BPS Invariants for 3-Manifolds at Rational Level $K$

Hee-Joong Chung

Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China

Abstract: We consider the Witten-Reshetikhin-Turaev invariants or Chern-Simons partition function at or around roots of unity $q = e^{2\pi i \frac{K}{s}}$ with rational level $K = \frac{r}{s}$ where $r$ and $s$ are coprime integers. From the exact expression for the $G = SU(2)$ Witten-Reshetikhin-Turaev invariants of Seifert manifolds at other roots of unity obtained by Lawrence and Rozansky, we provide an expected form of the structure of the Witten-Reshetikhin-Turaev invariants in terms of the homological blocks at other roots of unity. Also, we discuss the asymptotic expansion of knot invariants around roots of unity where we take a limit different from the standard limit in the volume conjecture.
1 Introduction

It is known that the Chern-Simons (CS) partition function agrees with the Reshetikhin-Turaev invariants, and they are often called the WRT (Witten-Reshetikhin-Turaev) invariants. Differently from the usual Chern-Simons theory where the level $K$ is taken to be an integer, $K$ can be a rational number in the Reshetikhin-Turaev construction. The Chern-Simons theory with integer level $K$ and its analytic continuation has been studied actively in (physics) literature, so it would be interesting to consider the case with $K$ being a rational number and its analytic continuation from rational $K$.

In this paper, we discuss the properties or behavior of the WRT invariant or CS partition function at or around roots of unity for closed 3-manifolds and knot complements in $S^3$.

In section 2, we consider the WRT invariant at other roots of unity in terms of homological blocks. It was conjectured that the WRT invariant for the standard root of unity $q = e^{2\pi i/3}$ where $K$ is an integer can be expressed so called the homological block, which is $q$-series invariants with integer powers and integer coefficients $[1, 2]$. The resurgence analysis on homological block has been discussed in $[3, 4]$. The number theory and CFT aspects of homological blocks have been considered in $[5]$. Generalization to $G = SU(N), N \geq 2$ for a certain class of Seifert manifolds with arbitrary number of singular fibers have been...
discussed in [6]. Also, homological blocks for knot complements has been studied in [7]. All these studies were about the standard root of unity or integer $K$.

From the integral formula of the $SU(2)$ WRT invariants for a certain class of Seifert manifolds at other roots of unity $q = e^{2\pi i \frac{1}{K}}$ with $K = \frac{r}{s}$ where $r$ and $s$ are coprime integers [8], we express them in terms of homological blocks. We discuss general structure of it and provide some examples.

In section 3, we consider the asymptotic expansion of knot invariants at general roots of unity. In the context of the volume conjecture, asymptotic expansion in the limit of the level $K \to \infty$ has been studied in many literature. This asymptotic expansion is the expansion around $q \to 1$ where the deviation from 1 is $e^{2\pi i \frac{1}{K}}$. The asymptotic expansion around general roots of unity that we consider is different from the standard limit in the context of volume conjecture. We consider the asymptotic expansion around given roots of unity $e^{2\pi i \frac{r}{s}}$ with rational level $K = \frac{r}{s}$ where the deviation from $e^{2\pi i \frac{r}{s}}$ is given by another expansion parameter. This was discussed in [9], also in [10–12] in a similar but a slightly different setup or in a different context. We calculate the leading order of the asymptotic expansion of superpolynomials for knots around the roots of unity.

In section 4, we summarize and discuss some future directions.

Note added: While preparing the manuscript, we found that [13] appeared, which overlaps with section 2 of this paper.

2 WRT invariants and homological blocks for $G = SU(2)$ at other roots of unity

In [8], in addition to the standard root of unity $q = e^{2\pi i \frac{1}{K}}$ where $K \in \mathbb{Z}$, Lawrence and Rozansky also considered the WRT invariant of a certain class of Seifert manifolds at other roots of unity by considering the Galois action. As in the case of the standard root of unity, we would like to express the WRT invariant at other roots of unity in terms of the homological blocks.

2.1 WRT invariant at other roots of unity in Chern-Simons matrix model

For Seifert manifolds $X(P_1/Q_1, \ldots, P_F/Q_F)$ with conditions that $P_j$’s and $Q_j$’s are coprime for each $j$ and $P_j$’s are pairwise coprime where $P_j, Q_j$’s are Seifert invariants, the finite sum expression for the WRT invariant in [8] is given by

$$Z_K(M_3) = B e^{\frac{\pi i}{2K}} \sum_{\beta = -P_K/K}^{PK} e^{-\frac{\pi i}{2K} \beta^2} \prod_{j=1}^{F} \frac{e^{\frac{\pi i \beta}{r_j}}} {\left( e^{\frac{\pi i \beta}{r_j}} - e^{-\frac{\pi i \beta}{r_j}} \right)^{F-2}}$$

(2.1)
where $H := P \sum_{i=1}^{3} \frac{Q_i}{P_i} = \pm |\text{Tor} H_1(M,\mathbb{Z})|$ and $P = \prod_{j=1}^{F} P_j$. Also,

$$B = -\frac{\text{sign } P}{4\sqrt{|P|}} e^{\frac{\pi i}{4} \text{sign } \left( \frac{4}{P} \right)},$$

$$\phi_F = 3 \text{sign } \left( \frac{H}{P} \right) + \sum_{j=1}^{F} \left( 12s(Q_j, P_j) - \frac{Q_j}{P_j} \right),$$

where $s(Q, P)$ is the Dedekind sum

$$s(Q, P) = \frac{1}{4P} \sum_{l=1}^{P-1} \cot \left( \frac{\pi l}{P} \right) \cot \left( \frac{\pi Q l}{P} \right)$$

for $P > 0$, which satisfies $s(-Q, P) = -s(Q, P)$. Here, we use the physics normalization

$$Z_K(S^1 \times S^2) = 1, \quad Z_K(S^3) = \sqrt{\frac{2}{K}} \sin \left( \frac{\pi}{K} \right).$$

With $x = e^{2\pi i \frac{1}{4KP}}$, (2.1) can be written as

$$Z_K(M_3) = \frac{B}{K} x^{P\phi_F} \sum_{\beta = -PK \atop K|\beta} \prod_{i=1}^{P} \frac{x^{2\beta i} - x^{-2\beta i}}{(x^{2\beta i} - e^{-2\beta i})^{F-2}}$$

Therefore, the expression (2.1) includes the part that is in $\mathbb{Q}[x]$. By considering the Galois action on $x$, $x$ is replaced with another primitive root of unity, $e^{2\pi i \frac{1}{4KP}}$, where $s$ is coprime to $4KP$. Due to this change, additional overall factor appears from $B$, but we will not consider it in this paper.

Up to such an overall factor, the integral expression for the WRT invariant at other roots of unity, $e^{2\pi i \frac{1}{4KP}}$ with $K = \frac{s}{8}$ for Seifert manifolds $X(P_1/Q_1,\ldots,P_F/Q_F)$ is given by

$$Z_K(S^1 \times S^2) \simeq \frac{B}{K} e^{-2\pi i \frac{1}{4KP}} \frac{\phi_3/4}{4\pi} \sum_{l=0}^{sH-1} \int_C dy f(y) e^{-2\pi i ly} - 2\pi i \sum_{m=0}^{2sP-1} \text{Res} \left( \frac{f(y)}{1 - e^{-2\pi i y}}, y = mK \right)$$

where

$$f(y) = e^{-2\pi i \frac{y}{4\pi P}} \prod_{j=1}^{F} \frac{e^{2\pi i \frac{y}{P_j}} - e^{-2\pi i \frac{y}{P_j}}}{e^{2\pi i \frac{y}{K}} - e^{-2\pi i \frac{y}{K}}}$$

and $C$ is a contour from $(-1+i)\infty$ to $(1-i)\infty$ for $\frac{P}{H} > 0$ and a clockwise rotation of it for $\frac{P}{H} < 0$. 

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2.2 WRT invariant at other roots of unity in terms of homological blocks

As we did for the case of the standard root of unity, we would like to analytically continue the level around $K = \frac{r}{s}$, and express the WRT invariant in terms of the homological blocks. We closely follow the calculation in [6]. For simplicity, we consider the case $F = 3$ and $\sum_{j=1}^{3} \frac{1}{r_j} < 1$.

The Gaussian integral part of (2.7) can be written as

$$\int_{\Gamma_t} dy e^{-\frac{\pi i}{3} \frac{y}{r/s} \left( y^2 + \frac{2\pi i}{r/s} y \right)^2 + 2\pi i \frac{r}{s} y \sum_{\chi} e^{\frac{\pi i}{3} (\frac{y}{r/s} - e^{-\frac{\pi i}{3} y})}}$$

(2.9)

where integration cycle $\Gamma_t$ is chosen in such a way that for each $t$ the integrand is convergent on both ends of infinity, i.e. when $r/s > 0$ and $P/H > 0$, $\Gamma_0$ is a line from $-(1+i)\infty$ to $(1+i)\infty$ through the origin and $\Gamma_t$ is parallel to $\Gamma_0$ and passes through a stationary phase point of $y = -2\pi i \frac{r}{s} t$. When $P/H < 0$, the contour is given by a clockwise rotation of $\Gamma_0$ by $\frac{\pi}{3}$ and similarly for $\Gamma_t$.

