Global Well-Posedness to the 3D Cauchy Problem of Nonhomogeneous Heat Conducting Navier–Stokes Equations with Vacuum and Large Oscillations

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Communicated by A. Constantin

Abstract. We study global well-posedness of strong solutions to the Cauchy problem of nonhomogeneous heat conducting Navier–Stokes equations with vacuum on the whole space \( \mathbb{R}^3 \). We derive the global existence and uniqueness of strong solutions provided that \( \|\rho_0\|_{L^\infty} \|\sqrt{\rho_0} u_0\|_{L^2}^2 \|\nabla u_0\|_{L^2}^2 \) is suitably small, with the smallness depending only on the viscosity coefficient \( \mu \) of the system under consideration. Moreover, we also obtain the large time decay rates of the solution. In particular, the smallness condition is independent of any norms of the initial data and allows the solution to have large oscillations. Furthermore, there is no need to require compatibility conditions for the initial data via time weighted techniques.

Mathematics Subject Classification. 76D05, 76D03.

Keywords. Nonhomogeneous heat conducting Navier–Stokes equations, Global well-posedness, Large time behavior, Vacuum, Large oscillations.

1. Introduction

Nonhomogeneous Navier–Stokes equations describe a fluid which is obtained by mixing two miscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance. We refer the reader to [9, Chapter 1] for the detailed derivation of such system. Due to their prominent roles in modeling many phenomena in physics, the nonhomogeneous Navier–Stokes equations have been studied extensively mathematically. Lions [9] obtained the global existence of weak solutions that allows the vacuum condition together with the density-dependent viscosity coefficient. The global existence of strong solutions with nonnegative density in three-dimensional (3D) bounded domains under some smallness assumptions was established in [1,2]. There are also very interesting investigations about global strong solution with density-dependent viscosity, please refer to [6,8,10,18] and references therein.

In the present paper, we are concerned with the nonhomogeneous heat conducting Navier–Stokes equations (see [11, p. 23]) in \( \mathbb{R}^3 \):

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u + \nabla P = 0, \\
c_v([\rho \theta]_t + \text{div}(\rho u \theta)) - \kappa \Delta \theta = 2\mu |\nabla u|^2, \\
\text{div} u = 0,
\end{cases}
\]

(1.1)

with the initial condition

\[
(\rho, \rho u, \rho \theta)(x, 0) = (\rho_0, \rho_0 u_0, \rho_0 \theta_0)(x), \quad x \in \mathbb{R}^3,
\]

(1.2)

This research was partially supported by National Natural Science Foundation of China (Nos. 11901474, 12071359), Exceptional Young Talents Project of Chongqing Talent (No. csc2021ycjh-bgzm0153), and the Innovation Support Program for Chongqing Overseas Returnees (No. cx2020082).
and the far field behavior

$$(\rho, \mathbf{u}, \theta) \to (0, \mathbf{0}, 0), \text{ as } |x| \to \infty, \ t > 0.$$  

Here the unknowns $\rho, \mathbf{u}, \theta, P$ are the fluid density, velocity, absolute temperature, and pressure, respectively. $\mathcal{D}(\mathbf{u})$ denotes the deformation tensor given by

$$\mathcal{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T).$$

The positive constant $\mu$ is the viscosity coefficient of the fluid, while $c_v$ and $\kappa$ are the heat capacity and the ratio of the heat conductivity coefficient over the heat capacity, respectively.

The system (1.1) has been attracted enormous attention in the last years, and before describing the main result of the present paper, we discuss briefly some of the known results which are closely related to our work. On the one hand, when the initial density is away from zero, Zhang and Tan [19] showed global existence and uniqueness of strong solutions under some smallness conditions. Their result was recently extended to the case of temperature-dependent viscosity by Guo and Li [5]. On the other hand, for the initial density allowing vacuum states, Zhong [20] proved global strong solutions under certain smallness condition and compatibility condition. This result was recently improved by Zhong [21], where the author established global existence and large time behavior of strong solutions for 3D problem with the Navier-slip boundary condition for velocity and Neumann boundary condition for temperature without using compatibility conditions via time weighted estimates. Later, Xu and Yu [16,17] have made serious efforts to tackle the case of density-temperature-dependent viscosity. We also mention the result of [15] on the nonhomogeneous heat conducting Navier–Stokes flows with the external force. Very recently, Zhong [22] provided global existence and large time behavior of strong solutions for 2D boundary value problem with large initial data and vacuum. It should be pointed out that the methods used in [20] depend heavily on the boundedness of domains and little is known for the global well-posedness of strong solutions to the Cauchy problem (1.1)–(1.3). The main goal of this paper is to study the global existence and large time behavior of strong solutions to (1.1)–(1.3) under suitable smallness condition. At the same time, we also give the decay rates of gradients of velocity and temperature.

For $1 \leq p \leq \infty$ and integer $k \geq 0$, the standard Sobolev spaces are denoted by:

$$L^p = L^p(\mathbb{R}^3), \ W^{k,p} = W^{k,p}(\mathbb{R}^3), \ H^k = H^{k,2}(\mathbb{R}^3).$$

Our main result can be stated as follows.

**Theorem 1.1.** Assume that the initial data $(\rho_0 \geq 0, \mathbf{u}_0, \theta_0 \geq 0)$ satisfies

$$\begin{cases}
\rho_0 \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3), & \nabla \rho_0 \in L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3), \\
\sqrt{\rho_0} \mathbf{u}_0 \in L^2(\mathbb{R}^3), & \nabla \mathbf{u}_0 \in L^2(\mathbb{R}^3), \ \text{div} \mathbf{u}_0 = 0, \\
\sqrt{\rho_0} \theta_0 \in L^2(\mathbb{R}^3), & \nabla \theta_0 \in L^2(\mathbb{R}^3).
\end{cases} \tag{1.4}$$

Then there exists a small positive constant $\varepsilon_0$ depending only on $\mu$ such that if

$$\|\rho_0\|_{L^\infty} \|\sqrt{\rho_0} \mathbf{u}_0\|_{L^2}^2 \|\nabla \mathbf{u}_0\|_{L^2}^2 \leq \varepsilon_0, \tag{1.5}$$
the problem (1.1)–(1.3) has a unique global strong solution \((\rho \geq 0, u, \theta \geq 0)\) such that for \(\tau > 0\),

\[
\begin{cases}
  \rho_t \in L^{\infty}(0, \infty; L^2 \cap L^3), \quad \rho \in L^{\infty}(0, \infty; L^{\frac{3}{2}} \cap L^\infty), \\
  \nabla \rho \in L^{\infty}(0, \infty; L^2 \cap L^6), \quad \rho \in C([0, \infty); L^p), \quad \frac{3}{2} \leq p < \infty, \\
  \nabla u \in L^{\infty}(0, \infty; L^2) \cap L^{\infty}(\tau, \infty; H^1) \cap L^2(\tau, \infty; W^{1,6}), \\
  \nabla \theta \in L^{\infty}(\tau, \infty; H^1) \cap L^2(\tau, \infty; W^{1,6}), \\
  \nabla u, \nabla \theta \in C([\tau, \infty); H^1), \\
  \nabla P \in L^{\infty}(\tau, \infty; L^2) \cap L^2(\tau, \infty; L^6), \\
  t\sqrt{\rho \theta}, t\nabla \theta \in L^{\infty}(0, \infty; L^2), \\
  e^{\frac{\sigma t}{2}} \nabla u, \quad t\nabla u, \quad t\nabla \theta, \quad \nabla \theta \in L^2(0, \infty; L^2),
\end{cases}
\]

(1.6)

where \(\sigma := 3\left(\frac{\alpha}{2}\right)^{\frac{1}{2}} \mu/\|\rho_0\|_{L^{\frac{3}{2}}}^\frac{1}{2}\). Moreover, there exists a positive constant \(C\) depending only on \(\mu, c_v, \kappa\), and the initial data such that for \(t \geq 1\),

\[
\begin{align*}
  &\left\| \nabla u(\cdot, t) \right\|_{H^1}^2 + \left\| \nabla P(\cdot, t) \right\|_{L^2}^2 + \left\| \sqrt{\rho \theta}(\cdot, t) \right\|_{L^2}^2 \leq Ce^{-\sigma t}, \\
  &\left\| \nabla \theta(\cdot, t) \right\|_{H^1}^2 + \left\| \sqrt{\rho \theta}(\cdot, t) \right\|_{L^2}^2 + \left\| \sqrt{\rho \theta}(\cdot, t) \right\|_{L^2}^2 \leq C\tau^{-2}.
\end{align*}
\]

(1.7)

**Remark 1.1.** Compared with the 3D Cauchy problem with density-temperature-dependent viscosity [16], the exponential decay-in-time property (1.7) is new. Meanwhile, there is no need to impose some compatibility condition on the initial data. Furthermore, it should be noted here that our smallness assumption (1.5) is independent of any norms of the initial data, which is different from that of in [16].

**Remark 1.2.** Since \(\varepsilon_0\) is dependent of \(\mu\), we cannot assert any more when \(\mu\) approaches to zero.

