Quantum Stochastic Dynamic and Quantum Measurement in Multi-Photon Optics

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Abstract. In this work a generic model of multi-photon optics is studied. It is considered \( m \) pump and \( n \) subharmonic fields. It shows how the multi-photon system, with direct and homodyne detection, can be obtained rigorously in the context of Measurement theory in Open Quantum System (see [1], [2] and [4]). This is done through the Quantum Stochastic Differential Equations with unbounded coefficients (see [9], and [4]). In particular the existence of the dynamics is proved, and a continuous measurements scheme is provided.

1. Introduction

In the theory of Open Quantum System the evolution of a quantum system \( \mathcal{H} \) in interaction with an external environment \( \mathcal{F} \) or external field is studied. From the perspective of the measurement over the system, it is possible to interpret the measuring instrument as part of the environment and make a powerful formulation of the quantum measurement theory (see [1] and [3]) using the Quantum Stochastic Calculus (see [12] and [13]).

The interaction between the quantum system \( \mathcal{H} \) and the environment \( \mathcal{F} \) is given by the Hudson-Parthasarathy equation or the Quantum Stochastic Schrödinger equation. In the classical theory of Hudson-Partha sarathy, all the coefficients are bounded operators but in many relevant cases of the quantum optics it involves unbounded coefficients such as the harmonic oscillator. The case of unbounded coefficients was studied by the authors Fagnola and Wills [9], which gave necessary conditions for the existence of solutions. In [4] a construction of the reduced dynamic of the evolution system-field and a continuous measurement scheme for unbounded coefficient is presented. This theory was applied to study of the degenerate parametric oscillator model, a case of the multi-photon optics with one pump field and two identical subharmonic field (see [5]). In this work we will extend the results of existence of the unitary dynamic to a generic multi-photon model.

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The multi-photon models are theoretically and experimentally important because in them quantum properly phenomena are verified; as squeezed light and quantum entanglement also plays a relevant role in quantum information and quantum communication.

It is considered an abstract model consisting in \( m \) pump fields with frequency \( w_1^p, \ldots, w_m^p \) and \( n \) subharmonic fields with frequency \( w_1^s, \ldots, w_n^s \).

The abstract model includes many channels that represent the devices of the measuring instrument (as photocounters and homodynek) and less.

In this work, the standard notation is used. For any separable complex Hilbert space \( \mathcal{H} \) let us introduce the following classes of operators in it: \( \mathcal{B}(\mathcal{H}) \), the space of bounded linear operators, \( \mathcal{U}(\mathcal{H}) \) the class of the unitary operators, \( \mathcal{T}(\mathcal{H}) \) the trace-class, \( \mathcal{S}(\mathcal{H}) := \{ \rho \in \mathcal{T}(\mathcal{H}) : \rho \geq 0, \text{Tr}(\rho) = 1 \} \) the set of statistical operators.

2. The Unitary System-Field Evolution

In quantum mechanics a system is represented by a complex and separable Hilbert space, that we denoted by \( \mathcal{H} \) and the environment is represented by the Symmetric Fock Space over \( L^2(\mathbb{R}_+; \mathbb{C}^d) \), that is denoted by \( \mathcal{F} \). Will be denoted by \( e(f) \) the exponential vector in the Fock space \( \mathcal{F} \) associated with the test function \( f \in L^2(\mathbb{R}_+; \mathbb{C}^d) \), it will be call coherent vector and vacuum vector to \( \psi(f) := \|e(f)\|^{-1}e(f) \) and \( e(0) = \psi(0) \) respectively. A comprehensive study of the construction of the Fock space is in [13].

The components of Right Hudson-Parthasarathy equation (Right H-P equation) on \( \mathcal{H} \otimes \mathcal{F} \) are system operators and quantum noise. The precise form is:

\[
\begin{align*}
\text{d}U(t) & = \left( \sum_{i=1}^{d} R_i \text{d}A_i^\dagger + \sum_{i,j=1}^{d} F_{ij} \text{d}\Lambda_{ij} + \sum_{j=1}^{d} N_j \text{d}A_j + K(t) \text{d}t \right) U(t) \\
U(0) & = 1
\end{align*}
\]

The coefficient \( R_i, F_{ij}, N_j \) and \( K \) are operators defined in the Hilbert space \( \mathcal{H} \) and the noise \( A_i^\dagger(t), \Lambda_{ij}(t) \) and \( A(t) \) in the Fock space \( \mathcal{F} \).

Using the notation: \( \Lambda_{i0}(t) = A_i^\dagger(t), \Lambda_{0j}(t) = A_j(t), \Lambda_{00}(t) = t, F_{i0} = R_i \) and \( F_{0j} = N_j \) is possible to write this equation in an abbreviate form

\[
\text{d}U(t) = \sum_{i,j \geq 0} F_{ij} \text{d}\Lambda_{ij}(t) U(t), \quad U(0) = 1.
\]

(2.1)

Analogously is possible to define the Left Hudson-Parthasarathy Equation by:

\[
\text{d}V(t) = \sum_{i,j \geq 0} V(t) F_{ji}^* \text{d}\Lambda_{ij}(t) \quad V(0) = 1.
\]

(2.2)
The existence of solution of (2.1) and (2.2) in the case that the elements of the matrix of coefficients are not all bounded was studied in [9]. The necessary conditions in order to define a Reduced Dynamic for the continual measurements are studied in [4], Chapter 4 and 5.

2.1. Hypothesis for the existence of solution

The following hypothesis over the matrix of the coefficients $F_{ij}$ are sufficient to ensure existence of the equations (2.1) and (2.2) ( [4], Hypothesis 3.4).

1. $F = (F_{ij}; 0 \leq i, j \leq d)$ is a matrix of closed operators in the initial space $\mathcal{H}$. By $F^*$ we denote the adjoint matrix, defined by $(F^*)_{ij} = F_{ji}^*$. We also define $\text{Dom}(F) := \bigcap_{i,j \geq 0} \text{Dom}(F_{ij})$, $\text{Dom}(F^*) := \bigcap_{i,j \geq 0} \text{Dom}(F_{ji}^*)$

2. For $1 \leq i, j \leq d$, we have $F_{ij} = S_{ij} - \delta_{ij}1$, where the $S_{ij}$ are the bounded operators on $\mathcal{H}$ satisfying the unitarity conditions

$$\sum_{k=1}^{d} S_{ki}^* S_{kj} = \sum_{k=1}^{d} S_{ik} S_{jk}^* = \delta_{ij}1.$$

3. There exist a dense subspace $D$ which is a core for $K, R_i, N_i$ and a dense subspace $\tilde{D}$ which is a core for $K^*, R_i^*, N_i^*, i = 1, \ldots, d$.

4. $\text{Dom}(N_i^*) \supset D \cup \tilde{D}$, $\text{Dom}(R_i) \supset D \cup \tilde{D}$, $\text{Dom}(N_i) \supset \text{Dom}(K)$, $\forall i \geq 1$.

5. $\forall k \geq 1$, $\forall u \in \text{Dom}(K)$: $S_{ki}u \in \text{Dom}(R_k^*)$, $\forall i \geq 1$.

6. We have $\forall u \in D$, $\forall v \in \tilde{D}$

$$2\text{Re} \langle Ku|u \rangle = -\sum_{k=1}^{d} ||R_k u||^2, \quad 2\text{Re} \langle K^* v|v \rangle = -\sum_{k=1}^{d} ||N_k^* v||^2.$$

7. $N_i^* u = -\sum_{k=1}^{d} S_{ki}^* R_k u$, $\forall u \in D \cup \tilde{D}$, $\forall i \geq 1$.

