Analytical solution for a class of network dynamics with mechanical and financial applications

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We consider networks with a specific type of nodes that can have either a discrete or continuous set of states. It is shown that no matter how complex the network is, its dynamical response to arbitrary inputs is defined in a simple way by its response to a monotone input. As illustrative applications, we propose and discuss a quasistatic mechanical model with objects interacting via friction forces, and a financial market model with avalanches and critical behavior induced by momentum trading strategies.

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I. INTRODUCTION

Dynamical processes on networks are used to model a wide variety of phenomena such as spreading of opinions through a population [1], propagation of infectious diseases [2], neural signaling in the brain [3], and cascading defaults in financial systems [4]. Similar dynamical processes on regular lattices are used for modeling phase transitions and critical phenomena in statistical mechanics [5], avalanches and propagation of cracks in earthquake fault systems [6], percolation phenomena [7, 8], crackling noise [9, 10], hysteresis and emerging memory effects in constitutive relationships of various materials and media [10]. The structure of the underlying network may strongly influence the dynamics, the response of the network to variations of the input and parameters, and the critical values of parameters such as the critical temperature of the random field Ising spin-interaction model [11], or the epidemic threshold for disease-spread models [12, 13]. Prediction of the response of a network to variations of the input or initial state is thus an important problem, which remains open for many real-world and randomly generated networks (e.g., networks with arbitrary degree distribution) [15].

Nodes of the above networks are often assumed to have a binary response modeled by Heaviside step functions [16]. In this paper, we consider networks with a different type of node characterized as Prandtl-Ishlinskii (PI) operators [17, 18]. We provide two motivating examples, one with a mechanical and the other with a financial background. We present an almost explicit solution for the input-state-output relationship for networks of PI operators at the nodes. Essentially, we demonstrate that a network of PI nodes is also a PI operator with, possibly, a discontinuous response. This fact sets a limitation on the class of systems that can be modeled by a network of connected PI nodes while simultaneously providing us with an effective tool for mapping the network topology to its dynamics in this class. In particular, for our second (financial) example, the network of binary PI nodes can be set to produce the same response to increasing inputs (the so-called primary response curve) as an arbitrary Ising spin model. However, this primary response curve defines the response of the PI network to all possible non-monotone inputs, while the response of the Ising model to non-monotone inputs is more complicated.

Depending on the process of interest, one can distinguish between the cases where (i) an initial perturbation of the state of a group of nodes propagates through the network due to node interactions towards a final equilibrium (or, non-equilibrium) network state, i.e., dynamics of an avalanche as in [16], and (ii) where the network is driven through a sequence of metastable equilibrium states by a slowly varying input (parameter), as in [15]. Here, we focus on the latter case. We make the standard assumption that the process of relaxation to metastable equilibrium states (avalanches) is much faster than variations of the input.

FIG. 1. (a) A mechanical analogy of the stop operator: a perfect elastic element (an ideal spring) and a Coulomb friction element (object on a dry surface) connected in series. When the displacement $x$ is varied so that the spring stress $\sigma$ is within the range $[-r, r]$ the spring stretches (or contracts) with $x$ causing linear changes in $\sigma$ while the object remains stationary on the surface. For larger variations of $x$, the spring strain and stress clamp at a value of $\pm r$ whereas the object moves relative to the surface following $x$. (b) A mechanical model with three nodes, each attached to a fixed left plate and a moving right plate by of two elastic springs, with interactions modeled by stop operators as in (a).
II. MECHANICAL EXAMPLE

In the mechanical context, the phenomenological PI model describes the hysteretic relationship between strain $x$ and stress $\sigma$ in elasto-plastic materials [19]; the same model is known as Maxwell-slip friction model in applications to modeling friction [20]. The simplest example of this model is Prandtl’s elastic-perfect plastic element [18], where the restriction that $-r \leq \sigma \leq r$ is combined with the assumption that Hooke’s law with Young’s modulus $e$ is obeyed within this interval. The operator $S_r$ that transforms the input time series $x(t)$ into the output time series $\sigma(t) = S_r[x](t)$ of Prandtl’s element with unit Young’s modulus is called the stop. Figure 1(a) shows the underlying mechanical model as a cascade connection of a Coulomb friction element and an ideal elastic element, as well as the clockwise parallelogram-shaped hysteresis loops in the $(x, \sigma)$ plane in response to sufficiently large oscillations of the strain $x$. The Coulomb friction model is the stop $\sigma = S_r[x]$ with infinite $e$ where the force $\sigma$ increases without motion until it reaches the limit value $\sigma = \pm r$ at which point motion starts and the force remains constant.

In the general PI model stops with different limits $r$ are superposed to obtain the relationship $\sigma(t) = \int_0^\infty S_r[x](t) d\mu(r)$ between the displacement (input) $x$ and the force (output) $\sigma$, where $\mu$ is some cumulative distribution function. According to this relationship, a new hysteresis loop is initiated in the $(x, \sigma)$ plane each time the input $x$ makes a turning point, see Fig. 2. Like the Ising spin model [21] and the Preisach model [22], the PI model has return point memory, which means that the moment the input repeats its past extremum value a hysteresis loop closes and the dynamics proceeds as if there were no such loop (the so-called memory deletion rule, see [18]). Moreover, in the PI model, the shape of all loops is defined explicitly by the primary response (PR) function $R(x) = 2 \int_0^{\infty} (\mu(\infty) - \mu(r)) dr$ (the function $R(2x)/2$ is known as the initial loading curve in the literature [18, 22]). Namely, for every loop, the arc where the input increases is a shifted initial segment of the graph of the PR function, while the arc of the loop where the input decreases is centrally symmetric to the arc where the input increases, see Fig. 2. These properties allow one to map an arbitrary piecewise monotone input $x(t)$ to the output $\sigma(t)$ graphically very simply using the PR curve. Equivalently, one can use the sequence of running main extrema $X_k(t)$ of the input $x(t)$ (see [24]),

