DIAMETER ESTIMATION OF GRADIENT $\rho$-EINSTEIN SOLITONS

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Abstract. Our aim in this article is to give a lower bound of the diameter of a compact gradient $\rho$-Einstein soliton satisfying some given conditions. We have also deduced some conditions of the gradient $\rho$-Einstein soliton with bounded Ricci curvature to become non-shrinking and non-expanding. Further, we have proved that a complete non-compact gradient shrinking or expanding Schouten soliton with non-constant potential and a boundedness condition on scalar curvature must be non-parabolic.

1. Introduction and preliminaries

The Ricci flow introduced by Hamilton [9] plays a significant role in Perelman’s proof of the Poincaré conjecture and currently it has been intensively used in the study of various geometric properties. For study on Ricci flow, see [5].

An important aspect in the investigation of the Ricci flow is the study of Ricci solitons. A gradient Ricci soliton is an $n (\geq 2)$-dimensional Riemannian manifold $(M, g)$ with Riemannian metric $g$, satisfying

\[ \text{Ric} + \nabla^2 f = \lambda g, \]

where $\nabla^2 f$ stands for the Hessian of $f \in C^\infty(M)$, the ring of smooth functions on $M$, $\text{Ric}$ is the Ricci curvature tensor and $\lambda \in \mathbb{R}$. A Ricci soliton $(M, g)$ is called expanding if $\lambda < 0$, steady if $\lambda = 0$ and shrinking if $\lambda > 0$. For some results of Ricci solitons see [12, 13, 18]. In general, it is natural to consider geometric flows of the following type on a $n (\geq 3)$-dimensional Riemannian manifold $(M, g)$:

\[ \frac{\partial g}{\partial t} = -2(\text{Ric} - \rho Rg), \]

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where $R$ denotes the scalar curvature of the metric $g$ and $\rho \in \mathbb{R}\setminus\{0\}$. The parabolic theory for these flows was developed by Catino et. al. [3], which was first considered by Bourguignon [2]. They called such a flow as Ricci-Bourguignon flows. they defined the following notion of $\rho$-Einstein solitons.

**Definition 1.1.** Let $(M, g)$ be a Riemannian manifold of dimension $n (\geq 3)$, and let $\rho \in \mathbb{R}$, $\rho \neq 0$. Then $M$ is called a $\rho$-Einstein soliton if there is a smooth vector field $X$ such that

\[
(2) \quad \text{Ric} + \frac{1}{2}L_X g - \rho Rg = \lambda g,
\]

where $L_X g$ represents the Lie derivative of $g$ in the direction of the vector field $X$.

If there exists a smooth function $f : M \to \mathbb{R}$ such that $X = \nabla f$ then the $\rho$-Einstein soliton is called a gradient $\rho$-Einstein soliton, denoted by $(M, g, f)$ and in this case $(2)$ takes the form

\[
(3) \quad \text{Ric} + \nabla^2 f - \rho Rg = \lambda g.
\]

The function $f$ is called a $\rho$-Einstein potential of the gradient $\rho$-Einstein soliton. As usual, a $\rho$-Einstein soliton is called steady for $\lambda = 0$, shrinking for $\lambda > 0$ and expanding for $\lambda < 0$. After rescaling the metric $g$ we may assume that $\lambda \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$. For more study on $\rho$-Einstein solitons, we refer to the interested reader [4, 16, 17] and also the references therein.

For particular value of the parameter $\rho$, a $\rho$-Einstein soliton is called

i) gradient Einstein soliton if $\rho = \frac{1}{2}$,

ii) gradient traceless Ricci soliton if $\rho = \frac{1}{n}$,

iii) gradient Schouten soliton if $\rho = \frac{1}{2(n-1)}$.

