On the Construction of Time-Symmetric Black Hole Initial Data*

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Abstract. We review the 3+1 - split which serves to put Einstein’s equations into the form of a dynamical system with constraints. We then discuss the constraint equations under the simplifying assumption of time-symmetry. Multi-Black-Hole data are presented and more explicitly described in the case of two holes. The effect of different topologies is emphasized.

Notation. Space-time is a manifold \( M \) with Lorentzian metric \( g \) of signature \((-++,+.)\). Greek indices are \( \in \{0,1,2,3\} \) and latin indices are \( \in \{1,2,3\} \). Indices from the beginning of the alphabet, like \( \alpha, \beta, \ldots \) and \( a, b, \ldots \), refer to orthonormal frames and indices from the middle, like \( \lambda, \mu, \ldots \) and \( l, m, \ldots \) to coordinate frames. The symbol \( \circ \) denotes the composition of maps. The relation \( := \) (=:) defines the left (right) hand side. The torsion and curvature tensors for the connection \( \nabla \) are defined by
\[
T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]
\]
and
\[
R(X,Y) := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad \text{respectively.}
\]
The covariant components of the Riemann and Ricci tensors are defined by
\[
R_{\alpha\beta\gamma\delta} := g(e_{\alpha}, R(e_{\gamma}, e_{\delta}) e_{\beta}) \quad \text{and} \quad R_{\alpha\gamma} := g^{\beta\delta} R_{\alpha\beta\gamma\delta}
\]

1 The 3+1 – Split

In this article we discuss the vacuum Einstein equations
\[
G^{\mu\nu} := R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 0,
\]
which form a system of ten quasi-linear second order differential equations for the ten functions \( g_{\mu\nu} \). However, the four equations \( G_{\mu0} = 0 \) do not involve the second time derivatives and hence constrain the set of initial data. To see this, recall that the twice contracted second Bianchi identity gives
\[
\partial_0 G_{0\nu} = - \partial_\nu G^{0\nu} - \Gamma_{\mu\lambda}^{\mu} G^{\lambda\nu} - \Gamma_{\mu\lambda}^{\nu} G^{\mu\lambda}.
\]
Since the right hand side contains at most second time derivatives the assertion follows. The ten Einstein equations therefore split into four constraints and six evolution equations \( G^{0k} = 0 \). That four equations constrain the initial data rather than guiding the evolution results in four dynamically undetermined functions.

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among the ten $g_{\mu\nu}$. The task is to parameterize the $g_{\mu\nu}$ in such a way that four dynamically undetermined functions can be cleanly separated from the other six. How this can be done via the 3+1 split is explained below. The four dynamically undetermined quantities will be the famous *lapse* (one function) and *shift* (three functions). It follows directly from (2) that the constraints will be preserved under this evolution.

The splitting of the Einstein equations will be formulated in a geometric fashion. We initially think of $(M, g)$ as given and satisfying the Einstein equations. Then we write down the evolution law for the intrinsic and extrinsic geometry of a spacelike 3-manifold $\Sigma$ as it moves through $M$. Together with the constraints they are equivalent to all Einstein equations. Finally this procedure is turned upside down by taking the evolution equations for $\Sigma$’s geometry as starting point. Only after their integration can we construct the ambient space-time.

### 1.1 3+1 Split Geometry

The topology of space-time (or the portion thereof) which we want to decompose into space and time must be a product $M \cong \Sigma \times \mathbb{R}$. We foliate $M$ by a one-parameter family of embeddings $e_t : \Sigma \to M$, $t \in \mathbb{R}$. For fixed $t$ the image of $e_t$ in $M$ is called $\Sigma_t$, or the $t$'th leaf of the foliation. All leaves are assumed spacelike. Hence there is a normalized timelike vector field, $n$, normal to all leaves. We choose one of the two possible orientations and thereby introduce the notions of future and past: A timelike vector $X$ is future pointing iff $g(X, n) < 0$ (recall signature convention). The tangent-bundle $T(M)$ can now be split into the orthogonal sum of the subbundle of spacelike vectors, $S(M)$, and the normal bundle, $N(M)$. The associated projection maps are given by

\[
S : T(M) \to S(M), \quad X \mapsto X + n g(n, X),
\]
\[
N : T(M) \to N(M), \quad X \mapsto -n g(n, X),
\]

which can be naturally continued to the cotangent bundle by setting $S^*(\omega) := \omega \circ S$ and then factorwise on tensor products and linearly on the whole tensor-bundle. Thus we obtain a split of the whole tensor bundle, where from now on the projection maps are simply called $S$ and $N$ for all tensors. Tensors in the image of $S$ are called *spatial*. It is easy to verify that

\[
h := Sg = g + n^b \otimes n^b,
\]

where $n^b := g(n, \cdot)$. Note that the restriction $h_t$ of $h$ to $T(\Sigma_t)$ is just the induced Riemannian metric on $\Sigma_t$. Identifying for the moment $\Sigma$ and $\Sigma_t$ via $e_t$ this leads to $h_t = e^*_t g$ (Exercise). For what follows it is however crucial to regard spatial tensors as tensors over $M$ and not over $\Sigma$. Otherwise covariant (or Lie-) derivatives in directions off $\Sigma$ would not make sense.

If $X, Y$ are any spatial vector fields we can write

\[
\nabla_X Y = S \nabla_X Y + N \nabla_X Y = D_X Y + n K(X, Y),
\]
where we defined the spatial covariant derivative, $D$, and the extrinsic curvature, $K$, by

\[ D_X := S \circ \nabla_X, \quad (7) \]

\[ K(X, Y) := -g(\nabla_X Y, n) = -g(\nabla_Y X, n) = g(\nabla_X n, Y). \quad (8) \]

The second equality in (8) – and hence the symmetry of $K$ – follows from the vanishing torsion of $\nabla$ and the fact that $[X, Y]$ is spatial. It is easy to prove that $K$ is indeed a tensor and that $D$ defines a connection on the tangent bundle of each leaf $\Sigma_t$. Extension via the Leibnitz rule leads to a unique connection on the bundle of spatial tensors, which can be directly defined by (7) with the extended meaning of $S$ described above. In fact it is just the Levi-Civita connection compatible with the metric $h$. To see this, we compute

\[ D_X h = S(\nabla_X (g + n^i \otimes n^j)) = 0, \]

since $\nabla_X g = 0 = S n^i$, so that $D$ is compatible with $h$. Vanishing torsion is also immediate:

\[ D_X Y - D_Y X - [X, Y] = S(\nabla_X Y - \nabla_Y X - [X, Y]) = 0, \]

by $[X, Y] = S[X, Y]$ and the vanishing torsion of $\nabla$.