$$\sigma(t) = \frac{R(2X_1(t))}{2} + \sum_{k \geq 1} (-1)^k R(|X_{k+1}(t) - X_k(t)|),$$

where we assume for simplicity a zero initial output of each stop $S_r$ and a positive input with the zero initial value $x(0) = 0$. Here, the running main extrema are defined consecutively by the relations $X_k(t) = \min_{r_{k-1} \leq \tau \leq t} x(\tau)$ for odd $k \geq 1$ and $X_0(t) = x(t)$ for even $k \geq 1$, where $r_0 = 0$ and $r_k$ is the last moment prior to $t$ when $x(r_k) = X_k$.

For any, possibly discontinuous, function $R(x)$ with $R(0) = 0$ that has bounded variation, the input-output relationship defined by Eq. (1) (equivalently, by Fig. 2) will be called the PI operator $I_R$ with PR function $R$ and will be denoted $\sigma(t) = I_R[x](t)$. In particular, the stop and PI model are PI operators.

In the PI model the stops do not interact. We now proceed with an example of a network of interacting stops modeling quasistatic one-dimensional dynamics of a mechanical system. The system consists of $N$ rigid fibers elongated along the $x$ direction and interacting due to friction between them. The fibers are stretched between two plates; the left plate is fixed, the right plate is subject to a time dependent quasistatic loading. In Fig. 1(b), each fiber is represented by a node ($N = 3$) attached to two plates by linear springs. The interaction between the nodes is modeled by Maxwell-slip friction elements. The balance of forces at each node can be written as

$$-k_i \xi_i + \tilde{k}_i (u - \xi_i) + \sum_{j=1,...,N; j \neq i} a_{ij} S_{r_{ij}}[\xi_j - \xi_i] = 0,$$

where: $\xi_i$ are displacements of the nodes; the displacement $u$ of the right plate is considered as the input (control); $k_i$ and $\tilde{k}_i$ are stiffnesses of the springs attached to the left and right plates, respectively; and all the initial displacements and forces are zero. According to the action-reaction principle the matrix $r_{ij}$ and the adjacency matrix $a_{ij}$, which quantifies the strength of the interactions between the nodes, are symmetric and nonnegative. In particular, $a_{ij} = 0$ when the two nodes do not interact. The system dissipates energy due to friction with the internal energy of the system $U = \frac{1}{2} \sum_i (k_i \xi_i^2 + \tilde{k}_i (u - \xi_i)^2) + \frac{1}{2} \sum_i \sum_{j \neq i} a_{ij} (S_{r_{ij}}[\xi_j - \xi_i])^2$.

The main observation we make about this system is that if in response to an increasing input $u$ each distance $|\xi_i - \xi_j|$ corresponding to a nonzero $a_{ij}$ grows monotonically, then the relationship between each displacement $\xi_i$ and the displacement $u$ of the plate is described by a PI operator $I_{R_{ij}}$ for all possible inputs $u(t)$ giving $\xi_i(t) = I_{R_{ij}}[u](t)$. This fact is rooted in the composition formula [24], which ensures that the cascade connection

![FIG. 2. Loops of the PI operator obtained from the primary response curve which is shown by the thick line. Each hysteresis branch (dotted, dashed, solid curves) is a shifted or shifted-and-rotated image of the corresponding segment of the primary response curve.](image-url)
σ = IR1[I_R2][u] of two PI operators with PR functions R1, R2 where R2 is monotone is also a PI operator IR1oR2 with the PR function (R1 o R2)(u) = R1(R2(u)). Substituting the relations \(\xi_i(t) = IR_i[u](t)\) in Eq. (2), using the composition formula, and replacing PI operators with their PR functions, we obtain the algebraic system \(\kappa_i u - (\kappa_i + \kappa_j) R_i(u) + \sum_{j \neq i} a_{ij} \phi_{ij}(R_j(u) - R_i(u)) = 0\) for the PR functions \(R_i\) of the PI operators \(I_{R_i}\) describing the displacements of nodes, where \(\phi_{ij}\) is the PR function of the stop \(S_i = I_{\phi_{ij}}\), see Fig. 4(a). The Browder-Minty property \(\Phi\) of these equations ensures that all the PR functions \(R_i\) are continuous and increasing. These functions are measurable from the system’s response to an increasing input \(u\), as \(\xi_i(u) = R_i(2u)/2\).

We remark that monotonicity of the relative displacements \(\xi_i - \xi_j\) with increasing \(u\) is a substantial condition for ensuring the PI relationships \(\xi_i(t) = IR_i[u](t)\) between the displacements of nodes and plates in system \(\Phi\) for arbitrary inputs \(u(t)\). Even in a system of 3 nodes the differences \(\xi_i - \xi_j\) can be non-monotone in \(u\), in which case the relationship between \(\xi_i\) and \(u\) loses the return point memory property and becomes more complex (see Fig. 6 in Appendix A). However, for example, if all the friction forces are relatively small compared to the forces of the springs, then the distances \(\xi_i - \xi_j\) are monotone and we have \(\xi_i(t) = IR_i[u](t)\) (see Fig. 5 in Appendix A).

