The convergence rate of multivariate operators on simplex in Orlicz space

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Abstract
The approximation of functions in Orlicz space by multivariate operators on simplex is considered. The convergence rate is given by using modulus of smoothness.

Keywords: Stancu-Kantorović operator, Meyer-König-Zeller operator, Convergence rate, Orlicz space

1. Introduction

Let \( \Phi(u) \) be a N-function, \( \Psi \) be the complementary function of \( \Phi \). We will say that \( \Phi \) satisfies the \( \Delta_2 \)-condition if \( \Phi(2u) \leq c\Phi(u) \) for any \( u \geq u_0 \geq 0 \) with some constant \( c \) independent of \( u \).

For \( \Delta = \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 1, x_1, x_2 \geq 0 \} \), the Orlicz space \( L^\Phi_\triangle(\triangle) \) corresponding to the function \( \Phi \) consists of all Lebesgue-measurable functions \( f(x) \) on \( \triangle \) such that integral \( \int_\Delta f(x)g(x)dx \) is finite for any measurable functions \( g(x) \) with \( \int_\Delta \Psi(g(x))dx < \infty \).

It is well-known that the space \( L^\Phi_\triangle(\Delta) \) becomes a complete normed space with Orlicz norm

\[
\|f\|_\Phi = \|f\|_{\Phi, \triangle} = \sup \left\{ \left| \int_\Delta f(x)g(x)dx \right| : \int_\Delta \Psi(g(x))dx \leq 1 \right\}.
\]

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It can be proved that

\[ \| f \|_\Phi = \inf_{\alpha > 0} \frac{1}{\alpha} \left\{ 1 + \int_\Delta \Phi(\alpha f(x)) \, dx \right\}. \]

See [1] for the above. For \( f \in L^*_\Phi(\Delta) \), we first extend \( f(x) \) from \( \Delta \) to \( D = [0, 1] \times [0, 1] \) according to \( f(x) = f(1-x) \), and then extend \( f(x) \) to \( \mathbb{R}^2 \) with period 1. The nonnegative function

\[ \Omega^2_{R^2}(f, r)_\Phi = \sup \{ \omega^2_h(f, r)_\Phi : h = (h_1, h_2) \in \mathbb{R}^2, |h| = 1 \} \]

of the variable \( r \geq 0 \) will be called the 2-th order modulus of continuity of the function \( f \in L^*_\Phi(\Delta) \) in the Orlicz norm \( \Phi \). Here, \( |h| = \sqrt{h_1^2 + h_2^2} \), and

\[ \omega^2_h(f, r)_\Phi = \sup_{|t| \leq r} \| f(x + th) + f(x - th) - 2f(x) \|_\Phi \]

is the 2-th order modulus of continuity in the direction \( h \) of the function \( f \).

For any Lebesgue-measurable function \( f(x) \) on \( \Delta \), the functional

\[ K_n(f; x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \tilde{p}_{n,k_1,k_2}(x)c_{n,k_1,k_2}\int_{\triangle_{k_1,k_2}} f(u) \, du \quad (1.1) \]

is called Meyer-König-Zeller-Kantorović operator on \( \Delta \); the functional

\[ K_{n,s}(f; x) = \sum_{k+l \leq n} b_{n,k,l,s}(x)(n+2)^2 \int_{I_{n,k,l}} f(u) \, du \quad (1.2) \]

is called Stancu-Kantorović operator on \( \Delta \), where \( x \in \Delta, n \in \mathbb{Z}^+, k, s, l \) are nonnegative integers, \( 0 \leq s < \frac{n}{2} \), and

\[ \tilde{p}_{n,k_1,k_2}(x) = \frac{(n + k_1 + k_2)!}{n!k_1!k_2!} x_1^{k_1} x_2^{k_2} (1 - x_1 - x_2)^{n+1}, \]

