LATTICE SIZE OF 2D AND 3D LATTICE POLYTOPES WITH
RESPECT TO THE UNIT CUBE

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ABSTRACT. We study the lattice size $l\square(P)$ of a lattice polytope $P$ with respect to the unit cube $\square$. The lattice size $l\square(P)$ is the smallest integer $l$ such that $P$ is contained in an $l$-dilate of the unit cube after some unimodular transformation $T$.

A similar invariant, $l\Sigma(P)$, where the unit cube is replaced with the standard $n$-dimensional simplex $\Sigma$, was studied by Schicho in the context of simplifying parametrizations of rational surfaces. Schicho gave an “onion skins” algorithm for mapping a lattice polygon $P$ into $l\Sigma$ for a small integer $l$. Castryck and Cools proved that this algorithm computes $l\Sigma(P)$ and gave a similar algorithm for finding $l\square(P)$ in the case when $P$ is a polygon.

We provide a new algorithm for computing $l\square(P)$ for a lattice polygon $P$, which does not require enumeration of lattice points in $P$. We also generalize our construction and explain a similar algorithm for computing the lattice size $l\square(P)$ of 3D lattice polytopes.

INTRODUCTION

The lattice size $l_X(P)$ of a non-empty lattice polytope $P \subset \mathbb{R}^n$ with respect to a set $X$ of positive Jordan measure was defined in [3] as the smallest $l$ such that $T(P)$ is contained in the $l$-dilate of $X$ for some unimodular transformation $T$. Note that when $X = [0, 1] \times \mathbb{R}^{n-1}$, the lattice size of a lattice polytope $P$ with respect to $X$ is the lattice width $w(P)$, a very important and well-studied invariant. Of particular interest are the cases when $X$ is either the standard $n$-dimensional simplex $\Sigma$ or the unit cube $\square = [0, 1]^n$. This paper is devoted to the study of $l\square(P)$ in dimensions 2 and 3.

The motivation for studying $l_X(P)$ comes from algebraic geometry, as explained below. Lattice size is also useful when dealing with questions that arise when studying lattice polytopes.

Let $k$ be a field. The Newton polytope $P = P(f)$ of a Laurent polynomial $f \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is the convex hull of the exponent vectors that appear in $f$. The total degree of $f$ can then be interpreted as the smallest $l$ such that $P$ is contained in the $l$ dilate $l\Sigma$ of the standard simplex $\Sigma$ after a shift by a lattice vector.

Let $A = (a_{ij})$ be a unimodular matrix, that is, all entries in $A$ are integers and det $A = \pm 1$. Then $A$ defines a monomial change of variables

$$x_i = u_1^{a_{i1}} u_2^{a_{i2}} \cdots u_n^{a_{in}}$$

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after which $f$ turns into a Laurent polynomial with Newton polytope $A(P)$.

Hence $\text{ls}_\Sigma(P)$ is the smallest total degree of a polynomial that $f$ can turn into under such monomial changes of variables. This monomial change of variables is a particular case of a birational map of the hypersurface defined by $f = 0$ in the algebraic torus $(k \setminus \{0\})^n$. Hence $\text{ls}_\Sigma(P)$ provides an upper bound for the lowest total degree of the hypersurface defined by $f = 0$ in $(k \setminus \{0\})^n$ under birational equivalence. Note that when finding $\text{ls}_\Box(P)$, the lattice size of $P$ with respect to the cube, we are minimizing the largest component of all the multi-degrees of the monomials that appear in $f$ over monomial changes of variables.

In [2] the lattice size of a lattice polygon with respect to the unit square is used to classify small lattice polygons and corresponding toric codes. This notion also appears implicitly in [1] and [5].

In [6] Schicho provided an “onion skins” algorithm for mapping a plane lattice polygon $P$ into $l\Sigma$ for a small integer $l$. In [3] Castryck and Cools proved that this algorithm computes $\text{ls}_\Sigma(P)$. The idea of this algorithm is that when one passes from a lattice polygon $P$ to the convex hull of its interior lattice points, its lattice size $\text{ls}_\Sigma(P)$ drops by 3 unless $P$ belongs to a list of exceptional cases. One can then compute $\text{ls}_\Sigma(P)$ by successively peeling off “onion skins” of the polygon. Castryck and Cools also developed in [3] a similar algorithm for computing $\text{ls}_\Box(P)$.

The downside of the “onion skins” algorithm is that it is quite time-consuming and, in particular, requires enumeration of interior lattice points of $P$. In this paper we provide a new algorithm for computing $\text{ls}_\Box(P)$ for lattice polygons $P$ and then extend our algorithm to 3D lattice polytopes. These new algorithms do not require enumeration of lattice points in $P$. In [4] we develop a similar algorithm for computing $\text{ls}_\Sigma(P)$, where $P$ is a lattice polygon.

Let $e_\Box(P)$ be the smallest $l$ such that $P$ is contained in $l\Box$ after a lattice translation. Here is our main result for the case of plane polygons.

