ON SPECTRAL MEASURES FOR CERTAIN UNITARY REPRESENTATIONS OF R. THOMPSON’S GROUP F.

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Abstract. The Hilbert space $H$ of backward renormalisation of an anyonic quantum spin chain affords a unitary representation of Thompson’s group $F$ via local scale transformations. Given a vector in the canonical dense subspace of $H$ we show how to calculate the corresponding spectral measure for any element of $F$ and illustrate with some examples. Introducing the "essential part" of an element we show that the spectral measure of any vector in $H$ is, apart from possibly finitely many eigenvalues, absolutely continuous with respect to Lebesgue measure. The same considerations and results hold for the Brown-Thompson groups $F_n$ (for which $F = F_2$).

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1. Introduction.

Let $F$ and $T$ be the Thompson groups as usual. In [10] an action of $F$ was shown to arise from a functor from the category $\mathcal{F}$ whose objects are natural numbers and whose morphisms are planar binary forests, to another category $\mathcal{C}$. Forests decorated with cyclic permutations of their leaves give a category $\mathcal{T}$ for which functors from $\mathcal{T}$ give actions of $T$.

The representations studied in [10] came from functors $\Phi$ to a trivalent tensor category (planar algebra) $\mathcal{C}$ in the sense of [12], based on a specific "vacuum vector" $\Omega$ in the 1-box space of the tensor category. A Thompson group element $g$ is represented by a pair of binary planar trees, drawn in the plane with one tree upside down on

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top of the other as below for an element of $F$ that we will call $X$:

\[ X = \]

The standard dyadic intervals defined by the leaves of the bottom tree are sent by $g$ (in the only affine way possible) to the corresponding intervals for the top tree.

If $\pi$ is the unitary representation defined by the (suitably normalised) trivalent vertex in $\mathcal{C}$, the coefficient

\[ \langle \pi(g)\Omega, \Omega \rangle \]

is simply equal to the pair of trees of $g$ interpreted as a planar diagram (tangle) for $\mathcal{C}$! (Or more correctly the pair of trees as drawn is a multiple of a single vertical straight line, and that multiple is $\langle \pi(g)\Omega, \Omega \rangle$.)

2. Definitions.

An $n$-ary planar forest is the isotopy class of a disjoint union of trees, whose vertices are $n + 1$-valent, embedded in $\mathbb{R}^2$, all of whose roots lie on $(\mathbb{R}, 0)$ and all of whose leaves lie on $(\mathbb{R}, 1)$. The isotopies are supported in the strip $([\mathbb{R}, [0, 1])$. The $n$-ary planar forests form a category in the obvious way with objects being $\mathbb{N}$ whose elements are identified with isotopy classes of sets of points on a line and whose morphisms are the forests which can be composed by stacking a forest in $(\mathbb{R}, [0, 1])$ on top of another, lining up the leaves of the one on the bottom with the roots of the other by isotopy then rescaling the $y$ axis to return to a forest in $(\mathbb{R}, [0, 1])$. The structure is of course actually combinatorial but it is very useful to think of it in the way we have described.

We will call this category $\mathcal{F}_n$ and from now on we will assume $n \geq 2$.

**Definition 2.1.** Fix $m \in \mathbb{N}$. For each $i = 1, 2, \ldots, m$ let $f_i$ be the planar $n$-ary forest with $m$ roots and $m + n - 1$ leaves consisting of straight lines joining $(k, 0)$ to $(k, 1)$ for $1 \leq k \leq i - 1$ and $(k, 0)$ to $(k + n - 1, 1)$ for $i + 1 \leq k \leq m$, and a single $n$-ary tree with root $(i, 0)$, leaves $(i, 1), (i + 1, 1), \ldots, (i + n - 1, 1)$, thus: $m$ at the
Note that any element of $\mathcal{F}_n$ is in an essentially unique way a composition of morphisms $f_i$, the only relation being $f_j f_i = f_i f_{j+n+1}$ for $i < j - n + 1$. The set of morphisms from 1 to $k$ in $\mathcal{F}_n$ is the set of $n$-ary planar rooted trees $\Sigma_n$ and is a directed set with $s \leq t$ iff there is an $f \in \mathcal{F}_n$ with $t = fs$.

Given a functor $\Phi : \mathcal{F}_n \to \mathcal{C}$ to a category $\mathcal{C}$ whose objects are sets, we define the direct system $S_\Phi$ which associates to each $t \in \Sigma_n$, $t : 1 \to k$, the set $S_t := \Phi(\text{target}(t)) = \Phi(k)$. For each $s \leq t$ we need to give $\iota_s^t$. For this observe that there is an $f \in \mathcal{F}_n$ for which $t = fs$ so we define, for $\kappa \in \Phi(\text{target}(s))$,$$
\iota_s^t(\kappa) = \Phi(f) \circ \kappa$$which is an element of $\text{Mor}_\mathcal{C}(\Phi(\text{target}(s)), \Phi(\text{target}(t)))$ as required. The $\iota_s^t$ trivially satisfy the axioms of a direct system.

As a slight variation on this theme, given a functor $\Phi : \mathcal{F}_n \to \mathcal{C}$ to any category $\mathcal{C}$, and an object $\omega \in \mathcal{C}$, form the category $\mathcal{C}^\omega$ whose objects are the sets $\text{Mor}_\mathcal{C}(\omega, \text{obj})$ for every object $\text{obj}$ in $\mathcal{C}$, and whose morphisms are composition with those of $\mathcal{C}$. The definition of the functor $\Phi^\omega : \mathcal{F}_n \to \mathcal{C}^\omega$ is obvious. Thus the direct system $S_{\Phi^\omega}$ associates to each $t \in \Sigma_n$, $t : 1 \to k$, the set $\text{Mor}_\mathcal{C}(\omega, \Phi(k))$. Given $s \leq t$ let $f \in \mathcal{F}_n$ be such that $t = fs$. Then for $\kappa \in \text{Mor}_\mathcal{C}(\omega, \Phi(\text{target}(s)))$,$$\iota_s^t(\kappa) = \Phi(f) \circ \kappa$$which is an element of $\text{Mor}_\mathcal{C}(\omega, \Phi(\text{target}(t)))$.

