Parametrically excited stability of a periodically stiffened beam under axial periodic excitation

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Abstract. The parametrically excited stability of a periodically stiffened beam under general periodic axial excitation is studied and the effect of periodic stiffeners on the beam stability is considered for the first time. The partial differential equation of motion of the beam with periodic stiffeners under axial excitation is given. The Galerkin method is used to convert the partial differential equation into ordinary differential equations with periodic time-varying parameters. The direct eigenvalue analysis method based on the Fourier expansion and generalized eigenvalue analysis is applied to solve the parametrically excited stability problem of the stiffened beam. A simply supported beam with periodic stiffeners under periodic axial excitation is considered for numerical investigation. The parametrically excited stability of the stiffened beam and the effects of stiffeners and excitation on the stability are illustrated by numerical results on unstable regions.

1. Introduction

Many kinds of stiffened structures are widely used in engineering. As beams subjected axial dynamic loading, loading coupled with structural deformation will result in vibration equations with time-varying parameters, and the parameters are considered as parametric excitations. The dynamic stability, especially the parametrically excited stability of the stiffened structures has attracted much interest. For instance, the vibration characteristics and parametrically excited instability of stiffened plates and shells with different boundary conditions under various excitations have been studied [1-8]. The vibration characteristics of rotating beams and centrifugally stiffened beams have been analyzed [9-11]. The transverse vibration and stability problems of beams with step change in cross-section have been solved [12-13]. The dynamic buckling and instability can be due to the unstable parametrically excited vibration of beams under axial periodic excitation [14-15]. The Hill equations or Mathieu equations have been used to describe systems with periodically parametric excitation. By using the Floquet theory, the stability of single Mathieu equation representing one-degree-of-freedom system has been fully investigated [16]. The direct eigenvalue analysis method dealing with the parametrically excited stability has been developed based on the Floquet theory and harmonic balance method and applied to cable systems with multi-degree-of-freedom to obtain unstable regions [17-19]. Nevertheless, the instability researches on beams were mainly based on single vibration mode and axial excitation chosen to be simple harmonic excitation. Therefore, the parametrically excited stability of stiffened structures with multiple modes coupling under general periodic excitation is a new subject which will shed light on relevant practical engineering problems.
The present study is on the dynamic stability of a stiffened beam, which cross-section is enlarged periodically due to stiffeners. The parametrically excited stability of the periodically stiffened beam with coupled mode vibration under general periodic axial excitation and the effects of the periodic stiffeners on the beam stability will be analyzed. First, the differential equation of motion of the stiffened beam under axial excitation is derived based on the Euler-Bernoulli beam theory. The partial differential equation is converted into ordinary differential equations with periodic time-varying parameters by using the Galerkin method. The non-dimensional treatment is adopted for general consideration. Second, the parametrically excited stability problem of the beam with periodic stiffeners under axial periodic excitation, described by the differential equations with periodic parameters, is solved by using the direct eigenvalue analysis method. Finally, numerical results on unstable regions are given to show the parametrically excited instability of the stiffened beam and the effects of the periodic stiffeners and excitation on the stability.

2. Vibration equations and stability analysis of stiffened beam

As shown in Figure 1, a periodically stiffened beam under general periodic axial loading can be treated as a parametrically excited system. According to the Euler-Bernoulli beam theory, the differential equation of motion of the beam with periodic stiffeners under axial excitation can be expressed as

\[ \frac{\rho A}{H} \frac{\partial^2 w}{\partial t^2} + c_b \frac{\partial w}{\partial t} + \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} \right) + F_x(t) \frac{\partial^2 w}{\partial x^2} = 0 \]  

(1)

\[ \frac{\rho A}{H} \frac{\partial^2 v}{\partial t^2} + c_b \frac{\partial v}{\partial t} + \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 v}{\partial x^2} \right) + F_x(t) \frac{\partial^2 v}{\partial x^2} = 0 \]  

