Computing differential equations for integrals associated to smooth Fano polytope

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Abstract We give an approximation algorithm of computing holonomic systems of linear differential equations for definite integrals of rational functions with parameters. We show that this algorithm gives a correct answer in finite steps, but we have no general stopping condition. We apply the approximation method to find differential equations for integrals associated to smooth Fano polytopes. These are interesting in the study of K3 surfaces and the toric mirror symmetry. In this class of integrals, we can apply Stienstra’s rank formula to our algorithm, which gives a stopping condition of the approximation algorithm.

Keywords Gröbner Basis · D-modules · Integration algorithm · Period map · Calabi-Yau variety

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1 Introduction

Let $D$ be the ring of differential operators with polynomial coefficients. A function $f(x_1, \ldots, x_n)$ is called a holonomic function when the annihilating ideal in $D$ of $f$ is a holonomic ideal (see, e.g., [18, Chapter 1]). Finding systems of differential equations for the definite integral $F(x_{m+1}, \ldots, x_n) = \int_C f(x) \, dx_1, \ldots, x_m$ where $C$ is a cycle
is a fundamental problem in the symbolic computation. Since the celebrated work of D. Zeilberger about 20 years ago, many algorithms have been proposed. Among these, algorithms given by Oaku [13,14] find the annihilating ideal for a rational function and construct generators of \((I + \partial_1 D+, \ldots, +\partial_m D) \cap D'\) for a holonomic left ideal \(I\) in \(D\). Here, \(D' = K(x_{m+1}, \ldots, x_n, \partial_{m+1}, \ldots, \partial_n)\) and \(K\) is \(C\).

We give an approximating variation of this algorithm, which improves its performance. We show that this approximation algorithm gives a correct answer in finite steps, but we have no general stopping condition. We apply this method to finding differential equations for integrals associated with smooth Fano polytopes. These are interesting in the study of K3 surfaces and the toric mirror symmetry [1,7,8,11]. For this class of integrals, we can apply Stienstra’s rank formula [19] to our algorithm, which gives a stopping condition.

2 Definite integrals associated with smooth Fano polytopes

A polytope \(P\) is called a smooth Fano \(d\)-polytope when it satisfies the following five conditions.

1. It is a lattice \(d\)-polytope.
2. The origin is in the interior of the polytope.
3. The dual polytope \(P^*\) is a lattice polytope.
4. All facets are simplices (simplicial polytope).
5. The vertices of each facet is a \(Z\) basis of \(Z^d\).

A polytope is called a reflexive polytope when it satisfies conditions 1, 2 and 3.

A classification of the reflexive polytopes in dimensions 3 and 4 is given in [8,9]. Figure 1 is a list of smooth Fano polytopes for dimensions 2 and 3. It can be obtained by extracting the smooth Fano polytopes from Kreuzer’s list of reflexive polytopes. We note that Øbro [16] gave an algorithm for finding all smooth Fano polytopes and also gave a list of them up to dimension 7 modulo isomorphism. Figure 1 is also generated by the program by Øbro.

Let \(\{a_1, \ldots, a_m\}\) be the vertices of a smooth Fano \(n\)-polytope \(P_{n,k}\) and \(a_{m+1}\) be the origin. The suffix \(k\) is the index in the list of the smooth Fano polytopes. We define \(f_{n,k}(x,t) = \sum_{i=1}^{m+1} x_i t^{a_i}\). We are interested in the definite integral with parameters associated with these polytopes defined by

\[
F_{n,k}(x) = \int_C f_{n,k}(x,t)^{-1} t_1^{-1}, \ldots, t_n^{-1} dt_1, \ldots, dt_n
\]

where \(x\) is a generic parameter vector in \(C^{m+1}\) and \(C\) is a cycle in \(H_n(T_x, C)\), \(T_x = \{t \in (C^*)^n \mid f_{n,k}(x,t) \neq 0\}\). This definite integral is called a period integral associated with \(P_{n,k}\). The function \(F_{n,k}(x)\) satisfies an \(A\)-hypergeometric system for \(A = \begin{pmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_{m+1} \end{pmatrix}\) and \(\beta = (-1, 0, \ldots, 0)^T\), and the system is reducible [1]. Since it is reducible, the function \(F_{n,k}(x)\) satisfies more differential equations than

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the \( A \)-hypergeometric differential equations. The quotient of \( D \) by the left ideal generated by these differential operators is a quotient module of the \( A \)-hypergeometric \( D \)-module. The quotient module is implicit in [1]. We are interested in an explicit expression for the quotient module.

