Microeconomics of the ideal gas like market models

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Abstract

We develop a framework based on microeconomic theory from which the ideal gas like market models can be addressed. A kinetic exchange model based on that framework is proposed and its distributional features have been studied by considering its moments. Next, we derive the moments of the CC model (Eur. Phys. J. B 17 (2000) 167) as well. Some precise solutions are obtained which conform with the solutions obtained earlier. Finally, an output market is introduced with global price determination in the model with some necessary modifications.

1 Introduction

Starting with an early attempt by Angle [1,2], a number of models based on kinetic theory of gas have been proposed to understand the emergence of the universal features of income and wealth distributions (see e.g. refs. [3,4,5]). The main focus of those models was to develop a framework that would give rise to gamma function-like behavior for the bulk of the distribution and a power-law for the richer section of the population. The CC-CCM models [6,7] have both of these features. The kinetic exchange model proposed by Dragulescu and Yakovenko [8] and later studied in more details by Guala [9], produces the gamma function-like behavior for the income distribution. We note that all of these models are generally based on some ad-hoc stochastic asset evolution.

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equations with little theoretical foundations for it. Our primary aim in this paper is to develop a consistent framework from which we can address this type of market models. Here we propose a model based on consumers’ optimization which can give rise to those particular forms of asset exchange equations used in refs. [6,8] as special cases. We then focus exclusively on the asset exchange equations and an analytically simple kinetic exchange model is proposed. Its distributional features are analyzed by considering its moments. The same technique is then applied to derive the moments of the distribution of income in the CC model [6] as well. We find that it provides a rigorous justification for the values of the parameters of the distribution, conjectured earlier in ref. [10]. A possible extension of the microeconomic settings of the basic model is also studied where we consider the output market explicitly with global price determination.

2 The model

We consider an $N$-agent exchange economy. Each of them produces a single perishable commodity. Each of these goods is different from all other goods. Money exists in this economy to facilitate transactions (existence of money is not formally explained here). Any commodity can enter as an argument in the utility function (see ref. [11] for a detailed discussion on the theory of utility) of any agent. These agents care for their future consumptions and hence they care about their savings in the current period as well. Each of these agents are endowed with an initial amount of money (the only type of non-perishable asset considered here) which is assumed to be unity for every agent for simplicity. At each time step, two agents meet randomly to carry out transactions according to their utility maximization principle. We also assume that the agents have time dependent preference structure. More precisely, we assume that the parameters of the utility function can vary over time [12][13]. In what follows, we analyze the trading outcomes when any two such agents meet in the market at some time-step $t$.

Suppose agent 1 produces $Q_1$ amount of commodity 1 only and agent 2 produces $Q_2$ amount of commodity 2 only and the amounts of money in their possession at time $t$ are $m_1(t)$ and $m_2(t)$ respectively. Since neither of the two agents possess the commodity produced by the other agent, both of them will be willing to trade and buy the other good by selling a fraction of their own productions as well as with the money that they hold. In general, at each time step there would be a net transfer of money from one agent to the other due to trade. Our aim is to understand how the amount of money held by the agents change over time. For notational convenience, we denote $m_i(t + 1)$ as $m_i$ and $m_i(t)$ as $M_i$ (for $i = 1, 2$).
We define the utility functions as follows. For agent 1, \( U_1(x_1, x_2, m_1) = x_1^{\alpha_1} x_2^{\alpha_2} m_1^{\alpha_m} \) and for agent 2, \( U_2(y_1, y_2, m_2) = y_1^{\alpha_1} y_2^{\alpha_2} m_2^{\alpha_m} \) where the arguments in both of the utility functions are consumption of the first (i.e. \( x_1 \) and \( y_1 \)) and second good (i.e. \( x_2 \) and \( y_2 \)) and amount of money in their possession respectively. For simplicity, we assume that the utility functions are of the Cobb-Douglas form with the sum of the powers normalized to 1 i.e. \( \alpha_1 + \alpha_2 + \alpha_m = 1 \), which corresponds to the constant returns to scale property (homogeneity of degree one)\([11]\). Let the commodity prices to be determined in the market be denoted by \( p_1 \) and \( p_2 \). Now, we can define the budget constraints as follows. For agent 1 the budget constraint is \( p_1 x_1 + p_2 x_2 + m_1 \leq M_1 + p_1 Q_1 \) and similarly, for agent 2 the constraint is \( p_1 y_1 + p_2 y_2 + m_2 \leq M_2 + p_2 Q_2 \). What these constraints mean is that the amount that agent 1 can spend for consuming \( x_1 \) and \( x_2 \) added to the amount of money that he holds after trading at time \( t+1 \) (i.e. \( m_1 \)) cannot exceed the amount of money that he has at time \( t \) (i.e. \( M_1 \)) added to what he earns by selling the good he produces (i.e. \( Q_1 \)). The same is true for agent 2.

