SOME APPROXIMATION RESULTS ON HIGHER ORDER GENERALIZATION OF BERNSTEIN TYPE OPERATORS DEFINED BY (p, q)-INTEGERS

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Abstract. In this paper, we introduce the higher order generalization of Bernstein type operators defined by (p, q)-integers. We establish some approximation results for these new operators by using the modulus of continuity.

1. Introduction and preliminaries

In 1912, S.N Bernstein [2] introduced the following sequence of operators $B_n : C[0, 1] \to C[0, 1]$ defined for any $n \in \mathbb{N}$ and for any $f \in C[0, 1]$ such as

$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right), \quad x \in [0, 1]. \quad (1.1)$$

In approximation theory, $q$-type generalization of Bernstein polynomials was introduced by Lupas [5].

For $f \in C[0, 1]$, the generalized Bernstein polynomial based on the $q$-integers is defined by Phillips [10] as follows

$$B_n,q(f; x) = \sum_{k=0}^{n} \left[ \binom{n}{k}_q \right] x^k \prod_{s=0}^{n-k-1} (1-q^s x) f \left( \frac{k}{n} \right), \quad x \in [0, 1]. \quad (1.2)$$

Recently, Mursaleen et al. [7] applied $(p, q)$-calculus in approximation theory and introduced first $(p, q)$-analogue of Bernstein operators(Revised) and defined as:

$$B_{n,p,q}(f; x) = \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} f \left( \frac{\left[ k \right]_p}{p^{k-n}[n]} \right) P_{n,k}(p, q; x), \quad 0 < q < p \leq 1, \quad x \in [0, 1] \quad (1.3)$$

where

$$P_{n,k}(p, q; x) = p^{k(k-1)/2} \left[ \binom{n}{k}_{p,q} \right] x^k \prod_{s=0}^{n-k-1} (p^s - q^s x). \quad (1.4)$$

They have also introduced and studied approximation properties based on $(p, q)$-integers given as: Bernstein-Stancu operators [8] and Bernstein-Shurer operators [9].

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We recall some basic properties of \((p, q)\)-integers.

The \((p, q)\)-integer \([n]_{p,q}\) is defined by
\[
[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \ldots, \quad 0 < q < p \leq 1.
\]

The \((p, q)\)-Binomial expansion is
\[
(x + y)^n_{p,q} := (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y)
\]
and the \((p, q)\)-binomial coefficients are defined by
\[
\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.
\]

For \(p = 1\), all the notions of \((p, q)\)-calculus are reduced to \(q\)-calculus. For details on \(q\)-calculus and \((p, q)\)-calculus, one can refer [11, 12, 1, 3] and [5], respectively.

In this paper we use the notation \([n]\) in place of \([n]_{p,q}\).

In [3], \((p, q)\)-derivative of a function \(f(x)\) is defined by
\[
D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0,
\]
and the formulae for the \((p, q)\)-derivative for the product of two functions is given as
\[
D_{p,q}(fg)(x) = f(px).D_{p,q}g(x) + \{D_{p,q}f(x)\}.g(qx), \quad \text{(1.6)}
\]
also
\[
D_{p,q}(fg)(x) = f(qx).D_{p,q}g(x) + \{D_{p,q}f(x)\}.g(px). \quad \text{(1.7)}
\]

Let \(r \in \mathbb{N} \cup \{0\}\) be a fixed number. For \(f \in C^n[0, 1]\) and \(m \in \mathbb{N}\), we define an operator of \(r^{th}\) order for \((p, q)\)- Bernstein type operators as follows:
\[
B^{[r]}_{n,p,q}(f; x) = \frac{1}{p^{(n-1)x}} \sum_{k=0}^{n} P_{n,k}(p, q; x) \sum_{i=0}^{r} \frac{1}{i!} f^{(i)} \left( \frac{[k]}{p^k q^n} \right) \left( x - \frac{[k]}{p^k q^n} \right)^i.
\]

In this paper, using the moment estimates from [6], we give the estimates of the central moments for operators defined by (1.3). We also study some approximation properties of an \(r^{th}\) order generalization of the operators defined by (1.8) using the techniques of the work on the higher order generalization of \(q\)-analogue [13]. Further, we study approximation properties and prove Voronovskaja type theorem for these operators.

If we put \(p = 1\), then we get the moments for \(q\)-Bernstein operators [6] and the usual generalization higher order \(q\)-Bernstein operators [13], respectively.
2. Main results

The following result is $(p, q)$-analogue of [1].

