GEOMETRIC PROPERTIES OF SOME TOTALLY ORDERED COMPACT SETS

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Abstract. In this paper, we show that there are a totally ordered compact $K$ separable, a Hausdorff topology $\tau'$ on $C(K)$ and two closed subspaces $Y_1, Y_2$ of $(C(K), \tau_p)$ such that $(C(K), \tau')$ is not universally measurable, $(C(K), \tau_p) = (Y_1, \tau_p) \oplus (Y_2, \tau_p)$, $(Y_1, \tau_p)$ is isomorphic to $(Y_2, \tau_p)$, $(Y_j, \tau_p) = (Y_j, \tau')$, $j \in \{1, 2\}$ and $\text{Bor}(C(K), \tau') \otimes \text{Bor}(C(K), \tau') \neq \text{Bor}(C(K)) \times C(K)$, $\tau' \otimes \tau'$, this is the main result of this work.

We start this work to construct totally ordered non-metrisable compact sets $K_\mu(E)$ from a reference set $E$ which is totally ordered, and from a positive Borel measure on $E$ satisfying some reasonable assumptions.

At first, we introduce a totally ordered set $E$, which is equipped with the order topology $\tau_0$, and a non-negative Borel measure on $(E, \tau_0)$. To these two elements we associate a compact set $K=K_\mu(E)$, obtained as spectrum of a certain $C^*$-subalgebra of $L^\infty(E, \mu)$. We impose some natural properties on $E$ and $\mu$ that make the compact $K$ not metrisable.

Afterward, we introduce a certain function $h_{\mu,E} = h$ defined on a subset of $K$ with values in $E$ or in $\overline{E} = E \cup \{-\infty, +\infty\}$. We study the surjectivity and the continuity of the function $h$; this function plays an important role for understanding the geometric properties of $K$.

We will see that $E$ is homeomorphic to $K$, When $E$ is a separable connected strongly Lindelöf space and if $\mu(I) > 0$ for every nonempty interval of $E$.

We study in detail a particular case when $E$ is a locally compact totally ordered group satisfying the following conditions:

$\forall a, b \in E$ such that $a < b$, $\exists c \in E$ verifying $a < c < b$. (*)

$\ a \leq b \iff a - b \leq 0, \ a, b \in E. \ (**)$

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Our main result (theorem 6) will be shown, for this case and we will see that the compact $K$ ($K$ is a Rosenthal compact set) has the following property:

$$\text{Bor}(C(K), \tau_p) \otimes \text{Bor}(C(K), \tau_p) = \text{Bor}(C(K)) \times C(K), \tau_p \otimes \tau_p$$

(with the continuous hypothesis) but $\text{Bor}(K) \otimes \text{Bor}(K) \neq \text{Bor}(K \times K)$.

Recall that if $L$ is a metrisable compact set, then $\text{Bor}(L) \otimes \text{Bor}(L) = \text{Bor}(L \times L)$.

We will see in part 6 (lemma 14) that if $L$ is a Rosenthal compact set, then $(C(L), \tau_p)$ is universally measurable. In theorem 6 the Hausdorff topology $\tau'$ on $C(K)$ satisfying $(Y_j, \tau_p) = (Y', \tau')$, $j \in \{1, 2\}$ and $(C(K), \tau_p) = (Y_1, \tau_p) \oplus (Y_2, \tau_p)$, $(Y_1, \tau_p)$ but $(C(K), \tau')$ is not universally measurable (here $Y_1$ and $Y_2$ are two closed subspaces of $(C(K), \tau_p)$).

Let $(E, \leq)$ be a totally ordered set. The order topology $\tau_0$ on $E$ is the topology generated by the intervals of the form $\{x \in E; x < a\}$ or of the form $\{x \in E; x > b\}$, $a, b \in E$. This topology is Hausdorff (if $x < a < y$ in $E$, $V_x = \{t \in E; t < a\}$ and $V_y = \{t \in E; t > a\}$ are disjoint neighborhoods of $x$ and $y$; if there is no $a$ between $x$ and $y$, $V_x = \{t \in E; t < y\}$ and $V_y = \{t \in E; t > x\}$ do the job).

We denote by $\tau_1$ (resp. $\tau_2$) the topology on $E$ generated by the intervals of the form $\{x \in E; x > a\}$ or $\{x \in E; x \leq b\}$ (resp. of the form $\{x \in E; x < a\}$ or $\{x \in E; x \geq b\}$), $a, b \in E$.

Let $(X, \tau)$ be a topological space and $X_1$ a subset of $X$. $(X_1, \tau)$ (since unless is said otherwise) denotes the topological space formed by the relative open subsets (a relative open subset of $X_1$ is of the form $X_1 \cap V$, where $V$ is an open subset of $(X, \tau)$).

1. Construction of totally ordered compact sets

In this part we construct totally ordered compact sets $K = K_\mu(E)$ non metrisables, in passing, we introduce the function $h$, this function help us to know the topological structure of $K$.

Let $(E, \leq)$ be a totally ordered set. If $E$ admits a maximum (resp. a minimum) we denote

$$+\infty = \max\{x \in E; x \in E\} \quad \text{(resp.} \quad -\infty = \min\{x \in E; x \in E\}).$$

Let $\mu$ be a positive Borel measure on $(E, \tau_0)$. One supposes that the measure $\mu$ verifies the following condition:

$$\mu(\{u \in E; u > t\}) > 0, \quad \forall t \in E \setminus \{+\infty\}. \quad (1.1)$$

Let $t \in E$ and let $f^t$ be the characteristic function of $\{u \in E; u > t\}$. Denote by $A$ the $C^*$-subalgebra of $L^\infty(\mu)$ (with unit) generated by the
family \( \{ f^t, t \in E \} \) (the scalars are complex scalars). Let \( K(E) \) be the set of characters on \( A \). By \cite{9}, \cite{24}, \( K_\mu(E) \) is \( \sigma(A^*, A) \) compact and \( A = C(K_\mu(E)) \).

As \( f^s f^t = f^{s+t} \) when \( s, t \in E \), the family \( \{ f^t \}_{t \in E} \) is stable under product, and \( A \) is simply the closure in \( L^\infty(\mu) \) of the vector space generated by the constant function 1 (the unit e of \( A \)) and the family \( \{ f^t \}_{t \in E} \). As a consequence, a character \( \theta \in K \) is fully determined by its values on the \( f^t \): if \( \theta(f^t) = \eta(f^t) \) for all \( t \in E \), then \( \theta = \eta \).

If \( \theta \in K_\mu(E) \) is a character and \( \chi_F \) the characteristic function of a subset \( F \subset E \), then \( \theta(\chi_F) = 0 \) or 1 because \( (\chi_F)^2 = \chi_F \). In particular, \( \theta(f^t) \in \{0, 1\} \) and furthermore, when \( s < t \in E \), we have \( \theta(f^s) \geq \theta(f^t) \) since \( f^s = f^t + \chi_{(s,t]} \).

For every \( t \in E \) let
\[
[t] = \left\{ t' \in E; \| f^t - f^{t'} \|_{L^\infty(\mu)} = 0 \right\} = \{ t' \in E; \mu(\{t \wedge t', t \vee t'\}) = 0 \}.
\]
In the following, we suppose that
\[
\{ [t]; t \in E \} \text{ is uncountable.} \tag{1.2}
\]

Since \( \| f^t - f^{t'} \|_{L^\infty(\mu)} \geq 1 \) for \( [t] \neq [t'] \), \( C(K_\mu(E)) = A \) is not a separable space, hence \( K_\mu(E) \) is not metrisable.

We define the function \( \psi_{\mu,E} : K_\mu(E) \times E \to \{0, 1\} \) by
\[
\psi_{\mu,E}(\theta, t) = \theta(f^t), \quad \theta \in K, \ t \in E.
\]
When we have that \(-\infty, +\infty \in E \), the function \( \psi_{\mu,E} \) is defined on \( K_\mu(E) \times [E \setminus \{-\infty, +\infty\}] \), with values in \( \{0, 1\} \) as was said above.

Let \( B_1^{\mu,E} \) be the subset of all characters \( \theta \in K \) such that the map \( (E, \tau_0) \ni t \mapsto \psi_{\mu,E}(\theta, t) \) is continuous; this means that the two sets \( \{ t \in E; \psi_{\mu,E}(\theta, t) = \varepsilon \} \), for \( \varepsilon = 0, 1 \), are both closed (and both open). Let now \( \theta \in K_\mu(E) \setminus B_1^{\mu,E} \); there are two types of discontinuities.

Case 1, \( \{ t \in E; \psi_{\mu,E}(\theta, t) = 0 \} \) is not closed. Let \( h_{\mu,E}(\theta) \) be a point in the closure of \( \{ t \in E; \psi_{\mu,E}(\theta, t) = 0 \} \) in \( E \) such that \( \psi_{\mu,E}(\theta, h_{\mu,E}(\theta)) = 1 \).

**Lemma 1.** The point \( h_{\mu,E}(\theta) \) is unique.

**Proof.** Suppose that there exists \( h'(\theta) \in E \) in the closure of the set \( E_0(\theta) = \{ t \in E; \psi_{\mu,E}(\theta, t) = 0 \} \), such that \( \psi_{\mu,E}(\theta, h'(\theta)) = 1 \) and \( h_{\mu,E}(\theta) < h'(\theta) \). The set \( \{ u \in E; u < h'(\theta) \} \) is then an open neighborhood of \( h_{\mu,E}(\theta) \), but \( h_{\mu,E}(\theta) \) is in the closure of \( E_0(\theta) \), hence there is \( u_1 \in E_0(\theta) \) such that \( u_1 < h'(\theta) \). This is impossible, because the map \( t \mapsto \psi_{\mu,E}(\theta, t) \) is a non-increasing function. Thus \( h_{\mu,E}(\theta) \) is unique.
Case 2. \( \{ t \in E; \psi_{\mu,E}(\theta,t) = 1 \} \) is not closed. We let now \( h_{\mu,E}(\theta) \) be the unique point in the closure of \( \{ t \in E; \psi_{\mu,E}(\theta,t) = 1 \} \) in \( E \) such that \( \psi_{\mu,E}(\theta,h_{\mu,E}(\theta)) = 0 \).

Let \( B_{2}^{\mu,E} = \{ \theta \in K_{\mu}(E) \setminus B_{1}^{\mu,E}; \{ t \in E; \psi_{\mu,E}(\theta,t) = 0 \} \) is not closed \} and \( B_{3}^{\mu,E} = \{ \theta \in K_{\mu}(E) \setminus B_{1}^{\mu,E}; \{ t \in E; \psi_{\mu,E}(\theta,t) = 1 \} \) is not closed \}.

When the space \( E \) is fixed, we write \( K = K_{\mu}(E), h_{\mu,E} = h, \psi = \psi_{\mu,E} \) and \( B_{j}^{\mu,E} = B_{j}, j \in \{ 1, 2, 3 \} \).

**Lemma 2.** For every \( \theta \in B_{2} \cup B_{3} \), we have
\[
h(\theta) = \sup \{ t \in E; \psi(\theta,t) = 1 \}.
\]
(1.3)

**Proof.** Remember that the map \( t \mapsto \psi(\theta,t) \) is a non-increasing function; let \( E_{j}(\theta) = \{ t \in E; \psi(\theta,t) = j \} \), for \( j = 0, 1 \).

If \( \theta \in B_{2} \), we have \( \psi(\theta,h(\theta)) = 1 \) while \( h(\theta) \) is in the closure of \( E_{0}(\theta) \); when \( v > h(\theta) \), the set \( \{ u \in E; u < v \} \) is a neighborhood of \( h(\theta) \), hence there is \( u_{1} \in E \) such that \( u_{1} < v \) and \( \psi(\theta,u_{1}) = 0 \), thus \( \psi(\theta,v) = 0 \); when \( \theta \in B_{2} \), the point \( h(\theta) \) is actually the largest element of \( E_{1}(\theta) \).

If \( \theta \in B_{3} \), then \( \psi(\theta,h(\theta)) = 0 \); if \( \psi(\theta,v) = 1 \), we have \( v < h(\theta) \), showing already that \( h(\theta) \) is an upper bound of \( E_{1}(\theta) \). If now we have \( a < h(\theta) \), then \( V = \{ t \in E; t > a \} \) is a neighborhood of \( h(\theta) \), implying that there is \( v_{1} \in E \) such that \( v_{1} > a \) and \( \psi(\theta,v_{1}) = 1 \). This shows that \( a < h(\theta) \) cannot be an upper bound of \( E_{1}(\theta) \). 

By an argument similar to that of Lemma 2 we see that:

**Lemma 3.** For every \( \theta \in B_{2} \cup B_{3} \):
\[
h(\theta) = \inf \{ t \in E; \psi(\theta,t) = 0 \}.
\]

Note that \( B_{2} \cap B_{3} = \emptyset, B_{2} = \{ \theta \in K \setminus B_{1}; \psi(\theta,h(\theta)) = 1 \} \) and \( B_{3} = \{ \theta \in K \setminus B_{1}; \psi(\theta,h(\theta)) = 0 \} \).

**Remark 1.** Assume that \((E,\tau_{0})\) is separable and that \((E,\leq)\) satisfies the condition (*) Then \((E,\tau_{0})\) has a countable basis.

**Proof.** Indeed, let \( (a_{n})_{n \in \mathbb{N}} \) be a dense sequence in \((E,\tau_{0})\). For every \( m,n \in \mathbb{N} \), consider \( V_{m,n} = \{ x \in E; a_{n} > x > a_{m} \} \) and let \( M = \{(m,n) \in \mathbb{N}^{2}; V_{m,n} \neq \emptyset \} \). It is easy to see that \((V_{m,n})_{(m,n) \in M}\) forms a basis of \((E,\tau_{0})\). 

**Example 1.** Let \( E \) be a separable uncountable abelian locally compact totally ordered group [12]. Assume that \( E \) satisfies the condition (*) and that the order topology coincides with the topology of \( E \). Choose \( \mu = m \) the Haar measure on \( E \). We observe that every non empty open
set in \( E \) is of strictly positive measure, and since \( \mu(\{t\}) = 0 \) for every \( t \in E \), the conditions (1.1) and (1.2) are satisfied. We shall study this example in detail in the last part.

**Example 2.** Let \( E = [0, 1] \) and \( \mu = m \) the Lebesgue measure, in corollary 12 we will see that \( K = K_m(E) \) is homeomorphic to the two-arrows space \([0, 1] \times \{0, 1\}\).

**Proposition 1.** The compact set \( K \) is totally ordered (i.e., the topology on \( K \) is defined by a total order relation).

**Proof.** We define on \( K \) the following relation, for all \( \theta_1, \theta_2 \in K \):

\[
\theta_1 \leq \theta_2 \iff \psi(\theta_1, t) = \theta_1(f^t) \leq \theta_2(f^t) = \psi(\theta_2, t), \quad \forall t \in E.
\]

Let us show that \((K, \leq)\) is totally ordered. Pick \( \theta_1, \theta_2 \in K \), \( \theta_1 \neq \theta_2 \). We may assume that there exists \( t_0 \in E \) such that \( \psi(\theta_1, t_0) = 0 \) and \( \psi(\theta_2, t_0) = 1 \). Let \( t \in E \). If \( t < t_0 \), \( \psi(\theta_2, t) = 1 \), hence \( \psi(\theta_1, t) \leq \psi(\theta_2, t) \). If \( t \geq t_0 \), \( \psi(\theta_1, t) = 0 \), hence \( \psi(\theta_1, t) \leq \psi(\theta_2, t) \), this implies that \( \theta_1 < \theta_2 \). It remains to show that the topology of \( K \) and the order topology are identical.

Since \( K \) is compact and \((K, \tau_0)\) is Hausdorff (see the introduction), it is enough to show that the order topology is weaker than the topology of \( K \). Given \( \theta \in K \), let

\[
Z = \{ \alpha \in K; \alpha < \theta \}.
\]

Suppose that \( Z \) is not empty and \( \alpha \in Z \). Since \( \alpha, \theta \) are \( \{0, 1\}\)-valued on the family \((f^t)_{t \in E}\) and \( \alpha < \theta \), it is clear that there is \( t \in E \) such that \( 0 = \alpha(f^t) < \theta(f^t) = 1 \), hence

\[
Z = \bigcup_{t \in E} \{ \alpha \in K; \psi(\alpha, t) = 0 \text{ and } \psi(\theta, t) = 1 \}.
\]

On the other hand, \( K \ni \alpha \mapsto \alpha(f^t) \) is continuous for the topology of \( K \), namely, the \( \sigma(A^*, A) \) topology; for every \( t \in E \) the set

\[
\{ \alpha \in K; \psi(\alpha, t) = 0 \text{ and } \psi(\theta, t) = 1 \}
\]

is an open subset of \( K \), hence \( Z \) is an open subset of \((K, \sigma(A^*, A))\). By the same proof, \( Z' = \{ \alpha \in K; \alpha > \theta \} \) is an open subset of \( K \).

**Definition 1.** Let \( K \) be a Hausdorff compact space. We say that \( C(K) \) admits an equivalent \( \tau_p\)-Kadec norm, if there exists an equivalent norm \( \rho \) on \( C(K) \) such that the strong topology and the \( \tau_p \) topology coincide on \( \{ f \in C(K); \rho(f) = 1 \} \).

**Corollary 1.** The space \( C(K) \) admits an equivalent \( \tau_p\)-Kadec norm.
Proof. This is a consequence of Proposition 1 and of Theorem A in [10]. ■

When \((X, \tau)\) is a topological space, we denote by \(\text{Bor}(X, \tau)\) the Borel \(\sigma\)-algebra of subsets of \(X\) that is generated by the class of open subsets of \((X, \tau)\).

**Corollary 2.** We have \(\text{Bor}(C(K), \|\cdot\|) = \text{Bor}(C(K)), \tau_p)\).

**Proof.** Corollary 1 shows us that there exists on \(C(K)\) an equivalent \(\tau_p\)-Kadec norm. By [13], it follows that \(\text{Bor}(C(K), \|\cdot\|) = \text{Bor}(C(K)), \tau_p)\). ■

2. **Surjectivity of \(h\)**

In this part, we study the surjectivity of \(h\), We find sufficient conditions for \(h\) to be surjective.

On the set \(\{[t]; t \in E\}\) we define an order relation by
\[
[t] < [u] \text{ if } t < u \text{ and } \mu([t, u]) > 0
\]
(we know that \([t] = [u]\) when \(\mu([t \land u, t \lor u]) = 0\)). It is obvious that the set \(\{[t]; t \in E\}\) is totally ordered by this relation.

**Definition 2.** A totally ordered set is said to be complete if every nonempty subset that has an upper bound, has a least upper bound.

One sees then that every nonempty subset \(F\) that has a lower bound has a greatest lower bound, by applying the preceding definition to the set \(F'\) of lower bounds of \(F\).

We denote by \(\theta'_{E} = \theta'\) the maximal element of \(K\) and by \(\theta''_{E} = \theta''\) the minimal element of \(K\). Note that \(\psi(\theta', t) = 1\) and \(\psi(\theta'', t) = 0\), for every \(t \in E \setminus \{-\infty, +\infty\}\). It is clear that \(\theta', \theta'' \in B_1\), the subset of \(K\) consisting of those \(\theta\) with \(t \mapsto \theta(f^t)\) continuous on \((E, \tau_0)\).

**Remark 2.** Denote by \(h_j\) the restriction of \(h\) to \(B_j\), \(j \in \{2, 3\}\); then \(h_j: B_j \to E\) is an injective function, for \(j \in \{2, 3\}\).

**Proof.** Let us show for example that \(h_2\) is an injective function. Let \(\theta_1, \theta_2 \in B_2\) such that \(\theta_1 < \theta_2\). There exists \(t \in E\) such that \(\psi(\theta_1, t) = 0\) and \(\psi(\theta_2, t) = 1\), it follows that \(h_2(\theta_1) < t \leq h_2(\theta_2)\), (by Lemma 2 we know that \(t \leq h_2(\theta_2)\)) hence \(h_2(\theta_1) \neq h_2(\theta_2)\). ■
Remark 3. Suppose that \((E, \leq)\) is complete. Let \(\theta \in B_1 \setminus \{\theta', \theta''\}\). It is clear that \(\{t \in E; \psi(\theta, t) = 1\}\) and \(\{u \in E; \psi(\theta, u) = 0\}\) are both nonempty, for otherwise \(\theta\) would coincide on every \(f^t, t \in E\), with either \(\theta'\) or \(\theta''\), implying that \(\theta = \theta'\) or \(\theta = \theta''\).

Suppose that \((E, \leq)\) is complete and let \(\theta \in B_1 \setminus \{\theta', \theta''\}\); there exists \(u' \in E\) with \(\psi(\theta, u') = 0\). Note that we have \(u' \geq u\) for every \(u \in E\) such that \(\psi(\theta, u) = 1\), hence \(\sup\{t \in E; \psi(\theta, t) = 1\}\) exists by completeness. Let us extend to \(B_1 \setminus \{\theta', \theta''\}\) the definition of \(h\) by setting \(h(\theta) = \sup\{t \in E; \psi(\theta, t) = 1\}\) (now, this formula holds for every \(\theta \in K \setminus \{\theta', \theta''\}\), see Lemma 2).

Lemma 4. Suppose that the order on \(E\) is complete. Then for every \(\theta \in B_1 \setminus \{\theta', \theta''\}\), one has \(\psi(\theta, h(\theta)) = 1\).

Proof. Let \(\theta \in B_1 \setminus \{\theta', \theta''\}\) and set \(E_1(\theta) = \{t \in E; \psi(\theta, t) = 1\}\). It is easy to see that \(h(\theta) = \sup E_1(\theta)\) belongs to the closure of \(E_1(\theta)\) for the order topology \(\tau_0\). But \(E_1(\theta) = \{t \in E; \theta(f^t) = 1\}\) is closed since \(\theta \in B_1\) and \(t \mapsto \theta(f^t)\) is continuous. It follows that \(h(\theta) \in E_1(\theta)\). ■

Denote \(\overline{E} = E \cup \{-\infty, +\infty\}\) (\(\overline{E} = E\), if \(-\infty, +\infty \in E\)). It is obvious that \(\overline{E}\) is totally ordered. Denote by \(e\) the unit element of \(C(K)\). When \(F\) is a subset of \(E\), denote by \(\chi_F\) the characteristic function of \(F\).

Lemma 5. Let \(t \in E \setminus \{-\infty\}\), then there exists a character \(\theta_{t,E} \in K\) (we denote \(\theta_{t,E} = \theta_t\) when \(E\) is fixed) such that for every \(u \in E\), we have \(\theta_t(f^u) = 1\) and only if \([u] \leq [t]\) (the value is 0 otherwise, of course).

Proof. We shall define a function \(\theta_t\) on \(A\) by setting first

\[
\begin{align*}
\theta_t(f^u) &= 1 & \text{if } [u] \leq [t], \\
\theta_t(f^u) &= 0 & \text{if } [u] > [t], \\
\theta_t(e) &= 1,
\end{align*}
\]

(2.1)

then extending it to the \(\mathbb{C}\)-vector subspace \(V\) of \(L^\infty(\mu)\) generated by the family \((f^t)_{t \in E}\) and the unit \(e\) of \(A\) (constantly equal to 1 on \(E\)).

