Classical and quantum Brownian motion in an electromagnetic field

Marco Patriarca1,* and Pasquale Sodano2

Received 8 May 2016, revised 2 November 2016, accepted 2 November 2016
Published online 6 December 2016

The dynamics of a Brownian particle in a constant magnetic field and time-dependent electric field is studied in the limit of white noise, using a Langevin approach for the classical problem and the path-integral Feynman-Vernon and Caldeira-Leggett framework for the quantum problem. A first goal of this study is to use the two-dimensional problem of a Brownian particle in a magnetic field to show that a proper reformulation of the oscillator model of quantum Brownian motion allows one to recover the classical limit of the dynamics correctly. Furthermore, the probability distribution in configuration space of an initial pure state represented by an asymmetrical Gaussian wave function is worked out and its general time evolution is decomposed in the superposition of basic processes: (a) the classical motion of the center of mass, (b) a rotation around the mean position, and (c) spreading processes along the principal axes.

1 Introduction

The problem of a Brownian particle in a magnetic field is of interest in many fields, from condensed matter (e.g. the Hall effect) to cosmology (e.g. the origin of cosmic rays). The present paper focuses on Brownian motion in a constant magnetic field and a spatially homogeneous, possibly time-dependent, electromagnetic field, a problem studied so far in various papers both in the classical [1–23] and in the quantum [7, 20, 22, 24–30] regime.

In the absence of noise and dissipation, a close analogy links a particle in a constant magnetic field to the harmonic oscillator, both at classical [31] and quantum [32] level. At classical level, the particle performs periodic harmonic motion with frequency \( \omega = \sqrt{k/m} \) for a harmonic oscillator of mass \( m \) and elastic constant \( k \) and \( \omega = |qB|/mc \), the cyclotron frequency, for a particle of charge \( q \) in a magnetic field \( B \) (\( c \) is the speed of light). At quantum level, the energy spectrum of both systems has equidistant energy levels \( E_n \) with energy spacing \( \Delta E = E_{n+1} - E_n = \hbar \omega \). Interestingly, even an arbitrarily small internal noise breaks this analogy, turning the harmonic-like motion of a particle in a magnetic field into one equivalent to that of a free Brownian particle in two dimensions, see e.g. Refs. [1, 2, 6, 8, 10] for the classical case and Refs. [28, 29, 33] for the quantum case.

In this paper we study the time evolution of the probability density in configuration space of a Brownian particle in a constant magnetic field and a homogeneous electric field, in the white noise approximation. We first consider the classical problem in Sec. 2, discussing some intriguing features such as the role of the Hall angle, the non-dissipative character of the uniform angular motion, and the formal equivalence with a complex Langevin equation. As for the quantum problem, studied in Sec. 3, the time evolution of an asymmetrical Gaussian wave packet is worked out in detail within the framework of the Feynman-Vernon [34] and Caldeira-Leggett [35] models. A first goal is to show how the classical limit is properly recovered using the invariant form of the oscillators Lagrangian and the corresponding correlated initial conditions discussed in Ref. [40]. Furthermore, the time evolution of the probability distribution in configuration space of an initial pure state represented by a general asymmetrical Gaussian wave function is analyzed in detail and shown to result from the superposition of different basic processes undergone by the probability wave packet, namely (a) the classical motion of the center of mass, (b) a rotation around the mean position, and (c) spreading processes along the principal axes. More detailed discussions of the wave packet dynamics as well as the study of other problems related to the one considered here will be done elsewhere.

1 NICPB–National Institute of Chemical Physics and Biophysics, Rävala 10, Tallinn 15042, Estonia
2 International Institute of Physics, Universidade Federal do Rio Grande do Norte, Natal-RN 59078-400, Brazil
* Corresponding author E-mail: marco.patriarca@kbfi.ee
2 Classical problem

In this section we consider the classical problem, both for introducing some relevant notation and in view of the classical limit considered in the following section. In the Langevin approach a classical Brownian particle in an electromagnetic field is described (in the limit of white noise) by the following stochastic equation [8],

$$m \frac{d^2 \mathbf{v}}{dt^2} = q \mathbf{E} + \frac{q}{c} \mathbf{v} \times \mathbf{B} - m \gamma \mathbf{v} + \mathbf{R}(t). \quad (1)$$

Here $\mathbf{v}(t) = (v_x(t), v_y(t), v_z(t)) = d\mathbf{r}(t)/dt$ is the particle velocity at time $t$; $\mathbf{r}(t) = (x(t), y(t), z(t))$ is the particle position; the first two terms on the right hand side are the environment forces, i.e., the dissipative force $-m \gamma \mathbf{v}$ ($\gamma$ is the friction coefficient) and the random force $\mathbf{R}(t)$; the first two terms represent the environment forces, i.e., the dissipative force $-m \gamma \mathbf{v}$ ($\gamma$ is the friction coefficient) and the random force $\mathbf{R}(t)$; the random force $\mathbf{R}(t) = (R_x(t), R_y(t), R_z(t))$, a Gaussian zero-mean $\delta$-correlated stochastic process ($i,j = x,y,z$),

$$\langle R_i(t) \rangle = 0, \quad \langle R_i(t) R_j(s) \rangle = (2m\gamma/\beta) \delta(t-s) \delta_{ij}, \quad (2)$$

where $\beta = 1/k_B T$ is the inverse temperature and $(\ldots)$ represents a statistical averaging over the stochastic force.

Decomposing the velocity as $\mathbf{v} = \mathbf{v}_|| + \mathbf{v}_\perp$, with $\mathbf{v}_||$ parallel and $\mathbf{v}_\perp$ perpendicular to the magnetic field, and analogously for the electric field, $\mathbf{E} = \mathbf{E}_|| + \mathbf{E}_\perp$, and the random force, $\mathbf{R} = \mathbf{R}_|| + \mathbf{R}_\perp$, Eq. (1) decouples,

$$m \frac{d \mathbf{v}_||}{dt} = q \mathbf{E}_|| - m \gamma \mathbf{v}_|| + \mathbf{R}_||(t), \quad \text{Eq. (3)}$$

$$m \frac{d \mathbf{v}_\perp}{dt} = q \mathbf{E}_\perp + \frac{q}{c} \mathbf{v}_\perp \times \mathbf{B} - m \gamma \mathbf{v}_\perp + \mathbf{R}_\perp(t). \quad \text{Eq. (4)}$$