**Theorem 0.1.** Let $P \subset \mathbb{R}^2$ be a lattice polygon with $e_\Box(P) = l$. If the lattice width of $P$ in the directions $(1, \pm 1)$ is at least $l$, then $\text{ls}_\Box(P) = l$. Furthermore, the lattice width of $P$ is the smaller of the lattice widths of $P$ in the directions $(1, 0)$ and $(0, 1)$.

This result provides a very fast algorithm for computing $\text{ls}_\Box(P)$: Start with $P$, find $e_\Box(P)$, check if the lattice width of $P$ in the directions $(1, 1)$ and $(1, -1)$ is at least $l$. If this is the case, conclude that $\text{ls}_\Box(P) = l$. If not, we pass to the polygon $AP$, where $A = \begin{bmatrix} 1 & 0 \\ 1 & \pm 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & \pm 1 \end{bmatrix}$, and repeat the algorithm. Notice that this algorithm will have at most $e_\Box(P) - 1$ steps.

The idea of our main 3D result, formulated in Theorem 3.7 is similar: one needs to check that the lattice width of $P$ in some directions is at least $e_\Box(P)$ in order to conclude that $\text{ls}_\Box(P) = e_\Box(P)$, but the situation is more complicated, as the number of directions one needs to check is not constant, in contrast to the plane case. As in the plane, this theorem leads to an algorithm for computing $\text{ls}_\Box(P)$ and $w(P)$. 
We also discuss fitting $P$ inside a small axis-parallel rectangle and recover a theorem from [3] that states that $P \subset \mathbb{R}^2$ can be mapped unimodularly inside the rectangle $[0, w(P)] \times [0, \text{ls}_\Box(P)]$. We also prove the generalization of this result for 3D lattice polytopes.

1. Definitions

Let $P \subset \mathbb{R}^n$ be a lattice polytope. Given a primitive direction $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, i.e. $\gcd(a_1, \ldots, a_n) = 1$, the lattice width of $P$ in the direction of $a$ is defined by

$$w_a(P) = \max_{x \in P} (x \cdot a) - \min_{x \in P} (x \cdot a),$$

where $x \cdot a$ denotes the standard dot-product. The lattice width $w(P)$ is the minimum of $w_a(P)$ over all primitive directions.

Recall that a square integer matrix $A$ is unimodular if $\det A = \pm 1$. For such a matrix $A$, transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, defined by $T(x) = Ax + v$, where $v \in \mathbb{Z}^n$, is called an affine unimodular transformation. Such transformations preserve the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

**Definition 1.1.** Let $P \subset \mathbb{R}^n$ be a lattice polytope and let $\Box = [0, 1]^n \subset \mathbb{R}^n$ denote the unit cube in $\mathbb{R}^n$. Then $\text{ls}_\Box(P)$, the lattice size of $P$ with respect to the unit cube, is the smallest $l$ such that $P$ is contained in the $l$-dilate $l\Box$ of the unit cube after an affine unimodular transformation $T$.

Let $e_\Box(P)$ be the smallest $l$ such that $P$ is contained in $l\Box$ after a lattice translation. Then $\text{ls}_\Box(P) = \min e_\Box(AP)$, where the minimum is taken over all unimodular matrices $A$ of size $n \times n$.

**Example 1.** Let $P \subset \mathbb{R}^2$ be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(2, 3)$. Then $e_\Box(P) = 3$ while $\text{ls}_\Box(P) = 2$.

To get from the first polygon to the second we apply the map

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For a linear function $f(x) = f(x_1, \ldots, x_n) = a_1x_1 + \cdots + a_nx_n$ with $a_1, \ldots, a_n \in \mathbb{Z}$ and a lattice polytope $P \subset \mathbb{R}^n$ denote

$$\Delta_P(f) = \max_{x \in P} f(x) - \min_{x \in P} f(x).$$
Note that $\Delta_P(f)$ is the lattice width $w_a(P)$ of $P$ in the lattice direction $a = (a_1, \ldots, a_n)$. When this does not create a confusion, we will write $\Delta(f)$ instead of $\Delta_P(f)$.

Clearly, $\Delta(f \pm g) \leq \Delta(f) + \Delta(g)$, $\Delta(af) = |a|\Delta(f)$, and 
$$e_{\square}(P) = \max\{\Delta(x_1), \ldots, \Delta(x_n)\}.$$ 
Also, for an $n \times n$ matrix $A = (a_{ij})$ we have 
$$e_{\square}(AP) = \max_{i=1 \ldots n} \{\Delta(a_{i1}x_1 + \cdots + a_{in}x_n)\} = \max_{i=1 \ldots n} \{w_{a_i}(P)\},$$
where $a_i = (a_{i1}, \ldots, a_{in})$ is the $i$th row of $A$.

## 2. Lattice size of polygons

In this section we explain a very fast algorithm for computing lattice size of lattice polygons $P \subset \mathbb{R}^2$ based on the following theorem.

**Theorem 2.1.** Suppose that $e_{\square}(P) = l$ and $\Delta_P(x \pm y) \geq l$. Then $\Delta(ax + by) \geq l$ for all primitive directions $(a, b) \in \mathbb{Z}^2$, except, possibly, for $(a, b) = (\pm 1, 0)$ or $(0, \pm 1)$. This implies that $l_{\square}(P) = l$ and $w(P) = \min\{\Delta_P(x), \Delta_P(y)\}$.

