Cosmic constraint on unified model of dark sectors in flat and non-flat varying gravitational constant theory

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Observations indicate that most universal matter are invisible and gravitational constant $G(t)$ maybe depends on the time. The theory of variation of $G$ (VG) is explored in this paper, with naturally resulting to the invisible components in universe. We utilize the observational data: lookback time data, model-independent gamma ray bursts data, growth function of matter linear perturbations, type Ia supernovae data with systematic errors, cosmic microwave background, and baryon acoustic oscillation data from the radial scale measurement and the peak-positions measurement, to restrict the unified model (UM) of dark components in VG theory. Using the best-fit values of parameters with the covariance matrix, constraints on the variation of $G$ are $(\frac{G}{G_0})_{z=3.5} \approx 3.5 \pm 0.0014$ and $(\frac{G}{G_0})_{\text{today}} \approx 0.7977^{+2.3566}_{-2.3566} \times 10^{-13} \text{yr}^{-1}$ in a flat geometry, the small uncertainties around constants. Limit on equation of state of dark matter is $w_{0dm} = 0.0151^{+0.0171}_{-0.0171}$ with assuming $w_{0de} = -1$ in the UM model, and dark energy is $w_{0de} = -0.9986^{+0.0011}_{-0.0011}$ with assuming $w_{0dm} = 0$ at prior. Restriction on UM parameters are $B_s = 0.7662^{+0.0127+0.0224}_{-0.0125-0.0265}$ and $\alpha = 0.0204^{+0.0201+0.0425}_{-0.0217-0.0398}$ with 1σ and 2σ confidence level. For the non-flat case, at 1σ confidence level the $\Lambda$CDM ($\Omega_k = 0$, $\beta = 0$ and $\alpha = 0$) is not included in VG-UM model, and larger errors are given: $\Omega_k = -0.0311^{+0.0259+0.0517}_{-0.0248-0.0501}$, $(\frac{G}{G_0})_{z=3.5} \approx 0.9917^{+0.0104}_{-0.0131}$ and $(\frac{G}{G_0})_{\text{today}} \approx 19.3678^{+21.8262}_{-21.8262} \times 10^{-13} \text{yr}^{-1}$.

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I. Introduction

Gravity theories are usually studied with an assumption that Newton gravity constant $G$ is a constant. But some observations hint that $G$ maybe depends on the time, such as observations from white dwarf star, pulsar, supernovae, and neutron star. In addition, cosmic observations predict that about 95% of the universal matter is invisible, including dark matter (DM) and dark energy (DE). The unified models of two unknown dark sectors (DM and DE) have been studied in several theories, e.g. in the standard cosmology, in the Hořava-Lifshitz gravity, in the RS and the KK higher-dimension gravity. In this paper, we study the unified model of dark components in theory of varying gravitational constant (VG). The attractive point in this model is that the variation of $G$ could result to the invisible components in universe, by relating the Lagrangian quantity from the generalized Born-Infeld theory to the VG theory. One source of DM and DE is explored. Using the Markov Chain Monte Carlo

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(MCMC) method \cite{12}, the cosmic constraints on unified model of DM and DE are performed in the framework of time-varying gravitational constant, with flat and non-flat geometry. The used cosmic data include the lookback time (LT) data \cite{13,14}, the model-independent gamma ray bursts (GRBs) data \cite{15}, the growth function (GF) of matter linear perturbations \cite{16–23}, the type Ia supernovae (SNIa) data with systematic errors \cite{24}, the cosmic microwave background (CMB) \cite{25}, and the baryon acoustic oscillation (BAO) data including the radial BAO scale measurement \cite{26} and the peak-positions measurement \cite{27–29}.

II. A time-varying gravitational constant theory with unified dark sectors in a non-flat geometry

We adopt the Lagrangian quantity of system

\[ L = \sqrt{g} \left( R \frac{G(t)}{G(t)} + 16\pi L_u \right) \]  

with a parameterized time-varying gravitational constant \( G = G_0 a(t)^{-\beta} \). \( t \) is the cosmic time, \( a = (1 + z)^{-1} \) is the cosmic scale factor, and \( z \) denotes the cosmic redshift. \( g \) is the determinant of metric, \( R \) is the Ricci scalar, and \( L_u = L_b + L_r + L_d \) corresponds to the Lagrangian density of universal matter including the visible ingredients: baryon \( L_b \) and radiation \( L_r \) and the invisible dark ingredients: \( L_d \). Utilizing the variational principle, the gravitational field equation can be derived \cite{30},

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} + G (\nabla_\mu \partial_\nu G^{-1} - g_{\mu\nu} \nabla_\sigma \partial^\sigma G^{-1}) \]  

in which \( R_{\mu\nu} \) is the Ricci tensor, \( T_{\mu\nu} \) is the energy-momentum tensor of universal matter that comprise the pressureless baryon (\( w_b = \frac{p_b}{\rho_b} = 0 \)), positive-pressure photon (\( w_r = \frac{p_r}{\rho_r} = \frac{1}{3} \)) and the unknown dark components (\( w_d = \frac{p_d}{\rho_d} \)). \( w \) is the equation of state (EoS), \( p \) is the pressure and \( \rho \) denotes the energy density, respectively.

