Ballistic annihilation in one dimension: a critical review

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Abstract. In this article, we review the problem of reaction annihilation $A + A \rightarrow \emptyset$ on a real lattice in one dimension, where $A$ particles move ballistically in one direction with a discrete set of possible velocities. We first discuss the case of pure ballistic annihilation, that is a model in which each particle moves simultaneously at constant speed. We then review ballistic annihilation with superimposed diffusion in one dimension. This model consists of diffusing particles each of which diffuses with a fixed bias, which can be either positive or negative with probability $1/2$, and annihilate upon contact. When the initial concentration of left- and right-moving particles is same, the concentration $c(t)$ decays as $t^{-1/2}$ with time, for pure ballistic annihilation. However, when the diffusion is superimposed decay is faster and the concentration $c(t) \sim t^{-3/4}$. We also discuss the nearest-neighbor distance distribution as well as crossover behavior.

1 Introduction

The dependence of chemical kinetics on the dynamics of the reacting particles has attracted considerable attention [1,2]. In all these works, one of the most important objective is to determine how the laws of motion influence the space-time evolution of the species concentrations. For irreversible diffusion-controlled reactions, when reactants diffuse and react on contact, the reaction creates spatial correlations that invalidate the mean-field approximation and hence the applicability of rate equations. For instance, consider the case of one-species annihilation (or aggregation) where one species reacts with itself via the bi-molecular reaction,

$$A + A \rightarrow C,$$

where $C$ is an inert species. The rate equation reads:

$$\dot{c}_A = -Kc_A^2.$$  \hfill (2)

According to the rate equation, the $A$ concentration $c_A(t)$ decays as $1/t$ at long times. However for the same process where $A$ walkers diffuse randomly in one dimension, the concentration decay is given by $(Dt)^{-1/2}$ [3]. Also the amplitude in Eq. (2) depends on the initial concentration, whereas the amplitude of the correct $1/\sqrt{t}$ decay only depends on the diffusion constant.

Beyond the immediate interest of such models for chemical reactions, the dynamical behavior of particle annihilation ($A + A \rightarrow \emptyset$) and particle aggregation ($A + A \rightarrow A$) have a direct one-to-one correspondence with the kinetic Ising and Potts model in one dimension. In that case, the zero-temperature Glauber dynamics can be mapped to a random walk problem for both the $q$ state nearest neighbor Potts model and Ising model ($q = 2$). In both cases domain walls perform random walks, and whenever the walkers meet, they either coagulate ($A + A \rightarrow A$) or annihilate ($A + A \rightarrow \emptyset$). Two domain walls will annihilate each other by contact, if the two spins that surround that domain wall are in same state. Specifically, for the Ising model ($q = 2$ state Potts model), the domain walls always annihilate. For the $q$-state Potts model with random initial conditions, they will annihilate with a probability $\frac{1}{q-1}$ and will coagulate with probability $\frac{q-2}{q-1}$. In either case, the domain walls perform pure random walks. The exponent for the concentration decay is identical to that of the domain growth exponent [4,5] or ordering exponent [6]. However, for several binary opinion dynamic models (which can be directly mapped to the Ising spin system) in one dimension, the corresponding walker pictures are much more complicated than a simple random walk. Sometimes the motion of the $A$ walkers can also have ballistic components [7–9]. Hence, understanding the ballistic annihilation with and without superimposed diffusion could also be useful for understanding these more complex dynamics.
Here, in this article, we review the problem of ballistic annihilation in one dimension. The main aim is to determine the decay law for the concentration of the particles for this single species reaction-annihilation problem. First, in Sect. II, we discuss the ballistic annihilation \((A + A \rightarrow \emptyset)\) where the \(A\) particles move in a purely ballistic way on the real line with velocities which can take only two values, \(v\) and \(-v\). We assume the particles to have always one of a discrete set of velocities. This model can also be simulated on a lattice, if we assume that all particles always move in one direction and if their positions are updated \(synchronously\). We shall thus refer to this model as the synchronous update model. In Sect. III, we talk about the ballistic annihilation with superimposed diffusion in one dimension. This model can be obtained, for instance, by modifying the update rule of the previous model to being \(asynchronous\). The conclusion and discussions are stated in the final section.

2 Synchronous Ballistic annihilation

2.1 Particles with only two velocities \(+v\) and \(-v\)

The purely ballistic model, that is the Ballistic annihilation under the synchronous update rule was first discussed by Elskens and Frisch [10]. In this model, particles are initially distributed randomly on a one dimensional lattice and move freely with independent initial velocities. In this single species annihilation problem whenever two particle meet they annihilate \((A + A \rightarrow \emptyset)\). The velocity distribution function, that is, the probability that the velocity is \(v\) or \(-v\), can be written as

\[
D(w) = q\delta(w + v) + p\delta(w - v), \quad p + q = 1. \tag{3}
\]

The main interest is to determine the decay law for the concentration decay of the particles.

