On the (im)possibility of extending the GRW model to relativistic particles

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Abstract
We investigate the relativistic properties of the distinguishable non-interacting relativistic GRW model presented in [1]. We discuss how the relativistic properties of this theory are contingent on the points of collapse being time-like to each other. We show that models describing indistinguishable or interacting particles require space-like points of collapse hence it is not possible to extend to such cases whilst retaining the desired relativistic and collapse dynamics of the original model.

1. Introduction
In quantum mechanics there are two forms of dynamics; unitary evolution, which is time reversible and preserves superpositions, which describes the evolution of isolated systems, and evolution described by positive operator valued measures (POVMs) which describes a system interacting with an external environment, such as when a measurement is performed. The measurement problem is the fact that quantum mechanics fails to provide a precise description of which form of evolution describes any one situation. From observation limits can be placed on which regimes may be described with unitary evolution or POVMs, but the theory itself does not provide these.

Spontaneous collapse models, first introduced by Ghirardi-Rimini-Weber [2], solve the measurement problem by giving a unique dynamics which completely describes the time evolution of the system at a non-relativistic level. This is done by introducing additional stochastic non-linear terms to the Schrödinger equation. These terms alter the form of unitary evolution such that there is a non-zero rate of the state describing a particle undergoing a spontaneous spatial localisation. This rate is proposed to be extremely low, such that a single particle may remain in a superposition for a long period of time, in line with what is seen experimentally. However for multiple particles which are entangled then any single particle spontaneously collapsing collapses all particles it is entangled with. This effectively increases the rate of collapse for systems with high numbers of particles, such that macroscopic bodies are localised on extremely short time scales, this is often called the amplification mechanism and it ensures macroscopic classicality. This removes the need for the theory to include a description of an external observer, as macroscopic measuring apparatus interacting with a microscopic system causes the microscopic system to become entangled and hence collapse, via the amplification mechanism. For a full review of spontaneous collapse models see [3].

In order for a spontaneous collapse model to be a successful description of the underlying physics then it must be consistent with special relativity. There is a tension between quantum mechanics and special relativity as quantum mechanics is non-local because space-like separated measurements of entangled systems must be correlated (as argued by EPR in [4]). A spontaneous collapse model should predict non-local correlations in order to remain consistent with experiment.

Note that this paper is concerned with collapse model’s consistency only with special relativity. Special relativity implies the prohibition of superluminal signalling, therefore a special relativistic collapse model would automatically be causal. A collapse model (or any quantum theory) aiming to...
be in agreement with general relativity would have to have a dynamical causal structure. See [5–9] for recent work in this area. From now on in this article we will use relativistic to mean consistent with special relativity.

In its original formulation the GRW model was not relativistic and described distinguishable particles for a discrete time processes. Continuous time collapse models have also been developed for instance in [10, 11]. There are various proposed models for relativistic collapse models: [12] where a prescription for the probability distribution of a matter density operator is Lorentz invariant, [13] which introduces a mediating pointer field, [14] in which collapse dynamics emerge by tracing out an environment from a relativistic quantum field theory, [15] which proposes that the terms modifying the conventional Schrödinger equation are functions of the stress-energy tensor. Pearle suggested a model in 1999 [16] and a proposed alteration of this in section 11 of [17] where energy is conserved by considering relational collapses.

In [18] a collapse model on a 1 + 1 lattice is presented and the authors suggest that it may be relativistic in the continuum limit. In this paper we will consider one of the most developed types of attempted relativistic collapse models, a relativistic GRW model. Both [1] and [19] propose relativistic extensions of GRW to described distinguishable non-interacting particles. As it is known that in reality particles are indistinguishable and interact with one another it should be considered if these models can be extended either to indistinguishable or interacting particles or both, whilst remaining relativistic. In this paper we show explicitly that this is not possible with [1]. We argue that the same difficulty occurs for [19] and therefore, that if it is possible to create a relativistic collapse model another route must be taken.

This paper is organised as follows: in section 2 requirements for a spontaneous collapse model to be relativistic are reviewed and the Tomogana-Schwinger formalism is discussed. In section 3 it is shown why the original GRW model does not meet the relativistic requirement. In section 4 Tumulka’s model is introduced and it is shown why it is relativistic. In section 5 the indistinguishable extension is attempted, it is shown it is either not relativistic or does not achieve macroscopic classicality, in section 6 it is shown that Tumulka’s model extended to interacting particles is also not relativistic.

2. COLLAPSE MODELS AND SPECIAL RELATIVITY

2.1. QUANTUM MECHANICS AND SPECIAL RELATIVITY

Standard quantum mechanics provides probability distributions for the values of observables that are measured. A relativistic quantum mechanics must predict that observers in any two inertial frames to have the same measurement statistics for the outcome of any experiment they can perform. This is the same conclusion reached in [20] by Aharonov and Albert. They state that for a system with observables $A, B, C...$ each with potential values $a_i, b_i, c_i...$ where $i$ runs over the potential values of each observable measured where observable $A$ is measured at time $t_a$ and found to have the value $a$, and other variables respectively, then agreement with special relativity implies that there is a covariant way of calculating $P(a, t_a, b, t_b, ...|c, t_c, d, t_d...)$ i.e.

$$P(a, t_a, b, t_b, ...|c, t_c, d, t_d...) = P(a', t_{a'}, b', t_{b'}, ...|c', t_{c'}, d', t_{d'}...). \text{ (1)}$$

where $a', b'$ etc. are the values of the observables in the coordinates of a different inertial frame.

Probability distributions in quantum mechanics are found from the state via the Born rule. For non-relativistic quantum mechanics states are functions over every point in spacetime. However if one wishes to have a relativistic quantum mechanics where the state undergoes instantaneous collapses then this is not possible. Instantaneous collapses are required to ensure that non-local observables (for example momentum or total charge) are conserved [20].

If one attempts to treat the state as a function over all of spacetime and describe the state as collapsing instantaneously in one inertial frame then this is equivalent to selecting a preferred frame, as the state cannot be normalised in every frame; see figure [1].

In order to offer a frame independent description of collapse of the state Aharonov and Albert proposed an alternative way of describing collapse [21], in which the state collapses instantaneously in every inertial frame.

