On Formal First Integrals for Singularities of Complex Vector Fields

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Abstract

We show that, for a generic germ of foliation of dimension one in \((\mathbb{C}^3, 0)\), the existence of a formal first integral implies the existence of a holomorphic one. This agrees with the same result in other settings obtained in the classic works [7, 6, 4]. Some properties of formal series and formal first integrals are explored.

1 Introduction

In [7] it is shown that the germ of a holomorphic foliation of codimension 1 in a neighborhood of \(0 \in \mathbb{C}^n\) with closed leaves off the origin and finitely many separatrices possesses a holomorphic first integral. For this, the authors manage to construct a formal first integral and a way to obtain from it a holomorphic one. A similar step is given in the article [6] where some other results about the existence of a holomorphic first integral are found (see also [4]).

Our main motivation comes from [3] where conditions are given for the existence of a holomorphic first integral for a generic germ of foliation of dimension one in \((\mathbb{C}^3, 0)\) but without the "formal to holomorphic step" letting open the question: Does in this scenario the existence of a formal first integral implies the existence of a holomorphic one?. It turns out that there is a positive answer that we resume in our Theorem (A).

In the statement of our theorem and in the rest of the article we will be based in the notation and results of [2, 3] that we write next for the sake of completeness.

Denote the ring of germs of holomorphic functions on \((\mathbb{C}^n, 0)\) by \(\mathcal{O}_n\), the ring of formal series on \((\mathbb{C}^n, 0)\) by \(\hat{\mathcal{O}}_n\), the group of formal diffeomorphisms of \((\mathbb{C}^n, 0)\) by \(\text{Diff}(\mathbb{C}^n, 0)\) and \(\text{Diff}(\mathbb{C}^n, 0)\) the subgroup of analytic diffeomorphisms (or just diffeomorphisms) of \(\text{Diff}(\mathbb{C}^n, 0)\). Given a germ of a holomorphic vector field \(\mathcal{X} \in \mathfrak{X}(\mathbb{C}^n, 0)\) we shall denote by \(\mathcal{F}(\mathcal{X})\) the germ of a one-dimensional holomorphic foliation on \((\mathbb{C}^n, 0)\) induced by \(\mathcal{X}\).

Definition 1. We shall say that \(\mathcal{F}(\mathcal{X})\) is non-degenerate generic if \(d\mathcal{X}(0)\) is non-singular, diagonalizable, and after some suitable change of coordinates \(\mathcal{X}\) leaves invariant the coordinate planes. Denote the set of germs of non-degenerate generic vector fields on \((\mathbb{C}^n, 0)\) by \(\text{Gen}(\mathfrak{X}(\mathbb{C}^n, 0))\).

Definition 2. We say that a germ of a holomorphic foliation \(\mathcal{F}(\mathcal{X})\) has a holomorphic first integral, if there is a germ of a holomorphic map \(F : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-1}, 0)\) such that:
(a) \( F \) is a submersion off some proper analytic subset. Equivalently if we write
\[ F = (f_1, \ldots, f_{n-1}) \]
in coordinate functions, then the \((n-1)\)-form \( df_1 \wedge \cdots \wedge df_{n-1} \) is non-identically zero.

(b) The leaves of \( F(\mathcal{X}) \) are contained in level curves of \( F \).

Further, a germ \( f \) of a meromorphic function at the origin \( 0 \in \mathbb{C}^n \) is called
\( F(\mathcal{X}) \)-invariant if the leaves of \( F(\mathcal{X}) \) are contained in the level sets of \( f \). This can be precisely stated in terms of representatives for \( F(\mathcal{X}) \) and \( f \), but can also be written as
\[ i_X(df) = X(f) \equiv 0. \]

**Definition 3** (condition \((\ast)\)). Let \( \mathcal{X} \) be a germ of a holomorphic vector field at the origin such that the origin \( 0 \in \mathbb{C}^m \), \( m \geq 3 \) is a nondegenerate singularity of \( \mathcal{X} \) (i.e. \( d\mathcal{X}(0) \) is non-singular). We say that \( \mathcal{X} \) satisfies condition \((\ast)\) if there is a real line \( L \subset \mathbb{C} \) through the origin, separating a certain eigenvalue \( \lambda \) from the others.

From (3) \( \iff \) (4) in Theorem 1 of [3] we get:

**Theorem 4.** Suppose that \( \mathcal{X} \in \text{Gen}(\mathcal{X}(\mathbb{C}^3, 0)) \) satisfies condition \((\ast)\) and let \( S_\mathcal{X} \) be the axis associated to the separable eigenvalue of \( \mathcal{X} \). Then, \( \text{Hol}(F(\mathcal{X}), S_\mathcal{X}, \Sigma) \) is periodic (in particular linearizable and finite) if and only if \( F(\mathcal{X}) \) has a holomorphic first integral.

We start with the following definition inspired by the definition of holomorphic first integral (Definition 2), though it will not be used until the end of the article is necessary to settle down the framework we use.

**Definition 5** (formal first integral). We say that a germ of a holomorphic foliation \( F(\mathcal{X}) \), were \( \mathcal{X} \in \mathcal{X}(\mathbb{C}^n, 0) \), has a formal first integral, if there is a formal map \( \hat{F} = (\hat{f}_1, \ldots, \hat{f}_{n-1}) \), with \( \hat{f}_1, \ldots, \hat{f}_{n-1} \in \hat{O}_n \), such that:

(a) The formal \((n-1)\)-form \( d\hat{f}_1 \wedge \cdots \wedge d\hat{f}_{n-1} \) is non-identically zero.

(b) \( \mathcal{X}(\hat{F}) \equiv 0 \), (i.e. \( \mathcal{X}(\hat{f}_i) \equiv 0 \) for all \( \hat{f}_i, i = 1, \ldots, n-1 \)).

Our main result is:

**Theorem (A).** Let \( F(\mathcal{X}) \) be the germ of a holomorphic foliation with \( \mathcal{X} \in \text{Gen}(\mathcal{X}(\mathbb{C}^3, 0)) \), if \( F(\mathcal{X}) \) possesses a formal first integral then it also possesses a holomorphic one.