In a non-flat geometry, one gets the evolutional equation of universe in VG theory

\[ H^2 + \frac{k}{a^2} = \frac{8\pi G_0}{3} a^{-\beta} \rho - \beta H^2 = \frac{8\pi G_0}{3} \rho_{\text{eff}}, \]  

\[ \frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2} = -8\pi G_0 a^{-\beta} p - \beta H^2 - \beta^2 H^2 - \beta^2 \frac{\ddot{a}}{a} = -8\pi G_0 p_{\text{eff}}. \]  

The effective energy conservation equation

\[ \dot{\rho}_{\text{eff}} + 3H (\rho_{\text{eff}} + p_{\text{eff}}) = 0 \]  

results to

\[ \dot{\rho} + 3H (\rho + \frac{2 + 2\beta}{2 + \beta} p) = \frac{\beta - \beta^2}{2 + \beta} H \rho. \]  

Integrating Eq. (6) can gain the energy density of baryon \( \rho_b \propto a^{-\frac{2+2\beta-\beta^2}{2+\beta}} \), the energy density of radiation \( \rho_r \propto a^{-\frac{2+2\beta-4\beta^2}{2+\beta}} \) and the energy density of curvature \( \rho_k \propto a^{-\frac{2+2\beta-8\beta^2}{2+\beta}} \). Relative to the constant-\( G \) theory, the evolutional equations of energy densities are obviously modified in VG theory, for the existence of VG parameter \( \beta \).

We concentrate on the Lagrangian density of dark components with the form \( L_d = -A \sqrt{|1 - (V(\varphi))^{\frac{1+\beta}{2}}|} \) from the generalized Born-Infeld theory \cite{31}, in which \( V(\varphi) \) is the potential. Relating this scalar field \( \varphi \) with the
time-varying gravitational constant by $\varphi(t) = G(t)^{-1}$, it is then found that the dark ingredients can be induced by the variation of $G$. The energy density of dark fluid in VG frame complies with

$$\rho_d = \rho_{0d}[B_s + (1 - B_s)a^{(-3 + \frac{6 \beta^2 - 2\beta - 6}{2\beta})}(1 + \alpha)]^{-1},$$  

(7)

here parameter $\beta$ reflects the variation of $G$, $\alpha$ and $B_s = \frac{6 + 6\beta}{\beta^2 + 2\beta + 6}$ are model parameters. Eq. (7) shows that the behavior of $\rho_d$ is like cold DM at early time$^1$ (for $a \ll 1$, $\rho_d \approx \rho_{0d}(1 - B_s)\frac{1}{H^2}a^{-3 + \frac{6 \beta^2 - 2\beta - 6}{2\beta}}$), and like cosmological-constant type DE at late time (for $a \gg 1$, $\rho_d \approx \rho_{0d}B_s^{\frac{1}{1 + \alpha}}$). Then Eq. (7) introduces a unified model (UM) of dark sectors in VG theory (called VG-UM). The Hubble parameter $H$ in a non-flat VG-UM model reads

$$H = \sqrt{\frac{H_0^2}{1 + \beta}} \{\Omega_0d[B_s a^{-\beta(1+\alpha)} + (1 - B_s)a^{(-3 + \frac{6 \beta^2 - 2\beta - 6}{2\beta})}(1 + \alpha)]\frac{1}{1 + \alpha} + \Omega_b a^{\frac{-2\beta^2 - 2\beta - 6}{2\beta}} + \Omega_r a^{\frac{-2\beta^2 - 2\beta - 6}{2\beta}} + \Omega_k a^{\frac{-2\beta^2 - 2\beta - 6}{2\beta}}\},$$  

(8)

with Hubble constant $H_0$ and dimensionless energy densities $\Omega_b = \frac{8\pi G_0 \rho_{0b}}{3H_0^2}$, $\Omega_r = \frac{8\pi G_0 \rho_{0r}}{3H_0^2}$, $\Omega_b = \frac{k}{H_0^2}$, and $\Omega_0d + \Omega_b + \Omega_r + \Omega_k = 1 + \beta$. Above equations are reduced to the standard forms in the constant-$G$ theory, for $\beta = 0$.

III. Data fitting

A. Lookback Time

Refs. [32, 33] define the LT as the difference between the current age $t_0$ of universe at $z = 0$ and the age $t_z$ of a light ray emitted at $z$,

$$t_L(z) = \int_0^z \frac{dz'}{(1 + z')H(z')}.$$  

(9)

Then the age $t(z_i)$ of an object can be expressed by the difference between the age of universe at $z_i$ and the age at $z_F$ (object was born) [13],

$$t(z_i) = \int_{z_i}^{\infty} \frac{dz'}{(1 + z')H(z')} - \int_{z_F}^{\infty} \frac{dz'}{(1 + z')H(z')} = t_L(z_F) - t_L(z_i).$$  

(10)

For an object born at $z_i$, the observed LT subjects to

$$t_L^{\text{obs}} = t_L(z_F) - t_L(z_i) = [t_0^{\text{obs}} - t(z_i)] - [t_0^{\text{obs}} - t_L(z_F)] = t_0^{\text{obs}} - t(z_i) - df.$$  

(11)

One defines

$$\chi^2_{\text{age}} = \sum_i \frac{[t_L(z_i) - t_L^{\text{obs}}(z_i, df)]^2}{\sigma^2_{T_i}} + \frac{[t_0 - t_0^{\text{obs}}]^2}{\sigma^2_{t_0^{\text{obs}}}},$$  

(12)

with $\sigma^2_{t_0^{\text{obs}}} + \sigma^2_{T_i} = \sigma^2_{T_i}$. $\sigma_{t_0^{\text{obs}}}$ is the uncertainty of the total universal age, and $\sigma_i$ is the uncertainty of the LT of galaxy $i$. Marginalizing the ‘nuisance’ parameter $df$ results to [34]

$$\chi^2_{LT}(p_i) = -2 \ln \int_0^{\infty} d(df) \exp(-\chi^2_{\text{age}}/2) = A - \frac{B^2}{C} + \frac{[t_0 - t_0^{\text{obs}}]^2}{\sigma^2_{t_0^{\text{obs}}}} - 2 \ln \left[\sqrt{\frac{\pi}{2C}} erf\left(\frac{B}{\sqrt{2C}}\right)\right],$$  

(13)

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$^1$ $\beta$ describes the effect on energy density of dark matter from variation of $G$. 


with \( A = \sum_i \frac{\Delta^2}{\sigma^2_i} \), \( B = \sum_i \frac{\Delta}{\sigma_i} \), \( C = \sum_i \frac{1}{\sigma_i} \) and \( \Delta = t_L(z_i) - [t_0^{obs} - t(z_i)] \). \( p_s \) denotes the theoretical model parameters. 