Let \{\(e_0, e_1, e_2, e_3\)\} be an orthonormal frame adapted to the foliation, i.e. $e_0 = n$, and \{\(e^0, e^1, e^2, e^3\)\} its dual. Then from (5) with \(n^i = e^0\) we have

\[ g = -e^0 \otimes e^0 + h = -e^0 \otimes e^0 + \sum_{a=1}^{3} e^a \otimes e^a. \quad (9) \]

The family of embeddings $t \mapsto e_t$ defines a vector field, $\partial/\partial t =: \partial_t$, which is easily characterized by its action on any smooth function $f$:

\[ \partial_t f := \frac{d}{dt} \bigg|_{t=0} f \circ e_t. \quad (10) \]

This vector field can be decomposed into normal and tangential components

\[ \partial_t = \alpha n + \beta = \alpha e_0 + \beta^a e_a, \quad (11) \]

with uniquely defined function $\alpha$ and spatial vector field $\beta$. They are called the lapse (function) and shift (vector field) respectively.

Let now \(\{x^\mu\}\) be an adapted local coordinate system on $M$ so that $x^0 = t$ and hence spatial fields $\partial_t := \partial/\partial x^0$. The flow lines of $\partial_t$ are then the lines of constant spatial coordinates $x^k$. Hence $(\alpha, \beta)$ are interpreted as normal and tangential components of the 4-velocity – measured in units of $t$ – with which the points of constant spatial coordinates move. To express the metric $g$ in terms of these coordinates we use an obvious matrix notation and write

\[
\begin{pmatrix}
\partial_t \\
\partial_k 
\end{pmatrix} = \begin{pmatrix}
\alpha \\
0 
\end{pmatrix} \begin{pmatrix}
\beta^a \\
A^a_k 
\end{pmatrix}
\begin{pmatrix}
e_0 \\
e_a
\end{pmatrix},
\]

\[
\begin{pmatrix}
e^0 \\
e^a
\end{pmatrix} = \begin{pmatrix}
dt & dx^k 
\end{pmatrix} \begin{pmatrix}
\alpha \\
0
\end{pmatrix} \begin{pmatrix}
\beta^a \\
A^a_k 
\end{pmatrix}.
\]

Introducing (13) into (9) yields the 3+1 split form of the metric $g$:

\[ g = -\alpha^2 dt \otimes dt + h_{ik} (dx^i + \beta^i dt) \otimes (dx^k + \beta^k dt), \quad (14) \]
where \( h_{ik} = h(\partial_i, \partial_k) = \sum_a A_i^a A_k^a \) and \( \beta^i A_k^a = \beta^a \). For the measure 4-form one easily obtains
\[
e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \alpha \sqrt{\det \{h_{ik}\}} \, d^4 x.
\]

In the ambient space-time the notion of time-derivative of spatial tensors is introduced via the Lie-derivative along the time flow generated by \( \partial_t \). But in order to render this an operation within the space of spatial tensor fields we must include a spatial projection. Using (11) we define the “doting” by
\[
\dot{h} := SL_\beta h = \alpha L_n h + SL_\beta h,
\]
where we also used \( L_\alpha h = \alpha L_n h \) and that \( L_n h = \) already spatial. This is true for any covariant spatial tensor and any smooth function \( \alpha \). To prove this, we first remark that by Leibnitz’ rule it suffices to prove it for a general spatial 1-form \( \omega \).

The first assertion now follows from
\[
L_\alpha n \omega = (i_\alpha \circ d + d \circ i_\alpha) \omega = \alpha i_n d \omega = \alpha L_n \omega.
\]
The second statement follows from the general formula
\[
i_n \circ L_v = i_{[n,v]} + L_v \circ i_n,
\]
showing that for \( v = n \) the left hand side annihilates any spatial tensor field. This identity also shows why we need the projector in the second expression on the right hand side of (15), since for \( v = \beta \) it shows that we would need \([n, \beta] \propto n \) for \( L_\beta h \) to be spatial. But this is generally false, as one easily shows that \([n, \beta^i \partial_k] \propto n \Rightarrow \partial_t (\beta^i) = 0 \).

We proceed by showing that \( L_n h \) is just twice the extrinsic curvature:
\[
K = \frac{1}{2} L_n h.
\]
To prove this relation, we take any spatial vector fields \( X, Y \) and compute:
\[
L_n h(X, Y) = \nabla_n (h(X, Y)) - h([n, X], Y) - h(X, [n, Y]) = h(\nabla_X n, Y) + h(X, \nabla_Y n) = 2K(X, Y),
\]
where we used the metricity of \( \nabla \), \((\nabla_n h)(X, Y) = (\nabla_n g)(X, Y) = 0\), and its vanishing torsion. Hence we arrive at
\[
K = \frac{1}{2\alpha} \left( \dot{h} - SL_\beta h \right).
\]
The projected Lie-derivative can be expressed in terms of the spatial covariant derivative in the usual way. In components with respect to a spatial coordinate frame this reads
\[
(SL_\beta h)_{ik} = D_i \beta_k + D_k \beta_i.
\]

1.2 Constraints and Equations of Motion

Using the splitting formula (6) for the connection \( \nabla \) in terms of \( D \) and \( K \) we can derive the so-called Gauss-Codazzi and Codazzi-Mainardi equations by a straightforward manipulation. In components with respect to \( \{e_\alpha\} \) and with \( R^{(3)} \) denoting the curvature of \( D \), they read respectively:
\[
R_{abcd} = R^{(3)}_{abcd} + K_{ac} K_{bd} - K_{ad} K_{bc},
\]
\[
R_{0abc} = D_c K_{ab} - D_b K_{ac}.
\]
From here it is easy to write down the constraints by noting that in orthonormal frames one has \( \sum_{a,b} R_{abab} = R + 2R_{00} = 2G_{00} \), i.e. the 00 component of the
Einstein tensor just depends on the spatial components of \( \nabla \)'s curvature. In fact, it is the sum of the spatial sectional curvatures of \( \nabla \). Further, \( G_{0\theta} = R_{0\theta} = \sum_a R_{0ab}a \). Hence we have the constraints, now written in components with respect a coordinate frame,

\[
G_{\mu\nu}n^\mu n^\nu = \frac{1}{2}(R^{(3)} - K_{ik}K^{ik} + (K^i_j)^2) = 0, \tag{20}
\]

\[
G_{\mu\nu}n^\mu = D^k(K_{ik} - h_{ik}K^j_j) = 0. \tag{21}
\]