\[ c_{n,k_1,k_2} = \frac{(n + k_1 + k_2)^2(n + k_1 + k_2 + 1)^2}{(n + k_1)(n + k_2)}, \]

\[ \triangle_{k_1,k_2} = \left[ \begin{array}{cc} k_1/n + k_2/n & k_1 + 1/n + k_2 + 1/n \end{array} \right] \times \left[ \begin{array}{cc} k_2/n + k_1/n & k_2 + 1/n + k_1 + 1/n \end{array} \right]. \]
For Theorem 1.1. Let \( L_p \) be the space of functions with different positions. The convergence rate of the operators (1.1) and (1.2) in convergence in space \( L_p \) has been studied (see [2, 3]). This paper intends to investigate their convergence in space \( L_p^\Phi(\Delta) \), and the main results are as follows.

**Theorem 1.1.** For \( f \in L_p^\Phi(\Delta) \), if a \( N \)-function \( \Phi(u) \) satisfies the \( \Delta_2 \)-condition, then

\[
\|K_n(f) - f\|_\Phi \leq C \left( \frac{1}{n} \|f\|_\Phi + \Omega_\Phi^2 \left( f, \sqrt{\frac{1}{n}} \right) \right).
\]

**Theorem 1.2.** For \( f \in L_p^\Phi(\Delta) \), if a \( N \)-function \( \Phi(u) \) satisfies the \( \Delta_2 \)-condition, then

\[
b_{n,k,l}(x) = \begin{cases} 
(1 - x_1 - x_2)p_{n-s,k,l}(x), & k + l \leq n - s, 0 \leq k, l < s; \\
(1 - x_1 - x_2)p_{n-s,k,l}(x) + x_1p_{n-s,k-l}(x), & k + l \leq n - s, s \leq k, 0 \leq l < s; \\
(1 - x_1 - x_2)p_{n-s,k,l}(x) + x_2p_{n-s,k,l-s}(x), & k + l \leq n - s, s \leq l, 0 \leq k < s; \\
(1 - x_1 - x_2)p_{n-s,k,l}(x) + x_1p_{n-s,k-l}(x) + x_2p_{n-s,k,l-s}(x,y), & k + l \leq n - s, s \leq k, s \leq l; \\
x_1p_{n-s,k-l}(x), & n - s < k + l \leq n, s \leq k, 0 \leq l < s; \\
x_2p_{n-s,k,l-s}(x), & n - s < k + l \leq n, 0 \leq k < s, s \leq l; \\
x_1p_{n-s,k-l}(x) + x_2p_{n-s,k,l-s}(x), & n - s < k + l \leq n, s \leq k, s \leq l.
\end{cases}
\]
\[ \| K_{n,s}(f) - f \|_\Phi \leq C \left( \frac{1}{n} \| f \|_\Phi + \Omega_{R^2}^2 \left( f, \sqrt{\frac{1}{n}} \right)_\Phi \right). \]

### 2. Lemmas

**Lemma 2.1.** \( K_n \) is a bounded linear operator, and \( \| K_n \|_\Phi \leq 2. \)

**Proof.** The linearity of \( K_n \) is obvious. The following proves \( \| K_n \|_\Phi \leq 2. \) After calculation, we can get

\[ \text{mes} \triangle_{k_1,k_2} = \frac{(n+k_1)(n+k_2)}{(n+k_1+k_2)^2(n+k_1+k_2+1)^2}, \]

\[ \int_{\triangle} \tilde{p}_{n,k_1,k_2}(x)dx = \frac{n+1}{(n+k_1+k_2+3)(n+k_1+k_2+2)(n+k_1+k_2+1)}. \]

