Let $F^*(K)$ be the set of all fuzzy complex numbers. In this paper some classical and measure-theoretical notions are extended to the case of complex fuzzy sets. They are fuzzy complex number-valued distance on $F^*(K)$, fuzzy complex number-valued measure on $F^*(K)$, and some related notions, such as null-additivity, pseudo-null-additivity, null-subtraction, pseudo-null-subtraction, autocontinuous from above, autocontinuous from below, and autocontinuity of the defined fuzzy complex number-valued measures. Properties of fuzzy complex number-valued measures are studied in detail.

1. Introduction

It is well known that additivity of a classical measure primly depicted measure problems under error-free condition. But when measure error was unavoidable, additivity could not fully depict the measure problems under certain condition. To overcome such difficulties, fuzzy measure has been developed. Research on fuzzy measures was very deep in those aspects: research based on a certain number of subsets of a classic set and a real value nonaddable measure (such as Choquet’s content theory [1], Sugeno’s measure theory [2]), research based on fuzzy sets and a real value measure (e.g., Zadeh’s addable measure [3]), and especially the research on fuzzy value measures which generalizes the set value measure theory.

Being a newly developing theory developed in the later 1960s, set value measure had been applied in many fields [4–6]. After the appearance of fuzzy numbers, people naturally thought of related measure and integral. In 1986 Zhang [7] defined a kind of fuzzy set measure on $R^n$, in 1998 Wu et al. generalized the codomain of fuzzy measure to fuzzy real number field and defined the Sugeno integral of fuzzy number fuzzy measure [8], and Guo et al. also defined the G fuzzy value measure integral of fuzzy value function [9] which generalized the Sugeno integral about fuzzy value fuzzy measure to fuzzy set [10]. In 1989, Buckley presented the concept of fuzzy complex number [11] which inspired that people needed to consider the measure and integral problem about fuzzy complex number.

At the beginning of the 90s, Guang-Quan [12–21] introduced fuzzy real distance and discussed the fuzzy real measure based on fuzzy sets and then gave the fuzzy real value integral and established fuzzy real valued measure theory on fuzzy set space. During 1991-1992, Buckley and Qu [22, 23] studied the problems of fuzzy complex analysis: fuzzy complex function differential and fuzzy complex function integral. During 1996–2001, Qiu et al. studied serially basic problems of fuzzy complex analysis theory, including the continuity of fuzzy complex numbers and fuzzy complex valued series [24], fuzzy complex valued functions and their differentiability [25], and fuzzy complex valued measure and fuzzy complex valued integral function [26, 27]. Wang and Li [28] gave the fuzzy complex valued measure based on the fuzzy complex number concept of Buckley, studied Lebesgue integral of fuzzy complex valued function, and obtained some important results.

As for applications of fuzzy complex number theory, Ramot et al. [29, 30] studied complex fuzzy sets and complex fuzzy logic, Dick [31] studied fuzzy complex logic more profoundly, Ha et al. [32] applied fuzzy complex set in statistical
learning theory and obtained a key theorem of statistical learning theory, Fu and Shen [33] studied modeling problems of fuzzy complex number, and Fu and Shen [34] applied fuzzy complex in pattern recognition and classification and obtained important results. Please see [35–37] for other applications.

This paper will extend the classical measure to fuzzy complex number-valued measure, which can better express the interactions among the attributes (cf. [32, 34–37]) and, thus, is expected to have extensive applications in information fusion technology, classification technology, machine learning, pattern recognition, and other fields. Section 2 is some preliminary notions (including fuzzy complex number, real fuzzy distance between two fuzzy real numbers, and two fuzzy complex numbers) and some basic operations and order relation of fuzzy complex. Section 3 is prepared for the next section. We defined, based on Ha’s work [38], the concepts of fuzzy complex distance and complex fuzzy set value complex fuzzy measure (an extension of fuzzy measure) on fuzzy complex number field. We also present, based on Zhang’s work [21], the concepts of null-additivity, pseudo-null-additivity, null-subtraction, pseudo-null-subtraction, autocontinuous from above, autocontinuous from below, and autocontinuity of fuzzy complex value fuzzy complex measure on complex fuzzy number set (this measure has the properties PGP and SA/SB). In Section 4, we deduced some important properties on complex fuzzy set value complex fuzzy measure which are generalizations of the corresponding results in measure theory; we also obtain some results on related integral theory.

2. Preliminaries

2.1. Fuzzy Complex Numbers. In this paper, R is the set of all real numbers set, K is the set of all complex numbers, X is an ordinary set, F*(R) is the set of all real fuzzy numbers on R, Δ(R) is the set of all interval numbers, (X, 𝒦) is a measurable space (thus 𝒦 is a σ-algebra), and F*(K) is the set of all fuzzy complex numbers on K. Let F*+(K) = {̃a ∈ 𝐹(𝐾) | ̃a ≥ 0, ̃a ∈ F*(R)} and let F*(K) = {̃a + ĩb | ̃a, ̃b ∈ F*+(R), i = \sqrt{-1}}.

Definition 1 (see [12]). Let ̃a, ̃b ∈ F*(R). Then the mapping (̃a, ̃b) : K → [0, 1] defined by (̃a, ̃b)(x + iy) = ̃a(x) ∧ ̃b(y) is a fuzzy complex number, where ̃a is called the real part of (̃a, ̃b) (written as Re(̃a, ̃b)) and ̃b is called the imaginary part (written as Im(̃a, ̃b)), and i = \sqrt{-1}. One will identify (̃a, 0) with ̃a and, thus, think fuzzy complex numbers are an extension of fuzzy real numbers. The set of all fuzzy complex numbers on K is denoted by F*(K).

