Phase states for a three-level atom interacting with quantum fields

A. B. Klimov  
Departamento de Física, Universidad de Guadalajara,  
Revolución 1500, 44420 Guadalajara, Jalisco, Mexico

L. L. Sánchez-Soto, J. Delgado, and E. C. Yustas  
Departamento de Óptica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain  
(Dated: February 9, 2020)

We introduce phase operators associated with the algebra su(3), which is the appropriate tool to describe three-level systems. The rather unusual properties of this phase are caused by the small dimension of the system and are explored in detail. When a three-level atom interacts with a quantum field in a cavity, a polynomial deformation of this algebra emerges in a natural way. We also introduce a polar decomposition of the atom-field relative amplitudes that leads to a Hermitian relative-phase operator, whose eigenstates correctly describe the corresponding phase properties. We claim that this is the natural variable to deal with quantum interference effects in atom-field interactions. We find the probability distribution for this variable and study its time evolution in some special cases.

PACS numbers: 42.50.Ct, 42.50.Dv, 42.50.Hz, 42.50.Fx

I. INTRODUCTION

Atomic coherence is essential for a proper understanding of many important effects appearing in the response of an atomic system to strong laser radiation. Perhaps the Mollow spectrum [1] for a strongly driven two-level atom has been the first milestone in quantum optics where atomic coherence plays a major role.

Once the population dynamics in two-level systems was understood, the interest in multilevel (and, especially, three-level) atoms became soon apparent. A significant amount of research has thus been devoted to their study and many effects based on quantum coherence such as spectroscopic dark states [2], electron shelving [3], narrow spectral lines [4], pulse matching [5], and anti-intuitive excitation [6], have been highlighted. These nonclassical features have an enormous variety of interesting and nontrivial consequences, including electromagnetically induced transparency [7], lasing without inversion [8], state-selective molecular excitation [9], and demonstrations of slow light [10] and fast light [11] to mention only a few examples.

For a three-level atom coupled to one or two quantized cavity field modes both the atomic dynamics and the statistical properties of the reemitted field have been discussed in detail [12] showing interesting phenomena, the most outstanding being, perhaps, the existence of collapses and revivals. The phase properties of this model have also been studied, mainly using the Pegg-Barnett formalism for the field, and the connection between revivals and phase has been put forward [13].

In spite of these achievements, even a cursory look at any classical model of atom-field interaction immediately reveals that the natural way of understanding the resonant behavior is in terms of the relative phase between the field and the atomic dipole. It is also clear that this relative phase is basic in the treatment of concepts such as interference or coherence that are ubiquitous in classical optics.

However, when one tries to extend this variable into the quantum domain one is immediately faced with two serious problems: first, there is no precise prescription to deal with the phase of a three-level atom [14]; second, we lack a satisfactory description of the atom-field interaction in terms of relative phase. The main goal of this paper is precisely to remedy these drawbacks by proposing a comprehensive theory of the relative phase for three-level atoms interacting with quantum fields.

Phase for three-level systems has been handled by invoking fuzzy concepts such as the phase of the associated wavefunction [15]. We emphasize that these notions, though well established in the classical limit, are not easily extrapolated into the realm of the quantum world. Since phase is considered to be a physical property, in the orthodox picture of quantum mechanics it must be associated with a selfadjoint operator or at least with a family of positive operator-valued measures [16]. In this spirit, phase operators for the algebra su(2), which describes two-level systems, have been previously worked out [17, 18, 19]. Since su(3) is the natural arena in which to deal with three-level atoms, we propose here an extension enabling us to introduce phase operators for these systems with a clear physical picture.

Concerning the aforementioned second problem, we stress that an operator describing the relative phase between field and atoms has resisted a quantum description [20, 21]. For this reason, its role has been played by the field quadratures [22], which in many respects properly account for phase relations and are free from the difficulties that
phase encounters in the quantum domain. When focusing on the relative phase between two subsystems, we think the best way to proceed, much in the spirit of our previous work on the subject [23], is to try a polar decomposition of the quantum amplitudes that parallels as much as possible the corresponding classical factorization.

For the relative phase between two quantum field modes this is quite a straightforward procedure and leads to a unitary solution [24]. For the case at hand, this polar decomposition seems to be more involved, mainly because, unlike for the case of two harmonic oscillators, the Hamiltonian cannot be cast in terms of su(3) generators, but rather in terms of a polynomial deformation of su(3). These nonlinear algebras have been examined very recently in quite different physical contexts [25]. Without embarking on mathematical subtleties, we shall exploit these previous results to perform the polar decomposition in an elegant way, obtaining a bona fide Hermitian operator representing the relative phase we wish to examine.

It is our firm belief that this Hermitian operator and its eigenstates should be a fundamental tool for examining all the issues related with quantum interference between atomic pathways. This leads to coherent effects as those mentioned earlier. In this paper we use this operator to introduce its associated probability distribution and to explain some relevant dynamical features of the model. Nevertheless we stress that given the variety of phenomena of both conceptual and practical importance arising in this field, it is outside the scope of this paper to provide a full account of them. Rather, our goal is to emphasize the role that the relative phase plays in atom-field interactions, a role that has previously gone almost unnoticed.

