THE LOCAL TO UNITY DYNAMIC TOBIT MODEL

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Abstract. This paper considers highly persistent time series that are subject to nonlinearities in the form of censoring or an occasionally binding constraint, such as are regularly encountered in macroeconomics. A tractable candidate model for such series is the dynamic Tobit with a root local to unity. We show that this model generates a process that converges weakly to a non-standard limiting process, that is constrained (regulated) to be positive. Surprisingly, despite the presence of censoring, the OLS estimators of the model parameters are consistent. We show that this allows OLS-based inferences to be drawn on the overall persistence of the process (as measured by the sum of the autoregressive coefficients), and for the null of a unit root to be tested in the presence of censoring. Our simulations illustrate that the conventional ADF test substantially over-rejects when the data is generated by a dynamic Tobit with a unit root, whereas our proposed test is correctly sized. We provide an application of our methods to testing for a unit root in the Swiss franc / euro exchange rate, during a period when this was subject to an occasionally binding lower bound.

Keywords: non-negative time series, dynamic Tobit, local unit root, unit root test.

1. Introduction

Since the 1950s nonlinear models have played an increasingly prominent role in the analysis and prediction of time series data. In many cases, as was noted in early work by Moran (1953), linear models are unable to adequately match the features of observed time series. The efforts to develop models that enjoy the flexibility afforded by nonlinearities, while retaining the tractability of linear models, have subsequently engendered an enormous literature (see e.g. Fan and Yao, 2003; Gao, 2007; Chan, 2009; and Terasvirta et al., 2010).

An important instance of nonlinearity arises when data is bounded by, truncated at, or censored below some threshold, since such phenomena cannot be adequately captured – even approximately – by a linear model. Many observed series are bounded below by construction, and may spend lengthy periods at or near their lower boundary, such as unemployment rates, prices, gross sectoral trade flows, and nominal interest rates. The non-negativity of interest rates, and the resulting constraints that this may impose on the efficacy of monetary policy, has received particular attention in recent years, as central bank policy rates have remained at or near the zero lower bound for a significant portion of the past two decades, across many economies (see e.g. Mavroeidis, 2021, and the works cited therein).

A tractable model for such series, which generates both their characteristic serial dependence and censoring, is the dynamic Tobit model. In its static formulation, the model originates...
with Tobin (1958). In its dynamic formulation, the model typically comes in one of two
categories, which we refer to as the latent and censored models. In the latent dynamic Tobit, an
unobserved process \( \{y_t^*\} \) follows a linear autoregression, with \( y_t = \max\{y_t^*, 0\} \) being observed;
whereas in the censored dynamic Tobit, \( \{y_t\} \) is modelled as the positive part of a linear function
of its own lags, and an additive error (see e.g. Maddala 1983, p. 186, or Wei 1999, p. 419).
In both models the right hand side may be augmented with other explanatory variables.
Relative to the latent model, the censored model has the advantage of being Markovian,
which greatly facilitates its use in forecasting. It has also been successfully applied to a range
of censored series, in both purely time series and panel data settings, including: the open
market operations of the Federal Reserve (Demiralp and Jordà, 2002; de Jong and Herrera,
2011); household commodity purchases (Dong et al., 2012); loan charge-off rates (Liu et al.,
2019); credit default and overdue loan repayments (Brezigar-Masten et al., 2021); and sectoral
bilateral trade flows (Bykhovskaya, 2023). Recently, Mavroeidis (2021) proposed the censored
and kinked structural VAR model to describe the operation of monetary policy during periods
when the zero lower bound may occasionally bind on the policy rate. If only the actual interest
rate (rather than some ‘shadow rate’) affects agents’ decision making, as assumed in closely
related work by Aruoba et al. (2022), then the univariate counterpart of this model is exactly
the censored dynamic Tobit.

The present work is concerned with the censored, rather than the latent, dynamic Tobit
model. As discussed by de Jong and Herrera (2011, p. 229), the censored model is arguably
more appropriate in settings where \( y_t = 0 \) results not from limits on the observability of
some underlying \( y_t^* \), but from an economic constraint on the values taken by \( y_t \). For example,
Bykhovskaya (2023, Supplementary Material, Appendix A) justifies the application of the
dynamic Tobit with reference to a game-theoretic model of network formation, in which the
zeros correspond to corner solutions of a constrained optimisation problem, i.e. where zeros
are systematically observed because of a non-negativity constraint on agents’ choices. In our
illustrative empirical application, to an exchange rate that is subject to a floor engineered by
a central bank (Section 5), the most recent values of the exchange rate are taken as sufficient
to describe the market equilibrium, and thus the conditional distribution of future rates. Our
focus on the censored model is also motivated by relatively greater need for the development of
relevant econometric theory in this area. In the latent model, the dynamics are simply those
of the latent autoregression, and so are readily understood by standard methods; whereas
in the censored model, the censoring affects the dynamics of \( \{y_t\} \) in a non-trivial manner,
making the analysis rather more challenging. Indeed, establishing the stationarity or weak
dependence of the censored dynamic Tobit is far from trivial, as can be seen from Hahn and
Kuersteiner (2010), de Jong and Herrera (2011), Michel and de Jong (2018), and Bykhovskaya
Motivated in part by recent work on modelling nominal interest rates near the zero lower bound, our concern is with the application of this model to series that are highly persistent, so that above the censoring point they exhibit the random wandering that is characteristic of integrated processes. The appropriate configuration of the dynamic Tobit model for such series, in which the autoregressive polynomial has a root local to unity, has not been considered in the literature to date – apart from the special case of a first-order model with an exact unit root, as in Cavaliere (2004) and Bykhovskaya (2023). Our results are thus entirely new to the literature.

Our principal technical contribution, within this setting, is to derive the limiting distributions of both the standardised regressor process, and the ordinary least squares (OLS) estimates of the parameters of the dynamic Tobit, when that model has an autoregressive root local to unity. The reader may find our focus on OLS surprising, as this method would usually provide inconsistent estimates in the presence of censoring. However, it turns out that in our setting consistency (for all model parameters) is restored, and we obtain a usable limit theory for the estimated sum of the autoregressive coefficients, which conventionally provides a measure of the overall persistence of a process (cf. Andrews and Chen, 1994; Mikusheva, 2007). While one may contemplate alternatively using maximum likelihood (ML) or least absolute deviations (LAD) to estimate the model, OLS enjoys the advantages of maintaining only weak distributional assumptions on the innovations (unlike ML), and avoiding the numerical minimisation of a non-convex criterion function (unlike LAD).

Our asymptotics provide the basis for practical unit root tests for highly persistent, censored time series. Motivated by our finding that OLS is consistent, we consider a test based on the (constant only) augmented Dickey–Fuller (ADF) $t$ statistic, but which employs critical values modified to reflect the censoring present in the data generating process. We show, via Monte Carlo simulations, that as our critical values are larger than the conventional ADF critical values, their use eliminates the significant over-rejection that may result from the naive application of the ADF test to censored data. (This tendency to over-reject the null of a unit root appears typical of models that incorporate unit roots and nonlinearities, having been also found by e.g. Hamori and Tokihisa, 1997; Kim et al., 2002; and Wang and De Jong, 2013.) Strikingly, the distribution of $t$ statistic, under censoring, is stochastically dominated by that obtained from the linear autoregressive model.2

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1de Jong and Herrera (2011) alternatively refer to this model as a ‘dynamic censored regression’, but the term ‘dynamic Tobit’ appears more commonly in the literature (see e.g. Hahn and Kuersteiner, 2010; Michel and de Jong, 2018; Bykhovskaya, 2023).

2This holds only with respect to the distributions: it is not true that if one simulates a linear and a Tobit model with the same underlying innovations, then the former $t$ statistic will always be larger than the latter.
Our work may be construed, more broadly, as extending the analysis of highly persistent time series, and the associated machinery of unit root testing, from a linear setting to a nonlinear setting appropriate to time series that are subject to a lower bound. In doing so, we complement the seminal work of Cavaliere (2005), which similarly sought to extend this machinery to the setting of bounded time series. Our contribution is to effect this extension within a class of nonlinear autoregressive models that have been widely applied to censored time series (as evinced by the works cited above), and which fall outside his framework.

On a technical level, the most closely related works to our own are those of Cavaliere (2004, 2005) and Cavaliere and Xu (2014), who develop the asymptotics of what they term ‘limited autoregressive processes’ with a near-unit root, which are (one- or two-sided) non-Markovian censored processes constructed by the addition of regulators to a latent linear autoregression. While their (one-sided) model has a superficial resemblance to the dynamic Tobit, there are important, but subtle differences between the two (see Section 2.4 for a discussion). Perhaps the most striking similarity is that both models, in the case of an exact unit root, give rise to processes that converge weakly to regulated Brownian motion; but when roots are merely local to unity, the limiting processes associated with these two models are distinct (see the discussion following Theorem 3.1).

Convergence to regulated Brownian motions has also been obtained previously in the setting of first-order threshold autoregressive models with an (exact) unit root regime and a stationary regime, as considered by Liu et al. (2011) and Gao et al. (2013). However, allowing for both near-unit roots and higher-order autoregressive terms introduces technical challenges that require us to take a markedly different approach from those employed in these earlier works. With respect to higher-order models, a major difficulty relates to the treatment of the differences \( \Delta y_t \). In a linear autoregressive model with a single unit root, and all other roots outside the unit circle, these would follow a stationary autoregression; but in our setting they instead follow a regime-switching autoregression, where the regime depends on the (lagged) level of \( y_t \), and so are inherently non-stationary. Accordingly, standard arguments for controlling the magnitude of \( \Delta y_t \), and deriving the limits of functionals thereof, are unavailing.

We provide a striking example in which \( \Delta y_t \) is explosive even though all but one of the autoregressive roots (the root at unity) lie outside the unit circle, due to the interactions between the two autoregressive regimes (see Appendix C). To preclude such cases, we develop a condition relating to the joint spectral radius of the autoregressive representation for \( \Delta y_t \), which is sufficient to control the magnitude of \( \Delta y_t \) and plays an essential role in our arguments. (This concept has been previously employed in the context of stationary autoregressive models, see e.g. Liebscher (2005); Saikkonen (2008).)

The remainder of this paper is organised as follows. Section 2 discusses the model and our assumptions. Asymptotic results and corresponding tests are derived in Section 3, while supporting Monte Carlo simulations are shown in Section 4. Section 5 applies our framework...
to the exchange rate between the Swiss franc and the euro during a period when this rate was subject to a lower bound. Finally, Section 6 concludes. All proofs appear in the appendices.

Notation. $C$, $C'$, $C''$, etc. denote generic constants that may take different values in different parts of this paper. All limits are taken as $T \to \infty$ unless otherwise specified. $p$ and $d$ respectively denote convergence in probability and distribution (weak convergence). We write `$X_T(r) \overset{d}{\to} X(r)$ on $D[0, 1]$' to denote that $\{X_T\}$ converges weakly to $X$, where these are considered as random elements of $D[0, 1]$, the space of cadlag functions on $[0, 1]$, equipped with the uniform topology. For $p \geq 1$ and $X$ a random variable, $\|X\|_p := (\mathbb{E}|X|^p)^{1/p}$.

2. The dynamic Tobit model with a near-unit root

2.1. Model and assumptions. Consider a time series $\{y_t\}$ generated by the dynamic Tobit model of order $k \geq 1$, written in augmented Dickey–Fuller (ADF) form,\(^3\)

\begin{equation}
(2.1) \quad y_t = \left[\alpha + \beta y_{t-1} + \sum_{i=1}^{k-1} \phi_i \Delta y_{t-i} + u_t\right]_+, \quad t = 1, \ldots, T,
\end{equation}

where $\Delta y_t := y_t - y_{t-1}$, and $[x]_+ := \max\{x, 0\}$ denotes the positive part of $x \in \mathbb{R}$. Let

\begin{equation}
(2.2) \quad B(z) := 1 - \beta z - (1 - z) \sum_{i=1}^{k-1} \phi_i z^i = (1 - \beta) z + (1 - z) \phi(z),
\end{equation}

where $\phi(z) := 1 - \sum_{i=1}^{k-1} \phi_i z^i$. We impose the following on the data generating process (2.1).

**Assumption A1.** $\{y_t\}$ is initialised by (possibly) random variables $\{y_{-k+1}, \ldots, y_0\}$. Moreover, $T^{-1/2} y_0 \overset{p}{\to} b_0$ for some $b_0 \geq 0$.

**Assumption A2.** $\{y_t\}$ is generated according to (2.1), where:

1. $\{u_t\}_{t \in \mathbb{Z}}$ is independently and identically distributed (i.i.d.) with $\mathbb{E}u_t = 0$ and $\mathbb{E}u_t^2 = \sigma^2$.
2. $\alpha = \alpha_T := T^{-1/2} a$ and $\beta = \beta_T = \exp(c/T)$ for some $a, c \in \mathbb{R}$.

**Assumption A3.** There exist $\delta_u > 0$ and $C < \infty$ such that:

1. $\mathbb{E}|u_t|^{2+\delta_u} < C$.
2. $\mathbb{E}|T^{-1/2} y_0|^{2+\delta_u} < C$, and $\mathbb{E}|\Delta y_i|^{2+\delta_u} < C$ for $i \in \{-k + 2, \ldots, 0\}$.

Figure 2.1 displays a typical sample path for the dynamic Tobit (2.1), under the preceding assumptions.

\(^3\)The autoregressive form is $y_t = [\alpha + \sum_{i=1}^{k} \beta_i y_{t-i} + u_t]_+$, where $\beta_1 = \beta + \phi_1$, $\beta_k = -\phi_{k-1}$, and $\beta_i = \phi_i - \phi_{i-1}$ for $i \in \{2, \ldots, k - 1\}$. In particular $\beta = \sum_{i=1}^{k} \beta_i$ corresponds to the sum of the autoregressive coefficients.
The main consequences of our assumptions may be summarised as follows.

(i) By the functional central limit theorem, A2.1 implies \( T^{-1/2} \sum_{t=1}^{[rT]} u_t \overset{d}{\to} \sigma W(r) \) on \( D[0, 1] \), where \( W(\cdot) \) is a standard Brownian motion. This convergence alone is sufficient to determine the asymptotics of \( T^{-1/2} y_{[rT]} \), and of the OLS estimators, when \( k = 1 \) (Theorems 3.1 and 3.3 below), but extending these results to \( k \geq 2 \) necessitates the slightly stronger conditions on \( \{u_t\} \) provided by A3.

(ii) In the absence of censoring, A2.2 would entail that \( y_t \) has an autoregressive root within an \( O(T^{-1}) \) neighbourhood of real unity. Just as in that case, we shall show that in the present setting \( T^{-1/2} y_{[rT]} \) converges weakly to a continuous process, albeit one that differs importantly from the diffusion process limit familiar from the uncensored case.

(iii) We require \( \alpha = O(T^{-1/2}) \) in A2.2, to ensure that the drift in \( \{y_t\} \) is of no larger order than the stochastic trend component. If this assumption were relaxed, so that e.g. \( \alpha \) were now a non-zero constant, the large-sample behaviour of \( \{y_t\} \) and the asymptotics of the OLS estimators would be quite different from those developed here. A fixed positive \( \alpha \) would generate an increasing linear trend, driving \( y_t \) ever further away from origin and making the censoring ultimately irrelevant; whereas a fixed negative \( \alpha \) can lead to \( \{y_t\} \) being stationary (see e.g. Bykhovskaya, 2023, Theorem 3).