We will now look at the problem in a little more detail. Initially let the position and velocity of the \(k\)th particle be \(x_k\) and \(v_k\). The particle positions are initially random and the velocities are independent. What we want to calculate is the average survival probability \(S(t)\) of the particles at time \(t\), as the concentration decay is identical to the decay of survival probability with time.

In this case two particles which are moving in the same direction move with the same velocity and, thus, cannot annihilate (Fig. 1a). Since, because of the annihilation process, particles cannot cross each other’s trajectories, it follows that the entire distribution of velocities of surviving particles to the right of a given surviving particle (let us denote the index by \(k\)) is independent of the whole distribution to its left.

The time at which any given particle, say a right moving particle, will annihilate is readily determined as follows, if the initial conditions are given: let us denote the given particle by the index 0 and place it at the origin. We further assign the index \(k\) to the \(k\)th particle to the right which is located at \(x(k)\), and compute

\[
S(k) = \sum_{l=0}^{k} \text{sign}(v_k). \tag{4}
\]

Denote by \(k_0\) the smallest value of \(k\) such that \(S(k_0) = 0\). Note that \(k_0\) is necessarily even. We then find that the annihilation time for the particle 0 is given by

\[
T_0 = x(k_0)/(2c). \tag{5}
\]

The probability density \(p(t)dt\) for \(T_0\) is, thus, given by

\[
p(t)dt = \sum_{k=1}^{\infty} \text{Prob} [S(k) = 0 \text{ for the first time}] \cdot \text{Prob} \left[2ct \leq \sum_{l=1}^{k} (x_{l+1} - x_l) \leq 2c(t + dt) \right]. \tag{6}
\]

This sum is readily evaluated as follows: the first factor in the product is a well-known problem in probability, it is the so-called gambler’s ruin problem, whereas the second is simply the expression for the distribution of the distance of the \(k\)th neighbor to the origin, given the nearest neighbor distance distribution. By the law of large numbers, if this distribution has a finite mean \((\langle l \rangle)\) and variance \(\sigma^2\), the second factor, for large \(k\), is a Gaussian of average \(k\langle l \rangle\) and variance \(k\sigma^2\).

The behavior of the first factor is somewhat non-trivial but well known, see for instance [12]. When \(p = q = 1/2\), the asymptotic behavior of this quantity goes a a power law \(k^{-3/2}\). It corresponds to the probability that an unbiased random walk remains to the right of the origin for \(k\) steps. On the other hand, if \(p > q\), the random walk is biased and with probability one eventually remains forever on the right side of the origin, in other words, right-going particles have a finite probability of surviving indefinitely.
For $p = 1/2$ and an initial exponential nearest neighbor distance distribution, the expression [6] can be evaluated without any approximation [10]. The survival probability is given by

$$S(t) = \int_{t}^{\infty} p(t') dt' = e^{-2\rho vt}[I_0(2\rho vt) + I_1(2\rho vt)],$$

(7)

where $I_0$ and $I_1$ are the modified Bessel’s functions and $\rho = \langle t \rangle^{-1}$. Asymptotically for $t \to \infty$, $S(t)$ decays in a power law fashion:

$$S(t) = \frac{1}{\sqrt{\pi \rho vt}} \left[ e^{-\rho vt} + O(t^{-1}) \right].$$

(8)

For $p > q$, on the other hand, the expressions for the probability of a random walk remaining for $k$ steps to the right of the origin become significantly more intricate and an explicit expression cannot be found. The asymptotic behavior, however, is readily evaluated: it is an exponential decay to zero of the concentration of the minority species (in this case, the left-going particles), and a corresponding exponential saturation of the majority species concentration to an asymptotic non-zero value of $p - q$. This exponential approach is further corrected by a power-law factor:

$$S(t) - S_\infty \approx (pq)^{1/4}(\rho vt)^{3/2}\exp(-2\rho vt),$$

(9)

where $r = 1 - 2\sqrt{q/p}$ is a quantity that measures the strength of the bias.

For the case $p = q = 1/2$, one can physically argue the decay law at long time as follows: $S(t)$ is mostly determined by the difference between the positive and negative velocity particles in an interval of time $2vt$. As the particles are initially distributed over the lattice independently with density $\rho$, the difference between the positive and negative velocity particles is of the order of $(2\rho vt)^{-1/2}$ for the population of order $2\rho vt$. Hence, the concentration of the particles at time $t$ (for $t \to \infty$) is given by

$$c(t) = \rho S(t) \simeq \sqrt{\frac{c(0)}{\pi vt}},$$

(10)

where $c(0) = \rho$ is the initial concentration of the particles.

