Transitions in collective response in multi-agent models of competing populations driven by resource level

Sonic H. Y. Chan,\textsuperscript{1} T. S. Lo,\textsuperscript{1} P. M. Hui,\textsuperscript{1} and N. F. Johnson\textsuperscript{2}

\textsuperscript{1}Department of Physics, The Chinese University of Hong Kong
Shatin, New Territories, Hong Kong
\textsuperscript{2}Department of Physics, University of Oxford,
Oxford OX1 3PU, United Kingdom

Abstract

We aim to study the effects of controlling the resource level in agent-based models. We study, both numerical and analytically, a Binary-Agent-Resource (B-A-R) model in which $N$ agents are competing for resources described by a resource level $1/2 \leq L < 1$, where $L = L/N$ with $L$ being the maximum amount of resource per turn available to the agents. Each agent picks the momentarily best-performing strategy for decision with the performance of the strategy being a result of the cumulative collective decisions of the agents. The agents may or may not be networked for information sharing. Detailed numerical simulations reveal that the system exhibits well-defined plateaux regions in the success rate which are separated from each other by abrupt transitions. As $L$ increases, the maximum success rate forms a well defined sequence of simple fractions. We analyze the features by studying the outcome time series, the dynamics of the strategies’ performance ranking pattern and the dynamics in the history space. While the system tends to explore the whole history space due to its competitive nature, an increasing $L$ has the effect of driving the system to a restricted portion of the history space. Thus the underlying cause of the observed features is an interesting self-organized phenomena in which the system, in response to the global resource level, effectively avoids particular patterns of history outcomes. We also compare results in networked population with those in non-networked population.

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I. INTRODUCTION

Looking around the world that we live in, we will find that there are complex systems everywhere. The study of these systems has become a hot research area in different disciplines such as physics, applied mathematics, biology, engineering, economics, and social sciences. In particular, agent-based models have become an essential part of research on Complex Adaptive Systems (CAS) [1]. For example, self-organized phenomena in an evolving population consisting of agents competing for a limited resource, have potential applications in areas such as engineering, economics, biology, and social sciences [1, 2]. The famous El Farol bar attendance problem proposed by Arthur [3, 4] constitutes a typical example of such a system in which a population of agents decide whether to go to a popular bar having limited seating capacity. The agents are informed of the attendance in the past weeks, and hence share common information, make decisions based on past experience, interact through their actions, and in turn generate this common information collectively. These ingredients are the key characteristics of complex systems [2]. The proposals of the binary versions of models of competing populations, either in the form of the Minority Game (MG) [5] or in a Binary-Agent-Resource (B-A-R) game [6, 7], have led to a deeper understanding in the research in agent-based models. For modest resource levels in which there are more losers than winners, the Minority Game proposed by Challet and Zhang [5, 8] represents a simple, yet non-trivial, model that captures many of the essential features of such a competing population. The MG, suitably modified, can be used to model financial markets and reproduce the stylized facts. The B-A-R model, which is a more general model in which the resource level is taken to be a parameter, has much richer behaviour. In particular, we will discuss the model and report the emergence of plateaux-and-jump structures in the average success rates in the population as the resource level is varied. We analyze the results within the ideas of the trail of histories in the history space and the strategy performance ranking patterns.

II. MODEL

The binary-agent-resource (B-A-R) model [6, 7, 9] is a binary version of Arthur’s El Farol bar attendance model [3, 4], in which a population of $N$ agents repeatedly decide whether to go to a bar with limited seating based on the information of the crowd size in recent
weeks. In the B-A-R model, there is a global resource level $L$ which is not announced to the agents. At each timestep $t$, each agent decides upon two possible options: whether to access resource $L$ (action ‘1’) or not (action ‘0’). The two global outcomes at each timestep, ‘resource overused’ and ‘resource not overused’, are denoted by 0 and 1. If the number of agents $n_1(t)$ choosing action ‘1’ exceeds $L$ (i.e., resource overused and hence global outcome 0) then the $N-n_1(t)$ abstaining agents win. By contrast if $n_1(t) \leq L$ (i.e., resource not overused and hence global outcome 1) then the $n_1(t)$ agents win. In order to investigate the behaviour of the system as $L$ changes, it is sufficient to study the range $N/2 \leq L \leq N$. The results for the range $0 \leq L \leq N/2$ can be obtained from those in the present work by suitably interchanging the role of ‘0’ and ‘1’ [7]. In the special case of $L = N/2$, the B-A-R model reduces to the Minority Game.