Taking trace of $(3)$ we obtain

\[
(4) \quad R + \Delta f - n\rho R = n\lambda.
\]

Thus for gradient Schouten soliton, $(4)$ takes the form

\[
(5) \quad R + \Delta f - \frac{nR}{2(n-1)} = n\lambda.
\]
The diameter estimation of Ricci soliton is an abuzz topic of research. One of the first result came from the work of Myers [15]. In particular, he showed that a complete $n$-dimensional Riemannian manifold $(M, g)$ with Ricci curvature satisfying $Ric \geq \lambda g$ for some positive constant $\lambda$ is compact with the diameter $diam(M)$ having an upper bound $\pi \sqrt{(n-1)/\lambda}$. After that many authors have investigated to find a bound for a manifold satisfying some curvature flow conditions, for example see [6, 7, 11]. For the complete literature in this topic see the survey article [19].

The paper is organized as follows: In the section 1, we have deduced a lower bound of the gradient $\rho$-Einstein soliton satisfying some curvature conditions. We have also found some conditions of the gradient $\rho$-Einstein soliton for being non-shrinking and non-expanding. Finally, section 2 deals with the non-parabolic behavior of Schouten soliton.

2. Diameter estimation

**Theorem 2.1.** Let $(M, g, f)$ be a complete non-compact gradient $\rho$-Einstein soliton with bounded Ricci curvature, i.e., $|Ric| \leq c$ for some constant $c$, $\rho R \geq c_1 \lambda$ for some constant $c_1$ and $\lim_{s_0 \to \infty} \int_0^{s_0} \nabla^2 f(X, X)$ is finite. Then the $\rho$-Einstein soliton is non-shrinking if $(1 + c_1) > 0$ and non-expanding if $(1 + c_1) < 0$.

*Proof.* Let us consider a length minimizing normal geodesic $\gamma : [0, s_0] \to M$ for some positive, arbitrarily large $s_0$. Take $p = \gamma(0)$ and $X(s) = \gamma'(s)$ for $s > 0$. Then $X$ is the unit tangent vector along $\gamma$. Now integrating (3) along $\gamma$, we get

$$\int_0^{s_0} Ric(X, X) = \int_0^{s_0} (\lambda + \rho R)g(X, X) - \int_0^{s_0} \nabla^2 f(X, X)$$

$$\geq \lambda(1 + c_1)s_0 - \int_0^{s_0} \nabla^2 f(X, X).$$

(6)

Again, the second variation of arc length implies that

$$\int_0^{s_0} \psi^2 Ric(X, X) \leq (n - 1) \int_0^{s_0} |\psi'(s)|^2 ds,$$

(7)
for every non-negative function \( \psi \) defined on \([0, s_0]\) with \( \psi(0) = \psi(s_0) = 0 \). We now choose the function \( \psi \) as follows:

\[
\psi(s) = \begin{cases} 
  s & s \in [0, 1] \\
  1 & s \in [1, s_0 - 1] \\
  s_0 - s & s \in [s_0 - 1, s_0].
\end{cases}
\]

Then, we have

\[
2(n - 1) + \sup_{B(p,1)} |\text{Ric}| + \sup_{B(\gamma(s_0),1)} |\text{Ric}| \geq (n - 1) \int_{0}^{s_0} |\psi'(s)|^2 ds + \int_{0}^{s_0} (1 - \psi^2)\text{Ric}(X,X) ds \\
\geq \int_{0}^{s_0} \psi^2 \text{Ric}(X,X) ds + \int_{0}^{s_0} (1 - \psi^2)\text{Ric}(X,X) ds \\
= \int_{0}^{s_0} \text{Ric}(X,X) ds.
\]

Combining (6) and (8), we obtain

\[
\lambda(1 + c_1)s_0 - \int_{0}^{s_0} \nabla^2 f(X,X) \leq 2(n - 1) + \sup_{B(p,1)} |\text{Ric}| + \sup_{B(\gamma(s_0),1)} |\text{Ric}| \\
\leq 2(n - 1) + 2c.
\]

(9)

