Distance-regular graphs of $q$-Racah type and the universal Askey-Wilson algebra

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Abstract
Let $\mathbb{C}$ denote the field of complex numbers, and fix a nonzero $q \in \mathbb{C}$ such that $q^4 \neq 1$. Define a $\mathbb{C}$-algebra $\Delta_q$ by generators and relations in the following way. The generators are $A, B, C$. The relations assert that each of

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

is central in $\Delta_q$. The algebra $\Delta_q$ is called the universal Askey-Wilson algebra. Let $\Gamma$ denote a distance-regular graph that has $q$-Racah type. Fix a vertex $x$ of $\Gamma$ and let $T = T(x)$ denote the corresponding subconstituent algebra. In this paper we discuss a relationship between $\Delta_q$ and $T$. Assuming that every irreducible $T$-module is thin, we display a surjective $\mathbb{C}$-algebra homomorphism $\Delta_q \to T$. This gives a $\Delta_q$ action on the standard module of $T$.

Keywords. Distance regular graph, $Q$-polynomial, Askey-Wilson relations, Leonard pair, subconstituent algebra.

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1 Introduction
The universal Askey-Wilson algebra $\Delta_q$ was introduced by the first author in [17]. The algebra $\Delta_q$ is a central extension of the Askey-Wilson algebra [20]. The algebra $\Delta_q$ is related to the $q$-Onsager algebra [17, Section 9], the algebra $U_q(\mathfrak{sl}_2)$ [18], and the double affine Hecke algebra of type $(C_1^\vee, C_1)$ [19]. In [8] the finite-dimensional irreducible $\Delta_q$-modules are classified up to isomorphism, under the assumption that $q$ is not a root of unity. In this paper we describe how $\Delta_q$ is related to the subconstituent algebra of a distance-regular graph that has $q$-Racah type.

Before we describe our results we set some conventions. An algebra is meant to be associative and have a 1. A subalgebra has the same 1 as the parent algebra. The field of complex numbers is denoted $\mathbb{C}$. Until the end of Section 5 we fix a nonzero $q \in \mathbb{C}$ such that $q^4 \neq 1$.

We now recall the universal Askey-Wilson algebra [17, Definition 1.2]. Define a $\mathbb{C}$-algebra $\Delta_q$ by generators and relations in the following way. The generators are $A, B, C$. The relations assert that each of

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

(1)

is central in $\Delta_q$. The algebra $\Delta_q$ is called the universal Askey-Wilson algebra.

We now recall some notions concerning distance-regular graphs (see Sections 3 and 4 for formal definitions). Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$. We say that
The paper is organized as follows. In Section 2 we recall some background concerning Leonard pairs and Leonard systems. In Section 3 we recall the basic theory concerning a distance-regular graph and its subconstituent algebras. In Section 4 we consider the irreducible Leonard pairs and Leonard systems. In Section 5 we describe our main results. In Section 6 we apply our results to the 2-homogeneous bipartite distance-regular graphs.

2 Leonard pairs and Leonard systems

In this section we recall some background concerning Leonard pairs and Leonard systems. For more information we refer the reader to [7], [15], [16]. Throughout this section fix an integer

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In this section we recall some background concerning Leonard pairs and Leonard systems. For more information we refer the reader to [7], [15], [16]. Throughout this section fix an integer
(i) Each of $A, A^*$ is a multiplicity-free element in $\text{End}(V)$.

(ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of $A$.

(iii) $\{E^*_i\}_{i=0}^d$ is an ordering of the primitive idempotents of $A^*$.

(iv) $E_i A^* E_j = \begin{cases} 0 & \text{if } |i - j| > 1 \\ \neq 0 & \text{if } |i - j| = 1 \end{cases}$ $(0 \leq i, j \leq d)$.

(v) $E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i - j| > 1 \\ \neq 0 & \text{if } |i - j| = 1 \end{cases}$ $(0 \leq i, j \leq d)$.

We call $d$ the diameter of $\Phi$.

Leonard pairs and Leonard systems are related as follows. Let $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E^*_i\}_{i=0}^d)$ denote a Leonard system on $V$. For $0 \leq i \leq d$ let $v_i$ denote a nonzero vector in $E_i V$. Then the sequence $\{v_i\}_{i=0}^d$ is a basis for $V$ that satisfies Definition 2.1(ii). For $0 \leq i \leq d$ let $v_i^*$ denote a nonzero vector in $E_i^* V$. Then the sequence $\{v_i^*\}_{i=0}^d$ is a basis for $V$ that satisfies Definition 2.1(i). Therefore the pair $A, A^*$ is a Leonard pair on $V$.