A modification has to be done to Theorem A in order to work in arbitrary dimension.

**Theorem (B).** Let \( F(\mathcal{X}) \) be the germ of a holomorphic foliation with \( \mathcal{X} \in \text{Gen}(\mathcal{X}(\mathbb{C}^n, 0)) \) satisfying condition \((\ast)\), if \( F(\mathcal{X}) \) possesses a formal first integral then it also possesses a holomorphic one.

Though we will always work in dimension \( n \), condition \((\ast)\) is something we cannot guarantee in dimension greater than 3.
2 Properties of Formal Series

Our first step is to study the properties we can get from the relationship \( \hat{f} \circ \hat{G} = \hat{f} \) where \( \hat{f} \in \hat{\mathcal{O}}_n \) and \( \hat{G} \in \hat{\mathcal{D}}\text{iff}(\mathbb{C}^n, 0) \), in this case we say that \( \hat{G} \) leaves \( \hat{f} \) invariant. Our work will guarantee that we only need to analyze the case where \( \hat{G} \) is linearizable and, as we stated in the following proposition this relationship characterizes both maps. First, we need the following definitions:

Definition 6. We say that \( \Lambda \in \mathbb{C}^n \) is multiplicative resonant if exist a multi-index \( Q = (q_1, \ldots, q_n) \in \mathbb{N}^n \), with \( |Q| = q_1 + \cdots + q_n \geq 1 \), such that

\[
\Lambda^Q := \lambda_1^{q_1} \cdots \lambda_n^{q_n} = 1.
\]

The existence of this kind of resonances is an obstruction for the linearization of a diffeomorphism as can be seen in [8].

Definition 7. We shall say that a monomial \( x^Q := x_1^{q_1} \cdots x_n^{q_n} \) is resonant with respect to \( \Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \) (or simply \((\lambda_1, \ldots, \lambda_n)\)-resonant) if \( |Q| \geq 1 \) and \( \Lambda^Q = 1 \).

Proposition 8. Let \( \hat{f} \in \hat{\mathcal{O}}_n \) and \( \hat{G} \in \hat{\mathcal{D}}\text{iff}(\mathbb{C}^n, 0) \) linearizable, if \( \hat{G} \) leaves \( \hat{f} \) invariant. Then, the linear part of \( \hat{G} \) is a diagonal matrix, \( \text{diag}(\lambda_1, \ldots, \lambda_n) \), whose elements are multiplicative resonant and, \( \hat{f} \) is the sum of only \((\lambda_1, \ldots, \lambda_n)\)-resonant monomials.

Proof. The one dimensional case appears as a proposition in [7], and for arbitrary dimension, let \( G(x) = Ax \) and \( \hat{f}(x) = \sum_{|I| \geq 1} a_I x^I \), where \( A \) is a non-singular \( n \times n \) matrix in Jordan form and \( x = (x_1, \ldots, x_n) \).

Consider first the diagonal case, \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \), in this case:

\[
G(x_1, \ldots, x_n) = (\lambda_1 x_1, \ldots, \lambda_n x_n),
\]

Thus,

\[
\hat{f} \circ G(x_1, \ldots, x_n) = \sum_{|I| \geq 1} a_I (\lambda_1 x_1)^{i_1} \cdots (\lambda_n x_n)^{i_n} = \sum_{|I| \geq 1} a_I x_1^{i_1} \cdots x_n^{i_n},
\]

which means that

\[
\lambda_1^{i_1} \cdots \lambda_n^{i_n} = 1, \text{ for all } I \text{ such that } a_I \neq 0, \text{ i.e. is multiplicative resonant.}
\]

If \( \hat{f} \neq 0 \) it is formed only by resonant monomials, furthermore there exist at most \( n \) “independent”, \( n \)-tuples \( I = (i_1, \ldots, i_n) \in \mathbb{N}^n \setminus 0 \) such that \( \lambda_1^{i_1} \cdots \lambda_n^{i_n} = 1 \).

In case we have \( n \) independent \( n \)-tuples, all \( \lambda_i \)'s are roots of the unity as we can see taking logarithm in each one of the \( n \) equalities \( \lambda_1^{i_1} \cdots \lambda_n^{i_n} = 1 \) and solving a linear system like the following:

\[
\begin{bmatrix}
i_{1,1} & \cdots & i_{1,n} \\
\vdots & \ddots & \vdots \\
i_{n,1} & \cdots & i_{n,n}
\end{bmatrix}
\begin{bmatrix}
\log \lambda_1 \\
\vdots \\
\log \lambda_n
\end{bmatrix}
= \begin{bmatrix}
2\pi ik_1 \\
\vdots \\
2\pi ik_n
\end{bmatrix},
\]

from its real part, which is a homogeneous linear system, we get that \( \log |\lambda_j| = 0 \) for all \( j \) and, from the imaginary part we obtain that the argument of each \( \lambda_j \) is a rational factor of \( 2\pi \).
Now, consider $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, in this case

$$G(x_1, x_2) = (\lambda x_1 + x_2, \lambda x_2)$$

and

$$f \circ G(x_1, x_2) = \sum_{|I| \geq 1} a_I (\lambda x_1 + x_2)^I = \hat{f}(x_1, x_2) = \sum a_j x_1^j x_2^j,$$

then $a_{i,j} = \sum_{k=0}^j C_{i+k,k} \lambda^{i+j-k} a_{i+k,j-k}$, where $C_{l,m} = \binom{l}{m}$.

