The Coleman-Weinberg Mechanism and First Order Phase Transitions

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Abstract

We study the Coleman-Weinberg phenomenon in a $U(N) \times U(N)$ symmetric scalar model in 4 dimensions. By comparing the numerical simulation results with the bare perturbation calculation in the weak bare coupling region, we demonstrate explicitly that a first order transition is induced by loop-effects. We also observed first order phase transitions in the strong coupling region.

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It is well known that a system with more than one relevant coupling can have a first order phase transition induced by quantum loop fluctuations – a phenomenon first discovered by Coleman and Weinberg [1, 2, 3]. The application of this Coleman-Weinberg mechanism led to a lower bound of 7-10 GeV on the Higgs mass in the Standard Model [4, 5] and to the suggestion that the QCD finite temperature chiral restoration phase transition might be first order for more than two quark flavor [6]. It has also been suggested that the Coleman-Weinberg mechanism might be relevant for chiral symmetry breaking in technicolor models [7]. Recently [8] it was pointed out that the Coleman-Weinberg mechanism might be used to place constraints on the top condensate model or strongly coupled extended technicolor models.

Let us consider the action (in Euclidean form)

$$S_\phi = \int d^4x \left\{ \frac{1}{2} tr \left( \partial_\mu \phi^\dagger \partial_\mu \phi \right) + \frac{m^2}{2} tr \left( \phi^\dagger \phi \right) + \lambda_1 \left( tr \phi^\dagger \phi \right)^2 + \lambda_2 tr \left( \phi^\dagger \phi \right)^2 \right\} ,$$

(1)

for a scalar field $\phi$, where $\phi$ is a $N \times N$ complex matrix. Such a model can be considered as a low energy effective action of a top condensate model or strongly coupled extended technicolor model (For simplicity, we ignore the fermions). It has $U(N) \times U(N)$ global symmetry under transformation: $\phi \to U \phi V^\dagger$, where $U$ and $V$ are $N \times N$ unitary matrices. The dynamical scale $\Lambda$ of a top condensate model or strongly coupled extended technicolor model provides a natural cut-off. At energies much lower than $\Lambda$ the $U(N) \times U(N)$ symmetry is spontaneously broken to $U(N)$. The vacuum expectation value (VEV) $v$ associated with this symmetry breaking is supposed to be responsible for generating the masses of the $W^\pm$ and $Z$. For the model in Eq. (1) to be a good low energy effective theory, there must be a large hierarchy between the VEV scale $v$ and the cut-off scale $\Lambda$.

However, this model is expected to have first order phase transitions because of the Coleman-Weinberg mechanism [4, 5]. As $m^2$ is tuned through the critical point, the VEV jumps from zero in the symmetric phase to a finite nonzero value in the broken phase. In certain portions of parameter space the transition may become strongly first order such that the condition $v/\Lambda \ll 1$ can no longer be maintained. Therefore this part of the parameter space should be excluded from the low energy effective model Eq. (1). It is hoped that this in turn will constrain the parameter space of the top condensate model or strongly-coupled extended technicolor model [8].

For the convenience of later discussions we first reproduce a geometric description of the Coleman-Weinberg mechanism developed by Yamagishi [10], which combines the effective potential approach originally used by Coleman and Weinberg and later developments in terms of renormalization group (RG) flows and fixed point structure in parameter space [3, 7, 9].

In the parameter region $\lambda_2 > 0$, spontaneous symmetry breaking happens along the diagonal of matrix $\phi$ with a minimum at $\phi_{ij} = \pm v \delta_{ij}$. The tree level potential in a zero-mass theory can be written as

$$U(\varphi) = N(N\lambda_1 + \lambda_2)\varphi^4 ,$$

(2)

