ASYMPTOTIC REPRESENTATIONS OF QUANTUM AFFINE SUPERALGEBRAS

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Abstract. We study representations of the $q$-Yangian, the upper Borel subalgebra with respect to RTT realization of the quantum affine superalgebra associated with the Lie superalgebra $\mathfrak{gl}(M, N)$. Following the work of Hernandez-Jimbo, we construct inductive systems of Kirillov-Reshetikhin modules by using a cyclicity result of tensor products of these modules we established recently, and realize their inductive limits as modules over the $q$-Yangian, extending the asymptotic construction of Hernandez-Jimbo to the super case. Then, we propose a new asymptotic construction on the inductive limits of the same inductive systems, resulting in modules over the full quantum affine superalgebra depending on an additional parameter. $q$-character and Gelfand-Tsetlin basis for these two kinds of modules are also investigated.

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1. Introduction

Let $q$ be a non-zero complex number which is not a root of unity. Let $\mathfrak{g} := \mathfrak{gl}(M, N)$ be the general linear Lie superalgebra. Let $U_q(\widehat{\mathfrak{g}})$ be the associated quantum affine superalgebra. (We refer to §2.2.2 for the precise definition.) This is a Hopf superalgebra neither commutative nor co-commutative, and it can be seen as a deformation of the universal enveloping algebra of the following affine Lie superalgebra:

$$L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] = \bigoplus_{1 \leq i, j \leq M+N} E_{ij} \otimes \mathbb{C}[t, t^{-1}].$$

Here the $E_{ij}$ for $1 \leq i, j \leq M + N$ are the elementary matrices in $\mathfrak{g}$.

In this paper, we are concerned with a distinguished family of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules, the so-called Kirillov-Reshetikhin modules. We would like to construct inductive
systems of these modules, and realize their inductive limits as modules over the upper Borel subalgebra of $U_q(\widehat{\mathfrak{g}})$, which is in our context the $\varphi$-Yangian $Y_q(\mathfrak{g})$. These limits may or may not carry an action of the full superalgebra $U_q(\widehat{\mathfrak{g}})$, and they may also be finite-dimensional.

1.1. Background. Our motivation for studying representations of quantum affine superalgebras comes, on the one hand, from the integrability structure of AdS/CFT correspondence where various quantum superalgebras related to $\mathfrak{sl}(2, 2)$ show up (see for example [BGM12]), and on the other hand, from the generalization of Hernandez-Jimbo's construction of asymptotic representations related to Baxter's $Q$-operators to the super case. In is paper, we concentrate on the second point.

In the early seventies, towards the study of the eight-vertex model, Baxter [Ba72] introduced the concept of $Q$-operators and $T$-$Q$ functional relations to determine the eigenvalues of transfer matrices. Ever since, various progress has been made towards understanding and generalizing Baxter’s $Q$-operators and $T$-$Q$ relations, notably the approach by using representation theory of quantum affine (super)algebras.

In a series of papers [BLZ96, BLZ97, BLZ99], Bazhanov-Lukyanov-Zamolodchikov (BLZ for short) generalized $T$, $Q$-operators from lattice models to integrable quantum field theory. The idea goes roughly as follows. Firstly, one gets a representation $W$ of the lower Borel subalgebra $B_-$ of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$, which is either a finite-dimensional evaluation module for the lattice model, or an infinite-dimensional vertex representation for the quantum field theory. $W$ is referred to as a quantum space. Secondly, one constructs an $L$-operator, which is an element of the completed tensor product $B_+ \otimes \text{End}W$. (In both integrable structures, $L$ is presumed to be the universal $R$-matrix $R \in B_+ \otimes B_-$ of $U_q(\widehat{\mathfrak{sl}}_2)$ with $B_-$ specialized to $\text{End}W$. See [BLZ99 Conjecture] for the statement and [BHK02, §3.3] for a proof.) Thirdly, $T$, $Q$-operators, as elements of $\text{End}W$, are defined as twisted traces of $L$ over various representations of $B_+$: finite-dimensional evaluation representations for $T$ and oscillator representations for $Q$. Baxter’s $T$-$Q$ relations are then deduced from tensor product decompositions of representations of $B_+$.

The oscillator representations are constructed quite explicitly in [BLZ99] by adapting the so-called oscillator realization of $B_+$. There are several generalizations on BLZ’s construction of oscillator representations and $T$-$Q$ relations when replacing $U_q(\widehat{\mathfrak{sl}}_2)$ by higher rank quantum affine (super)algebras. In the super case, this has been partly done: by Bazhanov-Tsuboi [BT08] for the quantum affine superalgebra $U_q(\mathfrak{sl}(2, 1))$, and later generalized by Tsuboi [TS12] for the quantum affine superalgebra $U_q(\mathfrak{gl}(M, N))$; by Kulish-Zeitlin [KZ05] and Ip-Zeitlin [IZ14] for the (twisted) quantum affine superalgebra $C^{(2)}_q(2)$. In both approaches, evaluation morphisms, oscillator realizations of Borel subalgebras and universal $R$-matrix remain the ingredients indispensable.

Recently, it was expected by Bazhanov-Lukyanov [BL13] that the $T$-$Q$ operators corresponding to the exceptional quantum affine superalgebra $U_q(\widehat{\mathfrak{D}}(2, 1; \alpha))$ should be related to the connection coefficients and the Wilson loop for various (perturbed) hypergeometric equations and there is need to construct both finite-dimensional ($T$-operator) and oscillator ($Q$-operator) representations for $U_q(\widehat{\mathfrak{D}}(2, 1; \alpha))$. However, evaluation morphisms from this quantum affine superalgebra to the finite type quantum superalgebra are not available, and
to construct finite-dimensional representations is already an interesting problem. Nevertheless, Drinfeld realizations corresponding to various Dynkin diagrams for this quantum affine superalgebra are known [HSTY08].

We remark also that in the super case, except for some small rank quantum affine superalgebras such as $C^{(2)}_q(2)$ [IZ14] and $U_q(gl(2,2))$ [Ga98], there is still no explicit formula of Damiani type [Da98] for the universal $R$-matrix of quantum affine superalgebras.

1.2. Asymptotic representations. Recently, Hernandez-Jimbo [HJ12] constructed the analogue of oscillator representations for an arbitrary (non-twisted) quantum affine algebra $U_q(\hat{g}''')$. Their construction is based on a well-studied family of finite-dimensional modules, the so-called Kirillov-Reshetikhin modules. They first constructed inductive systems of Kirillov-Reshetikhin modules, and then endowed their inductive limits with actions of the upper Borel subalgebra in an asymptotic way, resulting in oscillator representations for $U_q(\hat{g}'')$. The asymptotic construction eventually enables Frenkel-Hernandez [FH13] to interpret generalized T-Q relations in terms of representations and to prove a conjecture of Frenkel-Reshetikhin on the spectra of quantum integrable systems [FR99]. We refer to [He14] for a general review.

The advantage of Hernandez-Jimbo’s construction is that the asymptotic modules have simpler representation structures compared to finite-dimensional modules and they give rise to generalized T-Q relation in [FH13] whose proof does not need universal $R$-matrix. (The complete proof of Frenkel-Reshetikhin conjecture however does.)

In the present paper, we would like to apply Hernandez-Jimbo’s asymptotic construction to the quantum affine superalgebra $U_q(\hat{g})$. As we shall see, their inductive systems of Kirillov-Reshetikhin modules give rise to the oscillator representations of the upper Borel subalgebra $Y_q(\mathfrak{g})$ in [BT08, Ts12]. We also propose a new application of these inductive systems by realizing their inductive limits as one-parameter families of representations of the full quantum affine superalgebra.

1.3. Main results. To state in a neat way the resulting asymptotic representations in this paper, let us introduce several notations (slightly different from the main text).

Let $I_0 := \{1, 2, \cdots, M + N - 1\}$ be the set of Dynkin vertices for the Lie superalgebra $\mathfrak{g}$. For $r$ a Dynkin vertex and $f(z)$ a rational function with $f(0) \in \mathbb{C}^\times$, there is a simple $Y_q(\mathfrak{g})$-module, denoted by $S_r(f)$, which is generated by a highest $\ell$-weight vector $v$ satisfying $\phi_r(z)v = f(z)v$ and $\phi_i(z)v = v$ whenever $i \neq r$. Here the $\phi_i(z)$, as formal power series in $Y_q(\mathfrak{g})$, are quantum affine analogues of diagonal matrices in $\mathfrak{g}$.

Notably, for $r$ a Dynkin vertex, $a$ a non-zero complex number (called spectral parameter), and $k$ a positive integer (called lever of representation), the simple module $S_r(q^{k-1-zaq^2r})$ is called a Kirillov-Reshetikhin module. It is a finite-dimensional evaluation module, and its $Y_q(\mathfrak{g})$-module structure can be extended to that of $U_q(\mathfrak{g})$-module.

1.3.1. Inductive systems of Kirillov-Reshetikhin modules. From now on, fix the Dynkin vertex $r$ and the spectral parameter $a$. Our first main task in this paper is to construct an inductive system of the Kirillov-Reshetikhin modules $S_r(q^{k-1-zaq^2r})$ with respect to the level
of representation (see §4.4.5)

\[(1.1) \quad F_{k,l} : S_r(q^l \frac{1 - za}{1 - zaq^{2l}}) \rightarrow S_r(q^k \frac{1 - za}{1 - zaq^{2k}}) \quad \text{for} \ l < k.\]

The idea of construction is the same as that of Hernandez-Jimbo [11J12]: by using the cyclicity property of some particular tensor products of Kirillov-Reshetikhin modules. In our case, the following guarantees the existence of inductive systems (Theorem 3.11):

(A) Assume \( l_1 < l_2 < l_3 \). The tensor product \( S_r(\frac{1 - zaq^{2l_1}}{1 - zaq^{2l_2}}) \otimes S_r(\frac{1 - zaq^{2l_1}}{1 - zaq^{2l_2}}) \otimes S_r(\frac{1 - za}{1 - zaq^{2l}}) \), as a \( Y_q(\mathfrak{g}) \)-module, is of highest \( \ell \)-weight.

The proof relies on a more general cyclicity result in our previous paper [Zh14] and on some duality argument. (See [3] for details.)

One special feature of the above inductive system is that the structural maps \( F_{k,l} \) do not respect the \( U_q(\mathfrak{g}) \)-module structures. That is to say, given a generator \( x \) of the algebra, the two maps \( F_{k,l}x \) and \( xF_{k,l} \) are in general different. Nevertheless, one can establish stability and asymptotic properties of the \( F_{k,l} \), which enable us to express asymptotically these maps \( xF_{k,l} \), with \( l, x \) fixed and with \( k \) varying; they turn out to be Laurent polynomials in \( q^k \) of a particular form (see Propositions 4.4-4.7 for precise statements).

In the non-graded case, to prove these properties, Hernandez-Jimbo used some deep theory of \( q \)-characters of tensor products [11e10 Proposition 3.2], which is by no means available to us as we do not even have the notion of \( q \)-character for representations of quantum affine superalgebras. However, when working with the RTT realization of \( U_q(\mathfrak{g}) \), we are able to prove these two properties in a straightforward manner.

1.3.2. Asymptotic construction of Hernandez-Jimbo. The argument of Hernandez-Jimbo, which provides the inductive limit \( V_\infty \) of the inductive system (1.1) with a module structure of the upper Borel subalgebra, can be adapted to our situation without difficulty. Take the highest \( \ell \)-weight vectors \( v_k \) in \( S_r(q^k \frac{1 - za}{1 - zaq^{2k}}) \) as an example. They give rise to the same vector \( v_\infty \in V_\infty \). To get \( \phi_r(z)v_\infty \), note first of all \( \phi_r(z)v_k = q_r^k \frac{1 - za}{1 - zaq^{2k}} v_k \). By forgetting the term \( q_r^k \), and then by taking the limit \( \lim_{k \rightarrow \infty} q_r^k = 0 \), we obtain \( \phi_r(z)v_\infty = (1 - za)v_\infty \).

The stability and asymptotic properties of \( F_{k,l} \) explains in a conceptual way the validity of this argument for all vectors in \( V_\infty \). Eventually, we get a representation \( (\rho, V_\infty) \) of \( Y_q(\mathfrak{g}) \) having \( S_r(1 - za) \) as a simple sub-quotient, which is the desired oscillator representation of \( Y_q(\mathfrak{g}) \) within the framework of Bazhanov-Tsuboi [BT08, Ts12].

There are formal series \( \phi_r^-(z) \) in \( U_q(\mathfrak{g})[[z^{-1}]] \) such that \( \phi_r^-(z)v_k = q_r^{-k} \frac{1 - z^{-1}a^{-1}q_r^{-2k}}{1 - z^{-1}a^{-1}q_r^{-2k}} v_k \), giving rise to the expression \( \lim_{k \rightarrow \infty} q_r^{-2k} \), which is nonsense. For this reason, \( S_r(1 - za) \) does not carry an action of \( U_q(\mathfrak{g}) \).

1.3.3. Generic asymptotic construction. Starting from the same inductive system (1.1) and from the same stability and asymptotic properties, we propose in this paper a new limit process which endows the inductive limit \( V_\infty \) with \( U_q(\mathfrak{g}) \)-module structures. Again take the highest \( \ell \)-weight vectors \( v_k \) as an example. We have the asymptotic expressions \( \phi_r(z)v_k = q_r^k \frac{1 - za}{1 - zaq^{2k}} v_k \). Now, fix a non-zero complex number \( b \). By taking the limit \( \lim_{k \rightarrow \infty} q_r^k = b \), we obtain \( \phi_r(z)v_\infty = b^{-1} \frac{1 - za}{1 - zab^2} v_\infty \). Note that the expression \( \lim_{k \rightarrow \infty} q_r^{-k} = b^{-1} \) makes perfect
sense. Again, thanks to the stability and asymptotic properties, we get a representation \((\rho^+, V_\infty)\) of the full quantum affine superalgebra \(U_q(\hat{\mathfrak{g}})\) on the inductive limit. In particular, it has a simple sub-quotient \(S_r(b^{1-z_a/\Delta})\).

Informally, one can think of \((\rho^+, V_\infty)\) as \((\rho^0, V_\infty)\).

1.3.4. Category \(\mathcal{O}\) and \(q\)-character for representations of \(Y_q(\mathfrak{g})\). Now, following [HJ12], we introduce a category of representations of \(Y_q(\mathfrak{g})\) including all the Kirillov-Reshetikhin modules, \((\rho^+, V_\infty)\) and \((\rho^0, V_\infty)\) constructed above and we define the notions of \(q\)-character \(\chi_q\) and normalized \(q\)-character \(\tilde{\chi}_q\). Let us put together the main results obtained in this paper on category \(\mathcal{O}\) and (normalized) \(q\)-character.

(B) As in the non-graded case [He05, HJ12, MY14], there is a classification of simple \(Y_q(\mathfrak{g})\)-modules, simple \(U_q(\mathfrak{g})\)-modules, and finite-dimensional simple \(Y_q(\mathfrak{g})\)-modules in category \(\mathcal{O}\) in terms of rational functions (Lemma 6.6). Contrary to the non-graded case, there are finite-dimensional simple \(Y_q(\mathfrak{g})\)-modules which cannot be \(U_q(\mathfrak{g})\)-modules.

(C) In the case \(g = \mathfrak{gl}(1, 1)\), the explicit formula \(\tilde{\chi}_q(S_1(f))\) for all rational function \(f\) such that \(f(0) = 1\) is deduced. See Equation (6.39).

(D) If \(S\) is an evaluation module of a polynomial representation, or of the dual of a polynomial representation of \(U_q(\mathfrak{g})\), then \(\tilde{\chi}_q(S)\) is multiplicity-free, and it is a polynomial in the \([A_{i,x}]^{-1}\) where \(i \in I_0\) and \(x \in \mathbb{C}^\times\) (Corollaries 7.2, 7.5). Here as in the non-graded case, the \(A_{i,x}\) are the generalized simple roots, and \(U_q(\mathfrak{g})\) is the finite-type quantum superalgebra.

(E) Fix a Dynkin vertex \(r \in I_0\) and a spectral parameter \(a\). Then for all \(b \in \mathbb{C}^\times\)

\[
\tilde{\chi}_q(\rho^+, V_\infty) = \tilde{\chi}_q(\rho^b, V_\infty) = \lim_{k \to \infty} \tilde{\chi}_q(S_r(q^k - z_a/1 - zaq^{2k})).
\]

as formal power series in the \([A_{i,x}]^{-1}\) with coefficients 0 or 1. \((\rho^+, V_\infty)\) is a simple \(Y_q(\mathfrak{g})\)-module, and \((\rho^+, V_\infty)\) is simple provided some generic condition on \(b\). (Corollaries 8.8, 8.15)

(F) Assume that \(b\) is generic. Then related to the following chains of subalgebras:

\[
\begin{align*}
Y_q(\mathfrak{gl}(1, 0)) &\subset Y_q(\mathfrak{gl}(2, 0)) \subset \cdots \subset Y_q(\mathfrak{gl}(M - 1, 0)) \subset Y_q(\mathfrak{gl}(M, 0)) \\
\subset Y_q(\mathfrak{gl}(M, 1)) &\subset Y_q(\mathfrak{gl}(M, 2)) \subset \cdots \subset Y_q(\mathfrak{gl}(M, N - 1)) \subset Y_q(\mathfrak{gl}(M, N)) = Y_q(\mathfrak{g})
\end{align*}
\]

the simple modules in \((D)\) and \((E)\) admit Gelfand-Tsetlin bases (Propositions 7.1, 7.4 Theorem 8.1). We refer to [8.1] for the precise meaning of a Gelfand-Tsetlin basis. Notable properties are semi-simplicity and multiplicity-free when we consider the restrictions of these \(Y_q(\mathfrak{g})\)-modules to the above subalgebras.

The proof of \((D)-(F)\) is based on the compatibility of the structural maps \(F_{k,l}\) and \(q\)-characters (Lemmas 8.2, 8.12), and on the explicit computation of \(\tilde{\chi}_q(S)\) when \(S\) are the simple modules in \((D)\), by using Gelfand-Tsetlin bases. When \(S\) is an evaluation module of a polynomial representation, this is a straightforward generalization of a similar computation carried out by Frenkel-Mukhin [FM02, Lemma 4.7] in the non-graded case, and a closed formula for \(\tilde{\chi}_q(S)\) in terms of Young tableaux is obtained in Equation (7.41). When \(S\) is
an evaluation module of the dual of a polynomial representation, we do not have similar closed formula for $\tilde{\chi}_q(S)$.

The asymptotic constructions is quite Lie theoretic. It should eventually be done for more general quantum affine superalgebras like $U_q(\hat{D}(2, 1; \alpha))$, once we know how to define analogues of Kirillov-Reshetikhin modules, their character formulas and some cyclicity result of particular tensor products. (This is the reason why we are restricted to $U_q(\hat{g})$ in the present paper.) Moreover, such constructions should work for non-quantum algebras: semi-simple Lie algebras, current algebras and affine algebras, as we have good candidates for Kirillov-Reshetikhin modules such as Demazure modules and Weyl modules. We hope to return to these issues in future works.

1.4. Outline. This paper is organized as follows. In §2 we recall the definitions concerning quantum (affine) superalgebras and Yangians. Then we study the Gelfand-Tsetlin bases for certain finite-dimensional simple representations for the quantum superalgebra $U_q(\hat{g})$. §3 proves a cyclicity result of tensor products of Kirillov-Reshetikhin modules to be needed in the construction of inductive systems of Kirillov-Reshetikhin modules. In §4-5 we carry out in detail the asymptotic constructions for the quantum affine superalgebra $U_q(\hat{g})$, which are illustrated with explicit examples in §5.3. In §6 we introduce category $\mathcal{O}$ and $q$-character, discuss their general properties, and study in detail the case $\mathfrak{g} = \mathfrak{g}(1, 1)$ in §6.6. In §7 we compute the normalized $q$-character for some evaluation modules. Then in §8 we compute normalized $q$-character of asymptotic modules and establish Gelfand-Tsetlin bases for them.

About notations, in the introduction, for the sake of simplicity and for unifying the cases $r \leq M$ and $r > M$, we have used the nonstandard notation $S_r(f)$ for the Kirillov-Reshetikhin modules. In the main text, we adopt the more classical notation $W^{(r)}_{k, \alpha}$. Relevant results and proofs are usually divided into two parts depending on $r$.

Acknowledgments. The author is grateful to his supervisor David Hernandez and to Vyjayanthi Chari for their interest in the present work and for valuable discussions. Part of it was done while he was visiting Centre de Recherches Mathématiques in Montréal. He would like to thank Masaki Kashiwara for remarks on cyclicity of tensor products, and thank Eugene Mukhin and Weiqiang Wang for remarks on Gelfand-Tsetlin basis and $q$-character.

2. Preliminaries

In this section we first collect basic facts about the RTT realizations of the quantum affine superalgebra $U_q(\hat{g})$, the $q$-Yangian $Y_q(\mathfrak{g})$, and the quantum superalgebra $\mathcal{U}_q(\mathfrak{g})$. Next, we review part of the Schur-Weyl duality theory for tensor powers of the natural representation of $\mathcal{U}_q(\mathfrak{g})$, following Benkart-Kang-Kashiwara [BKK11]. Then we show the existence of Gelfand-Tsetlin basis for certain simple representations of $\mathcal{U}_q(\mathfrak{g})$: polynomial representations and their duals.

2.1. Conventions. Throughout this paper, all the vector superspaces and superalgebras are defined over the base field $\mathbb{C}$. We fix $q \in \mathbb{C}$ to be non-zero and not a root of unity. Fix
2.2 Quantum affine superalgebras.

Let $I := \{1, 2, \cdots, M + N\}$. Define the following maps:

$$| \cdot | : I \rightarrow \mathbb{Z}_2, i \mapsto |i| = \begin{cases} 0 & (i \leq M), \\ 1 & (i > M) \end{cases}$$

$d : I \rightarrow \mathbb{Z}, i \mapsto d_i := \begin{cases} 1 & (i \leq M), \\ -1 & (i > M). \end{cases}$

Set $q_i := q^{d_i}$. Set $P := \bigoplus_{i \in I} \mathbb{Z}e_i$. Let $(, ) : P \times P \rightarrow \mathbb{Z}$ be the bilinear form defined by $(\epsilon_i, \epsilon_j) = \delta_{ij}d_i$. Let $| \cdot | : P \rightarrow \mathbb{Z}_2$ be the morphism of abelian groups such that $|\epsilon_i| = |i|$. Set $I_0 := I \setminus \{M + N\}$. For $i \in I_0$, let $\alpha_i := \epsilon_i - \epsilon_{i+1}$. Introduce the root lattice $Q = \bigoplus_{i \in I_0} \mathbb{Z}\alpha_i \subset P$. Let $Q_{\geq 0} := \bigoplus_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i$.

In the following, only three cases of $|x| \in \mathbb{Z}_2$ are admitted: $x \in I$; $x \in P$; $x$ is a $\mathbb{Z}_2$-homogeneous, or of a vector superspace. If a vector space $V = \bigoplus_{i \in I} (\mathbb{Z} \alpha_i)$ carries a grading by an abelian group $P$, then we write $|x|_P = \alpha$ for $\alpha \in P$ and $x \in (V)_\alpha$.

Let $V := \bigoplus_{i \in I} \mathbb{C}v_i$ be the vector superspace with $\mathbb{Z}_2$-grading $|v_i| = |i|$. Then $\text{End} V$ is naturally a superalgebra. Let $E_{ij} \in \text{End} V$ be the linear endomorphism $v_k \mapsto \delta_{jk}v_i$. In particular, $|E_{ij}| = |i| + |j|$. Let $g$ be the general linear Lie superalgebra associated with $V$, which is, the vector superspace $\text{End} V$ endowed with the Lie bracket:

$$[f, g] := fg - (-1)^{|f||g|}gf$$

for $f, g \in \text{End} V$ homogeneous. In the following, $\text{End} V$ will always be viewed as a superalgebra, and $g$ as a Lie superalgebra, although they are the same as vector superspaces.

2.2 Quantum superalgebras. In this subsection, we review the RTT realizations of the quantum affine superalgebra $U_q(\hat{g})$, the $q$-Yangian $Y_q(\mathfrak{g})$, and the finite type quantum superalgebra $\mathfrak{u}_q(\mathfrak{g})$, following \cite{Zh14}.

2.2.1. The Perk-Schultz $R$-matrix. This is the element $R(z, w) \in (\text{End} V \otimes \text{End} V)[z, w]$:

$$R(z, w) := \sum_{i \in I} (zq_i - wq_i^{-1}) E_{ii} \otimes E_{ii} + (z - w) \sum_{i \neq j} E_{ii} \otimes E_{ij} + z \sum_{i < j} (q_i - q_i^{-1}) E_{ji} \otimes E_{ij} + w \sum_{i < j} (q_j - q_j^{-1}) E_{ij} \otimes E_{ji}. \tag{2.2}$$

$R(z, w)$ enjoys many fundamental properties (see \cite{Zh14} Proposition 3.6), notably the Yang-Baxter equation, and the following ice rule: for $a, b, c, d \in I$,

$$(2.3) \quad R_{ab,cd}(z, w) \neq 0 \implies \epsilon_a + \epsilon_b = \epsilon_c + \epsilon_d \in P$$

Here the $R_{ab,cd}(z, w) \in \mathbb{C}[z, w]$ are matrix coefficients defined by:

$$R(z, w)(v_c \otimes v_d) = \sum_{a,b \in I} R_{ab,cd}(z, w)(v_a \otimes v_b) \in V^\otimes 2[z, w].$$

Let us introduce $R := R(1, 0)$ and $R' := -R(0, 1)$. Then $R(z, w) = zR - wR'$.

2.2.2. Quantum affine superalgebra. This is the superalgebra $U_q(\hat{g})$ defined by:

(R1) generators $s_{ij}^{(n)}$, $\hat{t}_{ij}^{(n)}$ for $i, j \in I$ and $n \in \mathbb{Z}_{\geq 0}$;

(R2) $\mathbb{Z}_2$-grading $|s_{ij}^{(n)}| = |\hat{t}_{ij}^{(n)}| = |i| + |j|$. 

(R3) RTT-relations \[ \text{FRT89} \]

\[
\begin{align*}
(2.4) & \quad R_{23}(z, w)T_{12}(z)T_{13}(w) = T_{13}(w)T_{12}(z)R_{23}(z, w), \\
(2.5) & \quad R_{23}(z, w)S_{12}(z)S_{13}(w) = S_{13}(w)S_{12}(z)R_{23}(z, w), \\
(2.6) & \quad R_{23}(z, w)T_{12}(z)S_{13}(w) = S_{13}(w)T_{12}(z)R_{23}(z, w), \\
(2.7) & \quad t_{ij}^{(0)} = s_{ji}^{(0)} = 0 \quad \text{for } 1 \leq i < j \leq M + N, \\
(2.8) & \quad t_{ii}^{(0)} s_{ii}^{(0)} = 1 = s_{ii}^{(0)} t_{ii}^{(0)} \quad \text{for } i \in I.
\end{align*}
\]

Here \( T(z) = \sum_{i,j \in I} t_{ij}(z) \otimes E_{ij} \in (U_{q}(\mathfrak{g}) \otimes \text{End}\mathbf{V})[[z^{-1}]] \) and \( t_{ij}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} i_{ij}^{(n)} z^{-n} \in U_{q}(\mathfrak{g})[[z^{-1}]] \) (similar convention for \( S(z) \) with the \( z^{-n} \) replaced by the \( z^{n} \)).

Remark that (2.4)-(2.6) are operator equations in \((U_{q}(\mathfrak{g}) \otimes \text{End}\mathbf{V}^{\otimes 2})[[z, z^{-1}, w, w^{-1}]]\). Let us express for example Equation (2.5) in terms of matrix coefficients:

\[
\Delta(s_{ij}^{(n)}) = \sum_{a=0}^{n} \sum_{k \in I} (-1)^{\left|\left|a\right|+\left|j\right|\right|} R_{kl,ab}(z, w) s_{ai}(z) s_{bj}(w) s_{ij}^{(a)} s_{kj}^{(n-a)},
\]

\[
\Delta(t_{ij}^{(n)}) = \sum_{a=0}^{n} \sum_{k \in I} (-1)^{\left|\left|a\right|+\left|k\right|\right|} R_{cd,ij}(z, w) s_{ld}(w) s_{kc}(z) t_{ik}^{(a)} t_{kj}^{(n-a)},
\]

and antipode \( S : U_{q}(\mathfrak{g}) \rightarrow U_{q}(\mathfrak{g}) \)

\[
\sum_{i,j \in I} S(s_{ij}(z)) \otimes E_{ij} = \left( \sum_{i,j \in I} s_{ij}(z) \otimes E_{ij} \right)^{-1} \in (U_{q}(\mathfrak{g}) \otimes \text{End}\mathbf{V})[[z]],
\]

\[
\sum_{i,j \in I} S(t_{ij}(z)) \otimes E_{ij} = \left( \sum_{i,j \in I} t_{ij}(z) \otimes E_{ij} \right)^{-1} \in (U_{q}(\mathfrak{g}) \otimes \text{End}\mathbf{V})[[z^{-1}]].
\]

Here the RHS of the above formulas are well defined thanks to Relation (2.7).

2.2.3. \( q \)-Yangian. The subalgebra of \( U_{q}(\mathfrak{g}) \) generated by the \( s_{ij}^{(n)}, (s_{ii}^{(0)})^{-1} \) is called the \( q \)-Yangian, denoted by \( Y_{q}(\mathfrak{g}) \). As shown in \[ \text{Zh14} \] Proposition 3.10, these generators together with \( \mathbb{Z}_{2} \)-grading (R2) in \[ \text{2.2.2} \] Relation \[ \text{2.8} \] with the \( t_{ii}^{(0)} \) replaced by the \( (s_{ii}^{(0)})^{-1} \), Relation (2.7) without the \( t_{ij}^{(0)} \), and Relation (2.5) (or equivalently Relation (2.9)) give a full presentation for the superalgebra \( Y_{q}(\mathfrak{g}) \).

According to Formulas (2.10) and (2.12), \( Y_{q}(\mathfrak{g}) \) is a sub-Hopf-superalgebra of \( U_{q}(\mathfrak{g}) \).