If $\lambda^i \neq 1$ for all $i \in \mathbb{N}$ then $a_{i,0} \equiv 0$ and $a_{i,1} = \lambda^{i+1} a_{i,1} + C_{i+1,1} \lambda^i a_{i+1,0}$ implies $a_{i,j} \equiv 0$. Therefore, $\lambda^i = 1$ for some $i$ such that $a_{i,0} \neq 0$, then $a_{i-1,1} = \lambda a_{i-1,1} + C_{i-1} \lambda^{i-1} a_{i,0}$ implies $a_{i,0} = 0$, contradiction. The higher dimensional case works in the same way, because some part of $G$ will be of the form $(\ldots, \lambda x_j + x_{j+1}, \ldots, \lambda x_{j+k-1} + x_{j+k}, \lambda x_{j+k}, \ldots)$, for an eigenvalue $\lambda$, and making all $x_j = 0$ except for $x_{j+k-1}$ and $x_{j+k}$ we can apply the same analysis. Then, $G$ does not have this kind of Jordan’s blocks.

Finally, if $\hat{G} \in \text{Diff}(\mathbb{C}^n, 0)$ by hypothesis it is linearizable then, there is a formal change of coordinates such that $g^{-1} \circ \hat{G} \circ g(x) = d\hat{G}(0)x$ and we make the previous analysis over its linear part $G(x) = d\hat{G}(0)x$ concluding that, has to be a diagonal one with multiplicative resonant entries.

Now, using the proposition above we obtain another property, but in this occasion for a group of a formal diffeomorphism leaving invariant a set of formal series satisfying the following definition:

**Definition 9.** Let $\hat{f}_1, \ldots, \hat{f}_n \in \hat{O}_n$. We say that $\hat{f}_1, \ldots, \hat{f}_n$ are generically transverse if $df \land \cdots \land df = 0$.

The following proposition together with the previous part is one of the key parts of our work.

**Proposition 10.** Let $\hat{f}_1, \ldots, \hat{f}_n \in \hat{O}_n$ be generically transverse and $\mathcal{G}$ a group of formal diffeomorphisms ($\mathcal{G} \subset \text{Diff}(\mathbb{C}^n, 0)$) leaving invariant the set $\{\hat{f}_1, \ldots, \hat{f}_n\}$. Then the group $\mathcal{G}$ is periodic (in particular linearizable and finite).

In the demonstration of Proposition 10 we use the following theorem from [1], whose demonstration we put here to emphasize that is also valid in the formal case:

**Proposition 11.** A group $\mathcal{G} \subset \hat{\text{Diff}}(\mathbb{C}^n, 0)$ is linearizable if and only if there exists a vector field $\mathcal{X} = \mathcal{R} + \cdots$, where $\mathcal{R}$ is a radial vector field, such that $\mathcal{X}$ is invariant for every $\hat{G} \in \mathcal{G}$, i.e. $\hat{G}^* \mathcal{X} = \mathcal{X}$.

**Proof.**

($\Rightarrow$) Suppose that $\mathcal{G}$ is linearizable, i.e. there exists $g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $g \circ \mathcal{G} \circ g^{-1} = \{dG_0 | G \in \mathcal{G}\}$. Since $(A(\cdot))^* \mathcal{R} = \mathcal{R}$ for all $A \in \text{Gl}(n, \mathbb{C})$ (by a direct calculation $(A(\cdot))^* \mathcal{R}_z = dA(\cdot)_{A^{-1}z} \mathcal{R} A^{-1} z = z$), in particular for every
For this, suppose that 
then denoting \(X\) the terms after as we wanted. 
mal diffeomorphism (using Poincaré linearization theorem, \(\text{[5] Theorem 4.3}\)) and taking \(z = g(y)\) and multiplying by \(dg^{-1}(y)\) we have,
\[
dg^{-1}(g(y)) = d\hat{G}^{-1}(y)d_{gG^{-1}(y)}(g \circ \hat{G}^{-1}(y)),
\]
denoting \(X = dg^{-1}(g(y))\) we have \(\hat{G}^*X = X\). It is easy to see that \(X = R + \cdots\). For this, suppose that 
\[g(z) = Az + P_1(z) + P_{l+1}(z) + \cdots,\]
\[g^{-1}(z) = A^{-1}z + Q_\nu(z) + Q_{\nu+1}(z) + \cdots,\]
where \(A \in \mathcal{M}_n(\mathbb{C})\) and \(P_i, Q_\nu\) are polynomial vector fields of degree \(l\) and \(\nu\), then 
\[
dg^{-1}_z = A^{-1} + dQ_\nu(z) + dQ_{\nu+1}(z) + \cdots,
\]
\[
dg^{-1}_y = A^{-1} + dQ_\nu(y) + dQ_{\nu+1}(z) + \cdots,
\]
\[
dg^{-1}_y(g(z)) = (A^{-1} + dQ_\nu(y))g(z) + \cdots (Az + P_1(z) + \cdots)
\]
\[
X_z = z + A^{-1}(P_1(z) + P_{l+1}(z) + \cdots) + \cdots
\]
The terms after \(z\), if not 0, are of degree greater than one. Thus, \(X = R + \cdots\) as we wanted.
\((\Rightarrow)\) Since every eigenvalue of the linear part of \(X\) is 1, then \(X\) is in the Poincaré domain without resonances (additive resonances), therefore there exists a formal diffeomorphism (using Poincaré linearization theorem, \(\text{[5] Theorem 4.3}\)) \(g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)\) such that \(g^*X = R\), i.e. \(X = (dg^{-1}(\cdot))^{-1}g(\cdot)\).
We claim that \(g \circ \hat{G} \circ g^{-1}(y) = dG_0(y)\) for every \(G \in \mathcal{G}\). In fact, from the same procedure as before we can observe that 
\[
R_z = (g \circ \hat{G} \circ g^{-1})^*R_z.
\]
For this note that \(\hat{G}^*X = X \Rightarrow\)
\[
d\hat{G}_{\hat{G}^{-1}(y)}X_{\hat{G}^{-1}(y)} = X_z
\]
\[
d\hat{G}_{\hat{G}^{-1}(y)}dg^{-1}_{gG^{-1}(y)}g \circ \hat{G}^{-1}(y) = dg^{-1}_y g(y)
\]
taking \(z = g(y)\), we have 
\[
d\hat{G}_{\hat{G}^{-1}og^{-1}(z)}dg^{-1}_{gG^{-1}og^{-1}(z)}g \circ \hat{G}^{-1} \circ g^{-1}(z) = dg^{-1}_z(z)
\]
Therefore,
\[
(g \circ \hat{G} \circ g^{-1})^*R_z = d(g \circ \hat{G} \circ g^{-1})_{gG^{-1}og^{-1}(z)}R(g \circ \hat{G}^{-1} \circ g^{-1}(z)) = dg^{-1}_z(d\hat{G}_{\hat{G}^{-1}og^{-1}(z)}dg^{-1}_{gG^{-1}og^{-1}(z)}g \circ \hat{G}^{-1} \circ g^{-1}(z) = dg^{-1}_z dg^{-1}_z(z) \quad (by \ the \ the \ previous \ computation)
\]
\[
= z.
\]
Now, if we suppose that \( g \circ \hat{G} \circ g^{-1}(z) = Az + P_l(z) + P_{l+1}(z) + \cdots \), where \( P_j(z) \) is a polynomial vector field of degree \( j \), then it is easy to prove that
\[
(g \circ \hat{G} \circ g^{-1})^* \mathcal{R} = Az + lP_l(z) + (l + 1)P_{l+1}(z) + \cdots,
\]
In order to prove it, observe that \((g \circ \hat{G} \circ g^{-1})^* \mathcal{R}_z = \mathcal{R}_z \Rightarrow \)
\[
d(g \circ \hat{G} \circ g^{-1})_{g \circ \hat{G}^{-1} \circ g^{-1}(y)} \mathcal{R}(g \circ \hat{G}^{-1} \circ g^{-1}(y)) = \mathcal{R}(y)
\]
taking \( y = g \circ \hat{G} \circ g^{-1}(z) \) then
\[
d(g \circ \hat{G} \circ g^{-1})_{z} \mathcal{R}(z) = \mathcal{R}(g \circ \hat{G} \circ g^{-1}(z)),
\]
d\( g \circ \hat{G} \circ g^{-1} \), i.e. by hypothesis \( d(\hat{G}) \circ g^{-1} \),
\[
d(g \circ \hat{G} \circ g^{-1})_{z} \mathcal{R}(z) = d(\hat{G}) \circ g^{-1}(z),
\]
and therefore \( P_j(z) \equiv 0 \) for every \( j \geq 2 \).

