Modeling of mesoscopic superconducting suspensions by the method of integral equations

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Abstract. A mathematical model of the current density distribution in an axially symmetric system of superconductors in an external magnetic field is formulated within the framework of the integral form of London equations. The discretization of the model is carried out and the stability and convergence of the computational scheme are investigated. The distribution of magnetic induction, the density of currents in the ring and the sphere and the attractive force between them are calculated. The possibility of stable levitation of the sphere near the center of the ring is shown.

1. Introduction
For the first time the method of integral equations was applied to the calculation of the force characteristics of superconducting levitation systems in the work [1]. It was numerically solved Fredholm integral equations of the second kind for the density of fictitious magnetic charges induced on the surface of an axially symmetric superconducting body in the Meissner state, levitating in the magnetic field of two coaxial current-carrying coils. It was further shown [2] that the distribution of the vector magnetic potential in an axially symmetric two-connected superconducting body with a given magnetic flux is determined by the Fredholm integral equation of the first kind. On the basis of this approach, the force levitation of a superconducting sphere over a superconducting torus with a trapped magnetic flux was calculated [3], the force of interaction of two coaxial superconducting rings with a current was calculated [4], and discrete mathematical models were developed that allow numerically modeling arbitrary axially symmetric superconducting levitation systems [5]. The general theory of calculating the distribution of the density of surface currents and force characteristics of superconducting suspensions based on the method of integral equations was developed in [6]. In these works, the superconductor was considered as an ideal diamagnetic, and the penetration of the magnetic field into the superconductor in the Meissner state to the London penetration depth ($\lambda \sim 10^{-7}$ m) was neglected. This is permissible in calculating magnetic fields and current densities in macroscopic superconducting suspensions [7], whereas for mesoscopic superconducting suspensions [8, 9, 10], the dimensions of the structural elements of which are comparable with $\lambda$, this approximation is unacceptable. It is necessary to take into account the penetration of the magnetic field into the superconductor. Thus, in [11] a numerical analysis of the system of integral equations, describing the distribution of current density in axially symmetric superconducting suspensions within the London electrodynamics of superconductors, was carried out.
The purpose of this work is to calculate the magnetic field induction distribution and current density in an axially symmetric system of superconducting rings and spheres of micron sizes in an external uniform constant magnetic field, taking into account the London depth of penetration of the magnetic field into the superconductor. The force of attraction between them is also calculated. The possibility of stable levitation of the sphere near the geometric center of the system is shown.

2. The integral formulation of the mathematical model of current density distribution in axially symmetric superconductors

The distribution of the magnetic field and the current density in the system of simple-connected and multiply-connected superconducting bodies in the external constant magnetic field in the London approximation is described by a system of equations [12, 13]

\[ \nabla \times \vec{B} = \vec{j}, \]
\[ \nabla \cdot \vec{B} = 0, \]
\[ \mu_0 \lambda^2 \nabla \times \vec{j} + \vec{B} = 0, \]

where \( \vec{B} \) is the magnetic induction vector, \( \vec{j} \) is current density, \( \lambda \) is London depth of penetration of a magnetic field into a superconductor, \( \mu_0 \) is magnetic permeability of vacuum.

If in equation (3) we write magnetic induction as the rotor of the vector potential \( \vec{A} \), we obtain the following relationship between the current density and the vector potential

\[ j + \vec{A} + \nabla \chi = 0. \]

Here \( \chi \) is single-valued scalar function that determines gauge of the vector potential of the solution domain.

The choice of gauge is determined by the physical nature of the task. Thus, for superconductors in an external constant magnetic field, using the London gauge of the vector potential \( \nabla \cdot \vec{A} = 0, \vec{A} \cdot \vec{n} = 0 \) and \( \nabla \chi = 0 \) (\( \vec{n} \) is a vector normal to the surface of the superconductor), expression (4) takes the form

\[ \mu_0 \lambda^2 \vec{j} + \vec{A} = 0. \]

Outside the superconductor the vector potential satisfies the Laplace equation. At the boundary of the region, the required boundary conditions are set.

Integral equations for the density of currents, induced in a system of superconductors in a constant magnetic field, can be obtained by expressing the vector potential in equation (5) through the density of currents using known integral relations [14]. However, the kernel of these equations, being a simple layer potential, leads to a strong singularity of the equations. Therefore the numerical solution of such equations has significant difficulties.

The situation is simplified by the presence of axial symmetry in the system. In this case the vector potential \( \vec{A} \) has only one component \( A_\varphi \) in the cylindrical coordinate system \((\rho, \varphi, z)\).

