Yet another inverse function theorem

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Abstract

We prove a Nash-Moser type inverse function theorem in Fréchet spaces for functions with approximate inverses, allowing for a loss of derivatives proportional to $n$ in the way of Lojasiewicz and Zehnder.

1 Introduction

The goal of this paper is to obtain an inverse function theorem for functions with approximative inverses in graded Fréchet spaces, similar to the one that can be found in [Zeh,75]. There, a Nash-Moser inverse function theorem is proven for functions $\phi$ such that its derivative $D\phi$ does not have a right inverse, but only an approximation to it. In [Zeh,75], such a theorem is proven, under the assumption that the loss of derivatives does not depend on $n$. Thanks to Lojasiewicz and Zehnder (see [L-Z,79]), it is known that, when a right inverse for $D\phi$ exists, the loss of derivatives can be of the order of $(\lambda - 1)n$, as far as $\lambda < 2$. Our goal is to provide a bridge between these two theorems.

All along the article, $E$ and $F$ will be tame Fréchet spaces with graded norms, and $S_\theta$, for $\theta \geq 1$, will be the corresponding smoothing operators. The map $\phi$ of which we want to find the inverse will be a tame map, with tame derivative, that is, $|\phi(x)|_n \leq C|x|_n$ and $|\phi'(x)v|_n \leq C(|x|_n|v|_d + |v|_n)$. For such a function, we prove the following

**Theorem 0** Let $E$, $F$, and $\phi : E \to F$ be as above. Assume that there is a local approximate right inverse $L : (U \subset E) \times F \to E$ to $\phi'$, where $U$ is some neighborhood of $0 \in E$, satisfying

$$|(\phi'(x)L(x) - I)y|_n \leq C(|x|_n|y|_d + |y|_n)|x|_n$$
and
\[ |(\phi'(x)L(x) - I)y|_n \leq C(|x|_n |y|_d + |y|_n), \]
for some \(d\). Assume also that
\[ |L(x)y|_n \leq C(|x|_{n+d}|y|_d + |y|_{n+d}), \]
for any \((x,y) \in U \times F\) and \(n \geq 0\), and given \(d \geq 0\) and \(1 \leq \lambda < 2\). Assume also that for the Taylor rest
\[ R(x,v) = \phi(x + v) - \phi(x) - \phi'(x)v \]
there are \(C\) and \(l\) so that
\[ |R(x,v)|_n \leq C\left(|x|_n |v|^2 + |v||v|_n\right). \]
Moreover, assume that there is an \(m\) so that
\[ \sum_{l=0}^{\infty} (S_{\theta}(1 - \phi'(x)L(x)))^l y|_n \leq C(n)\theta^m(|x|_n |y|_d + |y|_n). \]
Then \(\phi\) has a local right inverse \(\psi\), defined on a neighborhood \(V = \{ y \in F : |y|_{s_0} < \delta \}\) of 0, and satisfying \(\phi \circ \psi = I\) on \(V\). The conditions we will impose on \(F\) will apply in a neighborhood of \(U = \{ |x|_l < 1 \}\), for some given \(l > 0\). The first condition we will require is that \(\phi\) be tame, that is (using that \(\phi(0) = 0\)) that there is a \(C\) for which:
\[ |S_{\theta}x|_n \leq C(n,k)\theta^{n-k}|x|_k \]
\[ |(1 - S_{\theta})x|_k \leq C(n,k)\theta^{-(n-k)}|x|_n \]
(1)
Recall that from [1] we get the interpolation inequalities:
\[ |x|_l \leq C(l,k,n) |x|_k^{1-\alpha} |x|_n^\alpha \quad \text{if } l = (1-\alpha)k + \alpha n. \]

2 The spaces
Let \(E\) and \(F\) be two Fréchet spaces, with graded norms (for the definitions, see Hamilton [Ham,82]). Moreover, we will assume that \(E\) and \(F\) are tame in the sense that there exist in each of them a collection of smoothing linear operators \(\{S_\theta \mid \theta \geq 1\}\), from the space into itself, such that, for \(0 \leq k \leq n\):
\[ |S_{\theta}x|_n \leq C(n,k)\theta^{n-k}|x|_k \]
\[ |(1 - S_{\theta})x|_k \leq C(n,k)\theta^{-(n-k)}|x|_n \]
(1)
Recall that from \cite{1} we get the interpolation inequalities:
\[ |x|_l \leq C(l,k,n) |x|_k^{1-\alpha} |x|_n^\alpha \quad \text{if } l = (1-\alpha)k + \alpha n. \]