**Proof.** We need to check that there is no unimodular matrix $A$ such that $e_{\square}(AP) < l$. If such a matrix existed then the lattice width of $P$ in the direction of each of its row vectors would be less than $l$, so it is enough to check that for all primitive lattice vectors $(a, b) \neq (\pm 1, 0) \text{ or } (0, \pm 1)$ we have $\Delta(ax + by) \geq l$.

Suppose first that $a$ and $b$ have same sign, that is, $\text{sgn}(a) \cdot \text{sgn}(b) \geq 0$. If we also have $|a| \geq |b|$, then $|a - b| = |a| - |b|$ and we get 
$$|a|l \leq \Delta(ax + ay) \leq \Delta(ax + by) + \Delta(ay - by) \leq \Delta(ax + by) + |a - b|l.$$ 
This implies that $\Delta(ax + by) \geq l(|a| - |a - b|) = l|b| \geq l$, provided that $b \neq 0$. Similarly, if $|a| \leq |b|$ we use 
$$|b|l \leq \Delta(bx + by) \leq \Delta(ax + by) + \Delta(-ax + bx) \leq \Delta(ax + by) + |a - b|l,$$
to get $\Delta(ax + by) \geq l(|b| - |a - b|) = l|a| \geq l$, provided that $a \neq 0$.

Next, suppose that $a$ and $b$ have opposite signs, that is, $\text{sgn}(a) \cdot \text{sgn}(b) \leq 0$. If $|a| \geq |b|$ then $|a + b| = |a| - |b|$ and we get 
$$|a|l \leq \Delta(ax - ay) \leq \Delta(ax + by) + \Delta(ay + by) \leq \Delta(ax + by) + |a + b|l,$$
which implies $\Delta(ax + by) \geq l(|a| - |a + b|) = l|b| \geq l$, if $b \neq 0$. Similarly, if $|a| \leq |b|$ then $|a + b| = |b| - |a|$ and 
$$|b|l \leq \Delta(bx - by) \leq \Delta(ax + by) + \Delta(ax + bx) \leq \Delta(ax + by) + |a + b|l.$$ 
Hence $\Delta(ax + by) \geq l(|b| - |a + b|) = l|a| \geq l$, if $a \neq 0$.

Finally, if either $a$ or $b$ is zero then the primitive direction is one of $(\pm 1, 0)$, $(0, \pm 1)$. \hfill \square
We next prove an interesting observation that if a lattice polygon \( P \subset I \) touches all four sides of the square \( l \) then \( l_s(P) = l \).

**Theorem 2.2.** Suppose that \( \Delta_P(x) = \Delta_P(y) = l \). Then \( l_s(P) = l \) and
\[
w(P) = \min\{l, \Delta_P(x + y), \Delta_P(x - y)\}.
\]

**Proof.** Consider a primitive direction \((a, b) \in \mathbb{Z}^2\). We have
\[
|a|l = \Delta(ax) \leq \Delta(ax + by) + \Delta(by) = \Delta(ax + by) + |b|l,
\]
which implies that \( \Delta(ax + by) \geq (|a| - |b|)l \geq l \), provided that \( |a| > |b| \). Similarly, we have
\[
|b|l = \Delta(by) \leq \Delta(ax + by) + \Delta(ax) = \Delta(ax + by) + |a|l,
\]
which gives \( \Delta(ax + by) \geq l \) if \( |a| < |b| \). If \( |a| = |b| \) then the primitive direction \((a, b)\) is one of \((\pm 1, 1), (1, \pm 1)\). Hence we may only have lattice width smaller than \( l \) in the directions \((1, 1)\) or \((1, -1)\), so \( w(P) = \min\{l, \Delta(x + y), \Delta(x - y)\} \). Since these two directions do not give rise to a unimodular matrix, we get \( l_s(P) = l \). \( \square \)

**Theorem 2.3.** There is an algorithm for finding \( l_s(P) \) of a lattice polygon \( P \), which is linear in \( e(S)(P) \) and does not require enumeration of lattice points in \( P \).

**Proof.** Note that if we are in a situation described either in Theorem 2.1 or Theorem 2.2 we have already found both \( l_s(P) \) and \( w(P) \).

If not, we use one of \( A = \begin{bmatrix} 1 & 0 \\ 1 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & \pm 1 \end{bmatrix} \) to pass to \( A(P) \) with strictly smaller \( e(S)(P) \) and repeat the process. This algorithm will terminate in at most \( e(S)(P) - 1 \) steps. The product of such \( A \)'s at all the steps will be the unimodular map that maps \( P \) into the smallest possible lattice square. \( \square \)

Our next goal is to unimodularly map \( P \) into the smallest axis-parallel rectangle. Recall that the product order on \( \mathbb{N}^2 \) is described by defining \((a_1, a_2) \leq (b_1, b_2)\) iff \( a_1 \leq b_1 \) and \( a_2 \leq b_2 \). Given a lattice polygon \( P \), let \( S_{1,1} \) be the set of pairs \((a, b) \in \mathbb{N}^2 \) such that the rectangle \([0, a] \times [0, b] \subset \mathbb{R}^2 \) contains a unimodular copy of \( P \). We now recover a theorem proved in [3].

