Form factors in the algebraic cluster model

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Abstract
I present a derivation of form factors in the algebraic cluster model for an arbitrary number of identical clusters. The form factors correspond to representation matrix elements which are derived in closed form for the harmonic oscillator and deformed oscillator limits. These results are relevant for applications in nuclear, molecular and hadronic physics.

Keywords: cluster model, electromagnetic form factors, algebraic methods

(Some figures may appear in colour only in the online journal)

1. Introduction

The dynamics of quantum many-body systems can be studied by applying external probes. The response of these systems to strong external fields leads to multiple excitation of the target system involving the excitation of many intermediate states. Examples are Coulomb excitation [1] and medium-energy proton scattering at forward angles [2] in nuclear physics, and electron scattering from polar molecules [3]. The standard approach to treat the coupling between target and projectile to all orders is that of a coupled-channel approach, which becomes complicated when the number of channels that has to be included is large. An alternative method is based on the eikonal or Glauber approximation in which the multiple scattering is summed to all orders and which gives a good description of the scattering at forward angles [2].

In an eikonal treatment of scattering from complex systems the scattering operator is expressed as an exponentiated multipole operator. If the eikonal treatment of the scattering problem is combined with an algebraic model of the quantum many-body system, the matrix elements of the eikonal scattering operator can be interpreted as representation matrix elements which can be obtained exactly to all orders in the coupling between target and projectile [4, 5]. In addition, there are special solutions in which these matrix elements can be derived in closed analytic form [6, 7].

In nuclear physics, these techniques have been applied to medium-energy proton scattering [6, 8], Coulomb excitation [9], subbarrier fusion [10] and electromagnetic excitation of α-cluster nuclei [11], and in molecular physics to medium-energy electron scattering from polar molecules [7].

The derivation of representation matrix elements and form factors for systems which are dominated by a single multipole, e.g. quadrupole oscillations in collective nuclei and the dipole degree of freedom in polar molecules, was studied for an arbitrary multipole in [12]. The aim of this contribution is to study a generalization to a system of coupled oscillators as is relevant for α-cluster nuclei. The results are valid for an arbitrary number of clusters.

2. The algebraic cluster model (ACM)

The ACM describes the relative motion of k clusters. It is based on the spectrum generating algebra of $U(\nu+1)$, where $\nu = 3(k - 1)$ represents the number of relative spatial degrees of freedom. As a special case the ACM contains the $U(4)$ vibron model for two-body problems ($k = 2$) with applications in diatomic molecules [13], nuclear clusters [14], and quark–antiquark configurations in mesons [15]. Further extensions of this idea are the $U(7)$ model for three-body clusters ($k = 3$) with applications to three-quark configurations in baryons [16], triatomic molecules [17] and nuclear clusters [11, 18], and the $U(10)$ model for four-body clusters ($k = 4$) which was introduced recently to describe the properties of the nucleus $^{16}\text{O}$ in terms of four-alpha clusters [19, 20].
The relevant degrees of freedom of a system of $k$-body clusters are given by the $k-1$ relative Jacobi coordinates

$$\vec{r}_j = \frac{1}{\sqrt{j(j+1)}} \left( \sum_{j=1}^{k} \vec{r}_j - j \vec{r}_{j+1} \right),$$

(1)

and their conjugate momenta, $\vec{p}_j$. Here $\vec{r}_j$ denotes the position vector of the $j$th cluster. The ACM is based on a bosonic quantization which consists in introducing $k-1$ vector boson operators (one for each relative coordinate) which are related to the coordinates and their conjugate momenta by

$$b_{j,m}^+ = \frac{1}{\sqrt{2}} \left( \rho_{j,m} - i \rho_{j,m}^* \right),$$

$$b_{j,m} = \frac{1}{\sqrt{2}} \left( \rho_{j,m} + i \rho_{j,m}^* \right),$$

(2)

with $m = -1, 0, 1$, and an additional auxiliary scalar boson, $s^\dagger, s$. The set of $(3k-2)^2$ bilinear products of creation and annihilation operators generates the Lie algebra of $U(3k-2)$. Since the building blocks of the ACM are bosons, all states of the system belong to the totally symmetric representation $[N]$ of $U(3k-2)$, where $N$ represents the total number of bosons.

