On the Lie ideals of C*-algebras

Leonel Robert

Abstract
Various questions on Lie ideals of C*-algebras are investigated. They fall roughly under the following topics: relation of Lie ideals to closed two-sided ideals; Lie ideals spanned by special classes of elements such as commutators, nilpotents, and the range of polynomials; characterization of Lie ideals as similarity invariant subspaces.

Keywords: Lie ideals, C*-algebras, commutators, nilpotents, polynomials, similarity invariant subspaces.

Introduction
This paper deals with Lie ideals in C*-algebras. Like other investigations on this topic ([Mie81, BKS08]), we use, and take inspiration from, Herstein’s work on the Lie ideals of semiprime rings. The abundance of semiprime ideals in a C*-algebra – e.g., the norm-closed ideals – plus a number of C*-algebra techniques – approximate units, polar decompositions, functional calculus – make it possible to further develop the results of the purely algebraic setting in the C*-algebraic setting.

The contributions in the present paper, though varied, revolve around the following themes: the commutator equivalence of Lie ideals to two-sided ideals; the study of Lie ideals generated by special elements such as nilpotents and projections and by the range of polynomials; the characterization of Lie ideals as subspaces invariant by similarities. These topics have been studied before, and this paper is a direct beneficiary of works such as [Mar10], [BKS08], and [BK09].

A selection of results in this paper follows: Let $A$ be a C*-algebra. We show below that the following are true:

(i) The closed two-sided ideal generated by the commutators of $A$ is also the C*-algebra generated by the commutators of $A$. (Theorem 1.3)

(ii) The closure of the linear span of the square zero elements agrees with the closure of the linear span of the commutators. If $A$ is unital and without 1-dimensional representations, then the linear span of the square zero elements agrees with the linear span of the commutators. (Corollary 2.3 and Theorem 4.2)
(iii) If $A$ is unital and has no bounded traces and $f$ is a nonconstant polynomial in noncommuting variables with coefficients in $\mathbb{C}$, then there exists $N$ such that every element of $A$ is a linear combination of at most $N$ values of $f$ on $A$. If $f(\mathbb{C}) = \{0\}$ (e.g., $f(x, y) = [x, y]$), then there exist C*-algebras where the least such $N$ can be arbitrarily large. (Corollary 3.10 and Example 3.11)

(iv) If $A$ is unital and either simple, or without bounded traces, or a von Neumann algebra, then a subspace $U$ of $A$ is a Lie ideal of $A$ if and only if $(1+x)U(1-x) \subseteq U$ for all square zero elements $x$ in $A$. (Corollaries 4.3 and 4.6.)

1 From pure algebra to C*-algebras

Let us fix some notation:

Throughout the paper $A$ denotes a C*-algebra.

Let $x$ and $y$ be elements in $A$. Then $[x, y]$ denotes the element $xy - yx$ (the commutator of $x$ and $y$). Let $X$ and $Y$ be subsets of $A$. Then $X + Y$, $XY$, and $[X, Y]$ denote the linear spans of the elements of the form $x + y$, $xy$, and $[x, y]$, with $x \in X$ and $y \in Y$, respectively. The linear span of $X$ is denoted by $\text{span}(X)$. The C*-algebra and the closed two-sided ideal generated by $X$ are denoted by $C^*(X)$ and $\text{Id}(X)$, respectively. (For the 2-sided ideal algebraically generated by $X$ we simply write $AXA$.) From the identity $[xy, a] = [x, ya] + [y, ax]$, used inductively, we deduce that

$$[X^n, A] \subseteq [X, A]$$

for any set $X \subseteq A$ and all $n \in \mathbb{N}$. We sometimes refer to this fact as the “linearizing property of $[\cdot, A]$”.

A subspace $L$ of $A$ is called a Lie ideal if it satisfies that $[L, A] \subseteq L$. We will make frequent use of the following elementary lemma:

**Lemma 1.1.** Let $L$ be a Lie ideal of $A$. Then $A[L, L]A \subseteq L + L^2$.

**Proof.** We have $[[L, L], A] \subseteq [L, L]$, by Jacobi’s identity. Thus,

$$[L, L]A \subseteq A[L, L] + [[L, L], A] \subseteq A[L, L] + [L, L],$$

Multiplying by $A$ on the left we get $A[L, L]A \subseteq A[L, L]$. Finally, from the identity $a[l_1, l_2] = [al_1, l_2] - [a, l_2]l_1$ we deduce that $A[L, L] \subseteq L + L^2$, as desired. □

The following theorem of Herstein is the basis of many of our arguments in this section (it holds for semiprime rings without 2-torsion):

**Theorem 1.2** ([Her70, Theorem 1]). Let $L$ be a Lie ideal of $A$. Then $[t, [t, L]] = 0$ implies $[t, L] = 0$ for all $t \in A$. 

2
Combining Herstein’s theorem and Lemma 1.1, we get the following theorem:

**Theorem 1.3.** The closed two-sided ideal generated by \([A,A]\) agrees with the \(C^*\)-algebra generated by \([A,A]\). In fact, \(\text{Id}([A,A]) = [A,A] + [A,A]^2\).

**Proof.** Let \(I = \text{Id}([[[A,A],[A,A]]])\). Then \([[x,y],[x,y],A/I]] = 0\) for all \(x,y \in A/I\). Herstein’s theorem implies that \([[x,y],A/I]] = 0\) for all \(x,y \in A/I\). That is, \([[A/I,A/I],A/I]] = 0\). Herstein’s theorem again implies that \([[A/I,A/I]] = 0\); i.e., \([A,A] \subseteq I\). On the other hand, \(I \subseteq [A,A] + [A,A]^2\), by Lemma 1.1. So,

\[
\text{Id}([A,A]) \subseteq I \subseteq [A,A] + [A,A]^2 \subseteq C^*([A,A]).
\]

Since \(C^*([A,A]) \subseteq \text{Id}([A,A])\), these inclusions must be equalities. \(\square\)

The following lemma is easily derived from the existence of approximately central approximate units for the closed two-sided ideals of \(A\):

**Lemma 1.4** ([Mie81, Lemma 1], [BKS08, Proposition 5.25]). Let \(I\) be a closed two-sided ideal of \(A\). Then

\[
[I,I] = [I,A] = I \cap [A,A].
\]

Brešar, Kissin, and Shulman show in [BKS08, Theorem 5.27] that \(\Bar{L}, A] = \Bar{[\text{Id}([L,A])],A}\) for any Lie ideal \(L\) of \(A\). In the theorem below we give a short proof of this important theorem:

**Theorem 1.5.** Let \(L\) be a Lie ideal of \(A\). Then

(i) \(\text{Id}([L,A]) = [L,A] + [L,A]^2\).

(ii) \(\text{Id}([L,A]), A] = [L,A] = [[L,A], A]\).

**Proof.** (i) We follow a line of argument similar to the proof of Theorem 1.3. Let \(M = [L,A]\) and \(I = \text{Id}([M,M])\). Let \(\Bar{L}\) and \(\Bar{M}\) denote the images of \(L\) and \(M\) in \(A/I\) by the quotient map. Then \(\Bar{M}, [\Bar{M}, A/I]] = 0\). By Herstein’s theorem, \([\Bar{M}, A/I]] = 0\); i.e., \([[\Bar{L}, A/I], A/I]] = 0\). By Herstein’s theorem again, \([\Bar{L}, A/I]] = 0\); i.e., \([L,A] \subseteq I\). On the other hand, \(I \subseteq \Bar{M} + \Bar{M}^2 = [L,A] + [L,A]^2\), by Lemma 1.1. So,

\[
\text{Id}([L,A]) \subseteq I \subseteq [L,A] + [L,A]^2 \subseteq C^*([L,A]).
\]

Since \(C^*([L,A]) \subseteq \text{Id}([L,A])\), all these inclusions must be equalities.

(ii) By (i) and the linearizing property of \([,A]\) recalled in (1.1), we have that

\[
\text{Id}([L,A]), A] = [L,A] + [L,A]^2, A] \subseteq [L,A], A].
\]

Thus, \(\text{Id}([L,A]), A] \subseteq [L,A], A] \subseteq [L,A]\). On the other hand,

\[
[L,A] \subseteq \text{Id}([L,A]) \cap [A,A] \subseteq \text{Id}([L,A]), A],
\]

(the second inclusion by Lemma 1.4). This completes the proof. \(\square\)
Lemma 1.6. Let \( L \) be a closed Lie ideal of \( A \) such that \( \text{Id}(L) = \text{Id}([L, A]) \) and \( L \subseteq [A, A] \). Then \( L = [\text{Id}(L), A] \).

Proof. The inclusion \( L \subseteq [\text{Id}(L), A] \) follows from \( L \subseteq [A, A] \cap \text{Id}(L) \) and Lemma 1.4. As for the opposite inclusion, we have \( [\text{Id}(L), A] = [\text{Id}([L, A]), A] \), by assumption, and \( [\text{Id}([L, A]), A] \subseteq L \), by Theorem 1.5.