![FIG. 3. Primary response curves](image)

**FIG. 3.** Primary response curves \(R(x)\) versus input \(x\) for several examples of PI operators: (a) stop (generates clockwise hysteresis loops); (b) play (counterclockwise loops); (c) binary PI operator (loops of both orientations); (d) continuous approximation of a binary PI.

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### III. FINANCIAL EXAMPLE

Next we turn to the main example of this paper where a PI network (with discontinuous PR functions) is used to describe momentum-trading strategies within a financial market. We shall demonstrate how such trading activity can induce both the long-term mispricings and sudden reversals that are characteristic of financial systems.

We consider a set of \(N\) agents. The state \(\chi_i\) of agent \(i (i = 1, \ldots, N)\) can be either 1 or -1. The ‘long’ state \(\chi_i = 1\) indicates that the \(i\)-th agent owns the asset and the ‘short’ state \(\chi_i = -1\) means the agent does not own the asset.

It is important to note that other agents who are not directly modeled play two important roles. Firstly, many will operate on short timescales, comparable with the arrival of new exogenous information, and translate this information into price changes. This allows us to consider the system as being slowly driven through metastable states. Secondly, they provide a pool of potential trading partners so that buyers and sellers amongst the \(N\) agents do not need to be matched up (as occurs in kinetic theory models of financial and economic systems).

The following drawup/drawdown rule for the \(N\) agents mimics traders who are trying to identify and profit from a perceived trend — so-called momentum traders. Such rules are used in actual trading algorithms (e.g. [28]) and are a special case of trading strategies considered in [29].

After switching to the long state \(\chi_i = 1\) (purchasing the asset) at time \(\tau\), the \(i\)-th trader tracks the asset price \(p(t)\) and the running maximum \(\max_{r \leq s \leq t} p(s)\) since time \(\tau\). The trader switches back to the short state \(\chi_i = -1\) at the first time \(\theta > \tau\) when the inequality \(p(t)/\max_{r \leq s \leq t} p(s) \leq \alpha_i^-\) is satisfied for some threshold value \(\alpha_i^- \in (0, 1)\). For example, if \(\alpha_i^- = 0.9\), then the trader sells at the moment when the price drops from its peak value by 10%. Using the log-price \(r(t) = \ln(p(t)/p(0))\), and defining \(\rho_i = \ln \alpha_i^-\) gives the selling condition \(\theta = \min\{t > \tau : r(t) - \max_{r \leq s \leq t} r(s) \leq \rho_i\}\).

This trader then adopts a similar strategy for deciding when to buy again. Namely, the trader tracks the ratio \(p(t)/\min_{r \leq s \leq t} p(s)\) and switches to the long state \(\chi_i = 1\) when this ratio exceeds a threshold value \(\alpha_i^+ > 1\). As above, we define the switching time \(\tau\) by \(\tau = \min\{t > \theta : r(t) - \min_{r \leq s \leq t} r(s) \geq \rho_i^+\}\) where \(\rho_i^+ = \ln \alpha_i^+\).

Following [29], we use the aggregated quantity \(\sigma = \sum_{i=1}^{N} \mu_i \chi_i\) to represent the overall sentiment of the market where the weights \(\mu_i > 0\) are a measure of the size or market impact of the agents.

In order to utilize the results of the previous section we must make the mild assumption that \(\rho_i^+ = -\rho_i := \rho_i\) for the switching thresholds of each trader. Then the relationship between \(r(t)\) and the state \(\chi_i(t)\) of each trader is defined by the binary PI operator \(\chi_i(t) = IR_i[r](t)\) whose PR function is the shift \(H_i(r) = H(r - \rho_i)\) of the Heaviside step function \(H(r)\) (see Fig. 3(c)). Moreover, as the superposition of PI operators is also a PI operator, the market sentiment is related to the logarithmic price by the PI operator \(\sigma(t) = IR[r](t)\) with the PR function \(R(r) = \sum_{i=1}^{N} \mu_i H_i(r)\).

We now introduce coupling between the agents by generalizing the drawup/drawdown strategy to make the \(i\)-th agent switch based upon the maximum drawdown and drawup of the aggregated quantity \(\xi_i = \sum_{j=1}^{N} a_{ij} \chi_j + b_ir\) with the threshold values \(\pm \rho_i\). That is, the input \(\xi_i\) replaces \(r\) in the trading strategy of the \(i\)-th trader leading to the network model

\[
\chi_i(t) = IR_i\left[\sum_{j=1}^{N} a_{ij} \chi_j(t) + b_ir(t)\right]; \quad \sigma = \sum_{j=1}^{N} \mu_j \chi_j, \quad (3)
\]

where \(b_i, \mu_i \geq 0, i = 1, \ldots, N\). The coefficients \(a_{ij} \geq 0\) of the adjacency matrix measure the (attracting) influence of state \(\chi_j\) of the \(j\)-th trader (node) upon the decision making of the \(i\)-th trader, and the PI operator \(IR_i\) at
each node has the step-shaped primary response curve. Using the composition formula for PI operators (as in the above mechanical example), one obtains a solution of model (3) in the form of the PI operator relationship $\chi_i(t) = I_{H_i} [r](t)$ between the state of each trader and the logarithmic price $r$, where the set of thresholds of the step response functions $H_i$ is a subset of the set of thresholds $\rho_i$ of the functions $H_i$. The composition formula known for PI operators with continuous PR functions [22] requires justification when applied to system (3) with discontinuous functions $H_i$. Omitting the details, this formula can be derived using the Kurzweil integral theory [30]. The PR curve $R(r) = \sum_{i=1}^{N} \mu_i H_i(r)$ of the PI relationship $\sigma(t) = I_R [r](t)$ between the logarithmic asset price and the market sentiment can be obtained by testing system (3) with an increasing input $r(t)$ (see Fig. 7(a)) in Appendix B, or by solving the algebraic system

$$\hat{H}_i(r) = H \left[ \sum_{j=1}^{N} a_{ij} \hat{H}_j(r) + b_i r - \rho_i \right]$$

(4)

derived from (3). A large jump in the PR curve $R$ in Fig. 7(a) corresponds to an avalanche: a change in the state of a node causes another node to change its state (via network connections) triggering the cascading effect which involves many other nodes.