Assuming that $\Re y > 0$ and $P > 0$, the rational function of the sine hyperbolic function can be expanded in terms of periodic function $\chi_{2P}(n)$,

$$\prod_{j=1}^{3} \frac{e^{\frac{\pi i}{3} y_j} - e^{-\frac{\pi i}{3} y_j}}{e^y - e^{-y}} = \sum_{n=0}^{\infty} \chi_{2P}(n) e^{-\frac{\pi i}{3} y}$$

(2.10)

and $\chi_{2P}(n)$ is in turn expressed in terms of another periodic function $\psi_{2P}^{(l)}(n)$

$$\chi_{2P}(n) = \sum_{c=0}^{3} \psi_{2P}^{(R_c)}(n)$$

(2.11)

where

$$\psi_{2P}^{(l)}(n) = \begin{cases} \pm 1 & \text{if } n \equiv \pm l \mod 2P \\ 0 & \text{otherwise} \end{cases}$$

(2.12)

and $R_0 = P(1 - (1/P_1 + 1/P_2 + 1/P_3))$, $R_1 = P(1 - (1/P_1 - 1/P_2 - 1/P_3))$, $R_2 = P(1 - (-1/P_1 + 1/P_2 - 1/P_3))$, and $R_3 = P(1 - (-1/P_1 - 1/P_2 + 1/P_3))$. Given $r$ and $s$, we consider the analytic continuation of $K$ from $\frac{r}{s}$. We also assume $H > 0$ and $\Im \frac{r}{s} < 0$ for convergence. We take a contour as in the case of standard root of unity, i.e. a line parallel to the imaginary axis of the $y$-plane that passes through $\Re y > 0$. Then the integral above gives

$$Z_{K=\frac{r}{s}}(M_3) = \frac{B}{2\pi i} q^{-\frac{\phi_3}{3}} \frac{1}{\sqrt{s}} \left( \frac{2i}{r/s} \right)^{1/2} H_{s-1} \sum_{t=0}^{\infty} e^{2\pi i K H t^2} \sum_{n=0}^{\infty} \chi_{2P}(n) e^{2\pi i \frac{r}{s} n q^{\frac{2}{r/s}}} \bigg| q^{-e^{2\pi i \frac{\phi_3}{3}}} \bigg|$$

(2.13)

1For the case of $F = 3$, $\sum_{j=1}^{3} \frac{1}{r_j} > 1$ can only happen when $(P_1, P_2, P_3) = (2, 3, 5)$ and in this case there is an additional term $2y^{1/3}$ when expanding the rational function of the sine hyperbolic function in (2.9). It is obvious how to keep track of the additional term after considering the case of $\sum_{j=1}^{3} \frac{1}{r_j} < 1$. 

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We note that in the case of the standard root of unity, the integral has the overall factor \( \frac{1}{\sqrt{s}} \). When considering other roots of unity, \( \frac{1}{\sqrt{s}} \) can be written as \( \frac{1}{\sqrt{s} \sqrt{r/s}} \) and we take \( \frac{1}{\sqrt{s} \sqrt{r/s}} \) as an overall factor rather than \( \frac{1}{\sqrt{r/s}} \).

As in the case of the standard root of unity, we expect that the expression that we obtained from the Gaussian integral part of the WRT invariant with analytic continuation of \( K = \frac{r}{s} \) contains the contributions from all flat connections. Also, the calculation can be done for other ranges of \( P, H, \) and \( \text{Im} \frac{r}{s} \), which we refer to section 2.7 in [6] for detailed explanation.

2.3 The case \( H = 1 \)

We consider the case \( H = 1 \). In this case, we have

\[
Z_{K=\frac{r}{s}}(M_3) = \frac{B}{24} q^{-\phi_{3/4}} \left( \frac{2iP}{r/s} \right)^{1/2} \sum_{l=0}^{s-1} e^{2\pi i l P^2} \sum_{n=0}^{\infty} \chi_{2P}(n) q^{n^2} \left| q^{-e^{2\pi i \frac{r}{s}}} \right|. \tag{2.14}
\]

Here, \( \sum_{l=0}^{s-1} e^{2\pi i l P^2} \) is the quadratic Gauss sum \( g(m; s) \) which is defined as

\[
g(m; s) = \sum_{n=0}^{s-1} e^{2\pi i mn^2/s} \tag{2.15}
\]

Therefore, in this case, we obtain

\[
Z_{K=\frac{r}{s}}(M_3) = \frac{B}{24} q^{-\phi_{3/4}} \left( \frac{2iP}{r/s} \right)^{1/2} g(P_r; s) \sum_{i=1}^{3} \tilde{\Psi}_P^{(R_i)}(q) \left| q^{-e^{2\pi i \frac{r}{s}}} \right|. \tag{2.16}
\]

where \( \tilde{\Psi}_P^{(l)}(q) \) is a false theta function

\[
\tilde{\Psi}_P^{(l)}(q) = \sum_{n=0}^{\infty} \psi_{2P}^{(l)}(n) e^{2\pi i \frac{n^2}{4P}} = \sum_{n=0}^{\infty} \psi_{2P}^{(l)}(n) q^{n^2}. \tag{2.17}
\]

which is the Eichler integral of the modular form \( \Psi_P^{(l)}(q) := \sum_{n=0}^{\infty} n \psi_{2P}^{(l)}(n) q^{n^2} \) of half-integer weight 3/2 [14]. In the limit \( q \rightarrow e^{2\pi i \frac{r}{s}} \), \( \tilde{\Psi}_P^{(l)}(q) \) becomes

\[
\tilde{\Psi}_P^{(l)}(e^{2\pi i \frac{r}{s}}) = \sum_{n=0}^{rP} \left( 1 - \frac{n}{rP} \right) \psi_{2P}^{(l)}(n) e^{2\pi i \frac{n^2}{4P}}. \tag{2.18}
\]

We refer to [14, 15] (see also [16, 17]) for derivation of this. \( s \) is coprime to \( 4rP \). So \( s \) is odd here.

Regarding the overall coefficient, the quadratic Gauss sum gives \( \sqrt{s} \) up to a phase factor. Therefore, \( \sqrt{s} \) is cancelled out and the resulting expression contains \( \frac{1}{\sqrt{r/s}} \) as an overall factor.

We also note that when \( s = 1 \), (2.16) becomes the WRT invariant with standard root of unity. Since in the case of \( H = 1 \), there is only one homological block, which is well defined in the region \( |q| < 1 \). Therefore, as in the case of the standard root of unity, we can expect that the limit of a single homological block can also be taken to the other roots of unity \( e^{2\pi i \frac{r}{s}} \) where \( r \) and \( s \) are coprime and above calculation indicates that it is so up to an overall factor.
2.4 The case $H \geq 2$

When $H \geq 2$, we decompose $\psi_{2P}^{(l)}(n)$ in terms of $\psi_{2PH}^{(l)}(n)$

$$\psi_{2P}^{(l)}(n) = \sum_{h=0}^{\lfloor \frac{H}{2} \rfloor - 1} \psi_{2PH}^{(2hP+l)}(n) - \sum_{h=0}^{\lfloor \frac{H}{2} \rfloor - 1} \psi_{2PH}^{(2(h+1)P+l)}(n) \tag{2.19}$$

where the floor and the ceiling function are given by

$$[x] = \max\{m \in \mathbb{Z} | m \leq x\}, \tag{2.20}$$

$$[x] = \min\{m \in \mathbb{Z} | m \geq x\}. \tag{2.21}$$

As done in the case of the standard root of unity [6], we can take $e^{2\pi i \frac{1}{P}}$ out of summation in (2.13). We repeat it for completeness. We take the representative in (2.13)

$$\sum_{t=1}^{sH-1} e^{2\pi i K \frac{t^2}{P}} \sum_{n=0}^{\infty} e^{2\pi i \frac{t}{P}} \psi_{2PH}^{(l)}(n) q^{\frac{1}{2H} n^2}, \tag{2.22}$$

which is nonzero when $n = 2HPm + l$ and $2HPm' - l$ with $m, m' \in \mathbb{Z}_{\geq 0}$. The $n = 2HPm + l$ part of (2.22) is

$$\sum_{t=1}^{sH-1} e^{2\pi i K \frac{t^2}{P}} \sum_{n=0}^{\infty} e^{2\pi i \frac{t}{P}} \psi_{2PH}^{(l)}(n) q^{\frac{1}{2H} n^2} = \sum_{t=1}^{sH-1} e^{2\pi i K \frac{t^2}{P}} e^{2\pi i \frac{l}{P}} \sum_{n=2HPm + l}^{\infty} \psi_{2PH}^{(l)}(n) q^{\frac{1}{2H} n^2} \tag{2.23}$$