(iv) The specific parametrisations in A2.2 are chosen merely for convenience: all of our results also hold when \( \alpha_T \) and \( \beta_T \) more generally satisfy \( T^{1/2} \alpha_T \to a \) and \( T(\beta_T - 1) \to c \). For ease of notation, we shall routinely suppress the \( T \) subscripts on \( \alpha_T \) and \( \beta_T \) throughout the following.

(v) Assumptions A1 and A3 imply that \( T^{-1/2} y_i \overset{p}{\to} b_0 \) for all \( i \in \{-k+1, \ldots, 0\} \).

2.2. Non-zero lower bound. Our machinery extends straightforwardly to the case where \( y_t \) is censored at some \( L \neq 0 \). Suppose (2.1) is modified to

\[
(2.3) \quad y_t = \max \left\{ L, \ x_t + \beta y_{t-1} + \sum_{i=1}^{k-1} \phi_i \Delta y_{t-i} + u_t \right\},
\]
and take $L = T^{1/2} \ell$ for some $\ell \in \mathbb{R}$, to allow the censoring point to be of the same order of magnitude as $\{y_t\}$. Defining $\tilde{y}_t := y_t - L$ and subtracting $L$ from both sides of (2.3), it may be verified that

$$\tilde{y}_t = \tilde{\alpha} + \beta \tilde{y}_{t-1} + \sum_{i=1}^{k-1} \phi_i \Delta \tilde{y}_{t-i} + u_t$$

where $\tilde{\alpha} := \alpha + (\beta - 1)L$. Thus, $\{\tilde{y}_t\}$ follows a dynamic Tobit with censoring at zero, with drift

$$T^{1/2} \tilde{\alpha} = T^{1/2}[\alpha + (\beta - 1)T^{1/2} \ell] \to a + c\ell =: \tilde{a}$$

and initialisation

$$\tilde{b}_0 = \lim_{T \to \infty} T^{-1/2} \tilde{y}_0 = \lim_{T \to \infty} T^{-1/2}(y_0 - L) = b_0 - \ell.$$ 

All our results hold in the setting of (2.3), with appropriate modifications. For simplicity, we work with $L = 0$ throughout the rest of the paper, except where otherwise indicated.

2.3. Alternative representation. It will be occasionally useful to rewrite (2.1) in a form that helps to clarify the connections between the dynamic Tobit and the linear autoregressive model. We can do this by defining

$$y_t^- := \left[ \alpha + \beta y_{t-1} + \sum_{i=1}^{k-1} \phi_i \Delta y_{t-i} + u_t \right]_-, \quad (2.4)$$

where $[x]_- := \min\{x, 0\}$. That is, when $y_t = 0$, $y_t^-$ records the value that $y_t$ would have taken had it not been censored at zero.

Since $[x]_+ = x - [x]_-$, we may then rewrite (2.1) as

$$y_t = \alpha + \beta y_{t-1} + \sum_{i=1}^{k-1} \phi_i \Delta y_{t-i} + u_t - y_t^-,$$ 

or equivalently, letting $L$ denote the lag operator, as

$$B(L)y_t = \alpha + u_t - y_t^-.$$ 

Thus, if we view $y_t^-$ as an additional noise term, (2.6) takes the form of a linear autoregression. The main challenge is that $y_t^-$ is itself a complicated nonlinear object, whose presence fundamentally alters the dynamics of $\{y_t\}$, even in the long run.

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4 We emphasise that this dependence of $L$ on $T$ should not be interpreted literally as specifying a model in which the censor point is a function of the sample size. Rather, the scaling is a mathematical device that allows us to obtain an improved asymptotic approximation to the finite sample distribution of $T^{-1/2}y_{(rT)}$, and hence of the test statistics that depend upon it. (See e.g. Assumption (A4) in Cavaliere (2005), where a similar device is used for this purpose.) If $L$ is in fact ‘small’ relative to a given sample size $T$, then this will be accommodated within our framework by $\ell$ being close to zero.
2.4. Connections with limited autoregressive processes. The representation (2.6) allows us to draw out the connections between our model and the limited autoregressive processes developed by Cavaliere (2004, 2005) and Cavaliere and Xu (2014). To put their model – for the special case of a process constrained to lie in \([0, \infty)\) – in a form comparable to ours, consider a latent process \(\{x_t^*\}\),

\[
x_t^* = \rho_T x_{t-1}^* + \varepsilon_t, \quad \rho_T = 1 + c/T,
\]

where \(\{\varepsilon_t\}\) is stationary. Define an observed process \(\{x_t\}\), whose increments are related to those of \(\{x_t^*\}\) via

\[
\Delta x_t = \Delta x_t^* + \xi_t
\]

where \(\xi_t > 0\) if and only if \(x_{t-1} + \Delta x_t^* < 0\), so as to ensure that \(x_t \geq 0\) for all \(t\). In particular, if we set

\[
\xi_t = -x_t^- := -[x_{t-1} + \Delta x_t^*]_-
\]

then \(\{x_t\}\) will be censored at zero. When \(c = 0\), by combining (2.7)–(2.9) we obtain

\[
x_t = x_{t-1} + \varepsilon_t - x_t^- = [x_{t-1} + \varepsilon_t]_+
\]

as a valid representation of a limited autoregressive process censored at zero.

Both (2.6) and (2.10) describe censored processes, but have some subtle, and yet important differences. Firstly, while \(\{y_t\}\) in (2.6) is a Markov process (with state vector \((y_t, \ldots, y_{t-k+1})\)), \(\{x_t\}\) in (2.10) will be Markov only if \(\{\varepsilon_t\}\) is i.i.d. The former is thus more suited to forecasting in the presence of higher-order dynamics.

Secondly, at a technical level, the differences between the models can be clearly illustrated by supposing that \(\alpha = 0\) and \(\beta = 1\). Then \(B(L) = (1 - L)\phi(L)\) in (2.6), which simplifies to

\[
y_t = y_{t-1} + \phi(L)^{-1}(u_t - y_t^-).
\]

In the dynamic Tobit, higher-order dynamics are captured by the (stationary) autoregressive polynomial \(\phi(L)\), whereas in the limited autoregressive model these enter via the weak dependence in \(\{\varepsilon_t\}\). To facilitate a comparison of the two models, suppose that \(\{\varepsilon_t\}\) follows the autoregression \(\phi(L)\varepsilon_t = u_t\). Then (2.10) becomes

\[
x_t = x_{t-1} + \phi(L)^{-1}u_t - x_t^-.
\]

We see immediately that if both models have only first-order dynamics \((k = 1)\), so that \(\phi(L) = 1\), then they exactly coincide (cf. Cavaliere, 2005, Remark 2.3). However, this no longer holds in the presence of higher-order dynamics \((k \geq 2)\). Suppose e.g. that \(\phi(L) = 1 - \phi_1 L\) for
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some $|\phi_1| < 1$. Then (2.11) becomes

$$y_t = y_{t-1} + \sum_{s=0}^{\infty} \phi_1^s u_{t-s} - \sum_{s=0}^{\infty} \phi_1^s y_{t-s}$$

whereas (2.12) yields

$$x_t = x_{t-1} + \sum_{s=0}^{\infty} \phi_1^s u_{t-s} - x_{t-s}.$$

Comparing (2.13) with (2.14), we see that the censoring affects the dynamics of $\{y_t\}$ and $\{x_t\}$ in different ways. Lagged values of $y_{t-s}$ have a direct effect on future $y_t$ (via $\sum_{s=0}^{\infty} \phi_1^s y_{t-s}$), whereas lagged $x_{t-s}$ have no such effect on $x_t$.

Thirdly, the differences between the two models also manifests itself through each giving rise to distinct classes of limiting processes. Even when $k = 1$, these coincide only in the special case of an exact unit root ($c = 0$): see the discussion following Theorem 3.1 below.

3. Asymptotic results

In this section we derive the weak limits of the standardised process $T^{-1/2}y_{[rT]}$ and the ordinary least squares (OLS) estimators of the parameters of the dynamic Tobit. The latter provides the basis for a unit root test for non-negative time series, a test that is both is straightforward to compute, and which does not require any assumption to be made on the distribution of the innovations, beyond the existence of sufficient moments. We first provide a separate treatment of the first-order model ($k = 1$), before progressing to the model with higher-order dynamics ($k \geq 1$). This facilitates a simplified exposition of the former case, which avoids the additional assumptions ($A_3$ and $A_4$) and technical concepts – notably the joint spectral radius – that are required when $k \geq 2$.

3.1. Limiting distribution of the regressor process. Let $\theta := (a, b_0, c)$, define the process

$$K_\theta(r) := b_0 + a \int_0^r e^{-cs} \, ds + \sigma \int_0^r e^{-cs} \, dW(s),$$

and denote its 'regulated' counterpart by

$$J_\theta(r) := e^{cr} \left\{ K_\theta(r) + \sup_{r' \leq r} \left[ -K_\theta(r') \right]_+ \right\}.$$  

Here the supremum on the r.h.s. regulates $J_\theta(r)$ to ensure that it is always non-negative: if $K_\theta(r)$ is negative, $\left[ -K_\theta(r) \right]_+ = -K_\theta(r) > 0$, so that $K_\theta(r) + \sup_{r' \leq r} \left[ -K_\theta(r') \right]_+ \geq 0$.

We first provide a result for the case $k = 1$, under which the model (2.1) reduces to

$$y_t = [\alpha + \beta y_{t-1} + u_t]_+.$$
Theorem 3.1. Suppose Assumptions A1 and A2 hold with $k = 1$ in (2.1). Then on $D[0, 1]$, 

$T^{-1/2}y_{[rT]} \overset{d}{\to} J_\theta(r).$ 

(3.4) 

The preceding is a new result, which relates to some of the previous literature as follows.

(i) The appearance of the supremum in the weak limit of (3.4) (see (3.2) above) is in line with the solution to the Skorokhod reflection problem (Revuz and Yor, 1999, p. 239).

(ii) Suppose that $a = b_0 = 0$. Then $e^{rt}K_\theta(r) = S_c(r) = \sigma \int_0^r e^{c(r-s)} dW(s)$, an Ornstein-Uhlenbeck process with autoregressive parameter $c$ (e.g. Chan and Wei, 1987; Phillips, 1987b), and (3.4) specialises to

$T^{-1/2}y_{[rT]} \overset{d}{\to} S_c(r) + \sup_{r' \leq r} [-e^{c(r-r')} S_c(r')]_+.$

on $D[0, 1]$. Whereas, if we take $\varepsilon_t = u_t$ in (2.7), the limited autoregressive process (2.7–(2.9) satisfies

$T^{-1/2}x_{[rT]} \overset{d}{\to} S_c(r) + \sup_{r' \leq r} [-S_c(r')]_+$

on $D[0, 1]$. Comparing the two preceding limits, we observe a subtle but crucial difference, due to the presence of the factor $e^{c(r-r')}$, showing that the asymptotics of the dynamic Tobit and limited autoregressive models are distinct even when $k = 1$.

(iii) When $a = b_0 = c = 0$, $J_\theta(r)$ coincides with a Brownian motion regulated from below at zero, which has the same distribution as a Brownian motion reflected at the origin, $\lfloor W(\cdot) \rfloor$, see e.g. Karatzas and Shreve (2012, p. 97). Another model that generates a process with this asymptotic distribution (upon rescaling by $T^{-1/2}$) is a first-order threshold autoregression with ‘unit root’ and ‘stationary’ regimes, as studied by Liu et al. (2011) and Gao et al. (2013). A special case of their model posits

$x_t = \beta(x_{t-1})x_{t-1} + u_t,$

where $\beta(x) = 1$ if $x \geq 0$, and $\beta(x) = 0$ otherwise. It follows that $x_t = [x_{t-1}]_+ + u_t$, and so $[x_t]_+ = [x_{t-1}]_+ + u_t]_+$, which corresponds to our setting (2.1) with $\alpha = 0$, $\beta = 1$, $k = 1$, and $y_t = [x_t]_+$. It is thus not surprising that, in this case, our Theorem 3.1 agrees exactly with the corresponding Theorem 3.1 of Liu et al. (2011).

(iv) The proof of Theorem 3.1 shows that the convergence (3.4) relies ultimately on $T^{-1/2} \sum_{t=1}^{[rT]} u_t \overset{d}{\to} \sigma W(r)$, as follows by the functional central limit theorem when $\{u_t\}$ is i.i.d. with mean zero and variance $\sigma^2$, as per A2. If more generally $\{u_t\}$ is weakly dependent with long-run variance $\omega^2$, then under appropriate regularity conditions (see e.g. Theorem 1.1 in Peligrad and Utev, 2005) we would instead have $T^{-1/2} \sum_{t=1}^{[rT]} u_t \overset{d}{\to} \omega W(r)$, with the consequence that (3.4) would continue to hold, albeit with $\omega$ replacing $\sigma$ in (3.1). However, the results in Section 3.2 below cannot be so straightforwardly generalised, as here the presence of weak dependence in $\{u_t\}$
would entail substantial changes to the limiting distributions (cf. the very different limits given in Theorems 2.1 and 2.3 of Liang et al. (2016), depending on whether the relevant innovation sequence is serially uncorrelated).

When \( k > 1 \), the other roots of the lag polynomial \( B(L) \) affect the behaviour of \( \{y_t\} \), and we need a further condition to ensure that the first differences \( \{\Delta y_t\} \) are well behaved. Let

\[
F_\delta := \begin{bmatrix}
\phi_1 \delta & \phi_2 & \cdots & \phi_{k-2} & \phi_{k-1} \\
\delta & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & & \\
& \ddots & \ddots & \ddots & \\
& & & 1 & 0
\end{bmatrix}.
\]

(3.5)

Under an appropriate condition on the matrices \( \{F_\delta \mid \delta \in [0,1]\} \), we can ensure \( \{\Delta y_t\} \) is stochastically bounded. To state that, we need the following (cf. Jungers (2009), Defn. 1.1):

**Definition.** The joint spectral radius (JSR) of a bounded collection \( \mathcal{A} \) of square matrices is

\[
\lambda_{\text{JSR}}(\mathcal{A}) := \limsup_{n \to \infty} \sup_{M \in \mathcal{A}^n} \lambda(M)^{1/n}
\]

where \( \lambda(M) \) denotes the spectral radius of \( M \), and \( \mathcal{A}^n := \{\prod_{i=1}^n A_i \mid A_i \in \mathcal{A}\} \).

Control over the JSR has been previously used to ensure the stationarity of regime-switching autoregressive models (e.g. Liebscher (2005); Saikkonen (2008)), and we shall utilise it in a similar manner here.

**Assumption A4.** \( \lambda_{\text{JSR}}(\{F_0, F_1\}) < 1 \).

Approximate upper bounds for the JSR can be computed numerically, to an arbitrarily high degree of accuracy, via semidefinite programming (Parrilo and Jadbabaie, 2008), making it reasonably straightforward to verify whether this condition is satisfied by given parameter values. The following result, which is proved in Appendix C, provides a sufficient condition for A4, which may be checked even more simply.