where $\varphi = \sqrt{tr(\phi^\dagger \phi)}/N$. Away from $\varphi = 0$ quantum fluctuation will contribute. However,
according to the solution of the RG equation for $U(\varphi)$, the loop corrections can be included by letting $\lambda_1, \lambda_2$ run with $\varphi$ while keeping the form of $U(\varphi)$ in Eq. (2) unchanged (here we have accounted the fact that the anomalous dimension, $\eta$, is zero in leading order for this model). If we define $t = \ln(\varphi/\mu)$ with $\mu$ a renormalization scale where $\lambda_1, \lambda_2$ are defined, the running of $\lambda_1(t), \lambda_2(t)$ will be governed by $\beta$ functions

$$
\frac{d\lambda_1}{dt} = \beta_1(\lambda_1, \lambda_2), \quad \frac{d\lambda_2}{dt} = \beta_2(\lambda_1, \lambda_2)
$$

with initial conditions $\lambda_1(0) = \lambda_1, \lambda_2(0) = \lambda_2$. A local minimum will develop and the Coleman-Weinberg phenomenon will be found if the running of $\lambda_1, \lambda_2$ crosses the “stability line”

$$
4(N\lambda_1 + \lambda_2) + N\beta_1 + \beta_2 = 0,
$$

in a region where

$$
N\lambda_1 + \lambda_2 > 0, \quad 4(N\beta_1 + \beta_2) + N\sum_{i=1}^{2} \beta_i \frac{\partial \beta_1}{\partial \lambda_i} + \sum_{i=1}^{2} \beta_i \frac{\partial \beta_2}{\partial \lambda_i} > 0.
$$

In the region $\lambda_2 < 0$ the symmetry breaking happens along one diagonal element $\phi_{ij} = \pm v\delta_{1i}\delta_{1j}$. In this case we write the tree level potential as

$$
U(\varphi) = (\lambda_1 + \lambda_2)\varphi^4,
$$

with $\varphi = \sqrt{\text{tr}(\varphi^\dagger\varphi)}$. Again the loop corrections can be included in the running of $\lambda_1, \lambda_2$. The Coleman-Weinberg phenomenon will occur if the running of $\lambda_1, \lambda_2$ crosses the “stability line”

$$
4(\lambda_1 + \lambda_2) + \beta_1 + \beta_2 = 0,
$$

in a region where

$$
\lambda_1 + \lambda_2 > 0, \quad 4(\beta_1 + \beta_2) + \sum_{i,j=1}^{2} \beta_i \frac{\partial \beta_j}{\partial \lambda_i} > 0.
$$

Using the one-loop perturbative $\beta$-functions

$$
\pi^2 \frac{d\lambda_1}{dt} = (N^2 + 4)\lambda_1^2 + 4N\lambda_1\lambda_2 + 3\lambda_2^2,
$$

$$
\pi^2 \frac{d\lambda_2}{dt} = 6\lambda_1\lambda_2 + 2N\lambda_2^2,
$$

the RG flow of $\lambda_1, \lambda_2$ is plotted in Figure 1 for $N = 2$. Notice that the arrows in Figure 1 indicate the direction of decreasing $\varphi$ (i.e. the infrared limit of the theory). The “stability lines” are indicated by the dotted lines. They are entirely inside the regions that satisfy the conditions of Eqs. (3) and Eqs. (8). Thus whenever the flow lines cross the “stability line” one expects the Coleman-Weinberg phenomenon. In the language of the RG flow the

\footnote{In Figure 1, the perturbative “stability lines” terminate in the strong coupling region (solid circles). This may merely mean the breaking down of perturbation theory. Indeed, we find in numerical simulation a first order phase transition at $\lambda_1 = -19.0, \lambda_2 = 40.0$.}
Coleman-Weinberg phenomenon occurs due to the absence of an infrared stable fixed point. The system is driven to a runaway RG trajectory in the infrared limit and the vacuum at $\varphi = 0$ becomes unstable [3, 4, 5].

