New study of Boer-Mulders function: implications for the quark and hadron transverse momenta

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In series of papers the Boer-Mulders function for a given quark flavour has been extracted: (1) from data on semi-inclusive deep inelastic scattering, using the simplifying, but theoretically inconsistent, assumption that it is proportional to the Sivers function for each quark flavour and (2) from data on Drell-Yan reactions. In earlier papers, using the latest semi-inclusive deep inelastic COMPASS deuteron data on the \(\langle \cos \phi_h \rangle\) and \(\langle \cos 2\phi_h \rangle\) asymmetries, we extracted the collinear \(x_B\)-dependence of the Boer-Mulders function for the sum of the valence quarks \(Q^V = u^V + d^V\) using a minimum number of model dependent assumptions, and found a significant disagreement with the analysis in (1).

In the present paper we provide a more complete analysis of the semi-inclusive deep inelastic scattering reaction, including a discussion of higher twist and interaction dependent terms, and also a comparison with the Boer-Mulders function extracted from data on the Drell-Yan reaction. We confirm that the proportionality relation of the BM function to the Sivers function, for each quark flavour, fails badly, but find that it holds rather well if applied to the non-singlet valence quark combination, \(Q^V\). We also find a good agreement with the results of the Drell-Yan analysis. Furthermore, we obtain interesting information on the quark transverse momentum densities in the nucleon and on the hadron transverse momentum dependence in quark fragmentation.

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I. INTRODUCTION

The Boer-Mulders (BM) function \([1]\) is an essential element in describing the internal structure of the nucleon. A non-zero BM function implies that inside an unpolarized nucleon there are transversely polarized quarks. It is a leading twist, chiral odd, transverse momentum dependent parton distribution. In a nucleon of momentum \(P\), and for a quark with transverse momentum \(k_\perp\), the BM function measures the difference between the number density of quarks polarized parallel and anti-parallel to \((P \times k_\perp)\). It describes the distribution of transversely polarized quarks \(q^\uparrow\) in an unpolarized proton \(p\). Different notations for it are found in literature:

\[
\Delta^N_{q^\uparrow/p}(x_B, k_\perp) \equiv \Delta_{s_q^\uparrow/p}(x_B, k_\perp) = -\frac{k_\perp}{m_p} h_\perp(x_B, k_\perp). \tag{1}
\]

First attempts to extract it from experiment were hindered by the scarcity of data and made the simplifying model assumption of its proportionality to the better known Sivers function \([2,3]\), an assumption motivated by model...
calculations [5–7]. However, this assumption applied for each quark separately, as explained in [8, 9], is theoretically inconsistent as it leads to gluons contributing in the evolution of non-singlet combinations of quark densities.

Other parametrizations for the BM function were obtained in [10, 11] from data on the pD and pp Drell-Yan (DY) processes. These processes are controlled by products of BM functions $h_{1}^{A}(x_{1}, k_{1}^{2}) \cdot h_{1}^{A}(x_{2}, k_{2}^{2})$, one from each of the initial hadrons in the reaction, and an additional condition, the positivity bound, is used to constrain some of the parameters. In [12] the TMD-evolution of the pion and proton BM functions was studied in the pion induced DY process $π → l^{+}l^{-} + X$.

In this paper, we show that the new COMPASS data on the unpolarized $⟨\cos φ_{h}⟩$ and $⟨\cos 2φ_{h}⟩$ asymmetries in semi-inclusive deep inelastic scattering (SIDIS) reactions for producing a hadron $h$ and its antiparticle $h$ at azimuthal angle $φ_{h}$, allows an extraction of the BM function with a minimum number of model dependent assumptions.

II. THE FORMALISM

As explained in [13] and [14] there is a great advantage in studying difference asymmetries $A^{h−h}$, effectively $A^{h}−A^{h}$, since both for the collinear and transverse momentum dependent (TMD) functions, only the flavour non-singlet valence quark parton densities (PDFs) and fragmentation functions (FFs) play a role and the gluon does not contribute. On a deuteron target an additional simplification occurs. Independently of the final hadron, only the sum of the valence-quark TMD functions $Q_{V} = u_{V} + d_{V}$ enters. In this paper we use SIDIS COMPASS data on a deuteron target, [15], and determine the BM TMD function only for $Q_{V}$, and with a minimum number of model assumptions.

