VISCOITY SOLUTIONS FOR NONLOCAL EQUATIONS
WITH SPACE-DEPENDENT OPERATORS

STEFANO BUCCHERI AND ULISSE STEFANELLI

ABSTRACT. We consider a class of elliptic and parabolic problems, featuring a specific nonlocal operator of fractional-laplacian type, where integration is taken on variable domains. Both elliptic and parabolic problems are proved to be uniquely solvable in the viscosity sense. Moreover, some spectral properties of the elliptic operator are investigated, proving existence and simplicity of the first eigenvalue. Eventually, parabolic solutions are proven to converge to the corresponding limiting elliptic solution in the long-time limit.

1. INTRODUCTION

The study of PDE problems driven by nonlocal operators is attracting an ever growing attention. This is in part motivated by the great relevance of such operators in applications, among which Lévy processes, differential games, and image processing, just to mention a few. The paramount example of nonlocal operator is the fractional laplacian, which can be defined in the following principal-value sense

\begin{equation}
( - \Delta )_s u ( x ) = \text{p.v.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy \quad \text{for } s \in (0, 1). \tag{1.1}
\end{equation}

In this paper we focus on elliptic and parabolic problems featuring a localized version of the classical fractional-laplacian operator, namely,

\begin{equation}
 u(x) \to \text{p.v.} \int_{\Omega(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy. \tag{1.2}
\end{equation}

In contrast with the classical fractional laplacian, the main feature in (1.2) is that integration is taken with respect to a \( x \)-dependent bounded set \( \Omega(x) \). Such a modification is inspired by the analysis of the hydrodynamic limit of kinetic equations [1] and of peridynamics [40]. We comment on these connection in Subsection 1.1 below.

In order to specify our setting further, let us recall that the homogeneous Dirichlet problem associated with the classical fractional laplacian \( ( - \Delta )_s \) (1.1) in a bounded set \( \Omega \subset \mathbb{R}^N \) requires to prescribe the nonlocal boundary condition \( u = 0 \) on \( \mathbb{R}^N \setminus \Omega \). Under such condition the fractional laplacian can be written as

\begin{equation}
( - \Delta )_s u(x) = \text{p.v.} \int_{\mathbb{R}^N} \frac{u(x) - u(y) \chi_{\Omega}}{|x - y|^{N+2s}} dy = \text{p.v.} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy + u(x) \int_{\Omega^c \setminus \Omega} \frac{1}{|x - y|^{N+2s}} dy
=: ( - \Delta )^\Omega_s u(x) + k(x) u(x), \tag{1.3}
\end{equation}

where \( \chi_{\Omega} \) indicates the characteristic function of \( \Omega \). The nonlocal operator \( ( - \Delta )^\Omega_s \) is usually called regional fractional laplacian. By indicating with \( d(x) \) the distance of \( x \in \Omega \) from the boundary \( \partial \Omega \), under mild regularity assumption on \( \partial \Omega \) the function \( k(x) \) can be proved to satisfy

\begin{equation}
\frac{\alpha}{d(x)^{2s}} \leq k(x) \leq \frac{\beta}{d(x)^{2s}}, \tag{1.4}
\end{equation}

for some \( 0 < \alpha < \beta \). We provide a proof of this property in Lemma 3.13. Differently from the classical fractional laplacian (1.1), the regional laplacian \( ( - \Delta )^\Omega_s \) acts on functions defined in \( \Omega \). Correspondingly, boundary conditions for the homogeneous Dirichlet problem for \( ( - \Delta )^\Omega_s \) can be directly prescribed on \( \partial \Omega \). The two operators \( ( - \Delta )_s \) and \( ( - \Delta )^\Omega_s \) differ especially in the vicinity of the boundary, as one can expect looking at (1.3).

Key words and phrases. Fractional laplacian, Perron method, principal eigenvalue, refiend maximum principle, half-relaxed limit, long-time behavior.
the boundary. While for any \( s \in (0, 1) \) the solution of the homogeneous Dirichlet problem associated with (1.1) behaves as \( d(x)^s \) as \( x \) approaches \( \partial \Omega \), the one associated to \( (\Delta)^s \) goes to zero as \( d(x)^{2s-1} \) for \( s \in (1/2, 1) \). In fact, for \( s \in (0, 1/2) \), the Dirichlet problem associated to the regional laplacian is not well-defined, independently of the regularity of \( \partial \Omega \). This can be explained via trace theory (the trace operator exists only if \( s > 1/2 \), see [43], [27]), or in term of the probabilistic process associated to \( (\Delta)^s \) (such a process reaches the boundary only if \( s > 1/2 \), see for instance [15]). Roughly speaking, we can say that the term \( k(x)u(x) \) regularizes the operator \( (\Delta)^s \) close to the boundary, by forcing a quantified convergence of solution to zero on approaching \( \partial \Omega \). The reader can find more on the relation between \( (\Delta)^s \) and \( (\Delta)^s \) in the recent survey [32] and in the references therein.

Inspired by position (1.2), by decomposition (1.3), and by property (1.4), we aim at considering more general nonlocal operators of the following form

\[
(1.5) \quad h(x)u(x) + \mathcal{L}_s(\Omega(x), u(x)),
\]

where \( \mathcal{L}_s(\Omega(x), u(x)) \) is defined as

\[
(1.6) \quad \mathcal{L}_s(\Omega(x), u(x_0)) = \text{p.v.} \int_{y \in \Omega(x)} \frac{u(x_0) - u(y)}{|x - y|^{N + 2s}} \, dy.
\]

In the following, we often use the change of coordinates \( z = x - y \). In this new reference frame we will write

\[
(1.7) \quad \tilde{\Omega}(x) = \{ z \in \mathbb{R}^N : x - z \in \Omega(x) \}.
\]

In (1.5), the function \( h(x) \in C(\Omega) \) is assumed to be given and to fulfill

\[
(1.8) \quad 0 < \frac{\alpha}{d(x)^{2s}} \leq h(x) \leq \frac{\beta}{d(x)^{2s}}, \quad \text{with} \quad 0 < \alpha \leq \beta,
\]

and the set-valued function \( x \to \Omega(x) \subset \Omega \) is assumed to satisfy

\[
(1.9) \quad \forall x \in \Omega : \lim_{y \to x} |\Omega(y) \Delta \Omega(x)| = 0,
\]

\[
(1.10) \quad \exists \zeta \in (0, 1/2), \forall x \in \Omega : \tilde{\Omega}(x) \cap B_r(0) = \Sigma \cap B_r(0) \text{ for all } r \leq \zeta d(x),
\]

for some given open set \( \Sigma \) such that

\[
(1.11) \quad \Sigma = -\Sigma \quad \text{and} \quad |\Sigma \cap (B_{2r}(0) \setminus B_r(0))| \geq q|B_{2r}(0) \setminus B_r(0)| \text{ for any } r > 0,
\]

with \( q \in (0, 1) \). Note that the symmetry of \( \Sigma \) is required for the well-definiteness of the operator on smooth functions and that assumption (1.9) ensure that operator \( \mathcal{L}_s \) from (1.6) is continuous with respect to his arguments. For a more extensive discussion on the hypothesis look at Section 1.2.

Our first goal is to show that the elliptic problem related to the nonlocal operator (1.5) given by

\[
(1.12) \begin{cases}
  h(x)u(x) + \mathcal{L}_s(\Omega(x), u(x)) = f(x) & \text{in } \Omega, \\
  u(x) = 0 & \text{on } \partial \Omega,
\end{cases}
\]

is uniquely solvable, for any \( f \in C(\Omega) \) such that

\[
(1.13) \quad \exists \eta_f \in (0, 2s), \ C > 0 : |f(x)|d(x)^{2s-\eta_f} \leq C.
\]

In the current setting it is natural to address problem (1.12) in the viscosity sense. Indeed, assumptions (1.9)-(1.11) imply that the operator (1.5) satisfies the comparison principle. Moreover the growth conditions (1.8) and (1.13) allow us to build a barrier for problem (1.12) of the form

\[
(1.14) \quad C_\alpha d(x)^\eta \quad \text{for some small } \eta > 0.
\]

Having comparison and barriers at hand, we can implement the Perron method and prove in Theorem 2.2 the existence of a unique viscosity solution for (1.12). Note that \( C_\alpha \) degenerates with \( \alpha \). In particular, if the term \( h(x) \) degenerates at the boundary, solvability of problem (1.12) may fail, at least for \( s \in (0, 1/2) \), see the above discussion.
A second aim of our paper is to address the spectral properties of the operator (1.5). We focus on the first eigenvalue associated to (1.5) with homogeneous Dirichlet boundary conditions and we show that there exists a unique $\lambda > 0$ such that the problem
\begin{equation}
\begin{cases}
h(x)v(x) + \mathcal{L}_s(\Omega(x), v(x)) = \lambda v(x) & \text{in } \Omega, \\
v(x) = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}
admits a strictly positive viscosity solution. Moreover such a solution is unique, up to multiplication by constants.

As our setting in nonvariational, the characterization of such first eigenvalue follows the approach of the seminal work [9]. More precisely, we define $\lambda$ as the supremum of the values $\lambda \in \mathbb{R}$ such that the problem
\begin{equation}
\begin{cases}
h(x)u(x) + \mathcal{L}_s(\Omega(x), u(x)) = \lambda u(x) + f(x) & \text{in } \Omega, \\
u(x) = 0 & \text{on } \partial \Omega, \\
u(x) > 0 & \text{in } \Omega,
\end{cases}
\end{equation}
admits a solution, for some given nonnegative and nontrivial $f \in C(\Omega)$ satisfying (1.13), see Theorems 2.3-2.4.

As we shall see, this characterization does not depend on the particular choice of $f$. Such a definition is slightly different from the usual one (see for instance [11], [17], [37] and [13] for the same approach in both local and nonlocal settings), being based on the concept of solution instead of that of supersolution.

A further focus of this paper is the study of the evolutionary counterpart of (1.12), namely, the parabolic problem
\begin{equation}
\begin{cases}
\partial_t u(t, x)(x) + h(x)u(x) + \mathcal{L}_s(\Omega(t, x), u(t, x)) = f(t, x) & \text{in } (0, T) \times \Omega, \\
u(t, x) = 0 & \text{on } (0, T) \times \partial \Omega, \\
u(0, x) = u_0(x) & \text{on } \Omega.
\end{cases}
\end{equation}

We prove existence and uniqueness of a global-in time viscosity solution to problem (1.17). See Theorem 2.5 below for the precise statement in a more general setting, where a time-dependent version of assumption (1.10) is considered. The behavior of the solution $u(x, t)$ for large times is then addressed in Theorem 2.6: Taking advantage of the characterization of $\lambda$ we prove that for any $\lambda < \lambda$ there exists $C_\lambda$ such that
\[|u(t, x) - u(x)| \leq C_\lambda \mathcal{P}(x)e^{-\lambda t},\]
where $\mathcal{P}$ is the normalized positive eigenfunction associated to $\lambda$, $u(x)$ is the solution of the elliptic problem (1.12) and $u(t, x)$ solves the parabolic problem (1.17) for the same time-independent forcing $f(x)$.

1.1. Relation with applications. As mentioned, the specific form of operator $\mathcal{L}_s$, in particular the dependence of the integration domain $\Omega(x)$ on $x$, occurs in connection with different applications.

A first occurrence of operators of the type of $\mathcal{L}_s$ is the study of the hydrodynamic limit of collisional kinetic equations with a heavy-tailed thermodynamic equilibrium. When posed in the whole space, the reference nonlocal operator in the limit is the classical fractional laplacian (1.1), see [34]. The restriction of the dynamics to a bounded domain with a zero inflow condition at the boundary asks for considering (1.2) instead [1]. In this connection, $\Omega(x)$ is defined to be the largest star-shaped set centered at $x \in \Omega$ and contained in $\Omega$. The heuristics for this choice is that a particle centered at $x$ is allowed to move along straight paths and is removed from the system as soon as it reaches the boundary. Hence, the possible interaction range of a particle sitting at $x$ is exactly $\Omega(x)$. In particular, the resulting hydrodynamic limit under homogeneous Dirichlet conditions features the nonlocal functional
\begin{equation}
a(x)u(x) + p.v. \int_{\Omega(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy =: a(x)u(x) + (-\Delta)^s_x u(x), \quad s \in (0, 1).
\end{equation}

Here, the function $a(x)$ has the following specific form
\begin{equation}
a(x) = \int_{\mathbb{R}^N} \frac{1}{|y|^{N+2s}} e^{-\frac{d(x, \sigma(y))}{|y|}} \, dy,
\end{equation}

with $d(x, \sigma(y))$ being the length of the segment joining $x \in \Omega$ with the closest intersection point between $\partial \Omega$ and the ray starting from $x$ with direction $\sigma(y) = \frac{x}{d(x, \sigma(y))}$. Clearly, if $\Omega$ is convex one has that $\Omega(x) = \Omega$ for all $x$ and the function $a(x)$ coincides, up to a constant, with the function $k(x)$ of (1.3) (see Lemma 3.12). In this case we recover exactly (up to a constant) the operator in (1.3). Note however that even in the case of a nonconvex domain $\Omega$, the function $a(x)$ satisfies the condition (1.8) (see Lemma 3.13).

A second context where nonlocal operators of the type of (1.6) arise is that of peridynamics [40]. This is a nonlocal mechanical theory, based on the formulation of equilibrium systems in integral instead of differential terms. Forces acting on the material point $x$ are obtained as a combined effect of interactions with other points in a given neighborhood. This results in an integral featuring a radial weight which modulates the influence of nearby points in terms of their distance [26, 41]. A reference nonlocal operator in this connection is

\begin{equation}
(1.20) \quad u(x) \mapsto \text{p.v.} \int_{B_\rho(x)} \frac{u(x) - u(y)}{|x - y|^{N + 2s}} dy.
\end{equation}

Here, $B_\rho(x)$ is the ball of radius $\rho > 0$ centered at $x$. In particular, the parameter $\rho$ measures the interaction range. Such operators have used to approximate the fractional laplacian in numerical simulations (see [25] and the reference therein), note also the parametric analysis in [16]. The operator $L_s$ in (1.6) corresponds hence to a natural generalization of the latter to the case of an interaction range which varies along the body, as could be the case in presence of a combination different material systems. This would correspond to choosing a varying $\rho(x)$.

1.2. Existing literature. To our knowledge, operator (1.5) has not been studied yet. Despite its simple structure when compared to the general operators usually allowed in the fully nonlinear setting, most of the available tools seem not to directly apply.

In this section, we aim at presenting a brief account of the literature in order to put our contribution in context. Following the seminal work [20], the existence theory of viscosity solutions, through comparison principle, barriers, and Perron method, has been generalized to a large class of elliptic and parabolic integral differential equations, see for instance [7, 5, 18, 21, 22] and [30].

Comparison principles for nonlocal problems in the viscosity setting can be found in [7], see also [30]. One of the key structural assumptions of these works reads, in our notation,

\begin{equation}
(1.21) \quad \int_{\mathbb{R}^N} |x_{\tilde{\Omega}(x)} - x_{\tilde{\Omega}(y)}||z|^{2-N-2s}dz \leq c|x - y|^2
\end{equation}

for some positive constant $c > 0$ (see assumption (35) in [7]). This type of condition allows the authors to implement the variable-doubling strategy of [20] for a large class of operators of so-called Lévy-Ito type. This is not expected here, for our set of assumptions does not imply (1.21) as the following simple argument shows: let $\Omega$ be the unitary ball centered at the origin and $\tilde{\Omega}(x) = B_{\rho(x)}$ for any $x \in \Omega$ with $\rho(x) = d(x)^{1/(2-2s)}$. Then, for any $x, y \in \Omega$ we get

\begin{equation}
\int_{\mathbb{R}^N} |x_{\tilde{\Omega}(x)} - x_{\tilde{\Omega}(y)}||z|^{2-N-2s}dz = \frac{\omega_N}{2 - 2s} |\rho(x)^{2-2s} - \rho(y)^{2-2s}| = \frac{\omega_N}{2 - 2s} |x - y|.
\end{equation}

Our alternative strategy is to assume that the operator is somehow translation invariant close to the singularity, with this property degenerating while approaching $\partial \Omega$, see assumption (1.10) above. This allows for some cancellations that bypass the problem of the singularity of the kernel. Instead of doubling variables, we use the inf/sup-convolution technique to prove that sum of viscosity subsolution is still a viscosity subsolution. Eventually, we also quote the interesting comparison result in [28]. There, the doubling of variable is combined with an optimal-transport argument. Such an approach, however, requires the uniform continuity of solutions, and it is not clear how to adapt it for general viscosity solutions.

As far as the construction of barriers is concerned, notice that the typical difficulty is to estimate from below a term of the type

\begin{equation}
\int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|x - y|^{N+2s}} dy,
\end{equation}

\begin{equation}
\int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|x - y|^{N+2s}} dy.
\end{equation}
which requires some regularity on the boundary of \( \Omega \). Of course we refer here to the standard case of the fractional laplacian, but the same idea can be extended to more general operators, see [5, Lemma 1]. In our case, since we impose a priori condition (1.8), we can actually deal with any open domain, paying the price of a poor control on the decay of the solution close to the boundary, see (1.14).

For an alternative approach to the Perron method, not relying on the comparison principle and therefore produces discontinuous viscosity solutions, we refer the interested reader to [35].

For a comprehensive overview on the numerous contributions to the regularity theory for viscosity solutions to nonlocal elliptic and parabolic equations, we address the reader to the rather detailed introductions of [39] and [31]. Here, we provide a small overview on some results more closely related to our work. The first fully PDE-oriented result about Hölder regularity for elliptic nonlocal operators has been obtained in [42]. A drawback of this approach is that it does not allow to consider the limit \( s \to 1 \). The first Hölder estimate which is robust enough to pass to the limit as \( s \to 1 \) has been obtained in [18] (see also [19]) and then generalized to parabolic equation in [21] and [22]. All these results apply to operators whose kernel is pointwise controlled from above and from below by the one of \((-\Delta)_s\). For results where such pointwise control is not available, we refer again to [39] and [31]. More in detail, our condition (1.11) is a simplified version of assumption (A3) in [39]. Condition (1.11) allows us to deduce an interior regularity estimate, in the spirit of the more general [35, Theorem 4.6].

As mentioned, we define the first eigenvalue of (1.5) following the approach in [9]. The advantage of this approach is that it is independent of the variational structure of the operator by directly relying on the maximum principle, as well as on the existence of positive (super-)solutions. For this reason it has been fruitfully used in the framework of viscosity solution for second order fully nonlinear differential equations, see for instance [11], [17], and [36].