The following is an improvement on Theorem 1.5 (ii) obtained by the same technique:

**Theorem 1.7.** Let \( K \) and \( L \) be Lie ideals of \( A \). Then \( [K, L] = [\text{Id}([K, L]), A] \).

Proof. Let \( M = [K, L] \). Notice that \( M \) is again a Lie ideal (by Jacobi’s identity). We will deduce that \( \overline{M} = [\text{Id}(M), A] \) from the previous lemma. We clearly have that \( \overline{M} \subseteq [A, A] \). Let \( I = \text{Id}([M, A]) \) and let \( \overline{K}, \overline{L}, \text{ and } \overline{M} \) denote the images of \( K, L, \text{ and } M \) in the quotient by this ideal. From \( [\overline{M}, A/I] = 0 \) and \( [\overline{K}, \overline{L}] = \overline{M} \) we get that \( [\overline{K}, \overline{L}], \overline{L} = 0 \). By Herstein’s theorem, \( [\overline{K}, \overline{L}] = 0 \); i.e, \( M = [K, L] \subseteq I \). It follows that \( \text{Id}(M) = \text{Id}([M, A]) \). By Lemma 1.6, \( \overline{M} = [\text{Id}(M), A] \), as desired.

**Remark 1.8.** The arguments in Theorems 1.5, 1.6, and 1.7 rely crucially on the fact that the closed two-ideals of a C*-algebra are semiprime. This makes it possible to apply Herstein’s theorem in the quotient by a closed two-sided ideal. Turning to non-closed Lie ideals, if we impose the semiprimeness of a suitable non-closed two-sided ideal at the outset, part of those same arguments still goes through. We may obtain in this way, for instance, the following result: If \( L \) is a Lie ideal of \( A \) such that the two-sided ideal generated by \([L, A], [L, A]\) is semiprime then (i) \( A[L, A]A = [L, A] + [L, A]^2 \), and (ii) \( A[L, A], A], A] = [[L, A], A] \). To get (i) we proceed as in Theorem 1.5 (i): Setting \( M = [L, A] \) and \( I = A[M, M]A \) and applying Herstein’s theorem in \( A/I \) in much the same way as we did in Theorem 1.5 (i) we arrive at \( [L, A] \subseteq I \). Then we have the inclusions \( A[L, A]A \subseteq I \subseteq [L, A] + [L, A]^2 \), which must in fact be equalities. To get (ii) we apply (i) and the linearizing property of \([, A]\):

\[
A[L, A], A], A] = [[L, A] + [L, A]^2, A] = [[L, A], A] .
\]

Next we discuss another variation on Theorem 1.5 for non-closed Lie ideals. This time we make use of the Pedersen ideal. Recall that the Pedersen ideal of a C*-algebra is the smallest dense two-sided ideal of the algebra (see Ped80, 5.6). Given a C*-algebra \( B \), we denote its Pedersen ideal by \( \text{Ped}(B) \).

**Lemma 1.9.** Let \( I \) be a closed two-sided ideal of \( A \). Then

\[
[\text{Ped}(I), \text{Ped}(I)] = [\text{Ped}(I), A] .
\]

Proof. Let \( P = \text{Ped}(I) \). The subspace \( P^2 \) is a dense two-sided ideal of \( I \). Since \( P \) is the minimum such ideal, we must have that \( P = P^2 \). From \( [P, A] = [P^2, A] \) and the identity \([xy, a] = [x, ya] + [y, ax]\) we get that \([P^2, A] \subseteq [P, P] \).
Theorem 1.10. Let $L$ be a Lie ideal of $A$ and let $P = \text{Ped}(\text{Id}([L, A]))$. Then

$$[P, P] = [L, P] = [[L, A], P].$$

Furthermore, if $L \subseteq P$ then $[L, A] = [P, P]$.

Proof. In the course of proving Theorem 1.5 we have shown that $\text{Id}([L, A]) = \text{Id}([[L, A], [L, A]])$. Therefore, the two-sided ideal $A([[L, A], [L, A]]A$ is dense in $\text{Id}([L, A])$. Since $P$ is the smallest such ideal, $P \subseteq A([[L, A], [L, A]]A$. Hence,

$$[P, P] \subseteq [A[[L, A], [L, A]]A, P] \subseteq [[L, A] + [L, A]^2, P] \subseteq [[L, A], P] \subseteq [L, P].$$

But $[L, P] \subseteq [P, P]$, by Lemma 1.9. Thus, the inclusions above must be equalities.

Suppose now that $L \subseteq P$. Then $[L, P] \subseteq [L, A] \subseteq [P, A] = [P, P]$, the latter equality by Lemma 1.9. Since $[L, P] = [P, P]$, these inclusions must be equalities.

Corollary 1.11. Among the Lie ideals $L$ such that $[L, A] = [A, A]$, the Lie ideal

$$[\text{Ped}(\text{Id}([A, A])), \text{Ped}(\text{Id}([A, A]))]$$

is the smallest.

Proof. Let $P = \text{Ped}(\text{Id}([A, A]))$. Then

$$[[P, P], A] = [[[P, P], \text{Id}([A, A])], A]$$

$$= [[[\text{Id}([A, A]), A], A], A]$$

$$= [[\text{Id}([A, A]), A], A]$$

$$= [A, A].$$

The second equality holds by Lemma 1.4 and the third and fourth by Theorem 1.5. Thus, $[P, P]$ is a Lie ideal satisfying that $[L, A] = [A, A]$.

Suppose now that $L$ is a Lie ideal such that $[L, A] = [A, A]$. By Theorem 1.10, $[P, P] = [L, P] \subseteq L$. So $L$ contains $[P, P]$.

It seems possible that under some $C^*$-algebra regularity condition, such as $A$ being pure (i.e., having almost unperforated and almost divisible Cuntz semigroup), it is the case that for every Lie ideal $L$ there exists a two-sided – possible non-closed – ideal $I$ such that $[L, A] = [I, A]$ (in the language of $\text{BKS08}$, $L$ and $I$ are called commutator equal). At present, we don’t even have an answer to the following question:

Question 1.12. Is there a $C^*$-algebra $A$ and a Lie ideal $L$ of $A$, such that $[L, A] \neq [I, A]$ for all two-sided (possibly non-closed) ideals $I$ of $A$?
We turn now to Lie ideals of \([A, A]\). A linear subspace \(U \subseteq A\) is called a Lie ideal of \([A, A]\) if \([U, [A, A]] \subseteq U\). Herstein’s [Her69, Theorem 1.12] implies that if \(A\) is simple and unital then a Lie ideal of \([A, A]\) is automatically a Lie ideal of \(A\) (this holds for simple rings without 2-torsion). In Theorem 1.15 below we show that the simplicity assumption can be dropped for closed Lie ideals of \([A, A]\). The key of the argument is again to apply a theorem of Herstein (Lemma 1.14 below) in the quotient by a suitable closed two-sided ideal.

**Lemma 1.13.** Let \(U\) be a Lie ideal of \([A, A]\). Let \(V = [U, U]\), \(W = [V, V]\), and \(X = [W, W]\). Then \(A[X, X]A \subseteq [U, U] + [U, U]^2\).

**Proof.** (Cf. [Her69, Lemma 1.7].) In the following inclusions we make use of Jacobi’s identity and the fact that \(U\) is a Lie ideal of \([A, A]\):

\[
[[U, U], A] \subseteq [U, [A, A]] \subseteq U,
\]

\[
[[U, U], [A, A]] \subseteq [[U, [A, A]], U] \subseteq [U, U].
\]

That is, \([V, A] \subseteq U\) and \(V\) is a Lie ideal of \([A, A]\). We deduce similarly that \([A, W] \subseteq V\) and that \(W\) and \(X\) are Lie ideals of \([A, A]\). Finally, since \(V \subseteq [A, A]\) we have that \([V, V] \subseteq V\); i.e., \(W \subseteq V\). We deduce similarly that \([X, X] \subseteq X\). Having made this preparatory remarks, we attack the lemma:

\[
[X, X]A \subseteq A[X, X] + [[X, X], A]
\]

\[
\subseteq A[X, X] + X
\]

\[
\subseteq AX + X.
\]

Hence, \(A[X, X]A \subseteq AX = A[W, W]\). Using now that \(a[w_1, w_2] = [aw_1, w_2] - [a, w_2]w_1\) we get that

\[
A[W, W] \subseteq [A, W] + [A, W]W
\]

\[
\subseteq V + VW
\]

\[
\subseteq V + V^2.
\]

Thus, \(A[X, X]A \subseteq V + V^2\), as desired.

**Lemma 1.14.** Let \(U\) be a Lie ideal of \([A, A]\). If \([U, U], A] = 0\) then \([U, A] = 0\).

**Proof.** See [Her69, Theorem 1.11] for the case of simple rings without 2-torsion. See [Row88, Exercise 17, page 344] for the extension to semiprime rings without 2-torsion (e.g., \(C^*\)-algebras).