We can expand the argument given in the previous paragraph to give a simple derivation for the concentration decay using scaling analysis [11]. If the size of the one-dimensional lattice is $L$, the initial number of particles is $c(0) L$. The difference between the number of left-moving and right-moving particles is of the order of

$$\Delta N = |N_+ - N_-| \sim \sqrt{N}.$$  

(11)

At long time ($t \to \infty$), all the particles which belong to the minority-velocity species are annihilated and hence $c(L, t \to \infty) \sim \Delta N/L \sim (c(0)/L)^{1/2}$. We can assume a scaling form for the concentration

$$c(L, t) \sim (c(0)/L)^{1/2} g(z),$$

(12)

with $z = L/\sqrt{vt}$. Following the above argument, we have

$$g(z) \to \text{constant} \quad (t \to \infty, z \to 0).$$

(13)

On the other hand in the short time limit, that is for $z \to \infty$, the concentration cannot depend on the size of the lattice, which implies

$$g(z) \propto z^{1/2}.$$  

(14)

Thus, we find

$$c(t) \sim \sqrt{\frac{c(0)}{vt}},$$

(15)

an expression similar to that of the solution given by equation (10).

### 2.2 The case of three velocities

Matters become rather more intricate in the case of three velocities. If we have particles moving with velocities $v_1, v_2,$ and $v_3$ we may suppose, using the Galileian invariance of the process, that one of the velocities is equal to zero, say the intermediate velocity. To limit the number of parameters, let us assume that the two extreme velocities are opposite, that is, we have velocities $\pm v$ and 0. Let the probability distribution be

$$D(w) = p_{-}\delta(w + v) + p_{0}\delta(w) + p_{+}\delta(w - v) \quad (p_{-} + p_{0} + p_{+} = 1).$$

(16)

This system had been studied numerically in [13]. Qualitatively, the behavior is the following: whenever the concentrations of one type of particles clearly dominates all others, only those particles that have higher concentrations will eventually survive. The phase diagram is, therefore, divided into three phases, in which either the $+$ particles, the $-$ particles or the immobile particles survive. Each phase has a boundary with the other two, and the three have only one point in common, which is numerically found to be very close to $p_{\pm} = 3/8$ and $p_0 = 1/4$.

We now describe the behavior of each species in different regions of the phase diagram. Within the regions in which only $\pm$ or immobile particles survive, the approach to equilibrium is exponential. In the boundary separating the $+$ from the $-$ region, we have

$$c_{\pm}(t) \simeq t^{-1/2}; \quad c_0(t) \simeq t^{-1}.$$  

(17)
the case of different problem is the case of ballistic aggregation in
is an amazingly involved [17]. The results fit closely the numerics.

The approach is not straightforward, however. In principle, it appears that any ballistic aggregation system can be in principle analyzed in this manner. However, even the detailed analysis of the three-velocity ballistic annihilation problem studied numerically in [13] is an amazingly involved tour de force. An altogether different problem is the case of ballistic aggregation in the case of continuous velocity distributions. This was treated within a Boltzmann equation approximation in [17]. The results fit closely the numerics.

3 Ballistic annihilation with superimposed diffusion

3.1 Description of the model

In this section, we review the problem of ballistic annihilation with superimposed diffusion for a two-velocity case, in which particles can have only the velocities, ±v. Initially the fraction of positive velocity particles are \( p = 1/2 + \varepsilon \) and the negative velocity particles are \( q = 1/2 - \varepsilon \). The particles are randomly distributed over the one dimensional lattice as usual. Most importantly, additionally to the ballistic particle motion we have a diffusive motion for the particle. This may, for instance, arise from an asynchronous update for the Monte Carlo Simulation. At each time step, one particle is chosen at random and moved in the direction of the corresponding velocity of that particle. After \( N \) such update one Monte Carlo time step is over, if the total number of particles over the lattice is \( N \). We consider the one species annihilation, that is whenever two particles meet they annihilate (\( A + A \rightarrow \emptyset \)) [11, 18].

The crucial difference between this model and the purely ballistic model is the possible annihilation between particles having the same velocity. In other words, the purely ballistic model involves two particle species (left-going and right-going) which cannot react. In the model with diffusion, however, the particles of the same species have another mechanism, namely diffusion, that allows them to annihilate. Since the two mechanisms scale differently, however, the resulting model is in a different universality class from either the ballistic or the purely diffusive model. A trivial example of the difference is given by the case in which \( p \) is significantly larger than \( q \): in a time of order one, the purely ballistic model reaches a limiting concentration, whereas the model involving diffusion decays as \( t^{-1/2} \).

For \( \varepsilon = 0 \), that is, for \( p = q \), at the beginning, the numbers of positive and negative velocity particles are equal on an average. When \( \varepsilon \neq 0 \left( -1/2 \leq \varepsilon \leq 1/2 \right) \) introduces inequality in the initial numbers of positive and negative velocity particles, and \( \varepsilon = \pm 1/2 \) means there is only one kind of particle.

