The Precise Formula in a Sine Function Form of the Norm of the Amplitude
and the Necessary and Sufficient Phase Condition for Any Quantum Algorithm with Arbitrary Phase Rotations

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Abstract.
In this paper we derived the precise formula in a sine function form of the norm of the amplitude in the desired state, and by means of the precise formula we presented the necessary and sufficient phase condition for any quantum algorithm with arbitrary phase rotations. We also showed that the phase condition: identical rotation angles $\theta = \phi$, is a sufficient but not a necessary phase condition.

1 Introduction

Quantum algorithms use two techniques: Fourier transforms and amplitude amplification. Grover’s search algorithm is based on the latter above. The problem addressed by Grover’s algorithm is to search a desired term (or marked term in [5], or target term in [4]) in an unordered database of size $N$. To accomplish this a quantum computer needs $O(\sqrt{N})$ queries by using Grover’s algorithm [4]. In Grover’s original version the algorithm consists of a sequence of unitary operations on a pure state, the algorithm is $Q = -I_0^{(\pi)} W I_\tau^{(\pi)} W$, where $W$ is Walsh-Hadamard transformation and $I_\tau^{(\pi)} = I - 2|x\rangle\langle x|$, which inverts the amplitude in the state $|x\rangle$; here $I_0^{(\pi)}$ and $I_\tau^{(\pi)}$ invert the amplitudes in the initial and desired basis states $|0\rangle$ and $|\tau\rangle$, respectively. To extend his original algorithm Grover in [4] replaced Walsh-Hadamard transformation with any quantum mechanical operation, then obtained the quantum search algorithm $Q = -I_\tau^{(\pi)} U^{-1} I_\tau^{(\pi)} U$, where $U$ is any unitary operation and $U^{-1}$ is equal to the adjoint (the complex conjugate of the transpose) of $U$. Grover thinks that it leads several new applications and broadens the scope for implementation. To further more generalize Grover’s algorithm it allows that the amplitudes are rotated by arbitrary phases, instead of being inverted. For

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example, quantum algorithm $Q = -I^{(\theta)}U^{-1}I^{(\phi)}U$, $\theta$ and $\phi$ are the rotation angles of the amplitudes in the initial basis state $|\gamma\rangle$ and in the desired basis state $|\tau\rangle$, respectively. Recently many authors have been contributing to the general quantum search algorithms with any unitary operations and arbitrary phase rotations.

For the general quantum search algorithms the following problems need to be solved.

1. What is the amplitude in the desired state after $k$ applications of $Q$?
2. What are rotation angles in the initial and the desired basis states to reach the desired state from the initial state? This problem is called the phase condition.
3. What is the optimal number of the iteration steps to find the desired state?
4. Which of the general algorithms is the most efficient?

In [6] we showed that the amplitude in the desired state for any quantum search algorithm which preserves a two-dimensional vector space can be exactly written as a polynomial form in $(\beta \lambda)$. From the precise formula in a polynomial form in $(\beta \lambda)$ we obtained some results in [6]. For example we found non-symmetric effects of different rotating angles and obtained the approximate formulas of the amplitude of the desired state and the optimal number of iteration steps to find the desired state. However from the precise formula it is not convenient to present a general phase condition. Specially for the algorithms with identical rotation angles Long et al. gave the approximate formulas of the amplitude in the desired state in [5]. In this paper we will give the precise formulas in a sine function form of the norm of the amplitude in the desired state for any quantum search algorithm with arbitrary rotation angles, which is necessary to present a sufficient and necessary phase condition.

To find the desired state with certainty Long et al. in [5] first presented a matching condition: identical rotation angles $\theta = \phi$. Then in [7] Hoyer gave the phase condition $\tan(\phi/2) = \tan(\phi/2)(1 - a)$. In [10] the recursion equation was used to study the quantum search algorithm, and it concluded that for different rotation angles: $\theta \neq \phi$ the algorithm fails to enhance the probability of measuring a marked state and therefore in order for the algorithm to apply, the two rotation angles must be equal, namely, $\theta = \phi$.

In this paper we will study the general quantum search algorithms with any unitary operations and arbitrary phase rotations. We will give the phase condition, which is necessary and sufficient, to find the desired state, so we can thoroughly solve the phase condition problem presented by Grover in [4]. We will also indicate that identical rotation angles $\theta = \phi$, which is the special case of our condition, is sufficient but not necessary to find the desired state, therefore it contradicts the conclusions obtained in [5] and [10]. Using the precise phase condition we can construct quantum algorithms with arbitrary rotations that succeed with certainty.

This paper is organized as follows. In section 4 we will derive the precise formula in a sine function form of the norm of the amplitude in the desired state with arbitrary phase rotations. In section 5 by means of the precise formula in the section 4 we will present the necessary and sufficient phase condition for any quantum algorithm with arbitrary phase rotations and give
the precise optimal number $k_0$ of applications of the algorithm $Q$ to find the desired state. In section 6 we will show that identical rotation angles $\theta = \phi$ is a sufficient but not a necessary phase condition to find the desired state. The section 7 will introduce the $|\sin|$ property and the periodicity and the monotone increasing property in the interval $[0, k_0]$ of the norm of the amplitude $|b_k|$ as a function of $k$. In section 8 and 9 for the algorithms with identical rotation angles $\theta = \phi$ and Grover’s algorithm we will give the reduced precise formulas in a sine function form of the norm of the amplitude and the reduced precise optimal numbers of iteration steps, respectively. In section 10 we will prove that the optimal number of iteration steps to find the desired state for Grover’s algorithm is less than the one for the algorithms with arbitrary identical rotation angles.

2 The algorithm \( Q = -I_\gamma U^{-1}I_\tau U \), where \( I_\gamma = I - 2 \cos \theta e^{i\theta} |\gamma\rangle \langle \gamma | \)
and \( I_\tau = I - 2 \cos \phi e^{i\phi} |\tau\rangle \langle \tau | \)

Grover studied the quantum search algorithm [4]: $Q = -I_\gamma U^{-1}I_\tau U$, where $U$ is any unitary operator and $U^{-1}$ is equal to the adjoint (the complex conjugate of the transpose) of $U$ and $I_x^{(\pi)} = I - 2| x \rangle \langle x |$, which inverts the amplitude in the state $| x \rangle$. Generally let $I_x = I - ae^{i\theta} | x \rangle \langle x |$. Then $I_x$ is unitary iff $(1 - ae^{i\theta})(1 - ae^{-i\theta}) = 1$. That is, $a = 2 \cos \theta$. Then $I_x = I - 2 \cos \theta e^{i\theta} | x \rangle \langle x |$. Please see [6]. If let $I'_x = I - (ae^{i\theta} + 1)| x \rangle \langle x |$, then $I'_x$ is unitary iff $a = \pm 1$. The case in which $a = -1$ is used in [3].

Let $|\gamma\rangle$ be the initial basis state and $|\tau\rangle$ the desired basis state. If we apply $U$ to $|\gamma\rangle$, then the amplitude of reaching state $|\tau\rangle$ is $U_{\gamma\tau}$. That is, $\langle \tau | U | \gamma \rangle = U_{\tau\gamma}$, and $\langle \gamma | U | \tau \rangle = U_{\gamma\tau}^*$, where $U_{\gamma\tau}^*$ is complex conjugate of $U_{\tau\gamma}$.

