Boson stars from a gauge condensate

Vladimir Dzhunushaliev
Dept. Phys. and Microel. Engineer., Kyrgyz-Russian Slavic University, Bishkek, Kievskaya Str. 44, 720021, Kyrgyz Republic

Kairat Myrzakulov and Ratbay Myrzakulov
Institute of Physics and Technology, 050032, Almaty, Kazakhstan

The boson star filled with two interacting scalar fields is investigated. The scalar fields can be considered as a gauge condensate formed by SU(3) gauge field quantized in a non-perturbative manner. The corresponding solution is regular everywhere, has a finite energy and can be considered as a quantum SU(3) version of the Bartnik - McKinnon particle-like solution.

Key words: boson star; scalar fields; regular solution

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I. INTRODUCTION

Non-Abelian solitons play an important role in gauge theories of elementary particle physics. Therefore it is reasonable to study gravitating gauge fields. The first example of gravitating non-Abelian solitons was discovered by Bartnik and McKinnon in the four-dimensional Einstein-Yang-Mills theory for the gauge group SU(2). Soon after the BK discovery it was realized that, apart from solitons, the Einstein-Yang-Mills model also contains non-Abelian black holes. In Ref. it was shown that the SU(3) branch of similar solutions exists also. Such solutions can be thought of as eigenstates of non-linear eigenvalue problems, which accounts for the discreteness of some their parameters.

The most surprising fact here is the existence of the Bartnik - McKinnon particles, which have not any analogue in flat space. From the physical point of view it proceeds from the fact that the classical Yang - Mills equations have not any topologically trivial regular solutions. In fact, it is connected with the confinement problem in quantum chromodynamics. The standard opinion will be that only quantized SU(3) gauge field may form regular objects, for example, a hypothesized flux tube connecting two interacting quarks.

However, it is known that in such sourceless non–Abelian gauge theories have no classical glueballs which otherwise would be an indication for the occurrence of confinement in the quantized theory. The reason simply is that nearby small portions of the Yang–Mills fields always point at the same direction in internal space and therefore must repel each other as similar charges.

Therefore, it is interesting to investigate gravitating quantum non-Abelian gauge fields. The main problem here is to describe quantum non-Abelian gauge fields in a non-perturbative manner. In Ref. is offered an approximate non-perturbative approach to quantization of SU(3) gauge field. It is shown that using some assumptions and approximations one can reduce the SU(3) Lagrangian to the Lagrangian describing two interacting scalar fields.

Thus, in this Letter we will investigate a boson star filled with two interacting scalar fields which can be considered as a gauge condensate. The boson star was first discovered theoretically by Kaup and by Ruffini and Bonazzola (for reviewing the boson star, see Refs. and other references are cited therein).

In Ref. the boson star is considered filled with the complex doublet of scalar fields coupled to an SU(2) non-abelian gauge field.

In Ref. the emission spectrum from a simple accretion disk model around a compact object is compared with the cases of a black hole and a boson star. It was found that, for certain values of the boson star parameters, it is possible to produce spectra similar to those which are generated when the central object is a black hole.

One can say that the presented here solution is a quantum SU(3) version of the Bartnik - McKinnon particle-like solution.
II. INITIAL EQUATIONS

Let us consider Einstein gravity interacting with two scalar fields $\phi, \chi$. The metric is

$$ds^2 = \left[1 + \frac{M(r)}{r}\right]dt^2 - \frac{\epsilon^\nu(r)}{1 + \frac{M(r)}{r}} dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2\right),$$ (1)

The Lagrangian for scalar fields $\phi$ and $\chi$ is

$$\mathcal{L} = \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi + \frac{1}{2} \nabla_\mu \chi \nabla^\mu \chi - V(\phi, \chi),$$ (2)

where $\mu, \nu = 0, 1, 2, 3$. The potential $V(\phi, \chi)$ is

$$V(\phi, \chi) = \frac{\lambda_1}{4} (\phi^2 - m_1^2)^2 + \frac{\lambda_2}{4} (\chi^2 - m_2^2)^2 - \frac{\lambda_2}{4} m_2^4 + \phi^2 \chi^2,$$ (3)

where $\frac{\lambda}{4} m_2^4$ is the constant which can be considered as a cosmological constant $\Lambda$. In Ref. [13] it is shown that these scalar fields present a quantum gauge field. In short it can be shown by the following way. In quantizing strongly interacting SU(3) gauge fields - via Heisenberg’s non-perturbative method [12] one first replaces the classical fields by field operators $A^B_\mu \to \hat{A}^B_\mu$. This yields the following differential equations for the operators

$$\partial_\mu \hat{F}^{B\mu\nu} = 0.$$ (4)