If we replace the binary PI operator $\chi_i(t) = I_{H_i} [\xi_i](t)$ at the nodes of model (3) by the simple input-output relationship $\chi_i(t) = H(\xi_i(t) - \rho_i)$ (a memoryless ideal switch), the response of the network to increasing inputs, i.e., the PR function, remains the same. Hence, plentiful results describing primary response functions of the networks of Heaviside switches (such as statistics of avalanches, critical parameters, etc., see e.g. [5]) are equally valid for the PI networks (3); see Fig. 4(b) in Appendix B. Equation (1) then describes explicitly the response of the PI network to arbitrary inputs in terms of its PR function $R$, while Eq. (3) links the network topology and structure (in terms of its adjacency matrix) with the PR function $R = \sum_{i=1}^{N} \mu_i H_i$.

We stress that (large) jumps of the network PR curve $R$ are due to avalanches caused by interaction of nodes rather than the discontinuity of the response function $H_i$ at the nodes. A similar discontinuous PR curve $R$ can be generated by a network of the PI nodes with continuous state, where each node has the continuous PR curve shown in Fig. 3(d). PI models of investment (supply) strategies with a continuous PR curve, such as the one shown in Fig. 3(b), have been proposed in economics literature [31]. The counterpart of Eq. (1) for a network model with such nodes leads to the PI operator which can have a discontinuous response due to avalanches.

### IV. A PRICING MODEL

The next step is to feed changes in sentiment back into the price to give a complete asset pricing model. The following simplifying assumptions allow us to compute analytical solutions and describe how the transition from a continuous to a discontinuous PR curve dramatically changes the market dynamics.

First we note that (3) is valid for arbitrary $r(t)$ and it is reasonable to reinterpret $r(t)$ in the definition of $\xi_i$ as being an exogenous Brownian information stream rather than the log-price. The log-price, now denoted $r^*(t)$, is assumed to be modified by the sentiment in a proportional way leading to $r^*(t) = r(t) + \kappa \sigma(t)$ where the important parameter $\kappa > 0$ quantifies the effect of momentum traders on the price (note that if, say, more momentum traders enter the market then $\kappa$ will increase, in which case $\kappa$ becomes a varying parameter). We now choose $\mu_i = 1/N, a_{ij} = \kappa/N$ and $b_i = 1$ so that $\chi_i(t) = I_{H_i}(r^*)(t)$ and all the agents react solely to the price as in the original drawup/drawdown description. Finally, the thresholds $\rho_i$ are chosen uniformly from an interval $[c, a]$. Figure 4 shows a simulation with $N = 10000$ and $[c, a] = [0.05, 0.45]$.

![FIG. 4. (a) Time series of the log-price $r^*(t)$ (dashed red line) and the exogenous Brownian information stream $r(t)$ (solid blue line). (b) Daily increments of the log-price $r^*(t)$ (dashed red line) and $r(t)$ (solid blue line). Plots (a), (b) were obtained for $N = 10000$ agents with thresholds uniformly distributed over the interval $[c, a] = [0.05, 0.45]$ for $\kappa = 0.15$. (c) Histogram of the daily log-price increments (red dots) and the exogenous Brownian information stream (blue squares) obtained from 50 simulations with the same parameters as in (b). The black curves is the analytic approximation for $r^*(t)$ (see Appendix C). (d) Same as (c) but for $\kappa = 0.21$, which is slightly above the critical $\kappa_c = 0.2$.](image-url)
subtle. Here the continuous PR curve of the operator \( \sigma = I_R[p] \) has the shape shown in Fig. 3(d). The dynamics can be reformulated as a random walk of a particle on a closed rectangular domain with motion along the right (left) boundary corresponding to increasing (decreasing) \( \sigma \) and motion on the interior and upper and lower boundaries corresponding to constant \( \sigma \). For a fixed \( \kappa < \kappa_c \) this model provides an analytic approximation (see Appendix C) to the distribution of log-price changes over a given time interval such as can be seen in Fig. 4(c).

The tails of these distributions have often been observed to follow approximate power laws \( p \) in actual markets but here they are close to a combination of Gaussian and error functions. As \( \kappa \) approaches the critical value \( \kappa_c \), the distribution becomes bimodal as in Fig. 4(d) where a smaller mode corresponding to large changes of the price separates from the main Gaussian mode.

The existence of a critical value together with the possibility of \( \kappa \) varying in time suggests a plausible mechanism for extreme market volatility and the associated bubbles/crashes and fat tails. As a particular asset class receives increased attention or is perceived to undergo some fundamental positive change, the price will rise to increase through the critical value \( \kappa \) to increase through the critical value \( \kappa_c \). For a fixed \( \kappa < \kappa_c \), the systemwide downward cascade.

