Study of COVID-19 anti-pandemic strategies by using optimal control

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Abstract

In this study, we present a new epidemiological model, with contamination from confirmed and unreported. We also compute equilibria and study their stability without intervention strategies. Optimal control theory has proven to be a successful tool in understanding ways to curtail the spread of infectious diseases by devising the optimal disease intervention strategies. We investigate the impact of distancing, case finding, and case holding controls while at the same time, we minimize the number of infected and dead individuals. The method consists of minimizing the cost functional related to infectious, death, and controls through some strategies to reduce the spread of the COVID19 epidemic.

Keywords— Optimal control strategy, Optimal control theory, Covid 19, Stability, Parameters estimates, Fitting COVID19 data

1. INTRODUCTION

The COVID-19 pandemic has continued to evolve for more than six months around the world. Many countries have applied containment measures, then deconfinement. Currently, because of the resurgence of cases, some of these countries are proceeding with re-containment measures.

The health and social distancing measures are not always respected by the populations. Among the confirmed cases, there are caregivers. That shows a security flaw in the quarantine procedures. There are many undetected cases in the people who favor the evolution of the pandemic. Hopes are on the discovery of a vaccine. But in the meantime, it is useful to come up with strategies that allow us to manage the pandemic better.

Several recent works have used SIR / SEIR models and other types of nonlinear differential equations (1, 2, 15) to understand the evolution of the pandemic but also to predict its subsequent evolution. Other techniques are also used such as machine learning, stochastic (1, 2, 16, 17), etc.

*Support of the Non Linear Analysis, Geometry and Applications (NLAGA) Project
There is some work dedicated to the application of optimal control to the pandemic. Many authors have used optimal control to study some diseases like HIV [8], [20].

We analyze an epidemiological differential equation model with the identification of its parameters and initial values, based upon reported case data from public health sources. The objective of this work is to develop control strategies to stem the evolution of the pandemic.

The paper is organized as follows. In section 2, we present the model. In section 3, a mathematical analysis of the model is performed. Then, in section 4, we introduce an optimal control problem to study. Thus in section 5 we show numerical results of the optimal control problem. We discuss the results in section 6. We explain the methods we use in this work in section 7. Finally, we give conclusions and perspectives in section 8.

2. MODEL FORMULATION

We consider the following differential equation model:

\[
\begin{align*}
\dot{S} &= \Lambda - \beta S (I + \epsilon I_1 + I_2) - \mu S, \\
\dot{I} &= \beta S (I + \epsilon I_1 + I_2) - (\mu + \alpha_1 + \alpha_2) I, \\
\dot{I}_1 &= \alpha_1 I - (\mu + d + \theta_1) I_1, \\
\dot{I}_2 &= \alpha_2 I - (\mu + d + \theta_2) I_2, \\
\dot{R} &= \theta_1 I_1 + \theta_2 I_2 - \mu R.
\end{align*}
\]

(1)
3 Mathematical Analysis

Table 1: The model variables

| Variable | Explanations for different classes |
|----------|-----------------------------------|
| $S(t)$  | Number of susceptible population at time $t$ |
| $I(t)$  | Number of infected population at time $t$ (i.e. asymptomatic infectious) |
| $I_1(t)$ | Number of infected reported population at time $t$ (i.e. symptomatic infectious with sever symptoms) |
| $I_2(t)$ | Number of infected unreported population at time $t$ (i.e., symptomatic infectious with mild symptoms) |
| $R(t)$  | Number of recovered adults satisfying undetectable criteria at time $t$ |

Table 2: Parameters model formulation and their description

| Parameter | Description |
|-----------|-------------|
| $\alpha_1$ | Rate at which asymptomatic infectious become reported symptomatic |
| $\alpha_2$ | Rate at which asymptomatic infectious become unreported symptomatic |
| $\beta$    | Rate of transmission |
| $\mu$      | Natural death rate of the population |
| $\Lambda$  | Recruitment rate |
| $\theta_1$ | Rate of recovery from reported population |
| $\theta_2$ | Rate of recovery unreported population |
| $d$        | Death rate of infected population due to COVID-19 coronavirus |

The system is supplemented by initial conditions

\[ S(t_0) = S_0 > 0, \quad I(t_0) = I_0 > 0, \quad I_1(t_0) = I_{10} = 0 \quad \text{and} \quad I_2(t_0) = I_{20} \geq 0, \]

with $t_0$ the starting time of the epidemic. Figure I depicts a flow diagram of the model. In this model, the confirmed are automatically quarantined. Even in general, there may have some stages before confirmed individuals become quarantined. In this work, we consider these two compartments as one. Security failing during the quarantine or the process of quarantine can expose susceptible people to contamination. That is modeled by the term $\beta S \epsilon I_1$. Then a proportion $\epsilon$ of confirmed $I_1$ can interact with susceptible.