However this equation will in its turn contain other, higher order Green’s functions. Repeating these steps leads to an infinite set of equations connecting Green’s functions of ever increasing order. This construction, leading to an infinite set of coupled, differential equations, does not have an exact, analytical solution and so must be handled using some approximation. The basic approach in this case is to give some physically reasonable scheme for cutting off the infinite set of equations for the Green’s functions. Starting with Eq. (4) one can generate an operator differential equation for the product $\hat{A}^B_\mu \hat{A}^C_\nu$ consequently allowing the determination of the Green’s function $\mathcal{G}^{BC}_{\mu\nu}$

$$\left\langle Q | \hat{A}^B(x) \partial_{\mu\nu} \hat{F}^{B\mu\nu}(x) | Q\right\rangle = 0.$$ (5)

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$$\left\langle Q | \hat{A}^B(x) \partial_{\mu\nu} \hat{F}^{B\mu\nu}(x) | Q\right\rangle = 0.$$ (5)
and
\[ \langle A^a_\mu(x)A^b_\nu(y)A^c_\alpha(z)A^d_\beta(u) \rangle = \lambda_2 \left[ \langle A^a_\mu(x)A^b_\nu(y) \rangle \langle A^c_\alpha(z)A^d_\beta(u) \rangle + \frac{\mu^2}{4} \left( \delta^{ab} \eta_{\mu\nu} \langle A^c_\alpha(z)A^d_\beta(u) \rangle + \delta^{cd} \eta_{\alpha\beta} \langle A^a_\mu(x)A^b_\nu(y) \rangle \right) + \frac{\mu^4}{16} \delta^{ab} \eta_{\mu\nu} \delta^{cd} \eta_{\alpha\beta} \right] \]

(permutations of indices) (11)

In this Letter we have changed \( \phi^a \to \phi \) and \( \chi^m \to \chi \).

We consider here the spherically symmetric case and the functions \( \phi, \chi \) are \( \phi(r), \chi(r) \). The Einstein and scalar field equations are

\[ R^\mu_\nu - \frac{1}{2} R g^\mu_\nu = \kappa T^\mu_\nu, \]  
\[ \frac{1}{\sqrt{-g}} \nabla_\mu \left( \sqrt{-g} g^{\mu\nu} \nabla_\nu \phi \right) = -\frac{\partial V(\phi, \chi)}{\partial \phi}, \]  
\[ \frac{1}{\sqrt{-g}} \nabla_\mu \left( \sqrt{-g} g^{\mu\nu} \nabla_\nu \chi \right) = -\frac{\partial V(\phi, \chi)}{\partial \chi}, \]

where \( \kappa \) is the gravitational constant; \( g_{\mu\nu} \) is the metric (11) and \( g \) is the corresponding determinant. After substituting metric (11) for Eq’s (12) - (14) and after algebraic transformations we have the following equations

\[ \nu' = \frac{x}{2} (\phi'^2 + \chi'^2), \]  
\[ M' = e^{\nu'} - 1 + \frac{x^2 + Mx}{4} (\phi'^2 + \chi'^2) - \]  
\[ \frac{x^2}{4} e^{\nu'} \left[ \frac{\lambda_1}{2} (\phi'^2 - m_1^2)^2 + \frac{\lambda_2}{2} \chi^2 (\chi^2 - 2m_2^2) + \phi'^2 \chi^2 \right], \]

\[ \phi'' + \phi' \left( \frac{2}{x} - \frac{\nu'}{2} + \frac{M' - M}{x + M} \right) = \frac{e^{\nu'}}{1 + \frac{M}{x}} \phi \left[ \chi^2 + \lambda_1 (\phi'^2 - m_1^2) \right], \]

\[ \chi'' + \chi' \left( \frac{2}{x} - \frac{\nu'}{2} + \frac{M' - M}{x + M} \right) = \frac{e^{\nu'}}{1 + \frac{M}{x}} \chi \left[ \phi^2 + \lambda_2 (\chi^2 - m_2^2) \right], \]

where \( \frac{d(\nu)}{dx} = (\cdots)' \); the following functions \( \phi \sqrt{x} \to \phi, \chi \sqrt{x} \to \chi \), constants \( m_{1,2} \sqrt{x} \to m_{1,2}, \lambda_{1,2}/2 \to \lambda_{1,2} \) and the dimensionless variable \( x = r/\sqrt{\kappa/2} \) are introduced.

