MULTI-POINT GREEN’S FUNCTIONS FOR SLE AND AN
ESTIMATE OF BEFFARA

BY GREGORY F. LAWLER* AND BRENT M. WERNESS

University of Chicago

In this paper we define and prove of the existence of the multi-
point Green’s function for SLE—a normalized limit of the probability
that an $SLE_\kappa$ curve passes near to a pair of marked points in the
interior of a domain. When $\kappa < 8$ this probability is non-trivial, and
an expression can be written in terms two-sided radial $SLE$. One of
the main components to our proof is a refinement of a bound first
provided by Beffara in [3]. This work contains a proof of this bound
independent from the original.

1. Introduction. The Schramm-Loewner evolution ($SLE$) is a random
process first introduced by Oded Schramm in [12] as a candidate for scaling
limits of models from statistical physics which are believed to be conformally
invariant. Since its introduction, $SLE$ has been rigorously established as
the scaling limit for a number of these processes, including the loop-erased
random walk [9], the percolation exploration process [14], and the interface
of the Gaussian free field [13]. For a general introduction to $SLE$ see, for
example, [5, 6, 15].

Chordal $SLE_\kappa$ for $\kappa > 0$ in the upper half-plane ($\mathbb{H}$) is a one-parameter
family of non-crossing random curves $\gamma : [0, \infty) \to \mathbb{H}$ with $\gamma(0) = 0$ and
$\gamma(\infty^-) = \infty$. Depending on $\kappa$, the geometry of the curve has several different
phases. When $0 < \kappa \leq 4$, the curves are simple (no self intersections). When
$\kappa > 4$, the curves are no longer simple, but they remain non-crossing. When
$\kappa \geq 8$, the curve is spacefilling, passing through every point in $\mathbb{H}$.

When examining geometric questions about the $SLE$ curves, such as the
almost sure Hausdorff dimension in [3], it is often useful to be able to provide
estimates on the probability that the process $\gamma(t)$ passes near a series of
marked points in $\mathbb{H}$. However, the non-Markovian nature of this process
makes estimating such probabilities difficult.

When trying to understand the probability that $SLE_\kappa$ gets near to some
point $z \in \mathbb{H}$ it is convenient to consider the conformal radius of $z$ in $H_t := \mathbb{H} \setminus \gamma(0, t)$, which we denote by $\Upsilon_t(z)$, instead of the Euclidean distance from
$z$ to $\gamma(0, t)$ (see Section 2.1 for the definition). This change does little to the

*Research supported by NSF grant DMS-0907143.

AMS 2000 subject classifications: Primary 60J67; secondary 82B27
geometry of the problem being considered since the conformal radius differs from the Euclidean distance by at most a universal multiplicative constant.

The Green’s function for $SLE_κ$ from 0 to $∞$ in $\mathbb{H}$ for $κ < 8$ is a form of the normalized probability of passing near to a point in $\mathbb{H}$. It is defined by

$$\lim_{ε \to 0} ε^{d-2} P\{Υ_∞(z) < ε\} = c_∗G_H(z; 0, ∞)$$

where $d := 1 + κ/8$ is the Hausdorff dimension of the $SLE_κ$ and $c_∗$ is some known constant depending on $κ$. The Green’s function was first computed in [11] (although they neither used this name nor definition), and the exact formula found there is given in Section 2.1. The existence of the limit requires some argument, and a form of it is proven in Lemma 2.10.

We wish to show that analogously that

$$\lim_{ε, δ \to 0} ε^{d-2} δ^{d-2} P\{Υ_∞(z) < ε; Υ_∞(w) < δ\}$$

exists and can be written as

$$c_2^2 G_H(z; 0, ∞)E_z^s[G_{H_{T_z}}(w; z, ∞)] + c_2^2 G_H(w; 0, ∞)E_w^s[G_{H_{T_w}}(z; w, ∞)]$$

where $E_z^s$ is the expectation of a particular form of $SLE$ called two-sided radial $SLE$, which can be understood as chordal $SLE$ conditioned to pass though the point $z$, and $G_{H_{T_z}}$ is the Green’s function for $SLE$ in the domain remaining at the time it does so. The form of the limit as the sum of two similar terms comes from the two possible orders that the curve can pass through $z$ and $w$, and each term individually can be thought of as an ordered Green’s function.

To prove this result, we will use techniques similar to those used in [3], where Beffara (in slightly different notation) established the estimate that there exists some $c > 0$ such that for any two points $z, w \in \mathbb{H}$ with $\text{Im}(z), \text{Im}(w) \geq 1$

$$P\{Υ_∞(z) < ε; Υ_∞(w) < ε\} < c ε^{2(2-d)} |z - w|^{d-2}.$$
We wish to show that curves that get near $z$ then near $w$ concentrates on curves like those in the left image. Estimating the probability of such curves is easy by repeated application of the Green’s function. However, such simple estimation gives the same order of magnitude to curves like those in the center image. This issue can be overcome as long as getting near to $w$ before $z$ decreases the probability that the SLE gets even closer to $w$ later on. This is often the case, however the right image shows an example where it is not. In this case, once the curve gets near to $z$, it is essentially guaranteed to pass near $w$. Controlling for these issues forms the bulk of this work.

In the process of proving the existence of the multi-point Green’s function for SLE, we also obtain an independent proof of a mild generalization of Beffara’s estimate—that there exists a $c > 0$ such that for any $z, w \in \mathbb{H}$ with $\text{Im}(z), \text{Im}(w) \geq 1$

$$\mathbb{P}\{\Upsilon_{\infty}(z) < \varepsilon; \, \Upsilon_{\infty}(w) < \delta\} < c \varepsilon^{2-d}\delta^{2-d}|z-w|^{d-2}.$$  

While it may be possible to derive some of the lemmas we require directly from the proof in [3], we include a complete proof of them, along with Beffara’s original estimate, so that the proof of our main result is completely self contained.

It is worth noting that Beffara’s estimate itself immediately yields an upper bound on the multi-point Green’s function. For a lower bound, and an application of the multi-point Green’s function to the proof of the existence of the “natural parametrization” of SLE, a parametrization of SLE by what can be thought of as a form $d$-dimensional arc length, see [10].

The paper is structured as follows. Section 2.1 begins by establishing the notation used throughout the paper, and to provide a few simple deterministic and random bounds required in the proofs that follow. Section 2.4 then gives a brief introduction to two-sided radial SLE and collects the facts about this process that we require to show the existence of the multi-point Green’s function. Section 3 provides a proof of the existence of the multi-point Green’s function assuming an estimate derived from our proof.
of Beffara’s estimate. The rest of the paper is dedicated to our independent proof of Beffara’s estimate. To aid in the presentation of this proof, we have separated the bounds required by the type of argument required: topological lemmas, probabilistic lemmas, and combinatorial lemmas. The proof of one of the topological lemmas is left to the appendix as the result is intuitive and the formal proof of it does little to aid the understanding of our main results.

In this paper we fix $\kappa < 8$ and constants implicitly depend on $\kappa$.

2. Preliminaries.

2.1. Notation. We set

$$a = \frac{2}{\kappa}, \quad d = 1 + \frac{\kappa}{8} = 1 + \frac{1}{4a},$$

$$\beta = \frac{8}{\kappa} - 1 = 4a - 1 > 0.$$

The Green’s function for chordal $SLE_\kappa$ (from 0 to $\infty$ in $\mathbb{H}$) is

$$G(x + iy) = G(re^{i\theta}) = r^{d-2} \sin^{4a} \frac{\theta}{4a} = y^{d-2} \sin^{\beta} \theta.$$  

The Green’s function can be defined for other simply connected domains as we now demonstrate. If $D$ is a simply connected domain, $z_1, z_2$ are distinct boundary points, let $\Phi_D: D \to \mathbb{H}$ be a conformal transformation with $\Phi_D(z_1) = 0, \Phi_D(z_2) = \infty$. This is unique up to a final dilation. If $w \in D$, we define

$$S_D(w; z_1, z_2) = \sin \arg \Phi_D(w),$$

which is independent of the choice of $\Phi_D$ and gives a conformal invariant. We let $\Upsilon_D(w)$ be (twice the) conformal radius of $w$ in $D$, that is, if $f: D \to D$ is a conformal transformation with $f(0) = w$, then $\Upsilon_D(w) = 2 |f'(0)|$. This satisfies the scaling rule

$$\Upsilon_{f(D)}(f(w)) = |f'(w)| \Upsilon_D(w).$$

It is easy to check that $\Upsilon_{\mathbb{H}}(x + iy) = y$, and, more generally,

$$\Upsilon_D(w) = \frac{\text{Im}(\Phi_D(w))}{|\Phi'_D(w)|}.$$

The Green’s function for $SLE_\kappa$ from $z_1$ to $z_2$ in $D$ is defined by

$$G_D(w; z_1, z_2) = \Upsilon_D(w)^{d-2} S(w; z_1, z_2)^\beta.$$
It satisfies the scaling rule

\[ G_D(w; z_1, z_2) = |f'(w)|^{2-d} G_{f(D)}(f(w); f(z_1), f(z_2)). \]

For a proof that the Green’s function so defined satisfies the limit claimed in the introduction, see Lemma 2.10.

Let \( \text{inrad}_D(w) = \text{dist}(w, \partial D) \) denote the inradius. Using the Koebe 1/4-theorem, we know that

\[ \frac{1}{2} \text{inrad}_D(w) \leq \Upsilon_D(w) \leq 2 \text{inrad}_D(w). \]

Therefore,

\[ G_D(w; z_1, z_2) \asymp \text{inrad}_D(w)^{d-2} S_D(w; z_1, z_2)^{\beta}, \]

where we write \( f_1 \asymp f_2 \) if there exists some constant \( c \) such that \( f_1 \leq c f_2 \) and \( f_2 \leq c f_1 \). We write

\[ \partial D = \partial_1 D \cup \partial_2 D \cup \{z_1, z_2\} \]

where \( \partial_1 D, \partial_2 D \) denote the two open arcs of \( \partial D \) with endpoints \( z_1, z_2 \). Let \( \hat{S}_D(w; z_1, z_2) \) be the minimum of the harmonic measures of \( \partial_1 D, \partial_2 D \) from \( w \). This is a conformal invariant, and a simple computation in \( \mathbb{H} \) shows that

\[ \hat{S}_D(w; z_1, z_2) = \frac{1}{\pi} \min \{\arg \Phi_D(w), \pi - \arg \Phi_D(w)\}, \]

and hence

\[ \hat{S}_D(w; z_1, z_2) \asymp S_D(w; z_1, z_2), \]

and

\[ G_D(w; z_1, z_2) \asymp \text{inrad}_D(w)^{d-2} \hat{S}_D(w; z_1, z_2)^{\beta}. \]

To bound the harmonic measure, it is often useful to use the Beurling estimate. We recall it here; for a proof see, for example, [2, Chapter V]. Let \( B_t \) be a standard Brownian motion and \( \tau_D \) denote the first exit time of some domain \( D \) for this Brownian motion.

**Proposition 2.1 (Beurling Estimate).** There is a constant \( c > 0 \) such that if \( z \in \mathbb{D} \) and \( K \) is a connected compact subset of \( \overline{\mathbb{D}} \) with \( 0 \in K \) and \( K \cap \partial \mathbb{D} \neq \emptyset \), then

\[ \mathbb{P}^z \{ B[0, \tau_D] \cap K = \emptyset \} \leq c |z|^{1/2}. \]

We may derive from this the following consequence.
Proposition 2.2. There is a constant $c > 0$ such that if $K$ is a connected compact subset of $\mathbb{H}$ with $K \cap \mathbb{R} \neq \emptyset$ and $z_0 \in \mathbb{H}$, $\epsilon > 0$ are such that $B_\epsilon(z_0) \cap K \neq \emptyset$ then for $w \in \mathbb{H}$,

$$\mathbb{P}^w\{B[0, \tau_{\mathbb{H} \setminus K}] \cap B_\epsilon(z_0) \neq \emptyset\} \leq c \left[\frac{\epsilon}{|z_0 - w|}\right]^{1/2}.$$ 

Proof. Consider the map

$$g(z) := \frac{\epsilon}{z - z_0}, \quad g : \mathbb{C} \setminus B_\epsilon(z_0) \to \mathbb{D}.$$ 

Let $K' = g(\mathbb{C} \setminus \mathbb{H} \cup [K \setminus B_\epsilon(z_0)])$, and note that $K'$ is a connected compact subset of $\mathbb{D}$ with $0 \in K'$ and $K' \cap \partial \mathbb{D} \neq \emptyset$. Thus by Proposition 2.1 we know

$$\mathbb{P}^{g(w)}\{B[0, \tau_{\mathbb{D}}] \cap K' = \emptyset\} \leq c |g(w)|^{1/2},$$

which by the conformal invariance of Brownian motion, and the definition of $g$ is the desired statement.

If $j = 1, 2$, let $\Delta_{D,j}(w; z_1, z_2)$ be the infimum of all $s$ such that there exists a curve $\eta : [0, 1) \to D$ contained in the disk of radius $s$ about $w$ with $\eta(0) = w$, $\eta(1^-) \in \partial_j D$. Note that

$$\text{inrad}_D(w) = \min \{\Delta_{D,1}(w; z_1, z_2), \Delta_{D,2}(w; z_1, z_2)\}.$$

We let

$$\Delta^*_D(w; z_1, z_2) = \max \{\Delta_{D,1}(w; z_1, z_2), \Delta_{D,2}(w; z_1, z_2)\}.$$ 

The Beurling estimate implies that there is a $c < \infty$ such that the probability a Brownian motion starting at $w$ reaches distance $\Delta^*_D(w; z_1, z_2)$ before leaving $D$ is bounded above by

$$c \left[\frac{\text{inrad}_D(w)}{\Delta^*_D(w; z_1, z_2)}\right]^{1/2}.$$ 

Therefore,

$$(2) \quad S_D(w; z_1, z_2) \asymp \hat{S}_D(w; z_1, z_2) \leq c \left[\frac{\text{inrad}_D(w)}{\Delta^*_D(w; z_1, z_2)}\right]^{1/2},$$

which gives us the upper bound

$$G_D(w; z_1, z_2) \leq c \text{inrad}_D(w)^{d-2+\frac{\beta}{2}} \Delta^*_D(w; z_1, z_2)^{-\frac{\beta}{2}}.$$ 

We will also need a fact which is a form of continuity of the Green’s function under a small perturbation of the domain. First consider the following two lemmas on the conformal radius.
Lemma 2.3. Let $B_r$ denote the closed disk of radius $e^{-r}$ about the origin. Suppose $D$ is a simply connected subdomain of $\mathbb{D}$ containing the origin and $e^{-r} < \text{inrad}_D(0)$. Suppose $B_t$ is a Brownian motion starting at the origin and let

$$\tau_D = \inf\{t : B_t \notin D\}, \quad \tau_{\mathbb{D}} = \inf\{t : B_t \notin \mathbb{D}\}, \quad \sigma_{r,D} = \inf\{t \geq \tau_D : B_t \in B_r\}.$$

Then,

$$P\{\tau_D < \sigma_{r,D} < \tau_{\mathbb{D}}\} = -\frac{1}{r} \log[\Upsilon_D(0)/2].$$

Proof. Let $f : D \to \mathbb{D}$ be the conformal transformation with $f'(0) > 0$. Since $\Upsilon_D(0)$ is twice the usual conformal radius, $-\log[\Upsilon_D(0)/2] = \log f'(0)$. Let $g(z) = \log||f(z)||/|z||$ which is a bounded harmonic function on $D$, and hence

$$\log f'(0) = g(0) = E[g(B_{\tau})] = -E[\log |B_\tau|].$$

For $e^{-r} \leq |w| < 1$, $-\log |w|/r$ is the probability that a Brownian motion starting at $w$ hits $B$ about the origin before leaving the $\mathbb{D}$. Therefore,

$$\log f'(0) = r P\{\tau_D < \sigma_{r,D} < \tau_{\mathbb{D}}\}.$$

Lemma 2.4. There exists a $c > 0$ such that for any two simply connected domains $D_1 \subseteq D_2$ and a point $w \in D_1 \cap D_2$ then

$$0 \leq \Upsilon_{D_2}(w) - \Upsilon_{D_1}(w) \leq c \text{diam}(D_2 \setminus D_1)$$

Proof. Without loss of generality, we may assume $\text{inrad}(D_2) = 1$. If $\text{inrad}(D_1) \leq 7/8$, then $\text{diam}(D_2 \setminus D_1) \geq 1/8$, and we can use the estimate $\text{inrad}(D) \times \Upsilon(D)$. If $\text{inrad}(D_1) \geq 7/8$, then we can use the previous lemma, conformal invariance, and the Koebe-(1/4) theorem to see $\Upsilon_{D_2}(w) - \Upsilon_{D_1}(w)$ is comparable to the probability that a Brownian motion starting at $w$ hits $D_2 \setminus D_1$ and returns to $\mathcal{B} = B_{1/16}(w)$, the disk of radius $1/16$ about $w$ without leaving $D_2$. Using the Beurling estimate, we see the probability of hitting $D_2 \setminus D_1$ is bounded above by $c \text{diam}(D_2 \setminus D_1)^{1/2}$ and using it again the probability of getting back to $\mathcal{B}$ before leaving $D$ is bounded by $c \text{diam}(D_2 \setminus D_1)^{1/2}$. 