8. The operators $K$ and $K^*$ are the infinitesimal generators of two strongly continuous contraction semigroups on $\mathcal{H}$.

9. There exist a positive self-adjoint operator $C$ on $\mathcal{H}$ and the constants $\delta > 0$ and $b_1, b_2 \geq 0$ such that ( [9], pp. 281–291) $\text{Dom}(C^{1/2}) \subset \text{Dom}(F)$ and

(a) for each $\epsilon \in (0, \delta)$, there exists a dense subspace $D_\epsilon \subset \tilde{D}$ such that $C^{1/2}D_\epsilon \subset \tilde{D}$ and each operator $F_{ij}^* C^{1/2}|_{D_\epsilon}$ is bounded, where $C_\epsilon = \frac{C}{1+\epsilon C}$;

(b) for all $0 < \epsilon < \delta$ and $u_0, \ldots, u_d \in \text{Dom}(F)$, the following inequality holds:

$$\sum_{i,j=0}^{d} \left( \langle u_i|C_\epsilon F_{ij} u_j \rangle + \langle F_{ji} u_i |C_\epsilon u_j \rangle + \sum_{k=1}^{d} \langle F_{ki} u_i |C_\epsilon F_{kj} u_j \rangle \right) \leq \sum_{i=0}^{d} \left( b_1 ||u_i|C_\epsilon u_i \rangle + b_2 ||u_i||^2 \right)$$
2.2. Hypothesis for unitary solution or Markov condition

Let $F_{ij}$ a matrix of operators such that the Hypothesis 2.1 are satisfied, hence exist a solution $U(t)$ of the equation (2.1). The family of operators $V(t)$ in $\mathcal{H} \otimes \mathcal{F}$, defined by $V(t) := U(t)^*$ is solution of the Left Quantum Stochastic Equation (2.2) ([4], Theorem 2.6).

For the physical interpretation of the evolution operator $U(t)$ a strongly continuous cocycle of unitary operators is needed ([1], Section 2.2 and references therein). The unitarity property, is associated to some property of a related quantum dynamical semigroup (QDS); so, we start with some notions on QDSs.

DEFINITION 1 A Quantum Dynamical Semigroup is called Markov or Conservative if, only if $T(t)[1] = 1$ holds $\forall t \geq 0$.

THEOREM 1 ([7], Theorem 3.22, Corollary 3.23) Let $A$ be the infinitesimal generator of a strongly continuous contraction semigroup in $\mathcal{H}$ and let $L_k$, $k = 1, \ldots$, be operators in $\mathcal{H}$ such that the domain of each operator $L_k$ contains the domain of $A$ and for every $u \in \text{Dom}(A)$ we have $2\text{Re} \langle u|Au \rangle + \sum_{k \geq 1} \|L_k u\|^2 = 0$.

For all $X \in \mathcal{B}(\mathcal{H})$, let us consider the quadratic form $\mathcal{L}[X]$ in $\mathcal{H}$ with domain $\text{Dom}(A) \times \text{Dom}(A)$ given by

$$\langle v|\mathcal{L}[X]u \rangle = \langle v|XAu \rangle + \langle Av|Xu \rangle + \sum_{k \geq 1} \langle L_k v|XL_k u \rangle.$$  \quad (2.3)

Then, there exists a QDS $T(t)$ solving the equation

$$\langle v|T(t)[X]u \rangle = \langle v|Xu \rangle + \int_0^t \langle v|\mathcal{L}[T(s)[X]]u \rangle \, ds$$  \quad (2.4)

with the property that $T(t)[1] \leq 1$, $\forall t \geq 0$, and such that for every $\sigma$-weakly continuous family $T'(t)$ of positive maps on $\mathcal{B}(\mathcal{H})$ satisfying Eqs. (2.3) and (2.4) we have $T(t)[X] \leq T'(t)[X]$, $\forall t \geq 0$, for all positive $X \in \mathcal{B}(\mathcal{H})$.

If moreover the QDS $T(t)$ is conservative, then it is the unique $\sigma$-weakly continuous family of positive maps on $\mathcal{B}(\mathcal{H})$ satisfying Eq. (2.4).

The QDS $T(t)$ defined in Theorem 1 is called the minimal quantum dynamical semigroup generated by $A$ and $L_k$, $k = 1, \ldots$. Sufficient conditions to assure Markovianity of a QDS are known ([7] ?). In the application we shall use the following result.

THEOREM 2 ([8], Theorem 9.6) Let $A$, $L_k$ be as in Theorem 1 and suppose that there exist two positive self-adjoint operators $Q$ and $Z$ in $\mathcal{H}$ with the following properties:
Dom(A) is contained in $\text{Dom}(Q^{\frac{1}{2}})$ and is a core for $Q^{\frac{1}{2}}$;
the linear manifold $\bigcap_{k \geq 1} L_k(\text{Dom}(A^2))$ is contained in $\text{Dom}(Q^{\frac{1}{2}})$;
Dom(A) ⊂ $\text{Dom}(Z^{\frac{1}{2}})$ and
\[-2\text{Re} \langle u|Au \rangle = \sum_{k=1}^{d} \|L_k u\|^2 = \|Z^{\frac{1}{2}} u\|^2, \quad \forall u \in \text{Dom}(A);
\]
Dom(Q) ⊂ $\text{Dom}(Z)$ and $\forall u \in \text{Dom}(Q^{\frac{1}{2}})$ we have $\|Z^{\frac{1}{2}} u\| \leq \|Q^{\frac{1}{2}} u\|$
there is a positive constant $b$ depending only on $A$, $L_k$, $Q$ such that, for all $u \in \text{Dom}(A^2)$, the following inequality holds
\[2\text{Re} \langle Q^{\frac{1}{2}} u|Q^{\frac{1}{2}} Au \rangle + \sum_{k=1}^{d} \|Q^{\frac{1}{2}} L_k u\|^2 \leq b\|Q^{\frac{1}{2}} u\|^2.
Then, the minimal quantum dynamical semigroup associated to $A$ and $L_k$ is Markov.

**THEOREM 3** ([8], Theorems 10.2, 10.3) Under Hypothesis 2.1, the left cocycle $V$ solving (2.2) is such that the family of operators $\tilde{T}(t)$ defined by
\[\langle v|\tilde{T}(t)[X]|u \rangle = \langle V(t)v \otimes e(0)|(X \otimes 1)V(t)u \otimes e(0)\rangle, \quad \forall u, v \in \mathcal{H}, \quad \forall X \in \mathcal{B}(\mathcal{H}).
\]
is the minimal QDS generated by $K^*$ and $N_k^*$, $k = 1, \ldots, d$.
Moreover, the following conditions are equivalent:
\bullet the process $V$ is an isometry;
\bullet the minimal QDS associated with $K^*$ and $N_k^*$ is conservative;

Now, both $U(t)$ and $V(t) = U(t)^*$ are isometries and, so, they are unitary operators.

**COROLLARY 1** Under Hypothesis 2.1 and if the minimal QDS generated by $K^*$ and $N_k$ is conservative, then the process $U$ solution of the equation (2.1) is unitary and it is the unique bounded solution.

