra and thus can be seen as the U\(^{op}\)-braided Lie algebra Der\(_{\mathfrak{M}^H}\)(A) = Der\(_{\mathfrak{R}H}\)(A) \cap \text{Hom}_{\mathfrak{M}^H}(A, A). The equality of U\(^{op}\)-braided Lie algebras

\[
\text{Der}\(_{\mathfrak{M}^H}\)(A) = \text{Der}\(_{\mathfrak{M}^H}\)(A)\text{inv}
\]

is immediate since the H-action on Der\(_{\mathfrak{M}^H}\)(A) is trivial so that the braiding \(\mathfrak{R}_H\) acts as \(\mathfrak{R}_H\). We thus have the U\(^{op}\)-braided Lie algebra isomorphism

\[
\text{Der}\(_{\mathfrak{M}^H}\)(A) = B \otimes \text{Der}\(_{\mathfrak{M}^H}\)(A)\text{inv}.
\]

The theorem is proven restricting this isomorphism to vertical vector fields. \(\square\)

Given a cotriangular Hopf algebra \(A\) the Galois object \(k = A\otimes A \subseteq A\) is a quantum principal bundle over the quantum homogeneous space \(B = k\). Here \((A, R) = (H, R_H)\) so that,

**Corollary 7.4.** Let \((A, R)\) be a cotriangular Hopf algebra with dual triangular Hopf algebra \((U, \mathfrak{R})\). The U\(^{op}\)-braided Lie algebra of infinitesimal gauge transformations aut\(_{\mathfrak{R}}\)(A) of the Galois object \(k = A\otimes A \subseteq A\) is isomorphic to the U\(^{op}\)-braided Lie algebra Der\(_{\mathfrak{M}^H}\)(A)\text{inv} of right-invariant vector fields which define the bicovariant differential calculus on \((A, R)\).

This theorem and the corollary show that Definition 7.1 of braided infinitesimal gauge transformations aut\(_{\mathfrak{R}}\)(A) captures a much richer structure – related to the bicovariant differential calculus – than that of the Lie algebra aut\(_{\mathfrak{R}}\)(A) in (7.1).

**Example 7.5.** Cotriangular quantum group \(\mathcal{O}(U_q(2))\) (the multiparametric quantum group \(\mathcal{O}(U_{q,r}(2))\) with \(r = 1\), see e.g. [32]). We recall that \(\mathcal{O}(U_q(2))\) is generated by the entries of the matrix \(T = (t_{ij}) := (a \ b \ c \ d)\), and by the inverse \(D\) of the quantum determinant \(D = ad - q^{-1}bc\). They satisfy the commutation relations

\[
\begin{align*}
ab &= q^{-1}ba; \quad ac = qca; \quad bd = qdb; \quad cd = q^{-1}dc; \quad bc = q^2cb; \quad ad = da \\
\end{align*}
\]

\[
\begin{align*}
aD^{-1} &= D^{-1}a; \quad bD^{-1} = q^{-2}D^{-1}b; \quad cD^{-1} = q^2D^{-1}c; \quad dD^{-1} = D^{-1}d.
\end{align*}
\]

The costructures are

\[
\Delta(t_{ij}) = \sum_{k=1,2} t_{ik} \otimes t_{kj}, \quad \Delta(D^{-1}) = D^{-1} \otimes D^{-1}, \quad \text{and} \quad \varepsilon(t_{ij}) = \delta_{ij}, \quad \varepsilon(D^{-1}) = 1, \quad \text{while the antipode is} \quad S(T) = D^{-1} \left( \begin{array}{cc} d & -q^{-1}b \\ -qc & a \end{array} \right), \quad S(D^{-1}) = D.
\]

The *-structure defining the real form \(\mathcal{O}(U_q(2))\) requires the deformation parameter to be a phase; the *-structure is then given by \(\left( \begin{array}{cc} a^* & b^* \\ c^* & d^* \end{array} \right) = D^{-1} \left( \begin{array}{cc} d & -q^{-1}b \\ -qc & a \end{array} \right), \quad (D^{-1})^* = D\).

From these relations it is easy to see that Aut\(_{\mathfrak{R}}\)(\(\mathcal{O}(U_q(2))\)) = \text{Char}(\mathcal{O}(\mathcal{O}(U_q(2)))) = U(1) \times U(1) \text{ with abelian gauge Lie algebra aut}_{\mathfrak{R}}\mathcal{O}(U_q(2)) = \mathbb{R}^2. This is half the vector space dimension of the commutative case (\(q = 1\)). Definition 7.1 gives on the other hand a braided gauge Lie algebra aut\(_{\mathfrak{R}}\)(\(\mathcal{O}(U_q(2))\)) that is non-abelian and 4-dimensional since the bicovariant differential calculus on \(\mathcal{O}(U_q(2))\) is 4-dimensional. Indeed, since \(\mathcal{O}(U_q(2)) = \mathcal{O}(U(2))_F\) is a twist deformation of the coordinate ring on \(U(2)\) [32], the calculus can be obtained as twist deformation of the calculus on \(U(2)\), hence from the linear space equalities Der(\(\mathcal{O}(U(2))\)) = Der(\(\mathcal{O}(U(2))\))_F = Der(\(\mathcal{O}(U(2))\))_F, cf. Theorem 5.10 the linear
space of vector fields has classical dimension. The four dimensional quantum Lie algebra is explicitly given in [5, Table 1 with conjugation iii) in (4.69) there. Notice that this is not a twist deformation of a principal bundle as in Example (6.3), rather \( \mathcal{O}(U_q(2)) \) it is a deformation of \( \mathcal{O}(U(2)) \) as a Hopf algebra. 

Remark 7.6. Braided derivations for a quantum principal bundle \( A \) over a quantum homogeneous space \( B = A^{coH} \) can be more intrinsically defined in terms of the cotriangular Hopf algebra \( (A, R) \), without referring to a dual triangular Hopf algebra \( (U, \mathcal{R}) \) giving \( (K, R) = (U^{op}, \mathcal{R}) \). Indeed, from Definition 5.3 of braided derivation, we have,

\[
(R_\alpha \triangleright a)(R^\alpha \triangleright \psi)(a') = \bar{R}_a(a_{(1)})R_\beta(a_{(2)})a_{(3)}R^\beta(a'_{(1)})\bar{R}^a(\psi(a'_{(2)}))\psi(a'_{(3)})
\]

for all \( a, a' \in A \), \( \psi \in \text{Hom}(A, A) \). Here we used the adjoint \( U^{op} \)-action on \( \text{Hom}(A, A) \), the properties of the universal matrix \( R \) and that \( \bar{R}(a \otimes a') = (\mathcal{R}, a \otimes a') = (R, a \otimes a') \), for all \( a, a' \in A \). Therefore, for \( (A, R) \) cotriangular we have the \( R \)-braided derivations

\[
\text{Der}^R(A) = \{ \psi : A \to A | \psi \text{ is } k \text{-linear,} \}
\]

\[
\psi(aa') = \psi(a)a' + \bar{R}_a(a_{(1)})R_\beta(a_{(2)})a_{(3)}R^\beta(a'_{(1)})\bar{R}^a(\psi(a'_{(2)}))\psi(a'_{(3)})
\]

with infinitesimal gauge transformations

\[
\text{aut}^R_B(A) = \{ u \in \text{Der}^R(A) | \delta(u(a)) = u(a_{(0)}) \otimes a_{(1)}, u(b) = 0, \text{ for all } a \in A, b \in B \}.
\]

These are expressed only in terms of the cotriangular Hopf algebra \( (A, R) \) and the Hopf–Galois extension \( B := A^{coH} \subseteq A \) with coaction given in (7.4).

When \( A \) is finite dimensional its cotriangularity is equivalent to the triangularity of its dual Hopf algebra \( U \) so that \( \text{Der}^R(A) = \text{Der}^{R^*}(A) \) and \( \text{aut}^R_B(A) = \text{aut}^{R^*}_{B^*}(A) \) as defined in Definition 7.4 with \( (K, R) = (U^{op}, \mathcal{R}) \).

8. Braided Hopf algebra gauge symmetry from twist deformation

Given a twist \( F \) of the Hopf algebra \( K \), the \( K \)-equivariant \( H \)-comodule algebra \( A \) is twisted to the \( K_F \)-equivariant \( H \)-comodule algebra \( A_F \) with multiplication defined in (3.1). The \( K \)-submodule algebra \( B = A^{coH} \) of \( A \) is twisted to the \( K_F \)-submodule algebra \( B_F = A_F^{coH} \). Furthermore \( B = A^{coH} \subseteq A \) is a Hopf–Galois extension (principal comodule algebra) if and only if \( B_F = A_F^{coH} \subseteq A_F \) is a Hopf–Galois extension (principal comodule algebra), see [2, Cor. 3.16 (Cor. 3.19)].