The basic idea behind this whole exercise is that both of the agents try to maximize their respective utility subject to their respective budget constraints and the invisible hand of the market that is the price mechanism works to clear the market for both goods (i.e. total demand equals total supply for both goods at the equilibrium prices). Ultimately we will study the money evolution equations in such a situation. Formally, agent 1’s problem is to maximize his utility subject to his budget constraint i.e. maximize \( U_1(x_1, x_2, m_1) \) subject to \( p_1 x_1 + p_2 x_2 + m_1 = M_1 + p_1 Q_1 \). Similarly for agent 2, the problem is to maximize \( U_2(y_1, y_2, m_2) \) subject to \( p_1 y_1 + p_2 y_2 + m_2 = M_2 + p_2 Q_2 \). Solving those two maximization exercises by Lagrange multiplier and applying the condition that the market remains in equilibrium, we get the competitive price vector \((\hat{p}_1, \hat{p}_2)\) as \( \hat{p}_i = (\alpha_i/\alpha_m)(M_1 + M_2)/Q_i \) for \( i = 1, 2 \) (see appendix A1).

We now examine the outcomes of such a trading process.

(a) At optimal prices \((\hat{p}_1, \hat{p}_2)\), \( m_1(t) + m_2(t) = m_1(t+1) + m_2(t+1) \) and this follows directly from Walras’ law\([11]\) saying that if all but one market clears then the rest also has to be cleared. That is, demand matches supply in all market at the market-determined price in equilibrium. Since money is also treated as a commodity in this framework, its demand (i.e. the total amount of money held by the two persons after trade) must be equal to what was supplied (i.e. the total amount of money held by them before trade). In any case, an algebraic proof is also given in the appendix (see A2).

(b) We now present the most important equation of money exchange in this model. We make a rather restrictive assumption that \( \alpha_1 \) in the utility function can vary randomly over time with \( \alpha_m \) remaining constant. It readily follows that \( \alpha_2 \) also varies randomly over time with the restriction
that the sum of $\alpha_1$ and $\alpha_2$ is a constant $(1-\alpha_m)$. In the money demand equations derived from the above-mentioned problem, we substitute $\alpha_m$ by $\lambda$ and $\alpha_1/(\alpha_1+\alpha_2)$ by $\epsilon$ to get the following money evolution equations as (see A3)

$$m_1(t+1) = \lambda m_1(t) + \epsilon (1-\lambda)(m_1(t) + m_2(t))$$

$$m_2(t+1) = \lambda m_2(t) + (1-\epsilon)(1-\lambda)(m_1(t) + m_2(t)).$$

For a fixed value of $\lambda$, if $\alpha_1$ (or $\alpha_2$, see A3) is a random variable with uniform distribution over the domain $[0, 1-\lambda]$, then $\epsilon$ is also uniformly distributed over the domain $[0, 1]$. It may be noted that $\lambda$ (i.e. $\alpha_m$ in the utility function) is the savings propensity used in the CC model [6].

(c) For the limiting value of $\alpha_m$ in the utility function (i.e. $\alpha_m \to 0$ which implies $\lambda \to 0$), we get the money transfer equation describing the random sharing of money without savings. This form of transfer equation has been used in Dragulescu and Yakovenko [8], Guala [9] and also in the model proposed later in this paper.

(d) It may be noted that at each time step, the price mechanism works only locally, i.e., it works to clear the markets for two commodities ($Q_1$ and $Q_2$) only. The markets considered here are perfectly competitive. Also, the set of competitive equilibria is a subset of the set of Pareto Optimal allocation or in other words, all competitive allocations are Pareto Optimal (see ref. [11] for the definition of **Pareto Optimality**). Hence, all the allocations achieved through such trading processes are Pareto Optimal. Also, since the exchange equations are not sensitive to the level of production, even if for some reason the level of production alters (due to production shock) the form of the transfer equations will remain the same provided the form of the utility function remains the same.

(e) Ref [13] also presents a microeconomic framework alongwith an asset evolution equation (see also [12]). But unlike here, the asset evolution equation for the $i$-th agent in ref. [13] depends on his own assets only.