The framework in the proof of Theorem 1.1 is the continuation argument (see, e.g., [6]). Thus our efforts are devoted to showing global \(a\ priori\) estimates on strong solutions to the system (1.1) in suitable higher-order norms, and the key ingredient is to get time-independent bounds on the \(L^1(0, T; L^\infty)\)-norm of \(\nabla u\). We should point out that the crucial techniques used in [21] cannot be adapted to the situation treated here, since his arguments depend heavily on the boundedness of domains. Here we mention difficulties to deal with the whole space case. First, for the 3D bounded domains case [21], the Poincaré type inequality plays an important role to get exponential decay-in-time property of \(\|\sqrt{\rho \theta}\|_{L^2}^2\), which in turn depends crucially on the boundedness of the domains. Applying Hölder’s and Sobolev’s inequalities, we find that \(\|\sqrt{\rho \theta}\|_{L^2}^2\) decays with the rate of \(e^{-2\sigma t}\) for \(\sigma = 3\left(\frac{\alpha}{2}\right)^{\frac{1}{2}} \mu/\|\rho_0\|_{L^{\frac{3}{2}}}^\frac{1}{2}\) (see (3.6)]. Next, the crucial dissipation estimate of the form \(\int_0^T \|\nabla \theta\|_{L^2}^2 \, dt\) (see Lemma 3.5 in [21]) is important to derive time weighted estimate of \(\|\sqrt{\rho \theta}\|_{L^2}^2\). However, his argument fails for 3D Cauchy problem since the method in [21] lies crucially in the \(L^2 L^\infty\)-norm of the temperature \(\theta\). To overcome this difficulty, motivated by [7, Lemma 6], we multiply the momentum equation (1.1) by \(u \theta\) to recover good bounds on the temperature \(\theta\) (see (3.31)]. Once these key estimates are obtained, we can obtain the desired global high-order \(a\ priori\) estimates of the solution.

The rest of this paper is organized as follows. In Sect. 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the proof of Theorem 1.1.

### 2. Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used frequently later.

We begin with the local existence and uniqueness of strong solutions whose proof can be performed by using energy methods (see, e.g., [12]).
Lemma 2.1. Assume that \((\rho_0, u_0, \theta_0)\) satisfies (1.4). Then there exist a small time \(T > 0\) and a unique strong solution \((\rho, u, \theta)\) to the problem (1.1)–(1.3) in \(\mathbb{R}^3 \times (0, T]\).

Next, the following Gagliardo-Nirenberg inequality (see [13]) will be useful.

Lemma 2.2. For \(p \in [2, 6], r \in (3, \infty),\) and \(q \in (1, \infty),\) there exists some generic constant \(C > 0\) which may depend on \(p, r,\) and \(s\) such that for \(f \in H^1(\mathbb{R}^3)\) and \(g \in L^q(\mathbb{R}^3) \cap D^{2,r}(\mathbb{R}^3)\), we have

\[
\|f\|_{L^p(\mathbb{R}^3)} \leq C \|f\|_{L^2(\mathbb{R}^3)} \|
abla f\|_{L^r(\mathbb{R}^3)}, \quad (2.1)
\]

and

\[
\|g\|_{L^\infty(\mathbb{R}^3)} \leq C \|g\|_{L^q(\mathbb{R}^3)} \|
abla g\|_{L^{r}(\mathbb{R}^3)}. \quad (2.2)
\]

Finally, we need the following regularity on the Stokes problem, whose proof can be found in [6].

Lemma 2.3. Suppose that \(F \in L^r(\mathbb{R}^3) \cap L^{\frac{3}{2}}(\mathbb{R}^3)\) with \(r \in [2, 6],\) then there exists some positive constant \(C\) depending only on \(\mu\) and \(r\) such that the unique weak solution \((u, P) \in D^1 \times L^2\) \((P\) is unique up to an additive constant\) to the following Stokes problem

\[
\begin{cases}
-\mu \Delta u + \nabla P = F, & x \in \mathbb{R}^3, \\
\text{div} \ u = 0, & x \in \mathbb{R}^3, \\
u \to 0, & |x| \to \infty,
\end{cases}
\]

satisfies

\[
\|\nabla u\|_{L^2} + \|P\|_{L^2} \leq C \|F\|_{L^{\frac{3}{2}}}, \quad (2.4)
\]

\[
\|\nabla^2 u\|_{L^r} + \|\nabla P\|_{L^r} \leq C \|F\|_{L^r}. \quad (2.5)
\]

3. Proof of Theorem 1.1

Before proceeding, we rewrite another equivalent form of the system (1.1) as the following

\[
\begin{align*}
\rho_t + u \cdot \nabla \rho &= 0, \\
\rho u_t + \mu u \cdot \nabla u - \mu \Delta u + \nabla P &= 0, \\
c_\nu \rho \theta_t + \mu u \cdot \nabla \theta - \kappa \Delta \theta &= 2 \mu |D(u)|^2, \\
\text{div} \ u &= 0.
\end{align*}
\]

In what follows, for simplicity, we denote by

\[
\int \cdot \, dx = \int_{\mathbb{R}^3} \cdot \, dx,
\]

and we sometimes use \(C(f)\) to emphasize the dependence on \(f\).

Lemma 3.1. It holds that

\[
\sup_{0 \leq t \leq T} \|\rho\|_{L^p} \leq \|\rho_0\|_{L^p}, \quad \text{for } \frac{3}{2} \leq p \leq \infty, \quad (3.2)
\]

and

\[
\sup_{0 \leq t \leq T} \|\sqrt{\rho} u\|_{L^2}^2 + \mu \int_0^T \|\nabla u\|_{L^2}^2 \, dx \leq \|\sqrt{\rho_0} u_0\|_{L^2}^2. \quad (3.3)
\]

Moreover, for \(\sigma := 3\left(\frac{\mu}{2}\right)^{\frac{1}{2}}\), there holds

\[
\sup_{0 \leq t \leq T} \left( e^{\sigma t} \|\sqrt{\rho} u\|_{L^2}^2 \right) + \int_0^T e^{\sigma t} \|\nabla u\|_{L^2}^2 \, dx \leq C \left( \mu, \|\rho_0\|_{L^\frac{3}{2}} \right) \left( \|\sqrt{\rho_0} u_0\|_{L^2}^2 \right). \quad (3.4)
\]
Proof. 1. Since (3.1) is a transport equation, we then directly obtain the desired (3.2). Moreover, applying standard maximum principle (see [3, p. 43]) to (3.1) along with \( \rho_0, \theta_0 \geq 0 \) shows
\[
\inf_{\mathbb{R}^3 \times [0,T]} \rho(x,t) \geq 0, \quad \inf_{\mathbb{R}^3 \times [0,T]} \theta(x,t) \geq 0.
\]

2. Multiplying (3.1)_2 by \( u \) and integration by parts leads to
\[
\frac{d}{dt} \| \sqrt{\rho} u \|_{L^2}^2 + 2\mu \| \nabla u \|_{L^2}^2 = 0.
\] (3.5)
Integrating (3.5) over \((0, T)\) leads to the desired (3.3).

3. By Hölder’s inequality, (3.2), and [14, Theorem], we have
\[
\| \sqrt{\rho} u \|_{L^2}^2 \leq \| \rho \|_{L^{\frac{3}{2}}} \| u \|_{L^{\frac{6}{5}}} \leq \frac{1}{3} \left( \frac{2}{\pi} \right)^{\frac{3}{2}} \| \rho_0 \|_{L^{\frac{3}{2}}} \| \nabla u \|_{L^2}^2,
\]
which combined with (3.5) leads to
\[
\frac{d}{dt} \| \sqrt{\rho} u \|_{L^2}^2 + 2\sigma \| \sqrt{\rho} u \|_{L^2}^2 \leq 0, \quad \text{for } \sigma := 3 \left( \frac{\pi}{2} \right)^{\frac{3}{2}} \mu / \| \rho_0 \|_{L^{\frac{3}{2}}}.
\]
This implies immediately that
\[
\| \sqrt{\rho} u(\cdot, t) \|_{L^2}^2 \leq \| \sqrt{\rho_0} u_0 \|_{L^2}^2 e^{-2\sigma t}, \quad \text{for any } t \geq 0.
\] (3.6)

Then we multiply (3.5) by \( e^{\sigma t} \) and use (3.6) to get
\[
\frac{d}{dt} \left( e^{\sigma t} \| \sqrt{\rho} u \|_{L^2}^2 \right) + 2\mu e^{\sigma t} \| \nabla u \|_{L^2}^2 = \sigma e^{\sigma t} \| \sqrt{\rho} u \|_{L^2}^2 \leq \sigma \| \sqrt{\rho_0} u_0 \|_{L^2}^2 e^{-\sigma t}.
\] (3.7)
Integrating (3.7) in time over \((0, T)\) yields
\[
\int_0^T e^{\sigma t} \| \nabla u \|_{L^2}^2 dt \leq C \left( \mu, \| \rho_0 \|_{L^{\frac{3}{2}}}, \| \sqrt{\rho_0} u_0 \|_{L^2} \right)
\]
which together with (3.6) gives the desired (3.4).