Therefore, taking limit as \( s_0 \to \infty \) on both sides of (9), we can write

\[
\lim_{s_0 \to \infty} \lambda(1 + c_1)s_0 - \lim_{s_0 \to \infty} \int_{0}^{s_0} \nabla^2 f(X,X) \leq 2(n - 1) + 2c.
\]

(10)

Now since \( \lim_{s_0 \to \infty} \int_{0}^{s_0} \nabla^2 f(X,X) \) is finite, hence, if \((1 + c_1) > 0\) and \(\lambda > 0\), then \(\lim_{s_0 \to \infty} \lambda(1 + c_1)s_0 = +\infty\), which contradicts the inequality (10). Thus \(\lambda \leq 0\), i.e., the \(\rho\)-Einstein soliton is non-shrinking. In a similar way we can show that if \((1 + c_1) < 0\) then \(\lambda \geq 0\), i.e., the \(\rho\)-Einstein soliton is non-expanding. \(\square\)

We know that for a non-trivial concave function \( f \in C^\infty(M) \), the function \((-f)\) is non-constant convex, also it implies that \( M \) is non-compact and \( \lim_{s_0 \to \infty} \int_{0}^{s_0} \nabla^2 f(X,X) \leq 0 \). Thus from the above Theorem 2.1 we can write the following corollary:

**Corollary 2.1.1.** Let \((M, g, f)\) be a complete gradient \(\rho\)-Einstein soliton with bounded Ricci curvature, i.e., \(|\text{Ric}| \leq c\) for some constant \(c\), \(\rho R \geq c_1 \lambda\) for some constant \(c_1\) and \(f\) is a non-constant concave function. Then the \(\rho\)-Einstein soliton is non-shrinking if \((1 + c_1) > 0\) and non-expanding if \((1 + c_1) < 0\).
Theorem 2.2. Let \((M, g, f)\) be a compact gradient \(\rho\)-Einstein soliton with \(c_2 g \leq \text{Ric} \leq c_3 g\).

Then for \(\rho > 0\),

\[
diam(M) \geq \max \left\{ \sqrt{\frac{2(f_{\text{max}} - f_{\text{min}})}{\lambda + n\rho c_3 - c_2}}, \sqrt{\frac{2(f_{\text{max}} - f_{\text{min}})}{c_3 - \lambda - n\rho c_2}}, \sqrt{\frac{8(f_{\text{max}} - f_{\text{min}})}{(np + 1)(c_3 - c_2)}} \right\},
\]

and for \(\rho < 0\),

\[
diam(M) \geq \max \left\{ \sqrt{\frac{2(f_{\text{max}} - f_{\text{min}})}{\lambda + n\rho c_2 - c_2}}, \sqrt{\frac{2(f_{\text{max}} - f_{\text{min}})}{c_3 - \lambda - n\rho c_3}}, \sqrt{\frac{8(f_{\text{max}} - f_{\text{min}})}{(np - 1)(c_2 - c_3)}} \right\},
\]

where the numbers \(c_2, c_3\) are denoted by

\[
c_2 = \inf_{x \in M} \{ \text{Ric}(v, v) : v \in T_x M, g(v, v) = 1 \},
\]

\[
c_3 = \sup_{x \in M} \{ \text{Ric}(v, v) : v \in T_x M, g(v, v) = 1 \}.
\]

Proof. Taking trace of \(c_2 g \leq \text{Ric} \leq c_3 g\), we obtain

\[
nc_2 \leq \text{Ric} \leq nc_3.
\]

As \(\rho > 0\), the above inequality yields

\[
n\rho c_2 \leq \rho \text{Ric} \leq n\rho c_3.
\]