As in [10] we consider the direct limit:
$$\lim_{\to} S_\Phi = \{(t, x) \text{ with } t \in \Sigma_n, x \in \Phi(\text{target}(t)) \} / \sim$$where $(t, x) \sim (s, y)$ iff there are $p, q \in \mathcal{F}_n$ with $pt = qs$ and $\Phi(p)(x) = \Phi(q)(y)$.

We use $\frac{t}{x}$ to denote the equivalence class of $(t, x)$ mod $\sim$.

The limit $\lim_{\to} S_\Phi$ will inherit structure from the category $\mathcal{C}$. For instance if the objects of $\mathcal{C}$ are Hilbert spaces and the morphisms are isometries then $\lim_{\to} S_\Phi$ will be a pre-Hilbert space which may be completed to a Hilbert space which we will also call the direct limit unless special care is required. We will denote this Hilbert space by $\mathcal{H}$. 

\[ \text{bottom right and m+n-1 at top right} \]
As was observed in [10], if we let $\Phi$ be the identity functor and choose $\omega = 1$, then the inductive limit consists of equivalence classes of pairs $\frac{t}{x}$ where $t \in T_n$ and $x \in \Phi(target(t)) = Mor(1, target(t))$. But $Mor(1, target(t))$ is nothing but $s \in T_n$ with $target(s) = target(t)$, i.e. trees with the same number of leaves as $t$. Thus the inductive limit is nothing but the Brown-Thompson group $F_n$ with group law

$$
\frac{r s}{s t} = \frac{r}{t}
$$

(see [3] for more information on the properties of these groups).

Moreover for any other functor $\Phi$, $\lim_{\to} S_{\Phi}$ carries a natural action of $F_n$ defined as follows:

$$
\frac{s}{t} \left( \frac{t}{x} \right) = \frac{s}{x}
$$

where $s, t \in T_n$ with $target(s) = target(t) = k$ and $x \in \Phi(k)$. A Brown-Thompson group element given as a pair of trees with $h$ leaves, and an element of $\lim_{\to} S_{\Phi}$ given as a pair (tree with $k$ leaves, element of $\Phi(k)$), may not be immediately composable by the above formula, but they can always be “stabilised” to be so within their equivalence classes.

The Brown-Thompson group action preserves the structure of $\lim_{\to} S_{\Phi}$ so for instance in the Hilbert space case the actions define unitary representations.

We end this section by introducing an alternative description of the Brown-Thompson groups $F_n$, which is a straightforward extension of the one in [1, Chapter 7] and [2] where Thompson’s groups $F$, $T$, and $V$ are considered. The $F_n$ thus become diagram groups in the sense of [5].

Let $h, k \in \mathbb{N}$, an $(h, k)$-strand diagram is a finite acyclic directed graph embedded in the unit square $[0, 1] \times [0, 1]$, with the following properties

1. the graph has $h$ univalent sources along the top of the square, and $k$ univalent sinks along the bottom of the square;
2. every other vertex is $n + 1$-valent, and is either a split or a merge (see Figure 2.1).

![Figure 2.1. A merge and a split:](image)

A reduction of an $(h, k)$-strand diagram is a move of type I and II shown in Figure 2.2.

![Figure 2.2. The three reduction moves.](image)
An \((h, k)\)-strand diagram is reduced if no more reductions are possible. Two \((h, k)\)-strand diagrams are equivalent if one can be obtained from the other by a sequence of reductions and inverse reductions. We observe that every reduced \((1, 1)\)-strand diagram is uniquely determined by the composition of a tree and an inverse tree (see [1, Theorem 7.1.6] for the case \(F = F_2\)). Thanks to the strand diagrams, we have the following description of the Brown-Thompson groups \(F_n\), cf. [2, Proposition 2.5] and [5].

**Theorem 2.1.** Reduced \((1, 1)\)-strand diagrams form a groupoid over the positive integers, the composition being concatenation followed by reduction, and the inverse being reflection about a horizontal line. Moreover, the isotropy group at the point 1 is isomorphic to the Brown-Thompson group \(F_n\).

Since the orientation of the edges is usually clear from the context, the arrows will be often suppressed.

As Belk and Matucci did for \(F = F_2\) in [2, Section 2.2], given a \((k, k)\)-strand diagram \(g\), we can close it, that is we may join the sinks to the sources, and obtain a graph \(g'\) which embeds into the annulus (see Figure 2.3). This graph is called the closure of \(g\).

**Figure 2.3.** The closure of a strand diagram. \(g = \text{ } \rightarrow \text{ } g' = \text{ } \)
A planar algebra is a collection of vector spaces \((P_n)\) whose elements can be combined in a multilinear way for every planar tangle - a planar tangle is a collection of disjoint “input” discs inside an “output” disc. The discs are connected by smooth non-intersecting curves known as strings. Elements of \(P_n\) “go into” an input disc with exactly \(n\) boundary points meeting the strings of the tangle. Once all the input discs have vectors assigned to them, the output vector is in \(P_m\) where \(m\) is the number of points on the output disc meeting the strings of the tangle. Planar tangles can be glued one into another to produce new planar tangles and the operations on vectors are compatible with gluing. A planar algebra with \(\dim(P_0) = 1\) has a “loop” parameter usually denoted \(d\) or \(\delta\) such that any closed string (not meeting a disc of the tangle) can always be removed provided a multiplicative factor of \(\delta\) is applied to the multilinear map. See [8] for details.