We will need some notion of closeness of two nested domains before we can state our lemma. Although the following definitions are very general, we will use it only in the case where the domains are the complements of a single curve considered up to two different times.
**Definition.** Given two nested simply connected domains \( D_1 \subseteq D_2 \subseteq \mathbb{H} \) with marked boundary points \( z_1 \in \partial D_1 \) and \( z_2 \in \partial D_2 \), we say \((D_1, z_1)\) and \((D_2, z_2)\) are \( R\)-close near \( z \) if the following holds. Let \( B_R(z) \) denote the connected component of \( B(z) \cap D_i \) which contains \( z \). Then,

- \( z_1 \in \partial B^{(1)}_R(z) \),
- \( z_2 \in \partial B^{(2)}_R(z) \), and
- \( D_2 \setminus D_1 \subseteq B_R(z) \).

**Lemma 2.5.** There exists \( c > 0 \) such that the following holds. Suppose \( z, w \in \mathbb{H} \), \( D_1 \subseteq D_2 \subseteq \mathbb{H} \) are simply connected domains, and \( z_1 \in \partial D_1 \), \( z_2 \in \partial D_2 \). If

- \( z, w \in D_1 \cap D_2 \),
- \((D_1, z_1)\) and \((D_2, z_2)\) are \( R\)-close near \( z \) for \( R \leq \text{inrad}_{D_1}(w) \wedge \frac{1}{2} |z - w| \),
- \( \infty \in \partial D_1 \cap \partial D_2 \),

then

\[
|G_{D_1}(w; z_1, \infty) - G_{D_2}(w; z_2, \infty)| \leq c \text{ inrad}_{D_1}(w)^{d-2-\frac{\beta+1}{2}} R^{-\frac{\beta+1}{2}}.
\]

One need not fix the point \( z \) in the beginning of this lemma by simply making the second bullet point of this lemma say that there exists some \( z \) so that the domains are \( R\)-close near \( z \), however we write it in this form since we will always use this lemma with a fixed \( z \) and \( w \) already in mind.

**Proof.** Recall that

\[
G_D(w; z_1, z_2) = \Upsilon_D(w)^{d-2} S_D(w; z_1, z_2) \beta
\]

where \( S(w; z_1, z_2) \) is the sine of the argument of \( w \) after applying the unique (up to scaling) conformal map, \( \Phi_D \), that sends \( D \) to \( \mathbb{H} \) while sending \( z_1 \) to \( 0 \) and \( z_2 \) to \( \infty \). Writing, as before,

\[
\partial D = \partial_1 D \cup \{z_1\} \cup \partial_2 D \cup \{z_2\}
\]

where the union is written in counter-clockwise order, this argument is a conformally invariant and can be computed by

\[
\arg \Phi_D(w) = \pi \cdot \mathbb{P}^w \{ B_\tau \in \partial_2 D \} \quad \text{where} \quad \tau = \inf \{ t : B_t \in \partial D \}
\]

where \( \mathbb{P}^w \) is the probability for a standard Brownian motion started at \( w \).

Consider our case. Write

\[
\partial D_1 = \partial_1 D_1 \cup \{z_1\} \cup \partial_2 D_1 \cup \{\infty\} \quad \text{and} \quad \partial D_2 = \partial_1 D_2 \cup \{z_2\} \cup \partial_2 D_2 \cup \{\infty\}
\]
again with the union written in counter-clockwise order. Note that the condition that \((D_1, z_1)\) and \((D_2, z_2)\) are \(R\)-close near \(z\) implies that

\[
\partial_1 D_1 \setminus B_R(z) = \partial_1 D_2 \setminus B_R(z) \quad \text{and} \quad \partial_2 D_1 \setminus B_R(z) = \partial_2 D_2 \setminus B_R(z).
\]

Define

\[
\tau_1 = \inf\{ t : B_t \in \partial D_1 \} \quad \text{and} \quad \tau_2 = \inf\{ t : B_t \in \partial D_2 \}
\]

and note that, since \(B_0 = w\), \(\tau_1 \leq \tau_2\).

We may write that

\[
|\arg \Phi_{D_1}(w) - \arg \Phi_{D_2}(w)| = |\pi \cdot \mathbb{P}^w \{B_{\tau_1} \in \partial D_1\} - \pi \cdot \mathbb{P}^w \{B_{\tau_2} \in \partial D_2\}| \\
\leq 2\pi \cdot \mathbb{P}^w \{B_t \in B_R(z) \text{ for some } t \leq \tau_2\}.
\]

where the last line follows since, if considered path-wise, the Brownian motion must enter \(B_R(z)\) if it is to hit a different side of the boundary in \(D_1\) versus \(D_2\) by equation (3). By the Beurling estimate (Proposition 2.2),

\[
|\arg \Phi_{D_1}(w) - \arg \Phi_{D_2}(w)| \leq c \left( \frac{R}{|z-w|} \right)^{1/2}.
\]

By noting that \(\text{inrad}_{D_1}(w) \leq c |z-w|\) by the choice of \(R\) and the definition of \(R\)-close, and splitting into the cases when \(\beta \geq 1\) versus \(\beta < 1\) we see

\[
|S_{D_1}(w; z_1, \infty)\beta - S_{D_2}(w; z_2, \infty)\beta| \leq c \left( \frac{R}{\text{inrad}_{D_1}(w)} \right)^{(\beta \land 1)/2}.
\]

Consider the term involving the conformal radius. By using Lemma 2.4 and recalling that \(d - 2 < 0\) and \(\Upsilon_{D_1}(w) \leq \Upsilon_{D_2}(w)\), we see

\[
|\Upsilon_{D_2}(w)^{d-2} - \Upsilon_{D_1}(w)^{d-2}| \leq (2 - d)\Upsilon_{D_1}(w)^{d-3}|\Upsilon_{D_2}(w) - \Upsilon_{D_1}(w)| \\
\leq c \Upsilon_{D_1}(w)^{d-2} \left( \frac{R}{\text{inrad}_{D_1}(w)} \right).
\]

Combining these, noting that \(R < \text{inrad}_{D_1}(w)\), gives

\[
|G_{D_1}(w; z_1, \infty) - G_{D_2}(w; z_2, \infty)| \\
\leq |\Upsilon_{D_1}(w)^{d-2} S_{D_1}(w; z_1, \infty)\beta - \Upsilon_{D_1}(w)^{d-2} S_{D_2}(w; z_2, \infty)\beta| \\
+ |\Upsilon_{D_1}(w)^{d-2} S_{D_2}(w; z_2, \infty)\beta - \Upsilon_{D_2}(w)^{d-2} S_{D_2}(w; z_2, \infty)\beta| \\
\leq c \Upsilon_{D_1}(w)^{d-2} \left( \frac{R}{\text{inrad}_{D_1}(w)} \right)^{(\beta \land 1)/2} + c \Upsilon_{D_1}(w)^{d-2} \left( \frac{R}{\text{inrad}_{D_1}(w)} \right) \\
\leq c \text{inrad}_{D_1}(w)^{d-2 - \frac{\beta \land 1}{2}} R^{\frac{\beta \land 1}{2}}.
\]

as desired. \(\square\)
2.2. Schramm-Loewner evolution. The chordal Schramm-Loewner evolution with parameter \( \kappa \) (from 0 to \( \infty \) in \( \mathbb{H} \) parametrized so that the half-plane capacity grows at rate \( a = 2/\kappa \)) is the random curve \( \gamma : [0, \infty) \to \mathbb{H} \) with \( \gamma(0) = 0 \) satisfying the following. Let \( H_t \) denote the unbounded component of \( \mathbb{H} \setminus \gamma(0, t] \), and let \( g_t \) be the unique conformal transformation of \( H_t \) onto \( \mathbb{H} \) with \( g_t(z) - z \to 0 \) as \( z \to \infty \). Then \( g_t \) satisfies the Loewner differential equation

\[
\frac{\partial_t g_t(z)}{g_t(z) - U_t}, \quad g_0(z) = z,
\]

where \( U_t = -B_t \) is a standard Brownian motion. For \( z \in \mathbb{H} \setminus \{0\} \), the solution of this initial value problem exists up to time \( T_z \in (0, \infty) \).

Suppose \( z \in \mathbb{H} \) and let

\[
Z_t = Z_t(z) = X_t + iY_t = g_t(z) - U_t.
\]

Then the Loewner differential equation becomes the SDE

\[
dZ_t = \frac{a}{Z_t} dt + dB_t.
\]

Let

\[
S_t = S_t(z) = S_{H_t}(z; \gamma(t), \infty) = \sin \arg Z_t,
\]

\[
\Upsilon_t = \Upsilon_t(z) = \Upsilon_{H_t}(z; \gamma(t), \infty) = \frac{Y_t}{|g_t'(z)|},
\]

\[
M_t = M_t(z) = G_{H_t}(z; \gamma(t), \infty) = \Upsilon_t^{d-2} S_t^\beta.
\]

Either by direct computation or by using the Schwarz lemma, we can see that \( \Upsilon_t \) decreases in \( t \) and hence we can define \( \Upsilon = \Upsilon_{T_z} \). If \( 0 < \kappa \leq 4 \), the \( SLE \) paths are simple and with probability one \( T_z = \infty \). If \( 4 < \kappa < 8 \), \( T_z < \infty \) and by (1) we know

\[
\Upsilon \asymp \text{dist} [z, \gamma(0, T_z) \cup \mathbb{R}] = \text{dist} [z, \gamma(0, \infty) \cup \mathbb{R}].
\]

Using Itô’s formula, we can see that \( M_t \) is a local martingale satisfying

\[
dM_t = \frac{a X_t}{X_t^2 + Y_t^2} M_t dB_t.
\]

We will need the following estimate for \( SLE \); see [1] for a proof. By a crosscut in \( D \) we will mean a simple curve \( \eta : (0, 1) \to D \) with \( \eta(0^+), \eta(1^-) \in \partial D \). We call \( \eta(0^+), \eta(1^-) \) the endpoints of the crosscut.

**Proposition 2.6.** There exists \( c < \infty \) such that if \( \eta \) is a crosscut in \( \mathbb{H} \) with \(-\infty < \eta(1^-) \leq \eta(0^+) = -1\), then the probability that an \( SLE_\kappa \) curve from 0 to \( \infty \) intersects \( \eta \) is bounded above by \( c \text{ diam}(\eta)^\beta \) where \( \beta = 4a - 1 \) is as defined in Section 2.1.
2.3. **Radial parametrization.** In order to prove the existence of multi-point Green’s functions, we will need to study the behavior of the SLE curve from the perspective of \( z \in \mathbb{H} \). To do so, it is useful to parametrize the curve so that the conformal radius seen from \( z \) decays deterministically. We fix \( z \in \mathbb{H} \) and let

\[
\sigma(t) = \inf \{ s : \Upsilon_s = e^{-2at} \}.
\]

Under this parametrization, the “starting time” is \(-\log(\Upsilon_0)/2a\) and the total lifetime of the curve is \(\log(\Upsilon_0/\Upsilon)/2a\). Let \(\Theta_t = \arg Z_{\sigma(t)}(z), \hat{S}_t = S_{\sigma(t)}(z) = \sin \Theta_t\). Using Itô’s formula one can see that \(\Theta_t\) satisfies

\[
d\Theta_t = (1 - 2a) \cot \Theta_t \, dt + d\hat{W}_t,
\]

where \(\hat{W}_t\) is a standard Brownian motion. Since \(a > 1/4\), comparison to a Bessel process shows that solutions to this process leave \((0, \pi)\) in finite time. This reflects that fact that chordal SLE\(_{\kappa}\) does not reach \(z\) for \(\kappa < 8\) and hence \(\Upsilon > 0\). Let

\[
\hat{M}_t = M_{\sigma(t)}(z) = e^{-2at(d-2)} \hat{S}_t^\beta = e^{-(2a-\frac{1}{2})t} \hat{S}_t^\beta.
\]

This is a time change of a local martingale and hence is a local martingale; indeed, Itô’s formula gives

\[
d\hat{M}_t = (4a - 1) \cot \Theta_t \, d\hat{W}_t.
\]

Using Girsanov’s theorem, see for example [4], we can define a new probability measure \(P^*\) which corresponds to paths “weighted locally by the local martingale \(\hat{M}_t\)”\). For the time being, we treat this as an arbitrary change of measure, however in Section 2.4 we will see that is precisely the change of measure which gives two-sided radial SLE. Intuitively, \(\hat{M}_t\) weights more heavily those paths whose continuations are likely to get much closer to \(z\). For more examples of the application of Girsanov’s theorem to the study of SLE, and a general outline of the way Girsanov’s theorem is used below, see [7].

In this weighting,

\[
d\hat{W}_t = (4a - 1) \cot \Theta_t \, dt + dW_t,
\]

where \(W_t\) is a standard Brownian motion with respect to \(P^*\). In particular, (7)

\[
d\Theta_t = 2a \cot \Theta_t \, dt + dW_t.
\]
Since $2a > 1/2$, we can see by comparison with a Bessel process that with respect to $\mathbb{P}^*$, the process stays in $(0, \pi)$ for all times. Using this we can show that $\hat{M}_t$ is actually a martingale, and the measure $\mathbb{P}^*$ can be defined by

$$\mathbb{P}^*[V] = \hat{M}_0^{-1} \mathbb{E}[\hat{M}_t 1_V] \quad \text{for} \quad V \in \mathcal{F}_t,$$

where $\mathcal{F}_t$ denotes the $\sigma$-algebra generated by $\{\hat{W}_s : 0 \leq s \leq t\}$. Much of the analysis of $SLE_\kappa$ as it gets close to $z$ uses properties of the simple SDE (7).

Recall that we assume that $a > 1/4$ and all constants can depend on $a$.

**Lemma 2.7.** There exists $c < \infty$ such that if $\Theta_t$ satisfies (7) with $\Theta_0 = x \in (0, \pi/2)$, then if $0 < y < 1$ and

$$\tau = \inf \{t : \Theta_t \in \{y, \pi/2\}\},$$

then

$$\mathbb{P}^*\{\Theta_\tau = y\} \leq c (y/x)^{1-4a}.$$

**Proof.** Itô’s formula shows that $F(\Theta_{t,\tau})$ is a $\mathbb{P}^*$-martingale where

$$F(s) = \int_s^{\pi/2} (\sin u)^{-4a} du, \quad \frac{F''(s)}{F'(s)} = -4a \cot s.$$

Note that $F(\pi/2) = 0$ and

$$F(s) \sim \frac{s^{1-4a}}{1-4a}, \quad s \to 0^+.$$

The optional sampling theorem implies that

$$F(x) = \mathbb{P}^*\{\Theta_\tau = y\} F(y).$$

**Lemma 2.8.** The invariant density for the SDE (7) is

$$f(x) = C_{4a} \sin^{4a} x, \quad 0 < x < \pi, \quad C_{4a} := \left[\int_0^\pi \sin^{4a} x dx\right]^{-1}.$$

**Proof.** This can be quickly verified and is left to the reader.

One can use standard techniques for one-dimensional diffusions to discuss the rate of convergence to the equilibrium distribution. We will state the one result that we need; see [10] for more details. If $F$ is a nonnegative function on $(0, \pi)$, let

$$I_F := C_{4a} \int_0^\pi F(x) \sin^{4a} x dx.$$
LEMMA 2.9. There exists \( u < \infty \) such that for every \( t_0 > 0 \) there exists \( c < \infty \) such that if \( F \) is a nonnegative function with \( I_F < \infty \) and \( t \geq t_0 \),

\[
| \mathbb{E}[F(\Theta_t)] - I_F | \leq c e^{-ut} I_F.
\]

Note that this estimate applies uniformly over all starting points \( x \).