II. PHASE FOR THREE-LEVEL SYSTEMS

We wish to explore in some detail the phase properties of a three-level system. To be specific we shall consider a \(\Lambda\) configuration, as shown in Fig. 1, with energy levels \(\omega_1 < \omega_2 < \omega_3\) (we shall use throughout all this paper units \(\hbar = 1\)) and with allowed dipole transitions \(1 \leftrightarrow 3\) and \(2 \leftrightarrow 3\), but not \(1 \leftrightarrow 2\).

In order to describe the kinematics of this system one must take into account that the natural generalization of the Bloch vector [22] (which is essential in understanding the behavior of a two-level system) comprises now the nine operators

\[
\hat{S}^{ij} = |j\rangle \langle i|,
\]

where \(|i\rangle\) denotes the eigenstate of the \(i\)th atomic level. One can easily check that they satisfy

\[
[\hat{S}^{ij}, \hat{S}^{kl}] = \delta_{il} \hat{S}^{kj} - \delta_{kj} \hat{S}^{il},
\]

which correspond to the commutation relations of the algebra \(u(3)\) [26].

Obviously, the three “diagonal” operators \(\hat{S}^{ii}\) measure level populations, while the “off-diagonal” ladder operators \(\hat{S}^{ij}\) generate transitions from level \(i\) to level \(j\). To emphasize this idea we define raising and lowering operators by

\[
\begin{align*}
\hat{S}^{ij}_+ &= \hat{S}^{ij} & \text{when } j > i, \\
\hat{S}^{ij}_- &= \hat{S}^{ji} & \text{when } j < i.
\end{align*}
\]

Because one has the trivial constraint \(\hat{S}^{11} + \hat{S}^{22} + \hat{S}^{33} = \hat{I}\), only two populations can vary independently. For this reason, it is customary to introduce two independent traceless operators

\[
\hat{S}^{13}_{z} = \frac{1}{2}(\hat{S}^{33} - \hat{S}^{11}), \quad \hat{S}^{23}_{z} = \frac{1}{2}(\hat{S}^{33} - \hat{S}^{22}),
\]

that measure atomic inversions between the corresponding levels. The operators \((\hat{S}^{ij}_+, \hat{S}^{ij}_-)(i \neq j)\) turn out to be the eight generators of the algebra \(su(3)\). Note that \((\hat{S}^{13}_z, \hat{S}^{13}_+)\) and \((\hat{S}^{23}_z, \hat{S}^{23}_+)\) form two \(su(2)\) subalgebras that physically correspond to the two dipoles that appear in the two allowed transitions \(1 \leftrightarrow 3\) and \(2 \leftrightarrow 3\). However, these two dipoles are not independent, since Eq. (2) imposes

\[
[\hat{S}^{13}_+, \hat{S}^{23}_+] = -\hat{S}^{12}_+, \quad [\hat{S}^{13}_-, \hat{S}^{23}_+] = 0,
\]

which constitutes a highly nontrivial coupling between them.

It is usual to take for granted the existence of the phase of the atomic dipoles, mainly because classically one has a clear picture of their meaning. However, strictly speaking, we do not have any prescription to deal with this variable at the quantum level. Nevertheless, as anticipated in the Introduction, one could expect to get a formalism very close to the one developed recently for two-level systems.
For definiteness let us first focus on the transition $1 \leftrightarrow 3$. If only these two levels were involved, such a transition would be described by a superposition of the form

$$|\Psi\rangle = \sin(\vartheta/2) |1\rangle + e^{i\varphi} \cos(\vartheta/2) |3\rangle. \quad (6)$$

This corresponds to a $1/2$ angular-momentum system. For this case \[27\] the Bloch vector is $\mathbf{S} = \hat{\sigma}/2$, $\hat{\sigma}$ being the Pauli matrices, and it is easy to work out that the mean values $s_j = \langle \Psi | S_j | \Psi \rangle$ are given by

$$s_x = \sin \vartheta \cos \varphi,$$
$$s_y = \sin \vartheta \sin \varphi,$$
$$s_z = \cos \vartheta. \quad (7)$$

This would support the naive belief that, when viewed in the Bloch sphere $\varphi$ and the eigenvalues of $\hat{\sigma}$, the phase of the transition $1 \leftrightarrow 3$ apparently does not take part in the transition) is an eigenstate of this operator, which introduces drastic changes in phase angle associated with the atomic dipole $1 \leftrightarrow 3$. If only these two levels were involved, such a transition would be described by a superposition of the form

$$|\Psi\rangle = \sin(\vartheta/2) |1\rangle + e^{i\varphi} \cos(\vartheta/2) |3\rangle. \quad (6)$$

One can work out that the unitary solution of Eq. (9) may be written as

$$\hat{E}^{13}_{\varphi} = |1\rangle \langle 3| + e^{i\varphi_0} |3\rangle \langle 1| - e^{-i\varphi_0} |2\rangle \langle 2|, \quad (10)$$

where the undefined factor $e^{i\varphi_0}$ appears due to the unitarity requirement of $\hat{E}^{13}_{\varphi}$. The main features of this operator are largely independent of $\varphi_0$, but for the sake of concreteness, we can make a definite choice. For example \[28\], if we assume a state such as the linear superposition \[3\], the complex conjugation of the wavefunction should reverse the sign of $\varphi^{13}$, which immediately leads to the condition $e^{i\varphi_0} = -1$. We conclude then that the operator we are looking for can be represented as

$$\hat{E}^{13}_{\varphi} = |1\rangle \langle 3| - |3\rangle \langle 1| + |2\rangle \langle 2|, \quad (11)$$