Let $Q$ be any quantum search algorithm such that
\[
Q \left( \begin{array}{c} |\gamma\rangle \\ U^{-1}|\tau\rangle \end{array} \right) = M \left( \begin{array}{c} |\gamma\rangle \\ U^{-1}|\tau\rangle \end{array} \right),
\]
where $M = \begin{pmatrix} \alpha & \beta \\ \lambda & \delta \end{pmatrix}$. That is, $Q$ preserves the vector space spanned by $|\gamma\rangle$ and $U^{-1}|\tau\rangle$. After $k$ applications of $Q$ as soon as the state $U^{-1}|\tau\rangle$ is obtained, then another operation of $U$ will put the state of the quantum computer to $|\tau\rangle$, the desired state.

In [3], we introduced the algorithm $Q = -I_\gamma U^{-1}I_\tau U$, where $I_\gamma = I - 2 \cos \theta e^{i\theta} |\gamma\rangle \langle \gamma |$ and $I_\tau = I - 2 \cos \phi e^{i\phi} |\tau\rangle \langle \tau |$, $\alpha = -(1 - 2 \cos \theta e^{i\theta} + 4 \cos \theta e^{i\theta} \cos \phi e^{i\phi} |U_{\tau\gamma}|^2)$, $\beta = 2U_{\tau\gamma} \cos \phi e^{i\phi}$, $\lambda = 2 \cos \theta e^{i\theta}(1 - 2 \cos \phi e^{i\phi})U_{\tau\gamma}^*$, $\delta = 2 \cos \phi e^{i\phi} - 1$. When $\theta = \phi = 0$, it reduces to Grover’s algorithm.

Please see [3] to know what the matrices $M$ are like for Grover’s, Long and et al.’s and Hoyer’s algorithms.

In this paper all discussions and derivations are based on the algorithm $Q = -I_\gamma U^{-1}I_\tau U$, however the results obtained in this paper also hold for Grover’s, Long and et al.’s and Hoyer’s
algorithms and any other quantum search algorithm which preserves a two-dimensional vector space provided that only $\alpha, \beta, \lambda$ and $\delta$ appear in the results.

3 The proof by induction of the precise formula in a polynomial form in $(\beta \lambda)$ of the amplitude for arbitrary phase rotations

For a quantum search algorithm $Q$ the key problem is what the amplitude is in the desired state after $k$ applications of $Q$. In [3] the approximated formula of the amplitude in the desired state was given. In [4] the first precise formula of the amplitude with arbitrary phase rotations was written in polynomial form in $(\beta \lambda)$. Though we obtained some interesting results in [6] by using the precise formula, it is not convenient to present a general phase condition to find the desired state. However the precise formula gave us hints in finding the precise formula in a sine function form of the amplitude.

Let $Q|\gamma\rangle = \alpha|\gamma\rangle + \beta(U^{-1}|\tau\rangle), Q(U^{-1}|\tau\rangle) = \lambda|\gamma\rangle + \delta(U^{-1}|\tau\rangle),$ and $Q^k|\gamma\rangle = a_k|\gamma\rangle + b_k(U^{-1}|\tau\rangle)$, where $a_k$ and $b_k$ are the amplitudes in the state $|\gamma\rangle$ and the desired state $U^{-1}|\tau\rangle$, respectively.

In [3] we showed that $a_k$ and $b_k$ can be exactly written as the following polynomial form in $(\beta \lambda)$, respectively. Let $[x]$ be the greatest integer which is or less than $x$.

$$b_k = \beta r_k$$

where $r_k = (c_{k0} + c_{k1}(\beta \lambda) + c_{k2}(\beta \lambda)^2 + ... + c_{k[(k-1)/2]}(\beta \lambda)^{(k-1)/2})$, $c_{kj} = \sum_{n=k-1-2j}^{0} l_{k(k-1-2j-n)}^{(j)} a^n \delta^{k-1-2j-n}$, and $l_{ki}^{(j)} = \binom{i+j}{j} \binom{k-i-j-1}{j}$. Note that $\binom{n}{0} = 1$, for any $n \geq 0$.

However we did not put a strict proof in [3]. We will put a proof by induction of the conclusion in appendix 1 of this paper.

4 For arbitrary phase rotations the precise formula in a sine function form of the norm of the amplitude $|b_k| = |\beta| \frac{|\sin k\Delta|}{\sin \Delta}$

Let $Q|\gamma\rangle = \alpha|\gamma\rangle + \beta(U^{-1}|\tau\rangle), Q(U^{-1}|\tau\rangle) = \lambda|\gamma\rangle + \delta(U^{-1}|\tau\rangle),$ and $Q^k|\gamma\rangle = a_k|\gamma\rangle + b_k(U^{-1}|\tau\rangle)$, where $a_k$ and $b_k$ are the amplitudes in the state $|\gamma\rangle$ and the desired state $U^{-1}|\tau\rangle$, respectively. For example, $Q^2|\gamma\rangle = (\alpha^2 + \beta \lambda)|\gamma\rangle + \beta(\alpha + \delta)(U^{-1}|\tau\rangle).$ Let us exactly compute $b_k$. From the precise formula in a polynomial form in $(\beta \lambda)$ of the amplitude in the section above $b_k = \beta r_k$. Clearly when $\beta = 0$, that is, $\cos \phi = 0$, $|b_k| = 0$ and the quantum algorithm becomes useless. Therefore we assume that $\beta \neq 0$, that is, $\cos \phi \neq 0$ in this paper.
4.1 A simpler iterated formula of the amplitude $b_k$

In $[6]$ the iterated formulas of $a_k$ and $b_k$ were given as follows. $a_{k+1} = (\alpha a_k + \lambda b_k)$ and $b_{k+1} = (\beta a_k + \delta b_k)$. Note that the iterated formula of $b_k$ ($a_k$) contains the term $a_i$ ($b_i$). From the iterated formula we can derive a simpler iterated formula as follows. From $b_k = \beta a_{k-1} + \delta b_{k-1}$. we obtain that $a_{k-1} = (b_k - \delta b_{k-1})/\beta$. Then $b_{k+1} = \beta a_k + \delta b_k = \beta(a_{k-1} + \lambda b_{k-1}) + \delta b_k$

$= \beta a_{k-1} + \beta \lambda b_{k-1} + \delta b_k = \beta(a_{k-1} + \lambda b_{k-1})/\beta + \beta \lambda b_{k-1} + \delta b_k$

$= (\alpha + \delta)b_k + (\beta \lambda - \alpha \delta)b_{k-1}$. Clearly the formula of $b_{k+1}$ does not contain the term $a_i$, therefore it is simpler than one in $[3]$.

From the section above $b_k = \beta r_k$, where $\beta$ does not contain $k$. So we only need to derive the precise formula in a sine function form of $r_k$. First we derive the iterated formula of $r_k$.

4.2 The iterated formula of $r_k$ in the amplitude $b_k = \beta r_k$

From the precise formula of the amplitude $b_k = \beta r_k$ above, after computing $b_1 = \beta$ and $b_2 = \beta(\alpha + \delta)$, then we obtain $r_1 = 1$, $r_2 = \alpha + \delta$, and $r_{k+1} = (\alpha + \delta)r_k + (\beta \lambda - \alpha \delta)r_{k-1}$.

For Grover’s algorithm $r_1 = 1$ and $r_2 = \alpha + 1, r_{k+1} = (\alpha + 1)r_k - r_{k-1}$.

Next let us derive the precise formula of $r_k$ using the iterated formula of $r_k$.