For any subsets A, B of R, write (A, B) ≡ A + iB = {x + iy | x ∈ A, y ∈ B}. The operation ∗ ∈ {+, −, ·} is described as follows:

\[ a_{1}^{∗} \text{ or } b_{1}^{∗} \] (written as Re(a_{1}^{∗} b_{1}^{∗})) for any a_{1}^{∗}, b_{1}^{∗} ∈ F*(K).

(2) c · c^{∗} = (a Re c^{∗} b Im c^{∗}) for any c_{1}^{∗} ∈ F*(K) and c = (a, b) ∈ K (c = a + ib, a, b ∈ R, i = \sqrt{-1}).

c’ ∈ F*(K) is said to be a fuzzy infinity [21] (written as ̃∞) if one of the supports of ̃a = Re c’ and ̃b = Im c’ is an unbound set. For any c_{1}^{∗}, c_{2}^{∗} ∈ F*(K), one makes the following appointments:

\[
\begin{align*}
&c_{1}^{∗} \leq c_{2}^{∗} \text{ if and only if } Re c_{1}^{∗} \leq Re c_{2}^{∗} \text{ and } Im c_{1}^{∗} \leq Im c_{2}^{∗}; \\
&c_{1}^{∗} = c_{2}^{∗} \text{ if and only if } c_{1}^{∗} \leq c_{2}^{∗} \text{ and } c_{2}^{∗} \leq c_{1}^{∗}; \\
&c_{1}^{∗} < c_{2}^{∗} \text{ if and only if } c_{1}^{∗} \leq c_{2}^{∗} \text{ and } Re c_{1}^{∗} \leq Re c_{2}^{∗} \text{ or } Im c_{1}^{∗} < Im c_{2}^{∗}; \\
&c_{1}^{∗} \geq 0 \text{ if and only if } Re c_{1}^{∗} \geq 0, \text{ Im } c_{1}^{∗} \geq 0.
\end{align*}
\]

One uses 𝒦′ to denote a family (which is obviously non-empty) of subsets of F*(K) that satisfies the following conditions:

(1) for each A ∈ 𝒦′, if B = \{inf ̃A_{0} | ̃A_{0} ⊂ A\} has upper bound, then sup ̃B ∈ F*(K);
(2) for each A ∈ 𝒦′, if ̃C = \{sup ̃A_{0} | ̃A_{0} ⊂ A\} has lower bound, then sup ̃B ∈ F*(K).

2.2. Fuzzy Distance of Fuzzy Numbers

Definition 2 (see cf. [21]). A mapping ̃ρ : F*(R) × F*(R) → F*(R) satisfying the following conditions is called a fuzzy metric or a fuzzy distance on F*(R):

(1) for any ̃a, ̃b ∈ F*(R), ̃ρ(̃a, ̃b) ≥ 0, and ̃ρ(̃a, ̃b) = 0 if and only if ̃a = ̃b;
(2) for any ̃a, ̃b ∈ F*(R), ̃ρ(̃a, ̃b) = ̃ρ(̃b, ̃a);
(3) for any ̃a, ̃b, ̃c ∈ F*(R), ̃ρ(̃a, ̃b) ≤ ̃ρ(̃a, ̃c) + ̃ρ(̃c, ̃b).

̃ρ(̃a, ̃b) is called the fuzzy distance of fuzzy real numbers ̃a and ̃b.

Example 3. The mapping ̃ρ : F*(R) × F*(R) → F*(R) defined by

\[
\begin{align*}
\tilde{\rho}(\tilde{a}, \tilde{b}) &= \bigcup_{\lambda \in [0, 1]} \lambda \left[ |a_{\lambda} - b_{\lambda}| + |a_{\lambda} - b_{\lambda}^{*}| \right] \\
&= \lambda \left[ |a_{\lambda} - b_{\lambda}| + |a_{\lambda} - b_{\lambda}^{*}| \right]
\end{align*}
\]

(1) is a fuzzy distance on F*(R).

Remark 4. Analogously, a mapping ̃ρ : F*(K) × F*(K) → F*(R) satisfying the following conditions is called a fuzzy metric or a fuzzy distance on F*(K):

(1) for any ̃a, ̃b ∈ F*(K), ̃ρ(̃a, ̃b) ≥ 0, and ̃ρ(̃a, ̃b) = 0 if and only if ̃a = ̃b;
(2) for any ̃a, ̃b ∈ F*(K), ̃ρ(̃a, ̃b) = ̃ρ(̃b, ̃a);
(3) for any ̃a, ̃b, ̃c ∈ F*(K), ̃ρ(̃a, ̃b) ≤ ̃ρ(̃a, ̃c) + ̃ρ(̃c, ̃b).

̃ρ(̃a, ̃b) is called the fuzzy distance of fuzzy complex numbers ̃a and ̃b.
Example 5. Let $\tilde{\rho}$ be a fuzzy distance on $F^*(R)$. Then the mapping $\rho': F^*(K) \times F^*(K) \rightarrow F^*(R)$ defined by

$$\rho'(c_1', c_2') = \tilde{\rho}(\text{Re } c_1', \text{Re } c_2') \vee \tilde{\rho}(\text{Im } c_1', \text{Im } c_2')$$  \hspace{1cm} (2)

is a fuzzy distance on $F^*(K)$.

Definition 6 (see cf. [12]). Let $\{c'_n\} \subset F^*(K)$ and let $c' \in F^*(K)$. $\{c'_n\}$ is said to converge to $c'$ according to a fuzzy metric $\tilde{\rho}'$ on $F^*(K)$ (written as $\lim_{n \rightarrow \infty} c'_n = c'$) if, for each $\epsilon > 0$, there exists a positive integer $N$ such that $\rho'(c'_n, c') < \epsilon$ for all $n \geq N$.

