ESTIMATES FOR THE SPECTRAL RADIUS OF A RANDOM WALK
ON THE MODULAR GROUP

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ABSTRACT. Using ideas of Nagnibeda (1999) and Gouëzel (2015), we estimate from
above and below the spectral radius of the random walk on the Cayley graph of the
modular group PSL(2, Z) associated to the system of generators \( \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\} \).

1. Introduction

The goal of this note is to give numerical estimates for the spectral radius of the
(uniform) random walk on the Cayley graph of the modular group PSL(2, Z) with
the standard generators \( r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

The main motivation for writing this note is that, even if it is likely that in
the present case, estimates (or even a complete description of the spectrum; see
Remark 1.3 below) are known by the experts, we could not find any clue in the
literature. This work arose as a spin-off of previous work of the author [Par20],
where the same techniques are used to give lower bounds for the bottom of the
spectrum of the combinatorial Laplace operator on the Cayley graph of a different
Fuchsian group (see [Par20, Appendix B]), which is equivalent to estimates from
above for the spectral radius of the corresponding random walk, as shows for-

mula (1) below. In particular, there is considerable text overlap in respect to some
background material for the upper bounds. We also stress that the estimates in the
present work do not intend to be optimal in any sense.

1.1. Estimates. For a finitely generated group \( G \) and \( S \) a system of generators of
\( G \), we denote the spectral radius of the corresponding random walk on the Cayley
graph by \( \rho(G, S) \). As suggested by S. Gouëzel (by personal communication), we
estimate \( \rho(G, S) \) from above following ideas of T. Nagnibeda [Nag99]. We also
follow ideas of S. Gouëzel [Gou15] to estimate \( \rho(G, S) \) from below. More precisely,
we prove the following.

Theorem 1.1. Let \( G = \text{PSL}(2, \mathbb{Z}) \) and \( S = \{ r, u \} \). Then, the spectral radius of the uniform
random walk on the corresponding Cayley graph satisfies

\[
0.975921 < \rho(G, S) < 0.976642.
\]

Remark 1.2. The spectral radius associated to a symmetric finite system of \( k > 1 \)
generators, is bounded from below by the spectral radius of (the random walk on)
a regular tree of degree \( d \), that is, by \( 2\sqrt{d - 1}/d \) (see, e.g., [Col98, Chapter II,
Section 7.2]). In our case, since \( |S\cup S^{-1}| = 3 \), this yields \( \rho(G, S) \geq 2\sqrt{2}/3 \approx 0.942809 \).
In particular, our estimates from below improves the trivial bound obtained by
comparison to the regular tree.

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Remark 1.3. It is worth to mention that $G = \text{PSL}(2, \mathbb{Z})$ is the free product of the cyclic groups generated by $r$ and $ur$, of order two and three, respectively (see, e.g., [Alp93]). It follows that the spectrum of the (uniform random walk on) $G$ associated to $S = \{r, ur\}$ can be completely described (see [Gut98, Theorem 4]; originally proved in [McL87]). In fact, in this case the point spectrum is $\{-\frac{5}{6}, 0\}$ and the (absolutely) continuous spectrum is

$$\left[ \frac{1}{6} - \frac{1}{6} \sqrt{13 + 8\sqrt{2}}, \frac{1}{6} - \frac{1}{6} \sqrt{13 - 8\sqrt{2}} \right] \cup \left[ \frac{1}{6} + \frac{1}{6} \sqrt{13 - 8\sqrt{2}}, \frac{1}{6} + \frac{1}{6} \sqrt{13 + 8\sqrt{2}} \right].$$

In particular, $\rho(G, S) = \frac{1}{6} + \frac{1}{6} \sqrt{13 + 8\sqrt{2}} \approx 0.988482$.

Occasionally, in applications (as in [Par20]), the quantity of interest is the bottom of the spectrum of the combinatorial Laplace operator of the Cayley graph, which we denote by $\mu_0(G, S)$. The the bottom of the spectrum $\mu_0(G, S)$ is related to the spectral radius $\rho(G, S)$ by the formula (see Section 2.1)

$$\mu_0(G, S) = |S \cup S^{-1}|(1 - \rho(G, S)).$$

As a direct consequence of formula (1) and Theorem 1.1 we get the following.

**Corollary 1.4.** Let $G = \text{PSL}(2, \mathbb{Z})$ and $S = \{r, u\}$. Then, the bottom of the spectrum of the combinatorial Laplace operator of the corresponding Cayley graph satisfies

$$0.070075449 < \mu_0(G, S) < 0.072236577$$

Remark 1.5. Compare with the upper bound given by the regular tree of degree three $\mu_0(G, S) \leq 3 - 2\sqrt{2} \approx 0.17157288$ (cf. Remark 1.2), and the corresponding bottom of the spectrum $\mu_0(G, S') = \frac{5}{6} - \frac{1}{6} \sqrt{13 + 8\sqrt{2}} \approx 0.034553462$, for the set of generators $S' = \{r, ur\}$ (cf. Remark 1.3).