4.3 $r_k$ in the amplitude $b_k = \beta r_k$ is exactly written as $r_k = \frac{z_1 - z_2}{z_1 - z_2}$, where $z_1 + z_2 = \alpha + \delta, z_1z_2 = -(\beta \lambda - \alpha \delta)$

From the iterated formula that $r_{k+1} = (\alpha + \delta)r_k + (\beta \lambda - \alpha \delta)r_{k-1}$ we can derive the precise formula in a sine function form of $r_k$. Let $r_{k+1} = (z_1 + z_2)r_k - z_1z_2r_{k-1}$, where $z_1 + z_2 = \alpha + \delta, z_1z_2 = -(\beta \lambda - \alpha \delta)$. Then we obtain that $r_{k+1}/r_{k-1} = \frac{z_1^{k+1} - z_2^{k+1}}{z_1 - z_2}$.

4.4 $r_k$ in the amplitude $b_k = \beta r_k$ is exactly written as $|r_k| = \frac{|\sin k \Delta|}{\sin \Delta}$

Let us study the equation $z^2 - (\alpha + \delta)z + (\alpha \delta - \beta \lambda) = 0$ which is the characteristic polynomial of the matrix $M$ which the algorithm $Q$ corresponds to. Let $|M|$ be the determinant of the matrix $M$. Then $|M| = \alpha \delta - \beta \lambda = | - I_4^{(x)}U^{-1}I_4^{(y)}U | = |I_4^{(x)}||I_4^{(y)}||U^{-1}U| = e^{ix}e^{iy}$, where $x$ and $y$ are the rotation angles.

Let $z_1$ and $z_2$ are the two roots of the equation. Let $z_1 = \rho_1 e^{i\psi_1}$ and $z_2 = \rho_2 e^{i\psi_2}$, where $\rho_1 > 0$ and $\rho_2 > 0$. From the result 1 in the appendix 3 clearly $|z_1z_2| = 1$, then $\rho_1 \rho_2 = 1$ and $\psi_1 + \psi_2 = 2(\theta + \phi)$. Let $z_1 = \rho e^{i\psi_1}$, where $\rho > 0$. Then $z_2 = \frac{1}{\rho} e^{i\psi_2}$. From the result 2 in the appendix 3 $z_1 + z_2 = \rho e^{i\psi_1} + \frac{1}{\rho} e^{i\psi_2} = 2\cos((\theta - \phi) - 2|U_{\gamma}|^2 \cos \theta \cos \phi)e^{i(\theta + \phi)}$.

Then $\rho \cos \psi_1 + \frac{1}{\rho} \cos \psi_2 = 2\cos((\theta - \phi) - 2|U_{\gamma}|^2 \cos \theta \cos \phi) \cos(\theta + \phi)$,
and $\rho \sin \psi_1 + \frac{1}{\rho} \sin \psi_2 = 2(\cos((\theta - \phi) - 2|U_{\tau_7}|^2 \cos \theta \cos \phi) \sin(\theta + \phi))$.

Then $\rho \cos \psi_1 + \frac{1}{\rho} \cos \psi_2 = \cos(\theta + \phi)$, $\rho^2 \cos \psi_1 + \cos \psi_2 = \cos(\theta + \phi)$, $\rho^2 \cos \psi_1 \sin(\theta + \phi) + \cos \psi_2 \sin(\theta + \phi) = \rho^2 \sin \psi_1 \cos(\theta + \phi) + \sin \psi_2 \cos(\theta + \phi) + \rho^2 \sin(\psi_1 - (\theta + \phi)) + \sin(\psi_2 - (\theta + \phi)) = 0$. Note that $\psi_1 + \psi_2 = 2(\theta + \phi)$. Therefore $(\rho^2 - 1) \sin(\psi_1 - (\theta + \phi)) = 0$.

There are two cases. Case 1. In the case $\rho \neq 1$. Thus $\sin(\psi_1 - (\theta + \phi)) = 0$. From that $\psi_1 + \psi_2 = 2(\theta + \phi)$ we obtain that $\psi_1 = \psi_2$. Let $z_1 = \rho e^{i\psi}$. Then $z_2 = \frac{1}{\rho} e^{i\psi}$. Then $z_1 + z_2 = (\rho + \frac{1}{\rho}) e^{i\psi} = \alpha + \delta$, from the results 4 and 5 in the appendix 3 we obtain $2 < \rho + \frac{1}{\rho} = |\alpha + \delta| \leq 2$. Therefore $\rho \neq 1$ is not possible.

Case 2. In the case $\rho = 1$. Let $z_1 = e^{i\psi_1}$ and $z_2 = e^{i\psi_2}$. Then $r_k = \frac{z_1 - z_2}{z_1 - z_2} = \frac{e^{ik\psi_1} - e^{ik\psi_2}}{e^{ik\psi_1} - e^{ik\psi_2}} = -2 \sin k \frac{\psi_1 + \psi_2}{2} \sin k \frac{\psi_1 - \psi_2}{2} + 2i \cos k \frac{\psi_1 + \psi_2}{2} \sin k \frac{\psi_1 - \psi_2}{2}

= \sin \frac{\psi_1 - \psi_2}{2} e^{i(k-1)\frac{\psi_1 + \psi_2}{2}}.

From that $|z_1 + z_2|^2 = |\alpha + \delta|^2$ we obtain that $\cos(\psi_1 - \psi_2) = 2(\cos(\theta - \phi) - 2|U_{\tau_7}|^2 \cos \theta \cos \phi)^2 - 1$. Since $\cos \phi \neq 0$, $\psi_1 \neq \psi_2$, please see the result 6 in the appendix 3. Without loss of generality, let $\psi_1 > \psi_2$, then $\psi_1 - \psi_2 = \arccos\{2(\cos(\theta - \phi) - 2|U_{\tau_7}|^2 \cos \theta \cos \phi)^2 - 1\} = \arccos\left(\frac{|\alpha + \delta|^2}{2} - 1\right)$.

Let $\Delta = \frac{\psi_1 - \psi_2}{2}$. Then $\sin \Delta = \sqrt{1 - |\alpha + \delta|^2 / 4} = \sqrt{1 - (\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi)^2 / 4} > 0$

Since $\cos \phi \neq 0$, $\psi_1 \neq \psi_2$. Therefore $r_k = \frac{\sin k\Delta}{\sin \Delta} e^{i(k-1)(\theta + \phi)}$, and $|r_k| = \frac{\sin k\Delta}{\sin \Delta}$.

4.5 The precise formula in a sine function form of the norm of the amplitude $|b_k|$.

From that $b_k = \beta r_k$ we obtain the following versions of $|b_k|$.

Let $p = |U_{\tau_7}|$ in this paper.

Version 1. $|b_k| = |\beta| \frac{\sin k\Delta}{\sin \Delta}$, where $|\beta| = 2p \cos \phi$. Please see the definition of $\Delta$ above.

Version 2. $|b_k| = |\beta| \frac{\sin(k(\arccos(\frac{|\alpha + \delta|^2}{2}) - 1))/2)}{\sqrt{1 - (\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi)^2 / 4}}$.

Version 3. $|b_k| = |\beta| \frac{\sin(k \arcsin(\sqrt{1 - (\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi)^2 / 4}))}{\sqrt{1 - (\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi)^2 / 4}}$.

Example 1. $\phi = 0$ and $\theta = \pi/2$. $|b_k| = 2p |\sin(k\pi/2)| << 1$. Please see the example in [4]. Therefore that $\phi = 0$ and $\theta = \pi/2$ can not be used to construct a working quantum algorithm.

Example 2. $\phi = \pi/6$ and $\theta = \pi/2$. $|b_k| = 2p |\sin(k\pi/3)| << 1$. Therefore that $\phi = \pi/6$ and $\theta = \pi/2$ can not be used either to construct a working quantum algorithm.