**Proposition 2.1.** For $n \geq 1$, $0 < q < p \leq 1$

\[
D_{p,q}(1 + x)^n_{p,q} = [n](1 + qx)^{n-1}_{p,q}.
\]  

**Proof.** By applying simple calculation on $(p, q)$-analogue, we have

\[
(1 + px)^n_{p,q} = p^{n-1}(1 + px)(1 + qx)^{n-1}_{p,q}; (1 + qx)^n_{p,q} = (p^{n-1} + q^n x)(1 + qx)^{n-1}_{p,q}.
\]  

(2.2)

Applying $(p, q)$-derivative and result (2.2) we get the desired result. □

**Lemma 2.2.** Let $B_{n,p,q}(f; x)$ be given by (1.3). Then for any $m \in \mathbb{N}$, $x \in [0,1]$ and $0 < q < p \leq 1$ we have

\[
B_{n,p,q}((t - x)^{m+1}_{p,q}; x) = \frac{p^{m+n}x(1 - x)}{[n]}D_{p,q}\left\{B_{n,p,q}((t - \frac{x}{p})^m_{p,q}; \frac{x}{p})\right\} + \frac{p^{m+n-1}m x(1-x)}{[n]}B_{n,p,q}((t - \frac{qx}{p})^{m-1}_{p,q}; \frac{qx}{p}) + \frac{m (p^n - q^n)}{[n]}B_{n,p,q}((t - x)^m_{p,q}; x).
\]

(2.3)

**Proof.** First of all by using (1.6) and Lemma 2.1, we have

\[
D_{p,q}\left(\frac{1}{p^{n-1}x}\sum_{k=0}^{n} (t - \frac{x}{p})^m_{p,q} P_{n,k}(p, q; \frac{x}{p})\right)
\]

\[
= \frac{1}{p^{n-1}x}\left(\sum_{k=0}^{n} (t - x)^{m}_{p,q} P_{n,k}(p, q; \frac{x}{p})\right) - \frac{m}{p}\sum_{k=0}^{n} \left(1 - \frac{qx}{p}\right)^{m-1}_{p,q} P_{n,k}(p, q; \frac{qx}{p}).
\]

(2.3)

Now in the same way by using (1.6) and Lemma 2.1, we have

\[
D_{p,q}\left\{P_{n,k}(p, q; \frac{x}{p})\right\} = D_{p,q}\left\{p^{\frac{k(k-1)}{2}}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} \left(\frac{x}{p}\right)^k \left(1 - \frac{x}{p}\right)^k\right\}
\]

\[
= p^{\frac{k(k-1)}{2}}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} \frac{1}{p^k x^{k-1}} \left(1 - \frac{qx}{p}\right)^{n-k}_{p,q} - \frac{1}{p^k x^{k}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} \frac{1}{p} x^{k-1} \left(1 - \frac{q}{p}\right)^{n-k-1}_{p,q}.
\]

(2.4)

By a simple calculation, we have

\[
\left(1 - \frac{qx}{p}\right)^{n-k}_{p,q} = \frac{1}{p^{n-k}(p - qx)_{p,q}^{n-k+1}} \frac{1}{p^{n-k}(1 - x)_{p,q}} (p^{n-k} - q^{n-k}x)(1 - x)_{p,q}^{n-k}.
\]

(2.5)

\[
\left(1 - \frac{q}{p}\right)^{n-k-1}_{p,q} = \frac{1}{p^{n-k-1}(1 - x)_{p,q}} (1 - x)_{p,q}^{n-k}.
\]

(2.6)
From (2.4), (2.5) and (2.6), we get

\[
D_{p,q} \left\{ P_{n,k} \left( p, q; \frac{x}{p} \right) \right\} = \frac{P_{n,k}(p, q; x)}{p^n x (1 - x)} \left( [k] (p^{n-k} - q^{n-k}) x - p^k [n - k] x \right),
\]
which implies that

\[
D_{p,q} \left\{ P_{n,k} \left( p, q; \frac{x}{p} \right) \right\} = \frac{P_{n,k}(p, q; x)}{p^n x (1 - x)} \left( p^{n-k} [k] - [n] x \right). \tag{2.7}
\]