An early result related to eigenvalues of nonlocal operators with singular kernel is in [8] where existence issues in presence of lower order terms are tackled. A closer reference is [24], where the principal eigenvalues of some fractional nonlinear equations, with inf/sup structure are studied. In this paper the authors prove, among other results, existence and simplicity of principal eigenvalues together with some isolation property and the antimaximum principle. Other results following the same line of investigation can be found in [37] and [13]. We point out that the operators considered in these works are just positively homogeneous (i.e. \( \mathcal{L}(u) \neq -\mathcal{L}(-u) \)), which gives rise to the existence of two principal half-eigenvalues, namely corresponding to differently signed eigenfunctions. We are not concerned with this phenomenon here. A common tool used in the previous works to prove existence of eigenvalues is a nonlinear version of the Krein-Rutmann theorem for compact operators, see again [37] and [13] and the references therein. Let us also mention the recent [12] and [14], that deal with a different kind of fractional operators.

To prove existence of the first eigenvalue, we follow a direct approach based on the approximation of problem (1.15). Unlike the previously quoted papers, we do not resort to a global regularity result to deduce either the compactness of the operator or the uniform convergence of approximating solutions. Instead, we combine the so-called half-relaxed-limit method, a version of the refined maximum principle and interior regularity result of [39, 35]. The half-relaxed limit method is a powerful tool to pass to the limit with no other regularity then uniform boundedness, see for instance [4] and the references therein. In general, the price to allow such generality is to handle discontinuous viscosity solutions. We however avoid this, for we are able to prove a refined maximum principle for (1.16) in the spirit of [9]. This, in turns, provides a comparison between sub and super solutions, eventually ensuring continuity.

Due to particular structure of our nonlocal operator, a key ingredient in our proof is a restriction procedure for (1.6) in subdomains of \( \Omega \), which is where the density assumption (1.11) is needed. Heuristically speaking, such assumption forces the operator to look the same at every scale, as for the kernel singularity and the behavior (1.8) of the term \( h \) (see Lemma 5.2). Eventually, our refined maximum principle allows us to show uniqueness of the first eigenvalue and its simplicity.

A general reference for the long-time behavior of solutions to nonlocal parabolic equation is the monograph [2]. We also refer to [10] and to the already mentioned [37]. In the latter work, a fractional operator with drift
term is considered and the viscosity solution of the associated homogeneous initial-boundary value problem is proved to converge to zero in the large-time limit.

Here, we consider a non-homogeneous parabolic equation with initial datum and homogeneous Dirichlet boundary condition. Moreover, we allow the coefficients of our operator to be time-dependent. Under suitable assumption on this time dependency, we prove that the parabolic solution converges to the stationary one exponentially in time. The exponential rate of convergence depends on the principal eigenvalue.

2. Statement of the main results

In this section, we collect our notation and state our main results.

Given any \( D \subset \mathbb{R}^M \), we indicate upper and lower semicontinuous functions on \( D \) as

\[
\text{USC}(D) = \left\{ u : D \to \mathbb{R} : \limsup_{z \to z_0} u(z) \leq u(z_0) \right\},
\]

\[
\text{LSC}(D) = \left\{ u : D \to \mathbb{R} : \liminf_{z \to z_0} u(z) \geq u(z_0) \right\}.
\]

We also write USC\(_D\)(\(D\)) and LSC\(_D\)(\(D\)) for the set of upper and lower semicontinuous functions that are bounded. Given a function \( u : D \to \mathbb{R} \) we indicate its upper and lower semicontinuous envelopes as

\[ u^*(z_0) = \limsup_{z \to z_0} u(z) \quad \text{and} \quad u_*(z_0) = \liminf_{z \to z_0} u(z), \]

respectively.

**Definition 2.1** (Viscosity solutions). *Elliptic case*: We say that \( u \in \text{USC}\(_\Omega\)(\(\Omega\)) (\(\in \text{LSC}\(_\Omega\)(\(\Omega\))) is a viscosity sub (super) solution to the equation

\[ h(x)u(x) + \mathcal{L}_s(\Omega, u(x)) = f(x) \]

if, whenever \( x \in \Omega \) and \( \varphi \in C^2(\Omega) \) are such that \( u(x) = \varphi(x) \) and \( u(y) \leq \varphi(y) \) for all \( y \in \Omega \), then

\[ h(x)\varphi(x) + \mathcal{L}_s(\Omega, \varphi(x)) \leq (\geq) f(x). \]

Moreover, \( u \in C(\overline{\Omega}) \) is a viscosity solution to problem (1.12) if it is both sub- and supersolution and satisfies the boundary condition \( u = 0 \) on \( \partial\Omega \) pointwise.

*Parabolic case*: We say that \( u \in \text{USC}\(_{(0,T)\times\Omega}\)(\((0,T)\times\Omega\)) (\(\in \text{LSC}\(_{(0,T)\times\Omega}\)((0,T)\times\Omega)) is a viscosity sub (super) solution to the equation

\[ \partial_t u(t,x) + h(x)u(t,x) + \mathcal{L}_s(\Omega, u(t,x)) = f(t,x) \quad \text{in } (0,T) \times \Omega \]

if, whenever \( (t,x) \in (0,T) \times \Omega \) and \( \varphi \in C^2((0,T) \times \Omega) \) are such that \( u(t,x) = \varphi(t,x) \) and \( u(\tau,y) \leq \varphi(\tau,y) \) for all \( \tau, y \in (0,T) \times \Omega \), then

\[ \partial_t \varphi(t,x) + h(x)\varphi(t,x) + \mathcal{L}_s(\Omega, \varphi(t,x)) \leq (\geq) f(t,x). \]

Moreover, \( u \in C([0,T] \times \overline{\Omega}) \) is a viscosity solution to (2.10) if it is both sub- and supersolution and satisfies the boundary and initial conditions

\[
\begin{align*}
\left\{ \begin{array}{ll}
\partial_t u(t,x) + h(x)u(t,x) + \mathcal{L}_s(\Omega, u(t,x)) &= f(t,x) & \text{in } (0,T) \times \Omega, \\
0 & \leq \varphi(t,x) & \text{on } \partial\Omega,
\end{array} \right.
\end{align*}
\]

pointwise.

We are now in the position of stating our main results.

**Theorem 2.2** (Well-posedness of the elliptic problem). *Let us assume (1.8)-(1.11), and (1.13). Then, problem (1.12) admits a unique viscosity solution \( u \). This satisfies \( |u(x)| \leq Cd(x)^\eta \) for some suitable small \( \eta > 0 \) and large \( C > 0 \).*
Theorem 2.3 (Well-posedness of the elliptic first-eigenvalue problem). Under the same assumption of Theorem 2.2, there exists a unique $\lambda > 0$ such that

$$
\begin{align*}
\begin{cases}
  h(x) v(x) + L_s(\Omega(x), v(x)) = \lambda v(x) & \text{in } \Omega, \\
  v(x) = 0 & \text{on } \partial \Omega,
\end{cases}
\end{align*}
$$

(2.2)

admits a nontrivial (strictly) positive viscosity solution. Such solution is unique, up to a multiplicative constant. Moreover, the first eigenvalue can be characterized as $\lambda = \sup E$, where

$$
E = \{ \lambda \in \mathbb{R} : \exists v \in C(\overline{\Omega}), v > 0 \text{ in } \Omega, v = 0 \text{ on } \partial \Omega \text{ such that } h v + L_s(v) = \lambda v + f \}.
$$

(2.3)

and $f \in C(\Omega)$ is any given positive and nonzero function satisfying (1.13). Note in particular, that the set $E$ is independent of $f$.

Theorem 2.4 (Well-posedness of the elliptic problem below the first eigenvalue). Under the same assumption of Theorem 2.2, for any $\lambda < \lambda$, the problem

$$
\begin{align*}
\begin{cases}
  h(x) u(x) + L_s(\Omega(x), u(x)) = \lambda u(x) + f(x) & \text{in } \Omega, \\
  u(x) = 0 & \text{on } \partial \Omega,
\end{cases}
\end{align*}
$$

(2.4)

admits a unique viscosity solution. In particular, we have $(-\infty, \lambda) = E$ where the set $E$ is defined in (2.3).

Let us now turn to the parabolic problem. Before stating our results, we present a time-depending generalization of the hypothesis of Theorem 2.2. We assume the set valued function $(t, x) \mapsto \Omega(t, x) \subset \Omega$ to fulfill the following assumptions

$$
\forall (t, x) \in (0, \infty) \times \Omega : \lim_{(\tau, y) \to (t, x)} |\Omega(\tau, y) \triangle \Omega(t, x)| = 0,
$$

(2.5)

$$
\forall T > 0, \exists \zeta \in (0, 1/2) : \forall (t, x) \in (0, T) \times \Omega : \Omega(t, x) \cap B_r(0) = \Sigma \cap B_r(0),
$$

(2.6)

for all $r \leq \zeta d(x)$ and $\Sigma$ as in (1.11). Recall that $\Omega(t, x) = \{ z \in \mathbb{R}^N : x - z \in \Omega(t, x) \}$. Moreover, we let $h \in C((0, \infty) \times \Omega)$ satisfy for $0 < \alpha \leq \beta$

$$
\frac{\alpha}{d(x)^{2s}} \leq h(t, x) \leq \frac{\beta}{d(x)^{2s}};
$$

(2.7)

and we assume that $f \in C((0, \infty) \times \Omega)$, $u_0 \in C(\Omega)$, and that there exists $\eta_1 \in (0, 2s)$ such that

$$
|f(x, t)| d(x)^{2s - \eta_1} \leq C,
$$

(2.8)

$$
|u_0(x)| d(x)^{-\eta_1} \leq C.
$$

(2.9)

Theorem 2.5 (Well-posedness of the parabolic problem). Let us fix $T \in (0, \infty)$. Under assumptions (1.11), (2.5)-(2.9) there exists a unique viscosity solution $u \in C((0, T) \times \Omega) \cap L^\infty((0, T) \times \Omega)$ of

$$
\begin{align*}
\begin{cases}
  \frac{\partial u(t, x)}{\partial t} + h(t, x) u(t, x) + L_s(\Omega(t, x), u(t, x)) = f(t, x) & \text{in } (0, T) \times \Omega, \\
  u(t, x) = 0 & \text{on } (0, T) \times \partial \Omega, \\
  u(0, x) = u_0(x) & \text{on } \Omega.
\end{cases}
\end{align*}
$$

(2.10)

Finally, we address the asymptotic behavior of the solution provided by Theorem 2.5 as $T \to \infty$. In order to do that, we require that all the time-dependent data in (2.10) suitably converge to their stationary counterparts in (1.12). In particular, we assume that, for some $\eta_2 \in (0, 2s)$ and $\lambda < \lambda$,

$$
|\Omega(t, x) \triangle \Omega(x)| d(x)^{2s - \eta_2} e^{\lambda t} \leq C_1,
$$

(2.11)

$$
|\Omega(t, x) \Delta \Omega(x)| d(x)^{-\eta_2} e^{\lambda t} \leq C_2.
$$

(2.12)

Assumption (2.6) must also be strengthened as follows

$$
\exists \zeta \in (0, 1/2), \forall (t, x) \in I \times \Omega : \tilde{\Omega}(t, x) \cap B_r(0) = \Sigma \cap B_r(0).
$$

(2.13)
Theorem 2.6 (Long-time behavior). Let us assume (1.8)-(1.10), (2.5)-(2.9), and (2.11)-(2.13). Let $u$ be the unique viscosity solution of the parabolic problem (2.10) on $(0, \infty)$, $v$ be the unique solution of the elliptic problem (1.12), and let $\lambda$ fulfill (2.11). Then, there exists $C = C(\lambda) > 0$ such that $|u(x, t) - v(x)| \leq Cd(x)^q e^{-\lambda t}$ for some small $\eta > 0$.

3. Preliminary material

We collect in this section some preliminary lemmas, which will be used in the proofs later on.

3.1. Background on viscosity theory. In the following, we will often make use of the continuity of the integral operator with respect to a suitable convergence of its variables. We state this property in its full generality, in order to be able to apply it in different contests throughout the paper.

Lemma 3.1 (Continuity of the integral operator). Let us consider a sequence of points $x_n \to \bar{x} \in \Omega$ and a family of bounded sets $\Theta(x_n), \Theta(\bar{x})$ such that $\chi_{\Theta(x_n)} \to \chi_{\Theta(\bar{x})}$ almost everywhere and, for some $\delta > 0$,

$$\Theta(x_n) \cap B_r(0) = \Sigma \cap B_r(0) \quad \text{for all } r \leq \delta \text{ and } n > 0,$$

with $\Sigma$ as in (1.11). Assume moreover that $\phi_n \to \phi$ pointwise with

$$|\phi_n(x_n) - \phi_n(x_n + z) - q_n \cdot z| \leq C|z|^2 \quad \text{for } |z| \leq \delta,$$

for some $q_n, q \subset \mathbb{R}^N$ and $C$ a positive constant that does not depend on $n$. Then

$$L_s(\Theta(x_n), \phi_n(x_n)) \to L_s(\Theta(\bar{x}), \phi(\bar{x})).$$

Proof. For any $r < \delta$, we can decompose the integral operator as follows

$$L_s(\Theta(x_n), \phi_n(x_n)) = L_s(\Theta(x_n) \setminus B_r(x_n), \phi_n(x_n)) + L_s(\Theta(x_n) \cap B_r(x_n), \phi_n(x_n)).$$

Notice that the first integral in the right-hand side of (3.2) is nonsingular and, for any fixed $r$, it readily passes to the limit as $n \to \infty$, thanks to the convergence of $\Theta(x_n)$ and $\phi_n$. Instead, the second one has to be meant in the principal value sense

$$L_s(\Theta(x_n) \cap B_r(x_n), \phi_n(x_n)) = \lim_{\rho \to 0} \rho \int_{|z| \leq r} \frac{\phi_n(x_n) - \phi_n(x_n + z) - q_n \cdot z}{|z|^{N+2s}} \chi_{\Sigma} dz,$$

where we have used that $\Sigma = -\Sigma$. Thanks to assumption (3.11) we get

$$|L_s(\Theta(x_n) \cap B_r(x_n), \phi_n(x_n))| \leq C \frac{\omega_N}{2 - 2s} r^{(2 - 2s)}.$$

Being a similar computation valid for $L_s(\Theta(\bar{x}), \phi(\bar{x}))$, we get that

$$|L_s(\Theta(x_n), \phi_n(x_n)) - L_s(\Theta(\bar{x}), \phi(\bar{x}))| \leq |L_s(\Theta(x_n) \setminus B_r, \phi_n(x_n)) - L_s(\Theta(\bar{x}) \setminus B_r, \phi(\bar{x}))| + 2C \frac{\omega_N}{2 - 2s} r^{(2 - 2s)}.$$

The assertion follows by taking the limit in the inequality above, first as $n \to \infty$ and then as $r \to 0$. □

Now we provide a suitably localized, equivalent definition of viscosity solutions, which will turn out useful in proving the comparison Lemma 4.3 below. Such equivalence is already known (see, for instance, [7]). For completeness, we give here a statement and a proof in the elliptic case.

Lemma 3.2 (Equivalent definition). We have that $u \in \text{USC}_b(\Omega) \subset \text{LSC}_b(\Omega)$ is a viscosity subsolution (super-solution, respectively) to the equation in (1.12) if and only if, whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u(x_0) = \varphi(x_0)$ and $u(y) \leq \varphi(y)$ for all $y \in \Omega$, then for all $B_r(x_0) \subset \Omega$ the function

$$(3.3) \quad \varphi_r(x) = \begin{cases} \varphi(x) & \text{in } B_r(x_0), \\ u(x) & \text{otherwise}, \end{cases}$$
satisfies

\begin{equation}
(3.4) \quad h(x_0)\varphi_r(x_0) + L_s(\Omega(x_0), \varphi_r(x_0)) \leq (\geq) f(x_0).
\end{equation}

A similar result hold in the parabolic case.

**Proof.** We prove only the equivalence in the case of subsolutions, the case of supersolution being identical.

Let us assume at first that \( u \in \text{USC}_b(\Omega) \) fulfills the conditions of Lemma 3.2. We want to check that is a viscosity subsolution in the sense of Definition 2.1. Let \( \varphi \in C^2(\Omega) \) be such that \( u - \varphi \) has a global maximum at \( x_0 \) and \( \varphi(x_0) = u(x_0) \). It then follows that, for any \( B_r(x_0) \subset \Omega \),

\[
f(x_0) \geq h(x_0)\varphi_r(x_0) + L_s(\Omega(x_0), \varphi_r(x_0))
\]

\[
= h(x_0)u(x_0) + \int_{\Omega(x) \setminus B_r(x_0)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy + \int_{\Omega(x) \cap B_r(x_0)} \frac{u(x) - \varphi(y)}{|x - y|^{N+2s}} \, dy
\]

\[
\geq h(x_0)u(x_0) + \int_{\Omega(x)} \frac{u(x) - \varphi(y)}{|x - y|^{N+2s}} \, dy
\]

\[
= h(x_0)\varphi(x_0) + L_s(\Omega(x_0), \varphi(x_0)),
\]

where the first inequality \( \geq \) comes from (3.4) and the second one follows as \( u \leq \varphi \in \Omega \).