The boundary conditions are

\[ \nu(0) = 0, \]  
\[ M(0) = 0, \]  
\[ \phi(0) = \phi_0, \quad \phi'(0) = 0, \]  
\[ \chi(0) = \chi_0, \quad \chi'(0) = 0. \]

### III. NUMERICAL INVESTIGATION

For the numerical calculations we choose the following parameters values

\[ \phi_0 = 1, \quad \chi_0 = \sqrt{0.6}, \quad \lambda_1 = 0.1, \quad \lambda_2 = 1.0. \]

We apply the methods of step by step approximation to finding of numerical solutions (the details of similar calculations can be found in Ref. [11]). For the numerical calculations we have to start from a small point \( x = \Delta = 0.01 \). For this
case the boundary conditions (24)-(27) are

\[
\nu(\Delta) = \nu_4 \Delta^4 / 4, \quad \nu_4 = \phi_0^2 + \chi_0^2 / 2, \\
M(0) = M_3 \Delta^3 / 3, \quad M_3 = -\frac{1}{4} \left[ \frac{\lambda_1}{2} (\phi_0^2 - m_1^2)^2 + \frac{\lambda_2}{2} (\chi_0^2 - m_2^2)^2 + \phi_0^2 \chi_0^2 \right],
\]

(24)

(25)

\[
\phi(0) = \phi_0 + \phi_2 \Delta^2 / 2, \quad \phi'(0) = \phi_2 \Delta, \quad \phi_2 = \frac{\phi_0}{3} \left[ \chi_0^2 + (\phi_0^2 - m_1^2) \right],
\]

(26)

\[
\chi(0) = \chi_0 + \chi_2 \Delta^2 / 2, \quad \chi'(0) = \chi_2 \Delta, \quad \chi_2 = \frac{\chi_0}{3} \left[ \phi_0^2 + (\chi_0^2 - m_2^2) \right].
\]

(27)

Step 1 On the first step we solve Eq. (17) (having zero approximations \( \nu_0(x) = 0, M_0(x) = 0 \)). The regular solution exists for a special value \( m_{1,i}^* \) only. For \( m_1 < m_{1,i}^* \) the function \( \chi_i(y) \to +\infty \), for \( m_1 > m_{1,i}^* \) the function oscillates and \( \chi_i(y) \to 0 \) (here the index \( i \) is the approximation number). One can say that in this case we solve a non-linear eigenvalue problem: \( \chi_i^*(y) \) is the eigenstate and \( m_{1,i}^* \) is the eigenvalue on this Step.

Step 2 On the second step we solve Eq. (15) using zero approximation \( \nu_0(x) = 0, M_0(x) = 0 \) and the first approximation \( \chi_1^*(y) \) for the function \( \chi(y) \) from the Step 1. For \( m_2 < m_{2,1}^* \) the function \( \phi_1(y) \to +\infty \) and for \( m_2 > m_{2,1}^* \) the function \( \phi_1(y) \to -\infty \). Again we have a non-linear eigenvalue problem for the function \( \phi_1(y) \) and \( m_{2,1}^* \).

Step 3 On the third step we repeat the first two steps that to have the good convergent sequence \( \phi_i^*(y), \chi_i^*(y) \). Practically we have made three approximations.

Step 4 On the next step we solve Eq’s (15) (16) which give us the functions \( \nu_1(x), M_1(x) \).

Step 5 On this step we repeat Steps 1-4 necessary number of times that to have the necessary accuracy of definition of the functions \( \nu^*(x), M^*(x), \phi^*(x), \chi^*(x) \).

After Step 5 we have the solution presented in Fig’s. 1-2. These numerical calculations give us the eigenvalues \( m_{1,i}^* \approx 1.6138771 \), \( m_{2,i}^* \approx 1.493441 \) and eigenstates \( \nu^*(y), M^*(x), \phi^*(x), \chi^*(x) \).