3. Mathematical Analysis

One of the most critical concerns about any infectious disease is its ability to invade a population. The basic reproduction number, $R_0$, is a measure of the potential for disease spread in a population. It represents the average number of secondary cases generated by an infected individual introduced into a susceptible population with no immunity to the disease in the absence of interventions to control the infection. If $R_0 < 1$, then on average, an infected individual produces less than one newly infected individual throughout his infection period. In this case, the infection may die out in the long run. Reversely, if $R_0 > 1$, each infected individual produces, on average more than one new infection. Hence the disease will be able to spread in a population. A significant value of $R_0$ may
indicate the possibility of a major epidemic. Using the next-generation operator technique described by (5) and subsequently analyzed by (21), we obtained the basic reproduction number.

### 3.1. Well–posedness of the model

In this section, we prove that the system (1) is epidemiologically meaningful. In other words, solutions of system (1) with positive initial data remain positive for all time \( t > 0 \).

Now, adding all equations in the differential system (1) gives

\[
\dot{N} = \Lambda - \mu N - d_1 I_1 - d_2 I_2 \leq \Lambda - \mu N.
\]

It then follows that, \( \lim_{t \to \infty} N(t) = \frac{\Lambda}{\mu} \) which implies that the trajectories of system (1) are bounded.

On the other hand, solving the differential inequality

\[
\dot{N} \leq \Lambda - \mu N,
\]

so that,

\[
0 \leq N(t) \leq \frac{\Lambda}{\mu} + (N(0) - \frac{\Lambda}{\mu})e^{-\mu t}.
\]

Thus, at \( t \to \infty, 0 \leq N(t) \leq \frac{\Lambda}{\mu} \). Therefore, all feasible solutions of model system (1) enter the region:

\[
D = \{ (S, I, I_1, I_2, R) \in \mathbb{R}_+^5 \}
\]

which is a positively invariant set of system (1). Furthermore, the model (1) is well-posed epidemiologically and we will consider dynamic behavior of model (1) on \( D \).

### 3.2. Equilibrium point

To obtain the disease-free equilibrium, \( I(t), I_1(t), I_2(t) \) and the right-hand-side of system (1) are set to zero. Then, the disease-free equilibrium will be given by

\[
E = (S_0, 0, 0, 0, 0) = \left( \frac{\Lambda}{\mu}, 0, 0, 0, 0 \right)
\]

### 3.3. Rate reproduction number

By using the next-generation operator method on the system (1), we establish the linear stability of \( E_0 \). Using the notation in (21), the matrices \( F \) and \( V \), for the new infection terms and the remaining transfer terms respectively, are given by (noting that \( S_0 = \frac{\Lambda}{\mu} \) at the DFE \( E_0 \))

\[
F = \begin{pmatrix} \beta S_0 & \epsilon \beta S_0 & \beta S_0 \\ \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \mu + \alpha_1 + \alpha_2 & 0 & 0 \\ 0 & \mu + d + \theta_1 & 0 \\ 0 & 0 & \mu + d + \theta_2 \end{pmatrix}.
\]

Thus,

\[
R_0 = \rho(FV^{-1}) = \frac{\beta S_0}{\mu + \alpha_1 + \alpha_2} = \frac{\beta \Lambda}{\mu(\mu + \alpha_1 + \alpha_2)}.
\]
Lemma 3.1. The DFE of the Covid 19-only model \((1)\), is locally asymptotically stable (LAS) if \(R_0 < 1\), and unstable if \(R_0 > 1\).

The threshold quantity of \(R_0\) is the reproduction number for COVID-19. It measures the average number of new Covid-19 infections generated by a single COVID-19 infected individual in a population where a certain fraction of infected individuals is treated.