The motion parallel to the magnetic field is equivalent to that of a one-dimensional Langevin particle acted upon only by the force $q \mathbf{E}_||(t)$ and will not be considered further. It is now useful to define the vectors

$$\mathbf{n}_\perp = \frac{\mathbf{v}_\perp}{|\mathbf{v}_\perp|}, \quad \mathbf{n}_|| = \mathbf{n}_\perp \times \frac{\mathbf{B}}{|\mathbf{B}|}, \quad (5)$$

see Fig. 1, and rewrite Eq. (4) as

$$\frac{d \mathbf{v}_\perp}{dt} = \frac{q}{m} \mathbf{E}_\perp + (\omega \mathbf{n}_|| - \gamma \mathbf{n}_\perp)|\mathbf{v}_\perp| + \frac{1}{m} \mathbf{R}_\perp(t), \quad \text{Eq. (6)}$$

$$= \frac{q}{m} \mathbf{E}_\perp + (\sin \chi_{II} \mathbf{n}_|| - \cos \chi_{II} \mathbf{n}_\perp)|\mathbf{v}_\perp| + \frac{1}{m} \mathbf{R}_\perp(t). \quad \text{Eq. (7)}$$

Here $\sin \chi_{II} = \omega/\Gamma$, $\cos \chi_{II} = \gamma/\Gamma$ are the force direction cosines in the $(\mathbf{n}_||, \mathbf{n}_\perp)$ frame and

$$\tan \chi_{II} = \omega/\gamma, \quad \Gamma = \sqrt{\gamma^2 + \omega^2}, \quad \text{Eq. (8)}$$

define the Hall angle [36] and the “effective damping constant”. Equation (7) has a simple physical interpretation, in that it displays the similarity between the Lorentz and the friction forces, which are both proportional to $|\mathbf{v}_\perp|$ and merge in an effective viscous force (still proportional to $|\mathbf{v}_\perp|$) with intensity $\Gamma$ and forming a constant angle $\chi_{II}$ (the Hall angle) with the vector $\mathbf{v}_\perp$ — see Fig. 1. Note that even if $\mathbf{n}_||(t)$ and $\mathbf{n}_\perp(t)$ are time-dependent, $\Gamma$ and $\chi_{II}$ are constant in time. The Hall angle $\chi_{II}$, best known from the Hall effect [36] where both a magnetic and an electric field are present, measures the relative strengths of Lorentz to friction force and therefore plays a key role also when only a magnetic field is present. The time scales associated to $\omega$ and $\gamma$ and the parameters $\Gamma$ and $\chi_{II}$ can be related to each other through the complex friction coefficient

$$\mathbf{\tilde{f}} = \gamma + i\omega = \Gamma \exp(i\chi_{II}), \quad \text{Eq. (9)}$$

relevant both in the classical and in the quantum problem; $\gamma$ and $\omega$ represent the real and imaginary parts of $\mathbf{\tilde{f}}$ while the Hall angle $\chi_{II}$ and the effective damping constant $\Gamma$ represent phase and modulus, respectively.

2.1 Constant $\mathbf{B}$ and homogeneous $\mathbf{E}(t)$

We assume a homogeneous electric field $\mathbf{E}(t)$, $\omega = qB/mc > 0$, and a constant magnetic field,

$$\mathbf{B} = (0, 0, B). \quad \text{Eq. (10)}$$
It is convenient to introduce the complex coordinate [31]

\[
Z(t) = x(t) + iy(t),
\]

and, analogously, the complex forces \(q\tilde{F}(t) = qE_x(t) + iqE_y(t), \tilde{R}(t) = R_x(t) + iR_y(t)\). Using Eq. (1), one finds that \(Z(t)\) satisfies the complex Langevin equation

\[
m\ddot{Z}(t) = -m\tilde{F}(t) + q\tilde{E}(t) + \tilde{R}(t).
\]

Here the Lorentz and friction forces are merged in the generalized friction force \(f_{\text{tot}} = -m\tilde{F}(t)\), where \(\tilde{F}\) is defined in Eqs. (8), see also Fig. 1. By integrations of Eq. (12) between \(t = t_a\) and \(t = t_b\), one obtains the complex coordinate \(Z(t_b)\) and coordinate \(Z(t_b)\),

\[
\dot{Z}(t_b) = \dot{Z}_0 \exp(-\tilde{F}t) + m^{-1}\tilde{F}[q\tilde{E} + \tilde{R}],
\]

\[
Z(t_b) = Z_0 + [\dot{Z}_0/\tilde{F}] [1 - \exp(-\tilde{F}t)] + m^{-1}\tilde{F}[q\tilde{E} + \tilde{R}].
\]

Here \(t = t_b - t_a\), \(Z_0 = Z(t_a) = x(t_a) + iy(t_a), \dot{Z}_0 = \dot{Z}(t_a) = \dot{x}(t_a) + iy(t_a);\) the functionals \(F\) and \(\tilde{F} = dF[f]/dt\) represent the inhomogeneous contributions of \(\tilde{E}\) and \(\tilde{R}\),

\[
F[f] = \int_{t_a}^{t_b} ds \{1 - \exp[-\tilde{F}(t_b - s)]\} \tilde{f}(s),
\]

where \(\tilde{f} = q\tilde{E} + \tilde{R}\).

2.2 Constant magnetic field

Here we consider the particular case of zero electric field. For convenience we rewrite the initial conditions on the complex variable \(Z\) as

\[
Z_0 = x_0 + iy_0, \quad \dot{Z}_0 = \dot{x}_0 + i\dot{y}_0 = v_0 \exp(i\varphi_0),
\]

where \(x_0 = x(t_a), \dot{x}_0 = \dot{x}(t_a),\) and analogously for \(y_0, \dot{y}_0,\) and we have introduced the initial velocity modulus \(v_0 = |\vec{v}_0|\) and the angle \(\varphi_0\) between \(\vec{v}_0\) and the x-axis,

\[
v_0 = \sqrt{\dot{x}_0^2 + \dot{y}_0^2}, \quad \tan \varphi_0 = \dot{y}_0/\dot{x}_0.
\]

