LAGRANGIAN ANTISURGERY

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Abstract. We describe an operation which, under certain conditions, modifies a Lagrangian submanifold $L$ such as to produce a new immersed Lagrangian submanifold $L'$, which as a smooth manifold is obtained by surgery along a framed sphere in $L$. Intuitively, this can be described as collapsing an isotropic disc with boundary on $L$ to a point. The operation inverse to that generalizes classical Lagrangian surgery. We also describe corresponding Lagrangian cobordisms.

1. Introduction

A fundamental question in symplectic geometry is what manifolds arise as the Lagrangian submanifolds of a given symplectic manifold $(M^{2n}, \omega)$. This question has different flavours and levels of difficulty depending e.g. on whether one asks for embedded or immersed Lagrangian submanifolds, or if one incorporates additional constraints such as exactness or monotonicity.

A natural attempt to construct new Lagrangian submanifolds is to modify given ones by some sort of surgery operation. There is one well known construction which resolves the transverse double points of a Lagrangian immersion $\iota : L \to M$ by replacing neighbourhoods of them by copies of $D^1 \times S^{n-1}$. For example, if $L$ is connected, oriented and immersed with a unique double point, then the resulting Lagrangian $L'$ is embedded and diffeomorphic to the connected sum $L \# (S^1 \times S^{n-1})$, provided that the surgery can be performed compatibly with the orientation. This operation, which we will refer to as Lagrangian 0-surgery, is due to Lalonde–Sikorav [LS91] for $n = 2$ and to Polterovich [Pol91] for general $n$.

Terminology and notation. In all of the following, “Lagrangian submanifolds” will generally be allowed to be immersed with a possibly positive number of transverse double points. We will usually not make a notational distinction between abstract smooth manifolds $L$ and their immersed images in $M$; that is, whenever we have a Lagrangian immersion $\iota : L \to M$, we will slightly abuse notation and denote its image $\iota(L) \subset M$ also by $L$.

1.1. Surgery of smooth manifolds. On the level of smooth manifolds, the passage from $L$ to $L'$ by Lagrangian 0-surgery replaces an embedded copy of $S^0 \times S^{n-1}$ by a copy of $D^1 \times S^{n-1}$. This is a special case of the following more general operation originally due to Milnor [Mil61]. Whenever a smooth $n$-dimensional manifold $L$ contains an embedding $\varphi : S^k \times D^{n-k} \to L$, one can cut out $\varphi(S^k \times D^{n-k})$ and replace it by a copy of $D^{k+1} \times S^{n-k-1}$ such as to obtain a new manifold

$$L' = (L \setminus \varphi(S^k \times D^{n-k})) \cup_{\varphi(S^k \times S^{n-k-1})} (D^{k+1} \times S^{n-k-1}).$$

Date: November 17, 2015.
This works because \( \partial(S^k \times D^{n-k}) = S^k \times S^{n-k-1} = \partial(D^{k+1} \times S^{n-k-1}) \). We say that the manifold \( L' \), which inherits a smooth structure from \( L \) in a canonical way, is obtained from \( L \) by \( k \)-surgery (a.k.a. surgery of index \( k+1 \)).

Surgery theory is closely connected to cobordism theory. The manifold \( L' \) resulting from \( k \)-surgery on a manifold \( L \) is cobordant to \( L \) via a cobordism

\[
V = ([0,1] \times L) \cup_{\{1\} \times \varphi(S^k \times D^{n-k})} D^{k+1} \times D^{n-k},
\]

i.e., a cobordism that arises from the cylinder \([0,1] \times L\) by attaching a \((k+1)\)-handle \( D^{k+1} \times D^{n-k} \) along \( \{1\} \times \varphi(S^k \times D^{n-k}) \). This cobordism is called the trace of the corresponding surgery.

1.2. Lagrangian antisurgery. Let now \( L \subset M \) be a Lagrangian submanifold containing an embedded copy of \( S^k \times D^{n-k} \). It is natural to ask if the manifold \( L' \) obtained by \( k \)-surgery on \( L \) can again be embedded or immersed into \( M \) as a Lagrangian submanifold. The answer to a strong version of this question is certainly negative: For example, a closed orientable manifold \( L \) that can be embedded in \( \mathbb{C}^n \) must have Euler characteristic \( \chi(L) = 0 \). However, \( k \)-surgery changes the Euler characteristic according to \( \chi(L') = \chi(L) + (-1)^{k+1} + (-1)^{n-k-1} \) and hence does not preserve its vanishing if \( n \) is even. So in this case no result of a single \( k \)-surgery on \( L \) admits a Lagrangian embedding into \( \mathbb{C}^n \).

In this paper we will describe a construction which implements \( k \)-surgery for Lagrangian submanifolds under certain conditions. Let \( L \subset M \) be a Lagrangian submanifold containing an embedding \( \varphi : S^k \times D^{n-k} \to L \) together with an isotropic surgery disc \( D \), that is, an embedded isotropic disc \( D \subset M \) intersecting \( L \) cleanly along \( S = \varphi(S^k \times \{0\}) \) and otherwise disjoint from \( L \) (this terminology is borrowed from \([\text{Dim}12]\)).

**Theorem 1.1.** The manifold \( L' \) obtained by \( k \)-surgery on \( L \) with respect to the embedding \( \varphi : S^k \times D^{n-k} \to L \) admits a Lagrangian immersion \( L' \to M \) whose image agrees with \( L \) outside an arbitrarily small neighbourhood of \( D \), and such that in this neighbourhood it has exactly one transverse double point. Moreover, there exists an immersed Lagrangian cobordism \( V : L' \sim L \) given by a Lagrangian immersion of the trace of the \( k \)-surgery into \( T^* \mathbb{R} \times M \).

The construction of \( L' \) and \( V : L' \sim L \), and hence the proof of Theorem 1.1, is the content of Section 2. We refer to the operation that passes from \( L \) to \( L' \) as Lagrangian \( k \)-antisurgery. The idea behind the terminology is that it creates a double point, in contrast to Lagrangian 0-surgery, which resolves a double point. To give a quick and intuitive description, one could say that Lagrangian \( k \)-antisurgery modifies a Lagrangian \( L \) by collapsing an isotropic \((k+1)\)-disc with boundary on \( L \) to a point.

The local model for the immersed Lagrangian \((k+1)\)-handle which enables us to build the cobordism \( V \), as well as the idea of implanting it along an isotropic disc, is inspired by a construction of Dimitroglou Rizell appearing in \([\text{Dim}12]\), which implements \( k \)-surgery for Legendrian submanifolds and builds corresponding Lagrangian cobordisms (in a different sense of the word).

1.3. Lagrangian cobordisms. The notion of Lagrangian cobordism appearing in Theorem 1.1 is that of Biran–Cornea \([\text{BC}13]\), adapted to the immersed setting in an obvious way: Two ordered collections \((l_i : L_i \to M)_{i=1}^r, (l'_j : L'_j \to M)_{j=1}^s\) of immersed Lagrangian submanifolds of \( M \) are called Lagrangian cobordant if there exists a smooth cobordism \((V; \coprod_i L_i, \coprod_j L'_j)\)
together with a Lagrangian immersion $V \to [0, 1] \times \mathbb{R} \times M \subset T^*\mathbb{R} \times M$ such that for some small $\delta > 0$, we have

$$V|_{[0,\delta) \times \mathbb{R}} = \prod_{i=1}^{r} [0, \delta) \times \{i\} \times L_i$$

and

$$V|_{(1-\delta, 1] \times \mathbb{R}} = \prod_{j=1}^{s} (1 - \delta, 1] \times \{j\} \times L'_j;$$

here we use the notation $V|_{U} := V \cap (U \times M)$ to denote the part of $V$ that lies over some subset $U \subset T^*\mathbb{R}$, and we identify $T^*\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ in the standard way. The Lagrangian submanifold $V \subset T^*\mathbb{R} \times M$ is called an immersed Lagrangian cobordism with negative ends $(L_i)_{i=1}^{r}$ and positive ends $(L'_j)_{j=1}^{s}$, and this relationship is denoted by $V : (L'_1, \ldots, L'_s) \leadsto (L_1, \ldots, L_r)$. In this article we will mainly deal with the case $r = s = 1$, i.e. with Lagrangian cobordisms $V : L' \leadsto L$

with a single positive and a single negative end.

Lagrangian cobordisms have recently attracted a lot of interest due to the fact that, provided certain monotonicity assumptions hold, they preserve Floer theoretic invariants and encode information about the Fukaya category, see [BC13, BC14] and also the recent [MW15]. So far, there have been essentially two known constructions of Lagrangian cobordisms, which are based on Hamiltonian isotopy resp. Lagrangian 0-surgery. Extending the toolkit for building new ones, such as those provided by Theorem 1.1 and Theorem 1.2 below, was one of the motivations for the present paper.

1.4. Reversing the construction. Lagrangian antisurgery constructs from a Lagrangian $L \subset M$ an new Lagrangian $L'$ with one (additional) double point. Changing perspectives, we can view $L$ as the result of resolving a double point of $L'$ by an operation which is an $(n - k - 1)$-surgery on the level of smooth manifolds, and which we therefore refer to as Lagrangian $(n - k - 1)$-surgery.

In the case $k = n - 1$, this reversed operation is the same as classical Lagrangian 0-surgery. That is, if $L'$ is the result of an $(n - 1)$-antisurgery on $L$, then $L$ can be obtained back from $L'$ by classical Lagrangian 0-surgery, and vice versa. We will discuss this point of view in Section 3.

1.5. Desingularization. The newly created double point of the Lagrangian $L'$ resulting from antisurgery can be resolved by 0-surgery; provided that $L$ is embedded, this yields again an embedded Lagrangian $L^2$ which is diffeomorphic to $L'\# P^n$ or $L'\# Q^n$ (with $P^n = S^1 \times S^{n-1}$ and $Q^n$ the mapping of an orientation-reversing involution of $S^{n-1}$). We will show that one can in fact simultaneously remove the singular locus of the immersed Lagrangian cobordism $V : L' \leadsto L$ produced by Theorem 1.1.

**Theorem 1.2.** There exists an embedded Lagrangian cobordism $V^2 : L^2 \leadsto L$ which coincides with the immersed cobordism $V : L' \leadsto L$ outside of an arbitrarily small neighbourhood of its singular locus. As a smooth manifold, $V^2$ is the concatenation of the trace of a $k$-surgery on $L$ and a 0-surgery on $L'$.

The construction constituting the proof will be given in Section 4.1. The singular locus of $V$ looks like a line of double points, and the passage from $V$ to $V^2$ replaces a neighbourhood of it by a Lagrangian 1-handle (essentially the Lagrangian 1-handle constructed in [BC13]).
1.6. Examples. We provide a few examples for the operation of going from \( L \) to \( L^2 \) and the corresponding cobordisms \( V^2 : L^2 \sim L \).

1.6.1. The case \( k = n - 1 \). In the case \( k = n - 1 \), \( L \) and \( L^2 \) are the results of two different ways of resolving the double point of \( L' \) by Lagrangian 0-surgery, and \( V^2 : L^2 \sim L \) is a Lagrangian cobordism between two such resolutions. We thus obtain the following statement as a corollary of Theorem 1.2.

Theorem 1.3 (See Theorem 5.1 for a more precise version). Any two Lagrangians \( L, L^2 \) obtained by resolving a double point of an immersed Lagrangian \( L' \) are Lagrangian cobordant by a cobordism \( V^2 : L^2 \sim L \) which is embedded if \( L' \) has precisely one double point.

As an application of this, we show that there exist Lagrangian cobordisms \( T^2_{Cl}(A') \sim T^2_{Cl}(A'') \) between Clifford and Chekanov tori in \( \mathbb{C}^2 \) with different area classes (the notation indicates that a disc of Maslov index 2 on the respective torus has area \( A' \) resp. \( A'' \)):

Corollary 1.4. For every \( A' < A' \), there exist (non-monotone) Lagrangian cobordisms \( T^2_{Cl}(A'') \sim T^2_{Cl}(A') \), \( T^2_{Cl}(A') \sim T^2_{Cl}(A'') \) and \( T^2_{Cl}(A') \sim T^2_{Cl}(A'') \), which as smooth manifolds are obtained from \([0,1] \times T^2\) by successively attaching a 2-handle and a 1-handle.

