Topological modes in relativistic hydrodynamics

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We show that gapless modes in relativistic hydrodynamics could become topologically nontrivial by weakly breaking the conservation of energy momentum tensor in a specific way. This system has topological semimetal-like crossing nodes in the spectrum of hydrodynamic modes that require the protection of a special combination of translational and boost symmetries in two spatial directions. We confirm the nontrivial topology from the existence of an undetermined Berry phase. One possible origin for the non-conservation terms of the energy momentum tensor is an external gravitational field which could be generated by a specific coordinate transformation from the flat spacetime. This suggests that topologically trivial modes could become topologically nontrivial by being observed in a special non-inertial reference frame. Finally we propose a holographic realization of this system.

\textbf{Introduction.} Hydrodynamics is the universal low energy theory for systems close to local thermal equilibrium at long distance and time. It could describe a variety of physical systems ranging from matter at large scales in the universe, the quark-gluon plasma \cite{1}, to Weyl semimetals \cite{2,3} and graphenes \cite{4} in the laboratory. At small momentum and frequency, perturbations of a hydrodynamic system away from the equilibrium would produce sound and transverse modes \cite{5}. These modes are gapless whose poles are at $\omega = k = 0$, which reflects the fact that energy momentum is conserved.

During the last decade, topologically nontrivial quantum states have been discovered in condensed matter physics \cite{6,7}. Later it has been found that many classical systems have nontrivial topological states too, including topological optical/sound systems (see e.g. \cite{8–10} and references therein), which have also been observed experimentally.

It raises the question if the gapless modes in relativistic hydrodynamics could also become topologically nontrivial under certain conditions. In this paper, we start from the relativistic hydrodynamics and show that after weakly breaking conservation of energy momentum, hydrodynamic modes could become topological semimetal-like nontrivial states that require the protection of a special spacetime symmetry.

\textbf{Effective Hamiltonian and spectrum in relativistic hydrodynamics.} We focus on the simplest hydrodynamic systems with no internal charges whose only conserved quantity is the energy momentum tensor that satisfies $\partial_\mu T^{\mu\nu} = 0$. Up to the first order in derivative, the constitutive equation for the energy momentum tensor in the Landau frame is

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P \Delta^{\mu\nu} - \eta \Delta^{\mu\alpha} \Delta^{\nu\beta} (\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} \eta_{\alpha\beta} \partial_\gamma u^\gamma) - \zeta \Delta^{\mu\nu} \partial_\alpha u^\alpha + O(\partial^3),$$

where $\Delta^{\mu\nu} = \eta^{\mu\nu} + u_\mu u_\nu$, $\epsilon$, $P$ are the energy densities and pressure and $\eta$, $\zeta$ are the shear and bulk viscosities.

With small perturbations away from equilibrium, the system would respond to the perturbations and develop hydrodynamic modes. There are four eigenmodes of the system. Two of them are the sound modes propagating in the direction of $k = (k_x, k_y, k_z)$ with the dispersion relation $\omega = \pm v_s k - i \Gamma_s k^2$, where $v_s = \sqrt{\frac{\partial P}{\partial \epsilon}}$ and $\Gamma_s = (\frac{4}{3} \eta + \zeta) / (\epsilon + P)$. The other two are transverse modes with $\omega = -i \frac{\epsilon}{\epsilon + P} k^2$. To the first order in $k$, dissipative terms disappear and the spectra of the four modes are real, which cross each other at $\omega = k = 0$. This spectrum looks similar to the spectrum of Dirac semimetals, except that we have two extra flat bands here.

To change the spectrum to a topological semimetal-like one, we need to add non-conservation terms of $T^{\mu\nu}$ into the conservation equations. As a first step, we develop the notion of an effective Hamiltonian in hydrodynamics. Substituting the constitutive equations for the perturbations $\delta T^{\mu\nu}$ into $\partial_\mu \delta T^{\mu\nu} = 0$, we could rewrite the equations into the form

$$i \partial_t \Psi = H \Psi$$

where we have defined

$$\Psi = \begin{pmatrix} \delta \epsilon \\ \delta \pi^x \\ \delta \pi^y \\ \delta \pi^z \end{pmatrix}, \quad H = \begin{pmatrix} 0 & k_x & k_y & k_z \\ k_x v_s^2 & 0 & 0 & 0 \\ k_y v_s^2 & 0 & 0 & 0 \\ k_z v_s^2 & 0 & 0 & 0 \end{pmatrix}$$

at leading order in $k$, i.e. omitting dissipative terms at $O(k^2)$.

