Filtering and estimation in stochastic volatility models with rationally distributed disturbances

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Abstract

This paper deals with the filtering problem for a class of discrete time stochastic volatility models in which the disturbances have rational probability density functions. This includes the Cauchy distributions and Student t-distributions with odd number of degrees of freedom. Using state space realizations to represent the rational probability density functions we are able to solve the filtering problem exactly. However the size of the involved state space matrices grows exponentially with each time step of the filter. Therefore we use stochastically balanced truncation techniques to approximate the high order rational functions involved. In a simulation study we show the applicability of this approach. In addition a simple method of moments estimator is derived.

Keywords: stochastic volatility, filtering, rational probability density function, state space realization, stochastically balanced truncation.


Introduction

In the area of financial time series the Black-Scholes model is often used for modelling the behaviour of the price of stocks, exchange rates and other financial time series. This is also the basis for much of the literature on pricing of derivative financial instruments such as options. However it is considered to be a well-known fact that although the volatility is assumed to be constant in the Black-Scholes model, in practice it is varying. This has led to the investigation of more general models in which the volatility is allowed to vary. One can broadly distinguish between two types of generalizations. One is the type of model in which the volatility is varying over time and its dynamic behaviour is described by some stochastic process. A problem with such models is that it is generally difficult to solve the volatility estimation problem for such models: the calculation of the conditional density of the volatility at some point in time, given the observations up till that same point in time, is usually a difficult task for which there are no closed form expressions. In the literature there are several proposals to approximate the conditional density, cf. e.g. [11], [14], [1]. The other, second type of model that is used is the ARCH model and its generalizations ([4], see also e.g. [7]), as applied to financial time series. These models have the advantage that the volatility is again time varying, and the conditional volatility (also called conditional heteroskedasticity in this context) is in fact prescribed by the model as a deterministic function of the past observations. By construction the problem of estimating the stochastic volatility has been solved in these models. However one could argue that this is at the expense of a less transparent model for the underlying data generating process. In the present paper a model of the first type will be presented, however with the advantage that for this model the volatility estimation problem can be solved, as we will show. Apart from the volatility to be time-varying another feature of financial time series that is often reported is that it has fat tails. In the literature there are many studies that try to deal with this phenomenon by specifying non-Gaussian disturbances. This goes back to the work of [15] who suggested to consider the class of stable distributions as possible distributions for the disturbances. An important example of stable distributions is given by the Cauchy distributions. More recent studies seem to favor other distributions, including Student t-distributions (cf e.g. [2] p. 19, [13]). In the approach followed in the present paper all disturbances are allowed which have a rational probability density function on the real line. This includes the Cauchy distributions and Student t-distributions with odd number of degrees of freedom. In fact it is well-known that the Gaussian
distribution can be approximated by a Student t-distribution of sufficiently high number of degrees of freedom. Therefore in a sense the corresponding Gaussian model is a limiting case of the class of models presented here. It should perhaps be stressed from the start that there is a price to be paid in the form of high complexity if one wants to use rational densities of higher (McMillan) degree. From the point of view of complexity in fact the estimation problem is easiest when the disturbances have Cauchy density. In a previous paper a matrix calculus was developed for performing various calculations with rational probability density functions ([10]) and applied to a filtering problem for a class of linear dynamical models. Here we extend this calculus and show how it can be fruitfully applied to the non-linear filtering problem of volatility estimation, in a specific class of stochastic volatility models.

The main extension of the calculus concerns a state-space formula for the composition of a proper rational function (which can be allowed to be a proper rational matrix function) with a proper rational function, under some minor condition that is required to ensure the resulting (rational) function is again proper.

It is shown that the conditional probability density functions for the state are all rational functions in this model class and we provide an explicit way to calculate these and hence solve the filtering problem exactly. However as the complexity of the resulting rational probability density functions increases very quickly over time, the exact filter cannot be implemented practically, except during a short period of time. An important innovation in this respect is the application of an approximation method stemming from stochastic systems theory, called the SBT (stochastically balanced truncation) method. This method allows to find a lower order positive rational density function which differs at each point on the real line by at most a given prescribed percentage of the original rational density function. In an application we use a tolerance level of 2% giving excellent results. (The bound used is well-known in stochastic systems theory and is based on the deep and elegant theory of Hankel norm approximation). In the implementation of the filter one needs to switch between various representations of the rational probability density functions. Numerically reliable methods are presented to perform these steps. The possibility to implement the various theoretical ideas in a numerically stable way is crucial for the success of the practical implementation and forms one of the key contributions of this paper to the practical usage of rational probability density functions in filtering problems. We provide the results of some applications to simulated data and to empirical FX (foreign exchange) data and present a number
of conclusions. A number of the technical results used are collected in an appendix.

1 The model class

Stochastic volatility models that we will consider are of the following form:

\[
\begin{align*}
X_{t+1} &= aX_t + W_t \\
Y_t &= V(X_t)U_t
\end{align*}
\]  

(1)

where for each \( t \in \mathbb{N} = \{1, 2, \ldots\} \), the random variables \( X_t, W_t, Y_t, U_t \) take their values in the real numbers, and where \( V(x) \) is a real-valued, positive polynomial function of \( x \in \mathbb{R} \); \( \{W_t, \, t \in \mathbb{N}\} \) and \( \{U_t, \, t \in \mathbb{N}\} \) are sequences of jointly stochastically independent real valued random disturbances with time-invariant probability density functions: for each \( t \in \mathbb{N} \), \( W_t \) has rational probability density function \( p_{W}(w) \), \( U_t \) has rational probability density function \( p_{U}(u) \). The initial state \( X_1 \) has rational probability density function \( p_{X_1}(x) \).

The parameter \( a \) is a real number that will be assumed to be unequal to zero for ease of exposition. In financial applications, the \( Y_t \) usually stand for the returns \( Y_t = \log(S_{t+1}/S_t) \) of some price process \( \{S_t, \, t \in \mathbb{N}\} \).

A number of remarks can be made about this model class.

(i) The family of rational probability density functions is a very rich class. It contains the stable class of Cauchy densities, it contains the Student distributions with odd number of degrees of freedom. Under relatively mild conditions, probability density functions can be approximated by rational probability density functions, as follows from results of the theory of rational approximation. It is well-known that the Gaussian probability density functions can be approximated for example by the Student t-distribution of sufficiently high number of degrees of freedom, therefore the Gaussian case appears in a certain sense as a limiting case of our model class.

(ii) The function \( V \) is a positive polynomial, i.e. for all \( x \in \mathbb{R} \), \( V(x) > 0 \). Here this is required for technical reasons. In the literature one finds other positive functions as specifications for \( V \) as well, for example an exponential function \( V(x) = \exp\left(\frac{x+2}{2}\right) \) (cf. e.g. [20]). If desired one can approximate the exponential function on any given finite interval by a positive polynomial. Generalization of the results presented here, to the case in which \( V \) is a non-negative polynomial, i.e. for all \( x \in \mathbb{R} \), \( V(x) \geq 0 \) is straightforward.
The parameters in the model as well as in the rational probability density functions of $W_t$ and $V_t$, $t = 1, 2, \ldots$, are assumed to be constants here. However they could be taken time-varying if desired. The resulting filter equations for that case form a straightforward extension of the filter equations presented in this paper.

2 The filter

We consider the following nonlinear filtering problem: Estimate at each time $t \in \mathbb{N}$ the volatility $V(X_t)$ from the sequence of observations $Y_t := \{Y_s, s \in \mathbb{N}, s \leq t\}$. (Note that we will use the same symbols $Y_t := \{Y_s, s \in \mathbb{N}, s \leq t\}$ for the random variables and their observed values. This is to avoid complicitating the notation any further. The interpretation of the symbols as random variables or observed values should be clear from the context). The solution of such a problem consists of finding for each $t \in \mathbb{N}$ the conditional probability density function of $X_t$ given $Y_t$, and deriving the desired estimate of $V(X_t)$ from this. Let the conditional density of some random variable $Z$ given $Y_t$ be denoted by $p_{Z|Y_t}$.

The filter consists of a set of recursive equations by which one can calculate the conditional probability density function of the state $X_t$ given the observations $Y_t$. The filter consists of a prediction step and an update step. In the prediction step one calculates the conditional density $p_{X_t, Y_t}$ of $X_t$ given the observations $Y_t$ starting from the conditional density $p_{X_t, Y_t}$ of $X_t$ given $Y_t$:

$$
p_{X_{t+1}|Y_t} = p_{aX_t|Y_t} * pW.
$$

Here $*$ denotes convolution.

In the update step one calculates the conditional density of $X_t$ given the observations $Y_t$ and the conditional density of $X_t$ given $Y_t$, using Bayes’ rule. Suppose the conditional probability density function $p_{X_t|Y_t-1}$ of $X_t$ given $Y_t$ is known and the observation $Y_t$ becomes available. The joint density of $(X_t, Y_t)$ can be obtained from the joint density of $(X_t, U_t)$ by a change of variables:

$$
\begin{pmatrix}
  X_t \\
  Y_t
\end{pmatrix} = \begin{pmatrix}
  X_t \\
  V(X_t)U_t
\end{pmatrix}.
$$

The inverse Jacobian determinant of this change of variables is $\frac{1}{V(X_t)}$, which is positive because $V$ is a positive polynomial. It follows that the joint
density of \((X_t, Y_t)\) is given by \(p_{X_t,Y_t|Y_{t-1}}(x, y) = \frac{p_{X_t,Y_t|Y_{t-1}}(x, \frac{y}{V(x)})}{V(x)} = p_{X_t|Y_{t-1}}(x)p_{Y_t|Y_{t-1}}(\frac{y}{V(x)})\). Substituting \(y = Y_t\), we obtain the following expression for the density of \(X_t|Y_t\):

\[ p_{X_t|Y_t}(x) = \frac{1}{c_t} p_{X_t|Y_{t-1}}(x)p_{Y_t|Y_{t-1}}(\frac{Y_t}{V(x)}) \frac{1}{V(x)}, \]

where

\[ c_t = \int_{-\infty}^{\infty} p_{X_t|Y_{t-1}}(x)p_{Y_t|Y_{t-1}}(\frac{Y_t}{V(x)}) \frac{1}{V(x)} dx. \]

Since \(p_{X_t,Y_t|Y_{t-1}}(x, Y_t) = p_{X_t|Y_t}(x)p_{Y_t|Y_{t-1}}(Y_t)\) it follows that

\[ c_t = \int p_{X_t,Y_t|Y_{t-1}}(x, Y_t) dx = p_{Y_t|Y_{t-1}}(Y_t). \]

Therefore we may evaluate the likelihood as

\[ p_{Y_1,Y_2,...,Y_T}(Y_1, Y_2, \ldots, Y_T) = c_1c_2 \cdots c_T. \]

Note that the value of the normalization constants \(c_1, c_2, \ldots, c_T\) easily follows from Proposition 3.1 without the need for explicit integration.

