DYNAMICS OF CONDUCTIVE/COOLING FRONTS:
CLOUD IMPLOSION AND THERMAL SOLITONS.

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ABSTRACT
We investigate the evolution of interfaces among phases of the interstellar medium with different temperature. It is found that, for some initial conditions, the dynamical effects related to conductive fronts are very important even if radiation losses, which tend to decelerate the front propagation, are taken into account. We have also explored the consequences of the inclusion of shear and bulk viscosity, and we have allowed for saturation of the kinetic effects. Numerical simulations of a cloud immersed in a hot medium have been performed; depending on the ratio of conductive to dynamical time, the density is increased by a huge factor and the cloud may become optically thick. Clouds that are highly compressed are able to stop the evaporation process even if their initial size is smaller than the Field length. In addition to the numerical approach, the time dependent evolution has been studied also analytically. Simple techniques have been applied to the problem in order to study the transition stages to a stationary state. The global properties of the solution for static and steady fronts and useful relations among the various physical variables are derived; a mechanical analogy is often used to clarify the physics of the results. It is demonstrated that a class of soliton-like solutions are admitted by the hydrodynamical equations appropriate to describe the conduction/cooling fronts (in the inviscid case) that do not require a heat flux at the boundaries. Solitons are shown to result from the exact balance of convective and conductive energy transport and we demonstrate that they can exist only as associated with the fast velocity mode (analogous to the positive Riemann invariant) of the system. Some astrophysical consequences are indicated along with some possible applications to the structure of the Galactic ISM and to extragalactic objects.
1. INTRODUCTION

The problem of the thermal evaporation of cool gas clouds embedded in a hot gas has received particular interest in the last years. Apart from the richness of physical aspects involved in the process, a strong astrophysical motivation for the study of this type of systems is certainly provided by the supposed ubiquitous presence of a hot phase of the interstellar medium (ISM) in the galactic disk and halo, coexisting with cooler phases. It is therefore crucial to understand the details of the physics to construct plausible models of the ISM.

The first systematic approach to the problem of the effects of thermal conduction on a cool cloud immersed in a hot medium has been undertaken by Cowie & McKee (1977). They obtained analytical solutions for the steady-state evaporation flow from the cloud; in addition, they pointed out that the classical diffusion approximation of thermal conduction breaks down when the mean free path of electrons becomes larger than the temperature scale (saturation). The relevance of this effect is quantified by the local parameter \( \sigma_T \), which expresses the ratio between classical and saturated conduction. Some points are worth to be mentioned: supposing that the flow is isobaric (or, equivalently, that its Mach number \( M \) is small) implies that classical conduction holds; secondly, if conduction is saturated, the usual hydrodynamic equations are not valid anymore, due to the large mean free paths involved. Giuliani (1984) pointed out that their solutions are obtained using a piece-wise method which leads to unphysical discontinuities of the derivatives of the variables at the transition points; his calculation allows instead for a gradual change from the classical to saturated regime.

In a subsequent paper McKee & Cowie (1977) generalized the steady state problem to include the effects of radiative losses, and they demonstrated the existence of a critical length at which radiative losses balance the conductive heating. A clear cut result for what concerns the steady state of conductive/cooling (CC) flows is given by McKee & Begelman (1990). They recognized that the key parameter governing the structure of the flow is the so-called Field length. This length corresponds to the largest wavelength of a perturbation stabilized by thermal conduction against thermal instability (Field 1965). Clouds smaller than the Field length will suffer evaporation, while clouds larger than the Field length will undergo evaporation if the pressure is lower than the saturated vapor pressure (Penston & Brown 1970), otherwise they would undergo condensation. Remarkably enough, this result is in agreement with the one found more than twenty years before by Zel’dovich & Pikelner (1969). These two studies, therefore, clearly define the necessary conditions that different ISM phases must fulfill in order to coexist in a stable steady state.

Aside from the consideration of radiative losses, a number of physical processes may affect the evolution of conduction fronts. Viscous stresses have been studied by Draine & Giuliani (1984) who show that they may have dramatic consequences on the solutions, in particular when conduction is highly saturated. This can be understood recalling that, if ion-electron temperature equipartition holds, electron and ion mean free paths are almost equal (Spitzer 1962). The main modifications to the results obtained by Cowie & McKee (1977) are a reduced Mach number of the evaporative outflow and value of \( \sigma_T \), and the appearance of a collisionless shock transition. They also have allowed, in analogy with saturated conduction, for viscosity saturation when velocity gradients become large. The impact of conduction on complex multi-phase systems resembling the actual ISM has been discussed by Balbus (1985) and Begelman and McKee (1990).

As it can be realized from the above discussion, most of the studies focused on the steady-state properties of the evaporation/condensation process. A remarkable exception is represented by the work of Doroshkevich and Zel’dovich (1981). They have studied the temporal evolution of CC fronts considering the effect of radiative losses, and have shown that, if the hot gas is freely cooling, a cooling wave propagates into the hot phase. Apart from the issue of the uncertain existence of the intermediate asymptotic regime they explore (for an extended discussion, see McKee & Begelman 1990), their thermal wave approach neglects all kind of hydrodynamic motions which may arise in the system. One of the aims of this paper is to demonstrate the relevance of such hydrodynamical effects which, almost unavoidably, are related to the presence of CC fronts, at least at the initial stages of their evolution. The basic principle on which our conviction is based is that, if the initial gradient of temperature at the interface between the hot and cold phase is large enough, the evaporation time \( \tau_k \) becomes smaller than the cloud characteristic dynamical time \( \tau_d \). In this case the pressure gradient, driving the motion, relaxes more slowly than the thermal one. This importance of the dynamical aspects has been emphasized also by Kovalenko & Shchekinov.
Thus, the interface behaves somewhat as an explosion site, where the existing pressure excess generates an outward flow (evaporation) and an inward one (implosion).

In a very elegant paper Elphick, Regev & Shaviv (1992) (but see also Elphick, Regev & Spiegel 1991) suggest a new approach to the study of the dynamics of CC fronts. They apply non-linear dynamics techniques, as the Lyapunov functionals, to the investigation of the dynamics and interaction of a system of localized structures constituted by the fronts in a thermally bistable medium. In spite of some approximations introduced into the problem (small thermal conductivity, ideal cooling function), their approach reveals extremely promising in the study of the dynamics of CC fronts. In the second part of the present paper we also make use of non-linear dynamics arguments to obtain the conditions under which a particular class of hydrodynamical non-linear waves, i.e., soliton-like, may describe astrophysical CC fronts. Although this subject can have an important impact on our understanding of the structure of the ISM (Adams & Fatuzzo 1992), it has not yet been studied in the context of conductive flows. The present paper will concentrate on the one dimensional case, but we feel that higher dimensional studies may discover an even larger variety of important physical aspects.

The paper is organized as follows. In §2 we show the result of numerical simulations where the effects of radiative losses, viscosity, and ionization are included along, of course, with thermal conduction. Saturation effects are also taken into account both for conduction and viscosity. In §3 we present some analytical calculations that are intended to represent a slightly different formulation of some of the aspects of the steady-state analysis, making use of the dynamical ideas put forward by this paper; §4 deals with soliton-like solutions, and in §5 a brief summary is given and some possible astrophysical consequences of the results are discussed.

2. EVOLUTION AND DYNAMICS OF CC FRONTS

Following the line presented in the Introduction, in this Section we mainly want to demonstrate that under the conditions which are thought to exist in the interstellar medium of galaxies, some regimes may result in strong non-linear, non-steady behaviour of the system. Therefore, rather than explore the entire range of parameters of the relevant physical quantities that may govern the mass, momentum and energy exchange among the various phases of interstellar medium, we concentrate on the cases that can more appropriately show the kind of dynamical features that are the main subject of the present work. Moreover, several other studies (see Introduction) have already investigated the static and steady-state approximations and we refer the reader to that papers for a general discussion.

The actual interstellar medium is far from being homogeneous and is characterized by the presence of strong gradients in the hydrodynamical quantities at the interfaces among the various phases by which it is constituted. Even when a pressure equilibrium is achieved between spatially contiguous phases, those gradients may induce bulk motions that eventually can drive the system away from the initial equilibrium. The mass and energy exchange taking place in these cases is governed, in general, by the combined action of thermal conductivity, radiative energy losses, viscosity and external heating provided to the system; we can refer appropriately to it as a conductive/cooling (CC) front.

In principle, a magnetic field may also play a non-secondary role in the evolution of CC fronts. However, as Balbus (1986) and Borkowski, Balbus & Fristrom (1990) have demonstrated, unless the magnetic field is perpendicular to the front propagation direction, it is not likely to appreciably modify the general evolution. On the other hand, if magnetic field is tangled and threads both hot and cold phases, the reduction of the conductive transport coefficient is not well known; some works indicate that this effect can be indeed a minor one, at least for low densities (Rosner & Tucker 1989; Tribble 1989). For these reasons, we will neglect magnetic fields in the following; the limits of this assumption will be discussed in §2.1.

The basic equations describing mass, momentum and energy conservation can be set in the following form:

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}),
\]

\[
\frac{D \mathbf{v}}{D t} = -\nabla \cdot \Pi,
\]

\[
\frac{D \rho}{D t} = -\nabla \cdot (\rho \mathbf{v}),
\]
\[
\frac{R}{\mu} \rho T \frac{DS}{Dt} = -\rho \mathcal{L}(p, T, x) + \nabla \cdot (\kappa \nabla T) + (\Pi - p\delta_{ij}) \frac{\partial v_i}{\partial x_j},
\tag{2.3}
\]

where \(D/ Dt\) is the Lagrangian derivative. The gas is supposed to be perfect and with solar abundances, with temperature \(T\), velocity \(v\) and density \(\rho\); the pressure \(p\) is obtained from the equation of state (2.4); \(s\) is the specific entropy; other symbols have usual meaning.

The equilibrium cooling function \(\mathcal{L}\) has the usual form

\[
\rho \mathcal{L}(\rho, T) = n^2 \Lambda(x, T) - n \mathcal{H}(\rho, T),
\tag{2.5}
\]

where \(n\) is the total particle number density, \(x\) is the fractional ionization of the medium and \(n\Lambda\) and \(\mathcal{H}\) are the cooling and heating rates per particle. We have calculated \(\mathcal{L}\) from the rates for microscopic processes given by Dalgarno & McCray (1972) and Black (1981) at low temperatures, while for the optically thin plasma case we have used the results obtained by Raymond et al. (1976).