Unlike in the previous section, we study this problem in discrete time and at each time step we choose a particle at random and move it one step to the right (or left) if the velocity of the particle is positive (or negative). In the asynchronous update, the random choice of particle leads to an effective diffusive motion of the particles with respect to their neighboring particles that have the same speed. This asynchronous update rule introduces diffusion so that, as remarked above, two particles with the same velocity can annihilate.

Let us here summarize the effects of this apparently minor change in the update rule. These are non-trivial and profound. For \( \varepsilon > 0 \left( \varepsilon \ll 1 \right) \), there are three different regimes (Fig. 2). In the first stage, we can neglect the concentration difference between left- and right-going particles. The common concentration \( c(t) \) then decays, as we will show, as \( t^{-3/4} \). This regime ends when the minority (that is left-going) particles begin to have as significantly different concentration fro the
majority. This is followed in short order by their complete disappearance. This happens at a time $t_1(\varepsilon)$ of the order of $\varepsilon^{-2}$ (Fig. 2). At this point, starts the second stage which is characterized by the decay of concentration as $t^{-1/4}$ (Fig. 2). This is due to the fact that the distribution of the positions of the surviving right going particles over the one dimensional lattice are far from being uniform, leading to an anomalous decay [19]. Finally, a third stage set in, when the distribution of the particles over the lattice becomes uniform due to the presence of diffusion in the system, which happens at a time $t_2(\varepsilon)$ that scales as $\varepsilon^{-4}$ (Fig. 2). In this third stage, the system contains only the right-moving particles moving ballistically, with superimposed diffusion. Due to Galilean invariance, this is equivalent to pure diffusive dynamics. Hence, the concentration decay as $(Dt)^{-1/2}$ asymptotically.

Even for $\varepsilon = 0$, there will be an inequality in the right and left going particle due to the random distribution of the particles at the beginning. For $\varepsilon = 0$, the concentration of an excess number of particles, $c_{ex}(0) \approx 1/\sqrt{N}$ (up to an $O(1)$ factor), where $N$ is the initial number of particles. Hence we can see both the first and second stages there, but the crossover time diverges with the system size. In the next subsection, we will discuss the initial $t^{-3/4}$ decay using dimensional analysis mostly for $\varepsilon = 0$.

### 3.2 Dimensional analysis for $\varepsilon = 0$: infinite system size

Due to the convection, after some time, particles organize themselves into domains of right- and left-moving particles. However because of asynchronous update rule, inside each domain of same-velocity particles, diffusive annihilation takes place. One could therefore with some plausibility argue that the diffusive annihilation mechanism leads to an effective time-dependent initial concentration $c_0(t) \sim (Dt)^{-1/2}$, which plays the role of $c(0)$ in Eq. (15), leading to the $t^{-3/4}$ decay of the concentration with time.

However, in this subsection, we present a more thorough approach to this system. Specifically, we shall use dimensional analysis, or scaling, to derive the $t^{-3/4}$ decay law, for $\varepsilon = 0$. We follow the argument presented in [18]. We define two dimensionless parameters:

$$x = \frac{v}{Dc(0)}, \quad (18a)$$
$$\tau = \frac{v^2 t}{D}, \quad (18b)$$

where $c(0)$ is the initial concentration, $D$ is the diffusion constant and $v$ is the velocity of the particles. The meaning of these quantities is as follows: $x$ is the ratio of two times: the time needed to cross the initial inter-particle separation ballistically, and that needed to cross the same distance diffusively. On the other hand, $\tau$ is a dimensionless time (Eq. 18b), given by the ratio of $t$ to the time scale $\tau$, above which ballistic motion dominates diffusion. Hence at times corresponding to $\tau \ll 1$, diffusion dominates, whereas for $\tau \gg 1$, drift does.

Now, the concentration of the particles as a function of $x$ and $\tau$, at time $t$ can be written as:

$$c(t) \approx c(0)\Phi(x, \tau). \quad (19)$$

If $x \ll 1$, $c(t)$ cannot depend on $v$ at small times, since then $\tau \ll 1$ and everything is dominated by diffusion. That implies that $c(t) \sim (Dt)^{-1/2}$ for small $t$. Hence, using Eqs. (18a) and (18b) we get

$$\Phi(x, \tau) = \frac{x}{\sqrt{\tau}}. \quad (20)$$

This should be valid until $\tau \sim 1$, as this is the crossover time between drift and diffusion.