As shown in [10] the convolution of two rational density functions is a rational function too. Therefore it follows easily that the conditional density functions defined above will all be rational, given our assumptions! A way to implement the filter using ideas from system theory will be presented in the next sections.

### 3 State-space calculus for rational probability density functions

#### 3.1 Introduction to the state-space calculus

The key idea is to identify rational densities with spectra of linear, dynamic, continuous time, finite dimensional systems. This allows us to use concepts and methods from systems theory; for an overview cf, e.g, [19].

Consider a rational non-normalized probability density function \(\rho(x)\). With it we associate a rational function \(\Phi(s)\) on the complex plane which is specified on the imaginary axis by

\[ \rho(x) = \Phi(ix), \quad \forall x \in \mathbb{R}. \]

Note that \(\Phi(\cdot)\) is a rational function which is nonnegative and integrable on the imaginary axis. Such a function will be called an integrable spectral
density in this paper. The function $\Phi$ has a representation as
\[
\Phi(s) = \frac{g_0 + g_1 s + \cdots + g_{2q} s^{2q}}{f_0 + f_1 s + \cdots + f_{2n} s^{2n}}; \quad n > q; \quad g_k, f_k \in \mathbb{C}
\]
with coprime polynomials $g(s) = g_0 + g_1 s + \cdots + g_{2q} s^{2q}$ and $f(s) = f_0 + f_1 s + \cdots + f_{2n} s^{2n}$. Since $\Phi(s)$ is strictly proper there exists a state space representation, i.e. a triple $[\tilde{F} \in \mathbb{C}^{2n \times 2n}, \tilde{G} \in \mathbb{C}^{2n \times 1}, \tilde{H} \in \mathbb{C}^{1 \times 2n}]$, such that
\[
\Phi(s) = \tilde{H}(sI_{2n} - \tilde{F})^{-1}\tilde{G}.
\]
Note that we need complex valued triples as $\rho$ may be non-symmetric. Further note that this representation is not unique. A state-space transformation $[\tilde{F}, \tilde{G}, \tilde{H}] \mapsto [T\tilde{F}T^{-1}, T\tilde{G}, HT^{-1}]$, where $T \in \mathbb{C}^{2n \times 2n}$ is a non-singular matrix, leads to a usually different state-space representation of the same function $\Phi$. As a shorthand notation for such a state space realization we will write:
\[
\Phi = \pi \left[ \begin{array}{c|c}
\tilde{F} & \tilde{G} \\
H & 0
\end{array} \right],
\]
where $\pi$ will be used in general to denote the mapping that maps a partitioned matrix $\left[ \begin{array}{c|c} A & B \\ C & D \end{array} \right]$ to the corresponding rational function $C(sI - A)^{-1}B + D$. It is assumed that the partitioning involved will be clear from the context in all cases. Clearly $\pi$ is invariant under state-space transformation.

Since $\Phi(ix) \geq 0$ holds for all $x \in \mathbb{R}$, there exists an additive as well as a multiplicative decomposition of $\Phi(s)$ of the form
\[
\Phi(s) = Z(s) + Z^*(s) = K(s)K^*(s)
\]
where $Z$ and $K$ are strictly proper. The rational transfer function $Z(s)$ is called a spectral summand and $K(s)$ is a spectral factor. Here for a rational complex function $G(s)$, $G^*(s)$ is defined as $G^*(s) = G(-\bar{s})$, where $\bar{z}$ denotes complex conjugation. In particular note that $\Phi^*(s) = \Phi(s)$ holds. Since $\Phi(s)$ has no poles on the imaginary axis, a stable summand $Z(s)$, and a stable factor $K(s)$ may be chosen, i.e. $K(s)$ and $Z(s)$ have no pole in the closed right half plane. From now on we always impose stability on $Z$ and $K$.

Since $Z, K$ are strictly proper rational functions, there exist state space representations:
\[
K = \pi \left[ \begin{array}{c|c}
A & B \\ C & 0
\end{array} \right]; \quad K^* = \pi \left[ \begin{array}{c|c}
-A^* & C^* \\ -B^* & 0
\end{array} \right]
\]
Here and elsewhere in this paper $M^*$ denotes the Hermitean transpose of a matrix $M$. It is important to note that the two above realizations may be chosen to share the $A$ and $C$ matrix.

Given these state space realizations for $Z(s)$ and $K(s)$, we may construct two alternative state space realizations for $\Phi$:}

\[
\begin{bmatrix} F & G \\ H & 0 \end{bmatrix} := \begin{bmatrix} A & 0 & M \\ 0 & -A^* & C^* \\ C & -M^* & 0 \end{bmatrix}, \Phi = \pi \begin{bmatrix} F & G \\ H & 0 \end{bmatrix}, \quad (4)
\]

\[
\begin{bmatrix} \tilde{F} & \tilde{G} \\ H & 0 \end{bmatrix} = \begin{bmatrix} A & -BB^* \\ 0 & -A^* \\ C & 0 \end{bmatrix} \begin{bmatrix} 0 \\ C^* \end{bmatrix}, \Phi = \pi \begin{bmatrix} \tilde{F} & \tilde{G} \\ H & 0 \end{bmatrix}, \quad (5)
\]

using standard formulas for the state space realizations of the sum and product of two rational functions, see Appendix A.2.

The co-degree of a proper rational function $G(s)$ is defined as the multiplicity of the zero of $G$ at infinity. Thus the co-degree of $\Phi$ is $2n - 2q$, see (2). Clearly the co-degree of $\Phi$ is twice the co-degree of its spectral factor $K$ and thus is even. For a more detailed discussion on the co-degree and the zeros of a rational function see Appendix A.1.

As shown in [10] the following proposition concerning the normalization constant and the moments of a rational density holds:

**Proposition 3.1** Let $X$ be a real random variable with non-normalized rational probability density function $\rho$ with corresponding spectral summand $Z$, hence $\Phi(ix) := \rho(x) = Z(ix) + Z^*(-ix)$, and let $(A, M, C)$ be a stable state-space realization of $Z$. Then $CM$ is real and positive and $\frac{\rho}{\pi CM}$ is the probability density function corresponding to $X$. The moments $\mathbb{E}(X^l)$ of $X$ exist for $l = 0, \ldots, k - 2$, where $k$ is the co-degree of $\Phi$ and the moments are given by $\mathbb{E}(X^l) = (-i)^l \frac{CA^lM}{CM}$, $l = 0, 1, 2, \ldots, k - 2$.

In [10] it was shown, i.a., that the operations of scaling and convolution of rational density functions can be translated into linear algebra operations on corresponding state-space realizations. For ease of reference these results are collected in the Appendix A.5.
3.2 The composition formula

A key step in the present paper is the construction of a realization of the rational density function given by \( p_U\left(\frac{y}{V(x)}\right) / V(x) \), where \( y \) is a fixed non-zero real number and \( V(x) \) is a positive polynomial on the real line, if a realization of \( \Phi(ix) = p_U(x) \) is known. Such a realization is constructed via the following proposition which gives a realization formula for the composition \( G = G_1 \circ g_2 \), \( G(s) = G_1(g_2(s)) \) of two proper rational complex functions \( G_1 \) and \( g_2 \). Here we will apply this result to \( G_1(ix) = p_U(yix)x \) and \( g_2(ix) = i/V(x) \). This implies that we could allow \( V(\cdot) \) to be a rational positive function such that \( 1/V(x) \) is strictly proper.

The only constraint on the pair \( G_1, g_2 \) will be that the direct feedthrough \( d_2 = \lim s \to -\infty g_2(s) \) is not a pole location of \( G_1 \), i.e. \( G_1(d_2) \neq \infty \), because otherwise the composition \( G_1 \circ g_2 \) would have a pole at infinity, or in other words it would not be proper rational function and therefore would not have a state space representation of the form that we use here. In fact in the proposition we will allow \( G_1 \) even to be a rational matrix function, corresponding to a multi-input, multi-output system in the system theoretic interpretation.

**Proposition 3.2** Let \( G_1, g_2 \) be proper rational functions with state space realizations \( (A_1 \in \mathbb{C}^{n_1 \times n_1}, B_1 \in \mathbb{C}^{n_1 \times m_1}, C_1 \in \mathbb{C}^{p_1 \times n_1}, D_1 \in \mathbb{C}^{p_1 \times m_1}) \) and \( (A_2 \in \mathbb{C}^{n_2 \times n_2}, b_2 \in \mathbb{C}^{n_2 \times 1}, c_2 \in \mathbb{C}^{1 \times n_2}, d_2 \in \mathbb{C}) \) respectively, so \( G_1(s) = D_1 + C_1(sI - A_1)^{-1} B_1 \), \( g_2(s) = d_2 + c_2(sI - A_2)^{-1} b_2 \). Assume that \( d_2 \) is not an eigenvalue of \( A_1 \). Then the composition \( G = G_1 \circ g_2 \) is again a proper rational (matrix) function with state space realization \( (A, B, C, D) \) given by the formulas

\[
\begin{align*}
A &= I_{n_1} \otimes A_2 + (A_1 - d_2 I_{n_1})^{-1} \otimes b_2c_2 \in \mathbb{C}^{n_1 n_2 \times n_1 n_2} \\
B &= -(A_1 - d_2 I_{n_1})^{-1} B_1 \otimes b_2 \in \mathbb{C}^{n_1 n_2 \times m_1} \\
C &= C_1(A_1 - d_2 I_{n_1})^{-1} \otimes c_2 \in \mathbb{C}^{p_1 \times n_1 n_2} \\
D &= D_1 - C_1(A_1 - d_2 I_{n_1})^{-1} B_1 \in \mathbb{C}^{p_1 \times m_1}
\end{align*}
\]  

(6)

Here \( \otimes \) denotes the Kronecker product, see e.g. [12], ch. 12.

**Proof:** Use will be made by the following inversion formula for rational matrices that is well known in system theory. Let \( \hat{A}, \hat{B}, \hat{C}, \hat{D} \), with \( \hat{D} \) invertible, denote the state space realization of a proper rational function \( \hat{G}(s) = \hat{D} + \hat{C}(sI - \hat{A})^{-1} \hat{B} \). Its inverse is given by \( (\hat{G}(s))^{-1} = \hat{D}^{-1} - \hat{D}^{-1} \hat{C}(sI - \hat{A} + \hat{B} \hat{D}^{-1} \hat{C})^{-1} \hat{B} \hat{D}^{-1} \).