2.2.4. Quantum superalgebra. It is the superalgebra \( \mathcal{U}_{q}(\mathfrak{g}) \) generated by \( s_{ij}, t_{ji} \) for \( 1 \leq i \leq j \leq M + N \), with \( \mathbb{Z}_{2} \)-degrees

\[
|s_{ij}| = |t_{ji}| = |i| + |j|
\]

and with RTT relations \[ \text{FRT90} \]

\[
R_{23} T_{12} T_{13} = T_{13} T_{12} R_{23}, \quad R_{23} S_{12} S_{13} = S_{13} S_{12} R_{23}
\]
Here, as usual, \( T = \sum_{i<j} t_{ji} \otimes E_{ji}, \quad S = \sum_{i<j} s_{ij} \otimes E_{ij} \in U_q(\mathfrak{g}) \otimes \text{End} V. \) \( U_q(\mathfrak{g}) \) is endowed with a Hopf superalgebra structure with similar coproduct as in Formulas (2.10)-(2.11).

The relationship between \( U_q(\bar{\mathfrak{g}}) \) and \( U_q(\mathfrak{g}) \) is explained as follows.

**Proposition 2.1.** (1) The assignment \( s_{ij} \mapsto s_{ij}^{(0)}, \quad t_{ji} \mapsto t_{ji}^{(0)} \) extends uniquely to a Hopf superalgebra morphism \( \iota : U_q(\mathfrak{g}) \to U_q(\bar{\mathfrak{g}}) \).

   (2) The assignment \( s_{ij}(z) \mapsto s_{ij} - zt_{ij}, \quad t_{ij}(z) \mapsto t_{ij} - z^{-1}s_{ij} \) extends uniquely to a superalgebra morphism \( \text{ev} : U_q(\bar{\mathfrak{g}}) \to U_q(\mathfrak{g}) \).

We understand that \( s_{ij} = t_{ij} = 0 \) in the superalgebra \( U_q(\mathfrak{g}) \) for \( 1 \leq i < j \leq M + N. \) The morphism \( \text{ev} \) is called an evaluation morphism. It is clear that \( \text{ev} \circ \iota = \text{Id}_{U_q(\mathfrak{g})} \).

2.2.5. **Structures of quantum superalgebras.** Let us gather together in this paragraph the main properties of \( U_q(\bar{\mathfrak{g}}), Y_q(\mathfrak{g}), U_q(\mathfrak{g}) \) which will be used later on.

(a) For \( a \in \mathbb{C}^{\times}, \) there is an automorphism of Hopf superalgebra

\[
\Phi_a : U_q(\bar{\mathfrak{g}}) \to U_q(\bar{\mathfrak{g}}), \quad s_{ij}^{(n)} \mapsto a^n s_{ij}^{(n)}, \quad t_{ij}^{(n)} \mapsto a^{-n} t_{ij}^{(n)}.
\]

\( \Phi_a \) restricts to an automorphism of the \( q \)-Yangian still denoted by \( \Phi_a : Y_q(\mathfrak{g}) \to Y_q(\mathfrak{g}). \) Let us define the evaluation morphism \( \text{ev}_a := \text{ev} \circ \Phi_a. \) So \( \text{ev}_1 = \text{ev} \).

(b) The following relations hold in \( U_q(\bar{\mathfrak{g}}) \) in view of Equations (2.4)-(2.8):

\[
\begin{align*}
(1) & \quad s_{ij}^{(0)} s_{jk}^{(0)} = q^{(\epsilon_i - \epsilon_j)} s_{jk}^{(0)} s_{ij}^{(0)}, \quad s_{ij}^{(0)} t_{jk}^{(n)} = q^{(\epsilon_i - \epsilon_j)} t_{jk}^{(n)} s_{ij}^{(n)} \quad (i,j,k) \in I,
(2) & \quad [s_{i,i+1}, t_{j,j+1}] q^{s_{i,i+1} - s_{j,j+1}} = \delta_{ij} (q_i - q_i^{-1}) (t_{ij}^{(0)} s_{i+1,i+1}^{(0)} - s_{ii}^{(0)} t_{i+1,i+1}^{(0)}) \quad (i,j \in I_0),
(3) & \quad [s_{ij}^{(1)}, t_{kj}^{(0)}] = (q_j - q_j^{-1}) t_{ij}^{(0)} s_{kj}^{(1)} \quad (1 \leq i < j \leq M + N).
\end{align*}
\]

Relation (2.15) gives rise to the weight grading on \( U_q(\bar{\mathfrak{g}}) \): for \( \alpha \in \mathbb{Q}, \)

\[
(U_q(\bar{\mathfrak{g}}))_\alpha = \{ x \in U_q(\bar{\mathfrak{g}}) \mid s_{ii}^{(0)} x (s_{ii}^{(0)})^{-1} = q^{(\epsilon_i - \epsilon_j)} x \quad \text{for} \quad i \in I \}.
\]

In particular, for \( i,j \in I \) we have \( [s_{ij}^{(n)}]_\mathbb{Q} = [t_{ij}^{(n)}]_\mathbb{Q} = \epsilon_i - \epsilon_j. \)

(c) Let \( f \in 1 + z^{-1} \mathbb{C}[z^{-1}] \) and \( g \in 1 + z \mathbb{C}[z]. \) There exists a superalgebra automorphism

\[
\phi_{(f,g)} : U_q(\bar{\mathfrak{g}}) \to U_q(\bar{\mathfrak{g}}), \quad t_{ij}(z) \mapsto ft_{ij}(z), \quad s_{ij}(z) \mapsto gs_{ij}(z).
\]

These automorphisms behave well under coproduct in the following way:

\[
(\phi_{(f,g)})_1 \otimes (\phi_{(f,g)})_2 \circ \Delta = \Delta \circ (\phi_{(f_1,f_2,g_1,g_2)}) : U_q(\bar{\mathfrak{g}}) \to U_q(\bar{\mathfrak{g}})^{\otimes 2}
\]

for \( f_1, f_2 \in 1 + z^{-1} \mathbb{C}[z^{-1}], \quad g_1, g_2 \in 1 + z \mathbb{C}[z]. \) Moreover \( \phi_{(f,g)} \) restricts to a superalgebra automorphism of \( q \)-Yangian denoted by \( \phi_q : Y_q(\mathfrak{g}) \to Y_q(\mathfrak{g}) \) as it does not depend on \( f. \)

(d) The following defines an isomorphism of Hopf superalgebras

\[
\Psi : U_q(\bar{\mathfrak{g}}) \to U_q(\bar{\mathfrak{g}})^{\text{cop}}, \quad s_{ij}^{(n)} \mapsto \varepsilon_{ji} t_{ji}^{(n)}, \quad t_{ij}^{(n)} \mapsto \varepsilon_{ji} s_{ji}^{(n)}.
\]

Here \( \varepsilon_{ij} := (-1)^{|i||j|+|i-j|} \) for \( i,j \in I. \)
We remark that the automorphism $\Psi$ and Relations (2.15)-(2.17) degenerate directly to the quantum superalgebra $U_q(g)$ thanks to Proposition 2.1. In particular, $U_q(g)$ is $\mathbb{Q}$-graded in the obvious way.

2.3. Schur-Weyl duality. There is a natural representation $\rho(1)$ of $U_q(g)$ on the vector superspace $V$ defined by the following matrix equations [Zh14 §4.4]:

$$(\rho(1) \otimes \text{Id}_{\text{End}V})(T) = (\text{Id}_{\text{End}V} \otimes \tau)(R^{-1}), \quad (\rho(1) \otimes \text{Id}_{\text{End}V})(S) = (\text{Id}_{\text{End}V} \otimes \tau)((R^t)^{-1}).$$

We would like to understand the $U_q(g)$-module structure of the tensor powers $V^\otimes s$.

2.3.1. Highest weight representations. Let $\lambda \in \mathbb{P}$. Up to isomorphism, there exists a unique simple $U_q(g)$-module, denoted by $L(\lambda)$, which is generated by a vector $v_\lambda$ satisfying:

$$|v_\lambda| = |\lambda|, \quad s_k v_\lambda = q^{(k,\lambda)} v_\lambda, \quad t_k v_\lambda = q^{-(k,\lambda)} v_\lambda, \quad s_{ij} v_\lambda = 0 \quad (i, j, k \in I, \ i < j).$$

For example, as $U_q(g)$-modules, $V \cong L(\epsilon_1)$ in view of the following equations:

$$\rho(1)(s_{ii}) = q_i E_{ii} + \sum_{j \neq i} E_{jj} = \rho(1)(t_{ii}^{-1}) \quad (\text{for } i \in I),$$

$$\rho(1)(s_{ij}) = (q_i - q_i^{-1}) E_{ij}, \quad \rho(1)(t_{ij}) = (q_i^{-1} - q_i) E_{ji} \quad (\text{for } 1 \leq i < j \leq M + N).$$

2.3.2. Characters. Usually a $U_q(g)$-module $V$ is $\mathbb{P}$-graded via the action of the $s_{ii}$:

$$(V)_\alpha := \{ x \in V \mid s_{ii} x = q^{(\epsilon_i, \alpha)} x \text{ for } i \in I \}. $$

From the $\mathbb{Q}$-grading on $U_q(g)$ we see that $(U_q(g))_\alpha(V)_\beta \subseteq (V)_{\alpha + \beta}$ for $\alpha \in \mathbb{Q}$ and $\beta \in \mathbb{P}$. Assume furthermore that all weight spaces $(V)_\alpha$ are finite-dimensional. Introduce characters

$$(2.20) \quad \chi(V) := \sum_{\alpha \in \mathbb{P}} \dim(V)_\alpha [\alpha] \in \mathbb{Z}[\mathbb{P}].$$

Here $\mathbb{Z}[\mathbb{P}]$ is the group ring of $\mathbb{P}$ over $\mathbb{Z}$.

2.3.3. Young combinatorics. Let us introduce several combinatorial objects before stating Schur-Weyl duality.

Definition 2.2. (1) Let $\mathcal{P} \subseteq \mathbb{P}$ be the subset consisting of $\lambda = \sum_{i \in I} \lambda_i \epsilon_i$ such that

(BKK1) $\lambda_i \geq 0$ for $i \in I$;
(BKK2) for $i \in I_0 \setminus \{M\}$, $\lambda_i \geq \lambda_{i+1}$;
(BKK3) if $\lambda_{M+j} > 0$ for some $1 \leq j \leq N$, then $\lambda_M \geq j$.

For such $\lambda \in \mathcal{P}$, define $ht(\lambda) := \sum_{i \in I} \lambda_i \in \mathbb{Z}_{\geq 0}$.

(2) A $g$-Young diagram is a Young diagram $Y$ such that $(M+1, N+1) \notin Y$. In other words, it is a finite subset $Y \subseteq \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ satisfying

$$(i, j + 1) \in Y \implies (i, j) \in Y \quad \text{for } i, j \in \mathbb{Z}_{>0},$$

$$\# \{ j \in \mathbb{Z}_{>0} \mid (i, j) \in Y \} \geq \# \{ j \in \mathbb{Z}_{>0} \mid (i+1, j) \in Y \} \quad \text{for } i \in \mathbb{Z}_{>0}. $$
Let $\mathcal{YD}$ be the set of $\mathfrak{g}$-Young diagrams.

(3) For $Y \in \mathcal{YD}$, a $\mathfrak{g}$-Young tableau of shape $Y$ is a map $f : Y \to I$ such that

- $(T1)$ if $(i, j), (i', j') \in Y$ and $i \leq i', j \leq j'$, then $f(i, j) \leq f(i', j')$;
- $(T2)$ if $(i, j), (i + 1, j) \in Y$ and $f(i, j) \leq M$, then $f(i, j) < f(i + 1, j)$;
- $(T3)$ if $(i, j), (i, j + 1) \in Y$ and $f(i, j) \geq M + 1$, then $f(i, j) < f(i + 1, j)$.

Let $\mathcal{B}(Y)$ be the set of $\mathfrak{g}$-Young tableaux of shape $Y$.

(4) For $\lambda = \sum_1 \lambda_i \epsilon_i \in \mathcal{P}$, define $Y^\lambda$ to be the $\mathfrak{g}$-Young diagram formed of such $(i, j)$ that

- $(Y1)$ if $1 \leq i \leq M$, then $j \leq \lambda_i$;
- $(Y2)$ if $i > M$, then $j \leq N$ and $\lambda_{M+j} \geq i - M$.

Let $\iota : \mathcal{P} \to \mathcal{YD}, \lambda \mapsto Y^\lambda$ be the bijective map thus obtained.

**Remark 2.3.** Later on, we shall consider $\mathfrak{gl}(a, b)$-Young diagrams and Young tableaux either for $(a, M, 1 \leq b \leq N)$ or for $(1 \leq a \leq M, b = 0)$. In this case, we have analogous definitions of $I_{a,b}, P_{a,b}, \mathcal{YD}_{a,b}, \mathcal{B}_{a,b}(Y), \mathcal{Y}_{a,b}$ by replacing $(M, N)$ with $(a, b)$ everywhere.

On the other hand, we shall make the obvious inclusions:

$$\mathcal{P}_{1,0} \subset \mathcal{P}_{2,0} \subset \cdots \subset \mathcal{P}_{M-1,0} \subset \mathcal{P}_{M,0} \subset \mathcal{P}_{M,1} \subset \cdots \subset \mathcal{P}_{M,N-1} \subset \mathcal{P}_{M,N} = \mathcal{P};$$

$$\mathcal{P}_{1,0} \subset \mathcal{P}_{2,0} \subset \cdots \subset \mathcal{P}_{M-1,0} \subset \mathcal{P}_{M,0} \subset \mathcal{P}_{M,1} \subset \cdots \subset \mathcal{P}_{M,N-1} \subset \mathcal{P}_{M,N} = \mathcal{P}.$$ 

**Theorem 2.4.** [BKK00] For all $s \in \mathbb{Z}_{\geq 1}$, the $\mathcal{U}_q(\mathfrak{g})$-module $V^\otimes s$ is completely reducible. More precisely, we have a decomposition into simple sub-$\mathcal{U}_q(\mathfrak{g})$-modules as follows:

$$V^\otimes s = \bigoplus_{\lambda \in \mathcal{P}} L(\lambda)^{c_\lambda}.$$ 

Here $c_\lambda \in \mathbb{Z}_{\geq 0}$ and $c_\lambda \neq 0$ if and only if $ht(\lambda) = s$. Furthermore, for $\lambda \in \mathcal{P}$

$$\chi(L(\lambda)) = \sum_{f \in \mathcal{B}(Y^\lambda)} \sum_{(i, j) \in Y^\lambda} \epsilon_{f(i, j)} \in \mathbb{Z}[\mathcal{P}].$$

Such representations $L(\lambda)$ with $\lambda \in \mathcal{P}$ are usually called polynomial representations, as they appear as simple submodules of tensor powers of the natural representation.

Let us end this paragraph with the following simple application of the character formula \[2.21\]. This result, on the asymptotic behaviour of weight spaces, serves as a motivation for our limit construction carried out later.

**Definition 2.5.** For $r \in I_0$, the $r$th fundamental weight $\varpi_r \in \mathcal{P}$ is defined by the formula

$$\varpi_r := \begin{cases} \sum_{j=1}^r \epsilon_j & (r \leq M), \\ - \sum_{j=M+N}^r \epsilon_j & (r > M). \end{cases}$$

**Corollary 2.6.** Suppose that $(L(l\varpi_r))_{l=1}^r \neq 0$ where $1 \leq r \leq M, l > 0$ and $\beta \in Q_{\geq 0}$. Then

$$\dim(L(k\varpi_r))_{k=1}^r \neq \dim(L(l\varpi_r))_{l=1}^r$$

for all $k \geq rl$. 


Proof. Note that \( k\varpi_r \in \mathcal{P} \) for all \( k \in \mathbb{Z}_{>0} \). Furthermore,
\[
Y^{k\varpi_r} = \{1, 2, \ldots, r\} \times \{1, 2, \ldots, k\} \subset \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}.
\]
In view of Formula (2.21), we only need to check the following: for all \( k > lr \) and for all \( f \in \mathcal{B}Y^{k\varpi_r} \) such that
\[
\sum_{(i,j) \in Y^{k\varpi_r}} \epsilon_f(i,j) = k\varpi_r - \beta,
\]
we must have \( f(i, 1) = i \) for \( 1 \leq i \leq r \). Let us fix such an \( f \).

The condition \((L(l\varpi_r))_{l\varpi_r - \beta} \neq 0\) says the existence of \( g \in \mathcal{B}(Y^{l\varpi_r}) \) such that
\[
l\varpi_r - \beta = \sum_{(i,j) \in Y^{l\varpi_r}} \epsilon_g(i,j).
\]
This says that
\[
\sum_{(i,j) \in Y^{k\varpi_r}} \epsilon_f(i,j) = (k - l) \sum_{i=1}^{r} \epsilon_i + \sum_{(i,j) \in Y^{l\varpi_r}} \epsilon_g(i,j) = \sum_{i=1}^{M+N} c_i \epsilon_i.
\]
In particular, \( c_i \geq k - l \) for \( 1 \leq i \leq r \). We prove by induction on \( 1 \leq i \leq r \) that: \( f(i, j) = i \) for all \( 1 \leq j \leq k - il \). For \( i = 1 \), this is obvious from the definition of Young tableau. Suppose the assertion for \( i - 1 \) true. If there exists \( 1 \leq j_0 \leq k - il \) such that \( f(i, j_0) > i \), consider the pre-image \( f^{-1}(i) \) of \( i \). Suppose that \((m, n) \in f^{-1}(i)\). Then we have the following:

1. \( m \leq i \), from the definition of Young tableau;
2. if \( m = i \), then \( 1 \leq n < j_0 \). In particular, \( n \leq k - il - 1 \);
3. if \( m < i \), then \( k - (i - 1)l < n \leq k \) as \( f(i - 1, j) = i - 1 \) for \( 1 \leq j \leq k - (i - 1)l \).

Note that \((m, n), (m', n) \in f^{-1}(i)\) forces \( m = m' \). By counting the number of \( n \) in \( f^{-1}(i) \) we get
\[
2f^{-1}(i) = c_i \leq k - il - 1 + (i - 1)l = k - l - 1,
\]
which contradicts with the fact that \( c_i \geq k - l \). Thus, \( f(i, j) = i \) for all \( 1 \leq j \leq k - il \) and
\( 1 \leq i \leq r \), as desired. \( \square \)

For example, when \( r = 1 \), all the weight spaces of \( L(k\varpi_1) \) are one-dimensional.

2.4. Gelfand-Tsetlin basis. Following Remark 2.3, let \( \mathbb{X} \) be the set of pairs \((a, b) \in \mathbb{Z}^2\) such that: either \((a = M, 1 \leq b \leq N)\); or \((1 \leq a \leq M, b = 0)\). Let \( k \mapsto (M_k, N_k) \) be the unique bijective map \( I \rightarrow \mathbb{X} \) such that \( M_k + N_k = k \) for \( k \in I \). For \( k \in I \), let \( \mathcal{U}_q(\mathfrak{g}_k) \) be the subalgebra of \( \mathcal{U}_q(\mathfrak{g}) \) generated by the \( s_{ij}, t_{ji} \) with \( 1 \leq i \leq j \leq k \). Then \( \mathcal{U}_q(\mathfrak{g}_k) \) is a sub-Hopf-superalgebra canonically isomorphic to \( \mathcal{U}_q(\mathfrak{g}(M_k, N_k)) \), and we have the following sequence of inclusions of Hopf superalgebras:
\[
\mathcal{U}_q(\mathfrak{g}_1) \subset \mathcal{U}_q(\mathfrak{g}_2) \subset \cdots \subset \mathcal{U}_q(\mathfrak{g}_{M+N-1}) \subset \mathcal{U}_q(\mathfrak{g}_{M+N}) = \mathcal{U}_q(\mathfrak{g}).
\]
Identify \( \mathfrak{g}_k \) with \( \mathfrak{g}(M_k, N_k) \) from now on.

Suppose \( 2 \leq k \leq M+N \). Let \( \lambda \in \mathcal{P}_{M_k,N_k} \). Let \( Y^\lambda = \mathbb{Y}_{M_k,N_k}(\lambda) \) be the \( \mathfrak{g}_k \)-Young diagram. If \( f \in \mathcal{B}_{M_k,N_k}(Y^\lambda) \) is a \( \mathfrak{g}_k \)-Young tableau of shape \( Y^\lambda \), then the subset \( f^{-1}(I_{M_k-1,N_k-1}) \subset Y^\lambda \) is a \( \mathfrak{g}_{k-1} \)-Young diagram. Moreover, the restriction of \( f \) to this subset gives us a \( \mathfrak{g}_{k-1} \)-Young tableau of corresponding shape.
Definition 2.7. (1) Suppose \(2 \leq k \leq M + N\). For \(\lambda \in \mathcal{P}_{M_k, N_k}\), let \(S_k(\lambda)\) be the set
\[
S_k(\lambda) := \{ \mathbb{Y}_{M_{k-1}, N_{k-1}}^{-1}(f^{-1}(I_{M_{k-1}, N_{k-1}})) \in \mathcal{P}_{M_{k-1}, N_{k-1}} \mid f \in \mathcal{P}_{M_k, N_k}(\mathbb{Y}_{M_k, N_k}(\lambda)) \}.
\]

(2) A sequence of weights \((\lambda^{(k)} : k \in I) \in \mathcal{P}^{M+N}\) is called a Gelfand-Tsetlin pattern if \(\lambda^{(k-1)} \in S_k(\lambda^{(k)})\) for all \(2 \leq k \leq M + N\). For \(\lambda \in \mathcal{P}\), let \(\mathcal{G}(\lambda)\) be the set of Gelfand-Tsetlin patterns \((\lambda^{(k)})\) with \(\lambda^{(M+N)} = \lambda\).

As in the non-graded case, Gelfand-Tsetlin patterns are in one-to-one correspondence with Young tableaux.

Lemma 2.8. Let \(\lambda \in \mathcal{P}\). The following defines a bijective mapping
\[
GT_\lambda : \mathcal{B}(\mathcal{Y}^\lambda) \longrightarrow \mathcal{G}(\lambda), \quad GT_\lambda(f)^{(k)} = \mathbb{Y}_{M_k, N_k}^{-1}(f^{-1}(I_{M_k, N_k})).
\]

Example 1. If \(N = 0\), then \(\mathcal{P}\) is the set of weights \(\sum_{i=1}^{M} \lambda_i \epsilon_i \in \mathcal{P}\) with \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M\). A Gelfand-Tsetlin pattern is a sequence of weights \((\sum_{i=1}^{k} \lambda_i^{(k)} \epsilon_i : 1 \leq k \leq M)\) with:

1. \(\lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \cdots \geq \lambda_k^{(k)}\) for \(1 \leq k \leq M\);
2. \(\lambda_1^{(k)} \geq \lambda_1^{(k-1)} \geq \lambda_2^{(k-1)} \) for \(2 \leq k \leq M\) and \(1 \leq i \leq k-1\).

This is the classical definition of Gelfand-Tsetlin pattern for \(\mathfrak{gl}_M\) (see [Mo06, pp.118]). When \(N > 0\), the situation becomes complicated. For example, take \((M, N) = (2, 1)\). Then the set \(\mathcal{G}(2\epsilon_1)\) has 5 elements, namely
\[(2\epsilon_1, 2\epsilon_1), (\epsilon_1, 2\epsilon_1, 2\epsilon_1), (\epsilon_1, \epsilon_1, 2\epsilon_1), (0, 2\epsilon_1, 2\epsilon_1), (0, \epsilon_1, 2\epsilon_1)\].

In particular, \((0, 0, 2\epsilon_1)\) is not permitted.

For \(k \in I\), \(\lambda \in \mathcal{P}_{M_k, N_k}\) and \(s \in \mathbb{Z}_2\), let \(L(\lambda; \mathfrak{g}_k)\) be the simple \(\mathcal{U}_q(\mathfrak{g}_k)\)-module generated by a highest weight vector of highest weight \(\lambda\) and of \(\mathbb{Z}_2\)-degree \(|\lambda|\); let \(C_s\) be the one-dimensional \(\mathcal{U}_q(\mathfrak{g}_k)\)-module of zero weight and of \(\mathbb{Z}_2\)-degree \(s\). The following is a direct consequence of Theorem 2.4.

Corollary 2.9. Let \(2 \leq k \leq M + N\) and let \(\lambda \in \mathcal{P}_{M_k, N_k}\). Then the \(\mathcal{U}_q(\mathfrak{g}_{k-1})\)-module 
\[
\text{Res}_{\mathcal{U}_q(\mathfrak{g}_k)}^{\mathcal{U}_q(\mathfrak{g}_{k-1})} L(\lambda; \mathfrak{g}_k)
\]
\[
= \bigoplus_{\lambda' \in S_k(\lambda)} L(\lambda'; \mathfrak{g}_{k-1}) \otimes C_{|\text{ht}(\lambda') - \text{ht}(\lambda)|[k]},
\]
as \(\mathcal{U}_q(\mathfrak{g}_{k-1})\)-modules.

Now we get an explicit description of Gelfand-Tsetlin basis as follows.

Corollary 2.10. For \(\lambda \in \mathcal{P}\), the \(\mathcal{U}_q(\mathfrak{g})\)-module \(L(\lambda)\) is equipped with a basis \((v_\lambda : \lambda \in \mathcal{G}(\lambda))\) satisfying: \(v_\lambda\) is contained in a sub-\(\mathcal{U}_q(\mathfrak{g}_k)\)-module isomorphic to
\[
L(\lambda^{(k)}; \mathfrak{g}_k) \otimes \mathcal{C}_{\sum_{i=k+1}^{M+N} \text{ht}(\mathcal{G}_{\lambda^{(k)}})^{-1}(i)[k]}
\]
for \(\lambda = (\lambda^{(k)}) \in \mathcal{G}(\lambda)\) and \(k \in I\).
2.5. Dual Gelfand-Tsetlin basis. We shall also need Gelfand-Tsetlin bases for $\mathcal{U}_q(\mathfrak{g})$-modules $L(\lambda)^*$ with $\lambda \in \mathcal{P}$. In general, if $H$ is a Hopf superalgebra and if $V$ is an $H$-module, then the vector superspace $V^* = \text{hom}(V, \mathbb{C})$ is endowed with an $H$-module structure by:

$$(a, f) \mapsto af : v \mapsto (-1)^{|a||f|} f(S(a)v)$$

for $\mathbb{Z}_2$-homogeneous $a \in H$ and $f \in V^*$. Here $S : H \rightarrow H$ is the antipode.

Let $(v^*_\lambda : \lambda \in \mathcal{GT}(\lambda))$ be a Gelfand-Tsetlin basis for $L(\lambda)$. Let $(v^*_\lambda : \lambda \in \mathcal{GT}(\lambda))$ be its dual basis for $L(\lambda)^*$. Since the embeddings $\mathcal{U}_q(\mathfrak{gl}_{k-1}) \hookrightarrow \mathcal{U}_q(\mathfrak{gl}_k)$ respect the Hopf superalgebra structures, and since the restricted modules $\text{Res}_{\mathcal{U}_q(\mathfrak{g})}^{\mathcal{U}_q(\mathfrak{gl}_k)} L(\lambda)$ are always completely reducible, we conclude that: $v^*_\lambda$ is contained in a simple sub-$\mathcal{U}_q(\mathfrak{gl}_k)$-module isomorphic to

$L^{(\lambda)}(\mathfrak{g}_k)^* \otimes \mathbb{C}^{\sum_{i=0}^{M+N} (-1)^{(GT^\lambda_{i+1})-1} (i|M)}$

for $\lambda \in \mathcal{GT}(\lambda)$ and $k \in I$. For this reason, call $(v^*_\lambda : \lambda \in \mathcal{GT}(\lambda))$ a dual Gelfand-Tsetlin basis.

Let us compute the highest weight of the $\mathcal{U}_q(\mathfrak{g})$-module $L(\lambda)^*$ when $\lambda \in \mathcal{P}$. Let $Y^\lambda = \mathcal{Y}(\lambda)$ be the associated $\mathfrak{g}$-Young diagram. For all $i, j \in \mathbb{Z}_{>0}$, define

$r_i := \sharp\{j \in \mathbb{Z}_{>0} \mid (i, j) \in Y^\lambda\}, \quad c_j := \sharp\{i \in \mathbb{Z}_{>0} \mid (i, j) \in Y^\lambda\}$;

$r'_i := \max(r_i - N, 0), \quad c'_j := \max(c_j - M, 0)$.

Then the highest and lowest weights $\lambda, \lambda_0$ of $L(\lambda)$ can be written as:

$$\lambda = \sum_{i=1}^{M} r_i \epsilon_i + \sum_{j=1}^{N} c_j \epsilon_{M+j}, \quad \lambda_0 = \sum_{i=1}^{M} r'_i \epsilon_{M+1-i} + \sum_{j=1}^{N} c'_{N+1-j} \epsilon_{M+j}.$$  

Moreover, as $\mathcal{U}_q(\mathfrak{g})$-modules, we have the identification

(2.22) $L(\lambda)^* \cong L(-\lambda_0)$.

In the non-graded case, $\lambda_0 = w_0(\lambda)$ where $w_0$ is a longest element of the Weyl group associated with the Lie algebra. In particular, $(\lambda + \mu)_0 = \lambda + \mu_0$ for $\lambda, \mu$ dominant. Such additive formula is no longer true in our case. For example, take $(M, N) = (2, 1)$. Then $(\epsilon_1)_0 = \epsilon_3$ and $(2\epsilon_1)_0 = \epsilon_2 + \epsilon_3$.

**Remark 2.11.**

1. For the quantum superalgebra $U_q(\mathfrak{g})$, Palev-Stoilova-Van der Jeugt [PSV94] established the Gelfand-Tsetlin bases for the so-called essentially typical representations. In our situation, for $\lambda \in \mathcal{P}$, $L(\lambda)$ is essentially typical if and only if $(M, N) \in Y^\lambda$. When this is the case, a combinatorial description of the set $\mathcal{GT}(\lambda)$ similar to the example above has been given (cf. Equations (9)-(10) in loc. cit). Moreover, explicit actions of the generators $e_i^\pm$ with respect to the Gelfand-Tsetlin bases were given therein.