Now consider the case $x \gg 1$. In that case, since time is discrete, $\tau \gg 1$ for all times. It may, therefore, appear that the dynamics is purely ballistic since $\tau \gg 1$; $c(t)$ is, thus, independent of $D$ in this regime, and it is given by $\sqrt{c(0)/vt}$, as discussed in the previous section. In this case, again using Eqs. (18a) and (18b), we get

$$\Phi(x, \tau) = \sqrt{\frac{x}{\tau}}. \quad (21)$$

The structure resulting from this dynamics is, therefore, that of the ballistic aggregation without diffusion studied in the previous Section. I this model, however, there form large domains of particles moving at the same velocity, which do not react in the pure ballistic model.

Now, due to the presence of the diffusion, the particles forming such domains can annihilate via diffusion. Since the particles in a domain have never interacted, we may roughly assume their separation to be given, in terms of the initial concentration, by $1/c(0)$. Particles having the same speed, therefore, react on a time scale $1/(Dc(0)^2)$, which in terms of the dimensionless quantities, is $\tau \sim x^2$. Hence, Eq. (21) will be valid until time $\tau \sim x^2$. The validity of (21) is, therefore, limited to finite, though large, values of $\tau$.

As explained above, the previous Eqs. (20) and (21) only hold for finite values of $\tau$. Now let us describe the large $\tau$ behavior for the full dynamics. In that case, we can write

$$\Phi(x, \tau) \approx x^\alpha \tau^\beta. \quad (22)$$

When $\tau \sim 1$ and $x \ll 1$, we can use both equation (20) and (22), which leads to

$$x \sim x^\alpha.$$
That gives \( \alpha = 1 \). Similarly, if \( x \gg 1 \) and \( \tau \sim x^2 \), we can use the Eq. (21) as well as (22), which lead us to
\[
\sqrt{\frac{x}{\tau}} \sim x^{-1/2} \sim x^{\alpha + 2\beta} = x^{1+2\beta},
\]
and that give us
\[
\beta = -\frac{3}{4}.
\]
Putting the value of \( \alpha \) and \( \beta \) in Eq. 22 we get
\[
\Phi(x, \tau) \simeq x^{\tau^{-3/4}}.
\]
From this, we finally get that for large time \((t \to \infty)\) the concentration of the particles \( c(t) \) will have the following form :
\[
c(t) \simeq v^{-1/2}D^{-1/4}t^{-3/4}.
\]
(23)

Now, we will see an alternative way to determine the decay law using dimensional analysis in a little different way. As we have already discussed the system can be completely characterized by the initial concentration \( c(0) \), velocity \( v \) and diffusion constant \( D \). From these parameters, one can get different combinations with the dimensions of concentrations, which are \( c(0) \), \( 1/vt \) and \( 1/\sqrt{Dt} \). From this, one can anticipate that these three concentration scales could be written multiplicatively to express the time-dependent concentration in a conventional scaling form. Hence, we can write the time-dependent concentration as follows :
\[
c(t) \sim c(0)^\rho \left( \frac{1}{vt} \right)^\sigma \left( \frac{1}{\sqrt{Dt}} \right)^{1-\rho-\sigma}.
\]
(24)
The exponents \( \rho \) and \( \sigma \) can be obtained by the following argument: first, let \( t < \tau_v \simeq D/v^2 \). Here, \( \tau_v \) is the characteristic time below which the diffusion dominates over drift. The particle then undergoes (weakly) biased diffusion, which leads to
\[
c(t) \sim (Dt)^{-1/2},
\]
(25)
independent of \( c(0) \). At \( t = \tau_v \), putting the expression of \( c(t) \) and \( \tau_v \) in Eq. (24), we obtain \( \rho = 0 \). Secondly, consider \( \tau_v < t < \tau_D \simeq 1/(Dc(0)^2), \) where \( \tau_D \) is the time needed for two neighboring same velocity particles to meet by diffusion. In this case, the dynamics is mostly ballistic and hence
\[
c(t) \sim \sqrt{c(0)/(vt)}.
\]
(26)
Again at \( t = \tau_D \), putting the expression of \( c(t) \) and \( \tau_D \) in Eq. (24), we obtain \( \sigma = 1/2 \). Using the value of \( \rho \) and \( \sigma \) in Eq. (24), one can get back the same expression for \( c(t) \) at large time which is given by Eq. (23).

Note that so far, we have not at all taken into account the possibility of unequal concentrations of left-going and right-going particles. Similarly, we have not considered the effect of a finite system size. We now turn to these issues.

### 3.3 The case of finite system size: second stage of dynamics

In an infinite system, it is readily seen that the arguments of Sect. 3.2 hold for all times. For finite systems, another regime appears, which we now describe.

When all the minority particles have been annihilated, the only reaction mechanism remaining is diffusive annihilation. One might thus be tempted to conclude that, for finite systems, we merely have an additional final stage in which ordinary diffusive annihilation occurs, with a decay of \( 1/\sqrt{Dt} \). As we show, matters are a bit less straightforward.