The function \(\kappa\) is the thermal conduction coefficient governed by atomic diffusion at low temperatures (below \(10^4\) K) and by electronic diffusion at high temperatures. The adopted form of \(\kappa\) is

\[
\kappa(x, T) = x\kappa_e T^{5/2} + (1 - x)\kappa_a T^{1/2},
\tag{2.6}
\]

where \(\kappa_e = 6.7 \times 10^{-6}\) and \(\kappa_a = 3.5 \times 10^3\) in cgs units. The stress tensor \(\Pi\) is

\[
\Pi_{ij} = -\eta \epsilon_{ij} - \zeta \delta_{ij} \nabla \cdot v + p\delta_{ij}.
\tag{2.7}
\]

The non-diagonal components of \(\Pi\) are

\[
\epsilon_{ij} = \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial v_l}{\partial x_l} \right),
\tag{2.8}
\]

where \(\eta = (2/3) Pr\kappa/R\) is the viscosity coefficient corresponding to shear motions with Prandtl number \(Pr\), whereas \(\zeta\) is the bulk viscosity.

The system (2.1)-(2.4) can be put in a non-dimensional form if the physical variables are appropriately scaled. This allows not only to write the equations in a simpler form, but it provides a first insight on the physical scales of the problem. A natural scaling is obtained if the cooling time \(\tau_c = [RT/(\gamma - 1)\mu n \Lambda]_0\) is taken as time unit and lengths are expressed in terms of \(\ell_F = (\kappa T/n^2 \Lambda)_0\), where the subscript 0 indicates that the quantities in parenthesis must be evaluated at appropriate values of the variables. The scale length \(\ell_F\) is usually named “Field length”; this length corresponds to the maximum wavelength of linear perturbations stabilized by thermal conduction. An additional characteristic time \(\tau_k = [p\ell^2/(\gamma - 1)\kappa T]_0\), where \(\ell\) is a fiducial size of the system, directly related to thermal conductivity, can be introduced.

An important point is that thermal conduction and viscosity are intrinsically kinetic processes and therefore it is crucial to understand the limits of any hydrodynamical approach dealing with such phenomena. This limits are set by the condition that the mean free path of the species (i.e., atoms or electrons) providing the energy and momentum transport must be larger than the scale length of the system, typically represented by the Field length for the cooler gas. As pointed out by several papers (Cowie & McKee 1977; Balbus & McKee 1982; McKee & Begelman 1990) this can be a quite common situation in the interstellar and intergalactic medium. A correct solution of the problem in this case would require the use of Boltzmann equation; unfortunately, this is possible only for simple cases where particular simplifications can be introduced (Draine & Giuliani 1984). A good phenomenological approximation is provided by the expression introduced by Giuliani (1984) for the thermal flux \(q\)

\[
q = -\kappa \left( \frac{\nabla T}{1 + \sigma_T} \right),
\tag{2.9}
\]
where \( \sigma_T \), known as the saturation parameter, is

\[
\sigma_T = \frac{|\kappa \nabla T|}{q_{sat}},
\]

and

\[
q_{sat} \sim \rho v_{th}^3,
\]

where \( v_{th} \) is the mean velocity for a Maxwellian distribution of the particle velocities. A similar relation holds also for the saturated viscosity (Draine & Giuliiani 1984). In strict analogy with conduction they make the following position for the saturated viscosity coefficient

\[
\eta_T = \eta \left(1 + \sigma_v\right),
\]

where \( \sigma_v = 2\eta \left(\frac{\epsilon_m}{\varphi_v p_i}\right) \).

In the previous equation the parameter \( \epsilon_m \) represents the largest positive eigenvalue of the matrix \( \epsilon_{ij} \), \( p_i \) is the usual ion isotropic pressure and \( \varphi_v \), describing the maximum temperature anisotropy, can be reasonably taken equal to 2. We adopted the same treatment also for the bulk viscosity coefficient.

Before going on and describe the numerical solutions, we believe it is instructive to have an insight of the relevant scales of the problem.

The characteristic lengths of the problem defined above are related by a very simple expression:

\[
\tau_k = \left(\frac{\ell}{\ell_F}\right)^2 \tau_c.
\]

Equation (2.13) describes a line separating the \( \tau_c-\tau_k \) plane in two regions (Fig. 1): the upper one \( (\tau_k > \tau_c) \) where cooling processes dominate over conductive ones \( (i.e., \text{condensation}) \) and the lower one \( (\tau_k < \tau_c) \), dominated by thermal conduction \( (i.e., \text{evaporation}) \). A common interpretation of the previous statement is probably based on the implicit assumption that the dynamical behaviour of the system can be neglected and the interface between the two phases can be treated as stationary. This is equivalent to impose that both \( \tau_c \) and \( \tau_k \) are much shorter than the dynamical response time of the system, which can be approximated as \( \tau_d \sim (\ell/\ell_F)c_0 \). This characteristic time has in effect a slightly more delicate definition when two gas phases with huge temperature differences are present. In this case, the actual value of \( \tau_d \) depends on the details of the geometry of the system, but since it is likely that the response of the colder phase to external pressure perturbations is slower, its dynamical time at least to provides an upper limit to this quantity. An usual astrophysical situation consists of a cold cloud, of size \( \ell \) and internal sound speed \( \bar{c}_s \), embedded in a hot medium, with sound speed \( c_s \). To simplify our considerations, we suppose that the cloud is initially in thermal equilibrium, \( \dot{L} = 0 \). If we substitute \( \ell = \bar{c}_s \tau_d \) in eq. (2.13), the locus of the points satisfying the condition \( \tau_c = \tau_d \) is

\[
\tau_k = \left(\frac{\bar{c}_s}{\ell_F}\right)^2 \tau_c.
\]

This curve intersects the one of eq. (2.13) in the point \( \tau_c = \ell_F/\bar{c}_s \) and, since \( \tau_c = \tau_k \) on the curve (2.13), at the intersection point is \( \tau_k = \tau_c = \tau_d \) (Fig. 1). In order to identify the regions of the \( \tau_c-\tau_k \) plane corresponding to a different \( \tau_k/\tau_d \) ratio, we substitute \( \ell = \bar{c}_s \tau_d \) in eq. (2.13), with the additional condition \( \tau_d = \tau_k \). These relations provide the locus of the points corresponding to \( \tau_k = \tau_d \):

\[
\tau_k = \left(\frac{\ell_F}{\bar{c}_s}\right)^2 \tau_c^{-1}.
\]
These three curves divide the $\tau_c-\tau_k$ plane into six regions with different ratios of the three characteristic times $\tau_c$, $\tau_k$, and $\tau_d$, as shown in Fig. 1. Therefore, once the temperature $T$ of the hot medium has been fixed, four different regimes are possible (if we neglect the patologic point at the intersection of the three curves), actually depending on $\bar{t}$: these cases are labeled in Fig. 1 as A, B, C and D.

Let us consider initially the subregions of this plane for which the relation between the characteristic times are described by strong inequalities ($\ll$ or $\gg$), represented by the dashed subregions of the $\tau_c-\tau_k$ plane. For a cloud with parameters corresponding to one of these subregions, the initial dynamics could be described in terms of the possible evolutionary modes ($i.e.$, isochoric, isobaric, isothermal). However, this description is not complete because, as discussed above, three different times $\tau_c$, $\tau_d$, $\tau_k$ govern the behaviour of the system, and yet it is possible to have different intermediate asymptotic regimes sharing a similar initial behaviour. Thus, in order to classify different regimes of the evolution of a cloud embedded in a hot intercloud gas it is necessary to take into account both the initial behaviour and intermediate asymptotics. We will assume, therefore, that during the initial stages of the evolution the relation between the largest characteristic times is not changed by the different dependence of those times on the dynamical variables. This holds for the sufficiently deep parts of the dashed regions of $\tau_c-\tau_k$ plane. The main features of the cloud evolution (initially in thermal equilibrium, as mentioned above) can be summarized as follows:

Region 1 ($\tau_k \ll \tau_c \ll \tau_d$): initial isochorical heating with subsequent cooling of the heated gas at $\tau_c < t \ll \tau_d$ — no evaporation;

Region 2 ($\tau_k \ll \tau_d \ll \tau_c$): isochorical heating with subsequent expansion of the heated gas at $\tau_d < t \ll \tau_c$ — evaporation;

Region 3 ($\tau_d \ll \tau_k \ll \tau_c$): isobarical expansion of the slowly heated gas — evaporation;

Region 4 ($\tau_d \ll \tau_c \ll \tau_k$): isobarical accretion of the cooling intercloud gas close to the interface onto the cloud — condensation;

Region 5 ($\tau_d \ll \tau_c \ll \tau_k$): isochorical cooling of the intercloud gas with subsequent slow accretion at $\tau_d < t \ll \tau_d$ — condensation;

Region 6 ($\tau_c \ll \tau_k \ll \tau_d$): isochorical cooling of intercloud gas with subsequent slow accretion — condensation.

A straight consequence is that only the initial evolution of a cloud with parameters lying in extreme regions of the $\tau_c-\tau_k$ plane (dashed in Fig. 1) can be considered as static or steady. The hydrodynamical motions which develop in the intermediate stages require a dynamical description. This statement is even stronger if the cloud initial parameters are located well into the undashed parts of Fig. 1, where it is impossible to separate the initial stages of the evolution from the intermediate asymptotics.

The main aim of this paper is to investigate these dynamical effects occurring at the interface among different phases of the ISM.

2.1 Numerical results

In the following we present the results concerning the numerical solution of eqs. (2.1)-(2.4). A number of simplifying assumptions have been made to solve the problem and we want to state them clearly to allow the reader to evaluate the limitations of our results.

We assume a) a spherical cloud immersed in a hot, homogeneous surrounding medium; b) initial pressure equilibrium $nT = \bar{n}\bar{T}$, where $n$ and $T$ are the density and temperature of the hot gas, respectively, and barred quantities refer to the cloud gas; c) the cloud is supposed to be initially in thermal equilibrium ($\mathcal{L} = 0$) in an isobarically stable state; the hot medium is instead allowed to cool down. In practice, given the ratio of the characteristic relevant times for the cases of interest here, the temperature of the hot medium is almost constant during the run; d) the ionization fraction is calculated at every time step by equating the collisional ionization and recombination rates. e) magnetic fields and non-equilibrium effects in the cooling function are neglected.

The main difficulty when solving the system (2.1)-(2.4) consists in the large difference among the time scales of the various processes. This is a typical situation encountered in stiff
problems, for which appropriate codes must be developed. For our purposes we have adapted the numerical code described by Kovalenko (1993). This is an implicit conservative code in Lagrangian variables that conserves mass, energy and, in absence of dissipation, entropy as well; the conservation of entropy means that the effects of scheme viscosity are negligible.

