Examples of space-times are given which contain scalar curvature singularities whereas the metric tensor is regular and continuous, but which are gravitationally strong. Thus the argument that such singularities are necessarily weak is incomplete; in particular the question of the gravitational strength of the null Cauchy horizon singularity which occurs in gravitational collapse remains open.

PACS: 04.20.Dw, 04.70.-s, 04.20.Jb

The picture of the internal structure of black holes formed by gravitational collapse has been greatly revised over the last decade. The BKL scenario which involves a crushing, space-like, oscillatory singularity has been replaced by a more subtle picture involving the null mass-inflation Cauchy horizon (CH) singularity which precedes a strong space-like central singularity. It has been argued repeatedly that the null CH singularity is gravitationally weak; that is, physical objects only suffer a finite tidal distortion on encountering this singularity. An appropriate mathematical model of this was given by Ellis and Schmidt and by Tipler. It is worth repeating the definitions here.

Consider a causal geodesic $\gamma: [t_0, 0) \rightarrow M$ in a space-time $(M, g)$ which approaches a singularity as $t \rightarrow 0$. For any $t_1 \in [t_0, 0)$, let $J_{t_1}(t)$ be the set of Jacobi fields orthogonal to and along $\gamma$ (at parameter value $t$) which vanish at $t = t_1$. Then the exterior product of any three (for time-like geodesics) or two (for null geodesics) independent Jacobi fields forms a volume element $V(t)$ along $\gamma$. The geodesic is said to terminate in a strong (weak) singularity if $\|V(t)\|$ is zero or infinite (non-zero and finite) in the limit $t \rightarrow 0$; the singularity itself is said to be strong (weak) if all causal geodesics terminate in a strong (weak) singularity.

According to a result of Tipler, a singularity will be weak if $t^2 R_{ab}(x(t)) k^a(t) k^b(t)$ vanishes in the limit $t \rightarrow 0$, where $k^a$ is tangent to the geodesic. Note that here the Ricci tensor must be viewed as a function of the parameter $t$, and not as a function of the space-time coordinates $x^a$. Thus, if the Riemann tensor components, when twice integrated along the geodesic give finite non-zero results, the singularity must be weak. Some authors have concluded that the same result holds if the Riemann tensor components, twice integrated as functions of the space-time coordinates, yield finite non-zero results. Thus a weak singularity has been deduced from the existence of a regular $C^0$ metric. We show here that this conclusion is not valid by producing examples of space-times which contain strong, regular $C^0$ singularities.

We consider a spherically symmetric space-time so that the line element may be written using double-null coordinates

$$ds^2 = -2e^{-2f}du dv + r^2(u,v)d\Omega^2. \tag{1}$$

Take $r = (v-u)/2$ to be the radius function of Minkowski space-time. The gravitational strength of a non-central ($r > 0$) singularity is completely determined by the limiting behaviour of $a(t)$, the norm of an arbitrary radial Jacobi field orthogonal to a radial time-like geodesic (here radial means lying in the 2-space spanned by $\partial_u$ and $\partial_v$). We have shown elsewhere that the tangential Jacobi fields orthogonal to arbitrary radial time-like and null geodesics have finite and non-zero norm at a non-central singularity. See for details. This function $a(t)$ obeys the covariant equation

$$\ddot{a} + 2e^{2f}f_{uv}a = 0, \tag{2}$$

where the overdot indicates differentiation along a time-like geodesic and the subscripts indicate partial derivatives. The coefficient of $a$ is given in terms of invariants by

$$F(x^a) := e^{2f}f_{uv} = \frac{E}{r^3} + 2\Psi_2 - \frac{R}{12},$$

where $E$ is the Lemaître-Misner-Sharp mass, $\Psi_2$ is the Newman-Penrose Weyl tensor term and $R$ is the Ricci scalar. If $a(t)$ is finite and non-zero in the limit $t \rightarrow 0$ (the location of the singularity), then this particular geodesic terminates in a weak singularity. Otherwise the geodesic terminates in a strong singularity. Thus from WKB analysis, if

$$\lim_{t \rightarrow 0} t^2 F(t) \neq 0,$$

then the geodesic terminates in a strong singularity, where the $t$ dependence indicates that the space-time invariant $F$ is evaluated along the geodesic.

The radial time-like geodesics of satisfy

$$\dot{u}v = \frac{1}{2}e^{2f}, \tag{3}$$

$$\ddot{u} - 2f_u\dot{u}^2 = 0, \tag{4}$$

$$\ddot{v} - 2f_v\dot{v}^2 = 0. \tag{5}$$
We consider a space-time with \( f = f(x) \), where \( x := u + v \). Then the geodesic equations reduce to the single second order equation

\[
\ddot{x} - 2f'\dot{x}^2 + 2f'\dot{e}^2 = 0
\]

which has the first integral

\[
\dot{x}^2 = 2e^{2f}(1 + ce^{2f}) =: y^2(t).
\]

To prove our result, we show that there exists a solution of (6) with the following properties.