In this way, in analogy to the electronic systems \cite{11} we have defined an effective Hamiltonian matrix $H$ whose eigenvalues give the spectrum of hydrodynamic modes.\footnote{Note that this effective Hamiltonian is different from the hydro-}
The four eigenvalues of the matrix Hamiltonian above give the sound modes \( \omega = \pm\nu_s \sqrt{k_x^2 + k_y^2 + k_z^2} \) and double copies of transverse modes \( \omega = 0 \). The form (2) is the “free” Hamiltonian matrix for a conserved energy momentum tensor.

**Topologically nontrivial modes.** To deform the spectrum of the hydrodynamic modes, we introduce non-conservation terms for the energy momentum tensor and make sure that the non-conservation terms are small enough to stay within the hydrodynamic limit. The non-conservation of energy and momentum could come from a certain external system which couples to the hydrodynamic system under study. We assume that the constitutive equations for perturbations of the hydrodynamic system being considered do not change.

We take a 4D hydrodynamic system and introduce non-conservation terms for \( T^{\mu\nu} \) as follows

\[
\partial_\mu \delta T^{\mu\nu} = m \delta T^{tx}, \quad \partial_\mu \delta T^{\mu x} = -m \nu_s^2 \delta T^{tt},
\]

\[
\partial_\mu \delta T^{xy} = b \nu_s \delta T^{tt}, \quad \partial_\mu \delta T^{xz} = -b \nu_s \delta T^{ty},
\]

where \( m \) terms gap the spectrum while \( b \) terms change the momentum position of the crossing nodes in the spectrum. Note that \( m \) terms mix the energy density and momentum density in \( x \) direction while \( b \) terms mix the momentum densities in \( y \) and \( z \) directions.

After substituting the fluctuations of constitutive equation into (3) we obtain

\[
i \partial_t \Psi = H \Psi
\]

where \( \Psi = (\delta \epsilon, \delta \pi^x, \delta \pi^y, \delta \pi^z)^T \) and

\[
H = \begin{pmatrix}
0 & k_x + im & k_y & k_z \\
(k_x - im)\nu_s^2 & 0 & 0 & 0 \\
k_y \nu_s^2 & 0 & 0 & ib \nu_s \\
k_z \nu_s^2 & 0 & -ib \nu_s & 0
\end{pmatrix}.
\]

\( H \) is similar to a Hermitian matrix as could be seen by redefining \( \delta \epsilon \rightarrow \frac{1}{\nu_s} \delta \epsilon \). Thus this effective \( H \) has real eigenvalues and the factor \( \nu_s \) could be ignored which could be taken back by an inverse transformation when necessary. The spectrum of the hydrodynamic modes for (5)

\[
\omega = \pm \frac{1}{\sqrt{2}} \sqrt{b^2 + k^2 + m^2 \pm \sqrt{(k_x^2 + m^2 - b^2)^2 + (k_y^2 + k_z^2)^2 + 2(k_y^2 + k_z^2)(k_x^2 + m^2 + b^2)}}
\]

Figure 1 shows this spectrum as a function of \( k_x \) for \( k_y = k_z = 0 \) in three different situations: \( m < b, m = b \) and \( m > b \) as well as for \( k_y > 0, k_z = 0 \) at \( m < b \). The effect of \( m \) terms is to gap the two sound modes. The effect of \( b \) terms is to lift and lower the two transverse flat bands to symmetric positions of opposite sides of the \( k \) axis. In this way, the modes have band crossings at nonzero values of \( k \) for \( m < b \).

From figure 1, we could see that for \( m < b \) there are four band crossing nodes at \( k_y = k_z = 0 \) while \( k_x \neq 0 \), and for these nodes \( \omega \neq 0 \). These four nodes are still points in the expanded space of \( \omega, k_x, k_y \) and \( k_z \) as can be seen from the fourth plot in figure 1. For \( m > b > 0 \), the system becomes critical with two nodes and for \( m > b \) the system becomes gapped again. This behavior is qualitatively similar to the topological phase transition of a topological semimetal [12].