3 The function
Let \(\phi : E \to F\) be a continuous function such that \(\phi(0) = 0\). We want to know which conditions on \(\phi\) assure us the existence of a local inverse \(\psi\), defined on a neighborhood \(V\) of \(0 \in F\) and satisfying \(\phi \circ \psi = I\) on \(V\). The conditions we will impose on \(F\) will apply in a neighborhood of \(U = \{ |x|_l < 1 \}\), for some given \(l > 0\). The first condition we will require is that \(\phi\) be tame, that is (using that \(\phi(0) = 0\)) that there is a \(C\) for which:
\[ |\phi(x)|_n \leq C|x|_{n+d_1}, \]
whenever \(x \in U\), for all \(n \geq 0\) and a given \(d_1\). Let us notice, though, that by renumbering the seminorms, we can get \(d_1 = 0\), and thus the requirement is:
\[ |\phi(x)|_n \leq C|x|_n. \]
Another requirement is that $\phi$ be Fréchet differentiable in $U$, and that its derivative can be bounded by

$$|\phi'(x)v|_n \leq C (|x|_{n+d_2}v|_d + |v|_{n+d_2})$$

for some $d_2 \geq 0$, and all $(x,v) \in U \times E$ and $n \geq 0$. Again, by reordering the seminorms we can get $d_2 = 0$. Thus what we ask for is that, for some $C$ and $d$:

$$|\phi'(x)v|_n \leq C (|x|_n|v|_d + |v|_n). \quad (3)$$

Moreover, we want $\phi'$ to have an approximate right inverse, that is, we ask for the existence of a function $L : (U \subset E) \times F \to E$, lineal with respect to $F$, satisfying:

$$|(\phi'(x)L(x) - I)y|_n \leq C (|x|_n|y|_d + |y|_n) |x|_n \quad (4)$$

and

$$|(\phi'(x)L(x) - I)y|_n \leq C (|x|_n|y|_d + |y|_n) \quad (5)$$

In particular, $L(0)$ is the right inverse of $\phi'(0)$. For $L$ we require (as in [L-Z,79]) the following growth condition to hold:

$$|L(x)y|_n \leq C (|x|_{\lambda+n+d}|y|_d + |y|_{\lambda+n+d}), \quad (6)$$

for any $(x,y) \in U \times F$ and $n \geq 0$, and given $d \geq 0$ and $\lambda \geq 1$. We also do need a bound for the Taylor rest

$$R(x,v) = \phi(x + v) - \phi(x) - \phi'(x)v.$$

Our hypothesis will be that:

$$|R(x,v)|_n \leq C (|x|_n|v|^2 + |v||v|_n). \quad (7)$$

All along the proof we will use an approximation to the inverse of $\phi'$ we will need to bound. Namely, what we need is that there is an $m$ so that:

$$|\sum_{l=0}^{\infty} (S_{\theta}(1 - \phi'(x)L(x)))^l y|_n \leq C(n)\theta^m(|x|_n|y|_d + |y|_n). \quad (8)$$

4 \hspace{1em} \textbf{the lemmas}

Let $1 \leq \lambda \leq \tau < 2$, take $\tau = \frac{\lambda + 2}{2}$. Let us consider the sequence $(\theta_p)_{p \in \mathbb{N}}$ defined by $\theta_0 = 2^{\tau^p}$. Observe that $\theta_p = \theta_{p+1}$.

We want to find the solution $x$ of the equation $\phi(x) = y$, for $y \in E$ small enough. To do so, we define the sequence $(x_p) = (x_p(y))_p$ by:

$$x_0 = 0$$

$$x_{p+1} = x_p + \Delta x_p$$

$$\Delta x_p = S_{\theta_p} L(x_p) \sum_{l=0}^{\infty} (S_{\theta_p}(1 - \phi'(x_p)L(x_p)))^l z_p$$

$$z_p = y - \phi(x_p).$$
Lemma 1 Let \( y \in E \) satisfy \(|y|_d \leq 1\). Let us assume that \(|x_j|_d < 1\) for \( j = 1 \div p \) (in order to have a well defined sequence). Then for any \( n \geq d\),

\[
|x_p|_n \leq K(n) \theta_p L(n)^d |y|_n
\]

where

\[
L(n) = \frac{n \lambda - 1}{\lambda \tau - 1} + \frac{1}{\lambda \tau - 1} \tau + \frac{m}{\tau - 1}.
\]

Proof: From \( \ref{2} \), we get that:

\[
|z_j|_n \leq |y|_n + |\phi(x_j)|_n \leq C(|y|_n + |x_j|_n)
\]

Thus, if \(|y|_d \leq 1\) and \(|x_j|_d < 1\), then \(|z_j|_d < 2C\).