Applying A: Mechanical model

In this section, we consider in more detail the mechanical model which is schematically illustrated in Fig. 1(b) and described by Eq. (2). This model can be used to represent a bunch of one-dimensional rigid fibers (shown as nodes in Fig. 1(b)) elongated along the horizontal axis, whose left and right ends are attached (by springs) respectively to the left and the right plates. The displacement of fiber \( i \) relative to the left plate is \( \xi_i \). We assume perfect elastic interactions between each fiber \( i \) and the left (and the right) plate with coefficients \( k_i \) and \( k_r \) correspondingly). Furthermore, we assume that each fiber is in contact with some other fibers along its length and there is Maxwell friction when they move with respect to one another. We model the friction force acting on the \( i \)-th fiber due to its relative displacement with respect to the \( j \)-th fiber by \( a_{ij}S_{r_{ij}}(\xi_j - \xi_i) \), where fiber interaction strengths \( a_{ij} \) are non-negative, and \( S_{r_{ij}} \) denotes the stop operator of half-width \( r_{ij} \geq 0 \) (see Fig. 1(a)) with input \( \xi_j - \xi_i \). Initially, all forces and displacements in the system are 0. The time-varying input of the system is the displacement \( u \) of the right plate relative to its initial position (see Fig. 1(b)); the left plate does not move. All the motions are quasistatic.

Equation (2), which describes the balance of forces for fiber \( i \), can be written as

\[
(k_i + \bar{k}_i)\xi_i + \sum_{j \in N_i} a_{ij}S_{r_{ij}}[\xi_i - \xi_j] = \bar{k}_iu,
\]

where \( N_i \) denotes the set of indices \( j \) for which \( a_{ij} \) is in the network with the adjacency matrix \( a_{ij} \) where each fiber is represented by a node).

Equation (A1) represents a piecewise linear system, which we can solve in each of the linear regimes while tracking the transitions from one linear regime to another. A switch between linear regimes occurs when any of the stop operators \( S_{r_{ij}} \) saturates (i.e., when the magnitude of the friction force between any pair of fibers \( i \) and \( j \) achieves its maximal possible value \( r_{ij} \) or de-saturates (the magnitude of the friction force becomes smaller than \( r_{ij} \)); we describe this by saying that link \( ij \) saturates or de-saturates. Before we consider the transitions between linear regimes in more detail, let us write Eq. (A1) in the form of a linear matrix equation

\[
M\vec{\xi} = \vec{K}u + \vec{D},
\]

where \( \vec{\xi} = [\xi_1, \ldots, \xi_n] \) and \( \vec{K} = [\bar{k}_1, \ldots, \bar{k}_n] \). The matrix \( M \) and vector \( D \) take specific values (given by Eqs. (A5) and (A6) below) for each of the linear regimes.

We introduce a new quantity \( O_{ij} \) which denotes the current reference point (the origin) for the interaction \( S_{r_{ij}}(\xi_i - \xi_j) \) between nodes \( i \) and \( j \). Specifically, \( O_{ij} \) is the value of \( \xi_i - \xi_j \) at which \( S_{r_{ij}}(\xi_i - \xi_j) = 0 \), provided that the relative displacement \( \xi_i - \xi_j \) approaches the value

V. CONCLUSIONS

To summarize, we have considered input driven dynamics on networks with PI operators at each node. Examples of such nodes are provided by models of plasticity and friction and some strategies exploited in momentum trading. We have shown that no matter how complex the network is, its response to arbitrary variations of the input is described by an effective PI operator and hence can be deduced in a simple and explicit way from networks response to a monotonically increasing input. Using these results we have derived the analytical form of fat-tailed price returns induced by momentum-based trading. Allowing the number of momentum traders (the parameter \( \kappa \)) to vary may yield new insights into the approximate power-law scaling observed in actual markets.

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Appendix A: Mechanical model

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\]

where \( \vec{\xi} = [\xi_1, \ldots, \xi_n] \) and \( \vec{K} = [\bar{k}_1, \ldots, \bar{k}_n] \). The matrix \( M \) and vector \( D \) take specific values (given by Eqs. (A5) and (A6) below) for each of the linear regimes.

We introduce a new quantity \( O_{ij} \) which denotes the current reference point (the origin) for the interaction \( S_{r_{ij}}(\xi_i - \xi_j) \) between nodes \( i \) and \( j \). Specifically, \( O_{ij} \) is the value of \( \xi_i - \xi_j \) at which \( S_{r_{ij}}(\xi_i - \xi_j) = 0 \), provided that the relative displacement \( \xi_i - \xi_j \) approaches the value
monotonically from its current value. Notice that $O_{ij} = -O_{ji}$. We also introduce a binary quantity $l_{ij}$ to represent the current state of link $ij$ (interaction between fibers $i$ and $j$),

$$l_{ij} = \begin{cases} 
1, & \text{if link } ij \text{ is unsaturated} \\
0, & \text{if link } ij \text{ is saturated} \end{cases}.$$  \hfill (A3)

We assume that initially $O_{ij} = 0$ for all the links and $l_{ij} = 1$ (all links are unsaturated). These quantities will be updated according to the rules described below when the variations in the input parameter $u$ become sufficiently large.