By using the Lemma 1 in [2], Jensen inequality of convex function and the Theorem 1.4 in [1], we obtain

\[ \| K_n(f) \|_\Phi = \inf_{\alpha > 0} \frac{1}{\alpha} \left\{ 1 + \int_{\triangle} \Phi \left( \alpha K_n(f; x) \right) dx \right\} \]

\[ = \inf_{\alpha > 0} \frac{1}{\alpha} \left\{ 1 + \int_{\triangle} \Phi \left( \alpha \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \tilde{p}_{n,k_1,k_2}(x)c_{n,k_1,k_2} \int_{\triangle_{k_1,k_2}} f(u)du \right) dx \right\} \]

\[ \leq \inf_{\alpha > 0} \frac{1}{\alpha} \left\{ 1 + \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \int_{\triangle} \tilde{p}_{n,k_1,k_2}(x)dx c_{n,k_1,k_2} \int_{\triangle_{k_1,k_2}} \Phi \left( \alpha f(u) \right) du \right\} \]

\[ \leq \inf_{\alpha > 0} \left\{ 1 + 2 \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \int_{\triangle_{k_1,k_2}} \Phi \left( \alpha f(u) \right) du \right\} \]

\[ \leq \inf_{\alpha > 0} \left\{ 1 + \int_{\triangle} \Phi \left( 2\alpha f(u) \right) du_1 du_2 \right\} \]

\[ = 2 \| f \|_\Phi \]

Namely

\[ \| K_n \|_\Phi \leq 2. \quad \Box \]

**Lemma 2.2.** \( K_{n,s} \) is a bounded linear operator, and \( \| K_{n,s} \|_\Phi \leq 12. \)
**Proof.** The linearity of $K_{n,s}$ is obvious. The following proves $\|K_{n,s}\|_{\Phi} \leq 12$.

After calculation, we can get

$$\text{mes}I_{n,k,l} = \text{mes}I_{n,k+s,l} = \text{mes}I_{n,k,l+s} = \frac{1}{(n+2)^2},$$

$$\int_{\Delta} p_{n-s,k,l}(x)dx = \frac{1}{(n-s+2)(n-s+1)}.$$

By using the Lemma 2.1 in [3], Jensen inequality of convex function and the Theorem 1.4 in [1], we obtain

$$\|K_{n,s}(f)\|_{\Phi} \leq \inf_{\alpha > 0} \frac{1}{\alpha} \left\{ 1 + \sum_{k+l \leq n-s} \int_{\Delta} p_{n-s,k,l}(x)dx \left( \int_{I_{n,k,l}} + \int_{I_{n,k+s,l}} \int_{I_{n,k,l+s}} \right) \Phi(\alpha f(u))du \right\},$$

$$\leq \inf_{\alpha > 0} \frac{1}{\alpha} \left\{ 1 + \frac{3(n+2)^2}{(n-s+2)(n-s+1)} \int_{\Delta} \Phi(\alpha f(u))du \right\},$$

$$\leq \inf_{\alpha > 0} \frac{1}{\alpha} \left\{ 1 + \int_{\Delta} \Phi(\alpha 12 f(u))du \right\},$$

$$= \|12f\|_{\Phi},$$

$$= 12\|f\|_{\Phi}.$$

Namely

$$\|K_{n,s}\|_{\Phi} \leq 12. \quad \square$$

**Lemma 2.3.** The following holds for (1.1).

$$K_n(1;x) = 1, \quad K_n \left( (u_i - x_i)^i; x \right) \leq \frac{C}{n}, \quad i = 1, 2.$$

**Lemma 2.4.** The following holds for (1.2).

$$K_{n,s}(1;x) = 1, \quad K_{n,s} \left( (u_i - x_i)^i; x \right) \leq \frac{C}{n}, \quad i = 1, 2.$$

The proof of the Lemma 2.3 and Lemma 2.4 can be obtained from the Lemma 3 in [2] and the Lemma 2.1 in [3].
Lemma 2.5. If we denote \( f_r \) the Steklov mean function for \( f \in L^*_\Phi(\Delta) \), i.e.

