On the semi-symmetric Lorentzian spaces.

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Abstract
We give some properties of semisymmetric pseudo-Riemannian manifolds. These are foliated manifolds and for the Lorentzian metric, Ricci operator has only real eigenvalues.

Keywords: Lorentzian manifolds, Semi-symmetric pseudo-Riemannian manifolds.

1. Introduction
A pseudo-Riemannian manifold \((M, g)\) is said globally symmetric (respectively, locally symmetric) if any point \(m \in M\) is a fixed point of a non-trivial involutive isometric \(\gamma_m\) (respectively, any point \(m \in M\) admits a neighborhood \(N_m\) and the symmetric geodesic map \(s_m : N_m \to N_m\) is an isometry, where \(s_m\) is defined by \(x \mapsto s_m(x) = \gamma_x(-1)\) such that \(\gamma_x\) is the geodesic curve from \(m = \gamma_x(0)\) to \(x = \gamma_x(1)\) for all \(x \in N_m\) [See [5]].

It is necessary that a globally symmetrical manifold is locally symmetric but is not sufficient unless the manifold is simply connected. So, that a manifold \((M, g)\) is locally symmetric if and only if its Riemann curvature tensor \(R\) is parallel, that is to say that \(\nabla R = 0\) where \(\nabla\) is the Levi-Civita connection associated to metric \(g\). Indeed; the necessary condition is obvious from the fact that \((ds_m)_m = -Id_{T_m M}\) and for the sufficient condition it is a result of the following theorem:

**Theorem 1.1.** [5] Let \((M, g)\) be a pseudo-Riemannian manifold. Let \(\nabla\) and \(R\) be the Levi Civita connection and the Riemann curvature tensor associated to the metric \(g\) respectively.

**Abstract**

Let \((x, y) \in M^2\) and \(\tau : T_x M \to T_y M\) be an isomorphism. Suppose that \(\tau R = R\) and \(\nabla R = 0\). Then, there exists a neighborhood \(V_x\) of \(x\) and a diffeomorphism \(\varphi : V_x \to V_y\) with values in a neighborhood \(V_y\) of \(y\) such that \((d\varphi)_x = \tau\).

E. Cartan in ([8, 7]) showed that a locally symmetric Riemannian manifold \((M, g)\) is locally homogeneous, more precisely, \(M = G/H_0\) where \(G\) is the connected component of \(id_M\) in \(Iso(M)\) the Lie group of isometries of \(M\) such that \(G\) acts transitively on \(M\). \(H_0\) is the isotropic subgroup at the point \(m \in M\). This structure is independent of the chosen point \(m\).

On the other hand, \(g := h_0 + T_m M\) is the Lie algebra of \(G\), where \(h_0\) is the Lie algebra of \(H_0\), \(T_m M\) is the tangent space of \(M\) at the point \(m\). The Lie bracket on \(g\) is given by:

\[
\begin{align*}
[X, Y] &:= R_m(X, Y), \quad \forall X, Y \in T_m M, \\
[A, X] &:= A(X), \quad \forall X \in T_m M, \quad \forall A \in h_0, \\
[A, B] &:= AB - BA, \quad \forall A, B \in h_0.
\end{align*}
\]

Moreover, \(R\) checked

\[
[R_m(x, y), R_m(z, t)] = R_{mm}(R_m(x, y)z, t) + R_m(z, R_m(x, y)t), \quad \forall (x, y, z, t) \in (T_m M)^4,
\]

**Definition 1.1.** a) Let \((V, g)\) be a pseudo-Riemannian vector space, \(\mathfrak{h}\) be a Lie subalgebra of \(so(V)\) and \(R\) be a curvature tensor on \(V\). If \(R(x, y) \in \mathfrak{h}\) for all \((x, y) \in V^2\), the triple \([V, \mathfrak{h}, R]\) is said holonomy system. If moreover \(R\) satisfies the relationship

\[
(A.R)(x, y) = [A, R(x, y)] - R(A(x), y) - R(x, A(y)) = 0, \quad \forall (x, y) \in V^2,
\]


Preprint submitted to Elsevier April 6, 2022
the holonomy system \([V, b, R]\) is said to be symmetric (see [13]).

b) A pseudo-Riemannian manifold \((M, g)\) is said to be semi-symmetric if its Riemann curvature tensor satisfies the equation [2].

Note that if, \((M, g)\) is a semi-symmetric pseudo-Riemannian manifold, then, at any point \(m \in M\), the triple \([T_m M, b(R_m), R_m]_g\) is symmetric, where \(b(R_m)\) is the vector space spanned by all \(R(x, y)\) for all \(x, y\) in \(T_m M\), moreover \(b(R_m)\) is a Lie algebra. This result allows K. Nomizu in 1968 ([12]) to conjecture that any semi-symmetric Riemannian manifold \((M, g)\) of dimension greater than or equal to 3 is locally symmetric (i.e., \(\nabla R = 0\)). But, in 1972, H. Takagi ([14]) gave a counter example of a Riemannian manifold of dimension 3 satisfying \(R(X, Y) \cdot R = 0\) and not \(\nabla R = 0\).

On the other hand, semi-symmetry is a generalization of two-symmetry (\(\nabla^2 R = 0\)) and of local symmetry. Several studies are done on semi-symmetry. In the Riemannian case, Z. I Szabo gave the complete classification of semi-symmetric Riemannian manifolds ([15, 16, 17]). For the other strictly pseudo-Riemannian metrics, there is no general study, the few cases studied are with additional conditions. We can mention the complete classification of four-dimensional semi-symmetric homogeneous Lorentzian manifolds given by the author, M. boucetta and A. Ikemakhen (See [2]). In [3], the author gave the complete classification of four dimensional semi-symmetric homogeneous neutral manifolds. In [9], G. Calvaruso and B. De Leo have studied the semi-symmetric Lorentzian three-manifolds admitting a parallel degenerate line field (See [10]). In [11], A. Haji-Badali and A. Zeim gave a complete classification of Semi-symmetric four dimensional neutral Lie groups.

The purpose of this work is to give a few characterizations of semi-symmetric Lorentzian manifolds where we give the following results:

1. Any semi-symmetric Lorentzian manifold, its Ricci operator has only real eigenvalues.
2. A semi-symmetric pseudo-Riemannian manifold \((M, g)\) is foliated manifold. In the case where the metric is Lorentzian, the restriction of the Ricci operator on each leaf has at most one non-zero real eigenvalue.
3. If \((M, g)\) is a simple leaf semi-symmetric Lorentzian manifold, then the Ricci operator is diagonalizable or isotropic. Moreover, if \(\lambda\) is a non-zero eigenvalue of Ricci and \(V_1 := \ker(Ric - \lambda id_{T_M})\) is Riemannian subspace of dimension \(\geq 3\), then, \(1 \leq \dim(\ker(Ric)) \leq 2\).

This paper is organized in the following way. In the second paragraph, we give generalities on curvature tensors on a pseudo-Riemannian vector space \(V\). Thus, characterizations of a semi-symmetric curvature tensor by decomposing the space \(V\) as a direct sum of the characteristic subspaces of the Ricci operator. We also give another so-called primitive decomposition under the action of primitive holonomy algebra. In third paragraph, we give some proprieties of Ricci decomposition and the primitive decomposition of tangent bundle of the semi-symmetric pseudo-Riemannian manifold. The last paragraph is devoted to studying semi-symmetric Lorentzian manifolds which is a simple leaf where Ricci is diagonalizable or isotropic and if Ricci admits a non-zero eigenvalue then it is diagonalizable. Moreover, if this eigenvalue has a multiplicity hore greater than or equal to 3 with Riemannian eigenspace, we will have \(1 \leq \dim(\ker(Ric)) \leq 2\).

