Abstract. Let $H$ be a compact $p$-adic analytic group without torsion element, whose Lie algebra is split semisimple and $\mathfrak{N}_H(G)$ be the full subcategory of the category of finitely generated modules over the Iwasawa algebra $\Lambda_G$ that are also finitely generated as $\Lambda_H$-modules, where $G = \mathbb{Z}_p \times H$. We show that if the class of a module $N$ in the Grothendieck group of $\mathfrak{N}_H(G)$ equals to the class of a completely faithful module, then $q(N)$ is also completely faithful, where $q(N)$ denotes the image of $N$ via the quotient functor. We also generalize Theorem 1.3 of Konstantin Ardakov in [1].

1. Introduction

In [6] the authors state the $GL_2$ main conjectures for elliptic curves over $\mathbb{Q}$. For such an elliptic curve set $F_\infty = \mathbb{Q}(E[p^{\infty}])$, i.e. the extension of $\mathbb{Q}$ with the coordinates of the $p$-division points of $E$. By Weil pairing, $\mathbb{Q}(\mu_{p^{\infty}}) \subset F_\infty$, hence $F_\infty$ contains $\mathbb{Q}^{cyc}$. Now set $G = \text{Gal}(F_\infty/\mathbb{Q})$ and $H = \text{Gal}(F_\infty/\mathbb{Q}^{cyc})$, we have that $\mathbb{Z} = G/H \simeq \mathbb{Z}_p$. One is particularly interested in the Pontrjagin-dual of the Selmer group of $E$, i.e. $X(E/F_\infty) = \text{Hom}(\text{Sel}(E/F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)$. There is an important left and right Ore set in $\Lambda_G$, $S^*$ and one can define the category of finitely generated $S^*$-torsion $\Lambda_G$-modules $\mathfrak{M}_H(G)$ (for details see [6] Chapter 2).The first conjecture in [6] states that under suitable assumptions, $X(E/F_\infty)$ is an object of $\mathfrak{M}_H(G)$. It also can be shown that a module $M \in \mathfrak{M}_H(G)$ if and only if $M/M(p)$ is finitely generated over $\Lambda_H$, where $M(p)$ denotes the $p$-primary submodule of $M$. Now we see that the category $\mathfrak{N}_G(H)$ contains all the quotient modules of the form $M/M(p)$, so it gives us a functor from the category $\mathfrak{M}_H(G)$ to $\mathfrak{M}_H(G)$. Hence we have a map $\varphi : K_0(\mathfrak{M}_H(G)) \to K_0(\mathfrak{N}_H(G))$.

If we assume that $G$ has no element of order $p$, then mainly as a consequence of Quillen’s theorem on localization sequence, we have a surjective map

$$\partial_G : K_1((\Lambda_G)_{S^*}) \to K_0(\mathfrak{M}_H(G)).$$

One can define the characteristic element of a module $M$ as an inverse image of $[M]$ with respect to this surjective map, so if the conjecture stated above is true, we can define the characteristic element of the dual of the Selmer group of $E$. The main conjecture states that if other conjectures hold (one is what we already mentioned and the other is that the $p$-adic $L$-function, $L_E$ exists in $K_1((\Lambda_G)_{S^*})$), then $L_E$ is a characteristic element of the dual, i.e. of $X(E/F_\infty)$.

Knowing that completely faithfulness is $K_0$-invariant in $K_0(\mathfrak{M}_H(G))$ in the sense of above can bring us closer to determine completely faithfulness via the characteristic element in $K_1(\Lambda(G)_{S^*})$. In [4] the authors find examples where the dual of the Selmer group is
completely faithful. The question whether or not this is true for any elliptic curve was raised in [5]. As a side effect of our investigation we give a slight generalization of a theorem of Konstantin Ardakov (see [1] Theorem 1.3), namely instead of \( G = \mathbb{Z} \times H \) where \( H \) is an open subgroup without torsion element and its Lie-algebra is split, semisimple we let its Lie-algebra be the direct product of a split, semisimple and an abelian Lie-algebra.

