OBLIQUE PROJECTION BASED STABILIZING FEEDBACK FOR NONAUTONOMOUS COUPLED PARABOLIC-ODE SYSTEMS

KARL KUNISCH
Institute for Mathematics and Scientific Computing, Karl-Franzens-Universität
Heinrichstr. 36, 8010 Graz, Austria

SÉRGIO S. RODRIGUES*
Johann Radon Institute for Computational and Applied Mathematics, ÖAW
Altenbergerstraße 69, A-4040 Linz, Austria

(Communicated by Irena Lasiecka)

Abstract. Global feedback stabilizability results are derived for nonautonomous coupled systems arising from the linearization around a given time-dependent trajectory of FitzHugh–Nagumo type systems. The feedback is explicit and is based on suitable oblique (nonorthogonal) projections in Hilbert spaces. The actuators are, typically, a finite number of indicator functions and act only in the parabolic equation. Subsequently, local feedback stabilizability to time-dependent trajectories results are derived for nonlinear coupled parabolic-ode systems of the FitzHugh–Nagumo type. Simulations are presented showing the stabilizing performance of the feedback control.

1. Introduction. In this work we focus on coupled systems consisting of a parabolic equation with controls, and an ordinary differential equation. This class of systems include well-known models describing electric excitations and the propagation of electric waves in nerve fibers and in heart tissue. In this context the system of equations is referred to as the monodomain equations. Given a bounded domain \( \Omega \subset \mathbb{R}^d \), where \( d \) is a positive integer, with smooth boundary \( \Gamma = \partial \Omega \), the controlled monodomain equations read

\[
\frac{\partial}{\partial t} v = \nu \Delta v - av^3 + bw^2 - cv - \mathcal{M}_1(v, w) + f + \sum_{i=1}^{M} u_i 1_{\omega_i}, \quad v(0) = v_0, \quad (1a)
\]

\[
\frac{\partial}{\partial t} w = -\delta w - \mathcal{M}_2(v, w), \quad w(0) = w_0, \quad (1b)
\]

with Neumann boundary conditions

\[
\left( \frac{\partial}{\partial n} v \right)_{\Gamma} = 0, \quad (1c)
\]

where \( f = f(x, t) \) is an external forcing, \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are suitable functions coupling the two equations, \( n \) is the unit outward normal vector to \( \Gamma \), and the constants in \( \{\nu, \delta, a, b, c\} \) are all given and strictly positive. Further, \( \{\omega_i \mid i \in \{1, 2, \ldots, M\}\} \) is

2010 Mathematics Subject Classification. 93D15, 93B52, 93C05, 93C20.

Key words and phrases. Exponential stabilization, explicit feedback, monodomain equations, finite-dimensional controller, oblique projections.

* Corresponding author.
a family of open sets of $\Omega$ and our $M$ actuators $\{1_{\omega_i} = 1_{\omega_i}(x) \mid i \in \{1, 2, \ldots, M\}\} \subset L^2(\Omega)$, are the indicator functions of the subsets $\omega_i$,

$$1_{\omega_i}(x) := \begin{cases} 
1 & \text{if } x \in \omega_i, \\
0 & \text{if } x \in \Omega \setminus \omega_i.
\end{cases}$$

Finally $u = u(t)$, taking values in $\mathbb{R}^M$, is a control function at our disposal.

In electrophysiology, see [11, Section 12.3.3], the variable $v$ models the transmembrane electric potential of the human heart and $w$ represents a gating variable. Typical models include the FitzHugh–Nagumo, the Rogers–McCulloch, and the Aliev–Panfilov model. See [7, 18, 21, 2, 8]. The coupling $M$ takes the form

$$M(w, v) := \begin{bmatrix} M_1(w, v) \\ M_2(w, v) \end{bmatrix} := \begin{bmatrix} dw + e vw \\ -\gamma v + \rho v^2 \end{bmatrix}$$

(2a)

where the constants $d, e, \gamma, \rho$ are also given, and are all strictly positive with the following exceptions defining the model:

- $e = \rho = 0$, for the FitzHugh–Nagumo model, [FN].
- $d = \rho = 0$, for the Rogers–McCulloch model, [RM].
- $d = 0$, for the Aliev–Panfilov model, [AP].

Assume that a desired heart rhythm is given as the solution of the (uncontrolled) system (1), corresponding to a suitable initial condition $(v_0, w_0) \in L^2(\Omega) \times L^2(\Omega)$,

$$\partial_t v = \nu \Delta v - av^3 + bv^2 - c v - M_1(w, v) + f, \quad v(0) = v_0,$$

$$\partial_t w = -\delta w - M_2(w, v), \quad w(0) = w_0.$$  

(3a)  

(3b)

(3c)

If $(v_0, w_0) \neq (v_0, w_0)$, then the behavior of (1) without control can be considerably different from the desired behavior of $(v, w)$, even if $(v_0, w_0) - (v_0, w_0)$ is small. Here we look for a feedback control $u$, so that the solution of (1) goes to the desired heart rhythm $(v, w)$, solving (3), provided the difference $(v_0, w_0) - (v_0, w_0)$ is small.

We will follow a standard idea: first we find a feedback operator which stabilizes $\text{globally}$ the linearization of (1) around $(v, w)$, then we use a suitable fixed point argument to conclude that the same feedback operator also stabilizes the nonlinear system locally.

More precisely, by direct computations, we can see that the (controlled) linearization around $(\bar{v}, \bar{w})$, satisfies

$$\frac{\partial y}{\partial t} + Ay + A_r y + \tilde{S}z - \sum_{i=1}^{M} u_i 1_{\omega_i} = 0, \quad y(0) = y_0$$

(4a)

$$\frac{\partial z}{\partial t} + Dz + R v = 0, \quad z(0) = z_0.$$  

(4b)

Moreover the difference $(y, z) := (v, w) - (\bar{v}, \bar{w})$, to the targeted solution, solves the system

$$\frac{\partial y}{\partial t} + Ay + A_r y + \tilde{S}z - \sum_{i=1}^{M} u_i 1_{\omega_i} = \mathcal{N}_1(y, z), \quad y(0) = y_0$$

(5a)

$$\frac{\partial z}{\partial t} + Dz + R v = \mathcal{N}_2(y, z), \quad z(0) = z_0.$$  

(5b)
with the linear operators
\[ y \mapsto Ay := -\nu \Delta y + y, \]
with \( (Ay, v)_{H^1(\Omega)', H^1(\Omega)} := \nu (\nabla y, \nabla v)_{L^2(\Omega)^d} + (y, v)_{L^2(\Omega)} \),
\[ z \mapsto Dz := \delta z, \]
\[ y \mapsto A_r y := -y - (-3a\nu^2 y + 2b\nu y - cy - d\nu y), \]
\[ z \mapsto \tilde{S} z := dz + e\nu z, \]
\[ y \mapsto \mathcal{R} y := (-\gamma + 2\rho\nu) y, \]
and the nonlinearities
\[ (y, z) \mapsto N_1(y, z) := -ay^3 - (-b + 3a\nu^2) y^2 - eyz, \]
\[ (y, z) \mapsto N_2(y, z) := -\rho y^2. \]

Without loss of generality we may suppose that the set of actuators \( \{1 \omega_i \mid i \in \{1, 2, \ldots, M\} \} \subset L^2(\Omega) \) is linearly independent. We also set \( U_M := \text{span}\{1 \omega_i \mid i \in \{1, 2, \ldots, M\}\} \). Notice the subscript \( r \) in the reaction operator \( A_r \), in order to distinguish it from the diffusion operator \( A \).

Let \( \alpha_i, i \in \{1, 2, 3, \ldots\} \), be the increasing sequence of repeated (Neumann) eigenvalues of \( A = -\nu \Delta + \text{Id} \) and let \( E_M := \text{span}\{e_i \mid i \in \{1, 2, \ldots, M\}\} \) be the space spanned by the first eigenfunctions of \( A \) in \( L^2(\Omega) \),
\[ 0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \alpha_{n+1} \rightarrow +\infty \quad \text{and} \quad A e_n = \alpha_n e_n. \]

For simplicity let us denote \( L^2 := L^2(\Omega) \) and \( H^1 := H^1(\Omega) \). We will show that a sufficient condition for \textit{global} stabilizability of system (4), and for \textit{local} stabilizability of system (5), is given by
\[ H = U_M \oplus E_M^\perp, \]
\[ \alpha_{M+1} > \Bigg( 6 + 4 \left| \frac{P_{E_M^\perp}^{U_M}}{\mathcal{L}(L^2)} \right|^2 \left| A_r \right|^2_{L^\infty(\mathbb{R}_0, \mathcal{L}(L^2, (H^1)'))} + \frac{\left| P_{E_M^\perp}^{U_M} \tilde{S} \right|_{L^\infty(\mathbb{R}_0, \mathcal{L}(L^2))}}{\delta} \right)^\frac{1}{2}, \]
where \( \mathcal{L}(L^2, (H^1)') \) stands for the space of bounded linear functionals from \( L^2 \) into \( (H^1)' \), and \( \mathcal{L}(L^2) \) stands for the space of bounded linear functionals from \( L^2 \) into itself. Further, \( P_{E_M^\perp}^{U_M} : L^2 \rightarrow U_M \) is the oblique projection in \( L^2 \) onto \( U_M \) along \( E_M^\perp \), and \( P_{E_M^\perp}^{U_M} = \text{Id} - P_{E_M^\perp}^{U_M} \) is the complementary oblique projection in \( L^2 \) onto \( E_M^\perp \) along \( U_M \).

The following regularity condition on the desired trajectory will be used.
\[ (\nu, \bar{w}) \in L^\infty(\mathbb{R}_0 \times \Omega) \times L^\infty(\mathbb{R}_0 \times \Omega). \]

Recall that in [13] it is shown that when \( \tilde{S} = 0 \), then (7) is a sufficient stabilizability condition for the parabolic system (4a).

The following Theorems 1.1 and 1.2 are the main results of this paper.

**Theorem 1.1** (Linearized monodomain equations). Let \( \lambda > 0 \) and \( s_0 \geq 0 \). If (7) and (8) holds true, then there are constants \( C \geq 1 \) and \( \mu > 0 \), independent of
(s_0, y_0, z_0), such that the solution of the linear system
\[
\frac{\partial}{\partial t} y + A y + A_r y + \bar{S} z - P_{U_M}^{E_M}(A y + A_r y + \bar{S} z - \lambda y) = 0, \quad y(s_0) = y_0, \quad (9a)
\]
\[
\frac{\partial}{\partial t} z + D z + R v = 0, \quad z(s_0) = z_0, \quad (9b)
\]
satisfies
\[
|\langle y(t), z(t) \rangle|_{L^2 \times L^2} \leq C e^{-\mu(t-s_0)} |\langle y(0), z_0 \rangle|_{L^2 \times L^2}, \quad t \geq s_0.
\]
for all (y_0, z_0) \in L^2 \times L^2.

**Theorem 1.2** (Monodomain equations). Let \( \lambda > 0 \) and \( s_0 \geq 0 \). If (7) and (8) holds true, then there are constants \( C \geq 1, \mu > 0, \) and \( \epsilon > 0, \) independent of \((s_0, y_0, z_0)\), such that the solution of the system
\[
\frac{\partial}{\partial t} y + A y + A_r y + \bar{S} z - P_{U_M}^{E_M}(A y + A_r y + \bar{S} z - \lambda y) = N_i(y, z), \quad y(s_0) = y_0, \quad (10a)
\]
\[
\frac{\partial}{\partial t} z + D z + R v = N_2(y, z), \quad z(s_0) = z_0, \quad (10b)
\]
satisfies
\[
|\langle y(t), z(t) \rangle|_{H^1 \times L^2} \leq C e^{-\mu(t-s_0)} |\langle y(0), z_0 \rangle|_{H^1 \times L^2}, \quad t \geq s_0,
\]
provided that \( |\langle y(0), z_0 \rangle|_{H^1 \times L^2} < \epsilon \).

**Remark 1.3.** Observe that (10) is exactly (5), with \( u \) given by
\[
\sum_{i=1}^{M} u_i 1_{\omega_i} = P_{U_M}^{E_M}(A y + A_r y + \bar{S} z - \lambda y). \quad (11)
\]
Furthermore, by definition \( E_M \) is an \( M \) dimensional space, which together with (35a) implies that the (ordered) family of actuators \( U_M := (1_{\omega_1}, 1_{\omega_2}, \ldots, 1_{\omega_M}) \) is linearly independent (with \( M \geq 1 \)). Thus, denoting the bijection \([U_M] : \mathbb{R}^M \rightarrow U_M\), with \( u \mapsto \sum_{i=1}^{M} u_i 1_{\omega_i} \), the control \( u \) as in (11) is uniquely defined, and we may write
\[
u = [U_M]^{-1} P_{U_M}^{E_M}(A y + A_r y + \bar{S} z - \lambda y).
\]

**Remark 1.4.** Observe that Theorem 1.2 provides a result on the stabilizability to trajectories, since it asserts that the solution \((v, w) = (\nu, \nu) + (y, z)\) of (1) goes exponentially to the targeted trajectory \((\nu, \nu)\), solving (3), provided (7) holds true, and \( |\langle v_0 - \nu_0, w_0 - \nu_0 \rangle|_{H^1 \times L^2} < \epsilon \).

Concerning (7b), it is proven in [20] for 1D domains of the form \( \Omega = (0, L) \subset \mathbb{R} \) that for any given \( r \in (0, 1) \) and any given \( M \geq 1 \) we can place \( M \) actuators \( 1_{\omega_i} = 1_{\omega_i^M} \), with \( \omega_i^M \subset (0, L) \), so that the operator norm \( \left| P_{U_M}^{E_M} \right|_{L(L^2)} \) remains bounded as \( M \) increases. Furthermore, the total volume covered by the actuators is equal \( rL \), and thus independent of \( M \). The extension of these results to multi-dimensional rectangles follows from the results in [13, Section 4.8], see also [13, Remark 3.9]. Note that, once the operator norm of the oblique projection remains bounded and \( \alpha_{M+1} \rightarrow +\infty \), then we can set \( M \) large enough so that (7) is satisfied. Thus for an arbitrary “coupling” quadruple \((d, e, \gamma, \rho)\), we can set \( M \) large enough so that system (4) is stable.

Recall also that in [5] the stabilizability of (1.1) is proven, with a Riccati based feedback, for the case \( \rho = 0 \). Comparing the results in this paper to those in [5], we can say that in [5] all the actuators are supported in an a priori given subset of the physical domain \( \Omega \) and stability of the Riccati based closed-loop system is
guaranteed under a condition on the coupling triple \((d, e, \gamma)\), see [5, Corollary 3.2 and Eq. (38)]. In this paper stability of the oblique projection based closed-loop system is obtained for an arbitrary coupling quadruple \((d, e, \gamma, \rho)\) under a boundedness condition on the operator norm of the oblique projection \(P_{\mathcal{E}_d}^{E_3}\) onto the span of the actuators, and the placement of the actuators is supposed to be at our disposal.

**Contents.** The rest of the paper is organized as follows. In Sect. 2 we derive results on the stability of linear coupled systems in an abstract setting. These results are applied in Sect. 3 to obtain stabilizability conditions for general linear coupled parabolic-ODE systems. In Sect. 4 we derive the stabilizability result for the concrete example of the linearized monodomain equations, in particular, it contains the proof of Main Theorem 1.1 in Sect. 4.1 and that of Main Theorem 1.2 in Sect. 4.2. Numerical simulations are presented in Section 5, for both the linearized system (9) and the nonlinear system (10), confirming the theoretical results and showing the performance of the explicit oblique projection stabilizing feedback control. Finally the Appendix gathers comments on the existence and uniqueness of weak solutions for the systems involved in the main text.