The potential function \(f\) has at least one point \(p\) where it attains its global minimum value, as \(M\) is compact. Let \(\gamma\) be a geodesic with \(\gamma(0) = p\). Then using (11) and (12) we calculate

\[
g(\nabla f, \gamma')(\gamma(s)) = g(\nabla f, \gamma')(\gamma(s)) - g(\nabla f, \gamma')(\gamma(0))
\]

\[
= \int_0^s \frac{\partial}{\partial s} g(\nabla f, \gamma')(\gamma(s)) ds
\]

\[
= \int_0^s \nabla \gamma^* g(\nabla f, \gamma')(\gamma(s)) ds
\]

\[
= \int_0^s \nabla^2 f(\gamma', \gamma')(\gamma(s)) ds
\]

\[
\leq (\lambda + n\rho c_3 - c_2)s.
\]

Integrating (13) we get

\[
f(\gamma(s)) - f(p) \leq \frac{(\lambda + n\rho c_3 - c_2)s^2}{2}.
\]
Since for every point \( x \in M \) there exists a minimizing geodesic joining \( p \) and \( x \), for all \( x \in M \) we have

\[
(14) \quad f(x) - f(p) \leq \left( \frac{\lambda + npc_3 - c_2}{2} \right) d^2(x, p),
\]

where \( d(x, p) \) is the distance between \( x \) and \( p \).

In particular, we obtain

\[
f_{\text{max}} - f_{\text{min}} \leq \left( \frac{\lambda + npc_3 - c_2}{2} \right) d^2,
\]

where \( d = \text{diam}(M) \), is the diameter of the manifold \( M \). This gives

\[
d^2 \geq \left( \frac{2(f_{\text{max}} - f_{\text{min}})}{\lambda + npc_3 - c_2} \right) s.
\]

Now we consider a point \( q \) at which \( f \) attains its global maximum. Let \( \gamma \) be a geodesic with \( \gamma(0) = q \). Then

\[
\begin{align*}
g(\nabla f, \gamma')(&\gamma(s)) = g(\nabla f, \gamma')(&\gamma(s)) - g(\nabla f, \gamma')(&\gamma(0)) \\
&= \int_0^s \frac{\partial}{\partial s} g(\nabla f, \gamma')(&\gamma(s)) ds \\
&= \int_0^s \nabla_{\gamma'} g(\nabla f, \gamma')(&\gamma(s)) ds \\
&= \int_0^s \nabla^2 f(\gamma', \gamma')(\gamma(s)) ds \\
&\geq \left( \lambda + npc_2 - c_3 \right) s.
\end{align*}
\]

Again integrating (15) we get

\[
f(\gamma(s)) - f(q) \geq \frac{(\lambda + npc_2 - c_3)}{2} s^2.
\]

Since for every point \( x \in M \) there exists a minimizing geodesic joining \( q \) and \( x \), for all \( x \in M \) we have

\[
f(x) - f(q) \geq \left( \frac{\lambda + npc_2 - c_3}{2} \right) d^2(x, q),
\]

where \( d(x, q) \) is the distance between \( x \) and \( q \).

This implies that

\[
f(q) - f(x) \leq \left( \frac{c_3 - \lambda - npc_2}{2} \right) d^2(x, q).
\]
In particular, we obtain
\[ f_{\text{max}} - f_{\text{min}} \leq \left( \frac{c_3 - \lambda - n\rho c_2}{2} \right) d^2, \]
which yields
\[ d^2 \geq \left( \frac{2(f_{\text{max}} - f_{\text{min}})}{c_3 - \lambda - n\rho c_2} \right). \]
Finally, adding (14) and (16) for \( x \) such that \( d(x, p) = d(x, q) \leq \frac{d}{2} \), we get
\[ f(q) - f(p) \leq \left( \frac{c_3 - \lambda - n\rho c_2}{2} \right) d^2(x, q) + \left( \frac{\lambda + n\rho c_3 - c_2}{2} \right) d^2(x, p) \leq \frac{(n\rho + 1)(c_3 - c_2)}{8} d^2. \]
This implies
\[ d^2 \geq \frac{8(f_{\text{max}} - f_{\text{min}})}{(n\rho + 1)(c_3 - c_2)}. \]
This proves the first part. For the second part, \( \rho < 0 \), the equation (11) implies that
\[ n\rho c_3 \leq \rho R \leq n\rho c_2, \]
and hence proceeding in a similar way as in the first case, we obtain the second part.  \( \square \)