**Theorem 2.4.** The set \( S_{1,1} \) admits a minimum with respect to the product order at the pair \((w(P), l_s(P))\).

**Proof.** We only need to show that \( P \) can be mapped unimodularly inside the rectangle \( R := [0, w(P)] \times [0, l_s(P)] \). After we run the algorithm of Theorem 2.3 we end up either in a situation described in the assumptions of Theorem 2.1, so \( P \) is unimodularly mapped inside \( R \), or we are under the assumptions of Theorem 2.2, where \( \Delta(x) = \Delta(y) = l_s(P) \) and \( w(P) = \min\{l_s(P), \Delta(x + y), \Delta(x - y)\} \). In the latter case, if \( w(P) = l_s(P) \), we have already mapped \( P \) inside \( R \). Otherwise, \( A = \begin{bmatrix} 1 & \pm 1 \\ 0 & 1 \end{bmatrix} \) will finish the task. \( \square \)
3. Lattice size in dimension 3

We now work on developing an algorithm for computing the lattice size of a lattice polytope $P$ in $\mathbb{R}^3$. Let $\Delta_P(x) = l_1 \leq \Delta_P(y) = l_2 \leq \Delta_P(z) = l$. Our goal now is to show that if in some directions the lattice width of $P$ is at least $l$, the same is true for almost all other directions.

**Theorem 3.1.** Suppose that $\Delta_P(mx + ny + z) \geq l$ for all $m, n \in \mathbb{Z}$. Then $\Delta_P(ax + by + cz) \geq l$ for all primitive directions $(a, b, c)$ with nonzero $c$, except, possibly, for directions $(a, b, c)$ that satisfy $|a| = |b| = 1$ and $|c| = 2$.

**Proof.** Fix a primitive direction $(a, b, c) \in \mathbb{Z}^3$ with nonzero $c$. Let $mc$ be the multiple of $c$ nearest to $a$ and $nc$ be the multiple of $c$ nearest to $b$. Consider the computation:

$$|c|l \leq \Delta(cm x + cny + cz)$$
$$\leq \Delta(ax + by + cz) + \Delta((mc - a)x + (nc - b)y)$$
$$\leq \Delta(ax + by + cz) + |mc - a|l + |nc - b|l$$

and its consequence:

$$\Delta(ax + by + cz) \geq |c|l - |mc - a|l - |nc - b|l.$$  

(3.1)

Suppose $c$ is odd. By our choice of $m$ and $n$ we have

$$|mc - a| \leq \frac{|c| - 1}{2} \text{ and } |nc - b| \leq \frac{|c| - 1}{2}.$$ 

With (3.1), this gives

$$\Delta(ax + by + cz) \geq |c|l - \frac{|c| - 1}{2}l - \frac{|c| - 1}{2}l = l.$$ 

Next let $c$ be even. Then

$$|mc - a| \leq \frac{|c|}{2} \text{ and } |nc - b| \leq \frac{|c|}{2}.$$ 

Notice that if $c \neq 2$, we cannot have equality in both of these inequalities. Indeed, if this were the case, for $c = 2k$, we would have that $k$ divides $\gcd(a, b, c)$. Thus, either $|mc - a| \leq (|c| - 2)/2$ or $|nc - b| \leq (|c| - 2)/2$ and, using this in (3.1), we get

$$\Delta(ax + by + cz) \geq |c|l - \frac{|c| - 2}{2}l = l.$$ 

This leaves only the case when $|c| = 2$. If either $a$ or $b$ is even then one of the differences $|mc - a|$, $|nc - b|$ is zero and the other one is either zero or one and it follows that $\Delta(ax + by + cz) \geq l$. 


For the remainder of the proof, suppose both \(a\) and \(b\) are odd and \(|c| = 2\). If \(\Delta(x + y) < l\), then
\[
\Delta(ax + by + cz) + l \geq \Delta(ax + by + cz) + \Delta(x + y) \\
\geq \Delta((a + 1)x + (b + 1)y + cz) \\
\geq 2\Delta\left(\frac{a + 1}{c} x + \frac{b + 1}{c} y + z\right) \\
\geq 2l
\]
where the final inequality holds since we assumed \(\Delta_p(mx + ny + z) \geq l\) for all \(m, n \in \mathbb{Z}\). It follows that \(\Delta(ax + by + cz) \geq l\). Similarly, we get the same result if \(\Delta(x - y) < l\).

Now, let \(\Delta(x \pm y) \geq l\). Suppose that neither \(|a|\) nor \(|b|\) is equal to one. Since both of them are odd, this means that each of their absolute values is at least 3. Suppose first that \(|a| \geq |b|\) and \(\text{sgn}(a) \cdot \text{sgn}(b) \geq 0\). We have
\[
\Delta(ax + by + cz) + \Delta((a - b)y - cz) \geq \Delta(ax + ay) \geq |a|l.
\]
Since \(\Delta((a - b)y - cz) \leq |a - b|l + |c|l = (|a| - |b| + |c|)l\), we get
\[
\Delta(ax + by + cz) \geq |a|l - (|a| - |b| + |c|)l = (|b| - |c|)l.
\]
Since \(|b| > |c| = 2\), the result holds. Using the inequality \(\Delta(x - y) \geq l\), we cover the case when \(\text{sgn}(a) = -\text{sgn}(b)\). The case when \(|b| \geq |a| > 1\) is also covered by switching the roles of \(a\) and \(b\) in the argument above.