As \(f^u = f^v\) in \(L^\infty(\mu)\) when \([u] = [v]\), every \(g \in V\) can be expressed as

\[
g = \sum_{j=1}^n \alpha_j f^{t_j} + \alpha_0 e,
\]

(2.2)

where \(\alpha_j \in \mathbb{C}\), \(t_j \in E\) are such that \([t_i] \neq [t_j]\) for \(i \neq j\), and \(f^{t_j} \neq e\), \(f^{t_j} \neq 0\) for all \(1 \leq j \leq n\) (as \(L^\infty\) classes; condition (\[\square\]) implies that \(f^t \neq 0\) when \(t \neq +\infty\)). Since \((E, \leq)\) is totally ordered, we may assume that \([t_1] < [t_2] < \ldots < [t_n]\); since \(f^{t_1} \neq e\) in \(A \subset L^\infty(\mu)\) and
e - f_{t_1} = \chi_{F_1}; \ F_0 \subset E; \text{we have } \mu(F_0) > 0; \text{it follows that } g = \alpha_0 \text{ on } F_0 \text{ and } |\alpha_0| \leq \|g\|_{L^\infty(\mu)}. \text{ Next, } [t_1] < [t_2] \text{ implies that } f_{t_1} - f_{t_2} = \chi_{F_1} \text{ with } F_1 = (t_1, t_2), \mu(F_1) > 0, \text{ and } g = \alpha_0 + \alpha_1 \text{ on } F_1; \text{ it follows that } \sum_{j=1}^n |\alpha_j| \leq \|g\|_{L^\infty(\mu)}. \text{ We proceed in this way to } f_{t_n} \neq 0, \text{ with the value } \sum_{j=0}^n |\alpha_j| \text{ on the set } F_n = \{ t \in E; t > t_n \} \text{ of positive measure. Finally,}

\[ \sup_{0 \leq k \leq n} \left| \sum_{j=0}^k \alpha_j \right| \leq \|g\|_{L^\infty(\mu)}. \] \tag{2.3}

It follows from (2.3) that \( e, f^{s_1}, \ldots, f^{s_m} \) are linearly independent in \( A \) when \( [s_i] \neq [s_j] \) for \( i \neq j \) and \( f^{s_j} \neq e, f^{s_j} \neq 0 \) for all \( 1 \leq j \leq m \).

Indeed, assuming as we may that \( s_1 < s_2 < \ldots < s_m \), the equality

\[ \sum_{j=1}^m \alpha_j f^{s_j} + \alpha_0 e = 0 \]

implies \( \sum_{j=0}^k \alpha_j = 0 \) for \( 0 \leq k \leq m \), hence \( \alpha_j = 0 \) for \( j = 0, \ldots, m \).

It follows that we may extend \( \theta_t \) in a natural way as a linear form on \( V \) by setting \( \theta_t(g) = \sum_{j=1}^n \alpha_j \theta_t(f^{s_j}) + \alpha_0 \) whenever \( g \) is expressed as in (2.2). Furthermore, \( \theta_t \) is a continuous function on \( V \) equipped with the \( L^\infty(\mu) \) norm: indeed, pick \( g = \sum_{j=1}^n \alpha_j f^{s_j} + \alpha_0 e \) in the previous form and assume \( t_1 < t_2 < \ldots < t_n \). We shall check that \( \theta_t(g) \) is equal to one of the sums \( \sum_{j=0}^k \alpha_j \), \( k \in \{ 0, \ldots, n \} \), depending on the position of \( t \): if \( [t] < [t_1] \), we have that \( \theta_t(f^{s_j}) = 0 \) for \( 1 \leq j \leq n \), hence \( \theta_t(g) = \alpha_0 \); if \( [t_k] \leq [t] < [t_{k+1}] \) and \( k \in \{ 1, \ldots, n-1 \} \), we see that \( \theta_t(f^{s_j}) = 1 \) when \( j \leq k \) and \( = 0 \) otherwise, thus \( \theta_t(g) = \alpha_0 + \alpha_1 + \ldots + \alpha_k \); finally, when \( [t] \geq [t_n] \), we have \( \theta_t(f^{s_j}) = 1 \) for every \( j \) and \( \theta_t(g) = \alpha_0 + \ldots + \alpha_n \). Therefore, we have by (2.3) that

\[ |\theta_t(g)| \leq \|g\|_{L^\infty(\mu)}. \]

Since the set \( \{ f^u; u \in E \} \cup \{ e \} \) forms a total set in \( A \), \( \theta_t \) extends (in a unique way) to a continuous linear form on \( A = C(K) \).

We check finally that \( \theta(t) \) is a character. First \( \theta_t(f^u f^v) = \theta_t(f^u) = \theta_t(f^v) \theta_t(f^u) \) because the values are 0 or 1. Let \( u, v \in E \) with \( [u] < [v] \); we have \( f^u f^v = f^v \) and we can check that \( \theta_t(f^u f^v) = \theta_t(f^v) \theta_t(f^u) \) by considering the cases \( [t] < [u] \) (values \( u \mapsto 0, v \mapsto 0 \), \( [u] \leq [t] < [v] \) (values \( u \mapsto 1, v \mapsto 0 \)), and \( [v] \leq [t] \) (values \( u \mapsto 1, v \mapsto 1 \)). This implies the result, by expanding products \( g_1 g_2 \) with \( g_1, g_2 \) expressed as in (2.2). It follows that \( \theta_t \in K. \]

Let us set \( h(\theta') = +\infty \) and \( h(\theta'') = -\infty \).

**Lemma 6.** Let \( t \in E \setminus \{ -\infty \} \). Then
Proof.

a). Assume that $\sup\{u \in E; u \in [t]\} = b \in [t]$ and $\{u \in E; u > b\}$ is not closed subset of $E$. Consider the character $\theta_t$ given by Lemma 3. We show that $\theta_t \in B_2^0$ and $h(\theta_t) = b$. Indeed, let $u \in E$. If $u > b$, then $[u] > [b]$ because $\sup\{v \in E; v \in [t]\} = b$. If $[u] > [b]$, it is obvious that $u > b$. We conclude that $u > b$ if and only if $[u] > [b]$, or if and only if $[u] > [t]$, since $[b] = [t]$.

Therefore, by the formula (2.1), we have $\psi(\theta_t, u) = 0$ if and only if $u > b$. Thus $\{u \in E; \psi(\theta_t, u) = 0\} = \{u \in E; u > b\}$ is not closed subset of $E$ by the hypothesis. We conclude that $\theta_t \in B_2$. Finally by Lemma 2

$$h(\theta_t) = \sup\{u \in E; \psi(\theta_t, u) = 1\} = \sup\{u \in E; u \leq b\} = b.$$ 

Conversely, assume that there is $\theta_t \in B_2$ such that $h(\theta_t) = b \in [t]$. By Lemma 2 we know that $\psi(\theta_t, v) = 1$ when $v < b$ and $\psi(\theta_t, v) = 0$ when $v > b$. Thus the set $E_0(\theta_t) = \{u \in E; \psi(\theta_t, u) = 0\}$ contains $\{u \in E; u > b\}$ and is contained in $\{u \in E; u \geq b\}$; but $b \notin E_0(\theta_t)$, otherwise $E_0(\theta_t)$ would be closed, contrary to the assumption $\theta_t \in B_2$. It follows therefore that $E_0(\theta_t) = \{u \in E; u > b\}$ is not closed subset of $E$. It remains to show that $\sup\{u \in E; u \in [t]\} = b$.

Let $u \in [t] = [b]$. We have then $\psi(\theta_t, u) = \psi(\theta_t, b) = 1$, and also $u \leq h(\theta_t) = b$ by Lemma 2. It follows that $b = \sup\{u \in E; u \in [t]\}$. □

b). The proof of b) is similar to that of a), replacing in Lemma 5 the formula

$$\theta_t(f^u) = \begin{cases} 1 & \text{if } [u] \leq [t], \\ 0 & \text{if } [u] > [t], \\ 1, \end{cases}$$

by the formula

$$\theta_t'(f^u) = \begin{cases} 1 & \text{if } [u] < [t], \\ 0 & \text{if } [u] \geq [t], \\ 1. \end{cases}$$

(2.4)
Lemma 4, and the latter set is \( \{E\} \).

Proof. The points \( t \in (a, b) \), \( t = a \), \( t = b \) and \( t = \pm \infty \) have \( d(b_1, b_2) \) such that \( \theta'' \) is not closed subset of \( E \).

Case 2, \( t < +\infty \): First note that \( b \neq -\infty \). Consider the character \( \theta_t \) given by Lemma 5. Let us show that \( \theta_t \in B_1 \). Note that \( t > b \) if and only if \( [u] > [b] \), because \( \sup\{v \in E; v \in [t]\} = b \). It implies that \( \{u \in E; \psi(\theta_t, u) = 0\} = \{u \in E; [u] > [b]\} = \{u \in E; u > b\} \); this is a closed subset of \( E \) by hypothesis, as well as \( \{u \in E; u \leq b\} \) (obviously), and the latter set is \( \{u \in E; \psi(\theta_t, u) = 1\} \); it follows that \( \theta_t \in B_1 \). By Lemma 4,

\[
\sup\{u \in E; \psi(\theta_t, u) = 1\} = \sup\{u \in E; u \leq b\} = b.
\]

We put \( \theta'' = \theta_t \). It is obvious that \( \theta'' \neq \theta'' \).

Conversely, assume that there exists \( \theta'' \subset B_1 \setminus \{\theta''\} \) such that \( h(\theta'') = b^+ \).

Case 1, \( \theta'' = \theta' \): We note that \( b = +\infty \), then \( \{t \in E; t > b\} = \emptyset \), which is closed subset of \( E \). On the other hand \( \sup\{v \in E; v \in [t]\} = b = +\infty \).

Case 2, \( \theta'' \neq \theta' \): Since \( \psi(\theta'', h(\theta'')) = 1 \) by Lemma 4, it follows that \( \sup\{u \in E; \psi(\theta'', u) = 0\} = \psi(\theta'', h(\theta''))^{-1}\{1\} = \{u \in E; u > b\} \) is a closed subset of \( E \), because \( \theta'' \in B_1 \). An argument similar to that of a) shows that \( \sup\{u \in E; u \in [t]\} = b \).

Corollary 3. Assume that the space \((E, \leq)\) is complete and that we have \( \mu([t \wedge t', t \vee t']) > 0 \) when \( t \neq t' \). Then \( h_{|B_1 \cup B_2} : B_1 \cup B_2 \to \overline{E} \) is onto \((-\infty, +\infty \notin E)\).

Proof. Note that the hypothesis implies that \( [t] = \{t\} \) for every \( t \in E \). Let \( a \in E \). If \( \{x \in E; x > a\} \) is not closed subset of \( E \), by Lemma 6a, there exists \( \theta \in B_2 \) such that \( h(\theta) = a \). If \( \{x \in E; x > a\} \) is closed subset of \( E \), by Lemma 6c, there exists \( \theta \in B_1 \) such that \( h(\theta) = a \). The points \( -\infty, +\infty \) are images of \( \theta', \theta'' \). Thus \( h_{|B_1 \cup B_2} \) is surjective.

Proposition 2. Assume \( E \) satisfies (*) and \( \mu([t \wedge t', t \vee t']) > 0 \) for \( t \neq t' \in E \). Then \( h_j : B_j \to E \setminus \{\infty, +\infty\} \) is onto, \( j \in \{2, 3\} \).

Proof. It suffices to show by Lemma 6 that the sets \( \{u \in E; u > t\} \) and \( \{u \in E; u < t\} \) are not closed subsets of \( E \), for every point \( t \in \overline{E} \setminus \{\infty, +\infty\} \). Indeed, let \( a, b \in E \) be such that \( t \in [a, b] \). Since \( E \) satisfies the condition (*), there exists \( c \in E \) such that \( a < c < t < b \), so \( c \in \{u \in E; u < t\} \), thus \( t \) is in the closure of \( \{u \in E; u < t\} \). We
Conclude that \( \{ u \in E; u < t \} \) is not a closed subset of \( E \). In the same way we can see that \( \{ u \in E; u > t \} \) is not a closed subset of \( E \). 

3. Continuity of \( h \)

In this part, we show that the function \( h \) is continuous on the subset of \( K \) where \( h \) is defined.

Recall that \( h(\theta') = +\infty \) and \( h(\theta'') = -\infty \). Let \( K_1 = (K \setminus B_1) \cup \{ \theta', \theta'' \} \). We denote again by \( h \) the restriction of \( h \) to \( K_1 \).

**Proposition 3.** Assume that \(-\infty, +\infty \notin E \). Then the map \( h : (K_1, \tau_0) \to (E, \tau_0) \) is continuous.

**Proof.** The topology of \( E \) is generated by the intervals of the form \( \{ u \in E; u < \alpha \} \) or of the form \( \{ u \in E; u > \alpha \} \), \( \alpha \in E \). Let \( \alpha \in E \) and let \( \theta_0 \) belong to the set \( W = \{ \theta \in K_1; h(\theta) < \alpha \} \). Let us show that there exists an open neighborhood of \( \theta_0 \) contained in \( W \). Note that \( \theta_0 \neq \theta' \).

By Lemma 3 we know when \( \theta_0 \neq \theta'' \) that \( \psi(\theta_0, \alpha) = 0 \) since \( h(\theta_0) < \alpha \) (and it is also true for \( \theta_0 = \theta'' \)), and we know that if \( \psi(\theta, \alpha) = 0 \), then \( h(\theta) \leq \alpha \). The set

\[
U = \{ \theta \in K_1; \psi(\theta, \alpha) = 0 \} = K_1 \cap \{ \theta \in K; \theta(f^\alpha) = 0 \}
\]

is open in \( K_1 \), and \( U \) is contained in \( \{ \theta \in K_1; h(\theta) \leq \alpha \} \). Consider \( V = U \setminus \{ \theta \in K_1; h(\theta) = \alpha \} \). Note that \( V \) is an open set in \( K_1 \), because \( \{ \theta \in K_1; h(\theta) = \alpha \} \) is finite by Remark 2, thus \( V \) is an open neighborhood of \( \theta_0 \) contained in \( W \).

Let \( \alpha \in E \) and let \( \theta_0 \) belong to \( W' = \{ \theta \in K_1; h(\theta) > \alpha \} \). Using now Lemma 3 we can show in a similar way that the set

\[
V' = \{ \theta \in K_1; \psi(\theta, \alpha) = 1 \} \setminus \{ \theta \in K_1; h(\theta) = \alpha \}
\]

is an open neighborhood of \( \theta_0 \) and \( V' \subset W' \). 

**Remark 4.** The space \((E, \tau_0)\) is connected if and only if \((E, \leq)\) is a complete and \( E \) satisfies the condition (*) [4] rem. (d), p. 58].

**Corollary 4.** Assume that \( E \) is connected and \(-\infty, +\infty \notin E \). Then \( h : K \to (E, \tau_0) \) is continuous.

**Proof.** It suffices to see that \( B_1 = \{ \theta', \theta'' \} \), because \( E \) is connected. 

By an argument similar to that of Proposition 3 and its corollary, we can show the following result:

**Proposition 4.** Assume that \(-\infty, +\infty \in E \). Then \( h : (K_1, \tau_0) \to (E, \tau_0) \) is continuous.
Corollary 5. Assume that $E$ is connected and $-\infty, +\infty \in E$. Then $h : K \to (E, \tau_0)$ is continuous.

Proposition 5. Assume that $(E, \leq)$ is complete and $-\infty, +\infty \notin E$. Then $h : K \to \overline{E}$ is continuous.

Proof. Let $\alpha \in E$ and let $\theta_0 \in W = h^{-1}(\{t \in \overline{E}; t < \alpha\}) = \{\theta \in K; h(\theta) < \alpha\}$. We shall find an open neighborhood of $\theta_0$ in $W$.

Case 1, $\theta_0 = \theta''$: Consider $t_0 \in E$ such that $t_0 < \alpha$ and let $V = \{\theta \in K; \psi(\theta, t_0) = 0\}$. Note that $\theta'' \in V$. It is obvious that $V \subset W$.

Case 2, $\theta_0 \in B_1 \setminus \{\theta''\}$: The set
$$\{u \in E; u < \alpha\} \cap \{t \in E; \psi(\theta_0, t) = 1\}$$
is an open neighborhood of $h(\theta_0)$, and because $\psi(\theta_0, h(\theta_0)) = 1$ by Lemma 4 it follows that there exists $a, b \in E$ ($a < b$) such that $h(\theta_0) \in [a, b[ \subset \{u; u < \alpha\} \cap \{t \in E; \psi(\theta_0, t) = 1\}$. Denote $V = \{\theta \in K; \psi(\theta, b) = 0\}$; the set $V$ is an open neighborhood of $\theta_0$, because $h(\theta_0) < b$ and $\psi(\theta_0, h(\theta_0)) = 1$. It remains to show that $V \subset W$. Indeed, let $\theta \in V = \{\theta \in K; \psi(\theta, b) = 0\}$ (hence $h(\theta) \leq b$). Let us show that $h(\theta) < b$. Assume that $h(\theta) = b$. Since $h(\theta_0) < h(\theta) = b$, $\theta_0 < \theta$, there exists $\gamma \in E$ such that $\psi(\theta_0, \gamma) = 0$ and $\psi(\theta, \gamma) = 1$, this implies that $a < h(\theta_0) < \gamma < h(\theta) = b$ ($\gamma < h(\theta)$ because $\psi(\theta, b) = 0$ and $\psi(\theta, \gamma) = 1$). We deduce that $\gamma \in ]a, b[ \subset \{u \in E; u < \alpha\} \cap \{t \in E; \psi(\theta_0, t) = 1\}$, which is impossible (note that $\psi(\theta_0, \gamma) = 0$), thus $h(\theta) < b$.

Now let us show that $b \leq \alpha$. Assume that $b > \alpha$. Since $a < h(\theta_0) < \alpha$, $\alpha \in ]a, b[$, hence $\alpha \in \{u \in E; u < \alpha\}$, this is impossible. It follows that $h(\theta) < b \leq \alpha$, i.e. $\theta \in W = \{\varphi \in K; h(\varphi) < \alpha\}$. We treat the case $\theta_0 \in B_2$ and the case $\theta_0 \in B_3$, as in Proposition 3.

By an argument similar to that of Proposition 5 we show:

Proposition 6. Assume that $(E, \leq)$ is complete and $-\infty, +\infty \in E$. Then $h : K \to E$ is continuous.

We define the measure $\mu'$ on $E$ by $\mu'([t]) = 1$, for every $t \in E$ ($\mu'$ is the counting measure). Let $A'$ be the $C^*$-subalgebra generated by $\{f^*, t \in E\}$ in $\mathcal{L}^\infty(E, \mu') = \ell^\infty(E)$ and let $K'_\mu(E) = K'$ be the set of characters on $A'$. Denote
$$h'(\theta) = \sup\{t \in E; \psi'(\theta, t) = 1\}$$
where $\psi'(\theta, t) = \theta(f^*), \theta \in K', t \in E \setminus \{-\infty, +\infty\}$. 

Lemma 8. Assume that \( E \) is separable, \( \tau \) with the order topology.

Remark 5. Let \((r, i) \in E \times \{0, 1\}\), \((s, j) \in E \times \{0, 1\}\),

\[(r, i) < (s, j) \text{ if and only if } r < s, \text{ or } r = s \text{ and } i < j.\]

Note that if \( E \) is totally ordered, then \( E \times \{0, 1\} \) has the same property.

Recall that \( \tau_1 \) (resp. \( \tau_2 \)) denotes the topology on a totally ordered set
generated by semi-open intervals \((a, b)\) (resp. \([a, b]\)). We need the
following classical lemmas:

**Lemma 7.** Let \( E' \) be a subset of \( E \). Then \((E', \tau_2)\) is homeomorphic to
\((E' \times \{1\}, \tau_0)\), viewed as topological subspace of \((E \times \{0, 1\}, \tau_0)\) equipped
with the order topology \( \tau_0 \) of the lexicographic order.

**Lemma 8.** Assume that \((E, \tau_0)\) is a strongly Lindelöf space. Then:
1) \((E, \tau_j)\) is a strongly Lindelöf space, \( j \in \{1, 2\} \).
2) \( \text{Bor}(E, \tau_0) = \text{Bor}(E, \tau_j), j \in \{1, 2\} \).
3) If \((E, \tau_0)\) is separable, then \((E, \tau_j)\) is separable, \( j \in \{1, 2\} \).
4) If every subset of \((E, \tau_0)\) is separable, then every subset of \((E, \tau_j)\)
is separable, \( j \in \{1, 2\} \).

**Remark 5.** Let \((X, \tau)\) be a strongly Lindelöf space and let \( Z \) be a
subspace of \((X, \tau)\). Then \((Z, \tau)\) is a strongly Lindelöf space.

**Proof of Lemma 7.** Consider \( F : (E', \tau_2) \rightarrow (E' \times \{1\}, \tau_0) \) the map
defined by \( F(t) = (t, 1), t \in E' \). We show that \( F \) is a homeomorphism.
Let \((a, j_0) \in E \times \{0, 1\}\) and let \( V = \{(t, 1) \in E' \times \{1\}; (t, 1) < (a, j_0)\}\)
be an open subset of \((E' \times \{1\}, \tau_0)\). We have

\[ F^{-1}(V) = \{t \in E'; (t, 1) < (a, j_0)\} = \{t \in E'; t < a\} \in (E', \tau_2). \]

Let \((a, j_0) \in E \times \{0, 1\}\) and let \( W = \{(t, 1) \in E' \times \{1\}; (t, 1) > (a, j_0)\}\).

**Case 1, \( j_0 = 1 \):** It is obvious that \( F^{-1}(V) = \{t \in E'; t > a\} \in (E_1, \tau_2). \)

**Case 2, \( j_0 = 0 \):** We observe that \( F^{-1}(V) = \{t \in E'; t \geq a\} \in (E', \tau_2). \)

We conclude that if \( V \) is an open subset of \((E' \times \{1\}, \tau_0)\), then \( F^{-1}(V) \)
is an open set of \((E', \tau_2)\) and conversely. ■

**Remark 6.** By an argument similar to that of Lemma 7, we show that
\((E', \tau_1)\) is homeomorphic to \((E' \times \{0\}, \tau_0)\).

**Proof of Lemma 8.**
1). We shall prove Lemma 8 for \( j = 1 \), the case \( j = 2 \) can be shown by a similar argument. The argument that allows us to show 1) is similar to [14, p. 58, 59].

Indeed, let \( \{ [a_i, b_i]; a_i < b_i, i \in I \} \) be a family of \( \tau_1 \)-open intervals in \( E \), let \( U = \bigcup_{i \in I} [a_i, b_i] \) and \( V = \bigcup_{i \in I} a_i, b_i \). We will show that there exists a countable subset \( I' \) of \( I \) such that \( U = \bigcup_{i \in I'} [a_i, b_i] \). Denote by \( J \) the subset of \( I \) consisting of those \( j \) such that \( b_j \notin V \).

Let us show that \( \{ b_j; j \in J \} \) is countable. Let \( b_j \neq b_{j'}, j, j' \in J \). Assume that there exists \( t \in [a_j, b_j] \cap [a_{j'}, b_{j'}] \). If \( b_j < b_{j'} \), then \( a_{j'} < t < b_j < b_{j'} \), it follows that \( b_j \in [a_{j'}, b_{j'}] \subset V \), which is impossible, hence \( [a_j, b_j] \cap [a_{j'}, b_{j'}] = \emptyset \), if \( b_j \neq b_{j'} \). By hypothesis \( (E, \tau_0) \) is a strongly Lindelöf space, then there exists a countable subset \( H \) of \( J \), such that \( \bigcup_{j \in I} [a_j, b_j] = \bigcup_{i \in H} [a_i, b_i] \). On the other hand, if \( b_j \neq b_{j'} \), \( j, j' \in J \), then \( [a_j, b_j] \cap [a_{j'}, b_{j'}] \) is empty; we conclude that \( \{ b_j; j \in J \} = \{ b_j; j \in H \} \), i.e. \( \{ b_j; j \in J \} \) is countable.

