CHARACTERIZATION OF 1-ALMOST GREEDY BASES

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Abstract. This article closes the cycle of characterizations of greedy-like bases in the “isometric” case initiated in [1] with the characterization of 1-greedy bases and continued in [2] with the characterization of 1-quasi-greedy bases. Here we settle the problem of providing a characterization of 1-almost greedy bases in Banach spaces. We show that a basis in a Banach space is almost greedy with almost greedy constant equal to 1 if and only if it has Property (A). This fact permits now to state that a basis is 1-greedy if and only if it is 1-almost greedy and 1-quasi-greedy. As a by-product of our work we also provide a tight characterization of almost greedy bases.

1. Introduction and background

Suppose $(X, \| \cdot \|)$ is an infinite-dimensional separable (real or complex) Banach space and $B = (e_n)_{n=1}^\infty$ is a (Schauder) basis for $X$. We will denote by $(e^*_n)_{n=1}^\infty$ the corresponding biorthogonal functionals. Assume that $B$ is semi-normalized, i.e., $0 < \inf_n \|e_n\| \leq \sup_n \|e_n\| < \infty$. Each $x \in X$ has a unique series expansion in terms of the basis,

$$x = \sum_{n=1}^\infty e^*_n(x)e_n.$$

The support of an $x \in X$ is the set $\text{supp}(x) = \{n \in \mathbb{N} : e^*_n(x) \neq 0\}$.

A greedy ordering for $x$ is an injective mapping $\rho : \mathbb{N} \to \mathbb{N}$ such that $\text{supp}(x) \subset \rho(\mathbb{N})$ and $|e^*_{\rho(i)}(x)| \geq |e^*_{\rho(j)}(x)|$ if $i \leq j$. For each $m \in \mathbb{N}$, an $m$-term greedy approximation to $x$, denoted by $G_m(x)$, is the $m$-th partial sum of the (formal) rearranged series $\sum_{i=1}^\infty e^*_{\rho(i)}(x)e_{\rho(i)}$ for some greedy ordering $\rho$ of $x$, i.e.,

$$G_m(x) = \sum_{i=1}^m e^*_{\rho(i)}(x)e_{\rho(i)}.$$
If the sequence \((e_n^*(x))_{n=1}^\infty\) contains several terms with the same absolute value then the greedy ordering for \(x\) is not unequivocally determined. However, there is a unique greedy ordering \(\rho\) for \(x\) fulfilling the extra properties that \(\rho(i) \leq \rho(j)\) if \(|e_{\rho(i)}^*(x)| = |e_{\rho(j)}^*(x)|\) and \(\rho(\mathbb{N}) = \mathbb{N}\) if \(x\) is finitely supported (notice that \(\rho(\mathbb{N}) = \text{supp}(x)\) is \(x\) is infinitely supported). We will refer to this ordering as the natural greedy ordering for \(x\). With this convention, the \(m\)-th natural greedy approximation to \(x\) is given by

\[
G_m(B, \mathcal{X})(x) := G_m(x) = \sum_{i=1}^m e_{\rho(i)}^*(x)e_{\rho(i)},
\]

where \(\rho\) is the natural greedy ordering.

Konyagin and Temlyakov [6] defined a basis to be greedy if \(G_m(x)\) is essentially the best \(m\)-term approximation to \(x\) using basis vectors, i.e., there exists a constant \(C \geq 1\) such that, for all \(x \in \mathcal{X}\) and \(m \in \mathbb{N}\),

\[
\|x - G_m(x)\| \leq C \inf\{\|x - \sum_{n \in A} \alpha_n e_n\| : |A| = m, \alpha_n \text{ scalars}\}. \tag{1.1}
\]

The smallest \(C\) in (1.1) is the greedy constant of the basis and will be denoted by \(C_g\). They then showed that greedy bases can be intrinsically characterized as unconditional bases with the additional property of being democratic, i.e., \(\|\sum_{n \in A} e_n\| \leq \Delta \|\sum_{n \in B} e_n\|\) whenever \(|A| = |B|\), for some constant \(\Delta \geq 1\). Recall also that a basis \((e_n)_{n=1}^\infty\) is unconditional if for \(x \in \mathcal{X}\) the series \(\sum_{i=1}^\infty e_{\pi(i)}^*(x)e_{\pi(i)}\) converges to \(x\) for any permutation \(\pi\) of \(\mathbb{N}\).