From Eqs. (14) and (2), separating \(\langle Z(t)\rangle\) into real and imaginary parts, one obtains the average coordinates,

\[
\langle x(t) \rangle = x_c = (v_0/\Gamma) \exp(-\gamma t) \cos(\omega t - \varphi_0 + \chi_H),
\]

\[
\langle y(t) \rangle = y_c = (v_0/\Gamma) \exp(-\gamma t) \sin(\omega t - \varphi_0 + \chi_H),
\]

where \(t = t_b - t_a\) and \((x_c, y_c)\) define the asymptotic position eventually approached for \(t \gg \gamma^{-1}\),

\[
x_c = \langle x(t \gg \gamma^{-1}) \rangle = x_0 + (v_0/\Gamma) \cos(\chi_H - \varphi_0),
\]

\[
y_c = \langle y(t \gg \gamma^{-1}) \rangle = y_0 - (v_0/\Gamma) \sin(\chi_H - \varphi_0).
\]

This provides a simple geometrical interpretation of the Hall angle \(\chi_H\) as the total deflection angle of the particle with respect to its initial velocity, while the effective friction coefficient \(\Gamma\) defines the total distance covered by the particle, \(d_0 = [(x_c - y_0)^2 + (y_c - x_0)^2]^{1/2} = v_0/\Gamma\). Comparison with the value \(d_0 = v_0/\Gamma\) for \(B = 0\) shows the role of \(\Gamma\) as an effective friction coefficient. In polar coordinates, choosing the origin in the asymptotic position \((x_c, y_c)\), from Eqs. (18) one obtains

\[
r(t) = \sqrt{[\langle x(t) \rangle - x_c]^2 + [\langle y(t) \rangle - y_c]^2} = (v_0/\Gamma) e^{-\gamma t},
\]

\[
\theta(t) = \arctan \left[\frac{\langle y(t) \rangle - y_c}{\langle x(t) \rangle - x_c}\right] = \varphi_0 - \chi_H - \omega t.
\]

By eliminating the time variable one finds that the shape of the trajectory is an exponential spiral,

\[
r(\theta) = (v_0/\Gamma) \exp[-\cotan \chi_H (\theta - \varphi_0 + \chi_H)].
\]

Notice in Eq. (20) that the particle approaches the asymptotic position \((x_c, y_c)\) with time scale \(\gamma^{-1}\)—as in the problem without magnetic field — and, at the same time, Eq. (21) shows a uniform angular motion with angular velocity \(\omega\) — as in the frictionless problem with a constant magnetic field. These complementary features are due to the fact that the Lorentz force acts perpendicularly to the particle velocity, leaving its modulus and therefore the relaxation dynamics unaffected, so that the only effect of the magnetic field is to bend the trajectory. Whereas the friction force—being anti-parallel to the particle velocity—changes the velocity modulus as if no magnetic field is present, without influencing the direction.

The position uncertainties can be expressed through the second moments

\[
\|\Delta x(t)^2\| = \|\langle x(t) - \langle x(t) \rangle \rangle^2\|,
\]

\[
\|\Delta y(t)^2\| = \|\langle y(t) - \langle y(t) \rangle \rangle^2\|,
\]

\[
\|\Delta x(t) \Delta y(t)\| = \|\langle [x(t) - \langle x(t) \rangle] \times [y(t) - \langle y(t) \rangle] \rangle\|,
\]

where \(\Delta x(t) = x(t) - \langle x(t) \rangle\) and \(\Delta y(t) = y(t) - \langle y(t) \rangle\) are the x and y displacements, which can be computed from
\(|\Delta Z(t)|^2\) and \(\langle |\Delta Z(t)|^2 \rangle\). One finds \(\langle \Delta x(t)^2 \rangle = \langle \Delta y(t)^2 \rangle\) and a radial mean square displacement \([8, 37]\)

\[
\langle \Delta r(t)^2 \rangle = \langle \Delta x(t)^2 \rangle + \langle \Delta y(t)^2 \rangle
= \frac{2 \cos \chi_{11}^2}{m \beta^2 \gamma^2} \left[ 2 \gamma t + 4 \cos \chi_{11} \left[ e^{-\gamma t} \cos(\omega t + \chi_{11}) \right.ight. \\
\left. - \cos \chi_{11} \right] + 1 - e^{-2\gamma t} \right].
\]

(26)

For \(\chi_{11} \to 0\) \((B \to 0)\) this expression reduces to the mean square displacement of a Langevin particle in a plane \([38]\), while in the asymptotic limit \(\gamma t \gg 1\) one obtains

\[
\langle \Delta r(t)^2 \rangle_i = 4 D_B t, \quad D_B = \gamma / (m \beta \Gamma^2) \equiv \text{Re} \left[ 1 / (m \beta \Gamma^2) \right].
\]

(27)

The latter equation resembles the Einstein relation, with the real part of \(1/\Gamma^2\) in place of \(1/\gamma\).

### 2.3 Constant electromagnetic field

It is easy to show from Eq. (14) that a homogeneous electric field modifies the mean position but not the mean square displacements. In the well known case of a constant electric field \(E\), with modulus \(E_0\) and components \(E_x = E_0 \cos \alpha, E_y = E_0 \sin \alpha\), the solution is the sum of the homogeneous solutions for \(E = 0\), Eqs. (18)–(19), and the inhomogeneous contribution \(\propto E_0\),

\[
\langle x(t) \rangle = \langle x(t) \rangle_{E=0} + qE_0 \cos(\alpha - 2 \chi_{11}) / m \Gamma^2,
+ (qE_0) / (m \Gamma^2) \exp(-\gamma t) \cos(\omega t + 2 \chi_{11} - \alpha)
+ V_x t,
\]

\[
\langle y(t) \rangle = \langle y(t) \rangle_{E=0} + qE_0 \sin(\alpha - 2 \chi_{11}) / m \Gamma^2
- (qE_0) / (m \Gamma^2) \exp(-\gamma t) \sin(\omega t + 2 \chi_{11} - \alpha)
+ V_y t,
\]

(28)

with drift velocities

\[
V_x = (qE_0 / m \Gamma) \cos(\alpha - \chi_{11}),
V_y = (qE_0 / m \Gamma) \sin(\alpha - \chi_{11}).
\]

(29)

The velocity modulus is \(V_0 = qE_0 / m \Gamma\), confirming the role of \(\Gamma\) as an effective friction coefficient.

### 3 Quantum problem

In this section we study the time evolution of a quantum wave packet between times \(t_a\) and \(t_b\) \((t_a < t_b)\).