From (2.3), (2.7), we have

\[
D_{p,q} \left( \sum_{k=0}^{n} \left( t - \frac{x}{p} \right)^m P_{n,k}(p, q; \frac{x}{p}) \right) = \frac{1}{p^{m+n+1}} \left[ m \right] \sum_{k=0}^{n} \left( t - \frac{x}{p} \right)^{m-1} P_{n,k}(p, q; \frac{x}{p})
+ \frac{1}{p^{m+n+1}} \sum_{k=0}^{n} (t - x)^m P_{n,k}(p, q; x) (p^{n-k} [k] - [n] x)
- \frac{1}{p^{m+n+1}} \sum_{k=0}^{n} (t - x)^m P_{n,k}(p, q; x)
\times \left( \left\lfloor \frac{n}{m} \right\rfloor p^m t - q^n x - \left\lfloor \frac{n}{m} \right\rfloor p^m (p^m - q^m) x \right).
\]

Hence we have

\[
D_{p,q} \left\{ B_{n,p,q} \left( (t - \frac{x}{p})^m; \frac{x}{p} \right) \right\} = - \frac{\left[ m \right]}{p} B_{n,p,q} \left( (t - \frac{x}{p})^{m-1}; \frac{x}{p} \right) + \frac{\left[ n \right]}{p^{m+n} x (1 - x)} B_{n,p,q} \left( (t - x)^{m+1}; x \right)
- \frac{\left[ m \right] (p^n - q^n)}{p^{m+n} (1 - x)} B_{n,p,q} \left( (t - x)^m; x \right).
\]

This complete the proof of Lemma 2.2. \hfill \Box

**Lemma 2.3.** Let \( B_{n,p,q} \left( (t - x)^m; x \right) \) be a polynomial in \( x \) of degree less than or equal to \( m \) and the minimum degree of \( \frac{1}{[n]} \) is \( \left\lfloor \frac{m+1}{2} \right\rfloor \). Then for any fixed \( m \in \mathbb{N} \) and \( x \in [0, 1] \), \( 0 < q < p \leq 1 \) we have

\[
B_{n,p,q} \left( (t - x)^m; x \right) = \frac{x (1 - x)^{m-2}}{\left[ n \right]^{\frac{m+1}{2}}} \sum_{k=0}^{m-2} b_{k,m,n}(p, q) x^k, \tag{2.8}
\]
such that the coefficients \( b_{k,m,n}(p, q) \) satisfy \( b_{k,m,n}(p, q) \leq b_m \), \( k = 1, 2, \ldots, m-2 \) and \( b_m \) does not depend on \( x, t, p, q; \) where \( \left\lfloor a \right\rfloor \) is an integer part of \( a \geq 0 \).

**Proof.** By induction it is true for \( m = 2 \). Assuming it is true for \( m \), then from Lemma 2.2 and equation (2.8) we have
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\[ B_{n,p,q} \left((t - x)^{m+1}; x\right) \]

\[
= \frac{p^{m+n-1}x(1-x)}{[n]^{1+\frac{m+n}{2}}} D_{p,q} \left\{ x \left(1 - \frac{x}{p}\right) \sum_{k=0}^{m-2} \frac{b_{k,m,n}(p,q)}{p_k} \left( \frac{q}{p} \right)^{k+1} x^{k+1} + \left( \frac{q}{p} \right)^{k+2} x^{k+2} \right\} \\
+ \frac{p^{m+n-1}m(x - 1)}{[n]^{1+\frac{m+n}{2}}} \sum_{k=0}^{m-3} \frac{b_{k,m-1,n}(p,q)}{p_k} \left( \frac{q}{p} \right)^{k+1} x^{k+1} - \left( \frac{q}{p} \right)^{k+2} x^{k+2} \\
+ \frac{[m](p^n - q^n)x(1-x)}{[n]^{1+\frac{m+n}{2}}} \sum_{k=0}^{m-2} b_{k,m,n}(p,q)x^{k+1} \\
= \frac{p^{m+n}x(1-x)}{[n]^{1+\frac{m+n}{2}}} \sum_{k=0}^{m-2} [k] b_{k,m,n}(p,q) \left( \frac{q}{p} \right)^{k} \left( x^{k} - \left\lfloor \frac{2}{p} \right\rfloor x^{k+1} \right) \\
+ \frac{p^{m+n+1}x(1-x)}{[n]^{1+\frac{m+n}{2}}} \sum_{k=0}^{m-2} \frac{b_{k,m,n}(p,q)}{p_k} \left( \frac{q}{p} \right)^{k} \left( x^{k} - \left\lfloor \frac{2}{p} \right\rfloor x^{k+1} \right) \\
+ \frac{p^{m+n-1}[m]x(1-x)}{[n]^{1+\frac{m+n}{2}}} \sum_{k=0}^{m-3} \frac{b_{k,m-1,n}(p,q)}{p_k} \left( \frac{q}{p} \right)^{k+1} x^{k+1} - \left( \frac{q}{p} \right)^{k+2} x^{k+2} \\
+ \frac{[m](p^n - q^n)x(1-x)}{[n]^{1+\frac{m+n}{2}}} \sum_{k=0}^{m-2} b_{k,m,n}(p,q)x^{k+1} \\
= \frac{x(1-x)}{[n]^{1+\frac{m+n}{2}}} \sum_{k=0}^{m-2} \left( p^{m+n-k}[k] + p^{m+n-k-1}q^k \right) b_{k,m,n}(p,q)x^{k} \\
- \frac{x(1-x)}{[n]^{1+\frac{m+n}{2}}} \sum_{k=0}^{m-1} \left( p^{m+n+1-k}[k-1]\right) b_{k-1,m,n}(p,q)x^{k} \\
+ \frac{x(1-x)}{[n]^{1+\frac{m+n}{2}}} \sum_{k=1}^{m-2} [m] p^{m+n-k-1}q^k b_{k-1,m-1,n}(p,q)x^{k} \\
- \frac{x(1-x)}{[n]^{1+\frac{m+n}{2}}} \sum_{k=1}^{m-1} [m] p^{m+n-k-1}q^k b_{k-2,m-1,n}(p,q)x^{k} \\
+ \frac{x(1-x)}{[n]^{1+\frac{m+n}{2}}} \sum_{k=1}^{m-1} [m](p^n - q^n) b_{k-1,m,n}(p,q)x^{k} \\
= \frac{x(1-x)}{[n]^{1+\frac{m+n}{2}}} \sum_{k=0}^{m-1} b_{k,m+1,n}(p,q)x^{k}
where