For instance tensors give a planar algebra where \(P_n = \otimes^n \mathbb{C}^k\) (the indices of the tensors go from 1 to \(k\)) and the planar tangle is interpreted as a scheme for contracting the indices of the tensors, with indices assigned to strings. The loop parameter here is \(k\).

If we take an \(n\)-ary planar forest \(f\) with \(p\) roots and \(q\) leaves and surround it by an annulus with roots attached to the inner circle and leaves to the outer one, and enlarge to discs all the vertices of the forest we have a planar tangle all but one of whose input discs have \(n + 1\) strings attached to their boundaries. (The edges of the forest are the strings of the planar tangle.) We then choose a vector \(R\) in the vector space \(P_{n+1}\) and insert it at all the vertices of the forest, obtaining a labelled tangle which produces a linear map \(\Phi_R(f)\) from \(P_p\) to \(P_q\) as below.

![Diagram of a planar tangle and a labelled tangle](image)

The axioms of a planar algebra are more than enough to show that the linear maps \(\Phi_R(f) : P_p \to P_q\) given by these tangles define a functor from \(n\)-ary forests to vector spaces and thus a linear representation of the Brown-Thompson group \(F_n\). There is in general no simpler description of the vector space on which the representation acts than as the direct limit of the \(P_n\) according to the directed set of rooted planar \(n\)-ary trees.

Nothing at all is required of the element \(R\) above. To make the linear representations unitary one requires more structure of the planar algebra. First each \(P_n\) is endowed with an antilinear involution \(*\) which is compatible with orientation reversing diffeomorphisms of tangles in the obvious way. We further require that the space
$P_0$ is one dimensional so there is a sesquilinear form $\langle \ , \ \rangle$ on each $P_n$ defined by

$$\langle X, Y \rangle = \begin{array}{c}
\end{array}$$

where we have suppressed the output disc of the tangle since it does not meet any strings. A planar algebra is called a *Hilbert planar algebra* if the resulting inner product is positive definite.

If $(P_n)$ is a Hilbert planar algebra and we choose an element $R \in P_{n+1}$ it is clear that the representation of the Brown-Thompson group $F_n$ which we have constructed using $R$ will be unitary if

$$\text{Equation 3.1.}$$

Once we have fixed $R$ with the unitarity condition as above we can suppress it, and the input and output discs, from the diagrams so that the above condition becomes simply:

$$\begin{array}{c}
\end{array}$$

There are many variants on how to make forests act on the $P_n$. For instance we can make the forest $f_{p,q}$ act from $P_{p+1}$ to $P_{q+1}$ by ignoring one boundary point and obtain another functor $\Pi_R$ as follows:

Forest $f : \begin{array}{c}
\end{array}$ \quad $\Pi_R(f) : P_4 \rightarrow P_7$

Both $\Phi_R$ and $\Pi_R$ give unitary representations $\phi_R$ and $\pi_R$ of $F_n$ respectively.

The functor $\Pi_R$ has the advantage that there is a canonical unit vector $\Omega \in P_2$ given by (loop parameter)$^{-1/2}$ times a string connecting the two boundary points of its disc. It is a simple exercise in the definitions to show then that the *coefficient*
\( \langle \pi_R(g)\Omega, \Omega \rangle \) is given by tying the bottom of the pair of trees defining \( g \in F_n \) to the top, blowing up the vertices and inserting \( R \) and \( R^* \) appropriately and evaluating the corresponding element of \( P_0 \). For instance if \( g \) is the \( X \) in the introduction we get:

**Figure 3.1.**

\[
\langle \pi_R(g)\Omega, \Omega \rangle = \frac{1}{\text{loop parameter}}
\]

It is these representations \( \pi_R \) that we will work with. The following theorem contains the only outcome of the above construction that we will use.

**Theorem 3.1.** Given a Hilbert planar algebra \( (P_n) \) and an element \( R \in P_{n+1} \) satisfying equation 3.1 there is a unitary representation \( \pi_R \) of the Brown-Thompson group \( \mathcal{F}_n \) on a Hilbert space \( \mathcal{H}_R \) together with a vector \( \Omega \in \mathcal{H}_R \) so that, if \( g = \frac{s}{t} \in \mathcal{F}_n \) is given by a pair of trees then the coefficient \( \langle \pi_R(g)\Omega, \Omega \rangle \) is equal to the evaluation in \( P_0 \) of the pair of trees interpreted in \( P_0 \) via insertions of \( R \) and \( R^* \) as an element of \( P_0 \) with the roots of \( s \) and \( t \) joined to each other as in figure 3.1.

Once \( R \) is fixed, the subscripts of \( \mathcal{H}_R \) etc. will be suppressed.

A word of warning: when calculating the moments \( \langle \pi(g^k)\Omega, \Omega \rangle \) we will use Guba-Sapir-Belk composition method, however one cannot replace the vertices of the diagrams until all the cancellations have been made since these cancellations do not hold in the planar algebra.

An interesting example is the following, if \( R \) is the (symmetric, self-adjoint) tensor with three indices taking values 1, 2 and 3:

\[
R_{i,j,k} = \begin{cases} 
\frac{1}{\sqrt{2}} & \text{if } i, j, k \text{ all distinct} \\
0 & \text{otherwise}
\end{cases}
\]

then \( R \) satisfies the unitarity condition and \( \langle \pi(g)\Omega, \Omega \rangle \) is the number of 3 edge-colourings of the planar graph given by the pair of trees with the top connected to the bottom.

The non-vanishing of \( \langle \pi(g)\Omega, \Omega \rangle \) for all \( g \in F \) is known to be equivalent to the four colour theorem!

4. **Some spectral measures.**

Although our general results on spectral measure hold in all cases we will be especially interested in this paper in the case where the planar algebra is that of a trivalent...
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Unfortunately this paper does not consider the unitary structure but the quantum $SO(3)$ category can easily be obtained from the Temperley-Lieb planar algebra whose unitarity goes back to [6]. Thus this planar algebra $(P_n)$ is a Hilbert planar algebra provided the loop parameter $d$ (to distinguish from the $\delta$ of the Temperley-Lieb) is in the set $\{(4\cos^2 \pi/k) - 1 \mid k = 6, 7, 8, 9, \ldots \} \cup [3, \infty)$.