#### 3.1 Initial state

It is assumed that the wave function \(\Psi(x_0, y_0, z_0, t_0)\) at the initial time \(t_0\) is known and factorized,

\[
\Psi(x_a, y_a, z_a, t_a) = \psi(x_a, y_a, t_a) \times \phi(z_a, t_a),
\]

(30)

so that the \(x-y\) motion, on which we concentrate, decouples from the \(z\)-motion. The problem is undetermined by a phase factor due to gauge invariance, if \(\psi \rightarrow \psi \times \exp(i \lambda)\) and at the same time \(A \rightarrow A + \nabla \lambda\), where \(A \equiv \lambda(x, y)\) is an arbitrary function of \(x\) and \(y\). We assume a constant magnetic field \(B = (0, 0, B) \equiv \nabla \times A\), derived from the vector potential

\[
A = (A_x, A_y, A_z) = ((\lambda - 1)By/2, (\lambda + 1)Bx/2, 0),
\]

(31)

with gauge parameter \(\lambda\). The initial wave function is assumed as an asymmetrical Gaussian \((\sigma_x \neq \sigma_y)\),

\[
\psi(x_a, y_a, t_a) = \sqrt{N_0} \exp \left[ -\frac{(x_a - x_0)^2}{4 \sigma_x^2} - \frac{(y_a - y_0)^2}{4 \sigma_y^2} \right.
+ \frac{i}{\hbar} \left[ q_x x_a + q_y y_a + \frac{\mu m\omega}{2} x_0 y_a \right].
\]

(32)

Without loss of generality it is assumed that no mixed term \(\chi xy\) is present in the exponent. The corresponding probability density is

\[
P(x_a, y_a, t_a) = |\psi(x_a, y_a, t_a)|^2
= N_0 \exp \left[ -(x_a - x_0)^2 / 2\sigma_x^2 - (y_a - y_0)^2 / 2\sigma_y^2 \right].
\]

(33)

Here \(N_0 = 1 / 2\pi \sigma_x \sigma_y\) is a normalization factor, the coordinates \((x_0, y_0) = ((x_a), (y_a))\) are the initial average positions (in this section \(\langle \ldots \rangle\) denotes a quantum average), and \(\sigma_x, \sigma_y\) are the corresponding standard deviations, \(\langle (x_a - x_0)^2 \rangle = \sigma_x^2\), \(\langle (y_a - y_0)^2 \rangle = \sigma_y^2\). The parameters \(q_x, q_y\) and \(\mu\) are related to the average initial velocity \(\bar{v} = (v_x, v_y) = \hbar / m = -i \nabla / m \hbar - eA / mc\), that from Eq. (32) is

\[
v_x = \frac{q_x}{m} + \frac{\mu - \lambda + 1}{2} \omega y_0,
\]

\[
v_y = \frac{q_y}{m} + \frac{\mu + \lambda - 1}{2} \omega x_0.
\]

(34)
Note that $\bar{\Psi}$ (as any other observable quantity) only depends on the difference
\begin{equation}
\alpha = \mu - \lambda, \tag{35}
\end{equation}
as a consequence of the gauge invariance of the problem.

The quantum treatment is based on the density matrix, that for a pure state is given, at $t = t_0$, by
\begin{equation}
\rho(x_a, \eta_a, x'_a, \eta'_a, t_0) = \psi(x_a, \eta_a, t_0)\psi^*(x'_a, \eta'_a, t_0). \tag{36}
\end{equation}

It is convenient to introduce the new coordinates
\begin{align}
X &= \frac{x' + x}{2}, \quad Y = \frac{y' + y}{2}, \quad \xi = \frac{x' - x}{2}, \quad \eta = \frac{y' - y}{2},
\end{align}
with inverse relations $x = X - \frac{\xi}{2}, y = Y - \frac{\eta}{2}, x' = X + \frac{\xi}{2}, y' = Y + \frac{\eta}{2}$. Then the initial density matrix is
\begin{align}
\rho(X_a, Y_a, \xi_a, \eta_a, t_0) & = \psi(X_a - \xi_a/2, Y_a - \eta_a/2)\psi^*(X_a + \xi_a/2, Y_a + \eta_a/2) \\
& = N_0 \exp \left\{ \frac{(X_a - x_a)^2}{2\sigma_x^2} - \frac{(Y_a - y_a)^2}{2\sigma_y^2} - \frac{\xi_a^2}{8\sigma_x^2} - \frac{\eta_a^2}{8\sigma_y^2} \\
& \quad - \frac{i}{\hbar} \left[ q_\xi \xi_a + p_\eta \eta_a + \frac{\mu_m x_0}{2} (\xi_a Y_a + X_a \eta_a) \right] \right\}. \tag{38}
\end{align}

### 3.2 Time evolution

The reduced density matrix $\rho(X, Y, \xi, \eta, t)$ evolves with time in a way similar to the wave function,
\begin{equation}
\rho(X_b, Y_b, \xi_b, \eta_b, t_b) = \int dX_adY_ad\xi_ad\eta_a J(b|a)\rho(X_a, Y_a, \xi_a, \eta_a, t_a), \tag{39}
\end{equation}
where the effective propagator $J(b|a)$ is conveniently expressed in terms of the effective action $S[X, Y, \xi, \eta]$ as
\begin{equation}
J(b|a) = J(X_b, Y_b, \xi_b, \eta_b, t_b|X_a, Y_a, \xi_a, \eta_a, t_a) \\
= \int_a^b DXDYD\xi D\eta \exp \left\{ \frac{i}{\hbar} S[X, Y, \xi, \eta] \right\}. \tag{40}
\end{equation}

Here $a$ represents the boundary conditions at time $t_a$: $X(t_a) = X_a, Y(t_a) = Y_a, \xi(t_a) = \xi_a$, and $\eta(t_a) = \eta_a$, and analogously for $b$. The effective action of an isolated system would simply be the difference of the two actions of the isolated system, $S[X, Y, \xi, \eta] = S_0[X - \xi/2, Y - \eta/2] - S_0[X + \xi/2, Y + \eta/2]$, and the effective propagator would factorize in the product of two propagators for $\psi(x, y, t)$ and $\psi^*(x', y', t)$. For a non-relativistic particle
\begin{equation}
S_0[x, y] = \int_{t_a}^{t_b} dt \left[ \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + \frac{e}{c} (\dot{x}A_x + \dot{y}A_y) \right]. \tag{41}
\end{equation}