We are left with the case when \(|c| = 2\) and at least one of \(|a|\), \(|b|\) is equal to one. If \(|b| = 1\) and \(|a| > 1\), then \(|a| \geq 3\). We can assume that \(c = 2\), passing, if necessary, from \((a, b, c)\) to \((-a, -b, -c)\), and get
\[
\Delta(ax + by + 2z) + \Delta(-by + (|a| - 2)z) \geq \Delta(ax + |a|z) = |a|\Delta(\text{sgn}(a)x + z) \geq |a|l.
\]
We then conclude that \(\Delta(ax + by + 2z) \geq (|a| - 1 - (|a| - 2))l = l\). The argument for \(|a| = 1\) and \(|b| > 1\) is analogous and leaves only the case with \(|c| = 2\) and \(|a| = |b| = 1\).

\[\Box\]

Our next goal is to show that one can ensure \(\Delta_p(ax + by + z) \geq l\) for all \((a, b) \in \mathbb{Z}^2\) by checking that this inequality holds true for a finite number of such pairs \((a, b)\).

**Lemma 3.2.** Let \(\Delta_p(x) = l_1 \leq \Delta_p(y) = l_2 \leq \Delta_p(z) = l\). Suppose that \(\Delta_p(x \pm y) \geq l_2\). If for \(a, b \in \mathbb{Z}\) we have \(\Delta_p(ax + by + z) < l\) then

1. If \(|a| \geq |b|\) then \(|b| \leq \frac{2l_1}{l_2 - 1}\);
2. If \(|b| \geq |a|\) then \(|a| \leq \frac{2l_2 - 1}{l_1 - 1}\).

**Proof.** Suppose \(\text{sgn}(a) \cdot \text{sgn}(b) \geq 0\) and \(|a| \geq |b|\). We have
\[
|a|\Delta(x + y) \leq \Delta(ax + by + z) + \Delta((a - b)y - z),
\]
which implies
\[ |a| l_2 < l + \Delta ((a-b)y - z) \leq l + |a-b| \Delta(y) + \Delta(z) \leq 2l + (|a| - |b|) l_2. \]

Rearranging the final inequality gives \(|a| l_2 - (|a| - |b|) l_2 < 2l\) which lets us conclude that \(|b| \leq (2l - 1)/l_2\). If \(|b| \geq |a|\), the computation is similar and we conclude that \(|a| \leq (2l - 1)/l_1\).

If \(\text{sgn}(a) = -\text{sgn}(b)\) and \(|a| \geq |b|\), then we use the remaining part of our hypothesis:
\[ |a| l_2 \leq \Delta(ax - ay) \leq \Delta(ax + by + z) + \Delta((a+b)y + z) < 2l + (|a| - |b|) l_2 \]

which again implies \(|a| \leq (2l - 1)/l_1\). The case \(\text{sgn}(a) = -\text{sgn}(b)\) and \(|b| \geq |a|\) is treated similarly.

**Lemma 3.3.** Let \( \Delta(x) = l_1 \leq \Delta(y) = l_2 \leq \Delta(z) = l \). If \( \Delta(ax + by + z) < l \) for \( a, b \in \mathbb{Z} \) then
\[ |a| \leq \frac{2l - 1 + |b| l_2}{l_1} \quad \text{and} \quad |b| \leq \frac{2l - 1 + |a| l_1}{l_2}. \]

**Proof.** Consider
\[ |a| l_1 = \Delta(ax) \leq \Delta(ax + by + z) + \Delta(by) + \Delta(z) \leq 2l - 1 + |b| l_2. \]

Dividing by \( l_1 \) we get
\[ |a| \leq \frac{2l - 1 + |b| l_2}{l_1}. \]

Starting from \(|b| l_2\) yields the other bound. \(\square\)

**Definition 3.4.** Let set \( S \) consist of all pairs \((a, b) \in \mathbb{Z}^2\) that satisfy
1. If \(|a| \geq |b|\) then \(|b| \leq \frac{2l - 1}{l_2} \) and \(|a| \leq \frac{2l - 1 + |b| l_2}{l_1}\);
2. If \(|b| \geq |a|\) then \(|a| \leq \frac{2l - 1}{l_1} \) and \(|b| \leq \frac{2l - 1 + |a| l_1}{l_2}\).

