NON-LEVEL O-SEQUENCES OF CODIMENSION 3 AND DEGREE OF THE
SOCLE ELEMENTS

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Abstract. It is unknown if an Artinian level O-sequence of codimension 3 and type \( r \) \((\geq 2)\) is unimodal, while it is known that any Gorenstein O-sequence of codimension 3 is unimodal. We show that some Artinian non-unimodal O-sequence of codimension 3 cannot be level. We also find another non-level case: if some Artinian algebra \( A \) of codimension 3 has the Hilbert function

\[ H : h_0 \ h_1 \ \cdots \ h_{d-1} \ \underbrace{h_d \ \cdots \ h_d}_{s\text{-times}} \ h_{d+s}, \]

such that \( h_d < h_{d+s} \) and \( s \geq 2 \), then \( A \) has a socle element in degree \( d + s - 2 \), that is, \( A \) is not level.

1. INTRODUCTION

Let \( X = \{P_1, \ldots, P_s\} \) be a set of \( s \) distinct points in the projective space \( \mathbb{P}^n(k) \) (where \( k = \overline{k} \) is an algebraically closed field). Then \( P_i \leftrightarrow \wp_i = (L_{i1}, \ldots, L_{in}) \subset R = k[x_0, x_1, \ldots, x_n] \) where the \( L_{ij}, \ j = 1, \ldots, n \) are \( n \) linearly independent linear forms and \( \wp_i \) is the (homogeneous) prime ideal of \( R \) generated by all the forms which vanish at \( P_i \). The ideal

\[ I = I_X := \wp_1 \cap \cdots \cap \wp_s \]

is the ideal generated by all the forms which vanish at all the points of \( X \).

Since \( R = \bigoplus_{i=0}^{\infty} R_i \) (\( R_i \): the vector space of dimension \( \binom{i+n}{n} \) generated by all the monomials in \( R \) having degree \( i \)) and \( I = \bigoplus_{i=0}^{\infty} I_i \), we get that

\[ A = R/I = \bigoplus_{i=0}^{\infty} (R_i/I_i) = \bigoplus_{i=0}^{\infty} A_i \]

is a graded ring. The numerical function

\[ H_X(t) = H_A(t) := \dim_k A_t = \dim_k R_t - \dim_k I_t \]

is called the \textit{Hilbert function} of the set \( X \) (or of the ring \( A \)).

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Given an O-sequence \( H = (h_0, h_1, \ldots) \), we define the first difference of \( H \) as

\[
\Delta H = (h_0, h_1 - h_0, h_2 - h_1, h_3 - h_2, \ldots).
\]

Let \( h \) and \( i \) be positive integers. Then \( h \) can be written uniquely in the form

\[
h = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}
\]

where \( m_i > m_{i-1} > \cdots > m_j \geq j \geq 1 \). This expansion for \( h \) is called the \( i \)-binomial expansion of \( h \). Also, define

\[
h^{(i)} = \binom{m_i + 1}{i + 1} + \binom{m_{i-1} + 1}{i - 1 + 1} + \cdots + \binom{m_j + 1}{j + 1},
\]

and \( 0^{(i)} = 0 \).

It is worth noting that \( R \) is a standard graded algebra since \( R = k[R_1] \), that is, \( R \) is generated (as a \( k \)-algebra) by its piece of degree 1. If \( I \) is a homogeneous ideal of \( R \), then \( R/I \) is again a standard graded \( k \)-algebra. Furthermore, if \( I \) has a height \( n+1 \) in \( R \), then \( A = R/I \) is an Artinian \( k \)-algebra, and hence \( \dim_k A < \infty \). Thus we can write \( A = k \oplus A_1 \oplus \cdots \oplus A_s \) where \( A_s \neq 0 \). We call \( s \) the socle degree of \( A \).

We associate to the graded Artinian algebra \( A \) a vector of non-negative integers which is an \((s+1)\)-tuple, called the \( h \)-vector of \( A \) and denoted by \( h(A) \). It is defined as follows.

\[
h(A) := (1, \dim_k A_1, \ldots, \dim_k A_s) = (h_0, h_1, \ldots, h_s) \quad \text{with} \quad h_s \neq 0.
\]

Moreover, an \( h \)-vector \((h_0, h_1, \ldots, h_s)\) is called unimodal if \( h_0 \leq \cdots \leq h_t = \cdots = h_{t'} \geq \cdots \geq h_s \).

Let \( \mathcal{F}_X \) be the graded minimal resolution of \( R/I_X \) (or \( X \)), i.e.,

\[
\mathcal{F}_X : \ 0 \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_1 \to R \to R/I_X \to 0.
\]

We can write

\[
\mathcal{F}_i = \bigoplus_{j=1}^{\gamma_i} R^{\beta_{ij}}(-\alpha_{ij})
\]

where \( \alpha_{i1} < \alpha_{i2} < \cdots < \alpha_{i\gamma_i} \). The numbers \( \alpha_{ij} \) are called the shifts associated to \( R/I_X \), and the numbers \( \beta_{ij} \) are called the graded Betti numbers of \( R/I_X \) (or \( X \)).

Now, we recall that if the last free module of the minimal free resolution of a graded ring \( A \) with Hilbert function \( H \) is of the form \( \mathcal{F}_n = R^3(-s) \) for some \( s > 0 \), then Hilbert function \( H \) and a graded ring \( A \) are called level. In particular, if \( \mathcal{F}_n = R^3(-s) \) in \( \mathcal{F}_X \), then we call \( X \) a level set of points in \( \mathbb{P}^n \). For a special case, if \( \beta = 1 \), then we call a graded Artinian algebra \( A \) Gorenstein.