An important case for us is \( F(x) = [\sin x]^{-\beta} = \sin 1^{-4a} x \). Let

\[
(9)\quad c_* = I_F = \frac{C_{4a}}{C_1} = \frac{2}{\int_0^\pi \sin^{4a} x \, dx}.
\]

We will take advantage of this uniform bound to give a concrete estimate on how well the Green’s function approximates the probability of getting near a point.

LEMMA 2.10. There exists \( u > 0 \) such that if \( D \) is a simply connected domain and \( z_1, z_2 \) are points in its boundary, \( r \leq 3/4 \), \( \gamma \) is an \( \text{SLE}_\kappa \) curve from \( z_1 \) to \( z_2 \), \( w \in D \), and \( D_\infty \) denotes the connected component of \( D \setminus \gamma(0, \infty) \) containing \( w \), then

\[
\begin{align*}
\mathbb{P}\{ \Upsilon_{D_\infty}(w) \leq r \cdot \Upsilon_D(w) \} &= c_* r^{2-d} S_D(w; z_1, z_2)^\beta \left[ 1 + O(r^u) \right] \\
&= c_* r^{2-d} \Upsilon_D(w)^{2-d} G_D(w; z_1, z_2) \left[ 1 + O(r^u) \right],
\end{align*}
\]

where \( c_* \) is as defined in (9). In particular, there exists \( c < \infty \) such that for all \( r \leq 3/4 \),

\[
\mathbb{P}\{ \Upsilon_{D_\infty}(w) \leq r \cdot \Upsilon_D(w) \} \leq c r^{2-d} S_D(w; z_1, z_2)^\beta.
\]

PROOF. By conformal invariance we may assume \( \Upsilon_D(w) = 1 \) and define \( t \) by \( r = e^{-2ut} \). Let \( \sigma = \inf\{s : \Upsilon_s = r\} \). Then,

\[
\begin{align*}
\mathbb{P}\{ \sigma < \infty \} &= \mathbb{E}\left[ \mathbf{1}\{ \sigma < \infty \} \right] \\
&= r^{2-d} \mathbb{E}\left[ M_t \hat{S}_t^{-\beta} \right] \\
&= r^{2-d} S_D(w; z_1, z_2)^\beta \mathbb{E}\left[ \hat{S}_t^{-\beta} \right] \\
&= c_* r^{2-d} S_D(w; z_1, z_2)^\beta \left[ 1 + O(e^{-ut}) \right] \\
&= c_* r^{2-d} \Upsilon_D(w)^{2-d} G_D(w; z_1, z_2) \left[ 1 + O(e^{-ut}) \right].
\end{align*}
\]

\( \square \)

Using (1) and (2), we immediately get the following lemma which is in the form that we will use.
Lemma 2.11. There exists $C < \infty$, such that if $D$ is a simply connected domain and $z_1, z_2$ are points in its boundary, $r \leq 3/4$, and $\gamma$ is an $\text{SLE}_\kappa$ curve from $z_1$ to $z_2$, then

$$
\mathbb{P} \{ \text{dist}[w, \gamma(0, \infty)] \leq r \cdot \text{inrad}_D(w) \} \leq C r^{2-d} \left[ \frac{\text{inrad}_D(w)}{\Delta^*_D(w; z_1, z_2)} \right]^{\beta/2}.
$$

2.4. Two-sided radial SLE. We call $\text{SLE}_\kappa$ under the measure $\mathbb{P}^*$ in the previous subsection two-sided radial $\text{SLE}_\kappa$ (from 0 to $\infty$ through $z$ in $\mathbb{H}$ stopped when it reaches $z$). Roughly speaking it is chordal $\text{SLE}_\kappa$ conditioned to go through $z$ (stopped when it reaches $z$). Of course this is an event of probability zero, so we cannot define the process exactly this way. We may provide a direct definition by driving the Loewner equation by the process defined in equation (7) rather than a standard Brownian motion. This definition uses the radial parametrization. We could also just as well use the capacity parametrization, in which case with probability one $T_z < \infty$.

One may justify the definition above examining its relationship to $\text{SLE}_\kappa$ conditioned to get close to $z$. This next proposition is just a restatement of the definition of the measure $\mathbb{P}^*$ when restricted to curves stopped at a particular stopping time.

Proposition 2.12. Suppose $\gamma$ is a chordal $\text{SLE}_\kappa$ path from 0 to $\infty$ and $z \in \mathbb{H}$. For $\varepsilon \leq \text{Im}(z)$, let $\rho_\varepsilon = \inf \{ t : \Upsilon_t(z) = \varepsilon \}$. Let $\mu, \mu^*$ be the two measures on $\{ \gamma(t) : 0 \leq t \leq \rho_\varepsilon \}$ corresponding to chordal $\text{SLE}_\kappa$ restricted to the event $\{ \rho_\varepsilon < \infty \}$ and two-sided radial $\text{SLE}_\kappa$ through $z$ respectively. Then $\mu, \mu^*$ are mutually absolutely continuous with respect to each other with Radon-Nikodym derivative

$$
\frac{d\mu^*}{d\mu} = \frac{G_{H_\mathbb{H}}(z; \gamma(\rho_\varepsilon), \infty)}{G_{\mathbb{H}}(z; 0, \infty)} = \frac{\varepsilon^{d-2} S_{H_\mathbb{H}}(z) \beta}{G_{\mathbb{H}}(z; 0, \infty)}.
$$

Note that as $\varepsilon \to 0$ the Radon-Nikodym derivative tends to infinity. This reflects the fact that $\mu^*$ is a probability measure and that the total mass of $\mu$ is of order $\varepsilon^{2-d}$ (see Lemma 2.10).

This proposition seems to indicate that there is a still a significant difference between two-sided radial $\text{SLE}_\kappa$ going though $z$ and $\text{SLE}_\kappa$ conditioned to get within a specific distance. However, by using the methods of Lemma 2.9 we get the following improvement.

Proposition 2.13. There exists $u > 0, c < \infty$ such that the following is true. Suppose $\gamma$ is a chordal $\text{SLE}_\kappa$ path from 0 to $\infty$ and $z \in \mathbb{H}$. For $\varepsilon \leq \text{Im}(z)$, let $\rho_\varepsilon = \inf \{ t : \Upsilon_t(z) = \varepsilon \}$. Suppose $\varepsilon' < 3\varepsilon/4$. Let $\mu', \mu^*$
be the two probability measures on \( \{ \gamma(t) : 0 \leq t \leq \rho_\varepsilon \} \) corresponding to chordal SLE_\( \kappa \) conditioned on the event \( \{ \rho_\nu < \infty \} \) and two-sided radial SLE_\( \kappa \) through \( z \) respectively. Then \( \mu', \mu^* \) are mutually absolutely continuous with respect to each other and the Radon-Nikodym derivative satisfies

\[
\left| \frac{d\mu^*}{d\mu'} - 1 \right| \leq c (\varepsilon' / \varepsilon)^u.
\]

From the definition, it is easy to show that there is a subsequence \( t_n \uparrow T_z \) with \( \gamma(t_n) \to z \). Stronger than this is true and in [8] it is proven that, for \( 0 < \kappa < 8 \), with probability one, two-sided radial gives a curve, by which we mean that with probability one \( \gamma(T_z-) = z \).

**Lemma 2.14.** Let \( \rho_\varepsilon = \inf \{ t : \Upsilon_t(z) = \varepsilon \} \). There exists \( \alpha > 0 \) so that for any \( z \in \mathbb{H} \) there exists \( c_z < \infty \), so that for any \( \varepsilon \) and \( R \) with \( \varepsilon \leq R \leq \text{Im}(z) \) we have

\[
\mathbb{P}^\ast \{ \gamma[\rho_\varepsilon, T_z] \not\subseteq B_R(z) \} \leq c_z \left( \frac{\varepsilon}{R} \right)^\alpha.
\]

**Proof.** This result was shown for two-sided radial through 0 from 1 to \(-1\) in \( \mathbb{D} \) in [8, Theorem 3]. Since \( c_z \) is allowed to depend on \( z \), the form in this lemma can be obtained by conformal invariance.

We will also need this bound in a chordal form, rather than two-sided radial form. In order to prove the chordal form, we need the following lemma.

**Lemma 2.15.** Let \( \rho_\varepsilon = \inf \{ t : \Upsilon_t(z) = \varepsilon \} \). There exists \( c < \infty \), such that if \( z \in \mathbb{H} \) and \( \varepsilon \leq \text{Im}(z)/2 \), \( 0 < \theta_0 \leq \pi/2 \),

\[
\mathbb{P} \{ S_{\rho_\varepsilon}(z) < \sin(\theta_0) \mid \rho_\varepsilon < \infty \} \leq c \theta_0^2.
\]

**Proof.** First note that by Proposition 2.12 and Lemma 2.10 we have that

\[
\mathbb{P} \{ S_{\rho_\varepsilon}(z) < \sin(\theta_0) \mid \rho_\varepsilon < \infty \} \leq c \mathbb{E}^\ast \{ S_{\rho_\varepsilon}^{-\beta}(z) 1 \{ S_{\rho_\varepsilon}(z) < \sin(\theta_0) \} \}.
\]

By applying the techniques from Lemma 2.9 with the function

\[
F(\theta) = \sin(\theta)^{-\beta} 1 \{ \sin(\theta) < \sin(\theta_0) \}
\]

and noting that the integral is

\[
\int_0^\pi \sin(\theta)^{-\beta} 1 \{ \sin(\theta) < \sin(\theta_0) \} \sin^{4\alpha} d\theta = 2 \int_0^{\theta_0} \sin(\theta) d\theta = O(\theta_0^2)
\]

we get the result. 

Lemma 2.16. Let $\rho_\varepsilon = \inf\{t : \Upsilon_t(z) = \varepsilon\}$. Fix $\varepsilon < \eta < R < 1$ and $z \in \mathbb{H}$, then there exists some $c$ depending only on $z$ and $\alpha > 0$ such that

$$\mathbb{P}\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z) \mid \rho_\varepsilon < \infty\} \leq c \left( \frac{\eta}{R} \right)^\alpha.$$ 

Proof. Let $0 < \theta < \pi/2$ be arbitrary; we will fix its precise value later. We apply Lemma 2.15 and Lemma 2.14 with the above to see that

$$\mathbb{P}\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z) \mid \rho_\varepsilon < \infty\} = \mathbb{P}\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z); S_{\rho_\varepsilon}(z) \geq \sin(\theta) \mid \rho_\varepsilon < \infty\} + \mathbb{P}\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z); S_{\rho_\varepsilon}(z) < \sin(\theta) \mid \rho_\varepsilon < \infty\} \leq c\mathbb{E}^*[S_{\rho_\varepsilon}^{-\beta}(z)1\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z); S_{\rho_\varepsilon}(z) \geq \sin(\theta)\}] + \theta^2$$

$$\leq c\theta^{-\beta}\mathbb{P}\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z)\} + \theta^2$$

$$\leq c\theta^{-\beta}\mathbb{P}\{\gamma[\rho_\eta, T_z] \not\subseteq B_R(z)\} + \theta^2$$

$$\leq c\theta^{-\beta}(\eta/R)^\alpha + \theta^2$$

where $c$ is being used generically. Thus by an appropriate choice of $\theta$, for example

$$\theta = (\eta/R)^{\frac{\alpha}{2\pi+\pi}},$$

we get the desired bound. \qed

3. Multi-point Green’s function. In this subsection we consider two distinct points $z, w \in \mathbb{H}$. To simplify notation, we write

$$\xi = \xi_{z, \varepsilon} = \inf\{t : \Upsilon_t(z) \leq \varepsilon\},$$

$$\chi = \chi_{\delta} = \inf\{t : \Upsilon_t(w) \leq \delta\}.$$ 

Although we will write $\xi, \chi$, it is important to remember that these quantities depend on $z, \varepsilon, w, \delta$. We let $\mathbb{P}, \mathbb{E}$ denote probabilities and expectations for $SLE_\kappa$ from 0 to $\infty$ in $\mathbb{H}$ and $\mathbb{P}^*, \mathbb{E}^*$ for the corresponding quantities for two-sided radial through $z$. The multi-point Green’s function, which we write

$$G(z, w) = G_{\mathbb{H}}(z, w; 0, \infty),$$

roughly corresponds to the probability that $SLE$ in $\mathbb{H}$ from 0 to $\infty$ goes through $z$ and then through $w$. This quantity is not symmetric. Although we do not have a closed form for this quantity, we can define it precisely.
DEFINITION. The multi-point Green’s function $G(z, w)$ is defined by

$$G(z, w) = G(z) \mathbb{E}^* [G_H(w; z, \infty)],$$

where $H$ is the unbounded component of $\mathbb{H} \setminus \gamma(0, T_z)$.

It is worth noting that if $w$ is swallowed by the two-sided radial SLE curve before reaching $z$, this Green’s function gives that event weight zero since the curve $w$ is unreachable no matter how close the curve was to $w$ before reaching $z$.

In [11], the exact formula for $G_H(z; 0, \infty)$ was found by considering the martingale $G_H(t; z, \gamma(t), \infty)$ and then using Itô’s formula and scaling to find the ODE that it satisfies, which could then be explicitly solved. When attempting the same technique here, a three real variable PDE results, which does not immediately seem to admit a closed form solution.

The justification for this definition comes from the following theorem. Implicit in the statement, is that the limit can be taken along any sequence of $\varepsilon, \delta$ going to zero.

THEOREM 1. If $z, w \in \mathbb{H}$, then

$$\lim_{\varepsilon, \delta \to 0^+} e^{d-2} \delta^{d-2} \mathbb{P}\{\xi < \chi < \infty\} = c^*_2 G(z, w),$$

where $c_*$ is as defined in (9).

When

$$d = \left(1 + \frac{\kappa}{8}\right) \wedge 2$$

is the dimension rather than simply $d = 1 + \kappa/8$, this theorem still defines an interesting quantity for $\kappa \geq 8$. Since the curve is space filling for $\kappa \geq 8$, the limit is trivial and

$$\lim_{\varepsilon, \delta \to 0^+} e^{d-2} \delta^{d-2} \mathbb{P}\{\xi < \chi < \infty\} = \mathbb{P}\{\xi_0 < \chi_0\} = c_* G(z, w).$$

This agrees with the above definition of $G(z, w)$ since we may take two-sided radial through $z$ for $\kappa \geq 8$ to be the measure on $\gamma$ stopped at the time the curve passes through $z$ and

$$G_D(w; z_1, z_2) = 1\{w \in D\}.$$ 

Since this case requires no further work, we will continue to assume that $\kappa < 8$.

We will need one lemma that will follow from our work on Beffara’s estimate, which we will prove in Section 4.
Lemma 3.1. There exists $\alpha > 0$, such that if $z, w \in \mathbb{H}$, then there exists $c = c_{z,w} < \infty$, such that for all $\varepsilon, \delta, r > 0$,

$$
P\{\xi < \chi < \infty; \text{inrad}_\xi(w) \leq r\} \leq c \varepsilon^{2-d} \delta^{2-d} r^\alpha.
$$

More precise results than this are obtained in this paper, but this is all that is required in this section.

Before going through the details of the proof, we briefly sketch the argument. To estimate

$$
P\{\xi < \chi < \infty\},
$$

we wish to show that this probability is carried mostly on curves which get within $\varepsilon$ of $z$ in conformal radius before decreasing the conformal radius of $w$ much at all. To show that the curves which do not do this are negligible, we use Lemma 3.1.

On the event that the curve stays bounded away from $w$, we know the Green’s function for getting to $w$ stays uniformly bounded, allowing us to use convergence of the conditioned measures $\mathbb{E}[\cdot | \xi < \infty]$ to $\mathbb{E}^*[\cdot]$, the two-sided radial measure, as measures on the SLE curve up until some fixed conformal radius $\eta \gg \varepsilon$.

This would be everything if it were not for the fact that the tip of the curves (the portion very near $z$) under the conditioned measure versus the two-sided radial measure have very different distribution. To handle this, we use Lemmas 2.14 and 2.16 to show that under both measures the tip stays close to $z$ most of the time in Euclidean distance, and then Lemma 2.5 tells us that the Green’s function for getting to $w$ is insensitive to these changes.

To aid in the understanding of the proof, Figure 2 shows diagrammatically the various distances considered and the approximate shape of a curve in the main term.