1.6.2. Monotone examples. It is desirable to have examples of Lagrangian cobordisms which satisfy the technical condition of monotonicity. In this article we call a Lagrangian \( L \) monotone if the homomorphism \( \omega : H_2(M, L) \rightarrow \mathbb{R} \) given by integration of \( \omega \) and the Maslov index \( \mu : H_2(M, L) \rightarrow \mathbb{Z} \) are positively proportional, i.e.

\[
\omega = \eta \mu
\]

for some \( \eta > 0 \).\(^1\) The reason for wanting this condition to be satisfied is essentially that Floer theory works best in that case, and thus such examples might lead to interesting applications. For example, the ends of monotone cobordisms are isomorphic objects in the Fukaya category, see [BC14].

While it is clear that monotonicity is generally not preserved when passing from \( L \) to \( L^2 \), it can be shown to be preserved in cases in which we have sufficient control over the size of the neighbourhood in \( M \) in which we perform the surgery. We provide a family of examples in Section 5.3.1 and prove the following:

Theorem 1.5 (See Theorem 5.3). For all \( n \) and \( k \) such that \( 2 \leq k \leq n - 3 \), there exists a monotone Lagrangian \( L \subset \mathbb{C}P^n \) diffeomorphic to \( S^1 \times S^{n-1} \) such that the result \( L^2 \times S^{n-1} \times S^{n-1} \) of \( k \)-antisurgery and subsequent removal of the double point, as well the cobordism \( V^2 : L^2 \sim L \), are monotone.

The Lagrangian \( L^2 \) resulting from this operation on \( L \cong S^1 \times S^{n-1} \) is diffeomorphic to \( S^{k+1} \times D^{n-k-1} \# 2P^n \) or \( S^{k+1} \times D^{n-k-1} \# P^n \# Q^n \) (depending on \( n \) and \( k \)). In particular, this yields examples of monotone Lagrangian cobordisms with a single positive and a single negative end which are non-diffeomorphic.

\(^1\)This definition is stronger than the usual one, in which proportionality of \( \omega \) and \( \mu \) is only required on the image of \( \pi_2(M, L) \rightarrow H_2(M, L) \). We use this stronger version as it is easier shown to be preserved under surgery in certain cases.
1.7. **Relation to other work.** As mentioned before, one important source of inspiration for our construction is a surgery construction for Legendrian submanifolds appearing in [Dim12]. The local model for the immersed Lagrangian handle we use can be traced back to [Arn80, ALP94], where it appears in a slightly different guise. It seems that the passage from $L$ to $L^\#$ in the case $n = 2$ and $k = 1$ is identical to an operation described in [Yau13]. The article [MW15] is an exploration of relations between Lagrangian surgery and Lagrangian cobordisms in a different direction.

**Acknowledgements.** I would like to thank Denis Auroux, Paul Biran, François Charette, Octav Cornea, Georgios Dimitroglou Rizell, Tobias Ekholm, Jonny Evans, Leonid Polterovich, Dmitry Tonkonog and Weiwei Wu for related discussions and comments. This article was mainly written while I was employed as a CIRGET postdoctoral fellow at the Centre de Recherches Mathématiques in Montreal, and partly during a stay at the Institut Mittag-Leffler in Stockholm. I thank both institutions for their hospitality.

## 2. Lagrangian antisurgery

In this section we will explain the construction of immersed Lagrangian $(k+1)$-handles $\Gamma \subset T^*\mathbb{R} \times T^*\mathbb{R}^n$, for $0 \leq k \leq n-1$, which will serve as the local models for the construction of the cobordisms appearing in Theorem 1.1. These handles are immersed Lagrangian cobordisms $\Gamma : \Lambda' \sim \Lambda$ diffeomorphic to $D^{k+1} \times D^{n-k}$ whose ends are Lagrangian submanifolds $\Lambda \approx S^k \times D^{n-k}$ and $\Lambda' \approx D^{k+1} \times S^{n-k-1}$ of $T^*\mathbb{R}^n$. The construction is inspired by a similar one in [Dim12].

### 2.1. Construction of $\Gamma$.** The handle $\Gamma$ will be defined as the union of the graphs of exact 1-forms $+dF$ and $-dF$, where $F : \mathfrak{U} \to \mathbb{R}$ is a function defined on a certain subset $\mathfrak{U} \subset \mathbb{R} \times \mathbb{R}^n$.

As a first step in defining $\mathfrak{U}$ and $F$, consider smooth functions $\sigma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying

1. $\sigma(x_0) = 0$ for $x_0 \leq \delta$,
2. $\sigma(x_0) = 1 + \epsilon$ for $x_0 \geq 1 - \delta$,
3. $\sigma'(x_0) > 0$ for $\delta < x_0 < 1 - \delta$,

and

1. $\rho(r^2) = 1$ for $r^2$ close to 0,
2. $\rho(r^2) = 0$ for $r^2 \geq 1 + 2\epsilon$,
3. $-1/(1 + \epsilon) < \rho'(r^2) \leq 0$ for all $r^2 \in \mathbb{R}_{\geq 0}$.
for certain small constants $\varepsilon, \delta > 0$. Denote by $r^2, s^2 : \mathbb{R}^n \to \mathbb{R}_{>0}$ the functions given by $r^2(x) = x_1^2 + \cdots + x_{k+1}^2$ and $s^2(x) = x_{k+2}^2 + \cdots + x_n^2$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then define a function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by

$$f(x_0, x) = r^2 + \sigma(x_0)\rho(r^2) - s^2 - 1$$

for $(x_0, x) \in \mathbb{R} \times \mathbb{R}^n$, where $r^2 \equiv r^2(x)$ and $s^2 \equiv s^2(x)$.

Let now $\mathcal{U} = \{(x_0, x) \in \mathbb{R} \times \mathbb{R}^n \mid f(x_0, x) \geq 0\}$ and define $F : \mathcal{U} \to \mathbb{R}$ by

$$F(x_0, x) = f(x_0, x)^{3/2}. \quad (1)$$

The restriction of $F$ to $\text{int}(\mathcal{U})$ is smooth, with differential given by

$$dF = \frac{3}{2} f(x_0, x)^{1/2} (\sigma'(x_0) \rho(r^2) dx_0 + (1 + \sigma(x_0)\rho'(r^2)) dr^2 - ds^2). \quad (2)$$

Note that $dF$ extends to a section of $T^*(\mathbb{R} \times \mathbb{R}^n)$ over $\mathcal{U}$ which vanishes along $\partial \mathcal{U} = \{f(x_0, x) = 0\}$; we will denote this extended section by $dF$ as well. The graphs $\Gamma_\pm \subset T^*(\mathbb{R} \times \mathbb{R}^n)$ of $\pm dF : \mathcal{U} \to T^*\mathbb{R}^n$ are Lagrangian submanifolds with boundary. The tangent spaces along the boundary are given by

$$TT_{\pm}|_{\partial \mathcal{U}} = T(N^*(\partial \mathcal{U}))|_{\partial \mathcal{U}},$$

where $N^*(\partial \mathcal{U})$ denotes the conormal bundle of $\partial \mathcal{U}$. Hence $\Gamma_+$ and $\Gamma_-$ fit together smoothly along $\partial \mathcal{U}$, in the sense that their union

$$\Gamma = \Gamma_+ \cup \Gamma_- = \{(x_0, x), \pm dF(x_0, x) \mid (x_0, x) \in \mathcal{U}\} \quad (3)$$

is a submanifold $T^*(\mathbb{R} \times \mathbb{R}^n)$ which is embedded near $\partial \mathcal{U}$. The locus where $\Gamma$ fails to be embedded is given by the points $(x_0, x) \in \mathcal{U}$ at which $dF$ vanishes (see Section 4.1).

$\Gamma$ is the immersed image of a $(k+1)$-handle $D^{k+1} \times D^{n-k}$, and moreover an immersed Lagrangian cobordism $\Gamma : \Lambda' \sim \Lambda$ in the sense of Section 1.3. To see the latter and to describe the ends, set $\mathcal{U}_{x_0} = \{x \in \mathbb{R}^n \mid (x_0, x) \in \mathcal{U}\}$ for $x_0 \in \mathbb{R}$ and define $F_{x_0} : \mathcal{U}_{x_0} \to \mathbb{R}$ to be the function given by $F_{x_0}(x) = F(x_0, x)$. Since $F_{x_0}$ is independent of $x_0$ if either $x_0 \leq \delta$ or $x_0 \geq 1 - \delta$, it follows that the part of $\Gamma$ lying over $(-\infty, \delta) \times \mathbb{R} \cup [1 - \delta, \infty) \times \mathbb{R} \subset T^*\mathbb{R}$ is

$$(-\infty, \delta) \times \{0\} \times \Lambda \cup [1 - \delta, \infty) \times \{0\} \times \Lambda' \quad (4)$$

with

$$\Lambda = \{(x, \pm dF_0(x)) \in T^*\mathbb{R}^n \mid x \in \mathcal{U}_0\},$$

$$\Lambda' = \{(x, \pm dF_1(x)) \in T^*\mathbb{R}^n \mid x \in \mathcal{U}_1\}. \quad (5)$$

This shows that $\Gamma$ is a Lagrangian cobordism (up to modifying the ends in an obvious way).

2.2. Isotropic surgery discs. We will now describe the situation in which it is possible to implant the local model described above, such as to produce from a given Lagrangian $L$ a new immersed Lagrangian $L'$ together with a Lagrangian cobordism $V : L' \sim L$.

The following is an adaptation of Definition 4.2 in [Dim12] to our setting.

**Definition 2.1.** Let $L \subset M$ be a Lagrangian submanifold and let $S \subset L$ be a embedded $k$-sphere with trivializable normal bundle. An isotropic surgery disc for $S$ consists of the following data:

1. An embedded isotropic $(k+1)$-disc $D \subset M$ such that
   - $\partial D = S$,
   - $\text{int} D \cap L = \emptyset$,
• any vector field \( X \) which is outward pointing normal to \( S = \partial D \) is nowhere contained in \( TL \).

(2) A symplectic subbundle \( E \) of \((TD)^\omega\) such that \( TD \oplus E = (TD)^\omega\), and a symplectic trivialization \( \Psi : D \times \C^{n-k-1} \to E \) such that the Lagrangian subbundle \( \Psi(S \times \R^{n-k-1}) \) of \( E|_S \) is contained in \( TL|_S \).

We will usually denote isotropic surgery discs simply by \( D \), omitting the bundle \( E \) and its trivialization \( \Psi \) from the notation.

An isotropic surgery disc \( D \) for a sphere \( S \subset L \) determines a homotopy class of trivializations of the normal bundle of \( S \subset L \) as follows: Let \( Y \subset TL|_S \) be any vector field that’s normal to \( S \subset L \) and such that \( \omega(X,Y) > 0 \) for any outward pointing normal vector field \( X \) to \( D \) (such a vector field \( Y \) exists due to the assumption on \( X \) in Definition 2.1). Then the subbundle \( \Psi(S \times \R^{n-k-1}) \oplus \R Y \) of \( TL|_S \) is complementary to \( TS \) and of rank \( n-k \), and thus it spans the normal bundle of \( S \subset L \). Since the space of all such vector fields \( Y \) is non-empty and contractible, the corresponding trivialization is determined up to homotopy.