In the B-A-R model, each agent shares a common knowledge of the past history of the most recent $m$ outcomes, i.e., the winning option in the most recent $m$ timesteps. The agents are essential identical, except for the details in the strategies that they are holding. The full strategy space thus consists of $2^{2^m}$ strategies, as in the MG. Initially, each agent randomly picks $s$ strategies from the pool of strategies, with repetitions allowed. The agents use these strategies throughout the game. At each timestep, each agent uses his momentarily best performing strategy with the highest virtual points. The virtual points for each strategy indicate the cumulative performance of that strategy: at each timestep, one virtual point (VP) is awarded to a strategy that would have predicted the correct outcome after all decisions have been made, whereas it is unaltered for those with incorrect predictions. Notice that in the literature, sometimes one VP is deducted for an incorrect prediction. The results reported here, however, come out to be the same. A random coin-toss is used to break ties between strategies. In the B-A-R model, the agents in the population may or may not be connected, by some kind of network. In the case of a networked population [10, 11, 12, 13, 14], each agent has access to additional information from his connected neighbours, such as his neighbours’ strategies and/or performance. Here, we will focus our discussion on the B-A-R model in a non-networked population and report some numerical results for networked population.

The B-A-R model thus represents a general setting of a competing population in which the resource level can be tuned. From a governmental management point of view, for example, one would like to study how a population may react to a decision on increasing or
decreasing a certain resource in a community. Will such a change in resource level lead to a large response in the community or the community will be rather insensitive? To evaluate the performance of an agent, one (real) point is awarded to each winning agent at a given timestep. A maximum of $L$ points per turn can therefore be awarded to the agents per timestep. An agent has a success rate $w$, which is the mean number of points awarded to the agent per turn over a long time window. The mean success rate $\langle w \rangle$ among the agents is then defined to be the mean number of points awarded per agent per turn, i.e., an average of $w$ over the agents. We are interested in investigating the details of how the success rates, including the mean success rate and the highest success rate among the agents, change as the resource level $L$ varies in the efficient phase, where the number of strategies (repetitions counted) in play is larger than the total number of distinct strategies in the strategy space.

III. RESULTS: PLATEAUX FORMATION AND ABRUPT JUMPS

The effects of varying $L$ were first studied by Johnson et al. [7]. These authors reported the dependence of the fluctuations in the number of agents taking a particular option, on the memory size $m$ for different values of $L$. For the MG (i.e., $L = N/2$) in the efficient phase (i.e., small values of $m$) the number of agents making a particular choice varies from timestep to timestep, with additional stochasticity introduced via the random tie-breaking process. The corresponding period depends on the memory length $m$. The underlying reason is that in the efficient phase for $L = N/2$, no strategy is better overall than any other. Hence there is a tendency for the system to restore itself after a finite number of timesteps, thereby preventing a given strategy’s VPs from running away from the others. As a result, the outcome bit-string shows the feature of anti-persistency or double periodicity [15, 16, 17, 18, 19, 20, 21]. Since a maximum of $L = N/2$ points can be awarded per turn, the mean success rate $\langle w \rangle$ over a sufficiently large number of timesteps is bound from above by $L/N = 1/2$.