1.2. **Structure of the paper.** In Section 2 we give some background on combinatorial group theory we need. In particular, in Section 2.1 we introduce the Markov operator and its spectral radius, which corresponds to $\rho(G, S)$, and the Laplace operators and its bottom of spectrum, which corresponds to $\mu_0(G, S)$. In Section 2.2, we state Nagnibeda’s ideas we need to give the upper bounds and, in Section 2.3, those of Gouëzel, for the lower bounds. In particular, we introduce compatible type functions and, in Section 3, we exhibit a particular family of compatible type functions we use in Section 4 to give estimates for the spectral radius. To conclude the proof of Theorem 1.1, we include some details of the computations for the lower bound in Appendices A and B.

2. **Combinatorial group theory**

In this section, we recall some aspects of combinatorial group theory we need and, in particular, we recall the definition of the spectral radius $\rho(G, S)$ and the bottom of the spectrum $\mu_0(G, S)$. The following discussion is completely general. For a complete introduction to this topic we refer the reader to Magnus–Karrass–Solitar’s book [MKS66].

Let $G$ be any group, and let $S$ be a subset of $G$. A **word** in $S$ is any expression of the form

$$w = s_1^{a_1}s_2^{a_2} \cdots s_n^{a_n}$$

where $s_1, \ldots, s_n \in S$ and $a_i \in \{+1, -1\}, i = 1, \ldots, n$. The number $l(w) = n$ is the **length** of the word.
Each word in $S$ represents an element of $G$, namely the product of the expression. The identity element can be represented by the empty word, which is the unique word of length zero. We say that two words are equivalent if they represent the same element in $G$.

**Notation.** As usual, we use exponential notation for abbreviation. Thus, for example, the word $s^3$ can also be denoted by $s^3$. We also use an overline to denote inverses, thus $\bar{s}$ stands for $s^{-1}$, and if $w = s_1^{\alpha_1} \cdots s_n^{\alpha_n}$, then $\bar{w} = s_n^{-\alpha_n} \cdots s_1^{-\alpha_1}$.

In these terms, a subset $S$ of a group $G$ is a system of generators if and only if every element of $G$ can be represented by a word in $S$. Henceforth, let $S$ be a fixed system of generators of $G$ and a word is assumed to be a word in $S$. A relator is a non-empty word that represent the identity element of $G$, that is, equivalent to the empty word.

Any word in which a generator appears next to its own inverse ($ss$ or $\bar{s}s$) can be simplified by omitting the redundant pair (trivial relator). We say that a word is reduced if it contains no such redundant pairs.

Let $v, w$ be two words. We say that $v$ is a subword of $w$ if $w = xvy$, for some words $x$ and $y$. If $x$ is the empty word we say that $v$ is a prefix of $w$. If $y$ is the empty word we say that $v$ is a suffix of $w$.

We say that a word is reduced in $G$ if it has no non-empty relators as subword. In particular, if a word is reduced in $G$, any of its subwords is also reduced in $G$.

For an element $g \in G$, we consider the word norm $|g|$ to be the least length of a word which is equals to $g$ when considered as a product in $G$, and every such word is called a geodesic, that is, if its length coincides with its word norm when considered as a product in $G$. In particular, a geodesic is always reduced in $G$ and a subword of a geodesic is also a geodesic.

For a relator, we call a subword that is also a relator, a subrelator. We say that a relator is primitive if it is not a trivial relator and every proper subword is reduced in $G$, that is, if it does not contain proper subrelators. In particular, a word is reduced in $G$ if and only if it contains no primitive relators as subword. Note that, if $S$ is a system of generators and $P$ is the set of all primitive relators on $S$, then $\langle S \mid P \rangle$ is a presentation of $G$.

The following decomposition result (see Figure 1) will be useful in Section 3.

**Lemma 2.1.** Let $v, w$ be two different equivalent geodesics. Then, there are geodesics $v_0, v_1, w_0, w_1$ and $x$ such that $v = v_0v_1x$ and $w = w_0w_1x$, and $v_1\bar{w}_1$ is a primitive relator (of even length).

**Proof.** Let $x$ be the largest common suffix of $v$ and $w$ (possibly $x$ is empty). Write $v = v'x$ and $w = w'x$. Let $w_1$ and $v_1$ be the smallest non-empty suffixes of $w'$ and $v'$ respectively such that $v_1$ and $w_1$ are equivalent. Such $v_1$ and $w_1$ exist since $v$ and $w$ are different words. Moreover, they have the same length since they are equivalent, that is, they are geodesics that evaluate to the same element in $G$. Write $v' = v_0v_1$ and $w' = w_0w_1$ (possibly $v_0$ and $w_0$ are empty). In particular $v_0$ and $w_0$ are equivalent, since the same holds for $v'$, $w'$ and $v_1, w_1$.

It remains to prove that $v_1\bar{w}_1$ is primitive. Suppose $z$ is a subrelator of $v_1\bar{w}_1$. Since $v_1$ and $w_1$ are geodesics, they and their subwords are reduced in $G$. Then $z = v_2\bar{w}_2$ for some non-empty suffixes $v_2$ and $w_2$ of $v_1$ and $w_1$ respectively. In particular, $v_2$ and $w_2$ are non-empty suffixes of $w'$ and $v'$ respectively and $v_2, w_2$ are equivalent. But, by definition, $v_1$ and $w_1$ are the smallest such suffixes and therefore
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\[ v_2 = v_1 \text{ and } w_2 = w_1. \text{ Thus, } v_1 \bar{w}_1 \text{ has no proper subrelators and, therefore, } v_1 \bar{w}_1 \text{ is primitive.} \]

As a direct consequence of Lemma 2.1, we have the following.