In [16], Stanley proved that any graded Artinian Gorenstein algebra of codimension 3 is unimodal. In fact, he proved a stronger result than unimodality using the structure theorem of Buchsbaum
and Eisenbud for the Gorenstein algebra of codimension 3 in [3]. Since then, the graded Artinian Gorenstein algebras of codimension 3 have been much studied (see [1], [6], [8], [12], [13], [15]). In [1], Bernstein and Iarrobino showed how to construct non-unimodal graded Artinian Gorenstein algebras of codimension higher than or equal to 5. Moreover, in [2], Boij and Laksov showed another method on how to construct the same graded Artinian Gorenstein algebras. Unfortunately, it has been unknown if there exists a graded non-unimodal Gorenstein algebra of codimension 4.

For unimodal Artinian Gorenstein algebras of codimension 4, it has been shown in [15] how to construct some of them using the link-sum method. We have also shown in [8] how to obtain some of unimodal Artinian Gorenstein algebras of any codimension \( n \geq 3 \).

For graded Artinian level algebras, it has been recently studied (see [1], [2], [6], [9]). Since every graded Artinian Gorenstein algebra of codimension 3 is unimodal, the following question in [6] is quite interesting.

**Question 1.1.** Is any level \( O \)-sequence of codimension 3 unimodal (Question 4.4, [6])?

In [6], we proved the following result. Let

\[
H : h_0 \ h_1 \ldots \ h_{d-1} \ h_d \ h_d \ldots
\]

with \( h_{d-1} > h_d \). If \( h_d \leq d + 1 \) with any codimension \( h_1 \), then \( H \) is not level (see Proposition 2.1).

The goal of this paper is to find an answer to Question 1.1 and we give an answer to this question under a certain condition. In fact, it suffices to find an answer to the following Question 1.2 (see Corollary 2.9).

**Question 1.2.** Let \( H \) be an \( O \)-sequence as in equation (1) with codimension 3. Is \( H \) NOT level?

As we mentioned above, it is shown that any Hilbert function \( H \) in equation (1) is not level when \( h_d \leq d + 1 \). In Section 2 we prove that any Hilbert function \( H \) with codimension 3 in equation (1) is not level when \( h_d \leq 2d + 2 \) (see Theorem 2.6). This provides an answer to Question 1.1 when \( h_d \leq 2d + 2 \) (see Corollary 2.9). Finally, in Section 3 we find the degree of the socle elements of a graded Artinian algebra of codimension 3 with Hilbert function

\[
H : h_0 \ h_1 \ldots \ h_{d-1} \underbrace{h_d \ldots \ h_d}_{s\text{-times}} \ h_{d+s},
\]

where \( h_d < h_{d+s} \) and \( s \geq 2 \). We prove that some graded algebra with Hilbert function \( H \) is not level and has a socle element in degree \( d + s - 2 \) (see Theorem 3.4).
In [6], we got an answer to Question 1.1 with the condition \( h_d \leq d + 1 \) as follows.

**Proposition 2.1** (Proposition 2.21, [6]). Let \( h = (1, n, h_2, \ldots, h_s) \) be the \( h \)-vector of an Artinian algebra with socle degree \( s \). Then \( h \) is **not** a level sequence if \( h_d = h_{d+1} \leq d + 1 \) and \( h_{d-1} > h_d \).

We shall expand the above proposition to a case \( h_d \leq 2d + 2 \) with codimension 3.

Let \( R = k[x_1, \ldots, x_n] \) and \( A = R/I \) where \( I \) is a homogeneous ideal of \( R \) having height \( n \). Then \( A \) has the minimal free resolution \( F \), as an \( R \)-module, of the form:

\[
0 \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow R \rightarrow A \rightarrow 0
\]

where \( F_j = \bigoplus_{t=1}^{\gamma_j} R^\beta_{j,j+1+t}(-(j + 1 + t)) \) are each free graded \( R \)-modules. In [5], Eliahou and Kervaire studied minimal free resolutions of certain monomial ideals. We recall some of their notations and results here.

**Definition 2.2.** Let \( T \in R = k[x_1, \ldots, x_n] \) be a term of \( R \). Then

\[
m(T) := \max\{i \mid x_i \text{ divides } T\}.
\]

In other words, \( m(T) \) is the largest index of an indeterminate that divides \( T \).

**Theorem 2.3** (Eliahou–Kervaire, [5]). Let \( I \) be a stable monomial ideal of \( R \) (e.g., a lex segment ideal). Denote by \( G(I)_d \) the elements of that set which have degree \( d \). Then

\[
\beta_{q,i} = \sum_{T \in G(I)_{i-q}} \left( m(T) - 1 \right)^{q-1} \frac{1}{q}.
\]

This beautiful theorem gives all the graded Betti numbers of the lex segment ideal just from an intimate knowledge of the generators of that ideal. Since the minimal free resolution of the ideal of a \( k \)-configuration in \( \mathbb{P}^n \) is extremal ([8], [11]), we may apply this result to those ideals. It is an immediate consequence of the Eliahou–Kervaire theorem that if \( I \) is either a lex-segment ideal or the ideal of a \( k \)-configuration in \( \mathbb{P}^n \) which has no generators of degree \( d \), then \( \beta_{q,i} = 0 \) whenever \( i - q = d \).

By the result of [14], the only way we can cancel graded Betti numbers is if there are the same graded Betti numbers in the adjacent free modules of the extremal minimal free resolution. Note that it is quite obvious for a case of \( n = 3 \).

