A Thom-Smale-Witten theorem on manifolds with boundary: the arbitrary metric case

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Abstract In this paper, we establish the canonical isomorphism between the Witten instanton complex and the Thom-Smale complex on manifolds with boundary with arbitrary Riemannian metric using Bismut-Lebeau’s analytic localization techniques.

Keywords Thom-Smale complex · Witten instanton complex · Witten Laplacian · Bismut-Lebeau’s analytic localization techniques

1 Introduction

In our previous work [11], we established the isomorphism between the Witten instanton complex and the Thom-Smale complex on a Riemannian manifold with boundary under the assumption that the metric is Euclidean around critical points. This paper is a continuation of [11].

In this paper, we work on a general metric, i.e., we drop off the assumption on the metric made in [11]. Our main result states that the isomorphism between the Witten instanton complex and the Thom-Smale complex still holds in the case of general metric. Moreover, we obtain both the linear growth of the upper part and the exponentially decay of the lower part of the spectrum of the Witten Laplacian $D^2_T$ (cf. (3.4)).

Our proof is more complicate than that of [11], though both proof consist of applying the Bismut-Lebeau analytic localization techniques along the lines of [14]. Compared to [11], there are mainly two difficulties involved for the general metric. The first is to calculate out explicitly the kernel of the approximate deformed de Rham operator $D^N_T$ (cf. Th.4). This is
done by choosing coordinates around critical points such that the Morse function has locally quadratic expressions and such that the metric is Euclidean only at the critical points. The second difficulty is to derive the exponential decay of the lower part of the spectrum of the Witten Laplacian. We tackle this problem by adapting and simplifying the argument given for critical submanifolds in [2, §9]. In our case, the critical submanifolds reduce to isolated critical points.

We adopt the same notations as in [11]. Let $M$ be a smooth $n$-dimensional compact manifold with boundary $\partial M$. A smooth function $f : M \to \mathbb{R}$ is called a Morse function if $f$ is a Morse function in the interior of $M$, $f|_{\partial M}$ is a Morse function on $\partial M$ and $f$ has no critical points on $\partial M$. Let $f$ be a Morse function on $M$. Let $C^j(f)$ (resp. $C^j(f|_{\partial M})$) be the set consisting of all critical points of $f$ (resp. $f|_{\partial M}$) with index $j$. Let $v$ be the outward normal vector field along $\partial M$. Denote by

$$C^j_-(f|_{\partial M}) = \{ p \in C^j(f|_{\partial M}) : (vf)(p) < 0 \},$$

$$C^j_+(f|_{\partial M}) = \{ p \in C^j(f|_{\partial M}) : (vf)(p) > 0 \},$$

and

$$c_j = \# C^j(f), \quad p_j = \# C^j_-(f|_{\partial M}), \quad q_j = \# C^j_+(f|_{\partial M}).$$

Set $C(f) = \bigsqcup_{j=0}^n C^j(f)$ and $C_-(f|_{\partial M}) = \bigsqcup_{j=0}^{n-1} C^j_-(f|_{\partial M})$.

Denote by

$$0 \leq \lambda^j_1(T) \leq \lambda^j_2(T) \leq \cdots \tag{1.1}$$

the eigenvalues of the Witten Laplacian $D^2_T$ with absolute boundary condition acting on $j$-forms.

In the following results, $M$ is a compact manifold with boundary endowed with an arbitrary Riemannian metric and $f$ is a Morse function on $M$.

**Theorem 1** There exists positive constants $a_1$ and $T_1$ such that for $T > T_1$, and $j = 0, 1, \ldots, n$, we have

$$\lambda^j_\ell(T) \geq a_1 T^2, \quad \text{for } \ell \geq c_j + p_j + 1. \tag{1.2}$$

Under the condition that the metric is Euclidean around critical points, the estimate (1.2) was obtained by Chang and Liu (cf. [4, §3, Th. 2]) by localization and the min–max principle and by Lu (cf. [11, Th. 0.1]) via elementary spectral estimates. In the case of arbitrary metric, Le Peutrec (cf. [10, Th. 3.5]) derived the estimate (1.2) by constructions of quasimodes and the WKB method. Here our method is again based on elementary spectral estimates.

As for the lower part of the spectrum of the Witten Laplacian, we have the following estimate.

**Theorem 2** There exist positive constants $a_2, a_3$ and $T_2$ such that for $T \geq T_2$ and $j = 0, 1, \ldots, n$,

$$\lambda^j_\ell(T) \leq a_2 e^{-a_3 T}, \quad \text{for } \ell \leq c_j + p_j. \tag{1.3}$$

The estimate (1.3) for $j = 0$ was obtained by D. Le Peutrec via WKB analysis (cf. [10, Th. 1.0.3]) and is new to our knowledge for $j > 0$.

It follows from Theorems 1 and 2 that for a given $C_0 > 0$, there exists $T_0 > 0$ such that for $T > T_0$, the number of eigenvalues of $D^2_T$ in $(0, C_0)$ equals $c_j + p_j$ (see Proposition 1). Let $F^c_{T,j}$ denote the $(c_j + p_j)$-dimensional vector space generated by the eigenspaces associated
to the eigenvalues lying in \([0, C_0]\). It is easy to see that \((F^C_{T,\bullet},dT)\) forms a complex, called
Witten instanton complex. Let \((C^\bullet, \partial)\) denote the Thom-Smale complex constructed in [9] (cf. (2.5)–(2.6)).