**Lemma 3.1.** If \( \sum_{i=1}^{k-1} |\phi_i| < 1 \), then Assumption A4 is satisfied.

**Remark 3.1.** (i) To give some intuition for why a condition on \( \{F_\delta\} \) is needed here, suppose that \( \alpha = 0 \) and \( \beta = 1 \). Then (2.5) entails

\[
\Delta y_t = \sum_{i=1}^{k-1} \phi_i \Delta y_{t-i} + u_t - y_t^-,
\]

and hence in the absence of censoring \( \{\Delta y_t\} \) would follow a linear autoregression, for which \( F_1 \) gives the associated companion form. Assumption A4 implies that the eigenvalues of \( F_1 \) are below 1 in modulus, and thus that the all roots of \( \phi(z) \) lie strictly outside the unit circle (see
e.g. Hamilton, 1994, Prop. 1.1). (Since \( \phi(0) = 1 \), it also follows that \( \phi(1) > 0 \).) In the presence of censoring, \( \{\Delta y_t\} \) may be shown to instead (when \( \alpha = 0 \) and \( \beta = 1 \)) follow a time-varying autoregression, in which it evolves jointly with an auxiliary process \( \{w_t\} \) as per

\[
w_t = \phi_1 \delta_{t-1} w_{t-1} + \sum_{i=2}^{k-1} \phi_i \Delta y_{t-i} + u_t,
\]

\[
\Delta y_{t-1} = \delta_{t-1} w_{t-1},
\]

for some (stochastic) sequence \( \delta_t \subset [0, 1] \) (see the proof of Lemma B.2 in Appendix B). \( F_\delta \) thus corresponds to the companion form autoregressive matrix for \( (w_t, \Delta y_{t-1}, \ldots, \Delta y_{t-k+2})^T \) when \( \delta_{t-1} = \delta \). Because \( \lambda_{JSR}(\{F_\delta \mid \delta \in [0, 1]\}) = \lambda_{JSR}(\{F_0, F_1\}) \), \( A4 \) is sufficient to ensure that this time-varying autoregressive system is stable, irrespective of the sequence \( \{\delta_t\} \).

(ii) As illustrated by Example 1 in Appendix C, merely requiring that \( \phi(z) \) have only stationary roots is not sufficient to guarantee the convergence (in distribution) of \( T^{-1/2} \bar{y}_{[rT]} \).

Due to the nonlinearity in the model it may be possible to induce explosive trajectories for both \( \Delta y_t \) and \( y_t \), via a succession of periods in which \( y_t > 0 \) alternates with \( y_t = 0 \) (see Figure C.1a). Thus additional conditions, such as \( A4 \), are needed to exclude such behaviour.

On the other hand, as illustrated by Example 2 in Appendix C, neither is Assumption \( A4 \) necessary for the convergence of \( T^{-1/2} \bar{y}_{[rT]} \). Finding a necessary condition thus remains a challenging open question.

**Theorem 3.2.** Suppose Assumptions \( A1-A4 \) hold. Then

\[
(3.6) \quad T^{-1/2} \bar{y}_{[rT]} \overset{d}{\to} \phi(1)^{-1} J_{\theta_\phi}(r) =: Y_{\theta_\phi}(r)
\]

on \( D[0, 1] \), where \( \theta_\phi := [a, \phi(1)b_0, \phi(1)^{-1}c] \).

The principal difference between Theorems 3.1 and 3.2 is that when \( k > 1 \), the stationary dynamics appear in the limit via the factor \( \phi(1) \). Notably, the local autoregressive parameter \( c \) is replaced by \( \phi(1)^{-1}c \) exactly as it would be if \( \{y_t\} \) were generated by a linear autoregression with a root local to unity (cf. Hansen, 1999, p. 599). Indeed, \( \phi(1) = 1 \) when \( k = 1 \), so in this case the two results coincide.

**Corollary 3.1.** For the Tobit model (2.3) with censoring point \( L \), \( T^{-1/2} \bar{y}_{[rT]} \overset{d}{\to} Y_{\tilde{\theta}_\phi}(r) \), where \( \tilde{\theta}_\phi := [a, \phi(1)\tilde{b}_0, \phi(1)^{-1}c] \).

### 3.2. OLS estimates.

We first consider the case where \( k = 1 \), as in the model (3.3), to develop intuition for our results.

When estimating (3.3) by OLS, we need to decide which deterministic terms should be included in the regression. In the absence of censoring, i.e. if the data generating process were simply a linear autoregression, the inclusion of a constant and a linear trend would render the distribution of the OLS estimator of \( \beta \) free of any nuisance parameters related to
the deterministic components. Unfortunately, the nonlinearity introduced by the censoring entails that $\alpha$ – or rather, the local parameter $a$ – will show up in the limiting distribution of $\hat{\beta}_T$, irrespective of which deterministics are included in the regression. To permit inferences to also be drawn on $a$, if required, we consider the OLS regression of $y_t$ on a constant and $y_{t-1}$, i.e.

$$\begin{bmatrix} \hat{\alpha}_T \\ \hat{\beta}_T \end{bmatrix} = \left( \sum_{t=1}^{T} \begin{bmatrix} 1 & y_{t-1} \\ y_{t-1}^2 \end{bmatrix} \right)^{-1} \sum_{t=1}^{T} \begin{bmatrix} 1 \\ y_{t-1} \end{bmatrix} y_t =: \mathcal{M}_T^{-1} m_T. \tag{3.7}$$

In the stationary dynamic Tobit model, OLS is inconsistent (see e.g. Bykhovskaya, 2023, Supplementary Material, Lemma B.1). However, as the following shows, when $\beta$ is local to unity, consistency is restored. The reason is that observations in the vicinity of zero accumulate only at rate $T^{1/2}$, so that a vanishingly small fraction of the sample is affected by the censoring.

**Theorem 3.3.** Suppose Assumptions A1 and A2 hold, with $k = 1$ in (2.1). Then

$$\begin{bmatrix} T^{1/2}(\hat{\alpha}_T - \alpha) \\ T(\hat{\beta}_T - \beta) \end{bmatrix} \overset{d}{\to} \begin{bmatrix} 1 \\ \int_0^1 J_\theta(r) \, dr \end{bmatrix}^{-1} \begin{bmatrix} v(1) - c \int_0^1 J_\theta(r) \, dr - b_0 - a \\ \sigma \int_0^1 J_\theta(r) \, dW(r) \end{bmatrix} \tag{3.8}$$

Remark 3.2. Letting $J_\theta^\mu(r) := J_\theta(r) - \int_0^1 J_\theta(s) \, ds$, an alternative expression for the limiting distribution of $\hat{\beta}_T$ is given by

$$b_\theta = \frac{J_\theta^\mu(1)^2 - J_\theta^\mu(0)^2 - \sigma^2}{2 \int (J_\theta^\mu(r))^2 \, dr} - c. \tag{3.9}$$

This agrees with the limiting distribution that would be obtained in the linear autoregressive model, except with $J_\theta(\cdot)$ taking the place of the usual Ornstein–Uhlenbeck process. (See Appendix A.2 for details.)

---

5Strictly speaking, this is true only if the autoregressive model is formulated in ‘unobserved components’ form (see e.g. Andrews and Chen (1994, Section 2.1) as

$$y_t = \mu + \delta t + y_t^* \quad \quad y_t^* = \beta y_{t-1}^* + u_t$$

so that the presence (or absence) of a linear drift in $y_t$ is independent of the value of $\beta$, and so can always be removed by deterministic detrending. By contrast, if the model is formulated ‘directly’ as

$$y_t = \alpha + \beta y_{t-1} + u_t,$$

then the linear trend that is present when $\beta = 1$ becomes an exponential trend when $\beta$ is local to unity. In the present (censored) setting, we may note that (3.3) is not equivalent to

$$y_t = [\mu + \delta t + y_t^*]_+ \quad \quad y_t^* = \beta y_{t-1}^* + u_t.$$

(This model is in fact the latent dynamic Tobit referred to in Section 1.)
For the case of general $k \geq 1$, let $\phi := (\phi_1, \ldots, \phi_{k-1})^T$, and

$$
\begin{align*}
(\hat{\alpha}_T, \hat{\beta}_T, \hat{\phi}_{1,T}, \ldots, \hat{\phi}_{k-1,T}) := \arg\min_{(a,b,f_1,\ldots,f_{k-1})} \sum_{t=1}^T \left( y_t - a - by_{t-1} - \sum_{i=1}^{k-1} f_i \Delta y_{t-i} \right)^2
\end{align*}
$$

denote the OLS estimators of the parameters of (2.1). Since, as the next results shows, the limiting distributions of $(\hat{\alpha}_T, \hat{\beta}_T)$ depend on $\phi(1)$, a consistent estimate of that quantity is needed to compute valid critical values for test statistics based on these estimators. The following also guarantees the consistency of $\hat{\phi}(1) := 1 - \sum_{i=1}^{k-1} \hat{\phi}_{i,T}$.

**Theorem 3.4.** Suppose Assumptions A1–A4 hold. Then $\hat{\phi}_T \xrightarrow{P} \phi$, and

$$
\left[ T^{1/2} (\hat{\alpha}_T - \alpha) \right] \xrightarrow{d} \left[ \frac{1}{\int Y_{\hat{\theta}}(r) \, dr} \int Y_{\hat{\theta}}(r) \, dr \int Y_{\hat{\theta}}^2(r) \, dr \right]^{-1} \left[ \phi(1)[Y_{\hat{\theta}}(1) - b_0 - c_\phi \int Y_{\hat{\theta}}(r) \, dr] - a \right]
$$

**Remark 3.3.** The parameters of the Tobit model (2.3) with censoring point $L$ can be estimated as in (3.10), with $\tilde{y}_t$ in place of $y_t$ and with $\tilde{\theta}_\phi$ replacing $\theta_\phi$ in (3.11).

The theorem shows that, despite the nonlinearity of the Tobit model, OLS is consistent for all parameters in the presence of a near unit root. The associated limit distribution theory for $\hat{\beta}_T$ provides a basis for the unit root tests developed in the next section, yielding a test that is both easy to compute, and semiparametric with respect to the distribution of the innovations. (For examples in which the erroneous imposition of normality can have undesirable consequences, in the context of a dynamic Tobit model, see the empirical illustrations in de Jong and Herrera (2011) and Bykhovskaya (2023).)

### 3.3. Unit root tests

The preceding results allow us to conduct asymptotically valid hypothesis tests on key parameters of the dynamic Tobit: in particular, to test the hypothesis of a unit root in this setting. This may be of interest for several reasons. For example, whether the variance of the errors made in forecasting $y_t$ remains bounded, or grows without bound at progressively longer forecast horizons, depends crucially on the presence of a unit root. In a setting with multiple series, one or more of which are non-negative, the presence of unit roots may also lead to spurious regressions or, more constructively, allow long-run equilibrium relationships to be identified from the (nonlinear) cointegrating relationships between the series (see Duffy et al., 2022).

In a linear autoregressive model, the presence of a unit root – equivalently, the sum of the autoregressive coefficients being unity ($\beta = 1$) – necessarily imparts a stochastic trend to $\{y_t\}$. However, in the dynamic Tobit the value of the intercept also matters. In particular, a fixed negative intercept ($\alpha < 0$ and not drifting toward zero) would continually push the process back towards the censoring point, thereby rendering it stationary (for $k = 1$, see Bykhovskaya, 2023, Theorem 3). Thus to the extent that the purpose of a test for a unit root is to test for
the presence of a stochastic trend in \( \{y_t\} \), rather than to detect a unit root per se, it may be considered more appropriate to test the null that \( \alpha = 0 \) and \( \beta = 1 \), as opposed to merely the restriction that \( \beta = 1 \), with it being desirable to reject this null in favour of a stationary alternative, when either \( \beta < 1 \) (exactly as in a linear model), or when \( \beta = 1 \) but \( \alpha < 0 \).\(^6\)

To construct our test statistics, we need an estimate of the error variance \( \sigma^2 \). We use
\[
\hat{\sigma}_T^2 := \frac{1}{T} \sum_{t=1}^T \hat{u}_t^2,
\]
where
\[
\hat{u}_t := y_t - \hat{\alpha}_T - \hat{\beta}_T y_{t-1} - \sum_{i=1}^{k-1} \hat{\phi}_{i,T} \Delta y_{t-i}.
\]
That is, \( \{\hat{u}_t\} \) are the OLS residuals, computed as if \( y_t \) were not subject to censoring. Let \( \mathcal{M}_T := \sum_{t=1}^T x_t x_t^T \), where \( x_t := (1, y_{t-1}, \Delta y_{t-1}, \ldots, \Delta y_{t-k+1})^T \).

**Corollary 3.2.** Suppose Assumptions A1 and A2 hold. If either: \( k = 1 \) in (2.1); or \( k > 1 \), and A3 and A4 hold, then \( \hat{\sigma}_T^2 \overset{d}{\to} \sigma^2 \) and
\[
(3.13) \quad t_{\alpha,T} := \frac{\hat{\alpha}_T - \alpha}{\hat{\sigma}_T \sqrt{\mathcal{M}_T^{-1}(1,1)}} \overset{d}{\to} \frac{a_{\theta_\phi}}{\sigma \sqrt{J_{\theta_\phi}^{-1}(1,1)}}, \quad t_{\beta,T} := \frac{\hat{\beta}_T - \beta}{\hat{\sigma}_T \sqrt{\mathcal{M}_T^{-1}(2,2)}} \overset{d}{\to} \frac{b_{\theta_\phi}}{\sigma \sqrt{J_{\theta_\phi}^{-1}(2,2)}},
\]
where \( \mathcal{M}_T^{-1}(i,j) \) denotes the \((i,j)\) element of \( \mathcal{M}_T^{-1} \).

This result allows us to conduct a one-sided test of a unit root versus a stationary alternative, which rejects when \( t_{\beta,T} \leq c \), where \( c \) is drawn from an appropriate quantile of the asymptotic distribution of \( t_{\beta,T} \). Under the null of a unit root \( a = c = 0 \), the limiting distribution of \( t_{\beta,T} \) in (3.13) depends (continuously) on the model parameters only through \( b_0 \phi(1)/\sigma \). This follows from the fact that for \( \theta_\phi = (0, \phi(1)b_0, 0) \),
\[
\sigma^{-1} J_{\theta_\phi}(r) = \sigma^{-1} \left\{ \phi(1)b_0 + \sigma W(r) + \sup_{r' \leq r} \left[ -\phi(1)b_0 - \sigma W(r') \right] \right\}_+
\]
\[
= \frac{\phi(1)b_0}{\sigma} \left\{ 1 + \frac{\sigma}{\phi(1)b_0} W(r) + \sup_{r' \leq r} \left[ -1 - \frac{\sigma}{\phi(1)b_0} W(r') \right] \right\}_+
\]
and thus \( \sigma \sqrt{J_{\theta_\phi}^{-1}(2,2)} \) and \( b_{\theta_\phi} \) (as defined in (3.9) above) do not change so long as \( b_0 \phi(1)/\sigma \) remains fixed. Table 3.1 tabulates the critical values corresponding to the relevant quantiles of the asymptotic distribution of \( t_{\beta,T} \), as a function of \( b_0 \phi(1)/\sigma \in [0,2.5] \). One can see that the smaller the ratio of the parameters, the larger the corresponding critical values, for every significance level. Further, as the final two lines of the table and the further discussion in Section 4.1 indicate, for values of \( b_0 \phi(1)/\sigma \) in excess of 2.5, the critical values coincide (to within two decimal places) with those of a conventional ADF \( t \) test (for a linear autoregression).

\(^6\)When \( k > 1 \), the above must be qualified somewhat, because of the possibility that the higher-order nonlinear dynamics of the system may generate explosive trajectories even when \( \beta < 1 \). For stationary alternatives that are local to unity in the sense that \( \beta = \exp(c/T) \) for some \( c < 0 \), this is excluded by A4. For non-local alternatives, some further condition (i.e. in addition to \( \beta < 1 \)) on the autoregressive system is needed to ensure stationarity: see e.g. de Jong and Herrera (2011) or Duffy et al. (2022, Sec. 3).
With the aid of the tabulated critical values, a test of \( H_0 : a = c = 0 \) may thus be carried out as follows. (For a general lower bound \( L \neq 0 \), first subtract it from the data, as described in Section 2.2.)