The above description is simple and elegant. However, the $\beta$-functions and consequently the RG flow in Figure 1 was calculated in the leading order renormalized perturbation theory. In principle it is possible that there might exist a nonperturbative infrared stable fixed point and the phase transition could become second order inside its attractive domain. For the purpose of Ref [8], this is an important question. If one stays around $\lambda_1 = \lambda_2 = 0$ the couplings run very slowly. It takes many decades to reach the VEV scale from the cut-off scale. Then the phase transition becomes practically second order and there can be a large hierarchy between $\Lambda$ and the VEV. It is only in the region where $\lambda_1$ is small and $\lambda_2$ is relatively large that the RG flow becomes fast and a small $\Lambda/v$ ratio becomes likely. This has been used [8] as an argument to exclude the large $\lambda_2$ region from the model in Eq. (1). However, should there be an infrared stable fixed point in the large parameter region, the model in Eq. (1) would have practically no constraints on its parameter values (besides the obvious condition $N\lambda_1 + \lambda_2 > 0$ for the bare action to be stable). Therefore, it was suggested [8] that a numerical simulation is needed to look for first order phase transitions in the strong coupling region.

Interest in numerical studies of the Coleman-Weinberg phenomenon has also been raised in a recent work of March-Russell [11]. He considered gauge-Higgs systems in 3 dimensions which are thought to have first order phase transitions due to the Coleman-Weinberg mechanism. He argued that the conventional picture based on the $d = 4 - \epsilon$ expansion result is unreliable for $\epsilon = 1$ and the phase transition for such systems in 3 dimensions could be second order[4]. The model in Eq. (1) can serve as a testing ground for this conjecture [4].

In 4 dimensions there is no problem associated with the $\epsilon$-expansion and the Coleman-Weinberg phenomenon is expected to exist at least in the weak coupling region. However, clear numerical evidence for such phenomenon has not been obtained so far. In the following, we report our first results in the numerical investigation of the model in Eq. (1) in 4 dimensions.

For the numerical simulations we set $N = 2$ in Eq. (1). The simple Metropolis algorithm is used with a uniform step size $\Delta \phi_{ij} = \Delta \phi$ which is tuned such that the acceptance rate is around 60%. Since there can be no symmetry breaking in a finite volume it is important to choose an order parameter invariant under $U(N) \times U(N)$. For a lattice with linear size $L$, we define

$$\bar{\phi}_{ij} = \frac{1}{L^4} \sum_x \phi_{ij}(x),$$

(10)

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3Numerical simulations of a spin model [12] and $SU(N), U(N)$ nonlinear sigma models [13] in 3 dimensions found evidences of first order phase transitions. These works seem to favor the $\epsilon$-expansion predictions.

4A Monte-Carlo simulation had been performed before for the model in Eq. (1) in 3 dimensions [14]. Unfortunately the result of Ref. [14] was wrong. The point shown to have two coexisting phases in that paper (Figure 2) actually is deep in the broken phase. Our investigation of this model in 3 dimensions found first order phase transitions and the results will be reported elsewhere.
and use $\langle tr(\bar{\phi}^i \bar{\phi}) \rangle$ as the order parameter. This order parameter is a finite number in the broken phase and vanishes like $O(1/L^4)$ in the symmetric phase.

For this exploratory investigation most simulations were done on relatively small lattices $(4^4, 6^4$ and $10^4$). To decide the order of phase transitions we looked for the hysteresis effects in the thermocycles: we perform a series of runs back and forth across the critical region. Each run uses the last configuration of the previous run as its initial configuration. In case of a first order phase transition, supercooling or superheating will produce the hysteresis effects. In addition, we looked at the histogram for the distribution of $tr(\bar{\phi}^i \bar{\phi})$. Since the histogram distribution is proportional to $\exp\{-L^4U_L(\bar{\phi})\}$, where $U_L(\bar{\phi})$ is the effective potential, one should observe a double peak structure throughout the critical region in a first order phase transition. There are more elaborate techniques [15] to determine the order of phase transitions which will not be used here.