Using the expression for \( A_\varphi \) of a line circular current [14] and integrating over the cross-sectional area \( S_0 \) of all superconductors, we obtain

\[ A_\varphi^{(\text{int})} = \frac{\mu_0}{2\pi} \int \int_{S_0} J_\varphi'(\rho', z') \sqrt{\frac{\rho'}{\rho}} f(m) dS', \]

where \( J_\varphi \) is the angular component of the current density,
\[ f(m) = \frac{1}{\sqrt{m}} \left[ (2-m)F(m) - 2E(m) \right], \]  

(7)

\( F \) and \( E \) are complete elliptic integrals of the first and second kind with the parameter

\[ m = \frac{4\rho \rho'}{(\rho + \rho')^2 + (z-z')^2}. \]  

(8)

In the presence of an external magnetic field, a vector potential \( A^{(\text{ext})}_\phi \) is added to the potential \( A^{(\text{int})}_\phi \). For the case of a homogeneous magnetic field with induction \( B_0 \), parallel to the \( Oz \) axis, this potential has the form [14]

\[ A^{(\text{ext})}_\phi = \frac{\rho B_0}{2}. \]  

(9)

Substituting (6) and (9) into (5) we obtain an integral equation describing the distribution of current density in an axially symmetric system of superconductors [11]

\[ \mu_0 \lambda^2 \rho J_\phi = \frac{B_0}{2\pi} \int \sqrt{\rho \rho'} f(m) J_\phi(\rho', z') dS' = -\rho A^{(\text{ext})}_\phi, \]  

(10)

where the kernel of the integral operator \( f(m) \) is defined by the formulas (7), (8). This is a Fredholm equation of the second kind with a kernel having an integrable singularity at the coinciding coordinates of the observation point and the source point. However, at the limit \( \rho \to 0 \) it degenerates into an equation of the first kind, which leads to poor conditionality of the problem [11]. To eliminate this problem, we introduce a new variable

\[ I_\phi(\rho, z) = \rho J_\phi(\rho, z), \]  

(11)

As a result, equation (10) is converted to the form

\[ \mu_0 \lambda^2 I_\phi = \frac{B_0}{2\pi} \int \sqrt{\rho \rho'} f(m) I_\phi(\rho', z') dS' = -\rho A^{(\text{ext})}_\phi, \]  

(12)

The solution of this equation is already a well-conditioned problem.

3. Discretization of integral equations for current density in axially symmetric superconductors

We divide the superconductor cross section into quadrangular cells and select the grid nodes in the centers of gravity of the cells. Let's define the cross-section dimensions \( \Delta_x, \Delta_y \) of the cell so that its area has a form \( \Delta S = \Delta_x \Delta_y + o(\Delta_x \Delta_y) \). We denote the value of the variable (11) in the grid nodes as \( I_{i,j} \), and the coordinates of the nodes as \( \rho_{i,j}, z_{i,j} \). In case of mismatched values of at least one of the coordinates \((i \neq i' \text{ or } j \neq j')\) the integral over \((i, j)\)-cell in equation (12) is replaced by the quadrature formula:

\[ \int_{\Delta x, \Delta y} \sqrt{\rho \rho'} f(m(I_{i,j})) I_\phi(\rho', z') dS' = \sqrt{\frac{\rho_{i,j}}{\rho'_{i,j}}} f(m_{i,j}^{i'}) I_{i,j} \Delta_x \Delta_y + o(\Delta_x \Delta_y). \]  

(13)

where

\[ m^{i'}_{i,j} = \frac{4 \rho_{i,j} \rho'_{i,j}}{(\rho_{i,j} + \rho'_{i,j})^2 + (z_{i,j} - z_{i',j})^2}. \]
When matching indices in pairs of coordinates we use the method of singularity selection:

$$\int_{\Delta_i \times \Delta_j} \frac{\rho_{i,j}}{r} \cdot f(m) I_q(\rho', z') dS' = I_{r,f} \int_{\Delta_i \times \Delta_j} f(m) dS' + o(\Delta_i \Delta_j). \tag{14}$$

We use an asymptotic expansion of expression (7) leaving only two principal terms:

$$f(m) \sim \ln 4 - 0.5 \ln(1 - m).$$

Then we have:

$$1 - m = \frac{(\rho - \rho')^2 + (z - z')^2}{(\rho + \rho')^2 + (z - z')^2} = \frac{x^2 + y^2}{4 \rho^2} + o(x^2 + y^2), \quad f(m) \sim \ln(8\rho) - 2 - \frac{1}{2} \ln(x^2 + y^2).$$

