SCATTERING FOR A MASSLESS CRITICAL NONLINEAR WAVE EQUATION IN 2 SPACE DIMENSIONS

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Abstract. We prove scattering for a massless wave equation which is critical in two space dimensions. Our method combines conformal inversion with decay estimates from Struwe’s previous work on global existence of a similar equation.

1. Introduction

We study the asymptotic behaviour of solutions to the nonlinear wave equation
\[ u_{tt} - \Delta u + u(e^{u^2} - 1 - u^2) = 0 \text{ on } \mathbb{R} \times \mathbb{R}^2 \]
with compactly supported initial data
\[ (u, u_t)|_{t=0} = (u_0, u_1) \in C_c^\infty(\mathbb{R}^2) \times C_c^\infty(\mathbb{R}^2). \]
Their initial energy is given by
\[ E_0 = \frac{1}{2} \int_{\mathbb{R}^2} \left( u_1^2 + |\nabla u_0|^2 + e^{u_0^2} - 1 - u_0^2 - \frac{1}{2} u_0^4 \right) dx. \]
Interest in this equation arises because it lies at the boundary of what one considers an energy-critical equation. For the defocusing nonlinear wave equation with power nonlinearity in dimension \( d \geq 3 \),
\[ u_{tt} - \Delta u + |u|^{p-2}u = 0 \text{ on } \mathbb{R} \times \mathbb{R}^d, \]
this border is marked by the Sobolev-critical power \( p^* = (d+2)/(d-2) \). In the subcritical case \( p < p^* \) as well as in the critical case \( p = p^* \) well-posedness in the energy space is known to hold. However, little is known for the supercritical case \( p > p^* \). In two space dimensions the embedding \( H^1(\mathbb{R}^2) \subset L^p(\mathbb{R}^2) \) for \( p < \infty \) renders every power nonlinearity subcritical. However, \( H^1(\mathbb{R}^2) \not\subset L^\infty(\mathbb{R}^2) \). Instead, we have the Trudinger-Moser inequality
\[ \sup_{u \in H^1_0(\Omega); \|\nabla u\|_{L^2(\Omega)}} \int_{\Omega} e^{\alpha u^2} dx \leq \begin{cases} C |\Omega|, & \alpha \leq 4\pi \\ \infty, & \alpha > 4\pi \end{cases} \]
for a smooth bounded domain \( \Omega \subset \mathbb{R}^2 \). Since
\[ \sup_{u \in H^1_0(\Omega); \|\nabla u\|_{L^2(\Omega)}} \int_{\Omega} e^{\alpha u^2} dx = \sup_{u \in H^1_0(\Omega); \|\nabla u\|_{L^2(\Omega)}} \int_{\Omega} e^{u^2} dx, \]
it seems that well-posedness of the initial value problem for the equation
\[ u_{tt} - \Delta u + ue^{u^2} = 0 \text{ on } \mathbb{R} \times \mathbb{R}^2 \]
may depend on the size of the initial energy
\[ E := \frac{1}{2} \int_{\mathbb{R}^2} \left( u_1^2 + |\nabla u_0|^2 + e^{u_0^2} - 1 \right) dx. \]

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For small data, global well-posedness for (5) was shown by Ozawa and Nakamura [6]. Ibrahim, Majdoub and Masmoudi proved global existence for data with energy $E \leq 2\pi$ which they define to be (sub-)critical [3]. Due to the dispersive nature of (5) they also expected $u$ to decay in time and to scatter towards a solution of the linear Klein-Gordon equation

$$u_{tt} - \Delta u + u = 0.$$  

Indeed, together with Nakanishi [4] they established scattering for the modified equation

$$u_{tt} - \Delta u + u \left( e^{u^2} - u^2 \right) = 0 \text{ on } \mathbb{R} \times \mathbb{R}^2,$$

as long as

$$E_{\text{mass}} = \frac{1}{2} \int_{\mathbb{R}^2} \left( u_1^2 + |\nabla u_0|^2 + e^{u_0^2} - 1 - \frac{1}{2} u_0^4 \right) \, dx \leq 2\pi,$$

leaving open the corresponding questions in the supercritical regime when $E > 2\pi$ or $E_{\text{mass}} > 2\pi$, respectively.

Surprisingly, in [9] Struwe was able to establish global existence for (5) for arbitrary smooth initial data using only energy estimates. Here, we show that also for scattering there is no restriction on the energy, at least when we consider the massless wave equation (1) for radially symmetric initial data.

**Theorem 1.1.** For any solution $u$ to the Cauchy problem (1), (2) with smooth compactly supported radial data $(u_0, u_1)$, $u_0(x) = u_0(|x|), u_1(x) = u_1(|x|)$, there exist $(v_0, v_1) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$, such that with the solution $v$ to the linear wave equation

$$v_{tt} - \Delta v = 0,$$

with Cauchy data $(v, v_t)|_{t=0} = (v_0, v_1)$ there holds

$$\| (u(t) - v(t), \partial_t u(t) - \partial_t v(t)) \|_{H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} \to 0, \quad \text{as } t \to \infty.$$

We therefore consider (1), (5) and (7) to be only critical problems, regardless of the size of the initial energy.