We choose the parameters $\lambda_1$ and $\lambda_2$ such that the bare lattice action is absolutely stable

$$\lambda_2 > 0, \quad N\lambda_1 + \lambda_2 > 0.$$  \hspace{1cm} (11)

According to the RG flow in Figure 1, $\lambda_1$ and $\lambda_2$ run very slowly in the weak coupling region. Qualitatively, a weak coupling bare action fixed to the right of the “stability line” in Figure 1, will need many decades of running in energy scale to get a renormalized theory to cross the “stability line”. The running becomes fast in the strong $\lambda_2$ region. On the lattice the running is controlled by the correlation length $\xi$ with $t \approx \ln \xi$, which is limited by the lattice size $\xi \leq L$. Therefore, to observe the Coleman-Weinberg phenomenon one either has to perform simulations on a large enough lattice 5 or has to move the bare theory very close to the “stability line” in order to get a renormalized theory to cross the “stability line”.

Our numerical results in the weak coupling region are shown in Figures 2 and 3 where the bare parameters are chosen to be very close to the “stability line” Eq. (11). The phase transition is clearly first order at $\lambda_1 = -0.22, \lambda_2 = 0.5$. Figure 2 shows thermocycles for the $4^4, 6^4$ and $10^4$ lattices. For the $4^4$ lattice each point in the thermocycle has 20000 sweeps as warm-up and 60000 sweeps in the measurement. For the $6^4$ and $10^4$ lattices they are 3000, 10000 and 400, 1200, respectively. Away from the critical point the finite size effects are small. They become larger close to the critical point: the hysteresis effect is barely visible on the $4^4$ lattice and becomes stronger on larger lattices as shown in Figure 2. Figure 3 shows the tunnelings between the broken and symmetric phases on the $4^4$ lattice at the “critical point”.

In the weak coupling region one may calculate the order parameter $\langle tr(\bar{\phi}^i \bar{\phi}) \rangle$ in the bare perturbation expansion and compare with the numerical results. In particular, for $\lambda_2 > 0$ the effective potential $U_L(\bar{\phi})$ has a global minimum at $\bar{\phi}_{ij} = \pm \nu \delta_{ij}$. If we define $\varphi \equiv \sqrt{tr(\bar{\phi}^i \bar{\phi})}/N$, the one-loop effective potential is given by

$$U(\varphi) = N \left( \frac{m^2}{2} \varphi^2 + (N\lambda_1 + \lambda_2)\varphi^4 \right) + \frac{1}{2L^d} \sum_{p \neq 0} \ln \left[ \hat{p}^2 + m^2 + 12(N\lambda_1 + \lambda_2)\varphi^2 \right]$$  \hspace{1cm} (12)

5We believe this might be the physical reason for the phenomenon observed in Ref. [13].
\[ + \frac{N^2 - 1}{2L^d} \sum_{p \neq 0} \ln \left[ \hat{p}^2 + m^2 + 4(N\lambda_1 + 3\lambda_2)\varphi^2 \right] + \frac{N^2}{2L^d} \sum_{p \neq 0} \ln \left[ \hat{p}^2 + m^2 + 4(N\lambda_1 + \lambda_2)\varphi^2 \right] , \]

where \( d = 4 \) and \( \hat{p}^2 = \sum_{\mu} \sin^2 p_\mu \). In the infinite volume limit the lattice sum should be replaced by a momentum space integral \( \frac{1}{L^d} \sum_{p \neq 0} \rightarrow \int \frac{d^dp}{(2\pi)^d} \).

At tree-level the effective potential Eq. (12) appears to indicate a second order phase transition at \( m^2 = 0 \) for arbitrary values of \( \lambda_1, \lambda_2 \) (provided \( N\lambda_1 + \lambda_2 > 0 \)). This is, however, changed at one-loop. For example, if we plot the one-loop effective potential at \( \lambda_1 = -0.22, \lambda_2 = 0.5 \) for various \( m^2 \) values. In contrast to the tree-level picture where there is always a single minimum, we will have two minima at \( \varphi = 0 \) and \( \varphi = v > 0 \). As \( m^2 \) is tuned from \(-1.0\) to \(-1.1\) the second minimum at \( \varphi > 0 \) changes from a local minimum to a global minimum: a picture ordinarily found at a first order phase transition.