3.4. Global stability of the disease-free equilibrium

We now turn to the global stability of the disease-free equilibrium \(E_0\). We prove that the disease-free equilibrium \(E_0\) is globally asymptotically stable under a certain threshold condition. To this aim, we use a result obtained by Kamgang and Sallet [11].

Let \(x_1 = (S, R)\) and \(x_2 = (I_2, I_1, I)\). We express the sub-system

\[
\dot{x}_1 = A_1(x_1, 0),(x - x^*_1), \quad \text{as}
\]

\[
\begin{cases}
\dot{S} = \Lambda - \mu S, \\
\dot{R} = -\mu R.
\end{cases}
\]

It is a linear system which is globally asymptotically stable at the equilibrium \(E_0\), corresponding to the DFE where the hypotheses \(H_1\) and \(H_2\) in [11] are satisfied.

The matrix \(A_2(x)\) is given by

\[
A_2(x) = \begin{pmatrix}
-(\mu + d_2 + \theta_2) & 0 & \alpha_2 \\
0 & -(\mu + d_1 + \theta_1) & \alpha_1 \\
0 & \beta S_0 - (\mu + \alpha_1 + \alpha_2)
\end{pmatrix}.
\]

The eigenvalues of the sub-matrix:

\[
J_0 = \begin{pmatrix}
-(\mu + d_1 + \theta_1) & \alpha_1 \\
0 & \beta S_0 - (\mu + \alpha_1 + \alpha_2)
\end{pmatrix}.
\]

Since \(J_0\) is a matrix of dimension 2, necessaries conditions for \(J_0\) to be stable is \(tr(J_0) < 1\) and \(det(J_0) > 0\). Note that \(tr(J_0) < 0\) gives \(\beta S_0 < \mu + \alpha_1 + \alpha_2\). Also, the condition \(det(J_0) > 0\) gives

\[
\beta S_0 \leq \mu + \alpha_1 + \alpha_2. \quad (8)
\]

Note that the inequality \((8)\) corresponds to \(R_0 \leq 1\). This achieves the proof.

We have the following result about the stability of the disease-free equilibrium.

Théorème 3.1. The disease-free equilibrium of system \((1)\) is globally asymptotically stable in \(D\) whenever \(R_0 \leq 1\). This implies the global asymptotic stability of the disease-free equilibrium on the nonnegative orthant \(\mathbb{R}_+^2\), i.e., the disease naturally dies out.

Proof. We consider the Lyapunov function defined by \(V(S, I, I) = I\). So, we have
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\[
\dot{V} = \dot{I} = \beta SI - (\mu + \alpha_1 + \alpha_2) I = I[\beta S - (\mu + \alpha_1 + \alpha_2)] = I(\mu + \alpha_1 + \alpha_2)[\frac{R_0 S}{S_0} - 1] \leq 0.
\]

Moreover \(\dot{V} = 0\) if \(I = 0\) or \(S = S_0\) and \(R_0 = 1\). Since we are in a positively invariant compact, by LaSalle’s invariance principle [12], the DFE is globally asymptotically stable in \(D\).

4. OPTIMAL CONTROL IN THE EPIDEMIC MODEL

Optimal control problems have generated a lot of interest from researchers all over the world. For instance (Imanov, 2011, see [9]) examined the application of the method of similar solutions in solving time optimal control problems with state constraints. Similarly, various techniques have been applied to study optimal control problems related to dynamical systems. However, we considered the aspect of optimal control to reduce the spread of COVID-19 disease through the combination of the aspects of the education campaign, quarantine, and treatment of infected individuals. This study intends to apply optimal control theory to minimize the spread disease by some control strategies and minimize the cost of applying controls in order to best combat the spread of COVID-19 disease.