3 **Stochastic model A: exchange with direct transfers**

We now consider an economy under government supervision. Suppose that the government believes in free market mechanism and at the same time, it also wishes to restore equality by taxing the richer ones. In other words, the government does not interfere with the process of market transactions but wishes to restore equality by redistributing income after each trading is performed. There are, say $N$ agents (where $N$ is macroscopically large) in the economy, each endowed with some amount of money (which is normalized to unity, for simplicity) at the beginning of all tradings. At each time-point the money exchange process takes place in two steps.
MC results of transfer model with $N = 100$. All simulations are done for 10 million time-steps and averaged over 1 million time-steps. $+$, $\times$, $*$ and $\square$ denotes the cases where $f=0$, 0.1, 0.2 and 0.3 respectively. \textit{Left panel}: Steady state income distributions of the transfer model. \textit{Right panel}: Steady state income distributions in semi-log scale with gamma pdfs. (dotted lines).

\( (a) \) Two agents are randomly selected and they trade in an absolutely random fashion. Note that this step directly followes from eqn. (1) above if we consider that $\lambda \to 0$ (i.e. if $\alpha_m$ in the utility function tends to zero). Hence,

\[
\begin{align*}
  m_i(t + 1/2) &= \epsilon[m_i(t) + m_j(t)] \\
  m_j(t + 1/2) &= (1 - \epsilon)[m_i(t) + m_j(t)].
\end{align*}
\]  
\( (2) \)

\( (b) \) The agents agree to split the excess income. Hence the agent with more money, transfers a fraction $f$ of the excess income to the agent with less money. It is reasonable to assume that $0 \leq f \leq 0.5$. If $m_i(t + 1/2) \geq m_j(t + 1/2)$, excess income $\delta = m_i(t + 1/2) - m_j(t + 1/2)$. Hence,

\[
\begin{align*}
  m_i(t + 1) &= m_i(t + 1/2) - (f \delta) \\
  m_j(t) &= m_j(t + 1/2) + (f \delta).
\end{align*}
\]  
\( (2a) \)

This process is repeated at each time step until the system reaches a steady state and the distribution $p(m)$ of income among the agents in the steady state are studied. The emergence of gamma function-like behavior is clearly seen in the figure [1]. It is seen computationally that as the fraction increases from 0 to 0.5, the distribution becomes a delta function starting from an exponential one. Left panel of figure [1] shows the pdf of income for several values of $f$.

\textbf{Qualitative features of the distribution}

Substituting for $\delta$, $m_i(t + 1/2)$ and $m_j(t + 1/2)$ in (2a) we get the reduced equations

\[
\begin{align*}
  m_i(t + 1) &= g[m_i(t) + m_j(t)].
\end{align*}
\]

5
\[ m_j(t+1) = (1-g)[m_i(t) + m_j(t)]. \] (3)

The expression of \( g \) in the above equations is \( g = f + (1-2f)\epsilon \). It may be noted that \( g \) is a linear transformation of an uniformly distributed variable \( \epsilon \). Hence, \( g \) is also uniformly distributed and its domain is \([f, 1-f]\). Consider now the \( i \)-th agent only and analyze its income equation. We denote expectation or average of a variable \( x \) by \( \langle x \rangle \) and variance of \( x \) as \( \Delta x \) where \( \Delta x = \langle (x-\langle x \rangle)^2 \rangle \). By taking expectations on both sides of the income equation, it can be easily shown that \( \langle m \rangle = 1 \) (see A4). To proceed further we derive the following formula (see A4) that relates variance of \( g \) to that of \( m \) i.e. the steady state money distribution,

\[ \Delta m = \frac{4\Delta g}{\frac{1}{2} - 2\Delta g}. \] (4)

Also note that \( g \sim \text{uniform}[f, 1-f] \) where \( 0 \leq f \leq 0.5 \) which implies \( \Delta g = (1-2f)^2/12 \). So when \( f = 0 \), \( \Delta g = 1/12 \) and this implies \( \Delta m = 1 \) which is indeed the variance of the distribution \( p(m) = e^{-m} \). Again, if \( f = 0.5 \), \( \Delta g = 0 \) implying \( \Delta m = 0 \) which in turn implies the resulting distribution is a \( \delta \) function.

**Fit with Gamma distribution**

We fit the resulting distribution to a Gamma function

\[ p(m) = \frac{m^{\alpha-1}e^{-\beta m}}{\Gamma(\alpha)\beta^{-\alpha}}. \] (5)

It is well known that the first two moments of this distribution are \( \alpha/\beta \) and \( \alpha/\beta^2 \) respectively. Comparing with what we have found, we get \( \alpha/\beta = 1 \) and \( 1/\beta = 4\Delta g/[\frac{1}{2} - 2\Delta g] \). This fit is shown in the right panel of figure [1].