The following lemma is a simple consequence of a lex segment ideal, so we shall omit the proof here.
Lemma 2.4. Let $I$ be the lex-segment ideal in $R = k[x_1, x_2, x_3]$ with Hilbert function $H = (h_0, h_1, \ldots, h_s)$ where $h_d = d + i$ and $1 \leq i \leq \frac{d^2 + d}{2}$. Then the last monomial of $I_d$ is

$$
x_1 x_2^{i-1} x_3^{d-i}, \quad \text{for} \quad 1 \leq i \leq d,
$$
$$
x_1^2 x_2^{i-(d+1)} x_3^{(2d-1)-i}, \quad \text{for} \quad d+1 \leq i \leq 2d-1,
$$

$$
\vdots
$$
$$
x_1^{d-1} x_2^{i-\frac{d^2+4}{2}} x_3^{\frac{d^2+d}{2}-i}, \quad \text{for} \quad \frac{d^2+d}{2} \leq i \leq \frac{d^2+d-2}{2},
$$
$$
x_1^d, \quad \text{for} \quad i = \frac{d^2+d}{2}.
$$

We need the following proposition to prove the main Theorem 2.6.

**Proposition 2.5.** Let $R = k[x_1, x_2, x_3]$ and let $H = (h_0, h_1, \ldots, h_s)$ be the h-vector of an Artinian algebra with socle degree $s$ and

$$
h_d = h_{d+1} = d + i, \quad h_{d-1} > h_d, \quad \text{and} \quad j := h_{d-1} - h_d
$$

for $i = 1, 2, \ldots, \frac{d^2 + d}{2}$. Then, for every $1 \leq k \leq d$ and $1 \leq \ell \leq d$,

$$
\beta_{1,d+2} = \begin{cases} 2k - 1, & \text{for} \quad (k - 1)d - \frac{k(k-3)}{2} \leq i \leq (k - 1)d - \frac{k(k-3)}{2} + (k - 1), \\ 2k, & \text{for} \quad (k - 1)d - \frac{k(k-3)}{2} + k \leq i \leq kd - \frac{(k - 1)k}{2}. \end{cases}
$$

$$
\beta_{2,d+2} = j + \ell, \quad \text{for} \quad (\ell - 1)d - \frac{(\ell - 2)(\ell - 1)}{2} < i \leq \ell d - \frac{(\ell - 1)\ell}{2}.
$$

**Proof.** Since we assume $h_d = d + i$, the monomials not in $I_d$ are the last $d + i$ monomials of $R_d$. By Lemma 2.4 the last monomial of $R_1 I_d$ is

$$
x_1 x_2^{i-1} x_3^{d-i+1}, \quad \text{for} \quad i = 1, \ldots, d,
$$
$$
x_1^2 x_2^{i-(d+1)} x_3^{2d-i}, \quad \text{for} \quad i = d+1, \ldots, 2d-1,
$$

$$
\vdots
$$
$$
x_1^{d-1} x_2^{i-\frac{d^2+4}{2}} x_3^{\frac{d^2+d}{2}-i}, \quad \text{for} \quad \frac{d^2+d}{2} \leq i \leq \frac{d^2+d-2}{2},
$$
$$
x_1^d x_3, \quad \text{for} \quad i = \frac{d^2+d}{2}.
$$

In what follows, the first monomial of $I_{d+1} - R_1 I_d$ is

$$
x_2^{d+1}, \quad \text{for} \quad i = 1,
$$
$$
x_1 x_2^{i-2} x_3^{(d+2)-i}, \quad \text{for} \quad i = 2, \ldots, d,
$$

$$
\vdots
$$
$$
x_1^{d-1} x_2 x_3, \quad \text{for} \quad i = \frac{d^2+d-2}{2},
$$
$$
x_1^{d-1} x_2^2, \quad \text{for} \quad i = \frac{d^2+d}{2}.
$$

(2)
Note that

\[(d + i)^d = (d + i) + k, \quad \text{for } i = (k - 1)d - \frac{k(k-3)}{2}, \ldots, kd - \frac{k(k-1)}{2}, \quad \text{and } k = 1, \ldots, d.\]

We now calculate the Betti number

\[\beta_{1,d+2} = \sum_{T \in G(T)_{d+1}} \binom{m(T) - 1}{1}.\]

Based on equation (2), we shall find this Betti number of two cases for \(i\) as follows.

**Case 1-1.** \(i = (k - 1)d - \frac{k(k-3)}{2}\) and \(k = 1, 2, \ldots, d.\)

Then, by equation (3), \(I_{d+1}\) has \(k\)-generators, which are

\[x_1^{k-1}x_2^{(d+2)-k}, x_1^{k-1}x_2^{(d+1)-k}x_3, \ldots, x_1^{k-1}x_2^{(d+3)-2k}x_3^{-1}.\]

By the similar argument, for \(i = (k - 1)d - \frac{k(k-3)}{2} + 1, \ldots, (k - 1)d - \frac{k(k-3)}{2} + (k - 1)\), \(I_{d+1}\) has \(k\)-generators including the element \(x_1^{k-1}x_2^{(d+2)-k}\). Hence we have that

\[\beta_{1,d+2} = \sum_{T \in G(T)_{d+1}} \binom{m(T) - 1}{1} = 2 \times (k - 1) + 1 = 2k - 1.\]

**Case 1-2.** \(i = (k - 1)d - \frac{k(k-3)}{2} + k = (k - 1)d - \frac{k(k-5)}{2}, \ldots, kd - \frac{k(k-1)}{2}\) and \(k = 1, 2, \ldots, d.\)

Then, by equation (3), \(I_{d+1}\) has \(k\)-generators, which are

\[x_1^kx_2^{i-((k-1)d-\frac{k(k-3)}{2})}x_3^{kd-\frac{k^2-3k-2}{2}}, \ldots, x_1^kx_2^{i-((k-1)d-\frac{k(k-5)}{2})}x_3^{kd-\frac{k(k-3)+1}{2}}.\]

Hence we have that

\[\beta_{1,d+2} = \sum_{T \in G(T)_{d+1}} \binom{m(T) - 1}{1} = 2 \times k = 2k.\]

Now we move on the Betti number:

\[\beta_{2,d+2} = \sum_{T \in G(T)_{d}} \binom{m(T) - 1}{2}.\]

Recall \(h_d = d + i\) and \(j := h_{d-1} - h_d\). The calculation in this case is much more complicated, and there are four cases based on \(i\) and \(j\).