**Proof of Theorem 1 given Lemma 3.1.** We first split according to how close we get to $w$ before getting close to $z$. Fixing some $r < |z - w|/2$, by Lemma 3.1 we see that for some $\alpha > 0$

$$
P\{\xi < \chi < \infty\} = P\{\xi < \chi < \infty; \text{inrad}_\xi(w) > r\}
$$

$$
+ P\{\xi < \chi < \infty; \text{inrad}_\xi(w) < r\} = P\{\xi < \chi < \infty; \text{inrad}_\xi(w) > r\} + O(\varepsilon^{2-d} \delta^{2-d} r^\alpha).
$$

Let $F_\xi$ denote the $\sigma$-algebra generated by the stopping time $\xi$. By applying Lemma 2.10 to $w$ in the domain $H_\xi$, we see if $\delta \leq r/2$,

$$
P\{\xi < \chi < \infty; \text{inrad}_\xi(w) > r | F_\xi\}
$$

$$
= 1\{\xi < \infty; \text{inrad}_\xi(w) > r\} c_\alpha \delta^{2-d} G_{H_\xi}(w; \gamma(\xi), \infty) [1 + O((\delta/r)^n)].
$$
Fig 2. A diagram of the proof of Theorem 1. Dotted circles represent conformal radii and solid circles refer to geometric radii. The bold curve gives an example of the approximate shape of a curve contributing to the leading order event.

Applying Lemma 2.10 to $z$ in $\mathbb{H}$ combined with the previous equation implies

$$c_n^{-2} \varepsilon^{d-2} \delta^{d-2} G_{\mathbb{H}}(z; 0, \infty)^{-1} \mathbb{P}\{\xi < \chi < \infty; \text{inrad}_\xi(w) > r\}$$

$$= [1 + O(\varepsilon^u + (\delta/r)^u)] \mathbb{E}[G_{H_{\xi}}(w; \gamma(\xi), \infty)1\{\text{inrad}_\xi(w) > r\} \mid \xi < \infty].$$

For simplicity of notation, given a stopping time $\tau$, we let

$$\mathbb{E}_{\tau}[\cdot] = \mathbb{E}[\cdot \mid \tau < \infty] \text{ and } G_{\tau,r} = G_{H_{\tau}}(w; \gamma(\tau), \infty)1\{\text{inrad}_{\tau}(w) > r\}.$$ and hence we may rewrite this as

$$\mathbb{P}\{\xi < \chi < \infty; \text{inrad}_\xi(w) > r\}$$

$$= c_n^2 \varepsilon^{2-d} \delta^{2-d} G_{\mathbb{H}}(z; 0, \infty)[1 + O(\varepsilon^u + (\delta/r)^u)] \mathbb{E}_\xi[G_{\xi,r}].$$

We wish to transform this expression from the conditioned measure to the two-sided radial measure, and from considering the situation at time $\xi$ (the time it first gets within conformal radius $\varepsilon$) to $T_2$ (the time under the two-sided radial measure that $z$ is first contained in the boundary of $H_{T_2}$). To do so we will pass through a series of steps.
Fix some $\eta, R$ so that $\varepsilon < \eta < R < |z - w|/2$. We wish to control the difference

$$|E^\varepsilon[G_{\xi, r}] - E^\varepsilon[G_{\xi, r}]| \leq E^\varepsilon[|G_{\xi, r} - G_{\xi, r}| \{\gamma[\xi, \xi] \subseteq B_R(z)\}]$$

$$+ E^\varepsilon[|G_{\xi, r} - G_{\xi, r}| \{\gamma[\xi, \xi] \not\subseteq B_R(z)\}]$$

By Lemma 2.5 and the fact that the inradius about $w$ cannot decrease between $\xi$ and $\xi$ if $\gamma[\xi, \xi] \subseteq B_R(z)$ we see that

$$E^\varepsilon[|G_{\xi, r} - G_{\xi, r}| \{\gamma[\xi, \xi] \subseteq B_R(z)\}] = O(r^{d-2-(\beta \wedge 1)/2} R^{(\beta \wedge 1)/2})$$

On the second term, the difference is no bigger than $O(r^{d-2})$ on an event, which by Lemma 2.16 is $O((\eta/R)\alpha')$ for some $\alpha' > 0$. Putting it all together yields

$$|E^\varepsilon[G_{\xi, r}] - E^\varepsilon[G_{\xi, r}]| = O(r^{d-2-(\beta \wedge 1)/2} R^{(\beta \wedge 1)/2} + r^{d-2}(\eta/R)\alpha')$$

By Lemma 2.13, we know for events in $F_{\xi}$ we have

$$\left| \frac{d\mathbb{P}^\varepsilon}{d\mathbb{P}_\xi} - 1 \right| = O((\varepsilon/\eta)^u)$$

and hence we have

$$|E^\varepsilon[G_{\xi, r}] - E^\varepsilon[G_{\xi, r}]| = O(r^{d-2}(\varepsilon/\eta)^u)$$

Analogously to before, consider splitting the difference

$$|E^\varepsilon[G_{\xi, r}] - E^\varepsilon[G_{T, r}]| \leq E^\varepsilon[|G_{\xi, r} - G_{T, r}| \{\gamma[\xi, T] \subseteq B_R(z)\}]$$

$$+ E^\varepsilon[|G_{\xi, r} - G_{T, r}| \{\gamma[\xi, T] \not\subseteq B_R(z)\}]$$

By Lemma 2.5 and the fact that the inradius about $w$ cannot decrease between $\xi$ and $T$ if $\gamma[\xi, T] \subseteq B_R(z)$, we again see

$$E^\varepsilon[|G_{\xi, r} - G_{T, r}| \{\gamma[\xi, T] \subseteq B_R(z)\}] = O(r^{d-2-(\beta \wedge 1)/2} R^{(\beta \wedge 1)/2})$$

The second term is on an event which is by Lemma 2.14 is $O((\eta/R)\alpha')$, and hence

$$|E^\varepsilon[G_{\xi, r}] - E^\varepsilon[G_{T, r}]| = O(r^{d-2-(\beta \wedge 1)/2} R^{(\beta \wedge 1)/2} + r^{d-2}(\eta/R)\alpha')$$

We may easily see that

$$\mathbb{P}^\varepsilon\{\text{inrad}_{T, r}(w) = 0\} \leq \sum_{k \geq 1} \mathbb{P}^\varepsilon\{\text{inrad}_{\xi, k}(w) = 0\} = 0$$
by the fact that $P^*$ is absolutely continuous with respect to $P$ until the stopping time $\xi_{1/k}$ combined with that fact that two-sided radial SLE generates a curve with probability one. Hence, since $G_{T_{z_1}}(w; z, \infty) \geq 0$, we have that

$$E^*[G_{T_{z_1}}(w; z, \infty) 1\{\text{inrad}_{T_{z_1}}(w) > r\}] \to E^*[G_{T_{z_1}}(w; z, \infty)] \text{ as } r \to 0.$$ 

Combining all these terms and by combining exponents, we see there exists some $b > 0$ such that

$$c^{d-2}\delta^{d-2}P\{\xi < \chi < \infty\} = c^2_G G_{H}(z; 0, \infty)[1 + O(\varepsilon^b + (\delta/r)^b)]E^*[G_{T_{z_1}}, r]$$

$$+ O(\varepsilon^b + (R/r)^b + (R/r)^b(\eta/R)^b + (\varepsilon/r)^b(\varepsilon/\eta)^b).$$

Thus by choosing $r, \eta, R$ so that as $\varepsilon, \delta \to 0$ we also have

$$r \to 0, \quad \delta/r \to 0, \quad \varepsilon/r \to 0, \quad R/r \to 0, \quad \eta/R \to 0, \quad \varepsilon/\eta \to 0$$

we see that

$$c^{d-2}\delta^{d-2}P\{\xi < \chi < \infty\} \to c^2_G G_{H}(z; 0, \infty)E^*[G_{T_{z_1}}(w; z, \infty)]$$

as desired.

This same argument generalizes to show that we can define higher-order Green’s functions of SLE as well (those that give normalized probabilities for passing through $k$ marked points in the interior) and that the resulting multi-point Green’s functions can be written in terms of expectations under the two-sided radial measure of lower order Green’s functions, for instance

$$\lim_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0} \varepsilon_1^{d-2}\varepsilon_2^{d-2}\varepsilon_3^{d-2}P\{\xi_{\varepsilon_1, z_1} < \xi_{\varepsilon_2, z_2} < \xi_{\varepsilon_3, z_3}\}$$

$$= c^2_G G_{H}(z_1; 0, \infty)E^*[G_{H_{T_{z_1}}}(z_2, z_3; z_1, \infty)]$$

where $E^*$ is the two sided radial measure passing through $z_1$.

Note that we may obtain the multi-point Green’s function as defined in the introduction by summing this over the case where it gets near to $z$ then $w$ and the case where it gets near to $w$ then $z$.

The remainder of this paper is dedicated to providing a proof of Lemma 3.1 and a sharpened version of Befarra’s estimate.

4. Proof of Befarra’s estimate and Lemma 3.1. To complete our proof of the existence of multi-point Green’s functions we require a proof of Lemma 3.1. We also wish to prove Befarra’s estimate which is the following theorem.
Theorem 2 (Beffara’s Estimate). There exists a $c > 0$ such that for all $z, w \in \mathbb{H}$ with $\text{Im}(z), \text{Im}(w) \geq 1$ we have that

\begin{equation}
\mathbb{P}\{\Upsilon_\infty(z) < \varepsilon, \Upsilon_\infty(w) < \delta\} \leq c \varepsilon^{2-d} \delta^{2-d} |z - w|^{d-2}
\end{equation}

The hard work will be establishing the result when $z, w$ are far apart. We will prove the following. We use the notation introduced in Section 2.1. For later convenience, we write this proposition in terms of the usual radius rather than the conformal radius, but it is easy to convert to conformal radius using the Koebe-1/4 theorem. We use the notation $\Delta_t(z) = \text{inrad}_{H_t}(z)$.

Proposition 4.1. For every $0 < \theta < \infty$, there exists $c < \infty$, such that if $z, w \in \mathbb{H}$ with $\text{Im}(z), \text{Im}(w) \geq \theta$ and $|z - w| \geq \theta/9$, then

\[ \mathbb{P}\{\Delta_\infty(z) \leq \varepsilon, \Delta_\infty(w) \leq \delta\} \leq c \varepsilon^{2-d} \delta^{2-d}. \]

Proof of Theorem 2 given Proposition 4.1. Without loss of generality we assume that $1 \leq \text{Im}(z) \leq \text{Im}(w)$. We first claim that is suffices to prove (10) when $1 = \text{Im}(z) \leq \text{Im}(w)$. Indeed, if this is true and $r > 1$, scaling implies that

\[ \mathbb{P}\{\Upsilon_\infty(rz) < \varepsilon, \Upsilon_\infty(rw) < \delta\} = \mathbb{P}\{\Upsilon_\infty(z) < \varepsilon/r, \Upsilon_\infty(w) < \delta/r\} \]
\[ \leq c (\varepsilon/r)^{2-d} (\delta/r)^{2-d} |z - w|^{d-2} \]
\[ < c \varepsilon^{2-d} \delta^{2-d} |rz - rw|^{d-2}. \]

Suppose $\varepsilon > |z - w|/10$. Then, using the one-point estimate Lemma 2.10, we get

\[ \mathbb{P}\{\Upsilon_\infty(z) < \varepsilon, \Upsilon_\infty(w) < \delta\} \leq \mathbb{P}\{\Upsilon_\infty(w) < \delta\} \]
\[ \leq c \delta^{2-d} \]
\[ \leq c 10^{2-d} \varepsilon^{2-d} \delta^{2-d} |w - z|^{d-2}. \]

A similar argument with $\delta$ shows that it suffices to prove (10) with $\text{Im}(z) = 1$ and $\varepsilon, \delta < |z - w|/10$. If $|z - w| \geq 1/9$, we can apply Proposition 4.1 directly. So for the remainder of the proof, we let $u = |z - w|$ and assume

\[ 1 = \text{Im}(z) \leq \text{Im}(w), \quad u \leq \frac{1}{9}, \quad \varepsilon, \delta \leq \frac{u}{10}. \]
We will use the growth and distortion theorems which we now recall (see, e.g., [5, Section 3.2]). Suppose \( f : \mathbb{D} \to \mathbb{C} \) is a univalent function with \( f(0) = 0, |f'(0)| = 1 \). Then if \( |\zeta| < 1 \),

\[
\frac{|\zeta|}{(1 + |\zeta|)^2} \leq |f(\zeta)| \leq \frac{|\zeta|}{(1 - |\zeta|)^2}.
\]

(11)

\[
\frac{1 - |\zeta|}{(1 + |\zeta|)^3} \leq |f'(\zeta)| \leq \frac{1 + |\zeta|}{(1 - |\zeta|)^3}.
\]

(12)

Let \( \tau = \inf\{t : |\gamma(t) - z| = 8u\} = \inf\{t : \Delta_t(z) = 8u\} \). The triangle inequality implies that \( 7u \leq \Delta_\tau(w) \leq 9u \). Lemma 2.10 implies that

\[
\mathbb{P}\{\tau < \infty\} \leq c u^{2-d}.
\]

(13)

Let \( g_t \) be the usual conformal map and let \( h = sg_t \) where \( s > 0 \) is chosen so that \( \text{Im}(h(z)) = 1 \). By the Schwarz lemma and the Koebe \((1/4)\)-theorem, \( 4u \leq \Upsilon_{H_t}(z) \leq 16u \), and since \( \Upsilon_\Xi(h(z)) = 1 \),

\[
\frac{1}{16u} \leq |h'(z)| \leq \frac{1}{4u}.
\]

Since \( h \) is a conformal transformation of the disk of radius \( 8u \) about \( z \), (11) implies

\[
\frac{4}{81} \leq (8/9)^2 u |h'(z)| \leq |h(w) - h(z)| \leq (8/7)^2 u |h'(z)| \leq \frac{16}{49}.
\]

Since \( \varepsilon \leq u/10 = (8u)/80 \), if \( |z - \zeta| \leq \varepsilon \), (11) implies

\[
|h(\zeta) - h(z)| \leq \left( \frac{80}{79} \right)^2 \varepsilon |h'(z)| \leq \left( \frac{80}{79} \right)^2 \varepsilon \frac{4u}{16u} \leq \frac{2\varepsilon}{7u}.
\]

Applying (12), we can see that

\[
|h'(w)| \leq \left( \frac{10/9}{(8/9)^3} \right) |h'(z)| \leq \left( \frac{10/9}{(8/9)^3} \right) \frac{1}{4u}.
\]

Applying (11) to the disk of radius \( 7u \) about \( w \) and using \( \delta \leq u/10 = (7u)/70 \), we see that for \( |\zeta - w| \leq \delta \),

\[
|h(\zeta) - h(w)| \leq \left( \frac{70}{69} \right)^2 \delta |h'(w)| \leq \left( \frac{70}{69} \right)^2 \delta \left( \frac{10/9}{(8/9)^3} \right) \frac{1}{4u} \leq \frac{3\delta}{7u}.
\]
Using the estimates of the previous paragraph, we can see by conformal invariance and the Markov property, that
\[
\mathbb{P}\{\Delta_\infty(z) \leq \varepsilon, \Delta_\infty(w) \leq \delta \mid \tau < \infty\}
\]
is bounded above by the supremum of
\[
\mathbb{P}\{\Delta_\infty(z') \leq \varepsilon', \Delta_\infty(w') \leq \delta'\},
\]
where the supremum is over
\[
\text{Im}(z') = 1, \quad \frac{4}{81} \leq |z' - w'| \leq \frac{16}{49}, \quad \varepsilon' \leq \frac{2\varepsilon}{7u}, \quad \delta' \leq \frac{3\delta}{7u}.
\]
Proposition 4.1 implies that there exists \(c'\) such that this supremum is bounded by
\[
c'(\varepsilon')^2 - d\delta'^2 - d \leq c\varepsilon^2 - d\delta^2 - du^2(d-2).
\]
If we combine this with (14), we get
\[
\mathbb{P}\{\Delta_\infty(z') \leq \varepsilon', \Delta_\infty(w') \leq \delta'\} \leq c\varepsilon^2 - d\delta^2 - du^2d-2,
\]
which is what we needed to prove. \(\square\)

By an analogous argument to how we obtained Theorem 2 from Proposition 4.1, we may obtain Lemma 3.1 from Proposition 4.2.

**Proposition 4.2.** For every \(0 < \theta < \infty\), there exists \(c < \infty\) and \(\alpha > 0\) such that if \(z, w \in \mathbb{H}\) with
\[
\text{Im}(z), \text{Im}(w) \geq \theta \quad \text{and} \quad |z - w| \geq \theta/9,
\]
then for \(\rho > \delta\)
\[
(14) \quad \mathbb{P}\{\Delta_\infty(z) \leq \varepsilon, \Delta_\infty(w) \leq \delta, \Delta_\sigma(w) \leq \rho\} \leq c\varepsilon^{2-d}\delta^{2-d}\rho^\alpha.
\]
where \(\sigma = \inf\{t : \Delta_t(z) \leq \varepsilon \text{ or } \Delta_t(w) \leq \delta\} \).