In this contribution, I study the ACM for identical clusters which is relevant to $\alpha$-cluster nuclei like $^{12}$C and $^{16}$O. For these systems, the Hamiltonian has to be invariant under the permutation group $S_k$ for $k$ identical objects. The most general one- and two-body Hamiltonian that describes the relative motion of a system of $k$ identical clusters, is a scalar under $S_k$, conserves angular momentum and parity as well as the total number of bosons, is given by

$$H = e_0 s^\dagger s - e_1 \sum_i b_i^\dagger \cdot \vec{b}_i + u_0 s^\dagger s \vec{s} \vec{s}$$

$$+ u_1 \sum_i s^\dagger b_i^\dagger \cdot \vec{b}_i + v_0 \left( \sum_i b_i^\dagger \cdot b_i^\dagger \vec{s} \vec{s} + \text{h.c.} \right)$$

$$+ \sum_L \sum_{i j' j} v_{ij'j}^{(L)} \left[ b_i^\dagger x b_j^\dagger \right]^{(L)} \cdot \left[ \vec{b}_{ij'} \cdot \vec{b}_j \right]^{(L)},$$

(3)

with $\vec{b}_{ij,m} = (-1)^{m-n} \vec{b}_{i-m}$ and $\vec{s} = s$. By construction, the $e_0$, $e_1$, $u_0$, $u_1$ and $v_0$ terms in equation (3) are invariant under $S_k$. The permutation symmetry imposes additional restrictions on the coefficients $v_{ij'j}^{(L)}$ of the last term.

The energy eigenvalues are obtained numerically by diagonalizing the Hamiltonian in a coupled harmonic oscillator basis. The corresponding wave functions are characterized by the total number of bosons $N$, angular momentum and parity $L^P$ and permutation symmetry $t$. In this contribution it is assumed that the identical clusters have no internal structure (like in the application to $\alpha$-cluster nuclei). As a consequence, the wave functions have to be completely symmetric under the permutation group $S_k$.

The ACM has a rich algebraic structure, which includes both continuous and discrete symmetries. It is of general interest to study limiting cases of the Hamiltonian of equation (3), in which the energy spectra and form factors can be obtained in closed form. In this contribution I consider two dynamical symmetries of the ACM Hamiltonian for the $k$-body problem which are related to the group lattice

$$U(3k-2) \supset \left\{ \begin{array}{c} U(3k-3) \\ SO(3k-2) \end{array} \right\} \supset SO(3k-3),$$

(4)

which are called the $U(3k-3)$ and $SO(3k-2)$ limits of the ACM, respectively. A geometric analysis shows that the $U(3k-3)$ limit corresponds for large $N$ to the (an)harmonic oscillator in $3(k-1)$ dimensions and the $SO(3k-2)$ limit to the deformed oscillator in $3(k-1)$ dimensions [19].

### 3. Transition form factors

Transition probabilities, charge radii, and other electromagnetic properties of interest can be obtained from the transition form factors. For electric transitions the form factors correspond to the matrix elements of the Fourier transform of the charge distribution

$$F(q) = \int d\vec{r} e^{i\vec{q} \cdot \vec{r}} \langle \alpha L' M' | \hat{\rho} (\vec{r}) | \alpha L M \rangle.$$

(5)

For an extended charge distribution in which the charges of the clusters are smeared by a Gaussian

$$\rho(\vec{r}) = \frac{Ze}{k} \left( \frac{\gamma}{\pi} \right)^{3/2} \sum_i e^{-\|(\vec{r} - \vec{n})\|^2},$$

(6)

the transition form factor reduces to

$$F(q) = Ze \sum_{M'} D_{M'M}^{(L)} (q) F(q) D_{M'M}^{(L)} (-q),$$

(7)

with

$$F(q) = e^{-i\vec{q} \cdot \vec{r}} \langle \alpha L'M' | e^{i\vec{q} \cdot \vec{r}} | \alpha L'M \rangle$$

$$= e^{-i\vec{q} \cdot \vec{r}} \langle \alpha L'M' | e^{-i\vec{q} \cdot \vec{r}} \rho_{k-1} | \alpha L'M' \rangle.$$

(8)

In the derivation I have used the symmetry of the wave functions, made a transformation to Jacobi coordinates and integrated over the center-of-mass coordinate.