To prepare for the proof of Theorem 1.1 we rewrite equation (1) abstractly as

$$u_{tt} - \Delta u + N = 0,$$

with the nonlinearity

$$N(u) := \left( e^{u^2} - 1 - u^2 \right) u.$$

The solution to (10) is given by the Duhamel formula

$$u(t) = \partial_t R(t) * u_0 + R(t) * u_1 + \int_0^t R(t - s) * N(u(s)) \, ds,$$

with $R$ the fundamental solution to (8). In Fourier space it reads

$$\mathcal{F}(R(t))(\xi) = \frac{\sin(|\xi| t)}{|\xi|}.$$
From the Duhamel formula (11), we read off how the initial data are propagated. We define
\[ v_0 := \mathcal{F}^{-1} \left( \hat{u}_0 - \int_0^\infty \frac{\sin(|\xi|s)}{|\xi|} \hat{N}(s) ds \right), \]
\[ v_1 := \mathcal{F}^{-1} \left( \hat{u}_1 + \int_0^\infty \cos(|\xi|s) \hat{N}(s) ds \right) \]
as initial data for the linear wave equation and call \( v \) the solution to the corresponding Cauchy problem. Using the Duhamel formula (11), one calculates
\[ (12) \quad \|u(t) - v(t)\|_{\dot{H}^1(\mathbb{R}^2)} = \| \int_t^\infty \frac{\sin(|\xi|(t-s))}{|\xi|} \hat{N}(s) ds \|_{\dot{H}^1(\mathbb{R}^2)}, \]
and a corresponding expression for the time derivative. To prove scattering we need to establish convergence of the integrals defining the initial data \((v_0, v_1)\) in the norm \(\dot{H}^1 \times L^2\). In the following lemma we reduce this problem to a bound on the nonlinearity \(N\).

**Lemma 1.2.** If \( \|N\|_{L^1((0,\infty);L^2(\mathbb{R}^2))} < \infty \), the integral
\[ \int_0^\infty \frac{\sin(|\xi|s)}{|\xi|} \hat{N}(s) ds \]
converges in \(\dot{H}^1\).

The lemma follows from equivalence of the norms
\[ \|u\|_{\dot{H}^1} \simeq \|\xi \hat{u}\|_{L^2}. \]
Thus, once \( N \in L^1_tL^2_x \) is established the assertion of Theorem 1.1 follows from (12).

In the case of the nonlinear Klein-Gordon equation we find similar representation formulæ and analogous results with the fundamental solution replaced by
\[ \mathcal{F}(R(t))\langle \xi \rangle = \frac{\sin(\langle \xi \rangle t)}{\langle \xi \rangle}, \]
where \( \langle \xi \rangle = \sqrt{1 + |\xi|^2} \). Then, scattering takes place in the norm \(H^1 \times L^2\).

This discussion highlights the significance of leaving out the cubic term in (1). Informally, for \( N(u) = u(e^{u^2} - 1) \) to be in \( L^1_tL^2_x \) we need to control \( \|u\|_{L^6_tL^\infty_x} \). However, \( L^1_tL^\infty_x \) is not an admissible Strichartz norm in two space dimensions. In this respect, we agree with \[4\]. In the course of our argument we will encounter further reasons that justify omission of the cubic term.

Moreover, we restrict our result to the massless equation (1). The reason for this is that the method of conformal inversion that we employ in section 3 to control the nonlinearity will fail for the massive equation.

Our work is organised as follows. In section 2 we derive estimates for the nonlinear term. As a by-product we obtain a scattering result for the massive equation (7) for small data where we only use standard \( L^p_tL^q_x \) Strichartz estimates, rather than the more elaborate estimates for Besov spaces in \[6\] and \[4\].

In section 3 we then prove Theorem 1.1 for large radially symmetric data. In a first step, by applying the method of conformal inversion as in [2] and adapting the decay estimates from [9], we find a hyperboloid contained inside the support of the solution \( u \), such that \( \|N\|_{L^1_tL^2_x} \) is bounded inside the hyperboloid. There, we need not assume the initial data to be radial. In the final step, we then use the radial symmetry of the data to bound \( \|N\|_{L^1_tL^2_x} \) in the complement of the hyperboloid. Thus, we conclude the proof of Theorem 1.1.
2. Scattering for small data

Scattering for \( u \) for small data was shown in [7]. Ibrahim et al. present a scattering result for data with initial energy \( E_{\text{mass}} \leq 2\pi \). However, the previous works rely on Besov space techniques and in the latter work a logarithmic inequality for \( \|u\|_{L^\infty} \). In this section, we show a more direct approach. We consider \( u_0, u_1 \in C_c^\infty(\mathbb{R}^2) \) with \( E_{\text{mass}} \) bounded by an absolute constant \( \varepsilon_0 \) to be determined later.

