Quasimomenta of string configurations

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Abstract. We sketch a reciprocal space analogue of the combinatorial bijection of Robinson-Schensted and Kerov-Kirillov-Reshetikhin (RSKKR) between magnetic configurations (the initial basis for quantum calculations of the eigenproblem of the Heisenberg Hamiltonian for a one-dim finite Heisenberg chain), and rigged string configurations (the classification labels for the exact results of Bethe Ansatz). Existence of such a bijection admits an interpretation of the exact quantum numbers of riggings as quasimomenta of \( l \)-strings. The extended size of an \( l \)-string results in selection rules for these quasimomenta, and thus in a division of the Brillouin zone into compact subzones of forbidden and allowed states of the system of coupled Bethe pseudoparticles. The forbidden Brillouin subzone for a particular \( l \)-string is evidently the effect of kinematical restrictions for motions of constituent Bethe pseudoparticles. These restrictions can be easily predicted in a combinatorially unique way due to completeness of the RSKKR bijection.

1. Introduction

The famous Bethe Ansatz (BA, [1]) provides exact solutions to the eigenproblem of the Hamiltonian for the one-dimensional Heisenberg magnetic ring with \( N \) nodes, each occupied by a spin \( 1/2 \) object. A complete set of exact eigenfunctions is classified by rigged string configurations [2]-[9] - combinatoric objects introduced by Kerov, Kirillov and Reshetikhin (KKR, [3]). Each rigged string configuration represents a correlated wave function - a superposition of magnetic configurations, the latter being the initial basis for quantum calculations. We argue in a parallel paper [9] that the set of all magnetic configurations with \( r \) reversed spins (and thus with the total magnetization \( M = N/2 - r \)), denoted by

\[
Q^{(r)} = \{j = (j_1, j_2, \ldots, j_r)| 1 \leq j_1 < j_2 < \ldots < j_r \leq N\},
\]

acquires the natural interpretation of a classical configuration space for the system of \( r \) Bethe pseudoparticles, each moving on the chain \( \tilde{N} = \{j = 1, 2, \ldots, N\} \). Thus a magnetic configuration presented by \( j \in Q^{(r)} \) is a quantum state of the system \( \tilde{r} = \{\alpha = 1, 2, \ldots, r\} \) of Bethe pseudoparticles, in which the pseudoparticle \( \alpha \in \tilde{r} \) is localized at the node \( j_\alpha \in \tilde{N} \). The dynamics of the system \( \tilde{r} \), imposed by the standard Heisenberg Hamiltonian, is such that the pseudoparticle \( \alpha \in \tilde{r} \) can move (or “jump”) to a nearest neighbor \((j_\alpha \pm 1) \mod N \) in the ring \( \tilde{N} \) provided that the “hard-core” condition \((j_\alpha \pm 1) \mod N \not= j_{(\alpha \pm 1) \mod r} \) is satisfied. Moreover, a single quantum transition involves exactly one such a jump, with all other pseudoparticles at fixed positions. The hard-core condition distinguishes some elements \( j \in Q^{(r)} \) in which all \( r \) pseudoparticles
are free to make any jump, from those in which some of them are “frozen” by the proximity of others. In this way, the classical configuration space \( Q^{(r)} \) acquires the geometry of locally \( r \)-dimensional hypercubic lattice, with \( F \)-dimensional boundaries, \( 1 \leq F < r \), corresponding to \( F \) islands of consecutive nodes occupied by Bethe pseudoparticles (the case \( F = r \) is generic).

We also consider there the RSKKRR bijection (the composition of KKR bijection [3] and the well known combinatoric correspondence of Robinson [10] and Schensted [11]) which maps the classical configuration space \( Q^{(r)} \) onto the set \( RC(N, r) \) of all rigged string configurations for fixed \( N \) and \( r \). We demonstrate there that the geometry of \( Q^{(r)} \) is naturally reflected - by the RSKKRR bijection - in the set \( RC(N, r) \) of exact BA solutions.

In the present paper we aim to propose a combinatoric assignment of definite quasimomenta to each string entering a rigged string configuration \((\nu \mathcal{L}) \in RC(N, r)\). The procedure bases on an observation that a simple combinatoric description of rigged strings in terms of some paths in the \((l, j)\) plane should be representation independent, and allows to replace the set \( \tilde{N} \) of nodes of the ring (interpreted as the “real lattice” for a single Bethe pseudoparticle) by the Brillouin zone \( B \), i.e. by the set of all quasimomenta, which are irreps of the translation group \( C_N \) (seen as the “reciprocal lattice”). In short, we replace the positional representation of the system \( \tilde{r} \) of Bethe pseudoparticles by the momentum one.