**Proposition 2.2.** Let \( \varphi = \varphi'yx \text{ and } \psi = \psi'zx \) be two equivalent geodesics such that \( y \bar{z} \) is reduced in \( G \). Then, \( y \bar{z} \) is a subword of some primitive relator (of even length).

**Proof.** Consider the decomposition given by Lemma 2.1. It is clear that \( y \) is a subword of \( v_1 \) and \( z \), of \( w_1 \). Then \( y \bar{z} \) is a subword of the primitive relator \( v_1 \bar{w}_1 \). \( \square \)

2.1. **Spectra.** Let \( G \) be a finitely generated group and \( S \subset G \) be a finite system of generators of \( G \). Let \( \ell^2(G) \) be the space of square-summable sequences on \( G \) with the inner product

\[ \langle h, h' \rangle := \sum_{g \in G} h_g h'_g, \]

for \( h, h' \in \ell^2(G) \), and define

- \( \mathcal{M}_S : \ell^2(G) \to \ell^2(G) \), the Markov operator on \( G \) associated to \( S \), by

\[
(M_S h)_g := \frac{1}{|S \cup \bar{S}|} \sum_{s \in S \cup \bar{S}} h_{gs}, \quad h \in \ell^2(G).
\]

We denote by \( \rho(G, S) \) the spectral radius of \( \mathcal{M}_S \), that is,

\[ \rho(G, S) = \sup \left\{ \frac{\langle M_S h, h \rangle}{\langle h, h \rangle}, \ h \in \ell^2(G) \right\}. \]

- \( \Delta_S : \ell^2(G) \to \ell^2(G) \), the Laplace operator on \( G \) associated to \( S \), by

\[
(\Delta_S h)_g := \sum_{s \in S \cup \bar{S}} (h_s - h_{gs}) \quad h \in \ell^2(G).
\]

We denote by \( \mu_0(G, S) \) the bottom of the spectrum of \( \Delta_S \), that is,

\[ \mu_0(G, S) = \inf \left\{ \frac{\langle \Delta_S h, h \rangle}{\langle h, h \rangle}, \ h \in \ell^2(G) \right\}. \]

**Remark 2.3.** It is clear that \( \Delta_S = |S \cup \bar{S}|(\text{id}_{\ell^2(G)} - M_S) \). In particular, the spectra \( \sigma(M_S) \) and \( \sigma(\Delta_S) \) are related by \( \sigma(\Delta_S) = |S \cup \bar{S}|(1 - \sigma(M_S)) \). Moreover, it is clear that \( \sigma(M_S) \) is symmetric. Thus, we deduce formula (1), that is,

\[ \mu_0(G, S) = |S \cup \bar{S}|(1 - \rho(G, S)). \]
Remark 2.4. The subjacent object in this discussion is the Cayley graph of \( G \) associated to \( S \). However we do not elaborate on this here.

2.2. Nagnibeda’s ideas. In order to give upper bounds for the spectral radius \( \rho(G, S) \), we follow ideas of Nagnibeda \cite{Nag99}, which are based in the following elementary result (c.f. \cite[Chapter II, Section 7.1]{Co98} and \cite[Section 1]{Nag99}).

Theorem 2.5 (Gabber–Galil’s lemma). Let \( G \) be a finitely generated group and \( S \) a finite system of generators of \( G \). Suppose there exists a function \( L : G \times S \to \mathbb{R}_+ \) such that, for every \( g \in G \) and \( s \in S \),

\[
L(g, s) = \frac{1}{L(gs, s^{-1})} \quad \text{and} \quad \frac{1}{|S \cup S|} \sum_{s \in S} L(g, s) \leq \delta, 
\]

for some \( \delta > 0 \). Then, \( \rho(G, S) \leq \delta \).

Proof. Since

\[
\left( \sqrt{L(g, s) h_g + L(gs, s^{-1}) h_{gs}} \right)^2 \geq 0, 
\]

we get that

\[
2 h_g h_{gs} \leq L(g, s) h_g^2 + L(gs, s^{-1}) h_{gs}^2. 
\]

Summing on \( g \in G \) and averaging on \( s \in S \cup \bar{S} \), we get

\[
\langle \mathcal{M}_h, h \rangle \leq \left( \frac{1}{|S \cup \bar{S}|} \sum_{s \in S \cup \bar{S}} L(g, s) \right) \langle h, h \rangle \leq \delta \langle h, h \rangle. 
\]

That is, \( \rho(G, S) \leq \delta \). \hfill \Box

Henceforth, let \( S \) be a symmetric finite system of generators of \( G \), that is, \( S = \bar{S} \).

For \( g \in G \), define \( S^+(g) := \{ s \in S : |gs| = |g| + 1 \} \), where \( | \cdot | \) is the word norm in \( G \) with respect to \( S \). For \( g \in G \) and \( s \in S \), we say that \( gs \) is a successor of \( g \) if \( s \in S^+(g) \) and that \( gs \) is a predecessor of \( g \) if \( s \in S^-(g) \). We also assume \( S^+(g) \cup S^-(g) = S \), for every \( g \in G \). Note that this is equivalent to say that every relator has even length.