**Proof of Proposition 11**: The idea is to use the above proposition, so that we need to find an invariant vector field \( \mathcal{X} \). First, consider the formal map \( H = (\hat{f}_1, \ldots, \hat{f}_n) \), for each \( \hat{G} \in \mathcal{G} \) by hypothesis \( \hat{f}_i \circ \hat{G} = f_i \) then we have \( H \circ \hat{G} = H \), and note that \( H \in \hat{\mathcal{D}}\text{iff}(\mathbb{C}^n, 0) \) because \( d\hat{f}_1 \wedge \cdots \wedge d\hat{f}_n \neq 0 \). Thus, this implies \( \mathcal{H} \circ \mathcal{G}^{-1} = \mathcal{H}, \mathcal{G} \circ \mathcal{H}^{-1} = \mathcal{H}^{-1} \) and \( d\mathcal{G}^{-1} = d\mathcal{H}^{-1} \).

Therefore define \( \mathcal{X} = (d\mathcal{H})^{-1} \mathcal{H} = d\mathcal{H}^{-1} \mathcal{H}(\cdot) \) which satisfies \( \hat{G}^* \mathcal{X} = \mathcal{X}, \)
\[
\hat{G}^* \mathcal{X}_z = d\hat{G}^{-1}_{z} \mathcal{X}^{G^{-1}_{z}},
\]
\[
= d\hat{G}^{-1}_{z} \mathcal{H}^{-1}_{\hat{H}(\hat{G}^{-1}_{z})} \mathcal{H}(G^{-1}_{z}),
\]
\[
= d\hat{G}^{-1}_{z} \mathcal{H}^{-1}_{\hat{H}(\hat{G}^{-1}_{z})} \mathcal{H}(z),
\]
\[
= \mathcal{H}^{-1}_{\hat{H}(\hat{G}^{-1}_{z})} \mathcal{H}(z), \] using \( d\mathcal{G}^{-1}_{\hat{H}(\hat{G}^{-1}_{z})} d\mathcal{H}^{-1}_{\hat{H}(\hat{G}^{-1}_{z})} = d\mathcal{H}^{-1}_{\hat{H}(\hat{G}^{-1}_{z})}, \)
\[
\hat{G}^* \mathcal{X}_z = \mathcal{X}_z.
\]

And, as before it is easy to see that \( \mathcal{X} = \mathcal{R} + \cdots \).

Then by the Proposition 11 we have that \( \mathcal{G} \) is linearizable. Furthermore, this implies that \( \mathcal{G} \) is in fact diagonalizable by Proposition 8 (because it preserves \( \{\hat{f}_i\} \)), and its diagonal form is made of roots of the unity because the transversally condition of \( \{\hat{f}_i\} \) implies the existence of \( n \) independent multi-indexes, which is the next step in the proof.

Working for simplicity in dimension two, write \( \hat{f}_1(x) = \sum \alpha_i x^i \) and \( \hat{f}_2(x) = \sum \beta_j x^j \) where \( x = (x_1, x_2) \), \( I = (i, j) \), and \( J = (r, s) \) then
\[
d\hat{f}_1 \wedge d\hat{f}_2 = (\sum \alpha_i \beta_j (i-1)x_1^{i-1} x_2^{j-1} x_1^{i+r-1} x_2^{j+s-1}) dx_1 \wedge dx_2,
\]
if there were no \( I, J \) independent such that \( \alpha_i \beta_j \neq 0 \) we would have \( d\hat{f}_1 \wedge d\hat{f}_2 = 0 \) contradicting the hypothesis, so there exists a couple \( I_0 = (i_0, j_0), J_0 = (r_0, s_0) \) with this condition. Consider \( \hat{G} \in \mathcal{G} \) and \( G = (d\hat{G})_0 \) its linear part given by a
diagonal matrix with eigenvalues $\lambda_1, \lambda_2$, the conditions $\hat{f}_i \circ G = \hat{f}_i$ for $i = 1, 2$ implies $\lambda_i^0 \lambda_j^0 = 1$ and $\lambda_i^0 \lambda_j^0 = 1$ respectively and, as before this implies that both are roots of the unity. Indeed, the previous analysis is more subtle, because we have to consider $\hat{f}_i \circ \hat{G} = (\hat{f}_i \circ g)(g^{-1} \circ \hat{G} \circ g) = (\hat{f}_i \circ g)(G) = (\hat{f}_i \circ g)$ where $g$ is a formal diffeomorphism who diagonalizes $G$, and then compare the first non-null therms in both sides.