\[
\begin{align*}
  b_{k,m+1,n}(p, q) &= \frac{1}{[n]^\alpha} \left( p^{m+n-k} [k] + p^{m+n-k-1} q^k \right) b_{k,m,n}(p, q) \\
  &- \frac{1}{[n]^\alpha} \left( p^{m+n+1-k} [k-1] + [2] p^{m+n-k-1} q^{k-1} \right) b_{k-1,m,n}(p, q) \\
  &+ \frac{1}{[n]^\alpha} [m] (p^n - q^n) b_{k-1,m,n}(p, q) + [m] p^{m+n-k-1} q^k b_{k-1,m-1,n}(p, q) \\
  &- [m] p^{m+n-k-1} q^k b_{k-2,m-1,n}(p, q).
\end{align*}
\]

Clearly

\[
\alpha = 1 + \left\lfloor \frac{m+1}{2} \right\rfloor - \left\lfloor \frac{m+2}{2} \right\rfloor,
\]

which lead us that either \(\alpha = 0\) or \(\alpha = 1\).

Since \(| b_{k,m,n}(p, q) | \leq b_m\), for \(k = m-1\), clearly we have

\[
\begin{align*}
  | b_{k,m+1,n}(p, q) | &\leq \frac{1}{[n]^\alpha} (p^{n+1} [m-1] + p^{n} q^{m-1}) b_m + \frac{1}{[n]^\alpha} (p^{n+2} [m-2] + [2] p^{n} q^{m-2}) b_m \\
  &+ \frac{1}{[n]^\alpha} [m] (p^n - q^n) b_m + [m] p^{n} q^{m-1} b_{m-1} \\
  &+ [m] p^{n} q^{m-1} b_{m-1} \\
  &= \frac{1}{[n]^\alpha} (p [m-1] + q^{m-1}) b_m + \frac{1}{[n]^\alpha} (p^2 [m-2] + [2] q^{m-2}) b_m \\
  &+ \frac{1}{[n]^\alpha} [m] b_m + [m] q^{m-1} b_{m-1} + [m] q^{m-1} b_{m-1} \\
  &= b_{m+1}, k = 1, 2, \ldots m-1,
\end{align*}
\]

and \(b_m\) does not depend on \(x, t, p, q\). This complete the proof. \(\square\)

From the Lemma 2.2 and Lemma 2.3 we have the following theorem.