(i) \( x(0) = 0 \)
(ii) \( 0 < e^f(0) < +\infty \)
(iii) \( f_{uv}(x) \to \infty \) as \( x \to 0 \)
(iv) \( t^2f_{uv}(t) \neq 0 \) as \( t \to 0^- \).

Property (iii) shows that there is a scalar curvature singularity at \( x = 0 \), which from the definition of \( x \) and \( r \) is non-central. Properties (i) and (ii) indicate that the metric is regular at the singularity and condition (iv) shows that the geodesic terminates in a strong curvature singularity. We show how to construct a \( C^0 \) metric from this solution below.

According to our ansatz, \( f_{uv} = f''(x) \), and from (6),

\[
e^{2f}(x(t)) = -\frac{1}{2c}(1 + (1 + 4cy^2(t))^{1/2}).
\]

We then calculate the derivative of \( f \) using \( f' = \dot{x}^{-1}\dot{f} \); the second derivative obeys

\[
f''e^{2f} = -y^{-1}zye^{2f} - 4z^{3/2}y^2 - 6z^{1/2}y^2,
\]

where \( z = (1 + 4cy^2(t))^{-1/2} \). Now take

\[
y(t) = 1 + t\sin(\frac{1}{t})
\]

for \( t \neq 0 \) and \( y(0) = 1 \). Then \( y \) is continuous on \( \mathbb{R} \), and so \( x(t) \) is defined, is differentiable and from (6), is monotone in a neighbourhood of \( t = 0 \). Thus the inverse function theorem may be applied to find \( t(x) \) in a neighbourhood of \( x = 0 \), and then (6) gives the metric function \( f \) as a continuous function of \( x \), regular at \( x = 0 \). The constant of integration may be chosen so that \( x(0) = 0 \). A straightforward calculation gives the right hand side of (6), and it is easily verified that the limit \( \lim_{t \to 0^-} t^2f''e^{2f} \) does not exist, and so as claimed, the singularity cannot be weak. Note that the method of construction of the metric, while somewhat complicated, guarantees that the singularity is gravitationally strong along every radial time-like geodesic running into it.

The key to constructing this example was to find \( y(t) \in C^0(\alpha,0) \cap C^2(\alpha,0) \) such that the limit \( \lim_{t \to 0^-} t^2H(y,\dot{y},\ddot{y},t) \) is non-zero, where \( H \) is given by the right hand side of (6), using (6) for \( f \). Given that continuous, non-differentiable functions vastly outnumber differentiable ones, for the ansatz used here, weak singularities would be the exception rather than the rule even for a regular \( C^0 \) metric.

As mentioned above, this result shows that recent arguments that the null CH singularity is gravitationally weak are incomplete. This incomplete argument was used in analytical and numerical studies of black holes by several authors, and also in a study of the generic nature of null singularities. We do not claim that these authors’ results are wrong, but that the argument is flawed. In fact in another study of the black hole interior, Brady et al. showed that in agreement with other authors the CH singularity is \( C^0 \) regular, and crucially, that the proper time \( t \) along timelike geodesics intersecting the singularity is a smooth function of the coordinates used. These are sufficient conditions to allow the use of Tipler’s result and to conclude that the singularity is weak. However in general, the question of the gravitational strength of the CH singularity remains open. It may be that the null nature, or some other feature, of the CH singularity enforces weakness; the singularity studied here is spacelike.

1. L. M. Burko and A. Ori (eds.), Internal Structure of Black Holes and Spacetime Singularities, (Institute of Physics Publishing, Bristol and Israel Physical Society, Jerusalem 1997).
2. V. A. Belinsky, I. M. Khalatnikov and E.M. Lifshitz, Sov. Phys. JETP 32, 169 (1970).
3. E. Poisson and W. Israel, Phys. Rev. Lett. 63, 1663 (1989).
4. A. Ori, Phys. Rev. Lett. 67, 789 (1991); Phys. Rev. Lett. 68, 2117 (1992).
5. P. R. Brady, S. Droz and S. M. Morsink, Phys. Rev. D58, 084034 (1998).
6. S. Hod and T. Piran, Gen. Rel. Grav. 30, 1555 (1998).
7. G. F. R. Ellis and B. G. Schmidt, Gen. Rel. and Grav. 8, 915 (1977).
8. F. J. Tipler, Phys. Lett 64A, 8 (1977).
9. A. Ori and E. E. Flanagan, Phys. Rev. D53, R1754 (1996).
10. B. C. Nolan, “Strengths of singularities in spherical symmetry”, e-print gr-qc/9902027 (1999).
11. L. M. Burko, Phys. Rev. Lett. 79, 4958 (1997).
12. L. M. Burko and A. Ori, Phys. Rev. D57, 7084 (1998).