**Unit root test procedure.**

1. Regress \( y_t \) on \((1, y_{t-1}, \Delta y_{t-1}, \ldots, \Delta y_{t-k+1})^T\) using OLS.
2. Calculate the \( t_{\beta,T} \) statistic (3.13) with \( \beta = 1 \).
3. Let \( \hat{\phi}(1) := 1 - \sum_{i=1}^{k-1} \hat{\phi}_{i,T} \) and \( \hat{b}_0 := T^{-1/2} y_1 \), where \( y_1 \) is the first observation in the sample. Compare \( t_{\beta,T} \) with the critical values in Table 3.1, for the row corresponding to the value nearest to \( \hat{b}_0 \hat{\phi}(1)/\hat{\sigma}_T \) (or use the conventional ADF critical values if this value exceeds 2.5).\(^7\)

As the simulations in the following section illustrate, our test indeed has the desirable properties outlined above, in the sense of tending to reject both when either \( \beta < 1 \), or when \( (\beta = 1, \alpha < 0) \), i.e. it has power to reject the null whenever \( \{y_t\} \) is stationary. In the event of a rejection, the reason for that rejection – i.e. whether this is due to \( \alpha < 0 \) or \( \beta < 1 \) – may be further investigated with the aid of LAD estimates of the model parameters, which by Bykhovskaya (2023, Theorem 6) are consistent and asymptotically normal in the stationary region. (By contrast, due to the relatively greater frequency of censoring, OLS is not consistent in the stationary region, as discussed in Section 4.3 of that work.)

**3.4. Specification testing.** An implication of the arguments given in the proof of Corollary 3.2, which is in line with alternative Tobit representation (2.5), is that the OLS residuals \( \{\hat{u}_t\} \) are consistent for \( \{u_t - y^-_t\} \). However, because \( y_t = 0 \) occurs relatively infrequently, \( \{u_t - y^-_t\} \) and \( \{u_t\} \) coincide sufficiently closely that e.g. \( T^{-1} \sum_{t=1}^{T} (u_t - y^-_t)^2 = T^{-1} \sum_{t=1}^{T} u_t^2 + o_p(1) \). In view of this, there appears to be some justification for continuing to use standard residual-based specification tests, such as for heteroskedasticity, serial correlation, or non-normality, when the dynamic Tobit is estimated by OLS (for an overview, see Kilian and Lütkepohl, 2017, Sec. 2.6–2.7). The proper second order asymptotics for such tests is outside of the focus of the present paper and is left for future research.

**4. Simulations**

We now illustrate how the values of the initial condition \( b_0 \) and the localising parameters \( a \) and \( c \) affect the distribution of \( t_{\beta} \), and compare the performance of a test based on critical

\(^7\)Alternatively one could use a parametric bootstrap procedure to estimate the quantiles of the null distribution of the test statistic, following the approach of Cavaliere and Xu (2014, Theorem 2), who prove the validity of a bootstrap procedure in a related setting with \( k = 1 \). Their proof could be directly transposed to our setting to justify a procedure based on \( y_t^{(r)} = \sqrt{T'} \hat{\phi}(1) y_{1}/\sqrt{T} \) and \( y_t^{(r)} = [y_{t-1}^{(r)} + u_t^r]_+ \), \( u_t^r \sim i.i.d. \mathcal{N}(0, \hat{\sigma}_T^2) \), \( t > 1, r = 1, \ldots, R \), where \( T' \geq T \) is the length of the simulated series and \( R \) is the number of simulated series.
Table 3.1. Critical values for the Tobit ADF test for different values of $b_0\phi(1)/\sigma$ (based on $10^7$ Monte Carlo simulations of $y_t = [y_{t-1} + u_t]_+ + y_0 = b_0\sqrt{T}$, $u_t \sim \text{i.i.d. } \mathcal{N}(0, 1)$, $T = 10^5$). The final line reports the critical values appropriate to an ADF t test in a linear autoregression.

| $b_0\phi(1)/\sigma$ | 1%    | 5%    | 10%   |
|---------------------|-------|-------|-------|
| 0.0                 | -4.69 | -3.77 | -3.34 |
| 0.1                 | -4.58 | -3.66 | -3.22 |
| 0.2                 | -4.49 | -3.57 | -3.14 |
| 0.3                 | -4.38 | -3.49 | -3.07 |
| 0.4                 | -4.25 | -3.40 | -3.00 |
| 0.5                 | -4.11 | -3.31 | -2.93 |
| 0.6                 | -3.97 | -3.22 | -2.87 |
| 0.7                 | -3.85 | -3.15 | -2.81 |
| 0.8                 | -3.75 | -3.08 | -2.75 |
| 0.9                 | -3.67 | -3.03 | -2.71 |
| 1.0                 | -3.60 | -2.99 | -2.68 |
| 1.1                 | -3.56 | -2.96 | -2.65 |
| 1.2                 | -3.52 | -2.94 | -2.63 |
| 1.3                 | -3.50 | -2.92 | -2.62 |
| 1.4                 | -3.48 | -2.90 | -2.61 |
| 1.5                 | -3.47 | -2.89 | -2.60 |
| 1.6                 | -3.46 | -2.89 | -2.59 |
| 1.7                 | -3.45 | -2.88 | -2.58 |
| 1.8                 | -3.45 | -2.87 | -2.58 |
| 1.9                 | -3.44 | -2.87 | -2.58 |
| 2.0                 | -3.44 | -2.87 | -2.57 |
| 2.5                 | -3.43 | -2.86 | -2.57 |
| ADF                 | -3.43 | -2.86 | -2.57 |

Values derived from Corollary 3.2 with one based on the conventional ADF critical values (Dickey and Fuller, 1979), when the data is subject to censoring.

4.1. Effect of $b_0$ and the connection to the conventional ADF test. Figure 4.1 depicts how a change in $b_0$ shifts the probability density (PDF) and the cumulative distribution (CDF) of $t_\beta$. As $b_0$ moves further above zero, the density shifts progressively to the right, as the probability that any trajectory of $K_\theta$ (initialised at $b_0$) will reach zero, and so be subject to censoring, correspondingly declines. Indeed, once $b_0$ is sufficiently large to make this probability negligible, the density becomes visually indistinguishable from that generated by a linear model (the solid green line in Figure 4.1), which is invariant to $b_0$. (Under the parametrisation used in the figure, this occurs when $b_0 = 2$; in general, this will depend on the magnitude of $\phi(1)b_0/\sigma$, in accordance with Theorem 3.2.) The rightward shift of these distributions, as $b_0$ grows, is similarly manifest in the critical values given in Table 3.1 above.
4.1. CDFs and PDFs of t-ratio $t_\beta$ under the Tobit and linear models. Data generating process is $y_t = [y_{t-1} + u_t]_+$, $y_0 = b_0 \sqrt{T}$ for Tobit model and $y^\ell_t = y^\ell_{t-1} + u_t$, $y^\ell_0 = 0$ for a linear model, $u_t \sim \text{i.i.d. } \mathcal{N}(0,1)$. Data is obtained from $10^7$ samples of length $T = 1000$.

Surprisingly, the CDFs exhibit stochastic ordering, in the sense that the distributions corresponding to higher values of $b_0$ first-order stochastically dominate those with lower values of $b_0$ (holding all other parameters constant). This is in line with the monotonically increasing (in $b_0 \phi(1)/\sigma$) quantiles in Table 3.1. In particular, the null distribution of the conventional ADF $t$ test – i.e. that appropriate to a linear autoregression – stochastically dominates the null distribution appropriate to a dynamic Tobit. Therefore, when data is generated by a dynamic Tobit with a unit root, the conventional ADF test will tend to over-reject: in the worst case, which occurs when $b_0 = 0$, we find that a nominal 5 per cent test will in fact reject 20 per cent of the time. The intuition is that the censoring causes the trajectories of $\{y_t\}$ to appear stationary, masking the presence of a unit root.

For the remainder of this section, all simulations are conducted with $y_0 = b_0 = 0$.

4.2. Effects of $a$ and $c$. Figure 4.2 shows how a change in local intercept $a$ (left panel) and local slope coefficient $c$ (right panel) affects the density of $t_\beta$. The means of these distributions across a range of values for $a$ and $c$ are also reported in Table 4.1. We can see that as $a$ or $c$ fall further below zero, the distribution of $t_\beta$ (both its mean and its entire probability mass) shifts leftward – with the opposite effect being observed when these parameters are progressively raised above zero.

4.3. Power. The preceding illustrates how changes in $a$ and/or $c$ may shift the distribution of $t_\beta$ in either direction, and so will affect the ability of the test to reject the null of a unit root.
Figure 4.2. Densities of t-ratio $t_\beta$ for various values of $a, c$. Data generating process is $y_t = \left[ \frac{a}{\sqrt{T}} + (1 + \frac{c}{T}) y_{t-1} + u_t \right]_+^+, y_0 = 0, u_t \sim \text{i.i.d. } \mathcal{N}(0, 1)$. Data is obtained from $10^6$ samples of time series of length $T = 1000$.

| $a$ | $c$ | -5 | -2 | -1 | 0 | 1 | 2 | 5 |
|-----|-----|----|----|----|---|---|---|---|
| -5  |    | -6.09 | -5.70 | -5.56 | -5.42 | -5.26 | -5.09 | -4.37 |
| -2  |    | -4.24 | -3.70 | -3.49 | -3.27 | -3.00 | -2.62 | 8.93  |
| -1  |    | -3.69 | -3.10 | -2.88 | -2.62 | -2.26 | -1.51 | 23.05 |
| 0   |    | -3.20 | -2.60 | -2.37 | -2.06 | -1.46 | 0.03  | 43.23 |
| 1   |    | -2.82 | -2.26 | -2.00 | -1.56 | -0.57 | 1.85  | 68.26 |
| 2   |    | -2.58 | -2.06 | -1.76 | -1.13 | 0.28  | 3.68  | 96.73 |
| 5   |    | -2.52 | -2.05 | -1.57 | -0.48 | 2.18  | 9.00  | 193.14 |

Table 4.1. Mean of t-ratio $t_\beta$ for various values of $a, c$. Data generating process is $y_t = \left[ \frac{a}{\sqrt{T}} + (1 + \frac{c}{T}) y_{t-1} + u_t \right]_+^+, y_0 = 0, u_t \sim \text{i.i.d. } \mathcal{N}(0, 1)$. Data is obtained from $10^6$ samples of time series of length $T = 1000$.

root (i.e. $H_0 : \alpha = 0, \beta = 1$). Power envelopes (rejection probabilities for a nominal 5 per cent, one-sided test), are displayed in Figure 4.3. These show that, on the one side, more negative values of $a$ and/or $c$ make it easier for the test to reject the null in favour of a stationary alternative. The power eventually reaches 100 per cent, indicating the consistency of the test against fixed alternatives in this region. This tendency to reject the null, as $a$ falls below zero, is in fact a desirable property of the test in this setting, since having $\alpha < 0$ in the dynamic Tobit implies (for $k = 1$) that $\{y_t\}$ is stationary, even when $\beta = 1$ (Bykhovskaya, 2023, Theorem 3).

On the other side, positive values of $c$ move $\{y_t\}$ into the explosive region, and our tendency to not reject in these cases is entirely consistent with the use of a one-sided test, as it is in a linear model (Figure 4.3b). Although we also fail to reject for sufficiently positive values of
(A) Power envelopes with respect to $a$.  (B) Power envelopes with respect to $c$.

Figure 4.3. Power envelopes with respect to $a$ and $c$. Data generating process $y_t = \left[ \frac{a}{\sqrt{T}} + \left( 1 + \frac{c}{T} \right) y_{t-1} + u_t \right]^+$, $y_0 = 0$, $u_t \sim \text{i.i.d. } \mathcal{N}(0,1)$. Data is obtained from $10^5$ samples of time series of length $T = 1000$.

$a$ (Figure 4.3a), in such cases an upward trend in $\{y_t\}$ would become discernable, and would carry the process away from the censoring point. Since $\{y_t\}$ would then make few (if any) visits to the censoring point, if one were interested in testing the null of a unit root against a trend stationary alternative, in this case, a conventional ADF test with intercept and trend would be appropriate.

5. Empirical illustration

In this section we illustrate the use of our methods through an application to testing for unit roots in nominal exchange rates, when these are subject to a one-sided bound. Unit roots are routinely detected in these series by conventional tests in empirical work (see e.g. Baillie and Bollerslev, 1989; Sarno and Valente, 2006, p. 3156; Hong and Phillips, 2010, p. 107). Their presence is also manifested in the robust performance of exchange rate forecasts based on random walks, which more elaborate models have struggled to beat consistently (Rossi, 2013). (For a discussion of the theoretical basis for the presence of a unit root in exchange rates, in the context of open economy New Keynesian models, we refer the reader to Section 2.1 of Engel, 2014.) Here we examine how censoring, as introduced by the deliberate action of a central bank to keep exchange rates above (or below) a nominated threshold, may alter our assessment of the evidence for or against a unit root, depending on whether that censoring is accounted for (cf. Cavaliere, 2005, Sec. 6.1).

In September 2011, in response to the ongoing appreciation of the Swiss franc, and with the policy rate effectively at the zero lower bound, the Swiss National Bank (SNB) instituted a floor on the euro–Swiss franc exchange rate of 1.20 francs per euro (Jordan, 2016; Hertrich, 2022). With the exchange rate well below the floor on the previous day, the immediate
The local to unity dynamic Tobit model was required to make the floor effective upon its introduction on 6 September. The floor remained in effect until the end of 15 January 2015. As can be seen from Figure 5.1, during that period, the exchange rate spent most of its time well above the floor (reaching a peak of 1.26 in May 2013), with two notable exceptions: the periods of April to August 2012, and from November 2014 until the end of the policy, both of which triggered action by the SNB to prevent the floor from being violated (see Figure 9 in Hertrich, 2022). (For a further discussion of the floor and its aftermath, see also von Schweinitz et al., 2021.) The observed trajectory of the exchange rate, during these episodes, is thus more plausibly consistent with a dynamic Tobit than it is with a linear autoregression.

In this setting, the presence of a unit root would entail that the exchange rate tends to make extended sojourns away from the floor, visits to which accumulate only at rate $T^{1/2}$, where $T$ denotes the duration of the policy. By contrast, in the stationary dynamic Tobit, visits to the floor would accumulate at rate $T$. Thus the failure to reject the null of a unit root would signal to the central bank that it may need to intervene relatively infrequently in the foreign exchange market, in order to make the floor effective, something that is in turn highly relevant to the future cost and viability of maintaining that floor.

Our data is drawn from the European Central Bank’s (ECB) daily reference exchange rate series (code: EXR.D.CHF.EUR.SP00.A) from the period during which the floor was operational (6 Sep 2011 to 15 Jan 2015), transformed by taking logarithms. To select the lag order $k$ used to compute $t_\beta$, we evaluated autoregressive models with $k \in \{1, \ldots, 15\}$ using the Akaike and Bayesian information criteria; both selected a model with only one lag. For this model, we obtained an ADF statistic slightly above $-2.87$: so that if the censoring were ignored, and this statistic referred to the conventional ADF critical values (Table 3.1), there would be just sufficient evidence to reject the null of a unit root at the five per cent level. On the other hand, the unit root is not rejected at conventional significance levels under the dynamic Tobit. By simulating the asymptotic distribution of $t_\beta$, given in Corollary 3.2, we
compute the $p$-value appropriate to the dynamic Tobit as either 0.2 (with $b_0 = 0$ imposed) or 0.18 (using $\hat{b}_0 = T^{-1/2}(y_0 - L)$ with $L = \log(1.20)$); similar results also obtain when $k = 2, 3$. This places $t_{\beta,T}$ much further from the critical region, thereby lending support to the hypothesis that the data is consistent with a dynamic Tobit model with a unit root.