The dotted line in Figure 2 indicates \( \langle tr(\bar{\phi}^\dagger \phi) \rangle \) as calculated in the bare perturbation theory. The agreement with the numerical result is reasonably well considering that \( \lambda_2 = 0.5 \) is not exactly weak coupling with our normalization. Numerically we found that the bare perturbation theory works within 20\% up to \( \lambda_1, \lambda_2 \approx 1 \).

Since very often first order phase transition happens at a finite correlation length where the cut-off effects can not be neglected (known as “no continuum limit”), one may wonder why the description of the Coleman-Weinberg phenomenon in the renormalized language could be qualitatively correct. The answer lies in the fact that the Coleman-Weinberg phenomenon happens in the infrared. Given a renormalized theory \( \lambda_{1R}, \lambda_{2R} \) fixed to the left of the “stability line” in Figure 1, one can always arrange (at least in the small coupling region) the bare theory sufficiently far away from the “stability line” along a trajectory such that it takes many decades in energy scale for the bare theory to “run” into the renormalized theory. In such arrangement the cut-off effects can be made small (the phase transition becomes weakly first order) and the use of the renormalized theory language becomes valid.

Of course, the cut-off effects can not be neglected in a parameter region where the phase transition is strongly first order. As pointed out in Ref. [8], such region must be excluded from the continuum theory because the renormalized theory is no longer meaningful.

In the strong coupling region we also found first order phase transitions. For example, in Figure 4 we show our simulation results at \( \lambda_1 = 0, \lambda_2 = 10 \) on the \( 10^4 \) lattice. The lower and upper curve correspond to Monte Carlo runs with disordered and ordered initial conditions. The fact that we are able to observe a first order phase transition here on a relatively small lattice conforms to the qualitative feature of the RG: although the point \( \lambda_1 = 0, \lambda_2 = 10 \) appears to be far away from the “stability line” in Figure 1, the running is fast due to the strong \( \lambda_2 \) coupling.

In conclusion, by comparing the numerical simulation results with the one-loop bare perturbation calculation in the weak coupling region, we have shown explicitly that in \( d = 4 \) this is similar in spirit to the argument for the triviality bound on the Higgs mass [16]. To actually map out the excluded region in the \((\lambda_1, \lambda_2)\) parameter space requires detailed work and is under investigation.
quantum fluctuations can induce a first order phase transition—commonly known as the Coleman-Weinberg phenomenon [1]. We also found first order phase transitions in the strong $\lambda_2$ coupling region. This might be taken as an indication that there is no infrared stable fixed point in the nonperturbative region. However, a systematic search (which is under investigation) is needed in order to confirm this point.

The numerical work to determine the complete phase diagram in both weak and strong coupling regions will be reported elsewhere [17].

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Figure Captions

Figure 1: Solid lines indicate the RG flow in $\lambda_1, \lambda_2$ parameter space. The arrows point to the infrared direction. The dotted lines are the “stability lines” calculated according to Eqs. (4) and (7) using the one-loop $\beta$-functions. They stop at end points (solid circle).

Figure 2: $\langle tr(\bar{\phi}^\dagger \phi) \rangle$ as a function of $m^2$ at $\lambda_1 = -0.22, \lambda_2 = 0.5$. The open circles indicate data on the $4^4$ lattice and open squares and open triangles are for $6^4$ and $10^4$ lattice respectively. Unless explicitly shown the statistical errors for data points are smaller than the size of symbols. The solid lines connect data points from $6^4$ and $10^4$ lattices to indicate the thermocycles. The dotted line gives the bare perturbation prediction.

Figure 3: Time history shows tunneling between two states at $\lambda_1 = -0.22, \lambda_2 = 0.5, m^2 = -0.875$. Data is obtained on the $4^4$ lattice. Tunneling becomes rare on larger lattices.

Figure 4: Time history shows two-states signal on the $10^4$ lattice at $\lambda_1 = 0, \lambda_2 = 10, m^2 = -29.625$. The lower and upper curve correspond to runs with disordered and ordered initial conditions.