Let \(P_\infty\) be the natural morphism from the de Rham complex \((\Omega^\bullet(M),d)\) to the Thom-Smale
complex \((C^\bullet, \partial)\), defined by integration on the closure of the unstable manifolds (cf.
§2):

\[
P_\infty(\alpha) = \sum_{p \in C(f) \cup C^-(f|_{\partial M})} [p]^* \int_{W_u(p)} \alpha \in C^\bullet,
\quad \text{for } \alpha \in \Omega^\bullet(M). \tag{1.4}
\]

Set

\[
P_{\infty,T}(\alpha) = P_\infty(e^{Tf} \alpha), \quad \text{for } \alpha \in F^C_{T,\bullet}. \tag{1.5}
\]

The main result of this paper is as follows.

**Theorem 3** The map \(P_{\infty,T} : (F^C_{T,\bullet},dT) \rightarrow (C^\bullet, \partial)\) is an isomorphism of complexes for \(T\)
large enough. In particular, the chain map \(P_\infty\) is a quasi-isomorphism between the de Rham
complex \((\Omega^\bullet(M),d)\) and the Thom-Smale complex \((C^\bullet, \partial)\).

The idea that the Thom-Smale complex could be recovered from the Witten instanton
complex goes back to Witten [13]. Besides, Witten’s idea to prove the classical Morse inequalities
by using spectral spaces was used by Demailly to prove holomorphic Morse inequalities, see
[12, Chap. 1].

For the sake of completeness, we state the following known Morse inequalities on man-
ifolds with boundary, which follow from the proof of Theorem 3 as in [11] and we omit
the proof here. Let \(\beta_j(M)\) denote the \(j\)th Betti number of the de Rham complex, i.e.,
\(\beta_j(M) = \dim H^j_{dR}(M, \mathbb{R})\).

**Corollary 1** For any \(k = 0, 1, \ldots, n\), we have

\[
\sum_{j=0}^{k} (-1)^{k-j} \beta_j(M) \leq \sum_{j=0}^{k} (-1)^{k-j} (c_j + p_j), \tag{1.6}
\]

with equality for \(k = n\).

Denote by \(\beta_j(M, \partial M)\) the \(j\)th Betti number of the relative de Rham complex with coefficients twisted by the orientation bundle, i.e., \(\beta_j(M, \partial M) = \dim H^j_{dR}(M, \partial M; o(TM))\).

**Corollary 2** For any \(k = 0, 1, \ldots, n\), we have

\[
\sum_{j=0}^{k} (-1)^{k-j} \beta_j(M, \partial M) \leq \sum_{j=0}^{k} (-1)^{k-j} (c_j + q_{j-1}), \tag{1.7}
\]

with equality for \(k = n\).

The organization of this paper is as follows. In Sect. 2, we introduce the Thom-Smale complex constructed by
Laudenbach in [23]. In Sect. 3, we describe briefly the Witten instanton complex. Section 4 is mainly devoted to various estimates of the components of the deformed
Dirac operators. In Sect. 5, we prove Theorems 1–3.
2 The Thom-Smale complex constructed by Laudenbach

In this section, we introduce the Thom-Smale complex constructed by Laudenbach in [9]. By [9, §2.1], there exists a vector field $X$ on $M$ satisfying the following conditions.

1. $(X f)(\cdot) < 0$ except at critical points in $C(f) \cup C_-(f|\partial M)$;
2. $X$ points inwards along $\partial M$ except in a neighborhood in $\partial M$ of critical points in $C_-(f|\partial M)$ where it is tangent to $\partial M$;
3. if $p \in C^j(f)$, then there exists a coordinate system $(x, U_p)$ such that on $U_p$ we have
   \[ f(x) = f(p) - \frac{x_1^2}{2} - \cdots - \frac{x_j^2}{2} + \frac{x_{j+1}^2}{2} + \cdots + \frac{x_n^2}{2}, \]
   (2.1)

   and
   \[ X = \sum_{i=1}^j x_i \frac{\partial}{\partial x_i} - \sum_{i=j+1}^n x_i \frac{\partial}{\partial x_i}; \]
   (2.2)

4. if $p \in C^j_-(f|\partial M)$, then there are coordinates $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$ on some neighborhood $U_p$ of $p$ such that on $U_p$,
   \[ f(x) = f(p) - \frac{x_1^2}{2} - \cdots - \frac{x_j^2}{2} + \frac{x_{j+1}^2}{2} + \cdots + \frac{x_{n-1}^2}{2} + x_n \]
   (2.3)

   and
   \[ X = \sum_{i=1}^j x_i \frac{\partial}{\partial x_i} - \sum_{i=j+1}^n x_i \frac{\partial}{\partial x_i}; \]
   (2.4)

5. $X$ is Morse-Smale in the sense that the global unstable manifolds and the local stable manifolds are mutually transverse. Denote by $W^u(p)$ (resp. $W^s_{loc}(p)$) the unstable manifold (resp. the local stable manifold) of $p$ which by definition, consists of all the flow lines of $X$ that emanate from $p$ (resp. end at $p$). Denote by $\overline{W}^u(p)$ the closure of $W^u(p)$ in $M$. It is a consequence of the above conditions (1)–(5) that the results [8, Prop. 2] about the structure of $W^u(p)$ still hold for $p \in C^j(f) \cup C^j_-(f|\partial M)$, i.e., $\overline{W}^u(p)$ is a $j$-dimensional submanifold of $M$ with conical singularities and $\overline{W}^u(p) \setminus W^u(p)$ is stratified by unstable manifolds of critical points of index strictly less than $j$.

