Spline-based surfaces in architecture and civil engineering

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Abstract. Spline is the most convenient approximation tool widely used for designing curvilinear technical surfaces in various branches of technology, architecture and civil engineering. To solve such problems, bicubic splines are used. The spline function as a polynomial of the third degree contains differential coefficients. Most often, this is the first derivative in the spline nodes. These coefficients entirely determine its shape. We propose using a fourth order integro-differential spline with additional integral coefficients to construct the surface. These coefficients characterize the shape of the spline curve and the surface between the nodes. As far as computing goes, this spline is not more complicated than traditional one, since it is constructed by solving tridiagonal linear equation system. However, additional coefficients are convenient for local modifications of the curve and surface shapes. This reduces the number of pieces of a composite spline surface. In the practice of constructing geometric shapes in architecture, there are problems when the surface is more complex in one direction than in another. For modeling such surfaces, instead of bicubic spline, we propose using a heterogeneous integro-differential spline. The latter is cubic in one direction of the grid of nodes, and the fourth order polynomial in another direction.

1. Introduction
The spline function is the most common universal mathematical apparatus in modern engineering geometry used for geometrical modeling of curvilinear surfaces of complex shape [1,2]. The spline function method is constantly being improved [3,4]. The method is primarily used where the surface shape is traditionally complex and often cannot be given by a single equation: for example, in the aircraft industry [5,6].

Initially, only the cubic interpolation spline existed and was used in the engineering practice of designing complex shapes for a long time. The spline of one parameter is a piecewise-smooth curve composed of segments of a cubic parabola smoothly joined to each other at the nodal points. Plot (segment) of the spline is given by the equation of the third order in the vector form

\[v(u) = v_i \varphi_1(u) + v_{i+1} \varphi_2(u) + [m_i \varphi_3(u) + m_{i+1} \varphi_4(u)]h_{i+1}\]  

(1)

where \(v_i\) and \(v_{i+1}\) are radius vectors of the beginning and end of the segment, \(m_i\) and \(m_{i+1}\) are the tangent vectors to the spline arc at the beginning and end of a segment, \(\varphi_1(u), \varphi_2(u)\varphi_3(u)\varphi_4(u)\) – Hermite weight functions:

\[\varphi_1(u) = 1 - 3u^2 + 2u^3, \varphi_2(u) = 3u^2 - 2u^3, \varphi_3(u) = u - 2u^2 + u^3, \varphi_4(u) = -u^2 + u^3 \]

\[0 \leq u \leq 1\] – depends on the parametrization. With end-to-end parametrization (for example, \(s\) – arc accumulated length)
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For the simple interpolation of a one-dimensional ordered array of \( N+1 \) point \( v_i, i = 0 \ldots N \) each spline link connects two adjacent points. A spline curve passes through all points and is piecewise smooth. \( N \) spline links are connected to each other at the \( i \)-points \( (i=1 \ldots N-1) \). Thus, the zero order of smoothness is ensured. The first order of smoothness is the simple equality of the first derivatives of the left-side and left-right functions at each \( i=1 \ldots N-1 \) point, that is the unity of the \( m_i \) coefficients’ array. The second order of smoothness is the continuity of the first and second derivatives at the nodes of the spline is ensured by solving a system of \( N \) linear equations with specifying the corresponding boundary conditions depending on the problem. The matrix of the system has a diagonal predominance is solved by the sweep method [1].

The surface of a point or line frame is interpolated by a two-dimensional spline, which is a composite piecewise-smooth surface of the second order of smoothness. This means that the coordinate lines of the surface are one-dimensional splines of the third degree of defect 1 by discontinuities of the third derivative at the intersection of the coordinate lines with the gluing lines of portions of the surface [2,6,7].

Each part of the spline is denoted (is unambiguously determined) by four coefficients which are vectors: Radius-vector site \( W_{ij} \), tangent vectors \( D_{ij}^{10} = \frac{\partial W_{ij}}{\partial s} \) and \( D_{ij}^{11} = \frac{\partial^2 W_{ij}}{\partial s^2} \) to coordinate lines in sites and twist vector \( D_{ij}^{11} = \frac{\partial^2 W_{ij}}{\partial s \partial t} \), that represents the second cross derivative with respect to both parameters from a two-dimensional vector function.