A. Parametrization of the TMD functions

The unpolarized TMD functions for $Q_{V}$ are parametrized in the standard way [16, 17]:

$$f_{Q_{V}/p}(x_{b}, k_{2}^{2}, Q^{2}) = Q_{V}(x_{b}, Q^{2}) \frac{e^{-k_{2}^{2}/(p_{2}^{2})}}{π(p_{2}^{2})},$$  \hspace{1cm} (2)

and

$$D_{h/q_{V}}(z_{h}, p_{2}^{2}, Q^{2}) = D_{q_{V}}^{h}(z_{h}, Q^{2}) \frac{e^{-p_{2}^{2}/(p_{2}^{2})}}{π(p_{2}^{2})},$$  \hspace{1cm} (3)

where $Q_{V}(x_{b}, Q^{2})$ is the sum of the collinear valence-quark PDFs:

$$Q_{V}(x_{b}, Q^{2}) = u_{V}(x_{b}, Q^{2}) + d_{V}(x_{b}, Q^{2})$$  \hspace{1cm} (4)

and $D_{q_{V}}^{h}(z_{h}, Q^{2})$ are the valence-quark collinear FFs:

$$D_{q_{V}}^{h}(z_{h}, Q^{2}) = D_{q_{V}}^{h}(z_{h}, Q^{2}) - D_{q_{V}}^{h}(z_{h}, Q^{2}),$$  \hspace{1cm} (5)

and $⟨k_{2}^{2}⟩$ and $⟨p_{2}^{2}⟩$ are parameters extracted from a study of the multiplicities in unpolarized SIDIS. There is some controversy in the literature, with several different published sets of values. It will turn out that this analysis favours a particular choice of these values.

The BM function is parametrized in a similar way:

$$Δf_{BM}^{Q_{V}}(x_{b}, k_{2}^{2}, Q^{2}) = Δf_{BM}^{Q_{V}}(x_{b}, Q^{2}) \sqrt{2e} \frac{k_{2}^{2}}{M_{BM}} \frac{e^{-k_{2}^{2}/(p_{2}^{2})}}{π(p_{2}^{2})},$$  \hspace{1cm} (6)

with

$$Δf_{BM}^{Q_{V}}(x_{b}, Q^{2}) = 2N_{BM}^{Q_{V}}(x_{b}) Q_{V}(x_{b}, Q^{2}).$$  \hspace{1cm} (7)
Here the $N_{BM}^{QV}(x_h)$ is an unknown function and $M_{BM}$, or equivalently $\langle k_1^2 \rangle_{BM}$:

$$\langle k_1^2 \rangle_{BM} = \frac{\langle k_1^2 \rangle M_{BM}^2}{(k_1^2) + M_{BM}^2},$$

is an unknown parameter.

Since the asymmetries under study involve a product of the BM parton density and the Collins FF, one requires also the transverse momentum dependent Collins function [18]:

$$\Delta^N D_{h/u_{\perp}}(z_h, p_\perp, Q^2) = \Delta^N D_{h/u_{\perp}}(z_h, Q^2) \sqrt{2e} \frac{p_\perp}{M_c} e^{-p_\perp^2/(p^2_c)} \pi(p_\perp^2),$$

where

$$\Delta^N D_{h/u_{\perp}}(z_h, Q^2) = 2 N_{C}^{h/u_{\perp}}(z_h) \Delta_{h}^{h}(z_h, Q^2).$$

The quantities $N_{C}^{h/u_{\perp}}(z_h)$ and $M_c$, or equivalently $\langle p_1^2 \rangle_c$:

$$\langle p_1^2 \rangle_c = \frac{\langle p_1^2 \rangle M_c^2}{(p_1^2) + M_c^2},$$

are known from studies of the azimuthal correlations of pion-pion, pion-kaon and kaon-kaon pairs produced in $e^+e^-$ annihilation: $e^+e^- \rightarrow h_1 h_2 + X$ and the sin($\phi_h + \phi_S$) asymmetry in polarized SIDIS [15–21].

Besides the BM-Collins contributions to the $\langle \cos \phi_h \rangle$ and $\langle \cos 2\phi_h \rangle$ unpolarized asymmetries, there exists also a contribution known as the Cahn effect [22, 23]. The Cahn effect is a purely kinematic effect, generated in the naive parton model by the quark intrinsic transverse momenta included in distribution and fragmentation functions. It is described by the unpolarized TMD functions $f^i_l(x_h, k_{\perp})$ and $D^h_l(z_h, p_{\perp})$, and is a sub-leading effect i.e. $1/Q^2$ contribution to $A_{UU}^{\cos 2\phi}$, and a $1/Q$ contribution to $A_{UU}^{\cos 4\phi}$.

### B. The difference asymmetries

For the differential cross section for SIDIS of unpolarized leptons on unpolarized nucleons in the considered kinematic region $P_T \simeq k_{\perp} \ll Q$ we use the expression:

$$\frac{d \sigma^h}{d x_B d Q^2 dz_h d^2 P_T} = \sigma_0^h \left\{ 1 + \frac{2(1 - y)}{[1 + (1 - y)^2]} \cos 2\phi_h A_{UU}^{\cos 2\phi_h} + \frac{2(2 - y)\sqrt{1 - y}}{[1 + (1 - y)^2]} \cos \phi_h A_{UU}^{\cos \phi_h} \right\}$$

where $\sigma_0$ is the $\phi_h$-independent part of the cross section. We consider the $\cos \phi_h$ and $\cos 2\phi_h$ azimuthal asymmetries $A_{UU}^{\cos \phi_h}$ and $A_{UU}^{\cos 2\phi_h}$, generated by the two contributions – the Cahn and the Boer-Mulders TMD mechanisms. The $\cos \phi_h$-asymmetry gets twist-3 Cahn and BM contributions as well as interaction dependent terms associated with quark-gluon-quark correlators [24], which will be discussed later. The $\cos 2\phi_h$-term is generated by a leading twist-2 BM effect and a twist-4 Cahn effect.