In the ACM, these matrix elements can be obtained algebraically by making the replacement

$$\sqrt{(k-1)/k} \rho_{k-1} \to q \rho_{k-1} / X_D,$$

(9)

where

$$D_{k-1,0} = b_{k-1,0}^\dagger s + s^\dagger b_{k-1,0}.$$

(10)

The coefficient $X_D$ is a normalization factor and is equal to the reduced matrix element of the dipole operator between the ground state with $L^P = 0^+$ and the first excited state with $L^P = 1^-$

$$X_D = \{ 1^- | [D_{k-1}] | 0^+_1 \}.$$
Therefore, in the ACM one has to evaluate the matrix elements of the transition operator
\[ \hat{U}(\epsilon) = e^{i\mathcal{D}_{1-0}}, \]
with \( \epsilon = -\frac{q\beta}{X_D} \) which can be interpreted as representation matrix elements of \( U(3k - 2) \), i.e. generalizations of the Wigner D-functions for \( SU(2) \). In general, these matrix elements can be derived from the transformation properties of a single boson. The transition operator \( \hat{U}(\epsilon) \) transforms the scalar boson and the \( \epsilon \)-component of the \( (k - 1) \)th Jacobi boson amongst each other, and does not affect the other bosons
\[ U(\epsilon) \left( b_{1-0}^{s^4}, b_{1-0}^{t} \right) U(-\epsilon) = \left( \cos \epsilon \sin \epsilon, \cos \epsilon \cos \epsilon \right) \left( b_{1-0}^{s^4}, b_{1-0}^{t} \right). \]

There are special solutions of the ACM in which these matrix elements can be derived in closed form. These solutions correspond to dynamical symmetries of the ACM Hamiltonian. Here I discuss two of them: the \( U(3k - 3) \) limit (harmonic oscillator) and the \( SO(3k - 2) \) limit (deformed oscillator). The general procedure to obtain the form factors for a single \( (2\alpha + 1) \)-dimensional oscillator was outlined in [12], and will be generalized here to the ACM for a system of \( k - 1 \) coupled three-dimensional oscillators.

### 3.1. Harmonic oscillator

In the absence of the \( v_0 \) term in equation (3), there is no coupling between different harmonic oscillator shells. The oscillator is harmonic if all terms, except \( c_0 \) and \( c_t \), are set to zero; otherwise it is anharmonic. This dynamical symmetry corresponds to the group reduction
\[ U(3k - 2) \supseteq U(3k - 3) \supseteq SO(3k - 3) \]
\[ \supseteq SO(3) \supseteq SO(2) \]
\[ \alpha, \quad L_t, \quad M \].

The label \( n \) represents the total number of oscillator quanta \( n = \sum n_i = 0, 1, \ldots, N \). The energy levels are grouped into oscillator shells characterized by \( n \) and parity \( P = (-1)^n \). The levels belonging to an oscillator shell are further classified by the symmetric irreducible representation \( \tau \) of \( SO(3k - 3) \) with \( \tau = n, n - 2, \ldots, 1 \) or 0 for \( n \) odd or even, the angular momentum \( L \) and its projection \( M \), and the permutation symmetry \( \alpha \). A denotes all additional labels that are needed for a unique classification scheme. This special case is called the \( U(3k - 3) \) limit of the ACM.