Conversely, let $A, A^*$ denote a Leonard pair on $V$. By [15, Lemma 1.3], each of $A, A^*$ is multiplicity-free. Let $\{v_i\}_{i=0}^d$ denote a basis for $V$ that satisfies Definition 2.1(ii). For $0 \leq i \leq d$ the vector $v_i$ is an eigenvector for $A$; let $E_i$ denote the corresponding primitive idempotent. Let $\{v_i^*\}_{i=0}^d$ denote a basis for $V$ that satisfies Definition 2.1(i). For $0 \leq i \leq d$ the vector $v_i^*$ is an eigenvector for $A^*$; let $E_i^*$ denote the corresponding primitive idempotent. Then $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E^*_i\}_{i=0}^d)$ is a Leonard system on $V$. We say that the Leonard pair $A, A^*$ and the Leonard system $\Phi$ are associated.

**Definition 2.3.** Let $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E^*_i\}_{i=0}^d)$ denote a Leonard system on $V$. For $0 \leq i \leq d$ let $\theta_i$ (resp. $\theta^*_i$) denote the eigenvalue of $A$ (resp. $A^*$) associated with $E_i$ (resp. $E_i^*$). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) the eigenvalue sequence (resp. dual eigenvalue sequence) of $\Phi$.

Let $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E^*_i\}_{i=0}^d)$ denote a Leonard system. We just defined the eigenvalue sequence and the dual eigenvalue sequence of $\Phi$. Associated with $\Phi$ are two more sequences of scalars, called the first split sequence and second split sequence [15, p. 155]. These sequences are denoted by $\{\phi_i\}_{i=1}^d$ and $\{\varphi_i\}_{i=1}^d$, respectively. The sequence $(\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$ is called the parameter array of $\Phi$. The Leonard system $\Phi$ is uniquely determined by its parameter array, up to isomorphism of Leonard systems [15, Theorem 1.9]. By [15, Lemma 5.2, Lemma 6.4] both

$$\varphi_i = (\theta^*_{i-1} - \theta^*_i) \sum_{h=0}^{i-1} (a_h - \theta_h), \quad (2)$$

$$\phi_i = (\theta^*_{i-1} - \theta^*_i) \sum_{h=0}^{i-1} (a_h - \theta_d-h) \quad (3)$$

for $1 \leq i \leq d$, where

$$a_h = \text{trace}(E^*_h A) \quad (0 \leq h \leq d). \quad (4)$$

**Definition 2.4.** Let $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E^*_i\}_{i=0}^d)$ denote a Leonard system on $V$, with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^d$. We say that $\Phi$ has $q$-Racah type whenever there exist nonzero $a, b$ in $\mathbb{C}$ such that both

$$\theta_i = aq^{2i-d} + a^{-1}q^{d-2i}, \quad \theta^*_i = bq^{2i-d} + b^{-1}q^{d-2i}$$

for $0 \leq i \leq d$. 

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Definition 2.5. A Leonard pair is said to have \textit{q-Racah type} whenever an associated Leonard system has q-Racah type.

Let $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ denote a Leonard system with parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$. Assume that $\Phi$ has q-Racah type. Let $c \in \mathbb{C}$ denote a root of the equation

$$\xi^2 - \kappa \xi + 1 = 0,$$

where $\kappa = 0$ for $d = 0$, and for $d \geq 1,$

$$\kappa = ab^{-1}q^{d-1} + a^{-1}bq^{1-d} + \frac{\phi_1}{(q - q^{-1})(q^d - q^{-d})}$$

with $a, b$ from Definition 2.4. From the form of (5) we see that $c \neq 0$. Moreover $c^{-1}$ is also a root of (5), and (5) has no further roots. By [7, Lemma 6.3], both

$$\varphi_i = a^{-1}b^{-1}q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})(q^i - abcq^{i-d-1})(q^i - abc^{-1}q^{i-d-1}),$$

$$\phi_i = ab^{-1}q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})(q^i - a^{-1}bcq^{i-d-1})(q^i - a^{-1}bc^{-1}q^{i-d-1})$$

for $1 \leq i \leq d$.

We now recall the Askey-Wilson relations [20]. We will work with the $\mathbb{Z}_3$-symmetric version [7, Theorem 10.1].

Theorem 2.6. [7, Theorem 10.1]. Let $A, A^*$ denote a Leonard pair on $V$ that has q-Racah type. Recall the scalars $a, b$ from Definition 2.4 and $c$ from below Definition 2.5. Then there exists a unique $A^c \in \text{End}(V)$ such that

$$A + \frac{qA^*A^c - q^{-1}A^cA^*}{q^2 - q^{-2}} = \frac{(a + a^{-1})(q^{d+1} + q^{-d-1}) + (b + b^{-1})(c + c^{-1})}{q + q^{-1}}I,$$  \hspace{1cm} (7)

$$A^* + \frac{qA^cA - q^{-1}A^cA^*}{q^2 - q^{-2}} = \frac{(b + b^{-1})(q^{d+1} + q^{-d-1}) + (c + c^{-1})(a + a^{-1})}{q + q^{-1}}I,$$  \hspace{1cm} (8)

$$A^c + \frac{qAA^c - q^{-1}A^cA}{q^2 - q^{-2}} = \frac{(c + c^{-1})(q^{d+1} + q^{-d-1}) + (a + a^{-1})(b + b^{-1})}{q + q^{-1}}I.$$  \hspace{1cm} (9)

3 Background on distance-regular graphs

In this section we review some basic concepts concerning distance-regular graphs. See Brouwer, Cohen and Neumaier [2] and Terwilliger [12,13] for more background information.