At first, we give some definitions that we will use later

**Definition 1.2.** Let \((M, g)\) be a locally connected pseudo-Riemannian manifold.

1. \((M, g)\) is said to be irreducible if, at each point \(m \in M\), the only subspace of \(T_m M\) which are invariant under the action of the holonomy group \(H_n(M)\) are \([0]\) and \(T_m M\), and reducible otherwise.
2. \((M, g)\) is said to be indecomposable if, at any point \(m \in M\), the only nondegenerate subspaces of \(T_m M\) which are invariant under the action of the holonomy Lie group \(H_n(M)\) are \([0]\) and \(T_m M\).
3. \((M, g)\) is said to be weakly irreducible if it is reducible and indecomposable.

2. Semisymmetrical curvature tensor on a pseudo-Riemannian vector space

2.1. Curvature tensor

Let \((V, \langle \cdot, \cdot \rangle)\) be a \(n\)-dimensional pseudo-Riemannian vector space. We identify \(V\) and its dual \(V^*\) by the means of \(\langle \cdot, \cdot \rangle\). This implies that the Lie algebra \(V \otimes V^*\) of endomorphisms of \(V\) is identified with \(V \otimes V\), the Lie algebra
so(V, ⟨ , ⟩) of skew-symmetric endomorphisms is identified with $V \wedge V$ and the space of symmetric endomorphisms is identified with $V \vee V$ (the symbol $\wedge$ is the outer product and $\vee$ is the symmetric product). For any $u, v \in V$,

$$(u \wedge v)w = \langle v, w \rangle u - \langle u, w \rangle v \quad \text{and} \quad (u \vee v)w = \frac{1}{2} ((v, w)u + (u, w)v).$$

On the other hand, $V \wedge V$ carries also a nondegenerate symmetric product also denoted by $\langle , \rangle$ and given by

$$(u \wedge v, w \wedge t) := \langle u \wedge v(w), t \rangle = \langle v, w \rangle(u, t) - \langle u, w \rangle(v, t).$$

We identify $V \wedge V$ with its dual by means of this metric.

We consider Bianchi’s linear mapping of the space $P = \vee^3(\wedge^2 V)$ given by:

$$B(a \wedge b) \vee (c \wedge d) = (a \wedge b) \vee (c \wedge d) + (b \wedge c) \vee (a \wedge d) + (c \wedge a) \vee (b \wedge d). \quad (4)$$

Let $\mathfrak{g}$ be a subalgebra of $so(V)$, we set:

$$R(\mathfrak{g}) := \ker(B) = \{ T \in \mathfrak{g} \vee \mathfrak{g} / B(T) = 0 \}$$

and

$$\mathfrak{g}_{sym} = \{ T \in R(\mathfrak{g})/\mathfrak{g}, T = 0 \}.$$

The set $R(\mathfrak{g})$ is called the space of all curvature tensors of type $\mathfrak{g}$ and any element of $R(so(V))$ is called curvature tensor on $V$. The set $\mathfrak{g}_{sym}$ is called the space of symmetric curvature tensors of type $\mathfrak{g}$.

According identifications cited above, we obtained:

**Lemma 2.1.** Let $(V, \langle , \rangle)$ be a pseudo-Riemannian vector space. Any curvature tensor $K \in R(so(V))$ can be identified with an element of $\otimes^4 V^*$ (i.e., a covariant 4-tensor on $V$) satisfying:

i) $K(a, b, u, v) = -K(b, a, u, v),$  
ii) $K(a, b, u, v) = -K(a, b, v, u),$  
iii) $K(a, b, u, v) + K(b, a, u, v) + K(u, a, b, v) = 0,$  
iv) $K(a, b, u, v) = K(u, v, a, b),$  

where $a, b, u, v \in V.$ Note that iv) is a result of i), ii) et iii).

**Lemma 2.2.** Let $(V, \langle , \rangle)$ be a pseudo-Riemannian vector space. Any curvature tensor $K \in R(so(V))$ can be identified with a symmetric bilinear map also denoted by $K : V \wedge V \rightarrow V \wedge V$ satisfying:

1. for all $u, v \in V$, $K(u \wedge v) = -K(v \wedge u),$  
2. for all $u, v, w \in V$, $K(u \wedge v)w + K(v \wedge w)u + K(w \wedge u)v = 0.$  

These relationships lead to:

$$\langle K(a \wedge b)u, v \rangle = \langle K(u \wedge v)a, b \rangle, \quad a, b, u, v \in V. \quad (5)$$

We often set:

$$K(u, v) := K(u \wedge v)$$

Let $(V, \langle , \rangle)$ be a pseudo-Riemannian vector space and let $K \in R(so(V))$ be a curvature tensor $V$. We set:

$$\mathfrak{h}(K) := (K \wedge^2 V) / \{ K(u, v) / u, v \in V \},$$

$\mathfrak{h}(K)$ is a vector subspace of $so(V)$. We set $\mathfrak{h}(K)$ the smallest Lie subalgebra of $so(V)$ containing $\mathfrak{h}(K)$ called primitive holonomy algebra of $K$, its Lie group is noted $\mathcal{H}(K)$ and called primitive holonomy group. The action of $\mathfrak{h}(K)$ and $\mathcal{H}(K)$ on the curvature tensor $K$ are given respectively by:

$$(A.K)(a, b) := [A, K(a, b)] - K(A(a), b) - K(a, A(b)), \quad (6)$$
and

\[(\sigma K)(a, b) := \sigma \circ K(\sigma^{-1}(a), \sigma^{-1}(b)) \circ \sigma^{-1}, \tag{7}\]

where \(A \in \mathfrak{h}(K), \sigma \in \mathcal{H}(K)\) and \(a, b \in V\).

The Ricci curvature associated to \(K\) is the symmetric bilinear form on \(V\) given by \(\text{Ric}_K(u, v) = \text{trace}(\tau(u, v))\), where \(\tau(u, v) : V \rightarrow V\) is given by \(\tau(u, v)(a) = K(u, a)v\). The Ricci operator is the symmetric endomorphism \(\text{Ric}_K : V \rightarrow V\) given by \(\langle \text{Ric}_K(u), v \rangle = \tau(u, v)\), for all \(u, v \in V\). We call \(K\) Einstein (resp. Ricci isotropic) if \(\text{Ric}_K = \lambda \text{Id}_V\) (resp. \(\text{Ric}_K \neq 0\) and \(\text{Ric}_K^2 = 0\)).

**Example 1.** if \(K = (u \wedge v) \vee (w \wedge t)\) then,

\[\text{Ric}_K = (u, w)t \vee v + (v, t)u \vee w - (v, w)t \vee u - (u, t)v \vee w.\]

Note that, if there is no ambiguity, we put \(\text{Ric} = \text{Ric}_K\) and \(\text{ric} = \text{Ric}_K\).

### 2.2. Semi-symmetrical curvature tensors

#### 2.2.1. Primitive decomposition

**Definition 2.1.** Let \((V, \langle \cdot, \cdot \rangle)\) be a \(n\)-dimensional pseudo-Riemannian vector space and let \(K\) a curvature tensor on \(V\). The curvature tensor \(K\) is called semi-symmetric if \(\mathfrak{h}(K)\) is a Lie algebra and \(K\) is a symmetric curvature tensor of type \(\mathfrak{h}(K)\), i.e. \(K \in \mathfrak{h}(K)_{\text{sym}}\).

**Remark 1.** In a pseudo-Riemannian space \((V, \langle \cdot, \cdot \rangle)\) of small dimension, to give the semi-symmetric curvature tensors on \(V\), it suffices to give the Lie subalgebras \(\mathfrak{g}\) of \(\text{so}(V)\) satisfying, \(\mathfrak{g} = \mathfrak{g}_{\text{sym}}\) (For example, see \([2,3]\)).

It is obvious that if the curvature \(K\) is constant, i.e. \(K = \lambda \text{Id}_V\), then it is semi-symmetric and if \(\lambda \neq 0\), we get \(\mathfrak{h}(K) = \text{so}(V)\). In particular, any curvature tensor on the space pseudo-riemannian of dimension 2 is semi-symmetric.