The modulus of the nonlinearity \( |N| = (e^{u^2} - u^2 - 1)|u| \) behaves like \( |u|^5 \) for small values of \( |u| \). For large values of \( |u| \) the exponential dominates. More precisely, we have the pointwise estimate

\[
|N| = |(e^{u^2} - u^2 - 1)|u| = |u|^3 \sum_{k=1}^{\infty} \frac{u^{2k}}{(k + 1)!}
\]

By Hölder’s inequality

\[
\|u^3 (e^{u^2} - 1)\|_{L^1_\Omega} \leq \|u\|_{L^3_\Omega} \|e^{u^2} - 1\|_{L^\infty_\Omega}.
\]

To control the norm of the exponential term we roughly estimate

\[
(e^{u^2} - 1)^{\frac{40}{9}} \leq e^{\frac{40}{9}u^2} - 1 \leq e^{4u^2} - 1.
\]

Then we can use a version of the Trudinger–Moser inequality [8].

\[
\sup_{\|u\|_{L^2} + \|\nabla u\|_{L^2} \leq 1} \int_\Omega (e^{4\pi u^2} - 1) \, dx \leq C_{TM}
\]

with a constant \( C_{TM} \) independent of the region \( \Omega \subset \mathbb{R}^2 \). By finite speed of propagation the support of \( u \) stays bounded locally uniformly in time. Since the energy is non-increasing in time, if \( \varepsilon_0 \leq 1/2 \), the condition \( \|u\|_{L^2} + \|\nabla u\|_{L^2} \leq 1 \) is satisfied for all times. Therefore we may combine (14) with (13) to obtain

\[
\|N\|_{L^1([0,T];L^2_\Omega(\mathbb{R}^2))} \leq C_{TM}\|u\|_{L^3([0,T];L^\infty_\Omega(\mathbb{R}^2))}^3 + e\|u\|_{L^1([0,T];L^2_\Omega(\mathbb{R}^2))}^2.
\]

We have chosen the power \( 40/9 \) for convenience. However, we are not free in our choice as we want to estimate \( u \) in \( L^q_t L^r_x \) with Strichartz estimates. Wave admissibility [3] demands that

\[
\frac{1}{q} + \frac{1}{2r} \leq \frac{1}{4},
\]

so we need \( q \geq 4 \). By Strichartz estimates (as in [7] Cor. 2.41, Lem. 2.43)

\[
f(T) := \|u\|_{L^\infty_t L^2_x(\mathbb{R}^2)}^q + \|u\|_{L^1_t L^2_x(\mathbb{R}^2)}^q \leq C_S \left( \|u_0, u_1\|_{H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} + \|N\|_{L^1([0,T];L^2_\Omega(\mathbb{R}^2))} \right),
\]

with a constant \( C_S \) that does not depend on the initial data. Then, by (15) and (16) we have

\[
f(T) \leq C_S \left( \|u_0, u_1\|_{H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} + C_{TM} f(T)^{\frac{40}{9}} + e f(T)^5 \right).
\]

The function \( f(T) \) is continuous and non-decreasing with \( f(0) = 0 \). Therefore there exists a time \( T_0 > 0 \) such that \( f(T) \leq 1 \) for \( 0 \leq T < T_0 \) and

\[
f(T) \leq C_S \left( \|u_0, u_1\|_{H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} + (e + C_{TM}) f(T)^{\frac{40}{9}} \right),
\]
for all times $T \in [0, T_0)$. Let $A = \min\{1, A_0\}$, where $A_0$ satisfies
\[
C_S(c + C_{TM})(2A_0)^{\frac{4}{3}} = \frac{1}{2} A_0 .
\]
Suppose $\|(u_0, u_1)\|_{H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} < \varepsilon_0$, where
\[
C_S\varepsilon_0 = \frac{1}{2} A ,
\]
then relation (17) implies $f(T) \leq A$ as long as $f(T) \leq 2A$. Hence, by continuity $f(T_0) \leq A$. By definition of $A$ and continuity again, $T_0$ can be arbitrarily extended and the bound $f(T) \leq A$ holds for all times. By (16) we have
\[
\|N\|_{L^1([0, \infty); L^2(\mathbb{R}^2))} \leq C_{TM} A^{\frac{4}{3}} + \varepsilon A^3 \leq \infty .
\]
Therefore $u$ scatters for $\|(u_0, u_1)\|_{H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} < \varepsilon_0$, in particular for $E_{\text{mass}} \leq \varepsilon_0$.

3. Scattering for large data

3.1. Conformal inversion. Suppose we are given initial data at time $a > 0$. We assume they are compactly supported inside a ball of radius $a/2$. By finite speed of propagation the solution is confined within the forward light cone emanating from the origin at time $a/2$, i.e.
\[
\sup \{u(t, \cdot) : \mathcal{B}_{1/2}(0) \} \geq a .
\]
We perform a conformal inversion
\[
\Phi : (t, x, u) \mapsto (T, X, U) ,
\]
as in (2), i.e. we define
\[
T := \frac{t}{1 - r^2}, \quad X := \frac{x}{1 - r^2}, \quad U := \Omega^{-\frac{1}{2}}u ,
\]
with the weight
\[
\Omega := \frac{1}{1 - r^2} = T^2 - R^2 ,
\]
where $r = |x|$, $R = |X|$. The conformal inversion leaves the structure of the d’Alembert operator invariant (4) Lemma 4.2 and
\[
(\partial_t^2 - \Delta X)U = \Omega^{-2}(\partial_t^2 - \Delta x)u .
\]
In fact, the conformal inversion can be regarded as a Kelvin transform of Minkowski space $(\mathbb{R}^{1, 2}, \eta)$ with metric $\eta_{\mu\nu} = \text{diag}(+1, -1, -1)$. This can be seen by writing the transformation as $G : x^\lambda \mapsto x^\lambda(x^\mu x^\nu \eta_{\mu\nu})^{-1} = x^\lambda(x, x)^{-1}_{\eta}$. One then calculates the differential,
\[
dG_x(y) = \frac{d}{dt}\bigg|_{t=0} G(x + ty) = \frac{d}{dt}\bigg|_{t=0} \left(\frac{x + ty}{\langle x, x \rangle_{\eta} + 2t(x, y)_{\eta} + t^2(y, y)_{\eta}}\right)\]
\[
= \frac{y}{\langle x, x \rangle_{\eta}} \frac{2x(x, y)_{\eta}}{\langle x, x \rangle_{\eta}^2} ,
\]
so that $((dG_x)_y, (dG_x)y)_{\eta} = (x, x)^{-2}_{\eta}(y, y)_{\eta}$ and the differential is a conformal transformation with respect to the metric $\eta$.