**Example 1** The vertices of \( P_{3,0} \) are \{\((1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)\}\). Then, we have, by definition,

\[
F_{3,0}(x) = \int_C (x_1 t_1 + x_2 t_2 + x_3 t_3 + x_4 t_4^{-1} t_2^{-1} t_3^{-1} + x_5)^{-1} t_4^{-1} t_2^{-1} t_3^{-1} dt_1 dt_2 dt_3,
\]

which is nothing but the \( A \)-hypergeometric integral for the matrix

\[
A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}
\]
and $\beta = (-1, 0, 0, 0)^T$. This satisfies the $A$-hypergeometric system $H_A(\beta)$ of which the holonomic rank is 4. As we will see later, the function $F_{3,0}(x)$ satisfies a system of rank 3.

We note that the problem of finding systems of differential equations for period integrals associated with $P_{n,k}$ has been considered in the study of the moduli space of the family of hypersurfaces $f_{n,k}(x, t) = 0$. Ishige [6] studied the case of $P_3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, -1), (-1, -1, -1)\}$ and Nagano [11] studied the cases of

\[
P_2 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, 0), (0, 0, -1)\},
\]
\[
P_4 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, -1), (-1, -1, -2)\},
\]
\[
P_5 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, 0), (-1, -1, -1)\},
\]
\[
P_r = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, -1, -1), (-1, 0, -1)\}.
\]

The correspondence of our table and Nagano’s table is as follows.

| $P_i$ | (dim, index) | $P_2$ | $P_4$ | $P_3$ | $P_5$ | $P_r$ |
|-------|--------------|-------|-------|-------|-------|-------|
| $P_1$ | (3,1)        |       |       |       |       |       |
| $P_2$ | (3,2)        |       |       |       |       |       |
| $P_3$ | (3,4)        |       |       |       |       |       |
| $P_5$ | (3,3)        |       |       |       |       |       |

$P_r$ is not a simplicial polytope. Nagano gives the series expansion of the function $F_{n,k}(x)$ and also a heuristic method for finding annihilating operators of the function by means of the method of undetermined coefficients; this method is enough for his purpose.

We will give an efficient algorithm to derive systems of differential equations for period integrals in this paper. Our method generates a basis of the associated cohomology groups (see, e.g., [18, p.233, Th 5.5.11]).

### 3 Computational bottleneck

Computation of differential equations satisfied by a definite integral with parameters can be reduced to the $D$-module theoretic integral of the annihilating ideal of the integrand [14,15,18]. This method works for any integrand which is a holonomic function. In particular, we can apply it to finding differential equations for the function $F_{n,k}$. However, it requires huge computational resources in general. In this section, we will explain what are bottlenecks of this algorithm in case of our problem.

We denote by $D$ the ring of differential operators of polynomial coefficients in $n$ variables

$$D = K \langle x_1, \ldots, x_m, x_{m+1}, \ldots, x_n, \partial_1, \ldots, \partial_m, \partial_{m+1}, \ldots, \partial_n \rangle$$

and by $D'$ that in $n - m$ variables

$$D' = K \langle x_{m+1}, \ldots, x_n, \partial_{m+1}, \ldots, \partial_n \rangle.$$
Let $I$ be a holonomic left ideal of $D$. The integration ideal of $I$ with respect to $x_1, \ldots, x_m$ is

$$J = (I + \partial_1 D + \ldots, + \partial_mD) \cap D',$$

which is known to be a holonomic left ideal of $D'$.

When the ideal $I$ is the annihilating ideal of a function $f(x_1, \ldots, x_n)$, the integration ideal $J$ annihilates the definite integral $\int_C f(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)dx_1, \ldots, dx_m$ for any $m$-cycle $C$ (see, e.g., [18, Chapter 5]).