2. For the Lie superalgebra $\mathfrak{g}$, there are analogs of such representations $L_0(\lambda)$ as $\lambda \in \mathcal{P}$ constructed as simple submodules of tensor powers of the natural representations of $\mathfrak{g}$ on $V$, also called covariant representations. (One can view $L(\lambda)$ as a deformation of $L_0(\lambda)$.) In [Mol10], Molev constructed Gelfand-Tsetlin bases for all the $L_0(\lambda)$ and deduced explicit formulas of the actions of the generators $E_{ij}$ with $i, j \in I, |i - j| = 1$ with respect to these bases. The main ingredients used by Molev are the Yangian $Y(\mathfrak{gl}_N)$ and the Mickelsson-Zhelobenko algebra $Z(\mathfrak{g}, \mathfrak{g}\mathfrak{l}_M)$.
(3) We shall give another characterization of Gelfand-Tsetlin basis for $L(\lambda)$ and dual Gelfand-Tsetlin basis for $L(\lambda)^*$ within the framework of representations of the quantum affine superalgebra $U_q(\hat{\mathfrak{g}})$. (See Propositions 7.1 and 7.4)

3. Kirillov-Reshetikhin module and tensor products

We prove a cyclicity result on tensor products of Kirillov-Reshetikhin modules over $U_q(\hat{\mathfrak{g}})$, based on our previous result [Zh14, Theorem 4.4] and on duality arguments.

3.1. Kirillov-Reshetikhin modules. Let us recall the definition of Kirillov-Reshetikhin modules $W_{k,a}^{(r)}$ with $r \in I_0, a \in \mathbb{C}^\times, k \in \mathbb{Z}_{>0}$. These are $U_q(\hat{\mathfrak{g}})$-modules

\[(3.23)\quad W_{k,a}^{(r)} := \text{ev}_a^* L(k\varpi_r) \otimes \mathbb{C}_{|k\varpi_r|}\]

where as usual $\mathbb{C}_s$ is the one-dimensional $U_q(\hat{\mathfrak{g}})$-module of $\mathbb{Z}_2$-degree $s \in \mathbb{Z}_2$.

Let $V$ be a $U_q(\hat{\mathfrak{g}})$-module. A non-zero vector $v \in V \setminus \{0\}$ is called a highest $\ell$-weight vector if $v$ is $\mathbb{Z}_2$-homogeneous and

\[s_{ij}^{(n)}v = 0 = s_{kk}^{(n)}v, \quad s_{ii}^{(n)}v, s_{kk}^{(n)}v \in \mathbb{C}v \quad (n \in \mathbb{Z}_{\geq 0}, \quad i, j, k \in I, \quad i < j)\]

$V$ is called a highest $\ell$-weight module if $V = U_q(\hat{\mathfrak{g}})v$ for some highest $\ell$-weight vector $v$. Similarly, the notions of lowest $\ell$-weight vector and lowest $\ell$-weight module are defined by replacing $(i < j)$ with $(i > j)$. Let $V, V'$ be two $U_q(\hat{\mathfrak{g}})$-modules and let $v, v'$ be highest/lowest $\ell$-weight vectors in $V, V'$ respectively. Write

\[s_{ii}(z)v = f_i(z)v, \quad s_{ii}(z)v' = f_i'(z)v' \quad \text{for } i \in I\]

Then $v \otimes v'$ is a highest/lowest $\ell$-weight vector such that

\[s_{ii}(z)(v \otimes v') = f_i(z)f_i'(z)v \otimes v' \quad \text{for } i \in I\]

For example, let $v_{k\varpi_r} \in L(k\varpi_r)$ be as in (2.3.1) and let $v_s \in \mathbb{C}_s$ be a non-zero vector with $s = |k\varpi_r|$. Then $w := \text{ev}_a^* v_{k\varpi_r} \otimes v_s \in W_{k,a}^{(r)}$ is a highest $\ell$-weight vector and

\[|w| = \overline{0}, \quad s_{ii}(z)w = (q^{(e_i,k\varpi_r)} - zaq^{-e_i,k\varpi_r})w, \quad t_{ii}(z)w = (q^{-e_i,k\varpi_r} - z^{-1}a^{-1}q^{e_i,k\varpi_r})w\]

The rest of this section is devoted to proving the following cyclicity result.

**Theorem 3.1.** Let $r \in I_0, a \in \mathbb{C}^\times$ and $l_1, l_2, l_3 \in \mathbb{Z}_{>0}$ such that $l_1 < l_2 < l_3$.

1. If $r \leq M$, then $W_{l_1,a}^{(r)} \otimes W_{l_2-1,aq^{-2l_1}}^{(r)} \otimes W_{l_3-2,aq^{-2l_2}}^{(r)}$ is of highest $\ell$-weight.
2. If $r > M$, then $W_{l_3-2,aq^{-2l_2}}^{(r)} \otimes W_{l_2-1,aq^{-2l_1}}^{(r)} \otimes W_{l_1,a}^{(r)}$ is of highest $\ell$-weight.

Let us recall another cyclicity result before turning to the proof.

**Theorem 3.2.** [Zh14, Theorem 4.2, Proposition 4.7] Let $r \in I_0, a \in \mathbb{C}^\times$ and $k \in \mathbb{Z}_{>0}$.

1. (I) \(\bigotimes_{j=1}^{k} W_{1,aq^{-2j}}^{(r)}\) is of highest $\ell$-weight.
2. (II) \(\bigotimes_{j=1}^{k} W_{1,aq^{2j}}^{(1)}\) is of lowest $\ell$-weight.
3.1.1. Reduction to the case \( r \leq M \). Following [Zh14, §4.2], let \( \mathfrak{g}' := \mathfrak{gl}(N, M) \). Let us define the quantum affine superalgebra \( U_q(\mathfrak{g}') \) in the same way as in 2.2.2 except that \( M, N \) are replaced by each other everywhere. For distinction, let \( s_{ij}^{(n)}, t_{ij}^{(n)} \) be the defining generators.

For \( i \in I \), set \( \overline{v}_i := M + N + 1 - i \) and \( |i|' := \begin{cases} 0 & (i \leq N) \\ 1 & (i > N) \end{cases} \). So that \( |s_{ij}^{(n)}| = |t_{ij}^{(n)}| = |i|^J + |j|^J \).

Let \( \varepsilon_{ij}^J = (-1)^{|i|^J(|i|^J + |j|^J)} \). By [Zh14, Proposition 4.3], the following

\[
(3.24) \quad f_{j,i} : U_q(\mathfrak{g}')^{\text{cop}} \rightarrow U_q(\mathfrak{g}), \quad s_{ij}^{(n)} \mapsto \varepsilon_{ji}^J s_{ji}^{(n)}, \quad t_{ij}^{(n)} \mapsto \varepsilon_{ji}^J t_{ji}^{(n)}
\]
defines an isomorphism of Hopf superalgebras.

Let \( V \) be a \( U_q(\mathfrak{g}) \)-module. If \( v \in V \) is a highest \( \ell \)-weight vector, then so is \( f_{j,i}^* v \in f_{i,j}^* V \).

Let us \( W^{(r)}_{k,a,j} \) be the corresponding Kirillov-Reshetikhin modules over \( U_q(\mathfrak{g}') \). Then, as \( U_q(\mathfrak{g}) \)-modules, there are isomorphisms \( W^{(N+M-r)}_{k,a,j} \approx f_{j,i}^* W^{(r)}_{k,a,j} \) for \( r \neq M \).

Hence, if (1) of Theorem 3.1 is proved, (2) follows automatically.

3.1.2. Dualities. Let \( V \) be a \( U_q(\mathfrak{g}) \)-module. On the dual space \( V^* = \text{hom}(V, \mathbb{C}) \), there are two structures of \( U_q(\mathfrak{g}) \)-modules. The first one is afforded by the Hopf superalgebra structure on \( U_q(\mathfrak{g}) \), the corresponding \( U_q(\mathfrak{g}) \)-module still denoted by \( V^* \) as in (2.13). The second one is the pull back \( \Psi^* V^* \), where \( \Psi \) is the superalgebra isomorphism given by (2.19). Let us write \( \Psi^* V^* \) as \( V^\vee \) for distinction. As usual, when \( V, W \) are finite-dimensional \( U_q(\mathfrak{g}) \)-modules, there are canonical isomorphisms of \( U_q(\mathfrak{g}) \)-modules

\[
(V \otimes W)^* \cong W^* \otimes V^*, \quad (V \otimes W)^\vee \cong V^\vee \otimes W^\vee.
\]

Let \( V \) be a \( U_q(\mathfrak{g}) \)-module. Let \( v \in V \setminus \{0\} \) be \( \mathbb{Z}_2 \)-homogeneous. We say that \( V \) is cogenerated by \( v \) if every non-zero submodule of \( V \) contains \( v \). In this case, \( v \) is also called a cogenerator and we write

\[
U_q(\mathfrak{g})v =: \text{socle}(V).
\]

Note that socle(\( V \)) is the simple sub-\( U_q(\mathfrak{g}) \)-module of \( V \).

**Remark 3.3.** Let \( V \) be a \( U_q(\mathfrak{g}) \)-module \( \mathbb{P} \)-graded with respect to the action of the \( s_{ii}^{(0)} \):

\[
(V)_{\lambda} = \{ x \in V \mid s_{ii}^{(0)} x = q^{(\lambda, \varepsilon_i)} x \text{ for } i \in I \}.
\]

Suppose that there exists \( \lambda_0 \in \mathbb{P} \) such that \( (V)_{\lambda_0} = \mathbb{C} v_0 \neq 0 \) and \( (V)_{\lambda} \neq 0 \) implies \( \lambda_0 - \lambda \in \mathbb{Q}_{\geq 0} \). Clearly, for such \( V \), the vector \( v_0 \) must be a highest \( \ell \)-weight vector. Let \( v_0^* \) be the linear form on \( V \) defined by: \( v_0^* \mapsto 1 \) and \( (V)_{\lambda} \mapsto 0 \) for \( \lambda \neq \lambda_0 \). Then by definition of dual modules, \( V \) is of highest \( \ell \)-weight if and only if \( v_0^* \) is a cogenerator in \( V^\vee \). Analogously, suppose that all the weights of \( V \) lie in the cone \( 1 + \mathbb{Q}_{\geq 0} \lambda_1 + \mathbb{Q}_{\geq 0} \lambda_2 \) and that \( (V)_{\lambda_1} = \mathbb{C} v_1 \) is one-dimensional. Then one can define \( v_1^* \in V^\vee \) similarly, and \( V \) is of lowest \( \ell \)-weight if and only if \( v_1^* \) is a cogenerator in \( V^\vee \).

**Remark 3.4.** Let \( s \in \mathbb{Z}_{>0} \). For \( 1 \leq i \leq s \), let \( V_i \) be a \( U_q(\mathfrak{g}) \)-module and let \( v_i \in V_i \) be a non-zero \( \mathbb{Z}_2 \)-homogeneous vector. Set \( S_i := U_q(\mathfrak{g})v_i \subseteq V_i \). If \( \bigotimes_{i=1}^s V_i \) is cogenerated by \( \bigotimes_{i=1}^s v_i \), then so is \( \bigotimes_{i=1}^s S_i \).
The proof of Theorem 3.1 (1) will go as follows: first we compute the duals \((W^{(r)}_{k,a})^\vee\) and rephrase (1) as a statement of cogenerators; next we realize the duals \((W^{(r)}_{k,a})^\vee\) as simple submodules arising from cogenerators and apply Theorem 3.2 and Remark 3.4 to conclude.

We close this subsection with the following convention. Let \(S_1, S_2\) be \(U_q(\hat{g})\)-modules. We say that \(S_1 \simeq S_2\) if there exists a one-dimensional \(U_q(\hat{g})\)-module \(D\) making the two \(U_q(\hat{g})\)-modules \(S_1\) and \(S_2 \otimes D\) isomorphic. Remark that \(\simeq\) does not change the property of being of highest \(\ell\)-weight, of lowest \(\ell\)-weight, or cogenerated.

### 3.2. Duals of Kirillov-Reshetikhin modules

Throughout this subsection, \(1 \leq r \leq M\). We shall determine \((W^{(r)}_{k,a})^\vee\) up to \(\simeq\).

#### 3.2.1. Dual of \(W^{(1)}_{1,a}\)

In view of (2.3.1) \(W^{(1)}_{1,a} = V\) as vector superspace. Let \(\rho_a\) be the corresponding representation of \(U_q(\hat{g})\) on \(V\). Similarly, the \(U_q(\hat{g})\)-module \((W^{(1)}_{1,a})^\vee\) gives rise to the representation \((\rho_a^\vee, V^*)\). Let \((v_i^* \in V^* : i \in I)\) be the dual basis of \((v_i \in V : i \in I)\). Then \(v_i^*\) is a highest \(\ell\)-weight vector in \((W^{(1)}_{1,a})^\vee\). In view of the \(P\)-grading on the simple module \((W^{(1)}_{1,a})^\vee\), it is enough to determine

\[
s_{ii}(z)v_i^* = v_i^*(S(t_{ii}(z^{-1}))v_1)v_i^*, \quad t_{ii}(z)v_i^* = v_i^*(S(s_{ii}(z^{-1})))v_i^*. \]

For this, let us introduce

\[
X(z) := (\rho_a \otimes \text{Id}_{\text{End}V})(S(z)), \quad Y(z) := (\rho_a \otimes \text{Id}_{\text{End}V})(T(z)).
\]

From Formulas (2.1.2, 2.1.3) we get

\[
X(z)^{-1} = (\rho_a \otimes \text{Id}_{\text{End}V})(S(\text{Id} \otimes \text{End}V)(S(z))), \quad Y(z)^{-1} = (\rho_a \otimes \text{Id}_{\text{End}V})(S(\text{Id} \otimes \text{End}V)(T(z))).
\]

We are led to determine the inverses of \(X(z), Y(z)\). In view of Proposition 2.1,

\[
Y(z) = -z^{-1}a^{-1}X(z) \in \text{End}(V^\otimes 2[[z^{-1}]]).
\]

Therefore, it is enough to find the inverse of \((X(z) \in \text{End}(V^\otimes 2[[z]]))\). By (2.3.1)

\[
X(z) = \sum_i (q_i - zaq_i^{-1})E_{ii} \otimes E_{ii} + (1 - za)\sum_{i \neq j} E_{ii} \otimes E_{jj} + \sum_{i < j} (q_i - q_i^{-1})E_{ij} \otimes E_{ij} + \sum_{i < j} za(q_i - q_i^{-1})E_{ji} \otimes E_{ji}.
\]

Let \(i, j \in I\). When \(i \neq j\),

\[
X(z)(v_i \otimes v_j) = (1 - za)v_i \otimes v_j, \quad X(z)^{-1}(v_i \otimes v_j) = (1 - za)^{-1}(v_i \otimes v_j).
\]

When \(i = j\),

\[
X(z)(v_i \otimes v_i) = (q_i - zaq_i)v_i \otimes v_i + (q_i - q_i^{-1})\sum_{s < i} v_s \otimes v_s + za(q_i - q_i^{-1})\sum_{t > i} v_t \otimes v_t.
\]

Let us introduce the following matrix \(M_{M,N}(z) \in \text{Mat}(M + N, \mathbb{C}[[z]])\):

\[
(M_{M,N}(z))_{ij} := \begin{cases} 
q_j^{-1} & \text{if } i < j, \\
q_j - zaq_j^{-1} & \text{if } i = j, \\
za(q_j - q_j^{-1}) & \text{if } i > j.
\end{cases}
\]
Let \((x_{ij}(z))_{i,j \in I}\) be the inverse of \(M_{M,N}(z)\). Then
\[
X(z)^{-1} = \sum_i x_{ii}(z) E_{ii} \otimes E_{ii} + (1 - za)^{-1} \sum_{i \neq j} E_{ii} \otimes E_{jj} + \sum_{i \neq j} (-1)^{|i|+|j|} x_{ij}(z) E_{ij} \otimes E_{ij}.
\]
As a result, in the \(U\)-module \(W_{1,a}^1\), we have
\[
l_{ii}(z)^* v_i^1 = v_i^1 \begin{cases} x_{11}(z^{-1}) & \text{if } i = 1, \\ (1 - z^{-1}a)^{-1} & \text{if } i > 1,
\end{cases} s_{ii}(z) v_i^1 = v_i^1 \begin{cases} -z^{-1}ax_{11}(z) & \text{if } i = 1, \\ -z^{-1}a(1 - z^{-1}a)^{-1} & \text{if } i > 1.
\end{cases}
\]

**Remark 3.5.** One can write down explicitly the inverse of \(M_{M,N}(z)\):
\[
(M^{-1}_{M,N}(z))_{ij} = \frac{\theta_i^{-1} \theta_j}{(1 - za)(1 - zaq^{-2M+2N})} \begin{cases} q_j^{-1} - q_i & \text{if } i < j, \\ q_j^{-1} - zaq^{-2M+2N} q_i & \text{if } i = j, \\ (q_j^{-1} - q_i) zaq^{-2M+2N} & \text{if } i > j
\end{cases}
\]
where the \(\theta_i\) for \(i \in I\) are defined inductively by: \(\theta_1 = 1\), \(\theta_{i+1} = q_i^{-1} q_i^{-1} \theta_i\).

By comparing the highest \(\ell\)-weight vectors we conclude that
\[
(W^{(1)})^r_{1,a} \simeq W^{(1)}_{1,a-1,q^{2M-2N}}.
\]

**3.2.2. Dual of \(W^{(r)}_{1,a}\).** Let \(b \in \mathbb{C}^\times\). It follows from Equation (3.26), Theorem 3.2 (II) and Remark 3.3 that the tensor product \(\bigotimes_{j=1}^r W^{(1)}_{1,bq^{-2j}}\) is cogenerated by \(v := \bigotimes_{j=1}^r v_j\), where \(v_j\) is a lowest \(\ell\)-weight vector in \(W^{(1)}_{1,bq^{-2j}}\) for \(1 \leq j \leq r\). In particular, the submodule \(U_q(\mathfrak{g})v\) is simple and it is exactly the socle of \(\bigotimes_{j=1}^r W^{(1)}_{1,bq^{-2j}}\). By comparing the lowest \(\ell\)-weight vectors, we see that
\[
W^{(r)}_{1,bq^{-2r}} \simeq \text{socle}(\bigotimes_{j=1}^r W^{(1)}_{1,bq^{-2j}}) = U_q(\mathfrak{g})v.
\]
On the other hand, the \(U_q(\mathfrak{g})\)-module \(\bigotimes_{j=1}^r W^{(1)}_{1,bq^{-2j}}\) is of lowest \(\ell\)-weight generated by \(v^*\) by Remark 3.3 and its simple quotient is the twisted dual of \(U_q(\mathfrak{g})v\). Making use of Equation (3.26) we conclude that
\[
(W^{(r)})^r_{1,a} \simeq W^{(r)}_{1,a-1,q^{2(M-N-r+1)}}.
\]

**3.2.3. Dual of \(W^{(r)}_{k,a}\).** Let \(b \in \mathbb{C}^\times\). By Theorem 3.2 the tensor product \(\bigotimes_{j=1}^k W^{(r)}_{1,bq^{-2j}}\) is of highest \(\ell\)-weight. We argue in a similar way as in the preceding paragraph (replacing lowest by highest):
\[
W^{(r)}_{k,bq^{2k}} \simeq \text{socle}(\bigotimes_{j=1}^k W^{(r)}_{1,bq^{2j}}).
\]
Similar duality argument shows that
\[
(W^{(r)})^r_{k,a} \simeq W^{(r)}_{k,a-1,q^{2(M-N-r+k)}}.
\]
3.3. Proof of Theorem 3.1 According to Remark 3.1, it is enough to prove (1). Fix $1 \leq r \leq M$ and $a \in \mathbb{C}^\times$. Let $l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ be such that $l_1 < l_2 < l_3$. Set $c := q^{2(M-N-r+1)}$. In view of Remark 3.3 (1) is equivalent to the following assertion:

(2) $(W_{l_1,a}^{(r)})^\vee \otimes (W_{l_2-l_1,aq^{-2l_1}}^{(r)})^\vee \otimes (W_{l_3-l_2,aq^{-2l_2}}^{(r)})^\vee$ is cogenerated by the tensor product of highest $\ell$-weight vectors.

On the other hand, Equations (3.27) and (3.28) say that

$$\left(\bigotimes_{j=0}^{l_1} W_{1,ca^{-1}q^{2j}}^{(r)}\right) \otimes \left(\bigotimes_{j=l_1}^{l_2-1} W_{1,ca^{-1}q^{2j}}^{(r)}\right) \otimes \left(\bigotimes_{j=l_2}^{l_1} W_{1,ca^{-1}q^{2j}}^{(r)}\right).$$

By Theorem 3.2, Remark 3.3 and Equation (3.27), the big tensor product $\bigotimes_{j=0}^{l_1-1} W_{1,ca^{-1}q^{2j}}^{(r)}$ is cogenerated by the tensor product of highest $\ell$-weight vectors. Hence (2) follows from Remark 3.4.

4. Asymptotic construction of Hernandez-Jimbo

In this section, following the idea of Hernandez-Jimbo in [HJ12], we construct inductive systems of Kirillov-Reshetikhin modules and endow $Y_q(\mathfrak{g})$-module structures on their inductive limits with the help of asymptotic algebras.

4.1. Highest $\ell$-weight modules. Let $V$ be a $Y_q(\mathfrak{g})$-module. Following Remark 3.1 we say that a non-zero $\mathbb{Z}_2$-homogeneous vector $v \in V$ is a highest $\ell$-weight vector if

$$s_{kk}^{(n)} v \in \mathbb{C} v, \quad s_{ij}^{(n)} v = 0 \quad \text{for } n \in \mathbb{Z}_{\geq 0}, \quad i, j, k \in I, \quad i < j.$$

To the end of this section, we will construct $Y_q(\mathfrak{g})$-modules $L_{r,a}^\pm$ where $r \in I_0$ and $a \in \mathbb{C}^\times$ such that $L_{r,a}^+$ contains a highest $\ell$-weight vector $v^+$ with

$$|v^+| = 0, \quad s_{jj}(z)v^+ = v^+ \begin{cases} 1 - za & \text{if } j \leq r, \\ 1 & \text{if } j > r, \end{cases}$$

and $L_{r,a}^-$ contains a highest $\ell$-weight vector $v^-$ with

$$|v^-| = 0, \quad s_{jj}(z)v^- = v^- \begin{cases} 1 & \text{if } j \leq r, \\ 1 - za & \text{if } j > r. \end{cases}$$

4.2. Asymptotic algebras. We propose in this subsection two versions of asymptotic algebras in the super case. Note that in the non-graded case, asymptotic algebras have been defined for all quantum affine algebras in [HJ12 §2.2].

Definition 4.1. Define $\tilde{Y}_q(\mathfrak{g})$ and $\widehat{Y}_q(\mathfrak{g})$ to be the two subalgebras of $Y_q(\mathfrak{g})$ generated by $\tilde{s}_{ij}^{(n)}$ and $\hat{s}_{ij}^{(n)}$ with $i, j \in I, n \in \mathbb{Z}_{\geq 0}$ respectively. Here

$$\tilde{s}_{ij}^{(n)} := s_{ij}^{(n)} (s_{jj}^{(0)})^{-1}, \quad \hat{s}_{ij}^{(n)} := (s_{ii}^{(0)})^{-1} s_{ij} \quad \text{for } i, j \in I, n \in \mathbb{Z}_{\geq 0}.$$

Call $\tilde{Y}_q(\mathfrak{g})$ and $\widehat{Y}_q(\mathfrak{g})$ asymptotic algebras.
Indeed, one can write out the full defining relations of asymptotic algebras in terms of the generators \( \hat{s}_{ij}^{(n)} \). Take \( \hat{Y}_q(\mathfrak{g}) \) for example. It is the superalgebra defined by

\begin{itemize}
  \item [(As1)] generators \( \hat{s}_{ij}^{(n)} \) for \( i, j \in I \) and \( n \in \mathbb{Z}_{\geq 0} \);
  \item [(As2)] \( \mathbb{Z}_2 \)-grading \( |\hat{s}_{ij}^{(n)}| = |i| + |j| \);
  \item [(As3)] defining relations (by taking \( \hat{s}_{ij}(z) := \sum_{n \in \mathbb{Z}_{\geq 0}} \hat{s}_{ij}^{(n)}z^n \in \hat{Y}_q(\mathfrak{g})[[z]] \))
\end{itemize}

\[
\sum_{a, b \in I} (-1)^{|a||b|+|j|} q^{(|e_i|+|e_j|-|j|)} R_{kl, ab}(z, w) \hat{s}_{ai}(z) \hat{s}_{bj}(w) = \sum_{c, d \in I} (-1)^{|c||d|+|l|} q^{(|e_c|+|e_k|-|l|)} R_{cd, ij}(z, w) \hat{s}_{id}(w) \hat{s}_{kc}(z)
\]

for \( i, j, k, l \in I \),

\( \hat{s}_{ii}^{(0)} = 1 \) for \( i \in I \), \( \hat{s}_{ji}^{(0)} = 0 \) for \( 1 \leq i < j \leq M + N \).

Remark also that \( \hat{Y}_q(\mathfrak{g}) \) and \( \hat{Y}_q^-(\mathfrak{g}) \) are \( \mathbb{Q} \)-homogeneous subalgebras as their generators are \( \mathbb{Q} \)-homogeneous. Endow \( \hat{Y}_q(\mathfrak{g}) \) and \( \hat{Y}_q^-(\mathfrak{g}) \) with the following \( \mathbb{C}[P] \)-module structures:

\[
[a]x = q^{(\alpha, \beta)}x \quad \text{for} \quad \alpha \in P, \beta \in \mathbb{Q}, x \in (\hat{Y}_q(\mathfrak{g}))_{\beta} \quad \text{or} \quad x \in (\hat{Y}_q^-(\mathfrak{g}))_{\beta}.
\]

From the Ice Rule \([2.3]\) and \([2.23]\) comes the following:

**Lemma 4.2.** As superalgebras, \( Y_q(\mathfrak{g}) \cong \hat{Y}_q(\mathfrak{g}) \times \mathbb{C}[P] \cong \hat{Y}_q^-(\mathfrak{g}) \times \mathbb{C}[P] \) where we identify the \( \{e_i\} \) with the \( \hat{s}_{ii}^{(0)} \) for \( i \in I \).

We adapt the notion of \( \mathbb{Q} \)-graded modules over asymptotic algebras in [HJ12, §2.4].

**Definition 4.3.** Let \( V \) be a module over \( \hat{Y}_q(\mathfrak{g}) \) (resp. over \( \hat{Y}_q^-(\mathfrak{g}) \)). We say that \( V \) is \( \mathbb{Q} \)-graded in the sense of Hernandez-Jimbo if there is a decomposition into a direct sum of sub-vector-superspaces \( V = \bigoplus_{\alpha \in \mathbb{Q}} V^{(\alpha)} \) such that

\[
\hat{s}_{ij}^{(n)} V^{(\alpha)} \subseteq V^{(\alpha+e_i-e_j)} \quad \text{(resp.} \hat{s}_{ij}^{(n)} V^{(\alpha)} \subseteq V^{(\alpha+e_i-e_j)} \text{)}
\]

for \( \alpha \in \mathbb{Q}, i, j \in I \) and \( n \in \mathbb{Z}_{\geq 0} \).

The following corollary is an application of the lemma above on the semi-direct product constructions of \( q \)-Yangian. It is parallel to [HJ12, Proposition 2.4].

**Corollary 4.4.** Let \( V = \bigoplus_{\alpha \in \mathbb{Q}} V^{(\alpha)} \) be a \( \mathbb{Q} \)-graded \( \hat{Y}_q(\mathfrak{g}) \)-module (resp. \( \hat{Y}_q^-(\mathfrak{g}) \)-module) in the sense of Hernandez-Jimbo. Then the module structure over the asymptotic algebra can be extended to that over \( Y_q(\mathfrak{g}) \) by setting

\[
\hat{s}_{ii}^{(0)} x = q^{(e_i, \alpha)} x
\]

for \( \alpha \in \mathbb{Q}, x \in V^{(\alpha)} \) and \( i \in I \).

### 4.3. Asymptotic construction of \( L_{r,a}^- \) with \( 1 \leq r \leq M \)

In this subsection, we construct the \( Y_q(\mathfrak{g}) \)-module \( L_{r,a}^- \) with \( 1 \leq r \leq M \) as a limit of the Kirillov-Reshetikhin modules \( (W_{k,a}^r : k \in \mathbb{Z}_{>0}) \) with \( \hat{Y}_q(\mathfrak{g}) \) the underlying asymptotic algebra. Fix \( a \in \mathbb{C}^\times \).

For \( k \in \mathbb{Z}_{>0} \), let \( v_k \) be a highest \( \ell \)-weight vector of the \( U_q(\mathfrak{g}) \)-module \( W_{k,a}^r \). For \( k, l \in \mathbb{Z}_{>0} \) such that \( l < k \), let \( Z^{(r)}(l < k, a) := \phi^*_N((1-z^{-1} a^{-1} q^{2i})^{-1},(1-z^{-1} a^{-1} q^{2i})^{-1}) W_{k-l,a}^{(r)} \).
Let \( v_{lk} \in \mathbb{Z}^r(l < k, a) \) be a highest \( \ell \)-weight vector.

The following lemma is a direct consequence of Theorem 5.11 (1).

**Lemma 4.5.** Let \( l, k \in \mathbb{Z}_{>0} \) with \( l < k \). Then the \( U_q(\hat{\mathfrak{g}}) \)-module \( W_{l,a}^{(r)} \otimes Z^{(r)}(l < k, a) \) is of highest \( \ell \)-weight. Moreover, its simple quotient is isomorphic to \( W_{k,a}^{(r)} \).

In consequence, for \( l < k \), there exists a unique morphism of \( U_q(\hat{\mathfrak{g}}) \)-modules

\[
\mathcal{F}_{k,l} : W_{l,a}^{(r)} \otimes Z^{(r)}(l < k, a) \to W_{k,a}^{(r)}, \quad v_l \otimes v_{lk} \mapsto v_k.
\]

Let \( F_{k,l} : W_{l,a}^{(r)} \to W_{k,a}^{(r)} \) be the restriction: \( x \mapsto \mathcal{F}_{k,l}(x \otimes v_{lk}) \).