**Proposition 2.1.** Let \((V, \langle \cdot, \cdot \rangle)\) be a pseudo-Riemannian vector space of dimension \(n\) and let \(K\) be a curvature tensor on \(V\) treated as an symmetrical endomorphism, \(K : a \wedge b \in \Lambda^2 V \mapsto K(a \wedge b) \in \Lambda^2 V\). Then, \(K\) is semi-symmetric iff \(K.K = 0\), i.e. \(K\) checks;

\[\{K(u, v), K(a, b)\} = K(K(u, v)a, b) + K(a, K(u, v)b), \quad u, v, a, b \in V. \tag{8}\]

Let \(K\) be a semisymmetric curvature tensor on \(n\)-dimensional pseudo-Riemannian vector space \((V, \langle \cdot, \cdot \rangle)\). Then \(\mathfrak{h}(K) = \mathfrak{h}(K)\) is the primitive holonomy Lie algebra of \(K\) and we set \(\mathcal{H}(K) := \mathcal{H}(K)\) its the primitive holonomy Lie group. Moreover, \(K\) is invariant by \(\mathcal{H}(K)\) and it is parallel under the action of \(\mathfrak{h}(K)\), i.e,

\[\sigma^*K = K \iff (\sigma^*K)(a, b)u = \sigma^*(K(\sigma^{-1}(a), \sigma^{-1}(b))(\sigma^{-1}(u))) = K(a, b)(u)\]

and

\[K(a, b).K = 0 \iff [K(a, b), K(u, v)] = K(K(a, b)(u), v) + K(u, K(a, b)(v)),\]

for all \(\sigma \in \mathcal{H}(K)\) and for all \((a, b, u, v) \in V^4\).

So the action of the primitive holonomy Lie algebra on the space \(V\) introduces a weakly decomposition of \(V\);

\[V = V_0 + V_1 + \ldots + V_s + V'_s\tag{9}\]

where \(V_0 = \{x \in V / \langle K(u, v)x = 0, \forall (u, v) \in V^2\}\), \(V'_s\) is the dual of the subspace \(V_0 \cap (V_1 + \ldots + V_s)\) and each subspace \(V_i + (V_i \cap V'_j)'\) is indecomposable subspace under the action of \(\mathfrak{h}(K)\) for all \(i \geq 1\).

**Definition 2.2.** The decomposition \(\mathfrak{h}(K)\) is called primitive decomposition of \(V\).

Moreover, the primitive decomposition satisfies the following properties;

1. for any \(i = 0, \ldots, s, V_i\) is \(\mathfrak{h}(K)\)-invariant,
2. for any \(i, j = 0, \ldots, s\) with \(i \neq j, K(V_i, V_j) = 0,\)
2.2.2. Ricci decomposition

**Lemma 2.3.** Let $K$ be a semi-symmetrical curvature tensor on the pseudo-Riemannian space $(V,\langle , \rangle)$. Then, its Ricci operator commutes with all the endomorphisms $K(u,v)$, that is:

$$K(u,v) \circ \text{Ric} = \text{Ric} \circ K(u,v), \quad \forall u, v \in V.$$  \hspace{1cm} (10)

**Proof.**

Soit $u, v, w, z \in V$.

$$(\text{Ric}(K(u,v)w), z) = \text{trace}(\langle z, K(u,v)w \rangle)$$

$$= \text{trace}(t \mapsto K(tz,t)(K(u,v)w))$$

$$= \text{trace}(t \mapsto (K(u,v)(K(z,t)w)) - \text{trace}(t \mapsto K(z,t)w)$$

$$- \text{trace}(t \mapsto K(u,v)z,t)w)$$

$$= \text{trace}([[K(u,v), \tau(z, w)]) - \text{trace}(\tau(K(u,v)z, w))$$

$$= (K(u,v) \circ \text{Ric}(w), z).$$  \hspace{1cm} \Box

A Ricci operator Ric satisfying the equation (10) is said semi-symmetric. In particular, for a pseudo-Riemannian manifold whose Ricci operator is parallel ($\nabla \text{Ric} = 0$), Ric is semi-symmetric. This type of manifolds was studied by C. Boubel (see [6]) by giving the following classification theorem:

**Theorem 2.1.** ([6]) Let $(M,g)$ be a pseudo-Riemannian manifold with a parallel Ricci curvature $\text{Ric}$ (i.e. $\nabla \text{Ric} = 0$) and let $\chi$ be the minimal polynomial of $\text{Ric}$. Then:

1. $\chi = \Pi_i P_i$ where:
   - $\forall i \neq j$, $P_i \cap P_j = 1$ (i.e. $P_i$ et $P_j$ are mutually prime),
   - $\forall i$, $P_i$ is irreducible or $P_i = X^2$.
2. There is a canonical family $(M_i)_i$ of pseudo-Riemannian manifolds such that the minimal polynomial of $\text{Ric}_i$ on each $M_i$ is $P_i$, and a local isometry $f$ mapping the Riemannian product $\Pi M_i$ onto $M$. $f$ is unique up to composition with a product of isometries of each factor $M_i$. If $M$ is complete and simply connected, $f$ is an isometry.

For the proof of the first result of this theorem above, C. Boubel only used the following hypothesis: On the tangent space $T_xM$ at each point $x \in M$, the Ricci operator $\text{Ric}_x$ commutes with the endomorphisms $R_x(u,v)$ for all $u, v \in T_xM$. A result which remains valid for spaces provided with semi-symmetrical curvatures and more particularly, for spaces with semi-symmetrical Ricci. Thus, the following result:

**Theorem 2.2.** Let $K$ be a semi-symmetric curvature tensor on a pseudo-Riemannian space $(V,\langle , \rangle)$ and let $\chi$ be the minimal polynomial of its Ricci operator $\text{Ric}$. Then the following properties are checked:

1. $\chi = \Pi_i P_i$ where:
   - $\forall i \neq j$, $P_i \cap P_j = 1$ (i.e. $P_i$ et $P_j$ are mutually prime),
   - $\forall i$, $P_i$ is irreducible or $P_i = X^2$.
2. $V$ splits orthogonally as
   $$V = E_0 \oplus E_1 \oplus \ldots \oplus E_r,$$  \hspace{1cm} (11)
   where $E_0 = \text{ker}(\text{Ric}^2)$ and $E_i = \text{ker}(P_i(\text{Ric}))$,
3. for any $u, v \in V$ and $i = 0, \ldots, r$, $E_i$ is $b(K)$-invariant,
4. for any $i, j = 0, \ldots, r$ with $i \neq j$, $K_{iE_i,AE_j} = 0$,
5. for any $i = 1, \ldots, r$, $\dim E_i \geq 2$.
6. The primitive holonomy algebra $b(K)$ is satisfied;
   $$b(K) = b_0(K) + b_1(K) + \ldots + b_r(K),$$
   where $b_i(R) := [K(X,Y) / X, Y \in E_i]$ is a Lie subalgebra for all $0 \leq i \leq r$.  

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Definition 2.3. The decomposition (17) is called Ricci decomposition of V.

Remark 2. If $P_0 = X$, we get that $\ker(\text{Ric}) = \ker(\text{Ric}^2)$. That’s why we always set $E_0 = \ker(\text{Ric}^2)$.

Proof.