A function \( t : G \to \mathbb{N} \) is called a type function on \( G \) and its value \( t(g) \) at \( g \in G \) is called the type of \( g \). We say that a type function \( t \) is compatible with \( S \), or simply that \( t \) is a compatible type function, if the following two conditions are equivalent:

1. \( t(g) = t(g') \);
2. \( |\{ s \in S^+(g) : t(gs) = k \}| = |\{ s' \in S^+(g') : t(g's') = k \}| \), for every \( k \in \mathbb{N} \).

That is, \( t \) is a compatible type function if the number of successors of each type of an element \( g \in G \) is completely determined by the type of \( g \).

For any type function \( t : G \to \mathbb{N} \) and positive valuation \( c : \mathbb{N} \to \mathbb{R}_+ \), we can consider a function \( L_c : G \times S \to \mathbb{R}_+ \) defined by

\[
L_c(g, s) = \begin{cases} 
    c_k, & \text{if } s \in S^+(g), \ k = t(gs), \\
    1/c_k, & \text{if } s \in S^-(g), \ k = t(g).
\end{cases}
\]

By definition, every \( L_c : G \times S \to \mathbb{R}_+ \) as above satisfies \( L_c(g, s) = 1/L_c(gs, s^{-1}) \), since \( s \in S^+(g) \) if and only if \( s^{-1} \in S^-(gs) \), and \( S = S^+(g) \cup S^-(g) \), for every \( g \in G \).

Moreover, for a compatible type function \( t \), we define for \( k = t(g) \in \mathbb{N} \), \( g \in G \),

\[
f_k(c) := \sum_{s \in S} L_c(g, s) = \sum_{s \in S^+(g)} c_{t(gs)} + \frac{|S^-(g)|}{c_k}.
\]
Note that this is well defined since $t$ is compatible with $S$ and therefore the sum depends only on $k$, the type of $g$.

As a direct consequence of Gabber–Galil’s lemma (Theorem 2.5), we get the following (c.f. [Nag99, Section 2]).

**Corollary 2.6** (Nagnibeda). Let $G$ be a finitely generated group and $S$ a finite, symmetric system of generators of $G$ without relators of odd length. Let $t : G \rightarrow \mathbb{N}$ be a compatible type function for $S$. Then,

$$\rho(G, S) \leq \frac{1}{|S|} \sup_{k \in \mathfrak{t}(G)} f_k(c),$$

for every $c : \mathbb{N} \rightarrow \mathbb{R}^+$, where $f_k$ is defined as above. $\square$

Then, every compatible (finite) type function gives lower bounds for the combinatorial spectrum.

### 2.3. Gouëzel’s ideas.

In order to give lower bounds for the spectral radius $\rho(G, S)$, we follow ideas of Gouëzel [Gou15]. The key tools are essentially the same compatible type functions, but Gouëzel’s techniques allows to neglect a finite number of elements of each type.

More precisely, following [Gou15, Definition 1.2], we say that $C : \mathbb{N} \rightarrow \mathbb{N}$ is a compatible type system for $(S)$ if $C$ is finite and for all $i, j \in C$, there is $\eta_{ij} \in \mathbb{N}$ such that, for all but finitely many $g \in G$ with $t(g) = j$, we have

$$|\{ s \in S^+(g) : t(gs) = i \}| = M_{ij}.$$

Then, [Gou15, Theorem 1.4] reads as follows.

**Theorem 2.7** (Gouëzel). Let $G$ be a finitely generated group and $S$ a finite, symmetric system of generators of $G$ without relators of odd length. Let $t : G \rightarrow \mathbb{N}$ be a compatible type system for $S$ and suppose that the associated matrix $M$ is Perron–Frobenius. Define the matrix $\tilde{M}$ by $\tilde{M}_{ij} = M_{ij}/p_i$, where $p_i$ is the number of predecessors of an element of type $i$, that is, $p_i = |S| - \sum_j M_{ij}$. It follows that $\tilde{M}$ is also Perron–Frobenius and let $\eta > 0$ be its dominating eigenvalue, which is simple, and let $v \in \mathbb{R}^{|\mathfrak{t}(G)|}$, an associated eigenvector with positive entries.

Let $D = \text{diag}(v)$, $\tilde{M} = D^{-1/2}MD^{1/2}$ and $\bar{M} = (\tilde{M} + \tilde{M}^T)/2$. Finally, let $\lambda$ be the maximal eigenvalue of the symmetric matrix $\bar{M}$. Then,

$$\rho(G, S) \geq 2\lambda / |S| \sqrt{\eta}.$$  

### 3. A compatible type function for the modular group

Until now, the discussion is completely general. We now specialize to the case of the modular group $G = \text{PSL}(2, \mathbb{Z})$ with generators $S = \{r, u\}$, where $r = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ and $u = \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$. The aim in the following is to give a compatible finite type function in this case, in order to give estimates for the spectral radius $\rho(G, S)$ using Corollary 2.6 and Theorem 2.7.