**Example 1.** The prototypical example for the situation described in Definition 2.1 is given by the Lagrangian \( \Lambda \subset T^*\R^n \) defined in (5) and the \( k \)-sphere

\[
S_0 = \{(x,y) \in T^*\R^n \mid x_1^2 + \cdots + x_{k+2}^2 = 1, x_{k+3} = \cdots = x_n = 0, y = 0\}; \tag{6}
\]

the obvious choice of isotropic surgery disc for \( S_0 \subset \Lambda \) is

\[
D_0 = \{(x,y) \in T^*\R^n \mid x_1^2 + \cdots + x_{k+2}^2 \leq 1, x_{k+3} = \cdots = x_n = 0, y = 0\} \tag{7}
\]

together with the symplectic subbundle

\[
E_0 = \langle \partial_{x_{k+2}}, \ldots, \partial_{x_n}, \partial_{y_{k+2}}, \ldots, \partial_{y_n} \rangle \tag{8}
\]
of \( TD_0^\omega \) and the identification \( \Psi_0 : D_0 \times \C^{n-k-1} \to E_0 \) taking \( D_0 \times \R^{n-k-1} \) to the subbundle \( \langle \partial_{x_{k+2}}, \ldots, \partial_{x_n} \rangle \) and \( D_0 \times \R^{n-k-1} \) to the subbundle \( \langle \partial_{y_{k+2}}, \ldots, \partial_{y_n} \rangle \).

Assume that we are in the situation of Definition 2.1, i.e. that we have a Lagrangian \( \Lambda \subset T^*\R^n \) with a sphere \( S \subset L \) and a corresponding isotropic surgery disc \( D \equiv (D,E,\Psi) \). Let \( \phi : D_0 \to D \) be a diffeomorphism; together with the symplectic trivialization \( \Psi : D \times \C^{n-k-1} \times E \), this determines an isomorphism of symplectic vector bundles \( T^*D_0 \oplus E_0 \cong T^*D \oplus E \) (here we use the notation of Example 1). An application of the isotropic neighbourhood theorem then yields an extension of \( \phi \) to a symplectomorphism

\[
\phi : \mathcal{W}_0 \to \mathcal{W}
\]

between appropriate Darboux-Weinstein neighbourhoods \( \mathcal{W}_0 \supset D_0 \) and \( \mathcal{W} \supset D \) of the discs in \( T^*\R^n \) resp. \( M \), and we may assume that this extension satisfies

\[
\phi(\Lambda \cap \mathcal{W}_0) = L \cap \mathcal{W}. \tag{9}
\]

To see this, note that the condition that the outward normal vector field to \( S \subset D \) is nowhere tangent to \( L \) guarantees that one can arrange \( D\phi(T\Lambda|_{S_0}) = TL|_S \); after adjusting \( \phi \) by a Hamiltonian isotopy and possibly shrinking the Weinstein neighbourhoods one obtains (9).
2.3. Implanting the local model. We now explain how to implant the local model and give the definition of Lagrangian antisurgery. To prepare for that, consider the neighbourhood of $D_0$ in $T^*\mathbb{R}^n$ given by

$$U_0 = \left\{ (x,y) \in T^*\mathbb{R}^n \mid r^2 < 1 + 2\varepsilon, \ s^2 < 2\varepsilon, \ ||y||^2 < 6\sqrt{2}\varepsilon(1 + 4\varepsilon) \right\}. \quad (10)$$

Denote by $U_0^c$ the complement of $U_0$ in $T^*\mathbb{R}^n$.

**Lemma 2.1.** We have $\Gamma \cap (T^*\mathbb{R} \times U_0^c) = \mathbb{R} \times (\Lambda \cap U_0^c) = \mathbb{R} \times (\Lambda' \cap U_0^c)$.

**Proof.** We first claim that for $((x_0, y_0), (y_0, y)) = ((x_0, x), \pm dF(x_0, x)) \in \Gamma$ with $r^2 < 1 + 2\varepsilon$, we already have $(x, y) \in U_0$. To see this, recall that the set $U \subset \mathbb{R} \times \mathbb{R}^n$ over which $\Gamma$ lives is characterized by $f(x_0, x) \geq 0$, where $f(x_0, x) = r^2 + \sigma(x_0)\rho(r^2) - s^2 - 1$. Since $r^2 \mapsto r^2 + \sigma(x_0)\rho(r^2) - 1$ is strictly increasing with value $2\varepsilon$ at $r^2 = 1 + 2\varepsilon$ for every $x_0 \in \mathbb{R}$, it follows that $s^2 < 2\varepsilon$. Moreover, one can read off from the expression for $dF(x_0, x)$ that the bound on $|y|^2$ is satisfied whenever $r^2 < 1 + 2\varepsilon$ and $s^2 < 2\varepsilon$.

Let now $((x_0, x), (y_0, y)) \in \Gamma \cap (T^*\mathbb{R} \times U_0^c)$. As a consequence of our claim, we obtain $r^2 \geq 1 + 2\varepsilon$, and hence the expression for $dF(x_0, x)$ simplifies to $dF(x_0, x) = \frac{1}{2}(r^2 - s^2 - 1)^{1/2}(dr^2 - ds^2)$, as $\rho(r^2) \equiv 0$ for $r^2 \geq 1 + 2\varepsilon$. Since this has vanishing $dx_0$ component and is independent of $x_0$, it follows that $((x_0, x), (y_0, y))$ lies in $\mathbb{R} \times (\Lambda \cap U_0^c)$ and in $\mathbb{R} \times (\Lambda' \cap U_0^c)$. Thus $\Gamma \times (T^*\mathbb{R} \times U_0^c)$ is contained in both these sets, and the reverse inclusions are obvious. $\square$

The neighbourhood $U_0$ of $D_0$ in (7) can be made arbitrarily small by letting the parameter $\varepsilon$ tend to zero. In particular, by choosing $\varepsilon$ sufficiently small we may assume that the closure $\overline{U_0}$ is contained in a Weinstein neighbourhood $W_0$ of $D_0$ as described in Section 2.2, i.e. such that we have a symplectic identification $\phi : W_0 \to W$ with a Weinstein neighbourhood $W$ of $D$.

**Definition 2.2.** Given such choices of $\varepsilon$ and $\phi$, we define the corresponding immersed Lagrangian $L' \subset M$ obtained from $L$ by Lagrangian $k$-antisurgery along the isotropic disc $D$ by

$$L' = (L \cap W_0) \cup \phi(\Lambda' \cap W_0), \quad (11)$$

and its immersed Lagrangian trace $V : L' \sim L$ by

$$V = \mathbb{R} \times (L \cap W_0) \cup (id \times \phi)(\Gamma \cap (T^*\mathbb{R} \times W_0)), \quad (12)$$

using the symplectomorphism $id \times \phi : T^*\mathbb{R} \times W_0 \to T^*\mathbb{R} \times W$.

The fact that the pieces which we glue fit together as required is a consequence of Lemma 2.1, which implies that $\phi(\Lambda' \cap (W_0 \setminus \overline{U_0})) = L \cap (W \setminus \overline{U})$ and $(id \times \phi)(\Gamma \cap (T^*\mathbb{R} \times (W_0 \setminus \overline{U_0}))) = \mathbb{R} \times (L \cap (W \setminus \overline{U}))$, where $U = \phi(U_0)$; hence the pieces of $\Lambda'$ resp. $\Gamma$ we glue overlap with corresponding pieces of $L$ resp. $\mathbb{R} \times L$ as required.

The Lagrangian submanifold $L' \subset M$ given by Definition 2.2 is the immersed image of the manifold obtained from $L$ by a $k$-surgery along $S$ with respect to the trivialization of the normal bundle of $S \subset L$ determined by the surgery disc $D$. The Lagrangian cobordism $V : L' \sim L$ is the immersed image of the trace corresponding to that surgery.
Lagrangian antisurgery produces from a Lagrangian submanifold \( L \) a new Lagrangian submanifold \( L' \) with an additional double point. Switching the roles of input and output, we can interpret \( L \) as the result of an operation which resolves a singularity of \( L' \) by replacing an immersed copy of \( D^{n-p} \times S^p \subset L' \) by an embedded copy of \( S^{n-p-1} \times D^{p+1} \subset L \).

**Definition 3.1.** We say that \( L \) is obtained from \( L' \) by Lagrangian \( p \)-surgery if \( L' \) is obtained from \( L \) by Lagrangian \((n-p-1)\)-antisurgery.

By this definition, a necessary condition for being able to perform Lagrangian \( p \)-surgery on a given Lagrangian \( L' \) is that it contains an immersed copy of \( D^{n-p} \times S^p \) which is obtained by implanting a suitable piece of the immersed Lagrangian \( \Lambda' \) in \( L' \). Observe that the part of \( \Lambda' \) lying over \( \{0\} \times \mathbb{R}^p \) is the image of a Whitney type immersion \( S^p \to T^*\mathbb{R}^p \cong \{0\} \times T^*\mathbb{R}^p \subset T^*\mathbb{R}^n \), obtained from the standard Whitney immersion

\[
S^p \to T^*\mathbb{R}^p, \quad (x, y) \mapsto x + iyx = (x, i\sqrt{1 - |x|^2}x)
\]

for \((x, y) \in S^p \subset \mathbb{R}^p \times \mathbb{R} \), by rescaling. This motivates the following definition:

**Definition 3.2.** Let \( L' \subset M \) be a Lagrangian submanifold containing the image of a Lagrangian immersion \( \iota : D^{n-p} \times S^p \to M \) which is an embedding away from \( \{0\} \times S^p \) and such that \( \tilde{S} = \iota(\{0\} \times S^p) \) has precisely one transverse double point. We call \( \iota(D^{n-p} \times S^p) \subset L' \) a Whitney degeneration if the following holds: There exists an embedded isotropic \( p \)-disc \( \tilde{D} \subset M \) with boundary in \( \tilde{S} \), together with a Weinstein neighbourhood \( \mathcal{N} \cong (T\tilde{D})^\omega/T\tilde{D} \oplus T^*\tilde{D} \) of \( \tilde{D} \) such that upon a suitable symplectic identification of \( \mathcal{N} \) with a subset of \( T^*\mathbb{R}^{n-p} \times T^*\mathbb{R}^p \), \( \tilde{S} \) is the image of a Whitney type immersion \( S^p \to \{0\} \times T^*\mathbb{R}^p \) (see Figure 2).

Containing a Whitney degeneration is in fact not a sufficient condition for a Lagrangian \( L' \) to be admissible for Lagrangian \( p \)-surgery. For example, the Whitney sphere \( S^n_{Wh} \subset T^*\mathbb{R}^n \) itself obviously contains Whitney degenerations \( \iota(D^{n-p} \times S^p) \) for every \( 0 \leq p \leq n-1 \), but it is not possible to perform Lagrangian \( p \)-surgery on \( S^n_{Wh} \) for any \( p > 0 \); This would lead to a compact embedded Lagrangian \( L \subset T^*\mathbb{R}^n \) with vanishing area class, which we know not to exist. Indeed, \( L \) would be diffeomorphic to \( S^{n-p-1} \times D^{p+1} \), which has \( H_1(L) = 0 \) for \( p \notin \{0, n-2\} \); in the case \( p = n-2 \), we would create an isotropic 2-disc whose boundary generates \( H_1(L) \cong \mathbb{Z} \), and hence the area class would vanish in this case as well.

### 3.1. Lagrangian 0-surgery

The case \( p = 0 \) of Definition 3.1 provides an alternative definition of Lagrangian 0-surgery. To see that it coincides with the usual notion (defined e.g. in Section 5.2 below), recall that the local model for that is the image of an embedding

\[
h_\gamma : \mathbb{R} \times S^{n-1} \to T^*\mathbb{R}^n, \quad (x, t) \mapsto (a(t)x, b(t)x),
\]
where \( \gamma(t) = (a(t), b(t)) \) is a certain curve in \( T^{*}\mathbb{R} \). This model bounds a Lagrangian disc given e.g. by the image of

\[
S^{n-1} \times [0, 1] \to T^{*}\mathbb{R}^n, \quad (x, s) \mapsto s(a(0)x, b(0)x).
\]

Hence any Lagrangian \( L \) resulting from Lagrangian 0-surgery of a Lagrangian \( L' \), as it is usually defined, bounds a corresponding Lagrangian disc \( D \); performing \((n-1)\)-antisurgery along \( D \) gives back the original \( L' \) up to Hamiltonian isotopy. Conversely, if \( L \) is any Lagrangian with a Lagrangian surgery disc \( D \) and \( L' \) is the result of \((n-1)\)-antisurgery along \( D \), one can get back to \( L \) by performing 0-surgery on \( L' \) in the usual sense.

