Hopf bifurcation of actuated micro-beam nonlinear vibrations in micro electro mechanical systems

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Abstract. In this paper, the effects of micro-beam stiffness changes to the dynamic of nonlinear vibrations are investigated. Nonlinear vibrations equation of an actuated micro-beam is derived based on Euler-Bernoulli beam theory. Galerkin method is adopted to simplify the nonlinear equation of the motion. The simpler equation transformed into a dynamical system and its eigen values are analysed. To show the dynamic of the system, the bifurcation and phase plane diagrams are drawn. The numerical result showed that the change of micro-beam stiffness exhibits a Hopf bifurcation.

1. Introduction
Technology has been very advanced; the tools are made small and smaller even in micro and nano sizes. These tools use Micro Electro Mechanical Systems (MEMS) or Nano Electro Mechanical Systems (NEMS). MEMS is the technology of microscopic merged devices that combine electrical and mechanical components [10]. NEMS is the natural evolution of MEMS that has a characteristic size of a nanometer order [9]. The evolution might have new effects, as demonstrated for nanotube-based NEMS [8]. Because of its micro or nano sizes, MEMS and NEMS are enormously used in various engineering applications. The other reasons due to its low energy consumption, high sensitivity, and easy-to-digital output [4]. The applications of the systems in engineering such as: micro-switches, transistors, accelerometers, etc [11]. Focus on MEMS, it has a core component named micro-beam. When the micro-beam is actuated electrically, it is vibrating nonlinearly.

Studying on the nonlinear vibrations of micro-beam is one of the most interesting topics for scientists [1]. Huang D et al, 2011, in [4] investigate the characteristics of micro-beam structure nonlinear vibration acting by multi-couplings factors. Huang D showed that the factors air damping force and alternating voltage excitation have a vital role in the characteristics of micro-beam non-linear vibrations. Huang said that to design and applicate micro-beam system, the effects of these two factors must be considered. In the same year, Chen C et al in [2] used the incremental harmonic balanced method to analyse the nonlinear dynamic stability of an electrically actuated microbeam. The investigation shows that symmetric electrostatic load can affect the dynamic stability. Two years earlier, He J H in [3] proposed variational iteration method to solve nonlinear equations and this variational method used by Bayat M et al, 2013, in [1] to approach the MEMS micro-beam actuated vibrations equation. Sadeghzadeh S and Kabiri K, 2016, in [10] implement the higher order Hamiltonian method to analyse an electrostatically actuated nonlinear micro-beam based micro electro mechanical oscillator.

In this paper, the effects of micro-beam stiffness change to the dynamic of MEMS micro-beam vibrations is considered. The nonlinear vibrations equation of MEMS micro-beam is formed based on
Euler-Bernoulli beam theory. To simplify the nonlinear equation of the motion, Galerkin method is adopted. The result equation is transformed into a dynamical system. And based of its eigen values, the dynamical system is normalized to show the bifurcation of the system.

2. Mathematical model

Figure 1 shows simplified typical MEMS model consisting of two fixed electrode and an unfixed single plate of a micro-cantilever or micro-beam. The micro-beam electrode is a flexible micro-plate capable to vibrate actuated by electricity on a certain voltage and serve as a mechanical resonator.

\[
\begin{align*}
EI \frac{\partial^4 w}{\partial x^4} + \rho bh \frac{\partial^2 w}{\partial t^2} + c \frac{\partial w}{\partial t} - kw_0 \frac{\partial w}{\partial t} &= \left[ N + \frac{Eb}{2l} \int_0^l \left( \frac{\partial w}{\partial x} \right)^2 dx \right] \frac{\partial^2 w}{\partial x^2} + f(x, t) \\
\end{align*}
\]

where \( E \) is the Young’s modulus, \( v \) is the Poisson’s ratio and \( \bar{E} \) is the effective modulus of the micro-beam then, based on [7], the value of \( \bar{E} \) depends on the thickness of the micro-beam, that is

\[
\bar{E} = \begin{cases} 
\frac{E}{1 - v^2}, & b \geq 5h \\
\frac{E}{1 - v^2}, & b < 5h.
\end{cases}
\]

According to theory of plate capacitor [7], the electrostatic force per unit length of the beam is

\[
f(x, t) = \frac{ebV^2}{2} \left( \frac{1}{(d - w)^2} - \frac{1}{(d + w)^2} \right) = \frac{2ebV^2}{(d^2 - w^2)^2} \cdot w \tag{2}
\]

The boundary conditions of equation (1) are:

\[
\begin{align*}
&w(0, t) = w(l, t) = 0 \\
&\frac{\partial w(0, t)}{\partial x} = \frac{\partial w(l, t)}{\partial x} = 0 \\
\end{align*}
\]

The nomenclature is given in Table 1.

