Stability analysis of a class of non-newtonian fluids based on generalized vector variational-like inequalities on riemannian manifold

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Abstract. Fluid mechanics is a branch of mechanics. It is the science of studying fluid phenomena and related mechanical behaviors. So far, the mutual soaking and fusion between fluid mechanics and other disciplines have formed many branches, many physicists and mathematicians are working on this field. The classical Newtonian fluid mechanics believe that in parallel flow, the shear force is proportional to the shear rate, and the proportional coefficient is called the viscosity coefficient. In recent years, with the increasing importance of non-Newtonian fluids, people have found that research on non-Newtonian flow is necessary. The variational inequalities based on Riemannian manifolds were first proposed by S.Z.Németh. S.Z.Németh studied the existence of solutions to variational inequalities on Hadamard manifolds. For the equivalence of non-differentiable vector optimization problems on Riemannian manifolds and generalized weak vector variational-like inequalities, we give some new conclusions.

1. Introduction
Fluid mechanics is a branch of mechanics. It is the science that studies fluid phenomena and related mechanical behaviors. So far, the mutual soaking and fusion between fluid mechanics and other disciplines have formed many branches. Many physicists and mathematicians are working on this field.

The classical Newtonian fluid mechanics believe that in parallel flow, the shear force is proportional to the shear rate, and the proportional coefficient is called the viscosity coefficient. On this basis, the famous N-S equation can be obtained. In recent years, people have become more and more aware of the important characteristics of non-Newtonian fluids. It has been found that there are a large number of fluids in daily life that do not obey Newton's law of constant viscosity, i.e. non-Newtonian fluids. Non-Newtonian fluids are ubiquitous in nature. For such fluids, under the action of stress, it will continuously change its motion state, and its constitutive relation is significantly different from Newton's law of constant viscosity [1-3]. However, there are currently very few results in this regard, therefore the study of non-Newtonian flow is necessary.

The variational inequality based on the Riemannian manifold was first proposed by S. Z. Németh. He studied the existence of solutions to the variational inequalities on Hadamard manifolds. For the equivalence of non-differentiable vector optimization problems on Riemannian manifolds and
generalized weak vector variational-like inequalities, we give some new conclusions. We mainly discuss
two kinds of nonlinear equations in Banach space, one is the nonlinear operator equation with concavity
and convexity, and the other is the nonlinear integral fusion differential equation. The main method used
is the semi-sequence method.

Our research on nonlinear operators with concavities and convexities is applied here, and we have
obtained the method of dealing with the nonlinear integral fusion differential equations discussed in this
paper.

In the fixed point and application of the ambiguity operator, we will discuss the condition that the
nonlinear monotone operator with concavity and convexity has a unique positive fixed point, give the
relationship of each concave operator, as well as each convex operator, and order convex. Based on this,
we prove that the necessary and sufficient conditions for the existence of the only positive fixed point
of the concave addition operator are given. This paper discusses the existence of unique positive fixed
points for monotone operators with ordered concave (convex) and mixed monotone operators with
concavity and convexity.

In Banach space, for nonlinear integral fusion differential equations, we study second-order nonlinear
integral-fitting differential equations (pulse integral-differential equations) with first-order differential
terms in Banach space. By establishing the comparison theorem, we can further discuss the existence of
the minimum and maximum solutions of the first-order nonlinear integral-differential equations with
nonlinear operators.

In the study of fluid motion, the vacuum state is often involved and the problem becomes difficult
and complicated. The existing results show that the Cauchy problem of the system of equations
containing the constant coefficient of the vacuum state is ill-posed. This ill-posedness is reflected in the
fact that the solution of this system has no continuous dependence on the initial value. When the initial
density has a tight support set, the system cannot have an overall regular solution. Based on physical
considerations, Liu Taiping, Xin Zhoupeng, and Yang Tong studied the Cauchy problem of viscosity-
dependent equations and proved its local well-posedness. In fact, only when the temperature and density
change within the appropriate range, the real fluid can be well described by the ideal fluid viscosity
coefficient Zhao and human constant. In the case of large changes in temperature and density, the
viscosity coefficient of the real fluid will vary greatly with temperature and density. On the other hand,
we know that the equations can be derived from the equation by second-order expansion, and we will
find that the viscosity coefficient is temperature-dependent during the derivation. For example, for a
hard ball collision model, the viscosity coefficient should be proportional to the square root of the
temperature. If we consider the motion of the helium fluid, the second law of thermodynamics can be
used to derive that the viscosity coefficient is density dependent. Therefore, when studying the problem
of initial density containing a vacuum, we need to consider the effect of density or temperature on the
viscosity coefficient. In addition, in geophysics, many of the mathematical models used to study fluid
motion are similar to those of viscosity-dependent density equations. For example, the study of shallow
water wave systems studies the existence of local smooth solutions of the boundary problem of shallow
water wave equations and the global smooth solution of initial values near equilibrium states. For the
shallow water wave equation or the overall existence of the large initial value weak solution of the
viscosity-dependent high-dimensional equations, there are still many problems that remain unresolved.