**Theorem 2.4.** Let \(m \in \mathbb{N}\) and \(0 < q < p \leq 1\). Then there exits a constant \(C_m > 0\) such that for any \(x \in [0, 1]\), we have

\[
| B_{n,p,q} ((t-x)_p^m; x) | \leq C_m \frac{x(1-x)}{[n]^\frac{m+1}{2}}.
\]

**Corollary 2.5.** Let \(m \in \mathbb{N}\) and \(0 < q < p \leq 1\). Then there exits a constant \(K_m > 0\) such that for any \(x \in [0, 1]\), we have

\[
B_{n,p,q} ((| t-x |)_p^m; x) \leq K_m \frac{x(1-x)}{[n]^\frac{m}{2}}. \tag{2.9}
\]
Proof. For an even \( m \), clearly we have
\[
B_{n,p,q}\left((t - x)^m_{p,q}; x\right) = C_m x(1 - x)^\frac{m+1}{n+1}.
\]

In case if \( m \) is odd, say \( m = 2u + 1 \), we have
\[
B_{n,p,q}\left((t - x)^{2u+1}_{p,q}; x\right) \leq \sqrt{B_{n,p,q}\left((t - x)^{4u+1}_{p,q}; x\right) \sqrt{B_{n,p,q}\left((t - x)^2_{p,q}; x\right)}}
\]
\[
\leq \sqrt{C_{4u} x(1 - x)^\frac{4u+1}{n}} \sqrt{C_2 x(1 - x)^\frac{1}{2}}
\]
\[
= \sqrt{C_{4u} x(1 - x)^\frac{2u}{n}} \sqrt{C_2 x(1 - x)^\frac{1}{2}}
\]
\[
= K_{2u+1} x(1 - x)^\frac{2u+1}{n}.
\]

This complete the proof. \( \square \)

**Theorem 2.6.** Let \( B^{[r]}_{n,p,q}(f; x) \) be an operator from \( C^r[0, 1] \to C^r[0, 1] \). Then for \( 0 < q < p \leq 1 \) there exits a constant \( M(r) \) such that for every \( f \in C^r[0, 1] \), we have
\[
\| B^{[r]}_{n,p,q}(f; x) \|_{C[0,1]} \leq M(r) \sum_{i=0}^{r} \| f^{(i)} \| = M(r) \| f \|_{C^r[0,1]}.
\]

**Proof.** Clearly \( B^{[r]}_{n,p,q}(f; x) \) is continuous on \([0, 1] \). From (1.8) we have
\[
B^{[r]}_{n,p,q}(f; x) = \sum_{i=0}^{r} \frac{(-1)^i}{i!} B_{n,p,q}\left((t - x)^i f^{(i)}(t); x\right).
\]

From the Corollary 2.5, we have
\[
| B_{n,p,q}\left((t - x)^i f^{(i)}(t); x\right) | \leq \| f^{(i)} \| B_{n,p,q}\left((t - x)^i; x\right)
\]
\[
\leq K_i \| f^{(i)} \| [n]^{-\frac{i}{2}}.
\]

Therefore
\[
\| B^{[r]}_{n,p,q}(f; x) \| \leq \sum_{i=0}^{r} \frac{(-1)^i}{i!} \| B_{n,p,q}\left((t - x)^i f^{(i)}(t); x\right) \|
\]
\[
\leq M(r) \sum_{i=0}^{r} \| f^{(i)} \|.
\]

This complete the proof. \( \square \)
3. Convergence properties of $B_{n,p,q}^{[r]}(f; x)$

The modulus of continuity of the derivative $f^{(r)}$ is given by

$$
\omega \left( f^{(r)}; t \right) = \sup \left\{ | f^{(r)}(x) - f^{(r)}(y) | : | x - y | \leq t, \ x, y \in [0, 1] \right\}. \quad (3.1)
$$

**Theorem 3.1.** Let $0 < q < p \leq 1$ and $r \in \mathbb{N} \cup \{0\}$ be a fixed number. Then for $x \in [0, 1], \ n \in \mathbb{N}$ there exits $D_r > 0$ such that for every $f \in C^r[0, 1]$ the following inequality holds

$$
| B_{n,p,q}^{[r]}(f; x) - f(x) | \leq D_r \frac{1}{|n|^2} \omega \left( f^{(r)}; \frac{1}{\sqrt{|n|}} \right). \quad (3.2)
$$

**Proof.** Let $r \in \mathbb{N}$. Then for $f \in C^r[0, 1]$ at a given point $t \in [0, 1]$, we have from the Taylor formula that

$$
f(x) = \sum_{i=0}^{r} \frac{f^{(i)}(t)}{i!} (x - t)^i + \frac{(x - t)^r}{(r - 1)!} \int_0^1 (1 - u)^{r-1} \left( f^{(r)}(t + u(x - t)) - f^{(r)}(t) \right) du.
$$