We need to show that the rational matrix \( G(s) \) with state space realization given by (6) is equal to \( G_1(g_2(s)) \). In order to do that we calculate
$G(s)$ as follows:

\[
G(s) = D + C(sI - A)^{-1}B \\
= D_1 - C_1(A_1 - d_2I_{n_1})^{-1}B_1 \\
\quad - (C_1(A_1 - d_2I_{n_1})^{-1} \otimes c_2) H^{-1}((A_1 - d_2I_{n_1})^{-1}B_1 \otimes b_2) \\
= D_1 - C_1(A_1 - d_2I_{n_1})^{-1}B_1 \\
- C_1(A_1 - d_2I_{n_1})^{-1}(I_{n_1} \otimes c_2) H^{-1}(I_{n_1} \otimes b_2)(A_1 - d_2I_{n_1})^{-1}B_1 \\
= D_1 + C_1 \left[ -(A_1 - d_2I_{n_1})^{-1} \right. \\
\left. - (A_1 - d_2I_{n_1})^{-1}(I_{n_1} \otimes c_2) H^{-1}(I_{n_1} \otimes b_2)(A_1 - d_2I_{n_1})^{-1} \right] B_1.
\]

where

\[
H = sI_{n_1n_2} - I_{n_1} \otimes A_2 - (A_1 - d_2I_{n_1})^{-1} \otimes b_2 c_2 \\
= sI_{n_1} \otimes I_{n_2} - I_{n_1} \otimes A_2 - (I_{n_1} \otimes b_2)(A_1 - d_2I_{n_1})^{-1}(I_{n_2} \otimes c_2).
\]

This expression has the form $D_1 + C_1(\tilde{G}(s))^{-1}B_1$, where $\tilde{G}(s) = \tilde{D} + \tilde{C}(sI - \tilde{A})^{-1}B$ and

\[
\tilde{D} = -(A_1 - d_2I_{n_1}), \quad \tilde{C} = (I_{n_1} \otimes c_2), \quad \tilde{A} = I_{n_1} \otimes A_2, \quad \tilde{B} = (I_{n_1} \otimes b_2).
\]

It follows that

\[
\tilde{G}(s) = -(A_1 - d_2I_{n_1}) + (I_{n_1} \otimes c_2)(sI_{n_1n_2} - I_{n_1} \otimes A_2)^{-1}(I_{n_1} \otimes b_2) \\
= I_{n_1} \otimes (d_2 + c_2(sI_{n_2} - A_2)^{-1}b_2) - A_1 \\
= I_{n_1} (d_2 + c_2(sI_{n_2} - A_2)^{-1}b_2) - A_1.
\]

and thus

\[
G(s) = D_1 + C_1 \left( I_{n_1} (d_2 + c_2(sI_{n_2} - A_2)^{-1}b_2) - A_1 \right)^{-1} B_1 = G_1(g_2(s)).
\]

\[\square\]

Remark. A special case of the composition formula can be found in the theory of phase-type distributions in statistics. See e.g. [17] and [18], equations (2.5.1), (2.5.2).

3.3 Transformations between the various representations of rational density functions

In order to implement the filter we need various different representations for the conditional densities, i.e. the integrable spectral density $\Phi$, the spectral summand $Z$, $Z + Z^* = \Phi$, and the spectral factor $K$, $KK^* = \Phi$. Thus we need procedures which compute such a representation from any of the others. The computation of $\Phi$ from $Z$ or $K$ follows from the formulas in
Appendix A.2. The computation of a spectral summand $Z$ from given $K$ or $\Phi$ is dealt with in the appendices A.3 and A.4 respectively. The most demanding task is the computation of a spectral factor given a spectral summand and this will be presented in the following subsection. It should be noted that this spectral factorization problem is a standard problem in systems theory. However most of the literature deals with the case where $\lim_{s \to \infty} \Phi(s) = R > 0$ holds, i.e. where there is no zero at infinity. Thus we found we had to develop a numerically robust procedure for the case where $\Phi(s)$ has a zero at infinity.

Let a spectral summand $Z(s) = C(sI_n - A)^{-1}M$ be given. Now the task is to compute $B$ such that $K(s) = C(sI_n - A)^{-1}B$ is a spectral factor, i.e. such that $\Phi(s) = Z(s) + Z^*(s) = K(s)K^*(s)$ holds. The basic tool for this conversion is the so called positive real lemma:

**Lemma 3.1** A stable rational function $Z(s) = C(sI_n - A)^{-1}M$ is positive real, i.e. $Z(ix) + Z^*(ix) \geq 0$ for all $x \in \mathbb{R}$, if and only if there exists a solution $P$ of the linear matrix inequality (LMI)

$$L(P) = \begin{bmatrix} -AP - PA^* & M - PC^* \\ M^* - CP & 0 \end{bmatrix} \geq 0 \quad (7)$$

If $P$ is a solution, then $\Phi(s) = Z(s) + Z^*(s) = K(s)K^*(s)$, where $K(s) = C(sI_n - A)^{-1}B$ and $B \in \mathbb{C}^{n \times r}$ is determined from

$$L(P) = \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix}^* \quad (8).$$

For a proof of this lemma see e.g. [5]. In addition we remark:

(i) The rank of $L(P)$ determines the column dimension of the function $K(s)$. In particular it can be shown that there always exist square factors $K$. Since we here deal exclusively with the scalar case, we are only interested in rank one solutions, i.e. in solutions $P$ where rank $L(P) = 1$.

(ii) By the asymptotic stability of $A$ it follows that any solution $P$ of the LMI is positive semidefinite.

(iii) The solution set $\mathcal{P} = \{P \mid L(P) \geq 0\}$ is convex and bounded. If the set $\mathcal{P}$ is non-empty it contains a minimal and a maximal element, $\underline{P}$ and $\overline{P}$ say, i.e. $\underline{P} \leq P \leq \overline{P}$ holds for all $P \in \mathcal{P}$. The minimum element corresponds to a minimum phase factor, $\underline{K}$ say, i.e. all zeros of $\underline{K}(s)$ are in the closed left half plane: $\underline{K}(s) = 0 \Rightarrow \Re(s) \leq 0$. Analogously $\overline{P}$ gives a maximum phase factor $\overline{K}$, i.e $\overline{K}(s) = 0 \Rightarrow \Re(s) \geq 0$. 

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By the positive real lemma it follows that the computation of the spectral factor is equivalent to the solution of the above LMI. A solution of this LMI will be constructed via the computation of what is known as a deflating subspace of the \((2n + 1) \times (2n + 1)\) dimensional pencil:

\[
\lambda E - N := \begin{bmatrix}
\lambda I_{2n} - F & G \\
-H & 0
\end{bmatrix} = \begin{bmatrix}
\lambda I_n - A & 0 & M \\
0 & \lambda I_n + A^* & C^*
\end{bmatrix}
\]

(9)

Note that the eigenvalues of this pencil are the zeros of \(\Phi(s)\). See Appendix A.1 for background material on pencils of the above form!

Suppose for the moment that we have given a (rank one) solution \(P = P^*\) of the LMI and the corresponding factor \(K(s) = C(sI_n - A)^{-1}B\). Furthermore let \(k = 2c\) be the co-degree of \(\Phi\) and thus \(c\) is the co-degree of \(K\). By some easy algebra it follows that

\[
\begin{bmatrix}
\lambda I_n - A \\
0
\end{bmatrix}
\begin{bmatrix}
P \\
I_n
\end{bmatrix}
\begin{bmatrix}
0 \\
-C^*
\end{bmatrix}
= \begin{bmatrix}
P B \\
I_n
\end{bmatrix}
\begin{bmatrix}
\lambda I_n + A^* \\
B^*
\end{bmatrix}
\]

This implies

(i)

\[
\begin{bmatrix}
P \\
I_n \\
0
\end{bmatrix}
\]

is a basis for a deflating subspace of the pencil \((\lambda E - N)\).

(ii)

\[
\begin{bmatrix}
\lambda I_n + A^* \\
B^*
\end{bmatrix}
\]

is the pencil corresponding to the zeros of \(K^*(s)\) and thus has a \((c + 1)\) dimensional infinite elementary divisor and an \((n - c)\) dimensional finite divisor corresponding to the finite zeros of \(K^*(s)\).

In order to construct a rank one solution of the LMI (7) we therefore have to compute an \((n + 1)\)-dimensional divisor of the pencil (9) which itself has a \((c + 1)\) dimensional infinite elementary divisor and an \((n - c)\) dimensional finite divisor. Let \(Z \in \mathbb{C}^{(2n+1)\times(n+1)}\) be a basis for the corresponding
deflating subspace, where in addition it is assumed that the first \( c+1 \) columns form a basis for the \( (c+1) \) dimensional deflating subspace corresponding to the \( (c+1) \) dimensional infinite elementary divisor. By the discussions in Appendix A.1 it follows that \( Z \) may be partitioned as

\[
Z = \begin{bmatrix}
0 & Z_{12} & Z_{13} \\
0 & Z_{22} & Z_{23} \\
z_{31} & 0 & z_{33}
\end{bmatrix}, \quad Z_{12}, Z_{22} \in \mathbb{C}^{n \times c}, Z_{13}, Z_{23} \in \mathbb{C}^{n \times (n-c)}.
\] (10)

Note that \( Z \) can be written as

\[
Z = \begin{bmatrix}
P & 0 \\
I_n & 0 \\
0 & 1
\end{bmatrix} T,
\]

where \( T \) is a \((n+1) \times (n+1)\) non-singular matrix. Hence the solution \( P \) is obtained from

\[
P = [Z_{12}, Z_{13}] [Z_{22}, Z_{23}]^{-1}.
\] (11)

The only remaining choice is the choice of the finite eigenvalues, which determine the zeros of the factor \( K^* \). E.g. in order to get the minimal solution \( \overline{P} \) (the minimum phase factor \( \overline{K} \)) one has to choose the \((n-c)\) anti stable eigenvalues \( \Re(\lambda_i) > 0 \). On the other hand choosing the stable eigenvalues \( \Re(\lambda_i) < 0 \) gives the maximum element \( \overline{P} \) and the maximum phase factor \( \overline{K} \).

