Exact Moderate and Large Deviations for Linear Processes

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\textbf{Abbreviated Title:} Exact Deviations for Linear Processes

\textbf{Abstract}

Large and moderate deviation probabilities play an important role in many applied areas, such as insurance and risk analysis. This paper studies the exact moderate and large deviation asymptotics in non-logarithmic form for linear processes with independent innovations. The linear processes we analyze are general and therefore they include the long memory case. We give an asymptotic representation for probability of the tail of the normalized sums and specify the zones in which it can be approximated either by a standard normal distribution or by the marginal distribution of the innovation process. The results are then applied to regression estimates, moving averages, fractionally integrated processes, linear processes with regularly varying exponents and functions of linear processes. We also consider the computation of value at risk and expected shortfall, fundamental quantities in risk theory and finance.

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1 Introduction and notations

Let \((\xi_i)_{i \in \mathbb{Z}}\) be a sequence of independent and identically distributed centered random variables with finite second moment and \(c_{ni}\) a sequence of constants. This paper focuses on the moderate and large deviations in non-logarithmic form for the linear process of the form

\[ S_n = \sum_{i=1}^{k_n} c_{ni} \xi_i. \]  

(1)

This class of linear processes is versatile enough to help analyzing regression estimates, moving averages that include long memory processes, linear processes with regularly varying coefficients and fractionally integrated processes.

Our goal is to find an asymptotic representation for the tail probabilities of the normalized sums defined by (1). Estimations of deviation probabilities occur in a natural way in many applied areas including insurance and risk analysis.

Specifically, we aim to find a function \(N_n(x)\) such that, as \(n \to \infty\),

\[ \frac{\mathbb{P}(S_n \geq x \sigma_n)}{N_n(x)} = 1 + o(1), \]

where \(\sigma_n^2 = \|S_n\|^2 = \mathbb{E} \xi_1^2 \sum_{i=1}^{k_n} c_{ni}^2\).

(2)

If \(x \geq 0\) is fixed, then (2) becomes the well-known central limit theorem by letting \(N_n(x) = 1 - \Phi(x)\), where \(\Phi(x)\) is the standard normal distribution function. In this paper we call \(\mathbb{P}(S_n/\sigma_n \geq x)\) the moderate or large deviation probabilities depending on the speed of convergence \(x = x_n \to \infty\). These tail probabilities of rare events can be very small. Here we call (2) the exact approximation, which is more accurate than the logarithmic version

\[ \frac{\log \mathbb{P}(S_n/\sigma_n \geq x)}{\log N_n(x)} = 1 + o(1), \]

(3)

which is often used in the literature in the context of large or moderate deviation. For example, suppose \(\mathbb{P}(S_n/\sigma_n \geq x) = 10^{-4}\) and \(N_n(x) = 10^{-5}\); then their logarithmic ratio is 0.8, which does not appear to be very different from 1, while the ratio for the exact version (2) is as big as 10. A multiplicative factor of this order can cause substantially different industrial standards in designing projects that can survive natural disasters. The logarithmic version (3) is incapable of effectively characterizing the differences between the tail probabilities.

As early as 1929, Khinchin considered the problem of moderate and large deviation probabilities in non-logarithmic form for independent Bernoulli random variables. The first large deviation probability result appeared in S. Nagaev (1965). A. Nagaev (1969) studied large deviation probabilities of i.i.d. random variables with regularly varying tails. Mikosch and A. Nagaev (1998) applied the large deviation probabilities for heavy-tailed random variables to insurance mathematics. The review work on this topic can be found in S. Nagaev (1979).
and Rozovski (1993). Rubin and Sethuraman (1965), Slastaikov (1978) and Frolov (2005) considered the moderate or large deviations for arrays of independent random variables. S. Nagaev (1979) presented the following very useful result: in (1) assume \( k_n = n, \ c_{ni} = 1, \) and that \( \xi_i \) has a regularly varying right tail. i.e.

\[
P(\xi_0 \geq x) = \frac{h(x)}{x^t} \text{ as } x \to \infty \text{ for some } t > 2,
\]

where \( h(x) \) is a slowly varying function (Bingham, Goldie and Teugels, 1987).

Namely, \( \lim_{x \to \infty} h(\lambda x)/h(x) = 1 \) for all \( \lambda > 0. \) If in addition, for some \( p > 2, \) \( \xi_0 \) has absolute moment of order \( p, \) then

\[
P(n \sum_{i=1}^{n} \xi_i \geq x\sigma_n) = (1 - \Phi(x))(1 + o(1)) + nP(\xi_0 \geq x\sigma_n)(1 + o(1)) \quad (5)
\]

for \( n \to \infty \) and \( x \geq 1. \) Note that (5) implies (2) with

\[
N_n(x) = (1 - \Phi(x)) + nP(\xi_0 \geq x\sigma_n). \quad (6)
\]

Hence if \( 1 - \Phi(x) = o(nP(\xi_0 \geq x\sigma_n)) \) (resp. \( nP(\xi_0 \geq x\sigma_n) = o(1 - \Phi(x)) \)), then in (2) we can also choose \( N_n(x) = 1 - \Phi(x) \) (resp. \( N_n(x) = nP(\xi_0 \geq x\sigma_n) \)).

The study of moderate and large deviation probabilities in non-logarithmic form for dependent random variables is still in its initial stage. Ghosh (1974) considered moderate deviations for \( m \)-dependent random variables. Chen (2001) obtained a moderate deviation result for Markov processes. Grama (1997) and Grama and Haeusler (2006) investigated the martingale case. Wu and Zhao (2008) studied moderate deviations for stationary processes which applies to many time series models. However the result in the latter paper can only be applied to linear processes with short memory and their transformations.

For analyzing linear processes with long memory and for obtaining other interesting applications, we study processes of type (1). Under mild conditions on the coefficients, we shall point out the zones in which the deviation probabilities can be approximated either by a standard normal distribution or by using the distribution of \( \xi_0. \) Our main result is that (5) holds in our case with

\[
N_n(x) = (1 - \Phi(x)) + \sum_{i=1}^{k_n} P(c_{ni}\xi_0 \geq x\sigma_n).
\]

The paper has the following structure. Section 2 presents a general moderate and large deviation result and various applications. Section 3 illustrates the results of a numerical study. In Section 4 we prove the results. In the Appendix we give some auxiliary results and we also mention some known facts needed for the proofs.

Before stating our results we introduce the notations that will be used throughout this paper: \( a_n \sim b_n \) means that \( \lim_{n \to \infty} a_n/b_n = 1, \) \( a_n = O(b_n) \) and also \( a_n \ll b_n \) means \( \limsup_{n \to \infty} a_n/b_n < \infty; \) \( a_n = o(b_n) \) if \( \lim_{n \to \infty} a_n/b_n = 0. \) By \( \|X\|_p \) we denote \( (E|X|^p)^{1/p}. \) The notation \( l(\cdot), \ h(\cdot) \) and \( \ell(\cdot) \) denote slowly varying functions. By convention \( 0/0 \) is interpreted as 0.
2 Main Results

Throughout the paper, we assume that:

Condition A. \( (\xi_i)_{i \in \mathbb{Z}} \) are i.i.d. centered random variables with finite second moment, \( \sigma^2 = \mathbb{E} \xi_0^2 \).

2.1 General linear processes

Our first results apply to general linear processes of type (1) with i.i.d. innovations. For \( c_n \), \( n > 0 \) and \( t > 0 \), we define

\[
B_{nt} = \sum_{i=1}^{k_n} c_i^t,
\]

and

\[
\sigma_n^2 = \text{var}(S_n) = B_{n2} \mathbb{E} \xi_0^2,
\]

\[
D_{nt} = B_{nt}^{-t/2} B_{nt}.
\]

The basic assumption in all our results is the uniform asymptotic negligibility of the variance of individual summands, namely

\[
\max_{1 \leq i \leq k_n} c_i^t / \sigma_n^2 \to 0.
\]

Our first theorem extends Nagaev’s result in (5) to general linear processes.

**Theorem 2.1** Assume that \( (\xi_i)_{i \in \mathbb{Z}} \) satisfies Condition A, and for a certain \( t > 2 \) it satisfies the right tail condition (4). Moreover, for a certain \( p > 2 \), \( \| \xi_0 \|_p < \infty \). Assume also that \( c_n \), \( n > 0 \) and (10) is satisfied. Let \( (x_n)_{n \geq 1} \) be any sequence such that for some \( c > 0 \) we have \( x_n \geq c \) for all \( n \). Then, as \( n \to \infty \),

\[
P(S_n \geq x_n \sigma_n) = (1 + o(1)) \sum_{i=1}^{k_n} P(c_i \xi_0 \geq x_n \sigma_n) + (1 - \Phi(x_n))(1 + o(1)).
\]

**Remark 2.1** To be precise, in relation (11) as well as in (12) and (13) below, by \( o(1) \) we understand a function which depends on \( x_n \) and on the underlying distribution, with the property that its limit as \( n \to \infty \) is zero. Each \( o(1) \) may represent a different function. The sequence \( (x_n)_{n \geq 1} \) may be bounded or may converge to infinity.

**Corollary 2.1** Under the conditions of Theorem 2.1 for \( x_n \geq a(\ln D_{nt}^{-1})^{1/2} \) with \( a > 2^{1/2} \) we have

\[
P(S_n \geq x_n \sigma_n) = (1 + o(1)) \sum_{i=1}^{k_n} P(c_i \xi_0 \geq x_n \sigma_n) \text{ as } n \to \infty.
\]
On the other hand, if \( 0 < x_n \leq b(\ln D_{nt}^{-1})^{1/2} \) with \( b < 2^{1/2} \), we have
\[
\mathbb{P}(S_n \geq x_n \sigma_n) = (1 - \Phi(x_n)) + o(1) \quad \text{as } n \to \infty.
\] (13)

**Remark 2.2** Notice that (12) and (13) assert different approximations for the tail probability \( \mathbb{P}(S_n \geq x \sigma_n) \): moderate behavior for \( x = x_n \) smaller than a threshold, when we can approximate this probability by using a normal distribution. On the other hand we have a large deviation type of behavior for \( x \) larger than another threshold. The behavior at the boundary \( \sqrt{2}(\ln D_{nt}^{-1})^{1/2} \) is more subtle and it depends on the slowly varying function \( h(\cdot) \). For the special case in which \( \lim_{x \to \infty} h(x) = h_0 > 0 \), we have
\[
\frac{\mathbb{P}(S_n \geq x \sigma_n)}{N_n(x)} = 1 + o(1), \quad \text{where } N_n(x) = (1 - \Phi(x)) + \frac{h_0}{(\sigma x)^t} D_{nt}.
\] (14)

If \( x \geq a(\ln D_{nt}^{-1})^{1/2} \) with \( a > 2^{1/2} \), then \( N_n(x) \sim h_0 D_{nt}/(\sigma x)^t \).