Fig. 2 shows the evolution of the relevant hydrodynamical variables describing the cloud-hot gas system as a function of radius. For sake of clarity, the displayed variables are normalized with the corresponding cloud ones and the velocity is normalized with the cloud internal sound speed $\bar{c}_s$. The initial conditions are the following: cloud temperature $\bar{T} = 10^4$ K, cloud density $\bar{n} = 1$ cm$^{-3}$, hot gas temperature $T = 3 \times 10^6$; the cloud size is $\ell = 3.68$ pc. From the pressure equilibrium condition, thus, the hot gas density is $n = 3.3 \times 10^{-3}$ cm$^{-3}$. This set of parameters has been chosen because they identify a C-type case according to the notation introduced in Fig. 1. As already pointed out, these cases are the most suitable one to elucidate the dynamical effects we are interested in; by the way, this particular initial condition is also a realistic one as far as the Galactic ISM is concerned. Values of the pressure of this order ($p/k = 10^4$ K cm$^{-3}$) are estimated in the Galactic disk, not only for the thermal component but also for the magnetic field and cosmic ray ones (Spitzer 1990). In addition, observational arguments indicate that most of the HI mass in the thin disk is contained in clouds of sizes 1-10 pc (Lockman, Hobbs & Shull 1986). The Field length for such a system is rather large: $\ell_F = 1.249 \times 10^3$ pc $\gg \ell$, as expected for a dynamical evaporation regime. It is also useful to compare the three relevant time scales for this numerical experiment: $\tau_c = 3 \times 10^7$ yr, $\tau_k = 8.6 \times 10^{-6} \tau_c$, $\tau_d = 0.01 \tau_c$. Under these conditions the electron mean free-path is $\lambda_e = 11.9$ pc and therefore thermal conduction is saturated; the global saturation parameter (McKee & Cowie 1977) is $\sigma_0 = 2 \lambda_e/\ell = 3.2$. The saturation of thermal conduction is included in the hydrodynamical code in the approximation described by eq. (2.9). As discussed in the Introduction, there are reasons to believe that when thermal conduction becomes saturated, this is the case also for viscosity. For sake of comparison, however, we present two numerical experiments that explore the cases $\varphi = \infty$ (“classical” viscosity) and $\varphi = 2$ (saturated viscosity).

Six different evolutionary stages, relative to the case $\varphi = \infty$, are reported in Fig. 2. At the beginning the strong temperature gradient present at the interface produces a huge localized thermal energy injection. In some sense, the interface behaves as an explosion site and the problem has some analogies, for example, with point explosions. However, the explosion point is not fixed in space, but it is comoving with the thermal front towards $r = 0$. As a result of the energy injection, two pressure perturbations are generated, propagating in opposite directions, as illustrated by the velocity profiles. The wave travelling with negative velocity becomes supersonic already at about $t = 0.15 \tau_d$, while the one travelling away from the cloud surface remains subsonic for all the subsequent evolution. This difference can be easily understood recalling that $c_s/\bar{c}_s \sim (T/\bar{T})^{1/2} \sim 17$. The shoulder-like feature shown by the temperature profile in the first three frames has the same width of the overpressurized region between the two pressure discontinuities. This feature is due to the density contrast between the left and the right boundary regions: radiative losses are enhanced where the density is higher and therefore the whole zone is strongly radiatively cooled. Another interesting phenomenon is represented by the compression of the cloud gas (for which $T \leq 10^4$ K). The spherical compression wave travelling towards the cloud center increases the density of the cool material and, at the same time, it is decelerated by the pressure gradient building up at the cloud center. Therefore a core is formed with characteristic density $\sim 100$ times the initial one. We will discuss this point in more detail below. When the pressure gradient inside the cloud has reached a critical value, the implosion is stopped and the velocity changes sign, i.e., the wave is reflected. This is evident from the fifth panel of Fig. 2, where also a pressure spike, which will later relax driving the reflected wave, is shown. The reflected wave will eventually superpose to the initial outward wave, which at the time $t = \tau_d$ has already reached $r \sim 3$. Note that the internal pressure is one order of magnitude larger than at $t = 0$. We have also followed the evolution for $t > \tau_d$ which can be summarized as follows. The mass loss in the outflow is reduced to very small values; the system settles down in a quasi-steady state, if one neglects the damped small oscillations of the interface around its equilibrium position, occurring with a period of the order of $\Delta t = 0.25 \tau_d$. These oscillations are the remnant of the initial bouncing and they are the result of the “elasticity” of the system. Eventually, at $t \sim \tau_c = 100 \tau_d$, the hot medium will start to cool considerably and a cooling wave will propagate into the ambient medium.

An important result is that even if the cloud size is smaller than the Field length, the
cloud will not be completely evaporated in cases like the present one (grouped under type C in Fig. 1) because of the strong compression taking place in its internal regions. In fact, the density and pressure increase induced by the compression enhances the radiative losses, thus decreasing, and eventually reversing, the front propagation velocity. A similar retarding effect due to cooling losses can be realized also from an inspection of the isobaric case, as we will show in § 3.1, and it is responsible for the transition to the steady state. However, the inclusion of the dynamics dramatically amplifies its consequences. In other words, the non-linear effects related to the dynamics (i.e., compression waves) modify the ratio between conductive heat flux and cooling, ultimately changing the Field length.

It is also instructive to look at the evolution of the local saturation parameter \(\sigma_T\) at various radii. In Fig. 3 the behaviour of \(\sigma_T\) is shown at three different evolutionary times. Clearly, inside the cloud the conduction is always not saturated. In the interface zone, instead, \(\sigma_T\) can be as high as \(\sim 100\), but at \(t = 0.6\tau_d\) it has already decreased to \(\sim 8\) and after that moment the conduction rapidly becomes classical. The peak values of \(\sigma_T\) are reached closely behind the interface, where the temperature gradients are larger; however, secondary structures appear (i.e., the peak at larger radius for \(t = 0.6\tau_d\) and the dip for \(t = \tau_d\)) that are mainly governed by the density changes corresponding to pressure jumps (see eqs. [2.10] and [2.11]).

Fig. 4 shows the evolution for the same parameters adopted in Fig. 2, but now allowing for saturated viscosity (\(\varphi = 2\)). The conclusions discussed in the previous case still hold qualitatively. The main difference is represented by the larger velocity of the outward wave caused by the decreased efficiency of the viscous dissipation of the pressure gradient. Therefore, being saturation of viscosity much higher in the hot medium than in the cloud, the gas compression is much lower (\(\sim 10\)). Saturated viscosity, however, does not seem to provide a mechanism capable to improve the cloud evaporation efficiency considerably.

For completeness, we have performed several other runs for different initial conditions. On this basis we can state that the numerical experiment described before has proven particularly exemplifying, as long as C-type initial states are considered. In other words, maintaining approximately the same value for the ratio \(\tau_d/\tau_c\), but changing the temperature of the hot medium, the results are qualitatively the same.

The cloud implosion process we have shown deserves some more detailed analysis given its astrophysical relevance. To provide a more intuitive tool to the discussion, we have reported in Fig. 5 (Plate 1) the evolutionary 2-D images of the cloud for the same case presented in Fig. 3. The use of density isocontours is aimed to point out the implosion effect. From Fig. 5 the creation of a central dense core is quite evident. The maximum value of central compression is reached at \(t = 0.7\tau_d\), when \(n_0(0) = 253\ cm^{-3}\) and the cloud size is 1 pc; the temperature in the cloud is in the range \(1.8 \times 10^3 < T < 2.5 \times 10^3\ K\). Initially the cloud has a mass of \(5\ M_\odot\), and at \(t = 0.7\tau_d\) it has decreased to only \(0.19\ M_\odot\). This corresponds to an average mass flux from the interface of about \(1.7 \times 10^{-6}\ M_\odot\ yr^{-1}\). After this moment, however, the rate of mass loss is rapidly quenched, as already discussed, and the cloud is able to survive, even if with a much lower residual mass.

Another consequence of the implosion process may be of some relevance. The initial column density of the gas for the case presented is \(N_H = 1.13 \times 10^{19}\ cm^{-2}\); at the moment of maximum compression \(N_H\) is increased by a factor 7.2. The cloud can become optically thick and, if some dust is present, molecule formation may take place in the interior shielded from energetic radiation.

We turn now to the discussion of the possible model limitations arising from the neglect of the magnetic field, \(B\), and non-equilibrium cooling. As far as \(B\) is concerned, we can estimate its dynamical influence in the following way. When \(B = 0\), the front propagation is stopped, as seen above, at the radius \(r_m\) at which the internal pressure of the cloud has reached a value \(p_m\), roughly equal to the pressure excess caused by conductive heating in the interface. The location of this point is determined by the specific characteristics of the dynamical evolution and energy balance of the front; in the example studied here, \(r_m \approx 0.2\). When a magnetic field is present, a simple but meaningful way to extend the previous analysis makes use of the virial theorem. For a spherical cloud, and neglecting thermal pressure, this is written as

\[
4\pi r^3 p_m = (1/3) r^3 B^2.
\]

Using the flux-freezing condition \(\pi r^2 B = \Phi = const.\), we obtain the equilibrium radius \(r_m\) for the
magnetic (and static) case:

\[ r_m = 0.9 \left( \frac{B_0^2}{8\pi p_m} \right)^{1/4} = 0.9 \beta_0^{1/4}, \]

where the subscript 0 indicates initial values of the quantities. Assuming equipartition between magnetic and thermal pressure, from Fig. 2 we derive \( \beta_0 \sim 0.01 \); therefore, in the magnetic case \( r_m = 0.28 \). According to this simple estimation, the main results obtained here should not be appreciably modified by the inclusion of \( B \), because its effects would become important roughly at the same stages as the thermal ones. This is not surprising if the initial condition prescribes equipartition. Of course, these arguments cannot substitute a much needed calculation in which magnetic fields are included. As already mentioned, the only other possibility to suppress compression requires the unlikely situation of a perfectly insulated cloud such that thermal conduction is almost completely inhibited.

Non-equilibrium effects can certainly be important for the prediction of radiative properties of the CC front. These effects have been studied for example by Ballet et al. (1986) and Slavin (1989). However, looking at Fig. 1 of Slavin (1989), in spite of large differences in the cooling functions, the non-equilibrium temperature profiles obtained differ very little from the equilibrium ones. This suggests that the dynamics, mainly governed here by temperature gradients, should hardly be affected. Thus, we believe that the equilibrium cooling function used provides at least a reasonable first approximation for the study tackled here.

3. STEADY-STATE PROPERTIES

The solutions discussed in the previous section illustrate the richness of physical effects that may descend from the inclusion of the dynamics in the study of CC fronts. Nevertheless, the cases we have described may appear somehow “extreme”, in the sense that the cold phase is strongly affected and eroded by the interaction with the hot gas. Such violent dynamical effects are driven by huge differences in the energy fluxes transported through the system: these models, in fact, correspond to situations in which heat input by thermal conduction and radiative losses are strongly out of balance. This naturally leads to the onset of convective motions, as, for example, it is well known from the theory of stellar interiors. In another formulation, according to the mapping of the \( \tau_c - \tau_k \) plane previously introduced, we could say that for these states the various time scales are very different and thus they are situated far from the bisectrix, where \( \tau_k = \tau_c \). Although only a minority of the real systems are so finely tuned to be located in the narrow region of the plane closed to the bisectrix, they provide an interesting sample of dynamical behaviours. In addition, such a bisectrix may represent the final stage of the evolution of a strongly dynamical system, after a transient period during which the pressure gradient has relaxed.