This procedure works provided that there are no zeros on the imaginary axis (except for the zero at infinity). Therefore for our implementation of the filter in addition we assume that \( p_U(x) \) and \( p_X_1(x) \) are strictly positive, which implies that all conditional densities in the filter will be strictly positive. However the numerical implementation still may run into trouble if there are zeros “close” to the imaginary axis!

The actual procedure is now as follows: Start with the pencil (9) and bring it to the staircase form (20,21). Apply a QZ transformation to the lower right \(((2n-2c) \times (2n-2c))\) dimensional block to bring the whole pencil into a QZ form, see e.g. [6]. So \( QEZ \) and \( QNZ \) are upper triangular matrices and \( Q \) and \( Z \) are both unitary. Next by a sequence of \( 2 \times 2 \) orthogonal transformations the diagonal elements corresponding to the \( n-c \) anti-stable (stable) eigenvalues are shifted to positions \( c+2, c+3, \ldots, n+1 \), without losing the triangular structure. The desired basis for the deflating subspace then is given by the first \( n+1 \) columns of the final \( Z \) matrix.

Finally compute \( P \) as described in (10, 11) and \( B \) from (7), (8) in Lemma A.1.
3.4 Description of the filter in terms of state space formulas

We can now describe how the filter could be calculated using state space formulas. Recall that for each rational probability density we associate three rational functions, namely the spectral density $\Phi$, the spectral summand $Z$ and the spectral factor $K$. Above it has been discussed how one can obtain the state space realization of $K$ given the realization of $Z$. In Appendix A.3 it is shown how to compute a state space realization of $Z$ given a state space realization of $K$ and in A.4 a realization of $Z$ is computed from a realization of $\Phi$. A state space realization of $\Phi$ given a state space realization for $Z$ or $K$ follows from the formulas in Appendix A.2. Therefore we can switch between these state space realizations as needed.

To start consider the probability density $p_{X|Y_1^{t-1}}$ or for $t = 1$ the density $p_{X_1}$. Calculate its spectral factor, $K_1$ say. Consider the spectral density $\Phi_U$ of $U$ and construct a state space realization for $G(s) = \Phi_U(-is)/(-is)$, where $Y_t$ is the observed output variable. Construct a state space realization of the spectral density function $g(s) = i/V(-is)$. Then use the composition formula to obtain a realization of the spectral density $G \circ g$ of $p_U(Y_t/V(x))/V(x)$. Calculate the realization of the spectral factor, $K_2$ say, of this density. Construct the product of $K_1$ and $K_2$ (see Appendix A.2), this gives the realization of the spectral factor of $c_t p_{X|Y_1^t}$. Calculate the corresponding spectral summand. Compute the normalization factor $c_t$ from Proposition 3.1 and compute the realization of the spectral summand of $p_{X|Y_1^t}$. This will be the input for the prediction step.

Calculate the realization of the spectral summand of $p_{X|Y_1^t}$ by using the formula for the scaling, see Proposition A.1, part (i) in Appendix A.5. Form the realization of the spectral summand of $p_W$. Construct the realization of the spectral summand of $p_{X_t+1|Y_1^t} = p_{X|Y_1^t} \ast p_W$ using the convolution formula given in Proposition A.1, part (ii), in Appendix A.5.

Now, as soon as a new observation $Y_{t+1}$ becomes available one can proceed to a new update step.

The behaviour of the co-degree and the state-space dimension and McMillan degree of the conditional densities in the filter can now be described. First consider the co-degree. Let $k_{1|0}$ denote the co-degree of $p_{X_1}$ and $k_{t|s}$ the co-degree of $p_{X|Y_1^s}$. We know that the co-degree of a rational probability density is even. Let $d$ denote the degree of the polynomial $V$, then the rational function $g$ constructed above has co-degree $d$. Because by assumption the probability density of $U$ is (strictly) positive, it follows that the co-degree of $G \circ g$ is $d$. Hence the co-degree of $p_{X|Y_1^t}$ is $k_{t|t} = k_{t|t-1} + d$. As non-zero scaling does not affect the co-degree and convolution of two
rational densities leads to a rational density with co-degree equal to the minimum of the two co-degrees of the arguments of the convolution (see the notes after Proposition A.1 in Appendix A.5) we find \( k_{t+1|t} = \min(k_{t|t}, k_W) \), where \( k_W \) denotes the co-degree of the rational density of \( W \). This makes that the co-degrees of the conditional densities \( p_{X_{t+1}|Y_t} \), \( t \geq 1 \) are bounded by \( k_W \). This will turn out to be important in the next section.

Now let us turn to an analysis of the state-space dimensions and the McMillan degree. For the composition and convolution of two rational functions the state dimension of the output is the product of the respective dimensions of the inputs (see Propositions 3.2, A.1) whereas for the product the dimensions add up, see Appendix A.2. Therefore in each step the state dimension of the realization of the conditional densities tends to increase dramatically. (Note that together with the result on co-degrees this suggests that the resulting conditional probability density functions will have a non-trivial numerator, even if one uses Student-t or Cauchy densities for the disturbances). Theoretically it is possible that the resulting state-space realization is not-minimal, in which case the McMillan degree would be smaller than the state-space dimension and a state-space reduction procedure could be applied. In practice we do not expect this to happen very often. However if this is the case approximately, one can profitably apply model reduction techniques, to keep the state-space dimensions manageable. We suggest to apply model reduction at each time step to approximate the high degree rational density \( p_{X_{t+1}|Y_t} \) by a lower degree rational density. This will be the topic of the next section.

4 Balancing and balanced model reduction

There are many possibilities for model reduction. The challenge here is that the approximant has to be nonnegative on the real axis. In terms of the corresponding spectral density this means that the approximant spectral density has to be nonnegative on the imaginary axis. In terms of the spectral summand this means that the approximant has to be positive real.

One well known method to achieve this is the so called “positive real balanced truncation” technique, see e.g. [3], which we will shortly explain here.

Note that the solution set \( \mathcal{P} \) contains two particular elements, namely the minimal and the maximal element, \( \underline{\mathcal{P}} \leq \mathcal{P} \) say, and it has been discussed in section 3.3 how to compute these elements. A state space realization \([A, M, C]\) of a spectral summand \( Z(s) = C(sI_n - A)^{-1}M \) is called positive
real balanced iff

\[ P = \overline{P}^{-1} = \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n), \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \]

holds. The \( \sigma_i \)'s are called the positive real singular values of \( Z \). Since \( P \leq \overline{P} \)
holds and since the squared singular values \( \sigma_i^2 \) are the eigenvalues of \( \overline{P}P^{-1} \), it
follows that these singular values are bounded by \( 0 \leq \sigma_i \leq 1 \). Furthermore
it is known (see [9], Theorem 4.1) that \( \sigma_1 = \cdots = \sigma_c = 1 \) and \( 1 > \sigma_j \)
for all \( j = c+1, \ldots, n \), holds, where \( k = 2c \) is the co-degree of \( \Phi = Z + Z^* \).

It is easy to see that a state space transformation
\[ [A, M, C] \rightarrow [TAT^{-1}, TM, CT^{-1}], \]
where \( T \in \mathbb{C}^{n \times n} \) is a non singular matrix, transforms \( P \) and \( \overline{P}^{-1} \) as \( P \rightarrow TP \) and \( \overline{P}^{-1} \rightarrow T^{-1}P^{-1}T^{-1} \). Therefore
such a balanced realization may be obtained by the following procedure. Suppose \( P \) and \( \overline{P} \)
are given and let \( P = P^{1/2}P^{1/2} \) and \( \overline{P} = \overline{P}^{1/2}\overline{P}^{1/2} \)
be some arbitrary factorization of these positive definite matrices. Here \( M^{1/2} \)
denotes a square root of a positive definite matrix \( M \geq 0 \), i.e. \( M = M^{1/2}(M^{1/2})^* \). In addition we use the notations \( M^{1/2} \)
and \( M^{-1/2} = (M^{1/2})^{-1} \) and \( M^{-1/2} = (M^{1/2})^{-1} \). Next let \( P^{1/2}\overline{P}^{1/2} = USV^* \)
unitary matrices, be a singular value decomposition (SVD). The state space transformation
\[ T = \Sigma^{-1/2}V^*\overline{P}^{-1/2} = \Sigma^{1/2}U^*P^{-1/2} \]
then gives the desired balanced realization, since
\[
\begin{align*}
TPT^* &= \Sigma^{1/2}U^*P^{-1/2} \quad P \quad P^{1/2}U\Sigma^{1/2} = \Sigma \\
T^{-1}\overline{P}^{-1}T^{-1} &= \Sigma^{1/2}V^*\overline{P}^{1/2}\overline{P}^{1/2}V\Sigma^{1/2} = \Sigma
\end{align*}
\]

Let \([\bar{A}, \bar{M}, \bar{C}]\) denote the balanced realization obtained by this procedure
and let these matrices by partitioned as

\[
\begin{bmatrix}
T & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
A & M \\
C & 0
\end{bmatrix}
\begin{bmatrix}
T^{-1} & 0 \\
0 & 1
\end{bmatrix} =
\begin{bmatrix}
\bar{A}_{11} & \bar{A}_{12} & \bar{M}_1 \\
\bar{A}_{21} & \bar{A}_{22} & \bar{M}_2 \\
\bar{C}_1 & \bar{C}_2 & 0
\end{bmatrix}
\]

The (positive real) balanced truncated model \( \hat{Z} \) is then defined as \( \hat{Z}(s) = \bar{C}_1(sI_m - \bar{A}_{11})^{-1}\bar{M}_1 \)
where \( m \) is the order of the reduced order system \( \hat{Z} \), i.e. \( \bar{A}_{11} \in \mathbb{C}^{n \times m}, \bar{M}_1 \in \mathbb{C}^{m \times 1} \) and \( \bar{C}_1 \in \mathbb{C}^{1 \times m} \).

It is important to note that
\[
\begin{bmatrix}
-\bar{A}\Sigma - \Sigma\bar{A}^* & \bar{M} - \Sigma\bar{C}^* \\
\bar{M}^* - \bar{C}\Sigma & 0
\end{bmatrix} \geq 0
\]
and the diagonal structure of $\Sigma$ implies that
\[
\begin{bmatrix}
-A_{11} \Sigma_{11} - \Sigma_{11} A_{11}^* & M_1 - \Sigma_{11} C_1^* \\
M_1^* - \bar{C}_1 \Sigma_{11} & 0
\end{bmatrix} \geq 0
\]
This ensures that the reduced order model $\hat{Z}$ is positive real, see Lemma 3.1!