Thus we will fix the trivalent vertex in this category as the element $R \in P_3$ which is a self-adjoint and rotationally invariant element of $P_3$ and consider the representation $\pi$ of $F$ given by Theorem 3.1 with its privileged unit vector $\Omega$.

In general if $u$ is a unitary operating on a Hilbert space with a vector $\psi$, the spectral theorem says there is a Radon measure $\mu_\psi$ on the circle $\mathbb{T}$ such that, for any continuous function $f : \mathbb{T} \to \mathbb{C}$,

$$\langle f(u)\psi, \psi \rangle = \int_\mathbb{T} f(\theta)d\mu_\psi(\theta).$$

In this section we determine the spectral measures for three elements of the Thompson group $F = F_2$ given by $\Omega$.

We recall some relations from [12]:

**Formula 4.1.**

$$\bigcirc = 0, \quad \bigtriangledown - \bigtriangledown = \frac{1}{d-1} \left( \bigtriangledown - \bigtriangledown \right), \quad \bigcirc = \bigotimes \quad \text{and} \quad \bigtriangleup = t \bigotimes,$$

where $t = (d - 2)(d - 1)^{-1}$, $d = \delta^2 - 1$ and $\delta = 2 \cos \pi/n$ for $n = 6, 7, 8, 9, \ldots$

In our examples, we will find a suitable representation of the element in $F$, calculate the moments, and finally reconstruct the measure.

**Example 4.1.** We let $A$ be the usual generator of $F$ given by the pair of trees picture

$$A = \bigotimes \bigotimes.$$

Using the Guba-Sapir-Belk multiplication or otherwise it is easy to see that $A^n$ is given by the pair of trees picture

$$A^n = \bigotimes \bigotimes.$$ 

So by the last formula of 4.1 we see that

$$\langle \pi(A)^n\Omega, \Omega \rangle = t^n.$$
Now let $\mu$ be a measure on $\mathbb{T}$ with moments $\mu_n$, i.e. $\int_{\mathbb{T}} e^{in\theta} d\mu = \mu_n$. Then if $\sum_{n=\infty}^{\infty} \mu_n e^{in\theta}$ converges to the smooth function $f$ then $\int_{\mathbb{T}} f(\theta) e^{-in\theta} d\mu = \mu_{-n}$ and since a Radon measure is determined by its values on the $e^{in\theta}$, $\mu$ is absolutely continuous w.r.t. Lebesgue measure and $\int_{\mathbb{T}} e^{in\theta} d\mu = \int_{0}^{\frac{2\pi}{2}} f(\theta) e^{in\theta} d\theta / 2\pi$.

Thus we see that the spectral measure of $\pi(A)$ is absolutely continuous and its density is given by the Poisson kernel

$$\sum_{n \in \mathbb{Z}} t^{|n|} e^{in\theta} = \frac{1 - t^2}{1 - 2t \cos \theta + t^2}.$$  

**Example 4.2.** Consider the following element of $F$

$$N = \begin{array}{c}
\end{array}$$

By applying an inverse move of type II, we may express $N$ as

$$N = \begin{array}{c}
\end{array}$$

In view of Theorem 2.1, this representation of $N$ is particularly convenient in calculating its powers because the consecutive occurrences of $S$ and $S^{-1}$ cancel out
We observe that the last diagram is reduced. 
We are now in a position to find the moments of $N$ in our representations. For $k \geq 1$ we have

$$\langle \pi(N)^k \Omega, \Omega \rangle \Omega = t^2 \left\{ \begin{array}{c} k + 1 \\
\vdots \end{array} \right\} = t^2 \langle \pi(A)^{-k-1} \Omega, \Omega \rangle \Omega ,$$

We already know the moments of the generator $A$ so in some sense we are done. But we would like to reinterpret the calculation of the moments of $A$ in a way that will work in more generality.

If we crop the above diagram under the red dashed line we get the following elements in the planar algebra

$$v = \bigwedge , \quad \tilde{E}_N^{k+1}v = \bigwedge .$$

In the planar algebra, we may see $\tilde{E}_N = \bigwedge$ acting as an endomorphism of the 1-dimensional vector space spanned by $v$, the action being $\tilde{E}_A v = tv$. Thus we can
interpret $\langle \pi(A)^{-k-1}\Omega, \Omega \rangle$ as the inner product

$$\langle \tilde{E}_k^{k+1}v, v \rangle = \langle t^{k+1}v, v \rangle = t^{k+1} \langle v, v \rangle = t^{k+1}$$

Therefore, we have $\langle \pi(N)^k\Omega, \Omega \rangle = t^{|k|+3}$ for all $k \in \mathbb{Z} \setminus \{0\}$ and $\langle \pi(N)^0\Omega, \Omega \rangle = 1$.

Then, the spectral measure of $g$ is $d\mu = fd\theta/(2\pi)$, where $f$ is the function

$$f(\theta) = \sum_{n=-\infty}^{\infty} \mu_n e^{in\theta} = \frac{2t^3 - 2t^4\cos \theta}{1 - 2t\cos \theta + t^2} - 2t^3 + 1$$

Example 4.3. We now consider the element $X \in F$ shown in [1], which we may see as the composition of three strand diagrams