4. Desingularization

Our aim in this section is to turn the immersed Lagrangian cobordism constructed in Definition \[2.2\] into an embedded one and thus prove Theorem \[1.2\].

4.1. The singular loci of \( \Gamma \) and \( \Lambda' \). The points where the cobordism \( \Gamma \) fails to be embedded are given by the \((x_0, x) \in \text{int} \mathfrak{U} \) where \( d\mathcal{F}(x_0, x) = 0 \), which is where the graphs of \( \pm d\mathcal{F} \) intersect each other. By [2], \( d\mathcal{F}(x_0, x) = 0 \) is equivalent to

\[
\begin{align*}
\sigma'(x_0)\rho(r^2)dx_0 &= 0, \\
(1 + \sigma(x_0)\rho'(r^2))dr^2 &= 0, \\
ds^2 &= 0.
\end{align*}
\]

(14)

The conditions imposed on \( \sigma \) and \( \rho \) imply that these equations are simultaneously satisfied if and only if \( x_0 \geq 1 - \delta \) and \( x = 0 \), and hence the singular locus of \( \Gamma \) is

\[
\Gamma^\delta = \{((x_0, 0), (0, 0)) \in T^* (\mathbb{R} \times \mathbb{R}^n) \mid x_0 \geq 1 - \delta\}. \tag{15}
\]

To see this, note that the conditions \( \sigma(x_0) \geq 0 \) and \(-1/ (1 + \epsilon) < \rho'(r^2) \leq 0 \) imply \( 1 + \sigma(x_0)\rho'(r^2) > 0 \), so the second equation only holds when \( d\mathcal{F} = 0 \); together with the third equation \( ds^2 = 0 \) we conclude that \( x = 0 \). Since \( \rho(0) = 1 \), the first equation simplifies to \( \sigma'(x_0)dx_0 = 0 \) and thus \( x_0 \leq \delta \) or \( x_0 \geq 1 - \delta \); since \((x_0, 0) \notin \mathfrak{U} \) for \( x_0 < 1 - \delta \), only the second of these possibilities leads to a solution of (14).

Recall that the part of \( \Gamma \) that lies over \([1 - \delta, \infty) \times \mathbb{R} \subset T^* \mathbb{R} \) is cylindrical of the form

\[
\Gamma|_{[1-\delta, \infty) \times \mathbb{R}} = [1-\delta, \infty) \times \{0\} \times \Lambda' \subset T^* \mathbb{R} \times T^* \mathbb{R}^n. \tag{16}
\]

(15) therefore also shows that the positive end \( \Lambda' \) of \( \Gamma \) has a double point at \( x = 0 \) and is embedded away from that. The tangent spaces

\[
\lambda_\pm = T_{(0, 0)}\Lambda'_\pm \tag{17}
\]

to the two sheets of \( \Lambda' \) at the double point are spanned by

\[
\begin{align*}
\partial_{x_i} \pm 3f_1(0)^{1/2}\partial_{y_i}, & \quad i = 1, \ldots, k + 1 \\
\partial_{x_i} \mp 3f_1(0)^{1/2}\partial_{y_i}, & \quad i = k + 2, \ldots, n
\end{align*}
\]

(18)

where \( f_1(0) = f(1, 0) \); since \( f_1(0) \neq 0 \), this shows that \( \lambda^+ \) and \( \lambda^- \) intersect transversely.

The transverse double point of \( \Lambda' \) can be removed by Lagrangian 0-surgery such as to produce an embedded Lagrangian submanifold \( \Lambda^\pm \subset T^* \mathbb{R}^n \). More interestingly, we will see
that one can resolve the singular locus $\Gamma^s$ of $\Gamma$ and turn the immersed cobordism $\Gamma : \Lambda \sim \Lambda'$ into an embedded Lagrangian cobordism $\Gamma^s : \Lambda \sim \Lambda$.

4.2. **Classical Lagrangian 0-surgery.** We recall briefly the usual definition of Lagrangian 0-surgery as given e.g. in [IC13, Section 6.1]: Let $\gamma$ be an embedded Lagrangian cobordism $\gamma : \Lambda \sim \Lambda$.

Given two Lagrangians $L_{\pm} \in M$ intersecting transversely at $p \in L_{-} \cap L_+$, one can implant this local model using a Darboux chart which identifies neighbourhoods of $p$ in $L_\pm$ with neighbourhoods of 0 in $\mathbb{R}^n \times \{0\}$ resp. $\{0\} \times \mathbb{R}^n$. The result is a new Lagrangian submanifold of $M$ which is commonly denoted by $L_- \# L_+$. Note that this notation keeps track of the order of the two Lagrangians, i.e. of which one gets identified with $\mathbb{R}^n \times \{0\}$ and which one with $\{0\} \times \mathbb{R}^n$. This is important, since the two ways of ordering usually lead to results which are smoothly non-isotopic and sometimes distinct even as smooth manifolds (e.g. orientable in one case, but non-orientable in the other case).

4.3. **Another Lagrangian 1-handle.** Let us now turn to the resolution of the singular locus $\Gamma^s$ of $\Gamma$. Essentially, we cut out a neighbourhood of $\Gamma^s$ and replace it with a copy of the Lagrangian 1-handle constructed in [IC13, Lemma 6.1.1], i.e. the "trace of surgery" cobordism $\mathbb{R}^n \# i\mathbb{R}^n \sim (\mathbb{R}^n, i\mathbb{R}^n)$ appearing there. In the following we will rephrase that construction somewhat in a way that's adapted to our situation.

Let $\eta_\pm : \mathbb{R} \to T^*\mathbb{R}$ be curves given by $\eta_\pm(x) = (t, \pm y(x))$, where $y : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a smooth function such that $y(x) > 0$ for $x < 0$ and $y(x) = 0$ for $x \geq 0$ (see Figure 4). Let $\lambda_\pm \subset T^*\mathbb{R}^n$ be two transversely intersecting Lagrangian subspaces. Then

$$W = \eta_+ \times \lambda_+ \cup \eta_- \times \lambda_-$$

is an immersed Lagrangian submanifold of $T^*\mathbb{R} \times T^*\mathbb{R}^n$ whose singular locus is $\mathbb{R}_{\geq 0} \times \{0\}$. We view it as an immersed Lagrangian cobordism $W : \lambda_- \cup \lambda_+ \sim (\lambda_-, \lambda_+)$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure3.png}
\caption{A curve $\gamma$ used to define Lagrangian 0-surgery}
\end{figure}
Proposition 4.1. There exists an embedded Lagrangian cobordism $W^\natural : \lambda_- \# \lambda_+ \sim (\lambda_-, \lambda_+)$ such that $W^\natural$ and $W$ coincide outside of an arbitrarily small neighbourhood of the singular locus $\mathbb{R}_{\geq 0} \times \{0\}$ of $W$.

Proof. We will show how to perform recall the construction of Biran–Cornea’s trace cobordism corresponding to the Lagrangian surgery of $\lambda_\pm$ such that the result agrees with $W$ outside of an arbitrarily small neighbourhood of the singular locus $\mathbb{R}_{\geq 0} \times \{0\}$ of $W$. We assume that $\lambda_- = \mathbb{R}^n \times \{0\}$ and $\lambda_+ = \{0\} \times \mathbb{R}^n$ for notational simplicity.

Choose a curve $\gamma : \mathbb{R} \to T^* \mathbb{R}$, $\gamma(t) = (a(t), b(t))$, as in Section 5.2 and let $L = \lambda_- \# \lambda_+ \subset T^* \mathbb{R}^n$ be the result of the corresponding 0-surgery of $\lambda_\pm$. Then define $\phi_\gamma : \mathbb{R} \times S^n \to T^* \mathbb{R}^{n+1}$ to be the composition of the map $\mathbb{R} \times S^n \to T^* \mathbb{R}^{n+1}$, $(t, x) \mapsto (a(t)x, b(t)x)$, with a rotation of the first factor of $T^* \mathbb{R}^{n+1} = T^* \mathbb{R} \times \mathbb{R}^n$ by $\pi/4$. Let $W' = \left(\text{im} \phi_\gamma\right)\mid_{\{(x, y) \in T^* \mathbb{R} \mid x \leq 0\}}$, be the part of the image of $\phi_\gamma$ that lies over the half-plane $\{(x, y) \in T^* \mathbb{R} \mid x \leq 0\}$. Note that $W'$ is a manifold with boundary, and the boundary is given by $L_0 = \{(0, 0)\} \times L$.

We will describe how to adjust $W'$ such that a cylindrical end $\mathbb{R}_{\geq 0} \times L$ can be glued on, and such that the resulting Lagrangian looks like $W = \eta_+ \times \lambda_+ \cup \eta_- \times \lambda_-$ outside a small neighbourhood of $\mathbb{R}_{\geq 0} \times \{0\}$.

To start, let $\mathcal{N}'$ be a Weinstein neighbourhood of the Lagrangian $\mathbb{R} \times L \subset T^* \mathbb{R}^{n+1}$ which is of the form $\mathcal{N} = T^* \mathbb{R} \times \mathcal{N}_L$, where $\mathcal{N}_L \subset T^* \mathbb{R}^n$ is a Weinstein neighbourhood of $L$. Consider a neighbourhood $U'_0$ of $L_0 = \partial W'$ in $W'$; by eventually shrinking it, we assume that $U'_0$ lies entirely in the Weinstein neighbourhood $\mathcal{N}$ and is the graph of a closed 1-form $\alpha_0$ over the subset $U_0 = (-3\varepsilon_0, 0] \times L$ of $\mathbb{R} \times L$ for some small $\varepsilon_0 > 0$. Note that $\alpha_0$ is exact because its restriction to $L_0$ vanishes (as $L_0$ is contained in $\mathbb{R} \times L$) and because $U'_0$ retracts onto $L_0$. We denote by $g_0 : U_0 \to \mathbb{R}$ the primitive of $\alpha_0$ which vanishes on $L_0$.

Next, set $U'_1 = W' \cap (T^* \mathbb{R} \times (T^* \mathbb{R}^n \backslash B_{2\kappa}^{2n}))$, where $\kappa$ is the parameter appearing in the definition of the curve $\gamma$ (see Section 5.2). Recalling the construction of $W'$, one sees that $U'_1$
is a subset of \((\ell_+ \times \lambda_-) \cup (\ell_- \times \lambda_+)\), where \(\ell_\pm = \{(x, y) \in T^*\mathbb{R} \mid x \pm y = \pm 0\}\). It follows that \(U_1'\) is the graph of an exact 1-form \(\alpha_1 = dg_1\) for a function \(g_1 : U_1 \to \mathbb{R}\) defined on the subset

\[ U_1 = \mathbb{R} \times (L \cap (T^*\mathbb{R}^n \setminus B_{2n}^{\mathbb{R}})) \]

of \(\mathbb{R} \times L\) whose restrictions to \(U_1 \cap (\mathbb{R} \times \lambda_+)\) resp. \(U_1 \cap (\mathbb{R} \times \lambda_-)\) depend only on the \(\mathbb{R}\)-coordinate. (Explicitly, we can take \(g_1 = \pm \frac{1}{x^2}\) on these subsets.)

We now define \(\alpha\) to be the 1-form on \(U_0 \cup U_1 \subset \mathbb{R} \times L\) that restricts to \(\alpha_i\) on \(U_i\); it is not hard to see that \(\alpha\) is well-defined (as the restriction to \(U_0' \cup U_1'\) of the canonical projection \(N \to \mathbb{R} \times L\) is one-to-one). Moreover, the primitives \(g_i\) of the \(\alpha_i\) piece together to a global primitive \(g\) of \(\alpha\). To see this, note that \(g_0\) and \(g_1\) agree on \(U_0 \cap U_1 \cap L_0\) (both vanish there); since \(U_0 \cap U_1\) retracts onto that set, they agree on all of \(U_0 \cap U_1\). We can therefore unambiguously define \(g : U_0 \cup U_1 \to \mathbb{R}\) to be the function that restricts to \(g_0\) on \(U_0\) and to \(g_1\) on \(U_1\).