3. Complex Fuzzy Set-Valued Complex Fuzzy Measures

The notion of complex fuzzy measure on family of classical sets was given in [26].

Definition 7 (see [26]). Let $R^\wedge = [0, +\infty]$ and let $C^\wedge = \{x + iy \mid x, y \in R^\wedge\}$. A fuzzy measure on a $\sigma$-algebra $A$ composed of subsets of $X$ is a mapping $\mu : A \rightarrow C^\wedge$ which satisfies the following conditions:

1. $\mu(\emptyset) = 0$;
2. if $A \subset B$, then $|\mu(A)| \leq |\mu(B)|$;
3. if $\{A_n\}_{n=1}^{\infty}$ is a monotonically non-increasing sequence of $\sigma$-measurable sets, then $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n \rightarrow \infty} \mu(A_n)$;
4. if $\{A_n\}_{n=1}^{\infty}$ is a monotonically increasing sequence of $\sigma$-measurable sets and $|\mu(A_n)| < +\infty$ for some $n_0$, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n \rightarrow \infty} \mu(A_n)$.

In this paper we need an expansion of this notion. First we defined the concept of fuzzy complex value distance.

Definition 8. A mapping $\tilde{\rho} : F^*(K) \times F^*(K) \rightarrow F^*(K)$ satisfying the following conditions is called a fuzzy complex value metric or a fuzzy complex value distance on $F^*(K)$:

1. for any $\tilde{a}, \tilde{b} \in F^*(K)$, $\tilde{\rho}(\tilde{a}, \tilde{b}) \geq 0$, and $\tilde{\rho}(\tilde{a}, \tilde{b}) = 0$ if and only if $\tilde{a} = \tilde{b}$;
2. for any $\tilde{a}, \tilde{b} \in F^*(K)$, $\tilde{\rho}(\tilde{a}, \tilde{b}) = \tilde{\rho}(\tilde{b}, \tilde{a})$;
3. for any $\tilde{a}, \tilde{b}, \tilde{c} \in F^*(K)$, $\tilde{\rho}(\tilde{a}, \tilde{b}) \leq \tilde{\rho}(\tilde{a}, \tilde{c}) + \tilde{\rho}(\tilde{c}, \tilde{b})$.

$\tilde{\rho}(\tilde{a}, \tilde{b})$ is called the fuzzy complex value distance of fuzzy complex numbers $\tilde{a}$ and $\tilde{b}$.

Remark 9. It can be easily seen that a mapping $\tilde{\rho} : F^*(K) \times F^*(K) \rightarrow F^*(K)$ is a fuzzy complex value metric on $F^*(K)$ if and only if $\tilde{\rho} = \tilde{\rho}_1 + i\tilde{\rho}_2$ for some two fuzzy metrics $\tilde{\rho}_1$ and $\tilde{\rho}_2$ on $F^*(K)$.

Definition 10. Let $\{\tilde{Z}_n\} \subset F^*(K), \tilde{Z} \in F^*(K)$, and $\tilde{\rho} : F^*(K) \times F^*(K) \rightarrow F^*(K)$ be a fuzzy complex value distance, and let $\tilde{\rho}(\tilde{Z}_n, \tilde{Z}) = \tilde{\rho}_1(\tilde{Z}_n, \tilde{Z}) + i\tilde{\rho}_2(\tilde{Z}_n, \tilde{Z})$ $(i = \sqrt{-1})$. If for each $\epsilon > 0$, there exists a positive integer $N$ such that $\tilde{\rho}_1(\tilde{Z}_n, \tilde{Z}) < \epsilon$ and $\tilde{\rho}_2(\tilde{Z}_n, \tilde{Z}) < \epsilon$ for all $n \geq N$ hold, then $\{\tilde{Z}_n\}$ is said to be convergent to $\tilde{Z}$ according to distance $\tilde{\rho}$, denoted by $(\tilde{\rho})\lim_{n \rightarrow \infty} \tilde{Z}_n = \tilde{Z}$.

Definition 11. Let $\tilde{\mu}$ be a fuzzy complex measure on $\sigma$-algebra $\mathcal{F}$ is said to have property (PGP) if, for each $\epsilon = \epsilon_1 + i\epsilon_2 > 0$, there exists a $\delta = \delta_1 + i\delta_2 > 0$ such that $|\tilde{\mu}(E) - \tilde{\mu}(F)| < \epsilon$ whenever $|\tilde{\mu}(E) - \tilde{\mu}(F)| < \delta$.
(\bar{\rho})\lim_{n} \bar{\mu}(\bar{B}_{n}) = 0$, there exists a subsequence \( \{\bar{B}_{n}\} \) of \( \{\bar{B}_{n}\} \) such that \( \bar{\mu}(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \bar{B}_{n}) = 0 \). It is said to have property (S/B) if \( (\bar{\rho})\lim_{n} \bar{\mu}(\bar{B}_{n}) = 0 \) for any \( \{\bar{B}_{n}\} \subset \mathcal{F} \).

### 4. Main Results

Let \( X \) be a set and \( F(X) \) the set of all fuzzy sets on \( X \). Then a subfamily \( \mathcal{F} \subset F(X) \) is admissible if and only if it satisfies the following conditions (see [21]):

1. \( X \in \mathcal{F} \);
2. if \( \overline{A}, \overline{B} \in \mathcal{F} \), then \( \overline{A} \cap \overline{B}, \overline{A} \cup \overline{B} \in \mathcal{F} \),

where \( (\overline{A} \cap \overline{B})(x) = \min(1, \overline{A}(x) + \overline{B}(x)) \) and \( (\overline{A} \cup \overline{B})(x) = \max(0, \overline{A}(x) - \overline{B}(x)) \) (for all \( x \in X \)).