In general we have something like $d\hat{f}_1 \wedge \cdots \wedge d\hat{f}_n \neq 0$ and $\hat{f}_i(x) = \sum_J a_{1J} x^J$ with $I = (i_1, \ldots, i_n)$, but the associativity of the wedge product permit us work with each couple, for instance

$$d\hat{f}_1 \wedge d\hat{f}_2 = \sum_{r<s} \left( \frac{\partial \hat{f}_1}{\partial x_r} \frac{\partial \hat{f}_2}{\partial x_s} - \frac{\partial \hat{f}_1}{\partial x_s} \frac{\partial \hat{f}_2}{\partial x_r} \right) dx_r \wedge dx_s,$$

each term of the sum works like the previous case and at list one of them should be not zero meaning that it exist a couple $(i_r, i_s), (j_r, j_s)$ independent and with this $I = (i_1, \ldots, i_r, \ldots, i_s, \ldots, i_n)$ and $J = (j_1, \ldots, j_r, \ldots, j_s, \ldots, j_n)$ are independent and its coefficients are not zero $a_{IJ} \neq 0$. Thus, the following sum is not zero,

$$\sum_{r<s} a_{IJ} x^{i_r + j_r - 1} x^{i_s + j_s - 1} dx_r \wedge dx_s,$$

where $I + J - (e_r + e_s) = (i_1 + j_1, \ldots, i_r + j_r - 1, \ldots, i_s + j_s - 1, \ldots, i_n + j_n)$. The wedge product with the next form, $d(\hat{f}_3(x)) = d(\sum_K a_{IK} x^K)$, will produce terms having $3 \times 3$ matrices related to the multi-indexes $I, J$ and $K$, and obviously the dependence of $K$ with $I, J$ would imply that all of them are zero. This process continues implying the existence of $n$ independent multi-indexes such that $\lambda_i^0 \cdots \lambda_i^0 = 1$ for each one of them, and the $\lambda_i$’s are roots of the unity as before.

Therefore, there exists $N \in \mathbb{N}$ such that $G^N = I$ and then $\langle \hat{G} \rangle$ (i.e. the group generated by $\hat{G}$) is finite. It remains to prove that $\mathcal{G}$ is commutative, consider $\hat{G}_1, \hat{G}_2 \in \mathcal{G}$ and note by $G_1, G_2$ their linear parts, then

$$\hat{G}_1 \circ \hat{G}_2 = g(g^{-1} \circ \hat{G}_1 \circ g)(g^{-1} \circ \hat{G}_2 \circ g)g^{-1}$$

$$= g(G_1 \circ G_2)g^{-1} \quad \text{they commute},$$

$$= g(G_2 \circ G_1)g^{-1},$$

$$= \hat{G}_2 \circ \hat{G}_1.$$  

\[ \square \]

### 3 First Integrals

We want to show that the existence of a formal first integral (in our setting) implies the existence of a holomorphic one. In order to establish this, we need first some properties of formal series.

#### 3.1 Formal Chain Rule

The aim of this paragraph is to show that the Chain Rule holds in the formal case we work with.
Lemma 12. Let $\hat{F} \in \mathcal{O}_n$ and $G \in \text{Diff}(\mathbb{C}^n, 0)$ be given. Then

$$d(\hat{F} \circ G) = d\hat{F} \cdot dG.$$ 

Proof. We start with $n = 1$, let $\hat{f} \in \mathcal{O}_1$ given by $\hat{f}(x) = \sum_{i=1}^{\infty} a_i x^i$, define $f_n \in \mathcal{O}_1$ by $f_n(x) = \sum_{i=1}^{n} a_i x^i$ and take $g \in \mathcal{O}_1$. We want to show that $d(\hat{f} \circ g) = d\hat{f}_n d\hat{g}$.

We already have that $d(f_n \circ g) = (df_n)_g d\hat{g}$, because they are holomorphic functions, and also, by the definition of the derivative of a formal series, we have $\lim_{n \to \infty} d f_n = d \hat{f}$. Therefore, we need to justify is that $\lim_{n \to \infty} (df_n)_g = (df)_g$ and $\lim_{n \to \infty} d(f_n \circ g) = d(f \circ g)$, both are consequence of the equality $\lim_{n \to \infty} f_n \circ g = \hat{f} \circ g$ and for this, we think in the coefficient of $x^k$ in $\hat{f} \circ g(x) = \sum_{i=1}^{\infty} a_i (\sum_{j=1}^{\infty} b_j x^j)^i$, where $g(x) = \sum_{j=1}^{\infty} b_j x^j$. This coefficient is formed after algebraic computation by some of the coefficients in $\sum_{i=1}^{k} a_i (\sum_{j=1}^{\infty} b_j x^j)^i$, indeed after $i, j = k$ all the elements in $\sum_{i=1}^{\infty} a_i (\sum_{j=1}^{\infty} b_j x^j)^i$ are of order greater than $k$, thus the same coefficients of $x^k$ belongs to both sides of $\lim_{n \to \infty} f_n \circ g = \hat{f} \circ g$.

Hence

$$d(\hat{f} \circ g) = d\hat{f}_n d\hat{g},$$

as we wanted.