The proofs of these results are based on a separate study of the behaviors of type (12) or (13), which is of independent interest. As a matter of fact, we shall see in the next two theorems that a result similar to (12) holds without the assumption of the finite moment of order \( p > 2 \) while the moderate deviation (13) does not require a regularly varying right tail.

**Theorem 2.2** Assume that \((\xi_i)_{i \in \mathbb{Z}}\) satisfies Condition A, and for a certain \( t > 2 \) it satisfies (4). Let \( c_n > 0 \) be a sequence of constants satisfying (10). Then, for any sequence \( x_n \geq C^t(\ln D_{nt}^{-1})^{1/2} \) with \( C_t > e^{1/2}(t + 2)/\sqrt{2} \) the large deviation result (12) holds.

As a counterpart to this result we shall formulate now the moderate deviation bound.

**Theorem 2.3** Assume that \((\xi_i)_{i \in \mathbb{Z}}\) satisfies Condition A and for a certain \( p > 2 \), \( \|\xi_0\|_p < \infty \). Assume that (10) is satisfied. If \( x_n^2 \leq 2 \ln(D_{nt}^{-1}) \) then the moderate deviation result (13) holds.

### 2.2 Applications to linear regression estimates

Many statistical procedures, such as estimation of regression coefficients, produce linear statistics of type (1). See for instance Chapter 9 in Beran (1994), for the case of parametric regression, or the paper by Robinson (1997), where kernel estimators are used for nonparametric regression. Here we consider the simple parametric regression model
\[ Y_i = \beta \alpha_i + \xi_i, \]
where \( \xi_i \) are i.i.d. centered errors with \( \mathbb{E}\xi_i^2 = a^2 \), \( (\alpha_i) \) is a sequence of positive real numbers and \( \beta \) is the parameter of interest. The least squares estimator \( \hat{\beta}_n \) of \( \beta \), based on a sample of size \( n \), satisfies
\[
S_n := \hat{\beta}_n - \beta = \frac{1}{\sum_{i=1}^n \alpha_i^2} \sum_{i=1}^n \alpha_i \xi_i,
\] (15)
so, the representation of type (1) holds with \(c_{ni} = \alpha_i / (\sum_{i=1}^{n} \alpha_i^2)\). Denote \(A_{nt} = \sum_{i=1}^{n} \alpha_i^2\). Notice that \(\text{var}(S_n) = \sigma^2 / A_{nt}\). Assume

\[
\lim_{n \to \infty} A_{n2}^{-1} \max_{1 \leq i \leq n} \alpha_i^2 = 0.
\]  \hspace{1cm} (16)

As an immediate consequence of Theorem 2.1, we obtain:

**Corollary 2.2**  
(i) Assume that \((\xi_i)_{i \in \mathbb{Z}}\) and \(x = x_n\) satisfies the conditions in Theorem 2.1. Under assumption (16), we have

\[
P(\hat{\beta}_n - \beta \geq x\sigma / A_{n2}^{1/2}) = (1 + o(1)) \sum_{i=1}^{n} P(\xi_i \geq x\sigma A_{n2}^{1/2} / \alpha_i) + (1 + o(1))(1 - \Phi(x)).
\]

(ii) If \(x > 0\) and \(x^2 \leq 2 \ln(A_{n2}^{1/2} / A_{nt})\), under the conditions in Theorem 2.1, we have

\[
P(\hat{\beta}_n - \beta \geq x\sigma / A_{n2}^{1/2}) = (1 + o(1))(1 - \Phi(x)).
\]

(iii) If \(x > 0\) and \(x^2 \geq C_t^2 \ln(A_{n2}^{1/2} / A_{nt})\) with \(C_t^2 > 2\), under the conditions in Theorem 2.1, then

\[
P(\hat{\beta}_n - \beta \geq x\sigma / A_{n2}^{1/2}) = (1 + o(1)) \sum_{i=1}^{n} P(\xi_i \geq x\sigma A_{n2}^{1/2} / \alpha_i).
\]

Similar results as in Theorems 2.2 and 2.3 can also be easily formulated. Theorems 2.1, 2.2 and 2.3 are also applicable to the nonlinear regression model \(y_i = g(x_i) + \xi_i, 1 \leq i \leq n\), where \(g(x)\) is an unknown function and \(\xi_i\) is the noise. Let \(x_i\) be the deterministic design points. Then the Nadaraya-Watson estimate \(\hat{g}_n\) satisfies

\[
\hat{g}_n(x) - \mathbb{E}\hat{g}_n(x) = \sum_{i=1}^{n} c_{ni}(x)\xi_i
\]

where, letting \(K\) be a kernel function and \(h_n\) be bandwidths, the weights

\[
c_{ni}(x) = K \left( x_i - x \right) / \sum_{i=1}^{n} K \left( x_i - x \right).
\]

Therefore it is of the type (1).

### 2.3 Application to moving averages

We now consider the sum \(S_n = \sum_{k=1}^{n} X_k\), where

\[
X_k = \sum_{j=-\infty}^{\infty} a_{k-j}\xi_j.
\]  \hspace{1cm} (17)
We assume that \( \sum_{i \in \mathbb{Z}} a_i^2 < \infty \), which is the necessary and sufficient condition for the existence of \( X_1 \). Observe that \( S_n = \sum_{i=-\infty}^{\infty} b_{ni} \xi_i \) is of form (1) with
\[
 b_{ni} = a_{1-i} + \cdots + a_{n-i} \tag{18}
\]
and \( k_n = \infty \). Assume \( b_{ni} > 0 \) for all \( i \) and let
\[
 U_{nt} = \left( \sum_i \hat{b}_{ni}^2 \right)^{-t/2} \sum_i \hat{b}_{ni}^t. \tag{19}
\]
In the corollary below, this quantity will replace \( D_{nt} \) from definition (9) and \( b_{ni} \) will replace the \( c_{ni} \) in Subsection 2.1. Define \( \sigma_n^2 = \mathbb{E} \xi_0^2 \sum_i b_{ni}^2 \). We know from Peligrad and Utev (1997) that under the assumption \( \sigma_n^2 \to \infty \) we have
\[
 \sigma_n^{-2} \sup_i b_{ni}^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{20}
\]
Therefore condition (10) is automatically satisfied. As a corollary of Theorems 2.1, 2.2 and 2.3 we obtain:

**Corollary 2.3** Assume that \( (X_n)_{n \geq 1} \) is defined by (17) and \( \sigma_n^2 \to \infty \).

(i) Assume that \( (\xi_i)_{i \in \mathbb{Z}} \) and \( x_n \) satisfy the conditions of Theorem 2.1 and \( b_{ni} > 0 \). Then (11) holds. Corollary 2.1 is also valid for the partial sum of (17).

(ii) Let \( (\xi_i)_{i \in \mathbb{Z}} \) be as in Theorem 2.2. Assume \( b_{ni} > 0 \). Then the large deviation result (12) holds for the sequence \( x_n \geq C_1 (\ln U_{nt}^{-1})^{1/2} \) with \( C_1 > e^{t/2} (t + 2)/\sqrt{2} \).

(iii) Assume \( (\xi_i)_{i \in \mathbb{Z}} \) is as in Theorem 2.3. Then the moderate deviation result (13) holds for \( x_n^2 \leq 2 \ln(U_{np}^{-1}) \).

Note that this corollary applies to general linear processes including the long memory processes with \( \sum_i |a_i| = \infty \). Asymptotic properties for long memory processes can be quite different from those of processes with short memory, partially because the variance of the partial sum goes to infinity at an order different than \( n \); see for example, Ho and Hsing (1997), Robinson (2003), Doukhan, Oppenheim and Taqqu (2003) among others. Hall (1992) gave a Berry-Esseen bound for the convergence rate in the central limit theorem.

We shall apply now this corollary to the important particular case of causal long-memory processes with
\[
a_i = \ell(i + 1)(1 + i)^{-r}, \quad i \geq 0, \quad \text{with} \quad 1/2 < r < 1, \quad \text{and} \quad a_i = 0 \quad \text{in rest}. \tag{21}
\]
Here \( \ell(\cdot) \) is a slowly varying function where the results can be given in a more precise form. Notice that in this particular case
\[
 X_k = \sum_{j=-\infty}^{k} a_{k-j} \xi_j. \tag{22}
\]
Let \( a_0 = 1 \). This case of long memory linear processes covers the well-known fractional ARIMA processes (cf. Granger and Joyeux; 1980, Hosking, 1981),
which play an important role in financial time series modeling and application. As a special case, let $0 < d < 1/2$ and $B$ be the backward shift operator with $B\varepsilon_k = \varepsilon_{k-1}$ and consider

$$X_k = (1 - B)^{-d}\xi_k = \sum_{i \geq 0} a_i \xi_{k-i}, \text{ where } a_i = \frac{\Gamma(i + d)}{\Gamma(d)\Gamma(i + 1)}.$$  

For this example we have $\lim_{n \to \infty} a_n/n^{d-1} = 1/\Gamma(d)$. Note that these processes have long memory because $\sum_{j \geq 0} |a_j| = \infty$.

**Corollary 2.4** Assume (21). If $(\xi_i)_{i \in \mathbb{Z}}$ satisfies the conditions of Theorem 2.1 then (11) holds. In particular (12) holds for $x_n > c_1(\ln n)^{1/2}$ with $c_1 > (t-2)^{1/2}$, while (13) holds, provided $0 < x_n < c_2(\ln n)^{1/2}$ with $c_2 < (t-2)^{1/2}$.

2.4 Application to risk measures

In risk theory and finance, value at risk (VaR) and expected shortfall (ES) play a fundamental role; see Jorion (2006), Holton (2003), McNeil et al (2005), Acerbi and Tasche (2002) among others. Mathematically, they are equivalent to quantiles and tail conditional expectations. In practice one is most interested in their extremal behavior which corresponds to tail quantiles. Despite their importance, however, their computation can be quite difficult and the related asymptotic justification is far from being trivial.