On the other hand, using \( \ref{8} \):

\[
|\sum_{l=0}^{\infty} (S_{\theta_p}(1 - \phi'(x_j) L(x_j)))^l z_p|_n \leq C \theta_p^m(|x_p|_n|z_p|_d + |z_p|_n) \leq C \theta_p^m(|x_p|_n + |y|_n)
\]

Let \( k = n - \frac{n-d}{\lambda} \). Then:

\[
|\Delta x_j|_n = |S_{\tau_j} L(x_j) \sum_{l=0}^{\infty} (S_{\tau_j}(1 - \phi'(x_j) L(x_j)))^l z_j|_n \leq
\]

\[
\leq C(n,k) \theta_j^k |x_j|_\lambda(n-k) + d \sum_{l=0}^{\infty} (S_{\tau_j}(1 - \phi'(x_j) L(x_j)))^l z_j|_d +
\]

\[
+ |\sum_{l=0}^{\infty} (S_{\tau_j}(1 - \phi'(x_j) L(x_j)))^l z_j|_\lambda(n-k) + d \leq
\]

\[
\leq C \theta_j^k (C \theta_j^m |x_j|_n + C \theta_j^m (|x_j|_n + |y|_n)) \leq
\]

\[
\leq C \theta_j^{k+m} (|x_j|_n + |y|_n).
\]
By using repeatedly this bound, and that $x_0 = 0$, we obtain:

$$
|x_{p+1}|_n \leq |x_p|_n + |\Delta x_p|_n \\
\leq (C\theta_p^{k+m} + 1)|x_p|_n + C\theta_p^{k+m}|y|_n \\
\leq (C\theta_p^{k+m} + 1)|x_{p-1}|_n + C\theta_p^{k+m}|y|_n + C\theta_p^{k+m}|y|_n \\
\leq \left( \prod_{j=0}^{p}(C\theta_j^{k+m} + 1) \right) |x_0|_n + |y|_n \sum_{j=p}^{0} \sum_{i=p}^{j+1} (C\theta_i^{k+m} + 1) \\
\leq |y|_n \sum_{j=p}^{0} (2C)^{j+1} \prod_{i=p}^{j+1} (2C\theta_i^{k+m}) \\
\leq |y|_n (p+1)(2C)^{p+1} \prod_{i=p}^{j+1} (2(k+m)^{\tau_i+1}) = |y|_n (p+1)(2C)^{p+1} \prod_{i=p}^{j+1} (2(k+m)^{\tau_i+1}).
$$

Let us notice that $(\tau - 1)L(n) = n + 1 - \frac{n - d}{\lambda} + m = k + m + 1 > k + m$, hence:

$$
\frac{(p+1)(2C)^{p+1} \prod_{i=p}^{j+1} (2(k+m)^{\tau_i+1})}{(2L(n))^n} \xrightarrow{p \to \infty} 0.
$$

Therefore there is a $K = K(n)$ so that

$$
|x_{p+1}|_n \leq (p+1)(2C)^{p+1} \prod_{i=p}^{j+1} (2(k+m)^{\tau_i+1}) \leq K2^L(n)^n = K\theta_{p+1}^{L(n)}.
$$

Our next goal will be to see that $|z_p|_d \xrightarrow{p \to \infty} 0$, whenever $y \in F$ is small enough. Namely, we will prove:

**Lemma 2** There are $M, s_0, \mu > 0$ so that if $|y| < s_0$ and $|x_j|_d < 1$ for $j = 0 \div p$ we have the bound:

$$
|z_p|_d \leq M\theta_p^{-\mu}|y|_s
$$

where $\mu = \frac{2 + \tau}{2 - \tau}(d + m)$.

