Convex hull-like property and supported images of open sets

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Dedicated to Professor Anthony To-Ming Lau, with esteem and friendship

Abstract. In this note, as a particular case of a more general result, we obtain the following theorem:

Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty bounded open set and let $f : \overline{\Omega} \to \mathbb{R}^n$ be a continuous function which is $C^1$ in $\Omega$. Then, at least one of the following assertions holds:

(a) $f(\Omega) \subseteq \text{conv}(f(\partial \Omega))$.

(b) There exists a non-empty open set $X \subseteq \Omega$, with $\overline{X} \subseteq \Omega$, satisfying the following property: for every continuous function $g : \Omega \to \mathbb{R}^n$ which is $C^1$ in $X$, there exists $\lambda > 0$ such that, for each $\lambda > \lambda$, the Jacobian determinant of the function $g + \lambda f$ vanishes at some point of $X$.

As a consequence, if $n = 2$ and $h : \Omega \to \mathbb{R}$ is a non-negative function, for each $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfying in $\Omega$ the Monge-Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = h,$$

one has

$$\nabla u(\Omega) \subseteq \text{conv}(\nabla u(\partial \Omega)).$$

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1. - Introduction

Here and in what follows, $\Omega$ is a non-empty relatively compact and open set in a topological space $E$, with $\partial \Omega \neq \emptyset$, and $Y$ is a real locally convex Hausdorff topological vector space. $\overline{\Omega}$ and $\partial \Omega$ denote the closure and the boundary of $\Omega$, respectively. Since $\overline{\Omega}$ is compact, $\partial \Omega$, being closed, is compact too.

Let us first recall some well-known definitions.

Let $S$ be a subset of $Y$ and let $y_0 \in S$. As usual, we say that $S$ is supported at $y_0$ if there exists $\varphi \in Y^* \setminus \{0\}$ such that $\varphi(y_0) \leq \varphi(y)$ for all $y \in S$. If this happens, of course $y_0 \in \partial S$.

Further, extending a maximum principle definition for real-valued functions, a continuous function $f : \overline{\Omega} \to Y$ is said to satisfy the convex hull property in $\overline{\Omega}$ (see [1], [2] and references therein) if

$$f(\Omega) \subseteq \text{conv}(f(\partial \Omega)),$$

$\text{conv}(f(\partial \Omega))$ being the closed convex hull of $f(\partial \Omega)$.

When $\dim(Y) < \infty$, since $f(\partial \Omega)$ is compact, $\text{conv}(f(\partial \Omega))$ is compact too and so $\overline{\text{conv}(f(\partial \Omega))} = \text{conv}(f(\partial \Omega))$.

A function $\psi : Y \to \mathbb{R}$ is said to be quasi-convex if, for each $r \in \mathbb{R}$, the set $\psi^{-1}([-\infty, r])$ is convex. Notice the following proposition:
PROPOSITION 1. - For each pair \( A, B \) of non-empty subsets of \( Y \), the following assertions are equivalent:

\[ (a_1) \quad A \subseteq \text{conv}(B) \, . \]

\[ (a_2) \quad \text{For every continuous and quasi-convex function } \psi : Y \to \mathbb{R} \text{, one has} \]

\[
\sup_A \psi \leq \sup_B \psi \, .
\]

PROOF. Let \((a_1)\) hold. Fix any continuous and quasi-convex function \( \psi : Y \to \mathbb{R} \). Fix \( \tilde{y} \in A \). Then, there is a net \( \{ y_\alpha \} \) in \( \text{conv}(B) \) converging to \( \tilde{y} \). So, for each \( \alpha \), we have

\[
y_\alpha = \sum_{i=1}^k \lambda_i z_i,
\]

where \( z_i \in B \), \( \lambda_i \in [0, 1] \) and \( \sum_{i=1}^k \lambda_i = 1 \). By quasi-convexity, we have

\[
\psi(y_\alpha) = \psi \left( \sum_{i=1}^k \lambda_i z_i \right) \leq \max_{1 \leq i \leq k} \psi(z_i) \leq \sup_B \psi
\]

and so, by continuity,

\[
\psi(\tilde{y}) = \lim_{\alpha} \psi(y_\alpha) \leq \sup_B \psi
\]

which yields \((a_2)\).