Here we shall apply Theorem 2.1 and provide approximate formulae for extremal quantiles and tail conditional expectations for $S_n$ defined by (1). Under the assumption $\lim_{x \to \infty} h(x) = h_0 > 0$, by (14) and Theorem 2.1

$$\mathbb{P}(S_n \geq x\sigma_n) = (1 + o(1))\frac{h_0}{(\sigma x)^t}D_{nt} + (1 - \Phi(x))(1 + o(1)).$$

Given the tail probability $\alpha \in (0, 1)$, let $q_{\alpha,n}$ be the upper $\alpha$-th quantile of $S_n$. Namely $\mathbb{P}(S_n \geq q_{\alpha,n}) = \alpha$. Elementary calculations show that $q_{\alpha,n}$ can be approximated by $x_{\alpha}\sigma_n$ in the sense that $\lim_{n \to \infty} x_{\alpha}\sigma_n/q_{\alpha,n} = 1$, where $x = x_{\alpha}$ is the solution to the equation

$$\frac{h_0}{(\sigma x)^t}D_{nt} + (1 - \Phi(x)) = \alpha.$$

In particular, if $\alpha \leq h_0 D_{nt}((a\sigma)^2 \ln D_{nt})^{-1/2}$ with $a > 2^{1/2}$, then, by Corollary 2.1, we can approximate $q_{\alpha,n}$ by $\sigma^{-1}(h_0 D_{nt}/\alpha)^{1/t}\sigma_n = \sigma^{-1}(B_{nt}h_0/\alpha)^{1/t}$. The
approximation is understood in the sense that \( \sigma^{-1}(B_n t_{0}/\alpha)^{1/t}/q_{\alpha,n} \rightarrow 1 \) as \( n \rightarrow \infty \), and the tail conditional expectation or expected shortfall is computed as

\[
\mathbb{E}(S_{n}|S_{n} \geq q_{\alpha,n}) = \frac{q_{\alpha,n}\mathbb{P}(S_{n} \geq q_{\alpha,n}) + \int_{q_{\alpha,n}}^{\infty} \mathbb{P}(S_{n} \geq w)dw}{\mathbb{P}(S_{n} \geq q_{\alpha,n})} \\
\approx q_{\alpha,n} + \frac{q_{\alpha,n}}{t-1} = \frac{t q_{\alpha,n}}{t-1} \sim \sigma^{-1} B_n^{1/t} t(h_0/\alpha)^{1/t}.
\]

We emphasize that, without the exact moderate deviation principle in Corollary 2.5, the validity of the above equivalence cannot be guaranteed. To the best of our knowledge, our example might be the only case that one can obtain explicit asymptotic expressions for VaR and ES for sums of dependent random variables.

### 2.5 Functionals of linear processes

In this subsection we shall use the result from (ii) of Corollary 2.5 to study the moderate deviation for nonlinear transformations of linear processes. Let \( K \) be a transformation which is measurable and \( \mathbb{E}K(X_0) = 0 \). Let

\[
H_n = \sum_{i=1}^{n} K(X_i) \text{ where } X_i \text{ is defined by (22)}.
\]

For example, if \( K(X_0) = I(X_0 \leq \tau) - \mathbb{P}(X_0 \leq \tau) \), then \( H_n/n \) becomes the empirical process. If \( X_i \) is short memory, namely \( a_i \) are absolutely summable, then we can apply the moderate deviation principle in Wu and Zhao (2008). However, the result in the latter paper is not applicable for long-range dependent processes. Despite its importance in risk analysis, the problem of moderate deviation under strong dependence has been rarely studied in the literature.

Here we shall establish such a principle in the context of nonlinear transforms of linear processes. First, we introduce some necessary notation for this section. Let \( F_n = (\cdots, \xi_{n-1}, \xi_n) \) be the shift process and define the projection operator \( P_{r'} = \mathbb{E}(\cdot|F_r) - \mathbb{E}(\cdot|F_{r'-1}) \). Denote the truncated processes \( X_{n,k} = \mathbb{E}(X_n|F_k) \). Now define the functions \( K_n(w) = \mathbb{E}[K(w + X_n - X_{n,0})] \) and \( K_\infty(w) = \mathbb{E}[K(w + X_n)] \). We consider transformations \( K \) with \( \kappa := K_\infty'(0) \neq 0 \). Define

\[
S_{n,1} = \sum_{i=1}^{n} [K(X_i) - \kappa X_i] = H_n - \kappa S_n, \text{ where } S_n = \sum_{i=1}^{n} X_i.
\]

Then \( H_n = \kappa S_n + S_{n,1} \). For a function \( g \), let \( g(w;\lambda) = \sup_{|y| \leq \lambda} |g(w+y)| \) be the local maximal function. Denote the collection of functions with second order partial derivatives by \( \mathcal{C}^2(\mathbb{R}) \). We need the following regularity condition.

**Condition B.** Let \( 2 \leq q < p \leq 2q \) and assume \( ||\xi_0||_{p} < \infty \). Assume \( K_n \in \mathcal{C}^2(\mathbb{R}) \) for all large \( n \) and that for some \( \lambda > 0 \),

\[
\sum_{i=0}^{2} ||K_{n-1}^{(i)}(X_{n,0};\lambda)||_{q} + ||\xi_1||_{p/q} K_{n-1}(X_{n,1})||_{q} + ||\xi_1 K_{n-1}'(X_{n,1})||_{q} = O(1).
\]
A version of Condition B with \( q = 2 \) is used in Wu (2006). We shall establish
the following moderate deviation result. For \( 1/2 < r < 1 \) and \( 1/2 < v < 1 \)
define
\[
\chi(v, r) = v \max(r - r/v, 1/2 - r, r - 1),
\]
\[
\omega(r) = \arg\min_{1/2 \leq v < 1} \chi(v, r) \quad \text{and} \quad \rho(r) = -\chi(\omega(r), r).
\]

**Theorem 2.4** Assume that Condition B holds with \( q = p\omega(r) \) and the conditions
of Corollary 2.5 (ii) are satisfied. Let \( c \) be such that \( 0 < c \leq p - 2 \) and
\( c < 2\rho(\omega) \). Then if \( x \leq c \ln n \), we have
\[
\mathbb{P}(H_n \geq |\kappa|\sigma_n x) = (1 - \Phi(x))(1 + o(1)) \quad \text{as} \quad n \to \infty.
\] (23)

**Remark 2.3** As mentioned in the proof of Theorem 2.4 in Section 4.8, (23) is
still valid if the normalizing constant \( |\kappa|\sigma_n \) therein is replaced by \( \sqrt{\var(H_n)} \).

**Remark 2.4** Theorem 2.4 only asserts a moderate deviation with the Gaussian
range. It is unclear whether the approximation of type (12) holds. We pose it
as an open problem.

**Remark 2.5** An explicit form for \( \omega(r) \) can be obtained. If \( r \geq 3/4 \), then \( \omega(r) = r \). If \( r < 3/4 \), then \( \omega(r) = r/(2r - 1/2) \). If \( 2\rho(\omega) \geq p - 2 \), then the moderate
deviation in (23) has the same range as for \( S_n \). The latter happens, for example,
if \( r = 3/4 \) and \( 2 < p < 16/5 \), since in this case \( 2\rho(3/4) \geq p - 2 \).

**Example 2.1** As an application to empirical processes, let \( K(X) = I(X \leq \tau) - \mathbb{P}(X \leq \tau) \), where \( \tau \in \mathbb{R} \) is fixed. Let \( X_n = \xi_n + \sum_{i=1}^{\infty} a_i \xi_{n-i} =: \xi_n + Y_{n-1} \),
where \( \|\xi_0\|_p < \infty, p > 2 \), and its density function \( f_{\xi} \) satisfies
\[
\sup_u |f_{\xi}(u) + |f'_{\xi}(u)|| < \infty.
\] (24)

Then \( K_1(w) = F_{\xi}(\tau - w) - F_X(\tau) \), where \( F_{\xi} \) is the distribution
function of \( \xi_i \). Under (24), we clearly have \( \sup_w |K'_1(w) + |K''_1(w)|| < \infty \). Observe that we have the identity: for \( n \geq 1 \),
\[
K_n(w) = \mathbb{E}K_1(w + a_1 \xi_{n-1} + a_2 \xi_{n-2} + \ldots + a_{n-1} \xi_1).
\]
Hence \( \sup_n \sup_w |K'_n(w) + |K''_n(w)|| < \infty \). So Condition B holds for any \( \lambda \)
since \( \xi_n \in L^p, p > 2 \).

### 3 A Numerical Study

In this section we shall design a numerical study of the accuracy of
the large deviation (12), normal approximation (13) and also the estimate (11). In
particular, we shall study the accuracy of the approximations in Corollary 2.4.
general it is very time-consuming to calculate tail probabilities by Monte-Carlo simulation, especially if they are small. One may need to carry out astronomically large amount of computations to obtain reasonably well approximations.

Here we shall approach the problem from a different angle. We let
\[ X_j = \sum_{i=1}^{\infty} a_i \xi_j - i, \]
where \( \xi_i, i \in \mathbb{Z} \), have Student’s t-distribution with degree of freedom \( \nu = 3 \), and \( a_i = i^{-0.9} \). Let \( S_n = \sum_{i=1}^{n} X_i \) with \( n = 300 \). Note that the characteristic function of \( \xi_i \) is
\[ \phi(t) = \left( \sqrt{\nu} |t| \right)^{\nu/2} K_{\nu/2}\left( \sqrt{\nu} |t| \right) \Gamma(\nu/2)^{\nu/2} - 1, \quad (25) \]
where \( K_{\nu/2} \) is the Bessel function (see Hurst (1995)). Then the characteristic function of \( S_n \) is
\[ \phi_{S_n}(t) = \prod_{j \in \mathbb{Z}} \phi(b_j t) \]
and by the inversion formula,
\[ \mathbb{P}(S_n \leq x) - \mathbb{P}(S_n \leq x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\sqrt{-1}y(x - x')} e^{\sqrt{-1}y} \phi_{S_n}(y) dy. \]
In the above equation let \( x' = 0 \). Since \( \xi_j \) is symmetric, \( \mathbb{P}(S_n \leq 0) = 1/2 \). In our numerical study we shall use (25) to compute the probability \( \mathbb{P}(S_n > x) \).