On applying $B_{n,p,q}^{[r]}(f; x)$, we get

$$
f(x) - B_{n,p,q}^{[r]}(f; x) = \sum_{k=0}^{n} \frac{x - \frac{[k]}{p^{k-n}[n]}}{(r - 1)!} \int_0^1 (1 - u)^{r-1} P_{n,k}(p, q; x)
$$

$$
\times \left[ f^{(r)} \left( \frac{[k]}{p^{k-n}[n]} + u \left( x - \frac{[k]}{p^{k-n}[n]} \right) \right) - f^{(r)} \left( \frac{[k]}{p^{k-n}[n]} \right) \right] du. \quad (3.3)
$$

Now from the definition and properties of modulus of continuity, we have

$$
| f^{(r)} \left( \frac{[k]}{p^{k-n}[n]} + u \left( x - \frac{[k]}{p^{k-n}[n]} \right) \right) - f^{(r)} \left( \frac{[k]}{p^{k-n}[n]} \right) | \leq \omega \left( f^{(r)}; u \left| x - \frac{[k]}{p^{k-n}[n]} \right| \right)
$$

$$
\omega \left( f^{(r)}; u \left| x - \frac{[k]}{p^{k-n}[n]} \right| \right) \leq \left( \sqrt{|n|} \left| x - \frac{[k]}{p^{k-n}[n]} \right| + 1 \right) \omega \left( f^{(r)}; \frac{1}{\sqrt{|n|}} \right). \quad (3.4)
$$

Now for every $0 \leq x \leq 1, \ 0 < q < p \leq 1, \ k \in \mathbb{N} \cup \{0\}, \ n \in \mathbb{N}$ and from (3.3) and (3.4), we get

$$
| B_{n,p,q}^{[r]}(f; x) - f(x) | \leq \frac{1}{r!} \omega \left( f^{(r)}; \frac{1}{\sqrt{|n|}} \right) \sum_{k=0}^{n} \left| x - \frac{[k]}{p^{k-n}[n]} \right|^r \left( \sqrt{|n|} \left| x - \frac{[k]}{p^{k-n}[n]} \right| + 1 \right) P_{n,k}(p, q; x)
$$
\[
= \frac{1}{r!}\omega \left( f^{(r)}; \frac{1}{\sqrt[n]{n}} \right) \left( \sqrt[n]{n} B_{n,p,q} \left( |x-t|^{r+1}; x \right) + B_{n,p,q} \left( |x-t|^r; x \right) \right). \tag{3.5}
\]

Using (3.9) and (3.5) for \( x \in [0, 1] \), we have

\[
|B_{n,p,q}^{[r]}(f; x) - f(x)| \leq \frac{1}{r!}(K_{r+1} + K_r) \left( \frac{1}{\sqrt[n]{n}} \right)^r \omega \left( f^{(r)}; \frac{1}{\sqrt[n]{n}} \right)
= D_r \left( \frac{1}{\sqrt[n]{n}} \right)^r \omega \left( f^{(r)}; \frac{1}{\sqrt[n]{n}} \right).
\]

In order to obtain the uniform convergence of \( B_{n,p,q}^{[r]}(f; x) \) to a continuous function \( f \), we take \( q = q_n \), \( p = p_n \) where \( q_n \in (0, 1) \) and \( p_n \in (q_n, 1] \) satisfying,

\[
\lim_n p_n = 1, \lim_n q_n = 1. \tag{3.6}
\]

**Corollary 3.2.** Let \( p = p_n, q = q_n, 0 < q_n < p_n \leq 1 \) satisfy (3.6) and \( f \in C^r[0, 1] \) for a fixed number \( r \in \mathbb{N} \cup \{0\} \). Then

\[
\lim_{n \to \infty} |n|^\frac{r}{2} \|B_{n,k}^{[r]}(f) - f\| = 0. \tag{3.7}
\]

We say that (cf. [13]) a function \( f \in C^r[0, 1] \) belongs to \( Lip_M(\alpha) \), \( 0 < \alpha \leq 1 \), provided

\[
|f(x) - f(y)| \leq M |x - y|^{\alpha}, \ (x, y \in [0, 1] \text{ and } M > 0). \tag{3.8}
\]

**Corollary 3.3.** Let \( p = p_n, q = q_n, 0 < q_n < p_n \leq 1 \) satisfy (3.6) and \( f \in C^r[0, 1] \) for a fixed number \( r \in \mathbb{N} \cup \{0\} \). If \( f^{(r)} \in Lip_M(\alpha) \) then

\[
\|B_{n,p,q}^{[r]}(f) - f\| = O \left( |n|^{-\frac{r+\alpha}{2}} \right). \tag{3.9}
\]