To allow this the state must be considered to be a functional on the 3D space-like hypersurfaces which make up the 4D manifold, instead of a function over all of spacetime. States are defined on space-like hypersurfaces, if we label a hypersurface as $\omega$ then we can write a state on it as $\psi_\omega(x)$. 

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The coordinate $x$ here labels the coordinates of the 3D surface $\omega$ but is a four vector $x \in \mathbb{M}^4$ as $\omega$ is understood to be embedded in 4D spacetime. So then every inertial observer has a state defined on their constant time 3D hypersurface. However each state may have different values at the same spacetime point $\psi_\omega(x) \neq \psi_\omega(x')$.

![Figure 1: A spacetime diagram showing the support of the state of a single particle in red dashed lines. Different constant time hypersurfaces are shown as blue dotted lines. Suppose that a collapse occurs along $\sigma_{t_1}$, to point P. Then the state on $\sigma_0'$ is not normalised as $\sigma_0'$ only intersects half the support of the state.](image)

In this framework every inertial observer can describe the time evolution of their system in terms of states on parallel constant time hypersurfaces within their frame using the Tomogana-Schwinger formalism. We will introduce this formalism and show that if collapses are excluded, then this description is Lorentz covariant if it is integrable. Then we will derive a condition for Lorentz covariance if collapses are time-like to each other and finally we will show that if collapses are space-like to each other then the dynamics is not Lorentz covariance and Eq. (1) is not satisfied.

2.2. The Tomogana-Schwinger formalism

The Tomogana-Schwinger formalism\cite{22,23} describes unitary evolution as maps between states defined on arbitrary space-like hypersurfaces without collapses. First we will introduce some additional notation for hypersurfaces. Let $\omega$ signify any generic space-like 3 dimensional hypersurface, let $\sigma_t$ denote a constant time hyperplane at time $t$ in an inertial frame $\mathcal{F}$ and hence $\sigma_{t'}$ is a constant time hyperplane in a different inertial frame $\mathcal{F}'$. Then suppose the state is defined on an $\omega$ in the manifold $\mathbb{M}^4$. In this article we restrict ourselves to considering Minkowski spacetime $\mathbb{M}^4$ as it is sufficient to see the relevant Lorentz transformation properties of the probability distributions. Then in inertial frame $\mathcal{F}$ which has coordinates $x$ on a hypersurface $\omega$ the state is $\psi_\omega(x) \in H$ and $H$ is a Hilbert space defined on $\omega$. In another inertial frame $\mathcal{F}'$ with coordinates $x'$ then on the same hyperplane $\omega$ the state is written $\psi'_\omega(x')$. A Dirac fermionic state under a Lorentz boost transforms as:

$$\psi_\omega(x) \rightarrow \psi'_\omega(x') = \Lambda \psi_\omega(x').$$

(2) where $\Lambda$ the spinor representation of the Lorentz group. In other words on the same hypersurface the states are equivalent up to a Lorentz transform.

In a frame $\mathcal{F}$ there is an arrow of time which defines a time ordering between any two points in $\mathbb{M}^4$. Analogously to the Schrödinger equation Tomogana and Schwinger defined the evolution of a state as it evolves between hypersurfaces, if there are no collapses between those surfaces:

$$\frac{\delta}{\delta \omega(x)} \psi_\omega(x) = -i \mathcal{H}(x) \psi_\omega(x)$$

(3) where $\frac{\delta}{\delta \omega(x)}$ is the functional derivative with respect to $\omega$ and $\mathcal{H}(x)$ is the Hamiltonian density. The functional derivation can be understood to be the variation in $\psi_\omega(x)$ with respect to a infinitesimal variation of $\omega$ about point a $x$, see figure 2.

![Figure 2: A diagram showing the infinitesimal variation, $\delta \omega$, of the hypersurface $\omega$ about the point $x$.](image)

The integrability condition for this system is that $[\mathcal{H}(x), \mathcal{H}(y)] = 0$ if $x$ and $y$ are space-like separated. This gives rise to an unitary evolution operator which relates two hypersurfaces:

$$U_{\omega_2}^{\omega_1} = T \exp[-i \int_{\omega_1}^{\omega_2} d^4 x \mathcal{H}(x)]$$

(4) such that $\psi_{\omega_2}(x) = U_{\omega_2}^{\omega_1} \psi_{\omega_1}(x)$, where $T$ means time ordering with respect to the frame $\mathcal{F}$. This
operator is frame independent although \( \mathcal{H}(x) \) is not Lorentz invariant, the only frame dependant terms from the time ordering are zero due to the integrability condition \[19\,24\]. Therefore we have that for a frame \( \mathcal{F} \):

\[
U^\omega \equiv T \exp[-i \int_{\omega}^{\omega_2} d^4 x' \mathcal{H}'(x')] = AU^\omega A
\]

(5a) \hspace{1cm} (5b)

2.3. The Tomogana-Schwinger formalism with collapses

Now we wish to extend this formalism to describe collapses of the state and derive conditions for theories with discrete collapses to be consistent with special relativity. In this article we will consider only collapses in the spatial basis as this is sufficient to explain the values of any experiment performed, as any observable can be coupled to position.

In a frame \( \mathcal{F} \) the spatial collapse of the state at \( x \in \mathcal{M}^4 \) is described as an operator \( \hat{L}_\omega(x) \) acting on the Hilbert space on the a space-like hypersurface \( \omega \) passing through \( x \). As a convention we choose the constant time hyperplane intersecting \( x \), labelled \( \sigma_t \) where \( t = x_0 \). This means that the collapse is described as occurring instantaneously in \( \mathcal{F} \), as discussed in section 2.1. \( \hat{L}_{\sigma_t}(x) \) localises the state about \( x \). The properties of this operator are model dependant however in general it is not unitary. In a different frame \( \mathcal{F}' \) the collapse operator \( \hat{L}'_{\sigma_t}(x') \) is defined on a constant time hypersurface \( \sigma'_t \).