In Figure 1 we report the ratios \( R(x) := \sum_i \mathbb{P}(b_{ni} \xi_0 \geq x)/\mathbb{P}(S_n > x) \) and \( g(x) := (1 - \Phi(x/\sigma_n))/\mathbb{P}(S_n > x) \); see (12) with \( c_{ni} = b_{ni} \). We can interpret \( R(x) \) (resp. \( g(x) \)) as tail (resp. Gaussian) approximation. As expected from Corollary 2.4 the Gaussian approximation is better if \( x \) is small, while the tail probability \( R(x) \) approximation is better when \( x \) is big. In the intermediate region we approximate by their sum.

4 Proofs
4.1 Preliminary approximations

Let \( (X_i)_{1 \leq i \leq n} \) be independent random variables. We shall approximate the tail distribution of partial sums by the tail of the sums of truncated random variables and a term involving the tail probabilities of individual summands. We use the following notations:
\[ S_n = \sum_{i=1}^{n} X_i, \quad S(j) = \sum_{i \neq j}^{n} X_i \]
and for \( x > 0 \) and \( \varepsilon > 0 \) we set
\[ X_i^{(\varepsilon x)} = X_i I(X_i < \varepsilon x), \quad S_n^{(\varepsilon x)} = \sum_{i=1}^{n} X_i^{(\varepsilon x)} \quad \text{and} \quad S_n^{(\varepsilon x)}(j) = \sum_{i \neq j}^{n} X_i^{(\varepsilon x)}. \quad (26) \]
Fig. 1. Tail approximation $R(x)$ (dashed curve), Gaussian approximation $g(x)$ (solid curve) and their sum (dotted curve) for long-memory processes with Student $t(3)$ innovations.

We shall prove the following key lemma that will be further exploited to approximate the tail distribution of $P(S_n \geq x)$ in terms of the sum of the truncated random variables and the tail distributions of the individual summands.

**Lemma 4.1** For any $0 < \eta < 1$, and $\varepsilon > 0$ such that $1 - \eta > \varepsilon$ we have

$$
|P(S_n \geq x) - P(S_n^{(\varepsilon x)} \geq x) - \sum_{j=1}^{n} P(X_j \geq (1 - \eta)x)| \leq
$$

$$
4 \left( \sum_{j=1}^{n} P(X_j \geq \varepsilon x) \right)^2 + 3 \sum_{j=1}^{n} P(X_j \geq \varepsilon x)(P(|S_n(j)| > \eta x))
$$

$$
+ \sum_{j=1}^{n} P((1 - \eta)x \leq X_j < (1 + \eta)x).
$$

**Proof.** We decompose the event $\{S_n \geq x\}$ according to $\max_{i \neq j} X_i < \varepsilon x$ or $\max_{i \neq j} X_i \geq \varepsilon x$, and the last one can happen if exactly one of the variables is larger than $\varepsilon x$ or at least two variables exceed $\varepsilon x$. Formally,
\[ P(S_n \geq x) = \sum_{j=1}^{n} P(S_n \geq x, X_j \geq \varepsilon x, \max_{i \neq j} X_i < \varepsilon x) \]

\[ + P\left( \bigcup_{1 \leq i \leq n-1} \bigcup_{i+1 \leq j \leq n} \{S_n \geq x, X_j \geq \varepsilon x, X_i \geq \varepsilon x\} \right) \]

\[ + P(S_n \geq x, \max_{1 \leq i \leq n} X_i < \varepsilon x) = A + B + C = \sum_{j=1}^{n} A_j + B + C. \]

The term \( B \) can be easily majorated by

\[ B \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P(X_j \geq \varepsilon x) P(X_i \geq \varepsilon x) \leq \left( \sum_{j=1}^{n} P(X_j \geq \varepsilon x) \right)^2. \]

We analyze now the first term. We introduce a new parameter \( \eta > 0 \). Since for any two events \( A \) and \( B \) we have \( |P(A) - P(B)| \leq P(AB^c) + P(A^cB) \), (here the prime stays for the complement), for each \( j \) we have

\[ |A_j - P(X_j \geq (1-\eta)x)| \leq P(S_n \geq x, X_j \geq \varepsilon x, X_j < (1-\eta)x) \]

\[ + P(X_j \geq (1-\eta)x, S_n < x) + P(X_j \geq (1-\eta)x, X_j < \varepsilon x) \]

\[ + P(X_j \geq (1-\eta)x, \max_{i \neq j} X_i \geq \varepsilon x) = I + II + III + IV. \]

We treat each term separately. By independence and since \( S_n \geq x \) and \( X_j < (1-\eta)x \) imply \( S_n(j) \geq \eta x \), we derive

\[ I \leq P(X_j \geq \varepsilon x) P(S_n(j) \geq \eta x). \]

The second term is treated in the following way:

\[ II \leq P((1-\eta)x \leq X_j < (1+\eta)x) + P(X_j \geq (1+\eta)x, S_n < x) \]

\[ \leq P((1-\eta)x \leq X_j < (1+\eta)x) + P(X_j \geq (1+\eta)x) P(-S_n(j) \geq \eta x). \]

Since \( 1-\eta > \varepsilon \) the third term is: \( III = 0 \). By independence, the forth term is

\[ IV = P(X_j \geq (1-\eta)x) P(\max_{i \neq j} X_i \geq \varepsilon x). \]

Overall, by the previous estimates and because \( 1-\eta > \varepsilon \), we obtain

\[ |A - \sum_{j=1}^{n} P(X_j \geq (1-\eta)x)| \leq 2 \sum_{j=1}^{n} P(X_j \geq \varepsilon x) (P(|S_n(j)| > \eta x) \]

\[ + \left( \sum_{j=1}^{n} P(X_j \geq \varepsilon x) \right)^2 + \sum_{j=1}^{n} P((1-\eta)x \leq X_j < (1+\eta)x). \]
It remains to analyze the last term, \( C \). Notice that
\[
|C - \mathbb{P}(S_n^{(\varepsilon x)} \geq x)| = \mathbb{P}(S_n^{(\varepsilon x)} \geq x) - \mathbb{P}(S_n \geq x, \max_{1 \leq i \leq n} X_i < \varepsilon x)
\]
\[
= \mathbb{P}(S_n^{(\varepsilon x)} \geq x, \max_{1 \leq i \leq n} X_i \geq \varepsilon x).
\]

Now we treat this term by the same arguments we have already used, by dividing the maximum in two parts:
\[
\mathbb{P}(S_n^{(\varepsilon x)} \geq x, \max_{1 \leq i \leq n} X_i \geq \varepsilon x) = \sum_{j=1}^{n} \mathbb{P}(S_n^{(\varepsilon x)}(j) \geq x, X_j \geq \varepsilon x, \max_{i \neq j} X_i < \varepsilon x)
\]
\[
+ \mathbb{P}(\bigcup_{1 \leq i \leq n-1} \bigcup_{i+1 \leq j \leq n} \{S_n^{(\varepsilon x)} \geq x, X_j \geq \varepsilon x, X_{ni} \geq \varepsilon x\}) = \sum_{j=1}^{n} F_j + G.
\]

The last term, \( G \) is majorated exactly as \( B \). As for the first term, we notice that because \( X_j \geq \varepsilon x \) the term \( X_j^{(\varepsilon x)} \) does not appear in the sum, and by independence we obtain
\[
F_j = \mathbb{P}(S_n^{(\varepsilon x)}(j) \geq x, X_j \geq \varepsilon x, \max_{i \neq j} X_i < \varepsilon x)
\]
\[
\leq \mathbb{P}(S_n^{(\varepsilon x)}(j) \geq x) \mathbb{P}(X_j \geq \varepsilon x).
\]

Now, clearly we have
\[
\mathbb{P}(S_n^{(\varepsilon x)}(j) \geq x) \leq \mathbb{P}(\max_{i} X_i \geq \varepsilon x) + \mathbb{P}(S_n^{(\varepsilon x)}(j) \geq x, \max_{i} X_i < \varepsilon x)
\]
\[
= \mathbb{P}(\max_{i} X_i \geq \varepsilon x) + \mathbb{P}(S_n(j) \geq x, \max_{i} X_i < \varepsilon x),
\]

implying that
\[
\sum_{j=1}^{n} F_j \leq \sum_{j=1}^{n} \mathbb{P}(X_{nj} \geq \varepsilon x)(\mathbb{P}(\max_{i} X_i \geq \varepsilon x) + \mathbb{P}(S_n(j) \geq x)).
\]

Overall,
\[
|C - \mathbb{P}(S_n^{(\varepsilon x)} \geq x)| \leq 2\left(\sum_{j=1}^{n} \mathbb{P}(X_j \geq \varepsilon x)\right)^2 + \sum_{j=1}^{n} \mathbb{P}(X_j \geq \varepsilon x)\mathbb{P}(S_n(j) \geq x).
\]

By gathering all the information above and taking into account that
\[
|\mathbb{P}(S_n \geq x) - \mathbb{P}(S_n^{(\varepsilon x)} \geq x) - \sum_{j=1}^{n} \mathbb{P}(X_j \geq (1 - \eta)x)| \leq
\]
\[
|A - \sum_{j=1}^{n} \mathbb{P}(X_j \geq (1 - \eta)x)| + |C - \mathbb{P}(S_n^{(\varepsilon x)} \geq x)| + |B|,
\]

the lemma is established. ♦

The following similar lemma is for the sum of infinite many terms.
Lemma 4.2 Let \(1 - \eta > \varepsilon > 0\) and \(x > 0\); let \(X_1, X_2, \ldots\), be independent random variables. Assume that the sum \(S = \sum_{i=1}^{\infty} X_i\) exists almost surely. Let \(S(j) = S - X_j\), \(X_i^{(\varepsilon x)} = X_i I(X_i < \varepsilon x)\). Then \(S^{(\varepsilon x)} = \sum_{i=1}^{\infty} X_i^{(\varepsilon x)}\) exists almost surely and

\[
|\Pr(S \geq x) - \Pr(S^{(\varepsilon x)} \geq x)| \leq \\
4 \left( \sum_{j=1}^{\infty} \Pr(X_j \geq \varepsilon x) \right)^2 + 3 \sum_{j=1}^{\infty} \Pr(X_j \geq \varepsilon x) \Pr(|S(j)| \geq \eta x) \\
+ \sum_{j=1}^{\infty} \Pr((1 - \eta)x \leq X_j < (1 + \eta)x).
\]

**Proof.** By Kolmogorov’s three-series theorem, \(S^{(\varepsilon x)} = \sum_{i=1}^{\infty} X_i^{(\varepsilon x)}\) converges almost surely. Let \(\Omega_0 \in \Omega\) with \(\Pr(\Omega_0) = 1\) be the set that both \(\sum_{i=1}^{\infty} X_i\) and \(\sum_{i=1}^{\infty} X_i^{(\varepsilon x)}\) converge. Hence on \(\Omega_0\), we understand \(S(\omega)\) as just the sum \(\sum_{i=1}^{\infty} X_i(\omega)\). Then following the proof of Lemma 4.1, we have Lemma 4.2. ◊

If \(S_n\) is stochastically bounded, i.e., \(\lim_{K \to \infty} \sup_n \Pr(|S_n| > K) = 0\), the approximation in Lemma 4.1 has a simple asymptotic form.