This proposition will follow immediately from the work required to show Proposition 4.1.

To prove the proposition, we will show that there exists a \(c < \infty\) such that (14) holds if \(|z - w| \geq 2\sqrt{2}\) and \(\text{Im}(z), (w) \geq 1\). By scaling one can easily deduce the result for all \(\theta > 0\) with a \(\theta\)-dependent constant. We fix
z, w with \(|z - w| \geq 2\sqrt{2}\) and \(\text{Im}(z), \text{Im}(w) \geq 1\), and denote by \(I\) some fixed vertical or diagonal line such that
\[
\text{dist}(z, I), \text{dist}(w, I) \geq 1
\]
and \(z, w\) lie in different components of \(\mathbb{H} \setminus I\). We will further assume, without loss of generality, that \(z\) is in the component of \(\mathbb{H} \setminus I\) which contains arbitrarily large negative real numbers in its boundary (more informally that \(z\) is in the left component).

4.1. An excursion measure estimate. Our main result will require an estimate of the “distance” between two boundary arcs in a simply connected domain. We will use excursion measure to gauge the distance; we could also use extremal distance, but we find excursion measure more convenient.

Suppose \(\eta\) is a crosscut in \(\mathbb{H}\) with \(-\infty < \eta(1^-) \leq \eta(0^+) \leq -1\). Let \(H_\eta\) denote the unbounded component of \(\mathbb{H} \setminus \eta\). Let \(E(\eta) = E_{\mathbb{H}_\eta}(\eta, [0, \infty))\) denote the excursion measure between \(\eta\) and \([0, \infty)\) in \(H_\eta\), the definition of which we now recall (see [5, Section 5.2] for more details). If \(z \in H_\eta\), let \(h_\eta(z)\) be the probability that a Brownian motion starting at \(z\) exits \(H_\eta\) at \(\eta\). For \(x \geq 0\), let \(\partial_y h_\eta(x)\) denote the partial derivative. Then
\[
E(\eta) = \int_0^\infty \partial_y h_\eta(x) \, dx.
\]

The excursion measure \(E_D(V_1, V_2)\) is defined for any domain and boundary arcs \(V_1, V_2\) in a similar way and is a conformal invariant. If \(V_2\) is smooth, then we can compute \(E_D(V_1, V_2)\) by a similar integral
\[
E_D(V_1, V_2) = \int_{V_2} \partial_n h_{V_1}(z) \, |dz|,
\]
where \(n\) denotes the inward normal. We need the following easy estimate.

**Lemma 4.3.** There exist \(c_1, c_2\) such that if \(\eta\) is a crosscut in \(\mathbb{H}\) with \(-\infty < \eta(1^-) \leq \eta(0^+) = -1\) and \(\text{diam}(\eta) \leq 1/2\), then
\[
c_1 \text{diam}(\eta) \leq E(\eta) \leq c_2 \text{diam}(\eta).
\]

**Sketch of proof.** In fact, we get an estimate
\[
\partial_y h_\eta(x) \asymp \frac{\text{diam}(\eta)}{(x + 1)^2}.
\]
The key estimate used here is the fact that if \( \text{Re}(z) \geq 0 \),
\[
h_{\eta}(z) \approx \frac{\text{Im}(z) \text{diam}(\eta)}{|z| + 1}.
\]

**Lemma 4.4.** There exists a \( C < \infty \) such that the following is true. Suppose \( H \subset \mathbb{C} \) is a half plane bounded by the line \( L = \partial H \), \( D \) is a simply connected subdomain of \( \mathbb{H} \) and \( z \in \partial D \) with \( d(z, L) > \frac{1}{2} \). Suppose \( I \) is a subinterval of \( L \cap \partial D \). Then for every \( \varepsilon < \frac{1}{4} \), the excursion measure between \( I \) and \( V := \partial D \cap \{w : |w - z| \leq \varepsilon\} \) is bounded above by \( C\varepsilon^{1/2} \).

**Proof.** Without loss of generality we assume that \( H = \mathbb{H}, z = i/2 \). Let \( h(w) \) denote the probability that a Brownian motion starting at \( w \) exits \( D \) at \( V \). Then the excursion measure is exactly
\[
\int_I \partial_y h(x) \, dx.
\]

Hence it suffices to give an estimate
\[
(16) \quad \partial_y h(x) \leq c \varepsilon^{1/2} [1 \wedge x^{-2}].
\]

For \( |x| \leq 4 \), this follows from the Beurling estimate. For \( |x| \geq 4 \), we first consider the excursion “probability” to reach \( \text{Re}(w) = x/2 \). By the gambler’s ruin estimate, this is bounded by \( O(|x|^{-1}) \). Then we need to consider the probability that a Brownian motion starting at \( z' \) with \( \text{Re}(z') = x/2 \) reaches the disk of radius 1 about \( z \) without leaving \( D \). By comparison with the same probability in the domain \( \mathbb{H} \), we see that this is bounded above by \( O(|x|^{-1}) \). Finally from there we need to hit \( V \) which contributes a factor of \( O(\varepsilon^{1/2}) \) by the Beurling estimate. Combining these estimates gives (16). □
Lemma 4.5. There exists $c > 0$ such that the following holds. Let $D$ be a simply connected domain, and let $\gamma$ be a chordal SLE$_{\kappa}$ path from $z_1$ to $z_2$ in $D$. Let $\eta : (0,1) \to D$ be a crosscut in $D$. Let $\xi : (0,1) \to D$ be another crosscut with $\xi(0^+) = z_1$, and let $D_1, D_2$ denote the components of $D \smallsetminus \xi$. Suppose $\eta \subset D_1$ and $z_2 \in \partial D_2$. Then,

$$\mathbb{P}\{\gamma(0,\infty) \cap \eta(0,1) \neq \emptyset\} \leq c \mathcal{E}_D(\eta,\xi)^\beta.$$ 

See Figure 3 for a diagram of the setup of this lemma.

**Proof.** By conformal invariance, we may assume that $D = \mathbb{H}$, $z_1 = 0, z_2 = \infty$, and it suffices to prove the result when $\mathcal{E}_D(\eta,\xi) \leq 1$ in which case the endpoints of $\eta$ are nonzero. Without loss of generality we assume that they lie on the negative real axis, and by scale invariance we may assume $\eta(1^-) \leq \eta(0^+) = -1$. Then monotonicity of the excursion measure implies that

$$\mathcal{E}_D(\eta,\xi) \geq \mathcal{E}_D(\eta).$$

Lemma 4.3 implies that if $\text{diam}(\eta) < 1/2$ then $\mathcal{E}_D(\eta) \propto \text{diam}(\eta)$. Since $\mathcal{E}_D(\eta) \leq 1$ one can see there is a $c_0$ so that $\text{diam}(\eta) \leq c_0$. The result then follows from Proposition 2.6.

Given the proof, the form of this lemma may seem odd as the curve $\xi$ is discarded halfway through; indeed, the result could be stated with $\mathcal{E}_D(\eta)$ rather than $\mathcal{E}_D(\eta,\xi)$ in the inequality. However, $\mathcal{E}_D(\eta)$ is hard to estimate directly and, in every case in this paper, the method of estimation is to find a curve $\xi$ and proceed as above.
4.2. Topological lemmas. The most challenging portion of this proof is gaining simultaneous control of the distances to the near and far edges of the curve. Luckily, we may eliminate a number of hard cases of the computations that follow by purely topological means. For clarity of presentation, we have isolated these topological lemmas here in a separate section. Let \( z, w, \mathcal{I} \) be as described in the paragraph around equation 15. We call \( \gamma \) a non-crossing curve (from 0 to \( \infty \) in \( \mathbb{H} \)) if is generated by the Loewner equation (4) with some driving function \( U_t \), and, as before, we let \( H_t \) be the unbounded component of \( \mathbb{H} \setminus \gamma(0, t] \) and \( \partial_1 H_t, \partial_2 H_t \) be the preimages (considered as prime ends) under \( g_t \) of \( (-\infty, U_t) \) and \( (U_t, \infty) \). We call a simple curve \( \omega : (0, \infty) \to H_t \) with \( \omega(0^+) = \gamma(t) \) and \( \omega(\infty) = \infty \) an infinite crosscut of \( H_t \). Such curves can be obtained as preimages under \( g_t \) of simple curves from \( U_t \) to \( \infty \) in \( \mathbb{H} \). An important observation is that infinite crosscuts of \( H_t \) separate \( \partial_1 H_t \) from \( \partial_2 H_t \).

We now define a particular crosscut of \( H_t \) contained in \( \mathcal{I} \) that separates \( z \) from \( w \).

**Definition.** Let \( \gamma \) be a non-crossing curve and let \( \mathcal{I}_t = \mathcal{I} \setminus \gamma(0, t] \). We denote by \( I_t = I_t(\mathcal{I}, z, w, \gamma) \) the unique open interval contained in \( \mathcal{I} \) such that the following four properties hold. For any \( t \leq t' \) we have:

- \( I_t \) is a connected component of \( \mathcal{I}_t \),
- \( I_t \subseteq I_{t'} \),
- \( H_t \setminus I_t \) has exactly two connected components, one containing \( z \) and one containing \( w \), and
- \( I_t = I_{t'} \) whenever \( \gamma(t, t'] \cap \mathcal{I} = \emptyset \).

We let \( H_t^z, H_t^w \) denote the components of \( H_t \setminus I_t \) that contain \( z \) and \( w \) respectively.

**Figure 4.** A few steps showing the behavior of \( I_t \) for some times \( 0 < t_1 < t_2 < t_3 \).
MULTI-POINT GREEN’S FUNCTIONS FOR SLE

Seeing that this notion is well defined is non-trivial, despite the intuitive nature what it should be (see Figure 4). To avoid breaking the flow of the document, the proof that it is well defined has been deferred to Appendix A.

**Lemma 4.6.** Suppose \( \gamma \) is a non-crossing curve with \( z, w \notin \gamma(0, \infty) \) and \( I_t = I_t(\mathcal{I}, z, w, \gamma) \) as above. Suppose \( \gamma(t) \in \overline{T_t} \). If \( I_t \) is not bounded, then

\[
\Delta^*_H(z, \gamma(t), \infty) \geq 1, \quad \Delta^*_H(w, \gamma(t), \infty) \geq 1.
\]

**Proof.** Suppose \( I_t \) is not bounded. Then \( I_t \) is an infinite crosscut of \( H \). Suppose that \( \Delta^*_H(z, \gamma(t), \infty) < 1 \). Then there is a crosscut \( \eta \) contained in a disc of radius strictly less than one centered on \( z \) which has one end point in \( \partial H_1 \) and one end point in \( \partial H_2 \). Hence \( \eta \) must intersect \( I_t \). However, \( \text{dist}(z, I_t) \geq \text{dist}(z, \mathcal{I}) \geq 1 \) which is a contradiction. Therefore, \( \Delta^*_H(z, \gamma(t), \infty) \geq 1 \). \( \square \)

**Lemma 4.7.** Suppose \( \gamma \) is a non-crossing curve with \( z, w \notin \gamma(0, \infty) \) and \( I_t = I_t(\mathcal{I}, z, w, \gamma) \) as above. Suppose \( \gamma(t) \in \overline{T_t} \). If \( I_t \) is bounded and \( H^*_t \) is bounded then

\[
\Delta^*_H(z, \gamma(t), \infty) \geq 1.
\]

**Proof.** Suppose \( I_t \) is bounded, \( H^*_t \) is bounded, and \( \Delta^*_H(z, \gamma(t), \infty) < 1 \). Then there is a crosscut \( \eta \) of \( H^*_t \) which has one end point in \( \partial H_1 \) and one end point in \( \partial H_2 \). Since \( H^*_t \) is bounded and \( \gamma(t) \in \overline{T_t} \) we may find an infinite crosscut \( \omega \) of \( H_t \) that never enters \( H^*_t \) (take a simple curve from \( \infty \) in \( H_t \) until it first hits \( I_t \) and then continue the curve along \( I_t \) to reach \( \gamma(t) \)). Since \( \eta \) and \( \omega \) do not intersect, we get a contradiction. \( \square \)

Given these simple observations, we can restrict the manner in which the various distances to the curve can be decreased.

**Lemma 4.8.** Suppose \( \gamma \) is a non-crossing curve with \( z, w \notin \gamma(0, \infty) \) and \( I_t = I_t(\mathcal{I}, z, w, \gamma) \) as above. Suppose \( t_0 \) is a time so that \( \gamma(t_0) \in \overline{T_{t_0}} \). Let \( \zeta = \inf\{t > t_0 \mid \gamma(t) \in I_{t-}\} \). Then at most one of the following holds:

- \( \Delta_{H_t,1}(z, \gamma(\zeta), \infty) < \Delta_{H_{t_0},1}(z, \gamma(t_0), \infty) \land 1 \), or
- \( \Delta_{H_t,2}(z, \gamma(\zeta), \infty) < \Delta_{H_{t_0},2}(z, \gamma(t_0), \infty) \land 1 \).

**Proof.** If \( \zeta = t_0 \), the above statement follows immediately, so we may assume \( \zeta > t_0 \). Consider the non-crossing loop \( \ell = \gamma[t_0, \zeta] \cup \mathcal{L} \) where \( \mathcal{L} \) is the line connecting \( \gamma(\zeta) \) and \( \gamma(t_0) \). Partition \( \mathbb{H} \) into two sets, the infinite
component of \( \mathbb{H} \setminus \ell \), which we will denote by \( A_\infty \), and the union of the finite components of \( \mathbb{H} \setminus \ell \) which we will denote by \( A_0 \). The point \( z \) is either in \( A_\infty \) or \( A_0 \). As the cases are similar, assume \( z \in A_\infty \). Since \( \ell \) is a non-crossing loop, we either have a curve \( \eta : [0,1] \to A_\infty \) with \( \eta(0) = z \) and \( \eta(1^-) \in \partial_1 H_\zeta \) or \( \eta(1^-) \in \partial_2 H_\zeta \), but not both. This yields that only one of the \( \Delta_{H_\zeta,j}(z, \gamma(\zeta), \infty) \) could have decreased past the minimum of 1 and its previous value.

\[ \text{Lemma 4.9.} \quad \text{Suppose } \gamma \text{ is a non-crossing curve with } z, w \notin \gamma(0, \infty) \text{ and } I_t = I_t(\mathcal{I}, z, w, \gamma) \text{ as above. Suppose } t_0 \text{ is a time so that } \gamma(t_0) \in \mathcal{T}_{t_0}, \text{ and let } \zeta = \inf \{ t > t_0 \mid \gamma(t) \in I_{t^-} \}. \supseteq \text{ Suppose for some } s < 1, \]
\[ \Delta^*_\zeta(z) \leq s < \Delta^*_{t_0}(z). \]

Then \( \Delta_{t_0}(z) \leq s \), and \( H^w_{t_0} \) and \( H^w_{\zeta} \) are bounded.

**Proof.** By the previous lemma, we have that either \( \Delta^1_\zeta(z) \geq \Delta^1_{t_0}(z) \land 1 \) or \( \Delta^2_\zeta(z) \geq \Delta^2_{t_0}(z) \land 1 \). This implies that \( \Delta^*_\zeta(z) \geq \Delta_{t_0}(z) \land 1 \), and hence \( \Delta_{t_0}(z) \land 1 \leq s \) which is the first assertion.

We now prove that \( H^w_{\zeta} \) is bounded. Assume first that both \( H^w_{\zeta} \) and \( H^z_{\zeta} \) are unbounded. Then \( I_{\zeta} \) is unbounded and by Lemma 4.6 we have that
\[ \Delta^*_H_{\zeta}(z, \gamma(t), \infty) \geq 1 \]
which is a contradiction. Thus one of \( H^w_{\zeta} \) or \( H^z_{\zeta} \) is bounded. If \( H^z_{\zeta} \) is bounded, then by Lemma 4.7 we have
\[ \Delta^*_H_{\zeta}(z, \gamma(t), \infty) \geq 1 \]
which is again a contradiction. Thus \( H^w_{\zeta} \) is bounded, as desired.