Comment. Versions 1 and 2 and 3 of $b_k$ are also the same for Grover’s, Long et al.’s, Hoyer’s algorithms and any other quantum search algorithm which preserves a two-dimensional vector space.
5 The necessary and sufficient phase condition \( \sin \Delta \leq |\beta| \) for arbitrary phase rotations and the precise optimal number of iteration steps to find the desired state

To construct a successful quantum search algorithm we need to know what the rotation angles are to find the desired state. This is called the phase condition. For arbitrary phase rotations Long et al. first obtained an important result for phase condition [5], that is, identical rotation angles \( \theta = \phi \). Bilham also obtained the same result using recursion equations [10]. In [7] Hoyer gave the phase condition \( \tan(\varphi/2) = \tan(\phi/2)(1 - a) \). Here we will give a necessary and sufficient phase condition for arbitrary phase rotations to find the desired state with certainty. The conditions given by Long et al., Hoyer and Bilham in [10][7][5] are the special cases of our phase condition.

5.1 The necessary and sufficient phase condition \( \sin \Delta \leq |\beta| \)

An algorithm can search the desired state with certainty, that is, there exists a \( k \) such that \( |b_k| = 1 \), if and only if \( \sin \Delta \leq |\beta| \), where \( \sin \Delta = \sqrt{1 - |\alpha + \delta|^2/4} = \sqrt{1 - (\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi)^2} \).

Let us derive the conclusion. If \( b_k = 1 \) for some \( k \), that is, \( |b_k| = |\beta| \), then \( |\sin k\Delta| = \frac{\sin \Delta}{|\beta|} \). Clearly \( \frac{\sin \Delta}{|\beta|} \leq 1 \) since \( |\sin k\Delta| \leq 1 \). Therefore \( \sin \Delta \leq |\beta| \) and the optimal number of iteration steps to find the desired state with certainty \( k_o = \frac{1}{\Delta} \arcsin \frac{\sin \Delta}{|\beta|} \) such that \( |b_k| = 1 \). Conversely if \( \sin \Delta \leq |\beta| \), then \( \frac{\sin \Delta}{|\beta|} \leq 1 \). Let \( k_o = \frac{1}{\Delta} \arcsin \frac{\sin \Delta}{|\beta|} \). Clearly \( k_o \Delta = \arcsin \frac{\sin \Delta}{|\beta|} \) and \( \sin k_o \Delta = \frac{\sin \Delta}{|\beta|} \). Therefore \( |b_{k_o}| = |\beta| \frac{|\sin k_o \Delta|}{\sin \Delta} = 1 \).

Sometimes it is convenient to use \( \frac{\sin \Delta}{|\beta|} \leq 1 \) instead of \( \sin \Delta \leq |\beta| \). Another version of the phase condition is \( \sqrt{1 - |\alpha + \delta|^2/4} \leq |\beta| \). Also \( k_o = \frac{\arcsin(\sqrt{1 - |\alpha + \delta|^2/4})}{\arcsin(\sqrt{1 - |\alpha + \delta|^2/4})} \).

The phase condition also holds for Long et al.’s and Hoyer’s algorithms and any other quantum search algorithm which preserves a two-dimensional vector space.

Example 3. When \( \theta = \pi/2 \), for any \( \phi \), \( C_2 = \frac{1}{2p} \gg 1 \), therefore \( |b_k| < 1 \).

Therefore that \( \theta = \pi/2 \) can not be used either to construct a working quantum algorithm, please see the section 3.3 in [6].

From the general phase condition we specially get the following two corollaries whose derivations were put in the appendix 4. It is convenient to use the two corollaries to check if an algorithm satisfies the phase condition.

5.2 Two corollaries of the phase condition

Corollary 1.
Let $\theta$ and $\phi$ be in the same quadrant or $|\theta - \phi| < \pi/2$ and $\cos \theta \cos \phi < 0$. Then $\sin \Delta \leq |\beta|$, that is, the algorithm can search the desired state with certainty, if and only if $|\theta - \phi| \leq \arccos(2p^2 \cos \theta \cos \phi + \sqrt{1 - 4p^2 \cos^2 \phi})$.

Corollary 2.
Let $\theta$ and $\phi$ be in the same quadrant or $\cos \theta \cos \phi < 0$ and $\sin \theta \sin \phi < 0$. Then $\sin \Delta > |\beta|$, that is, the algorithm cannot search the desired state with certainty, if $|\sin(\theta - \phi)| > |\beta|$.

6 Identical rotation angles $\theta = \phi$ is a sufficient but not a necessary phase condition to find the desired state

For any quantum search algorithm with arbitrary rotation angles what are the rotation angles to find the desired state with certainty? In [5] Long et al. first presented identical rotation angles $\theta = \phi$ to find the desired state with certainty. In [10] Bilham also derived the phase condition. In this section we will show that the phase condition: identical rotation angles $\theta = \phi$, is sufficient but not a necessary to find the desired state with certainty.

6.1 Identical rotation angles $\theta = \phi$ is sufficient to find the desired state

Given $\theta = \phi$, then $\sin \Delta = 2p |\cos \phi| \sqrt{1 - p^2 \cos^2 \phi}$, and clearly $\sin \Delta \leq |\beta|$. Therefore when $\theta = \phi$ by the phase condition the quantum algorithm $Q$ can search the desired state with certainty except that $I_y = I_z = I$.

6.2 Identical rotation angles $\theta = \phi$ is not necessary to find the desired state

We will use the following examples to show that the condition $\theta = \phi$ is not necessary.

Example 4. Assume that $\phi = 0$. Then $\sin \Delta \leq |\beta|$, that is, the algorithm can search the desired state with certainty, if and only if $|\sin \theta| \leq \frac{2p^2}{|1 - 2p^2|}$.

Let us show the result above holds. From the phase condition when $\phi = 0 \sin \Delta \leq \frac{2p^2}{|1 - 2p^2|}$. Let $\frac{\sin \Delta}{|\beta|} \leq 1$ Then we obtain $4p^4 \cos^2 \theta \geq (1 - 4p^2) \sin^2 \theta$, $\sin^2 \theta \leq \frac{4p^4}{(1 - 2p^2)^2}$, and $|\sin \theta| \leq \frac{2p^2}{|1 - 2p^2|}$.

For example, when $p = 0.5, \phi = 0$ and $\theta = \pi/3$, $\sin \frac{\pi}{3} = \sqrt{3}/2 < \frac{2p^2}{|1 - 2p^2|} = 1$, therefore the phase condition is satisfied. In the case $k_0 = 1$.

Example 5. Assume that $\theta = 0$. It is not hard to show that $\sin \Delta \leq |\beta|$, that is, the algorithm can search the desired state with certainty, if and only if $\cos^2 \phi \geq 1/(1 + 4p^4)$.

From the examples clearly identical rotation angles $\theta = \phi$ is not necessary to find the desired state with certainty. Therefore it contradicts the conclusions obtained in [10] and [5]. However identical rotation angles $\theta = \phi$ is a special but an important phase condition.
7 The $|\sin|$ property and the periodicity and the monotone increasing property in the interval $[0, k_0]$ of the norm of the amplitude $|b_k|$ as a function of $k$

For a successful quantum search algorithm $Q$ it is necessary that the norm of the amplitude in the desired state is amplified after each application of $Q$. Therefore to construct a quantum search algorithm we need to study the properties of the amplitude in the desired state. In this section we will show that any quantum search algorithm with any unitary operations and arbitrary rotation angles is a $|\sin|$ algorithm, here we mean the norm of the amplitude in the desired state can be written as $|\sin|$ form after $k$ applications of $Q$. We will also show that though $|b_k|$ periodically changes between 0 and 1 when $k \to \infty$, fortunately as $k$ increases from 0 to the optimal number $k_0$ of iteration steps, $|b_k|$ behaves like $\sin x$ in $[0, \pi/2]$.