In the case considered here of a non-isolated system, the effective action contains an additional term, the influence phase $\Phi_2$, left from the integration of the environment degrees of freedom $[34, 39]$,
\begin{equation}
S[X, Y, \xi, \eta] = S_0 [X - \xi/2, Y - \eta/2] \\
- S_0 [X + \xi/2, Y + \eta/2] + h \Phi_2[X, Y, \xi, \eta]. \tag{42}
\end{equation}

Here we focus on the dynamics of the probability density $P(X, Y, t)$ in configuration space, which is given by the diagonal elements of $\rho(x, x', y, y', t)$, i.e., $P(X, Y, t) = \rho(x, x', y, y, t)|_{x=x'=X; y=y'=Y}$. If the coordinates $X, Y, \xi, \eta$ are used, one can obtain $P$ setting $\xi = \eta = 0$, i.e., $P(X, Y, t) = \rho(X, Y, \xi = 0, \eta = 0, t)$. Then the probability density at time $t_b$, from Eqs. (40), is
\begin{align}
P(X_b, Y_b, t_b) & = \rho(X_b, Y_b, 0, 0, t_b) \\
& = \int dX_adY_ad\xi_ad\eta_a J(b|a) \rho(X_a, Y_a, \xi_a, \eta_a, t_a) \\
& \quad \times J(X_b, Y_b, 0, 0, t_b|X_a, Y_a, \xi_a, \eta_a, t_a) \\
& \quad \times \rho(X_a, Y_a, \xi_a, \eta_a, t_a). \tag{43}
\end{align}

We now proceed to obtain $\Phi_2[X, Y, \xi, \eta]$ and $J(b|a)$.

### 3.3 Influence functional and phase

The influence phase $\Phi$ — or equivalently the influence functional $\mathcal{F} = \exp(i\Phi)$ — is the central quantity in the study of a non-isolated quantum system, since it describes the effect of the interaction with the environment. In the framework of the Feynman-Vernon model [34] a dissipative environment is represented in one dimension by an infinite set of harmonic oscillators interacting with the central particle. Here the simple hypothesis is made that a two-dimensional environment is described by a straightforward extension of the one-dimensional case, so that the central particle interacts with a set of two-dimensional harmonic and isotropic
The action of the total system is
\[
S_{\text{tot}}[x, y; \{x_n\}, \{y_n\}] = S_0[x, y] + \frac{m}{2} \sum_n \int_{t_a}^{t_b} dt \{ \dot{x}_n^2 + \dot{y}_n^2 + \omega_n^2 [(x_n - x)^2 + (y_n - y)^2] \},
\]
where \( (x, y) \) and \( \{x_n, y_n\} \) \( (n = 1, 2, \ldots) \) represent the coordinates of the central particle and of the oscillators, respectively. It has to be noticed that the oscillators have their equilibrium positions coinciding with the central particle position, as requested by general symmetry constraints and that, for consistency, the form of the total Lagrangian in Eq. (44) also determines the specific form of the density matrix of the bath oscillators (assumed to be initially at thermal equilibrium) needed for the calculation of the influence phase. Skipping details which will be illustrated elsewhere and referring the reader to Ref. [40] for the detailed study of the one-dimensional problem, here we only mention that the initial density matrix of the generic dimensional problem, here we only mention that the tails which will be illustrated elsewhere and referring for the calculation of the influence phase. Skipping de-summation is due to the magnetic field, the real term \( \propto \gamma \) is related to dissipation, and the last imaginary term to thermal fluctuations. This quadratic functional can be calculated exactly to obtain
\[
J(b|a) = f(t) \exp \left\{ i \tilde{S}/\hbar \right\},
\]
where \( f(t) \) is a normalization factor and \( \tilde{S} \) the effective action \( S \) computed along the “classical” trajectories defined by
\[
\frac{1}{m} \frac{\delta S}{\delta \xi(t)} = \ddot{X}(t) + \gamma \dot{X}(t) - \omega \dot{Y}(t) + \frac{2i\gamma}{\beta} \xi(t) = 0,
\]
\[
\frac{1}{m} \frac{\delta S}{\delta \eta(t)} = \ddot{Y}(t) + \gamma \dot{Y}(t) + \omega \dot{X}(t) + \frac{2i\gamma}{\beta} \eta(t) = 0,
\]
subject to the boundary conditions \( a \) and \( b \). Integrating by parts Eq. (48) and setting \( \xi_b = \eta_b = 0 \) (we are interested in the probability density), one has
\[
\left( \tilde{S} \right)_{\xi_b=\eta_b=0} = m \left[ X_{\xi_a} + \dot{Y}_{\eta_a} - (\omega/2)(Y_{\xi_a} - X_{\eta_a}) \right] - \frac{i \gamma}{\beta} \int_{t_a}^{t_b} dt \left[ \xi^2(s) + \eta^2(s) \right]_{\xi_b=\eta_b=0}.
\]
Thus, only \( \xi(t) \) and \( \eta(t) \) defined in Eqs. (52) and (53) with final conditions \( \xi_b = \eta_b = 0 \) are needed to compute \( \left( \tilde{S} \right)_{\xi_b=\eta_b=0} \). They can be found, in analogy with the classical case, solving the equation obtained introducing the coordinates \( Z = X + iY, \xi = \xi + im, \) given by
\[
\dot{Z}(t) + \tilde{P} \dot{Z}(t) = -\frac{2i\gamma}{\beta} \xi(t), \quad \dot{\xi}(t) - \dot{\tilde{P}} \xi(t) = 0.
\]
We skip the details and provide the result, reading
\[
\langle \tilde{S} \rangle_{t=\hbar, \eta_{a}} = m \left\{ [\tilde{H}(t)\xi_{a} + \tilde{J}(t)\eta_{a}(X_{b} - X_{a})
+ [\tilde{H}(t)\eta_{a} - \tilde{J}(t)\xi_{a}](Y_{b} - Y_{a})
+ \frac{\omega}{2} (X_{a}\eta_{a} - Y_{a}\xi_{a}) + \frac{2\text{i}m\gamma}{\hbar} \epsilon(t) (\xi_{a}^{2} + \eta_{a}^{2}) \right\},
\]
where
\[
\tilde{H}(t) + i\tilde{J}(t) = \tilde{F}[1 - \exp(-\tilde{F}t)]
\]
\[
\epsilon(t) = \frac{1}{M} \left[ t + \frac{2}{\Gamma} e^{-\gamma t} \cos(\omega t + \chi_{H}) \right] - \frac{2\gamma}{\Gamma^{2}} + \frac{1 - e^{-2\gamma t}}{2\gamma}. \tag{57}
\]
\[
M(t) = 1 - 2e^{-\gamma t} \cos(\omega t) + e^{-2\gamma t}. \tag{58}
\]