By the definition of \( \zeta \) and \( I_\zeta \), we know \( \gamma(t_0, \zeta) \) is contained in precisely one of \( H^z_{t_0} \) or \( H^w_{t_0} \). Since
\[ \Delta^*_\zeta(z) < 1 \leq \Delta^*_{t_0}(z) \]
by assumption, we know \( \gamma(t_0, \zeta) \subseteq H^w_{t_0} \). Assume that \( H^w_{t_0} \) were unbounded. Then there is a curve \( \eta \) from \( w \) to \( \infty \) contained in \( H^w_{t_0} \). Since \( H^w_{\zeta} \) is bounded \( \eta \cap \partial H^w_{\zeta} \) is non-empty. By definition,
\[ \partial H^w_{\zeta} \subseteq \gamma(0, t_0] \cup \gamma(t_0, \zeta] \cup I_\zeta. \]
We now show \( \eta \) cannot intersect any of the three sets on the right. Since \( \eta \) is in \( H^w_{t_0} \), we know \( \eta \cap (\gamma(0, t_0] \cup I_\zeta) = \emptyset \) and moreover, since \( I_\zeta \subseteq I_{t_0} \), that \( \eta \cap I_\zeta = \emptyset \). Since \( \gamma(t_0, \zeta) \subseteq H^z_{t_0} \), we know \( \eta \cap \gamma(t_0, \zeta) = \emptyset \). Thus we have a contradiction, and \( H^w_{t_0} \) must be bounded, as desired.
4.3. Main SLE estimates. We now use the above topological restrictions to help us establish the needed SLE estimates. Let $T_z$ (resp. $T_w$) denote the first time that $z$ (resp. $w$) is not in $H_t$ and let $T = T_z \wedge T_w$ denote the first time that one of $z, w$ is not in $H_t$. Note that if the curve is to approach $z$ and $w$ to within $\varepsilon$ and $\delta$ as desired, it must do so before $T_z \lor T_w$.

We also define the following recursive set of stopping times. Let $\tau_0 = 0$. Given $\tau_j < T$, define $\hat{\tau}_j$ as the infimum over times $t > \tau_j$ such that

$$\Delta_t(z) \leq \frac{1}{2} \Delta_{\tau_j}(z) \text{ or } \Delta_t(w) \leq \frac{1}{2} \Delta_{\tau_j}(w).$$

Given this, let $\tau_{j+1}$ be the infimum over times $t > \hat{\tau}_j$ such that $\gamma(t) \in \overline{T_{\tau_j}}$. These times are understood to be infinite when past $T$ and hence at least one of the points can no longer be approached by the curve. The sequence of stopping times $\{\tau_k\}_{k \geq 0}$ are called renewal times. We let $R_{k+1} = 0$ if $\tau_{k+1} < \infty$ and $\Delta_{\tau_{k+1}}(z) \leq \frac{1}{7} \Delta_{\tau_k}(z)$; in this case, we can see that $\Delta_{\tau_{k+1}}(w) \geq \frac{1}{7} \Delta_{\tau_k}(w)$. If $\tau_{k+1} < \infty$ and $\Delta_{\tau_{k+1}}(w) \leq \frac{1}{7} \Delta_{\tau_k}(w)$, we set $R_{k+1} = 1$. We set $R_{k+1} = \infty$ if $\tau_{k+1} = \infty$. Less formally, the renewal times encode when our curve halved it’s distance to either $z$ or $w$ and then returned to $I_t$, while $R_k$ specifies which point we halved the distance to. Let $\mathcal{F}_k = \mathcal{F}_{\tau_k}$.

**Lemma 4.10.** There exist $c < \infty, \alpha > 0$ such that for all $k \geq 0$, $r \leq 1/2$,

$$\mathbb{P}\{R_{k+1} = 0; \Delta_{\tau_{k+1}}(z) \leq r \Delta_{\tau_k}(z) \mid \mathcal{F}_k\} \leq c \mathbb{1}\{\tau_k < T\} \Delta_{\tau_k}(z)^{\alpha} r^{2-d}.$$

**Proof.** We assume $\tau_k < T$ and we write $\tau = \tau_k$, $\xi = \xi(z; r\Delta(z))$. First, consider the event that either $I_\tau$ is not bounded or both $I_\tau$ and $H_\tau^z$ are bounded. By Lemmas 4.6 and 4.7, we have $\Delta^*(z) \geq 1$. Thus by Lemma 2.11, we get

$$\mathbb{P}\{\xi < \infty \mid \mathcal{F}_k\} \leq c r^{2-d} \Delta_\tau(z)^{\beta/2}.$$

Suppose that $I_\tau$ is bounded and $H_\tau^z$ is bounded. We split into two cases: $\Delta^*_\tau(z) \leq \sqrt{\Delta_\tau(z)}$ and $\Delta^*_\tau(z) > \sqrt{\Delta_\tau(z)}$. If $\Delta^*_\tau(z) > \sqrt{\Delta_\tau(z)}$, then Lemma 2.11 implies

$$\mathbb{P}\{\xi < \infty \mid \mathcal{F}_k\} \leq c r^{2-d} \Delta_\tau(z)^{\beta/4}.$$

Suppose $\Delta^*_\tau(z) \leq \sqrt{\Delta_\tau(z)}$. Then there exist simple curves $\eta_1, \eta_2 : [0, 1) \to H_\xi$ contained in the disk of radius $2\Delta^*_\tau(z)$ about $z$ with $\eta(0) = z$ and $\eta(1+) \in \partial J H_\tau$. At the time $\xi$ we can consider the line segment $L$ from $\gamma(\xi)$ to $z$. There exists a crosscut of $H_\xi$, $\check{\eta}$, contained in $L \cup \eta_1$ or in $L \cup \eta_2$, one of whose endpoints is $\gamma(\xi)$, that disconnects $I_\xi$ from infinity. Using Lemma 4.4, we see that

$$\mathcal{E}_{H_\xi}(\check{\eta}, I_\xi) \leq c \Delta^*_\tau(z)^{1/2} \leq c \Delta_\tau(z)^{1/4}.$$
Thus, using Lemma 4.5 we see that
\[ P\{ \xi < \tau_{k+1} < \infty \mid F_k \} \leq c \Delta_r(z)^{\beta/4} P\{ \xi < \infty \mid F_k \} \leq c r^{2-d} \Delta_r(z)^{\beta/4}. \]

**Remark.** The proof of the last lemma was not difficult given the estimates we have derived. However, it is useful to summarize the basic idea. If \( \Delta_r(z) \) is not too small, then it suffices to estimate
\[ P\{ R_{k+1} = 0; \Delta_{\tau_{k+1}}(z) \leq r \Delta_{\tau_k}(z) \mid F_k \} \]
by
\[ P\{ \xi < \infty \mid F_k \}. \]
However, if \( \Delta_{\tau_k}(z) \) is not much bigger than \( \Delta_{\tau_k}(z) \) this estimate is not sufficient. In this case, we need to use
\[ P\{ \xi < \infty \mid F_k \} P\{ \tau_{k+1} < \infty \mid F_k, \xi < \infty \}. \]

The above argument provides a good bound on the probability that the near side gets even closer. To complete our argument, we must also provide a bound limiting the probability that the far side can get closer as well.

**Lemma 4.11.** There exists \( c < \infty \) such that for all \( k \geq 0, s \leq 1/4 \), if
\[ \xi^* = \inf\{ t > \tau_k \mid \Delta^*_t(z) \leq s \} \quad \text{and} \quad \eta^* = \inf\{ t > \xi^* \mid \gamma(t) \in I_t^- \}, \]
then
\[ P\{ \eta^* < \infty, \Delta_{\eta^*}^*(z) \leq s \mid \Delta_{\tau_k}^*(z) > s, F_{\tau_k} \} \leq cs^{\beta/2} \]

**Proof.** Assume \( \Delta_{\tau_k}^*(z) > s \). If \( \eta^* < \infty \) we may define
\[ \varpi = \sup\{ t < \eta^* \mid \gamma(t) \in I_t^- \} \]
to be the previous time that \( \gamma \) crossed \( I_{t^-} \) before \( \eta^* \). Note that \( \tau_k \leq \varpi < \xi^* < \eta^* \) and \( \Delta_{\varpi}^*(z) > s \). By considering the two times \( \varpi \) and \( \eta^* \) in Lemma 4.9 we see that \( H^w_{\gamma} \) is bounded.

Consider the situation at time \( \xi^* \). By the definition of the stopping times, there must be a curve \( \nu : (0,1) \to H_{\xi^*}^- \) which contains \( z \), is never more than distance 2s from \( z \), has \( \nu(0^+) \in \partial_1 H_{\xi^*} \) and \( \nu(1^-) \in \partial_2 H_{\xi^*} \) such that \( \nu \) separates \( I_{\xi^*}^- \) and hence \( w \) from infinity. Since \( \nu \) is at least distance 1/2 from \( I_{\xi^*}^- \) we know from Lemma 4.4 that the excursion measure between \( \nu \) and \( I_{\xi^*}^- \) in \( H_{\xi^*}^- \) is bounded above by \( Cs^{1/2} \). Then an application of Lemma 4.5 tells us that the probability of \( \gamma \) returning to \( I_{\xi^*}^- \) is bounded above by \( Cs^{3/2} \) which gives the lemma. \( \square \)
The following two lemmas combine the methods of the above two bounds.

**Lemma 4.12.** There exist \( c < \infty, \alpha > 0 \) such that for all \( k \geq 0, r \leq 1/2, s \leq 1/4, \)

\[
\mathbb{P}\left\{ R_{\tau_{k+1}} = 0; \Delta_{\tau_{k+1}} (z) \leq r \Delta_{\tau_k} (z); \Delta_{\tau_{k+1}}^* (w) \leq s \mid \mathcal{F}_k \right\} 
\leq c \{ \tau_k < T \} \Delta_{\tau_k} (z)^\alpha \left[ s^\alpha + 1 \{ \Delta_{\tau_k}^* (w) \leq s \} \right] r^{2-d}.
\]

**Proof.** If \( \Delta_{\tau_k}^* (w) \leq s \) then the desired statement reduces to Lemma 4.10. Thus, we may assume that \( \Delta_{\tau_k}^* (w) > s. \)

Let \( \zeta^* = \zeta_{\tau_k}^* \) be the infimum over times \( t > \tau_k \) so that \( \Delta_{\tau_k}^* (w) \leq s \) and \( \gamma (t) \in I_\gamma \). Let \( \sigma = \sigma_k = \inf \{ t > \tau_k \mid \Delta_{\gamma}(z) \leq r \Delta_{\tau_k} (z) \} \).

By Lemma 4.9 we know if \( \Delta_{\tau_k}^* (w) > s \) and \( \zeta^* < \infty \), there is a time \( \omega, \tau_k \leq \omega < \zeta^* \) for which there is a curve connecting \( \partial_1 H_\omega \) to \( \partial_2 H_\omega \) passing through \( \gamma (\omega) \) contained in a disk of radius \( 2s \) about \( w \) separating \( I_\omega \) from infinity. Then, by Lemma 4.5, we have that

\[
\mathbb{P}\{ \zeta^* < \infty \mid \Delta_{\tau_k}^* (z) > s, \mathcal{F}_{\tau_k} \} \leq cs^\alpha.
\]

Combining the above two bounds gives the desired result. \( \square \)

**Lemma 4.13.** There exist \( c < \infty, \alpha > 0 \) such that for all \( k \geq 0, r \leq 1/2, s \leq 1/4 \),

\[
\mathbb{P}\left\{ R_{\tau_{k+1}} = 0; \Delta_{\tau_{k+1}} (z) \leq r \Delta_{\tau_k} (z); \Delta_{\tau_{k+1}}^* (z) \leq s \mid \mathcal{F}_k \right\} 
\leq c \{ \tau_k < T \} \Delta_{\tau_k} (z)^\alpha \left[ s^\alpha + 1 \{ \Delta_{\tau_k}^* (z) \leq s \} \right] r^{2-d}.
\]

**Proof.** If \( \Delta_{\tau_k}^* (z) \leq s \) or \( s \geq 1/4 \), the conclusion reduces to Lemma 4.10. Thus we may assume that \( \Delta_{\tau_k}^* (z) > s, s \leq 1/4 \). Let \( E \) denote the event

\[
E = \left\{ R_{\tau_{k+1}} = 0; \Delta_{\tau_{k+1}} (z) \leq r \Delta_{\tau_k} (z); \Delta_{\tau_{k+1}}^* (z) \leq s; \Delta_{\tau_k}^* (z) > s \right\}.
\]

Let \( \sigma = \inf \{ t \mid \Delta_{\gamma}(z) \leq r \Delta_{\tau_k} (z) \}. \)
and note that on the event $E$,
\[
\tau_{k+1} = \inf\{t > \sigma \mid \gamma(t) \in I_t \}.
\]
Define $\xi$ to be the infimum over times $t \geq \sigma$ such that there is a curve $\eta : (0,1) \to H_\delta$ with $\eta(0^+) = \gamma(t)$ and $\eta(1^-) \in \partial H_\delta$ with $\eta$ contained entirely in the ball of radius $2s$ about $z$, and $\eta$ separating $I_t$ from $\infty$.

We now claim that given $F_\sigma$ either $\xi < \tau_{k+1}$ or $\Delta^*_{\tau_{k+1}}(z) > s$. To see this, suppose neither holds. Since $\Delta^*_{\tau_{k+1}}(z) \leq s$, for every $s < s' \leq 2s \leq 1/2$, there is a crosscut $\eta$ of $H_{\tau_{k+1}}$ going through $z$ whose endpoints are in $\partial_1 H_{\tau_{k+1}}, \partial_2 H_{\tau_{k+1}}$, respectively and which is contained in the disk of radius $s'$ about $z$. By Lemma 4.9 we know $\eta$ must disconnect $I_{\tau_{k+1}}$ from $\infty$ since $H^w_{\tau_{k+1}}$ must be bounded. We can choose such an $\eta$ such that at least one end point of $\eta$ is not in $\gamma[0,\tau_k]$, for otherwise all such $\eta$ would be a crosscuts of $H_{\tau_k}$ separating $w$ from infinity which would imply that $\Delta^*_{\tau_k}(z) \leq s$.

Let $\zeta = \sup\{t \leq \tau_{k+1} \mid \gamma(t) \in \partial_I\} > \tau_k$ and note that $\tau_k < \zeta < \tau_{k+1}$. If $\zeta \geq \sigma$ we are done since this $\eta$ demonstrates that $\xi < \tau_{k+1}$.

Thus assume $\zeta < \sigma$. In this case we will construct another curve which satisfies what we want at the time $\sigma$. Since $\zeta < \sigma$ we know the curve $\eta$ defined above disconnects $I_\sigma$ from infinity in $H_\sigma$. By the definition of $\sigma$ as the first time that $\Delta_\sigma(z) \leq s$, the straight open line segment, $L$, from $\gamma(\sigma)$ to $z$ is contained in $H_\sigma$. Additionally, since $\Delta_\sigma(z) \leq \Delta^*_\sigma(z) \leq s$ we know $\eta(0,1) \cup L$ is contained entirely in the ball of radius $2s$ about $z$. Thus we may find a curve $\tilde{\eta}$ contained in $\eta(0,1) \cup L$ which separates $I_\sigma$ from infinity in $H_\sigma$ with $\eta(0^+) = \gamma(t)$ and $\eta(1^-) \in \partial H_\delta$ and with $\tilde{\eta}$ contained entirely in the ball of radius $2s$ about $z$, proving that $\xi = \sigma < \tau_{k+1}$. Thus we have reached a contradiction.

On the event $E$ we know $\Delta^*_{\tau_{k+1}}(z) \leq s$ and thus the above argument tells us $\xi < \tau_{k+1}$. We have therefore shown that if $\Delta^*_k(z) > 2s, s \leq 1/4$, then
\[
P(E \mid F_k) \leq P\{\sigma \leq \xi < \tau_{k+1} < \infty \mid F_k\}.
\]
We may now argue as in the second part of the proof of Lemma 4.10 to obtain $P\{\sigma < \infty \mid F_k\} \leq c \Delta_{\tau_k}(z)^\alpha$ and $P\{\tau_{k+1} < \infty \mid F_\xi\} \leq c s^\alpha$.

4.4. Combinatorial estimates. We have now completed the bulk of the probabilistic estimates. Most of what remains is a combinatorial argument to sum up the bounds proven above across all possible ways that the $SLE$ curve may approach $z$ and $w$ in turn.