**Proposition 3.5.** The set \( S \) is finite and its size does not exceed \( \frac{64 l^2}{l_1 l_2} \).

**Proof.** If \((a, b) \in S\) then if \(|a| \leq |b|\) we have \(|a| < \frac{2l}{l_1}\) and \(|b| < \frac{4l}{l_2}\) and, similarly, if \(|b| \leq |a|\), we have \(|b| < \frac{2l}{l_2}\) and \(|a| < \frac{4l}{l_1}\), which demonstrates that \( S \) is finite and the bound of the proposition follows. \(\square\)

This shows that there are at most finitely many pairs \((a, b)\) that may satisfy \( \Delta(ax + by + z) < l \) under the assumptions of Lemmas [3.2 and 3.3] We next consider a particular case when set \( S \) is very small. The proof of this proposition is a direct application of Lemmas [3.2 and 3.3].

**Proposition 3.6.** Let \( \Delta_P(x) = \Delta_P(y) = \Delta_P(z) = l \) and suppose that \( \Delta_P(x \pm y) \geq l \). Then we may have \( \Delta_P(ax + by + z) < l \) only for pairs \((a, b)\), where \((|a|, |b|)\) is in the set \(\{(1, 1), (1, 2), (2, 1)\}\).

We now formulate our main result.
**Theorem 3.7.** Let $\Delta_P(x) = l_1 \leq \Delta_P(y) = l_2 \leq \Delta_P(z) = l$. Suppose that $\Delta_P(x \pm y) \geq l_2$ and $\Delta_P(ax + by + z) \geq l$ for all $(a, b) \in S$. Then $ls(P) = l$.

*Proof.* By Lemmas 3.2 and 3.3, we have that $\Delta(mx + ny + z) \geq l$ for all $(m, n) \in \mathbb{Z}^2$. Applying Theorem 3.1, we know that $\Delta(ax + by + cz) \geq l$ for all primitive $(a, b, c) \in \mathbb{Z}^3$ except, possibly, for triples $(a, b, c)$ with $|c| = 0$ or 2. If there existed a unimodular matrix $A$ with $e_\square(AP) < l$ then the width of $P$ in the direction of each of its rows would have been less than $l$. Hence the rows of $A$ would have to have last components equal to 0 or 2, but then its determinant would be even. \hfill $\Box$

Theorem 3.7 leads to an algorithm for computing lattice size $ls(P)$ for a lattice polytope $P \subset \mathbb{R}^3$. Without loss of generality, switching $x, y,$ and $z$, we can assume that $\Delta_P(x) \leq \Delta_P(y) \leq \Delta_P(z)$. Applying the 2D algorithm of Theorem 2.3 to the projection of $P$ to the $(x, y)$-plane we can ensure that $\Delta_P(x \pm y) \geq l_2$, where $\Delta_P(x) = l_1 \leq \Delta_P(y) = l_2 \leq \Delta_P(z) = l$. According to Theorem 2.1, this implies that $\Delta(ax + by) \geq l_2$ for all primitive $(a, b) \in \mathbb{Z}^2$ except, possibly, $(a, b) = (\pm 1, 0)$.

Our next step is to check whether for all directions $(a, b)$ in the set $S$ we have $\Delta_P(ax + by + z) \geq l$. If this is the case then by Theorem 3.7 we conclude that $ls(P) = l$. Otherwise, we find $(a, b) \in S$ such that $\Delta_P(ax + by + z) < l$ and repeat the process with $P$ replaced with $P' = AP$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{bmatrix}$. Since $\Delta_P(x') + \Delta_P(y') + \Delta_P(z') < l_1 + l_2 + l$, the algorithm will terminate after finitely many steps, that is, we will arrive at a situation where the assumptions of Theorem 3.7 are satisfied.

Since by Proposition 3.5 each step takes $O(l^2)$ operations and we have $O(l)$ steps, this algorithm will take $O(l^3)$ operations, or using our notation from before, $O(e_\square(P)^3)$ operations.

A more careful analysis of the algorithm will help us bring this bound down. Let’s first assume that $l_2 < l$. As before, we use the 2D algorithm to ensure that $\Delta_P(x \pm y) \geq l_2$. Note that each step in the 2D algorithm takes at most 4 checks and reduces $l_1 + l_2 + l$ by at least 1.

Next we search through $S$ to find a direction $(m, n, 1) \in S$ that corresponds to the smallest possible lattice width $l'$ of $P$ over all the directions in $S$ and let $z' = mx + ny + z$. We can assume that $l' < l$ for otherwise the algorithm will terminate at this step.

If $l' \geq l_2$ then $\Delta(x) = l_1 \leq \Delta(y) = l_2 \leq \Delta(z') = l'$ and $\Delta(x + y) \geq l_2$. Now pick an arbitrary $(a, b) \in \mathbb{Z}^2$. If $(a + m, b + n, 1) \in S$ then $\Delta(ax + by + z') = \Delta((a + m)x + (b + n)y + z) \geq l'$. Otherwise, $\Delta(ax + by + z') = \Delta((a + m)x + (b + n)y + z) \geq l > l'$, and using Theorem 3.7 we conclude that in the case $l' \geq l_2$ we have $ls(P) = l'$.
Hence we only need to go through \( S \) once to either terminate the algorithm or to reduce \( l \) by at least \( l - l_2 \). That is, we are using \( O \left( \frac{e^2}{l_1 l_2} \right) \) operations to make \( l - l_2 \) steps, which is \( O \left( \frac{e^2}{l_1 l_2 (l - l_2)} \right) \) operations per step.