**4 Stochastic model B: exchange with savings**

We consider the following asset equations with a savings parameter \( \lambda \)

\[ m_i(t+1) = \lambda m_i(t) + \epsilon(1-\lambda)[m_i(t) + m_j(t)] \]

\[ m_j(t+1) = \lambda m_j(t) + (1-\epsilon)(1-\lambda)[m_i(t) + m_j(t)]. \] (6)

It may be noted that this is the equation used in the CC model [6] and has been derived by the utility maximization principle above (eqn. (1)). The CC model has been studied extensively though the exact form of the distribution is still unknown. In ref. [10], it has been conjectured that the steady state distribution is approximately a Gamma distribution [eqn. (5)] with parameter \( \alpha = (1 + 2\lambda)/(1-\lambda) \). Here, we give a simple derivation for this by fixing the average amount of money per agent to unity.
As is done in the last model (section 2), we consider the $i$-th agent only

$$m_i(t + 1) = \lambda m_i(t) + \epsilon(1 - \lambda)[m_i(t) + m_j(t)].$$

By taking expectation on both sides it may be shown that $\langle m \rangle = 1$ (see A5). Next, by applying the variance operator on both sides and simplifying, we get

$$\Delta m = \lambda^2(\Delta m + 1) + 2(1 - \lambda)^2(\Delta \epsilon + \frac{1}{4})(\Delta m + 2) + \lambda(1 - \lambda)(\Delta m + 2) - 1.$$

Note that here $\epsilon \sim \text{uniform}[0, 1]$ which implies $\Delta \epsilon = 1/12$. Substituting this value for $\Delta \epsilon$ and after rearranging terms we get,

$$\Delta m = \frac{(1 - \lambda)^2}{(1 - \lambda)(1 + 2\lambda)}.$$  \(7\)

The proof is given in appendix (see A5). Clearly if $\lambda \neq 1$, $\Delta m = (1 - \lambda)/(1 + 2\lambda)$ as in ref. [10]. Hence $\Delta m = 1$ for $\lambda = 0$, which is indeed the case for an exponential distribution and for $0 \leq \lambda < 1$, the distribution is approximated by eqn. (5) with $\alpha = (1 + 2\lambda)/(1 - \lambda)$ and $\beta = \alpha$ as is conjectured in ref. [10]. For $\lambda \to 1$, however, by applying l'Hôpital's rule we get $\Delta m = 0$. That explains why the steady state distribution tends to a delta function as the rate of savings i.e. $\lambda \to 1$ as widely observed in simulations [3][6][10][14].

5 A generalized framework: trading in global market

So far we have assumed a situation where at each period two agents meet and they carry out transactions according to their utility considerations. The price mechanism works only locally, between these two agents to match supply and demand of these two agents. We try to generalize the model to include an expanding market where prices will be determined globally and the market will be cleared globally. Here, we depict the market as a black box where agents interact with each other indirectly through the market. In the demand side, each agent maximizes intertemporal utility subject to budget constraint and allocates income accordingly between present and future consumption. In the basic model (section 2), present consumption was expressed as a function of the two commodities consumed. But here we relax that assumption and represent both present and future consumption by the amount of money spend on present and future consumptions. What the agents save for future consumption earns them interest income. Hence the market grows over time. On the production side, they can invest in production and get their returns accordingly. Formally, there are $N$ number of agents in the economy each taking part in the consumption and production activities in each period. Agents decide about future in discrete framework. The income stream generated by
an agent is also discrete. We assume that each agent when deciding about the extent of present and future consumption, treats the problem as a two-period choice problem with the present (today or this week or this month etc.) as the first period and everything after that as the second period. In each period, each agent invests a part of his current money-holding to produce some goods and he gets back some return from the market by selling them. Here we make another assumption that the market is perfectly competitive which means that the number of agents is so large that an agent by itself can not influence the pricing mechanism of the market.

We analyze a typical agent’s behavior at any time step \( t \) in the following three steps.