Consider now $g \in \mathcal{O}_2$ and the same $\hat{f}$ that before. In this case the chain rule is consequence of the previous one, because if we fix one of the variables for example $y = y_0$, then $g(\cdot, y_0) \in \mathcal{O}_1$ and $\frac{\partial}{\partial x}(f \circ g) = d\hat{f}_n g(\hat{f}(x, y_0)) + d\hat{g}(x, y_0)$ by the previous case.

For the two dimensional case take $\hat{F} \in \mathcal{O}_2$ and $G(x, y) = (g_1(x, y), g_2(x, y))$ given by $F(x) = \sum_{i} a_{i}x^i$ and $g_1, g_2 \in \mathcal{O}_2$, then we have

$$\hat{F} \circ G(x, y) = \sum_{i} a_{i}(g_1(x, y))^i (g_2(x, y))^j,$$

$$= \sum_{i} (g_1(x, y))^i \left( \sum_{j} a_{i,j} (g_2(x, y))^j \right), \text{ note } \hat{F}_i(x) = \sum_{j} a_{i,j} x^j,$$

$$= \sum_{i} (g_1(x, y))^i \hat{F}_i(g_2(x, y)).$$

So, $\hat{F} \circ G$ can be written as a sum of the product of a two formal series $(g_1(x, y))^i$ and $\hat{F}_i(g_2(x, y))$, whose derivatives are known by the previous case. Now note that $\hat{F} \circ G$ is a formal series then is derivation is made term by term, and in the previous paragraph we only rearrange those terms, thus

$$\frac{\partial}{\partial x}(\hat{F} \circ G)(x, y) = \sum_{i} \frac{\partial}{\partial x} \left( (g_1(x, y))^i \hat{F}_i(g_2(x, y)) \right),$$

$$= \sum_{i} \left( i g_1^{-1} \frac{\partial g_1}{\partial x} \hat{F}_i(g_2) + g_1^i \frac{\partial \hat{F}_i}{\partial g_2} \right)(x, y),$$

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\[
\sum_i \left( ig_i^{-1} \frac{\partial g_i}{\partial x} \sum_j a_{i,j} g_j^2 + g_i \left( \sum_j j a_{i,j} g_j^{-1} \right) \frac{\partial g_2}{\partial x} \right) (x, y),
\]

\[
= \sum_{i,j} \left( i a_{i,j} (g_1(x,y))^{-1} (g_2(x,y))^{-1} \frac{\partial g_1}{\partial x} + j a_{i,j} (g_1(x,y))^{-1} (g_2(x,y))^{-1} \right) \frac{\partial g_2}{\partial x},
\]

\[
= \frac{\partial \hat{F}}{\partial x} \bigg|_G \frac{\partial g}{\partial x} (x, y).
\]

In conclusion, the chain rule works for the case \( \hat{F} \in \hat{O}_2 \) and \( G \) as before, and the process above is easily generalized to greater dimension. \( \square \)

### 3.2 Algebraic criterion

The following lemma and proposition are, at first glance, mostly \( n \) dimensional versions of Lemma 1 and Proposition 1 in [3]. Nevertheless, there is a big difference which turns out to be an important property as we will see later.

**Lemma 13.** Let \( \Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \setminus 0 \) and, let \( N_{n-1 \times n} \) be a matrix with entries in \( \mathbb{N} \) and linearly independent lines, satisfying

\[ N \Lambda^t = 0 \in \mathbb{C}^{n-1}. \]

Then there are \( k_1, \ldots, k_n \in \mathbb{Z} \) and \( \lambda \in \mathbb{C}^* \) such that

\[(\lambda_1, \ldots, \lambda_n) = (k_1, \ldots, k_n) \lambda. \]

**Proof.** The proof consists in the solution of a linear system, take

\[ N = \begin{bmatrix}
  n_{1,1} & \cdots & n_{1,n-1} & n_{1,n} \\
  \vdots & \ddots & \vdots & \vdots \\
  n_{n-1,1} & \cdots & n_{n-1,n-1} & n_{n-1,n}
\end{bmatrix}, \quad A = \begin{bmatrix}
  n_{1,1} & \cdots & n_{1,n-1} \\
  \vdots & \ddots & \vdots \\
  n_{n-1,1} & \cdots & n_{n-1,n-1}
\end{bmatrix}
\]

the independence allows to take \( n - 1 \) independent columns, suppose the first ones, and form the matrix \( A \) which is invertible, thus multiplying by \( A^{-1} \) the system \( N \Lambda^t = 0 \) we get,

\[ \begin{bmatrix}
  1 & \cdots & 0 & \tilde{k}_1 \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & 1 & \tilde{k}_{n-1}
\end{bmatrix}
\begin{bmatrix}
  \lambda_1 \\
  \vdots \\
  \lambda_n
\end{bmatrix} = \begin{bmatrix}
  0 \\
  \vdots \\
  0
\end{bmatrix}_{n-1 \times 1},
\]

and we have \( n - 1 \) equation of the form \( \lambda_i + \tilde{k}_i \lambda_n = 0 \), then

\[(\lambda_1, \ldots, \lambda_n) = (-\tilde{k}_1, \ldots, -\tilde{k}_{n-1}, 1) \lambda_n,
\]

we know exactly who are the \( \tilde{k}_i \)'s, because they satisfy

\[ \begin{bmatrix}
  n_{1,1} & \cdots & n_{1,n-1} \\
  \vdots & \ddots & \vdots \\
  n_{n-1,1} & \cdots & n_{n-1,n-1}
\end{bmatrix}
\begin{bmatrix}
  \tilde{k}_1 \\
  \vdots \\
  \tilde{k}_{n-1}
\end{bmatrix} = \begin{bmatrix}
  n_{1,n} \\
  \vdots \\
  n_{n-1,n}
\end{bmatrix}.
\]
and by the Cramer rule, \( \hat{k}_i = \frac{|A|}{|A_i|} \), where \( |\cdot| \) means determinant and \( A_i \) is the matrix \( A \) changing the column \( i \) by \([n_1 \ldots n_{i-1} n_n]^t\). Finally we get,

\[
(\lambda_1, \ldots, \lambda_n) = (|A_1|, \ldots, |A_{n-1}|, -|A|) \lambda,
\]

with \( \lambda = -\lambda_n/|A| \) and \( k_i = |A_i|, k_n = -|A| \in \mathbb{Z} \) for \( i = 1, \ldots, n-1 \) as we wanted.