Cubic spline has the following form

\[
W_{33} = \begin{pmatrix}
W_{ij} & W_{i+1j} & D_{ij}^{10} & D_{i+1j}^{10} \\
W_{ij+1} & W_{i+1j+1} & D_{ij+1}^{10} & D_{i+1j+1}^{10} \\
D_{ij}^{11} & D_{i+1j}^{11} & D_{ij}^{11} & D_{i+1j+1}^{11} \\
D_{ij+1}^{11} & D_{i+1j+1}^{11} & D_{ij+1}^{11} & D_{i+1j+1}^{11}
\end{pmatrix}
\]

\[
\varphi(u) = (\varphi_1(u), \varphi_2(u), \varphi_3(u), \varphi_4(u)) h_i, \varphi(u) = (\varphi_1(v), \varphi_2(v), \varphi_3(v), \varphi_4(v)) l_j,
\]

Hermite weight functions with parameters \( u = \frac{s-s_i}{s_{i+1}-s_i}, v = \frac{t-t_j}{t_{j+1}-t_j} \).

When one of the parameters is fixed (\( u \) or \( v \)) in (2), the two-dimensional spline is a one-dimensional spline — a curved line (1) which is left from the remaining fixed parameter (the so-called partial spline).

Today, bi-cubic splines are used for surface approximation. That means that they have the same structure in the directions of both parameters and both partial splines are cubic. There are generic stress splines, B-splines and their generalization NURBS [3], as well as Coons surfaces that have developed the usual interpolation spline functions and are highlighted [8]. Some kinds of splines used are used to numerically solve equations but not for constructing surfaces. [9, 10]

To design the surface the coefficients are calculated by the same algorithm (in this case, the sweep method for a system of equations with a three-diagonal matrix). The geometrical meaning of all coefficients is clear, the properties are studied, and the CAD user is relieved of the obligation to choose the direction of parameterization (for example, in which direction should the array be ordered according to one parameter, and which one in another if they are the same).

2. Methods

Analysis of tasks for designing curvilinear surfaces shows that sometimes the requirements for surface sections in one direction are higher than for sections in another. For example, it is known that the aerodynamic profile in the direction of the incident flow must have a smoothness order not lower than 2. While the cross section of the same surface by the plane perpendicular to the incoming flow can be
the first order of smoothness. This means that it may not make sense to build a spline of 3×3 order. A less critical section may be, for example, a quadratic spline of the first order of smoothness.

Consider heterogeneous splines that have partial splines of different degrees in two topologically perpendicular directions.

3. Results

Construct the spline of 3×2 order.

We propose to use the integro-differential spline (ID-spline) as a quadric spline [11]. For simplicity, we consider the scalar function \( S(x) \). At each segment of the parameter \( x \), the spline has the form

\[
S(u) = S_i \psi_1(u) + S_{i+1} \psi_2(u) + \int_t^{t+1} \psi_3(u) \, du / h_i \tag{3}
\]

where \( S_i \) and \( S_{i+1} \) are the initial and final values of the function, \( \int_t^{t+1} \) - the content of the surface under the function graph (figure 1a) that is determined by the numerical integration, \( \psi(u) \) - basic functions of the second order \( \psi_1(u) = (1 - u)(1 - 3u) \), \( \psi_2(u) = (3u - 2u) \), \( \psi_3(u) = 6u(1 - u) \)

The classic cubic spline at each site is determined by the equation

\[
S(u) = S_i \varphi_1(u) + S_{i+1} \varphi_2(u) + (S'_i \varphi_1(u) + S'_{i+1} \varphi_2(u)) h_i, \tag{4}
\]

where \( \varphi_j(u), \ j = 1..4 \) - basic functions (1), \( S'_i \varphi_1(u), S'_{i+1} \varphi_2(u) \) - the first derivatives of the spline, that is, the tangents of the angles of inclination of the tangents at the ends of the spline (figure 1b). To set the first spline, 3 coefficients are needed; for the second, 4 coefficients.