where $e^{2\pi i \frac{t}{P}(2HPm + l)} = e^{2\pi i \frac{t}{P} l}$ is used. The $n = 2HPm - l$ part of (2.22) is given by

$$\sum_{t=1}^{sH-1} e^{2\pi i K \frac{t^2}{P}} \sum_{n=2HPm - l}^{\infty} e^{-2\pi i \frac{t}{P}} \psi_{2PH}^{(l)}(n) q^{\frac{1}{2H} n^2} = \sum_{t=1}^{sH-1} e^{2\pi i K \frac{t^2}{P}} e^{-2\pi i \frac{l}{P}} \sum_{n=2HPm - l}^{\infty} \psi_{2PH}^{(l)}(n) q^{\frac{1}{2H} n^2}, \tag{2.24}$$

which can be written as

$$\sum_{t'=1}^{sH-1} e^{2\pi i rsPH - 4\pi irPt' + 2\pi i \frac{t'}{P} t'^2} e^{2\pi i \frac{t'}{P} l} \sum_{n=2HPm - l}^{\infty} \psi_{2PH}^{(l)}(n) q^{\frac{1}{2H} n^2} \tag{2.25}$$

with $t' = sH - t$. Since we take a limit to $K = \frac{r}{s}$ with $r$ and $s$ being coprime integers, $e^{2\pi i rsPH - 4\pi irPt' + 2\pi i \frac{t'}{P} t'^2}$ becomes $e^{2\pi i \frac{P}{s} t'^2}$. Thus, (2.22) can be expressed as

$$\sum_{t=1}^{sH-1} e^{2\pi i K \frac{t^2}{P}} e^{2\pi i \frac{l}{P}} \sum_{n=0}^{\infty} \psi_{2PH}^{(l)}(n) q^{\frac{1}{2H} n^2}. \tag{2.26}$$
Therefore, (2.13) can be written as

\begin{equation}
Z_K = (M_3) \simeq \left[ \sum_{t=0}^{sH-1} e^{2\pi i t K / H^2} \sum_{m=0}^{3} \left( \sum_{h=0}^{H-1} e^{2\pi i \phi (2hP + R_m)} \tilde{\Psi}_{HP}^{(2hP+R_m)} (q) - \sum_{h=0}^{H-1} e^{2\pi i \phi (2(h+1)P-R_m)} \tilde{\Psi}_{HP}^{(2(h+1)P-R_m)} (q) \right) \right]_{q \sim e^{2\pi i \phi / H}}.
\end{equation}

The expression (2.27) can be further organized. We first consider the case that \( H \) is odd. The case for even \( H \) can be done similarly. We take \( t = 0, 1, \ldots, Hs - 1 \) as \( t = vH + u \) where \( u = 0, 1, \ldots, H - 1 \) and \( v = 0, 1, \ldots, s - 1 \). Splitting \( u = 0 \) part and the other part \( u = 1, \ldots, H - 1 \), then (2.27) can be written as

\begin{equation}
\sum_{v=0}^{s-1} \sum_{u=0}^{H-1} e^{2\pi i \phi P \pi H (v+u)^2} \sum_{m=0}^{3} \left( \sum_{h=0}^{H-1} \tilde{\Psi}_{HP}^{(2hP+R_m)} (q) - \sum_{h=0}^{H-1} \tilde{\Psi}_{HP}^{(2(h+1)P-R_m)} (q) \right) + \sum_{v=0}^{s-1} \sum_{u=1}^{H-1} e^{2\pi i \phi P \pi H u^2} \sum_{m=0}^{3} \left( \sum_{h=0}^{H-1} e^{-4\pi i \phi Puv} e^{2\pi i \phi (2hP + R_m)} + e^{4\pi i \phi Puv} e^{2\pi i \phi (2hP + R_m)} \tilde{\Psi}_{HP}^{(2hP+R_m)} (q) - \sum_{h=0}^{H-1} e^{-4\pi i \phi Puv} e^{2\pi i \phi (2(h+1)P-R_m)} \tilde{\Psi}_{HP}^{(2(h+1)P-R_m)} (q) \right) \right]_{q \sim e^{2\pi i \phi / H}}.
\end{equation}

We can do similarly for the sum over \( s \) as we did for \( u \). We note that \( s \) is an odd number in this paper. Firstly, we split \( v = 0 \) part and rewrite the sum over \( v = 1, \ldots, s - 1 \) as sum over \( v = 1, \ldots, \frac{s-1}{2} \). Then, the third and the fourth line have a common factor
\( (e^{-4\pi i \frac{z}{P} \nu v} + e^{4\pi i \frac{z}{P} \nu v}), \) and (2.28) becomes

\[
\begin{align*}
&= \sum_{v=0}^{s-1} \left( e^{2\pi i \frac{z}{P} w v} \sum_{m=0}^{3} \left( \sum_{h=0}^{H-1} \tilde{\Psi}^{(2hP+R_m)}(q) - \sum_{h=0}^{H-1} \tilde{\Psi}^{(2(h+1)P-R_m)}(q) \right) \right) \\
+ &\sum_{u=1}^{H-1} \left[ e^{2\pi i \frac{z}{P} \nu u^2} \sum_{m=0}^{3} \left( \sum_{h=0}^{H-1} \left( e^{-2\pi i \frac{w}{P}(2hP+R_m)} + e^{2\pi i \frac{w}{P}(2hP+R_m)} \right) \tilde{\Psi}^{(2hP+R_m)}(q) \\
&\quad - \sum_{h=0}^{H-1} \left( e^{-2\pi i \frac{w}{P}(2(h+1)P-R_m)} + e^{2\pi i \frac{w}{P}(2(h+1)P-R_m)} \right) \tilde{\Psi}^{(2(h+1)P-R_m)}(q) \right) \right] \\
+ &\sum_{v=1}^{s-1} \sum_{u=1}^{H-1} \left[ e^{2\pi i \frac{z}{P} \nu v^2} \sum_{m=0}^{3} \left( \sum_{h=0}^{H-1} \left( e^{-2\pi i \frac{w}{P}(2hP+R_m)} + e^{2\pi i \frac{w}{P}(2hP+R_m)} \right) \tilde{\Psi}^{(2hP+R_m)}(q) \\
&\quad - \sum_{h=0}^{H-1} \left( e^{-2\pi i \frac{w}{P}(2(h+1)P-R_m)} + e^{2\pi i \frac{w}{P}(2(h+1)P-R_m)} \right) \tilde{\Psi}^{(2(h+1)P-R_m)}(q) \right) \right]|_{\nu e^{2\pi i \frac{w}{P}}}.
\end{align*}
\]

(2.29)

Since \( \sum_{v=-1}^{s-1} e^{2\pi i \frac{z}{P} \nu v^2} (e^{-4\pi i \frac{z}{P} \nu \nu v} + e^{4\pi i \frac{z}{P} \nu \nu v}) = \sum_{v=-1}^{s-1} e^{2\pi i \frac{z}{P} \nu v^2} e^{4\pi i \frac{z}{P} \nu v}, \) we obtain

\[
Z_{K=\frac{z}{s}}(M_3) = \sum_{v=0}^{s-1} \left( e^{2\pi i \frac{z}{P} \nu v^2} \sum_{m=0}^{3} \left( \sum_{h=0}^{H-1} \tilde{\Psi}^{(2hP+R_m)}(q) - \sum_{h=0}^{H-1} \tilde{\Psi}^{(2(h+1)P-R_m)}(q) \right) \right) \\
+ \sum_{v=0}^{s-1} \sum_{u=1}^{H-1} \left[ e^{2\pi i \frac{z}{P} \nu (vH+u)^2} \sum_{m=0}^{3} \left( \sum_{h=0}^{H-1} \left( e^{-2\pi i \frac{w}{P}(2hP+R_m)} + e^{2\pi i \frac{w}{P}(2hP+R_m)} \right) \tilde{\Psi}^{(2hP+R_m)}(q) \\
&\quad - \sum_{h=0}^{H-1} \left( e^{-2\pi i \frac{w}{P}(2(h+1)P-R_m)} + e^{2\pi i \frac{w}{P}(2(h+1)P-R_m)} \right) \tilde{\Psi}^{(2(h+1)P-R_m)}(q) \right) \right] |_{\nu e^{2\pi i \frac{w}{P}}}.
\]

(2.30)

When \( H \) is even, we also have similar results,

\[
Z_{K=\frac{z}{s}}(M_3) = \sum_{v=0}^{s-1} \left( e^{2\pi i \frac{z}{P} \nu v^2} \sum_{m=0}^{3} \sum_{h=0}^{H-1} \left( \tilde{\Psi}^{(2hP+R_m)}(q) - \tilde{\Psi}^{(2(h+1)P-R_m)}(q) \right) \right) \\
+ \sum_{v=0}^{s-1} \sum_{u=1}^{H-1} \left[ e^{2\pi i \frac{z}{P} \nu (vH+u)^2} \sum_{m=0}^{3} \sum_{h=0}^{H-1} \left( e^{-2\pi i \frac{w}{P}(2hP+R_m)} + e^{2\pi i \frac{w}{P}(2hP+R_m)} \right) \tilde{\Psi}^{(2hP+R_m)}(q) \\
&\quad - \left( e^{-2\pi i \frac{w}{P}(2(h+1)P-R_m)} + e^{2\pi i \frac{w}{P}(2(h+1)P-R_m)} \right) \tilde{\Psi}^{(2(h+1)P-R_m)}(q) \right] \right] |_{\nu e^{2\pi i \frac{w}{P}}}.
\]

(2.31)
Comparing with the case of the standard root of unity, the factor \( \sum_{v=0}^{s-1} e^{2\pi i \frac{v}{s} P} (vH+u)^2 \) is different. Other than that, the structure is the same as in the case of the standard root of unity. As a consistency check, when \( s = 1 \) we obtain the result with the standard root of unity.