7.1 The $|\sin|$ property of the norm of the amplitude $|b_k|$ as a function of $k$

From the section 4 above the norm of the amplitude $|b_k|$ in the desired state can be written as $|\sin|$ form. Fix $U_{\tau \gamma}$ and the rotation angles $\theta$ and $\phi$, $|b_k|$ is only a $|\sin|$ function of $k$. Therefore it is easy to understand why the curves of $|b_k|$ in the Fig. 3 and Fig. 4 on page 30 in [5] look like the ones of $|\sin|$. In [5] Long et al. could not explain the phenomenon.

7.2 The periodicity of the norm of the amplitude $|b_k|$ as a function of $k$

Fix the rotation angles $\theta$, $\phi$ and $|U_{\tau \gamma}|$, $|b_k|$ is a periodic function of $k$. Let $T$ be the period of $|b_k|$ as a $|\sin|$ function of $k$. Then $T = \frac{\pi}{\sin \Delta} = \frac{\pi}{\arcsin \sqrt{1 - |\alpha + \delta|^2}/4}$. Therefore when $U_{\tau \gamma}$ and the rotation angles $\theta$ and $\phi$ are fixed, then when $k$ tends to infinite, $|b_k|$, like $|\sin|$, changes between 0 and 1 with the period $T$.

For the identical rotation angles $\theta = \phi$, the period $T = \frac{\pi}{\sin \Delta} = \frac{\pi}{\arcsin(2p \cos \phi \sqrt{1 - p^2 \cos^2 \phi})}$. For example, let $p = 0.1$ and $\phi = \pi/4$. Then $T = 22.2$. In the case the curve of $|b_k|$ looks like the one in Fig. 3 in [5]. Let $\phi = 9\pi/20$. Then the curve of $|b_k|$ looks like the one in Fig. 4 in [5].

For Grover’s algorithm the period $T = \frac{\pi}{\sin \Delta} = \frac{\pi}{\arcsin(2p \sqrt{1 - p^2})}$. Let $p = 0.1$ and 0.01. Then $T = 15.7$ and 157, respectively.

7.3 The monotone increasing property in the interval $[0, k_0]$, where $k_0$ is the optimal number of iteration steps, of the norm of the amplitude $|b_k|$ as a function of $k$

When the phase condition $\sin \Delta \leq |\beta|$ is satisfied, the optimal number of iteration steps to search the desired state with certainty $k_0 = \frac{\pi}{\sin \Delta}$ is

$0 < k_0 \Delta = \arcsin \frac{|\sin \Delta|}{|\beta|} \leq \pi/2$. Therefore when $0 < k \leq k_0, 0 < k \Delta \leq k_0 \Delta \leq \pi/2, |\sin k \Delta| =
\[
sin k\Delta, |b_k| = |\beta| \frac{\sin k\Delta}{\sin \Delta}. \text{ Therefore } |b_k| \text{ as a function of } k \text{ is strictly monotone increasing as } k \text{ increases from 0 to } k_0. \text{ This result is first reported in this paper.}
\]

8 In the case identical rotation angles \( \theta = \phi \) the precise formula in a sine function form of the norm of the amplitude and the precise optimal number of iteration steps

In [5] using many transformations Long et al. derived the approximate formulas of the amplitude in the desired state and the optimal number of iteration steps to find the desired state for identical rotation angles.

8.1 The norm of the amplitude in the desired state

When \( \theta = \phi \) the precise formula of the norm of the amplitude \( |b_k| \) in the desired state can be reduced. In the case let \( b_{kl} \) be the amplitude in the desired state. Then we have

Version 1. \( |b_{kl}| = |\beta| \frac{\sin k\Delta}{\sin \Delta} \), where \( \Delta = \frac{\psi_1 - \psi_2}{2} \) and \( \psi_1 - \psi_2 = \arccos\{2(1 - 2p^2 \cos^2 \phi)^2 - 1\} \).

Remark 1. Clearly \( \Delta \) is small since \( p \) is very small. Therefore \( \frac{\sin k\Delta}{\sin \Delta} < k \) and \( |b_{kl}| < k|\beta| = 2kp|\cos \phi| \).

Version 2. \( |b_{kl}| = \left| \frac{\sin k \arcsin(2p|\cos \phi|\sqrt{1 - p^2 \cos^2 \phi})}{\sqrt{1 - p^2 \cos^2 \phi}} \right| \) since \( \sin \Delta = 2p|\cos \phi| \sqrt{1 - p^2 \cos^2 \phi} \).

Remark 2. Let \( T \) be the period of \( |b_{kl}| \) as a \( |\sin| \) function of \( k \). Then \( T = \frac{\pi}{\arcsin(2p|\cos \phi|\sqrt{1 - p^2 \cos^2 \phi})} \).

8.2 The optimal number of iteration steps to find the desired state

Let \( k_{ol} \) be the optimal number of iteration steps to search the desired state with certainty. Then \( k_{ol} = \arcsin \sqrt{1 - p^2 \cos^2 \phi} / \arcsin(2p|\cos \phi|\sqrt{1 - p^2 \cos^2 \phi}) \).

9 For Grover’s Algorithm the amplitude in the desired state is exactly written as \( \beta \frac{\sin k\xi}{\sin \xi} \) and the optimal number of iteration steps is exactly \( \arcsin \sqrt{1 - p^2} / \arcsin 2p\sqrt{1 - p^2} \)

9.1 The precise formula in a sine function form of the amplitude in the desired state

For Grover’s algorithm \( \alpha = 1 - 4|U_{r\gamma}|^2, \beta = 2U_{r\gamma}, \lambda = -2U_{r\gamma}^*, \delta = 1 \). Therefore for Grover’s algorithm the equation \( z^2 - (\alpha + \delta)z + (\alpha\delta - \beta\lambda) = 0 \) becomes the equation \( z^2 - (\alpha + 1)z + 1 = 0 \)
which has two conjugate roots since the coefficients are real. Since the product of the two roots is 1 the norms of the two roots are 1. Let \( z_1 = e^{i\xi} \) and \( z_2 = e^{-i\xi} \). From \( z_1 + z_2 = e^{i\xi} + e^{-i\xi} = 2 \cos \xi = \alpha + 1 \), obtain \( \cos \xi = (\alpha + 1)/2 = 1 - 2|U_{\tau_G}|^2 = 1 - 2p^2 \). Then \( z_1^k - z_2^k = e^{ik\xi} - e^{-ik\xi} = 2i \sin k\xi \), \( r_k = \frac{z_1^k - z_2^k}{z_1 - z_2} = \frac{\sin k\xi}{\sin \xi} \), which can also be obtained from \( r_k = \frac{\sin k\psi_1 - \sin k\psi_2}{\sin \psi_1 - \sin \psi_2} \) in the section 4 of this paper by letting \( \psi_1 = \xi \) and \( \psi_2 = -\xi \).

Let \( b_{kg} \) be the amplitude in the desired state for Grover’s algorithm. Then we have

Version 1. \( b_{kg} = \beta \frac{\sin k\xi}{\sin \xi} \), \( |b_{kg}| = |\beta| \frac{|\sin k\xi|}{|\sin \xi|} = 2p |\sin k\xi| \), where \( \cos \xi = 1 - 2p^2 \).