Let $X$ denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of the matrices whose rows and columns are indexed by $X$ and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of the column vectors whose coordinates are indexed by $X$ and whose entries are in $\mathbb{C}$. Observe that $\text{Mat}_X(\mathbb{C})$ acts on $V$ by left multiplication. We endow $V$ with the Hermitian inner product $\langle \cdot, \cdot \rangle$ that satisfies $\langle u, v \rangle = \overline{u^tv}$ for $u, v \in V$, where $t$ denotes transpose and $\overline{\cdot}$ denotes complex conjugation. Let $W, W'$ denote nonempty subsets of $V$. These subsets are said to be \textit{orthogonal} whenever $\langle w, w' \rangle = 0$ for all $w \in W$ and $w' \in W'$.

Let $\Gamma$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $X$, path-length distance function $\partial$ and diameter $D = \max\{\partial(x, y) \mid x, y \in X\}$. For a vertex $x \in X$ and integer $i \geq 0$ define

$$\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}.$$
The graph $\Gamma$ is said to be distance-regular whenever for all integers $h, i, j$ ($0 \leq h, i, j \leq D$) and vertices $x, y \in X$ with $\partial(x, y) = h$, the number $p_{ij}^h = |\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of $x, y$. The constants $p_{ij}^h$ are called the intersection numbers of $\Gamma$. From now on assume that $\Gamma$ is distance-regular with diameter $D \geq 3$.

We recall the Bose-Mesner algebra of $\Gamma$. For $0 \leq i \leq D$ let $A_i$ denote the matrix in $\text{Mat}_X(\mathbb{C})$ with $(x, y)$-entry

$$
(A_i)_{xy} = \begin{cases} 
1; & \text{if } \partial(x, y) = i \\
0; & \text{if } \partial(x, y) \neq i
\end{cases} \quad (x, y \in X).
$$

We call $A_i$ the $i$-th distance matrix of $\Gamma$. Note that $A_0 = I$, where $I$ is the identity matrix in $\text{Mat}_X(\mathbb{C})$. The matrix $A = A_1$ is the adjacency matrix of $\Gamma$. We observe that $J = \sum_{i=0}^{D} A_i$, where $J \in \text{Mat}_X(\mathbb{C})$ has all entries 1. Moreover $A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h$ for $0 \leq i, j \leq D$. Let $M$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A$. By [2, p. 127] the matrices $\{A_i\}_{i=0}^{D}$ form a basis for $M$. We call $M$ the Bose-Mesner algebra of $\Gamma$. By [2, p. 45], $M$ has a basis $\{E_i\}_{i=0}^{D}$ such that (i) $E_0 = |X|^{-1} J$; (ii) $I = \sum_{i=0}^{D} E_i$; (iii) $E_i E_j = \delta_{ij} E_i$ for $0 \leq i, j \leq D$. By [1] p. 59, 64] the matrices $\{E_i\}_{i=0}^{D}$ are real and symmetric. We call $\{E_i\}_{i=0}^{D}$ the primitive idempotents of $\Gamma$. We call $E_0$ trivial. Since $\{E_i\}_{i=0}^{D}$ form a basis for $M$, there exist complex scalars $\{\theta_i\}_{i=0}^{D}$ such that

$$
A = \sum_{i=0}^{D} \theta_i E_i. \quad (10)
$$

By (10) and since $E_i E_j = \delta_{ij} E_i$ we have

$$
\theta_i = \sum_{i=0}^{D} \theta_i E_i = (0 \leq i \leq D). \quad (11)
$$

We call $\theta_i$ the eigenvalue of $\Gamma$ corresponding to $E_i$. Note that the eigenvalues $\{\theta_i\}_{i=0}^{D}$ are mutually distinct since $A$ generates $M$. Moreover $\{\theta_i\}_{i=0}^{D}$ are real, since $A$ and $\{E_i\}_{i=0}^{D}$ are real. The vector space $V$ decomposes as

$$
V = \sum_{i=0}^{D} E_i V \quad (\text{orthogonal direct sum}).
$$

We now recall the Krein parameters. Let $\circ$ denote the entry-wise product in $\text{Mat}_X(\mathbb{C})$. Observe $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leq i, j \leq D$, so $M$ is closed under $\circ$. Therefore there exist complex scalars $q_{ij}^h$ such that

$$
E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \leq i, j \leq D). \quad (12)
$$