To finally adjust \(W'\) as required, let \(Y : \mathbb{R} \to \mathbb{R}\) be the anti-derivative of \(y\) with \(Y(0) = 0\) (where \(y : \mathbb{R} \to \mathbb{R}_{\geq 0}\) is the function appearing in the definition of \(\eta_\pm\)) and let \(\zeta : U_1 \to \mathbb{R}\) be a cut-off function with \(\zeta \equiv 0\) near \(U_1 \cap ((-3\varepsilon_0, 0] \times B_{3\varepsilon_0}^\mathbb{R})\) (represented by the right shaded rectangle in Figure 5) and \(\zeta \equiv 1\) away from a small neighbourhood of that set. Then consider the function \(G : U_0 \cup U_1 \to \mathbb{R}\) whose restriction to \(U_1 \cap (\mathbb{R} \times \lambda_+)\) is \(G = \pm \zeta Y - g\) and whose restriction to \(U_0 \cap U_1\) is \(g\) (this definition yields a smooth function because \(U_0 \cap U_1 \subset (-3\varepsilon_0, 0] \times B_{3\varepsilon_0}^\mathbb{R})\)). Note that since the restriction of \(G\) to the complement of a small neighbourhood of \((U_0 \cup U_1) \cap ((-3\varepsilon_0, 0] \times B_{3\varepsilon_0}^\mathbb{R}) \subset \mathbb{R} \times L\) depends only on the \(\mathbb{R}\)-coordinate, the Hamiltonian isotopy on \(N|_{U_0 \cup U_1}\) induced by \(G\) moves the corresponding region in \(U_0' \cup U_1'\) only in the direction of the fibres of \(T^*\mathbb{R}\), and its time one map takes this region to \(\eta_+ \times \lambda_+ \times \eta_- \times \lambda_-\) by construction of \(G\). Moreover, the image of the neighbourhood \(U_0'\) of \(\partial W'\) under the time one map is such that a cylindrical end \(\mathbb{R}_{\geq 0} \times L\) can be glued on smoothly.

This isotopy so far only moves \(U_0' \cup U_1' \subset W'\). To fix that, consider the subsets \(\tilde{C} \subset C\) of \(U_0 \cup U_1\) given by \(C = ((-4\kappa, -\varepsilon_0) \times B_{2\varepsilon_0}^\mathbb{R}) \cap (U_0 \cup U_1)\) and \(\tilde{C} = ((-3\kappa, -2\varepsilon_0) \times B_{2\varepsilon_0}^\mathbb{R}) \cap (U_0 \cup U_1)\) (we assume here that \(\varepsilon_0\) was chosen small enough such that \(2\varepsilon_0 < 3\kappa\)) and choose a cut-off function \(\xi : U_0 \cup U_1 \to \mathbb{R}\) such that \(\xi \equiv 0\) on \(\tilde{C}\), \(\xi \equiv 1\) on \((U_0 \cup U_1) \setminus C\), and such that the restriction of \(\xi\) to \(C \cap ((-4\kappa, -3\kappa) \times (B_{2\varepsilon_0}^\mathbb{R} \setminus B_{2\varepsilon_0}^\mathbb{R}))\) depends only on the \(\mathbb{R}\)-coordinate of \(\mathbb{R} \times L\); see Figure 5 in which this last region is represented by the little shaded rectangle. The
Hamiltonian isotopy on $N|U_0 \cup U_1$ induced by $\xi G : U_0 \cup U_1 \to \mathbb{R}$ and applied to $U_0' \cup U_1' \subset W'$ is equal to that induced by $G$ in a neighbourhood of $\partial W'$, and it extends to a Lagrangian isotopy defined on all of $W'$ (including the parts which are not graphical over $\mathbb{R} \times L$) which away from a neighbourhood of the origin moves $\ell_\pm \times (\lambda_\pm \cap B_{2\kappa})$ in the direction of the fibres of $T^* \mathbb{R}$ with the same speed with which the corresponding parts of $\ell_\pm \times (\lambda_\pm \cap (T^n \mathbb{R}^n \setminus B_{2\kappa}^n)) \subset U_1'$ move.

We define $W^\sharp$ to be the image of $W'$ under the time one map of this isotopy. \hfill \Box

4.4. **Surgery of the singular locus of** $\Gamma$. Observe that, in suitable Darboux coordinates, a neighbourhood of the singular locus of $\Gamma$ is given by $(\eta_- \times \lambda_-) \cup (\eta_+ \times \lambda_+)$, with $\lambda_\pm = T_{(0,0)}A_\pm$ and with curves $\eta_\pm$ as described in the previous subsection (up to a shift to the right by $1 - \delta$); explicitly, we can take $\eta_\pm(x_0) = (x_0, \pm \frac{3}{2} (\sigma(x_0) - 1)^{1/2} \sigma'(x_0))$.

Having set up such an identification, we can use Proposition 4.1 to replace this neighbourhood with a corresponding piece of $W^\sharp : \lambda_- \# \lambda_+ \sim (\lambda_- \lambda_+)$, constructed with respect to a sufficiently small value of the parameter $\kappa$. The outcome of this operation is an embedded Lagrangian cobordism $\Gamma^\sharp : \Lambda^\sharp \sim \Lambda$, where $\Lambda^\sharp$ is the result of resolving the double point of $\Lambda'$ by Lagrangian 0-surgery. By choosing $\kappa$ sufficiently small, we can guarantee that $\Gamma^\sharp$ and $\Gamma$ coincide outside of an arbitrarily small neighbourhood of $\Gamma^s$ in $T^* \mathbb{R}^{n+1}$.

Assume now that we have a Lagrangian $L \subset M$ with a sphere $S \subset L$ and an isotropic surgery disc $D$ for $S$. Repeating the construction in Section 2.3 but replacing $\Gamma$ with $\Gamma^\sharp$ and consequently $\Lambda$ with $\Lambda^\sharp$, we produce a Lagrangian cobordism $V^\sharp : L^\sharp \sim L$, where $L^\sharp$ is the Lagrangian obtained by resolving the double point created when performing antisurgery on $L$ along the isotropic disc $D$. $V^\sharp$ is embedded if $L$ is embedded, and it can be arranged to agree with with the corresponding $V : L' \sim L$ outside of an arbitrarily small neighbourhood of its singular locus by choosing the parameter $\kappa$ sufficiently small. Topologically, this cobordism is the concatenation of the traces corresponding to first performing $k$-surgery on $L$ and then 0-surgery on the result $L'$ of the first step; in other words, $V^\sharp$ is obtained from $[0, 1] \times L$ by first attaching a $(k + 1)$-handle and then a 1-handle.

This proves Theorem 1.2.

4.5. **Orientability.** The result of abstract $k$-surgery on an orientable manifold $L$ is always orientable if $k \geq 1$, since the $D^{k+1} \times S^{n-k-1}$ we glue in has a connected boundary in that case (or rather, every component has a connected boundary – this includes the case $k = n - 1$). In the case $k = 0$, orientability depends on whether $D^1 \times S^{n-1}$ is glued consistently along its two boundary components. Let

$$P^n = S^1 \times S^{n-1},$$

$$Q^n = D^1 \times S^{n-1}/\sim,$$

where $\sim$ identifies $\{1\} \times S^{n-1}$ with $\{-1\} \times S^{n-1}$ using an orientation reversing involution of $S^{n-1}$ (i.e., $Q^n$ is the mapping torus of such an involution). The result of 0-surgery on $L$ is diffeomorphic to $L \# P^n$ in the orientable case and to $L \# Q^n$ in the non-orientable case.
Returning to the Lagrangian setting, assume that the $L$ we start with is orientable. When passing from $L$ to $L'$ by $k$-antisurgery, we replace an embedded copy of $S^k \times D^{n-k}$ (a subset of $\Lambda$) by an immersed copy of $D^{k+1} \times S^{n-k-1}$ with a transverse double point (a subset of $\Lambda'$), and we resolve this double point by 0-surgery when passing from $L'$ to $L^\sharp$. Thus $L^\sharp$ is obtained from $L$ by replacing an embedded copy of $S^k \times D^{n-k-1}$ by an embedded copy (a subset of $\Lambda^\sharp$) of either

$$(D^{k+1} \times S^{n-k-1})\#P^n \text{ or } (D^{k+1} \times S^{n-k-1})\#Q^n$$

in the case $k < n - 1$. The first possibility leads to $L^\sharp$ being orientable, the second to $L^\sharp$ being non-orientable; the next propositions tell when which of these alternatives holds.

**Proposition 4.2.** If $n$ is even, then $L^\sharp$ is orientable for odd $k$ and non-orientable for even $k$. If $n$ is odd, we can arrange both of these possibilities.

**Proof.** Which one of the alternatives holds depends on the sign $(-1)^{(n-1)/2+1} \Lambda' \cdot \Lambda'_+ \cdot \Lambda'_+$, where $\Lambda'_- \cdot \Lambda'_+$ denotes the intersection index with respect to the symplectic orientation of the ambient manifold, see [Pol91]. If $n$ is odd, we can reverse this sign by switching the role of $\Lambda'_-$ and $\Lambda'_+$ (i.e. by reversing the choice of which sheet gets identified with $\mathbb{R}^n$ and which with $i\mathbb{R}^n$ when we implant the local model for Lagrangian 0-surgery) and thus realize both possibilities. If $n$ is even, switching $\Lambda'_-$ and $\Lambda'_+$ does not affect the sign.

To compute $\Lambda'_- \cdot \Lambda'_+$, let $\omega_\pm$ be the orientations of $\Lambda'_\pm$ which the projections $\Lambda'_\pm \to \mathbb{R}^n$ to the zero-section match up with the standard orientation of $\mathbb{R}^n$. These are given by the ordered bases of $\Lambda'_\pm$ listed in [18]. It follows from this description that the orientation $\omega_+ \oplus \omega_-$ is given by the ordered basis $(\partial/\partial x_1 + \partial/\partial y_1, \ldots, \partial/\partial x_{k+1} + \partial/\partial y_{k+1}, \partial/\partial x_{k+2} - \partial/\partial y_{k+2}, \ldots, \partial/\partial x_n - \partial/\partial y_n)$ of $\mathbb{R}^{2n} = T_{(0,0)} T^* \mathbb{R}^n$. The matrix taking this basis to the standard symplectic basis $(\partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/\partial x_n, \partial/\partial y_n)$ has determinant $(-1)^{(n-1)/2+k+1} 2^n$, and thus $\omega_+ \oplus \omega_- = (-1)^{(n-1)/2+k+1} \omega_\omega$, where $\omega_\omega$ denotes the symplectic orientation. In order to determine $\Lambda'_- \cdot \Lambda'_+$, we must choose orientations of $\Lambda'_\pm$ which induce the same orientation of $\Lambda'$. One such choice is given by $\omega_+$ for $\Lambda'_+$ and $-\omega_-$ for $\Lambda'_-$. Since $\omega_+ \oplus (-\omega_-) = (-1)^{(n-1)/2+k} \omega_\omega$, we obtain $\Lambda'_- \cdot \Lambda'_+ = (-1)^{(n-1)/2+k}$.

It follows that $(-1)^{(n-1)/2+1} \Lambda'_- \cdot \Lambda'_+ = (-1)^{k+1}$, and thus the claimed statement follows from [Pol91, Theorem 4].