Let us review the $D$-module theoretic integration algorithm following Oaku [14] to discuss bottlenecks with timing data. We define the ring isomorphism of $D \to F$ by

$$\mathcal{F}(x_i) = \begin{cases} -\partial_i & (1 \leq i \leq m) \\ x_i & (m < i \leq n) \end{cases} \quad \mathcal{F}(\partial_i) = \begin{cases} x_i & (1 \leq i \leq m) \\ \partial_i & (m < i \leq n) \end{cases}$$

and we call it the Fourier transformation. The inverse map of $\mathcal{F}$ is called the inverse Fourier transformation and denoted by $\mathcal{F}^{-1}$. We use the symbols $\text{in}_{(-w, w)}$ and $\text{ord}_{(-w, w)}$ defined in [18] to express the initial form and the order with respect to the weight vector $(-w, w)$. The generic $b$-function of $I$ with respect to the weight $w \in \mathbb{R}^n$ is the monic generator of the ideal $\text{in}_{(-w, w)}(I) \cap \mathbb{C}[s]$ in $\mathbb{C}[s]$ where $s = \sum_{i=1}^n w_i x_i \partial_i$.

**Algorithm 1** [Computation of the integration ideal, [14, 15, 18]]

- Input: Generators of a holonomic left $D$ ideal $I$.
  - Weight vector $w = (w_1, \ldots, w_m, w_{m+1}, \ldots, w_n)$ satisfying $w_1, \ldots, w_m > 0, w_{m+1} = \ldots = w_n = 0$.
- Output: Generators of the integration ideal of $I$ with respect to $x_1, \ldots, x_m$.

1. (a) Compute a Gröbner basis $G = \{ h_1, \ldots, h_l \}$ of $\mathcal{F}(I)$ with respect to the order $<_{(-w, w)}$.
   (b) Compute the generic $b$-function $b(s)$ of $\mathcal{F}(I)$ with respect to $w$. If there is no non-negative integral root of $b(s) = 0$, then output 1.
   - Let $s_0$ be the maximal non-negative integral root of $b(s) = 0$.
   - Put $m_i = \text{ord}_{(-w, w)}(h_i)$, $B_d = \{ \partial_1^{i_1} \ldots \partial_m^{i_m} | i_1 w_1 + \ldots, + i_m w_m \leq d \}$, $r = \# \{(i_1, \ldots, i_m) | i_1 w_1 + \ldots + i_m w_m \leq s_0 \}$.

2. Elements in $\mathcal{F}^{-1}(B)$ can be expressed as

$$\sum_{i_1 w_1 + \ldots + i_m w_m \leq s_0} c_{i_1, \ldots, i_m} \partial_1^{i_1} \ldots \partial_m^{i_m} \times (c_{i_1, \ldots, i_m} \in D').$$

We regard it as an element of $(D')'$. In other words, we consider a vector $(c_{i_1, \ldots, i_m} h_i w_1 + \ldots, + i_m w_m \leq s_0)$ of which entries belong to $D'$. Let $M$ be the left $D'$-module generated by them.
3. Compute a Gröbner basis of $M$ with respect to a POT (Position Over Term) order such that the element standing for $x_1^0, \ldots, x_m^0$ is the minimum. Collect elements in the Gröbner basis such that all entries except the entry standing for $x_1^0, \ldots, x_m^0$ is 0. We regard these elements as elements of $D'$. Output them.

Example 2 (Period integral associated to a smooth Fano 2-polytope) We will derive a system of differential equations for the period integral

$$F_{2,0}(x) = \int_C (x_1 t_1 + x_2 t_2 + x_3 t_1^{-1} t_2^{-1} + x_4)^{-1} t_1^{-1} t_2^{-1} dt_1 dt_2$$

associated to the smooth Fano polytope $P_{2,0}$ by the algorithm we have presented.

1. Obtain the annihilating ideal $I$ of $f_{2,0}(x, t)^{-1} t_1^{-1} t_2^{-1} = (x_1 t_1 + x_2 t_2 + x_3 t_1^{-1} t_2^{-1} + x_4)^{-1} t_1^{-1} t_2^{-1}$ by Oaku’s algorithm [13] or the Briançon-Maisonobe algorithm [2,10].

2. For the ideal $I$, we compute the integration ideal of $I$ with respect to $t_1, t_2$ $J = (I + \partial_{t_1} D + \partial_{t_2} D) \cap D'$. Here, $D = K\langle t_1, t_2, x_1, x_2, x_3, x_4, \partial_{t_1}, \partial_{t_2}, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4} \rangle$ and $D' = K\langle x_1, x_2, x_3, x_4, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4} \rangle$.

