Hierarchy Construction of Quantum Hall States and Non-Commutative Chern-Simons Theory

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Abstract

In this paper, we study the non-commutative Chern-Simons description of the hierarchy of quantum Hall states. Our method is based on the framework suggested by Susskind in hep-th/0101029. By using the area preserving diffeomorphism gauge symmetry of quasiparticle fluid, we show that non-commutative Chern-Simons description of the hierarchy construction of quantum Hall states with generic filling fraction can be realized in Susskind’s approach. The relationship between our model and the previous work on the effective field theory of quantum Hall states is also discussed.

1 Introduction

After the discovery of fractional Quantum Hall Effect (FQHE) in 1982, many efforts have been made on the theoretical explanation of this phenomenon. As pointed in [1] (see also [2][3]), the most well established theoretical idea in the domain of the FQHE is the Laughlin wave function describing the states with filling fraction

$$\Psi_{\text{laugh}} = \prod_{i<j} (Z_i - Z_j)^{\frac{\nu}{2}} \exp\left(-\frac{1}{2} \sum |Z_i|^2 \right). \quad (1)$$

The main developments on the states with more generic filling fraction are the hierarchy construction [4] as well as the composite fermion proposal [5]. In the hierarchy construction, the condensation of quasiparticles changes the filling fraction of the system naturally. Any states with odd denominator filling fraction could be realized in this manner. On the other hand, the composite fermion proposal suggests that electron combining with 2p magnetic flux quanta forms a composite fermion. The FQHE is explained as the integer Quantum Hall Effect of the composite fermion. The composite fermion approach

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have made great progress on explaining the experiment results. However, the physical mechanism of forming the composite fermion remains a great puzzle.

Many effective field theories based on these two approaches were established (see for example \[7\][8][9]). An interesting observation is that the Chern-Simons theory plays a crucial role in both the two approaches. For example, the Chern-Simons term provides the mechanism of transition between electron and composite fermion, and the coefficients in the effective Chern-Simons action of hierarchy model is suggested to describe the topological order of the corresponding state. Despite the successes of these effective on describing the physics of FQHE, the origin of the Chern-Simons term remains unclear. A reasonable idea on this item in which the Chern-Simons action arises naturally from the Lorentz force of magnetic field is suggested in \[10\] by Bahcall and Susskind. Inspired by the development of matrix model in M theory, this idea is developed more systematically and more soundly in \[6\]: 1) The dynamics of 2D-electron fluid in perpendicular external strong magnetic field is a Chern-Simons gauge theory based the group of area preserving diffeomorphisms (APD’s). This gauge theory of APD’s captures many of the long distance features of the Quantum Hall system; 2) In order to capture the discrete or granular character of the electron system, we need to discretize the APD’s. A well known way to do so is method of the non-commutative field theory. Thus, a non-commutative version of Chern-Simons theory for FQHE is achieved. This scenario can be thought of as a micro-theory to describe FQHE based on the dynamics of noncommutative fluids of charged particle (e.g., see a recent review \[11\]). The corresponding matrix model has also been established. The subsequent discussions \[12\][13][14][15] have shown the consistence between this description and the Laughlin wave function’s with $\nu = \frac{1}{2m+1}$. Following this great success, we should pursue a further challenge problem whether the hierarchy construction of $\nu$ could be derived from this fluid dynamics theory or not. In other words, we should pursue the micro dynamical origin of the hierarchy construction in FQHE. The purpose of this paper is reformulating the hierarchy construction in the Susskind’s framework. The main point is the condensation of quasiparticles which are the excitations of original Quantum Hall Fluid could also be analyzed in the Susskind’s approach to get a non-commutative Chern-Simons description.