**Notation.** We write \(\mathbb{R}\) and \(\mathbb{N}\) for the sets of real numbers and nonnegative integers, respectively, and we define \(\mathbb{R}_r := (r, +\infty)\), for \(r \in \mathbb{R}\), and \(\mathbb{N}_0 := \mathbb{N} \setminus \{0\}\).

We denote by \(\Omega \subseteq \mathbb{R}^d\) a bounded open connected subset, with \(d \in \mathbb{N}_0\).

For a normed space \(X\), we denote by \(|\cdot|_X\) the corresponding norm, by \(X'\) its dual, and by \(\langle \cdot, \cdot \rangle_{X', X}\) the duality between \(X'\) and \(X\). The dual space is endowed with the usual dual norm: \(|f|_{X'} := \sup\{\langle f, x \rangle_{X', X} \mid x \in X \text{ and } |x|_X = 1\}\). In case \(X\) is a Hilbert space we denote the inner product by \(\langle \cdot, \cdot \rangle_X\).

For an open interval \(I \subseteq \mathbb{R}\) and two Banach spaces \(X, Y\), we write \(W(I, X, Y) := \{f \in L^2(I, X) \mid \partial_t f \in L^2(I, Y)\}\), where the derivative \(\partial_t f\) is taken in the sense of distributions. This space is endowed with the natural norm \(|f|_{W(I, X, Y)} := (|f|^2_{L^2(I, X)} + |\partial_t f|^2_{L^2(I, Y)})^{1/2}\). In case \(X = Y\), we write \(H^1(I, X) := W(I, X, X)\).

The time derivative (in the distribution sense) of a vector function \(v\) taking values in a Banach space \(X\) will be denoted by \(\dot{v} := \frac{d}{dt} v\).

If the inclusions \(X \subseteq Z\) and \(Y \subseteq Z\) are continuous, where \(Z\) is a Hausdorff topological space, then we can define the Banach spaces \(X \times Y, X \cap Y,\) and \(X + Y\), endowed with the norms \(|(a, b)|_{X \times Y} := (|a|^2_X + |b|^2_Y)^{1/2}\); \(|a|_{X \cap Y} := |(a, a)|_{X \times Y};\) and \(|a|_{X + Y} := \inf_{(a_x, a_y) \in X \times Y} \{(|a|^X, |a|^Y)|_{X \times Y} \mid a = a^X + a^Y\}\), respectively. We can show that, if \(X\) and \(Y\) are endowed with a scalar product, then also \(X \times Y, X \cap Y,\) and \(X + Y\) are. In case we know that \(X \cap Y = \{0\}\), we say that \(X + Y\) is a direct sum and we write \(X \oplus Y\) instead.

Again, if \(X\) and \(Y\) are endowed with a scalar product, then also \(W(I, X, Y)\) is. The space of continuous linear mappings from \(X\) into \(Y\) will be denoted by \(\mathcal{L}(X, Y)\). When \(X = Y\) we simply write \(\mathcal{L}(X) := \mathcal{L}(X, X)\).

If the inclusion \(X \subseteq Y\) is continuous, we write \(X \hookrightarrow Y\). We write \(X \overset{d}{\hookrightarrow} Y\), respectively \(X \overset{c}{\hookrightarrow} Y\), if the inclusion is also dense, respectively compact.

\(\mathcal{C}_{[a_1, \ldots, a_d]}\) denotes a nonnegative function of nonnegative variables \(a_j\) that increases in each of its arguments.

Finally, \(C, C_i, i = 0, 1, \ldots\), stand for unessential positive constants.

**2. Stability of coupled evolutionary systems.** We shall present a stability result for a coupled abstract system in a form which is convenient for our purposes.
We shall utilize a separable Hilbert space $H$, which we consider as a pivot space, $H = H'$. We introduce a closed subspace $G \subseteq H$, and two evolutionary systems
\begin{align}
\dot{v} &= -\mathcal{A}v, & v(s_0) &= v_0 \in G, \\
\dot{w} &= -\mathcal{D}w, & w(s_0) &= w_0 \in H,
\end{align}
in the Hilbert spaces $G$ and $H$, where for suitable Hilbert spaces $\mathcal{V} \subset G$ and $\mathcal{W} \subset H$, we suppose that $\mathcal{A}(t) \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ and $\mathcal{D}(t) \in \mathcal{L}(\mathcal{W}, \mathcal{W}')$ are operators in $G$ and $H$, respectively, with domains $D(\mathcal{A}) = D(\mathcal{A}(t)) \subseteq G$ and $D(\mathcal{D}) = D(\mathcal{D}(t)) \subseteq H$, independent of $t$.

Furthermore, we assume that
\begin{align*}
\mathcal{A}(t) &\xrightarrow{d} \mathcal{V} \xrightarrow{d} \mathcal{G} \xrightarrow{d} \mathcal{V}' \xrightarrow{d} D(\mathcal{A})' & \text{and} & \mathcal{D}(t) &\xrightarrow{d} \mathcal{W} \xrightarrow{d} \mathcal{H} \xrightarrow{d} \mathcal{W}' \xrightarrow{d} D(\mathcal{D})'.
\end{align*}

Given linear operators $\mathcal{S} = \mathcal{S}(t) \in \mathcal{L}(H, G)$ and $\mathcal{R} = \mathcal{R}(t) \in \mathcal{L}(G, H)$, we will consider the coupled system
\begin{align}
\dot{v} &= -\mathcal{A}v - \mathcal{S}w, & v(s_0) &= v_0, \tag{15a} \\
\dot{w} &= -\mathcal{D}w - \mathcal{R}v, & w(s_0) &= w_0, \tag{15b}
\end{align}
whose stability properties are the main focus in this section. Namely, we look for a suitable condition, so that the stability of (15) follows from the stability of each of the systems (12) and (13) separately.

Let us set
\[ H := G \times H, \quad \mathcal{V} := \mathcal{V} \times \mathcal{W}, \quad \text{and} \quad \mathcal{M} := \begin{bmatrix} -\mathcal{A} & -\mathcal{S} \\ -\mathcal{R} & -\mathcal{D} \end{bmatrix}. \tag{16} \]

Note that, since $G \subseteq H$ is a closed subspace we may write
\[ H' = H, \quad G' = G, \quad \text{and} \quad \mathcal{H} = \mathcal{H}', \]
and we have
\[ \mathcal{M} \in \mathcal{L}(\mathcal{V}, \mathcal{V}'), \quad D(\mathcal{M}) = D(\mathcal{A}) \times D(\mathcal{D}), \quad D(\mathcal{M}) \xrightarrow{d} \mathcal{V} \xrightarrow{d} \mathcal{H} \xrightarrow{d} \mathcal{V}' \xrightarrow{d} D(\mathcal{M})'. \]

**Assumption 2.1.** For every bounded time interval $I = (s_0, s_1) \subset \mathbb{R}_0$, we have the existence and uniqueness of:

- a $(G, \mathcal{V})$-weak solution $v$ for system (12), for any given $v_0 \in G$,
- a $(H, \mathcal{W})$-weak solution $w$ for system (13), for any given $w_0 \in H$,
- a $(\mathcal{H}, \mathcal{V})$-weak solution $z = (v, w)$ for system (15) for any given $z_0 \in \mathcal{H}$.

Since the existence and uniqueness of solutions is not the focus of this paper, it is simply assumed by us. In the Appendix, we briefly recall the standard procedure on the construction of weak solutions as a weak limit of suitable Galerkin approximations. In particular we will see that the above assumption is satisfied for a general class of operators $\mathcal{A}$, $\mathcal{D}$, $\mathcal{S}$, and $\mathcal{R}$. Applications to the linearized monodomain equations will be given in Sect. 4.

**Assumption 2.2.** The operators $\mathcal{S}$ and $\mathcal{R}$ are essentially bounded:
\[ \mathcal{S} \in L^\infty(\mathbb{R}_0, \mathcal{L}(H, G)) \quad \text{and} \quad \mathcal{R} \in L^\infty(\mathbb{R}_0, \mathcal{L}(G, H)). \]

For the unique solutions of (12) and (13), we introduce the notation $U^{\mathcal{A}}_{(t, s_0)} v_0 := v(t)$ and $U^\mathcal{D}_{(t, s_0)} w_0 := w(t)$, and note that $U^{\mathcal{A}}_{(t, s_0)} v_0 = U^{\mathcal{A}}_{(t, s)} U^{\mathcal{A}}_{(s, s_0)} v_0 = U^{\mathcal{A}}_{(t, s)} v(s)$ for $t \geq s \geq s_0 \geq 0$. 

Assumption 2.3. There are pairs $(C_A, \mu_A)$ and $(C_D, \mu_D)$, both in $[1, +\infty) \times (0, +\infty)$ and independent of $s_0 \geq 0$, so that
\[ |U^{-A}_{(t,s_0)}v_0|_G \leq C_A e^{-\mu_A(t-s_0)} |v_0|_G \text{ and } |U^{-D}_{(t,s_0)}w_0|_H \leq C_D e^{-\mu_D(t-s_0)} |w_0|_H, \] (17)
for all $t \geq s_0 \geq 0$.

Theorem 2.4. Under Assumptions 2.3 and 2.2, and if the inequality
\[ \xi := \frac{C_A C_D \|S\|\|R\|}{\mu_A \mu_D} < 1 \] (18)
holds true, then for all $\varepsilon$ sufficiently small there exists a constant $D_c \geq 1$ such that the weak solution of system (15) satisfies
\[ |(v(t), w(t))|_{G \times H} \leq D_c e^{-\varepsilon(t-s_0)} |(v_0, w_0)|_{G \times H} \] (19)
independently of $(v_0, w_0)$.

The proof is given below. Before we derive some auxiliary results, where Assumptions 2.2 and 2.3 are assumed to hold true.

Lemma 2.5. The weak solution of system (15) satisfies the estimate
\[ |v|_{L^1(\mathbb{R}, L^1(\mathbb{R}, G))} \leq \frac{C_A}{\mu_A} |v_0|_G + \frac{C_A C_D \|S\|}{\mu_A \mu_D} |w_0|_H + \frac{C_A C_D \|S\|\|R\|}{\mu_A \mu_D} |v|_{L^1(\mathbb{R}, L^1(\mathbb{R}, G))}. \] (20)

where $\|S\| := \|S\|_{L^\infty(\mathbb{R}, L^1(\mathbb{R}, G))}$ and $\|R\| := \|R\|_{L^\infty(\mathbb{R}, L^1(\mathbb{R}, G))}$.

Proof. Using Duhamel formula, we integrate the equation in (15b) and obtain, for $t \geq s_0$,
\[ w(t) = U^{-D}_{(t,s_0)}w_0 - \int_{s_0}^{t} U^{-D}_{(t,s)}(R(s)v(s)) \, ds, \]
and
\[ |w(t)|_H \leq C_D e^{-\mu_D(t-s_0)} |w_0|_H + C_D \int_{s_0}^{t} e^{-\mu_D(t-s)} |R(s)v(s)|_H \, ds. \] (21)
Therefore, for $t \geq s_0$, we obtain
\[ |v(t)|_G = \left| U^{-A}_{(t,s_0)}v_0 - \int_{s_0}^{t} U^{-A}_{(t,s)}(S(s)w(s)) \, ds \right|_G \leq C_A e^{-\mu_A(t-s_0)} |v_0|_G + C_A \|S\| \int_{s_0}^{t} e^{-\mu_A(t-s)} |w(s)|_H \, ds \] (22)
\[ \leq C_A e^{-\mu_A(t-s_0)} |v_0|_G + C_A C_D \|S\| \int_{s_0}^{t} e^{-\mu_A(t-s)} e^{-\mu_D(s-s_0)} |w_0|_H \, ds \]
\[ + C_A C_D \|S\|\|R\| \int_{s_0}^{t} e^{-\mu_A(t-s)} \int_{s_0}^{s} e^{-\mu_D(s-\tau)} |v(\tau)|_G \, d\tau \, ds. \] (23)
Let us consider first the case $\mu_A \neq \mu_D$. In this case we obtain
\[ \int_{s_0}^{t} e^{-\mu_A(t-s)} e^{-\mu_D(s-s_0)} |w_0|_H \, ds = |w_0|_H e^{-\mu_A t + \mu_D s_0} \int_{s_0}^{t} e^{(\mu_A - \mu_D)s} \, ds \]
\[ = \frac{|w_0|_H}{\mu_A - \mu_D} \left( e^{(\mu_A - \mu_D)t} - e^{(\mu_A - \mu_D)s_0} \right) \]
\[ = \frac{|w_0|_H}{\mu_A - \mu_D} \left( e^{-\mu_D(t-s_0)} - e^{-\mu_A(t-s_0)} \right) \] (24)
If inequality (26) holds true, then the weak solution of system (15) satisfies
\[ |(v, w)|_{L^\infty(R_0, G \times H)} \leq D_c |(v_0, w_0)|_{G \times H}, \tag{27} \]
for a suitable constant $D_c \geq 1$, independent of $(v_0, w_0)$.