To illustrate evolution with collapses consider in a frame \( \mathcal{F} \) two hypersurfaces \( \sigma_0, \sigma_f \) before and after a collapse at a point \( x \). The state \( \psi_{\sigma_f} \) is given by evolving the state to a hyperplane of collapse, applying the collapse operator and normalising then evolving to \( \sigma_f \):

\[
\psi_{\sigma_f} = \frac{U_{\sigma_f}^\dagger \hat{L}_{\sigma_t}(x) U_{\sigma_0}^\dagger \psi_{\sigma_0}}{\left\| \hat{L}_{\sigma_t}(x) U_{\sigma_0}^\dagger \psi_{\sigma_0} \right\|^2}.
\]

(6)

It is necessary that all points of collapse between \( \sigma_0, \sigma_f \) are known in order to construct such a map between them as in general \( \hat{L}_{\sigma_t}(x) \psi_\omega \neq \psi_\omega \) for any \( \omega \). Therefore in order to relate states in different frames on their respective constant time hypersurfaces all collapses between those hypersurfaces must be known.

2.4. Relativistic Condition for Collapse Models

We will now consider a relativistic condition for a quantum mechanics with collapses. Assume that the dynamics are Markovian such that only the most recent point of collapse must be considered to specify the probability distribution of the position of the next collapse. Then in a frame \( \mathcal{F} \) assuming the point of last collapse \( y \) and the state of the entire system immediately after collapse \( \psi_{\sigma_0} \) are known (for simplicity we will assume \( y \) occurs at \( t = 0 \)) the probability distribution of the position of the next collapse is:

\[
P(x|y, \psi_{\sigma_0}) = \left\| \hat{L}_{\sigma_t}(x) U_{\sigma_0}^\dagger \psi_{\sigma_0} \right\|^2
\]

(7)

where \( \sigma_t \) is the surface intersecting \( x \). This conditional probability distribution is in the same form as the one in eq. [4] where \( \psi_{\sigma_0} \) gives all the possible information about the system at \( (y, t_0) \).

Figure 3: The blue dotted lines show the space-like hypersurfaces. Then \( \sigma_0 \) labels the surface on which the initial state of the system is defined and \( \sigma'_0 \) is the equivalent hypersurface in a different inertial frame. The point \( y \) is the initial point of collapse. The black dashed line is the future light cone of \( y \). The point \( x \) is space-like to \( y \) and lies between \( \sigma_0 \) and \( \sigma'_0 \) and hence would effect evolution between the states defined on those surfaces.

For a Lorentz transformed inertial frame \( \mathcal{F}' \) with coordinates \( x' \) the initial conditions are the point of last collapse \( y' \) and the state on the hyperplane \( \sigma_{0'} \). Therefore special relativity requires that:

\[
P(x|y, \psi_{\sigma_0}) = P(x'|y', \psi_{\sigma_{0'}}).
\]

(8)

This condition is stronger than only requiring that the equations of motion transform covariantly, as in order to check this condition one must be able to compare the initial conditions (here the
state and position of the previous collapse) between different inertial frames, as noted in [19].
This has consequences when considering collapse models for multiple particles.
In order to verify eq. 8 the map between \( \psi_{\sigma_0} \) and \( \psi'_{\sigma_0'} \) must be known, therefore positions of all collapses between those surfaces must be known.
For a series of time-like collapses this condition is met as there can be no collapses between \( \sigma_0 \) and \( \sigma_0' \) hence they are related by:

\[
\psi'_{\sigma_0'} = \Lambda^\dagger U^\sigma_{\sigma_0'} \psi_{\sigma_0} \tag{9}
\]
To specify the transformation of \( \hat{L}_{\sigma_0}(x) \) we consider the fact that special relativity also requires that the conditional probability is given by:

\[
P(x_1, x_2 | y, \psi_{\sigma_0}) = P(x_1', x_2' | y', \psi'_{\sigma_0'}) \tag{10}
\]
where \( x_1, x_2 \) and \( y \) are all time-like to each other.
The conditional probability is given by:

\[
P(x_1, x_2 | y, \psi_{\sigma_0}) = \left| \left| \hat{L}_{\sigma_0}(x_2) U^\sigma_{\sigma_0} \hat{L}_{\sigma_0}(x_1) U^\sigma_{\sigma_0} \psi_{\sigma_0} \right| \right|^2 \tag{11}
\]
where \( \sigma_0 \) is the hypersurface of collapse intersecting \( x_1 \) in frame \( \mathcal{F} \) and \( \sigma_2 \) intersects \( x_2 \). Assuming that the Hamiltonian is covariant so that eq. 5b holds then eq. 11 can be written as:

\[
P(x_1', x_2' | y', \psi'_{\sigma_0'}) = \left| \left| \hat{L}_{\sigma_0'}(x_2') U^\sigma_{\sigma_0'} \hat{L}_{\sigma_0'}(x_1') U^\sigma_{\sigma_0'} \psi'_{\sigma_0} \right| \right|^2 = \left| \left| \hat{L}_{\sigma_0'}(x_2') \Lambda^\dagger U^\sigma_{\sigma_0'} U^\sigma_{\sigma_0'} \hat{L}_{\sigma_0'}(x_1') \Lambda \hat{L}_{\sigma_0'}(x_1') \Lambda^\dagger U^\sigma_{\sigma_0'} U^\sigma_{\sigma_0} \psi_{\sigma_0} \right| \right|^2 \tag{12}
\]
Where eq. 5b has been used to transform the unitary operators and \( \sigma_1, \sigma_2 \) are hypersurfaces of collapse intersecting \( x_1 \) and \( x_2 \) in frame \( \mathcal{F}' \). So by inspection the condition for eq. 10 to hold is:

\[
\hat{L}_{\sigma_0'}(x') = \Lambda^\dagger U^\sigma_{\sigma_0'} \hat{L}_{\sigma_0}(x) U^\sigma_{\sigma_0'} \Lambda. \tag{13}
\]
Eq. 13 requires that the collapse operator transforms covariantly and that the collapse can be described by a operator acting on any space-like hypersurface intersecting \( x \). This is equivalent to requiring that the collapse happens instantaneously in all inertial frames. With these conditions spatial collapses which are time-like to one another may be described in a way that is consistent with special relativity.
For a theory with spatial collapses which are space-like to each other the initial state in different inertial frames can no longer necessarily be related to each other by eq. 9 as there might be collapses in the region enclosed between \( \sigma_0 \) and \( \sigma_0' \), as shown in figure 3. To verify eq. 8 the position of all collapses in this region must be known, since this region includes points which are in the future of \( y \) in \( \mathcal{F} \). This is equivalent to requiring knowledge of future points of collapse. Stochastic theories cannot meet this requirement, as the points of collapse are a single realisation of a random process and hence cannot be determined with certainty.
So spontaneous collapse models can meet condition 8 when collapses are time-like to each other but for space-like collapses the initial condition for observers in two frames cannot be compared so the condition is not satisfied. As we shall see, this observation is crucial for showing that Tumulka’s model cannot be extended to indistinguishable or interacting particles.