The content is organized as follows. We review the argument of \[6\] first in Section 2. Then filling fraction hierarchy is constructed by considering the condensation of quasiparticles in Section 3. In Section 4, the effects of interaction between quasiparticles is discussed. we show that these interaction is related to the fractional statistics of quasiparticles. The full noncommutative Chern-Simons theory as well as the corresponding matrix model is established. Finally, the relationship with previous results is discussed in Section 5.
Non-commutative Chern-Simons description of quantum Hall effect

In order to illuminate the clue we will follow and the notations, we briefly recall the procedure in [6]. A collection of identical electrons indexed by \( \alpha \), moving on a plane can be described by the Lagrangian

\[
L = \sum_{\alpha} \left( \frac{m}{2} \dot{x}_\alpha^2 - V(x) \right),
\]

where \( V \) is the potential energy. Assuming the system behaves like a fluid we can pass to a continuum description by replacing the discrete label \( \alpha \) by a pair of continuous coordinates \( y_1, y_2 \). These coordinates label the material points of the fluid. The system of particles is thereby replaced by a pair of continuum fields \( x_i(y,t) \) with \( i = 1, 2 \). Without loss of generality we can choose the coordinates \( y \) so that the number of particles per unit area in \( y \) space is constant and given by \( \rho_0 \). The real space density is

\[
\rho = \rho_0 \left| \frac{\partial y}{\partial x} \right|,
\]

where \( \left| \frac{\partial y}{\partial x} \right| \) is the Jacobian connecting the \( x \) and \( y \) coordinate systems. One further assume that under this fluid approximation the potential terms could be written as

\[
V = V \left( \rho_0 \left| \frac{\partial y}{\partial x} \right| \right).
\]

The Lagrangian becomes

\[
L = \int d^2y \rho_0 \left[ \frac{m}{2} \dot{x}^2 - V \left( \rho_0 \left| \frac{\partial y}{\partial x} \right| \right) \right].
\]

The fluid is dissipationless and the above Lagrangian has a gauge symmetry originated from relabeling the particles. Consider the area preserving diffeomorphism from \( y \) to \( y' \) with unit Jacobian. It induce the fields transformation \( x'(y') = x(y) \) which keeps the Lagrangian invariant if the boundary terms has no contribute (e.g. the periodical boundary condition). The infinitesimal transformation takes the form

\[
\delta x_a = \epsilon_{ij} \frac{\partial x_a}{\partial y_i} \frac{\partial \Lambda}{\partial y_j}.
\]

Under the assumption that the potential \( V \) in (4) has a minimum at \( \rho = \rho_0 \), [6] defined a field \( A \) by

\[
x_i = y_i + \epsilon_{ij} \frac{A_j}{2\pi \rho_0}.
\]
The gauge transformation becomes

\[ \delta A_i = 2\pi \rho_0 \frac{\partial \Lambda}{\partial y_i} + \frac{\partial A_i}{\partial y_i} \frac{\partial \Lambda}{\partial y_m} \epsilon_{i,m}. \]  

(8)

which has the form as the first order truncations of a U(1) gauge transformation on non-commutative plane. Thus approximately \( A \) could be regarded as a U(1) gauge field on the non-commutative \( y \)-plane. Further discussion shows that (5) will give the standard Maxwell action of \( A \) under proper gauge conditions.

If there is a background magnetic field \( B \), the Lagrangian gets an extra term

\[ L' = \frac{eB}{2} \int \rho_0 d^2 y \epsilon_{ab} \dot{x}_a x_b. \]  

(9)

Substituting (7) into (9) and dropping total time derivatives gives the desired Chern-Simons (CS) Lagrangian

\[ L' = \frac{eB}{8\pi^2 \rho_0} \int d^2 y \epsilon_{ab} \dot{A}_a A_b = \frac{1}{4\pi \nu_1} \int d^2 y \epsilon_{ab} \dot{A}_a A_b. \]  

(10)

If \( B \) is sufficiently large, the dynamics will be dominated by CS term and the Maxwell terms could be ignored.

For (9), the conserved quantity related to the gauge symmetry is

\[ qJ^0 = \frac{eB}{2\pi} \left( \frac{1}{2} \epsilon_{ij} \epsilon_{ab} \frac{\partial x_b}{\partial y_j} \frac{\partial x_a}{\partial y_i} - 1 \right), \]  

(11)

where \( q \) is charge of the quasiparticles, that is, the vortices of fluid. To linear level, (11) becomes

\[ \frac{1}{2\pi \nu_1} \nabla \times A = qJ^0(y). \]  

(12)

Take \( J^0 = \delta^2(y) \), one can derive the quantization of the charge \( q \) as in [6]

\[ q = n, \]

where \( n \) is integer.