1. In the proof of the theorem [2.1] C. Boubel only used the commutativity of the Ricci operator with the each endomorphisms $R(u, v)$ which remains valid for the first result of our theorem.
2. Let $\chi = \Pi P$, be the irreducible decomposition of $\chi$.
   Then:
   
   $$V = \bigoplus_i E_i, \quad \text{where} \quad E_i = \ker(P_i(\text{Ric})).$$

3. Let $(u, v) \in V^2$. So $K(u, v)$ commutes with $\text{Ric}$, then each $E_i$ is invariant by $K(u, v)$.
4. Let $u \in E_i$, $v \in E_j$ and $a, b \in V$ such that $i \neq j$. Since $K(a, b)(E_i) \subset E_i$ and $\langle E_i, E_j \rangle = 0$, we obtain
   
   $$0 = \langle K(a, b)u, v \rangle = \langle K(u, v)a, b \rangle$$

   and then, $K(u, v) = 0$.
5. We assume that it exists $i \in \{1, \ldots, r\}$ such that $\dim E_i = 1$. We choose an element $e$ in $E_i$ checking $\langle e, e \rangle = \epsilon$ with $\epsilon^2 = 1$ and we complete it to an orthonormal basis $(e, e_1, \ldots, e_{n-1})$ with $\langle e, e_i \rangle = \epsilon_i$ and $\epsilon_i^2 = 1$. For all $a, b \in V$, we get that $K(a, b)$ is skewsymmetric endomorphism which leaves $E_i$ invariant. Then $K(a, b)e = 0$, while

   $$\epsilon a_i = \langle \text{Ric}(e), e \rangle = \epsilon(\langle K(e, e)e, e \rangle + \sum_{i=1}^{n-1} \epsilon_i \langle K(e, e_i)e, e_i \rangle) = 0,$$

   which is absurd and this completes the proof of the theorem. \hfill \Box

This proposition reduces the determination of semi-symmetric curvature tensors on pseudo-riemannian vector spaces to the determination of two classes of semi-symmetric curvature tensors:

1. Einstein semi-symmetric curvature tensors ( with complexification in the case of non-real complex eigenvalues)
2. Semi-symmetric curvature tensors with Ricci operator is satisfied $\text{Ric}^2 = 0$.

2.3. Semi-symmetrical Lorentzian space

Proposition 2.2. (13) Let $K$ be a semi-symmetric curvature tensor on a Lorentzian vector space $(V, \langle \cdot, \cdot \rangle)$. Then all eigenvalues of $\text{Ric}_K$ are real. Denoted by $\alpha_1, \ldots, \alpha_r$ the non null eigenvalues and $E_1, \ldots, E_r$ the corresponding eigenspaces. Then:

1. $V$ splits orthogonally as $V = E_0 \oplus E_1 \oplus \ldots \oplus E_r$, where $E_0 = \ker(\text{Ric}^2)$,
2. for any $i = 0, \ldots, r$, $E_i$ is $\mathfrak{h}(K)$-invariant,
3. for any $i, j = 0, \ldots, r$ with $i \neq j$, $K_{|E_iE_j} = 0$,
4. for any $i = 1, \ldots, r$, $\dim E_i \geq 2$.

Moreover, the primitive holonomy algebra $\mathfrak{h}(K)$ splits orthogonally:

$$\mathfrak{h}(K) = \mathfrak{h}_0(K) + \mathfrak{h}_1(K) + \ldots + \mathfrak{h}_r(K),$$

where $\mathfrak{h}_i(K) := \{K(u, v) / u, v \in E_i\}$ is a Lie subalgebra for all $0 \leq i \leq r$.

More precisely, one of the following two situations occurs:

a) Ricci is diagonalizable: $E_0 = \ker(\text{Ric}).$

b) Ricci is of isotropic type: $\ker(\text{Ric}) \subsetneq E_0 = \ker(\text{Ric}^2)$.
Proof. This is a special case of theorem \ref{prop:ricciminimal} and it suffices to show that Ricci has only real eigenvalues, this is equivalent to showing that Ricci’s minimal polynomial has no non-real roots.

Suppose that Ric admits a non-real eigenvalue $z = a + ib$ where $b \neq 0$. Then, $z$ and $\bar{z}$ are roots of an irreducible factor of degree 2 of the minimal Ricci polynomial.

Let $e$ and $\overline{e}$ in $V$ such that

$$\langle e, e \rangle = -\langle \overline{e}, \overline{e} \rangle = 1, \quad \langle e, \overline{e} \rangle = 0, \quad \text{Ric}(e) = ae - b\overline{e} \quad \text{and} \quad \text{Ric}(\overline{e}) = be + a\overline{e}.$$  

Then, the subspace $E = \text{span}[e, \overline{e}]$ and its orthogonal subspace $E^\perp$ are stable by Ric and by endomorphisms $\text{K}(u, v)$ for all $u$ and $v$ in $V$. Let $\mathbb{B} = (e_1, \ldots, e_{n-2})$ be an orthonormally basis of $E^\perp$.

Then, we have:

$$b = \langle \text{Ric}(\overline{e}), e \rangle = \langle \text{K}(\overline{e}, e)e, e \rangle - \langle \text{K}(\overline{e}, \overline{e})e, \overline{e} \rangle + \sum_{i=1}^{n-2} \langle \text{K}(\overline{e}, e_i)e, e_i \rangle = 0,$$

which contradicts the fact that $b \neq 0$.

\[ \square \]

Remark 3. In \cite{ref}, the author had demonstrated this proposition with different techniques.

Lemma 2.4. Let $\text{K}$ be a semi-symmetric curvature tensor on the Lorentzian vector space $(V, \langle \cdot, \cdot \rangle)$. Then the primitive decomposition of $V$ is given as:

$$V = (V_0 + V_1 + V_0^\perp) \oplus V_2 \oplus \ldots \oplus V_i,$$

where $V_0^\perp$ is the dual subspace of $V_0 \cap V_i$.

Therefore, $V_i$ is a Riemannian subspace of dimension greater than or equal to 2 for any $i \geq 2$.

And so,

1. If $E_0$ is a riemannian space, then $V_0 = E_0$ is also riemannian space and for all $i \geq 1$, the $V_i$ is Einstein space.
2. If $\text{K}$ has a tensor Ricci of isotropic type, i.e. $E_0 = \ker(\text{Ric^2})$, then $E_0 = V_0 + V_1 + V_0^\perp$ such that $V_0^\perp$ is of dimension 1 and for all $i \geq 2$, the $V_i$ is Einstein Riemannian space.

3. Semi-symmetric pseudo-Riemannian differential manifolds

Let $(M, g)$ be a pseudo-Riemannian differential manifold of dimension $n$. Let $\nabla$, $R$, $\text{ric}$ and $\text{Ric}$ are respectively, the Levi-Civita connection, the Riemann curvature, the tensor and the Ricci operator associated to the metric $g$. We set $\mathcal{X}(M)$ the set of vector fields on $M$.

So $(M, g)$ is semisymmetric manifold. Then, at each point $m \in M$, the restriction $R_m$ of $R$ on $T_mM$ is a semi-symmetric curvature tensor and the minimal polynomial of $\text{Ric}_m$ is of the form $X = \prod_{i=0}^r P_i$ where the polynomials $(P_i)$ are mutually prime and for any $i$, $P_i$ is irreducible or $P_i = X^2$. We can assume that $P_i$ is irreducible for all $i \geq 1$.

We define the distributions:

$$E_0(m) := \ker(\text{Ric}_m^2) \quad \text{et} \quad E_i(m) := \ker(P_i(\text{Ric}_m)) \quad \text{for} \quad i \geq 1,$$

and we have the following proposition:

Proposition 3.1. The distributions $(E_i)_i$ checked the following properties:

For all $i, j \geq 1$ with $i \neq j$, we have:

$$\nabla_{E_i}E_j \subset E_i, \quad \nabla_{E_j}E_i \subset E_0 + E_j, \quad \nabla_{E_i}E_j \subset E_i, \quad \nabla_{E_j}E_0 \subset E_0, \quad \nabla_{E_i}E_0 \subset E_0 + E_i.$$  

(12)
Proof.
Let \( m \in M \). The theorem \ref{2.2} induces that:

\[
T_m M = E_0(m) \oplus E_1(m) \oplus \ldots \oplus E_r(m).
\]  

(13)

In the first step, we will show that for \( i \geq 1 \) and \( X \in E_i^\perp, \nabla X E_i \subset E_i \);

Let \( i \geq 1 \) and \( X \in E_i^\perp \). First, we show \( \nabla_X (R(E_i, E_i)) E_i \subset E_i \);

Let us take \( Y, Z, T \in E_i \), by the second Bianchi identity, we get

\[
\nabla_X R(Y, Z, T) := (\nabla_X R)(Y, Z)T
\]

\[
= -\nabla_Y R(Z, X, T) - \nabla_Z R(X, Y, T)
\]

\[
= -\nabla_Y (R(Z, X)T) + R(\nabla_Y Z, X)T + R(Z, \nabla_Y X)T + R(Z, X)\nabla_Y T
\]

\[
- \nabla_Z (R(Y, X)T) + R(\nabla_Z Y, X)T + R(Y, \nabla_Z X)T + R(Y, X)\nabla_Z T
\]

\[
= R(\nabla_Y Z, X)T + R(\nabla_Z Y, X)T + R(\nabla_Z Y, X)T + R(\nabla_Y Z, X)T.
\]

By theorem \ref{2.2}, we get; \( R(V, V)(E_i) \subset E_i \) and \( \nabla_X R(Y, Z, T) \in E_i \).