**Theorem 1.15.** A (norm) closed Lie ideal of \([A, A]\) is a Lie ideal of \(A\).
Proof. Let \( U \) be a closed Lie ideal of \([A, A]\). Consider the sets \( V = [U, U] \), \( W = [V, V] \) and \( X = [W, W] \). Let \( I = \text{Id}([X, X]) \). Let \( \bar{U} \) denote the image of \( U \) in \( A/I \) by the quotient map. Define \( \bar{V}, \bar{W}, \) and \( \bar{X} \) similarly. Then \( [\bar{X}, \bar{X}] = 0 \), which, by Lemma 1.14 implies that \( [\bar{X}, A/I] = 0 \). That is, \( [\bar{W}, \bar{W}], A/I] = 0 \). Again by Lemma 1.14 we get that \( [\bar{W}, A/I] = 0 \). Two more applications of Lemma 1.14 then yield that \( [\bar{W}, A/I] = 0 \). That is, \( [\bar{V}, A/I] = 0 \). Hence, \( \text{Id}([U, U]) \subseteq I \subseteq [U, U] + [U, U]^2 \subseteq \text{Id}([U, U]) \).

In the second inclusion we have used Lemma 1.13. Since \( \text{Id}([U, U]) \subseteq \text{Id}([U, A]) \), all these must be equalities. Taking commutators with \( A \) and using (1.1) we get
\[
\text{Id}([U, A]) = [\bar{U}, A/I] = [\bar{U}, A] = \text{Id}([U, A]),
\]
Lemma 1.4 on the other hand, implies that
\[
[U, A] \subseteq \text{Id}([U, A]) \cap [A, A] = \text{Id}([U, A]),\]
Hence, \( [U, A] \subseteq U \); i.e., \( U \) is a Lie ideal of \( A \).

2 Nilpotents and polynomials

In this section we look at closed Lie ideals spanned by nilpotents and by the range of polynomials.

For each natural number \( k \geq 2 \) let \( N_k \) denote the set of nilpotent elements of \( A \) of order exactly \( k \). Since the set \( N_k \) is invariant by unitary conjugation (and by similarity), the closed subspace \( \text{span}(N_k) \) is a Lie ideal of \( A \) (see [Ped80] and Theorem 2.6 below).

The following lemma is surely well known:

Lemma 2.1. Every element of \( N_k \) is a sum of \( k-1 \) commutators for all \( k \geq 2 \).

Proof. Let \( x \) be a nilpotent of order at most \( k \) (i.e., in \( \bigcup_{j \leq k} N_j \)). Let \( x = v|x| \) be the polar decomposition of \( x \) in \( A^{**} \). Let \( \bar{x} = |x|^\frac{1}{2}v|x|^\frac{1}{2} \) (the Aluthge transform of \( x \)). Observe that \( x = |x|^\frac{1}{2}v|x|^\frac{1}{2} + \bar{x} \). Also,
\[
\bar{x}^{k-1}(\bar{x}^{k-1})^* = |x|^\frac{1}{2}x^{k-1}v^*(x^{k-2})^*v|x|^\frac{1}{2} = 0,
\]
where we have used that \( |x|^\frac{1}{2}x^{k-1} = 0 \) (since \( |x|^\frac{1}{2} \in C^*(x*x) \) and \( (x*x)x^{k-1} = 0 \)). Thus \( \bar{x} \) is a nilpotent of order at most \( k-1 \). Continuing this process inductively we arrive at the desired result.

For each \( k \in \mathbb{N} \) let \( I_k \) denote the intersection of the kernels of all representations of \( A \) of dimension at most \( k \). Notice that \( I_1 = \text{Id}([A, A]) \) and that \( I_1 \supseteq I_2 \supseteq \ldots \). It is not hard to show that \( I_k \) is the smallest closed two-sided ideal the quotient by which is a \( k \)-subhomogeneous \( C^* \)-algebra (i.e., one whose irreducible representations are at most \( k \)-dimensional).
Theorem 2.2. \( \text{span}(N_k) = [I_{k-1}, A] \) for all \( k \geq 2 \).

Proof. It is well known that \( \text{Id}(N_k) = I_{k-1} \) (e.g., see [AM03, Lemma 6.1.3]). We must then show that \( \text{span}(N_k) = [\text{Id}(N_k), A] \). Let \( I = \text{Id}([N_k, A]) \). Let \( x \in N_k \). Since \( [x, A] \subseteq I \), the quotient map sends \( x \) to the center of \( A/I \). But the center, being a commutative \( C^* \)-algebra, cannot contain nonzero nilpotents. Thus, \( x \in I \). This shows that \( N_k \subseteq \text{Id}(N_k), A] \). On the other hand, \( N_k \subseteq [A, A] \) by Lemma 2.1. Thus, \( \text{span}(N_k) = [\text{Id}(N_k), A] \) by Lemma 1.6.

Corollary 2.3. \( \text{span}(N_2) = [A, A] \).

Proof. The previous theorem implies that \( \text{span}(N_2) = [\text{Id}([A, A]), A] \). On the other hand, \( [\text{Id}([A, A]), A] = [A, A] \), by Theorem 1.5 (ii) applied with \( L = A \).

The following corollary is merely a restatement of Corollary 2.3

Corollary 2.4. A positive bounded functional on \( A \) is a trace if and only if it vanishes on \( N_2 \).

Question 2.5. Is \( [A, A] = \text{span}(N_2) \)? Is \( \text{span}(N_2) \) a Lie ideal?

We will return to these questions in Section 3.

Combining Corollary 2.3 and Theorem 1.15 of the previous section we can prove the following \( C^* \)-algebraic version of a theorem of Amitsur for simple rings ([Ami56, Theorem 1]):

Theorem 2.6. A closed subspace \( U \) of \( A \) is a Lie ideal if and only if \( (1 + x)U(1 - x) \subseteq U \) for all \( x \in N_2 \).

Proof. Say \( U \) is a Lie ideal. Let \( u \in U \) and \( x \in N_2 \). Then

\[
(1 + x)u(1 - x) = u + [x, u] + \frac{1}{2}[x, [x, u]] \in U.
\]

Suppose now that \( (1 + x)U(1 - x) \subseteq U \) for all \( x \in N_2 \). Let \( u \in U \) and \( x \in N_2 \). Then

\[
[x, u] - xux = (1 + x)u(1 - x) - u \in U,
\]

\[
[x, u] + xux = -(1 - x)u(1 + x) + u \in U.
\]

Hence \( [u, x] \in U \). That is, \( [U, N_2] \subseteq U \). Passing to the span of \( N_2 \) and taking closure we get from Corollary 2.3 that \( [U, [A, A]] \subseteq U \). That is, \( U \) is a closed Lie ideal of \( [A, A] \). By Theorem 1.15 \( U \) is a Lie ideal of \( A \).

Let \( f(x_1, \ldots, x_n) \) be a polynomial in noncommuting variables with coefficients in \( \mathbb{C} \). Let us denote by \( f(A, \ldots, A) \), or \( f(A) \) for short, the range of \( f \) on \( A \). (If \( A \) is non-unital we assume that \( f \) has no independent term.) Since the set \( f(A) \) is
invariant by similarity, \( \text{span}(f(A)) \) is a Lie ideal. It is shown in [BK09] Theorem 2.3] that even \( \text{span}(f(A)) \) is Lie ideal.

In the sequel by a polynomial we always understand a polynomial in noncommuting variables with coefficients in \( \mathbb{C} \).

Recall that for each \( k \in \mathbb{N} \) we let \( I_k \) denote the intersection of the kernels of all representations of \( A \) of dimension at most \( k \). In the following theorem we use the conventions \( I_0 = A \) and \( M_0(\mathbb{C}) = \{0\} \). We regard every polynomial as an identity on \( M_0(\mathbb{C}) \). By a nonconstant polynomial we mean one with positive degree in at least one of its variables.

**Theorem 2.7.** Let \( f \) be a nonconstant polynomial. Suppose that \( f(A) \subseteq [A, A] \). Then \( \text{span}(f(A)) = [I_k, A] \), where \( k \geq 0 \) is the largest number such that \( f \) is an identity on \( M_k(\mathbb{C}) \) (such a number must exist since no polynomial is an identity on all matrix algebras).

**Proof.** Let \( I = \text{Id}([f(A), A]) \). Then \( A/I \) is a subhomogeneous C*-algebra, since it satisfies the (nontrivial) polynomial identity \( f(x_1, \ldots, x_n, y) \) (see [Bla06] Proposition IV.1.4.6]). The range of \( f \) on \( A/I \) is both in the center of \( A/I \) and in \( [A/I, A/I] \), as \( f(A) \subseteq [A, A] \). But in a subhomogeneous C*-algebra the center and the closure of the span of the commutators have zero intersection (since this is true in every finite dimensional representation). Hence, \( f(A/I) = \{0\} \); i.e., \( f(A) \subseteq I \). Thus, \( \text{Id}(f(A)) = I = \text{Id}([f(A), A]) \). By assumption, we also have that \( f(A) \subseteq [A, A] \). It follows that \( \text{span}(f(A)) = [I, A] \) by Lemma 1.6.