Because the equation has strong coupling and singularity, and we allow the appearance of a vacuum,
these have brought us great difficulties [4-5]. In order to overcome this difficulty, we first regularize the
viscous terms with singularity. Prove the existence of the solution without vacuum, and then return to
the original problem through the process of taking the limit, which proves the existence and uniqueness
of the strong solution. The equation we discuss is as follows:

\[
\rho_t + (\rho u)_x + \text{div } u = \nabla u \quad (x,t) \in \Omega_T \tag{1}
\]

\[
(\rho u)_t + (\rho u^2)_x + P_x = f \quad (x,t) \in \Omega_T \tag{2}
\]
The unknown quantity $\rho, u, \eta, P(\rho)$ represents the fluid density, velocity, density and pressure of the particles in the mixture, respectively. $a > 0, \gamma > 1, 1/3 < p < 2$. The given function $f$ represents external potential energy, such as gravity and buoyancy. In particular, if $f$ is the gravitational potential, then the $\alpha$ is the gravity constant, $\gamma > 0$ is the viscosity coefficient, $\eta$ is a constant.

For the above problem, because the equation has strong coupling and singularity, similar to the proof of the second chapter, we can consider the regularization of the viscous term with singularity, which shows the existence of the solution without vacuum [6-7]. The limit process proves the existence and uniqueness of a strong solution to the problem.

2. Main Result

Here, we use the following related concepts: the concept of Riemann manifold, such as the local Lipschitz function on the Riemannian manifold $M$, the tangent space $T_x M$, the tangent bundle $T M$, the cotangent space $T^*_y M$, the coordinate card, the parallel transfer $P_{\delta_y}$, and so on.

Definition 1 Let function $f: M \rightarrow R$ satisfy the local Lipschitz condition near point $y \in M$, and the subset $\partial f(y) = \langle \zeta \in T_y M^*| f^\circ (y,v) \rangle \leq \langle \zeta, v \rangle, v \in T_y M$ of the cotangent space $T_y M^*$ of point $y$ is called the generalized subdifferential of the function $f$ at a point $y$.

Lemma 1 assumes that $\alpha > 0, u \in C^0_{\infty} \Omega, \Omega \subset R$, then $F(-\infty D^\alpha u(x)) = (-i\omega)^\alpha u(\omega); F(\omega D^\alpha u(x)) = (i\omega)^\alpha u(\omega)$, where $F$ is the Fourier transform operator $u(\omega) = F(u)$, which is $u(\omega) = \int_{R} \text{Re}^{i\omega x} u(x) dx$.

Through the zero-extension transformation of the function, all the approximation formats described above can be applied to the bounded region. Let $u(\bar{x})$ be a function that extends from the bounded region $(x_L, x_R)$ and extends the condition of the above corresponding theorem [8].

When $p = 0$, the operator defined by the formula degenerates into the lower triangle operator, then all the eigenvalues of $A^\alpha_p$ are greater than 1. In fact, it is known from the above lemma that $\lambda(A^\alpha_p) = \left(\frac{2}{3}\right)^\alpha$. This is also when the second-order Lubich formula is used directly to discretize spatial fractional derivatives and their application to fractional evolution equations will result in unstable formats [9].

Theorem 2.1 Let $M$ be a finite dimensional Riemannian manifold, $K \subset M$ for $\tau$ is an invariant convex set. For $\forall i \in J$, $f_i: K \rightarrow R$ is a local Lipschitz on $K$, and is an invariant convex function for $\Gamma$, then $x \in K$ is a valid solution if and only if $x$ is also the solution of $W^m, p$. 

\[(\rho, u)(x, 0) = (\rho_0, u_0)(x) \quad x \in [0, 1] \quad (3)\]

\[u(0, t) = u(1, t) = 0 \quad t \in [0, T] \quad (4)\]

\[P(0, t) = P(1, t) = 0, \quad t \in [0, T] \quad (5)\]
3. Proofs of Main Result

In this section, we complete the proofs of Theorems 2.1 by establishing a series of lemmas on some priori estimates. We begin with the following discussion on the estimates in $H^2([0, T])$.

First, we will use some of the following conclusions [9].