On the other hand,

\[
\nabla X R(Y, Z, T) = \nabla_X (R(Y, Z)T) - R(\nabla_X Y, Z)T - R(Y, \nabla_X Z)T - R(Y, Z)\nabla_X T
\]

\[
= \nabla_X (R(Y, Z)T) - R(\nabla_X Y, Z)T - R(Y, \nabla_X Z)T
\]

\[
+ R(Z, \nabla_X T)Y + R(\nabla_X T, Y)Z.
\]

which proves that, \( \nabla_X R(Y, Z, T) \in E_i \).

Now we will show that \( \nabla_X \text{Ric}(Y) \subset E_i \).

We choose an orthogonal basis \((e_1, ..., e_n)\) associated to the decomposition \([13]\) where \( e_k = (e_k, e_k) \) such that \( e_k^2 = 1 \).

Let \( U \in E_i^\perp \). If \( e_k \in E_i \), we have already seen that \( \nabla_X (K(Y, e_k)e_k) \in E_i \) and if \( e_k \in E_i^\perp \), we get \( R(Y, e_k) = 0 \). Consequently, we have

\[
\langle \nabla_X (\text{Ric}(Y)), U \rangle = -\langle \text{Ric}(Y), \nabla_X U \rangle
\]

\[
= \sum_{k=1}^{n} e_k \langle R(Y, e_k)e_k, \nabla_X U \rangle
\]

\[
= -\sum_{k=1}^{n} e_k \langle \nabla_X (R(Y, e_k)e_k), U \rangle
\]

\[
= 0.
\]

then \( \nabla_X (\text{Ric}(Y)) \subset E_i \).

If \( P_i(t) = t^2 + at + b \) whit \( b \neq 0 \), then for all \( Y \in E_i \), we have \( Y = -\frac{1}{b}(\text{Ric}^2(Y) + a\text{Ric}(Y)) \).

Consequently, \( \nabla_X Y \subset E_i \).

If \( P_i(t) = t - \lambda_i \) whit \( \lambda_i \neq 0 \), then for all \( Y \in E_i \), we have \( Y = -\frac{1}{\lambda_i}\text{Ric}(Y) \).

Consequently \( \nabla_X Y \subset E_i \).

So, \( \nabla_X E_i \subset E_i \), which shows that \( \nabla_X E_i \subset E_i \) and \( \nabla_{E_j} E_i \subset E_i \), for all \( i, j \geq 1 \) whit \( i \neq j \).

The other results are obtained immediately because the metric \( g \) is parallel(i.e. \( \nabla g = 0 \)). \( \square \)

Corollary 3.1. Let \((M, g)\) be a connected semi-symmetric pseudo-Riemannian manifold. Let \( X = \prod_i P_i \) be the minimal polynomial of Ric. If we set \( E_0 := \ker(\text{Ric}^2) \) and for all \( i \geq 1 \), \( E_i := \ker(P_i(\text{Ric})) \). Then, for all \( i \geq 1 \), the distributions \( E_0 \) et \( E_0 + E_i \) are involutive.

We set \( N_0 \) the integral submanifolds of \( E_0 \).
Remark 4. The distributions $E_0$ and $E_0 + E_i$ are involutive but not necessarily parallele.

On the other way, for a semisymmetrical pseudo-Riemannian manifolds $(M, g)$, the action of the primitive holonomy Lie algebra $h(R_m)$ gives the primitive decomposition of tangent space $T_m M$ on point $m \in M$ as

$$T_m M = V_0(m) + V_1(m) + \ldots + V_r(m) + V_{r+1}(m)$$

(14)

where $V_0(m) = \{x \in T_m M \mid R(u, v)x = 0$, for all $(u, v) \in T_m^2 M\}$ and the space $V_r(m)$ is the dual subspace of $V_0(m)$ and $\{V_1(m) + \ldots + V_r(m)\}$.

With a similar proof of the proposition [3.1], we check that the distributions $V_i$ have the following properties:

**Proposition 3.2.** For all $i, j \geq 1$, let $i \neq j$:

$$\nabla V_i \subset V_i, \nabla V_i \subset V_0 + V_r, \nabla V_i \subset V_0, \nabla V_0 \subset V_0 + V_i.$$  

(15)

Now we go back to the Ricci decomposition and for all $1 \leq i \leq r$, we consider the distributions $F_i$ spanned by the vector fields of the forms

$$X_1, \nabla X_1, X_2, \nabla X_2, \ldots, \nabla X_k, X_{k+1}, \ldots,$$

(16)

where the vector fields $(X_k)_{k \geq 1}$ are belong to $E_i$.

In the same way, let us the subspaces: $F_i$, $\tilde{F}_i$, and $F_0$ checking:

$$F = F_1 + F_2 + \ldots + F_r,$$

$$TM = F \oplus (F \cap F^\perp) \oplus F_0,$$

$$\tilde{F}_i = F_i \oplus (F_i \cap F_i^\perp),$$

where $(F \cap F^\perp)$ and $(F_i \cap F_i^\perp)$ are respectively, the dual subspaces of $F \cap F^\perp$ and $F_i \cap F_i^\perp$.

Remark 5.

It’s obvious that:

$$\forall i \geq 1, \quad F_0 \cup (F \cap F^\perp) \cup (F \cap F^\perp) \subseteq E_0 \quad \text{and} \quad E_i \subseteq F_i \subseteq E_0 + E_i.$$  

For $i, j \geq 1$, we put $X_1, \ldots, X_k$, (resp.$Y_1, \ldots, Y_l$, etc.) the vector fields belong to $E_i$ (resp. $E_j$).

In the first step, we show the following lemma:

**Lemma 3.1.** For all $0 \leq i \neq j \neq 0$, the vector fields of the forms

$$\nabla X_i, \nabla Y_i, \nabla Y_j, \nabla Y_{j+1}$$

are tangent to $F_j$, i.e, we have

$$\nabla E_i F_j \subseteq F_j.$$  

As a consequence, for all $j \geq 1$, the distribution $F_j$ is parallel and the integral submanifolds are totally geodesic.

**Proof.** We can prove this lemma by induction:

Let $0 \leq i \neq j \geq 1$. So $R(X_1, Y_1)Y_2 = 0$, then

$$\nabla X_1, \nabla Y_1, 2 = \nabla Y_1, \nabla X_2, + \nabla \{X_1, Y_1\} Y_2 = \nabla Y_1, Y_2 + \nabla Y_2, Y_2 - \nabla X_1, Y_2,$$

where the vector fields $Y_2^* := \nabla X_1, Y_2$ and $Y_1^* := \nabla X_1, Y_1$ are tangent to $E_j$ and the vector fields $X_1^* := \nabla Y_1, X_1$ is tangent to $E_i$ for $i \geq 1$ and for $i = 0$, we get $X_1^* \in E_0 + E_j$ and $\nabla X_1, Y_2$ belong to $F_j$. This assumes that the three terms above are tangent to $F_j$. So $\nabla X_1, \nabla Y_2 \in F_j$ is checked.