We call the $q_{ij}^h$ the Krein parameters of $\Gamma$. These parameters are real and nonnegative [2, p. 48–49]. Let $\{E_i\}_{i=1}^{D}$ denote an ordering of the nontrivial primitive idempotents of $\Gamma$. The ordering $\{E_i\}_{i=1}^{D}$ is called Q-polynomial whenever for all $0 \leq i, j \leq D$ the Krein parameter $q_{ij}^1$ is zero if $|i - j| > 1$ and nonzero if $|i - j| = 1$. The graph $\Gamma$ is called Q-polynomial (with respect to the given ordering $\{E_i\}_{i=1}^{D}$) whenever the ordering $\{E_i\}_{i=1}^{D}$ is Q-polynomial. Until the end of Section 5 we assume that $\Gamma$ is Q-polynomial with respect to $\{E_i\}_{i=1}^{D}$.

We recall the dual Bose-Mesner algebras of $\Gamma$ [12, p. 378]. For the rest of this paper, fix a vertex $x \in X$. For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with $(y, y)$-entry

$$
(E_i^*)_{yy} = \begin{cases} 
1; & \text{if } \partial(x, y) = i \\
0; & \text{if } \partial(x, y) \neq i
\end{cases} \quad (y \in X).
$$

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We call $E^*_i$ the \textit{i-th dual idempotent} of $\Gamma$ \textit{with respect to} $x$. Observe that $I = \sum_{i=0}^{D} E^*_i$ and $E^*_i E^*_j = \delta_{ij} E^*_i$ for $0 \leq i, j \leq D$. Therefore the matrices $\{E^*_i\}_{i=0}^{D}$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\Mat_X(\mathbb{C})$. We call $M^*$ the \textit{dual Bose-Mesner algebra} of $\Gamma$ \textit{with respect to} $x$.

For $0 \leq i \leq D$ let $A^*_i = A^*_i(x)$ denote the diagonal matrix in $\Mat_X(\mathbb{C})$ with $(y,y)$-entry
\[(A^*_i)_{yy} = |X|(E^*_i)_{xy} \quad (y \in X).
\]
We call $A^*_i$ the \textit{dual distance matrix} corresponding to $E^*_i$ and $x$. By \cite{12} p. 379 the matrices $\{A^*_i\}_{i=0}^{D}$ form a basis for $M^*$. We abbreviate $A^* = A^*_1$ and call this the \textit{dual adjacency matrix} of $\Gamma$ \textit{with respect to} $x$. The matrix $A^*$ generates $M^*$ \cite{12} Lemma 3.11.

Since $\{E^*_i\}_{i=0}^{D}$ form a basis for $M^*$, there exist complex scalars $\{\theta^*_i\}_{i=0}^{D}$ such that
\[A^* = \sum_{i=0}^{D} \theta^*_i E^*_i. \tag{13}\]
By \cite{13} and since $E^*_i E^*_j = \delta_{ij} E^*_i$ we have
\[A^* E^*_i = E^*_i A^* = \theta^*_i E^*_i \quad (0 \leq i \leq D). \tag{14}\]
We call $\theta^*_i$ the \textit{dual eigenvalue} of $\Gamma$ corresponding to $E^*_i$. Note that the dual eigenvalues $\{\theta^*_i\}_{i=0}^{D}$ are mutually distinct since $A^*$ generates $M^*$. Moreover $\{\theta^*_i\}_{i=0}^{D}$ are real, since $A^*$ and $\{E^*_i\}_{i=0}^{D}$ are real. The vector space $V$ decomposes as
\[V = \sum_{i=0}^{D} E^*_i V \quad \text{(orthogonal direct sum)}.
\]

Let $T = T(x)$ denote the subalgebra of $\Mat_X(\mathbb{C})$ generated by $M$ and $M^*$. We call $T$ the \textit{subconstituent algebra} \textit{(or Terwilliger algebra)} of $\Gamma$ \textit{with respect to} $x$ \cite{12} p. 380. Note that $T$ is generated by $A$, $A^*$. Since each of $A$ and $A^*$ is real and symmetric, $T$ is closed under the conjugate-transpose map. Hence $T$ is semisimple \cite{12} Lemma 3.4.

\section{4 \textbf{T-modules}}

We continue to discuss the subconstituent algebra $T$ of our distance-regular graph $\Gamma$. In this section we consider the $T$-modules. By a $T$-\textit{module} we mean a subspace $W$ of $V$ such that $SW \subseteq W$ for all $S \in T$. We refer to $V$ itself as the \textit{standard module} for $T$. A $T$-module $W$ is said to be \textit{irreducible} whenever $W$ is nonzero and contains no $T$-modules other than 0 and $W$.

Let $W$ denote a $T$-module. Since $T$ is closed under the conjugate-transpose map, the orthogonal complement $W^\perp$ is also a $T$-module. Therefore $V$ decomposes into an orthogonal direct sum of irreducible $T$-modules.