\[\square\]

Lemma 3.2. There exist positive constants \( C \) and \( L \) depending only on \( \mu \) such that, for any \( t \in (0, T) \),
\[
\sup_{0 \leq s \leq t} \| \nabla u \|_{L^2}^2 + \frac{1}{2L} \| \rho_0 \|_{L^\infty} \int_0^t \| \nabla^2 u \|_{L^2}^2 ds \leq \| \nabla u_0 \|_{L^2}^2 + C \| \rho_0 \|_{L^\infty} \int_0^t \| \nabla^2 u_0 \|_{L^2} \left( \sup_{0 \leq s \leq t} \| \nabla u \|_{L^2} \right) \int_0^t \| \nabla^2 u \|_{L^2}^2 ds.
\] (3.9)

Proof. 1. Multiplying (3.1) by \( u \) and integrating the resulting equation over \( \mathbb{R}^3 \), we get from Cauchy-Schwarz inequality that
\[
\frac{\mu}{2} \frac{d}{dt} \| \nabla u \|_{L^2}^2 + \| \sqrt{\rho} u \|_{L^2}^2 = - \int \rho u \cdot \nabla u \cdot u dx \leq \frac{1}{2} \int \rho |u|^2 dx + \frac{1}{2} \int \rho |u|^2 |\nabla u|^2 dx,
\]
which yields that
\[
\mu \frac{d}{dt} \| \nabla u \|_{L^2}^2 + \| \sqrt{\rho} u \|_{L^2}^2 \leq \int \rho |u|^2 \| \nabla u \|_{L^2}^2 dx.
\] (3.10)
Integrating (3.10) over \((0, t)\) gives rise to
\[
\mu \sup_{0 \leq s \leq t} \| \nabla u \|_{L^2}^2 + \int_0^t \| \sqrt{\rho} u(s) \|_{L^2}^2 ds \leq \mu \| \nabla u_0 \|_{L^2}^2 + \mu \int_0^t \| \nabla u \|_{L^2}^2 dx ds.
\] (3.11)

2. Recall that \((u, P)\) satisfies the following Stokes system
\[
\begin{cases}
-\mu \Delta u + \nabla P = -\rho u_t - \rho u \cdot \nabla u, \quad x \in \mathbb{R}^3, \\
\text{div } u = 0, \quad x \in \mathbb{R}^3, \\
u \rightarrow 0, \quad |x| \rightarrow \infty.
\end{cases}
\] (3.12)
Applying Lemma 2.3 with \( F = -\rho u_t - \rho u \cdot \nabla u \), we obtain from (3.2) that
\[
\| \nabla^2 u \|^2_{L^2} + \| \nabla P \|^2_{L^2} \leq C(\mu) \left( \| \rho u_t \|^2_{L^2} + \| \rho u \cdot \nabla u \|^2_{L^2} \right)
\leq L \| \rho_0 \|_{L^\infty} \| \sqrt{\rho} u_t \|^2_{L^2} + C \| \rho_0 \|_{L^\infty} \| \sqrt{\rho} u \cdot \nabla u \|^2_{L^2}
\] (3.13)
for some \( C \) and \( L \) depending only on \( \mu \). Integrating (3.13) multiplied by \( \frac{1}{2L \| \rho_0 \|_{L^\infty}} \) with respect to \( t \) and adding the resulting inequality to (3.11), we derive that
\[
\sup_{0 \leq s \leq t} \| \nabla u \|^2_{L^2} + \frac{1}{2L \| \rho_0 \|_{L^\infty}} \left[ \int_0^t \| \nabla^2 u \|^2_{L^2} ds + \frac{1}{2} \int_0^t \| \sqrt{\rho} u_t(s) \|^2_{L^2} ds \right]
\leq \| \nabla u_0 \|^2_{L^2} + \tilde{C} \int_0^t \int \rho |u|^2 \| \nabla u \|^2_{dxds}
\] (3.14)
for some \( \tilde{C} \) depending only on \( \mu \). By (3.2), Hölder’s inequality, Sobolev’s inequality, and (3.3), we have
\[
\int \rho |u|^2 \| \nabla u \|^2_{dx} \leq \| \rho \|^2_{L^\infty} \| \sqrt{\rho} u \|_{L^2} \| u \|_{L^6} \| \nabla u \|^2_{L^3}
\leq C \| \rho_0 \|^2_{L^\infty} \| \sqrt{\rho} u_0 \|_{L^2} \| \nabla u_0 \|^2_{L^2}
\leq C \| \rho_0 \|^2_{L^\infty} \| \sqrt{\rho_0} u_0 \|_{L^2} \| \nabla u_0 \|^2_{L^2}.
\]
As a consequence, we derive that
\[
\tilde{C} \int_0^t \int \rho |u|^2 \| \nabla u \|^2_{dxds} \leq C(\mu) \| \rho_0 \|^2_{L^\infty} \| \sqrt{\rho_0} u_0 \|_{L^2} \left( \sup_{0 \leq s \leq t} \| \nabla u \|_{L^2} \right) \int_0^t \| \nabla^2 u \|^2_{L^2} ds,
\] (3.15)
which combined with (3.14) implies the desired (3.9) and finishes the proof of Lemma 3.2.

**Lemma 3.3.** For \( L \) as that in Lemma 3.2, there exists a positive constant \( \varepsilon_0 \) depending only on \( \mu \) such that
\[
\sup_{0 \leq t \leq T} \| \nabla u \|^2_{L^2} + \frac{1}{2L \| \rho_0 \|_{L^\infty}} \left[ \int_0^T \| \nabla^2 u \|^2_{L^2} dt + \frac{1}{2} \int_0^T \| \sqrt{\rho} u_t(s) \|^2_{L^2} ds \right] \leq 2 \| \nabla u_0 \|^2_{L^2},
\] (3.16)
provided that
\[
\| \rho_0 \|_{L^\infty} \| \sqrt{\rho_0} u_0 \|^2_{L^2} \| \nabla u_0 \|^2_{L^2} \leq \varepsilon_0.
\] (3.17)

**Proof.** Define functions \( E(t) \) and \( \Phi(t) \) as follows
\[
E(t) := \sup_{0 \leq s \leq t} \| \nabla u \|^2_{L^2} + \frac{1}{2L \| \rho_0 \|_{L^\infty}} \left[ \int_0^t \| \nabla^2 u \|^2_{L^2} ds + \frac{1}{2} \int_0^t \| \sqrt{\rho(s)} \|^2_{L^2} ds \right],
\]
\[
\Phi(t) := \| \rho_0 \|_{L^\infty} \| \sqrt{\rho_0} u_0 \|^2_{L^2} \sup_{0 \leq s \leq t} \| \nabla u \|^2_{L^2},
\]
where \( L \) is the same as in (3.9). In view of the regularities of \( u \), one can obtain that both \( E(t) \) and \( \Phi(t) \) are continuous functions on \((0, T)\). By (3.9), there is a positive constant \( M \) depending only on \( \mu \) such that
\[
E(t) \leq \| \nabla u_0 \|^2_{L^2} + M \sqrt{\Phi(t)} E(t).
\] (3.18)

We set
\[
\varepsilon_0 := \min \left\{ \frac{1}{16M}, \frac{1}{32M^2} \right\},
\]
and suppose that
\[
\| \rho_0 \|_{L^\infty} \| \sqrt{\rho_0} u_0 \|^2_{L^2} \| \nabla u_0 \|^2_{L^2} \leq \varepsilon_0.
\]
We claim that
\[
\Phi(t) < \min \left\{ \frac{1}{2M}, \frac{1}{4M^2} \right\}, \quad 0 \leq t \leq T.
\]
Otherwise, by the continuity and monotonicity of $\Phi(t)$, there is a $T_0 \in (0, T]$ such that

$$\Phi(T_0) = \min \left\{ \frac{1}{2M}, \frac{1}{4M^2} \right\}. \tag{3.19}$$

On account of (3.19), it follows from (3.18) that

$$E(T_0) \leq \|\nabla u_0\|_{L^2}^2 + \frac{1}{2} E(T_0),$$

and hence

$$E(T_0) \leq 2\|\nabla u_0\|_{L^2}^2.$$  
Recalling the definition of $E(t)$ and $\Phi(t)$, we deduce from the above inequality that

$$\Phi(T_0) \leq \|\rho_0\|_{L^\infty} \|\sqrt{\rho_0} u_0\|_{L^2}^2 E(T_0) \leq 2\|\rho_0\|_{L^\infty} \|\sqrt{\rho_0} u_0\|_{L^2}^2 \|\nabla u_0\|_{L^2}^2 \leq 4 \varepsilon_0 = \min \left\{ \frac{1}{4M}, \frac{1}{8M^2} \right\},$$

which contradicts with (3.19).