**Proposition 4.1** Assume that \(S_n\) is stochastically bounded, the variables are centered and \(x_n \to \infty\). Then for any \(0 < \eta < 1\), and \(\varepsilon > 0\) such that \(1 - \eta > \varepsilon\), we have

\[
|\Pr(S_n \geq x_n) - \Pr(S_n^{(\varepsilon x_n)} \geq x_n)| - \sum_{j=1}^{n} \Pr(X_j \geq (1 - \eta)x_n)| \leq \quad (27)
\]

\[
o(1) \sum_{j=1}^{n} \Pr(X_j \geq \varepsilon x_n) + \sum_{j=1}^{n} \Pr((1 - \eta)x_n \leq X_j < (1 + \eta)x_n),
\]

where \(o(1)\) depends on the sequence \(x_n\), \(\eta\) and \(\varepsilon\) and converges to 0 as \(n \to \infty\).

**Proof.** We just notice that for independent centered random variables, if \(S_n\) is stochastically bounded, by Lévy inequality (Inequality 1.1.3 in de la Peña and Giné 1999), we have \(\max_{1 \leq i \leq n} |X_i|\) is stochastically bounded too. By taking into account that \(|S_n(j)| \leq |S_n| + \max_{1 \leq i \leq n} |X_i|\), and using the fact that \(x_n \to \infty\) as \(n \to \infty\) we obtain

\[
\sum_{j=1}^{n} \Pr(X_j \geq \varepsilon x_n) \Pr(|S_n(j)| \geq \eta x_n) \leq \max_{1 \leq j \leq n} \Pr(|S_n(j)| \geq \eta x_n) \sum_{j=1}^{n} \Pr(X_j \geq \varepsilon x_n)
\leq \left( \Pr(|S_n| \geq \eta x_n/2) + \Pr(\max_{1 \leq i \leq n} |X_i| \geq \eta x_n/2) \right) \sum_{j=1}^{n} \Pr(X_j \geq \varepsilon x_n)
\leq o(1) \sum_{j=1}^{n} \Pr(X_j \geq \varepsilon x_n)\] as \(n \to \infty\).
Then, by independence
\[
\mathbb{P}\left( \max_{1 \leq j \leq n} |X_j| \geq \varepsilon x_n \right) = \mathbb{P}(|X_1| \geq \varepsilon x_n) + \sum_{k=2}^{n} \mathbb{P}\left( \max_{1 \leq j \leq k-1} |X_j| < \varepsilon x_n \right) \mathbb{P}(|X_k| \geq \varepsilon x_n)
\]
\[
\geq \mathbb{P}\left( \max_{1 \leq j \leq n} |X_j| < \varepsilon x_n \right) \sum_{k=1}^{n} \mathbb{P}(|X_k| \geq \varepsilon x_n),
\]
which gives
\[
\left( \sum_{j=1}^{n} \mathbb{P}(|X_j| \geq \varepsilon x_n) \right)^2 \leq \mathbb{P}\left( \max_{1 \leq j \leq n} |X_j| \geq \varepsilon x_n \right) \sum_{j=1}^{n} \mathbb{P}(|X_j| \geq \varepsilon x_n)
\]
\[
= o(1) \sum_{j=1}^{n} \mathbb{P}(|X_j| \geq \varepsilon x_n) \quad \text{as} \quad n \to \infty,
\]
since \( x_n \to \infty \) as \( n \to \infty \) and \( \max_{1 \leq j \leq n} |X_j| \) is stochastically bounded.

**Remark 4.1** Based on Lemma 4.2, it is easy to verify that Proposition 4.1 is still valid if we extend the sums up to infinity.

### 4.2 Proof of Theorem 2.2

It is convenient to normalize by the variance of partial sum and we shall consider without restricting the generality that
\[
\mathbb{E} \xi_0^2 = 1, \quad \sum_{i=1}^{k_n} c_{ni}^2 = 1 \quad \text{and} \quad \max_{1 \leq i \leq k_n} c_{ni}^2 \to 0. \quad (28)
\]
Then we have \( \sum_{i=1}^{k_n} c_{ni}^4 \leq \max_{1 \leq i \leq k_n} c_{ni}^4 \to 0 \) implying that \( D_n^{-1} \to \infty \). Moreover, the sequence \( \sum_{i=1}^{k_n} c_{ni} \xi_i \) is stochastically bounded and we analyze the two terms of the right side and the last term of the left side in Proposition 4.1. Let \( x_n \to \infty \) as \( n \to \infty \). In order to ease the notation we shall denote \( x = x_n \), but we keep in mind that \( x \) depends on \( n \) and tends to infinite with \( n \). By taking into account that \( x/c_{ni} \geq x \to \infty \) and \( h \) is a slowly varying function we notice first that for any \( a > 0 \)
\[
\lim_{x \to \infty} \max_{1 \leq i \leq k_n} \left| h(ax/c_{ni}) - h((1 + \gamma)x/c_{ni}) \right| = 0.
\]
We derive for any \( |\gamma| < 1 \) fixed
\[
\left| \sum_{i=1}^{k_n} c_{ni}^t \left( h\left( \frac{x}{c_{ni}} \right) - h((1 + \gamma)\frac{x}{c_{ni}}) \right) \right| \leq \sum_{i=1}^{k_n} c_{ni}^t h\left( \frac{x}{c_{ni}} \right) |1 - \frac{h((1 + \gamma)x/c_{ni})}{h(x/c_{ni})}| = o(1) \sum_{i=1}^{k_n} c_{ni}^t h\left( \frac{x}{c_{ni}} \right), \quad \text{as} \quad n \to \infty,
\]
\[
16
\]
implying that

\[
\frac{\sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_i \geq (1 \pm \eta)x)}{\sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_i \geq x)} = \frac{\sum_{i=1}^{k_n} c_{ni}^t h((1 \pm \eta)x/c_{ni})}{(1 \pm \eta)^t \sum_{i=1}^{k_n} c_{ni}^t h(x/c_{ni})} \to 1
\]

when \( n \to \infty \) followed by \( \eta \to 0 \).

Then, we also have

\[
\frac{\sum_{i=1}^{k_n} \mathbb{P}((1 - \eta)x \leq c_{ni}\xi_i < (1 + \eta)x)}{\sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_i \geq x)} \to 0 \text{ as } n \to \infty \text{ and } \eta \to 0.
\]

Similarly, for every \( \varepsilon > 0 \) fixed we have that

\[
\frac{\sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_i \geq \varepsilon x)}{\sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_i \geq x)} = \frac{\sum_{i=1}^{k_n} c_{ni}^t h(\varepsilon x/c_{ni})}{\varepsilon^t \sum_{i=1}^{k_n} c_{ni}^t h(x/c_{ni})} \to \frac{1}{\varepsilon^t} \text{ as } n \to \infty,
\]

and then,

\[
\sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_i \geq \varepsilon x) \ll \sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_i \geq x) \text{ as } n \to \infty.
\]

So far, for any \( \varepsilon > 0 \) fixed, by letting \( n \to \infty \) first and after that, passing with \( \eta \) to 0, we deduce by the above consideration combined with Proposition 4.1 that

\[
\mathbb{P}(S_n \geq x) = \sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_i \geq x)(1 + o(1)) + \mathbb{P}(S_n^{(\varepsilon x)} \geq x) \text{ as } n \to \infty. \tag{29}
\]

It remains to study the term \( \mathbb{P}(S_n^{(\varepsilon x)} \geq x) \). We shall base this part of the proof on Corollary 1.7 in S. Nagaev (1979), given in the Appendix, which we apply with \( m > t \), that will be selected later. Because we assume \( E(\xi_2^2) = 1 \) and \( \sum_{i=1}^{k_n} c_{ni}^2 = 1 \), we have for all \( y, B_n^2(-\infty, y) \leq 1 \), and therefore, Theorem 5.1 implies:

\[
\mathbb{P}(S_n^{(\varepsilon x)} \geq x) \leq \exp(-\alpha^2 x^2/2e^m) + (A_n(m; 0, \varepsilon x)/(\beta e^{m-1}x^m))^{\beta/\varepsilon}.
\]

with \( \alpha = 1 - \beta = 2/(m + 2) \). Then, obviously, it is enough to show that for \( x = x_n \) as in Theorem 2.2 we can select \( \varepsilon > 0 \) such that

\[
\exp(-\alpha^2 x^2/2e^m) + \left(\frac{A_n(m; 0, \varepsilon x)}{\beta e^{m-1}x^m}\right)^{\beta/\varepsilon} = o(1) \sum_{i=1}^{k_n} c_{ni}^t h\left(\frac{x}{c_{ni}}\right) \text{ as } n \to \infty. \tag{30}
\]

Let \( x = x_n \geq C[\ln(D_{nt}^{-1})]^{1/2} \) where \( C > e^{m/2}(m + 2)/\sqrt{2} \). As we mentioned at the beginning of the proof, we clearly have \( x_n \to \infty \).