At the beginning of this stage, that is, when all minority particles have disappeared, the distribution of all the surviving particles (which are a fraction of the majority particles) over the one dimensional lattice is far from being uniform. This leads to the next decay law. In this subsection, we describe by simple arguments, the spatial distribution of the particles over the one-dimensional lattice, and argue that it leads to \( t^{-1/4} \) decay. We follow the arguments given in [18], to which the reader is referred for details.

Let us start with the behavior of the surviving particles in ballistic annihilation with synchronous updating. Once the initial condition is set at random, each particle has a unique annihilation partner. Otherwise, the particle survives forever. Initially, the distance between the particle (positive velocity) situated at the 4th lattice site and the unique annihilation partner of that particle be \( \pi_+(k) \). \( k \) runs from 0 to \( L \), where \( L \) is the length of the one-dimensional lattice. Once the initial configuration is fixed each particle survives until it encounters its reaction partner. Hence, the collision time is
\[
\tau_+(k) = \frac{\pi_+(k)}{2}. \tag{27}
\]
Here, we have considered the positive velocity particles. The entire description is also valid for the negative velocity particles. In that case we would say that the initial distance of the unique annihilation partner be \( \pi_-(k) \).

We can now try to determine the structure of the set \( \Sigma_+ \), which is defined as
\[
\Sigma_+ = \{k : \tau_+(k) > t\}. \tag{28}
\]
Let us consider that \( k \) is the distance between two positive velocity particles. Without loss of generality, we may assume that we have two particles, one at 0 and the other one at \( k > 0 \).

If \( k > t \), then the two intervals \([0, t]\) and \([k, k + t]\) are disjoint. The probability that both sites belong to \( \Sigma_+ \) is
simply the product of either site belonging to \( \Sigma_t \) and hence, no dependence on \( k \) exists. This is independent of the fact whether we consider \( \varepsilon = 0 \) for the initial configuration or not.

On the other hand, for \( k < t \), the situation is little different. Let us assume a virtual random walk that takes a step to the right whenever there is positive velocity particle in the initial configuration and takes a step to the left if the velocity of the particle is negative. For \( \varepsilon = 0 \), the walk is symmetric. Hence, for \( \varepsilon = 0 \), the probability of having two particles separated by a distance \( k < t \) and both surviving a time \( t \) is equivalent to the the probability that a symmetric random walk, which starts at the origin and takes a step to the right whenever there is positive velocity particles due to the presence of diffusion, eliminates all (well, actually, most of) the particles with distance less than \( \sqrt{t} \). Hence, the corresponding set of surviving particles becomes a fractal set of dimension 1/2 with lower cutoff \( \sqrt{t} \) and upper cutoff \( t \).

For \( \varepsilon > 0 \), the argument expanded so far is still valid. The main difference is that now the virtual random walk is biased. That virtual walker now takes a step to the right with probability 1/2 + \( \varepsilon \), whereas a step to the left has probability 1/2 - \( \varepsilon \). For \( k < t \), the probability that two particles, one at 0 and another at \( k \) both belong to \( \Sigma_t \), is still given by the probability (say \( p_c(k) \)), that a random walk (biased now), which starts at the origin and takes a step to the right, does not return to the origin before time \( k \).

The probability \( p_c(k) \) saturates to a positive value \( p_c(\infty) \) as \( k \to \infty \). This saturation happens when \( k \sim k_c(\varepsilon) \approx \varepsilon^{-2} \) as \( \varepsilon \to 0 \) [20, 21] and when \( \varepsilon \to 0 \) we have \( p_c(\infty) \approx \varepsilon \). The set \( \Sigma_t \) is, therefore, a fractal set with a cutoff which is either at \( t \) or at \( \varepsilon^{-2} \), whichever is smaller.

From the arguments given above, we see that the particles surviving at time \( t_1 \) (Fig. 2), at which only one species (either \( +e \) or \( -e \) velocity particles) survives, lie on a fractal of dimension \( d_f = 1/2 \). This is also true for both \( \varepsilon = 0 \) and \( \varepsilon > 0 \) in the initial condition.

The fact that \( \Sigma_t \) is a fractal further implies that the second stage of the dynamics is purely diffusive (as only one species survives), starting from an initial condition where the particles are distributed on a fractal of dimension \( d_f \) (0 < \( d_f < 1 \)). The probability distribution function for this initial inter-particle distances for two nearest particles has, for \( x \gg 1 \), a behavior similar to that of the distribution

\[
P(x) = \frac{1}{\zeta(\lambda)} \sum_{l=1}^{\infty} l^{-\lambda} \delta(x - l),
\]

where \( \lambda = d_f + 1 \). For the purposes of the asymptotic estimations we shall be dealing with, \( P(x) \) and the exact probability distribution can be used interchangeably. Considering this discrete distribution given by Eq. (29), and following the formalism developed in [19], one can obtain the number of particles \( n(t) \) at time \( t \) (normalized by initial number of particles \( n(0) \)) which is given by