We study the twist deformation of infinitesimal gauge transformations when \( K \) has triangular structure \( R \) using the results in 5.4. In this case \( K_F \) has triangular structure \( R_F \) and the \( K \)-braided Lie algebra \( (\text{aut}^R_B(A_F), [ , ]_F) \) is twisted to the \( K_F \)-braided Lie algebra \( (\text{aut}^{R_F}_B(A_F), [ , ]_F) \) with bracket \( [ , ]_F = [ , ]_F \) as in Proposition 5.14. This is a braided Lie subalgebra of \( \text{Der}^R(A)_F, [ , ]_F \). We can also consider the \( K_F \)-braided Lie algebra \( (\text{aut}^{R_F}_B(A_F), [ , ]_F) \):

Proposition 8.1. The \( (K_F, R_F) \)-braided Lie algebras \( (\text{aut}^{R_F}_B(A_F), [ , ]_F) \) and \( (\text{aut}^{R_F}_B(A_F), [ , ]_F) \) are isomorphic via the map

\[
\mathcal{D} : \text{aut}^{R_F}_B(A_F) \to \text{aut}^{R_F}_B(A_F),
\]

the restriction of the isomorphism \( \mathcal{D} : \text{Der}^R(A)_F, [ , ]_F \to \text{Der}^{R_F}(A)_F, [ , ]_F \) of braided Lie algebras, in Theorem 5.13 to \( \text{aut}^{R_F}_B(A)_F \subseteq \text{Der}^R(A)_F \).
Proof. Since $B_\mathbb{F}$ is a $K_\mathbb{F}$-module (and a $K$-module since the $K_\mathbb{F}$ and $K$ algebra structures coincide), the isomorphism $\mathcal{D}$ maps vertical derivations into vertical derivations: $\mathcal{D}(u)(b) = 0$ for any $u \in \text{aut}^R_{B_\mathbb{F}}(A_\mathbb{F})$ and $b \in B_\mathbb{F}$. Equivariance of $\mathcal{D}(u)$ under the $H$-coaction is straightforward from $H$-equivariance of $u$ and the compatibility of the $K$-action (or the $K_\mathbb{F}$-action) with the $H$-coaction (cf. [2.11]). Verticality and $H$-equivariance of $\mathcal{D}^{-1}(u_\mathbb{F})$ for any $u_\mathbb{F} \in \text{aut}^R_{B_\mathbb{F}}(A_\mathbb{F})$ are similarly proven. As a consequence $\mathcal{D} : \text{aut}^R_B(A)_\mathbb{F} \rightarrow \text{aut}^R_{B_\mathbb{F}}(A_\mathbb{F}) \subseteq \text{Der}^R_{B_\mathbb{F}}(A_\mathbb{F})$ is injective and surjective. \hfill \qed

We can also consider the $(K, \mathbb{R})$-braided Hopf algebra $U(\text{aut}^R_B(A))$. Under a twist $F$ of $K$ we have the $(K_\mathbb{F}, \mathbb{R}_\mathbb{F})$-braided Hopf algebra $U(\text{aut}^R_B(A)_\mathbb{F})$:

**Corollary 8.2.** The twist $U(\text{aut}^R_B(A))_F$ of the $(K, \mathbb{R})$-braided Hopf algebra $U(\text{aut}^R_B(A))$ of the Hopf–Galois extension $B = A^{coH} \subseteq A$ is isomorphic to the $(K_\mathbb{F}, \mathbb{R}_\mathbb{F})$-braided Hopf algebra $U(\text{aut}^R_{B_\mathbb{F}}(A_\mathbb{F}))$ of the Hopf–Galois extension $B_\mathbb{F} = A^{coH}_{\mathbb{F}} \subseteq A_\mathbb{F}$.

**Proof.** From Proposition 5.18 the braided Hopf algebra $U(\text{aut}^R_B(A))_F$ is isomorphic to $U(\text{aut}^R_B(A)_F)$. As a corollary of Proposition 8.1 this latter is isomorphic to the $(K_\mathbb{F}, \mathbb{R}_\mathbb{F})$-braided Hopf algebra $U(\text{aut}^R_{B_\mathbb{F}}(A_\mathbb{F}))$. \hfill \qed

8.1. **Gauge transformations of twisted principal bundles.** Braided Lie algebra and Hopf algebra gauge symmetries arise in particular by considering twist deformations of commutative principal bundles of affine varieties. In this case $K = U(\text{Lie}(L))$ is a commutative Hopf algebra and $H = \mathcal{O}(G)$ a commutative Hopf algebra with $L$ and $G$ affine algebraic groups and $\mathcal{O}(G)$ the coordinate ring of $G$. The $L$-equivariant principal bundle of affine varieties $P \rightarrow M = P/G$ gives the $(K, \mathbb{R} = 1 \otimes 1)$-equivariant Hopf–Galois extension $B = A^{coH} \subseteq A$, where $A = \mathcal{O}(P)$, $B = \mathcal{O}(M)$ are the coordinate rings of $P$ and $M$. The Lie algebra $\text{aut}_B(A)$ and the Hopf algebra $U(\text{aut}_B(A))$ of infinitesimal gauge transformations are braided Lie and Hopf algebras with trivial braiding $\mathbb{R} = 1 \otimes 1$. Under a twist $F$ of $K$ we obtain the $(K_\mathbb{F}, \mathbb{R}_\mathbb{F} = F_\mathbb{R}^{-1})$-braided Lie and Hopf algebras $\text{aut}^R_{B_\mathbb{F}}(A_\mathbb{F})$ and $U(\text{aut}^R_{B_\mathbb{F}}(A_\mathbb{F}))$ of the Hopf–Galois extension $B_\mathbb{F} = A^{coH}_{\mathbb{F}} \subseteq A_\mathbb{F}$.

Due to the isomorphisms $\text{aut}^R_B(A)_F \simeq \text{aut}^R_{B_\mathbb{F}}(A_\mathbb{F})$ and $U(\text{aut}^R_B(A))_F \simeq U(\text{aut}^R_{B_\mathbb{F}}(A_\mathbb{F}))$ of Proposition 8.1 and Corollary 8.2 these braided Lie and Hopf algebras coincide as linear spaces with the initial Lie and Hopf algebras $\text{aut}_B(A)$ and $U(\text{aut}_B(A))$. This has to be compared with the (unbraided) gauge Lie and Hopf algebras $\text{aut}_B(A_F)$ and $U(\text{aut}_B(A_F))$ as defined in (7.1) which are usually much smaller as vector spaces (cf. Example 7.5).

In the following we illustrate this constructions with a few explicit examples considering twists arising from actions of tori, $L = \mathbb{R}^2$ (abelian twists), and from actions of $L = \mathbb{R}_{>0} \rtimes \mathbb{R}$, the affine group $ax + b$ of the real line (Jordanian twists).

Let us start with a $\mathbb{R}^2$-equivariant principal bundle $P \rightarrow M = P/G$ as before. The universal enveloping algebra $K$ of $\mathbb{R}^2$ is generated by two commuting elements $H_1, H_2$. Due to the torus action, these act as derivations on $A$ and on $B$. The twist

$$F = e^{\pi i \theta (H_1 \otimes H_2 - H_2 \otimes H_1)}$$

of $K$, with $\theta \in \mathbb{R}$, then defines the algebras $A_\theta := A_F$ and $B_\theta := B_\mathbb{F}$ with multiplication given in (1.10). The element $F$ actually belongs to a topological completion of the algebraic tensor product $K \hat{\otimes} K$ which does not play a role here, see Remark 4.2. The algebras $A_\theta$ and $B_\theta$ are equivalently obtained considering the 2-cocycle $\gamma : \mathcal{O}(\mathbb{R}^2) \otimes \mathcal{O}(\mathbb{R}^2) \rightarrow \mathbb{C}, \ t \otimes t' \mapsto \gamma(t \otimes t') := \langle F, t \otimes t' \rangle$ and the 2-cocycle deformation of $B$ via the coaction of $\mathcal{O}(\mathbb{R}^2)$ dual to the action of $K$. 43
Example 8.3. Trivial bundle. Consider the trivial Hopf–Galois extension \( B \subseteq B \otimes H \), with \( B = \mathcal{O}(M) \) and \( H = \mathcal{O}(G) \) the coordinate algebras of an algebraic affine variety \( M \) and a group \( G \). The gauge group of this bundle consists of affine variety maps \( \mathbb{T}^2 \to G \), that is, algebra maps \( H \to B \). The Lie algebra of infinitesimal gauge transformations is the free \( B \)-module

\[
\text{aut}_{B}(B \otimes H) = B \otimes \text{Lie}(G) \tag{8.2}
\]

with \( \text{Lie}(G) \) the right-invariant vector fields on \( G \). Equivalently (8.2) is the Lie algebra of vertical derivations of \( A = B \otimes H \). If \( \{ \chi^i \} \) \((i = 1, \ldots, \dim(G))\) is a basis of the tangent space \( T_e G \) at the unit of \( G \), then \( \{ X^i \} \), with \( X^i(h) = \chi^i(h) \cdot h \), \( h \in H \), is a basis of right-invariant vector fields on \( G \). The Lie bracket is \( [b_i X^i, b_j' X^j] = b_i b_j'[X^i, X^j] \), for all \( b_i, b_j' \in B \).