### 4.3.1. First properties of the \( F_{k,l} \)

As in Remark 5.3, the \( U_q(\hat{\mathfrak{g}}) \)-modules \( W_{k,a}^{(r)} \) are \( \mathbb{P} \)-graded with respect to the action of the \( s_{ii}^{(0)} \). Also, Theorem 2.4 gives a combinatorial description of dimensions of the weight spaces \( (W_{k,a}^{(r)})_\lambda \) for \( \lambda \in \mathbb{P} \). Let \( (W_{k,a}^{(r)}, \rho^k) \) be the representation associated to the \( U_q(\hat{\mathfrak{g}}) \)-module \( W_{k,a}^{(r)} \).

**Proposition 4.6.** The maps \( F_{k,l} : l < k \in \mathbb{Z}_{>0} \) verify the following properties:

1. for \( \beta \in \mathbb{Q}_{\geq 0} \), \( F_{k,l}(W_{l,a}^{(r)})_{l\varpi_r - \beta} \subseteq (W_{k,a}^{(r)})_{k\varpi_r - \beta} \);
2. \( F_{k,l} : W_{l,a}^{(r)} \to W_{k,a}^{(r)} \) is injective;
3. for \( l < k < u \in \mathbb{Z}_{>0} \), \( F_{u,k}F_{k,l} = F_{u,l} \);
4. there exists a strictly increasing function \( t : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0} \) such that

\[
\rho^k(s_{ij}^{(n)})F_{k,l}(W_{l,a}^{(r)}) \subseteq F_{k,t(l)}(W_{t(l),a}^{(r)}),
\]

for all \( i, j \in I, n \in \mathbb{Z}_{\geq 0} \) and \( l, k \in \mathbb{Z}_{>0} \) with \( t(l) < k \).

**Proof.** (1) comes easily from the action of the \( s_{ii}^{(0)} \) on \( v_{lk} \).

For (2), consider the \( s_{ii}^{(0)} \) with \( i \in I_0 \). We have

\[
\Delta(s_{ii}^{(0)}) = s_{ii}^{(0)} \otimes 1 + s_{ii}^{(0)}(s_{i+1,i+1}^{(0)})^{-1} \otimes s_{ii}^{(0)}.
\]

As \( s_{ii}^{(0)}v_{lk} = 0 \), we see that \( F_{k,l} \) respects the action of the \( s_{ii}^{(0)} \) with \( i \in I_0 \). We are left to verify the following: let \( \beta \in \mathbb{Q}_{\geq 0} \setminus \{0\} \) and \( x \in (W_{l,a}^{(r)})_{l\varpi_r - \beta} \) be such that \( s_{ii}^{(0)}x = 0 \) for all \( i \in I_0 \), then \( x = 0 \). Note that \( W_{l,a}^{(r)} = ev^a_\lambda L(l\varpi_r) \) with \( L(l\varpi_r) \) the simple \( \mathfrak{u}_q(\mathfrak{g}) \)-module of highest weight \( l\varpi_r \). Let such \( x \in L(l\varpi_r) \) be given (we use the quantum superalgebra \( \mathfrak{u}_q(\mathfrak{g}) \) and its generators \( s_{ij}, t_{ij} \)). Then \( s_{i+1,i+1}x = 0 \) for all \( i \in I_0 \). It follows from Equation 2.17 and Proposition 2.1 that \( s_{ij}x = 0 \) for all \( 1 \leq i < j \leq M + N \). \( x \) is a highest weight vector for the \( \mathfrak{u}_q(\mathfrak{g}) \)-module \( L(l\varpi_r) \), which is impossible.

For (3), remark first that the \( U_q(\hat{\mathfrak{g}}) \)-module \( S := W_{l,a}^{(r)} \otimes Z^{(r)}(l < k, a) \otimes Z^{(r)}(k < u, a) \) is of highest \( \ell \)-weight by Theorem 5.11 (1). In consequence, \( \mathcal{F}_{u,k} \circ (\mathcal{F}_{k,l} \otimes \text{Id}_{Z^{(r)}(k < u, a)}) \) is the unique morphism of \( U_q(\hat{\mathfrak{g}}) \)-modules \( S \to W_{u,a}^{(r)} \) sending \( v_l \otimes v_{lk} \otimes v_{ku} \) to \( v_u \). On the
other hand, that \( Z^{(r)}(l < k, a) \otimes Z^{(r)}(k < u, a) \) is of highest \( \ell \)-weight gives us a surjective morphism of \( U_{q}(\mathfrak{g}) \)-modules

\[
G : Z^{(r)}(l < k, a) \otimes Z^{(r)}(k < u, a) \rightarrow Z^{(r)}(l < u, a), \quad v_{lk} \otimes v_{ku} \mapsto v_{lu}.
\]

It follows from the uniqueness of \( F \) that

\[
\mathcal{F}_{u,l} \circ (\text{Id}_{W_{l,a}^{(r)}} \otimes G) = \mathcal{F}_{u,k} \circ (\mathcal{F}_{k,l} \otimes \text{Id}_{Z^{(r)}(l < u, a)})).
\]

Applying the above equation to \( W_{l,a}^{(r)} \otimes v_{lk} \otimes v_{ku} \) we get \( F_{u,l} = F_{u,k} \circ F_{k,l} \).

(4) comes from the asymptotic behaviour of dimensions of weight subspaces observed in Corollary 2.6.

We remark that in the non-graded case [HJ12], (4) is proved by using a deep property [Hei10 Proposition 3.2] of \( q \)-character concerning tensor product of two vectors (not necessarily two modules). In particular, it was shown [HJ12 Lemma 4.4] that \( t(l) = l + 1 \).

Let us fix \( t : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0} \) in (4). Thanks to (2) the following operators

\[
F_{k,l}^{-1}(\rho^{k}(s_{ij}^{(n)}))F_{k,l} : W_{l,a}^{(r)} \rightarrow W_{t(l),a}^{(r)}
\]

are well-defined for \( i, j \in I, n \in \mathbb{Z}_{>0} \) and \( k, l \in \mathbb{Z}_{>0} \) such that \( k > t(l) \). Also, \( (W_{k,a}^{(r)}, F_{k,l}) \) is an inductive system of vector superspaces. Let \( (W_{\infty}, F_{k}) \) be its inductive limit with \( F_{k} : W_{k,a}^{(r)} \rightarrow W_{\infty} \) the structural maps. As the \( F_{k,l} \) are injective, so are the \( F_{k} \).

4.3.2. Asymptotic properties of the \( F_{k,l} \).

**Proposition 4.7.** For \( i, j \in I, n \in \mathbb{Z}_{>0} \) and \( l \in \mathbb{Z}_{>0} \), there exist uniquely two linear maps \( A_{ij}^{(n)}(l), B_{ij}^{(n)}(l) : W_{l,a}^{(r)} \rightarrow W_{t(l),a}^{(r)} \) such that

\[
F_{k,l}^{-1}(\rho^{k}(s_{ij}^{(n)}))F_{k,l} = A_{ij}^{(n)}(l) + q^{-2k}B_{ij}^{(n)}(l)
\]

for all \( k > t(l) \).

**Proof.** It is enough to establish the existence of these maps \( A_{ij}^{(n)}(l), B_{ij}^{(n)}(l) \), as their uniqueness follows automatically from assumption. For this, we follow essentially the argument of [HJ12 Proposition 4.5].

By definition (Proposition 2.1), \( \rho^{k}(s_{ij}^{(n)}) = 0 \) in the following cases: either \( (n \geq 2) \), or \( (n = 1, i < j) \), or \( (n = 0, i > j) \).

**Case A:** \( i = j \). For \( \beta \in Q_{\geq 0} \) and \( x \in (W_{l,a}^{(r)})_{\ell \omega_{r}, -\beta} \), we have \( F_{k,l}(x) \in (W_{k,a}^{(r)})_{k\omega_{r}, -\beta} \), meaning

\[
\rho^{k}(s_{jj}(z))F_{k,l}(x) = (q^{(ej,k\omega_{r}, -\beta)} - zaq^{-2k(ej,k\omega_{r}, -\beta)})F_{k,l}(x).
\]

It follows that \( F_{k,l}^{-1}(\rho^{k}(s_{jj}(z)))F_{k,l}(x) = F_{t(l),l}(x) - q^{-2k(ej,k\omega_{r})}zaq^{2(ej,\beta)}F_{t(l),l}(x) \).

**Case B:** \( i < j \). It is enough to consider \( s_{ij}^{(0)} \). Note that

\[
\Delta(s_{ij}^{(0)}) = \sum_{i' = 1}^{j} s_{ii'}^{(0)} \otimes s_{ij}^{(0)}.
\]
For $x \in W_{t,a}^{(r)}$, since $s_{ij}^{(0)}v_{lk} = 0$ for $i' < j$, we get

$$\rho^k(s_{ij}^{(0)})F_{k,l}(x) = s_{ij}^{(0)}F_{k,l}(x \otimes v_{lk}) = F_{k,l}(s_{ij}(x \otimes v_{lk})).$$

In other words, $\rho^k(s_{ij}^{(0)})F_{k,l}(W_{t,a}^{(r)}) \subseteq F_{k,l}(W_{t,a}^{(r)})$ and hence $F_{k,l}^{-1}(\rho^k(s_{ij}^{(0)})F_{k,l}) = F_{t(l),l}\rho^l(s_{ij}^{(0)})$. **Case C**: $i > j$. It is enough to consider $\rho^k(s_{ij}^{(1)}) = -a^k(t_{ij}^{(0)})\rho^k((s_{jj}^{(0)})^{-1})$. Observe furthermore that $s_{jj}^{(0)}v_{lk} = 0$ if $j > r$ and $j' \neq j$. Hence, when $j > r$, we have

$$\rho^k(s_{ij}(z))F_{k,l}(x) = F_{k,l}(s_{ij}(z)x \otimes v_{lk}) = F_{k,l}(s_{ij}(z)x).$$

for $x \in W_{t,a}^{(r)}$. This says in particular that $F_{k,l}^{-1}(\rho^k(s_{ij}^{(1)})F_{k,l}) = F_{t(l),l}\rho^l(s_{ij}^{(1)})$. We are left to consider the case $i = j + 1$.

**Case C.1**: $i \leq r$. Observe that $t_{ij}^{(0)}v_{lk} = 0$ for $j < i' \leq i$. In view of the coproduct for $t_{ij}^{(0)}$, we find that for $x \in W_{t,a}^{(r)}$:

$$\rho^k(t_{ij}^{(0)})F_{k,l}(x) = \rho^k(t_{ij}^{(0)})F_{k,l}(x \otimes v_{lk}) = F_{k,l}(t_{ij}^{(0)}(x \otimes v_{lk})).$$

It follows that $F_{k,l}^{-1}(\rho^k(s_{ij}^{(1)})F_{k,l}) = -a^k(t_{ij}^{(0)})F_{t(l),l}\rho^l(t_{ij}^{(0)}(s_{jj}^{(0)})^{-1}) = a^2(k-1)t_{t(l),l}\rho^l(s_{ij}^{(1)}).$

**Case C.2**: $(i,j) = (r+1,r)$. Introduce $f_{t} := (t_{t+1,r+1})^{-1}t_{t+1,r}$. Then in the superalgebra $U_q(\tilde{\mathfrak{g}})$ by Equation (2.16)

$$[s_{i',i'+1}^{(0)}, f_{t}] = \delta_{i',r}(q - q^{-1})(s_{t+1,r}^{(0)} - s_{t,r}^{(0)}).$$

for $i' \in I_0$. In order to establish Equation (4.30), it is enough to prove the following assertions AS($\beta$) for all $\beta \in \mathbb{Q}_{\geq 0}$.

**AS($\beta$)**: for $x \in W_{t_{t(l)},t_{l(t)}}^{(r)}$, there exist $x',x'' \in W_{t_{t(l)},t_{l(t)}}^{(r)}$ such that for all $k > t(l)$,

$$F_{k,l(t)}^{-1}(\rho^k(f_{t})F_{k,l}(x)) = q^kx' + q^{-k}x''.$$
Here the second equality comes from Case B and the definition of $F_{k,l}$. We have therefore found $C_{i}', D_{i}' \in (W_{t(l),a}^{(r)})_{t(l)\varpi_r}$ such that
\[
\rho^{t(l)}(\tilde{s}_{i',i'+1}^{(0)})w_k = q^k C_{i}' + q^{-k} D_{i}'
\]
for all $k > t(l)$ and $i' \in I_0$. It follows that
\[
\rho^{t(l)}(\tilde{s}_{i',i'+1}^{(0)})q^{k+1}w_k - q^{k+1}w_{k+1} - q^{k+3}w_{k+1} + q^{k+2}w_{k+2} = 0
\]
for all $i' \in I_0$. As we see in the proof of Proposition 4.6 this says that
\[
q^{k+2}w_k - q^{k+1}w_{k+1} - q^{k+3}w_{k+1} + q^{k+2}w_{k+2} = 0
\]
for all $k > t(l)$, since $w_k \in (W_{t(l),a}^{(r)})_{t(l)\varpi_r}$. In matrix form, this reads
\[
\begin{pmatrix} w_{k+2} \\ w_{k+1} \end{pmatrix} = \begin{pmatrix} q+q^{-1} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_{k+1} \\ w_k \end{pmatrix}.
\]
The $2 \times 2$ matrix above is diagonalizable with eigenvalues $q, q^{-1}$. We can therefore find $\mathbf{x}', \mathbf{y}' \in (W_{t(l),a}^{(r)})_{t(l)\varpi_r}$ which are linear combinations of $w_{t(l)+1}, w_{t(l)+2}$ such that
\[
w_k = q^k \mathbf{x}' + q^{-k} \mathbf{x}''
\]
for all $k > t(l)$. This proves the assertion AS(0). Suppose $m > 0$ and that the assertions AS(β) for $\ell(\beta) < m$ have been proved. Fix $\beta \in \mathbb{Q}_{\geq 0}$ with $\ell(\beta) = m$. Let $x \in (W_{t(l),a}^{(r)})_{t(l)\varpi_r}$. As before, take $w_k := F_{k,t(l)}^{-1}(f_r)F_{k,t(l)}(x) \in (W_{t(l),a}^{(r)})_{t(l)\varpi_r}$. Then
\[
\rho^{t(l)}(\tilde{s}_{i',i'+1}^{(0)})w_k = \rho^{t(l)}(\tilde{s}_{i',i'+1}^{(0)})F_{k,t(l)}^{-1}F_{k,t(l)}(x) = F_{k,t(l)}^{-1}\rho^{k}(\tilde{s}_{i',i'+1}^{(0)})F_{k,t(l)}(x) = \delta_{i',r}(q - q^{-1})(q^{-k+\beta,\alpha_r} - q^{-k-(\beta,\alpha_r)})F_{t(l),t(l)}(x)
\]
and that the assertions AS(β) for $\ell(\beta) < m$ have been proved. Fix $\beta \in \mathbb{Q}_{\geq 0}$ with $\ell(\beta) = m$. Let $x \in (W_{t(l),a}^{(r)})_{t(l)\varpi_r}$. As before, take $w_k := F_{k,t(l)}^{-1}(f_r)F_{k,t(l)}(x) \in (W_{t(l),a}^{(r)})_{t(l)\varpi_r}$. Then
\[
\rho^{t(l)}(\tilde{s}_{i',i'+1}^{(0)})w_k = q^k C_{i}' + q^{-k} D_{i}'
\]
for all $k > t(l)$. This concludes the proof.

Case C.3: $i > r + 1$ and $j = r$. This comes from Case C.2 and Equation 2.17:
\[
\begin{array}{c}
t_r^{(0)} - t_{r+1}^{(0)} = (q - q^{-1})t_r^{(0)}.
\end{array}
\]
Remark that $(|r + 1| + |r|)(|i| + |r + 1|) = 0$, so there is no sign on the left hand side.

Case C.4: $j < r < i$. This comes from Cases C.1-C.3 and the following relation in $U_q(\tilde{g})$:
\[
\begin{array}{c}
t_r^{(0)} - t_{r+1}^{(0)} = (q - q^{-1})t_r^{(0)}.
\end{array}
\]
This concludes the proof.

From the above proof, we see that: if $i < j$ then $A_{ij}^{(n)}(l) = F_{t(l),l}^{(0)}(s_{ij}^{(n)})$ and $A_{ij}^{(n)}(l) = 0$ for $n > 0$; if $j < i \leq r$ then $A_{ij}^{(n)}(l) = 0$; if $j < i$ then $A_{ij}^{(n)}(l) = 0$ for $n \neq 1$. □
4.3.3. \( \hat{Y}_q(\mathfrak{g}) \)-module structure on \( W_\infty \). Since \( t \) is strictly increasing, we get another inductive system of vector superspaces \( (W^{(r)}_{l(t),a}, F_{l(t),a}) \) over the linearly ordered set \( \mathbb{Z}_{>0} \). Furthermore, the following defines a morphism of inductive systems

\[
F_{l(k),k} : W^{(r)}_{k,a} \to W^{(r)}_{l(k),a}
\]

which induces an isomorphism of inductive limits. Identify their inductive limits.

Let us show that \( (A^{(n)}_{ij}(k) : W^{(r)}_{k,a} \to W^{(r)}_{l(k),a}) \) gives a morphism of inductive system of vector superspaces. In other words,

\[
F_{l(t+1),t}A^{(n)}_{ij}(l) = A^{(n)}_{ij}(l + 1)F_{l+1,t} : W^{(r)}_{t,a} \to W^{(r)}_{t+1,a}.
\]

Indeed, for all \( k > t(l + 1) \), we have

\[
F_{l(t+1),t}A^{(n)}_{ij}(l) + q^{-2k}F_{l(t+1),t}P^{(n)}_{ij}(l) = F_{l(t+1),t}F_{l,t}^{-1}k^{(n)}F_{k,l} = F_{k,t+1,l}B^{(n)}_{ij}F_{l+1,t} = A^{(n)}_{ij}(l + 1)F_{l+1,t} + q^{-2k}B^{(n)}_{ij}(l + 1)F_{l+1,t}.
\]

The desired equations follow. We obtain therefore linear endomorphisms of \( W_\infty \):

\[
A^{(n)}_{ij} := \lim_{l \to \infty} A^{(n)}_{ij}(l) : W_\infty \to W_\infty.
\]

Moreover, \( A^{(n)}_{ij} \) is \( \mathbb{Z}_2 \)-homogeneous of degree \(|i| + |j|\) in view of Equation (4.30).

Now, as in [HJ12 §4], the assignment \( \tilde{s}^{(n)}_{ij} \mapsto A^{(n)}_{ij} \) defines a representation of the superalgebra \( \hat{Y}_q(\mathfrak{g}) \) on the vector superspace \( W_\infty \). In other words, the \( A^{(n)}_{ij} \) respect the condition (As3) in §4.1. Clearly, \( A^{(0)}_{ij} = 0 \) for \( i > j \). By considering the operators

\[
F_{k,t+1,l}^{-1}k^{(n)}(s^{(n)}_{ij} s^{(n')}_{ij'})F_{k,l} = (F_{k,t+1,l}^{-1}k^{(n)}(s^{(n)}_{ij} s^{(n')}_{ij'})F_{k,t+1,l})(F_{k,t+1,l}^{-1}k^{(n)}(s^{(n')}_{ij'} s^{(n')}_{ij'})F_{k,l})
\]

and by using Equation (4.30), we see that (As3) is indeed true for the \( A^{(n)}_{ij} \).

4.3.4. \( Y_q(\mathfrak{g}) \)-module structure on \( W_\infty \). Proposition 4.6 (1) implies that \( W_\infty \) is endowed with a \( Q \)-grading: for \( \alpha \in \mathbb{Q}, x \in (W_\infty)^{(\alpha)} \) if there exists \( k \in \mathbb{Z}_{>0} \) such that \( x \in F_k(W^{(r)}_{k,a})_{a + k \infty} \).

Now Equation (4.30) says that this is a \( Q \)-grading in the sense of Hernandez-Jimbo. In conclusion, \( W_\infty \) becomes a \( Y_q(\mathfrak{g}) \)-module thanks to Corollary 4.1.

From the proof of Proposition 4.7 we see that for \( \alpha \in \mathbb{Q}, x \in (W_\infty)^{(\alpha)} \) and \( j \in I \),

\[
|x| = |\alpha|, \quad s_{ij}(z)x = x \begin{cases} q^{(\epsilon_j,\alpha)} & (j \leq r), \\ q^{(\epsilon_j,\alpha)} - zaq^{(\epsilon_j,\alpha)} & (j > r).
\end{cases}
\]

Moreover, \( W_\infty \) contains a highest \( \ell \)-weight vector \( v_\infty := F_k y_k \) with the same action of the \( s_{ii}(z) \) as that of the highest \( \ell \)-weight vector \( v^- \in L^-_{r,a} \) in §4.1. For this reason, set \( L^-_{r,a} \) to be the \( Y_q(\mathfrak{g}) \)-module \( W_\infty \) thus obtained.
4.4. **Asymptotic construction of** $L_{r,a}^+$ **with** $1 \leq r \leq M$. **This** is parallel to the construction of $L_{r,a}^-$. **We** start from an inductive system of Kirillov-Reshetikhin modules, establish stability (Proposition 4.6) and asymptotic (Proposition 4.7) properties for this inductive system, and endow $Y_q(\hat{g})$-module structure on the inductive limit through an asymptotic algebra. **As** the proofs of these properties are identical to the case of $L_{r,a}$, we omit them in this subsection.

The Kirillov-Reshetikhin modules involved will be $W_{k,aa^2k}^{(r)}$ for $k \in \mathbb{Z}_{>0}$. For $l < k$, set

$$Z_{kl} := \phi^*_r((1-z^{-1}a^{-1}q^{-2l})^{-1},(1-zaq^{2l})^{-1})W_{k-l,aa^2k}^{(r)}.$$ 

Let $v_k \in W_{k,aa^2k}^{(r)}$ (resp. $v_{kl} \in Z_{kl}$) be a highest $\ell$-weight vector.

For $l < k$, the $U_q(\hat{g})$-module $Z_{kl} \otimes W_{t,aa^2l}^{(r)}$ is of highest $\ell$-weight with simple quotient isomorphic to $W_{k,aa^2k}^{(r)}$. This affords a unique morphism of $U_q(\hat{g})$-modules

$$\mathcal{F}_{k,l} : Z_{kl} \otimes W_{t,aa^2l}^{(r)} \rightarrow W_{k,aa^2k}^{(r)}, \quad v_k \otimes v_l \mapsto v_k.$$ 

Let $F_{k,l} : W_{t,aa^2l}^{(r)} \rightarrow W_{k,aa^2k}^{(r)}$ be the restriction: $x \mapsto \mathcal{F}_{k,l}(v_k \otimes x)$. Then the $F_{k,l}$ verify all the properties in Proposition 4.6 with $W_{k,a}^{(r)}$ replaced by $W_{k,aa^2k}^{(r)}$ everywhere. Furthermore, a detailed analysis as in the proof of Proposition 4.7 shows that: for all $i, j \in I, n \in \mathbb{Z}_{\geq 0}$ and $l \in \mathbb{Z}_{>0}$, there exist linear operators $A_{ij}^{(n)}(l), B_{ij}^{(n)}(l) : W_{t,aa^2l}^{(r)} \rightarrow W_{k,aa^2k}^{(r)}$ such that

$$F_{k,l}^{-1} A_{ij}^{(n)}(l) F_{k,l} = A_{ij}^{(n)}(l) + q^{2k} B_{ij}^{(n)}(l) \quad \text{for all } k > t(l).$$

As before, the $A_{ij}^{(n)}(l)$ give rise to a morphism of inductive systems of vector superspaces $(W_{k,aa^2k}^{(r)}, F_{k,l})$ and $(W_{t,aa^2l}^{(r)}, F_{l(t),l(l)})$. We obtain therefore a representation of the asymptotic algebra $\hat{Y}_q(\hat{g})$ on the inductive limit:

$$(W_{r,a}^+, F_k) := \lim_{\rightarrow} (W_{k,aa^2k}^{(r)}, F_{k,l}), \quad A_{ij}^{(n)}(l) \mapsto A_{ij}^{(n)}(l).$$

Also $W_{r,a}^+$ is $Q$-graded in the sense of Hernandez-Jimbo: for $\alpha \in Q$, $x \in (W_{r,a}^+)^{(\alpha)}$ if $x \in F_k(W_{k,aa^2k}^{(r)})_{\alpha + k\omega}$, for some $k \in \mathbb{Z}_{>0}$. In this way, we get a representation of $Y_q(\hat{g})$ on $W_{r,a}^+$ by Corollary 4.4. For $\alpha \in Q, x \in (W_{r,a}^+)^{(\alpha)}$ and $j \in I$,

$$|x| = |\alpha|, \quad s_{jj}(z)x = x \begin{cases} q^{(\alpha,\epsilon_j)} - zaq^{-1}(\alpha,\epsilon_j) & (j \leq r), \\ q^{(\alpha,\epsilon_j)} & (j > r). \end{cases}$$

The vector $v_\infty := F_k(v_k) \in W_{r,a}^+$ verifies the same conditions as $v^+ \in L_{r,a}$ in 4.1. Let $L_{r,a}^+$ be the $Y_q(\hat{g})$-module $W_{r,a}^+$ thus obtained.

4.5. **Construction of** $L_{r,a}^{\pm}$ **for** $M + 1 \leq r \leq M + N - 1$. **For** this purpose, let $\mathfrak{g}' = \mathfrak{gl}(N, M)$ be as in 3.1.1. Recall that we have defined the quantum affine superalgebra $U_q(\hat{g}')$ by the RTT generators $s_{ij;J}, v_{ij;J}$ as in 3.1.1. Let $Y_q(\mathfrak{g}')$ be the subalgebra of $U_q(\hat{g}')$ generated by
the $s_{ij,j}^{(n)}$. Then the isomorphism $f_{J,I}$ defined by Formula 3.24 restricts to an isomorphism of Hopf superalgebras which we write as $f_{J,I}$

$$f_{J,I} : Y_q(\mathfrak{g}')^\text{cop} \rightarrow Y_q(\mathfrak{g}), \quad s_{ij,j}^{(n)} \mapsto \varepsilon_{ji} s_{ji,j}^{(n)}.$$ 

We construct as in 3.34.4 the $Y_q(\mathfrak{g}')$-modules $L_{r,a;J}^\pm$ as limits of corresponding Kirillov-Reshetikhin modules for $1 \leq r \leq N$ and $a \in \mathbb{C}^\times$. Now the $Y_q(\mathfrak{g}')$-modules $L_{r,a}^\pm$ for $M + 1 \leq r \leq M + N - 1$ and $a \in \mathbb{C}^\times$ are realized as:

$$L_{r,a}^\pm \cong (J_{J,I})^* L_{M+N-r,a;J}^\pm.$$ 

Consider $L_{r,a}^\pm$ with $M + 1 \leq r \leq M + N - 1$ for example. For $l < k$, define

$$Z_{kl} := \phi^*_{(1-z^{-1}a^{-1}q^{2l})^{-1},(1-zaq^{-2l})^{-1}} W_{k-l,aq^{-2l}}^{(r)}.$$ 

Then the tensor product $Z_{kl} \otimes W_{l,a}^{(r)}$ is of highest $l$-weight. This affords an inductive system $(W_{k,a}, F_{k,l})$ where $F_{k,l} : W_{l,a}^{(r)} \rightarrow W_{k,a}^{(r)}$ comes from the surjective map

$$F_{k,l} : Z_{kl} \otimes W_{l,a}^{(r)} \rightarrow W_{k,a}^{(r)}.$$ 

The rest is completely parallel to 4.4.

Following [FH13], the $Y_q(\mathfrak{g}')$-modules $L_{r,a}^\pm$ are called positive/negative prefundamental modules. We shall see that they are always simple. Contrary to the non-graded case where all prefundamental modules are infinite-dimensional, $L_{r,a}^\pm$ is finite-dimensional if and only if $r = M$. This says that there are “more” finite-dimensional simple $Y_q(\mathfrak{g}')$-modules than finite-dimensional simple $U_q(\mathfrak{g})$-modules, as we have seen in [Zh14 §5] on representation theory of $Y_q(\mathfrak{g}((1,1)))$. See also Proposition 6.7.

4.6. Relation with Tsuboi’s work. The $Y_q(\mathfrak{g})$-modules $L_{r,a}^\pm$ have been constructed by Tsuboi in a different way. In [Ts12], Tsuboi proposed the notion of a contracted quantum superalgebra. This is the superalgebra defined by generators $\hat{s}_{ii}, \hat{s}_{i-1}, \hat{i}_{ii}, \hat{s}_{jk}, \hat{i}_{kj}$ for $i,j,k \in I, j < k$ with $\mathbb{Z}_2$-degrees

$$|\hat{s}_{ii}| = |\hat{s}_{i-1}| = |\hat{i}_{ii}| = 0, \quad |\hat{s}_{jk}| = |\hat{i}_{kj}| = |j| + |k|$$

subject to the following relations (take $\hat{T} = \sum_{jk} \hat{i}_{jk} \otimes E_{jk}, \hat{S} = \sum_{jk} \hat{s}_{jk} \otimes E_{jk}$)

$$R_{23}\hat{T}_{12}\hat{T}_{13} = \hat{T}_{13}\hat{T}_{12}R_{23}, \quad R_{23}\hat{S}_{12}\hat{S}_{13} = \hat{S}_{13}\hat{S}_{12}R_{23}, \quad R_{23}\hat{T}_{12}\hat{S}_{13} = \hat{S}_{13}\hat{T}_{12}R_{23}, \quad \hat{s}_{ii}\hat{s}_{i-1}^{-1} = 1 = \hat{s}_{i-1}^{-1}\hat{s}_{ii}.$$

Let $\hat{U}_q(\mathfrak{g})$ be the superalgebra obtained. Then the proof of Proposition 2.1 implies that

$$e\hat{v}_a : Y_q(\mathfrak{g}) \rightarrow \hat{U}_q(\mathfrak{g}), \quad s_{ij}(z) \mapsto \hat{s}_{ij} - za\hat{i}_{ij}, \quad (s_{ii}^{(0)})^{-1} \rightarrow \hat{s}_{ii}^{-1}$$

defines a morphism of superalgebras for $a \in \mathbb{C}^\times$. Let $\hat{L}_r^+$ be the simple $\hat{U}_q(\mathfrak{g})$-module generated by a highest weight vector $v^+$:

$$|v^+| = 0, \quad \hat{s}_{jk}v^+ = 0, \quad \hat{s}_{ii}v^+ = v^+, \quad \hat{i}_{ii}v^+ = v^+ \begin{cases} 1 & (i \leq r), \\ 0 & (i > r). \end{cases}$$
Similarly, let $\hat{L}_{r}^{-}$ be the simple $\hat{\mathcal{U}}_{d}(\mathfrak{g})$-module generated by a highest weight vector $v^{-}$:

$$
|v^{-}| = 0, \quad \hat{s}_{jk}v^{-} = 0, \quad \hat{s}_{ii}v^{-} = v^{-}, \quad \hat{t}_{ii}v^{-} = v^{-} \begin{cases} 0 & (i \leq r), \\
1 & (i > r). 
\end{cases}
$$

Then, as $\mathcal{Y}(\mathfrak{g})$-modules, $L_{r,a}^{\pm} \cong ev_{a}^{*}\hat{L}_{r}^{\pm}$. Tsuboi constructed the $\hat{L}_{r}^{\pm}$ via the $q$-oscillator realizations of the contracted quantum superalgebra $\hat{\mathcal{U}}_{d}(\mathfrak{g})$, which is a generalization of the construction carried out in [BYOS] for $U_{q}(\widehat{\mathfrak{sl}}(2,1))$ to the case $U_{q}(\mathfrak{g}(M,N))$.