Substituting equation (2) into equation (1) and then we can transform the result into

\[
\frac{\partial^4 w}{\partial y^4} + \beta_1 \frac{\partial^2 w}{\partial t^2} + \beta_2 \frac{\partial w}{\partial t} = \left[ \beta_3 + \beta_4 \int_0^1 \left( \frac{\partial w}{\partial y} \right)^2 dy \right] \frac{\partial^2 w}{\partial y^2} + \beta_5 \frac{w}{(d^2 - w^2)^2} \tag{5}
\]

where

\[
\beta_1 = \frac{pbh l^4}{EI}, \beta_2 = \frac{(c - kw_0) l^4}{EI}, \beta_3 = \frac{N l^2}{EI}, \beta_4 = \frac{bh}{2l^2}, \beta_5 = \frac{2ebV^2 l^4}{EI}, y = x 
\]
The value of $y$ become the element of $[0,1]$ and the boundary conditions (3) and (4) become

$$w(0, t) = w(1, t) = 0$$

(6)

$$\frac{\partial w(0, t)}{\partial y} = \frac{\partial w(1, t)}{\partial y} = 0$$

(7)

**Table 1. Nomenclature**

| $I$     | the cross-sectional second moment of inertia (moment of inertia about the longitudinal axis of the micro-beam) |
|---------|-----------------------------------------------------------------------------------------------------------|
| $\rho$  | the mass density of the micro-beam along the $x$ direction                                               |
| $b$     | the width of the micro-beam                                                                                |
| $h$     | the thickness of the micro-beam                                                                            |
| $c$     | the damping coefficient                                                                                    |
| $k$     | the stiffness of the micro-beam                                                                            |
| $w_0$   | The maximum lateral displacement                                                                          |
| $N$     | the axial force/the tensile or compressive axial load                                                     |
| $l$     | the length of micro-beam                                                                                   |
| $e$     | the dielectric constant of the interface/vacuum permittivity (8.85 pf/m)                                 |
| $d$     | gap size                                                                                                |
| $V$     | the actuating voltage                                                                                     |

3. **Transformation into system of differential equations**

We assumed that the solution of equation (5) could be introduced as;

$$\omega(y, t) = \sum_{i=0}^{n} \phi_i(y) u_i(t)$$

Where $\phi_i(y)$ is the $i^{th}$ eigen function of micro-beam that satisfied the appropriate boundary condition [6], $u_i(t)$ is the $i^{th}$ deflection coordinate, and $n$ is the degree of freedom of the micro-beam.

In this paper, we consider only a single degree of freedom ($n = 1$), thus the deflection function would become

$$\omega(y, t) = \phi(y) u(t).$$

(8)

The trial function is

$$\phi(y) = C \sin(\pi y),$$

(9)

where $C$ is a real constant.

Substitute equation (9) into equation (5) and assume that $u_2 = u_1'$ we get a system of differential equation

$$u_1' = u_2$$

$$u_2' = -\frac{\beta_5 \pi^2 - \pi^4 + \beta_6}{\beta_1} u_1 + \frac{2\beta_5 C^2 \sin^2(\pi y) - \beta_4}{\beta_1^2} \pi^4 C^2 \sin^2(\pi y) u_1^3 + O(u_1^5)$$

(10)

The Jacobian matrix for the equilibrium $u_1 = 0$ is

$$J = \begin{bmatrix} \frac{\beta_5 \pi^2 - \pi^4 + \beta_6}{\beta_1} & 1 \\ \frac{-\beta_5 \pi^2 - \pi^4 + \beta_6}{\beta_1} & -\frac{\beta_4}{\beta_1^2} \end{bmatrix}$$

(11)

For if $E = 169 \text{ GPa}$, $I = 1.33 \times 10^{-7} \mu m$, $\rho = 2330 \text{ kg/m}^2$, $b = 4 \mu m$, $h = 2 \mu m$, $N = 15 \text{ MPa}$, $V = 3.94 V$, $c = 5 \times 10^{-11} \text{ Ns/m}$, $d = 2.2 \mu m$, and $w_0 = 2 \mu m$ we got that the eigenvalues of $J$ would be complex. And the eigenvalues would purely imaginer if $k = 7 \times 10^{-5} N/m$, the real value of eigen would be negative if $k < 7 \times 10^{-5} N/m$, and would be positive if $k > 7 \times 10^{-5} N/m$. 