In the above $P_T$ and $\phi_h$ are the transverse momentum and azimuthal angle of the final hadron in the $\gamma^*\text{-nucleon}$ c.m. frame, $z_h$, $Q^2$ and $y$ are the usual measurable SIDIS quantities:

$$z_h = \frac{(P \cdot P_h)}{(P \cdot q)}, \quad Q^2 = -q^2, \quad q = l - l', \quad y = \frac{(P \cdot q)}{(P \cdot l)}$$

where $l$ and $l'$, $P$ and $P_h$ are the 4-momenta of the initial and final leptons, and the initial and final hadrons. Note that

$$Q^2 = 2MEx_hy$$

where $M$ is the target mass (in this paper the deuteron mass) and $E$ the lepton laboratory energy.
Further we shall work with the so called difference asymmetries \( A_{h^+, h^-} \) that have the following general structure:

\[
A_{h^+, h^-} = \frac{\Delta \sigma_{h^+} - \Delta \sigma_{h^-}}{\sigma_{h^+} - \sigma_{h^-}},
\]

where \( \sigma_{h^+}, \sigma_{h^-} \) and \( \Delta \sigma_{h^+, h^-} \) are the unpolarized and polarized cross sections respectively. The difference asymmetries are not a new measurement, but they are expressed in terms of the usual asymmetries \( A_{h^+}, h^- \):

\[
A_{h^+} = \frac{\Delta \sigma_{h^+}}{\sigma_{h^+}}, \quad A_{h^-} = \frac{\Delta \sigma_{h^-}}{\sigma_{h^-}},
\]

and the ratio \( r \) of the unpolarized SIDIS cross sections for production of \( h^- \) and \( h^+ \), \( r = \sigma_{h^-} / \sigma_{h^+} \):

\[
A_{h^+, h^-} = \frac{1}{1-r} \left( A_{h^+} - r A_{h^-} \right).
\]

As mentioned above, the advantage of using the difference asymmetries is that, based only on charge conjugation (C) and isospin (SU(2)) invariance of the strong interactions, they are expressed purely in terms of the best known valence-quark distributions and fragmentation functions; sea-quark and gluon distributions do not enter. For a deuteron target there is the additional simplification that, independently of the final hadron, only the sum of the valence-quark distributions enters.

In the following we use the asymmetries \( A_{UU}^{\cos \phi_h} \) and \( A_{UU}^{\cos 2\phi_h} \) as defined in Eq. \([12]\) and used in the COMPASS paper \([13]\). [Note that several different definitions \([25]\) of these asymmetries exist in the literature, some of them even differing between COMPASS publications \([26]\)]. If the \( Q^2 \)-dependence of the collinear PDF’s and FFs can be neglected, the \( x_a \)-dependent difference asymmetries are related to the theoretical functions via:

\[
A_{UU}^{\cos \phi_h, h^-}(x_a) = \sqrt{\frac{\langle k^2 \rangle}{\langle Q^2(x_a) \rangle}} \left\{ N_{B M}^V(x_a) C_{B M}^h + C_{\text{Cahn}}^h \right\},
\]

\[
A_{UU}^{\cos 2\phi_h, h^-}(x_a) = \left\{ N_{B M}^V(x_a) \hat{C}_{B M}^h + \frac{\langle k^2 \rangle}{\langle Q^2(x_a) \rangle} C_{\text{Cahn}}^h \right\},
\]

where \( \langle Q^2(x_a) \rangle \) is some mean value of \( Q^2 \) for the corresponding \( x_a \)-bin, and the coefficients \( C_{B M}, C_{\text{Cahn}}, \hat{C}_{B M} \) and \( \hat{C}_{\text{Cahn}} \) are dimensionless constants given by integrals over various products of the unpolarized or Collins FFs and, crucially, whose values depend on the parameters \( \langle k^2 \rangle, \langle p_T^2 \rangle, M_{B M} \) and \( M_C \). For a finite range of integration over \( P_T^2 \), corresponding to the experimental kinematics, \( a \leq P_T^2 \leq b \), they are given by the expressions:

\[
C_{\text{Cahn}}^h = -2 \frac{\int d z_h \, z_h \, [D_{Q V}^h(z_h); S_1(a, b; \langle P_T^2 \rangle)/\eta_z + z_h^2 \rangle^{1/2}}{\int d z_h \, \left[ D_{Q V}^h(z_h); S_0(a, b; \langle P_T^2 \rangle) \right]} \cdot \\frac{\lambda^2 \lambda^2}{\lambda^2 \lambda^2} \frac{\lambda^2 \lambda^2}{\lambda^2 \lambda^2}
\]

\[
C_{B M}^h = 4 \frac{\int d z_h \, \left[ z_h^2 \lambda^2 \lambda^2 \lambda^2 \lambda^2 \right]}{\int d z_h \, \left[ D_{Q V}^h(z_h); S_0(a, b; \langle P_T^2 \rangle) \right]} \cdot \frac{\lambda^2 \lambda^2}{\lambda^2 \lambda^2} \frac{\lambda^2 \lambda^2}{\lambda^2 \lambda^2}
\]

\[
\hat{C}_{\text{Cahn}}^h = \frac{\int d z_h \, \left[ z_h^2 / \eta_z + z_h^2 \right]}{D_{Q V}^h(z_h); S_0(a, b; \langle P_T^2 \rangle) \right]} \cdot \frac{\lambda^2 \lambda^2}{\lambda^2 \lambda^2} \frac{\lambda^2 \lambda^2}{\lambda^2 \lambda^2}
\]

\[
\hat{C}_{B M}^h = 2 \frac{\int d z_h \, \left[ z_h^2 / \eta_z + z_h^2 \right]}{D_{Q V}^h(z_h); S_0(a, b; \langle P_T^2 \rangle) \right]} \cdot \frac{\lambda^2 \lambda^2}{\lambda^2 \lambda^2} \frac{\lambda^2 \lambda^2}{\lambda^2 \lambda^2}
\]

where, with \( \tau = \) either \( P_T^2 \) or \( P_T^2 \); \( t \):

\[
S_n(a, b; \tau) = \int_a^b dP_T^2 P_T^2 e^{-P_T^2/\tau} \tau^{1+n/2}.
\]
$$\langle P_T^2 \rangle = \langle p_T^2 \rangle + z_h^2 \langle k_T^2 \rangle.$$  
(25)

Here $[D_{q\bar{q}}^h(z_h)]$ and $[\Delta^N D_{q\bar{q}V}^h(z_h)]$ are combinations of the collinear and Collins FFs:

$$[D_{q\bar{q}}^h(z_h)] = e_u^2 D_{uv}^h + e_d^2 D_{d\bar{v}}^h,$$  
(26)

$$[\Delta^N D_{q\bar{q}V}^h(z_h)] = e_u^2 \Delta^N D_{uvV}^h + e_d^2 \Delta^N D_{d\bar{v}V}^h,$$  
(27)

and

$$\eta = \langle p_T^2 \rangle / \langle k_T^2 \rangle, \quad \lambda_c = M_C^2 / (\langle p_T^2 \rangle + M_C^2), \quad \lambda_{BM} = M_{BM}^2 / (\langle k_T^2 \rangle + M_{BM}^2).$$  
(28)

### C. The parameters $\langle k_T^2 \rangle$, $\langle p_T^2 \rangle$, $M_{BM}^2$, and $M_C^2$

As mentioned, there is a wide range of values for these parameters given in the literature. The parameters $\langle k_T^2 \rangle$ and $\langle p_T^2 \rangle$ are basic as they enter the normalization functions in all TMD asymmetries. At present the experimentally obtained values are controversial:

1) $\langle k_T^2 \rangle \approx 0.25 \text{ GeV}^2$ and $\langle p_T^2 \rangle \approx 0.20 \text{ GeV}^2$, extracted from the old EMC [31] and FNAL [32] data on the Cahn effect in the SIDIS $\cos \phi_h$ asymmetry.

2) $\langle k_T^2 \rangle = 0.18 \text{ GeV}^2$ and $\langle p_T^2 \rangle = 0.20 \text{ GeV}^2$, based on a study of the old HERMES data on the $\cos \phi_h$ and $\cos 2\phi_h$ asymmetries in SIDIS. These values were used in the extraction of the BM functions in [3].

An analysis [34] of the more recent available data on multiplicities in SIDIS from HERMES [33] and COMPASS [30] separately, gives quite different values:

3) $\langle k_T^2 \rangle = 0.57 \pm 0.08 \text{ GeV}^2$ and $\langle p_T^2 \rangle = 0.12 \pm 0.01 \text{ GeV}^2$, extracted from HERMES data

4) $\langle k_T^2 \rangle = 0.61 \pm 0.20 \text{ GeV}^2$ and $\langle p_T^2 \rangle = 0.19 \pm 0.02 \text{ GeV}^2$, extracted from COMPASS data.