The property of being unconditional is easily seen to be equivalent to that of being suppression unconditional, which means that there is a constant \(K\) such that for all \(x \in \mathcal{X}\) and all \(A \subset \mathbb{N}\), \(A\) finite,

\[
\|x - P_A(x)\| \leq K \|x\|, \tag{1.2}
\]

where \(P_A(x) = \sum_{n \in A} e_n^*(x)e_n\) is the natural projection onto the linear span of \(\{e_n : n \in A\}\). The smallest \(K\) in (1.2) coincides with the least constant \(K\) such that for all \(x \in \mathcal{X}\) and \(A \subset \mathbb{N}\) finite we have

\[
\|P_A(x)\| \leq K \|x\|.
\]

This optimal constant is called the suppression unconditional constant of the basis and is denoted by \(K_{su}\). If \(\mathcal{B}\) is unconditional and \(K \geq 1\) is such that \(K_{su} \leq K\) we say that \(\mathcal{B}\) is \(K\)-suppression unconditional.

A basis which is democratic and unconditional is superdemocratic, i.e., there exists a best constant \(\Gamma \geq 1\) (\(\Gamma\)-superdemocratic) such that
the inequality
\[ \left\| \sum_{n \in A} \varepsilon_n e_n \right\| \leq \Gamma \left\| \sum_{n \in B} \theta_n e_n \right\| \]
holds for any two finite sets of integers \( A \) and \( B \) of the same cardinality, and any choice of signs \( (\varepsilon_n)_{n \in A} \) and \( (\theta_n)_{n \in B} \).

Another key concept in the theory, introduced as well by Konyagin and Temlyakov \[6\], is that of quasi-greedy basis. A basis is quasi-greedy if for \( x \in X \), \( \lim_{m \to \infty} G_m(x) = x \), that is, the series \( \sum_{i=1}^{\infty} e_{\rho(i)}^*(x) e_{\rho(i)} \) converges to \( x \), where \( \rho \) is the natural greedy ordering for \( x \). Subsequently, Wojtaszczyk \[7\] proved that these are precisely the bases for which the greedy operators \( (G_m)_{m=1}^{\infty} \) are uniformly bounded (despite the fact that they are nonlinear on \( x \)), i.e., there exists a constant \( C \geq 1 \) such that for all \( x \in X \) and \( m \in \mathbb{N} \),
\[ \|G_m(x)\| \leq C\|x\|. \tag{1.3} \]

We will denote by \( C_w \) the smallest constant in (1.3).

Obviously, if (1.3) holds then there is a (possibly different) least constant \( C_\ell \geq 1 \) such that
\[ \|x - G_m(x)\| \leq C_\ell\|x\|, \quad x \in X, \ m \in \mathbb{N}. \tag{1.4} \]

Some authors call \( C_w \) the quasi-greedy constant of the basis, whereas others give that name to the number
\[ C_{qg} = \max\{C_w, C_\ell\}. \]

For the time being, and while a satisfactory consensus is reached, by analogy with unconditional bases we will call the number \( C_\ell \) in (1.4) the suppression quasi-greedy constant of the basis. Thus from now on we will use \( C \)-suppression quasi-greedy to refer to a quasi-greedy basis such that (1.4) holds with a constant \( C \geq C_\ell \). Regardless of one’s preferences for the right quasi-greedy constant, if \( B \) is \( K \)-suppression unconditional then \( B \) is quasi-greedy with \( C_{qg} \leq K \).

As Wojtaszczyk pointed out in \[7\], the choice of the greedy ordering for each \( x \in X \) with which to construct the greedy operators \( (G_m)_{m=1}^{\infty} \) plays no relevant role in the theory. Indeed, if a basis \( (e_n)_{n=1}^{\infty} \) is quasi-greedy then \( x = \sum_{i=1}^{\infty} e_{\rho(i)}^*(x) e_{\rho(i)} \) for all \( x \in X \) and all possible greedy orderings \( \rho \) of \( x \). Similarly, if \( (e_n)_{n=1}^{\infty} \) is \( C \)-suppression quasi-greedy then for all \( x \in X \) and \( m \in \mathbb{N} \) we have
\[ \|x - G_m(x)\| \leq C\|x\|, \tag{1.5} \]
for any \( m \)-term greedy approximation \( G_m(x) \) to \( x \).