(2)

where \( w \) is the beam transverse displacement, \( \rho \) is the mass density, \( A' \) is the cross-sectional area of stiffened parts of the beam, \( A \) is the cross-sectional area of uniform parts of the beam, \( E \) is the elastic modulus, \( f \) and \( I \) are the second moments of areas of stiffened and uniform parts of the beam, respectively, \( c_b \) is the structural damping coefficient, \( F_x(t) \) is the general periodic (or harmonic) axial excitation, \( x \) is the axial coordinate, \( t \) is time, \( N_p \) is the number of stiffeners, \( d \) is the width of cross-section of the beam, \( H \) and \( h_0 \) are the heights of cross-sections of stiffened and uniform parts of the beam, respectively, and \( b \) is the axial length of a stiffener. Equations (1) and (2) can be rewritten in a non-dimensional form

\[ P(z) \frac{\partial^2 \nu}{\partial t^2} + \frac{c_b}{\rho L} \frac{\partial \nu}{\partial t} + \omega_0^2 \rho \left( \frac{P^3(z) \partial^4 \nu}{\partial z^4} \right) + f_x(t) \frac{\partial^2 \nu}{\partial z^2} = 0 \]  

(3)

where \( \nu = w/d \), \( z = x/L \), \( L \) is the beam length, and \( P(z) \) is a piecewise function. There is \( P(z) = 1 \) for uniform parts and \( P(z) = H/h_0 \) for stiffened parts. Introducing a non-dimensional time \( \tau = \omega_0 t \), Equation (3) can be rewritten as

\[ P(z) \frac{\partial^2 \nu}{\partial \tau^2} + \frac{c_d}{\rho L} \frac{\partial \nu}{\partial \tau} + \rho \left( \frac{P^3(z) \partial^4 \nu}{\partial z^4} \right) + f_x(t) \frac{\partial^2 \nu}{\partial z^2} = 0 \]  

(4)
\[ L = \frac{Eh^2}{12\rho L^3}, \quad c_d = \frac{c_b}{\rho dh_0L^2}, \quad f_X(t) = \frac{F_X(t)}{\rho dh_0L^2\omega_0^2} \]

The time-varying function \( f_X(t) \) is periodic due to the periodic excitation with \( T=2\pi/\omega \) and can be expanded into a Fourier series with coefficients \( A_k \) and \( \theta_k \)

\[ f_X(t) = A_0 + \sum_{k=1}^{m} A_k \sin(k\omega t + \theta_k) = A_0 + \sum_{k=1}^{m} A_k \sin(k\frac{\omega}{\omega_0} t + \theta_k) \]

The displacement \( v \) in Equation (4) can be expanded into

\[ v(z, \tau) = \sum_{j=1}^{n} q_j(\tau)\psi_j(z) \]

where \( q(\tau) \) is the time function, \( \psi_j(z) \) is the shape function, and \( n \) is an integer. For simply supported boundary condition, \( \psi_j(z)=\sin(j\pi z) \). Based on the Galerkin method, substituting Equation (7) into Equation (5), multiplying the equation with \( \sin(\omega \tau) \), they can be assembled into a matrix equation as

\[ M \frac{d^2Q}{dt^2} + C \frac{dQ}{dt} + K(\tau)Q = 0 \]

where \( Q=[q_1, q_2, \ldots, q_n]^T \) is a generalized displacement vector, \( M, C, K(\tau) \) are generalized mass, damping, and stiffness matrices which can be determined by Equation (4) with \( \psi_j(z) \). \( K(\tau) \) is a periodic functional matrix because of \( f_X(t) \). Equation (8) describes a parametrically excited system for the stiffened beam with coupled multiple modes vibration.

Based on the direct eigenvalue analysis method in the reference [18], the stiffness matrix \( K(\tau) \) can be expressed as

\[ K(\tau) = \frac{1}{2}K_{0\theta} + \sum_{j=1}^{m_k} (K_{0j} \sin j\omega \tau + K_{0j} \cos j\omega \tau) \]

The equation for state system \( Q \) and \( \dot{Q} \) can be derived from system Equation (8), and the corresponding coefficient matrix \( B(\tau) \) can be expressed as Fourier series by Equation (9). According to the Floquet theory, the generalized displacement \( Q \) is expressed as the product of periodic and exponential components. The periodic component can be expressed as Fourier series. After these series are substituted into the state equation, algebraic equations can be obtained by balancing each harmonic term, and the equations yield a matrix eigenvalue problem. By using the eigenvalues, the stability of the generalized displacement \( Q \) and the parametrically excited vibration \( v \) of the stiffened beam can be determined directly.