3. **Observables and Instruments**

Let $B_1, B_2, \ldots B_{d'}$ be commuting selfadjoint operators on $\mathbb{C}^d$ and let us choose the complete orthonormal system $\{z_i, i = 1, \ldots, d\}$ in $\mathbb{C}^d$ such that it diagonalizes all the operators $B_1, \ldots, B_{d'}$ and such that its first $d''$ components, $0 \leq d'' \leq d$, span the intersection of the null spaces of these operators; then, we have $B_k = \sum_{i=d''+1}^{d} B_{ki}|z_i\rangle\langle z_i|$, $B_{ki} \in \mathbb{R}$ and let $h_1, \ldots h_{d'} \in L^2_{\text{loc}}(\mathbb{R}^+; \mathbb{C}^d)$ such that
\[
\text{Im} \langle h_i(t)|h_j(t) \rangle = 0 \quad B_i h_j(t) = 0, \quad \forall t \geq 0 \quad \forall i, j = 1, \ldots, d'
\] (3.5)
we can write, on the exponential domain,

\[ X(k, t) = \sum_{i=1}^{d'} \int_0^t \left( \langle z_i | h_k(s) \rangle \, dA_i(s) + \langle h_k(s) | z_i \rangle \, dA_i^\dagger(s) \right) + \sum_{i=d'+1}^{d} B_{ki} \Lambda_{ii}. \]

(3.6)

In quantum optical systems the continuous measurement of observables of the type \( \int_0^t \left( \langle h_i(s) | z_k \rangle \, dA_i(s) + \langle h_k(s) | z_i \rangle \, dA_i^\dagger(s) \right) \) can be obtained by homodyne detection, while terms like \( \int_0^t \, d\Lambda_{ii}(s) \, ds \) are realized by direct detection (is possible to obtain a more general construction, see [1] and [3]).

A key point in the whole construction is that even in the Heisenberg picture the observables \( X(k, t) \) continue to be represented by commuting operators and, so, they can be jointly measured also at different times.

**DEFINITION 2** Given the observables \( X(k, t) \) as in the equation (3.6) and real numbers \( \kappa_k, k = 1, \ldots, d \), is defined the **Characteristic Operators** by:

\[ \hat{\Phi}_t(\kappa) := \exp \left( \sum_{k \geq 1} \kappa_k X(k, t) \right) \quad \text{with} \quad \kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{R}^d. \]

Let \( U(t) \) be a right solution of the equation (2.1) then this representing the system-field dynamics and define \( \forall T \geq 0 \) the “output” characteristic operator by \( \hat{\Phi}_T^{\text{out}}(\kappa) := U(T)^{\dagger} \hat{\Phi}_T(\kappa) U(T) \). The key property giving the commutativity of the observables is \( \hat{\Phi}_T^{\text{out}}(1_{[0, t]}(\kappa)) = \hat{\Phi}_t^{\text{out}}(\kappa), \ 0 \leq t \leq T \).

**THEOREM 4** ([1], Proposition 3.1) There exists a measurement space \( (\Omega, \mathcal{D}) \), a projection valued measure \( \xi \) on \( (\Omega, \mathcal{D}) \), a family of commuting functions \( \{ \tilde{X}(k, t; \cdot), k = 1, \ldots, d', \ t \geq 0 \} \) on \( \Omega \), such that for any choice of \( n \), \( 0 = t_1 < \cdots < t_n = t \), \( \kappa_k \in \mathbb{R} \) one has:

\[ \hat{\Phi}_t(\kappa) = \int_\Omega \exp \left( \sum_{l=1}^n \sum_{k=1}^{d'} \kappa_k \left[ \tilde{X}(k, t_l; \omega) - \tilde{X}(k, t_{l-1}; \omega) \right] \right) \xi(d\omega) \]

Let \( s \in S(\mathcal{H} \otimes \mathcal{F}) \) be the initial system-field state. The **characteristic functional** of the process \( \tilde{X} \) (the “Fourier transform” of its probability law) is given by

\[ \Phi_t(\kappa) = \text{Tr} \left\{ \hat{\Phi}_t(\kappa) U(t) s U(t)^{\dagger} \right\} = \text{Tr} \left\{ \hat{\Phi}_T^{\text{out}}(\kappa) s \right\}. \]

(3.7)

All the probabilities describing the continuous measurement of the observables \( X(k, t) \) are contained in \( \Phi_t(\kappa) \); let us give explicitly the construction of the joint probabilities for a finite number of increments.
The measurable functions \( \{ \tilde{X}(k,t; \cdot), k = 1, \ldots, d', t \geq 0 \} \), introduced in Theorem 4, represent the output signal of the continuous measurement. Let us denote by 
\[
\Delta \tilde{X}(t_1, t_2) = \left( \tilde{X}(1, t_2) - \tilde{X}(1, t_1), \ldots, \tilde{X}(m, t_2) - \tilde{X}(m, t_1) \right)
\]
the vector of the increments of the output in the time interval \((t_1, t_2)\) and by \(\xi(dx; t_1, t_2)\) the joint projection valued measure on \(\mathbb{R}^d\) of the increments \(X(k, t_2) - X(k, t_1), k = 1, \ldots, d\) and we can write.

\[
\Phi_t(k) = \text{Tr} \left\{ \exp \left( i \sum_{l=1}^{d'} \sum_{k=1}^{n} \nu_l \alpha \left[ X(k, t_l) - X(k, t_{l-1}) \right] \right) U(t) \mathbb{1} U(t)^* \right\}
\]

\[
= \int_{\mathbb{R}^{nd'}} \left( \prod_{l=1}^{n} e^{i \sum_{k=1}^{d'} \nu_l x_l} \right) \mathbb{P}_s \left[ \Delta \tilde{X}(t_0, t_1) \in dx^1, \ldots, \Delta \tilde{X}(t_{n-1}, t_n) \in dx^n \right],
\]

where the physical probabilities are given by

\[
\mathbb{P}_s \left[ \Delta \tilde{X}(t_0, t_1) \in A_1, \ldots, \Delta \tilde{X}(t_{n-1}, t_n) \in A_n \right] = \text{Tr} \left\{ \left( \prod_{l=1}^{n} \xi(A_j; t_{l-1}, t_l) \right) U(t) \mathbb{1} U(t)^* \right\}
\]

4. The Subharmonic and Pump Field

4.1. The Creator, Annihilations and Number Operators

Consider the Hilbert space \(\ell^2(\mathbb{N})\) with its canonical base \(\{e_k\}_{k \in \mathbb{N}}\). The creator and annihilation operators are defined in \(\ell^2(\mathbb{N})\) by:

\[
\text{Dom}(a^\dagger) = \text{Dom}(a) = \left\{ \{u_k\} \in \ell^2(\mathbb{N}); \sum_{k \geq 1} k |u_k| \leq \infty \right\}
\]

\[
a^\dagger e_k = \sqrt{k + 1} \ e_{k+1}, \quad ae_k = \sqrt{k} \ e_{k-1} \quad ae_0 = 0
\]

The operator number is defined by:

\[
\text{Dom}(a^\dagger a) = \left\{ \{u_k\} \in \ell^2(\mathbb{N}); \sum_{k \geq 1} k^2 |u_k| \leq \infty \right\}, \quad a^\dagger a \ e_k = k \ e_k
\]

The fundamental commutation rule is \([a; a^\dagger] = \mathbb{1}\)
4.2. **The Subharmonic and Pump Fields**

In this paper, \(m\) pump fields and \(n\) subharmonic fields are considered. These fields are modeled as independent modes of the creator and annihilation operators. The tensorial product of \(\ell^2(N)\) \(m\)-th, denoted by \(\mathcal{H}_p\), and the tensorial product of \(\ell^2(N)\) \(n\)-th, denoted by \(\mathcal{H}_s\), will be considered. The total space is the tensorial product between \(\mathcal{H}_s\) and \(\mathcal{H}_p\), i.e.