The associated eigenstates are

$$|\varphi^{13}_0\rangle = |2\rangle, \quad (12)$$

and the eigenvalues of $\varphi^{13}$ are 0 and $\pm \pi/2$, respectively. This is a remarkable result. It shows that the eigenvectors $|\varphi^{13}_0\rangle$ look like the standard ones for a quantum dipole or a spin 1/2. However, the “spectator” level $|2\rangle$ (i.e., it apparently does not take part in the transition) is an eigenstate of this operator, which introduces drastic changes in the dynamics. In other words, the phase of the transition $1 \leftrightarrow 3$ “feels” the state $|2\rangle$.

Perhaps, the most striking feature of this theory is that a measurement of the phase of the transition gives only three possible values: 0 and $\pm \pi/2$. While this kind of statement seems rather reasonable when dealing with spin systems, they are scarcely recognized when dealing with atoms.

An analogous reasoning for the transition $2 \leftrightarrow 3$ gives the corresponding operator $\hat{E}^{23}_{\varphi}$

$$\hat{E}^{23}_{\varphi} = |2\rangle \langle 3| - |3\rangle \langle 2| + |1\rangle \langle 1|, \quad (13)$$
with eigenvectors

\[ |\varphi_0^{23}\rangle = |1\rangle, \]

\[ |\varphi_\pm^{23}\rangle = \frac{1}{\sqrt{2}}(|3\rangle \pm i|2\rangle), \]

and the same eigenvalues as before.

Note that

\[ \hat{E}_{\varphi}^{13} \hat{E}_\varphi^{23} \neq \hat{E}_\varphi^{12}, \]

which clearly displays the quantum nature of this phase.

To any smooth function \( F(\varphi^{ij}) \) of the dipole phase \( \varphi^{ij} \) we can associate the operator

\[ F(\hat{\varphi}^{ij}) = \sum_{r=0,\pm} |\varphi_r^{ij}\rangle \langle \varphi_r^{ij}|, \]

where the sum runs over the eigenvalues 0 and \( \pm \pi/2 \). The mean value of this operator function can be computed as

\[ \langle F(\hat{\varphi}^{ij}) \rangle = \sum_r F(\varphi_r^{ij}) P(\varphi_r^{ij}), \]

where \( P(\varphi_r^{ij}) \) is the probability distribution

\[ P(\varphi_r^{ij}) = \text{Tr} \left[ \hat{\rho} |\varphi_r^{ij}\rangle \langle \varphi_r^{ij}| \right], \]

for any state described by the density matrix \( \hat{\rho} \). We shall make use of these results in next Section.

### III. EXPLORING THE ROLE OF THE RELATIVE PHASE

#### A. Quantum dynamics of a three-level atom coupled to a two-mode field

To maintain the discussion as self contained as possible we briefly review the basic aspects of the interaction of a three-level atom with a two-mode field. We shall merely outline what will be essential for our later discussion of phase properties. We write the Hamiltonian for this system as

\[ \hat{H} = \hat{H}_A + \hat{H}_F + \hat{V}, \]

where

\[ \hat{H}_A = \sum_i \omega_i \hat{S}_i^{ii}, \]

\[ \hat{H}_F = \omega_a \hat{a}^\dagger \hat{a} + \omega_b \hat{b}^\dagger \hat{b}, \]

\[ \hat{V} = g_a (\hat{a} \hat{S}_+^{13} + \hat{a}^\dagger \hat{S}_-^{13}) + g_b (\hat{b} \hat{S}_+^{23} + \hat{b}^\dagger \hat{S}_-^{23}). \]

Here \( \hat{H}_A \) describes the dynamics of the free atom and \( \hat{H}_F \) represents the cavity modes of frequency \( \omega_a \) and \( \omega_b \), with annihilation operators \( \hat{a} \) and \( \hat{b} \), respectively. Finally, in the interaction term \( \hat{V} \), written in the dipole and rotating-wave approximations, we assume that the allowed transition 1 \( \leftrightarrow \) 3 couples (quasi)resonantly to the mode \( a \) and the transition 2 \( \leftrightarrow \) 3 couples to the mode \( b \) with coupling constants \( g_a \) and \( g_b \) that will be taken as real numbers.

The bare basis for the total system is \( |i\rangle_A \otimes |n_a,n_b\rangle_F \), where \( |n_a,n_b\rangle_F \) is the usual two-mode Fock basis. However, one can check that the two excitation-number operators

\[ \hat{N}_a = \hat{a}^\dagger \hat{a} - \hat{S}_+^{11} + 1, \quad \hat{N}_b = \hat{b}^\dagger \hat{b} - \hat{S}_+^{22} + 1, \]

are conserved quantities. In consequence, we can rewrite the Hamiltonian (19) as

\[ \hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}, \]

where

\[ \hat{H}_0 = \sum_i \omega_i \hat{S}_i^{ii}, \quad \hat{H}_{\text{int}} = g_a (\hat{a} \hat{S}_+^{13} + \hat{a}^\dagger \hat{S}_-^{13}) + g_b (\hat{b} \hat{S}_+^{23} + \hat{b}^\dagger \hat{S}_-^{23}), \]

and the same eigenvalues as before.
where

\[ \hat{H}_0 = \omega_a \hat{N}_a + \omega_b \hat{N}_b + (\omega_3 - \omega_a - \omega_b) \hat{I}, \]

\[ \hat{H}_{\text{int}} = -\Delta_a \hat{S}_{11}^{11} - \Delta_b \hat{S}_{22}^{22} + g_a (\hat{a} \hat{S}_{13}^{13} + \hat{a}^{\dagger} \hat{S}_{23}^{13}) + g_b (\hat{b} \hat{S}_{23}^{23} + \hat{b}^{\dagger} \hat{S}_{23}^{23}). \]