Version 2. \( |b_{kg}| = \frac{|\sin k \arccos(1-2p^2)|}{\sqrt{1-p^2}} \).

Version 3. \( |b_{kg}| = \frac{|\sin k \arcsin 2p \sqrt{1-p^2}|}{\sqrt{1-p^2}} \).

Remark 1. Fix \( U_{\tau_G} \), \( b_{kg} \) is a sin function of \( k \).

Remark 2. Let \( T \) be the period of \( |b_{kg}| \) as a \( |\sin| \) function of \( k \). Then \( T = \pi/ \arcsin(2p \sqrt{1-p^2}) \).

Remark 3. \( |b_{kg}| = |\beta \frac{\sin k\xi}{\sin \xi}| < 2kp \) since \( \frac{\sin k\xi}{\sin \xi} < k \) when \( \xi \) is small.

9.2 The precise formula of the optimal number of iteration steps and the derivation of Grover’s approximate formula \( \frac{\pi}{4p} \)

Let \( k_{og} \) be the optimal number of iteration steps to search the desired state with certainty for Grover’s algorithm. Then \( k_{og} = \arcsin \sqrt{1-p^2}/ \arcsin 2p \sqrt{1-p^2} \) using the version 3 of \( |b_{kg}| \) and letting \( |b_{kg}| = 1 \).

In \( \text{[4]} \) Grover only gave an approximate optimal number of applications of Grover’s algorithm to find the desired state. Let us show how the approximate formula \( \frac{\pi}{4p} \) is derived from our precise formula.

Clearly \( k_{og} \) can not generate the Taylor’s polynomial at the point \( p = 0 \) since \( \arcsin 2p \sqrt{1-p^2} = 0 \) when \( p = 0 \). However \( 4p/\pi \) is the Taylor’s polynomial of degree 1 generated by \( 1/k_{og} \) at the point \( p = 0 \). Therefore \( k_{og} \approx \frac{\pi}{4p} \).

We can also give another explanation for \( \frac{\pi}{4p} \). Clearly \( \pi/2 \) is the Taylor’s polynomial of degree 0 generated by \( \arcsin \sqrt{1-p^2} \) at \( p = 0 \). By using the Taylor’s polynomial of degree 1 \( \arcsin 2p \sqrt{1-p^2} = 2p \sqrt{1-p^2} = 2p \). Therefore \( k_{og} \approx \frac{\pi}{4p} \).

Remark 4. \( k_{og} \to \infty \) when \( p \to 0 \).
The optimal number of iteration steps for Grover’s algorithm is less than the one for the algorithms with arbitrary identical rotation angles $\theta = \phi$ when $|U_{\tau\gamma}|$ is fixed

In the paper [3] in the first-order approximate formula of the amplitude it showed that Grover’s algorithm is optimal among algorithms with arbitrary phase rotations. Now we will strictly show that Grover’s algorithm has the less number of iteration steps than the algorithms with $\theta = \phi$ when $|U_{\tau\gamma}|$ is fixed.

$k_{og}$ and $k_{ol}$ are before defined as the optimal number of iteration steps to search the desired state with certainty for Grover’s algorithm and for the algorithm with $\theta = \phi$, respectively. $k_{og} = \arcsin \left( \frac{\sqrt{1 - p^2}}{\arcsin(2p \sqrt{1 - p^2})} \right)$ and $k_{ol} = \arcsin \left( \frac{\sqrt{1 - p^2 \cos^2 \phi}}{\arcsin(2p \cos \phi \sqrt{1 - p^2 \cos^2 \phi})} \right)$. When $\phi = 0$ or $\pi$ clearly $k_{ol}$ is reduced to $k_{og}$.

Let show $k_{og} < k_{ol}$ when $0 < \phi < \pi/2$ or $\pi/2 < \phi < \pi$. When $p$ is fixed $k_{ol}$ is a function of $\phi$. And $k_{ol}$ is symmetric about $\phi = \pi/2$. Let $(k_{ol})'_\phi$ be the derivative of $k_{ol}$. After computing the derivative we obtain that $(k_{ol})'_\phi > 0$ when $0 < \phi < \pi/2$ therefore $k_{ol}$ is strictly monotone increasing as $\phi$ increases. And $k_{ol}$ is strictly monotone decreasing as $\phi$ increases when $\pi/2 < \phi < \pi$ since $(k_{ol})'_\phi < 0$. Therefore always $k_{og} < k_{ol}$ when $0 < \phi < \pi/2$ or $\pi/2 < \phi < \pi$. $k_{og} = k_{ol}$ only when $\phi = 0$ or $\pi$.

For example $\phi = \pi/3$. Then $k_{ol} = \arcsin \left( \frac{\sqrt{1 - p^2/4}}{\arcsin(p \sqrt{1 - p^2/4})} \right)$. It is not hard to see $k_{ol} > k_{og}$ since $\arcsin \left( \frac{\sqrt{1 - p^2/4}}{\arcsin(p \sqrt{1 - p^2/4})} \right)$ and $\arcsin(p \sqrt{1 - p^2/4}) < \arcsin(2p \sqrt{1 - p^2})$.

Let $|U_{\tau\gamma}|$ be 0.1. We obtained the following the table 1 using MATLAB. From the table 1

The table 1

| $\theta = \phi$ | $0$ | $\pi/6$ | $\pi/3$ | $2\pi/3$ | $5\pi/6$ | $\pi$ |
|-----------------|-----|--------|--------|---------|--------|------|
| $k_{ol}$        | 7   | 8      | 15     | 15      | 8      | 7    |

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Appendix 1

The proof by induction of the precise formula in a polynomial form in $\beta\lambda$ of the amplitude for arbitrary phase rotations

Proof. In [3] the iterated formulas of $a_k$ and $b_k$ were given as follows. $a_{k+1} = (a a_k + \lambda b_k)$ and $b_{k+1} = (\beta a_k + \delta b_k)$.

From the iterated formula of the amplitude by induction hypothesis

$$b_{k+1} = \beta a_k + \delta b_k = \beta (a^k + d_{k1}(\beta \lambda) + d_{k2}(\beta \lambda)^2 + ... + d_{k[k/2]}(\beta \lambda)^{[k/2]})$$

$$+ \delta (c_{k0} + c_{k1}(\beta \lambda) + c_{k2}(\beta \lambda)^2 + ... + c_{k[(k-1)/2]}(\beta \lambda)^{[(k-1)/2]})$$

$$= \beta ((a^k + \delta c_{k0}) + (d_{k1} + \delta c_{k1})(\beta \lambda) + (d_{k2} + \delta c_{k2})(\beta \lambda)^2 + ...).$$
Since \( l^{(0)}_{ki} = 1 \) clearly \( c_{k0} = \sum_{n=0}^{\infty} l^{(0)}_{k(k-1-n)} \alpha^n \delta^{k-1-n} = \alpha^{k-1} + \alpha^{k-2} \delta + \alpha^{k-3} \delta^2 + ... + \delta^{k-1} \), and \( \alpha^k + \delta c_{k0} = \alpha^k + \alpha^{k-1} \delta + \alpha^{k-2} \delta^2 + ... + \delta^k = c_{(k+1)0} \).