In the new variables $T, X$ equation (4) becomes
\[
\partial_T^2 U - \Delta U + \Omega^{-2} U(e^{3U/2} - 1 - \Omega U^2) = 0 .
\]
Note that we changed the direction of time. The transformed function $U$ has support inside the set
\[
\sup U = \{(T, X) : T - R \leq \frac{2}{a}, \quad T^2 - R^2 \leq a\} .
\]
For the following arguments we fix $a$. This is not a restriction. In fact, for any initial data with compact support we may shift the initial time such that the support of the initial data at the starting time is contained inside our fixed cone. We choose $a = 1$. This leads to $\Omega \leq 1$ for $T \leq 1$.

3.2. Energy-Flux relation in conformal coordinates. For the remainder of the argument we closely follow [9]. We multiply (18) with $U_T$. Then we obtain

$$ (19) \partial_T e - \text{div } m = TP $$

with the scaled energy density

$$ e := \frac{1}{2} \left( U_T^2 + |\nabla U|^2 + \Omega^{-3}(e^{\Omega U^2} - 1 - \Omega U^2 - \frac{1}{2}\Omega^2 U^4) \right), $$

the momentum density

$$ m := U_T \nabla U, $$

and the remainder

$$ P := \Omega^{-4} \left( \Omega U^2 (e^{\Omega U^2} - 1 - \Omega U^2) - 3(e^{\Omega U^2} - 1 - \Omega U^2 - \frac{1}{2}\Omega^2 U^4) \right) $$

$$ = U^8 \sum_{k=0}^{\infty} \frac{(\Omega U^2)^k}{(k+4)!} (k+1) \geq 0. $$

The power series expansion of $P$ shows that the right hand side of (19) is positive. Therefore the scaled energy is non-increasing as we approach the origin. Note, that removing the mass term is crucial at this point. Without doing so, we are left with an additional term $-2\Omega^{-2}U^2$ in $P$ that spoils the definite sign of the remainder. Furthermore, the same observation holds for the $u^3$-term in the original equation.

For $T_0 < 1$ we integrate (19) over the forward light cone $\{ R \leq T \}$ where we truncate by the initial data surface and the support of $U$, i.e. we integrate over $K := \{(T,X) \in \text{supp } U, T_0 \leq T, |X| = R \leq T \}$.

Its boundary $\partial K$ has four components. The first one is the initial data surface. It contributes the energy $E_a$ on the initial data surface. The second is the boundary of the support of $U$ inside $\{ R < T \}$. Its contribution vanishes. The third boundary is the mantle of the light cone, $M_{T_0}^2 := \{(T,X) : T_0 \leq T \leq 1, |X| = R = T \}$.

We write

$$ V(Y) := U(|Y|, Y) $$

for the restriction of $U$ to the mantle. We call the quantity

$$ \int_{M_{T_0}^2} \frac{1}{2} \left( \Omega^3 |\nabla V|^2 + e^{\Omega V^2} - \frac{1}{2}\Omega^2 V^4 \right) dY, $$

the flux of $U$ through the mantle $M_{T_0}^2$. The last boundary yields the energy in new coordinates,

$$ E(T_0) := \int_{B_{R_0}(0)} e dX. $$

Putting everything together, we find

$$ E(T_0) + \frac{1}{\sqrt{2}} \text{Flux}(M_{T_0}^1) = E_a - \int_K PT dX dT, $$

in particular we have the energy inequality

$$ E(T_0) + \frac{1}{\sqrt{2}} \text{Flux}(M_{T_0}^1) \leq E_a. $$
Therefore the limit \( \lim_{T \to 0} E(T, B_T(0)) \) exists and the flux decays,
\[
\text{Flux}(M_1^T) := \sup_{0 < S < T} \text{Flux}(M_S^T) \to 0, \quad T \to 0 .
\]

Moreover, the remainder term \( PT \) is bounded by the initial energy,
\[
\int_K PT dX dT \leq E_a .
\]

3.3. **Pointwise estimates for the average on the mantle.** We derive pointwise estimates for the spherical averages
\[
\nabla = \nabla(T) = \frac{1}{2\pi} \int_0^{2\pi} V(e^{i\phi} T) d\phi .
\]

By Hölder’s inequality
\[
|\nabla(T)| \leq |\nabla(T_1)| + \int_T^{T_1} |\nabla(S)| dS \leq |\nabla(T_1)| + \left( \int_T^{T_1} |\nabla|S|^{2} dS \cdot \int_T^{T_1} \frac{dS}{S} \right)^{\frac{1}{2}}
\]
\[
\leq |\nabla(T_1)| + \pi^{-\frac{1}{4}} \text{Flux}^\ast(M_1^{T_1}) \log^\frac{1}{2} \left( \frac{T_1}{T} \right) .
\]