### 4.6. Monotonicity

Assume that the Lagrangian $L$ we start with is monotone in the sense of the definition given in the introduction (Section 1.6.2). It is of interest to know under what conditions the Lagrangian $L^\sharp$ and the cobordism $\mathcal{V}^k: L^\sharp \simeq \Lambda^\sharp$ are monotone as well. Let us assume that $0 \leq k \leq n - 2$, in which case $\Lambda^\sharp$ is diffeomorphic to either $(D^{k+1} \times S^{n-k-1})\#P^n$ or $(D^{k+1} \times S^{n-k-1})\#Q^n$, and thus

$$H_1(\Lambda^\sharp) \cong \mathbb{Z} \quad \text{and} \quad H_2(T^*\mathbb{R}^n, \Lambda^\sharp) \cong \mathbb{Z}.$$ 

There is a preferred generator $\sigma \in H_2(T^*\mathbb{R}^n, \Lambda^\sharp)$ characterized by the positivity of its symplectic area, and we denote by $\tau = \partial \sigma \in H_1(\Lambda^\sharp)$ its boundary. We will denote by $\sigma$ and $\tau$ also the corresponding elements of $H_2(M, L^\sharp)$ and $H_1(L^\sharp)$.
In order to preserve monotonicity during the passage from $L$ to $L^2$, we need to ensure that the symplectic area and the Maslov index of $\sigma$ are related by

$$\omega(\sigma) = \eta_L \mu(\sigma),$$  \hspace{1cm} (20)

where $\eta_L$ is the given monotonicity constant of $L$. Whenever $2 \leq k \leq n - 3$, the second part of Proposition 4.3 below implies that (20) is actually a sufficient condition for the monotonicity of both $L^2$ and the cobordism $V^2 : L^2 \sim L$: It says essentially that $H_2(M, L^2)$ and $H_2(T^*\mathbb{R} \times M, V^2)$ are obtained from $H_2(M, L)$ by adjoining the additional generator $\sigma$.

We will provide a computation of the Maslov index $\mu(\sigma)$ in Proposition 4.4. As for the area $\omega(\sigma)$, consider first the generator $\sigma' \in H_2(T^*\mathbb{R}, \Lambda') \cong \mathbb{Z}$ of positive area. A simple computation shows that $\omega(\sigma') = 2^3/2$ (the area of one of the teardrops in Figure 6). The 0-surgery by which we pass from $\Lambda'$ to $\Lambda^2$ modifies this area by an amount which we can make arbitrarily small, i.e.

$$\omega(\sigma) = 2^3/2 \pm \alpha$$

where $\alpha > 0$ can be chosen arbitrarily small (cf. Figure 7 where $\alpha$ is the area of one of the triangle-like pieces; the two different signs $\pm$ correspond to the two ways of performing the 0-surgery).

**Proposition 4.3.** Assume that $2 \leq k \leq n - 3$.

1. There is a natural isomorphism $H_1(L) \oplus \mathbb{Z} \cong H_1(L^2)$ which sends $1 \in \mathbb{Z}$ to $\tau \in H_1(L^2)$. Moreover, the inclusions $L, L^2 \hookrightarrow V^2$ induce an injection $H_1(L) \hookrightarrow H_1(V^2)$ resp. an isomorphism $H_1(L^2) \cong H_1(V^2)$.

2. There's a natural isomorphism $H_2(M, L) \oplus \mathbb{Z} \cong H_2(M, L^2)$ which sends $1 \in \mathbb{Z}$ to $\sigma \in H_2(M, L^2)$. Moreover, the map $H_2(M, L^2) \rightarrow H_2(T^*\mathbb{R} \times M, V^2)$ induced by inclusion is surjective.

**Proof.** The assertions in (1) are simple consequences of the definitions of (abstract) surgery and the corresponding trace cobordisms. They actually hold on the level of fundamental groups, as one can see by using the van Kampen theorem.

As for (2), let $X = L \smallsetminus (S^k \times D^{n-k})$ be the subset of $L$ obtained by removing the region along which we surger, and note that $X$ is also a subset of $L^2$. Consider the maps $H_2(M, X) \rightarrow H_2(M, L)$ and $H_2(M, X) \oplus \mathbb{Z} \rightarrow H_2(M, L^2)$ which on $H_2(M, X)$ are induced by inclusion and where the second map takes $1 \in \mathbb{Z}$ to the class $\sigma \in H_2(M, L^2)$ described above. A bit of either meditation or diagram chasing shows that these maps are both isomorphisms and thus yield a natural isomorphism $H_2(M, L) \oplus \mathbb{Z} \rightarrow H_2(M, L^2)$.

We make this precise for $i_* : H_2(M, X) \rightarrow H_2(M, L)$. Its surjectivity can be read off from the three rightmost squares of the diagram below (or seen by applying one of the 4-lemmas):

$$\begin{array}{cccc}
H_2(X) & \xrightarrow{j_X} & H_2(M) & \xrightarrow{p_X} & H_2(M, X) & \xrightarrow{i_X} & H_1(X) & \xrightarrow{i_X} & H_1(M) \\
\downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \downarrow \\
H_2(L) & \xrightarrow{j_L} & H_2(M) & \xrightarrow{p_L} & H_2(M, L) & \xrightarrow{i_L} & H_1(L) & \xrightarrow{i_L} & H_1(M)
\end{array}$$

For injectivity, consider the following part of the Mayer-Vietoris sequence for $X \subset L, Y = S^k \times D^{n-k} \subset L$ (so $X \cup Y = L, X \cap Y \simeq S^k \times S^{n-k-1}$): $\cdots \rightarrow H_2(S^k \times S^{n-k-1}) \rightarrow H_2(X) \oplus H_2(S^k \times D^{n-k}) \rightarrow H_2(L) \rightarrow 0$. Let $a \in H_2(M, X)$ be an element in the kernel of
That there exists an element \( b \) in \( H_2(M) \) such that \( p_X(b) = a \), and for this there must exist in turn an element \( c \in H_2(L) \) such that \( j_2(c) = b \) by commutativity and exactness. Using the piece of the Mayer-Vietoris sequence from above and the fact that the composition \( H_2(S^n \times D^{n-k}) \rightarrow H_2(L) \rightarrow H_2(M) \) vanishes, we can assume that \( c = i_X(c') \) for some \( c' \in H_2(X) \). Hence \( j_X(c') = j_L(i_X(c')) = b \) and hence \( a = p_X(b) = p_X(j_X(c')) = 0 \).

For the second part of (2), consider the following diagram, in which the horizontal sequences belong to the long exact sequences for the pairs \((M, L^2)\) and \((C \times M, V^2)\) and where the vertical maps are induced by inclusions:

\[
\begin{array}{cccc}
H_2(M) & \rightarrow & H_2(M, L^2) & \rightarrow \ H_1(L^2) & \rightarrow \ H_1(M) \\
\cong & & \cong & & \cong \\
H_2(C \times M) & \rightarrow & H_2(C \times M, V^2) & \rightarrow \ H_1(V^2) & \rightarrow \ H_1(C \times M)
\end{array}
\] (21)

Using the 4-lemma, it follows that \( H_2(M, L^2) \rightarrow H_2(C \times M, V^2) \) is surjective. \( \square \)

**Proposition 4.4.** Assume that \( 0 \leq k \leq n - 2 \). The Maslov index of \( \sigma \in H_2(T^*\mathbb{R}^n, \Lambda^2) \) is given by \( \mu(\sigma) = 1 - k \) resp. \( \mu(\sigma) = n - k - 1 \), where the two cases correspond to the two different ways of resolving the double point of \( \Lambda' \) by Lagrangian 0-surgery.

**Proof.** Denote by \( \mathbb{R}_i \) the \( x_i \)-coordinate subspace of \( \mathbb{R}^n \) and by \( T^*\mathbb{R}_i \) the \( (x_i, y_i) \)-coordinate subspace of \( T^*\mathbb{R}^n \), for \( i = 1, \ldots, n \). To compute \( \mu(\sigma) \), we will represent \( \sigma \) by a disc that lies in \( T^*\mathbb{R}_n \) and compute how the tangent spaces to \( \Lambda^2 \) twist as we traverse its boundary.

The formula [2] shows that the differential of \( F_1 = F(1, \cdot) \) at points of the form \( x = (0, \ldots, 0, x_n) \in \mathbb{R}_n \) is given by \( dF_1(x) = -3(\varepsilon - x_n^2)^{1/2} \sum_{i=k+2}^n x_i dx_i \), and hence

\[ \Lambda' \cap T^*\mathbb{R}_n = \{ (x_n, \mp 3(\varepsilon - x_n^2)^{1/2}, x_n) \in T^*\mathbb{R}_n \mid x_n^2 \leq \varepsilon \}, \]

as depicted in Figure [3] where the blue segment corresponds to \( +dF_1 \) and the red segment to \( -dF_1 \). Differentiating (2) shows that the tangent space to \( \Lambda'_\pm = \text{graph}(\pm dF_1) \) over \( x = (0, \ldots, 0, x_n) \in \mathbb{R}_n \) is spanned by

\[
\begin{align*}
\partial_{x_i} \pm 3f_1(x)^{1/2} \partial_{y_i}, & \quad i = 1, \ldots, k + 1 \\
\partial_{x_i} \pm 3f_1(x)^{1/2} \partial_{y_i}, & \quad i = k + 2, \ldots, n - 1 \\
\partial_{x_n} + 3\left( f_1(x)^{1/2} - f_1(x)^{-1/2} x_n^2 \right) \partial_{y_n}, & \quad i = n,
\end{align*}
\]

where \( f_1 = f(1, \cdot) \). Note that the last vector is proportional to \( f_1(x)^{1/2} \partial_{x_n} \mp 3 \left( f_1(x) - x_n^2 \right) \partial_{y_n} \), so it approaches a multiple of \( \partial_{y_n} \) as \( x_n \rightarrow \varepsilon \) (which is the conormal direction to \( \{ f_1(x) = 0 \} \subset \mathbb{R}_n \) at \( x = (0, \ldots, 0, \pm \varepsilon^{1/2}) \)). Moreover, [22] shows that all these tangent spaces split as direct sums of 1-dimensional subspaces of the \( T^*\mathbb{R}_i \); in particular, the tangent spaces to \( \Lambda'_\pm \) at the origin are of the form \( \lambda_\pm = \lambda_1^\pm \times \cdots \times \lambda_n^\pm \) where the \( \lambda_i^\pm \) are 1-dimensional subspaces of \( T^*\mathbb{R}_i \).

To see how the 0-surgery by which we pass from \( \Lambda' \) to \( \Lambda^2 \) affects the picture, recall from Section [5.2] that to obtain \( \Lambda^2 \) we use symplectomorphisms \( \Phi : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n \) which near the origins identify \( \mathbb{R}^n \) with \( \Lambda'_\pm \) and \( i\mathbb{R}^n \) with \( \Lambda'_\mp \) (the \( \pm \)-cases correspond to the two different ways of performing the surgery). After perturbing \( \Lambda' \) a bit such that it agrees with \( \lambda_+ \cup \lambda_- \) near the origin, we may assume that \( \Phi \) is linear and of the form \( \phi_1^\pm \times \cdots \times \phi_n^\pm \) with linear
maps $\phi_{\pm} : T^*\mathbb{R} \to T^*\mathbb{R}_i$ taking $\mathbb{R} \times \{0\}$ to $\lambda^i_{\pm}$ and $\{0\} \times \mathbb{R}$ to $\lambda^i_{\mp}$. Using such an identification, we glue in a Lagrangian copy of $D^1 \times S^{n-1}$ which is given by a portion of the image of the map $h_\gamma : \mathbb{R} \times S^{n-1} \to T^*\mathbb{R}^n$ defined in (19). Figure 7, in which the left picture corresponds to $\Phi^-$ and the right one to $\Phi^+$, shows what $\Lambda^I \cap T^*\mathbb{R}_n$ looks like.