(i) Each agent’s problem is to maximize utility subject to his budget constraint. For simplicity, we assume that the utility function is of Cobb-Douglas type. Briefly, at time \( t \) the \( i \)-th agent’s problem is to maximize 

\[
    u(f, c) = f^\lambda c^{1-\lambda}
\]

subject to 

\[
    \frac{f}{1+r} + c = m(t)
\]

where \( f \) is the amount of money kept for future consumption, \( c \) is the amount of money to be used for current consumption, \( m(t) \) is the amount of money holding at time \( t \) and \( r \) is the interest rate prevailing in the market. This is a standard utility maximization problem and solving it by Lagrange multiplier, we get the optimal allocation as 

\[
    c^* = (1 - \lambda)m(t)\quad \text{and} \quad f^* = (1 + r)\lambda m(t)
\]

(ii) The \( i \)-th agent invests \( (1 - \lambda_i)m_i(t) \) in the market and produces an output vector \( y_i(t) \) which he sells in the market at some market determined price vector \( p_t \) which is same for everybody. By the assumption of perfect competition we get that 

\[
    (1 - \lambda_i)m_i(t) = p(t)y_i(t).
\]

The argument is roughly the following. If \( l.h.s \geq r.h.s \), then it is not optimal to produce because cost is higher than revenue. On the other hand, if \( r.h.s \geq l.h.s \), then there exists what is called supernormal profit which attracts more agents to produce more. But that leads to a fall in price and hence the economy comes to the equilibrium only when \( l.h.s = r.h.s \). Summing up the above equation over all agents, we get 

\[
    \sum_i (1 - \lambda_i)m_i(t) = p(t)\sum_i y_i(t).
\]

The above equation can be rewritten as 

\[
    M(t)V(t) = p(t)Y(t), \quad (8)
\]

where \( M(t) \) is the total money in the system and \( V(t) \) is equivalent to the velocity of money at time \( t \). Clearly, \( V(t) \) depends on the parameter of the utility functions \( \lambda_i \) for all agents. It may be noted that the derived equation is analogous to the Fisher equation of ‘quantity theory of money’
For an alternative interpretation of the Fisher equation in the context of CC type exchange models, see ref [16].

(iii) We have considered a closed economy. During the exchange process money is neither created nor destroyed. After all trading are done, each agent has whatever they saved for future consumption and the interest income earned from it added to some fraction $\alpha_i(t)$ of the total amount of money invested in production of current consumption. Briefly

\[ m_i(t + 1) = (1 + r)\lambda_i m_i(t) + \alpha_i(t)\sum_i(1 - \lambda_i)m_i(t) \]

or from eqn. (8),

\[ m_i(t + 1) = (1 + r)\lambda_i m_i(t) + \alpha_i(t)p(t)Y(t). \]  

Let us assume $r = 0$ and $\alpha_i(t)p(t)Y(t) = \epsilon(t)$. We get the following reduced equation,

\[ m_i(t + 1) = \lambda_i m_i(t) + \epsilon(t). \]

5.1 Steady state distribution of money and price

The money exchange process of each agent is governed by eqn. (10) where $\epsilon(t)$ can be assumed to be white noise. Then it has been shown that this process produces the gamma function-like part as well as the power-law tail [17]. We now consider the more general version of it viz. eqn. (9). Clearly this is an autoregressive process of order 1 with $(1 + r)\lambda_i < 1$ assuming that the last term is white noise. If we take expectation over the whole expression we get

\[ [1 - (1 + r)\lambda_i]\langle m_i \rangle = \langle \alpha_i(t) \rangle \langle p(t) \rangle \langle Y(t) \rangle. \]

We rewrite the equation in terms of average money holding (without subscript) and denoting $\langle \alpha_i(t) \rangle \langle p(t) \rangle \langle Y(t) \rangle$ by a finite constant $C$, as

\[ \lambda = \frac{1}{1 + r}(1 - \frac{C}{m}) \]

which implies $d\lambda \propto dm/m^2$. Since $P(m)dm = \rho(\lambda)d\lambda$ where $P(m)$ is the distribution of money and $\rho(\lambda)$ is the distribution of $\lambda$, we have

\[ P(m) = \rho \left( \frac{1}{1 + r}(1 - \frac{C}{m}) \right) \frac{1}{m^2}. \]  

(11)

For example if $\lambda$ is distributed uniformly then the distribution of money has a power law feature with the exponent being 2. Similar argument is present in refs. [4]17. Ref. [17] also presents examples of the emergence of gamma function-like behavior in the distribution of money for various types of noise terms.
We now focus on eqn. (8). We rewrite it without subscript as following.

\[ p = \frac{V}{Y/M} \]

Note that \( M \) is of the order of \( N \), the number of agents whose total production is \( Y \). In short run fluctuations in output takes place for several reasons. We assume that both \( V \) and \( Y/M \) are distributed uniformly. It may be shown in that case that, the distribution of price is a power law,

\[ f(p) \sim p^{-2}. \] (12)

Hence in this model price also may have a power law fluctuation. It may be noted that there is no clear evidence supporting the existence of a power law in commodity price fluctuation. But it has been verified in stock price fluctuations (see e.g. ref. [18]).