To show the reverse implication, let us assume that \( u \in \text{USC}_b(\Omega) \) is a viscosity subsolution to (1.12) according to Definition 2.1. Let \( \varphi \in C^2(\Omega) \) be such that \( u - \varphi \) has a maximum at \( x_0 \in \Omega \) and \( \varphi(x_0) = u(x_0) \) and, for any \( B_r(x_0) \subset \Omega \), let \( \varphi_r \) be the auxiliary function defined in (3.3). As a first step, we modify \( \varphi_r \) as \( \varphi_{r,n}(x) = \varphi_r(x) + \frac{1}{n}|x - x_0|^2 \) and notice that, for any \( n \in \mathbb{N} \), the function \( u - \varphi_n \) has a strict local maximum at \( x_0 \) and

\[
u - \varphi_{r,n} \leq -\frac{\nu^2}{4n} \quad \text{in} \quad \Omega \setminus B_{\frac{r}{n}}(x_0).
\]

Let \( \psi_1, \psi_2 \in C^\infty(\Omega) \) be a partition of unity associated to the concentric balls \( B_{\frac{r}{n}}(x_0) \) and \( B_{\frac{2r}{n}}(x_0) \), namely, \( 0 \leq \psi \leq 1 \), \( \psi_1 = 1 \) in \( B_{\frac{r}{n}}(x_0) \), \( \psi_1 = 0 \) in \( \Omega \setminus B_{\frac{2r}{n}}(x_0) \), and \( \psi_1 + \psi_2 = 1 \). Let us finally set

\[
\zeta_n = \psi_1 \varphi_{r,n} + \rho_m * (\psi_2 \varphi_{r,n})
\]

where \( \rho_m \) is a mollifier and \( \{m_n\}_{n \in \mathbb{N}} \) is a sequence of numbers converging to 0 to be suitably determined. Notice that \( \zeta_n(x) = \varphi_{r,n}(x) + \frac{1}{n}|x - x_0|^2 \) in \( B_{\frac{r}{n}}(x_0) \) and that \( \zeta_n(x_0) = u(x_0) \) for \( n \) large. Moreover, thanks to the properties of mollifiers, for any \( n \in \mathbb{N} \) there exists \( m_n \in \mathbb{N} \) such that

\[
u - \zeta_n \leq -\frac{\nu^2}{8n} \quad \text{in} \quad \Omega \setminus B_{\frac{r}{n}}(x_0).
\]

Then, \( \zeta_n \in C^2 \) is a good test function for Definition 2.1 and we have

\[
\tilde{h}(x_0)u(x_0) + L_s(\Omega(x_0), \zeta_n(x_0)) \leq f(x_0).
\]

Taking the limit as \( n \to \infty \) we prove the assertion applying Lemma 3.1. Note indeed that \( \zeta_n \to \varphi \) in \( C_2(B_{\frac{r}{n}}(x_0)) \) and \( \zeta_n \to u \) pointwise in \( \Omega \setminus B_{\frac{r}{n}}(x_0) \).

In proving the stability of families of viscosity solutions, a suitable notion of limit for sequences of upper semicontinuous functions has to be considered, see for instance [18]. We introduce the following.

**Definition 3.3** (\( \Gamma \)-convergence). A sequence of upper-semicontinuous functions \( v_n \) is said to \( \Gamma \)-converge to \( v \) in \( D \subset \mathbb{R}^M \) if

\[
(3.5) \quad \text{for all converging sequences} \ z_n \to \tilde{z} \text{ in } D : \quad \lim_{n \to \infty} \sup_{n} v_n(z_n) \leq v(\tilde{z})
\]

\[
(3.6) \quad \text{for all} \ \tilde{z} \in D \ 	ext{there exists a sequence} \ z_n \to \tilde{z} : \quad \lim_{n \to \infty} v_n(z_n) = v(\tilde{z}).
\]
This concept corresponds (up to a sign change) to a localized version of the classical $\Gamma$-convergence notion, see [23], hence the same name.

Clearly, uniformly converging sequences in $\Omega$ are also $\Gamma$-converging. Moreover, $\Gamma$-convergence readily ensues in connection with the upper-semicontinuous envelope of a family of functions. Both examples will play a role in the sequel.

The following stability result is an adaptation of the classical one provided in Proposition 4.3 of [20] (see also Lemma 4.5 in [18]).

**Lemma 3.4 (Stability).** Let us consider $v \in \text{USC}_b((0, T) \times \Omega)$ and $f \in C((0, T) \times \Omega)$. Assume moreover that 

i) \( \{v_n\} \subset \text{USC}_b((0, T) \times \Omega) \) $\Gamma$–converges to $v$ in $(0, T) \times \Omega$, 

ii) \( f_n \rightarrow f, h_n \rightarrow h \) locally uniformly, and $|\Omega_n \Delta \Omega| \rightarrow 0$ as $n \rightarrow \infty$, 

iii) \( \partial_t v_n(t, x) + h_n(t, x)v_n(t, x) + \mathcal{L}_a(\Omega_n(t, x), v_n(t, x)) \leq f_n(t, x) \) in $(0, T) \times \Omega$ in the viscosity sense.

Then, if $\Omega(t, x)$ satisfies (2.5), (2.6), it follows that 
\[
\partial_t v(t, x) + h(t, x)v(t, x) + \mathcal{L}_a(\Omega(t, x), v(t, x)) \leq f(t, x)
\]
in the viscosity sense.

**Remark 3.5.** An elliptic version of the Lemma holds true by assuming (1.9)-(1.11). Notice that the stationary case can be straightforwardly obtained from the evolutionary one upon interpreting $u : \Omega \rightarrow \mathbb{R}$ as a trivial time-dependent function $\tilde{u}(t, x) = u(x)$ on $(0, T) \times \Omega$. In fact, if such a function is touched from above or from below by a smooth function $\phi \in C^2((0, T) \times \Omega)$ at some $(t, x) \in (0, T) \times \Omega$, we have that $\partial_t \phi(t, x) = 0$. We hence conclude that such time-dependent representation $\tilde{u}(t, x)$ of a subsolution (or supersolution) $u(x)$ of the elliptic problem is subsolution (supersolution, respectively) of its parabolic counterpart.

**Proof of Lemma 3.4.** Let us assume that $v - \varphi$ has a strict global maximum, equal to 0, at $(\bar{t}, \bar{x}) \in (0, T) \times \Omega$. Taking $\varphi_\theta = \varphi + \theta([t - \bar{t}]^2 + |x - \bar{x}|^2)$, we have that also the $\text{sup}(v - \varphi_\theta)$ is reached only at $(\bar{t}, \bar{x})$. Owing to the assumption i), we know that there exists a sequence of points $(\tau_n, y_n) \subset (0, T) \times \Omega$ such that 
\[
(\tau_n, y_n, v_n(\tau_n, y_n)) \to (\bar{t}, \bar{x}, v(\bar{t}, \bar{x})).
\]
Thanks to the penalization in the definition of $\varphi_\theta$ and assumption i), for $n$ large enough we have that 
\[
v_n(\tau_n, y_n) - \varphi_\theta(\tau_n, y_n) \leq \sup_{(0, T) \times \Omega} (v_n - \varphi_\theta) = v_n(t_n, x_n) - \varphi(\tau_n, y_n) = \epsilon_n,
\]
for some $\{(t_n, x_n)\} \in (0, T)$ such that, up to a not relabeled subsequence, $(t_n, x_n) \to (\bar{t}, \bar{x}) \in (0, T) \times \Omega$. Using again the $\Gamma$–convergence of $v_n$, we find 
\[
(v - \varphi_\theta)(\bar{t}, \bar{x}) \geq \lim_{n \to \infty} \sup_{(0, T) \times \Omega} (v_n - \varphi_\theta)(t_n, x_n) \geq \lim_{n \to \infty} (v_n - \varphi_\theta)(\tau_n, y_n) = (v - \varphi)(\bar{t}, \bar{x}) = 0.
\]
Since the supremum of $v - \varphi$ is strict, this implies that $(\bar{t}, \bar{x}) = (\bar{t}, \bar{x})$ and that $\epsilon_n \rightarrow 0$. Moreover, setting $\varphi_n = \varphi_\theta + \epsilon_n$, it follows that $\sup_{(0, T) \times \Omega} (v_n - \varphi_\theta) = (v_n - \varphi_\theta)(t_n, x_n) = 0$.

In conclusion, we have proved that $v_n - \varphi_n$ has a global maximum at $(t_n, x_n)$ (for $n$ large enough) and $v_n(t_n, x_n) = \varphi_n(t_n, x_n)$, that $\epsilon_n \rightarrow 0$, and that $(t_n, x_n) \to (\bar{t}, \bar{x})$. Since each $v_n$ is a subsolution at $(\bar{t}, \bar{x})$ we get 
\[
\partial_t \varphi_n(t_n, x_n) + h_n(t_n, x_n)v_n(t_n, x_n) + \mathcal{L}_a(\Omega_n(t_n, x_n), \varphi_n(t_n, x_n)) \leq f_n(t_n, x_n).
\]
The first two terms in the left-hand side and the one in the right-hand side easily pass to the limit for $n \to \infty$, thanks to the continuity of $h$ and $f$ and to the definition of $\varphi_n$. In order to deal with the integral operator notice that 
\[
\mathcal{L}_a(\Omega_n(t_n, x_n), \varphi_n(t_n, x_n)) = \mathcal{L}_a(\Omega(t_n, x_n), \varphi(t_n, x_n)).
\]
Since $\varphi$ is smooth, we can use Lemma 3.1, with $\Theta(x_n) = \Omega_n(t_n, x_n)$ and $\phi_n(\cdot) = \varphi(t_n, \cdot)$, and pass to the limit with respect to $n$. Eventually, we get 
\[
\partial_t \varphi(\bar{t}, \bar{x}) + h(\bar{t}, \bar{x})v(\bar{t}, \bar{x}) + \mathcal{L}_a(\Omega(\bar{t}, \bar{x}), \varphi(\bar{t}, \bar{x})) \leq f(\bar{t}, \bar{x}),
\]
for any \( \theta > 0 \). Taking the limit (using Lemma 3.1 again) as \( \theta \to 0 \), we obtain the desired result. \( \square \)

The previous stability result highlights the robustness of the notion of viscosity solution in relation to limit procedures. Notice that, given any uniformly bounded sequence of viscosity (sub/super) solutions of a certain family of equations, one can always find a \( \Gamma \)–limit and this is the candidate (sub/super) solution for the limiting equation. Such candidate is given by the lower/upper half relaxed limit

\[
\overline{u}(x) = \sup \{ \limsup_{n \to \infty} v_n(x_n) : x_n \to x \}, \quad \underline{v}(x) = \inf \{ \liminf_{n \to \infty} v_n(x_n) : x_n \to x \}.
\]

It is easy to check that \( v_n \Gamma \)-converges to \( \overline{u} \) and \( -v_n \Gamma \)-converges to \( -\underline{v} \). The key point here is that no compactness on the sequence \( v_n \) is required for the existence of \( \overline{u} \) and \( \underline{v} \), as boundedness suffices. As we shall see in Section 5, this is a particularly powerful tool when dealing with equations that satisfy a comparison principle.

### 3.2. Sup- and infconvolution

In the sequel we often need to determine the equation (or inequality) solved by the difference of sub- or supersolutions. Note that, since such functions are not smooth, the property of being a sub- or supersolution may be not preserved by taking differences. To deal with this difficulty, we need to use suitable regularization of the involved functions. Let us start recalling the definition of supconvolution of \( u \in \text{USC}_b((0, T) \times \Omega) \), namely,

\[
u^\epsilon(t, x) = \sup_{(\tau, y) \in (0, T) \times \Omega} \left\{ u(\tau, y) - \frac{1}{\epsilon} (|x - y|^2 + |t - \tau|^2) \right\}.
\]

Notice that, since \( u \) is upper semicontinuous and bounded, for \( \epsilon \) small enough the supremum above is reached inside \((0, T) \times \Omega\). To be more precise, let us adopt the following notation: for any \((t, x) \in (0, T) \times \Omega\), let \((t^\epsilon, x^\epsilon)\) be a point with the following property

\[
u^\epsilon(t, x) = u(t^\epsilon, x^\epsilon) - \frac{1}{\epsilon} (|x - x^\epsilon|^2 + |t - t^\epsilon|^2).
\]

Then,

\[
(\ref{eq:supconvolution}) \quad (|x - x^\epsilon|^2 + |t - t^\epsilon|^2) \leq 2\epsilon\|u\|_{L^\infty((0, T) \times \Omega)}.
\]

Moreover, by construction, it results that the parabola

\[
P(t, x) = u(t^\epsilon, x^\epsilon) - \frac{1}{\epsilon} (|t - t^\epsilon|^2 + |x - x^\epsilon|^2)
\]

touches \( u^\epsilon \) from below at \((\bar{t}, \bar{x})\). This shows that the supconvolution is semiconvex in \((0, T) \times \Omega\). Such a property is particularly useful in order to pointwise evaluate some viscosity inequality related to subsolutions. Indeed, let us assume that \( u^\epsilon \) is touched from above by a smooth function at \((\bar{t}, \bar{x})\). Then, thanks to its semiconvexity property, we deduce that \( u^\epsilon \in C^{1,1}(\bar{t}, \bar{x}) \), namely there exist \( q \in \mathbb{R}^{N+1} \) and \( C > 0 \) such that, in a neighborhood of \((\bar{t}, \bar{x})\),

\[
(\ref{eq:semiconvex}) \quad \left| u^\epsilon(t, x) - u^\epsilon(\bar{t}, \bar{x}) - q \cdot \frac{t - \bar{t}}{x - \bar{x}} \right| \leq C(|t - \bar{t}|^2 + |x - \bar{x}|^2).
\]

This means that the time derivative and the fractional operator can be evaluated pointwise at \((\bar{t}, \bar{x})\) (see Lemma 3.6 below).

Similarly, the infconvolution of a function \( v \in \text{LSC}((0, T) \times \Omega) \cap L^\infty((0, T) \times \Omega) \) is defined as

\[
u_\epsilon(t, x) = \inf_{(\tau, y) \in (0, T) \times \Omega} \left\{ u(\tau, y) + \frac{1}{\epsilon} (|x - y|^2 + |t - \tau|^2) \right\},
\]

and we let \((t_\epsilon, x_\epsilon)\) be the point where

\[
u_\epsilon(t, x) = u(t_\epsilon, x_\epsilon) - \frac{1}{\epsilon} (|x - x_\epsilon|^2 + |t - t_\epsilon|^2).
\]
The property of the inf convolution correspond to those of the sup convolution up to the trivial transformation $v_*=\theta v$. Omitting further details for the sake of brevity, we limit ourselves in proving the following inequality on sup convolutions.

**Lemma 3.6.** Let us assume (1.11), (2.5), and (2.6) and that $u(t,x)$ is a viscosity subsolution to (2.10) and let $\bar{u}(t,x)$ be its sup convolution. If $\bar{u}$ is touched from above by some smooth function at $(\bar{t}, \bar{x})$, the following inequality holds in a classical sense

\begin{equation}
\partial_t \bar{u}(\bar{t}, \bar{x}) + h(\bar{t}, \bar{x})\bar{u}(\bar{t}, \bar{x}) + \mathcal{L}_s(\bar{t}, \bar{x}), \bar{u}(\bar{t}, \bar{x})) \leq f(\bar{t}, \bar{x}),
\end{equation}

where $(\bar{t}, \bar{x})$ satisfies (3.9).

**Proof.** Let us assume that $\bar{u}$ is touched from above by a smooth $\varphi$ at $(\bar{t}, \bar{x})$. Let us recall that there exists $(\bar{t}, \bar{x})$ such that

\[ u(\bar{t}, \bar{x}) = u(\bar{t}, \bar{x}) - \frac{1}{\epsilon}|(\bar{t} - \bar{t})|^2 + |\bar{x} - \bar{x}|^2 \quad \text{and} \quad (\bar{t}, \bar{x}) \to (\bar{t}, \bar{x}) \quad \text{as} \quad \epsilon \to 0.
\]

By definition of sup convolution we have that

\[ u(\bar{t} + \bar{t} - \bar{t}, x + \bar{x} - \bar{x}) \geq u(t, y) - \frac{1}{\epsilon}|(\bar{t} - \bar{t})|^2 + |\bar{x} - \bar{x}|^2.
\]

Choosing $(t, y) = (t, x)$ it follows

\[ u(\bar{t} + \bar{t} - \bar{t}, x + \bar{x} - \bar{x}) \geq u(t, x) - \frac{1}{\epsilon}|(\bar{t} - \bar{t})|^2 + |\bar{x} - \bar{x}|^2.
\]

Then, by defining

\[ \varphi(t, x) = \varphi(t + \bar{t} - \bar{t}, x + \bar{x} - \bar{x}) + \frac{1}{\epsilon}|(\bar{t} - \bar{t})|^2 + |\bar{x} - \bar{x}|^2,
\]

we infer that $\varphi$ touches from above $u$ at $(\bar{t}, \bar{x})$. Since $u$ is a viscosity subsolution to (2.10), it follows that

\begin{equation}
\partial_t \varphi_r(\bar{t}, \bar{x}) + h(\bar{t}, \bar{x})\varphi_r(\bar{t}, \bar{x}) + \mathcal{L}_s(\bar{t}, \bar{x}), \varphi_r(\bar{t}, \bar{x})) \leq f(t, x).
\end{equation}

Now notice that by construction of $\varphi$ it results $\partial_t \varphi_r(\bar{t}, \bar{x}) = \partial_t \varphi_r(\bar{t}, \bar{x})$ and

\[ \varphi_r(\bar{t}, \bar{x}) - \varphi_r(\bar{t}, \bar{x}) = \varphi_r(\bar{t}, \bar{x}) - \varphi_r(\bar{t}, \bar{x} + z),
\]

that implies

\[ \mathcal{L}_s(\bar{t}, \bar{x}, \varphi_r(\bar{t}, \bar{x})) = \int_{\Omega(\bar{t}, \bar{x})} \frac{\varphi_r(\bar{t}, \bar{x}) - \varphi_r(\bar{t}, \bar{x} + z)}{|z|^{N+2s}} dz = \mathcal{L}_s(\bar{t}, \bar{x}, \varphi_r(\bar{t}, \bar{x})).
\]

Then, (3.15) becomes

\[ \partial_t \varphi_r(\bar{t}, \bar{x}) + h(\bar{t}, \bar{x})\varphi_r(\bar{t}, \bar{x}) + \mathcal{L}_s(\bar{t}, \bar{x}, \varphi_r(\bar{t}, \bar{x})) \leq f(\bar{t}, \bar{x}).
\]

Since $\bar{u}$ is touched from above by a smooth function at $(\bar{t}, \bar{x})$, we know that $\bar{u} \in C^{1,1}(\bar{t}, \bar{x})$ (see (3.11)) and then $\partial_t \varphi_r(\bar{t}, \bar{x}) = \partial_t \varphi_r(\bar{t}, \bar{x})$. Recalling assumption (2.6) and since $(\bar{t}, \bar{x}) \to (\bar{t}, \bar{x}) \in (0, T) \times \Omega$, we deduce that there exists $\delta > 0$ such that, for any $r < \delta$, we can decompose the nonlocal operator as follows

\[ \mathcal{L}_s(\bar{t}, \bar{x}, \varphi_r(\bar{t}, \bar{x})) = \int_{\Sigma \cap B_r(0)} \frac{\varphi(\bar{t}, \bar{x}) - \varphi(\bar{t}, \bar{x} + z)}{|z|^{N+2s}} dz + \int_{\Omega(\bar{t}, \bar{x}) \setminus B_r(0)} \frac{\varphi(\bar{t}, \bar{x}) - \varphi(\bar{t}, \bar{x} + z)}{|z|^{N+2s}} dz.
\]

The integral on $\Sigma \cap B_r(0)$ is well defined converges to zero as $r \to 0$, due to the smoothness of $\varphi$ and the symmetry of $\Sigma$. To deal with the second integral we apply Lemma 3.1 with $\bar{x} = \bar{x}$, $\Theta(\bar{x}) = \Omega(\bar{t}, \bar{x}) \setminus B_r(0)$ and $\phi_n(\bar{t}, \bar{x}) = \phi(\bar{t}, \bar{x})$. We deduce that $\mathcal{L}_s(\bar{t}, \bar{x}, \varphi_r(\bar{t}, \bar{x})) \to \mathcal{L}_s(\bar{t}, \bar{x}, \varphi(\bar{t}, \bar{x}))$ as $r \to 0$. This completes the proof of the Lemma. 

\[ \square \]
A similar inequality holds for inconvolution. For the sake of later reference, we state it here below without proof. This can be obtained by straightforwardly adapting the argument of Lemma 3.6.