Recently, the importance of determining the values of $\langle k_T^2 \rangle$ and $\langle p_T^2 \rangle$ was specially stressed [37]. Two quite different parametrizations for both the Sivers [21,38] and Collins [21,39] functions, with comparable accuracies of the fits to the data exist, but using very different values of the Gaussian widths $\langle k_T^2 \rangle$ and $\langle p_T^2 \rangle$ of the unpolarized distributions.

We shall attempt to fit the SIDIS data using 5 different sets of the parameters $\langle k_T^2 \rangle$, $\langle p_T^2 \rangle$, $M_{BM}^2$, and $M_C^2$. For $M_{BM}^2$ we try the values 0.34, 0.19 and 0.80 GeV$^2$, which correspond to the values for the Sivers $k_T^2$ obtained in [27] and [28,29]. The value of $M_C^2$ is taken from the known parametrizations of the Collins function [20] and [21].

The coefficients $C_{BM}, \hat{C}_{BM}, \hat{C}_{BM}$, are given in Table I, grouped together in Sets corresponding to the values of these parameters, with $\rho \equiv -C_{BM}/\hat{C}_{BM}$.

### III. THE COMPASS ASYMMETRIES

As mentioned earlier, we extract $\mathcal{N}_{BM}(x_a)$ from the difference asymmetries $A_j^{h^-} \rightarrow h^-$, related in [8] to the corresponding usual asymmetries $A_j^h$ and $A_j^\perp$ for positive and negative charged hadron production measured in COMPASS [15] via the relation [41]:

$$A^{h^-} \rightarrow h^- = \frac{1}{1-r} \left( A^{h+} - r A^{h^+} \right), \quad J = \cos \phi_h, \quad \cos 2\phi_h.$$  
(29)

Here $r$ is the ratio of the unpolarized $x_a$-dependent SIDIS cross sections for production of negative and positive hadrons $r = \sigma^{h^-}(x_a) / \sigma^{h^+}(x_a)$ measured in the same kinematics [41]. In the COMPASS kinematics to each value of $\langle Q^2 \rangle$ corresponds a definite value of $\langle x_a \rangle$, thus fixing the $Q^2$ interval we fix also the $x_a$-interval. As shown in [8], in the whole $Q^2$-range covered by COMPASS, $Q^2 \simeq [1 - 11] \text{ GeV}^2$, there is almost no $Q^2$-dependence both in the valence-quark distributions $u_v$ and $d_v$ and in the FFs, i.e. in the whole $x_a$-interval. Thus, we consider it reasonable
TABLE I: $C_{Cahn}, C_{BM}, \hat{C}_{Cahn}, \hat{C}_{BM}$ and $\rho$ calculated for different sets of $\langle k_{\perp}^2 \rangle$, $\langle p_{\perp}^2 \rangle$, $M_{BM}^2$ and $M_{C}^2$ [GeV$^2$]. The parametrizations for the collinear FFs are from AKK'2008 [40], and for the Collins functions – for sets I – IV – from [19] and [21], and for set V – from [20] and [21]. The integrations are according to COMPASS kinematics: $0.01 \leq P_T^2 \leq 1$ GeV$^2$ and $0.2 \leq z_h \leq 0.85$ [15].

to use the following fitting interval $x_B \in [0.006, 0.1]$ corresponding to $Q^2 \in [1.26, 11.24]$ GeV$^2$. (The $Q^2$-evolution will be separately discussed in Sec. V.)

In our analysis we use smooth fit functions of $x_B$ to the measured asymmetries $A^{\cos \phi,h+}_{UU,d}$, $A^{\cos \phi,h-}_{UU,d}$, $A^{\cos 2\phi,h+}_{UU,d}$ and $A^{\cos 2\phi,h-}_{UU,d}$. Then the difference asymmetries are calculated from Eq. (29). Our input functions are shown in Figs. [1] and [2]. The error for the difference asymmetries is calculated as a composed error implied by Eq. (29):

$$
\Delta A_{UU,d}^{h^+-h^-} = \frac{1}{1 - r}\sqrt{\left(\Delta A_{UU,d}^{h^+}\right)^2 + r^2 \left(\Delta A_{UU,d}^{h^-}\right)^2},
$$

where ($\Delta A^j$) are the errors of the asymmetries of the fit parameters. In the analysis both statistical and systematic experimental errors are included.

FIG. 1: Fit to the ordinary asymmetries. $\chi^2_{d.o.f.} = 0.49, 1.99, 0.47$ and 1.56 for $A^{\cos \phi,h+}_{UU,d}$, $A^{\cos \phi,h-}_{UU,d}$, $A^{\cos 2\phi,h+}_{UU,d}$ and $A^{\cos 2\phi,h-}_{UU,d}$, respectively. Only statistical errors are shown.
IV. NUMERICAL RESULTS ON THE BM FUNCTION, $\langle k_\perp^2 \rangle$ AND $\langle p_\perp^2 \rangle$

Here we present the strategy of our analysis and the obtained results.