The order $m$ of the reduced order model may be chosen such that the approximation error does not exceed an a priori given bound. In [8], equation (4.30), the following relative error bound for the spectral densities is given:
\[
|\Phi(ix) - \hat{\Phi}(ix)|/\Phi(ix) \leq \left( \prod_{k=m+1}^{n} \frac{(1 + \sigma_k)^2}{(1 - \sigma_k)^2} \right)^{-1} - 1 \quad \text{for all } x \in \mathbb{R} \quad (12)
\]
where $\Phi = Z + Z^*$ and $\hat{\Phi} = \hat{Z} + \hat{Z}^*$. Let $k = 2c$ be the co-degree of $\Phi(s)$. By the discussion above it follows that this bound is finite if and only if $m \geq c$ holds. Furthermore note that for $m \geq c$ the reduced order spectrum $\hat{\Phi}$ also has co-degree $2c$, see [9], Theorem 6.1.

From (12) it is easy to derive an error bound for the corresponding probability density functions. For simplicity assume that $\Phi$ is normalized, i.e. $\int \Phi(ix)dx = 1$ and thus $p(x) = \Phi(ix)$ is a pdf. Let $\hat{p}(x) = \Phi(ix)/(\int \Phi(ix)dx)$ denote the approximation of $p(x)$ and let $0 < \tau < 1$ denote the error bound on the right hand side of (12). From (12) we obtain $\Phi(ix)(1 - \tau)) \leq \hat{\Phi}(ix) \leq \Phi(ix)(1 + \tau)$ and thus
\[
(1 - \tau) \leq \int_{-\infty}^{\infty} \hat{\Phi}(ix)dx \leq (1 + \tau). \quad (13)
\]
Therefore it follows that
\[
|p(x) - \hat{p}(x)|/p(x) \leq \frac{1 + \tau}{1 - \tau} - 1 = \frac{2\tau}{1 - \tau} \quad \text{for all } x \in \mathbb{R}. \quad (14)
\]
It should be noted that in our experiments we observe that (13) is only a rough upper bound for the “integrated” approximation error and thus (14) is a conservative upper error bound.

In our implementation of the filter a model reduction step is included after each prediction step. This means after we have computed a realization of the spectral summand of $p_{X_{t+1}|Y_t}$, we apply the above described scheme to get a realization of an approximant. This will be used instead of $p_{X_{t+1}|Y_t}$. The order $m$ of the reduced order model is chosen such that the above error bound (14) does not exceed a given threshold $1 > \tau > 0$. Note that the co-degree of $p_{X_{t+1}|Y_t}$ is bounded by the co-degree of $p_{W_t}$ and that the reduction step does not alter the co-degree!
5 Autocovariance function and estimation

In this section we analyze the properties of the processes \((Y_t)\) and \((|Y_t|)\). In particular it will be shown, given some suitable assumptions, that \((Y_t)\) is a white noise process and that \((|Y_t|)\) is an ARMA process. The mean and the auto covariance function of \((|Y_t|)\) may be easily computed from the model parameters in particular from the coefficients of the polynomial \(V(x)\) and from the moments of the noise processes \((W_t)\) and \((U_t)\). This enables the use a simple method of moments to estimate the model parameters.

The standing assumptions in this section are as follows:

(i) \(V(x) = v_0 + v_1 x + \cdots v_d x^d\) is a non negative polynomial \((V(x) \geq 0\) for all \(x \in \mathbb{R}\)) and it has order \(d\).

(ii) The processes \((W_t)\) and \((U_t)\) are two i.i.d processes, which are independent from each other. The moments \(M_{W}(k) := \mathbb{E} W^k_t\) exist for all \(0 \leq k \leq m_W\) and \(m_W \geq 2d\) holds. The moments \(M_{U}(k) := \mathbb{E} U^k_t\) exist for all \(0 \leq k \leq m_U\) and \(m_U \geq 2\) holds.

(iii) The parameter \(a\) is bounded by \(|a| < 1\).

Note that within this section it is not needed that \(W_t\) and \(U_t\) have rational probability density functions.

The main result is given in the following Proposition:

**Proposition 5.1** Under the assumptions (i), (ii) and (iii) there exists a strictly stationary solution \((X_t, Y_t)\) of the model (1).

The moments \(M_X(k) := \mathbb{E} X^k_t\) exist up to order \(m_X = m_W\) and may be computed recursively from the relations (starting with \(M_X(0) = 1\))

\[
M_X(k) = \frac{1}{1 - a^k} \sum_{l=0}^{k-1} \binom{k}{l} (k-l)^a M_{W}(k-l) M_X(l) ; 1 \leq k \leq m_W \quad (15)
\]

The process \((Y_t)\) is a white noise process.

The process \(Z_t = |Y_t|\) is an ARMA process of order less than or equal to \(d + 1\).

**Proof:** Let \(M_{t,k} = \sum_{j=1}^{k} |a|^{j-1} |W_{t-j}|\). Since \(M_{t,k}\) is monotonically increasing with \(k\), and since \(\mathbb{E} M_{t,k} \leq \mathbb{E} |W_t| / (1 - |a|)\) is bounded, we conclude that \(\lim_{k \to \infty} M_{t,k}\) and \(X_t := \lim_{k \to \infty} \sum_{j=1}^{k} a^{j-1} W_{t-j}\) exist a.s. Furthermore \(\mathbb{E} X_t = \mathbb{E} W_t / (1 - a)\).
Now suppose that $E X_t^l$ exists for $1 \leq l < k \leq m_w$. From (1) it follows that

$$X_{t+1}^k - a^k X_t^k = \sum_{l=0}^{k-1} \binom{k}{l} a^l X_t^l W_t^{k-l}.$$ 

Since $\sum_{l=0}^{k-1} \binom{k}{l} a^l X_t^l W_t^{k-l}$ has a finite mean it follows analogously that $E X_t^k$ exists. By taking expectations on both sides of the above equation and by using the independence of $X_t$ and $W_t$ one obtains (15).

Furthermore for $k \geq 0$, $X_{t+k+1}^i = (a X_{t+k} + W_{t+k})^i X_t^j = \sum_{l=0}^i \binom{i}{l} a^l W_{t+k}^{i-l} X_{t+k}^l X_t^j$ and thus $E(X_{t+k+1}^i) = \sum_{l=0}^i \binom{i}{l} a^l M_W(i-l) E(X_{t+k}^l X_t^j)$.

Define $\vec{X}_t = (1, X_t, \ldots, X_t^d)'$, $\vec{M}_X = E\vec{X}_t = (1, M_X(1), \ldots, M_X(d))'$, $\vec{V} = (v_0, \ldots, v_d)'$ and

$$F = \begin{pmatrix}
(\binom{0}{0}) a^0 M_W(0) & 0 & \cdots & \cdots & 0 \\
(\binom{1}{0}) a^0 M_W(1) & (\binom{1}{1}) a M_W(0) & \ddots & & \\
(\binom{2}{0}) a^0 M_W(2) & (\binom{2}{1}) a M_W(1) & (\binom{2}{2}) a^2 M_W(0) & \ddots & \\
& \ddots & & \ddots & 0 \\
(\binom{d}{0}) a^0 M_W(d) & \cdots & \cdots & \cdots & (\binom{d}{d}) a^d M_W(0)
\end{pmatrix}$$

Using these notations the above relations may be written as: $\vec{M}_X = \vec{F} \vec{M}_X$ and $E\vec{X}_{t+k+1}^l = \vec{F} E\vec{X}_{t+k}^l \vec{V}$.

First consider the process $(V(X_t))$. It is immediate to see that $E V(X_t) = \vec{V}' \vec{M}_X$ and $E V(X_{t+k}) V(X_t) = \vec{V}' \vec{F}^k (E\vec{X}_t \vec{X}_t') \vec{V}$, for $k \geq 0$. Note that $F$ has eigenvalues $1, a, \ldots, a^d$ and that $e = (1, 0, \ldots, 0)$ and $\vec{M}_X$ are the left and the right eigenvectors corresponding to the eigenvalue 1. Furthermore $e E\vec{X}_t \vec{X}_t' = \vec{M}_X'$. This implies that the auto-covariance function of $V(X_t)$ is given by

$$\text{Cov}(V(X_{t+k}), V(X_t)) = \left\{ \begin{array}{ll}
\vec{V}' (E \vec{X}_t \vec{X}_t' - \vec{M}_X \vec{M}_X') \vec{V} & \text{for } k = 0 \\
\vec{V}' (F - \vec{M}_X e)^{k-1} \left( (F - \vec{M}_X e) E\vec{X}_t \vec{X}_t' \vec{V} \right) & \text{for } k > 0
\end{array} \right.$$ 

From the above representation it follows that $(V(X_t))$ is an ARMA process of order less than or equal to $d + 1$. Note that $(F - \vec{M}_X e)$ has eigenvalues $0, a, \ldots, a^d$.

Next consider the process $(Y_t)$. We have $E Y_t = E V(X_t) E U_t = 0$, by the independence of $X_t$ and of $U_t$. The auto covariance function of $(Y_t)$ is given by

$$E Y_{t+k} Y_t = E (V(X_{t+k}) V(X_t)) E (U_{t+k} U_t) = \left\{ \begin{array}{ll}
E V(X_t)^2 E U_t^2 & \text{for } k = 0 \\
0 & \text{for } k > 0
\end{array} \right.$$
Finally let us consider $|Y_t|$. The mean value of $|Y_t|$ is $\mathbb{E}|Y_t| = \mathbb{E}V(X_t)\mathbb{E}|U_t|$ and the second moments are given by $\mathbb{E}|Y_t|^2 = \mathbb{E}V(X_t)^2\mathbb{E}U_t^2$ and $\mathbb{E}|Y_{t+k}|Y_t| = \mathbb{E}(V(X_{t+k})V(X_t))(\mathbb{E}|U_t|)^2$ for $k > 0$. This implies

$$\text{Cov}(|Y_{t+k}|, |Y_t|) = \begin{cases} \mathbb{E}V(X_t)^2\mathbb{E}U_t^2 - (\mathbb{E}V(X_t)\mathbb{E}|U_t|)^2 & \text{for } k = 0 \\
\text{Cov}(V(X_{t+k}), V(X_t))(\mathbb{E}|U_t|)^2 & \text{for } k > 0 \end{cases}$$

Of course analogous calculations apply for $|Y_t|^k$, provided, that sufficiently many moments of $W_t$ and of $U_t$ exist.