\( \text{erfc}(x) = 1 - \text{erf}(x) \) is the complementary error function of \( x \). The observed universal age at today \( t_0^{obs} = 13.75 \pm 0.13 \) Gyr \([35]\) is used, and the observed data on the galaxies age are listed in Table I.

| \( z_i \) | 0.10 | 0.25 | 0.60 | 0.70 | 0.80 | 1.27 | 0.1117 | 0.1174 | 0.222 | 0.2311 | 0.3559 | 0.452 | 0.575 | 0.644 | 0.676 | 0.833 | 0.836 | 0.922 | 1.179 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( t_i \) | 10.65 | 8.89 | 4.53 | 3.93 | 3.41 | 1.60 | 10.2 | 10.0 | 9.0 | 7.6 | 7.0 | 6.0 | 6.0 | 5.8 | 5.5 | 4.6 | 3.5 | 4.3 | 3.5 | 3.4 | 3.5 | 3.6 | 3.2 | 3.0 | 3.6 | 3.2 | 2.8 | 3.0 | 2.5 | 3.0 | 2.6 | 2.5 |

**TABLE I:** The 38 data points of galaxy age \([13, 14]\). The first 6 data are from Ref.\([13]\).

**B. Gamma Ray Bursts**

In GRBs observation, the famous Amati’s correlation can be expressed \([36, 37]\), \( \log \frac{E_{iso}}{c_{GRB}} = a + b \log \frac{E_{p,i}}{m_{proton}} \). \( E_{iso} = 4\pi d_L^2 S_{bolo} / (1 + z) \) and \( E_{p,i} = E_{p,obs} (1 + z) \) are the isotropic energy and the cosmological rest-frame spectral peak energy, respectively. \( d_L \) is the luminosity distance and \( S_{bolo} \) is the bolometric fluence of GRBs. Ref. \([38]\) introduced a model-independent quantity of distance measurement,

\[
\tau_p(z_i) = \frac{r_p(z)}{r_p(z_0)}, \quad r_p(z) = \frac{(1 + z)^{1/2} H_0}{c} r(z), \quad r(z) = \frac{d_L(z)}{1 + z}
\]

with \( z_0 \) being the lowest GRBs redshift. For GRBs constraint, \( \chi^2_{GRBs} \) has a form

\[
\chi^2_{GRBs}(p_s) = [\Delta \tau_p(z_i)] \cdot (Cov^{-1}_{GRBs})_{ij} \cdot [\Delta \tau_p(z_i)]
\]

in which \( \Delta \tau_p(z_i) = \tau_p^{data}(z_i) - \tau_p(z_i) \), and \( (Cov^{-1}_{GRBs})_{ij} \) is the covariance matrix. Using 109 GRBs data, Ref. \([15]\) obtained 5 model-independent datapoints listed in table \([II]\) where \( \sigma(\tau_p(z))^+ \) and \( \sigma(\tau_p(z))^- \) are the 1\( \sigma \) errors. The \( \{\tau_p(z_i)\} \) correlation matrix is \([15]\)

\[
(Cov_{GRB}) = \begin{pmatrix}
1.0000 & 0.7780 & 0.8095 & 0.6777 & 0.4661 \\
0.7780 & 1.0000 & 0.7260 & 0.6712 & 0.3880 \\
0.8095 & 0.7260 & 1.0000 & 0.6046 & 0.5032 \\
0.6777 & 0.6712 & 0.6046 & 1.0000 & 0.1557 \\
0.4661 & 0.3880 & 0.5032 & 0.1557 & 1.0000
\end{pmatrix},
\]

with the covariance matrix

\[
(Cov_{GRB})_{ij} = \sigma(\tau_p(z_i)) \sigma(\tau_p(z_j))(Cov_{GRB})_{ij},
\]

where \( \sigma(\tau_p(z_i)) = \sigma(\tau_p(z_i))^+ \), if \( \tau_p(z) \geq \tau_p^{data} \); \( \sigma(\tau_p(z_i)) = \sigma(\tau_p(z_i))^- \), if \( \tau_p(z) < \tau_p^{data} \).

**C. Growth Function of Matter Linear Perturbations**

The \( \chi^2_{GF} \) from growth function of matter linear perturbations \( f \) can be constructed

\[
\chi^2_{GF}(p_s) = \sum_i \frac{[f_{th}(p_s, z_i) - f_{obs}(z_i)]^2}{\sigma^2(z_i)},
\]

(18)
where the used observational values of $f_{\text{obs}}$ are listed in Table II. $f$ is defined via $f(a) = \frac{aD'(a)}{D(a)}$, with $D = \frac{2}{H_0} \frac{\rho}{\rho_0}$. $'$ denotes derivative with respect to $a$. So in theory, $f$ can be gained by numerically solving the following differential equation in VG theory

$$D''(a) + \left[ \frac{3}{a} + \frac{H'(a)}{H(a)} + \frac{\beta}{a} \right] D'(a) + \frac{6 + 2\beta + \beta^2}{2 + \beta} \frac{H_0^2 \Omega_m}{2H(a)a^2} a^{-2-2\beta-\beta^2} = 0,$$

with $D(a) \simeq a$ as the initial condition for $a \approx 0$. Comparing with the most popular ΛCDM model, the effective current matter density can be written, $\Omega_m = \Omega_b + (1 + \beta - \Omega_k - \Omega_\Lambda)(1 - B_\Lambda)$ for VG-UM. Obviously, for $\beta = 0$ it is consistent with the form of $\Omega_m$ in UM of constant $G$ theory [39, 41].

