Determination of the area of robust stability of the system with a PID controller

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Abstract. An approach has been proposed to determine the area of change in the parameters of the controlled plant by the example of a system with PID controller. The search for ranges of parameter changes is based on the solution of the nonlinear programming problem, which is obtained as a conclusion of V.L. Kharitonov’s theorem on the robust stability of linear systems.

1. Introduction.

The parameters of automatic control systems (ACS) may change over time for reasons such as aging, wear, heating, deformation, etc. In addition, the parameters of a plant may not be determined accurately enough when developing controllers. In the general case, traditional methods of analysis and synthesis of automatic control systems suggest that, when considering a specific ACS with set nominal values of the parameters, small changes in these parameters do not significantly change the nature of the movement. However, the parameters may not always vary within sufficiently small limits. Then it becomes necessary to study the ACS to maintain its predetermined properties for all parameter values defined within any limits. In this case, they speak of the analysis of the robustness of the considered property of automatic control systems [1,2,3]. The most important among the properties of an ACS is its stability [4, 5, 6]. Thus, this paper discusses the robust stability of an automatic control system [7, 8]. In the conditions of changing the parameters of the controlled plant, it is possible to assign the following problems:

First problem. Along with the known model of an automatic control system with its nominal parameters, the maximum deviations of the real parameters of the controlled plant are known. We need to determine how the change in the parameters will affect the stability of the ACS.

Second problem. There is a known model of an automatic control system. It is necessary to determine in which ranges it is possible to change the parameters of the controlled plant so that the ACS remains stable. It is important to determine on the basis of what criteria stability is considered. So, for example, considering the Nyquist criterion, it is necessary to study the behavior of an amplitude-phase-frequency characteristic. The use of the root criterion involves the study of the relative position of the roots and poles of the transfer function, etc. Also, from a practical perspective, it should be borne in mind that the search for ranges of changes in the controlled plant parameters is not limited only to stability criteria. The specification of parameter tolerances makes its own adjustments to the solution of this problem. On the one hand, the value of these tolerances should not be small, since this requires precision equipment, and on the other hand, a large value of the tolerances
will not be able to provide interchangeability of parts. It should not also be forgotten about the quality indicators of the transient process (overshoot, settling time, etc.).

There is no single approach to the analysis of robust stability of automatic control systems; depending on the nature of the system under study and the type of disturbances, the problem is solved in different ways. In the general case, the following approaches can be distinguished:

- a root approach based on the mutual arrangement of zeros and poles [9];
- a frequency approach associated with the construction of frequency characteristics [10];
- an algebraic approach, for example, a method based on V.L. Kharitonov’s polynomials [2];
- using methods of the theory of sensitivity [11];
- using the concept of system structural stability in phase space [10];
- using the H-theory [12, 13].

In this paper, we solve the problem of determining the range of values of the parameters of the controlled plant within which the ACS should be robustly stable. Among the many approaches listed above, we chose the simplest approach based on the theorem of V.L. Kharitonov, who determined the formation of the theory of robustness as a whole.

Let us consider this approach by a specific example – a system with a PID controller.

2. Materials and methods
We consider an automatic control system with the following block diagram, Figure 1 [14, 15, 16].

\[
\begin{align*}
G_{con}(p) & \xrightarrow{k_{ac}} \frac{k_{ac}}{p} \xrightarrow{k_{pl}} \frac{k_{pl}}{T_{pl1}p + 1} \xrightarrow{k_{pl2}} \frac{1}{T_{pl2}p + 1}
\end{align*}
\]

**Figure 1.** The block diagram of the ACS with a PID controller

Here \(k_{ac}, k_{ob}\) are the gain factors of the links of the controlled plant; \(T_{pl1}, T_{pl2}\) are time constants;

\[G_{con}(p) = k_p + k_i \frac{1}{p} + k_d p,\]

where \(p\) is the differentiation operator; \(k_p, k_i, k_d\) are PID controller parameters.