**Proof.** From (20) and (18) we obtain
\[ (1 - \xi)|v|_{L^1(R_0, G)} < \frac{C_A}{\mu_A} |v_0|_G + \frac{C_A C_D \|S\|}{\mu_A \mu_D} \|w_0\|_H. \]
Then, from (21), we arrive at
\[
|w|_{L^\infty(R_0, H)} \leq C_D |w_0|_H + C_D \|R\| \|v|_{L^1(R_0, G)} \leq C_D |w_0|_H + (1 - \xi)^{-1} C_D \|R\| \left( \frac{C_A}{\mu_A} |v_0|_G + \frac{C_A C_D \|S\|}{\mu_A \mu_D} |w_0|_H \right).
\]
Finally, from (22), we derive
\[ |v|_{L^\infty(R_{s_0}, G)} \leq C_A |v_0|_G + \frac{C_A \|S\| \|w\|_{L^\infty(R_{s_0}, H)}}{\mu_A}, \]
which ends the proof. \(\square\)

**Proof of Theorem 2.4.** If \((v, w)\) solves (15), then \((\overline{v}(t), \overline{w}(t)) := e^{\varepsilon(t-s_0)}(v(t), w(t))\) solves
\[
\dot{\overline{v}} = -(A - \varepsilon)\overline{v} - S\overline{w}, \quad \overline{v}(s_0) = v_0, \quad (28a)
\]
\[
\dot{\overline{w}} = -(D - \varepsilon)\overline{w} - R\overline{v}, \quad \overline{w}(s_0) = w_0. \quad (28b)
\]
We also have that \(z\) solves \(\dot{z} = -Az, \quad z(s_0) = z_0\), if, and only if, \(\overline{z}(t) := e^{\varepsilon(t-s_0)}z(t)\) satisfies
\[
\dot{\overline{z}} = -(A - \varepsilon)\overline{z}, \quad \overline{z}(s_0) = z_0, \quad \text{and} \quad |\overline{z}|_G \leq C_A e^{-(\mu_A - \varepsilon)(t-s_0)} |z_0|_G. \]
Similarly \(u\) solves \(\dot{u} = -D u, \quad u(s_0) = u_0\), if, and only if, \(\overline{u}(t) := e^{\varepsilon(t-s_0)}u(t)\) satisfies
\[
\dot{\overline{u}} = -(D - \varepsilon)\overline{u}, \quad \overline{u}(s_0) = u_0, \quad \text{and} \quad |\overline{u}|_H \leq C_D e^{-(\mu_D - \varepsilon)(t-s_0)} |u_0|_H. \]
Now if (18) holds true, then it also holds
\[
\xi_\varepsilon := \frac{C_A C_D \|S\| \|R\|}{(\mu_A - \varepsilon)(\mu_D - \varepsilon)} < 1 \quad (29)
\]
for a small enough \(\varepsilon \in (0, \min\{\mu_A, \mu_D\})\). From Corollary 2.6 it follows that
\[
|\langle \overline{v}, \overline{w} \rangle|_{L^\infty(R_{s_0}, G \times H)} \leq D_c \left( |v_0, w_0|_{G \times H} \right) (30)
\]
for a suitable constant \(D_c \geq 1\) independent of \((v_0, w_0)\), from which we can conclude (19). \(\square\)

Let us now consider a perturbation of system (15) as follows, where \((\eta^\flat, \eta^\sharp) \in L^2(\mathbb{R}_0, G \times H),\)
\[
\dot{v} = -Av - S\eta^\flat + \eta^\flat, \quad v(s_0) = v_0, \quad (31a)
\]
\[
\dot{w} = -Dw - Rv + \eta^\sharp, \quad w(s_0) = w_0. \quad (31b)
\]
where \((\eta^\flat, \eta^\sharp) \in L^\infty(\mathbb{R}_0, G \times H)\) satisfies, for suitable \(C_\eta \geq 1\) and \(\mu_\eta > 0,\)
\[
\left| (\eta^\flat, \eta^\sharp) \right|_{L^\infty(\mathbb{R}_t, G \times H)} \leq C_\eta e^{-\mu_\eta(t-s)} \left| (\eta^\flat, \eta^\sharp) \right|_{L^\infty(\mathbb{R}_s, G \times H)}, \quad t \geq s \geq s_0 \geq 0. \quad (31c)
\]

**Corollary 2.7.** Under the assumptions of Theorem 2.4 the weak solution of system (31) satisfies
\[
|\langle v(t), w(t) \rangle|_{G \times H} \leq D_\eta e^{-\varepsilon_\eta(t-s_0)} \left( \left| \langle v_0, w_0 \rangle \right|_{G \times H} + \left| (\eta^\flat, \eta^\sharp) \right|_{L^\infty(\mathbb{R}_{s_0}, G \times H)} \right), \quad (32)
\]
for all \(t \geq s_0 \geq 0\) and all \(0 < \varepsilon_\eta < \min\{\varepsilon, \mu_\eta\}\), with \(\varepsilon\) satisfying (29), and with \(D_\eta = D_c + \frac{D_c C_\eta}{\varepsilon - \varepsilon_\eta}\) independent of \((s_0, v_0, w_0)\). Here \(D_c\) is as in Theorem 2.4.

**Proof.** With \(\mathcal{M}\) as in (16), by Duhamel formula we find that
\[
|\langle v(t), w(t) \rangle|_{G \times H} \leq \left| U_{(t,s_0)}^{\mathcal{M}}(v_0, w_0) \right|_{G \times H} + \left| \int_{s_0}^{t} U_{(t,s)}^{\mathcal{M}}(\eta^\flat, \eta^\sharp)(s) \, ds \right|_{G \times H}
\]
and, by Theorem 2.4, since \( \varepsilon_{\eta} < \min\{\varepsilon, \mu_{\eta}\} \), and proceeding as in (24),

\[
\begin{align*}
|\langle v(t), w(t) \rangle|_{G \times H} - D_{c}e^{-\epsilon(t-s_{0})}|\langle v_{0}, w_{0} \rangle|_{G \times H} & \leq D_{c}C_{n} \int_{s_{0}}^{t} e^{-\varepsilon(t-s)}e^{-\mu_{\eta}(s-s_{0})}\left|\langle \eta^{\delta}, \eta^{\delta} \rangle\right|_{L^{\infty}(\mathbb{R}_{0}, G \times H)} ds \\
& \leq D_{c}C_{n}\left|\langle \eta^{\delta}, \eta^{\delta} \rangle\right|_{L^{\infty}(\mathbb{R}_{0}, G \times H)} \int_{s_{0}}^{t} e^{-\varepsilon(t-s)}e^{-\varepsilon_{\eta}(s-s_{0})} ds \\
& \leq \frac{D_{c}C_{n}}{\varepsilon - \varepsilon_{\eta}} \left( e^{-\varepsilon_{\eta}(t-s_{0})} - e^{-\varepsilon(t-s_{0})} \right) \left|\langle \eta^{\delta}, \eta^{\delta} \rangle\right|_{L^{\infty}(\mathbb{R}_{0}, G \times H)} \\
& \leq \frac{D_{c}C_{n}}{\varepsilon - \varepsilon_{\eta}} \left( 1 - e^{-(\varepsilon_{\eta}-\varepsilon)(t-s_{0})} \right) e^{-\varepsilon_{\eta}(t-s_{0})} \left|\langle \eta^{\delta}, \eta^{\delta} \rangle\right|_{L^{\infty}(\mathbb{R}_{0}, G \times H)},
\end{align*}
\]

from which we obtain (32), with \( D_{\eta} = D_{c} + \frac{D_{c}C_{n}}{\varepsilon - \varepsilon_{\eta}} \).

3. Stability of the coupled parabolic-ODE closed-loop system. Here we show that the explicit oblique projections based feedback control proposed in [13] for stabilization of nonautonomous linear parabolic equations is also able to stabilize a general class of nonautonomous linear coupled parabolic-ODE systems, where the control acts (only) in the parabolic component. We will prove that the system

\[
\begin{align*}
\dot{y}(t) + Ay(t) + A_{\varepsilon}y(t) + \bar{S}(t)w(t) - K_{U_{M}}(t)(y(t), w(t)) &= 0, \quad y(s_{0}) = y_{0}, \quad (33a) \\
\dot{w}(t) + D(t)w(t) + R(t)y(t) &= 0, \quad w(s_{0}) = w_{0}, \quad (33b)
\end{align*}
\]

with the explicit feedback

\[
(y, w) \mapsto K_{U_{M}}(y, w) := D^{F_{U_{M}}}K_{U_{M}} \left( Ay + A_{\varepsilon}y - \lambda y + \bar{S}w \right), \quad (33c)
\]

is stable, under suitable assumptions on the linear span of the actuators \( U_{M} = \text{span}\{1_{\omega_{1}}, 1_{\omega_{2}}, \ldots, 1_{\omega_{M}}\} \) and on the operators in (33).

3.1. Assumptions. We look at system (33) as an evolutionary system in \( H \times H \), where \( H \) is our pivot separable Hilbert space, \( H' = H \).

First we present our assumptions on the operators in (33). They will guarantee that the assumptions in the abstract setting of Section 2 are fulfilled. Most of the assumptions are standard, and concern the existence and uniqueness of weak solutions for auxiliary systems. The only nonstandard assumption is the main stabilizability condition given below in Assumption 3.7. It can be seen as a generalization of the condition presented in [13] for the case of parabolic equations.

Let \( V \) be another Hilbert space with \( V \subset H \).

Assumption 3.1. \( A \in L(V \to V') \) is symmetric and \( (y, z) \mapsto \langle Ay, z \rangle_{V', V} \) is a complete scalar product in \( V \).

From now we will suppose that \( V \) is endowed with the scalar product \( (y, z)_{V} := \langle Ay, z \rangle_{V'}, V \), which still makes \( V \) a Hilbert space. Necessarily, \( A : V \to V' \) is an isometry.

Assumption 3.2. The inclusion \( V \subseteq H \) is dense, continuous, and compact.

Necessarily, we have that

\[
\langle y, z \rangle_{V', V} = (y, z)_{H}, \quad \text{for all } (y, z) \in H \times V,
\]
and also that the operator $A$ is densely defined in $H$, with domain $D(A)$ satisfying

$$D(A) \xrightarrow{\text{d.c.}} V \xrightarrow{\text{d.c.}} H \xrightarrow{\text{d.c.}} V' \xrightarrow{\text{d.c.}} D(A)'$$.

Further, $A$ has a compact inverse $A^{-1} : H \to D(A)$, and we can find a nondecreasing system of (repeated) eigenvalues $(\alpha_n)_{n \in \mathbb{N}_0}$ and a corresponding complete basis of eigenfunctions $(e_n)_{n \in \mathbb{N}_0}$:

$$0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \alpha_{n+1} \to +\infty \quad \text{and} \quad Ae_n = \alpha_n e_n.$$  

We can define, for every $\beta \in \mathbb{R}$, the fractional powers $A^\beta$, of $A$, by

$$A^\beta \sum_{n=1}^{+\infty} y_n e_n := \sum_{n=1}^{+\infty} \alpha_n^\beta y_n e_n,$$

and the corresponding domains $D(A^{|\beta|}) := \{ y \in H \mid A^{|\beta|} y \in H \}$, and $D(A^{-|\beta|}) := D(A^{|\beta|})'$. We have that $D(A^\beta) \xrightarrow{\text{d.c.}} D(A^{\beta_1})$, for all $\beta > \beta_1$, and we can see that $D(A^0) = H$, $D(A^1) = D(A)$, $D(A^2) = V$.

For the time-dependent operators we assume the following:

**Assumption 3.3.** For almost every $t > 0$ we have $A_t(t) \in \mathcal{L}(H, V')$, and we have a uniform bound, that is, $A_t \in \mathcal{L}\left(\mathbb{R}_0, \mathcal{L}(H, V')\right)$.

**Assumption 3.4.** We have $\mathcal{W} \xrightarrow{d} H \xrightarrow{d} \mathcal{W}'$, and $\mathcal{D} \in \mathcal{L}(\mathcal{W}, \mathcal{W}')$ as in (14). For any $s_0 \geq 0$ and $w_0 \in H$ there is one, and only one, $(H, \mathcal{W})$-weak solution for

$$\dot{w}(t) + \mathcal{D}(t)w(t) = 0, \quad w(s_0) = w_0,$$

and there are constants $C_D \geq 1$ and $\mu_D > 0$, independent of $(s_0, w_0)$, such that the solution $w$ satisfies $|w(t)|_H \leq C_D e^{-\mu_D(t-s)}|w(s)|_H$, for all $t \geq s \geq s_0 \geq 0$.

**Assumption 3.5.** The coupling operators $\tilde{S}$ and $\mathcal{R}$ are both in $\mathcal{L}\left(\mathbb{R}_0, \mathcal{L}(H)\right)$.

**Assumption 3.6.** There exists one, and only one, solution $(v, w) \in W_{\text{loc}}(\mathbb{R}_0, V \times \mathcal{W}, V' \times \mathcal{W}')$ for the system

$$\begin{align*}
\dot{v} + P_{M}^{U_M} Av + P_{M}^{U_M} A_t v + P_{M}^{U_M} \tilde{S} w &= 0, \quad v(s_0) = v_0 \in E_M^L, \\
\dot{w} + \mathcal{D} w + \mathcal{R} v &= 0, \quad w(s_0) = w_0 \in H,
\end{align*}$$

(34a)  

(34b)

Note that in case $K_{U_M}(t) = 0$, we have that (33) is in the form of system (15), with $G = H$ and $\mathcal{V} = V$. Note also that (34) is in the form of system (15), with $G = E_M^L = P_{E_M^L} H$ and $\mathcal{V} = E_M^L \cap V$. We suppose that $E_M^L \subset H$ and $E_M \subset H$ are endowed with the norm inherited from $H$, and we denote (cf. Lemma 2.5)

$$
\| P_{E_M}^{U_M} \tilde{S} \|_{\mathcal{L}\left(\mathbb{R}_0, \mathcal{L}(H)\right)} \quad \text{and} \quad \| \mathcal{R} \| := |\mathcal{R}|_{\mathcal{L}\left(\mathbb{R}_0, \mathcal{L}(H)\right)}.
$$

**Assumption 3.7.** The linear span $U_M$ of the actuators satisfies

$$H = U_M \oplus E_M^L,$$

(35a)

$$\alpha_{M+1} > \inf_{\substack{y \in \mathbb{R}_2, \\ 2 > \gamma_1 + \gamma_2}} \left( \frac{\gamma_1^{-1} \Xi_1 + \gamma_2^{-1} \left( 2 + 2 \| P_{E_M}^{U_M} \|_{\mathcal{L}(H)}^2 \Xi_2 \right)}{(2 - \gamma_1 - \gamma_2)} + \frac{C_D P_{E_M}^{U_M} \Xi_1 \| \mathcal{R} \|}{(2 - \gamma_1 - \gamma_2) \mu_D} \right),$$

(35b)
where
\[
\Xi_1 := \sup_{(t,\tilde{y}) \in \mathbb{R}_0 \times (E^\perp_{\tilde{M}} \cap V)} \frac{|\langle A(t)\tilde{y}, \tilde{y} \rangle_{V',V}|^2}{|\tilde{y}|^2_{V'} |\tilde{y}|^2_V} \leq \left| P_{E^\perp_{\tilde{M}}} A_t P_{E^\perp_{\tilde{M}}} \right|^2_{L^\infty(\mathbb{R}_0, \mathcal{L}(H,V'))},
\]
(36a)
\[
\Xi_2 := \left| P_{E^\perp_{\tilde{M}}} A_t P_{E^\perp_{\tilde{M}}} \right|^2_{L^\infty(\mathbb{R}_0, \mathcal{L}(H,V'))}.
\]
(36b)

**Remark 3.8.** Note that (7b) is a particular case of (35b), with \( \gamma_1 = \gamma_2 = \frac{1}{2} \).

**Remark 3.9.** Recall that from [13] (by taking \( y_0 \in E^\perp_{\tilde{M}} \) in [13, Theorem 3.6]) we know that when \( \tilde{S} = 0 \), then the system \( \dot{v} + P_{E^\perp_{\tilde{M}}} A v + P_{E^\perp_{\tilde{M}}} A_t v = 0 \) is stable provided (35) holds true, with \( R = 0 \). In particular we observe that Assumptions 2.1 and 2.2, and estimates (17) hold true for system (15) with \( A = A + P_{E^\perp_{\tilde{M}}} A_t \) and \( S = P_{E^\perp_{\tilde{M}}} \tilde{S} \), that is, they hold true for system (34).

### 3.2. The stability result.

The main result of this section, which will lead to main Theorems 1.1 and 1.2, is the following.

**Theorem 3.10.** Under Assumptions 3.1–3.7, the coupled system (33) is stable: there are constants \( C_c \geq 1 \) and \( \mu_c > 0 \) such that
\[
|\langle y(t), w(t) \rangle|_{H \times H} \leq C_c e^{-\mu_c (t-s_0)} |\langle y_0, w_0 \rangle|_{H \times H}, \quad t \geq s_0.
\]
with \((C_c, \mu_c)\) independent of \((s_0, y_0, w_0)\).

The proof is presented below, in section 3.2.2.

#### 3.2.1. Auxiliary results.

Here we derive some auxiliary results we will use in the proof of Theorem 3.10. We suppose that the Assumptions 3.1–3.7 do hold true.

Notice that \( G = E^\perp_M \) is a closed subspace of \( H \). Let us set \( V = G \cap \tilde{V} \), which we endow with the norm inherited from \( V \). Observe that \( V \) is a closed subspace of \( V \), and \( A \) maps \( V \) onto \( \tilde{V} \).