![FIG. 1: The functions \( \phi^*(x), \chi^*(x) \)](image)

![FIG. 2: The profiles of metric components \( g_{tt}(x) = 1 + \frac{\nu(x)}{M(x)} \) and \( g_{rr}(x) = \frac{e^{\nu(x)}}{1+x} \).](image)

It is easy to see that the asymptotical behavior of the solution is

\[
\nu(x) \approx \max_{x \to \infty} \left\{ \lambda_1 m_1^2 \phi_\infty^2 e^{-2x/\sqrt{2\lambda_1 m_1^2} / x}, \left( m_1^2 - \lambda_2 m_2^2 \right) \chi_\infty^2 e^{-2x/\sqrt{2\lambda_2 m_2^2} / x} \right\},
\]

(28)

\[
M(x) \approx M_\infty x, \quad M_\infty = e^{\nu(\infty)} - 1,
\]

(29)

\[
\phi(x) \approx m_1 + \phi_\infty e^{-x/\sqrt{2\lambda_1 m_1^2} / x},
\]

(30)

\[
\chi(x) \approx \chi_\infty e^{-x/\sqrt{2\lambda_2 m_2^2} / x},
\]

(31)
where $\nu_\infty, \phi_\infty, \chi_\infty$ are constants. From Fig. 2 we see that at the infinity $g_{tt} \to \text{const} \neq 1$. One can avoid this problem in the following way. The numerical calculations show that the asymptotical behavior of the function $M(x)$ is

$$M(x) \approx M_\infty x - a$$

(32)

where $a$ is a constant. In this case $g_{tt}$ metric component is

$$g_{tt} = 1 + \frac{M}{x} \approx e^{\nu_\infty} - \frac{a}{x}.$$  

(33)

At the infinity we can rescale the time

$$\tilde{t} = e^{\nu_\infty/2} t$$

(34)

then $g_{tt}$ will be

$$\tilde{g}_{tt} \approx 1 - \frac{a e^{-\nu_\infty}}{x}.$$  

(35)

The dimensionless energy density is

$$e(x) = T_0^0 = \left(\frac{\kappa}{2}\right)^2 \varepsilon(r) = \frac{1}{4} \left(1 + \frac{M(x)}{x}\right) e^{-\nu(x)} \left(\phi'^2 + \chi'^2\right) + \frac{\lambda_1}{8} \left(\phi^2 - m_1^2\right)^2 + \frac{\lambda_2}{8} \chi^2 \left(\chi^2 - 2m_2^2\right)^2 + \frac{1}{4} \phi^2 \chi^2$$

(36)

and it is presented in Fig. 3. One can calculate the mass of this object in the following manner. The usual definition of the mass $m$ gives us

$$\sqrt{\kappa} m = 4\pi \int_0^\infty x^2 T_0^0 dx \approx 2.65$$

(37)

here the mass $m$ has the dimension $cm^{-1}$.

![FIG. 3: The profile of energy density $e(x)$.](image)

IV. DISCUSSION AND CONCLUSIONS

In this Letter we have considered two gravitating scalar fields. As the result we have obtained the particle-like solution. The authors point of view (following Ref. [11]) is that this object is either a star filled with a gravitating quantum SU(3) gauge condensate (if the solution is stable) or a quantum fluctuation in the surrounding gauge condensate (if the solution is unstable). The problem of stability of the presented solution is not simple as the solution is numerical and it is the eigenfunction of non-linear equations system. We hope to investigate this problem
in the future research. Nevertheless one can note a remark on this problem. Eq. 36 and Fig. 3 shows that in the center of the solution there is a negative potential energy density – the term $-\frac{1}{2} m^2_2$ which plays the role of $\Lambda$ term. Such exotic matter can act in a gravitationally repulsive way. In this case the interplay between repulsive forces on small distances and attractive forces on big distances could lead to the stability of the gravitating ball filled with the gauge condensate.

In contrast with the Bartnik - McKinnon solution and non-Abelian black holes the presented solution has a flat space limit which is presented in Ref. [11]. Another interesting feature is that in both cases the solutions are eigenstates of a nonlinear eigenvalue problem.

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