3.4 Probability density

The effective propagator (49), with the effective action (56), and the initial conditions (38) can be used in Eq. (43) to compute the probability density \(P(X_{b}, Y_{b}, t_{b})\). The first two integrations give
\[
P(X_{b}, Y_{b}, t_{b}) = f(t) \int d\xi_{a} d\eta_{a} \exp \left\{ -A_{0}\xi_{a}^{2} - B_{0}\eta_{a}^{2}
+ 2C_{0}\xi_{a}\eta_{a} + D_{0}\xi_{a} + E_{0}\eta_{a} \right\},
\]
where
\[
A_{0}(t) = \frac{m^{2}}{2\hbar^{2}} \left[ \sigma_{a}^{2} \tilde{H}(t)^{2} + \sigma_{a}^{2} \tilde{J}(t)^{2} + \frac{h}{2m\sigma_{a}^{2}} + \frac{2\gamma\epsilon(t)}{m\beta} \right],
\]
\[
B_{0}(t) = \frac{m^{2}}{2\hbar^{2}} \left[ \sigma_{a}^{2} \tilde{H}(t)^{2} + \sigma_{a}^{2} \tilde{J}(t)^{2} + \frac{h}{2m\sigma_{a}^{2}} + \frac{2\gamma\epsilon(t)}{m\beta} \right],
\]
\[
C_{0}(t) = \frac{m^{2}}{2\hbar^{2}} \left[ \sigma_{a}^{2} \tilde{J}(t)^{2} - \sigma_{a}^{2} \tilde{J}(t)^{2} + \frac{h}{2m\sigma_{a}^{2}} + \frac{2\gamma\epsilon(t)}{m\beta} \right],
\]
\[
D_{0}(t) = \frac{im}{h} \left[ \tilde{H}(t)(X_{b} - x_{0}) - \tilde{J}(t)(Y_{b} - y_{0}) - v_{x} \right],
\]
\[
E_{0}(t) = \frac{im}{h} \left[ \tilde{J}(t)(X_{b} - x_{0}) + \tilde{H}(t)(Y_{b} - y_{0}) - v_{y} \right]. \tag{60}
\]
where \(\tilde{J}(t) = \tilde{J}(t) + \frac{1}{2}(\pm\alpha - 1)\omega\) and \(\alpha\) is the gauge parameter given by Eq. (35). The last integrations give
\[
P(X_{b}, Y_{b}, t_{b}) = N(t) \exp \left\{ -A_{1}X_{b}^{2} - B_{1}Y_{b}^{2}
+ 2C_{1}X_{b}Y_{b} + D_{1}X_{b} + E_{1}Y_{b} \right\}, \tag{62}
\]
where \(N(t)\) is a suitable normalization factor, \(X_{b}'\) and \(Y_{b}'\) are the coordinates relative to the initial mean position, i.e., \(X_{b}' = X_{b} - x_{0}\) and \(Y_{b}' = Y_{b} - y_{0}\), and the time-dependent coefficients are given by
\[
A_{1} = F_{0}(A_{0}, B_{0}, C_{0}) \left[ B_{0}\tilde{H}^{2} + 2C_{0}\tilde{J} + A_{0}\tilde{J}^{2} \right],
\]
\[
B_{1} = F_{0}(A_{0}, B_{0}, C_{0}) \left[ B_{0}\tilde{H}^{2} - 2C_{0}\tilde{J} + A_{0}\tilde{J}^{2} \right],
\]
\[
C_{1} = F_{0}(A_{0}, B_{0}, C_{0}) \left[ (A_{0} - B_{0})\tilde{H} + C_{0}(\tilde{H}^{2} - \tilde{J}^{2}) \right],
\]
\[
D_{1} = 2F_{0}(A_{0}, B_{0}, C_{0}) \left[ (B_{0}\tilde{H} + C_{0}\tilde{J})v_{x} + (C_{0}\tilde{H} - A_{0}\tilde{J})v_{y} \right],
\]
\[
E_{1} = 2F_{0}(A_{0}, B_{0}, C_{0}) \left[ (C_{0}\tilde{H} - A_{0}\tilde{J})v_{x} + (A_{0}\tilde{H} - C_{0}\tilde{J})v_{y} \right]. \tag{63}
\]
where \(F_{0}(A_{0}, B_{0}, C_{0}) \equiv (m/\hbar)^{2}/(A_{0}B_{0} - C_{0}^{2})\).

3.5 Mean trajectory

The mean coordinates at the generic time \(t_{b}\) are
\[
\langle X \rangle_{t} = \int dX_{b} dY_{b} P(X_{b}, Y_{b}, t_{b}) = x_{0} + \frac{\tilde{H}v_{x} + \tilde{J}v_{y}}{2(\tilde{H}^{2} + \tilde{J}^{2})},
\]
\[
\langle Y \rangle_{t} = \int dX_{b} dY_{b} P(X_{b}, Y_{b}, t_{b}) = y_{0} + \frac{\tilde{H}v_{y} - \tilde{J}v_{x}}{2(\tilde{H}^{2} + \tilde{J}^{2})}. \tag{64}
\]

By using the expressions of \(\tilde{H}(t)\) and \(\tilde{J}(t)\) in Eqs. (57), these equations consistently reduce to the classical solutions (18) and (19), with \(t\) replaced by \(t - t_{b}\) and \(x_{0}, y_{0}, v_{x}, v_{y}\) by the corresponding quantum mean values \(\langle x \rangle, \langle y \rangle\), \(\langle v_{x} \rangle \equiv \langle p_{x} \rangle / m, \langle v_{y} \rangle \equiv \langle p_{y} \rangle / m\), respectively, at \(t = t_{b}\).