**Proof:** By induction on $p$. Observe that $x_{p+1} = x_p + \Delta x_p$, so $\phi(x_{p+1}) = \phi(x_p) + \phi'(x_p)\Delta x_p + R(x_p, \Delta x_p)$, and hence, if we write $A = 1 - \phi'(x_p)L(x_p)$,
we have that:

\[
\begin{align*}
z_{p+1} &= y - \phi(x_{p+1}) = y - \phi(x_p) - \phi'(x_p) \Delta x_p - R(x_p, \Delta x_p) = \\
&= z_p - \phi'(x_p) S_{\theta_p} L(x_p) \sum_{l=0}^{\infty} (S_{\theta_p} (1 - \phi'(x_p) L(x_p)))^l z_p - R(x_p, \Delta x_p) = \\
&= z_p - \phi'(x_p) L(x_p) \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p + \\
&+ \phi'(x_p) (1 - S_{\theta_p}) L(x_p) \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p - R(x_p, \Delta x_p) = \\
&= z_p + (1 - \phi'(x_p) L(x_p)) \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p - \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p + \\
&+ \phi'(x_p) (1 - S_{\theta_p}) L(x_p) \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p - R(x_p, \Delta x_p) = \\
&= z_p + \sum_{l=0}^{\infty} S_{\theta_p} A (S_{\theta_p} A)^l z_p + (1 - S_{\theta_p}) A \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p - \\
&- \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p + \phi'(x_p) (1 - S_{\theta_p}) L(x_p) \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p - \\
&- R(x_p, \Delta x_p) = \\
&= z_p + \sum_{l=0}^{\infty} (S_{\theta_p} A)^{l+1} z_p + (1 - S_{\theta_p}) A \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p - \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p + \\
&+ \phi'(x_p) (1 - S_{\theta_p}) L(x_p) \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p - R(x_p, \Delta x_p) = \\
&= \phi'(x_p) (1 - S_{\theta_p}) L(x_p) \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p + (1 - S_{\theta_p}) A \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p - \\
&- R(x_p, \Delta x_p) \\
&= (a) (b) (c)
\end{align*}
\]

\[ \text{(c)} \]

\[ \text{(d)} \]
To bound (a), we use \([3 4 5]\) and \([6]\). Then, for any \(s \geq d\), and \(s_0 = \lambda s + d\):

\[
|\phi'(x_p)(1 - S_{\theta_p})L(x_p)\sum_{l=0}^{\infty}(S_{\theta_p} A)^l z_p|d \leq C |(x_p|d)(1 - S_{\theta_p})L(x_p)\sum_{l=0}^{\infty}(S_{\theta_p} A)^l z_p|d +
\]

\[
+|(1 - S_{\theta_p})L(x_p)\sum_{l=0}^{\infty}(S_{\theta_p} A)^l z_p|d|d \leq 2C |(1 - S_{\theta_p})L(x_p)\sum_{l=0}^{\infty}(S_{\theta_p} A)^l z_p|d \leq C \theta_p^{-(s-d)}|L(x_p)\sum_{l=0}^{\infty}(S_{\theta_p} A)^l z_p|d \leq C \theta_p^{-(s-d)}(|x_p|s_0)\sum_{l=0}^{\infty}(S_{\theta_p} A)^l z_p|d + |s_0)\sum_{l=0}^{\infty}(S_{\theta_p} A)^l z_p|s_0) \leq C \theta_p^{-(s-d)}(|x_p|s_0)\sum_{l=0}^{\infty}(S_{\theta_p} A)^l z_p|d + |s_0)\sum_{l=0}^{\infty}(S_{\theta_p} A)^l z_p|s_0) \leq C \theta_p^{-(s-d-m)}(|x_p|s_0)\sum_{l=0}^{\infty}(S_{\theta_p} A)^l z_p|d + |s_0)\sum_{l=0}^{\infty}(S_{\theta_p} A)^l z_p|s_0) \leq C \theta_p^{-(s-d-m)}(K_{\theta_p} L(s_0) + 1)|y|s_0 \leq C \theta_p^{-(s-d-m-L(s_0))}|y|s_0.
\]

But

\[
s - d - m - L(s_0) = s - d - m - (\lambda s + d)\left(\frac{\lambda - 1}{\lambda \tau - 1} - \frac{1}{\lambda \tau - 1}\right) - \frac{m}{\tau - 1} = \frac{s \tau - \lambda}{\tau - 1} - \frac{d + \lambda}{\lambda \tau - 1} - \frac{m}{\tau - 1}.
\]

Thus this expression tends to infinity as \(s \to \infty\), so that we can choose \(s\) so as to get \(s - d - m - L(s_0) \geq \mu \tau\). For \(\tau = \frac{2 + \lambda}{2}\) we need \(s \geq \frac{P_0(d,\lambda)}{2(2-\lambda)^2}\).