Flux decays towards the origin by (20). So there exists a time \( T_0 \leq 1 \) such that for smaller times \( 0 < T \leq T_0 \) we have
\[
\text{Flux}^\ast(M_1^{T_0}) \leq \text{Flux}^\ast(M_1^{T_0}) \leq \frac{1}{8} .
\]

With this explicit bound on the flux we can fix a second time \( T_1, 0 < T_1 \leq T_0 \) such that \( 8|\nabla(T_0)| \leq \log^{1/2}(1/T) \) for \( 0 < T \leq T_1 \). By \( T_0 \leq 1 \) we have \( \log(T_0/T) \leq \log(1/T) \). Therefore,
\[
4|\nabla(T)| \leq \log^{\frac{1}{2}}(\frac{1}{T}) \forall 0 < T \leq T_1 .
\]

3.4. **Decay of energy.** We introduce polar coordinates \( R, \phi \). The energy law (19) becomes
\[
\partial_T (Re) - \partial_R(Rm) = \frac{1}{R} \partial_\phi(U_T U_\phi) + RTP ,
\]
where now
\[
e := \frac{1}{2} \left( U_T^2 + U_R^2 + R^{-2} U_\phi^2 + \Omega^3 \left( e^{\Omega U^2} - 1 - \Omega U^2 - \frac{1}{2} \Omega^2 U^4 \right) \right) ,
\]
\[
m := U_T U_R .
\]

We multiply equation (18) with \( X \cdot \nabla U \). Then
\[
\partial_T (X \cdot m)
\]
\[
= - \text{div} \left( X \cdot \nabla U \nabla U - \frac{X}{2} (|\nabla U|^2 - U_T^2 + \Omega^{-3}(e^{\Omega U^2} - 1 - \Omega U^2 - \frac{1}{2} \Omega^2 U^4)) \right)
\]
\[
+ U_T^2 - \Omega^{-3}(e^{\Omega U^2} - 1 - \Omega U^2 - \frac{1}{2} \Omega^2 U^4) = - R^2 P .
\]

In polar coordinates,
\[
\partial_T (R^2 m)
\]
\[
= - \frac{1}{2} \partial_R \left( R^2(U_T^2 + U_R^2 + R^{-2} U_\phi^2 + \Omega^{-3}(e^{\Omega U^2} - 1 - \Omega U^2 - \frac{1}{2} \Omega^2 U^4)) \right)
\]
\[
+ R \left( U_T^2 - \Omega^{-3}(e^{\Omega U^2} - 1 - \Omega U^2 - \frac{1}{2} \Omega^2 U^4) \right)
\]
\[
= \partial_\phi(U_R U_\phi) - R^3 P .
\]
Multiplying (15) with \((U - \nabla)\) we obtain

\[
\begin{align*}
\partial_T (U_T (U - \nabla)) & - \text{div}(\nabla U (U - \nabla)) \\
+ |\nabla U|^2 - U_T^2 + U_T \nabla T + \Omega^{-2} U (U - \nabla)(e^{\Omega U^2} - 1 - \Omega U^2) &= 0 .
\end{align*}
\]

Or, again in polar coordinates,

\[
\begin{align*}
\partial_T (RU_T (U - \nabla)) & - \partial_R (RU_R (U - \nabla)) \\
+ R \left(|\nabla U|^2 - U_T^2 + U_T \nabla T + \Omega^{-2} U (U - \nabla)(e^{RU^2} - 1 - \Omega U^2)\right) &= \frac{1}{R} \partial_\phi ((U - \nabla) U_\phi) .
\end{align*}
\]

We rescale the energy identity (23) with \(\frac{R}{T}\). Then

\[
(26) \quad \partial_T (\frac{R^2}{T} e) - \partial_R (\frac{R^2}{T^2} m) + \frac{R^2}{T^2} e - \frac{R}{T} m = \partial_\phi (\frac{1}{T} U_T U_\phi) + R^2 P .
\]

We divide both (24) and (26) by \(T\). Then

\[
\begin{align*}
\partial_T (\frac{R^2}{T} m) \\
- \frac{1}{2} \partial_R \left( \frac{R^2}{T} (U_T^2 + U_R^2 - R^{-2} U_\phi^2 + \Omega^{-3} (e^{RU^2} - 1 - \Omega U^2 - \frac{1}{2} \Omega^2 U^4)) \right) \\
+ \frac{R^2}{T^2} m + \frac{R}{T} \left( U_T^2 - \Omega^{-3} (e^{RU^2} - 1 - \Omega U^2 - \frac{1}{2} \Omega^2 U^4) \right) \\
= \partial_\phi (\frac{1}{T} U_R U_\phi) - \frac{R^2}{T^2} P .
\end{align*}
\]