Dilworth et al. introduced in \[4\] a property for bases basis that is intermediate between quasi-greedy and greedy. They defined a basis
\( B = (e_n)_{n=1}^{\infty} \) to be almost greedy if there exists a constant \( C \geq 1 \) such that for all \( x \in X \) and \( m \in \mathbb{N} \),

\[
\|x - g_m(x)\| \leq C \inf \{ \|x - P_A(x)\| : |A| = m \}. \tag{1.6}
\]

Comparison with (1.1) shows that this is formally a weaker condition: in (1.1) the infimum is taken over all possible linear combinations that we can form with \( m \) basis elements whereas in (1.6) only projections of \( x \) onto \( m \)-term subsets of \( B \) are considered. The least constant in (1.6) is the almost greedy constant of the basis and is denoted by \( C_{ag} \).

If \( B \) is an almost greedy basis and \( C \) is a constant such that \( C_{ag} \leq C \) we say that \( B \) is \( C \)-almost greedy.

In this paper we are concerned with greedy-like bases in the “isometric” case, i.e., in the case that the constants that arise in the context of greedy bases in the three above-mentioned different forms are 1. This study was initiated in [1], where the authors obtained the following characterization of 1-greedy bases.

**Theorem 1.1 ([1, Theorem 3.4]).** A basis \( B \) for a Banach space \( X \) is 1-greedy if and only if \( B \) is 1-suppression unconditional and satisfies Property (A).

In order to explain this characterization we need a few more definitions.

Given a basis \( B = (e_n)_{n=1}^{\infty} \) for \( X \) and \( x, y \in X \) we say that \( y \) is a greedy permutation of \( x \) if we can write

\[
x = z + t \sum_{n \in A} \varepsilon_n e_n \quad \text{and} \quad y = z + t \sum_{n \in B} \theta_n e_n \tag{1.7}
\]

for some \( z \in X \), some sets of integers \( A \) and \( B \) of the same finite cardinality with \( \supp(z) \cap (A \cup B) = \emptyset \), some signs \( (\varepsilon_n)_{n \in A} \) and \( (\theta_n)_{n \in B} \), and some scalar \( t \) such that \( \sup_n |e_n^* (z)| \leq t \). If, in addition, \( A \cap B = \emptyset \), we say that \( y \) is a disjoint greedy permutation of \( x \). In other words, \( y \) is obtained from \( x \) by moving those terms of \( x \) (or some of them) whose coefficients are maximum in absolute value to gaps in the support of \( x \). We are also allowed to change the sign of (some of) the terms we move. Then, the basis \( B \) is said to satisfy Property (A) if \( \|x\| = \|y\| \) whenever \( y \) is a disjoint greedy permutation of \( x \).

Property (A) can be relaxed by allowing a factor of distortion in the norm of vectors which are a disjoint greedy permutation of each other, which motivates the next concept.

**Definition 1.2.** A basis \( B \) of a Banach space \( X \) is said to be symmetric for largest coefficients if there is a constant \( C \geq 1 \) (\( C \)-symmetric for largest coefficients) such that if \( x \) and \( y \) are finitely supported vectors
in \( B \) we have \( \|y\| \leq C\|x\| \) whenever \( y \) is a disjoint greedy permutation of \( x \).

Note that for \( C = 1 \) this is just the above mentioned Property (A). It is not hard to prove that if \( B \) is \( C \)-symmetric for largest coefficients and \( y \) is a greedy permutation of \( x \) then \( \|y\| \leq C^2\|x\| \). In particular, if \( B \) has Property (A) and \( y \) is a greedy permutation of \( x \) then \( \|y\| = \|x\| \) (which is the way Property (A) was originally defined in [1]).

The concept of symmetry for largest coefficients is referred to as Property (A) with constant \( C \) in [3], where the authors show the following generalization of Theorem 1.1.

**Theorem 1.3** ([3, Theorem 2]). Let \( B \) be a basis of a Banach space \( X \).

(i) If \( B \) is \( C \)-greedy, then it is \( C \)-suppression unconditional and \( C \)-symmetric for largest coefficients.