It is well known (see, e.g., [Alp93]) that $\langle r, u \mid r^2, (ru)^3 \rangle$ is a presentation of $G$. Since we have the relator $r^2$, for the sake of simplicity, we omit henceforth $\bar{r}$, since as element in $G$ it coincides with $r$. The set of primitive relators is then (up to include the variants with $\bar{r}$ instead of $r$) given by

$$\{ r^2, (ru)^3, (r\bar{u})^3, (ur)^3, (\bar{u}r)^3 \}.$$
In particular, every relator has even length and we can apply previous discussion, that is, both Corollary 2.6 and Theorem 2.7.

Let $S(g)$ be the set of all suffixes of geodesics for $g \in G$. Then, by the description of the primitive relators, as a direct consequence of Proposition 2.2, we have the following.

**Corollary 3.1.** The following cases cannot happen:

- $u, \bar{u} \in S(g)$,
- $ur, \bar{u}r \in S(g)$,
- $ar, \bar{a}^2 \in S(g)$,
- $ar, a \in S(g)$, for $a = u$ or $\bar{u}$.

**Proof.** Neither $u^2$, $ur\bar{u}$ nor $\bar{u}ru$ are subwords of a primitive relator. □

Let $S_n(g)$ be the set of all suffixes of length $n \in \mathbb{N}$ of geodesics for $g \in G$ and define, by recurrence, $S'_1(g) = S_1(g)$ and

$$S'_{n+1}(g) = \begin{cases} S_{n+1}(g) & \text{if } S_{n+1}(g) \neq \emptyset, \\ S'_n(g) & \text{if } S_{n+1}(g) = \emptyset. \end{cases}$$

Note that any injective function $j : S'_n(g) \rightarrow \mathbb{N}$ defines a (finite) type function. Namely, $t = j \circ S'_n : G \rightarrow \mathbb{N}$, which we call the suffix type function of level $n$.

**Lemma 3.2.** The suffix type function of level 2 is compatible with $S$.

**Proof.** Being compatible with $S$ means that the type $t(g)$ of $g \in G$ completely determines the number of successors of each type. Then, it is enough to show that $S'_2(g)$ completely determines the multiset $\{S'_2(gs) : s \in S^+(g)\}$.

By Corollary 3.1, we have that $\{u, \bar{u}\} \not\subset S_1(g)$, that $|S_2(g)| \leq 2$ and that $S_2(g) = 2$ if and only if $S_2(g) = \{ar, r\bar{a}\}$ or $\{ra, a^2\}$, for $a = u$ or $\bar{u}$. It follows then, that

$$S'_1(g) \in \{\emptyset, \{r\}, \{u\}, \{\bar{u}\}, \{u, r\}, \{\bar{u}, r\}\} \quad \text{and}$$

$$S'_2(g) \in \{\emptyset, \{r\}, \{a\}, \{ra\}, \{ar\}, \{a^2\}, \{ra, r\bar{a}\}, \{ra, a^2\}\} \quad \text{a=}=\text{u}, \bar{u}.$$

Moreover, it is clear that $s \in S^+(g)$ if and only if $\bar{s} \not\in S_1(g)$.

Let $a = u$ or $\bar{u}$.

- If $S'_2(g) = \emptyset$, $g = \text{id}$ and, evidently, $S'_2(gs) = \{s\}$, for $s \in S = S^+(g)$.
- If $S'_2(g) = \{r\}$, $g = r$ and $S'_2(gb) = \{rb\}$, for $b \in \{u, \bar{u}\} = S^+(r)$.
- If $S'_2(g) = \{a\}$, $g = a$ and $S^+(g) = \{r, a\}$; $S'_2(gr) = \{ar\}$ and $S'_2(ga) = \{a^2\}$.
- If $S'_2(g) = \{ra\}$, then $S^+(g) = \{r, a\}$; $S'_2(gr) = \{ar, r\bar{a}\}$ and $S'_2(ga) = \{a^2\}$.
- If $S'_2(g) = \{ar\}$, $S^+(g) = \{u, \bar{u}\}$; $S'_2(gr) = \{ra, \bar{a}r\}$ and $S'_2(ga) = \{r\bar{a}\}$.
- If $S'_2(g) = \{a^2\}$, then $S^+(g) = \{r, a\}$; $S'_2(gr) = \{ar\}$ and $S'_2(ga) = \{a^2\}$.
- If $S'_2(g) = \{ar, r\bar{a}\}$, then $S^+(g) = \{\bar{a}\}$ and $S'_2(ga) = \{r, a^2\}$.
- If $S'_2(g) = \{ra, a^2\}$, $S^+(g) = \{r, a\}$; $S'_2(gr) = \{ar\}$ and $S'_2(ga) = \{a^2\}$.