6. Conclusion

This paper extends local to unity asymptotics to the setting of a dynamic Tobit model. Censoring fundamentally changes the analysis and requires new tools to derive the asymptotics. We obtain novel limit theorems for convergence to regulated processes, that is, to processes constrained to lie above a threshold. The effect of that censoring on the limiting distribution of our test statistics varies according to the proximity of the initialisation to the censoring point, with the distributions associated with the linear model re-emerging as that initialisation moves sufficiently far from the censoring point.

Our results underpin the development of a unit root test appropriate to censored data generated by a dynamic Tobit model. In contrast to the setting of a linear model, here the presence of a stochastic trend entails restrictions on both $\alpha$ and $\beta$, since $\alpha < 0$ (with $\beta = 1$) can be consistent with stationarity (and is consistent with stationarity for $k = 1$). Nonetheless, a test of this null can still be effected by the usual $t_{\beta}$ statistic, using adjusted critical values. We provide an empirical illustration of our methods to testing for a unit root in nominal exchange rates, when these are subject to a one-sided bound.

The results of this paper could be developed further in a number of directions. One possibility would be to extend our results beyond the local to unity setting, by allowing for moderate deviations from a unit root (Giraitis and Phillips, 2006), with the aim of establishing (uniformly) valid confidence intervals for $\beta$, as per Mikusheva (2007, 2012) in the linear model. As we depart from the linear model, new types of asymptotics emerge, which may involve $c$ in different ways (cf. Bykhovskaya and Phillips (2018) where $c$ is no longer a constant but varies with time).

The analysis of $c \to \pm \infty$ for the censored model may be undertaken in conjunction with the derivation of the asymptotics of least absolute deviations regression or maximum likelihood estimators of the model, which would enjoy consistency across a wider domain than does OLS. For such extensions, the results of Section 3.1, regarding the asymptotics of $\{y_t\}$, are likely to be of fundamental importance.

For another direction in which our analysis could be extended, recall from Section 1 that the dynamic Tobit provides the kernel of more elaborate, multivariate models that allow for the possibility of censoring and/or some other threshold-related nonlinearity. The analysis of (exact) unit roots and cointegration, in the setting of such a model, is developed by Duffy et al. (2022), with the aid of our results.
Appendix A. Proofs of results for $k = 1$.

A.1. Limiting distribution of $T^{-1/2}y_{[nr]}$.

**Lemma A.1.** Suppose that $x_t = [x_{t-1} + v_t]_+$, for $t = 1, \ldots, T$, and $T^{-1/2}(x_0 + \sum_{s=1}^{[rT]} v_s) \xrightarrow{d} V(r)$ on $D[0,1]$. Then on $D[0,1]$,

$$T^{-1/2}x_{[rT]} \xrightarrow{d} V(r) + \sup_{r' \leq r} [-V(r')]_+.$$

**Proof.** By Bykhovskaya (2023, Supplementary Material, Lemma D.8) and Cavaliere (2004, Lemma 1),

$$x_t = [x_{t-1} + v_t]_+ = x_0 + \sum_{s=1}^{t} v_s + \sup_{t' \in \{0, \ldots, t\}} \left[-x_0 - \sum_{s=1}^{t'} v_s\right]_+.$$

Defining $V_T(r) := T^{-1/2}(x_0 + \sum_{s=1}^{[rT]} v_s)$, we have that $V_T(r) \xrightarrow{d} V(r)$ on $D[0,1]$ by the hypotheses of the lemma. Since

$$T^{-1/2}x_{[rT]} = V_T(r) + \sup_{r' \in [0,r]} [-V_T(r')]_+$$

and the supremum on the r.h.s. is a continuous functional of $V_T(\cdot)$, the result follows by the continuous mapping theorem (CMT).

**Proof of Theorem 3.1.** Multiplying (3.3) by $\beta^{-t}$, and defining $x_t := \beta^{-t}y_t$, we have

$$x_t = \beta^{-t}y_t = \beta^{-t}[\beta y_{t-1} + \alpha + u_t]_+ = [\beta^{-(t-1)} y_{t-1} + \beta^{-t}(\alpha + u_t)]_+ = [x_{t-1} + v_t]_+,$$

where $v_t := \beta^{-t}(\alpha + u_t)$. Under A1, $T^{-1/2}x_0 = T^{-1/2}y_0 \rightarrow b_0$, and so

$$\frac{1}{T^{1/2}} \left(x_0 + \sum_{s=1}^{[rT]} v_s\right) = \frac{x_0}{T^{1/2}} + \frac{a}{T} \sum_{s=1}^{[rT]} e^{-cs/T} + \frac{1}{T^{1/2}} \sum_{s=1}^{[rT]} e^{-cs/T} u_s$$

(A.1)

$$\xrightarrow{d} b_0 + a \int_0^r e^{-cs} \, ds + \sigma \int_0^r e^{-cs} \, dW(s) = K_\theta(r)$$

on $D[0,1]$, where $K_\theta$ is as defined in (3.1), $\theta = (a, b_0, c)$, and the weak convergence follows by the martingale central limit theorem. It follows by Lemma A.1 that on $D[0,1]$,

$$T^{-1/2}y_{[rT]} = \beta^{[rT]} T^{-1/2}x_{[rT]} = e^{[rT]/T} T^{-1/2} x_{[rT]} \xrightarrow{d} e^{cr} \left[K_\theta(r) + \sup_{r' \leq r} [-K_\theta(r')]_+\right].$$

**A.2. OLS asymptotics.** Recall from (2.4) that, in the AR(1) model, $y_t^+ = [\alpha + \beta y_{t-1} + u_t]_-$, and that in this case (2.5) specialises to

$$y_t = [\alpha + \beta y_{t-1} + u_t]_+ = \alpha + \beta y_{t-1} + u_t - y_t^-.$$

**Lemma A.2.** Suppose Assumptions A1 and A2 hold with $k = 1$. Then

(i) $T^{-1/2} \sum_{t=1}^T (u_t - y_t^-) \xrightarrow{d} J_\theta(1) - b_0 - c \int_0^1 J_\theta(r) \, dr - a$;
(ii) \( \sum_{t=1}^{T}(y_t^-)^2 = o_p(T) \); and

(iii) \( T^{-1} \sum_{t=1}^{T}(\Delta y_t)^2 \overset{p}{\to} \sigma^2 \).

Proof. (i). We first note from (A.2) that

\[
\sum_{t=1}^{T}(u_t - y_t^-) = \sum_{t=1}^{T}(y_t - \alpha - \beta y_{t-1}) = \sum_{t=1}^{T}(y_t - y_{t-1}) - (\beta - 1) \sum_{t=1}^{T} y_{t-1} - T \alpha
\]

\[
= y_T - y_0 - (\beta - 1) \sum_{t=1}^{T} y_{t-1} - T^{1/2} a.
\]

Since \( T(\beta - 1) = c + o(1) \), it follows from Theorem 3.1 and the CMT that, under \( A_1 \),

\[
\frac{1}{T^{1/2}} \sum_{t=1}^{T}(u_t - y_t^-) = \frac{1}{T^{1/2}}(y_T - y_0) - \frac{c + o(1)}{T^{3/2}} \sum_{t=1}^{T} y_{t-1} - a
\]

\[
\overset{d}{\to} J_\theta(1) - b_0 - c \int_0^1 J_\theta(r) \, dr - a.
\]

(ii). Since \( \beta \geq 0 \) and \( y_{t-1} \geq 0 \),

\[
0 \geq y_t^- = (\alpha + \beta y_{t-1} + u_t)1\{\alpha + \beta y_{t-1} + u_t \leq 0\} \geq v_t1\{v_t \leq -\beta y_{t-1}\},
\]

where \( v_t := \alpha + u_t \). Hence

\[
\max_{1 \leq t \leq T} |y_t^-| \leq \max_{1 \leq t \leq T} |v_t| \leq \frac{|a|}{T^{1/2}} + \max_{1 \leq t \leq T} |u_t| = o_p(T^{1/2})
\]

where the final equality holds since \( \{u_t\} \) is i.i.d. with finite variance, under Assumption \( A2.1 \). Further, by the result of part (i),

\[
\sum_{t=1}^{T} y_t^- = \sum_{t=1}^{T} u_t + O_p(T^{1/2}) = O_p(T^{1/2}).
\]

Hence

\[
\sum_{t=1}^{T}(y_t^-)^2 \leq \max_{1 \leq t \leq T}|y_t^-| \sum_{t=1}^{T}|y_t^-| = -\max_{1 \leq t \leq T}|y_t^-| \sum_{t=1}^{T} y_t^- = o_p(T^{1/2})O_p(T^{1/2}) = o_p(T).
\]

(iii). Since

\[
\sum_{t=1}^{T} \alpha^2 = a^2 = O(1), \quad \sum_{t=1}^{T} \alpha(\beta - 1)y_{t-1} = O_p(1), \quad \sum_{t=1}^{T}[(\beta - 1)y_{t-1}]^2 = O_p(1)
\]

by Theorem 3.1 and the CMT;

\[
\sum_{t=1}^{T} \alpha(u_t - y_t^-) = O_p(1),
\]
by Lemma A.2(i); and
\[
\frac{1}{T} \sum_{t=1}^{T} (u_t - y_t^-)^2 = \frac{1}{T} \sum_{t=1}^{T} u_t^2 - \frac{2}{T} \sum_{t=1}^{T} y_t^- u_t + \frac{1}{T} \sum_{t=1}^{T} (y_t^-)^2 \xrightarrow{p} \sigma^2,
\]
by the law of large numbers, the Cauchy–Schwarz (CS) inequality, and the result of part (ii); and
\[
\left| \sum_{t=1}^{T} (\beta - 1)y_{t-1}(u_t - y_t^-) \right| \leq \sqrt{\sum_{t=1}^{T} (\beta - 1)^2 \sum_{t=1}^{T} (u_t - y_t^-)^2} = O_p(\sqrt{T}),
\]
by the preceding; it follows that
\[
\frac{1}{T} \sum_{t=1}^{T} (\Delta y_t)^2 = \frac{1}{T} \sum_{t=1}^{T} (\alpha + (\beta - 1)y_{t-1} + u_t - y_t^-)^2 = \frac{1}{T} \sum_{t=1}^{T} (u_t - y_t^-)^2 + o_p(1) \xrightarrow{p} \sigma^2.
\]

**Proof of Theorem 3.3.** We have from (A.2) that
\[
\begin{bmatrix}
\hat{\alpha}_T - \alpha \\
\hat{\beta}_T - \beta
\end{bmatrix} = \left( \sum_{t=1}^{T} \begin{bmatrix} 1 & y_{t-1} \\ y_{t-1} & y_{t-1}^2 \end{bmatrix} \right)^{-1} \sum_{t=1}^{T} \begin{bmatrix} 1 \\ y_{t-1} \end{bmatrix} (u_t - y_t^-)
\]
where
\[
\frac{1}{T^{1/2}} \sum_{t=1}^{T} (u_t - y_t^-) \xrightarrow{d} J_\theta(1) - b_0 - c \int_0^1 J_\theta(r) \, dr - a
\]
by Lemma A.2(i). To obtain the weak limit of \( \sum_{t=1}^{T} y_{t-1}(u_t - y_t^-) \), we note that since only one of \( y_t \) and \( y_t^- \) can be nonzero, \( y_t y_t^- = 0 \), and hence \( y_t^- y_{t-1} = -y_t^- \Delta y_t \). Thus by the CS inequality and Lemma A.2(ii)–(iii),
\[
\sum_{t=1}^{T} |y_t^- y_{t-1}| = \sum_{t=1}^{T} |y_t^- \Delta y_t| \leq \left[ \sum_{t=1}^{T} (y_t^-)^2 \sum_{t=1}^{T} (\Delta y_t)^2 \right]^{1/2} = o_p(T).
\]
It follows by the preceding and Liang et al. (2016, Theorem 2.1) that
\[
\frac{1}{T} \sum_{t=1}^{T} y_{t-1}(u_t - y_t^-) = \frac{1}{T} \sum_{t=1}^{T} y_{t-1} u_t + o_p(1) \xrightarrow{d} \sigma \int_0^1 J_\theta(r) \, dW(r).
\]
In view of (A.4)–(A.6), a final appeal to Theorem 3.1 and the CMT yields
\[
\begin{bmatrix}
T^{1/2}(\hat{\alpha}_T - \alpha) \\
T(\hat{\beta}_T - \beta)
\end{bmatrix} \xrightarrow{d} \begin{bmatrix} 1 \\ \int J_\theta(r) \, dr \end{bmatrix}^{-1} \begin{bmatrix} J_\theta(1) - b_0 - c \int J_\theta(r) \, dr - a \\
\sigma \int J_\theta(r) \, dW(r) \end{bmatrix}.
\]
Verification of (3.9). By the Frisch-Waugh-Lovell theorem, and using partial summation (as in the proof of Theorem 3.1 in Phillips (1987a)) we have

\[ \hat{\beta}_T - 1 = \frac{\sum_{t=1}^T y_{t-1}^{\mu} y_t^{\mu}}{\sum_{t=1}^T (y_{t-1}^{\mu})^2} = 1 = \frac{\sum_{t=1}^T \Delta y_t^{\mu}}{\sum_{t=1}^T (y_{t-1}^{\mu})^2} \]

(A.7)

\[ \hat{\beta} = \frac{(y_{t-1}^{\mu} + \Delta y_t^{\mu})^2 - (y_{t-1}^{\mu})^2 - (\Delta y_t^{\mu})^2}{2 \sum_{t=1}^T (y_{t-1}^{\mu})^2} = \frac{(y_{t-1}^{\mu})^2 - (y_t^{\mu})^2 - \sum_{t=1}^T (\Delta y_t^{\mu})^2}{2 \sum_{t=1}^T (y_{t-1}^{\mu})^2} \]

(3.9) then follows (noting the centring here is around 1, rather than \( \beta \)) by Theorem 3.1, Lemma A.2(iii), and the CMT. □

Proof of Corollary 3.2 (\( k = 1 \)). Once we have shown that \( \hat{\sigma}_T^2 \overset{p}{\to} \sigma^2 \), the limiting distributions of the \( t \) statistics will follow from Theorem 3.1 and the CMT. Noting from (A.2) that

\[ \hat{u}_t = y_t - \hat{\alpha}_T - \hat{\beta}_T y_{t-1} = (\alpha - \hat{\alpha}_T) + (\beta - \hat{\beta}_T) y_{t-1} + (u_t - y_t), \]

and that \( \sum_{t=1}^T \hat{u}_t = 0 \) and \( \sum_{t=1}^T y_{t-1} \hat{u}_t = 0 \), we have

\[ \sum_{t=1}^T [u_t^2 - (u_t - y_t^-)^2] = \sum_{t=1}^T [\hat{u}_t - (u_t - y_t^-)][\hat{u}_t + (u_t - y_t^-)] \]

\[ = \sum_{t=1}^T [(\alpha - \hat{\alpha}_T) + (\beta - \hat{\beta}_T) y_{t-1}][\hat{u}_t + (u_t - y_t^-)] \]

\[ = (\alpha - \hat{\alpha}_T) \sum_{t=1}^T (u_t - y_t^-) + (\beta - \hat{\beta}_T) \sum_{t=1}^T y_{t-1} (u_t - y_t^-) \]

\[ = O_p(T^{-1/2}) O_p(T^{1/2}) + O_p(T^{-1}) O_p(T) = O_p(1) \]

where the orders of \( \sum_{t=1}^T (u_t - y_t^-) \) and \( \sum_{t=1}^T y_{t-1} (u_t - y_t^-) \) follow from (A.5) and (A.6) above, and the rates of convergence of \( \hat{\alpha}_T \) and \( \hat{\beta}_T \) from Theorem 3.3. Hence, by (A.3),

\[ \hat{\sigma}_T^2 = \frac{1}{T} \sum_{t=1}^T \hat{u}_t^2 = \frac{1}{T} \sum_{t=1}^T (u_t - y_t^-)^2 + O_p(T^{-1}) \overset{p}{\to} \sigma^2. \] □