(ii) Conversely, if \( B \) is \( K \)-suppression unconditional and and \( C \)-symmetric for largest coefficients, then it is \( K^2C \)-greedy.

In turn, 1-quasi-greedy bases have been recently characterized in [2].

**Theorem 1.4** ([2, Theorem 2.1]). Let \( B \) be a basis in a Banach space \( X \). The following are equivalent:

(i) \( B \) is quasi-greedy with \( C_{qg} = 1 \).

(ii) \( B \) is quasi-greedy with \( C_w = 1 \).

(iii) \( B \) is suppression unconditional with suppression unconditional constant \( K_{su} = 1 \).

Our aim is to complete the description of “isometric” greedy-like bases by providing the following characterization of almost greedy bases in the optimal case that \( C_{ag} = 1 \).

**Theorem 1.5** (Main Theorem). A basis in a Banach space is 1-almost greedy if and only if it has Property (A).

### 2. Proof of the Main Theorem

The proof of Theorem 1.5 will rely on the following result. Notice its analogy with Theorem 1.3.

**Proposition 2.1.** Let \( B \) be a basis in a Banach space \( X \).

(i) If \( B \) is \( C \)-almost greedy then \( B \) is \( C \)-suppression quasi-greedy and \( C \)-symmetric for largest coefficients.

(ii) Conversely, if \( B \) is \( K \)-suppression quasi-greedy and \( C \)-symmetric for largest coefficients then \( B \) is \( CK \)-almost greedy.
In order to show Proposition 2.1 we will need the full force of the hypotheses. Part (i) uses an equivalent re-formulation of the condition defining almost greedy bases which will give us for free a small chunk of what we aim to prove.

**Lemma 2.2.** Suppose $\mathcal{B} = (e_n)_{n=1}^{\infty}$ is $C$-almost greedy. Then for $x \in X$, $m \in \mathbb{N}$, and any $m$-term greedy approximation $G_m(x)$ of $x$ we have

$$\|x - G_m(x)\| \leq C \inf \left\{ \|x - P_A(x)\| : 0 \leq |A| \leq m \right\}.$$  

**Proof.** Our aim is to prove that

$$\|x - G_m(x)\| \leq C \|x - P_A(x)\|,$$  

for $x \in X$, $m \in \mathbb{N}$, $G_m(x)$ $m$-term greedy approximation to $x$, and $A \subset \mathbb{N}$ with $|A| \leq m$.

To that end, we start by assuming two extra conditions. Suppose first that $|e^*_{\rho(m+1)}(x)| < |e^*_{\rho(m)}(x)|$, where $\rho$ is the natural greedy ordering for $x$. In this case $G_m(x) = G_m(x)$, and we say that $G_m(x)$ is a strictly greedy approximation to $x$. Suppose also that $x$ is finitely supported. Then there is $A \subset B \subset \mathbb{N}$ with $|B| = m$ such $P_A(x) = P_B(x)$. Therefore

$$\|x - G_m(x)\| = \|x - G_m(x)\| \leq C \|x - P_B(x)\| = C \|x - P_A(x)\|,$$

and we are done.

To obtain (2.1) in the general case we use a standard perturbation argument. Let $E = \{\rho(i) : 1 \leq i \leq m\}$. Given $\delta > 0$ there is $y \in X$ so that $\|x - y\| \leq \delta$, $G_m(y)$ is a strictly greedy approximation to $y$, and $G_m(x) = P_E(x) = P_E(y) = G_m(y)$. Then there is some finitely supported $z$ in $X$ such that $\|y - z\| \leq \delta$, $G_m(z)$ is a strictly greedy approximation to $z$, and $G_m(z) = P_E(z)$. Hence,

$$\begin{align*}
\|x - G_m(x)\| &\leq \|x - z\| + \|z - G_m(z)\| + \|G_m(z) - G_m(x)\| \\
&\leq C \|z - P_A(z)\| + (1 + \|P_E\|) \|x - z\| \\
&\leq C(\|z - x\| + \|x - P_A(x)\| + \|P_A(x) - P_A(z)\|) \\
&\quad + (1 + \|P_E\|) \|x - z\| \\
&\leq C \|x - P_A(x)\| + (1 + C + \|P_E\| + C \|P_A\|) \|x - z\| \\
&\leq C \|x - P_A(x)\| + 2(1 + C + \|P_E\| + C \|P_A\|) \|x - z\|.
\end{align*}$$