Since \( (E, \tau_0) \) is a strongly Lindelöf space, there exists a countable subset \( I_1 \) of \( I \), such that \( V = \bigcup_{i \in I_1} [a_i, b_i] \). Let \( I_2 = I_1 \cup H \); this set is a countable subset of \( I \). Let

\[
W = \bigcup_{i \in I_2} [a_i, b_i].
\]

We check that \( W = U \), this will end the proof. It is clear that \( W \supset V \) since \( I_2 \supset I_1 \). Furthermore, if \( t \in U \setminus V \), then \( t \) must be an endpoint \( b_i \) of some \( [a_i, b_i] \), for an \( i \in I \). This means that \( i \in J \) and \( b_i = t \) belongs to the family \( \{ b_j; j \in H \} \), hence to \( \bigcup_{j \in H} [a_j, b_j] \subset W \). The space \( (E, \tau_1) \) is therefore a strongly Lindelöf space. \( \blacksquare \)

2). It is clear by 1) that \( \text{Bor}(E, \tau_0) = \text{Bor}(E, \tau_1) \). \( \blacksquare \)

3). Assume \( (E, \tau_0) \) is separable. Let \( (x_n)_{n \geq 0} \) be a dense sequence in \( (E, \tau_0) \) and \( M = \{ b \in E; \exists a \in E \text{ such that } [a, b] = \{ b \} \} \).

**Step 1**: Let us show that \( M \) is countable.

By 1), \( (E, \tau_1) \) is a strongly Lindelöf space, it follows that \( \bigcup_{b \in M} \{ b \} = \bigcup_{b \in M} b = M \) is countable. \( \blacksquare \)

**Step 2**: Let us show that \( M_2 = \{ x_n; n \in \mathbb{N} \} \cup M \) is dense in \( (E, \tau_1) \).

Consider \( [a, b] \) a nonempty open interval of \( (E, \tau_1) \). If \( [a, b] = \{ b \} \), then \( b \in M \) and \( [a, b] \cap M_2 \neq \emptyset \). If \( [a, b] \neq \emptyset \), then there exists \( x_n \in [a, b] \), i.e. \( x_n \in [a, b] \cap M_2 \). We conclude that \( M_2 \) is dense in \( (E, \tau_1) \). \( \blacksquare \)

4). We need only observe that clearly, any subset \( E' \) of a strongly Lindelöf space \( E \) is strongly Lindelöf for the induced topology, and then apply 3) to \( E' \). \( \blacksquare \)
Remark 7. Suppose that \((E, \tau_0)\) is a strongly Lindelöf space. Then \(E \times \{0, 1\}\) has the same property.

Proof. By Lemma 8, the space \((E, \tau_j)\) is a strongly Lindelöf space, when \(j \in \{1, 2\}\). Using Lemma 7 and Remark 6, one obtains that \((E \times \{0\}, \tau_0)\) and \((E \times \{1\}, \tau_0)\) are strongly Lindelöf spaces, therefore \(E \times \{0, 1\} = E \times \{0\} \cup E \times \{1\}\) has the same property. □

Remark 8. Suppose that \((E, \tau_0)\) is a separable strongly Lindelöf space (resp. every subset of \((E, \tau_0)\) is separable). Then \(E \times \{0, 1\}\) is separable (resp. every subset of \(E \times \{0, 1\}\) is separable).

Proof. By Lemma 8, \((E, \tau_j)\) is separable, \(j \in \{1, 2\}\) (resp. every subset of \((E, \tau_j)\) is separable, \(j \in \{1, 2\}\)). On the other hand, by Lemma 7 and Remark 6, \((E, \tau_1)\) is homeomorphic to \(E \times \{0\}\) and \((E, \tau_2)\) is homeomorphic to \(E \times \{1\}\), it follows that \(E \times \{0\}, E \times \{1\}\) are separables (resp. every subset of \(E \times \{0\}\) or of \(E \times \{1\}\) is separable), hence \(E \times \{0, 1\} = E \times \{0\} \cup E \times \{1\}\) is separable (resp. every subset of \(E \times \{0, 1\}\) is separable). □

Proposition 7. i) Assume that every subset of \((E, \tau_0)\) is separable. Then \(B_1\) is a strongly Lindelöf space.

ii) Assume that \((E, \tau_0)\) is separable and that \((E, \leq)\) satisfies the condition (\(
\star\)). Then \(B_1\) has a countable basis.

Proof.

i). Let \(B\) be a subset of \(E\) and let \((u_n)_{n \geq 0}\) be a dense sequence in \((B, \tau_0)\). For every \(u \in B\) denote \(V_u = \{\theta \in B_1; \psi(\theta, u) = 1\}\). Let us show that \(\bigcup_{u \in B} V_u = \bigcup_{n \in \mathbb{N}} V_{u_n}\).

Let \(\theta \in \bigcup_{u \in B} V_u\). There exists \(u \in B\) such that \(\theta \in V_u\), hence \(u \in H_\theta = \{t \in B; \psi(\theta, t) = 1\}\). Thus there exists \(n \in \mathbb{N}\) such that \(u_n \in H_\theta\), this implies that \(\theta \in V_{u_n}\). It follows that \(\bigcup_{u \in B} V_u = \bigcup_{n \in \mathbb{N}} V_{u_n}\).

If \(V'_u = \{\theta \in B_1; \psi(\theta, u) = 0\}\), \(u \in B\), by a similar argument, we show that \(\bigcup_{u \in B} V'_u = \bigcup_{n \in \mathbb{N}} V'_{u_n}\). For every \(u, v \in B\) let

\[W_{u,v} = \{\theta \in B_1; \psi(\theta, u) = 1 \text{ and } \psi(\theta, v) = 0\}\]

Show that \(\bigcup_{u,v \in B} W_{u,v} = \bigcup_{m,n \in \mathbb{N}} W_{u_m,u_n}\).

Let \(\theta \in W_{u,v}\). Note that \(u \in H_\theta = \{t \in B; \psi(\theta, t) = 1\}\) and \(v \in H'_\theta = \{t \in B; \psi(\theta, t) = 0\}\), hence there \(u_m \in H_\theta\) and \(u_n \in H'_\theta\). It is obvious that \(\theta \in W_{u_m,u_n}\), this implies that \(\bigcup_{u,v \in B} W_{u,v} = \bigcup_{m,n \in \mathbb{N}} W_{u_m,u_n}\). □
\[\text{\textit{Step 1}}: \text{ Let us show that the map } h_3 : (B_3, \tau_0) \to (E, \tau_1) \text{ is continuous.} \]

Since \[h_3 : (B_3, \tau_0) \to (E, \tau_0) \text{ is continuous (by Proposition 3), it suffices to show that } (h_3)^{-1}(V) \text{ is an open subset of } B_3, \text{ when } V = \{t \in E; t \leq \alpha\}, \alpha \in E. \]

Let \(\alpha \in E\) and let \(V = \{t \in E; t \leq \alpha\}\). Consider \(\theta_0 \in (h_3)^{-1}(V) \) and \(V_1 = \{\theta \in B_3; \psi(\theta, h_3(\theta_0)) = 0\}\). It is clear that \(V_1\) is an open neighborhood of \(\theta_0\). On the other hand, if \(\theta \in V_1\), \(h(\theta) \leq h(\theta_0) \leq \alpha\), hence \(\theta \in (h_3)^{-1}(V)\). This implies that \(V_1 \subseteq (h_3)^{-1}(V)\). Thus \((h_3)^{-1}(V)\) is an open subset of \((B_3, \tau_0)\).

\[\text{\textit{Step 2}}: \text{ Let us show that the map } h_3 : (B_3, \tau_0) \to (h_3(B_3), \tau_1) \text{ is a homeomorphism.} \]

Consider \(t \in E\) and \(V_t = \{\theta \in B_3; \psi(\theta, t) = 1\}\). By definition of \(B_3\), \(V_t = \{\theta \in B_3; h_3(\theta) > t\}\), it follows that \(h_3(V_t) = \{u \in h_3(B_3); u > t\}\) is an open subset of \((h_3(B_3), \tau_1)\). In the same way, if we let \(W_t = \)
\{ \theta \in B_3 ; \psi(\theta, t) = 0 \} = \{ \theta \in B_3 ; h_3(\theta) \leq t \}, \text{ we see that } h_3(W) = \{ u \in h_3(B_3) ; u \leq t \} \text{ is an open set of } (h_3(B_3), \tau_1). \text{ Thus } h_3 \text{ is a homeomorphism.}

By Lemma 8 and Remark 5, \((h_3(B_3), \tau_1)\) is a strongly Lindelöf space, hence \((B_3, \tau_0)\) has the same property. ■

II). By Lemma 8, every subset of \((h_3(B_3), \tau_1)\) is separable, using step 2, one obtains that every subset of \((B_3, \tau_0)\) is separable. ■

**Corollary 6.** Assume that \((E, \tau_0)\) is a strongly Lindelöf space, \((E, \leq)\) is complete and every subset of \((E, \tau_0)\) is separable. Then \(K\) is separable.

**Proof.** First let us show that \((B_1, \tau_0)\) is separable. By an argument similar to that of Proposition 8-II), one shows that the map \(h_1 : (B_1, \tau_0) \to (h(B_1), \tau_2), \theta \to h(\theta)\) is a homeomorphism. By Lemma 8, \((h(B_1), \tau_2)\) is separable, hence \((B_1, \tau_0)\) is separable. On the other hand, by Proposition 8, \((B_j, \tau_0)\) is separable, \(j = 0, 1\). Since \(K = B_1 \cup B_2 \cup B_3\), \(K\) is separable. ■

**Corollary 7.** Assume that \((E, \tau_0)\) is a separable space and satisfies the condition \((\ast)\). Then \(K\) is a separable strongly Lindelöf space.

**Proof.** By Proposition 7, \(B_1\) has a countable basis. On the other hand by Remark 1, \((E, \tau_0)\) has a countable basis, this implies that \((E, \tau_0)\) is a strongly Lindelöf space and every subset of \((E, \tau_0)\) is separable, therefore by Proposition 8, \(B_2, B_3\) are separable strongly Lindelöf spaces. Thus \(K = B_1 \cup B_2 \cup B_3\) is a separable strongly Lindelöf space. ■

**Remark 9.** Let \((L, <)\) non-empty totally ordered set; assume that the order topology of \(L\) is compact and separable. By 19 every subset of \(L\) is separable and the order topology of \(L\) is strongly Lindelöf.

**Corollary 8.** Assume that \((E, \tau_0)\) is separable and connected. Then \(K\) is separable space.

Let \(P\) be a Polish space. We denote by \(B_1(P)\) the space of first class functions (with real values or complex values) on \(P\).

We denote by \(\tau_p\) the topology of pointwise convergence on \(B_1(P)\).

**Remark 10.** Let \((E, \tau_0)\) be a separable strongly Lindelöf space. Suppose that the map \(h_j : B_j \to E \setminus \{-\infty, +\infty\} \) is onto, for \(j \in \{2, 3\}\). Then \(K \setminus B_1\) is separable.

**Proof.** Note first that \((E \setminus \{-\infty, +\infty\}, \tau_0)\) is a separable strongly Lindelöf space. By Lemma 8, \((E \setminus \{-\infty, +\infty\}, \tau_1)\) is a separable space. On the other hand by step 2 of Proposition 8, the map \(h_3 : (B_3, \tau_0) \to \)
(E \ \{-\infty, +\infty\}, \tau_1) is a homeomorphism, therefore \((B_3, \tau_0)\) is separable. By a similar argument, one shows that \((B_2, \tau_0)\) is separable.

As \(K - B_1 = B_2 \cup B_3\), \(K - B_1\) is separable. 

**Definition 3.** Let \(L\) be a compact Hausdorff space. We say that \(L\) is a Rosenthal compact set, if there exists a Polish space \(P\) such that \(L\) continuously embedded in \((B_1(P), \tau_P)\) \[\mathbb{N}\].

**Definition 4.** Let \(X\) be a Hausdorff space. \(X\) is said to be an analytic set, if \(X\) is a continuous image of a Polish space \[\mathbb{P}\] chap. II, p. 96].

**Proposition 9.** Suppose that \((E, \tau_0)\) is an analytic set and suppose that \(-\infty, +\infty \notin E\). Then \(K\) is a Rosenthal compact set.

**Proof.** Since \((E, \tau_0)\) is analytic, there exists a Polish space \(P\) and a surjective continuous map \(\phi : P \to (E, \tau_0)\). Let \(\theta \in K\). Consider the map \(\sigma_\theta : P \to \{0, 1\}, x \in P \to \psi(\theta, \phi(x))\). Note at first that if \(\theta \in B_1\), \(\sigma_\theta\) is a continuous function. Let \(\theta \in B_2\). Prove that \(\sigma_\theta\) is a first class function.

It is clear that

\[(\sigma_\theta)^{-1}\{1\} = \{x \in P; \psi(\theta, \phi(x)) = 1\} = \{x \in P; \phi(x) \leq h(\theta)\}\]

which is a closed subset of \(P\). On the other hand

\[(\sigma_\theta)^{-1}\{0\} = \{x \in P; \psi(\theta, \phi(x)) = 0\} = \{x \in P; \phi(x) > h(\theta)\}\]

is an open subset of \(P\). Therefore the sets \((\sigma_\theta)^{-1}\{1\}\), \((\sigma_\theta)^{-1}\{0\}\) are \(G_\delta\) subsets of \(P\). It follows that \(\sigma_\theta\) is a first class function. If \(\theta \in B_3\), by a similar argument, one shows that \(\sigma_\theta\) is a first class function. One defines the map \(\sigma : K \to B_1(P)\), by \(\sigma(\theta) = \sigma_\theta, \theta \in K\). It is obvious that \(\sigma\) is into and continuous, hence \(K\) is a Rosenthal compact set. 

**Proposition 10.** Suppose that \((E, \leq)\) is complete and \((E, \tau_0)\) is a strongly Lindelöf space. Then \(K\) is a strongly Lindelöf space.

**Proof.** We can suppose that \(-\infty, +\infty \notin E\). Denote \(K_2 = B_1 \cup B_2 - \{\theta', \theta''\}\) and consider the map \(h_{|K_2} : (K_2, \tau_0) \to (h(K_2), \tau_2)\) (note that \(\sup\{t; \psi(\theta, t) = 1\}\) exists for all \(\theta \in K_2\) by Lemma \[\mathbb{2}\] and Remark \[\mathbb{3}\]). By hypothesis, \((E, \tau_0)\) is a strongly Lindelöf space, hence by Lemma \[\mathbb{8}\] \((E, \tau_2)\) has the same property, therefore \((h(K_2), \tau_2)\) is a strongly Lindelöf space. Thus \((K_2, \tau_0)\) is a strongly Lindelöf space, because \(h_{|K_2}\) is a homeomorphism (By an argument similar to that of Proposition \[\mathbb{8}\]).

Observe finally that \(K = K_2 \cup B_3 \cup \{\theta', \theta''\}\), \(K_2\) is a strongly Lindelöf space and \(B_3\) is a strongly Lindelöf space by Proposition \[\mathbb{8}\] hence \(K\) is a strongly Lindelöf space. 

Let \(L\) be a Hausdorff compact space. Denote \(M^+(L)\) the Radon probability measures space on \(L\).
For every Banach space $X$, we denote by $B_X$ the closed unit ball of $X$.

**Definition 5.** [2, chap. IX.8] A topological space $(X, \tau)$ is said to be a completely regular space, if $(X, \tau)$ can be embedded in a compact Hausdorff space.

It is clear that every topological subspace of a completely regular space is a completely regular space.

Recall that $\mu'$ is the counting measure defined on $E$ and $h'(\theta) = \sup\{t \in E; \psi'(\theta, t) = 1\}$, $\theta \in K' \setminus B'_1$, where $\psi'(\theta, t) = \theta(f^*)$, $t \in E$. Denote by $B'_1$ the set of $\theta \in K'$ such that the map $t \in E \to \psi'(\theta, t) \in \{0, 1\}$ is continuous, $B'_2 = \{\theta \in K' \setminus B'_1; \psi'(\theta, h'(\theta)) = 1\}$ and by $B'_3 = \{\theta \in K' \setminus B'_1; \psi'(\theta, h'(\theta)) = 0\}$.

**Remark 11.** Suppose that $(E, \leq)$ is complete $(-\infty, +\infty \notin E)$. Then $(E, \tau_j)$ is a completely regular, $j \in \{1, 2\}$.

**Proof.** We shall show this remark for $j = 2$. Consider $\mu = \mu'$. By Corollary 3, $h'(K_2) = E$. On the other hand, in the proof of Proposition 2, we have shown that $h' : B'_2 \cup B'_1 \setminus \{\theta, \theta''\} = K_2 \to (E, \tau_2)$ is a homeomorphism. Thus $(E, \tau_2)$ is a completely regular space.

Let

$$E_1 = \{a \in E; \{x \in E; x > a\} \text{ is not closed subset of } E\}$$

and

$$E_2 = \{a \in E; \{x \in E; x < a\} \text{ is not closed subset of } E\}.$$ 

**Remark 12.** The space $(E_2, \tau_1)$ is completely regular space.

**Proof.** Consider $\mu = \mu'$. By Lemma 3(b), the restriction of $h'$ (in Remark 11) to $B'_3$ is onto on $E_2$ and by step 2 of Proposition 3 this map is homeomorphism. On the other hand, $B'_3$ is a completely regular space, hence $(E_2, \tau_1)$ has the same property.

**Remark 13.** Suppose that $(E, \leq)$ satisfies the condition $(\ast)$. Since the point $a$ is in the closure of the set $\{x \in E; x < a\}$, for every $a \in E$, the set $\{x \in E; x < a\}$ is not closed. We conclude that $E = E_2$.

**Proposition 11.** Suppose that $E$ is a Polish space and $\mu(\{a\}) = 0$ for every $a \in E$ (i.e., $\mu$ is diffuse on the Borel $\sigma$-algebra of $E$). Then for every $\xi \in C(K)^*$ the map $t \in (E, \tau_0) \to \xi(f^t)$ is a first-class function.

**Proof.** It is suffices to show Proposition 11 when $\xi \in B_{C(K)^*}$.

**Step 1:** Let us show that for every $g \in L^1(E, \mu)$ the map $t \in E \to (g, f^t) = \int_E g(x)f^t(x)d\mu(x)$ is continuous.
Let \( g \in L^1(E, \mu) \) and let \((t_n)_{n \geq 0}\) be a sequence in \( E \) such that \( t_n \to t \) in \((E, \tau_0)\). Consider \( y \in E \) such that \( y \neq t \). We shall show that 

\[
\lim_{n \to +\infty} f^{t_n}(y) = f^t(y).
\]

We show that there does not exist two subsequences \((t_{m_k})_{k \geq 0}\) and 
\((t_{n_k})_{k \geq 0}\) such that 
\( f^{t_{m_k}}(y) = 1 \) and \( f^{t_{n_k}}(y) = 0 \) for every \( k \in \mathbb{N} \). Assume there are such subsequences. We have then \( t_{n_k} \geq y > t_{m_k} \) for every \( k \in \mathbb{N} \), by passing to the limit we obtain \( y = t \), which is impossible.

Thus we distinguish two cases:

Case 1: there exists \( n_0 \in \mathbb{N} \) such that \( f^{t_n}(y) = 1 \) for every \( n \geq n_0 \):
Note that \( y > t_n \), \( \forall n \geq n_0 \), hence \( y \geq t \). Since \( y \neq t \), \( y > t \), i.e.
\[
0 \lim_{n \to +\infty} f^{t_n}(y) = f^t(y).
\]

Case 2: there exists \( n_0 \) such that \( f^{t_n}(y) = 0 \) for every \( n \geq n_0 \):
This implies that \( y \leq t_n \) for every \( n \geq n_0 \), by passing to the limit, we obtain \( y \leq t \). Thus 
\[
\lim_{n \to +\infty} f^{t_n}(y) = f^t(y).
\]

By hypothesis on \( \mu \), 
\[
(g, f^{t_n}) = \int_{E \setminus \{t\}} g(y) f^{t_n}(y) \, d\mu(y).
\]

By Lebesgue’s dominated convergence theorem, we have

\[
(g, f^{t_n}) = \int_{E \setminus \{t\}} g(y) f^{t_n}(y) \, d\mu(y) \\
\lim_{n \to +\infty} \int_{E \setminus \{t\}} g(y) f^t(y) \, d\mu(y) \\
= (g, f^t).
\]

**Step 2:** Let \( \xi \in M^+(K) \). Let us show that the map \( t \in E \to \xi(f^t) \in \mathbb{R} \) is a first class function.

By Hahn-Banach theorem there exists \( \xi' \in B_{L^\infty(\mu)^*} \) which extends \( \xi \) with \( ||\xi|| = ||\xi'|| \). Since \( B_{L^1(\mu)} \) is \( w^* \)-dense in \( B_{L^\infty(\mu)^*} = B_{L^1(\mu)^*} \),
there is a net \((g_\alpha)_{\alpha \in I}\) in the unit ball of \( L^1(\mu) \) such that \( g_\alpha \to \xi' \) for the topology \( w^* \) in \([L^\infty(\mu)]^* \). There exist \( \xi_1, \xi_2 \in M^+(K) \) such that \( g_\alpha^+ \to \xi_1 \in M^+(K) \) and \( g_\alpha^- \to \xi_2 \in M^+(K) \), \( \sigma[C(K)^*, C(K)] \), hence \( \xi = \xi_1 - \xi_2 \), because \( g_\alpha = g_\alpha^+-g_\alpha^- \), for all \( \alpha \) and \( M^+(K) \) is \( \sigma(C(K)^*, C(K)) \)
compact. By using Proposition \( \[ \] K \) is a Rosenthal compact set. On the other hand, by \( \[ \] \), \( M^+(K) \) is a Rosenthal compact set, and by \( \[ \] \) \( M^+(K) \) is angelic. Therefore, there exist two subsequences \( (g_{j_k}^+)_{k \geq 0}, (g_{i_k}^-)_{k \geq 0} \), such that 
\( g_{j_k}^+ \to \xi_1 \) and \( g_{i_k}^- \to \xi_2 \), for the topology \( w^* \),
i.e. \( h_k = g_{j_k}^+ - g_{i_k}^- \to \xi_1 - \xi_2 = \xi \) for the topology \( w^* \). Thus, 
\( (h_k, f^t) \to \xi(f^t) \) for every \( t \in E \).
Finally, by step 1, the map \( t \in E \to (h_k, f^t) \) is continuous for every \( k \), which implies that \( t \in E \to \xi(f^t) \) is a first class function. ■

If \( \xi \) in the closed unit ball of \([C(K)]^*\), it suffices to see that there exists \( u^+, u^-, v^+, v^- \in M^+(K) \) such that \( \xi = (u^+ - u^-) + i(v^+ - v^-) \), by using step 2, we obtain that the map \( t \in E \to \xi(f^t) \) is a first class function. ■

Denote \( J = [E \cup \{+\infty\}] \times \{1\} \cup [E \cup \{-\infty\}] \times \{0\} \) \((-\infty, +\infty \notin E\). As \( J \) is a subset of \( \mathbb{E} \times \{0, 1\} \) and \( \mathbb{E} \times \{0, 1\} \) is totally ordered (for the lexicographic order), \( J \) is totally ordered. We denote again by \( \tau_0 \) the order topology of \( J \).

Let \( J_1 \) be a subset of \( J \). Recall that the notation \((J_1, \tau_0)\) means that the topology \( \tau_0 \) on \( J_1 \) is the topology formed of relative open subsets of \( J_1 \) in \((J, \tau_0)\). Let

\[
J_1 = [h(B_2 \cup B_1 \setminus \{\theta''\})] \times \{1\} \cup [h(B_3 \cup \{\theta''\}) \times \{0\}].
\]

In following Theorem \( J_1 \) is equipped by the relative topology \( \tau_0 \).

**Theorem 1.** Assume that \((E, \leq)\) is complete and \(-\infty, +\infty \notin E\). Then \( K \) is topological homeomorphic and order isomorphic to \( J_1 \).