We have carried out extensive numerical simulations on the B-A-R model to investigate the dependence of the success rate on $L$ for $N/2 \leq L \leq N$. Unless stated otherwise, we consider systems with $N = 1001$ agents and $s = 2$. Figure 1 shows the results of the mean success rate $\langle w \rangle$, dark solid line) as a function of $L$ in a typical run for (a) $m = 3$ and (b) $m = 1$, together with the range corresponding to one standard deviation about $\langle w \rangle$ in
FIG. 1: Statistics of the success rate $w$ as a function of resource level $L$ with $N = 1001$, $s = 2$, (a) $m = 3$ and (b) $m = 1$. For each value of $m$, the agents have the same set of strategies for all $L$.

the success rates among the $N$ agents ($\sigma_w$, dotted lines) and the spread in the success rates given by the highest and the lowest success rates ($w_{\text{max}}$ and $w_{\text{min}}$, thin solid lines) in the population.

By taking a larger value of $N$ than most studies in the literature, we are able to analyze
the dependence on $L$ and $m$ in greater detail and discover new features. In particular, these quantities all exhibit abrupt transitions (i.e., jumps) at particular values of $L$. Between the jumps, the quantities remain essentially constant and hence form steps or ‘plateaux’. We refer to these different plateaux as states or phases, since it turns out that the jump occurs when the system makes a transition from one type of state characterizing the outcome bit-string to another. The origins of the plateaux are due to (i) the finite number of states allowed in each $m$, and (ii) insensitive to $L$ inside each state. The success rates within each plateau (state) come out to the the same. For different runs, the results are almost identical. At most, there are tiny shifts in the $L$ values at which jumps arise due to (i) different initial strategy distributions among the agents in different runs, and (ii) different random initial history bit-strings used to start the runs. In other words, a uniform distribution of strategies among the agents (e.g., in the limit of a large population) gives stable values of $L$ at the jumps. The results indicate that a community does not react to a slight change in resource level that lies within a certain plateau, but the response will be abrupt if the change in resource level swaps through critical values between transitions.

| $w_{\text{max}}$ | 1-bits:0-bits | Range of $L$ | Period |
|------------------|----------------|-------------|--------|
| 1/2              | 8:8            | $\sim 510-600$ | length 16 |
| 12/17            | 12:5           | $\sim 600-620$ | length 17 |
| 17/23            | 17:6           | $\sim 620-640$ | length 23 |
| 5/6              | 5:1            | $\sim 640-695$ | 111110 |
| 6/7              | 6:1            | $\sim 695-745$ | 11111110 |
| 7/8              | 7:1            | $\sim 745-755$ | 11111111 |
| 1                | 1:0            | $\sim 755-1000$ | 1 |

TABLE I: The states characterized by $w_{\text{max}}$ for the B-A-R model with $N = 1001$, $m = 3$ and $s = 2$. The results are obtained from the numerical data shown in Fig. 1(a).

The most striking feature in Fig. 1(a) is that the values of the plateaux in the highest success rate $w_{\text{max}}$, are given by simple fractions, e.g., 1, 7/8, 6/7, 5/6, 12/17, 1/2, etc. Figure 1(b) shows that the features in the success rates for the simpler case of $m = 1$ are similar to those in Fig. 1(a), except that the plateaux in $w_{\text{max}}$ take on fewer values, i.e., 1, 3/4, 1/2 as $L$ decreases. These values are closely related to the statistics in the outcome
TABLE II: The states characterized by $w_{\text{max}}$ for the B-A-R model with $N = 1001$, $m = 1$ and $s = 2$. The results are obtained from the numerical data shown in Fig. 1(b).