and

\[
\begin{align*}
\partial_T \left( \frac{R}{T} U_T (U - \nabla) \right) & - \partial_R \left( \frac{R}{T} U_R (U - \nabla) \right) \\
+ \frac{R}{T} \left( |\nabla U|^2 - U_T^2 + U_T \nabla T - U_T \frac{U - \nabla}{T} \right) \\
+ \Omega^{-2} U (U - \nabla)(e^{RU^2} - 1 - \Omega U^2) \\
= \partial_T \left( \frac{R}{T} U_T (U - \nabla) + \frac{(U - \nabla)^2}{2T} \right) - \partial_R \left( \frac{R}{T} U_R (U - \nabla) \right) \\
+ \frac{R}{T} \left( |\nabla U|^2 - U_T^2 + U_T \nabla T + \Omega U - \nabla \frac{U - \nabla}{T} \right) + \Omega^{-2} U (U - \nabla)(e^{RU^2} - 1 - \Omega U^2) \\
= \partial_\phi \left( \frac{U - \nabla}{RT} U_\phi \right) .
\end{align*}
\]
Adding (26) and (27) with one half of (28) yields
\[
\partial_T \left( \frac{R^2}{T} (e + m + \frac{1}{2} U - \nabla + (U - \nabla)^2) \right) \\
- \partial_R \left( \frac{R^2}{T} (e + m - R^{-2}U_\phi^2 - \Omega^{-3}(e^{\Omega U^2} - 1 - \Omega U^2 - \frac{1}{2} \Omega^2 U^4) \\
+ \Omega U \frac{U - \nabla}{2R} \right) \\
\right) \\
= \partial_R \left( \frac{1}{T} (U_R + U_T + \frac{U - \nabla}{2R}) U_\phi \right) \\
+ \frac{R}{T} \left( \frac{3}{2} \Omega^{-3}(e^{\Omega U^2} - 1 - \Omega U^2 - \frac{1}{2} \Omega^2 U^4) \\
- \frac{1}{2} \Omega^{-2}U(U - \nabla)(e^{\Omega U^2} - 1 - \Omega U^2) \right) + R^2(1 - \frac{R}{T}) P. \\
\right)
\]

Lemma 3.1. For any time \( T_2 \) with \( 0 < T_2 < T_1 \) we have
\[
\int_{K_{T_2}} (1 \pm \frac{R}{T})(e \pm m) + \frac{(U - \nabla)^2}{2T^2} \frac{dXdT}{T} \leq C(1 + E_a + T_2^2 E_a^3), \\
\]
where \( K_{T_2} \) is the truncated light cone
\[
K_{T_2} := \{(T, X); T \leq T_2, |X| \leq T \}. \\
\]

Proof. Fix \( T_2 < T_1 \). We integrate equation (26) over the truncated cone \( K_{T_2} \). Then
\[
I_+ = \int_{K_{T_2}} \left( (1 + \frac{R}{T})(e + m) + \frac{(U - \nabla)^2}{2T^2} \right) \frac{dXdT}{T} \leq II + IV + V, \\
\]
where we label the terms \( I_+ \), \( II \), \( IV \) and \( V \) as in [9] pp6-9. Our only modification to the proof given there lies in how we handle the error
\[
V = \int_{K_{T_2}} \left( -\frac{1}{2} U_T \nabla_T - V_T \frac{U - \nabla}{2T} + RT(1 - \frac{R}{T}) P \right. \\
+ \frac{3}{2} \Omega^{-3}(e^{\Omega U^2} - 1 - \Omega U^2 - \frac{1}{2} \Omega^2 U^4) \\\n\left. - \frac{1}{2} U(U - \nabla) \Omega^{-2}(e^{\Omega U^2} - 1 - \Omega U^2) \right) \frac{dXdT}{T}. \\
\]
By (21),
\[
\int_{K_{T_2}} R \left( 1 - \frac{R}{T} \right) P dX dT \leq \int_{K_{T_2}} T P dX dT \leq E_a. \\
\]
For the remaining terms we add and subtract in \( \nabla \),
\[
\frac{3}{2} \Omega^{-3} \left( e^{\Omega U^2} - 1 - \Omega U^2 - \frac{1}{2} \Omega^2 U^4 \right) - \frac{1}{2} U(U - \nabla) \Omega^{-2} \left( e^{\Omega U^2} - 1 - \Omega U^2 \right) \\
= \frac{3}{2} \Omega^{-3} \left( e^{\Omega U^2} - 1 - \Omega U^2 - \frac{1}{2} \Omega^2 U^4 \right) - \frac{1}{2} U(U - \nabla) \Omega^{-2} \left( e^{\Omega U^2} - 1 - \Omega U^2 \right) \\
- \frac{3}{2} \Omega^{-3} \left( e^{\Omega U^2} - 1 - \Omega U^2 \right) + \frac{3}{2} \Omega^{-3}(e^{\Omega U^2} - 1 - \Omega U^2 - \frac{1}{2} \Omega^2 U^4) \\\n= f(U, \nabla) + \frac{3}{2} \Omega^{-3}(e^{\Omega U^2} - 1 - \Omega U^2 - \frac{1}{2} \Omega^2 U^4) . \\
\]
We can compensate the second term with the pointwise bound from (22),
\[
\frac{3}{2}\Omega^{-3}(e^{\Omega V^2} - 1 - \Omega V^2) - \frac{1}{2}\Omega^2 V^4 = \frac{3}{2} \sum_{k=1}^{\infty} \frac{\Omega^{k-3} V^{2k}}{k!} = \frac{3}{2} V^6 \sum_{k=0}^{\infty} \frac{(\Omega V^2)^k}{(k+3)!} \leq \frac{3}{2} V^6 e^{\Omega V^2} \leq C \log^3 \left( \frac{1}{T} \right) \frac{1}{T^{1/3}} \leq C \log^3 \left( \frac{1}{T} \right) \frac{1}{T} ,
\]
where we used \( \Omega \leq 1 \). Then
\[
\int_{K \tau_2} \log \left( \frac{1}{T} \right) \frac{1}{T} \frac{dX}{dT} \leq C \int_0^T \log \left( \frac{1}{T} \right) dT \leq C < \infty .
\]
Recalling that
\[
f(U, V) = \frac{3}{2} \sum_{k=1}^{\infty} \frac{\Omega^{k-3} (U^{2k} - V^{2k})}{k!} - \frac{1}{2} U(U - V) \sum_{k=2}^{\infty} \frac{\Omega^{k-2} U^{2k}}{k!} ,
\]
we observe that \( f(-U, -V) = f(U, V) \). Furthermore, if \( U \) and \( V \) have opposite sign, say \( U \geq 0, V \leq 0 \), then \( U(U - V) \geq U^2 \). Comparing coefficients we see that the second power series dominates the first and \( f \) is negative. Therefore, we only need to analyse the case \( U, V > 0 \).