**Remarks on the larger number of singular fibers**

As discussed in [6], the calculation for the case of larger number of singular fibers \( F \geq 4 \) is parallel to the case of \( F = 3 \), so the structures are the same. Therefore, from above calculation, when \( F \geq 4 \) we can see that we also have the same structure of expression as in the case of \( F = 3 \) above. Thus, for the case of \( F \geq 4 \), we can just use the formulas in Appendix A and B in [6] and take \( q \downarrow e^{2\pi i \frac{t}{r}} \) limit\(^2\) and replace \( e^{2\pi i K \frac{r}{P} t^2} \) with \( \sum_{v=0}^{s-1} e^{2\pi i \frac{v}{s} P} (vH+u)^2 \).

### 2.5 Properties and general structure

In the case of the standard root of unity, the summation variable \( t = 0, 1, \ldots, H - 1 \) was regarded as abelian flat connections where the holonomy is diag \( (e^{2\pi i \frac{t}{r}}, e^{-2\pi i \frac{t}{r}}) \). Therefore, \( (t, -t) \) was related to \( (-t, t) \mod H \) by the Weyl group action. Accordingly, they Weyl orbit of \( (t, -t) \) corresponds to the equivalent abelian flat connection where \( t = 0 \), in particular, corresponds to the trivial flat connection.

In the case of other roots of unity \( K = \frac{t}{s} \) where \( s \) is not equal to 1, the summation variable \( t \) goes from 0 to \( sH - 1 \). But as we saw in previous sections, \( e.g. \) in (2.30) and (2.31), it was possible to write expression as sum over \( u = 0, 1, \ldots, H - 1 \) and \( v = 0, 1, \ldots, s - 1 \). And we saw that, other than the limit \( q \downarrow e^{2\pi i \frac{t}{r}} \), the difference from the case of the standard root of unity is \( \sum_{v=0}^{s-1} e^{2\pi i \frac{v}{s} P} (vH+u)^2 \) while it was just \( e^{2\pi i \frac{r}{P} u^2} \) when \( s = 1 \). That is, given that \( Z_K(M_3) = \sum_t e^{2\pi i K \frac{r}{P} t^2} Z_t \) schematically in the case of the standard root of unity where \( Z_t \) is the contribution from the abelian flat connection \( t \), when \( K = \frac{t}{s} \) with \( r \) and \( s \) are coprime integers and \( s \neq 1 \), (2.30) and (2.31) indicate that \( Z_K(M_3) = \sum_{v=0}^{s-1} \sum_u e^{2\pi i \frac{v}{s} P} (vH+u)^2 Z_u \).

We also note that upon \( u \rightarrow u + H \), \( \sum_{v=0}^{s-1} e^{2\pi i \frac{v}{s} P} (vH+u)^2 \) becomes \( \sum_{v=0}^{s-1} e^{2\pi i \frac{v}{s} P} ((v+1)H+u)^2 \), but one can see that it is the same with \( \sum_{v=0}^{s-1} e^{2\pi i \frac{v}{s} P} (vH+u)^2 \), and also the summand is the same. Therefore, the expression is invariant under \( u \rightarrow u + H \). In addition, upon \( u \rightarrow -u \), the summand in (2.30) and (2.31) are the same, but \( \sum_{v=0}^{s-1} e^{2\pi i \frac{v}{s} P} (vH+u)^2 \) becomes \( \sum_{v=0}^{s-1} e^{2\pi i \frac{v}{s} P} (vH-u)^2 \). However, from \( \sum_{v=0}^{s-1} e^{2\pi i \frac{v}{s} P} (vH-u)^2 = e^{2\pi i \frac{1}{s} P} u^2 + \sum_{v=1}^{s-1} e^{2\pi i \frac{v}{s} P} (vH-u)^2 \), rewriting expression with \( v' = s - v \) and renaming \( v' \) to \( v \), the RHS gets back to \( \sum_{v=0}^{s-1} e^{2\pi i \frac{v}{s} P} (vH+u)^2 \). Thus, the expression is invariant under \( u \leftrightarrow -u \). Therefore, for the case of other roots of unity, it is consistent that the Weyl orbit of \( (u, -u) \mod H \) labels the abelian flat connection.

We also note that \( u \leftrightarrow -u \) provide the complex conjugate of \( (u, -u) \) at the level of holonomy. Therefore, as in the case of the standard root of unity, we can say that the contributions from the abelian flat connection and from the conjugate abelian flat connection are the same, though in the case of \( SU(2) \) these two abelian flat connections are

\(^2\)The \( q \downarrow e^{2\pi i \frac{t}{r}} \) limit of false theta functions and their derivatives has been discussed in [18, 19].
We use similar notations as in [6]. We denote the Weyl orbit of \( (u, -u) \in (\mathbb{Z}_H)^2 / \mathbb{Z}_2 \) as \( W_u \). When \( H \) is even, since Weyl orbits \( W_u \) and \( W_{u + \frac{H}{2}} \) that are related by the action of the center give the same contribution to the WRT invariant up to an overall factor \( e^{\pi i r} \), we group \( W_u \) and \( W_{u + \frac{H}{2}} \) by orbits under the action of the center, which we denote by \( C_a \) where \( a \) is a label for the abelian flat connection. The range of \( a \) is \( a = 0, 1, \ldots, \frac{H-2}{2} \) when \( H \) is a multiple of 2 but not of 4, and \( a = 0, 1, \ldots, \frac{H}{2} \) when \( H \) is a multiple of 4. We also denote elements in the Weyl orbit \( W_u \) by \( \tilde{a} \) and a representative of any of \( W_u \) in \( C_b \) by \( \tilde{b} \).

With the notation above, the \( S \)-matrix is the same with the one in the case of the standard root of unity

\[
S_{ab} = \frac{1}{\sqrt{\text{gcd}(2, H)}} \sum_{W_u \in C_a} \sum_{\tilde{a} \in W_u} e^{2\pi i k(\tilde{a}, \tilde{b})} |\text{Tor} H_1(M_3, \mathbb{Z})|^2
\]

with

\[
lk(u, u') = \frac{P}{H} \sum_{j=1}^{2} u_j u'_j = \frac{2P}{H} u_1 u'_1
\]

where \( u = (u_1, -u_1) \) and \( u' = (u'_1, -u'_1) \). We often use the notation \( lk(a, b) := lk(\tilde{a}, \tilde{b}) \).

Then, the WRT invariant is given by

\[
Z_{SU(2)}(M_3) = \frac{B}{2t} q^{-\phi/4} \frac{1}{\sqrt{s}} \left( \frac{2i}{r/s} \frac{P}{P} \right)^{1/2} \sqrt{\text{gcd}(2, H)} H \sum_{v=0}^{s-1} \sum_{a,b} e^{2\pi i v \frac{P}{H} \frac{\pi}{4}} S_{ab} \hat{Z}_b(q) \bigg|_{q^2 e^{2\pi i \frac{H}{2}}}
\]

when \( H \) is odd or a multiple of 4.