| $z_i$ | 0.15 | 0.22 | 0.32 | 0.35 | 0.41 | 0.55 | 0.60 | 0.77 | 0.78 | 1.4 |
|------|------|------|------|------|------|------|------|------|------|-----|
| $f_{\text{obs}}$ | 0.51 ± 0.11 | 0.60 ± 0.10 | 0.654 ± 0.18 | 0.70 ± 0.18 | 0.50 ± 0.07 | 0.75 ± 0.18 | 0.73 ± 0.07 | 0.91 ± 0.36 | 0.70 ± 0.08 | 0.90 ± 0.24 |
| Ref. | [16, 17] | [18] | [19] | [20] | [18] | [21] | [18] | [22] | [18] | [23] |

**Table III:** The observational data of growth function $f_{\text{obs}}$.

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### D. Type Ia Supernovae

We use the Union2 dataset of SNIa, published in Ref. [24]. In VG theory, the theoretical distance modulus $\mu_{th}(z)$ is written as $\mu_{th}(z) = 5 \log_{10}[D_L(z)] + \frac{15}{4} \log_{10} \frac{\rho_0}{\rho_c} + \mu_0$ with the Hubble-free luminosity distance $D_L(z) = \frac{H_0}{c}(1+z)^2 A(z)$ and $\mu_0 = 5 \log_{10}(\frac{H_0^{-1}}{\text{Mpc}}) + 25 = 42.38 - 5 \log_{10} h$.

$D_A(z) = \frac{c}{(1+z)\sqrt{|\Omega_k|}} \sinh[\sqrt{|\Omega_k|} \int_0^z \frac{dz'}{H(z')}]$ is the proper angular diameter distance, here $\sinh(\sqrt{|\Omega_k|} x)$ respectively denotes $\sin(\sqrt{|\Omega_k|} x)$, $\sqrt{|\Omega_k|} x$ and $\sinh(\sqrt{|\Omega_k|} x)$ for $\Omega_k < 0$, $\Omega_k = 0$ and $\Omega_k > 0$. Cosmic constraint from SNIa observation can be done by a calculation on [42, 49]

$$\chi^2_{\text{SNIa}}(p_s) = \sum_{SN1a} \frac{[\mu_{th}(p_s, z_i) - \mu_{\text{obs}}(z_i)]^2}{\sigma^2_{\mu_i}},$$

where $\mu_{\text{obs}}(z_i) = m_{\text{obs}}(z_i) - M$ is the observed distance moduli, with the absolute magnitude $M$. The nuisance parameter $M' = \mu_0 + M$ can be marginalized over analytically, $\chi^2_{\text{SNIa}}(p_s) = -2 \ln \int_{-\infty}^{+\infty} \exp \left[ -\frac{1}{2} \chi^2_{\text{SNIa}}(p_s, M') \right] dM'$ resulting to [50, 52]

$$\chi^2_{\text{SNIa}}(p_s) = A - \frac{B^2}{C},$$

where

| Number | $z$ | $\tau_p(z)$ | $\sigma(\tau_p(z))$ | $\sigma(\tau_p(z))^{-1}$ |
|--------|----|-------------|-----------------|------------------|
| 0      | 0.0331 | 1.0000 | --- | --- |
| 1      | 1.0000 | 0.9320 | 0.1711 | 0.1720 |
| 2      | 2.0700 | 0.9180 | 0.1720 | 0.1718 |
| 3      | 3.0000 | 0.7795 | 0.1630 | 0.1629 |
| 4      | 4.0480 | 0.7652 | 0.1936 | 0.1939 |
| 5      | 8.1000 | 1.1475 | 0.4297 | 0.4389 |

**Table II:** Distances calculated by using the 109 GRBs data via Amati’s correlation [15].
with
\[
A = \sum_{SN1a} \left\{ 5 \log_{10}[D_L(p_s, z_i)] + \frac{15}{4} \log_{10} \left( \frac{G}{G_0} - m_{\text{obs}}(z_j) \right) \right\} \cdot C_{ij}^{-1} \cdot \left\{ 5 \log_{10}[D_L(p_s, z_j)] + \frac{15}{4} \log_{10} \left( \frac{G}{G_0} - m_{\text{obs}}(z_j) \right) \right\}
\]
\[
B = \sum_{SN1a} C_{ij}^{-1} \cdot \left\{ 5 \log_{10}[D_L(p_s, z_j)] + \frac{15}{4} \log_{10} \left( \frac{G}{G_0} - m_{\text{obs}}(z_j) \right) \right\}
\]
\[
C = \sum_{SN1a} C_{ii}^{-1}.
\]

The inverse of covariance matrix \(C_0^{-1}\) with systematic errors can be found in Refs. [24, 56].

E. Cosmic Microwave Background

\(\chi^2_{CMB}\) has a form
\[
\chi^2_{CMB}(p_s) = \Delta d_i [\text{Cov}^{-1}(d_i(p_s), d_j(p_s))][\Delta d_i]^t,
\]
with \(\Delta d_i(p_s) = d_i^{\text{theory}}(p_s) - d_i^{\text{obs}}\). 9-year WMAP gives \(d_i^{\text{obs}} = [l_A(z_*) = 302.04; R(z_*) = 1.7246; z_* = 1090.88]\), and the corresponding inverse covariance matrix is [25]
\[
\text{Cov}^{-1} = \begin{pmatrix}
3.182 & 18.253 & -1.429 \\
18.253 & 11887.879 & -193.808 \\
-1.429 & -193.808 & 4.556
\end{pmatrix}.
\]

\(z_*\) is the redshift at decoupling epoch of photons via \(z_* = 1048 [1 + 0.00124(\Omega_b h^2)^{-0.738}] [1 + g_1(\Omega_m h^2)^{g_2}]\) with \(g_1 = 0.0783(\Omega_b h^2)^{-0.238} (1 + 39.5(\Omega_b h^2)^{0.763})^{-1}\) and \(g_2 = 0.560 (1 + 21.1(\Omega_b h^2)^{1.81})^{-1}\); \(l_A(p_s; z_* = (1+z_*) \frac{\pi D_S(p_s; z_*)}{r_s(z_*)}\)

is the acoustic scale; \(R(p_s; z_* = \sqrt{\Omega_m H_0^2}(1+z_*)D_A(p_s; z_*)/c\) is the CMB shift parameter.