Appendix B. Proofs of results for general \( k \)

B.1. Limiting distribution of \( T^{-1/2} y_{[n:]} \): \( AR(k) \) case. Let \( \rho \) denote the inverse of the root of \( B(z) \) closest to real unity, which for \( T \) sufficiently large must be real because \( A2 \) permits \( B(z) \) to have only one root local to unity. Thus \( B(z) \) factorises as

\[ B(z) = (1 - \beta) z + \phi(z)(1 - z) = \psi(z)(1 - \rho z) \]

(B.1)
for $z \in \mathbb{C}$ where $\psi(z) = 1 - \sum_{i=1}^{k-1} \psi_i z^i$. Under A2.2, $\beta = \beta_T \to 1$ and, thus, $\rho = \rho_T \to 1$, from which it follows that $\psi_i \to \phi_i$ for $i \in \{1, \ldots, k-1\}$, as $T \to \infty$. Thus for $T$ sufficiently large, $\rho$ is real and positive (as we shall maintain throughout the following), and such a condition as A4 also holds when each $\phi_i$ in (3.5) is replaced by $\psi_i$. Moreover, taking $z = \rho_T^{-1}$ in the preceding, it follows that

$$0 = B(\rho_T^{-1}) = (1 - \beta_T)\rho_T^{-1} + \phi(\rho_T^{-1})(1 - \rho_T^{-1})$$

(B.2)

$$\Leftrightarrow T(\rho_T - 1) = \phi(\rho_T^{-1})^{-1}T(\beta_T - 1) \to \phi(1)^{-1}c =: c_\phi.$$  

The factorisation (B.1) also permits us to rewrite the model (2.6) for $\{y_t\}$ in terms of the quasi-differences $\Delta \rho y_t$,

$$\psi(L)\Delta \rho y_t = \alpha + u_t - y_t^-,$$

where $\Delta \rho := 1 - \rho L$. With the aid of this representation we establish the following preliminary lemmas.

**Lemma B.1.** Suppose Assumption A2 holds, and define

(B.4)  

$$x_t := \psi(\rho^{-1})\rho^{-t}y_t \quad \xi_t := \sum_{s=1}^{t} \rho^{-s}(\alpha + u_s) - \gamma_\rho(L)[\rho^{-t}\Delta \rho y_t - \Delta \rho y_0]$$

for $t \in \{1, \ldots, T\}$, where $\gamma_\rho(L)$ is the $k - 2$ order polynomial such that $\psi(L) = \psi(\rho^{-1}) + \gamma_\rho(L)\Delta \rho$ (with $\psi(L) = 1$ and $\gamma_\rho(L) = 0$ when $k = 1$). Then for $t \in \{1, \ldots, T\}$,

(B.5)  

$$x_t = [x_{t-1} + \Delta \xi_t]_+.$$

**Proof.** Observe that for any series $\{\eta_s\}$,

(B.6)  

$$\sum_{s=1}^{t} \rho^{-s}\Delta \rho \eta_s = \sum_{s=1}^{t} \rho^{-s}(\eta_s - \rho \eta_{s-1}) = \sum_{s=1}^{t} \rho^{-s}\eta_s - \sum_{s=1}^{t} \rho^{-s-1}\eta_{s-1} = \eta_t - \rho^t \eta_0.$$

Applying this to $\psi(L)\Delta \rho y_s = \alpha + u_s - y_s^-$ (from (B.3) above), we obtain

$$\psi(L)y_t - \rho^t\psi(L)y_0 = \sum_{s=1}^{t} \rho^{-s}(\alpha + u_s) - \sum_{s=1}^{t} \rho^{-s}y_s^-.$$  

Using $\psi(L) = \psi(\rho^{-1}) + \gamma_\rho(L)\Delta \rho$, and rearranging yields

$$\psi(\rho^{-1})y_t + \sum_{s=1}^{t} \rho^{-s}y_s^- = \sum_{s=1}^{t} \rho^{-s}(\alpha + u_s) - \gamma_\rho(L)(\Delta \rho y_t - \rho^t \Delta \rho y_0) + \rho^t\psi(\rho^{-1})y_0.$$  

Finally, multiplying by $\rho^{-t}$ and recalling the definitions of $(x_t, \xi_t)$ from (B.4), we have

(B.7)  

$$x_t + \sum_{s=1}^{t} \rho^{-s}y_s^- = x_0 + \xi_t.$$  

---

8Formally, one should write $\psi_{i,T}$, since the coefficients of the polynomial $\psi(z)$ depend on $\beta$, which in turn varies with $T$. We omit this dependence on $T$ for ease of notation.
To proceed from (B.7) to show that \((x_t, \xi_t)\) satisfy (B.5), we note that for \(t \geq 1,\)

\[
\xi_{t+1} = x_{t+1} - x_0 + \sum_{s=1}^{t+1} \rho^{-s} y_s = x_{t+1} - x_0 + \rho^{-(t+1)} y_{t+1} + \sum_{s=1}^{t} \rho^{-s} y_s
\]

\[
= x_{t+1} + \rho^{-(t+1)} y_{t+1} + \xi_t - x_t
\]

where the first and last equalities follow from (B.7). Hence

\[
x_{t+1} + \rho^{-(t+1)} y_{t+1} = x_t + \Delta \xi_{t+1}.
\]

From (2.1) and (2.4), at most one of \(y_{t+1}\) and \(y_{t-1}\) can be nonzero, and must have opposite signs. Since \(\psi(\rho^{-1}) \to \phi(1) > 0\) (due to stationarity), we must have \(\psi(\rho^{-1}) \rho^{-t} > 0\) for all \(T\) sufficiently large. The same must also be true for \(x_{t+1} = \psi(\rho^{-1}) \rho^{-(t+1)} y_{t+1}\) and \(\rho^{-(t+1)} y_{t+1}\).

Hence,

\[
x_{t+1} = [x_t + \Delta \xi_{t+1}]_+
\]

for \(t \geq 1\). Plugging \(\xi_0 = 0\) into (B.7) when \(t = 1\), we have

\[
x_1 + \rho^{-1} y_1 = x_0 + \xi_1 = x_0 + \Delta \xi_1
\]

and thus \(x_1 = [x_0 + \Delta \xi_1]_+,\) by the same argument. \(\square\)

**Lemma B.2.** Suppose Assumptions A1–A4 hold. Then there exists a \(C < \infty\) such that

\[
\max_{-k+2 \leq t \leq T} (\|\Delta \rho y_t\|_2 + \|\Delta y_t\|_2) < C.
\]

**Proof.** We have from (B.3) that

\[
\Delta \rho y_t + y_t^{-1} = \sum_{i=1}^{k-1} \psi_i \Delta \rho y_{t-i} + \alpha + u_t =: w_t,
\]

for \(t \in \{1, \ldots, T\}\). Since, as noted in the proof of Lemma B.1, only one of \(y_t\) and \(y_t^{-1}\) can be nonzero, and have opposite signs,

\[
\Delta \rho y_t > 0 \implies y_t > \rho y_{t-1} \geq 0 \implies y_t^{-1} = 0.
\]

It follows that either \(w_t > 0\), in which case \(\Delta \rho y_t = w_t\); or \(w_t \leq 0\), in which case \(\Delta \rho y_t \in [w_t, 0]\). Hence there exists a \(\delta_t \in [0, 1]\) such that \(\Delta \rho y_t = \delta_t w_t\) for \(t \in \{1, \ldots, T\}\). Taking \(w_0 = \Delta \rho y_0\) and \(\delta_0 = 1\), (B.8) is equivalent to a dynamical system defined by

\[
w_t = \psi_1 \delta_{t-1} w_{t-1} + \sum_{i=2}^{k-1} \psi_i \Delta \rho y_{t-i} + \alpha + u_t,
\]

(B.9)

\[
\Delta \rho y_{t-1} = \delta_{t-1} w_{t-1}
\]

(B.10)
for $t \in \{1, \ldots, T\}$, for an appropriate sequence $\{\delta_t\} \subset [0, 1]$. Defining

$$w_t := \begin{bmatrix} w_t \\ \Delta_\rho y_{t-1} \\ \vdots \\ \Delta_\rho y_{t-k+2} \end{bmatrix}, \quad v_t := \begin{bmatrix} \alpha + u_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad F_\delta(\psi) := \begin{bmatrix} \psi_1 \delta & \psi_2 & \cdots & \psi_{k-2} & \psi_{k-1} \\ \delta & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \cdots & \cdots \\ 1 & 0 \end{bmatrix}$$

where $\psi := (\psi_1, \ldots, \psi_{k-1})$, we can write the companion form of (B.9)–(B.10) as

$$w_t = F_{\delta_{t-1}}(\psi)w_{t-1} + v_t$$

for $t \in \{1, \ldots, T\}$, with the initialisation $w_0 := (\Delta_\rho y_0, \Delta_\rho y_{-1}, \ldots, \Delta_\rho y_{-k+2})^T$.

Since $\psi \rightarrow \phi = (\phi_1, \ldots, \phi_{k-1})^T$ as $T \rightarrow \infty$, $F_\delta(\psi) \rightarrow F_\delta(\phi) = F_\delta$, where $F_\delta$ is as defined in (3.5). By Proposition 1.8 in Jungers (2009) and the continuity of the JSR,

$$\lambda_{JSR}(\{F_\delta(\psi) \mid \delta \in [0, 1]\}) = \lambda_{JSR}(\{F_0(\psi), F_1(\psi)\}) \rightarrow \lambda_{JSR}(\{F_0, F_1\}) < 1$$

under A4. It follows that there exists a norm $\|\cdot\|_*$ and a $\gamma \in [0, 1)$ such that (for all $T$ sufficiently large),

$$\|w_t\|_* \leq \|F_{\delta_{t-1}}\|_*\|w_{t-1}\|_* + \|v_t\|_* \leq \gamma\|w_{t-1}\|_* + \|v_t\|_*$$

whence by backward substitution,

$$\|w_t\|_* \leq \sum_{s=0}^{t-1} \gamma^s\|v_{t-s}\|_* + \gamma^t\|w_0\|_*.$$

By the equivalence of norms on finite-dimensional spaces, there exists a $C < \infty$ such that

$$|\Delta_\rho y_{t-1}| \leq C \left[ \sum_{s=0}^{t-1} \gamma^s(|\alpha| + |u_{t-s}|) + \gamma^t \sum_{i=-k+2}^0 |\Delta_\rho y_i| \right].$$

Deduce that for any $p \geq 1$,

$$\|\Delta_\rho y_{t-1}\|_p \leq \frac{C|\alpha|}{1-\gamma} + C \sum_{s=0}^{t-1} \gamma^s\|u_{t-s}\|_p + C \sum_{i=-k+2}^0 \|\Delta_\rho y_i\|_p$$

$$\leq \frac{C|\alpha|}{1-\gamma} + C \max_{1 \leq s \leq t} \|u_s\|_p + C \sum_{i=-k+2}^0 \|\Delta_\rho y_i\|_p. \quad \text{(B.11)}$$

Now for each $i \in \{-k+2, \ldots, 0\}$, $\Delta_\rho y_i = (1-\rho)[y_0 - \sum_{j=i+1}^0 \Delta y_j] + \rho \Delta y_i$, and hence there exists a $C' < \infty$ such that

$$\|\Delta_\rho y_i\|_{2+\delta_u} \leq C' \left[ \frac{1}{T^{1/2}} \|T^{-1/2}y_0\|_{2+\delta_u} + \frac{1}{T} \sum_{j=i+1}^0 \|\Delta y_j\|_{2+\delta_u} + \|\Delta y_i\|_{2+\delta_u} \right],$$
where we have used that $1 - \rho = O(T^{-1})$. Taking $p = 2 + \delta_u$ in (B.11), it follows under A3 that $\max_{-k+2 \leq t \leq T} \| \Delta \rho y_t \|_{2+\delta_u}$ is bounded uniformly in $T$. To obtain the corresponding result for $\Delta y_t$, we use (B.6) with $\eta_t = y_t$ to write

$$
\Delta y_t = \Delta \rho y_t + (\rho - 1)y_{t-1} = \Delta \rho y_t + (\rho - 1) \left[ \sum_{s=1}^{t-1} \rho^{t-1-s} \Delta \rho y_s + \rho^{t-1} y_0 \right]
$$

whence there exists a $C'' < \infty$ such that

$$
\| \Delta y_t \|_{2 + \delta_u} \leq \| \Delta \rho y_t \|_{2 + \delta_u} + C'' \left[ T^{-1} \max_{1 \leq s \leq t} \| \Delta \rho y_s \|_{2 + \delta_u} + T^{-1/2} \| T^{-1/2} y_0 \|_{2 + \delta_u} \right],
$$

from which, under A3, the result follows.

**Proof of Theorem 3.2.** When $k = 1$, the result follows by Theorem 3.1; we therefore suppose $k \geq 2$. We first note that by (B.2) above, $\rho^{[rT]} \to e^{c_\rho r}$ uniformly in $r \in [0,1]$. Hence for $(x_t, \xi_t)$ as in Lemma B.1,

$$
T^{-1/2} \left[ x_0 + \sum_{s=1}^{[rT]} \Delta \xi_s \right] = T^{-1/2} x_0 + T^{-1/2} \xi_{[rT]}
$$

(B.12)

\[= (1) \psi(\rho^{-1}) T^{-1/2} y_0 + T^{-1/2} \sum_{s=1}^{[rT]} \rho^{-s}(\alpha + u_s) + o_p(1) \xrightarrow{(2)} K_{\theta_\phi}(r),\]

on $D[0,1]$, where $\theta_\phi = (a, \phi(1)b_0, \phi(1)^{-1}c)$, = (1) holds since Lemma B.2 implies that $\max_{0 \leq t \leq T} | \Delta \rho y_t | = o_p(T^{1/2})$, and $\xrightarrow{(2)}$ holds by the same arguments as which yielded (A.1) above and by recalling that $\psi(\rho^{-1}) \to \phi(1)$. Hence by Lemma B.1, $x_t$ and $v_t := \Delta \xi_t$ satisfy the requirements of Lemma A.1. We thus have

$$
T^{-1/2} y_{[rT]} = \psi(\rho^{-1})^{-1} \rho^{[rT]} T^{-1/2} x_{[rT]}
$$

\[\xrightarrow{d} \phi(1)^{-1} e^{c_\rho r} \left\{ K_{\theta_\phi}(r) + \sup_{r' \leq r} [-K_{\theta_\phi}(r')] \right\} = \phi(1)^{-1} J_{\theta_\phi}(r)
\]

on $D[0,1]$.