We shall estimate each term in the left hand side of (30) separately. Because, by the definition of \( \alpha \) we have \( C > e^{m/2}\alpha^{-1}\sqrt{2} \), we can select \( 0 < \eta < 1 \) such that \( C^2\alpha^2/2e^m = (1 - \eta)^{-2} \).
Taking into account the fact that for any $c > 0$ and $d > 0$ we have $y^d \exp(-cy)$ = $o(\exp(-c(1 - \eta)y))$ as $y \to \infty$, by the definition on $x$ and $\eta$, we obtain:

\[ x^{(t-2\eta)/(1-\eta)} \exp(-\frac{\alpha^2 x^2}{2e^m}) = o(1) \exp(-\frac{\alpha^2 x^2}{2e^m}(1 - \eta)) \]

\[ = o(1)(\sum_{i=1}^{k_n} c_{ni}^t)^{\alpha^2(1-\eta)/2e^m} = o(1)(\sum_{i=1}^{k_n} c_{ni}^t)^{(1-\eta)^{-1}}. \]

Applying now the Hölder inequality we clearly have,

\[ \sum_{i=1}^{k_n} c_{ni}^t = \sum_{i=1}^{k_n} c_{ni}^{2\eta} \leq (\sum_{i=1}^{k_n} c_{ni}^2)^{\eta} (\sum_{i=1}^{k_n} (t-2\eta)/(1-\eta))^{1-\eta}. \] (31)

Taking into account that $\sum_{i=1}^{k_n} c_{ni}^2 = 1$, we obtain overall

\[ \exp(-\frac{\alpha^2 x^2}{2e^m}) = o(1)x^{-(t-2\eta)/(1-\eta)} \sum_{i=1}^{k_n} c_{ni}^{(t-2\eta)/(1-\eta)}. \]

Since $t > 2$, $(t-2\eta)/(1-\eta) > t$. Then, by combining this observation with the properties of slowly varying functions we have

\[ \exp\left(\frac{\alpha^2 x^2}{2e^m}\right) = o(1)\sum_{i=1}^{k_n} c_{ni} h\left(\frac{x}{c_{ni}}\right). \]

We select $\varepsilon$ by analyzing the second term in the left hand side of (30). Notice that by integration by parts formula, for every $z > y > 0$,

\[ \mathbb{E}[\xi_{0}^m I(0 \leq \xi_0 < z)] = -z^m \mathbb{P}(\xi_0 \geq z) + m \int_{0}^{z} u^{m-1} \mathbb{P}(\xi_0 \geq u)du \leq y^m + m \int_{y}^{z} u^{m-1} \mathbb{P}(\xi_0 \geq u)du. \]

Replacing $z = \varepsilon x/c_{ni}$, taking into account condition (4), the properties of slowly varying functions, and the facts that $x/c_{ni} \to \infty$ and $m > t$, we have

\[ \mathbb{E}[\xi_{0}^m I(0 \leq c_{ni} \xi_0 < \varepsilon x)] \leq y^m + 2m \int_{y}^{\varepsilon x} u^{m-t-1} h(u)du = O\left((\frac{x}{c_{ni}})^{m-t} h\left(\frac{x}{c_{ni}}\right)\right) \]

for $y$ sufficiently large. It follows that

\[ A_n(m; 0, \varepsilon x) = \sum_{i=1}^{k_n} c_{ni}^m \mathbb{E}[\xi_{0}^m I(0 \leq c_{ni} \xi_0 < \varepsilon x)] \]

\[ \ll \sum_{i=1}^{k_n} c_{ni}^m \left(\frac{x}{c_{ni}}\right)^{m-t} h\left(\frac{x}{c_{ni}}\right) = x^{m-t} \sum_{i=1}^{k_n} c_{ni}^t h\left(\frac{x}{c_{ni}}\right). \]
Choose $\varepsilon$ with $0 < \varepsilon < \beta$. Then the second term has the order

$$
\left( \frac{A_n(m; 0, \varepsilon x)}{\beta \varepsilon m^{-1} x^m} \right)^{\beta/\varepsilon} \ll \left( \frac{x^{m-t} \sum_{i=1}^{k_n} c_{ni} h(x_c)}{x^{m} \beta \varepsilon h(x_c)} \right)^{\beta/\varepsilon} = o \left( \sum_{i=1}^{k_n} \frac{c_{ni} h(x)}{x^t} \right).
$$

Overall we obtain for any $x \geq C(\ln(\sum_{i=1}^{k_n} c_{ni}^p)^{-1})^{1/2}$ with $C > e^{m/2}(m + 2)/\sqrt{2}$,

$$
P(S_n \geq x) = (1 + o(1)) \sum_{i=1}^{k_n} P(c_{ni} \xi_0 \geq x) \text{ as } n \to \infty,
$$

where $m > t$. Since $C_t > e^{t/2}(t + 2)/\sqrt{2}$ we can select and fix $m > t$ such that $C_t > e^{m/2}(m + 2)/\sqrt{2}$. \(\Diamond\)

### 4.3 Proof of Theorem 2.3

For simplicity we normalize by the variance of $S_n$ and assume \(28\). This result easily follows from Theorem 1.1 in Frolov (2005) when moments strictly larger than 2 are available. This theorem is given for convenience in the Appendix (Theorem 5.2). Because we assume the existence of moments of order $p > 2$, we have

$$\Lambda_n(u, s, \varepsilon) \leq u \sum_{j=1}^{k_n} c_{nj}^2 \mathbb{E}[\xi_0^2 I(|\xi_0| > \varepsilon/s) \leq e^{2-p} u s^{p-2} D_{np} \mathbb{E}[\xi_0]^p].$$

where $D_{np} = \sum_{j=1}^{k_n} |c_{nj}|^p$. Then, for $x^2 \leq 2 \ln(1/D_{np})$,

$$\Lambda_n(x^4, x^5, \varepsilon) \leq e^{2-p} x^{4+5(p-2)} D_{np} \mathbb{E}[\xi_0]^p \leq e^{2-p} D_{np} (2 \ln(1/D_{np}))^{(5p-6)/2} \mathbb{E}[\xi_0]^p,$

which converges to 0 since $D_{np} \leq \max_{1 \leq j \leq k_n} |c_{nj}|^{p-2} \to 0$ by \(10\). Notice also that the $L_{np}$ in Theorem 5.2 satisfies $L_{np} \leq D_{np} \mathbb{E}[\xi_0]^p \to 0$. The latter implies $x^2 - 2 \ln(L_{np}) - (p-1) \ln(\ln(L_{np})) \to -\infty$ provided $x^2 \leq 2 \ln(D_{np}^{-1})$. Then the result is immediate from Theorem 5.2. \(\Diamond\)

### 4.4 Proof of Theorem 2.1

Again for simplicity we normalize by the variance and assume \(28\). Without loss of generality we may assume $2 < p < t$. This is so because if $p \geq t$ with $\mathbb{E}[|\xi_0|^p] < \infty$ then we can find a $p'$ such that $2 < p' < t$ and $\mathbb{E}[|\xi_0'|^p] < \infty$. We shall consider a sequence $x_n$ which converges to 0. So, let $x = x_n \to \infty$.

Starting from the relation \(29\) and applying Proposition 5.1 to the second term in the right hand side we obtain for any $\varepsilon > 0$ and $x^2 \leq \varepsilon c_c \ln(D_{np}^{-1})$ with $c_c < 1/\varepsilon$ and for all $n$ sufficiently large $P(S_n^{(x)} \geq x) = (1 - \Phi(x))(1 + o(1))$. We notice now that by \(31\) applied with $\eta = (t-p)/(t-2)$ and simple considerations,

$$D_{nt} \ll D_{np} \ll (D_{nt})^{(p-2)/(t-2)}. \quad (32)$$

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So far, by using this last relation, we showed by (29) and the above considerations that (11) holds for $0 < x \leq C \ln(D_{n^t})^{1/2}$ with $C$ an arbitrary positive number. On the other hand, because $1 - \Phi(x) \leq (2\pi)^{-1/2}x^{-1} \exp(-x^2/2)$, by Theorem 2.2 and by the arguments leading to the proof of relation (30), there is a constant $c_1 > 0$ such that for $x > c_1 \ln(D_{n^t})^{1/2}$, we simultaneously have

$$
P(S_n \geq x) = (1 + o(1)) \sum_{i=1}^{k_n} P(c_{ni} \xi_0 \geq x)
$$

and

$$1 - \Phi(x) = o(\sum_{i=1}^{k_n} P(c_{ni} \xi_0 \geq x)).$$

Then (11) holds for all $x > 0$ since $C$ is arbitrarily large and can be selected such that $c_1 < C$.

Now if the sequence $x_n$ is bounded we apply first Theorem 2.3 and obtain the moderate deviation result in (13). Then, because $x_n \geq c > 0$ we notice that, by the arguments leading to the proof of relation (30), the second part in the right hand side of (11) is dominant, so the first part is negligible as $n \to \infty$. ♦

4.5 Proof of Corollary 2.1

Again without loss of generality we normalize by the variance and assume (28). The ideas involved in the proof of this corollary already appeared in the previous proofs, so we shall mention only the changes. We start from (11). To prove (12) we have to show that

$$1 - \Phi(x) = o(\sum_{i=1}^{k_n} P(c_{ni} \xi_0 \geq x))$$

for $x \geq a \ln D_{n^t}^{1/2}$ with $a > 2^{1/2}$. First we shall use the relation $1 - \Phi(x) \leq (2\pi)^{-1/2}x^{-1} \exp(-x^2/2)$. Then, we adapt the proof we used to establish the first part of (30), when we compared $\exp(-\alpha^2x^2/2e^m)$ to $\sum_{i=1}^{k_n} P(c_{ni} \xi_0 \geq x)$. The main difference is that now we take $m = 0$ and $\alpha = 1$.

For the proof of (13), we use the inequality $1 - \Phi(x) \geq (2\pi)^{-1/2}(1 + x)^{-1} \exp(-x^2/2)$. By (31) and (32) we have for every $0 < \varepsilon < t - 2$,

$$\sum_{i=1}^{k_n} P(c_{ni} \xi_0 \geq x) \ll \sum_{i=1}^{k_n} c_{ni}^{t-\varepsilon} x^{2-\varepsilon} \ll \frac{1}{x^{t-\varepsilon}} (D_{n^t})^{(t-2-\varepsilon)/(t-2)}.$$  

Then, it is easy to see that, because $\varepsilon$ can be made arbitrarily small, for $1 < x \leq b \ln D_{n^t}^{1/2}$ with $b < 2^{1/2}$ we have

$$\sum_{i=1}^{k_n} P(c_{ni} \xi_0 \geq x) = o(1 - \Phi(x)).$$

When $0 < x \leq 1$ we apply Theorem 2.3 ♦
4.6 Proof of Corollary 2.3

As in the other proofs, for simplicity we assume $E\xi_0^2 = 1$.