We first have the following result.

**Theorem 15.** Every fuzzy complex measure \( \bar{\mu} \) on a admissible class \( \mathcal{F} \subset F(Z) \) is a complex fuzzy value fuzzy complex measure on \( \mathcal{F} \).

**Proof.** We only prove the upper continuity and lower continuity of \( \bar{\mu} \). Suppose \( \{\overline{A}_{n}\} \subset \mathcal{F} \subset F(Z) \), \( \overline{A}_{n} \cap \bigcap_{n=1}^{\infty} \overline{A}_{n} \in \mathcal{F} \), and \( \bar{\mu}(\overline{A}_{n}) \neq \overline{\infty} \) for some \( n_{0} \). By monotonicity of \( \bar{\mu} \), we have

\[
0 \leq \Re \bar{\mu}(\overline{A}_{n}) \leq \Re \bar{\mu}(\overline{A}_{n_0}) \quad \text{and} \quad 0 \leq \Im \bar{\mu}(\overline{A}_{n}) \leq \Im \bar{\mu}(\overline{A}_{n_0})
\]

for any \( n \geq n_0 \). Since \( \overline{A}_{n_0} \cap \overline{A}_{n} \neq \overline{\infty} \), \( (\bigcap_{n=n_0}^{\infty} \overline{A}_{n}) \), we have

\[
\Re \bar{\mu} \left( \bigcap_{n=n_0}^{\infty} \overline{A}_{n} \right) = (\bar{\rho}) \lim_{n} \Re \bar{\mu} \left( (\overline{A}_{n_0} \cap \overline{A}_{n}) \cap \overline{A}_{n} \right)
\]

\[
= (\bar{\rho}) \lim_{n} \Re \bar{\mu} \left( \overline{A}_{n_0} \cap \overline{A}_{n} \right) + (\bar{\rho}) \lim_{n} \Re \bar{\mu} \left( \overline{A}_{n} \right)
\]

\[
= \Re \bar{\mu} \left( \overline{A}_{n_0} \cap \left( \bigcap_{n=n_0}^{\infty} \overline{A}_{n} \right) \right) + (\bar{\rho}) \lim_{n} \Re \bar{\mu} \left( \overline{A}_{n} \right).
\]

(3)

Thereby

\[
\Re \bar{\mu} \left( \overline{A}_{n_0} \right) + \Re \bar{\mu} \left( \bigcap_{n=n_0}^{\infty} \overline{A}_{n} \right)
\]

\[
= \Re \bar{\mu} \left( \overline{A}_{n_0} \cap \left( \bigcap_{n=n_0}^{\infty} \overline{A}_{n} \right) \right) + (\bar{\rho}) \Re \bar{\mu} \left( \overline{A}_{n} \right).
\]

(4)

It follows that \( (\bar{\rho})\lim_{n} \Re \bar{\mu}(\overline{A}_{n}) = \Re \bar{\mu}(\bigcap_{n=n_0}^{\infty} \overline{A}_{n}) \). Similarly, we can prove \( (\bar{\rho})\lim_{n} \Im \bar{\mu}(\overline{A}_{n}) = \Im \bar{\mu}(\bigcap_{n=n_0}^{\infty} \overline{A}_{n}) \), which means that \( \bar{\mu} \) is upper continuous.

Assume \( \{\overline{B}_{n}\} \subset \mathcal{F} \subset F(Z) \) and \( \overline{B}_{n} \supset \bigcap_{n=1}^{\infty} \overline{B}_{n} \in \mathcal{F} \). Then \( \bigcup_{n=1}^{\infty} \overline{B}_{n} \cap \overline{B}_{n} \in \mathcal{F} \). Theorem 15. Every fuzzy complex measure \( \bar{\mu} \) on a admissible class \( \mathcal{F} \subset F(Z) \) is a complex fuzzy value fuzzy complex measure on \( \mathcal{F} \).

**Theorem 16.** Every complex fuzzy set-value complex fuzzy measure \( \bar{\mu} \) on a fuzzy \( \sigma \)-algebra \( \mathcal{F} \subset F(Z) \) is exhaustive.

**Proof.** Suppose \( \{\overline{A}_{n}\} \subset \mathcal{F} \) is a disjoint sequence; then

\[
\bigcup_{n=1}^{\infty} \overline{A}_{n} \cap \bigcap_{n=1}^{\infty} \overline{A}_{n} = \emptyset.
\]

(6)

Assume \( \bigcup_{n=1}^{\infty} \bigcap_{n=1}^{\infty} \overline{A}_{n} \neq \emptyset \); then \( \Re(\bigcup_{n=1}^{\infty} \bigcap_{n=1}^{\infty} \overline{A}_{n}(z_0)) > 0 \) for some \( z_0 \in Z \) or \( \Im(\bigcup_{n=1}^{\infty} \bigcap_{n=1}^{\infty} \overline{A}_{n}(z_0)) > 0 \) for some \( z_0 \in Z \); that is, \( \Re(\bigcup_{n=1}^{\infty} \bigcap_{n=1}^{\infty} \overline{A}_{n}(z_0)) > 0 \) for all \( n \geq 1 \) or \( \Im(\bigcup_{n=1}^{\infty} \bigcap_{n=1}^{\infty} \overline{A}_{n}(z_0)) > 0 \) for all \( n \geq 1 \). Without loss of generality, we assume the first. Then there are two distinct indices \( k_{n_1} \) and \( k_{n_2} \) such that \( \overline{A}_{k_{n_1}}(z_0) > 0 \) and \( \overline{A}_{k_{n_2}}(z_0) > 0 \) (and, thus, \( \Re(\bigcup_{n=1}^{\infty} \bigcap_{n=1}^{\infty} \overline{A}_{n}(z_0)) > 0 \), which conflicts with the fact that \( \{\overline{A}_{n}\} \) is a disjoint sequence. Therefore \( \bigcup_{k=n_{1}}^{\infty} \overline{A}_{k} \cap \bigcap_{n=1}^{\infty} \bigcap_{n=1}^{\infty} \overline{A}_{n} = \emptyset \) holds. Thus