**Proof.**

One defines \( T : K \to (J_1, \tau_0) \), by

\[
T(\theta) = \begin{cases} 
(h(\theta), 1), & \text{if } \theta \in B_2 \cup (B_1 \setminus \{\theta''\}) \\
(h(\theta), 0), & \text{if } \theta \in B_3 \cup \{\theta''\}
\end{cases}, \quad \theta \in K.
\]

Let us show that \( T \) is a homeomorphism. Since the restriction of \( h \) to \( B_j \) is injective, \( j \in \{1, 2, 3\} \), \( T \) is injective. So suffices to show that \( T \) is continuous, because \( K \) is a Hausdorff compact space.

Let \((\alpha, \beta) \in J \) and let \( V = \{(x, y) \in J_1; (x, y) < (\alpha, \beta)\}\) be an open subset of \( J_1 \). We prove that \( T^{-1}(V) \) is an open subset of \( K \). We distinguish two cases:

**Case 1.** \( \beta = 0 \): It is obvious that \( T^{-1}(V) = (h)^{-1}(\{t \in \mathbb{E}; t < \alpha\}) \) which is an open subset of \( K \), by Proposition 5.

**Case 2.** \( \beta = 1 \): Let \( \theta \in T^{-1}(V) \). We shall find an open neighborhood \( C \) of \( \theta \) in \( T^{-1}(V) \). We distinguish again two cases:

**Case 2-a.** \( h(\theta) < \alpha \): Put \( C = h^{-1}(\{t \in \mathbb{E}; t < \alpha\}) \) (note that \( \theta \in C \)). if \( \varphi \in C, T(\varphi) = (h(\varphi), j) < (\alpha, 1), \ (j \in \{0, 1\} \), hence \( C \subset T^{-1}(V) \). On the other hand, by Proposition 5 \( C \) is an open subset of \( K \).

**Case 2-b.** \( h(\theta) = \alpha > -\infty \): Note that \( \theta \neq \theta' \), because \( \theta \in T^{-1}(V) \). Denote \( C = \{\varphi \in K; \psi(\varphi, h(\theta)) = 0\} \). \( C \) is an open subset of \( K \).

Let us show that \( C \) contains \( \theta \).
Since \( \theta \in T^{-1}(V) \), \( T(\theta) = (h(\theta), j) < (h(\theta), 1) \), hence \( j = 0 \). By definition of \( T \psi(h, \theta)) = 0 \), i.e. \( \theta \in C \).

It remains to show that \( C \subset T^{-1}(V) \). Let in fact \( \varphi \in C \); this means that \( \psi(\varphi, h(\theta)) = 0 \), we conclude that \( h(\varphi) \leq h(\theta) \), if \( h(\varphi) = h(\theta) = \alpha \), then \( \psi(\varphi, h(\theta)) = \psi(\varphi, h(\theta)) = 0 \). It follows that \( T(\varphi) = (h(\varphi), 0) < (\alpha, 1) \). Thus \( \varphi \in T^{-1}(V) \).

If \( h(\varphi) < h(\theta) = \alpha \), then \( T(\varphi) = (h(\varphi), j) < (\alpha, 1) \) \( (j \in \{0, 1\}) \), hence \( \varphi \in T^{-1}(V) \). Therefore \( C \subset T^{-1}(V) \).

Let now \( (\alpha, \beta) \in J \) and let \( V = \{(x, y) \in J_1; (x, y) > (\alpha, \beta)\} \) be an open subset of \( J_1 \). Show that \( T^{-1}(V) \) is an open subset of \( K \).

**Case 1, \( \beta = 1 \):** By Proposition 5, \( T^{-1}(V) = h^{-1}\{t \in \mathbb{E}; t > \alpha\} \) is an open subset of \( K \).

**Case 2, \( \beta = 0 \):** Consider \( \theta \in T^{-1}(V) \). We distinguish two cases:

**Case 2-c, \( h(\theta) > \alpha \):** The set \( C = h^{-1}\{t \in \mathbb{E}; t > \alpha\} \) is an open subset of \( K \), by Proposition 5 which contains \( \theta \). The condition \( h(\varphi) > \alpha \) implies that \( T(\varphi) = (h(\varphi), j) > (\alpha, 0) \), hence \( C \subset T^{-1}(V) \).

**Case 2-d, \( h(\theta) = \alpha < +\infty \):** Note that \( \theta \neq \theta'' \), because \( \theta \in T^{-1}(V) \). Choose \( C = \{\varphi \in K; \psi(\varphi, h(\theta)) = 1\} \). The hypothesis \( \theta \in T^{-1}(V) \) implies that \( T(\theta) = (h(\theta), j) > (h(\theta), 0) \), hence \( j = 1 \), this means that \( \psi(\theta, h(\theta)) = 1 \). It follows that \( C \) contains \( \theta \). Show that \( C \subset T^{-1}(V) \). Let \( \varphi \in C \). It is clear that \( h(\varphi) \geq h(\theta) \), if \( h(\theta) = h(\varphi) \), then \( \psi(\varphi, h(\varphi)) = \psi(\varphi, h(\theta)) = 1 \), hence \( T(\varphi) = (h(\varphi), 1) > (\alpha, 0) \), i.e. \( \varphi \in T^{-1}(V) \). Finally if \( h(\theta) < h(\varphi) \), then \( T(\varphi) = (h(\varphi), 1) > (h(\theta), 0) = (\alpha, 0) \), this implies that \( \varphi \in T^{-1}(V) \). Thus \( C \subset T^{-1}(V) \).

Let us show that \( T \) is an order isomorphism.

Let \( \theta, \varphi \in K \) such that \( \theta < \varphi \). There exists \( t \in E \) such that \( \psi(\theta, t) = 0 \) and \( \psi(\varphi, t) = 1 \). This implies that

\[
h(\varphi) \geq t \geq h(\theta). \tag{4.1}
\]

Obviously if \( h(\theta) < h(\varphi) \), then \( T(\theta) = (h(\theta), j_1) < (h(\varphi), j_2) = T(\varphi) \).
If \( h(\theta) = h(\varphi) \), by (4.1) \( h(\theta) = h(\varphi) = t \), hence \( \theta \in B_3^2 \) and \( \varphi \in B_2 \), this implies that \( T(\theta) = (h(\theta), 0) < (h(\varphi), 1) = T(\varphi) \).

Conversely, suppose that \( T(\theta) < T(\varphi) \). If \( h(\theta) < h(\varphi) \), then \( \theta < \varphi \), if \( h(\theta) = h(\varphi) \), since \( T(\theta) = (h(\theta), j_1) < T(\varphi) = (h(\varphi), j_2) \), \( j_1 = 0 \) and \( j_2 = 1 \). Thus \( \theta \in B_3^2 \) and \( \varphi \in B_2 \), hence \( \theta < \varphi \). ■

**Corollary 9.** Suppose that \( E \) is connected and \( \mu([t \wedge t', t \vee t']) > 0 \), for \( t \neq t' \). Then \( K \) is topological homeomorphic and order isomorphic to

\[
J = ([E \cup \{+\infty\}] \times \{1\} \cup [E \cup \{-\infty\}] \times \{0\}, \tau_0).
\]

**Proof.** Since \( E \) is connected, by Remark 4, \( (E, \leq) \) is complete. On the other hand, by Proposition 2 \( h_j \) is surjective onto \( E \setminus \{-\infty, +\infty\}, \)
j \in \{2, 3\}, it follows that \( T \) (defined in Theorem [3]) is surjective (note that \( B_1 = \{\theta', \theta''\} \)). ■

**Corollary 10.** Let \( E \) be an abelian group (-\( \infty, +\infty \notin E \)). Assume that \( E \) separable and connected. Then \( K \) is topological homeomorphic and order isomorphic to 
\[
([E \cup \{+\infty\}] \times \{1\} \cup [E \cup \{-\infty\}] \times \{0\}, \tau_0).
\]

**Proof.** The hypothesis that \( E \) is connected implies that \((E, \leq)\) is complete. On the other hand, \( E \) satisfies the condition (*) (see Example [1]), shows that the measure \( \mu = m \) verifies the hypothesis of Corollary [9], hence \( K \) is isomorphic to \((\{E \cup \{+\infty\}] \times \{1\} \cup [E \cup \{-\infty\}] \times \{0\}, \tau_0)\).

■

**Lemma 9.** Assume that \((E, \leq)\) is complete \((E \) is a separable strongly Lindelöf space\). Then:
1) \( J = ([E \cup \{+\infty\}] \times \{1\} \cup ([E \cup \{-\infty\}] \times \{0\}, \tau_0) \) is compact.
2) \((J, \tau_0)\) is a separable space.

**Proof.**
1). Consider \( \mu = \mu' \) the counting measure. By using Theorem [1], we obtain that the set \([E \cup \{+\infty\}] \times \{1\} \cup ([h'(B_3) \cup \{-\infty\}] \times \{0\}) \) is compact in \((J, \tau_0)\), because this space is homeomorphic to \( K' \). By a similar argument (by changing the statement of Theorem [1]) we may show that the set \([h'(B_2) \cup \{+\infty\}] \times \{1\} \cup ((E \cup \{-\infty\}] \times \{0\}) \) is compact in \((J, \tau_0)\), thus 
\[
\left( ([E \cup \{+\infty\}] \times \{1\} \cup ([h'(B_3) \cup \{-\infty\}] \times \{0\}) \right) 
\cup \left( ([h'(B_2) \cup \{+\infty\}] \times \{1\} \cup ((E \cup \{-\infty\}] \times \{0\}) \right) = J
\]
is compact. ■

2). Since \( E \) is a separable strongly Lindelöf space, \((E, \tau_0)\) has the same property. By Lemma [8], \((E, \tau_2)\) is a separable strongly Lindelöf space. On the other hand Lemma [7] shows us that \((E \times \{1\}, \tau_0)\) is a separable subspace of \((E \times \{0, 1\}, \tau_0)\). Note now that \([E \cup \{+\infty\}] \times \{1\} = E \times \{1\} \setminus \{(-\infty, 1)\}, \) hence \(([E \cup \{+\infty\}] \times \{1\}, \tau_0)\) is separable. By a similar argument, one shows that \(([E \cup \{-\infty\}] \times \{0\}, \tau_0)\) is a separable of \( E \times \{0, 1\} \). It follows that \([E \cup \{+\infty\}] \times \{1\} \cup ([E \cup \{-\infty\}] \times \{0\}) = J \) is a separable space of \( E \times \{0, 1\} \). Let \( i : (J, \tau_0) \to (J, \tau_0) \) be the identity map of \( J \) \(((J, \tau_0)\), where is a topological subspace of \( E \times \{0, 1\} \). It is clear that \( i \) is continuous, thus \((J, \tau_0)\) is separable. ■

By an argument similar to that of Theorem [1], one shows the following theorem
Theorem 2. Suppose that \((E, \leq)\) is complete and \(-\infty, +\infty \in E\). Then \(K\) is topological homeomorphic and order isomorphic to

\[
((h(B_2 \cup B_1 \setminus \{\theta''\}) \times \{1\}) \cup [(h(B_3 \cup \{\theta''\}) \times \{0\}, \tau_0).]
\]

By an argument similar to that of Corollary 9 one shows the following corollary:

Corollary 11. Suppose that \(E\) is connected, that \(-\infty, +\infty \in E\) and that \(\mu([t \land t',t \lor t']) > 0\), for \(t \neq t'\). Then \(K\) is topological homeomorphic and order isomorphic to \(((E \setminus \{-\infty\}) \times \{1\} \cup [E \setminus \{+\infty\}] \times \{0\}, \tau_0).\)

Theorem 3.

Remark 14. Let \((E, \tau_0)\) be a separable strongly Lindelöf space. Suppose that \((E, \leq)\) is complete and \(-\infty, +\infty \in E\). Then \(K\) is a separable.

5. Construction of a totally ordered compact set homeomorphic to \(K_\mu(E)\)

This part is devoted to find sufficient conditions for \(K\) to be isomorphic to \(E\).

Recall that \(E_1 = \{t \in E; \{x \in E; x > t\}\) is not closed in \(E\}, that \(E_2 = \{t \in E; \{x \in E; x < t\}\) is not closed in \(E\}\}, h(\theta') = +\infty \) and \(h(\theta'') = -\infty\).

Theorem 4. Assume that \((E, \leq)\) is complete, that \(-\infty, +\infty \in E\), \(E \setminus \{+\infty, -\infty\} = E_1 \cup E_2\) and for every \(t \in E_1 \cup E_2\), there exists \(b \in E_1\) and \(a \in E_2\) such that \([t] = \{a,b\}, b > a\). Then the map \(h : K \to E\) is a homeomorphism.

Proof.

Step 1: Show that \(h : K \to E\) is onto.

Let \(b \in E_1\). By hypothesis, there exists \(a \in E_2\) such that \([b] = \{b, a\}\) and \(a < b\). By Lemma 6, there is \(\theta_b \in B_2\) such that \(h(\theta_b) = b\). In the same way, for every \(a' \in E_2\), there is \(\theta_{a'} \in B_3\) such that \(h(\theta_{a'}) = a'\).

Step 2: Let us show that the map \(h : K \to E\) is into.

Suppose there exist \(\theta_1, \theta_2 \in K\) such that \(h(\theta_1) = h(\theta_2) = t_0\). Since the restriction of \(h\) to \(B_1 \cup B_2\) (resp. to \(B_3\)) is injective, by step 1 of Proposition 10 (resp. by Remark 2), it is enough to treat the following cases:

Case 1, \(\theta_1 \in B_2\) and \(\theta_2 \in B_3\):

Case 1-a, \(t_0 \in E_1\): By hypothesis, there exists \(a \in E_2\) such that \([t_0] = \{a, b\}\) and \(b = t_0 > a\). On the other hand, since \(a \in [t_0] = [b]\)
and \( \theta_2 \in B_3 \), \( \psi(\theta_2, a) = \psi(\theta_2, b) = \psi(\theta_2, h(\theta_2)) = 0 \), this implies that 
\( a \geq h(\theta_2) = b \), it is impossible because \( b = t_0 > a \).

Case 1-b, \( t_0 = a \in E_2 \): By hypothesis, there exists \( b \in E_1 \) such that \( [t_0] = \{a, b\} \) and \( t = t_0 < b \). Observe that \( \psi(\theta_1, b) = \psi(\theta_1, a) = \psi(\theta_1, h(\theta_1)) = 1 \). Thus \( b \leq h(\theta_1) = t_0 \), which is impossible.

Case 2, \( \theta_1 \in B_1 \setminus \{\theta', \theta''\} \) and \( \theta_2 \in B_3 \):

Case 2-c, \( t_0 = b \in E_1 \): By hypothesis, there exists \( a \in E_2 \) such that \( [t_0] = \{a, b\} \) and \( a < b \). Note that \( \psi(\theta_2, a) = \psi(\theta_2, b) = 0 \), hence \( a \geq h(\theta_2) = t_0 = b \), this is impossible.

Case 2-d, \( t_0 = a \in E_2 \): There exists \( b \in E_1 \) such that \( [t_0] = \{a, b\} \) and \( t_0 = a < b \). In this case, we have \( \psi(\theta_1, b) = \psi(\theta_1, a) = 1 \), by Lemma 4, hence \( b \leq h(\theta_1) = a \), which is impossible.

Finally, by Proposition 6, \( h \) is continuous, therefore \( h \) is a homeomorphism.

**Remark 15.** Under the hypothesis of Theorem 4, we have \( B_1 = \{\theta', \theta''\} \).

**Proof.** Suppose that \( \theta \in B_1 \setminus \{\theta', \theta''\} \). Note that \( h(\theta) = t_0 \in E \setminus \{-\infty, +\infty\} \). In the proof of Theorem 4 step 1, we showed that there exists \( \varphi \in B_2 \cup B_3 \) such that \( h(\varphi) = h(\theta) = t_0 \). We deduce that \( \theta = \varphi \), which is impossible.

**Remark 16.** Under the hypothesis of Theorem 4, by step 1, one notes that if \( \theta \in B_2 \) (resp. \( \theta \in B_3 \)) \( h(\theta) \in E_1 \) (resp. \( h(\theta) \in E_2 \)).

Consider \( E = [0, 1] \times \{0, 1\} \) equipped with the lexicographic order and \( \Phi : [0, 1] \to [0, 1] \times \{0, 1\} \) the map defined by \( \Phi(t) = (t, 1) \), \( t \in [0, 1] \). \( \Phi \) is Borel. Indeed, let \( F : ([0, 1], \tau_2) \to ([0, 1] \times \{1\}, \tau_0) \) the map defined by \( F(t) = (t, 1) \). In the proof of Lemma 7, we showed that \( F \) is a homeomorphism. On the other hand by Lemma 8, \( \text{Bor}([0, 1], \tau_2) = \text{Bor}([0, 1], \tau_0) \), hence \( \Phi \) is Borel.

Put \( \mu = \Phi(m_1) \), where \( m_1 \) the Lebesgue measure on \([0, 1] \).

**Corollary 12.** \( h : K \to (E = [0, 1] \times \{0, 1\}, \mu) \) is a homeomorphism.

**Proof.** Put \( E_1 = [0, 1] \times \{1\} \setminus \{(1, 1)\} \) and \( E_2 = [0, 1] \times \{0\} \setminus \{(0, 0)\} \). Let \( b = (t_0, 1) \in E_1 \). Suppose that \( b_1 = (t_1, j_1) \in E \times \{0, 1\} \) and \( t_0 > t_1, j_1 \in \{0, 1\} \). One has
\[
\mu([b_1, b]) = \mu(\{(t, j); (t_1, j_1) < (t, j) \leq (t_0, 1)\})
= m_1(\{t \in E; (t_1, j_1) < (t, 1) \leq (t_0, 1)\})
\geq m_1(\{t \in E; t_1 < t < t_0\}) > 0.
\]

If \( b_1 = (t_1, j_1) \) and \( t_0 < t_1 \), by a similar argument, one shows that \( \mu([b, b_1]) > 0 \).
Consider \( a = (t_0, 0) \). It is clear that \( \mu([a,b]) = 0 \), hence \([b] = \{a, b\}\) (by above).

It remains to show that \( V = \{(t, j) \in E \times \{0, 1\}; ((t, j) > (t_0, 1))\} \) is not closed in \( E \) (by a similar argument, one shows that \( V = \{(t, j) \in E \times \{0, 1\}; ((t, j) < (t_0, 0))\} \) is not closed in \( E \)).

Let \((t_n)_{n \geq 0}\) be a sequence in \([0,1]\) such that for every \( n \in \mathbb{N} \), \( t_n > t_0 \) and \( t_n \to t_0 \). Since \((t_n, 1) \in V\) for every \( n \in \mathbb{N} \), it is enough to show that \((t_n, 1) \to_{n \to +\infty} (t_0, 1) \notin V\).

Let \( W = \{(u,j) \in E \times \{0, 1\}; (u,j) < (u_0,j_0)\} \) be an open neighborhood of \((t_0, 1)\). Since \( t_0 < u_0 \) (because \((t_0, 1) \in W\)), there exists \( n_0 \in \mathbb{N} \) such that \( t_n < u_0 \) for every \( n \geq n_0 \), it follows that \((t_n, 1) \in W\).

Consider \( W = \{(u,j) \in E \times \{0, 1\}; (u,j) > (u_0,j_0)\} \) an open neighborhood of \((t_0, 1)\). Obviously \( u_0 \geq u_0 \), hence \( t_n > t_0 \geq u_0 \). It follows that \((t_n, 1) \in W\), for every \( n \). By Theorem 4, \( h\) is a homeomorphism.

In the following of this part, one supposes that \((E, \tau_0)\) is a connected strongly Lindelöf space, \(-\infty, +\infty \in E\) and \([t] = \{t\}\) for all \( t \in E\). One defines the map \( \Pi_1 : E \to K\) by

\[
\Pi_1(t) = \begin{cases} 
\theta_t, & \text{if } t \in ]-\infty, +\infty[ \\
\theta', & \text{if } t = +\infty \\
\theta'' & \text{if } t = -\infty.
\end{cases}
\]

By Lemma 6, \( \theta_t \in B_2\), and \( h(\theta_t) = t, \forall t \in E \setminus \{-\infty, +\infty\}\).

**Lemma 10.** The map \( \Pi_1 \) is Borel.

**Proof.** Consider \( V_\theta = \{\theta \in K; \psi(\theta, a) = 1\} \) and \( V'_\theta = \{\theta \in K; \psi(\theta, a) = 0\} \). It suffices to show \((\Pi_1)^{-1}(V_\theta)\) and \((\Pi_1)^{-1}(V'_\theta)\) are Borel subsets of \( E\) for every \( a \in E \setminus \{-\infty, +\infty\}\), because by Proposition 10 \( K\) is a strongly Lindelöf space (note that \((E, \tau_0)\) is connected implies that \((E, \leq)\) is complete and satisfies the condition (*) (cf. [14], rem. (d), p. 58)).

Pick \( a \in E \setminus \{-\infty, +\infty\}\). One has

\[
(\Pi_1)^{-1}(V_\theta) = \{t \in E \setminus \{-\infty, +\infty\}; \psi(\theta_t, a) = 1\} \cup +\infty = \{t \in E \setminus \{-\infty, +\infty\}; a \leq h(\theta_t)\} \cup +\infty = \{t \in E \setminus \{-\infty, +\infty\}; a \leq t\} \cup +\infty.
\]

Thus \((\Pi_1)^{-1}(V_\theta)\) is Borel. By a similar argument, one shows that \((\Pi_1)^{-1}(V'_\theta)\) is Borel. \( \blacksquare \)

Put \( \mu_1 = \Pi_1(\mu)\). Note that \( \mu_1\) is a measure on \( K\).

**Lemma 11.** Let \( \theta_0 \in B_2 \) and \( \theta_1 \in B_3\). Then:
I) The sets \( \{ \varphi \in K; \varphi > \theta_0 \} \) and \( \{ \varphi \in K; \varphi < \theta_1 \} \) are not closed subsets in \( K \).

II) For every \( \varphi_0 \in B_2 \), there exists \( \varphi_1 \in B_3 \) such that \( [\varphi_0] = \{ \varphi_0, \varphi_1 \} \) and \( h(\varphi_0) = h(\varphi_1) \).

Proof.

I). We shall show that the set \( \{ \varphi \in K; \varphi > \theta_0 \} \) is not closed, the proof is similar for \( \{ \varphi \in K; \varphi < \theta_1 \} \).

Step 1: Let \( \varphi \in K \) such that \( \varphi > \theta_0 \). Show that \( h(\varphi) > h(\theta_0) \).

Suppose that \( h(\varphi) = h(\theta_0) \). Since the restriction of \( h \) to \( B_j \) is injective, \( j \in \{2,3\} \), \( \varphi \in B_3 \), this is impossible because \( \psi(\varphi,h(\varphi)) = 0 \), \( \psi(\theta_0,h(\theta_0)) = 1 \) and \( \varphi > \theta_0 \). It follows that \( h(\varphi) > h(\theta_0) \). ■

Step 2: Show that \( \{ \varphi \in K; \varphi > \theta_0 \} \) is not closed in \( K \).