Note that \( l_2(l - l_2) \geq l - 1 \) since \( l_2(l - l_2) \) is an upside down parabola when considered as a function in \( l_2 \) and since \( 1 \leq l_2 \leq l - 1 \) the smallest value occurs at \( l_2 = l - 1 \) and \( l_2 = 1 \). Hence we get \( \frac{e^2}{l_1 l_2 (l - l_2)} \leq \frac{l^2}{l_1 (l - 1)} \leq \frac{e^2}{l - 1} = O(1) \).

If \( l_2 = l \) then \( S \) is of size at most \( \frac{64e^2}{l_1 l_1} \leq 64l = O(1) \). We conclude that each reduction of \( l_1 + l_2 + l \) by 1 requires \( O(1) \) operations and hence the algorithm is quadratic in \( e (\square) \).

**Theorem 3.8.** There exists an algorithm for finding the lattice size of a 3D lattice polytope \( P \), which takes \( O(e (\square)^2) \) operations and does not require enumeration of lattice points in \( P \).

**Corollary 3.9.** The algorithm of Theorem 3.8 can be used to find the lattice width of \( P \).

**Proof.** After we run this algorithm we end up in a situation where

\[
\Delta(x) = l_1 \leq \Delta(y) = l_2 \leq \Delta(z) = l = ls(\square)(P), \Delta(x \pm y) \geq l_2
\]

and \( \Delta_P(ax + by + z) \geq l \) for all \((a,b) \in \mathbb{Z}^2 \). By Theorem 2.1 this implies that \( \Delta(ax + by) \geq l_2 \) for all primitive directions \((a,b)\), except, possibly, for \((a,b) = (\pm 1,0) \) or \((0,\pm 1)\). Also, by Theorem 3.7 we have that \( \Delta(ax + by + cz) \geq l \) for all primitive directions \((a,b,c)\) with \( c \neq 0 \), except, possibly, for the case when \(|a| = |b| = 1 \) and \(|c| = 2 \). We denote

\[
E = \{(a,b,c) \in \mathbb{Z}^3 | |a| = |b| = 1 \text{ and } |c| = 2 \}\.
\]

Now to find \( w(P) \) one needs to find the minimum of the lattice width of \( P \) with respect to all directions in \( E \) and then pick the smaller one of this minimum and \( l_1 \). \( \square \)

Based on our 2D results, one may hope that in the 3D case the set of directions \( S \) that one needs to check would consist of directions whose components have absolute value of at most 1. The example below demonstrates that the conditions \( \Delta_P(x \pm y \pm z) \geq l \), \( \Delta_P(x \pm z) \geq l \), \( \Delta_P(y \pm z) \geq l \), and \( \Delta_P(x \pm y) \geq l_2 \) do not necessarily imply that \( ls(\square)(P) = l \).

**Example 2.** Let \( P \) be the convex hull of the following set of points in \( \mathbb{R}^3 \)

\[
\{(1,0,0), (1,1,0), (0,4,0), (2,4,0), (0,1,1), (0,4,1), (0,1,10)\}.
\]

Then \( \Delta(x) = 2 \), \( \Delta(y) = 4 \), \( \Delta(z) = 10 \), \( \Delta(x + y + z) = 10 \), \( \Delta(x - y + z) = 13 \), \( \Delta(x + y - z) = 15 \), \( \Delta(x - y - z) = 12 \), \( \Delta(x + z) = 10 \), \( \Delta(x - z) = 12 \), \( \Delta(y + z) = 11 \),...
\[ \Delta(y - z) = 13, \Delta(x + y) = 5, \Delta(x - y) = 5, \text{but} \Delta(2x + y + z) = 9, \] 
so one can apply \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \) to get \( e_\square(AP) = 9. \)

**Definition 3.10.** Let \( P \subset \mathbb{R}^3 \) be a lattice polytope. Then its lattice size \( w_2(P) \) with respect to the set \( [0, 1] \times [0, 1] \times \mathbb{R} \) is the smallest \( k \in \mathbb{Z} \) such that \( P \) is contained \([0, k] \times [0, k] \times \mathbb{R} \) after an affine unimodular transformation \( T. \)

Now we observe that it is impossible for \( P \) to have lattice width less than \( l_2 \) with respect to two directions in \( E. \) Indeed, if \((a_1, b_1, c_1)\) and \((a_2, b_2, c_2)\) are two such distinct directions then all the components of the sum of these two directions are even, and hence we get

\[ \Delta(a_1x + b_1y + c_1z) + \Delta(a_2x + b_2y + c_2z) \geq 2\Delta \left( \frac{a_1 + a_2}{2} x + \frac{b_1 + b_2}{2} y + \frac{c_1 + c_2}{2} z \right) \geq 2l_2, \]

where, for the last inequality, we observe that half-sum of two distinct directions in \( E \) cannot be in \( E. \)