Now, let \((a_2)\) hold. Let \( x_0 \in A \). If \( x_0 \not\in \text{conv}(B) \), by the standard separation theorem, there would be \( \psi \in Y^* \setminus \{0\} \) such that \( \sup_{\text{conv}(B)} \psi < \psi(x_0) \), against \((a_2)\). So, \((a_1)\) holds. \( \triangle \)

Clearly, applying Proposition 1, we obtain the following one:

PROPOSITION 2. - For any continuous function \( f : \overline{\Omega} \to Y \), the following assertions are equivalent:

\[ (b_1) \quad f \text{ satisfies the convex hull property in } \overline{\Omega} \, . \]

\[ (b_2) \quad \text{For every continuous and quasi-convex function } \psi : Y \to \mathbb{R} \text{, one has} \]

\[
\sup_{x \in \Omega} \psi(f(x)) = \sup_{x \in \partial \Omega} \psi(f(x)) .
\]

In view of Proposition 2, we now introduce the notion of convex hull-like property for functions defined in \( \Omega \) only.

DEFINITION 1. - A continuous function \( f : \Omega \to Y \) is said to satisfy the convex hull-like property in \( \Omega \) if, for every continuous and quasi-convex function \( \psi : Y \to \mathbb{R} \), there exists \( x^* \in \partial \Omega \) such that

\[
\lim_{x \to x^*} \sup \psi(f(x)) = \sup_{x \in \Omega} \psi(f(x)) .
\]

We have

PROPOSITION 3. - Let \( g : \overline{\Omega} \to Y \) be a continuous function and let \( f = g|_{\Omega} \).

Then, the following assertions are equivalent:

\[ (c_1) \quad f \text{ satisfies the convex hull-like property in } \Omega \, . \]

\[ (c_2) \quad g \text{ satisfies the convex hull property in } \overline{\Omega} \, . \]

PROOF. Let \((c_1)\) hold. Let \( \psi : Y \to \mathbb{R} \) be any continuous and quasi-convex function. Then, by Definition 1, there exists \( x^* \in \partial \Omega \) such that

\[
\lim_{x \to x^*} \sup \psi(f(x)) = \sup_{x \in \Omega} \psi(f(x)) .
\]

But

\[
\lim_{x \to x^*} \sup \psi(f(x)) = \psi(g(x^*))
\]
and hence 
\[ \sup_{x \in \partial \Omega} \psi(g(x)) = \sup_{x \in \Omega} \psi(g(x)). \]

So, by Proposition 2, (c2) holds.

Now, let (c2) hold. Let \( \psi : Y \to \mathbb{R} \) be any continuous and quasi-convex function. Then, by Proposition 2, one has 
\[ \sup_{x \in \partial \Omega} \psi(g(x)) = \sup_{x \in \Omega} \psi(g(x)). \]

Since \( \partial \Omega \) is compact and \( \psi \circ g \) is continuous, there exists \( x^* \in \partial \Omega \) such that 
\[ \psi(g(x^*)) = \sup_{x \in \partial \Omega} \psi(g(x)). \]

But 
\[ \psi(g(x^*)) = \lim_{x \to x^*} \psi(f(x)) \]
and, by continuity again,
\[ \sup_{x \in \Omega} \psi(g(x)) = \sup_{x \in \Omega} \psi(g(x)) \]
and so 
\[ \lim_{x \to x^*} \psi(f(x)) = \sup_{x \in \Omega} \psi(f(x)) \]
which yields (c1).

After the above preliminaries, we can declare the aim of this short note: to establish Theorem 1 below jointly with some of its consequences.

**THEOREM 1.** - For any continuous function \( f : \Omega \to Y \), at least one of the following assertions holds:

(i) \( f \) satisfies the convex hull-like property in \( \Omega \).