Compared with Tsuboi’s oscillation realizations, our results in [4.3-4.4] can be rephrased as asymptotic realizations of the $\hat{\mathcal{U}}_{d}(\mathfrak{g})$-modules $L_{r}^{\pm}$.

5. Generic asymptotic construction

In this section, based on the inductive systems of Kirillov-Reshetikhin modules and their stability and asymptotic properties in previous section, we propose a new asymptotic construction which realizes the inductive limits as modules over the full quantum affine superalgebra $\mathcal{Y}(\hat{\mathfrak{g}})$ instead of the $q$-Yangian $\mathcal{Y}(\mathfrak{g})$.

Let us fix two parameters $a,b \in \mathbb{C}^{\times}$ and a Dynkin node $r \in I_{0}$. To the end of this section, we shall construct a $U_{q}(\hat{\mathfrak{g}})$-module, written as $L_{r,a}(b)$, which has a non-zero vector $v$ of $\mathbb{Z}_{2}$-degree $0$ such that

$$
\hat{s}_{ij}^{(n)}v = \hat{t}_{ij}^{(n)}v = 0 \quad \text{for} \ n \in \mathbb{Z}_{\geq 0}, 1 \leq i < j \leq M + N
$$

and in the case $1 \leq r \leq M$,

$$
(5.32) \quad s_{ij}(z)v = v \begin{cases} b - z a b & (i \leq r), \\
1 - z a b^{2} & (i > r), \end{cases} \quad t_{ii}(z)v = v \begin{cases} b^{-1} - z^{-1} a^{-1} b^{-1} & (i \leq r), \\
1 - z^{-1} a^{-1} b^{-2} & (i > r), \end{cases}
$$

whereas in the case $M + 1 \leq r < M + N$,

$$
(5.33) \quad s_{ij}(z)v = v \begin{cases} 1 - z a b & (i \leq r), \\
1 - z a & (i > r), \end{cases} \quad t_{ii}(z)v = v \begin{cases} 1 - z^{-1} a^{-1} & (i \leq r), \\
1 - z^{-1} a^{-1} b^{-1} & (i > r). \end{cases}
$$

5.1. Construction of $L_{r,a}(b)$ with $1 \leq r \leq M$. Let $V_{k} := W_{k,aq^{2k}}^{(r)}$ and fix $v_{k} \in V_{k}$ a vector of highest $\ell$-weight for $k \in \mathbb{Z}_{>0}$. Let $(F_{k,t} : V_{l} \to V_{k})_{l \leq k}$ be the inductive system of vector superspaces constructed in [4.2]. Let $(V_{\infty}, F_{l} : V_{l} \to V_{\infty})$ be its inductive limit. Define $v_{\infty} := F_{l}(v_{1}) \in V_{\infty}$. Then from $F_{k,t}(v_{l}) = v_{k}$ we see that $F_{l}(v_{l}) = v_{\infty}$ for all $l \in \mathbb{Z}_{>0}$.

Let $S$ be the subset of $U_{q}(\hat{\mathfrak{g}})$ consisting of the RTT generators $s_{ij}^{(n)}, t_{ij}^{(n)}$ for $i,j \in I, n \in \mathbb{Z}_{\geq 0}$. Let $\rho^{k}$ be the representation of $U_{q}(\hat{\mathfrak{g}})$ on $V_{k}$ for $k \in \mathbb{Z}_{>0}$. Let $t : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be a strictly increasing function such that (Proposition [4.6])

$$
\rho^{k}(s_{ij}^{(n)})F_{k,t}V_{l} \subseteq F_{k,t+l}(V_{l}) \quad \text{for} \ k > t(l).
$$

By definition of $V_{k} = ev_{aq^{2k}}^{*}L(k^{-1})$ we see that

$$
\rho^{k}(t_{ij}(z)) = -z^{-1} a^{-1} q^{-2k} \rho^{k}(s_{ij}(z)) \in \text{End}V_{k}[z^{-1}] .
$$
It follows that $\rho^k(s)F_{k,l}V_l \subseteq F_{k,t(l)}V_{t(l)}$ for $k > t(l)$ and $s \in S$. Since the $F_{k,l}$ are injective, the operators

$$F_{k,t(l)}^{-1}\rho^k(s)F_{k,l} : V_l \rightarrow V_{t(l)}$$

for $s \in S, k > t(l)$ are well-defined.

**Lemma 5.1.** Let $l \in \mathbb{Z}_{>0}$ and $s \in S$. Then there exists uniquely a Hom$(V_l, V_{t(l)})$-valued Laurent polynomial $P_{t,s}(z) \in \text{Hom}(V_l, V_{t(l)})[z, z^{-1}]$ such that

$$F_{k,t(l)}^{-1}\rho^k(s)F_{k,l} = P_{t,s}(z)|_{z=q^k} \text{ for } k > t(l).$$

Furthermore, the coefficients of $z^n$ in $P_{t,s}(z)$ are non-zero only if $-2 \leq n \leq 3$.

**Proof.** Assume without loss of generality $s = s^{(n)}_{ij}$. Since $\rho^k(s^{(0)}_{ii})F_{k,t(l)}V_{t(l)} \subseteq F_{k,t(l)}V_{t(l)}$

$$F_{k,t(l)}^{-1}\rho^k(s)F_{k,l} = F_{k,t(l)}^{-1}\rho^k(s^{(0)}_{ii}s^{(n)}_{ij})F_{k,l} = (F_{k,t(l)}^{-1}\rho^k(s^{(0)}_{ii})F_{k,t(l)})(F_{k,t(l)}^{-1}\rho^k(s^{(n)}_{ij})F_{k,l}).$$

On the other hand, by definition of the $F_{k,l}$

$$F_{k,t(l)}^{-1}\rho^k(s^{(0)}_{ii})F_{k,l} = q^{(k-t(l))\epsilon_{r,s}}\rho^{(t(l))}(s^{(0)}_{ii}).$$

Combining with Equation (4.31), we find a Laurent polynomial $P_{t,s}(z)$ with the desired property. Clearly such $P_{t,s}$ is unique. $\square$

Now we argue as in §4.3.3 By using the defining property of the $P_{t,s}$ we see that for all $s \in S$ and $l \in \mathbb{Z}_{>0}$:

$$F_{t(l+1),t(l)}P_{t,s}(z) = P_{t+1,s}(z)F_{t+1,l} \in \text{Hom}(V_l, V_{t(l+1)})[z, z^{-1}].$$

In other words, if we write

$$P_{t,s}(z) := \sum_{n=-2}^{3} P_{t,s}[n]z^n, \quad P_{t,s}[n] \in \text{Hom}(V_l, V_{t(l)}),$$

then for all $-2 \leq n \leq 3$, $P_{t,s}[n]$ induces a morphism of inductive systems of vector spaces:

$$(P_{t,s}[n])_l : (V_l, F_{k,l}) \rightarrow (V_{t(l)}, F_{l(t),t(l)}).$$

Let $P_s[n]$ be the inductive limit of the above morphism. Then $P_s[n] \in \text{End}(V_{\infty})$ as both inductive systems give rise to the same inductive limit. As a result, the following assignment

$$s \mapsto \sum_{n=-2}^{3} P_s[n]b^n \in \text{End}(V_{\infty}) \quad \text{for } s \in S$$

defines a representation of the quantum affine superalgebra $U_q(\mathfrak{g})$ on $V_{\infty}$. Let us compute the action of $s^{(n)}_{ij}, t^{(n)}_{ij}$ on $v_\infty$ for $1 \leq i \leq j \leq n$. First, if $i < j$,

$$F_{k,t(l)}^{-1}\rho^k(s^{(n)}_{ij})F_{k,l}v_l = F_{k,t(l)}^{-1}\rho^k(t^{(n)}_{ij})F_{k,l}v_l = 0.$$
In other words, \( P_{L, (s_{ij}(z))} v_l = 0 = P_{L, (t_{ij}(z))} v_l \). Henceforth \( s_{ij}^{(n)} v_\infty = t_{ij}^{(n)} v_\infty = 0 \). Next, assume \( i = j \), in view of the following equation

\[
F^{-1}_{k, t(l)} p^k(s_{ij}(z)) F_{k, t(l)} v_l = v_{t(l)} \left\{ \begin{array}{ll}
q^k - zaq^k & (i \leq r), \\
1 - zaq^{2k} & (i > r),
\end{array} \right.
\]

\[
F^{-1}_{k, t(l)} p^k(t_{ij}(z)) F_{k, t(l)} v_l = v_{t(l)} \left\{ \begin{array}{ll}
q^{-k} - z^{-1}a^{-1}q^{-k} & (i \leq r), \\
1 - z^{-1}a^{-1}q^{-2k} & (i > r).
\end{array} \right.
\]

In a similar way, we get \( s_{ii}(z) v_\infty \) and \( t_{ii}(z) v_\infty \) by regarding \( q^k \) in the above equation as \( b \). The \( U_q(\hat{g}) \)-module \( V_\infty \) is the desired \( L_{r, a}(b) \), as \( v_\infty \) verifies Equation (5.32).

5.2. Construction of \( L_{r, a}(b) \) with \( r > M \). In this case, by abuse of language, set \( V_k := W_{k, a}^{(r)} \). Following (1.14), let \( v_k \in V_k \) be a highest \( \ell \)-weight vector and let \( F_{k, l} : V_l \rightarrow V_k \) be the structural maps of the inductive system of vector superspaces \( (V_k, F_{k, l}) \) so that \( F_{k, l} v_l = v_k \) for \( l < k \). Let \( (V_\infty, F_l) : V_l \rightarrow V_\infty \) be the inductive limit. Let \( v_\infty := F_1(v_1) \in V_\infty \).

As before, choose a strictly increasing function \( t : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0} \) so that the following operators \( \rho^k \) denotes the representation of \( U_q(\hat{g}) \) on \( V_k \)

\[
F^{-1}_{k, l(t)} p^k(s) F_{k, l} : V_l \rightarrow V_{t(l)}
\]

with \( s \in S, k > t(l) \) are well-defined. The following lemma is proved in a similar way as Lemma 5.1.

**Lemma 5.2.** Let \( l \in \mathbb{Z}_{>0} \) and \( s \in S \). Then there exists uniquely a \( \text{Hom}(V_l, V_{t(l)}) \)-valued Laurent polynomial \( P_{l, s}(z) \in \text{Hom}(V_l, V_{t(l)})(z, z^{-1}) \) such that

\[
F^{-1}_{k, l(t)} p^k(s) F_{k, l} P_{l, s}(z) = P_{l, s}(z) |_{z = q^k} \quad \text{for } k > t(l).
\]

Furthermore, the coefficients of \( z^n \) in \( P_{l, s}(z) \) are non-zero only if \( -2 \leq n \leq 1 \).

The rest is also parallel to the preceding subsection. We get a representation of \( U_q(\hat{g}) \) on the vector superspace \( V_\infty \). Moreover, \( v_\infty \) is killed by \( s_{ij}(z), t_{ij}(z) \) whenever \( 1 \leq i < j \leq n \). Also by replacing \( q^k \) and \( v_l, v_{t(l)} \) in the following equation with \( b \) and \( v_\infty \), we conclude that \( v_\infty \) verifies Equation (5.33):

\[
F^{-1}_{k, l(t)} p^k(s_{ii}(z)) F_{k, l} v_l = v_{t(l)} \left\{ \begin{array}{ll}
1 - za & (i \leq r), \\
q^k - zaq^{-k} & (i > r),
\end{array} \right.
\]

\[
F^{-1}_{k, l(t)} p^k(t_{ii}(z)) F_{k, l} v_l = v_{t(l)} \left\{ \begin{array}{ll}
1 - z^{-1}a^{-1} & (i \leq r), \\
q^{-k} - z^{-1}a^{-1}q^{-k} & (i > r).
\end{array} \right.
\]

The \( U_q(\hat{g}) \)-module \( V_\infty \) is the desired \( L_{r, a}(b) \).

5.3. Examples. Let us give three examples to illustrate the general construction. In this subsection \( g = \mathfrak{gl}(2, 1) \). We construct \( L_{2, a}^+, L_{2, a}(b) \) explicitly. Fix \( a, b \in \mathbb{C}^x \).
5.3.1. Construction of $L^-_{2,a}$. For $k \in \mathbb{Z}_{>0}$, let $v_4 \in W_{k,a}^{(2)}$ be a lowest $\ell$-weight vector. Define

$$v_3 := s_{23}(0)v_4, \quad v_2 := s_{12}(0)v_3, \quad v_1 := s_{23}(0)v_2.$$ 

Then by Theorem 2.4, $(v_1, v_2, v_3, v_4)$ constitute a basis for the $U_q(\mathfrak{g})$-module $W_{k,a}^{(2)}$. Moreover, from Relation 2.5 we deduce the explicit action of the $s_{ij}(z)$. Let $\rho^k$ be the representation corresponding to the $U_q(\mathfrak{g})$-module $W_{k,a}^{(2)}$. Then

$$\begin{cases}
\rho^k(s_{11}(z)) = (q^k - zaq^{-k})(E_{11} + E_{22}) + (q^{k-1} - zaq^{-k+1})(E_{33} + E_{44}), \\
\rho^k(s_{22}(z)) = (q^k - zaq^{-k})(E_{11} + E_{33}) + (q^{k-1} - zaq^{-k+1})(E_{22} + E_{44}), \\
\rho^k(s_{33}(z)) = (1 - za)E_{11} + (q^{-1} - zaq)(E_{22} + E_{33}) + (q^{-2} - zaq^2)E_{44}, \\
\rho^k(s_{12}(z)) = q^kE_{23}, \\
\rho^k(s_{13}(z)) = q^{-1}(q - q^{-1})^{-1}(E_{24} - E_{13}), \\
\rho^k(s_{23}(z)) = q^{-1}E_{12} + q^{-2}E_{34}, \\
\rho^k(s_{21}(z)) = za(q - q^{-1})q^{-k}E_{32}, \\
\rho^k(s_{31}(z)) = zaq(q - q^{-1})^2(q^{k+1} - q^{-k-1})E_{42} - zaq(q - q^{-1})^2(q^k - q^{-k})E_{31}, \\
\rho^k(s_{32}(z)) = zaq(q - q^{-1})(q^k - q^{-k})E_{21} + zaq^2(q - q^{-1})(q^{k+1} - q^{-k-1})E_{43}.
\end{cases}$$

Here the $E_{ij} : v_l \mapsto \delta_{ij}v_i$ are the linear transformations on the underlying vector superspace $W = \bigoplus_{j=1}^4 \mathbb{C}v_j$ of $W_{k,a}^{(2)}$. Observe that for all $1 \leq i, j \leq 3$, there exist $A_{ij}(z), B_{ij}(z) \in (\text{End}W)[[z]]$ such that

$$\rho^k(s_{ij}(z)) = A_{ij}(z) + q^{-2k}B_{ij}(z) \quad \text{for all } k \in \mathbb{Z}_{>0}.$$ 

By using these $A_{ij}(z)$, we get the $Y_q(\mathfrak{g})$-module $L^-_{2,a} = W$ defined by:

$$\begin{cases}
\rho^-(s_{11}(z)) = E_{11} + E_{22} + q^{-1}(E_{33} + E_{44}), \\
\rho^-(s_{22}(z)) = E_{11} + E_{33} + q^{-1}(E_{22} + E_{44}), \\
\rho^-(s_{33}(z)) = (1 - za)E_{11} + (q^{-1} - zaq)(E_{22} + E_{33}) + (q^{-2} - zaq^2)E_{44}, \\
\rho^-(s_{12}(z)) = E_{23}, \\
\rho^-(s_{13}(z)) = q^{-1}(q - q^{-1})^{-1}(E_{24} - E_{13}), \\
\rho^-(s_{23}(z)) = q^{-1}E_{12} + q^{-2}E_{34}, \\
\rho^-(s_{21}(z)) = 0, \\
\rho^-(s_{31}(z)) = zaq^2(q - q^{-1})^2E_{42} - zaq(q - q^{-1})^2E_{31}, \\
\rho^-(s_{32}(z)) = zaq(q - q^{-1})E_{21} + zaq^3(q - q^{-1})E_{43}.
\end{cases}$$

Remark 5.3. It is straightforward to check that the representation $(W, \rho^-)$ is simple. Following [2.4], let $Y_q(\mathfrak{g}_2)$ be the subalgebra of $Y_q(\mathfrak{g})$ generated by the $s_{ij}^{(n)}(s_{ii})^{-1}$ with $n \in \mathbb{Z}_{>0}$ and $1 \leq i, j \leq 2$. Then as superalgebras $Y_q(\mathfrak{g}_2) \cong Y_q(\mathfrak{gl}(2,0))$. Furthermore,

$$W = \mathbb{C}v_1 \oplus \mathbb{C}v_4 \oplus (\mathbb{C}v_2 \oplus \mathbb{C}v_3)$$.
is a Krull-Schmidt decomposition of the $Y_q(\mathfrak{g}_2)$-module $\text{Res}^{Y_q(\mathfrak{g})}_{Y_q(\mathfrak{g}_2)} W$ into indecomposable submodules. Note that the third factor is not simple. Hence the underlying $Y_q(\mathfrak{g}_2)$-module structure on $W$ is not semi-simple.

5.3.2. Construction of $L_{2,a}(b)$ and $L^+_2, a$. For $k \in \mathbb{Z}_{>0}$, let $w_4 \in W_{k, aq^{2k}}^{(2)}$ be a lowest $\ell$-weight vector. Define

$$w_3 := \check{s}_{23}^{(0)} w_4, \quad w_2 := \check{s}_{12}^{(0)} w_3, \quad w_1 := \check{s}_{23}^{(0)} w_2.$$ 

Then $(w_1, w_2, w_3, w_4)$ is a basis for $W_{k, aq^{2k}}^{(2)}$. In the following, we identify the underlying vector superspaces of the $W^{(r)}_{k, aq^{2k}}$ for $k \in \mathbb{Z}_{>0}$ with the vector superspace $W = W_{1, aq^2}$ by using this preferred basis. Let $\rho^k$ be the representation associated to the $U_q(\mathfrak{g})$-module $W_{k, aq^{2k}}^{(2)}$. We have

$$\rho^k(s_{11}(z)) = (q^k - zaq^k)(E_{11} + E_{22}) + (q^{k-1} - zaq^{k+1})(E_{33} + E_{44}),$$

$$\rho^k(s_{22}(z)) = (q^k - zaq^k)(E_{11} + E_{33}) + (q^{k-1} - zaq^{k+1})(E_{22} + E_{44}),$$

$$\rho^k(s_{33}(z)) = (1 - zaq^{2k})E_{11} + (q^{-1} - zaq^{2k+1})(E_{22} + E_{33}) + (q^{-2} - zaq^{2k+2})E_{44},$$

$$\rho^k(s_{12}(z)) = q^k E_{23},$$

$$\rho^k(s_{13}(z)) = q^k(q - q^{-1})^{-1}(qE_{24} - E_{13}),$$

$$\rho^k(s_{23}(z)) = q^k E_{12} + q^k E_{34},$$

$$\rho^k(s_{21}(z)) = zaq^k(q - q^{-1})^2 E_{32},$$

$$\rho^k(s_{31}(z)) = zaq^{k-1}(q - q^{-1})^2(q^{k+1} - q^{-k-1})E_{42} - zaq^k(q - q^{-1})^2(q^k - q^{-k})E_{31},$$

$$\rho^k(s_{32}(z)) = zaq^k(q - q^{-1})(q^k - q^{-k})E_{21} + zaq^k(q - q^{-1})(q^{k+1} - q^{-k-1})E_{43}.$$ 

Now by replacing $\rho^k, q^k$ in the above formulas with $\rho, b$ respectively, we get a representation $\rho$ of $U_q(\mathfrak{g}) W := \bigoplus_{i=1}^4 Cw_i$ for $W_{1, aq^2}^{(2)}$. The corresponding $U_q(\mathfrak{g})$-module is $L_{2,a}(b)$.

As before, for $1 \leq i, j \leq 3$, there exist $A_{ij}(z), B_{ij}(z) \in \text{End} W[[z]]$

$$\rho^k(s_{ij}(z)) = A_{ij}(z) + q^{2k} B_{ij}(z) \quad \text{for all } k \in \mathbb{Z}_{>0}.$$ 

Similarly, we get the $Y_q(\mathfrak{g})$-module $L^+_{2,a} = W$:

$$\rho^+(s_{11}(z)) = (1 - za)(E_{11} + E_{22}) + (q^{-1} - zaq)(E_{33} + E_{44}),$$

$$\rho^+(s_{22}(z)) = (1 - za)(E_{11} + E_{33}) + (q^{-1} - zaq)(E_{22} + E_{44}),$$

$$\rho^+(s_{33}(z)) = E_{11} + q^{-1}(E_{22} + E_{33}) + q^{-2}E_{44},$$

$$\rho^+(s_{12}(z)) = E_{23},$$

$$\rho^+(s_{13}(z)) = (q - q^{-1})^{-1}(qE_{24} - E_{13}),$$

$$\rho^+(s_{23}(z)) = E_{12} + E_{34},$$

$$\rho^+(s_{21}(z)) = za(q - q^{-1})^2 E_{32},$$

$$\rho^+(s_{31}(z)) = -zaq^{-2}(q - q^{-1})^2 E_{42} + za(q - q^{-1})^2 E_{31},$$

$$\rho^+(s_{32}(z)) = -za(q - q^{-1})E_{21} - zaq^{-1}(q - q^{-1})E_{43}.$$
Remark 5.4. The $Y_q(\mathfrak{g})$-module $L^\pm_{r,a}$ defined above is easily seen to be simple. Furthermore, contrary to Remark 3.3 it is semi-simple as a $Y_q(\mathfrak{g}_2)$-module.

We point out that $L^-_{2,a}$ and $L^+_{2,a}$ have been constructed by Bazhanov-Tsuboi [BT08 Appendix B.3] as representations of the upper Borel subalgebra of $U_q(\hat{\mathfrak{g}})$ defined by Drinfeld-Jimbo generators, bearing the name $\overline{W}_3^+(x)$ and $\overline{W}_3^-(x)$ respectively after scattering.

6. Category $\mathcal{O}$ and $q$-character

We have constructed in [45] the asymptotic modules: $L^\pm_{r,a}$ as modules over $Y_q(\mathfrak{g})$ and $L_{r,a}(b)$ as modules over $U_q(\hat{\mathfrak{g}})$, as certain limits of Kirillov-Reshetikhin modules $W_{k,a}$. In this section, we introduce a category $\mathcal{O}$ of $Y_q(\mathfrak{g})$-modules including these three kinds of modules following Hernandez-Jimbo [HJ12] and study the Frenkel-Reshetikhin $q$-character for this category.

6.1. Quantum Berezinian. The quantum affine superalgebra admits another system of generators, the so-called Drinfeld generators, $X^\pm_{i,n}, K_{j,\pm s}$ where $i \in I_0, j \in I, n \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geq 0}$, arising from Gauss decomposition:

$$
\begin{align*}
S(z) &= (\sum_{i < j} f_{ji}(z) \otimes E_{ji} + 1 \otimes \text{Id}_V)(\sum_{i} K_i^+(z) \otimes E_{ii})(\sum_{i < j} e_{ij}^+(z) \otimes E_{ij} + 1 \otimes \text{Id}_V), \\
T(z) &= (\sum_{i < j} f_{ji}(z) \otimes E_{ji} + 1 \otimes \text{Id}_V)(\sum_{i} K_i^-(z) \otimes E_{ii})(\sum_{i < j} e_{ij}^-(z) \otimes E_{ij} + 1 \otimes \text{Id}_V).
\end{align*}
$$

For example, $K_1^+(z) = s_{11}(z)$ and $K_1^-(z) = t_{11}(z)$. Now for $i \in I_0 = I$, we have

$$
X^+_{i}(z) = e^+_{i,i+1}(z) - e^-_{i,i+1}(z) = \sum_n X^+_{i,n}z^n, \quad X^-_{i}(z) = f^-_{i,i+1}(z) - f^+_{i,i+1}(z) = \sum_n X^-_{i,n}z^n.
$$

We refer to [Zh14, §3.4] for more details on the relations and on the coproduct of these Drinfeld generators. Recall the definition of the $d_i, \theta_i$ in [21] and in Remark 3.3. For $i \in I$, define the quantum Berezinian

$$
(6.34) \quad C_i(z) := \prod_{j=1}^{i} (K^+_{j}(z\theta_j^{-1}))^{d_j} \in Y_q(\mathfrak{g})[[z]].
$$

Here $K^+_{j}(z) = \sum_{s \in \mathbb{Z}_{\geq 0}} K^+_{j,s}z^s \in Y_q(\mathfrak{g})[[z]].$

The following result comes from [Zh14 Theorem 3.2, Proposition 3.13].

Corollary 6.1. Let $k \in I$.

1. $\Delta(C_k(z)) - C_k(z) \otimes C_k(z) \in \sum_{\alpha \in \mathbb{Q}_{\geq 0} \setminus \{0\}} (Y_q(\mathfrak{g}))_\alpha \otimes (Y_q(\mathfrak{g}))_{-\alpha}[z].$

2. For all $i, j \in I$ such that $i, j \leq k$, we have $C_k(z)s_{ij}(w) = s_{ij}(w)C_k(z)$ and $C_k(z)t_{ij}(w) = t_{ij}(w)C_k(z)$ as formal power series in $U_q(\hat{\mathfrak{g}})$.

Let $C_q(\hat{\mathfrak{g}})$ be the subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by the $(s_{ii}^{(0)})^{-1}$ and by the coefficients of the $C_i(z)$. Then $C_q(\hat{\mathfrak{g}})$ is indeed a commutative subalgebra of $Y_q(\mathfrak{g})$. Let $C_{i,0} \in Y_q(\mathfrak{g})$ be
6.2. **Weights and $\ell$-weights.** By using the commutative subalgebra $C_q(\mathfrak{g})$ of $Y_q(\mathfrak{g})$, let us introduce the notion of a *weight* and an *$\ell$-weight*.

Denote $\mathfrak{P} := (\mathbb{C}^\times)^I$. Endow it with an additive abelian group structure:

$$(a_i)_{i \in I} + (b_i)_{i \in I} := (a_i+b_i)_{i \in I}.$$  

For $i \in I$, define $\epsilon'_i \in \mathfrak{P}$ by $(\epsilon'_i)_j = \begin{cases} 1 & (j < i), \\ q & (j \geq i). \end{cases}$ Then the assignment $\epsilon_i \mapsto \epsilon'_i$ extends uniquely to an injective homomorphism of abelian groups $\mathfrak{P} \longrightarrow \mathfrak{P}$. From now on, we view $\mathfrak{P}$ as a free abelian subgroup of $\mathfrak{P}$ and identify $\epsilon_i = \epsilon'_i$ for $i \in I$. In this way, $\mathfrak{Q} \subset \mathfrak{P} \subset \mathfrak{P}$. Moreover, $\alpha_i \in \mathfrak{P}$ for $i \in I_0$ is the $I$-tuple $(q^b)_j$.

Similarly, denote $\hat{\mathfrak{P}} := (\mathbb{C}[[z]]^\times)^I$. Endow it with a multiplicative abelian group structure:

$$(f_i)_{i \in I}(g_i)_{i \in I} := (f_i g_i)_{i \in I}.$$  

Let $\varpi : \hat{\mathfrak{P}} \longrightarrow \mathfrak{P}$ and $\sigma : \mathfrak{P} \longrightarrow \hat{\mathfrak{P}}$ be two maps defined as follows:

$$\varpi(f_i(z))_{i \in I} = (f_i(0))_{i \in I}, \quad \sigma(a_i)_{i \in I} = (a_i)_{i \in I}.$$  

These are homomorphisms of abelian groups. Furthermore, $\varpi \circ \sigma = \text{Id}_{\mathfrak{P}}$. For the precise statements of results, we shall not identify $\mathfrak{P}$ as a sub-abelian-group of $\hat{\mathfrak{P}}$ by $\sigma$.

Let us introduce the analogues of $\epsilon_i, \alpha_j \in \mathfrak{P}$ in $\hat{\mathfrak{P}}$ to be used later.

**Definition 6.2.** For $i \in I$ and $a \in \mathbb{C}$, define $X_{i,a} \in \hat{\mathfrak{P}}$ by

$$(X_{i,a})_j(z) = \begin{cases} 1 & (j < i), \\ q & (j \geq i). \end{cases}$$

For $i \in I_0$, define $A_{i,a} := X_{i,a}^{-1}X_{i+1,a} \in \hat{\mathfrak{P}}$. Call $A_{i,a}$ a generalized simple root if $a \neq 0$.

By definition $\varpi(X_{i,a}) = \epsilon_i$ and $\varpi(A_{i,a}) = \alpha_i$.