The detunings are defined as

\[ \Delta_a = \omega_{31} - \omega_a, \quad \Delta_b = \omega_{32} - \omega_b, \]

with \( \omega_{ij} = \omega_i - \omega_j \). It is a straightforward calculation to check that

\[ [\hat{H}_0, \hat{H}_{\text{int}}] = 0. \]

Therefore, both the free Hamiltonian \( \hat{H}_0 \) and the interaction Hamiltonian \( \hat{H}_{\text{int}} \) are constants of motion. \( \hat{H}_0 \) determines the total energy stored in the system, which is conserved by the interaction. This allows us to factor out \( \exp(-i \hat{H}_0 t) \) from the evolution operator and drop it altogether. Thus, the problem can be reduced to study the restriction of \( \hat{\Psi} \) to each subspace \( \mathcal{H}^{(N_a, N_b)} \) with fixed values of the pair of excitation numbers \( (N_a, N_b) \). In each one of these subspaces \( \mathcal{H}^{(N_a. N_b)} \) there are three basis vectors that can be written as

\[ |i; n_a = N_a - \mu_i, n_b = N_b - \nu_i \rangle, \]

where the values of \( \mu_i \) and \( \nu_i \) are defined as

\[ (\mu_1, \mu_2, \mu_3) = (0, 1, 1), \quad (\nu_1, \nu_2, \nu_3) = (1, 0, 1). \]

Note that when \( N_a = 1 \) and \( N_b = 0 \) or \( N_a = 0 \) and \( N_b = 1 \) some states may have negative photon occupation number and must be eliminated. In the subspace \( \mathcal{H}^{(N_a, N_b)} \), \( \hat{H}_{\text{int}} \) is represented by the \( 3 \times 3 \) matrix

\[ \hat{H}_{\text{int}}^{(N_a, N_b)} = \begin{pmatrix} 0 & g_b \sqrt{N_b} & g_a \sqrt{N_a} \\ g_b \sqrt{N_b} & -\Delta_b & 0 \\ g_a \sqrt{N_a} & 0 & -\Delta_a \end{pmatrix}. \]

Let us assume that at \( t = 0 \) the atomic wave function can be written as the coherent (normalized) superposition

\[ |\Psi(0)\rangle_A = \sum_i c_i |i\rangle_A, \]

while the field is in a two-mode coherent state

\[ |\Psi(0)\rangle_F = |\alpha_a, \alpha_b\rangle = \sum_{n_a, n_b=0}^{\infty} Q_{n_a} Q_{n_b} |n_a, n_b\rangle_F. \]

Here \( Q_n \) is the Poissonian weighting factor of the coherent state (with zero phase) with mean number of photons \( \bar{n} \):

\[ Q_n = \sqrt{e^{-\bar{n}} \frac{\bar{n}^n}{n!}}. \]

At a later time \( t \) the state vector for the atom-field system in the interaction picture can be expressed as

\[ |\Psi(t)\rangle = \sum_{N_a, N_b=0}^{\infty} \sum_{i,j=1}^{3} Q_{N_a - \mu_i} Q_{N_b - \nu_j} c_j \times U_{ij}^{(N_a, N_b)}(t) |i, N_a - \mu_i, N_b - \nu_j\rangle, \]

where \( U_{ij}^{(N_a, N_b)}(t) \) are the matrix elements of the evolution operator in the subspace \( \mathcal{H}^{(N_a, N_b)} \)

\[ U_{ij}^{(N_a, N_b)}(t) = \langle i; N_a - \mu_i, N_b - \nu_i | \exp[-i\hat{H}_{\text{int}}^{(N_a, N_b)} t] | j; N_a - \mu_j, N_b - \nu_j \rangle, \]

which can be calculated exactly as can be seen, e.g. in Ref [2]. This state describes completely the system evolution and will be the basis for our phase analysis in the following.
B. Polar decomposition of the relative atom-field amplitudes: deformed su(3) dynamics

As stated in the Introduction, our goal is to describe the atom-field relative phase by resorting to a polar decomposition of the corresponding complex amplitudes. To this end, let us define the operators

\[ \hat{X}^{13}_+ = \hat{a} \hat{S}^{13}_+, \quad \hat{X}^{13}_z = \hat{S}^{13}_z; \]

\[ \hat{X}^{23}_+ = \hat{b} \hat{S}^{23}_+, \quad \hat{X}^{23}_z = \hat{S}^{23}_z. \]

These operators satisfy most of the usual su(3) commutation relations provided (5) is recast as

\[ [\hat{X}^{13}_+, \hat{X}^{23}_-] = -\hat{Y}^{12}_+, \]

where

\[ \hat{Y}^{12}_+ = -\alpha \hat{b} \hat{S}^{12}_+. \]

However, some of them must be modified in the following way

\[ [\hat{X}^{13}_+, \hat{X}^{13}_-] = \hat{N}_a (1 - 2S^{11}_a - S^{22}_a), \]
\[ [\hat{X}^{23}_+, \hat{X}^{23}_-] = \hat{N}_b (1 - 2S^{22}_a - S^{11}_a), \]
\[ [\hat{Y}^{12}_+, \hat{Y}^{12}_-] = \hat{N}_a \hat{N}_b (S^{11}_a - S^{22}_a), \]

which corresponds to a polynomial deformation of the algebra su(3). The essential point for us is that one can develop a theory in very close analogy with the standard su(3) algebra.