The transfer function of a closed-loop system with a PID controller is written as:

\[
W(p) = \frac{(k_p p + k_i + k_d p^2)k_{ac}k_{pl}}{p^2(T_{pl1}p + 1)(T_{pl2}p + 1) + (k_p p + k_i + k_d p^2)k_{ac}k_{pl}}.
\]

The characteristic polynomial of the considered ACS:

\[
Q(p) = T_{pl1}T_{pl2}p^4 + (T_{pl1} + T_{pl2})p^3 + (k_d k_{ac}k_{pl} + 1)p^2 + k_p k_{ac}k_{pl}p + k_i k_{ac}k_{pl}.
\]

Let \(p = j\omega\), then:

\[
Q(j\omega) = T_{pl1}T_{pl2}\omega^4 - j(T_{pl1} + T_{pl2})\omega^3 - (k_d k_{ac}k_{pl} + 1)\omega^2 + jk_p k_{ac}k_{pl}\omega + k_i k_{ac}k_{pl}.
\]

Hence, the imaginary and real parts of \(Q(j\omega)\) are respectively equal:
We find the ranges in which the parameters of the controlled plant can vary:

\[
T_{pl1} < T_{pl1} < \overline{T}_{pl1};
\]

\[
T_{pl2} < T_{pl2} < \overline{T}_{pl2};
\]

\[
k_{pl1} < k_{pl1} < \overline{k}_{pl1};
\]

\[
k_{ac} < k_{ac} < \overline{k}_{ac};
\]

Then the coefficients of the polynomial \(Q(j\omega)\), given there is positivity of all the parameters of the controlled plant (a necessary condition for stability), change within the following ranges:

\[
T_{pl1}T_{pl2} < T_{pl1}T_{pl2} < \overline{T}_{pl1}\overline{T}_{pl2};
\]

\[
T_{pl1} + T_{pl2} < T_{pl1} + T_{pl2} < \overline{T}_{pl1} + \overline{T}_{pl2};
\]

\[
(k_{d}k_{ac}\overline{k}_{pl1} + 1) < (k_{d}k_{ac}\overline{k}_{pl1} + 1) < (k_{d}\overline{k}_{ac}\overline{k}_{pl1} + 1);
\]

\[
k_{p1}k_{ac}\overline{k}_{pl1} < k_{p1}k_{ac}\overline{k}_{pl1} < k_{p1}\overline{k}_{ac}\overline{k}_{pl1};
\]

Hence, the lower and upper boundaries of the change in the imaginary and real parts of the characteristic polynomial are expressed by the formulas:

\[
u = \frac{1}{T_{pl1}T_{pl2}}(k_{d}k_{ac}\overline{k}_{pl1} + 1)\omega^2 + k_{i}k_{ac}\overline{k}_{pl1};
\]

\[
u = -\left(\frac{T_{pl1}}{T_{pl1} + T_{pl2}}\right)\omega^2 + k_{p}k_{ac}\overline{k}_{pl1}\omega;
\]

\[
\overline{u} = \frac{1}{T_{pl1}T_{pl2}}(k_{d}k_{ac}\overline{k}_{pl1} + 1)\omega^2 + k_{i}\overline{k}_{ac}\overline{k}_{pl1};
\]

\[
\overline{\nu} = -\left(\frac{T_{pl1}}{T_{pl1} + T_{pl2}}\right)\omega^2 + k_{p}\overline{k}_{ac}\overline{k}_{pl1}\omega.
\]

We compose the Kharitonov polynomials [2]:

\[
Q_{1} = \overline{u} + j\overline{\nu};
\]

\[
Q_{2} = \overline{u} + \overline{\nu};
\]

\[
Q_{3} = u + \overline{\nu};
\]

\[
Q_{3} = u + \nu.
\]

Given there are (3) and (4), we obtain:

\[
Q_{1}(\lambda) = k_{i}\overline{k}_{ac}\overline{k}_{pl1} + k_{p}k_{ac}\overline{k}_{pl1}\lambda + (k_{d}k_{ac}\overline{k}_{pl1} + 1)\lambda^2 + (\overline{T}_{pl1} + \overline{T}_{pl2})\lambda^3 + \overline{T}_{pl1}\overline{T}_{pl2}\lambda^4;
\]

\[
Q_{2}(\lambda) = k_{i}\overline{k}_{ac}\overline{k}_{pl1} + k_{p}\overline{k}_{ac}\overline{k}_{pl1}\lambda + (k_{d}\overline{k}_{ac}\overline{k}_{pl1} + 1)\lambda^2 + (\overline{T}_{pl1} + \overline{T}_{pl2})\lambda^3 + \overline{T}_{pl1}\overline{T}_{pl2}\lambda^4;
\]
For robust stability of the system, it is enough that the Kharitonov polynomials are stable. By definition, a polynomial is called a stable polynomial if all its zeros are in the left semi-plane on the complex plane [2].