When \( H \) is a multiple of 2 but not of 4, the WRT invariant can be written as

\[
Z_{SU(2)}(M_3) = \frac{B}{2t} q^{-\phi/4} \frac{1}{\sqrt{s}} \left( \frac{2i}{r/s} \frac{P}{P} \right)^{1/2} \sqrt{2} \sum_{v=0}^{s-1} \sum_{a,b} e^{2\pi i v \frac{P}{H} \frac{\pi}{4}} (Y \otimes S_{ab})_{ab} \hat{Z}_b(q) \bigg|_{q^2 e^{2\pi i \frac{H}{2}}}
\]
where \( \hat{a}, \hat{b} = 0, \ldots, \frac{H}{2}, Y = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) so that \( (Y \otimes S_{ab})_{\hat{a}\hat{b}} = \begin{pmatrix} S_{ab} & S_{ab} \\ S_{ab} & S_{ab} \end{pmatrix} \), and \( e^{2\pi i \frac{Y}{2}(\nu H + \hat{a})^2} = (e^{2\pi i \frac{Y}{2}(\nu H + 0)^2}, \ldots, e^{2\pi i \frac{Y}{2}(\nu H + \frac{H}{2})^2}, e^{2\pi i \frac{Y}{2}(\nu H + 0)^2}e^{\pi i r}, \ldots, e^{2\pi i \frac{Y}{2}(\nu H + \frac{H}{2})^2}e^{\pi i r}) \). Since \( Z_a = Z_{\frac{H}{H-a}} \) for \( a = 0, 1, \ldots, \frac{H}{2}, \) in the notation of (2.35) we have \( Z_a(q) = Z_{\frac{H}{H-a}}(q) \), \( \hat{a} = 0, \ldots, \frac{H}{2} \) with \( \hat{a}_a(q) = (Y \otimes S_{ab})_{\hat{a}\hat{b}}Z_b(q) \) where \( Z_a(q) = Z_a(q) \) for \( a = 0, 1, \ldots, \frac{H}{2} \) and \( Z_\hat{a}(q) = Z_{\frac{H}{H-a}}(q) \) for \( a = \frac{H}{2}, \frac{H}{2} - 1, \ldots, \frac{H}{2} \). Similarly, we denote \( \tilde{Z}_b(q) = \tilde{Z}_b(q) \) for \( b = 0, 1, \ldots, \frac{H}{2} \) and \( \tilde{Z}_\hat{b}(q) = \tilde{Z}_{\frac{H}{H-a}}(q) \) for \( b = \frac{H}{2}, \frac{H}{2} - 1, \ldots, \frac{H}{2} \).

We expect that (2.34) and (2.35) hold for general closed 3-manifolds.

### 2.6 Examples

In this section, we provide some examples. We mostly quote examples from [6].

- **H = 2** from \((P_1, P_2, P_3) = (3, 5, 7)\)

For the case of the standard root of unity, the WRT invariant can be expressed as

\[
Z_K(M_3) = \frac{B}{2i} q^{-\phi_3/4} \left( \frac{105i}{K} \right)^{1/2} (1 + e^{\pi i K})(\tilde{Z}_0 + \tilde{Z}_1) \bigg|_{q = e^{2\pi i \frac{1}{K}}}.
\]

(2.36)

where

\[
\tilde{Z}_0(q) = \tilde{\Psi}_2^{(34)} + \tilde{\Psi}_2^{(106)} + \tilde{\Psi}_2^{(134)} + \tilde{\Psi}_2^{(146)},
\]

(2.37)

\[
\tilde{Z}_1(q) = -\tilde{\Psi}_2^{(64)} - \tilde{\Psi}_2^{(76)} - \tilde{\Psi}_2^{(104)} - \tilde{\Psi}_2^{(176)}.
\]

(2.38)

For other roots of unity, we have

\[
Z_{K=\frac{5}{2}}(M_3) = \frac{B}{2i} q^{-\phi_3/4} \left( \frac{105i}{r/s} \right)^{1/2} \omega (1 + e^{\pi i r})(\tilde{Z}_0 + \tilde{Z}_1) \bigg|_{q = e^{2\pi i \frac{1}{r/s}}}.
\]

(2.39)

For example, when \( K = \frac{5}{2} = \frac{3}{11}, \frac{4}{11}, \frac{7}{11}, \frac{8}{11}, \omega = -i, i, 1, 1 \), respectively. Or we may put it in the form of

\[
Z_{K=\frac{5}{2}}(M_3) = \frac{B}{2i} q^{-\phi_3/4} \left( \frac{70i}{r/s} \right)^{1/2} \sum_{a,b=0}^{1} \omega_a Y_{ab} \tilde{Z}_b \bigg|_{q = e^{2\pi i \frac{1}{r/s}}}.
\]

(2.40)

where \( Y_{ab} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) and \((w_0, w_1) = (\omega, \omega e^{\pi i r})\).

- **H = 3** from \((P_1, P_2, P_3) = (2, 5, 7)\)

The WRT invariant at the standard root of unity can be written as

\[
Z_K(M_3) = \frac{B}{i} q^{-\phi_3/4} \left( \frac{70i}{K} \right)^{1/2} \sum_{a,b=0}^{1} e^{2\pi i K CS_a S_{ab} \tilde{Z}_b} \bigg|_{q = e^{2\pi i \frac{1}{K}}}.
\]

(2.41)
where
\[
\hat{Z}_0 = -\hat{\Psi}_{210}^{(39)} - \hat{\Psi}_{210}^{(81)} - \hat{\Psi}_{210}^{(129)} + \hat{\Psi}_{210}^{(249)} ,
\]
\[
\hat{Z}_1 = \hat{\Psi}_{210}^{(11)} - \hat{\Psi}_{210}^{(31)} + \hat{\Psi}_{210}^{(59)} + \hat{\Psi}_{210}^{(101)} + \hat{\Psi}_{210}^{(109)} + \hat{\Psi}_{210}^{(151)} + \hat{\Psi}_{210}^{(199)} + \hat{\Psi}_{210}^{(241)} ,
\]
\[(CS_0, CS_1) = (0, \frac{1}{3}), \text{ and the } S\text{-matrix is}
\]
\[
S_{ab} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} .
\]

For other roots of unity, we have
\[
Z_{K \in Z}(M_3) = B \frac{q^{-\phi_3/4}}{r/s} \left( \frac{105i}{2K} \right)^{1/2} \sum_{a,b=0}^{1} \omega_a S_{ab} \hat{Z}_b \bigg|_{q^{-\phi_3/4}} ,
\]
\[
Z_K = z(M_3) = \frac{B}{2i} \left( \frac{105i}{2K} \right)^{1/2} \sum_{a,b=0}^{1} e^{2\pi i KCS_a S_{ab} \hat{Z}_b} \bigg|_{q^{-\phi_3/4}} ,
\]
\[(CS_0, CS_1) = (0, \frac{1}{3}). \text{ And the homological blocks are given by}
\]
\[
\hat{Z}_0 = -\hat{\Psi}_{420}^{(64)} - \hat{\Psi}_{420}^{(76)} - \hat{\Psi}_{420}^{(104)} - \hat{\Psi}_{420}^{(176)} + \hat{\Psi}_{420}^{(244)} + \hat{\Psi}_{420}^{(316)} + \hat{\Psi}_{420}^{(344)} + \hat{\Psi}_{420}^{(356)} ,
\]
\[
\hat{Z}_1 = \hat{\Psi}_{420}^{(34)} + \hat{\Psi}_{420}^{(106)} + \hat{\Psi}_{420}^{(134)} + \hat{\Psi}_{420}^{(146)} + \hat{\Psi}_{420}^{(274)} - \hat{\Psi}_{420}^{(286)} - \hat{\Psi}_{420}^{(314)} - \hat{\Psi}_{420}^{(386)} .
\]

For example, when \( K = \frac{r}{s} = \frac{5}{7}, \frac{7}{11}, \frac{11}{13}, \frac{15}{17} \), \((\omega_1, \omega_2) = (i, e^{\frac{7\pi i}{11}}), (-i, e^{\frac{7\pi i}{11}}), (1, e^{\frac{11\pi i}{17}}), (1, 1)\), respectively.

\[H = 4 \text{ from } (P_1, P_2, P_3) = (3, 5, 7)\]

The WRT invariant with \( K \in \mathbb{Z} \) is given by
\[
Z_K(M_3) = \frac{B}{2i} \left( \frac{105i}{2K} \right)^{1/2} \left( Z_0 + e^{\frac{\pi i}{3} K} Z_1 + Z_2 \right) = \frac{B}{2i} \left( \frac{105i}{2K} \right)^{1/2} \sum_{a,b=0}^{1} e^{2\pi i KCS_a S_{ab} \hat{Z}_b} \bigg|_{q^{-\phi_3/4}} ,
\]
\[(CS_0, CS_1) = (0, \frac{1}{3}). \]

For example, when \( K = \frac{2}{7}, \frac{3}{11}, \frac{7}{13}, \frac{4}{17}, (\omega_0, \omega_1) = (i, -i), (-i, 1), (1, -1), (-1, -1)\), respectively, with \( \omega_0 = \omega_2 \).
\[ H = 5 \quad \text{from} \quad (P_1, P_2, P_3) = (2, 3, 11) \]

The WRT invariant with an integer \( K \in \mathbb{Z} \) is given by

\[
Z_K(M_3) = \frac{B}{i} q^{-\phi/4} \left( \frac{33i}{K} \right)^{1/2} \sum_{a,b=0}^{2} e^{2\pi i KCS_a} S_{ab} \hat{Z}_b \bigg|_{q = e^{2\pi i \phi}}. \tag{2.51}
\]

where \( K \) is an integer and \((CS_0, CS_1, CS_2) = (0, \frac{1}{3}, \frac{2}{3})\). The homological blocks are

\[
\hat{Z}_0 = \tilde{\Psi}_{330}^{(5)} + \tilde{\Psi}_{330}^{(115)} + \tilde{\Psi}_{330}^{(215)} + \tilde{\Psi}_{330}^{(325)} 
\]

\[
\hat{Z}_1 = -\tilde{\Psi}_{330}^{(17)} + \tilde{\Psi}_{330}^{(83)} - \tilde{\Psi}_{330}^{(127)} - \tilde{\Psi}_{330}^{(337)} + \tilde{\Psi}_{330}^{(193)} + \tilde{\Psi}_{330}^{(203)} + \tilde{\Psi}_{330}^{(247)} + \tilde{\Psi}_{330}^{(347)} 
\]

\[
\hat{Z}_2 = -\tilde{\Psi}_{330}^{(49)} + \tilde{\Psi}_{330}^{(61)} - \tilde{\Psi}_{330}^{(71)} - \tilde{\Psi}_{330}^{(149)} - \tilde{\Psi}_{330}^{(181)} - \tilde{\Psi}_{330}^{(259)} + \tilde{\Psi}_{330}^{(269)} + \tilde{\Psi}_{330}^{(379)}
\]

and the S-matrix is

\[
S_{ab} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & A & B \\ 2 & B & A \end{pmatrix} \tag{2.55}
\]

where \( A = \frac{1}{2} (\sqrt{5} - 1) \) and \( \frac{1}{2} (\sqrt{5} - 1) \).