Let \( L = \mathbb{T}^2 \) act on \( M \). With the twist (8.1) the Hopf–Galois extension \( B \subseteq B \otimes H \), is deformed to \( B_\theta \subseteq A_\theta = B_\theta \otimes H \). Here we use \( \ast \) to denote the multiplications \( \ast \) in \( B_\mathbb{F} = B_\theta \) and \( A_\mathbb{F} = A_\theta \). The corresponding infinitesimal gauge transformations form the braided Lie algebra and free \( B_\theta \)-module

\[
\text{aut}_{B_\theta}^R(B_\theta \otimes H) = \mathcal{D} (\text{aut}_{B}(B \otimes H)_\theta) = B_\theta \otimes \text{Lie}(G) \tag{8.3}
\]

where \( \mathcal{D} \) is the Drinfeld double. For the last equality we first observe that \( \mathcal{D}(X^i) = X^i \) for \( X^i \in \text{Lie}(G) \subseteq \text{Der}(A) \). Indeed, the twist acts only on \( B \) while the right-invariant vector field \( X^i \) acts only on \( H \) and so, for all \( b \otimes h \in A_\theta = B_\theta \otimes H \),

\[
\mathcal{D}(X^i)(b \otimes h) = (X^i(S(\mathcal{F}_\theta^i))\mathcal{F}_\alpha \triangleright (b \otimes h)) = (X^i)(S(\mathcal{F}_\theta^i))\mathcal{F}_\alpha \triangleright b \otimes X^i(h) = b \otimes X^i(h) = X^i(b \otimes h).\]

Then \( [X^i, X^j]_{R_\mathbb{F}} = \mathcal{D}([X^i, X^j]_\mathbb{F}) = [X^i, X^j] \) since \( [X^i, X^j]_\mathbb{F} = [X^i, X^j] \). From (7.28) the braided Lie bracket (defined in (7.3)) explicitly reads, for all \( b_i, b_j' \in B \),

\[
[b_i \ast X^i, b_j' \ast X^j]_{R_\mathbb{F}} = b_i \ast b_j' \ast [X^i, X^j],
\]

being \( (b_i \ast X^i)(a) = b_i \ast X^i(a) \) for all \( a \in (B_\theta \otimes H) \), cf. (5.27).

From Proposition 7.4 the universal enveloping algebra \( U(B_\theta \otimes \text{Lie}(G)) \) of this braided Lie algebra is a \( K_\mathbb{F} \)-braided Hopf algebra (since \( K \) is abelian \( K_\mathbb{F} \) coincides with \( K \) as algebra but has triangular structure \( R_\mathbb{F} \)). The coproduct, counit and antipode are uniquely determined on the generators \( b_i \ast X \in B_\theta \otimes \text{Lie}(G) \) as

\[
\Delta(b_i \ast X) = b_i \ast X \boxtimes 1 + 1 \boxtimes b_i \ast X, \quad \varepsilon(b_i \ast X) = 0, \quad S(b_i \ast X) = -b_i \ast X.
\]

We briefly compare this result with the approach to the gauge group defined in equation (6.2) and based on algebra maps (not considering twist deformations of the structure Hopf algebras as in Proposition 6.4). An algebra map \( \Gamma : B_\theta \otimes H \to B_\theta \otimes H \) which is the identity on \( B_\theta \otimes 1 \) is determined by its restriction to \( G_{1 \otimes H} : 1 \otimes H \to B_\theta \otimes H \). Using the algebra map property of \( \Gamma \) we further have, for all \( b \in B_\theta, h \in H \),

\[
\Gamma(1 \otimes h)(b \otimes 1) = \Gamma(1 \otimes h)\Gamma(b \otimes 1) = \Gamma((1 \otimes h)(b \otimes 1)) = \Gamma((b \otimes 1)(1 \otimes h)) = (b \otimes 1)\Gamma(1 \otimes h)
\]

that shows that \( G_{1 \otimes H} : 1 \otimes H \to Z(B_\theta) \otimes H \), where \( Z(B_\theta) \) is the center of \( B_\theta \). This is drastically different from the commutative case. For example if \( M = \mathbb{T}^2 \) then \( B_\theta = \mathcal{O}(\mathbb{T}^2) \) and, for \( \theta \) irrational, \( Z(B_\theta) = \mathbb{C} \) and the gauge group is just the structure group \( G \).

Example 8.4. The instanton bundle on the sphere \( S^4_\theta \). The spheres \( S^7 \) and \( S^4 \) are the homogeneous spaces \( S^7 = \text{Spin}(5)/\text{SU}(2) \) and \( S^4 = \text{Spin}(5)/\text{Spin}(4) \simeq \text{Spin}(5)/\text{SU}(2) \times \)
SU(2), hence the Hopf fibration $S^7 \to S^4$ is a Spin(5)-equivariant SU(2)-principal bundle. Then, the right-invariant vector fields $X \in \mathfrak{so}(5) \simeq \text{spin}(5)$ on Spin(5) project to the right cosets $S^7$ and $S^4$ and generate the $\mathcal{O}(S^7)$-module of vector fields on $S^7$ and the $\mathcal{O}(S^4)$-module of vector fields on $S^4$. This latter is the submodule of right SU(2)-invariant vector fields on $S^7$. A convenient generating set for the $\mathcal{O}(S^7)$-module is given by the following right SU(2)-invariant vector fields (cf. [20]):

$$H_1 = \frac{1}{4}(z_1 \partial_1 - z_1 \partial_2^* - z_2 \partial_2 + z_3 \partial_3^* + z_4 \partial_4 - z_4 \partial_4^*)$$
$$H_2 = \frac{1}{4}(-z_1 \partial_1 + z_1 \partial_2^* + z_2 \partial_2 - z_3 \partial_3^* - z_4 \partial_4 + z_4 \partial_4^*)$$

(8.4)

$$E_{10} = \frac{1}{\sqrt{2}}(z_1 \partial_3 - z_2 \partial_4 + z_3 \partial_4^* + z_4 \partial_4^*)$$
$$E_{01} = \frac{1}{\sqrt{2}}(z_2 \partial_3 - z_3 \partial_4^* + z_4 \partial_4 - z_1 \partial_1^*)$$
$$E_{11} = -z_4 \partial_3 + z_3 \partial_3^*$$
$$E_{11} = -z_1 \partial_2 + z_2 \partial_2^*$$

(8.5)

Here $z_1, z_2, z_3, z_4$ generate the coordinate ring of $S^7$ with $z_1 z_1^* + z_2 z_2^* + z_3 z_3^* + z_4 z_4^* = 1$, the partial derivatives $\partial_{\mu}, \partial_{\mu}^*$, are defined by $\partial_{\mu}(z_\nu) = \delta_{\mu\nu}$ and $\partial_{\mu}(z_\nu^*) = 0$ and similarly for $\partial_{\mu}^*$, $\mu, \nu = 1, 2, 3, 4$. The above vector fields are chosen so that their commutators close the Lie algebra $\mathfrak{so}(5)$ in the form

$$[H_1, H_2] = 0; \quad [H_j, E_r] = r_j E_r; \quad [E_r, E_{-r}] = r_1 H_1 + r_2 H_2; \quad [E_r, E_s] = N_{rs} E_{r+s}.$$  

(8.6)

Then, the elements $H_j, j = 1, 2$, are the generators of the Cartan subalgebra, and $E_r$ is labelled by $r = (r_1, r_2) \in \Gamma = \{(+1, 0), (0, +1), (+1, +1)\}$, one of the eight roots. Also, $N_{rs} = 0$ if $r+s$ is not a root and $N_{rs} \in \{1, -1\}$ otherwise. The $*$-structure is given by $H_j^* = H_j$ and $E_r^* = E_{-r}$. Since the $*$-structure on vector fields $X$ is defined by $X^*(f) = -(X(f^*))^*$ for any function $f$, one accordingly checks that for the vector fields in (8.3) and (8.5), $E_{-r}(z_\mu) = -(E_r(z_\mu)^*)^*$ and $H_j(z_\mu) = -(H_j(z_\mu^*))^*$. The general right SU(2)-invariant vector field on $S^7$, that is $H$-equivariant derivation of $\mathcal{O}(S^7)$, is

$$X = b_1 H_1 + b_2 H_2 + \sum_r b_r E_r$$

(8.7)

with $b_j, b_r \in \mathcal{O}(S^4)$. The derivations $X$ are real, $X^* = X$, if and only if $b_j^* = b_j$ and $b_r^* = b_{-r}$.