For simplicity, let us focus first on the allowed transition \( 1 \leftrightarrow 3 \) and notice that in every three-dimensional invariant subspace \( \mathcal{H}^{(N_a, N_b)} \) the state \( |1; N_a, N_b - 1\rangle \) plays the role of a vacuum state since

\[ \hat{X}^{13}_+ |1; N_a, N_b - 1\rangle = 0. \]

In this subspace the operator \( \hat{X}^{13}_+ \) is diagonal and we can work out again a polar decomposition similar to (3), namely

\[ \hat{X}^{13}_+ = \sqrt{\hat{X}^{13}_+ \hat{X}^{13}_+} \hat{E}^{13}_\Phi, \]

where the operator \( \sqrt{\hat{X}^{13}_+ \hat{X}^{13}_+} \) is diagonal and

\[ \hat{E}^{13}_\Phi \hat{E}^{13*}_\Phi = \hat{E}^{13*}_\Phi \hat{E}^{13}_\Phi = \hat{I}, \]

\[ [\hat{E}^{13}_\Phi, \hat{N}_a] = [\hat{E}^{13}_\Phi, \hat{N}_b] = 0. \]

The first equation ensures that the operator \( \hat{E}^{13}_\Phi = \exp(i \Phi^{13}) \), representing the exponential of the relative phase between the field and the dipole \( 1 \leftrightarrow 3 \), is unitary. The second one guarantees that we may study its restriction to each invariant subspace \( \mathcal{H}^{(N_a, N_b)} \).

Much in the same way as we did in Sec. II, the operator \( \hat{E}^{13}_\Phi \) solution of (33) can be expressed in \( \mathcal{H}^{(N_a, N_b)} \) as

\[ \hat{E}^{13}_\Phi = |1; N_a, N_b - 1\rangle \langle 3; N_a - 1, N_b - 1| \]
\[ - |3; N_a - 1, N_b - 1\rangle \langle 1; N_a, N_b - 1| \]
\[ + |2; N_a - 1, N_b\rangle \langle 2; N_a - 1, N_b|, \]

As one would expect, it acts as a ladder-like operator

\[ \hat{E}^{13}_\Phi |3; N_a - 1, N_b - 1\rangle = |1; N_a, N_b - 1\rangle, \]
\[ \hat{E}^{13}_\Phi |2; N_a - 1, N_b\rangle = |2; N_a - 1, N_b|, \]
and thus has eigenvectors

\[ |\Phi_0^{13}\rangle = |2; N_a - 1, N_b\rangle, \]
\[ |\Phi_{\pm}^{13}\rangle = \frac{1}{\sqrt{2}}(|3; N_a - 1, N_b - 1\rangle \pm i|1; N_a, N_b - 1\rangle), \]

while the eigenvalues of \( \hat{\Phi}^{13} \) are 0 and \( \pm \pi/2 \), respectively.

Obviously, a similar reasoning for the transition 2 \( \leftrightarrow \) 3 gives the corresponding operator \( \hat{E}_{\Phi}^{23} \) as

\[
\hat{E}_{\Phi}^{23} = |2; N_a - 1, N_b\rangle \langle 3; N_a - 1, N_b - 1| \\
- |3; N_a - 1, N_b - 1\rangle \langle 2; N_a - 1, N_b| \\
+ |1; N_a, N_b - 1\rangle \langle 1; N_a, N_b - 1|,
\]

with eigenvectors

\[ |\Phi_0^{23}\rangle = |1; N_a, N_b - 1\rangle, \]
\[ |\Phi_{\pm}^{23}\rangle = \frac{1}{\sqrt{2}}(|3; N_a - 1, N_b - 1\rangle \pm i|2; N_a - 1, N_b\rangle), \]

and the same eigenvalues as before. The states (43) and (45) are the basis for our subsequent analysis of the dynamics of the relative phase.

\[ \text{C. Relative-phase distribution function} \]

For any state, the information one can reap using a measurement of some observable is given by the statistical distribution of the measurement outcomes. Once again, let us first focus on the relative phase between the field mode \( a \) and the dipole transition 1 \( \leftrightarrow \) 3. According to Eq. (18) it seems natural to define the probability distribution function of a state described by the density matrix \( \hat{\rho}_a \) as

\[
P(N_a, N_b, \Phi_r^{13}, t) = \text{Tr}[\hat{\rho}(t) |\Phi_r^{13}\rangle \langle \Phi_r^{13}|],
\]

where the vectors \( |\Phi_r^{13}\rangle \) are given in Eq. (43) and the subscript \( r \) runs the three possible eigenvalues 0, and \( \pm \pi/2 \). This expression can be interpreted as a joint probability distribution for the relative phase and the excitation operators \( \hat{N}_a \) and \( \hat{N}_b \). From this function, we can derive the distribution for the relative phase as the marginal distribution

\[
P(\Phi_r^{13}, t) = \sum_{N_a, N_b=0}^{\infty} P(N_a, N_b, \Phi_r^{13}, t).
\]