By using this representations of $X$, its powers assume the following form

If we crop the figure under the red dashed line, we get these two elements of the planar algebra:

$$\xi = \\quad \eta = \left\{ \begin{array}{c} k \\ \end{array} \right.$$
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$\eta = \tilde{E}_X^k \xi$. Let us diagonalise the linear map $\tilde{E}_X$ and find a formula for all the moments of $X$. With respect to this basis, the map $\tilde{E}_X$ is represented by the following matrix

$$\tilde{E}_X = \begin{pmatrix} t & 0 & 1 \\ 0 & 1 & (d-1)^{-1} \\ 0 & 0 & (1-d)^{-1} \end{pmatrix}.$$ 

which has eigenvalues and eigenvectors

$$\lambda_1 = t, \quad v_1 = w_1$$

$$\lambda_2 = 1, \quad v_2 = w_2$$

$$\lambda_3 = (1-d)^{-1}, \quad w_3 = -v_1 - v_2/d + v_3$$

With respect to the basis $\{w_1, w_2, w_3\}$, the linear map $\tilde{E}_X$ is represented by the diagonal matrix $\text{diag}(t, 1, (1-d)^{-1})$. Simple computations show that $\xi = tw_1 + d^{-1}w_2 + (1-d)^{-1}w_3$ and thus, for $k \geq 1$, we have $\eta = \tilde{E}_X^k \xi = t^{k+1}w_1 + d^{-1}w_2 + (1-d)^{-k-1}w_3$.

As in the previous example, we interpret (2) as $\langle \pi(X)^k \Omega, \Omega \rangle = (\tilde{E}_X^k \xi, \xi)$. Thus the moments are

$$\langle \pi(X)^k \Omega, \Omega \rangle = t^{k+1} \frac{1}{d} - \frac{1}{(1-d)^{k+1}} \frac{1}{d(1-d)^{k+1}}$$

Now there are two cases to handle depending on whether $d = 2$ (which entails that $t = 0$) or not. In the former case we have $\langle \pi(X)^k \Omega, \Omega \rangle = (1 + (-1)^k)/2$. This means that the measure is actually $\mu = (\delta_t + \delta_{-t})/d$, where $\delta_a$ denotes the Dirac measure with support $\{a\}$. In the latter case the spectral measure is given by the sum of $\delta_t/2$ and $fd\theta/(2\pi)$, where

$$f(\theta) = \frac{2t^2 - 2t^2 \cos \theta}{1 + t^2 - 2t \cos \theta} = \frac{(t + d^{-1})(2 - 2d - 2 \cos \theta)}{(1-d)^2 - 2(1-d) \cos \theta + 1} + \frac{2td^2 - 2td + 1 - d^2 + 2d}{d(1-d)}$$

The aim of the next sections is to generalize the method used in these examples to any element of the Thompson group and to a broader class of representations.

5. THE ESSENTIAL PART OF AN ELEMENT OF $F^n$.

Motivated by the calculations of the previous section we will give a more or less canonical decomposition of an element $g \in F^n$ which will allow us to control the “width” of $g^k$ for large $k$.

As in [2] Section 3 we will work with the groupoid of equivalence classes of $(h, k)$-strand diagrams. Given an $(h, k)$-strand diagram $f$ we denote by $[f]$ the corresponding element of the groupoid.
Theorem 5.1. Every $g$ in $F_n$ can be written in the form

\[ g = S^i E_g S^{-1} \]

where $S$ is a $(1,m)$-strand diagram, $E_g$ is a reduced $(m,m)$-strand diagram such that if we concatenate two copies of $E_g$ the resulting diagram is already reduced.

The Example 4.2 shows that, in general, $S$ does not need to be a tree, but just a strand diagram. As in [2, Section 4] one can define three types of cycles in annular strand diagrams, that is free loops (directed cycle with no vertices), split loops (a directed cycle with splits, but no merges), and merge loops (a directed cycle with merges, but no splits).

Before proving the previous theorem, we state a couple of results whose proofs are straightforward extensions of those in [2].

Proposition 5.1. (cf. [2, Proposition 3.2, p. 249]) Let $f$ be a $(p,p)$-diagram, let $f'$ be its closure, and let $g'$ be a closed diagram obtained by applying a reduction to $f'$. Then there exist a $(q,q)$-diagram $g$ whose closure is $g'$, and an $(p,q)$-strand diagram $h$ such that $[f] = [h][g][h]^{-1}$.

Proposition 5.2. (cf. [2, Proposition 4.1, p. 252]) Let $f$ be any reduced closed strand diagram. Then

1. Every component of $f$ has at least one directed cycle.
2. Every directed cycle in $f$ is either a free loop, a split loop, or a merge loop.
3. Any two directed cycles in $f$ are disjoint, and no directed cycle intersects itself.

Proof of Theorem 5.1. Let $g'$ be the closure of $g \in F_n$. There exists a finite number of reduction moves of type I, II, and III, say $k$, that yield a reduced annular strand diagram. By Proposition 5.1 there is a family of elements of the groupoid $g_0 = g, g_1, \ldots, g_k$ and $h_1, \ldots, h_k$ such that, for all $i = 0, \ldots, k - 1$, we have $[g_i] = [h_{i+1}] [g_{i+1}] [h_{i+1}]^{-1}$, the closure of $g_{i+1}$ is obtained from the closure of $g_i$ by applying a reduction move, the closure of $g_k$ is the reduced annular strand diagram corresponding to $g'$.

Therefore, we have

\[ [g] = [h_1][g_1][h_1]^{-1} = [h_1][h_2][g_2][h_2]^{-1}[h_1]^{-1} = \ldots = [h_1] \cdots [h_k][g_k][h_k]^{-1} \cdots [h_1]^{-1} \]
We set $[S] := [h_1] \cdots [h_k]$ and $[E_g] := [g_k]$.

That no cancellation can occur taking powers of $E_g$ is clear. Take the universal cover $\pi : \tilde{A} \to A$ of an annulus $A$ containing $E_g$. Then $\pi^{-1}(E_g)$ consists of an infinitely long chain of copies of $E_g$. If there were any cancellation between vertices in $E^p_g$, $\pi$ would send them to a cancellation (perhaps wrapping round $A$ several times) in $\tilde{E}_g$ inside $A$. But $E_g$ is supposed to be reduced in the annulus. □

The following lemma is interesting in itself and provides the last tool we need for the proof of the main result of this section.