(ii) There exists a non-empty open set \( X \subseteq \Omega \), with \( \overline{X} \subseteq \Omega \), satisfying the following property: for every continuous function \( g : \Omega \to Y \), there exist \( \lambda > 0 \) such that, for each \( \lambda > \lambda \), the set \( (g + \lambda f)(X) \) is supported at one of its points.

**2. Proof of Theorem 1**

Assume that (i) does not hold. So, we are assuming that there exists a continuous and quasi-convex function \( \psi : Y \to \mathbb{R} \) such that
\[ \limsup_{x \to z} \psi(f(x)) < \sup_{x \in \Omega} \psi(f(x)) \] (1)
for all \( z \in \partial \Omega \).

In view of (1), for each \( z \in \partial \Omega \), there exists an open neighbourhood \( U_z \) of \( z \) such that 
\[ \sup_{x \in U_z \cap \Omega} \psi(f(x)) < \sup_{x \in \Omega} \psi(f(x)). \]

Since \( \partial \Omega \) is compact, there are finitely many \( z_1, ..., z_k \in \partial \Omega \) such that 
\[ \partial \Omega \subseteq \bigcup_{i=1}^{k} U_{z_i}. \] (2)

Put 
\[ U = \bigcup_{i=1}^{k} U_{z_i}. \]

Hence 
\[ \sup_{x \in U \cap \Omega} \psi(f(x)) = \max_{1 \leq i \leq k} \sup_{x \in U_{z_i} \cap \Omega} \psi(f(x)) < \sup_{x \in \Omega} \psi(f(x)). \]
Now, fix a number \( r \) so that
\[
\sup_{x \in U \cap \Omega} \psi(f(x)) < r < \sup_{x \in \Omega} \psi(f(x))
\] (3)
and set
\[
K = \{ x \in \Omega : \psi(f(x)) \geq r \}.
\]
Since \( f, \psi \) are continuous, \( K \) is closed in \( \Omega \). But, since \( K \cap U = \emptyset \) and \( U \) is open, in view of (2), \( K \) is closed in \( E \). Hence, \( K \) is compact since \( \Omega \) is so. By (3), we can fix \( \bar{x} \in \Omega \) such that \( \psi(f(\bar{x})) > r \). Notice that the set \( \psi^{-1}([-\infty, r]) \) is closed and convex. So, thanks to the standard separation theorem, there exists a non-zero continuous linear functional \( \varphi : Y \to \mathbb{R} \) such that
\[
\varphi(f(\bar{x})) < \inf_{y \in \psi^{-1}([-\infty, r])} \varphi(y).
\] (4)

Then, from (3) and (4), it follows
\[
\varphi(f(\bar{x})) < \inf_{x \in \Omega \setminus K} \varphi(f(x))
\]
Now, choose \( \rho \) so that
\[
\varphi(f(\bar{x})) < \rho < \inf_{x \in \Omega \setminus K} \varphi(f(x))
\]
and set
\[
X = \{ x \in \Omega : \psi(f(x)) < \rho \}.
\]
Clearly, \( X \) is a non-empty open set contained in \( K \). Now, let \( g : \Omega \to Y \) be any continuous function. Set
\[
\lambda = \inf_{x \in X} \frac{\varphi(g(x)) - \inf_{z \in K} \varphi(g(z))}{\rho - \varphi(f(x))}.
\]
Fix \( \lambda > \lambda \). So, there is \( x_0 \in X \) such that
\[
\frac{\varphi(g(x_0)) - \inf_{z \in K} \varphi(g(z))}{\rho - \varphi(f(x_0))} < \lambda.
\]
From this, we get
\[
\varphi(g(x_0)) + \lambda \varphi(f(x_0)) < \lambda \rho + \inf_{z \in K} \varphi(g(z))
\] (5)
By continuity and compactness, there exists \( \hat{x} \in K \) such that
\[
\varphi(g(\hat{x}) + \lambda f(\hat{x})) \leq \varphi(g(x)) + \lambda f(x)
\] (6)
for all \( x \in K \). Let us prove that \( \hat{x} \in X \). Arguing by contradiction, assume that \( \varphi(f(\hat{x})) \geq \rho \). Then, taking (5) into account, we would have
\[
\varphi(g(x_0)) + \lambda \varphi(f(x_0)) < \lambda \varphi(f(\hat{x})) + \varphi(g(\hat{x}))
\]
contradicting (6). So, it is true that \( \hat{x} \in X \), and, by (6), the set \( (g + \lambda f)(X) \) is supported at its point \( g(\hat{x}) + \lambda f(\hat{x}) \).