4. Normalization

Devine new variable \( \alpha = k - 7 \times 10^{-5} \) substitute to equation (10), we got
\[
 u' = A(\alpha)u + F(u, \alpha) \tag{12}
\]
where
\[
 A(\alpha) = \begin{bmatrix}
 0 & 1 \\
 -1.027 \cdot 10^{22} & 107.296 \alpha
\end{bmatrix}
\tag{13}
\]
and
\[
 F(u, \alpha) = \begin{bmatrix}
 0 \\
 F_1 u_1^3 + O(u_1^5)
\end{bmatrix}
\tag{14}
\]
Where \( F_1 \) is the function of \( y \), that is
\[
 F_1 = -9.691762116 \times 10^{17} + 4.71704240010^{-19} \sin^2(\pi y).
\]

One of the eigenvalues of matrix \( A \) is
\[
 \lambda(\alpha) = 53.64860665 \alpha + \sqrt{2878,115270 \alpha^2 - 1.1027488348 \times 10^{22}}
\]
(15)
The eigenvector corresponds to \( \lambda(\alpha) \) is
\[
 v = \begin{bmatrix}
 1 \\
 p \\
 \lambda(\alpha) \\
 p
\end{bmatrix}
\]

Where \( p = (1 - 2i)(2\lambda(\alpha) - 107.2961373 \alpha) \). And one of the eigenvalues of \( A^T \) is \( \bar{\lambda}(\alpha) \). The eigenvector corresponds to \( \bar{\lambda} \) is
\[
 w = \left( \begin{bmatrix}
 1 + 2i(\bar{\lambda}(\alpha) - 107.2961373 \alpha) \\
 1 + 2i
\end{bmatrix}
\right)
\]
Thus for \( \alpha \) sufficiently small we have
\[
 \langle v, w \rangle = 1
\]
and
\[
 \langle v, \bar{w} \rangle = 0.
\]

If \( z = \langle v, u \rangle \), since \( \langle v, w \rangle = 1 \) and \( \langle v, \bar{w} \rangle = 0 \) we have that \( u = zw + \bar{z} \bar{w} \), we have
\[
 \langle v, F(zw + \bar{z} \bar{w}) \rangle = \frac{1}{p} (g_1 z^3 + g_2 z^2 \bar{z} + g_3 z \bar{z}^2 + g_4 \bar{z}^3) + O((z + \bar{z})^5)
\]
where
\[
 g_1 = (-11 - 2i) \bar{\lambda} F_1 (\bar{\lambda} - 107.2961373 \alpha) \\
 g_2 = (15 + 30i) \bar{\lambda} F_1 (\bar{\lambda} - 107.2961373 \alpha) (\bar{\lambda} - 107.2961373 \alpha) \\
 g_3 = (15 + 30i) \bar{\lambda} F_1 (\bar{\lambda} - 107.2961373 \alpha) (\bar{\lambda} - 107.2961373 \alpha) \\
 g_4 = (-11 + 2i) \bar{\lambda} F_1 (\bar{\lambda} - 107.2961373 \alpha)
\]
Thus equation (12) can be transform into
\[
 \dot{z} = (\lambda(\alpha)z + g(z, \bar{z}, \alpha)
\]
where
\[
 g(z, \bar{z}, \alpha) = \langle v_1, F(zv_2 + \bar{z} \bar{v}_2) \rangle
\]
For if \( g_{kl}(\alpha) = \frac{\partial^{k+l} g(z, \bar{z}, \alpha)}{\partial z^k \partial \bar{z}^l} \bigg|_{z=0} \) we have
\[
 g_{20}(\alpha) = 0 \\
 g_{11}(\alpha) = 0 \\
 g_{02}(\alpha) = 0 \\
 g_{21}(\alpha) = \frac{2}{p} g_2
\]
Thus, we have first Lyapunov coefficient at \( \alpha = 0 \) is
\[
 l_1(0) = \frac{Re(i=g_{20}g_{11} + Im(\lambda(0))g_{21})}{2 Im^2(\lambda(0))} > 0.
\]
According to theorem in the generic of Hopf bifurcation [5], system (16) can be change into
\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = \begin{bmatrix}
\beta & -1 \\
1 & \beta
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} + (z_1^2 + z_2^2) \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}.
\] (17)

Based on [5] and [12], equation (17) is the normal form of subcritical Hopf bifurcation.