Without loss of generality, assume that $\delta = 2^{-m}$ and $\epsilon = 2^{-n}$ and let
\[
\xi_z = \xi_{z,\epsilon} = \inf\{t : \Delta_t(z) \leq 2^{-n}\}, \quad \xi_w = \xi_{w,\delta} = \inf\{t : \Delta_t(w) \leq 2^{-m}\},
\]
\[ \xi = \xi_z \vee \xi_w = \inf\{ t : \Delta_t(z) \leq 2^{-n}, \Delta_t(w) \leq 2^{-m} \}. \]

These are similar to \( \chi \) and \( \xi \) from the previous sections, however now the times denote the first time that the curve gets within a small Euclidean distance of the point, rather than a small conformal radius of the point. Let \( \sigma \) be the minimal \( \tau_k \) such that \( \Delta_{\tau_k}(z) \leq 2^{-n+1} \) or \( \Delta_{\tau_k}(w) \leq 2^{-m+1} \). Let \( k_\sigma \) be the index so that \( \sigma = \tau_{k_\sigma} \). If such a renewal time does not exist, let \( k_\sigma = \infty \) and \( \sigma = \infty \). Note that if \( \xi \) is finite, then so is \( \sigma \).

Let \( V_{z,k}, V_z \) denote the events (and their indicator functions)

\[ V_{z,k} = \{ k_\sigma = k, R_\sigma = 0 \}, \quad V_z = \bigcup_{k=1}^\infty V_{z,k}. \]

We define \( V_w \) analogously. By the definition of \( \sigma \), on the event the event \( V_z \),

\[ \Delta_{\tau_{k_\sigma-1}}(z) \geq 2^{-n+1}, \quad \Delta_{\tau_{k_\sigma-1}}(w) \geq 2^{-m+1}, \quad \Delta_\sigma(z) < 2^{-n+1}. \]

Also,

\[ \Delta_\sigma(w) > 2^{-m}, \]

for if \( \Delta_\sigma(w) \leq 2^{-m} \), there would have been a renewal time after \( \tau_{k_\sigma-1} \) but before \( \tau_k = \sigma \). Note that

\[ \{ \xi < \infty \} \subset [V_z \cap \{ \xi_w < \infty \}] \cup [V_w \cap \{ \xi_z < \infty \}]. \]

We will concentrate on the event \( V_z \cap \{ \xi_w < \infty \} \); similar arguments handle the event \( V_w \cap \{ \xi_z < \infty \} \).

Define the integers \((i_l, j_l)\) by stating that at the renewal time \( \tau_l \),

\[ 2^{-i_l} < \Delta_{\tau_l}(z) \leq 2^{-i_l+1}, \quad 2^{-j_l} < \Delta_{\tau_l}(w) \leq 2^{-j_l+1}. \]

If \( \sigma < \infty \), we write \((i_\sigma, j_\sigma) = (i_{k_\sigma}, j_{k_\sigma})\). On the event \( k_\sigma = k, R_\sigma = 0 \), there is a finite sequence of ordered triples

\[ \pi = [(i_0, j_0, 0), (i_1, j_1, R_1), \ldots, (i_{k-1}, j_{k-1}, R_{k-1}), (i_k, j_k, R_k) = (i_\sigma, j_\sigma, 0)], \]

\[ i_l, j_l \in \{1, 2, 3, \ldots\}, \quad R_l \in \{0, 1\}. \]

We have the following properties for \( 0 \leq l \leq k - 1 \):

- If \( R_{l+1} = 0 \), then \( i_{l+1} \geq i_l + 1 \) and \( j_l \leq j_{l+1} \leq j_l + 1 \).
- If \( R_{l+1} = 1 \), then \( i_l \leq i_{l+1} \leq i_l + 1 \) and \( j_{l+1} \geq j_l + 1 \).
We call any sequence of triples satisfying these two properties a **legal** sequence of length \( k \). For any \( i, j, k \), let \( S_k(i, j, 0) \) denote the collection of legal finite sequences of length \( k \) whose final triple is

\[(i_k, j_k, R_k) = (i, j, 0).\]

If \( \pi \) is a legal finite sequence of length \( k \), let \( V_{z,\pi} \) be the event that \( k_{\sigma} = k, R_{\sigma} = 0 \) and the renewal times up to and including \( \sigma \) give the sequence \( \pi \). Figure 5 illustrates this definition.

![Diagram](image)

**Fig 5.** A curve \( \gamma \) (shown in bold) in \( V_{z,\pi} \) with \( \pi = [(0, 0, 0), (0, 1, 1), (2, 1, 0), (3, 1, 0)] \).

Define \( K_l \) for \( 1 \leq l \leq k \) by

\[ K_l = \begin{cases} i_{l-1} & \text{if } R_l = 0, \\ j_{l-1} & \text{if } R_l = 1. \end{cases} \]

The next proposition gives the fundamental estimate.

**Proposition 4.14.** There exist \( c \) and an \( \alpha > 0 \) such that the following holds. Let \( i, j, k \) be integers and let \( \pi \in S_k(i, j, 0) \). Then

\[ \mathbb{P}[V_{z,\pi} \cap \{\xi_w < \infty\}] \leq c^k 2^{(m+n)(d-2)} e^{-\alpha(i+j-n)} \prod_{l=1}^{k} e^{-\alpha K_l}. \]
Proof. Note that on the event $V_z$ we may say by Lemma 2.11 that
\[
\mathbb{P}\{\xi_w < \infty \mid \mathcal{F}_k\} \leq c \left( \frac{2^{-j}}{\Delta^*_k(w)} \right)^{\beta/2} 2^{(m-j)(d-2)}.
\]
We will proceed by splitting the event $V_{z,\pi}$ into the case where $\Delta^*_k(w) \geq 2^{-j}$ and the case where it is not.

Before doing so, we will discuss how to estimate $\mathbb{P}[V_{z,\pi}]$ without any further conditions, as it is important to our bounds below. By the definition of $S_k(i,j,0)$ we have that
\[
\pi = [(i_0,j_0,0),(i_1,j_1,R_1),\ldots,(i_k,j_k,0)].
\]
where the sequence of triples is a legal sequence as described above.

We will estimate this probability by applying Lemma 4.10 to approximate the probability of each step, which is to say the probability that given that the SLE at time $\tau_l$ yields the triple $(i_l,j_l,R_l)$ we get the triple $(i_{l+1},j_{l+1},R_{l+1})$ at the time $\tau_{l+1}$. As the two cases are similar, we assume that $R_{l+1} = 0$. Since $K_{l+1} = i_l$, we know the distance to $z$ at time $\tau_l$ is less than $2^{-K_{l+1}}$. We wish the distance from $z$ to the SLE curve to decrease by at least a factor of $2^{-j_{l+1}}$. The probability of this is shown by Lemma 4.10 to be of the order $ce^{-\alpha K_{l+1} 2^{(d-2)(i_l - i_{l+1} + j_l - j_{l+1})}}$ as $j_{l+1}$ can be at most one greater than $j_l$.

To get the probability of $V_{z,\pi}$, we need only multiply through each of these $k$ individual probabilities to get that
\[
\mathbb{P}[V_{z,\pi}] \leq \prod_{l=1}^{k} e^{-\alpha K_l} 2^{(d-2)(i_l - i_{l-1} + j_l - j_{l-1})}
\]
\[
= c^k \exp \left\{ \log(2)(d-2) \sum_{l=1}^{k} (i_l - i_{l-1} + j_l - j_{l-1}) \right\} \prod_{l=1}^{k} e^{-\alpha K_l}
\]
\[
= c^k 2^{(d-2)(i_0 + j_0)} \prod_{l=1}^{k} e^{-\alpha K_l}
\]
where we have absorbed the $2^{(d-2)(i_0 + j_0)}$ (bounded above by a constant given the restrictions of $z$ and $w$) into the $c^k$ term in the last line by redefining $c$.

We now return to our main estimate. Note that, when $\Delta^*_k(w) \geq 2^{-j/2}$,
we have

\[ P[V_z, \pi \cap \{ \Delta_{\tau_k}^*(w) \geq 2^{-j/2}\} \cap \{ \xi_w < \infty \}] \]
\[ \leq c P[V_z, \pi \cap \{ \Delta_{\tau_k}^*(w) \geq 2^{-j/2}\}] 2^{-\beta j/4} 2^{(m-j)(d-2)} \]
\[ \leq c^k 2^{-\beta j/4} 2^{(m-j)(d-2)} \sum_{l=0}^{k-1} P[V_z, \pi \cap \{ \Delta_{\tau_l}^*(w) \geq 2^{-j/2}, \Delta_{\tau_{l+1}}^*(w) < 2^{-j/2}\}] \]
\[ \leq k c^k 2^{-\alpha j/2} 2^{(i+j)(d-2)} \prod_{l=1}^{k} e^{-\alpha K_l}. \]

where the third line follows from the above discussion and the last line holds for some choice of \( \mu > 0 \).

Thus we need only understand the event

\[ P[V_z, \pi \cap \{ \Delta_{\tau_k}^*(w) < 2^{-j/2}\} \cap \{ \xi_w < \infty \}] \]
\[ \leq c P[V_z, \pi \cap \{ \Delta_{\tau_k}^*(w) < 2^{-j/2}\}] 2^{(m-j)(d-2)}. \]

For the event \( \{ \Delta_{\tau_k}^*(w) < 2^{-j/2}\} \) there must be at least one \( l \) such that \( \Delta_{\tau_l}^*(w) \geq 2^{-j/2} \) and \( \Delta_{\tau_{l+1}}^*(w) < 2^{-j/2} \). By using Lemma 4.12 for that single step if \( R_l = 1 \) or Lemma 4.13 if \( R_l = 0 \) and 4.10 for all other steps we have that

\[ P[V_z, \pi \cap \{ \Delta_{\tau_k}^*(w) < 2^{-j/2}\}] \]
\[ \leq c^k 2^{-\beta j/4} 2^{(m-n)(d-2)} \prod_{l=1}^{k} e^{-\alpha K_l}. \]
By combining this with the above event we see that
\[
\mathbb{P}[V_{z,\pi} \cap \{ D_{\tau_k}^* (w) < 2^{-j/2} \} \cap \{ \xi_w < \infty \}] \\
\leq kc^k 2^{(m-j)(d-2)2-\alpha j/2} 2^{(i+j)(d-2)} \prod_{l=1}^k e^{-\alpha K_l} \\
\leq c^k 2^{(m+n)(d-2)2-\mu (j+i-n)} \prod_{l=1}^k e^{-\alpha K_l}.
\]
for some choice of $\mu$ and where $c$ is being used generically to absorb the leading $k$. Thus by choosing $\mu$ and $\alpha$ to be the same (which we can do by taking the minimum for both) we get the desired result.

We will now show how this proposition implies the main theorem. The proof rests upon the following combinatorial lemma.

**Lemma 4.15.** For every $\alpha > 0$, there exist $c$ and a $u > 0$ such that for all $k$
\[
\sum_{\pi \in \delta_k(i,j,0)} k \prod_{l=1}^k e^{-\alpha K_l} \leq c e^{-uk^2}.
\]

**Proof.** We fix $\alpha$ and allow all constants to depend on $\alpha$. Let
\[
\Sigma_k = \sum_{[m]_k} k \prod_{l=1}^k e^{-\alpha m_l},
\]
where the sum is over all strictly increasing finite sequences of positive integers, written as $[m]_k := [m_1, m_2, \ldots, m_k]$. We first claim that
\[
\Sigma_k \leq c_1 e^{-\alpha k^2/4}.
\]
Consider the following recursive relation:
\[
\Sigma_k = \sum_{[m]_k} \prod_{l=1}^k e^{-\alpha m_l} \\
\leq \sum_{[m]_{k-1}} \sum_{m_k = k} e^{-\alpha m_k} \prod_{l=1}^{k-1} e^{-\alpha m_l} \\
= \Sigma_{k-1} \sum_{j=k}^{\infty} e^{-\alpha j} \\
\leq c_2 \Sigma_{k-1} e^{-\alpha k}.
\]
Applying this bound inductively to \( \Sigma_k \) yields
\[
\Sigma_k \leq c_2^k \exp \left\{ -\alpha \sum_{i=1}^k i \right\} \leq c_1 e^{-\alpha k^2/4}
\]
as desired.

To choose a legal sequence in \( S_k(i,j,0) \) there are \( 2^{k-1} \) ways to choose the values \( R_1, \ldots, R_{k-1} \). Given the values of \( R_1, \ldots, R_{k-1} \) we choose the increases of the integers. If \( R_l = 0 \), then \( i_l > i_{l-1} \) and \( j_l = j_{l-1} \) or \( j_l = j_{l-1} + 1 \). The analogous inequalities hold if \( R_1 = 1 \). There are \( 2^k \) ways to choose whether \( j_l = j_{l-1} \) or \( j_l = j_{l-1} + 1 \) (or the corresponding jump for \( i_l \) if \( R_1 = 1 \)). In the other components we have to increase by an integer. We therefore get that the sum is bounded above by
\[
2^{k-1} \max_{0 \leq l \leq k-1} 2^{k-l} \Sigma_{l}^{k-l-1} \Sigma_{k-l-1} \leq c^k \max_{0 \leq l \leq k-1} e^{-\alpha l^2/4} e^{-\alpha(k-l-1)^2/4} \leq ce^{-\alpha k^2}.
\]

By combining Proposition 4.14 and Lemma 4.15, there exist \( c \) such that
\[
\sum_{k=1}^{\infty} \sum_{\pi \in S_k(i,j,0)} \mathbb{P}[V_{z,\pi} \cap \{ \xi_w < \infty \}] \leq c 2^{(m+n)(d-2)} e^{-\alpha(j+i-n)},
\]
and hence by summing over \( i \geq n - 1, j \geq 0 \) we get
\[
\mathbb{P}[V_z \cap \{ \xi < \infty \}] = \sum_{i=n-1}^{\infty} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \sum_{\pi \in S_k(i,j,0)} \mathbb{P}[V_{z,\pi} \cap \{ \xi_w < \infty \}] \leq c 2^{(m+n)(d-2)} = c \varepsilon^{2-d} \delta^{2-d}.
\]

By the symmetry of \( z, w \) we have the bound
\[
\mathbb{P}[V_w \cap \{ \xi < \infty \}] \leq c \varepsilon^{2-d} \delta^{2-d}
\]
and hence
\[
\mathbb{P}\{ \Delta_\infty(z) \leq \varepsilon, \Delta_\infty(w) \leq \delta \} = \mathbb{P}\{ \xi < \infty \} \leq c \varepsilon^{2-d} \delta^{2-d}
\]
as required to complete the proof of Proposition 4.1, and hence the proof of Beffara’s estimate.
With the proof set up in this way, we may now rapidly complete our proof of the existence of the multi-point Green’s function. By mirroring the proof above, we may conclude that for \( \rho = e^{-\ell} \) (and hence for all \( \rho \)) that

\[
P[V_z \cap \{ \xi < \infty, \Delta_\sigma(w) \leq \rho \}] \leq P[V_z \cap \{ \xi_w < \infty, \Delta_\sigma(w) \leq \rho \}]
= \sum_{i=n-1}^{\infty} \sum_{j=\ell}^{\infty} \sum_{k=1}^{\infty} \sum_{\pi \in S_k(i,j,0)} P[V_{z,\pi} \cap \{ \xi_w < \infty \}]
\leq c_2^{2(m+n)(d-2)} e^{-\alpha \ell} = c_2 e^{2-d} \beta^{2-d} \rho^\alpha.
\]

This proves Proposition 4.2, and hence completes the proof of the existence of the multi-point Green’s function.

APPENDIX A: THE EXISTENCE OF THE \( I_T \)

The aim of this appendix is to prove the existence of the separating set \( I_T \) desired above.

**Definition.** Let \( \gamma \) be a curve in the upper half plane and let \( z, w, I \) be a pair or distinct points in \( \mathbb{H} \) separated by the line \( I \). Let \( I_T = I \setminus \gamma(0,t] \). We will denote by \( I_T \) the unique open interval contained in \( I \) such that the following four properties hold. For any \( t \leq t' \) we have:

- \( I_T \) is a connected component of \( I_T \),
- the \( I_T \) are decreasing, which is to say \( I_T \subseteq I_T' \),
- \( H_T \setminus I_T \) has exactly two connected components, one containing \( z \) and one containing \( w \), and
- \( I_T = I_T' \) whenever \( \gamma(t,t'] \cap I = \emptyset \).

It may, at first glance, seem simple to define such sets inductively. However, in general the set of times that a curve \( \gamma \) crosses \( I \) may be uncountable and have no well defined notion of “the previous crossing.” To avoid this issue and show this notion is well defined, we require a few topological lemmas.