The difference is that in dimension 3 we know that \( k_1 \cdot k_2 \cdot k_3 < 0 \), so we can make one of them negative and the others positive by changing the \( \lambda \). However, in dimension \( n > 3 \), the only thing we know about the signs of the \( k_i \) is that they cannot be all positive nor negative thanks to the condition \( n_1k_1 + \cdots + n_nk_n = 0 \).

Here we have an example in dimension 4 where \( k_1 \cdot k_2 \cdot k_3 \cdot k_4 > 0 \), take

\[
N = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix},
\]

if satisfies \( NA^t = 0 \) for some \( \Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) then,

\[
(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-1, 1, 1, -1) \lambda.
\]

**Proposition 14.** Suppose that \( \mathcal{X} \in \text{Gen}(\mathfrak{X}(\mathbb{C}^n, 0)) \) has a formal first integral, then \( \mathcal{F}(\mathcal{X}) \) can be given in local coordinates by a vector field of the form

\[
\mathcal{X}(x) = k_1x_1(1 + a_1(x)) \frac{\partial}{\partial x_1} + \cdots + k_nx_n(1 + a_n(x)) \frac{\partial}{\partial x_n},
\]

where \( k_1, \ldots k_n \in \mathbb{Z} \) and \( a_1, \ldots, a_n \in \mathcal{M}_n \). In particular if \( n = 3 \), \( \mathcal{X} \) satisfies condition (⋆).

**Proof.** We are considering \( \mathcal{X} \in \text{Gen}(\mathfrak{X}(\mathbb{C}^n, 0)) \), and by definition, after a change of coordinates can be written in the form

\[
\mathcal{X}(x) = \lambda_1x_1(1 + a_1(x)) \frac{\partial}{\partial x_1} + \cdots + \lambda_nx_n(1 + a_n(x)) \frac{\partial}{\partial x_n},
\]

suppose now that \( \hat{F} = (\hat{f}_1, \ldots, \hat{f}_{n-1}) \) is the formal first integral, then \( \mathcal{X}(\hat{f}_i) \equiv 0 \) for \( i = 1, \ldots, n-1 \). If \( \hat{f}_i(x) = \sum_{|I| > p_i} a_I x^I \) then

\[
\frac{\partial \hat{f}_i}{\partial x_r} = \sum_{|I| > p_i} (i_r)a_I x_1^{i_1} \cdots x_{r-1}^{i_{r-1}} x_r x_{r+1}^{i_{r+1}} \cdots x_n^{i_n},
\]
and

\[ X(\hat{f}_i) = \sum_{r=1}^{n} \lambda_r x_r(1 + a_r(x)) \left( \sum_{|I| > p_i} (i_r)_I a_I x_{i_1}^1 \cdots x_{i_n}^n \right), \]

\[ = \sum_{|I| > p_i} \sum_{r=1}^{n} i_r \lambda_r a_I (1 + a_r(x)) x_I, \]

\[ = \sum_{|I| > p_i} \sum_{r=1}^{n} i_r \lambda_r \delta_I (1 + a_r(x)) x^I, \]

\[ = \sum_{|I| > p_i} a_I \left( \sum_{r=1}^{n} i_r \lambda_r \right) x^I + \sum_{|I| > p_i} a_I \left( \sum_{r=1}^{n} i_r \lambda_r a_I (x) \right) x^I, \]

\[ J^p X(\hat{f}_i) = \sum_{|I| > p_i} a_I \left( \sum_{r=1}^{n} i_r \lambda_r \right) x^I = 0, \]

then \( \sum_{r=1}^{n} i_r \lambda_r = 0 \) for each \( I = (i_1, \ldots, i_n) \) when \( a_I \neq 0 \). For the demonstration of Proposition 10 we know that there are \( n - 1 \) \( n \)-tuples satisfying this condition, with them we can form the matrix \( N \) as in Lemma 13 and we are done.

### 3.3 Holonomy and formal first integrals

We know that holonomy maps leave invariant the level sets of a holomorphic first integral. What we want to obtain is a similar invariant relation in the case of a formal one, for simplicity we work in dimension 3 but small changes are needed for the general case. Consider the foliation given by

\[ X(x_1, x_2, x_3) = px_1 a_1(x) \frac{\partial}{\partial x_1} + qx_2 a_2(x) \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}, \]

where \( a_1, a_2 \in M_3 \) and \( p, q \in \mathbb{Q} \), be \( S := (x_1 = x_2 = 0) \) and \( \Sigma := (x_3 = 1) \). Now consider the closed loop \( \gamma : [0, 1] \to S \) given by \( \gamma(t) = (0, 0, e^{2\pi i t}) \) and let \( \Gamma_{(x_1, x_2)}(t) = (\Gamma_1(x_1, x_2, t), \Gamma_2(x_1, x_2, t), \gamma(t)) \) be its lifting along the leaves of \( \mathcal{F}(\mathcal{X}) \) starting at \( (x_1, x_2, 1) \in \Sigma \). In particular, the map \( h \in \text{Diff}(\mathbb{C}^2, 0) \) given by \( \Gamma_{(x_1, x_2)}(1) = (h(x_1, x_2), 1) \) is a generator of \( \text{Hol}(\mathcal{F}(\mathcal{X}), S, \Sigma) \). Since \( \Gamma_{(x_1, x_2)}(t) \) belongs to a leaf of \( \mathcal{F}(\mathcal{X}) \), then

\[ \frac{\partial}{\partial t} \Gamma_{(x_1, x_2)}(t) = \alpha X(\Gamma_1(x_1, x_2, t), \Gamma_2(x_1, x_2, t), \gamma(t)). \]