Generators of $J$ obtained by Algorithm 1 are

$$(x_4^3 + 27 x_1 x_2 x_3) \partial_{x_4}^2 + 3 x_2^2 \partial_{x_4} + x_4, \ 9 x_2 x_3 \partial_{x_4}^2 - x_4^2 \partial_{x_1} \partial_{x_4} - x_4 \partial_{x_1},$$

$$9 x_1 x_3 \partial_{x_4}^2 - x_4^2 \partial_{x_2} \partial_{x_4} - x_4 \partial_{x_2}, \ -9 x_1 x_2 \partial_{x_4}^2 + x_2^2 \partial_{x_3} \partial_{x_4} + x_4 \partial_{x_3}, \ -3 x_3 \partial_{x_4}^2 - x_4 \partial_{x_1} \partial_{x_2},$$

$$-3 x_2 \partial_{x_4}^2 - x_4 \partial_{x_1} \partial_{x_3}, \ -3 x_1 \partial_{x_4}^2 - x_4 \partial_{x_2} \partial_{x_3}, \ -\partial_{x_4}^3 + \partial_{x_1} \partial_{x_2} \partial_{x_3}, \ x_4 \partial_{x_4} + 3 x_1 \partial_{x_1} + 1,$$

$$-x_4 \partial_{x_4} - 3 x_2 \partial_{x_2} - 1, \ x_4 \partial_{x_4} + 3 x_3 \partial_{x_3} + 1.$$  

The four elements from the last are elements in the $A$-hypergeometric ideal $H_A(\beta)$ and other elements are not. Here, we put $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$, $\beta = (1, 0, 0)^T$.

We present timing data obtained by applying these algorithms to our problem. The timing data are taken on the computer algebra system Risa/Asir [12] version 20100206 and the library package nk_restriction.rr running on a machine with Xeon X5470 (3.33GHz) CPU and 3.6 GB memory except the computation of annihilating ideal, which is computed by the computer algebra system Singular [5] and the library package dmod.lib on the same machine. It is known that the Singular implementation of it is the best one among several implementations. In the table below, the field “Ann” stands for Briançon-Maisonobe algorithm of computing annihilating ideal [2,10] implemented in Singular, the field “$\langle -u, u \rangle$-GB and generic-b” for the step 1 (a), (b) of Algorithm 1, the field “base” for the step 1 (c), (d) and the step 2 of Algorithm 1, and the field “GB” for the step 3 of Algorithm 1. Each entry means seconds. The symbol “—” means that we could not obtain the result in one day. We could not compute differential equations for $F_{3,12}, \ldots, F_{3,17}$ in one day.
The data tell us that if we can accelerate the steps “Ann” (computation of annihilating ideal) and “$\langle -w, w \rangle$-GB and generic-b” (computation of $(−w, w)$-Gröbner basis and generic $b$-function), then we may be able to speed up the computation in this list. In the next section, we will suggest a new efficient method to find differential equations satisfied by integrals associated to a class of smooth Fano polytopes.

4 An integration algorithm with approximate annihilating ideals

**Definition 1 (Approximate annihilating ideal)** Let $f, g$ be polynomials. The $i$-th approximate annihilating ideal of the rational function $\frac{f}{g}$ is the ideal generated by the elements of which $(0, 1)$-order is less than or equal to $i$ in $\text{Ann}_D \frac{f}{g}$. We denote it by $\text{Ann}^{(i)}_D \frac{f}{g}$.

The following method is simple, but is very useful. This has been used by many people for mathematical experiments.

**Algorithm 2** [Computation of the approximate annihilating ideal, see, e.g., [3]]

- **Input:** Rational function $\frac{f}{g}$ and order $i$.
- **Output:** Generators of the $i$-th approximate annihilating ideal $\text{Ann}^{(i)}_D \frac{f}{g}$.