\[
f_r(x) = \frac{1}{r^d} \int_{[-r/2,r/2]^d} f(x + u + v) \, ds \, dt,
\]

then

\[
\|f_r\|_\Phi \leq C \|f\|_\Phi, \tag{2.1}
\]

\[
\|f - f_r\|_\Phi \leq C \Omega^2_{\mathcal{R}^2}(f, r)_\Phi, \tag{2.2}
\]

\[
\left\| \frac{\partial f_r}{\partial x_1} \right\|_\Phi \leq C \left( \|f_r\|_\Phi + \left\| \frac{\partial^2 f_r}{\partial x_1^2} \right\|_\Phi \right), \tag{2.3}
\]

\[
\left\| \frac{\partial f_r}{\partial x_2} \right\|_\Phi \leq C \left( \|f_r\|_\Phi + \left\| \frac{\partial^2 f_r}{\partial x_2^2} \right\|_\Phi \right), \tag{2.4}
\]

\[
\left\| \frac{\partial^2 f_r}{\partial x_1^2} \right\|_\Phi + \left\| \frac{\partial^2 f_r}{\partial x_2^2} \right\|_\Phi + \left\| \frac{\partial^2 f_r}{\partial x_1 \partial x_2} \right\|_\Phi \leq \frac{C}{r^2} \Omega^2_{\mathcal{R}^2}(f, r)_\Phi. \tag{2.5}
\]

(2.1), (2.2), (2.5) can be directly verified, and the proof of (2.3), (2.4) is similar to that of the Lemma 1a in [4]. If N-function \( \Phi \) satisfies the \( \Delta_2 \)-condition, then \( L^*_\Phi \) is separable. This leads to the following conclusion [5].

Lemma 2.6. If N-function \( \Phi \) satisfies the \( \Delta_2 \)-condition, then

\[
\left\| \sup_{u_1 \neq x_1} \frac{1}{u_1 - x_1} \int_{x_1}^{u_1} \left\| \frac{\partial^2 f_r(\xi, x_2)}{\partial \xi^2} \right\|_\Phi \, d\xi \right\|_\Phi \leq C \left\| \frac{\partial^2 f_r}{\partial x_1^2} \right\|_\Phi, \tag{2.6}
\]

\[
\left\| \sup_{u_2 \neq x_2} \frac{1}{u_2 - x_2} \int_{x_2}^{u_2} \left\| \frac{\partial^2 f_r(x_1, \eta)}{\partial \eta^2} \right\|_\Phi \, d\eta \right\|_\Phi \leq C \left\| \frac{\partial^2 f_r}{\partial x_2^2} \right\|_\Phi, \tag{2.7}
\]

\[
\left\| \sup_{u_2 \neq x_2} \frac{1}{u_2 - x_2} \int_{x_2}^{u_2} \left( \sup_{u_1 \neq x_1} \frac{1}{u_1 - x_1} \int_{x_1}^{u_1} \left\| \frac{\partial^2 f_r(\xi, \eta)}{\partial \xi \partial \eta} \right\|_\Phi \, d\xi \right) \, d\eta \right\|_\Phi \leq C \left\| \frac{\partial^2 f_r}{\partial x_1 \partial x_2} \right\|_\Phi. \tag{2.8}
\]
3. Proof of the main results

The proof of the Theorem 1.1 and Theorem 1.2 is similar, so only the Theorem 1.1 is proved below.