Consider these two epidemic models with controls \(u_1(t)\), \(u_2(t)\), \(u_3(t)\), death \(D(t)\), recovered \(R_1(t)\) from infected reported and recovered \(R_2(t)\) from infected unreported, given by the following two models:

- **Model 1:**

  \[
  \begin{align*}
  \dot{S} &= \Lambda - \beta S (I + \epsilon (1-u_1) I_1 + I_2 (1-u_1)) - \mu S, \\
  \dot{I} &= \beta S (I + \epsilon (1-u_1) I_1 + I_2 (1-u_1)) - (\mu + \alpha_1 (1 + u_2) + \alpha_2 (1 - u_3)) I, \\
  \dot{I}_1 &= \alpha_1 (1 + u_2) I - (\mu + d(1-u_3) + \theta_1 (1 + u_3)) I_1, \\
  \dot{I}_2 &= \alpha_2 (1 - u_3) I - (\mu + d(1-u_3) + \theta_2 (1 + u_3)) I_2, \\
  \dot{R}_1 &= \theta_1 (1 + u_3) I_1 - \mu R_1, \\
  \dot{R}_2 &= \theta_2 (1 + u_3) I_2 - \mu R_2, \\
  \dot{D} &= d(1-u_3) (I_1 + I_2).
  \end{align*}
  \]  

(9)
4.1 Modeling the optimal control problem

In this subsection, we present the optimal control problem we intend to solve. Two strategies are proposed to analyze the spread of the viruses when some controls are applied.

Let’s set \( u(t) = (u_1(t), u_2(t), u_3(t)) \in [0, 1]^3 \), \( x(t) = (S(t), I(t), I_1(t), I_2(t), R_1(t), R_2(t), D(t)) \) and \( x_0 = (S_0, I_0, I_{10}, I_{20}, R_{10}, R_{20}, D_0) \).

Our objective functional to be minimized is as follows:

\[
J(x, u) = \int_0^T \left[ a_1 I(t) + a_2 I_1(t) + a_3 I_2(t) + a_4 D(t) + \frac{a_5}{2} u_1^2(t) + \frac{a_6}{2} u_2^2(t) + \frac{a_7}{2} u_3^2(t) \right] dt
\]

We assume that the relative intervention costs are nonlinear and take a quadratic form in the controls. The coefficients, \( a_i \), \( i = 1 \cdots 7 \), are balancing factors according to the size and the importance.

\[ \text{Model 2:} \]
\[
\begin{align*}
\dot{S} &= \Lambda - \beta S(I + \epsilon(1 - u_1)I_1 + (1 - u_1)I_2) - \mu S, \\
\dot{I} &= \beta S(I + \epsilon(1 - u_1)I_1 + (1 - u_1)I_2) - (\mu + \alpha_1(1 + u_2 + u_3) + \alpha_2(1 - u_2 - u_3)) I, \\
\dot{I}_1 &= \alpha_1(1 + u_2 + u_3) I - (\mu + d(1 - u_3) + \theta_1(1 + u_2 + u_3)) I_1, \\
\dot{I}_2 &= \alpha_2(1 - u_2 - u_3) I - (\mu + d(1 - u_3) + \theta_2(1 + u_2 + u_3)) I_2, \\
\dot{R}_1 &= \theta_1(1 + u_2 + u_3) I_1 - \mu R_1, \\
\dot{R}_2 &= \theta_2(1 + u_2 + u_3) I_2 - \mu R_2, \\
\dot{D} &= d(1 - u_3)(I_1 + I_2).
\end{align*}
\]
of the objective functional. Thus, we seek optimal controls variables $u^*$ and states variables $x^*$ such that

$$J(x^*, u^*) = \min_{u(t) \in [0, 1]^3} J(x, u)$$

subject to: $x(t)$ satisfies the DE model (9) or (10) (12)

$$x(0) = x_0$$

By minimizing the functional, we want to reduce, at the same time, the infectious asymptomatic, the reported and unreported symptomatic, the death, and the controls. Further, we will propose a constructed functional $J(x, u)$. For the construction of that functional, refer to section 7.

4.2. Existence of an optimal control solution

We analyze sufficient conditions for the existence of a solution to the optimal control problem (13). Using a result in Fleming and Rishel ([6]) and Hattaf and Yousfi ([7]), the existence of the optimal control can be obtained.

**Théorème 4.1.** There exists an optimal control $u^*$ and corresponding state $x^*$ to the problem (13).

**Proof.** The existence of an optimal control is guaranteed by Corollary 4.1 of Fleming ([6]) due the following

1. the convexity of the integrand of $J$ with respect to $u$;
2. a priori boundedness of the state solutions;
3. Lipschitz property of the state system with respect to the state variables.

Since the functional is continuously differentiable in $t, x, u$, and with the bounded domains of the state $x$ and the control $u$, there exist an optimal control and state $(x^*, u^*)$ that minimize the functional (11).