**Lemma 3.7.** Let us assume that $v(t,x)$ is a viscosity supersolution to (2.10) and let $v_e(t,x)$ be its inconvolution. If $v_e$ is touched from below by some smooth function $(t,\bar{x})$, the following inequality holds in a classical sense

$$
\partial_t v_e(t,\bar{x}) + h(t_e, \bar{x}_e)v_e(t,\bar{x}) + \mathcal{L}_s(\Omega(t_e, \bar{x}_e), v_e(t,\bar{x})) \leq f(t_e, \bar{x}_e),
$$

where $(t_e, x_e)$ satisfies (3.17).

In the following Lemma we eventually state that the difference of a super- and a subsolution is still a supersolution.

**Lemma 3.8 (Difference).** Let us consider $h_1(t,x), h_2(t,x)$ that satisfy (2.7), $\{\Omega_1(t,x)\}, \{\Omega_2(t,x)\}$ that satisfy (2.5), (2.6) and two functions $u \in USC_b((0,T) \times \Omega), v \in LSC_b((0,T) \times \Omega)$ that solve in the viscosity sense

$$
\partial_t u(t,x) + h_1(t,x)u(t,x) + \mathcal{L}_s(\Omega_1(t,x), u(t,x)) \leq f_1(t,x) \quad \text{in } (0,T) \times \Omega
$$

$$
\partial_t v(t,x) + h_2(t,x)v(t,x) + \mathcal{L}_s(\Omega_2(t,x), v(t,x)) \geq f_2(t,x) \quad \text{in } (0,T) \times \Omega,
$$

respectively. Then $w = u - v$ solves in the viscosity sense

$$
\partial_t w(t,x) + h_1(t,x)w(t,x) + \mathcal{L}_s(\Omega_1(t,x), w(t,x)) \leq \hat{f}(x,t) \quad \text{in } (0,\infty) \times \Omega
$$

where

$$
\hat{f}(x,t) = f_1(t,x) - f_2(t,x) + M|h_1(t,x) - h_2(t,x)| + 2M \int_{|z| \geq \frac{1}{2D(x)}} \frac{|\chi_{\Omega_1(t,x)} - \chi_{\Omega_2(t,x)}|}{|z|^{N+2s}} \, dz,
$$

with $M = \max\{|u|_{L^\infty((0,T) \times \Omega)}, |v|_{L^\infty((0,T) \times \Omega)}\}$.

**Proof.** Recalling definitions (3.8) and (3.12), let us consider the function $w^e(t,x) = u^e(t,x) - v_e(t,x)$ and assume that it is touched from above by a $\varphi \in C^2((0,\infty) \times \Omega)$ at point $(t,\bar{x})$. This means that

$$
w^e(t,\bar{x}) - v_e(t,\bar{x}) = \varphi(t,\bar{x}) \quad \text{and} \quad w^e(t,\bar{x}) \leq \varphi \quad \text{in } \Omega.
$$

This latter fact, together with the semiconvexity property of both $w^e$ and $-v_e$, implies that $w^e$ and $-v_e$ are $C^{1,1}(\bar{T}, \bar{x})$ (see (3.11)). We are hence in the position of applying Lemmas 3.6 and 3.7 and evaluating the inequalities satisfied by $w^e$ and $v_e$ pointwise. We have that

$$
\partial_t w^e(t,\bar{x}) + h_1(t,\bar{x})w^e(t,\bar{x}) + \mathcal{L}_s(\Omega_1(t,\bar{x}), w^e(t,\bar{x})) \leq f_1(t,\bar{x}),
$$

and that

$$
\partial_t v_e(t,\bar{x}) + h_2(t,\bar{x})v_e(t,\bar{x}) + \mathcal{L}_s(\Omega_2(t,\bar{x}), v_e(t,\bar{x})) \geq f_2(t,\bar{x}).
$$

Recalling the ordering assumption between $w^e$ and $\varphi$ and combining the two inequalities above, we infer that, for $\epsilon$ small enough,

$$
\partial_t \varphi_e(t,\bar{x}) + h_1(t,\bar{x})\varphi_e(t,\bar{x}) + \mathcal{L}_s(\Omega_1(t,\bar{x}), \varphi_e(t,\bar{x}))
\leq \partial_t w^e(t,\bar{x}) + h_1(t,\bar{x})w^e(t,\bar{x}) + \mathcal{L}_s(\Omega_1(t,\bar{x}), w^e(t,\bar{x}))
\leq f_1(t,\bar{x}) - f_2(t,\bar{x}) + M|h_1(t,\bar{x}) - h_2(t,\bar{x})|
+ 2M \int_{\mathbb{R}^N} \frac{|\chi_{\Omega_1(t,\bar{x})} - \chi_{\Omega_2(t,\bar{x})}|}{|z|^{N+2s}} \, dz.
$$

Let us stress that, for $\epsilon$ small enough, the integral term in the right hand side above is finite. Indeed, thanks to the assumption (2.6) and since $(\bar{t}, \bar{x}) \to (\bar{t}, \bar{x})$ and $(t_e, \bar{x}_e) \to (\bar{t}, \bar{x})$ as $\epsilon \to 0$, then

$$
\exists \epsilon_0 > 0 : \forall \epsilon \in (0, \epsilon_0) \quad B_{\frac{1}{2}d(z)} \cap \tilde{\Omega}(\bar{t}, \bar{x}) = B_{\frac{1}{2}d(z)} \cap \tilde{\Omega}(t_e, \bar{x}_e) = B_{\frac{1}{2}d(z)} \cap \Sigma.
$$
This implies that
\[
\int_{\mathbb{R}^N} \frac{|\chi_{\tilde{\Omega}_1(t', x')} - \chi_{\tilde{\Omega}_2(t', x')}|}{|z|^{N+2s}} dz = 2M \int_{|z| \geq \frac{1}{4} \delta(x)} \frac{|\chi_{\tilde{\Omega}_1(t', x')} - \chi_{\tilde{\Omega}_2(t', x')}|}{|z|^{N+2s}} dz 
\]
\[
\leq \left( \frac{2}{\zeta d(x)} \right)^{N+2s} |\tilde{\Omega}_1(t', x') \Delta \tilde{\Omega}_2(t', x')|. 
\]

Since (3.17) is true any time that \( w' \) is touched from above by a smooth \( \varphi \) at some point in \((0, T) \times \Omega\), we can conclude that \( w' \) solves in the viscosity sense
\[
\partial_t w'(t, x) + h_s(t', x')w'(t, x) + \mathcal{L}_s(\Omega_1(t', x'), w'(t, x)) \leq f'(t, x),
\]
where
\[
h_s(t, x) = h_1(t', x'),
\]
\[
\Omega_s(t, x) = \Omega_1(t', x'),
\]
\[
f'(t, x) = f_1(\tilde{t}, \tilde{x}) - f_2(\tilde{t}, \tilde{x}) + M|h_1(\tilde{t}, \tilde{x}) - h_2(\tilde{t}, \tilde{x})| + 2M \int_{|z| \geq \frac{1}{4} \delta(x)} \frac{|\chi_{\tilde{\Omega}_1(t', x')} - \chi_{\tilde{\Omega}_2(t', x')}|}{|z|^{N+2s}} dz
\]
and the point \( t', x' \) is related to \((t, x)\) through (3.9) and (3.10). Thanks to Lemma 3.4, we can pass to the limit in (3.17) as \( \epsilon \to 0 \) and obtain the desired result. \( \square \)

For later purpose we also explicitly state an elliptic version of Lemma 3.8.

**Corollary 3.9.** Assume (1.8)-(1.10), that \( f_1, f_2 \in C(\Omega) \) satisfy (1.13) and that \( u \in USC_b(\Omega), v \in LSC_b(\Omega) \) solve
\[
h(x)u(x) + \mathcal{L}_s(\Omega(x), u(x)) \leq f_1(x) \quad \text{in } \Omega
\]
\[
h_2(x)v(x) + \mathcal{L}_s(\Omega(x), v(x)) \geq f_2(x) \quad \text{in } \Omega,
\]
respectively. Then \( w = u - v \) solves
\[
\partial_t w(x) + h_s(t, x)w(x) + \mathcal{L}_s(\Omega(x), w(x)) \leq f_1(x) - f_2(x) \quad \text{in } \Omega.
\]

**Remark 3.10.** For the sake of brevity, we do not provide a proof of Corollary 3.9, see Remark 3.5.

### 3.3. Regularity

By adapting the regularity theory for fully nonlinear integro-differential equations from [18, Sec. 14] we can prove the following.

**Theorem 3.11 (Hölder regularity).** Let us assume (1.8)-(1.11), that \( f \in C(\Omega) \cap L^\infty(\Omega) \), and that \( u \in C(\Omega) \cap L^\infty(\Omega) \) solves in the viscosity sense
\[
h(x)u(x) + \mathcal{L}_s(\Omega(x), u(x)) = f(x) \quad \text{in } \Omega.
\]
Then, for any open sets \( \Omega' \subset \subset \Omega'' \subset \subset \Omega \), it follows that
\[
\|u\|_{C^\gamma(\Omega')} \leq \tilde{C},
\]
where \( \gamma \in (0, 1) \) and \( \tilde{C} = \tilde{C}(\|f\|_{L^\infty(\Omega)}, s, \zeta, d(\Omega', \Omega''), \|u\|_{L^\infty(\Omega)}) \).

**Proof.** We claim that
\[
\mathcal{L}_s(\Sigma(x), u(x)) = \tilde{f}(x) \quad \text{in } \Omega
\]
in the viscosity sense, where \( \Sigma(x) = \Sigma + x \) and \( \tilde{f} \in C(\Omega) \) is a suitable function such that \( \tilde{f}(x) \approx d(x)^{-2s} \) close to \( \partial \Omega \). Once such a property is verified, the proof of the Lemma follows from [35, Theorem 4.6]. See also [39, Theorem 7.2], the parabolic problem is treated.

In order to prove the claim, we follow the ideas of [18, Sec. 14]. Let us assume that, any time \( u \) is touched from above with a smooth function at some \( x \in \Omega \), \( u \) belongs to \( C^{1,1}(x) \). Using Lemma 3.1, we deduce that
\[
h(x)u(x) + \mathcal{L}_s(\Omega(x), u(x)) \leq f(x)
\]
pointwise for any such a \( x \in \Omega \). Thanks to assumption (1.10), the nonlocal operator can be estimated as follows
\[
\mathcal{L}_s(\Omega(x), u(x)) = \int_{\Sigma \cap B_{\delta(d(x))}} \frac{u(x) - u(x + z)}{|z|^{N+2s}} dz + \int_{\Omega(x) \setminus B_{\delta(d(x))}} \frac{u(x) - u(x + z)}{|z|^{N+2s}} dz + \int_{\Sigma \setminus B_{\delta(d(x))}} \frac{u(x) - u(x + z)}{|z|^{N+2s}} dz
\]
\[
= \int_{\Sigma} \frac{u(x) - u(x + z)}{|z|^{N+2s}} dz + \int_{\Omega(x) \setminus B_{\delta(d(x))}} \frac{u(x) - u(x + z)}{|z|^{N+2s}} dz - \int_{\Sigma \setminus B_{\delta(d(x))}} \frac{u(x) - u(x + z)}{|z|^{N+2s}} dz
\]
\[
= \mathcal{L}_s(\Sigma(x), u(x)).
\]

Let us set
\[
\tilde{f}(x) = f(x) - \int_{\Omega(x) \setminus B_{\delta(d(x))}} \frac{u(x) - u(x + z)}{|z|^{N+2s}} dz + \int_{\Sigma \setminus B_{\delta(d(x))}} \frac{u(x) - u(x + z)}{|z|^{N+2s}} dz.
\]

This proves that
\[
\mathcal{L}_s(\Sigma(x), u(x)) \leq \tilde{f}(x),
\]
assuming that \( u \) belongs to \( C^{1,1}(x) \). Let us apply this argument to the sup convolution \( u^\varepsilon \). By definition of the sup convolution, we recall that, any time \( u^\varepsilon \) is touched from above by a smooth function \( \varphi \) at \( x \in \Omega \), then \( u^\varepsilon \in C^{1,1}(x) \). Moreover, thanks to Theorem 3.6, we have that
\[
h(x^\varepsilon)u(x) + \mathcal{L}_s(\Omega(x^\varepsilon), u(x)) \leq f(x^\varepsilon)
\]
pointwise for any such a \( x \in \Omega \). Then, thanks to the argument above, it follows that
\[
\mathcal{L}_s(\Sigma(x^\varepsilon), u^\varepsilon(x)) \leq \tilde{f}(x^\varepsilon).
\]

Eventually, thanks to the stability property of viscosity solution (see Lemma 3.4), we can pass to the limit in the inequality above to conclude that
\[
\mathcal{L}_s(\Sigma(x), u(x)) \leq \tilde{f}(x),
\]
in the viscosity sense. Similarly, one can check that
\[
\mathcal{L}_s(\Sigma(x), u(x)) \geq \tilde{f}(x),
\]
and the proof of the initial claim follows. \( \square \)

3.4. Equivalence with the fractional laplacian. We now present two technical lemmas, shedding light on the relation between the operator in (1.12) and the classical fractional laplacian.

**Lemma 3.12** (Equivalence). The function defined in (1.19) can be equivalently written as
\[
a(x) = \Gamma(2s + 1) \int_{S(x)^c} \frac{1}{|z|^{N+2s}},
\]
where \( S(x) \) is the largest star-shaped subset of \( \Omega \) centered at \( x \) and \( S(x)^c = \Omega - S(x) \).

If \( \Omega \) is convex, the fractional laplacian \( (-\Delta)_s \) defined in (1.3) is equivalent to the elliptic operator defined in (1.18), as
\[
\Gamma(2s + 1)(-\Delta)_s \varphi(x) = a(x) \varphi(x) + (-\Delta)_s^* \varphi(x)
\]
on suitably smooth function \( \varphi \).

**Proof.** We firstly notice that
\[
a(x) = \int_{\mathbb{R}^N} \frac{1}{|y|^{N+2s}} \frac{e^{-\frac{d(x,y)^2}{|y|^2}}}{|y|^2} dy
\]
\[
= \int_{\omega_{N-1}} \int_0^\infty \frac{1}{\rho^{4+2s}} \frac{e^{-\frac{d(x,\sigma)}{\rho}}}{\rho^2} d\rho d\sigma
\]
\[
= \int_{\omega_{N-1}} \frac{1}{d(x,\sigma)^{2s}} d\sigma \int_0^\infty \rho^{2s-1} e^{-\frac{\rho}{d(x,\sigma)^2}} d\rho = \Gamma(2s) \int_{\omega_{N-1}} \frac{d\sigma}{d(x,\sigma)^{2s}}
\]
where we recall that \( d(x, \sigma(y)) \) denotes the distance between \( x \) and the first point reached on \( \partial \Omega \) by the ray from \( x \) with direction \( \sigma(y) = y/|y| \). On the other hand, we have that
\[
\int_{S(x)} \frac{1}{|z|^{N+2s}} \, dz = \int_{\omega_{N-1}} \int_{d(x, \sigma)} \rho^{-1-2s} \, d\rho \, d\sigma = \frac{1}{2s} \int_{\omega_{N-1}} \frac{d\sigma}{d(x, \sigma)^{2s}}.
\]

The conclusion follows from the fact that \( \Gamma(2s+1) = \Gamma(2s)2s \).

Assume now that \( \Omega \) is convex. Then \( S(x) \equiv \Omega \) for any \( x \in \Omega \) and
\[
a(x) = \Gamma(2s+1) \int_{\{x \in \Omega\}^c} \frac{1}{|z|^{N+2s}} \, dz = \Gamma(2s+1) \int_{\Omega^c} \frac{1}{|x-y|^{N+2s}} \, dy.
\]
Then, recalling (1.18), we have that, for any \( \varphi \in C_c^\infty(\Omega) \),
\[
a(x) \varphi(x) + (-\Delta)_x^s \varphi(x) = \Gamma(2s+1) \left[ \int_{\Omega^c} \frac{1}{|x-y|^{N+2s}} \varphi(x) \, dy + \int_{\Omega} \varphi(x) - \varphi(y) \right] \left[ \int_{\Omega^c} \frac{1}{|x-y|^{N+2s}} \, dy \right] = \Gamma(2s+1) \, p.v. \int_{\Omega^c} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} \, dy = \Gamma(2s+1) (-\Delta)_x^s \varphi(x). \tag*{□}
\]

We now introduce a class of domains for which the function \( k(x) \) defined in (1.3) satisfies the bounds (1.4). To this aim, we assume that the complement \( \Omega^c \) satisfies a uniform positive density condition, namely that there exists \( \rho_0 > 0 \) and \( \kappa > 0 \) such that
\[
|B_{\rho} (\bar{x}) \cap \Omega^c| \geq \kappa |B_{\rho}(\bar{x})| \quad \text{for all } \bar{x} \in \partial \Omega \text{ and } \rho \in (0, \rho_0).
\]

Let us stress that (3.18) is weaker than the exterior cone condition.

**Lemma 3.13 (Bounds on \( h \)).** Let \( \Omega \) be an open bounded set of \( \mathbb{R}^N \) with \( \Omega^c \) satisfying (3.18). Then, the function
\[
h(x) = \int_{\Omega^c} \frac{1}{|x-y|^{N+2s}} \, dy
\]
satisfies the bounds (1.4).