![Figure 1. Parabolic and cubic splines.](image)

Now let’s look at the two-dimensional case. There are two parallel boundaries which are set along the variable \( x \) for a portion of the surface of an explicit form defined by a two-dimensional function \( S(x,y) \) on a rectangular grid \( [x_i, x_{i+1}] \times [y_j, y_{j+1}] \) with parameters \( x \) and \( y \) for the corresponding values of the second variable \( y_j \) and \( y_{j+1} \). Those boundaries are supposed to be quadratic ID-splines (figure 2).
They are marked as $S_j(x)$ and $S_{j+i}(x)$. They are partial $S(x,y)$ with $y = \text{const}$. To get the entire batch, you need to interpolate these partial splines using the $y$ parameter. To do this, we design three separate splines interpolating separately the three coefficients of the given boundaries $S_j(x) - S_{j+i}(x)$, $S_j(x_{i+1}) - S_{j+i}(x_{i+1})$, and $I_{i+1}^{t_i}(y_j) - I_{i+1}^{t_i}(y_{j+1})$.

Interpolate parameters by cubic splines of the third degree. Interpolation of the first two coefficients — edge points — results in partial splines — portion bounds $S(x,y)$ along parameter $y$. They are marked as $S_j(y), S_{i+1}(y)$ on the Figure 2. The third parameter interpolates with the spline function $I_{i+1}^{t_i}(y)$. Each of those three splines has four parameters. Thus, the surface portions defined by twelve coefficients.

$$S_{2,3} = \varphi(v) G \psi^T(u), \quad (5)$$

where

$$G = \begin{pmatrix}
S_{ij} & S_{i+1j} & I_{i+1}^{t_i}(y_i) \\
S_{ij+1} & S_{i+1j+1} & I_{i+1}^{t_i}(y_{i+1}) \\
m_{ij} & m_{ij+1} & m_{i+1}^{t_i+1}(y_i) \\
m_{ij+1} & m_{ij+1} & m_{i+1}^{t_i+1}(y_{i+1})
\end{pmatrix}.$$

$$\varphi(v) = (\varphi_1(v), \varphi_2(v), \varphi_3(v)l_j, \varphi_4(v)l_j, \psi(u) = (\psi_1(u), \psi_2(u), \frac{\psi_3(u)}{h_i}).$$

Basic functions $\psi(u)$ and $\varphi(v)$ of a quadratic and cubic splines are respectively defined above,

$$u = \frac{x-x_i}{h_i}, \quad v = \frac{y-y_i}{l_j}, \quad h_i = x_{i+1} - x_i, l_j = y_{j+1} - y_j,$$

$$m_{ij} = \frac{\partial S(x_i,y)}{\partial y}, I_{i+1}^{t_i}(y_i) = \int_{x_i}^{x_{i+1}} S(x,y) dx, m_{i+1}^{t_i+1} = \frac{\partial I_{i+1}^{t_i}(y)}{\partial y}. $$

To join two portions of such a spline function along a gluing line - a quadratic spline, it is necessary and sufficient to calculate four coefficients $m_y$ and two $mI^{t_i+1}(y)$ with the others known. For the joining of two portions along the cubic spline, it is necessary and sufficient to calculate two coefficients $I_{i+1}^{t_i}(y)$ and two $mI^{t_i+1}(y)$ with the rest known. In the first case, it is the system of equations with a two-diagonal matrix and the equation is solved step by step. In the second case, the matrix is three-diagonal and the system is solved by the sweep method for a usual cubic spline.
The considered interpolation method is easier than the traditional one, but this advantage is not great at the modern level of development of computer technology. However, it can be used if you suddenly need other special features of ID splines, for example, constructing a surface with control of its shape locally. Here and in other cases, it is possible not to simplify the sections in one of the directions, but to complicate, not using a quadratic spline, but a fourth-degree spline \( S_{3,4}(u,v) \). The cell of the surface will be set by 20 coefficients.

Along one of the two families of coordinate lines, a section of the surface \( S_{3,4}(u,v) \) is a partial spline itself \( S_3(u) \) (4) along the other family – integral differential spline \( S_4(v) \):

\[
S(v) = S_j \psi_1(v) + S_{j+1} \psi_2(v) + (m_j \psi_3(v) + m_{j+1} \psi_4(v)) S_{\jmath} + I_{\jmath}^{i+1} \psi_5(v)/l_j
\]