**Lemma 5.1.** Let $A$ and $B$ be composable reduced strand diagrams. There is a sequence of cancellations in $AB$ between vertices in $A$ and vertices in $B$ so that the result $\tilde{A}B$ of performing these cancellations admits no more cancellations.

**Proof.** It suffices to take a sequence $S$ of cancellations between vertices in $A$ and $B$ that is maximal. By contradiction suppose that, after the cancellations in $S$ have been performed, there is a cancellation between vertices $v$ and $w$, which must be both in $A$ or both in $B$ by the maximality of $S$.

First suppose this cancellation is a move of type II. Wolog suppose $v$ is the merge and that both $v$ and $w$ are in $A$. Then in $AB$ the edge emerging from $v$ must connect $v$ to a vertex $u$ of $A$ other than $w$, otherwise $A$ itself would admit cancellations. ($v$ can never be connected to a vertex of $B$ as we will see.) But since $u$ does not appear in $\tilde{A}B$, there is a cancellation in $S$ involving $u$. This cancellation $s$ must be with a vertex of $B$. So whether $s$ is of type I or II, after performing $s$ there is an edge between $v$ and a vertex in $B$ (possibly a sink). But no subsequent cancellations can result in an edge connecting $v$ to a vertex outside $B$. Thus the edge in $AB$ connecting $v$ to $w$ is impossible, a contradiction.

Now suppose the cancellation between $v$ and $w$ is type I and let $v$ be the split. One of the edges emanating from $v$ must have been originally connected to some other vertex $u$ of $A$. Now argue as before. □

**Theorem 5.2.** Let $g, h, \tilde{h}$ be three elements of $F_n$. Then, there exists a $k \in \mathbb{N}$ and two strand diagrams $S_+, S_-$, such that for all $p \geq k$ the following diagram is reduced

\[
\begin{array}{c}
S_+ \\
\tilde{E}_g \\
\vdots \\
\tilde{E}_g \\
S_-
\end{array}
\]

where $\tilde{E}_g$ is a reduced $(m,m)$-strand diagram.
Proof. By using the representation of $g$ in Lemma 5.1, we see that the consecutive occurrences of $S^{-1}$ and $S$ cancel out for all $p \in \mathbb{N}$. Therefore, we get

\[
g^p = \text{ and } hg^p\tilde{h} = \]

Now let $\hat{S}$ and $\tilde{S}$ be the reduced strand diagrams associated with $hS$ and $S^{-1}\tilde{h}$, respectively. Consider the pairs $A_p := \hat{S}$, $B_p := (E_g)^p$ and apply Lemma 5.1 to this family of pairs. Since $A_p$ is always the same diagram (and has a finite number of vertices), for $p$ greater than a certain $k_1 \in \mathbb{N}$ there are no more cancellations. Therefore, we have $\hat{S}(E_g)^p = S'_+(E_g)^{p-k_1+1}$ for all $p \geq k_1$, where $S'_+$ is a reduced strand diagram. Similarly, if we consider the pairs $A_p := (E_g)^p$, $B_p := \tilde{S}$ we see that, for $p$ greater than a certain $k_2 \in \mathbb{N}$, there are no more cancellations and $(E_g)^p\tilde{S} = (E_g)^{p-k_2+1}S'_-$. At this stage some additional reductions might be possible. By construction these reductions can only occur between vertices of $S'_+$ and $S'_-$. We observe that if a merge in $S'_+$ is connected by a straight line to a split in $S'_-$, a reduction move of type II occurs and $n - 1$ new parallel edges appear in $E_g$. Similarly if a split in $S'_+$ is connected by $n$ parallel edges to a merge in $S'_-$, we may cancel $n - 1$ parallel edges of $E_g$ thanks to a move of type I. Once these reductions are performed, one easily gets the desired representation of $hg^p\tilde{h}$. \hfill $\square$

We observe that the natural number $k$ and the strand diagrams $S_{\pm}$, $E_g$ depend on both $h$ and $\tilde{h}$. Moreover, $E_g$ is equal to $E_g$ up to parallel edges. When $h = \tilde{h} = 1$, we call the element $E_g$ the essential part of $g$. 

6. Absolute continuity of the spectral measure.

We keep the notation of the previous sections. Given an element \( g \in F_n \), a vector \( \psi \in H \), there is a linear functional \( I_{g,\psi} \) which maps \( f \in C(T) \) to \( \langle f(\pi(g))\psi, \psi \rangle \). As already mentioned, this functional is determined by the so-called spectral measure, which we denote by \( \mu_{g,\psi} \), or simply by \( \mu_{\psi} \) if the element \( g \) is clear from the context.

The aim of this section is to classify the spectral measures which arise from a certain family of representations of \( F_n \).

**Theorem 6.1.** With the notations of the previous sections, suppose that \( \Phi(k) \) is a finite dimensional Hilbert space for all \( k \in \mathbb{N} \) and let \( \Omega \) be a vector in \( S_\iota \), that is \( \Omega \) is of the form \( (\iota,x) \), where \( \iota \) is the tree with one vertex and no edges and \( x \in \Phi(1) \). Let \( g \in F_n \) and \( \psi \in [\pi(F)\Omega]^\perp \), with \( \{g_i\}_{i \in I} \subset F_n \), \( \alpha_i \in \mathbb{C} \). Then, the spectral measure associated with \( g \) and \( \psi \) can be decomposed as \( \mu_{g,\psi} = \mu_1 + \mu_2 \), where \( \mu_1 \) is a measure which is absolutely continuous with respect to the Lebesgue measure and \( \mu_2 \) is a pure point measure with finite support.

First of all we are going to prove the theorem under the assumption that \( \psi = \pi(h)\Omega \). For the sake of clarity, our proof is in turn divided into a series of preliminary lemmas. Our plan is to find a formula for the moments of \( g \), which will allow us to reconstruct the measure thanks to the theory of distributions.