**Case 2-1.** \((\ell - 1)d - \frac{(\ell-2)(\ell-1)}{2} < i < \ell d - \frac{(\ell-1)\ell}{2}\) and \(\ell = 1, 2, \ldots, d.\)

Then the last monomial of \(I_d\) is

\[x_1^\ell x_2^{i-(\ell-1)d+\frac{(\ell-3)\ell}{2}}x_3^{\ell d-(\ell-1)\ell-i}.\]
(a) \((k - 1)d - \frac{(k-1)k}{2} < i + j < kd - \frac{k(k+1)}{2}\) and \(k = \ell, \ell + 1, \ldots, d\).

Then the first monomial of \(I_d - R_1I_{d-1}\) is

\[
\frac{x^i_1x^j_2 - (k-1)d - \frac{(k-2)(k+1)}{2}}{x^j_3} \left( kd - \frac{(k-1)(k+2)}{2} \right) - (i+j),
\]

and hence we have \((j + k)\)-generators in \(I_d\) as follows:

\[
x^i_1x^j_2 - (k-1)d - \frac{(k-2)(k+1)}{2} x^j_3 - (i+j), \ldots, x^i_1x^{d-k}_2, \ldots,
\]

\[
x^{(k-1)\ell - (k-1)}_1 x^{(k-1)\ell - (k-1)}_2 x^{(k-1)\ell - (k-1)}_3, \ldots, x_1^{k-1}x^{d-(k-1)}_2 x_3^{d-(k-1)},
\]

and thus

\[
\beta_{2,d+2} = \sum_{T \in G(I_d)} \left( \frac{m(T) - 1}{2} \right) = j + \ell,
\]

(b) \(i + j = (k - 1)d - \frac{(k-1)k}{2}\) and \(k = \ell + 1, \ldots, d\).

Then the first monomial of \(I_d - R_1I_{d-1}\) is

\[
x^{k-1}_1x^{d-(k-1)}_2,
\]

and hence we have \((j + k)\)-generators in \(I_d\) as follows:

\[
x^{k-1}_1x^{d-(k-1)}_2, x^{k-1}_1x^{d-(k-1)}_2x^{d-(k-1)}_3, \ldots, x_1^{k-1}x^{d-(k-1)}_3,
\]

\[
x^{\ell + 1}_1x^{(d-1) - \ell}_2, x^{\ell + 1}_1x^{(d-1) - \ell}_2x^{(d-1) - \ell}_3, \ldots, x_1^{\ell + 1}x^{(d-1) - \ell}_3,
\]

\[
x^{\ell}_1x^{d-\ell}_2 \ldots, x_1^{\ell + i - (\ell - 1)d + \frac{(\ell - 1)k}{2} - \frac{(\ell - 1)\ell}{2} - i} x^{d-(\ell - 1)\ell + \frac{(\ell - 1)k}{2} - \frac{(\ell - 1)\ell}{2}}_3,
\]

and thus

\[
\beta_{2,d+2} = \sum_{T \in G(I_d)} \left( \frac{m(T) - 1}{2} \right) = j + \ell.
\]

Case 2-2. \(i = \ell d - \frac{(\ell-1)\ell}{2}\) and \(\ell = 1, 2, \ldots, d\).

Then the last monomial of \(I_d\) is

\[
x^{\ell}_1x^{d-\ell}_2.
\]

(a) \((k - 1)d - \frac{(k-1)k}{2} < i + j < kd - \frac{k(k+1)}{2}\) and \(k = \ell + 1, \ldots, d\).

Then the first monomial of \(I_d - R_1I_{d-1}\) is

\[
x^i_1x^j_2 - (k-1)d - \frac{(k-2)(k+1)}{2} x^j_3 - (i+j)
\]

\[
x^{k-1}_1x^{d-(k-1)}_2 x^{d-(k-1)}_3, \ldots, x_1^{k-1}x^{d-(k-1)}_3,
\]

and thus

\[
\beta_{2,d+2} = \sum_{T \in G(I_d)} \left( \frac{m(T) - 1}{2} \right) = j + \ell.
\]
and hence we have \((j + k)\)-generators in \(I_d\) as follows:
\[
x_1^{(k-1)} d^{(k-1) - (i)} x_2^{(k-1)} d^{(k-1) - (i)} x_3^{(k-1)} d^{(k-1) - (i)} , \ldots , x_1^{k-1} d^{k-1} \}
\]
\[
\vdots
\]
\[
x_1^{\ell+1} x_2^{(d-1) - \ell} x_3^{\ell+1} x_2^{(d-2) - \ell} x_3^{\ell+1} x_2^{(d-1) - \ell} x_3^{\ell} \}
\]
\[
x_1^{\ell} x_2^{d - \ell} \}
\]
and thus
\[
\beta_{2,d+2} = \sum_{T \in \mathcal{G}(I_d)} \left( m(T) - 1 \right) = j + \ell.
\]

(b) \(i + j = (k - 1)d - \frac{(k-1)k}{2}\) and \(k = \ell + 1, \ldots, d\).

Then the first monomial of \(I_d - R_1 I_{d-1}\) is
\[
x_1^{(k-1)} d^{(k-1) - (i)}
\]
and hence we have \((j + k)\)-generators in \(I_d\) as follows:
\[
x_1^{(k-1)} d^{(k-1) - (i)} x_2^{(k-1)} d^{(k-1) - (i)} x_3^{(k-1)} d^{(k-1) - (i)} , \ldots , x_1^{k-1} d^{k-1} \}
\]
\[
\vdots
\]
\[
x_1^{\ell+1} x_2^{(d-1) - \ell} x_3^{\ell+1} x_2^{(d-2) - \ell} x_3^{\ell+1} x_2^{(d-1) - \ell} x_3^{\ell} \}
\]
\[
x_1^{\ell} x_2^{d - \ell} \}
\]
and thus
\[
\beta_{2,d+2} = \sum_{T \in \mathcal{G}(I_d)} \left( m(T) - 1 \right) = j + \ell,
\]
as we wished. \(\square\)

Now we are ready to prove the main theorem in this section.