3. Schouten solitons

A Riemannian manifold \( M \) is parabolic if every subharmonic function \( u \) on \( M \) with \( u^* = \sup_M u < \infty \), must be constant \([8, 20]\), equivalently, if every positive superharmonic function \( u \) on \( M \) is constant. Otherwise \( M \) is said to be non-parabolic. The Green function \( G(x, y) \) on \( M \) is defined by (see, \([8]\))
\[ G(x, y) = \frac{1}{2} \int_0^\infty k(t, x, y) dt, \]
where \( k(t, x, y) \) is the heat kernel of \( M \). If \( p \) is a fixed point on \( M \) and \( M \) is non-parabolic then there is a unique, minimal, positive Green function and is denoted by \( G(p, x) \). The function \( l(x) \) is defined by \( l(x) = [n(n - 2)\omega_n \cdot G(p, x)]^{\frac{1}{n-1}} \), where \( \omega_n \) is the volume of the unit ball in the \( n \) dimensional Euclidean space \( \mathbb{R}^n \). Also the asymptotic volume ratio of \( M \) is defined as
\[ V_M = \lim_{r \to \infty} \frac{\text{Vol}(B_r(p))}{\omega_n r^n}, \]
where \( B_r(p) \) is the open ball with radius \( r \) and center at \( p \). For more details see, \([20]\) and also references therein.

In this section first we state one theorem and two lemmas from \([1]\) and \([20]\), which will be used to prove our results:
Theorem 3.1. [1] Let \((M, g, f)\) be a complete non-compact non-steady Schouten soliton such that the potential function \(f\) is not-constant. Then for \(\lambda > 0\) (resp., \(\lambda < 0\)), \(f\) attains a global minimum (resp., maximum) and also \(f\) is unbounded above (resp., below). Furthermore,

\[
0 \leq \lambda R \leq 2(n - 1)\lambda^2,
\]

\[
2\lambda(f - f_0) \leq |\nabla f| \leq 4\lambda(f - f_0),
\]

with \(f_0 = \min_{p \in M} f(p)\), if \(\lambda > 0\) (resp., \(f_0 = \max_{p \in M} f(p)\), if \(\lambda < 0\)).

Lemma 3.2. [20] If \((M, g)\) is an \(n\)-dimensional complete and non-compact Riemannian manifold such that it is not-parabolic with non-negative Ricci curvature, then

\[
\lim_{r \to \infty} \frac{\int_{l \leq r} |\nabla l|^3}{r^n} = (V_M)^{\frac{1}{n}}\omega_n.
\]

Lemma 3.3. [20] If \((M, g)\) is an \(n(\geq 3)\)-dimensional complete Riemannian manifold with non-negative Ricci curvature and the volume growth is maximal (resp., not maximal), then \(|\nabla l| \leq 1\) (resp., \(\lim_{r \to \infty} \sup_{t(x) = r} |\nabla l|(x) = 0\)).

Theorem 3.4. Let \((M, g, f)\) be a complete non-compact gradient shrinking Schouten soliton of dimension \(n(> 4)\) with \(R \leq k < \frac{(n-1)(n-4)}{n-2}\) for some real constant \(k\) and the potential function \(f\) is positive non-constant. Then all the ends of \(M\) are non-parabolic.