1. Put $P = \sum a_\alpha \partial^\alpha$ where $a_\alpha$ are polynomials to be determined and the sum runs over all $\alpha$ such that the order of $\partial^\alpha$ (i.e. $\alpha_1+, \ldots, +\alpha_n$) is less than or equal to $i$. 

| Dim | Index | Ann | $\langle -w, w \rangle$-GB and generic-b | Base | GB |
|-----|-------|-----|------------------------------------------|------|----|
| 2   | 0     | $< 1$ | 0.004 | 0.04 | 0.004 |
| 2   | 1     | $< 1$ | 0.022 | 0.04 | 0.008 |
| 2   | 2     | $< 1$ | 0.035 | 0.03 | 0.019 |
| 2   | 3     | 2     | 0.58  | 0.11 | 0.19  |
| 2   | 4     | 180   | 93    | 1.6  | 14    |
| 3   | 0     | 1     | 0.052 | 0.048 | 0.020 |
| 3   | 1     | 1     | 0.20  | 0.09  | 0.052 |
| 3   | 2     | 3     | 0.36  | 0.16  | 0.10  |
| 3   | 3     | 5     | 1.6   | 0.25  | 0.14  |
| 3   | 4     | 14    | 1.4   | 0.25  | 0.013 |
| 3   | 5     | 4,165 | 1,710 | 726  | 5,726 |
| 3   | 6     | 2,383 | 5,037 | 1,007 | 6,194 |
| 3   | 7     | 135   | 92    | 87   | 257   |
| 3   | 8     | 1,551 | 1,618 | 499  | 2,227 |
| 3   | 9     | 16    | 13    | 34   | 86    |
| 3   | 10    | 1,492 | 852   | 183  | 994   |
| 3   | 11    | 6,213 | 1,588 | 406  | 1,839 |
| 3   | 12    | —     | —     | —    | —     |
| 3   | 17    | —     | —     | —    | —     |

........
2. Apply the differential operator $P$ to the rational function $\frac{f}{g}$ and derive a relation for polynomials $a_\alpha$ so that $P \cdot \frac{f}{g} = 0$. The polynomials $a_\alpha$ can be determined by a syzygy computation in the ring of polynomials.

**Example 3** (Computation of the approximate annihilating ideal) We will compute the first approximate annihilating ideal for

$$\tilde{f}_{2,0}(t, x) := f_{2,0}(t, x)^{-1}t_1^{-1}t_2^{-1} = \frac{1}{x_1t_1^2t_2 + x_2t_1t_2^2 + x_3 + x_4t_1t_2}.$$

1. Apply $P = a_1 \partial_{t_1} + a_2 \partial_{t_2} + a_3 \partial_{x_1} + a_4 \partial_{x_2} + a_5 \partial_{x_3} + a_6 \partial_{x_4} + a_7$ ($a_i$ are polynomials to be determined) to $\tilde{f}_{2,0}(t, x)$. Then, the numerator $n(t, x)$ is

$$n(t, x) = (-t_2x_4 - t_2^2x_2 - 2t_1t_2x_1)a_1 + (-t_1x_4 - 2t_1t_2x_2 - t_1^2x_1)a_2$$
$$-t_1^2t_2a_3 - t_1t_2^2a_4 - a_5 - t_1t_2a_6 + (t_1t_2x_4 + x_3 + t_1t_2^2x_2 + t_1^2t_2x_1)a_7.$$

2. We want to determine $a_i$ so that $n(t, x) = 0$. Let $c_i$ be the coefficient of $a_i$ of $n(t, x)$. Any element of the syzygy module $\text{Syz}(c_1, \ldots, c_7)$ gives $\{a_i\}$ such that $n(t, x) = 0$ and if a set $\{a_i\}$ makes $n(t, x) = 0$, then it is an element of the syzygy. The syzygy can be obtained by the Gröbner basis computation in the ring of polynomials and it is generated by

$$(-1, 0, 0, 0, t_2x_4 + t_2^2x_2, 2x_1, 0), (0, -1, 0, 0, t_1x_4 + t_1^2x_1, 2x_2, 0),$$
$$(0, 0, -1, 0, 0, t_1, 0), (0, 0, 0, -1, 0, t_2, 0), (0, 0, 0, 0, -t_1t_2, 1, 0),$$
$$(0, 0, 0, 0, x_3 + t_2x_2 + t_1x_1, 1),$$
$$(0, 0, 0, 0, t_1t_2x_4 + x_3 + t_1t_2^2x_2 + t_1^2t_2x_1, t_1t_2).$$