We have a problem on graphing the phase portrait of system (10) since it has a form of sinus function that depends on \( y \) variable meanwhile the system is of \( u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \) variable and has a parameter \( k \). If we fixed the sinus function or the \( y \) variable, we cannot get any conclusion about the dynamic of the system. But system (17) is only depends on \( z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \) variable and a parameter \( \beta \). Here we can take some value of \( \beta \), that is for \( \beta < 0, \beta = 0, \beta > 0 \) to see the dynamic of the system. Figure 2 showed the bifurcation diagram of system (17).

![Figure 2. Bifurcation diagram of system (17).](image)

Figure 2a. depicts the phase portrait of system (17) when parameter \( \beta < 0 \). Here we can see that the equilibrium point is unstable, and the system has a stable limit cycle. In Figure 2b., parameter \( \beta = 0 \), the equilibrium point stills unstable, but the limit cycle starts to vanish. And in Figure 2c., parameter \( \beta > 0 \), the equilibrium point is stable, and there is no limit cycle.

5. Conclusion

The model of nonlinear vibrations equation of a micro-beam that is derived by Euler-Bernoulli beam theory and then simplified by Galerkin method is exhibit a Hopf bifurcation. The bifurcation showed up when the parameter \( k \) that act as the stiffness of the micro-beam is equal to \( 7 \times 10^{-5} \text{N/m} \). The type of Hopf is subcritical, that is the equilibrium point is stable if \( k > 7 \times 10^{-5} \text{N/m} \) and unstable when \( k < 7 \times 10^{-5} \text{N/m} \). The stable limit cycle around the equilibrium point appears when the equilibrium point is unstable.

References

[1] Bayat M et al 2014 Nonlinear Vibration of an Electrostatically Actuated Microbeam Latin American Journal of Solids and Structures 11 pp 534 – 544
[2] Chen C et al 2011 Nonlinear Dynamic Stability of an Electrically Actuated Microbeam Structure in MEMS Proceedings of the ASME International Mechanical Engineering Congress & Exposition (IMECE)
[3] He J H 1999 Variational Iteration Method: a Kind of Nonlinear Analytical Technique: Some Examples International Journal of Non-Linear Mechanics 34(4) pp 699–708
[4] Huang D et al 2011 Nonlinear Vibration Analysis of MEMS Micro-Beam Structure Acting by Multi-Couplings Factors Proceedings of the ASME International Mechanical Engineering Congress & Exposition (IMECE)
[5] Kuznetsov Y A 1998 *Elements of Applied Bifurcations Theory* Second Edition (New York: Springer-Verlag)

[6] Nayfeh A H and Mook D T 1979 *Nonlinear Oscillations* (New York: John Wiley & Sons Inc)

[7] Pelesko J A and Bernstein D H 2003 *Modeling MEMS and NEMS* (Florida: Chapman & Hall/CRC)

[8] Pugno N 2004 *Tunneling Current-Voltage Controls, Oscillations and Instability of Nanotube- and Nanowire-Based Nanoelectromechanical Systems* Glass Physics and Chemistry Special Issue: Nanoparticles, Nanostructures and Nanocomposites 31 pp 535–544

[9] Pugno N 2005 *Non-linear Statics and Dynamics of Nanoelectromechanical Systems Based on Nanoplates and Nanowires* Proceedings of the Institution of Mechanical Engineers Part N: Journal Nanoengineering and Nanosystems 219 Issue 1 pp 29 – 40

[10] Sadeghzadeh S and Kabiri A 2016 *Application of Higher Order Hamiltonian Approach to the Nonlinear Vibration of Micro Electro Mechanical Systems* Latin American Journal of Solids and Structures 13 pp 478 – 497

[11] Senturia S D 2001 *Microsystem Design* (Dordrecht: Kluwer Academic Publisher)

[12] Wiggins S 2003 *Introduction to Applied Nonlinear Dynamical Systems and Chaos* Second Edition (New York: Springer-Verlag)