\[
(\bar{\rho}) \lim_{n} \Re \bar{\mu} \left( \overline{A}_{n} \right) \leq (\bar{\rho}) \lim_{n} \Re \bar{\mu} \left( \bigcap_{k=n_{1}}^{\infty} \overline{A}_{k} \right)
\]

\[
= \Re (\emptyset) = 0,
\]

(7)

\[
(\bar{\rho}) \lim_{n} \Im \bar{\mu} \left( \overline{A}_{n} \right) \leq (\bar{\rho}) \lim_{n} \Im \bar{\mu} \left( \bigcap_{k=n_{1}}^{\infty} \overline{A}_{k} \right)
\]

\[
= \Im (\emptyset) = 0.
\]

Then we get \( (\bar{\rho})\lim_{n} \Re \bar{\mu}(\overline{A}_{n}) \leq 0 \).

On the other hand, since \( \bar{\mu} \) is complex fuzzy set-value complex fuzzy measure on the fuzzy \( \sigma \)-algebra \( \mathcal{F} \), \( (\bar{\rho})\lim_{n} \Re \bar{\mu}(\overline{A}_{n}) \geq 0 \), so \( (\bar{\rho})\lim_{n} \Re \bar{\mu}(\overline{A}_{n}) = 0 \).
Theorem 17. If \( \mu(\mathcal{F}) = \{\mu(A) \mid A \in \mathcal{F}\} \in \mathcal{A}^* \), then

\[
\begin{aligned}
\text{Re} \, \mu \left( \lim_{n \to \infty} A_n \right) &\leq \langle \rho \rangle \lim_{n \to \infty} \text{Re} \, \mu \left( A_n \right) \\
&\leq \langle \rho \rangle \lim_{n \to \infty} \text{Re} \, \mu \left( A_n \right) \\
&\leq \text{Re} \, \mu \left( \lim_{n \to \infty} A_n \right), \\
\text{Im} \, \mu \left( \lim_{n \to \infty} A_n \right) &\leq \langle \rho \rangle \lim_{n \to \infty} \text{Im} \, \mu \left( A_n \right) \\
&\leq \langle \rho \rangle \lim_{n \to \infty} \text{Im} \, \mu \left( A_n \right) \leq \text{Im} \, \mu \left( \lim_{n \to \infty} A_n \right). 
\end{aligned}
\]

Proof. Since \( \bigcap_{k=1}^{\infty} A_n \neq \emptyset \) about \( k \), \( \text{Re} \, \mu \left( \bigcap_{k=1}^{\infty} A_n \right) = \langle \rho \rangle \lim_{n \to \infty} \text{Re} \, \mu \left( A_n \right) \) and \( \text{Im} \, \mu \left( \bigcap_{k=1}^{\infty} A_n \right) = \langle \rho \rangle \lim_{n \to \infty} \text{Im} \, \mu \left( A_n \right) \).

As \( \text{Re} \, \mu \left( \bigcap_{k=1}^{\infty} A_n \right) \leq \text{Re} \, \mu \left( A_n \right) \) and \( \text{Im} \, \mu \left( \bigcap_{k=1}^{\infty} A_n \right) \leq \text{Im} \, \mu \left( A_n \right) \) for all \( n \geq k \), \( \text{Re} \, \mu \left( \bigcap_{k=1}^{\infty} A_n \right) \leq \text{Re} \, \mu \left( A_n \right) \) and \( \text{Im} \, \mu \left( \bigcap_{k=1}^{\infty} A_n \right) \leq \text{Im} \, \mu \left( A_n \right) \). Therefore

\[
\begin{aligned}
\text{Re} \, \mu \left( \lim_{n \to \infty} A_n \right) &= \text{Re} \, \mu \left( \bigcup_{k=1}^{\infty} A_n \right) \\
&\leq \langle \rho \rangle \lim_{k \to \infty} \text{Re} \, \mu \left( A_n \right) \\
&= \langle \rho \rangle \lim_{n \to \infty} \text{Re} \, \mu \left( A_n \right) \\
&\leq \langle \rho \rangle \lim_{n \to \infty} \text{Im} \, \mu \left( A_n \right) \\
&= \langle \rho \rangle \lim_{n \to \infty} \text{Im} \, \mu \left( A_n \right) ,
\end{aligned}
\]

which means \( \text{Re} \, \mu \left( \lim_{n \to \infty} A_n \right) \leq \langle \rho \rangle \lim_{n \to \infty} \text{Re} \, \mu \left( A_n \right) \).

Similarly, \( \text{Im} \, \mu \left( \lim_{n \to \infty} A_n \right) \leq \langle \rho \rangle \lim_{n \to \infty} \text{Im} \, \mu \left( A_n \right) \). \( \square \)

From properties of upper limits and lower limits we can see the following theorem holds.