The set of differential operators standing for them

$$-\partial_{t_1} + (t_2x_4 + t_2^2x_2)\partial_{x_3} + 2x_1\partial_{x_4}, -\partial_{t_2} + (t_1x_4 + t_1^2x_1)\partial_{x_3} + 2x_2\partial_{x_4},$$
$$-\partial_{x_1} + t_1\partial_{x_4}, -\partial_{x_2} + t_2\partial_{x_4}, -t_1t_2\partial_{x_3} + \partial_{x_4},$$
$$x_3\partial_{x_3} + (x_4 + t_2x_2 + t_1x_1)\partial_{x_4} + 1, (t_1t_2x_4 + x_3 + t_1t_2^2x_2 + t_1^2t_2x_1)\partial_{x_4} + t_1t_2$$

is a set of generators of the first approximate annihilating ideal.

We note that $f_{2,0}t_1t_2$ is non-singular, which implies that the annihilating ideal is generated by the first order operators. However, when $k > 0$, the polynomials $f_{2,k}t_1t_2$ and $f_{3,k}t_1t_2$ have singularities. It is an interesting open question that at which order the approximate annihilating ideal equals to the annihilating ideal. Castro and Ucha [3] proved that if a polynomial $f$ in two variables is weighted homogeneous, then we have $\text{Ann}_D^{(1)} \frac{1}{f} = \text{Ann}_D \frac{1}{f}$.

Utilizing approximate annihilating ideals, we give an algorithm to find an approximate integration ideal. The syzygy computation in the ring of polynomials is faster than the computation of the annihilating ideal in $D$ in general. Then, we can expect...
that our approximation algorithm runs faster than the standard one. However, we need
a stopping condition for the approximation procedure.

**Algorithm 3 (Integration algorithm with approximate annihilating ideals)**

- Input: Rational function $\frac{f}{g}$ and $m \in \mathbb{N}$.
- Output: Subideal $J'$ of the integration ideal of $\text{Ann}_D \frac{f}{g}$. We call $J'$ an approximate
  integration ideal.

1. Compute $J = \text{Ann}^{(m)}_D \frac{f}{g}$ (Algorithm 2)
2. If $J$ is holonomic, then apply the $(D$-module theoretic) integration algorithm
   (Algorithm 1) to $J$ and put it $J'$, else put $J' = \langle 0 \rangle$.

The following fact is well-known and is implicitly utilized by several people.

**Lemma 1** There exists a natural number $m$ such that the integration ideal of $\text{Ann}^{(m)}_D \frac{f}{g}$
agrees with the integration ideal of $\text{Ann}_D \frac{f}{g}$.

**Proof** Since $D$ is a Noetherian ring, there exists $m$ such that $\text{Ann}^{(m)}_D \frac{f}{g} = \text{Ann}_D \frac{f}{g}$.
The theorem follows from this fact. $\Box$

**Example 4** We use the example by Castro and Ucha [3]. Consider the polynomial $f = x^4 + y^5 + xy^4$ (Reiffen curves). We have

$$\text{Ann}_D^{(1)} \frac{1}{f} \subsetneq \text{Ann}_D^{(2)} \frac{1}{f} = \text{Ann}_D \frac{1}{f}.$$ 

The integration ideal of $\text{Ann}_D^{(1)} \frac{1}{f}$ with respect to $x$ is $J^{(1)} = D \cdot \{y P\}$ and the integration
ideal of $\text{Ann}_D \frac{1}{f}$ is $J = D \cdot \{P\}$. Here, $P = (-27y^4 + 256y^3)\partial_y^3 + (-432y^3 + 3456y^2)\partial_y^2 + (-1896y^2 + 12336y)\partial_y - 2184y + 10920$. The integration ideal of
$\text{Ann}_D^{(2)} \frac{1}{f}$ agrees with $J$.

Let us come back to our problem of computing differential equations for period
maps associated to smooth Fano polytopes. In applications, we do not need to find
the exact integration ideal and we may find an approximate integration ideal of which
holonomic rank agrees with that of the exact ideal. For this purpose, the method given
by Stienstra gives a way to find the approximation order “m”. We follow the notation
of the paper of Stienstra [19, p.434].