Let us now show that \( I = I_k \), with \( k \geq 0 \) as in the statement of the theorem. Let \( \pi: A \to M_l(\mathbb{C}) \) be a representation of \( A \) with \( l \leq k \). By assumption, \( f(M_l(\mathbb{C})) = \{0\} \). Hence, \( f(A) \subseteq \ker \pi \), and so \( I = \text{Id}(f(A)) \subseteq \ker \pi \). Since, by definition, \( I_k \) is the intersection of the kernels of all such \( \pi \), we get that \( I \subseteq I_k \). To prove the opposite inclusion notice first that \( A/I \) must be a \( k \)-subhomogeneous C*-algebra. For suppose that there exists an irreducible representation \( \pi: A/I \to M_m(\mathbb{C}) \), with \( m > k \). Since \( f \) is an identity on \( A/I \) and \( \pi \) is onto, we get that \( f \) is an identity on \( M_m(\mathbb{C}) \). This contradicts our choice of \( k \). Hence, every irreducible representation of \( A/I \) has dimension at most \( k \); i.e., \( A/I \) is \( k \)-subhomogeneous. Since \( I_k \) may be alternatively described as the smallest closed two-sided ideal the quotient by which is \( k \)-subhomogeneous, \( I_k \subseteq I \). \( \square \)

Let \( s_k \) denote the standard polynomial in \( k \) noncommuting variables. That is,

\[
s_k(x_1, \ldots, x_k) = \sum_{\sigma \in S_k} \text{sign}(\sigma)x_{\sigma(1)} \cdots x_{\sigma(k)},
\]

where \( S_k \) denotes the symmetric group on \( k \) elements. The Amitsur-Levitzky theorem states that \( s_{2k} \) is a polynomial identity of minimal degree on \( M_k(\mathbb{C}) \) ([AL50]). Define \( \pi_1(x, y) = [x, y] \) and

\[
\pi_{k+1}(x_1, \ldots, x_{2k+1}) = [\pi_k(x_1, \ldots, x_{2k}), \pi_k(x_{2k+1}, \ldots, x_{2k+1})]
\]
for all \( k \geq 1 \). The following two special cases of the previous theorem are worth remarking upon:

**Corollary 2.8.** \( \text{span}(\sigma_{2k}(A)) = [I_k, A] \) and \( \text{span}(\pi_k(A)) = [A, A] \) for all \( k \geq 1 \).

**Proof.** Let \( k \in \mathbb{N} \). It is well known that \( s_{2k} \) is expressible as a sum of commutators in the algebra of polynomials in \( 2k \) noncommuting variables. Hence, \( s_{2k}(A) \subseteq [A, A] \). We can thus apply Theorem 2.7 to \( s_{2k} \). By the Amitsur-Levitsky theorem, \( s_{2k} \) is a polynomial identity of \( M_k(\mathbb{C}) \) but not of \( M_{k+1}(\mathbb{C}) \). Thus, by Theorem 2.7 \( \text{span}(\sigma_{2k}(A)) = [I_k, A] \).

The polynomial \( \pi_k \) is an identity on \( \mathbb{C} \) but not on \( M_2(\mathbb{C}) \). (In fact, by [Her70, Theorem 2], if \( \pi_k \) is a polynomial identity on a semiprime ring without 2-torsion then the ring must be commutative.) Thus, by Theorem 2.7 \( \text{span}(\pi_k(A)) = [A, A] \). \( \square \)

Let's now give a characterization of the polynomials whose range is contained in \( [A, A] \). Following [BK09], we say that two polynomials \( f \) and \( g \) (in noncommuting variables, with coefficients in \( \mathbb{C} \)) are cyclically equivalent if \( f - g \) is a sum of commutators in the ring \( \mathbb{C}(X_1, X_2, \ldots) \) of polynomials in noncommuting variables. If a polynomial is cyclically equivalent to 0 then its range is clearly in \( [A, A] \). On the other hand, if \( A \) has no bounded traces then \( A = [A, A] \) (see [CP79]) and so any polynomial has range in \( [A, A] \). The general case is a mixture of these two. In the following theorem we maintain the conventions that \( I_0 = A \), \( M_0(\mathbb{C}) = \{0\} \), and that every polynomial is an identity on \( M_0(\mathbb{C}) \).

**Theorem 2.9.** Let \( k \geq 0 \) be the smallest number such that the closed two-sided ideal \( I_k \) has no bounded traces (set \( k = \infty \) if this is never the case). Let \( f \) be a nonconstant polynomial.

(i) If \( k = \infty \) then \( f(A) \subseteq [A, A] \) if and only if \( f \) cyclically equivalent to 0.

(ii) If \( k < \infty \) then \( f(A) \subseteq [A, A] \) if and only if \( f \) is cyclically equivalent to a polynomial identity on \( M_k(\mathbb{C}) \).

**Proof.** Let us first prove the forward implications. If \( f \) is cyclically equivalent to 0 then clearly \( f(A) \subseteq [A, A] \). Suppose that \( k < \infty \) and that \( f \) is cyclically equivalent to a polynomial \( g \) which is an identity on \( M_k(\mathbb{C}) \). Then \( g(A) \subseteq I_k \) and \( I_k = [I_k, I_k] \), since \( I_k \) has no bounded traces. Thus, \( g(A) \subseteq [A, A] \). But \( (f - g)(A) \subseteq [A, A] \). Thus, \( f(A) \subseteq [A, A] \), as desired.

Let us suppose now that \( f(A) \subseteq [A, A] \). We will follow closely the proof of [BK09, Theorem 4.5] where the result is obtained for the range of polynomials on matrix algebras. If the independent term of \( f \) is nonzero then \( 1 \in f(A) \subseteq [A, A] \). Hence, \( A \) has no bounded traces; i.e., \( k = 0 \). Since, by convention, any polynomial is an identity on \( M_0(\mathbb{C}) \), we are done. Let us assume now that \( f \) has no independent term. Let \( f = \sum_{i=1}^{m} f_i \) be the decomposition of \( f \) into multihomogeneous polynomials. Then, by the proof of [BK09, Theorem 2.3], \( f_i(A) \subseteq \text{span}(f(A)) \) for all \( i \). This
Suppose that the degree of \( f \) on \( x_1 \) is 1. Then \( f \) is cyclically equivalent to a polynomial of the form \( x_1 g(x_2, \ldots, x_n) \). Hence \( A g(A) \subseteq [A, A] \), which in turn implies that \( \text{Id}(g(A)) \subseteq [A, A] \). If \( g \) is 0, then \( f \) is cyclically equivalent to 0 and we are done. If \( g \) is constant and nonzero, then \( A = \text{Id}(g(A)) \subseteq [A, A] \). That is, \( A = [A, A] \), \( k = 0 \), and \( f \) is an identity on \( M_0(\mathbb{C}) \); again we are done. If \( g \) is nonconstant then \( \text{Id}(g(A)) = I_{k'} \) for some \( k' \) and furthermore \( g \) is an identity on \( M_{k'}(\mathbb{C}) \) (see the proof of Theorem 2.7). From \( I_{k'} \subseteq [A, A] \) and Lemma 1.4 we deduce that \( I_{k'} = [I_{k'}, I_{k'}] \). Hence, \( I_{k'} \) has no bounded traces; i.e., \( k' \geq k \). It follows that \( g \) is an identity on \( M_k(\mathbb{C}) \), and since \( f = x_1 g \), so is \( f \). This completes the first step of the induction.

Suppose now that \( f(x_1, \ldots, x_n) \) is a multihomogeneous polynomial whose variable of smallest degree, \( x_n \), has degree \( d \), with \( d > 1 \). Consider the polynomial

\[
g(x_1, \ldots, x_n, x_{n+1}) = f(x_1, \ldots, x_{n-1}, x_n + x_{n+1}) - f(x_1, \ldots, x_{n-1}, x_{n+1}) - f(x_1, \ldots, x_n).
\]

Then \( g(A) \subseteq [A, A] \) and the degree of \( g \) on \( x_n \) is less than \( d \). By induction, \( g \) is cyclically equivalent to a polynomial identity on \( M_k(\mathbb{C}) \) (if \( k < \infty \)) or cyclically equivalent to 0 (if \( k = \infty \)). Since \( f(x_1, \ldots, x_n) = \frac{1}{2^d-2} g(x_1, \ldots, x_n, x_n) \), the same holds for \( f \).

3 Finite sums and sums of products

Recall the following basic fact: a dense two-sided ideal in a unital C*-algebra must agree with the whole C*-algebra (because it would intersect the ball of radius one centered at the unit, all whose elements are invertible). It follows that if \( A \) is unital and \( A = \text{Id}(X) \) then \( A = AXA \). Here we exploit this fact to obtain quantitative versions of some of the results from the previous sections.

**Theorem 3.1.** Let \( A \) be unital and let \( L \) be a Lie ideal of \( A \) such that \( \text{Id}([L, A]) = A \). Suppose that \( L \) is linearly spanned by a set \( \Gamma \subseteq A \); i.e., \( L = \text{span}(\Gamma) \). Suppose furthermore that there exists \( M \in \mathbb{N} \) such that for all \( l \in \Gamma \) and \( z \in A \) the commutator \([l, z]\) is a linear combination of at most \( M \) elements of the set \( \Gamma \). The following are true:

(i) There exists \( N \) such that every element of \( A \) is expressible as a linear combination of \( N \) elements of \( \Gamma \) and \( N \) products of two elements of \( \Gamma \).