**DEFINITION 3**

Subharmonic Space : \(\mathcal{H}_s := \bigotimes_{k=1}^{n} \ell^2(N)\)

Pump Space : \(\mathcal{H}_p := \bigotimes_{k=1}^{m} \ell^2(N)\)

Total Space : \(\mathcal{H} := \mathcal{H}_s \otimes \mathcal{H}_p\)

For any \(s = \{s_i\}_{i=1}^{n} \subset N\), finite sequence the natural number and it's defined the vector \(e(s) := \bigotimes_{k=1}^{n} e_{s_i}\), where \(e_{s_i}\) indicate the element \(s_i\)-th of the canonical base of \(\ell^2(N)\), then \(e(s)\) is a generic element of the canonical orthogonal base of \(\mathcal{H}_s\),

Analogously, one can define a generic element of the canonical base of \(\mathcal{H}_p\) by \(e(p) := \bigotimes_{k=1}^{m} e_{p_j}\) where \(p = \{p_j\}_{j=1}^{m} \subset N\)

Finally, given sequences \(s = \{s_i\}_{i=1}^{n}\) and \(p = \{p_j\}_{j=1}^{m}\) we defined a generic element of the canonical base of \(\mathcal{H}\) by: \(e(s, p) := e(s) \otimes e(p)\)

The following is the formal definition of the subharmonic and pump fields.

**DEFINITION 4** For \(1 \leq k \leq n\) let

\[
\text{Dom}(a_k) = \text{Dom}(a_k^\dagger) = \bigotimes_{k=1}^{k-1} \ell^2(N) \otimes \cdots \otimes \text{Dom}(a) \otimes \cdots \otimes \ell^2(N) \\
\text{Dom}(a_k) = \bigotimes_{k=1}^{k-1} \ell^2(N) \otimes \cdots \otimes \text{Dom}(a^\dagger) \otimes \cdots \otimes \ell^2(N)
\]

\[
a_k := \mathbf{1} \otimes \cdots \otimes a \otimes \cdots \mathbf{1} \quad \text{and} \quad a_k^\dagger := \mathbf{1} \otimes \cdots \otimes a^\dagger \otimes \cdots \mathbf{1}
\]

The action of the \(k\)-th subharmonic field over a element of the canonical base of \(\mathcal{H}_s\) is on the \(k\)-th component of the vector, i.e. formally:

\[
a_k e(s) = \sqrt{s_k} e(s - \delta_{ik}) \quad \text{and} \quad a_k^\dagger e(s) = \sqrt{s_k + 1} e(s + \delta_{ik})
\]

where the sequences \(s \pm \delta_{ik} = \{s_i \pm \delta_{ik}\} \)
For the $k$-th number operator, of the equation (4.13), it follows that

$$a_k^\dagger a_k(e(s)) = s_k e(s) \quad \text{with Dom}(a_k^\dagger a_k) = \ell^2(N) \otimes \cdots \otimes \ell^2(N)$$

For $1 \leq k \leq m$, an identical construction is possible for pump fields in $\mathcal{H}_p$ thus get the following

$$b_k e(s, p) = \sqrt{p_k} e(s, p - \delta_{ik}) \quad \text{and} \quad b_k^\dagger e(s, p) = \sqrt{p_k + 1} e(s, p + \delta_{ik}) \quad (4.12)$$

**REMARK 1:**

The space $\mathcal{H}_s$ and $\mathcal{H}_p$ are included in $\mathcal{H}$ the natural form, therefore it is possible to identify $a_k$ with $a_k \otimes 1$ and $b_k$ with $1 \otimes b_k$ on the space $\mathcal{H}$. Therefore, of the equation (4.13), it follows that

$$a_k e(s, p) = \sqrt{s_k} e(s - \delta_{ik}, p) \quad \text{and} \quad a_k^\dagger e(s, p) = \sqrt{s_k + 1} e(s + \delta_{ik}, p) \quad (4.13)$$

$$b_k e(s, p) = \sqrt{p_k} e(s, p - \delta_{ik}) \quad \text{and} \quad b_k^\dagger e(s, p) = \sqrt{p_k + 1} e(s, p + \delta_{ik}) \quad (4.14)$$

With a direct check, we obtain the following commutation rules:

$$[a_k, a_k^\dagger] = [b_k, b_k^\dagger] = 1 \quad (4.15)$$

and all other possible commutation between $a_i, a_j^\dagger, b_k, b_l^\dagger$ are null, for all choice of $i, j, k$ and $l$.

### 5. The Multi-Photon Model

#### 5.1. The Hamiltonian Operator

It’s considered that the pump fields arrive with a frequency $w_k^p$ and that sub-harmonic fields emerge with a frequency $w_k^s$. Due to energy conservation the sum of the frequencies $w_k^p$ and $w_k^s$ must be the same.

Due to physical considerations the term Hamiltonian contains free type of energies; the first due to the number of photons pumped $N_p$, the second due to the number of photons emerging $N_s$ and the third to the interaction $I$. The total Hamiltonian is the sum of the three terms:

$$H = N_s + N_p + igI \quad \text{where } g \text{ is not a null constant and} \quad (5.16)$$
\[ N_s = \sum_{i=1}^{n} w_s^i a_i^\dagger a_i, \quad N_p = \sum_{j=1}^{m} w_p^j b_j^\dagger b_j, \quad I = \left( \prod_{i=1}^{n} a_i^\dagger \prod_{j=1}^{m} b_j - \prod_{i=1}^{n} a_i \prod_{j=1}^{m} b_j^\dagger \right) \] (5.17)

**REMARK 1:** It is important to consider the condition: \[ \sum_{i=1}^{n} w_s^i = \sum_{j=1}^{m} w_p^j. \]

### 5.2. The Channels

The construction of the mathematical model for the evolution of the system with measurement should include the interaction system-instrument and the loss. This model consists in one photocounter for any field, one homodynes detectors for any subharmonic field, loss in the mirrors and thermal dissipation. These are described by finite number of channel of the following form:

**The Photocounter Channels:** \[ R_i^{(1)} = \alpha_i^{(1)} a_i, \quad R_j^{(2)} = \alpha_j^{(2)} b_j \]

**The Homodyne Detectors:** \[ R_i^{(3)} = \alpha_i^{(3)} a_i \]

**The Pumps Channels:** \[ R_i^{(4)} = \alpha_i^{(4)} b_j \]

**The Loss Channels:** \[ R_i^{(5)} = \alpha_i^{(5)} a_i, \quad R_j^{(6)} = \alpha_j^{(6)} b_j, \quad R_i^{(7)} = \alpha_i^{(7)} a_i^\dagger, \quad R_j^{(8)} = \alpha_j^{(8)} b_j^\dagger \]

where \( i = 1, \ldots, n \), \( j = 1, \ldots, m \) and the \( \alpha_k^{(l)} \) are the complex numbers for any \( l = 1, \ldots, 8 \) and \( \forall k \). For a physical consideration about the channel you can see [2] and [10].