To obtain the dynamical equations one starts again from the defining equation of the curvature and manipulates the expression for \( R_{0a0b} \). Observing that \( \nabla_n(g(e_a, \nabla e_{\alpha})n) = L_nK_{ab} \) one arrives at

\[
R_{0a0b} = -L_nK_{ab} + K_{ac}K^c_b + a_a a_b + D_a a_b, \tag{22}
\]

where \( a := \nabla a \). Note also that \( a^\nu = L_n n^\nu \) (Exercise: Prove it). Despite appearance, the last term in (22) is also symmetric.\(^1\) Now, \( R_{ab} = -R_{0a0b} + \sum_c R_{cacb} \), so that with (18) we have

\[
R_{ab} = R^{(3)}_{ab} + L_nK_{ab} + K_{ab}K^c_c - 2K_{ac}K^c_b - a_a a_b - D_a a_b. \tag{23}
\]

This is almost the evolution equation we wish to obtain. As in (15) we have \( \dot{K} = \alpha L_nK + S \beta K \), and since we want to write down the final equation in a coordinate basis, we can simplify the terms involving \( a \) by noting that \( a_i = (L_n n^\nu)(\partial \nu) = n^\nu(\partial_\nu \frac{1}{\alpha} (\partial_\nu - \beta)) = \partial_\nu \alpha / \alpha. \) Hence\(^2\)

\[
\dot{K}_{ik} = \alpha(2K_{ij}K^j_k - K_{ik}K^j_j + R_{ik} - R^{(3)}_{ik}) + L_\beta K_{ik} + D_i D_k \alpha, \tag{24}
\]

where in the vacuum case we consider here one sets \( R_{ik} = 0 \). Note that in a coordinate frame dotting just means taking the partial derivative of the components, i.e., \( L_\partial h_{ij} = \partial h_{ij} / \partial t \).

The dynamical formulation is now complete. The constraints are given by eqs. (20)(21) and the six evolution equations of second order are written as twelve equations of first order, given by (15) and (24). The six dynamical components of \( g \) are the \( h_{ij} \), whereas there are no evolution equations for the four functions \( \alpha, \beta \).

The initial value problem thus takes the following form: 1.) choose a 3-manifold \( \Sigma \) with local coordinates \( \{x^\nu\} \), 2.) find a Riemannian metric \( h_{ij} \) and a symmetric covariant tensor field \( K_{ij} \) on \( \Sigma \) which satisfy (20)(21), 3.) choose any convenient

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\(^1\) This is due to \( n \) being hypersurface-orthogonal. To see this, we first note the identity

\[
(L_n - \frac{1}{2} a_\alpha a_\alpha d\nu \wedge n^\nu \wedge n^\nu = d a_\alpha a_\alpha \wedge n^\nu \) (Exercise: Prove it). Now, hypersurface-orthogonality of \( n \) \( \Rightarrow d n^\nu \wedge n^\nu = 0 \Rightarrow d a^\nu \wedge n^\nu = 0 \Leftrightarrow S d a^\nu = 0 \Leftrightarrow D_b a^a b = 0. \)

\(^2\) Be aware that some authors define the extrinsic curvature with opposite sign, for example E. Seidel in his lecture. Hence the discrepancy between our eqns. (17)(24) with his (5)(6) respectively. In our convention, which agrees with Hawking & Ellis, a positive \( K^i_j \) implies volume expansion under deformations in normal direction. Also note that “dotting” does not commute with index raising. Hence notations like \( K^{ij} \) are ambiguous. For example, denoting the index raising operation by a superscript \( \sharp \), (16) immediately gives \( (L_nK^i_j - (L_nK)^i_j) = -4K^j_j K^{ik}. \)
functions $\alpha(t, x^k), \beta^i(t, x^k)$, 4.) evolve $h_{ij}$ and $K_{ij}$ via (15)(24) by using the choices made in the previous step, 5.) take the solution curve $h_{ij}(t, x^k)$ and the functions from 3.) to construct the space-time metric according to (14), where $x^0 = t$. The $g$ so constructed solves Einstein’s equations. An important theorem guarantees that for suitably specified data a maximal evolution $(M, g)$ exists which is unique up to diffeomorphisms (Choquet-Bruhat and Geroch 1969). See also Choquet-Bruhat and York (1980) for a review and further references.

Regarding step 2.), we remark that all topologies $\Sigma$ allow some initial data, i.e., there are no topological obstructions to (20)(21) (Witt 1986). This might change if geometrically special data are sought (see below). To illustrate step 3.), we mention the so-called maximal slicing condition on $\alpha$. To derive it, we compute $L_n(h^{ij}K_{ij}) = -2K_{ij}K_{ij} + h_{ij}L_nK_{ij} = -R_{00} - K_{ij}K_{ij} + \Delta \alpha/\alpha$, where $\Delta = D_iD_i$ and where we used $R_{00} = \sum_{\alpha} R_{\alpha 0 \alpha}$ and (22) to replace $L_nK_{ij}$. Hence

$$L_{\beta}(K^i_i) = (\Delta - K_{ij}K_{ij} - R_{00})\alpha + L_\beta(K^i_i),$$

(25)

where we left in $R_{00}$ for generality. Note that the strong energy condition implies $R_{00} \geq 0$ through the Einstein equations. The hypersurface $\Sigma \subset M$ is called maximal if the trace of its extrinsic curvature – the so-called mean curvature – is zero, i.e., $K^i_i = 0$. This is equivalent to $\Sigma$ being a stationary point of all the 3-dimensional volume functionals for domains in $\Sigma$ and fixed boundaries. 3 (Exercise: Prove this using (16).) Now, given a maximal slice $\Sigma \subset M$, (25) gives the following simple condition on $\alpha$ if the evolution is to preserve maximality: $O\alpha = 0$ with elliptic operator $O = \Delta - K_{ij}K_{ij} - R_{00}$. (Exercise: Assuming the strong energy condition, prove that any smooth function $\alpha$ in the kernel of $O$ cannot have a positive local maximum or negative local minimum on $\Sigma$.) In the vacuum case one can use (20) to write $O = \Delta - R^{(3)}$, i.e., purely in terms of the intrinsic geometry of $\Sigma$, where clearly $R^{(3)} \geq 0$.