3. The Original GRW Model
The original GRW model is a model for \( N \) distinguishable particles, however in order see that it is not relativistic it is sufficient to consider the one particle case. We define the model as follows.
Consider a one particle state on some initial hypersurface \( \psi_{\sigma_0}(z, t = 0) \in H \), where \( z \in \mathbb{R}^3 \) is the coordinate of the particle. The unitary evolution operator is of the form of eq. 4. Define for a collapse at \( t \) the collapse operator as:

\[
\hat{L}_{\sigma_0}(z) := \left( \frac{\alpha}{\pi} \right)^{\frac{3}{2}} \exp\left( -\frac{\alpha(z - \hat{q})^2}{2} \right) \tag{14}
\]
where \( \alpha \) is a free parameter of the model and \( \sigma_2 \) is the surface intersecting the point \( (z, t) \), and \( \hat{q} \) is the position operator on \( H \).
The model also has a stochastic Poisson process \( T \) with an average \( \tau \), which gives the time interval between two points of collapse. Let a realisation of the this process be labelled \( \Delta t \). In a frame \( \mathcal{F} \) given a collapse at point \( y \in M^4 \) where \( y^0 = 0 \) then the next collapse must occur on the hypersurface \( \sigma_{\Delta t} \), as shown in figure 2. The probability distribution for a collapse at a point \( x \in \mathbb{R}^3 \) on this surface is:

\[
P(x | y, \Delta t, \psi_{\sigma_0}) = \left| \left| \hat{L}_{\sigma_{\Delta t}}(x) U^{\sigma_{\Delta t}}_{\sigma_{\sigma_0}} \psi_{\sigma_0} \right| \right|^2 \tag{15}
\]
This distribution is normalised such that:

\[
\int_{\mathbb{R}^3} d^3x \, P(x | y, \Delta t, \psi_{\sigma_0}) = 1. \tag{16}
\]
As eq. [14] does not meet condition eq. [13] then the model is not Lorentz covariant.

4. TUMULKA’S RELATIVISTIC COLLAPSE MODEL FOR DISTINGUISHABLE PARTICLES

In [1], Tumulka suggested a relativistic version of the GRW model in which the domain of the probability distribution for the position of the collapse is not a flat hyperplane but a hyperboloid. The model describes non-interacting distinguishable particles, as such the only distinction between the N-particle model and N one-particle models is that in the former the initial wavefunction may have entangled particles. Here we will briefly describe the one-particle theory.

4.1. SINGLE PARTICLE MODEL

The unitary evolution for this model is given by the Dirac equation for a single particle. The model also includes a stochastic Poisson process \( T \), with a mean \( \tau \), with a realisation labelled \( \Delta t \). Instead of defining the probability distribution for the position of a collapse on a flat hyperplane, it is defined on a Lorentz invariant future pointing hypersurface of the system is defined. For the GRW model \( \alpha \) where \( \hat{q} \) is the position operator on the Hilbert space defined on \( \Sigma, x = f(x, \Sigma) \), where \( f \) is a function that maps a 4-vector point in \( \mathbb{M}^4 \) to a 3-vector, \( \text{s-dist}_\Sigma \) is the Lorentz invariant distance on \( \Sigma \), and \( K_\Sigma(z) \) is a normalisation constant defined such that:

\[
\int_{\Sigma} d\Sigma \left\| \hat{L}_\Sigma(x) \psi_{\Sigma} \right\|^2 = 1.
\]

Thus the conditional probability distribution for there to be a collapse at position \( x \in \Sigma \) is:

\[
P(x|y, \Delta t, \psi_{\sigma_0}) = \left\| \hat{L}_\Sigma(x) U_{\Sigma}^{\sigma_0} \psi_{\sigma_0} \right\|^2 \quad (19)
\]

The distribution \( (19) \) completely defines the collapse dynamics as once the position \( x \) is known then \( \psi_{\sigma_t} \), a state on the constant time hypersurface at \( t = x_0 \), can be defined using eq. [6]. Then the process can be iterated to find the position of each subsequent collapse.

Since the model specifies that collapses only occur on \( \Sigma \) then the collapses are time-like to another hence the relationship eq. [7] holds, as for a one particle model there are not collapses between \( \Sigma \) and \( \sigma_0 \). As the unitary dynamics are given by the Dirac equation they satisfy condition eq. [5b].

The collapse operator on a constant time hypersurface can be defined as:

\[
\hat{L}_{\sigma_x}(x) = U_{\Sigma}^{\sigma_x} \hat{L}_\Sigma(x) U_{\sigma_x}^{\Sigma}.
\]

It is shown in appendix [B] that the operator \( (20) \) meets the condition eq. [13]. Therefore this model is consistent with special relativity. In order for this model to localise the particle in any inertial frame, it must be true that \( \hat{L}_{\sigma_x}(x) \) can be approximated as:

\[
\hat{L}_{\sigma_x}(x) \approx K_{\sigma_x}(\hat{q}) f_\alpha(x, \hat{q})
\]

where \( f_\alpha(x) \) is a function sharply peaked about \( x = 0 \) with a width proportional to \( 1/\alpha \) and \( K_{\sigma_x}(\hat{q}) \) is a normalisation function. If this were not true then the collapse would not localise the state in the position basis in each inertial frame.
The extension to N distinguishable non-interacting particles is done by defining N single particles on N different spacetime manifolds, see [1]. This model does not go far beyond the single particle model, as the particles are non-interacting and distinguishable. If the state is initially entangled then the model is non-local, but if the state is not initially entangled then there is no way of generating entanglement as there is no interaction.