Thus, given only the value of $S'_2(g)$ we can tell $S'_2(gs)$, $s \in S^+(g)$ and therefore, suffix type functions of level 2 are compatible with $S$. □

We summarize the proof of the previous lemma by the following diagram which shows each possible $S'_2(gs)$, for $g \in G$, with its respective multiset of $S'_2(gs)$,
s ∈ S+(g), and where a = u or ⠠:

\[
S_2^*(g) \rightarrow S_2^*(gs), s \in S^+(g)
\]

\[
\emptyset \rightarrow \{ r \}, \{ u \}, \{ \overline{u} \}
\]

\[
\{ r \} \rightarrow \{ ru \}, \{ \overline{ru} \}
\]

\[
\{ a \} \text{ or } \{ a^2 \} \rightarrow \{ ar \}, \{ a^2 \}
\]

\[
\{ ra \} \text{ or } \{ ra, a^2 \} \rightarrow \{ ar, \overline{ra} \}, \{ a^2 \}
\]

\[
\{ ar \} \rightarrow \{ ra, \overline{ar} \}, \{ \overline{ra} \}
\]

\[
\{ ar, \overline{ra} \} \rightarrow \{ \overline{ra}, \overline{a^2} \}
\]

It is not difficult to see in the previous diagram that there are different suffix types which share the types of the successors. Namely \{ a \}, \{ a^2 \} and \{ ra \}, \{ ra, a^2 \}. This allows us to reduce the number of types. Furthermore, it is clear that distinguishing \( u \) and \( \overline{u} \) in the previous description has no major benefit for the Nagnibeda-type estimates from Corollary 2.6. This motivates the definition of the following type function. Let \( T : G \rightarrow \{ 0, \ldots, 5 \} \) be the type function defined as follows:

\[
T(g) = \begin{cases} 
0 & \text{if } S^*_2(g) = \emptyset, \\
1 & \text{if } S^*_2(g) = \{ r \}, \\
2 & \text{if } S^*_2(g) = \{ a \} \text{ or } \{ a^2 \}, a = u \text{ or } \overline{u}, \\
3 & \text{if } S^*_2(g) = \{ ra \} \text{ or } \{ ra, a^2 \}, a = u \text{ or } \overline{u}, \\
4 & \text{if } S^*_2(g) = \{ ar \}, a = u \text{ or } \overline{u}, \\
5 & \text{if } S^*_2(g) = \{ ar, \overline{ra} \}, a = u \text{ or } \overline{u}.
\end{cases}
\]

From the previous discussion, we deduce the following.

**Theorem 3.3.** The type function \( T : G \rightarrow \{ 0, \ldots, 5 \} \) is compatible with \( S \). Moreover,

- Type 0 elements have one type 1 and two type 2 successors;
- Type 1 elements have two type 3 successors;
- Type 2 elements have one type 2 and one type 4 successor;
- Type 3 elements have one type 2 and one type 5 successor;
- Type 4 elements have one type 3 and one type 5 successor; and
- Type 5 elements have one type 3 successor.

Thus, we have a compatible type function with a full description of the types of the successors for each type. We can then finally apply Nagnibeda's (Corollary 2.6) and Gouëzel's (Theorem 2.7) estimates for the spectral radius \( \rho(G, S) \).

### 4. Estimates for the spectral radius

In this section we finish the proof of Theorem 1.1.

#### 4.1. Upper bound

By Theorem 3.3, the \( f_k \)'s of Corollary 2.6 are given by:

\[
\begin{align*}
& f_0(c) = c_1 + 2c_2, & f_1(c) = 2c_3 + 1/c_1, & f_2(c) = c_2 + 4 + 1/c_2, \\
& f_3(c) = c_2 + c_5 + 1/c_3, & f_4(c) = c_3 + c_5 + 1/c_4, & f_5(c) = c_3 + 2/c_5.
\end{align*}
\]

It follows that \( \mu_0(G, S) \geq |S| - \max_k f_k(c) \), for every \( c = (c_1, \ldots, c_5) \in \mathbb{R}_+^5 \). Thus, the problem can be reduced to find the optimal such bound. This can be solved numerically: we get that \( \overline{c} \in \mathbb{R}_+^5 \) with \( \overline{c}_1 = 1, \overline{c}_2 \approx 0.732625567, \overline{c}_3 \approx 0.792704707, \overline{c}_4 \approx 0.832345202, \overline{c}_5 \approx 0.935795167 \).
is a (local) minimum for \( \max_k f_k(c) \), and \( \max_k f_k(\xi) < 2.929924551 \).
Finally, since \(|S| = 3\), by Corollary 2.6, it follows that
\[
\rho(G, S) < 0.976642.
\]

4.2. **Lower bound.** Using the type function of Theorem 3.3, by discarding all type 0 and type 1 elements, that is, \(id\) and \(r\), respectively, we get a type system with Perron-Frobenius matrices \(M\) and \(\tilde{M}\), where
\[
M = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0
\end{pmatrix} \quad \text{and} \quad \tilde{M} = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1/2 & 1/2 & 0
\end{pmatrix}.
\]

Following Theorem 2.7, we compute the dominating eigenvalue \(\eta\) and corresponding positive eigenvector \(v \in \mathbb{R}^4_+\) of \(\tilde{M}\), which are
\[
\eta = \frac{\sqrt{5} + 1}{2} \quad \text{and} \quad v = \left(\frac{\sqrt{5} + 3}{2}, \frac{\sqrt{5} + 1}{2}, \frac{\sqrt{5} + 1}{2}, 1\right)^T.
\]