3. Applications

The first application of Theorem 1 shows a strongly bifurcating behaviour of certain equations in \( \mathbb{R}^n \).

THEOREM 2. - Let \( \Omega \) be a non-empty bounded open subset of \( \mathbb{R}^n \) and let \( f : \Omega \to \mathbb{R}^n \) a continuous function.
Then, at least one of the following assertions holds:

\( (d_1) \) \( f \) satisfies the convex hull-like property in \( \Omega \).

\( \triangle \)
(d₂) There exists a non-empty open set $X \subseteq \Omega$, with $\overline{X} \subseteq \Omega$, satisfying the following property: for every continuous function $g : \Omega \to \mathbb{R}^n$, there exist $\lambda > 0$ such that, for each $\lambda > \lambda$, there exist $\hat{x} \in X$ and two sequences $\{y_k\}, \{z_k\}$ in $\mathbb{R}^n$, with

$$\lim_{k \to \infty} y_k = \lim_{k \to \infty} z_k = g(\hat{x}) + \lambda f(\hat{x}),$$

such that, for each $k \in \mathbb{N}$, one has

(j) the equation

$$g(x) + \lambda f(x) = y_k$$

has no solution in $X$;

(jj) the equation

$$g(x) + \lambda f(x) = z_k$$

has two distinct solutions $u_k, v_k$ in $X$ such that

$$\lim_{k \to \infty} u_k = \lim_{k \to \infty} v_k = \hat{x}.$$

PROOF. Apply Theorem 1 with $E = Y = \mathbb{R}^n$. Assume that $(d_1)$ does not hold. Let $X \subseteq \Omega$ be an open set as in (ii) of Theorem 1. Fix any continuous function $g : \Omega \to \mathbb{R}^n$. Then, there is some $\lambda > 0$ such that, for each $\lambda > \lambda$, there exists $\hat{x} \in X$ such that the set $(g + \lambda f)(X)$ is supported at $g(\hat{x}) + \lambda f(\hat{x})$. As we observed at the beginning, this implies that $g(\hat{x}) + \lambda f(\hat{x})$ lies in the boundary of $(g + \lambda f)(X)$. Therefore, we can find a sequence $\{y_k\}$ in $\mathbb{R}^n \setminus (g + \lambda f)(X)$ converging to $g(\hat{x}) + \lambda f(\hat{x})$. So, such a sequence satisfies (j).

For each $k \in \mathbb{N}$, denote by $B_k$ the open ball of radius $\frac{1}{k} \hat{x}$ centered at $\hat{x}$. Let $k$ be such that $B_k \subseteq X$. The set $(g + \lambda f)(B_k)$ is not open since its boundary contains the point $g(\hat{x}) + \lambda f(\hat{x})$. Consequently, by the invariance of domain theorem ([3], p. 705), the function $g + \lambda f$ is not injective in $B_k$. So, there are $u_k, v_k \in B_k$, with $u_k \neq v_k$ such that

$$g(u_k) + \lambda f(u_k) = g(v_k) + \lambda f(v_k).$$

Hence, if we take

$$z_k = g(u_k) + \lambda f(u_k),$$

the sequences $\{u_k\}, \{v_k\}, \{z_k\}$ satisfy (jj) and the proof is complete. △

REMARK 1. - Notice that, in general, Theorem 2 is no longer true when $f : \Omega \to \mathbb{R}^m$ with $m > n$. In this connection, consider the case $n = 1, m = 2, \Omega = [0, \pi]$ and $f(\theta) = (\cos \theta, \sin \theta)$ for $\theta \in [0, \pi]$. So, for each $\lambda > 0$, on the one hand, the function $\lambda f$ is injective, while, on the other hand, $\lambda f([0, \pi])$ is not contained in $\text{conv} \{f(0), f(\pi)\}$.