If a link $ij$ is unsaturated ($l_{ij} = 1$), then the value of $S_{r_{ij}}$ is given by $(\xi_i - \xi_j - O_{ij})$. In the case when link $ij$ is saturated ($l_{ij} = 0$), the value of $S_{r_{ij}}$ is given by $r_{ij} \text{sgn}(\xi_i - \xi_j - O_{ij})$. Therefore, using the notation $O_{ij}$ and $l_{ij}$, we can rewrite Eq. (A1) as

$$
(k_{ij} + \tilde{k}_{ij})\xi_i + \sum_{j \in N_i} l_{ij} a_{ij} (\xi_i - \xi_j - O_{ij}) + \sum_{j \in N_i} (1 - l_{ij}) a_{ij} r_{ij} \text{sgn}(\xi_i - \xi_j - O_{ij}) = \tilde{k}_{ij} u.
$$

Equation (A4) can be written in matrix form (A5) where the elements of $M$ and $\tilde{D}$ are given by

$$M_{ij} = \begin{cases} 
-k_{ij} - \sum_{j \in N_i} a_{ij} l_{ij}, & \text{if } i \neq j \\
-k_{ij} - \sum_{j \in N_i} a_{ij} l_{ij}, & \text{if } i = j \end{cases}.$$  \hfill (A5)

and

$$\tilde{D}_i = \sum_{j \in N_i} a_{ij} (l_{ij} O_{ij} - (1 - l_{ij}) r_{ij} \text{sgn}(\xi_i - \xi_j - O_{ij})).$$  \hfill (A6)

For example, if we consider three fibers connected as in Fig. 1(b), then Eq. (A2) takes the form

Suppose we want to calculate the values of $\xi_i$ as the input $u$ varies. The solution of Eq. (A2) is given by

$$\bar{\xi} = M^{-1}(\tilde{K}u + \tilde{D}).$$  \hfill (A8)

However, we need to update $M$ and $\tilde{D}$ each time a link saturates or de-saturates.

The condition for the saturation of an unsaturated link $ij$ is $\xi_i - \xi_j = O_{ij} \pm r_{ij}$. We note that when we check this condition for all pairs of $i$ and $j$, then it is sufficient to consider only one of the two cases, for example,

$$\xi_i - \xi_j = O_{ij} + r_{ij},$$  \hfill (A9)

since the other case is captured due to $O_{ij} - r_{ij} = -(O_{ji} + r_{ji})$. Using the link saturation condition (A9) and Eq. (A8) we obtain the values of $u_{ij}$ at which the link between nodes $i$ and $j$ saturates:

$$u_{ij} = \frac{O_{ij} + r_{ij} + (M^{-1} \tilde{D})_i - (M^{-1} \tilde{D})_j}{(M^{-1} K)_i - (M^{-1} K)_j}.$$  \hfill (A10)

Hence, we can calculate $\xi_i$ from Eq. (A8) for all $u$ (without the need to update $M$ and $\tilde{D}$) until $u$ passes through any of $u_{ij}$ values. When $u$ reaches any of $u_{ij}$, this will indicate that we transition to a new linear regime, and thus have to calculate new $M$ and $\tilde{D}$ as $l_{ij}$ changes from 1 to 0 at this point.

The de-saturation of a saturated link $ij$ occurs when $\xi_i - \xi_j$ has a turning point (passes through a local maximum or minimum value). There are two ways this can happen. First, due to complex interactions between the nodes, a link $ij$ may de-saturate due to the saturation of another link $mn$ (this happens in the example shown in Fig. 6 as described in steps 6 and 13 of Table II). Second, saturated links may de-saturate when the input $u$ has a turning point. (Interestingly, saturated links may remain saturated when $u$ makes a turning point; this happens in the example shown in Fig. 6 as described in step 4 of Table II where $S_{r_{ij2}}$ does not de-saturate.) In both cases, we need to determine whether $\xi_i - \xi_j$ has a turning point by evaluating the sign of the derivative of $\xi_i - \xi_j$ with
respect to $u$. The derivative is obtained from Eq. (A8) and is given by

$$\left(\xi_i - \xi_j\right)'_u = (M^{-1}\dot{K})_i - (M^{-1}\dot{K})_j,$$

(A11)

In the first case, we need to evaluate the sign of $(\xi_i - \xi_j)'_u$ before and after the saturation of $m$. Moreover, a change in $t_{ij}$ will affect matrix $M$ and therefore further changes in $(\xi_i - \xi_j)'_u$ (and thus in $l_{ij}$) are possible. This means that we need to iterate the evaluation of $(\xi_i - \xi_j)'_u$, $l_{ij}$ and $M$ until $l_{ij}$ reaches a steady state.

In the second case, we need to find a partition of previously saturated links into a set of links that remain saturated and a set that becomes de-saturated. These sets should ensure the consistency condition on $(\xi_i - \xi_j)'_u$, when $u$ makes a turning point, that $(\xi_i - \xi_j)'_u$ should change the sign for links that remain saturated, and not for links that become de-saturated. Similar to the first case, finding the set of de-saturating links may be not straightforward because of the dependency of $(\xi_i - \xi_j)'_u$ on $l_{ij}$. However, this can be done numerically by simply looping through all possible partitions and finding the one that leads to consistency.

Finally, for the resulting set of links that became de-saturated we calculate the new $O_{ij}$ from $\xi_i - \xi_j$, $r_{ij}$, and the current $O_{ij}$:

$$O_{ij}^{\text{new}} = \xi_i - \xi_j - \text{sgn}(\xi_i - \xi_j - O_{ij})r_{ij}.\quad (A12)$$

Figures 5 and 6 present examples of systems consisting of three interacting fibers (nodes). For the system presented in Fig. 5 all three relative displacements $|\xi_i - \xi_j|$ grow monotonically in response to an increasing (decreasing) input $u$ (see the lower panel). Hence, the position of each node $\xi_i$ is related to the displacement of the right plate $u$ by a PI operator $\xi_i(t) = I_{R[i]}u(t)$. Indeed, all the hysteresis loops in Fig. 5 (see the upper panel) are centrally symmetric, which is the characteristic property of PI operators.