By virtue of the claim we showed in the above, we derive from (3.18) that

$$E(t) \leq 2\|\nabla u_0\|_{L^2}^2, \quad 0 < t \leq T,$$

provided that (3.17) holds true. This implies (3.16) and completes the proof of Lemma 3.3. \hfill \Box

Lemma 3.4. Let the condition (3.17) be satisfied, then there exists a positive constant $C$ depending only on $\mu$ and the initial data such that, for $i \in \{1, 2\}$,

$$\sup_{0 \leq t \leq T} (t^i \|\nabla u\|_{L^2}^2) + \int_0^T t^i (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2) \, dt \leq C. \tag{3.20}$$

Moreover, for $\sigma$ as that of in Lemma 3.1, one has that

$$\sup_{0 \leq t \leq T} (e^{\sigma t} \|\nabla u\|_{L^2}^2) + \int_0^T e^{\sigma t} (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2) \, dt \leq C. \tag{3.21}$$

Proof. 1. Adding (3.13) multiplied by $\frac{1}{2L}$ to (3.10), we infer from (3.2), H"older’s inequality, Sobolev’s inequality, and (3.16) that

$$\mu \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\rho} u_t\|_{L^2}^2 + \frac{1}{2L} \|\nabla^2 u\|_{L^2}^2 \leq C \int \rho |u|^2 |\nabla u|^2 \, dx$$

$$\leq C \|\rho\|_{L^\infty} \|u\|_{L^4}^2 \|\nabla u\|_{L^4}^2$$

$$\leq C \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}^2 \|\nabla u\|_{L^2}^3 \|\nabla u\|_{L^2}^3$$

$$\leq C \|\nabla u\|_{L^2}^2 + \frac{1}{4L} \|\nabla^2 u\|_{L^2}^2,$$

which implies that

$$\frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^2. \tag{3.22}$$

Multiplying (3.22) by $t$ yields that

$$\frac{d}{dt} (t \|\nabla u\|_{L^2}^2) + t (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2) \leq Ct \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2. \tag{3.23}$$

From (3.8), one has

$$\int_0^T t \|\nabla u\|_{L^2}^2 \, dt = \int_0^T te^{-\sigma t} e^{\sigma t} \|\nabla u\|_{L^2}^2 \, dt \leq \sup_{0 \leq t \leq T} (te^{-\sigma t}) \int_0^T e^{\sigma t} \|\nabla u\|_{L^2}^2 \, dt \leq C, \tag{3.24}$$
which combined with (3.23) and Gronwall’s inequality leads to
\[
\sup_{0 \leq t \leq T} (t\|\nabla u\|_{L^2}^2) + \int_0^T t \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \right) dt \leq C. \tag{3.25}
\]
For \( i = 2 \), we can obtain similar result and omit the details for simplicity.

2. Multiplying (3.22) by \( e^{\sigma t} \), we derive that
\[
\frac{d}{dt} \left( e^{\sigma t} \|\nabla u\|_{L^2}^2 \right) + e^{\sigma t} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \right) \leq C e^{\sigma t} \|\nabla u\|_{L^2}^2 + \sigma e^{\sigma t} \|\nabla u\|_{L^2}^2
\]
\[
\leq C e^{\sigma t} \|\nabla u\|_{L^2}^2. \tag{3.26}
\]
Integrating (3.26) over \((0, T]\) together with (3.8) leads to (3.21).

Next, we show the regularity of the temperature \( \theta \) as follows.

**Lemma 3.5.** Let the condition (3.17) be satisfied, then there exists a positive constant \( C \) depending only on \( \mu, \ c_v, \ \kappa, \) and the initial data such that for \( i \in \{0, 1, 2\} \),
\[
\sup_{0 \leq t \leq T} \left( t^i \|\nabla^i \theta\|_{L^2}^2 \right) + \int_0^T t^i \|\nabla^i \theta\|_{L^2}^2 dt \leq C. \tag{3.27}
\]

**Proof.** 1. Multiplying (3.1)_3 by \( \theta \) and integrating the resulting equation over \( \mathbb{R}^3 \) yield that
\[
c_v \frac{d}{dt} \|\nabla \theta\|_{L^2}^2 + 2\kappa \|\nabla \theta\|_{L^2}^2 \leq \tilde{C} \int |\nabla u|^2 \theta dx \tag{3.28}
\]
for some positive constant \( \tilde{C} \) depending only on \( \mu \). To estimate \( \int |\nabla u|^2 \theta dx \), motivated by [7, Lemma 6], we multiply (3.1)_2 by \( u \theta \) and integrate the resulting equation over \( \mathbb{R}^3 \) to obtain that
\[
\tilde{C} \int |\nabla u|^2 \theta dx \leq C(\mu) \int \left( \rho |u|^i \theta + \rho |u|^i |\nabla u|^i \theta + |u| |\nabla u|^i \theta + P |u|^i |\nabla \theta| \right) dx
\]
\[
\leq \kappa \|\nabla \theta\|_{L^2}^2 + C(\mu, \kappa) \|\sqrt{\rho} u\|_{L^2}^2 + C(\mu, \kappa) \int (\rho |u|^2 \theta^2 + |u|^2 |\nabla u|^2 + P^2 |u|^2) dx
\]
\[
\leq \kappa \|\nabla \theta\|_{L^2}^2 + C \|\sqrt{\rho} u\|_{L^2}^2 + C \|u\|_{L^6} \|\nabla u\|_{L^2} \|\sqrt{\rho} \theta\|_{L^6} \|\nabla u\|_{L^2} \|\sqrt{\rho} \theta\|_{L^6} \|\nabla u\|_{L^2}
\]
\[
+ C \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} + C \|P\|_{L^2} \|\nabla P\|_{L^2} \|\nabla u\|_{L^2} \|\nabla u\|_{H^1}
\]
\[
+ C \|P\|_{L^2} \|\nabla P\|_{L^2} \|\nabla u\|_{L^2} \tag{3.29}
\]
due to Hölder’s inequality, (2.1), (2.2), and Sobolev’s inequality. Moreover, we get from Lemma 2.3, (3.2), and (3.16) that
\[
\|P\|_{L^2} \|\nabla P\|_{L^2} \leq C \left( \|\rho u\|_{L^\frac{8}{5}} + \|\rho u \cdot \nabla u\|_{L^\frac{8}{3}} \right) \left( \|\rho u\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} \right)
\]
\[
\leq C \|\rho\|_{L^\frac{1}{2}} \left( \|\sqrt{\rho} u\|_{L^2} + \|\sqrt{\rho} \cdot \nabla u\|_{L^2} \right) \|\nabla \theta\|_{L^\infty} \left( \|\sqrt{\rho} u\|_{L^2} + \|\sqrt{\rho} \cdot \nabla u\|_{L^2} \right)
\]
\[
\leq C \|\sqrt{\rho} u\|_{L^2}^2 + C \|\rho u \cdot \nabla u\|_{L^2}^2
\]
\[
\leq C \|\sqrt{\rho} u\|_{H^1}^2 + C \|\nabla u\|_{H^1}^2, \tag{3.30}
\]
which combined with (3.28), (3.29), and (3.16) implies that
\[
c_v \frac{d}{dt} \|\sqrt{\rho} \theta\|_{L^2}^2 + \kappa \|\nabla \theta\|_{L^2}^2 \leq C \|\nabla u\|_{H^1} \|\sqrt{\rho} \theta\|_{L^2}^2 + C \|\nabla u\|_{H^1}^2 \|\sqrt{\rho} u\|_{L^2}^2 \tag{3.31}
\]
This together with Gronwall’s inequality, (3.3), and (3.16) leads to
\[
\sup_{0 \leq t \leq T} \|\sqrt{\rho} \theta\|_{L^2}^2 + \int_0^T \|\nabla \theta\|_{L^2}^2 dt \leq C. \tag{3.32}
\]
2. Multiplying (3.31) by $t$ gives that
\[
c_v \frac{d}{dt}(t\|\sqrt{\rho_t}\|_{L^2}^2) + \kappa t\|\nabla \theta\|_{L^2}^2
\leq C\|\nabla u\|_{H^1}^2(t\|\sqrt{\rho_t}\|_{L^2}^2) + C t\|\nabla u\|_{L^2}^2 + C t\|\nabla u\|_{L^2}^2 + c_v \|\sqrt{\rho_t}\|_{L^2}^2
\leq C\|\nabla u\|_{H^1}^2(t\|\sqrt{\rho_t}\|_{L^2}^2) + C t\|\nabla u\|_{L^2}^2 + C t\|\nabla u\|_{L^2}^2 + c_v \|\rho\|_{L^2}^2 \theta \|_{L^2}^2
\leq C\|\nabla u\|_{H^1}^2(t\|\sqrt{\rho_t}\|_{L^2}^2) + C t\|\nabla u\|_{L^2}^2 + C t\|\nabla u\|_{L^2}^2 + c_v \|\nabla \theta\|_{L^2}^2
\] (3.33)
due to Hölder’s inequality, (3.2), and Sobolev’s inequality. Thus, we obtain from (3.33), Gronwall’s inequality, (3.20), (3.24), and (3.32) that
\[
\sup_{0 \leq t \leq T} t\|\nabla \theta\|_{L^2}^2 + \int_0^T t\|\nabla \theta\|_{L^2}^2 dt \leq C.
\] (3.34)

For $i = 2$ in (3.27), we can also obtain the desired and omit the details for simplicity.