Then, we get
\[
D = \text{diag}(v), \quad \tilde{M} = D^{-1/2}MD^{1/2} \quad \text{and} \quad \tilde{M} = \frac{\hat{M} + \hat{M}^T}{2} = \frac{1}{2} \left(\begin{array}{cccc}
2 & \sqrt{5} - 1 & \sqrt{5} + 1 & 0 \\
\sqrt{5} - 1 & 0 & 1 & \sqrt{5} + 2 \\
\sqrt{5} + 1 & 1 & 0 & \sqrt{5} + 1 \\
0 & \sqrt{5} + 2 & \sqrt{5} + 1 & 0
\end{array}\right).
\]

Finally, we compute numerically the dominant eigenvalue of the symmetric matrix \(\tilde{M}\), which is \(\lambda > 1.860673779029\). Thus, by Theorem 2.7, we get that
\[
\rho(G, S) \geq \frac{2\lambda}{3\sqrt{\eta}} > 0.975180.
\]

Note that this is not precisely the estimate from below given in Theorem 1.1. We present it, however to exhibit the computations in detail in a simpler case. In Appendix A we show that the suffix type function of level 3 is compatible with \(S\) and give the corresponding numerical lower bound using Theorem 2.7. This is not yet the value given in Theorem 1.1. In Appendix B we show that the suffix type function of level 4 is compatible with \(S\) and that the corresponding type system allows to show the lower bound in Theorem 1.1.

This concludes the proof of Theorem 1.1

**Remark 4.1.** The nature of Nagnibeda’s estimates suggest that it is not possible to improve the upper bound using suffix type function of higher level. In our case of interest, this is supported by our numerical experiments. Furthermore, as shown by Nagnibeda [Nag04, Section 3] in the case of surface groups, the corresponding upper bounds obtained using compatible type functions correspond to the spectral radius of a random walk on the tree of geodesics of the group and, in particular, do not depend on the choice of the type function.
A. Lower bounds using a suffix type function of level 3

Similar to Lemma 3.2, suffix type functions of level 3 are also compatible. The following diagram shows each possible $S_3^*(g)$, for $g \in G$, with its respective multiset of $S_3^*(g), s \in S^+(g)$. We include also the (reduced) suffix type function.

$$S_3^*(g) \rightarrow S_3^*(gs), s \in S^+(g)$$

$$t(g) \rightarrow t(gs), s \in S^+(g)$$

- $\emptyset \rightarrow \{r\}, \{u\}, \{\bar{u}\}$
- $\{r\} \rightarrow \{ru\}, \{r\bar{u}\}$
- $\{a\} \rightarrow \{ar\}, \{a^2\}$
- $\{ra\} \rightarrow \{rar, \bar{a}r\}, \{ra^2\}$
- $\{ar\} \rightarrow \{ara, \bar{a}r\}, \{ar\bar{a}\}$
- $\{a^2\} \rightarrow \{a^2r\}, \{a^3\}$
- $\{rar, \bar{a}r\} \rightarrow \{ar\bar{a}, r\bar{a}\}$
- $\{ra^2\} \rightarrow \{ar^2\}, \{a^3\}$
- $\{ar\bar{a}\} \rightarrow \{r\bar{a}, ara\}, \{r\bar{a}^2\}$
- $\{a^2r\} \rightarrow \{ara, r\bar{a}r\}, \{ar\bar{a}\}$
- $\{a^3\} \rightarrow \{a^2r\}, \{a^3\}$
- $\{ar\bar{a}, r\bar{a}^2\} \rightarrow \{r\bar{a}r, ara, \bar{a}r^2\}, \{r\bar{a}^2, \bar{a}^3\}$
- $\{r\bar{a}r, ara, \bar{a}r^2\} \rightarrow \{\bar{a}r, ra^2\}$
- $\{ra^2, a^3\} \rightarrow \{a^2r\}, \{a^3\}$

where $a = u$ or $\bar{u}$.

Types 0 to 5 represents the finite set of words of length strictly bounded by 3, so we can discard them and we get a type system with Perron-Frobenius matrices $M$ and $\tilde{M}$, where

$$M = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

and

$$\tilde{M} = \begin{pmatrix}
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}. $$

Following Theorem 2.7, we can proceed as in Section 4.2. For $\tilde{M}$, we compute the Perron–Frobenius eigenvalue $\eta = \frac{\sqrt{5} + 1}{2}$ and an associated positive eigenvector

$$v = \begin{pmatrix}
\frac{1}{2} \left( 2 + \sqrt{5} \right), \frac{\sqrt{5} + 1}{2}, \frac{1}{2} \left( 3 + \sqrt{5} \right), 2 + \sqrt{5}, 2 + \sqrt{5}, \frac{\sqrt{5} + 1}{2}, \frac{1}{2}, \frac{1}{2}
\end{pmatrix}. $$
Then, we get $D = \text{diag}(v)$, $\hat{M} = D^{-1/2}MD^{1/2}$ and $\tilde{M} = \frac{\hat{M} + \hat{M}^\top}{2}$ which equals

$$
\begin{pmatrix}
0 & 0 & \sqrt{\sqrt{5} - 1} & \sqrt{2} & 0 & \frac{\sqrt{51 + 2}}{2} & 0 & 0 \\
0 & 0 & \sqrt{\sqrt{5} + 1} & \sqrt{\sqrt{5} - 1} & \sqrt{2} & 0 & 0 & 0 \\
\sqrt{\sqrt{5} - 1} & \sqrt{\sqrt{5} + 1} & 0 & \sqrt{51 + 2} & 0 & 0 & 0 & 0 \\
\sqrt{2} & \sqrt{\sqrt{5} + 1} & 0 & 1 & 0 & 0 & \sqrt{\sqrt{5} - 2} \\
0 & \sqrt{\sqrt{5} + 1} & 0 & 1 & 2 & 0 & 0 & \sqrt{\sqrt{5} - 2} \\
\sqrt{5 - 2} & 0 & 0 & 0 & 0 & \frac{\sqrt{51 + 2}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{51 + 2}}{2} & 0 & 0 \\
0 & 0 & 0 & \sqrt{5 - 2} & \sqrt{5 - 2} & \frac{\sqrt{51 + 2}}{2} & 0 & 0
\end{pmatrix}
$$