**B.2. OLS asymptotics: AR(k) case.** Since, by the implication of Assumption A4, all the roots of $\phi(z) = 1 - \sum_{i=1}^{k-1} \phi_i z^i$ lie strictly outside the unit circle, there exists a sequence $\{\varphi_i\}_{i=0}^\infty$ with $\varphi_0 = 1$ and $\sum_{i=0}^\infty |\varphi_i| < \infty$ such that $\phi^{-1}(z) := \sum_{i=0}^\infty \varphi_i z^i$ satisfies $\phi^{-1}(z) \phi(z) = 1$ for all $|z| \leq 1$. Moreover, there exists a $C < \infty$ and a $\gamma_\phi \in (0,1)$ such that $|\varphi_i| < C \gamma_\phi^i$ for all $i \geq 0$. (See e.g. Brockwell and Davis (1991, Section 3.3).)
Lemma B.3. Let \( \phi_m^{-1}(z) := \sum_{i=0}^{m} \varphi_i z^i \), where \( m \geq 1 \). Then there exists a \( C < \infty \), independent of \( m \), and \( \{d_{m,i}\}_{i=1}^{k-1} \) such that
\[
\phi_m^{-1}(z)\phi(z) = 1 - z^m \sum_{i=1}^{k-1} d_{m,i} z^i =: 1 - d_m(z)z^m
\]
for all \( |z| \leq 1 \) and \( m \in \mathbb{N} \), and where \( |d_{m,i}| \leq C \gamma_m \).

Proof. Since
\[
1 = \phi^{-1}(z)\phi(z) = \phi(z) \left[ \phi_m^{-1}(z) + \sum_{i=m+1}^{\infty} \varphi_i z^i \right]
\]
for \( |z| \leq 1 \), it follows that
\[
1 - \phi(z) \sum_{i=m+1}^{\infty} \varphi_i z^i = \phi(z)\phi_m^{-1}(z) = \left( 1 - \sum_{i=1}^{k-1} \varphi_i z^i \right) \sum_{i=0}^{m} \varphi_i z^i =: 1 - \sum_{i=1}^{m+k-1} \varphi_{m,i} z^i,
\]
using that \( \varphi_0 = 1 \). Matching coefficients on the left and right hand sides, we obtain \( \varphi_{m,i} = 0 \) for all \( i \in \{1, \ldots, m\} \); while for \( i \in \{m+1, \ldots, m+k-1\} \), \( \varphi_{m,i} = \sum_{j=0}^{(m+1)} \varphi_j \varphi_{i-j} \). Taking \( d_{m,i} := \varphi_{m,m+i} \sum_{j=0}^{i-1} \varphi_j \varphi_{m+i-j} \) for \( i \in \{1, \ldots, k-1\} \), and noting that
\[
|d_{m,i}| \leq \sum_{j=i}^{k-1} |\varphi_j||\varphi_{m+i-j}| \leq C \gamma_m \sum_{j=1}^{k-1} |\varphi_j|
\]
yields the result.

Lemma B.4. Suppose Assumptions A1–A4 hold. Then for each \( s \in \{0, \ldots, k-1\} \),

(i) \( T^{-1/2} \sum_{t=1}^{T} (u_t - y_t^-) \overset{d}{\to} \phi(1)[Y_{\theta}\phi(1) - c_{\phi} \int Y_{\theta}(r) - b_0] - a; \)

(ii) \( \sum_{t=1}^{T} (y_t^-)^2 = o_p(T); \)

(iii) \( T^{-1} \sum_{t=1}^{T} \Delta y_t \Delta y_{t-s} \overset{p}{\to} \sigma^2 \sum_{n=0}^{\infty} \varphi_n \varphi_{n+s}; \) and

(iv) \( \sum_{t=1}^{T} y_t \Delta y_{t-s} = O_p(T). \)

Proof. (i). Applying the factorisation (B.1) to (2.6), we get
\[
(B.13) \quad u_t - y_t^- = -\alpha - (\beta - 1)y_{t-1} + \phi(L)\Delta y_t
\]
and hence, recalling that \( \phi(1) = 1 - \sum_{i=1}^{k-1} \phi_i \), \( \alpha = T^{1/2}a \), and \( T(\beta - 1) \to c \) under A2,
\[
\frac{1}{T^{1/2}} \sum_{t=1}^{T} (u_t - y_t^-) = -a - \frac{T(\beta - 1)}{T^{3/2}} \sum_{t=1}^{T} y_{t-1} + \frac{1}{T^{1/2}} \left( \sum_{t=1}^{T} \Delta y_t - \sum_{i=1}^{k-1} \phi_i \sum_{t=1}^{T} \Delta y_{t-i} \right)
\]
\[
= -a - \frac{T(\beta - 1)}{T^{3/2}} \sum_{t=1}^{T} y_{t-1} + \frac{y_T - y_0}{T^{1/2}} - \sum_{i=1}^{k-1} \phi_i \frac{y_{t-i} - y_{0-i}}{T^{1/2}} \overset{d}{\to} -a - c \int_{0}^{1} Y_{\theta}(r) \, dr + \phi(1)[Y_{\theta}(1) - b_0] \]
where convergence holds by Assumption A1, Theorem 3.2 and the CMT, recalling that $c_\phi = \phi(1)^{-1}c$.

(ii). The argument is analogous to that used to prove Lemma A.2(ii). Rewriting the factorisation (B.1) as

$$
\beta z + (1 - z) \sum_{i=1}^{k-1} \phi_i z^i = 1 - (1 - \rho z) \left[ 1 - \sum_{i=1}^{k-1} \psi_i z^i \right] = \rho z + \sum_{i=1}^{k-1} \psi_i z^i (1 - \rho z)
$$

we have from (2.6) that

$$
y_t - \left[ \beta y_{t-1} + \sum_{i=1}^{k-1} \psi_i \Delta y_{t-i} + \alpha + u_t \right] = [\rho y_{t-1} + v_t],
$$

where $v_t := \sum_{i=1}^{k-1} \psi_i \Delta y_{t-i} + \alpha + u_t$. Since $\rho y_{t-1} \geq 0$, it follows that

$$0 \geq y_t^- = (\rho y_{t-1} + v_t)1\{\rho y_{t-1} + v_t \leq 0\} \geq v_t 1\{\rho y_{t-1} + v_t \leq 0\}.$$

By Lemma B.2, $\|\Delta y_t\|_{2+\delta_u}$ is bounded uniformly in $t$. Hence, under A3, so too is

$$
\|v_t\|_{2+\delta_u} \leq \sum_{i=1}^{k-1} \psi_i \|\Delta y_{t-i}\|_{2+\delta_u} + |\alpha| + \|u_t\|_{2+\delta_u},
$$

whence it follows that $\max_{1 \leq t \leq T} |y_t^-| \leq \max_{1 \leq t \leq T} |v_t| = o_p(T^{1/2})$. Hence, using the result of part (i),

$$
\sum_{t=1}^{T} (y_t^-)^2 \leq \max_{1 \leq t \leq T} |y_t^-| \sum_{t=1}^{T} |y_t^-| = -\max_{1 \leq t \leq T} |y_t^-| \sum_{t=1}^{T} y_t^- = -\max_{1 \leq t \leq T} |y_t^-| \left( \sum_{t=1}^{T} u_t + O_p(T^{1/2}) \right) = o_p(T).
$$

(iii). Let $s \in \{0, \ldots, k - 1\}$. By Lemma B.3, applying $\phi_{t-1}^{-1}(L) = \sum_{i=0}^{t-1} \varphi_i L^i$ to both sides of (B.13) yields

$$
\Delta y_t - d_{t-1}(L) \Delta y_1 = \phi_{t-1}^{-1}(L) \phi(L) \Delta y_t = \sum_{i=0}^{t-1} \varphi_i L^i [(\beta - 1) y_{t-1} + \alpha + u_t - y_t^-]
$$

and hence for $t \in \{1, \ldots, T\}$,

$$
\Delta y_t = \sum_{i=0}^{t-1} \varphi_i (\alpha + u_{t-i}) + (\beta - 1) \sum_{i=0}^{t-1} \varphi_i y_{t-1-i} - \sum_{i=0}^{t-1} \varphi_i y_{t-i}^- + d_{t-1}(L) \Delta y_1
$$

(B.14) $w_t + r_{1,t} + r_{2,t} + r_{3,t}$. 


Decompose
\[ w_t = \sum_{i=0}^{t-1} \varphi_i (\alpha + u_{t-i}) = \sum_{i=0}^{\infty} \varphi_i u_{t-i} - \sum_{i=t}^{\infty} \varphi_i u_{t-i} + \frac{a}{T^{1/2}} \sum_{i=0}^{t-1} \varphi_i y_t =: \eta_t + r_{4,t} + r_{5,t}. \]

The sequence \( \{\eta_t\} \) is a stationary linear process whose coefficients decay exponentially; hence by Phillips and Solo (1992, Thm. 3.7 and Rem. 3.9),

\[ \frac{1}{T} \sum_{t=s+1}^{T} \eta_t \eta_{t-s} \xrightarrow{p} \sigma^2 \sum_{n=0}^{\infty} \varphi_n \varphi_{n+s}. \]

Since for all \( i \), \( |\varphi_i| < C \gamma_i^\rho, \gamma_i \in (0, 1) \), we have

\[ \mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} r_{1,t}^2 \right] = \frac{\sigma^2}{T} \sum_{t=1}^{T} \sum_{i=0}^{\infty} \varphi_i^2 = O(T^{-1}), \quad \frac{1}{T} \sum_{t=1}^{T} r_{2,t}^2 = \frac{\sigma^2}{T^2} \sum_{t=1}^{T} \left( \sum_{i=0}^{t-1} \varphi_i \right)^2 = O(T^{-1}). \]

It follows by the CS inequality that \( \frac{1}{T} \sum_{t=s+1}^{T} (\eta_t r_{t,t-s} + \eta_{t-s} r_{t,t}) = o_p(1) \) for \( \ell \in \{4, 5\} \) and \( \frac{1}{T} \sum_{t=s+1}^{T} r_{\ell,t} r_{\ell',t-s} = o_p(1) \) for \( \ell, \ell' \in \{4, 5\} \), and hence

\[ \frac{1}{T} \sum_{t=s+1}^{T} w_t w_{t-s} = \frac{1}{T} \sum_{t=s+1}^{T} \eta_t \eta_{t-s} + o_p(1) \xrightarrow{p} \sigma^2 \sum_{n=0}^{\infty} \varphi_n \varphi_{n+s} \]

for each \( s \in \{0, \ldots, k-1\} \). Thus, if we can show that

\[ \sum_{t=1}^{T} r_{1,t}^2 = O_p(1) \quad \sum_{t=1}^{T} r_{2,t}^2 = o_p(T) \quad \sum_{t=1}^{T} r_{3,t}^2 = O_p(1). \]

it will follow that

\[ \frac{1}{T} \sum_{t=1}^{T} \Delta y_t \Delta y_{t-s} = (1) \frac{1}{T} \sum_{t=s+1}^{T} \Delta y_t \Delta y_{t-s} + o_p(1) \]

\[ = (2) \frac{1}{T} \sum_{t=s+1}^{T} w_t w_{t-s} + o_p(1) \xrightarrow{(3)} \sigma^2 \sum_{n=0}^{\infty} \varphi_n \varphi_{n+s}. \]

where \( = (1) \) holds by Lemma B.2, \( = (2) \) by (B.14)–(B.16) and the CS inequality, and \( \xrightarrow{(3)} \) by (B.15).

It remains to prove (B.16). Since \( \beta - 1 = O(T^{-1}) \), there exists a \( C < \infty \) such that

\[ |r_{1,t}| \leq C \sum_{i=0}^{t-1} |\varphi_i||y_{t-1-i}| \leq \left( \sum_{i=0}^{\infty} |\varphi_i| \right) \frac{C}{\sqrt{T}} \max_{0 \leq t \leq T-1} \left| y_t / \sqrt{T} \right| = O_p(T^{-1/2}) \]

uniformly in \( t \) by Theorem 3.2; the first part of (B.16) follows immediately. Next,

\[ \sum_{t=1}^{T} r_{2,t}^2 = \sum_{t=1}^{T} \left( \sum_{i=0}^{t-1} \varphi_i y_{t-i} \right)^2 = \sum_{t=1}^{T} \sum_{i=0}^{t-1} \sum_{j=0}^{t-1} \varphi_i \varphi_j y_{t-i} y_{t-j} \]
\[ y_t = \alpha + \beta y_{t-1} + \phi^T \Delta y_{t-1} + u_t - y_t^- \]

the centred and rescaled OLS estimators \( \hat{\mu}_T^T := (\hat{\alpha}_T, \hat{\beta}_T, \hat{\phi}_T^T) \) of \( \mu^T = (\alpha, \beta, \phi^T) \) are equal to

\[
\begin{bmatrix}
T^{1/2} (\hat{\alpha}_T - \alpha) \\
T (\hat{\beta}_T - \beta) \\
\hat{\phi}_T - \phi
\end{bmatrix}
= D_{2,T} (\hat{\mu}_T - \mu) = (D_{1,T}^{-1} M_T D_{2,T}^{-1})^{-1} D_{1,T}^{-1} m_T
\]

where \( D_{1,T} := \text{diag}\{T^{1/2}, T, I_{k-1}T\} \), \( D_{2,T} := \text{diag}\{T^{1/2}, T, I_{k-1}\} \),

\[
D_{1,T}^{-1} M_T D_{2,T}^{-1} = \begin{bmatrix}
1 & T^{-3/2} \sum_{t=1}^T y_{t-1} & T^{-1/2} \sum_{t=1}^T \Delta y_{t-1}^T \\
T^{-3/2} \sum_{t=1}^T y_{t-1} & T^{-2} \sum_{t=1}^T y_{t-1}^2 & T^{-1} \sum_{t=1}^T y_{t-1} \Delta y_{t-1}^T \\
T^{-3/2} \sum_{t=1}^T \Delta y_{t-1} & T^{-2} \sum_{t=1}^T y_{t-1} \Delta y_{t-1} & T^{-1} \sum_{t=1}^T \Delta y_{t-1} \Delta y_{t-1}^T
\end{bmatrix}
\]
and

\[(B.19) \quad D_{1,T}^{-1} m_T = \begin{bmatrix}
  T^{-1/2} \sum_{t=1}^{T} (u_t - y_t^-) \\
  T^{-1} \sum_{t=1}^{T} y_{t-1}(u_t - y_t^-) \\
  T^{-1} \sum_{t=1}^{T} \Delta y_{t-1}(u_t - y_t^-)
\end{bmatrix}.
\]

It remains to determine the weak limits of the elements of (B.18) and (B.19). We consider (B.18) first. By Theorem 3.2 and the CMT,

\[(B.20) \quad Y_T := \begin{bmatrix}
  1 \\
  T^{-3/2} \sum_{t=1}^{T} y_{t-1} \\
  T^{-2} \sum_{t=1}^{T} y_{t-1}^2
\end{bmatrix} \xrightarrow{d} \begin{bmatrix}
  1 \\
  \int Y_{\theta_\phi}(r) \, dr \\
  \int Y_{\theta_\phi}^2(r) \, dr
\end{bmatrix} := Y_{\theta_\phi}.
\]

By Theorem 3.2 and Lemma B.4(iv),

\[(B.21) \quad \Xi_T := \begin{bmatrix}
  T^{-1/2} \sum_{t=1}^{T} \Delta y_{t-1}^T \\
  T^{-1} \sum_{t=1}^{T} y_{t-1} \Delta y_{t-1}^T
\end{bmatrix} = O_p(1).
\]

