The Cauchy problem for higher-order linear partial differential equation

Guangqing Bi a)†, Yuekai Bi b)

a) School of Electronic and Information Engineering, BUAA, Beijing 100191, China
b) E-mail: yuekaiffly@163.com

For the linear partial differential equation

\[ \mathcal{P}(\partial_x, \partial_t) u = f(x, t), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^1, \]

with \( \mathcal{P}(\partial_x, \partial_t) \) being \( \prod_{i=1}^{m} (\partial_{t}^{2} - a_i \mathcal{P}(\partial_x)) \) or \( \prod_{i=1}^{m} (\partial_{t}^{2} - a_i^2 \mathcal{P}(\partial_x)) \), the authors give the analytic solution of the Cauchy problem using the abstract operators \( e^{t \mathcal{P}(\partial_x)} \) and \( \frac{\sinh(t \mathcal{P}(\partial_x))^{1/2}}{\mathcal{P}(\partial_x)^{1/2}} \). By representing the operators with integrals, explicit solutions are obtained with an integral form of a given function.

Keywords: cauchy problem, partial differential equation, abstract operators

MSC(2000): 35G10; 35G05

1. Introduction and main results

In 1997, Guangqing Bi first introduced the concept of abstract operators in reference [1], and determined the algorithms of abstract operators \( e^{t \mathcal{P}(\partial_x)} \) and \( \frac{\sinh(t \mathcal{P}(\partial_x))^{1/2}}{\mathcal{P}(\partial_x)^{1/2}} \). Using this type of operators, in reference [2] the author has obtained the following results:

**Theorem BI1.** Let \( a_1, a_2, \ldots, a_m \) be arbitrary real or complex numbers different from each other, \( \mathcal{P}(\partial_x) \) be a partial differential operator of any order, then \( \forall f(x, t) \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^1) \), we have

\[
\begin{cases}
  \prod_{i=1}^{m} (\partial_{t}^{2} - a_i \mathcal{P}(\partial_x)) u = f(x, t), & x \in \mathbb{R}^n, \quad t \in \mathbb{R}^1, \\
  \left. \frac{\partial^j u}{\partial t^j} \right|_{t=0} = 0, & j = 0, 1, 2, \ldots, m - 1.
\end{cases}
\]

\[ u(x, t) = \int_{0}^{t} \int_{0}^{t-\tau} \frac{(t - \tau - \tau')^{m-2}}{(m-2)!} \sum_{j=1}^{m} \frac{a_j^{m-1}}{(a_j - a_i)} e^{\tau'a_j \mathcal{P}(\partial_x)} f(x, \tau) \, d\tau' \, d\tau. \]
**Theorem BI2.** Let \( a_1, a_2, \ldots, a_m \) be arbitrary real or complex numbers different from each other, \( P(\partial_x) \) be a partial differential operator of any order, then \( \forall f(x,t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^1) \), we have

\[
\begin{cases}
\left. \prod_{i=1}^m \left( \frac{\partial^2}{\partial t^2} - a_i^2 P(\partial_x) \right) u \right|_{t=0} = f(x,t), & x \in \mathbb{R}^n, \ t \in \mathbb{R}^1, \\
\left. \frac{\partial^j u}{\partial t^j} \right|_{t=0} = 0, & j = 0, 1, 2, \ldots, 2m - 1.
\end{cases}
\]

Then \( u(x,t) = \int_0^t \int_0^{t-\tau} \frac{(t-\tau)^2 - \tau^2}{(2m-3)!} \sum_{j=1}^m a_j^{2m-2} \sinh(\tau a_j P(\partial_x)^{1/2}) a_j P(\partial_x)^{1/2} f(x,\tau) d\tau' d\tau. \) (4)

In reference [3] Guangqing Bi has obtained the following results:

**Theorem BI3.** Let \( m \in \mathbb{N} \) and \( m > 1 \), \( P(\partial_x) \) be a partial differential equation of any order. Then \( \forall f(x,t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^1) \), we have

\[
\begin{cases}
\left. \left( \frac{\partial^2}{\partial t^2} - P(\partial_x) \right)^m u \right|_{t=0} = f(x,t), & x \in \mathbb{R}^n, \ t \in \mathbb{R}^1, \\
\left. \frac{\partial^j u}{\partial t^j} \right|_{t=0} = 0, & j = 0, 1, 2, \ldots, 2m - 1.
\end{cases}
\]