The \( m \) terms in (5) do not gap the four band crossing nodes in the \( m < b \) case, however, if we have extra \( m \) terms in the \( y \) or \( z \) directions, the gaps will open no matter how small the \( y \) or \( z \) mass parameters are. The spectrum in this case looks the same as the bottom right one in figure 1. In this situation, there are no crossing nodes anymore. This means that the nodes should be topologically nontrivial under the protection of symmetries that forbid the \( m \) terms in the \( y \) and \( z \) directions. We will see later that the symmetry needed here is a spe-

dynamic Hamiltonian, e.g. in section 2.4 of [5] in the sense that this effective Hamiltonian matrix \( \Psi^T (-k) H \Psi(k) \) gives the “kinetic” part of the Hamiltonian while the latter only counts in the potential part associated with the source terms.
cial combination of translational and boost symmetry in $y$ and $z$ directions. In this sense, the system (3) experiences a symmetry protected topological phase transition that happens at the critical point $m = b$.

Note that for the hydrodynamical modes $\omega(k = 0)$ is not zero anymore due to the non-conservation of energy, i.e., energy is constantly pumped into or out of the system. These crossing nodes at $m < b$ are dissipative when order $k^2$ terms are taken into account. This is different from the $\omega = k = 0$ nodes which are real poles in hydrodynamics with unbroken translational symmetries.

The new $m$ and $b$ terms above are not dissipative so they only change the shape of the spectrum while do not introduce any imaginary parts in the dispersion relation. In contrast, momentum dissipation terms in e.g. [13–15] are dissipative terms.

**Possible origin for non-conservation terms of $T^{\mu\nu}$**. The simplest way to have the non-conservation terms of $T^{\mu\nu}$ in (3) is to introduce an external rank two symmetric tensor field $f_{\mu\nu}$. There are at least two possible origins for $f_{\mu\nu}$.

The first possibility is to consider an external symmetric tensor matter field $f_{\mu\nu}$ that couples to the energy momentum tensor of the system and contributes an effective $f_{\mu\nu}T^{\mu\nu}$ term to the Lagrangian of the system. With this extra term the energy momentum of the system will not be conserved as it can be transferred to the external system.

Omitting terms at order of $O(k^2)$ or higher, we get the non-conservation equation for $T^{\mu\nu}$ [16]

$$\partial_\mu T^{\mu\nu} = T^{\rho\mu} \partial_{\nu} f_{\rho\mu} - 2 \partial_\mu f_{\mu\nu}. \quad (6)$$

With this equation, we could switch on the following components of $f_{\mu\nu}$ to get the form of effective Hamiltonian (5)

$$f_{tt} = f_{xx} = mx, \quad f_{tx} = f_{xt} = \frac{1}{2} mt (v^2 + 1), \quad f_{ty} = f_{yt} = -\frac{1}{2} bv_z z, \quad f_{tz} = f_{zt} = \frac{1}{2} bv_z y. \quad (7)$$

The second and more natural possibility is for this external field to be a gravitational field $h_{\mu\nu}$. Then the whole metric field is $g_{\mu\nu} = h_{\mu\nu} + \eta_{\mu\nu}$ and the energy momentum tensor is conserved as $\nabla_\mu T^{\mu\nu} = 0$ in the new spacetime so that $\partial_\mu T^{\mu\nu} = 0$ does not hold anymore. Expanding this equation in $h_{\mu\nu}$ we get

$$\partial_\mu \delta T^{\mu\nu} = -\frac{1}{2} \partial_\alpha h^{\delta\alpha\nu} - \frac{1}{2} \eta^{\nu\beta} (2 \partial_\mu h_{\alpha\beta} - \partial_\beta h_{\mu\alpha}) \delta T^{\mu\alpha}. \quad (8)$$

Again, we have assumed that $O(h_{\mu\nu}) \sim O(k)$ and only kept leading order in $k$ terms. To get the exact $m$ and $b$ terms in the effective Hamiltonian (5), $h_{\mu\nu}$ should be chosen to be the same as $f_{\mu\nu}$ in (7). \(^2\)

This graviton field $h_{\mu\nu}$ could come from sources of massive matter and more interestingly it could also come from a coordinate transformation from the flat Minkowski metric by $x'_\mu = x_\mu + \xi_\mu$ with

$$\xi_\mu = \left( \frac{mxt}{2}, \frac{mx^2}{4} + \frac{mt^2}{4} v_z^2, -\frac{b}{4} vt, \frac{b}{4} vht \right). \quad (9)$$

This is an intriguing result as usually a nontrivial gravitational field could not be transformed to a flat spacetime globally but only locally. It could be checked that this new metric field has all components of the Riemann tensor vanishing at leading order, thus could be transformed to the flat spacetime. Though equivalent to a flat spacetime, $h_{\mu\nu}$ could still be viewed as a non-trivial gravitational field according to the equivalence principle.