For the system shown in Fig. 6 the relative displacement $\xi_1 - \xi_2$ between the nodes 1 and 2 changes non-monotonically when the input increases (decreases), see the lower panel. As a result, testing the system with non-monotone inputs reveals that the relationship between the input $u$ and displacement $\xi_1$ time series is not a PI relationship and is more complex. In particular, this relationship does not have the return point memory property: when the input $u$ goes, for example, from $-100$ to $-80$ and back to $-100$ the loop does not close as shown by the bold line in the upper panel of Fig. 6 see also steps 4 through 10 in Table 1.

Appendix B: Financial model

We consider an example of the network model (3) of interacting momentum traders, see Fig. 3 for the parameters of the network. Figure 7(a) presents the PR curve of the PI relationship $\sigma(t) = I_{R[i]}[r](t)$ between the logarithmic asset price and the market sentiment (a solution of the model). This PR curve has been obtained simply by testing system (3) with an increasing input $r(t)$.

Fig. 5. An example where $\xi_i - \xi_j$ are monotone in $u$, hence $\xi_i(t) = I_{R[i]}[u](t)$. In this example the system has three fibers (nodes). Each node interacts with the other two and the forces of interaction between them are 1 (i.e., all $\alpha_{ij} = 1$). The nodes all have the same $r_{ij} = 1$. The left springs’ stiffness parameters are $K = \{1, 1, 1\}$ and the right springs’ stiffness parameters are $\tilde{K} = \{0, 1, 10\}$. Initially all displacements are zeros. The upper panel shows variations of the position $\xi_1$ of the first node in response to the input $u$ which starts at 0 and varies monotonically between the following turning points: $\{0, -100, -80, -100, -90, -97, -75\}$. Plots of $\xi_2$, $\xi_3$ against $u$ (not shown), as well as plots of any weighted sum of $\xi_i$, also demonstrate symmetric loops. The lower panel shows the monotonic growth of the displacements $|\xi_i - \xi_j|$ for a decreasing input $u$ starting at 0.

Fig. 6. An example where $\xi_1 - \xi_2$ is non-monotone for a decreasing input $u$ (the lower panel) and so the relationship $\xi_1(u)$ loses the return point memory property (the upper panel). The network structure, parameters and the variation of $u$ are the same as in Fig. 5 except that $K = \{1, 10, 1\}$. The sequence of saturations for the lower panel is as follows: $S_{r_{12}}$ (at $u \approx -1.56$), $S_{r_{13}}$ (at $u \approx -2.02$), $S_{r_{12}}$ (at $u = -33$). The plots have corners at these points.
TABLE I. Table presenting the sequence of input values \( u \) and the corresponding \( \xi_1 \) values, at which nodes (stop operators \( S_{r_{ij}} \)) saturate/desaturate for the example shown in the upper panel of Fig. 6. Each saturation/de-saturation creates a corner point of the piecewise linear trajectory.

The histogram in Fig. 7(b) shows statistics of avalanche sizes for the PR curve calculated from 1000 realizations of random networks and node thresholds. A large jump in the PR curve corresponds to a big avalanche involving many nodes.

Appendix C: Pricing model

In this section, we discuss in more detail the pricing model \( r^*(t) = r(t) + \kappa \sigma(t) \), where \( r^* \) is the log-price of the asset, \( r \) is the exogenous Brownian information stream, the parameter \( \kappa \) quantifies the effect of momentum traders on the price, and the sentiment of the market \( \sigma \) is defined as the arithmetic mean of the states \( \chi_i \) of momentum traders,

\[
\sigma = \frac{1}{N} \sum_{i}^{N} \chi_i(t). \tag{C1}
\]

Dynamics of the states are driven by the log-price according to the PI input-output relationship, \( \chi_i(t) = I_{H_i}[r^*](t) \), which closes the model. Here the PR function \( H_i(r^*) = H(r^* - \rho_i) \) is the step function with threshold \( \rho_i \) chosen uniformly from \([c, a]\).

Testing the system with an increasing input, we see that in the continuum limit \( N \rightarrow \infty \) the exogenous Brownian input and the variables \( \sigma \) and \( r^* \) are related by the formulas

\[
\sigma(t) = I_{H}[r + \kappa \sigma](t), \quad r^*(t) = r(t) + \kappa I_{H}[r^*](t) \tag{C2}
\]

where the PR function of the PI operator \( I_{H} \) has the profile shown in Fig. 3(d) with \( \rho_1 = c \) and \( \rho_2 = a \). According to our results, these relationships can be easily solved explicitly:

\[
\sigma(t) = I_{H}[r](t), \quad r^*(t) = r(t) + \kappa \sigma(t) \tag{C3}
\]

and two cases are possible. In the subcritical case, \( \kappa < \kappa_c = (a - c)/2 \), the PR function \( R \) in these relationships also has the shape shown in Fig. 3(d) with the same \( \rho_1 = c \), but with a smaller \( \rho_2 = a - 2\kappa > \rho_1 \). In the supercritical case \( \kappa > \kappa_c \), the function \( R \) is the step function with the threshold \( c \). That is, in the supercritical case, due to a global avalanche, all the traders switch their state simultaneously causing \( \sigma \) jump between the values \( \pm 1 \).