For a general state as in Eq. (42), one has

\[
P(N_a, N_b, \Phi_r^{13}, t) = |\langle \Phi_r^{13}|\Psi(t)\rangle|^2,
\]

which, through direct calculation, gives

\[
P(\Phi_0^{13}, t) = \sum_{N_a, N_b=0}^{\infty} \left| \sum_{j=1}^{3} Q_{N_a-\mu_j} Q_{N_b-\nu_j} c_j U_{2j}^{(N_a,N_b)}(t) \right|^2,
\]
\[
P(\Phi_{\pm}^{13}, t) = \sum_{N_a, N_b=0}^{\infty} \left| \sum_{j=1}^{3} Q_{N_a-\mu_j} Q_{N_b-\nu_j} c_j [U_{3j}^{(N_a,N_b)}(t) \pm iU_{3j}^{(N_a,N_b)}(t)] \right|^2.
\]
Much in the same way one also gets analogous results for the transition 2 ↔ 3:

\[
P(\Phi_{0}^{23}, t) = \sum_{N_{a}, N_{b}=0}^{\infty} \left| \sum_{j=1}^{3} Q_{N_{a}-\mu_{j}} Q_{N_{b}-\nu_{j}} c_{j} U_{ij}^{(N_{a}, N_{b})}(t) \right|^{2},
\]

\[
P(\Phi_{\pm}^{23}, t) = \sum_{N_{a}, N_{b}=0}^{\infty} \left| \sum_{j=1}^{3} Q_{N_{a}-\mu_{j}} Q_{N_{b}-\nu_{j}} c_{j} \left[ U_{ij}^{(N_{a}, N_{b})}(t) \pm i U_{ij}^{(N_{a}, N_{b})}(t) \right] \right|^{2}.
\]

This is our basic and compact result to analyze the evolution of the relative phase.

We have numerically evaluated this distribution for the three allowed values of the relative phase for the case when the atom is initially in the ground state |1⟩ and modes a and b are in a coherent state with a mean number of photons \( \bar{n}_{a} \) and \( \bar{n}_{b} \), respectively. For computational simplicity we have used the rescaled time

\[
\tau = \frac{g_{a} t}{2\pi \sqrt{\bar{n}_{a}}},
\]

in all the plots and have assumed that \( g_{a} = g_{b} \), which is not a serious restriction.

In Fig. 2 we have plotted a typical situation of a weak field, in which the number of excitations in the system is small, say \( \bar{n}_{a} \sim \bar{n}_{b} \sim 1 \). The pattern shows an almost oscillatory behavior, which can be easily understood if we retain only the two first terms in the sums over \( N_{a} \) and \( N_{b} \) and use the explicit form of the evolution operator \( \mathcal{U} \). A relevant and general feature that is apparent from this figure is that the probabilities associated with \( \Phi_{0}^{23} \) always oscillate out of phase, a point previously demonstrated for the case of the Jaynes-Cummings model [20].

Perhaps more interesting is the case of strong-field dynamics, when the number of excitations in the system is large and so \( \bar{n}_{a} \) or \( \bar{n}_{b} \), or both, are large. In Fig. 3 we have plotted the relative-phase probabilities for \( \bar{n}_{a} = 50 \) and (a) \( \bar{n}_{b} = 0.5 \), and (b) \( \bar{n}_{b} = 50 \) photons, with the atom initially in the ground state |1⟩. When \( \bar{n}_{b} = 0.5 \), the distribution \( P(\Phi_{0}^{23}, t) \) (which is the probability of finding the atom in the level |2⟩) is almost negligible, while \( P(\Phi_{\pm}^{23}, t) \) show collapses and revivals. One may interpret this physically as follows: the transition 1 ↔ 3 is so intense due to stimulated processes in mode a that there is no population transfer to level |2⟩, which originates a regular oscillation of the dipole 1 ↔ 3 with the corresponding collapses and revivals in the relative phase. The well-known (nearly) time-independent behavior in the time windows between collapse and revival is also clear. The probability of finding the atom in the level |1⟩, \( P(\Phi_{0}^{23}, t) \), tends to be 1/2 (except at the revivals), which confirms that the transition 1 ↔ 3 is almost saturated.

When \( \bar{n}_{b} \) grows, the position of the collapses and revivals changes, according to standard estimates [13]. When \( \bar{n}_{a} = \bar{n}_{b} = 50 \), \( P(\Phi_{0}^{23}, t) \) is centered at 1/4, while \( P(\Phi_{\pm}^{23}, t) \) is centered at 1/2, showing that the populations tend to be equidistributed because now the transition 2 ↔ 3 is almost saturated too.