6 Summary and discussion

Our primary focus was to develop a minimal microeconomic framework to derive the asset equations used in the ideal gas like market models. We see that the framework considered above can very easily reproduce the exchange equations used in the CC model (with fixed savings parameter). In a certain limit, it also produces the exchange equations with complete random sharing of monetary assets. Based on this model we have proposed an ideal gas like model of income distribution and we have shown that it captures the gamma function-like behavior of the real income distribution quite well. As discussed above, the framework considered here and the resulting exchange equations differ significantly from those considered in [12,13]. The utility function (in the basic microeconomic model considered above) deals with the behavior of the agents in an exchange economy. However, it also captures the behavior of traders of put and call options of the same stock in a stock market. The price of call and put options of a particular stock generally vary inversely, depending on strike prices and expiration dates. An exception to this generalization is periods of symmetric volatility in the stock’s price, when the simultaneous purchase of call and put options, a straddle, may be profitable. An option’s price (particularly the log of proportional return) is readily identified with its utility. Further, \( \lambda \) may be slightly re-interpreted from determining the utility of savings to determining the utility of protecting savings from risky trading. An interesting question would be whether the stationary distribution of the CC model returns in this model of option trading or not. If not, what modification of the CC model might?

Next, we have analyzed the Monte Carlo simulation results by considering the
first two moments of the income distribution. The same has been done to analyze the income distributions produced by the CC model. Only the moment considerations in both the models show the transition from exponential to delta function with changes in the parameter values of the respective models (the rate of transfer in case of the transfer model and the rate of savings in case of CC model). Moreover, the values of the income distribution parameters, conjectured in ref. [10], have been derived here only by considering those moments. Next, the initial microeconomic model is generalized by incorporating an output market (with global price determination) explicitly. The asset evolution equation in this context is seen to be represented well by an autoregressive process which very easily produces a power law distribution of assets. It has already been discussed in ref. [17] in details how such a process can generate insightful results regarding the distribution of monetary assets. In the same context, an aggregative equation is derived which is analogous to the Fisher equation. From this equation, a power law in price fluctuation is also derived. Taken together, these models provide a link between the standard microeconomic settings (individual optimization and output market) and the asset exchange equations used in the ideal gas like market models.

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7 Appendix

A1: Demand functions derived from the utility maximization problem are as follows. For agent 1,

\[ x_1^* = \alpha_1 \frac{(M_1 + p_1 Q_1)}{p_1}, \quad x_2^* = \alpha_2 \frac{(M_1 + p_1 Q_1)}{p_2}, \quad m_1^* = \alpha_m (M_1 + p_1 Q_1). \]

Similarly for agent 2,

\[ y_1^* = \alpha_1 \frac{(M_2 + p_2 Q_2)}{p_1}, \quad y_2^* = \alpha_2 \frac{(M_2 + p_2 Q_2)}{p_2}, \quad m_2^* = \alpha_m (M_2 + p_2 Q_2). \]

Now, we equate demand and supply of both commodities (i.e. \( x_1^* + y_1^* = Q_1 \) and \( x_2^* + y_2^* = Q_2 \)). By substituting the values of \( x_1^*, x_2^*, y_1^*, \) and \( y_2^* \) and by solving these two equations we get market clearing prices \((\hat{p}_1, \hat{p}_2)\) where

\[
\hat{p}_1 = \frac{\alpha_1 (M_1 + M_2)}{\alpha_m Q_1} \quad \text{and} \quad \hat{p}_2 = \frac{\alpha_2 (M_1 + M_2)}{\alpha_m Q_2}.
\]
A2: Consider the demand functions for money (i.e. $m_1^*, m_2^*$) at optimal prices $(\hat{p}_1, \hat{p}_2)$. Clearly,

$$m_1^* + m_2^* = \alpha_m(M_1 + M_2) + \alpha_m(\hat{p}_1 Q_1 + \hat{p}_2 Q_2).$$

Substituting the value of $(\hat{p}_1, \hat{p}_2)$ in the above equation and using that $(\alpha_1 + \alpha_2 + \alpha_m) = 1$, we get the desired result that

$$m_1^* + m_2^* = (M_1 + M_2).$$

A3: From A1 we get in equilibrium,

$$m_1^* = \alpha_m(M_1 + \hat{p}_1 Q_1).$$

Substituting the value of $\hat{p}_1$ in the above equation we get $m_1^* = \alpha_m M_1 + \alpha_1(M_1 + M_2)$ which can be written as

$$m_1^* = \alpha_m M_1 + \frac{\alpha_1}{\alpha_1 + \alpha_2} (1 - \alpha_m)(M_1 + M_2).$$