**Proof.** To start with, notice that, we have \( \Omega^c \subset B_d(x) \) for all \( x \in \Omega \). Then,
\[
h(x) \leq \int_{B_d(x)^c} \frac{1}{|x-y|^{N+2s}} \, dy = \int_{|z| \geq d(x)} \frac{1}{|z|^{N+2s}} \, dz = \frac{\omega_N}{2s} \, \frac{d(x)^{2s}}{d(x)^{2s}}
\]
whence the upper bound in (1.4).

To prove the lower bound let us take \( x \in \Omega \) such that \( d(x) \leq \rho_0 \). We then have
\[
h(x) = \int_{\Omega^c} \frac{1}{|x-y|^{N+2s}} \, dy \geq \int_{\Omega^c \cap B_{d(x)}(\bar{x})} \frac{1}{|x-y|^{N+2s}} \, dy
\]
where \( \bar{x} \in \partial \Omega \) is such that \( d(x) = |x - \bar{x}| \). Using that if \( y \in B_{d(x)}(\bar{x}) \) then \( |x-y| \leq |x| + |y| \leq 2d(x) \) and taking advantage of (3.18) (recall that \( d(x) \leq \rho_0 \)), the inequality above becomes
\[
h(x) \geq \frac{1}{(2d(x))^{N+2s}} |\Omega^c \cap B_{d(x)}(\bar{x})| \geq C \, \frac{1}{d(x)^{2s}}.
\]

4. Existence and uniqueness

The existence of viscosity solutions follows by applying the classical Perron method. We give here full details of this construction in the parabolic case, hence proving Theorem 2.5. The proof of Theorem 2.2 follows the same lines, being actually simpler. We comment on it at the end of the section.

As a first step toward the implementation of the Perron method, we start by providing a suitable barrier for the elliptic problem.
Lemma 4.1 (Barriers). Let us assume that the set valued function \( x \to \Omega(x) \) satisfies (1.10), (1.11). Then there exists a positive \( \tilde{\eta} = \tilde{\eta}(N, s, \zeta, \alpha) \) such that, for any \( \eta \in (0, \tilde{\eta}] \), the function \( u_\eta(x) = d(x) \eta \) solves the inequality
\[
\frac{\alpha}{d(x) t} u_\eta(x) + \mathcal{L}_s(\Omega(x), u_\eta(x)) \geq \frac{\alpha}{2} d(x)^{\eta - 2s} \text{ in } \Omega
\]
in the viscosity sense.

Remark 4.2. Notice that under assumptions (1.11) and (2.6), the function \( u_\eta \), with \( \eta < \tilde{\eta}(N, s, \zeta, \alpha) \) also satisfies in the viscosity sense
\[
\partial_t u_\eta(x) + \frac{\alpha}{d(x) t} u_\eta(x) + \mathcal{L}_s(\Omega(t, x), u_\eta(x)) \geq \frac{\alpha}{2} d(x)^{\eta - 2s} \text{ in } (0, T) \times \Omega,
\]
since for all \( t \in (0, \infty) \) the set valued function \( x \to \Omega_t(x) = \Omega(t, x) \) satisfies (1.10) with \( \zeta_T \). If we moreover assume (2.13), the barrier is uniform in time.

Proof of Lemma 4.1. Fix \( x \in \Omega \) and assume that there exists \( \varphi \in C^2(\Omega) \) such that \( u_\eta - \varphi \) has a minimum in \( \Omega \) at \( x \) and that \( u_\eta(x) = \varphi(x) \). We have to check (see Lemma 3.2) that for any \( B_r(x) \subset \Omega \)
\[
\frac{\alpha}{d(x) t} u_\eta(x) + \mathcal{L}_s(\Omega(x), \varphi_r(x)) \geq \frac{\alpha}{2} d(x)^{\eta - 2s}.
\]
Since \( \varphi \) touches \( u_\eta \) from below at \( x \), we deduce [3, Prop. 2.14] that there exists a unique \( \tilde{x} \in \partial \Omega \) such that \( d(x) = |x - \tilde{x}| \). To simplify notation, from now on we use a system of coordinates that is centered at \( \tilde{x} \), so that \( d(x) = |x| \). Notice that
\[
\varphi(x) - \varphi(x + z) \geq u_\eta(x) - u_\eta(x + z) \geq (|x|^\eta - |x + z|^\eta) \quad \text{for all } z \in \tilde{\Omega}(x),
\]
the last inequality following from the fact that \( d(x + z) \leq |x + z| \). We then have that
\[
\mathcal{L}_s(\Omega(x), \varphi_r(x)) \geq \int_{\tilde{\Omega}(x)} \frac{|x|^\eta - |x + z|^\eta}{|z|^{N+2s}} dz
\]
\[
= \left( \int_{\{|z| \leq \zeta |x| \}} \frac{|x|^\eta - |x + z|^\eta}{|z|^{N+2s}} dz + \int_{\{|z| > \zeta |x| \}} \frac{|x|^\eta - |x + z|^\eta}{|z|^{N+2s}} dz \right) =: I_1 + I_2,
\]
where we have used that \( \tilde{\Omega}(x) \cap B_{\zeta |x|} = \Sigma \cap B_{\zeta |x|} \) (see assumption (1.10)). Thanks to Taylor expansion, we get that
\[
|x + z|^\eta = |x|^\eta + \eta |x|^{\eta - 2} z \cdot z + \frac{1}{2} \eta (\eta - 2) |x|^\eta |z|^2 + \eta |z|^{\eta - 2} |z|^2 \leq |x|^\eta + \eta |x|^{\eta - 2} z \cdot z + 2^{\eta - 3} \eta |x|^{\eta - 2} |z|^2 \quad \text{for } |z| \leq \zeta |x|
\]
with \( \zeta = x + tz \) for some \( t \in (0, 1) \) and the inequality follows from neglecting a negative term and from the fact that \( |\xi| \geq (1 - \zeta) |x| \geq \frac{1}{2} |x| \). This implies that
\[
I_1 \geq -\int_{\{|z| \leq \zeta |x| \}} (\eta |x|^{\eta - 2} z \cdot z + \eta 2^{\eta - 3} |x|^{\eta - 2} |z|^2) \frac{1}{|z|^{N+2s}} dz
\]
\[
= -\eta 2^{\eta - 3} |x|^{\eta - 2} \int_{\{|z| \leq \zeta |x| \}} |z|^{2-N-2s} dz \geq -\eta C |x|^{\eta - 2s},
\]
where, in the second line, we have used that the set \( \{|z| \leq \zeta |x| \} \cap \Sigma \) is radially symmetric and that the first order term of the expansion vanishes in the principal value sense. On the other hand, one has that
\[
I_2 \geq |x|^{\eta - 2s} \int_{\{|y| \geq \zeta \}} \frac{1 - (1 + |y|)^\eta}{|y|^{N+2s}} dy.
\]
Combining this last inequality with (4.3) and (4.4), we obtain that
\[
\mathcal{L}_s(\Omega(x), \varphi_r(x)) \geq -g(\eta) d(x)^{\eta - 2s},
\]
where
\[ g(\eta) = C \left( \eta + \int_{\{|y| \geq \epsilon\}} \frac{1 - (1 + |y|)^\eta}{|y|^{N+2s}} \, dy \right). \]

Notice that, thanks to the Lebesgue Dominated Convergence Theorem, the integral in the brackets above goes to zero as \( \eta \to 0 \). It follows that
\[ \frac{\alpha}{d(x)^{2s}} u_\eta(x) + \mathcal{L}_s(\Omega(x), \phi_s(x)) - f(x) \geq (\alpha - g(\eta))d(x)^{\eta-2s}. \]

At this point it is enough to choose \( \eta \leq \bar{\eta} \) satisfying \( \alpha - g(\bar{\eta}) = \alpha/2 \) in order to conclude the proof. \( \square \)

Let us now provide a comparison principle for equation (2.10). This relies on Lemma 3.8, which is in turn based on the regularization of sub/super-solutions through sup/inf convolution.

**Lemma 4.3 (Comparison).** Assume (1.11), (2.5)-(2.7), that \( T \in (0, \infty) \), that \( u(t, x) \) and \( v(t, x) \) are sub- and supersolutions to (2.10), respectively, that they are ordered on the boundary, namely \( u \leq v \) on \((0, T) \times \partial\Omega\), and that \( u(0, \cdot) \leq v(0, \cdot) \) on \( \Omega \). Then,
\[ u \leq v \text{ in } (0, T) \times \Omega. \]

**Proof.** Given \( \delta > 0 \), let us introduce the function \( u_\delta(t, x) = u(t, x) - \frac{\delta}{T-t} \) and notice that it is a viscosity subsolution to (2.10), namely,
\[ \partial_t u_\delta(t, x) + h(t, x)u_\delta(t, x) + \mathcal{L}_s(\Omega(t, x), u_\delta(t, x)) - f(t, x) \leq \frac{-\delta}{T-t} < 0. \]

We firstly show that \( u_\delta \leq v \), for any \( \delta > 0 \), and then conclude the proof by taking the limit as \( \delta \) goes to 0. Using Lemma 3.8, we deduce that \( w = u_\delta - v \) solves in the viscosity sense
\[ \partial_t w(t, x) + h(t, x)w(t, x) + \mathcal{L}_s(\Omega(t, x), w(t, x)) \leq 0 \text{ in } \Omega. \]

Let us assume by contradiction that \( \sup_{(0, T) \times \Omega} w = M > 0 \). Due to the ordering assumption on the parabolic boundary \((0, T) \times \partial\Omega\) and on the initial conditions, and the behavior of \( u_\delta \) as \( t \to T^- \), \( M \) is attained inside at \((\tilde{t}, \tilde{x}) \in (0, T) \times \Omega\). This implies that the constant function \( M \) touches from above \( w \) at the point \((\tilde{t}, \tilde{x})\), and then it is an admissible test function for \( w \) to be a subsolution. It follows that
\[ \frac{\alpha}{d^{2s}(\tilde{x})} M \leq h(\tilde{t}, \tilde{x})M \leq 0, \]
that is clearly a contradiction. Then \( u_\delta - v = w \leq 0 \) for any \( \delta > 0 \) and the assertion follows. \( \square \)

**Corollary 4.4 (Elliptic comparison).** Assume (1.8)-(1.11) that \( u(x) \) and \( v(x) \) are sub- and supersolutions to (1.12), respectively, and that \( u \leq v \) on \( \partial\Omega \). Then,
\[ u \leq v \text{ in } \Omega. \]

The proof of this corollary can be easily deduced from that of Corollary 3.9 and we omit the details for the sake of brevity.

We are now ready to present a first existence result, which relies on the possibility of finding suitable barriers for the parabolic problem. We will later check that such barriers can be easily obtained from Lemma 4.1.

**Theorem 4.5 (Existence, given barriers).** Assume (1.11), (2.5)-(2.7). Let \( T \in (0, \infty) \) and \( \underline{l}(t, x) \) and \( \bar{l}(t, x) \) be sub- and supersolution to (2.10), respectively, with \( \underline{l} = \bar{l} = 0 \) on \((0, T) \times \partial\Omega\). Then, for any \( u_0 \in C(\Omega) \) such that \( \underline{l}(0, x) \leq u_0(x) \leq \bar{l}(0, x) \) for all \( x \in \Omega \), problem (2.10) admits a unique viscosity solution.

**Proof.** We aim at applying Perron’s method. Let us set
\[ A = \left\{ w \in \text{USC}(\Omega \times (0, T)) : \underline{l}(t, x) \leq w(t, x) \leq \bar{l}(t, x) \text{ for } (t, x) \in (0, T) \times \partial\Omega, \right. \]
\[ \left. w \text{ is a subsolution to (2.10), and } w(x, 0) \leq u_0(x) \right\}. \]
Since $\bar{f} \in A \neq \emptyset$, we can set
\[
 u(t, x) = \sup_{w \in A} w(t, x).
\]
By definition, it follows that for any $(\bar{t}, \bar{x}) \in (0, T) \times \Omega$ there exists a sequence $\{v_n\} \subset A$ that $\Gamma$-converges to the upper semicontinuous envelope $u^*$ at $(\bar{t}, \bar{x})$. We can use Lemma 3.4 to show that $u^*$ is a subsolution to the equation in (2.10). Moreover $u^*(\cdot, 0) \leq u_0(\cdot)$ in $\Omega$ and $u^*(t, \cdot) \leq 0$ on $\partial\Omega$ for any $t \in (0, T)$. In fact, assume by contradiction that there exists $\bar{x} \in \Omega$ such that $u^*(x, 0) - u_0(x) = \xi > 0$. This would mean that there exist $\{x_n\} \subset \Omega$, $\{t_n\} \subset [0, T)$ and $\{w_n\} \subset A$ such that
\[
 x_n \to \bar{x}, \quad t_n \to 0 \quad \text{and} \quad w_n(x_n, t_n) \to u_0(\bar{x}) + \xi,
\]
that is in contradiction with the definition of $u$. Similarly we check that $u^*(t, \cdot) \leq 0$ on $\partial\Omega$. This implies that $u^* \in A$ and, by definition of $u$, we get that $u = u^*$.

Now we claim that the lower-semicontinuous envelope $u_*$ is a supersolution to (2.10) and that $u_*(x, 0) \geq u_0(x)$ for all $x \in \Omega$. Once the claim is proved, we can apply the comparison principle of Lemma 4.3 to the subsolution $u$ and the supersolution $u_*$ to infer that $u \leq u_*$.

This implies that $u_* = u^* = u$ is a viscosity solution to (2.10) that satisfies the boundary and initial conditions in the classical sense.

Let us hence prove that $u_*$ is a supersolution with $u_*(x, 0) \geq u_0(x)$. By contradiction, we assume there exists $\phi \in C^2((0, T) \times \Omega)$ such that $u_* - \phi$ has a strict global minimum at $(t_0, x_0)$, $u_*(t_0, x_0) = \phi(t_0, x_0)$ and
\[
 \partial_t \phi(t_0, x_0) + h(t_0, x_0)u_*(t_0, x_0) + \int_{\Omega(t_0, x_0)} \frac{\phi(t_0, x_0) - \phi(z, t_0)}{|x_0 - z|^{N+2\sigma}} \, dz < f(t_0, x_0).
\]
This means that there exists $\epsilon > 0$ such that the function
\[
 F(t, x) = \partial_t \phi(t, x)h(x)u_*(t, x) + h(t, x)\phi(t, x) + L_\sigma(\Omega(t), x, \phi(t)) - f(t, x)
\]
satisfies $F(t_0, x_0) = -\epsilon$. Since such a function is continuous at $(t_0, x_0)$, there exists $r > 0$ such that $F(t, x) < -\frac{\epsilon}{2}$ for all $(t, x) \in B_r(t_0, x_0)$ where $B_r(t_0, x_0) = \{(t-t_0)^2 + |x-x_0|^2\}^{\frac{1}{2}} < r \subset (0, T) \times \Omega$. Let us define
\[
 \delta_1 = \inf_{x \in \Omega \cap B_r(t_0, x_0)} (v - \phi)(t, x) > 0, \quad \delta_2 = \frac{\epsilon}{4 \sup_{x \in B_r(x_0)} h(x)},
\]
and set $\delta = \min\{\delta_1, \delta_2\}$. With this choice of $r$ and $\delta$ we define
\[
 V = \begin{cases} 
 \max\{v, \phi + \delta\} & \text{in } B_r(t_0, x_0), \\
 v & \text{otherwise}.
\end{cases}
\]
Notice that, since $v$ is upper semicontinuous, the set $\{v - \phi - \delta < 0\}$ is open and nonempty. (since, by definition of lower semicontinuous envelope, there exists a sequence $x_n \to x_0$ such that $v(z_n) \to u_*(x_0) = \phi(x_0)$ as $n \to \infty$). Moreover $\{v - \phi - \delta < 0\} \subset B_r(t_0, x_0)$ thanks to the choice of $\delta_1$. We want to prove that $V$ is a subsolution. Let us consider now $\psi \in C^2((0, T) \times \Omega)$ such that $V - \psi$ has a global maximum at $(\tau_0, y_0)$ and $\psi(\tau_0, y_0) = V(\tau_0, y_0)$. If $V(\tau_0, y_0) = v(\tau_0, y_0)$, since $V \geq v$, it results that $v - \psi$ has a global maximum at $(\tau_0, y_0)$ and $v(\tau_0, y_0) = \psi(\tau_0, y_0)$. Using that $v$ is a subsolution we get that
\[
 \partial_t \psi(\tau_0, y_0) + h(y_0)V(\tau_0, y_0) + L_\sigma(\Omega(\tau_0, y_0), \psi(\tau_0, y_0)) = \partial_t \psi(\tau_0, y_0) + h(y_0)v(\tau_0, y_0) + L_\sigma(\Omega(\tau_0, y_0), \psi(\tau_0, y_0)) \leq f(\tau_0, y_0).
\]
Let us now focus on the case $V(\tau_0, y_0) = \phi(\tau_0, y_0) + \delta \neq v(\tau_0, y_0)$. This implies that
\[
 \partial_t \psi(\tau_0, y_0) = \partial_t \phi(\tau_0, y_0),
\]
and that $(\tau_0, y_0) \in B_r(t_0, x_0)$. Then we have that
\[
 \phi + \delta - \psi \leq V - \psi \leq 0 \text{ in } B_r(t_0, x_0),
\]
where we have used the fact that \( \phi + \delta \leq V \) in \( B_r(t_0, x_0) \). Moreover, we readily check
\[
\phi + \delta - \psi \leq \phi + \delta - v \leq 0 \quad \text{in} \quad (0, T) \times \Omega \setminus B_r(t_0, x_0),
\]
since \( v \equiv V \leq \psi \) in \( (0, T) \times \Omega \setminus B_r(t_0, x_0) \) and thanks to the definition of \( \delta_1 \). As effect of the two inequalities above, we deduce that \( \phi + \delta \leq \psi \) in \( (0, T) \times \Omega \). It follows that
\[
\begin{align*}
\partial_t \psi(t_0, y_0) + h(t_0, y_0) V(t_0, y_0) + L_s(\Omega(t_0, y_0), \psi(t_0, y_0)) & \\
\leq & \partial_t \phi(t_0, y_0) + h(t_0, y_0)(\phi(t_0, y_0) + \delta) + L_s(\Omega(t_0, y_0), \phi(t_0, y_0)) \\
\leq & f(t_0, y_0) - \frac{\epsilon}{2} + \frac{\epsilon}{4} < f(t_0, y_0),
\end{align*}
\]
where the last inequality comes from the choice of \( r \) and \( \delta \). This leads to a contradiction since it implies that \( V \in A \) and that \( V > v \geq u \) somewhere in \( B_r(t_0, x_0) \). This proves that \( u_* \) is a supersolution to (2.10).