Projected back to the \( x \)-plane, it is shown in [6] that the quasiparticle carries fractional electronic change \( e_{qp} = q \nu_1 e \) as well as magnetic flux \( \Phi_{qp} = q(2\pi/e) \). The statistic phase for exchanging two quasiparticles is \( \phi_{qp} = \pi \nu_1 \), that is, the quasiparticle obeys the fractional statistics.

If \( q = 0 \) there is no vortices excitation, one can incorporate the constraint to the lagrangian by introducing a time component of \( A \):

\[ L' = \frac{eB\rho_0}{2} \epsilon_{ab} \int d^2 y \left[ \left( \dot{x}_a - \frac{1}{2\pi \rho_0} \{ x_a, A_0 \} \right) x_b - \frac{\epsilon_{ab}}{2\pi \rho_0} A_0 \right]. \]  

(13)
where \( \{F(y), G(y)\} = \epsilon_{ij} \partial_i F \partial_j G \) denotes the Poisson bracket. Furthermore, because of the observation that (8) and (13) have the form as the first order truncations of a U(1) CS theory on non-commutative plane, as well as considerations about the discreteness of electrons, it is argued in [6] that the theory describing the quantum Hall system should be the non-commutative CS theory

\[
L = -\frac{1}{4\pi \nu_1} \int d^2y \epsilon^{\mu \nu \rho} \left( \hat{A}_\mu \ast \partial_\nu \hat{A}_\rho + \frac{2i}{3} \hat{A}_\mu \ast \hat{A}_\nu \ast \hat{A}_\rho \right),
\]

with the usual Moyal star product defined in terms of the non-commutativity parameter

\[
[y_1, y_2] = i\theta_1 = \frac{i}{2\pi \rho_0},
\]

and

\[
\nu_1 = \frac{2\pi}{eB\rho_0}.
\]

This non-commutative CS theory can be reformulated via the following matrix model,

\[
L_M = \frac{eB}{2} \epsilon_{ab} Tr \left( x_a + i[x_a, A_0]_m \right) x_b - eB\theta Tr A_0.
\]

where \( x_a \) and \( A_0 \) are infinite dimension hermitian matrices. By considering the operation of exchange two particles in the matrix model, one can find the important result that the statistic phase of the particles which constitute the fluid is related to the filling fraction as

\[
\phi = \frac{\pi}{\nu_1}.
\]

Thus, for the electron fluid the level parameter in (14) should be quantized as

\[
\nu_1 = \frac{1}{p_1},
\]

where \( p_1 \) is an odd integer.

In absence of the quasiparticle excitations, the static solution \( x_i = y_i \) implies that the electron density on the physical \( x \)-plane is \( \rho = \rho_0 \). Therefore, one gets an explanation of the filling fraction of the fractional quantum Hall effect

\[
\nu = \frac{2\pi \rho}{eB} = \frac{1}{p_1}.
\]

### 3 The quasiparticle condensation and the hierarchy construction

In order to describe the quantum Hall effects with generic filling fraction, we should consider the condense of quasiparticles on the quantum Hall fluid.
First consider the dynamics of one quasiparticle. According to the quantization condition of $q$, for the element quasiparticle, we have $q = \pm 1$. Regarding the quasiparticle as "electron" on the $y$-plane, and now the $A_i$ plays the role of electromagnetic field. Analog with the electromagnetic coupling, one may take the form as

$$L_{\text{int}} = q \int d^2 y A_\mu J^\mu + \ldots$$

The dots refer to possible higher order correction from the non-commutativity of $y$-plane. We will come back to this issue in the next section. For a single quasiparticle, the background value of $A_i$ is zero, therefore this term vanishes.