6 Simulation results

All simulation and estimation results presented here are based on the following specifications:

\begin{align*}
X_{t+1} &= aX_t + W_t \\
Y_t &= \Psi V(\sigma X_t)U_t
\end{align*} \hspace{1cm} (16)

This is a slight reformulation of the model (1). The idea is to fix the function $V(x)$ and the distributions of $W_t$ and of $U_t$, which leaves the three parameters $a, \Psi$ and $\sigma$ for estimation.

The function $V(x)$ is chosen as

$$V(x) = (1 + \frac{x}{2d})^d + 0.1$$

which is a rough approximation of $\exp(x/2)$. The additional constant 0.1 is added to ensure $V(x) > 0$.

The inputs $W_t$ and $U_t$ are assumed to have scaled t-distributions. This means that $c_W W_t$ has a t-distribution with $n_W$ degrees of freedom and the scaling constant chosen such that $\mathbb{E}W_t^2 = 1$ holds. Analogously $c_U U_t$ has a t-distribution with $n_U$ degrees of freedom and the scaling is such that $\mathbb{E}U_t^2 = 1$ holds. Throughout this section the integer parameters $d, n_W$ and $n_U$ are fixed and given by $d = 4$, $n_W = 9$ and $n_U = 3$. This implies in particular that the assumptions of Proposition 5.1 are fulfilled.

The first part of this section deals with the estimation of the parameters $(a, \Psi, \sigma)$. Table 1 shows the moments of the process $(|Y_t|)$ for some combinations of the parameters $a, \Psi = 1.0, \sigma$. 

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In a small simulation study we have investigated the performance of a simple method-of-moments estimation, where 10 lags of the auto-covariance function of $|Y_t|$ have been used. To be more precise let

$$m(a, \Psi, \sigma) := (\mathbb{E}|Y_t|, \text{Var}|Y_t|, \text{Cov}(|Y_{t+1}|, |Y_t|), \ldots, \text{Cov}(|Y_{t+10}|, |Y_t|))$$

and let $\hat{m}_T$ be the sample estimate of this vector of moments given a sample of size $T$. Then the estimates $(\hat{a}, \hat{\Psi}, \hat{\sigma})$ are computed by minimizing

$$\|m(a, \Psi, \sigma) - \hat{m}_T\|^2$$

The results for 1000 simulation runs for (simulated) data series of length $T = 1000$ are collected in table 2. Both the mean estimation error (mean) and the standard deviation (std) over these 1000 simulation are shown in dependence of the true parameters. Note e.g. that the estimate of $a$ shows a significant bias especially for small $a$ and $\sigma$. However this is only a first rough estimation scheme and other enhanced estimates will be investigated in future.

It has been mentioned in section 2 that the filter is able to compute the likelihood. However the computation of the filter is presently too time demanding to implement a maximum likelihood estimation based on the filter.

Next we test the filter on some real world data. In particular we consider data which also have been analysed by [14]. The authors consider five exchange rate data series and study the empirical performance of stochastic volatility models. Here we only consider the Dollar/Yen exchange rate data, which consists of $T = 1102$ weekly observations from 3 January 1973 until 9 February 1994.

The parameters of the model (16) have been estimated by the method of moments as described in the previous section. Here 25 lags of the auto covariance are used and the obtained estimates are $\hat{a} = 0.957$, $\hat{\sigma} = 0.309$ and $\hat{\Psi} = 0.921$. Figure 1 shows the sample ACF and the fitted ACF. Next the filter is run on this data set to compute a one step ahead prediction of $|Y_{t+1}|$. The result is shown in figure 2. Note that, since the stationary solution of the state $X_t$ is not rationally distributed as far as we know, we have simply assumed that $X_1$ has a scaled t-distribution with $n_X = 9$ degrees of freedom and the scaling was chosen such that the variance of $X_1$ is equal to $1/(1-\alpha^2)$ i.e. equal to variance of the stationary solution.

Finally we consider some simulated data. The parameters were chosen as $\alpha = 0.9$, $\Psi = 2$ and $\sigma = 1.5$. The simulation and the filter were initialized with a scaled t-distributed random variable $X_1$, where the degrees of freedom
is $n_X = 9$ and the scaling parameter is chosen such that the variance of $X_1$ is equal to $1/(1-a^2)$. The length of the simulated series is $T = 100$.

Figure 3 shows the simulated trajectory of $Y_t$ and the one step ahead prediction of $|Y_t|$, i.e. $E(|Y_t| | Y_{t-1}) = \Psi(E|U_t|)E(V(\sigma X_t) | Y_{t-1})$. The conditional expectation $E(V(\sigma X_t) | Y_{t-1})$ is computed from the conditional probability density function $p_{X_1|Y_{t-1}}$, which is computed by the filter. See also Propositions 3.1 and 5.1.

Figure 4 shows the conditional probability density function $p_{X_{t+1}|Y_t}$, for $t = 100$. In each time step of the filter balanced model reduction is used as described above. Let $\hat{p}_{X_{t+1}|Y_t}$ denote the approximation of the conditional pdf $p_{X_{t+1}|Y_t}$. The order $m$ of the reduced order system is chosen such that the relative error $|p_{X_{t+1}|Y_t}(x) - \hat{p}_{X_{t+1}|Y_t}|/p_{X_{t+1}|Y_t}(x)$ is at most 0.02, i.e. we allow at most an error of 2 percent. See equation (14). For this specific example typical model orders are $n = 85$ and $m = 9$, which means that the state dimension is almost reduced by a factor 10. If one compares the conditional expectation of $X_t$ and of $V(X_t)$ given the observations $Y_t$, then in this example the relative error is of the order $10^{-14}$. These numbers indicate the excellent quality of the used approximation scheme.

Finally figure 5 shows the evolution of the conditional densities $p_{X_{t+1}|Y_t}$ over time.

7 Conclusion

The exact filter for a class of stochastic volatility models is derived. A standard stochastic volatility model in which the disturbances are Gaussian and the volatility function involved is exponential can be viewed as a limiting case. The complexity of the exact filter increases in the sense that the matrices that are used to represent the rational probability density functions tend to grow quickly. An approximate filter is presented in which at each time step the conditional probability density function of the state, which is rational, is replaced by an approximating rational probability density function, using the SBT method (stochastically balanced truncation). Using a well-known error bound the approximating rational probability density function can be chosen such that on each point of the real line the relative error is less than a given percentage (the tolerance level involved can be chosen by the user). In some simulated and empirical applications we find that using a tolerance level of as low as 2 percent still leads to an enormous reduction in complexity, keeping the order of the rational functions well within bounds.
that are considered tractable with modern computers. Lower tolerance lev-
els could also be achieved if desired, but then larger matrices will have to be
handled. The model presented is very flexible, especially with respect to the
specification of the probability density functions for the disturbances. Here
one can vary between very heavy-tailed disturbances (with Cauchy density
for instance) and less heavy-tailed disturbances (with Student-t densities
that are approximating Gaussian densities for example). In the applica-
tions in this paper we have stayed as close as possible to the traditional Gaussian
model. However the possibility of specifying more heavy-tailed densities
seems one of the most interesting features of this class of models. Exploring
those possibilities is an interesting topic for future research. Also valuation
of financial derivatives in a market in which the asset price movements can
be described by a stochastic volatility model of the type investigated here,
is an interesting topic for future research. More generally the methodol-
ogy of working with rational density functions in filtering problems in the
way presented here could have a much wider range of applications, as the
methodology is really general and flexible and numerically stable methods
for various operations involved are now provided. Preliminary experience
with the methodology shows especially striking results deriving from the
application of the SBT approximation method. It is to be expected that
this can also be successfully applied to the linear filtering problems with
rationally distributed disturbances considered in [10].

A Results from system theory

A.1 Numerical calculation of the co-degree and of the zeros
of a strictly proper rational function

Consider a strictly proper scalar rational function\(^1\)

\[
G(s) = C(sI_n - A)^{-1}B = \frac{a_0 + a_1 s + \cdots + a_q s^q}{b_0 + b_1 s + \cdots + b_n s^n}
\]

where

\[q < n \text{ and } a(s) = a_0 + a_1 s + \cdots + a_q s^q, \ b(s) = b_0 + b_1 s + \cdots + b_n s^n \text{ are coprime.}\]

The co-degree of \(G\) is defined as \((n - q)\), i.e. as the multiplicity of the
infinite zero of \(G(s)\). Since \(G\) is strictly proper the co-degree is positive.

\(^1\)In this section \(G\) is an arbitrary, not necessarily stable, transfer function. We will use
results obtained in this section e.g. for a spectrum \(\Phi = KK^*\) and for its factor \(K^*\).
The Taylor series expansion of \( G(s) \) at infinity is given by
\[
G(s) = G_0 + G_1 s^{-1} + G_2 s^{-2} + \cdots \\
= D + CBS^{-1} + CABs^{-2} + \cdots
\]

Therefore the co-degree of \( G \) is related to the Markov parameters of \( G \) as follows.

**Lemma A.1** The co-degree of \( G(s) = C(sI_n - A)^{-1}B \) is equal to \( c \) iff

- \( CA^{c-1}B \neq 0 \) and \( CA^{i-1}B = 0 \) for all \( 1 \leq i < c \).

Note that a naive check on \( CA^{i-1}B = 0 \) in order to compute the co-degree is numerically unstable since \( A \) might have eigenvalues of modulus larger than one and thus round off errors would “explode”.

The (finite) zeros of the transfer function \( G(s) \) are the (finite) eigenvalues of the pencil:

\[
\lambda E - N := \begin{bmatrix} \lambda I_n - A & B \\ -C & D \end{bmatrix}
\]

Therefore the co-degree and the finite zeros of \( G \) may be computed from the eigenstructure of the above pencil. We will make use of the following concepts, see e.g. [21]. A pencil \( (\lambda E - N) \) is called regular if it is square and if \( \det(\lambda E - N) \) is not constant. The zeros of \( \det(\lambda E - N) \) are the eigenvalues of the pencil. Suppose there exist full column rank matrices \( X, Y \in \mathbb{C}^{n \times k} \), \( k \leq n \) and matrices \( \bar{E}, \bar{N} \in \mathbb{C}^{k \times k} \) such that

\[
(\lambda E - N)X = Y(\lambda E_k - \bar{N})
\]

holds. The space spanned by the columns of \( X \) is called a deflating subspace of the pencil \( (\lambda E - N) \). This is a generalization of the concept of invariant subspaces to arbitrary pencils. The \( k \)-dimensional pencil \( (\lambda E - N) \) is called a divisor of \( (\lambda E - N) \). If \( E \) is non singular, then \( (\lambda E - N) \) is called a finite divisor of \( (\lambda E - N) \). In this case \( (\lambda E - N) \) has \( k \) finite eigenvalues which are of course also eigenvalues of \( (\lambda E - N) \). If there exist two non singular matrices \( S, T \in \mathbb{C}^{k \times k} \) such that

\[
S(\lambda E_k - \bar{N})T = \begin{pmatrix}
-1 & \lambda & 0 & \cdots & 0 \\
0 & -1 & \lambda & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & \lambda \\
0 & \cdots & \cdots & 0 & -1
\end{pmatrix}
\]
then \((\lambda E - N)\) is called an elementary infinite divisor.