When the pressure can be considered almost constant, the stress tensor reduces to \( \Pi = \text{const} \), and we are left with the energy equation (2.3). This allows us to study the transition from the last stages of the dynamical evolution into the steady state. In the following we will derive a law for the damped motion of the CC front based on simple flux balance principles. The same approach will prove to be quite powerful also in order to understand the soliton-like solutions discussed below.

3.1 Slowing of the front due to cooling

For \( \Pi = \text{const} \) motions are subsonic and the energy equation (2.3) reduces to the standard heat equation modified by the presence of cooling. To outline the main physical aspects, we study the simple planar case, for which eq. (2.3) becomes

\[ \frac{R}{(\gamma - 1)\mu_p} \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left[ \kappa \frac{\partial T}{\partial x} \right] - \rho \mathcal{L}(p, T, x); \]

for isochorical processes the previous equation can be set in a non-dimensional form as described for the system (2.1)-(2.4)

\[ \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left[ \kappa \frac{\partial T}{\partial x} \right] - \mathcal{L}(p, T, x); \quad (3.1) \]
to simplify the notation we have maintained for $\kappa/\rho$ the same symbol $\kappa$. In order to restrict the number of assumptions, we just take $H = 0$ and assume that $\Lambda$ does not depend on the ionization, which is a good assumption for temperatures $> 10^4$ K. To simplify the notation, we replace $L(p, T, x)$ with $\lambda(T)$ in eq. (3.1). The problem then is completely defined when two boundary conditions (BCs) are assigned along with the initial condition (the problem is of parabolic nature)

$$
\begin{aligned}
T(-\infty, t) &= 0; \\
T(1, t) &= 1; \\
T(x, 0) &= 0 \quad x < 0.
\end{aligned}
$$

The discontinuity in the initial temperature profile will generate a thermal front, whose position we define as the point $x_f$ at which $T = 0$, propagating towards $-\infty$. A numerical solution of eq.(3.1) with the cooling function described in §2 is shown in Fig. 6. The effect of the cooling on the dynamics of the front is evident, along with the deceleration of the front to $\dot{x}_f = 0$, which occurs at $t \sim 0.15$ and $x = -0.36$. We interpret these results as the tendency of the system towards the state in which conductive flux and volume energy losses are balanced, thus leading to a truly stationary state. In order to substantiate this statement we have performed some additional analytical calculations.

From a dimensional analysis of eq. (3.1) we obtain an approximate expression for $x_f$ when radiative losses are neglected:

$$
\frac{T}{t} \approx T^{\beta+1} x_f^2,
$$

or $x_f \propto t^{1/2}$, having taken $\kappa \propto T^\beta$. Note that the usual requirement for a fixed initial energy input (Zel’doovich & Raizer 1965) here has been released. It is well known that this problem, with different BCs, admits an exact self-similar solution given by the differential equation

$$
\frac{d}{d\xi} \left( f^{\beta} \frac{df}{d\xi} \right) + \frac{\beta df}{2 d\xi} = 0.
$$

where $f$ and $\xi$ are appropriate self-similar quantities (Zel’dovich & Raizer 1965). As shown in §2, the inclusion of cooling processes introduces different scaling laws. An expression for the shape of the front can be obtained from eq. (3.1), looking for solutions of the form $T = T(x - vt)$, thus reducing the original partial differential equation to an ODE. The solution, valid for $L = 0$, is

$$
T(x) \sim \beta v|x_f - x|^{1/\beta}.
$$

To study the transition to a steady state we use flux balance arguments. The energy input $E_k$ due to thermal conduction can now be dissipated by radiative energy losses $E_c$. Hence the front must decelerate and eventually be stop, for a certain value $x_f^*$ implicitly defined by

$$
|E_k| = \frac{\partial T}{\partial t} = \lambda(T) = |E_c|;
$$

integrating eq. (3.5) over a column of gas of unit area we have

$$
\int_0^{x_f^*} \frac{\partial T}{\partial t} dx = \int_0^{x_f^*} \lambda(T) dx.
$$

The shape of non-linear conductive front (eq.[3.4]) is characterized by an almost constant profile and a steep cutoff close to $x_f$. Therefore, a good approximation, when evaluating the rhs integral, is to assume $T = \text{const}$, which gives

$$
|E_c| \sim x_f^*.
$$
If we use eq. (3.4), and the relation \( \dot{x}_f = (1/2)x_f^{-1} \), we can write the lhs integrand of eq. (3.6) as

\[
\frac{\partial T}{\partial t} = \frac{1}{4x_f^2}(x_f - x)^{1/2} \left[ (x_f - x)^{-1} - \frac{\beta}{x_f} \right].
\]  

(3.8)

It is easy then to show that

\[
|\mathcal{E}_k| = \frac{1}{4(1 + \beta)}x_f^{1/2} - 2,
\]

(3.9)

or, equating to eq. (3.7),

\[
x_f^* \sim [4(1 + \beta)]^{(1/3 - 1)}.
\]

(3.10)

For electron conductivity, \( \beta = 5/2 \) and \( x_f^* \sim 0.36 \), in excellent agreement with the numerical solution shown in Fig. 6.

### 3.2 Static CC Configurations

As demonstrated above, the energy flux drive the system towards an equilibrium state in which the energy input (due to thermal conductivity) and output (due to radiative losses) are equal. In a CC front, energy is transported by conductive \( q_\kappa = -\kappa \nabla T \) and convective \( q_v = \rho \epsilon v \) flux, where \( \epsilon \) is the specific thermal energy. In equilibrium must be

\[
\oint (q_v + q_\kappa) dS = \int \rho \mathcal{L} \cdot dV,
\]

(3.11)

where the integration is performed in the rhs over the volume of the system, and in the lhs over the surface delimitating the volume, in the appropriate frame of reference in which \( \partial/\partial t = 0 \). We shall consider an ensamble of scalar variables \( \varphi \), depending on \( x \), \( \varphi = \varphi(x) \), and velocity \( v = (v(x), 0, 0) \), \( x \in (-\infty, +\infty) \).

To illustrate our method we consider first the simplest case of a static medium, \( v = 0 \). This method is based on the local analysis of the differential equations governing the system, and on the global properties of the energy exchange which determine the characteristics of the flow at the boundaries (i.e., \( \pm \infty \)). A similar approach has been developed recently by Elphick et al. (1991, 1992) who have investigated the interactions of planar CC fronts. We will concentrate here on somewhat different aspects of the problem, namely the general characteristics of the system for which steady-state solutions of CC fronts do exist. For the particular case \( v = 0 \), the system (2.1)-(2.4) reduces to the equation

\[
\frac{d}{dx} \kappa(T) \frac{dT}{dx} = \rho \mathcal{L}(\rho, T),
\]

(3.12)

with the BCs

\[
\begin{align*}
T(-\infty) &= T_1, \\
T(+\infty) &= T_2;
\end{align*}
\]

(3.13)

the medium is assumed to be in thermal equilibrium at \( x = \pm \infty \), \( \mathcal{L}(\rho_1, T_1) = \mathcal{L}(\rho_2, T_2) = 0 \).

Since in the static case \( \rho(T) \propto 1/T \) for isobaric and \( \rho(T) = \text{const} \) for isochoric case, respectively, we can re-write eq. (3.12) in a simpler form

\[
\frac{d}{dx} \kappa(T) \frac{dT}{dx} = \tilde{\mathcal{L}}(T).
\]

(3.14)

We define the non-dimensional variable \( \theta = T/T_2 \) and the functions \( \kappa(\theta) = \kappa(T)/\kappa(T_2) \), \( \tilde{\mathcal{L}}(\theta) = \rho \mathcal{L}(T)/[\rho_2 \kappa(T_2) T_2] \), and introduce the new variable \( \eta(\theta) \) defined by \( d\eta/d\theta = \kappa(\theta) \). Given the univocal mapping \( \eta \leftrightarrow \theta \), eq. (3.12) can be written also as

\[
\frac{d^2 \eta}{dx^2} = \tilde{\mathcal{L}}(\eta).
\]

(3.15)
When \( d\tilde{L}/d\eta = 0 \), it must be \(|d^2 \tilde{L}/d\eta^2| > 0\). It follows that
\[
\frac{dq^2}{d\eta} = 2\tilde{L}(\eta),
\] (3.16)
with \( q \equiv d\eta/dx = 0 \) at the two points \( \eta = \eta_1 \) and \( \eta = \eta_2 \). The existence of a static solution is therefore guaranteed if the medium is thermally stable \((d\tilde{L}/d\eta > 0)\) at \( x = \pm\infty \) and \( \int_{\eta_1}^{\eta_2} \tilde{L}d\eta = 0 \).

The last condition on the integral is equivalent to eq. (3.11) and expresses the fact that cooling losses are exactly balanced by heating. There is an odd number of stationary points \((\tilde{L} = 0)\) pertaining to thermally unstable states with \( d\tilde{L}/d\eta < 0 \); to simplify our analysis we consider the case with a single stationary point, \( \eta_0 \), for which, \( d\tilde{L}/d\eta_0 < 0 \).

Near the \( i \)-th thermally stable point, the solution of eq. (3.15) can be linearized
\[
\frac{d^2\eta}{dx^2} = \tilde{\tilde{L}}_\eta(\eta)\delta\eta,
\] (3.17)
where \( \delta\eta = \eta - \eta_i \), \( \tilde{\tilde{L}}_\eta = d\tilde{L}/d\eta = (d\tilde{L}/d\theta)/\kappa \). Substituting \( \delta\eta \propto \exp(\omega x) \) into eq. (3.17), we obtain
\[
\omega = \pm \sqrt{\tilde{L}_\eta}.
\] (3.18)

The previous relation describes a saddle point. The characteristic behaviour of the phase trajectories near such a stationary point in the \( q-\eta \) plane is plotted in Fig. 7a-b:

\[
q \sim \pm (\tilde{\tilde{L}}_\eta)^{1/2}|\eta - \eta_i|, \text{ for } \tilde{\tilde{L}}_\eta > 0,
\]
and
\[
q \sim (|\tilde{\tilde{L}}_\eta|/3)^{1/2}|\eta - \eta_i|^{3/2}, \text{ for } \tilde{\tilde{L}}_\eta = 0.
\]

A peculiar case corresponds to \( \tilde{\tilde{L}}(\eta) \sim |\eta - \eta_i|^\alpha \) with \( \alpha < 1 \), so that \( \tilde{\tilde{L}}_{\eta_i} = +\infty \). In this case the linearized equation (3.17) is not valid, and a correct solution can be obtained integrating directly eq.(3.16), which gives (Fig. 7c)
\[
q \sim \pm|\eta - \eta_i|^{1+\alpha/\alpha}.
\]

The thermally unstable point \( \eta_0 \) is a center, and as a result, the phase diagram of the system described by eq.(3.12)-(3.13) has the form plotted in Fig. 8a, and the eigenfunction of this system, \( T = T(x) \) (Fig. 8b), corresponds to the separatrix. It is easy to see that the topology of the phase space allows for the existence of a non-trivial solution \((T \neq \text{const})\) with \( T(-\infty) = T(+\infty) = T_2 \) (or \( T_1 \)), as shown in Fig. 9a-b. A final possibility corresponds to a periodic structure constituted by clouds and intercloud gas (Fig. 9c).