F. Baryon Acoustic Oscillation

The radial (line-of-sight) BAO scale measurement from galaxy power spectra can be depicted by
\[
\Delta z_{BAO}(z) = \frac{H(z) r_s(z_d)}{c}.
\]

Two observed values are \(\Delta z_{BAO}(z = 0.24) = 0.0407 \pm 0.0011\) and \(\Delta z_{BAO}(z = 0.43) = 0.0442 \pm 0.0015\), respectively [26].

Here \(r_s(z)\) is the comoving sound horizon size \(r_s = c \int_0^t \frac{c_s dt}{a}\). \(c_s\) is the sound speed of the photon–baryon fluid, \(c_s^{-2} = 3 + \frac{4}{3} \times (\frac{\Omega_b}{\Omega_m}) a. \ z_d\) denotes the drag epoch, \(z_d = \frac{1281(\Omega_m h^2)^{-0.419}[1 + b_1(\Omega_b h^2)]^{b_2}}{1 + 0.607(\Omega_m h^2)^{0.674}}[1 + b_1(\Omega_b h^2)]^{b_2}\) with \(b_1 = 0.313(\Omega_m h^2)^{-0.419} + 0.607(\Omega_m h^2)^{0.674}\) and \(b_2 = 0.238(\Omega_m h^2)^{0.223}\).

The measurement of BAO peak positions can be performed by the WiggleZ Dark Energy Survey [27], the Two Degree Field Galaxy Redshift Survey [28] and the Sloan Digital Sky Survey [29]. Introducing \(D_V(z) = (1 +\)}
z^2D_A^2(z) \left( \frac{r_s(z_d)}{D_V(z,p_s)} \right)^{1/3}, \text{ the observational data from BAO peak positions can be exhibited by}

\begin{equation}
X = \begin{pmatrix}
\frac{r_s(z_d)}{D_V(0.106)} & -0.336 \\
\frac{r_s(z_d)}{D_V(0.2)} & -0.1905 \\
\frac{r_s(z_d)}{D_V(0.35)} & -0.1097 \\
\frac{r_s(z_d)}{D_V(0.44)} & -0.0916 \\
\frac{r_s(z_d)}{D_V(0.53)} & -0.0726 \\
\frac{r_s(z_d)}{D_V(0.73)} & -0.0592
\end{pmatrix}, \quad V^{-1} = \begin{pmatrix}
4444 & 0 & 0 & 0 & 0 & 0 \\
0 & 30318 & -17312 & 0 & 0 & 0 \\
0 & -17312 & 87046 & 0 & 0 & 0 \\
0 & 0 & 0 & 23857 & -22747 & 10586 \\
0 & 0 & 0 & -22747 & 128729 & -59907 \\
0 & 0 & 0 & 10586 & -59907 & 125536
\end{pmatrix}
\end{equation}

with the inverse covariance matrix $V^{-1}$ shown in Ref. [57].

Thus, the $\chi^2_{BAO}$ can be constructed in a form,

$$
\chi^2_{BAO}(p_s) = \frac{[\Delta z_{BAO}(z = 0.24) - 0.0407]^2}{0.0011^2} + \frac{[\Delta z_{BAO}(z = 0.43) - 0.0442]^2}{0.0015^2} + X^t V^{-1} X.
$$

$X^t$ denotes the transpose of $X$.

**IV. Cosmic constraints on unified model of dark sectors in VG theory with flat and non-flat geometry**

Taking a joint likelihood analysis $L \propto e^{-\chi^2/2}$ as the products of the separate likelihoods, the total $\chi^2$ is adopted

$$
\chi^2 = \chi^2_{LT} + \chi^2_{GRBs} + \chi^2_{GF} + \chi^2_{SNIa} + \chi^2_{CMB} + \chi^2_{BAO}.
$$

**A. Flat-universe case**

![FIG. 1: 1σ and 2σ contours of flat VG-UM (left) and ΛCDM (right) model parameters.](image)

In order to obtain the stringent constraint on VG theory, we utilize the cosmic data different from Ref. [30], to calculate the total likelihood. Concretely, the LT data, the GRBs data, the GF data, the SNIa data with systematic error and the BAO data from radial measurement are not used in Ref. [30]. After calculation, the 1-dimension
distribution and the 2-dimension contours of VG-UM model parameters are illustrated in Fig. 2 for a flat universe. A stringent constraint on VG parameter is \( \beta = 0.0008^{+0.0034}_{-0.0032} + 0.0063^{+0.0069}_{-0.0064} \). Obviously, a small uncertainty around zero for parameter \( \beta \) is obtained, at 1\( \sigma \) and 2\( \sigma \) regions. In VG theory, the constraint on UM model parameters are \( B_s = 0.7662^{+0.0127+0.0248}_{-0.0125-0.0269}, \alpha = 0.0204^{+0.0201+0.0425}_{-0.0217-0.0398} \), \( h = 0.7058^{+0.0096+0.0202}_{-0.0101-0.0192} \) and \( 100\Omega_b h^2 = 2.324^{+0.055+0.118}_{-0.054-0.107} \). At 1\( \sigma \) confidence level, the value of \( \alpha = 0 \) is not excluded, which demonstrates that the \( \Lambda \)CDM model can not be distinguished from VG-UM model by the joint cosmic data. Besides the mean values with limits, the best-fit values of VG-UM model parameters are determined and exhibited in Table IV too. As a reference, the \( \Lambda \)CDM model is calculated by using the combined observational data appeared in section III, where the best-fit values and the mean values with limits on \( \Lambda \)CDM model are laid in Table IV. In \( \Lambda \)CDM model, one receives \( \Omega_{0\text{de}} = 0.7101^{+0.0126+0.0270}_{-0.0135-0.0282} \) that is compatible to the effective result of \( \Omega_{0\text{de}} \) in VG-UM model.