Now, in the general case:

So $R(X_1, Y_1) = 0$, we obtain:

$$\nabla X_1, \nabla Y_1, 2 = \nabla Y_1, \nabla X_2, + \nabla \{X_1, Y_1\} Y_2 = \nabla Y_1, Y_2 + \nabla Y_2, Y_2 - \nabla X_1, Y_2.$$  

As the vector field $Y_1^* := \nabla X_1, Y_1$ is tangent to $E_j$, and the vector field $X_1^* := \nabla Y_1, X_1$ is tangent to $E_i$, by the induction hypothesis, we get that $\nabla X_1, \nabla Y_1, 2$ is tangent to $F_j$. \qed
Lemma 3.2. The subspaces $F_j$, $F_{j+1}$,... and $F_r$ are pairwise orthogonal.

Proof. By induction, we show that the vector field $\nabla_{Y_j} \nabla_{Y_{j+1}} \cap \nabla_{Y_{j+k}}$ for $k \geq 0$ is orthogonal to $E_i$.

For $k = 0$ and $k = 1$, it's obvious.

Now, suppose that result is true for any vector fields of the form $\nabla_{Y_j} \nabla_{Y_{j+1}} \cap \nabla_{Y_{j+k}}$. Let $X$ be a vector fields in $E_i$. Then, the vector fields $\nabla_{Y_j} X$ is also tangent to $E_i$. By induction, we get

$$g(\nabla_{Y_j} X, \nabla_{Y_{j+1}} \cap \nabla_{Y_{j+k}}) = -g(\nabla_{Y_j} X, \nabla_{Y_{j+1}} \cap \nabla_{Y_{j+k}}) = 0.$$

This proves the statement.

Using induction again, we can prove that the vector fields on the form $\nabla_{Y_j} \nabla_{Y_{j+1}} \cap \nabla_{Y_{j+k}}$ are orthogonal to those which are of the form $\nabla_{Y_j} \nabla_{Y_{j+1}} \cap \nabla_{Y_{j+k}}$. For the case $k = 0$, the proof is given above. Now, for the vectors fields on the form $\nabla_{Y_j} \nabla_{Y_{j+1}} \cap \nabla_{Y_{j+k}}$ are orthogonal to the vector fields on the form $\nabla_{Y_j} \nabla_{Y_{j+1}} \cap \nabla_{Y_{j+k}}$, so the induction hypothesis and lemma(3.2), we get

$$g(\nabla_{Y_j} \nabla_{Y_{j+1}} \cap \nabla_{Y_{j+k}}, \nabla_{Y_j} \nabla_{Y_{j+1}} \cap \nabla_{Y_{j+k+1}}) = -g(\nabla_{Y_j} \nabla_{Y_{j+1}} \cap \nabla_{Y_{j+k}}, \nabla_{Y_j} \nabla_{Y_{j+1}} \cap \nabla_{Y_{j+k+1}}) = 0,$$

whih the vector fields on the form $\nabla_{Y_j} \nabla_{Y_{j+1}} \cap \nabla_{Y_{j+k}}$ are tangent to $F_j$. This gives completely the proof of lemma.

Corollary 3.2. $(F \cap F_i) \cap F_0$ and $\tilde{F}_i = F_i \cap (F_j) \cap F_i$ are involutive distributions, for all $j \geq 1$.

Proof. We have already seen the relations $\nabla_{E_i} F_j \subseteq F_j$ only for the cases $i \geq 0$ and $j \geq 1$. It remains to show the relationships $\nabla_{E_i} F_0 \subseteq F_0$ for $j > 0$. The first is obvious, since

$$g(\nabla_{E_i} F_0, F_k) = -g(F_0, \nabla_{E_i} F_k) = -g(F_0, F_k) = 0, \text{ for all } k > 0.$$

Finally, for the formula $\nabla_{E_i} F_0$, we get

$$g(\nabla_{E_i} F_0, F_i) = -g(F_0, \nabla_{E_i} F_i) = g(F_0, F_i) = 0.$$

Remark 6. The distribution $F_j \cap (F_j) \cap F_0$ is involutive non degenerate.

Proposition 3.3. Let $(M, g)$ be a semi-symmetric, locally connected pseudo-Riemannian manifold. Then, $M$ is a foliated manifold and the minimal polynomial of the restriction of the Ricci operator on each leaf has one of the following forms $X, X^2, P, XP$ or $X^2P$ where $P$ is an irreducible polynomial and it is prime with the polynomial $X$.

In the case where the tangent bundle of $M$ admits only one involutive subbundle nondegenerate, $M$ is called a simple leaf.

Lemma 3.3. Let $(M, g)$ be a simple leaf semisymmetric, locally connected, pseudo-Riemannian manifold. Then its a tangent bundle of $M$ admits only one involutive subbundle nondegenerate, $M$ is called a simple leaf.

1. $TM = F_1 \oplus (F_1 \cap (F_1)^\perp)^\perp = E_0 \oplus E_1$.
2. $TM = E_0$.

4. The simple leaf semi-symmetrical Lorentzian manifold

In this section we will give some properties of semi-symmetric Lorentzian spaces.
4.1. Ricci decomposition

In the Lorentzian case, the comparison of primitive and Ricci decompositions produces the following Proposition;

**Proposition 4.1.** Let \((M, g)\) be a simple leaf semisymmetric, locally connected, Lorentzian manifolds. Then, we get one of the following situations:

1. \(E_0 = V_0\), i.e. \(\mathcal{N}_0\) is a flat submanifolds.
2. The tangent bundle over \(M\) is on the forme \(E_0 = V_0 + V_1 + (V_0 \cap V_1)\), where \((V_0 \cap V_1)\) is the dual subspace of \((V_0 \cap V_1)\). In this situation, we get one of the two following cases:
   (a) \(E_0 = V_0 \oplus V_1\).
   (b) There exists an isotropic vector fields \(p\) with its dual vector fields \(q\) such that \(E_0 = V_0 + V_1 + \text{span}[q]\). In this case, the tensor curvature holds \(R^2 = 0\).

**Proof.** The primitive and Ricci decompositions imply that

\[ V_0 = E_0 \] or \(V_0 \not\subset E_0\).

If \(V_0 \not\subset E_0\), this means that the curvature tensor is non-zero on \(\mathcal{N}_0\) and the action of the primitive holonomy algebra \(h_0(R)\) on \(E_0\) induces a non-trivial decomposition:

\[ E_0 = V_0 + V_1 + (V_0 \cap V_1)\] ,

where \(V_0 = \{x \in E_0/ \forall h \in h_0(R), h(x) = 0\}\), \(V_1 := h_0(R)(E_0) \not\subset \{0\}\) and \((V_0 \cap V_1)\) is the dual subspace of \(V_0 \cap V_1\).

By proposition 3.2, we show that the distribution \(V_0 + V_1\) is involutive and since \(M\) is a simple leaf, we get that \(TM = E_0\).

So, if \(V_0 \cap V_1 \not\subset \{0\}\) is a non-trivial subspace, then it is generated by an isotropic parallel vector fields \(p\) and according to the classification of weakly irreducible holonomy algebras of Lorentzian manifolds given by L. B. Berger and A. Ikemakhen in [4], for any point \(m \in M\), \(h_0(R_m)\) is a Lie subalgebra of type 2 or 4. Consequently, each element \(K \in h_0(R_m)\) is written in the form

\[ K = K_0 - p_m \wedge X, \]

where \(X \in \tilde{E}_0(m) \cong E_0(m)/(p,q)\) and \(K_0 \in so(\tilde{E}_0(m)) \cong \wedge^2 \mathbb{R}^r\), such that \(r = \dim(E_0) - 2\) and \(q\) is an isotropic vector fields which \(g(p,q) = 1\).