For agent 2 the corresponding equation is

$$m_2^* = \alpha_m M_2 + \frac{\alpha_2}{\alpha_1 + \alpha_2} (1 - \alpha_m)(M_1 + M_2).$$

Denoting $\alpha_m$ by $\lambda$ and $\alpha_1/(\alpha_1 + \alpha_2)$ by $\epsilon$, we get the desired asset equations of CC model. Since the parameters add up to 1, if $\alpha_1$ is uniform over $[0,1-\lambda]$ then so is $\alpha_2$.

A4: For the $i$-th agent, the rule of trading is the following

$$m_i(t+1) = g[m_i(t) + m_j(t)].$$

Applying expectation operator on both sides, we get

$$\langle m_i \rangle = \langle g \rangle [\langle m_i \rangle + \langle \frac{1}{N} \sum_j m_j \rangle].$$

Writing the above equation without subscript and using the fact that $\langle g \rangle = \frac{1}{2}$, we get $\langle m \rangle = \frac{1}{2}[\langle m \rangle + 1]$. This in turn gives $\langle m \rangle = 1$.

Also, in the steady state $\Delta m_i = \Delta [g(m_i + m_j)] = \langle x^2 \rangle - (\langle x \rangle)^2$, where $x = [g(m_i + m_j)]$. Note that $\langle x \rangle = 1$. Hence

$$\Delta m = \langle g^2 \rangle \langle m_i^2 + m_j^2 + 2m_i m_j \rangle - 1.$$ 

Using the fact that $m_i$ and $m_j$ are uncorrelated and $\Delta g = \langle g^2 \rangle - 1/4$, we get

$$\Delta m = (\Delta g + \frac{1}{4})(2\Delta m + 4) - 1.$$
Simplifying, we get
\[ \Delta m = \frac{4\Delta g}{\frac{1}{2} - 2\Delta g}. \]

**A5:** For the \( i \)-th agent, the rule of trading is the following
\[ m_i(t + 1) = \lambda m_i(t) + \epsilon(1 - \lambda)[m_i(t) + m_j(t)]. \]

Following **A4**, by taking expectations on both sides we get \( \langle m \rangle = 1 \). Also, in the steady state, \( \Delta m_i = \langle x^2 \rangle - (\langle x \rangle)^2 \) where \( x = \lambda m_i + \epsilon(1 - \lambda)(m_i + m_j) \).

Using the fact that \( \langle x \rangle = 1 \), we get
\[ \Delta m_i = \lambda^2 \langle m_i^2 \rangle + (1 - \lambda)^2 \langle \epsilon^2(m_i + m_j)^2 \rangle + 2\lambda(1 - \lambda)\langle \epsilon \rangle \langle m_i(m_i + m_j) \rangle - 1. \]

Using the result from **A4** that \( \langle \epsilon^2(m_i + m_j)^2 \rangle = (\Delta \epsilon + 1/4)(2\Delta m + 4) \) and the result that \( \Delta \epsilon = 1/12 \), we get the following equation after rearranging terms
\[ \Delta m = \lambda^2(\Delta m + 1) + \frac{2}{3}(1 - \lambda)^2(\Delta m + 2) + \lambda(1 - \lambda)(\Delta m + 2) - 1. \]

Simplifying the above expression we get the desired result,
\[ \Delta m = \frac{(1 - \lambda)^2}{(1 - \lambda)(1 + 2\lambda)}. \]

8 **Appendix - annex**

This section is for record and an appeal; not for publication. We had the rare fortune to have an economist as a referee who has been extremely supportive of such econophysics research, unusually encouraging (see the excerpts below), very caring yet critical (more than 3 pages of referee reports; even checking some of the results independently), suggesting several improvements to make the presentation more acceptable to the economists as well. There have also been some suggestions for future research to establish the points.