**Proof.** From (3.2) and (3.8), we have

\[
\|B_{n,p,q}^{[r]}(f) - f\| \leq D_r M \frac{1}{|n|^\frac{r}{2}} \frac{1}{|n|^\frac{r}{2}}.
\]

\[\square\]

**Theorem 3.4.** Let \( 0 < q < p \leq 1 \). Suppose that \( f \in C^{r+2}[0, 1], \) where \( r \in \mathbb{N} \cup \{0\} \) is fixed then we have

\[
\left| B_{n,p,q}^{[r]}(f; x) - f(x) \right| - \frac{(-1)^r f^{(r+1)}(x) B_{n,p,q} \left( (t-x)^{r+1}; x \right)}{(r+1)!} \right.
- \left. \frac{(-1)^r f^{(r+2)}(x) B_{n,p,q} \left( (t-x)^{r+2}; x \right)}{(r+2)!} \bigg| \right| \leq \left( K_{r+2} + K_{r+4} \right) \frac{x(1-x)}{|n|^{\frac{r}{2}+1}} \sum_{i=0}^{r} \frac{1}{i!(r+2-i)!} \omega \left( f^{(r+2-i)}; \frac{1}{|n|^{\frac{1}{2}}} \right).
\]
Proof. Let \( f \in C^{r+2}[0,1] \) and \( x \in [0,1] \) for a fixed number \( r \in \mathbb{N} \cup \{0\} \) we have \( f^{(i)} \in C^{r+2-i}[0,1], \ 0 \leq i \leq r \). Then by Taylor formula we can write

\[
f^{(i)}(t) = \sum_{i=0}^{r+2-i} \frac{f^{(i+j)}(x)}{j!} (t-x)^j + R_{r+2-i}(f; t; x), \tag{3.10}
\]

where

\[
R_{r+2-i}(f; t; x) = \frac{f^{(r+2-i)}(\zeta^{n-k-1}_t) - f^{(r+2-i)}(x)}{(r+2-i)!} (t-x)^{r+2-i},
\]

and

\[
| \zeta_t - x | < | t - x |.
\]

Therefore from (1.8) and (3.10) we have

\[
B_{n,p,q}^{[r]}(f; x) = \sum_{k=0}^{n} P_{n,k}(p, q; x) \sum_{i=0}^{r} \frac{(-1)^i}{i!} \frac{f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x)}{(r+2-i)!} (t-x)^{r+2-i}
\]

\[
= I_1 + I_2, \ \text{where} \ t = \frac{[k]}{p^{k-n}[n]}
\]

Which implies that

\[
| B_{n,p,q}^{[r]}(f; x) - I_1 | = | I_2 |
\]

\[
= \left| \sum_{k=0}^{n} P_{n,k}(p, q; x) \sum_{i=0}^{r} \frac{(-1)^i}{i!} \frac{f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(\zeta_t)}{(r+2-i)!} (t-x)^{r+2-i} \right|
\]

\[
= \left| B_{n,p,q} \left( \sum_{i=0}^{r} \frac{(-1)^i}{i!} \frac{f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x)}{(r+2-i)!} (t-x)^{r+2-i} \right) \right|.
\]

We use the well-known inequality

\[
\omega(f, \lambda \delta) \leq (1 + \lambda^2) \omega(f, \delta),
\]

\[
| f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x) | \leq \omega \left( f^{(r+2-i)}, | \zeta_t - x | \right)
\]

\[
\leq \omega \left( f^{(r+2-i)}, | t - x | \right)
\]

\[
\leq \omega \left( f^{(r+2-i)}, [n]^{-\frac{1}{2}} \right) (1 + [n](t-x)^2).
\]

Hence

\[
| I_2 | \leq B_{n,p,q} \left( \sum_{i=0}^{r} \frac{(-1)^i}{i!} \frac{f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x)}{(r+2-i)!} \right) \ | t - x |^{r+2}, x
\]

\[
\leq B_{n,p,q} \left( \sum_{i=0}^{r} \frac{1}{i!(r+2-i)} \omega \left( f^{(r+2-i)}, [n]^{-\frac{1}{2}} \right) (1 + [n](t-x)^2) | t - x |^{r+2}, x \right)
\]
\[ \sum_{i=0}^{r} \frac{1}{n^{(r+2-i)}} \omega \left( f^{(r+2-i)} \right) \left( t - x \right)^{r+2} \left( t - x \right)^{r+1} \]

\[ \leq \sum_{i=0}^{r} \frac{1}{n^{(r+2-i)}} \omega \left( f^{(r+2-i)} \right) \left( K_{r+2} \left( t - x \right)^{r+2} + K_{r+3} \left( t - x \right)^{r+1} \right) \]

\[ = (K_{r+2} + K_{r+3}) \sum_{i=0}^{r} \frac{1}{n^{(r+2-i)}} \omega \left( f^{(r+2-i)} \right). \]