(ii) There exists \( K \) such that every single commutator \([x, y]\) in \( A \) is expressible as a linear combination of \( K \) elements of \( \Gamma \).

**Proof.** We have shown in the proof of Theorem 1.5 (i) that \( \text{Id}([L, A]) = \text{Id}([L, L]) \). (Indeed, after setting \( I = \text{Id}([[L, A], [L, A]]) \), we proceeded to show that \([L, A] \subseteq \))
\[ I, \text{ which implies that } \text{Id}([L, A]) \subseteq I \subseteq \text{Id}([L, L]). \] Clearly, these inclusions must be equalities.) Therefore, \( A = \text{Id}([L, L]) = \text{Id}([\Gamma, \Gamma]). \) Since \( A \) is unital, it is algebraically generated as a two-sided ideal by \([\Gamma, \Gamma]\). Hence,

\[ 1 = \sum_{i=1}^{n} x_{i}[k_{i}, l_{i}]y_{i}, \]

for some \( x_{i}, y_{i} \in A \) and \( k_{i}, l_{i} \in \Gamma \). Let \( a \in A \). Then

\[ a = \sum_{i=1}^{n} (ax_{i})[k_{i}, l_{i}]y_{i}. \]

It suffices to show that each term of the sum on the right is a linear combination of a fixed number of elements of \( \Gamma \) and of products of two elements of \( \Gamma \). We have the following identity (derived from the arguments in the proof of Lemma 1.1 (i)):

\[ x[l, m]y = [xyl, m] - [xy, m]l + [xm, yl] - [x, [y, l]]m + [xl, my] - [x, [m, y]]l, \]

for all \( x, y \in A \) and \( l, m \in \Gamma \). Observe that each of the terms on the right side are of either one of the following forms: \([z, l]\), \([z, l]'\), \([z, [z', l]]\), or \([z, [z', l]]'\), where \( z, z' \in A \) and \( l, l' \in \Gamma \). Recall now that, by assumption, the commutators \([z, l]\), with \( z \in A \) and \( l \in \Gamma \), are expressible as linear combinations of at most \( M \) elements of \( \Gamma \). This implies that elements of either one of the forms mentioned before are linear combinations of either \( M \) or \( M^2 \) elements of \( \Gamma \) or products of two elements of \( \Gamma \).

(ii) Let \( x \in A \). By (i), \( x = \sum_{i=1}^{N} \lambda_{i}l_{i} + \sum_{i=1}^{N} \mu_{i}m_{i}n_{i} \) for some scalars \( \lambda_{i}, \mu_{i} \) and some \( l_{i}, m_{i}, n_{i} \in \Gamma \). Let \( y \in A \). Then,

\[ [x, y] = \sum_{i=1}^{N} \lambda_{i}[l_{i}, y] + \sum_{i=1}^{N} \mu_{i}[m_{i}, n_{i}y] + \sum_{i=1}^{N} \mu_{i}[n_{i}, ym_{i}]. \]

Appealing to the fact that every commutator of the form \([l, z]\), with \( l \in \Gamma \) and \( z \in A \) is a linear combination of at most \( M \) elements of \( \Gamma \), we deduce that the right side is a linear combination of \( 3MN \) elements of \( \Gamma \).

**Theorem 3.2.** Let \( A \) be unital and without 1-dimensional representations. Then there exists \( N \in \mathbb{N} \) such that every element of \( A \) is expressible as a sum of the form

\[ \sum_{i=1}^{N} [a_{i}, b_{i}] + \sum_{i=1}^{N} [c_{i}, d_{i}] : [c_{i}', d_{i}']. \]

**Proof.** The quotient \( A/\text{Id}([A, A]) \) is a commutative \( C^* \)-algebra. If it were nonzero, it would have non-trivial 1-dimensional representations. But we have assumed that \( A \) has no 1-dimensional representations, Thus, \( A = \text{Id}([A, A]). \) The previous theorem is then applicable to \( L = [A, A] \) and \( \Gamma = \{[x, y] \mid x, y \in A\} \), yielding the desired result. \[ \square \]
We can link the constant $N$ in Theorem 3.2 to a certain notion of “divisibility” studied in [RR13]. A unital C$^*$-algebra $A$ is called weakly $(2, N)$-divisible if there exist $x_1, \ldots, x_N \in N_2$ and $d_1, \ldots, d_N \in A$ such that

$$1 = \sum_{i=1}^N d_i^* x_i x_i d_i.$$  

(The definition of weakly $(2, N)$-divisible in [RR13] is in terms of the Cuntz semi-group of $A$ but can be seen to be equivalent to this one.) A unital C$^*$-algebra without 1-dimensional representations must be weakly $(2, N)$-divisible for some $N$ ([RR13, Corollary 5.4]). This fact, combined with the following proposition, gives another proof of Theorem 3.2.

**Proposition 3.3.** If $A$ is unital and weakly $(2, N)$-divisible then every element of $A$ is expressible as a sum of the form

$$a = \left( \sum_{i=1}^N d_i^* x_i^* x_i d_i \right) \cdot a = \sum_{i=1}^N [d_i^* x_i^* x_i d_i] + \sum_{i=1}^N x_i d_i a d_i^* x_i^*.$$

Proof. Suppose that $1 = \sum_{i=1}^N d_i^* x_i x_i d_i$, with $x_i \in N_2$ for all $i$. Let $a \in A$. Then

$$a = \left( \sum_{i=1}^N d_i^* x_i^* x_i d_i \right) \cdot a = \sum_{i=1}^N [d_i^* x_i^* x_i d_i] + \sum_{i=1}^N x_i d_i a d_i^* x_i^*.$$

It thus suffices to show that $xbx^*$ is a product of 2 commutators for all $x \in N_2$ and $b \in A$. Say $x = v|x|$ is the polar decomposition of $x$ in $A^*$. Then $xbx^* = (xb|x|\frac{1}{2}) \cdot (|x|\frac{1}{2}v^*)$. But both $xb|x|\frac{1}{2}$ and $|x|\frac{1}{2}v^*$ belong to $N_2$. (Let us prove this for the latter: We have $|x|\frac{1}{2} \in C^*(x^*x) \subseteq |x|Ax$. Multiplying by $v$ on the left we get that $v|x|\frac{1}{2} \in Ax$. Since $x$ is a square zero element, we deduce that $v|x|\frac{1}{2}$, and its adjoint, are square zero elements as well.) By Lemma 2.1 both $xb|x|\frac{1}{2}$ and $|x|\frac{1}{2}v^*$ are commutators. \hfill \Box

**Remark 3.4.** If $1 \in B \subseteq A$ and $B$ is weakly $(2, N)$-divisible then so is $A$. This observation can be used to find upper bounds on $N$ for specific examples (e.g., when $B$ is a dimension drop C$^*$-algebra; see [RR13, Example 3.12]).

Let $P \subseteq A$ denote the set of projections of $A$. Let us apply Theorem 3.1 to span($P$). To see that this is a Lie ideal, recall that the linear span of the idempotents is Lie ideal and that, by a theorem of Davidson (see paragraph after [MM98, Theorem 4.2]), every idempotent is a linear combination of five projections. In Davidson’s theorem, the number of projections can be reduced to four:

**Lemma 3.5.** Every idempotent of $A$ is a linear combination of four projections

Proof. Let $e \in A$ be an idempotent and let $p \in A$ denote its range projection. Then $e = p + x$, with $x \in pA(1 - p)$. Let us show that $x$ is a linear combination of three
projections. It suffices to assume that $\|x\| < \frac{1}{2}$. For each $x \in pA(1 - p)$ such that $\|x\| < \frac{1}{2}$ let us define

$$q(x) = \left( \frac{1 + \sqrt{1 - 4xx^*}}{2} \right) \begin{pmatrix} x^{*} & x \\ (1 - p)x & (1 - p)A(1 - p) \end{pmatrix}.$$

A straightforward computation shows that $q(x)$ is a projection and, furthermore, that

$$x = \frac{1 + i}{4} q(x) + \frac{-1 + i}{4} q(-x) - \frac{i}{2} q(ix).$$

**Theorem 3.6.** Suppose that the C*-algebra $A$ is unital and that $\text{Id}([P, A]) = A$. The following are true:

(i) There exists $N$ such that every element of $A$ is expressible as a linear combination of $N$ projections and $N$ products of two projections.

(ii) There exists $K$ such that every commutator $[x, y]$, with $x, y \in A$, is expressible as a linear combination of $K$ projections.

**Proof.** Both (i) and (ii) will follow once we show that Theorem 3.1 is applicable to the Lie ideal span$(P)$ and the generating set $P$. It suffices to show that a commutator of the form $[p, z]$, with $p$ a projection, is a linear combination of projections with a uniform bound on the number of terms. But

$$[p, z] = (p + pz(1 - p)) - (p + (1 - p)zp),$$

where $p + pz(1 - p)$ and $p + (1 - p)zp$ are idempotents. Each of them is a linear combination of four projections by Lemma 3.5.