Now, defined the operator

\[ R := \sum_{k=1}^{4} \left( \sum_{i=1}^{n} R_i^{(2k-1)*} R_i^{(2k-1)} + \sum_{j=1}^{m} R_j^{(2k)*} R_j^{(2k)} \right) \] (5.18)

Of the equation (5.18) and equation (4.15), it follows that

\[ R = \sum_{k=1}^{4} \left( \sum_{i=1}^{n} |\alpha_i^{(2k-1)}|^2 a_i^\dagger a + \sum_{j=1}^{m} |\alpha_j^{(2k)}|^2 b_j^\dagger b \right) + \left( \sum_{i=1}^{n} |\alpha_i^{(7)}|^2 + \sum_{j=1}^{m} |\alpha_j^{(8)}|^2 \right) \mathbb{I} \]

### 5.3. The Evolution Equation

In order to define the operators \( R_i \) of the right H-P equations, the operators \( R_j^{(l)} \) will be used for the definition of the channels. So, the operators \( R_i \)
coincide with the operators $R_j^{(l)}$ in ascending order of the index $l$. A precise and complicated definition is

For $k = 1, 3, 5, 7$ and $\frac{k-1}{2}(n+m) + 1 \leq i \leq \frac{k+1}{2}n + \frac{k-1}{2}m$ we defined

$$R_i = R_j^{(k)} \quad \text{with} \quad j = i - \frac{k-1}{2}(n+m)$$

For $k = 2, 4, 6, 8$ and $\frac{k}{2}n + \frac{k-2}{2}m + 1 \leq i \leq \frac{k-1}{2}(n+m)$ we defined

$$R_i = R_j^{(k)} \quad \text{with} \quad j = i - \frac{k}{2}n + \frac{k-2}{2}m$$

Note that the equation (5.18) implies that $R = \sum_{i \geq 1} R_i^* R_i$. In this model $S_{ij} = \delta_{ij} 1$ and due of the Hypothesis 2.1, one has

$$F_{ij} = 0 \quad N_j = -R_j^* \quad \forall j \quad \text{and} \quad K = -iH - \frac{1}{2}R$$

Therefore the Hudson-Parthasarathy equation has the form:

$$dU(t) = \left( \sum_{i \geq 1} R_i dA_i(t) - \sum_{i \geq 1} R_i^* dA_i(t) + K dt \right) U(t), \quad U(0) = 1$$

5.4. Existence of the Dynamic

The proof of the existence of solution of the Hudson-Parthasarathy equation requires to check the Hypothesis 2.1. Just take $D = \tilde{D}$ given by the linear span of the basis $\{e(s, p); s = \{s_i\}_{i=1}^n, p = \{p_j\}_{j=1}^m \subset \mathbb{N}\}$ due to the construction of the H-P equation, the first seven points of the Hypothesis 2.1 are satisfies. The proof is short and is the same as in (5), Theorem 6.1.

The Hypothesis 2.1.(6) implies that $K$ and $K^*$ are dissipative operators and, therefore, they are generators of contraction semigroups (14), Corolary 4.4). Thus, the point Hypothesis 2.1.(8) is satisfied.

The very strong condition is the point (9) in the Hypothesis 2.1. For the demonstration, it takes several previous lemmas, which are shown in the appendix at the end of the article.

**PROPOSITION 1** Let $N := N_s + N_p$ with $N_s$ and $N_p$ as in the equation (5.10), then for any function $f$, is satisfies

$$a_k f(N) = f(N + w_k^s 1)a_k, \quad f(N)a_k^\dagger = a_k^\dagger f(N + w_k^s 1) \quad \forall k = 1, \ldots, n$$

and

$$b_k f(N) = f(N + w_k^p 1)b_k, \quad f(N)b_k^\dagger = b_k^\dagger f(N + w_k^p 1) \quad \forall k = 1, \ldots, m$$
Proof. Of the equations (4.13) and (4.14), for any \(e(s,p)\) element of the base of \(\mathcal{H}\), is obtain that:

\[
Ne(s,p) = \lambda(s,p)e(s,p) \quad \text{with} \quad \lambda(s,p) := \sum_{i=1}^{n} w_i^s s_i + \sum_{j=1}^{m} w_j^p p_j \quad (5.19)
\]

Therefore the number \(\lambda(s,p)\) is the eigenvalue associate with the vector \(e(s,p)\).

This allow to define the operator \(f(N)\) in the elements of the base by:

\[
f(N)e(s,p) := f(\lambda(s,p))e(s,p) \quad (5.20)
\]

analogously is defined:

\[
f(N + w_k^s \mathbb{1})e(s,p) := f(\lambda(s,p) + w_k^s)e(s,p) \quad (5.21)
\]

Then for equations (4.13), (5.20) and (5.21) is obtained that

\[
f(N + w_k^s \mathbb{1})a_k e(s,p) = f(N + w_k^s \mathbb{1})\sqrt{s_k}e(s - \delta_{ik},p) = \sqrt{s_k}f(\lambda(s - \delta_{ik},p) + w_k^s)e(s - \delta_{ik},p)
\]

but, for the equation (5.19) it follows that \(\lambda(s - \delta_{ik},p) = \lambda(s,p) - w_k^s\). Therefore

\[
f(N + w_k^s \mathbb{1})a_k e(s,p) = f(\lambda(s,p))\sqrt{s_k}e(s - \delta_{ik},p) = af(N)e(s,p)
\]

This prove the first equality of the Proposition. The other proofs are identical.

\[\square\]

In order to simplify the notation, we will use the letter \(q\) to indicate the number \(\lambda(s,p)\) in the equation (5.19), i.e.

\[
q := \lambda(s,p) = \langle e(s,p) | Ne(s,p) \rangle = \sum_{i=1}^{n} w_i^s s_i + \sum_{j=1}^{m} w_j^p p_i \quad (5.22)
\]

**PROPOSITION 2** The operator \(N\), defined in the Proposition, commutes with the Hamiltonian operator \(H\), defined in equation (5.16).

Proof. The operator \(H = N + I\), then \([H,N] = [N,N] + [I,N] = [I,N]\), therefore only be proof that \([I,N] = 0\)
\[ [I, N] = \left[ \prod_{i=1}^{n} a_i^\dagger \prod_{j=1}^{m} b_j - \prod_{i=1}^{n} a_i \prod_{j=1}^{m} b_j^\dagger, N_s + N_p \right] \]

Analyzing each term, one has:

\[
\left[ \prod_{i=1}^{n} a_i^\dagger \prod_{j=1}^{m} b_j, N_s \right] = \prod_{i=1}^{n} a_i^\dagger \prod_{j=1}^{m} b_j - \sum_{k=1}^{n} w_k^s a_k^\dagger a_k - \sum_{k=1}^{m} w_k^s a_k a_k^\dagger \prod_{i=1}^{n} a_i^\dagger \\
= \prod_{j=1}^{m} b_j \sum_{k=1}^{n} w_k^s \left( \prod_{i=1}^{n} a_i^\dagger a_k^\dagger a_k - a_k a_k^\dagger \prod_{i=1}^{n} a_i^\dagger \right) \\
= \prod_{j=1}^{m} b_j \sum_{k=1}^{n} w_k^s \prod_{i=1}^{n} a_i^\dagger \left[ a_k a_k^\dagger - a_k a_k^\dagger \prod_{i=1}^{n} a_i^\dagger \right] \\
= -\sum_{k=1}^{n} w_k^s \left( \prod_{j=1}^{m} b_j \prod_{i=1}^{n} a_i^\dagger \right)
\]

Analogously, for the terms:

\[
\left[ \prod_{i=1}^{n} a_i^\dagger \prod_{j=1}^{m} b_j, N_p \right] = \sum_{k=1}^{m} w_k^p \left( \prod_{j=1}^{m} b_j \prod_{i=1}^{n} a_i^\dagger \right)
\]

Therefore, for Remark 5.1, it follows that:

\[
\left[ \prod_{i=1}^{n} a_i^\dagger \prod_{j=1}^{m} b_j, N_s \right] + \left[ \prod_{i=1}^{n} a_i^\dagger \prod_{j=1}^{m} b_j, N_p \right] = 0
\]

To proceed in the other two cases, one must proceed in the identical form.

\[ \Box \]

**Proposition 3** For any \( u \in D \) Re \( \langle a_k u | [C_e, a_k] u \rangle \leq 32^{n+m} \langle u | C_e u \rangle \) and similarly for \( a_k^\dagger, b_k \) and \( b_k^\dagger \).