On the other hand, in the $x$-plane viewpoint the quasiparticle provides an extra electric charge $e_{qp} = q\nu e$. The electromagnetic interaction for this extra charge is not taken account in the origin lagrangian (13). Noting that the background configuration for the quasiparticle is

$$x_i = y_i,$$

the corresponding interacting term should be

$$L'_{\text{int}} = -\frac{e_{qp}}{2} \epsilon_{ab} \hat{X}_a X_b = -\frac{2\pi \rho_0}{2} \epsilon_{ab} \hat{Y}_a Y_b,$$

where $X$ and $Y$ are the position of quasiparticle on $x$-plane as well as $y$-plane. The magnetic field $B$ on $x$-plane has induced a effective "magnetic field" on $y$-plane

$$B = 2\pi \rho_0.$$

In presence of this background magnetic field $B$, the coordinate operator will get a non-commutativity

$$[\hat{y}_1, \hat{y}_2] = \frac{i}{qB} = \frac{i}{2\pi \rho_0 q}.$$  

It is consistent with the the non-commutativity of $y$-plane

$$[y_1, y_2] = \frac{i}{2\pi \rho_0}.$$ 

Now if there is a quasiparticle fluid on the $y$-plane, one can repeat the procedure in Section 2. The quasiparticles are labeled by a pair of continuous coordinates $z_i$, and the position of quasiparticle on the $y$-plane becomes the continuum fields $y_i(z,t)$. Without loss of generality we can choose the coordinates $y$ so that the number of particles per unit area in $y$ space is constant and given by $\rho'_0$. Furthermore, one assumes that the potential energy from the Coulomb interaction becomes a function of density under this fluid approximation, that is, analog of (4) holds for the quasiparticles. Therefore, a
gauge symmetry originated from relabeling holds in this description. The corresponding
infinitesimal transformation is
\[ \delta y_a = \epsilon_{ij} \frac{\partial y_a}{\partial z_i} \frac{\partial N'}{\partial z_j}. \]  
(21)

Under the assumption that \( \rho' = \rho'_0 \) is a local minimum, one defines
\[ y_i = z_i + \epsilon_{ij} \frac{a_j}{2\pi \rho'_0}, \]  
and the gauge transformation becomes
\[ \delta a_i = 2\pi \rho'_0 \frac{\partial N'}{\partial z_i} + \frac{\partial a_i}{\partial z_l} \frac{\partial N'}{\partial z_m} \epsilon_{l,m}. \]  
(23)

This suggests that the theory describing the small deviation from the minimum should
be a U(1) gauge theory on the noncommutative \( z \)-plane.

If we ignore kinetic term as well as the interaction between the quasiparticles, we could
get the second level non-commutative CS action following the same argument as that in
Section 2
\[ L_2 = -\frac{q}{4\pi \nu_2} \int d^2 z \epsilon_{\mu\nu} \left( a_\mu \star \partial_\nu a_\rho + \frac{2i}{3} a_\mu \star a_\nu \star a_\rho \right), \]  
(24)
with the usual Moyal star product defined in terms of the non-commutativity parameter
\[ [z_1, z_2] = i\theta_2 = \frac{i}{2\pi \rho'_0}. \]  
(25)

The level parameter \( \nu_2 \) is defined as
\[ \nu_2 = \frac{2\pi \rho'_0}{B} = \frac{\rho'_0}{\rho_0}. \]  
(26)

Since we ignore the interaction between quasiparticles, the quasiparticle should be re-
garded as an boson on the \( y \)-plane. In order to give the right statistics in the corresponding
matrix model, the level parameter in (24) should be quantized as
\[ \nu_2 = \frac{1}{p_2}, \]  
(27)
where \( p_2 \) is a even positive integer since both \( \rho'_0 \) and \( \rho_0 \) are positive.

In our present case, the constraint (11) takes the form
\[ eB \left( \frac{1}{2} \epsilon_{ij} \epsilon_{ab} \frac{\partial x_b}{\partial y_j} \frac{\partial x_a}{\partial y_i} - 1 \right) = q\rho'_0. \]  
(28)

Then the electron density on the physical \( x \)-plane becomes
\[ \rho = \rho_0 \left| \frac{\partial y}{\partial x} \right|. \]
\[ \rho_0^2 = \frac{1}{2} \epsilon_{ij} \epsilon_{ab} \frac{\partial x_i}{\partial y_j} \frac{\partial x_a}{\partial y_b} \]  
\[ = \rho_0 \left( 1 + \frac{2\pi \rho \rho_0'}{eB} \right)^{-1} \]  
\[ = \rho_0 \left( 1 + q\nu_1\nu_2 \right)^{-1}. \quad (29) \]