The maximal slicing condition plays an important rôle in numerical evolution schemes, since – by definition – the evolving maximal slices $\Sigma_t \subset M$ approach slowest the regions of strongest spatial compression. In this sense they have the tendency to avoid singularities. For further information see section 2.3 of E. Seidel’s lecture. Finally we note that since not all topologies $\Sigma$ allow for metrics with $R^{(3)} \geq 0$, there exist topological obstructions to maximal initial data sets (Witt 1986).

2 Time-Symmetric Initial Data

Suppose a hypersurface $\Sigma \subset M$ has vanishing extrinsic curvature, $K = 0$. From (6) we then have $\nabla_X Y = D_X Y$ for all vector fields $X, Y$ tangent to $\Sigma$. In particular, if $\gamma : I \to \Sigma$ is a curve with tangent vector field $\gamma'$ over $\gamma$, then

\footnote{The standard terminology is that such stationary points are called “maximal” if the ambient geometry is Lorentzian and “minimal” if it is Riemannian, irrespectively of whether they really are true maxima or minima respectively. True extrema are called stable maximal (minimal) surfaces.}
\[ \nabla_{\gamma'} \gamma' = D_{\gamma'} \gamma' \text{ and } \gamma \text{ is a geodesic in } \Sigma \text{ iff it is a geodesic in } M. \] Submanifolds for which this is true are called \textit{totally geodesic}. This is a stronger condition than maximality. In general, constant mean curvature data play an important role in the solution theory for the constraints (see York 1973, Ó Murchadha and York 1974). Here we shall vastly shortcut the general procedure by imposing the condition that \( \Sigma \) be totally geodesic. One can then show that the maximal development, \( M \), from these data allows an isometry fixing \( \Sigma \) pointwise and exchanging the two components of \( M - \Sigma \). Hence such data are called \textit{time symmetric}. For such cases the constraints reduce to the simple condition that \((\Sigma, h)\) has vanishing Ricci-scalar:

\[ R^{(3)}(h) = 0, \quad (26) \]

where for later convenience we explicitly indicated the metric as argument of \( R^{(3)} \). A general idea for solving (26) is to prescribe \( h \) up to an overall conformal factor \( \Phi \), and let (26) determine the latter. So setting \( h = \Phi^4 h' \), with fourth power just for convenience, we have by the conformal transformation law for the Ricci-scalar

\[ R^{(3)}(\Phi^4 h') = -8\Phi^{-5} (\Delta_{h'} - \frac{1}{8} R^{(3)}(h')) \Phi =: -8\Phi^{-5} C_{h'} \Phi = 0, \quad (27) \]

where \( \Delta_{h'} \) is the Laplacian for the metric \( h' \). We are interested in \( C^2 \) solutions satisfying \( \Phi > 0 \) and where \((\Sigma, h)\) has no boundaries at finite distances, i.e. \( \Sigma \) should be topologically complete in the metric topology defined by the distance function induced by \( h \). The last condition is equivalent to \((\Sigma, h)\) being geodesically complete (theorem of Hopf-Rinow-DeRahm, see e.g. Spivak 1979). In addition, we shall only be interested in manifolds whose ends are asymptotically flat. Allowing the manifold \( \Sigma \) to have more ends or to be otherwise topologically more complicated allows for a greater variety of solutions. Note that to each of \( n \) asymptotically flat ends there corresponds an ADM-mass of which \( n - 1 \) are independent (see below).

\textbf{Brill Waves.} One may ask whether simple asymptotically flat solutions to \( C_{h'} \Phi = 0 \) exist on \( \Sigma = \mathbb{R}^3 \). There are no (regular!) black-hole solutions with this simple topology, but there are solutions representing localized gravitational waves of non-zero total ADM energy (Araki 1959). In the axisymmetric case they were investigated in detail by Brill (1959). Solutions of this kind are collectively called “Brill waves”. One takes (from now on in the usual shorthand suppressing the \( \otimes \))

\[ h' = \exp(\lambda q(z, \rho))(dz^2 + d\rho^2) + \rho^2 \, d\varphi^2, \quad (28) \]

where the profile-function \( q \) must for \( r \to \infty \) fall off like \( r^{-2} \) and like \( r^{-3} \) in its first derivatives in order for \( h \) to turn out asymptotically flat. \( q \) characterizes the geometry in the meridional cross section \((z\rho\text{-plane})\) of the toroidal gravitational wave. Regularity on the axis also requires \( q \) and \( \partial_\rho q \) to vanish for \( \rho = 0 \). The
parameter $\lambda \in \mathbb{R}_+$ is sometimes introduced to independently parameterize the overall amplitude. Equation (26) for $\Phi(z, \rho)$ takes the particularly simple form
\[
(\Delta_f + \frac{1}{4} \lambda \Delta^{(2)}q) \Phi = 0,
\] (29)
where $\Delta_f$ is the flat Laplacian and $\Delta^{(2)} = \partial^2/\partial z^2 + \partial^2/\partial \rho^2$. Given $q$, everywhere positive solutions for $\Phi$ exist provided $\lambda$ is below some critical value depending on the choice $q$ (Araki 1959). To see uniqueness, assume the existence of two solutions $\Phi_1$ and $\Phi_2$ and set $h_i = \Phi_i^2 h'$, $i = 1, 2$. Then $\Phi_3 := \Phi_1/\Phi_2$ is also $C^2$, positive and tends to 1 at infinity. But (27) immediately implies $\Delta h_i \Phi_3 = 0$, and since also $R^{(3)}(h_i) = 0$ this is equivalent to $\Delta h_2 \Phi_3 = 0$. Hence $\Phi_3 = 1$ due to the fact that the only bounded harmonic functions are the constant ones.