2. Preliminaries

2.1. Iwasawa algebras. Let \( p \) be a prime number. We will work with modules over Iwasawa algebras

\[ \Lambda_G := \varprojlim_{N \triangleleft_o G} \mathbb{Z}_p[G/N] \]

where \( G \) is a compact \( p \)-adic analytic group.

The Iwasawa theory for elliptic curves in arithmetic geometry provides the main motivation for the study of Iwasawa algebras, for example when \( G \) is a certain subgroup of the \( p \)-adic analytic group \( \text{GL}_2(\mathbb{Z}_p) \). Homological and ring-theoretic properties of these Iwasawa algebras are useful for understanding the structure of the Pontryagin dual of Selmer groups and other modules over the Iwasawa algebras. For more information, see for example [3] or [5].

2.2. Fractional ideals and prime c-ideals.

2.2.1. Fraction Ideals. Let \( R \) be a Noetherian domain, then it is well-known that \( R \) has a skewfield of fractions, \( Q(R) \). Recall that a right \( R \)-submodule \( I \) of \( Q(R) \) is called fractional right \( R \)-ideal if it is non-zero and there is a \( q \in Q(R) \), such that \( q \neq 0 \) and \( I \subseteq qR \). One can define fractional left \( R \)-ideals similarly. When it is obvious what ring we mean, we just call it a fractional right or left ideal. If we have a fractional right ideal \( I \), one can define its inverse

\[ I^{-1} := \{ q \in Q(R) \mid qI \subseteq R \} \]

Of course we can define the inverse for fractional left ideals. Let us consider the dual of \( I \), i.e. \( I^* = \text{Hom}_R(I, R) \). This is a left \( R \)-module and there is a natural isomorphism \( u : I^{-1} \to I^* \) that sends an element \( i \in I^{-1} \) to the homomorphism induced by left multiplication by \( i \). The following lemma is useful to compute \( I^{-1} \).

**Lemma 2.2.1.** Let \( R \) be a Noetherian domain and \( I \) be a non-zero right ideal of \( R \). Then \( I^{-1}/R \simeq \text{Ext}^1(R/I, R) \).

2.2.2. Prime c-ideals. Let \( I \) be a fractional right ideal. The reflexive closure of \( I \) is \( \overline{I} := (I^{-1})^{-1} \). This is also a fractional right ideal and it contains \( I \). \( I \) is called reflexive if it is the same as its reflexive closure, i.e. \( I = \overline{I} \). One can say equivalently that \( I \to (I^*)^* \) is an isomorphism. The next proposition will be quite useful.

**Proposition 2.2.2.** Let \( R \to S \) be a ring extension such that \( R \) is noetherian and \( S \) is flat as a left and right \( R \)-module. Then there is a natural isomorphism

\[ \psi^i_M : S \otimes_R \text{Ext}_R^i(M, R) \to \text{Ext}_S^i(M \otimes_R S, S) \]

for all finitely generated right \( R \)-modules and all \( i \geq 0 \). A similar statement holds for left \( R \)-modules. If moreover \( S \) is a noetherian domain, then

(i) \( \overline{I} \cdot S = \overline{I} \cdot S \) for all right ideals \( I \) of \( R \),
(ii) if $J$ is a reflexive right $S$-ideal, then $I \cap R$ is a reflexive right $R$-ideal.

**Proof.** We omit the proof of this proposition. It can be found in [2] Proposition 1.2. □

Another important result is the following

**Proposition 2.2.3.** Let $R$ be a noetherian domain and $I$ be a proper $c$-ideal of $R$. If there is an element $x \in R$ such that $x$ is non-zero, central in $R$, $R/x \cdot R$ is a domain and $x \in I$ then $I = x \cdot R$.