We use the Hurwitz matrices and write them for each polynomial (4):

\[
\Delta Q_1 = \begin{pmatrix}
\frac{(T_{p11} + T_{p12})}{T_{p11} T_{p12}} & k_p k_{ac} k_{pl} & 0 & 0 \\
& (k_d k_{ac} k_{pl} + 1) & k_i k_{ac} k_{pl} & 0 \\
& 0 & (T_{p11} + T_{p12}) & k_p k_{ac} k_{pl} & 0 \\
& 0 & 0 & (k_d k_{ac} k_{pl} + 1) & k_i k_{ac} k_{pl}
\end{pmatrix}
\]

\[
\Delta Q_2 = \begin{pmatrix}
\frac{(T_{p11} + T_{p12})}{T_{p11} T_{p12}} & k_p k_{ac} k_{pl} & 0 & 0 \\
& (k_d k_{ac} k_{pl} + 1) & k_i k_{ac} k_{pl} & 0 \\
& 0 & (T_{p11} + T_{p12}) & k_p k_{ac} k_{pl} & 0 \\
& 0 & 0 & (k_d k_{ac} k_{pl} + 1) & k_i k_{ac} k_{pl}
\end{pmatrix}
\]

\[
\Delta Q_3 = \begin{pmatrix}
\frac{(T_{p11} + T_{p12})}{T_{p11} T_{p12}} & k_p k_{ac} k_{pl} & 0 & 0 \\
& (k_d k_{ac} k_{pl} + 1) & k_i k_{ac} k_{pl} & 0 \\
& 0 & (T_{p11} + T_{p12}) & k_p k_{ac} k_{pl} & 0 \\
& 0 & 0 & (k_d k_{ac} k_{pl} + 1) & k_i k_{ac} k_{pl}
\end{pmatrix}
\]

\[
\Delta Q_4 = \begin{pmatrix}
\frac{(T_{p11} + T_{p12})}{T_{p11} T_{p12}} & k_p k_{ac} k_{pl} & 0 & 0 \\
& (k_d k_{ac} k_{pl} + 1) & k_i k_{ac} k_{pl} & 0 \\
& 0 & (T_{p11} + T_{p12}) & k_p k_{ac} k_{pl} & 0 \\
& 0 & 0 & (k_d k_{ac} k_{pl} + 1) & k_i k_{ac} k_{pl}
\end{pmatrix}
\]

Since the coefficients of the polynomials are positive, for the stability of the Kharitonov polynomials it is sufficient to apply the Lienar-Shipar criterion, according to which the following inequalities are necessary and sufficient:

\[
\left(\frac{(T_{p11} + T_{p12})(k_d k_{ac} k_{pl} + 1) - T_{p11} T_{p12} k_p k_{ac} k_{pl}}{T_{p11} T_{p12}}\right) k_p k_{ac} k_{pl} < 0
\]

\[
-(T_{p11} + T_{p12}) k_i k_{ac} k_{pl} (T_{p11} + T_{p12}) > 0
\]

\[
\left(\frac{(T_{p11} + T_{p12})(k_d k_{ac} k_{pl} + 1) - T_{p11} T_{p12} k_p k_{ac} k_{pl}}{T_{p11} T_{p12}}\right) k_p k_{ac} k_{pl} < 0
\]

\[
-(T_{p11} + T_{p12}) k_i k_{ac} k_{pl} (T_{p11} + T_{p12}) > 0
\]

\[
\left(\frac{(T_{p11} + T_{p12})(k_d k_{ac} k_{pl} + 1) - T_{p11} T_{p12} k_p k_{ac} k_{pl}}{T_{p11} T_{p12}}\right) k_p k_{ac} k_{pl} < 0
\]

\[
-(T_{p11} + T_{p12}) k_i k_{ac} k_{pl} (T_{p11} + T_{p12}) > 0
\]
We complete the replacement. Let:

\[ T_{pl1} = T_{pl1} - h_{T11}; \quad T_{pl2} = T_{pl2} - h_{T12}; \]

\[ T_{pl2} = T_{pl2} - h_{T21}; \quad T_{pl2} = T_{pl2} + h_{T22}; \]

\[ k_{pl} = k_{pl} - h_{kpl1}; \quad k_{pl} = k_{pl} + h_{kpl2}; \]

\[ k_{ac} = k_{ac} - h_{kim1}; \quad k_{ac} = k_{ac} + h_{kim2}; \]

where

\[ h_{T11}, h_{T12}, h_{T21}, h_{T22}, h_{kpl1}, h_{kpl2}, h_{kim1}, h_{kim2} \geq 0. \] (5)

