Computing Forgotten Topological Index of Extremal Cactus Chains

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Abstract

The $F$-index is the whole of 3D squares of vertex degrees in a chart $G$. It as of late turned into the subject of a few investigates because of its extraordinary capability of uses. The point of this paper is to register the $F$-record of triangular prickly plant chain, square desert flora chains, 6-sided cactus restraints and polyomino restraints. In addition, we decided the extremal chains in the desert plant chains and polyomino fastens as for the $F$-index.

Keywords: Forgotten index, Zagreb index, cactus chain, polyomino chain
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1 Introduction

The topological indices expect a fundamental activity in synthetic graph hypothesis, especially in QSAR and QSPR assessments. Graph speculation has given a variety of significant mechanical assemblies to the theoretical physicists, for instance, the topological indices, topological systems and counting polynomials. The information on the substance constitution of iota is addressed by a sub-nuclear chart. The vertices and edges of sub-nuclear charts are contrasting with the atoms of the blends and substance bonds, separately. A portrayal of the structure or a condition of the particles is extraordinarily valuable in complex preliminaries. Topological lists and physico-invention possessions are used for the illustrating, pharmacologic, toxicological, characteristic
and various possessions of engineered blends. A segment of the topological index that are most investigated in compound charts theory are Wiener index.

The topological index is addressed as a lone number that is partners the particular properties of an outline. The degree based topological index paly a fundamental occupation in substance graph speculation and particularly in theoretical science. The first and most analyzed record is the Wiener list that relies upon partition. Its old name is way record anyway later on it was renamed as Wiener file [32]. Let $G$ be a basic and associated diagram with request $n$ and size $m$. The level of a vertex is quantity of edges that are episode to it and meant by $\Theta(u)$. Zagreb files are two most seasoned most contemplated gradation built topological lists [33]. The first and second Zagreb lists are characterized as:

$$M_1(G) = \sum_{v \in V} (\Theta(v))^2 = \sum_{uv \in E} (\Theta(u) + \Theta(v)),$$

$$M_2(G) = \sum_{uv \in E} [\Theta(u) \times \Theta(v)].$$

These two Zagreb records have been used and broke down to examination of sub-nuclear multifaceted nature, ZE- isomerism, chirality, and hetero-organizations. Progressively approximately their physico-manufactured solicitations and logical possessions canister be establish in [15, 16], independently.

Gutman et al. [14] decided the harsh formulae for the total $\pi$-electron vitality. One of them is the central Zagreb file and $F$− list was experienced on which this imperativeness depends, yet that was completely ignored. This rundown is used just in scarcely any business identified with the essential general Zagreb list and zeroth-request general Randić list. Furtula et al. [13] renamed this topological record as a disregarded topological list and they gained some fascinating results. The $F$-list is portrayed as:

$$F(G) = \sum_{v \in V} (\Theta(v))^3.$$  

The $M_1^\alpha(G)$ is characterized as :

$$M_1^\alpha(G) = \sum_{v \in V} \Theta(v)^\alpha = \sum_{uv \in E} [(\Theta(u))^{(\alpha-1)} + (\Theta(v))^{(\alpha-1)}].$$

Where $\alpha \neq 0, \alpha \in \mathbb{R}, \alpha \neq 1$. Li et al. [22, 25] demarcated the Special wide-ranging Randić index as:

$$R_0^\alpha(G) = \sum_{u \in V} (\Theta(u))^\alpha,$$

for all real $\alpha$.

In the event that $\alpha = 3$, at that point zeroth-request general Randić file is equivalent to the $F$-list. A few peak possessions of $F$-record for sub-atomic trees and approximately poorer and superior limits for the $F$-list were prearranged in [1]. Hosamani figured some upper limits for the $F$-list and its a few applications [19]. Nilanjan et al. [12] processed the specific recipes for the chart activities regarding the $F$-file. Yarahmaadi et al. [30, 31] processed the unconventional availability record of the chain hexagonal cacti.

Li et al. [23] decided the Hosoya polynomials of chains. A few properties of the prickly plant chains are given in [20, 24]. Xu et al. [27] processed the PI file for polyomino chains. Randić record and whole network file of polyomino chains are now concentrated in [28, 29]. The general Randić record of extremal polyomino chains decided in [3–10].
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2 Results on Cactus Chains

In this segment, we figure the specific equations for the desert flora binds regarding the $F$-record and furthermore decided the extremal fastens as for the $F$-list. A prickly plant diagram is a straightforward associated chart in which nope edge fabrics in supplementary than individual cycle. Each square of a desert flora chart is whichever a cycle or an edge. On the off chance that all squares of a desert plant are patterns of a similar size $m$, at that point these sort of prickly plant charts are supposed to be $m$-uniform. The quantity of cycles in a $m$-uniform desert flora chain is baptized measurement of manacle. Besides, any $m$-uniform desert plant manacle of measurement more prominent than solitary has precisely two fatal cycles. Every residual cycle in chain are called inner cycles.