Theorem 18. Let \( \mu : F(Z) \to F^*_+(K) = \{\mu, iB : \mu, B \in F^*_+(R), i = \sqrt{-1} \} \) and \( E \in F(Z) \). If \( \mu \) is 0-addable and upper continuous on \( F(Z) \), then, for any \( \{A_n\} \subset F(Z) \) satisfying \( A_{n+1} \supseteq A_n \) \((n = 1, 2, \ldots)\), \( \langle \rho \rangle \lim \mu \left( A_n \right) = 0 \) and \( \mu \left( E \cup A_{n_0} \right) \neq 0 \), then

\[
\begin{aligned}
\langle \rho \rangle \lim \mu \left( E \cup A_n \right) &= \mu \left( E \cup A_n \right) \\
&= \mu \left( E \cup A_n \right) = \mu \left( E \cup A_n \right) .
\end{aligned}
\]

Proof. Let \( A = \bigcap_{n=1}^{\infty} A_n \). Since \( \mu \) is upper continuous,

\[
\text{Re} \, \mu \left( A \right) = \text{Re} \, \mu \left( \bigcap_{n=1}^{\infty} A_n \right) = \langle \rho \rangle \lim \text{Re} \, \mu \left( A_n \right) = 0,
\]

\[
\text{Im} \, \mu \left( A \right) = \text{Im} \, \mu \left( \bigcap_{n=1}^{\infty} A_n \right) = \langle \rho \rangle \lim \text{Im} \, \mu \left( A_n \right) = 0.
\]

Hence \( \langle \rho \rangle \lim \mu \left( E \cup (A \cap B_n) \right) = \mu \left( E \right) \). \( \square \)
Similarly, we have the following.

**Theorem 21.** Suppose that $A \in \mathcal{F} \subset F(Z)$, $\mu$ is a complex fuzzy set-value complex fuzzy measure on $F(Z)$ which is pseudo-zero subtractable about $A$, and $\mu(A) \neq \emptyset$. If $(\bar{\mu})\lim_n \mu(\bar{A} \cap \bar{B}_n) = \bar{\mu}(\bar{A})$ for any $\{\bar{B}_n\} \subset \mathcal{F}$ satisfying $\bar{B}_n \uparrow \bar{A}$, then $(\bar{\mu})\lim_n \mu(\bar{C} \cap \bar{B}_n) = \bar{\mu}(\bar{C})$ for any $\bar{C} \in \bar{A} \cap \bar{F}$.

**Theorem 22.** Let $\bar{\mu}$ be a complex fuzzy set-value complex fuzzy measure on a fuzzy algebra $\mathcal{F}$ of $F(Z)$, where $\mu(\mathcal{F}) = \{\mu(A) : A \in \mathcal{F}\} \subset A^\ast$. If $\mu$ is autoc. 1, then $\bar{\mu}$ possesses (P.G.P) property.

**Proof.** Suppose that $\bar{\mu}$ does not possess (P.G.P) property; then there exists an $\varepsilon_0 = \varepsilon_0' + i\varepsilon_0''$ (where $\varepsilon_0' , \varepsilon_0''$ are positive real numbers) such that, for any natural numbers $n, m$, there exist $\{\bar{E}_n\}, \{\bar{F}_n\} \subset \mathcal{F}$ such that

\[ \text{Re} \bar{\mu}(\bar{E}_n) \lor \text{Re} \bar{\mu}(\bar{F}_n) < \frac{1}{n}, \]
\[ \text{Im} \bar{\mu}(\bar{E}_n) \lor \text{Im} \bar{\mu}(\bar{F}_n) < \frac{1}{m}, \]
\[ \text{Re} \bar{\mu}(\bar{E}_n \lor \bar{F}_n) \not< \varepsilon_0', \quad \text{Im} \bar{\mu}(\bar{E}_n \lor \bar{F}_n) \not< \varepsilon_0''. \]  

Thus $\bar{\mu}(\bar{E}_n \lor \bar{F}_n) \not< \varepsilon_0$ and, thus, $(\bar{\mu})\lim_n \text{Re} \bar{\mu}(\bar{E}_n) = (\bar{\mu})\lim_n \text{Im} \bar{\mu}(\bar{F}_n) = 0$ and $(\bar{\mu})\lim_n \text{Re} \bar{\mu}(\bar{F}_n) = (\bar{\mu})\lim_n \text{Im} \bar{\mu}(\bar{F}_n) = 0$. From upper autocontinuity of $\bar{\mu}$, we have

\[ (\bar{\mu})\lim_n \text{Re} \bar{\mu}(\bar{E}_n \lor \bar{F}_n) = 0, \quad (\bar{\mu})\lim_n \text{Im} \bar{\mu}(\bar{E}_n \lor \bar{F}_n) = 0. \]

Therefore $\bar{\mu}(\bar{E}_n \lor \bar{F}_n) < \varepsilon_0$ for some $n_0 \geq 1$. This conflicts with the hypothesis.

**Theorem 23.** Suppose that $\bar{\mu}$ possesses (P.G.P) property. If $\{\bar{E}_n\} \subset \mathcal{F}$ and $(\bar{\mu})\lim_n \bar{\mu}(\bar{E}_n) = 0$, then there exists a sequence $\{\delta_n\}$ of real numbers satisfying $\delta_n > 0$ (for all $n$) and $\delta_n \rightarrow 0$ and a subsequence $\{\bar{E}_{n_k}\}$ of $\{\bar{E}_n\}$ such that $\mu(\bigcup_{k=1}^{\infty} \bar{E}_{n_k}) < \delta_k$ (for all $k \geq 1$). Furthermore, $\mu$ possesses (S.A.P) property.