Given the evenness of the obtained function, we present the integral in (14) as:

$$\int_{\Delta_i \times \Delta_j} f(m) dS' = \Delta_i \Delta_j \left[ \ln \frac{16\rho}{\sqrt{\Delta_i^2 + \Delta_j^2}} - \frac{1}{2} \left( 1 + \frac{\Delta_i}{\Delta_j} \arctan \frac{\Delta_j}{\Delta_i} + \frac{\Delta_j}{\Delta_i} \arctan \frac{\Delta_i}{\Delta_j} \right) \right] + o(\Delta_i \Delta_j). \tag{15}$$

As a result, the numerical model corresponding to equation (12) will have the form:

$$\lambda^2 I_{i,j} + \frac{1}{2\pi} \sum_{r,f} Q^{r,f}_{i,j} \Delta_i \Delta_j I_{r,f} = - \frac{\rho_{i,j} A^{\text{ext}}_{i,j}}{\mu_0}, \tag{16}$$

where according to the equalities (13), (14) and (15):

$$Q^{r,f}_{i,j} = \begin{cases} \sqrt{\frac{\rho_{i,j}}{\rho_{r,f}}} f(m^{r,f}_{i,j}), & i \neq i' \quad \text{and} \quad j \neq j', \\ \ln \left( \frac{16\rho}{\sqrt{\Delta^2 + \Delta^2}} \right) - \frac{1}{2} \left( 1 + \frac{\Delta_i}{\Delta_j} \tan \frac{\Delta_j}{\Delta_i} + \frac{\Delta_j}{\Delta_i} \tan^{-1} \frac{\Delta_i}{\Delta_j} \right), & i = i', \quad j = j'. \end{cases}$$

Using the solution of equation (16) and the expression for the induction of a line circular current, we can then calculate the z-component of the force, acting on the individual body of the system:

$$F_z = \mu_0 \sum_{(i,j) \in \Omega} I_{i,j} \frac{\Delta_i}{\sqrt{\rho_{i,j}}} \sum_{(i',j') \in i,j} \frac{z_{i,j} - z_{i',j'}}{\rho_{i,j} + \rho_{i',j'}} \frac{\rho_{i,j}}{\sqrt{\rho_{i,j}}} g(m^{r,f}_{i,j}) I^{r,f}_{i,j} \frac{\Delta^2}{\rho_{i,j}}. \tag{17}$$

Here $$\Omega$$ is the set of grid nodes in the section of a given body, and the notation is used:

$$g(m) = \frac{1}{\sqrt{m}} \left[ \frac{(2-m)E(m)}{1-m} - 2K(m) \right].$$

For regularization of the calculation of the integral in the expression (17), we take into account that the z-component of the self-action force of the circular current is zero. Therefore, the corresponding term is excluded from the sum (17).
4. Computational experiment with discrete model for current density in axially symmetric superconductors and its discussion

To calculate the distribution of current densities, magnetic field strength, total current and forces in mesoscopic current-carrying superconducting elements, an additional software module to the integrated computer simulation system of superconducting suspension modeling was developed [15]. With its help, the adequacy of the developed discrete model was checked, its convergence and accuracy were investigated. COMSOL Multiphysics was used for graphical representation of the results of calculations.

The system of coaxial plane ring and sphere (Figure 1) is considered as a test problem. The center of the sphere of radius \( R \) was aligned with the origin of the coordinate system, the center of the ring is located on the axis \( Oz \) at the point \( z = -h \). The ring has a rectangular cross-section with width \( a \) and height \( b \), the inner radius of the ring is \( r \). The calculation was carried out for the parameter values \( R = 1 \ \mu m, b = 1 \ \mu m, a = 2 \ \mu m, r = 1.5 \ \mu m, H_0 = 1 \times 10^4 \ \text{A/m} \) and for two values of the penetration depth \( \lambda = 0.09 \ \mu m \) and \( \lambda = 0.45 \ \mu m \).

![Figure 1. The geometric model of the ring-sphere system and its orientation in an external constant magnetic field \( B_0 \).](image)

To study the convergence and accuracy of the numerical model, the total current induced by a given magnetic field in a mesoscopic superconducting ring and sphere. The \( z \)-component \( F_z \) of the force acting on the ring in a model with a different number of degrees of freedom were calculated. The calculation results are given in Table 1. The numbers of cells used along the long and short sides of the rectangular section of the ring and the numbers of cells along the angular and radial coordinates in the section of the sphere are also given in Table 1. The condition number \( \nu_C \) of the model in the infinity norm was also calculated.