This result suggests that in a specific non-inertial frame, we could observe hydrodynamic modes that are topologically protected even when they are topologically trivial in the original inertial frame.

Note that in this case, with nonzero components of $h_{\mu\nu}$ the constitutive equations for $T_{\mu\nu}$ could also be written into a covariant form thus leading to extra terms compared to the original constitutive equations. However, it can be explicitly checked that these extra terms do not contribute to the equation (3) at leading $k \sim m, b$ order.

Now we could work out the symmetry of the system (3) as the isometry of the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, i.e. coordinate transformations that leave $g_{\mu\nu}$ unchanged. The symmetry could be viewed as the Lie transformation of the Poincare symmetry generated by the vector (9). Among the ten generators of this new isometry [16], two of them are responsible for forbidding $m$ terms in the $y$ and $z$ directions and protecting the nontrivial topological states, which are generated by $x^\alpha \rightarrow x^\alpha + \epsilon^\alpha$, where $\epsilon^\alpha = a_y \chi_y + a_z \chi_z$ with $\chi_y = (0, \frac{bby}{4}, 0, 1)$, $\chi_z = \left( \frac{bby}{4}, 0, \frac{bby}{4}, 1 \right)$ and $a_y, a_z$ two infinitesimal constants. $(\chi_y$ or $\chi_z)$ is a special combination of the $y$ (or $z$) direction translational symmetry and the boost symmetry of the $t-y$ (or $t-z$) direction.

Finally we mention another possible circumstance to have this nonzero $h_{\mu\nu}$, which could arise in analog gravity systems, i.e. certain materials could give rise to effective hydrodynamic equations as if there exists a nontrivial gravitational field.

**Topological invariant.** For systems protected by a certain symmetry, we could calculate the topological invariant at a high symmetric point in the momentum space, which is

\(^2\) We assume that we are working in a large but finite volume of spacetime and $m, b$ are so small that $mx, by, bz, mt \ll 1$ in the finite volume. We ignore the boundary effects as the volume is large.
$k_y = k_z = 0$ in this case. There is a charge conjugation symmetry for the solutions and we could focus on the lower two nodes in the left top plot of figure 1. Here as we are in zero effective residual spatial dimension, the calculation of the topological invariant is different from the Berry phase or Berry curvature for nodal line or Weyl semimetals. For the left node at $k_x = k_1$ in the left top plot of figure 1, the green solution at the left limit $k_x \to k_{1-}$ and the right limit $k_x \to k_{1+}$ are denoted as $|n_1\rangle$ and $|n_2\rangle$ separately. We could define a Berry phase between the two states $e^{-i\alpha} = \langle n_1|n_2\rangle$, to denote the topological invariant here. If the Berry phase is an undetermined one, i.e. $|n_1\rangle$ and $|n_2\rangle$ are orthogonal to each other, the system would be topologically nontrivial as the two states cannot be connected without passing through a singularity, which means the lower band and the upper band could not be separated by small perturbations.

In (3), $|n_1\rangle = \frac{1}{\sqrt{2}}(0,0,-i,1)$ and $|n_2\rangle = \frac{1}{\sqrt{1+i^2}}(-i\sqrt{m_1^2+k_2^2},1,0,0)$. Therefore $\langle n_1|n_2\rangle = 0$, which means that the Berry phase is undetermined. From the argument above the two bands cannot be separated easily by a gap without going through a topological phase transition. Similar behavior of an undetermined Berry phase has also happened for the holographic nodal line semimetals [17, 18].

This result confirms that the four nodes in figure 1 are topologically nontrivial protected by a special combination of translational and boost symmetry in the $y$ and $z$ directions. At the same time, the Berry phase accumulated through the whole circle around this node would be trivial indicating that it is indeed topologically trivial without the symmetry.

**Transport properties.** We can follow the calculations in [5, 19] to compute the heat transport for this system to uncover more observational effects. We obtain

$$\kappa_{xx}(\omega, k_x) = -\frac{i\omega(\epsilon + P)}{T \left( (k_x^2 + m^2)v_s^2 + i\frac{n}{\epsilon + P}\omega k_x^2 - \omega^2 \right)},$$

$$\kappa_{yy}(\omega, k_x) = \kappa_{zz}(\omega, k_x) = -\frac{k_x^2\eta + i\omega(\epsilon + P)}{T \left( b^2v_s^2 + (i\omega + \frac{n}{\epsilon + P}k_x^2)^2 \right)},$$

$$\kappa_{yz}(\omega, k_x) = -\kappa_{zy}(\omega, k_x) = \frac{(\epsilon + P)dv_s}{T \left( b^2v_s^2 + (i\omega + \frac{n}{\epsilon + P}k_x^2)^2 \right)}.$$ 