Letting $\delta$ tend to zero we obtain (2.1) and the proof is over. \qed

The proof of Part (ii) in Proposition 2.1 relies on some stability properties of quasi-greedy bases with respect to the multiplication by certain bounded sequences.
Lemma 2.3. Suppose \((e_n)_{n=1}^\infty\) is \(C\)-suppression quasi-greedy. Then for any \(N \in \mathbb{N}\), any \(N\)-tuple \((a_i)_{i=1}^N\) such that \(|a_i|\) is non-increasing, and any injective map \(\rho: \{1, \ldots, N\} \to \mathbb{N}\) we have

\[
\left\| \sum_{i=1}^N \lambda_i a_i e_{\rho(i)} \right\| \leq C \left\| \sum_{i=1}^N a_i e_{\rho(i)} \right\|,
\]

for all multipliers \((\lambda_i)_{i=1}^N\) such that \(0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \leq 1\).

Proof. Applying (1.5) to \(\sum_{i=1}^N a_i e_{\rho(i)}\) we get the claim for all \((\lambda_i)_{i=1}^N\) in the set

\[
S = \left\{ (0, \ldots, 0, 1, 1, \ldots, 1): 0 \leq m \leq N \right\}.
\]

Therefore the statement holds for all \(N\)-tuples in the convex hull \(\text{co}(\mathcal{S})\) of \(\mathcal{S}\). But \(\text{co}(\mathcal{S})\) is precisely the collection of all \((\lambda_i)_{i=1}^N\) of the desired form. \(\square\)

From this lemma we infer the following (non-linear) multiplier boundedness theorem.

Theorem 2.4. Suppose \((e_n)_{n=1}^\infty\) is \(C\)-suppression quasi-greedy. Then for any \(x \in \mathbb{X}\) and any greedy ordering \(\rho\) of \(x\),

\[
\left\| \sum_{i=1}^\infty \lambda_i e_{\rho(i)}^* (x) e_{\rho(i)} \right\| \leq C \|x\|,
\]

whenever \((\lambda_i)_{i=1}^\infty\) is a non-decreasing sequence of scalars with \(0 \leq \lambda_i \leq 1\) for all \(i\).

Proof. Lemma 2.3 yields that for all \(M, N \in \mathbb{N}\) with \(M \leq N\) we have

\[
\left\| \sum_{i=M}^N \lambda_i e_{\rho(i)}^* (x) e_{\rho(i)} \right\| \leq C \left\| \sum_{i=M}^N e_{\rho(i)}^* (x) e_{\rho(i)} \right\|.
\]

Hence \(\sum_{i=1}^\infty \lambda_i e_{\rho(i)}^* (x) e_{\rho(i)}\) is a Cauchy series and therefore converges. Moreover

\[
\left\| \sum_{i=1}^\infty \lambda_i e_{\rho(i)}^* (x) e_{\rho(i)} \right\| \leq C \limsup_N \left\| \sum_{i=1}^N e_{\rho(i)}^* (x) e_{\rho(i)} \right\| = C \|x\|.
\]

\(\square\)

We will also need the following re-formulation of the symmetry for largest coefficients.
Proposition 2.5. Suppose \((e_n)_{n=1}^{\infty}\) is \(C\)-symmetric for largest coefficients. Then, for any \(x \in X\),

\[
\|x\| \leq C \left\| x - P_A(x) + t \sum_{n \in B} \varepsilon_n e_n \right\|,
\]
whenever \(A\) and \(B\) are such that \(0 \leq |A| \leq |B| < \infty\) and \(B \cap \text{supp} \, x = \emptyset\), \(|\varepsilon_n| = 1\) for all \(n \in B\), and \(|e_n^*(x)| \leq t\) for all \(n \in \mathbb{N}\).

Proof. Assume, without lost of generality, that \(A \subset \text{supp} \, x\). By density, it suffices to prove the result when \(x\) is finitely supported. In this case, \(B \subset D\) such that \(|D| = |B|\), \(D \cap B = \emptyset\) and \(D \cap \text{supp} \, x = A\).