The introduction of the referee report reads:

“Stochastic particle system models in which particles exchange a positive quantity that is conserved (sum of quantity over all particles remaining constant) have been shown to have stationary distributions that resemble in some ways distributions of personal labor income and related income measures (e.g., household income). A challenge facing researchers interested in particle systems is
to synthesize these findings with conventional economics. Working in the opposite direction (economic theory to a plausible function for income distribution) have been generations of economists. Since Pareto’s time, economists have struggled to variously derive distributions of labor income, household income, personal asset income, small business net income, and large business net income from neoclassical micro-economics. These efforts have succeeded in plausibly explaining the mean of such distributions. Explaining the second moment is a work in progress. The most successful effort - at least the most widely accepted as indicated by the frequency of its appearance in introductory textbooks - is the derivation of the log-normal distribution as a model of personal income. This derivation has a well known problem with its second moment but the simplicity of the stochastic generator is nevertheless appealing.”

“Finding how micro-economic theory implies the distribution of labor income is what the ascent of the tallest peak of the Himalayas was to mountain climbers before Tenzing Norgay stepped on to its summit. [This paper] attempt this feat. If it succeeds, it is a landmark paper. Just making progress toward its objective, just ascending to a ‘col’ no one else has yet gotten to, is a notable paper deserving publication. A number of economists who later won the Nobel Prize in economics attempted to solve the problem of deriving a function for labor income distribution, when they were young and ambitious. Their efforts were published. No one thinks that anyone has plausibly solved the problem yet. Until recently economists did not consider particle system models of income distribution.”

“... [this] paper represents a clever synthesis of conservative particle system model and familiar micro-economic concepts and equations. Such a synthesis, even an empirically implausible one, is important. ...”

The report further mentions:

“... The paper’s integration of basic equations of mathematical economics with the CC model is interesting and worth publishing. The paper’s implausibilities as economics will be apparent to economists and will stimulate tinkering with the paper’s model. ... I [had the] advantage from having seen the paper before publication.”

We are aware that our effort here is modest and requires further developments. We would like to appeal to the economists, and the referee in particular, to join the effort in establishing the ideal gas like market models (e.g., CC-CCM models) with microeconomic foundations. This derivation of the income distribution from microeconomic theory is, as the referee mentioned, long overdue.
References

[1] J. Angle, *The surplus theory of social stratification and the size distribution if personal wealth*, Social Forces 65 (1986) 293.

[2] J. Angle, *The inequality process as a wealth maximizing algorithm*, Physica A 367 (2006) 388.

[3] *Econophysics of Wealth Distribution*, Eds. A. Chatterjee and B. K. Chakrabarti, New Economic Windows Series, Springer, Milan, 2005

[4] A. Chatterjee, B. K. Chakrabarti, *Kinetic exchange models for income and wealth distributions*, Eur. Phys. J. B 60 (2007) 135.

[5] V. Yakovenko, J. B. Rosser, *Statistical mechanics of money, wealth and income*, Rev. Mod. Phys. (in press), arXiv: 0905.1518.

[6] A. Chakraborti, B. K. Chakrabarti, *Statistical mechanics of money: How saving propensity affects its distribution*, Eur. Phys. J. B 17 (2000) 167.

[7] A. Chatterjee, B. K. Chakrabarti, S. S. Manna, *Pareto law in a kinetic model of market with random saving propensity*, Physica A 335 (2004) 155.

[8] A. A. Dragulescu, V. M. Yakovenko, *Statistical mechanics of money*, Eur. Phys. J. B 20 (2001) 585.

[9] S. Guala., *Taxes in a simple wealth distribution model by inelastically scattering particles*, arXiv: 0807.4484

[10] M. Patriarca, A. Chakraborti, K. Kaski, *A statistical model with a standard gamma distribution*, Phys. Rev. E 70 (2004) 016104.

[11] A. Mas-Colell, M. D. Whinston, J. R. Green, *Microeconomic Theory*, Oxford University Press, New York, 1995

[12] T. Lux, *Emergent statistical wealth distribution in simple monetary exchange models: a critical review*, in ref. [3] p.51.

[13] J. Silver, E. Slud, K. Takamoto, *Statistical equilibrium wealth distribution in an exchange economy with stochastic preference*, J. Econ. Theo. 106 (2002) 417.

[14] A. Chatterjee, B. K. Chakrabarti, R. B. Stinchcombe, *Master equation for a kinetic model of trading market and its analytic solution*, Phys. Rev. E 72 (2005) 026126.

[15] N. G. Mankiw, *Macroeconomics*, Worth Publishers, New York, 2003

[16] Y. Wang, N. Ding, *Dynamic process of money transfer models*, in ref. [3] p. 126.

[17] U. Basu, P. K. Mohanty, *Modelling wealth distribution in growing markets*, Eur. Phys. J. B 65 (2008) 585.

[18] D. Sornette, *Why Stock Markets Crash*, Princeton University Press, Princeton, New Jersey, 2004