**Remark 3.7.** If $B$ is a unital C*-subalgebra of $A$ and $\text{Id}([P_B, B]) = B$, then

$$1 = \sum_{i=1}^{n} x_i[p_i, q_i]z_i,$$

for $x_i, y_i, z_i \in B$ and projections $p_i, q_i \in P_B$. It follows that the constants $N$ and $K$ that one finds for $B$ following the proof of Theorem 3.1 applied to $L = \text{span}(P_B)$ also work for the C*-algebra $A$. This observation can be used to obtain concrete estimates of these constants in cases where $B$ is rather simple.

An element of a C*-algebra is called full if it generates the C*-algebra as a closed two-sided ideal. Recall also that a unital C*-algebra is said to have real rank zero if its invertible selfadjoint elements are dense in the set of selfadjoint elements. By [Dav96, Theorem V.7.3], this is equivalent to asking that every hereditary C*-subalgebra of $A$ has an approximate unit consisting of projections.
Corollary 3.8. Suppose that $A$ is unital and either contains two full orthogonal projections or has real rank zero and no 1-dimensional representations. Then there exist $N$ and $K$ such that (i) and (ii) of the previous theorem hold for $A$.

Proof. Let us show in both cases that $\text{Id}([P, A]) = A$.

Say $p$ is a projection such that $p$ and $1 - p$ are full; i.e., $A = \text{Id}(p) = \text{Id}(1 - p)$. Then

$$A = \text{Id}(p) \cdot \text{Id}(1 - p) = A p A (1 - p) A = \text{Id}(p A (1 - p)).$$

On the other hand, $\text{Id}(p A (1 - p)) = \text{Id}([p, A])$. Indeed,

$$p A (1 - p) = [p, A] (1 - p) \subseteq \text{Id}([p, A]),$$

and conversely

$$[p, A] = \{ p a (1 - p) - (1 - p) a p \mid a \in A \} \subseteq \text{Id}(p A (1 - p)).$$

(We have $(1 - p) a p \in \text{Id}(p A (1 - p))$ since closed two-sided ideals are selfadjoint.) Hence, $A = \text{Id}(p A (1 - p)) = \text{Id}([p, A])$, as desired.

Suppose now that $A$ has real rank zero and no 1-dimensional representations, i.e., $A = \text{Id}([A, A])$. Since $\text{Id}([A, A]) = \text{Id}(N_2)$ (where, as before, $N_2$ denotes the set of nilpotents of order two), $A = \text{Id}(N_2)$. Furthermore, since $A$ is unital there exist $x_1, \ldots, x_n \in N_2$ such that $A = \text{Id}(x_1, \ldots, x_n)$, for it suffices to choose these elements such that $\sum_{i=1}^n a_i x_i b_i$ is invertible for some $a_i, b_i \in A$. Since $A$ has real rank-zero, the hereditary subalgebras $x_i A x_i$ have approximate units consisting of projections for all $i$. Using this, we can find projections $p_i \in x_i A x_i$ for $i = 1, \ldots, n$ such that $A = \text{Id}(p_1, \ldots, p_n)$. We claim that $p_i$ is Murray-von Neumann subequivalent to $1 - p_i$ for all $i$. To prove this, let $x_i = v_i [x_i]$ be the polar decomposition of $x_i$ in $A^{**}$. Since $p_i \in x_i A x_i$, we have $p_i \leq v_i^* v_i$. On the other hand, $x_i^2 = 0$ implies that $v_i^* v_i$ and $v_i v_i^*$ are orthogonal projections. Hence, $v_i v_i^* \leq 1 - p_i$. It follows that $p_i = (v_i v_i^*) (p_i p_i)$ and $(v_i p_i) (p_i v_i) = v_i p_i v_i^* \leq v_i v_i^* \leq 1 - p_i$. This proves the claim.

We now have that $\text{Id}(p_i) \subseteq \text{Id}(1 - p_i)$ for all $i = 1, \ldots, n$. Hence,

$$\text{Id}(p_i) = \text{Id}(p_i) \cdot \text{Id}(1 - p_i) = \text{Id}(p_i A (1 - p_i)) = \text{Id}([p_i, A]),$$

for all $i = 1, \ldots, n$. So $A = \text{Id}(p_1, \ldots, p_n) = \text{Id}([p_1, A], \ldots, [p_n, A])$, as desired.

Next we turn to the Lie ideals generated by polynomials already investigated in the previous section. As before, by a polynomial we understand a polynomial in noncommuting variables with coefficients in $\mathbb{C}$.

Theorem 3.9. Let $k \in \mathbb{N}$. Suppose that the $C^*$-algebra $A$ is unital and has no representations of dimension less than or equal to $k$. Let $f$ be a nonconstant polynomial such that $f(A) \subseteq [A, A]$ and which is not a polynomial identity on $M_k(\mathbb{C})$. The following are true:
(i) There exists $N$ such that each element of $A$ is expressible as a linear combination of $N$ values of $f$ on $A$ and $N$ products of two values of $f$ on $A$.

(ii) There exists $K$ such that each commutator $[x, y]$ in $A$ is expressible as a linear combination of $K$ values of $f$ on $A$.

Proof. Both (i) and (ii) will follow from Theorem 3.1 applied to the Lie ideal $\text{span}(f(A))$, with generating set $f(A)$, once we show the hypotheses of that theorem are valid in this case.

Since all representations of $A$ have dimension at least $k + 1$, we have $A = I_k$, where $I_k$ is as defined in the previous section. Also, by the proof of Theorem 2.7, $\text{Id}(f(A)) = I_{k'}$, where $k'$ is the largest number such that $f$ is an identity on $M_{k'}(\mathbb{C})$. But $f$ is not an identity on $M_k(\mathbb{C})$, so we must have that $k' \leq k$. Hence $\text{Id}(f(A)) = A$. Furthermore, as argued in the proof of Theorem 2.7, $\text{Id}([f(A), A]) = \text{Id}(f(A))$. Thus, $A = \text{Id}([f(A), A])$.

To complete the proof, it remains to show that there is a uniform bound on the number of terms expressing a commutator $[f(\overline{\sigma}), y]$ as a linear combination of elements of $f(A)$. This is indeed true, and can be derived from the proof of [BK09] Theorem 2.3] (showing that $\text{span}(f(A))$ is a Lie ideal). We only sketch the argument here: Say $f = \sum_{i=1}^{m} f_i$ is the decomposition of $f$ into a sum of multihomogeneous polynomials. Then, as argued in the proof of [BK09] Theorem 2.3], relying on [BK09] Lemma 2.2], each evaluation $f_i(\overline{\sigma})$ is expressible as a linear combination of at most $(d + 1)^n$ values of $f$. Here $d$ is the maximum of the degrees of $f$ on its variables and $n$ the number of variables. It thus suffices to prove the desired result for each $f_i$, or put differently, to assume that $f$ is multihomogeneous. If $f$ is a constant polynomial then $[f(\overline{\sigma}), y] = 0$ and the desired conclusion holds trivially. Let us assume that $f$ is multihomogeneous and has nonzero degree. We can furthermore reduce ourselves to the multilinear case. For suppose that $f$ has degree $d > 1$ on $x_n$. Let

$$g(x_1, \ldots, x_n, x_{n+1}) = f(x_1, \ldots, x_{n-1}, x_n + x_{n+1})$$

$$- f(x_1, \ldots, x_{n-1}, x_{n+1}) - f(x_1, \ldots, x_n). \quad (3.1)$$

Then the degree of $g$ on $x_n$ is less than $d$ and $f(x_1, \ldots, x_n) = \frac{1}{2^{d-2}} g(x_1, \ldots, x_n, x_n).$ This reduces the proof to $g$. Continuing in this way, we arrive at a multilinear polynomial. Finally, if $f$ is multilinear then the identity

$[f(a_1, \ldots, a_n), y] = f([a_1, y], \ldots, a_n) + f(a_1, [a_2, y], \ldots, a_n) + \cdots + f(a_1, \ldots, [a_n, y])$

shows that there is a uniform bound on the number of terms expressing $[f(\overline{\sigma}), y]$ as a linear combination of values of $f$. \qed

A theorem of Pop ([Pop02, Theorem 1]) says that if $A$ is unital and without bounded traces then there exists $M \in \mathbb{N}$ such every element of $A$ is a sum of
$M$ commutators. Combining this with the previous theorem yields the following corollary:

**Corollary 3.10.** Let $A$ be unital and without bounded traces. Let $f$ be a nonconstant polynomial. Then there exists $N \in \mathbb{N}$ such that each element of $A$ is expressible as a linear combination of $N$ values of $f$ on $A$.

**Proof.** Since $A$ has no bounded traces it has no finite dimensional representations. Hence $I_k = A$ for all $k \in \mathbb{N}$. Furthermore, $f(A) \subseteq [A, A]$ by Pop’s theorem. Thus, by the preceding theorem, every commutator is a linear combination of $K$ values of $f$. On the other hand, every element of $A$ is a sum of $M$ commutators (by Pop’s theorem). So every element of $A$ is a linear combination of $KM$ values of $f$. $\square$

In [BK10], Brešar and Klep reach the conclusion of the preceding corollary for $K(H)$ and $B(H)$ (the compact and bounded operators on a Hilbert space) and for certain rings obtained as tensor products.