If $S \subseteq \mathbb{R}^n$ is a non-empty open set, $x \in S$ and $h : S \to \mathbb{R}^n$ is a $C^1$ function, we denote by $\det(J_h(x))$ the Jacobian determinant of $h$ at $x$.

Another important consequence of Theorem 1 is as follows:

THEOREM 3. - Let $\Omega$ be a non-empty bounded open subset of $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^n$ be a $C^1$ function.

Then, at least one of the following assertions holds:

(ε₁) $f$ satisfies the convex hull-like property in $\Omega$ .

(ε₂) There exists a non-empty open set $X \subseteq \Omega$, with $\overline{X} \subseteq \Omega$, satisfying the following property: for every continuous function $g : \Omega \to \mathbb{R}^n$ which is $C^1$ in $X$, there exists $\lambda > 0$ such that, for each $\lambda > \lambda$, one has

$$\det(J_{g + \lambda f}(\hat{x})) = 0$$

for some $\hat{x} \in X$.

PROOF. Assume that (ε₁) does not hold. Let $X$ be an open set as in (ii) of Theorem 1. Let $g : \Omega \to \mathbb{R}^n$ be a continuous function which is $C^1$ in $X$. Then, there is some $\lambda > 0$ such that, for each $\lambda > \lambda$, there exists $\hat{x} \in X$ such that the set $(g + \lambda f)(X)$ is supported at $g(\hat{x}) + \lambda f(\hat{x})$. By remarks already made, we infer that
the function $g + \lambda f$ is not a local homeomorphism at $\hat{x}$, and so \( \det(J_{g+\lambda f}(\hat{x})) = 0 \) in view of the classical inverse function theorem. \( \square \)

In turn, here is a consequence of Theorem 3 when \( n = 2 \).

**THEOREM 4.** Let \( \Omega \) be a non-empty bounded open set of \( \mathbb{R}^2 \), let \( h : \Omega \to \mathbb{R} \) be a continuous function and let \( \alpha, \beta : \Omega \to \mathbb{R} \) be two \( C^1 \) functions such that \( |\alpha_x \beta_y - \alpha_y \beta_x| + |h| > 0 \) and \( (\alpha_x \beta_y - \alpha_y \beta_x)h \geq 0 \) in \( \Omega \). Then, any \( C^1 \) solution \((u, v) \in \Omega \) of the system

\[
\begin{cases}
    u_x v_y - u_y v_x = h \\
    \beta_y u_x - \beta_x u_y - \alpha_y v_x + \alpha_x v_y = 0
\end{cases}
\]  

(7)

satisfies the convex hull-like property in \( \Omega \).

**PROOF.** Arguing by contradiction, assume that \((u, v)\) does not satisfy the convex hull-like property in \( \Omega \). Then, by Theorem 3, applied taking \( f = (u, v) \) and \( g = (\alpha, \beta) \), there exist \( \lambda > 0 \) and \((\hat{x}, \hat{y}) \in \Omega \) such that

\[ \det(J_{g+\lambda f}(\hat{x}, \hat{y})) = 0 . \]

On the other hand, for each \((x, y) \in \Omega \), we have

\[ \det(J_{g+\lambda f}(x, y)) = (u_x v_y - u_y v_x)(x, y)\lambda^2 + (\beta_y u_x - \beta_x u_y - \alpha_y v_x + \alpha_x v_y)(x, y)\lambda + (\alpha_x \beta_y - \alpha_y \beta_x)(x, y) \]

and hence

\[ h(\hat{x}, \hat{y})\lambda^2 + (\alpha_x \beta_y - \alpha_y \beta_x)(\hat{x}, \hat{y}) = 0 \]

which is impossible in view of our assumptions. \( \square \)

We conclude by highlighting two applications of Theorem 4.