**Proof.** For proof see for example [1] Lemma 2.2. □

2.3. **Pseudo-null modules.** Let $R$ be an arbitrary ring and $M$ be an $R$-module. Recall from [2] (section 1.3) that $M$ is called pseudo-null if $\text{Ext}^0_R(N, R) = \text{Ext}^1_R(N, R) = 0$ for any submodule $N \subseteq M$. The category of pseudo-null modules $\mathcal{C}$ is a full subcategory of $\text{mod}(R)$, i.e. the category of finitely generated right $R$-modules and is "localizing" in the sense that it is a Serre subcategory, i.e. if $0 \to A_1 \to A \to A_2 \to 0$ is a short exact sequence of right $R$-modules, then $A$ lies in $\mathcal{C}$ if and only if $A_1$ and $A_2$ lie in $\mathcal{C}$ and any $R$-module has a largest unique submodule contained in $\mathcal{C}$. Let $R$ be a noetherian domain. Since $\mathcal{C}$ is a Serre subcategory we can consider the quotient category and the quotient functor$$q: \text{mod}(R) \to \text{mod}(R)/\mathcal{C}.$$

2.4. **Completely faithful modules.** We recall from [2] what a completely faithful module is. One can consider the annihilator ideal of an object $M$ in the quotient category

$$\text{Ann}(M) := \sum \{\text{Ann}_R(N) \mid q(N) \cong M\}$$

and $M$ is said to be completely faithful if $\text{Ann}(L) = 0$ for any non-zero subquotient $L$ of $M$. Now let $G = H \times Z$ where $Z = \mathbb{Z}_p$ and $H$ a compact $p$-adic analytic group without torsion element, whose Lie algebra is split semisimple. Let $\text{mod}(\Lambda_G)$ be the category of finitely generated left $\Lambda_G$-modules and consider $\mathfrak{N}_H(G)$, the full subcategory of all left $\Lambda_G$-modules that are finitely generated as $\Lambda_H$-modules.

Note that if $M \in \mathfrak{N}_H(G)$ then it has no pseudo-null submodules if and only if it has no $\Lambda_H$ torsion element. Recall [1] Theorem 1.3 for it is of particular importance:

**Theorem 2.4.1.** Let $p \geq 5$ and $G$ as above. If $M$ has no non-zero pseudo-null submodules, then $q(M)$ is completely faithful if and only if $M$ is torsion-free over $\Lambda_Z$.

2.5. **The Grothendieck group of $\mathfrak{N}_H(G)$**. Recall the definition of the Grothendieck group $K_0(\mathcal{A})$ of a skeletally small abelian category from [3] (Definition 6.1.1). However we will have a module category $\mathfrak{N}_H(G)$ in which case $K_0(\mathfrak{N}_H(G))$ is the abelian group presented as having one generator $[A]$ for each isomorphism class of modules and one relation for every short exact sequence, i.e. $[A_2] = [A_1] + [A_3]$ in $K_0(\mathcal{A})$ whenever we have a short exact sequence of the form

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

where $A_1, A_2, A_3 \in \mathfrak{N}_H(G)$. We will use a useful fact in $K$-theory that basically tells us how many relations we have whenever two modules have the same class in $K_0(\mathfrak{N}_H(G))$. 
However we will state a more general version. For details see [9] (Ex. 6.4.)

**Lemma 2.5.1.** Let \( A \) be a small abelian category. If \([A_1] = [A_2]\) in \( K_0(A)\) then there are short exact sequences in \( A \)

\[
0 \rightarrow C \rightarrow K \rightarrow D \rightarrow 0
\]

\[
0 \rightarrow C \rightarrow L \rightarrow D \rightarrow 0
\]

such that \( A_1 \oplus K = A_2 \oplus L \).

2.6. Completely faithfulness and \( K_0 \) invariance.

**Theorem 2.6.1.** Let \( p \geq 5 \). Let \( H \) be a torsion-free compact \( p \)-adic analytic group whose Lie algebra is split semisimple over \( \mathbb{Q}_p \) and let \( G = \mathbb{Z}_p \times H \). Let \( M, N \in \mathfrak{N}_H(G) \) and let \( q(M) \) be completely faithful in the sense of 2.3. If \([M] = [N]\) in \( K_0(\mathfrak{N}_H(G))\) then \( q(N) \) is also completely faithful.