By the step 1, \( \{ \varphi \in K; \varphi > \theta_0 \} = \{ \varphi \in K; h(\varphi) > h(\theta_0) \} \). Suppose that this subset is closed in \( K \). We shall show that the set - \( \{ t \in E; t > h(\theta_0) \} \) is closed, in this case the proof will be finished because \( E \) is connected, hence \( \{ t \in E; t > h(\theta_0) \} \) is not closed. ■

By Proposition \( \Box \) the map \( h : K \rightarrow E \) is continuous and the set \( \{ \varphi \in K; h(\varphi) > h(\theta_0) \} \) is a compact subset of \( K \), it follows that \( h(\{ \varphi \in K; h(\varphi) > h(\theta_0) \}) = \{ h(\varphi); h(\varphi) > h(\theta_0), \varphi \in K \} \) is compact in \( E \). On the other hand, by Proposition \( \Box \) \( h \) is surjective, this implies that \( \{ h(\varphi); h(\varphi) > h(\theta_0) \} = \{ t \in E; t > h(\theta_0) \} \) is compact, hence it is closed in \( E \). ■

II). Let \( \varphi_0 \in B_2 \). By Lemma \( \Box \) there exists \( \varphi_1 \in B_3 \) such that \( h(\varphi_0) = h(\varphi_1) \), it is clear that \( \varphi_0 > \varphi_1 \) (note that \( \psi(\varphi_0,h(\varphi_0)) = 1 \) and \( \psi(\varphi_1,h(\varphi_1)) = 0 \) ). On the other hand

\[
\mu_1([\varphi_1,\varphi_0]) = \mu(\{ t \in E; \varphi_1 < \Pi_1(t) \leq \varphi_0 \})
\leq \mu(\{ t \in E; h(\varphi_1) \leq h(\Pi_1(t)) \leq h(\varphi_0) \})
= \mu(\{ t \in E; h(\varphi_1) \leq t \leq h(\varphi_0) \}) = 0.
\]

Thus \( \varphi_1 \in [\varphi_0] \).

Let \( \varphi \in K \setminus \{ \theta', \theta'', \varphi_0, \varphi_1 \} \). Let us show that \( \varphi \notin [\varphi_0] \).

Case 1, \( \varphi > \varphi_0 \): Since the restriction of \( h \) to \( B_j \) is injective, \( j = 2,3 \), \( h(\varphi) > h(\varphi_0) \). Observe that

\[
\mu_1([\varphi_0,\varphi]) = \mu(\{ t \in E; \varphi_0 < \Pi_1(t) \leq \varphi \})
\geq \mu(\{ t \in E; h(\varphi_0) < h(\Pi_1(t)) < h(\varphi) \})
= \mu(\{ t \in E; h(\varphi_0) < t \leq h(\varphi) \}) > 0, \tag{5.1}
\]

(because \( [t] = \{ t \} \), for all \( t \in E \)) hence \( \varphi \notin [\varphi_0] \).

Case 2, \( \varphi < \varphi_0 \): Let us to show that \( h(\varphi) < h_1(\varphi_0) \). Assume that \( h(\varphi) = h_1(\varphi_0) \), hence \( h(\varphi) = h(\varphi_0) = h(\varphi_1) \), since the restriction of \( h \)
to $B_j$ is injective, $j = 2, 3$ and $\varphi \in B_0 \cup B_1$, either $\varphi = \varphi_0$ or $\varphi = \varphi_1$, which is impossible. Thus $h(\varphi) < h(\varphi_0)$. By a similar argument, one shows that $\mu([\varphi, \varphi_0]) > 0$. We deduce that $[\varphi_0] = \{\varphi_0, \varphi_1\}$. 

Recall that $\psi_{\mu_1, K}(\eta, \varphi) = f^\varphi(\eta)$, where $f^\varphi$ is the characteristic function of $\{\theta \in K; \theta > \varphi\}$. Let $A_1$ be the $C^*$-subalgebra (with unit) generated by the family $\{f^\varphi, \varphi \in K\}$ in $L^\infty(\mu_1)$. Note that $A_1 = C(K_{\mu_1}(K))$. By lemmas 2 and 4, $h(\varphi, \varphi, \psi, K) = \sup \{\varphi \in K; \psi_{\mu_1, K}(\eta, \varphi) = 1\}$ exists for every $\eta \in K(K) (\theta''_K = \max(K_{\mu_1}(K))$ and $\theta''_K = \min(K_{\mu_1}(K))$.

**Corollary 13.** The map $h_{\mu_1, K}: K_{\mu_1}(K) \to K_{\mu}(E) = K$ is a homeomorphism.

**Proof.** Consider the reference set $\tilde{E} = K$, $E_1 = B^\mu_2 = B_2$, $E_2 = B^\mu_3 = B_3$. By Lemma 11 and Theorem 4 we obtain that $h_{\mu_1, K}: K_{\mu_1}(K) \to K$ is a homeomorphism (note that $\mu_1$ satisfies the condition (11)).

Let $\Pi_2: K \to K_{\mu_1}(K)$ the map defined by

$$
\Pi_2(\varphi) = \begin{cases} 
\theta_{\varphi, K} \in B^\mu_{\mu_1, K}, & \text{if } \varphi \in E_1 = B^\mu_2 = B_2 \\
\theta'_{\varphi, K} \in B^\mu_3 = B_3 & \text{if } \varphi \in E_2 = B^\mu_3 = B_3 \\
\Pi_2(\theta'_E) = \theta'_K \text{ and } \Pi_2(\theta''_E) = \theta''_K, & \Pi_2(\varphi) = \theta_{\varphi, K}
\end{cases}
$$

Where $\theta_{\varphi, K}, \theta'_{\varphi, K}$ the elements founded by lemma 6. Note that $\Pi_2(B_j) = B^\mu_{\mu_1, K}$, $j \in \{2, 3\}$.

**Lemma 12.** The map $\Pi_2: K \to K_{\mu_1}(K)$ is Borel.

**Proof.** By Proposition 10, $K_{\mu_1}(K)$ is a strongly Lindelöf space, it suffices then to show that $(\Pi_2)^{-1}(V_\theta), (\Pi_2)^{-1}(V'_\theta)$ are Borel subsets of $K$, where $V_\theta = \{\eta \in K_{\mu_1}(K); \psi_{\mu_1, K}(\eta, \theta) = 1\}$ and $V'_\theta = \{\eta \in K_{\mu_1}(K); \psi_{\mu_1, K}(\eta, \theta) = 0\}$. We can suppose that $\theta \in B_2$. We have

$$(\Pi_2)^{-1}(V_\theta) = \{\varphi \in B_2; \psi_{\mu_1, K}(\theta_{\varphi, K}, \theta) = 1\} \\
\cup \{\varphi \in B_3; \psi_{\mu_1, K}(\theta'_{\varphi, K}, \theta) = 1\} \cup \{\theta'\}$$

$$= \{\varphi \in B_2; \theta \leq h_{\mu_1, K}(\theta_{\varphi, K})\} \\
\cup \{\varphi \in B_3; \theta < h_{\mu_1, K}(\theta'_{\varphi, K})\} \cup \{\theta'\}$$

$$= \{\varphi \in B_2; \theta \leq \varphi\} \cup \{\varphi \in B_3; \theta < \varphi\} \cup \{\theta'\}$$

Thus $(\Pi_2)^{-1}(V_\theta)$ is a Borel subset of $K$. By a similar argument, one shows that $(\Pi_2)^{-1}(V'_\theta)$ is a Borel subset of $K$. 

Let $\mu_2 = \Pi_2(\mu_1)$. 


Lemma 13. Suppose that $\mu(\lvert t \land t' \land t \lor t' \rvert) > 0$, for every $t \neq t'$. Let $\eta_0 \in B^K_2$ and $\eta_1 \in B^K_3$. Then:

I) The sets $\{\eta \in K_{\mu_1}(K); \eta > \eta_0\}$, $\{\eta \in K_{\mu_1}(K); \eta < \eta_1\}$ are not closed subsets of $K_{\mu_1}(K)$.

II) For every $\lambda_0 \in B^\mu_2(K)$, there exists $\lambda_1 \in B^\mu_3(K)$ such that $[\lambda_0] = \{\lambda_0, \lambda_1\}$ and $h(\mu_{1,K}(\lambda_0)) = h(\mu_{1,K}(\lambda_1))$.

Proof.

I). Let us show that $\{\eta \in K_{\mu_1}(K); \eta > \eta_0\}$ is not closed. Suppose that there exists $\eta \in K_{\mu_1}(K)$ such that $\eta > \eta_0$. There exists $\theta \in K$ such that $\psi_{\mu_1,K}(\eta, \theta) = f^\theta(\eta) = 1$ and $\psi_{\mu_1,K}(\eta_0, \theta) = f^\theta(\eta_0) = 0$. Since $\eta_0 \in B^\mu_2(K)$, $h_{\mu_1,K}(\eta) \geq \theta > h_{\mu_1,K}(\eta_0)$. It follows that $h_K(\eta) > h_K(\eta_0)$. We conclude that $\{\eta \in K(K); \eta > \eta_0\} = \{\eta \in K(K); h_{\mu_1,K}(\eta) > h_{\mu_1,K}(\eta_0)\}$.

Assume now that $\{\eta \in K_{\mu_1}(K); \eta > \eta_0\} = \{\eta \in K_{\mu_1}(K); h_{\mu_1,K}(\eta) > h_{\mu_1,K}(\eta_0)\}$ is closed. The map $h_{\mu_1,K}: K_{\mu_1}(K) \to K$ is a homeomorphism by Corollary [13], it follows that

$h_{\mu_1,K}[\{\eta \in K_{\mu_1}(K); h_{\mu_1,K}(\eta) > h_{\mu_1,K}(\eta_0)\}] = \{h_{\mu_1,K}(\eta); h_K(\eta) > h_{\mu_1,K}(\eta_0), \eta \in K_{\mu_1}(K)\}$

is a compact subset of $K$. Since $h_{\mu_1,K}$ is onto, the set

$\{h_{\mu_1,K}(\eta) \in K; h_{\mu_1,K}(\eta) > h_{\mu_1,K}(\eta_0), \eta \in K_{\mu_1}(K)\} = \{\varphi \in K; \varphi > h_{\mu_1,K}(\eta_0)\}$

is closed in $K$. On the other hand, by Remark [16], $h_{\mu_1,K}(\eta_0) \in B_2$, using Lemma [14] we obtain that $\{\varphi \in K; \varphi > h_{\mu_1,K}(\eta_0)\}$ is not closed, which is impossible. Thus $\{\eta \in K_{\mu_1}(K); \eta > \eta_0\}$ is not closed. By a similar argument, one shows that the set $\{\eta \in K_{\mu_1}(K); \eta < \eta_1\}$ is not closed. ■

II). Let $\lambda_0 \in B^\mu_2(K)$. Since $h_3 : B_3 \to E \setminus \{-\infty, +\infty\}$ is onto, there exists $\theta_1 \in B_3$ such that

$h(\theta_1) = h(\mu_{1,K}(\lambda_0)).$ (5.2)

On the other hand, by Corollary [13] and Remark [16], there exists $\lambda_1 \in B^\mu_3(K)$ such that $\mu_{1,K}(\lambda_1) = \theta_1$, this implies by (5.2) that

$h(\mu_{1,K}(\lambda_0)) = h(\mu_{1,K}(\lambda_1)).$ (5.3)

We shall prove that $\lambda_1 < \lambda_0$.

Suppose that $\lambda_1 \geq \lambda_0$. It follows that $h(\mu_{1,K}(\lambda_1)) = \theta_1 \geq h(\mu_{1,K}(\lambda_0))$. Since $\theta_1 \in B_3$, $\psi(\theta_1, h(\theta_1)) = 0$, hence $\psi_{\mu_{1,K}}(h(\mu_{1,K}(\lambda_0)), h(\theta_1)) = 0$, this implies that $h(h_{\mu_{1,K}}(\lambda_0)) < h(h(\theta_1))$ because by Corollary [13] and Remark [16] $h_{\mu_{1,K}}(\lambda_0) \in B_2$, it is impossible by (5.2) and (5.3). Thus $\lambda_1 < \lambda_0$.

Let us show that $\lambda_1 \in [\lambda_0]$. 
Observe that

\[
\mu_2(\lambda_0, \lambda_1) = \mu_2(\{\eta \in K_{\mu_1}(K); \lambda_0 < \eta \leq \lambda_1\})
\]

\[
\leq \mu_1(\{\theta \in K; h_{\mu_1,K}(\lambda_0) < h_{\mu_1,K}(\Pi_2(\theta)) \leq h_{\mu_1,K}(\lambda_1)\})
\]

\[
= \mu_1(\{\theta \in K; h_{\mu_1,K}(\lambda_0) \leq \theta \leq h_{\mu_1,K}(\lambda_1)\})
\]

\[
= \mu(\{t \in E; h_{\mu_1,K}(\lambda_0) \leq t \leq h_{\mu_1,K}(\lambda_1)\})
\]\n
We conclude that \(\lambda_0 \notin [\lambda_1]\). Let us show that \(\eta \in K_{\mu_1}(K) \setminus \{\theta_K', \theta''_K, \lambda_0, \lambda_1\}\). We show that \(\eta \notin [\lambda_0]\).

**Case 1**: \(\eta > \lambda_0\).

**Step 1**: We shall show that \(h(h_{\mu_1,K}(\eta)) > h(h_{\mu_1,K}(\lambda_1)) = h(h_{\mu_1,K}(\lambda_0))\).

Indeed, suppose that \(h(h_{\mu_1,K}(\eta)) = h(h_{\mu_1,K}(\lambda_1)) = h(h_{\mu_1,K}(\lambda_0))\). By Corollary 13 and Remark 16, \(h_{\mu_1,K}(\lambda_0) \in B_2\) and \(h_{\mu_1,K}(\lambda_1) \in B_3\). We distinguish two cases:

**Case 1-a, \(\eta \in B_2^{\mu_1,K}\)**: By Corollary 13 and Remark 16, \(h_{\mu_1,K}(\eta) \in B_2\). On the other hand, the restriction of \(h\) to \(B_2\) is injective, hence \(h_{\mu_1,K}(\eta) = h_{\mu_1,K}(\lambda_0)\). This is impossible because the restriction of \(h_{\mu_1,K}\) to \(B_2^{\mu_1,K}\) is injective by Remark 2.

**Case 1-b, \(\eta \in B_3^{\mu_1,K}\)**: Since \(h_{\mu_1,K}(\eta) \in B_3\) and the restriction of \(h\) to \(B_3\) is injective, \(h_{\mu_1,K}(\eta) = h_{\mu_1,K}(\lambda_1)\), this implies that \(\eta = \lambda_1\), this is impossible (we have shown previously that \(\lambda_0 > \lambda_1\)). We conclude that \(h(h_{\mu_1,K}(\eta)) > h(h_{\mu_1,K}(\lambda_1)) = h(h_{\mu_1,K}(\lambda_0))\). 

**Step 2**: Let us show that \(\mu_2(\lambda_0, \eta) > 0\).

We have

\[
\mu_2(\lambda_0, \eta) = \mu_2(\{\omega \in K_{\mu_1}(K); \lambda_0 < \omega < \eta\})
\]

\[
\geq \mu_2(\{\omega \in K_{\mu_1}(K); h_{\mu_1,K}(\lambda_0) < h_{\mu_1,K}(\omega) < h_{\mu_1,K}(\eta)\})
\]

\[
= \mu_1(\{\theta \in K; h_{\mu_1,K}(\lambda_0) < h_{\mu_1,K}(\Pi_2(\theta)) \leq h_{\mu_1,K}(\eta)\})
\]

\[
= \mu_1(\{\theta \in K_{\mu_1}(K); h_{\mu_1,K}(\lambda_0) < \theta \leq h_{\mu_1,K}(\eta)\})
\]

\[
\geq \mu_1(\{\theta \in K_{\mu_1}(K); h_{\mu_1,K}(\lambda_0) < \theta < h_{\mu_1,K}(\eta)\})
\]

\[
= \mu(\{t \in E; h(h_{\mu_1,K}(\lambda_0)) < h(\theta) \leq h(h_{\mu_1,K}(\eta))\})
\]

\[
= \mu(\{t \in E; h(h_{\mu_1,K}(\lambda_0)) < t < h(h_{\mu_1,K}(\eta))\}) > 0. 
\]
Thus $\eta \notin [\lambda_0]$.

Case 2, $\eta < \lambda_0$: By a similar argument, one shows that $\eta \notin [\lambda_0]$. It follows that $[\lambda_0] = \{\lambda_0, \lambda_1\}$. ■

Let $A_2$ be the $C^*$-subalgebra which is generated by the family $\{f^n; \eta \in K_{\mu_1}(K)\}$ (where $f^n$ the characteristic function of $\{\xi \in K_{\mu_1}(K); \xi > \eta\}$) in $L^\infty(K_{\mu_1}(K), \mu_2)$, $(A_2 = C(K_{\mu_2}(K_{\mu_1}(K))))$.

Recall that $B^\mu_2(K_{\mu_1}(K))$ is the set of $\xi \in K_{\mu_2}(K_{\mu_1}(K))$ such that the map $\eta \in (K_{\mu_1}(K)) \setminus \{\theta', \theta''\} \rightarrow \psi_{\mu_2, K_{\mu_1}(K)}(\xi, \eta) = \xi(f^n)$ is continuous and $B_2^\mu_2(K_{\mu_1}(K)) = \{\xi \in K_{\mu_2}(K_{\mu_1}(K)) \setminus B_1^\mu_2(K_{\mu_1}(K)); \psi_{\mu_1, K}(\xi, h_{K_{\mu_1}(K)}(\xi)) = 3 - j\}$, $j = 2, 3$, where $h_{\mu_2, K_{\mu_1}(K)}(\xi) = \sup \{\eta \in K_{\mu_1}(K); \psi_{\mu_2, K_{\mu_1}(K)}(\xi, \eta) = 1\}$.

**Corollary 14.** The map $h_{\mu_2, K_{\mu_1}(K)} : K_{\mu_2}(K_{\mu_1}(K)) \rightarrow K_{\mu_1}(K)$ is a homeomorphism.

**Proof.** Consider the reference set $\tilde{E} = K_{\mu_1}(K)$, $(E_1) = B_2^\mu_2(K_{\mu_1}(K))$, $E_2 = B_3^\mu_2(K_{\mu_1}(K))$. By Theorem 4 and Lemma 13 $h_{\mu_2, K_{\mu_1}(K)} : K_{\mu_2}(K_{\mu_1}(K)) \rightarrow K_{\mu_1}(K)$ is a homeomorphism. ■

6. General properties of $(C(L), \tau_p)$ when $L$ is a Rosenthal compact set

In this part, we show that the function $\psi : (K \times (E, \tau_0)) \rightarrow \{0, 1\}$ is not Borel under some assumptions. Let $L$ be Hausdorff compact. In [27] one shows under the Martin’s Axiom that $(C(L), \tau_p)$ is universally measurable. During this part, we show that $(C(L), \tau_p)$ is universally measurable without Martin’s Axioms, if $L$ is a Rosenthal compact set. Finally, we show that if $K$ is separable, then $(C(K, \tau_p)$ is not measure-compact.

**Definition 6.** Let $(X, \tau)$ be a Hausdorff space. We say that $(X, \tau)$ is a Radon space [21, p. 117], if for every positive finite Borel measure on $(E, \tau)$ is a Radon measure.

**Definition 7.** Let $(X, \tau)$ be a Hausdorff space. A positive finite Borel measure $\nu$ on $(X, \tau)$ is said to be normal if for every net $(O_\alpha)$ of open sets is increasing to open set $O$, $\lim \nu(O_\alpha) = \nu(O)$.

**Remark 17.** Let $(X, \tau)$ be a strongly Lindelöf space. Then every positive Borel measure on $(X, \tau)$ is normal.

**Remark 18.** Let $(X, \tau)$ be a Hausdorff topological space and let $X_1 \subset X$. Then $\text{Bor}(X_1, \tau) = \{C \cap X_1; C \in \text{Bor}(X, \tau)\}$. 
Remark 19. Let $L$ be a compact strongly Lindel"of space. Then $K$ is a Radon space.

Proof. As $L$ is a strongly Lindel"of space, by Remark 17 every positive finite Borel measure on $K$ is normal. On the other hand, $K$ is universally measurable, by [22, th. 3.2] every normal positive finite Borel measure on $K$ is a Radon measure, hence $L$ is a Radon space. ■

Proposition 12. Let $(E, \tau_0)$ be a strongly Lindel"of space such that $-\infty, +\infty \notin E$; let $E'$ be a subset of $E$ such that $(E', \tau_1)$ be completely regular space. Assume that there exists a measure $\nu$ strictly positive on $(E', \tau_0)$ such that $\nu$ is diffuse on $E$. Then $(E', \tau_1)$ is not universally measurable space.

Proof. By Lemma 8 $\text{Bor}(E, \tau) = \text{Bor}(E, \tau_1)$. This implies by Remark 18 that $\text{Bor}(E', \tau_0) = \text{Bor}(E', \tau_1)$. Hence, there exists $V \in \text{Bor}(E', \tau_1)$ such that $0 < \nu(V) < +\infty$. One defines the measure $\nu_1$ on $E'$ by, $\nu_1(U) = \nu(U \cap V), U \in \text{Bor}(E', \tau_1)$. $\nu_1$ is a finite Borel measure on $(E', \tau_1)$. On the other hand, by Lemma 8 $(E, \tau_1)$ is a strongly Lindel"of space, hence $(E', \tau_1)$ has the same property by Remark 5 we conclude by Remark 17 that $\nu_1$ is normal. Suppose now that $(E', \tau_1)$ is universally measurable. By [22, th. 3.2], $\nu_1$ is a Radon measure. It follows that there exists a compact set $L_0$ of $(E', \tau_1)$ such that $\nu_1(L_0) > 0$. Put $x_0 = \max \{t; t \in L_0\}$. By hypothesis on $\nu$, we have $\nu_1(L_0 - \{x_0\}) > 0$, then there exists a compact set $L_1$ of $L_0 - \{x_0\}$ satisfying $\nu_1(L_1) > 0$. Denote $x_1 = \max \{t; t \in L_1\}$. By induction one constructs a strictly decreasing sequence $(x_n)_{n \geq 0}$ of $L_0$. Let $\mathcal{U}$ be a non trivial ultrafilter on $\mathbb{N}$. Since $L_0$ is compact, there exists $a \in L_0$ such that $x_n \to a$ in $(E', \tau_1)$. Let $O = \{t \in E'; t \leq a\}$. $O$ is an open neighborhood of $a$, this implies that $\{n \in \mathbb{N}; x_n \in O\} = \{n \in \mathbb{N}; x_n \leq a\} \in \mathcal{U}$. It is impossible because $x_k \geq a$, for every $k \in \mathbb{N}$ and the sequence $(x_n)_{n \geq 0}$ is strictly decreasing. ■

By Remark 11 and Proposition 12 one has the following corollary:

Corollary 15. Let $(E, \tau_0)$ be a strongly Lindel"of space. Assume that $\nu$ strictly positive on $(E, \tau_0)$ and diffuse on $E$, $(E, \leq)$ is complete and $-\infty, +\infty \notin E$. Then $(E, \tau_1)$ is not universally measurable.

Recall that

$$E_2 = \{a \in E; \{x \in E; x < a\} \text{ is not a closed subset of } E\}.$$

Corollary 16. Assume that $(E, \tau_0)$ is a strongly Lindel"of space, that $-\infty, +\infty \notin E$, that $E_2 = E$, the measure $\mu$ is diffuse on $E$ and that $\mu([t \wedge t', t \vee t']) > 0$ for $t \neq t'$. Then $B_3$ is not universally measurable.
Proof. By Remark 12, \((E, \tau_1)\) is completely regular. On the other hand Proposition 12 shows us that \((E, \tau_1)\) is not universally measurable and by step 2 of Proposition 8-I), \(B_3\) is homeomorphic to \((h(B_3), \tau_1)\). But \(h(B_3) = E\) by Lemma 6 we conclude that \(B_3\) is not universally measurable. \(\blacksquare\)

Remark 20. In Corollary 16, we can replace the hypothesis \(E_2 = E\), by the hypothesis \(h_3\):

\[ h_3 : B_3 \to E \] is onto.

Corollary 17. Let \((E, \tau_0)\) be a strongly Lindelöf space, and suppose that \(-\infty, +\infty \notin E\). Assume that there exists a diffuse measure \(\nu\) on \(E\) and there exists a compact set \(L_0\) of \((E, \tau_0)\) verifying \(0 < \nu(L_0) < +\infty\). Then \((E, \tau_1)\) does not embed in any analytic set.