To clarify the differences among these four cases we recur to a mechanical analogy. We can re-write eq. (3.15) in the following form:
\[
\frac{d^2\eta}{dx^2} = -\nabla_\eta \Phi(\eta),
\] (3.19)
where \( \nabla_\eta \equiv \partial/\partial\eta; \Phi(\eta) = -\int_{\eta_i}^{\eta} \tilde{L}(\eta')d\eta' \) can be seen as a “potential”. In the framework of this analogy, the characteristic size of the region with \( T \sim T_2 \) is just the “time interval” \( \Delta x \) needed for the mechanical system (3.19) to make a complete orbit around the turning point \( \eta = \eta_2 \):
\[
\Delta x(\eta_2) \sim \int_{\eta_i}^{\eta_2} \frac{d\eta}{q(\eta)},
\] (3.20)
where \( \eta' > \eta_1 \), for example \( \eta' = \eta_0 \). From eq. (3.16), the integral is equal to

\[
\Delta x(\eta_2) \sim \int_{\eta_0}^{\eta_2} \frac{d\eta}{\sqrt{2|\Phi(\eta)|}};
\]

hence, \( \Delta x(\eta_2) \) is determined by the behaviour of \( \Phi(\eta) \) near \( \eta = \eta_2 \). Three different possibilities can be envisaged:

- **a)** Let \( \tilde{\mathcal{L}}(\eta) = \tilde{\mathcal{L}}_\eta(\eta - \eta_2) + \tilde{\mathcal{L}}_{\eta\eta}(\eta - \eta_2)^2/2 + \ldots \), with finite values of \( \tilde{\mathcal{L}}_\eta, \tilde{\mathcal{L}}_{\eta\eta}, \ldots \). In this case \( \Phi(\eta) \sim (\eta - \eta_2)^2 + \ldots \) and the integral (3.21) has a logarithmic singularity:

\[
\Delta x(\eta_2 - \epsilon) \sim \ln \epsilon, \quad \text{for } \epsilon \to 0.
\]

This corresponds to the solution shown in Fig. 8a-b.

- **b)** \( \tilde{\mathcal{L}}_\eta(\eta_2) = 0, \tilde{\mathcal{L}}_{\eta\eta}(\eta_2) \neq 0 \); \( \Phi(\eta) \sim (\eta - \eta_2)^3 + \ldots \) and the integral (3.21) has a power-like discontinuity at \( \eta = \eta_2 \):

\[
\Delta x(\eta_2 - \epsilon) \sim \epsilon^{-1/2}.
\]

Qualitatively, the profile of \( T(x) \) is the same as in the previous case.

- **c)** \( \mathcal{L}(\eta) \sim -{(\eta_2 - \eta)\alpha}, \alpha < 1 \), so that \( \mathcal{L}_\eta(\eta_2) = \infty; \Phi(\eta) \sim U_0(\eta_2 - \eta)^{\alpha+1} \). The integral (3.21) has a finite value, and the \( T(x) \) profile is similar to the one plotted in Fig. 9.

Thus, static CC fronts corresponding to a i) finite cloud surrounded by hot intercloud gas, ii) finite intercloud layer separating two infinite cold regions, are possible. In addition, a configuration constituted by iii) a sequence of flaky cloud-intercloud regions is also possible if the cooling function \( \tilde{\mathcal{L}}(\eta) \) has a singular behaviour near the stationary points \( \eta_1 \) and \( \eta_2 \). These are the only solutions for which at least one of the two gas components has a finite size. Note that this requires \( \tilde{\mathcal{L}}_\eta \neq 0 \) and \( \kappa(\eta_i) = 0 \) at the stationary points.

We conclude that the existence and the structure of steady-state CC fronts is determined by the global relation (3.11) and by the local properties of the cooling function and of the conductivity: eq.(3.11) expresses the balance of the different mechanisms of energy transport through the system, whereas \( \tilde{\mathcal{L}}(T) \) and \( \kappa(T) \) determine the behaviour of the temperature near the stationary (saddle) points. An important feature is that, when \( \kappa(T) = 0 \), the thermal conductivity is able to support the energy transport near \( T = T_2 \) only over a finite interval \( \Delta x \). In Appendix A we clarify the relationship between this conclusion and the Field length.

### 3.3 Steady-state CC fronts

The variety of solutions describing steady-state CC fronts with non-zero velocity is, of course, richer than the static one. This is due to the fact that, in general, the interaction of hot and cold gas is likely to generate evaporation or condensation flows. Convective motions, in addition, lead to a symmetry breaking because, even if the temperature gradient is the same as in the static case, the upstream heat flux differs from the downstream one. In terms of the phase space \( (q, \eta) \), this means that a convective heat flux changes the topology of the plane, and the velocity can be seen as a bifurcation parameter. Starting again from the integral equation (3.11) we want to obtain the relation between the velocity and the global temperature and density distribution for the steady-state case.

Let us consider a steady-state flow corresponding to an evaporation front travelling with velocity \( u_0 \). All the hydrodynamical variables can be written in the form \( a = a(x + u_0 t) \). In the frame of reference comoving with the evaporation wave the appropriate equations are:

\[
\nabla(\rho u) = 0, \quad (3.22)
\]

\[
\rho u \nabla u + \nabla p = 0, \quad (3.23)
\]
\[
\frac{R}{\gamma - 1} \rho u \nabla T = -p \nabla u + \nabla (\kappa \nabla T) - L(\rho, T),
\]  
(3.24)

\[
p = \frac{R}{\mu} \rho T.
\]  
(3.25)

Here \(u\) is the velocity in the comoving frame reference, \(\nabla = d/dX\), \(X = x + u_0 t\). We introduce the following non-dimensional variables

\[
\begin{align*}
\xi &= X/\ell_{F_1}, \\
\rho &= \rho/\rho_1, \quad T = T/T_1, \\
p &= \gamma p/(\rho_1 c_1^2), U = u/c_1, \quad \lambda = \ell_{F_1} L/(\gamma c_1^2 \rho_1),
\end{align*}
\]  
(3.26)

where the subscript 1 refers to the unperturbed state at \(X = -\infty\), and \(c_1 = \sqrt{R\gamma T_1/\mu}\).

The first integrals of motion are the following:

\[
\begin{align*}
\rho U &= U_0 = J, \\
\rho U^2 + p &= U_0^2 + 1, \\
\frac{\gamma}{\gamma - 1} (pv - 1) + \frac{1}{2} (U^2 - U_0^2) &= -\frac{\lambda_s(\xi)}{U_0},
\end{align*}
\]  
(3.27)-(3.29)

where \(J = U_0\) is the mass flux, \(v = \rho^{-1}\) is the specific volume, and

\[
\lambda_s(\xi) = \int_{-\infty}^{\xi} \lambda(\rho(\xi), T(\xi)) d\xi.
\]  
(3.30)

Writing this system for \(\xi = +\infty\), and adding the equation of state

\[
\lambda(\rho, T) = 0
\]  
(3.31)

for the gas at \(\xi = +\infty\), we can obtain the values of \(p, \rho, U\) at \(\xi = +\infty\); \(U_0\) at \(\xi = -\infty\) is equal to the velocity of the evaporation front for given values of \(\rho_1, \rho_1, T_1\) at \(\xi = -\infty\) and \(T_2\) at \(\xi = +\infty\). Note that equation (3.31) is not valid in the intermediate range \(-\infty < \xi < +\infty\).

Let us assume that \(\lambda(\rho, T) = 0\) has a shape similar to a van der Waals-type equation, shown in Fig. 10, usually appropriate for a two-phase ISM model (Field, Goldsmith & Habing, 1969). We stress, however, that the results obtained in this Section do not depend on the specific form of the cooling function adopted but only on the number of its zeros. Thus,

\[
p = \begin{cases} 
(c_1^2/\gamma) \rho & \text{for } \xi = -\infty; \\
(c_2^2/\gamma) \rho & \text{for } \xi = +\infty,
\end{cases}
\]  
(3.32)

where \(c_2 \gg c_1\) for the differences between warm \((T_2 \simeq 10^4 K)\) and cold \((T_1 \simeq 10^2 K)\), or hot \((T_2 \simeq 10^6 K)\) and warm \((T_1 \simeq 10^4 K)\) ISM phases. In non-dimensional form the equation of state for the hot gas \((\xi = +\infty)\) is

\[
p_2 = c_2^2 v_2^{-1}.
\]  
(3.33)

The dynamical integrals of motion (3.27)-(3.29) can be reduced to

\[
p_2 = 1 + \gamma U_0^2 (1 - v_2),
\]  
(3.34)

and

\[
\frac{\gamma}{\gamma - 1} (p_2 v_2 - 1) + \frac{1}{2} (1 - p_2) v_2 - \frac{\gamma}{2} U_0^2 (1 - v_2) = Q,
\]  
(3.35)
where \( Q \equiv -\lambda_s/U_0, \lambda_s = \lambda_s(+\infty) \). Equation (3.33) is sometimes referred to as the Mikhelson equation (Zel’dovich & Kompaneets 1955), while equation (3.34) is similar to a Hugoniot adiabatic for \( \lambda_s = 0 \). Combining equations (3.33) and (3.34) with the equation of state for the hot gas (3.32), we obtain

\[
\frac{c_s^2}{v_2} = 1 + \gamma U_0^2(1 - v_2),
\]

and

\[
\frac{\gamma}{\gamma - 1} (c_s^2 - 1) + \frac{1}{2} (1 - \frac{c_s^2}{v_2})v_2 - \frac{\gamma U_0^2}{2} (1 - v_2) = Q.
\]

Equation (3.35) has a solution for \( \gamma U_0^2 \ll c_s^{-2} \)

\[
v_2 = c_s^2 (1 + \gamma U_0^2 c_s^2 + o(\gamma U_0^2 c_s^2));
\]

after substitution into eq.(3.27) we obtain

\[
U_2 = c_s^2 U_0 (1 + \gamma U_0^2 c_s^2 + o(\gamma U_0^2 c_s^2)).
\]

The assumed approximation \( \gamma U_0^2 \ll c_s^{-2} \) corresponds to a subsonic flow at \( \xi = +\infty \). Under such conditions eq.(3.36) can be reduced to

\[
\frac{2\gamma}{\gamma - 1} c_s^2 U_0 + \gamma U_0^3 c_s^4 = -2\lambda_s,
\]

whose solution is

\[
U_0 \simeq -\frac{\gamma - 1}{\gamma} \frac{\lambda_s}{c_s^2}.
\]

For \( \lambda_s > 0 \) eq. (3.40) describes an evaporation wave (net heating, \( U_0 > 0 \)), whereas for \( \lambda_s < 0 \) it describes a condensation wave (net cooling, \( U_0 < 0 \)). It can be shown (Appendix B) that this represents a subsonic flow at \( \xi = +\infty \) in accordance with the assumption made above.