**Proof.** Note that Re \( \langle a_k u | [C_e, a_k] u \rangle \leq |\langle a_k u | [C_e, a_k] u \rangle| \) and can write the vector \( u \) as sum of the elements of the base \( \mathcal{H} \). The results is an immediate consequence of the Lemma \[ \Box \]
PROPOSITION 4 For any choice of \( u_0 \) and \( u \) in \( D \) one has

\[
2\Re \langle u \| [C, a_k] u_0 \rangle \leq 32^{n+m} \mathcal{L} \| u \|^2 + \| u_0 \|^2
\]

where \( \mathcal{L} \) is as in REMARK 2 of the Appendix.

Proof. Let \( \{ \gamma_l^0 \} \) and \( \{ \gamma_l \} \) two family of sequence in \( \mathbb{C} \) such that

\[
u_0 = \sum_l \gamma_l^0 e(s_l, p_l), \text{ and } u = \sum_l \gamma_l e(s_l, p_l), \text{ with } s_l = \{ s^l_i \}_1^n, \text{ and } p_l = \{ p^l_j \}_1^m
\]

Hence

\[
2\Re \langle u \| [C, a_k] u_0 \rangle = 2\Re \left( \sum_{ij} \gamma_l e(s_l, p_j^l) [C, a_k] \sum_l \gamma_l^0 e(s_l, p_l^j) \right)
\]

\[
= 2\Re \left( \sum_l \gamma_l e(s_l, p_j^l) \sum_l \gamma_l^0 \frac{L_k^a(q^l)}{\sqrt{s_k^{l+1}}} e(s_l - \delta_{lk}, p_j^l) \right)
\]

\[
= 2\Re \sum_l \gamma_l \gamma_l^{0, l+1} \frac{L_k^a(q^l)}{\sqrt{s_k^{l+1}}}
\]

but note that for any even complex numbers \( z, w \) one has \( 2\Re(z\overline{w}) \leq |z|^2 + |w|^2 \) and for any even sequences \( \{ a_i \}, \{ b_i \} \) the positive numbers one has

\[
\sum_i a_i b_i \leq \sum_i a_i \sum_i b_i.
\]

Therefore

\[
2\Re \langle u \| [C, a_k] u_0 \rangle \leq \sum_l \left| L_k^a(q^l) \right|^2 \sum_l \| \gamma_l \|^2 + \sum_l \| \gamma_l^0 \|^2
\]

For Lemma 4 one has

\[
\sum_l \left| L_k^a(q^l) \right|^2 \leq 32^{n+m} \mathcal{L}, \text{ the result is obtained.}
\]

A analogous results is obtained for \( a_k^\dagger, b_k \) and \( b_k^\dagger \).

\( \square \)

THEOREM 5 Hypothesis \( 2.1.(9) \) is satisfied.

Proof. Just take \( \delta = \frac{1}{2r^2(n+m)} \) as in the Lemma 3, \( C := N^{2(n+m)} \) and \( D_{\epsilon} = D, \forall \epsilon \in ]0, \delta[ \). Clearly \( C_{\epsilon^2} D_{\epsilon} \subset D \) being that all vectors of the canonical base are eigenvector of the operator \( C_{\epsilon^2} \). Clearly the operator \( a_k, a_k^\dagger, b_k \) and \( b_k^\dagger \) are relatively bounded with respect to \( K \)

\[
K = -i(N_s + N_p + \frac{g}{2} I) - \frac{1}{2} R
\]

and the component of \( K \) “more” unbounded is the operator \( I \). Therefor if \( IC_{\epsilon^2} \big|_{D_{\epsilon}} \) is bounded, then \( F_{ij} C_{\epsilon^2} \big|_{D_{\epsilon}} \) is bounded \( \forall i, j \geq 0 \)
Let \( u \in D \) and \( \gamma_\ell \) a sequence such that \( u = \sum_\ell \gamma_\ell e(s_\ell, p_\ell) \), with \( s_\ell = \{s_\ell^1\}^n_1 \) and \( p_\ell = \{p_\ell^1\}^m_1 \), then

\[
IC_{\gamma'}^2 u = \sum_\ell \gamma_\ell \mathcal{L}_0^{\frac{1}{2}}(q_\ell) \left( \prod_{i,j} \sqrt{(s_i^1 + 1)p_j^1} - \prod_{i,j} \sqrt{s_i^1(p_j^1 + 1)} \right) e'
\]

with \( e' = (e(s_i^1 + 1, p_j^1 - 1) - e(s_i^1 - 1, p_j^1 + 1)) \). Now \( s_i^1 + 1, p_j^1 + 1 \leq q_\ell, \forall i, j; \) therefore

\[
\|IC_{\gamma'}^2 u\| \leq \left\| \sum_\ell 2\gamma_\ell \frac{q_\ell^{n+m}}{1 + \epsilon q_\ell^{2(n+m)}} \sqrt{q_\ell^{n+m}} e' \right\| \leq \sum_\ell 2|\gamma_\ell| \frac{q_\ell^{\frac{3}{2}(n+m)}}{1 + \epsilon q_\ell^{2(n+m)}} \|e'\|
\]

the sequence \( \frac{q_\ell^{\frac{3}{2}(n+m)}}{1 + \epsilon q_\ell^{2(n+m)}} \rightarrow 0 \), hence

\[
\|IC_{\gamma'}^2 u\| \leq C \sum_\ell \|\gamma_\ell\| = C\|u\| \text{ for same value of the constant } C.
\]

A consequence of the Proposition 2 is that by explicitly computing the left hand side of the inequality (b) of the point (9) in the Hypothesis 2.1, one has:

\[
2\text{Re} \sum_{i \geq 1} \sum_{l=1}^8 \left\langle u_i + \frac{1}{2} R^{(l)}_i u_0 \big| C_\ell, R^{(l)}_i u_0 \right\rangle \leq \sum_{i \geq 0} \left( b_1 \left\langle u_i | C_\ell u_i \right\rangle + b_2 \|u_i\|^2 \right)
\]

For \( l = 1 \) one has

\[
2\text{Re} \sum_{k \geq 1} \left\langle u_k + \frac{1}{2} R^{(1)}_k u_0 \big| C_\ell, R^{(1)}_k u_0 \right\rangle = 2\text{Re} \sum_{k \geq 1} \left\langle u_k \big| C_\ell, \alpha^{(1)}_k a_k u_0 \right\rangle + \text{Re} \sum_{k \geq 1} \left\langle \alpha^{(1)}_k a_k u_0 \big| C_\ell, \alpha^{(1)}_k a_k u_0 \right\rangle
\]

for Proposition 4

\[
2\text{Re} \sum_{k \geq 1} \left\langle u_k \big| C_\ell, \alpha^{(1)}_k a_k u_0 \right\rangle \leq \sum_{k \geq 1} 2\text{Re} (\alpha^{(1)}_k) (\mathcal{L}\|u_k\|^2 + \|u_0\|^2)
\]