Next we construct examples showing that if $f(C) = \{0\}$ then the number $N$ in Corollary 3.10 can be arbitrarily large. Taking $f(x, y) = [x, y]$ this shows that in Pop’s theorem the number of commutators can be arbitrarily large. Taking $f(x_1, \ldots, x_6) = [x_1, x_2] + [x_3, x_4] \cdot [x_5, x_6]$ this shows that the $N$ in Theorem 3.2 can be arbitrarily large as well.

**Example 3.11.** Let $f$ be a polynomial in $n$ noncommuting variables such that $f(C) = \{0\}$. Let $K \in \mathbb{N}$. We will construct a $C^*$-algebra $A$, unital and without bounded traces, and an element $e \in A$ not expressible as a linear combination of $K$ values of $f$. Let $S^2$ denote the 2-dimensional sphere. Let $\eta \in M_2(C(S^2))$ be a rank one non-trivial projection (i.e., one not Murray-von Neumann equivalent to a constant rank one projection). Choose $N \geq 2Kn$. Let $\eta_N = \eta^{\otimes N} \in M_{2N}(C((S^2)^N))$. It is well known that the vector bundle associated to $\eta_N^{\otimes N}$ has non-trivial Euler class. In particular, any $N$ sections of the vector bundle associated to $\eta_N$ have a common vanishing point.

Let $X = \prod_{i=1}^{\infty} (S^2)^N$. Let $1_X$ denote the unit of $C(X)$. Let $e$, $p$, and $q$ be projections in $C(X, B(\ell^2(\mathbb{N})))$ defined as follows:

$$e = \text{diag}(1_X, 0, 0, \ldots),$$
$$q(x_1, x_2, \ldots) = \text{diag}(0, \eta_N(x_1), \eta_N(x_2), \ldots),$$
$$p(x_1, x_2, \ldots) = \text{diag}(1_X, \eta_N(x_1), \eta_N(x_2), \ldots),$$

where $x_i \in (S^2)^N$ for all $i = 1, 2, \ldots$. The following facts are known (see [DD63, Théorème 6] and [Rør03, Section 4]):

i. $q^{\oplus N+1}$ is a properly infinite projection (i.e., $q^{\oplus N+2}$ is Murray-von Neumann subequivalent to $q^{\oplus N+1}$),

17
ii. $e$ is not Murray-von Neumann subequivalent to $q^\oplus N$. Thus, for any $N$ elements of $qC(X, B(\ell^2(N)))e$ (i.e., “sections” of $q$) there exists $x \in X$ on which they all vanish.

Since $p = e \oplus q$, we have that $p^\oplus N+1$ is also a properly infinite projection. Let us define $A = pC(X, B(\ell^2(N)))p$. Notice first that $A$ cannot have bounded traces, since its unit is stably properly infinite. Let us show that $e \in A$ cannot be approximated within a distance less than one by a linear combination of $K$ elements of $f(A)$. Suppose, for the sake of contradiction, that

\[ \|e - \sum_{i=1}^{K} \lambda_i f(\overline{a}_i)\| < 1. \]

Multiplying by $e$ on the left and on the right we get

\[ \|e - \sum_{i=1}^{K} \lambda_i e f(\overline{a}_i)\| < 1. \]  

(3.2)

Say $\overline{a}_i = (a_{i,1}, \ldots, a_{i,n})$ for $i = 1, \ldots, K$. Since $p = e \oplus q$, we may regard each $a_{i,j} \in A$ as an “$e \times q$” matrix:

\[ a_{i,j} = \begin{pmatrix} b_{i,j} & c_{i,j} \\ d_{i,j} & e_{i,j} \end{pmatrix} \in \begin{pmatrix} eC(X, B(\ell^2))e & eC(X, B(\ell^2))q \\ qC(X, B(\ell^2))e & qC(X, B(\ell^2))q \end{pmatrix} \]

for all $i = 1, \ldots, K$ and $j = 1, \ldots, n$. Since $N \geq 2nK$, there exists $x \in X$ such that $c_{i,j}(x) = d_{i,j}(x) = 0$ for all $i, j$. But $eC(X, B(\ell^2))e \cong \mathbb{C}$ and $f(\mathbb{C}) = 0$. So $e f(\overline{a}_i(x))e = f(\overline{a}_i(x)) = 0$ for all $i = 1, \ldots, K$. Evaluating at $x \in X$ in (3.2) we then get $\|e(x) - 0\| < 1$, which is clearly impossible.

**Remark 3.12.** The previous example shows also that the existence of a unit cannot be dropped neither in Theorem 3.2 nor in Corollary 3.10. Indeed, consider $A = \bigoplus_{N=1}^{\infty} A_N$, with $A_N$ as in the example above. Then $A$ has no bounded traces (whence no 1-dimensional representations) but $A \neq \text{span}(f(A))$ for any polynomial $f$ in noncommuting variables such that $f(\mathbb{C}) = 0$.

### 4 Similarity invariance and the span of $N_2$

Let $U \subseteq A$ be a linear subspace. In this section we investigate the equivalence between the following two properties of $U$:

(i) $(1 + x)U(1 - x) \subseteq U$ for all $x \in N_2$,

(ii) $U$ is a Lie ideal.
We have seen in Theorem 2.6 that if \( U \) is closed then (i) and (ii) are indeed equivalent. Furthermore, the proof of (ii) \( \Rightarrow \) (i) in Theorem 2.6 is valid for any subspace \( U \) of \( A \). Thus, we are interested in the implication (i) \( \Rightarrow \) (ii) when \( U \) is not necessarily closed. In the closed case, the proof of (i) \( \Rightarrow \) (ii) in Theorem 2.6 can be split into two steps: In the first step we showed that \( U \) is a Lie ideal of \([A, A]\). This was done as follows: (i) readily implies that \([U, N_2] \subseteq U\). Then using that \([A, A] \subseteq \text{span}(N_2)\) (by Corollary 2.3) and that \( U \) is closed, we arrived at \([U, [A, A]] \subseteq U\). In the second step we appealed to Theorem 1.8, showing that a closed Lie ideal of \([A, A]\) is a Lie ideal of \( A \).

Let us first address the passage from \([U, N_2] \subseteq U\) to \([U, [A, A]] \subseteq U\) in the non-closed case. Let \( A_+ \) denote the positive elements of \( A \). Let us define

\[
N_2^c = \{ x \in A \mid xe = fx = x \text{ for some } e, f \in A_+ \text{ such that } ef = 0 \}.
\]

One readily checks that \( N_2^c \subseteq N_2 \). Let us show that \( N_2^c \) is dense in \( N_2 \). Let \( x \in N_2 \). Observe that for each \( \phi \in C_0(0, 1] \) we have \( \phi(|x|)\phi(|x^*|) = 0 \), since \( \phi(|x|) \in C^*(x^*x) \) and \( \phi(|x^*|) \in C^*(xx^*) \). Let us choose \( \phi_1, \phi_2, \ldots \in C_0(0, 1] \), an approximate unit of \( C_0(0, 1] \) such that \( \phi_{n+1} \phi_n = \phi_n \) for all \( n \). Then \( \phi_n(|x^*|)x\phi_n(|x|) \in N_2^c \) for all \( n \), since we can set \( e = \phi_{n+1}(|x|) \) and \( f = \phi_{n+1}(|x^*|) \). Furthermore, \( \phi_n(|x^*|)x\phi_n(|x|) \rightarrow x \).

Thus, \( N_2^c \) is dense in \( N_2 \).

Let us define

\[
SN_2^c = \bigcup_{x \in N_2 \cup \{0\}} (1 + x)N_2^c(1 - x).
\]

Notice that we still have \( SN_2^c \subseteq N_2 \).

**Lemma 4.1.** \( \text{span}(SN_2^c) \) is a Lie ideal.

**Proof.** It suffices to show that \([A, x] \subseteq \text{span}(SN_2^c)\) for all \( x \in N_2^c \). For then, conjugating by the algebra automorphism \( a \mapsto (1 + y)a(1 - y) \), with \( y \in N_2 \), and using the invariance of \( SN_2^c \) under such automorphisms, we get that \([A, (1 + y)(1 - x)] \subseteq \text{span}(SN_2^c)\) for all \( x \in N_2^c \) and \( y \in N_2 \), as desired.