Let $V$ be a $Y_q(\mathfrak{g})$-module. For $a = (a_i)_{i \in I} \in \mathfrak{P}$, define the weight space

$$(V)_a := \{v \in V \mid C_{i,0}v = a_i v \text{ for } i \in I\}.$$  

Note that when $a \in \mathfrak{P}$, the definition of weight space $(V)_a$ is the same as the one given in Remark 3.3 if we view $U_q(\hat{\mathfrak{g}})$-modules therein as $Y_q(\mathfrak{g})$-modules. Similarly, for $f = (f_i(z))_{i \in I}$, define the $\ell$-weight space

$$(V)_f := \{v \in V \mid (C_i(z) - f_i(z))^n x = 0 \text{ for } i \in I \text{ and } n \gg 0\}.$$
6.3. Category \( \mathcal{O} \). We say that a \( Y_q(\mathfrak{g}) \)-module \( V \) is in category \( \mathcal{O} \) if:

(BGG1) \( V \) has a weight space decomposition \( V = \bigoplus_{\alpha \in \mathfrak{P}} V_\alpha \);

(BGG2) for all \( \alpha \in \mathfrak{P} \) we have \( \dim(V)_\alpha < \infty \);

(BGG3) there exist a finite number of weights \( \lambda_1, \ldots, \lambda_s \in \mathfrak{P} \) such that \( (V)_{\alpha} \neq 0 \) implies \( \alpha \in \bigcup_{j=1}^{s} (\lambda_j - \mathbb{Q}_{\geq 0}) \).

As usual \( \mathcal{O} \) is a tensor category (stable under direct sum, sub-quotient and tensor product).

Example 2. For \( \lambda \in \mathfrak{P} \) and \( a \in \mathbb{C}^\times \), the evaluation modules \( \text{ev}^a_L(\lambda), \) viewed as \( Y_q(\mathfrak{g}) \)-modules, are in category \( \mathcal{O} \). Furthermore, the \( Y_q(\mathfrak{g}) \)-modules \( L_{r,a}^\pm \) for \( 1 \leq r \leq M + N - 1 \) are in category \( \mathcal{O} \). Indeed, the weights of these modules lie in \(-\mathbb{Q}_{\geq 0}\), and for \( \alpha \in \mathbb{Q}_{\geq 0} \),

\[
\dim(L_{r,a}^\pm)_{\alpha} = \lim_{k \to \infty} \dim(W_{k,a}^{(r)}k_{\mathfrak{w}r,\alpha})
\]

where the limit at the RHS exists thanks to Corollary \( 3.1 \) in the case \( r \leq M \) and thanks to \( 3.1.1 \) in the case \( r > M \).

Example 3. Let \( r \in I_0 \) and \( a, b \in \mathbb{C}^\times \). Then \( L_{r,a}(b) \), viewed as a \( Y_q(\mathfrak{g}) \)-module, is in category \( \mathcal{O} \). Consider the case \( r \leq M \). Let us come back to the situation of \( \S 5.1 \). Let \( \alpha \in \mathbb{Q}_{\geq 0} \). For \( l \in \mathbb{Z}_{>0} \) and \( x \in (V_l)_{\mathfrak{w}r,-\alpha} \), we have

\[
F_{k,l}(s)^{0}_{ii}(0)F_{k,l}(x) = F_{l(l),l}(x) \begin{cases} q^{k-(\epsilon_i,\alpha)} & (i \leq r), \\ q^{-\epsilon_i,\alpha} & (i > r). \end{cases}
\]

It follows that \( s^{0}_{ii}F_l(x) = F_l(x) \begin{cases} b^{\epsilon_i} & (i \leq r), \\ q^{-\epsilon_i} & (i > r). \end{cases} \) In other words, \( F_l(x) \in (L_{r,a}(b))_{\lambda_{r,b},-\alpha} \) where \( \lambda_{r,b} \in \mathfrak{P} \) is given by \( (\lambda_{r,b})_j = b^{\min(j-r)} \). In consequence, \( (L_{r,a}(b))_{\lambda} \neq 0 \) only if \( \lambda = \lambda_{r,b} - \alpha \) for some \( \alpha \in \mathbb{Q}_{\geq 0} \), in which case

\[
\dim(L_{r,a}(b))_{\lambda_{r,b},-\alpha} = \lim_{k \to \infty} \dim(W_{k,aq^{2k}}^{(r)}k_{\mathfrak{w}r,\alpha}) < \infty.
\]

This says that \( L_{r,a}(b) \) is in category \( \mathcal{O} \).

When \( r > M \), similarly one can find \( \lambda_{r,b} \in \mathfrak{P} \) with \( (\lambda_{r,b})_j = b^{\max(j-r,0)} \) such that the above statements on weight spaces of \( L_{r,a}(b) \) in the case \( r \leq M \) remain true.

6.4. Character and \( q \)-character. To define classical character and \( q \)-character, let us first introduce the target rings. Let \( \mathcal{E} \subset \mathbb{Z}^{\mathfrak{P}} \) be the set of maps \( c : \mathfrak{P} \longrightarrow \mathbb{Z} \) satisfying \( c(\alpha) = 0 \) for all \( \alpha \) outside a finite union of sets of the form \( \lambda - \mathbb{Q}_{\geq 0} \). Endow \( \mathcal{E} \) with a ring structure:

\[
[a] := \delta_{\alpha,\cdot}, \quad [\alpha][\beta] := [\alpha + \beta], \quad (c + d)(\alpha) := c(\alpha) + d(\alpha).
\]

In particular, we see that \( \mathcal{E} \) contains the group ring \( \mathbb{Z}[\mathfrak{P}] \). For \( V \) a \( Y_q(\mathfrak{g}) \)-module in category \( \mathcal{O} \), define its classical character as in \( 2.3.2 \)

\[
\chi(V) := \sum_{\alpha \in \mathfrak{P}} \dim(V)_\alpha [\alpha] \in \mathcal{E}.
\]

Let \( \mathcal{E}_r \subset \mathbb{Z}^{\mathfrak{P}} \) be the set of maps \( c : \mathfrak{P} \longrightarrow \mathbb{Z} \) satisfying \( c(f) = 0 \) for all \( f \) such that \( \mathfrak{w}(f) \) is outside a finite union of sets of the form \( \mu - \mathbb{Q}_{\geq 0} \) and such that for each \( \alpha \in \mathfrak{P} \), there are
Remark 6.5. Let \( V, W \) be in category \( \mathcal{O} \). In general, it is not true that \((V)_f \otimes (W)_g \subseteq (V \otimes W)_{fg}\) for \( f, g \in \hat{\mathfrak{g}}\). However, if \( v \in V \) is a highest \( \ell \)-weight vector (see \([1.1]\)), then \( v \in (V)_f \) for a unique \( f \in \hat{\mathfrak{g}} \) and for all \( g \in \hat{\mathfrak{g}} \)

\[ v \otimes (W)_g \subseteq (V \otimes W)_{fg}. \]
For example, if $V, W$ verify such weight conditions that their normalized characters exist (Remark 6.3), then so does $V \otimes W$ and
\[
\tilde{\chi}(V \otimes W) = \tilde{\chi}(V)\tilde{\chi}(W), \quad \tilde{\chi}_q(V \otimes W) = \tilde{\chi}_q(V)\tilde{\chi}_q(W).
\]

6.5. Simple modules in category $\mathcal{O}$. As in the non-graded case [HJ12, Theorem 3.11], we also have a classification of simple modules in category $\mathcal{O}$ in terms of highest $\ell$-weight.

Following [Zh14 §5.1.3], let $R$ be the subset of $(1 + z\mathbb{C}[[z]])^{I_0}$ consisting of $I_0$-tuples of power series $(f_i(z) : i \in I_0)$ such that $f_i(z) \in \mathbb{C}(z)$ for all $i \in I_0$.

**Lemma 6.6.** (1) For all $f = (f_i(z) : i \in I_0) \in R$, there exists uniquely a simple $Y_q(\mathfrak{g})$-module generated by a highest $\ell$-weight vector $v$ with
\[
|v| = 0, \quad s_{ii}(z)v = (\prod_{j=i}^{M+N-1} f_j(z))v \quad \text{for } i \in I_0, \quad s_{M+N,M+N}(z)v = v.
\]
Moreover, such a module is in category $\mathcal{O}$. Let $V(f)$ be the $Y_q(\mathfrak{g})$-module thus obtained.

(2) All simple modules in category $\mathcal{O}$ can be factorized uniquely into $V(f) \otimes \mathbb{C}_f$ with $f \in R$ and $\mathbb{C}_f$ one-dimensional.

**Proof.** The proof of Part (2) is standard as in the non-graded case. Let $S$ be a simple module in category $\mathcal{O}$. Then $S$ must be a highest $\ell$-weight $Y_q(\mathfrak{g})$-module. Let $v \in S$ be a highest $\ell$-weight vector with weight $\lambda \in \mathfrak{p}$. Then it is enough to show that
\[
K_{i+1}^+(z)K_i^+(z)^{-1}v \in \mathbb{C}(z)v \quad \text{for } i \in I_0.
\]
This comes essentially from the fact that $(S)_{\lambda - \alpha_i}$ is finite-dimensional combined with the following relations and $\mathbb{Q}$-grading on $U_q(\mathfrak{g})$:
\[
|K_{i+1}^+|_q = 0, \quad |X_{i,n}^\pm|_q = \pm \alpha_i, \quad (Y_q(\mathfrak{g}))_{\alpha}(S_\mu) \subseteq S_{\alpha + \mu},
\]
\[
\sum_{n \in \mathbb{Z}_{>0}} [X_{i,n}^+, X_{i,n}]z^n = (q_i - q_i^{-1})(K_{i+1}^+(z)K_i^+(z)^{-1} - K_{i+1,0}^+(z)K_i^+(z)^{-1}) - K_{i+1,0}^+(z)K_i^+(z)^{-1}),
\]
\[
[h_i, X_{i,n}^-] = X_{i,n+1}^-, \quad |h_i|_q = 0,
\]
where $h_i \in Y_q(\mathfrak{g})$ is some properly chosen element (see [Zh14, Appendix A]).

The proof of Part (1) is the same as that of [HJ12, Theorem 3.11] or [Zh14, Lemma 5.1] by realizing $V(f)$ as a simple sub-quotient of certain tensor product of the $L_{r,a}^\pm$ and one-dimensional $Y_q(\mathfrak{g})$-modules.

As in [Zh14 §5.1.1], one-dimensional $Y_q(\mathfrak{g})$-modules are factorized uniquely into the form
\[
\mathbb{C}_s \otimes \mathbb{C}_a \otimes \mathbb{C}_f, \quad s \in \mathbb{Z}_2, \quad a = (a_i)_{i \in I} \in \mathfrak{p}, \quad f \in 1 + z\mathbb{C}[[z]].
\]
Here $\mathbb{C}_a$ is the usual sign module, $\mathbb{C}_a = \mathbb{C}w$ is the $Y_q(\mathfrak{g})$-module with
\[
|v| = \overline{0}, \quad s_{ii}(z)v = a_i \delta_{ij}v,
\]
and $\mathbb{C}_f = \mathbb{C}w$ is the $Y_q(\mathfrak{g})$-module such that
\[
|v| = \overline{0}, \quad s_{ii}(z) = f(z)\delta_{ij}v.
\]
Example 5. For \( r \in I_0 \) and \( a \in \mathbb{C}_\times \), let \( \varpi_{r,a}^{+} \in \mathbb{R} \) be such that
\[
(\varpi_{r,a}^{+})_j := (1 - za\delta_{jr})^{\pm 1} \in 1 + z\mathbb{C}[z].
\]
Then the \( Y_q(\mathfrak{g}) \)-module \( L_{r,a}^{\pm} \) has a simple sub-quotient isomorphic to \( V(\varpi_{r,a}^{\pm}) \otimes \mathbb{C}_{f_{r,a}^{\pm}} \), where \( f_{r,a}^{+} = 1 \) and \( f_{r,a}^{-} = 1 - za \).

Example 6. For \( r \in I_0 \) and \( a, b \in \mathbb{C}_\times \), the \( Y_q(\mathfrak{g}) \)-module \( L_{r,a}(b) \) has a simple sub-quotient isomorphic to \( V(f) \otimes D \) with \( D \) one dimensional and \( f \in \mathbb{R} \) defined by
\[
f_j = 1 \text{ for } j \neq r, \quad f_r = \begin{cases} 
\frac{1 - za}{1 - zab} & (r \leq M), \\
\frac{1 - za}{1 - zab - z} & (r > M).
\end{cases}
\]

We end this subsection with the following observations.

Proposition 6.7. Let \( f = (f_i : i \in I_0) \in \mathbb{R} \). Consider the \( Y_q(\mathfrak{g}) \)-module \( V(f) \).

1. \( V(f) \) is finite-dimensional if and only if for all \( i \in I_0 \setminus \{ M \} \) there exists \( P_i(z) \in 1 + z\mathbb{C}[z] \) such that \( f_i(z) = P_i(zq^{-1}) \).

2. \( V(f) \) admits a \( U_q(\mathfrak{g}) \)-module structure extending that of \( Y_q(\mathfrak{g}) \) up to tensor product by one-dimensional modules if and only if for all \( i \in I_0 \), seen as a meromorphic function, \( f_i(z) \) is regular at \( z = \infty \) and \( f_i(\infty) \in \mathbb{C}_\times \).

The proof of this proposition is again standard as in the non-graded case, by using the Drinfeld generators of \( U_q(\mathfrak{g}) \). See [HJ12, §3.2] for Part (1), [Hed05, Lemma 4.11] and [MY11, Theorem 3.6] for Part (2). For example, As seen from their constructions in [HJ15] the \( L_{r,a}(b) \) are \( U_q(\mathfrak{g}) \)-modules, whereas the \( L_{r,a}^{\pm} \) are not. Note that this proposition also gives a classification of finite-dimensional simple \( U_q(\mathfrak{g}) \)-modules in terms of highest \( \ell \)-weight, as done in [Zh13] with a quite different approach.

6.6. Category \( \mathcal{O} \) of the \( q \)-Yangian \( Y_q(\mathfrak{gl}(1,1)) \). In this subsection, we compute the normalized \( q \)-character for all simple modules \( V(f) \) in category \( \mathcal{O} \) of the \( q \)-Yangian \( Y_q(\mathfrak{gl}(1,1)) \). This serves as the first non-trivial example for the study of normalized \( q \)-character of the asymptotic modules. Note that in this case \( \mathbb{R} \) is the set of rational functions of the form \( \frac{P(z)}{Q(z)} \) where \( P(z), Q(z) \in 1 + z\mathbb{C}[z] \) are co-prime.

6.6.1. Prime simple modules. Let \( a, b \in \mathbb{C} \) be such that \( a \neq b \). As we see in [Zh14, §5.2.2], the simple module \( V(\frac{1 - z a}{1 - z b}) \) is two-dimensional with basis \( v_1, v_2 \) and with the action of the \( s_{ij}(z) \) given as follows:
\[
(s_{ij}(z))_{1 \leq i, j \leq 2} = \begin{pmatrix}
1 - za & q^{-1} - zaq & (q^{-1} - q(b-a)) & E_{11} + q^{-1} - zaq \\
1 - zb & 1 - zb & E_{22} & E_{11} + q^{-1} - zaq \\
-\frac{1}{1 - zb} E_{21} & E_{11} + q^{-1} - zaq & E_{22} & E_{11} + q^{-1} - zaq
\end{pmatrix}
\]
where the \( E_{ij} : v_k \mapsto \delta_{jk} v_i \) are endomorphisms of \( V(\frac{1 - z a}{1 - z b}) \). Let us compute the action of \( K_{i}^{\pm}(z) \). By the Gauss decomposition,
\[
K_{1}^{+}(z) = s_{11}(z) = \frac{1 - za}{1 - zb} E_{11} + \frac{q^{-1} - zaq}{1 - zb} E_{22},
K_{2}^{+}(z) = s_{22}(z) + s_{21}(z) s_{11}(z)^{-1} s_{12}(z)
\]
It follows that (§6.1)

\[ C_1(z) = \frac{1 - za}{1 - zb} E_{11} + \frac{q^{-1} - zaq}{1 - zb} E_{22}, \quad C_2(z) = \frac{1 - za}{1 - zb} \text{Id}. \]

By definition of normalized \( q \)-character in Remark 6.3, (6.38)

\[ \tilde{\chi}_q(V(\frac{1 - za}{1 - zb})) = 1 + [A_1, aq]^{-1}. \]

In particular, the normalized \( q \)-character of \( V(\frac{1 - za}{1 - zb}) \) depends only on \( a \in \mathbb{C} \).

6.6.2. General simple modules. Let \( 1 \neq f \in \mathbb{R} \). Then there exists a unique factorization

\[ f(z) = \prod_{i=1}^{s} \frac{(1 - za_i)}{(1 - zb_j)} \]

where \( a_i, b_j \in \mathbb{C}_\times \) and \( a_i \neq b_j \) for all \( 1 \leq i \leq s, 1 \leq j \leq t \). Let us set \( a_i = 0 \) for \( s < i \leq \max(s, t) \) and \( b_i = 0 \) for \( t < i \leq \max(s, t) \). Then \( a_i \neq b_j \) for all \( 1 \leq i, j \leq \max(s, t) \).

From [Zh14 §5.2.1] we have the factorization

\[ V(f) \cong \bigotimes_{i=1}^{\max(s, t)} V(\frac{1 - za_i}{1 - zb_i}) \]

In view of Remark 6.5 this amounts to

(6.39) \[ \tilde{\chi}_q(V(f)) = \prod_{i=1}^{\max(s, t)} (1 + [A_1, a_i q]^{-1}). \]

**Remark 6.8.** (1) Let \( 1 \neq f \in \mathbb{R} \). Then \( \tilde{\chi}_q(V(f)) \) is a polynomial in the \( [A_1, a]^{-1} \) with \( a \in \mathbb{C}_\times \) if and only if \( f(\infty) \neq 0 \). Retain the above factorization for \( f \in \mathbb{R} \). Then \( V(f) \) has multiplicity-free \( q \)-character if and only if \( a_i \neq a_j \) for all \( 1 \leq i < j \leq \max(s, t) \).

(2) \( \tilde{\chi}_q(V(\frac{1}{1 - za})) = \sigma \tilde{\chi}(V(\frac{1}{1 - za})) \). In other words, the normalized character and normalized \( q \)-character for the simple module \( V(\frac{1}{1 - za}) \) coincide. In the non-graded case, as proved by Hernandez-Jimbo [HJ12 Theorem 7.5] in the simply-laced case and later by Frenkel-Hernandez [FH13 Theorem 4.1] in full generality, this is true for the positive prefundamental modules within the framework of Hernandez-Jimbo, and it gives rise to Baxter’s T-Q relation for an arbitrary non-twisted quantum affine algebra.

(3) As in [Zh14 §5], let \( \mathcal{F} \) be the category of finite-dimensional representations of \( Y_q(\mathfrak{g}) \). It is not true that \( \mathcal{F} \) must be a subcategory of \( \mathcal{O} \). Nevertheless, the \( q \)-character \( \chi_q(V) \in \mathbb{Z}[\hat{\mathfrak{p}}] \subset \mathcal{E}_\ell \) is still well-defined for \( V \) in \( \mathcal{F} \) and it induces a ring homomorphism

\[ \chi_q : K_0(\mathcal{F}) \to \mathbb{Z}[\hat{\mathfrak{p}}] \]
where $K_0(\mathcal{F})$ is the usual Grothendieck ring of the tensor category $\mathcal{F}$. In order to have an injective $q$-character map as in the non-graded case, we need to take into account the $\mathbb{Z}_2$-grading. This is done by extending $\chi_q$ to the super $q$-character map $s.\chi_q$:

$$ s.\chi_q(V) := \sum_{f \in \Phi} s.\dim(V)_f [f] \in \mathbb{Z}[\hat{\mathfrak{p}}] \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon]/(\varepsilon^2) $$

where $s.\dim W := \dim W_{\mathcal{F}} + \varepsilon \dim W^\tau$ is the superdimension associated with a vector superspace [Se96, §1]. The resulting map $s.\chi_q : K_0(\mathcal{F}) \rightarrow \mathbb{Z}[[\mathfrak{p}]] \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon]/(\varepsilon^2)$ is an injective ring homomorphism. In a similar way as in Remark 6.3 normalized super $q$-character $s.\tilde{\chi}_q$ can be defined, and Equation (6.39) becomes

$$(6.40) \quad s.\tilde{\chi}_q(V(f)) = \prod_{i=1}^{\max(s,t)} (1 + \varepsilon [A_{1,a_i}]^{-1}).$$

(4) If we work directly with category $\mathcal{O}$, then $s.\chi_q(V)$ is still well-defined and it induces an injective ring homomorphism $s.\chi_q : K_0^f(\mathcal{O}) \rightarrow \mathcal{E}_F \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon]/(\varepsilon^2)$ where $K_0^f(\mathcal{O})$ is certain completion of the usual Grothendieck ring $K_0(\mathcal{O})$. Already in the non-graded case [HJ12, Proposition 3.12] concerning the injective $q$-character morphism, a completed version $K_0^f(\mathcal{O})$ of Grothendieck ring is implicitly used. We are grateful to David Hernandez for this comment.

7. $q$-CHARACTER OF EVALUATION REPRESENTATIONS

In this section, we study the (normalized) $q$-character of the evaluation modules $ev^a_\lambda L(\lambda)$ and $ev^a_\lambda L(\lambda)^*$ for $\lambda \in \mathfrak{p}$, following the idea of Frenkel-Mukhin [FM02, Lemma 4.7] relating $\ell$-weight spaces to (dual) Gelfand-Tsetlin bases.

7.1. $q$-character and Gelfand-Tsetlin bases. For $k \in I$, let $U_q(\widehat{\mathfrak{g}}_k)$ be the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by the $s^{(n)}_{ij}, l^{(n)}_{ij}$ with $i, j \leq k$. Then as seen in Corollary 6.1 the coefficients of $C_k(z)$ are central elements in $U_q(\widehat{\mathfrak{g}}_k)$. Moreover the following are isomorphic as superalgebras:

$$ U_q(\widehat{\mathfrak{g}}_k) \cong U_q(\mathfrak{gl}(M_k, N_k)) $$

where $M_k, N_k$ are defined at the beginning of §2.4. Also from Proposition 2.1

$$ \text{ev}_a U_q(\widehat{\mathfrak{g}}_k) \subseteq U_q(\mathfrak{g}_k). $$

Hence, for $V$ a $U_q(\mathfrak{g})$-module and for $a \in \mathbb{C}^\times$, as $U_q(\widehat{\mathfrak{g}}_k)$-modules

$$ \text{Res}_{U_q(\widehat{\mathfrak{g}}_k)} \text{ev}_a V \cong \text{ev}_a' \text{Res}_{U_q(\mathfrak{g}_k)} V. $$

At the RHS, $\text{ev}_a : U_q(\widehat{\mathfrak{g}}_k) \cong U_q(\mathfrak{gl}(M_k, N_k)) \rightarrow U_q(\mathfrak{gl}(M_k, N_k)) \cong U_q(\mathfrak{g}_k)$.

**Proposition 7.1.** Let $\lambda \in \mathfrak{p}$. Let $V(\lambda; a) := \phi^*_{(1-z^{-1}a^{-1})-1,(1-z_a)-1} \text{ev}_a^* L(\lambda)$. Then

$$(7.41) \quad \chi_q(V(\lambda; a)) = \sum_{f \in \mathbb{Z}(\mathfrak{Y}^\lambda)} [\prod_{(i,j) \in \mathfrak{Y}^\lambda} \chi_{f(i,j),aq^2(i-j)-1}].$$

In particular, $V(\lambda; a)$ has multiplicity-free $q$-character.
Proof. The idea is the same as that of [EM02, Lemma 4.7]. For completeness, let us explain briefly the main steps. Let \((v_\lambda : \lambda \in GL(\lambda))\) be a basis of \(L(\lambda)\) verifying Corollary 2.10. Fix a Gelfand-Tsetlin pattern \(\lambda = (\lambda^{(k)})_{k \in I} \in GL(\lambda)\) and let \(f := GT^{-1}_\lambda(\lambda)\) be the associated Young tableau. For \(k \in I\), there exists a sign module \(C_k\) such that \(v_\lambda\) is contained in a simple sub-\(U_q(\mathfrak{g}_k)\)-module \(S\) isomorphic to \(\phi^*_\lambda(1-z^{-1}a^{-1})^{-1}(1-z)\) for completeness, let us explain briefly the main steps. Let \((v_\lambda : \lambda \in GL(\lambda))\) be a basis of \(L(\lambda)\) verifying Corollary 2.10. Fix a Gelfand-Tsetlin pattern \(\lambda = (\lambda^{(k)})_{k \in I} \in GL(\lambda)\) and let \(f := GT^{-1}_\lambda(\lambda)\) be the associated Young tableau. For \(k \in I\), there exists a sign module \(C_k\) such that \(v_\lambda\) is contained in a simple sub-\(U_q(\mathfrak{g}_k)\)-module \(S\) isomorphic to \(\phi^*_\lambda(1-z^{-1}a^{-1})^{-1}(1-z)\) for all \(v \in S\). In particular, \(v_\lambda\) is stable by \(C_k(z)\). Now \(C_k(z)\) can be computed via a highest \(\ell\)-weight vector \(v^+\) of \(S\). Indeed, if \(\lambda(k) = \sum_{i=1}^{k} x_i \epsilon_i\), then

\[
C_k(z)v^+ = \prod_{j=1}^{k} \left(K^+_j(z^{-1})\right)^{d_j} v^+ = \prod_{j=1}^{k} \left(s_{jj}(z^{-1})\right)^{d_j} v^+ = v^+ \prod_{j=1}^{k} \left(\frac{q^{x_j} - za \theta^{-1}_j q^{-x_j} \theta_j}{1 - za \theta^{-1}_j}\right)^{d_j} = v^+ g_k(z).
\]

Here the second equation above comes from Gauss decomposition and from the fact that \(v^+\) is a highest \(\ell\)-weight vector of the \(U_q(\mathfrak{g}_k)\)-module \(S\). It is enough to show

\[
g_k(z) = \prod_{(i,j) \in Y^\lambda} x_{f(i,j),aq^{2(i-j)-1}} \prod_{f(i,j) \leq k} q^{1 - zaq^{2(i-j)-2}} - zaq^{2(i-j)}.
\]

For example, let us assume \(k > M\). Then \(f(i,j) \leq k\) if and only if \((1 \leq i \leq M, 1 \leq j \leq x_i)\) or \((1 \leq j \leq k - M, M + 1 \leq i \leq M + x_M + j)\). It follows that

\[
RHS = \prod_{i=1}^{M} q^{x_i} - zaq^{-1}\frac{1 - zaq^{2(i-1) - x_i}}{1 - zaq^{2(i-1)}} \times \prod_{i=1}^{M} q^{-x_M + j} - zaq^{2(M-j) + x_M + j}\frac{1 - zaq^{2(M-j)}}{1 - zaq^{2(M-j)}}^{-1} = \prod_{j=1}^{k} \left(\frac{q^{x_j} - za \theta^{-1}_j q^{-x_j} \theta_j}{1 - za \theta^{-1}_j}\right)^{d_j} = g_k(z).
\]

That \(V(\lambda; a)\) has multiplicity-free \(q\)-character comes from Definition 2.2 and from the fact that the \(X_{i,b}\) with \(i \in I, b \in \mathbb{C}^\times\) are algebraically independent in \(E_\ell\). As an immediate corollary, we have

Corollary 7.2. \(\tilde{x}_q(V(\lambda; a)) \in \mathbb{Z}[A_{i,b}^{-1} : i \in I_0, b \in \mathbb{C}^2]\) for \(\lambda \in \mathcal{P}\) and \(a \in \mathbb{C}^\times\).

Proof. Let \(\lambda = \sum_{i \in I} x_i \epsilon_i \in \mathcal{P}\). Let \(Y^\lambda\) be the associated \(g\)-Young diagram. Let \(f^+ \in \mathcal{B}(Y^\lambda)\) be the \(g\)-Young tableau corresponding to the Gelfand-Tsetlin pattern

\[
\text{GT}_\lambda(f^+)^{(k)} = \sum_{i=1}^{k} x_i \epsilon_i \quad \text{for } k \in I.
\]
Then for all \( f \in \mathcal{B}(Y^\lambda) \) and \((i, j) \in Y^\lambda\), we have \( f(i, j) \geq f^+(i, j) \). The rest is clear from Equation (7.41) and Definition 6.2.

**Example 7.** Let \((M, N) = (2, 1)\) and \( g = \mathfrak{gl}(2, 1) \). Then

\[
\bar{\chi}_q(W_{k, aq^{2k}}^{(2)}) = 1 + [A_{2, aq}^{-1}] + [A_{1, aq}]^{-1} + [A_{1, aq}]^{-1} + [A_{2, aq}]^{-1},
\]

\[
\bar{\chi}_q(W_{k, aq^{2k}}^{(1)}) = 1 + (1 + [A_{2, aq}^{-1}]) \sum_{l=1}^{k} [A_{1, aq^{2j-1}]}^{-1}.
\]

The following combinatorial property of normalized \( q \)-character will be used later on.

**Lemma 7.3.** Let \( a, b \in \mathbb{C}^\times \) and \( k \in \mathbb{Z}_{>0} \). Let \( 1 \leq r \leq M \). Write

\[
\bar{\chi}_q(W_{k, aq}^{(r)}) = \sum_{\Phi \in \mathcal{P}} c_{\Phi} [\Phi], \quad \bar{\chi}_q(W_{k, aq^2}^{(r)}) = \sum_{\Phi \in \mathcal{P}} d_{\Phi} [\Phi],
\]

Assume that \( b^2 \notin q^{2Z_{>0}} \). Then for all \( \Phi, \Phi_1, \Phi_2 \in \hat{\mathcal{P}} \) such that

\[
\Phi = \Phi_1 \Phi_2, \quad c_{\Phi} = c_{\Phi_1} = d_{\Phi_2} = 1,
\]

we must have \( \Phi_2 = 1 \).