Finally let us prove that \( u_*(x, 0) \geq u_0(x) \) for all \( x \in \Omega \). Again assume by contradiction that there exists \( \bar{x} \in \Omega \) such that
\[
(4.8)
\]
Our aim is to build a barrier from below for \( u(t, x) \) in a neighborhood of \((0, \bar{x})\) (hence a barrier for \( u_* \), as well), contradicting (4.8). Thanks to the continuity of \( u_0 \), for any \( \epsilon > 0 \) there exists \( \delta_\epsilon < \frac{1}{2}d(\bar{x}) \) such that
\[
|u_0(x) - u_0(\bar{x})| \leq \epsilon \quad \text{if} \quad |x - \bar{x}| \leq \delta_\epsilon.
\]
Take now a function \( \eta(x) \in C^\infty_c(B_1(0)) \) with \( 0 \leq \eta \leq 1 \) and \( \eta(0) = 1 \), and define
\[
\tilde{w}(t, x) = a\eta\left(\frac{\bar{x} - x}{\delta_\epsilon}\right) - b - K\delta_\epsilon^{-2s}t,
\]
with \( a = u_0(\bar{x}) - \epsilon + \|u_0\|_{L^\infty(\Omega)}, b = \|u_0\|_{L^\infty(\Omega)} \) and \( K > 0 \) to be chosen below. Thanks to the choice of \( a \) and \( b \) it is easy to check that \( \tilde{w}(0, x) \leq u_0(x) \). Moreover, recalling that \( \text{supp}(\eta(\cdot/\delta_\epsilon)) \subset B_{\delta_\epsilon}(\bar{x}) \) and that the integral operator scales as \( \delta^{-2s} \), we get that
\[
\partial_t \tilde{w} + h\tilde{w} + L_s(\Omega(t, x), \tilde{w}(t, x)) - f \leq -K\delta_\epsilon^{-2s} + \delta^{-2s}C(\eta) + ||f||_{L^\infty(\Omega)} \leq 0,
\]
where the last inequality follows by letting \( K > C(\eta) + \delta_\epsilon^{2s}[aC(d(\bar{x})) + ||f||_{L^\infty(\Omega)}] \). Hence, \( \tilde{w} \in A \) and, by definition of \( u(t, x), \tilde{w}(t, x) \leq u(t, x) \). Now, for any \( \epsilon > 0 \), there exists \( \delta_\epsilon \) (possibly smaller then \( \delta_\epsilon \)) such that
\[
u_0(\bar{x}) - 2\epsilon \leq \tilde{w}(t, x) \leq u(t, x) \quad \text{for} \quad |\bar{x} - x| \leq \delta_\epsilon \quad t \in [0, \delta_\epsilon].
\]
Then the same inequality holds for \( u_* \), contradicting (4.8). 

\[ \square \]

**Proof of Theorem 2.5.** Let us argue for \( T < \infty \) first. Choose \( \eta \leq \min\{\bar{\eta}, \eta_1\} \) where \( \eta_1 \) is from (2.8)-(2.9) and \( \bar{\eta} \) from Lemma 4.1 and Remark 4.2, and set \( \tilde{I}(t, x) = Qd(x)^\eta \) where \( Q \) is a positive constant to be chosen later. Whenever a smooth function \( \varphi \) touches \( \tilde{I} \) from above at \((t_0, x_0)\) we deduce that
\[
\begin{align*}
\partial_t \varphi_r(t_0, x_0) + h(t_0, x_0)\varphi_r(t_0, x_0) + L_s(\Omega(t, x), \varphi_r(t_0, x_0)) - f(t_0, x_0) & \\
\geq & \frac{\alpha}{d(x)^{2s}}\varphi_r(t_0, x_0) + L_s(\Omega(t, x), \varphi_r(t_0, x_0)) - |f(t_0, x_0)| \\
\geq & \left(Q\frac{\alpha}{2} - |f(t_0, x_0)||d(x)_0|^{2s-\eta}\right)d(x_0)^{\eta-2s} \geq 0.
\end{align*}
\]
The first inequality comes from the fact that \( \partial_t \varphi_r(t_0, x_0) \) must be zero and from assumption (2.7) whereas the second inequality follows by construction of \( \tilde{I} \) and by Lemma 4.1. The third inequality follows from the assumption on \( f \) (see (2.8)) and by taking \( Q \) large enough. This proves that \( \tilde{I}(t, x) \) is a supersolution of (2.10). Similarly, we can show that \( \tilde{I}(t, x) = -\tilde{I}(t, x) \) is a subsolution. By possibly taking an even larger value of \( Q \) if necessary, we deduce that \( \tilde{I}(0, x) \leq u_0(x) \leq \tilde{I}(0, x) \), thanks to assumption (2.8) on \( u_0 \) and to the choice of \( \eta \). At this point, we can apply Theorem 4.5 and conclude the proof.
The limiting case $T = \infty$ can be tackled by passing to the limit in the sequence $\{u_n\}$ of solutions of problem (2.10) in $(0, n) \times \Omega$. Thanks to Lemma 4.3 we have that

$$u_n(t, x) \equiv u_m(t, x) \quad \text{in} \quad (0, \min\{n, m\}) \times \Omega.$$ 

Then, for any $(t, x) \in (0, \infty) \times \Omega$, we can uniquely define $u(t, x) = u_{[t]+1}(t, x)$, where $[t]$ is the integer part of $t$. From the comparison principle applied on each domain $(0, n) \times \Omega$, this uniquely defines a solution for all times.

As mentioned above, we are not giving the details of the proof of Theorem 2.2. Indeed, the elliptic case of Theorem 2.2 follows again from by Perron method, by means of the barriers from Lemma 4.1. Here, one is asked to use an elliptic version of the comparison Lemma 4.3, which can be deduced using Corollary 3.9.

5. THE EIGENVALUE PROBLEM

In this section, we focus on the eigenvalue problem associated to the operator (1.5). Before discussing our specific notion of eigenvalue, we prepare some technical tools.

**Lemma 5.1** (Strong maximum principle). Assume (1.8)-(1.11) and let $u \in \text{LSC}_b(\Omega)$ solve

$$h(x)u(x) + \mathcal{L}(\Omega(x), u(x)) \geq 0$$

in the viscosity sense in $\Omega$ and $u \geq 0$ in $\partial \Omega$. Then, either $u \equiv 0$ or $u > 0$ in $\Omega$.

**Proof.** Notice that, thanks to the comparison principle, we have that $u \geq 0$ in $\Omega$. Let us assume that $u(x_0) = 0$ at some $x_0 \in \Omega$ and that, by contradiction, $u(y_0) > 0$, for some $y_0 \in \Omega$.

If $y_0 \in \Omega(x_0)$, since $x_0$ is a minimum for $u$, there exists $\varphi \in C^2(\Omega)$ such that $\varphi(x_0) = u(x_0) = 0$, $\varphi(x_0) \leq u(x_0)$ in $\Omega$. Moreover, since $u \in \text{LSC}_b(\Omega)$, we can choose $\varphi$ nonnegative and nontrivial in $\Omega(x_0)$. Since $\varphi$ is an admissible test function for $u$ at point $x_0$ and it follows that

$$\int_{\Omega(x_0)} \frac{-\varphi(x_0 + z)}{|z|^{N+2s}} \geq 0.$$ 

This is however contradicting the fact that $\varphi \geq 0$ is nontrivial in $\Omega(x_0)$ and proves that $u(x_0) = 0$ implies $u \equiv 0$ in $\Omega(x_0)$.

If $y_0 \notin \Omega(x_0)$, thanks to assumption (1.11) and the fact that $\Sigma$ is open, there exists a finite set of points $\{x_i\}_{i=0}^K \subset \Omega$ such that $x_i \in \Omega(x_{i-1})$ for $i = 1, \cdots, K$ and $y_0 \in \Omega(x_K)$. Using inductively the previous part we deduce that $u = 0$ in each $\Omega(x_i)$, that is $u(y_0) = 0$, which is again a contradiction.

The next technical Lemma allows us to restrict the operator to a subdomain of $\Omega$. This requires to modify both the sets $\Omega(x)$ and the function $h$. Thanks to the assumptions, in particular the density bound for $\Sigma$ in (1.11), it turns out the the restricted operator satisfies the same properties of the original one.

**Lemma 5.2** (Localization). Let $f \in C(\Omega)$ and assume that $v$ solves in a viscosity sense

$$h(x)v(x) + \mathcal{L}_s(\Omega(x), v(x)) \leq f(x) \quad \text{in} \quad \Omega.$$ 

If the the open set $O \subset \Omega$ is such that $v \leq 0$ in $\Omega \setminus O$, then $v$ also solves in the viscosity sense

$$j(x)v(x) + \mathcal{L}_s(\Xi(x), v(x)) \leq f(x) \quad \text{in} \quad O,$$

where $\Xi(x) = \Omega(x) \cap O$ and $j(x) = h(x) + \int_{\Omega(x) \setminus O} \frac{dy}{|x-y|^{N+2s}}$. By additionally assuming that $O$ coincides with some ball $\bar{B} \subset \Omega$ and by setting $\bar{d}(x) = \text{dist}(x, \partial \bar{B})$, it holds true that

$$c_1 \bar{d}(x)^{-2s} \leq j(x) \leq c_2 \bar{d}(x)^{-2s} \quad x \in \bar{B}.$$
Proof. Let us assume that \( \max_{\Omega}(v - \varphi) = (v - \varphi)(\bar{x}) = 0 \) and that \( B_r(\bar{x}) \subset \subset \Omega \). It is possible to extend \( \varphi \) to all \( \Omega \) so that \( \max_{\Omega}(v - \varphi) = (v - \varphi)(\bar{x}) = 0 \) (with a slight abuse of notation, we still indicate the extension by \( \varphi \)). Then, we have

\[
 f(\bar{x}) \geq h(\bar{x})v(\bar{x}) + \int_{B_r(\bar{x}) \cap \Omega(\bar{x})} \frac{\varphi(\bar{x}) - \varphi(y)}{|\bar{x} - y|^{N+2s}} |\bar{x} - y|^{N+2s} dy + \int_{\Omega(\bar{x}) \setminus B_r(\bar{x})} \frac{v(\bar{x}) - v(y)}{|\bar{x} - y|^{N+2s}} |\bar{x} - y|^{N+2s} dy
\]

\[
 = \left[ h(\bar{x}) + \int_{\Omega(\bar{x}) \setminus \Omega(\bar{x})} \frac{dy}{|x - y|^{N+2s}} \right] v(\bar{x}) + \int_{B_r(\bar{x}) \cap \Omega(\bar{x})} \frac{\varphi(\bar{x}) - \varphi(y)}{|\bar{x} - y|^{N+2s}} dy + \int_{\Omega(\bar{x}) \setminus B_r(\bar{x})} \frac{v(\bar{x}) - v(y)}{|\bar{x} - y|^{N+2s}} dy.
\]

where the last inequality comes from the fact that \( v \leq 0 \) in \( \Omega \setminus O \).

Let us consider now the case \( O \equiv \tilde{B} \). The estimate from above in (5.1) can be deduced as in (3.19). We omit the details. To show the estimate from below, fix \( k > 1 \) so that

\[
c - \frac{2}{\omega_N} |B_1(0) \cap A(k)| \geq \frac{1}{2} c,
\]

where \( c \) is the constant in (1.11) and \( A(k) = \{ y \in \mathbb{R}^N : -k^{-1} \leq y_1 \leq 0 \} \). We also point out that the symmetry of \( \Sigma \) implies, for \( B_{k^2}^+(0) = B_r(0) \cap \{ z \leq 0 \} \) and \( r > 0 \), that

\[
(\Sigma \cap B_{k^2}^+(0)) \geq \frac{c}{2} |B_r(0)|.
\]

Moreover, without loss of generality, we assume that \( \tilde{B} = \{|y| \leq 1\} \) (this is always true up to a translation and dilation) and take \( x \in \{|y| \leq 1\} \) such that \( k\tilde{d}(x) < \zeta d(x) \). Let us take now a system of coordinates with origin in the center of \( \tilde{B} \) such that \( |x| = -x_1 = 1 - d(x) \). It follows that \( \tilde{\Omega}(x) \cap B_{k\tilde{d}(x)}(0) = \Sigma \cap B_{k\tilde{d}(x)}(0) \) and

\[
\int_{\Omega(x) \setminus \tilde{B}} \frac{dy}{|x - y|^{N+2s}} \geq \int_{\tilde{\Omega}(x) \cap B_{k\tilde{d}(x)}(0) \setminus \{ y_1 > -1 \}} \frac{dy}{|x - y|^{N+2s}} = \int_{\Sigma \cap B_{k\tilde{d}(x)}(0) \setminus \{ z_1 > -d(x) \}} \frac{dz}{|z|^{N+2s}} \geq \tilde{d}(x)^{-n-2s} |\Sigma \cap B_{-d(x)}^{-k\tilde{d}(x)}(0) \setminus \{ z_1 > -d(x) \}|.
\]

We get that

\[
|\Sigma \cap B_{-d(x)}^{k\tilde{d}(x)}(0) \setminus \{ z_1 > -d(x) \}| \geq |\Sigma \cap B_{-d(x)}^{k\tilde{d}(x)}(0) | - |\{-d(x) \leq z_1 < 0 \} \cap B_{-d(x)}^{k\tilde{d}(x)}(0) | \geq \frac{c}{2} |B_{k\tilde{d}(x)}| - (k\tilde{d}(x))^N |B_1(0) \cap A(k)| \geq \frac{\omega_N}{4} c(k\tilde{d}(x))^N,
\]

where we have used (5.2) and the definition of \( k \) in the last two inequality respectively. Putting together (5.3) and (5.4) and recalling the condition \( k\tilde{d}(x) < \zeta \tilde{d}(x) \), we deduce that

\[
\int_{\tilde{\Omega}(x) \setminus \tilde{B}} \frac{dy}{|x - y|^{N+2s}} \geq c_1 \tilde{d}(x)^{-2s} \quad \text{for all } x \in \tilde{B} \text{ with } k\tilde{d}(x) \leq \zeta \text{dist}(\tilde{B}, \Omega).
\]

This, together with the definition of \( j(x) \), completes the proof of the Lemma.

\[
\square
\]

Lemma 5.3 (Refined Maximum Principle). Assume (1.8) and (11.1). Let \( \lambda > 0, 0 \leq f \in C(\Omega) \), and assume that \( u \in \text{LSC}_b(\Omega) \), with \( u > 0 \) in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \), satisfies

\[
h(x)u(x) + L_\lambda(\Omega(x), u(x)) \geq \lambda u(x) + f(x).
\]

Moreover, let \( v \in \text{USC}_b(\Omega) \), with \( v \leq 0 \) on \( \partial \Omega \), satisfy

\[
h(x)v(x) + L_\lambda(\Omega(x), v(x)) \leq \lambda v(x).
\]

If \( f \) is non trivial then \( v \leq 0 \). If \( f \equiv 0 \) and there exists \( x_0 \in \Omega \) such that \( v(x_0) > 0 \) then \( v = tu \) for some \( t > 0 \).
We have that to prove that either
\[ h(x)z_t(x) + \mathcal{L}_s(\Omega(x), z_t(x)) \leq \lambda z_t(x). \]
Notice that for all \( \rho > 0 \) and any \( t > 0 \) such that
\[ t > \sup_{d(x) > \rho} \frac{v(x)}{\inf_{d(x) > \rho} u(x)} \]
(recall that \( u > 0 \) in \( \Omega \)) we have that
\[ \{ x \in \Omega : d(x) \geq \rho \} \subset \{ x \in \Omega : z_t < 0 \}. \]
We now use Lemma 5.2 to restrict (5.5) to \( \Omega_\rho = \{ x \in \Omega : d(x) < \rho \} \) and get
\[ j(x)z_t + \mathcal{L}_s(\Xi(x), z_t) \leq \lambda z_t \quad \text{in} \quad \Omega_\rho, \]
where \( j(x) = h(x) + \int_{\Omega(x) \setminus \Omega_\rho} \frac{\partial z}{\partial s} \) and \( \Xi(x) = \Omega(x) \cup \Omega_\rho \). Taking \( \rho \) such that \( \rho < \left( \frac{\lambda}{\mu} \right)^{\frac{1}{\alpha}} \) and using the coercivity assumption (1.8) on \( h(x) \), it follows that \( j(x) - \lambda > 0 \). Then, since \( z_t \leq 0 \) on \( \partial \Omega_\rho \), we can apply the comparison principle to (5.6) and deduce that \( z_t \leq 0 \) in \( \Omega_\rho \). This means that \( z_t \leq 0 \) in \( \Omega \).

Let us focus on the case \( 0 \neq f \geq 0 \) and assume, by contradiction, that there exists \( x_0 \in \Omega \) such that \( v(x_0) > 0 \). Then, up to a multiplication with a positive constant, we have that \( v(x_0) > u(x_0) \).
Let us set
\[ \tau = \inf \{ t : z_t \leq 0 \text{ in } \Omega \} \]
and recall that \( \tau > 1 \) since \( v(x_0) > u(x_0) \). As \( z_\tau \leq 0 \) we get
\[ h(x)z_\tau(x) + \mathcal{L}_s(\Omega(x), z_\tau(x)) \leq \lambda z_\tau(x) \leq 0 \quad \forall t \geq \tau. \]
We can apply the strong maximum principle of Lemma 5.1 to prove that either \( z_\tau \equiv 0 \) or \( z_\tau < 0 \). This latter case is not possible since it would contradict the definition of \( \tau \). Having that \( z_\tau \equiv 0 \) we get \( v_\tau := v = \mu u \). We have
\[ h(x)v_\tau(x) + \mathcal{L}_s(\Omega(x), v_\tau(x)) \leq \mu v_\tau(x) \]
by assumption and, since \( v_\tau = \mu u_\tau \),
\[ h(x)v_\tau(x) + \mathcal{L}_s(\Omega(x), v_\tau(x)) \geq \mu v_\tau(x) + f(x). \]
By combining these two inequality, using Corollary 3.9 and recalling that \( f \) is nontrivial, we obtain a contradiction. Hence, \( v \leq 0 \).