Infinitesimal gauge transformations are $H$-equivariant derivations $X$ which are vertical: $X(b) = 0$ for $b \in \mathcal{O}(S^4)$. As shown in [21], §4, they form the $\mathcal{O}(S^4)$-module generated by

$$K_1 = 2x H_2 + \beta^* \sqrt{2} E_{01} + \beta \sqrt{2} E_{0-1}$$
$$W_{01} = \sqrt{2}(\beta H_1 + \alpha^* E_{11} + \alpha E_{-11})$$
$$W_{10} = \sqrt{2}(\alpha H_2 - \beta^* E_{11} + \beta E_{-11})$$
$$W_{11} = 2x E_{11} + \alpha \sqrt{2} E_{01} - \beta \sqrt{2} E_{10}$$
$$W_{1-1} = -2x E_{-11} + \beta^* \sqrt{2} E_{10} + \alpha \sqrt{2} E_{0-1}$$

(8.8)

where $\alpha = 2(z_1 z_1^* + z_2^* z_4^*), \beta = 2(z_2 z_2^* - z_1 z_3^*), x = z_1 z_1^* + z_2 z_2^* - z_3 z_3^* - z_4 z_4^*$ satisfy the relation $\alpha^* \alpha + \beta^* \beta + x^2 = 1$ and are the coordinates of $\mathcal{O}(S^4) \subset \mathcal{O}(S^7)$. The commutators of the generators in (8.8) define the gauge Lie algebra since $[bX, b'X'] = bb'[X, X']$ for
any \( b, b' \in \mathcal{O}(S^4) \) and \( X, X' \in \text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7)) \). These generators satisfy \( K_j^* = K_j, \ W_i^* = W_{-i} \). They transform under the adjoint representation of \( \text{so}(5) \), in particular

\[
H_j \triangleright K_i = [H_j, K_i] = 0, \quad H_j \triangleright W_r = [H_j, W_r] = r_j W_r.
\] (8.9)

Due to \( \text{Spin}(5) \)-equivariance we have the decomposition, see [17, §4],

\[
\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7)) = \bigoplus_{n \in \mathbb{Z}_0} [d(2, n)]
\] (8.10)

where \([d(2, n)]\) is the representation of \( \text{so}(5) \) as derivations on \( \mathcal{O}(S^7) \) of highest weight vector \( \alpha^* W_{11} \) of weight \((n + 1, 1)\) and dimension \( d(2, n) = \frac{1}{2}(n + 1)(n + 4)(2n + 5) \). These derivations are combinations of the derivations in \((8.8)\) with spherical harmonics of degree \( n \) on \( S^4 \) (harmonic polynomials on \( \mathbb{R}^5 \) of homogeneous degree \( n \)).

The right-invariant vector fields \( H_1 \) and \( H_2 \) of \( \text{Spin}(5) \) are the vector fields of a maximal torus subgroup \( T^2 \subset \text{Spin}(5) \). They define the universal enveloping algebra \( K \) of the abelian Lie algebra \([H_1, H_2] = 0\). Their action \((8.3)\) on \( \mathcal{O}(S^7) \) commutes with the \( \mathcal{O}(SU(2)) \) right coaction on \( \mathcal{O}(S^7) \). The associated twist in \((8.1)\) corresponds to the torus 2-cocycle of \([2\text{ Ex. } 3.21]\), hence it gives the \( \mathcal{O}(SU(2))-\text{Hopf–Galois} \) extension \( \mathcal{O}(S^4) = \mathcal{O}(S^7)^{\text{cot}(SU(2))} \subset \mathcal{O}(S^7) \) introduced in [27]. In the following we use the subscript \( \theta \) instead of \( F \) for twisted algebras and their multiplications to conform with the literature. The twisted algebra \( \mathcal{O}(S^4) \) is generated by \( \alpha, \alpha^*, \beta, \beta^* \) with \( \alpha \ast \alpha^* + \beta \ast \beta^* + x \ast x = 1 \) where the only nontrivial commutation relations are

\[
\alpha \ast \beta = e^{2\pi i \theta} \beta \ast \alpha, \quad \alpha \ast \beta^* = e^{-2\pi i \theta} \beta^* \ast \alpha
\] (8.11)

and their complex conjugates. They are obtained from the general definition of twisted multiplication \([4.10]\) using that the coordinates \( \alpha, \beta, x \) are eigen-functions of \( H_1 \) and \( H_2 \).

The twist deformation of the gauge Lie algebra \( \text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7)) \) via the action \((8.9)\) is the \( \mathcal{O}(S^4) \)-module and braided Lie algebra \( (\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7)))_{\mathcal{F}} \) and from Proposition \(8.1\), we have \( \text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7)) = \mathcal{D}(\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7)))_{\mathcal{F}} \) and from Corollary \(5.20\) and equation \((5.28)\) we can conclude that (we denote the multiplications \( \ast \) in \( \mathcal{B}_\mathcal{F} = B_\theta \) and \( \mathcal{A}_\mathcal{F} = A_\theta \), and the module structures by \( \ast \)):

**Proposition 8.5.** The braided Lie algebra \( \text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7)) \) of infinitesimal gauge transformations of the \( \mathcal{O}(SU(2))-\text{Hopf–Galois} \) extension \( \mathcal{O}(S^4) \subset \mathcal{O}(S^7) \) is generated, as an \( \mathcal{O}(S^4) \)-module, by the elements

\[
\tilde{K}_j := \mathcal{D}(K_j) = K_j, \quad \tilde{W}_r := \mathcal{D}(W_r) = W_r e^{\pi i \theta(r_1 H_2 - r_2 H_1)}, \quad j = 1, 2, \quad r \in \Gamma
\] (8.13)

with bracket

\[
[\tilde{K}_1, \tilde{K}_2]_{\mathcal{F}} = \mathcal{D}([K_1, K_2]) \quad [\tilde{K}_j, \tilde{W}_r]_{\mathcal{F}} = \mathcal{D}([K_j, W_r])
\]

\[
[\tilde{W}_r, \tilde{W}_s]_{\mathcal{F}} = e^{-i\pi \theta/(s)} \mathcal{D}([W_r, W_s])
\]

The braided Lie bracket of generic elements \( \tilde{X}, \tilde{X}' \) in \( \text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7)) \) and \( b, b' \in \mathcal{O}(S^4) \) is given by

\[
[b \ast \tilde{X}, b' \ast \tilde{X}']_{\mathcal{F}} = b \ast (R_a \triangleright b') \ast (R^a \triangleright \tilde{X}, \tilde{X}')_{\mathcal{F}}
\] (8.14)

with \( \mathcal{O}(S^4) \)-module structure as in \((5.27)\).
In [7] the Lie brackets $[K_j, W_i]$ and $[W_i, W_j]$ are found. We now compute the brackets of the generators $\tilde{K}_j, \tilde{W}_i$ of the braided gauge Lie algebra $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^4))$. For example, the bracket $[\tilde{W}_{-1}, \tilde{W}_0] = \sqrt{2}\beta W_{-1} - \sqrt{2}\alpha^*(K_1 + K_2)$ is the same as $e^{-\pi i \theta}[W_{-1}, W_0]_F = e^{\pi i \theta} \sqrt{2}\beta \cdot \tilde{W}_{-1} - \sqrt{2}\alpha^* (K_1 + K_2)$, where we used the module structure of $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^4))_F$ given by $\eqref{5.2.1}$ and that the coordinates of the sphere $S_4$ are eigen-functions of $H_1$ and $H_2$. Then, by applying the map $\mathcal{D}$ we have that $[\tilde{W}_{-1}, \tilde{W}_0]_F = e^{2\pi i \theta} \sqrt{2}\beta \cdot \tilde{W}_{-1} - \sqrt{2}\alpha^*(K_1 + K_2)$. We list the independent braided Lie algebra brackets in the following table:

| Bracket | Expression |
|---------|------------|
| $[\tilde{K}_1, \tilde{K}_2]_{\mathcal{D}}$ | $= \sqrt{2}(\alpha^* \cdot \tilde{W}_{10} - \alpha \cdot \tilde{W}_{01})$ |
| $[\tilde{K}_1, \tilde{W}_{01}]_{\mathcal{D}}$ | $= -\sqrt{2}\beta \cdot \tilde{K}_2 + 2x \cdot \tilde{W}_{01}$ |
| $[\tilde{K}_2, \tilde{W}_{-1}]_{\mathcal{D}}$ | $= -2x \cdot \tilde{W}_{-1} + \sqrt{2}e^{\pi i \theta} \beta^* \cdot \tilde{W}_{01}$ |
| $[\tilde{K}_1, \tilde{W}_{10}]_{\mathcal{D}}$ | $= \sqrt{2}\beta \cdot \tilde{W}_{11} - \sqrt{2}\alpha^* \beta^* \cdot \tilde{W}_{01}$ |
| $[\tilde{K}_1, \tilde{W}_{11}]_{\mathcal{D}}$ | $= 2x \cdot \tilde{W}_{11} - \sqrt{2}\alpha^* \beta^* \cdot \tilde{W}_{01}$ |
| $[\tilde{K}_2, \tilde{W}_{01}]_{\mathcal{D}}$ | $= \sqrt{2}\beta \cdot \tilde{W}_{11} + \sqrt{2}e^{\pi i \theta} \alpha^* \cdot \tilde{W}_{01}$ |
| $[\tilde{K}_2, \tilde{W}_{-1}]_{\mathcal{D}}$ | $= 2x \cdot \tilde{W}_{-1} - \sqrt{2}\alpha^* \beta^* \cdot \tilde{W}_{01}$ |
| $[\tilde{K}_2, \tilde{W}_{10}]_{\mathcal{D}}$ | $= 2x \cdot \tilde{W}_{10} - \sqrt{2}\alpha^* \tilde{K}_1$ |
| $[\tilde{K}_2, \tilde{W}_{11}]_{\mathcal{D}}$ | $= 2x \cdot \tilde{W}_{11} + \sqrt{2}e^{\pi i \theta} \alpha^* \cdot \tilde{W}_{01}$ |
| $[\tilde{W}_{01}, \tilde{W}_{-1}]_{\mathcal{D}}$ | $= \sqrt{2}\beta \cdot \tilde{W}_{11} + \sqrt{2}e^{\pi i \theta} \alpha^* \cdot (\tilde{K}_2 - \tilde{K}_1)$ |
| $[\tilde{W}_{01}, \tilde{W}_{10}]_{\mathcal{D}}$ | $= \sqrt{2}\beta \cdot \tilde{W}_{10} - \sqrt{2}\alpha^* \beta^* \cdot \tilde{W}_{01}$ |
| $[\tilde{W}_{01}, \tilde{W}_{11}]_{\mathcal{D}}$ | $= \sqrt{2}\beta \cdot \tilde{W}_{11}$ |
| $[\tilde{W}_{10}, \tilde{W}_{-1}]_{\mathcal{D}}$ | $= \sqrt{2}\beta \cdot \tilde{W}_{11} + \sqrt{2}\alpha^* \tilde{W}_{01}$ |
| $[\tilde{W}_{10}, \tilde{W}_{11}]_{\mathcal{D}}$ | $= \sqrt{2}\alpha^* \cdot \tilde{W}_{11}$ |
| $[\tilde{W}_{-1}, \tilde{W}_{01}]_{\mathcal{D}}$ | $= \sqrt{2}e^{2\pi i \theta} \beta \cdot \tilde{W}_{01} - \sqrt{2}e^{\pi i \theta} \alpha^* \cdot (\tilde{K}_1 + \tilde{K}_2)$ |
| $[\tilde{W}_{-1}, \tilde{W}_{10}]_{\mathcal{D}}$ | $= \sqrt{2}e^{-2\pi i \theta} \beta \cdot \tilde{W}_{01}$ |
| $[\tilde{W}_{-1}, \tilde{W}_{11}]_{\mathcal{D}}$ | $= \sqrt{2}\beta \cdot \tilde{W}_{10} + \sqrt{2}e^{\pi i \theta} \beta^* \cdot (\tilde{K}_1 + \tilde{K}_2)$ |
| $[\tilde{W}_{10}, \tilde{W}_{-1}]_{\mathcal{D}}$ | $= \sqrt{2}\alpha^* \beta \cdot \tilde{W}_{01} - \sqrt{2}\alpha^* \tilde{W}_{01}$ |
| $[\tilde{W}_{10}, \tilde{W}_{11}]_{\mathcal{D}}$ | $= \sqrt{2}\alpha^* \beta \cdot \tilde{W}_{01} + \sqrt{2}\alpha^* \beta \cdot \tilde{W}_{01}$ |

\textbf{Table 1}
The remaining brackets are obtained via \(\ast\)-conjugation using \( ([\tilde{X}, \tilde{X}]_{R^F})^\ast = [\tilde{X}^\ast, \tilde{X}^\ast]_{R^F} \) and \((b \ast \tilde{X})^\ast = R_{\tilde{F}} b \ast R_{\tilde{F}}^{\ast\ast} \tilde{X}^\ast\), where \(R_F = F_{21} \bar{F} = \bar{F}^2\). These expressions hold since \(F\) is compatible with \(\ast\)-conjugation: \(F^\ast \cap \ast = (S \otimes S)F_{21}\), (for details see e.g. [3] Sect. 8). They can also readily be obtained by computing the brackets of the missing generators starting from the undeformed brackets.

From Proposition 5.4 we have that the universal enveloping algebra \(U(\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))))\) of this \(K_F\)-braided Lie algebra is a \(K_F\)-braided Hopf algebra. The algebra is the quotient of the tensor algebra of \(\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7)))\) modulo the ideal generated by the braided Lie algebra relations \([8, 14]\). The coproduct, counit and antipode are uniquely determined on the generators \(b \ast \tilde{X} \in \text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7)))\) as

\[
\Delta(b \ast \tilde{X}) = b \ast \tilde{X} \otimes 1 + 1 \otimes b \ast \tilde{X}, \quad \varepsilon(b \ast \tilde{X}) = 0, \quad S(b \ast \tilde{X}) = -b \ast \tilde{X}.
\]

Let us explicitly check the coproduct for the derivations \(\tilde{W}_t\). For \(a_s \in \mathcal{O}(S^7)\) an eigenfunction of \(H_t\) of eigenvalue \(s_j\), the derivation \(\tilde{W}_t\) acts as

\[
\tilde{W}_t(a_s) = (\mathcal{F}^s \triangleright W_t)(\mathcal{F}^s \triangleright a_s) = e^{-i\pi r/s}W_t(a_s).
\]

On the product of two such eigen-functions \(a_s, a_m \in \mathcal{O}(S^7)\), we can explicitly see that \(\tilde{W}_t\) acts as a braided derivation, with respect to the braiding \(R_F = F_{21} \bar{F} = \bar{F}^3\):

\[
\tilde{W}_t(a_s \ast a_m) = e^{-i\pi r/s + (s+m)/s} \tilde{W}_t(a_s a_m)
\]

\[
= e^{-i\pi r/s + (s+m) + (s+m)/s} \left[ W_t(a_s)W_t(a_m) + sW_t(a_s)W_t(a_m) \right]
\]

\[
= e^{-i\pi r/s + (s+m) + (s+m)/s} \left[ e^{-i\pi s/4} W_t(a_s) \ast a_m + e^{i\pi s/4} a_s \ast W_t(a_m) \right]
\]

\[
= e^{-i\pi r/s} W_t(a_s) \ast a_m + e^{-i\pi r/s} W_t(a_s) \ast a_m + e^{-i\pi r/s} W_t(a_s) \ast a_m + e^{-i\pi r/s} W_t(a_m) \ast a_m
\]

\[
= \Delta(\tilde{W}_t)(a_s \ast a_m).
\]

Example 8.6. The quantum orthogonal bundle on the homogeneous space \(S^4\). We describe the principal bundle \(\mathcal{O}(SO(5, \mathbb{R})) \rightarrow \mathcal{O}(SO(5, \mathbb{R}))/\mathcal{O}(SO(4, \mathbb{R})) = S^4\) as the Hopf–Galois extension \(\mathcal{O}(S^4) \subset \mathcal{O}(SO(5, \mathbb{R}))\). Here \(\mathcal{O}(SO(5, \mathbb{R}))\) is the Hopf algebra generated by the commuting entries of the matrix \(N = (n_{jK})\), \(J, K = 1, 2 \ldots 5\), modulo the relations

\[
N^t QN = Q, \quad N^t QN^t = Q, \quad \det(N) = 1, \quad \text{where} \quad Q := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

The costructures on the generators are \(\Delta(N) = N \otimes N, \varepsilon(N) = 1, S(N) = N^{-1}\) and \(\ast\)-structure \(N^\ast = (n^\ast_{jK}) = Q N^t Q^t\), that is, \(n^*_{jK} = n_{jK}\) with \(1' = 3, 2' = 4, 3' = 1, 4' = 2, 5' = 5\). The Hopf algebra \(\mathcal{O}(SO(4, \mathbb{R}))\) is the quotient via the Hopf ideal generated by \(n_{55} - 1, n_{25}, n_{35}\) for \(\mu, \nu = 1, 2, 3, 4\) so that we have the Hopf algebra projection

\[
\pi: \mathcal{O}(SO(5, \mathbb{R})) \longrightarrow \mathcal{O}(SO(4, \mathbb{R})), \quad N \longmapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}.
\]

This gives the coaction \(\delta = (1 \otimes \pi)\Delta: \mathcal{O}(SO(5, \mathbb{R})) \rightarrow \mathcal{O}(SO(5, \mathbb{R})) \otimes \mathcal{O}(SO(4, \mathbb{R}))\). The base space algebra \(\mathcal{O}(S^4) \subset \mathcal{O}(SO(5, \mathbb{R}))\) is then generated by the coinvariant elements

\[
\alpha = \sqrt{2} n_{15}, \quad \beta = \sqrt{2} n_{25}, \quad \alpha^* = \sqrt{2} n_{35}, \quad \beta^* = \sqrt{2} n_{45}, \quad x = n_{55}.
\]

The \(\sqrt{2}\) rescaling is chosen so that \(\alpha, \beta, x\) satisfy the 4-sphere relation \(\alpha\alpha^* + \beta\beta^* + x^2 = 1\) (coming from the reality and orthogonality conditions \(N^t N^* = 1_3\)).
The $O(S^4)$-module freely generated by the right-invariant vector fields on the group manifold $SO(5, \mathbb{R})$ is the $O(S^4)$-module of $SO(4, \mathbb{R})$-equivariant derivations of the principal bundle $SO(5, \mathbb{R}) \to S^4$. A convenient basis is

\[
\begin{align*}
H_1 & := n_{1K} \partial_{1K} - n_{3K} \partial_{3K} & H_2 & := n_{2K} \partial_{2K} - n_{4K} \partial_{4K} \\
E_{10} & := n_{5K} \partial_{1K} - n_{1K} \partial_{5K} & E_{-10} & := n_{3K} \partial_{2K} - n_{5K} \partial_{1K} \\
E_{01} & := n_{5K} \partial_{1K} - n_{2K} \partial_{5K} & E_{0-1} & := n_{4K} \partial_{2K} - n_{5K} \partial_{2K} \\
E_{11} & := n_{2K} \partial_{3K} - n_{1K} \partial_{2K} & E_{-1-1} & := n_{3K} \partial_{2K} - n_{4K} \partial_{1K} \\
E_{1-1} & := n_{4K} \partial_{3K} - n_{1K} \partial_{2K} & E_{-11} & := n_{3K} \partial_{4K} - n_{2K} \partial_{1K}
\end{align*}
\]  

(8.18)

with summation on $K = 1, \ldots, 5$ understood, and $\partial_{ij}(n_{KL}) = \delta_{IK}\delta_{jL}$, for $I, J, K, L = 1, 2, \ldots, 5$. These ten generators close the Lie algebra $[8,6]$ of $so(5)$ and satisfy the reality conditions $H^*_j = H_j$ and $E^*_r = E_r$. As in (8.7), the generic real equivariant derivation is of the form $X = b_1 H_1 + b_2 H_2 + \sum_r b_r E_r$, with $b^*_j = b_j$, $b^*_r = b_r$ in $O(S^4)$ and now $H_j, E_r$ in (8.18).