This model for single particles uses the Dirac equation. The Dirac equation for a single particle leads to solutions with negative energy so is not physical. Solutions to the Dirac equation describe particles and anti-particles which are elements of a Fock space, hence in order to be physically realistic the Tumulka collapse model must be extended to indistinguishable particles.

5. Attempt at an indistinguishable particle extension

Here we present an attempted extension to the model presented in [1] to non-interacting indistinguishable particles. Initially we will consider collapses which are time-like to each other.

In an N particle model without interaction the number of particles remains constant. As we are considering fermions the Hilbert space of the system is an N-particle anti-symmetric Fock space for a fixed number of particles.

$$H = S_- H_	ext{Fock}^{\otimes N}(1)$$

where $H(1)$ the one particle Hilbert space, $S_-$ is a tensor operator which anti-symmetrises the tensor product of N Hilbert spaces. A state of this system is a completely anti-symmetric combination of single particle states $\Psi_\omega$. The unitary evolution of this model is given by the Dirac equation. The collapse operator is the number operator with a weighting function. On a hyperboloid $\Sigma$ the collapse operator is:

$$\mathcal{J}_\Sigma(x) := \int_{-\infty}^{\infty} d\Sigma(y) \hat{K}_\Sigma(y) \exp(-\frac{\alpha}{2} s\text{dist}_\Sigma^2(x,y)) \hat{a}^\dagger(y) \hat{a}(y)$$

where $\hat{a}^\dagger(y)$ and $\hat{a}(y)$ are the creation and annihilation operators on the anti-symmetric Fock space. The model proceeds much the same way as the the single particle case, given an initial state on a constant time hypersurface $\Psi_{\sigma_0}$ and a point of collapse $y$ and a stochastic interval $\Delta t$ then probability distribution for the position of the next collapse is:

$$P(x|y, \Delta t, \Psi_{\sigma_0}) = \left| \left| \hat{J}_\Sigma(x) U_{\sigma_0}^\Sigma \Psi_{\sigma_0} \right| \right|^2$$

with $\Sigma$ defined as before.

This indistinguishable particle extension with collapses which are time-like to each other is consistent with special relativity as condition eq. 13 is met by the unitary operator given by eq. 45, condition eq. 13 is met as the collapse operator has the same form as the single particle operator. Finally as the collapses are time-like to each other then there are no collapses between $\sigma_0$ and $\Sigma$ and therefore $U_{\sigma_0}^\Sigma$ can be specified hence the relationship eq. 9 can be used.

However this model has an issue. If given a point of collapse $y$ the only region that the subsequent collapse can occur is in the future light cone of $y$ then macroscopic classicality is not recovered.

This can be seen with a simple example, with two macroscopic objects. Suppose there is a system made up a large number of indistinguishable particles $N$, where $N$ is an even number. The initial state of the system is two macroscopic objects, i.e. two areas with high densities of particles, with a large distance separation between the centre of mass of these two areas, labelled $2d$, see figure 6. Assume that initially each object is in a spatial superposition, separated by a distance $2d$, where $r \ll d$. For simplicity we will work in one dimension but the argument can be extended to three dimensions. We will work in the framework of second quantization.

The initial state of the system on a constant time hypersurface $\sigma_0$ is:

$$|\Psi_{\sigma_0}\rangle = \frac{1}{2} (\hat{A}_1 + \hat{A}_2)(\hat{B}_1 + \hat{B}_2)|0\rangle$$

where $|0\rangle$ is the vacuum of a N particle anti-symmetric Fock space and:

$$\hat{A}_1 = \prod_{n=0}^{N/2} \hat{g}(-d - r, n)$$
$$\hat{A}_2 = \prod_{n=0}^{N/2} \hat{g}(-d + r, n)$$
$$\hat{B}_1 = \prod_{n=0}^{N/2} \hat{g}(d - r, n)$$
$$\hat{B}_2 = \prod_{n=0}^{N/2} \hat{g}(d + r, n)$$
The initial state is:

\[ \hat{N}_0 = \frac{N}{2} \left| \Psi_{\sigma_0} \right\rangle \]

and similarly for the right part of the system \( \hat{N}_B |\Psi_{\sigma_0}\rangle = N/2 |\Psi_{\sigma_0}\rangle \). However the initial state is not in an eigenstate of the number operator for the region to the left of \(-d\):

\[ \hat{N}_A |\Psi_{\sigma_0}\rangle = \int_{-\infty}^{-d} dx \hat{a}^\dagger(x) \hat{a}(x) (\hat{A}_1 \hat{B}_1 + \hat{A}_2 \hat{B}_2 + \hat{A}_1 \hat{B}_2 + \hat{A}_2 \hat{B}_1) |0\rangle = \frac{N}{2} \hat{A}_1 |\Psi_{\sigma_0}\rangle \]

which is not proportional to \( |\Psi_{\sigma_0}\rangle \). \( |\Psi_{\sigma_0}\rangle \) is also not an eigenstate of \( \hat{N}_A, \hat{N}_B_1 \) and \( \hat{N}_B_2 \). This implies there are two objects each in a superposition over two areas, not one object in a superposition over four areas nor four objects each in a localised position.