In the case of other roots of unity, we have

\[
Z_K = \frac{B}{i} q^{-\phi/4} \left( \frac{33i}{K} \right)^{1/2} \sum_{a,b=0}^{2} \omega_a S_{ab} \hat{Z}_b \bigg|_{q = e^{2\pi i \phi}}. \tag{2.56}
\]

For example, when \( K = \frac{r}{s} \), \( (\omega_1, \omega_2, \omega_3) = (1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}), (-1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}), (i, e^{\frac{4\pi i}{3}}, e^{\frac{2\pi i}{3}}), (\sqrt{5}, \sqrt{5}, \sqrt{5}) \), respectively.

\[ H = 6 \quad \text{from} \quad (P_1, P_2, P_3) = (5, 7, 11) \]

The WRT invariant with \( K \in \mathbb{Z} \) is given by

\[
Z_K(M_3) = \frac{B}{2i} q^{-\phi/4} \left( \frac{385i}{3K} \right)^{1/2} \left( Z_0 + e^{2\pi i K \frac{1}{3}} Z_1 + e^{\pi i K} Z_2 + e^{2\pi i K \frac{5}{3}} Z_3 \right) \tag{2.57}
\]

\[
Z_0 = \tilde{\Psi}_{2310}^{(288)} + \tilde{\Psi}_{2310}^{(372)} - \tilde{\Psi}_{2310}^{(552)} + \tilde{\Psi}_{2310}^{(1212)}, \tag{2.59}
\]

\[
Z_1 = -\tilde{\Psi}_{2310}^{(328)} + \tilde{\Psi}_{2310}^{(988)} + \tilde{\Psi}_{2310}^{(1168)} + \tilde{\Psi}_{2310}^{(1252)} - \tilde{\Psi}_{2310}^{(1828)} + \tilde{\Psi}_{2310}^{(1868)} - \tilde{\Psi}_{2310}^{(1912)} - \tilde{\Psi}_{2310}^{(2092)}, \tag{2.60}
\]

\[
Z_2 = \tilde{\Psi}_{2310}^{(218)} + \tilde{\Psi}_{2310}^{(308)} + \tilde{\Psi}_{2310}^{(442)} + \tilde{\Psi}_{2310}^{(482)} - \tilde{\Psi}_{2310}^{(1058)} - \tilde{\Psi}_{2310}^{(1142)} - \tilde{\Psi}_{2310}^{(1322)} + \tilde{\Psi}_{2310}^{(1982)}, \tag{2.61}
\]

\[
Z_3 = -\tilde{\Psi}_{2310}^{(1098)} + \tilde{\Psi}_{2310}^{(1758)} + \tilde{\Psi}_{2310}^{(1938)} + \tilde{\Psi}_{2310}^{(2022)}, \tag{2.62}
\]
which are related to $Z_a$'s as $Z_0 = Z_3 = \hat{Z}_0 + \hat{Z}_1 + \hat{Z}_2 + \hat{Z}_3$ and $Z_1 = Z_2 = 2\hat{Z}_0 - \hat{Z}_1 + 2\hat{Z}_3 - \hat{Z}_2$.

When $K$ is rational, we have

$$Z_{K=n}(M_3) = \frac{B}{2\pi} q^{-\phi_3/4} \left(\frac{385i}{3r/s}\right)^{1/2} \sum_{a=0}^{3} \omega_a Z_a$$

$$= \frac{B}{2\pi} q^{-\phi_3/4} \left(\frac{385i}{K}\right)^{1/2} \sum_{a,b=0}^{3} \omega_a \left(\left[\begin{array}{c} 1 \\ 1 \\ 1 \
\end{array}\right] \otimes S_{ab}\right)_{ab} \hat{Z}_b(q)$$

(2.63) (2.64)

where $\omega_a = (\omega_0, \omega_1, \omega_3, \omega_2)$. For example, when $K = \frac{r}{s} = \frac{4}{17}, \frac{7}{11}, \frac{13}{3}, \frac{23}{7}$, $(\omega_0, \omega_1) = (1, e^{\frac{4\pi i}{r}}), (-1, e^{\frac{4\pi i}{r}}), (\sqrt{3}, 0), (i, e^{\frac{\pi i}{r}})$, respectively, where $\omega_0 = e^{\pi i r} \omega_3$ and $\omega_1 = e^{\pi i r} \omega_2$.

3 Asymptotic expansion of knot invariants at roots of unity

There are various versions of the volume conjecture, such as the parametrized volume conjecture, quantum volume conjecture, etc.\(^3\) Or it can be also generalized to the case with the parameter $t$ for the grading of the categorification or with the parameter $a$ in HOMFLY polynomial.

The original volume conjecture was considered in the limit that $K \to \infty$ where the Chern-Simons level $K$ and the color $n$ of the Jones polynomial is identified $K = n$. In the parametrized volume conjecture, both $K$ and $n$ are taken to infinity $K, n \to \infty$ while the ratio $n/K$ being fixed. The quantum invariants are expressed in terms of $q = e^{\frac{2\pi i}{r}}$, and in the limit $K \to \infty$, $q \to 1$.

In this section, we consider the asymptotic expansion in the limit that $q$ goes to the root of unity

$$q \to \zeta^*_r := e^{2\pi i \frac{s}{r}}$$

(3.1)

for a few knot polynomials where $\frac{s}{r} \in \mathbb{Q}$ where $r$ and $s$ are coprime integers.

**Limit of $q$ to the root of unity**

The limit for the asymptotic expansion of the invariants with respect to the root of unity that we will discuss in this section is different from the standard $q \to 1$ limit with the level $K \to \infty$. For the standard limit in the volume conjecture, the expansion parameter was $\epsilon = \frac{2\pi i}{K}$ and $\epsilon \to 0$ or $K \to \infty$ has been considered.

However, when we consider the asymptotic expansion around general roots of unity $e^{2\pi i \frac{1}{r}} = e^{2\pi i \frac{s}{r}}$, it looks more natural to introduce another expansion parameter, $h$, with respect to general root of unity, $e^{2\pi i \frac{s}{r}}$. This was considered in [9], in [10, 11] in a similar but a slightly different setup, and in [12] in the context of Nahm sum.

Therefore, we take $q$ as

$$q = e^{h/r} e^{2\pi i \frac{s}{r}} = e^{h/r} \zeta^*_r$$

(3.2)

which goes to $\zeta^*_r$ upon $h \to 0$. So, in this limit, $q^r \to 1$. We note that we can also consider the asymptotic expansion around the root of unity $e^{2\pi i \frac{1}{r}} = e^{2\pi i \frac{1}{r}}$ where $s$ is set to $1$, not taking $K = r \to \infty$ but taking $h \to 0$.

\(^3\)See [20] for the review.
3.1 Asymptotic expansion of $q$-Pochhammer symbol at the root of unity

We note that many knot or link invariants can be expressed in terms of $q$-Pochhammer symbols and monomials. Examples include trefoil knot $3_1$, figure-eight knot $4_1$, $5_{1,2}$, $6_1$, some other twist knots, the Hopf links [21–25]. Therefore, we study the behavior of $q$-Pochhammer symbol in the limit of $q$ to the root of unity.

**Asymptotic expansion of $(x; q)_\infty$ at $q \to 1$**

Before discussing the case of general roots of unity, we review the case that $q \to 1$. We denote the deviation of $q$ from 1 as $e^{\epsilon}$, which is $e^{\frac{2\pi i}{K}}$ in this case.