### B. Non-flat universe

![1σ and 2σ contours of parameters in non-flat VG-UM (left) and \( \Lambda \)CDM (right) model, respectively.](image)

For a non-flat geometry, restriction on dimensionless curvature density is \( \Omega_k = -0.0311^{+0.0259+0.0517}_{-0.0248-0.0501} \) in the varying-\( G \) theory with containing unified dark sectors. Obviously, at 1\( \sigma \) error a closed universe \( \Omega_k < 0 \) is predicted. Furthermore, the uncertainty of \( \Omega_k \) is more enlarged than other results. For example, using the same data to constrain...
TABLE V: The mean values with limits and the best-fit values of model parameters in a non-flat geometry.

| Parameter | Mean values with limits (VG-UM) | Best fit (VG-UM) | Mean values with limits (ΛCDM) | Best fit (ΛCDM) |
|-----------|---------------------------------|-----------------|-------------------------------|-----------------|
| $\Omega_k$ | $-0.0311^{+0.0025+0.0017}_{-0.0248-0.0021}$ | -0.0225 | $-0.0002^{+0.0024+0.0052}_{-0.0024-0.0048}$ | -0.0004 |
| $\beta$   | $-0.0387^{+0.0321+0.0646}_{-0.0310-0.0592}$ | -0.0277 | 0 | 0 |
| $B_s$     | $0.7550^{+0.0142+0.0270}_{-0.0143-0.0302}$ | 0.7620 | — | — |
| $\alpha$  | $0.0982^{+0.0035+0.1186}_{-0.0052-0.1287}$ | 0.0682 | 0 | 0 |
| $h$       | $0.6936^{+0.0145+0.0283}_{-0.0135-0.0267}$ | 0.6992 | $0.6916^{+0.0109+0.0197}_{-0.0101-0.0193}$ | 0.6930 |
| $100\Omega_k h^2$ | $2.2991^{+0.0506+0.1106}_{-0.0506-0.1132}$ | 2.2978 | $2.2689^{+0.0412+0.0815}_{-0.0410-0.0776}$ | 2.266 |
| $\Omega_{0dc}$ | $0.7053^{+0.0161+0.0338}_{-0.0155-0.0300}$ | 0.7126 | $0.7098^{+0.0144+0.0265}_{-0.0140-0.0294}$ | 0.7126 |

other models it has $\Omega_k = -0.0002^{+0.0024+0.0052}_{-0.0024-0.0048}$ (with model parameter $\Omega_{0dc} = 0.7098^{+0.0144+0.0265}_{-0.0140-0.0294}$) in ΛCDM model, $\Omega_k = -0.0001^{+0.0025+0.0052}_{-0.0025-0.0056}$ (with model parameters $B_s = 0.7665^{+0.0101+0.0194}_{-0.0099-0.0205}$ and $\alpha = 0.0209^{+0.0186+0.0401}_{-0.0189-0.0373}$) in constant-$G$ UM. Taking the ΛCDM model as a reference, it is shown that the influence on the fitting value of $\Omega_k$ is small by the added parameter $B_s$ and $\alpha$ as seen in constant-$G$ UM model, while the influence on the value of $\Omega_k$ is large by the added VG parameter $\beta$ as indicated in VG-UM model. In addition, the different constraint results are received, when one compares the flat-geometry VG-UM model with the non-flat-geometry VG-UM model. From table V in the non-flat universe one reads VG parameter $\beta = -0.0387^{+0.0321+0.0646}_{-0.0310-0.0592}$ which shows that at 1σ confidence level VG parameter $\beta$ is less than zero, while the value of $\beta$ is around zero at 1σ confidence level in the flat universe. Still, relative to a small uncertainty of $\beta$ in the flat geometry, a very large error of VG parameter $\beta$ is given in the non-flat geometry. Other parameters are $B_s = 0.7550^{+0.0142+0.0270}_{-0.0143-0.0302}$ and $\alpha = 0.0982^{+0.0035+0.1186}_{-0.0052-0.1287}$ in the non-flat VG-UM. We then find at 1σ confidence level, the flat ΛCDM model ($\Omega_k = 0, \beta = 0$ and $\alpha = 0$) is not included in VG-UM model. This result in VG theory is different from the popular point that the complicated cosmological model is usually degenerate with the ΛCDM model.

V. Behaviors of $\mathcal{G}$ with the confidence level in flat and non-flat VG-UM theory

![FIG. 3: The best-fit evolutions of $\frac{\mathcal{G}}{\Omega}$ and $\frac{\mathcal{G}}{\Omega}$ with their confidence level in flat (upper) and non-flat (lower) universe.](image)