Proof. Because

\[
f_r(u) - f_r(x) = (u_1 - x_1) \frac{\partial f_r(x)}{\partial x_1} + (u_2 - x_2) \frac{\partial f_r(x)}{\partial x_2} + \int_{x_1}^{u_1} (u_1 - \xi) \frac{\partial^2 f_r(\xi, x_2)}{\partial \xi^2} \, d\xi
\]

\[+ \int_{x_2}^{u_2} (u_2 - \eta) \frac{\partial^2 f_r(x_1, \eta)}{\partial \eta^2} \, d\eta + \int_{x_1}^{u_1} \int_{x_2}^{u_2} \frac{\partial^2 f_r(\xi, \eta)}{\partial \xi \partial \eta} \, d\xi \, d\eta,
\]
so

\[
|K_n(f_r; x) - f_r(x)| \leq |K_n((u_1 - x_1); x)| \left| \frac{\partial f_r(x)}{\partial x_1} \right| + |K_n((u_2 - x_2); x)| \left| \frac{\partial f_r(x)}{\partial x_2} \right| + |K_n((u_1 - x_1)^2; x)| \left( \sup_{u_1 \neq x_1} \frac{1}{u_1 - x_1} \int_{x_1}^{u_1} \left| \frac{\partial^2 f_r(\xi, x_2)}{\partial \xi^2} \right| \, d\xi \right) +
\]

\[
|K_n((u_2 - x_2)^2; x)| \left( \sup_{u_2 \neq x_2} \frac{1}{u_2 - x_2} \int_{x_2}^{u_2} \left| \frac{\partial^2 f_r(x_1, \eta)}{\partial \eta^2} \right| \, d\eta \right) +
\]

\[
|K_n((u_1 - x_1)|u_2 - x_2|; x)| \left( \sup_{u_2 \neq x_2} \frac{1}{u_2 - x_2} \int_{x_2}^{u_2} \left( \sup_{u_1 \neq x_1} \int_{x_1}^{u_1} \left| \frac{\partial^2 f_r(\xi, \eta)}{\partial \xi \partial \eta} \right| \, d\xi \right) \, d\eta \right).
\]

Noticing

\[
|K_n((u_1 - x_1)|u_2 - x_2|; x)| \leq \frac{1}{2} |K_n((u_1 - x_1)^2; x)| + |K_n((u_2 - x_2)^2; x)|,
\]

we continue the above estimation using the Lemma 2.3, Lemma 2.5 and Lemma 2.6.

\[
\|K_n(f_r) - f_r\|_\Phi \leq \frac{C}{n} \left( \left\| \frac{\partial f_r}{\partial x_1} \right\|_\Phi + \left\| \frac{\partial f_r}{\partial x_2} \right\|_\Phi + \left\| \frac{\partial^2 f_r}{\partial x_1^2} \right\|_\Phi + \left\| \frac{\partial^2 f_r}{\partial x_2^2} \right\|_\Phi + \left\| \frac{\partial^2 f_r}{\partial x_1 \partial x_2} \right\|_\Phi \right) + \frac{1}{r^2} \Omega^2_{R^2}(f, r)_\Phi.
\]
If \( r = \sqrt{\frac{1}{n}} \), then
\[
\|K_n(f_r) - f_r\|_\Phi \leq \frac{C}{n} \left( \|f\|_\Phi + n\Omega_{R^2}^2 \left( f, \sqrt{\frac{1}{n}} \right)_\Phi \right).
\]
For \( f \in L^*_\Phi(\triangle) \), using the Lemma 2.1 and Lemma 2.5 we get
\[
\|K_n(f) - f\|_\Phi \leq \|K_n(f) - K_n(f_r)\|_\Phi + \|K_n(f_r) - f_r\|_\Phi + \|f_r - f\|_\Phi
\leq 3\|f_r - f\|_\Phi + \|K_n(f_r) - f_r\|_\Phi
\leq C\Omega_{R^2}^2(f, r)_\Phi + C \left( \frac{1}{n} \|f\|_\Phi + \Omega_{R^2}^2 \left( f, \sqrt{\frac{1}{n}} \right)_\Phi \right).
\]

4. Remark

If \( \Phi(u) = u^p \ (1 < p < \infty) \), then \( L^*_\Phi(\triangle) = L_p \). Thus the corresponding conclusions in [2] and [3] can be obtained from the Theorem 1.1 and Theorem 1.2.

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