**Proof.** Let \( Y = \{(i, j) \in \mathbb{Z}^2 | 1 \leq i \leq r, 1 \leq j \leq k\} \) be the \( g \)-Young diagram associated with \( L(\kappa \mathcal{W}_g) \). Let \( \mathcal{B} = \mathcal{B}(Y) \) be the set of \( g \)-Young tableaux of shape \( Y \). In view of Proposition 7.1, it is enough to prove the following:

1. Let \( f, g, h \in \mathcal{B} \) be such that in the group \( \hat{\mathcal{P}} \):

\[
\prod_{(i, j) \in Y} \mathcal{X}_{g(i, j), aq^{2i-j-1}}^{-1} \mathcal{X}_{f(i, j), aq^{2i-j-1}} = \prod_{(i, j) \in Y} \mathcal{X}_{i, abq^{2i-j}}^{-1} \mathcal{X}_{h(i, j), abq^{2i-j-1}}.
\]

then \( h(i, j) = i \) for \((i, j) \in Y\).

Remark that the \( \mathcal{X}_{i, x} \) with \( i \in I, x \in \mathbb{C}^\times \) generate a free abelian subgroup of \( \hat{\mathcal{P}} \). Let us compare the index \( x \) of the \( \mathcal{X}_{i, x} \) on both sides. For \( 1 \leq i \leq r \) by assumption

\[
abq^{2(i-k-1)} = aq^{2(i-j-s-1)} \notin \{aq^{2(i'-j')-1} | (i', j') \in Y\}.
\]

Hence the \( \mathcal{X}_{i, abq^{2i-k-1}} \) for \( 1 \leq i \leq r \) on the RHS must disappear. Furthermore, \( i \leq h(i, k) \) and \( h(i, k) \leq h(i', k) \) for \( 1 \leq i \leq i' \leq k \). It follows that \( h(i, k) = i \) for \( 1 \leq i \leq r \). By definition of a \( g \)-Young tableau, this says that \( h(i, j) = i \) for all \((i, j) \in Y\).

As we see in Example 7 (by taking \( g = \mathfrak{gl}(2, 1), r = 2, b = q^{-1} \)), the Lemma above is false if we remove the condition on \( b \). More generally, let \( 1 \leq s < r \leq M \) and define three \( g \)-Young tableaux \( f, g, h \) of shape \( Y \) as follows:

\[
f(i, j) = \begin{cases} M + 1 & (i \geq r - s, j = k), \\ i & \text{otherwise,} \end{cases} \quad g(i, j) = \begin{cases} M + 1 & (i > r - s, j = k), \\ r - s & (i = r - s, j = k), \\ i & \text{otherwise,} \end{cases}
\]

\[
h(i, j) = \begin{cases} M + 1 & (i = r, j = k), \\ i & \text{otherwise.} \end{cases}
\]
Then (1) in the proof of the Lemma is satisfied by taking $b = q^{-a}$.

### 7.2. $q$-character and dual Gelfand-Tsetlin bases

Our next task is to consider such $U_q(\mathfrak{g})$-modules $\text{ev}_\lambda^* L(\lambda)^*$ with $\lambda \in P$.

**Proposition 7.4.** Let $\lambda \in P$. Let $(\nu_\lambda^\Delta : \Delta \in \mathcal{S}\mathcal{T}(\lambda))$ be a dual Gelfand-Tsetlin basis for the $\mathcal{U}_q(\mathfrak{g})$-module $L(\lambda)^*$ as in (2.3). Then there exist $f_{i,\Delta}(z) \in \mathbb{C}[z]$ for $i \in I$ and $\Delta \in \mathcal{S}\mathcal{T}(\lambda)$ such that in the $\mathcal{U}_q(\mathfrak{g})$-module $\text{ev}_\lambda^* L(\lambda)^*$

$$C_i(z)\nu_\lambda^\Delta = f_{i,\Delta}(z)\nu_\lambda^\Delta \in \mathbb{C}[z]$$

Furthermore, if $f_{i,\Delta}(z) = f_{i,\mu}(z)$ for $\Delta, \mu \in \mathcal{S}\mathcal{T}(\lambda)$, then $\lambda^{(i)} = \mu^{(i)}$. In particular, $\text{ev}_\lambda^* L(\lambda)^*$, viewed as a $Y_q(\mathfrak{g})$-module in category $\mathcal{O}$, has multiplicity-free $q$-character.

Contrary to Proposition 7.1, we do not have a closed formula for $\chi_q(\text{ev}_\lambda^* L(\lambda)^*)$.

**Proof.** The existence of the $f_{i,\Delta}(z)$ comes from the defining properties of the dual Gelfand-Tsetlin basis in (2.3). We only need to show that $f_{i,\Delta} = f_{i,\mu}$ implies $\lambda^{(i)} = \mu^{(i)}$.

For $\Delta \in \mathcal{S}\mathcal{T}(\lambda)$, let $\lambda^{(i)}_b$ be the lowest weight of the $\mathcal{U}_q(\mathfrak{g}_b)$-module $L(\lambda^{(i)}_b; \mathfrak{g}_b)$ defined in (2.3). Then $\lambda^{(i)}_b = \mu^{(i)}_b$ if and only if $\lambda^{(i)} = \mu^{(i)}$. In the following, we show that $\lambda^{(i)}_b = \mu^{(i)}$ under the condition $f_{i,\Delta} = f_{i,\mu}$.

Assume first $1 \leq i \leq M$. Write $\lambda^{(i)}_b = \sum_{s=1}^i a_{i+1-s} \epsilon_s$ and $\mu^{(i)}_b = \sum_{s=1}^i a'_{i+1-s} \epsilon_s$. Then $a_1 \geq a_2 \geq \cdots \geq a_i, a'_1 \geq a'_2 \geq \cdots \geq a'_i$.

Since $\nu_\Delta^\lambda$ is contained in a sub-$\mathcal{U}_q(\mathfrak{g}_s)$-module of $L(\lambda)^*$ isomorphic to $L(-\lambda^{(i)}_b; \mathfrak{g}_s) \otimes \mathbb{C}[\lambda - \lambda^{(i)}]$, we get an explicit formula for $f_{i,\Delta}$:

$$f_{i,\Delta} = \prod_{s=1}^i (q^{-a_{i+1-s}} - zaq^{2s-1} + a_{i+1-s}), \quad f_{i,\mu} = \prod_{s=1}^i (q^{-a'_{i+1-s}} - zaq^{2s-1} + a'_{i+1-s}).$$

Note that $s - 1 + a_{i+1-s} < t - 1 + a'_{i+1-t}$ for $1 \leq s < t \leq i$. Hence $a_s = a'_s$ for $1 \leq s \leq i$.

Next assume $i = M + k$ with $1 \leq k \leq N$. Let us write

$$\lambda^{(M+k)}_b = \sum_{l=1}^M x_{M+1-l} \epsilon_l + \sum_{j=1}^k y_{k+1-j} \epsilon_{M+j}, \quad \mu^{(M+k)}_b = \sum_{l=1}^M x'_{M+1-l} \epsilon_l + \sum_{j=1}^k y'_{k+1-j} \epsilon_{M+j}.$$

Then the $x_l, y_j$ (similar for the $x'_l, y'_j$) verify the following conditions:

(A) $x_1 \geq x_2 \geq \cdots \geq x_M$ and $y_1 \geq y_2 \geq \cdots \geq y_k$;

(B) if $y_j \leq r$, then $x_{r+1} = 0$ (with the convention $x_{M+1} = x_{M+2} = \cdots = 0$).

Our aim is to show that $x_l = x'_l$ and $y_j = y'_j$ for $1 \leq l \leq M$ and $1 \leq j \leq k$. Note that $f_{M+k,\Delta} = f_{M+k,\mu}$ implies $\frac{N(z)}{D(z)} = \frac{N'(z)}{D'(z)}$ where

$$N(z) = \prod_{l=1}^M (1 - zaq^{2(l-1)+2x_{M+1-l}}), \quad D(z) = \prod_{j=1}^k (1 - zaq^{2(M-k-1+j)-2y_j)}$$
and $N'(z), D'(z)$ are similarly defined using $x'_i, y'_j$. We shall prove $N(z) = N'(z)$. If one of the fractions $\frac{N(z)}{D(z)}, \frac{N'(z)}{D'(z)}$ is in reduced form, then by counting the zeros and the poles we get $N(z) = N'(z)$. Suppose therefore neither $\frac{N(z)}{D(z)}$ nor $\frac{N'(z)}{D'(z)}$ is reduced.

Let $P$ (resp. $Z$) the set of poles (resp. zeros) not including $\infty$ of the rational function $\frac{N(z)}{D(z)}$. Let $P_0$ (resp. $Z_0$) be the set of zeros of $D(z^{-1})$ (resp. $N(z^{-1})$). Define similarly $P', Z', P'_0, Z'_0$. Then

\[ P = P', \quad Z = Z', \quad P \subseteq P_0, \quad Z \subseteq Z_0. \]

Moreover, there exists $1 \leq j_1 < j_2 < \cdots < j_s \leq k$ such that

\[ P_0 \setminus P = \{ aq^{2(M-k-1+jt-yj)} M \leq t \leq s \}. \]

One can find a unique $1 \leq i_s \leq M$ such that $aq^{2(i_s-1+x_{M+1-i_s})} = aq^{2(M-k-1+j_s-yj)}$. So

\[ yj = M - k + j_s - i_s - x_{M+1-i_s} \leq M - k + j_s - i_s. \]

Now Condition (B) above says that $x_{M-k+j_s-i_s+1} = 0$. In view of Condition (A) ($j_s \leq k$)

\[ x_{M+1-i_s} = 0, \quad yj = M - k + j_s - i_s. \]

Again Condition (A) says that $x_{M+1-l} = 0$ for $1 \leq l \leq i_s$. It follows that

\[ \{ aq^{2(l-1)+x_{M+1-l}} ] i_s < l \leq M \} \subseteq Z, \quad N(z) = \prod_{l \leq i_s} (1 - zaq^{2(l-1)}) \prod_{l > i_s} (1 - zaq^{2(l-1)+x_{M+1-l}}). \]

Similar analysis of the rational function $\frac{N'(z)}{D'(z)}$ leads to $i'_s$ with the above properties. We are therefore led to show that $i_s = i_s'$. Assume the contrary: say $i_s > i_s'$. Then observe first from $Z = Z'$ that

\[ x_{M+1-l} = x'_{M+1-l} \quad \text{for} \quad l > i_s. \]

Next, we have $aq^{2(i_s-1+x'_{M+1-i_s})} \in Z' = Z$. Hence there must be $1 \leq j \leq M$ such that

\[ j - 1 + x_{M+1-j} = i_s - 1 + x'_{M+1-i_s}. \]

Since $aq^{2(i_s-1+x_{M+1-i_s})} \not\in Z$ by assumption, $j \neq i_s$. When $j < i_s$, $x_{M+1-j} = 0$ and the above equation can not hold. So $j > i_s$. It follows that $x_{M+1-j} = x'_{M+1-j}$ and

\[ i_s - 1 + x'_{M+1-i_s} = j - 1 + x'_{M+1-j}, \quad \text{which is impossible as} \quad x'_{M+1-i_s} \leq x'_{M+1-j}. \]

This completes the proof. \hfill \Box

As in Proposition 7.4, all the $\ell$-weight spaces of $\text{ev}_a^* L(\lambda)^*$ are one-dimensional of the form $\mathbb{C}v_\lambda$ with $\lambda \in \mathcal{T}(\lambda)$.

**Corollary 7.5.** $\check{\chi}_q(\text{ev}_a^* L(\lambda)^*) \in \mathbb{Z}[A_i, b_i] : i \in I_0, b \in aq^{2Z+1}]$ for $\lambda \in \mathcal{P}$ and $a \in \mathbb{C}^\times$.

**Proof.** First observe from Definition 6.2 that

\[ (A_i, a)_i = q \frac{1 - zaq^{-1}}{1 - zaq}, \quad (A_i, a)_j = 1 \quad \text{for} \quad j \neq i. \]

As in the proof of Corollary 7.2, let $\lambda \in \mathcal{P}$. Let $Y^\lambda$ be the associated $\mathfrak{g}$-Young diagram. Let $f^-(\mathcal{B}(Y^\lambda))$ be the $\mathfrak{g}$-Young tableau such that the associated Gelfand-Tsetlin vector $v_{\mathcal{T}(f^-)} \in L(\lambda)$ is a lowest weight vector. For simplicity, let us denote $\mu := \mathcal{T}(f^-)$. In view of Proposition 7.4, it is enough to show that for all $i \in I$ and for all $\lambda \in \mathcal{T}(\lambda)$, the power
series $f_{i,\mu}(z)f_{i,\lambda}(z)^{-1}$ can be written as a product of $q^{1-2bq_{ij}}$ with $b \in \mathbb{C}^\times$. We consider the case $i = M + k$ with $1 \leq k \leq N$. (The case $i \leq M$ is similar and simpler.)

Let $g := GT^{-1}(\lambda)$. Then as in the proof of Corollary 7.2 we have

$$g(x, y) \leq f^-(x, y) \quad \text{for } (x, y) \in Y^\lambda.$$  

As before, for $\theta \in \mathcal{P}_{M, N}$, let $\theta_b$ be the lowest weight of the $U_q(\mathfrak{g}_b)$-module $L(\theta; \mathfrak{g}_k)$. Write

$$\mu_b^{(M+k)} = \sum_{j=1}^{M} x_{M+1-j}^j + \sum_{l=1}^{k} y_{k+1-j}^l \epsilon_{M+j}, \quad \lambda_b^{(M+k)} = \sum_{j=1}^{M} x_{M+1-j}^j + \sum_{l=1}^{k} y_{k+1-j}^l \epsilon_{M+j}.$$

Then as in the proof of Proposition 7.1 we have

$$f_{i,\mu} = \prod_{j=1}^{M} (q^{-x_{M+1-j}} - zaq^{2(j-1)+x_{M+1-j}}) \times \prod_{l=1}^{k} (q^{y_{k+1-l}} - zaq^{2(M-l)-y_{k+1-l}})^{-1},$$

$$f_{i,\lambda} = \prod_{j=1}^{M} (q^{-x_{M+1-j}^j} - zaq^{2(j-1)+x_{M+1-j}^j}) \times \prod_{l=1}^{k} (q^{y_{k+1-l}} - zaq^{2(M-l)-y_{k+1-l}})^{-1}.$$

It is therefore enough to show that $x_j \leq x_j'$ and $y_l \leq y_l'$ for $1 \leq j \leq M$ and $1 \leq l \leq k$.

Let us verify $y_l \leq y_l'$ for $1 \leq l \leq k$. (The case for $x$ is parallel.) For this, we use the description in [2,5] on the relationship between Young diagrams and lowest weight vectors. Namely, we have (Definition 2.7)

$$y_{k+1-j} = \sharp \{x \in \mathbb{Z}_{>0} \mid (x, N + 1 - j) \in Y^\lambda\} = \sharp \{f^-(1)^{-1}(M + k)\},$$

$$y_{k+1-j}' = \sharp \{x \in \mathbb{Z}_{>0} \mid (x, k + 1 - j) \in g^{-1}(I_{M, N})\}.$$  

Suppose that $(x, N + 1 - j) \in Y^\lambda$. Then $f^-(x, N + 1 - j) = M + k$. It follows that $g(x, N + 1 - j) \leq M + k$. Hence $g(x, k + 1 - j) \leq g(x, N + 1 - j) \leq i$. In other words,

$$\{x \in \mathbb{Z}_{>0} \mid (x, N + 1 - j) \in Y^\lambda\} \subseteq \{x \in \mathbb{Z}_{>0} \mid (x, k + 1 - j) \in g^{-1}(I_{M, N})\}.$$

This says that $y_l \leq y_l'$, as desired.

**Example 8.** Let $(M, N) = (1, 2)$ and $\mathfrak{g} = \mathfrak{gl}(1, 2)$. Consider the $U_q(\mathfrak{g})$-module $W^{(2)}_{k,a}$. Let us compute its normalized $q$-character by using dual Gelfand-Tsetlin basis. The $\mathfrak{g}$-Young diagram in this case is

$$Y = \{(i, 1) \in \mathbb{Z}^2 \mid 1 \leq i \leq k\}.$$  

There are $2k + 1$ $\mathfrak{g}$-Young tableaux of shape $Y$, namely $f_i, g_j$ with $0 \leq i \leq k, 0 \leq j \leq k - 1$:

$$f_i(l, 1) = \begin{cases} 2 & (1 \leq l \leq i), \\ 3 & (i < l \leq k) \end{cases}, \quad g_j(l, 1) = \begin{cases} 1 & (l = 1), \\ 2 & (1 < l \leq j + 1), \\ 3 & (j + 1 < l \leq k) \end{cases}.$$  

Let $v_i, w_j \in W^{(2)}_{k,a}$ be dual Gelfand-Tsetlin vectors associated with $f_i, g_j$. Then $v_0$ is a highest $\lambda$-weight vector. The action of $C_q(\mathfrak{g})$ on these vectors becomes:

$$C_1(z)v_i = (1 - za)v_i, \quad C_1(z)w_j = (q^{-1} - zaq)w_j,$$

$$C_2(z)v_i = (1 - za)(q^j - zaq^{-1})^{-1}v_i, \quad C_2(z)w_j = (1 - za)(q^{j+1} - zaq^{-j-1})^{-1}w_j.$$
It follows that
\[ \widetilde{\chi}_q(W^{(2)}_{k,a}) = 1 + (1 + [A_{1,aq}]^{-1}) \sum_{l=1}^{k} \prod_{j=1}^{l} [A_{2,aq^{-2j+1}}]^{-1}. \]

8. Gelfand-Tsetlin basis of asymptotic representations

In this section, we study the normalized $q$-character for the asymptotic modules $L_{r,a}^+$ and $L_{r,a}(b)$ constructed in previous sections \[4\] and we establish Gelfand-Tsetlin basis for these modules (provided that $b$ is generic).

8.1. Gelfand-Tsetlin basis. For $k \in I$, let $Y_q(g_k)$ be the subalgebra of $Y_q(g)$ generated by the $s_{ij}^{(n)}(s_{ii}^{(0)})^{-1}$ with $i, j \leq k$. Then there exists canonical isomorphisms of superalgebras
\[ Y_q(gl(M_k, N_k)) \cong Y_q(g_k) \subset U_q(g_k) \cong U_q(gl(M_k, N_k)). \]

We have therefore a chain of sub-superalgebras of $Y_q(g)$:
\[ Y_q(g_1) \subset Y_q(g_2) \subset \cdots \subset Y_q(g_{M+N-1}) \subset Y_q(g_{M+N}) = Y_q(g). \]

Let $V$ be a $Y_q(g)$-module. A basis $(v_{ij})_{j \in \Lambda}$ of $V$ is called a Gelfand-Tsetlin basis if

- (B1) $V$ is semi-simple as a $Y_q(g_k)$-module for all $k \in I_0$ and simple as a $Y_q(g)$-module;
- (B2) for all $j \in \Lambda$ there exists a sequence $(S_{s})_{s \in I}$ such that: $S_s$ is a simple sub-$Y_q(g_s)$-module; $S_s \subset S_{s+1}$ for $1 \leq s < M + N$; $v_j \in S_1$;
- (B3) for all $1 \leq k \leq M + N$ and for $S$ a simple sub-$Y_q(g_s)$-module, the decomposition of $\text{Res}_{Y_q(g_{s-1})} Y_q(g_s)$ into simple sub-$Y_q(g_{s-1})$-modules is multiplicity-free.

Now we can state the central result of this section.

**Theorem 8.1.** Let $r \in I_0, a, b \in \mathbb{C}^\times$ be such that $b \notin \pm q^\mathbb{Z}$. Then the $Y_q(g)$-modules $L_{r,a}^+$ and $L_{r,a}(b)$ admit Gelfand-Tsetlin bases.

The idea of proof goes as follows. Fix $r \in I_0$. There exists an inductive system $(V_k, F_{k,l})$ constructed in \[4\] where $V_k$ are properly defined Kirillov-Reshetikhin modules. The representations of $Y_q(g)$ on $L_{r,a}^+$ and on $L_{r,a}(b)$ are built upon the same underlying space, namely the inductive limit $(V_\infty, F_k)$. For all $k \in \mathbb{Z}_{>0}$, the $Y_q(g)$-module $V_k$ admits a Gelfand-Tsetlin basis. Furthermore the structural maps $F_{k,l} : V_l \rightarrow V_k$ respect Gelfand-Tsetlin bases on both sides. From this we construct directly a basis of $V_\infty$ which serves at the same time as a Gelfand-Tsetlin basis for $L_{r,a}^+$ and for $L_{r,a}(b)$.

The next two subsections are devoted to the detailed proof of this theorem in cases $r \leq M$ and $r > M$. Along the proof, we shall find the simple sub-$Y_q(g_k)$-modules in (B2) in a combinatorial way (by using semi-infinite Young tableaux).

Before the proof, let us make the following convention. For $1 \leq s \leq M + N$, related to the $q$-Yangian $Y_q(g_s) \cong Y_q(gl(M_s, N_s))$ we can define in exactly the same way the category $O_s$, the group $\overline{M}_s, N_s$, the generalized simple roots $A_{i,a,s} \in \overline{M}_s, N_s$ for $1 \leq i < s$, the ring $(\mathcal{E}_s)$, the $q$-character $\chi_q^{s}$ and the normalized $q$-character $\tilde{\chi}_q^{s}$. We identify $\overline{M}_s, N_s$ (resp.
(\mathcal{E}_\ell)_s) with a subset of \( \mathfrak{F} \) (resp. \( \mathcal{E}_\ell \)) in the following natural way:

\[
f = (f_i : 1 \leq i \leq k) \mapsto f' = (f'_i : 1 \leq i \leq M + N), \quad f'_i = \begin{cases} f_i & (i \leq k), \\ 1 & (i > k). \end{cases}
\]

Then under this identification, \( A_{i,a,s} = A_{i,a} \) for \( 1 \leq i < k \) as seen from Definition 6.2

8.2. Proof of Theorem 8.1 when \( r \leq M \). Fix \( a \in \mathbb{C}^\times \). Let us be in the situation of the first paragraph of \([5.1]\) so that we have an inductive system of vector superspaces \((V_k, F_{k,l})\) with inductive limit \((V_\infty, F_k)\). Note that we have fixed a highest \( \ell \)-weight vector \( v_k \in V_k \) for all \( k \in \mathbb{Z}_{>0} \) such that \( F_{k,l}v_k = v_k \) whenever \( k > l \). Moreover, we take \( v_\infty = F_1(v_1) \). Denote \((\rho^k, V_k), (\rho^+, V_\infty), (\rho^b, V_\infty)\) the representations of \( \mathfrak{Y}_g(\mathfrak{g})\)-modules \( W^{(r)}_{k,aq^2k}, L^+_r, L_{r,a}(b) \). Let \( t : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0} \) be a strictly increasing function such that the \( F_{k,l}(l)\rho^k(s_{ij}^{(n)})F_{k,l} \) for \( k > t(l) \) are well-defined. (See Proposition 4.9)

8.2.1. Compatibility of Gelfand-Tsetlin bases and structural maps. Fix \( l, k \in \mathbb{Z}_{>0} \) such that \( l < k \). Choose for all \( l > 0 \) a Gelfand-Tsetlin basis \((v_\underline{\lambda} : \underline{\lambda} \in \mathfrak{Y}(l\varpi_r))\) of \( V_l \) satisfying the properties of Corollary 2.10. (Note that \( V_k \cong L(k\varpi_r) \) as \( \mathfrak{U}_q(\mathfrak{g})\)-modules.)

**Lemma 8.2.** In the above situation, we have: for all \( \underline{\lambda} = (\lambda^{(i)}) \in \mathfrak{Y}(l\varpi_r) \),

\[
F_{k,l}v_{\underline{\lambda}} \in \mathbb{C}^\times v_{\underline{\mu}}
\]

with \( \underline{\mu} = (\mu^{(i)}) \in \mathfrak{Y}(k\varpi_r) \) defined by \( \mu^{(i)} = \lambda^{(i)} + (k-l) \)

\[
\begin{cases} \varpi_i & (i \leq r), \\ \varpi_r & (i > r). \end{cases}
\]

*Proof.* This comes from Remark 6.5 combined with (the proof of) Proposition 7.1 on relationship between Gelfand-Tsetlin vectors and \( \ell \)-weight spaces and on multiplicity-free property of \( q \)-character, in view of the definition of \( F_{k,l} \) in 4.4 \( \square \)

8.2.2. Semi-infinite Young tableau and Gelfand-Tsetlin bases. Based on Lemmas 8.2 and 2.8, we introduce another index set of Gelfand-Tsetlin basis of \( V_k \). This will be convenient for the statement of results.

Let \( Y^{(r)} \) be the subset of \( \mathbb{Z}^2 \) consisting of \((i, j)\) such that \( 1 \leq i \leq r \) and \( j < 0 \). Let \( \mathcal{B}^{(r)} \) be the set of functions \( f : Y^{(r)} \to I \) satisfying (T1)-(T3) in Definition 2.2 and: for all \( 1 \leq i \leq r \), there exists \( j < 0 \) such that \( f(i, j) = i \). Such an \( f \) is also called a semi-infinite Young tableau. For \( k > 0 \), let \( \mathcal{B}^{(r)}_k \) be the subset of \( \mathcal{B}^{(r)} \) consisting of such \( f \) that \( f(i, j) = i \) for \( j < -k \). We have therefore a chain of subsets of \( \mathcal{B}^{(r)} \):

\[
\mathcal{B}^{(r)}_1 \subseteq \mathcal{B}^{(r)}_2 \subseteq \mathcal{B}^{(r)}_3 \subseteq \cdots, \quad \mathcal{B}^{(r)} = \bigcup_{k=1}^\infty \mathcal{B}^{(r)}_k.
\]

There is a canonical bijective map \( \pi_k : \mathcal{B}(Y^{k\varpi_r}) \to \mathcal{B}^{(r)}_k \) sending \( g \), a \( \mathfrak{g} \)-Young tableau of shape \( Y^{k\varpi_r} \), to a semi-infinite Young tableau \( f \) where \( f(i, j) = \begin{cases} i & (j < -k), \\ g(i, j + k + 1) & (j \geq -k). \end{cases} \)

Let \( f_0 \in \mathcal{B}^{(r)} \) be such that \( f_0(i, j) = i \). By using Lemma 2.8 Corollary 2.10 and Lemma
one can construct by induction on \( k > 0 \), a Gelfand-Tsetlin basis \( \{ v[k, f] | f \in \mathcal{B}_k^r \} \) of \( V_k \) verifying the following properties.

(a) \( v[k, f_0] = v_k \) for all \( k > 0 \).
(b) Let \( l < k \) and \( f \in \mathcal{B}_l^r \). Then \( F_{k,l}v[l, f] = v[k, f] \).
(c) \( v[k, f] \in \mathbb{C}^\times v_\mu \) with \( \mu = GT_{k,r}(\pi_k^{-1} f) \in \mathcal{G}(k \varpi_r) \).

For \( f \in \mathcal{B}^r \), define \( v[f] := F_k v[k, f] \) whenever \( f \in \mathcal{B}_k^r \). Note that \( v[f] \in V_\infty \) is well-defined thanks to (b). Moreover \( v[f_0] = v_\infty \), and \( \{ v[f] | f \in \mathcal{B}^r \} \) is a basis of \( V_\infty \).

By definition, \( v[f] \) is the semi-infinite Young tableau obtained from \( f \) by replacing any \( f(i, j) \in \mathcal{B}_l^r \) with \( i > l \) and \( f(i, j) \in \mathcal{B}_l^r \) whenever \( f(i, j) > s \) or \( g(i, j) > s \). Clearly

\[
v[f] \in S_1(f) \subseteq S_2(f) \subseteq \cdots \subseteq S_{M+N-1}(f) \subseteq S_{M+N}(f) = V_\infty.
\]

For \( 1 \leq i \leq \min(r, s) \), define \( l_i := \{ j < 0 | f(i, j) > s \} \). Set \( \lambda_s(l, f) := \sum_{i=1}^{\min(r, s)} (l_i - l) \epsilon_i \).

Let \( f_{is} \in \mathcal{B}^r \) be the semi-infinite Young tableau obtained from \( f \) by replacing any \( f(i, j) \leq s \) with \( i \). At last, let \( \Phi^1, \Phi^+, \Phi^0 \in \mathfrak{p}^\ast \) such that

\[
\rho^1(C_i(z))v_l = (\Phi^1)_i(z)v_l, \quad \rho^+(C_i(z))v_\infty = (\Phi^+)_i(z)v_\infty, \quad \rho^0(C_i(z))v_\infty = (\Phi^0)_i(z)v_\infty.
\]

The precise formulas for \( \Phi^1, \Phi^+, \Phi^0 \) will be given in the proof of Lemma 8.7.

**Lemma 8.3.** Let \( f \in \mathcal{B}_l^r \) and \( s \in I \). The sub-vector-superspace \( F_{l}^{-1}S_s(f) \subseteq V_l \) is a simple \( Y_q(\mathfrak{g}_s) \)-module isomorphic to \( ev^\ast \lambda_l L(\lambda_s(l, f); \mathfrak{g}_s) \otimes D \) where \( D \) is a sign module. Moreover, \( v[l, f_{ls}] \) is a highest \( l \)-weight vector of the \( Y_q(\mathfrak{g}_s) \)-module \( F_{l}^{-1}S_s(f) \).

**Proof.** Indeed, \( F_{l}^{-1}S_s(f) \) is the sub-vector-superspace of \( V_l \) spanned by the \( v[l, g] \) where \( g \in \mathcal{B}_l^r \) and \( g(i, j) = f(i, j) \) whenever \( g(i, j) > s \) or \( f(i, j) > s \). In view of (c), the map \( \pi_l \), and the correspondence between Young tableaux and Gelfand-Tsetlin patterns in Lemma 2.8, the lemma is evident. \( \square \)

**Lemma 8.4.** For \( s \in I \) and \( f \in \mathcal{B}^r \), the sub-vector-superspace \( S_s(f) \) of \( V_\infty \) is stable by \( \rho^x(Y_q(\mathfrak{g}_s)) \) whenever \( x \in \{+, b\} \).

**Proof.** By definition, \( S_s(f) \) is stable by the \( \rho^x(s_i^{(0)}_{ij}) \). (See also Lemma 8.7 below for more general statements.) It is enough to show that \( S_s(f) \) is stable by the \( \rho^x(s_i^{(n)}_{ij}) \) where \( n \in \mathbb{Z}_{\geq 0} \) and \( 1 \leq i, j \leq s \). Let us assume \( f \in \mathcal{B}^r \).