Next we will show \( d_{kj} + \delta c_{kj} = c_{(k+1)j} \). Note that \( l^{(j)}_{kj} = \begin{pmatrix} k-j+i \cr j \end{pmatrix} \begin{pmatrix} j+i-1 \cr j-1 \end{pmatrix} + \begin{pmatrix} j+i-1 \cr j \end{pmatrix} \begin{pmatrix} k-j-i \cr j \end{pmatrix} = l^{(j)}_{k(k-1)} + l^{(j)}_{k(k-2)j} \alpha^{k-2j-1} \delta + l^{(j)}_{k(k-2)j} \alpha^{k-2j} \delta^2 + ... + l^{(j)}_{k(k-2)j} \alpha^{k-2j-1} \delta^2 + ...
\)
\( = l^{(j)}_{k0} \alpha^{k-2j} + l^{(j)}_{k(k-2)j} \alpha^{k-2j} \delta + l^{(j)}_{k(k-2)j} \alpha^{k-2j} \delta^2 + ... + l^{(j)}_{k(k-2)j} \alpha^{k-2j} \delta^2 + ...
\)
\( = l^{(j)}_{k(k-1)0} \alpha^{k-2j} + l^{(j)}_{k(k-2)j} \alpha^{k-2j} \delta + l^{(j)}_{k(k-2)j} \alpha^{k-2j} \delta^2 + ...
\)
\( + l^{(j)}_{(k+1)(k-2)j} \alpha^{k-2j} \delta + l^{(j)}_{(k+1)(k-2)j} \alpha^{k-2j} \delta^2 + ...
\)
\( = c_{(k+1)j} \).

When \( k = 2m + 1 \), \( a_k = \alpha^k + d_{k1}(\beta \lambda) + d_{k2}(\beta \lambda)^2 + ... + d_{km}(\beta \lambda)^m \),
\( b_k = \beta(c_{k0} + c_{k1}(\beta \lambda) + c_{k2}(\beta \lambda)^2 + ... + c_{km}(\beta \lambda)^m) \),
note that \( d_{kj} + \delta c_{kj} = c_{(k+1)j} \), where \( 1 \leq j \leq m \), we finished the proof of the case.

When \( k = 2m \), \( a_k = \alpha^k + d_{k1}(\beta \lambda) + d_{k2}(\beta \lambda)^2 + ... + d_{k(m-1)(\beta \lambda)}^{m-1} + d_{km}(\beta \lambda)^m \),
\( b_k = \beta(c_{k0} + c_{k1}(\beta \lambda) + c_{k2}(\beta \lambda)^2 + ... + c_{k(m-1)(\beta \lambda)^{m-1}} \),
Note that \( d_{kj} + \delta c_{kj} = c_{(k+1)j} \), where \( 1 \leq j \leq m-1 \), and note that \( d_{km} = c_{(k+1)m} = 1 \), we also finished the proof of the case.

As well we can by induction derive the precise formula \( a_k \) of the amplitude in the initial state \( |\gamma\rangle \) after \( k \) applications of \( Q \).

Therefore the proof is complete.

Appendix 2.

From the iterated formula that \( r_{k+1} = (\alpha + \delta)r_k + (\beta \lambda - \alpha \delta)r_{k-1} \) we can derive its precise formula in the sine function form of \( b_k \). Let \( r_{k+1} = (z_1 + z_2) \alpha r_k - z_1 z_2 r_{k-1} \), where \( z_1 + z_2 = \alpha + \delta \), \( z_1 z_2 = -(\beta \lambda - \alpha \delta) \). Note that \( r_1 = 1 \) and \( r_2 = \alpha + \delta = z_1 + z_2 \). Then \( r_{k+1} = z_1 r_k + z_2 r_k - z_1 z_2 r_{k-1} \), thus we obtain
\( r_{k+1} - z_1 r_k = z_2 r_k - z_1 z_2 r_{k-1} = z_2(r_k - z_1 r_{k-1}) \) \( (1) \)
\( r_{k+1} - z_2 r_k = z_1 r_k - z_1 z_2 r_{k-1} = z_1(r_k - z_2 r_{k-1}) \). \( (2) \)

From \( (2) \) \( r_{k+1} - z_1 r_k = z_2(r_k - z_1 r_{k-1}) = z_2^2(r_{k-1} - z_1 r_{k-2}) = ... = z_2^{k-1}(r_2 - z_1 r_1) \) obtain
\( r_{k+1} - z_1 r_k = z_2^{k-1}(r_2 - z_1 r_1) \) \( (4) \).
From (3) \( r_{k+1} - z_2 r_k = z_1 (r_k - z_2 r_{k-1}) = z_2^2 (r_{k-1} - z_2 r_{k-2}) = \ldots = z_1^{k-1} (r_2 - z_2 r_1), \)

so that \( r_{k+1} = z_1^{k-1} (r_2 - z_2 r_1) \), (5)

From \( z_1 \times (4) - z_2 \times (5) \) obtain that \( z_1 r_{k+1} - z_2 r_{k+1} = z_1^k (r_2 - z_2 r_1) - z_2^k (r_2 - z_1 r_1) = z_1^k (z_1 + z_2 - z_2) - z_2^k (z_1 + z_2 - z_2) = z_1^{k+1} - z_2^{k+1} \). Note that \( r_2 = z_1 + z_2 \) and \( r_1 = 1 \).

From the result 6 in the Appendix 3 we know that \( z_1 \neq z_2 \) provided that \( \cos \phi \neq 0 \), then

\[
\rho_{k+1} = \frac{z_1^{k+1} - z_2^{k+1}}{z_1 - z_2}.
\]

Appendix 3.

Several results

\[
\alpha = e^{i2\theta} - (e^{i2\phi} + 1)(e^{i2\phi} + 1)|U_{\gamma}|^2, \beta = (e^{i2\phi} + 1)|U_{\gamma}|^2, \lambda = -e^{i2\phi}(e^{i2\theta} + 1)|U_{\gamma}|, \delta = e^{i2\phi}.
\]

Result 1. \( \alpha \beta - \lambda \delta = e^{i2\phi} \).

Proof. \( \beta \lambda - \alpha \delta = -e^{i2\phi}(e^{i2\theta} + 1)(e^{i2\phi} + 1)|U_{\gamma}|^2 \).

Let \( \rho = |U_{\gamma}| \) in this paper.

Result 2. \( \alpha + \beta = 2(\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi)e^{i(\theta + \phi)} = 2((1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi) e^{i(\theta + \phi)}. \)

Proof. \( \alpha + \beta = e^{i2\theta} + e^{i2\phi} - 4 \cos \theta \cos \phi e^{i(\theta + \phi)} p^2 = 2(\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi e^{i(\theta + \phi)}) + 2(\cos(\theta + \phi) - 2p^2 \cos \theta \cos \phi e^{i(\theta + \phi)}) + 2(\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi e^{i(\theta + \phi)}) + 2(\cos(\theta + \phi) - 2p^2 \cos \theta \cos \phi e^{i(\theta + \phi)}) = 2((1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi) e^{i(\theta + \phi)} \).

Result 3. \( |\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi| \leq 1 \) and the equality holds if and only if \( \cos \theta = \cos \phi = 0 \).

Proof. Note that \( \cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi = (1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi \), and \( -1 < 1 - 2p^2 < 1 \) whenever \( 0 < p < 1 \). When \( \cos \theta \cos \phi = 0 \) clearly \( |(1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi| \leq 1 \). Let us prove that \( |(1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi| < 1 \) whenever \( \cos \theta \cos \phi \neq 0 \). There are two cases.

Case 1: \( \cos \theta \cos \phi > 0 \).

Case 1.1: \( 0 < p < \sqrt{2}/2 \). Then \( 0 < 1 - 2p^2 < 1 \). Then \( 0 < (1 - 2p^2) \cos \theta \cos \phi < \cos \theta \cos \phi \), so \( \sin \theta \sin \phi < (1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi < \cos \theta \cos \phi + \sin \theta \sin \phi = \cos(\theta - \phi) \).