**Remark 2.6.2.** One can think of \( G \) as \( \Gamma_1 \) which is the first inertia subgroup of \( \text{GL}_n(\mathbb{Z}_p) \) i.e.

\[
\Gamma_1 = \{ \gamma \in \text{GL}_n(\mathbb{Z}_p) | \gamma \equiv 1 \mod (p) \}
\]

In this case \( G = Z \times H \) where \( Z \cong \mathbb{Z}_p \) is the centre of \( G \) and \( H \) is an open subgroup of \( \text{SL}_n(\mathbb{Z}_p) \) that is normal in \( G \).

2.7. Some well known facts in commutative algebra. We will briefly mention some additional tools we use in order to prove 2.6. First suppose \( R \) is a commutative ring, the support of an \( R \)-module \( M \) denoted by \( \text{Supp}_R(M) \) is the set of prime ideals, \( P \) of \( R \) such that the localized module \( M_P \neq 0 \). If \( M \) is finitely generated then \( \text{Supp}_R(M) \) is exactly the set of prime ideals containing \( \text{Ann}_R(M) \).

Note that \( M \) is torsion-free if and only if \( M \) has no \( \mathbb{N} \leq M \) \( R \)-submodule such that \( \text{Ann}_R(N) \neq 0 \).

We will use the usual notation for the set of all prime ideals of a ring \( R \) by \( \text{Spec}(R) \). It is also well known that the nilradical is the set of nilpotent elements and also the intersection of all prime ideals of \( R \).

3. Theorem 2.6

Now we are ready to prove what we stated in section 2.6

3.1. The proof Theorem 2.6

**Proof.** By Lemma 2.5 we have short exact sequences 2.5.1 such that \( M \oplus K = N \oplus L \). By Theorem 2.3 it is enough to prove that \( N \) is \( \Lambda_Z \) torsion-free. In order to do that recall from 2.7 that it suffices to show that \( \text{Ann}_{\Lambda_Z}(N') = 0 \) for all \( N' \Lambda_G \)-submodule of \( N \). First we show that \( \text{Ann}_{\Lambda_Z}(N) = 0 \). Since all \( P \in \text{Supp}_{\Lambda_Z}(N) \) contains \( \text{Ann}_{\Lambda_Z}(N) \) and the nilradical of \( \Lambda_Z \) is zero i.e. \( N'(\Lambda_Z) = 0 \) and it is also the intersection of all prime ideals of \( \Lambda_Z \), so it is enough to prove that \( \text{Supp}_{\Lambda_Z}(N) = \text{Spec}(\Lambda_Z) \). For that we localize in \( \Lambda_H \) with the \((0)\) ideal. By Goldie’s theorem we get a skewfield what we denote by \( Q(H) \). If \( S \) is a \( \Lambda_G \)-module from the category \( \mathfrak{N}_H(G) \) then after localization we get a finite dimensional vector space, \( Q(S) \) over \( Q(H) \). Since localization is exact and \( \Lambda_Z \) is central in \( \Lambda_G \) the statement of Lemma 2.5 is still true, so we still have the
We will prove that after localization with $P$ can deduce that the elements of $(\Lambda \otimes Q)$ action on it and let $P$ be arbitrary. Since localization is exact and $\Lambda$ is central in $\Lambda_G$ we get short exact sequences

$$0 \longrightarrow C_P \longrightarrow K_P \longrightarrow D_P \longrightarrow 0$$

such that $Q(M)_P \oplus K_P = Q(N)_P \oplus L_P$.

**Lemma 3.1.1.** Let $Q(M)$ be a finite dimensional vectorspace over $Q(H)$ with a $\Lambda_Z$ action on it and let $P$ be an arbitrary prime ideal of $\Lambda_Z$. Then $Q(M)_P$ is also finite dimensional over $Q(H)$ where $Q(M)_P$ denotes the localized module of $Q(M)$ with $P$.