For such \(s\) we have seen that:

\[
|\phi'(x_p)(1 - S_{\theta_p})L(x_p)\sum_{l=0}^{\infty}(S_{\theta_p} A)^l z_p|d \leq C \theta_p^{-(s-d-m-L(s_0))}|y|s_0 \leq C \theta_p^{-(s-d-m-L(s_0))}|y|s_0 \leq C \theta_p^{-\mu}|y|s_0.
\]
To bound (b) we proceed in a similar way than as for (a), and we get that:

\[
|\sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p|_d \leq C_{\theta_p} - (s_0 - d)|\sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p|_{s_0} \\
\leq C_{\theta_p} - (s_0 - d)\left(\sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p|_d + \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p|_{s_0}\right) \\
\leq C_{\theta_p} - (s_0 - d)\left(C_{\theta_p}^m |x_p|_{s_0} + C_{\theta_p}^m (|x_p|_{s_0}|z_p|_d + |z_p|_{s_0})\right) \\
\leq C_{\theta_p} - (s_0 - d - m)(|x_p|_{s_0} + |z_p|_{s_0}) \\
\leq C_{\theta_p} - (s_0 - d - m)(K_{\theta_p} L(s_0) + 1)|y|_{s_0} \\
\leq C_{\theta_p} - (s_0 - d - m - L(s_0))|y|_{s_0}.
\]

Since \(s_0 = \lambda s + d\), this expression tends to infinity faster than the former one. Thus for the same \(s\) as before we have that:

\[
|\sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p|_d \leq C_{\theta_p} - (s_0 - d - m - L(s_0))|y|_{s_0} \leq C_{\theta_p+1}^{-\mu}|y|_{s_0}.
\]

To bound the Taylor series we use \(\sum\), getting:

\[
|R(x_p, \Delta x_p)|_d \leq C(|x_p|_d|\Delta x_p|_d^2 + |\Delta x_p|_d^2) \leq 2C|\Delta x_p|_d^2 \\
\leq C(\theta_p^d)^2|L(x_p)|\sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p|_d^2 \\
\leq C(\theta_p^d)^2 (|x_p|_d (\sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p|_d^2 + \sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p|d)^2 \\
\leq C(\theta_p^d)^2 \left(\sum_{l=0}^{\infty} (S_{\theta_p} A)^l z_p|_{d}^2 \right) \\
\leq C(\theta_p^d)^2 (|x_p|_d |z_p|_d + |z_p|_d)^2 \\
\leq C\theta_p^{2d+2m}|z_p|_{d}^2.
\]  \(9\)

Then, as \(-2 < -\tau\),

\[
2d + 2m - 2\mu = 2(d + m) - \frac{-2\tau}{2 - \tau} = \frac{-4}{2 - \tau} (d + m)\tau < \frac{-2 - \tau}{2 - \tau} (d + m)\tau = -\mu\tau
\]

Now from the induction hypothesis we get that:

\[
|R(x_p, \Delta x_p)|_d \leq C\theta_p^{2d+2m}|z_p|_{d}^2 \leq CM^2\theta_p^{2d+2m}\theta_p^{-2\mu}|y|_{s_0}^2 \leq CM^2\theta_p+1^{-\mu}|y|_{s_0}^2.
\]
Hence what we have seen is that

$$|z_{p+1}|d \leq C(1 + M^2 |y|_{s_0})\theta_{p+1}^{-\mu}|y|_{s_0},$$

For some $C$ not depending on $p$. WLOG, $M > C$, and we choose $\delta$ so that $\delta \leq \min\{1, \frac{\mu}{M}\}$. Then the lemma is satisfied at least for $|y|_{s_0} < \delta$.

Lemma 3 There is a $\delta > 0$ so that, for $|y|_{s_0} < \delta$, $|x_j|d < 1$ for all $j \geq 0$.