3. Numerical results on parametrically excited instability

A simply supported beam with two or three periodic stiffeners under general periodic axial excitation is considered. The constant part of \( f_X(\tau) \) is \( A_0=0.3f_{cr} \), where \( f_{cr} \) is the static critical compression force of buckling, and \( \theta_0=0 \). The first six non-dimensional natural frequencies of the beam without stiffeners are 1, 4, 9, 16, 25, and 36. The unstable regions on the plane of dimensionless excitation frequency (\( \omega_0 \)) and amplitude \( A_1 \) for the uniform and stiffened beams are shown in Figures 2-8, which consist of unstable points as shaded part. An unstable point indicates that the excitation with corresponding frequency and amplitude [Equation (6)] will induce the beam vibration unstable. The unstable region is mainly distributed around the twice natural frequencies of the beam. By comparing Figures 3, 4, 6 and 7 with Figure 2, it is obtained that the unstable region in high excitation frequencies has a certain reduction by using the periodic stiffeners.

By comparing Figures 3(a) with 3(b) and Figures 6(a) with 6(b), it is obtained that the unstable region is reduced by increasing the axial length of stiffeners. And by comparing Figures 6(a) with 3(a)
and Figures 6(b) with 3(b), it is obtained that the unstable region is reduced remarkably by increasing the number of stiffeners. Furthermore, the decrement of unstable region by increasing the number of stiffeners is more obvious than that by increasing the axial length of stiffeners, for example, by comparing Figures 3 and 6 with Figure 2. Thus, the parametrically excited stability of the beam under periodic axial excitation can be improved by increasing the axial length and number of periodic stiffeners, and increasing the number of periodic stiffeners is a more efficient way.

Figures 4 and 7 show the unstable regions on the plane of excitation frequency and amplitude for the beam with periodic stiffeners of equal axial lengths and different heights. By comparing Figure 7 with Figure 2, it is obtained that the unstable region around the twice \((3i+M)\)-order natural frequencies of the beam \((M=1, 2 \text{ and } i=0, 1, 2, \ldots)\) is reduced remarkably by increasing the height of periodic stiffeners. Thus, the parametrically excited stability of the beam under periodic axial excitation can be improved by increasing the height of periodic stiffeners.

Figures 5 and 8 show the unstable regions on the plane of excitation frequency and amplitude for the beam with two or three periodic stiffeners under multiple harmonic excitations [given in Equation (6)]. By comparing Figures 5(a) with 3(b) and Figures 8(a) with 6(b), it is obtained that the unstable region for periodic excitation with \(A_1\) is similar to that with \(A_1\) and \(A_2\), and some small parts of the unstable region are enlarged by excitation with \(A_2\). Thus, the main parts of the unstable region and then the parametrically excited stability of the beam with periodic stiffeners are determined based on the basic frequency of the periodic axial excitation.

**Figure 2.** Unstable region of uniform beam under periodic excitation with \(A_1\).

![Figure 2](image)

**Figure 3.** Unstable regions of stiffened beam with two stiffeners of height \(H/h_0=1.5\) under periodic excitation with \(A_1\).

![Figure 3](image)
Figure 4. Unstable regions of stiffened beam with two stiffeners of axial length $b=0.2$ under periodic excitation with $A_1$.

Figure 5. Unstable regions of stiffened beam with two stiffeners of axial length $b=0.2$ and height $H/h_0=1.5$ under periodic excitation with $A_1$ and/or $A_2$.

Figure 6. Unstable regions of stiffened beam with three stiffeners of height $H/h_0=1.5$ under periodic excitation with $A_1$. 
Figure 7. Unstable regions of stiffened beam with three stiffeners of axial length $b=0.2$ under periodic excitation with $A_1$. 

Figure 8. Unstable regions of stiffened beam with three stiffeners of axial length $b=0.2$ and height $H/h_0=1.5$ under periodic excitation with $A_1$ and/or $A_2$. 

4. Conclusions
The parametrically excited stability of a periodically stiffened beam in coupled multiple modes vibration under periodic axial excitation has been studied by using the newly proposed direct eigenvalue analysis method. The periodically enlarged cross-section by stiffeners for improving the beam stability has been considered. The partial differential equation of motion of the beam with periodic stiffeners under general periodic axial excitation has been derived based on the Euler-Bernoulli beam theory and converted into ordinary differential equations by using the Galerkin method. The parametrically excited stability of the stiffened beam under periodic excitation has been solved by using the direct eigenvalue analysis method based on the Fourier expansion and generalized eigenvalue analysis. It has been concluded that: (a) the unstable region of the uniform beam is distributed around the twice natural frequencies of the structure; (b) the unstable region is reduced by using the periodic stiffeners and the parametrically excited stability is improved by increasing the axial length, height and number of stiffeners; and (c) the basic frequency and amplitude of the periodic axial excitation determine the main distribution and parametrically excited stability of the stiffened beam.

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