Here \( S, \psi \) are the initial and final function meaning, \( I_{\jmath}^{i+1} \) - the area of the surface under the graph as it was before, \( \psi(v) \) - basic functions of the fourth order \( \psi_4(v) = (1 + 5)(1 - v)^2(1 - 3v), \psi_5(v) = v(2 + 5v)(1 - v)^2/2 \),

\[
\psi_4(v) = 30v^2(1 - v)^2, \quad v = \frac{y - y_j}{y_{j+1} - y_j}, \quad l_j = y_{j+1} - y_j
\]

Thus, the two-dimensional spline function it has the form

\[
S_{3,4}(u,v) = \varphi(u)G\psi^T(v), \quad G = \begin{pmatrix}
S_{ij} & S_{i+1,j} & m_{ij}^x & m_{i+1,j}^x & l_{i+1,j}^{i+1}(y_j) \\
S_{ij+1} & S_{i+1,j+1} & m_{ij+1}^x & m_{i+1,j+1}^x & m_{i+1,j+1}^{i+1}(y_{j+1}) \\
m_{ij}^y & m_{ij+1}^y & m_{ij+1}^{xy} & m_{i+1,j+1}^{xy} & m_{i+1,j+1}^{i+1}(y_{j+1})
\end{pmatrix},
\]

\( \varphi(u) \) and \( \psi(v) \) - matrices of basic functions of 3 and 4 orders, respectively.

Here we have marked \( m_{ij}^x \) - the first partial derivative of the spline function with respect to the parameter \( x \), \( m_{ij}^y \) – with respect to the parameter \( y \), \( m_{ij}^{xy} \) – the second derivative. \( ml_{ij}^{i+1}(y_j) \) – the first partial derivative with respect to the parameter \( x \) of the integral-function with respect to the parameter \( y \).

It should be noted that to ensure the smoothness of the surface \( S_3 \), modeled in the direction of a partial quadratic spline, the integral parameters must be found by solving systems of equations. When \( S_{3,4} \) is used, the change of \( I_{i+1}^{i+1}(y_j) \) after solving the system of linear equations will lead to the rupture of the second derivatives but will retain the first derivatives. The second order of smoothness is provided by the task \( I_{i+1}^{i+1}(y_j) \) until the solution of the system of linear equations. If necessary, you can provide a third order of smoothness. In this case, \( I_{i+1}^{i+1}(y_j) \) must be calculated by the system of equations derived from the condition of continuity of the third-spline derivatives \( S_i \).

We add that when designing surfaces in engineering practice, surfaces are used in a parametric form, such as (1). When moving from a scalar spline function to a parametric vector spline, the geometric meaning of the vector parameters is very clear. However, the surface \( W_{33}(u,v) \) has a feature that somewhat limits its scope: partial quadratic splines are flat. This deficiency is devoid of \( W_3(u,v) \) surface.

4. Discussion
And now let’s consider the possibility of designing the surface of the coating of an architectural structure – an indoor football stadium (figure 3).
Figure 3. The project of indoor sports facilities.

The use of complex shapes for the construction of building envelopes has led to the emergence of a new field of research, that is often referred to as construction-aware design, fabrication-aware-design, shape rationalization or architectural geometry [12]. Different kinds of kinematic surfaces are used to construct envelopes, taking into account some predefined conditions for the construction [13-19]. These surfaces are used as special construction methods. A spline surface with a small number of nodes can be considered a kinematic piecewise smooth surface [11]. An ID-spline can be used as a universal method.

The surface of the main overhang is a frame-kinematic surface, closed in the direction of one of the families of the carcass lines.

At the first design stage, the inner and outer edges of the coating are represented by one-dimensional ordered point frames. Then both arrays are interpolated by fourth-dimensional one-dimensional parametric splines. These two spatial lines serve as guides for the kinematic surface. The cubic splines $W_4(v)$ will be the generators of this surface.

The designer sets the spline $W_3(u)$ at several nodal points. By changing the shape of these several key curves, it is possible at this stage to approximately determine the shape of the surface itself. Next, a spline of two variables is constructed (figure 4). After that, if necessary, you can change the shape of the surface, operating with differential and integral coefficients of the spline.

Figure 4. Heterogeneous spline $W_{34}(u, v)$, modeling the overhang.

5. Conclusions
Thus, our non-uniform spline allows expanding the capabilities of the universal method of designing complex technical surfaces based on spline functions. All calculations in the considered algorithms are easily programmed. The proposed spline can be easily added to the existing computer-aided design systems as an additional tool.
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