Let \( x \in N_2^c \). Let \( e \) and \( f \) be positive elements such that \( xe = fx = x \) and \( ef = 0 \). Using functional calculus on \( e \), let us find positive contractions \( e_0, e_1, e_2, e_3 \in C^*(e) \) such that \( e_0e_1 = e_1, e_1e_2 = e_2, e_2e_3 = e_3 \) and \( xe_3 = x \). Similarly, let us find positive contractions \( f_0, f_1, f_2, f_3 \in C^*(f) \) such that \( f_0f_1 = f_1, f_1f_2 = f_2, f_2f_3 = f_3 \) and \( f_3x = x \). Note that \( xe_i = f_jx = x \) and \( e_if_j = 0 \) for all \( i, j = 0, 1, 2, 3 \). Now let \( a \in A \). Then

\[
ax - xa = ax - e_1ax + e_1ax - xaf_1 + xaf_1 - xa
= (1 - e_1)ax + [e_1af_1, x] - xa(1 - f_1).
\]

The term \((1 - e_1)ax\) is in \( N_2^c \). Indeed, \( 1 - e_2 \) and \( e_3 \) act as multiplicative units on the left and on the right of \((1 - e_1)ax\) and \((1 - e_2)e_3 = 0 \). We check similarly that
For infinite von Neumann algebras, the following corollary is [Mie81, Theorem 2]. (Miers also considered closed subspaces of von Neumann algebras, which we have already dealt with in Theorem 2.6)
Corollary 4.3. Suppose that $A$ is either unital and without bounded traces or a von Neumann algebra. Then a subspace $U$ of $A$ is a Lie ideal if and only if $(1 + x)U(1 - x) \subseteq U$ for all $x \in N_2$.

Proof. That a Lie ideal satisfies the similarity invariance of the statement has already been shown in the proof of Theorem 2.6. So let us suppose that $U$ is a subspace such that $(1 + x)U(1 - x) \subseteq U$ for all $x \in N_2$. As remarked at the start of this section, this implies that $[U, N_2] \subseteq U$. Let us consider first the case that $A$ is unital and without bounded traces. Then span($N_2$) = $[A, A]$, since $A$ is unital and has no 1-dimensional representations (since it has no bounded traces). Thus, $[U, [A, A]] \subseteq U$. Furthermore, $[A, A] = A$, by Pop’s theorem. Hence, $[U, A] \subseteq U$; i.e., $U$ is a Lie ideal.

Suppose now that $A$ is a von Neumann algebra. Let us show again that span($N_2$) = $[A, A]$ and that if $U$ is a Lie ideal of $[A, A]$ then it is a Lie ideal of $A$. The latter is [Mie81, Lemma 3] and can be proven as follows: In a von Neumann algebra we have $A = Z(A) + [A, A]$, where $Z(A)$ denotes the center of $A$ (if $A$ is infinite, because $A = [A, A]$, and if $A$ is finite, by [FdlH80, Theorem 3.2]); so $[U, A] = [U, [A, A]]$ for any subset $U$ of $A$. Let us now show that span($N_2$) = $[A, A]$. The ideal $Id([A, A])$ is closed in the ultraweak topology of $A$, being the intersection of the kernels of all 1-dimensional representations of $A$ (which are always ultraweakly continuous). Thus, $Id([A, A])$ is also a von Neumann algebra. In particular, it is unital. Since it has no 1-dimensional representations,

$$span(N_2) = [Id([A, A]), Id([A, A])] \supseteq [[A, A], [A, A]].$$

From $A = [A, A] + Z(A)$ we get that $[A, A] = [[A, A], [A, A]]$. Hence, span($N_2$) = $[A, A]$.

The passage from $U$ being a Lie ideal of $[A, A]$ to being a Lie ideal of $A$ can also be made assuming that $A$ is unital and that $[U, A]$ is full:

Lemma 4.4. Suppose that $A$ is unital. If $U$ is a Lie ideal of $[A, A]$ such that $Id([U, A]) = A$ then $[A, A] \subseteq U$ (so $U$ is a Lie ideal of $A$).

Proof. Let $V = [U, U], W = [V, V], X = [W, W]$. We have shown in the proof of Theorem 1.13 that $Id([U, A]) = Id([X, X]).$ So $A = Id([X, X])$. Since $A$ is unital, the set $[X, X]$ generates $A$ algebraically as a two-sided ideal. But $A[X, X]A \subseteq [U, U] + [U, U]^2$, by Lemma 1.13. Hence, $A = [U, U] + [U, U]^2$. Then,

$$[A, A] = [[U, U] + [U, U]^2, A] = [[U, U], A] \subseteq [U, [U, A]] \subseteq U.$$

Theorem 4.5. Suppose that $A$ is unital and without 1-dimensional representations. Let $U$ be a subspace of $A$ such that $Id([U, A]) = A$. If $(1 + x)U(1 - x) \subseteq U$ for all $x \in N_2$ then $[A, A] \subseteq U$.

Proof. The similarity invariance of $U$ implies that $[U, N_2] \subseteq U$ and by Theorem 4.2 we get that $[U, [A, A]] \subseteq U$. The previous lemma then shows that $[A, A] \subseteq U$.  

21
Corollary 4.6. Let $A$ be simple and unital. A subspace $U$ of $A$ is a Lie ideal if and only if $(1 + x)U(1 - x) \subseteq U$ for all $x \in N_2$.

Proof. Since $A$ is simple we have either $\text{Id}([U, A]) = 0$ or $\text{Id}([U, A]) = A$. If $\text{Id}([U, A]) = 0$ then $U$ is a subset of the center, which by the simplicity of $A$ is $\mathbb{C}$. If $\text{Id}([U, A]) = A$ then by the previous theorem $[A, A] \subseteq U$. In either case it follows that $U$ is a Lie ideal of $A$.  

Amitsur’s $\text{[Ami56] Theorem 1}$ (that a similarity invariant subspace of a simple algebra must be a Lie ideal) requires the existence of a nontrivial idempotent in the algebra. An example in $\text{[Ami56]}$ shows that this hypothesis cannot be dropped. Corollary $\text{4.6}$ shows, however, that for simple unital $C^*$-algebras this assumption is not necessary (even though they may well fail to have any nontrivial idempotents).

References

[Ami56] S. A. Amitsur, Invariant submodules of simple rings, Proc. Amer. Math. Soc. 7 (1956), 987–989.

[AL50] A. S. Amitsur and J. Levitzki, Minimal identities for algebras, Proc. Amer. Math. Soc. 1 (1950), 449–463.

[AM03] Pere Ara and Martin Mathieu, Local multipliers of $C^*$-algebras, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2003.

[Bla06] B. Blackadar, Operator algebras, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006. Theory of $C^*$-algebras and von Neumann algebras; Operator Algebras and Non-commutative Geometry, III.

[BK09] Matej Brešar and Igor Klep, Values of noncommutative polynomials, Lie skew-ideals and tracial Nullstellensätze, Math. Res. Lett. 16 (2009), no. 4, 605–626.

[BK10] Matej Brešar and Igor Klep, A note on values of noncommutative polynomials, Proc. Amer. Math. Soc. 138 (2010), no. 7, 2375–2379.

[BKS08] Matej Brešar, Edward Kissin, and Victor S. Shulman, Lie ideals: from pure algebra to $C^*$-algebras, J. Reine Angew. Math. 623 (2008), 73–121.

[CP79] Joachim Cuntz and Gert Kjaergård Pedersen, Equivalence and traces on $C^*$-algebras, J. Funct. Anal. 33 (1979), no. 2, 135–164.

[Dav96] Kenneth R. Davidson, $C^*$-algebras by example, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996.

[DD63] Jacques Dixmier and Adrien Douady, Champs continus d’espaces hilbertiens et de $C^*$-algèbres, Bull. Soc. Math. France 91 (1963), 227–284 (French).

[FdlH80] Th. Fack and P. de la Harpe, Sommes de commutateurs dans les algèbres de von Neumann finies continues, Ann. Inst. Fourier (Grenoble) 30 (1980), no. 3, 49–73 (French).

[Her69] I. N. Herstein, Topics in ring theory, The University of Chicago Press, Chicago, Ill.-London, 1969.

[Her70] I. N. Herstein, On the Lie structure of an associative ring, J. Algebra 14 (1970), 561–571.

[Mar10] L. W. Marcoux, Projections, commutators and Lie ideals in $C^*$-algebras, Math. Proc. R. Ir. Acad. 110A (2010), no. 1, 31–55.
[MM98] L. W. Marcoux and G. J. Murphy, *Unitarily-invariant linear spaces in C∗-algebras*, Proc. Amer. Math. Soc. **126** (1998), no. 12, 3597–3605.

[Mie81] C. Robert Miers, *Closed Lie ideals in operator algebras*, Canad. J. Math. **33** (1981), no. 5, 1271–1278.

[Ped80] Gert K. Pedersen, *The linear span of projections in simple C∗-algebras*, J. Operator Theory **4** (1980), no. 2, 289–296.

[Pop02] Ciprian Pop, *Finite sums of commutators*, Proc. Amer. Math. Soc. **130** (2002), no. 10, 3039–3041 (electronic).

[RR13] Leonel Robert and Mikael Rørdam, *Divisibility properties for C∗-algebras*, Proc. Lond. Math. Soc. (3) **106** (2013), no. 6, 1330–1370.

[Rør03] Mikael Rørdam, *A simple C∗-algebra with a finite and an infinite projection*, Acta Math. **191** (2003), no. 1, 109–142.

[Row88] Louis H. Rowen, *Ring theory. Vol. I*, Pure and Applied Mathematics, vol. 127, Academic Press, Inc., Boston, MA, 1988.

LEONEL ROBERT, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISIANA AT LAFAYETTE, LAFAYETTE, LOUISIANA 70506

E-mail: lrobert@louisiana.edu