| $w_{\text{max}}$ | 1-bits:0-bits | Range of $L$ | Period |
|------------------|----------------|--------------|--------|
| 1/2              | 2:2            | $\sim 520-650$ | 1100   |
| 3/4              | 3:1            | $\sim 680-750$ | 1110   |
| 1                | 1:0            | $\sim 750-1000$ | 1      |

bit-string. For large $N$ and $m = 1$, the outcome bit-string shows a period of 4 bits. For values of $L$ with $w_{\text{max}} = 1/2$, it turns out that the fraction of the outcome 1 in a period is exactly 1/2. For the range of $L$ corresponding to $w_{\text{max}} = 3/4$, there are three 1’s in a period of 4. At the range corresponding to $w_{\text{max}} = 1$, there are persistently 1. Thus, the system passes through different states with different ratios of 1 and 0 in the outcome bit-string as $L$ varies. For $m = 3$, we have also carried out detailed analysis of the outcome bit-string. For later discussions, we summarize in Tables I and II the values of $w_{\text{max}}$, the range of $L$ corresponding to the observed value of $w_{\text{max}}$, the ratio of number of occurrence of 1-bits to 0-bits and the period in the outcome bit-string, as obtained numerically from the data shown in Fig. 1. Hereafter, $w_{\text{max}}$ is used to label the state at a given $L$.

IV. WHY IS IT SO? THE HISTORY SPACE AND BIT-STRING PATTERNS

A. The history space

We analyze the underlying mechanism for the emergence of plateaux and jumps in the performance of the system in terms of the path that a system passes through in the history space and the strategy performance ranking as the system evolves [11]. The history space consists of all the possible history bit-strings for a given value of $m$. For $m = 3$, it includes $2^3$ bit-strings of 0’s and 1’s. Figure 2(a) shows the history space for $m = 3$, together with the possible transitions (arrows) from one history to another. It will prove convenient to group the possible history bit-strings for a given $m$ into columns, in the way shown in Fig. 2(a). We label each column by a parameter $\zeta$, which is the number of 0’s in the 3-bit history (histories) concerned. One immediate advantage of this labelling scheme is that the
FIG. 2: (a) The history space in the B-A-R model with $m = 3$. The nodes correspond to the $2^m$ possible histories. The transition between nodes are indicated by the arrows, together with the outcome needed for the transitions to occur. The histories can be grouped into columns labeled by a parameter $\zeta$ which gives the number of ‘0’ bits in the histories. (b) For the $\zeta_{\text{max}} = 1$ states at $L \lesssim (1 - 1/2^s)N$, the system follows a path restricted to histories in the $\zeta = 0$ and $\zeta = 1$ columns, with $n$ loops at the $\zeta = 0$ history.

different states characterized by $w_{\text{max}}$ turn out to involve paths in a restricted portion of the full history space. For example, the state with $w_{\text{max}} = 1$ is restricted to the $\zeta = 0$ portion of the history space, i.e., the 111 history bit-string leads to an outcome of 1 and hence persistent self-looping at the node 111 in history space. The states with $w_{\text{max}} = 7/8$, 6/7, and 5/6 correspond to different paths in the history space restricted to the $\zeta = 0$ and $\zeta = 1$ groups of histories, as shown in Fig. 2(b). The states with $w_{\text{max}} = 17/23$ and 12/17 have paths extended to include $\zeta = 2$ histories. The state with $w_{\text{max}} = 1/2$ has paths that visit the whole history space ($\zeta = 0, 1, 2, 3$). The classification is summarized in Table III. One can regard this behaviour as the system, in response to the global resource level $L$, effectively avoiding certain nodes in the history space and hence avoiding particular patterns of historical outcomes. Thus, the resource level may be used as a control on the population’s response, in a way that restricts the agents from using every part of their strategies by suppressing the occurrence of some of the history bit-strings.
TABLE III: Classification of states by means of column number $\zeta$ for $m = 3$. The states characterized by $w_{\text{max}}$ have paths in the history space covering nodes in the columns $\zeta = 0, \cdots, \zeta_{\text{max}}$.