$T \setminus Z_\epsilon$ is defined as $\mathbb{C}^{n-1} \setminus \{x_1, \ldots, x_{n-1} \cdot f(x_1, \ldots, x_{n-1}) = 0\}$ where
$f(x_1, \ldots, x_{n-1})$ is defined as follows from the set of points $A = \left( \begin{array}{c} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_l \end{array} \right)$:

$$f(x_1, \ldots, x_{n-1}) = \sum_{i=1}^{l} u_i x^{a_i} \quad (u_i \in \mathbb{C}).$$ 

We put $g = x_1, \ldots, x_{n-1} \cdot f(x_1, \ldots, x_{n-1})$. 
Algorithm 4 (Algorithm finding a set of differential operators for integrals associated to a class of smooth Fano polytopes)

1. Evaluate the dimension \( r = \dim W_n H^{n-1}(T \setminus Z_s) \) by Stienstra’s method [19, p.435 (57), p.448 Th.10 (iv)].
2. Put \( i = 1 \).
3. Compute the integration ideal \( J \) for \( \text{Ann}^{(i)} D(1/g) \) with respect to \( x_1, \ldots, x_{n-1} \).
4. If the holonomic rank \( J \) is equal to \( r \), then stop and output \( J \), else \( i \leftarrow i + 1 \) and go to the step 3.

Theorem 1

1. Suppose that \( A \) is smooth Fano (reflexive Gorenstein) and admits a unimodular regular triangulation and, moreover,

\[
r = \dim W_n H^{n-1}(T \setminus Z_s) = \dim H^{n-1}(T \setminus Z_s)
\]  

Then, Algorithm 4 outputs a set of differential operators of which holonomic rank agrees with \( r \).

2. If Algorithm 4 stops, then it implies the equality (1).

Proof (1) Since the annihilating ideal \( J = \text{Ann}_D (1/g) \) is finitely generated, there exists \( i \) such that \( \text{Ann}^{(i)} D(1/g) = \text{Ann}_D (1/g) \). The integration module of \( D/J \) is isomorphic to \( H^{n-1}(T \setminus Z_s) \) [15]. Therefore, our algorithm outputs the integration ideal of which rank is equal to \( r \). (2) We have \( W_{n-1} H^{n-1}(T \setminus Z_s) \subseteq W_n H^{n-1}(T \setminus Z_s) \subseteq H^{n-1}(T \setminus Z_s) \) in general. The claim (2) follows from this inclusion. \( \square \)

Example 5 We consider the case \( P_{2,0} \). Vertices of the polytope \( P_{2,0} \) are \( a_1 = (1, 0), a_2 = (0, 1), a_3 = (-1, -1) \). We put \( a_4 = (0, 0) \) and

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0
\end{pmatrix} = (a_{ij}).
\]

Put \( T = \{1, 2, 3, 4, 12, 13, 23, 124, 234, 134\} \). Here, \( i_1, \ldots, i_t \) denotes the simplex spanned by \( a_{i_1}, \ldots, a_{i_t} \). Then, \( T \) is a unimodular triangulation of the set of points \( \{a_1, \ldots, a_4\} \). Following Stienstra [19], we consider the ring \( R_{A,T} = \mathbb{Z}[c_1, c_2, c_3, c_4] / J \) where \( c_i \) is a variable standing for the point \( a_i \) and the ideal \( J \) of the ring \( \mathbb{Z}[c_1, c_2, c_3, c_4] \) is

\[
J = \langle c_1 + c_2 + c_3 + c_4, c_1 - c_3, c_2 - c_3, c_1 c_2 c_3 \rangle.
\]

The first 3 linear polynomials stand for \( A \) and the last monomial stands for a simplex which is not in the triangulation \( T \). Put

\[
core T = \{\text{intersection of the maximal simplices}\}
\]
and

\[ c_{\text{core}} = \prod_{i \in \text{core T}} c_i. \]

In this example, we have \( c_{\text{core}} = c_4 \). Stienstra’s theorem claims that the rank of \( R_{A,T}/\text{Ann}_{R_{A,T}} c_{\text{core}} \) as \( \mathbb{Z} \)-module gives a lower bound of the holonomic rank of the approximate integration ideal. In this example, the rank is 2. The first approximate integration ideal gives this lower bound for this example and then we obtain the answer.