The following theorem is a consequence of the maximum principle.

**Théorème 4.2.** Given an optimal control $u^*$ and corresponding state $x^*$ solutions to the problem (13), then there exist adjoint variable $p$ such that $u^*, x^*$ satisfy the Pontryagin’s Maximum Principle.

**Proof.** Since the problem (13) with the model (10) generalize the problem with the model (10), we only perform a proof for the problem (13) with the model (10). The theorem is a direct application of Pontryagin’s maximum principle (13). Then the Hamiltonian of the problem (13) is given as follows:

$$H = a_1 I(t) + a_2 I_1(t) + a_3 I_2(t) + a_4 D(t) + \frac{a_5}{2} u_1^2(t) + \frac{a_6}{2} u_2^2(t) + \frac{a_7}{2} u_3^2(t) + \sum_{i=1}^{n} p_i g_i$$

where $g_i$, $i = 1 \cdots 7$ denotes the right side of the differential equation of the $i$ the state variables, and $p = (p_1(t), p_2(t), p_3(t), p_4(t), p_5(t), p_6(t), p_7(t))$ the associated adjoints for the states $x$. Then, we obtain


\[
H = a_1 I(t) + a_2 I_1(t) + a_3 I_2(t) + a_4 D(t) + \frac{a_5}{2} u_1^2(t) + \frac{a_6}{2} u_2^2(t) + \frac{a_7}{2} u_3^2(t) + p_7(t)(d - u_3(t))(I_1(t) + I_2(t)) + p_3(t)(\alpha_1 I(t)(u_2(t) + u_3(t) + 1) - I_1(t)(d - u_3(t)) + \mu + \theta_1 u_2(t) + \theta_2 u_3(t) + 1) + p_4(t)(\alpha_2 I(t)(-u_2(t) - u_3(t) + 1) - I_2(t)(d - u_3(t)) + \mu + \theta_2 u_2(t) + u_3(t) + 1)) + p_1(t)(-\beta S(t)(\epsilon I_1(t) - 1 - u_1(t)) + I_2(t)(1 - u_1(t)) + I(t)) + \Lambda - \mu S(t) + p_2(t)(\beta S(t)(\epsilon I_1(t)(1 - u_1(t)) + I_2(t)(1 - u_1(t)) + I(t)) - I(t)(\mu + \alpha_1 u_2(t) + u_3(t) + 1) - \alpha_2 I(t)(-u_2(t) - u_3(t) + 1) + p_5(t)(\theta_1 I_1(t)(u_2(t) + u_3(t) + 1) - \mu R_1(t)) + p_6(t)(\theta_2 I_2(t)(u_2(t) + u_3(t) + 1) - \mu R_2(t))
\]

Therefore we can derive the following:

\[
\begin{align*}
\dot{p}_1 &= -\frac{\partial H}{\partial I_1}, \\
\dot{p}_2 &= -\frac{\partial H}{\partial I_2}, \\
\dot{p}_3 &= -\frac{\partial H}{\partial I_3}, \\
\dot{p}_4 &= -\frac{\partial H}{\partial R_1}, \\
\dot{p}_5 &= -\frac{\partial H}{\partial R_2}, \\
\dot{p}_6 &= -\frac{\partial H}{\partial D}
\end{align*}
\]

with \( p_i(T) = 0 \) for \( i = 1, 2, 3, 4, 5, 6, 7 \) evaluated at the optimal controls and the corresponding states, which results in adjoint system of theorem (4.2). The Hamiltonian \( H \) is minimized with respect to the controls at the optimal controls; therefore, we differentiate \( H \) with respect to \( u_1, u_2, \) and \( u_3 \) on the set \( \Gamma \) respectively, thereby obtaining the following optimality conditions:

\[
\frac{\partial H}{\partial u_1} = 0, \quad \frac{\partial H}{\partial u_2} = 0, \quad \frac{\partial H}{\partial u_3} = 0
\]