For $p \in C^j(f) \cup C^j_-(f|\partial M)$, let $[p]$ be the real line generated by $p$, and let $[p]^*$ be the line dual to $[p]$. As in [9], set

\[ C^j = \bigoplus_{p \in C^j(f) \cup C^j_-(f|\partial M)} [p]^*. \]  

(2.5)

The boundary morphism $\partial$ from $C^j$ to $C^{j+1}$ is given by

\[ \partial[p]^* = \sum_{q \in C^{j+1}(f) \cup C^{j+1}_-(f|\partial M)} n(q, p)[q]^* \]

(2.6)

with $n(q, p) \in \mathbb{Z}$. It is a consequence of [9, §2.2, Prop.] that $(C^*, \partial)$ is a chain complex.
3 The Witten instanton complex

Let $g^{TM}$ be a Riemannian metric on $M$, and let $\nabla^{TM}$ be the Levi-Civita connection associated with the metric $g^{TM}$. Let $\nabla^{\Lambda(T^*M)}$ be the connection on $\Lambda(T^*M)$ induced by the connection $\nabla^{TM}$. Denote by $\{\cdot, \cdot\}$ the metric on $\Lambda(T^*M)$ induced by $g^{TM}$.

Let $o(TM)$ be the orientation bundle over $M$, and let $dv_M$ be the density (or Riemannian volume form) on $M$, i.e., $dv_M$ is a smooth section of the line bundle $\Lambda^n(T^*M) \otimes o(TM)$ (cf. [1, p. 29], [3, p. 88]). Denote by $\Omega^i(M)$ the space of smooth differential $i$-forms on $M$. Set $\Omega(M) = \bigoplus_{i=0}^n \Omega^i(M)$. Let $E$ denote the set of square integrable sections of $\Lambda(T^*M)$ over $M$. For $\alpha, \beta \in E$, set

$$\langle \alpha, \beta \rangle_E = \int_M \langle \alpha, \beta \rangle(x) dv_M(x). \quad (3.1)$$

We denote by $\| \cdot \|_E$ the norm on $E$ induced by the inner product (3.1).

Let $d$ denote the exterior differential derivative on $\Omega(M)$, and let $\delta$ be the dual of $d$ with respect to the metric (3.1). Set

$$d_T = e^{-Tf} d \cdot e^{Tf}, \quad \delta_T = e^{Tf} \delta \cdot e^{-Tf}. \quad (3.2)$$

The deformed de Rham operator $D_T$ is given by

$$D_T = d_T + \delta_T. \quad (3.3)$$

The Witten Laplacian on manifolds is defined by

$$D^2_T = (d_T + \delta_T)^2 = d_T \delta_T + \delta_T d_T. \quad (3.4)$$

Define the domain of the weak maximal extension of $d_T$ by

$$\text{Dom}(d_T) = \{ w \in E, d_T w \in E \}, \quad (3.5)$$

where $d_T w$ is calculated in the sense of distribution. We denote by $d^*_T$ the Hilbert space adjoint of $d_T$. Every smooth differential form $w$ has a natural decomposition into the norm and the tangent components along $\partial M$,

$$w = w_{\text{tan}} + w_{\text{norm}}. \quad (3.6)$$

where $w_{\text{tan}}$ does not contain the factor $v$. Integrations by parts shows

$$\text{Dom}(d^*_T) \cap \Omega(M) = \{ w \in \Omega(M), w_{\text{norm}} = 0 \text{ on } \partial M \},
\text{Dom}(d_T) \cap \Omega(M) = \{ w \in \Omega(M), w_{\text{norm}} = 0 \text{ on } \partial M \}. \quad (3.7)$$

In view of (3.7), the domain of the extension $D_T = d_T + \delta_T^*$ of the deformed de Rham operator is:

$$\text{Dom}(D_T) \cap \Omega(M) = \{ w \in \Omega(M), w_{\text{norm}} = 0 \text{ on } \partial M \}. \quad (3.8)$$

We define the self-adjoint extension of $D^2_T$ by $D^2_T = d_T d^*_T + d^*_T d_T$. Then

$$\text{Dom}(D^2_T) \cap \Omega(M) = \{ w \in \Omega(M), \langle d_T w \rangle_{\text{norm}} = 0 \text{ on } \partial M \}. \quad (3.9)$$

Following the argument after [14, Prop. 5.5], one easily get Morse inequalities (1.6) once the following Proposition holds.

**Proposition 1** For any $C_0 > 0$, there exits $T_0 > 0$ such that when $T \geq T_0$, the number of eigenvalues in $[0, C_0]$ of $D^2_T\big|_{\text{Dom}(D^2_T) \cap \Omega(M)}$ equals $c_j + p_j$.\hfill \&
4 Analysis of the deformed Dirac operators

4.1 Localization of the problem

We restate the following result from [11, §5.1]. Note that the estimate [11, (5.1)] is independent of the choice of the metric. The readers are referred to [11, Prop. 5.1] for the proof.

Denote by $U$ the union of all $U_p$'s in the conditions (3)–(4) in Sect. 2.