**THEOREM 5.** Let \( \Omega \) be a non-empty bounded open subset of \( \mathbb{R}^2 \), let \( h : \Omega \to \mathbb{R} \) be a continuous non-negative function and let \( w \in C^2(\Omega) \) be a function satisfying in \( \Omega \) the Monge-Ampère equation

\[ w_{xx}w_{yy} - w_{xy}^2 = h . \]

Then, the gradient of \( w \) satisfies the convex hull-like property in \( \Omega \).

**PROOF.** It is enough to observe that \((w_x, w_y)\) is a \( C^1 \) solution in \( \Omega \) of the system (7) with \( \alpha(x, y) = -y \) and \( \beta(x, y) = x \) and that such \( \alpha, \beta \) satisfy the assumptions of Theorem 4. \( \square \)

**THEOREM 6.** Let \( \Omega \) a non-empty bounded open subset of \( \mathbb{R}^2 \) and let \( \beta : \Omega \to \mathbb{R} \) be a \( C^1 \) function. Assume that there exists another \( C^1 \) function \( \alpha : \Omega \to \mathbb{R} \) so that the function \( \alpha_x \beta_y - \alpha_y \beta_x \) vanishes at no point of \( \Omega \).

Then, for any function \( u \in C^1(\Omega) \cap C^0(\overline{\Omega}) \) satisfying in \( \Omega \) the equation

\[ \beta_y u_x - \beta_x u_y = 0 , \]

(8)

one has

\[ \sup_{\Omega} u = \sup_{\partial \Omega} u \]

and

\[ \inf_{\Omega} u = \inf_{\partial \Omega} u . \]

**PROOF.** Observe that the function \((u, 0)\) satisfies the system (7) with \( h = 0 \) and that the assumptions of Theorem 4 are fulfilled. So, \((u, 0)\) satisfies the convex hull-like property in \( \Omega \). Since \( u \in C^0(\overline{\Omega}) \), the conclusion follows from Proposition 3. \( \square \)

**REMARK 2.** Observe that when \( \Omega \) is also star-shaped, the conclusion of Theorem 6 holds for any harmonic function \( \beta \) whose gradient vanishes at no point of \( \Omega \). Indeed, if \( \beta : \Omega \to \mathbb{R} \) is such a function, the
differential form $\beta_y dx - \beta_x dy$ is exact since $\beta_{yy} = -\beta_{xx}$ and $\Omega$ is star-shaped. So, there is a $C^1$ function $\alpha : \Omega \to \mathbb{R}$ such that $\alpha_x = \beta_y$ and $\alpha_y = -\beta_x$. Hence, $\alpha_x \beta_y - \alpha_y \beta_x = \beta_x^2 + \beta_y^2 > 0$ in $\Omega$. That is to say, $\alpha$ satisfies the assumption of Theorem 6, and the claim follows.

**REMARK 3.** - Of course, if $u$ is a solution of the equation (8) and $g : \mathbb{R} \to \mathbb{R}$ is any $C^1$ function, the composite function $g \circ u$ is a solution of (8) too. On the basis of this remark and in view of Theorem 6, we formulate the following final conjecture:

**CONJECTURE 1.** - Let $\Omega$ a non-empty bounded open subset of $\mathbb{R}^2$ and let $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ be such that, for every $C^1$ function $g : \mathbb{R} \to \mathbb{R}$, one has

$$
\sup_{\Omega} g \circ u = \sup_{\partial \Omega} g \circ u
$$

and

$$
\inf_{\Omega} g \circ u = \inf_{\partial \Omega} g \circ u .
$$

Then, there exist two $C^1$ functions $\alpha, \beta : \Omega \to \mathbb{R}$ such that the function $\alpha_x \beta_y - \alpha_y \beta_x$ vanishes at no point of $\Omega$ and $u$ is a solution in $\Omega$ of the equation

$$
\beta_y u_x - \beta_x u_y = 0 .
$$

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