**Proof.** For any real numbers $\varepsilon', \varepsilon'', \delta', \delta'' > 0$, let $\varepsilon = \varepsilon' + i\varepsilon''$, $\delta_1 = \delta'_1 + i\delta''_1$, $\delta_1 \in (0, \varepsilon')$, and $\delta_1'' \in (0, \varepsilon'')$. Since $\mu$ possesses (P.G.P) property, there exists a $\delta_1 \in (0, \varepsilon)$ such that $\text{Re} \bar{\mu}(\bar{E} \lor \bar{F}) < \varepsilon'$ and $\text{Im} \bar{\mu}(\bar{E} \lor \bar{F}) < \varepsilon''$. Since $\bar{\mu}(\bar{E} \lor \bar{F}) < \varepsilon'$, there exists an $n_1$ such that $\text{Re} \bar{\mu}(\bar{E}_n \lor \bar{F}) < \varepsilon'$, and $\bar{\mu}(\bar{E}_n \lor \bar{F}) < \varepsilon''$. Therefore $\bar{\mu}(\bar{E}_n \lor \bar{F}) < \varepsilon$. Since $\mu$ possesses (P.G.P) property, there exists a $\delta_2 \in (0, \delta_1' + i\delta_1'')$ such that $\delta_2_1 \in (0, \delta_1' + i\delta_1'')$ and $\delta_2'_1 \in (0, \delta_1' + \varepsilon'/2)$, $\delta_2''_1 \in (0, \delta_1'' + \varepsilon''/2)$. Similarly, $\text{Re} \bar{\mu}(\bar{E} \lor \bar{F}) < \delta'_1$ and $\text{Im} \bar{\mu}(\bar{E} \lor \bar{F}) < \delta''_1$. Therefore $\bar{\mu}(\bar{F}) < \delta'_1$ and $\bar{\mu}(\bar{F}) < \delta''_1$. Hence $\text{Re} \bar{\mu}(\bar{E}_n \lor \bar{F}) < \delta'_1$, $\bar{\mu}(\bar{E}_n \lor \bar{F}) < \delta''_1$, and $\text{Re} \bar{\mu}(\bar{E}_n \lor \bar{F}) < \varepsilon'$ and $\text{Im} \bar{\mu}(\bar{E}_n \lor \bar{F}) < \varepsilon''$.

For $\delta_2 = \delta_2' + i\delta_2'' > 0$ and there exists $\delta_3 = \delta_3' + i\delta_3'' > 0$, $\delta'_3 \in (0, \delta'_3 + \varepsilon'/2)$, and $\delta''_3 \in (0, \delta''_3 + \varepsilon''/2)$ such that $\text{Re} \bar{\mu}(\bar{E} \lor \bar{F}) < \delta'_3$, $\text{Im} \bar{\mu}(\bar{E} \lor \bar{F}) < \delta''_3$.

Thus, we have that $\bar{\mu}$ possesses (S.A.P) property.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgments**

This work was supported by the International Science and Technology Cooperation Foundation of China (Grant no. 2012DFA11270) and the National Natural Science Foundation of China (Grant no. 11071151).

**References**

[1] G. Choquet, “Theory of capacities,” *Annales de l’Institut Fourier*, vol. 5, pp. 131–295, 1953.

[2] M. Sugeno, *Theory of fuzzy integrals and its applications [Ph.D. thesis]*, Tokyo Institute of Technology, 1974.

[3] L. A. Zadeh, “Fuzzy sets as a basis for a theory of possibility,” *Fuzzy Sets and Systems*, vol. 1, no. 1, pp. 3–28, 1978.

[4] H. Hermes, “Calculus of set valued function and control,” *Journal of Mathematics and Mechanics*, vol. 18, no. 1, pp. 47–60, 1968.

[5] N. S. Papageorgiou, “Contributions to the theory of set-valued functions and set-valued measures,” *Transactions of the American Mathematical Society*, vol. 304, pp. 245–265, 1982.

[6] E. Klein and A. C. Thompson, *Theory of Correspondences*, John Wiley & Sons, New York, NY, USA, 1984.

[7] W. Zhang, “Measure fuzzy numbers,” *Chinese Science Bulletin*, vol. 23, pp. 9–10, 1986.

[8] C. X. Wu, D. L. Zhang, C. M. Guo, and C. Wu, “Fuzzy number fuzzy measures and fuzzy integrals. (1). Fuzzy integrals of functions with respect to fuzzy number fuzzy measures,” *Fuzzy Sets and Systems*, vol. 98, no. 3, pp. 355–360, 1998.

[9] C. M. Guo, D. L. Zhang, and C. X. Wu, “Fuzzy-valued fuzzy measures and generalized fuzzy integrals,” *Fuzzy Sets and Systems*, vol. 97, no. 2, pp. 255–260, 1998.

[10] C. X. Wu, D. L. Zhang, B. Zhang et al., “Fuzzy number fuzzy measure and fuzzy integrals (1): fuzzy-valued functions with respect to fuzzy number fuzzy measure on fuzzy sets,” *Fuzzy Sets and Systems*, vol. 107, no. 2, pp. 219–226, 1999.
[11] J. J. Buckley, "Fuzzy complex numbers," *Fuzzy Sets and Systems*, vol. 33, no. 3, pp. 333–345, 1989.

[12] Z. Guang-Quan, "Fuzzy limit theory of fuzzy complex numbers," *Fuzzy Sets and Systems*, vol. 46, no. 2, pp. 227–235, 1992.

[13] Z. Guang-Quan, "Fuzzy continuous function and its properties," *Fuzzy Sets and Systems*, vol. 43, no. 2, pp. 159–171, 1991.