We now recall the notion of endpoint, dual endpoint, diameter, and dual diameter. Let $W$ denote an irreducible $T$-module. Observe that $W$ is a direct sum of the nonzero spaces among $E^*_0 W, \ldots, E^*_D W$. Similarly, $W$ is a direct sum of the nonzero spaces among $E_0 W, \ldots, E_D W$. Define $\rho = \min\{|i| \quad 0 \leq i \leq D, E^*_i W \neq 0\}$ and $\tau = \min\{|i| \quad 0 \leq i \leq D, E_i W \neq 0\}$. We call $\rho$ and $\tau$ the \textit{endpoint} and \textit{dual endpoint} of $W$, respectively. Define $d = |\{i | \quad 0 \leq i \leq D, E^*_i W \neq 0\}| - 1$ and $d^* = |\{i | \quad 0 \leq i \leq D, E_i W \neq 0\}| - 1$. We call $d$ and $d^*$ the \textit{diameter} and \textit{dual diameter} of $W$, respectively. By \cite{11} Corollary 3.3 the diameter of $W$ is equal to the dual diameter of $W$. By \cite{12} Lemma 3.9, Lemma 3.12 $\dim E^*_i W \leq 1$ for $0 \leq i \leq D$ if and only if $\dim E_i W \leq 1$ for $0 \leq i \leq D$. In this case $W$ is said to be \textit{thin}.
Lemma 4.1. [12] Lemma 3.4, Lemma 3.9, Lemma 3.12. Let $W$ denote an irreducible $T$-module with endpoint $\rho$, dual endpoint $\tau$, and diameter $d$. Then $\rho, \tau, d$ are nonnegative integers such that $\rho + d \leq D$ and $\tau + d \leq D$. Moreover the following (i)–(iv) hold.

(i) $E_i^*W \neq 0$ if and only if $\rho \leq i \leq \rho + d$, $(0 \leq i \leq D)$.

(ii) $W = \sum_{h=0}^{d} E_{\rho+h}^* W$ (orthogonal direct sum).

(iii) $E_i W \neq 0$ if and only if $\tau \leq i \leq \tau + d$, $(0 \leq i \leq D)$.

(iv) $W = \sum_{h=0}^{d} E_{\tau+h} W$ (orthogonal direct sum).

Let $W$ and $W'$ denote irreducible $T$-modules. These $T$-modules are called isomorphic whenever there exists a vector space isomorphism $\sigma : W \to W'$ such that $(\sigma S - S \sigma)W = 0$ for all $S \in T$. Assume that the $T$-modules $W$ and $W'$ are isomorphic. Then they have the same endpoint, dual endpoint and diameter. Now assume that the $T$-modules $W$ and $W'$ are not isomorphic. Then $W$ and $W'$ are orthogonal [3, Lemma 3.3]. Let $\Psi$ denote the set of isomorphism classes of irreducible $T$-modules. The elements of $\Psi$ are called types. For $\psi \in \Psi$ let $V_\psi$ denote the subspace of $V$ spanned by the irreducible $T$-modules of type $\psi$. Call $V_\psi$ the homogeneous component of $V$ of type $\psi$. Observe that $V_\psi$ is a $T$-module. We have the decomposition

$$V = \sum_{\psi \in \Psi} V_\psi \quad \text{(orthogonal direct sum).} \quad (15)$$

For $\psi \in \Psi$ define $e_\psi \in \text{Mat}_X(\mathbb{C})$ such that $(e_\psi - I)V_\psi = 0$ and $e_\psi V_\lambda = 0$ for all $\lambda \in \Psi$ with $\lambda \neq \psi$. By construction $I = \sum_{\psi \in \Psi} e_\psi$. Moreover $e_\psi e_\lambda = \delta_{\psi,\lambda} e_\psi$ for $\psi, \lambda \in \Psi$. According to the Wedderburn theory [3, Chapter IV] the elements $\{e_\psi\}_{\psi \in \Psi}$ form a basis for the center of $T$.

5 The main results

We continue to discuss the subconstituent algebra $T$ of our distance-regular graph $\Gamma$. In this section we obtain our main results, which concern how $T$ is related to the universal Askey-Wilson algebra $\Delta_q$.

Recall the $Q$-polynomial ordering $\{E_i\}_{i=1}^{D}$ of the nontrivial primitive idempotents of $\Gamma$.

Definition 5.1. [3] Section 1]. Our $Q$-polynomial structure $\{E_i\}_{i=1}^{D}$ is said to have $q$-Racah type whenever there exist scalars $u, u^*, v, v^*$ in $\mathbb{C}$ with $u, u^*, v, v^*$ nonzero such that both

$$\theta_i = w + u q^{2i-D} + v q^{2i-D-2i}, \quad \theta_i^* = w^* + u^* q^{2i-D} + v^* q^{2i-D-2i}$$

for $0 \leq i \leq D$.