\[
n(t) = -\frac{\Gamma(1 - \lambda)}{2\zeta(\lambda)\Gamma(\frac{3 - \lambda}{2})} \left( \frac{\lambda - 1}{2} \right) + \frac{\Gamma(1 - \lambda)}{4(1 - \lambda)[\zeta(\lambda)]^2} \tau^{-\lambda - 1} + \frac{\zeta(\lambda - 1)}{2\zeta(\lambda)\sqrt{\pi}} \tau^{-1/2} + O(\tau^{-\lambda/2}),
\]

where \( \tau = D_{eff} t \). See Appendix in [18] for details. In our case the fractal dimension \( d_f = 1/2 \), and hence \( \lambda = 3/2 \). The leading term of the decay law is \( t^{-1/4} \) and the coefficient of this leading term is positive, as \( \Gamma(1 - \lambda) \) is negative for all \( \lambda > 1 \). Putting the value of \( \lambda \) in Eq. (30), we get

\[
n(t) = \frac{\sqrt{\pi}}{\zeta(3/2)\Gamma(3/4)} \tau^{-1/4} + \frac{(\sqrt{\pi})}{\zeta(3/2)^2} \left( \frac{\sqrt{\pi}}{2\sqrt{\pi}\zeta(3/2)} \right) \tau^{-1/2} + O(\tau^{-3/4}) \approx 0.5537 \tau^{-1/4} + 0.102 \tau^{-1/2} + O(\tau^{-3/4}).
\]

Now, let us progress to determine the crossover times \( t_1(\varepsilon) \) and \( t_2(\varepsilon) \). \( t_1(\varepsilon) \) is the time at which the dynamics have a crossover from \( t^{-3/4} \) initial behavior, to the long-time \( t^{-1/4} \) behavior. Hence at \( t_1(\varepsilon) \), we can write

\[
at_1(\varepsilon)^{-3/4} = b(\varepsilon)t_1(\varepsilon)^{-1/4},
\]

where the coefficient \( a \) does not depend on \( \varepsilon \), as the decay at the beginning does not depend on the initial concentration. However \( b(\varepsilon) \sim \varepsilon \), as \( p_c(\infty) \sim \varepsilon \) for \( \varepsilon \to 0 \). Simplifying Eq. (32), we obtain

\[
t_1(\varepsilon) \sim \varepsilon^{-2}.
\]

For \( \varepsilon \to 0 \), this crossover time \( t_c \) will diverge and, hence, will scale with system size \( L \) for finite \( L \). We will still be able to see the \( t^{-1/4} \) decay due to the concentration of an excess number of particles, \( c_{ex}(0) \approx 1/\sqrt{N} \) for setting the initial configuration at random.

Though it is possible to see the \( t^{-1/4} \) decay for \( \varepsilon = 0 \) it is difficult as there remain very few particles at this late stage. However \( \varepsilon \neq 0 \), it is, respectively, easier to detect the \( t^{-1/4} \) (which is the leading term) decay of concentration on the fractal, as the remaining particles are more in this situation. As the dynamics is entirely diffusive at this stage, the fractal structure will eventually die out to uniform distribution. So, there will be another crossover after which the concentration will decay as \( t^{-1/2} \) at very long time. If we consider the
crossover time for the second crossover as \( t_2(\varepsilon) \), we can write

\[
b(\varepsilon)t_2(\varepsilon)^{-1/4} = c t_2(\varepsilon)^{-1/2},
\]

where the coefficient \( c \) is not a function of \( \varepsilon \) as it does not depend on the initial concentration. Simplifying the above expression, we get

\[
t_2(\varepsilon) \sim \varepsilon^{-4}.
\]

It is indeed rather challenging to measure this second crossover time \( t_2(\varepsilon) \), as the number of remaining particles is very low and the time very large.

In Sect. 3.4, we will show some of the numerical evidence supporting the theoretical claims described in this subsection and in Sect. 3.2.

### 3.4 Numerical evidences and scaling

If initially there are \( N \) number of particles, then at time \( t_1(\varepsilon) \), there remain \( N\varepsilon^{3/2} \) particles and at \( t_2(\varepsilon) \), the number of particles remain only \( N\varepsilon^{5/2} \). This makes some numerically challenging features of the model: if we wish to have a clean separation of time scales, we need at least \( \varepsilon \sim 10^{-2} \). In a different language, if we wish to have more or less hundred particles at time \( t_2(\varepsilon) \) (for being able to observe the final exponent reliably), we need to start with \( N \sim 10^7 \), which is unrealistic. Hence, to drag out the features of the system explained in the previous subsection one have to rely on various simulations with different values of the parameters and the corresponding scaling laws.