Thus, one gets the filling fraction
\[ \nu = \frac{2\pi \rho}{eB} = \nu_1 \left( 1 + q\nu_1\nu_2 \right)^{-1} \]
\[ = \frac{1}{p_1 + \frac{q}{p_2}}. \quad (30) \]

The same procedure could be carried on level by level, and we could reformulate the hierarchy construction [4] naturally in Susskind's approach
\[ \nu = \frac{1}{p_1 \pm \frac{q}{p_2}}. \quad (31) \]

Since that \( p_1 \) is odd and \( p_i (i > 1) \) is even, any filling fraction with odd denominator could be realized in this manner.

### 4 Interaction between the quasiparticles

Now we incorporate the interaction between the quasiparticles. There are two kinds of interactions in our system. The first one is the Coulomb interaction of electronic charges. This interaction could be approximated a potential which is a function of the quasiparticle density as assumed before. If \( B \) is large, this term could be ignored when one consider the long range behavior. The second one comes from the \( y \)-plane magnetic field produced by the quasiparticle fluid itself in [12]. The mean value of this magnetic field is
\[ B_{self} = 2\pi \nu_1 q\rho'_0, \quad (32) \]
thus the total magnetic field felt by quasiparticle should be
\[ B_{total} = 2\pi \nu_1 q\rho'_0 + 2\pi \rho_0, \quad (33) \]
and the total filling fraction is altered as
\[ \tilde{\nu}_2 = \frac{2\pi \rho'_0}{qB_{total}} = \frac{1}{q^2 \nu_1 + \frac{q}{\nu_2}}. \quad (34) \]

Then the corresponding matrix model has the statistic phase
\[ \Delta \Gamma = \frac{\pi}{\nu_2} = \pi \left( q^2 \nu_1 + \frac{q}{\nu_2} \right). \quad (35) \]
Since $1/\nu_2$ is even integer, the phase is equivalent to $\pi \nu_1$ which is just the statistic phase of quasiparticle obtained in [6]. This result is not surprising. In [6], the derivation of quasiparticle statistics is just based on considering the back reaction of quasiparticle.

Now we will decide the action of the full theory. As in [6], the initial Lagrangian comes from the Lorentz force in the magnetic field is

$$L = q \rho_0' \int d^2z \left[ -\frac{B}{2} \epsilon_{ij} y_j + \hat{A}_i + \frac{\theta_1}{2} \epsilon_{jk} \hat{A}_j \frac{\partial \hat{A}_k}{\partial y_i} \right] \dot{y}_i - \hat{A}_0,$$

(36)

where

$$\hat{A}_\mu(z) = A_\mu(y(z)).$$

(37)

Noting that the field strength on the non-commutative $y$-plane is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i(A_\mu \ast A_\nu - A_\nu \ast A_\mu)$, we should make sure this field strength is felt by the quasiparticle in order to obtain the right statistics. Therefore, additional term is added to the $A_\mu j^\mu$ coupling. Here we just write down the linear term appeared before we take the full non-commutative assumption as in (14).

The conserved quantity related to the gauge transformation (23) is

$$\int d^2y \Pi_i \delta y_i,$$

(38)

where $\Pi_i$ is the canonical conjugate to $y_i$

$$\Pi_i = q \rho_0' \left( -\frac{B}{2} \epsilon_{ij} y_j + \hat{A}_i + \frac{\theta_1}{2} \epsilon_{jk} \hat{A}_j \frac{\partial \hat{A}_k}{\partial y_i} \right).$$

(39)