Very interesting physical phenomena arise when one considers coherent superpositions of atomic states, because it is then possible to cancel absorption or emission under certain conditions, i.e., the atom is effectively transparent to the incident field even in the presence of resonant transitions. A semiclassical analysis [21], in which the fields are treated as c-numbers and described by the complex Rabi frequencies \( \Omega_{a} e^{-i\theta_{a}} \) and \( \Omega_{b} e^{-i\theta_{b}} \) (note that \( \theta_{a} \) and \( \theta_{b} \) are the ‘phases’ of the respective fields), easily shows that when the initial atomic state is a superposition of the two lower levels of the form

\[
|\Psi(0)\rangle_{A} = \frac{1}{\sqrt{2}} (|1\rangle + e^{i\varphi} |2\rangle),
\]

coherent trapping occurs whenever

\[
\Omega_{a} = \Omega_{b}, \quad \theta_{a} - \theta_{b} - \varphi = \pm \pi.
\]

In other words, when these conditions are fulfilled the population is trapped in the lower states and there is no absorption.

To corroborate this behavior valid in the strong-field limit, let us note that, although the transition 1 ↔ 2 is dynamically forbidden, one can still define phase eigenvectors for it:

\[
|\Phi_{0}^{12}\rangle = |3, N_{a} - 1, N_{b} - 1\rangle,
\]

\[
|\Phi_{\pm}^{12}\rangle = \frac{1}{\sqrt{2}} (|2; N_{a} - 1, N_{b}\rangle \pm i|1; N_{a}, N_{b} - 1\rangle).
\]
In Fig. 4 we have plotted the probabilities $P(\Phi_{12}, t)$ when the atom is initially in a trapped state like (52) with $\bar{n}_a = \bar{n}_b = 50$. We see that $P(\Phi_{12}, t)$, which is the probability of finding the atom in the upper state, shows the remarkable behavior of trapping, except for the presence of very small superimposed oscillations. Note also that $P(\Phi_{\pm}, t)$ also shows the same kind of behavior.

IV. CONCLUSIONS

In this paper we have investigated an appropriate operator for the quantum description of the relative phase in the interaction of a three-level atom with quantum fields. To this end, we have resorted to a proper polar decomposition of the corresponding amplitudes, which has been justified on physical grounds as well as using the theory of polynomial deformations of $\text{su}(3)$.