More than 50 years prior, the desert flora charts were known as Husimi trees [18, 26].

2.1 Triangular Cactus Chains

A triangular desert flora is a diagram together tetragons are patterns of request 3 i.e., 3-unvarying prickly plant chains. In the triangular desert plant $T_n$, if each pattern of request 3 has all things considered twofold expurgated vertices and apiece expurgated vertex is pooled by accurately twofold patterns of request 3, at that point $T_n$ is known as a chain three-cornered prickly plant. The quantity of pinnacles and quantity of authorities of fetter wedge-shaped prickly plant of extent $n$ are $2n + 1$ and $3n$, separately. Clearly, all equivalent extent chain triangular prickly plants are isomorphic. A diagram $T_5$ is appeared in Figure 1. Next, we process the $F$-record of triangular desert flora chain.

Theorem 1. For $n \geq 2$, the $F$-index is $F(T_n) = 72n - 48$.

The windmill diagram signified by $W_n$ is the wedge-shaped desert plant with $n$ trios that all the trios portion a vertex. The quantity of pinnacles and superiorities of windmill diagram $W_n$ are $2n + 1$ and $3n$ individually. A case of the windmill diagram is appeared in Figure 1. In the accompanying hypothesis, we process the $F$-record of windmill chart $W_n$.

Theorem 2. For $n \geq 1$, the $F$-index is $F(W_n) = 8n^3 + 16n$.

2.2 Square Cactus Chains

A Square Cactus Chains is a diagram with tetragons are $C_4$ i.e., 4-undeviating. The quantity of vertices and the quantity of edges of the square prickly plant are $3n + 1$ and $4n$, separately. On the off chance that the twofold vertices are nearby in $C_4$, at that point such tetragonal is an ortho- tetragons and is meant by $S_n$. In the event that twofold pinnacles are not immediate in $C_4$, at that point such tetragons are christened para-tetragons and is indicated by $Q_n$ (These phrasings engaged after the hypothesis of benzenoid hydrocarbons [11]). The ortho and para restraints of tetragons are appeared in Figure 2.

Theorem 3. For $n \geq 2$, the $F$-index is $F(S_n) = F(Q_n) = 80n - 48$.

2.3 Hexagonal Cactus Chains

A Hexagonal Cactus cawsers is a 6-undeviating prickly plant. The quantity of vertices and the edges in a 6-sided desert plant cawsers are $5n + 1$ and $6n$, separately. On the off chance that each hexagonal prickly plant has
all things considered twofold expurgated vertices and apiece expurgated vertex is pooled by precisely twofold hexagons, at point that desert flora is called hexagonal desert flora hawsers. Twofold vertices are nearby in $C_6$ at that juncture they are on ortho-station, on the off chance that two vertices have separation two, at that point they are on meta-position, and in the event that the separation flanked by twofold vertices is three, at that point that are on para-location. An inner 6-sided in a chain 6-sided prickly plant is baptized ortho-6-sdied, meta-6-sdied, or para-6-sdied if its bowdlerized -vertices are in ortho, meta, and para- location, separately. A normal hawser partakes every single inside hexagon of a hexagonal desert plant chains are of a similar kind.

**Theorem 4.** For $n \geq 3$, the $F$-index of hexagonal chains are $F(O_n) = F(M_n) = F(R_n) = 96n - 48$.

**Theorem 5.** For $n \geq 3$, the square (ortho, para) cactus chains has the minimum $F$-index than the hexagonal (ortho, meta, para) cactus chains.

### 3 Results on polyomino chains

A polyomino framework is a 2-associated chart thru the culmination goalmouth that every single inside aspect is encircled by $C_4$ of length one. The birthplace of polyomino framework goes to the work done by Klarner [21]. The historical backdrop of polyomino framework is ridiculous and extensive that is begun from 20-th century yet they stayed promoted in the current time at first by Golomb [17]. Presently, they are branded by arithmeticians, scientists, physicists and partake been utilized in numerous solicitations [2]. A polyomino fetter is a polyomino context, in which the linking of the focuses of its neighboring standard structures a way $c_1c_2 \cdots c_n$, where $c_i$ is the focal point of the $i$th tetragonal [31].

Let $B_n$ be the arrangement of polyomino handcuffs thru stretch $n$. A $C_4$ of a polyomino chain has it is possible that a couple of neighboring squares. On the off chance that a tetragonal makes them neighbor tetragonal, it is
A polyomino restraint entails of a disarray of fragments \( S_1, S_2, S_3, ..., S_r, r \geq 1 \) and \( l_1, l_2, l_3, ..., l_r \) be situated the disarray of extents of fragments, somewhere the entirety of these extents is identical to \( n + r - 1 \).

First to compute the \( F \)-index, we delineate a stricture \( \alpha(S_i), 1 < i < r \), as follows:

\[
\alpha(S_i) = \begin{cases} 
1 & \text{for } l(S_i) = 2, \\
0 & \text{for } l(S_i) > 2,
\end{cases}
\]

and \( \alpha(S_1) = \alpha(S_r) = 0 \).