The amplification mechanism will cause a collapse of one of the objects almost immediately. Suppose that the collapse is at spacetime point \((t,-d+r)\), where \(t\) is so small that \(U_{\sigma_0}^t \approx I\). Then following eq. [6] and eq. [20] we find the state immediately after the collapse, on constant time hypersurface \(\sigma_t\) to be:

\[ |\Psi_{\sigma_t}\rangle = \frac{\hat{J}_{\sigma_t}(-d+r) |\Psi_{\sigma_0}\rangle}{\| \hat{J}_{\sigma_t}(-d+r) |\Psi_{\sigma_0}\rangle \|^2} \]

where:

\[ \hat{J}_{\sigma_t}(-d+r) |\Psi_{\sigma_0}\rangle = \int_{-\infty}^{-d} dy K(y) f_n(-d+r-y) \hat{a}^\dagger(y) \hat{a}(y) |\Psi_{\sigma_0}\rangle \]

where a similar approximation to eq. [21] has been made such that \(f_n(x)\) is a function sharply peaked about \(x = 0\) with a width proportional to \(1/\alpha\) and \(K(y)\) is a normalisation function. To evaluate this consider just the term:

\[ \hat{J}_{\sigma_t}(-d+r) \hat{A}_1 \hat{B}_1 |0\rangle = \frac{1}{2} \int_{-\infty}^{\infty} dy K(y) f_n(-d+r-y) \times \]

\[ \hat{a}^\dagger(y) \hat{a}(y) \prod_{n=0}^{N/2} \prod_{m=0}^{N/2} \hat{g}(-d-r,n) \hat{g}(d-r,m) |0\rangle \]
The contributions from the $\hat{g}(-d - r, n)$ and $\hat{g}(d - r, m)$ operators are weighted by factors of $f_\alpha(-2r + n\epsilon/4)$ and $f_\alpha(2d - 2r + n\epsilon/4)$ respectively. As $-2r + n\epsilon/4 \gg 1/\alpha$ and $2d - 2r + n\epsilon/4 \gg 1/\alpha$ then $f_\alpha(-2r + n\epsilon/4) \approx 0$ and $f_\alpha(2d - 2r + n\epsilon/4) \approx 0$. Hence:

$$\hat{J}_\alpha(-d + r)\hat{A}_1\hat{B}_1|0\rangle \approx 0.$$ 

A similar suppression occurs for $\hat{J}_\alpha(-d + r)\hat{A}_1\hat{B}_2|0\rangle$. However the terms $\hat{A}_2\hat{B}_1 + \hat{A}_2\hat{B}_2$ are not suppressed as the $f_\alpha$ is approximately 1 for the part of the state centred on $-d + r$. Therefore we are left with:

$$\hat{J}_\alpha(-d + r)|\Psi_{\sigma_0}\rangle \approx \frac{N}{4} \hat{A}_2(\hat{B}_1 + \hat{B}_2)|0\rangle$$

therefore:

$$|\Psi_{\sigma_1}\rangle \approx \frac{1}{\sqrt{2}} \hat{A}_2(\hat{B}_1 + \hat{B}_2)|0\rangle$$

Figure 6: Schematic spacetime diagram showing the evolution of a pair of space-like separated macroscopic objects separated by distance $d$. If there is a collapse at point $x$ the next collapse must occur on the hyperboloid (dotted blue line), and therefore the object on the right will stay in a superposition.

We turn now to the question of whether it is possible for this relativistic collapse model to be extended to distinguishable, but interacting particles.

6. Attempt at a Distinguishable Interacting Particle Extension

For this extension we will treat the particles as distinguishable hence the Hilbert space does not have to be a Fock space. The state for an $N$ particle model is:

$$|\Psi_\omega(z_1, z_2...z_N, t)\rangle \in \mathcal{H} = H_1 \otimes H_2... \otimes H_N$$

where $z_i$ labels the position of each particle. By definition, for an interacting theory it is not possible to express the Hamiltonian density as a factorised operator:

$$\mathcal{H} \neq \mathcal{H}_1 \otimes \mathcal{H}_N \otimes \mathcal{H}_2 \otimes \mathcal{H}_{N-2}... \otimes \mathcal{H}_N$$

where $\mathcal{H}_i$ are Hamiltonian densities which act on one particle states. Therefore it is not possible to write the associated unitary evolution operator as a separable tensor product on each sub-Hilbert space:

$$U_{\omega_1}^{\omega_2} \neq U_{1,\omega_1}^{\omega_2} \otimes ... \otimes U_{i,\omega_1}^{\omega_2}... \otimes U_{N,\omega_1}^{\omega_2},$$
where \( U_{\omega_1,\omega_2}^{\omega_2,\omega_1} \) acts only on the \( i^{th} \) particle. Instead the evolution between two surfaces \( \omega_1 \) and \( \omega_2 \) must be given by unitary operator which acts over all of \( H \). As a consequence of this, in order to specify the evolution between two surfaces on different hypersurfaces, all points of collapse between them must be known. For \( N \) distinguishable particles there are \( N \) series of collapses with points of collapse that may be space-like to each other. Hence the relationship \([9]\) does not hold and the model is not consistent with special relativity.

7. Conclusion

In this work we have considered spontaneous collapse models and their consistency with special relativity. We have shown that although Tumulka’s model for one particle is consistent with special relativity extensions to either interacting or indistinguishable particles are not possible. The question is then if any spontaneous collapse model can be relativistic, whether that model describes point like collapses such as in GRW or other models such as continuous spontaneous collapse. The two requirements that collapses must be time-like to each other to preserve frame independent probability distributions and that collapses must be space-like to ensure that macroscopic objects remain localised seem to imply a contradiction and therefore that such a model is not possible. Recent work by Adler \([25]\) supports this idea. If collapse models are not consistent with special relativity the effect of violations of the Lorentz symmetry should be investigated in order to be confronted by experiment.

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Appendices

A. Lorentz Invariant Hyperboloid

Define \( F(x) \) to be the future of the point \( x \in \mathbb{M}^4 \), i.e. the region contained within the future light cone and its boundary. Then the hyperboloid about a point \( y \), parametrised by \( r \), is:

\[
H_r(y) = \{ x \in \mathbb{F}(y) | (y^\mu - x^\mu)(y_\mu - x_\mu)^2 = r^2 \} \tag{43}
\]

Then the space-like distance on a hyperboloid to be the shortest path length between two points \( x \) and \( y \) can be defined:

\[
s-dist(x, y) := \inf \{ L[\gamma] | \gamma(a) = x, \gamma(b) = y \} \tag{44}
\]

where \( \inf \) denotes infimum, and \( L[\gamma] \) is the length of a smooth path \( \gamma \) on \( H_r(y) \). s-dist\((x, y)\) is Lorentz invariant. A Hilbert space \( H \) can be defined on this hypersurface as the space of all solutions to the the Dirac equation:

\[
i \hbar \gamma^\mu (\nabla_\mu - i e \frac{A}{\hbar} \gamma_\mu \psi) = m \psi \tag{45}
\]

where \( x \in \mathbb{M}^4 \), \( e \) is the charge, \( m \) is the mass, \( A_\mu \) is the electromagnetic vector potential. Working in the interaction picture, the inner product on \( H \) is given by:

\[
\langle \psi_\omega | \psi_\omega \rangle := \int d\omega \bar{\psi}(x)n_\mu(x)\gamma^\mu \psi(x) \tag{46}
\]

where the domain of the integral is over the surface \( \omega \), \( x \in \mathbb{M}^4 \), \( d\omega \) is the volume element induced on \( \omega \) by the metric of \( \mathbb{M}^4 \), \( n_\mu(x) \) is the future pointing normal vector to \( \omega \) and \( \bar{\psi} = \gamma^0 \psi^\dagger \).