The wave functions for the \( U(3k - 3) \) limit are given by
\[ |(N)\tau \alpha L, M \rangle = \frac{B_{\alpha \tau}}{\sqrt{(N - n)!}} \left( \sum_{k=1}^{N(n)} b_{1-0}^{s^4} \right)^{\frac{\tau}{2}} |(\tau)\alpha \tau L, M \rangle, \]
with
\[ B_{\alpha \tau} = (\frac{-\tau}{\tau})^{\frac{\tau}{2}} \frac{(2\tau + 3k - 5)!!}{(n + \tau + 3k - 5)!!(n - \tau)!!}. \]

The matrix elements of the transition operator \( \hat{U}(\epsilon) \) can be derived by using the transformation properties of equation (13)
\[ U_{\text{Nada}}(\epsilon) = \langle (N)\tau \alpha L, M | \hat{U}(\epsilon) | (N)000000 \rangle = B_{\alpha \tau} A_{\text{Nada}} \frac{N!}{(N - n)!} (\sinh \epsilon)^n (\cos \epsilon)^{N-n} \]
with
\[ A_{\text{Nada}} = \frac{1}{\tau !} \langle (\tau)\alpha \tau L, 0 | \left( b_{1-0}^{s^4} \right)^{\frac{\tau}{2}} | 0 | 000000 \rangle. \]

In general, the coefficients \( A_{\text{Nada}} \) have to calculated explicitly. Only for the case of two-body clusters they have been derived in closed form [7].

For large \( N \), the \( U(3k - 3) \) limit corresponds to the (an) harmonic oscillator [19]. The coefficient \( e \) is given by \( e = -q\beta/X_D \) with \( X_D = \sqrt{3}N \). In the large \( N \) limit which is taken such that \( n/N \approx 1 \) and \( q\beta \) remains finite, the transition matrix elements reduce to
\[ U_{\text{Nada}}(\epsilon) \rightarrow B_{\alpha \tau} A_{\text{Nada}} \left( -\frac{\text{aq} \beta}{\sqrt{3}} \right)^{\frac{\tau}{2}} e^{-\frac{q\beta}{\sqrt{3}}} \].

As an example, the elastic form factor is given by
\[ U_{\text{el}}(\epsilon) = (\cos \epsilon)^N \rightarrow e^{-\frac{q\beta}{\sqrt{3}}} = 1 - \frac{1}{6} q^2 \beta^2 + \cdots. \]

Figure 1 shows a comparison of the elastic form factor in the \( U(3k - 3) \) limit calculated for finite \( N = 10 \) and in the large \( N \) limit. The scale parameter \( \beta \) is related to the rms radius of the system
\[ \left( \frac{r^2}{\sigma^2} \right) = -6 \frac{dF_{\text{el}}(\epsilon)}{d\epsilon^2} |_{\epsilon=0} = \beta^2 + \frac{3}{2\gamma}. \]

The probability that a state belonging to a given oscillator shell \( n \) can be excited from the ground state with \( n = 0 \) is
given by the binomial distribution

\[
P_n(\epsilon) = \sum_{\text{rad}} \left| U_{n\text{rad},n}(\epsilon) \right|^2 = \binom{N}{n} \left( \sin^2 \epsilon \right)^n \left( \cos^2 \epsilon \right)^{N-n}.
\]

(22)

For \( \epsilon = 0 \) only the ground state is excited. With increasing values of \( \epsilon \) all higher oscillator shells are successively excited until for \( \epsilon = \pi/2 \) all strength is concentrated in the highest oscillator shell with \( n = N \). The excitation probability is symmetric around \( \epsilon = \pi/2 \), and is a periodic function with period \( \pi \) which implies that for \( \epsilon = \pi \) all strength is again concentrated in the ground state. This behavior is an artefact of the finiteness of the model space. The range of \( q \) values shown in the figures is up to \( q = 4 \) \((1/\text{fm})\) which corresponds to \( \epsilon = -q\beta/\sqrt{3N} \approx -1.83 \). The excitation probabilities of all states of a given oscillator shell show the same dependence on \( \epsilon \) (or \( q \)), the only difference is in the numerical factor \( B_{N\alpha} A_{\text{rad}} \).