Then \( u(x,t) = \int_0^t \int_0^{t-\tau} \frac{((t-\tau)^2 - \tau^2)^m-2}{(2m-2)!! (2m-4)!!} \sinh(\tau' P(\partial_x)^{1/2}) f(x,\tau) d\tau' d\tau. \) (6)

By combining the abstract operators and Laplace transform, the authors have obtained the following results in reference [4]:

**Theorem BI4.** Let \( m \in \mathbb{N} \) and \( m > 1 \), \( P(\partial_x) \) be a partial differential equation of any order. Then \( \forall f(x,t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^1), \ \varphi_j(x) \in C^\infty(\mathbb{R}^n) \), we have

\[
\begin{cases}
\left. \left( \frac{\partial^2}{\partial t^2} - P(\partial_x) \right)^m u \right|_{t=0} = \varphi_j(x), & x \in \mathbb{R}^n, \ t \in \mathbb{R}^1, \\
\left. \frac{\partial^j u}{\partial t^j} \right|_{t=0} = 0, & j = 0, 1, 2, \ldots, 2m - 1.
\end{cases}
\]

Then \( u(x,t) = \int_0^t \int_0^{t-\tau} \frac{((t-\tau)^2 - \tau^2)^m-2}{(2m-2)!! (2m-4)!!} \sinh(\tau' P(\partial_x)^{1/2}) f(x,\tau) \tau' d\tau' d\tau + \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} P(\partial_x)^k \sum_{j=0}^{2m-1-2k} \frac{\partial^{2m-1-2k-j}}{\partial t^{2m-1-2k-j}} \int_0^t \frac{(t^2 - \tau^2)^{m-2}}{(2m-2)!! (2m-4)!!} \sinh(\tau P(\partial_x)^{1/2}) \varphi_j(x) d\tau. \) (8)

Using the same method, we have obtained the following results in this paper:
Theorem 1. Let \( a_1, a_2, \ldots, a_m \) be arbitrary real or complex roots different from each other for \( b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m = 0 \), and \( P(\partial_x, \partial_t) \) be a partial differential operator defined by

\[
P(\partial_x, \partial_t) = \sum_{k=0}^{m} b_k P(\partial_x)^{m-k} \frac{\partial^k}{\partial t^k}, \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}^1, \ 1 < m \in \mathbb{N}.
\]

Where \( P(\partial_x) \) is a partial differential operator of any order. Then \( \forall f(x,t) \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_t^1) \), \( \varphi_r(x) \in C^\infty(\mathbb{R}^n) \), we have

\[
\begin{align*}
\begin{cases}
P(\partial_x, \partial_t)u = f(x,t), & x \in \mathbb{R}^n, \ t \in \mathbb{R}^1. \\
\frac{\partial^r u}{\partial t^r} \bigg|_{t=0} = \varphi_r(x), & r = 0, 1, 2, \ldots, m - 1.
\end{cases}
\end{align*}
\]

(9)

\[
u(x,t) = \int_0^t \int_0^{t-r} \frac{(t - \tau - \tau')^{m-2}}{(m-2)!} \sum_{j=1}^{m} \frac{a_j^{m-1}}{\prod_{i=1}^{m} (a_j - a_i)} e^{\tau a_j P(\partial_x)} f(x, \tau) d\tau' d\tau
\]

\[+ \sum_{k=1}^{m} b_k P(\partial_x)^{m-k} \sum_{r=0}^{k-1} \frac{\partial^{k-1-r}}{\partial t^{k-1-r}} \int_0^t \frac{(t - \tau)^{m-2}}{(m-2)!} \frac{\prod_{i=1}^{m} (a_j - a_i)}{e^{\tau a_j P(\partial_x)} \varphi_r(x) d\tau}
\]

(10)

Theorem 2. Let \( a_1, a_2, \ldots, a_m \) be arbitrary real or complex roots different from each other, satisfy \( \sum_{k=0}^{m} b_{2k} x^{2k} = \prod_{i=1}^{m} (x^2 - a_i^2) \), and \( P(\partial_x, \partial_t) \) be a partial differential operators defined by

\[
P(\partial_x, \partial_t) = \sum_{k=0}^{m} b_{2k} P(\partial_x)^{m-k} \frac{\partial^{2k}}{\partial t^{2k}}, \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}^1, \ 1 < m \in \mathbb{N}.
\]