**Lemma A.1.** Let \( U \) be a connected open set in \( \mathbb{C} \) separated by a smooth simple curve \( \eta : [0,1] \to \overline{U} \). Let \( V \subset U \) be a connected open subset. Then for any points \( z, w \in V \), there exists a curve \( \xi : [0,1] \to V \) from \( z \) to \( w \) which intersects \( \eta \) a finite number of times.

**Proof.** This proof mirrors the classic proof that a connected open set is path connected. Define an equivalence relation on \( V \) where points \( z, w \in V \) are equivalent if \( z \) can be connected to \( w \) by a curve \( \xi \) which intersects \( \eta \) a
finite number of times. This can readily be shown to satisfy the requirements of an equivalence relation.

Let $V_\alpha$ denote the open connected components of $V \setminus \eta$. If $z, w$ are both in the same $V_\alpha$ then they may be connected by a curve which does not intersect $\eta$, hence each $V_\alpha$ is contained entirely in a single equivalence class.

Consider a disc, $D$, contained in $V$ centered on a point $\eta(t_0)$ for some $t_0 \in (0, 1)$ with components $V_\alpha$ and $V_\beta$ on either side of $\eta$ near this point. Since $\eta$ is smooth and simple, by choosing $D$ sufficiently small we may find a diffeomorphism $\phi$ so that $\phi(D) = \mathbb{D}$ and $\phi(\eta \cap D) = \{it : t \in (-1, 1)\}$. Connect $-1/2$ to $1/2$ by the straight line between them, which only intersects the image of $\eta$ once. Taking the image of this line under $\phi^{-1}$ gives a curve $\xi$ satisfying the conditions of the equivalence relation connecting two points, one in $V_\alpha$ and one in $V_\beta$. Thus components of $V \setminus \eta$ which are directly separated by $\eta$ are in the same equivalence class. Since $V$ is connected, the only equivalence class is $V$ itself.

Suppose $U$ is a connected open set in $\mathbb{C}$ separated by a curve $\eta : (0, 1) \to U$ into two components $U_1, U_2$ with points $z \in U_1$ and $w \in U_2$. Let $V$ be a connected subset of $U$. Define $D_V(z, w; \eta)$ to be the the set of connected components of $V \cap \eta$ which disconnects $z$ from $w$ in $V$.

**Corollary A.2.** Let $U$ be a connected open set in $\mathbb{C}$ separated by a smooth simple curve $\eta : [0, 1] \to \overline{U}$ into two components $U_1, U_2$ with $z \in U_1$ and $w \in U_2$. Let $V \subset U$ be a connected open subset containing $z$ and $w$. Then $|D_V(z, w; \eta)|$ is finite and odd.

**Proof.** To see that the number is finite, take the curve $\xi$ between $z$ and $w$ as in the above lemma and note that any $\eta_i$ which separates $z$ from $w$ must intersect $\xi$.

To see that it is odd, consider the connected components of $V' := V \setminus \bigcup_{\gamma \in D_V(z, w; \eta)} \gamma$. There are exactly $|D_V(z, w; \eta)| + 1$ such components. $\eta$ separates $U$ into two components, and hence the components of $V'$ are alternately contained in $U_1$ and $U_2$. Since the component containing $z$ is in $U_1$ and the component containing $w$ is in $U_2$ there must be an even number of components of $V'$, which makes $|D_V(z, w; \eta)|$ odd.

This general topological lemma has the following consequence in our setting. To simplify notation, we will define $D_t = D_{H_t}(z, w, \mathcal{I})$.

**Corollary A.3.** Fix $0 \leq t' \leq t < \infty$. Then a connected component $I$ of $\mathcal{I}_t$ separates $z$ from $w$ in $H_{t'}$ if and only if the number of elements of $D_t$ contained in $I$ is odd.
Proof. The ‘only if’ direction is precisely Corollary A.2. Thus we wish to show that if the number of elements of $D_t$ contained in $I$ is odd then $I$ separates $z$ from $w$.

Assume not, so the number of elements of $D_t$ contained in $I$ is odd but $I$ does not separate $z$ from $w$. $H_t \setminus I$ has two components, one of which contains both $z$ and $w$. Consider any curve $\eta$ connecting $z$ to $w$. Without loss of generality assume that $\eta$ crosses each element of $D_t$ exactly once by simply removing any portion of the curve between the first and last times that it crosses each element of $D_t$. Since $\eta$ crosses each element of $D_t$ contained in $I$ precisely once we know $\eta$ crosses $I$ an odd number of times and hence it must start and end in different components of $H_t \setminus I$ which contradicts the fact that it connects $z$ to $w$. \[ \square \]

We may now use this to prove that $I_t$ is well defined.

Proof of well-definedness of $I_t$. For a component $I$ of $\mathcal{I}_t$ and $t' < t$ let $C_{t'}(I)$ denote the component of $\mathcal{I}_{t'}$ which contains $I$. We claim there exists a unique component of $\mathcal{I}_t$, which we will denote $I_t$, such that for all $0 \leq t' \leq t$ we have $C_{t'}(I_t) \in D_{t'}$. Note that such an $I_t$ clearly satisfies all the conditions of the definition.

First we prove existence. Let $\{J_i\}_{i=1}^\infty$ be the connected components of $\mathcal{I}_t$. Assume that none satisfy the above condition, which is to say that for each $i$ there exists a $t_i \leq t$ so that $C_{t_i}(J_i)$ does not separate $z$ from $w$ in $H_{t_i}$. Now $\{C_{t_i}(J_i)\}_{i=1}^\infty$ covers $\mathcal{I}_t$ since the $J_i$ did as well, and moreover since by construction the $C_{t_i}(J_i)$ are either contained in each other or disjoint we may find a sub-collection $\{C_{t_{i_k}}(J_{i_k})\}_{k=1}^\infty$ which covers $\mathcal{I}_t$ with all elements pairwise disjoint. By Corollary A.3 there are an even number of elements of $D_t$ contained in $C_{t_{i_k}}(J_{i_k})$ for each $k$. However, since they cover disjointly, this implies that $|D_t|$ is even, which contradicts Corollary A.2 completing the proof of existence.

Now we establish uniqueness. Let $I_t^{(1)}, I_t^{(2)}, \ldots I_t^{(\ell)}$ denote the components of $\mathcal{I}_t$ such that for all $0 \leq t' \leq t$ we have $C_{t'}(I_t^{(i)}) \in D_{t'}$, and assume that $\ell > 1$. Define

$$t_0 = \sup \{ t' : \exists i \neq j \text{ s.t. } C_{t'}(I_t^{(i)}) = C_{t'}(I_t^{(j)}) \}.$$ 

By this definition, it is clear that $\gamma(t_0) \in \mathcal{I}$. Moreover, there exists a $t_1 < t_0$ such that $\gamma(t_1, t_0) \cap \mathcal{I} = \emptyset$ since if there did not then $\gamma(t_0)$ is a limit point of $\gamma(0, t_0) \cap \mathcal{I}$ which implies that an earlier time would have separated all the $I_t^{(i)}$ from each other contradicting the choice of $t_0$. The components of $\mathcal{I}_{t_0}$ are precisely those of $\mathcal{I}_{t_1}$ except for a single component, call it $J$, which is
split into $J_1, J_2$ in $I_{t_0}$ by $\gamma(t_0)$. By the choice of $t_0$, $J$ is $C_{t_0}(I^{(i)}_t)$ for some $i$ and both of $J_1, J_2$ are $C_{t_0}(I^{(i)}_t)$ for some $i$. This is a contradiction since by Corollary A.3 each of $J, J_1, J_2$ must contain an odd number of elements of $\mathcal{D}_t$. \hfill \Box$

APPENDIX B: THE PDE FOR THE GREEN’S FUNCTION

We outline here the derivation of the PDE which governs the ordered version of the multi-point Green’s function. From 1, we know that for $z, w \in \mathbb{H}$ with

\[ \xi = \xi_\varepsilon = \xi_{z, \varepsilon} = \inf \{ t : \Upsilon_t(z) \leq \varepsilon \}, \]
\[ \chi = \chi_\delta = \chi_{w, \delta} = \inf \{ t : \Upsilon_t(w) \leq \delta \}, \]

we have that

\[ G_{\mathbb{H}}(z, w; 0, \infty) = \frac{1}{c_2} \lim_{\varepsilon, \delta \to 0^+} \varepsilon^{d-2} \delta^{d-2} \mathbb{P} \{ \xi < \chi < \infty \}. \]

By the domain Markov property, and conformal invariance of $SLE$, one can deduce that

\[ M_t := \mathbb{E}[G_{\mathbb{H}}(z, w; 0, \infty) \mid \mathcal{F}_t] \]
\[ = G_{H_t}(z, w; 0, \infty) \]
\[ = |Z_t'(z)|^{2-d} |Z_t'(w)|^{2-d} \cdot G_{\mathbb{H}}(Z_t(z), Z_t(w); 0, \infty) \]

is a local martingale, where $Z_t$ is the unique conformal map defined by equation 5 which maps $H_t$ to $\mathbb{H}$, sending $\gamma(t)$ to 0. We will find the SDE which $M_t$ satisfies and use that the drift must zero to find the differential equation that $G(x_1, y_1, x_1, y_2) := G_{\mathbb{H}}(x_1 + iy_1, x_2 + iy_2; 0, \infty)$ must satisfy.

From equation 5, we know that

\[ dZ_t(z) = \frac{a}{Z_t(z)} \frac{X_t(z)}{Z_t(z)} dt + dB_t, \]

and hence, letting $Z_t(z) = X_t(z) + iY_t(z)$, we see that

\[ dX_t(z) = \frac{a X_t(z)}{X_t(z)^2 + Y_t(z)^2} dt + dB_t \]
\[ dY_t(z) = \frac{a Y_t(z)}{X_t(z)^2 + Y_t(z)^2} dt \]

To compute the SDE for $|Z_t'(z)|$, we must use the logarithm. First note that

\[ dZ_t'(z) = -\frac{a Z_t'(z)}{Z_t(z)^2} dt \]
and hence that
\[ d[\log Z_i'(z)] = \frac{dZ_i'(z)}{Z_i'(z)} = -\frac{a}{Z_i(z)^2} dt. \]

We may thus recover the norm of the absolute value by consider the real part, yielding
\[ d|Z_i'(z)|^{2-d} = a(d - 2)|Z_i'(z)|^{2-d} \frac{X_i(z)^2 - Y_i(z)^2}{(X_i(z)^2 + Y_i(z)^2)^2} dt. \]

From these, we may compute the equation for \( M_t \). Note that only \( X_i(z) \) and \( X_i(w) \) have non-zero diffusion coefficients. Surpressing the arguments of \( G \) in the notation, We obtain the following
\[
dM_t = M_t \left[ a(d - 2) \frac{X_i(z)^2 - Y_i(z)^2}{(X_i(z)^2 + Y_i(z)^2)^2} + a(d - 2) \frac{X_i(w)^2 - Y_i(w)^2}{(X_i(w)^2 + Y_i(w)^2)^2} \right.
+ \frac{aX_i(z)}{X_i(z)^2 + Y_i(z)^2} \frac{\partial x_1 G}{G} + \frac{aX_i(w)}{X_i(w)^2 + Y_i(w)^2} \frac{\partial x_2 G}{G} - \frac{aY_i(z)}{X_i(z)^2 + Y_i(z)^2} \frac{\partial y_1 G}{G} - \frac{aY_i(w)}{X_i(w)^2 + Y_i(w)^2} \frac{\partial y_2 G}{G}
+ \frac{1}{2} \frac{\partial x_1 x_1 G}{G} + \frac{1}{2} \frac{\partial x_2 x_2 G}{G} + \frac{\partial x_1 x_2 G}{G} \right] dt + M_t \left[ \frac{\partial x_1 G}{G} + \frac{\partial x_2 G}{G} \right] dB_t
\]

Collecting together the drift terms and specializing to \( t = 0 \) yields
\[
0 = a(d - 2) \frac{x_1^2 - y_1^2}{(x_1^2 + y_1^2)^2} G + a(d - 2) \frac{x_2^2 - y_2^2}{(x_2^2 + y_2^2)^2} G
+ a \frac{x_1 \partial x_1 G - y_1 \partial y_1 G}{x_1^2 + y_1^2} + a \frac{x_2 \partial x_2 G - y_2 \partial y_2 G}{x_2^2 + y_2^2}
+ \frac{1}{2} \partial x_1 x_1 G + \frac{1}{2} \partial x_2 x_2 G + \partial x_1 x_2 G.
\]

This PDE has a particularly nice structure. Let
\[
L_i = a(d - 2) \frac{x_i^2 - y_i^2}{(x_i^2 + y_i^2)^2} + a \frac{x_i \partial x_i G - y_i \partial y_i G}{x_i^2 + y_i^2} + \frac{1}{2} \partial x_i x_i.
\]

This can be seen to be precisely the differential operator which arises in the computation of the single point Green’s function, but now we have a copy for both \( z \) and \( w \). With this we can rewrite the equation for the multi-point Green’s function as
\[
(L_1 + L_2 + \partial x_1 x_2)G = 0.
\]
Given this simple form it may be reasonable to look for solutions which are, in some sense, asymptotically $G(z)G(w)$. Additionally, it is worth noting that this extends to arbitrary $n$-point Green’s functions by

$$\left[\sum_{i=1}^{n} L_i + \sum_{1 \leq i < j \leq n} \partial_{x_i, x_j}\right] G = 0$$

as one might expect.

The boundary conditions of this equation are not clear, and their determination may provide bounds of intrinsic interest.

The above equation shows the PDE in its most symmetric form, however in order to find an explicit solution, it may be useful to exploit the scaling rule for the Green’s function to reduce this to an equation for a function of three real variables. There is no unique way to do so, and no such reductions have lead to a particularly simple equation. A reasonable example would be to scale the above equation so that $y_1 = 1$ in which case we can find a three real variable function $\hat{G}$ so that

$$G(x_1, y_1, x_2, y_2) = y_1^{2(d-2)} \hat{G}\left(\frac{x_1}{y_1}, \frac{x_2}{y_1}, \frac{y_2}{y_1}\right).$$

From this the PDE can be derived, however the result is not illuminating.

ACKNOWLEDGEMENTS

We would like to thank the anonymous reviewer of this paper for a very careful reading and for pointing out a number of errors in an earlier draft of this paper. We would also like to thank Mohammad Rezaei for helpful comments.

REFERENCES

[1] Alberts, T. and Kozdron, M. (2008). Intersection probabilities for a chordal SLE path and a semicircle. Electron. Commun. Probab. 13 448–460.
[2] Bass, R. (1995). Probabilistic techniques in analysis. Probability and its Applications (New York). Springer-Verlag, New York.
[3] Beffara, V. (2008). The dimension of the SLE curves. Ann. Probab. 36 1421–1452.
[4] Karatzas, I. and Shreve, S. (1991). Brownian motion and stochastic calculus, Second ed. Graduate Texts in Mathematics 113. Springer-Verlag, New York.
[5] Lawler, G. (2005). Conformally invariant processes in the plane. Mathematical Surveys and Monographs 114. American Mathematical Society, Providence, RI.
[6] Lawler, G. (2009). Schramm-Loewner evolution (SLE). In Statistical mechanics. IAS/Park City Math. Ser. 16 231–295. Amer. Math. Soc., Providence, RI.
[7] Lawler, G. (2010). Fractal and multifractal properties of SLE. preprint.
[8] Lawler, G. (2011). Continuity of radial and two-sided radial SLE at the terminal point. arXiv: 1104.1620.
[9] Lawler, G., Schramm, O. and Werner, W. (2004). Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.* **32** 939–995.
[10] Lawler, G. and Zhou, W. (2010). SLE curves and natural parametrization. arXiv:1006.4936.
[11] Rohde, S. and Schramm, O. (2005). Basic properties of SLE. *Ann. of Math. (2)* **161** 883–924.
[12] Schramm, O. (2000). Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.* **118** 221–288.
[13] Schramm, O. and Sheffield, S. (2009). Contour lines of the two-dimensional discrete Gaussian free field. *Acta Math.* **202** 21–137.
[14] Smirnov, S. (2001). Critical percolation in the plane: conformal invariance, Cardy’s formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.* **333** 239–244.
[15] Werner, W. (2004). Random planar curves and Schramm-Loewner evolutions. In *Lectures on probability theory and statistics. Lecture Notes in Math.* **1840** 107–195. Springer, Berlin.

Department of Mathematics
University of Chicago
5734 University Avenue
Chicago, IL 60637-1546
E-mail: lawler@math.uchicago.edu
bwerness@math.uchicago.edu