Our hypothesis gives

\[
\|x - P_A(x) + u\| \leq C \left\| x - P_A(x) + t \sum_{n \in B} \varepsilon_n e_n \right\| \tag{2.2}
\]
for all \(u\) in the set

\[
U = \left\{ t \sum_{n \in D} \theta_n e_n : |\theta_n| = 1 \right\}.
\]

Consequently, \(2.2\) holds for any \(u \in \text{co}(U)\). Since

\[
\text{co}(U) = \left\{ \sum_{n \in D} s_n e_n : |s_n| \leq t \right\},
\]
we have that \(P_A(x) \in \text{co}(U)\), and we are done. \(\square\)

We are now in a position to complete the proof of Proposition 2.1 and, immediately after, that of Theorem 1.5.

Proof of Proposition 2.1. (i) Assume \(B\) is \(C\)-almost greedy. To prove that \(B\) is \(C\)-suppression quasi-greedy, just take \(A = \emptyset\) in Lemma 2.2.

In order to get that \(B\) is \(C\)-symmetric for largest coefficients, we pick \(x, y \in X\) such that \(y\) is a disjoint greedy permutation of \(x\). Let \(t, A, B, (\varepsilon_n)_{n \in A}, (\theta_n)_{n \in B}\), and \(z\) be as in (1.7) and consider

\[
u = z + t \sum_{n \in A} \varepsilon_n e_n + t \sum_{n \in B} \theta_n e_n.\]

Let \(m = |A| = |B|\). Then \(G_m(u) := t \sum_{n \in A} \varepsilon_n e_n\) is a \(m\)-term greedy approximation to \(u\), and \(P_B(u) = t \sum_{n \in B} \theta_n e_n\). Hence, by Lemma 2.2

\[
\|y\| = \|u - G_m(u)\| \leq C\|u - P_B(u)\| = C\|x\|.
\]

(ii) Let \(x \in X\), \(m \in \mathbb{N}\), and \(A \subset \mathbb{N}\) with \(|A| = m\). For \(n \in \mathbb{N}\) put \(a_n = e_n^*(x)\). Pick out \(B \subset \mathbb{N}\) of cardinality \(m\) such that \(P_B(x) = \sum_{n \in B} a_n e_n \) is a disjoint greedy permutation of \(x\). Then \(P_B(x) = x\) and hence \(x = P_B(x) \in \text{co}(U)\).
\[ \mathcal{G}_m(x) . \] Denote \( t = \min\{ |a_n| : n \in B \} \) and \( q = |A \setminus B| = |B \setminus A| \). By Proposition \( \text{Proposition 2.5} \)

\[
\| x - \mathcal{G}_m(x) \| \leq C \left\| x - \mathcal{G}_m(x) - P_{A \setminus B}(x) + t \sum_{n \in B \setminus A} \text{sign}(a_n)e_n \right\|
\]

\[
= C \left\| x - P_{A \cup B}(x) + \sum_{n \in B \setminus A} \frac{t}{|a_n|}a_n e_n \right\| .
\]

Since the \( |a_n| \leq t \leq |a_k| \) for \( n \in \mathbb{N} \setminus (A \cup B) \) and \( k \in B \setminus A \), there is a greedy ordering \( \rho \) for \( x - P_A(x) \) such that \( B \setminus A = \{ \rho(i) : 1 \leq i \leq q \} \).

Define a sequence \( (\lambda_i)_{i=1}^\infty \)

\[
\lambda_i = \begin{cases} t/|a_{\rho(i)}| & \text{if } 1 \leq i \leq q, \\ 1 & \text{if } q < i. \end{cases}
\]

We infer that \( (\lambda_i)_{i=1}^\infty \) is non-decreasing. By Theorem \( \text{Theorem 2.4} \)

\[
\left\| x - P_{A \cup B}(x) + \sum_{n \in B \setminus A} \frac{t}{|a_n|}a_n e_n \right\| = \left\| \sum_{i=1}^\infty \lambda_i a_{\rho(i)} e_{\rho(i)} \right\| \leq K\| x - P_A(x) \| .
\]

Combining we obtain \( \| x - \mathcal{G}_m(x) \| \leq CK\| x - P_A(x) \| , \) as desired. \( \square \)

**Remark 2.6.** The same technique used in the proof of Part (ii) of Proposition \( \text{Proposition 2.1} \) enables us to amalgamate some of the steps in the proof, achieved in \( \text{[3]} \). of Theorem \( \text{[1.3]} \). This slight improvement leads to show that, if \( \mathcal{B} \) is \( K \)-suppression unconditional and \( C \)-symmetric for largest coefficients, then \( \mathcal{B} \) is \( CK \)-greedy.