**Computation of $\mu(\sigma)$ for $\Phi^-$.** Let $\ell_-$ be the loop on $\Lambda^I \cap T^*\mathbb{R}_n$ as shown in Figure 7 traversed in counterclockwise direction, which is a representative of $\tau \in H_1(\Lambda^3)$. The black segment of $\ell_-$ is the image of $\Phi_- \circ h : I \times S^{n-1} \to T^*\mathbb{R}^n$, where $I \subset \mathbb{R}$ is a small interval containing 0. Differentiation of (19) shows that the tangent space $\Lambda^3$ at $\Phi_- \circ h(t,(0,\ldots,0,-1)) = \Phi_-(0,\ldots,0,-a(t),0,\ldots,0,-b(t))$ is spanned by

$$\Phi_- \circ Dh(\partial_{x_j}) = \phi^0_n(a(t)e_j + ib(t)e_j), \quad j = 1,\ldots,n-1,$$

$$\Phi_- \circ Dh(\partial_t) = \phi^0_n(-\dot{a}(t)e_n - i\dot{b}(t)e_n).$$

(23)

where the $e_j$ denote the standard basis vectors of $\mathbb{R}^n \times \{0\} \subset T^*\mathbb{R}^n$.

The formulas in (22) and (23) show that the path in the Lagrangian Grassmannian $Gr_L(T^*\mathbb{R}^n)$ induced by $\ell_-$ splits as a product of paths in $Gr_L(T^*\mathbb{R}_1) \times \cdots \times Gr_L(T^*\mathbb{R}_n) \subset Gr_L(T^*\mathbb{R}^n)$. The arrows in Figure 8 indicate how these 1-dimensional subspaces turn as we traverse the corresponding pieces of $\ell_-$; we read off from these pictures that the Maslov index of $\sigma$ is given by $\mu(\sigma) = 1 - k$.

**Computation of $\mu(\sigma)$ for $\Phi^+$.** The surgery using $\Phi^+$ creates a loop $\ell_+$ on $\Lambda^3 \cap T^*\mathbb{R}_n$ as depicted on the right hand side of Figure 7. Again, the associated path in the Lagrangian Grassmannian lies in $Gr_L(T^*\mathbb{R}_1) \times \cdots \times Gr_L(T^*\mathbb{R}_n) \subset Gr_L(T^*\mathbb{R}^n)$. Figure 9 indicates what the $n$ components look like, and one can read off from it that the Maslov index of the loop
shown is $2(n-k-1)$. Since this loop represents twice the generator $\tau \in H_1(\Lambda^3)$, we have $\mu(\sigma) = n - k - 1$ in this case.

\[\mu(\sigma) = n - k - 1\]

\[\mu(\sigma) = n - k - 1\]

**Figure 8.** Computation of $\mu(\sigma)$ for the resolution using $\Phi_-$.  

\[\mu(\sigma) = n - k - 1\]

\[\mu(\sigma) = n - k - 1\]

**Figure 9.** Computation of $\mu(\sigma)$ for the resolution using $\Phi_+$.  

\[\mu(\sigma) = n - k - 1\]
5. Examples

5.1. The case $k = n - 1$. Assume that a Lagrangian $L$ possesses a Lagrangian surgery disc $D$ and let $L'$ be the result of $(n - 1)$-antisurgery on $L$ along $D$. As discussed in Section 3.1 this is equivalent to saying that $L$ is the result of resolving a double point of $L'$ by Lagrangian 0-surgery. By the discussion in Section 4.1 there exists a cobordism $V^2 : L^3 \sim L$ between $L$ and any other resolution $L^3$ of the same double point of $L'$ by Lagrangian 0-surgery. This proves most of the following statement:

**Theorem 5.1.** Any two Lagrangians $L$, $L'$ obtained by resolving a double point of an immersed Lagrangian $L'$ are Lagrangian cobordant by a cobordism $V^2 : L^3 \sim L$ which (as a smooth manifold) is obtained from the cylinder $[0,1] \times L$ by successively attaching an $n$-handle and a 1-handle. If $L'$ has precisely one double point, then $V^2$ is embedded. Moreover, if $H_1(M) = 0$ and $L$ and $L'$ are both monotone with the same monotonicity constant, then $V^2$ is monotone.

**Proof.** The only statement that does not follow directly from the discussion before is the one about monotonicity. It is easy to see from the description of the cobordism $V^2$ that the map $H_1(L) \oplus H_1(L^3) \to H_1(V^2)$ induced by the inclusion of the ends is surjective; in fact, there exist a generator $\gamma \in H_1(L^3)$ such that the restriction of this map to $H_1(L) \oplus H_1(L^3) \to H_1(V^2)$ is surjective. Consider now the following commutative diagram, where the horizontal arrows come from the exact sequences of the various pairs and the vertical ones are induced by inclusions:

$$
\begin{array}{ccc}
H_2(M) \oplus H_2(M, L) & \to & H_1(L) \oplus H_1(L^3) \\
\downarrow & & \downarrow \\
H_2(T^*\mathbb{R} \times M) & \to & H_1(T^*\mathbb{R} \times M, V^2) \\
\downarrow & & \downarrow \\
H_1(T^*\mathbb{R} \times M) & \to & H_1(V^2) \\
\end{array}
$$

Since the first and third vertical maps are surjective and the fourth is an isomorphism as $H_1(M) = 0$ by assumption, it follows from the 4-lemma that the map $H_2(M, L) \oplus H_2(M, L^3) \to H_2(T^*\mathbb{R} \times M, V^2)$ is also surjective. Hence if $L$ and $L'$ are both monotone with the same monotonicity constant, $V^2$ is also monotone. \qed

5.1.1. Clifford and Chekanov tori. Consider the Whitney sphere $S^2_{Wh}$ in $\mathbb{C}^2$. Resolving its double point by any Lagrangian 0-surgery produces a Lagrangian torus which is automatically monotone since the boundary of the Lagrangian disc created when performing this surgery generates one summand of $H_1$ of the torus. In fact, the two topologically different types of resolving the double point yield the Clifford torus $T^2_{Cl}$ in one case and the Chekanov torus $T^2_{Ch}$ in the other case.

This is easy to see if one recalls that each of the three Lagrangians can be obtained by rotating certain curves $\gamma : S^1 \to \mathbb{C}$, i.e. as

$$L_\gamma = \{ (e^{is})e^{it}, (e^{is})e^{-it} \in \mathbb{C}^2 \mid s, t \in [0, 2\pi] \}.$$ 

To obtain $S^2_{Wh}$ like that, use a figure 8 curve $\gamma_{Wh}$ with double point at the origin and symmetric with respect to the origin (to be precise, this yields the image of the standard Whitney immersion $S^2 \to T^*\mathbb{R}^2 \cong \mathbb{C}^2$ given in [13] under a linear symplectomorphism).
Resolving its double point by 0-surgery has the same result as resolving the double point of the figure 8 curve and then rotating the resulting curve, which can be assumed to be still symmetric with respect to the origin. The two different ways of performing this surgery yield a connected curve $\gamma_{Cl}$ encircling the origin in one case, and a disconnected curve $\gamma_{Ch}$ whose components do not encircle the origin in the other case, see Figure 10. The corresponding Lagrangian tori are $T^2_{Cl}$ resp. $T^2_{Ch}$ up to Hamiltonian isotopy.

![Figure 10. The curves used for constructing $S^2_{Wh}$, $T^2_{Cl}$ and $T^2_{Ch}$.](image)

To make the discussion a bit more quantitative, we denote by $T^2_{Cl}(A)$ and $T^2_{Ch}(A)$ the Clifford and Chekanov tori for which a Maslov 2 disc has area $A > 0$, i.e. for which the monotonicity constant is $\frac{1}{2}A$; moreover, we denote by $S^2_{Wh}(A)$ the Whitney sphere for which a generator of $H_2(\mathbb{C}^2, S^2_{Wh}) \cong \mathbb{Z}$ has area $A$. In all cases, $A$ is half of the area bounded by the respective curves in Figure 10. One can hence infer from this figure that a necessary condition for being able to obtain $T^2_{Cl}(A')$ and $T^2_{Ch}(A'')$ from 0-surgery on $S^2_{Wh}(A)$ is that $A'' < A < A'$. It is easy to see that this is also a sufficient condition (i.e. that one can get arbitrarily small Chekanov tori and arbitrarily large Clifford tori from a given Whitney sphere), and hence we obtain the following statement as a corollary of Theorem 5.1.

**Corollary 5.2.** For every $A'' < A'$, there exist (non-monotone) Lagrangian cobordisms $T^2_{Ch}(A'') \sim T^2_{Cl}(A')$, $T^2_{Cl}(A'') \sim T^2_{Cl}(A')$ and $T^2_{Ch}(A'') \sim T^2_{Ch}(A')$, which as smooth manifolds are obtained from $[0,1] \times T^2$ by successively attaching a 2-handle and a 1-handle.

The fact that the monotonicity constant for the two ends of any cobordism $T^2_{Ch}(A') \sim T^2_{Cl}(A)$ constructed by our method must differ can also be seen as follows: If they were the same, the cobordism would also be monotone by Theorem 5.1 and hence preserve e.g. the count of pseudoholomorphic of Maslov index 2 as first observed in [Che97] (see also [BC13, BC14]); however, it is well known that these counts are different for $T^2_{Cl}$ and $T^2_{Ch}$.

A similar argument shows that one cannot build a Lagrangian cobordism between the *monotone* Clifford and Chekanov tori in $\mathbb{C}P^2$ or $S^2 \times S^2$ by this method, since the monotonicity constant of any monotone Lagrangian there is determined by that of the ambient manifold (in particular, it’s the same for Clifford and Chekanov).

5.2. **The case** $k = 0$. Lagrangian 0-antisurgery can be performed on any Lagrangian $L$ in any symplectic manifold $M$, since isotropic surgery discs of dimension one are always for free: They are simply embedded paths $\gamma : D^1 \to M$ hitting $L$ cleanly at their ends and nowhere else. We give two simple applications in the lowest dimensions $n = 1, 2$ (for $n = 1$, this case coincides of course with the previously discussed case $k = n - 1$).
5.2.1. Curves. Consider a simple closed curve $\alpha$ on a surface $\Sigma$ and a surgery disc $\gamma : D^1 \to \Sigma$ for $\alpha$. Let $\beta$ be a simple closed curve which is obtained from $\gamma$ by connecting $\gamma(0)$ and $\gamma(1)$ by a segment on $\alpha$. The geometric intersection number $i(\alpha, \beta)$ can either be 0 or 1, depending on whether $\gamma$ approaches $\alpha$ from the same or two different sides at its ends. If $i(\alpha, \beta) = 0$, the result of first antisurgering $\alpha$ along $\gamma$ and then resolving the double point is a curve $\tilde{\alpha}$ isotopic but generally not Hamiltonian isotopic to $\alpha$ (for one of the two possible resolutions; the other one leads to a non-connected result). If $i(\alpha, \beta) = 1$, the result is the squared Dehn twist of $\alpha$ about a curve $\tilde{\beta}$ isotopic to $\beta$. The corresponding Lagrangian cobordisms

$$\tilde{\alpha} \simeq \alpha \quad \text{resp.} \quad \tau^2_\beta(\alpha) \simeq \alpha$$

are non-orientable in both cases.

5.2.2. Surfaces. Lagrangian 0-antisurgery applied to a Lagrangian surface $L$ in a symplectic 4-manifold $M$ yields a Lagrangian immersion of a surface $L'$ with $\chi(L') = \chi(L) - 2$. For example, if we start with a torus in $T^*\mathbb{R}^2$, iterating this procedure $g-1$ times produces of a genus $g$ surface with exactly $g-1$ double points. By resolving these double points, we get Lagrangian embeddings into $T^*\mathbb{R}^2$ of all non-orientable surfaces whose Euler characteristic is divisible by 4, and corresponding non-orientable cobordisms. The existence of such embeddings and immersions was of course known before, see e.g. [ALP94].