**Lemma 3.11.** There exists a unique \((E^\perp_{\tilde{M}}, V \cap E^\perp_{\tilde{M}})\)-weak solution for the system
\[
\dot{v} + P_{E^\perp_{\tilde{M}}} A v + P_{E^\perp_{\tilde{M}}} A_t v = 0, \quad v(s_0) = v_0 \in E^\perp_{\tilde{M}}.
\]
(37)

Moreover, we have
\[
|v(t)|^2_H \leq e^{-\mu (t-s_0)} |v_0|^2_H, \quad \text{for a suitable } \mu > \frac{C_D \| P_{E^\perp_{\tilde{M}}} \tilde{S} \| \| R \|}{\mu_D}.
\]
(38)

**Proof.** The existence and uniqueness is proven in [13, Section 3.1]. Note that we have \( P_{E^\perp_{\tilde{M}}} A v = A P_{E^\perp_{\tilde{M}}} v = Av \) for any \( v \in E^\perp_{\tilde{M}} \). Now, proceeding as in [13, Section 3.1], by multiplying the equation by \( 2v \), we obtain
\[
\frac{d}{dt} |v|^2_H = -2 |v|^2_V - 2 \langle P_{E^\perp_{\tilde{M}}} A_t v, v \rangle_{V',V}
\]
\[
= -2 |v|^2_V - 2 \langle P_{E^\perp_{\tilde{M}}} A_t v, v \rangle_{V',V} - 2 \langle P_{E^\perp_{\tilde{M}}} P_{E^\perp_{\tilde{M}}} A_t v, v \rangle_{V',V}
\]
and for any given positive constants \( \gamma_1 \) and \( \gamma_2 \),
\[
\frac{d}{dt} |v|^2_H \leq -(2 - \gamma_1 - \gamma_2) |v|^2_V + \gamma_1^{-1} |v|^2_H + \gamma_2^{-1} \left| P_{E^\perp_{\tilde{M}}} P_{E^\perp_{\tilde{M}}} \right|_{L(V',V)}^2 |v|^2_H
\]
with $\Xi_1$ and $\Xi_2$ as in (36). Now from $|v|_H^2 \geq \alpha_{M+1} |v|_H^2$, because the solution $v$ takes its values in $E_M^\perp$, we obtain
\[
\frac{d}{dt} |v|_H^2 \leq - \left( (2 - \gamma_1 - \gamma_2) \alpha_{M+1} - \gamma_1 \gamma_2 \right) \Xi_1 \Xi_2 - \gamma_1 \gamma_2 \left| P_{E_M^\perp} P_{EM} \right|_{L(V')}^2 |v|_H^2, \tag{39}
\]
and from (35b), it follows that we can choose $\gamma_1$ and $\gamma_2$ such that
\[
\mu := (2 - \gamma_1 - \gamma_2) \alpha_{M+1} - \gamma_1 \gamma_2 \Xi_1 \Xi_2 - \gamma_1 \gamma_2 \left| P_{E_M^\perp} P_{EM} \right|_{L(V')}^2 > \frac{C_D \left\| P_{E_M^\perp} S \right\|_{\mathcal{L}(R)}}{\mu_D},
\]
which implies (38).

**Corollary 3.12.** Given $q \in H^1(\mathbb{R}, E_M)$ and $(v_0, w_0) \in E_M^\perp \times H$, there exists a unique solution for system
\[
\dot{v} + P_{E_M^\perp} A(v + q) + P_{E_M^\perp} A(v + q) + P_{E_M^\perp} \tilde{S}w + P_{E_M^\perp} \dot{q} = 0, \quad v(s_0) = v_0, \quad w(s_0) = w_0.
\]

If there is a pair of constants $C_q \geq 1$ and $\lambda_q > 0$ such that
\[
\|q, \dot{q}\|_{L^\infty(\mathbb{R}, H \times H)} \leq C_q e^{\lambda_q (t-s)} \|q, \dot{q}\|_{L^\infty(\mathbb{R}, H \times H)}, \quad \text{for all} \quad t \geq s \geq s_0 \geq 0,
\]
then there are constants $D_q \geq 1$ and $\varepsilon_q \in (0, \min\{\mu, \mu_D, \lambda_q\})$, such that the solution satisfies
\[
\|v(t), w(t)\|_{H \times H} \leq D_q e^{-\varepsilon_q (t-s_0)} \left( \|v_0, w_0\|_{H \times H} + \|q, \dot{q}\|_{L^\infty(\mathbb{R}, H \times H)} \right). \tag{41}
\]
Furthermore, the constants $D_q$ and $\varepsilon_q$ are independent of $(s_0, v_0, w_0)$. The constants $\mu$ and $\mu_D$ are as in (38) and Assumption 3.4.

**Proof.** When $q = 0$, the existence and uniqueness of the solution, in the space $W_{loc}(\mathbb{R}, E_M^\perp \cap V \times W_h(\mathbb{R}, E_M^\perp \cap V') \times W_h')$, is given by Assumption 3.6. Setting $A = P_{E_M^\perp} A + P_{E_M^\perp} A$, from Lemma 3.11 we have that the solution of
\[
\dot{v} + A v = 0, \quad v(s_0) = v_0 \in E_M^\perp
\]
satisfies $|v(t)|_{E_M^\perp} \leq C_A e^{-\mu t} |v_0|_{E_M^\perp}$ with $(C_A, \mu_A) = (1, \mu)$, where $\mu$ is as in (38). Then, we observe that for small enough $\varepsilon \in (0, \min\{\mu_A, \mu_D\})$ we have
\[
\varepsilon_q = \frac{C_A \mu_D \|P_{E_M^\perp} S\|_{\mathcal{L}(H)}}{(\mu_A - \varepsilon)(\mu_D - \varepsilon)} < 1.
\]
Observe also that system (40) is in the form of system (31), by setting $\eta^A = R q$ and $\eta^A = P_{E_M^\perp} A q + P_{E_M^\perp} A q + P_{E_M^\perp} \dot{q}$. Furthermore we find
\[
|\eta^A|^2_{L^\infty(\mathbb{R}, H)} \leq \|R\| \|q\|_{L^\infty(\mathbb{R}, H)} \leq \|R\| C_q e^{\lambda_q (t-s)} \|q, \dot{q}\|_{L^\infty(\mathbb{R}, H \times H)}; \\
|\eta^A|^2_{L^\infty(\mathbb{R}, H)} \leq \left| P_{E_M^\perp} A + P_{E_M^\perp} A \right|_{L(E_M, H)} \|q\|_{L^\infty(\mathbb{R}, H)} + \left| P_{E_M^\perp} \right|_{L(E_M, H)} \|\dot{q}\|_{L^\infty(\mathbb{R}, H)}; \\
C_1 C_q e^{\lambda_q (t-s)} \|q, \dot{q}\|_{L^\infty(\mathbb{R}, H \times H)},
\]
for a suitable constant $C_1 \geq 1$. Recall that $E_M \subset D(A) \subset H$ is a finite-dimensional space. Then from Corollary 2.7, it follows (41), for all $\varepsilon_q \in (0, \min\{\varepsilon, \lambda_q\})$ and a suitable constant $D_q \geq 1$. 

3.2.2. Proof of Theorem 3.10. Let us set \( q_0 := P_{E_M}y_0, v_0 := P_{E_M^+}y_0, \) and \( q(t) := e^{-\lambda(t-s_0)}q_0. \) Then we also set the solution \((v, w)\) for system (40), given by Corollary 3.12, with the initial condition \((v, w)(s_0) = (v_0, w_0)\). We start by showing that \((y, w) := (q + v, w)\) solves (33). Indeed, with \( y = q + v, \) we can rewrite (40a) as
\[
0 = \dot{y} - \dot{q} + P_{E_M^+}Ay + P_{E_M^+}^{\perp}A_v y + P_{E_M^+}^{\perp} \tilde{S}w + P_{E_M^+}^{\perp} q = \dot{y} + A y + A_y y + \tilde{S}w + \tilde{P}_{E_M} q
\]
that is,
\[
0 = \dot{y} + A y + A_y y + \tilde{S}w - P_{E_M}^{\perp}Ay - P_{E_M}^{\perp}A_v y - P_{E_M}^{\perp} \tilde{S}w - P_{E_M}^{\perp} q = \dot{y} + A y + \tilde{S}w - \lambda y,
\]
because \( \dot{q} = -\lambda q \) and \( P_{E_M}^{\perp} q = P_{E_M}^{\perp} P_{E_M} y = P_{E_M}^{\perp} y. \) Therefore, we conclude that \((y, w)\) solves (33), with \( K \) as in (33c). Finally, we notice that \((v(t), w(t))\) is orthogonal to \((q(t), 0)\) in \( H \times H, \) because \( v(t) \in E_M^+ \) is orthogonal to \( q(t) \in E_M \) in \( H. \) Therefore \( ||(y(t), w(t))||_{H \times H} = ||(v(t), w(t))||_{H \times H} + ||(q(t), 0)||_{H \times H} \) and, using (41) in Corollary 3.12, we arrive at
\[
|| (y(t), w(t)) ||_{H \times H} \leq D_q e^{-\varepsilon_q(t-s_0)} \left( ||(v_0, w_0)||_{H \times H} + ||(q_0)||_{L^\infty(\mathbb{R}, H) \times \mathbb{R})} + e^{-\lambda(t-s_0)} |q_0|_H \right)
\]
\[
\leq D_q e^{-\varepsilon_q(t-s_0)} \left( ||(v_0, w_0)||_{H \times H} + (1 + \lambda^2) \frac{1}{2} |q_0|_H \right) + e^{-\lambda(t-s_0)} |q_0|_H
\]
with \( \varepsilon_q < \lambda. \) Hence, Theorem 3.10 follows with \( \mu_c = \varepsilon_q \) and \( C_c = 1 + D_q(1 + \lambda^2)^\frac{1}{2}. \)

4. Applications. Stabilization of the monodomain equations. Assume that the desired trajectory \((\overline{v}, \overline{w})\) is given as the solution of the uncontrolled system (1). As mentioned in the Introduction, the difference \((y, z) := (v, w) - (\overline{v}, \overline{w})\) satisfies system (10), with the operators defined as in (6). In this section we prove Theorems 1.1 and 1.2 which concern the stabilization of systems (9) and (10), respectively.

4.1. Proof of main Theorem 1.1. Theorem 1.1 will follow from Theorem 3.10. For this purpose we discuss Assumptions 3.1–3.7, for the choice
\[
H = L^2(\Omega), \quad V = H^1(\Omega), \quad \text{and} \quad D(A) = \left\{ h \in H^2(\Omega) \mid (\frac{\partial}{\partial n} h)_{\Gamma} = 0 \right\}.
\]
Assumptions 3.1 and 3.2 are satisfied and due to (8) Assumptions 3.3 and 3.5 hold true. It is easy to see that Assumption 3.4 holds with \( \mathcal{W} = H \) and \( (C_D, \mu_D) = (1, \delta). \)

Proposition 4.1. If (8) and Assumption 3.7 are satisfied, then Assumption 3.6 holds.

The proof follows from standard arguments. A few details are given in the Appendix. Thus, under condition (8) and Assumption 3.7, all the assumptions in Section 3.1 are satisfied. By Theorem 3.10 this implies the following Corollary, which in turn implies Theorem 1.1. From Remark 3.8 we recall that condition (7), which is assumed in Theorem 1.1, implies that Assumption 3.7 is satisfied.

Corollary 4.2. Let \( \lambda > 0 \) and \( s_0 \geq 0. \) If (8) and Assumption 3.7 hold true, then there are constants \( C \geq 1 \) and \( \mu > 0, \) such that the solution of the linear system (9) satisfies
\[
||(y(t), z(t))||_{H \times H} \leq Ce^{-\mu(t-s_0)} ||(y_0, z_0)||_{H \times H}, \quad t \geq s_0,
\]
for all \((y_0, z_0) \in H \times H. \) Here the constants \( C \) and \( \mu \) are independent of the triple \((s_0, y_0, z_0).\)
4.2. Proof of main Theorem 1.2. To deal with the nonlinear systems, we will need strong solutions. We can prove that such solutions exist due to the fact that the reaction operator \( A_r \) defined as in (6) satisfies

\[
A_r \in L^\infty(\mathbb{R}_0, \mathcal{L}(V, H)),
\]

if (8) holds.

Proposition 4.3. Let (8) and Assumption 3.7 be satisfied, let \( \mu \) be as in Corollary 4.2, and \((y_0, z_0) \in V \times H\), with \( s_0 \geq 0 \). Then the solution for (9) is strong, that is, \((y, z) \in W^{\text{loc}}(\mathbb{R}_{s_0}, D(A) \times H, H \times H)\). Moreover, we have

\[
\sup_{s \geq s_0} \left| e^{\mu t(s-s_0)} (y, z) \right|_{W((s,s+1), D(A) \times H, H \times H)} \leq C |(y_0, z_0)|_V H.
\]

The proof, which relies in part on known arguments, is sketched in the Appendix.

The next lemma gathers estimates on our nonlinearity \( \mathcal{N} = \mathcal{N}(y) = \mathcal{N}'(t, y) \). Let us write, for simplicity,

\[
H = H \times H, \quad V = V \times H, \quad \text{and} \quad D(A) = D(A) \times H.
\]

Lemma 4.4. There exists a constant \( \tilde{C}_1 \geq 0 \) such that for all pairs \( p = (y, z) \) and \( \tilde{p} = (\tilde{y}, \tilde{z}) \) in \( D(A) \times H \),

\[
|\mathcal{N}(p) - \mathcal{N}(\tilde{p})|_H^2 \leq \tilde{C}_1 |p - \tilde{p}|_V^2 (1 + |p|_V^2 + |\tilde{p}|_V^2) \left( |p|_{D(A)}^2 + |\tilde{p}|_{D(A)}^2 \right) + \tilde{C}_1 |p - \tilde{p}|_{D(A)}^2 (|p|_V^2 + |\tilde{p}|_V^2).
\]

Further for any given \( \zeta > 0 \) there exists \( \tilde{C}_2 \geq 0 \) such that

\[
(n(p) - n(\tilde{p}), p - \tilde{p})_H \leq \zeta |p - \tilde{p}|_V^2 + \tilde{C}_2 (1 + |p|_V^2 + |\tilde{p}|_V^2) (1 + |p|_{D(A)}^2 + |\tilde{p}|_{D(A)}^2) |p - \tilde{p}|_H^2,
\]

with

\[
\{\varepsilon_1, \varepsilon_2\} \in [0, +\infty) \quad \text{and} \quad \{\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\} \in [2, +\infty).
\]

Proof. Recall that from (6), we have \( \mathcal{N}(y, z) = (-ay^3 + (b - 3a\mu)y^2 - eyz, -\rho y^2) \), which we write as

\[
\mathcal{N} = \sum_{j=1}^{4} G_j, \quad \text{with} \quad \begin{cases} 
G_1(y, z) := (-ay^3, 0), & G_2(y, z) := ((b - 3a\mu)y^2, 0), \\
G_3(y, z) := (0, -\rho y^2), & G_4(y, z) := (eyz, 0).
\end{cases}
\]

We will argue that (43) holds with \( \mathcal{N} \) replaced by \( G_j \) for \( j = 1, \ldots, 4 \). This implies that (43) holds for \( \mathcal{N} = \sum_{j=1}^{4} G_j \). From [19, Section 5, Examples 1 and 2], we can conclude that (43) is satisfied with \( \mathcal{N} \) replaced by \( G_1 \) and \( G_2 \) involving only the PDE component \( y \) of the solution and the PDE coordinate of the nonlinearity. Recall that the component \( y \) of the solution lives in \( D(A) \), for (almost) every time \( t \geq 0 \). This regularity is not necessarily satisfied by the ODE component \( z \), so we must check the details with the monomial term \( yz \) in \( G_4 \). On the other hand \( G_3 \) involves the ODE component of the nonlinearity, so also in this case we will check the details.