A. Compatibility extraction of the Boer-Mulders function

We extract $N_{BM}(x_B)$ from relations (18) and (19) of the difference asymmetries. Relations (18) and (19) provide 2 independent equations for the extraction of $N_{BM}(x_B)$ for each set of the parameters in Table I. The analysis shows that the 2 extractions are compatible with each other, within errors, for the parameters values $\{\langle k_\perp^2 \rangle, \langle p_\perp^2 \rangle\}$:

\begin{align}
\langle k_\perp^2 \rangle &= 0.18 \text{ GeV}^2, \quad \langle p_\perp^2 \rangle = 0.20 \text{ GeV}^2 \\
\langle k_\perp^4 \rangle &= 0.25 \text{ GeV}^2, \quad \langle p_\perp^2 \rangle = 0.20 \text{ GeV}^2.
\end{align}

with a slight preference for (32). Note that these values for $\langle k_\perp^2 \rangle$ and $\langle p_\perp^2 \rangle$ agree with those obtained in [33] and with the theoretical considerations [42–44].

In Fig. 3 we present our results for sets I, III and V. The plots for sets II and IV overlap with those for sets I and III, respectively, which implies that our analysis is not sensitive to $M_{BM}$. Consequently, in the following we shall refer to sets I, III and V, only.

FIG. 3: $N_{BM}^{Q\nu}(x_B)$ extracted from the difference asymmetries, Eqs. (18) and (19), for sets I, III and V. Plots for sets II and IV overlap with those for sets I and III, respectively.
We obtain a simple linear fit to the extracted averaged $N^{Q\nu}_{BM}$ for the parameter Set III, Eq. (32):

$$N^{Q\nu}_{BM}(x_B) = ax_B + b,$$

$$a = -1.69 \pm 0.08, \quad b = 0.215 \pm 0.006.$$  \hspace{1cm} (33)

**B. Direct test for $\langle k^2_{\perp} \rangle$ and $\langle p^2_{\perp} \rangle$**

Interestingly, there is a second way to utilize equations (18) and (19) which directly fixes the values of the parameters $\langle k^2_{\perp} \rangle$, $\langle p^2_{\perp} \rangle$, $M_B$ and $M_C$ in Table I. Eliminating $N^{Q\nu}_{BM}(x_B)$ from Eqs. (18) and (19) and using the variable $\rho$ we obtain:

$$A(x_B) = B(x_B),$$  \hspace{1cm} (34)

where

$$A(x_B) = \sqrt{\langle Q^2(x_B) \rangle} A_{UU,d}^{\cos \phi_h, h^+ - h^-} + \rho A_{UU,d}^{\cos 2\phi_h, h^+ - h^-},$$  \hspace{1cm} (35)

$$B(x_B) = C_{Cahn} + \rho \langle k^2 \rangle / \langle Q^2(x_B) \rangle C_{Cahn},$$  \hspace{1cm} (36)

and the explicit expression for $\rho$ is:

$$\rho = \frac{C_{BM}}{C_{BM}} \left[ z^2 \lambda_{BM} \Delta N_{BM}(z_B) \right] \left[ S_2(a, b; \langle P^2 \rangle_{BM}) + \langle \eta \lambda_B - z^2 \lambda_{BM} \rangle S_1(a, b; \langle P^2 \rangle_{BM}) / (z^2 \lambda_{BM} + \eta \lambda_B)^{3/2} \right] / \left[ z^2 \lambda_{BM} + \eta \lambda_B \right] S_1(a, b; \langle P^2 \rangle_{BM}) / (z^2 \lambda_{BM} + \eta \lambda_B)^{3/2}$$  \hspace{1cm} (37)

Figure 4 compares the two functions $A(x_B)$ and $B(x_B)$ for sets I, III and V.

![Figure 4](image)

**FIG. 4:** The test of Eq. (34) for sets I, III and V. Plots for sets II and IV overlap with those for sets I ($\chi^2_{d.o.f.} = 0.572$) and III ($\chi^2_{d.o.f.} = 0.151$), respectively.

One sees from Fig. 4 that the COMPASS data on $A_{UU}^{\cos \phi_h}$ and $A_{UU}^{\cos 2\phi_h}$, while roughly compatible with Set I again favour the parameter values of Set III, Eq. (32).