Solving for \( u_1^*, u_2^*, \) and \( u_3^* \), we obtain

\[
\begin{align*}
u_1^*(t) &= \frac{1}{a_5}(-p_1(t)(-\beta S(t)(-\epsilon I_1(t) - I_2(t))) - p_2(t)(\beta S(t)(-\epsilon I_1(t) - I_2(t)))) \\
u_2^*(t) &= \frac{1}{a_6}(-p_3(t)(\alpha_1 I(t) - \theta_1 I_1(t)) - p_5(t)(\theta_1 I_1(t)) - p_4(t)(-\theta_2 I_2(t) - \alpha_2 I(t)) - p_6(t)(\theta_2 I_2(t) - p_2(t)(\alpha_2 I(t) - \alpha_1 I(t)))) \\
u_3^*(t) &= \frac{1}{a_7}(-p_7(t)(d(I_1(t) + I_2(t))) - p_3(t)(\alpha_1 I(t) - (\theta_1 - d) I_1(t)) - p_4(t)(-(\theta_2 - d) I_2(t) - \alpha_2 I(t)) - p_5(t)(\theta_1 I_1(t)) - p_6(t)(\theta_2 I_2(t) - p_2(t)(\alpha_2 I(t) - \alpha_1 I(t))))
\end{align*}
\]

This end the proof \( \square \)

5. **Numerical simulation of optimal controls**

In this section, we show numerical simulation of optimal controls problem. We use ACADO an optimal control solver tool to solve the problems. ACADO solver use direct method that is it starts by discretizing the problem to get a non linear problem (NLP) and at the end solve the NLP problem. The parameters of the model are estimated by fitting data of the cumulative cases of Senegal.
country. That method of fitting cumulative data cases has been presented in [15], [1] and [2]. Details is given in the appendix section. The values of the parameters are: $\Lambda = 0.0914 N/100$, with 

$N = 16743927$ the total population of Senegal. $\beta = 1.04756 \cdot 10^{-5}, \mu = 0.000219, \alpha_1 = 0.110064,$ $\alpha_2 = (1 - f) \alpha_1 / f$, with $f = 0.8; \theta_1 = 1/7, \theta_2 = 1/7, \ d = 0.00194523, \epsilon = 0.02$. The initial conditions are: $t_0 = 0.0166363, I_0 = 190.612, I_{10} = 98.3121, I_{20} = 24.578, S_0 = N - I_0, R_{10} = 0, R_{20} = 0, D_0 = 0$.

We constructed and use the following functional $J(x, u) = ck(I(t) + I_1(t) + I_2(t) + D(t)) + \frac{1}{2}(u_1^2 + u_2^2 + u_3^2)$. With $c = 4.9 \cdot 10^{-5}$ and $k = 1$.

We consider different strategies:

1. We solve the optimal control problem (13) with the model (9).

2. We solve the optimal control problem (13) with the model (10).

The functional are constructed based on the general one (11). See Section 7 for more details. The figures 2 show results of the model (1) where we do not consider controls. The figures 4 and 3 show the results related to the strategy 1, while the figures 6 and 5 show the results related to the strategy 2.

![Figure 2: Plot of the differential equation (1). There is no controls strategies.](image-url)
Figure 3: Plot of controls of the first strategy.
Figure 4: Plot of states of the first strategy.
Figure 5: Plot of controls of the second strategy.
Figure 6: Plot of states of the second strategy.
Figure 7: Comparative plot of states with and without controls.
Figure 8: Zoom of comparative plot of states with and without controls showing the effect of optimal controls strategies on the infected and death population.
6. DISCUSSION

The functional we minimize has two parts. The first part is composed of infected and death terms, while the second part is composed of the control terms. The size scale of these two parts is very different. Hence the choice of the coefficients \(a_1, \cdots, a_7\), as balance, has a great influence on the result.

We construct a function with an economic sens by using some data. More details on the construction of that function can be found in the appendix.

We see in the figures 2, 4 and 6 that for all the two strategies the infected asymptomatic individuals \(I\), the infected reported individuals \(I_1\) and the infected unreported individuals \(I_2\) are reduced compared to the case without controls. Also, the death cases are reduced with controls in comparison to the situation without controls.

The second strategy is better than the first one. Indeed, in the second strategy, there is no epidemic.

The results in the figures 7 and 8 show a difference in the number of infected (Reported/ Unreported) individuals without controls compared to the number with optimal strategies. Due to the control strategies 1 and/or 2, the number of infected individuals (Reported/ Unreported) decreases and reaches the turning point of the asymptomatic infectious cases later than without controls. At the same time, optimal strategies reduced the maximal number of infected (Reported/ Unreported) people compared to the case without controls. In other words, the maximal value of the peak decreases, and the time of the peak is postponed by applying controls.