In the large \( N \) limit, the excitation probability reduces to the familiar Poisson distribution for the harmonic oscillator [1]

\[
P_n(\epsilon) \to \frac{1}{n!} \left( \frac{q^2 \beta^2}{3} \right)^n e^{-q^2 \beta^2}.
\]

(23)

Figure 2 shows the results for \( P_n \) for the case of four-cluster systems. The top panel shows the result for the sum over all states according to equation (23). In the bottom panel, the sum is restricted to states which are symmetric \((t = [k])\) under the permutation group \( S_k \), as is relevant for the case of \( \alpha \)-cluster nuclei. The curves for a given oscillator shell have the same shape as in the top panel, but they are multiplied by a factor of 1 for \( n = 0 \), 1/3 for \( n = 2 \) and 2/9 for \( n = 3 \). The \( n = 1 \) shell is absent since it does not contain a symmetric state.

### 3.2. Deformed oscillator

For the (an)harmonic oscillator, the number of oscillator quanta \( n \) is a good quantum number. However, when \( v_0 \neq 0 \) in equation (3), the oscillator shells with \( \Delta n = \pm 2 \) are mixed, and the eigenfunctions are spread over many different oscillator shells. A dynamical symmetry that involves the mixing between oscillator shells, is provided by the reduction

\[
\left( U(3k - 2) \supset SO(3k - 2) \supset SO(3k - 3) \right)_{[N]} \supset \left( SO(3) \supset SO(2) \right)^{\sigma L_t \text{, } M}.
\]

(24)

The label \( \sigma = N, N - 2, \ldots, 1 \) or 0 for \( N \) odd or even, respectively, characterizes the symmetric representations of \( SO(3k - 2) \), and \( \tau = 0, 1, \ldots, \sigma \) those of \( SO(3k - 3) \). The remaining quantum numbers are the same as for the harmonic oscillator.

The wave functions for the \( SO(3k - 2) \) limit are given by

\[
\left[ [\sigma] \sigma L_t, M \right] = B_{N\sigma} \left( P^\dagger \right)^{\sigma L_t} \left[ [\sigma] \sigma L_t, M \right],
\]

(25)

Figure 2. Excitation probability in the \( U(3k - 3) \) limit of the ACM (harmonic oscillator) for \( k = 4 \) clusters calculated with \( \beta = 2.50 \text{ fm} \) and \( N = 10 \) for \( n = 0 \) (solid black line), \( n = 1 \) (solid green line), \( n = 2 \) (solid blue line) and \( n = 3 \) (solid red line). In the top panel the sum is performed over all states of a given oscillator shell, whereas in the bottom panel only the symmetric states are taken into account.

with

\[
B_{N\sigma} = (-)^{\frac{N\sigma}{2}} \sqrt{\frac{(2\sigma + 3k - 4)!!}{(N + \sigma + 3k - 4)!!(N - \sigma)!!}}.
\]

(26)

and \( P^\dagger \) is the pair creation operator in the boson space

\[
P^\dagger = s^\dagger s^\dagger \sum_{i,j} b_i^\dagger b_j^\dagger.
\]

(27)

The state with \( N = \sigma \) can be written as

\[
\left[ [\sigma] \sigma \alpha L_t, M \right] = \sum_{\tau = -\frac{\sigma - 1}{2}}^{\frac{\sigma + 1}{2}} F_j(\sigma, \tau) \left( s^\dagger \right)^{\sigma - \tau - 2j} \left( P^\dagger \right)^{\tau L_t} \left[ [\sigma] \sigma \alpha L_t, M \right],
\]

(28)
with

\[ F_j(\sigma, \tau) = \sqrt{\frac{(\sigma - \tau)! (2\sigma + 3k - 5)!!}{(\sigma + 3k - 6)!! (\sigma + \tau + 3k - 5)!}} \]

\[ \left( -\frac{1}{2} \right)^{j} \frac{(2\sigma + 3k - 6 - 2j)!!}{(\sigma - \tau - 2j)!}. \]  

(29)