The analysis of the obtained data showed that the greatest influence on the accuracy of the calculation of the total current has the number of cells along the long side of the ring section and along the depth of the sphere. The calculated value of the total force depends mainly on the total number of degrees of freedom of the model. The condition number of the system increases approximately proportionally to \( \lambda^{-2} \) with decreasing depth of penetration and does not depend on the number of degrees of freedom of the numerical model. At the same time for the considered relations of parameters conditionality remains good. This circumstance is associated with the used variable transformation (11). The calculation of the force on the ring and on the sphere showed that their numerical values in the model completely coincide, which indicates the conservatism of the model.
Table 1. Influence of the number of degrees of freedom on the calculated characteristics of the model

| $\lambda, \mu$m | $h, \mu$m | Number of degrees of freedom | $v_C$ | $I_{\text{ring}}$, $10^{-2}$ A | $I_{\text{ph}}$, $10^{-3}$ A | $F_z$, $10^{-11}$ N |
|-----------------|------------|-------------------------------|-------|-----------------------------|-----------------------------|---------------------|
|                 | ring       | sphere                        |       |                             |                             |                     |
| 0.09            | 30×40      | 30×40                         | 146   | 5.17826                     | 5.51458                     | 1.97469             |
|                 | 60×20      | 60×20                         | 146   | 5.17978                     | 5.50601                     | 1.97063             |
|                 | 43×14      | 43×14                         | 141   | 5.17500                     | 5.50772                     | 1.96308             |
|                 | 30×20      | 30×20                         | 140   | 5.17715                     | 5.51871                     | 1.97344             |
|                 | 15×20      | 15×20                         | 133   | 5.16142                     | 5.55886                     | 1.97199             |
|                 | 30×10      | 30×10                         | 134   | 5.16622                     | 5.51303                     | 1.94964             |
| 0.45            | 60×20      | 60×20                         | 146   | 5.18656                     | 4.66771                     | 0                   |
|                 | 43×14      | 43×14                         | 7.33  | 3.61749                     | 4.09105                     | 0.97125             |
|                 | 30×10      | 30×10                         | 7.33  | 3.61623                     | 4.0886                      | 0.96507             |

The Runge-Romberg estimation of accuracy and convergence showed the linear rate of convergence of the model and the relative error in determining the magnitude of the force about 1.5% (at the lowest number of degrees of freedom) and about 0.5% (at the largest one). The error in determining the total current is about four times less than these values.

The force of attraction between the ring and the sphere at different distances between them was calculated for the values $\lambda = 0.09$ and $\lambda = 0.45$ microns (Figure 2). From the picture it can be seen that a stable levitation of the sphere near the center of the ring is possible.

![Figure 2](image_url)
The results of calculations of the magnetic induction distribution in the section passing through the axis of symmetry of the ring and the sphere are presented in Figures 3-4 ($\lambda = 0.09 \, \mu m, h = 0$).

![Figure 3](image3.png)

**Figure 3.** The lines of magnetic flux and vectors of magnetic flux density in close proximity to the superconducting ring and sphere in an external constant magnetic field.

![Figure 4](image4.png)

**Figure 4.** The value of $B_z$ magnetic induction component on a line passing perpendicular to the axis of symmetry of the system through the center of the sphere.

Figures 5-6 present the results of calculations of the current density in the sections of the sphere and ring ($\lambda = 0.09 \, \mu m, h = 0$).
Figure 5. Isolines for current density (in A/m²) in the sphere (a) and in the ring (b).

Figure 6. The value of the current density on the line passing perpendicular to the axis of symmetry of the system through the center of the sphere.

5. Conclusion
A mathematical model of current density distribution in the London approximation in an axially symmetric system of several superconducting bodies in an external constant magnetic field is formulated in the framework of the method of integral equations. The discretization of the model is carried out by the method of feature selection. Stability and convergence of the computational scheme for the case of a system of superconducting ring and sphere of micron sizes are investigated. The distributions of current density and magnetic induction in the system are calculated. The force of attraction between the ring and the sphere is also calculated for different values of the depth of
penetration of the magnetic field into the superconductor. The possibility of stable suspension of the sphere near the center of the ring is shown by computational experiment. This type of suspension can be used in quantum magnetomechanic experiments.

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