Concerning \( G_4 \) we proceed as follows: estimate (43) is trivially satisfied for \( e = 0 \), thus we consider only the case \( e \neq 0 \), then with \( (\zeta_y, \zeta_z) := (y - \tilde{y}, z - \tilde{z}) \in D(A) \times H \), we find

\[
\frac{1}{2t} |G_4(p) - G_4(\tilde{p})|_H^2 = |yz - \tilde{y}\tilde{z}|_H^2 \leq 2 (|\zeta_y|_H^2 + |\zeta_z|_H^2).
\]
second we employ the Sobolev embedding $H^{\frac{3}{2}} \hookrightarrow L^\infty$ \cite[Lemma 13.2]{1} in the first one. For the second we employ the Agmon embedding $H^\frac{3}{2}(\Omega) \hookrightarrow L^1(\Omega)$ \cite[Theorem 4.57]{6}, the interpolation inequality $|h|_{H^\frac{3}{2}(\Omega)} \leq C |h|_{L^2(\Omega)}^\frac{1}{2} |h|_{H^1(\Omega)}^\frac{1}{2}$ \cite[Proposition 2.3 and Theorem 9.6]{16}, and the Young inequality $ab = (\frac{1}{\varsigma} a) (\varsigma b) \leq \frac{1}{4} (\frac{1}{\varsigma} a)^4 + \frac{3}{4} (\varsigma b)^4$ \cite{25}, with $(a, b, \varsigma) \in \mathbb{R}^3$. We obtain

$$\frac{1}{\varsigma} \| G_4(p) - G_4(\tilde{p}) \|_{H}^2 \leq C_1 \left( |\zeta|_{D(A)}^2 |z|_H^2 + |\zeta|_{D(A)}^2 |\tilde{y}|_D(A) \right),$$

$$\frac{1}{\varsigma} \| G_4(p) - G_4(\tilde{p}) \|_{H}^2 \leq C_2 \left( |p - \tilde{p}|_{D(A)}^2 |p|_H^2 + |p - \tilde{p}|_{D(A)}^2 |\tilde{p}|_D(A) \right),$$

for any given $\varsigma > 0$, and for suitable constants $C_1 > 0$, $C_2 > 0$ and $C_3 = C_3(\varsigma) > 0$. Therefore we can conclude that the inequalities in (43) also hold true for $G_3$.

Finally, concerning $G_3$ we proceed as follows: First

$$\| G_3(p) - G_3(\tilde{p}) \|_{H}^2 \leq |y - \tilde{y}|_H^2 |y + \tilde{y}|_\infty \leq C_1 (|p - \tilde{p}|_H^2 (|p|_{D(A)}^2 + |\tilde{p}|_{D(A)}^2),$$

for $C_1 > 0$ independent of $p$ and $\tilde{p}$ in $D(A)$, second we find that

$$\frac{1}{\varsigma} \| G_3(p) - G_3(\tilde{p}) \|_{H}^2 \leq (y^2 - \tilde{y}^2, z - \tilde{z})_H = (\zeta y + \tilde{y}, \zeta z)_H \leq |y + \tilde{y}|_\infty |\zeta y|_L^2 \leq C_1 |y + \tilde{y}|_{D(A)} |p - \tilde{p}|_H^2 \leq C_1 (1 + |y|_{D(A)}^2 + |\tilde{y}|_{D(A)}^2) |p - \tilde{p}|_H^2.$$

Thus (43) holds true also for $\mathcal{N}$ replaced by $G_3$. 

\begin{lemma}
If (8) holds, then the feedback operator in (33c) is bounded: $\mathcal{K}_{U_M} \in L^\infty(\mathbb{R}_0, \mathcal{L}(H \times H, H))$.
\end{lemma}

\begin{proof}
Indeed, proceeding as in \cite[Proof of Theorem 3.7]{13} we find

$$\| \mathcal{K}_{U_M} \|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H \times H, H))} \leq \left| P_{F_M^T} P_{F_M} \right|_{L^\infty(V', H)} (\alpha_3^{\frac{1}{2}} + |A_e|_{L^\infty(\mathbb{R}_0, \mathcal{L}(V', H'))} + \left| \tilde{S} \right|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H, V'))} + \lambda \alpha_1^{-\frac{1}{2}})$$

and from (8) and (6) we have $A_e \in L^\infty(\mathbb{R}_0, \mathcal{L}(H, V'))$ and $\tilde{S} \in L^\infty(\mathbb{R}_0, \mathcal{L}(H))$. 
\end{proof}

\begin{remark}
We are now prepared to provide the proof for Theorem 1.2, which relies on fixed point arguments. With two modifications, we can rely on the arguments utilized in \cite[Section 3.2]{5} (see also \cite[Section 3]{19}). First, the feedback operator in these references is obtained as the solution of a suitable differential Riccati equation, in place of $\mathcal{K}_{U_M}$ as in (33c). However, thanks to the estimates in
\end{remark}
Corollary 4.2 and Proposition 4.3 we can replace the Riccati-based feedback operator in [5, Section 3.2] by $K_{U^M}$. Second the conditions on the nonlinearity $N$ in [19, Section 3] are slightly stronger than those available here.

In fact, in [19, Section 3], instead of property (43b), the following stronger assumption is assumed:

$$
\langle G_f (p) - G_f (\hat{p}), p - \hat{p} \rangle_H 
\leq C_2 (1 + |p|^2 + |\hat{p}|^2) \left( 1 + |p|^2_{D(A)} + |\hat{p}|^2_{D(A)} \right)^{3/2} |p - \hat{p}|_{V} |p - \hat{p}|_H 
+ \tilde{C}_2 (1 + |p|^2 + |\hat{p}|^2_{D(A)} (1 + |p|^2_{D(A)} + |\hat{p}|^2_{D(A)}) |p - \hat{p}|^2_H .
$$

(44)

It is clear that that the last estimate implies (43b), because $2 |p - \hat{p}|_V |p - \hat{p}|_H \leq \beta^{-1} |p - \hat{p}|_H + \beta |p - \hat{p}|^2_V$, for arbitrary $\beta > 0$.

Since Assumption 3.7 is implied by (7), Theorem 1.2 follows from the following.

**Theorem 4.7.** Let (8) and Assumption 3.7 hold true. Then there are constants $C \geq 1$ and $\epsilon > 0$, independent of $(s_0, y_0, z_0)$, that the solution of the system (10) satisfies

$$
|(y(t), z(t))|_V \leq C e^{-\mu (t-s_0)} |(y_0, z_0)|_V, \quad t \geq s_0,
$$

(45)

provided that $(y_0, z_0)|_V < \epsilon$, where $\mu > 0$ is as in Corollary 4.2.

**Proof.** We follow the main argument sketched in [19, Section 3] and [3, Section 4]. Let us consider the system

$$
\dot{y} = -Ay - Ay - \tilde{S}z + K_{U^M} (y, z) + g_1, \quad y(s_0) = y_0 \in V, \quad (46a)
$$

$$
\dot{z} = -\delta z - R y + g_2, \quad z(s_0) = z_0 \in H. \quad (46b)
$$

with $g \in L^2((s_0, s_0 + T), H)$. Multiplying the dynamics by $Lz$, with $L = \begin{bmatrix} A & 0 \\ 0 & \delta \end{bmatrix}$ and $D(L) = D(A)$, gives us

$$
\langle (y(t), z(t)), (y, z) \rangle_V + \langle (y(t), z(t)), g \rangle_{L^2((s_0, s_0 + T), D(A))} 
\leq D_3(T) \left( \langle (y(t), z(t)), (y, z) \rangle_V + \langle g \rangle_{L^2((s_0, s_0 + T), H)}^2 \right), \quad (47)
$$

with the constant $D_3(T)$ depending on $|A_r|_{L^\infty (R_0, C(H, H))}$, $|K_{U^M}|_{L^\infty (R_0, C(H, H))}$, and $T$, and independent of $s_0$ and $(y(t), z(t))$. Here we use Lemma 4.5.

Next, we write (46) as

$$
\nu + L \nu + C \nu - \tilde{S} \nu = g, \quad \nu(s_0) = \nu_0 \in V
$$

(48)

with $\nu_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$, $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$, $C = \begin{bmatrix} A_r & \tilde{S} \\ 0 & R \end{bmatrix}$, and $\tilde{S} = \begin{bmatrix} \tilde{S}(1, 1) & \tilde{S}(1, 2) \\ \tilde{S}(1, 1) & \tilde{S}(1, 2) \end{bmatrix}$. In particular, notice that $|\tilde{S}|_{L^\infty (R_0, C(H, H))} = |K_{U^M}|_{L^\infty (R_0, C(H, H))}$. Here $\tilde{S}(1, 1)y := K_{U^M}(y, 0)$ and $\tilde{S}(1, 2)z := K_{U^M}(0, z)$.

We will look for a solution of the nonlinear system in a subset $Z^\mu_0 \subset Z^\mu$ of the Banach space

$$
Z^\mu := \left\{ \nu \in L^2_{loc} (R_0, H) \mid \nu \right|_{Z^\nu} < \infty \right\}
$$

endowed with the norm $|\nu|_{Z^\nu} := \sup_{r \geq s_0} |e^{\mu (r-s_0)} \nu|_{W((r,r+1), D(A), H)}$. We also set

$$
Z_{loc}^\mu := \left\{ \nu \in L^2_{loc} (R_0, H) \mid \|e^{\mu (r-s_0)} \nu\|_{W((r,r+1), D(A), H)} < \infty, \text{ for all } r \geq s_0 \right\}.
$$
For a given constant \( \varrho > 0 \) we define the subset \( \mathcal{Z}_\varrho^\mu \) as follows.

\[
\mathcal{Z}_\varrho^\mu := \{ \mathbf{v} \in \mathcal{Z}^\mu \mid \|\mathbf{v}\|_{\mathcal{W}(\Omega)}^2 \leq \varrho \|\mathbf{v}_0\|_{\mathcal{W}(\Omega)}^2 \}.
\]

Further we define the mapping \( \Psi : \mathcal{Z}_\varrho^\mu \to \mathcal{Z}_\varrho^\mu \), \( \hat{\mathbf{v}} \mapsto \mathbf{v} \), taking a given vector \( \hat{\mathbf{v}} \) to the solution \( \mathbf{v} \) of

\[
\hat{\mathbf{v}} + \mathbf{L}\mathbf{v} + \mathbf{C}\mathbf{v} - \mathbf{f}\mathbf{v} = \mathbf{N} (\hat{\mathbf{v}}), \quad \mathbf{v}(s_0) = \mathbf{v}_0.
\]

5. Numerical simulations. Here we present simulation results for (9) and (10). As actuators we will take suitable (piecewise constant) indicator functions \( \{1_{\omega_i} \mid i \in \{1, 2, \ldots, M\} \} \subset \mathcal{H} \). In this section \( \mathcal{H} := L^2(\Omega) \). In Figure 1 we can see the placement of (rectangular) actuators’ regions. Recall also that the possibly repeated eigenvalues \( \alpha_i \) of the Laplacian, under Neumann boundary conditions increase likely linearly, as we would expect from Weyl asymptotic formula [24]. See also [12, 10].

The following simulations were carried out using a finite elements based spatial approximation of our equations, with a Crank–Nicolson time discretization.

We will set the parameters in the FitzHugh–Nagumo, Rogers–McCulloch, and Aliev–Panfilov models so that we will have 3 constant steady states \( \{\tilde{v}_i, \tilde{w}_i\}, i \in \{1, 2, 3\} \) and so that the first (pde) components are \( \tilde{v}_1 = 0, \tilde{v}_2 = 1, \) and \( \tilde{v}_3 = 2 \). As we will see:
where \( \lfloor T \rfloor \in \mathbb{N} \). Consider the time-dependent vector function \( \mathbf{v}(t) \) and the second one is spatially and temporally dependent and is defined as follows: we define our reference trajectory \( \mathbf{v}_m(t,x) \) as in (2)-[FN] or [RM]:

\[
(v_1, \hat{w}_1) = (0, 0), \quad (v_2, \hat{w}_2) = (1, 10), \quad (v_3, \hat{w}_3) = (2, 20). \tag{52a}
\]

— for \( \mathbb{W} \) as in (2)-[AP]:

\[
(v_1, \hat{w}_1) = (0, 0), \quad (v_2, \hat{w}_2) = (1, 5), \quad (v_3, \hat{w}_3) = (2, 0). \tag{52b}
\]

We will argue that for all 3 models:

- the linearizations around the steady state \( (\hat{v}_1, \hat{w}_1) \) and around the steady state \( (\hat{v}_3, \hat{w}_3) \) are stable,
- the linearization around the steady state \( (\hat{v}_2, \hat{w}_2) \) is unstable.

We will consider the systems (9) and (10) with two choices as reference trajectories. The first one is the unstable steady state

\[
(\mathbf{v}(t), \mathbf{w}(t)) = (\hat{v}_2, \hat{w}_2), \tag{53a}
\]

the second one is spatially and temporally dependent and is defined as follows: we consider the time-dependent vector function \( \mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2) : \mathbb{R}_0 \to \mathbb{R}^2 \),

\[
\mathcal{T}_1(t) = \begin{cases} 
\hat{v}_2 + (t - 2 [\frac{1}{2}]) (\hat{v}_3 - \hat{v}_2) & \text{if } 0 \leq t - 2 [\frac{1}{2}] \leq 1, \\
\hat{v}_3 + (t - 2 [\frac{1}{2}]) - 1 (\hat{v}_2 - \hat{v}_3) & \text{if } 1 < t - 2 [\frac{1}{2}] \leq 2,
\end{cases}
\]

\[
\mathcal{T}_2(t) = \begin{cases} 
\hat{w}_2 + (t - 2 [\frac{1}{2}]) (\hat{w}_3 - \hat{w}_2) & \text{if } 0 \leq t - 2 [\frac{1}{2}] \leq 1, \\
\hat{w}_3 + (t - 2 [\frac{1}{2}]) - 1 (\hat{w}_2 - \hat{w}_3) & \text{if } 1 < t - 2 [\frac{1}{2}] \leq 2,
\end{cases}
\]

where \( [s] \in \mathbb{N} \) stands for the smallest integer which is smaller than \( s \leq [s] < s + 1 \). Then we define our reference trajectory \( (\mathbf{v}(t,x), \mathbf{w}(t,x)) \) as

\[
(\mathbf{v}(t,x_1,x_2), \mathbf{w}(t,x_1,x_2)) = (\mathcal{T}_1(t) + \mathcal{X}_1(t,x_1,x_2), \mathcal{T}_2(t) + \mathcal{X}_2(t,x_1,x_2)), \tag{53b}
\]

with

\[
\mathcal{X}_1(t,x_1,x_2) := 0.001 \left| \sin(\pi t) \left( \cos(\pi t + 0.1(x_1 - x_2)) + (x_2 - 1) \sin(0.2 x_1) \right) \right|_\mathbb{R},
\]

\[
\mathcal{X}_2(t,x_1,x_2) := 0.1 \left| \sin(\pi t) \left( \sin(\pi t + 0.1(x_1 - x_2)) + 0.1 (x_2^2 - 3 x_2) \right) \right|_\mathbb{R}.
\]

Notice that \( (\mathbf{v}, \mathbf{w}) \) is periodic in time with period 2, \( (\mathbf{v}(t + 2,x), \mathbf{w}(t + 2,x)) = (\mathbf{v}(t,x), \mathbf{w}(t,x)) \) for all \( t \geq 0 \). Furthermore, for any given nonnegative integer \( m \in \mathbb{N} \),

\[
(\mathbf{v}(2m,x), \mathbf{w}(2m,x)) = (\hat{v}_2, \hat{w}_2) \text{ and } (\mathbf{v}(2m + 1,x), \mathbf{w}(2m + 1,x)) = (\hat{v}_3, \hat{w}_3).
\]
is, the trajectory \((\overline{v}, \overline{w})\) “oscillates” between the unstable steady state \((\overline{v}_2, \overline{w}_2)\) and the stable steady state \((\overline{v}_3, \overline{w}_3)\).