Let us now consider the case \( f \equiv 0 \) and \( v(x_0) > 0 \) for some \( x_0 \in \Omega \). Upon multiplying by a positive constant, we can assume that \( v(x_0) > u(x_0) \). Following exactly the same argument and notation of the previous step we obtain that either \( z_\tau \equiv 0 \) or \( z_\tau < 0 \). The latter option again leads to a contradiction. Hence, \( z_\tau \equiv 0 \), which corresponds to the assertion. \( \square \)

**Theorem 5.4.** Assume (1.8)-(1.11). Given \( \lambda > 0 \) and a nonzero \( 0 \leq f \in C(\Omega) \) satisfying (1.13), let us assume that there exists \( 0 \leq u \in \text{LSC}_b(\Omega) \) such that
\[
\begin{cases} 
    h(x)u(x) + \mathcal{L}_s(\Omega(x), u(x)) \geq \lambda u(x) + f(x) & \text{in } \Omega, \\
    u(x) = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Then, for any \( \mu \leq \lambda \) and \( |g| \leq f \), there exists a solution to
\[
\begin{cases} 
    h(x)v(x) + \mathcal{L}_s(\Omega(x), v(x)) = \mu v(x) + g(x) & \text{in } \Omega, \\
    v(x) = 0 & \text{on } \partial \Omega.
\end{cases}
\]
If moreover \( g \) is nonnegative and nontrivial then \( v > 0 \).
Proof. Let us set $v_0 = 0$ and recursively define the sequence $\{v_n\}$ of solutions to
\[
\begin{aligned}
&\left\{ \begin{array}{ll}
h(x)v_n(x) + \mathcal{L}_s(\Omega(x), v_n(x)) = \mu v_{n-1} + g(x) & \text{in } \Omega, \\
v_n(x) = 0 & \text{on } \partial \Omega.
\end{array} \right.
\end{aligned}
\]
Notice that the existence of each $v_n$ is ensured by Theorem 2.5. We now prove that $|v_n| \leq u$ by induction on $n$. Let $n = 1$. As $|g| \leq f$ the comparison principle from Corollary 4.4 ensures that $|v_1| \leq u$. Assume that $|v_{n-1}| \leq u$. Since $|\mu v_{n-1} + g(x)| \leq \lambda u(x) + f(x)$, we can use the comparison principle (see Corollary 4.4) to deduce that $|v_n| \leq u$. In case $g \geq 0$ a similar argument shows that $0 \leq v_n \leq v_{n+1}$.

This implies that $|\mu v_{n-1} + g(x)| \leq \lambda u + f \leq Cd(x)^{\eta-2s}$, where we have used assumption (1.13) for the last inequality. Using Lemma 4.1, we can conclude that there exist a large $Q$ (independent of $n$) such that $\tilde{v}(x) = Qd(x)^{\eta}$, with $\eta = \min\{\bar{\eta}, \eta_f\}$, solves
\[
h(x)\tilde{v}(x) + \mathcal{L}_s(\Omega(x), \tilde{v}(x)) \geq \mu v_{n-1} + g(x) \quad \text{in } \Omega.
\]
Thanks again to the comparison principle, we deduce that $v_n \leq Qd(x)^{\eta}$. Similarly, it follows that $v_n \geq -Qd(x)^{\eta}$. Let us consider then the half-relaxed limits of the sequence $v_n$
\[
\overline{v}(x) = \sup \left\{ \limsup_{n \to \infty} v_n(x_n) : x_n \to x \right\}, \quad \underline{v}(x) = \inf \left\{ \liminf_{n \to \infty} v_n(x_n) : x_n \to x \right\}.
\]
Notice that by construction both $\overline{v}$ and $\underline{v}$ vanish on $\partial \Omega$. Taking advantage of Lemma 3.4, we deduce that $\overline{v}$ and $\underline{v}$ are respectively sub and super-solution to (5.7). Moreover, Corollary 3.9 implies that $w = \overline{v} - \underline{v}$ solves
\[
h(x)w(x) + \mathcal{L}_s(\Omega(x), w(x)) \leq \mu w(x) \quad \text{in } \Omega, \quad \text{and } w = 0 \quad \text{on } \partial \Omega.
\]
Since $u$ satisfies
\[
h(x)u(x) + \mathcal{L}_s(\Omega(x), u(x)) \geq \mu u(x) + f(x) \quad \text{in } \Omega,
\]
we may use Lemma 5.3 to conclude that $w \leq 0$, namely $\overline{v} \leq \underline{v}$. Due to the natural order between the two functions, we deduce that $v = \overline{v} = \underline{v}$ is a viscosity solution to (5.7). If $g \geq 0$ and not trivial, by using the strong maximum principle we easily deduce that $v > 0$. \(\square\)

Let us assume that the nontrivial $0 \leq f \in C(\Omega)$ satisfies (1.13) and recall the definition of the set
\[
\mathcal{E}_f = \{ \lambda \in \mathbb{R} : \exists v \in C(\overline{\Omega}), v > 0 \text{ in } \Omega, v = 0 \text{ on } \partial \Omega, \text{ such that } hv + \mathcal{L}_s(\Omega, v) = \lambda v + f \}.
\]
Moreover, let
\[
(5.8) \quad \lambda_f = \sup \mathcal{E}_f.
\]
As we shall see, $\lambda_f$ does not depend on the particular choice of $f$. By definition and thanks to Theorem 5.4 we deduce that
\[
\text{if } g \leq f \text{ then } \lambda_g \leq \lambda_f.
\]
The following Lemma shows us that $\lambda_f$ is finite and that $\mathcal{E}_f$ is a left semiline.

Lemma 5.5. Assume (1.8)-(1.11) and that $0 \leq f \in C(\Omega)$ is nonzero and satisfies (2.8). Then, $\lambda_f$ is positive and finite and $\mathcal{E}_f$ is a left semiline with $\mathcal{E}_f \neq \mathbb{R}$.

Proof. Notice that for any
\[
\lambda \in (-\infty, \frac{\alpha}{\operatorname{diam}(\Omega)^{2s}})
\]
the operator
\[
u \mapsto [h(x) - \lambda]|u + \mathcal{L}(\Omega(x), u)
\]
fulfills assumptions (1.10)-(1.8). We can apply the existence results and the strong maximum principle of the previous chapter to deduce that $(-\infty, \frac{\alpha}{\operatorname{diam}(\Omega)^{2s}}) \subset \mathcal{E}_f$. Moreover, if $\lambda \in \mathcal{E}_f$, Theorem (5.4) assures that any $\mu < \lambda$ belongs to $\mathcal{E}_f$ as well. This proves that $\mathcal{E}_f$ is a left semiline.

To show that $\mathcal{E}_f \neq \mathbb{R}$ let us take $\lambda < \lambda_f$. Since $\mathcal{E}_f$ is a left semiline, there exists some $v \in C(\overline{\Omega})$ with $v = 0$ on $\partial \Omega$ and strictly positive in $\Omega$, such that
\[
h(x)v(x) + \mathcal{L}_s(\Omega(x), v(x)) = \lambda v(x) + f(x) \quad \text{in the viscosity sense in } \Omega.
\]
Now we want to restrict this inequality to a ball $B \subset \subset \Omega$ such that $f > 0$ in $B$. In order to do it, for any $x \in B$, we define $\Xi(x) = \Omega(x) \cap B$. Taking advantage of the positivity $v$, we can apply Lemma 5.2 to $-v$ and deduce that

$$j(x)v + \mathcal{L}_s(\Xi(x), v) \geq \lambda v + f(x) \quad \text{in the viscosity sense in } B,$$

where

$$j(x) = h(x) + \int_{\Omega(x) \setminus B} \frac{dy}{|x - y|^{N+2s}}.$$

Thanks to Lemma 5.2, we have that $j(x)$ satisfies (2.7) (by possibly changing the constants) and that the family $\{\Xi(x)\}$ satisfies the same kind of assumptions of $\{\Omega(x)\}$. Then, for any positive continuous function $g$ with compact support in $B$ there exists a unique viscosity solution to

$$\begin{cases}
    j(x)w(x) + \mathcal{L}_s(\Xi(x), w(x)) = g(x) & \text{in } B, \\
    w(x) = 0 & \text{on } \partial B.
\end{cases}$$

Thanks to the strong maximum principle of Lemma 5.1 and the fact that $g$ has compact support in $B$, it follows that $0 < g \leq C_0w$ for some positive constant $C_0$. If $C_0 < \lambda$, we would get that

$$j(x)w(x) + \mathcal{L}_s(\Xi(x), w(x)) \leq \lambda w(x) \quad \text{in the viscosity sense in } B.$$ 

Applying Lemma 5.3 to (5.9) and (5.10), it would follow $w \leq 0$, which is a contradiction. This proves that $\lambda \leq C_0$. Since the constant $C_0$ does not depend on $\lambda$, we finally deduce

$$\lambda_f = \sup \{ E_f \leq C_0 \}.$$

This concludes the proof of the Lemma.

Let us provide now the proof of the well-posedness of the first-eigenvalue problem.

\textbf{Proof of Theorem 2.3.} We split the argument into subsequent steps.

\textbf{Step 1:} Let us show at first that for a given $f \in C(\Omega)$ that satisfies (2.8) and the additional condition

$$f(x) \geq \theta > 0 \quad \text{in } \Omega,$$

problem (2.2) admits a solution $v_f > 0$ with $\lambda_f := \sup E_f$. Let $\{\lambda_n\}$ be a sequence that converges to $\lambda_f$ and consider the associate sequence $\{v_n\} \subset C(\overline{\Omega})$, $v_n > 0$ of solutions to

$$\begin{cases}
    h(x)v_n(x) + \mathcal{L}_s(\Omega(x), v_n(x)) = \lambda_n v_n(x) + f(x) & \text{in } \Omega, \\
    v_n(x) = 0 & \text{on } \partial \Omega.
\end{cases}$$

We first claim that $\|v_n\|_{L^\infty(\Omega)} \to \infty$. Indeed, let us assume by contradiction that there exists $k > 0$ such that $|v_n| \leq k$. Thanks to Lemma 4.1, we deduce that there exists $Q = Q(k)$ such that the function $\overline{\mathcal{L}}(x) = Qd(x)^\eta$, with $\eta = \min\{\overline{\eta}, \eta_f\}$, solves in the viscosity sense

$$h(x)\overline{L}(x) + \mathcal{L}_s(\Omega(x), \overline{L}(x)) \geq \lambda_n v_n(x) + f(x) \quad \text{in } \Omega \quad \forall n > 0.$$

Thanks to Corollary 4.4 and the sign of $v_n$, we have that

$$0 < v_n(x) \leq Qd^\eta(x),$$

where the right-hand side does not depend on $n$. Using Lemma 3.4, we deduce that $\underline{v}(x) = \inf \{\liminf_{n \to \infty} v_n(x_n) : x_n \to x\}$ is a supersolution to

$$\begin{cases}
    h(x)v(x) + \mathcal{L}_s(\Omega(x), v(x)) = \lambda_f v(x) + f(x) & \text{in } \Omega, \\
    v(x) = 0 & \text{on } \partial \Omega.
\end{cases}$$

Since $v_n > 0$ also $\underline{v} \geq 0$ in $\Omega$ and Lemma 5.1 assures that $\underline{v} > 0$ in $\Omega$. Then, Theorem 5.4 provides a solution $\underline{v}_\infty > 0$ to (5.14). Taking $\epsilon > 0$ such that $f \geq \frac{1}{2} \lambda_f + \epsilon \underline{v}_\infty$, which is possible thanks to (5.11), it follows that $\underline{v}_\infty = 2\underline{v}_\infty$ satisfies

$$h(x)\underline{v}_\infty + \mathcal{L}_s(\Omega(x), \underline{v}_\infty(x)) \geq (\lambda_f + \epsilon)\underline{v}_\infty(x) + f(x) \quad \text{in } \Omega.$$
Taking again advantage of Theorem 5.4 we reach a contradiction with respect to the definition of $\lambda_f$.

We have then proved that $\|u_n\|_{L^\infty(\Omega)} \to \infty$. Setting $u_n = v_n\|v_n\|_{L^\infty(\Omega)}^{-1}$ we obtain that

$$
\begin{cases}
h(x)u_n(x) + \mathcal{L}_a(\Omega, x, u_n(x)) = \lambda_n u_n(x) + \frac{f(x)}{\|v_n\|_{L^\infty(\Omega)}} & \text{in } \Omega, \\
u_n(x) = 0 & \text{on } \partial \Omega.
\end{cases}
$$

Since $0 < u_n \leq 1$, we deduce as in (5.13) that $u_n \leq Qd^\alpha(x)$ and then

$$
\begin{align*}
\overline{\mu}(x) &= \sup \{\limsup_{n \to \infty} u_n(x_n) : x_n \to x\}, \\
\underline{\mu}(x) &= \inf \{\liminf_{n \to \infty} u_n(x_n) : x_n \to x\},
\end{align*}
$$

are well defined and vanish on the boundary. Lemma 3.4 assures us that they are sub and super solution to

$$
(5.15)
\begin{cases}
h(x)u(x) + \mathcal{L}_a(\Omega, x, u(x)) = \lambda_f u(x) & \text{in } \Omega, \\
u(x) = 0 & \text{on } \partial \Omega.
\end{cases}
$$

By definition of $u_n$, there exists a sequence of points $\{x_n\} \subset \Omega$ such that $u_n(x_n) = 1$. Thanks to the uniform bound $u_n \leq Qd^\alpha(x)$, we deduce that there exists $\Omega' \subset \subset \Omega$ and that $\{x_n\} \subset \Omega'$. Using Lemma 3.11, it follows that

$$
\|u_n\|_{C^\gamma(\Omega')} \leq \tilde{C}(C, s, \zeta, d(\Omega', \Omega'), Q\|d\|_{L^\infty(\Omega)}),
$$

with $\Omega' \subset \subset \Omega'' \subset \subset \Omega$. Then, up to a not relabeled subsequence, $u_n$ uniformly converges to a continuous function $u \in C(\Omega')$. Furthermore, $u \equiv \overline{\mu} \equiv \underline{\mu}$ in $\Omega'$ and $u_n(x_n) \to u(\bar{x}) = 1$ for some $\bar{x} \in \Omega'$. This entails on the one hand that $\underline{\mu}$ is a non negative and nontrivial supersolution to (5.15), so that Lemma 5.1 implies $\underline{\mu} > 0$. On the other hand, we obtain that there exists $\bar{x} \in \Omega'$ such that $\overline{\mu}(\bar{x}) > 0$.

By applying again Lemma 5.3 directly to $\overline{\mu}$ and $\underline{\mu}$, we conclude that there exist $t > 0$ such that $\overline{\mu} = t \underline{\mu}$. This implies that both functions are continuous. Moreover, since $t > 0$ we deduce that both $\underline{\mu}$ and $\overline{\mu}$ are at the same time sup- and super-solutions. Hence, $\underline{\mu}$ and $\overline{\mu}$ are eigenfunctions related to $\lambda_f$.

Now we want to get rid of assumption (5.11). Notice that we used it to show that $\|v_n\|_{L^\infty(\Omega)}$ must diverge. Then, we have to prove that $\|v_n\|_{L^\infty(\Omega)} \to \infty$, assuming that the nontrivial positive continuous function $f$ solely satisfies (1.13). Again, let us argue by contradiction and suppose that $\|v_n\|_{L^\infty(\Omega)} \leq k$. This would lead again to the existence of a non trivial $v_\infty$ solution to (5.14). Setting $g = \sup \{f, \theta\}$, the previous argument provides us with $\lambda_g > 0$ and $v_\infty > 0$ solution to (2.2). Since $f \leq g$, by construction we deduce that $\lambda_f \leq \lambda_g$. Assume that $\lambda_f < \lambda_g$ and take $\mu \in (\lambda_f, \lambda_g)$. Then, thanks to the definition of $\lambda_g$, the fact that $f \leq g$ and by using Theorem 5.4, it results that the following problem

$$
\begin{cases}
h(x)z(x) + \mathcal{L}_a(\Omega, x, z(x)) = \mu z(x) + f(x) & \text{in } \Omega, \\
z(x) = 0 & \text{on } \partial \Omega,
\end{cases}
$$

admits a positive solution. This however contradicts the fact that $\lambda_f$ is a supremum. On the other hand, if $\lambda_f = \lambda_g$, Lemma 5.3, applied to $v_\infty$ and $v_g$, would imply $v_g \leq 0$, which is again a contradiction.

**Step 2:** Let us assume that there exists another couple $(\mu, w) \in (0, \infty) \times C(\Omega)$ that solves (2.2) with $w > 0$. If $\mu < \lambda_f$, we deduce that there exists $\lambda \in (\mu, \lambda_f)$ and, by definition of $\lambda_f$ and Lemma 5.5, a function $u \in C(\Omega)$, with $u > 0$ in $\Omega$, solving

$$
\begin{cases}
h(x)u(x) + \mathcal{L}_a(\Omega, x, u(x)) = \lambda u(x) + f(x) & \text{in } \Omega, \\
u(x) = 0 & \text{on } \partial \Omega.
\end{cases}
$$

On the other hand, since $\mu < \lambda$ and $w > 0$ we have that

$$
\begin{align*}
h(x)w(x) + \mathcal{L}_a(\Omega, x, w(x)) & \leq \lambda w(x) & \text{in } \Omega, \\
w(x) & = 0 & \text{on } \partial \Omega.
\end{align*}
$$

Being in the same setting of Lemma 5.3, we deduce that $w \leq 0$, which is a contradiction.
This proves that \( \mu \geq \lambda_f \). Assume by contradiction, that \( \mu > \lambda_f \). Take \( \epsilon > 0 \) small enough in order to have \( \lambda_f < \mu - \epsilon \) and a nonnegative nontrivial continuous function \( g(x) \) such that \( \epsilon w \geq g \) in \( \Omega \). It follows that \( w \) solves
\[
 h(x)w(x) + \mathcal{L}_s(\Omega(x), w(x)) \geq (\mu - \epsilon)w(x) + g(x) \quad \text{in } \Omega.
\]
Using Theorem 5.4 we deduce that there exists \( w_g \) solution to
\[
\begin{cases}
 h(x)w_g(x) + \mathcal{L}_s(\Omega(x), w_g(x)) = (\mu - \epsilon)w_g(x) + g(x) & \text{in } \Omega, \\
 w_g(x) = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Applying the refined comparison principle of Lemma 5.3 between \( v_f \) and \( w_g \) it follows that \( v_f \leq 0 \), which is a contradiction. We eventually conclude that \( \lambda_f = \mu \). This also implies that \( \lambda_f = \lambda_g = \overline{\lambda} \) for all \( f, g \) that satisfy (2.8).