Where \( P(\partial_x) \) be a partial differential operator of any order, then \( \forall f(x,t) \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_t^1) \), \( \varphi_r(x) \in C^\infty(\mathbb{R}^n) \), we have

\[
\begin{align*}
\begin{cases}
P(\partial_x, \partial_t)u = f(x,t), & x \in \mathbb{R}^n, \ t \in \mathbb{R}^1. \\
\frac{\partial^r u}{\partial t^r} \bigg|_{t=0} = \varphi_r(x), & r = 0, 1, 2, \ldots, 2m - 1.
\end{cases}
\end{align*}
\]

(11)

\[
u(x,t) = \int_0^t \int_0^{t-r} \frac{(t - \tau - \tau')^{2m-3}}{(2m-3)!} \sum_{j=1}^{m} \frac{a_j^{2m-2}}{\prod_{i=1}^{m} (a_j^2 - a_i^2)} \sinh(\tau a_j P(\partial_x)^{1/2}) \frac{f(x, \tau)}{a_j P(\partial_x)^{1/2}} d\tau' d\tau
\]

\[+ \sum_{k=1}^{m} b_{2k} P(\partial_x)^{m-k} \sum_{r=0}^{k} \frac{\partial^{2k-1-r}}{\partial t^{2k-1-r}} \int_0^t \frac{(t - \tau)^{2m-3}}{(2m-3)!} \frac{\prod_{i=1}^{m} (a_j^2 - a_i^2)}{\sinh(\tau a_j P(\partial_x)^{1/2})} \frac{f(x, \tau)}{a_j P(\partial_x)^{1/2}} d\tau' d\tau
\]

\[\times \sum_{j=1}^{m} \frac{a_j^{2m-2}}{\prod_{i=1}^{m} (a_j^2 - a_i^2)} \sinh(\tau a_j P(\partial_x)^{1/2}) \frac{f(x, \tau)}{a_j P(\partial_x)^{1/2}} \varphi_r(x) d\tau.
\]

(12)
2. Proof of theorems

According to the Theorem BI1 and Theorem BI2, we just need to prove the following corollary of Theorem 1 and Theorem 2:

**Corollary 1.** Let \( a_1, a_2, \ldots, a_m \) be arbitrary real or complex roots different from each other for \( b_0 + b_1x + b_2x^2 + \cdots + b_mx^m = 0 \), and \( P(\partial_x, \partial_t) \) be a partial differential operators defined by

\[
P(\partial_x, \partial_t) = \sum_{k=0}^{m} b_k P(\partial_x)^{m-k} \frac{\partial^k}{\partial t^k}, \quad x \in \mathbb{R}^n, \; t \in \mathbb{R}^1, \; 1 < m \in \mathbb{N}.
\]

Where \( P(\partial_x) \) be a partial differential operator of any order, then for \( \forall \varphi_r(x) \in C^\infty(\mathbb{R}^n) \), we have

\[
\left\{
\begin{array}{l}
P(\partial_x, \partial_t)u = 0, \\
\frac{\partial^r u}{\partial t^r} \bigg|_{t=0} = \varphi_r(x), \\
\end{array}
\right.
\]

(13)

\[
u(x, t) = \sum_{k=1}^{m} b_k P(\partial_x)^{m-k} \sum_{r=0}^{k-1} \frac{\partial^{k-1-r}}{\partial t^{k-1-r}} \int_{0}^{t} \left( \frac{t - \tau}{(m-2)!} \sum_{j=1}^{m} a_j^{-1} e^{\tau a_j P(\partial_x)} \varphi_r(x) d\tau \right).
\]

(14)

**Proof.** Considering initial conditions, the Laplace transform of the Eq (13) with respect to \( t \) is

\[
\sum_{k=0}^{m} b_k P(\partial_x)^{m-k} (s^k U(x, s) - \sum_{r=0}^{k-1} s^{k-1-r} \varphi_r(x)) = 0,
\]

(15)

\[
\prod_{i=1}^{m} (s - a_i P(\partial_x)) U(x, s) - \sum_{k=1}^{m} b_k P(\partial_x)^{m-k} \sum_{r=0}^{k-1} s^{k-1-r} \varphi_r(x) = 0.
\]