| $\zeta_{\text{max}}$ | States |
|----------------------|--------|
| 3                    | $\frac{8}{16}$ |
| 2                    | $\frac{12}{17}, \frac{17}{23}$ |
| 1                    | $\frac{5}{7}, \frac{6}{7}, \frac{7}{8}$ |
| 0                    | 1      |

B. Outcome Bit-string statistics at different resource levels

As the game proceeds, the system evolves from one history bit-string to another. This can be regarded as transitions between different nodes (i.e., different histories) in the history space. For $L = N/2$ in the efficient phase, it has been shown that the conditional probability of an outcome of, say, 1 following a given history is the same for all histories. For $L \neq N/2$, the result still holds for states characterized by $w_{\text{max}} = 1/2$. Note that a history bit-string can only make transitions to history bit-strings that differ by the most recent outcome, e.g., 111 can only be make transitions to either 110 or 111, and thus many transitions between two chosen nodes in the history space are forbidden. In addition, these allowed transitions do not in general occur with equal probabilities. This leads to specific outcome (and history) bit-string statistics for a state characterized by $w_{\text{max}}$.

We have carried out detailed analysis of the outcomes following a given history bit-string for $m = 3$, and for each of the possible states over the whole range of $L$, i.e., we obtain the chance of getting 1 (or 0) for every 3-bit history by counting from the outcome bit-string. Table IV gives the relative numbers of occurrences of each outcome for every history bit-string. For the state with $w_{\text{max}} = 1/2 = 8/16$, for example, the outcomes 0 and 1 occur with equal probability for every history bit-string, as in the MG. For the other states, the results reveal several striking features. It turns out that $w_{\text{max}}$ is simply given by the relative frequency of an outcome of 1 in the outcome bit-string, which in turn is governed by the resource level $L$. For example, a 0 to 1 ratio of 5 : 12 in the outcome bit-strings corresponds to the state with $w_{\text{max}} = 12/17$. In Table IV, we have intentionally grouped the history bit-strings into rows according to the label $\zeta$ in Fig. 2. We immediately notice that for every
TABLE IV: Outcome statistics for the B-A-R model at various $w_{\text{max}} = 8/16, 12/17, 17/23, 5/6, 6/7, 7/8,$ and $1$. The parameter $\zeta$ labels groups of histories as defined in Fig. 2.

| $w_{\text{max}}$ | $\zeta = 3$ | $\zeta = 2$ | $\zeta = 1$ | $\zeta = 0$ |
|-------------------|-------------|-------------|-------------|-------------|
| $\rightarrow 0 \rightarrow 1$ | $000$ | $001$ | $010$ | $101$ |
| $000$ | $000$ | $001$ | $010$ | $100$ |
| $001$ | $011$ | $101$ | $110$ | $011$ |
| $010$ | $101$ | $110$ | $111$ | $010$ |
| $100$ | $110$ | $111$ | $111$ | $100$ |

| $\rightarrow 0 \rightarrow 0$ | $000$ | $001$ | $010$ | $101$ |
| $000$ | $000$ | $001$ | $010$ | $100$ |
| $001$ | $011$ | $101$ | $110$ | $011$ |
| $010$ | $101$ | $110$ | $111$ | $010$ |
| $100$ | $110$ | $111$ | $111$ | $100$ |

| $\rightarrow 0 \rightarrow 1$ | $000$ | $001$ | $010$ | $101$ |
| $000$ | $000$ | $001$ | $010$ | $100$ |
| $001$ | $011$ | $101$ | $110$ | $011$ |
| $010$ | $101$ | $110$ | $111$ | $010$ |
| $100$ | $110$ | $111$ | $111$ | $100$ |

TABLE IV: Outcome statistics for the B-A-R model at various $m = 3$ states. The table shows the relative number of occurrence of each outcome following every possible history bit-string for the states characterized by $w_{\text{max}} = 8/16, 12/17, 17/23, 5/6, 6/7, 7/8,$ and $1$. The parameter $\zeta$ labels groups of histories as defined in Fig. 2.

possible state in the B-A-R model, the relative frequency of each outcome is a property of the group of histories having the same label $\zeta$ rather than the individual history bit-string, i.e., all histories in a group have the same relative fraction of a given outcome. This observation is important in understanding the dynamics in the history space for different states in that
it is no longer necessary to consider each of the $2^m$ history bit-strings in the history space. Instead, it is sufficient to consider the four groups of histories (for $m = 3$) as shown in Fig. 2(a). Analysis of results for higher values of $m$ show the same feature.