For the calculation of transition form factors one has to derive the matrix elements of \( \hat{U}(\epsilon) \) between the ground state and an arbitrary final state. Since the \( \hat{D}_{k-1} \) is a generator of \( SO(3k - 2) \) the transition operator only connects to states which belong to the \( SO(3k - 2) \) ground state band with \( \sigma = N \). The matrix element for the excitation from the ground state can be expressed in terms of a Gegenbauer polynomial [12]

\[ U_{NN\text{el}_L}(\epsilon) = \left\langle \left| N \right\rangle \sigma \tau a_L, M = 0 \left| \hat{U}(\epsilon) \right| \left| N \right\rangle 00000 \right\rangle \]

\[ = A_{\text{el}_L} \sqrt{\frac{(2\tau + 3k - 6)!!}{(3k - 6)!!}} \sqrt{\frac{N!(3k - 5)!}{(N - \tau)!(2\tau + 3k - 5)!}} \sqrt{\frac{(N + 3k - 5)!}{(N + \tau + 3k - 5)!}} \]

\[ (\text{isinc}) \sum_{N_L}^N \left( \frac{\epsilon}{N_L} \right) \cos(\epsilon). \]  

(30)

In the large \( N \) limit, the \( SO(3k - 2) \) limit corresponds to a deformed oscillator in \( 3(k - 1) \) dimensions [19]. The coefficient \( \epsilon \) is given by \( \epsilon = -q\beta/X_D \) with \( X_D = \sqrt{N(N + 3k - 4)/(k - 1)} \). In the large \( N \) limit which is taken such that \( \tau/N \ll 1 \) and \( q\beta \) remains finite, the transition matrix element can be expressed in terms of a spherical Bessel function for \( k \)-body clusters with \( k \) even

\[ U_{NN\text{el}_L}(\epsilon) \rightarrow A_{\text{el}_L} \sqrt{3k - 5}!! (2\tau + 3k - 5)!! \]

\[ \frac{(-i)^j J_{\tau + \frac{5}{2} - k} \left( q\beta \sqrt{k - 1} \right)}{(q\beta \sqrt{k - 1})^{\frac{5}{2} - k}}. \]  

(31)

and in terms of a cylindrical Bessel function for \( k \) odd

\[ U_{NN\text{el}_L}(\epsilon) \rightarrow A_{\text{el}_L} \sqrt{3k - 5}!! (2\tau + 3k - 5)!! \]

\[ \frac{(-i)^j J_{\tau + \frac{3}{2} - k} \left( q\beta \sqrt{k - 1} \right)}{(q\beta \sqrt{k - 1})^{\frac{3}{2} - k}}. \]  

(32)

In the \( SO(3k - 2) \) limit, the elastic form factor is given by

\[ U_{\text{el}}(\epsilon) = \frac{N!(3k - 5)!}{(N + 3k - 5)!} C_N^{\frac{3k-1}{2}} (\cos \epsilon) \]

\[ \left\{ \begin{array}{ll}
(3k - 5)!! \frac{J_{\tau + \frac{5}{2} - k} \left( q\beta \sqrt{k - 1} \right)}{(q\beta \sqrt{k - 1})^{\frac{5}{2} - k}}, & \text{for } \text{even} \\
(3k - 5)!! \frac{J_{\tau + \frac{3}{2} - k} \left( q\beta \sqrt{k - 1} \right)}{(q\beta \sqrt{k - 1})^{\frac{3}{2} - k}}, & \text{for } \text{odd} 
\end{array} \right. \]  

(33)

Figure 3 shows a comparison of the elastic form factor in the \( SO(3k - 2) \) limit calculated for finite \( N \) and in the large \( N \) limit for three- and four-body clusters, respectively. In this case, the elastic form factor shows an oscillatory behavior. The results in the large \( N \) limit are closer to the exact calculations for the case of three-body clusters than they are for four-body clusters. This behavior can be understood qualitatively by first expressing the Gegenbauer polynomial in terms of a hypergeometric function and next making an expansion in powers of \( q\beta \)

\[ U_{\text{el}}(\epsilon) = zF_1 \left( -\frac{N}{2}, \frac{N + 3k - 4}{2}, \frac{3k - 3}{2}; \sin^2 \epsilon \right) \]

\[ = 1 - \frac{1}{6} (q\beta)^2 + \frac{k - 1}{24 (3k - 1)} (q\beta)^4 \]

\[ \times \left( 1 - \frac{2(3k - 4)}{3N^2} + O \left( \frac{1}{N^3} \right) \right) \cdots. \]  