Let us set some notations. As in Section 4 we denote the moments by \( \mu_p := \langle \pi(g^p)\psi, \psi \rangle = \langle \pi(g^p h)\Omega, \pi(h)\Omega \rangle = \langle \pi(h^{-1}g^p h)\Omega, \Omega \rangle \)

Thanks to Theorem 5.2 the element \( h^{-1}g^p h \) can be expressed in terms of some diagrams \( \bar{E}_g \), which depend on \( g \) and \( h \). By the Jordan-Chevalley decomposition, the operator \( \Phi(\bar{E}_g) \) can be decomposed as \( \Phi(\bar{E}_g) = x_{ss} + x_n \), where \( x_{ss} \) is the semisimple part, \( x_n \) is the nilpotent part, \( x_r = 0 \) for some \( r \in \mathbb{N} \cup \{0\} \), and \( [x_n,x_{ss}] = 0 \). We denote by \( \{v_i\}_{i \in A} \) an orthonormal basis of eigenvectors for \( x_{ss} \) and by \( \{\lambda_i\}_{i \in A} \) the corresponding eigenvalues, that is \( x_{ss}v_i = \lambda_i v_i \). We observe that \( |\lambda_i| \leq 1 \) for all \( i \in A \) because \( \Phi(\bar{E}_g) \) may be expressed as the composition of isometries and co-isometries. By using Theorem 5.2 the elements \( h^{-1}g^p h \) have the following form

\[
(3) \quad h^{-1}g^p h = \left\{ \begin{array}{c}
S_+ \\
\vdots \\
\bar{E}_g \\
S_-
\end{array} \right\} \quad \text{with} \quad p - k + 1 \quad \forall \ p \geq k
\]

where \( S_\pm \) are some strand diagrams and \( k \) is a suitable natural number.

As done in the examples, we want to use the former diagram to compute the moments
of $g$. If we crop the diagram in correspondence of the red dashed line, we may express $\mu_p$ as $\langle \Phi(\tilde{E}_g)^p \rangle_{\xi, \eta}$, where $\xi = \Phi(S_-) x$ and $\eta = \Phi(S_+) x$. We have $\xi = \sum_{i \in A} \xi_i v_i$ and $\eta = \sum_{i \in A} \eta_i v_i$.

The following lemma has to do with the form of the moments.

**Lemma 6.1.** For $p \geq h$ we have that

$$
\mu_p = \sum_{q=0}^{r-1} \sum_{l \in A} \binom{p-h+1}{q} \xi_i \lambda_i^{p-h+1-q} \langle x_n^q v_i, \eta \rangle
$$

(4)

$$
= \sum_{q=0}^{\min \{r-1, p-h+1 \}} \sum_{l \in A} c_{l,q} \lambda_i^{p-h+1-q} \prod_{k=h-1}^{h+q-2} (p-k)
$$

where $c_{l,q} := \xi_i \langle x_n^q v_i, \eta \rangle / q!$ and $\binom{p}{q} = 0$ if $q > p$.

**Proof.** By using the Jordan-Chevalley decomposition of $\Phi(\tilde{E}_g)$ we have

$$
\mu_p = \langle (x_{ss} + x_n)^{p-h+1} \xi, \eta \rangle = \sum_{q=0}^{p-h+1} \binom{p-h+1}{q} \langle x_n^q x_{ss}^{p-h+1-q} \xi, \eta \rangle
$$

$$
= \sum_{q=0}^{r-1} \sum_{l \in A} \binom{p-h+1}{q} \xi_i \lambda_i^{p-h+1-q} \langle x_n^q v_i, \eta \rangle .
$$

\[\square\]

The linear functional $I_{g,\psi} : C(\mathbb{T}) \to \mathbb{C}$ is determined by its restriction to the Laurent polynomials in $e^{i\theta}$, here denoted by $\mathbb{C}[e^{i\theta}, e^{-i\theta}]$. In particular, we have $I_{g,\psi}(e^{ik\theta}) := \mu_k$ and $I_{g,\psi}(e^{-ik\theta}) := \mu_{-k} = \bar{\mu}_k$ for $k \geq 0$.

We may partition $A$ into two subsets

$$
A_1 = \{ l \in A \mid |\lambda_l| < 1 \} \quad A_2 = \{ l \in A \mid |\lambda_l| = 1 \} .
$$

and define three linear functionals on $C(\mathbb{T})$ by

$$
I_{g,1,\psi}(e^{i\theta}) := \begin{cases} 
\sum_{q=0}^{\min \{r-1, p-h+1 \}} \sum_{l \in A_1} c_{l,q} \lambda_i^{p-h+1-q} \prod_{k=h-1}^{h+q-2} (p-k) & \text{for } p \geq h \\
0 & \text{for } 0 \leq p < h
\end{cases}
$$

$$
I_{g,2,\psi}(e^{i\theta}) := \begin{cases} 
\sum_{q=0}^{\min \{r-1, p-h+1 \}} \sum_{l \in A_2} c_{l,q} \lambda_i^{p-h+1-q} \prod_{k=h-1}^{h+q-2} (p-k) & \text{for } p \geq h \\
0 & \text{for } 0 \leq p < h
\end{cases}
$$

$$
I_{g,3,\psi}(e^{i\theta}) := \begin{cases} 
0 & \text{for } p \geq h \\
\mu_p & \text{for } 0 \leq p < h
\end{cases}
$$

Accordingly, we have the decomposition $I_{g,\psi} = I_{g,1,\psi} + I_{g,2,\psi} + I_{g,3,\psi}.$