Therefore \( |(1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi| < 1 \).

Case 1.2: \( \sqrt{2}/2 \leq p < 1 \). Then \( -1 < 1 - 2p^2 \leq 0 \), \( -\cos(\theta + \phi) = -\cos \theta \cos \phi + \sin \theta \sin \phi < (1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi \leq \sin \theta \sin \phi \). As well \( |(1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi| < 1 \).
Case 2: \( \cos \theta \cos \phi < 0 \).

Case 2.1: \( 0 < p < \sqrt{2}/2 \). Then \( 0 < 1 - 2p^2 < 1 \). \( \cos \theta \cos \phi < (1 - 2p^2) \cos \theta \cos \phi < 0, \cos(\theta - \phi) < (1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi < \sin \theta \sin \phi \). As well \(|(1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi| < 1 \).

Case 2.2: \( \sqrt{2}/2 \leq p < 1 \). Then \(-1 < 1 - 2p^2 \leq 0 \). \( 0 \leq (1 - 2p^2) \cos \theta \cos \phi < -\cos \theta \cos \phi, \sin \theta \sin \phi \leq (1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi < -\cos(\theta + \phi) \).

As well \(|(1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi| < 1 \).

From the cases 1 and 2 when \( \cos \theta \cos \phi \neq 0 \) clearly \(|(1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi| < 1 \).

Now let us to prove the second part.

If \(|(1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi| = 1 \) then from the above clearly \( \cos \theta \cos \phi = 0 \). Then \(|(1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi| = |\sin \theta \sin \phi| = 1 \). Then \(|\sin \theta| = |\sin \phi| = 1 \), that is, \( \cos \theta = \cos \phi = 0 \).

Conversely if \( \cos \theta = \cos \phi = 0 \) then \(|\sin \theta| = |\sin \phi| = 1 \). Then \(|(1 - 2p^2) \cos \theta \cos \phi + \sin \theta \sin \phi| = 1 \).

We finished the proof.

Result 4. \(|\alpha + \beta| = 2\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi | \leq 2 \).

It is trivial from the results 2 and 3.

Result 5. Assume that \( p > 0 \). Then \( p + \frac{1}{p} \geq 2 \), and the equality holds if and only if \( p = 1 \).

Result 6. Given that \( z_1 + z_2 = \alpha + \beta, z_1 z_2 = \alpha \delta - \beta \lambda \). Then \( z_1 \neq z_2 \) if \( \beta \neq 0 \) (that is, \( \cos \phi \neq 0 \)).

Proof. Assume that \( z_1 = z_2 \). Let \( z_1 = z_2 = \rho e^{i\psi} \). From \(|z_1 z_2| = 1 \) we obtain \( \rho = 1 \). From \( 2 e^{i\psi} = \alpha + \beta \), we obtain that \(|\alpha + \beta| = 2 \), that is, \( |\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi| = 1 \). From the result 3 in the Appendix 3 it means that \( \cos \theta = \cos \phi = 0 \). It contradicts that \( \cos \phi \neq 0 \). Therefore in the case \( z_1 \neq z_2 \).

Appendix 4.

From the general phase condition we can derive the following corollaries. Let \( p = |U_{\tau_1}| \).

Corollary 1.

Let \( \theta \) and \( \phi \) be in the same quadrant or \(|\theta - \phi| < \pi/2 \) and \( \cos \theta \cos \phi < 0 \). Then \( \sin \Delta \leq |\beta| \), that is, the algorithm can search the desired state with certainty, if and only if \(|\theta - \phi| \leq \arccos(2p^2 \cos \theta \cos \phi + \sqrt{1 - 4p^2 \cos^2 \phi}) \).

Proof. If \( \theta \) and \( \phi \) be in the same quadrant then \( \cos \theta \cos \phi > 0 \) and \( \sin \theta \sin \phi > 0 \), then \( \cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi = (1 - 2a^2) \cos \theta \cos \phi + \sin \theta \sin \phi > 0 \). If \( |\theta - \phi| < \pi/2 \) and \( \cos \theta \cos \phi < 0 \) then \( \cos(\theta - \phi) > 0 \) and as well \( \cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi > 0 \).

\( (\Rightarrow) \). If \( \sin \Delta \leq |\beta| \) then \( \sqrt{1 - (\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi)^2} \leq |\beta| \) and \( 1 - 4p^2 \cos^2 \phi \leq (\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi)^2 \). In the conditions given \( \sqrt{1 - 4p^2 \cos^2 \phi} \leq |\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi| = \cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi \), and so \( \cos(\theta - \phi) \geq \sqrt{1 - 4p^2 \cos^2 \phi} + 2p^2 \cos \theta \cos \phi \). Therefore \(|\theta - \phi| \leq \arccos(2p^2 \cos \theta \cos \phi + \sqrt{1 - 4p^2 \cos^2 \phi}) \).

\( (\Leftarrow) \). Clearly \( \cos(\theta - \phi) \geq \sqrt{1 - 4p^2 \cos^2 \phi} + 2p^2 \cos \theta \cos \phi \) and \( \cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi \geq \sqrt{1 - 4p^2 \cos^2 \phi} \). Given that \( \cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi > 0 \), obtain \( (\cos(\theta - \phi) - 2p^2 \cos \theta \cos \phi)^2 \geq \)
1−4p²cos²φ, and 4p²cos²φ ≥ 1−(cos(θ−φ)−2p²cosθcosφ)² ≥ 0 by the result 3 in the appendix.
Therefore sin Δ ≤ |β|. We finished the proof.

In [3] we had the result as follows. If |θ−φ| < |β| then the algorithm Q maybe search the desired state with certainty. This is the first-order approximate phase condition. Next we will continue studying what will happen when |θ−φ| > |β|.

Corollary 2.
Let θ and φ be in the same quadrant or cos θ cos φ < 0 and sin θ sin φ < 0. Then sin Δ > |β|, that is, the algorithm can not search the desired state with certainty, if |sin(θ−φ)| > |β|.

Proof. Note when θ and φ are in the same quadrant, cos θ cos φ > 0 and sin θ sin φ > 0.

1−(cos(θ−φ)−2p²cosθcosφ)²
= sin²(θ−φ)+4p²cos(θ−φ)cosθcosφ−4p⁴cos²θcos²φ
= sin²(θ−φ)+4p²cosθcosφ(1−p²)cosθcosφ+sinθsinφ > sin²(θ−φ) in the conditions given by the corollary. Therefore if |sin(θ−φ)| > |β| then sin Δ > |β|. We finished the proof.

References
[1] P. Shor, in Proc. of the 35th Annual Symposium on Foundation of Computer Science, 1994, IEEE Computer Society Press. Los Alamos, CA, 1994, pp124-134.

[2] L.K. Grover, Phys. Rev. Lett. 79, 325-328(1997).

[3] L.K. Grover, Phys. Rev. Lett. 79, 4709-4712(1997).

[4] L.K. Grover, Phys. Rev. Lett. 80, 4329-4332(1998).

[5] G.L. Long et al., Phys. Lett. A.262, 27-34(1999).

[6] Dafa Li and Xinxin Li, Phys. Lett. A.287/5-6, pp304-316(2001). Also in e-print quant-ph/0105033.

[7] P. Hoyer, Phy. Rev. A. 62,052304(2000).

[8] G.L. Long et al., Phys. Rev. A, Vol. 61, 042305, 2000.

[9] G.L. Long, Phys. Rev. A, Vol. 64, 022307, 2001.

[10] Eli Bilham et. al, 63, 012310, 2000.