3.6 Position uncertainties

For the second central moments, providing the quantum uncertainties on the particle coordinates, one obtains
\[
\langle \Delta X^{2} \rangle_{t} = \int dX_{b} dY_{b} [X_{b} - \langle X \rangle_{t}]^{2} P(X_{b}, Y_{b}, t_{b})
\]
\[
= 2 \left( \frac{\hbar}{m} \right)^{2} B_{0}\tilde{H}^{2} - 2C_{0}\tilde{J} + A_{0}\tilde{J}^{2},
\]
\[
\langle \Delta Y^{2} \rangle_{t} = \int dX_{b} dY_{b} [Y_{b} - \langle Y \rangle_{t}]^{2} P(X_{b}, Y_{b}, t_{b})
\]
\[
= 2 \left( \frac{\hbar}{m} \right)^{2} B_{0}\tilde{H}^{2} + 2C_{0}\tilde{J} + A_{0}\tilde{J}^{2},
\]
\[
\langle \Delta X \Delta Y \rangle_{t} = \int dX_{b} dY_{b} [X_{b} - \langle X \rangle_{t}] [Y_{b} - \langle Y \rangle_{t}] P(X_{b}, Y_{b}, t_{b})
\]
\[
= 2 \left( \frac{\hbar}{m} \right)^{2} (B_{0} - A_{0})\tilde{H}^{2} + C_{0}(\tilde{J}^{2} - A_{0}\tilde{J}^{2}) \tag{65}
\]
\]}
With the help of Eqs. (57), one obtains

\[ \langle \Delta X^2 \rangle_t = \Delta \tau^2 / 2 + (g_{\text{r}} / m\sigma_0)^2 + (g_{\text{b}} / m\sigma_0)^2 \]
\[ + \sigma^2_1 [1 - (1 + \alpha)\omega g_{\text{r}}]^2 + \sigma^2_1 [(1 - \alpha)\omega g_{\text{b}}]^2, \]
\[ \langle \Delta Y^2 \rangle_t = \Delta \tau^2 / 2 + (g_{\text{r}} / m\sigma_0)^2 + (g_{\text{b}} / m\sigma_0)^2 \]
\[ + \sigma^2_2 [1 - (1 - \alpha)\omega g_{\text{r}}]^2 + \sigma^2_2 [(1 + \alpha)\omega g_{\text{b}}]^2, \]
\[ \langle \Delta X \Delta Y \rangle_t = (g_{\text{r}} / m)^2 (\sigma^2_1 - \sigma^2_2) \]
\[ + 4 (g_{\text{r}}^2 - g_{\text{b}}^2) \rho \left( \mathscr{R} - \rho \alpha / \omega \right) \left( \sigma^2_1 - \sigma^2_2 \right) \]
\[ + \alpha \left( \sigma^2_1 + \sigma^2_2 \right) / 2, \]

where \( g_r(t) = (M \mathscr{R}) / (2 \Gamma^2) \) and \( g_b(t) = (M \mathscr{R}) (2 \Gamma^2) \). In the asymptotic limit \( \gamma t \gg 1 \), one recovers the classical limit also for the mean square displacements,

\[ \langle \Delta X^2 \rangle_t \to 2 D_B t, \quad \langle \Delta Y^2 \rangle_t \to 2 D_B t, \quad \langle \Delta X \Delta Y \rangle_t \to \text{const}. \]

(66)

3.7 Evolution of the wave packet

The Gaussian probability wave packet obtained above may not be easy to visualize in space and time, apart in the asymptotic limit. However, a simple interpretation and visualization of its time evolution can be given. Adapting the boundary surface method [41] to the present two-dimensional problem, one can visualize the probability density \( P(x, y, t) \) in space through an iso-probability line defined by \( P(x, y, t) = \text{const} \), which is defined in such a way that it encloses an assigned fraction of the total probability, e.g. 90%. From Eq. (62), one can alternatively define the iso-probability curve \( P(X, Y, t) \propto \exp[-g(X, Y)] \) through the condition

\[ g(X, Y) = A_1 X^2 + B_1 Y^2 - 2 C_1 X Y - D_1 X - E_1 Y = \text{const}. \]

(68)

This equation defines an ellipse in the \( X-Y \)-plane with time-varying center and principal axis orientation and lengths. In order to visualize this ellipse it is convenient first to move to the reference frame \( (X_1, Y_1) \) where the ellipse center is at rest, by introducing the coordinates relative to the average position and therefore to the classical solution,

\[ X_1 = X - \langle X \rangle_t , \quad Y_1 = Y - \langle Y \rangle_t. \]

(69)

The new boundary curve,

\[ A_1 X_1^2 + B_1 Y_1^2 - 2 C_1 X_1 Y_1 = \text{const}, \]

(70)

defines an ellipse with center in the origin. The ellipse can be further simplified by eliminating the mixed term \( \propto X_1 Y_1 \), by bringing it to its normal form. The time-dependent coefficient \( C_1 \) implies a rotation in time of the principal axes of the ellipse. One can move to a reference frame \( X_2-Y_2 \) rotating with the ellipse, defined by

\[ X_2 = X_1 \cos \theta + Y_1 \sin \theta, \quad Y_2 = -X_1 \sin \theta + Y_1 \cos \theta, \]

(71)

where \( \theta(t) \) is a suitable function of time. The new boundary curve is defined by

\[ A_2 X_2^2 + B_2 Y_2^2 - 2 C_2 X_2 Y_2 = \text{const} , \]
\[ 2 A_2 = \left[ A_1 + B_1 + (A_1 - B_1 \cos(2\theta) - 2 C_1 \sin(2\theta)) \right] , \]
\[ 2 B_2 = \left[ A_1 + B_1 - (A_1 - B_1 \cos(2\theta) + 2 C_1 \sin(2\theta)) \right] , \]
\[ 2 C_2 = \left[ (A_1 - B_1 \sin(2\theta) + 2 C_1 \cos(2\theta)) \right] . \]

(72)

The mixed term (and the rotational motion) can be removed by setting \( C_2 = 0 \), which defines an angle \( \theta(t) \) which can be written as the sum \( \theta(t) = \theta_1(t) + \theta_2(t) \), with

\[ \tan \theta = 2 C_1 / (B_1 - A_1) \equiv - \tan(\theta_1 + \theta_2), \]
\[ \tan \theta_1 = 2 \mathscr{R} (\omega^2 - \mathscr{R}^2), \]
\[ \tan \theta_2 = \frac{2 \mathscr{R} (\sigma^2_1 I - \sigma^2_2 X)}{\sigma^2_1 Y - \sigma^2_2 X} = \sigma^2_1 X + \sigma^2_2 Y + \left( \frac{\hbar}{2m} \right) (\sigma^2_1 - \sigma^2_2). \]

(73)

In the new variables, the boundary curve reads

\[ A_2(t) X_2^2 + B_2(t) Y_2^2 = \text{const}, \]

(74)

which represents an ellipse with the \( X_2 \)- and \( Y_2 \)-principal axis proportional to \( 1 / \sqrt{A_2} \) and \( 1 / \sqrt{B_2} \), respectively.