Finally, we compute numerically the dominant eigenvalue of the symmetric matrix $\tilde{M}$, which is $\lambda > 1.861945657376$. Thus, by Theorem 2.7, we get that

$$
\rho(G, S) \geq \frac{2\lambda}{3\sqrt{\eta}} > 0.975847.
$$

\[\square\]

B. **Lower bounds using a suffix type function of level 4**

Similar to Lemma 3.2 and Appendix A, suffix type functions of level 4 are also compatible. The following diagram shows each possible $S'_4(g)$, for $g \in G$, with its respective multiset of $S'_4(gs), s \in S^+(g)$. We include also the (reduced) suffix type function.

$$
\begin{array}{ll}
S'_4(g) & \rightarrow S'_4(gs), s \in S^+(g) \\
\emptyset & \rightarrow \{r\}, \{u\}, \{\tilde{u}\} \\
\{r\} & \rightarrow \{ru\}, \{r\tilde{u}\} \\
\{a\} & \rightarrow \{ar\}, \{a^2\} \\
\{ra\} & \rightarrow \{rar, \tilde{a}r\}, \{ra^2\} \\
\{ar\} & \rightarrow \{ara, r\tilde{a}\}, \{ar\tilde{a}\} \\
\{a^2\} & \rightarrow \{a^2r\}, \{a^3\} \\
\{rar, \tilde{a}r\} & \rightarrow \{rar, \tilde{a}r\}, \{ar\tilde{a}\} \\
\{ra^2\} & \rightarrow \{ra^2r\}, \{ra\} \\
\{ar\tilde{a}\} & \rightarrow \{ar\tilde{a}, a^2ra\}, \{ar\tilde{a}\} \\
\{a^2r\} & \rightarrow \{a^2ra, a\tilde{a}r\}, \{a^2r\tilde{a}\} \\
\{a^3\} & \rightarrow \{a^3r\}, \{a^4\} \\
\{rara, ar\} & \rightarrow \{rar, a^2r\tilde{a}, ra^2r, a^3\} \\
\{ra^2r\} & \rightarrow \{a^2ra, a\tilde{a}r\}, \{a^2r\tilde{a}\} \\
\{ra\} & \rightarrow \{a^2r\}, \{a^4\} \\
\{ar\}, a^2ra & \rightarrow \{ra, ara\} \\
\{a\tilde{a}\} & \rightarrow \{a^2r\}, \{a^3\}
\end{array}
$$

$$
\begin{array}{ll}
t(g) & \rightarrow t(gs), s \in S^+(g) \\
0 & \rightarrow 1, 2, 3 \\
1 & \rightarrow 3, 5 \\
2 & \rightarrow 4, 5 \\
3 & \rightarrow 6, 7 \\
4 & \rightarrow 6, 8 \\
5 & \rightarrow 9, 10 \\
6 & \rightarrow 1, 11 \\
7 & \rightarrow 12, 13 \\
8 & \rightarrow 14, 15 \\
9 & \rightarrow 14, 16 \\
10 & \rightarrow 17, 18 \\
11 & \rightarrow 19, 20 \\
12 & \rightarrow 14, 16 \\
13 & \rightarrow 17, 18 \\
14 & \rightarrow 11 \\
15 & \rightarrow 12, 13
\end{array}
$$
\{a^2 r \bar{a}\} \to \{arar, a^2 ra, \bar{a}r\} \quad 16 \to 14, 15
\{a^3 r\} \to \{a^2 ra, \bar{a}rr, a^2 \bar{r}\} \quad 17 \to 14, 16
\{a^4\} \to \{a^3 r, \bar{a}r\} \quad 18 \to 17, 18
\{arar, a^2 ra, \bar{a}r\} \to \{r\bar{a}ra, \bar{a}ra, \bar{a}^2 ra\} \quad 19 \to 21
\{\bar{a}r a^2, ra^2\} \to \{ra^2, \bar{a}^2 r, \{ra^2, ra^3\} \quad 20 \to 12, 18
\{ra^2, ra^3\} \to \{\bar{a}r, \bar{a}^2 r, ra^2, \bar{a}r^2, ra^3\} \quad 21 \to 19, 20

where \(a = u \) or \( \bar{u} \).

Types 0 to 10 represents the finite set of words of length strictly bounded by 4, so we can discard them and we get a type system with \(11 \times 11\) Perron-Frobenius matrices. Similarly to Section 4.2 and Appendix A, by Theorem 2.7, we can conclude that \(\rho(G, S) > 0.975921\). □

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