Under the assumptions in Theorem 2.1, for any $0 \leq t \leq T$, we have the following estimates

\begin{align}
\int_0^T u^{2n} \, dx + n(2n-1) \int_0^1 \int_0^R \rho^1 \nabla u^{2n-1} \cdot \nabla u \, dx \, dt & \leq C_1(T) \quad (6) \\
\|u(t)\|_{L^2}^2 + \iint_{\Omega} u_x^2 \, dx \, dt & \leq C_2(T) \quad (7)
\end{align}

Proofs of theorem 2.1

Differentiating (1.1) with respect to $t$, multiplying the resulting equation by $u$, in $L^2([0, 1])$, performing an integration by parts, and using Lemma 1, we have

\begin{align}
\frac{\partial q_i}{\partial t} - L_i(t) q_i & \leq \sum_{j=1}^N b_{ij} q_j \quad (x, t) \in D_M \\
A_i r_i(x, t) + q_i(0, x) & \leq \iint_{\Omega} \sum_{k=1}^n P_{ij}(x, t) \lambda_i(y, t) \, dx \, dt \quad (9)
\end{align}

One can get

$q(x, t) < 0, \quad (x, t) \in D$

Let $r = (r_1, r_2, \cdots, r_n)$, in which $r_i = q_i e^{-q_i}$, from (8) and (9),

\begin{align}
(r_i)_t - M_i(t) r_i & < \sum_{i=1}^n \sigma_{ij} r_j \quad (t, x) \in D \\
A_i r_i(x, t) - \|M_i(t)\|^2 & < \iint_{\Omega} \sum_{i=1}^n \sigma_{ij}(x, t) r_i(x, t) \, dx \, dt \quad (t, x) \in D
\end{align}

We choose $\sigma$ to be big enough, for every $i \ (i = 1, 2, \cdots, n)$, because of $r(x, 0) = q(x, 0) = 0 \quad (x \in \Omega)$, there exists a $\varepsilon > 0$, such that $r(t, x) > 0 \quad x \in \Omega, \ o \leq t \leq \varepsilon$. Let

\[\Gamma = \{t; t < M, \forall x \in \Omega, 0 \leq S \leq t, r(s, x) < 0\}\]

Then there exists $t$, such that $t = \sup \Gamma$ and $0 \leq t \leq M$.

If $x \in \Omega$, then $\sigma_i(x, t), \Gamma_i(t) \sigma_i(x, t) \leq 0$, that is
If \( x \in \Omega \), then \( \sigma_i(x,t) \leq 0 \), that is \( \sigma_i(x,t) = \max \tilde{\Gamma}_i(t) \sigma_i(x,t) \), that is

\[
(\sigma_i)_t - \Gamma_i(t) \sigma_i \left|_{(x,t)} \right. \leq \sum_{k=1}^{n} \delta_{ij} r_i \left|_{(x,t)} \right.
\]

Using the principle of Hopf strong maximum of second-order parabolic partial differential equations

\[
\sigma_i(x) \frac{\partial r_i(x,t)}{\partial x} + \lambda_i(t,x) = \alpha_i \frac{\partial r_i(x,t)}{\partial x} > 0
\]

Integrating (13) with respect to \( t \), applying the interpolation inequality, we conclude

\[
\int_{0}^{t} \sum_{i=1}^{n} \sigma_{ij}(x,t) q_i(x,t) dx dt < \| M_i(t) \|^2 \quad (t,x) \in D
\]

On the other hand, by (14), we derive from the assumption (15) and (16) we get

\[
\int_{0}^{T} u_i^2(x) dx \leq C_2(T)
\]

Using Young's inequality and Sobolev's embedding theorem \( W^{1,1} \hookrightarrow L^\infty \), and Lemma 1, applying embedding theorem, we derive from (12), (15) - (17) that

\[
\| u_x \|_{L^2}^2 \leq C_1(T) \left( \| u_x \| + \| u_{xx} \| \right) \leq C_2(T)
\]

Which along with Lemma 1 gives Theorem2.1. The proof is completed.

4. Conclusion
In this paper, the general parabolic equations are divided into three categories according to the range of the second-order terms of the parabolic equation: degenerate parabolic equations, uniform parabolic equations, and singular diffusion equations. Although molecular diffusion and heat conduction are their common physical backgrounds, however, the difference in diffusion rate still makes the properties of the equation solution, such as the smoothness of the solution, have a large difference. This paper discusses the equation as a model. When \( n>1 \), it represents the degenerate parabolic equation; when \( n=1 \), it represents Consistent parabolic equations; and when \( n<1 \), represent singular diffusion equations. This paper studies the changes in the three types of equations when the nonlinear properties of the equation change, and study the continuity of the initial values. Dependence.

From the conclusions obtained, in the case of the initial value problem. An integrable solution can be obtained. This shows that if the initial value problem has a solution in other senses, the decay rate of
the solution is not enough to cause the integral term to converge. Therefore, in essence, the above problem is that the solution can be related to $L^1(\mathbb{R})$. The critical value of the product. If you abandon the requirement that the solution belongs to $L^1(\mathbb{R})$, for example, limit the solution to $L^\infty(\mathbb{R})$, or ask the solution to be weighted and belong to $L^1(\mathbb{R})$. Since the singularity of the equation is stronger at this time, we can try other methods, but until now, we have not overcome the difficulty of giving singularity due to zero.

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