**Theorem 6.** Let \( n \geq 2 \) be an integer, \( B_n \in \mathbb{B}_n \) be a polyomino restraint per \( n \) tetragons and entailing of \( r \) fragments signified by \( S_1, S_2, ..., S_r, \) \( r \geq 1 \) with lengths \( l_1, l_2, l_3, ..., l_r \). Then \( F(B_n) = 54n + 18r - 40 \).

**Proof.** The superiority customary of \( B_n \) is divided into twofold detachments: \( E_1 \) and \( E_2 \). The detachments \( E_1 \) comprehends, every such edge which are expurgated athwart by conservative run line going through the focuses of \( S_i \) for \( 1 \leq i \leq r \) (Figure 6(a)). The detachments \( E_2 \) comprises of such edges that are not constituent of \( E_1 \), i.e., \( E_2 = E(B_n) \setminus E_1 \), the components of \( E_2 \) are portrayed in Figure 6(b) as a conventional run streaks. At that point, we get
For 1 < i < r, we have:
\[ \sum_{uv \in E_i \cap E(S_i)} [(\Theta(u))^2 + (\Theta(v))^2] = \sum_{i=1}^{r} \sum_{uv \in E_i \cap E(S_i)} [(\Theta(u))^2 + (\Theta(v))^2]. \]

Therefore, we have
\[ \sum_{uv \in E_1} [(\Theta(u))^2 + (\Theta(v))^2] = \sum_{i=1}^{r} (l_i - 2) + 46 \sum_{i=2}^{r-1} \alpha(S_i) + 58r - 24. \]

Also we have:
\[ \sum_{uv \in E_2} [(\Theta(u))^2 + (\Theta(v))^2] = \sum_{i=1}^{r} \sum_{uv \in E_2 \cap E(S_i)} (\Theta(u))^2 + (\Theta(v))^2). \]

For 1 < i < r, we have
\[ \sum_{uv \in E_2 \cap E(S_i)} [(\Theta(u))^2 + (\Theta(v))^2] = \sum_{i=1}^{r} (2l_i - 4) - 46 \sum_{i=2}^{r-1} \alpha(S_i) + 14, \]
\[ \sum_{uv \in E_2 \cap E(S_1)} [(\Theta(u))^2 + (\Theta(v))^2] = 18(2l_1 - 4) + 33, \]
\[ \sum_{uv \in E_2 \cap E(S_r)} [(\Theta(u))^2 + (\Theta(v))^2] = 18(2l_r - 4) + 33. \]

Therefore, we get
\[ \sum_{uv \in E_2} [(\Theta(u))^2 + (\Theta(v))^2] = 18 \sum_{i=1}^{r} (2l_i - 4) - 46 \sum_{i=2}^{r-1} \alpha(S_i) + 14r + 38. \]

\[ F(B_n) = \sum_{uv \in E(B_n)} [(\Theta(u))^2 + (\Theta(v))^2] = \sum_{uv \in E_1} [(\Theta(u))^2 + (\Theta(v))^2] + \sum_{uv \in E_2} [(\Theta(u))^2 + (\Theta(v))^2] = 54n + 18r - 40. \]
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Theorem 7. For \( n \geq 2 \), then the F-index of linear and zig-zag chains are

- \( F(L_n) = 54n - 22 \).
- \( F(Z_n) = 72n - 58 \).

Theorem 8. For \( n \geq 2 \), then \( F(L_n) \leq F(B_n) \leq F(Z_n) \), with right(left) equality if and only if \( B_n \cong Z_n \) (\( L_n \cong B_n \)).

Proof. From the Theorems 6 and 7, it can be easily seen that \( B_n \) is thoroughgoing(minutest) if and only if \( r \) is thoroughgoing(minutest). Clearly, we have \( B_n \cong Z_n \) (\( L_n \cong B_n \)) if and only if \( r \) is thoroughgoing(minutest).

We presently figure and afterward look at the F-record of crisscross and easy chair nanoribbons with \( n \) hexagons. The charts of crisscross and easy chair nanoribbons are meant by \( Z'_n \) and \( A_n \), individually. The formulae of the F-list of these charts are the followings:

Theorem 9. For \( n \geq 2 \), we have

- \( F(Z'_n) = 70n - 22 \).
- \( F(A_n) = 70n + 58 \).

From Theorem 9, we get the minimum F-index of nanoribbons which is stated in the following theorem.

Theorem 10. For \( n \geq 2 \), we have \( F(Z'_n) < F(A_n) \).

4 Conclusion

In this paper, we partake figured the F-index of triangular cactus manacles, tetragonal cactus restraints, hexagonal cactus restraints and the polyomino restraints. We correspondingly strongminded the tiniest cactus restraints by means of reverence to the F-index. Nevertheless, we strongminded the extremal polyomino restraints per veneration to F-index.

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