We now concentrate on the properties of the phase plane (Hayashi 1985 provides an excellent background on this topic). It has to be pointed out that the flow considered is approximately isobaric, as derived from eq. (3.40):

\[
p = \rho c^2 = 1 - \gamma U_0^2 c^2 + o(\gamma U_0^2 c^2)
\]

This leads to the following form of the energy equation (3.24)

\[
\frac{1}{\gamma - 1} U_0 \frac{dT}{d\xi} + \frac{dU}{d\xi} - \frac{d}{d\xi} \kappa \frac{dT}{d\xi} = -\lambda(\rho, T).
\]

Combining the mass and momentum integrals, (3.27)-(3.28), with the equation of state (3.25) we find

\[
T = -U^2 + \frac{1 + U_0^2}{U_0} U,
\]

which gives an estimation of the temperature valid for subsonic flows

\[
T \simeq \frac{U}{U_0}.
\]

After substitution into eq. (3.42), we can re-write the same equation as

\[
\frac{d^2 \eta}{d\xi^2} - \frac{\gamma}{\gamma - 1} U_0 \frac{d\eta}{d\xi} = \lambda(\rho, \eta),
\]

(3.44)
In analogy with the static case, we will assume that \( \lambda(\rho, \eta) \) has three stationary points (Fig. 11a): two of them correspond to thermally stable states (\( \xi = -\infty \) and \( \xi = +\infty \)) whereas the third (\( \xi = \xi_0 \)) corresponds to a thermally unstable one. For small perturbations around the stationary points we have:

\[
\frac{d^2 \delta \eta}{d\xi^2} - \frac{\gamma}{\gamma - 1} \frac{U_0 d\delta \eta}{\kappa d\xi} = \lambda_0 \delta \rho + \lambda_\eta \delta \eta, \tag{3.45}
\]

where \( \delta \rho \simeq -\rho^2 \delta \eta / \kappa \) for isobaric perturbations. Taking \( \delta \eta \propto \exp(\omega \xi) \) the characteristic equation corresponding to (3.45) is

\[
\omega^2 - \frac{\gamma}{\gamma - 1} \omega + \left( \frac{\rho^2}{\kappa} \lambda_0 - \lambda_\eta \right) = 0, \tag{3.46}
\]

and the eigenvalues are

\[
\omega_{1,2} = \frac{1}{2} \frac{\gamma}{\gamma - 1} \frac{U_0}{\kappa} \pm \sqrt{\frac{1}{4} \left( \frac{\gamma}{\gamma - 1} \right)^2 \frac{U_0^2}{\kappa^2} - \left( \frac{\rho^2}{\kappa} \lambda_0 - \lambda_\eta \right)}. \]

Since for a thermally stable gas \( \rho^2 / \kappa \lambda_0 - \lambda_\eta < 0 \), it is easy to see that the nature of the thermally stable points corresponds to a saddle point (Fig. 11b). At the intermediate point \( \eta_0 \), instead, \( \rho^2 / \kappa \lambda_0 - \lambda_\eta > 0 \). When condition (3.40) holds, the discriminant is negative because \( U_0^2 = O(\ell_F/\ell_C)^2 c_s^{-4} \), whereas \( U_0 = O(\ell_F/\ell_C) c_s^{-2} \) and \( \lambda_0, \lambda_\eta = o(\ell_F/\ell_C) \). Thus \( \eta_0 \) is an unstable spiral point (Fig. 11b), and the phase diagram differs qualitatively from the static case. The main difference is that evaporation (Fig. 11c) and condensation (Fig. 11d) waves are not symmetric anymore. Thus, a steady-state CC front can exist only as a switch-on wave between two stable phases (cold/hot), and periodic structures like those plotted in Fig. 9c are not realisable.

To better understand this difference we apply again the mechanical analogy used above. Equation (3.44) can be written as follows:

\[
\frac{d^2 \eta}{d\xi^2} - \frac{\gamma}{\gamma - 1} \frac{U_0 d\eta}{\kappa d\xi} = -\nabla_\eta \Phi(\eta). \tag{3.47}
\]

In this case the potential \( \Phi(\eta) \) is an asymmetric function of \( \eta \) (Fig. 12a-b); therefore, for the evaporative case (\( d\eta / d\xi > 0 \)) a negative “friction force”, like the one represented by the second term on the lhs of eq. (3.50), increases the “energy” of the “particle” during its motion from \( \eta = \eta_1 \) to \( \eta = \eta_2 \). Since \( \Phi(\eta_1) = 0 \), we have \( \Phi(\eta_2) > \Phi(\eta_1) \) for an evaporation wave, and \( \Phi(\eta_2) < \Phi(\eta_1) \) for a condensation wave. This means that it is impossible for this “particle” to reach \( \eta = \eta_2 \) from \( \eta = \eta_1 \) and turn back, because such transitions lead to a difference between the initial and final values of the “potential” at \( \eta = \eta_1 \). Out of analogy, this is due, of course, to the radiative losses occurred in the gas during the phase transitions. The eigensolution of eq. (3.47) corresponds to the equality between the “potential” barrier and the “work” made by the negative “friction force”. Indeed, multiplying eq. (3.47) by \( d\eta / d\xi \) and integrating over \( \xi (-\infty, +\infty) \) with the BCs \( (d\eta / d\xi)_{\pm \infty} = 0 \), we obtain

\[
\int_{\eta_1}^{\eta_2} \frac{\gamma}{\gamma - 1} \frac{U_0}{\kappa} \left( \frac{d\eta}{d\xi} \right)^2 d\xi = \left\{ \begin{array}{ll} \Phi(\eta_2) - \Phi(\eta_1), & \text{for evaporation;} \\ \Phi(\eta_1) - \Phi(\eta_2), & \text{for condensation.} \end{array} \right.
\]

An artificial way to maintain a steady-state CC front with a structure of a hot (cold) gaseous layer surrounded by cold (hot) gas (similar to that of the static CC configuration plotted in Fig. 9) is to inject a fixed heat flux at a certain \( \xi \). In such a way, the additional heat flux behind the condensation wave provides heating to the cooled gas and evaporates it, restoring the initial state (Fig. 13a). The specific value of the heat flux necessary to maintain such a CC front is determined by the value of \( d\eta / d\xi \) at the top of the phase trajectory coming out of the critical point \( \eta = \eta_2 \) (in Fig. 13a it corresponds to the point S). In the case of an evaporation wave, a negative heat flux behind the CC front dilutes the excess of thermal energy, provides a cooling mechanism.
to the hot gas and favors the condensation process (Fig. 13b). However, this requires that the
heat source moves together with the CC front.

4. EXISTENCE OF SOLITON-LIKE SOLUTIONS

The previous results demonstrate that a CC front structure similar to a soliton (conden-
sation - evaporation [CE] wave or evaporation - condensation [EC] wave) is possible if supported
by an artificial heat source. However, the hydrodynamical integrals contain non-linear terms that
may cause the same effect. According to eq. (3.43)

$$
\Delta T = -2U \Delta U + \frac{1 + U_0^2}{U_0} \Delta U.
$$

The two terms on the rhs produce a perturbation of $T$ in opposite directions. This perturbation
is small only if $|U_0| \ll 1$, as in the static case. Thus, we expect that the non-linear term $U^2$ in
eq. (3.43) may result in a qualitative change of the phase plane topology that allows soliton-like
solutions without any artificial heat source at the boundary.

Let us consider the energy equation (3.24), describing a stationary wave with velocity $U_0$

$$
\frac{U_0}{\gamma - 1} \frac{dT}{d\xi} = -p \frac{dU}{d\xi} + \frac{d}{d\xi} \kappa \frac{dT}{d\xi} - \lambda(\rho, T),
$$

and with BCs $T(-\infty) = T(+\infty) = T_1$. Note that $U_0$ in eq. (4.1) is in general different from the
$U_0$ determined in the previous Section for evaporation or condensation waves. Using the relations
among $T$, $p$ and $U$ determined by the first integrals we have

$$
p = U_0^2 + 1 - U_0 U = \epsilon_0 - U_0 U,
$$

$$
T = -U^2 + \frac{1 + U_0^2}{U_0} U = -U^2 + \frac{\epsilon_0}{U_0} U,
$$

where $\epsilon_0 = 1 + U_0^2$ is the specific enthalpy. Equation (4.3) allows us to reduce eq. (4.1) to a form
containing only $T$. For this purpose we solve eq. (4.3) to find $U = U(T)$:

$$
U = \frac{1}{2} \frac{\epsilon_0}{U_0} \left(1 \pm \sqrt{1 - \frac{4U_0^2}{\epsilon_0^2} T}\right) \equiv f^\pm(T),
$$

where $\pm$ correspond to the “fast” and “slow” velocity mode, respectively. After substitution of eq.
(4.2) into eq. (4.1), using equation (4.4), we can re-write eq. (4.1) as follows

$$
\frac{d^2T}{d\zeta^2} - \frac{U_0}{2} \left(\frac{\gamma + 1}{\gamma - 1} - F^\pm\right) \frac{dT}{d\zeta} - \kappa \lambda(\rho(T), T) = 0,
$$

where $d\zeta = d\xi/\kappa$, and

$$
F^\pm = \pm \left(1 - \frac{4U_0^2}{\epsilon_0^2} T\right)^{-1/2}.
$$

We further assume that the cooling function has two equilibrium points: $T_1 = 1$, and $T_2$
defined by $\lambda(\rho, T) = 0$. These two points are stationary points on the phase plane $T-(dT/d\zeta)$, that
are $(T_1, 0)$ and $(T_2, 0)$. The solution corresponding to a solitary (EC or CE) wave is possible if one
of the points is a center and the other one is a saddle point. The characteristic equation for linear
perturbations $\delta T \propto \exp(\omega \zeta)$ for the two stationary points is

$$
\omega^2 - \frac{U_0}{2} \left(\frac{\gamma + 1}{\gamma - 1} - F^\pm\right) \omega - \left(\frac{dL}{dT}\right)_i = 0,
$$

(4.6)
where \( i = 1, 2 \) and \( dL/dT = \kappa(T_i)(d\lambda/dT)_i \). Hence, the eigenvalues are

\[
\omega_{1,2}^{(i)} = \frac{U_0}{4} \left( \frac{\gamma + 1}{\gamma - 1} - F_i^\pm \right) \pm \sqrt{\frac{U_0^2}{16} \left( \frac{\gamma + 1}{\gamma - 1} - F_i^\pm \right)^2 + \left( \frac{dL}{dT} \right)_i^2}.
\]  

(4.7)

The thermally stable point with \( (dL/dT)_i > 0 \) is a saddle point, whereas the nature of the thermally unstable point with \( (dL/dT)_i < 0 \) depends on the value of \( (\gamma + 1/\gamma - 1 - F_i^\pm) \). We analyze now in detail the case that corresponding to soliton-like solutions which are the subject of this Section. To be specific, we suppose that the saddle point is identified with \( (\gamma + 1/\gamma - 1 - F_i^\pm) \) in detail the case that corresponding to soliton-like solutions which are the subject of this Section.