**Proof.** Let $Q(M)_S$ be the $S$-torsion submodule of $Q(M)$ where $S = \Lambda_Z \setminus P$. This is a $Q(H)$ subspace of $Q(M)$ since $\Lambda_Z$ is central and the set $S$ is multiplicatively closed. Also $Q(M)_S \cdot S^{-1} = 0$. One can see that the quotient submodule $Q(M)/Q(M)_S$ is $S$ torsion-free since $s \cdot m \in Q(M)_S \iff \exists s_1 \in S$ such that $s_1 \cdot s \cdot m = 0$ but that implies that $m \in Q(M)_S$.

We have a short exact sequence of $\Lambda_S$-modules

$$0 \longrightarrow Q(M)_S \longrightarrow Q(M) \longrightarrow Q(M)/Q(M)_S \longrightarrow 0$$

After localization with $P$ one can easily see that $Q(M) \cdot S^{-1} \simeq (Q(M)/Q(M)_S) \cdot S^{-1}$. We will prove that $(Q(M)/Q(M)_S) \cdot S^{-1} \simeq Q(M)/Q(M)_S$ as $Q(H)$ vector spaces.

We can consider localization of a module as tensoring it by the localized ring so one can deduce that the elements of $(Q(M)/Q(M)_S) \cdot S^{-1}$ are of the form $(x \otimes \frac{1}{s})$ where $x \in Q(M)/Q(M)_S$ and $s \in S$. Let us observe first that multiplication with an arbitrary element $s \in S$ is an injective linear transormation on the vector space $Q(M)/Q(M)_S$ hence it is an isomorphism so one can see that using the surjectivity property that every $x \in Q(M)/Q(M)_S$ can be written as $s \cdot y$ for some $y \in Q(M)/Q(M)_S$ so every element is of the form $(y \otimes 1)$ in $Q(M)/Q(M)_S \cdot S^{-1}$. Since both $Q(M)$ and $Q(M)_S$ are finite dimensional, it follows that $Q(M)/Q(M)_S$ is also finite dimensional. Let $e_1, \ldots, e_n$ be a basis in $Q(M)/Q(M)_S$. We have just seen that $(e_i \otimes 1)$ is a generating system for $Q(M)/Q(M)_S \cdot S^{-1}$. In order to see that they are linearly independent one easily see that if there is a linear combination of these is also of the form $(y \otimes 1)$, it is zero if and only if $y$ is $S$-torsion, but $Q(M)/Q(M)_S$ is $S$-torsion free. 

So all the the modules in the short exact sequences remain finite dimensional over $Q(H)$ after localization with $P$ by Lemma 3.1. Then we see that $L_P = Q(M)_P \oplus K_P$ so we have

$$0 \longrightarrow C_P \longrightarrow K_P \longrightarrow D_P \longrightarrow 0$$

$$0 \longrightarrow C_P \longrightarrow Q(M)_P \oplus K_P \longrightarrow D_P \longrightarrow 0$$

but that since these are finite dimensional vector spaces over $Q(H)$ we have that $Q(M)_P = 0$ which cannot be since $M$ is completely faithful.

So now we see that $\text{Ann}_{\Lambda_Z}(N) = 0$. We are left to prove that $N$ has no $N'$ $\Lambda_G$-submodule such that $\text{Ann}_{\Lambda_Z}(N') \neq 0$. Let us suppose that there is one. It means that there is a $P \in \text{Spec}(\Lambda_Z)$ prime ideal such that $Q(N')_P = 0$. By using Lemma 3.3 again we see
that \( \dim_{Q(H)}Q(M)_F < \dim_{Q(H)}Q(M) \) but that cannot be since \( M \) is completely faithful hence \( \Lambda_Z \) torsion-free. \( \square \)