**Lemma 3.6.** Let the condition (3.17) be satisfied, then there exists a positive constant $C$ depending only on $\mu$, $c_v$, $\kappa$, and the initial data such that, for $i \in \{1, 2\}$,
\[
\sup_{0 \leq t \leq T} t\|\nabla u_t\|_{L^2}^2 + \int_0^T t\|\nabla u_t\|_{L^2}^2 + \|\sqrt{\rho_t}\|_{L^2}^2 dt \leq C.
\] (3.35)

Moreover, for $\sigma$ as that in Lemma 3.1, one has that, for $\zeta(T) := \min\{1, T\}$,
\[
\sup_{\zeta(T) \leq t \leq T} e^{\sigma t}\|\nabla u_t\|_{L^2}^2 + \int_{\zeta(T)}^T e^{\sigma t}\|\nabla u_t\|_{L^2}^2 dt \leq C.
\] (3.36)

**Proof.** 1. Differentiating (3.1)$_2$ with respect to $t$, we arrive at
\[
\rho u_{tt} + \rho u \cdot \nabla u_t - \mu A u_t = -\nabla P_t + \rho_t(u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u.
\] (3.37)

Multiplying (3.37) by $u_t$, integrating the resulting equation over $\mathbb{R}^3$ and using (1.1)$_1$ yield that
\[
\frac{1}{2} \frac{d}{dt} \int \rho|u_t|^2 dx + \mu \int |\nabla u_t|^2 dx
= \int \text{div}(\rho u)|u_t|^2 dx + \int \text{div}(\rho u) u \cdot \nabla u_{tt} - \int \rho u_t \cdot \nabla u \cdot u_t dx =: J_1 + J_2 + J_3.
\] (3.38)

By virtue of Hölder’s inequality, Sobolev’s inequality, (3.2), and (3.16), we deduce that
\[
|J_1| = \left| - \int \rho u \cdot \nabla |u_t|^2 dx \right|
\leq 2\|\rho\|_{L^2}^2 \|u\|_{L^6} \|\nabla u_t\|_{L^2} \|\nabla u_t\|_{L^2}
\leq C\|\rho\|_{L^2}^2 \|\nabla u\|_{L^2} \\|\sqrt{\rho_t}\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla u_t\|_{L^2}
\leq C\|\rho\|_{L^2}^2 \|\nabla u\|_{L^2} \\|\sqrt{\rho_t}\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla u_t\|_{L^2}
\leq \mu \|\nabla u_t\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2;
\]
\[
|J_2| = \left| - \int \rho u \cdot \nabla (u \cdot \nabla u_t) dx \right|
\leq \int (\rho |u| \|\nabla u\|_{L^2}^2 |u_t| + \rho |u|^2 \|\nabla^2 u\|_{L^2} |u_t| + \rho |u|^2 \|\nabla u\|_{L^2} |u_t|) dx
\leq C\|u\|_{L^6} \|\nabla u\|_{L^2} \|\nabla u\|_{L^6} \|u_t\|_{L^6} + C\|u\|_{L^6}^2 \|\nabla^2 u\|_{L^2} \|u_t\|_{L^6} + C\|u\|_{L^6} \|\nabla u\|_{L^6} \|u_t\|_{L^2}
\leq C\|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2} \|u_t\|_{L^6} + C\|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2} \|u_{tt}\|_{L^2}
\[
\begin{align*}
&\leq \frac{\mu}{6}\|\nabla u_t\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2; \\
&\left|J_3\right| \leq \|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2} \leq C\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^2 \|\sqrt{\rho}u_t\|_{L^2}^2 \\
&\leq C\|\rho\|_{L^\infty}^3 \|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^2}^3 \\
&\leq \frac{\mu}{6}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2.
\end{align*}
\]
Substituting the above estimates into (3.38), we derive that
\[
\frac{d}{dt}\|\sqrt{\rho}u_t\|_{L^2}^2 + \mu \|\nabla u_t\|_{L^2}^2 \leq C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2,
\tag{3.39}
\]
which multiplied by \(t^i\) (\(i \in \{1, 2\}\)) together with Gronwall's inequality, (3.16), and (3.20) yields that
\[
\sup_{0 \leq t \leq T} (t^i\|\sqrt{\rho}u_t\|_{L^2}^2) + \int_0^T t^i\|\nabla u_t\|_{L^2}^2 dt \leq C. \tag{3.40}
\]

2. We obtain from (3.13), (3.2), Hölder's inequality, Sobolev's inequality, and (3.16) that
\[
\|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 \leq C\left(\|\rho u_t\|_{L^2}^2 + \|\rho \cdot \nabla u\|_{L^2}^2\right)
\leq C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\rho\|_{L^\infty}^2 \|u\|_{L^6}^2 \|\nabla u\|_{L^2}^2 \\
\leq C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \|\sqrt{\rho}u_t\|_{L^2}^2 \|\nabla^2 u\|_{L^2}^2 \\
\leq C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|\nabla^2 u\|_{L^2}^2,
\]
which implies that
\[
\|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 \leq C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2. \tag{3.41}
\]
Multiplying (3.1)_3 by \(\theta_t\) and integrating the resulting equation over \(\mathbb{R}^3\) give rise to
\[
\frac{\kappa}{2} \frac{d}{dt}\int |\nabla \theta|^2 dx + c_v \int \rho |\theta_t|^2 dx = -c_v \int \rho (u \cdot \nabla \theta) \theta_t dx + 2\mu \int |\mathcal{D}(u)|^2 \theta_t dx =: I_1 + I_2. \tag{3.42}
\]
By (3.2), (2.2) with \(q = r = 6\), and Sobolev's inequality, we get that
\[
|I_1| \leq c_v \rho \|\theta_t\|_{L^\infty}^2 \|\nabla u\|_{L^\infty} \|\theta\|_{L^2} \\
\leq C\|\rho\|_{L^\infty}^2 \|\sqrt{\rho}u_t\|_{L^2}^2 \|\nabla u\|_{L^2} \|\theta\|_{L^2} \\
\leq C\|\rho\|_{L^\infty}^2 \|\sqrt{\rho}u_t\|_{L^2}^2 \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \|\theta\|_{L^2} \\
\leq \frac{c_v}{2} \|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{H^1}^2 \|\theta\|_{L^2}^2. \tag{3.43}
\]
From Hölder’s inequality, Sobolev’s inequality, and (2.1) with \(p = 3\), one has
\[
I_2 = 2\mu \frac{d}{dt}\int |\mathcal{D}(u)|^2 \theta_t dx - 2\mu \int (|\mathcal{D}(u)|^2)_t \theta_t dx \\
\leq 2\mu \frac{d}{dt}\int |\mathcal{D}(u)|^2 \theta_t dx + C \int |\nabla u| \nabla u_t |dx \\
\leq 2\mu \frac{d}{dt}\int |\mathcal{D}(u)|^2 \theta_t dx + C \|\theta\|_{L^6} \|\nabla u\|_{L^3} \|\nabla u_t\|_{L^2} \\
\leq 2\mu \frac{d}{dt}\int |\mathcal{D}(u)|^2 \theta_t dx + C \|\nabla \theta\|_{L^2} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \|\nabla u_t\|_{L^2} \\
\leq 2\mu \frac{d}{dt}\int |\mathcal{D}(u)|^2 \theta_t dx + C \|\nabla u_t\|_{L^2}^2 + C \|\nabla u\|_{H^1}^2 \|\theta\|_{L^2}^2, \tag{3.44}
\]
Substituting (3.43) and (3.44) into (3.42), we obtain that
\[ B'(t) + c_v \| \sqrt{\rho} \theta \|_{L^2}^2 \leq C \| \nabla u \|_{H^1}^2 \| \nabla \theta \|_{L^2}^2 + C \| \nabla u_t \|_{L^2}^2, \]
where
\[ B(t) := \int (\kappa \| \nabla \theta \|^2 - 4\mu \| \mathcal{O}(u) \|^2 \theta) \, dx \]
satisfies
\[ \frac{\kappa}{2} \| \nabla \theta \|_{L^2}^2 - C \| \nabla u \|_{L^2}^2 - C \| \sqrt{\rho} u_t \|_{L^2}^2 \leq B(t) \leq C \| \nabla \theta \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 + C \| \sqrt{\rho} u_t \|_{L^2}^2. \]
Here we have used
\[ 4\mu \int |\mathcal{O}(u)|^2 \theta \, dx \leq C \| \nabla \theta \|_{L^2} \| \nabla u \|_{L^2} \| \nabla u \|_{L^2}^2 \]
\[ \leq C \| \nabla \theta \|_{L^2} \| \nabla u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2}^2 \]
\[ \leq C \| \nabla \theta \|_{L^2} \| \nabla u \|_{L^2}^2 + | \sqrt{\rho} u_t \|_{L^2} \| \nabla u \|_{L^2}^2 \]
due to (2.1) with \( p = 3 \), (3.41), and (3.16). Multiplying (3.45) by \( t^i \) (\( i \in \{1, 2\} \)) together with (3.46) leads to
\[ \frac{d}{dt} (t^i B(t)) + c_v t^i \| \sqrt{\rho} \theta \|_{L^2}^2 \leq C \| \nabla u \|_{H^1}^2 (t^i \| \nabla \theta \|_{L^2}^2) + Ct^i \| \nabla u_t \|_{L^2}^2 \]
\[ + Ct^{i-1} (t^i \| \nabla \theta \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 + \| \sqrt{\rho} u_t \|_{L^2}^2), \]
which combined with Gronwall’s inequality, (3.46), (3.40), (3.20), (3.27), and (3.16) yields that
\[ \sup_{0 \leq t \leq T} (t^i \| \nabla \theta \|_{L^2}^2) + \int_0^T t^i \| \sqrt{\rho} \theta \|_{L^2}^2 \, dt \leq C. \]
This along with (3.40) gives rise to (3.35).