If \( b_2^{(1)} = \text{Max} \left( 2\text{Re} (\alpha^{(1)}_k) \mathcal{L}; 2\text{Re} (\alpha^{(1)}_k) \right) \) then

\[
2\text{Re} \sum_{k \geq 1} \left\langle u_k \big| C_\ell, \alpha^{(1)}_k a_k u_0 \right\rangle \leq \sum_{k \geq 0} b_2^{(1)} \|u_k\|^2
\]
also
\[
\text{Re} \sum_{k \geq 1} \left\langle \alpha_k^{(1)} a_k u_0 \right| [C_\epsilon, \alpha_k^{(1)} a_k] u_0 \right\rangle = \sum_{k \geq 1} |\alpha_k^{(1)}|^2 \text{Re} \left\langle a_k u_0 \right| [C_\epsilon, a_k] u_0 \right\rangle
\]

For Proposition 3
\[
\leq \sum_{k \geq 1} 32^{n+m} |\alpha_k^{(1)}|^2 \langle u_0 | C_\epsilon u_0 \rangle \leq b_1^{(1)} \sum_{k \geq 0} \langle u_k | C_\epsilon u_k \rangle
\]
where \(b_1^{(1)} := \sum_{k \geq 1} 32^{n+m} |\alpha_k^{(1)}|^2\)

Ad Analogous procedure is performed for \(a^\dagger, b\) and \(b^\dagger\). Then just make
\[
b_1 = \max_{i=1,\ldots,8} \left( b_1^{(i)} \right) \quad \text{and} \quad b_2 = \max_{i=1,\ldots,8} \left( b_2^{(i)} \right)
\]
and with this the proof is finished.

\[\Box\]

**COROLLARY 2** the equations (2.1) and (2.2) admit solution.

**Proof.** Is a immediate conclusion of the Theorem 5.

\[\Box\]

### 5.5. Unitarity Solution

The solution to the equation (2.1) and (2.2) does not imply, necessarily, that these are unitary. The unitarity of the solution is linked to the Markov Property of the minimal quantum semigroup defined in the Theorem 3. The unitary property is necessary of to have a solution with physical sense.

**THEOREM 6** The family of the operator \(U(t)\) and \(V(t)\), solution of the equations (2.1) and (2.2) respectively, are unitary process of the operators.

**Proof.** The proof is an application of the Theorem 2 where the operator \(A = K^*\) and the operators \(L_k = -\sum_{l=1}^{8} R_k^{(l)*}\). The operators \(Q\) and \(Z\) are defined.

\[
Q = k_1(N_s + N_p) + k_2 1 \quad \text{and} \quad Z = R
\]
where
\[ k_1 = \max_{k, l} \{ |\alpha_k^{(l)}|^2 \} \quad \text{and} \quad k_2 = \sum_k |\alpha_k^{(7)}|^2 + |\alpha_k^{(8)}|^2 \]

The proof continues as in the [5, Proposition 6.1]

\[ \square \]

5.6. Measurement in the Multi-Photon Model

For the construction of measurements we take the \( n + m \) photo-counter and the \( n \) homodyne detectors. By the equations (3.5), we defined the operator

\[ B_k = |z_k\rangle\langle z_k| \quad \text{if} \quad k = 1, \ldots, n + m \]

(5.23)

with \( B_k = 0 \) if \( k \neq 1, \ldots, n + m \)

and the functions:

\[ \langle h_k | z_i \rangle = \delta_{ik} e^{i(\theta_k - w^s_k)} \quad \text{if} \quad k = n + m + 1, \ldots, 2n + m \]

(5.24)

with \( h_k = 0 \) for \( k \neq n + m + 1, \ldots, 2n + m \)

According to the equations (5.23) and (5.24), the observable defined in the equation (3.6) take the form

\[ X(k, t) = \Lambda_{kk}(t) \quad \text{for} \quad k = 1, \ldots, n + m \]

(5.25)

and for \( k = n + m + 1, \ldots, 2n + m \)

\[ X(k, t) = \int_0^t \sum_{k \geq 0} \left( e^{-i(\theta_k - w^s_k)t} dA_k(s) + e^{i(\theta_k - w^s_k)t} dA_k^\dagger(s) \right) \]

(5.26)

In order to describe the coherent monochromatic lasers pumping the \( b_k \)-mode, we have to take a coherent state of the field with \( f \)-function given by:

\[ \langle f | z_i \rangle = \frac{i e^{-i w^s_i t}}{\alpha_k^{(i)}} 1_{(0,T)}(t) \quad i = 2n + m + 1, \ldots, 2(n + m) \quad \text{and} \quad k = i - 2n - m \]

(5.27)
6. Appendix

In this section we will provide several necessary lemmas for the proof of the proposition 3 and 4.

The operator $C$ is defined by $C := N^{2(n+m)}$ with an appropriate domain and for any $\epsilon > 0$ the operator $C_\epsilon$, defined by $C_\epsilon := \frac{C}{(1+\epsilon C)^2}$ is bounded.

The action of the operator $C_\epsilon$ over the elements of the base $\mathcal{H}$, due to the Proposition 1 and the equation (5.22), is given by

$$C_\epsilon e(s,p) = \frac{q^{2(n+m)}}{(1+\epsilon q^{2(n+m)})^2} e(s,p) \quad (6.28)$$

**DEFINITION 5** Now, we define the following auxiliaries functionals

$$\mathcal{L}_0(q) = \langle e(s,p) | C_\epsilon e(s,p) \rangle$$

$$\mathcal{L}_k^a(q) := \langle a_k e(s,p) | [C_\epsilon, a_k] e(s,p) \rangle \quad \mathcal{L}_k^a(q) := \langle a_k e(s,p) | [C_\epsilon, a_k] e(s,p) \rangle$$

$$\mathcal{L}_k^b(q) := \langle b_k e(s,p) | [C_\epsilon, b_k] e(s,p) \rangle \quad \mathcal{L}_k^b(q) := \langle b_k e(s,p) | [C_\epsilon, b_k] e(s,p) \rangle$$

Note that the functionals of Definition 5 depends of the choice of the sequences $s = \{s_i\}_{i=1}^n$, $p = \{p_j\}_{j=1}^m$, because that $q$ depends of these sequences.

**LEMMA 1**

$$\mathcal{L}_0(q) = \frac{q^{2(n+m)}}{(1+\epsilon q^{2(n+m)})^2}$$

$$\mathcal{L}_k^a(q) = s_k (\mathcal{L}_0(q - w_k^a) - \mathcal{L}_0(q)) \quad \mathcal{L}_k^a(q) = (s_k + 1) (\mathcal{L}_0(q + w_k^a) - \mathcal{L}_0(q))$$

$$\mathcal{L}_k^b(k) := p_k (\mathcal{L}_0(q - w_k^b) - \mathcal{L}_0(q)) \quad \mathcal{L}_k^b(k) := (p_k + 1) (\mathcal{L}_0(q + w_k^b) - \mathcal{L}_0(q))$$

**Proof.** The first equality is immediate. Of the equation (5.19) one has

$$Ne(s_i - \delta_{ik}, p_j) = \left( \sum_{i \neq k} w_k^a s_i + w_k^a (s_i - 1) + \sum_{j=1}^m w_k^{p_j} p_i \right) e(s_i - \delta_{ik}, p_j)$$

$$= (q - w_k^a) e(s_i - \delta_{ik}, p_j).$$
therefore $C(e(s_i - \delta_{ik}, p_j) = L_0(q - w^*_k)e(s_i - \delta_{ik}, p_j)$.

\[
\mathcal{L}_k^a(q) = \langle a_k e(s, p) \mid [C, a_k] e(s, p) \rangle 
= \langle \sqrt{s_k} e(s_i - \delta_{ik}, p_j) \mid C_\epsilon \sqrt{s_k} e(s_i - \delta_{ik}, p_j) - a_k L_0(q)e(s, p) \rangle 
= \langle \sqrt{s_k} e(s_i - \delta_{ik}, p_j) \mid \sqrt{s_k} (L_0(q - w^*_k) - L_0(q))e(s_i - \delta_{ik}, p_j) \rangle 
= s_k (L_0(q - w^*_k) - L_0(q))
\]

The other case are analogous.

\[\square\]

**Remark 2:** Due to that \(\int_{\mathbb{R}^+} L_0(x) dx \leq \infty\), we can define the constant \(L = \sum_q L_0(q)\)

**Lemma 2** For any \(r > 0\) and \(k \in \mathbb{N}\), exist \(\epsilon > 0\) such that

\[
\frac{1}{(1 + \epsilon(x - r)^{2k})^2} \leq \frac{16^k}{(1 + \epsilon x^{2k})^2} \quad \forall x \in \mathbb{R}^+
\]

**Proof.**

If \(x \geq 2r\), then \(\frac{1}{x - r} \leq \frac{2}{x}\) and this implies \(\frac{1}{1 + \epsilon(x - r)^{2k}} \leq \frac{4^k}{1 + \epsilon x^{2k}}\) and this implies the result.