3 Black-Hole Data

A substantial variety of time-symmetric black-hole data can already be obtained by solving (27) when $h'$ is flat, i.e., where the 3-metric, $h$, on the spatial slice at the moment of time-symmetry is conformally flat. One can obtain manifolds with any number of asymptotically flat ends, and then reduce this number by a process which is best described by calling it “plumbing” (see below). We shall devote the rest of this paper to the description of such solutions and techniques. Note that for flat $h'$ we are left with the simple harmonic equation involving only the flat Laplacian:
\[
\Delta_f \Phi = 0.
\] (30)

In general it is difficult to infer from given initial data whether they correspond to a spacetime with black holes, i.e. with event horizons. However, in the examples to follow it is easy to see that that there will be apparent horizons, since for time symmetric data apparent horizons correspond precisely to minimal surfaces $S \subset \Sigma$. Proposition 9.2.8 of Hawking and Ellis (1973) now implies the existence of an event horizon whose intersection with $\Sigma$ is on, or outside, the outermost apparent horizon for any regular predictable spacetime that develops from data satisfying the strong energy condition. Concerning the topology of apparent horizons we remark the following: Using the formula for the second variation of the area functional and the theorem of Gauss-Bonnet, one shows that for ambient metrics with non-negative Ricci scalar any connected component of an orientable stable minimal surface of finite volume must be a

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4 The condition on $S$ being an apparent horizon is that the congruences of outgoing null rays from $S$ must have zero divergence. Analytically this translates into $\text{tr}_2(\kappa) = \pm (\text{tr}(K) - K(\nu, \nu))$ where $\kappa, \nu$ are respectively the extrinsic curvature and normal of $S$ in $\Sigma$. The upper sign is valid for past apparent horizons, and the lower one for future apparent horizons. $\text{tr}_2$ is the 2-dimensional trace using the induced metric of $S$ and $\text{tr}$ the 3-dimensional trace using $h$. For time-symmetric initial data ($K = 0$) this condition states that $\kappa$ is traceless and hence $S$ minimal in $\Sigma$. 

topological 2-sphere (Gibbons 1972). Allowing also for non-orientable apparent
horizons, one deduces from this that for metrics $h$ of $\Sigma$ with $R^{(3)}(h) \geq 0$ a
connected component of an apparent horizon is either $S^2$ or $RP^2$, the latter
being the (non-orientable) 2-dimensional real projective space. If $\Sigma$ is orientable
$RP^2 \subset \Sigma$ is one-sided, as in the example below.

### 3.1 Schwarzschild Data

We start by noting that the most general non-trivial solution of (30) on $\Sigma = \mathbb{R} - \{0\}$ is given by $\Phi(x) = 1 + \frac{m}{r}$ with $r = \|x\|$ and $m \in \mathbb{R}_+$. We cannot have any higher multipole moments because then $\Phi$ necessarily has zeros on $\Sigma$. Just removing $\Phi^{-1}(0)$ from $\Sigma$ does not work since these points are at finite distance so that the resulting space would not be (geodesically) complete. This is also
the reason why $m$ must be positive. Hence we obtain for the metric $h$ in polar
coordinates

$$ h = \left(1 + \frac{m}{2r}\right)^4 (dr^2 + r^2 d\Omega^2), \quad (31) $$

with $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. Now, it is easy to verify that the following two
diffeomorphisms, $I$ and $\tilde{I}$, of $\Sigma$ are involutive (i.e., square to the identity) isome-
tries:

$$ I(r, \theta, \varphi) := \left(\frac{m^2}{4r}, \theta, \varphi\right), \quad (32) $$

$$ \tilde{I}(r, \theta, \varphi) := \left(\frac{m^2}{4r}, \pi - \theta, \varphi + \pi\right). \quad (33) $$

The map $I$ is called an inversion on the sphere $r = m/2$, whereas $\tilde{I}$ is that
inversion plus an additional antipodal map on the spheres of constant $r$. We shall
sometimes refer to them as inversions of the first and second kind respectively. $\tilde{I}$
has no fixed points while $I$ fixes each point of the sphere $S = \{x \mid r = \frac{m}{2}\}$ (which,
as set, is also left invariant by $\tilde{I}$). As fixed point set of an isometry $\tilde{S}$ must be
totally geodesic, hence minimal and therefore an apparent horizon. Its surface
area is $A = 16\pi m^2$, and it separates the two isometric regions $r > m/2$ and
$r < m/2$. The metric (31) corresponds to the spatial part of the Schwarzschild
metric of mass $m$ in isotropic coordinates, which cover both asymptotically flat
regions (I and III) on the Kruskal manifold. Using this isotropic form, one can
read off $\alpha = (1 - m/2r)/(1 + m/2r)$, $\beta = 0$ and verify that with this choice the
static form of (24) with $K = 0$ is satisfied (Exercise).

The manifold $\Sigma$ has two isometric ends and we can get rid of one by suitably
identifications. For this we take the quotient $\tilde{\Sigma}$ of $\Sigma$ with respect to the free
action of $\tilde{I}$. The freeness guarantees that the quotient will be a manifold, and,
by being an isometry, the metric descends to a smooth metric on the quotient. $\tilde{\Sigma}$

---

5 Proof: Consider the unique geodesic $\gamma$ starting on and tangentially to $S$. It cannot
leave $S$ since if it would, its image under $I$ would be a different geodesic with the
same initial conditions, which contradicts the uniqueness theorem for ODE’s.
can be pictured by cutting $\Sigma$ along $S$, throwing away one piece, and identifying opposite points on the inner boundary $S$ on the retained piece. Hence topologically $\tilde{\Sigma}$ is the real projective space, $\mathbb{RP}^3$, minus a point. The projection of $S$ into $\tilde{\Sigma}$ is a totally geodesic, one-sided (i.e. non-orientable) surface diffeomorphic to $\mathbb{RP}^2$. $\tilde{\Sigma}$ is orientable, smooth, complete and with one end which is isometric to, and hence has the same ADM mass as, either end in $\Sigma$. This demonstrates how the introduction of more ends or other topological features makes it possible to define non-trivial black-hole data. One may also combine Brill waves with a black hole to model a single distorted black hole. This is further discussed in section 2.2 of E. Seidel’s lecture.