**Proof of Theorem 1.3** Let \( \mathcal{B} = (e_n)_{n=1}^\infty \) be a basis for a Banach space \( \mathbb{X} \). Appealing to Proposition \( \text{Proposition 2.1} \) we need only show that if \( \mathcal{B} \) has Property (A) then \( \mathcal{B} \) is 1-suppression quasi-greedy. Let \( x \in \mathbb{X} \), \( k \notin \text{supp}(x) \), and \( s \) be a scalar such that \( |e_n^*(x)| \leq |s| \) for all \( n \in \mathbb{N} \). Applying Proposition \( \text{Proposition 2.5} \) in the case in which \( A = \emptyset \) and \( B = \{ k \} \), we obtain \( \| x \| \leq \| x + se_k \| . \) From here, we conclude the proof using a straightforward induction argument. \( \square \)

As an on-the-spot corollary we can re-state Theorem \( \text{Theorem 1.1} \) involving the three kinds of greedy-like bases in the isometric case.

**Corollary 2.7.** A basis \( \mathcal{B} \) of a Banach space \( \mathbb{X} \) is 1-greedy if and only if it is 1-almost greedy and \( 1 \)-quasi-greedy.

**Remark 2.8.** Dilworth et al. gave a first characterization of almost greedy bases as those bases that are quasi-greedy and democratic with
coarse estimates of the constants involved (see [4, Theorem 3.3]). Proposition 2.1 reflects the fact that if we wish to attain the optimality in the almost greedy constant of a basis we need to replace the property of being democratic with that of being symmetric for largest coefficients. At the same time, in order to be able to strictly remain within the class of almost greedy bases we must use the suppression quasi-greedy constant $C_\ell$ instead of $C_w$ or $C_{qg}$. Indeed, as soon as the quasi-greedy constant $C_w$ comes into play and attains the value 1, the basis is 1-suppression unconditional (see Theorem 1.4) and therefore we have trespassed on greedy territory! This observation, in combination with the likeness in the definition of $C_\ell$ and the definitions of greedy and almost greedy constants, conveys that $C_\ell$ might be the fair candidate to be called the quasi-greedy constant of the basis.

**Remark 2.9.** An idea that was implicit in [1] and that is present as well in the work of Dilworth et al. [3], is that being 1-superdemocratic cannot supplant Property (A) to determine the isometric properties of greedy-like bases, even when combined with other stronger features of the basis. Indeed, the basis $B$ in the Banach space of Example 5.4 of [1] is 1-lattice unconditional and 1-superdemocratic but fails to be 1-greedy because $B$ does not have Property (A). The same reason, failure of Property (A), is the one that, according to Theorem 1.5, prevents a basis from being 1-almost greedy.

**Remark 2.10.** Although throughout this paper the word *basis* meant *Schauder basis* we would like to stress that all we have said here, as well as the other characterizations of greedy-like bases in the isometric cases, remains valid in the more general framework of semi-normalized bounded biorthogonal systems, that is, sequences $(e_n, e_n^*)_{n=1}^{\infty} \subset X \times X^*$ such that

1. $X = \text{span}\{e_n : n \in \mathbb{N}\}$.
2. $e_n^*(e_k) = 1$ if $k = n$ and $e_n^*(e_k) = 0$ otherwise.
3. $\sup\{\max\{\|e_n\|, \|e_n^*\|\} : n \in \mathbb{N}\} < \infty$.

This kind of bases is quite common and there are important examples such as the trigonometric system in the space $L_1$ which belong to this class and yet they are not Schauder bases. Following Wojtaszczyk’s approach in his study of greedy-like bases for bounded biorthogonal systems [7], in this paper we have written down the proofs so that they work in this more general setting.
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