We have applied this algorithm and obtain the following two tables. The field “rank of \( H_A(\beta) \)” is the holonomic rank of the hypergeometric system \( H_A(\beta) \) and the field “rank of \( W_n H^{n-1} \)” is the lower bound evaluated by Stienstra’s method. The field “AppAnn” in the second table is the time to find the first order approximate annihilating ideal. It is an interesting observation that although our polynomials have singularities in most cases, we need only first order operators. We could not solve the cases from \( P_{3,12} \) to \( P_{3,17} \) except \( P_{3,15} \) in 17 days. Note that our method is faster than the exact method and the case of \( P_{3,15} \) cannot be solved by the exact method, but it can be solved by our approximation method.

| Dim | Index | Rank of \( H_A(\beta) \) | Rank of \( W_n H^{n-1} \) |
|-----|-------|--------------------------|--------------------------|
| 2   | 0     | 3                        | 2                        |
| 2   | 1     | 4                        | 2                        |
| 2   | 2     | 4                        | 2                        |
| 2   | 3     | 5                        | 2                        |
| 2   | 4     | 6                        | 2                        |
| 3   | 0     | 4                        | 3                        |
| 3   | 1     | 6                        | 4                        |
| 3   | 2     | 6                        | 4                        |
| 3   | 3     | 6                        | 4                        |
| 3   | 4     | 6                        | 4                        |
| 3   | 5     | 8                        | 5                        |
| 3   | 6     | 8                        | 5                        |
| 3   | 7     | 8                        | 5                        |
| 3   | 8     | 8                        | 5                        |
| 3   | 9     | 8                        | 5                        |
| 3   | 10    | 8                        | 5                        |
| 3   | 11    | 8                        | 5                        |
| 3   | 12    | 10                       | 6                        |
| 3   | 13    | 10                       | 6                        |
| 3   | 14    | 10                       | 6                        |
| 3   | 15    | 10                       | 6                        |
| 3   | 16    | 12                       | 7                        |
| 3   | 17    | 12                       | 7                        |
5 Conclusion

We have succeeded to efficiently find differential equations for period maps standing for $P_{3,5}$ to $P_{3,11}$ by our new approximation method. Explicit expressions of differential equations for period maps and programs used in this paper can be found in http://www.math.kobe-u.ac.jp/OpenXM/Math/sfano

| Dim | Index | AppAnn | $<(-w,w)>$-GB and generic-b | Base | GB |
|-----|-------|--------|-----------------------------|------|----|
| 3   | 5     | 0.12   | 1,253                       | 10   | 5.8 |
| 3   | 6     | 0.18   | 11,781                      | 2,130| 67  |
| 3   | 7     | 0.18   | 32                          | 2.3  | 0.71|
| 3   | 8     | 0.18   | 4,014                       | 69   | 8.7 |
| 3   | 9     | 0.18   | 9.1                         | 1.5  | 0.34|
| 3   | 10    | 0.18   | 248                         | 3.4  | 1.8 |
| 3   | 11    | 0.21   | 572                         | 6.9  | 4.3 |
| 3   | 12    | —      | —                           | —    | —   |
|     |       |        |                             | —    | —   |
|     |       |        |                             | —    | —   |
| 3   | 15    | 0.33   | 196,570                     | 164  | 432 |
|     |       |        |                             | —    | —   |
|     |       |        |                             | —    | —   |
| 3   | 17    | —      | —                           | —    | —   |

Let us briefly discuss two future research plans of this work.

The first one is to utilize the construction of all irreducible quotients of $A$-hypergeometric $D$-module by Saito [17]. To do this, we need to give a new integration algorithm to fit to his construction.

The second one is to improve the final step of computing a $(-w, w)$-Gröbner basis. Let $R$ be the ring of differential operators with rational function coefficients with the variables $x_1, \ldots, x_m, x_{m+1}, \ldots, x_n$ and $R'$ the ring of differential operators with the variables $x_{m+1}, \ldots, x_n$. Chyzak [4] gives an approximation algorithm to find elements of $(RI + \sum_{i=1}^{m} \partial_i R) \cap R'$ by utilizing the method of undetermined coefficients. Since we have to find an approximate basis of the cohomology group to use Stienstra’s criterion, we cannot use the $R$ and we need to use $D$. By this reason, we cannot try this idea with the current implementation Mgfun, which uses $R$, of this algorithm in the Maple. However, Chyzak’s algorithm itself can be easily modified for $D$. It is our future project to give an efficient implementation of this approximation algorithm for $D$ to apply to our problem.

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