The initial condition in all the simulations, unless otherwise explicitly stated, is the constant
\[
(v_0, w_0) = (-1, -1).
\] (54)

In the feedback (33c), we have set the parameter
\[
\lambda = 1.
\] (55)

5.1. **The linearized FitzHugh–Nagumo model.** Here we consider the linear system (9), with the parameters in (6) set as
\[
\begin{align*}
\nu &= 1, & \delta &= 0.01, \\
a &= 1, & b &= 3, & c &= 1, & \gamma &= 0.1, & \rho &= 0.
\end{align*}
\] (56)

We can see that the vectors as in (52a) are constant steady states of the uncontrolled and unforced nonlinear coupled system (1) (i.e., with \(u = 0\) and \(f = 0\)). Indeed we can see that for a constant vector \((\overline{v}, \overline{w})\) we have \(\Delta \overline{v} = 0\) and that
\[
\xi(\overline{v}, \overline{w}) = \begin{bmatrix} \xi_1(\overline{v}, \overline{w}) \\ \xi_2(\overline{v}, \overline{w}) \end{bmatrix} = \begin{bmatrix} a\overline{v}^3 - b\overline{w}^2 + c\overline{v} + d\overline{w} + e\overline{w}\overline{v} \\ \delta\overline{w} - \gamma\overline{v} + \rho\overline{w}\overline{v} \end{bmatrix}
\]
satisfies, for \(\overline{v} \in \{0, 1, 2\}\),
\[
\begin{align*}
\xi(0, \overline{w}) &= \begin{bmatrix} d\overline{w} \\ \delta\overline{w} \end{bmatrix}, & \xi(1, \overline{w}) &= \begin{bmatrix} a - b + c + d\overline{w} + e\overline{w} \\ \delta\overline{w} - \gamma + \rho \end{bmatrix}, & \xi(2, \overline{w}) &= \begin{bmatrix} 8a - 4b + 2c + d\overline{w} + 2e\overline{w} \\ \delta\overline{w} - 2\gamma + 4\rho \end{bmatrix}.
\end{align*}
\] (57a)

Therefore with the parameters as in (56) we find that
\[
\xi(0, 0) = \xi(1, 10) = \xi(2, 20) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]
which shows that the vectors as in (52a) are constant steady states for system (1).

Observe also that the linearization (4) is stable around \((0, 0)\) and \((2, 20)\), and is unstable around \((1, 10)\). Indeed, the linearization around a constant vector \((\overline{v}, \overline{w})\) reads
\[
\frac{\partial}{\partial t} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} -3a\overline{v}^2 + 2b\overline{w} - c - e\overline{w} \\ \gamma - 2\rho\overline{v} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix},
\] (58)

and we find that with the parameters as in (56):

- For \((\overline{v}, \overline{w}) = (0, 0)\), the eigenvalues of the matrix \(\begin{bmatrix} -c & -d \\ \gamma & -\delta \end{bmatrix}\) are characterized by \((-1 - \lambda)(-\delta - \lambda) + d\gamma = 0\), that is, \(\lambda^2 + (1 + \delta)\lambda + \delta + d\gamma = 0\). Therefore, \(2\lambda = -(1 + \delta) \pm \sqrt{(1 + \delta)^2 - 4(\delta + d\gamma)}\). It is clear that both eigenvalues have strictly negative real part, because \((1 + \delta)^2 - 4(\delta + d\gamma) < (1 + \delta)^2\).
- For \((\overline{v}, \overline{w}) = (2, 20)\), the eigenvalues of the matrix \(\begin{bmatrix} -12a + 4b - c & -d \\ \gamma & -\delta \end{bmatrix}\) are again characterized by \((-1 - \lambda)(-\delta - \lambda) + d\gamma = 0\), that is, they both have strictly negative real part.
• For $\hat{v}, \hat{w} = (1, 10)$, the eigenvalues of the matrix \[
\begin{bmatrix}
-3a + 2b - c & -d \\
\gamma & -\delta
\end{bmatrix}
\]
are characterized by $(2 - \lambda)(-\delta - \lambda) + d\gamma = 0$, that is, $\lambda^2 + (\delta - 2)\lambda - 2\delta + d\gamma = 0$. Thus $2\lambda = (2 - \delta) \pm \sqrt{(2 - \delta)^2 - 4(-2\delta + d\gamma)} = (2 - \delta) \pm \sqrt{(2 + \delta)^2 + 4(\delta + d\gamma)}$. It is clear that the largest eigenvalue have strictly positive real part, because $(2 + \delta)^2 + 4(\delta + d\gamma) > 0$ and $(2 - \delta) > 0$.

5.1.1. The case of the constant unstable steady state. Here we consider the time-independent trajectory $(\overline{v}, \overline{w}) = (1, 10)$, as in (53a), which is the constant unstable steady state for the FitzHugh–Nagumo model.

In the figures below the annotation “dyn=feed” refers to the dynamics under the feedback control and “dyn=free” to the free dynamics (without control). The annotation “model=...” specifies the type of the monodomain model defined by the selection of parameters (cf. (2)).

In Figure 2 we see the performance of our feedback control, which is able to stabilize system (9). The slope of $\log(|(y, z)|_{H \times H})$ is approximately $3.3 - 3.8$ for time $t \geq 5$, which means that $|(y, z)|_{H \times H}$ decreases exponentially with rate approximately $-\delta = -0.02$.

Recall that the explicit feedback has been constructed to stabilize the PDE component, and as a corollary the stability of the ODE component follows as well. Consequently we cannot expect to obtain a rate of decay for the ODE component which is better (smaller) than $-\delta$.

In Figure 2 we can see also the curve illustrating the squared norm $|y|^2_H$ of the PDE component versus the squared norm $|z|^2_H$ of the ODE component. The configuration at initial time ($t = 0$) is the starting point of the curve located at $(|y0|^2_H, |z0|^2_H) \approx (42, 42)$ (the red dot). The configuration at final time ($t = 30$ in case of Figure 2) is the end point of the curve (the green dot).

Further notice that the free dynamics is exponentially unstable, the norm is increasing exponentially with rate approximately $\frac{1}{2} \frac{129}{30} = 2$.

5.1.2. The case of a time-dependent trajectory. We consider the time-dependent trajectory $(\overline{v}, \overline{w})$ as in (53b). In Figure 3 we observe that after time $t \geq 5$ the norm of the solution of the linear system (9) decreases exponentially with rate approximately $-\delta = -0.01$. The free dynamics is exponentially unstable with norm increasing exponentially with rate approximately 1.

5.2. The nonlinear FitzHugh–Nagumo model. We consider again the time-dependent reference trajectory as in (53b), but now for the nonlinear system (10). In Figure 4 we see that the norm of the controlled nonlinear system also decreases.
exponentially to zero with rate $-0.01$. The free dynamics is not stable, the norm of the solution does not decrease as time increases.

5.3. The linearized Rogers–McCulloch model. Here, in system (9), we take the parameters

\[ \nu = 1, \quad \delta = 0.01, \]
\[ a = 1, \quad b = 4, \quad c = 2, \]
\[ d = 0, \quad e = 0, \quad \gamma = 0.1, \quad \rho = 0. \]

Note that from (57), with the parameters as in (59), we find

\[ \xi(0, 0) = \xi(1, 10) = \xi(2, 20) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \]

which shows that the vectors as in (52a) are constant steady states for system (1). For the linearization around a constant steady state (58), we find that with the parameters as in (59):

- For $(\bar{v}, \bar{w}) = (0, 0)$, the eigenvalues of the matrix $\begin{bmatrix} -c & -d \\ \gamma & -\delta \end{bmatrix}$ with $d = 0$ are given by $\{-c, -\delta\}$. Thus both eigenvalues are strictly negative.
- For $(\bar{v}, \bar{w}) = (2, 20)$, the eigenvalues of $\begin{bmatrix} -12a + 4b - c - 2 & -d - 2e \\ \gamma & -\delta \end{bmatrix}$ are characterized by $(0 - \lambda)(-\delta - \lambda) + 2e\gamma = 0$, that is, $\lambda^2 + \delta\lambda + 2e\gamma = 0$. Therefore, $2\lambda = -\delta \pm \sqrt{\delta^2 - 8e\gamma}$, and we can conclude that both eigenvalues have a strictly negative real part because $\delta^2 - 8e\gamma < \delta^2$. 
For \((\hat{v}, \hat{w}) = (1, 10)\), the eigenvalues of the matrix
\[
\begin{bmatrix}
-3a + 2b - c - 1 & -d - e \\
\gamma & -\delta
\end{bmatrix}
\]
are characterized by
\[(2 - \lambda)(-\delta - \lambda) + e\gamma = 0,\]
that is,
\[
\lambda^2 + (\delta - 2)\lambda - 2\delta + e\gamma = 0.
\]
Thus, \(2\lambda = (2 - \delta) \pm \sqrt{(2 - \delta)^2 + 4\delta},\) because from (59) we have \(e\gamma = \delta\). It is clear that the largest eigenvalue has a strictly positive real part.

5.3.1. The case of the constant unstable steady state. Here the reference trajectory \((\bar{v}, \bar{w}) = (1, 10)\) is as in (53a). In Figure 5 we can see that the explicit feedback is able to stabilize the system \((9)\) exponentially with rate approximately \(-\delta = -0.01\), while the free dynamics is exponentially unstable with rate approximately 2.

5.3.2. The case of a time-dependent trajectory. We consider the time-dependent trajectory \((\bar{v}, \bar{w})\) as in (53b). In Figure 6 we can see that the explicit feedback is able to stabilize the system \((9)\) exponentially with rate approximately \(-\delta = -0.01\), while the free dynamics is exponentially unstable with rate approximately 1.5.

5.4. The nonlinear Rogers–McCulloch model. Here, we consider the time-dependent trajectory \((\bar{v}, \bar{w})\) as in (53b), and consider the corresponding nonlinear system \((10)\) with the parameters as in (59). In Figure 7 we see that our explicit feedback is able to exponentially stabilize system \((10)\). The norm of the controlled system decreases exponentially with rate approximately \(-\delta = -0.01\). Notice also that the free dynamics is unstable.
5.5. The linearized Aliev–Panfilov model. Here we consider the linear system (9), with the parameters
\[ \begin{align*}
\nu &= 1, & \delta &= 0.01, \\
a &= 1, & b &= 2.5, & c &= 1, \\
d &= 0, & e &= 0.1, & \gamma &= 0.1, & \rho &= 0.05.
\end{align*} \tag{60} \]

Note that from (57), with the parameters as in (60), we find
\[ \xi(0,0) = \xi(1,5) = \xi(2,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \]
which shows that the vectors as in (52b) are constant steady states for system (1). For the linearization around a constant steady state (58), we find that with the parameters as in (59):
- For \((\hat{v}, \hat{w}) = (0,0)\), the eigenvalues of the matrix \( \begin{bmatrix} -c & -d \\ \gamma & -\delta \end{bmatrix} \) are \{-c, -\delta\}. They are both strictly negative.
- For \((\hat{v}, \hat{w}) = (2,0)\), the eigenvalues of \( \begin{bmatrix} -12a + 4b - c & -d - 2e \\ \gamma - 0.2 & -\delta \end{bmatrix} \) are characterized by \((-3 - \lambda)(-\delta - \lambda) + 2e(\gamma - 0.2) = 0\), that is, \(\lambda^2 + (3 + \delta)\lambda + \delta = 0\). Therefore, \(2\lambda = -(3 + \delta) \pm \sqrt{(3 + \delta)^2 - 4\delta}\), and we have that both eigenvalues have a strictly negative real part.
- For \((\hat{v}, \hat{w}) = (1,5)\), the eigenvalues of \( \begin{bmatrix} -3a + 2b - c - \frac{\delta}{2} & -d - d_2 \\ 0 & -\delta \end{bmatrix} \) are given by \(\lambda \in \{-\delta, \frac{1}{2}\}\), that is, the largest eigenvalue is strictly positive.

5.5.1. The case of the constant unstable steady state. Here we consider the time-independent trajectory \((v(t), w(t)) = (1,5)\), as in (53a). In Figure 8 we see that our explicit feedback is able to stabilize system (9). The norm of the solution decreases exponentially with rate approximately \(-\delta = -0.01\). The norm of the free dynamics is exponentially increasing with rate approximately 0.5.

5.5.2. The case of a time-dependent trajectory. We consider again the linear system (9), with the time-dependent trajectory \((v, w)\) as in (53b). In Figure 9 we confirm that also in this case the feedback is able to stabilize system (9). The norm decreases exponentially with rate approximately \(-\delta = -0.01\). The free dynamics is not stable. The norm of the solution is not converging to zero, and is “close” to a periodic behavior.
5.6. The nonlinear Aliev–Panfilov model. We take the time-dependent trajectory \((v, w)\) as in (53b), and the corresponding nonlinear system (10), again with the parameters as in (60). For Figure 10 we ran the simulation for time \(t \in [0, 30]\) as in previous examples. In this situation the results do not allow us to conclude that the feedback is stabilizing. The norm seems to decrease but, we cannot yet clearly see a rate of stability.

Therefore we ran the simulation for a larger time interval and the results are shown in Figure 11. Again we observe that an exponentially decreasing rate is achieved. In fact, it is approximately \(-\delta = -0.01\) (the slope for time \(t \geq 100\) is approximately \(-0.02\)).

We also observe that the norm corresponding to the free dynamics is not converging to zero, that is the free dynamics is not stable.
5.7. On the sign of the leading coefficient $a$. In the above examples the linearization based feedback control is able to stabilize the nonlinear system for the initial condition $(-1, -1)$, as in (54).

Notice that in the models above, the sign of the leading coefficient $a = 1$ of $av^3 - b^2v^2 + cv$ is positive, and in this case we may say that is has the good sign, in the sense that such sign somehow favors stability of the nonlinear system, for example it may guarantee the existence of global solutions (cf. [9, Chapter 5, Remark 5.1], for the uncoupled parabolic equation). The positive sign of $a$ may explain why the feedback still works for the nonlinear system for such a “large” initial condition.

Recall that simulations in [19], for a single uncoupled parabolic equation, show that the feedback is able to stabilize the nonlinear system only if the initial condition is small enough. This is also the statement of Theorem 4.7, concerning the coupled system.