2. Strings within the momentum representation

Each magnetic configuration \( f \in Q^{(r)} \) is a mapping \( F : \tilde{N} \to \{+,-\} \), i.e. a word of the length \( N \) in the alphabet \( \{+,-\} \) of single-node spins, with the weight \( wt(f) \equiv (w_1, w_2) = (N-r, r) \), where \( w_1 \) and \( w_2 \) is the number of occurrences of the letter “+” and “-”, respectively, in the word \( f \). We recall after [9, 12] that the RSKKRR bijection \( \rho : Q^{(r)} \rightarrow RC(N, r) \) is specified, for each \( f \in Q^{(r)} \), by a path in the \((l, j)\)-plane.

A path [4], [5] of the length \( N \) is defined recursively with respect to \( j \in \tilde{N} \) as a continuous line, consisting of consecutive segments such that (i) the segment at the step \( j - 1 \) joins the point \((j-1, l)\) with one of the two points: either \((j, l + 1)\) for the step up, or \((j, l - 1)\) for the step down, (ii) the path starts at \((0, 0)\). Thus a path \( p \) can be presented as

\[
p = (p_1, p_2,\ldots, p_N), \quad p_j \in \{\text{up}, \text{down}\}, \quad j \in \tilde{N}.
\]

In order to define the path \( p(f) \), we write down the magnetic configuration \( f \), treated as the word of the length \( N \) in the alphabet 2 of single-node spins, in the following form

\[
f = u_1 v_1 u_2 v_2 \ldots u_a v_a u_{a+1}.
\]

We thus distinguish here, within the word \( f \), some subwords \( v_i \) of consecutive \( 2A_i = |v_i| \) letters, \( i = 1, 2, \ldots, a \). A subword \( v_i \), referred by us to as a minimal compensated subword, is defined by the following conditions: (i) \( wt(v_i) = (A_i, A_i) \), i.e. all letters “-” are compensated by some next “+” within \( v_i \), (ii) the length \( 2A_i \) is minimal with respect to compensation property, which means that for each proper subword \( v_i^{(j)} \) of \( v_i \), consisting of its first \( j \) letters, one has \( w_1 > w_2 \) for \( wt(v_i^{(j)}) = (w_1, w_2) \), \( i = 1, 2, \ldots 2A_i - 1 \) (so that the compensation is reached at the end of \( v_i \) only). We refer to all letters \( f(j), i \in \tilde{N} \), which enter any minimal compensated subword \( v_i \), as to nodes bounded into strings, and all the remaining letters of \( f \) as unbounded ones. The unbounded letters between \( v_{i-1} \) and \( v_i \) form the (possibly empty) subword \( u_i, i = 1, 2, \ldots, a + 1 \), with \( u_1 \) and \( u_{a+1} \) formed from the leftmost and rightmost part of \( f \), respectively.

The path \( p(f) \) is now specified by the formula

\[
p_j(f) = \begin{cases} \text{up} & f(j) = "-", \\ \text{down} & f(j) = "-", \quad \text{or if } f(j) = "-", \quad \text{and } f(j) \in u(f), \\ & f(j) = "-", \quad \text{and } f(j) \in v(f), \end{cases}
\]
where 
\[ u(f) = u_1 u_2 \ldots u_a u_{a+1} \]  
(5) 
and 
\[ v(f) = v_1 v_2 \ldots v_a \]  
(6) 
are subwords of \( f \). The decomposition (3) is unique, and the corresponding path \( p(f) \) looks like a mountain range, with cuspidal local maxima and minima. Each internal maximum (for \( 1 < j < N \)) corresponds to a string, each minimal compensated subword \( v_i \) represents a multipyramid, i.e. a segment of the mountain range with the same initial and final heights and all intermediate heights larger, and each subword \( u_i \) represents a segment of the "sea of holes" for all multipyramids, in a form of a straight line parallel to the diagonal \( j = L \).

The subword \( v(f) \), with the weight \( wt(v(f)) = (r', r') \), defines the string configuration \( \nu \vdash r' \), or \( \nu = (m_1, m_2, \ldots) \), with \( m_l \) being the number of \( l \)-strings, and

\[ \sum_{i=1}^{a} A_i = \sum_{l \geq 1} m_l = r', \]  
(7) 
whereas the word \( u(f) \), taken together with all words \( v_i, i = 1, 2, \ldots, a \) defines riggings \( L_{l_i}, l = 1, 2, \ldots, v \in \tilde{m}_l \equiv \{1, 2, \ldots, m_l\} \) for all \( l \)-strings. The collection \( L = \{L_{l_i}\} \) and string configuration \( \nu \vdash r' \) determine the rigged string configuration \( \nu L = \rho(f) \).