An alternative characterisation of the co-degree now is as follows:

**Lemma A.2** The co-degree of \(G(s)\) is positive and it is equal to \(c\) iff the pencil (17) has an elementary infinite divisor of dimension \((c + 1)\) and a finite divisor of dimension \((n - c)\).

**Proof:** To prove this lemma the pencil is transformed to a so-called staircase form as defined in [21]. This will also give a numerically robust way to analyze the co-degree and the eigenstructure of the above pencil.

Let \(U_1 \in \mathbb{C}^{n \times n}\) be a row compression of \((-B)\), i.e. \(U_1\) is a unitary matrix such that \(U_1^*(-B) = [b, 0, \ldots, 0]^*\) and \(b > 0\). (Note that \(B \neq 0\).) Apply this state space transformation and define

\[
\begin{bmatrix}
U_1^* & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
A & -B \\
C & -0
\end{bmatrix}
\begin{bmatrix}
U_1 \\
0 \\
1
\end{bmatrix} =
\begin{bmatrix}
A_1 & -B_1 \\
C_1 & -0
\end{bmatrix}
\]

Note that \(CB = C_1B_1\) and thus the first element of \(C_1\) is zero iff \(c > 1\).

In the next step let

\[U_2 = \begin{bmatrix}
1 & 0 \\
0 & U_2
\end{bmatrix}\]

where \(\bar{U}_2 \in \mathbb{C}^{n-1 \times n-1}\) is a row compression of the last \(n - 1\) entries of the first column of \(A_1\). Apply this state space transformation to get

\[
\begin{bmatrix}
U_2^* & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
A_1 & -B_1 \\
C_1 & -0
\end{bmatrix}
\begin{bmatrix}
U_2 \\
0 \\
1
\end{bmatrix} =
\begin{bmatrix}
A_2 & -B_2 \\
C_2 & -0
\end{bmatrix}
\]

By construction the \((1, 1)\) element of \(A_1\) and the first elements of \(C_1\) and of \(B_1\) are not affected by this transformation. Furthermore note that the last \(n - 1\) elements of \(B_2\) and the last \(n - 2\) elements of the first row of \(A_2\) are zero. In addition we have \(CAB = C_2A_2B_2 = 0\) iff \(c > 2\). Thus \(c > 2\) holds iff the second element of \(C_2\) is zero.

Now this procedure is repeated until a nonzero element pops up in the \(k\)-th position of \(C_k\). This is a possible way to estimate the co-degree of \(G\).
After $c + 1$ steps of this kind we end up with a matrix of the form:

$$
\begin{bmatrix}
* & * & \cdots & \cdots & \cdots & * & \cdots & \cdots & * & \oplus \\
\oplus & * & \cdots & \cdots & \cdots & * & \cdots & \cdots & * & 0 \\
0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & \oplus & * & \cdots & \cdots & * & 0 \\
0 & \cdots & \cdots & 0 & 0 & * & \cdots & \cdots & * & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & * & \cdots & \cdots & * & 0 \\
0 & \cdots & \cdots & 0 & 0 & \alpha & * & \cdots & \cdots & * & 0
\end{bmatrix}
$$

(19)

The horizontal and vertical lines partition the above matrix into blocks of size $c$, $n - c$ and 1 respectively. Two particular elements of the above matrix, namely the $(c + 1, 1)$ and the $(n + 1, c)$ element, are denoted with $\beta$ and $\alpha$ respectively. Note that $\beta > 0$ and $\alpha \neq 0$ holds.

Note that (for $j < c$) the columns $[1, \ldots, j + 1]$ of the matrix $U = U_1 U_2 \cdots U_{c+1}$ form an orthogonal basis of the column space of $[B, AB, \ldots, A^j B]$.

By a permutation of rows and columns we bring the last column to the first position and the last row to the $(c + 1)$-th position. Finally apply the Givens rotation

$$
\tilde{Q} = \begin{bmatrix}
\alpha^* & \beta^* \\
-\beta & \alpha
\end{bmatrix}
\frac{1}{\sqrt{\alpha^*\alpha + \beta^*\beta}} =: \begin{bmatrix}
q_{11} & q_{12} \\
q_{21} & q_{22}
\end{bmatrix}
$$

to the rows $c+1$ and $c+2$. If $Q$ and $Z$ denote the concatenation of all these unitary row and column operations, then we have

$$
Q \begin{bmatrix}
A \\
C
\end{bmatrix}
\begin{bmatrix}
-B \\
0
\end{bmatrix} Z =
\begin{bmatrix}
\oplus & * & \cdots & \cdots & * & \cdots & * \\
0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & \oplus & * & \cdots & * \\
0 & \cdots & \cdots & 0 & 0 & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \tilde{N}_{11} & \tilde{N}_{12} \\
0 & \cdots & \cdots & 0 & 0 & \tilde{N}_{22}
\end{bmatrix}
$$

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and
\[
Q \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} Z = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \vdots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 1 & 0 & 0 & \cdots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & q_{12} & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 & q_{22} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & 1 & \ddots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 1
\end{bmatrix}
=: \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} \\ 0 & \bar{E}_{22} \end{bmatrix}
\]

(20)

Now this block upper triangular form displays the eigenstructure of the pencil \((\lambda E - N)\). Since \(\bar{N}_{11} \in \mathbb{C}^{c+1 \times c+1}\) is an upper-triangular non-singular matrix it follows that \((\lambda \bar{E}_{11} - \bar{N}_{11})\) is an \((c + 1)\) dimensional elementary infinite divisor of the pencil. Furthermore note that \(Z\) may be partitioned as
\[
Z = \begin{bmatrix} 0 & \bar{U} \\ 1 & 0 \end{bmatrix}
\]
and that the first \(c + 1\) columns of \(Z\) form a basis for the deflating subspace corresponding to this infinite divisor. The same holds true if we only take the first \(j + 1\) columns, for \(0 \leq j \leq c\). To be more precise consider the \((j + 1 \times j + 1)\) dimensional left upper sub-block of \((\lambda \bar{E}_{11} - \bar{N}_{11})\). By the triangular structure of the matrices \(\bar{E}_{11}\) and \(\bar{N}_{11}\) it follows that this sub-block defines an infinite elementary divisor and that the first \(j + 1\) columns of \(Z\) span the corresponding deflating subspace.

Since \(\bar{E}_{22} \in \mathbb{C}^{(n-c) \times (n-c)}\) is non singular it follows that \((\lambda \bar{E}_{22} - \bar{N}_{22})\) is an \((n - c)\) dimensional finite divisor of the pencil. \(\square\)

A.2 Elementary operations on rational functions

Let two strictly proper rational function \(G_i = C_i(sI - A_i)^{-1}B_i\) with state space realizations \((A_i, B_i, C_i), i = 1, 2\) be given.
A state space realization for \( G_1(s) = B_1(-sI - A_1)^{-1}C_1^* \) is given by

\[
G_1^* = \pi \begin{bmatrix}
-A_1^* & C_1^* \\
-B_1^* & 0
\end{bmatrix}
\]

The sum \( G_1 + G_2 \) has a state space realization:

\[
G_1 + G_2 = \pi \begin{bmatrix}
A_1 & 0 & B_1 \\
0 & A_2 & B_2 \\
C_1 & C_2 & 0
\end{bmatrix}
\]

The product \( G_1G_2 \) has a state space realization:

\[
G_1G_2 = \pi \begin{bmatrix}
A_1 & B_1B_2 & 0 \\
0 & A_2 & B_2 \\
C_1 & 0 & 0
\end{bmatrix}
\]

If \( C_1B_1 = 0 \) then \( G_1(ys)s \) is strictly proper and a state space realization is given by

\[
G_1(ys)s = \pi \begin{bmatrix}
A_1y^{-1} & A_1B_1y^{-1} - B_1 \\
0 & 0
\end{bmatrix}
\]

### A.3 Computation of a spectral summand from a spectral factor

Suppose we have given a (stable) spectral factor \( K(s) = C(sI - A)^{-1}B \) and that we want to compute a spectral summand of \( \Phi(s) = K(s)K^*(s) \):

Let \( P \) be the solution of the Lyapunov equation

\[
AP + PA^* + BB^* = 0
\]

and define \( M = PC^* \). The state space transformation \( T \)

\[
T = \begin{bmatrix}
I_n & P \\
0 & I_n
\end{bmatrix}
\]

then gives

\[
\begin{bmatrix}
I_n & P \\
0 & I_n
\end{bmatrix} \begin{bmatrix}
A & -BB^* \\
0 & -A^*
\end{bmatrix} \begin{bmatrix}
I_n & -P \\
0 & I_n
\end{bmatrix} = \begin{bmatrix}
A & -AP - PA^* - BB^* \\
0 & -A^*
\end{bmatrix} = \begin{bmatrix}
A & 0 \\
0 & -A^*
\end{bmatrix},
\]

\[
\begin{bmatrix}
I_n & P \\
0 & I_n
\end{bmatrix} \begin{bmatrix}
0 & C^* \\
C^* & 0
\end{bmatrix} = \begin{bmatrix}
PC^* \\
M C^*
\end{bmatrix},
\]

\[
\begin{bmatrix}
C & 0 \\
0 & I_n
\end{bmatrix} \begin{bmatrix}
I_n & -P \\
0 & I_n
\end{bmatrix} = \begin{bmatrix}
C & -CP \\
C & -M^*
\end{bmatrix},
\]

and thus \( Z(s) = C(sI_n - A)^{-1}M \) is a (stable) spectral summand of \( \Phi(s) \).
A.4 Computation of a spectral summand from an integrable spectral density

Let $\Phi = H(sI_{2n} - F)^{-1} G$ be given. First compute a Schur decomposition of $F$ such that the stable eigenvalues of $F$ appear on the first $n$ positions, i.e.

$$\bar{F} = V^* F V = \begin{bmatrix} \bar{F}_{11} & \bar{F}_{12} \\ 0 & \bar{F}_{22} \end{bmatrix}$$

where $V$ is a unitary matrix, $\bar{F}$ is an upper triangular matrix and $\bar{F}_{11} \in \mathbb{C}^{n \times n}$ is asymptotically stable.