i) First, if \( U \leq V \) then
\[
f(U, V) \leq \frac{1}{2} V^2 \Omega^{-3} (e^{\Omega V^2} - 1 - \Omega V^2) \leq \frac{1}{2} V^6 e^{\Omega V^2} ,
\]
which we estimate with the bound on \( |V| \) as above.

ii) Second, if \( V < U \leq 4V \) then
\[
f(U, V) \leq \frac{3}{2} \Omega^{-3} \left( e^{16\Omega V^2} - 1 - 16\Omega V^2 - \frac{1}{2}(16\Omega)^2 V^4 \right) \leq \frac{3}{2} 16^3 V^6 e^{16\Omega V^2} \leq C \log^3 \left( \frac{1}{T} \right) \frac{1}{T} ,
\]
where the factor 4 in (22) together with \( \Omega \leq 1 \) ensure that the power in \( 1/T \) stays smaller than 1.

iii) For the remaining case \( U > 4V \) we write \( V = \alpha U \), i.e. \( \alpha < 1/4 \). Then we analyse the power series
\[
f(U, V) = \frac{1}{4} (U^6 - V^6) + \frac{3}{2} \sum_{k=4}^{\infty} \frac{\Omega^{k-3} (U^{2k} - V^{2k})}{k!} - \frac{1}{2} U(U - V) \sum_{k=2}^{\infty} \frac{\Omega^{k-2} U^{2k}}{k!} .
\]
For the leading term we use \( \alpha < 1/4 \) to compare with \( (U - V)^6 \),
\[
U^6 - V^6 = U^6(1 - \alpha)^6 \leq C U^6 (1 - \alpha)^6 = C(U - V)^6 .
\]
Then, by the Poincaré-Sobolev inequality, on each time slice
\[
\int_{B_T(0)} \frac{(U - \mathbf{V})^6}{T} \, dX \leq \frac{C}{T} \left( \int_{B_T(0)} |\nabla U|^2 \, dX \right)^6 \\
\leq CT \left( \int_{B_T(0)} |\nabla U|^2 \, dX \right)^3 \\
\leq CTE_a^3.
\]
Integration in time yields a term bounded by $T^{-2\alpha}E_a^3$. The remaining power series is negative, as
\[
3\sum_{k=4}^{\infty} \frac{\Omega^k (U^{2k} - \mathbf{V}^{2k})}{k!} - \frac{1}{2} U(U - \mathbf{V}) \sum_{k=2}^{\infty} \frac{\Omega^{k-2} U^{2k}}{k!} \\
= U^6 \left( -\frac{1}{2} (1 - \alpha) \\
+ \sum_{k=1}^{\infty} \frac{(\Omega U^2)^k (1 - \alpha^{2(k+3)})}{(k+3)!} \left( \frac{1}{2} \left( 3(1 - \alpha^{2(k+3)}) - (1 - \alpha)(k+3) \right) \right) \right) \\
\leq 0.
\]
Note, that this calculation further motivates the exclusion of $u^3$ in the original equation.

With those modifications Struwe’s original proof yields the desired statement.

\[\square\]

3.5. Bound inside a hyperboloid. We fix a time $0 < T_\varepsilon < T_1$ such that
\[
\text{Flux}(u, M^{T_\varepsilon}) + \int_{K^{T_\varepsilon}} \left( 1 \pm \frac{R}{T} \right) (e \pm m) + \frac{(U - \mathbf{V})^2}{T^2} \, dX \, dT < \varepsilon.
\]
In the same fashion as in Struwe’s Lemma 4.3 we obtain

**Lemma 3.2.** There exists $\varepsilon > 0$ and a constant $C < \infty$ such that for any $0 < T \leq 4^{-1} T_\varepsilon$ there holds
\[
\int_{K^T} e^{4U^2} \, dX \, dT \leq CT.
\]

The region $\Phi^{-1}(K^T)$ is a hyperboloid. Its asymptote is the cone $\{ r = t - 1/(2T) \}$. In the following we fix $T \leq 4^{-1} T_\varepsilon$. Let $t_0 = 1/T$, the smallest time inside the hyperboloid. Furthermore, we denote $D = \Phi^{-1}(K^T)$.

Using the above Lemma we obtain decay of the nonlinearity in $L^2_t L^2_x$ locally in time.