Then we obtain the constraint analogy with (12)

$$\epsilon_{kl} \frac{\partial}{\partial z_l} \left( -\frac{B}{2} \epsilon_{ij} y_j + \hat{A}_i + \frac{\theta_1}{2} \epsilon_{jm} \hat{A}_j \frac{\partial \hat{A}_m}{\partial y_i} \right) \frac{\partial y_i}{\partial z_k}$$

$$= -\frac{B}{2} \epsilon_{ij} \epsilon_{kl} \frac{\partial y_j}{\partial z_l} \frac{\partial y_i}{\partial z_k} + \epsilon_{kl} \frac{\partial \hat{A}_i}{\partial z_l} \frac{\partial y_i}{\partial z_k} + \frac{\theta_1}{2} \epsilon_{kl} \epsilon_{jm} \frac{\partial \hat{A}_j}{\partial z_l} \frac{\partial \hat{A}_m}{\partial z_k}$$

$$= -B (1 + q \nu_1 \nu_2).$$

(40)

Now we introduce the axillary field $a_0$ and take the constraint as equation of motion for $a_0$. The Lagrangian becomes

$$L = q \rho_0' \int d^2z \left[ -\frac{B}{2} \epsilon_{ij} y_j + \hat{A}_i + \frac{\theta_1}{2} \epsilon_{jk} \hat{A}_j \frac{\partial \hat{A}_k}{\partial y_i} \right] \left( \dot{y}_i - \frac{1}{2\pi \rho_0} \{ y_i, a_0 \} \right)$$

$$+ \frac{B}{2\pi \rho_0} (1 + q \nu_1 \nu_2) a_0 - \hat{A}_0.$$

(41)

Expanding around the vacuum

$$y_i = z_i + \epsilon_{ij} \frac{a_j}{2\pi \rho_0},$$

(42)
one deduced the Lagrangian for the fluctuation as

\[
L = -\frac{q}{4\pi\nu_2} \int d^2z \epsilon_{\mu\nu\rho} \left( a_\mu \partial_\nu a_\rho + \frac{\theta_2}{3} a_\mu \{ a_\nu, a_\rho \} \right) \\
+ \frac{q}{2\pi} \int d^2z \left( \epsilon_{ij} \left( \hat{A}_i + \frac{\theta_1}{2} \epsilon_{lk} \hat{A}_l \frac{\partial \hat{A}_k}{\partial y_l} \right) f_{0j} - 2\pi \rho_0 \hat{A}_0 + Bq\nu_1 \nu_2 a_0 \right),
\]

(43)

where

\[
f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + \theta_2 \{ a_\mu, a_\nu \}.
\]

(44)

In this case, the constraint can be rewritten as

\[
\frac{B}{4\pi \rho_0^2} \epsilon_{ij} f_{ij} + \epsilon_{ij} D_i \hat{A}_j + \epsilon_{ij} \frac{\theta_1}{2} \{ \hat{A}_i, \hat{A}_j \} = Bq\nu_1 \nu_2,
\]

(45)

where

\[
D_i \hat{A}_j = \frac{\partial}{\partial z_i} \hat{A}_j + \theta_2 \{ a_i, \hat{A}_j \}.
\]

(46)

Again, as in [6], the above formula as well as the discreteness of quasiparticles implies the full noncommutative assumption. Together with the results in Section 2, the full system is described by

\[
L = L_1 + L_2 + L_{\text{int}}
\]

(47)

\[
L_1 = \frac{1}{4\pi\nu_1} \int d^2y \epsilon_{\mu\nu\rho} \left( A_\mu \ast \partial_\nu A_\rho + \frac{2i}{3} A_\mu \ast A_\nu \ast A_\rho \right)
\]

(48)

\[
L_2 = -\frac{q}{4\pi\nu_2} \int d^2z \epsilon_{\mu\nu\rho} \left( a_\mu \ast \partial_\nu a_\rho + \frac{2i}{3} a_\mu \ast a_\nu \ast a_\rho \right)
\]

(49)

\[
L_{\text{int}} = \frac{q}{2\pi} \int d^2z \left( \epsilon_{ij} \left( \hat{A}_i + \frac{\theta_1}{2} \epsilon_{lk} \hat{A}_l \frac{\partial \hat{A}_k}{\partial y_l} \right) \ast f_{0j} - 2\pi \rho_0 \hat{A}_0 + Bq\nu_1 \nu_2 a_0 \right),
\]