(Note that for calculating $\chi^2$ in Fig. 4 we use the COMPASS data points:

$$\chi^2 = \sum_{i=1}^{N} \frac{(A(x_i) - B(x_i))^2}{(\Delta A(x_i))^2}$$  \hspace{1cm} (38)

divided by the degrees of freedom $d.o.f. = N - l$, $l$ is the number of free parameters in the fit, in this fit $l = 0$. Here $A(x_i)$ and $B(x_i)$ are values of the experimental points, $\Delta A(x_i)$ are the errors at $x_i$ calculated from Eq. (30).
V. A WORD OF CAUTION: EVOLUTION, INTERACTION TERMS AND HIGHER TWIST

In the analysis presented above, we have not commented on the question of evolution. Clearly the use of Eq. (7) i.e. \( f_{0}^{QV}(x_{B}, Q^{2}) = 2 N_{0}^{QV}(x_{B}) QV(x_{B}, Q^{2}) \) imposes, incorrectly, the \( Q^{2} \)-evolution of the unpolarized PDFs onto the BM function. However the magnitude of this evolution should be a reasonable measure of the genuine BM evolution. Based on this argument, we have checked that, for the data under analysis, \( Q^{2} \in [1.26, 11.24] \) GeV\(^2\), evolution effects are completely negligible. This is shown in Fig. 5 where we compare the evolved and the non-evolved results for sets I and III for \( N_{BM}^{QV}(x_{B}) \) extracted from Eqs. (18) and (19).

The expression Eq. (18), which we have used above for the asymmetry \( A_{UU}^{2\varphi_{b}, \vec{h} \rightarrow \vec{h}}(x_{B}) \) is incomplete. There are so called interaction dependent terms [24], linked to the quark-gluon-quark correlators, which have been left out. These terms are unknown, but, roughly speaking we expect to have

\[
|\text{interaction dependent terms}| \sim \alpha_s |\text{terms in Eq. (18)}| \tag{39}
\]

and thus should be negligible or at least very small compared to the terms kept in Eq. (18).

Finally, it should be noted that the expression Eq. (19) for the asymmetry \( A_{UU}^{\cos \varphi_{b}, \vec{h} \rightarrow \vec{h}}(x_{B}) \) is unusual in that it contains a combination of a twist-2 BM term \( N_{BM}^{QV}(x_{B}) \hat{C}_{BM}^{h} \) with a twist-4 Cahn term \( \langle k_{2}^{2} \rangle \langle Q^{2} \rangle \hat{C}_{Cahn}^{h} \), and one might wonder whether there might exist important twist-4 BM terms which are not accounted for in Eq. (19). That this is not so can be understood from the following argument. In Fig. 6 we compare the BM and Cahn contributions to Eq. (19). Remarkably, the twist-4 Cahn contribution is bigger in magnitude than the twist-2 BM contribution! This peculiar situation is due to two factors. Firstly the twist-4 pre-factor in the Cahn term, \( \langle k_{2}^{2} \rangle \langle Q^{2} \rangle \hat{C}_{Cahn}^{h} \), is not really small for the values of \( Q^{2} \) in our data. Secondly, the Cahn factor \( \hat{C}_{Cahn}^{h} \) is anomalously large because it depends on the unpolarized PDFs and FFs. Any twist-4 BM type contribution would thus be expected to be negligibly small by comparison.
VI. COMPARISON TO OTHER BOER-MULDERS PARAMETRIZATIONS

Our valence Boer-Mulders function $\Delta f^Q_{BM}(x_B)$,

$$\Delta f^Q_{BM}(x_B, Q^2) = 2 N^Q_{BM}(x_B) Q V(x_B, Q^2),$$

is shown in Fig. 7 where it is compared to $\Delta f^Q_{BM}(x_B)$ calculated using two other parametrizations of BM functions available in literature – the BM functions published in [3, 4] and in [10, 11]. The BM function published in [3, 4] is extracted from the $\cos 2\phi_h$-asymmetry in SIDIS, using the simplifying, but theoretically inconsistent, assumption that it is proportional to the Sivers function for each quark flavour separately. The parametrizations in [10, 11] are extracted from the azimuthal $\cos 2\phi$-asymmetry of the final lepton pair in unpolarised Drell-Yan processes. We compare our result to the parametrization in [11], obtained from the combined analysis of the $pp$ and $pd$ DY processes.

FIG. 7: Comparison of $\Delta f^Q_{BM}$ for Sets I and III with the result of Barone et al. (left) [3] (light gray) and [4] (dark gray) and with Lu et al. [11] – middle (Set I) and right (Set III). We use respectively, GRV1998 [46] and MSTW2008 [47] parametrizations for the collinear PDFs.

It is seen that, both for SET I and III, there is a significant difference between our predictions and those of refs. [3, 4], and a good agreement with the results in [11] from DY data. This suggests that the BM functions in [3, 4] are incorrect.
VII. TEST OF THE BOER-MULDERS TO SIVERS RELATION

In refs. [2–4] the BM functions were assumed proportional to the Sivers functions for each quark and anti-quark flavor $q$ separately:

$$\Delta f^q_{BM}(x_B, k_\perp) = \lambda_q \Delta f^q_{Siv}(x_B, k_\perp)$$

(41)

which, as implied by the results in Fig. 7 above, is badly violated, in agreement with conclusions reached in our earlier paper [14]. We here return to the question of the proportionality between the BM and Sivers functions, but now only for the valence-quark contributions $Q_V = u_V + d_V$:

$$\Delta f^{Q_V}_{BM}(x_B, k_\perp) = \lambda \Delta f^{Q_V}_{Siv}(x_B, k_\perp)$$

(42)

For the Sivers function we use an analogous parametrization to the BM, Eq. (6), but with the replacements $M_{BM} \rightarrow M_{Siv}$ and $N_{BM} \rightarrow N_{Siv}$. Then Eq. (42) implies

$$M_{BM} = M_{Siv},$$

and

$$N_{Q_V}^{BM}(x_B) = \lambda N_{Q_V}^{Siv}(x_B)$$

(43)

which we shall now test.