It is known [26] that the dilogarithm function and the function $\text{Li}_2(x; q)$ are related as

$$\lim_{\epsilon \to 0} (e^{\epsilon} \text{Li}_2(x; e^{-\epsilon})) = \text{Li}_2(x). \quad (3.3)$$

where

$$\text{Li}_2(x; q) = \sum_{n=1}^{\infty} \frac{x^n}{n(1 - q^n)} \quad (3.4)$$

with $x, q \in \mathbb{C}$ with $|x|, |q| < 1$. Also, it can be shown [26] that

$$- \log(x; q)_\infty = \text{Li}_2(x; q). \quad (3.5)$$

Therefore, from the generating function of Bernoulli polynomial $B_m(x)$

$$\frac{e^{\beta t}}{1 - e^t} = - \sum_{m=0}^{\infty} B_m(\beta) \frac{t^{m-1}}{m!} \quad (3.6)$$

the asymptotic expansion of $\text{Li}_2(q^c x; e^\epsilon)$ as $\epsilon \to 0$ is given by

$$\text{Li}_2(q^c x; e^\epsilon) \simeq - \sum_{m=0}^{\infty} \frac{B_m(c)}{m!} \text{Li}_{2-m}(x) \epsilon^{m-1}, \quad (3.7)$$

so

$$(q^c x; q)_\infty \sim \exp \left( \sum_{m=0}^{\infty} \frac{B_m(c)}{m!} \text{Li}_{2-m}(x) \epsilon^{m-1} \right). \quad (3.8)$$

For example, (3.8) up to $\epsilon^3$ is given by

$$(q^c x; q)_\infty \sim \exp \left( \frac{1}{\epsilon} \text{Li}_2(x) + \left( c - \frac{1}{2} \right) \log \frac{1}{1-x} + \frac{\epsilon}{12} (6c^2 - 6c + 1) \frac{x}{1-x} 
+ \frac{\epsilon^2}{12} (2c^3 - 3c^2 + c) \frac{x}{(1-x)^2}
+ \frac{\epsilon^3}{720} (30c^4 - 60c^3 + 30c^2 - 1) \frac{x+x^2}{(1-x)^3} + \cdots \right). \quad (3.9)$$
Asymptotic expansion of \((x; q)_\infty\) at the root of unity

Asymptotic expansion of \((q^c x; q)_\infty\) has been discussed in [9, 11, 12, 27], but has not been applied to the knot polynomials yet. We express \((q^c x; q)_\infty\) as

\[
(q^c x; q)_\infty = \prod_{a=0}^{r-1} (q^{c+a} x; q^r)_\infty = \exp \left( - \sum_{a=0}^{r-1} \text{Li}_2 (q^{c+a} x; q^r) \right). \tag{3.10}
\]

By using (3.4), we have

\[
- \sum_{a=0}^{r-1} \text{Li}_2 (q^{c+a} x; q^r) = - \sum_{a=0}^{r-1} \sum_{n=1}^\infty \frac{x^n}{n} q^{(c+a)n} (1 - q^m). \tag{3.11}
\]

Since \(q^r = e^h\), (3.11) becomes

\[
- \sum_{a=0}^{r-1} \sum_{n=1}^\infty \frac{(\zeta^s(c+a)x)^n}{n} e^{nh(c+a)/r} = \sum_{a=0}^{r-1} \sum_{n=1}^\infty \sum_{m=0}^\infty \frac{(\zeta^r(c+a)x)^n}{n^2-m} B_m \left( \frac{c+a}{r} \right) \frac{h^{m-1}}{m!} \tag{3.12}
\]

where (3.6) is also used. From \(B_m (x+y) = \sum_{l=0}^m \binom{m}{l} B_l (x) y^{m-l}\) where \(\binom{m}{l}\) is the binomial coefficient, (3.12) becomes

\[
\sum_{a=0}^{r-1} \sum_{n=1}^\infty \sum_{m=0}^\infty \frac{(\zeta^r(c+a)x)^n}{n^2-m} \sum_{l=0}^m \binom{m}{l} B_l (c/r) \frac{a^{m-l} h^{m-1}}{m!}. \tag{3.13}
\]

By using the identity \(\text{Li}_n (x^k) = k^{n-1} \sum_{j=0}^{k-1} \text{Li}_n \left( e^{2\pi i j/k} x \right) \) (e.g. in [28]), from (3.13) we obtain

\[
\sum_{m=0}^\infty \frac{B_m (c/r)}{m!} \text{Li}_{2-m}(x) r^{m-1} h^{m-1} + \sum_{m=1}^{\infty} \sum_{l=0}^{m-1} \sum_{a=0}^{r-1} \text{Li}_{2-m}(\zeta^r(c+a)x) \binom{m}{l} B_l (c/r) \frac{a^{m-l} h^{m-1}}{m!}. \tag{3.14}
\]

Therefore, asymptotic expansion of \((q^c x; q)_\infty\) around \(\zeta^r\) is given by

\[
(q^c x; q)_\infty \sim \exp \left( \sum_{m=0}^\infty \frac{B_m (c/r)}{m!} \text{Li}_{2-m}(x) r^{m-1} h^{m-1} \right. \\
+ \left. \sum_{m=1}^{\infty} \sum_{a=0}^{r-1} \sum_{l=0}^{m-1} \binom{m}{l} B_l (c/r) \text{Li}_{2-m}(\zeta^r(c+a)x) \frac{a^{m-l} h^{m-1}}{m!} \right). \tag{3.15}
\]

For example, up to \(O(h^2)\), (3.15) is given by

\[
(q^c x; q)_\infty \sim \exp \left( \frac{1}{hr} \text{Li}_2(x^r) - \frac{(c/r - 1)}{2} \log(1 - x^r) + \sum_{a=0}^{r-1} \frac{a}{r} \log(1 - \zeta^r(c+a)x) \right. \\
+ \frac{1}{2} \left( \left( \frac{c}{r} \right)^2 - \left( \frac{c}{r} \right) + \frac{1}{2} \right) x^r \left( 1 - x^r \right) h + \sum_{a=0}^{r-1} \frac{a}{2r} \frac{\zeta^r(c+a)x}{1 - \zeta^r(c+a)x} \left( \frac{a}{r} + 2 \left( \frac{a}{r} - \frac{1}{2} \right) \right) h + O(h^2) \right) \tag{3.16}
\]

where we see that the leading order of expansion (3.15) is given by

\[
\exp \left( \frac{1}{hr} \text{Li}_2(x^r) \right). \tag{3.17}
\]
3.2 Examples: superpolynomial of knots

We consider the asymptotic expansion for the superpolynomials of unknot, trefoil knot, and figure-eight knot in totally symmetric representation $S^n$. Upon specialization, for example, $a = q^N$, we can recover the Poincaré polynomial of a knot for $G = SU(N)$. There is subtlety in such specialization [21, 29, 30] but there is no problem for examples here. We can also take $t = -1$ limit, then it would give the HOMFLY polynomial or the Jones polynomial upon $a = q^2$.

With the limit $\hbar \to 0$ in $q = e^{\hbar/r} \zeta_r^s$, we also need to consider the limit for $x$ in superpolynomials. In the superpolynomials with totally symmetric representation $S^n$, $x$ is given by $x = q^n$. Since we deformed $q$ from $\zeta_r^s$ by $e^{2\pi i \xi r}$, we may consider $x = e^{\hbar n/r} e^{2\pi i m r}$ where as $\hbar \to 0$ and $n \to \infty$, another parameter $\xi$ is taken to $\xi \to \infty$ such that $\hbar n = u$ is fixed and $\hbar \xi = m \in \mathbb{Z}$ is also fixed. When getting back to the undeformed original $x = \zeta_r^s$, we can take $\hbar \to 0$ and $\xi \to \infty$ such that $\hbar \xi = n$.

Unknot

The unnormalized superpolynomial of unknot is

$$P_{S^0}(a, q, t) = a^{-\frac{3}{2}} q^{\frac{3}{2}} (-t)^{-\frac{3}{2}} \frac{(a(-t)^{3}; q)_n}{(q; q)_n}.$$  

The leading term of asymptotic expansion is given by

$$P_{S^0}(a, q, t) \sim e^{\frac{1}{\hbar r}} \frac{1}{r^2} \int x^* \log y \frac{dx'}{x'}$$

so that the asymptotic expansion is expressed as

$$P_{S^0}(a, q, t) \sim e^{\frac{1}{\hbar r}} \tilde{W}$$

such that $hn = u$ is fixed and $h \xi = m \in \mathbb{Z}$ is also fixed. We also note that in this limit $x' \to e^{hn} = e^n$.

With above setup and also (3.15), we calculate the leading order term of the perturbative expansion of superpolynomials around root of unity $e^{2\pi i \xi r}$. 

Unknot

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such that $hn = u$ is fixed and $h \xi = m \in \mathbb{Z}$ is also fixed. We also note that in this limit $x' \to e^{hn} = e^n$. 