3. Multiplying (3.39) by \( e^{\sigma t} \), we have
\[ \frac{d}{dt} (e^{\sigma t} \| \sqrt{\rho} u_t \|_{L^2}^2) + \mu e^{\sigma t} \| \nabla u_t \|_{L^2}^2 \leq Ce^{\sigma t} \| \sqrt{\rho} u_t \|_{L^2}^2 + Ce^{\sigma t} \| \nabla^2 u \|_{L^2}^2 + \sigma e^{\sigma t} \| \sqrt{\rho} u_t \|_{L^2}^2 \]
\[ \leq Ce^{\sigma t} (\| \sqrt{\rho} u_t \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^2). \]
This combined with Gronwall’s inequality and (3.21) implies (3.36).

Lemma 3.7. Let the condition (3.17) be satisfied, then there exists a positive constant \( C \) depending only on \( \mu, c_v, \kappa, \) and the initial data such that
\[ \int_0^T \| \nabla u \|_{L^\infty} \, dt \leq C. \]

Proof. 1. One gets from (2.2) with \( (q, r) = (2, 4) \), Lemma 2.3, and (3.16) that
\[ \| \nabla u \|_{L^\infty} \leq C \| \nabla u \|_{L^2}^\frac{1}{2} \| \nabla^2 u \|_{L^4}^\frac{1}{4} \]
\[ \leq C (\| \rho u_t \|_{L^4} + \| \rho u \cdot \nabla u \|_{L^4})^\frac{1}{2} \]
\[ \leq C \| \rho u_t \|_{L^4}^\frac{1}{2} + C \| \rho u \cdot \nabla u \|_{L^4}^\frac{1}{2}. \]

2. By Hölder’s inequality, Sobolev’s inequality, and (3.2), we have
\[ \| \rho u_t \|_{L^4} \leq \| \rho \|_{L^\infty}^\frac{1}{4} \| \sqrt{\rho} u_t \|_{L^2}^\frac{1}{2} \| u_t \|_{L^6}^\frac{1}{3} \leq C \| \sqrt{\rho} u_t \|_{L^2}^\frac{1}{2} \| \nabla u_t \|_{L^2}^\frac{1}{2}, \]
which implies for any $0 \leq a < b < \infty$,
\[
\int_a^b \left\| \rho u_t \right\|_{L^4}^2 dt \leq C \int_a^b \left\| \sqrt{\rho} u_t \right\|_{L^2}^8 \left\| \nabla u \right\|_{L^2} dt.  \tag{3.51}
\]

As a consequence, if $T \leq 1$, we obtain from (3.51), Hölder’s inequality, and (3.35) that
\[
\int_0^T \left\| \rho u_t \right\|_{L^4}^2 dt \leq C \int_0^T t^{\frac{5}{12}} \left\| \sqrt{\rho} u_t \right\|_{L^2}^\frac{7}{2} \cdot t^{-\frac{1}{2}} \cdot t^\frac{1}{2} \left\| \nabla u_t \right\|_{L^2} dt
\leq C \left( \sup_{0 \leq t \leq T} \int_0^T t \left\| \sqrt{\rho} u_t \right\|_{L^2}^2 dt \right)^\frac{1}{12} \left( \int_0^T t^{\frac{1}{2}} dt \right)^\frac{5}{12} \left( \int_0^T t \left\| \nabla u_t \right\|_{L^2}^2 dt \right)^\frac{1}{4}
\leq CT^{\frac{5}{12}} \leq C. \tag{3.52}
\]

If $T > 1$, one deduces from (3.52), (3.51), Hölder’s inequality, (3.36), and (3.35) that
\[
\int_0^T \left\| \rho u_t \right\|_{L^4}^2 dt = \int_0^1 \left\| \rho u_t \right\|_{L^4}^2 dt + \int_1^T \left\| \rho u_t \right\|_{L^4}^2 dt
\leq C + C \int_1^T \left\| \sqrt{\rho} u_t \right\|_{L^2}^\frac{7}{2} \left\| \nabla u_t \right\|_{L^2}^\frac{1}{2} dt
\leq C + C \left( \sup_{1 \leq t \leq T} \int_1^T e^{\sigma t} \left\| \sqrt{\rho} u_t \right\|_{L^2}^2 dt \right)^\frac{1}{12} \left( \int_1^T \left( e^{-\frac{5}{6}t} t^{-\frac{1}{2}} \right)^\frac{3}{2} dt \right)\frac{5}{12} \left( \int_1^T t \left\| \nabla u_t \right\|_{L^2}^2 dt \right)^\frac{1}{4}
\leq C + C \left( \int_1^T e^{-\frac{5}{6}t} dt \right)^\frac{3}{4} \leq C, \tag{3.53}
\]
where $\sigma$ is the same as that in Lemma 3.1. Hence, we infer from (3.52) and (3.53) that
\[
\int_0^T \left\| \rho u_t \right\|_{L^4}^2 dt \leq C. \tag{3.54}
\]

3. By (3.2), (2.2) with $(q, r) = (6, 6)$, and (2.1) with $p = 4$, we have
\[
\left\| \rho u \cdot \nabla u \right\|_{L^4} \leq \left\| \rho \right\|_{L^\infty} \left\| u \right\|_{L^\infty} \left\| \nabla u \right\|_{L^4}
\leq C \left\| u \right\|_{L^6}^\frac{1}{2} \left\| \nabla u \right\|_{L^6}^\frac{1}{2} \left\| \nabla u \right\|_{L^2}^\frac{1}{2} \left\| \nabla^2 u \right\|_{L^2}^\frac{1}{2}
\leq C \left\| \nabla u \right\|_{L^2}^3 \left\| \nabla^2 u \right\|_{L^2}^\frac{5}{2},
\]
which together with (3.41) and Young’s inequality yields that
\[
\left\| \rho u \cdot \nabla u \right\|_{L^4}^2 \leq C \left\| \nabla u \right\|_{L^2}^\frac{1}{2} \left\| \nabla^2 u \right\|_{L^2}^\frac{5}{2}
\leq C \left\| \nabla u \right\|_{L^2}^\frac{1}{2} \left( \left\| \sqrt{\rho} u_t \right\|_{L^2}^\frac{5}{2} + \left\| \nabla u \right\|_{L^2}^\frac{5}{2} \right)
\leq C \left\| \nabla u \right\|_{L^2}^\frac{1}{2} \left\| \sqrt{\rho} u_t \right\|_{L^2}^\frac{5}{2} + C \left\| \nabla u \right\|_{L^2}^\frac{7}{2}
\leq C \left\| \sqrt{\rho} u_t \right\|_{L^2}^2 + C \left\| \nabla u \right\|_{L^2}^\frac{5}{2} + C \left\| \nabla u \right\|_{L^2}^\frac{7}{2}
\leq C \left\| \sqrt{\rho} u_t \right\|_{L^2}^2 + C e^{\sigma t} \left\| \nabla u \right\|_{L^2}^2 + e^{-\frac{1}{2} \alpha t} + e^{-2 \sigma t}. \tag{3.55}
\]

Thus, we obtain from (3.55), (3.16), and (3.4) that
\[
\int_0^T \left\| \rho u \cdot \nabla u \right\|_{L^4}^2 dt \leq C. \tag{3.56}
\]
This together with (3.54) and (3.50) leads to (3.49). \qed
Lemma 3.8. Let the condition (3.17) be satisfied, then there exists a positive constant $C$ depending only on $\mu$, $c_v$, $\kappa$, and the initial data such that
\[
\sup_{0 \leq t \leq T} \left( \|\nabla \rho\|_{L^2 \cap L^6} + \|\rho_t\|_{L^2 \cap L^3} \right) \leq C. \tag{3.57}
\]