Definition 5.2. The graph $\Gamma$ is said to have $q$-Racah type whenever $\Gamma$ has a $Q$-polynomial structure of $q$-Racah type.

From now on, assume that our $Q$-polynomial structure $\{E_i\}_{i=1}^{D}$ has $q$-Racah type, as in Definition 5.1 In Section 3 we defined the adjacency matrix $A$ and the dual adjacency matrix $A^*$. We now adjust $A$ and $A^*$. Define scalars $a, b \in \mathbb{C}$ such that $a^2 = u/v$ and $b^2 = u^*/v^*$. Note that $a$ and $b$ are nonzero. Define matrices $A, B$ in $\text{Mat}_X(\mathbb{C})$ as follows:

$$A = \frac{A - w I}{av}, \quad B = \frac{A^* - w^* I}{bv^*}. \quad (16)$$
Lemma 5.3. The following hold for $0 \leq i \leq D$.

(i) The eigenvalue of $A$ associated with $E_i$ is
\[ aq^{2i-D} + a^{-1}q^{D-2i}. \]  
(ii) The eigenvalue of $B$ associated with $E_i^*$ is
\[ bq^{2i-D} + b^{-1}q^{D-2i}. \]

Proof. (i) By (10) and since (17) is equal to $(\theta_i - w)/(av)$.
(ii) By (13) and since (18) is equal to $(\theta_i^* - w^*)/(bw^*)$.$\square$

We mentioned in Section 3 that $T$ is generated by $A, A^*$. By this and (16) we see that $T$ is generated by $A, B$.

Lemma 5.4. Let $W$ denote a thin irreducible $T$-module with endpoint $\rho$, dual endpoint $\tau$, and diameter $d$. Then $(A, \{E_i\}_{i=\tau}^{\tau+d}, B, \{E_i^*\}_{i=\rho}^{\rho+d})$ acts on $W$ as a Leonard system of $q$-Racah type. For this Leonard system the eigenvalue sequence is
\[ a(W)q^{2i-d} + a(W)^{-1}q^{d-2i} \quad (0 \leq i \leq d), \] where $a(W) = aq^{2\tau+d-D}$. Moreover the dual eigenvalue sequence is
\[ b(W)q^{2i-d} + b(W)^{-1}q^{d-2i} \quad (0 \leq i \leq d), \] where $b(W) = bq^{2\rho+d-D}$.

Proof. By [13, Theorem 4.1] and since $W$ is thin, the sequence $(A, \{E_i\}_{i=\tau}^{\tau+d}, A^*, \{E_i^*\}_{i=\rho}^{\rho+d})$ acts on $W$ as a Leonard system. By this and (16), the sequence $\Phi = (A, \{E_i\}_{i=\tau}^{\tau+d}, B, \{E_i^*\}_{i=\rho}^{\rho+d})$ acts on $W$ as a Leonard system. Using Lemma 5.3 one checks that $\Phi$ has eigenvalue sequence (19) and dual eigenvalue sequence (20). Consequently $\Phi$ has $q$-Racah type.$\square$

Given $\psi \in \Psi$, let $W$ denote an irreducible $T$-module of type $\psi$. Then the endpoint, dual endpoint, and diameter of $W$ are independent of the choice of $W$, and depend only on $\psi$. Assume that $W$ is thin. Then the scalars $a(W)$, $b(W)$ from Lemma 5.4 are independent of the choice of $W$, and depend only on $\psi$. Sometimes we write $a(W) = a(\psi)$ and $b(W) = b(\psi)$. Sometimes we write $d(\psi)$ for the diameter of $W$.

Definition 5.5. Assume that each irreducible $T$-module is thin. Define
\[ a = \sum_{\psi \in \Psi} a(\psi)e_\psi, \quad b = \sum_{\psi \in \Psi} b(\psi)e_\psi, \quad \Lambda = \sum_{\psi \in \Psi} \epsilon_d(\psi)+1e_\psi. \]

Lemma 5.6. Assume that each irreducible $T$-module is thin. Then the matrices $a$, $b$, $\Lambda$ from Lemma 5.5 are invertible. Their inverses are
\[ a^{-1} = \sum_{\psi \in \Psi} a(\psi)^{-1}e_\psi, \quad b^{-1} = \sum_{\psi \in \Psi} b(\psi)^{-1}e_\psi, \quad \Lambda^{-1} = \sum_{\psi \in \Psi} q^{-d(\psi)-1}e_\psi. \]

Proof. By Definition 5.5 and the discussion below (15).$\square$
Lemma 5.7. Let $W$ denote a thin irreducible $T$-module. By Lemma 5.4 the sequence $(A, \{E_i\}_{i=1}^{r+1}, B, \{E_i^\dagger\}_{i=p}^{r+1})$ acts on $W$ as a Leonard system of $q$-Racah type. Let $c = c(W)$ denote the corresponding scalar from Section 2. The scalar $c$ is defined up to reciprocal. We sometimes write $c = c(\psi)$, where $\psi$ denotes the type of $W$.