**Lemma 6.2.** The functional $I_{g,1,\psi} : C(\mathbb{T}) \to \mathbb{C}$ is of the form $I_{g,1,\psi}(f) = \int_0^{2\pi} f(\theta) \bar{f}_1(\theta) d\theta / (2\pi)$ for some $f_1 \in C(\mathbb{T})$. 
Proof. By formula (1) and the hypothesis \(|\lambda_i| < 1\) for all \(i \in A_1\), we have that the following series is absolutely convergent

\[
\sum_{p \in \mathbb{Z}} I_{g,1,\psi}(e^{ip\theta}) e^{ip\theta} = \sum_{p \geq h} \sum_{q=0}^{\min\{r-1,p-h+1\}} \sum_{l \in A_1} c_{l,q} \lambda_l^{p-h+1-q} \prod_{k=h-1}^{h+q-2} (p-k)
\]

\[
+ \sum_{p \geq h} \sum_{q=0}^{\min\{r-1,p-h+1\}} \sum_{l \in A_1} \bar{c}_{l,q} \lambda_l^{p-h+1-q} \prod_{k=h-1}^{h+q-2} (p-k)
\]

We denote by \(f_1(\theta)\) the corresponding function in \(C(\mathbb{T})\). It follows that the functional \(I_{g,1,\psi}\) is induced by the measure \(\tilde{f}_1 d\theta/(2\pi)\).

Lemma 6.3. It holds

\[
I_{g,2,\psi}(\cdot) = \sum_{q=0}^{r-1} \sum_{l \in A_2} c_{l,q} (-1)^q \delta_{\lambda_l} (e^{i(p-h+1)\theta}) = \sum_{q=0}^{\min\{r-1,p-h+1\}} \sum_{l \in A_2} c_{l,q} \lambda_l^{p-h+1-q} \prod_{k=h-1}^{h+q-2} (p-k) = I_{g,2,\psi}(e^{ip\theta}).
\]

We are at last in a position to prove the main result of this paper.

Proof of Theorem 6.1. By the previous discussion, it follows that \(I_{g,\psi}\) is a distribution of order strictly greater than 0, unless

\[
\sum_{q=0}^{r-1} \sum_{l \in A_2} c_{l,q} (-1)^q \delta_{\lambda_l} (e^{i(-h+1)\theta}) = 0
\]

Since a measure is a distribution of order 0, then we have proven the theorem under the hypothesis that \(\psi = \pi(h)\Omega\).

Suppose now that \(\psi\) is a generic vector in \([\pi(F)\Omega]^\neq\). The Hilbert space \(\mathcal{H}\) can be decomposed into \(\mathcal{H}_{ac} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{pp}\), where \(\mathcal{H}_{ac} = \{v \in \mathcal{H} \mid \mu_{g,v} \text{ is absolutely continuous}\}\), \(\mathcal{H}_{sc} = \{v \in \mathcal{H} \mid \mu_{g,v} \text{ is continuous singular}\}\), \(\mathcal{H}_{pp} = \{v \in \mathcal{H} \mid \mu_{g,v} \text{ is pure point}\}\). So far we have shown that there is a family of vectors in \(\mathcal{H}_{ac} \oplus \mathcal{H}_{pp}\) that span a dense subspace of \(\mathcal{H}\). This means that the spectral measure \(\mu_{g,\psi}\) cannot contain a component which is continuous singular.

All we have to do now is to prove that the support of \(\mu_2\) is finite. Thanks to von
Neumann’s ergodic theorem we know that, for any contraction $V$ on an Hilbert space and any $\lambda \in \mathbb{T}$, the sequence $\Sigma_N(\lambda^{-1}V) := \sum_{i=0}^{N} (\lambda^{-1}V)^i / N$ converges in the strong operator topology to the projection onto the eigenspace $\text{Ker}(V - \lambda)$. By Theorem 5.2 we may express $h_2^{-1}g^ph_1$ as $S_+ \tilde{E}(h_1, h_2)^p S_-$ for $p \geq k$, where $S_+, \tilde{E}(h_1, h_2)$, $S_-$ are suitable strand diagrams. Despite the fact that $\Phi(\tilde{E}(h_1, h_2))$ depends on $h_1, h_2 \in F_n$, its spectrum is only a function of $\pi(g)$. We claim that $\Sigma_N(\lambda^{-1}\pi(g))$ converges to 0 for every $\lambda$ not in $\text{sp}(\Phi(\tilde{E}(h_1, h_2))) \cap \mathbb{T}$. To this end it is enough to show that the limit is 0 in the weak topology and actually it suffices to show $\langle \Sigma_N(\lambda^{-1}\pi(g))\pi(h_1)\Omega, \pi(h_2)\Omega \rangle \rightarrow 0$ for every $h_1, h_2 \in F_n$. Simple computations lead to

$$
\langle \Sigma_N(\lambda^{-1}\pi(g))\pi(h_1)\Omega, \pi(h_2)\Omega \rangle = \left\langle \sum_{i=0}^{N} \left( \frac{\lambda^{-1}\pi(g)}{N} \right)^i \pi(h_1)\Omega, \pi(h_2)\Omega \right\rangle
$$

$$
= \left\langle \sum_{i=0}^{N} \left( \frac{\lambda^{-1}\left(\Phi(\tilde{E}(h_1, h_2))\right)}{N} \right)^i \xi_1, \xi_2 \right\rangle
$$

$$
= \langle \Sigma_N(\lambda^{-1}\Phi(\tilde{E}(h_1, h_2)))\xi_1, \xi_2 \rangle
$$

Since $\Phi(\tilde{E}(h_1, h_2))$ is an endomorphism of a finite dimensional vector space, it has finitely many eigenvalues and $\Sigma_N(\lambda^{-1}\Phi(\tilde{E}(h_1, h_2)))$ has non-trivial limit only for $\lambda \in \text{sp}(\Phi(\tilde{E}(h_1, h_2)))$. From this discussion it follows that the support of the pure point component is finite. \hfill \Box

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ON SPECTRAL MEASURES FOR CERTAIN UNITARY REPRESENTATIONS OF $F$

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