Proof. Taking spatial derivative $\nabla$ on the transport equation (3.1) leads to
\[
\partial_t \nabla \rho + u \cdot \nabla^2 \rho + \nabla u \cdot \nabla \rho = 0. \tag{3.58}
\]
For $q \in [2, 6]$, multiplying (3.58) by $q|\nabla \rho|^{q-2}\nabla \rho$ and integrating the resulting equation over $\mathbb{R}^3$ give rise to
\[
\frac{d}{dt} \int |\nabla \rho|^q dx + q \int u \cdot \nabla^2 \rho \cdot |\nabla \rho|^{q-2} \nabla \rho dx = -q \int \nabla u \cdot \nabla \rho \cdot |\nabla \rho|^{q-2} \nabla \rho dx.
\]
Integration by parts together with $\text{div} \ u = 0$ indicates that
\[
q \int u \cdot \nabla^2 \rho \cdot |\nabla \rho|^{q-2} \nabla \rho dx = \int u \cdot \nabla (|\nabla \rho|^q) dx = -\int |\nabla \rho|^q \text{div} \ u dx = 0.
\]
Thus, we get that
\[
\frac{d}{dt} \|\nabla \rho\|_{L^q}^q \leq q \int |\nabla u| |\nabla \rho|^q dx \leq q \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^q}^q.
\]
This implies that
\[
\frac{d}{dt} \|\nabla \rho\|_{L^q} \leq \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^q},
\]
which combined with Gronwall’s inequality and (3.49) gives that
\[
\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^q} \leq C. \tag{3.59}
\]
Noticing the following facts
\[
\|\rho_t\|_{L^2} = \|u \cdot \nabla \rho\|_{L^2} \leq \|\nabla \rho\|_{L^3} \|u\|_{L^6} \leq C \|\nabla \rho\|_{L^2}^{\frac{1}{2}} \|\nabla \rho\|_{L^6}^{\frac{1}{2}} \|\nabla u\|_{L^2},
\]
\[
\|\rho_t\|_{L^3} = \|u \cdot \nabla \rho\|_{L^3} \leq \|\nabla \rho\|_{L^6} \|u\|_{L^6} \leq C \|\nabla \rho\|_{L^6} \|\nabla u\|_{L^2},
\]
which together with (3.59) and (3.16) yields that
\[
\sup_{0 \leq t \leq T} \|\rho_t\|_{L^2 \cap L^3} \leq C. \tag{3.60}
\]
Thus, the desired (3.57) follows from (3.59) and (3.60). \hfill \Box

Lemma 3.9. Let the condition (3.17) be satisfied, then there exists a positive constant $C$ depending only on $\mu$, $c_v$, $\kappa$, and the initial data such that
\[
\sup_{0 \leq t \leq T} \left( t^2 \|\sqrt{\rho} \theta_t\|_{L^2}^2 \right) + \int_0^T t^2 \|\nabla \theta_t\|_{L^2}^2 dt \leq C. \tag{3.61}
\]

Proof. Differentiating (3.1) with respect to $t$ and using (1.1), we arrive at
\[
c_v \rho \theta_{tt} + \rho u \cdot \nabla \theta_t - \kappa \Delta \theta_t = c_v \text{div} \ (\rho u) \theta_t - c_v \rho u \cdot \nabla \theta - c_v \rho u_t \cdot \nabla \theta + 2 \mu (|\mathbf{D}(u)|^2)_t. \tag{3.62}
\]
Multiplying (3.62) by $\theta_t$ and integrating (by parts) over $\mathbb{R}^3$ yield that
\[
\frac{c_v}{2} \frac{d}{dt} \int \rho |\theta_t|^2 dx + \kappa \int |\nabla \theta_t|^2 dx
\]
\[
= c_v \int \text{div} (\rho u) |\theta_t|^2 dx + c_v \int \text{div} (\rho u) (u \cdot \nabla \theta) \theta_t dx - c_v \int \rho (u_t \cdot \nabla \theta) \theta_t dx
\]
\[
+ 2 \mu \int (|\mathbf{D}(u)|^2)_t \theta_t dx =: \sum_{k=1}^{4} J_k. \tag{3.63}
\]
By Hölder’s inequality, Sobolev’s inequality, (2.2), (3.2), (3.20), (3.57), and (3.41), we have
\[
|J_1| = \left| -c_v \int \rho \mathbf{u} \cdot \nabla |\theta| \, dt \right|
\leq 2c_v \|\rho\|_{L^\infty} \|\mathbf{u}\|_{L^\infty} \|\sqrt{\rho} \theta\|_{L^2} \|\nabla \theta\|_{L^2}
\leq C \|\mathbf{u}\|_{L^6} \|\nabla \mathbf{u}\|_{L^6} \|\sqrt{\rho} \theta\|_{L^2} \|\nabla \theta\|_{L^2}
\leq \frac{K}{8} \|\nabla \theta\|_{L^2}^2 + C \|\nabla \mathbf{u}\|_{H^1} \|\sqrt{\rho} \theta\|_{L^2}^2;
\]
\[
|J_2| \leq c_v \int |\rho_t| \|\mathbf{u}\| \|\nabla \theta\| |\theta_t| \, dx
\leq c_v \|\rho_t\|_{L^3} \|\mathbf{u}\|_{L^\infty} \|\nabla \theta\|_{L^2} \|\theta_t\|_{L^6}
\leq C \|\mathbf{u}\|_{L^6} \|\nabla \mathbf{u}\|_{L^6} \|\sqrt{\rho} \theta\|_{L^2} \|\nabla \theta\|_{L^2}
\leq \frac{K}{8} \|\nabla \theta\|_{L^2}^2 + C \|\nabla \theta\|_{L^2}^2 \|\nabla \mathbf{u}\|_{L^2}^2 + C \|\sqrt{\rho} \mathbf{u}\|_{L^2} \|\nabla \theta\|_{L^2}^2;
\]
\[
|J_3| \leq \|\rho\|_{L^\infty} \|\sqrt{\rho} \mathbf{u}\|_{L^3} \|\nabla \theta\|_{L^2} \|\theta_t\|_{L^6}
\leq \|\rho\|_{L^\infty} \|\sqrt{\rho} \mathbf{u}\|_{L^2} \|\sqrt{\rho} \mathbf{u}\|_{L^6} \|\nabla \theta\|_{L^2} \|\nabla \theta\|_{L^2}
\leq \frac{K}{8} \|\nabla \theta\|_{L^2}^2 + C \|\sqrt{\rho} \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2}^2 + C \|\nabla \mathbf{u}\|_{L^2}^2.
\]

Substituting the above estimates into (3.63), we derive that
\[
c_v \frac{d}{dt} \|\sqrt{\rho} \theta\|_{L^2}^2 + \kappa \|\nabla \theta\|_{L^2}^2 \leq C \|\nabla \mathbf{u}\|_{H^1} \|\sqrt{\rho} \theta\|_{L^2}^2 + C \|\sqrt{\rho} \mathbf{u}\|_{L^2} \|\nabla \theta\|_{L^2} \|\nabla \mathbf{u}\|_{L^2}^2
\leq C \|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2}^2 + C \|\nabla \mathbf{u}\|_{L^2}^2 \|\nabla \theta\|_{L^2}^2.
\] (3.64)

Multiplying (3.64) by \(t^2\) yields that
\[
c_v \frac{d}{dt} (t^2 \|\sqrt{\rho} \theta\|_{L^2}^2) + \kappa t^2 \|\nabla \theta\|_{L^2}^2
\leq C \|\nabla \mathbf{u}\|_{H^1} (t^2 \|\sqrt{\rho} \theta\|_{L^2}^2) + C t \|\sqrt{\rho} \mathbf{u}\|_{L^2} \|\nabla \theta\|_{L^2} (t \|\nabla \mathbf{u}\|_{L^2})
\leq C t^2 \|\nabla \mathbf{u}\|_{L^2}^2 + C \|\sqrt{\rho} \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{H^1} \|\nabla \theta\|_{L^2}^2.
\] (3.65)

which combined with Gronwall’s inequality, (3.40), (3.16), and (3.48) leads to (3.61).

Lemma 3.10. Let the condition (3.17) be satisfied, then there exists a positive constant \(C\) depending only on \(\mu, c_v, \kappa,\) and the initial data such that
\[
\sup_{0 \leq t \leq T} \left[ t^2 (\|\nabla \mathbf{u}\|_{H^1}^2 + \|\nabla P\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) \right]
\leq C.
\] (3.66)
Moreover, for $\sigma$ as that in Lemma 3.1 and $\zeta(T)$ as in (3.36), one has that
\[
\sup_{\zeta(T) \leq t \leq T} \left[ e^{\sigma t} \left( \| \nabla u \|^2_{W^{1,1}} + \| \nabla P \|^2_{L^2} \right) \right] \leq C. \tag{3.67}
\]

**Proof.**
1. We obtain from (3.41), (3.40), and (3.20) yields that, for $i \in \{1, 2\}$,
\[
\sup_{0 \leq t \leq T} \left[ t^i \left( \| \nabla u \|^2_{W^{1,1}} + \| \nabla P \|^2_{L^2} \right) \right] \leq C. \tag{3.68}
\]

We derive from (3.41), (3.36), and (3.21) that
\[
\sup_{\zeta(T) \leq t \leq T} \left[ e^{\sigma t} \left( \| \nabla u \|^2_{W^{1,1}} + \| \nabla P \|^2_{L^2} \right) \right] \leq C. \tag{3.69}
\]