Proof: Again by induction on $j$. For $j = 0$ the requirements are trivially satisfied. Assume that the requirements are satisfied for a certain $j$. Then if $|x_j|d < 1$, $j = 0 \div p$, we have that $|\Delta x_j|d \leq C\theta_j^{\mu}\theta_j^{\mu}z_{p}|d|$, if we follow the same procedure as in [9]. Because of lemma 2, we have that $|\Delta x_j|d \leq C\theta_j^{-\mu}|y|_{s_0}$, therefore $|x_{p+1}|d \leq \sum_{j=0}^{p} |\Delta x_j|d \leq C\left(\sum_{j=0}^{p} \theta_j^{-\mu}\right)|y|_{s_0}$. Using that $\mu - d - m = (d + m)\frac{\mu - \mu}{d} > 0$, if we choose $\delta < \min\{\delta, \left(\sum_{j=0}^{\infty} \theta_j^{-\mu}\right)^{-1}\}$, we get

$$|x_{p+1}|d \leq \left(\sum_{j=0}^{\infty} \theta_j^{-\mu}\right)|y|_{s_0} \leq \left(\sum_{j=0}^{\infty} \theta_j^{-\mu}\right) \delta < 1. \quad (10)$$

Hence lemmas 1 and 2 are true for any $|y|_{s_0} \leq \delta$, with no restrictions on the sequence $(x_p)\_p$.

Next we will try to improve the estimate of lemma 2, changing $\mu$ by any $a > 0$.

Lemma 4 For any $a > 0$ there are constants $C = C(a)$ and $n = n(a)$ so that

$$|z_{p}|d \leq C |y|_{n(a)}\theta_{p}^{-a}$$

for any $p \geq 0$ and any $y$ with $|y|_{s_0} < \delta$.

Proof: Because of lemma 2, the statement of the lemma is satisfied for $0 < a \leq \mu$. Let $a \geq \mu$ and assume the statement is true for this $a$. We will see that the statement is also satisfied for $a + d + m$.

We use that

$$|z_{p+1}|d \leq \left|\phi'(x_p)(1 - S_{\theta_p})L(x_p)\sum_{l=0}^{\infty} (S_{\theta_p}A)^l z_{p}|d| +$$

$$+ \left|(1 - S_{\theta_p})A \sum_{l=0}^{\infty} (S_{\theta_p}A)^l z_{p}|d| + |R(x_p, \Delta x_p)|d\right|$$

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and we choose $n_0 = \lambda n + d$ with $n - d - m - L(n_0) > \tau(a + d + m)$. Then,

$$\left|(1 - S_{\theta_p})L(x_p)\sum_{l=0}^{\infty} (S_{\theta_p}A)^l z_p|d \leq C\theta_p^{-(a+d+m)}|y|_{n_0}.$$  

$$\left|(1 - S_{\theta_p})A \sum_{l=0}^{\infty} (S_{\theta_p}A)^l z_p|d \leq C\theta_p^{-(a+d+m)}|y|_{n_0}.$$  

We can bound the Taylor rest using

$$|R(x_p,\Delta x_p)|_{d} \leq C|\Delta x_p|_{d}^2 \leq C\theta_p^{2d+2m}|z_p|_{d}^2$$

and applying lemma 4 for this $a$ to this formula. by doing so, we get

$$|R(x_p,\Delta x_p)|_{d} \leq C\theta_p^{-2(a-d-m)}|y|_{n(a)}^2 \leq C\theta_p^{-2(a-d-m)}|y|_{2n(a)}$$

where, for the last inequality, we have used that $|y|_0 \leq 1$ and the convexity inequalities. On the other hand, $2a-2(d+m) \geq \tau(a+d+m) \iff a \geq \mu$. Therefore $|R(x_p,\Delta x_p)|_{d} \leq C\theta_p^{-2(a-d-m)}|y|_{2n(a)}$. If we choose $C = \max\{C,\theta_p^{(a-d-m)}\}$ we have $|z_0|_{d} \leq |y|_{d} \leq C\theta_0^{-2(a-d-m)}|y|_{n(a+d+m)}$, and, using the induction, also that

$$|z_{p+1}|_{d} \leq C|y|_{n(a+d+m)}\theta_p^{-(a+d+m)}$$

for any $p \geq 0$, where $n(a + d + m) = \max\{2n(a), n_0\}$.