**Theorem 2.6.** Let \(H\) and \(j\) be as in Proposition 2.5. Then for every \(-(d - 1) \leq i \leq d + 2\), \(H\) is not level.

**Proof.** By Proposition 2.5, this theorem holds for \(-(d - 1) \leq i \leq 1\). It suffices to prove this theorem for \(2 \leq i \leq d + 2\). By Proposition 2.5 we have that
\[
\beta_{1,d+2} = \begin{cases} 2, & \text{for } i = 2, \ldots, d, \\ 3, & \text{for } i = d + 1, d + 2 \end{cases}
\]
and
\[
\beta_{2,d+2} = \begin{cases} j + 1, & \text{for } i = 2, \ldots, d, \\ j + 2, & \text{for } i = d + 1, d + 2 \end{cases}
\]
Hence if \(j \geq 2\), then \(H\) is not level since \(\beta_{2,d+2} > \beta_{1,d+2}\). It is enough, therefore, to show the case \(j = 1\).
First, assume \( i = 2, 3, \ldots, d \). Then, by equation 4, we have \( \beta_{1,d+2} = \beta_{2,d+2} = 2 \). Moreover, we see that \( h_{d-1} = d + i + j = d + i + 1 \) and \( h_d = h_{d+1} = d + i \).

Now suppose \( A = R/I, \) \( R = k[x_1, x_2, x_3], \) is a level algebra with \( h \)-vector \( (h_0, h_1, \ldots, h_{d-1}, h_d, h_{d+1}) \) where \( h_d = h_{d+1} \) and the ideal \( I \) has \( h^{(d+1)}_{d+1} = (d + i)^{(d+1)} = (d + i + 1) \)-generators in degree \( d + 2 \). Let \( J = (I_{\leq d+1}) \). Then the Hilbert function of \( R/J \) begins

\[
\begin{align*}
h_0, h_1, \ldots, h_{d-1}, d + i, d + i, d + i + 1, \ldots
\end{align*}
\]

Note that \( d + i - 1, d + i, d + i + 1 \) in degrees \( d, d + 1, d + 2 \) have the maximal growth, and so, by Theorem 3.4 in [6], \( R/J \) has one dimensional socle element in degree \( d \). Since \( R/J \) and \( R/I \) agree in degree \( \leq d + 1 \), \( R/I \) has such a socle element. It follows that in order for \( R/I \) to be level, \( I \) must have at most \( (d + i) \)-generators in degree \( d + 2 \). Then both copies \( R(−(d + 2)) \) of the last free module of the minimal free resolution of \( R/I \) cannot be canceled. Therefore, the Hilbert function \( H \) cannot be level.

Second, assume \( i = d + 1 \). By equation 4, we have \( \beta_{1,d+2} = \beta_{2,d+2} = 3 \).

Suppose the ideal \( I \) has \( h^{(d+1)}_{d+1} = (2d + 1)^{(d+1)} = (2d + 2) \)-generators in degree \( d + 2 \) and let \( J = (I_{\leq d+1}) \). Then the Hilbert function of \( R/J \) begins

\[
\begin{align*}
h_0, h_1, \ldots, h_{d-1}, 2d + 1, 2d + 1, 2d + 2, \ldots
\end{align*}
\]

Note also that \( 2d, 2d + 1, 2d + 2 \) in degrees \( d, d + 1, d + 2 \) have the maximal growth. Therefore, by Theorem 3.4 in [6] again, \( R/J \) has one dimensional socle element in degree \( d \), so does \( R/I \) by the same argument as above. Thus three copies \( R(−(d + 2)) \) of the last free module of the minimal free resolution of \( R/I \) cannot be canceled. Therefore, the Hilbert function \( H \) cannot be level.

Finally, assume \( i = d + 2 \). By the similar argument to the case \( i = d + 1 \), \( H \) is not level either, as we wished.

Theorem 2.6 shows that Question 1.2 is true if \( h_d \leq 2d + 2 \). The following example shows a case \( j = 1 \) and \( i = d + 2 \) \( (h_d = 2d + 2) \) of this theorem.

**Example 2.7.** Let \( I \) be the lex-segment ideal in \( R = k[x_1, x_2, x_3] \) with Hilbert function

\[
\begin{align*}
\mathbf{H} : & \quad 1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 17 \quad 16 \quad 16 \quad 0 \quad \rightarrow
\end{align*}
\]

Note that \( h_7 = 16 = 2 \times 7 + 2 = 2d + 2 \), which satisfies the condition in Theorem 2.6 and \( j = h_6 - h_7 = 17 - 16 = 1 \). Hence any Artinian ring with Hilbert function \( \mathbf{H} \) cannot be level.
We now give another example for \( i = d + 3 \) \((h_d = 2d + 3)\) which does not satisfy the condition in Theorem 2.6.

**Example 2.8.** Let \( I \) be the lex-segment ideal of \( R \) with Hilbert function

\[
H : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 18 \ 17 \ 17 \ 0 \rightarrow .
\]

Note that \( h_7 = 17 = 2 \times 7 + 3 = 2d + 3 \) and \( j = 18 - 17 = 1 \). Hence, by Proposition 2.5 we have \( \beta_{1,d+2} = 4 \) and \( \beta_{2,d+2} = 3 \), that is, \( \beta_{1,d+2} > \beta_{2,d+2} \). This means that we cannot say if Hilbert function \( H \) is level only based on shifts and Betti numbers. In other words, for \( i \leq d + 2 \) (or \( h_d \leq 2d + 2 \)), we can decide if \( H \) is **not** level using shifts and Betti numbers.