First, we see the decay of concentration for \( \varepsilon = 0 \) which shows the \( t^{-3/4} \) decay law at the beginning and then a crossover to \( t^{-1/4} \) decay (Fig. 3).

As we know, initial concentration for the excess particles \( c_{ex}(0) = 1/\sqrt{N} \). If the excess particles, which decay due to diffusive annihilation lie on a fractal of dimension \( 1/2 \) from the very beginning of the dynamics, then they will decay following the Eq. (31). The plot of Eq. 31 with the numerical data for \( c_{ex}(t) \) (Fig. 3) shows perfect agreement, supporting this assumption.

Next, we will show the distribution of inter-particle distances between two neighboring particles at late times, when the particles decay as \( t^{-1/4} \). \( P_s(l) \) and \( P_d(l) \) are the distribution functions of inter-particle distances between two neighboring particles with same and different velocities (Fig. 4). \( P_s(l) \sim l^{-3/2} \) for large \( l \) (Inset at the bottom of Fig. 4), which is evidence that inside a domain of same velocity particles, the particles are distributed over a fractal of dimension \( 1/2 \). For small \( l \), \( P_d(l) \sim l \) due to the diffusion-limited annihilation [22]. The distribution \( P_d(l) \) is almost flat at the beginning and then has an exponential decay (Top right inset of Fig. 4).

At the part of the exponential decay, the value of \( P_d(l) \) suddenly increases as two domains or fractals of same velocity particles are moving apart from each other making the probability of having some large value of \( l \) very high. \( P(l) \) is the general distribution function for inter-particle distances between two neighboring particles, independent of their velocity (Main plot of 4).

Next, we will show the numerics for \( 0 < \varepsilon < 1/2 \). Though for \( \varepsilon = 0 \), there exist three different dynamical regimes, due to the finiteness of the system, the regimes are not that well separated and hence, the crossover times are not very clean. We will mostly discuss the scaling behaviors involving \( \varepsilon \) and \( t \), which hold for \( |\varepsilon| \ll 1 \), for two crossovers in this article. For details of the numerics, one should consult [18].
The scaling function for the first two dynamical regimes can be written as

\[ c(\varepsilon, t) \sim \varepsilon^{2\delta} f(\varepsilon^2 t), \quad \text{for } \varepsilon \ll 1, \tag{35} \]

where \( f(x) \to x^{-\delta} \) with \( \delta = 3/4 \), for \( x \ll 1 \) and \( f(x) \to x^{-1/4} \) when \( x \gg 1 \). The raw data as well as the scaled data using \( \delta = 3/4 \) are shown in Fig. 5. The collapse is good for the first two dynamical regimes where the exponent values are \( 3/4 \) and \( 1/4 \), respectively.

The scaling law for the second and third dynamical regime (where \( \varepsilon^4 t \) is the relevant dimensionless quantity) can be written in the following form

\[ c(\varepsilon, t) \sim \varepsilon^{4\eta} g(\varepsilon^4 t), \quad \text{for all } \varepsilon \ll 1, \tag{36} \]

where \( g(x) \to x^{-\eta} \) with \( \eta = 1/2 \), for \( x \to \infty \) and \( g(x) \to x^{-1/4} \) when \( x \ll 1 \).

The quality of collapse of the scaled data is good for second and third dynamical region where the exponent values are \( 1/4 \) and \( 1/2 \) respectively (Fig. 6). Data collapse cannot be obtained for the exponentially decaying part, as the scaling theory applies only for the power law regions.

4 Conclusions

In this article, we have reviewed the problem of ballistic annihilation with discrete velocities in one dimension. First we discussed the decay of the concentration of the particles under the synchronous update rule. After that, we described the properties and characteristics of the dynamics when diffusion is superimposed, for instance by making the update rule asynchronous. There are several open questions on ballistic annihilation which could be addressed in future. One of the important issues that could be studied is the consequence of initial particle distributions on the decay law for the ballistic annihilation with superimposed diffusion. If initially the particles are distributed over a fractal of fractal dimension \( d \neq 1 \), then the concentration of the particles should still decay in a power law fashion, but with a different exponent depending on the value of \( d \), the dimension of the system. Following the argument presented in the section 3.3, the concentration decay should again go through a crossover in time, before it finally starts decaying with exponent \( 1/2 \). Checking this hypothesis could be a part of a future project.

The problems of two species ballistic annihilation [23,24] with superimposed diffusion in one and higher dimension are also open for the future study. On the other hand, the annihilation problem for the persistent random walkers [25–27] is also very interesting that could be studied in future. One can view the problem of ballistic annihilation as a limiting case for \( A + A \to \emptyset \), where \( A \) walkers are the persistent walkers. This problem can be studied under both the synchronous and asynchronous update rule in future. The generalization of the above results to three or more velocities is also a challenging problem.

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Author contributions

All authors contributed equally to the paper.

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