Where \( U(x, s) = Lu(x, t) \). Let \( G_m(\partial_x, t) = L^{-1}[\prod_{i=1}^{m} (s - a_i P(\partial_x))], \) by solving \( U(x, s) \), we have its inverse Laplace transform:

\[
u(x, t) = L^{-1} U(x, s) = \sum_{k=1}^{m} b_k P(\partial_x)^{m-k} \sum_{r=0}^{k-1} L^{-1} \frac{a_j^{-1}}{\prod_{i=1}^{m} (s - a_i P(\partial_x))} \varphi_r(x)
\]

(15)

Now let us solve \( G_m(\partial_x, t) \). Considering initial conditions, the Laplace transform of the Eq (11) with respect to \( t \) is

\[
\prod_{i=1}^{m} (s - a_i P(\partial_x)) U(x, s) = F(x, s), \quad F(x, s) = Lf(x, t).
\]
By solving $U(x, s)$ and using the convolution theorem, we have its inverse Laplace transform:

$$u(x, t) = L^{-1}U(x, s) = L^{-1}\left(\frac{1}{\prod_{i=1}^{m}(s - a_iP(\partial_x))}F(x, s)\right) = G_m(\partial_x, t) * f(x, t).$$

By comparing (2) with $u(x, t) = G_m(\partial_x, t) * f(x, t)$, we have the expression of the abstract operator $G_m(\partial_x, t)$:

$$G_m(\partial_x, t) = \int_0^t \frac{(t - \tau)^{m-2}}{(m-2)!} \sum_{j=1}^{m} \frac{a_j^{m-1}}{\prod_{i\neq j}(a_j - a_i)} e^{\tau a_j P(\partial_x)} d\tau. \quad (16)$$

Applying (16) to (15), thus the Corollary 1 is proved.

**Corollary 2.** Let $a_1, a_2, \ldots, a_m$ be arbitrary real or complex roots different from each other, which satisfy $\sum_{k=0}^{m} b_{2k} x^{2k} = \prod_{i=1}^{m} (x^2 - a_i^2)$, and $P(\partial_x, \partial_t)$ be a partial differential operator defined by

$$P(\partial_x, \partial_t) = \sum_{k=0}^{m} b_{2k} P(\partial_x)^{m-k} \frac{\partial^{2k}}{\partial t^{2k}}, \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}, \ 1 < m \in \mathbb{N}.$$ 

Where $P(\partial_x)$ is a partial differential operator of any order, then $\forall \varphi_r(x) \in C^\infty(\mathbb{R}^n)$, we have

$$\left\{ \begin{array}{l} P(\partial_x, \partial_t)u = 0, \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}. \\ \frac{\partial^r u}{\partial t^r} \bigg|_{t=0} = \varphi_r(x), \quad r = 0, 1, 2, \ldots, 2m - 1. \end{array} \right. \quad (17)$$

$$u(x, t) = \sum_{k=0}^{m} b_{2k} P(\partial_x)^{m-k} \sum_{r=0}^{2k-1} \frac{\partial^{2k-1-r}}{\partial t^{2k-1-r}} \int_0^t \frac{(t - \tau)^{2m-3}}{(2m-3)!} \int_0^{\tau} \frac{e^{\tau a_j P(\partial_x)}}{a_j P(\partial_x)^{1/2}} \varphi_r(x) d\tau. \quad (18)$$

**Proof.** Considering initial conditions, the Laplace transform of the Eq (17) with respect to $t$ is

$$\sum_{k=0}^{m} b_{2k} P(\partial_x)^{m-k} s^{2k} U(x, s) - \sum_{r=0}^{2k-1} s^{2k-1-r} \varphi_r(x) = 0,$$

$$\prod_{i=1}^{m} (s^2 - a_i^2 P(\partial_x)) U(x, s) - \sum_{k=1}^{m} b_{2k} P(\partial_x)^{m-k} \sum_{r=0}^{2k-1} s^{2k-1-r} \varphi_r(x) = 0.$$
Where $U(x, s) = Lu(x,t)$. Let $G_m(\partial_x, t) = L^{-1}[1/ \prod_{i=1}^{m}(s^2 - a_i^2P(\partial_x))]$, by solving $U(x, s)$, we have its inverse Laplace transform:

$$u(x, t) = L^{-1}U(x, s) = \sum_{k=1}^{m} b_{2k} P(\partial_x)^{m-k} \sum_{r=0}^{2k-1} L^{-1} \frac{s^{2k-1-r}}{\prod_{i=1}^{m}(s^2 - a_i^2P(\partial_x))} \varphi_r(x)$$

$$= \sum_{k=1}^{m} b_{2k} P(\partial_x)^{m-k} \sum_{r=0}^{2k-1} \frac{\partial^{2k-1-r}}{\partial t^{2k-1-r}} G_m(\partial_x, t)\varphi_r(x). \tag{19}$$

Now let us solve the $G_m(\partial_x, t)$. Considering initial conditions, the Laplace transform of the Eq (3) with respect to $U$ is

$$\prod_{i=1}^{m}(s^2 - a_i^2P(\partial_x))U(x, s) = F(x, s), \quad F(x, s) = Lf(x, t).$$

By solving $U(x, s)$ and using the convolution theorem, we have its inverse Laplace transform:

$$u(x, t) = L^{-1}U(x, s) = L^{-1} \frac{1}{\prod_{i=1}^{m}(s^2 - a_i^2P(\partial_x))} F(x, s) = G_m(\partial_x, t) * f(x, t).$$

By comparing (14) with $u(x, t) = G_m(\partial_x, t) * f(x, t)$, we have the expression of the abstract operator $G_m(\partial_x, t)$:

$$G_m(\partial_x, t) = \int_{0}^{t} \frac{(t-\tau)^{2m-3}}{(2m-3)!} \sum_{j=1}^{m} a_j^{2m-2} \sinh(\tau a_j P(\partial_x)^{1/2}) \prod_{i\neq j}(a_j^2 - a_i^2) \left[ \frac{a_j}{a_j P(\partial_x)^{1/2}} \right] d\tau. \tag{20}$$

Applying (20) to (19), thus the Corollary 2 is proved.

3. Examples

**Theorem 3.** Let $a_1, a_2, \ldots, a_m$ be arbitrary real roots different from each other, which satisfy $\sum_{k=0}^{m} b_{2k}x^{2k} = \prod_{i=1}^{m}(x^2 - a_i^2)$, and $P(\partial_x, \partial_t)$ be a partial differential operator defined by

$$P(\partial_x, \partial_t) = \sum_{k=0}^{m} b_{2k} \Delta_{n}^{m-k} \frac{\partial^{2k}}{\partial t^{2k}} \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^1, \quad 1 < m \in \mathbb{N}.$$ 

Where $\Delta_n$ is an $n$-dimensional Laplacian, and $n-2 = 2\nu + 1$, $\nu \in \mathbb{N}$, then $\forall f(x,t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}_t^1)$, $\varphi_r(x) \in C^\infty(\mathbb{R}^n)$, we have

$$\begin{cases}
P(\partial_x, \partial_t)u = f(x,t), & x \in \mathbb{R}^n, \quad t \in \mathbb{R}^1. \\
\frac{\partial^r u}{\partial t^r} \bigg|_{t=0} = \varphi_r(x), & r = 0, 1, 2, \ldots, 2m-1. \tag{21}
\end{cases}$$
u(x, t) = \int_0^t \int_0^{t-\tau} \frac{(t-\tau)^{2m-3}}{(2m-3)!} \sum_{j=1}^{m} a_j^{2m-2} \prod_{i \neq j} (a_j^2 - a_i^2) \\
\times \left[ \prod_{i=0}^{\nu} f(\xi', \tau) dS_{n,j} \right] \\
+ \sum_{k=1}^{m} b_{2k} P(\partial_x)^{m-k} \sum_{r=0}^{2k-1} \left( \frac{\partial^{2k-1-r}}{\partial t^{2k-1-r}} \right) \int_0^t (t-\tau)^{2m-3} \prod_{i \neq j} (a_j^2 - a_i^2) \\
\times \left[ \prod_{i=0}^{\nu} \varphi_r(\xi) dS_{n,j} \right] \\
+ \sum_{l=0}^{\nu-1} \frac{a_j^{2l+1}}{(2l+1)!} \Delta_n^l f(x, \tau) \] (22)