4. **Theorem [2,4]**

4.1. **Generalization of Theorem [2,4]** In [1] the author proved that whenever \( G = Z \times H \) where \( Z = \mathbb{Z}_p \) and \( M \) is finitely generated \( \Lambda_G \)-module which has no non-zero pseudo-null submodule then \( q(M) \) is completely faithful if and only if \( M \) is \( \Lambda_Z \) torsion-free. We will generalize it to \( G = Z_1 \times H_1 \) where \( Z_1 \cong \mathbb{Z}_p \) and \( H_1 \) is such that its Lie-algebra is the direct product of a split semisimple and an abelian Lie-algebra. In this case \( H_1 = \mathbb{Z}_p^n \times H \) where \( H \) is such that its Lie-algebra is split semisimple. Let us denote \( Z_2 = \mathbb{Z}_p^n \). We can choose topological generator for \( Z_2 \cong Z_1 \) i.e. \( g_1 \) such that \( < \overline{g_1} > \cong \mathbb{Z}_p \). Now let \( z = g_1 - 1 \). Let \( \Lambda_Z \) be the Iwasawa algebra over the group \( Z = Z_1 \times Z_2 \).

**Theorem 4.1.1.** Let \( p \geq 5 \). Let \( G = Z_1 \times H_1 \) where \( Z_1 \) and \( H_1 \) as above. Let \( M \) be a finitely generated torsion \( \Lambda_G \)-module such that it has no non-zero pseudo-null submodules. Then \( q(M) \) is completely faithful if and only if \( M \) is \( \Lambda_Z \) torsion-free.

**Proposition 4.1.2.** Let \( G \) be a torsionfree compact \( p \)-adic analytic group. Then \( \Lambda_G \) is a maximal order.

**Proof.** see [1] Theorem 4.1. We will use two localization of \( \Lambda_G \) in order to prove the next proposition. Let \( S \) be the Ore set \( S = \{ s \in \Lambda_G \mid s \text{ is regular mod } P_H \} \) and let \( T \) be another Ore set such that \( T = \{ t \in \Lambda_G \mid \Lambda_G/\Lambda_G \cdot t \text{ finitely generated over } \Lambda_H \} \).

The definition of these two Ore sets can be found in [1] 3.1 and in [6] Definition 2 respectively, although both are the work of Venjakob. By Proposition 2.6 in [6] one can identify \( T \) to be those elements that are regular modulo \( \mathcal{N} \) where

\[ \mathcal{N} := \text{preimage of } \mathcal{N}(\Omega(G/J)) \text{ in } \Lambda_G \]

and \( J \) denotes an arbitrary pro-\( p \) subgroup of \( H \) which is normal in \( G \). For details see [6] section 2.

The special case of the next proposition can be found in [1] Theorem 4.3

**Proposition 4.1.3.** If \( I \) is a prime \( c \)-ideal of \( \Lambda_G \) then it is \( p \cdot \Lambda_G \) or \( \Lambda_G/I \) is finitely generated over \( \Lambda_H \).

**Proof.** First case: If \( I \cap T = \emptyset \). It can be seen that \( S \subseteq T \) so that means \( I \cap S = \emptyset \) also. So by Proposition 3.4 in [1] the localized ideal \( I_S \) in \( \Lambda_{G,H} \) equals to \( p \cdot \Lambda_{G,H} \), but it means that \( p \in I = I_S \cap \Lambda_G \) so by Proposition 2.2.2 \( I = p \cdot \Lambda_G \).

Second case: If \( I \cap T \neq \emptyset \). It means that \( \Lambda_G/I \) is \( T \)-torsion in the sense of [6] (section 2). So by Proposition 2.3 in [6] \( \Lambda_G/I \) if finitely generated over \( \Lambda_H \). \( \square \)

The next proposition of of crucial importance.