By using the best-fit values of model parameters with their covariance matrix, in flat and non-flat VG-UM theory
the best-fit evolutions of $\frac{\dot{G}}{G_0}$ with their confidence level (the shadow region) are illustrated in Figure 3. "Dot" denotes the derivative with respect to $t$. Fig. 3 provides prediction that the today’s value $(\frac{\dot{G}}{G_0})_{\text{today}} \approx 0.7977^{+2.3566}_{-2.3566} \times 10^{-13} \text{yr}^{-1}$ in a flat geometry. This restriction on $(\frac{\dot{G}}{G_0})_{\text{today}}$ is more stringent than other results seen in table VII. Also, using the best-fit value of parameter $\beta$ with error the shapes of $(\frac{\dot{G}}{G_0})_{z=3.5}$ are exhibited. Taking high redshift $z = 3.5$ as another reference, we find $(\frac{\dot{G}}{G_0})_{z=3.5} \approx 1.0003^{+0.0014}_{-0.0016}$ and $(\frac{\dot{G}}{G_0})_{z=3.5} \approx -0.4012^{+1.1797}_{-1.1797} \times 10^{-12} \text{yr}^{-1}$ in the flat VG-UM model. The upper panel of Fig. 3 reveal that the behaviors of $G$ and its derivative are around the constant-$G$ theory in the flat geometry. But the behavior of $G$ has the more departure from constant-$G$ theory for a non-flat case. In the non-flat geometry, limits on the variation of $G$ at today is $(\frac{\dot{G}}{G_0})_{\text{today}} \approx 19.3678^{+21.8262}_{-21.8262} \times 10^{-13} \text{yr}^{-1}$, and at $z = 3.5$ they are $(\frac{\dot{G}}{G_0})_{z=3.5} \approx 0.9917^{+0.0104}_{-0.0131}$ and $(\frac{\dot{G}}{G_0})_{z=3.5} \approx 101.828^{+119.524}_{-119.524} \times 10^{-13} \text{yr}^{-1}$. It tends to hint, $\dot{G} > 0$. Finally, from the best-fit evolutions of $\frac{\dot{G}}{G_0}$, they give the different evolutionary tendency in flat and non-flat universe, respectively. Since the monotonicity of $\frac{\dot{G}}{G}$ is $-\beta H$ depends on the symbol of $\beta$, it is important to sternly constrain the value of $\beta$.

### VI. Behaviors of EoS with the confidence level in flat and non-flat VG-UM theory

The EoS of UM in VG theory is demonstrated

$$w_{VG-UM}(z) = \frac{P_{VG-UM}}{\rho_{VG-UM}} = \frac{\beta - 3}{3} \frac{B_s}{B_s + (1 - B_s)(1 + z)^{(1+\alpha)(3-\beta)}},$$

For both flat and non-flat universe, we point out at early time $w_{VG-UM} \sim 0$ (DM), and in the future $w_{VG-UM} \sim -1$ (DE), as illustrated in Figure 4 (left). If the dark sectors are thought to be separable, it is interested to investigate the properties of both dark components in VG-UM model. Considering that the behavior of dark matter is known

| Observations | Limits $(\text{yr}^{-1})$ |
|--------------|--------------------------|
| Pulsating white dwarf G117-B15A [2] | $|\frac{\dot{G}}{G_0}| \leq 4.1 \times 10^{-10}$ |
| Nonradial pulsations of white dwarfs [3] | $-2.5 \times 10^{-10} \leq \frac{\dot{G}}{G_0} \leq 4 \times 10^{-11}$ |
| Millisecond pulsar PSR J0437-4715 [4] | $|\frac{\dot{G}}{G_0}| \leq 2.3 \times 10^{-11}$ |
| Type-Ia Supernovae [5] | $\frac{\dot{G}}{G_0} \leq 10^{-11}$ |
| Neutron star masses [6] | $\frac{\dot{G}}{G_0} = (-0.6 \pm 4.2) \times 10^{-12}$ |
| Helioseismology [58] | $|\frac{\dot{G}}{G_0}| \leq 1.6 \times 10^{-12}$ |
| Lunar laser ranging experiment [59] | $\frac{\dot{G}}{G_0} = (4 \pm 9) \times 10^{-13}$ |
| Big Bang Nuclei-synthesis [60] | $-3.0 \times 10^{-13} < \frac{\dot{G}}{G_0} < 4.0 \times 10^{-13}$ |

Table VI: The best-fit values of $\frac{\dot{G}}{G_0}$ and $\frac{\dot{G}}{G}$ with their confidence level in VG-UM model, for flat and non-flat universe.

Table VII: Limits on the variation of $G$. 

| | Flat | Non-flat |
|---|---|---|
| $(\frac{\dot{G}}{G_0})_{z=3.5}$ | $1.0003^{+0.0014}_{-0.0016}$ | $0.9917^{+0.0104}_{-0.0131}$ |
| $(\frac{\dot{G}}{G_0})_{\text{today}}$ | $0.7977^{+2.3566}_{-2.3566} \times 10^{-13} \text{yr}^{-1}$ | $19.3678^{+21.8262}_{-21.8262} \times 10^{-13} \text{yr}^{-1}$ |
| $(\frac{\dot{G}}{G_0})_{z=3.5}$ | $-4.012^{+11.797}_{-11.797} \times 10^{-13} \text{yr}^{-1}$ | $101.828^{+119.524}_{-119.524} \times 10^{-13} \text{yr}^{-1}$ |
i.e. its EoS $w_{dm} = 0$ ($\rho_{dm} = \rho_{0dm} a^{-3} \left(\frac{\alpha}{2} \right) - 3 \frac{\alpha}{2}$), the EoS of dark energy in VG-UM model subjects to,

$$w_{de} = \frac{p_{de}}{\rho_{de}} = \frac{p_{VG-UM}}{\rho_{VG-UM} - \rho_{dm}} = -\frac{A}{\rho_{VG-UM} - \rho_{dm} \rho_{VG-UM}'}. \tag{30}$$

Using the best-fit values of model parameters and the covariance matrix, the evolution of $w_{de}$ with confidence level in VG-UM is plotted in Fig. 4 (middle), for flat and non-flat universe. If one deems the behavior of dark energy is the cosmological constant i.e. $w_{\Lambda} = -1$ ($p_{\Lambda} = -\rho_{\Lambda}$), the EoS of dark matter in VG-UM model obeys

$$w_{dm} = \frac{p_{dm}}{\rho_{dm}} = \frac{p_{VG-UM} - p_{\Lambda}}{\rho_{VG-UM} - \rho_{\Lambda}} = -\frac{\rho_{A} \rho_{VG-UM}'}{\rho_{VG-UM} - \rho_{A} \rho_{VG-UM}'} \tag{31}$$

which is drawn in Fig. 4 (right) with the confidence level, for flat and non-flat universe.