We now pass to Question 1.1 and the following corollary answers to this question for \( i \leq d + 2 \) (or \( h_d \leq 2d + 2 \)).

**Corollary 2.9.** Let \( H = \{h_i\}_{i \geq 0} \) be an O-sequence with \( h_1 = 3 \). If

\[
h_{d-1} > h_d, \quad h_d \leq 2d + 2, \quad \text{and} \quad h_{d+1} \geq h_d
\]

for some degree \( d \), then \( H \) is not level.

**Proof.** Note that, by the proof of Theorem 2.6 any graded ring with Hilbert function

\[
H' : h_0 \ h_1 \ \cdots \ h_{d-1} \ h_d \ h_d \rightarrow
\]

has a socle element in degree \( d - 1 \).

Now let \( A = \bigoplus_{i \geq 0} A_i \) be a graded ring with Hilbert function \( H \). If \( A_{d+1} = \{f_1, f_2, \ldots, f_{h_{d+1}}\} \) and \( I = (f_{h_{d+1}}, \ldots, f_{h_{d+1}}) \bigoplus_{j \geq d+2} A_j \), then a graded ring \( B = A/I \) has Hilbert function

\[
h_0 \ h_1 \ \cdots \ h_{d-1} \ h_d \ h_d,
\]

and hence \( B \) has a socle element in degree \( d - 1 \) by Theorem 2.6. Since \( A_i = B_i \) for every \( i \leq d \), \( A \) also has the same socle element in degree \( d - 1 \) as \( B \), and thus \( H \) is not level as we wished. \( \square \)

**Example 2.10.** Consider an O-sequence

\[
H : 1 \ 3 \ 6 \ 10 \ 14 \ 18 \ 17 \ 16 \ h_8 \ \cdots .
\]

Then, there are only 3 possible O-seuquences such that \( h_8 \geq h_7 = 16 \) since \( h_8 \leq h_7^{(7)} = 16^{(7)} = 18 \). By Theorem 2.6 \( H \) is not level if \( h_8 = h_7 = 16 \). The other two non-unimodal O-sequences, by Corollary 2.9

\[
1 \ 3 \ 6 \ 10 \ 14 \ 18 \ 17 \ 16 \ 17 \ \cdots \ \text{and}
\]

\[
1 \ 3 \ 6 \ 10 \ 14 \ 18 \ 17 \ 16 \ 18 \ \cdots .
\]
cannot be level.

3. Degree Of The Socle Elements Of Graded Artinian Algebras

In this section, we are interested in another non-level O-sequences of codimension 3:

(5) \( H : h_0 \ h_1 \ \cdots \ h_{d-1} \ \ h_d \ \cdots \ h_{d+s} \)

where \( s \geq 2 \) and \( h_d < h_{d+s} \). In particular, we shall prove that some graded algebra with Hilbert function \( H \) of codimension 3 in equation (5) has a socle element in degree \( d + s - 2 \), and hence cannot be level.

First, we recall the definitions of type vectors and \( k \)-configurations in \( \mathbb{P}^n \).

**Definition 3.1** \((n\text{-type vectors}, \text{Definition 2.1, [7]})\).  
1) A 0\text{-type vector} will be defined to be \( T = 1 \). It is the only 0\text{-type vector}. We shall define \( \alpha(T) = -1 \) and \( \sigma(T) = 1 \).

2) A 1\text{-type vector} is a vector of the form \( T = (d) \) where \( d \geq 1 \) is a positive integer. For such a vector we define \( \alpha(T) = d = \sigma(T) \).

3) A 2\text{-type vector}, \( T \), is

\[ T = ((d_1), (d_2), \ldots, (d_m)) \]

where \( m \geq 1 \), the \( (d_i) \) are 1\text{-type vectors}. We also insist that \( \sigma(d_i) = d_i < \alpha(d_{i+1}) = d_{i+1} \).

For such a \( T \) we define \( \alpha(T) = m \) and \( \sigma(T) = \sigma((d_m)) = d_m \).

Clearly, \( \alpha(T) \leq \sigma(T) \) with equality if and only if \( T = ((1), (2), \ldots, (m)) \). For simplicity in the notation we usually rewrite the 2\text{-type vector} \( ((d_1), \ldots, (d_m)) \) as \( (d_1, \ldots, d_m) \).

4) Now let \( n \geq 3 \). An \( n\text{-type vector}, T \), is an ordered collection of \( (n-1)\text{-type vectors}, \ T_1, \ldots, T_s \), i.e.

\[ T = (T_1, \ldots, T_s) \]

for which \( \sigma(T_i) < \alpha(T_{i+1}) \) for \( i = 1, \ldots, s-1 \).

For such a \( T \) we define \( \alpha(T) = s \) and \( \sigma(T) = \sigma(T_s) \).

**Definition 3.2** \((k\text{-configuration in } \mathbb{P}^n, \text{Definition 4.1, [7]})\).

\( S_0 \): The only element in \( S_0 \) is \( H := 1 \rightarrow \). It is the Hilbert function of \( \mathbb{P}^0 \), which is a single point. That is the only \( k \)-configuration in \( \mathbb{P}^0 \).

\( S_1 \): Let \( H \in S_1 \). Then \( \chi_1(H) = T = (e) \) where \( e \geq 1 \). We associate to \( H \) any set of \( e \) distinct points in \( \mathbb{P}^1 \). Clearly any set of \( e \) distinct points in \( \mathbb{P}^1 \) has Hilbert function \( H \).