Multi-Schwarzschild Data. Taking $\Sigma = \mathbb{R}^3 - \{c_1, \cdots, c_n\}$ the generalization of (31) is easily obtained with $n$ poles of strengths $a_i \in \mathbb{R}_+$ at “positions” $c_i$:

$$\Phi(x) = 1 + \sum_{i=1}^{n} \frac{a_i}{r_i}, \quad (34)$$

where $r_i := \|x - c_i\|$. For each $i$ we can introduce inverted polar coordinates $r'_i = a_i^2/r_i$ to probe the region $r_i \to 0$ by letting $r_i' \to \infty$. Doing this shows that the metric is asymptotically of the form (31) with certain mass parameters $m = m_i$ given below. The same is true for the region $r \to \infty$ with mass $M$. Hence one obtains $n + 1$ asymptotically flat ends. The internal masses and the overall mass are given by ($r_{ij} := \|c_j - c_i\|$)

$$m_i = 2a_i(1 + \chi_i), \quad \text{where} \quad \chi_i := \sum_{j \neq i} \frac{a_j}{r_{ij}}, \quad \text{and} \quad M = 2 \sum_i a_i. \quad (35)$$

In terms of the parameters $a_i, r_{ij}$ the binding energy takes the simple form

$$\Delta M := M - \sum_{i=1}^{n} m_i = -2 \sum_{i=1}^{n} a_i \chi_i = -2 \sum_{i=1}^{n} \sum_{j \neq i} \frac{a_i a_j}{r_{ij}} < 0. \quad (36)$$

Note that there are as many independent masses as there are generators of the second homology group of $\Sigma$. These generators may be represented by stable minimal surfaces associated to each internal end. Their surfaces areas clearly satisfy $A_i > 16\pi(2a_i)^2$, since the right hand side represents the minimal area in the strictly smaller metric (31) for just one hole with parameter $m = 2a_i$. But there is also an upper bound for the area, given by the recently proven Riemannian Penrose inequality$^6$ (Huisken and Ilmanen 1997), which in our context reads $A \leq 16\pi (2a_i)^2(1 + \chi_i)^2$. Assuming the existence of an event horizon

---

$^6$ The proof of Huisken and Ilmanen (1997) applies to all asymptotically flat Riemannian 3-Manifolds whose Ricci scalar satisfies $R \geq 0$. They prove that the area $A$ of the outermost stable minimal surface bounding an end and the ADM mass $m$ of that end satisfy $A \leq 16\pi m^2$. It implies the positive mass theorem for data with $R \geq 0$. 

Time-Symmetric Initial Data

(see above), the Area Theorem (see my other contribution to this volume) implies that the area of the hole in the \(i\)'th end cannot evolve below \(A_i\), which, using the first inequality above, implies in particular that the energy which is bound in the final hole is greater than \(2a_i = m_i/(1 + \chi_i)\). The difference of the (conserved) ADM mass \(m_i\) to the mass of the final hole is therefore bounded above by \(m_i\chi_i/(1 + \chi_i)\). In other words, the fraction of energy being radiated is less than \(\chi_i/(1 + \chi_i)\). This still allows for total conversion into radiation if one chooses \(\chi_i \to \infty\).

It would be of course more interesting to express (36) in terms of physical variables, like the individual masses \(m_i\), and more geometrically defined distance functions than \(r_{ij}\), like e.g. the proper geodesic distance of the minimal surfaces in the \(i\)'th and \(j\)'th throat. Note that for small mass-to-separation ratios we may in a first approximation replace \(a_i\) by \(\frac{1}{2}m_i\) and \(r_{ij}\) by the geodesic distance of the \(i\)'th and \(j\)'th apparent horizons and get the familiar Newtonian formula.

The location of minimal surfaces is interesting for a variety of reasons. It somewhat simplifies in the case of just two holes, which is automatically axisymmetric. Then the variational principle for the minimal surfaces reduces to a geodesic principle for curves in the \(z\rho\)-half-plane (cylindrical coordinates). The appropriately parameterized solution curves just describe a motion of a point particle in the potential \(-\frac{1}{2}\rho^2\Phi\) (Cadež 1974). However, general analytic solution still do not exist. Numerical studies by Bishop (1982) for equal masses \(a_1 = a_2 =: a\) show the very interesting behaviour above the critical value \(a/r_{12} \approx 1/1.53\), where two more minimal surfaces appear, each of which enclosing the previous two. Initially they coincide, but for increasing \(a/r_{12}\) they separate with the inner one rapidly increasing in area whereas the outermost staying almost constant. See also Gibbons (1984) for a related discussion.

For the data discussed below the difficulty of determining location and size of minimal surfaces is absent, but somewhat as trade-off the concept of individual mass now becomes slightly more problematic.

**Different Topologies for Multi-Hole Data.** There are other generalizations of the single hole case. The ones we discuss now will preserve the existence of involutive isometries like (32-33), but now for each apparent horizon. The manifolds they exist on have two or even just one end. The construction is somewhat involved (Lindquist 1963) and uses the method of images to construct solutions to (30). This method was introduced by Misner (1963) for the time symmetric case and later generalized to more general situations (e.g. Bowen and York 1980, Bowen 1984). (There is also a recent alternative proposal by Brandt...
and Brügmann (1997).) For the general understanding it will be sufficient to explain the construction for just two holes. Note that the ADM definition of mass cannot be applied to the individual hole if it does not have an asymptotically flat end associated to it. But there exist alternative proposals for mass due to Lindquist (1963) and Penrose (1982) which can be employed here. (See also the general review by Penrose (1984).) But it should be pointed out that these definitions do not always apply in more general situations. For example, for the applicability of Penrose’s mass definition within time-symmetric hypersurfaces the metric of this hypersurface must be conformally flat (Tod 1983, Beig 1991).

3.2 Two Hole Data

Just as in electrostatics, we shall use the method of images to construct special solutions to (30). This is done by placing image masses in an auxiliary, fictitious space so as to enforce special properties of \( \Phi \). The properties which will be enforced here are such that the inversions (32)(33) on 2-spheres become isometries.