(50)

where

\[
f(y) \ast g(y) = \exp \left( \frac{i}{2} \theta_1 \epsilon_{ij} \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \xi^j} \right) f(y + \xi) g(y + \zeta) |_{\xi = \zeta = 0}
\]

(51)

\[
f(z) \ast g(z) = \exp \left( \frac{i}{2} \theta_2 \epsilon_{ij} \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \xi^j} \right) f(z + \xi) g(z + \zeta) |_{\xi = \zeta = 0}
\]

(52)

\[
f(y)^\ast g(y) = \sum_{n=0}^{\infty} \frac{1}{(n + 1)!} \left( \frac{i}{2} \theta_1 \epsilon_{ij} \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \xi^j} \right)^n f(y + \xi) g(y + \zeta) |_{\xi = \zeta = 0}.
\]

(53)

Taking the \( \delta A_0 \) and \( \delta a_0 \) variations of (47) to give the constraints, the leading terms are

\[
\frac{1}{4\pi\nu_1} \epsilon_{ij} F_{ij} = q\rho_0 \left( 1 + \frac{1}{4\pi\rho_0^2} \epsilon_{ij} f_{ij} \right)^{-1}
\]

(54)

\[
\left( 1 + \frac{1}{4\pi\rho_0^2} \epsilon_{ij} f_{ij} \right) \left( B + \frac{1}{2} \epsilon_{ij} \hat{F}_{ij} \right) = B \left( 1 + q\nu_1 \nu_2 \right).
\]

(55)
Substituting the constraints back into (50), we find Lagrangian becomes

\[
L = \frac{1}{4\pi\nu_1} \int d^2y \epsilon_{\mu\nu\rho} \left( A_\mu \ast \partial_\nu A_\rho + \frac{2i}{3} A_\mu \ast A_\nu \ast A_\rho \right) - \frac{1}{4\pi\tilde{\nu}_2} \int d^2z \epsilon_{\mu\nu\rho} \left( a_\mu \ast \partial_\nu a_\rho + \frac{2i}{3} a_\mu \ast a_\nu \ast a_\rho \right) - \int d^2z q\rho' A_0. \tag{56}
\]

If we separate out the matrix model of quasiparticle, the coefficient $1/4\pi\tilde{\nu}_2$ would relate to the correct statistics of quasiparticles as we hope.

There are some subtleties for the corresponding matrix model since the two kind of non-commutativity in our case. In a naive construction, we can replace the fields in (41) by matrices straightforwardly and arrive at

\[
L = q Tr \{ \left( -\frac{B}{2} \epsilon_{ij} y_j + A_i + \frac{i}{2} \epsilon_{jk} A_j [A_k, \epsilon_{il} y_l] \right) (\dot{y}_i + i[y_i, a_0]) + \frac{B}{2\pi\rho_0} (1 + q\nu_1\nu_2) a_0 - A_0 \}, \tag{57}
\]

where

\[
A_i = 2\pi\rho_0 \epsilon_{ij} (x_j - y_j). \tag{58}
\]

When the number of electron and quasiparticle are $K$ and $\nu_1 K$ respectively, the matrices $x_i$ are $K \times K$ while $y_i$ are $\nu_1 K \times \nu_1 K$. We may simply take $y_i \otimes I_{\nu_1 \times \nu_1}$ as $y_i$ in the above expression.

### 5 Comparison with K-matrix theory

Our result is related to the effective Wen and Zee’s K-matrix Chern-Simons theory introduced in [7]. If we convert the Lagrangian (49) and (50) to $y$-plane by using the relation (22). Although the full theory looks rather complicated after this operation, the leading term is rather simple and interesting. It is

\[
L = \frac{1}{4\pi\nu_1} \int d^2y \epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \frac{q}{4\pi\nu_2} \int d^2y \epsilon_{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \frac{q}{2\pi} \int d^2y \left( \epsilon_{\mu\nu\rho} A_\mu \partial_\nu a_\rho - 2\pi\rho' A_0 \right) + \frac{q}{2\pi} \int d^2y Bq\nu_1\nu_2 a_0 (1 - \epsilon_{ij} \partial_i a_j). \tag{59}
\]

The result is analogy with the action for one generation K-matrix Chern-Simons theory. Especially, we find the coefficients in our result is fit with those in [7] which is used to describe the K-matrix as well as topological order. The main differences in between is the meaning of the gauge fields. The gauge fields are the coordinates of the particle in our case while they are related to the current of the particle in K-matrix Chern-Simons
theory. We employed the Lagrangian picture of fluid and K-matrix Chern-Simons theory is explained in the Eulerian picture.