B. Lorentz Covariance of Collapse Operator of Original Tumulka Model

We will directly demonstrate that the requirement eq. \([8]\) is met for this model. To see if eq. \([10]\) meets the condition of eq. \([8]\) consider a Lorentz transformed frame \( \mathfrak{F}' \) with coordinates \( y' = \{ y'_0, y'_1, y'_2, y'_3 \} \). In this frame the position of the previous collapse is \( y' \), which along with \( \Delta t \) specifies \( \Sigma = H_c\Delta t \). As it is a scalar \( \Delta t \) is a frame independent quantity.

The initial state at \( t' = 0 \) which is \( \psi_{\sigma_0} \) can be related to \( \psi_{\sigma_0} \) by \( \psi_{\sigma_0} = U_{\sigma_0}^{\sigma_0} \psi_{\sigma_0} \) as once the position of the collapse at \( y \) is known then are no collapses between the surface \( \sigma_0 \) and \( \sigma_0' \) as they only occur on \( \Sigma \), see figure \([1]\). As before we will work in the interaction picture. Then:

\[
P(x'|y', \Delta t, \psi_{\sigma_0}') = \int_{\Sigma} d\Sigma' n_\mu(z') L_{\Sigma'}(x', z') \bar{\psi}(z') \gamma^\mu L_{\Sigma'}(x', z') \psi(z') \tag{47}
\]

The domain of integration is \( \Sigma \) which is Lorentz invariant, and \( n_\mu(x)\bar{\psi}(x)\gamma^\mu \psi(x) \) is also an Lorentz
invariant quantity. In $\mathcal{F}$ with coordinates $x$ there is a metric on the 3 dimensional $\Sigma$ induced by the metric on $\mathbb{M}^4$. For $\mathcal{F}'$ is this also true. Then there exists Jacobian matrix $J$ between the coordinate system induced on $\Sigma$ in frame $\mathcal{F}$ and the coordinate system induced on $\Sigma$ in frame $\mathcal{F}'$ such that the volume elements can be related by:

$$d\Sigma' = \det(J)d\Sigma$$  \hspace{1cm} (48)$$

Then due to the normalisation condition the function $L'_\Sigma(x', z')$ can be written (up to a phase):

$$L'_\Sigma(x', z') = \frac{\exp(-\frac{\alpha}{2} s\text{-dist}^2(x', z'))}{\left(\int_{\Sigma} d\sigma' \exp(-\frac{\alpha}{2} s\text{-dist}^2(x', z'))\right)^{\frac{1}{2}}} \frac{\exp(-\frac{\alpha}{2} s\text{-dist}^2(x, z))}{\left(\int_{\Sigma} d\sigma' \exp(-\frac{\alpha}{2} s\text{-dist}^2(x, z))\right)^{\frac{1}{2}}}$$  \hspace{1cm} (49)$$

as $s\text{-dist}(x, z) = s\text{-dist}(x', z')$. Therefore:

$$P(x'|y', \Delta t, \psi_{\sigma_0}) = \int_{\Sigma} \det(J)d\Sigma L'_\Sigma(x', z') n_\mu(z)\bar{\psi}(z)\gamma^\mu\psi(z)$$

$$= \int_{\Sigma} \det(J)d\Sigma \times$$

$$\exp(-\frac{\alpha}{2} s\text{-dist}^2(x, z)) n_\mu(z)\bar{\psi}(z)\gamma^\mu\psi(z)$$

$$\left(\int_{\Sigma} \det(J)d\Sigma \exp(-\frac{\alpha}{2} s\text{-dist}^2(x, z))\right)^{\frac{1}{2}}$$

$$= P(x|y, \Delta t, \psi_{\sigma_0})$$

With the last line being reached because the Jacobians cancel.

C. Tumulka’s Model for Indistinguishable Particles with Space-like Collapses

To see that Tumulka’s model for space-like collapses with indistinguishable particles is not viable consider the joint probability distribution for two collapse points $x_1, x_2 \in \mathbb{M}^4$ which are space-like separate from one another. This distribution must also be Lorentz covariant, as it is a probability distribution of the value of observables. The probability distribution for points of future collapse can no longer be conditioned on a single point of collapse. In the spirit of the distinguishable particle model, assume there are multiple series of collapse points, such that the subsequent point of collapse in each series is defined to be on a hyperboloid defined by the previous point. To see that under such an assumption Lorentz covariance is lost, it is sufficient to consider only two series.

![Figure 7: Spacetime diagram for Tumulka’s collapse model with space-like collapses. The blue dotted lines show the hypersurface of the initial state $\sigma_0$ and two hyperboloids on which the next collapses must occur $\Sigma_1$ and $\Sigma_2$. The initial condition in this frame is given by latest points of collapse for each particle, $y_1$ and $y_2$, and the state on $\sigma_0$. $x_1$ is a possible position for the next collapse. The probability distribution for a collapse at $x$ is not known as the map between $\sigma_0$ and $\Sigma_1$ cannot be specified as there may be a collapse on $\Sigma_2$. Suppose there are two initial points of collapse $y_1, y_2$, and consider a frame $\mathcal{F}$ were they are simultaneous so $y_1^0 = y_2^0$. Define a initial state on the constant time hypersurface that intersects the points $y_1$ and $y_2$, $\Psi_{\sigma_0}$. For each series of collapses there will be a set of realisations of a Poisson process to give realisations of the time intervals which define the position of the hyperboloids $\Delta t_1, \Delta t_2...$ From this initial condition define two hyperboloids $\Sigma_1 = H_{c\Delta t_1}(y_1)$ and $\Sigma_2 = H_{c\Delta t_2}(y_2)$ Then the joint probability distribution is given by:

$$P(x_1, x_2|y_1, y_2, \Delta t_1, \Delta t_2, \Psi_{\sigma_0}) = P(x_2|y_1, y_2, \Delta t_1, \Delta t_2, \Psi_{\sigma_0})\times$$

$$P(x_1|y_1, y_2, \Delta t_1, \Delta t_2, \Psi_{\sigma_0})$$

In the frame $\mathcal{F}$ the second term on the LHS can be written $P(x_1, |y_1, \Delta t_1, \Psi_{\sigma_0})$ as $y_2$ and $\Delta t_2$ define $\Sigma_2$ which has no effect on the position of $x_1$ that is not already accounted for by conditioning on $\Psi_{\sigma_0}$. If it is assumed that:

$$\Psi_{\Sigma_1} = U_{\sigma_0}^\Sigma \Psi_{\sigma_0}$$  \hspace{1cm} (52)$$
Then eq. \[51\] will be proportional too:

\[
P(x_2, |x_1, y_1, y_2, \Delta t_1, \Delta t_2, \Psi_{\sigma_0}) \propto \left\| \hat{L}(x_2) U_{\Sigma_2} \hat{L}(x_1) U_{\Sigma_1} \Psi_{\sigma_0} \right\|^2. \tag{53}
\]

For the assumption of eq. \[52\] to be true there must be no collapses in the region between the hypersurfaces. As the hypersurface \(\Sigma_2\) intersects with this region, (see figure 7) on which there may be a collapse, then this assumption is not justified. There is a similar problem with the operator \(U_{\Sigma_1}\). Therefore this model cannot specify the probability distributions for the next set of collapses given the previous set and hence is not a viable model.

8. References

[1] Roderich Tumulka. A relativistic version of the Ghirardi–Rimini–Weber model. *Journal of Statistical Physics*, 125(4):821–840, 2006.

[2] Gian Carlo Ghirardi, Alberto Rimini, and Tullio Weber. Unified dynamics for microscopic and macroscopic systems. *Physical Review D*, 34(2):470, 1986.

[3] Angelo Bassi and GianCarlo Ghirardi. Dynamical reduction models. *Physics Reports*, 379(5-6):257–426, 2003.

[4] Albert Einstein, Boris Podolsky, and Nathan Rosen. Can quantum-mechanical description of physical reality be considered complete? *Physical review*, 47(10):777, 1935.

[5] Joseph F Fitzsimons, Jonathan A Jones, and Vlatko Vedral. Quantum correlations which imply causation. *Scientific reports*, 5:18281, 2015.

[6] Adrien Feix and Časlav Brukner. Quantum superpositions of common-cause and direct-cause causal structures. *arXiv preprint arXiv:1606.09241*, 2016.

[7] Jean-Philippe W MacLean, Katja Ried, Robert W Spekkens, and Kevin J Resch. Quantum-coherent mixtures of causal relations. *Nature communications*, 8:15149, 2017.

[8] Lucien Hardy. Towards quantum gravity: a framework for probabilistic theories with non-fixed causal structure. *Journal of Physics A: Mathematical and Theoretical*, 40(12):3081, 2007.

[9] Časlav Brukner. Quantum causality. *Nature Physics*, 10(4):259, 2014.

[10] Gian Carlo Ghirardi, Philip Pearle, and Alberto Rimini. Markov processes in hilbert space and continuous spontaneous localization of systems of identical particles. *Phys. Rev. A*, 42:78–89, Jul 1990.

[11] Roderich Tumulka. On spontaneous wave function collapse and quantum field theory. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 462, pages 1897–1908. The Royal Society, 2006.

[12] Daniel Bedingham, Detlef Dürr, GianCarlo Ghirardi, Sheldon Goldstein, Roderich Tumulka, and Nino Zanghí. Matter density and relativistic models of wave function collapse. *Journal of Statistical Physics*, 154(1-2):623–631, 2014.

[13] Daniel J Bedingham. Relativistic state reduction dynamics. *Foundations of Physics*, 41(4):686–704, 2011.

[14] Antoine Tilloy. Interacting quantum field theories as relativistic statistical field theories of local beables. *arXiv preprint arXiv:1702.06325*, 2017.

[15] Oreste Nicrosini and Alberto Rimini. Relativistic spontaneous localization: a proposal. *Foundations of Physics*, 33(7):1061–1084, 2003.

[16] Philip Pearle. Relativistic collapse model with tachyonic features. *Physical Review A*, 59(1):80, 1999.

[17] Jonathan Oppenheim and Benni Reznik. Fundamental destruction of information and conservation laws. *arXiv preprint arXiv:0902.2361*, 2009.

[18] Fay Dowker and Joe Henson. Spontaneous collapse models on a lattice. *Journal of Statistical Physics*, 115(5-6):1327–1339, 2004.

[19] Heinz-Peter Breuer and Francesco Petruccione. Relativistic formulation of quantum-state diffusion. *Journal of Physics A: Mathematical and General*, 31(1):33, 1998.

[20] Yakir Aharonov and David Z Albert. States and observables in relativistic quantum field theories. *Physical Review D*, 21(12):3316, 1980.
[21] Yakir Aharonov and David Z Albert. Can we make sense out of the measurement process in relativistic quantum mechanics? *Physical Review D*, 24(2):359, 1981.

[22] Sin-itiro Tomonaga. On a relativistically invariant formulation of the quantum theory of wave fields. *Progress of Theoretical Physics*, 1(2):27–42, 1946.

[23] Julian Schwinger. Quantum electrodynamics. A covariant formulation. *Physical Review*, 74(10):1439, 1948.

[24] Z Koba. On the integrability condition of tomonaga-schwinger equation. *Progress of Theoretical Physics*, 5(1):139–141, 1950.

[25] Stephen L Adler. Connecting the dots: Mott for emulsions, collapse models, colored noise, frame dependence of measurements, evasion of the “free will theorem”. *Found. Phys.*, 48:1557–1567, 2018.