Definition 5.8. Assume that each irreducible $T$-module is thin. Define
\[ c = \sum_{\psi \in \Psi} c(\psi)e_\psi. \]

Lemma 5.9. Assume that each irreducible $T$-module is thin. Then the matrix $c$ from Definition 5.8 is invertible. The inverse is
\[ c^{-1} = \sum_{\psi \in \Psi} c(\psi)^{-1}e_\psi. \]

Proof. By Definition 5.8 and the discussion below (15).

Lemma 5.10. Assume that each irreducible $T$-module is thin. Recall the matrices $a, b, \Lambda$ from Definition 5.5 and the matrix $c$ from Definition 5.8. Then $a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, \Lambda^{\pm 1}$ act on each irreducible $T$-module $W$ as
\[ a(\psi)^{\pm 1}I, \quad b(\psi)^{\pm 1}I, \quad c(\psi)^{\pm 1}I, \quad q^{\pm d(\psi)^{\pm 1}}I, \]
respectively, where $\psi$ is the type of $W$.

Proof. Follows from the discussion below (15).

Lemma 5.11. Each of the matrices $a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, \Lambda^{\pm 1}$ is a central element of $T$.

Proof. By our discussion at the end of Section 4, each of $\{e_\psi\}_{\psi \in \Psi}$ is a central element of $T$. The result follows in view of Definition 5.5 and Definition 5.8.

Theorem 5.12. Assume that each irreducible $T$-module is thin. Then there exists a unique $C \in T$ such that
\begin{align*}
A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} &= \frac{(a + a^{-1})(\Lambda + \Lambda^{-1}) + (b + b^{-1})(c + c^{-1})}{q + q^{-1}}, \quad (21) \\
B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} &= \frac{(b + b^{-1})(\Lambda + \Lambda^{-1}) + (c + c^{-1})(a + a^{-1})}{q + q^{-1}}, \quad (22) \\
C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} &= \frac{(c + c^{-1})(\Lambda + \Lambda^{-1}) + (a + a^{-1})(b + b^{-1})}{q + q^{-1}}. \quad (23)
\end{align*}

Proof. Define $C$ such that (23) holds. Then $C \in T$ since $T$ contains $A, B$ as well as $a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, \Lambda^{\pm 1}$. We now verify (21), (22). To do this, it suffices to show that (21), (22) hold on each irreducible $T$-module. But this follows from the construction and Theorem 2.6.

Theorem 5.13. Assume that each irreducible $T$-module is thin. Then each of
\[ A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} \]
is central in $T$. The matrices $A, B$ are from (16) and $C$ is from Theorem 5.12.
Proof. In each equation (21)–(23), the matrix on the right is a central element of $T$ by Lemma 5.11.

Theorem 5.14. Assume that each irreducible $T$-module is thin. Then there exists a surjective $\mathbb{C}$-algebra homomorphism $\Delta_q \to T$ that sends

$$A \mapsto A, \quad B \mapsto B, \quad C \mapsto C.$$  

The matrices $A, B$ are from (16) and $C$ is from Theorem 5.12.

Proof. By Theorem 5.13 and since $T$ is generated by $A, B$.

Theorem 5.15. Assume that each irreducible $T$-module is thin. Then the standard module $V$ becomes a $\Delta_q$-module on which the $\Delta_q$-generators $A, B, C$ act as follows:

$$\begin{array}{c|ccc}
\Delta_q\text{-generator} & A & B & C \\
\text{action on } V & \Delta_q & \Delta_q & \Delta_q
\end{array}$$

The matrices $A, B$ are from (16) and $C$ is from Theorem 5.12.

Proof. Pull back the $T$-module structure on $V$, using the homomorphism $\Delta_q \to T$ from Theorem 5.14.

6 2-Homogeneous bipartite distance-regular graphs

In this section we work out an example. Throughout this section we assume that $\Gamma$ is bipartite and 2-homogeneous in the sense of Nomura [10], but not a hypercube. For a thorough description of such $\Gamma$, see [4], [5]. By [4] Theorem 42 $\Gamma$ is $Q$-polynomial. Let $\{E_i\}_{i=1}^D$ denote a $Q$-polynomial ordering of the nontrivial primitive idempotents of $\Gamma$. Let $\{\theta_i\}_{i=0}^D$ and $\{\theta^*_i\}_{i=0}^D$ denote the corresponding eigenvalue and dual eigenvalue sequences. The following lemma summarizes the properties of $\Gamma$ that we need.

Lemma 6.1. [4] Corollary 43. The following (i), (ii) hold.

(i) There exists a nonzero $q \in \mathbb{C}$ such that $q^4 \neq 1$ and

$$\theta_i = \frac{q^{D-2} + q^{2-D}}{q^2 - q^{-2}}(q^{D-2i} - q^{2i-D}) \quad (0 \leq i \leq D). \quad (25)$$

(ii) $\theta^*_i = \theta_i \quad (0 \leq i \leq D)$.