Proof. For \(a = \frac{n-4}{4} - \frac{k(n-2)}{4(n-1)} > 0\), using the equation (5) and the Theorem 3.1 we calculate

\[
\Delta f^{-a} = -af^{-a-1}\Delta f + a(a+1)f^{-a-2}|\nabla f|^2
\]

\[
\leq -a\left\{ \frac{n}{2} + \frac{nR}{2(n-1)} - R \right\} f^{-a-1} + a(a+1)\{2(f - f_0)\} f^{-a-2}
\]

\[
= \left\{ -a\left( \frac{n}{2} + \frac{nR}{2(n-1)} - R \right) + 2a(a+1) \right\} f^{-a-1} - 2a(a+1)f_0f^{-a-2}
\]

\[
\leq \left\{ -a\left( \frac{n}{2} + \frac{nR}{2(n-1)} - R \right) + 2a(a+1) \right\} f^{-a-1}
\]

\[
\leq a\left\{ \frac{k(n-2)}{2(n-1)} - \frac{n}{2} + 2(a+1) \right\} f^{-a-1} = 0.
\]

Hence it follows that \(f^{-a}\) is a positive superharmonic function which converges to zero at infinity. This proves that (see, [8]) any end of \(M\) and hence \(M\) is non-parabolic. \(\square\)
The following corollaries immediately follows from Lemma 3.2, Lemma 3.3 and Theorem 3.4.

**Corollary 3.4.1.** Let \((M, g, f)\) be a complete non-compact gradient shrinking Schouten soliton of dimension \(n(> 4)\) with \(R \leq k < \frac{(n-1)(n-4)}{n-2}\) for some real constant \(k\), \(\text{Ric} \geq 0\) and the potential function \(f\) is non-constant with \(\min_{p \in M} f(p) = f_0 \geq 0\). Then the following relation holds:

\[
\lim_{r \to \infty} \frac{\int_{l \leq r} |\nabla l|^3}{r^n} = (V_M)^{-\frac{1}{n-2}} \omega_n.
\]

**Corollary 3.4.2.** Let \((M, g, f)\) be a complete non-compact gradient shrinking Schouten soliton of dimension \(n(> 4)\) with not maximal volume growth, \(R \leq k < \frac{(n-1)(n-4)}{n-2}\) for some real constant \(k\), \(\text{Ric} \geq 0\) and the potential function \(f\) is non-constant with \(\min_{p \in M} f(p) = f_0 \geq 0\). Then \(\lim_{r \to \infty} \sup_{t(x) = r} |\nabla l|(x) = 0\). Furthermore, if it has maximal volume growth, then \(|\nabla l| \leq 1\).

**Theorem 3.5.** Let \((M, g, f)\) be a complete non-compact gradient expanding Schouten soliton with \(-(n-1) < k_1 \leq R\) for some real constant \(k_1\) and the potential function \(f\) is positive non-constant with \(f^{-b}\) bounded above. Then all the ends of \(M\) are non-parabolic.

**Proof.** For \(b = \frac{n-2}{2} + \frac{k_1(n-2)}{2(n-1)} > 0\), using the equation (3) and the Theorem 3.1 we calculate

\[
\Delta f^{-b} = -bf^{-b-1}\Delta f + b(b+1)f^{-b-2}|\nabla f|^2
\]

\[
\geq -b\left\{- \frac{n}{2} + \frac{nR}{2(n-1)} - R\right\}f^{-b-1} + b(b+1)(f_0 - f)f^{-b-2}
\]

\[
= \left\{- b\left( - \frac{n}{2} - \frac{(n-2)R}{2(n-1)} - b(b+1)\right)f^{-b-1} + b(b+1)f_0 f^{-b-2}\right\}
\]

\[
\geq \left\{- b\left( - \frac{n}{2} - \frac{(n-2)R}{2(n-1)} - b(b+1)\right)f^{-b-1}\right\}
\]

\[
\geq b\left\{ \frac{k_1(n-2)}{2(n-1)} + \frac{n}{2} - (b+1)\right\}f^{-b-1} = 0.
\]

Hence it follows that \(f^{-b}\) is a subharmonic function which is bounded above. This proves that (see, [8]) any end of \(M\) and hence \(M\) is non-parabolic. \(\square\)

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