**Proposition 4.1.4.** If \( I \) is a prime \( c \)-ideal in \( \Lambda_G \) such that \( \Lambda_G/I \) is finitely generated over \( \Lambda_H \). Then \( \Lambda_Z \cap I \neq \emptyset \) or \( I \cap \Lambda_H \neq \emptyset \).
Proof. The proof of this proposition is quite similar to that of in [1] Prop. 4.5 with a little additional argument. We again have an increasing chain of finitely generated \( \Lambda_{H_1} \)-modules
\[
\Lambda_{H_1} = A_0 \subset A_1 \subset A_2 \ldots
\]
where \( A_i = \bigoplus_{k=0}^i \Lambda_{H_k} z^k \). The image of this chain in \( \Lambda_G/I \) must stabilize by the Noetherian property of \( \Lambda_G/I \) that we assumed. So again \( I \cap A_n \neq 0 \) for some \( n \). Let us consider the minimal such \( n \), if it is zero, then we are done, if it is not, then we have a non-zero polynomial
\[
a = a_n z^n + \cdots + a_0 \in I.
\]
By minimality of \( n a_n \neq 0 \). Writing again \( Q(H) \) for the skewfield of fractions of \( \Lambda_{H_1} \), we consider the polynomial ring \( Q(H_1)[z] \). Note that \( Q(I \cap \Lambda_{H_1}) \) is a two-sided ideal in \( Q(H_1)[z] \) and \( a_n^{-1}a \in Q(I \cap \Lambda_{H_1}[z]). \) Consider an element \( u \in Q(H_1) \) and look at the commutator \( [u, a_n^{-1}a] \). It has strictly smaller degree than \( n \) and it is still in the ideal \( Q(I \cap \Lambda_{H_1}). \) So with clearing the common denominator we get an element which is in \( I \cap A_{n-1} \), so it must be zero by minimality of \( n \). It means that
\[
a_n^{-1}a_i Z(Q(H_1))
\]
for all \( i < n \). Since \( \Lambda_{Z_2} \) is central in \( \Lambda_{H_1} \) and the center of \( \Lambda_H \) is \( \mathbb{Q}_p \) by Theorem 4.4 in [1], it follows that \( a_n^{-1}a \in Q(Z_2) \) i.e. in the field of fractions of \( \Lambda_{Z_2} \), therefore we can find an \( m \geq 0 \) such that \( p^m a_n^{-1}a \in \Lambda_Z \). Now \( I \) is a prime ideal, \( a_n \) is not in \( I \) and \( p^m a = a_n (p^m a_n^{-1}) a Q \in I \), we see that \( I \cap a_n = \emptyset \). □

Now we are ready to prove Theorem 4.1. It is essentially the same as the proof of the special version in [1] with a little modification.

4.2. proof of Theorem 4.1.

Proof. First by Proposition 4.1.1 \( \Lambda_G \) is a maximal order. By Proposition 4.1.1. in [5] and the fact that \( M \) is \( \Lambda_G \)-torsion, \( q(M) = M_0 \oplus M_1 \) where \( M_0 \) is completely faithful and \( M_1 \) is locally bounded. Let us consider \( X \leq M \) the \( \Lambda_Z \)-torsion submodule of \( M \). Like in [1] we show that \( q(X) = M_1 \). Since \( \Lambda_G \) is noetherian we can find a finitely generated \( \Lambda_G \)-submodule \( \mathcal{N} \) of \( M \) such that \( q(\mathcal{N}) = M_1 \). Let \( M_0, \mathcal{N}_0 \) be the maximal pseudo-null submodule of the modules \( M \) and \( \mathcal{N} \). Since \( \mathcal{N}_0 \leq M_0 = 0 \), the annihilator of \( M_1 \) equals to the annihilator of \( \mathcal{N} \). Since \( M_1 \) is locally bounded, \( \mathcal{N} \) is a \( \Lambda_G \)-torsion, bounded object in mod(\( \Lambda_G \)). So by Lemma 4.3 (i) in [5] \( \text{Ann}(q(\mathcal{N})) \) is a prime-c ideal. Therefore by Proposition 4.1.4 there is a central element \( x \in \Lambda_Z \) such that \( x \in \text{Ann}(q(\mathcal{N})), \)

\[
x \Lambda_G \subseteq \text{Ann}(q(\mathcal{N}))
\]

but it means that \( \mathcal{N} \subseteq X \). Hence \( M_1 = q(\mathcal{N}) \subseteq q(X) \) but \( q(X) \subseteq M_1 \) is true also since \( \Lambda_Z \) is central. Of course \( q(M) \) is completely faithful if and only if \( M_1 = 0 \) but now we see that it happens if and only if \( \mathcal{N} = 0 \) since \( \mathcal{N} \) has no non-zero pseudo-null submodule. □

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