(34)

In the large \( N \) limit, one has

\[ U_{\text{el}}(\epsilon) \rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (k - 1)^n (3k - 5)!!}{(2n)!! (2n + 3k - 5)!!} (q\beta)^{2n} \]
Figure 4. Excitation probability in the $SO(3k - 2)$ limit of the ACM (deformed oscillator) for $k = 4$ clusters calculated with $\beta = 2.50$ fm and $N = 10$ for $\tau = 0$ (solid black line), $\tau = 1$ (solid green line), $\tau = 2$ (solid blue line) and $\tau = 3$ (solid red line). In the top panel the sum is performed over all states of a given $\tau$-multiplet, whereas in the bottom panel only the symmetric states are taken into account.

\[ P_\tau(\epsilon) = 1 - \frac{1}{6} (q\beta)^2 + \frac{k - 1}{24(3k - 1)} (q\beta)^4 - \cdots. \]  

(35)

Whereas up to first order in $(q\beta)^2$ the results do not depend on $N$, for the second order term there is a $N$ dependent factor that moreover depends on the number of clusters $k$, which is smaller for three than for four clusters. The above equations show that also in this case, the scale parameter $\beta$ is related to the rms radius according to equation (21).

Since the dipole operator is a generator of $SO(3k - 2)$ only states belonging to the ground state band with $\tau = N$ can be excited from the ground state. The probability that a state belonging to the $\tau$ multiplet of the ground state band can be excited from the ground state with $\tau = 0$ is given by

\[ P_\tau(\epsilon) = \sum_{\alpha LR} |\langle 0|N\alpha LR(\epsilon)\rangle|^2. \]  

(36)

Just as for the harmonic oscillator, with increasing values of $\epsilon$ all higher $\tau$ multiplets are successively excited. The symmetry properties of $P_\tau$ are the same as for the harmonic oscillator. The range shown in the figures corresponds to $\epsilon = -q\beta/\sqrt{N(N+5)/2} = -1.15$ for three-body clusters and $\epsilon = -q\beta/\sqrt{N(N+8)/3} = -1.29$ for four-body clusters. The excitation probabilities of all states of a given $\tau$-multiplet show the same dependence on $\epsilon$ (or $q$), the only difference is in the numerical factor $A_{tot}$.

In the large $N$ limit, the excitation probability reduces to

\[ P_\tau(\epsilon) \to \frac{(3k - 5)!!(2\tau + 3k - 5)(\tau + 3k - 6)!}{(3k - 6)! \tau!} \left( \frac{j_{\tau+\frac{3k-5}{2}}(q\beta \sqrt{k-1})}{(q\beta \sqrt{k-1})^{\frac{3k-5}{2}}} \right)^2, \]  

(37)

for $k$ even and

\[ P_\tau(\epsilon) \to \frac{(3k - 5)!!(2\tau + 3k - 5)(\tau + 3k - 6)!}{(3k - 6)! \tau!} \left( \frac{J_{\tau+\frac{3k-5}{2}}(q\beta \sqrt{k-1})}{(q\beta \sqrt{k-1})^{\frac{3k-5}{2}}} \right)^2, \]  

(38)

for $k$ odd. Figure 4 shows the results for $P_\tau$ for the case of four-cluster systems. The top panel shows the result for the sum over all states, whereas in the bottom panel the sum is restricted to states which are symmetric ($t = k$) under the permutation group $S_k$. The curves for a given $\tau$-multiplet have the same shape as in the top panel, but they are multiplied by a factor of 1 for $\tau = 0$, $1/4$ for $\tau = 2$ and $11/36$ for $\tau = 3$. The $\tau = 1$ multiplet is missing since it contains no symmetric states.