**Proposition 2** There exist constants $a_4 > 0$, $T_3 > 0$ such that for any $s \in \text{supp}(D_T)$ with $\text{supp}(s) \subset M \setminus U$ and $T \geq T_3$, we have

$$\|D_T s\|_E \geq a_4 T \|s\|_E.$$  \hspace{1cm} (4.1)

The above result indicates that the eigenvectors of the deformed de Rham operator with small eigenvalues “concentrate” near critical points in $C(f)$ and $C(f|_{\partial M})$.

4.2 A Taylor expansion of the deformed de Rham operator near critical points

Fix $p \in C(f) \cup C_-(f|_{\partial M})$. Set $N = T_p M$ and $U_\varepsilon = \{Z \in N, \|Z\|_{\varepsilon, T} < \varepsilon\}$. Since $M$ is compact, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the map $Z \in N \rightarrow \exp_p Z \in M$ is a diffeomorphism for $U_\varepsilon$ onto a neighborhood $U_\varepsilon$ of $p$ in $M$. From now on, we identify $U_\varepsilon$ with $U_\varepsilon$. We will use the notation $Z$ instead of $\exp_p Z$. In the sequel, we always take for simplicity $U_p = U_\varepsilon$ in the conditions (3)–(4) in Sect. 2.

Let $dv_N$ be the volume element of the vector space $N$, and let $k(Z)$ be the smooth positive function defined on $U_\varepsilon$ by the equation

$$dv_M(Z) = k(Z)dv_N(Z).$$  \hspace{1cm} (4.2)

The function $k$ has a positive lower bound on $U_{\varepsilon_0/2}$ and $k(0) = 1$.

Denote by $\pi$ the projection from the vector space $N$ onto the point $p$. Set $\Lambda(N^*) = \Lambda(T_p^* M)$. Let $V$ be the pullback of the vector space $\Lambda(N^*)$ via the map $\pi$. If $Z \in U_\varepsilon$, then the fibre $\Lambda(T^* M)_Z$ and $V$ are identified by the parallel transport with respect to the connection $\nabla^{\Lambda(T^* M)}$ along the curve $t \mapsto tZ$, $t \in [0, 1]$. In this way, the space of smooth sections of $\Lambda(T^* M)$ over $U_\varepsilon$ can be identified with the space of $V$-valued smooth functions on $N$, defined in the neighborhood $U_\varepsilon$.

Let $E$, $E_\varepsilon$ be the sets of square integrable sections of $V$ over $N$ and $U_\varepsilon$, respectively. If $s_1, s_2 \in E$, set

$$\langle s_1, s_2 \rangle_E = \int_N \langle s_1, s_2 \rangle(Z)dv_N(Z).$$  \hspace{1cm} (4.3)

We denote by $\| \cdot \|_E$ the norm on $E$ induced by the product (4.3). Clearly, if $s \in E$ has compact support in $U_\varepsilon$, we can and we will identify $s$ with an element of $E$ which has compact support in $U_\varepsilon$.

Let $e_1, \ldots, e_n$ be an orthonormal basis of the vector space $N$. Denote by $\nabla_{e_j}$ the derivative along the vector $e_j$.

**Definition 1** Let $D^N$ be the operator acting on $E$:

$$D^N = \sum_{j=1}^n c(e_j)\nabla_{e_j}.$$  \hspace{1cm} (4.4)
Clearly, $D^N$ is a Dirac operator on $E$, i.e., $D^N$ is formally self-adjoint with respect to the product (4.3).

Using the identification of $\Lambda(T^*M)$ with $V$ on $U_\varepsilon$, we can now consider the connection $\nabla^{\Lambda(T^*M)}$ as a connection on $V$ over $U_\varepsilon$. Similarly, the deformed de Rham operator $DT$ now acts on $E_\varepsilon$. By (4.2), the operator $k^{1/2}DTk^{-1/2}$ is formally self-adjoint on $E_\varepsilon$ with respect to the product (4.3).

Denote by $\nabla f$ the gradient vector field of $f$ with respect to the metric $g_{TM}$. Then, by [1, Prop. 1.28], we have

$$\nabla f = v + O(|Z|^2)$$

(4.5) with

$$v(Z) = \sum_{\alpha=1}^n \frac{\partial f}{\partial Z_\alpha} \frac{\partial}{\partial Z_\alpha}, \quad |Z| = \left(\sum_{\alpha=1}^n Z_\alpha^2\right)^{1/2}.$$  \hspace{1cm} (4.6)

Set

$$D_T^N = D^N + T\hat{c}(v).$$  \hspace{1cm} (4.7)

Then we have the following analogue of [2, Th. 8.18], [5, Lemma 2.4].

**Theorem 4** The following asymptotic formula holds on $E_\varepsilon$ as $T \to +\infty$,

$$k^{1/2}DTk^{-1/2} = D_T^N + O(|Z|^2 \partial + |Z| + T|Z|^2),$$

(4.8)

where $\partial$ denotes a 1st-order differential operator.