Therefore

\[ |B_{n,p,q}^{[r]}(f; x) - I_1| \leq (K_{r+2} + K_{r+3}) \frac{x(1-x)}{n^{r+1}} \sum_{i=0}^{r} \frac{1}{i!(r+2-i)!} \omega \left( f^{(r+2-i)} \right). \]

Now we simplify for \( I_1 \)

\[ I_1 = \sum_{k=0}^{n} P_n,k(p, q; x) \sum_{i=0}^{r} \frac{1}{p^{k-n}[n] \cdot i!} \sum_{l=i}^{r} f^{(l)}(x) \left( \frac{k}{p^{k-n}[n]} - x \right)^{l-i} \]

\[ = \sum_{k=0}^{n} P_n,k(p, q; x) \sum_{i=0}^{r} \frac{1}{i!} \sum_{l=i}^{r} f^{(l)}(x) \left( \frac{k}{p^{k-n}[n]} - x \right)^{l} \]

\[ + \sum_{k=0}^{n} P_n,k(p, q; x) \sum_{i=0}^{r} \frac{1}{i!} f^{(r+1)}(x) \left( \frac{k}{p^{k-n}[n]} - x \right)^{r+1} \]

\[ + \sum_{k=0}^{n} P_n,k(p, q; x) \sum_{i=0}^{r} \frac{1}{i!} f^{(r+2)}(x) \left( \frac{k}{p^{k-n}[n]} - x \right)^{r+2} \]

\[ = \sum_{k=0}^{n} P_n,k(p, q; x) \sum_{l=0}^{r} \frac{f^{(l)}(x)}{l!} \left( \frac{k}{p^{k-n}[n]} - x \right)^{l} \sum_{i=0}^{r} \left( \frac{l}{i} \right) (-1)^i \]

\[ + \frac{f^{(r+1)}(x)}{(r+1)!} \sum_{k=0}^{n} P_n,k(p, q; x) \left( \frac{k}{p^{k-n}[n]} - x \right)^{r+1} \sum_{i=0}^{r} \left( \frac{r+1}{i} \right) (-1)^i \]

\[ + \frac{f^{(r+2)}(x)}{(r+2)!} \sum_{k=0}^{n} P_n,k(p, q; x) \left( \frac{k}{p^{k-n}[n]} - x \right)^{r+2} \sum_{i=0}^{r} \left( \frac{r+2}{i} \right) (-1)^i. \]

For \( n \in \mathbb{N}, \ r \in \mathbb{N} \cup \{0\} \) we have

\[ \sum_{i=0}^{r} \left( \frac{r+1}{i} \right) (-1)^i = (-1)^r, \ \sum_{i=0}^{r} \left( \frac{r+2}{i} \right) (-1)^i = (r+1)(-1)^r. \]

Therefore

\[ I_1 = f(x) + \frac{(-1)^r f^{(r+1)}(x) B_{n,p,q} \left((t-x)^{r+1}; x\right)}{(r+1)!} \]

\[ + \frac{(-1)^r f^{(r+2)}(x) B_{n,p,q} \left((t-x)^{r+2}; x\right)}{(r+2)!}. \]

This complete the proof. \( \square \)
Corollary 3.5. Let \( p = p_n, \ q = q_n, \ 0 < q_n < p_n \leq 1 \) satisfy (3.6) and \( f \in C^2[0,1] \) for a fixed number \( r \in \mathbb{N} \cup \{0\} \). Then for every \( x \in [0,1] \) we have

\[
\left| B_{n,p_n,q_n}^{[r]}(f;x) - f(x) - \frac{f''(x)}{2} x(1-x) \right| \leq K \omega \left( f'', [n]^{-\frac{4}{2}} \right),
\]

where \( K = \frac{K_2 + K_4}{2} \). Moreover,

\[
\lim_{n \to \infty} \omega \left( f''^{[n]} - f''(x) \right) = \frac{x(1-x)}{2} f''(x)
\]

uniformly on \([0,1]\).

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