Another important thing is that the Chern-Simons description of edge excitation in quantum Hall state introduced in [7] would receive a natural explanation in our framework. To show this point, let us take the simplest case where there is no quasiparticle excitation as an example. Considering the system in [7], an extra electric field $E$ is introduced to confine the quantum Hall fluid below the $x$-axes. The Lagrangian of single particle becomes

$$L = \rho_0 \int d^2 y \left( \frac{eB}{2} \epsilon_{ab} \dot{x}_a x_b - eE x_2 \right).$$

(60)

The particles will get an extra velocity from the electric field. Thus instead of (7), we would expand around the vacuum configuration as

$$x_i = y_i + \frac{\epsilon_{ij} E_j t + \epsilon_{ij} A_j}{2\pi \rho_0}.$$  

(61)

Then the resulting $y$-plane Chern-Simons theory remains the same formula as (10)

$$L = \frac{eB}{8\pi^2 \rho_0} \int d^2 y \epsilon_{ab} \dot{A}_a A_b,$$

(62)

which is the standard $y$-plane Chern-Simons Lagrangian in the temporal gauge. When we discuss the physics on the $x$-plane, the transformation

$$x_i = y_i + \frac{\epsilon_{ij} E_j t}{B},$$

(63)

should be performed. That is just the transformation for physical gauge choice in [7]. In presence of a boundary of the system, the area preserving diffeomorphism gauge symmetry is restricted naturally to those which leave the boundary invariant as demanded in [7]. Especially, our approach automatically gives out the velocity of edge excitation $v = \frac{E}{B}$ without comparison to other computation which is needed in [7].

From the above evidences, we conclude that the K-matrix Chern-Simons theory could be regarded as certain commutative limit of our theory. The main differences are the meaning of gauge fields as well as the additional terms coming from non-commutativity. These differences offer reasonable candidates for the experimental deviation [19] to the results of tunneling exponent predicted by K-matrix theory. We will discuss this subject in a subsequent work.

6 Conclusion

In this paper we have reviewed the interesting way of construct non-commutative Chern-Simons description for FQHE with $\nu = 1/p$ in [6]. It is based on the area preserving
diffeomorphisms of continuous electron indexes under fluid approximation. We have applied the same method on the quasiparticles and reformulated the hierarchy construction of FQHE suggested in [4]. The crucial point is that the coordinates of quasiparticle becomes the new gauge field on $z$-plane, where $z$-plane comes from a continuum description of the quasiparticle indexes rather than a simple coordinate choice. It is different from the case that one directly adds another kind of particles on $x$-plane which is a direct superposition of two (or more) independent Laughlin droplets. We have also shown that the interaction between quasiparticles implies the correct statistics of quasiparticles. The full non-commutative Chern-Simons theory is also built. However, the corresponding matrix model remains unclear because the coupling of the two kinds non-commutativity particles living on different plane. Also, the case is different in superposition of two independent Laughlin droplets where no such coupling exists. We have offered some naive proposal on the matrix model. Our work is closely related to the K matrix theory in [7]. We have provided an element derivation of the Chern-Simons theory for hierarchy description of quantum Hall state. The K-matrix Chern-Simons theory could be regarded as its commutative limit. On some related discussions of noncommutative Chern-Simons description of hierarchies FQH states, see [20][21].

We have mentioned in Section 1 that there is another approach to FQHE with generic filling fraction, the composite fermion assumption. It is argued in [16] that the hierarchy construction and composite fermions may be complementary views of the same phenomena rather than mutually excluding descriptions. Some researches such as [18] [17] have tried to relate the Chern-Simons actions in [7] and those in the composite fermion approach [8]. Since the composite fermion approach is favored by the experiment results, we hope to relate the two sides under Susskind’s framework in our future works.

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