**Proof** We recall the proof for the reader’s convenience. For $Z \in U_\varepsilon$, $X \in N$, let $\tilde{X}$ be the parallel transport of the vector $X$ with respect to the connection $\nabla^{TM}$ along the curve $t \mapsto tZ$, $t \in [0, 1]$, i.e.,

$$\nabla^TM \tilde{X} = 0.$$  \hspace{1cm} (4.9)

Then

$$k^{1/2}DTk^{-1/2} = \sum_{\alpha=1}^n c(e_\alpha)\nabla_{\tilde{e}_\alpha}^{\Lambda(T^*M)} + T\hat{c}(\nabla f) - \frac{1}{2}k^{-1}(\tilde{e}_\alpha k)(Z)c(e_\alpha).$$

(4.10)

By [1, Prop. 1.28],

$$\tilde{e}_\alpha(Z) = e_\alpha + O(|Z|^2), \quad k(Z) = 1 + O(|Z|^2).$$

(4.11)

Set

$$w = \nabla^{\Lambda(T^*M)} - \nabla.$$  \hspace{1cm} (4.12)

By [1, Prop. 1.18],

$$w_p = 0.$$  \hspace{1cm} (4.13)

Then the formula (4.8) follows immediately from (4.5), (4.7) and (4.11)–(4.13). The proof of Theorem 4 is complete. \hfill $\blacksquare$
Set
\[ R_T = k^\frac{1}{2} D_T k^{-\frac{1}{2}} - D_T^N. \] (4.14)

Then \( R_T \) is a first order differential operator acting on the space \( \mathbf{E}_{v_0} \). By (4.8), we know that as \( T \to +\infty \),
\[ R_T = O(|Z|^2 \partial + |Z| + T|Z|^2). \] (4.15)

We now calculate out explicitly the kernel of the operator \( D_T^N \) on \( C^\infty (N, \Lambda(N^*)) \) under suitably chosen coordinate system \((Z, U_p)\) around critical points in \( C^1_j (f) \cup C^1_j (f|_{\partial M}) \).

If \( p \in C^j_j (f) \), by [6, §4.2], there exists coordinate system \((Z, U_p)\) around \( p \) such that the metric \( g^{TM} \) is Euclidean at the point \( p \) and that on \( U_p \),
\[ f(Z) = f(p) - \frac{\lambda_1}{2} Z_1^2 - \cdots - \frac{\lambda_j}{2} Z_j^2 + \frac{\lambda_{j+1}}{2} Z_{j+1}^2 + \cdots + \frac{\lambda_n}{2} Z_n^2, \] (4.16)
with \( \lambda_1, \ldots, \lambda_n \) positive. Set \( e_\alpha = \frac{\partial}{\partial Z_\alpha} \). Then \( \{e_1, \ldots, e_n\} \) forms an orthonormal basis of the vector space \( N \). The vector \( v \) in (4.6) has the form
\[ v(Z) = -\sum_{\alpha=1}^j \lambda_\alpha Z_\alpha e_\alpha + \sum_{\alpha=j+1}^n \lambda_\alpha Z_\alpha e_\alpha. \] (4.17)

Let \( e^1, \ldots, e^n \) denote the dual basis associated to \( e_1, \ldots, e_n \). The following result comes from [11, Prop.3.1]:

**Theorem 5** The kernel of the operator \((D_T^N)^2\) is of one dimension and is spanned by
\[ \beta_p = \exp\left(-\frac{T}{2} \sum_{\alpha=1}^n \lambda_\alpha Z_\alpha^2 \right) e^1 \wedge \ldots \wedge e^j. \] (4.18)

Moreover, all nonzero eigenvalues of \( D_T^2 \) are \( \geq 2T \).

If \( p \in C^j_j (f|_{\partial M}) \), by [7, (3.27)], there exists coordinates \((Z, U_p)\) around \( p \) such that the metric \( g^{TM} \) is Euclidean at the point \( p \) and that on \( U_p \),
\[ f(Z) = f(p) - \frac{\mu_1}{2} Z_1^2 - \cdots - \frac{\mu_j}{2} Z_j^2 + \frac{\mu_{j+1}}{2} Z_{j+1}^2 + \cdots + \frac{\mu_{n-1}}{2} Z_{n-1}^2 + \mu_n Z_n, \] (4.19)
with \( \mu_1, \ldots, \mu_n \) positive. Set \( e_\alpha = \frac{\partial}{\partial Z_\alpha} \). Then \( \{e_1, \ldots, e_n\} \) forms an orthonormal basis of the vector space \( N \). The vector \( v \) in (4.6) has the form
\[ v(Z) = -\sum_{\alpha=1}^j \mu_\alpha Z_\alpha e_\alpha + \sum_{\alpha=j+1}^{n-1} \mu_\alpha Z_\alpha e_\alpha + \mu_n e_n. \] (4.20)

Let \( e^1, \ldots, e^n \) denote the dual basis associated to \( e_1, \ldots, e_n \). The following result comes from [11, Prop.3.3]:

**Theorem 6** The kernel of the operator \((D_T^N)^2\) is of one dimension and is spanned by
\[ \beta_p = \exp\left(-\frac{T}{2} \sum_{\alpha=1}^{n-1} \mu_\alpha Z_\alpha^2 - T \mu_n Z_n \right) e^1 \wedge \ldots \wedge e^j. \] (4.21)

Moreover, all nonzero eigenvalues of \( D_T^2 \) are \( \geq 2T \).
4.3 A decomposition of the deformed de Rham operator

Set
\[ R_p = C^\infty(\{p\}, \mathbb{R}) \simeq \mathbb{R}, \text{ for } p \in C(f) \cup C_-(f|_M); \] (4.22)
and
\[ F_j = \bigoplus_{p \in C^j(f) \cup C^j_-(f|_M)} R_p, \quad F = \bigoplus_{j=0}^n F_j. \] (4.23)

Denote by \( e_p \) the generator of the real line \( R_p \) for each \( p \in C(f) \cup C(f|_M) \). We now equip the vector space \( F \) with a metric \( \langle \cdot, \cdot \rangle \) such that for any \( p, q \in C(f) \cup C_-(f|_M) \) with \( p \neq q \), \( e_p \) and \( e_q \) are orthonormal to each other and that
\[ \langle e_p, e_p \rangle = 1 \] (4.24)
for each \( p \in C(f) \cup C(f|_M) \). Let \( \| \cdot \|_F \) be the norm on \( F \) induced by this metric.