While a detailed illustration of the wave packet dynamics will be presented elsewhere, the equations above clearly show that it can be interpreted in a simple way as a superposition of (a) a translational motion defined by the classical solution, (b) a rotational motion with angular velocity \( \dot{\theta}(t) \), where \( \theta \) is defined by Eq. (73), and (3) a spreading process defined by the second moments \( 1 / A_2 \) and \( 1 / B_2 \) of the ellipse, once it is recast in normal form.
4 Conclusions

We used the oscillator model of quantum dissipative systems as an effective framework for studying the dynamics of a quantum Brownian particle in configuration space. Due to the quadratic nature of the system Lagrangian, the center of mass of the probability density is expected to move like the classical particle and the mean square displacement to become asymptotically equal to the classical one. In general, not all the formulations of the oscillator model found in the literature fulfill this requirements. We have shown, taking the two-dimensional problem of a quantum Brownian particle in a magnetic field as a working example, that using the prescriptions given in Ref. [40], concerning the invariance properties of the bath-oscillators sector of the total Lagrangian and the corresponding forms of the initial density matrices of the bath oscillators, all the expected results are obtained.

Furthermore, we have found that the probability wave packet in configuration space, besides moving on average according to the classical solution, can be visualized as if it rotates around its center of mass with an angular velocity depending on the system parameters and, at the same time, undergoes a standard diffusive spreading. Further work is needed for understanding this effect.

Acknowledgments. M.P. acknowledges support from the Estonian Science Foundation Grant no. 9462 and by institutional research funding IUT (IUT-39) of the Estonian Ministry of Education and Research.

Key words. Quantum Brownian motion, magnetic field, Feynman-Vernon model, Caldeira-Leggett model.

References

[1] J. B. Taylor, Phys. Rev. Lett. 6, 262 (1961).
[2] B. Kurşunoğlu, Ann. Phys. 17, 259 (1962).
[3] B. Kurşunoğlu, Phys. Rev. 132, 211 (1963).
[4] R. L. Liboff, Phys. Rev. 141, 222 (1966).
[5] Karmeshu, J. Phys. Soc. Japan 34, 1467 (1973).
[6] Karmeshu, Phys. Fluids 17, 1828 (1974).
[7] N. Xiang, Phys. Rev. E 48, 1590 (1993).
[8] J. Singh and S. Dattagupta, Pramana J. Phys. 47, 199 (1996).
[9] D. Lemons and D. Kaufman, IEEE Trans. on Plasma Sci. 27(5), 1288 (1999).
[10] R. Czopnik and P. Garbaczewski, Phys. Rev. E 63, 021105 (2001).
[11] R. Czopnik and P. Garbaczewski, Acta Physica Pol. B 32(5), 1437 (2001).
[12] I. Dodin and N. Fisch, Phys. Rev. E 72, 046602 (2005).
[13] T. Simoes and R. Lagos, Physica A 355, 274 (2005).
[14] J. I. Jiménez-Aquino and M. Romero-Bastida, Phys. Rev. E 74, 041117 (2006).
[15] J. I. Jiménez-Aquino and M. Romero-Bastida, Phys. Rev. E 76, 021106 (2007).
[16] J. I. Jiménez-Aquino, M. Romero-Bastida, and A. P. G. Noyola, Revista Mexicana de Física E 54(10), 81 (2008).
[17] F. Paraan, M. Solon, and J. Esguerra, Phys. Rev. E 77, 022101 (2008).
[18] N. Voropajeva and T. Örd, Phys. Lett. A 372, 2167 (2008).
[19] L. Hou, Z. Mišković, A. Piel, and P. Shukla, Physics of Plasmas 16(5), 053705 (2009).
[20] S. Dattagupta, J. Kumar, S. Sinha, and P. A. Sreeram, Phys. Rev. E 81, 031136 (2010).
[21] R. Lagos and T. Simoes, Physica A 390(9), 1591 (2011).
[22] S. Dattagupta, Diffusion. Formalism and Applications (CRC Press Taylor & Francis, Boca Raton, 2014).
[23] P. Friz, P. Gassiat, and T. Lyons, Trans. Am. Math. Soc. 367(11), 7939 (2015).
[24] H. Furuse, J. Phys. Soc. Japan 28(3), 559 (1970).
[25] A. K. Das, Z. Phys. B 40, 353 (1981).
[26] A. K. Das, Physica A 110, 489 (1982).
[27] A. M. Jayannavar and N. Kumar, J. Phys. A: Math. Gen. 14, 1399 (1981).
[28] Y. Marathe, Phys. Rev. A 39, 5927 (1989).
[29] S. Dattagupta and J. Singh, Pramana J. Phys. 47, 211 (1996).
[30] A. Mitra, arXiv:1007.0168 (2010).
[31] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields, Theoretical Physics Vol. 2 (Pergamon Press, Oxford, 1951).
[32] L. D. Landau and E. M. Lifshitz, Quantum mechanics, Theoretical Physics Vol. 3 (Pergamon Press, Oxford, 1965).
[33] X. L. Li, G. W. Ford, and R. F. O’Connel, Phys. Rev. A 41, 5287 (1990).
[34] R. P. Feynman and E. L. Vernon, Ann. Phys. (N.Y.) 24, 118 (1963).
[35] A. O. Caldeira and A. J. Leggett, Physica A 121, 587 (1983).
[36] N. W. Ashcroft and N. D. Mermin, Solid State Physics (Saunders College, Philadelphia, 1976).
[37] J. H. Williamson, J. Phys. A 1, 629 (1968).
[38] M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. 17, 323 (1945).
[39] R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integrals (McGraw-Hill, N.Y., 1965).
[40] M. Patriarca, Il Nuovo Cimento B 111, 61 (1996).
[41] P. W. Atkins, Physical Chemistry (Oxford University Press, Oxford, 1986).