Within this combinatoric picture of strings, based essentially on the positional representation, each \( l \)-string is an extended object on the path \( p(f) \), which occupies the size \( 2l \), and can move by jumps over that part of the ring \( \tilde{N} \) which is not occupied by this string, nor by other strings. Moreover, this string can also move by sliding over both sides of larger \( l' \)-strings \( (l' > l) \), until their relative heights on the path \( p(f) \) coincide (cf. [9] for detail).

This picture should be representation independent, i.e. it should work in momentum representation (e.g. in the basis of wavelets [13]), with the "real space" \( \tilde{N} \) replaced by the Brillouin zone

\[ B = \{k = 0, \pm 1, \pm 2, \ldots, \pm (N/2 - 1), N/2 \} \quad \text{for } N \text{ even} \]
\[ \pm (N - 1)/2, \quad \text{for } N \text{ odd} \]  
(8) 
counting irreps
\[ \Gamma_k(j) = \exp(2\pi i kj/N), \quad j \in \tilde{N} \]  
(9) 
of the translation symmetry group \( C_N [13] \) by quasimomenta \( k \).

In the momentum representation, the kinematical restrictions resulting from the size of an \( l \)-string define the forbidden region in the Brillouin zone, concentrated symmetrically over its center \( k = 0 \). All quasimomenta outside this region are admissible for \( l \)-strings.

### 3. Allowed and forbidden Brillouin subzones for \( l \)-strings

Let us consider first an isolated, single \( l \)-string, that is the string configuration \( \nu = \{l\} \). Then the rigged string configuration is

\[ \nu L = \begin{array}{cccc} L_i & \mathcal{L} \end{array} P_l = N - 2l, \]  
(10) 
with the rigging \( L \) varying in the range

\[ 0 \leq L \leq P_l, \]  
(11)
such that $P_l$ is the total number of $l$-holes, and $L$ - the number of holes to the left of the $l$-string on the corresponding path $p(f)$. In this simple case, one has a single minimal compensated subword $v = \ldots - + + \cdots +$, $|v| = 2l$, and $f = u_1vu_2$, such that $|u_1| = L$ and $|u| = P_l$ for $u = u_1u_2$. The number $P_l$ of holes is here equal to the total length of the word $u$, whereas the rigging is the length of the subword $u_1$, i.e. the number of the letters to the left of the minimal compensated subword $v$. The total number of riggings is

$$|z(\nu)| = P_l + 1 = N - 2l + 1.$$  \hspace{0.5cm} (12)

In the momentum representation, the Brillouin zone is divided as

$$B = B_f \cup B_a, \quad B_f \cap B_a = \emptyset,$$  \hspace{0.5cm} (13)

with

$$B_f = \{ |k| \leq k_e \}$$  \hspace{0.5cm} (14)

being the region forbidden for the $l$-string, and

$$B_a = \{ |k| > k_e \}$$  \hspace{0.5cm} (15)

being the set of quasimomenta which are admissible. The quasimomentum

$$k_e = l - 1$$  \hspace{0.5cm} (16)

defines the edge of the forbidden Brillouin zone, and is the quantity which replaces the number $P_l$ of $l$-holes in the momentum representation. The admissible value $k$ of quasimomentum is related to the rigging $L$ by the formula

$$k = (L + l) \mod B,$$  \hspace{0.5cm} (17)

where $k \mod B$ means reduction of the integer $k$ to the Brillouin zone $B$, i.e. reduction mod $N$ to the range indicated by Eq. (8). These results for single $l$-strings are readily extended to an arbitrary rigged string configuration $\nu \mathcal{L}$. Each $l$-string $(l, v)$, $l = 1, 2, \ldots, v \in \tilde{m}_l$, corresponds to quasimomentum

$$k(\nu \mathcal{L}; l) = (L_{iv} + Q_l) \mod B,$$  \hspace{0.5cm} (18)

where $Q_l$ is the number of boxes in the first $l$ columns of the Young diagram $\nu \vdash r'$. The integer $Q_l$ determines precisely the kinematical restrictions imposed on the quasimomentum of a string, resulting from the hard-core property of constituent Bethe pseudoparticles. These restrictions define the forbidden Brillouin zone for $l$-strings, of the size

$$|B_f(\nu, l)| = 2Q_l - 1,$$  \hspace{0.5cm} (19)

so that the allowed quasimomenta belong to $B_a(\nu, l)$, of the size

$$|B_a(\nu, l)| = P_l + 1 = N - 2Q_l + 1.$$  \hspace{0.5cm} (20)