We extract $N_{Siv}(x_B)$ from the difference Sivers asymmetries $A_{UT,h^{+}-h^{-}}^{Siv}(x_B)$ using the single-spin asymmetries presented by COMPASS for $h^\pm$ on deuterons [29]. The expression for $A_{UT,h^{+}-h^{-}}^{Siv}(x_B)$ is [14]:

$$A_{UT,d}^{Siv,h^{+}-h^{-}}(x_B) = \sqrt{\frac{e\pi}{2}} K_{Siv}^{h} N_{Siv}^{Q_V}(x_B),$$

(44)

$$K_{Siv}^{h} = \frac{\langle k_\perp^2 \rangle_s}{M_{S} \langle k_\perp^2 \rangle_s} \int dz_h \frac{z_h \langle |D_h^{q_{V}}| \rangle_s}{\sqrt{\langle (P^2_T)_s \rangle_s}},$$

$$\langle P^2_T \rangle_s = \langle p_{T}^2 \rangle + z_h^2 \langle k_\perp^2 \rangle_s$$

(45)

In Fig. 8 we show the measured single-spin $A_{Siv}^{h^{\pm}}$ and difference $A_{Siv}^{h^{+}-h^{-}}$ Sivers asymmetries, and in Fig. 9 we show the extracted BM and Sivers functions $N_{Q_V}^{BM}(x_B)$ and $N_{Q_V}^{Siv}(x_B)$ for Set I with and Set III. We see that Eq. (43) holds, confirming the results of [14], and that $\lambda \approx 1.0$ for both Set I and Set III.

FIG. 8: The Sivers asymmetries: $A_{Siv}^{h^{\pm}}$ (left) and $A_{Siv}^{h^{+}-h^{-}}$ (right).
FIG. 9: Test of proportionality for BM to Sivers functions, Eq. (43). Both for Set I (left) and Set III (right), $N_{BM}^{QV}(x_B)$ overlaps $N_{Siv}^{QV}(x_B)$, which implies $\lambda \approx 1$ for both sets.

VIII. CONCLUSIONS

In a combined analysis of the $\cos \phi_h$ and $\cos 2\phi_h$ azimuthal asymmetries in unpolarized SIDIS, measured most recently by COMPASS, we determined 1) the BM function $f_{BM}^{QV}(x_B, Q^2)$ for the sum of the valence quarks $Q_V = u_V + d_V$ and 2) obtained information on the average transverse momenta $\langle k^2_\perp \rangle$ and $\langle p^2_\perp \rangle$, which play a role in the transverse momentum dependent Parton Distribution Functions and Fragmentation Functions respectively. The results are obtained using the standard assumption of factorization of transverse momentum and $x_B$ dependence, with the transverse momentum dependence given by a Gaussian. The analysis is based on a study of the so called difference asymmetries between hadron $h$ and $\bar{h}$.

We have compared our results to the existing ones in the literature. For the BM function we agree with the results obtained from an analysis of DY processes \cite{10,11} but disagree strongly with the results obtained in a model analysis of the $\cos 2\phi_h$ asymmetry in SIDIS in \cite{3,4}.

Our favoured values for $\langle k^2_\perp \rangle$ and $\langle p^2_\perp \rangle$ agree with those obtained in a previous analysis of the $\cos \phi_h$ and $\cos 2\phi_h$ modulations in SIDIS \cite{30,33} i.e. $(\langle k^2_\perp \rangle = 0.18 \text{ GeV}^2; \langle p^2_\perp \rangle = 0.20 \text{ GeV}^2)$ and $(\langle k^2_\perp \rangle = 0.25 \text{ GeV}^2; \langle p^2_\perp \rangle = 0.20 \text{ GeV}^2)$, and disagree with the later, larger values $(\langle k^2_\perp \rangle = 0.57 \text{ GeV}^2; \langle p^2_\perp \rangle = 0.12 \text{ GeV}^2)$ obtained from a study of multiplicities \cite{34}.

Finally we note that future data on the $\langle \cos \phi_h \rangle$ and $\langle \cos 2\phi_h \rangle$ asymmetries on protons, for charged pions or kaons, will allow access to the BM function for the valence quarks $u_V$ and $d_V$ separately, in the same, approximately model independent manner \cite{14}.

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