Using (8.17), these derivations restricted to $O(S^4)$ coincide with those in (8.21), (8.3) restricted to $O(S^4)$. This implies that the verticality condition $X(b) = 0$ for each $b \in O(S^4)$, is the same as that imposed on the derivations of the instanton example. Therefore, infinitesimal gauge transformations, that is, $SO(4, \mathbb{R})$-equivariant derivations $X$ which are also vertical, are generated, as an $O(S^4)$-module, by the vector fields $K_j, W_r$ defined as in (8.8) but now with $H_j, E_r$ as in (8.18).

Due to $SO(5)$-equivariance of the bundle $SO(5) \to S^4$ we have the direct sum decomposition [7 Prop. 4.3],

$$\text{aut}_{O(S^4)}(O(SO(5, \mathbb{R}))) = \bigoplus_{n \in N_0} [d(2,n)] \oplus [d(2, n-1)]$$  

(8.19)

of the gauge Lie algebra. Here $[d(2,n)]$, $[d(2, n-1)]$ are the representations of $so(5)$ with highest weight vectors $\alpha^n W_{11}$, $\alpha^{n-1}(\sqrt{2} \pi W_{11} + \alpha W_{01} - \beta W_{10})$ of weights $(n, 1)$, $(n+1, 1)$, respectively. They consist of derivations on $O(SO(5, \mathbb{R}))$ which are combinations of the derivations in (8.8) with spherical harmonics of degree $n$ on $S^4$.

The twist deformation of the Hopf–Galois extension $O(S^4) \subset O(SO(5, \mathbb{R}))$ gives the $O(SO_\theta(4, \mathbb{R}))$-Hopf–Galois extension $O(S^4_\theta) \subset O(SO_\theta(5, \mathbb{R}))$. It can be obtained in two equivalent ways (cf. [2 Sect. 4.1 and 4.1.1] where, dually, 2-cocycles are used):

- On the one hand, using the Cartan subalgebra of $so(5)$ given by $H_1$ and $H_2$ we twist with

$$F := e^{2\pi i \theta(H_1 \otimes H_2 - H_2 \otimes H_1)}, \quad \theta \in \mathbb{R},$$  

(8.20)

the Hopf algebra $O(SO(5, \mathbb{R}))$ to $O(SO_\theta(5, \mathbb{R}))$ and then obtain $O(SO_\theta(4, \mathbb{R}))$ as the quotient via the Hopf ideal generated by $n_{55} - 1, n_{5\nu}, n_{\mu5}$ for $\mu, \nu = 1, 2, 3, 4$. Defining the right $O(SO_\theta(4, \mathbb{R}))$-coaction $\delta = (\text{id} \otimes \pi) \Delta$, with $\pi : O(SO_\theta(5, \mathbb{R})) \to O(SO_\theta(4, \mathbb{R}))$, $N \mapsto (\delta_0^N \delta_1^0)$, cf. (8.11), we obtain the quantum homogeneous space $O(S^4_\theta) \subset O(SO_\theta(5, \mathbb{R}))$ as the coinvariant subalgebra. The generators of $O(S^4_\theta)$ are in (8.17) and satisfy the unit radius and the commutation relations (8.11).

- On the other hand we can look at the inclusion $O(S^4) \subset O(SO(5, \mathbb{R}))$ as a Hopf–Galois extension, and twist it. In this case we forget the Hopf algebra structure of $O(SO(5, \mathbb{R}))$ and just use that we have a left and a right action of $U_H := U(so(4))$ on the total space, this is the left $U_H^\circ \otimes U_H$ action of Section (7.1). We first consider the twist in (8.20), with $H_1$ and $H_2$ that are the generators of the Cartan subalgebra of $so(4) \subset so(5)$, and twist
the Hopf–Galois extension using the $U_H$-action. The structure Hopf algebra $\mathcal{O}(SO(4, \mathbb{R}))$ is twisted to the Hopf algebra $\mathcal{O}(SO_0(4, \mathbb{R}))$ and the total space $\mathcal{O}(SO(5, \mathbb{R}))$ is twisted as a left $U_H$-module algebra. The resulting Hopf–Galois extension is still equivariant with respect to the $U_H$-action. We then consider a second twist deformation using (8.20) as a twist of $K = U^{op}$ (as in the first paragraph of Section 8). This gives the Hopf–Galois extension $\mathcal{O}(S^7_0) \subset \mathcal{O}(SO_0(5, \mathbb{R}))$.

This second approach is well adapted to describe the gauge Lie algebra of the Hopf–Galois extension $\mathcal{O}(S^7_0) \subset \mathcal{O}(SO_0(5, \mathbb{R}))$. The first twist deformation uses the $U_H$-action which acts trivially on $SO(4, \mathbb{R})$-equivariant derivations and hence on infinitesimal gauge transformations. The gauge Lie algebra is therefore undeformed. This same result follows from Proposition 8.5 where the 2-cocycle $\gamma$ is associated with the maximal torus $T^2 \subset SO(4, \mathbb{R})$ of the Cartan subalgebra and defined by the twist (8.20). Under the second twisting the Hopf algebra $K = U^{op}$ becomes the Hopf algebra $K_F$ with nontrivial braiding $R_F = \overline{F}^2$ and the gauge Lie algebra correspondingly becomes a $K_F$-braided Lie algebra. According to Proposition 5.14 the braided Lie bracket on the generators reads as in (8.12) where $K_j, W_j$ are defined as in (8.8) but now with $H_j, E_r$ in (8.18). We next apply the isomorphism $\mathcal{D}$ of Proposition 8.1 and obtain the braided gauge Lie algebra $\text{aut}_\mathcal{O}(S^7_0)(\mathcal{O}(SO_0(5, \mathbb{R})))$ of the $(\mathcal{O}(SO(4, \mathbb{R}))$-Hopf–Galois extension $\mathcal{O}(S^7_0) \subset \mathcal{O}(SO_0(5, \mathbb{R})))$.

Proposition 8.7. The braided gauge Lie algebra $\text{aut}_\mathcal{O}(S^7_0)(\mathcal{O}(SO_0(5, \mathbb{R})))$ is described in Proposition 8.5 by replacing $\mathcal{O}(S^7_0)$ with $\mathcal{O}(SO_0(5, \mathbb{R})))$.

In particular, the Lie algebra bracket of $\text{aut}_\mathcal{O}(S^7_0)(\mathcal{O}(SO_0(5, \mathbb{R})))$ is determined by the brackets in Table 1 of the previous Example 8.5. The difference between the gauge transformations of the noncommutative instanton and orthogonal bundles on $S^7_0$ is in their actions on the total spaces $\mathcal{O}(S^7_0)$ and $\mathcal{O}(SO_0(5, \mathbb{R}))$, respectively. This is clear when comparing the two different direct sums (8.10) and (8.19) describing the linear space structure of the braided Lie algebras $\text{aut}_\mathcal{O}(S^7_0)(\mathcal{O}(S^7_0))$ and $\text{aut}_\mathcal{O}(S^7_0)(\mathcal{O}(SO_0(5, \mathbb{R})))$.

Example 8.8. A principal line bundle over the $\kappa$-Minkowski spacetime (Jordanian twist deformation). Let $P = \mathbb{R}_{>0} \times \mathbb{R}^{n+1}$ be the affine group of dilatations and translations of the $\mathbb{R}^{n+1}$. It is the group of invertible matrices $(u^i_j)$ with $u \in \mathbb{R}_{>0}, x \in \mathbb{R}^{n+1}$. We consider the principal bundle $P \to P/\mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ acts by right multiplication of $(u^i_j)$. From $(u^0_1) = (x^{-1}_1, 0, 1)^* (0^0_1)$ the base space is $\mathbb{R}^{n+1}$ and the bundle is trivial. Dually we have the Hopf–Galois extension $B = A^{coH} \subset A$ of coordinate rings on these affine algebraic varieties. The coordinate ring $A$ is generated by the elements $u, x^I$ while the coordinate ring $B = \mathcal{O}(\mathbb{R}^{n+1})$ is generated by $x^I := x^I u^{-1}, I = 0, 1, \ldots, n$. The infinitesimal gauge transformations form the abelian Lie algebra $\mathcal{O}(\mathbb{R}^{n+1}) \otimes \text{Lie}(\mathbb{R}_{>0}) \simeq \mathcal{O}(\mathbb{R}^{n+1})$. The Lie algebra of right-invariant vector fields of the group $P$ is spanned by

$$u \frac{\partial}{\partial u}, \quad u \frac{\partial}{\partial x^I}, \quad I = 0, 1, \ldots, n.$$ 

The vector field $u \frac{\partial}{\partial u} + \sum I x^I \frac{\partial}{\partial x^I}$ is vertical and span, with coefficients in $B = \mathcal{O}(\mathbb{R}^{n+1})$, the Lie algebra $\text{Lie}(\mathbb{R}_{>0})$ of infinitesimal gauge group transformations.