**Proof of part (ii).** Because the Fuk-Nagaev inequality (Theorem 5.1) and the inequalities in Lemma 4.1 and Proposition 4.1 are still valid for the case $k_n = \infty$ (see Remark 5.1 in the Appendix, Lemma 4.2 and Remark 4.1 in Subsection 4.1), all the arguments in the proof of Theorem 2.2 hold under the conditions of this corollary.

**Proof of part (iii).** The result (iii) in this corollary is obtained on the same lines as of Theorem 2.3. The modification of the proof is rather standard but computationally intensive. There are several ideas behind this proof. The infinite series is decomposed as a sum up to $k_n$ and the rest $R_n$. The sequence $k_n$ is selected independently of $x_n$ such that the rest of the series $R_n$ is negligible for the moderate deviation result. This is possible because the coefficients $b_{ni}$, defined as $b_{ni} = a_{1-i} + \ldots + a_{n-i}$ with $\sum_{i \in \mathbb{Z}} a_i^2 < \infty$, have some regularity properties. For instance by the Hölder’s inequality,

$$b_{ni}^2 \leq n(a_{1-i}^2 + \ldots + a_{n-i}^2)$$

and so, for any $k > n$,

$$\sum_{|i| \geq k} b_{ni}^2 \leq n^2 \sum_{|i| \geq k-n-1} a_{1-i}^2. \quad (33)$$

We then note that the existence of moments of order $p > 2$ for $\xi_0$ and $\sum_{i \in \mathbb{Z}} a_i^2 < \infty$ imply that $X_0$ also has finite moments of order $p$. Indeed, by Rosenthal inequality (see for instance Theorem 1.5.13 in de la Peña and Giné, 1999), there is a constant $C_p$ such that

$$E|\sum_{j=n}^{m} a_j \xi_j|^p \leq C_p[(\sum_{j=n}^{m} a_j^2)^{p/2} + E|\xi_0|^p \sum_{j=n}^{m} |a_j|^p]$$

which implies that $E|\sum_{j=n}^{m} a_j \xi_j|^p \to 0$ as $m \to \infty$, and therefore $X_0$ exists in $L_p$.

For $k_n$ a sequence of integers, denote $R_n = \sum_{|i| > k_n} b_{ni} \xi_i$ and note that $R_n$ is also well defined in $L_p$. Again by Rosenthal inequality we obtain

$$E|R_n|^p \leq C_p[(\sum_{|i| > k_n} b_{ni}^2)^{p/2} + E|\xi_0|^p \sum_{|i| > k_n} |b_{ni}|^p]. \quad (34)$$

We select now $k_n$ large enough such that

$$\sum_{|i| > k_n} b_{ni}^2 \leq ||\xi_0||^2 \sum_{j} |b_{nj}|^p)^{2/p}.$$

This is possible by relation (33) and the fact that $\sum_{i} a_i^2 < \infty$. With this selection we obtain

$$E|R_n|^p \leq 2C_p E|\xi_0|^p \sum_{i} |b_{ni}|^p. \quad (35)$$
Write now

\[ S_n = \sum_{|i| \leq k_n} b_{ni} \xi_i + R_n. \]

We view \( S_n \) as the sum of \( k_n + 1 \) independent random variables and then apply Theorem 5.2 as in the proof of Theorem 2.3. By taking into account (35), the term \( L_{np} \) from Theorem 5.2 is

\[ L_{np} = \frac{1}{\sigma_n^p} \left( \sum_{|i| \leq k_n} |b_{ni}|^p \mathbb{E}(|\xi_0|^p I(|\xi_0| > 0)) + \mathbb{E}(R_n^p I(R_n > 0)) \right) \]

\[ \leq \frac{2C_p + 1}{\sigma_n^p} \sum_i |b_{ni}|^p \mathbb{E}(|\xi_0|^p) = (2C_p + 1)U_{np} \mathbb{E}(|\xi_0|^p) = L'_{np}. \]

Because we assume the existence of moments of order \( p \), by (35) we have

\[ \Lambda_n(u, s, \epsilon) \leq \frac{u}{\sigma_n^p} \left[ \sum_{|j| \leq k_n} b_{nj}^2 \mathbb{E}(|b_{nj}|^2 I(|b_{nj}| > \epsilon \sigma_n/s) + \mathbb{E}R_n^p I(|R_n| > \epsilon \sigma_n/s) \right] \]

\[ \leq \frac{us^{p-2}}{\sigma_n^p \epsilon^{p-2}} \sum_{|j| \leq k_n} |b_{nj}|^p \mathbb{E}(|\xi_0|^p) + \mathbb{E}|R_n|^p \leq \frac{us^{p-2}}{\epsilon^{p-2}} L'_{np}. \]

Therefore, for \( x^2 \leq 2 \ln(1/L'_{np}) \leq 2 \ln(1/L_{np}) \),

\[ \Lambda_n(x^4, x^5, \epsilon) \leq \epsilon^{2-p} x^{4+5(p-2)} L'_{np} \leq \epsilon^{2-p} (2 \ln(1/L'_{np}))^{(5p-6)/2} L'_{np}. \]

Finally note that by (20) we obtain

\[ U_{np} \leq \sup_j |b_{nj}|^{p-2} (\sum b_{nj}^2)^{(p-2)/2} \to 0, \]

and consequently \( L'_{np} \to 0 \). Therefore, \( \Lambda_n(x^4, x^5, \epsilon) \to 0 \). Note also that the quantity \( L_{np} \) in Theorem 5.2 satisfies \( L_{np} \leq L'_{np} \to 0 \). Therefore if \( x^2 - 2 \ln(L'_{np})^{-1} - (p - 1) \ln \ln(L'_{np})^{-1} \to -\infty \) we have that \( x^2 - 2 \ln(L^{-1}_{np}) - (p - 1) \ln \ln(L^{-1}_{np}) \to -\infty \) and the result holds for such a positive \( x \).

It remains to show that \( x^2 \leq 2 \ln(U^{-1}_{np}) \) implies \( x^2 - 2 \ln(L'_{np})^{-1} - (p - 1) \ln \ln(L'_{np})^{-1} \to -\infty \), which holds provided that

\[ 2 \ln((U^{-1}_{np})^{1/L'_{np}}[\ln(L'_{np})^{-1}]^{(1-p)/2}) \to -\infty. \]

This last divergence is equivalent to

\[ (U^{-1}_{np})^{1/L'_{np}}[\ln(L'_{np})^{-1}]^{(1-p)/2} \to 0. \]

Clearly, because \( L'_{np} = (2C_p + 1)U_{np} \mathbb{E}(|\xi_0|^p) \) and the fact that we have shown that \( L'_{np} \to 0 \) the result follows.

**Proof of part (i).** The proof is similar to the proof of Proposition 2.1 and Corollary 2.1. We have only to show that Proposition 5.1 is still valid in this
context if we let \( k_n = \infty \). The proof is similar to the proof of (iii) but more involved, since the sequence of truncated variables is not centered. Denote

\[
X'_{ni} = b_{ni}\xi_i I(b_{ni}\xi_i \leq \varepsilon x\sigma_n) = b_{ni}\xi_i'.
\]

For \( k_n \) a sequence of integers, denote \( R'_n = \sum_{|i|>k_n} b_{ni}\xi_i' \) and note that \( R'_n \) is also well defined in \( L_p \). By Rosenthal inequality, after centering we obtain

\[
\mathbb{E}|R'_n|^p \leq C'_p \left( \sum_{|i|>k_n} b_{ni}^2 \right)^{p/2} + \mathbb{E}|\xi_0|^p \sum_{|i|>k_n} |b_{ni}|^p + |\mathbb{E}(R'_n)|^p.
\]

Because \( x = c > 0 \) and the fact that \( \mathbb{E}(X'_{ni}) = -\mathbb{E}(b_{ni}\xi_i I(b_{ni}\xi_i > \varepsilon x\sigma_n)) \) we obtain

\[
|\mathbb{E}(R'_n)| \leq \frac{1}{\varepsilon x\sigma_n} \sum_{|i|>k_n} b_{ni}^2 \leq \frac{1}{\varepsilon c\sigma} \sum_{|i|>k_n} b_{ni}^2.
\]

We select now \( k_n \), depending on \( c, \varepsilon \) and the distribution of \( \xi_0 \) and the coefficients \((a_k)\), large enough such that

\[
(\sum_{|i|>k_n} b_{ni}^2)^{p/2} + \left( \frac{1}{\varepsilon c\sigma} \sum_{|i|>k_n} b_{ni}^2 \right)^p \leq \mathbb{E}|\xi_0|^p \sum_{i} |b_{ni}|^p,
\]

and so

\[
\mathbb{E}|R'_n|^p \leq 2C'_p \mathbb{E}|\xi_0|^p \sum_{i} |b_{ni}|^p.
\]

Write now \( S'_n = \sum_{|i|\leq k_n} b_{ni}\xi_i' + R'_n \) and view \( S'_n \) as the sum of \( k_n+1 \) independent random variables and then apply Proposition 5.1. Similar computations as in the proof of the point (iii) show that \( L_{np} \) in Proposition 5.1 is bounded by

\[
L_{np} \leq \frac{2C'_p + 1}{\sigma_n^p} \mathbb{E}|\xi_0|^p \sum_{i} |b_{ni}|^p = (2C'_p + 1)U_{np}\mathbb{E}|\xi_0|^p.
\]

Then, by Proposition 5.1 if \( x^2 \leq c \ln((2C'_p + 1)U_{np}\mathbb{E}|\xi_0|^p)^{-1} \) for \( c < 1/\varepsilon \), we have \( x^2 \leq c \ln(L_{np}^{-1}) \) for \( c < 1/\varepsilon \) and

\[
\mathbb{P} \left( \sum_{i} X'_{ni} \geq x\sigma_n \right) = (1 - \Phi(x))(1 + o(1)).
\]

It remains to notice that because \( U_{np} \to 0 \), we also have the result for \( x^2 \leq c \ln(U_{np})^{-1} \) for any \( c < 1/\varepsilon \), for all \( n \) sufficiently large. \( \diamond \)

### 4.7 Proof of Corollary 2.4

This Corollary follows from Corollary 2.3 via Lemma 5.1 in the Appendix. It remains to give an explicit form of the intervals moderate deviation and large deviation boundaries. Without loss of generality, we assume that \( \mathbb{E}\xi_0^2 = 1 \). For proving the large deviation part of this corollary we have to analyze the
condition on \( x \) from part (i) of Corollary 2.3 namely \( x > a([\ln(n)]^{1/2}) \) with \( a = \sqrt{2} \). By Lemma 5.1

\[
B_{n^2} = \sum_i b_{n^2}^i \sim c_r n^{3-2r}I_2(n)
\]

and

\[
C_1t^*(n)(1-r)^{t+1} \leq \sum_{j=1}^{\infty} b_{n^j}^i \leq C_2t^*(n)(1-r)^{t+1}.
\]