To be specific, we suppose that the saddle point is identified with \( T = T_1 \), and \( T = T_2 > T_1 \) is the center. The phase diagram of the system in this case is plotted in Fig. 14a; the relative temperature eigenfunction is shown in Fig. 14b.

It is easy to see that the point \((T_2, 0)\) can be a center only for the fast mode solution:

\[
U = \frac{1}{2} \frac{\epsilon_0}{U_0} \left( 1 + \sqrt{1 - \frac{4U_0^2}{\epsilon_0 T}} \right) \equiv f^+,
\]

(4.8)

with the necessary condition \( \gamma + 1/\gamma - 1 - F^+ = 0 \), equivalent to the equation:

\[
\frac{1}{\sqrt{1 - \frac{dU^2}{\epsilon_0 T}}} = \frac{\gamma + 1}{\gamma - 1},
\]

(4.9)

whose solution is

\[
U_0^2 = - \left[ 1 - \frac{(\gamma + 1)^2}{2\gamma} \right] \pm \sqrt{\left( 1 - \frac{(\gamma + 1)^2}{2\gamma} \right)^2 - 1}.
\]

(4.10)

This solution determines two values of \( U_0^2 > 0 \) if the sound speed at the center point satisfies the inequality \( c_2^2 > 4\gamma/(\gamma + 1)^2 \), valid for \( T_2 > T_1 = 1 \). In this case, there exist two different soliton-like solutions corresponding to supersonic and subsonic solitary waves of EC type. The FWHM of the soliton can be estimated directly from eq. (4.5). At the top of the soliton \( dT/d\xi = 0 \) and \( d^2T/d\xi^2 < 0 \), \( \kappa \lambda \neq 0 \), hence

\[
\Delta x \sim \sqrt{\frac{\kappa \lambda}{|d^2T/d\xi^2|}}.
\]

All quantities should be taken for \( T = T_* \), where \( T_* \) is the temperature at the soliton top (or bottom for a CE soliton); \( T_* \) can be estimated at the point where \( \Phi = 0 \). (\( \Phi \) is defined by eq. [3.19]).

A similar analysis for the case with a saddle point at \( T = T_2 \) (thermally stable phase) and a center point at \( T = T_1 \) demonstrates the existence of a CE soliton-like solution with a reversed shape with respect to the EC case.

These soliton-like solutions correspond to the separatrix of the phase plane (see Fig. 14a) like the hydrodynamical solitons described by the Korteweg-de Vries equation. However, their physical nature is quite different. In fact, they are the result of the simultaneous action of four processes: thermal conductivity \( \nabla(\kappa \nabla T) \), the cooling \( \mathcal{L}(\rho, T) \), convective transport \( \rho \mathbf{u} \nabla T \) and advection \( p \nabla \mathbf{u} \). Indeed, the thermal conductivity determines the second-order characteristic equation (4.6) and the set of solutions near the stationary points; the radiative cooling determines the number of stationary points and the local topology of the phase space in their vicinity. The most interesting role is played by the last two processes: their conspiracy is able to restore the symmetry of the phase space broken by the hydrodynamical motions for slow isobaric flows (e.g., eq. [3.33]). The sum of the hydrodynamical terms in eq. (4.1), taking into account eqs. (4.2) and (4.3), can be written as:

\[
\frac{U_0}{\gamma - 1} \frac{dT}{d\xi} + p \frac{dU}{d\xi} = \frac{U_0}{2} \left( \frac{\gamma + 1}{\gamma - 1} + \frac{\epsilon_0}{U_0} \frac{dU}{dT} \right) \frac{dT}{d\xi}.
\]
Thus, the first term on the rhs represents an asymmetric heat flux and hence destroys the symmetry of the phase space present in the isobaric case (\(dU/dT = 0\)). The second term describes a stabilization or amplification of the convective heat flux depending on the sign of \(dU/dT\). For the fast mode \(dU/dT < 0\), thus the convective heat flux is stabilized and balanced for an appropriate value of \(U_0\) at a certain \(T = T_1\) (or \(T = T_2\)). This provides the soliton-like shape of the wave. The soliton temperature profile, however, is slightly asymmetric as it can be realized by rewriting eq. (4.5) in the following form

\[
q \frac{dq}{dT} - \frac{U_0}{2} \left( \frac{\gamma + 1}{\gamma - 1} - F \right) - \kappa \lambda = 0.
\]

It is clear that the upper and lower parts of the \((T, q)\) plane are asymmetric: at the points where \(\frac{dq}{d\zeta} = 0\), in fact, \(T(q)\) for \(q > 0\) differs from \(T(q)\) for \(q < 0\).

5. SUMMARY AND POSSIBLE APPLICATIONS

In this paper we have studied the dynamics of thermal conduction fronts in a multi-phase medium with parameters suitable to describe the general ISM. The inclusion of radiative losses affects substantially both the dynamics and the structure of the conductive/cooling front. However, a number of additional physical effects like viscosity, ionization, saturation of the kinetic processes have been shown to be able to modify appreciably the overall behaviour of the system, when included. The presence of several, often very different, scales in the problem determines regimes in which the response of the system is strongly time dependent and non-linear. CC fronts may induce relevant dynamical effects if the conductive time is much shorter that the sound crossing time of the cloud. This condition is, in practice, equivalent to require that the cloud size is smaller than the Field length for the hot gas.

As an example, we have explored in detail a realistic dynamical regime by means of numerical simulations. The adopted initial condition consists of a spherical cloud in equilibrium with a surrounding hot medium at a pressure \(p/k = 10^4\ \text{cm}^{-3}\ \text{K}\), with a ratio of intercloud-to-cloud gas temperature of \(T/\bar{T} = 300\). This set-up may fairly well reproduce a typical interstellar Galactic environment. The main results are summarized as follows:

1. For the case \(\varphi = \infty\) (classical viscosity) a huge pressure gradient is created at the interface hot/cool gas which drives an outward and an inward flow. In some sense, the interface behaves like an explosion site. The supersonic wave travelling towards the cloud center compresses the cloud and at the same time tends to evaporate it. A dense (> 100 times the initial density) core is formed; the subsequent increase in pressure is able to stop the compression wave and to reflect it back.

2. We have shown that after the pressure gradient has relaxed, the front is decelerated, and the system tends to a steady-state. Using simplifying positions as neglecting the viscosity and thermal conduction saturation, the final stages of this process have been investigated exploiting flux balance arguments. These findings have been substantiated by their agreement with exact numerical solutions.

3. Clouds of size smaller than the Field length can be able eventually to stop the evaporation process due to the density and pressure increase following the compression phase; the net results is that the enhanced radiative losses decelerate the CC front (non-linear effect).

4. In the first evolutionary stages, thermal conduction is highly saturated with a value of \(\sigma_T \sim 100\), but it rapidly becomes classical after about one half of the dynamical time. The role of the dynamics in decreasing the saturation level of the kinetic effects is an important piece of information to add to the results for the steady cases obtained by McKee & Cowie (1977) and Draine & Giuliani (1984).

5. If saturated viscosity is allowed, \(\varphi = 2\), the compression factor is decreased by a factor of \(\sim 10\), but the main features of the flow remain qualitatively the same.

The global properties of static and steady-state CC front solutions have been derived from an analysis of their behaviour in the phase plane. Different thermal phases individuate stationary states whose topological nature is determined by the stability properties of those states.
on the behaviour of the cooling function at the stationary points, it has been shown that in the static case three different configurations are possible: i) finite cloud surrounded by hot intercloud gas, ii) finite intercloud layer separating two infinite cold regions, and iii) a periodic sequence of cloud-intercloud regions. The last possibility is related to the fact that in this case thermal conduction can transport energy only over a finite spatial interval. The addition of a velocity field produces a symmetry breaking in the topology of the solutions and, therefore, periodic structures are not allowed.

It has been shown that solitary wave solutions are admitted by the set of equations describing CC fronts, even in the case of no heat flux at the boundaries, thanks to the intrinsic non-linear character of these equations. Physically, those soliton-like solutions are the result of the exact balance between convective and conductive heat flux, which restores the symmetry breaking occurring, in general, when the flow is not isobaric. These solitary waves have the remarkable property that, differently from an evaporation front that leaves the gas at a higher temperature behind it, they change the temperature of the medium affected by their passage but after this transient the gas returns to its initial state.

Finally, we add here that it is possible to demonstrate the existence of self-similar solutions of the equations describing CC fronts in absence of bulk motions. Their properties are currently under investigation and they will be discussed elsewhere. The preliminary results show that, contrary to the suggestion of Doroshkevich & Zel’dovich, cooling media do not necessarily induce condensation waves, but they may excite evaporation modes in the system.

The most severe limitation of our model descends from the spherical geometry used in the numerical calculations. The main reason for this choice is, of course, dictated by simplicity requirements; real ISM clouds can largely deviate from this idealisation. Some of the findings of this paper can be modified by more refined and multi-dimensional calculations; the striking nature of the implosion effect, in particular, can be partially an artifact of the geometry. However, we believe that the basic physics (compression, density increase, slowing of the front, wave reflection) underlying the process, and outlined throughout the paper, must be realized also by more complex configurations. In addition, this simplification highly facilitates the comparison with analogous classical works in the field, based on the same assumption.

Some mathematical problems related to the existence of solitons need some more accurate analysis. For example, it is not clear to us if the the non-linear terms present in the hydrodynamic equations are likely to destroy closed trajectories in the phase space around the center point for large amplitudes. In addition, the possibility of closed paths seems to depend on the nature of the system, which in turn is determined by its characteristic equation (Guckenheimer & Holmes 1983). Finally, the stability of solitons for the Korteweg-de Vries equation has been demonstrated by Benjamin (1972), but the analogous question for the thermal solitons we are presenting here is completely open.