In order to deform the principal bundle $P \to P/\mathbb{R}_{>0}$, let $K$ be the universal enveloping algebra of the right-invariant vector fields of the group $P$ and, for $\kappa \in \mathbb{R}$, define the Jordanian twist (see e.g. [8])

$$F = \exp \left( u \frac{\partial}{\partial u} \otimes \sigma \right),$$  

(8.21)
where \( \sigma = \ln \left( 1 + \frac{1}{\kappa} P_0 \right) \), \( P_0 = i u \frac{\partial}{\partial x^0} \) and \( [u \frac{\partial}{\partial x^0}, P_0] = i P_0 \). This twist is a formal power series in \( \frac{1}{\kappa} P_0 \). However, due to the affine algebraic nature of \( A \), only a finite number of powers of \( \frac{1}{\kappa} P_0 \) are relevant and we can avoid defining the topological completion of the tensor product \( K \otimes K \). We deform the \( K \)-equivariant Hopf–Galois extension \( B = A^{coH} \subseteq A \) to the \( K_F \)-equivariant one \( B_F = A_E^{coH} \subseteq A_F \). The total space algebra \( A_F \) is the polynomial ring generated by the coordinates \( u, u^{-1} \) with relations \( uu^{-1} = u^{-1}u = 1 \), and \( n + 1 \) coordinates \( x^I, I = 0, 1, \ldots, n \). The only nontrivial commutation relation is

\[
  x^0 \star u^{-1} - u^{-1} \star x^0 = -\frac{i}{\kappa} x^0 .
\]  

(8.22)

The base algebra \( B_F \) is generated by the coordinates \( x^I = x^I u^{-1} = x^I \star u^{-1} \), \( I = 0, 1, \ldots, n \), with commutation relations

\[
x^0 \star x^j - x^j \star x^0 = -\frac{i}{\kappa} x^j , \quad x^I \star x^j - x^j \star x^I = 0
\]  

(8.23)

for \( j, l = 1, \ldots, n \). This is the algebra of the \((n + 1)\)-dimensional \( \kappa \)-Minkowski quantum space. A \(*\)-conjugation on \( A_F \) is defined by \( u^* = u, (x^I)^* = x^I \). It follows that \((x^j)^* = x^j \) and \((x^0)^* = x^0 + \frac{i}{\kappa} x^j \), so that \((x^0 + \frac{i}{\kappa} x^j)^* = x^0 + \frac{1}{2\kappa} x^j \) and this real generator has the same commutation relations as \( x^0 \) in (8.23).

Since the vector field \( u \frac{\partial}{\partial x^0} \) entering the twist commutes with the vertical derivation \( X := u \frac{\partial}{\partial x^0} + \sum_I x^I \frac{\partial}{\partial x^I} \), we have the braided vertical \( \mathbb{R}_{>0} \)-equivariant derivation \( \mathcal{D}(X) = X \) satisfying \( [X, X]_{R_F} = \mathcal{D}(\delta([X, X])) = 0 \). Infinitesimal gauge transformations are given by \( b \star X \) with \( b \in B_F = \mathcal{O}(\mathbb{R}^{n+1})_F \). From (5.28) we see that the braided gauge Lie algebra is abelian: \( [b \star X, b' \star X]_{R_F} = b \star b' \star [X, X]_{R_F} = 0 \).

**Example 8.9.** A non-abelian principal bundle over the \( \kappa \)-Minkowski spacetime. The previous example generalises to a non-abelian setting. For example let the total space be \( P = \mathbb{R}_{>0} \times \mathbb{R}^{n+1} \times SO(1, n) \), the \( n + 1 \)-dimensional Poincaré-Weyl group, the semidirect product of the Poincaré group with the group \( \mathbb{R}_{>0} \) of dilatations. This is the group of invertible matrices \( (u^0 x^I) \) with \( u \in \mathbb{R}_{>0} \), \( x \in \mathbb{R}^{n+1} \) and \( T = (t_{I,J})_{I,J = 0,1,\ldots,n} \in SO(1, n) \).

Associated with the principal bundle \( P \to \mathbb{R}^{n+1} = P/G \), where \( G = \mathbb{R}_{>0} \times SO(1, n) \) we have the Hopf–Galois extension \( B = A^{coH} \subseteq A \) generated by the coordinate functions \( u, x^I, t_{I,J} \). As in the previous example, with the twist in (8.21) the only nontrivial commutation relation in the deformed algebra \( A_F \) is the one in (8.22); the base space algebra \( B_F \) is the \( \kappa \)-Minkowski in (8.23). The infinitesimal gauge transformations form the braided Lie algebra

\[
\text{aut}^B_{B_F}(A_F) = \mathcal{D}(\text{aut}_{A_F}(A_F)) = \mathcal{O}(\mathbb{R}^{n+1})_F \otimes \mathcal{D}(\text{Lie}(\mathbb{R}_{>0} \times SO(1, n))_F)
\]  

\[
= \mathcal{O}(\mathbb{R}^{n+1})_F \otimes \text{Lie}(\mathbb{R}_{>0} \times SO(1, n)) .
\]

The last equality follows from the commutativity of the vector field \( u \frac{\partial}{\partial x^0} \), entering the twist, with the right \( G \)-invariant (\( H \)-equivariant) vertical vector fields \( X \in \text{Lie}(\mathbb{R}_{>0} \times SO(1, n)) \) generating the gauge transformations

\[
u \frac{\partial}{\partial u} + x^I \frac{\partial}{\partial x^I} , \quad t_{0K} \frac{\partial}{\partial t_{0K}} + t_{jK} \frac{\partial}{\partial t_{jK}} , \quad t_{iK} \frac{\partial}{\partial t_{iK}} - t_{jK} \frac{\partial}{\partial t_{iK}} .
\]

Here \( i, j = 1, 2,\ldots,n \) and sum over \( I, K = 0, 1,\ldots,n \) is understood. Indeed this commutativity implies \( \mathcal{D}(X) = X \) and \( [X, X']_F = [X, X'] \), so that \( [X, X']_{R_F} = \mathcal{D}([X, X'])_F = [X, X'] \) for all \( X, X' \in \text{Lie}(\mathbb{R}_{>0} \times SO(1, n)) \). Moreover, \( [b \star X, b' \star X']_{R_F} = b \star b' \star [X, X'] \), for all \( b, b' \in \mathcal{O}(\mathbb{R}^{n+1})_F \).
Example 8.10. Formal deformation quantization of smooth principal bundles and their gauge groups. The twist deformations presented in this section for principal bundles that are affine algebraic varieties can be also considered for smooth $L$-equivariant principal bundles $P \to \mathbf{P}/G$. In this case, see [2] Ex. 3.24, we have the gical $K[[\hbar]]$-equivariant $H[[\hbar]]$-Hopf–Galois extension $B[[\hbar]] \simeq A[[\hbar]]^{\text{co}H[[\hbar]]} \subseteq A[[\hbar]]$, where $K[[\hbar]]$, $B[[\hbar]]$, $H[[\hbar]]$ and $A[[\hbar]]$ are the formal power series extension of the $\mathbb{C}$-modules $K = C^\infty(L)$, $B = C^\infty(P/G)$, $H = C^\infty(G)$ and $A = C^\infty(P)$. A twist $F$ in $K[[\hbar]] \otimes K[[\hbar]] \simeq C^\infty(L \times L)[[\hbar]]$ then leads to the noncommutative topological $H[[\hbar]]_F$-Hopf–Galois extension $B[[\hbar]]_F \simeq A[[\hbar]]_F^{\text{co}H[[\hbar]]}_F \subseteq A[[\hbar]]_F$. The associated braided gauge Lie algebra is $\text{aut}_{B[[\hbar]]}_F(A[[\hbar]]_F)$, where $\text{aut}_{B[[\hbar]]}_F(A[[\hbar]]_F) \subseteq \text{Der}_{\text{Re}}(A[[\hbar]]_F) \subseteq \text{Hom}(A[[\hbar]]_F, A[[\hbar]]_F)$, this latter being the linear space of continuous algebra automorphisms.