7. MATERIAL AND METHODS

7.1. Estimation of parameters

The estimation of the parameters of the model (1) is done by using techniques in [15], [1] and [2]. We fit the cumulative data with an exponential function \(TNI(t) = b \exp(ct) - a\). In addition, we assume that the cumulative function can be given in integral form as \(TNI(t) = \alpha_1 \int_{t_0}^{t} I(s)ds + TNI_0\).

Then \(TNI(t_0) = TNI_0 = b \exp(ct_0) - a\). Thus, we obtain \(t_0 = \frac{\ln(TNI_0 + a) - \ln(b)}{c}\).

Also, we have:

\[
I(t) = TNI(t) = bc \exp(ct).
\]

Then \(I(t_0) = \frac{bc}{\alpha_1} \exp(ct_0) = \frac{c}{\alpha_1} (TNI_0 + a) = I_0\) and \(\frac{I(t)}{I(t_0)} = \exp(c(t - t_0))\). Hence, we obtain

\[
I(t) = I(t_0) \exp(c(t - t_0)),
\]

then \(\dot{I}(t) = cI(t)\) and \(\dot{I}(t_0) = cI(t_0)\).

Let's set \(\delta_1\) and \(\delta_2\) such that \(I_1 = \delta_1 I\) and \(I_2 = \delta_2 I\). Then replacing in the second an third equation of the following system:

\[
\begin{align*}
\dot{I} &= \beta S (I + cI_1 + I_2) - (\mu + \alpha_1 + \alpha_2) I, \\
\dot{I}_1 &= \alpha_1 I - (\mu + d + \theta_1) I_1, \\
\dot{I}_2 &= \alpha_2 I - (\mu + d + \theta_2) I_2,
\end{align*}
\]

then

\[
\begin{align*}
\dot{I} &= \beta S (I + \delta_1 I + \delta_2 I) - (\mu + \alpha_1 + \alpha_2) I, \\
\dot{I}_1 &= \alpha_1 I - (\mu + d + \theta_1) I_1, \\
\dot{I}_2 &= \alpha_2 I - (\mu + d + \theta_2) I_2,
\end{align*}
\]
we obtain
\[ \delta_1 = \frac{\alpha_1}{c + \mu + d + \theta_1} = \frac{I_{10}}{I_0} \] \hspace{1cm} (18)
\[ \delta_2 = \frac{\alpha_2}{c + \mu + d + \theta_2} = \frac{I_{20}}{I_0} \] \hspace{1cm} (19)

Then introducing (19) in the first equation of (17), we obtain:
\[ c + \mu + \alpha_1 + \alpha_2 = \beta S_0 (1 + \epsilon \delta_1 + \delta_2) \]

Hence
\[ \beta = \frac{c + \mu + \alpha_1 + \alpha_2}{S_0 (1 + \epsilon \delta_1 + \delta_2)} \] \hspace{1cm} (20)

Replacing (19) in (20), we obtain:
\[ \beta = \frac{(c + \mu + \alpha_1 + \alpha_2)(c + \mu + d + \theta_1)(c + \mu + d + \theta_2)}{S_0 ((c + \mu + d + \theta_1)(c + \mu + d + \theta_2) + \epsilon \alpha_1 (c + \mu + d + \theta_1) + \alpha_2 (c + \mu + d + \theta_2))} \] \hspace{1cm} (21)

To estimate the death rate, we have:
\[ D_1(t) = \int_{t_0}^{t} dI_1(s)ds = \int_{t_0}^{t} d\delta_1 I_1(s)ds \]
\[ = \frac{d}{c + \mu + d + \theta_1} \left( b \exp(ct) - a - TN I_0 \right). \] \hspace{1cm} (22)

With \( D_1 \), the death from reported cases.

We consider that 80\% of cases can be detected. Then \( f = 0.8 \) and \( \alpha_2 = \frac{1 - f}{f} \alpha_1 \), with \( \alpha_1 \) estimated above. We set the infectious period to medical values 1/7 for all infected reported and unreported. The pandemic death rate \( d \) is estimated by using reported death data. We consider the same value for death from unreported cases.