From this vector equation one has \( \gamma'(t) = \alpha \gamma(t) \), thus \( \alpha = 2\pi i \). Furthermore,

\[ \frac{\partial \Gamma_1}{\partial t} = 2p\pi i \Gamma_1(x_1, x_2, t)a_1(\Gamma), \]

\[ \frac{\partial \Gamma_2}{\partial t} = 2q\pi i \Gamma_2(x_1, x_2, t)a_2(\Gamma). \]

**Remark 15.** Note that by Proposition 13 we can take the vector field \( \mathcal{X} \) as before.
Suppose that \( F = (\hat{f}_1, \hat{f}_2) \), with \( \hat{f}_1, \hat{f}_2 \in \mathcal{O}_3 \), is a formal first integral of the foliation \( \mathcal{F}(\mathcal{X}) \), this means \( \mathcal{X}(\hat{F}) \equiv 0 \) and then,

\[
0 = px_1a_1(x_1, x_2, x_3) \frac{\partial \hat{f}_1}{\partial x_1} + qx_2a_2(x_1, x_2, x_3) \frac{\partial \hat{f}_1}{\partial x_2} + x_3 \frac{\partial \hat{f}_1}{\partial x_3},
\]

by evaluating \( \Gamma \) and multiplying by \( 2\pi i \),

\[
0 = 2\pi i \Gamma(x_1, x_2, 1) \frac{\partial \hat{f}_1}{\partial x_1} + 2\pi i \Gamma(x_2, 1) \frac{\partial \hat{f}_1}{\partial x_2} + 2\pi i \gamma(t) \frac{\partial \hat{f}_1}{\partial x_3}
\]

\[
0 = \frac{\partial \Gamma_1}{\partial t} \frac{\partial \hat{f}_1}{\partial x_1} + \frac{\partial \Gamma_2}{\partial t} \frac{\partial \hat{f}_1}{\partial x_2} + \gamma'(t) \frac{\partial \hat{f}_1}{\partial x_3}.
\]

The last line (which also has for \( \hat{f}_2 \)) implies that \( \hat{f}_1 \circ \Gamma \) is constant in \( t \), then,

\[
\hat{f}_1 \circ \Gamma(x_1, x_2, 1) = \hat{f}_1 \circ \Gamma(x_1, x_2, 0),
\]

\[
\hat{f}_1(h(x_1, x_2), 1) = \hat{f}_1(x_1, x_2, 1).
\]

In conclusion, we obtain the relation we were looking for:

\[
\hat{F}(h(x_1, x_2), 1) = \hat{F}(x_1, x_2, 1)
\]

### 3.4 From formal to holomorphic first integral

Now we are in conditions to prove our first main result:

**Proof of Theorem A.** By definition of formal first integral \( df_1 \wedge df_2 \neq 0 \) and by Proposition 1 in [2], the vector field \( \mathcal{X} \) can be written as:

\[
\mathcal{X}(x) = mx_1(1 + a_1(x)) \frac{\partial}{\partial x_1} + nx_2(1 + a_2(x)) \frac{\partial}{\partial x_2} - kx_3(1 + a_3(x)) \frac{\partial}{\partial x_3},
\]

were \( m, n, k \in \mathbb{Z}_+ \) and \( a_1, a_2, a_3 \in \mathcal{M}_3 \) in particular satisfies condition (*) (the vector field we took in the previous section is this one multiplied by \( -(1 + a_3(x))^{-1} \) who defines the same foliation).

The equalities \( \hat{f}_i(h(x_1, x_2), 1) = \hat{f}_i(x_1, x_2, 1) \), for \( i = 1, 2 \), from the end of the previous section allows us to think only in two variables, making a change of coordinates \( g(x_1, x_2, x_3) = (x_1, x_2, x_3 + 1) \) we have that,

\[
(\hat{f}_i \circ g)^{-1}(h(x_1, x_2), 1) = (\hat{f}_i \circ g)^{-1}(x_1, x_2, 1),
\]

\[
(\hat{f}_i \circ g)(h(x_1, x_2), 0) = (\hat{f}_i \circ g)(x_1, x_2, 0),
\]

noting \( \hat{f}_i \circ g(x_1, x_2, 0) = \hat{f}_i(x_1, x_2) \) we can use now the second and third sections, but first, we have to guarantee that they are generically transverse. In general \( df_1 \wedge df_2 \neq 0 \) does not imply \( (df_1 \wedge df_1)_{x_3 = 1} = df_1 \wedge df_2 \neq 0 \).

Again, using the previous section we have

\[
\hat{f}_1 \circ \Gamma(x_1, x_2, t) = \hat{f}_1 \circ \Gamma(x_1, x_2, 0),
\]

\[
\hat{f}_1(\Gamma(x_1, x_2, t)) = \hat{f}_1(x_1, x_2, 1),
\]
thus $d(\hat{f}_1 \circ \Gamma) \wedge d(\hat{f}_2 \circ \Gamma) = d\hat{f}_1 \wedge d\hat{f}_2$, if the second part is null then the first one too. Hence, fixing $(x_1, x_2)$ over the path $\Gamma(x_1, x_2)(t)$ we have that $d\hat{f}_1 \wedge d\hat{f}_2 \equiv 0$ and, $\Gamma$ is defined for $(x_1, x_2)$ in a small neighborhood of 0 in $x_3 = 1$ whose saturated is a neighborhood of 0 in $C^3$ (see, [10]) then $d\hat{f}_1 \wedge d\hat{f}_2 \equiv 0$ contradicting our hypothesis.

With this in mind, by Proposition [10] we have that $\text{Hol}(\mathcal{F}(\mathcal{X}), S, \Sigma)$ is periodic because it preserves $\{\tilde{f}_1, \tilde{f}_2\}$ and, its generated by one germ of diffeomorphism. Therefore, the Theorem [2] implies that $\mathcal{F}(\mathcal{X})$ has a holomorphic first integral.

As for arbitrary dimension we have:

Proof of Theorem B. The proof goes on as the previous one but now we use Theorem 5 in [9] which needs the condition ($\ast$).

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