Corollary 6.2. The above $Q$-polynomial structure has $q$-Racah type, with $w = w^* = 0$ and

$$v = v^* = -u = -u^* = \frac{q^{D-2} + q^{2-D}}{q^2 - q^{-2}}. \quad (26)$$

Note 6.3. The scalar $q$ in [5] corresponds to our $q^2$.

Recall the subconstituent algebra $T = T(x)$ from Section 3. The irreducible $T$-modules are described in the following lemma.
Lemma 6.4. [5, Theorem 4.1, Lemma 4.2]. Let $W$ denote an irreducible $T$-module, with endpoint $\rho$. Then the following (i)–(iv) hold.

(i) $0 \leq \rho \leq \lfloor D/2 \rfloor$.

(ii) Up to isomorphism, $W$ is the unique irreducible $T$-module with endpoint $\rho$.

(iii) $W$ is thin.

(iv) $W$ has dual endpoint $\rho$ and diameter $D - 2\rho$.

In Section 5 we displayed a $\Delta_q$ action on the standard module of $T$. In this section we describe the $\Delta_q$ action in more detail. Fix $i \in \mathbb{C}$ such that $i^2 = -1$.

Lemma 6.5. Recall the scalars $a, b$ from above Lemma 5.3. Then $a, b \in \{i, -i\}$. Moreover,

$$A = A \frac{q^2 - q^{-2}}{a(q^{D-2} + q^{-2} - D)}, \quad B = A^* \frac{q^2 - q^{-2}}{b(q^{D-2} + q^{-2} - D)}. \quad \quad (27)$$

Proof. By (26) and the construction $a^2 = u/v = -1$, so $a \in \{i, -i\}$. Similarly $b \in \{i, -i\}$. By Corollary 6.2 and (16) we obtain (27).

Lemma 6.6. Let $W$ denote an irreducible $T$-module. Then

$$a(W) = a, \quad b(W) = b. \quad \quad (28)$$

Moreover

$$c(W) \in \{i, -i\}. \quad \quad (29)$$

Proof. Let $\rho$ denote the endpoint of $W$. By Lemma 6.4, $W$ has dual endpoint $\rho$ and diameter $d = D - 2\rho$. Moreover $W$ is thin. By Lemma 5.4 we obtain $a(W) = a$ and $b(W) = b$. We now show (29). First assume that $d = 0$. Setting $\kappa = 0$ in (5) we find $c(W)^2 = -1$, so (29) holds. Next assume that $d \geq 1$. By Lemma 5.4, the sequence $(A, \{E_i\}_{i=0}^{\rho-1}, B, \{E_i^*\}_{i=\rho}^{\rho+1})$ acts on $W$ as a Leonard system, which we denote by $\Phi$. To find $c(W)$ we apply (3), (4) to $\Phi$. We first obtain $\phi_1(\Phi)$ using (3). By (3) and since $\Gamma$ is bipartite,

$$\phi_1(\Phi) = -(\theta_0(\Phi) - \theta_1^*(\Phi))\theta_d(\Phi) = -(bq^{-d} + b^{-1}q^d - bq^{2-d} - b^{-1}q^{-2-d})(aq^d + a^{-1}q^{-d}) = ab(q - q^{-1})(q^{d-1} + q^{-d-1})(q^d - q^{-d}).$$

By this and (5) we obtain $\kappa(\Phi) = 0$. Therefore $c(W)^2 = -1$ in view of (5), and (29) follows.

Lemma 6.7. We have $a = aI$ and $b = bI$.

Proof. We show that $a = aI$. The element $a$ is given in Definition 5.5. By (28), $a(\psi) = a$ for all $\psi \in \Psi$. We mentioned at the end of Section 4 that $I = \sum_{\psi \in \Psi} e_\psi$. By these comments $a = aI$. One similarly shows that $b = bI$.

Lemma 6.8. We have

$$a + a^{-1} = 0, \quad b + b^{-1} = 0, \quad c + c^{-1} = 0. \quad \quad (30)$$

Proof. The first two equations follow from Lemma 6.7 and $a, b \in \{i, -i\}$. The third equation follows from Definition 5.8, Lemma 5.9, and (29).
Theorem 6.9. Let $C$ be as in Theorem 5.12. Then

$$A + q \frac{BC - q^{-1}CB}{q^2 - q^{-2}} = 0,$$
$$B + q \frac{CA - q^{-1}AC}{q^2 - q^{-2}} = 0,$$
$$C + q \frac{AB - q^{-1}BA}{q^2 - q^{-2}} = 0.$$

(31)  
(32)  
(33)

Proof. Evaluate (21)–(23) using Lemma 6.8.

Remark 6.10. The equations (31)–(33) can be obtained directly from [5, Lemma 3.3].

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