By Lemma B.4(iii),

\[(B.22) \quad \Omega_T := T^{-1} \sum_{t=1}^{T} \Delta y_{t-1} \Delta y_{t-1}^T \xrightarrow{p} \Omega
\]

where \(\Omega_{ij} := \sigma^2 \sum_{n=0}^{\infty} \varphi_n \varphi_{n+i-j}\). It follows by the partitioned matrix inversion formula and the continuity of matrix inversion that

\[(B.23) \quad (D_{1,T}^{-1} M_T D_{2,T}^{-1})^{-1} = \begin{bmatrix}
  Y_T & \Xi_T \\
  \phi(1) & \Omega_T
\end{bmatrix}^{-1} = \begin{bmatrix}
  Y_T^{-1} & -Y_T^{-1} \Xi_T \Omega_T^{-1} \\
  0 & \Omega_T^{-1}
\end{bmatrix} + o_p(1).
\]

We turn next to (B.19). By Lemma B.4(i),

\[(B.24) \quad Z_T^{(a)} := \frac{1}{T^{1/2}} \sum_{t=1}^{T} (u_t - y_t^-) \rightarrow \phi(1) \left[ Y_{\theta_\phi}(1) - c_\phi \int_0^1 Y_{\theta_\phi}(r) \, dr - b_0 \right] - a := Z_T^{(a)}.
\]

Next, since only one of \(y_t\) and \(y_t^-\) can be nonzero, \(y_{t-1}y_t^- = -\Delta y_t y_t^-\), and hence

\[(B.25) \quad Z_T^{(b)} := \frac{1}{T} \sum_{t=1}^{T} y_{t-1}(u_t - y_t^-) = \frac{1}{T} \sum_{t=1}^{T} y_{t-1} u_t + \frac{1}{T} \sum_{t=1}^{T} \Delta y_t y_t^- = (1) \frac{1}{T} \sum_{t=1}^{T} y_{t-1} u_t + o_p(1) \xrightarrow{d (2)} \int_0^1 Y_{\theta_\phi}(r) \, dW(r) := Z_T^{(b)}\]

where \(\xrightarrow{d (2)}\) holds by Theorem 3.2 and Liang et al. (2016, Theorem 2.1), and \(= (1)\) since

\[(B.26) \quad \left| \frac{1}{T} \sum_{t=1}^{T} \Delta y_t y_t^- \right|^2 \leq \frac{1}{T} \sum_{t=1}^{T} (\Delta y_t)^2 \frac{1}{T} \sum_{t=1}^{T} (y_t^-)^2 = o_p(1)
\]
by the CS inequality and Lemma B.4(ii)–(iii). Finally, for \( s \in \{1, \ldots, k - 1\} \),

\[
\frac{1}{T} \sum_{t=1}^{T} \Delta y_{t-s}(u_t - y_t^-) = \frac{1}{T} \sum_{t=1}^{T} \Delta y_{t-s} u_t - \frac{1}{T} \sum_{t=1}^{T} \Delta y_{t-s} y_t^- = \frac{1}{T} \sum_{t=1}^{T} \Delta y_{t-s} u_t + o_p(1),
\]

by the same argument as which yielded (B.26). Further, by Lemma B.2,

\[
\mathbb{E} \left( \frac{1}{T} \sum_{t=1}^{T} \Delta y_{t-s} u_t \right)^2 = \frac{\sigma^2}{T^2} \sum_{t=1}^{T} \mathbb{E}(\Delta y_{t-s})^2 = O(T^{-1}).
\]

Hence

\[
(B.27) \quad \frac{1}{T} \sum_{t=1}^{T} \Delta y_{t-s}(u_t - y_t^-) = O_p(T^{-1/2}) + o_p(1) = o_p(1).
\]

Letting \( Z_T := (Z_T^{(\alpha)}, Z_T^{(\beta)})^T \) and \( Z_{\theta_0} := (Z_{\theta_0}^{(\alpha)}, Z_{\theta_0}^{(\beta)})^T \), it follows from (B.19), (B.24), (B.25) and (B.27) that

\[
D_{1,T}^{-1} m_T = \begin{bmatrix} Z_T \\ o_p(1) \end{bmatrix} \xrightarrow{d} \begin{bmatrix} Z_{\theta_0} \\ 0 \end{bmatrix}.
\]

Therefore, recalling (B.17), and using (B.20)–(B.23), we obtain

\[
D_{2,T}(\hat{\mu}_T - \mu) = \begin{bmatrix} \mathcal{Y}_T^{-1} - \mathcal{Y}_T^{-1} \mathcal{Z}_T \mathcal{Z}_T^{-1} + o_p(1) \end{bmatrix} \begin{bmatrix} Z_T \\ o_p(1) \end{bmatrix}
\]

\[
= \begin{bmatrix} \mathcal{Y}_T^{-1} Z_T \\ 0 \end{bmatrix} + o_p(1) \xrightarrow{d} \begin{bmatrix} \mathcal{Y}_{\theta_0}^{-1} Z_{\theta_0} \\ 0 \end{bmatrix}.
\]

**Proof of Corollary 3.2** \((k \geq 2)\). We first show that \( \hat{\sigma}_T^2 \xrightarrow{p} \sigma^2 \). Adapting the argument from the \( k = 1 \) case, we have

\[
\sum_{t=1}^{T} [\hat{u}_t^2 - (u_t - y_t^-)^2] = \sum_{t=1}^{T} [\alpha - \hat{\alpha}_T] + (\beta - \hat{\beta}_T)y_{t-1} + (\phi - \hat{\phi}_T)^T \Delta y_{t-1} [\hat{u}_t + (u_t - y_t^-)]
\]

\[
= (\alpha - \hat{\alpha}_T) \sum_{t=1}^{T} (u_t - y_t^-) + (\beta - \hat{\beta}_T) \sum_{t=1}^{T} y_{t-1}(u_t - y_t^-)
\]

\[
+ (\phi - \hat{\phi}_T)^T \sum_{t=1}^{T} \Delta y_{t-1} (u_t - y_t^-)
\]

\[
= O_p(T^{-1/2})O_p(T^{1/2}) + O_p(T^{-1})O_p(T) + o_p(1)o_p(T) = o_p(T)
\]

where the the orders of the sums follow from Lemma B.4(i), (B.25) and (B.27), and the rates of convergence of the OLS estimators from Theorem 3.4. It follows that

\[
\frac{1}{T} \sum_{t=1}^{T} \hat{u}_t^2 = \frac{1}{T} \sum_{t=1}^{T} (u_t - y_t^-)^2 + o_p(1) = \frac{1}{T} \sum_{t=1}^{T} u_t^2 + o_p(1) \xrightarrow{p} \sigma^2
\]
by Lemma B.4(ii), the LLN and the CS inequality.

To complete the proof, we note that since \( \phi(1)^{-1}J_{\theta_\phi}(r) = Y_{\theta_\phi}(r) \),

\[
Y_{\theta_\phi} = \begin{bmatrix}
1 & 1 \\
0 & \phi(1)^{-1}
\end{bmatrix}
\begin{bmatrix}
\int_0^1 Y_{\theta_\phi}(r) \, dr \\
\int_0^1 J_{\theta_\phi}(r) \, dr \\
\int_0^1 J_{\theta_\phi}^2(r) \, dr
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & \phi(1)^{-1}
\end{bmatrix}.
\]

It follows that the result of Theorem 3.4 can be rewritten as

\[
\begin{bmatrix}
T^{1/2}(\hat{\alpha}_T - \alpha) \\
T(\hat{\beta}_T - \beta)
\end{bmatrix} \overset{d}{\to} \begin{bmatrix}
a_{\theta_\phi} \\
\phi(1)b_{\theta_\phi}
\end{bmatrix}.
\]

By (B.20) and (B.23), the upper left 2 \times 2 block of \( D_{1,T}^{-1}M_TD_{2,T}^{-1} \) converges to \( Y_{\theta_\phi} \). Thus, \( D_{2,T}M_T^{-1}D_{1,T} \) converges to \( Y_{\theta_\phi}^{-1} \), and so \( T\left(M_T^{-1}(1,1) \overset{d}{\to} J_{\theta_\phi}^{-1}(1,1) \right) \) and \( T^2\left(M_T^{-1}(2,2) \overset{d}{\to} \phi(1)^2J_{\theta_\phi}^{-1}(2,2) \right) \). Hence, the result follows by the CMT.

**Appendix C. Role of the joint spectral radius**

**Example 1 (Stationary roots of \( \phi(z) \) are not sufficient for Theorem 3.2).** Let \( \beta = 1 \) and

\[
B(z) = (1 - z)\phi(z), \quad \phi(z) = (1 - z + 0.9z^2)(1 - 1.3z + 0.9z^2).
\]

The roots of \( \phi(z) \) are larger than 1 in absolute value (roots are approximately \( 0.56 \pm 0.9i \) and \( 0.7 \pm 0.77i \)). However, simulations in Figure C.1 indicate that \( \{\Delta y_t\} \) is not stochastically bounded and \( \{y_t\} \) grows exponentially, preventing the result of Theorem 3.2 from holding.

![Figure C.1](image_url)

(A) The rescaled by \( \sqrt{T} \) process explodes.  (B) First differences are not bounded.

**Figure C.1.** Stationary roots of \( \phi(z) \) are not sufficient for Theorem 3.2. Data generating process corresponds to Example 1 with \( T = 1000, y_0 = 0, u_t \sim \text{i.i.d.} \mathcal{N}(0, 1) \).
Example 2 (Assumption A4 is not necessary for Theorem 3.2). Let
\[ B(z) = (1 - z)\phi(z), \quad \phi(z) = 1 - 1.3z + 0.8z^2. \]
That is, \( k = 3, \beta = 1, \phi_1 = 1.3 \) and \( \phi_2 = -0.8 \). The roots of \( \phi(z) \) are larger than 1 in absolute value (roots are approximately \( 0.8 \pm 0.77i \)). However, the largest (in modulus) eigenvalue of \( F_1F_1F_0 \) is \( -1.04 \), and so \( \lambda_{\text{ISR}}(\{F_0, F_1\}) \geq |-1.04|^{1/3} > 1 \). On the other hand, simulations in Figure C.2 indicate that \( \{\Delta y_t\} \) is stochastically bounded. Thus, Assumption A4 is not necessary for Theorem 3.2.

Proof of Lemma 3.1. Let \( \sum_{i=1}^{k-1} |\phi_i| = \Phi < 1 \) and \( M \in A^n = \{\prod_{j=1}^n A_j \mid A_j \in \{F_0, F_1\}\} \). Let us show that the spectral radius of \( M, \lambda(M) \), is at most \( \Phi^{n/(k-1)} \), where \( [\cdot] \) is a floor function. Then,
\[
\lambda_{\text{ISR}}(\{F_0, F_1\}) = \limsup_{n \to \infty} \sup_{M \in A^n} \lambda(M)^{1/n} \leq \limsup_{n \to \infty} \left( \Phi^{n/[k-1]} \right)^{1/n} = \Phi^{1/[k-1]} < 1.
\]

First, suppose \( n = k - 1 \), so that \( M = F_{\delta_{k-1}} \cdots F_{\delta_1} \), where \( \delta_j \in \{0, 1\}, j = 1, \ldots, k - 1 \). Let us show that for any vector \( x = (x_1, \ldots, x_{k-1})^T \in \mathbb{R}^{k-1}, \|Mx\|_\infty \leq \Phi \|x\|_\infty \), where \( \|x\|_\infty := \max_{i=1,\ldots,k-1} \{|x_i|\} \), i.e. the maximum norm \( \ell_\infty \). Let \( x^s := F_{\delta_s} \cdots F_{\delta_1}x, s = 1, \ldots, k - 1 \).

We prove by induction that for any \( s = 1, \ldots, k - 1 \), \( |x_i^s| \leq \Phi \|x\|_\infty \) for \( i = 1, \ldots, s \) and \( |x_i^s| \leq \|x\|_\infty \) for \( i > s \). To verify induction base, note that since
\[
F_\delta x = \left( \delta \phi_1 x_1 + \sum_{i=2}^{k-1} \phi_i x_i, \delta x_1, x_2, \ldots, x_{k-2} \right)^T,
\]
\(|x_1| \leq \Phi \|x\|_\infty \) and \( |x_i^s| \leq |x_{i-1}^s| \leq \|x\|_\infty \) for \( i > 1 \). To verify induction step, suppose that the claim holds for \( s \) and let us prove it for \( s + 1 \). Using \( x^{s+1} = F_{\delta_{s+1}}x^s \) and (C.1), we get
|x_{1}^{s+1}| \leq \Phi\|x^{*}\|_{\infty} \leq \Phi\|x\|_{\infty}$, where the last inequality follows from induction hypothesis. Next, $|x_{2}^{s+1}| = |\delta x_{1}^{s}| \leq |x^{*}_{1}| \leq \Phi\|x\|_{\infty}$ and for $i > 2$ we have $|x_{i}^{s+1}| = |s_{i}|$. Thus, for $i > s + 1$ we have $|x_{i}^{s+1}| = |x_{i-1}^{s}| \leq \|x\|_{\infty}$, while for $2 < i \leq s + 1$, $|x_{i}^{s+1}| = |s_{i}| \leq \Phi\|x\|_{\infty}$.

It follows from the preceding that since $Mx = x^{k-1}$, all coordinates of $Mx$ are bounded by $\Phi\|x\|_{\infty}$ and $\|Mx\|_{\infty} \leq \Phi\|x\|_{\infty}$.

Now consider general $n$ and $M \in \mathcal{A}^{n}$, $M = F_{\delta_{n}} \ldots F_{\delta_{1}}$. We know that for any $x \in \mathbb{R}^{k-1}$, $\|F_{\delta_{k-1}} \ldots F_{\delta_{1}} x\|_{\infty} \leq \Phi\|x\|_{\infty}$. Thus, $\|F_{\delta_{k-1}} \ldots F_{\delta_{1}} x\|_{\infty} \leq \Phi\|F_{\delta_{k-1}} \ldots F_{\delta_{1}} x\|_{\infty} \leq \Phi\|x\|_{\infty}$ and letting $\hat{n} = \left\lfloor \frac{n}{k-1} \right\rfloor$ by iterative back-substitution we get $\|F_{\delta_{\hat{n}}(k-1)} \ldots F_{\delta_{1}} x\|_{\infty} \leq \Phi^{\hat{n}}\|x\|_{\infty}$.

Finally note from (C.1) that applying $F_{\delta}$ cannot increase the maximum norm, so that

$$\|Mx\|_{\infty} = \|F_{\delta_{n}} \ldots F_{\delta_{1}} x\|_{\infty} \leq \|F_{\delta_{\hat{n}}(k-1)} \ldots F_{\delta_{1}} x\|_{\infty} \leq \Phi^{\hat{n}}\|x\|_{\infty}.$$ 

If $x$ is an eigenvector of $M$ with the corresponding eigenvalue $\lambda$, we must have $Mx = \lambda x$ and $\|Mx\|_{\infty} = |\lambda| \cdot \|x\|_{\infty}$. Since $\|Mx\|_{\infty} \leq \Phi^{\hat{n}}\|x\|_{\infty}$, we must also have $|\lambda| \leq \Phi^{\hat{n}}$. Therefore, the spectral radius of $M$, $\lambda(M)$, is at most $\Phi^{\hat{n}} \left\lfloor \frac{n}{k-1} \right\rfloor$, which completes the proof.

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