From the phase states obtained in this procedure we have defined a probability distribution function for the relative phase and studied its time evolution, showing how the formalism could be applied to understanding more involved phenomena.
[1] B. R. Mollow, Phys. Rev. A 5, 1522 (1972); ibid. 12, 1919 (1975).
[2] E. Arimondo and G. Orriols, Lett. Nuovo Cim. 17, 333 (1976); G. Alzetta, A. Gozzini, L. Moi, and G. Orriols, Nuovo Cim. B 36, 5 (1976); R.M. Whitley and C. R. Stroud, Phys. Rev. A 14, 1498 (1976); E. Arimondo, Prog. Opt. 35, 257 (1996).
[3] G. C. Hegerfeldt and M. B. Plenio, Phys. Rev. A 46, 373 (1992).
[4] G. C. Hegerfeldt and M. B. Plenio, Phys. Rev. A 52, 3333 (1995); M. B. Plenio, J. Mod. Optics 43, 753 (1996); P. Zhou and S. Swain, Phys. Rev. Lett. 77, 3995 (1996); E. Paspalakis and P. L. Knight, ibid. 81, 293 (1998); C. H. Keitel, ibid. 83, 1307 (1999).
[5] R. J. Cook and B. W. Shore, Phys. Rev. A 20, 539 (1979); F. T. Hioe and J. H. Eberly, ibid. 25, 2168 (1982); S. E. Harris, Phys. Rev. Lett. 72, 52 (1994); J. H. Eberly, M. L. Pons, and H. R. Haq, ibid. 72, 56 (1994).
[6] J. Oreg, F. T. Hioe, and J. H. Eberly, Phys. Rev. A 29, 690 (1984).
[7] S. E. Harris, Physics Today 50, 36 (1997); K. J. Boller, A. Imamoglu, and S. E. Harris, Phys. Rev. Lett. 66, 2593 (1991); S. E. Harris, ibid. 70, 552 (1993); E. Paspalakis, S. Q. Gong, and P. L. Knight, Opt. Comm. 152, 293 (1998); E. Paspalakis, N. J. Kylstra, and P. L. Knight, Phys. Rev. Lett. 82, 2079 (1999).
[8] O. A. Kocharovskaya and Ya. I. Khanin, Pis'ma Zh. Eksp. Teor. Fiz. 48, 581 (1988); S. E. Harris, Phys. Rev. Lett. 62, 1033 (1989); M. O. Scully, S. Y. Zhu, and A. Gavrielides, ibid. 62, 2813 (1989); A. Imamoglu, J. E. Field, and S. E. Harris, ibid. 66, 1154 (1991).
[9] U. Gaubatz, P. Rudecki, M. Becker, S. Schiemann, M. Külz and K. Bergmann, Chem. Phys. Lett. 149, 463 (1988).
[10] L. V. Hau, S. E. Harris, Z. Dutton, and C. H. Behroozi, Nature 397, 594 (1999); M. M. Kash, V. A. Sautenkov, A. S. Zibrov, L. Hollberg, G. R. Welch, M. D. Lukin, Y. Rostovtsev, E. S. Fry, and M. O. Scully, Phys. Rev. Lett. 82, 5229 (1999); D. Budker, D. F. Kimball, S. M. Rochester, and Y. Y. Yamashita, ibid. 83, 1767 (1999).
[11] R. Y. Chiao, Phys. Rev. A 48, R34 (1993); L. J. Wang, A. Kuzmich, and A. Dogariu, Nature 406, 277 (2000).
[12] H. I. Yoo and J. H. Eberly, Phys. Rep. 118, 239 (1985).
[13] E. I. Aliskenderov, H. Trung Dung, and A. S. Shumovsky, Quantum Opt. 3, 241 (1991); H. Trung Dung, R. Tanasi, and A. S. Shumovsky, Quantum Opt. 3, 255 (1991).
[14] B. C. Sanders, H. de Guise, S. D. Bartlett, and W. Zhang, Phys. Rev. Lett. 86, 369 (2001).
[15] S. J. Buckle, S. M. Barnett, P. L. Knight, M. A. Lauder, and D. T. Pegg, J. Mod. Opt. 33, 119 (1986).
[16] C. W. Helstrom, Quantum Detection and Estimation Theory (Academic, New York, 1976).
[17] J. M. Lévy-Leblond, Rev. Mex. Fis. 22, 15 (1973).
[18] A. Vourdas, Phys. Rev. A 41, 1653 (1990).
[19] D. Ellinas, J. Math. Phys. 32, 135 (1990).
[20] A. Luis and L. L. Sánchez-Soto, Phys. Rev. A 56, 994 (1997); Opt. Commun. 133, 159 (1997).
[21] J. Delgado, E. C. Yustas, L. L. Sánchez-Soto, and A. B. Klimov, Phys. Rev. A 63, 063801 (2001).
[22] L. Allen and J. H. Eberly, Optical Resonance and Two-Level Atoms (Dover, New York, 1987).
[23] J. Delgado, A. Luis, L. L. Sánchez-Soto, and A. B. Klimov, J. Opt. B: Quantum Semiclass. Opt. 2, 33 (2000).
[24] A. Luis and L. L. Sánchez-Soto, Phys. Rev. A 48, 4702 (1993); L. L. Sánchez-Soto and A. Luis, Opt. Commun. 105, 84 (1994); A. Luis, L. L. Sánchez-Soto, and R. Tanasi, Phys. Rev. A 51, 1634 (1995).
[25] P. W. Higgs, J. Phys. A 12, 309 (1979); E. K. Sklyanin, Funct. Anal. Appl. 16, 263 (1982); M. Roček, Phys. Lett. B 255, 554 (1991); D. Bonatsos, C. Daskaloyannis, and G. A. Lalazissis, Phys. Rev. A 47, 3448 (1993); V. P. Karassiov, J. Phys. A 27, 153 (1994); V. P. Karassiov and A. B. Klimov, Phys. Lett. A 191, 117 (1994); C. Quesne, J. Phys. A 28, 2847 (1995); B. Abdesselam, J. Beckers, A. Chakrabarti, and N. Debergh, J. Phys. A 29, 3075 (1996); N. Debergh, ibid. 31, 4013 (1998).
[26] R. Gilmore, Lie Groups, Lie Algebras, and Some of Their Applications (Wiley, New York, 1974).
[27] C. Cohen-Tannoudji, B. Diu, F. and Laloë, Quantum Mechanics (Addison, New York, 1990).
[28] A. Luis and L. L. Sánchez-Soto, Prog. Opt. 44, 421 (2000).
[29] M. O. Scully and M. S. Zubairy, Quantum Optics (Cambridge U. Press, Cambridge, 1999).
FIG. 1: The energy scheme of a three-level Λ-type atom interacting with two single-mode quantum fields, coupling the two ground states to a common excited atomic state.

FIG. 2: The probability distribution function for the six allowed values of the relative phase as a function of the rescaled time $\tau$ in the case of a weak field with $\bar{n}_a = \bar{n}_b = 1$.

FIG. 3: The probability distribution function for the six allowed values of the relative phase as a function of the rescaled time $\tau$ in the strong-field limit with: a) $\bar{n}_a = 50$, $\bar{n}_b = 0.5$ and b) $\bar{n}_a = 50$, $\bar{n}_b = 50$.

FIG. 4: The probability distribution function for the allowed values of the relative phase $\hat{\Phi}^{12}$ as a function of the rescaled time $\tau$ for a trapped state with $\bar{n}_a = \bar{n}_b = 50$. 
\[ P(\phi_{13}, \tau) \]
\[ P(\phi_{0, 13}, \tau) \]
\[ P(\phi_{13}, \tau) \]
\[ P(\phi_{0, 13}, \tau) \]
\[ P(\phi_{0, 23}, \tau) \]
\[ P(\phi_{23}, \tau) \]
\[ P(\phi + 13, \tau) \]

\[ P(\phi - 13, \tau) \]

\[ P(\phi + 23, \tau) \]

\[ P(\phi - 23, \tau) \]

\[ P(\phi - 23, \tau) \]

\[ (a) \]
\[
P(\phi^{+13}, \tau) \quad P(\phi^{013}, \tau) \quad P(\phi^{-13}, \tau)
\]

\[
(2, (-\phi) d) \quad (2, \phi^{0} d) \quad (2, \phi^{-1} d)
\]
\[ P(\phi + 12, \tau) \]
\[ P(\phi - 12, \tau) \]
\[ P(\phi^0, \tau) \]
\[ P(\phi^1, \tau) \]