For \( k > u \), the preceding lemma says that \( F_{k}^{-1}S_s(f) \subseteq V_k \) is stable by the \( \rho^k(s_i^{(n)}_{ij}) \) with \( 1 \leq i, j \leq s \). It follows that (\( v < k < t(k) < u \))

\[
F_{u,t(k)}^{-1}\rho^u(s_i^{(n)}_{ij})F_{u,k}F_{k}^{-1}S_s(f) \subseteq F_{u,t(k)}^{-1}\rho^u(s_i^{(n)}_{ij})F_{u,1}^{-1}S_s(f) \subseteq F_{u,t(k)}^{-1}F_{t(k)}^{-1}S_s(f).
\]

On the other hand, we know from [4,4] the following asymptotic formula

\[
F_{u,t(k)}^{-1}B_{u,k} = A_k + q^{2u}B_k \quad \text{for } u > t(k)
\]
where $A_k, B_k : V_k \to V_{t(k)}$ are linear operators depending only on $i, j, n$ and inductively on $k$. As $u$ can be arbitrarily large, we must have

$$A_k F_{ik} S_s(f) \subseteq F_{ik} S_s(f), \quad B_k F_{ik} S_s(f) \subseteq F_{ik} S_s(f)$$

for all $k > v$. But this says exactly that $\rho^+(s_{ij}^{(n)})$ (resp. $\rho^b(s_{ij}^{(n)})$), being defined as inductive limit of the $A_k$ (resp. $A_k + b^2 B_k$), stabilizes $S_s(f)$, as desired. \hfill $\square$

Based on the proof of the above lemma, the following lemma is clear.

**Lemma 8.5.** Let $f \in \mathcal{B}(r)$ and let $s \in I$. Then $v[f, s] \in S_s(f)$ is a highest $\ell$-weight vector for the $Y_q(\mathfrak{g}_s)$-modules $(\rho^+, S_f)$ whenever $x \in \{+, b\}$. Moreover

$$\rho^b(s_{ii}(z))v[f, s] = v[f, s] \begin{cases} b(q^{-li} - zaq^l) & (1 \leq i \leq \min(r, s)), \\ 1 - zaq^2 & (\min(r, s) < i \leq s), \end{cases}$$

$$\rho^+(s_{ii}(z))v[f, s] = v[f, s] \begin{cases} q^{-li} - zaq^l & (1 \leq i \leq \min(r, s)), \\ 1 & (\min(r, s) < i \leq s). \end{cases}$$

Here as usual $l_i := \sharp\{j < 0 \mid f(i, j) > s\}$ for $1 \leq i \leq \min(r, s)$.

**Remark 8.6.** Lemmas 8.3, 8.5 can be rephrased as follows. The original inductive system $(V_k, F_k, t)$ admits a sub-inductive-system of vector superspaces of the form:

$$(ev)_{aq^2k} L(\lambda_s(k, f); \mathfrak{g}_s), F_k, t), \quad \lambda_s(k, f) = \sum_{i=1}^{\min(r, s)} (k - l_i) \epsilon_i.$$
\[(\Phi^+)_i(z) = \begin{cases} \prod_{j=1}^i (1 - za\theta_j^{-1}) & (i \leq r), \\ \prod_{j=1}^r (1 - za\theta_j^{-1}) & (i > r), \end{cases} \]

\[(\Phi^b)_i(z) = \begin{cases} b^i \prod_{j=1}^i (1 - za\theta_j^{-1}) & (i \leq r), \\ b^i \prod_{j=1}^r (1 - za\theta_j^{-1}) \times \prod_{j=r+1}^i (1 - za\theta_j^{-1} b^2) & (i > r), \end{cases} \]

we see from the construction of \(\rho^b\) in [5,1] that (by replacing \(q^k\) with \(b\) everywhere)

\[\rho^b(C_i(z)v[f]) = (\Phi^b\Phi^b)_i(z)_{|q^k \to b} v[f] = (\Phi^b\Phi^b)_i(z)v[f].\]

Finally, for \(\rho^+(C_i(z)v[f])\), from the above formulas of \(\Phi^k, \Phi^+\) we deduce that there exists a formal power series \(h_i(w) \in 1 + w\mathbb{C}[[w]]\) for \(i \in I\) such that for all \(k > t(l)\),

\[(\Phi^k)_i(z)(\Phi^k)_i(0)^{-1} = (\Phi^+)_i(z)h_i(zq^k), \quad (\Phi^+_i)_i(0) = 1\]

\[F_{k,l(t)}^{-1}\rho^k(C_i(z)\phi_i(l))v[l,f] = h_i(zq^k)(\Phi^+_\Phi^+_i)_i(z)(\Phi^+_\Phi^+_i)_i(0)^{-1}v[l,f].\]

From Gauss decomposition observe that \(C_i(z)\phi_i(l) \in \widehat{\mathcal{Y}}(\mathfrak{g})[[z]]\). Now from the construction of \(\rho^+\) in [4,4] we obtain (by taking \(q^k = 0\))

\[\rho^+(C_i(z)\phi_i(l))v[f] = (\Phi^+\Phi^+_i)_i(z)(\Phi^+\Phi^+_i)_i(0)^{-1}v[f].\]

It is therefore enough to compute \(\rho^+(C_i(0))v[f]\). For this, note that in \((\rho^i, V_i)\),

\[|v[l,f]|_{\mathbf{P}} = l \varpi_r + \sum_{i=1}^r \sum_{j=1}^l (\epsilon f_{i,j,k} - \epsilon_i).\]

Hence in \((\rho^+, \mathcal{V}_\infty)\), the \(\mathbb{Q}\)-degree of \(v[f]\) is given by the second term of the RHS. From the proof of Proposition 5.1 we get an explicit expression of the \((\Phi^+)_i(0)\):

\[\rho^+(C_i(0))v[f] = (\Phi^+_\Phi^+_i)_i(0)v[f] = (\Phi^+_\Phi^+_i)_i(0)v[f].\]

In consequence, \(\rho^+(C_i(z))v[f] = (\Phi^+_\Phi^+_i)_i(z)v[f]\), as desired. \(\square\)

We arrive at the following important consequence of the preceding lemmas: simplicity of \(S_{M+N}(f)\) for \(\rho^+\) and \(\rho^b\).

**Corollary 8.8.** Let \(1 \leq r \leq M\) and \(a, b \in \mathbb{C}^\times\). Then

\[\widetilde{\chi}_q(L^+_{r,a}) = \widetilde{\chi}_q(L_{r,a}(b)) = \lim_{k \to \infty} \widetilde{\chi}_q(W_{k,a,q^{2k}})\]

as formal power series in the \([A_{i,x}]^{-1}\) where \(i \in I_0\) and \(x \in aq^{2N+1}\). Furthermore, \(L^+_{r,a}\) is a simple \(Y_q(\mathfrak{g})\)-module. \(L_{r,a}(b)\) is simple if \(b \notin \pm q^{2N+1}\).

**Proof.** Let us explain firstly that the limit of normal \(q\)-characters above makes sense. For \(V\) a module in category \(\mathcal{O}\) with well-defined normalized character (Remark 6.3), set

\[\widetilde{\chi}_q(V) = \sum_{\Phi \in \Phi} d(\Phi, V)[\Phi] \in \mathcal{E}_\ell.\]
Then according to Proposition 7.3 for \( k \in \mathbb{Z}_{>0} \) and \( \Phi \in \hat{\mathfrak{H}} \), either \( d(\Phi, V_{k}) = 0 \), or \( d(\Phi, V_{k}) = 1 \), in which case \( \Phi \) is a product of the \( A_{i,x}^{+1} \) for \( i \in I_{0}, x \in aq^{2Z+1} \). Furthermore, Lemma 8.2 indicates that for \( l < k \)

\[
d(\Phi, V_{l}) \leq d(\Phi, V_{k}).
\]

Combining with the facts that \( \tilde{\chi}_{q}(V) \) is a refinement of \( \tilde{\chi}(V) \) and that \( \lim_{k \to \infty} \tilde{\chi}(V_{k}) \) exists (Equation (6.37)), we conclude that so does \( \lim_{k \to \infty} \tilde{\chi}_{q}(V_{k}) \), and it is a formal power series in the \([A_{i,x}]^{-1} \) where \( i \in I_{0}, x \in aq^{2Z+1} \) with coefficients 0 or 1.

Next the character formulas are immediate consequences of Lemma 8.7:

\[
\tilde{\chi}_{q}(L_{r,a}^{+}) = \tilde{\chi}_{q}(L_{r,a}(b)) = \sum_{f \in B(r)} [\Phi_{f}] = \lim_{k \to \infty} \tilde{\chi}_{q}(W_{r,a}^{(r)}).
\]

Thirdly we show that \( L_{r,a}^{+} \) is a simple \( Y_{q}(\mathfrak{g}) \)-module, imitating the proof of [HJ12, Theorem 6.1]. According to Proposition 6.7 \( L_{r,a}^{+} \) has a simple sub-quotient, denoted by \( S_{r,a}^{+} \), which contains a highest \( \ell \)-weight vector \( w \) such that \( s_{ii}(z)w = w \begin{cases} 1 - za & (\leq r) \\ 1 & (> r) \end{cases} \). Furthermore, for \( x \in \mathbb{C}^{\times} \), there exists a simple \( Y_{q}(\mathfrak{g}) \)-module \( S_{r,a}^{-} \) in category 0 containing a highest \( \ell \)-weight vector \( u_{x} \) such that \( s_{ii}(z)u_{x} = u_{x} \begin{cases} 1 & (\leq r) \\ 1 - zx & (> r) \end{cases} \). Let us show that \( \tilde{\chi}_{q}(L_{r,a}^{+}) = \tilde{\chi}_{q}(S_{r,a}^{+}) \). In other words, if \( l > 0 \) and \( d(\Phi, V_{l}) = 1 \), then \( \Phi \) must appear in \( \tilde{\chi}_{q}(S_{r,a}^{+}) \). Indeed, for all \( k > l \), we can find a one-dimensional \( Y_{q}(\mathfrak{g}) \)-module \( D \) such that:

(F) \( D \otimes V_{k} \) is a simple sub-quotient of \( S_{r,a}^{+} \otimes S_{r,a}^{-2k} \).

It follows from Remark 5.5 that \( \Phi \) must appear in \( \tilde{\chi}_{q}(S_{r,a}^{+}) \tilde{\chi}_{q}(S_{r,a}^{-2k}) \). In other words, there exists \( \Phi_{k}^{+} \) (resp. \( \Phi_{k}^{-} \)) in \( \tilde{\chi}_{q}(S_{r,a}^{+}) \) (resp. \( \tilde{\chi}_{q}(S_{r,a}^{-2k}) \)) such that

\[
\Phi = \Phi_{k}^{+}\Phi_{k}^{-}.
\]

Since \( \varpi(\Phi), \varpi(\Phi_{k}^{+}) \in -\mathbb{Q}_{\geq 0} \), the set \( \{ \Phi_{k}^{+} : k > l \} \) is finite. The pigeonhole principle indicates that there exists an infinite sequence \( (k_{n})_{n>0} \) of integers such that

\[
l < k_{1} < k_{2} < k_{3} < \cdots, \quad \Phi_{k_{1}}^{+} = \Phi_{k_{2}}^{-} = \Phi_{k_{3}}^{-} = \cdots.
\]

Lemma 8.2 below says that \( \Phi_{k_{1}}^{-} \in \sigma_{2}^{\mathfrak{H}} \), so does \( \Phi_{k_{1}}^{+} \Phi_{k_{1}}^{-} \). The later, as a product of the \( A_{i,x}^{+1} \), must be 1. So \( \Phi = \Phi_{k_{1}}^{+} \Phi_{k_{1}}^{-} \) appears in \( \tilde{\chi}_{q}(S_{r,a}^{+}) \), as desired.

At last we show that \( L_{r,a}(b) \) is simple provided that \( b \notin \pm q^{2Z} \). For this, let \( S(r,a,b) \) be a simple sub-quotient of \( L_{r,a}(b) \) containing a highest \( \ell \)-weight vector \( u \) such that \( s_{ii}(z)u = u \begin{cases} b - zab & (\leq r) \\ 1 - zab^{2} & (> r) \end{cases} \). Let \( l > 0 \). There exists a one-dimensional \( Y_{q}(\mathfrak{g}) \)-module \( D \) such that:

(F1) \( D \otimes V_{l} \) is a simple sub-quotient of \( S(r,a,b) \otimes S(r,ab^{2},b^{-1}q^{d}) \).

Assume \( d(\Phi, V_{l}) = 1 \). There exist a monomial \( \Phi_{1} \) in \( \tilde{\chi}_{q}(S(r,a,b)) \) and a monomial \( \Phi_{2} \) in \( \tilde{\chi}_{q}(S(r,ab^{2},b^{-1}q^{d})) \) such that \( \Phi = \Phi_{1}\Phi_{2} \). On the other hand, from the definition of \( S(r,a,b) \)
and from the normalized $q$-character of $L_{r,a}(b)$ we deduce that there exists $k > l$ such that

$$d(\Phi_1, W_{k,ag^{2k}}^{(r)}) = 1 = d(\Phi_2, W_{k,bg^{2k}}^{(r)}).$$

Note that $d(\Phi, W_{k,ag^{2k}}^{(r)}) = 1$ from Lemma 8.2. Now Lemma 7.3 forces $\Phi_2 = 1$. Hence $\Phi = \Phi_1$ appears in $\tilde{\chi}_q(S(r,a,b))$. It follows that $L_{r,a}(b) \cong S(r,a,b)$ is simple.

\fi
\medskip
\textbf{Lemma 8.9.} Let $W$ be a $Y_q(\mathfrak{g})$-module in category $\mathcal{O}$ admitting normalized character. Let $(k_n)_{n \in \mathbb{Z}_>0}$ be a strictly increasing sequence of integers. For $n \in \mathbb{Z}_{>0}$ take $W_n := \Phi^*_{q^{2kn}} W$. If $\Phi \in \Psi$ is such that $d(\Phi, W_n) > 0$ for all $n$, then $\Phi \in \sigma(\Psi)$.

\begin{proof}
Let $\theta : \Psi \rightarrow \Psi$ be the group isomorphism $(f_1(z)) \mapsto (f_1(zq))$. Then for $n \in \mathbb{Z}_{>0}$,

$$\tilde{\chi}_q(W_n) = \sum_{\Psi \in \Psi} d(\Psi, W)[\theta^{2kn} \Psi].$$

One can therefore find $\Phi_n \in \Psi$ such that $\Phi = \theta^{2kn} \Phi_n$ and $d(\Phi_n, W) > 0$. As $W$ is in category $\mathcal{O}$ and the $\varpi(\Phi_n)$ are the same, $\{\Phi_n \mid n > 0\}$ is a finite set. By pigeonhole principle, there exists a sub-sequence $(l_n)_{n \in \mathbb{Z}_{>0}}$ of strictly increasing integers such that $\Phi_{l_1} = \Phi_{l_n}$ for all $n > 0$. It follows that $\Phi_{l_1} = \theta^{2l_2 - 2l_1} \Phi_{l_1}$. This forces $\Phi_{l_1} \in \sigma(\Psi)$, so does $\Phi$.

\end{proof}

\textbf{Remark 8.10.} Corollary 8.8 says that $L_{r,a}(b)$ is simple if $b \in \pm q^{Z^{-r}}$. In the case $r = M$, it is possible to determine the $b \in \pm q^{Z^{-r}}$ making $L_{r,a}(b)$ non-simple as $L_{M,a}(b)$ is of dimension $2^{MN}$. In the case $r < M$, if $b \in \pm q^k$ with $k \geq 0$, then $L_{r,a}(b)$ has a finite-dimensional simple sub-quotient isomorphic to $W_{k,ag^{2k}}^{(r)}$ (up to tensor product by one-dimensional modules, with the convention that $W_{0,a}^{(r)}$ is the trivial module). Since $L_{r,a}(b)$ is infinite-dimensional, it is not simple. It remains to find integers $1 \leq s < r$ such that $L_{r,a}(\pm q^{-s})$ is not simple. Unfortunately, limited by Lemma 7.3 we still do not know the answer.

The proof of Corollary 8.8 applied perfectly if we replace $S_{M+N}(f)$ with more general $S_s(f)$, we obtain the following corollary, whose proof is omitted, on normalized $q$-character formulas and on simplicity of asymptotic modules.

\textbf{Corollary 8.11.} Let $f \in \mathcal{B}^{(r)}_I, s \in I$ and let $b \in \mathcal{B}^\infty$.

1. The $Y_q(\mathfrak{g}_s)$-modules $(\rho^+, S_s(f))$ and $(\rho^b, S_s(f))$ are in category $\mathcal{O}_s$ and

$$\tilde{\chi}_q^{s}(\rho^+, S_s(f)) = \tilde{\chi}_q^{s}(\rho^b, S_s(f)) = \lim_{k \rightarrow \infty} \tilde{\chi}_q^{s}(e^{q^s} L(\lambda_a(k,f); \mathfrak{g}_s))$$

as formal power series in the $[A_{i,x}]^{-1}$ with $1 \leq i < s$ and $x \in aq^{2Z+1}$.

2. The $Y_q(\mathfrak{g}_s)$-module $(\rho^+, S_s(f))$ is simple, while the $Y_q(\mathfrak{g}_s)$-module $(\rho^b, S_s(f))$ is simple if $b \notin \pm q^{Z}$. 

8.2.3. \textbf{End of proof of Theorem 8.4} when $r \leq M$. Assume $b \notin \pm q^{Z}$. Let $x \in \{+, b\}$. Consider the $Y_q(\mathfrak{g})$-module $(\rho^x, V_\infty)$. According to Corollary 8.11, its basis $\{v[f] \mid f \in \mathcal{B}^{(r)}\}$ together with the chains (one for each $f \in \mathcal{B}^{(r)}$)

$$v[f] \in S_1(f) \subseteq S_2(f) \subseteq S_3(f) \subseteq \cdots \subseteq S_{M+N-1}(f) \subseteq S_{M+N}(f) = V_\infty$$
satisfies \((B1)-(B2)\) in the definition of a Gelfand-Tsetlin basis in \[\text{8.1}\]. Observe that for \(f, g \in \mathcal{B}(r)\) and \(s \in I\), by definition of \(S_s(f)\), Lemma \[\text{8.5}\] and Corollary \[\text{8.11}\] the following statements are equivalent:

1. \(v[f] \in S_s(g)\);  
2. \(\{(i, j) \in Y^{(r)} \mid f(i, j) > s\} = \{(i, j) \in Y^{(r)} \mid g(i, j) > s\}\);  
3. \((\rho^x, S_s(f))\) and \((\rho^x, S_s(g))\) are isomorphic as \(Y_q(\mathfrak{g}_a)\)-modules;  
4. \(S_s(f) = S_s(g)\).

Now \((B3)\) on multiplicity-free property of the decomposition of \(\text{Res}^{Y_q(\mathfrak{g}_a)}_{Y_q(\mathfrak{g}_{s-1})}(\rho^x, S_s(f))\) into simple \(Y_q(\mathfrak{g}_{s-1})\)-modules, which are the \(S_{s-1}(g)\), is obvious.

\[\square\]

8.3. **Proof of Theorem \[\text{8.1}\]** when \(r > M\). The proof is almost identical to the case \(r \leq M\). We explain the main steps.

Take \(V_l := W_{l,a}^{(r)}\) which is a tensor product of \(ev_s^*L(l\varpi_r)\) with a sign module. We borrow the notations from \[\text{8.2}\] so that \(v_k \in V_l, v_\infty \in V_\infty, F_{k,l}, F_k, \rho^k, \rho^+, \Phi^k, \Phi^+\) and the strictly increasing function \(t\) will be used freely. In the present case,

\[
(\Phi^k)_i(z) = \begin{cases} 
\prod_{j=1}^i (1 - za\theta_j^{-1})^{-d_j} & (i \leq r), \\
\prod_{j=1}^i (1 - za\theta_j^{-1})^{-d_j} \times \prod_{j=r+1}^i (q^k - za\theta_j^{-1}q^{-k})^{-1} & (i > r),
\end{cases}
\]

\[
(\Phi^b)_i(z) = \begin{cases} 
\prod_{j=1}^i (1 - za\theta_j^{-1})^{-d_j} & (i \leq r), \\
\prod_{j=1}^i (1 - za\theta_j^{-1})^{-d_j} \times \prod_{j=r+1}^i (b - za\theta_j^{-1}b^{-1})^{-1} & (i > r),
\end{cases}
\]

\[
(\Phi^+)_i(z) = \begin{cases} 
\prod_{j=1}^i (1 - za\theta_j^{-1})^{-d_j} & (i \leq r), \\
\prod_{j=1}^i (1 - za\theta_j^{-1})^{-d_j} & (i > r).
\end{cases}
\]

Set \(Y^{(r)}\) to be the subset of \(\mathbb{Z}^2\) consisting of \((i, j)\) such that \(1 \leq j \leq M + N - r\) and \(i > 0\), and define \(B^{(r)}\) to be the set of functions \(f : Y^{(r)} \to I\) satisfying 

\( (T1)-(T3) \) in Definition \[\text{2.2}\] \(\) and, for all \(1 \leq j \leq M + N - r\), there exists \(i > 0\) such that \(f(i, j) = r + j\). Let \(B_k^{(r)}\) be the subset of \(B^{(r)}\) consisting of \(f\) such that \(f(k + 1, j) = r + j\) for all \(1 \leq j \leq M + N - r\).

For \(s \in I\) and \(f \in B_l^{(r)}\), the subset \(Y_s(l, f)\) of \(Y^{(r)}\) consisting of \((i, j)\) such that \(i \leq l\) and \(f(i, j) \leq s\) is easily seen to be a \(\mathfrak{g}_a\)-Young diagram. Set \(\mu_s(l, f) := \bigwedge_{M_s,N_s}(Y_s(l, f))\). (See Remark \[\text{2.3}\]). Let \(\lambda_s(l, f) \in P_{M_s,N_s}\) be the highest weight of the simple \(\mathfrak{u}_q(\mathfrak{g}_a)\)-module \(L(\mu_s(l, f))\). Then we have the following observation.

1. If \(s \leq r\), then \(\lambda_s(l, f) = \lambda_s(k, f)\) whenever \(l < k\) and \(f \in B_l^{(r)}\).
2. If \(s > r\), then \(\lambda_s(k, f) = \lambda_s(l, f) - (k - l) \sum_{j=r+1}^s \epsilon_j\) whenever \(l < k\) and \(f \in B_l^{(r)}\).

Set \(f_0 \in B^{(r)}\) to be such that \(f(i, j) = j + r\) for all \(1 \leq j \leq M + N - r\).

According to \[\text{2.5}\] for all \(l > 0\), there is a dual Gelfand-Tsetlin basis \(\{v_{l,f} \mid f \in B_l^{(r)}\}\) of \(V_l\) such that for all \(s \in I\) and \(f \in B_l^{(r)}\) the vector \(v_{l,f}\) is contained in a simple sub-\(Y_q(\mathfrak{g}_a)\)-module of \(V_l\) of the form:

\[ev_s^*L(\lambda_s(l, f); \mathfrak{g}_a) \otimes D \cong ev_s^*L(\mu_s(l, f); \mathfrak{g}_a) \otimes D\]

where \(D\) is a sign module (irrelevant to us). Now from Proposition \[\text{7.3}\] Corollary \[\text{7.5}\] their proofs, and the construction of \(F_{k,l}\) in \[\text{1.5}\] we conclude that
Lemma 8.12. For \( l < k, f \in B^{(r)}_l \) we have \( F_{k,l}v[l, f] \in \mathbb{C}^X v_{k,f} \).

Now we can choose by induction on \( k \) a basis \( \{v[k, f] \mid f \in B^{(r)}_k\} \) of \( V_k \) such that

(a) \( F_{k,l}v[l, f] = v[k, f] \) whenever \( k > l \) and \( f \in B^{(r)}_k \);
(b) \( v[l, f_0] = v_l \) for all \( l > 0 \);
(c) \( v[l, f] \in \mathbb{C}^X v_{l,f} \) for \( f \in B^{(r)}_l \).

Define \( v[f] := F_kv[k, f] \in V_{\infty} \) whenever \( f \in B^{(r)}_k \). As before, for \( f \in B^{(r)} \) and \( s \in I \), let \( S_s(f) \) be the sub-vector-superspace of \( V_{\infty} \) spanned by the \( v[g] \) where \( g \in B^{(r)} \) verifies \( g(i, j) = f(i, j) \) whenever one of them is bigger than \( s \). Now Corollary 8.8 stays the same in our situation, as the proof of which requires only the multiplicity-free property of normalized \( q \)-character of evaluation modules. As in \( \S \) 2.3, we conclude the proof of Theorem 8.1 in the case \( r > M \). \( \square \)

Remark 8.13. Remark 8.6 remains true in our situation. Let \( s \in I \) and let \( f \in B^{(r)}_l \). Write

\[
\lambda_s(l, f) = \sum_{i=1}^s x_i \varepsilon_i.
\]

Let \( v[f]_{\downarrow s} \in S_s(f) \) be a highest \( \ell \)-weight vector for both \( (\rho^+, S_s(f)) \) and \( (\rho^b, S_s(f)) \). If \( s \leq r \), then \( S_s(f) \) is finite-dimensional and for \( 1 \leq i \leq s \)

\[
\rho^+(s_{ii}(z))v[f]_{\downarrow s} = \rho^b(s_{ii}(z))v[f]_{\downarrow s} = (q_{i}^{x_i} - zaq_{i}^{-x_i})v[f]_{\downarrow s}.
\]

If \( s > r \), then \( S_s(f) \) is infinite-dimensional and

\[
\rho^+(s_{ii}(z))v[f]_{\downarrow s} = v[f]_{\downarrow s} \begin{cases} \frac{q_{i}^{x_i} - zaq_{i}^{-x_i}}{q^{-x_i-l}} & (1 \leq i \leq r), \\ \frac{q_{i}^{x_i} - zaq_{i}^{-x_i}}{q^{-x_i-l} - zab^{-1}q^{l+x_i}} & (r < i \leq s). \end{cases}
\]

\[
\rho^b(s_{ii}(z))v[f]_{\downarrow s} = v[f]_{\downarrow s} \begin{cases} \frac{q_{i}^{x_i} - zaq_{i}^{-x_i}}{q^{-x_i-l}} & (1 \leq i \leq r), \\ \frac{q_{i}^{x_i} - zaq_{i}^{-x_i}}{q^{-x_i-l} - zab^{-1}q^{l+x_i}} & (r < i \leq s). \end{cases}
\]

In this case, \( (\rho^+, S_s(f)) \) and \( (\rho^b, S_s(f)) \) can be seen as the asymptotic modules obtained from the following sub-inductive-system

\[
(ev^{s}_a L(\lambda_s(k, f); g_s) \otimes D_k, F_{k,l})
\]

of the original one \( (V_k, F_{k,l}) \) by carrying out the asymptotic constructions in \( \S \) and \( \S \). Here the \( D_k \) are some properly defined sign modules.

8.4. Consequences on normalized \( q \)-character. Let us go back to \( \S \). The following corollary is now direct in view of Corollary 8.8 (which holds for all \( r \in I_0 \)) applied to \( S_{M+N}(f) \) and Examples 5.6

Corollary 8.14. Let \( f = (f_i : i \in I_0) \in R \). Assume that \( f_i(\infty) \neq 0 \) for all \( i \in I_0 \). Then \( \tilde{\chi}_q(V(f)) \) is a formal power series in the \([A_i, x]^{-1}\) where \( i \in I_0 \) and \( x \in \mathbb{C}^X \). In particular, if \( S \) is a finite-dimensional simple \( U_q(\mathfrak{g}) \)-module, then the normalized \( q \)-character \( \tilde{\chi}_q(S) \) is a polynomial in the \([A_i, x]^{-1}\).
Let \( r \in I_0 \) be such that \( M < r < M + N \). Consider the \( U_q(\hat{\mathfrak{g}}) \)-module \( L_{r,a}(b) \). By pulling it back with respect to \( f_{J,I} : U_q(\hat{\mathfrak{g}}') \to U_q(\hat{\mathfrak{g}}) \), we get a \( U_q(\hat{\mathfrak{g}}') \)-module \( f_{J,I}^* L_{r,a}(b) \) such that

\[
s_{ii,J}(z)f_{J,I}^* v = f_{J,I}^* v \begin{cases} 
  b - zab^{-1} & (i \leq M + N - r), \\
  1 - za & (i > M + N - r)
\end{cases}
\]

where \( v \in L_{r,a}(b) \) is a highest \( \ell \)-weight vector of \( L_{r,a}(b) \). By comparing the normalized characters and highest \( \ell \)-weights of \( f_{J,I}^* L_{r,a}(b) \) and \( L_{M+N-r,ab^{-2},J}(b) \) we conclude that

**Corollary 8.15.** Let \( M < r < M + N \) and \( a, b \in \mathbb{C}^\times \). Assume that \( b \notin \pm q^{2\mathbb{Z}} - M - N + r \). Then the \( U_q(\hat{\mathfrak{g}}) \)-module \( L_{r,a}(b) \) is simple.

Let us rephrase Lemma [8,8] and Corollary [8,15] in another way. Fix \( i \in I_0 \) and \( a \in \mathbb{C}^\times \). For \( b \in \mathbb{C} \setminus \{a\} \), set \( \theta_{i,a,b} \in \mathbb{R} \) to be

\[
(\theta_{i,a,b})_j = 1 \quad (j \neq i), \quad (\theta_{i,a,b})_i = \frac{1 - za}{1 - zb}
\]

Then whenever \( b \) is generic (for example \( b \notin q^{2\mathbb{Z}} \)), we have

\[
\tilde{\chi}_q(V(\theta_{i,a,b})) = \tilde{\chi}_q(V(\theta_{i,a,0})).
\]

In other words, when \( b \) is generic, \( \tilde{\chi}_q(V(\theta_{i,a,b})) \) is independent of \( b \). Note that a similar phenomenon happens for more general rational functions than the \( \frac{1 - za}{1 - zb} \), as we have observed in Equation (6.39) concerning the \( q \)-Yangian \( Y_q(\mathfrak{gl}(1,1)) \).

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