With these bounds on $|z_p|_{d}$ we can obtain new bounds on $|z_p|_{n}$, using the convexity inequalities. To do so, we proceed in the following way:

**Lemma 5** For any $n \geq 0$ and any $b > 0$ there are $C = C(n,b)$ and $\sigma(n,b)$ so that for any $y \in E$ with $|y|_{n_0} < \delta$ and any $p \geq 0$ we have that:

$$|\Delta x_p|_{n} \leq C|y|_{\sigma(n,b)}\theta_p^{-b}$$

$$|z_p|_{n} \leq C|y|_{\sigma(n,b)}\theta_p^{-b}$$

**Proof:** From the convexity inequalities,

$$|\Delta x_p|_{n} \leq C|\Delta x_p|_{0}^{1/2}|\Delta x_p|_{2n}^{1/2}$$

and, using lemma 1,

$$|\Delta x_p|_{2n} \leq |x_{p+1}|_{2n} + |x_p|_{2n} \leq C\left(\theta_p^{L(2n)} + \theta_p^{L(n)}\right) |y|_{2n} \leq C\theta_p^{\tau L(2n)}|y|_{2n}.$$  

From lemma 4 we know that $|\Delta x_p|_{0} \leq C\theta_p^{-a+m}|z_p|_{d} \leq C|y|_{n(a)}\theta_p^{-a+m}$, for any $a > 0$. By taking $a = 2b + m + \tau L(2n)$ and $\sigma(b,n) = \max\{n(a), 2n\}$ what we get from the previous inequality is that:

$$|\Delta x_p|_{n} \leq C\theta_p^{-a+m}\theta_p^{-L(2n)}|y|_{\sigma(b,n)} = C\theta_p^{-a+b+m-L(2n)}|y|_{\sigma(b,n)} = C\theta_p^{-b}|y|_{\sigma(b,n)}$$

10
Likewise, for $z_p$ we use that $|z_p|_n \leq C|z_p|_{\frac{n}{d}}|z_p|_{\frac{n}{2n}}$. Then we use the definition of $z_p$ and lemma 1 obtaining that $|z_p|_{2n} \leq |y|_{2n} + C|x_p|_{2n} \leq C|y|_{2n} + \theta_p L(n)|y|_{2n} \leq C\theta_p L(n)|y|_{2n}$. Hence, since the norms are increasing, we see that:

$$|z_p|_n \leq C|z_p|_{\frac{n}{d}}|z_p|_{\frac{n}{2n}} \leq C\theta_p^{-\frac{n}{d}}\theta_p^{L(n)}|y|_{\sigma(b,n)} = C\theta_p^{-\frac{n}{d}}\theta_p^{-L(n)}|y|_{\sigma(b,n)} \leq C\theta_p^{-b}|y|_{\sigma(b,n)}$$

with the same $\sigma(b,n)$ as before.

**Theorem 6** Under the hypothesis 1, 2, 3, 4, 5, 6, 7, and 8, $\phi$ has a local right inverse $\psi$, defined on a neighborhood $V = \{y \in F | |y|_{s_0} < \delta\}$ of 0, and satisfying $|\psi(y)|_d \leq C|y|_{s_0}$.

**Proof:** Using that $|x_p - x_{p+l}|_n \leq C \sum_{j=p}^{p+l-1} |\Delta x_p|_n |y|_{\sigma(b,n)}$ and lemma 5, we get that:

$$|x_p - x_{p+l}|_n \leq C(n,b)|y|_{\sigma(b,n)} \sum_{j=p}^{\infty} \theta_j^{-b}$$

and thus, for $y \in V$, $(x_p)_p$ is a Cauchy sequence. Let $x = \lim_{p \to \infty} x_p$. As, again because of lemma 5, $z_p = y - \phi(x_p) \to 0$, and since $\phi$ is continuous, $y = \lim_{p \to \infty} \phi(x_p) = \phi(x)$. For $y \in V$, we define $\psi(y) = \lim_{p \to \infty} x_p(y)$, and thus $\phi \circ \psi(y) = I$. **10** gives us the desired bound.

**References**

[Ham,82] Richard S. Hamilton, The inverse function theorem of Nash and Moser, Bulletin of the American Mathematical Society Volume 7, number 1 (1982), pp. 65-222.

[L-Z,79] S. Lojasiewicz and E. Zehnder, An inverse function theorem in Fréchet spaces, Journal of functional analysis 33 (1979), pp. 165-174.

[Zeh,75] E. Zehnder, Generalized implicit function theorems with applications to some small divisor problems I, Communications on pure and applied mathematics Vol. 28 (1975), pp. 91-140.