Then the restriction of the curvature on the space \(E_0(m)\) checks

\[ \forall (X, Y) \in E_0(m)\], \(\exists E \in \tilde{E}_0(m)\) such that \(R_m(X, Y) = \tilde{R}(X, Y) - p_m \wedge E,\)

where \(\tilde{X}\) and \(\tilde{Y}\) are the projection respectively of the \(X\) and \(Y\) on the subspace \(\tilde{E}_0(m)\) and \(\tilde{R}\) is the restriction of \(R_m\) on \(\tilde{E}_0(m)\) and so \(\tilde{R}\) is a semisymmetric curvature tensor on the irreducible Riemannian space \(\tilde{E}_0(m)\) where \(\text{Ric}_m = 0\). Then

\[ \tilde{R} = 0 \]

and \(R_m(X, Y) = -p_m \wedge E.\)

Thus \(R^2 = 0.\)

**Proposition 4.2.** Let \((M, g)\) be a connected simple leaf semi-symmetric Lorentzian manifold. Then the Ricci operator admits at most an one non-zero real eigenvalue. If \(\lambda\) is such an eigenvalue, then the tangent space of \(M\) at any point \(m \in M\) will be of the form:

\[ T_m M = \ker \left( \text{Ric}_m - \lambda(m)IId_{E_0} \right) \oplus V_0(m). \] (17)

Moreover, if \(\dim(E_1) \geq 3\), then the function \(m \in M \mapsto \lambda(m)\) is of class \(C^\infty\) and it depends only on \(\mathcal{N}_0\) the flat integral submanifolds of \(V_0 = E_0\).
Proof.
Let \((M, g)\) be a connected, semi-symmetric, simple leaf Lorentzian manifold. Let \(\lambda\) be a non-zero eigenvalue of \(\text{Ric}\).

The equation ([7]) is a result of Lemma 3.2 and Proposition 4.1.

Now, we consider the codifferential \(\delta\) on \(M\) given by:

\[
\delta(\alpha)(Y_1, \ldots, Y_r) = -\sum_i (\nabla_{X_i} \alpha)(X_i, Y_1, \ldots, Y_r),
\]

where \(\alpha\) is a \((r + 1)\)-differential form, \((X_1, \ldots, X_n)\) is an orthonormal frame on \(M\) and \((Y_1, \ldots, Y_r)\) a family of \(r\) vector fields (See [5], page 34).

The Ricci tensor verifies:

\[
\delta(\text{ric})(Y_1, \ldots, Y_r) = -\frac{1}{2} d(s),
\]

where \(s\) is the scalar curvature, \(d\) is the exterior differential over \(M\) (See proposition 1.94 [5] page 43).

Let \((X_1, \ldots, X_n)\) be a orthonormal frame of \(TM\) whit the \(X_i\) are tangent to \(E_0\) for \(i \leq \nu = \dim(E_0)\) and for \(i > \nu\), the \(X_i\) are tangent to \(E_1\).

Let us \(j > \nu\), we get:

\[
0 = \frac{1}{2} d(s)(X_j) + \delta(\text{ric})(X_j)
\]

\[
= \frac{1}{2} X_j(s) - \sum_i (\nabla_{X_i} \text{ric})(X_i, X_j)
\]

\[
= \frac{1}{2} X_j((n - \nu)\lambda) - \sum_i (\text{ric}(X_i, X_j)) - \text{ric}(\nabla_{X_i} X_j) - \text{ric}(X_i, \nabla_{X_j} X_j),
\]

\[
= \frac{n -\nu}{2} X_j(\lambda) - \sum_i (\text{ric}(X_i, X_j)) + \lambda X_i g(X_i, X_j) - \text{ric}(\nabla_{X_i} X_j) - \text{ric}(X_i, \nabla_{X_j} X_j),
\]

\[
= \frac{n -\nu}{2} X_j(\lambda).
\]

So \(n - \nu \geq 3\) implies that \(X_j(\lambda) = 0\) and therefore, \(\lambda\) only depends on the integral submanifolds of \(E_0\).

\[\square\]

4.2. Basic formulas

In this subsection, we consider \((M, g)\) a simple leaf semi-symmetric Lorentzian manifold of dimension \(n\) such that the Ricci admits a non-zero eigenvalue \(\lambda\) of multiplicity \(n - r\) such that \(E_0\) is a Lorentzian subspace.

So at any point \(m \in M\), the tangent space splits as:

\[
T_m M = E_0(m) \oplus E_1(m),
\]

where

\[
E_0(m) = \ker(\text{Ric}_m) \quad \text{and} \quad E_1(m) = \ker(\text{Ric}_m - \lambda(m) id_{T_m M}).
\]

and \(N_0\) the integral submanifold of \(E_0\) is flat.

The real number \(r = \dim(E_0(m))\) called the nullity index of the curvature at the point \(m\). The multiplicity of eigenvalue \(\lambda\) also called the co-nullity index.

So \(N_0\) the integral submanifold of \(E_0\) is flat, then we can choose a fram \((e_1, \ldots, e_i)\) of \(E_0\) such that, for all \(i \geq 2, e_i = g(e_i, e_i) = -g(e_i, e_1) = 1, \) and \(g(e_i, e_j) = 0, \) for all \(1 \leq i \neq j \leq r.\)

Let us \((u_1, \ldots, u_r)\) a local coordinate system associated to \((e_1, \ldots, e_r)\), i.e. \(e_i = \frac{\partial}{\partial u_i}\) for all \(1 \leq i \leq r.\)

On the other hand, \(R(E_0, E_0)E_1 = 0,\) then we can also choose \((X_1, \ldots, X_{n-r})\) an orthonormal fram of \(E_1\) such that, for all \(1 \leq j \leq n - r, X_j\) is parallel on \(N_0, \) i.e. \(\nabla_{X_j} X_j = 0.\)
For all vector fields $e_i$ and for all vector fields $X$, we set

$$\nabla_X e_i = A_i(X) + \sum_{j=1}^{r} B_i^j(X) e_j,$$  \hspace{1cm} (19)

where $A_i(X)$ is the orthogonal projection of $\nabla_X e_i$ on $E_1$. $A_i$ is $(1, 1)$-tensor on $M$ which is zero on $E_0$ and $B_i^j$ are covariant tensors on $M$ which they have the value zero on $E_0$.

Moreover, we get

$$B_i^j(X) = e_j g(\nabla_X e_i, e_i) = -e_j g(e_i, \nabla_X e_i) = -e_j e_i B_i^j(X).$$ \hspace{1cm} (20)

**Definition 4.1.** The field tensors $A_i$ and $B_i^j$ are called the second fundamental forms corresponding to the system $\{e_1, ..., e_r\}$.

We define the $(0, 2)$-tensor $B^i$ by:

$$B^i(X, Y) := -g(A_i(X), Y),$$

$$B^i(X, e_j) := B^j_i(e_j, X) = B^i(e_i, e_i) = 0,$$ \hspace{1cm} (21)

where $X$ and $Y$ are vectors fields tangent to $E_1$.

**Lemma 4.1.** The second fundamental forms $A_i$ and $B_i^j$ and the curvature $R$ satisfy the following properties:

$$2 \text{trace}(A_i) = \frac{(n - r) \partial \lambda}{\partial u_i}, \hspace{1cm} 1 \leq i \leq r,$$ \hspace{1cm} (22)

$$\nabla_e A_i(X) = -A_j \circ A_i(X),$$ \hspace{1cm} (23)

$$\nabla_e B_i^j(X) = -B_i^k(A_k(X)),$$ \hspace{1cm} (24)

$$(\nabla_e R)(X, Y) = R(Y, A_i(X)) + R(A_i(Y), X),$$ \hspace{1cm} (25)

$$R(X, Y)A_i(Z) + R(Y, Z)A_i(X) + R(Z, X)A_i(Y) = 0,$$ \hspace{1cm} (26)

for $X, Y, Z \in E_1$.

**Proof.** The formula (22) is result of the formula (18). Indeed;

$$\delta(v(s)(e_i)) = -\frac{1}{2} \delta(s)(e_i)$$

Let us $X, Y, Z \in E_1$.

Formulas (23) and (24) follow from the equation $R(e_i, X)e_j = 0$.