Where $S_{n,j} = 2(2\pi)^{\nu+1}(a_j \tau')^{n-1}$, $S_{n,j} = 2(2\pi)^{\nu+1}(a_j \tau)^{n-1}$, and $\xi' \in \mathbb{R}_n$ is the integral variable. The integral is on the hypersphere $(\xi'-x_1)^2 + (\xi'-x_2)^2 + \cdots + (\xi'-x_n)^2 = (a_j \tau')^2$, and $dS_{n,j}$ is its surface element. $\xi \in \mathbb{R}_n$ is the integral variable on the hypersphere $(\xi_1-x_1)^2 + (\xi_2-x_2)^2 + \cdots + (\xi_n-x_n)^2 = (a_j \tau)^2$, and $dS_{n,j}$ is its surface element.

**Proof.** According to (30) in reference [4], we have:

$$\frac{\sinh(\frac{t a_j \Delta_n^{1/2}}{a_j \Delta_n^{1/2}}) f(x)}{a_j \Delta_n^{1/2}} = t \int_0^t \int_0^{t-\tau} \frac{(a_j^2 \Delta_n)^{\nu}}{S_{n,j}} \varphi_r(\xi) dS_{n,j} dt$$

$$+ \sum_{l=0}^{\nu-1} \frac{a_j^{2l+1}}{(2l+1)!} \Delta_n^l f(x), \quad \nu = \frac{n-3}{2}. \] (23)

Where $\Delta_n$ is an n-dimensional Laplacian, $n-2 = 2\nu + 1$, $S_{n,j} = 2(2\pi)^{\nu+1}(a_j t)^{n-1}$. $\xi \in \mathbb{R}_n$ is the integral variable on the hypersphere $(\xi_1-x_1)^2 + (\xi_2-x_2)^2 + \cdots + (\xi_n-x_n)^2 = (a_j t)^2$, and $dS_{n,j}$ is its surface element.

In Theorem 2, let $P(\partial_x) = \Delta_n$, then Theorem 3 is proved by the substitution of (23).

Similarly, we can easily obtain explicit solutions of the Cauchy problem of more complex partial differential equations. For the initial-boundary value problem, the operator $P(\partial_x)$ must have the characteristic function related to boundary conditions, in order to expand the known function $f(x, \tau)$, $\varphi_r(x)$ in [10] or [12] by using the characteristic function of $P(\partial_x)$. Therefore, if the determination of the characteristic function can be ascribed to the Sturm-Liouville problem of given boundary conditions, then this initial-boundary value problem is solvable.

$P(\partial_x)$ in Theorem BI4, Theorem 1 and Theorem 2 can be variable-coefficient partial differential operators. For instance, if $P(\partial_x)$ is a self-adjoint operator defined in a Hilbert space,
then the abstract operator
$$\frac{\sinh(tP(\partial_x)^{1/2})}{P(\partial_x)^{1/2}} \quad \text{and} \quad \cosh(tP(\partial_x)^{1/2}) = \frac{\partial}{\partial t} \frac{\sinh(tP(\partial_x)^{1/2})}{P(\partial_x)^{1/2}}$$
can act on the Hilbert space, which also is a bounded operator in the Hilbert space. In this case, we can attach proper boundary conditions to the initial value problems in (7), (9) and (11). Therefore, the given function \( f(x, t), \varphi_r(x) \) becomes a function with boundary conditions, and can be expressed in a Hilbert space within the given domain. In order to solve the corresponding initial-boundary value problem, we need to solve the characteristic value problem of \( P(\partial_x) \) under given boundary conditions to determine a set of orthogonal functions, which generates a linear manifold of a Hilbert space, thus \( f(x, t), \varphi_r(x) \) can be expressed in the Hilbert space.

**References**

[1] G.Q. Bi, Applications of abstract operator in partial differential equation(i), *Pure and Applied Mathematics*. 13(1997), 7-14.

[2] G.Q. Bi, Applications of abstract operator in partial differential equation(ii), *Chinese Quarterly Journal of Mathematics*. 14(1999), 80-87.

[3] G.Q. Bi, Operator methods in high order partial differential equation, *Chinese Quarterly Journal of Mathematics*. 16(2001), 88-101.

[4] G.Q. Bi, Y.K. Bi, [arXiv:1008.3808](http://arxiv.org/abs/1008.3808)