5.3. Middle-dimensional cases. The question of whether the Lagrangian $L^2$ obtained from a monotone Lagrangian $L$ by $k$-antisurgery and subsequent desingularization is still monotone is easiest when $k$ is in the range $2 \leq k \leq n-3$, in which case it reduces to controlling the symplectic area $\omega(\sigma)$ of one new generator $\sigma \in H_2(M, L^2)$ by Proposition 1, 3 (cf. the discussion in Section 4.6). Below we will discuss a family of monotone Lagrangians $L$ in $\mathbb{C}P^n$ bounding Lagrangian surgery discs with Weinstein neighbourhoods that are large enough for us to be able to adjust $\omega(\sigma)$ such that the resulting $L^2$ and $V^2 : L^2 \simeq L$ are monotone as well.

5.3.1. Monotone examples in $\mathbb{C}P^n$. We will view $\mathbb{C}P^n$ as a compactification of $D^*\mathbb{R}P^n$, the unit cotangent bundle of $\mathbb{R}P^n$ with respect to the round metric, which is obtained by collapsing the cogeodesic orbits on the unit cosphere bundle to points (i.e. by a symplectic cut in the sense of [Ler95]). For $r \in (0, 1)$, let $S_r \subset D^*\mathbb{R}P^n$ be the sphere of radius $r$ in the codisc fibre over $p = [0 : \cdots : 0 : 1] \in \mathbb{R}P^n$. Flowing $S_r$ around by the cogeodesic flow yields a Lagrangian

$$L_r \cong S^1 \times S^{n-1}$$

which as a Lagrangian submanifold of $\mathbb{C}P^n$ is monotone if and only if $r = \frac{n-1}{n+1}$ (see the proof of Theorem 5.3). By construction, the sphere $S_r \subset L_r$ bounds a Lagrangian disc $D_r$. If we perform $k$-antisurgery along some $(k+1)$-dimensional subdisces of $D_r$ and then remove the double point thus created, we obtain an embedded Lagrangian $L^2_r \subset \mathbb{C}P^n$ which is diffeomorphic to

$$(S^{k+1} \times S^{n-k-1}) \# 2P^n \quad \text{resp.} \quad (S^{k+1} \times S^{n-k-1}) \# P^n \# Q^n$$

(depending on $n$ and $k$, see Section 4.5), and an embedded Lagrangian cobordism $V^2 : L^2_r \simeq L_r$. 

Theorem 5.3. For all $n$ and $k$ such that $2 \leq k \leq n - 3$, we can perform this construction in such a way that the Lagrangian $L^k_r \subset \mathbb{C}P^n$ and the Lagrangian cobordism $V^2 : L^k_r \leadsto L_r$ obtained from the monotone $L_r$ are monotone as well.

Proof. The restricted codisc bundle $D^*(\mathbb{R}P^n \setminus \mathbb{R}P^{n-1})$ lying over the complement of a hyperplane $\mathbb{R}P^{n-1} \subset \mathbb{R}P^n$ is symplectomorphic to $D^n(1) \times D^n(\frac{\pi}{2}) \subset T^*\mathbb{R}^n$ by a symplectomorphism

$$\phi : D^*(\mathbb{R}P^n \setminus \mathbb{R}P^{n-1}) \to D^n(1) \times D^n(\frac{\pi}{2})$$

which identifies the fibre $D^*_p\mathbb{R}P^n$ with $D^n(1) \times \{0\}$. Explicitly, if we identify $\mathbb{R}P^{n-1} = \{[x_1 : \cdots : x_n : 0] \in \mathbb{R}P^n\}$, such a symplectomorphism is induced by the diffeomorphism $\mathbb{R}P^n \setminus \mathbb{R}P^{n-1} \to D^n(\frac{\pi}{2})$ given by $[x_1 : \cdots : x_{n+1}] \mapsto \arcsin(\sqrt{x_1^2 + \cdots + x_{n+1}^2})(x_1, \ldots, x_n)$, where $[x_1 : \cdots : x_{n+1}]$ is the unique representative of a point in $\mathbb{R}P^n \setminus \mathbb{R}P^{n-1}$ for which $x_1^2 + \cdots + x_{n+1}^2 = 1$ and $x_{n+1} > 0$.

In this identification, we have

$$L_r \cap D^*(\mathbb{R}P^n \setminus \mathbb{R}P^{n-1}) = (N^*S^{n-1}(r))_{<\frac{\pi}{2}}$$

(24)

where $(N^*S^{n-1}(r))_{<\frac{\pi}{2}} \subset D^n(1) \times D^n(\frac{\pi}{2})$ is the part of the conormal bundle of the radius $r$ sphere $S^{n-1}(r) \subset \mathbb{R}^n$ consisting of all conormal vectors of length less than $\frac{\pi}{2}$; see Figure 11 for a schematic picture. It is easy to determine from this picture the value of $r$ for which $L_r$ is monotone: A generator of $H_2(\mathbb{C}P^n)$ has first Chern number $n + 1$ and is represented in Figure 11 by a rectangle going from top to bottom (in the sense that such a rectangle becomes a sphere generating $H_2(\mathbb{C}P^n)$ when we compactify), so its symplectic area is $2\pi$; the monotonicity constant of the resulting $\mathbb{C}P^n$ is hence $\eta_{\mathbb{C}P^n} = \frac{2\pi}{n+1}$. The Lagrangian $L_r$ bounds a family of discs of Maslov index 2 and area $(1 - r)\pi$, represented by the gray strips in Figure 11. Combining these facts, it follows that $L_r$ is monotone if and only if $r = \frac{n+1}{2\pi}$, in which case its monotonicity constant is $\eta_{L_r} = \frac{1}{2}\eta_{\mathbb{C}P^n} = \frac{\pi}{n+1}$.

Let us now perform $k$-antisurgery on the monotone $L_r$ for some $2 \leq k \leq n - 3$ along a $(k + 1)$-dimensional subdisc of the Lagrangian disc $D_r$, and then remove the new double point of $L'_r$ by 0-surgery. As discussed before, Proposition 4.3 implies that the resulting $L^k_r$ and $V^2 : L^k_r \leadsto L_r$ are also monotone under the assumption $2 \leq k \leq n - 3$, provided that the

![Figure 11. $D^*(\mathbb{R}P^n \setminus \mathbb{R}P^{n-1})$](image-url)
generator $\sigma \in H_2(\mathbb{C}P^n, L_r')$ satisfies $\omega(\sigma) = \eta_L, \mu(\sigma)$. For one of the ways of performing the 0-surgery we obtain $\mu(\sigma) = n - k - 1$ by Proposition 4.4 and thus we need 

$$\omega(\sigma) = \frac{n - k - 1}{n + 1} \pi$$

(25) to guarantee monotonicity. This is the case in which we perform the 0-surgery using a symplectomorphism $\Phi_+: T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ which identifies $\mathbb{R}^n \times \{0\}$ with $\lambda_+ = T_{(0,0)}\Lambda_+'$ and $\{0\} \times \mathbb{R}^n$ with $\lambda_- = T_{(0,0)}\Lambda_-'$, as in the proof of Proposition 4.4. (The other possibility $\Phi_-$ leads to a negative Maslov index, so there is no chance of getting something monotone.)

Consider the intersection of $L_r'$ with $(D^n(1) \times D^n(\frac{\pi}{2})) \cap T^*\mathbb{R}^n$ as shown in the left part of Figure 12, where $T^*\mathbb{R}^n$ is the $(x_n, y_n)$-coordinate subspace of $T^*\mathbb{R}^n$. We claim that, after potentially decreasing the parameter $\varepsilon$ appearing in the model for the antisurgery, we can perform the 0-surgery in such a way that the intersection of the resulting $L_r^\circ$ with $(D^n(r) \times D^n(\frac{\pi}{2})) \cap T^*\mathbb{R}^n$ bounds any given area in $(0, 2r\pi)$, by adjusting the curve with respect to which the 0-surgery is performed such that it bounds the right amount of area (see the right part of Figure 12). Since this area equals $2\omega(\sigma)$, it follows that we can arrange $\omega(\sigma)$ to take any given value in $(0, r\pi)$; in particular, we can satisfy (25) for any $2 \leq k \leq n - 3$ since $r = \frac{n - 1}{n + 1}$.

In the rest of the proof we will justify the claim made above. Let $U \subset L_r'$ be a neighbourhood of the double point in $L_r$ given by the union of two balls in $\lambda_\pm$ (we assume we perturbed $L_r'$ slightly such that near the double point it agrees with $\lambda_+ \cup \lambda_-$, as in the proof of Proposition 4.4). Then consider a curve $\gamma_+$ in the upper half of $(D^n(r) \times D^n(\frac{\pi}{2})) \cap T^*\mathbb{R}^n$ which connects two of the free ends of $(L_r' \setminus U) \cap (D^n(r) \times D^n(\frac{\pi}{2})) \cap T^*\mathbb{R}^n$, such as the upper red curve in the right part of Figure 12 and encloses the required area.

Consider now the $SO(n)$ action obtained by pushing forward by $\Phi_+$ the standard $SO(n)$ action on $T^*\mathbb{R}^n$ given by $A(x, y) = (Ax, Ay)$ for $A \in SO(n)$. Observe that the orbit $SO(n)\gamma_+$ of $\gamma_+$ under this action is the image under $\Phi_+$ of a local model for 0-surgery as described in Section 5.2. The reason is that we can consider the model Lagrangian constructed there with

![Figure 12. $L_r' \cap T^*\mathbb{R}^n$ vs. $L_r^\circ \cap T^*\mathbb{R}^n$](image-url)
respective to a curve $\gamma \subset T^*\mathbb{R}$ (i.e. the image of the map $h_\gamma : \mathbb{R} \times S^{n-1} \to T^*\mathbb{R}^n$) as the orbit of $\gamma$, viewed now as living in $T^*\mathbb{R}_n \subset T^*\mathbb{R}^n$, under the standard $SO(n)$ action on $T^*\mathbb{R}^n$.

To see that we can actually use the orbit $SO(n)\gamma_+$ to perform 0-surgery on $L'_r$, it remains to show that it is contained in $D_n'(r) \times D_n'(\frac{\pi}{2})$, and that it does not intersect $L'_r$ away from the attaching region near the double point (so that we actually obtain an embedded Lagrangian $L'_r^\natural$; the point here is that in contrast to the general construction in Section 4.1, we are not allowing ourselves to perform the modification in an arbitrarily neighbourhood of the singular locus, so an additional argument is needed).

A simple computation shows that $A \in SO(n)$ acts by $(x, y) \mapsto (Ax, A'y)$, where the matrix $A'$ also lies in $SO(n)$ (it is obtained from $A$ by multiplying the last $n - k - 1$ columns of $A$ by $-1$, and then the last $n - k - 1$ rows by $-1$). The first required statement follows immediately from that. As for the second, note that points $(x, y)$ in the relevant part of $A'$ satisfy $\|y\| < c(\varepsilon)\|x\|$, where $c(\varepsilon)$ is a constant for fixed $\varepsilon$, and $c(\varepsilon) \to 0$ as $\varepsilon \to 0$; this can be deduced from the expression (2). Hence if we adjust the curve $\gamma_+$ such that outside of a small neighbourhood of the origin all points $(x_n, y_n) \in \gamma_+$ satisfy $|y_n| > c(\varepsilon)|x_n|$, the fact that $A, A' \in SO(n)$ implies that $SO(n)\gamma_+$ cannot intersect $L'_r$ except where it should. Note that since we can let $\varepsilon$ tend to $0$, this requirement on $\gamma_+$ does not restrict the area we can enclose.

An almost identical argument shows that the Lagrangian 1-handle corresponding to the 0-surgery with respect to the curve $\gamma_+$ can be glued in such a way that the resulting cobordism $V^2: L^\natural \rightarrow L_r$ is embedded. \hfill \qed

Remark 5.4. The Lagrangian $L_r \cong S^1 \times S^{n-1}$ can also be viewed as the Lagrangian circle bundle over the trace of the polarization of $CP^n$ given by the quadric $\Sigma = \{z_1^2 + \cdots + z_{n+1}^2 = 0\}$, in the sense of [Bir01] [Bir06]. This construction also yields examples of Lagrangians with Lagrangian surgery discs in other polarized Kähler manifolds.

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