**Step 3.** In this last step we show that solutions of (2.2) are unique up to a multiplicative constant. Assume that \( w \) is a nontrivial solution to (2.2) and let \( v_f > 0 \) be the solution provided by Step 1. Since \( w \) is nontrivial we can always assume, up to a multiplication with a (not necessarily positive) constant, that \( w(x_0) > 0 \). Then, we can use the second part of Lemma 5.3 to conclude that \( w = tv_f \) for some constant \( t \).

**Proof of Theorem 2.4.** Since \( \lambda < \overline{\lambda} \), thanks to the assumptions on \( f \) and using characterization (5.8), we deduce that there exists a function \( v > 0 \) in \( \Omega \) solving
\[
\begin{cases}
 h(x)v(x) + \mathcal{L}_s(\Omega(x), v(x)) = \lambda v(x) + |f(x)| & \text{in } \Omega, \\
v(x) = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Clearly, \( v \) is a supersolution for (2.4) and we can take advantage of Theorem 5.4 to conclude that (2.4) admits a solution. To deal with uniqueness we assume that (2.4) has two solutions \( v \) and \( z \) and set \( w = v - z \). Using Corollary 3.9 it follows that \( w \) solves
\[
\begin{cases}
 h(x)w(x) + \mathcal{L}_s(\Omega(x), w(x)) = \lambda w(x) & \text{in } \Omega, \\
w(x) = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Applying Lemma 5.3 to \( v \) and \( w \), we conclude that \( w \leq 0 \). The same conclusion holds for \( -w = z - v \), so that \( w = 0 \) and \( v \equiv z \). \( \square \)

6. **Asymptotic Analysis**

In this last section, we address the large time behavior of the solution to (2.10).

**Theorem 6.1.** Let us assume (1.8)-(1.11), that \( g_0 \in C(\Omega) \) with \( \text{supp}(g_0) \subset \subset \Omega \), and that there exist constants \( \eta_\gamma, C_\gamma > 0 \) such that the continuous nonnegative function \( g : (0, \infty) \times \Omega \to \mathbb{R}^+ \) satisfies
\[
g(t, x)d(x)^{2s-\eta}e^{\lambda t} \leq C_\gamma \quad \text{for some } \lambda < \overline{\lambda},
\]
where \( \overline{\lambda} \) is the first eigenvalue provided by Theorem 2.3. Then, if \( w \in C((0, \infty) \times \overline{\Omega}) \cap L^\infty((0, \infty) \times \Omega) \) satisfies in the viscosity sense
\[
-g(x, t) \leq \partial_t w(t, x) + h(x)w(t, x) + \mathcal{L}_s(\Omega(x), w(t, x)) \leq g(x, t) \quad \text{in } (0, \infty) \times \Omega,
\]
coupled with boundary and initial conditions
\[
\begin{cases}
w(t, x) = 0 & \text{on } (0, T) \times \partial \Omega, \\
w(0, x) = g_0(x) & \text{on } \Omega,
\end{cases}
\]
one has \( |w(x, t)| \leq Q_\lambda d(x)^{\overline{\eta}}e^{-\lambda t}, \) for all \( \overline{\eta} \leq \min\{\overline{\eta}, \eta^\alpha\} \) and some \( Q_\lambda > 0 \) with \( Q_\lambda \to \infty \) as \( \lambda \to \overline{\lambda} \).
Proof. Let us consider $\varphi_\lambda$ solving
\begin{equation}
\begin{cases}
  h(x)\varphi_\lambda(x) + \mathcal{L}_s(\Omega(x), \varphi_\lambda(x)) = \lambda \varphi_\lambda(x) + C_g d(x)^{\eta_s - 2s} & \text{in } \Omega, \\
  \varphi_\lambda(x) = 0 & \text{on } \partial \Omega, \\
  \varphi_\lambda(x) > 0 & \text{in } \Omega.
\end{cases}
\end{equation}

Such a function exists since $\lambda < \bar{\lambda}$ and thanks to the characterization of Theorem 2.3. We want to prove that $\overline{w}(t, x) = e^{-\lambda t} \varphi_\lambda(x)$ solves the viscosity sense
\[
\partial_t \overline{w}(t, x) + h(x)\overline{w}(t, x) + \mathcal{L}_s(\Omega(x), \overline{w}(t, x)) \geq g(x, t) \quad \text{in } (0, \infty) \times \Omega.
\]

In order to achieve this, let $\phi \in C^2(\Omega \times (0, \infty))$ and $(t, x) \in (0, \infty) \times \Omega$ such that $\overline{w}(t, x) = \phi(t, x)$ and that $\overline{w}(\tau, y) \geq \phi(\tau, y)$ for $(\tau, y) \in (0, \infty) \times \Omega$. We need to check (see Lemma (3.2)) that for any $B_r(x) \subset \Omega$
\[
\partial_t \phi_r(t, x) + h(x)\phi_r(t, x) + \mathcal{L}_s(\Omega(x), \phi_r(t, x)) \geq g(x, t).
\]

By construction of $\phi_r$ we have that $\partial_t \phi_r(t, x) \geq -\lambda e^{-\lambda t} \varphi_\lambda(x)$. Moreover, the function $\psi^d(y) = \phi e^{\lambda t}$ touches $\varphi_\lambda$ at $x$ from below. Then, we infer that
\[
h(x)\phi_r(t, x) + \mathcal{L}_s(\Omega(x), \phi_r(t, x)) = e^{-\lambda t} \left[ h(x)\psi^d_r(x) + \mathcal{L}_s(\Omega(x), \psi^d_r(x)) \right] \\
\geq e^{-\lambda t} [\lambda \varphi_\lambda(x) + C_g d(x)^{\eta_s - 2s}],
\]

where the last inequality follows from the definition of $\varphi_\lambda$. By collecting the information obtained we get that
\[
\partial_t \phi_r(t, x) + h(x)\phi_r(t, x) + \mathcal{L}_s(\Omega(x), \phi_r(t, x)) - g(x, t) \\
\geq -\lambda e^{-\lambda t} \varphi_\lambda(x) + e^{-\lambda t} [\lambda \varphi_\lambda(x) + C_g d(x)^{\eta_s - 2s}] - g(x, t) \\
= e^{-\lambda t} d(x)^{\eta_s - 2s} [C_g - g(x, t)e^{\lambda t} d(x)^{2s - \eta_s}] \geq 0,
\]

where the last inequality comes from assumption (6.1). Similarly, we can prove that $\underline{w}(t, x) = -e^{-\lambda t} \varphi_\lambda(x)$ solves
\[
g(x, t) \leq \partial_t \underline{w}(t, x) + h(x)\underline{w}(t, x) + \mathcal{L}_s(\Omega(t, x), \underline{w}(t, x)) \leq 0 \quad \text{in } (0, \infty) \times \Omega.
\]

Since $g_0$ has a compact support we can assume $|g_0| \leq \varphi_\lambda$, for otherwise we can consider $k\varphi_\lambda$ instead of $\varphi_\lambda$ for large $k > 0$. Therefore, we can use the comparison principle to deduce that
\[
|w(t, x)| \leq e^{-\lambda t} \varphi_\lambda(x).
\]

At this point, notice that the right-hand side of the first equation in (6.2) can be estimated as follows
\[
\lambda \varphi_\lambda(x) + C_g d(x)^{\eta_s - 2s} \leq \lambda \| \varphi_\lambda \|_{L^\infty(\Omega)} + C_g d(x)^{\eta_s - 2s} \leq C_{\lambda, g} d(x)^{\eta_s - 2s}.
\]

This implies that there exists $Q = Q(C_{\lambda, g})$ large enough so that $\tilde{t} = Qd(x)^{\eta_s}$ is a super solution to (6.2) (see Lemma 4.1). Then, we can use the comparison principle to conclude that $\varphi_\lambda(x) \leq \tilde{t}$, which concludes the proof. \hfill \Box

We conclude by presenting a proof of Theorem 2.6.

Proof of Theorem 2.6. Using Lemma 3.8, it follows that $w(t, x) = u(t, x) - v(x)$ solves in the viscosity sense
\begin{equation}
\partial_t w(t, x) + h(x)w(t, x) + \mathcal{L}_s(\Omega(x), w(t, x)) \leq \tilde{f}(x, t) \quad \text{in } (0, \infty) \times \Omega
\end{equation}
where
\[
\tilde{f}(x, t) = |f(x, t) - f(x)| + M|h(x, t) - h(x)| + 2M \int_{\mathbb{R}^N} \frac{|X_{\tilde{\Omega}(t, x)} - X_{\tilde{\Omega}(x)}|}{|z|^{N+2s}} dz.
\]

Similarly, we can apply again Lemma 3.8 to $-w(t, x) = v(x) - u(t, x)$ to deduce that
\begin{equation}
\partial_t w + h(t, x)w + \mathcal{L}_s(\Omega(x), w) \geq -\tilde{f}(x, t) \quad \text{in } (0, \infty) \times \Omega.
\end{equation}

Notice now that
\[
\int_{\mathbb{R}^N} \frac{|X_{\tilde{\Omega}(t, x)} - X_{\tilde{\Omega}(x)}|}{|z|^{N+2s}} dz = \int_{|z| \geq \frac{1}{2} d(x)} \frac{|X_{\tilde{\Omega}(t, x)} - X_{\tilde{\Omega}(x)}|}{|z|^{N+2s}} dz.
\]
\[ \frac{C}{d(x)^{N+2s}} |\Omega(t, x)\Delta \Omega(x)| \leq \frac{Ce^{-\lambda \tau}}{d(x)^{2s-\eta}}, \]

where we have used assumptions (1.10) and (2.6) to deduce the equation in the first line, and assumption (2.12) to deduce the last inequality in the second line. Thanks to inequalities (6.3) and (6.4), the assertion follows by a direct application of Theorem 6.1.

ACKNOWLEDGEMENT

S.B. is supported by the Austrian Science Fund (FWF) projects F65, P32788 and FW506004. U.S. is supported by the Austrian Science Fund (FWF) projects F 65, W 1245, I 4354, I 5149, and P 32788 and by the OeAD-WTZ project CZ 01/2021.

REFERENCES

[1] P. Aceves-Sánchez, C. Schmeiser, Fractional diffusion limit of a linear kinetic equation in a bounded domain, Kinet. Relat. Mod., 10 (2017), 541–551.
[2] F. Andreu-Vaillo, J. Mazón, J. D. Rossi, J. J. Toledo-Melero, Nonlocal Diffusion Problems, Mathematical surveys and monographs, vol. 165. American Mathematical Soc., 2021.
[3] M. Bardi, I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Modern Birkhäuser Classic, 1997.
[4] G. Barles, An Introduction to the Theory of Viscosity Solutions for First-Order Hamilton–Jacobi Equations and Applications, In: Hamilton-Jacobi Equations: Approximations, Numerical Analysis and Applications. Lecture Notes in Mathematics, vol 2074. Springer, Berlin, Heidelberg.
[5] G. Barles, E. Chasseigne, C. Imbert, On the Dirichlet Problem for Second-Order Elliptic Integro-Differential Equations, Indiana Univ. Math. J., 57 (2008), 213–246.
[6] G. Barles, E. Chasseigne, and C. Imbert, Hölder continuity of solutions of second-order non-linear elliptic integro-differential equations, J. Eur. Math. Soc., 13 (2011), 1–26.
[7] G. Barles, C. Imbert, Second-order elliptic integro-differential equation: viscosity solutions’ theory revisited, Ann. Inst. H. Poincaré Anal. Non Linéaire, 25 (2008), 567–585.
[8] B. Barrios, L. Del Pezzo, J. García-Melián, A. Quaas, A priori bounds and existence of solutions for some nonlocal elliptic problems, Rev. Mat. Iberoam., 34 (2018), 195–220.
[9] H. Berestycki, L. Nirenberg, S. R. S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators in general domains, Comm. Pure Appl. Math., 47 (1994), 47–82.
[10] H. Berestycki, J. M. Roquejoffre, L. Rossi, The periodic patch model for population dynamics with fractional diffusion, Discrete Contin. Dyn. Syst., 4 (2011), 1-13.
[11] I. Birindelli, D. Danilov, First eigenvalue and maximum principle for fully nonlinear singular operators, Adv. Differential Equations, 11 (1) (2006), 91-119.
[12] I. Birindelli, G. Galise, D. Schiera, Maximum principles and related problems for a class of nonlocal extremal operators, preprint, arXiv:2107.07303, (2021).
[13] A. Biswas, Principal eigenvalues of a class of nonlocal integro-differential operators, J. Differential Equations, 268 (2020), 5257-5282.
[14] A. Biswas, M. Modasiya, Mixed local-nonlocal operators: maximum principles, eigenvalue problems and their applications, preprint, arXiv:2110.06746 (2021).
[15] K. Bogdan, K. Burdzy, Z. Chen, Censored stable processes, Probab. Theory Related Fields, 127 (2003), 89–152.
[16] O. Burkovska, C. Glusa, M. D’Elia, An optimization-based approach to parameter learning for fractional type nonlocal models, Comput. Math. Appl., in press.
[17] J. Busca, M. Esteban, A. Quaas, Nonlinear eigenvalues and bifurcation problems for Pucci’s operator, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 22 (2005), 1–26.
[18] L. Caffarelli, L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Commun. Pur. Appl. Math., 2009, 62, 597–638.
[19] L. Caffarelli, L. Silvestre, Regularity Results for Nonlocal Equations by Approximation, Arch. Rational. Mech. Anal., 200 (2011) 59–88.
[20] M. Crandall, H. Ishii, P.L. Lions, User’s guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. (N.S.), 27 (1992), 1–67.
[21] H. Chang Lara, G. Dávila, Regularity for solutions of nonlocal parabolic equations, Calc. Var. Partial Differential Equations, 49 (2014) 139-172.
[22] H. Chang Lara, G. Dávila, Hölder estimates for nonlocal parabolic equations with critical drift, J. Differential Equations, 260 (2016) 4237-4284.
[23] G. Dal Maso, An introduction to Γ-convergence. Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston Inc., Boston, MA, 1993.
[24] G. Dávila, A. Quaas, E. Topp, Existence, nonexistence and multiplicity results for nonlocal Dirichlet problems, J. Differential Equation 266 (2019) 5971–5997.

[25] S. Duo, H. Wang, Y. Zhang, A comparative study on nonlocal diffusion operators related to the fractional Laplacian, Discrete Contin. Dyn. Syst. Ser. B 24 (2019), 231-256.

[26] E. Emmrich, R. B. Lehoucq, D. Puhst, Peridynamics: a nonlocal continuum theory, In: M. Griebel, M. A. Schweitzer (eds), Meshfree Methods for Partial Differential Equations VI, Lecture Notes in Computational Science and Engineering, vol 89, 45–65, Springer, Berlin, Heidelberg, 2008.

[27] M. M. Fall, Regional fractional Laplacians: Boundary regularity, preprint, https://arxiv.org/abs/2007.04808.

[28] N. Guillen, C. Mou, A. ´Swie ¸ch, Coupling Lévy measures and comparison principles for viscosity solutions, Trans. Amer. Math. Soc., 372 (2019), 7327–7370.

[29] C. Imbert, A nonlocal regularization of first order Hamilton-Jacobi equations, J. Differential Equations 211 (2005), 214–246.

[30] E.R. Jakobsen, K.H. Karlsen, A maximum principle for semicontinuous functions applicable to integro-partial differential equations, NoDEA Nonlinear Differential Equations Appl., 13 (2006), 137–165.

[31] M. Kassmann, M. Rang, R. W. Schwab, Integro-differential equations with nonlinear directional dependence, Indiana Univ. Math. J., 63 (2014), 1467–1498.

[32] A. Lischke, G. Panga, M. Gulian, F. Song, C. Glusa, X. Zheng, Z. Mao, W. Cai, M. M. Meerschaert, M. Ainsworth, G. E. Karniadakis, What is the fractional Laplacian? A comparative review with new results, J. Comput. Phys., 2020, 404, https://doi.org/10.1016/j.jcp.2019.109009.

[33] A. Mellet, Fractional diffusion limit for collisional kinetic equations: a moments method, Indiana Univ. Math. J., 59 (2010) 1333-1360.

[34] A. Mellet, S. Mischler, C. Mouhot, Fractional diffusion limit for collisional kinetic equations, Arch. Ration. Mech. Anal., 2011, 199, 493-525, https://doi.org/10.1007/s00205-010-0354-2.

[35] C. Mou, Perron’s method for nonlocal fully nonlinear equations, Anal. PDE 10 (2017) 1227–1254.

[36] A. Quaas, B. Sirakov, Principal eigenvalues and the Dirichlet problem for fully nonlinear elliptic operators, Adv. Math., 218 (2008) 105–135.

[37] A. Quaas, A. Salort, A. Xia, Principal eigenvalues of fully nonlinear integro-differential elliptic equations with a drift term, ESAIM Control Optim. Calc. Var., 26, 2020.

[38] X. Ros-Oton, J. Serra, Boundary regularity for fully nonlinear integro-differential equations, Duke Math. J. 165 (2016), 2079-2154.

[39] R. W. Schwab and L. Silvestre, Regularity for parabolic integro-differential equations with very irregular kernels, Anal. PDE 9 (2016), 727-772.

[40] S. A. Silling, Reformulation of elasticity theory for discontinuities and long-range forces, J. Mech. Phys. Solids, 48 (2000), 175-209.

[41] S. A. Silling, R. B. Lehoucq, Peridynamic theory of solid mechanics, Adv. Appl. Mech., 44 (2010), 73-166.

[42] L. Silvestre, Hölder estimates for solutions of integro-differential equations like the fractional Laplace, Indiana Univ. Math. J., 55 (2006) 1155–1174.

[43] L. Tartar, An Introduction to Sobolev Spaces and Interpolation Spaces, Lect. Notes Unione Mat. Ital., 3, Springer-Verlag, Berlin, Heidelberg, 2007

(Stefano Buccheri) FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, A-1090 VIENNA, AUSTRIA

Email address: stefano.buccheri@univie.ac.at

(Ulisse Stefanelli) FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, A-1090 VIENNA, AUSTRIA, & VIENNA RESEARCH PLATFORM ON ACCELERATING PHOTOREACTION DISCOVERY, UNIVERSITY OF VIENNA, WAHRINGER-STRASSE 17, A-1090 VIENNA, AUSTRIA, & ISTITUTO DI MATEMATICA APPLICATA E TECNOLOGIE INFORMATICHE E. MAGenes, VIA FERRATA 1, I-27100 PAVIA, ITALY.

Email address: ulisse.stefanelli@univie.ac.at