In this way, the total quasimomentum of an exact BA eigenstate reads

$$k(\nu \mathcal{L}) = \sum_{l \geq 1} (m_l Q_l + \sum_{v \in \tilde{m}_l} L_{iv}) \mod B.$$  \hspace{0.5cm} (21)
We consider in some detail an example of distribution of rigged string configurations over the Brillouin zone for the case \( N = 10, \nu = \{21^2\} \). A rigged string configuration \( \nu \mathcal{L} = \rho(f) \) has now the form
\[
\nu \mathcal{L} = \begin{array}{c}
| \; | \\
L \\
| \; | \\
L_1 \\
| \; | \\
L_2 \\
\end{array}
\text{with } 0 \leq L \leq 2, \\
4 \leq L_1 \leq L_2 \leq 4,
\]
so that \( L \) is the rigging of the 2-string, and \( L_1 \leq L_2 \) are the riggings of the two 1-strings, the latter counted from the left to the right in the open chain \( \tilde{N} = 10 \). The quasimomentum for this case is
\[
k(\nu \mathcal{L}) = (L + L_1 + L_2 + 10) \mod B
\]
This case corresponds to \( r' = 4 \) Bethe pseudoparticles, coupled into a 2-string \( (m_2 = 1) \) of the form
\[
\nu_2 = \ldots ++
\]
and two \( (m_1 = 2) \) 1-strings, each of the form
\[
\nu_1 = - + .
\]
The total number of holes for the 2-string is \( P_2 = 10 - 2 \cdot 4 = 2 \), whereas that for each of the 1-strings is \( P_1 = 10 - 2 \cdot 3 = 4 \). Thus the total number of different riggings is
\[
|z(\{21^2\})| = \begin{pmatrix} P_1 + m_1 \\ m_1 \end{pmatrix} \begin{pmatrix} P_2 + m_2 \\ m_2 \end{pmatrix} = \begin{pmatrix} 2 + 1 \\ 1 \end{pmatrix} \begin{pmatrix} 4 + 2 \\ 2 \end{pmatrix} = 45.
\]
The Brillouin zone for this case with appropriate distribution of strings is
\[
B = \{\begin{array}{ccccccccccccc}
-4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
7 & 5 & 4 & 2 & 2 & 4 & 5 & 7 & 7 & 45 \text{ altogether}
\end{array}\}
\]
and consists of the center \( \{0\} \), the boundary \( \{5\} \), and the interior \( \{\pm 1, \pm 2, \pm 3 \pm 4\} \).

4. Conclusions
Existence of RSKKR bijection between magnetic configurations (the initial basis for quantum calculations), and rigged string configurations (the exact solutions of BA), can be exploited not only in the positional, but also in the momentum representation. As a result, each string acquires its own quasimomentum. Kinematically, an \( l \)-string in a string configuration \( \nu \) can take on any quasimomentum outside a forbidden region of the Brillouin zone. This forbidden region is concentrated symmetrically around the center \( k = 0 \), and its size is determined by the combinatoric parameter dependent on \( \nu \). The total quasimomentum of a rigged string configuration \( \nu \mathcal{L} \) can be readily evaluated by merely combinatoric data on \( \nu \mathcal{L} \).

It is well known that each exact solution \( \nu \mathcal{L} \) of BA equations has a definite quasimomentum \( k(\nu \mathcal{L}) \) - the sum of pseudomomenta of Bethe pseudoparticles. The fact that each string has its own conserved quasimomentum is a manifestation of integrability of the one-dimensional Heisenberg magnet.

The distribution of exact BA solutions of a set \( z(\nu) \) of all riggings of a particular string configuration \( \nu \) is considerably inhomogeneous. An interesting question arises what is the band structure of BA solutions, and, in particular, what is the relation between energy bands (admittedly composite) and string configurations.

We point out here that the assignment of a definite quasimomentum to each string in an exact BA solution has only a combinatoric meaning, similar to RSKKR bijection, and resulting just from the existence of this bijection. Clearly, exact BA solutions are much more complicated
wave packets than naive pictures stemming from shapes of paths, overlapped multipyramids, etc. Still, it is amasing for us that a merely combinatoric bijection is able to reproduce precisely all quantum numbers of exact BA eigenstates, in simple and transparent terms of quasimomenta of strings.

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