With the formulae above, when $m = b = 0$, all components of the DC heat transport diverge. For generic $m$ and $b$, we have vanishing DC heat transport $\kappa_{xx}(0,0), \kappa_{yy}(0,0)$ and $\kappa_{zz}(0,0)$ while $\kappa_{zy}(0,0) = -\kappa_{zy}(0,0) = \frac{e^2}{4\pi T}$. These $m$ and $b$ terms eliminate the unphysical divergence of DC heat transports and lead to interesting vanishing DC heat transport behavior. $\mathcal{O}(k^2)$ effects. $\mathcal{O}(k^2)$ terms lead to dissipative effects and give rise to imaginary parts of frequency in the spectrum. The $\mathcal{O}(k^2)$ terms make the effective Hamiltonian matrix non-Hermitian. Here we still keep terms at $m \sim b$ order while not $m^2 \sim b^2$ order assuming that $m \sim k^2$.

From the eigenvalues of Hamiltonian with $\mathcal{O}(k^2)$ effects included we find that the real part has not changed while imaginary parts appear. At the four nodes, the imaginary parts are not zero indicating that the four nodes are dissipative in comparison to nondissipative nodes at $\omega = 0$ in the usual hydrodynamics. The imaginary part for each of the band has a jump at the crossing nodes at $k_x = 0$ in the $k_x$ axis, i.e. the imaginary parts of the same band are different at the left and right limits of the singular node. This behavior is similar to the behavior of the eigenstates when calculating the Berry phase and thus provides another piece of evidence of the existence of a symmetry protected topological singular node.

**Ward identities and holographic realization.** The physics of hydrodynamics has been studied extensively in holography for strongly coupled systems [20–22]. We aim to construct a holographic system possessing the same non-conservation equation of (3), thus providing an example of this system in the strongly coupled limit.

Holographically we could also perform a coordinate transformation to get a non-inertial frame version of AdS/CFT correspondence which has the metric $g^{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ at the boundary. This system should have the same spectrum of the hydrodynamic modes. As a first step for a confirmation, we need to show that our holographic system indeed has the non-conservation of (3). For this purpose, we will first obtain the Ward identities for $T^{\mu\nu}$ in the non-conserved hydrodynamic system from (3) and match these identities to those in the holographic non-inertial frame system.

In the case that these non-conservation terms come from a gravitational field, we could start from the covariant conservation equation $\nabla_{\mu}T^{\mu\nu} = 0$ and differentiate it with respect to $g_{\lambda\rho}$ to obtain the Ward identities in the momentum space of the boundary system. We could as well choose to treat $h_{\mu\nu}$ as an external effective matter field and perform the same procedure starting from (6). The final results are exactly the same. To the first order in $h_{\mu\nu}$ the Ward identities are

$$k_{\mu}G^{\mu\nu,\lambda\rho}(k) + i \left[ T^{(1)}_{\mu\nu}G^{\mu\nu,\lambda\rho}(k) + T^{(1)}_{\mu\nu}G^{\mu\nu,\lambda\rho}(k) \right] + \text{contact terms} = 0,$$

where the explicit form of the contact terms is omitted. With nonzero $h_{\mu\nu}$ several components of $T^{(1)}_{\mu\nu}$ would be nonzero and contribute extra terms to the Ward identities of $k_{\mu}G^{\mu\nu,\lambda\rho}(k) + \text{contact terms} = 0$ in hydrodynamics systems with conserved $T^{\mu\nu}$ [23].

The Ward identities (10) could be reproduced from the holographic non-inertial frame system where we start from the usual AdS Schwarzschild black hole and perform coordinate transformations so that the boundary metric
becomes $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We have checked that the holographic Ward identities match exactly to the hydrodynamic Ward identities (10). The details will be presented in [16] and the hydrodynamic modes and Green functions will be systematically studied in future work.

Outlook. Besides the new features in transport coefficients of this system and the sudden increase of amplitude at the crossing nodes at nonzero $\omega$ and $k$, there would possibly be other experimental features that could be observed in laboratories, e.g. graphene and Weyl semimetals in a non-inertial frame. We expect the results obtained in this paper could have further applications for providing more stable hydrodynamic modes. Finally, it is possible that topologically trivial systems other than hydrodynamic systems, e.g. electronic systems, would also become topologically nontrivial in a certain non-inertial frame.

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