Below we will consider the “$a$−” model where we take a negative leading coefficient $a$. Notice that our results are independent of the sign of $a$. We will consider the following parameters

\[
\begin{align*}
\nu &= 1, & \delta &= 0.01, \\
\alpha &= -1, & b &= 0, & c &= -1, \\
\gamma &= 0.1, & \rho &= 0.05.
\end{align*}
\]

which roughly can be seen a variant of the Aliev–Panfilov model with the negative sign for $a$ and $c$ and with $b = 0$. In such situation we may expect the uncontrolled solution of the uncoupled corresponding parabolic system to blow up in finite time. See [9, Chapter 5], [14], and references therein. Therefore we may expect the solution of the corresponding coupled system to explode as well.

5.7.1. The linearized “$a$−” model. We consider the time-dependent trajectory $(\bar{v}, \bar{w})$ as in (53b), with $(1, 1)$ in the place of the steady states: that is with $(\bar{v}_2, \bar{w}_2) := (1, 1) =: (\bar{v}_3, \bar{w}_3)$.

In Figure 12 we see that the norm of the controlled solution of system (9) decreases exponentially with rate approximately $-\delta = -0.01$. The norm of the free dynamics increases exponentially with rate approximately 4.

5.7.2. The nonlinear “$a$−” model. Here we consider the nonlinear system (10). We will test with the resized initial conditions

\[
(v_0, w_0)_{\varepsilon} = \varepsilon(v_0, w_0), \quad \text{with} \quad \varepsilon \in \{1, 0.1, 0.05\}.
\]
First in Figure 13, for the free dynamics for the nonlinear system (10), we observe that the norm of the solution blows up in finite time, for all initial conditions as in (62).

Next we consider the corresponding systems under the action of the feedback control. In this case, in Figure 14, we can see that our feedback is not able to stabilize the system for the initial conditions \((-1, -1)\) and \((-0.1, -0.1)\). But, it is able to stabilize the system if we take the smaller initial condition \((-0.05, -0.05)\) which is consistent with our local stabilization result for the nonlinear system, as stated in Theorem 1.2.
6. Final remarks.

6.1. On the stability conditions (7). We have shown that stability of the explicit closed-loop system holds provided the conditions in (7) are satisfied. Condition (7a), $H = U_M \oplus E_{\not M}$ is necessary and sufficient for the oblique projection $P_{U_M}^{E_{\not M}}$ to be well defined. This condition is not very restrictive. Indeed, in 1D it is enough that the indicator functions actuators are linearly independent for both Dirichlet and Neumann boundary conditions [20, Lemmas 4.3 and 5.1].

Condition (7b), on the other hand is not so transparent. But for rectangular domains we have investigated whether, for large enough $M$, the condition can be satisfied for $M$ actuators $1_{\omega_i}$, $i \in \{1, 2, \ldots, M\}$, whose supports are constrained to cover altogether an a priori fixed arbitrarily small volume, $\text{vol} \left( \bigcup_{i=1}^M \omega_i \right) \leq r \text{vol}(\Omega)$, with $r \in (0, 1)$ independent of $M$. This question was answered affirmatively for 1D, in [20, Theorems 4.4, 4.5, and 5.2], for both Dirichlet and Neumann boundary conditions.

Therefore, from [13, Section 4.8.1], we also know that we can place suitable $M^d$ actuators in higher-dimensional rectangles in $\mathbb{R}^d$, $d \geq 2$, so that an analogous condition to (7b) is satisfied, which again implies the stability of the closed-loop system.

Finally for general (nonrectangular) domains $\Omega \subset \mathbb{R}^d$, $d \geq 2$, we do not know whether we can construct $M$ actuators $1_{\omega_i}$ (covering an a priori fixed total volume) so that (7b) is satisfied. For this, it would be enough to prove that it is possible to construct/place such actuators in such a manner that the operator norm of the oblique projection $P_{U_M}^{E_{\not M}}$ will remain bounded as $M$ increases, because $\alpha_{M+1} \rightarrow +\infty$. This possibility was conjectured in [13, Section 4.8.2]. This is an interesting open problem, because simulations in [13] and in this manuscript show the stabilizing performance of the feedback in nonrectangular 2D domains.

6.2. On robustness. One of the reasons for the importance of feedback control in applications is its gain in robustness against noise and disturbances. Simulations in [13], for a single parabolic equation, show that our feedback control can attenuate state estimation errors of the form $\eta(t,x) = \eta_1(t,x)y(t,x) + \eta_2(t,x)$, where $\eta_1(t,x)$ and $\eta_2(t,x)$ reach small magnitudes. In particular, the component $\eta_1(t,x)y(t,x)$ can be seen as a disturbance of the reaction operator $A_r$. In addition to the numerical evidence, it would be interesting to investigate the robustness properties of the explicit proposed feedback, due to the importance of such properties for automatic control applications. Also other types of noise and plant (dynamics) disturbances could be of interest, because the type of such errors may depend on the problem/applications.

6.3. On the number of actuators. Here we consider the same setting as in Section 5.7.1. Recall that in Figure 12, we have the results corresponding to the case of 28 actuators as in Figure 1.

Now in Figure 15, we present the corresponding results in the case we take only 4 actuators as in Figure 15. We constructed the actuators so that the volume covered by the 4 actuators in Figure 15 is equal to the total volume covered by the 28 actuators in Figure 1. In either case the total volume covered by the actuators is approximately 5.4% of the volume of $\Omega$. Though the volume covered by the actuators is the same, we conclude that the explicit feedback corresponding to
the 4 actuators as in Figure 15 is not able to stabilize system (9), while the explicit feedback corresponding to the 28 actuators as in Figure 1 is, as shown by Figure 12.

![Figure 15. Norm of solution. Linearization around the time-dependent trajectory.](image)

From the results in [13] we can conclude that once we know that
\[
\|P_{E_M} \|_{L(H)}^2
\]
remains bounded with respect to \( M \), then increasing both \( \lambda \) and the number of actuators allows to achieve any prescribed rate of exponential stability in the case of a single uncoupled parabolic equation. In Figure 16 we observe that 4 actuators chosen as in Figure 15 are not able to stabilize the uncoupled parabolic equation, while 28 actuators as in Figure 1 are able to stabilize the uncoupled parabolic equation with rate approximately 1, which is the best rate we can get, independently of \( M \), because we have chosen \( \lambda = 1 \) in our feedback law, see (55).

![Figure 16. PDE only. With \( d = d_2 = \gamma = \rho = 0 \) (uncoupled system) and \( z_0 = 0 \). Solution norm \( \|(y, z)\|_{H \times H} = |y|_H^2 \). Linearization around the time-dependent trajectory.](image)

As we said before, besides the number of actuators, another important question concerns the (optimal) placement of the actuators, say so that
\[
\|P_{E_M} \|_{L(H)}^2
\]
is as small as possible. In Figure 1 and Figure 15 we simply tried to “spread” the actuators over \( \Omega \) as “uniformly” as possible. However, the results in [20], show that the best way of “spreading” the actuators is a nontrivial question already in the 1D case.

By curiosity, the norm (computed numerically, see [13]) of the projection with the 4 actuators as in Figure 15 is given by
\[
\|P_{E_M} \|_{L(H)}^2 \approx (0.0327)^{-1} \approx 30,
\]
while the
norm of the projection with 28 actuators as in Figure 1 is given by $\|PE^\perp M_\Omega\|^2_{L(H)} \approx (1.9367 \cdot 10^{-4}) \approx 5000$, which is considerably bigger.

From [13, Sections 4.2 and 5.1], we know that there are particular examples of parabolic equations where the necessary and sufficient number of actuators is known, regardless of the location of the actuators.

Even if two different placements of the actuators lead to the stability of the closed-loop linear system, a better placement of the actuators may lead to a better performance of the feedback control, see [13, Section 5.1].

6.4. On the stabilization to periodic solutions. While our results apply for general bounded trajectories, in our numerical examples we have taken periodic trajectories as targeted solutions. We will show here that for this case our feedback law is also periodic. Indeed, suppose that our targeted solution $(v, w)$ for system (3) is time-periodic: for a given period $T > 0$

$$(v(s + T), w(s + T)) = (v(s), w(s)), \quad \text{for all } s \geq 0. \quad (63)$$

In that case we observe that the feedback law is also periodic

$$K_{U_\Omega}(s + T) = K_{U_\Omega}(s). \quad (64)$$

Indeed, for all $y, z \in H \times H$,

$$K_{U_\Omega}(s + T)(y, z) = P_{U_\Omega}^E \left( Ay + A_r(s + T)y - \lambda y + \bar{S}(s + T)z \right)$$

$$= P_{U_\Omega}^E \left( Ay + A_r(s)y - \lambda y + \bar{S}(s)z \right) = K_{U_\Omega}(s)(y, z),$$

because, from (6), we see that $A_r(s + T)y = A_r(s)y$ and $\bar{S}(s + T) = \bar{S}(s)$, when $(v, w)$ is $T$-periodic, as in (63).

Recalling that the monodomain equations provide a model for the heart rhythm we expect the healthy solution to be periodic, that is, it makes sense to consider at the very beginning that $(v, w)$ satisfies (63).

Then it is natural to ask whether we can take advantage of the time periodicity of $A + A_r + \bar{S}$.

In [4, 17] the stabilization of time-periodic parabolic systems is proven under suitable conditions on the set of actuators. Such conditions take advantage of the fact that the operators in the parabolic equation are time-periodic, which allows to derive suitable properties of the asymptotic behavior of the solutions from the spectral properties of the periodic map or Floquet map.

The feedback law proposed in [4, 17] is, however, not explicit. It is given implicitly and involves the solution of a suitable periodic Riccati equation. The computation of such a solution can be an expensive numerical task.

It would be interesting to know whether we can construct an ad-hoc, still explicit, feedback operator in this situation. Recall that in [13, Section 6.5], an ad-hoc explicit feedback is proposed for the case of time-independent targeted trajectories (i.e., steady states).

Appendix A. In Assumptions 2.1, 3.4, and 3.6, the existence of weak solutions was assumed for abstract nonautonomous systems. As announced, we now give some comments on this subject. We also give the proofs of Propositions 4.1 and 4.3.
On the existence of weak solutions. Let $\mathcal{H}$ be a separable Hilbert space, which we consider as a pivot space $\mathcal{H} = \mathcal{H}'$. Given another Hilbert space $\mathcal{V} \xrightarrow{d} \mathcal{H}$, and a family of linear mappings $\mathcal{L}(t) \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ it follows that the domain of $\mathcal{L}(t)$, defined as $D(\mathcal{L}(t)) := \{ h \in \mathcal{H} \mid \mathcal{L}(t)h \in \mathcal{H} \}$, satisfies

$$D(\mathcal{L}(t)) \xrightarrow{d} \mathcal{V} \xrightarrow{d} \mathcal{H} \xrightarrow{d} \mathcal{V}' \xrightarrow{d} D(\mathcal{L}(t)).$$

The domain $D(\mathcal{L}(t))$ is assumed to be endowed with the natural graph norm $|h|_{D(\mathcal{L}(t))} := \left( |h|_{\mathcal{H}}^2 + |\mathcal{L}(t)h|_{\mathcal{H}}^2 \right)^{1/2}$. Further, for time dependent operators, we assume that the domain $D(\mathcal{L}(t))$ is independent of time:

$$D(\mathcal{L}(t)) = D(\mathcal{L}(0)), \quad \text{for all } t \geq 0.$$

Therefore we simply write $D(\mathcal{L}) := D(\mathcal{L}(t))$. Set the time interval $I := (s_0, s_1)$. 

**Definition A.1.** We say that $z(t)$ is a $(\mathcal{V}, \mathcal{H})$-weak solution, in $I \times \mathcal{H}$, for the evolutionary system

$$\dot{z} = -\mathcal{L}z, \quad z(s_0) = z_0 \in \mathcal{H}, \quad \text{for all } t \in I,$$

if $z \in W^1(I, \mathcal{V}, \mathcal{V}')$, $z(s_0) = z_0$, and $\langle \dot{z} + \mathcal{L}z, \phi \rangle_{\mathcal{V}', \mathcal{V}} = -\langle z, \dot{\phi} \rangle_{\mathcal{H}} + \langle \mathcal{L}z, \phi \rangle_{\mathcal{V}', \mathcal{V}} = 0$ for all $\phi \in \{ f \in C^\infty(I, \mathcal{V}) \mid \text{supp } f \subseteq I \}$. 

Recall that $W^1(I, \mathcal{V}, \mathcal{V}') \rightarrow C([s_0, s_1], \mathcal{H})$. In case that for any nonempty interval $J = (t_1, t_2) \subseteq I$, the $(\mathcal{V}, \mathcal{H})$-weak solution, in $J \times \mathcal{H}$, for the linear system (65) exists and is unique, for all initial conditions $z(t_1) = h$, then we denote it by $z(t) = U_{(t_1, t)}^h$, for $t \in J$. Necessarily, for all $h \in \mathcal{H}$, $z_0 \in \mathcal{H}$, and $t_1 \geq t \geq t_1 \geq s_0$, we have that

$$U_{(t_1, t)}^h h = h \quad \text{and} \quad U_{(t_1, t_1)}^{\mathcal{L}} z_0 = U_{(t_1, s_1)}^{\mathcal{L}} U_{(t_1, t_1)}^{\mathcal{L}} z_0.$$

**Definition A.2.** We say that $(\mathcal{V}, \mathcal{H}, \mathcal{H})$ has the Galerkin approximation property if there exists a complete orthonormal basis $\{ \psi_i \mid i \in \mathbb{N}_0 \} \subset \mathcal{V}$ so that

1. for all $h \in \mathcal{H}$, $h = \sum_{i=1}^{\infty} \langle h, \psi_i \rangle_{\mathcal{H}\mathcal{H}} \psi_i := \lim_{N \rightarrow \infty} \sum_{i=1}^{N} \langle h, \psi_i \rangle_{\mathcal{H}\mathcal{H}} \psi_i$;
2. for all $z_0 \in \mathcal{H}$ and $N \in \mathbb{N}_0$, by setting $z_0^N := \sum_{i=1}^{N} \langle z_0, \psi_i \rangle_{\mathcal{H}\mathcal{H}} \psi_i$ and considering the solution $z^N$ of

$$\dot{z}^N = -Q^N \mathcal{L}z^N, \quad z^N(s_0) = z_0^N \in \mathcal{H},$$

the sequence $(z^N)_{N \in \mathbb{N}_0}$ remains bounded: $|z^N|_{W^1(I, \mathcal{V}, \mathcal{V}')} \leq D |z_0|_{\mathcal{H}}$, with $D$ independent of $(N, z_0)$;
3. we have $Q^N \mathcal{L}z^N \rightarrow \mathcal{L}z$ in $\mathcal{V}'$, provided $z^N \rightarrow z$ in $\mathcal{V}$.

**Lemma A.3.** Let $(\mathcal{V}, \mathcal{H}, \mathcal{H})$ have the Galerkin approximation property. If there is a constant $B_{\mathcal{L}} \geq 0$ so that

$$|\langle \mathcal{L}h, h \rangle_{\mathcal{V}', \mathcal{V}}|_{\mathbb{R}} \leq B_{\mathcal{L}} |h|_{\mathcal{H}}^2, \quad \text{for all } h \in \mathcal{V},$$

then there exists one, and only one, $(\mathcal{V}, \mathcal{H})$-weak solution for system (65).