4. Summary and conclusions

In this contribution, I showed how the derivation of transition form factors for systems of two- and three-body clusters can be generalized to an arbitrary number of $k$ clusters. The derivation was carried out in explicit form for two dynamical symmetries of the ACM both for finite systems (finite number of bosons $N$) and infinite systems (large $N$ limit). The ACM is based on the algebraic quantization of the relative Jacobi variables for few-body systems. The ensuing $U(3k - 2)$ spectrum generating algebra incorporates all vibrational and rotational degrees of freedom from the beginning, and takes into account the permutation symmetry of identical clusters in an exact manner.

First I discussed the $U(3k - 2) \supset U(3k - 3)$ limit which corresponds to the harmonic oscillator. With increasing value of the coupling strength $\epsilon$ the different oscillator shells are excited successively before they fall off exponentially (in the large $N$ limit). The relative transition matrix elements to states belonging to the same oscillator shell only depend on a geometric factor $B_{rel}A_{rad}$ and not on the coupling strength $\epsilon$. Similarly, in the $SO(3k - 2)$ limit (deformed oscillator) the different $\tau$
multiplets are excited successively. In this case the form factors show an oscillatory behavior since in the large $N$ limit they are given by Bessel functions. Just as for the harmonic oscillator, the relative matrix elements to states belonging to the same $\tau$ multiplet only depend on a geometric factor ($A_{\text{out}}$).

The present results for transition form factors are of general interest since the ACM has found interesting applications in many different areas of physics. Future work includes possible applications of the ACM for four-body systems in molecular physics ($X_4$ molecules), nuclear physics ($^{16}\text{O}$ as a cluster of four $\alpha$ particles), and hadronic physics ($q^4 - q$ multiquark configurations). As a final comment, it is important to stress that the ACM provides a general framework to study the full rotational and vibrational structure of many-body systems which is not restricted to the case of identical particles discussed in this contribution. It can be applied to other situations as well, such as nonidentical particles and/or other geometric configurations [21].

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References

[1] Alder K and Winther A 1975 Electromagnetic excitation (North-Holland: Amsterdam)

[2] Amado R D, McNeil J A and Sparrow D A 1982 Phys. Rev. C 25 13

[3] Collins L A and Norcross D W 1978 Phys. Rev. A 18 467

[4] Balster G J, van Roosmalen O S and Dieperink A E L 1983 J. Math. Phys. 24 1392

[5] Wenes G, Dieperink A E L and van Roosmalen O S 1984 Nucl. Phys. A 424 81

[6] Ginocchio J N 1984 Nucl. Phys. A 421 369c

[7] Bijker R, Amado R D and Sparrow D A 1986 Phys. Rev. A 33 247

[8] Wenes G, Ginocchio J N, Dieperink A E L and van der Cammen B 1986 Nucl. Phys. A 459 631

[9] Wenes G, Yoshinaga N and Dieperink A E L 1985 Nucl. Phys. A 443 472

[10] Balantekin A B, Bennett J R and Takigawa N 1991 Phys. Rev. C 44 145

[11] Bijker R and Iachello F 2000 Phys. Rev. C 61 067305

[12] Bijker R and Iachello F 2002 Ann. Phys., NY 298 334

[13] Bijker R and Ginocchio J N 1992 Phys. Rev. C 45 3030

[14] Iachello F and Levine R D 1995 Algebraic Theory of Molecules (Oxford: Oxford University Press)

[15] Iachello F and Jackson A D 1982 Phys. Lett. B 108 151

[16] Iachello F and Jackson A D 1983 Phys. Lett. B 131 281

[17] Bijker R, Iachello F and Levine A 1994 Ann. Phys., NY 236 69

[18] Bijker R, Iachello F and Leviatan A 2000 Ann. Phys., NY 284 89

[19] Bijker R, Iachello F and Leviatan A 1995 Phys. Rev. A 52 2786

[20] Bijker R, Dieperink A E L and Leviatan A 2000 Phys. Rev. A 52 2786

[21] Bijker R and Leviatan A 1998 Few-Body Syst. 25 89