For the state characterized by $w_{\text{max}} = 1$, the outcome bit-string is persistently 1 and the path in the history space is repeatedly $111 \rightarrow 1$ (in the format of history $\rightarrow$ outcome). Therefore, the path simply corresponds to an infinite number of loops around the history node 111. Since the path is restricted to the $\zeta = 0$ node, we will also refer to this state as $\zeta_{\text{max}} = 0$ state. The system is effectively frozen into one node in the history space. In this case, there are effectively only two kinds of strategies in the whole strategy set, which differ by their predictions for the particular history 111. The difference in predictions for the other $(2^m - 1)$ history bit-strings become irrelevant. Obviously, the ranking in the performance of the two effective groups of strategies is such that the group of strategies that suggest an action ‘1’ for the history 111, outperforms the group that suggests an action ‘0’. For a uniform initial distribution of strategies, there are $N/2^s$ agents taking the action ‘0’ and $(1 - 1/2^s)N$ agents taking the action ‘1’, since half of the strategies predict 0 and half of them predict 1. To sustain a winning outcome of 1, the criterion is that the resource level $L$ should be higher than the number of agents taking the action ‘1’. Therefore, we have for the state with $w_{\text{max}} = 1$ that

$$\langle w \rangle = 1 - \frac{1}{2^s}$$  \hspace{1cm} (1)

and

$$L_{\text{min}} = \left(1 - \frac{1}{2^s}\right)N.$$  \hspace{1cm} (2)

These results are in agreement with numerical results. For $s = 2$, $\langle w \rangle = 3/4$ for $L > 3N/4$. Note that Eqs. (1) and (2) are valid for any values of $m$.

Table IV shows that the states with $w_{\text{max}} = 5/6, 6/7, 7/8$ have very similar features in terms of the bit-string statistics. They differ only in the frequency of giving an outcome of 1 following the history of 111. Note that the $\zeta = 2$ and $\zeta = 3$ histories do not occur. The results imply that as the system evolves, the path in history space for these states is restricted to the two groups of histories labeled by $\zeta = 0$ and $\zeta = 1$. The statistics show that the outcome bit-strings for the states with $w_{\text{max}} = 5/6, 6/7$ and $7/8$ exhibit only one 0-bit in a period of 6, 7 and 8 bits, respectively. We refer to these states collectively as $\zeta_{\text{max}} = 1$ states, since the portion of allowed history space is bounded by the $\zeta = 1$ histories.
Graphically, the path in history space consists of a few self-loops at the node 111, i.e., from 111 to 111, then passing through the \( \zeta = 1 \) group of histories once and back to 111, as shown in Fig. 2(b). The states with \( w_{\text{max}} = 1/2, 12/17, 17/23 \) involve the other groups of histories and exhibit complicated looping among the histories. We refer to them collectively as higher (i.e., \( \zeta_{\text{max}} > 1 \)) states.

V. THE \( \zeta_{\text{max}} = 1 \) STATES

The observed values of \( w_{\text{max}} \) for \( \zeta_{\text{max}} = 1 \) states can also be derived by following the evolution of the strategy performance ranking pattern as the game proceeds. We summarize the main ideas here. Details can be found in Ref. [22]. The major result is that for given \( m \), the values of \( w_{\text{max}} \) in a \( \zeta_{\text{max}} = 1 \) state can only take on

\[
    w_{\text{max}} = \frac{m + n}{m + n + 1},
\]

(3)

where \( 2 \leq n \leq m + 1 \). Taking \( m = 3 \) for example, we have \( 2 \leq n \leq 4 \) and hence \( n = 2, 3, 4 \). The values of \( w_{\text{max}} \) are 5/6, 6/7, and 7/8, exactly as observed in the numerical simulations.