Solve the Lyapunov equation

$$-\bar{F}_{11} P + P \bar{F}_{22} + \bar{F}_{12} = 0$$

and set

$$A = \bar{F}_{11}, \quad M = [I, P]V^* G \quad C = HV(I_n, 0)^*$$

to get a stable spectral summand $Z = C(sI - A)^{-1} M$.

A.5 Operations on rational densities

In [10] it was show that the operations of translation, scaling, multiplication and convolution of rational densities can be translated into linear algebra operations on corresponding state-space realizations of spectral summands. For ease of reference here we give some of these results which are needed for the implementation of the filter. Note that multiplication of two rational functions could be implemented via their summands. However using spectral factors seems to be numerically more reliable. Thus in our implementation of the filter we have chosen this approach.

**Proposition A.1** Let $X_1$ and $X_2$ denote stochastically independent random variables with rational density functions $p_1$, $p_2$. For $j = 1, 2$, let $Z_j(s)$ denote the corresponding stable spectral summand, with a state-space realization $[A_j, M_j, C_j]$ with state-space dimension $n_j$.

(i) For $a \neq 0$ the random variable $X = aX_1$ has a rational density whose spectral summand has a state space realization given by $[A, M, C] = [aA_1, M_1, C_1]$ if $a > 0$ and $[A, M, C] = [-aA_1^*, M_1^*, C_1^*]$ if $a < 0$. 

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(ii) The sum $X = X_1 + X_2$ has a rational density function $p = p_1 \star p_2$, i.e. the convolution of $p_1$ and $p_2$, and the spectral summand of $p$ has a state-space realization given by $[A, M, C]$ where $A = A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2$, $M = M_1 \otimes M_2$ and $C = C_1 \otimes C_2$.

We finish this subsection with a note on the co-degree of the convolution of two rational probability density functions. Note that for two independent random variables, $X_1, X_2$ say, it holds that

$$
E|X_1 + X_2|^r < \infty \text{ if and only if } E|X_1|^r < \infty \text{ and } E|X_2|^r < \infty.
$$

see e.g. [16], Problem 4.6.11. Together with 3.1 this implies that the co-degree of the convolution of two rational densities is equal to the minimum of the co-degrees of these two densities. This fact is used in the text to track the co-degrees of the conditional probability density functions arising in the filter.

References

[1] D. Brigo and B. Hanzon, On some filtering problems arising in mathematical finance, Insurance: Mathematics and Economics, vol. 22, 1998, pp. 53-64.

[2] J. Y. Campbell, A. W. Lo and A. C. MacKinlay, The Econometrics of Financial Markets, Princeton University Press, Princeton New Jersey, 2nd ed, 1997.

[3] U. B. Desai and D. Pal, A transformation approach to stochastic model reduction, IEEE Transactions on Automatic Control, vol. AC-29, nr. 12, December 1984, pp. 1097-1100.

[4] R.F. Engle, Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of the U.K. Inflation, Econometrica, vol. 50, 1982, pp. 987-1008.

[5] P. L. Faurre, Stochastic Realization Algorithms, pp. 1–25 in: R.K. Mehra and D.G. Lainiotis, System Identification: Advances and Case Studies, Academic Press, New York, 1976.

[6] G. Golub and C. VanLoan, Matrix Computations, John Hopkins University Press, Maryland, 2nd ed., 1989.
[7] C. Gourieroux, *ARCH Models and Financial Applications*, Springer, New York, 1997.

[8] M. Green, *A relative error bound for balanced stochastic truncation*, IEEE Transactions of Automatic Control, vol. AC-33, nr. 10, pp. 961-965, October 1988.

[9] M. Green, *Balanced Stochastic Realizations*, Linear Algebra and its Applications, vol. 98, 1988, pp. 211-247.

[10] B. Hanzon and R.J. Ober, *A State-Space Calculus for Rational Probability Density Functions and Applications to Non-Gaussian Filtering*, SIAM J. Control and Optimization, vol. 40, nr.3, 2001, pp. 724-740.

[11] A. Harvey, E. Ruiz and N. Shephard, *Multivariate Stochastic Variance Models*, Review of Economic Studies, vol. 61, 1994, pp. 247-264.

[12] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, Academic Press, Orlando, Florida, 1985.

[13] A. Lucas, *Outlier robust unit root analysis*, Thesis Publishers, Amsterdam, 1996.

[14] R. Mahieu and P. Schotman, *An Empirical Application of Stochastic Volatility Models*, Journal of Applied Econometrics, vol. 13, June 1998, pp. 333-360.

[15] B. Mandelbrot, *The variation of certain speculative prices*, J. Business, vol. 36, 1963, pp. 394-419.

[16] V.K. Rohatgi, *An Introduction to Probability Theory and Mathematical Statistics*, John Wiley & Sons, New York, 1976.

[17] M.F. Neuts, *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*, The Johns Hopkins University Press, Baltimore, 1981.

[18] M.W. Fackrell, *Characterization of Matrix-exponential Distributions*, PhD thesis, School of Applied Mathematics, Adelaide, 2003. http://thesis.library.adelaide.edu.au/uploads/approved/adt-SUA20051207.123257/public/02whole.pdf

[19] Wilson J. Rugh, *Linear system theory*, Prentice-Hall, Upper Saddle River, NJ, 2nd ed., 1996.
[20] S. Taylor, *Modelling Financial Time Series*, John Wiley and Sons, London, 1986.

[21] P.M. Van Dooren, *The Generalized Eigenstructure Problem in Linear System Theory*, IEEE Transactions on Automatic Control, vol. AC-26, nr. 1, February 1981, pp. 111-129.
\[ \sigma = 0.5 \quad \sigma = 1 \]

|                  | \(a = 0.5\) | \(a = 0.9\) |
|------------------|-------------|-------------|
| \(E|Y_t|\)         | 0.7202      | 0.7809      |
|                  | 0.7797      | 1.0279      |
| \text{Var}(|Y_t|) | 0.8506      | 1.3303      |
|                  | 1.2994      | 4.3120      |
| \text{Corr}(|Y_{t+1}|,|Y_t|) | 0.0209       | 0.0619      |
|                  | 0.1133       | 0.2270      |

Table 1: Moments of the process \(|Y_t|\).

\[ \text{This table shows the moments of the absolute values of the outputs } |Y_t| \text{ for some parameter values } a, \Psi = 1, \sigma. \]
\begin{table}[h]
\centering
\begin{tabular}{llcc}
\hline
 & \multicolumn{1}{c}{\(\sigma = 0.5\)} & \multicolumn{1}{c}{\(\sigma = 1\)} \\
\hline
\(\hat{a} - a\) & mean & \(a = 0.5\) & -0.3154 & 0.0297 \\
 & & \(a = 0.9\) & -0.0522 & -0.0000 \\
 & \text{std} & \(a = 0.5\) & 0.2203 & 0.2321 \\
 & & \(a = 0.9\) & 0.2322 & 0.0591 \\
\hline
\(\Psi - \Psi\) & mean & \(a = 0.5\) & -0.0347 & 0.0074 \\
 & & \(a = 0.9\) & -0.0350 & -0.0644 \\
 & \text{std} & \(a = 0.5\) & 0.0642 & 0.0726 \\
 & & \(a = 0.9\) & 0.1010 & 0.4151 \\
\hline
\(\hat{\sigma} - \sigma\) & mean & \(a = 0.5\) & -0.1343 & -0.2232 \\
 & & \(a = 0.9\) & -0.0428 & 0.0142 \\
 & \text{std} & \(a = 0.5\) & 0.4956 & 0.4751 \\
 & & \(a = 0.9\) & 0.3886 & 0.4262 \\
\hline
\end{tabular}
\caption{Simulation results for the MM estimator.}
\end{table}

This table shows the results for 1000 simulation runs for (simulated) data series of length \(T = 1000\). Both the mean estimation error (mean) and the standard deviation (std) over these 1000 simulation are shown in dependence of the true parameters.
Figure 1: Dollar/Yen exchange rate: sample ACF (green) and fitted ACF of the absolute values.
Figure 2: Dollar/Yen exchange rate: absolute values of the exchange rates (black) and the corresponding one step ahead predictions as given by the filter (yellow).
Figure 3: Simulated data: time series plot of the simulated trajectories of the noise process $U_t$ (blue) and of the outputs $Y_t = V(X_t)U_t$ (green). The gray shaded area is bounded by $\pm V(X_t)E[U_t]$, i.e., the conditional expectation of the absolute values of $Y_t$ given the state $X_t$. The dashed black line shows the corresponding conditional expectation of $|Y_t|$ given the past observations as computed by the filter, (i.e. the one step ahead forecasts of $|Y_t|$).
Figure 4: Simulated data: This figure shows the conditional pdf $p_{X_t|Y_t}$ for $t = 100$. Note that also the approximant probability density function, $\hat{p}_{X_{t+1}|Y_t}$ say, as computed by the positive real balanced truncation method is plotted. However on this scale the full order probability density function and the low order approximation can hardly be distinguished, since for the relative approximation error $|p_{X_t|Y_t}(x) - \hat{p}_{X_{t+1}|Y_t}(x)|/p_{X_t|Y_t}(x) \leq 0.0051$ holds by (14). The state space dimension of the realization of the spectral summand of $p_{X_{t+1}|Y_t}$ is $n = 85$ and the spectral summand of $\hat{p}_{X_{t+1}|Y_t}$ has order $m = 9$. The co-degree of the corresponding spectral densities $\Phi_{X_{t+1}|Y_t}(ix) = p_{X_{t+1}|Y_t}(ix)$ and $\hat{\Phi}_{X_{t+1}|Y_t}(ix) = \hat{p}_{X_{t+1}|Y_t}(ix)$ is 10. The vertical black line marks the true value $X_{t+1}$ and the dashed black line marks the corresponding estimate, i.e. the conditional expectation $E(X_{t+1}|Y_t)$.
Figure 5: Simulated data: This plot shows the evolution of the conditional densities $p_{X_{t+1}|Y_t}$. Each “column” shows the conditional $p_{X_{t+1}|Y_t}$ for a given time $t$, where high values are coded with red and low values of this pdf are coded with blue. In addition the solid black line shows the trajectory of $X_t$ and the blue line marks the corresponding one step ahead predictions, i.e. the mean of the conditional densities $p_{X_{t+1}|Y_t}$. 