Proof. Consider the identity map \(i : (E, \tau_0) \to (E, \tau_1)\). By Lemma 8 \(i\) is Borel. Suppose now that there exists an analytic set \(Y\) such that \((E, \tau_1)\) is a topological subspace of \(Y\). Consider the measure \(\nu_1\) defined on \(L_0\) by \(\nu_1(B) = \nu(B), B \in \text{Bor}(L_0)\). Since \((L_0, \tau_0)\) is a strongly Lindelöf space, \(\nu_1\) is normal. On the other hand \((L_0, \tau_0)\) is universally measurable, by [22, th.3.2] \(\nu_1\) is a Radon measure. By [23, th. 14], there is a compact subset \(L_1\) of \(L_0\) of measure strictly positive such that the restriction of \(i\) to \(L_1\) is continuous (with values in \(Y\)). It follows that \(i(L_1) = L_1\) is compact in \((E, \tau_1)\) and \(\nu_1(L_1) > 0\). using Proposition 12 for \(E' = L_1\), we obtain that \(L_1\) is not universally measurable, which is impossible. \(\blacksquare\)

Corollary 18. Let \((E, \tau_0)\) be a strongly Lindelöf such that \(-\infty, +\infty \notin E\). Suppose that \(h_3\) is onto and the hypothesis on \(\nu\) in corollary 17 are satisfied for \(\nu = \mu\). Then \(B_3\) does not embed in any analytic set.

Proof. By Proposition 8-I) \(B_3\) is homeomorphic to \((E, \tau_1)\). For getting the corollary it suffices to use Corollary 17.

Proposition 13. Suppose that \((E, \tau_0)\) is separable, satisfying the condition \((*)\), and the measure \(\mu\) is diffuse on \(E\), \(h_3 : B_3 \to E\) is surjective, \(-\infty, +\infty \notin E\) and \(\mu([t \land t', t \lor t']) > 0\) for \(t \neq t'\). Then the map \(\psi : (K \times (E, \tau_0)) \to \{0, 1\}\) is not Borel.

Proof. 

Step 1: Let us show that \(\text{Bor}(K) \otimes \text{Bor}(E) = \text{Bor}(K \times E)\).

It is obvious that \(\text{Bor}(K) \otimes \text{Bor}(E) \subset \text{Bor}(K \times E)\). We show the reverse inclusion. Let \((O_k)_{k \geq 0}\) be a countable basis of \((E, \tau_0)\) (a such basis exists by Remark 1) and let \(V = \bigcup_{i \in I} U_i \times W_i\), where the \(U_i\) are open subsets of \(K\) and the \(W_i\) are open subsets of \(E\). For every \(i \in I\),
there exists a subset $M_i$ of $\mathbb{N}$ such that $W_i = \bigcup_{k \in M_i} O_k$, hence $V = \bigcup_{k \in \mathbb{N}} \left( \bigcup_{i \in Y_k} U_i \right) \times O_k$, where $Y_k = \{i; k \in M_i\}$. Since $\bigcup_{i \in Y_k} U_i$ is an open subset of $K$, then

$$V \in \text{Bor}(K) \otimes \text{Bor}(E).$$

It follows that $\text{Bor}(K \times E) \subset \text{Bor}(K) \otimes \text{Bor}(E)$. ■

**Step 2:** Let us show that every Borel subset of $K$ is universally measurable.

By Corollary 7, $K$ is a strongly Lindelöf space, using Remark 19, one obtains that $K$ is a Radon space. We conclude that every Borel subset of $K$ is relatively universally measurable in $K$, by [22, th. 3.2] every Borel subset of $K$ is universally measurable. ■

Suppose now that $\psi$ is Borel. Consider the map $\sigma : K \setminus B_1 \to K \times E$ defined by $\sigma(\theta) = (\theta, h(\theta))$, $\theta \in K \setminus B_1$. This map $\sigma$ is Borel because $\text{Bor}(K) \otimes \text{Bor}(E) = \text{Bor}(K \times E)$ and $h$ is continuous by Proposition 3. It follows that $\psi \circ \sigma$ is Borel, hence $(\psi \circ \sigma)^{-1}\{0\} = B_3$ is Borel subset of $K$. By the step 2, $B_3$ is universally measurable, this is impossible by Remark 20. ■

Let $(Y, \tau)$ be a Hausdorff topological space. We denote by $\text{Bair}(Y)$ the $\sigma$-algebra generated by the continuous functions on $Y$ and by $P_\sigma(Y)$ the set of probability measures on $(Y, \text{Bair}(Y))$.

**Definition 8.** ([16], [17], [28]) Let $X$ be a completely regular space:

a) $X$ is measure-compact, if for every measure $\nu \in P_\sigma(X)$ and every net $(f_\alpha)_{\alpha \in I}$ of bounded continuous decreasing functions on $(X, \tau)$ such that $f_\alpha \to 0$, $\lim_{\alpha} \int_X f_\alpha d\nu = 0$.

b) $X$ is strongly measure-compact, if for every measure $\nu \in P_\sigma(X)$ and every $\varepsilon > 0$, there is a compact $L$ of $X$ such that $\nu^*(L) > 1 - \varepsilon$, where

$$\nu^*(L) = \inf \{\nu(H); H \in \text{Bair}(X) \text{ and } L \subset H\}.$$

Note that if $X$ is strongly measure-compact, then $X$ is measure-compact.

**Remark 21.** Suppose that $(E, \leq)$ satisfies the condition $(\ast)$. Then $(E, \tau_0)$ and $(E, \tau_1)$ are regular spaces.

**Proof.** Let us show for example that $(E, \tau_1)$ is regular. Indeed, let $V = [a, b]$ an open neighborhood of $x$. Since $(E, \leq)$ verifies the condition $(\ast)$, there is $c \in E$ such that $a < c < x \leq b$, hence $[c, b] = [c, b] \subset V$ and $[c, b]$ is an open neighborhood of $x$. We can use the same argument if $V = (a, |b|$, $a < b$. ■
Proposition 14. Let $(E, \tau_0)$ be a separable strongly Lindelöf space satisfying the condition $(\ast)$. Suppose there is a probability measure $\nu$ on $(E, \tau_0)$, diffuse on $E$ and $-\infty, +\infty \notin E$. Then $(E, \tau_1)$ is not strongly measure-compact.

Proof. Since $E$ satisfies $(\ast)$, $E = E_2$. By Remark 12, $(E, \tau_1)$ is completely regular.

Step 1: Show that every open subset of $(E, \tau_1)$ is in $\text{Bair}(E, \tau_1)$.

Pick $a \in E$. The set $\{t \in E; t \leq a\}$ is open and closed in $(E, \tau_1)$, hence the characteristic function of $\{t \in E; t \leq a\}$ is continuous on $(E, \tau_1)$. It follows that $\{t \in E; t \leq a\} \in \text{Bair}(E, \tau_1)$.

Since $(E, \tau_0)$ is strongly Lindelöf and regular (by Remark 21), $(a)$ is $G_\delta$ of $(E, \tau_0)$. Thus there exists a sequence of open subsets $(O_n)_{n \geq 0}$ of $(E, \tau_0)$ such that $(a) = \bigcap_{n \geq 0} O_n$. For every $n \geq 0$, there is an open interval $I_n$ of $O_n$ such that $a \in I_n \subset O_n$, this implies that there exists $a_n \in E$ such that $]a_n, a[ \subset I_n \subset O_n$. Therefore $(a) = \bigcap_{n \geq 0} ]a_n, a[.$

On the other hand, $]a_n, a[ = \{t \in E; t \leq a\} - \{t \in E; t \leq a_n\} \in \text{Bair}(E, \tau_1)$.

Step 2: Let $L$ be a compact subset of $(E, \tau_1)$. Let us show that $\nu(L) = 0$.

Suppose $\nu(L) > 0$. By Proposition 12, $L$ is not universally measurable, which is impossible.

Step 3: Let us show that $(E, \tau_1)$ is not strongly measure-compact.

Note first by step 1 that $\text{Bor}(E, \tau_1) = \text{Bair}(E, \tau_1)$, hence $\nu \in P_0(E, \tau_1)$.

Suppose now that $(E, \tau_1)$ is a strongly measure-compact. Then there exists a compact $L$ subset of $(E, \tau_1)$ such that $(\nu)^*(L) > 0$. By step 1, $L \in \text{Bair}(E, \tau_1)$, this implies that $\nu(L) = (\nu)^*(L) > 0$, this is impossible by step 2.

Note that if $X$ a Lindelöf space, then $X$ is measure-compact [28]. In [16]-[17] one shows that there exists a measure-compact space which is not strongly measure-compact. The following remark gives another type of example.

Remark 22. There exists a Lindelöf space $Y$, separable paracompact which is not strongly measure-compact.

Proof. It suffices to apply Proposition 14 to $E = ]0, 1[$, $\nu$ is the Lebesgue measure, hence $Y = (]0, 1[, \tau_1)$ is not strongly measure-compact.
Let $L$ be a Hausdorff compact space. In [27] one shows under the Martin’s Axiom that $(C(L), \tau_p)$ is universally measurable. In the following Lemma, one shows that $(C(L), \tau_p)$ is universally measurable without Martin’s Axioms if $L$ is a Rosenthal compact set.

**Proposition 15.** Let $L$ be a Rosenthal compact set. Then $(C(L), \tau_p)$ is universally measurable.

**Proof.** We shall show Lemma 15 if $C(L)$ is the real continuous functions space. The proof is similar when one replaces $\mathbb{R}$ by $C$. By [22, th. 3.2] it suffices to show that every normal probability measure on $(C(L), \tau_p)$ is a Radon measure. Let $\nu$ a normal probability measure on $(C(L), \tau_p)$.

Denote, for every $n \in \mathbb{N}^*$, $B_n = \{ g \in C(L); \| g \|_{C(L)} \leq n \}$ and $\nu_n$ the measure defined by $\nu_n(C) = \nu(C \cap B_n)$, $C \in \Bor(C(L), \tau_p)$. Since $\nu_n(C) \to_{n \to +\infty} \nu(C)$ and $B_n$ is $\tau_p$ closed, it is enough to show that $\nu_n$ is a Radon measure for all $n \geq 1$.

**Step 1:** Let us show that for every $n \geq 1$, $\nu_n$ is a normal measure.

Let $n \in \mathbb{N}^*$ and let $(F_\alpha)$ be a net of closed subsets of $(C(L), \tau_p)$ which is decreasing to a closed set $F$ of $(C(L), \tau_p)$. Since $B_n$ is $\tau_p$-closed, the net $(F_\alpha \cap B_n)$ is decreasing to $B_n \cap F$ and $\nu$ is normal, it follows that

$$\lim_{\alpha} \nu_n(F_\alpha) = \lim_{\alpha} \nu(F_\alpha \cap B_n) = \nu(F \cap B_n) = \nu((\bigcap_{i \in I} F_i) \cap B_n) = \nu_n(F).$$

**Step 2:** Pick $n \in \mathbb{N}$. Let us show that $\nu_n$ is a Radon measure.

We can suppose that $\text{supp}(\nu_n)$ is included in the closed unit ball of $(C(L), \| \cdot \|)$. Denote $V$ the vector subspace spanned by $L$ in the dual of $(C(L), \| \cdot \|)$. For every $v \in V$ let $H_v$ the hyperplane defined by $\nu$ in $C(L)$, i.e.

$$H_v = \{ f \in C(L); (f, v) = v(f) = 0 \}.$$

Consider the set $I'$ formed of all elements $v$ of $V$ such that $\nu_n(H_v) = 1$ and $E_{\nu_n}$ is the intersection of $H_v$ when $v \in I'$ (note that $0 \in I'$).

Let us show that $\nu_n(E_{\nu_n}) = 1$. Let $\tilde{I} = \{ S; S \in I \}$ and $H_S = \cap_{v \in S} H_v, S \in \tilde{I}$. It is clear that $(H_S)_{S \in \tilde{I}}$ is a net of decreasing closed subsets of $(C(L), \tau_p)$; since $\nu_n$ is normal,

$$\nu_n(E_{\nu_n}) = \nu_n(\bigcap_{v \in I'} H_v) = \inf_{S \in \tilde{I}} \nu_n(H_S) = 1,$$

because $\nu_n(H_S) = 1$, for every $S \in \tilde{I}$. For $y \in L$ put

$$[y] = \{ x \in L; (f, y) = (f, x), \forall f \in E_{\nu_n} \}$$

and $L_{\nu_n} = \{ [y]; y \in L \}$ ($L_{\nu_n}$ as a topological subspace of $[-1, +1]^{E_{\nu_n}}$, where $[y](f) = (f, y), f \in E_{\nu_n}$). We shall show that $L_{\nu_n}$ is compact.
It is enough to show that $L_{\nu_n}$ is closed in $[-1,+1]^{E_{\nu_n}}$. Let $(y_\alpha)_{\alpha \in I}$ be a net in $L_{\nu_n}$ such that $[y_\alpha] \to z \in [-1,+1]^{E_{\nu_n}}$. Since $L$ is compact, there exists $y \in L$ such that $y_\alpha \to y \in L$, this implies that $[y_\alpha](f) = (f, y_\alpha) \to (f, y) = [y](f)$ for every $f \in E_{\nu_n}$. On the other hand, $[y_\alpha](f) = (f, y_\alpha) \to z(f)$, it follows that $z(f) = [y](f)$ for every $f \in E_{\nu_n}$. We conclude that $z = [y].$

**Step 2-a):** Suppose that $[y] = [y']$, $\nu_n$-almost-everywhere, $(y, y' \in L)$ and show that $[y] = [y']$ everywhere.

Indeed, for almost all $f \in C(L)$, $(y, f) = (y', f)$. Thus $(y-y', f) = 0$ for almost all $f \in C(L)$, i.e. the hyperplane $H_{y-y'}$ is of measure one, hence $H_{y-y'}$ contains $E_{\nu_n}$. We deduce that for every $f \in E_{\nu_n}$, $(f, y) = (f, y')$. ■

We define the map $\Sigma : L \to L_{\nu_n}$, by $\Sigma(y) = [y], y \in L$ (it is clear that $\Sigma$ is continuous) and the map $U : L_{\nu_n} \to L^1(E_{\nu_n}, \nu_n)$ by $U([x])(f) = f(x), x \in L, f \in E_{\nu_n}$. Note that $U$ is into by step 3.

**Step 2-b):** Sow that $U$ is continuous. For that let $F$ be a closed subset of $L^1(E_{\nu_n}, \nu_n)$ and let $[x]$ be a point in the closure of $U^{-1}(F)$. There exists a net $(x_\alpha)_{\alpha \in I}$ in $U^{-1}(F)$ such that $x_\alpha \to [x]$. Since $L$ is compact, there exists $x' \in L$ such that $x_\alpha \to x'$ in $L$, by continuity of $\Sigma$, $x_\alpha \to [x']$ in $L_{\nu_n}$. It follows that $[x'] = [x].$

On the other hand, by [3] $L$ is angelic, hence there is a countable subsequence $(x_{i_k})_{k \geq 0}$ such that $x_{i_k} \to_{k \to +\infty} x'$. By the continuity of $\Sigma$ we have $[x_{i_k}] \to_{k \to +\infty} [x'] = [x]$ in $L_{\nu_n}$. By dominated convergence theorem, $U([x_{i_k}]) \to_{n \to +\infty} U([x])$ in $L^1(E_{\nu_n}, \nu_n)$. Since $U([x_{i_k}]) \in F$, for every $k \in \mathbb{N}, U([x]) \in F$. It follows that $U$ is continuous. ■

Thus $L_{\nu_n}$ is metrisable, because $U$ is a homeomorphism on its range. Therefore, $(C(L_{\nu_n}), || . ||)$ is a Polish space, and $(C(L_{\nu_n}), \tau_p)$ is defined by a separating family of seminorms, by [2] TG. 9, 68-6, th. 14 $\text{Bor}(C(L_{\nu_n}), || . ||) = \text{Bor}(C(L_{\nu_n}), \tau_p)$. Since $(C(L_{\nu_n}), || . ||)$ is a Radon space [21] chap. II, th. 9 and $(E_{\nu_n}, \tau_p)$ is a closed subspace of $(C(L_{\nu_n}), \tau_p)$, $(E_{\nu_n}, \tau_p)$ is a Radon space. We deduce that $\nu_n$ is a Radon measure. ■

For every locally topological vector space $X$, we denote by $\text{Cyl}(X)$ the $\sigma$-algebra generated by continuous linear forms on $X$. Recall that $(X, \text{weak})$ coincides with $\text{Cyl}(X)$ [6] th. 2.3].

**Lemma 14.** Let $L$ be a Rosenthal compact set. Then $(C(L), \tau_p)$ is measure-compact if and only if $(C(L), \text{weak})$ is measure-compact.

**Proof.** Show first that $\text{Cyl}((C(L), \text{weak})) = \text{Cyl}((C(L), \tau_p))$. (It follows by [6] th. 2.3] that

$$\text{Bair}(C(L), \text{weak})) = \text{Bair}(C(L), \tau_p)).$$

(6.1)
Indeed, it is clear that $\text{Cyl}((C(L), \tau_p)) \subset \text{Cyl}((C(L), \text{weak}))$. Let us show the converse inclusion. Let $y^* \in M^+(L)$. By [25], $M^+(L)$ is a Rosenthal compact set, hence by [3] it is angelic, we deduce that there exists a sequence $(y^*_n)_{n \geq 0}$ of finite support on $L$ such that $y^*_n \to y^*$ for the $w^*$-topology. For each $n \in \mathbb{N}$, $y^*_n$ is measurable with respect to $\text{Cyl}((C(L), \tau_p))$, this implies that $y^*$ is measurable with respect to $\text{Cyl}((C(L), \tau_p))$. If $y^* \in B_{(C(L))^{\tau_p}}$, there exists $\mu_1, \mu_2, \mu_3, \mu_4 \in M^+(L)$ such that $y^* = \mu_1 - \mu_2 - i(\mu_3 - \mu_4)$, hence $y^*$ is measurable with respect to $\text{Cyl}((C(L), \tau_p))$, thus $\text{Cyl}((C(L), \text{weak})) \subset \text{Cyl}((C(L), \tau_p))$.

Suppose now that $(C(L), \tau_p)$ is measure-compact. Let $\nu$ be a measure defined on $\text{Cyl}((C(L), \text{weak}))$. By [25] th. 2, $\nu$ extends to a normal measure $\tilde{\nu}$ on $(C(L), \tau_p)$. On the other hand, by Proposition [15] $(C(L), \tau_p)$ is universally measurable, hence $\tilde{\nu}$ is a Radon measure on $(C(L), \tau_p)$. Let $\varepsilon > 0$ and let $C \in \text{Bor}(C(L), \tau_p)$. There exists a $\tau_p$-compact set $H \subset C$ such that $\tilde{\nu}(H) > \tilde{\nu}(C) - \varepsilon$. For every integer $n \in \mathbb{N}^*$, consider $B_n = \{ g \in C(L); \|g\|_{(C(L))} \leq n \}$. Since $\tilde{\nu}(H \cap B_n) \to_{n \to +\infty} \tilde{\nu}(H)$, there is $n_0 \in \mathbb{N}^*$ such that $\tilde{\nu}(H \cap B_{n_0}) > \tilde{\nu}(C) - \varepsilon$.

Note that $H \cap B_{n_0}$ is an uniformly bounded $\tau_p$-compact subset of $H$, hence by [11] $H \cap B_{n_0}$ is a weakly compact subset. By [21] chap. 1, th. 16, one obtains that $\tilde{\nu}$ extends to a Radon-measure $\nu'$ on $(C(L), \text{weak})$. In particular $\nu$ extends to a normal measure on $(C(L), \text{weak})$. Note that, if $f$ is a bounded continuous function on $(C(L), \text{weak})$, then we have
\[
\int_{C(L)} f(x) \, d\tilde{\nu}(x) = \int_{C(L)} f(x) \, d\nu'(x) = \int_{C(L)} f(x) \, d\nu(x).
\]
Let now $(f_\alpha)_{\alpha \in I}$ be a decreasing net of bounded continuous functions on $(C(L), \text{weak})$ such that $f_\alpha \to 0$ for the pointwise topology. By [25] th. 2 $\int_{C(L)} f_\alpha(x) \, d\nu(x) = \int_{C(L)} f_\alpha(x) \, d\tilde{\nu}(x) \to 0$. It follows that $(C(L), \text{weak})$ is measure-compact.

Conversely, suppose that $(C(L), \text{weak})$ is measure-compact. Since, by above $\text{Bair}(C(L), \text{weak}) = \text{Bair}(C(L), \tau_p)$, it follows that $(C(L), \tau_p)$ is measure compact. ■

**Proposition 16.** Suppose that $K$ is separable. Then $(C(K), \tau_p)$ is not measure-compact.

**Proof.** It is clear that for every $\theta \in K = B_1 \cup B_2 \cup B_3$, the map $t \to \psi(\theta, t)$ is measurable.
By an argument similar to that of Proposition 11, we show that for every \( \xi \in C(K)^* \), the map \( t \in E \to \xi(f^t) \) is measurable.

Suppose now that \((C(K), \tau_p)\) is measure-compact. Lemma 13 shows us that \((C(K), \text{weak})\) is measure-compact. Since the map \( t \to f_t \) is scalarly measurable and \((C(K), \text{weak})\) is measure-compact, by Theorem 6, there exists a strongly measurable function \( g \) with values in \( C(K) \) such that for every \( \xi \in C(K)^* \), \( \xi(f^t) = \xi(g(t)) \), for almost all \( t \in E \).

Let \((\theta_n)_{n \geq 0}\) be a dense sequence in \( K \). For every \( n \in \mathbb{N} \) there exists a measurable subset \( D_n \) of \( E \) such that \( \mu(D_n) = 0 \) and \( \theta_n(f^t) = \theta_n(g(t)) \), for all \( t \in (D_n)^c \). Denote \( D = \bigcup_{n \geq 0} D_n \). Since \( \theta_n(f^t) = \theta_n(g(t)) \), for all \( n \in \mathbb{N} \) and all \( t \in D^c \), \( \theta(f^t) = \theta(g(t)) \), for all \( \theta \in K \) and all \( t \in D^c \). It follows that \( f^t = g(t) \), for almost all \( t \in E \), this is impossible, because \( \|f_t - f_t^t\|_{L^\infty(E, \mu)} \geq 1 \), if \( t \neq t' \) and \( g \) has almost values in a separable subspace of \( C(K) \). Thus \((C(K), \tau_p)\) is not measure-compact. ■

7. The case of a separable connected abelian group verifying the condition (**)

In this part, \( E \) is a separable connected abelian group verifying the condition (**)? and \( E \) is locally compact.

By Remark 1, \( E \) has a countable basis. As \( E \) is locally compact, by [21, chap. II, th. 6], \( E \) is a Polish space.

Note that \(-\infty, +\infty \notin E \). For \( g \in C(K) \subset L^\infty(E, \mu) \), \( \theta \in K \) and \( t \in E \), one defines \( g_t \in C(K) \) by \( g_t(u) = g(u - t) \), for almost all \( u \in E \) and \( \theta_t \in K \) by \( \theta_t(r) = \theta(rt) \), \( r \in C(K) \).

Let \( f \) be the characteristic function of \( \{t \in E; t > 0\} \).

**Lemma 15.** For every \( \theta \in B_2 \), \( h(\theta_u) = h(\theta) - u \), for all \( u \in E \).

**Proof.** Since \( E \) satisfies the condition (**), \( f_t = f^t \), for every \( t \in E \). Let \( \theta \in B_2 \) and let \( t, u \in E \) such that \( \psi(\theta_u, t) = 1 \). Hence we have \( \psi(\theta_u, t) = \theta_u(f_t) = \theta[(f_t)u] = \theta(f_{t+u}) = \psi(\theta, t + u) = 1 \). It follows that \( t + u \leq h(\theta) \). The condition (**)? implies that \( t \leq h(\theta) - u \). By Lemma 2 one has \( h(\theta_u) = \sup \{t; \psi(\theta_u, t) = 1\} \leq h(\theta) - u \).