As a result, we obtain the system of inequalities:

\[(T_{pl1} + h_{T12} + T_{pl2} + h_{T22})(k_{d}(k_{ac} - h_{kim1})(k_{pl} - h_{kpl1}) + 1) - (T_{pl1} + h_{T12})(T_{pl2} + h_{T22})k_{p}(k_{ac} - h_{kim1})(k_{pl} - h_{kpl1}) k_{p}(k_{ac} - h_{kim1})(k_{pl} - h_{kpl1}) \]

\[
- (T_{pl1} + h_{T12} + T_{pl2} + h_{T22})^{2}k_{i}(k_{ac} + h_{kim2})(k_{pl} + h_{kpl2}) > 0
\] (6)

\[
(T_{pl1} - h_{T11} + T_{pl2} - h_{T21})(k_{d}(k_{ac} - h_{kim1})(k_{pl} - h_{kpl1}) + 1) - (T_{pl1} + h_{T11})(T_{pl2} - h_{T21})k_{p}(k_{ac} + h_{kim2})(k_{pl} - h_{kpl2}) k_{p}(k_{ac} + h_{kim2}) \]

\[
- (T_{pl1} - h_{T11} + T_{pl2} - h_{T21})^{2}k_{i}(k_{ac} + h_{kim1})(k_{pl} - h_{kpl1}) > 0
\]

\[
(T_{pl1} - h_{T11} + T_{pl2} - h_{T21})(k_{d}(k_{ac} + h_{kim2})(k_{pl} + h_{kpl2}) + 1) - (T_{pl1} - h_{T11})(T_{pl2} - h_{T21})k_{p}(k_{ac} + h_{kim1})(k_{pl} + h_{kpl2}) k_{p}(k_{ac} + h_{kim1}) \]

\[
- (T_{pl1} - h_{T11} + T_{pl2} - h_{T21})^{2}k_{i}(k_{ac} - h_{kim1})(k_{pl} - h_{kpl1}) > 0
\]

\[
(T_{pl1} + h_{T12} + T_{pl2} + h_{T22})(k_{d}(k_{ac} + h_{kim2})(k_{pl} + h_{kpl2}) + 1) - (T_{pl1} + h_{T12})(T_{pl2} + h_{T22})k_{p}(k_{ac} - h_{kim1})(k_{pl} - h_{kpl1}) k_{p}(k_{ac} - h_{kim1}) \]

\[
- (T_{pl1} + h_{T12} + T_{pl2} + h_{T22})^{2}k_{i}(k_{ac} - h_{kim1})(k_{pl} - h_{kpl1}) > 0
\]
Next, we introduce the objective function:

$$I = \left( h_{T_{11}} + h_{T_{12}} + h_{T_{21}} + h_{T_{22}} + h_{k_{p11}} + h_{k_{p12}} + h_{k_{i1}} + h_{k_{i2}} \right)^{-1}. \quad (7)$$

Since the problem of searching for ranges (2) is reduced to finding the maximum values of \( h_{T_{11}}, h_{T_{12}}, h_{T_{21}}, h_{T_{22}}, h_{k_{p11}}, h_{k_{p12}}, h_{k_{i1}}, h_{k_{i2}} \) taking into account restrictions (5) and (6), we obtain the problem of minimizing the function \( I \) under the indicated restrictions:

$$\min_{h_{T_{11}}, h_{T_{12}}, h_{T_{21}}, h_{T_{22}}, h_{k_{p11}}, h_{k_{p12}}, h_{k_{i1}}, h_{k_{i2}}} I. \quad (8)$$

Thus, in order to determine the ranges of variation of the parameters of the controlled plant of an automatic system with a PID controller, it is sufficient to solve the nonlinear programming problem (8), (7), (6), (5).

### 3. Results.

Let the initial automatic system with a PID controller be set by the parameters defined in Table 1.

**Table 1.** System parameter

| Parameter | \( k_{ac} \) | \( k_{pl} \) | \( T_{pl1} \) | \( T_{pl2} \) | \( k_{p} \) | \( k_{i} \) | \( k_{d} \) |
|-----------|-------------|-------------|-------------|-------------|-----------|-----------|-----------|
| Value     | 0.1         | 1           | 3           | 2           | 0.5       | 0.04      | 0.01      |

The transient process for this system is shown in Figure 2.