The formulas (25) and (26) follow from the second Bianchi identity, indeed:

$$(\nabla_e R)(X, Y) = \nabla_Y R(X, e_i) + (\nabla_Y R)(X, e_i)$$

$$= [\nabla_X, R(X, e_i)] - R(\nabla_X e_i, Y) - R(e_i, \nabla_X Y)$$

$$+ [\nabla_Y, R(X, e_i)] - R(\nabla_Y X, e_i) - R(X, \nabla_Y e_i)$$

$$= -R(\nabla_X e_i, Y) - R(X, \nabla_Y e_i)$$

$$= R(Y, A_i(X)) + R(A_i(Y), X)$$

So $R(X, Y)(e_i) = R(Z, X)(e_i) = R(Y, Z)(e_i) = 0$, we get

$$0 = (\nabla_X R)(Y, Z)(e_i) + (\nabla_Y R)(Z, X)(e_i) + (\nabla_Z R)(X, Y)(e_i)$$

$$= -R(Y, Z)(\nabla_X e_i) - R(Z, X)(\nabla_Y e_i) - R(X, Y)(\nabla_Z e_i)$$

$\square$

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**Corollary 4.1.** \( (\nabla_e R) \) is a curvature tensor on \( M \) and \( \nabla_e R(X,Y) \in \mathfrak{h}_1(R) \) for all \( X, Y \in E_1 \).

**Proposition 4.3.** There exists a function \( \mu_i \) of class \( C^\infty \) on \( M \) such that

\[
\nabla_e R(X,Y) = -2\mu_i R(X,Y),
\]

where \( X, Y \in E_1 \).

**Proof.**

Let \( m \) be a point in \( M \). So the primitive holonomy group \( \mathcal{H}_1(R_m) \) acts irreducibly on \( E_1(m) \). Therefore, \( [E_1(m), R_m, \mathcal{H}_1(R_m)] \) and \( [E_1(m), (\nabla_e R)_m, \mathcal{H}_1(R_m)] \) are two Riemannian irreducible symmetric holonomy systems. According to the corollary of Theorem 6 in ([13]), there exists a real \( \mu_i(m) \) satisfying

\[
(\nabla_e R)_m = -2\mu_i(m) R_m.
\]

So the tensors \( \nabla_e R \) and \( R \) are of class \( C^\infty \), then the function \( \mu_i \) is of class \( C^\infty \).

**Lemma 4.2.** Any second fundamental form \( A_i \) on \( E_1 \) has one of the following forms \( A_i = \mu_i I \) or else \( n \in 2\mathbb{N} \) and \( A_i = \mu_i I + \lambda_i J \) with \( J^2 = -I \), where \( \mu_i \) and \( \lambda_i \) are functions of class \( C^\infty \). The skewsymmetric endomorphism \( J \) is a uniquely determined and is independent of \( i \) and the choice of the system \( (e_1, ..., e_r) \) and commutes with each elements of \( \mathcal{H}_1(R_m) \) at any point \( m \in M \).

**Proof.** The proof is similar as that of Lemma 4.4 in ([15]) considering the Lie algebra \( \mathcal{H}_1(R_m) \) and the Riemannian irreducible symmetric holonomy system \( S_m := [E_1, R_m, \mathcal{H}_1(R_m)] \).

From the formula 22, we get

\[
\mu_i = -\frac{1}{2\lambda} \frac{\partial \lambda}{\partial u_i}, \quad 1 \leq i \leq r.
\]

So, if \( A_i = \mu_i I + \lambda_i J \), since \( \nabla_e A_i = -A_i^2 \), we get

\[
\nabla_e J = 0,
\]

\[
\nabla_e \mu_i = \lambda_i^2 - \mu_i^2.
\]

\[
-2\lambda_i \mu_i = \frac{\partial \lambda_i}{\partial u_i}.
\]

Then

\[
\lambda_i \frac{\partial \lambda_i}{\partial u_i} = \lambda_i \frac{\partial \lambda}{\partial u_i}.
\]

**Lemma 4.3.** If the co-nullity index satisfies \( n - r \geq 3 \), we get that

\[
r \in \{1, 2\}.
\]

**Proof.**

By Lemma 3.3 we get

\[
TM = F_1 \oplus (F_1 \cap (F_1)^\perp) = E_0 \oplus E_1,
\]

and

\[
F_1 = \text{span}\{pr_0(\nabla_X Y) / X, Y \in E_1\}
\]

where \( pr_0 : TM \to E_0 \) is the orthogonal projection in \( E_0 \). We will show that

\[
r = \text{dim}(\text{span}(pr_0(\nabla_X Y) / X, Y \in E_1)).
\]

Necessarily, we have \( r \geq 1 \). Let \( (e_1, ..., e_r) \) be an orthonormal fram of \( E_0 \).
First, we show \( \dim(\text{span}\{\text{pr}_0(\nabla_X X)/ X \in E_1\}) = 1 \); Let us \( X, Y \in E_1 \). If \( X \) and \( Y \) are orthogonal unit field. Let \( A_i = \mu_i I + \lambda_i J \) be a second fundamental form, where \( J \) is skew-symmetric satisfying \( J^2 = -I \). Then
\[
g(\nabla_X X, e_i) = -\mu_j g(X, X) = -\mu_j g(Y, Y) = g(\nabla_Y Y, e_i).
\]
consequently, \( \text{pr}_0(\nabla_X X) = \text{pr}_0(\nabla_Y Y) \).

If \( X \) and \( Y \) are not orthogonal. Since \( \dim(E_i) \geq 3 \), we can choose \( Z \) orthogonal to both \( X \) and \( Y \).

Then, \( \text{pr}_0(\nabla_X X) = \text{pr}_0(\nabla_Z Z) = \text{pr}_0(\nabla_Y Y) \), thus
\[
\dim(\text{span}\{\text{pr}_0(\nabla_X X)/ X \in E_1\}) = 1.
\]
Moreover, there is no isotropic vector field in \( F_1 \) unless \( \dim(E_0) = 2 \).

If any second fundamental forms are of the form \( A_i = \mu_i I \), we get that
\[
\dim(\text{span}\{\text{pr}_0(\nabla_X X)/ X, Y \in E_1\}) = 1.
\]
Indeed, for all orthogonal fields \( X \) and \( Y \) in \( E_1 \), we have
\[
g(\nabla_X X, e_i) = -\mu_j g(X, Y) = g(\nabla_Y Y, e_i) = 0.
\]
Hence \( \text{span}\{\text{pr}_0(\nabla_X Y)/ X, Y \in E_1\} = \text{span}\{\text{pr}_0(\nabla_X X)/ X \in E_1\} \).

If there is a second fundamental form \( A_i = \mu_i I + \lambda_i J \) where \( \lambda_i \neq 0 \) and \( J \) is skew-symmetric endomorphism checking \( J^2 = -I \).

Then \( \dim(E_i) = 2I \) and we can choose \( \{X_1, Y_1 = J(X_1), ..., X_l, Y_l = J(X_l)\} \) an orthonormal frame of \( E_1 \).

Let \( A_i = \mu_i I + \lambda_i J \) be the second fundamental form associated to the vector fields \( e_k \). For all \( i \neq j \), we get
\[
\begin{align*}
g(\nabla_X Y_j, e_k) &= g(\nabla_X X_j, e_k) = -\lambda_j \\
g(\nabla_X Y_k, e_j) &= g(\nabla_Y Y_k, e_j) = g(\nabla_X X_k, e_j) = g(\nabla_Y Y_k, e_j) = 0.
\end{align*}
\]
Thus
\[
\dim(\text{span}\{\text{pr}_0(\nabla_X Y)/ X, Y \in E_1, g(X, Y) = 0\}) = 1
\]
and \( F_1 = \text{span}\{\text{pr}_0(\nabla_X Y)/ X, Y \in E_1, g(X, Y) = 0\} + \text{span}\{\text{pr}_0(\nabla_X X)/ X \in E_1\} \).

Consequently, we get
\[
r = \dim\left(\text{span}\{\text{pr}_0(\nabla_X Y)/ X, Y \in E_1\}\right) \leq 2.
\]

\[\square\]

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