A set of \( e \) distinct points in \( \mathbb{P}^1 \) will be called a \( k\text{-configuration in } \mathbb{P}^1 \) of type \( T = (e) \).
Remark 3.3. Now we shall introduce some non-level O-sequences based on type vectors.

(a) Let $X$ be a $k$-configuration in $\mathbb{P}^2$ of type $T = (d_1, \ldots, d_\alpha)$ with $d_{i+1} - d_i \geq 3$ for some $i = 1, \ldots, \alpha - 1$. Since $X$ is a $k$-configuration in $\mathbb{P}^2$ of type $T = (d_1, \ldots, d_\alpha)$, we have the minimal free resolution of $R/I_X$ is

$$
0 \to R(-(d_1 + \alpha)) \oplus \cdots \oplus R(-(d_i + \alpha - i + 1)) \oplus \cdots \oplus R(-(d_\alpha + 1)) \\
\to R(-\alpha) \oplus R(-(d_1 + \alpha - 1)) \oplus \cdots \oplus R(-(d_i + \alpha - i)) \oplus \cdots \oplus R(-d_\alpha) \\
\to R \to R/I_X \to 0
$$

by Theorem 2.6 in [10]. Since $d_{i+1} - d_i \geq 3$, we have that $d_i + \alpha - i + 1 < d_{i+1} + \alpha - (i+1)$, which means that $R(-(d_i + \alpha - i + 1))$ of the last free module cannot be canceled. Hence the Hilbert function $H_X$ is not level.

(b) Let $X$ be a $k$-configuration in $\mathbb{P}^3$ of type $T = (T_1, \ldots, T_\alpha)$ and let $F_X$ be the minimal free resolution of the coordinate ring of $X$. If either $T_i$ is the 2-type vector as in this remark (a) or $\sigma(T_i) + 2 < \alpha(T_{i+1})$ for some $i = 1, \ldots, \alpha - 1$, then the Hilbert function $H_X$ is not level. To show this, we shall use the same notation as in Theorem 3.2, [11]. If $T_i$ is a 2-type vector as in (a) for some $i$, then $H_X$ is obviously not level by the same idea as in (a). Now
assume that $\sigma(T_i) + 2 < \alpha(T_i+1)$. Then we have

$$
\begin{align*}
\varepsilon_i + 2 + \tilde{d}_{i\alpha(T_i)} &= \alpha(T_i) - i + \alpha + 2 + \sigma(T_i) - \alpha(T_i) \\
&= \sigma(T_i) - i + \alpha + 2, \quad \text{and} \\
\varepsilon_{i+1} + 1 &= \alpha(T_{i+1}) - (i + 1) + \alpha + 1 \\
&= \alpha(T_{i+1}) - i + \alpha.
\end{align*}
$$

Since $\sigma(T_i) + 2 < \alpha(T_{i+1})$, we see that $\varepsilon_i + 2 + \tilde{d}_{i\alpha(T_i)} < \varepsilon_{i+1} + 1$, and hence $R(- (\varepsilon_i + 2 + \tilde{d}_{i\alpha(T_i)}))$ in the last free module of $F_X$ cannot be canceled, that is, the Hilbert function $H_X$ is not level.

(c) Let $T = (T_1, \ldots, T_a)$ be the 3-type vector with $T_i = (d_{i1}, \ldots, d_{i\alpha(T_i)})$ for every $i$. If $\alpha(T_i) = \sigma(T_{i-1}) + 1$ and $d_{i1} \geq 3$ for some $i$, then the Hilbert function of a $k$-configuration $X$ in $\mathbb{P}^3$ of type $T$ is not level. To show this, we shall use the same notation as in Theorem 3.2, \[11\] again. Then we have

$$
\begin{align*}
\varepsilon_{i-1} + 2 + \tilde{d}_{i-1\alpha(T_{i-1})} &= \alpha(T_{i-1}) - (i - 1) + \alpha + 2 + d_{i-1\alpha(T_{i-1})} - \alpha(T_{i-1}) \\
&= d_{i-1\alpha(T_{i-1})} + \alpha - i + 3, \quad \text{and} \\
\varepsilon_i + 1 &= \alpha(T_i) + \alpha - i + 1.
\end{align*}
$$

Since $\alpha(T_i) = \sigma(T_{i-1}) + 1 = d_{i-1\alpha(T_{i-1})} + 1$, we see that $\varepsilon_i + 1 < \varepsilon_{i-1} + 2 + \tilde{d}_{i-1\alpha(T_{i-1})}$. Moreover, we have

$$
\begin{align*}
\varepsilon_i + 1 + \tilde{d}_{i1} &= \alpha(T_i) + \alpha - i + 1 + d_{i1} - 1 \\
&= \alpha(T_i) + \alpha - i + d_{i1} \\
&> d_{i-1\alpha(T_{i-1})} + \alpha - i + 3 \\
&= \varepsilon_{i-1} + 2 + \tilde{d}_{i-1\alpha(T_{i-1})}.
\end{align*}
$$

In other words,

$$
\varepsilon_i + 1 < \varepsilon_{i-1} + 2 + \tilde{d}_{i-1\alpha(T_{i-1})} < \varepsilon_i + 1 + \tilde{d}_{i1},
$$

and hence $R(- (\varepsilon_{i-1} + 2 + \tilde{d}_{i-1\alpha(T_{i-1})}))$ in the last free module of $F_X$ cannot be canceled, that is, the Hilbert function $H_X$ is not level.

Now we are ready to discuss about the degree of the socle elements of some Artinian algebra with Hilbert function $H$ in equation \[5\].