![Graphs showing the evolution of EoS with confidence level in flat and non-flat VG-UM model.](image)

**FIG. 4**: The evolutions of EoS with confidence level in flat (upper) and non-flat (lower) VG-UM model. Evolution of $w_{VG-UM}(z)$ (left), evolution of $w_{de}(z)$ in VG-UM model with assuming $w_{dm} = 0$ at prior (middle), and evolution of $w_{dm}(z)$ in VG-UM model with assuming $w_{de} = -1$ at prior (right).

|                | $w_{0VG-UM}$ | $w_{0dm}$ (with $w_{0de} = -1$) | $w_{0de}$ (with $w_{0dm} = 0$) |
|----------------|--------------|---------------------------------|---------------------------------|
| Flat           | $-0.7685_{+0.0124}^{0.0011}$ | $0.0151_{+0.0171}^{-0.0124}$ | $-0.9986_{+0.0011}^{-0.0011}$ |
| Non-flat       | $-0.7690_{+0.0111}^{0.0011}$ | $0.0281_{+0.0258}^{-0.0258}$ | $-1.0092_{+0.0104}^{-0.0104}$ |

**TABLE VIII**: The best-fit values of $w_{0VG-UM}$, $w_{0dm}$ and $w_{0de}$ with their confidence level hinted by VG-UM model, for flat and non-flat universe.

Fig. 4 gives the current values $w_{0dm} = 0.0151_{+0.0171}^{-0.0124}$ in the flat VG-UM model and $w_{0dm} = 0.0281_{+0.0258}^{-0.0258}$ in the non-flat universe, which have the larger uncertainties than $w_{0dm} = 0.0016_{+0.0016}^{-0.0016}$ calculated in the non-unified model of constant-$G$ theory by Ref. 61. At confidence level, a small-positive pressure of DM is indicated by non-flat VG-UM model, or it is almost hinted in flat model. For the current value of $w_{0de}$, it approximates to -1 with a very small uncertainty, for both flat ($w_{0de} = -0.9986_{+0.0011}^{-0.0011}$) and non-flat ($w_{0de} = -1.0092_{+0.0104}^{-0.0104}$) universe. In addition, at high redshift the uncertainty of $w_{de}$ (or $w_{dm}$) is enlarged (or narrowed). For the best-fit evolution, $w_{de}$ and $w_{dm}$ are variable with the time, and at the early time of universe they tend to have the small deviation from zero pressure.
VII. Conclusions

Observations anticipate that $G$ may be variable and most universal energy density are invisible. The attractive properties of this study is that the variation of $G$ naturally results to the invisible components in universe. The VG could provide a solution to the originated problem of DM and DE. We apply recently observed data to constrain the unified model of DE and DM in the framework of VG theory for flat and non-flat universe. Using the LT, the GRBs, the GF, the SNIa with systematic error, the CMB from 9-year WMAP and the BAO data from measurement of radial and peak positions, uncertainties of VG-UM parameter space are obtained in flat and non-flat universe. In addition, the uncertainty of $\Omega_k$ where at $1\sigma$ and peak positions, uncertainties of VG-UM parameter space are obtained in flat and non-flat universe.

In a flat geometry, constraint on mean value of VG parameter is $\beta = 0.0008^{+0.0034+0.0063}_{-0.0032-0.0064}$ with a small uncertainty around zero, and restrictions on UM model parameters are $B_s = 0.7662^{+0.0127+0.0248}_{-0.0125-0.0269}$ and $\alpha = 0.0204^{+0.0201+0.0425}_{-0.0217-0.0398}$ with $1\sigma$ and $2\sigma$ confidence level. But for a non-flat universe, the different constraint result on parameter $\beta$ is obtained, where at $1\sigma$ confidence level $\beta = 0$ (constant $G$ theory) are excluded. Restriction on dimensionless curvature density is $\Omega_k = -0.0311^{+0.0259+0.0517}_{-0.0248-0.0501}$ in the non-flat VG-UM universe. Obviously, at $1\sigma$ error a closed universe $\Omega_k < 0$ is indicated. In addition, the uncertainty of $\Omega_k$ is more enlarged than other results. At $1\sigma$ confidence level, the flat $\Lambda$CDM model ($\Omega_k = 0$, $\beta = 0$ and $\alpha = 0$) is not included in VG-UM model.

Using the best-fit values of VG-UM parameters and their covariance matrix, the limits on today’s value are $(\frac{\dot{G}}{G})_{today} = (-3.154, 1.559) \times 10^{-13}$ and $(\frac{\ddot{G}}{G})_{today} \simeq 19.3678^{+21.8262}_{-21.8262} \times 10^{-13} yr^{-1}$ in flat and non-flat geometry, respectively. At redshift $z = 3.5$, one finds $(\frac{\dot{G}}{G})_{z=3.5} \simeq 1.0003^{+0.0014}_{-0.0016}$ and $(\frac{\ddot{G}}{G})_{z=3.5} \simeq 0.9917^{+0.0104}_{-0.0131}$ for flat and non-flat universe. If one considers that the DM and the DE could be separable, EoS of DE and DM in unified model are discussed by combing with the fitting results. It is shown that $w_{0dm} = 0.0151^{+0.0017}_{-0.0171}$ and $w_{0dm} = 0.0281^{+0.0258}_{-0.0258}$ for flat and non-flat universe with assuming $w_{0de} = -1$ in VG-UM theory, while it has $w_{0de} = -0.9986^{+0.0011}_{-0.0011}$ and $w_{0de} = -1.0092^{+0.0104}_{-0.0104}$ for flat and non-flat universe, with assuming $w_{0dm} = 0$ at prior.

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