For the birth rate, we use 32.9\% of year 2018, from \url{https://fr.wikipedia.org/wiki/Démographie_du_Sénégal}. Then the recruitment is \( \Lambda = 32.9\% N/365 \) by day. The death rate is 7.9\% by year at 2018.

7.2. Construction of the functional

In order to have a functional with economic sense, we consider what follows:

- We lost money when people are infected or death. Let’s note that cost \( a \) by individual and by day, associated to \( I, I_1, I_2 \) and \( D \). Then \( a = a_1 = a_2 = a_3 = a_4 \).
- We lost money when we perform test on susceptible individuals. Let’s note that cost \( a_6 \) by day, associated to controls \( u_2 \).
- We spend money to provide treatment. Let’s note that cost \( a_7 \) by day, associated to the control \( u_3 \).
- We spend money to carry out health campaigns and education. Let’s note that cost \( a_5 \) by day, associated to the control \( u_1 \).
Then the functional \( J(1, 2, 3) \) become: 
\[
J(u_1, u_2, u_3) = a(I(t) + I_1(t) + I_2(t) + D(t)) + a_5 \frac{u_1^2}{2} + a_6 \frac{u_2^2}{2} + a_7 \frac{u_3^2}{2}.
\]
We consider that \( J(u_1, u_2, u_3) \leq C \), with \( C \) a maximal expense. Then we can rewrite the functional with proportion coefficients \( a = \frac{a_1}{C}, a_5 = \frac{a_5}{C}, a_6 = \frac{a_6}{C}, a_7 = \frac{a_7}{C} \):
\[
J(u_1, u_2, u_3) = a' (I(t) + I_1(t) + I_2(t) + D(t)) + a_5 \frac{u_1^2}{2} + a_6 \frac{u_2^2}{2} + a_7 \frac{u_3^2}{2} \leq 1.
\]
To characterize the loss of money due to infected and death individuals, we use the GDP per capita. The Gross domestic product (GDP) per capita is an indicator of the level of economic activity. It is the value of GDP divided by the number of inhabitants of a country. This indicator is sometimes used to roughly measure per capita income. See [https://fr.wikipedia.org/wiki/Produit_intérieur_brut_par_habitant](https://fr.wikipedia.org/wiki/Produit_intérieur_brut_par_habitant).
Now considering the 2018 GDP per capita and per day of the Senegal country evaluated to 2456.334 FCFA calculated from 1522 $ per capita, per year. The data come from [https://www.populationdata.net/pays/senegal/](https://www.populationdata.net/pays/senegal/). We set \( a = 2456.334 \).
A COVID-19 test in Senegal country is evaluated to 50000 FCFA by individual. If we fix a number of test to perform at 1000 by day, then we have \( a_6 = 50000000 \). We see that \( a = a_6 \cdot 4.91267 \cdot 10^{-5} \).
We set \( c = 4.91267 \cdot 10^{-5} \). We choose to fix \( a_5, a_7 \) to the same value of \( a_6 \). Then the functional becomes:
\[
J(u_1, u_2, u_3) = ca_5' (I(t) + I_1(t) + I_2(t) + D(t)) + a_6 \frac{u_2^2}{2} + a_7 \frac{u_3^2}{2} \leq 1.
\]
Thus considering that the costs are proportional to there respective control, we write:
\[
J(u_1, u_2, u_3) = c \frac{u_2^2}{2} (I(t) + I_1(t) + I_2(t) + D(t)) + \frac{u_1^2}{2} + \frac{u_2^2}{2} + \frac{u_3^2}{2}
\]
Finally to generalize the functional we obtain:
\[
J(u_1, u_2, u_3) = ck(I(t) + I_1(t) + I_2(t) + D(t)) + \frac{u_1^2}{2} + \frac{u_2^2}{2} + \frac{u_3^2}{2}.
\]

8. Conclusion and perspectives

In this work, we solve optimal control problems. A new epidemic model, with confirmed contamination, has been presented. We use distancing, case finding, and case holding controls to reduce the spread of the epidemic. We mathematically analyze the model and estimates the parameters used to solve the optimal control problems. In this particular research, the trend of population dynamics is important. It can be easily seen that by increasing educational campaigns, disease tests, and financial support ensure drug for infected individuals, we can successfully decrease the number of infected and death.
In further work, we intend to use models with additional compartments as quarantine and treatment.

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