Eq. (3) is a result of the collective response of the agents stemming from their strategy selection and decision-making processes, which in turn are coupled to the strategy ranking pattern. To close the feedback mechanism, the strategy ranking pattern must evolve in such a way so as to be consistent with the path in the history space. Recall that the \( \zeta_{\text{max}} = 1 \) states are those in which the system only visits the \( \zeta = 0 \) and \( \zeta = 1 \) histories, as shown in Fig. 2(b). In this situation, only \( (m + 1) \) history bits in a strategy are relevant, despite each strategy has \( 2^m \) entries, i.e., strategies that only differ in their predictions for the histories which do not occur are now effectively identical. Therefore, many strategies have tied performance.

The key point is that a complete path of the \( \zeta_{\text{max}} = 1 \) states in history space corresponds to one in which there are \( n \) turns around the \( \zeta = 0 \) history, i.e., from 111 \( \rightarrow \) 1 for \( n \) turns, then breaks away to visit each of the \( \zeta = 1 \) histories once and returns to the \( \zeta = 0 \) history (see Fig. 2(b)). The value of \( n \) that is consistent with the condition \( L < 3N/4 \) turns out to be \( 2 \leq n \leq m + 1 \). We have argued that for \( L > 3N/4 \), the system loops around the \( \zeta = 0 \) history indefinitely, since the number of agents \( (3N/4) \) persistently taking the option 1 is smaller than \( L \), in the large population limit. For \( L < 3N/4 \), the situation is
as follows. As the system starts to loop around the $\zeta = 0$ history, the strategies start to split in performance, with the group of strategies predicting $111 \rightarrow 1$ ($111 \rightarrow 0$) becomes increasingly better (worse). There is an overlap in performance between these two groups, as a result of their predictions when the system passes through the $\zeta = 1$ histories. As the number of loops at the $\zeta = 0$ history increases, the overlap in performance decreases and the number of agents taking option 1 increases towards $3N/4$, as a result of the strategy selection process. The condition $L < 3N/4$, thus imposes an upper bound on $n$. When the number of turns at the $\zeta = 0$ history exceeds a certain value fixed by $L$, the option 1 becomes the losing option and the system breaks away to the $\zeta = 1$ histories as the system makes transition from the history 111 to the history 110. The lower bound on $n$ is set by the restriction that the strategies in the group predicting $111 \rightarrow 1$ must perform better on the average than the group predicting $111 \rightarrow 0$, as $L > N/2$. With this understanding on the allowed paths in the history space, we can evaluate the number of winners in each turn of the path and obtain Eq. (3) for $w_{\text{max}}$, with $n$ being the number of loops that the system makes at the $\zeta = 0$ history and $2 \leq n \leq m + 1$.

VI. B-A-R IN CONNECTED POPULATIONS

We have also studied the B-A-R model in a connected population. The agents are assumed to be connected in the form of a classical random graph (CRG), i.e., random network. In a CRG, each node (representing an agent here) has a probability $p$ of connecting to another node. The agents interact in the following way. When an agent is linked to other agents, he checks if the strategies of his connected neighbours perform better than his own strategies. If so, he will choose the one with the highest VP among his neighbours and then use it for decision in that turn. If not, he will use his own best strategy. We have carried out detailed numerical calculations of $\langle w \rangle$ and $w_{\text{max}}$. Figures 3 and 4 show how $\langle w \rangle$ and $w_{\text{max}}$ depend on $L$ for various connecting probability $p$, with the $p = 0$ results corresponding to those in a non-networked population.