Conversely, let \( t \in E \) such that \( \psi(\theta, t) = 1 \). For every \( u \in E \), we have \( \psi(\theta_u, t - u) = \theta_u(f_{t-u}) = \theta((f_{t-u})u) = \theta(f_t) = \psi(\theta, t) = 1 \), hence \( h(\theta_u) \geq t - u \), since \( E \) satisfies the condition (**)?, \( h(\theta_u) + u \geq t \). By Lemma 2 we obtain that \( h(\theta_u) + u \geq h(\theta) \), i.e. \( h(\theta_u) \geq h(\theta) - u \), we deduce that \( h(\theta_u) = h(\theta) - u \). ■

**Remark 23.** In the previous lemma, one can replace \( B_2 \) by \( B_3 \) (using Lemma 3).
Remark 24. \( B_2 \) is notempty.

Proof. Indeed, suppose that \( B_2 = \emptyset \). Proposition\[7\] tells us that \( B_3 = \emptyset \), hence \( K = B_1 \) which is metrisable (by Proposition\[7\]ii), It is impossible, because \( (E, \mu) \) satisfies the condition (1.2). ■

Lemma 16. For every \( \theta \in K \setminus B_1 \):

\[
\sup \{ -t \in E; \psi(\theta, t) = 0 \} = -\inf \{ t \in E; \psi(\theta, t) = 0 \}.
\]

Proof. Consider \( \theta \in K \setminus B_1 \). Put

\[
\beta = -\inf \{ t \in E; \psi(\theta, t) = 0 \}.
\]

Let \( t \in E \) such that \( \psi(\theta, t) = 0 \). By definition of \( -\beta \) we have \( -\beta \leq t \). The condition (**) implies that \( -\beta + (\beta - t) \leq t + (\beta - t) \), hence \( -t \leq \beta \), to complete the proof, it suffices to show that for every \( z \in E \) such that \( -t \leq z \), and if \( \psi(\theta, t) = 0 \), then \( \beta \leq z \). Let a such \( z \in E \). Since \( E \) satisfies the condition (**), \( t \geq -z \), for every \( t \in E \) such that \( \psi(\theta, t) = 0 \). We deduce that \( \inf \{ t \in E; \psi(\theta, t) = 0 \} \geq -z \), i.e. \( \beta \leq z \). ■

For every \( \theta \in K \), one defines \( \hatsm{\theta} \in K \), by \( \hatsm{\theta}(g) = \theta(g) \), where \( g(x) = g(-x) \), for almost \( x \in E \), \( g \in C(K) \subset L^\infty(E, \mu) \).

Proposition 17. Let \( \theta \in K \setminus B_1 \). Then:

i) \( h(\hatsm{\theta}) = -h(\hatsm{\theta}) \).

ii) \( \varphi \in B_2 \), if and only if \( \hatsm{\varphi} \in B_3 \).

iii) \( \hatsm{\hatsm{\theta}} = \theta'' \).

Proof.

i). The measure \( \mu = m \) is invariant by translation, then \( \mu(\{t\}) = 0 \), for every \( t \in E \). Thus we have

\[
\mu\left( \{ y \in E; (f_{-t})(y) = 0 \} \right) \\
= \mu(\{ y \in E; f_{-t}(y) = 0 \}) \\
= \mu(\{ y \in E; f(t - y) = 0 \}) = \mu(\{ y \in E; y \geq t \}) \\
= \mu(\{ y \in E; e(y) - f_t(y) = 0 \}).
\]

By a similar argument we show that

\[
\mu(\{ y \in E; (f_{-t})(y) = 1 \}) = \mu(\{ y \in E; e(y) - f_t(y) = 1 \}).
\]

Thus for every \( t \in E \)

\[
(f_{-t}) = e - f_t.
\] (7.1)
Let \( \theta \in K \setminus B_1 \) and \( t \in E \). By (7.1),
\[
\psi(\theta, -t) = \theta(f_{-t}) - \psi(\theta, t) = 1 - \psi(\theta, t),
\]
\text{i. e.}
\[
\psi(\theta, -t) = 0 \text{ if and only if } \psi(\theta, t) = 1. \tag{7.2}
\]

Using (7.2) and Lemma 2, one obtains
\[
h(\theta) = \sup \{ t \in E; \psi(\theta, t) = 1 \} = \sup \{ t \in E; \psi(\theta, -t) = 0 \}
\]
\[
= \sup \{ -t \in E; \psi(\theta, t) = 0 \}. \tag{7.3}
\]

On the other hand, by Lemma 16 and Lemma 3
\[
\sup \{ -t \in E; \psi(\theta, t) = 0 \} = - \inf \{ t \in E; \psi(\theta, t) = 0 \} = -h(\theta).
\]

Therefore \( h(\theta) = -h(\theta) \). \( \blacksquare \)

ii). By (7.2) and (i), \( \psi(\varphi, h(\varphi)) = 1 \) if and only if \( \psi(\varphi, h(\varphi)) = \psi(\varphi, -h(\varphi)) = 0 \). Thus \( \varphi \in B_2 \) if and only if \( \varphi \in B_3 \). \( \blacksquare \)

iii). Obviously by (7.1). \( \blacksquare \)

**Lemma 17.** Let \( \theta \in B_j \) and \( t \in E \). Then \( \theta_t \in B_j, j \in \{2, 3\} \).

**Proof.** Suppose that \( \theta \in B_2 \). By Lemma 15 we have \( \psi(\theta_t, h(\theta)) = \psi(\theta_t, h(\theta) - t) = \psi(\theta, h(\theta)) = 1 \), hence \( \theta_t \in B_2 \). By a similar argument, one shows that \( \theta_t \in B_3 \), if \( \theta \in B_3 \). \( \blacksquare \)

**Proposition 18.** For \( \theta \in B_2 \), the map \( U'_\theta : E \to K \times K, t \to (\theta, \theta_t) \) is not Borel.

**Proof.** Suppose that there exists \( \theta \in B_2 \) such that \( U'_\theta \) is a Borel function. Choose a subset \( H \) of \( E \) which is not in the Borel \( \sigma \)-algebra of \( E \). We define the open subset \( V \) of \( K \times K \) by
\[
V = \bigcup_{u \in H} \left[ \{ \varphi \in K; \psi(\varphi, -u + h(\theta)) = 1 \} \times \{ \varphi \in K; \psi(\varphi, -u + h(\theta)) = 0 \} \right].
\]
By Lemma 17, for every $t \in E$, $\theta_t \in B_2$ and $(\check{\theta})_t \in B_3$ (because $\check{\theta} \in B_3$, by Proposition 17). Thus we have

$$(U''_\theta)^{-1}(V)$$

$$= \bigcup_{u \in H} \{ t \in E; \psi(\theta_t, -u + h(\theta)) = 1 \text{ and } \psi((\check{\theta})_t, -u + h(\check{\theta})) = 0 \}$$

$$= \bigcup_{u \in H} \{ t \in E; h(\theta_t) \geq -u + h(\theta) \text{ and } h((\check{\theta})_t) \leq -u + h(\check{\theta}) \}$$

$$= \bigcup_{u \in H} \{ t \in E; h(\theta_\check{t}) - t \geq -u + h(\theta) \text{ and } h(\check{\theta}_\check{t}) - t \leq -u + h(\check{\theta}) \}$$

$$= \bigcup_{u \in H} \{ t \in E; -u \leq t \leq -u \} = \bigcup_{u \in H} \{ u \} = H,$$

because $h(\theta_t) = h(\theta) - t$ and $h((\check{\theta})_t) = h(\check{\theta}) - t$, for every $t \in E$, by Lemma 15. This implies that $H$ is a Borel subset of $E$, which is impossible. ■

**Corollary 19.** The Borel $\sigma$-algebra of the product space $K \times K$ strictly contains $\text{Bor}(K) \otimes \text{Bor}(K)$.

**Proof.** Consider the maps $\delta_1 : t \in E \rightarrow \theta_t \in B_2 \subset K$, $\delta_2 : t \in E \rightarrow (\check{\theta})_t \in B_3 \subset K$.

Let us show that $\delta_1, \delta_2$ are Borel functions. Let $u, t \in E$. It is clear that the subsets of the form

$$(\delta_1)^{-1}[\{ \varphi \in K; \psi(\varphi, u) = 0 \}], \quad (7.3)$$

$$(\delta_1)^{-1}[\{ \varphi \in K; \psi(\varphi, t) = 1 \}],$$

$$(\delta_2)^{-1}[\{ \varphi \in K; \psi(\varphi, u) = 0 \}],$$

$$(\delta_2)^{-1}[\{ \varphi \in K; \psi(\varphi, t) = 1 \}],$$

are Borel subsets of $E$ (note for example that

$$(\delta_1)^{-1}[\{ \varphi \in K; \psi(\varphi, u) = 0 \}] = \{ t \in E; \psi(\theta_t, u) = 0 \} = \{ t \in E; h(\theta_t) = h(\theta) - t < u \} = \{ t \in E; t > h(\theta) - u \}).$$

On the other hand by Proposition 10, $K$ is a strongly Lindelöf space, hence $\delta_1$ and $\delta_2$ are Borel functions, we deduce that for every $S \in \text{Bor}(K) \otimes \text{Bor}(K)$, $(U'_\theta)^{-1}(S) \in \text{Bor}(E)$.

Suppose now that $\text{Bor}(K \times K) = \text{Bor}(K) \otimes \text{Bor}(K)$. By above, $U'_\theta$ is a Borel function, this is impossible by Proposition 18. ■

L. Lindenstrauss and C. Stegall [15] showed that there exists a Banach space $X$ which does not contain $\ell^\infty$ isomorphically and a map
σ : \{0,1\}^N → X scalarly measurable, but σ is not weakly equivalent to a measurable function. In the following proposition, one gives an other type of examples. In the fact, the considered function (in Proposition 19) is locally integrable in the sense of Riemann.

**Proposition 19.** There exists a Banach space X and a function g : E → X such that

I) X does not contain \(\ell^\infty\) isomorphically.

II) g is scalarly measurable.

III) g cannot be weakly equivalent to a strongly measurable function.

IV) g is locally integrable in the sense of Riemann.

**Proof.**

I). Consider X = C(K). Since for all \(\xi \in B_{C(K)^*}\) there exist \(\xi_1, \xi_2, \xi_3, \xi_4 \in M^+(K)\) such that \(\xi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4)\), by [3]–[8] \(B_{C(K)^*}\) is angelic, hence X does not contain \(\ell^\infty\) isomorphically, because by [18] the closed unit ball of \((\ell^\infty)^*\) is not angelic. ■

II). Let \(g : E \to (C(K) \subset L^\infty(E)\) the function defined by \(g(t) = f_t\) for almost all \(t \in E\). By Proposition [11] g is scalarly measurable. ■

III). In the proof of Proposition [16] we saw that g cannot be weakly equivalent to a strongly measurable function. ■

IV). Let \(a, b \in E\) such that \(a < b\). Note that

\[
\int_a^b \chi_{[t,+\infty]}(\omega) \, dt = \begin{cases} (a \lor \omega) - a, & \text{si } \omega \leq b \\ b - a, & \text{si } \omega > b. \end{cases}
\]

Consider \(t_0 = a, t_1, \ldots, t_n = b \in [a, b]\) such that \(t_0 < t_1 < \ldots < t_n\). For \(t_k(\omega) \leq \omega < t_{k+1}(\omega)\), we have \(\sum_{k=1}^{n} \chi_{[t_k,+\infty]}(\omega)(t_k - t_{k-1}) = \sum_{k=1}^{n} (t_k - t_{k-1}) = t_k(\omega) - a\). One deduces that

\[
\sum_{k=1}^{n} \chi_{[t_k,+\infty]}(\omega)(t_k - t_{k-1}) = \begin{cases} t_k(\omega) - a, & \text{si } t_k(\omega) \leq \omega < t_{k+1}(\omega) \\ 0, & \text{si } \omega < a \\ \sum_{k=1}^{n} (t_k - t_{k-1}) = b - a, & \text{si } \omega > b. \end{cases}
\]

Thus

\[
\left\|\sum_{k=1}^{n} \chi_{[t_k,+\infty]}(\omega)(t_k - t_{k-1}) - \int_a^b \chi_{[t,+\infty]}(\omega) \, dt\right\|_{L^\infty(E)} \leq \sup_{\omega \in [a,b]} |\omega - t_k(\omega)| \leq \sup_{1 \leq k \leq n} |t_k - t_{k-1}|.
\]
It follows that $g$ is integrable in the sense of Riemann on $[a, b]$. ■

Let $M$ be the subset of $C((K)$ formed by even functions with values in $\{0, 1, 2\}$ ($g$ is even function on $K$ if $g = \check{g}$).

**Theorem 5.** There exists a Hausdorff topology $\tau'$ on $C(K)$, a $C^*$-subalgebra $Y_1$ of $C(K)$ and a Banach subspace $Y_2$ of $C(K)$ such that

1) $\tau_p$ is finer than $\tau'$.
2) $(C(K), \tau')$ is not universally measurable.
3) $Y_1$ and $Y_2$ are $\tau_p$ closed and $(C(K), \tau_p) = (Y_1, \tau_p) \oplus (Y_2, \tau_p)$.
4) $(Y_1, \tau_p)$ is isomorphic to $(Y_2, \tau_p)$.
5) $(Y_j, \tau_p) = (\theta, \tau')$, $j \in \{1, 2\}$.
6) $(\theta, \tau_p)$ contains a closed discrete uncountable subset which is contain in $M$.
7) $\text{Bor}(C(K), \tau') \otimes \text{Bor}(C(K), \tau') \neq \text{Bor}(C(K) \times C(K), \tau' \times \tau')$.
8) The projection $P : (C(K), \tau') \rightarrow (Y_1, \tau')$ is not a Borel function.
9) The set \{H : $E \rightarrow (C(K), \tau'); H$ is a Borel function\} is not a vector subspace of $C(K)^E$.

**Proof.**

1). 

**Step 1:** Let $\theta \in B_2$. Let us show that \{$\theta_t; t \in E$\} = $B_2$.

First observe that $B_2 \neq \emptyset$ by Remark 24. Let $\varphi \in B_2$. There exists $t \in E$ such that $h(\varphi) = h(\theta_t)$ (because $h(\theta_t) = h(\theta) - t$ by Lemma 15).

By Lemma 17, $\theta_t \in B_2$. Since $h_2 : B_2^0 \rightarrow E$ is into, by Remark 2, $\varphi = \theta_t$. ■

**Step 2:** Show that there exists a sequence in $B_2$ which is dense in $K$.

Since $h_2$ is onto, there is a sequence $(\theta_n)_{n \geq 0}$ in $B_2$ such that $(h(\theta_n))_{n \geq 0}$ is dense in $E$.

We will show that $(\theta_n)_{n \geq 0}$ is dense in $K$. For that, let $t \in E$. Denote $V_t = \{\theta \in K; \psi(\theta, t) = 1\}$. Choose $\theta \in K \setminus B_1$ such that $t < h(\theta)$, there exists $\theta_m \in B_2$ satisfying $t < h(\theta_m)$, because the sequence $(h(\theta_n))_{n \geq 0}$ is dense in $E$. Since $\psi(\theta_m, h(\theta_m)) = 1$ and $t < h(\theta_m)$, $\theta_m \in V_t$. Let $u \in E$ and let $U_u = \{\theta \in K; \psi(\theta, u) = 0\}$. Choose $\theta \in K \setminus B_1$ such that $u > h(\theta)$, there is $\theta_{m'} \in B_2$ satisfying $u > h(\theta_{m'})$. It follows that $\psi(\theta_{m'}, u) = 0$, because $\psi(\theta_{m'}, h(\theta_{m'})) = 1$. Thus $\theta_{m'} \in U_u$.

Let $t, u \in E$ and let

$$\theta_0 \in W_{t,u} = \{\theta \in K; \psi(\theta, t) = 1 \text{ and } \psi(\theta, u) = 0\}.$$ 

We note that $t < u$. Since $E$ satisfies the condition $(\ast)$ and $h$ is onto, there exists $\theta_1 \in K \setminus B_1$ such that $t < h(\theta_1) < u$. We deduce that there exists $\theta_n \in B_2$ verifying $t < h(\theta_n) < u$. Thus $\theta_n \in W_{t,u}$. ■
Let $\tau'$ be the topology defined on $C(K)$ by the condition that $u_\alpha \to u$ in $(C(K), \tau')$, if and only if $u_\alpha(\theta) \to u(\theta)$ for all $\theta \in B_2 \cup B_1$. Therefore $\tau_p$ is finer than $\tau'$, and by step 2 $\tau'$ is Hausdorff.

2). Put $D = \{f_i; t \in E\} \subset (C(K), \tau')$.

**Step 1:** Show that $(D, \tau')$ is a strongly Lindelöf space.

Pick $\theta \in B_2$ and define the map $\pi : (D, \tau') \to B_2$, by $\pi(f_i) = \theta_i$, $t \in E$. It suffices to show that $\pi$ is a homeomorphism, because $B_2$ is strongly Lindelöf.

Let us first show that $\pi$ is injective. For that, let $u, v \in E$ such that $\theta_u = \theta_v$. By Lemma 15 $h(\theta) - u = h(\theta) - v$, hence $u = v$. Let $u \in E$. Put $V_u = \{\theta \in B_2; \psi(\theta, u) = 0\}$ and $W_u = \{\theta \in B_2; \psi(\theta, u) = 1\}$. Thus we have $\pi^{-1}(V_u) = \{f_i; \psi(\theta_i, u) = 0\} = \{f_i; \theta_i(f_u) = 0\} = \{f_i; \theta_u(f_i) = 0\}$ is an open subset of $(D, \tau')$. In the same way, we have $\pi^{-1}(W_u) = \{f_i; \theta_u(f_i) = 1\}$ is an open subset of $(D, \tau')$. We deduce that $\pi$ is continuous.

We shall show that $\pi^{-1}$ is continuous. Indeed, let $t_1, t_2 \in E$. Denote $W_1 = \{f_i; f_i(\theta_{t_1}) = 1\}$ and $W_2 = \{f_i; f_i(\theta_{t_2}) = 0\}$. It is enough to show that $\pi(W_j)$ is an open subset of $B_2$, $j \in \{1, 2\}$, because the open sets of the previous form generated the topology $\tau'$.

Note that $\pi(W_1) = \{\theta_i; f_i(\theta_{t_1}) = 1\} = \{\theta_i; f_i(\theta_i) = 1\}$ and $\pi(W_2) = \{\theta_i; f_i(\theta_{t_2}) = 0\} = \{\theta_i; f_{t_2}(\theta_i) = 0\}$. We deduce that $\pi(W_1)$, $\pi(W_2)$ are open subsets of $B_2$.

Thus $\pi$ is a homeomorphism. □

Let $F_1$ be the closure of $D$ in $(C(K), \tau')$ and let $\sigma_1 : (E, \tau_0) \to F_1 \subset (C(K), \tau')$ the map defined by $\sigma_1(t) = f_t$, $t \in E$. We will show that $\sigma_1$ is Borel.

Indeed, let $O$ be an open subset of $(D, \tau')$. There exists a sequence $(O_i)_{i \in I}$ of $D$ such that $O = \bigcup_{i \in I} O_i$ and each $O_i$ is a finite intersection of open subsets, under the form $W_1, W_2$, where $W_1 = \{f_i; f_i(\theta_{t_1}) = 1\}$, $W_2 = \{f_i; f_i(\theta_{t_2}) = 0\}$, $t_1, t_2 \in E$. Note that for every $\theta \in B_2 \cup B_1$ the map $t \in E \to f_i(\theta) = \psi(\theta, t)$ is Borel, hence $(\sigma_1^{-1})(O_i)$ is Borel, for all $i \in I$. By step 1, $(D, \tau')$ is strongly Lindelöf, one concludes that there is a countable subset $I_1$ of $I$ such that $O = \bigcup_{i \in I_1} O_i$. It follows that $(\sigma_1)^{-1}(O)$ is a Borel subset of $(E, \tau_0)$. Finally if $O'$ is an open subset of $F_1$, then $(\sigma_1)^{-1}(O') = (\sigma_1)^{-1}(O' \cap D)$ and $O = O' \cap D$ is an open subset of $(D, \tau')$.

**Step 2:** Show that every compact of $F_1$ is metrisable.

Note that if $g \in F_1$, $g(\theta) \in \{0, 1\}$.

Let $L$ be a compact subset of $F_1$ and let $(\theta_n)_{n \geq 0}$ be a sequence in $B_2$ which is dense in $K$ (a such sequence exists by step 2 of 1). Define the
map $\xi : L \to \{0, 1\}^\mathbb{N}$ by $\xi(g) = (g(\theta_n))_{n \geq 0}$, $g \in L$. It is clear that $\xi$ is continuous and injective, this implies that $L$ is metrisable. $\blacksquare$

Observe now that the topology $\tau'$ is defined by a family of semi-norms which separates the points of $C(K)$, hence $(C(K), \tau')$ is completely regular space. Since every subspace of a completely regular space is completely regular, $(F_1, \tau')$ is completely regular.

Suppose that $(C(K), \tau')$ is universally measurable.

Step 3: Show that $F_1$ is universally measurable.

By [22 th. 3.2], it is enough to show that every normal probability measure on $F_1$ is a Radon measure. Let $\nu$ be a normal probability measure on $F_1$. Consider $\nu'$ the measure defined on $(C(K), \tau')$ by $\nu'(B) = \nu(B \cap F_1)$, where $B \in \text{Bor}(C(K), \tau')$. Let us show that $\nu'$ is a normal measure on $(C(K), \tau')$.

Let $i : F_1 \to (C(K), \tau')$ be the canonical injection. Since $i$ is continuous and $\nu$ is normal, by [22], $i(\nu) = \nu'$ is a normal measure. On the other hand $(C(K), \tau')$ is universally measurable and $\nu'$ is normal, hence by [22 th. 3.2], $\nu'$ is a Radon measure. Now we let $B'$ be a Borel subset of $F_1$. Note that $B'$ is a Borel subset of $(C(K), \tau')$.

Thus we have

$$
\nu(B') = \nu'(B') = \sup \{\nu'(L'); L' \text{ compact in } B'\}
= \sup \{\nu(L'); L' \text{ compact in } B'\}.
$$

Therefore $\nu$ is a Radon measure. It follows that $F_1$ is universally measurable. $\blacksquare$

Let $L$ be a compact subset of $(E, \tau_0)$ of strictly positive measure. We can suppose that $\mu(L) = 1$. Consider the map $\eta_1 : L \to F_1$, the restriction of $\sigma_1$ to $L$ and $\nu_1 = \eta_1(\mu_L)$, where $\mu_L$ the measure defined on $L$ by

$$
\mu_L(C) = \mu(C) = m(C), \quad C \in \text{Bor}(L) \subset \text{Bor}(E).
$$

Step 4: Show that $\nu_1$ is a normal measure on $F_1$.