**Figure 2.** The transient process of the automatic system with a PID controller

Having solved problems (8), (7), (6), (5), we obtained the boundaries of the change in the parameters of the controlled plant, which are presented in Table 2.

**Table 2.** Limits of change of parameters of object of regulation

|                  | Lower boundary | Upper boundary |
|------------------|----------------|----------------|
| \( k_{ac} \)     | 0              | 0.1000         |
| \( k_{pl} \)     | 0              | 1.0000         |
| \( T_{pl1} \)    | 1.1779         | 9.4802         |
| \( T_{pl2} \)    | 0              | 2.0000         |
From Table 2 we can observe that the lower boundaries of the gain factors are zero. This is because the smaller the gain factor is, the less the oscillation is. With the values of the parameters corresponding to the upper boundaries, the system is at the stability boundary.

4. Conclusion
Thus, an approach has been developed to determine the permissible values of the parameters of the controlled plant at which the automatic system will be robustly stable, based on the reduction of the original problem to the nonlinear programming problem. The system of restrictions is obtained on the basis of the theorem of V.L. Kharitonov on robust stability.

The condition on which the approach is based is sufficient but not necessary for the system under consideration, since the coefficients of the polynomial are interdependent.

In general, any automatic control system must satisfy not only the requirement for stability, but also meet a number of other quality criteria. In this case, one can add some quality criterion to problem (8), (7), (6), (5), and, having solved the multicriteria problem, obtain solutions taking into account other characteristics of the transient process.

References
[1] Dorf R, Bishop R 2012 Modern control systems (Moscow: Laboratory of Basic Knowledge Publ.) 832.
[2] Kim D P 2003 The theory of automatic control. Volume 1. Linear systems (Moscow: FIZMATLIT Publ.) 288.
[3] Oshchepkov A Yu 2013 Automatic control systems: theory, application, modeling in MATLAB (St. Petersburg: Lan' Publ.) 208.
[4] Lukyanov A V, Krakovsky Yu M, Arshinsky L V and Kutsyi N N 2018 The development of software for controlling a safety system of the machines using vibration analysis Far East Journal of Mathematics 1 Sciences (EJMS). 103(2) 441-450.
[5] Kucyi N N, Lukyanov A V, Kargapol’cev S K and Tikhii I I 2018 Training of neural network based PWM controllers Advances and Applications in Discrete Mathematics 19(4) 359-371.
[6] Galayaev A A, Lysenko P V 2019 Energy-Optimal Control of Harmonic Oscillator Automation and Remote Control, 80(1) 16–29.
[7] Kogan M M 2016 Design of optimal and robust control with $H_{\infty/\gamma}$ 0 performance criterion Automation and Remote Control 77(8) 1317–1333.
[8] Zhirabok A N, Suvorov A Yu 2014 A method for constructing robust diagnostic observers Automation and Remote Control 75(2) 208–218.
[9] Efimov S V, Zamyatin S V, Sukhodoev M S and Gaivoronsky S A 2008 Determination of the desired location region of the dominant poles of a closed system taking into account its zeros The News of Tomsk Polytechnic University 312(5) 57-61.
[10] Andronov A A, Witt A A and Khaikin S E 1959 Theory of oscillations (Moscow: Fiz.-mat. literature Publ.) 916.
[11] Rosenwasser E N, Yusupov R M 1971 Methods of the theory of sensitivity in automatic control (Leningrad: Energia Publ.) 345.
[12] Egupov N D 2000 Methods of the modern theory of automatic control (Moscow: Bauman MGTU) 748.
[13] Egupov N D 2011 Methods of robust, neuro-fuzzy and adaptive control (Moscow: Bauman MGTU) 744.
[14] Kutsyi N N, Livshits A V 2018 Searchless algorithm for parametric optimization of a PI-controller with semi-permanent integration Advanc en des in Differential Equation and Control Processes 19(2) 69-82.
[15] Morozov M V 2019 On Small Perturbations of a Periodic Homogeneous Differential Inclusion with an Asymptotically Stable Set Automation and Remote Control 80(5) 834–839.
[16] Aleksandrov A. G. 2018 Design of Controllers by Indices of Precision and Speed. III. Control-Stable Multidimensional Plants Automation and Remote Control 79(2) 241–257.