Let \( E^1 \) (resp. \( E^1 \)) be the set of sections of \( \Lambda^i(T^*M) \) over \( M \) (resp. \( \Lambda^i(N^*) \) on the vector space \( N \)) which lie in the first Sobolev space. Let \( \| \cdot \|_{E^1} \) (resp. \( \| \cdot \|_{E^1} \)) be a Sobolev norm on \( E \) (resp. \( E^1 \)).

For \( r \in (0, 2\varepsilon) \), let \( \gamma_r : \mathbb{R} \to [0, 1] \) be a family of smooth cut-off functions such that \( \gamma_r(x) = 1 \) if \( |x| \leq r/2 \) and that \( \gamma_r(x) = 0 \) if \( |x| \geq r \). For any \( p \in C^j(f) \), set
\[ \eta_p(x) = \gamma_r(|x|). \] (4.25)

For \( q \in C^j_-(f|_M) \), set
\[ \sigma_q(x) = \gamma_b(|x|) \text{ with } b^2 = \varepsilon^2/n. \] (4.26)
and
\[ \eta_q(x) = \sigma_q(x', x_n^2), \quad x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R_+}. \] (4.27)

Then \( \eta_q \) is a smooth cut-off function such that \( \eta_q(x) = 1 \) if \( |x| \leq b/2 \) and that \( \eta_q(x) = 0 \) if \( |x| \geq \varepsilon \). Moreover, we have
\[ \frac{\partial \eta_q}{\partial x_n}(x', 0) = 0. \] (4.28)

We can consider \( \eta_p \) (resp. \( \eta_q \)) as a smooth function defined on \( M \) with values in \( \mathbb{R_+} \), which vanishes on \( M \setminus U_p \) (resp. \( M \setminus U_q \)). For any \( p \in C^j(f) \) and \( q \in C^j_-(f|_M) \), set
\[ \alpha_{p,T} = \int_{U_p} \eta_p^2(Z) \exp\left(-T \sum_{a=1}^n \lambda_a Z_a^2 \right) dZ_1 \wedge \cdots \wedge dZ_n, \]
\[ \alpha_{q,T} = \int_{U_q} \eta_q^2(Z) \exp\left(-T \sum_{a=1}^{n-1} \mu_a Z_a^2 - 2T \mu_n Z_n \right) dZ_1 \wedge \cdots \wedge dZ_n. \] (4.29)

Clearly, there exists \( c > 0 \) such that as \( T \to +\infty \),
\[ \alpha_{p,T} = \left(\frac{\pi}{T}\right)^{\frac{n}{2}} \lambda! + O(e^{-cT}), \quad \alpha_{q,T} = \frac{1}{2T} \left(\frac{\pi}{T}\right)^{\frac{n-1}{2}} \mu! + O(e^{-cT}) \] (4.30)
with $\lambda! = \lambda_1 \cdots \lambda_n$ and $\mu! = \mu_1 \cdots \mu_n$. Set
\[
\rho_{p,T} = \frac{\eta_p(x)}{\sqrt{\alpha_p}} \exp \left( -\frac{T}{2} \sum_{a=1}^{n} \lambda_a x_a^2 \right) dx_1 \wedge \cdots \wedge dx_j,
\]
\[
\rho_{q,T} = \frac{\eta_q(x)}{\sqrt{\beta_q}} \exp \left( -\frac{T}{2} \sum_{a=1}^{n-1} \mu_a x_a^2 - T \mu_n x_n \right) dx_1 \wedge \cdots \wedge dx_j.
\]
(4.31)

Clearly,
\[
\rho_{p,T} \in \text{Dom} \left( D^2_T \right) \cap \Omega^j(M).
\]
(4.32)

By (3.9), (4.28) and (4.31), we obtain
\[
\rho_{q,T} \in \text{Dom} \left( D^2_T \right) \cap \Omega^j(M).
\]
(4.33)

Let $I_T : F \to E$ be the linear map from $F$ to $E$ by sending the generator $e_p$ to $\rho_{p,T}$ for each $p \in C(f) \cup C(f|_{\partial M})$. Set $J_T : F \to E$ by $J_T = k^{-\frac{1}{2}} I_T$. One verifies directly that the maps $I_T$ and $J_T$ are isometries. Set
\[
E^j_T = J_T(F_j), \quad E_T = \bigoplus_{j=0}^{n} E^j_T; \quad E_T = \bigoplus_{j=0}^{n} I_T(F_j).
\]
(4.34)

Clearly,
\[
\dim E_T = \sum_{j=0}^{n} (c_j + p_j).
\]
(4.35)

**Proposition 3** There exists $a_5 > 0$ and $a_6 > 0$ such that for any $s \in E_T$ and any $T \geq 1$, we have
\[
\| D_T^s \|_E \leq a_5 e^{-a_6 T} \| s \|_E, \\
\| R_T s \|_E \leq a_5 e^{-a_6 T} \| s \|_E.
\]
(4.36)

**Proof** The first inequality in (4.36) follows immediately from Theorems 5, 6 and (4.31). By integration by parts, we obtain the second inequality of (4.36). \qed