With above setup and also (3.15), we calculate the leading order term of the perturbative expansion of superpolynomials around root of unity $e^{2\pi i \xi r}$.
Then, from
\[ y = e^{\frac{\partial \tilde{W}}{\partial x}} \]  
we have
\[ y^r = a^{-\frac{1}{2}}(t)^{-\frac{1}{2}r} \frac{1 - a^r(-t)^{3r}x^r}{1 - x^r} \]  
(3.24)

This is the same with the super A-polynomial of unknot
\[ y = a^{-\frac{1}{2}}(t)^{-\frac{1}{2}} \frac{1 - a^{-3}x}{1 - x} \]  
(3.25)

with \( x \rightarrow x^r, \ y \rightarrow y^r, \ a \rightarrow a^r, \) and \(-t \rightarrow (-t)^r\).

**Trefoil knot**

The case of trefoil knot can be done similarly. The superpolynomial for the trefoil knot is given by
\[ P_{3_1}^{Sn}(a, q, t) = \sum_{k=0}^{r} \left( \frac{a(-t)^3}{q} \right)_n (a^{-1}q^{-t}q)_k \frac{a^z q^{r(z^r+1)k} (-t)^{2k} q}{(q; q)_k (q; q)_{n-k}} \]  
(3.26)

and from the asymptotic expansion we have
\[ \tilde{W} = \frac{1}{r^2} \left( \log x^r \log a^r + \log z^r \log x^r - \frac{3}{2} \log x^r \log(-t)^r + \log(-t)^{3r} \log z^r - \frac{\pi^2}{6} \right. \]

\[ + \ Li_2(z^r) + Li_2(x^r z^{-r}) - Li_2(a^r(-t)^{3r}x^r) + Li_2(a^r(-t)^{3r}) - Li_2(a^r(-t)^r) + Li_2(a^r(-t)^r) \]  
(3.27)

From the condition
\[ 1 = e^{\frac{\partial \tilde{W}(a, t; x)}{\partial x}}, \quad y = e^{\frac{\partial \tilde{W}(a, t; x)}{\partial x}} \]  
(3.28)

we have
\[ 1 = x^r(-t)^{2r} (1 - x^r z^{-r})(1 - a^r(-t)^r z^{-r}), \quad y^r = a^\frac{z^r}{r}(1 - a^r(-t)^3 x^r) \]  
(3.29)

Solving these, we obtain the super A-polynomial
\[ 0 = a^{3r/2}(-t)^{5r/2} x^{3r} \]  
\[ + a^{r/2} \left( 1 - (-t)^{2r} x^r + 2(-t)^{3r} (1 - a^r(-t)^r) x^{2r} - a^r(-t)^{5r} x^{3r} + a^{2r}(-t)^{6r} x^{4r} \right) y^r \]
\[ + a^{r/2}(-t)^{3r/2} (-1 + x^r) y^{2r} \]  
(3.30)

This is the standard super A-polynomial with all the variables are replaced with \( r \)-th power of them.
Figure-eight knot

Similarly, from the superpolynomial,

\[
\mathcal{P}_4^{\text{sn}}(a, q, t) = \sum_{k=0}^{n} \frac{(a(-t)^3; q)_n}{(q; q)_k(q; q)_{n-k}} (aq^{-1}(-t); q)_{k} (aq^n(-t)^3; q)_{k} a^{-k} a^{-\frac{2}{3}} q^3 q^{k(1-n)} (-t)^{-2k} (-t)^{-\frac{3}{2}n},
\]

we obtain

\[
\mathcal{W} = \frac{1}{r^2} \left( - \log z^r \log a^r - \frac{1}{2} \log x^r \log a^r - \log z^r \log x^r - \log z^r \log(-t)^{2r} - \log x^r \log(-t)^{3r/2} \right.
\]

\[
+ \text{Li}_2(a^r (-t)^{3r}) - \text{Li}_2(a^r (-t)^rz^r) + \text{Li}_2(a^r (-t)^{r'})
\]

\[
- \text{Li}_2(a^r x^r (-t)^{3r}z^r) + \text{Li}_2(z^{r'}) + \text{Li}_2(x^r z^{-r}) - \frac{\pi^2}{3} \right).
\]

This gives

\[
1 = \frac{(1 - x^r z^{-r})(1 - a^r(-t)^rz^r)(1 - a^r(-t)^{3r}x^r z^r)}{a^r(-t)^{2r}x^r(1 - z^r)}, \quad y^r = \frac{(1 - a^r(-t)^{3r}x^r z^r)}{a^r(-t)^{3r/2}z^r(1 - x^r z^{-r})}
\]

and the A-polynomial is given by

\[
0 = a^{3r/2}(-t)^{5r/2}x^{2r}(1 - a^r(-t)^{3r}x^r)
\]

\[
+ a^r(-1 + (-t)^r)(1 + (-t)^r)x^r + 2a^r(-t)^{3r}(1 - (-t)^r)x^{2r}
\]

\[
+ 2a^r(-t)^{4r}(1 + (-t)^r)x^{3r} - a^2r(-t)^{6r}(1 + (-t)^r)x^{4r} + a^2r(-t)^{8r}x^{5r}y^r
\]

\[
+ (1 - a^r(-t)^r)(1 + (-t)^r)x^r + 2a^r(-t)^{2r}(1 - (-t)^r)x^{2r} + 2a^r(-t)^{4r}(1 - (-t)^r)x^{3r}
\]

\[
+ a^2r(-t)^{5r}(1 + (-t)^r)x^{4r} - a^{3r}(-t)^{7r}x^{5r}y^r + a^2r(-t)^{4r}(-1 + x^r)x^{2r}y^{3r},
\]

which is the same with the standard super-A-polynomial with variables replaced by their \(r\)-th powers as expected.

**Remarks**

From above examples, we see that the volume function or twisted superpotential is given by the standard twisted superpotential with variables replaced with \(r\)-th power of them up to the overall \(1/r^2\) factor, and also the A-polynomial takes the same form, which could be expected already from (3.17).

If we want higher order terms in the expansion, we can use (3.15) for \(q\)-Pochhammer symbols. For monomials, we can express them in terms of theta functions, which is in turn expressed in terms of \(q\)-Pochhammer symbols. So in this way full asymptotic expansion could also be obtained.
From above calculations, we expect in general that in the limit
\[ q = e^{h/r} \zeta^r \to \zeta^r, \quad x = e^{h n/r} e^{2 \pi i \hbar \xi s / \tau} \to e^{u/r} e^{2 \pi i s m / \tau} \]  
(3.35)
as \( h \to 0, n \to \infty \), and \( \xi \to \infty \) such that \( h n = u \) is fixed and \( h \xi = m \in \mathbb{Z} \) is also fixed, the asymptotic behavior of the knot invariants would take a form
\[
P_n(a, q, t) \simeq \exp \left( \frac{1}{h \tau} S_{-1}(u, a, (-t)^r) + \sum_{k=0}^{\infty} S_k(u, a, t; r, \zeta^s, m) h^k \right) \]
(3.36)

In the context of the 3d-3d correspondence, the \( q \)-Pochhammer symbol appears as \( S^1 \times D^2 \) partition function of the chiral multiplet with Neumann or Dirichlet boundary condition [31]. Therefore, \( u \) parameter that appears in the limit of \( x \) in (3.35) would be proportional to mass parameter of 3d \( \mathcal{N} = 2 \) theory. Also, since the A-polynomial is a function of \( x^r \), which is \( x^r = e^u \), the A-polynomial discussed in this paper actually agrees with the A-polynomial obtained in the standard limit if \( \epsilon n \) is fixed to \( u \). Therefore, matter contents of the theory is equally captured in either limits.

The physical meaning of the asymptotic expansion around the root of unity and its relation to the corresponding 3d \( \mathcal{N} = 2 \) theory is an interesting problem and we leave it as future work.

4 Discussion

We discussed the WRT invariants at other roots of unity in terms of homological blocks. The structure was the same with a few difference. One is obviously the limit \( q \searrow e^{2 \pi i \hat{\phi}} \), another is about the factor \( \sum_{v=0}^{s-1} e^{2 \pi i \frac{v H}{r} (v H + u)^2} \). The properties in the case of the standard root of unity, such as the symmetries from the action of the center and the complex conjugation were also available. In the second half, we discussed the asymptotic expansion of knot invariants around general roots of unity, which is different from the standard limit in the volume conjectures. We calculate the leading order of the asymptotic expansions of superpolynomials around the roots of unity \( e^{2 \pi i \hat{\phi}} \), and saw that the volume function/twisted superpotential and A-polynomial are the same with those in the context of standard volume conjecture upon replacement of variables by their \( r \)-th powers.

There are several interesting directions. It would be interesting to generalize \( SU(2) \) case for higher rank case and also to the case of two-variable series for knot complements in [7]. Also, detailed resurgence analysis for the case of rational \( K \) would be also interesting. In addition, the physical understanding for the case of rational \( K \) via the 3d-3d correspondence would be an important direction to study.

More thorough study for the asymptotic expansion discussed in section 3 is needed. For example, systematic calculations of higher order terms in the asymptotic expansion around the roots of unity, such as via quantization of A-polynomial, would be an interesting direction. It would be important to study mathematical meaning or applications of such expansion and also physical meaning of it in the context of the 3d-3d correspondence.
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