We start by drawing two 2-spheres \( S_i := S(a_i, c_i), \) \( i = 1, 2 \), with radii \( a_i \) and centered at \( c_i \). The spheres are non-intersecting and outside each other, so that \( r_{12} > a_1 + a_2 \). On \( \mathbb{R}^3 - \{c_i\} \) we have the diffeomorphisms \( I_i \) and \( \tilde{I}_i \), which in polar coordinates at \( c_i \) take the forms (32) and (33) respectively. These induce involutions on the space of functions, defined by

\[
J_i(f) := \frac{a_i}{r_i} f \circ I_i \quad \text{and} \quad \tilde{J}_i(f) := \frac{a_i}{r_i} f \circ \tilde{I}_i
\]  

respectively, where \( f \) is any function. The crucial property of these maps is

\[
\Delta_t \circ J_i = (a_i/r_i)^4 J_i \circ \Delta_t \quad \text{and} \quad \Delta_t \circ \tilde{J}_i = (a_i/r_i)^4 \tilde{J}_i \circ \Delta_t,
\]  

which in particular implies that the image of a harmonic function will again be harmonic, although with different singularity structure. The image of the constant function, \( f \equiv 1 \), under either of these maps is just \( f' = a_i/r_i \), i.e., the pole of strength \( a_i \) at \( c_i \). Moreover, given the unit pole \( f(x) = 1/\|x - d\| \) at \( d \) outside \( S_i \), then its image under \( J_i \) is

\[
J_i(f) = \frac{a_i}{\|c_i - d\|} \frac{1}{\|x - I_i(d)\|},
\]  

and correspondingly for \( \tilde{J}_i \). It represents a pole of strength \( a_i/\|c_i - d\| < 1 \) at the image point \( I_i(d) \) (resp. \( \tilde{I}_i(d) \)).

Writing down the metric \( h = \Phi^4 ds^2 \) in polar coordinates centered at \( c_i \), one easily verifies that \( I_i \ (\tilde{I}_i) \) is an isometry of \( h \) if \( \Phi \) is invariant under \( J_i \) (\( \tilde{J}_i \)). The construction of such an invariant \( \Phi \) is by brute force: One averages the function \( \Phi_0 \equiv 1 \) over the free product of the groups generated by \( J_1, J_2 \) (\( \tilde{J}_1, \tilde{J}_2 \)). The elements of this free-product-group are strings of alternating \( J_1 \)'s and \( J_2 \)'s, where for each string length \( n \geq 1 \) there are the two different elements.
By definition, the string of length 0 is the identity element. Hence one sets

$$\Phi_N := 1 + \sum_{n=1}^{N} \sum_{i_1 \cdots i_n} J_{i_1} \circ \cdots \circ J_{i_n} (\Phi_0),$$

where the first sum is over the two different elements of length $n$. On $\mathbb{R}^3 - \{\text{image points}\}$ the sequence $\Phi_N$ converges to a smooth function $\Phi$ for $N \to \infty$. Convergence follows because at level $N$ the strengths of the new poles are suppressed by at least a factor of $q^{N-1}$, where $q = \sup_{i,j} a_i / (r_{ij} - a_j) < 1$.

Note also that all image poles in $S_i$ lie in fact in the interior of the concentric but smaller sphere of radius $a'_i := a_i^2 / (r_{ij} - a_j)$. Cutting out the interiors of $S(a'_i, c_i)$ for $i = 1, 2$ thus leaves the spheres $S_i$ with small collar neighborhoods the two sides of which are isometrically mapped into each other by $I_i$ (or $\tilde{I}_i$).

Using two copies of the manifold so obtained we can pairwise identify these collar neighborhoods using these isometries so that an Einstein-Rosen manifold with two bridges results. Their topology is that of the twice punctured “handle” $S^1 \times S^2$ with each puncture corresponding to an asymptotically flat end. This construction generalizes to any number $N$ of holes (or bridges), where as manifold one obtains the twice punctured connected sum of $N - 1$ handles. (For the notion of connected sums see e.g. Giulini 1994.) For two holes of equal mass one may also just identify $S_1$ and $S_2$ and get Misner’s wormhole (Misner 1960) if one uses inversions of the first kind, or its non-orientable counterpart if one uses inversions of the second kind (Giulini 1990). Both manifolds just have one end. In the second case one has the additional possibility to just close “close-off” the spheres $S_i$ individually by identifying its antipodal points using $\tilde{I}_i$ (Giulini 1992). The manifold has the topology of the once punctured connected sum of two real projective spaces $\mathbb{RP}^3$. It is orientable and has only one asymptotically flat end. It can be seen as the generalization to two holes of the once punctured $\mathbb{RP}^3$ obtained above. This construction also generalizes to any number $N$ of holes and one obtains the once punctured connected sum of $N \mathbb{RP}^3$’s. These manifolds are doubly covered by the $N$-bridge manifolds discussed above.

### 3.3 Analytic Expressions

In the case of two holes there exists a geometrically adapted coordinate system – so called spherical bi-polar coordinates – which allows to write down explicit expressions. We take $a_1 = a_2 = a$, $c_1 = de_z$ and $c_2 = -de_z$. Taking the $a_i$’s equal means that the holes are of equal size (individual mass). We thus consider a two parameter family of configurations labeled e.g. by mass (overall or individual) and separation. All image poles are on the $z$-axis whose strengths $a_n$ and locations $d_n$ (positively counted $z$ coordinate) satisfy the coupled recursion relations

$$a_n = a_{n-1} - \frac{a}{d + d_{n-1}}, \quad d_n = d \mp \frac{a^2}{d + d_{n-1}}.$$

$$\Phi_N := 1 + \sum_{n=1}^{N} \sum_{i_1 \cdots i_n} J_{i_1} \circ \cdots \circ J_{i_n} (\Phi_0),$$
where the upper (lower) sign is valid for inversions of the first (second) kind. Using instead of $a, d$ the parameters $c, \mu_0$ defined by $a := c/ \sinh \mu_0$, $d := c \coth \mu_0$ we can solve the recursion relations by

$$a_n = \frac{c}{\sinh n\mu_0}, \quad d_n = c \tanh n\mu_0,$$

for the upper sign, and for the lower sign

$$a_n = \frac{c}{\sinh n\mu_0}, \quad d_n = c \coth n\mu_0 \quad \text{for } n \text{ even},$$
$$a_n = \frac{c}{\cosh n\mu_0}, \quad d_n = c \tanh n\mu_0 \quad \text{for } n \text{ odd}. \quad (44)$$

In the $xz$-plane we introduce bi-polar coordinates via $\exp(\mu - i\eta) = (\xi + c)/(\xi - c)$ with $\xi = z + ix$. By construction the lines of constant $\mu$ intersect those of constant $\eta$ orthogonally. Both families consist of circles; those in the first family are centered on the $z$-axis with radius $c/\sinh \mu$ at $|z| = c \coth \mu$; and those in the second family on the $x$-axis with radius $c/\sin \eta$ at $|x| = c \cot \eta$. Rotating this system around the $z$-axis with azimuthal angle $\phi$ leads to the spherical bi-polar coordinates. Explicitly one obtains

$$x = c \frac{\sin \eta \cos \phi}{\cosh \mu - \cos \eta}, \quad y = c \frac{\sin \eta \sin \phi}{\cosh \mu - \cos \eta}, \quad z = c \frac{\sinh \mu}{\cosh \mu - \cos \eta}. \quad (45)$$