Let $E_T^\perp$ (resp. $E_T^\perpp$) as the orthogonal complement of $E_T$ in $E$ (resp. of $E_T$ in $E$). Then $E_T, E_T^\perp$ have orthogonal splittings:
\[
E = E_T \oplus E_T^\perp, \quad E_T = E_T \oplus E_T^\perpp.
\]
(4.37)

Let $p_1$ (resp. $p_{1\perp}$) denote the orthogonal projections from $E$ onto $E_T$ (resp. from $E_T$ onto $E_T^\perp$). Then (3.1), (4.2) and (4.3) imply that
\[
p_1 = k^{-\frac{1}{2}} p_{1\perp} k^{\frac{1}{2}}.
\]
(4.38)

Set $p_{1\perp} = 1 - p_1, p_{1\perp} = 1 - p_{1\perp}$.

Also we have another orthogonal splitting about $E_T$ in Dom($D_T$):
\[
\text{Dom}(D_T) = E_T \oplus F_T.
\]
(4.39)

where $F_T$ is the orthogonal complement of $E_T$ in Dom($D_T$). Then $F_T \subset E_T^\perp$. Denote by $p_2, p_{2\perp}$ the orthogonal projections from Dom($D_T$) onto $E_T$ and $F_T$, respectively.
Following Bismut-Lebeau [2, §9], we decompose the deformed de Rham operator $D_T$ according to the splittings (4.37) and (4.39):

\[
D_{T,1} = p_1 D_T p_2, \quad D_{T,2} = p_1 D_T p_2^\perp, \\
D_{T,3} = p_1^\perp D_T p_2, \quad D_{T,4} = p_1^\perp D_T p_2^\perp. 
\]

(4.40)

Then

\[
D_T = D_{T,1} + D_{T,2} + D_{T,3} + D_{T,4}.
\]

(4.41)

### 4.4 Estimates of the components of the deformed Dirac operators

Set $E_T^{1,\perp} = E_T^1 \cap E^1$. The following estimates ([14, Prop. 5.6]) still hold for the operators $D_{T,j}$’s.

**Proposition 4**

1. As $T \to +\infty$, 
   \[
   J_T^{-1} D_{T,1} J_T = O(e^{-cT}); 
   \]
   (4.42)

2. There exist positive constants $b_1$ and $b_2$ such that for any $s \in E_T^{1,\perp} \cap \text{Dom}(D_T), s' \in E_T$ and any $T \geq 1$, we have
   \[
   \|D_{T,2}s\|_E \leq b_1 e^{-b_2 T} \|s\|_E, \\
   \|D_{T,3}s'\|_E \leq b_1 e^{-b_2 T} \|s'\|_E. 
   \]
   (4.43)

**Proof** We adapt the proof from [2, §9]. Combining (4.8) and (4.38), we obtain

\[
J_T^{-1} D_{T,1} J_T = I_T^{-1} p_1 (D_T^N + R_T) I_T. 
\]

(4.44)

Then (4.42) follows from (4.36) and (4.44). To prove the second inequality in (4.43), it suffice to prove it for $s' = J_T e_p$. By (4.8) and (4.38), we get

\[
D_{T,3}s' = k^{1/2} p_1^\perp (D_T^N + R_T) \rho_{p,T}. 
\]

(4.45)

Then (4.2), (4.36) and (4.45) imply the second inequality in (4.43) for $s' = J_T e_p$. Clearly,

\[
D_{T,2}s = \sum_{p \in C(f) \cup \mathcal{C}(f|_{\partial M})} \{D_T s, k^{1/2} \rho_{p,T}\}_E \cdot k^{1/2} \rho_{p,T}, \\
= \sum_{p \in C(f) \cup \mathcal{C}(f|_{\partial M})} \{k^{1/2} s, (D_T^N + R_T) \rho_{p,T}\}_E \cdot k^{1/2} \rho_{p,T}. 
\]

(4.46)

Then the first inequality in (4.43) follows from (4.36) and (4.46). The proof of Proposition 4 is complete. \qed

For the component $D_{T,4}$, we first establish the following analogue of [2, Prop. 9.12]. Note that the spaces $E_T, E_T^\perp$ implicitly depend on $\varepsilon$.

**Lemma 1** There exists $\varepsilon \in (0, \frac{\varepsilon_0}{4}), b_3 > 0$ and $T_4 > 0$, such that for any $s \in E_T^{1,\perp} \cap \text{Dom}(D_T)$ with $\text{supp}(s) \subset U_2$ and any $T \geq T_4$, we have

\[
\|D_T s\|_E \geq b_3 T \|s\|_E. 
\]

(4.47)
Proof We adapt our proof from [2, pp. 109–115]. Define \( s \in E^1 \) by \( s' = k^\frac{1}{2} s \). By (4.38), \( p_1 s' = 0 \). By (4.8), we find
\[
\| DT s \|_E^2 = \| k^\frac{1}{2} DT k^{-\frac{1}{2}} s' \|_E^2 \geq \frac{1}{2} \| DT_N s' \|_E^2 - \| RT s' \|_E^2. 
\]
(4.48)

Denote by \( E'_T \) the image of \( F \) in \( E \) by the linear map \( e_p \in F \to \beta_p \in E \). Let \( p'_1 \) be the orthogonal projection operator from \( E \) onto \( E'_T \). By Theorems 5 and 6, we find
\[
\| DT_N s' \|_E^2 = \| DT_N (s' - p'_1 s') \|_E^2 \geq 2T \| s' - p'_1 s' \|_E^2. 
\]
(4.49)