[14] Z. Guang-Quan, "Fuzzy number-valued fuzzy measure and fuzzy number-valued fuzzy integral on the fuzzy set," *Fuzzy Sets and Systems*, vol. 49, no. 3, pp. 357–376, 1992.

[15] Z. Guang-Quan, "The structural characteristics of the fuzzy number-valued fuzzy measure on the fuzzy $\sigma$-algebra and their applications," *Fuzzy Sets and Systems*, vol. 52, no. 1, pp. 69–81, 1992.

[16] Z. Guang-Quan, "Convergence of a sequence of fuzzy number-valued fuzzy measurable functions on the fuzzy number-valued fuzzy measure space," *Fuzzy Sets and Systems*, vol. 57, no. 1, pp. 75–84, 1993.

[17] Z. Guang-Quan, "The convergence for a sequence of fuzzy integrals of fuzzy number-valued functions on the fuzzy set," *Fuzzy Sets and Systems*, vol. 59, no. 1, pp. 43–57, 1993.

[18] Z. Guang-Quan, "On fuzzy number-valued fuzzy measures defined by fuzzy number-valued fuzzy integrals I," *Fuzzy Sets and Systems*, vol. 45, no. 2, pp. 227–237, 1992.

[19] Z. Guang-Quan, "On fuzzy number-valued fuzzy measures defined by fuzzy number-valued fuzzy integrals II," *Fuzzy Sets and Systems*, vol. 48, no. 2, pp. 257–265, 1992.

[20] G.-Q. Zhang, *Fuzzy Measure Theory*, Guiyang, Guizhou Science and Technology Press, 1994.

[21] G.-Q. Zhang, *Fuzzy Number-Valued Measure Theory*, Tsinghua University Press, Beijing, China, 1998.

[22] J. J. Buckley and Y. Qu, "Fuzzy complex analysis I: differentiation," *Fuzzy Sets and Systems*, vol. 41, no. 3, pp. 269–284, 1991.

[23] J. J. Buckley, "Fuzzy complex analysis II: integration," *Fuzzy Sets and Systems*, vol. 49, no. 2, pp. 171–179, 1992.

[24] J. Qiu, C. Wu, and F. Li, "On the restudy of fuzzy complex analysis: part I. the sequence and series of fuzzy complex numbers and their convergences," *Fuzzy Sets and Systems*, vol. 115, no. 3, pp. 445–450, 2000.

[25] J. Qiu, C. Wu, and F. Li, "On the restudy of fuzzy complex analysis: part II. The continuity and differentiation of fuzzy complex functions," *Fuzzy Sets and Systems*, vol. 120, no. 2, pp. 517–521, 2001.

[26] J. Qiu, F. Li, and L. Su, "Complex fuzzy measure and complex fuzzy integral," *Journal of Lanzhou University (Natural Science Edition)*, vol. 32, pp. 155–158, 1996.

[27] C. Wu and J. Qiu, "Some remarks for fuzzy complex analysis," *Fuzzy Sets and Systems*, vol. 106, no. 2, pp. 231–238, 1999.

[28] G. Wang and X. Li, "Generalized Lebesgue integrals of fuzzy complex valued functions," *Fuzzy Sets and Systems*, vol. 127, no. 3, pp. 363–370, 2002.

[29] D. Ramot, R. Milo, M. Friedman, and A. Kandel, "Complex fuzzy sets," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 2, pp. 171–186, 2002.

[30] D. Ramot, M. Friedman, G. Langholz, and A. Kandel, "Complex fuzzy logic," *IEEE Transactions on Fuzzy Systems*, vol. 11, no. 4, pp. 450–461, 2003.

[31] S. Dick, "Toward complex fuzzy logic," *IEEE Transactions on Fuzzy Systems*, vol. 13, no. 3, pp. 405–414, 2005.

[32] M. Ha, W. Pedrycz, and L. Zheng, "The theoretical fundamentals of learning theory based on fuzzy complex random samples," *Fuzzy Sets and Systems*, vol. 160, no. 17, pp. 2429–2441, 2009.

[33] X. Fu and Q. Shen, "A novel framework of fuzzy complex numbers and its application to compositional modelling," in *Proceedings of the IEEE International Conference on Fuzzy Systems*, pp. 536–541, Jeju Island, Republic of Korea, August 2009.

[34] X. Fu and Q. Shen, "Fuzzy complex numbers and their application for classifiers performance evaluation," *Pattern Recognition*, vol. 44, no. 7, pp. 1403–1417, 2011.

[35] S.-Q. Ma, F.-C. Chen, Q. Wang, and Z.-Q. Zhao, "The design of fuzzy classifier base on sugeno type fuzzy complex-value integral," in *Proceedings of the 7th International Conference on Computational Intelligence and Security (CIS’11)*, pp. 338–342, Hainan, China, December 2011.

[36] S.-Q. Ma, F.-C. Chen, Q. Wang, and Z.-Q. Zhao, "Sugeno type fuzzy complex-value integral and its application in classification," *Procedia Engineering*, vol. 29, pp. 4140–4151, 2012.

[37] S.-Q. Ma, F.-C. Chen, and Z.-Q. Zhao, "Choquet type fuzzy complex-valued integral and its application in classification," in *Fuzzy Engineering and Operations Research*, vol. 147 of *Advances in Intelligent and Soft Computing*, pp. 229–237, Springer, Berlin, Germany, 2012.

[38] M. Ha, L. Yang, and C. Wu, *Generalized Fuzzy Set Valued Measure Introduction*, Science Press, Beijing, China, 2009.