It follows from (3.13) and (1.3) that
\[
\begin{cases}
  -\kappa \Delta \theta = 2\mu |\nabla u|^2 - c_v \rho \theta_t - c_v \rho u \cdot \nabla \theta, \quad x \in \mathbb{R}^3, \\
  \lim_{|x| \to \infty} \theta = 0. \tag{3.70}
\end{cases}
\]

Hence the standard $L^p$-estimate to the elliptic equation (see, e.g., [4, Chapter 9]) gives rise to
\[
\| \nabla^2 \theta \|^2_{L^2} \leq C(\kappa, \mu, c_v) \left( \| \nabla^2 u \|^2_{L^2} + \| \rho \theta \|^2_{L^2} + \| \rho u \cdot \nabla \theta \|^2_{L^2} \right)
\leq C\| \nabla u \|^2_{L^4} + C\| \rho \|^2_{L^\infty} \| \sqrt{\rho} \theta_t \|^2_{L^2} + C\| \rho \|^2_{L^\infty} \| u \|^2_{L^6} \| \nabla \theta \|^2_{L^2},
\]
\[
\leq C\| \nabla u \|^2_{L^4} \| \nabla^2 u \|^2_{L^2} + C\| \sqrt{\rho} \theta_t \|^2_{L^2} + C\| \nabla u \|^2_{L^2} \| \nabla \theta \|^2_{L^2} \| \nabla^2 \theta \|^2_{L^2}.
\]
due to Hölder’s inequality, Sobolev’s inequality, (3.2), (2.1), and (3.16). Thus, one gets that
\[
\| \nabla \theta \|^2_{H^{1,1}} \leq C\| \nabla u \|^2_{H^{1,1}} \| \nabla \theta \|^2_{H^{1,1}} + C\| \sqrt{\rho} \theta_t \|^2_{L^2} + C\| \nabla \theta \|^2_{L^2}, \tag{3.71}
\]
which along with (3.68), (3.61), (3.48), and (3.27) yields that
\[
\sup_{0 \leq t \leq T} (\| \nabla \theta \|^2_{H^{1,1}}) \leq C. \tag{3.72}
\]

2. We get from (3.12) and Lemma 2.3 that
\[
\| \nabla u \|^2_{L^6} + \| \nabla P \|^2_{L^6} \leq C\| \rho u \|^2_{L^6} + C\| \rho u \cdot \nabla u \|^2_{L^6}.
\]
We obtain from Sobolev’s inequality and (3.2) that
\[
\| \rho u \|^2_{L^6} \leq \| \rho \|^2_{L^\infty} \| u \|^2_{L^6} \leq C\| \nabla u \|^2_{L^2}. \tag{3.73}
\]
Moreover, we have
\[
\| \rho u \cdot \nabla u \|^2_{L^6} \leq \| \rho \|^2_{L^\infty} \| u \|^2_{L^6} \| \nabla u \|^2_{L^6} \leq C\| \nabla u \|^2_{L^6} \| \nabla u \|^2_{L^6} \leq C\| \nabla u \|^2_{L^2} \| \nabla^2 u \|^2_{L^2}. \tag{3.74}
\]
Thus, we get that
\[
\| \nabla u \|^2_{W^{1,6}} + \| \nabla P \|^2_{L^6} \leq C\| \nabla u \|^2_{W^{1,6}} + C\| \nabla u \|^2_{L^2} + C\| \nabla u \|^2_{L^2} \| \nabla^2 u \|^2_{L^2} \leq C\| \nabla^2 u \|^2_{L^2} + C\| \nabla u \|^2_{L^2} + C\| \nabla^2 u \|^2_{L^2},
\]
which together with (3.35), (3.16), (3.68), and (3.20) implies that, for $i \in \{1, 2\}$,
\[
\int_0^T t^i \left( \| \nabla u \|^2_{W^{1,6}} + \| \nabla P \|^2_{L^6} \right) dt 
\leq C \int_0^T t^i \| \nabla u \|^2_{L^2} dt + C \int_0^T t^i \| \nabla u \|^2_{L^2} dt + C \int_0^T \| \nabla u \|^2_{L^2} \| \nabla^2 \nabla u \|^2_{L^2} (t^i - 1) \| \nabla^2 u \|^2_{L^2} dt 
\leq C + C \left( \sup_{0 \leq t \leq T} \| \nabla u \|^2_{L^2} \right) \left( \sup_{0 \leq t \leq T} t^i \| \nabla^2 \nabla u \|^2_{L^2} \right) \int_0^T t^i - 1 \| \nabla^2 \nabla u \|^2_{L^2} dt \leq C. \tag{3.75}
\]
3. We infer from (3.70) and $L^p$-estimate to the elliptic equation that
\[
\|\nabla^2 \theta\|_{L^6} \leq C(\kappa, \mu, c_0) \left( \|\nabla u\|^2_{L^6} + \|\rho \theta_t\|^2_{L^6} + \|\rho u \cdot \nabla \theta\|_{L^6}^2 \right).
\]
Applying (2.2) and the following Gagliardo-Nirenberg inequality
\[
\|\nabla f\|_{L^2(\mathbb{R}^3)} \leq C\|\nabla^2 f\|^\frac{3}{2}_{L^2(\mathbb{R}^3)} \|f\|^\frac{1}{2}_{L^\infty(\mathbb{R}^3)},
\]
we have
\[
\|\nabla u\|^2_{L^6} = \|\nabla u\|^2_{L^{12}} \leq C\|\nabla^2 u\|_{L^6}^2 \|u\|^2_{L^\infty} \leq C\|\nabla^2 u\|_{L^6}^2 \|\nabla u\|_{L^6} \|u\|_{L^6} \leq C\|\nabla^2 u\|^2_{L^6} \|\nabla u\|^2_{L^6}.
\]
Similarly to (3.73) and (3.74), we arrive at
\[
\|\rho \theta_t\|^2_{L^6} + \|\rho u \cdot \nabla \theta\|^2_{L^6} \leq C\|\nabla \theta_t\|^2_{L^2} + C\|\nabla u\|^2_{L^2} \|\nabla^2 u\|^2_{L^2} \|\nabla^2 \theta\|^2_{L^2}.
\]
Hence, there holds
\[
\|\nabla \theta\|^2_{W^{1,6}} \leq C\|\nabla \theta\|^2_{L^6} + C\|\nabla^2 \theta\|^2_{H^1} + C\|\nabla \theta_t\|^2_{L^2} + C\|\nabla u\|^2_{L^2} \|\nabla^2 u\|_{L^2} \|\nabla^2 \theta\|^2_{L^2}
\]
\[
\leq C\|\nabla^2 \theta\|^2_{L^2} + C\|\nabla^2 u\|^2_{L^6} \|\nabla u\|^2_{H^1} + C\|\nabla \theta_t\|^2_{L^2} + C\|\nabla \theta\|^2_{H^1} \|\nabla \theta\|^2_{L^2}
\]
\[
\leq C\|\nabla \theta\|^2_{H^1} \|\nabla u\|^2_{H^1} + C\|\nabla \theta\|^2_{L^2} + C\|\nabla^2 u\|^2_{L^6} \|\nabla \theta\|^2_{H^1}
\]
\[
+ C\|\nabla \theta\|^2_{L^2} + C\|\nabla \theta\|^2_{H^1} + C\|\nabla \theta\|^2_{L^2} \|\nabla \theta\|^2_{L^2}
\]
(3.76)
due to (3.71) and the following fact
\[
\|\sqrt{\rho} \theta_t\|^2_{L^2} \leq \|\rho\|^\frac{1}{2}_{L^\infty} \|\theta_t\|^2_{L^6} \leq C\|\nabla \theta_t\|^2_{L^2}.
\]
Thus, we infer from (3.76), (3.75), (3.68), (3.61), (3.72), (2.7), (3.3), and (3.16) that
\[
\int_0^T t^2 \|\nabla \theta\|^2_{W^{1,6}} dt \leq C \sup_{0 \leq t \leq T} \left[ t^2 (\|\nabla u\|^2_{H^1} + \|\nabla^2 \theta\|^2_{L^2}) \right] \int_0^T \|\nabla u\|^2_{H^1} dt + C \int_0^T t^2 \|\nabla \theta\|^2_{L^2} dt
\]
\[
+ C \sup_{0 \leq t \leq T} (t \|\nabla u\|^2_{H^1}) \int_0^T t \|\nabla^2 u\|^2_{L^6} dt + C \int_0^T t^2 \|\nabla \theta_t\|^2_{L^2} dt \leq C.
\]
This finishes the proof of Lemma 3.10. \qed

Proof of Theorem 1.1. With all the a priori estimates established in Lemmas 3.1–3.10, we can immediately obtain the existence result of Theorem 1.1 by standard arguments as those in [6]. Here we omit the details for simplicity. Moreover, the decay rate (1.7) follows from (3.36), (3.67), (3.27), (3.61), and (3.72). \qed

Acknowledgements. The author would like to express his gratitude to the reviewers for careful reading and helpful suggestions which led to an improvement of the original manuscript.

Declaration

Conflict of interest The author declares that he has no conflict of interest.

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(accepted: December 9, 2021; published online: December 22, 2021)