Hence to compute \( w_2(P) \), we need to find the minimum \( m \) of the lattice width of \( P \) with respect to all the directions in \( E \) and then we pick the smallest two numbers out of \( m, l_1, \) and \( l_2. \) If the two smallest ones are \( l_1 \) and \( l_2 \) then \( w_2(P) = l_2. \) If we have \( l_1 \leq m < l_2 \) then \( w_2(P) = m \) since we can apply to \( P \) a unimodular map \( A = \begin{bmatrix} 1 & 0 & 0 \\ a & b & c \\ 0 & 0 & 1 \end{bmatrix} \), where \((a, b, c)\) is a direction with \(|a| = |b| = 1\) and \(|c| = 2\) such that the lattice width of \( P \) in that direction equals \( m. \)

In the case \( m < l_1 \leq l_2 \) we would have \( w_2(P) = l_1 \) and we would use the same map \( A \) as before, but with first two rows switched. We have proved:

**Theorem 3.11.** Let \( P \subset \mathbb{R}^3 \) be a lattice polytope. Then there exists an algorithm for finding \( w(P), w_2(P), \) and \( \text{ls}_2(P) \) which is quadratic in \( e_\square(P) \) and does not require enumeration of lattice points in \( P. \)

The product order on \( \mathbb{N}^3 \) is described by defining

\[(a_1, b_1, c_1) \leq (a_2, b_2, c_2) \iff a_1 \leq a_2, \ b_1 \leq b_2, \text{ and } c_1 \leq c_2.\]

Given a lattice polytope \( P \subset \mathbb{R}^3 \), let \( S_{1,1,1} \) be the set of triples \((a, b, c) \in \mathbb{Z}^3 \) such that the rectangle \([0, a] \times [0, b] \times [0, c] \subset \mathbb{R}^3 \) contains a unimodular copy of \( P. \)

**Theorem 3.12.** The set \( S_{1,1,1} \) admits a minimum with respect to the product order at the triple \((w(P), w_2(P), \text{ls}_2(P))\).

**Proof.** We only need to show that \( P \) can be mapped unimodularly inside \( R = [0, w(P)] \times [0, w_2(P)] \times [0, \text{ls}_2(P)] \). If after we run the algorithm we end up in the situation where \( l_1 \leq l_2 \leq m \) then \( P \) is already inside \( R. \) If we get \( l_1 \leq m < l_2, \) then the map \( A \) as above will map \( P \) inside \( R. \) Finally, if \( m \leq l_1, \) then this map \( A \) with first two rows switched will finish the task. \( \square \)
Example 3. In the view of Theorem 2.2 it is natural to ask whether for a lattice polytope $P \subset \mathbb{R}^3$ with $\Delta_P(x) = \Delta_P(y) = \Delta_P(z) = l$ we can conclude that $l_{\square}(P) = l$. The answer is negative, as demonstrated by the following example. Let $P$ be the convex hull of the set
\[
\{(0, 3, 1), (5, 2, 3), (4, 0, 4), (2, 5, 4), (1, 3, 0), (3, 4, 5)\},
\]
so we have $\Delta_P(x) = \Delta_P(y) = \Delta_P(z) = 5$. If we apply $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}$ to this set and shift by $\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$ the image is the convex hull of
\[
\{(0, 1, 2), (4, 4, 4), (1, 2, 0), (4, 0, 3), (1, 3, 4), (4, 0, 2)\},
\]
for which $\Delta(x) = \Delta(y) = \Delta(z) = 4$.

4. Lattice Size of $n$-Dimensional Lattice Polytopes.

We include for completeness the generalization of the standard width algorithm for computing lattice size of a lattice polytope $P \subset \mathbb{R}^n$ with respect to the unit cube $[0,1]^n$, which was explained in [2]. This algorithm is quite time-consuming, but it works in any dimension $n$.

Let $e_{\square}(P) = l$. If there exists a unimodular $n \times n$ matrix $A$ such that $e_{\square}(AP) \leq l - 1$ then the lattice width of $P$ in the direction of each row vector of $A$ is at most $l - 1$, since $w_{e_i}(AP) \leq l - 1$ for standard basis vectors $e_i$ and
\[
w_{e_i}(AP) = \max_{x \in P} e_i \cdot (Ax) - \min_{x \in P} e_i \cdot (Ax) = \max_{x \in P} (A^T e_i) \cdot x - \min_{x \in P} (A^T e_i) \cdot x = w_{A^T e_i}(P).
\]
Let $M$ be the center of mass of $P$ and let $R$ be the radius of the largest circle $C$ centered at $M$ that fits inside $P$. We shift $P$ so that the origin is at $M$. If $||v|| > \frac{l - 1}{2R}$ then
\[
w_v(P) \geq w_v(C) = 2||v||R > l - 1.
\]
Hence if we want to find $A$ such that $l_1(AP) \leq l - 1$ we only need to consider lattice vectors $v$ with $||v|| \leq \frac{l - 1}{2R}$ and check if we can find $n$ of them that can be used as rows to form a unimodular matrix $A$. The algorithm would then search through all possible size $n$ collections of primitive lattice vectors in $\mathbb{Z}^n$ with norm at most $\frac{l - 1}{2R}$ and check if such a collection forms a parallelepiped of volume 1. The output is a unimodular matrix $A$ with the smallest $e_{\square}(AP)$, which implies $l_{\square}(P) = e_{\square}(AP)$.

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