Gradient Estimates for a Nonlinear Diffusion Equation on Complete Manifolds*

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Abstract This paper deals with the gradient estimates of the Hamilton type for the positive solutions to the following nonlinear diffusion equation:

$$u_t = \Delta u + \nabla \phi \cdot \nabla u + a(x)u \ln u + b(x)u$$

on a complete noncompact Riemannian manifold with a Bakry-Emery Ricci curvature bounded below by $-K$ ($K \geq 0$), where $\phi$ is a $C^2$ function, $a(x)$ and $b(x)$ are $C^1$ functions with certain conditions.

Keywords Gradient estimate, Bakry-Emery Ricci curvature, Nonlinear diffusion equation

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1 Introduction

The notion of Bakry-Emery Ricci tensor associated with a diffusion operator was introduced by Bakry [1], which we recall as follows.

Definition 1.1 Given an n-dimensional Riemannian manifold $(M, g)$ and a $C^2$ function $\phi$ on $M$, one has a diffusion operator $L := \Delta + \nabla \phi \cdot \nabla$, where the $\Delta$ and $\nabla$ are the Laplace operator and the gradient operator on $M$ respectively. Then the Bakry-Emery Ricci tensor associated with the diffusion operator $L$ is defined as the following symmetric 2-tensor:

$$\widetilde{\text{Ric}} := \text{Ric} - \nabla^2 \phi - \frac{\nabla \phi \otimes \nabla \phi}{m - n},$$

where the constant $m \geq n$; if $m = n$, we assume $\phi = 0$. Denote by $\text{Ric}_\infty$ the limit $\lim_{m \to \infty} \widetilde{\text{Ric}} = \text{Ric} - \nabla^2 \phi$.

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In this note, we want to study the gradient estimates of the Hamilton type for the positive solution to the following nonlinear diffusion equation:

$$u_t = Lu + a(x)u \ln u + b(x)u$$  \tag{1.1}

on a complete noncompact Riemannian manifold with the above Bakry-Emery Ricci curvature bounded below by $-K$ ($K \geq 0$), where $a(x)$ and $b(x)$ are $C^1$ functions with certain conditions (for details, see Theorem 1.3).

The elliptic case of the equation (1.1) with $\phi = 0$, namely

$$\triangle u + au \ln u + bu = 0,$$  \tag{1.2}

was first considered by Ma [6] in the case that $a$ and $b$ are constants and $a < 0$ when he studied the gradient Ricci Soliton. He also pointed out that it is interesting to consider the gradient estimates for the positive solutions to the corresponding parabolic equation

$$u_t = \triangle u + au \ln u + bu.$$  \tag{1.3}

Later, Yang [12] studied the above parabolic equation and obtained the gradient estimate of Li-Yau type (see [3]) for the solutions to (1.3). Here we should also mention that Li [5] studied earlier the following equation:

$$u_t = \triangle u + bu^\alpha$$  \tag{1.4}

for some $\alpha > 0$, and got the gradient estimates and the Harnack inequality which generalize the corresponding estimates of Li-Yau [3].

When $a = b = 0$, the equation was studied by Li [5]. He obtained a gradient estimate of the Li-Yau type.

There is another kind of gradient estimates developed by Hamilton [2]. He considered the heat equation on compact manifolds and obtained the following estimate, which we call the gradient estimate of Hamilton type.

**Theorem 1.1** Let $M$ be a compact manifold without boundary and with Ricci curvature bounded below by $-K$, $K \geq 0$. Suppose that $u$ is any positive solution to the heat equation $u_t = \triangle u$ with $u \leq C$ for all $(x, t) \in M \times (0, +\infty)$. Then

$$\frac{\lvert \nabla u \rvert^2}{u^2} \leq \left( \frac{1}{t} + 2K \right) \left( \ln \frac{C}{u} \right).$$

In [10], Souplet and Zhang extend the above gradient estimate to noncompact manifolds.

**Theorem 1.2** (Souplet-Zhang) Let $M$ be an $n$-dimensional complete noncompact manifold with the Ricci curvature bounded below by $-K$, $K \geq 0$. Suppose that $u$ is any positive solution to the heat equation $u_t = \triangle u$ in $Q_{2R,2T} = B(x_0, 2R) \times [t_0 - 2T, t_0]$, and $u \leq C$ in $Q_{2R,2T}$. Then one has in $Q_{R,T}$,

$$\frac{\lvert \nabla u \rvert}{u} \leq C_1 \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K} \right) \left( 1 + \ln \frac{C}{u} \right),$$

where $C_1$ is some positive constant depending only on the dimension $n$ of $M$. 