Although this paper is not particularly dedicated to a specific astrophysical situation, a number of consequences worth to be explored can be suggested.

A result that has important and immediate implications for our understanding of the ISM descends from the cloud implosion induced by thermal conduction. Due to the unavoidable density increase, the cloud tends to become optically thick, providing a site for molecule formation. A clear evidence of molecular clouds embedded in neutral hydrogen, both surrounded by hot, x-ray emitting plasma, has been provided by the ROSAT shadowing experiments (Burrows & Mendenhall 1991; Snowden et al. 1991) in the direction of the Draco complex. The phenomenon we have studied requires cloud properties which are by far typical of the HI clouds populating the Galaxy. Therefore, it may take place whenever very hot gas ($T \sim 3 \times 10^6 - 10^7$ K) is suddenly injected by some process (the most natural being supernova explosions) in their surroundings. In a couple of papers, Bertoldi (1989) and Bertoldi & McKee (1990) have discussed an analogous cloud compression mechanism based on the action of an ionization-shock induced by the radiation field of newly born stars. Our mechanism is rather important in the regions in which the radiation field in not particularly enhanced, but some hot gas is present. A prototype of these regions can be the Galactic halo, where hot gas is suggested by several different observations (Sembach & Savage 1992; for a review see Spitzer 1990) along with neutral HI clouds. Those clouds can be a product of a thermal instability occurring in a fountain flow under particular conditions (Houck & Bregman 1990, Ferrara & Einaudi 1992, Li & Ikeuchi 1992). However, it has been pointed out (Ferrara & Einaudi 1992) that for the
prevailing conditions of the hot flow, turbulence is more easily generated than a condensation; in this case the clouds should have a different origin. If the predictions of the McKee & Ostriker (1977) model are correct, suitable conditions for the implosion mechanism can easily be found in the disk as well.

Some other aspects of the cloud/hot gas interaction deserve additional study. For example, clouds engulfed by a strongly anisotropic hot gas flow can be rocket-accelerated as a whole, by the same mechanism providing implosion in the spherically symmetric case. This may have important consequences for the observed degree of HI turbulence required by some models to support the extended neutral Galactic component (Lockman & Gehman 1991; Ferrara 1993). Also, the interaction of a cloud with the hot surrounding medium may induce turbulent motions inside the cloud. These motions are likely to be supersonic, as can be realized from the results of §2.1. Therefore, even in the presence of a substantial dissipation, the energy flux due to conductivity may be able to sustain a remarkably high degree of supersonic turbulence for a long time interval in the cloud.

Finally, we would like to mention that soliton solutions may lead to a very different picture of the interstellar medium. The temperature fluctuations related to their wavy nature can create time-dependent patterns which can propagate and survive for long times due to their intrinsic stability. This may have some relevance for the interfaces producing the highly ionized species (C IV, N V and O VI) detected either in the disk and in the halo by absorption line measures. The study of extragalactic environments where two-phase media are found, (e.g., protogalactic clouds in a hot intergalactic medium, emitting clouds in the Broad Line Region of AGNs, cooling flows) constitutes an appealing application of the theory developed here. However, still more physical insight can be achieved by a dedicated analysis in which the three dimensional properties of soliton solutions are considered.

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APPENDIX A

The solutions found for static CC fronts are intimately connected with the Field length. To see this, we recall that eq. (3.14) for \( \kappa(T) = 0 \) has the form

\[
\kappa_T \left( \frac{dT}{dx} \right)^2_{T=T_2} = \tilde{\mathcal{L}}(T_2) = 0,
\]

and

\[
\kappa_T \left( \frac{d^2T}{dx^2} \right)_{T=T_2} = (\tilde{\mathcal{L}}_T)_{T_2}.
\]

For \( \kappa_T(T_2) \neq 0 \) and \( \tilde{\mathcal{L}}_T(T_2) \neq 0 \), this results in a finite size of the region with \( T \simeq T_2 \), of the order
of

\[ \Delta x(T_2) \sim \sqrt{\frac{T_2 \kappa(T_2)}{L_T(T_2)}}. \]

The fact that \( \Delta x(T_2) \) comes out to be of the order of the Field length \( \ell_F \) is not surprising. This is in agreement with the discussed interpretation of \( \ell_F \) as the length over which cooling and conductive energy fluxes are able to establish equilibrium.

The previous statement can be analogously demonstrated also in the case \( \kappa(T) \neq 0 \), in which conductivity provides heat transport over the whole system. At \( \eta = \eta_0 \) (center), eq.(3.16) has the following solution

\[ q^2 = q_0^2 - |L_{\eta_0}|(\eta - \eta_0)^2, \]

where \( q_0^2 = 2 \int_{\eta_1}^{\eta_0} \mathcal{L}(\eta)d\eta \). It follows that

\[ \eta = \eta_0 \pm q_0^2 / (2\ell^2) \text{sin} \frac{x}{\lambda_{\eta_0}}, \]

with \( \ell_{\eta_0} = \ell_F(\eta_0) \). Similarly, for \( \eta = \eta_2 \) (saddle), the same equation has the solution

\[ \eta = \eta_2 [1 - e^{-x/\ell_{\eta_2}}], \]

with \( \ell_{\eta_2} = \ell_F(\eta_2) \).

**APPENDIX B**

We show here that the flow described by eq. (3.39 is subsonic at \( \xi = +\infty \). First we recall that

\[ |U_2| = |U_0|v_2 \simeq \frac{\gamma - 1}{\gamma} |\lambda_s|; \]

If \( |\lambda_s| \sim \lambda \), then \( |U_2| \ll c_2 \) and the flow is subsonic. This can be shown the following way. We can express the non-dimensional cooling function \( \lambda \) in the equivalent form

\[ \lambda = \frac{\ell_F \mathcal{L}_m \mathcal{L}}{c_1 p_1 \Delta \mathcal{L}_m} \sim \frac{\ell_F \hat{\lambda}}{\ell_c}, \]

where \( \mathcal{L}_m \neq 0 \) is the absolute value of the cooling function at a certain temperature \( T_1 < T < T_2 \); for example, we can assume that \( \mathcal{L}_m \) is the maximum of \( |\mathcal{L}| \) in the interval \( (T_1, T_2) \); \( \ell_c = c_1 \tau_c \) is the wavelength whose inverse frequency is equal to the cooling time; \( \hat{\lambda} = \mathcal{L} / \mathcal{L}_m \) is the cooling function normalized to its maximal value \( \mathcal{L}_m \). The integral \( \int_{-\infty}^{\infty} \lambda(T) d\zeta \sim \lambda \); this means that \( |\lambda_s| \sim \ell_F / \ell_c \).

Taking into account that \( \kappa \propto v_T / (\sigma_{el} n) \), where \( v_T \sim c_1 \) is the thermal velocity, \( \sigma_{el} \) is the cross section for elastic scattering of particles providing heat transfer, and \( \mathcal{L} \propto (\sigma_{el} n v_T) \Delta E p_{in} \), where \( \Delta E \) is the mean energy radiated by two colliding particles in inelastic processes, and \( p_{in} \) is the mean probability for inelastic processes, we obtain \( \ell_F / \ell_c \sim (p_{in} \Delta E / kT_1)^{1/2} \). This estimate demonstrates that \( \ell_F / \ell_c \) is much smaller than unity because inelastic processes are sufficiently less frequent than elastic ones. For example, in a free-free radiating gas at \( T \sim 10^6 \) K, \( p_{in} \Delta E / kT \sim v_T / c \) where \( c \) is the light speed (Landau & Lifshits 1975). These arguments demonstrate that evaporation or condensation in a stationary flow are always subsonic.

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**FIGURE CAPTIONS**

**Figure 1** Different possible regimes for an inhomogeneous medium as a function of the ratio between the characteristic conductive and cooling times. The linear curve corresponds to eq. (2.13), the cubic curve corresponds to eq. (2.14), and the negative slope curve corresponds to eq. (2.14a); the vertical line corresponds to a fixed temperature of the hot phase; dashed regions correspond to states with large differences in the values of the characteristic times.

**Figure 2** Evolutionary stages of a CC front as a function of radius in units of $\ell$ for the case $\varphi = \infty$. The parameters adopted are $\bar{n} = 1.0$ cm$^{-3}$, $T = 10^4$ K, $\bar{\ell} = 3.68$ pc, $T = 3 \times 10^6$ K. In each panel the flow Mach number (solid line), the density (dotted), and the log of temperature (dashed) and pressure (long-dashed), all normalized to the cloud values, are reported. Note that in the last four panels the density is divided by a factor of 10, as indicated by the label $n/10$.

**Figure 3** Evolution of the local saturation parameter $\sigma_T$ as a function of the cloud radius for the same case shown in Fig. 2.

**Figure 4** The same as Fig. 2, but $\varphi = 2$.

**Figure 5** The same as Fig. 2, density isocontours.

**Figure 6** Evolution of an isobaric thermal conduction front (numbers correspond to time elapsed from $t = 0$) with a real cooling function. *Solid curves* full solution, *dashed curves* no cooling included.

**Figure 7** Segments of phase trajectories near the stationary point $\eta_1$ for a) $\dot{L}_{\eta_1} > 0$, b) $\dot{L}_{\eta_1} = 0$, and c) $\dot{L}_{\eta_1} = +\infty$.

**Figure 8** a) Phase diagram corresponding to the system (3.12), (3.13); b) eigensolution of (3.12), (3.13).

**Figure 9** Phase trajectories and eigenfunctions of eq. (3.12) with a) $T(-\infty) = T(+\infty) = T_2$ (cold cloud immersed in a hot intercloud gas); b) $T(-\infty) = T(+\infty) = T_1$ hot bubble in a cold infinite gas; c) periodic structure of cloud - intercloud gas.

**Figure 10** Van der Waals-type equation of state for interstellar gas. The dotted lines correspond to the intermediate region $-\infty < \xi < +\infty$.

**Figure 11** a) Schematic dependence of the cooling function on $\xi$: for an evaporation wave cooling function is asymmetric and heating dominates ($\lambda_s < 0$); b) structure of the integral curves near the critical points; c) phase diagram for an evaporation CC front and d) for a condensation one.

**Figure 12** Asymmetric “potential” curve. The straight line shows the increase in potential energy due to a negative “friction”.

**Figure 13** Special case of a solitary a) condensation-evaporation and b) evaporation-condensation wave with fixed conductive heat flux at the moving boundary $S$.

**Figure 14** a) Phase diagram and b) soliton-like profile of $T$ for the CC front described by eq. (4.5) with $\gamma + 1/\gamma - 1 = F(T_2)$; the thermally stable phase is located at $T = T(\pm\infty) = T_1$. 