Then, for certain constants \( K_1 \) and \( K_2 \) and because \( U_{n^t}^{-1} = B_{n^2}/B_{nt} \), we have for \( n \) sufficiently large

\[
K_1 + \ln n^{(t-2)/2} \leq \ln U_{n^t}^{-1} \leq K_2 + \ln n^{(t-2)/2}.
\]

So, the asymptotic result (12) holds for \( x \geq c_1([\ln(n)]^{1/2}) \) where \( c_1 > (t-2)^{1/2} \). Furthermore, (13) holds for \( 0 < x \leq c_2([\ln(n)]^{1/2}) \) where \( c_2 < (t-2)^{1/2} \). \( \diamond \)

4.8 Proof of Theorem 2.4

Without restricting the generality we assume \( \kappa > 0 \), since similar computations can be done when \( \kappa < 0 \). Let \( A_n = \sum_{i=0}^{\infty} \theta_i \). Using the argument of Theorem 5 in Wu (2006), under Condition B, we have

\[
\|P_0(K(X_n) - \kappa X_n)\|_q = O(\theta_n), \text{ where } \theta_n = |a_{n_1}|^{p/q} + |a_{n_2}|A_n^{1/2}.
\]

Let \( \theta_i = 0 \) if \( i \leq 0 \) and \( \Theta_n = \sum_{i=1}^{n} \theta_i \). Then by Theorem 1 in Wu (2007), there exists a constant \( B_0 \geq 1 \) such that

\[
\frac{\|S_{n,1}\|_q^2}{B_0^2} \leq \sum_{i \in \mathbb{Z}} (\Theta_{n+i} - \Theta_i)^2 \leq 2n(\Theta_{n+2n}^2) + \sum_{i=n+1}^{\infty} (\Theta_{n+i} - \Theta_i)^2. \quad (36)
\]

By Karamata’s theorem, \( A_n \sim (2r - 1)^{-1} n^{1-2r}I(n)^2 \), and if \( i > n \), \( \Theta_{n+i} - \Theta_i = O(n\theta_i) \) and \( \sum_{i=n+1}^{\infty} \theta_i^2 = O(n\theta_i^2) \). Let \( \ell(\cdot) \) be a slowly varying function and \( \beta \in \mathbb{R} \). Again by Karamata’s theorem, there exists another slowly varying function \( \ell_0(\cdot) \) such that \( \sum_{i=0}^{\infty} i^{-\delta} \ell(i) = O(1+n^{1-\beta})\ell_0(n) \). Hence by (36), there exists a slowly varying function \( \ell_1(\cdot) \) such that

\[
\|S_{n,1}\|_q = O(\sqrt{n})(1+n^{1-rp/q} + n^{1-r+(1-2r)/2})\ell_1(n). \quad (37)
\]

For \( n \geq 3 \) let \( g_n = [\ln(n)]^{-1} \). Then

\[
\mathbb{P}(S_n \geq (x + g_n)\sigma_n) - \mathbb{P}(H_n \geq \kappa \sigma_n) \leq \mathbb{P}(|S_{n,1}| \geq \kappa g_n\sigma_n). \quad (38)
\]

Since \( x^2 \leq c\ln n \) and \( gn = \ln(n)^{-1} \), we have that \( 1 - \Phi(x + g_n) \sim 1 - \Phi(x) \). Hence by Corollary 2.5 (23) follows from (38) in view of

\[
\mathbb{P}(|S_{n,1}| \geq \kappa g_n\sigma_n) \leq \frac{\|S_{n,1}\|_q}{|\kappa|g_n\sigma_n^q} = \frac{O(\sqrt{n^q})(1+n^{q-rp} + n^{(3/2-r)-q})\ell_1^q(n)}{g_n^q(n^{3/2-2r}I(n))^q} \quad (39)
\]

\[
= n^{-r}(\frac{\ell_q^q(n)}{g_n^q\ell(n)}) = \frac{o(n^{-c/2})}{\ln n} = o(xe^{-x/2}) = o[1 - \Phi(x)],
\]

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since \(c/2 < pp(r)\). Here we note that \(\ell_1(n)/(g_n l(n))\) is also slowly varying in \(n\) and \(x \leq c \ln n\). By (37) and (39), it is easily seen that the normalizing constant \(\kappa_{\eta n}\) can be replaced by \(\sqrt{\text{var}(H_n)}\). The proof of the upper bound is similar and it is left to the reader. \(\Diamond\)

5 Appendix

The following Theorem is a slight reformulation of Fuk–Nagaev inequality (see Corollary 1.7, S. Nagaev, 1979):

**Theorem 5.1** Let \(X_1, \ldots, X_n\) be independent random variables. Assume \(m \geq 2\). Suppose \(E X_i = 0\), \(i = 1, \ldots, n\), \(\beta = m/(m + 2)\), and \(\alpha = 1 - \beta = 2/(m + 2)\).

For \(y > 0\), define \(X_i(y) = X_i I(X_i \leq y)\), \(A_n(m; 0, y) := \sum_{i=1}^{k_n} E[X_i^m I(0 < X_i < y)]\) and \(B_n^2(\infty, y) := \sum_{i=1}^{k_n} E[X_i^2 I(X_i < y)]\). Then for any \(x > 0\) and \(y > 0\)

\[
P\left(\sum_{i=1}^{k_n} X_i(y) \geq x\right) \leq \exp\left(-\frac{\alpha x^2}{2 m B_n^2(-\infty, y)}\right) + \left(\frac{A_n(m; 0, y)}{\beta x y^{m-1}}\right)^{\beta x/y}.
\]

(40)

**Remark 5.1** Let \(X_1, X_2, \ldots\), be independent random variables. Assume that the sum \(S = \sum_{i=1}^{\infty} X_i\) exists almost surely. By the same argument as in Lemma 4.2 \(\sum_{i=1}^{\infty} X_i(y)\) converges almost surely for all \(y > 0\). By passing to the limit in (40) we note that this version of Fuk-Nagaev inequality is still valid for \(P(\sum_{i=1}^{\infty} X_i(y) \geq x)\).

We shall also use the following result which is an immediate consequence of Theorem 1.1 in Frolov (2005).

**Theorem 5.2** Let \((X_{nj})_{1 \leq j \leq k_n}\) be an array of row-wise independent centered random variables. Let \(p > 2\) and denote \(S_n = \sum_{j=1}^{k_n} X_{nj}\), \(\sigma_n^2 = \sum_{j=1}^{k_n} E X_{nj}^2\), \(M_{np} = \sum_{j=1}^{k_n} E X_{nj}^p I(X_{nj} \geq 0) < \infty\), \(L_{np} = \sigma_n^{-p} M_{np}\) and denote

\[
\Lambda_n(u, s, \epsilon) = \frac{u}{\sigma_n^2} \sum_{j=1}^{k_n} E X_{nj}^2 I(X_{nj} \leq -\epsilon \sigma_n / s).
\]

Furthermore, assume \(L_{np} \to 0\) and \(\Lambda_n(x^4, x^5, \epsilon) \to 0\) for any \(\epsilon > 0\). Then if \(x \geq 0\) and \(x^2 - 2 \ln(L_{np}^{-1}) - (p - 1) \ln(L_{np}^{-1}) \to -\infty\), we have

\[
P(S_n \geq x \sigma_n) = (1 - \Phi(x))(1 + o(1)).
\]

For truncated random variables by following the proof of Theorem 1.1 in Frolov (2005) we can present his relation (3.17) as a proposition.

**Proposition 5.1** Assume the conditions in Theorem 5.2 are satisfied. Fix \(\epsilon > 0\). Define

\[
X_{nj}^{(\varepsilon \sigma_n)} = X_{nj} I(X_{nj} \leq \varepsilon \sigma_n) \quad \text{and} \quad S_n^{(\varepsilon \sigma_n)} = \sum_{j=1}^{k_n} X_{nj}^{(\varepsilon \sigma_n)}.
\]

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Then if \( x^2 \leq c \ln(L^{-1}) \) with \( c < 1/\varepsilon \), for all \( n \) sufficiently large we have
\[
P \left( S_n^{(\varepsilon \sigma_n)} \geq x\sigma_n \right) = (1 - \Phi(x))(1 + o(1)).
\]

The following facts about the series are going to be used to analyze a class of linear processes:

**Lemma 5.1** Assume \( a_i = (i)^{-r} \) with \( 1/2 < r < 1 \). Let \( b_j := b_{nj} := \sum_{i=1}^{j} a_i \) if \( 1 \leq j \leq n \) and \( b_{nj} := \sum_{i=j-n+1}^{j} a_i \) if \( j > n \). Then, for two positive constants \( C_1 \) and \( C_2 \), we have
\[
C_1(l^t(n)n^{(1-r)t+1}) \leq \sum_{j=1}^{\infty} b_{nj} \leq C_2(l^t(n)n^{(1-r)t+1}),
\]
for any \( t \geq 2 \). In the case \( t = 2 \), \( \sum_{j=1}^{\infty} b_{nj}^2 = c_r n^{3-2r}l^2(n) \) with
\[
c_r = \int_{0}^{\infty} [x^{1-r} - \max(x - 1, 0)^{1-r}]^2 dx / (1 - r)^2.
\]

**Proof.** It is easy to see that \( b_{nj} \ll j^{1-r}l(j) \) for \( j \leq 2n \) and \( b_{nj} \ll n(j - n)^{-r}l(j) \) for \( j > 2n \) from the Karamata theorem (see part 1 of Lemma 5.4 in Peligrad and Sang (2012)). Therefore,
\[
\sum_{j=1}^{2n} b_{nj}^i \leq \sum_{j=1}^{2n} b_{nj}^i + \sum_{j=2n+1}^{\infty} b_{nj}^i \ll \sum_{j=1}^{2n} j^{(1-r)t}l(j) + \sum_{j=2n+1}^{\infty} n^{t(j-n)} - rt(j) = O(l^t(n)n^{(1-r)t+1}).
\]
The proof in the other direction is similar. The result of case \( t = 2 \) is well known. See for instance Theorem 2 in Wu and Min (2005). ♦

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