Since \( p_1 s' = 0 \), we have
\[
p'_1 s' = \sum_{p \in C(f) \cup C(f | h, M)} c^{-1}_p \langle s', (1 - \rho) \beta_p \rangle_E \cdot \beta_p 
\]
(4.50)

with \( c_p = \| \beta_p \|_E^2 \). It follows from (4.50) that there exist \( C_1 > 0 \) satisfying
\[
\| p_1 s' \|_E^2 \leq \frac{C_1}{\sqrt{T}} \| s' \|_E^2. 
\]
(4.51)

Substituting (4.51) into (4.49), we obtain
\[
\| DT_N s' \|_E^2 \geq \left( T - \frac{2C_1}{\sqrt{T}} \right) \| s' \|_E^2. 
\]
(4.52)

We denote by \( \Delta^N \) the flat Laplacian operator on the vector space. Clearly,
\[
(DT_N^2) = \Delta^N + T \left( DT_N, \hat{c} (v) \right) + T^2 |v|^2. 
\]
(4.53)

Then there exists \( C_2 > 0 \) such that for any \( \alpha \in (0, 1) \),
\[
\| DT_N s' \|_E^2 \geq \alpha \| \Delta^N s', s' \|_E - C_2 \alpha T \| s' \|_E^2. 
\]
(4.54)

We now fix \( \alpha \in (0, 1) \) such that \( C_2 \alpha \leq \frac{1}{2} \). Combining (4.52) and (4.54), we get
\[
\| DT_N s' \|_E^2 \geq \frac{\alpha}{2} \| \Delta^N s', s' \|_E + \left( \frac{T}{2} - \frac{2C_0}{\sqrt{T}} \right) \| s' \|_E^2. 
\]
(4.55)

By elliptic estimates, there exists \( C_3 > 0, C_4 > 0 \) such that
\[
\frac{\alpha}{2} \| \Delta^N s', s' \|_E \geq C_3 \| s' \|_E^2 - C_4 \| s' \|_E^2. 
\]
(4.56)

Substituting (4.56) into (4.55), we get
\[
\| DT_N s' \|_E^2 \geq C_3 \| s' \|_E^2 + \left( \frac{T}{2} - \frac{2C_1}{\sqrt{T}} - C_4 \right) \| s' \|_E^2. 
\]
(4.57)

Clearly,
\[
\| RT s' \|_E^2 \leq \left( T \varepsilon^6 + \varepsilon^2 \right) \| s' \|_E^2 + \varepsilon^2 \| s' \|_E^2. 
\]
(4.58)

Substituting (4.57) and (4.58) into (4.52), we find
\[
\| DT_N s' \|_E^2 \geq \left( \frac{C_3}{2} - \varepsilon^2 \right) \| s' \|_E^2 + \left( \frac{1}{4} T - T \varepsilon^6 - \frac{C_1}{\sqrt{T}} - \frac{C_4}{2} - \varepsilon^2 \right) \| s' \|_E^2. 
\]
(4.59)

Then (4.47) follows immediately from (4.59). The proof of Lemma 1 is complete. \( \square \)
We now fix $\varepsilon \in (0, \varepsilon_0]$ as in Lemma 1. Then we have the following analogue of [2, Prop.9.13].

**Lemma 2** There exist $b_4 > 0$ and $T_5 > 0$, such that for any $s \in E^1 \cap \text{Dom}(D_T)$ which vanishes outside $U_{\varepsilon}$ and any $T \geq T_5$, we have
\[
\|D_T s\|_E \geq b_4 T \|s\|_E.
\]  
(4.60)

**Proof** The readers are referred to [2, Prop. 9.13] for the proof. \qed

By Lemmas 1–2 and an argument of partition of unity, the following analogue of [2, Th. 9.11] holds (cf. [2, pp. 115–117]).

**Proposition 5** There exist $\varepsilon \in (0, \varepsilon_0]$, $b_5 > 0$ and $T_6 > 0$, such that for any $s \in E^{1,\perp}_T \cap \text{Dom}(D_T)$ and any $T \geq T_6$, we have
\[
\|D_T s\|^2_E \geq b_5 T \|s\|^2_E.
\]  
(4.61)

From Proposition 5, we obtain the following analogue of [2, Th.9.14].

**Proposition 6** There exist $b_6 > 0$, $T_7 > 0$, such that for any $T \geq T_7$, $s \in E^{1,\perp}_T$, then
\[
\|D_{T,4s}\|^2_E \geq b_6 T \|s\|^2_E.
\]  
(4.62)

### 5 Proof of our main Theorem

#### 5.1 Proof of Theorems 1 and 2

Using (4.42), (4.43), and the min–max principle [12, (C.3.3)], we obtain (1.3). By (4.43), (4.62) and the min–max principle, we get (1.2). The proof of Theorems 1 and 2 is complete.

#### 5.2 Proof of Proposition 1

Note that Propositions 4 and 6 play the same role as [11, Prop.5.4] in [11]. By a routine process as in [11], we finish the proof of Proposition 1.

#### 5.3 Proof of Theorem 3

By Proposition 3, we can carry on nearly word by word the proof of [11, Th. 1.3] to complete the proof of Theorem 3.

**Acknowledgements** The author would like to thank Professor Xiaonan Ma for his kind advices.

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