**Proof.** A solution $z$ is given by a weak-limit, in $W^1(I, \mathcal{V}, \mathcal{V}')$, of the Galerkin approximations solving (66), $z^N \rightarrow z$. The uniqueness follows from $\frac{d}{dt} |d|_{\mathcal{H}}^2 = -2 \langle \mathcal{L}d, d \rangle_{\mathcal{V}', \mathcal{V}} \leq 2B_{\mathcal{L}} |d|_{\mathcal{H}}^2$, where $d$ is the difference between two solutions, which implies $|d(t)|_{\mathcal{H}}^2 \leq e^{2B_{\mathcal{L}}(t-s_0)} |d(s_0)|_{\mathcal{H}}^2 = 0$, for all $t \in I$. \(\square\)
The notion of weak solutions is classical. We refer to [15, Chapter 1, Section 6],
[22, Chapter 1, Section 3], and [23, Chapter 3, Sections 1.3, 1.4, and 3.2].

**Proof of Proposition 4.1.** We suppose that Assumptions 3.1–3.5, and 3.7 hold
true. Recall that \( H = L^2(\Omega), V = H^1(\Omega) \) and \( D(A) = \{ f \in H^2(\Omega) \mid \frac{\partial f}{\partial n} = 0 \} \). Let us take the spaces

\[ \mathcal{S} = (E_M^L \cap H) \times H, \quad \mathfrak{V} = (E_M^L \cap V) \times H, \quad \mathfrak{D}(\mathfrak{L}) = (E_M^L \cap D(\Delta)) \times H, \]

and the linear operator, in matrix form,

\[ \mathfrak{L} \in \mathcal{L}(\mathfrak{V}, \mathfrak{W}), \quad \begin{bmatrix} v \\ w \end{bmatrix} \mapsto \begin{bmatrix} P_{E_M^L} A + P_{E_M^L} A_r \mathcal{R} \\ P_{E_M^L} \tilde{S} \mathcal{D} \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}. \] (67)

Next we prove that system (65) has the Galerkin property. We set “the” sequence
of eigenfunctions \( e_i, i \geq 1 \), of \( A = -\Delta + \text{Id} \) under Neumann boundary conditions.
Then we set the basis \( \{ e_i \}_{i > M, j \geq 1} \) \( \subset \mathfrak{V} \) for the space \( \mathcal{S} \). It is clear that (1)
in Definition A.2 is satisfied. Next we consider the system

\[ \dot{z}^N = -Q^N \mathcal{L} z^N = \begin{bmatrix} -A - Q^N P_{E_M^L} A_r & -Q^N P_{E_M^L} \tilde{S} \\ -Q^N \mathcal{R} & -\delta \text{Id} \end{bmatrix}, \quad z^N(s_0) = z^N_0 \in \mathfrak{S}, \]

with \( Q^N \begin{bmatrix} f \\ g \end{bmatrix} := \begin{bmatrix} \sum_{i=1}^N (f, e_{i+1}) H e_{i+1} \\ \sum_{i=1}^N (g, e_{i}) H e_{i} \end{bmatrix} \).

Since \( Q^N \) is an orthogonal projection in \( \mathfrak{S} \), with \( z^N = \begin{bmatrix} z^N_1 \\ z^N_2 \end{bmatrix} \) we have \( \langle Q^N P_{E_M^L} A z^N_1, z^N_1 \rangle_V = \langle P_{E_M^L} A z^N_1, z^N_1 \rangle_H = \| z^N_1 \|_V^2 \) and \( \langle Q^N D z^N_2, z^N_2 \rangle_H = \delta \| z^N_2 \|_H^2 \). \( \tilde{S} \) holds true for system (68).

From Remark 3.9 we know that the system \( \dot{z}^N = -A z^N \) is stable if \( \tilde{S} = 0 \). Therefore, we can conclude that the assumptions in Lemma 2.5 are satisfied. Further,
from Assumption 3.7, it follows that the assumption in Corollary 2.6 is also satisfied. Thus, from Corollary 2.6, the Galerkin approximations satisfy

\[ \| z^N \|_{L^\infty(R_{0, \delta})} \leq C \| z^N_0 \|_{\mathcal{S}}. \]

This inequality, together with

\[ \begin{align*}
\frac{d}{dt} \| z^N \|_{\mathcal{S}}^2 & \leq -2 \| z^N_1 \|_V^2 - 2\delta \| z^N_2 \|_H^2 + 2 \| Q^N P_{E_M^L} A_r \|_{L(H, V')} \| z^N_2 \|_H \| z^N_1 \|_V \\
& \quad + 2 \| Q^N P_{E_M^L} \tilde{S} \|_{L(H, H)} \| z^N_2 \|_H \| z^N_1 \|_H + 2 \| Q^N \mathcal{R} \|_{L(H, H)} \| z^N \|_H \| z^N_2 \|_H \\
& \leq -\| z^N \|_V^2 - \delta \| z^N_2 \|_H^2 + C_1 \| P_{E_M^L} A_r \|_{L(H, V')} \| z^N_1 \|_H^2 \\
& \quad + \frac{C_2}{\delta} \| P_{E_M^L} \tilde{S} \|_{L(H, H)} \| z^N_1 \|_H^2 + C_3 \| \mathcal{R} \|_{L(H, H)} \| z^N_2 \|_H^2,
\end{align*} \]
Proof of Proposition 4.3. From (42) and \((y(s_0), z(s_0)) \in V \times H\), the fact that the solution for system (9) is strong follows by standard arguments. Indeed, recalling (33c), system (9) becomes
\[
\begin{align*}
y(t) &= -Ay - A_r y - \tilde{S}z + K_{U, \delta}(y, z), \\
z(t) &= -\delta z - R y, \tag{69a}
\end{align*}
\]
Multiplying the dynamics by \(\mathcal{L}z\), where \(\mathcal{L} = \begin{bmatrix} A & 0 \\ 0 & \delta \end{bmatrix}\), \(\mathcal{F} = H \times H\), \(\mathfrak{N} = V \times H\), and \(D(\mathcal{L}) = D(A) \times H\), we obtain (47) with \(g = 0\). Using (47) we can now conclude that \((y, z) \in W((s_0, s_0 + T), D(\mathcal{L}), \mathcal{F}) \subseteq C((s_0, s_0 + T), D(\mathcal{L}), \mathcal{F})\). The same argument leads us to
\[
|(y, z)|_{2\infty, ((s, s + 1), \mathfrak{N})}^2 + |(y, z)|_{2L^2((s, s + 1), D(\mathcal{L}))}^2 \leq D_4 |(y(s), z(s))|_{D(\mathcal{L})}^2, \quad s \geq s_0 \geq 0, \tag{70}
\]
with \(D_4 = D_3(1)\) independent of \(s\) and \(s_0\).

Furthermore, we have the following smoothing property
\[
|(y(s + 1), z(s + 1))|_{D(\mathcal{L})}^2 \leq D_5 |(y(s), z(s))|_{D(\mathcal{L})}^2, \quad \text{for all} \quad s \geq s_0 \geq 0, \tag{71}
\]
with \(D_5\) independent of \(s\). This can be derived as follows. We consider, for \(t \geq s\), the function \((\tilde{y}(t), \tilde{z}(t)) := (t - s)(y(t), z(t))\), which still solves system (46) with different data:
\[
\begin{align*}
\dot{\tilde{y}} &= -A\tilde{y} - A_r \tilde{y} - \tilde{S}\tilde{z} + K_{U, \delta}(\tilde{y}, \tilde{z}) + y, \quad \tilde{y}(s) = 0 \in V, \\
\dot{\tilde{z}} &= -\delta\tilde{z} - R\tilde{y} + z, \quad \tilde{z}(s) = 0 \in H,
\end{align*}
\]
from which (cf. estimate (47) with \(g = (y, z)\)) we can obtain the estimate
\[
|(\tilde{y}, \tilde{z})|_{2L^2((s, s + 1), \mathfrak{N})}^2 + |(\tilde{y}, \tilde{z})|_{2L^2((s, s + 1), D(\mathcal{L}))}^2 \leq D_6 \left(|y|^2_{L^2((s, s + 1), \mathfrak{N})} + |z|^2_{L^2((s, s + 1), \mathfrak{N})}\right) \leq D_6 C |(y, z)(s)|_{D(\mathcal{L})}^2,
\]
where the last inequality follows from Corollary 4.2 (with \(s\) in the role of \(s_0\)). In particular (71) holds true.

For simplicity let us write
\[
\Psi_s := |(y, z)|_{2L^2((s, s + 1), \mathfrak{N})}^2 + |(y, z)|_{2L^2((s, s + 1), D(\mathcal{L}))}^2.
\]
Then from estimates (71) and (70) together with Corollary 4.2, it follows, for all \( s \geq s_0 \geq 0 \):

- if \( s \leq s_0 + 1 \), then
  \[
  \Psi_s \leq D_4 (|y(s)|^2 + |z(s)|^2) \leq D_4^2 (|y(s_0)|^2 + |z(s_0)|^2) \leq D_4^2 e^{2\mu(s-s_0)} (|y(s_0)|^2 + |z(s_0)|^2),
  \]

- if \( s \geq s_0 + 1 \), then
  \[
  \Psi_s \leq D_4 D_5 (|y(s-1)|^2 + |z(s-1)|^2) \leq D_4 D_5 e^{-2\mu(s-1-s_0)} (|y(s_0)|^2 + |z(s_0)|^2).
  \]

We finish the proof by showing the estimate in Proposition 4.3. It is clear now that we have

\[
\sup_{s \geq s_0} e^{2\mu(s-s_0)} (|y(s)|^2 + |z(s)|^2) \leq \tilde{C} (|y(s_0)|^2 + |z(s_0)|^2),
\]

with \( \tilde{C} = e^{2\mu} \max \{D_4^2, D_4 D_5 C\} \). By direct computations we also obtain

\[
\int_s^{s+1} e^{2\mu(t-s_0)} |(y(t), z(t))|^2_{D(\Sigma)} \, dt \leq e^{2\mu(s+1-s_0)} \int_s^{s+1} |(y(t), z(t))|^2_{D(\Sigma)} \, dt
\]

\[
\leq e^{2\mu(s+1-s_0+s-s_0)} \tilde{C} (|y(s_0)|^2 + |z(s_0)|^2) = e^{2\mu} \tilde{C} (|y(s_0)|^2 + |z(s_0)|^2),
\]

Using the dynamics in (69), we also obtain for \((\hat{y}, \hat{z}) := e^{\mu(t-s_0)} (y(t), z(t))\),

\[
|\hat{y}|_{L^2((s,s+1),\mathcal{H})} \leq \mu |\hat{y}|_{L^2((s,s+1),\mathcal{H})} + \left| e^{\mu(t-s_0)} (y, z) \right|_{L^2((s,s+1),D(\Sigma))},
\]

Therefore, recalling that \( \mathcal{W} \rightarrow \mathcal{H} \), we conclude from estimates (72), (73), and (74) that

\[
\left| e^{\mu(t-s_0)} (y, z) \right|_{W((s,s+1),D(\Sigma),\mathcal{H})} \leq D_8 (|y(s_0)|^2 + |z(s_0)|^2), \quad \text{for all} \quad s \geq s_0,
\]

with \( D_8 \) independent of \( s \geq s_0 \), which is equivalent to the estimate in Proposition 4.3.

\[\square\]

Acknowledgments. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No [668998]).

REFERENCES

[1] S. Agmon, Lectures on Elliptic Boundary Value Problems, Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965, URL https://bookstore.ams.org/chel-369-h/.
[2] R. R. Aliev and A. V. Panfilov, A simple two-variable model of cardiac excitation, Chaos, Solitons & Fractals, 7 (1996), 293–301.
[3] V. Barbu, S. S. Rodrigues and A. Shirikyan, Internal exponential stabilization to a nonstationary solution for 3D Navier-Stokes equations, SIAM J. Control Optim., 49 (2011), 1454–1478.
[4] V. Barbu and G. Wang, Feedback stabilization of periodic solutions to nonlinear parabolic-like evolution systems, Indiana Univ. Math. J., 54 (2005), 1521–1546.
[5] T. Breiten, K. Kunisch and S. S. Rodrigues, Feedback stabilization to nonstationary solutions of a class of reaction diffusion equations of FitzHugh-Nagumo type, SIAM J. Control Optim., 55 (2017), 2684–2713.
[6] F. Demengel and G. Demengel, Functional Spaces for the Theory of Elliptic Partial Differential Equations, Springer, London, EDP Sciences, Les Ulis, 2012.
[7] R. FitzHugh, Impulses and physiological states in theoretical models of nerve membrane, Biophysical J., 1 (1961), 445–466.
[8] M. Hieber, N. Kajiwara, K. Kress and P. Tolksdorf, Strong time periodic solutions to the bidomain equations with FitzHugh–Nagumo type nonlinearities, Analysis of PDEs, (2017), URL https://arxiv.org/abs/1708.05304.
[9] B. Hu, *Blow-Up Theories for Semilinear Parabolic Equations*, Lecture Notes in Mathematics, 2018, Springer, Heidelberg, 2011.

[10] V. Y. Ivrii, Second term of the spectral asymptotic expansion of the Laplace-Beltrami operator on manifolds with boundary, *Funct. Anal. Appl.*, 14 (1980), 98–106.

[11] J. Keener and J. Sneyd, *Mathematical Physiology. Vol. II: Systems Physiology*, Second edition, Interdisciplinary Applied Mathematics, 8/I. Springer, New York, 2009.

[12] P. Kröger, Upper bounds for the Neumann eigenvalues on a bounded domain in euclidean space, *J. Funct. Anal.*, 106 (1992), 353–357.

[13] K. Kunisch and S. S. Rodrigues, Explicit exponential stabilization of nonautonomous linear parabolic-like systems by a finite number of internal actuators, *ESAIM Control Optim. Calc. Var.*, (2018).

[14] H. A. Levine, Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $P_u = -Au + F(u)$, *Arch. Ration. Mech. Anal.*, 51 (1973), 371–386.

[15] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod and Gauthier-Villars, Paris, 1969.

[16] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Die Grundlehren Math. Wiss. Einzeldarstellungen, vol. I. Springer-Verlag, 1972.

[17] A. Lunardi, Stabilizability of time-periodic parabolic equations, *SIAM J. Control Optim.*, 29 (1991), 810–828.

[18] J. Nagumo, S. Arimoto and S. Yoshizawa, An active pulse transmission line simulating nerve axon, in *Proceedings of the IRE*, 50 (1962), 2061–2070.

[19] D. Phan and S. S. Rodrigues, Stabilization to trajectories for parabolic equations, *Math. Control Signals Syst.*, 30 (2018), Art. 11, 50 pp.

[20] S. S. Rodrigues and K. Sturm, On the explicit feedback stabilisation of 1D linear nonautonomous parabolic equations via oblique projections, *IMA J. Math. Control Inform.*, (2018).

[21] J. M. Rogers and A. D. McCulloch, A collocation-Galerkin finite element model of cardiac action potential propagation, *IEEE Trans. Biomed. Eng.*, 41 (1994), 743–757.

[22] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, Reprint of the 1984 edition, AMS Chelsea Publishing, Providence, RI, 2001, [http://www.ams.org/bookstore-getitem/item=CHEL-343-H](http://www.ams.org/bookstore-getitem/item=CHEL-343-H).

[23] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung), *Math. Ann.*, 71 (1912), 441–479.

[24] W. H. Young, On classes of summable functions and their Fourier series, *Proc. R. Soc. Lond.*, 87 (1912), 225–229.

Received November 2018; revised March 2019.

*E-mail address*: karl.kunisch@uni-graz.at

*E-mail address*: sergio.rodrigues@ricam.oeaw.ac.at