Together with (42-44) this gives

$$\frac{a_n}{\|x \pm d_n e_z\|} = \frac{[\cosh \mu - \cos \eta]^{1/2}}{[\cosh(\mu \pm 2n\mu_0) - \varepsilon \cos \eta]^{1/2}}. \quad (46)$$

where $\varepsilon = 1$ if one uses inversions of the first kind and $\varepsilon = -1$ if one uses those of the second kind. The final expression for the metric in $(\mu, \eta, \phi)$-coordinates can now be written down:

$$h = \left[1 + \sum_{n=1}^{\infty} \left( \frac{a_n}{\|x + d_n e_z\|} + \frac{a_n}{\|x - d_n e_z\|} \right)^4 \right] dx \cdot dx \quad (47)$$

$$= \left[ \sum_{n \in \mathbb{Z}} (\cosh(\mu + 2n\mu_0) - \varepsilon^n \cos \eta)^{-1/2} \right] \left( d\mu^2 + d\eta^2 + \sin^2 \eta d\phi^2 \right). \quad (48)$$

It nicely exhibits the isometries $(\mu, \eta, \phi) \mapsto (\mu + 2\mu_0, \eta, \phi)$ for $\varepsilon = 1$ and $(\mu, \eta, \phi) \mapsto (\mu + 2\mu_0, \pi - \eta, \phi)$ for $\varepsilon = -1$. The extrinsic curvature matrix for the surfaces of constant $\mu$ with respect to an orthonormal basis in $\eta$ and $\phi$ direction is given by $2\Phi^{-1} \partial \Phi / \partial \mu$ times the unit matrix. Hence $K$ has only a trace part (the surfaces of constant $\mu$ are totally umbilic) and vanishes iff $\mu = \pm \mu_0$. Hence in both cases, $\varepsilon = \pm 1$, the apparent horizons are also totally geodesic (this we already knew for $\varepsilon = 1$).
Next we turn to the expressions for the masses. We shall follow Lindquist (1963) and define the mass of the first hole by appropriately applying (35): We sum all the "bare masses" \(a_i\) in \(S_1\), each enhanced by an interaction factor \(1 + \chi_i\), which includes the interactions of each pole in \(S_1\) with any pole in \(S_2\), but not with poles in \(S_1\). This we write as

\[
m_1 = 2 \sum_{i \in S_1} a_i \left(1 + \sum_{j \in S_2} \frac{a_j}{r_{ij}}\right),
\]

with the obvious meaning of "\(\varepsilon\)". Since \(m_1 = m_2\) we write \(m\) for the individual mass and \(M\) for the overall mass. The latter is just the sum of all \(2a_i\). Using (42-44) one obtains (quantities referring to \(\varepsilon = -1\) carry a tilde)

\[
m = 2c \sum_{n=1}^{\infty} \frac{n}{\sinh n\mu_0}, \quad M = 4c \sum_{n=1}^{\infty} \frac{1}{\sinh n\mu_0},
\]

for \(\varepsilon = 1\), and for \(\varepsilon = -1\)

\[
\tilde{m} = 2c \sum_{n=1}^{\infty} \frac{2n}{\sinh 2n\mu_0} + 2c \sum_{n=0}^{\infty} \frac{2n + 1}{\cosh(2n + 1)n\mu_0},
\]
\[
\tilde{M} = 4c \sum_{n=1}^{\infty} \frac{1}{\sinh 2n\mu_0} + 4c \sum_{n=0}^{\infty} \frac{1}{\cosh(2n + 1)n\mu_0}.
\]

As mentioned above, we define the distance of the holes as the geodesic distance of the apparent horizons \(\mu = \pm \mu_0\). The shortest geodesic connecting these two surfaces is \(\eta = \pi\). For \(\varepsilon = 1\) its length, \(l\), may be expressed in closed form:

\[
l = 2c \left(1 + 2m\mu_0\right),
\]

with \(m\) from (50). I have not been able to find such a compact expression in the case \(\varepsilon = -1\).

Like \(\tilde{l}\), many quantities of interest cannot be evaluated in closed form. In these cases it may be useful to expand in powers of \(m/l\). Numerical studies show that additional outer apparent horizons form (i.e. the holes merge) for values above \(m/l \simeq 0.26\) (Smarr et al 1976), so that good convergence holds up to the merging ratio.

**Comparing \(\varepsilon = 1\) to \(\varepsilon = -1\).** We have seen that mathematically these two cases differ by allowing different topologies. But are there more physical aspects in which they differ? A natural question is how for fixed “physical” variables \(m = \tilde{m}\) and \(l = \tilde{l}\) the total energies \(M\) and \(\tilde{M}\) differ (Giulini 1990). One finds

\[
\frac{\tilde{M} - M}{M} = \left(\frac{m}{2l}\right)^2 + \mathcal{O}(3),
\]
showing that for $\varepsilon = -1$ the holes are slightly tighter bound (i.e. they attract stronger), although the additional energy gained until merge is only about $10^{-2} M$. This result is qualitatively unchanged if one uses Penrose’s instead of Lindquist’s definition of mass.

Another difference shows up in the deformation of the apparent horizons upon (adiabatic) approach of the two holes. One can define an intrinsic deformation parameter as follows: Regard $(\eta, \phi)$ as polar coordinates. The poles are the zeros of the Killing field $\partial_\phi$. Define $C_\eta$ as twice their geodesic distance. Among the orbits of $\partial_\phi$ is one of greatest length, $C_\phi$. The deformation parameter is $D := (C_\eta - C_\phi)/C_\eta$. One obtains (Giulini 1990)

$$D = \frac{3}{2} \left(\frac{m}{2l}\right)^3 + \mathcal{O}(4), \quad (55)$$
$$\tilde{D} = \frac{3}{2} \left(\frac{m}{2l}\right)^2 + \mathcal{O}(3). \quad (56)$$

The power of 2 in (56) seems in conflict with the usual “tidal-force” interpretation. The shapes themselves are also different. Like eggs with the thick ends pointing towards each other in the first case, and prolonged symmetrically (with respect to reflections on the equator $\eta = \pi/2$) in the second.

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