**Lemma 3.3.** Let $t_2 \geq t_1 \geq t_0$. Then
\[
\int_{D \cap \{ t_1 \leq t \leq t_2 \}} |N(u)|^2 \, dx \, dt \leq C t_1^{-2}.
\]
Proof. Inside $D_{t_1}^{t_2} = D \cap \{t_1 \leq t \leq t_2\}$ we have $t + r \geq t$ and $t - r \geq 1/(2T)$. Therefore, $\Omega \leq C/t_1$ with a constant $C$ that is uniform over $D_{t_1}^{t_2}$. Then, we calculate

\[
\int_{D_{t_1}^{t_2}} |u(e^{u^2} - 1 - u^2)|^2 \, dx \, dt = \int_{\Phi(D_{t_1}^{t_2})} \Omega U^2(e^{\Omega U^2} - 1 - \Omega U^2)^2 \Omega^{-3} \, dX \, dT \leq \int_{\Phi(D_{t_1}^{t_2})} \Omega^2 U^{16} e^{2\Omega U^2} \, dX \, dT \leq \frac{C}{t_1^4} \int_{\Phi(D_{t_1}^{t_2})} e^{3\Omega U^2} \, dX \, dT \leq C T \Omega^{-3} \frac{t_2}{t_1^2}.
\]

□

We conclude

Lemma 3.4. Inside $D$ the nonlinearity is bounded in $L_{t_1}^1 L_{t_2}^2$, i.e.

\[
\int_{t_0}^\infty \left( \int_{D \cap (\{t\} \times \mathbb{R}^2)} |N|^2 \, dx \right)^{\frac{1}{2}} \, dt < \infty.
\]

Proof. Divide $[t_0, \infty]$ into intervals $I_n = [t_0 2^n, t_0 2^{n+1}]$. Then, by Hölder’s inequality and Lemma 3.3

\[
\int_{t_0}^\infty \left( \int_{D \cap (\{t\} \times \mathbb{R}^2)} |N|^2 \, dx \right)^{\frac{1}{2}} \, dt = \sum_{n=0}^{\infty} \int_{I_n} \left( \int_{D \cap (\{t\} \times \mathbb{R}^2)} |N|^2 \, dx \right)^{\frac{1}{2}} \, dt \leq \sum_{n=0}^{\infty} (t_0 2^n)^{\frac{1}{2}} \left( \int_{I_n} \int_{D \cap (\{t\} \times \mathbb{R}^2)} |N|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \sum_{n=0}^{\infty} C t_0^{\frac{1}{2}} 2^{-n} < \infty.
\]

3.6. The case of radial data. In the previous section we have obtained control of the nonlinearity inside a hyperboloid $\Phi^{-1}(K^T)$, where $T \leq 4^{-1} T_\epsilon$. Let $t_0 = 1/T$, the smallest time in the hyperboloid. Now fix $T$ and choose $d > \frac{1}{2T}$. Let

\[ A_{t_1} = \{(t, x); t \geq t_1, t - d \leq |x| \leq t\}. \]

Then, there exists a time $t_1 := t_0$ such that

\[ \{(t, x); t \geq t_1, |x| \leq t\} \subset (\Phi^{-1}(K^T) \cap \{(t, x); t \geq t_1\}) \cup A_{t_1}, \]

i.e. the thinned cone $A_{t_1}$ covers the region where we have not yet obtained control over the nonlinearity.

In the following, we will restrict ourselves to the case of radial solutions. We will show that we can bound the nonlinearity inside $A_{t_1}$ in $L_{t_1}^1 L_{t_2}^2$. 

In the case of radially symmetric data we employ the following bound. Let \( t > t_1 \) fixed, \( t - d \leq r \leq t \). Recall, that \( u \) is compactly supported within \( B_t(0) \). Then,
\[
|u(t, r)| \leq \int_r^t |\partial_s u(t, s)| ds \\
\leq \int_{t-d}^t |\partial_s u(t, s)| ds \\
\leq \left( \int_{t-d}^t |\partial_s u(t, s)|^2 s \, ds \right)^{1/2} \left( \int_{t-d}^t \frac{1}{s} \, ds \right)^{1/2} \\
\leq CE^\frac{1}{2} \left( \log \left( \frac{t}{t-d} \right) \right)^{1/2}.
\]
Therefore there exists \( t_2 \geq t_1 \) such that for all \( t \geq t_2 \)
\[
|u(t, r)| \leq \frac{C}{t^2},
\]
with a constant \( C \) independent of \( t \geq t_2 \).

**Lemma 3.5.** Let \( t_2 \) as above. Then \( \mathcal{N} \) is bounded in \( L^1_t L^2_x \) inside \( A_{t_2} \).

**Proof.** Again we estimate
\[
|N(u)| = |u|(|e^{u^2} - 1 - u^2|) \leq \frac{1}{2} |u|^5 e^{u^2}
\]
pointwise. Then,
\[
\int_{B_t(0)\setminus B_{t-d}(0)} u^{10} e^{2u^2} \, dx \leq Ct \cdot t^{-5} = Ct^{-4}.
\]
Therefore,
\[
\int_{t_2}^\infty \left( \int_{B_t(0)\setminus B_{t-d}(0)} u^{10} e^{2u^2} \, dx \right)^{1/2} \, dt \leq C \int_{t_2}^\infty t^{-2} \, dt < \infty.
\]

Combining Lemma 3.3 with Lemma 3.5 we obtain \( \|\mathcal{N}\|_{L^1([t_2, \infty) ; L^2(\mathbb{R}^2))} < \infty \). Using Lemma 1.2 we conclude the proof of Theorem 1.1.

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