Let $(U_n)$ be a net of open subsets in $F_1$ which is increasing to an open subset of $F_1$. Observe that for every Borel subset $B$ of $F_1$, $\nu_1(B) = \mu(\{t \in L; f_t \in B\}) = \mu(\{t \in L; f_t \in B \cap D\})$. 
By step 1 of 2), \((D, \tau')\) is strongly Lindelöf, hence there exists a countable subset \(I_1\) of \(I\) such that \(\bigcup_{i \in I_1} U_i \cap D = \bigcup_{i \in I_1} U_i \cap D\). Thus

\[
\nu_1(\bigcup_{i \in I} U_i) = \mu(\{t \in L; f_t \in (\bigcup_{i \in I} U_i) \cap D\})
\]

\[
= \mu(\{t \in L; f_t \in \bigcup_{i \in I} U_i \cap D\}) = \mu(\{t \in L; f_t \in \bigcup_{i \in I} U_i \cap D\})
\]

\[
= \nu_1(\bigcup_{i \in I} U_i) = \sup_{i \in I} \nu_1(U_i) \leq \sup_{i \in I} \nu_1(U_i).
\]

This implies that \(\nu_1\) is a normal measure on \(F_1\). \(\blacksquare\)

By step 3 of 2), \(F_1\) is universally measurable, since \(\nu_1\) is a normal probability measure on \(F_1\), by \([22, \text{th. 3.2}]\), \(\nu_1\) is a Radon measure, we deduce that there exists a compact subset \(G_1\) of \(F_1\) such that \(\nu_1(G_1) > \frac{1}{2}\). We denote by \(\eta_2\) the restriction of \(\sigma_1\) to \((\eta_1)^{-1}(G_1) \subseteq L\). The map \(\eta_2\) is Borel with values in \(G_1\) and \(m((\eta_1)^{-1}(G_1)) = \nu_1(G_1) > \frac{1}{2}\). The step 2 of 2) shows us that \(G_1\) is metrisable. By \([3]\), there exists a compact subset \(L_1\) of \((\eta_1)^{-1}(G_1), \tau_0\) such that \(\mu(L_1) > \frac{1}{3}\) and the restriction of \(\sigma_1\) to \(L_1\) is continuous.

Since \(\mu\) is diffuse on \(E\), by an argument similar to that of Proposition \([12]\) one constructs a strictly decreasing sequence \((t_n)_{n \geq 0}\) in \(L_1\). There exists \(t_0 \in L_1\) such that \(t_n \to t_0\) (modulo subsequence).

On the other hand, the restriction of \(\sigma_1\) to \(L_1\) is continuous, it follows that \(f_{t_n} \xrightarrow{n \to +\infty} f_{t_0}\) in \((C(K), \tau')\). For \(s = h(\theta) - t_0\), we have \(f_{t_n}(\theta_s) \xrightarrow{n \to +\infty} f_{t_0}(\theta_s) = \psi(\theta_s, t_0) = \psi(\theta, h(\theta)) = 1\). Thus for \(n\) large enough, \(\psi(\theta_s, t_n) = 1\), hence \(t_n \leq h(\theta_s) = h(\theta) - s = t_0\). For seeing a contradiction, it suffices to note that the sequence \((t_n)_{n \geq 0}\) is strictly decreasing. \(\blacksquare\)

3). Let \(Y_1\) be the vector subspace formed by even functions in \(C(K) \subseteq L^\infty(E)\) and \(Y_2\) be the vector space formed by odd functions in \(C(K) \subseteq L^\infty(E)\). Observe that if \(g \in C(K)\), \(\forall \hat{g} \in C(K)\), because \(\hat{f}_t = e - f_{-t}, \forall t \in E\).

Let us show that \(Y_1\) is \(\tau_p\) closed in \(C(K)\). For that let \((r_\alpha)_{\alpha \in I}\) be a net in \(Y_1\) such that \(r_\alpha \to r\) in \((C(K), \tau_p)\). We must prove that \(r \in Y_1\).

For \(\alpha \in I\) and \(\theta \in K\), we have \(r_\alpha(\theta) = \hat{r_\alpha}(\theta) = r_\alpha(\theta)\), by passing to the limit we obtain that \(\hat{r}(\theta) = r(\theta)\), hence \(r = r \in Y_1\). By a similar argument, we show that \(Y_2\) is \(\tau_p\) closed in \(C(K)\).
On the other hand, for \( g \in C(K) \), \( g = (g + \hat{g})/2 \oplus (g - \hat{g})/2 \), this implies that \( (C(K), \tau_p) = (Y_1, \tau_p) \oplus (Y_2, \tau_p) \) (note that the map \( g \in C(K) \to \hat{g} \in C(K) \) is continuous).  

4). Define the operator \( U : (Y_1, \tau_p) \to (Y_2, \tau_p) \), by \( U(g) = 2g - f - g \), \( g \in Y_1 \). Let us show that \( U \) has values in \( Y_2 \). For that let \( t \in G \). Observe that
\[
(2gf - g)(-t) = 2g(t)f(-t) - g(t),
\]
but \( f(-t) = e - f(t) \), hence
\[
(2gf - g)(-t) = 2g(t)(e - f(t) - g(t)) = -2g(t)f(t) - g(t).
\]
Since \( 2(2gf - g)f - (2gf - g) = g, U^{-1}(g) = 2gf - g, g \in Y_2 \). We observe that \( U, U^{-1} \) are continuous. Thus \( U \) is an isomorphism.

5). Show that \( (Y_1, \tau') = (Y_2, \tau_p) \). For \( \theta \in B_2 \cup B_1 \) and \( g \in Y_1 \) we have \( g(\hat{\theta}) = (\hat{g})(\theta) = g(\theta) \). Since \( \{ \hat{\theta}; \theta \in B_2 \cup B_1 \} = B_3 \cup B_1 \) by Proposition \[17\] it follows that \( (Y_1, \tau') = (Y_1, \tau_p) \).

Let us show that \( (Y_2, \tau') = (Y_2, \tau_p) \). Let \( (g_\alpha)_{\alpha \in I} \) be a net in \( Y_2 \) and let \( g \in Y_2 \) such that \( g_\alpha(\theta) \to g(\theta) \), for every \( \theta \in B_2 \cup B_1 \). We must prove that \( g_\alpha \to g \) in \( Y_2, \tau_p \). Note that \( U^{-1}(g_\alpha)(\theta) = (2g_\alpha \times f - g_\alpha)(\theta) \) and \( U^{-1}(g)(\theta) = (2g \times f - g)(\theta) \), hence \( U^{-1}(g_\alpha)(\theta) \to U^{-1}(g)(\theta) \), for every \( \theta \in B_2 \cup B_1 \). Since \( (Y_1, \tau') = (Y_1, \tau_p) \), \( U^{-1}(g_\alpha)(\theta) \to U^{-1}(g)(\theta) \) in \( (Y_1, \tau_p) \), we deduce that \( g_\alpha \to g \) in \( (Y_2, \tau_p) \), because \( U \) is a homeomorphism. Thus \( (Y_2, \tau') = (Y_2, \tau_p) \).

6). For every \( t \in E \) denote \( w_t = f_t + (\hat{f}_t) = f_t + 1 - f_{-t} \) and \( V = \{ \omega_t; t \in E \} \). We shall show that \( (V, \tau_p) \) is discrete and closed in \( (Y_1, \tau_p) \). Indeed, let \( t \in E \); choose \( \theta_1 \in B_2 \) and \( \theta_2 \in B_3 \) such that \( h(\theta_1) = h(\theta_2) = t \). Denote
\[
W = \{ \omega_s; |\omega_s(\theta_j) - \omega_t(\theta_j)| < \frac{1}{2}, j \in \{1, 2\} \}.
\]
We observe that \( W \) is an open subset of \( (V, \tau_p) \).

Let us show that \( W = \{ \omega_t \} \).

Case 1, \( t < 0 \): Let \( \omega_s \in W \). Note that \( \omega_t(\theta_1) = 1 + 1 - 0 = 2 \) and \( \omega_t(\theta_2) = 0 + 1 - 0 = 1 \), hence \( \omega_s(\theta_1) = 2 \) and \( \omega_s(\theta_2) = 1 \), this implies that \( \psi(\theta_1, s) = 1, \psi(\theta_1, -s) = 0 \) and \( \psi(\theta_2, s) = \psi(\theta_2, -s) \).

The case \( \psi(\theta_2, s) = \psi(\theta_2, -s) = 1 \) is excluded, because it will lead that \( t > 0 \), which is impossible. We have then \( \psi(\theta_1, s) = 1 \) and \( \psi(\theta_2, s) = \psi(\theta_2, -s) = 0 \). It follows that \( s \leq h(\theta_1) = t \leq s \). It means that \( t = s \).
Case 2, \( t > 0 \): Let \( \omega_s \in W \). Note that \( \omega_1(\theta_1) = 1 + 1 - 1 = 1 \) and \( \omega_2(\theta_2) = 0 + 1 - 1 = 0 \), hence \( \omega_s(\theta_1) = \psi(\theta_1, s) + 1 - \psi(\theta_1, -s) = 1 \) and \( \omega_s(\theta_2) = \psi(\theta_2, s) + 1 - \psi(\theta_2, -s) = 0 \). Therefore \( \psi(\theta_1, s) = \psi(\theta_1, -s) \) and \( \psi(\theta_2, s) = 0 \) (note that \( \psi(\theta_2, -s) = 1 \)). The case \( \psi(\theta_1, s) = \psi(\theta_1, -s) = 0 \) leads that \( t < 0 \), which is impossible, hence \( \psi(\theta_1, s) = 1 \) and \( \psi(\theta_2, s) = 0 \). It follows that \( s = t \).

Case 3, \( t = 0 \): Let \( \omega_s \in W \). It is clear that \( \omega_0(\theta_1) = \omega_0(\theta_2) = 1 \), which implies that \( \psi(\theta_1, s) = \psi(\theta_1, -s) \) and \( \psi(\theta_2, s) = \psi(\theta_2, -s) \). Note that the case \( \psi(\theta_1, s) = \psi(\theta_1, -s) = 0 \) and the case \( \psi(\theta_2, s) = \psi(\theta_2, -s) = 1 \) are excluded. Thus \( \psi(\theta_1, s) = 1 \) and \( \psi(\theta_2, s) = 0 \). This implies that \( s = 0 \). Therefore \((V, \tau_p)\) is a discrete space in \((Y_1, \tau_p)\).

It remains to show that \( V \) is a closed subset in \((Y_1, \tau_p)\).

Let \((\omega_{t\alpha})_{\alpha \in I}\) be a net in \( V \) such that \( \omega_{t\alpha} \to g \in Y_1 \). Choose a fixed point \( \theta \) in \( B_2 \). The sequence \((\theta_{t\alpha})_{\alpha \in I}\) is a net in \( K \), hence, there exists \( \theta_0 \in K \) such that \( \theta_{t\alpha} \to \theta_0 \in K \) (with respected to an ultrafilter \( \mathcal{U} \) on \( I \)).

Case 1, \( t\alpha \to +\infty \): Let \( \varphi \in K \setminus \{\theta', \theta''\} \). There exist \( v \in E^+ \) and \( u \in E^- \) such that \( -u > v, \psi(\varphi, u) = 1 \) and \( \psi(\varphi, v) = 0 \). Put \( W = \{t \in E; t > v \text{ and } t < u\} \). We observe that \( S = \{\alpha \in I; t\alpha \in W\} \) belongs to \( \mathcal{U} \), hence for all \( \alpha \in S \), \( f_{t\alpha}(\varphi) = \psi(\varphi, t\alpha) = 0 \) and \( f_{-t\alpha}(\varphi) = \psi(\varphi, -t\alpha) = 1 \), it follows that \( \omega_{t\alpha}(\varphi) = 0 + 1 - 1 = 0 \), for every \( \alpha \in S \). We conclude that \( g(\varphi) = 0 \). It is obvious that \( \omega_{t\alpha}(\theta') = \omega_{t\alpha}(\theta'') = 1 \), which implies that \( g(\theta') = g(\theta'') = 1 \). Therefore, we have \( \{\theta', \theta''\} = g^{-1}(\{1\}) \), hence this subset is closed and open, because \( g \) has values in \( \{0, 1, 2\} \) (note that \( K = \{\theta', \theta''\} \cup g^{-1}(\{0, 2\}) \)). Since \( \{\theta'\} = \{\theta', \theta''\} \setminus \{\theta''\}, \{\theta'\} \) is an open subset, this means that there is \( t_0 \in E \) such that \( \{\theta'\} = \{\theta \in K; \psi(\theta, t_0) = 1\} \), this is impossible, because the set \( \{\theta \in K; \psi(\theta, t_0) = 1\} \) contains the set \( \{\theta \in K; h(\theta) < t_0\} \).

Case 1, \( t\alpha \to -\infty \): By an argument similar to the previous case, we show that this case can be excluded, because \( g(\varphi) \in \{0, 1, 2\} \), for every \( \varphi \in K \).

Since the cases 1 and 2 are excluded and \( E \) is a locally compact, there exists \( t_0 \in E \) such that \( t\alpha \to t_0 \in E \). By Lemma 15, \( h(\theta_{t\alpha}) = h(\theta) - t\alpha \), which implies that \( t\alpha \to h(\theta) - h(\theta_0) = t_0 \) (because \( h \) is continuous by Proposition 5).

Show that \( g = \omega_{t_0} \). Pick \( \varphi \in K \setminus B_1 \).

Case 1, \( t_0 > 0 \).

Case 1-a), \( -t_0 < h(\varphi) < t_0 \): Let \( W = \{t \in E; -t < h(\varphi) < t\} \). \( W \) is an open neighborhood of \( t_0 \), hence
\[
S = \{\alpha \in I; -t\alpha < h(\varphi) < t\} \in \mathcal{U}.
\]
For every $\alpha \in S$ we have $f_{t_{\alpha}}(\varphi) = \psi(\varphi, t_{\alpha}) = \psi(\varphi, t_{0}) = 0$ and $\psi(\varphi, -t_{\alpha}) = \psi(\varphi, -t_{0}) = 1$, this implies that $\omega_{t_{\alpha}}(\varphi) = \omega_{t_{0}}(\varphi) = 0 + 1 - 1 = 0$ for all $\alpha \in S$. Thus $\omega_{t_{\alpha}}(\varphi) \to \omega_{t_{0}}(\varphi) = g(\varphi)$.

Case 1-b), $h(\varphi) < -t_{0}$. Put

$$W = \{ t \in E; h(\varphi) < -t < 0 \}.$$  

$W$ is an open neighborhood of $t_{0}$, hence $S = \{ \alpha \in I; h(\varphi) < -t_{\alpha} < 0 \}$ belongs to $U$. It is clear that $\omega_{t_{\alpha}}(\varphi) = \omega_{t_{0}}(\varphi) = 0 + 1 - 0 = 1$ (because $t_{\alpha} > -t_{\alpha}$ and $\psi(\varphi, -t_{\alpha}) = 0$, hence $\psi(\varphi, t_{0}) = 0$). We deduce that $\omega_{t_{\alpha}}(\varphi) \to \omega_{t_{0}}(\varphi) = 1 = g(\varphi)$.

Case 1-c), $h(\varphi) > t_{0}$: By an argument similar to case 1-b), one shows that $\omega_{t_{\alpha}}(\varphi) \to \omega_{t_{0}}(\varphi) = g(\varphi)$.

Case 2, $t_{0} < 0$: The case $t_{0} < h(\varphi) < -t_{0}$, $h(\varphi) < -t_{0}$ and the case $h(\varphi) > t_{0}$ will be are treated by arguments similar to the previous cases.

Case 3), $h(\varphi) = t_{0}$, $t_{0} \in E$: Without losing the generality, we can suppose that $\varphi \in B_{2}$. In the proof of 1), we showed that $B_{2}$ is dense in $K$. Hence there exists a sequence $(\varphi_{n})_{n \geq 0}$ in $B_{2}$ such that $\varphi_{n} \rightarrow_{n \rightarrow +\infty} \varphi$. We can choose the sequence $(\varphi_{n})_{n \geq 0}$ in $B_{2} \setminus \{ \varphi \}$, because $K$ does not contain isolated points. Let us show that for any large enough $t_{0} < h(\varphi_{n})$ (note that the restriction of $h$ to $B_{2}$ is injective, hence $h(\varphi_{n}) \neq t_{0}$).

Indeed, suppose that for every $n \in \mathbb{N}$, there exists $m_{n} \in \mathbb{N}$ such that $t_{0} > h(\varphi_{m_{n}})$. Since $\psi(\varphi_{m_{n}}, h(\varphi_{m_{n}})) = 1$, $\psi(\varphi_{m_{n}}, t_{0}) = 0$. By passing to the limit, we obtain that $\psi(\varphi, t_{0}) = \psi(\varphi, h(\varphi)) = 0$, which is impossible, hence there exist $n_{0} \in \mathbb{N}$ such that $t_{0} < h(\varphi_{n})$ for all $n \geq n_{0}$.

Applying the case 1-c, to $\varphi = \varphi_{n}$, we obtain $\omega_{t_{\alpha}}(\varphi_{n}) \to \omega_{t_{0}}(\varphi_{n}) = g(\varphi_{n})$ for every $n \geq n_{0}$. It follows that $\omega_{t_{\alpha}}(\varphi) = g(\varphi)$, because $g$ is continuous.

Case 4, $h(\varphi) > t_{0} = 0$: Put

$$W = \{ t \in E; h(\varphi) > t \text{ and } -h(\varphi) < t \}.$$  

$W$ is a open neighborhood of $t_{0} = 0$, hence

$$S = \{ \alpha \in I; h(\varphi) > t_{\alpha} \text{ and } -h(\varphi) < t_{\alpha} \} \in U.$$  

If $\alpha \in S$, $\omega_{t_{\alpha}}(\varphi) = \omega_{t_{0}}(\varphi) = 1 + 1 - 1 = 1$. We conclude that $\omega_{t_{\alpha}}(\varphi) = 1 \to \omega_{t_{0}}(\varphi) = 1 = g(\varphi)$.

The case $h(\varphi) < t_{0} = 0$ is treated by a similar argument. If $\varphi \in B_{1}$, it is clear that $\omega_{t_{\alpha}}(\varphi) \to \omega_{t}(\varphi)$. Thus $V$ is a closed subset in $(Y_{1}, \tau_{p})$.

7). We will show that the map $\sigma_{2} : t \in E \to (f_{t}) \in (C(K), \tau')$ is Borel.
Note that for every $t \in E$, $(f_t) = e - f_{-t}$. Consider the map $\Delta : E \to E$ defined by $\Delta(t) = -t$, $t \in E$, $\Delta$ is a Borel function. Thus $\sigma_2 = e - \sigma_1 \circ \Delta$, $\sigma_2$ is a Borel function, because in the proof of 2) we showed that $\sigma_1 : t \in E \to f_t$ is Borel.

Consider the map $\phi_1 : t \in E \to (f_t, (f_t)) \in C(K) \times C(K)$. It is obvious that if $W \in \text{Bor}(C(K), \tau') \otimes \text{Bor}(C(K), \tau')$, then $\phi_1^{-1}(W)$ is a Borel subset of $E$.

Suppose now that $\text{Bor}(C(K), \tau') \otimes \text{Bor}(C(K), \tau') = \text{Bor}(C(K) \times C(K), \tau' \times \tau')$. Therefore the map $\phi_1 : E \to (C(K) \times C(K), \tau' \times \tau')$, $t \to (f_t, (f_t))$ is Borel.

Define the map $\phi_2 : (C(K) \times C(K), \tau' \times \tau') \to (C(K), \tau')$, by $\phi_2(g, u) = g + u$, $(g, u) \in C(K) \times C(K)$. Observe that $\phi_2$ is continuous, hence it is Borel. It follows that $\phi_2 \circ \phi_1$ is Borel. We deduce that the map $t \in E \to w_t = f_t + (f_t) \in (Y_1, \tau') = (Y_1, \tau_p)$ is Borel. But $\text{Bor}(C(K), \tau_p) = \text{Bor}(C(K), ||.||)$ by Corollary 2, hence $\text{Bor}(Y_1, \tau_p) = \text{Bor}(Y_1, ||.||)$, which implies that the map $t \in (E, \tau_0) \to \omega_t \in Y_1$ is strongly measurable, this is impossible, because in the proof of 6), we saw that $(V, \tau_p)$ is uncountable discrete space, this means that the map $t \in E \to \omega_t$ cannot be almost everywhere valued in a separable subspace of $C(K)$.

8). Suppose that the projection $P : (C(K), \tau') \to (Y_1, \tau') = (Y, \tau_p)$ is Borel. Since the map $\sigma_1 : t \in E \to f_t \in (C(K), \tau')$ is Borel, the map $t \in E \to P \circ \sigma_1(t) = Pf_t = w_t/2 \in (C(K), \tau')$ is Borel, but in the proof of 7) we saw that this is impossible. Thus $P$ is not a Borel function.

9). Put $H_1 = \sigma_1 : t \in E \to f_t \in (C(K), \tau')$ and $H_2 = \sigma_2 : t \in E \to (f_t) \in (C(K), \tau')$. We proved previously that $H_1, H_2$ are Borel and in the end of the proof of 8) we proved that $H_1 + H_2$ is not Borel.

**Definition 9.** Let $(X, \tau)$ be a topological Hausdorff space. $(X, \tau)$ is called a $K$-analytic set if $(X, \tau)$ is a continuous image of a set belonging to the family $K_{\sigma \delta}$.

By [20] th. 2.7.1, every $K$-analytic set is a Lindelöf space. On the other hand if $L$ is separable and compact such that $(C(L), \tau_p)$ is a Lindelöf space, then $L$ is metrisable [1] (under $MA+\neg CH$).

**Corollary 20.** $(C(K), \tau')$ is not $K$-analytic.

**Proof.** Suppose that $(C(K), \tau')$ is $K$-analytic. By [4] $(C(K), \tau')$ is universally measurable, which is impossible by Theorem 5.2.)
Corollary 21. There exists two subspaces $Z_1$, $Z_2$ of ($C(K), \tau_p$), $\tau_p$-closed such that $Z_1 \cap Z_2 = \{0\}$, $(Z_j, \tau')$ is universally measurable, $j \in \{1, 2\}$ and $(Z_1, \tau')$ is isomorphic to $(Z_2, \tau')$, but $(Z_1 \oplus Z_2, \tau')$ is not universally measurable.

Proof. Let $Z_j = Y_j$, $j \in \{1, 2\}$. By Theorem 5, one has $(Z_j, \tau') = (Z_j, \tau_p)$, $Z_j$ is $\tau_p$-closed in ($C(K), \tau_p$), $j \in \{1, 2\}$ and $(Z_1, \tau')$ is isomorphic to $(Z_2, \tau')$. On the other hand, Lemma 15 shows us that $(C(K), \tau_p)$ is universally measurable, hence $(Z_j, \tau') = (Z_j, \tau_p)$ is universally measurable (because a closed subset of an universally measurable set is universally measurable). Since $(Z_1 \oplus Z_2, \tau') = (C(K), \tau')$, by Theorem 2, $(Z_1 \oplus Z_2, \tau')$ is not universally measurable. \hfill \blacksquare

Remark 25. By Corollary [2] $\text{Bor}(C(K), \tau_p) \otimes \text{Bor}(C(K), \tau_p)$ is equal to $\text{Bor}(C(K), \|\|) \otimes \text{Bor}(C(K), \|\|)$.

On the other hand by [26, th. 3] (with the continuous hypthothesis) one has

$$\text{Bor}(C(K), \|\|) \otimes \text{Bor}(C(K), \|\|) = \text{Bor}(C(K) \times C(K), \|\|).$$

Thus

$$\text{Bor}(C(K) \times C(K), \tau_p \times \tau_p) \subset \text{Bor}(C(K) \times C(K), \|\|) = \text{Bor}(C(K), \|\|) \otimes \text{Bor}(C(K), \|\|) = \text{Bor}(C(K), \tau_p) \otimes \text{Bor}(C(K), \tau_p).$$

But

$$\text{Bor}(C(K), \tau_p) \otimes \text{Bor}((C(K), \tau_p) \subset \text{Bor}(C(K) \times C(K), \tau_p \times \tau_p),$$

one deduces that

$$\text{Bor}(C(K), \tau_p) \otimes \text{Bor}((C(K), \tau_p) = \text{Bor}(C(K) \times C(K), \tau_p \times \tau_p).$$

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