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APPENDIX A. PROOF OF PROPOSITION 4.14

From the expression $(\Delta \otimes \text{id})\mathbf{F}$ obtained from the twist condition (1.4) for $\mathbf{F}$, it is immediate to see that the two expressions of the map $D$ in (4.28) coincide. Then, $D^{-1}(k) = \mathbf{F}^{-1} k S_T(\mathbf{F}_a) u_\mathbf{F}$ is the inverse of $D$. An equivalent expression for $D^{-1}$ is then given by $D^{-1}(k) = (F^\alpha \triangleright_{\text{ad}_F} k) F_a$. Indeed, this follows by recalling that $F$ is a twist for $K$ if and only if $\mathbf{F}$ is a twist for $K_F$ and observing that the element $\mathbf{u}_\mathbf{F}$ in the Hopf algebra $K_F$ (analogous to the element $u_F$ in $K$) is

$$u_F = S_T(F^\beta) F_\beta = u_F S(F^\beta) S(F^\gamma) F_\alpha F_\beta = u_F S(F^\gamma) F^\alpha F_\beta = u_F .$$

Next we show that $D$, or equivalently its inverse $D^{-1}$, is a morphism of $K_F$-modules: $D^{-1}(h \triangleright_{\text{ad}_F} k) = h \triangleright_{\text{ad}} D^{-1}(k)$. On the one hand we compute

$$D^{-1}(h \triangleright_{\text{ad}_F} k) = ((F^\alpha h) \triangleright_{\text{ad}_F} k) F_a = (F^\alpha h)_1 S_T((F^\alpha h)_2) \mathbf{u}_\mathbf{F} F_a
= F^\beta F^\alpha_1 h_1 \mathbf{F}^\gamma \mathbf{F}_\gamma \mathbf{u}_\mathbf{F} F_a$$

which, by using the twist condition (1.3) gives

$$D^{-1}(h \triangleright_{\text{ad}_F} k) = F^\alpha h_1 \mathbf{F}^\gamma \mathbf{F}_\gamma \mathbf{u}_\mathbf{F} F_\beta F_\alpha
= F^\beta h_1 \mathbf{F}^\gamma \mathbf{F}_\gamma \mathbf{u}_\mathbf{F} F_\beta F_\alpha
= F^\alpha h_1 \mathbf{F}^\gamma \mathbf{F}_\gamma \mathbf{u}_\mathbf{F} F_\beta F_\alpha$$

when using that $S(F^\beta) \mathbf{u}_\mathbf{F} F_\beta = 1$. Finally, since $F$ is unital,

$$D^{-1}(h \triangleright_{\text{ad}_F} k) = h_1 \mathbf{F}^\gamma \mathbf{F}_\gamma \mathbf{u}_\mathbf{F} S(h_2) .$$

On the other hand, since $S_T(\cdot) u_F = u_F S(\cdot)$, we have

$$h \triangleright_{\text{ad}} D^{-1}(k) = h_1 \mathbf{F}^\gamma \mathbf{F}_\gamma S_T(\mathbf{F}_a) u_F S(h_2) = h_1 \mathbf{F}^\gamma k S_T(\mathbf{F}_a) S(h_2)$$

and the two expressions coincide.
It is easy to prove that the map $D$ is an algebra map:

$$D(h \triangleright k) = D((\bar{F}^i \triangleright h) (\bar{F}_a \triangleright k)) = \left( \bar{F}^\beta \triangleright ((\bar{F}^i \triangleright h) (\bar{F}_a \triangleright k)) \right) \bar{F}_\beta,$$

and using the twist condition, this simplifies to

$$D(h \triangleright k) = (\bar{F}^\beta \triangleright h) \left( (\bar{F}^\beta(\bar{F}_a) \triangleright k)) \right) \bar{F}_\beta = (\bar{F}^\beta \triangleright h) \bar{F}_\beta(\bar{F}_a) \triangleright k) \bar{F}_\alpha = (\bar{F}^\beta \triangleright h) \bar{F}_\beta(\bar{F}_a) \triangleright k) \bar{F}_\alpha = D(h)D(k).$$

The proof that $D^{-1}$ is also a coalgebra morphism, $(D^{-1} \otimes D^{-1}) \circ \Delta_F = \Delta_{K_F} \circ D^{-1}$, is more involved. We first observe that $\bar{F}^\beta k S(\bar{F}_a) = D^{-1}(k \bar{u}_F)$ for each $k \in K_F$. Then, using the twist condition (1.3),

$$(F^\mu \triangleright \text{id}) \otimes F_a = F^i F^\nu k S(\bar{F}^\gamma F^\mu F_{\nu(1)}) \otimes F^\mu F_{\nu(2)} = D^{-1}(F^\nu k S(F^\mu F_{\nu(1)}) \bar{u}_F) \otimes F^\mu F_{\nu(2)},$$

and the coproduct in $K_F$ can be rewritten as

$$\Delta_{K_F}(k) = F^\beta \triangleright \text{id} (k^1(S(R_{F,a})) \otimes (F^\mu R^\alpha) \triangleright \text{id} k^2),$$

with $R_F = F_{21} \bar{F} = F_a R^\beta F^\nu \otimes F^\alpha R^\gamma F_a$. Then, since $D^{-1}$ is a $F^\nu$-module morphism,

$$\Delta_F(k) = F^\mu k^1(\bar{F}^\nu \triangleright \text{id}) \bar{u}_F(S(R_{F,a}) \bar{u}_F \otimes (R^\alpha \triangleright \text{id} D^{-1}(F^\mu k^2 F_{\nu(1)}))$$

$$= F^\mu k^1(\bar{F}^\nu \triangleright \text{id}) \bar{u}_F(S(F^\nu R^\beta F_a) \bar{u}_F \otimes \left( (F^\alpha R^\gamma F_{\nu(1)}) \triangleright \text{id} D^{-1}(F^\mu k^2 F_{\nu(1)}) \right).$$

To prove that $D^{-1}$ is also a coalgebra morphism, we have hence to show that

$$F^\mu k^1(\bar{F}^\nu \triangleright \text{id}) \bar{u}_F(S(F^\mu R^\beta F_a) \bar{u}_F \otimes \left( (F^\alpha R^\gamma F_{\nu(1)}) \triangleright \text{id} D^{-1}(F^\mu k^2 F_{\nu(1)}) \right),$$

where

$$\Delta(D^{-1}(k)) = F^\gamma(1) k^1(\bar{F}^\nu \triangleright \text{id}) \bar{u}_F(S(F^\gamma) S(R_{F,a}) S(F_{\nu(2)}) \otimes F^\nu (2) k^2 F_{\nu(2)} F_{\nu(1)} S(F^\nu) S(F_{\nu(3)}),$$

as deduced from $\Delta(u_F) = F(\bar{u}_F \otimes \bar{u}_F) (S \otimes S)(F_{21}) = F^\nu u_F S(F_{\nu(2)}) \otimes F_{\nu(3)} F_{\nu(3)}$, obtained with some computations. Then the right hand side of the identity we wish to show becomes

$$F^\nu(1)(\bar{F}^\gamma \triangleright \text{id}) \bar{u}_F \otimes \left( (F^\mu R^\alpha F_{\nu(1)}) \triangleright \text{id} \bar{F}^\nu(k^2) F_{\nu(1)} S(F^\nu) S(F_{\nu(3)}),$$

$$= F^\nu(1)(\bar{F}^\gamma \triangleright \text{id}) \bar{u}_F S(F^\mu R^\alpha F_{\nu(2)} S(F_{\nu(2)}) \otimes \left( (F_{\nu(3)} R^\alpha F_{\nu(1)}) \triangleright \text{id} \bar{F}^\nu(k^2) F_{\nu(1)} S(F^\nu) S(F_{\nu(3)}),$$

$$= F^\nu(1)(\bar{F}^\gamma \triangleright \text{id}) \bar{u}_F S(F^\mu R^\alpha F_{\nu(2)} S(F_{\nu(2)}) \otimes \left( (F_{\nu(3)} R^\alpha F_{\nu(1)}) \triangleright \text{id} \bar{F}^\nu(k^2) F_{\nu(1)} S(F^\nu) S(F_{\nu(3)}).$$
\[
F^\gamma k_1 \mathcal{D}_k \left( F^\alpha R_\alpha F_\beta \right) u_F \otimes (F_\mu R_\alpha) \triangleright_{ad} \left( F^\alpha S(\gamma) \right) \\
= F^\gamma k_1 \mathcal{D}_k \left( F^\alpha R_\alpha F_\beta \right) u_F \otimes (F_\mu R_\alpha) \triangleright_{ad} \left( F^\alpha S(\gamma) \right)
\]

where we have used the twist condition (4.4) to get the second last identity. Next, being \( F \) the inverse of \( F \), using again the twist condition (4.4), this expression simplifies to

\[
F^\gamma k_1 \mathcal{D}_k \left( F^\alpha R_\alpha F_\beta \right) u_F \otimes (F_\mu R_\alpha) \triangleright_{ad} \left( F^\alpha S(\gamma) \right)
\]

From the definition of \( D^{-1} \), it is easy to see that the above can be rewritten as

\[
F^\gamma k_1 \mathcal{D}_k \left( F^\alpha R_\alpha F_\beta \right) u_F \otimes (F_\mu R_\alpha) \triangleright_{ad} \left( D^{-1}(\gamma) \right),
\]

and thus coincides with the left hand side, \((\text{id} \otimes D^{-1}) \Delta_F(k)\). This concludes the proof.

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