**Theorem 3.4.** Let $H$ be as in equation \[5\] and $T = (T_1, \ldots, T_a)$ be the 3-type vector corresponding to the Hilbert function whose first difference is $H$. If $h_d = \cdots = h_{d+s-1} = d + s + (i - 1)$ and
\( h_{d+s} = d + s + i \) where \( 1 \leq i \leq \alpha(T_{\alpha-1}) \), then

\[
T_{\alpha} = (1, 2, \ldots, d + s, d + s + 1),
\]

\[
T_{\alpha-1} = \begin{cases} 
(\ldots, d + s - 2), & \text{for } i = 1, \\
(\ldots, d + s - (i + 1), d + s - (i - 2), \ldots, d + s), & \text{for } i = 2, \ldots, \alpha(T_{\alpha-1}).
\end{cases}
\]

In particular, the O-sequence \( H \) is not level and any Artinian graded algebra with Hilbert function \( H \) has a socle element in degree \( d + s - 2 \).

**Proof.** It suffices to prove this theorem for \( i = 1 \) and 2, respectively, since we can use the same argument for the rest of the cases \( i \geq 3 \) as for \( i = 2 \).

**Case 1.** If \( i = 1 \), that is, \( h_d = \cdots = h_{d+s-1} = d + s \) and \( h_{d+s} = d + s + 1 \), then from the following equation,

\[
H : \begin{array}{cccccccc}
h_0 & h_1 & h_2 & \cdots & h_d & h_d & h_{d+s} \\
1 & 2 & 3 & \cdots & d + s - 1 & d + s & d + s + 1 \\
1 & a & \cdots & 1 & 0 & 0 & \\
1 & \cdots & 1 & \\
\end{array}
\]

where \( 1 \leq a \leq 3 \), we have that

\[
T_{\alpha} = (1, 2, 3, \ldots, d + s, d + s + 1)
\]

\[
T_{\alpha-1} = (\ldots, d + s - 2).
\]

In what follows

\[
\alpha(T_{\alpha}) = d + s + 1 = (d + s - 2) + 3 = \sigma(T_{\alpha-2}) + 3 > \sigma(T_{\alpha-1}) + 2,
\]

and hence, by Remark 3.3 (b), \( H \) is not level.

Recall that \( F_X \) is the minimal free resolution of a coordinate ring \( R/I_X \) of a \( k \)-configuration \( X \) in \( \mathbb{P}^d \) with Hilbert function \( G \) such that \( \Delta G = H \) and we shall use the same notation as in Theorem 3.2 in [11] for the rest of the proof. Since the non-cancelable shift of the last
free module of \( \mathcal{F}_X \) is
\[
\varepsilon_{\alpha-1} + 2 + \tilde{d}_{\alpha-1}(\mathcal{T}_\alpha-1) \\
= \left[ \alpha(\mathcal{T}_\alpha-1) - (\alpha - 1) + \alpha \right] + 2 + d_{\alpha-1}(\alpha(\mathcal{T}_\alpha-1)) - \alpha(\mathcal{T}_\alpha-1) \\
= d_{\alpha-1}(\alpha(\mathcal{T}_\alpha-1)) + 3 \\
= (d + s - 2) + 3,
\]
any algebra with Hilbert function \( H \) has a socle element in degree \( d + s - 2 \).

**Case 2.** If \( i = 2 \), that is, \( h_d = \cdots = h_{d+s-1} = d + s + 1 \) and \( h_{d+s} = d + s + 2 \), then from the following equation
\[
H : h_0 h_1 h_2 \cdots h_d \quad h_{(d+s-1)}-th \quad h_d \quad h_{d+s} \\
1 \quad 2 \quad 3 \quad \cdots \quad d+s-1 \quad d+s \quad d+s+1 \\
1 \quad a \quad \cdots \quad 2 \quad 1 \quad 1 \\
1 \quad 2 \quad \cdots \quad 2 \quad 1 \quad 1 \\
\cdots
\]
where \( 2 \leq a \leq 3 \), we have that
\[
\mathcal{T}_\alpha = (1, 2, 3, \ldots, d+s, d+s+1) \\
\mathcal{T}_{\alpha-1} = (\ldots, d+s-3, d+s).
\]
Since the difference of the last two 2-type vectors of \( \mathcal{T}_{\alpha-1} \) is 3, by Remark 3.3 (a), any Artinian algebra with Hilbert function \( H \) is not level. Furthermore, the non-cancelable shift of the last free module of \( \mathcal{F}_X \) is
\[
\varepsilon_{\alpha-1} + 2 + \tilde{d}_{\alpha-1}(\alpha(\mathcal{T}_{\alpha-1})-1) \\
= \left[ \alpha(\mathcal{T}_{\alpha-1}) - (\alpha - 1) + \alpha \right] + 2 + d_{\alpha-1}(\alpha(\mathcal{T}_{\alpha-1})-1) - \alpha(\mathcal{T}_{\alpha-1}) - 1) \\
= d_{\alpha-1}(\alpha(\mathcal{T}_{\alpha-1})-1) + 4 \\
= (d + s - 2) + 4,
\]
and thus any Artinian algebra with Hilbert function \( H \) has a socle element in degree \( d+s-2 \).

By continuing this process for \( 3 \leq i \leq \alpha(\mathcal{T}_{\alpha-1}) \), we complete the proof, as we wished.  \( \square \)

**Remark 3.5.** Using the same argument as in the proof of Corollary 2.9, Theorem 3.4 holds when \( h_{d+s} > h_d \), in general.

**Example 3.6.** Consider an O-sequence \( H : 1 \quad 3 \quad 6 \quad 8 \quad 9 \quad 9 \quad 10 \). Then \( d = 4 \) and \( s = 3 \), and so \( h_{d+s} = 10 = 7 + 3 \), that is, \( i = 3 \). Note that \( \alpha(\mathcal{T}_2) = 4 \). Applying Theorem 3.4 to this
case, we conclude any graded Artinian algebra with Hilbert function $H$ is not level and has a socle element in degree $d + s - 2 = 5$.

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