If \(0 \leq x < 2r\) then inequality is equivalent \(\epsilon (x^{2k} - 4^k(x - r)^{2k}) \leq 4^k - 1\) the maximal of the polynomial \(f(x) = x^{2k} - 4^k(x - r)^{2k}\) is in \(x_M = \frac{2^{2k-1}r}{2^{2k-1} - 1}\), therefore is \(f(x_M) = \frac{2^{4k^2}}{(2^{2k-1} - 1)^{2k}} r^{2k} = (2r)^{2k}\)

Therefore \(\forall x \in [0, 2r]\) one has that \(\epsilon (x^{2k} - 4^k(x - r)^{2k}) \leq \epsilon (2r)^{2k}\) then just take \(\epsilon < \frac{1}{2^{2k} r}\) and proof is finished.

\[\square\]

**Lemma 3** For any \(i, j \in \mathbb{N}\) exist \(\epsilon > 0\) such that
LEMMA 4 For any \( \{s_i\} \) and \( \{p_j\} \) one has:

\[
|\mathcal{L}_k^a(q)| \leq 32^{n+m}L_0(q), \quad |\mathcal{L}_k^a(q)| \leq 32^{n+m}L_0(q)
\]

\[
|\mathcal{L}_k^b(q)| \leq 32^{n+m}L_0(q), \quad |\mathcal{L}_k^b(q)| \leq 32^{n+m}L_0(q)
\]

Proof. We provide the first inequality

\[
|\mathcal{L}_k^a(q)| = |\langle L_0(q - w_k^s) - L_0(q)\rangle| \quad \text{For Lemma}\ 4
\]

\[
eq s_k \left| \frac{(q - w_k^s)^{2(n+m)}}{(1 + \epsilon(q - w_k^s)^{2(n+m)})^2} - \frac{q^{2(n+m)}}{(1 + \epsilon q^{2(n+m)})^2} \right| =
\]

\[
s_k \left| \frac{(q - w_k^s)^{n+m}}{1 + \epsilon(q - w_k^s)^{2(n+m)}} - \frac{q^{n+m}}{1 + \epsilon q^{2(n+m)}} \right| \left( \frac{q^{n+m}}{1 + \epsilon(q - w_k^s)^{2(n+m)}} + \frac{q^{n+m}}{1 + \epsilon q^{2(n+m)}} \right)
\]

Now

\[
\left( \frac{(q-w_k^s)^{n+m}}{1+\epsilon(q-w_k^s)^{2(n+m)}} + \frac{q^{n+m}}{1+\epsilon q^{2(n+m)}} \right)
\]

\[
= \frac{(q-w_k^s)^{n+m}(1+\epsilon q^{2(n+m)})+(1+\epsilon(q-w_k^s)^{2(n+m)})q^{n+m}}{(1+\epsilon(q-w_k^s)^{2(n+m)})(1+\epsilon q^{2(n+m)})}
\]

\[
= \frac{(q-w_k^s)^{n+m}+q^{n+m}}{(1+\epsilon(q-w_k^s)^{2(n+m)})(1+\epsilon q^{2(n+m)})} \quad (6.29)
\]

\[
\leq \frac{2q^{n+m}(1+\epsilon q^{2(n+m)})}{(1+\epsilon(q-w_k^s)^{2(n+m)})(1+\epsilon q^{2(n+m)})} \quad (6.29)
\]

\[
= \frac{2q^{n+m}}{1+\epsilon(q-w_k^s)^{2(n+m)}}
\]
Analogously one can show

\[
\begin{align*}
  s_k \left| \frac{(q-w_k^s)^{n+m}}{1+\epsilon(q-w_k^s)^{2(n+m)}} - \frac{q^{n+m}}{1+\epsilon q^{2(n+m)}} \right| \\
  \leq s_k \left| \frac{(q-w_k^s)^{n+m} - q^{n+m}}{1+\epsilon(q-w_k^s)^{2(n+m)}} \right| \leq \frac{2^{n+m-1}q^{n+m}}{1+\epsilon(q-w_k^s)^{2(n+m)}}
\end{align*}
\]  

(6.30)

Therefore by equations (6.29) and (6.30) and Lemma 3 the result is obtained. The other cases are identical.

\[\square\]

Bibliography

[1] Alberto Barchielli, Continual Measurements in Quantum Mechanics and Quantum Stochastic Calculus. In S. Attal, A. Joye, C.-A. Pillet (eds.), Open Quantum Systems III, Lecture Notes in Mathematics 1882 (Springer, Berlin, 2006), pp. 207–291.

[2] Alberto Barchielli, Direct and heterodyne detection and other applications of quantum stochastic calculus to quantum optics, Quantum Optics 2 (1990), 423–441.

[3] V.P. Belavkin, Nondemolition measurements, nonlinear filtering and dynamic programming of quantum stochastic process, Lecture Notes in Control and Information Sciences Modelling and Control of System (1988), No 121, 245–265.

[4] Ricardo Castro Santis, Quantum stochastic calculus with unbounded coefficient, 1 ed., Lambert Academic Publisher, 2010

[5] R. Castro Santis and A. Barchielli Quantum stochastic differential equations and continuous measurements: unbounded coefficients, Reports on Mathematical Physics (accepted).

[6] E. B. Davies, Quantum Theory of Open Systems. Academic Press, London, 1976.

[7] F. Fagnola, Quantum Markov Semigroups and Quantum Flows Proyecciones 18, n.3 (1999).

[8] F. Fagnola, Quantum Stochastic Differential Equations and Dilation of Completely Positive Semigroups. In S. Attal, A. Joye, C.-A. Pillet (eds.), Open Quantum Systems II, Lecture Notes in Mathematics 1881 (Springer, Berlin, 2006), pp. 183–220.

[9] F. Fagnola, S. J. Wills, Solving quantum stochastic differential equations with unbounded coefficients, J. Funct. Anal. 198 (2003) 279–310.

[10] Zhe-Yu, Jeff Ou, Multi-Photon Quantum Interference, Springer Verlag, 2006.

[11] A. S. Holevo, Statistical Structure of Quantum Theory, Lecture Notes in Physics m 67. Springer, Berlin, 2001.

[12] R.L. Hudson and K.R. Parthasarathy, Quantum Itô formula and stochastic evolutions, Communications in Mathematical Physics (1992), No 93, 301–323.

[13] K.R. Parthasarathy, An Introduction to Quantum Stochastic Calculus (Birkhäuser, Basel, 1992).

[14] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations (Springer, Berlin, 1983).