We observe that the features of plateaux-and-jumps persist in networked populations. There are differences in the details. While $w_{\text{max}}$ takes on the same set of values, the threshold in the resource level ($L_{\text{min}}$) to sustain a particular level of success rate is found to be higher in a networked population than in a non-networked one, as shown in Fig. 3 and Fig. 4.
FIG. 3: The mean success rate $\langle w \rangle$ as a function of resource level $L$ for $N = 1001$, $s = 2$, and $m = 1$. Different lines denote different values of $p$ in a classical random graph (CRG). Data for different $L$ are obtained from the same connected population and distribution of strategies among the agents.

For a given resource level, allowing agents to share information on strategy performance may actually worsen the global performance of the system. This is particularly clear if we inspect the results of $\langle w \rangle$ in the range of $0.5 < L/N < 0.75$ in Fig. 3. The addition of a small number of links will lower $\langle w \rangle$. This behaviour is consistent with the crowd-anticrowd theory [6, 23, 24] in that the links enhance the formation of crowds. The situation can be quite different in a high resource limit $L/N \approx 1$, where a small number of links may actually be beneficial. The reason is that for high resource level, some resource is left unused as $N/4$ agents will not have access to a strategy that predicts the winning option. Using the links, some of these agents switch from losers to winners and thus enhance the overall performance of the population [10].
FIG. 4: The highest success win rate $w_{\text{max}}$ among the agents as a function of resource level $L$ for $N = 1001$, $s = 2$, and $m = 1$. Different lines denote different values of $p$ in a classical random graph (CRG). Data are obtained from the same configurations as in Figure 3.

VII. SUMMARY

We have studied numerically and analytically the effects of a varying resource level $L$ on the success rate of the agents in a competing population within the B-A-R model. We found that the system passes through different states, characterized either by the mean success rate $\langle w \rangle$ or by the highest success rate in the population $w_{\text{max}}$, as $L$ decreases from the high resource level limit. Transitions between these states occur at specific values of the resource level. We found that different states correspond to different paths covering a subspace within the whole history space. Just below the high resource level is a range of $L$ that gives $m$ states corresponding to the fractions $w_{\text{max}} = (m+n)/(m+n+1)$, with $2 \leq n \leq m+1$. This result is in excellent agreement with that obtained by numerical simulations. The paths of these states in the history space are restricted to those $m$-bit histories with at most one-bit of 0 and with $n$ loops around the 111...history. This result is derived by considering the coupling of the restricted history subspace that the system visits, the strategy performance
ranking pattern, and the strategy selection process. While our analysis can also be applied to the $\zeta_{\text{max}} > 1$ states, the dynamics and the results are too complicated to be included here.

Our analysis also serves to illustrate the sensitivity within multi-agent models of competing populations, to tunable parameters. By tuning an external parameter, which we take as the resource level in the present work, the system is driven through different paths in the history space which can be regarded as a ‘phase space’ of the system. The feedback mechanism, which is built-in through the decision making process and the evaluation of the performance of the strategies, makes the system highly sensitive to the resource level in terms of which states the system decides to settle in or around. These features are quite generally found in a wide range of complex systems. The ideas in the analysis carried out in the present work, while specific to the B-A-R model used, should also be applicable to other models of complex systems.

While we have focused on the B-A-R model in our discussions, many of the underlying ideas are general to a wider class of complex systems. For example, one may regard the resource level $L$ as a handle in controlling a driving force in the system. With $L = N/2$, i.e., in the MG, and the random initial distribution of strategies and random initial history, the system is allowed to diffuse from an initial node in the history space to visit all the possible histories. A deviation of the resource level from $N/2$ acts like a driving force in the history space. Thus, there is always a competition between diffusive and driven behaviour, resulting in the non-trivial behaviour in the B-A-R model and its variations. For this reason, the present B-A-R system provides a fascinating laboratory for studying correlated, non-Markovian diffusion on a non-trivial network (i.e., history space).

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