We first consider the subcritical case which is more relevant to normal market conditions and more interesting. Plausible ranges of the parameters \( a, c \) and \( \kappa \) can be estimated as follows. A momentum trader reacting to price changes on the order of say 1% would trade too frequently, incurring significant transaction costs, with most of the trading being driven by random fluctuations rather than actual changes in the price trend. Conversely, thresholds of the order of 50% would result in very infrequent trading that misses many moderately-sized trends.
The parameter $\kappa$ can be estimated by considering the total influence of momentum traders on the asset price. A reasonable estimate of the difference in price between a market with maximum positive momentum ($\sigma = 1$) and negative momentum ($\sigma = -1$) is 20-50% ceteris paribus (although it may go much higher during an asset bubble as new speculators enter the market: during such an event the distribution of threshold values may also move lower as agents’ investing time horizons shorten). The values $[a, c] = [0.05, 0.45]$ and $\kappa = 0.15$ that have been used in computations presented in Fig. 4(a)-(c) are consistent with these estimates while still being comfortably within the subcritical regime.

Our objective is to calculate the profile of the daily log-price increments histogram shown in Fig. 4(c). For this purpose, we first find the stationary distribution of the stochastic process $\sigma(t) = I_R[r](t)$. The shape of the PR curve $R$ allows us to describe this process as a random walk of a particle in a rectangle, where the vertical coordinate of the particle is $\sigma$, while the horizontal coordinate is an auxiliary variable $w$, see Fig. 8. The motion of the particle $(w(t), \sigma(t))$ is driven by the Brownian input $r(t)$. For simplicity, we describe the random walk in a discrete time/state setting. In this case, the particle lives on a rectangular mesh with $n_x$ columns and $n_y$ rows and the Brownian input is represented by a random walk $r$ which at every time step with equal probability makes one step left or one step right along a uniform mesh on the real line. First assume that the input $r$ moves left at some moment. Then, if the particle was not on the left side of the rectangle (left column of the mesh), it also moves one step left to a neighboring node; it moves one step down from any node of the left side, except from the lower corner; and, if the particle was in the lower left corner of the rectangle, it remains there. Similarly, when $r$ moves right, so does the particle if it was not on the right side of the rectangle; it moves one step up from any node of the right side, except from the upper corner; and, it remains in the upper right corner if it was there (see Fig. 8). In this model, the horizontal and vertical step of the rectangular mesh are related by $|\Delta w| = (\kappa_c - \kappa)|\Delta \sigma|$, the horizontal step equals the step of the input mesh, $|\Delta w| = |\Delta r|$, and the number of rows and columns in the rectangular mesh are related by $cn_y = 2(\kappa_c - \kappa)n_x$. These relationships ensure that the increment of the log-price equals $\Delta r^{*} = \Delta r + \kappa \Delta \sigma$ where $\Delta r$ and $\Delta \sigma$ are the increments of the input and the vertical coordinate of the particle at the same time step, respectively.

A simple calculation shows that the probability density of the stationary distribution for the random walk $(w, \sigma)$ linearly decreases on the lower (upper) side of the rectangle from the lower left to the lower right (upper right to upper left) corner and is uniform on the rest of the rectangle. In the continuous time/state limit $(n_x, n_y \to \infty)$, when the input $r(t)$ becomes the continuous Brownian motion, the density function of the stationary probability distribution for the random process $(w(t), \sigma(t))$ on the rectangle $\Pi = \{0 \leq w \leq c, 0 \leq \sigma \leq 2\}$ is

$$
\rho_{st}(w, \sigma) = \frac{(c - w)\delta(\sigma) + w\delta(\sigma - 2) + \kappa_c - \kappa}{c(a - 2\kappa)},
$$

where $\delta$ denotes the Dirac delta function. We note that in the continuum limit the process $w$ becomes the reflected Brownian motion on the interval $[0, c]$ (with reflecting boundary condition at both ends).

**FIG. 8.** Rectangular random walk $(w, \sigma)$. At each time step, a particle makes one of two possible moves with equal probability as shown by arrows. It moves either to a neighboring node or, if it is at the upper right or lower left corner, possibly to the same node.

Calculations of the profile of the histogram for daily log-price increments $\Delta r_n^* = r^*(t_n + \tau) - r^*(t_n)$, where $\tau = 1$ day is a fixed time interval and $t_n = n\tau$, will be performed in the continuous time/state setting. Assuming ergodicity, statistics of the increments $\Delta r_n$ obtained from a typical long trajectory of the processes $r$ and $(w, \sigma)$ can be approximated by the probability density function of the random variable

$$
\Delta r^* = r^*(\tau) - r^*(0) = r(\tau) + \kappa(\sigma(\tau) - \sigma(0)),
$$

where the stationary process $(w(t), \sigma(t))$ bounded by the rectangle $\Pi$ is driven by the Brownian input $r(t)$ (with $r(0) = 0$) and has the law (C4). The following calculations are based on the assumption that the maximal increment of the Brownian input $r$ during one day remains bounded by the quantity $c/2$ with a probability close to 1:

$$
P\left(\max_{0 \leq t \leq \tau} |r(t)| \geq c/2\right) \ll 1.
$$

For the plots shown in Fig. 11 the variance of the Brownian input $r(T)$ at the end of the time interval $T = 40$ years (with 250 trading days per year) has been set to 1. Hence, for one trading day $r(\tau) \sim N(0, \Sigma^2)$ with the standard deviation $\Sigma = 0.01$. Since $c/2 = 2.5\Sigma$ for these plots, $P(|r(\tau)| \leq c/2) = 0.988$, which agrees with (C6). We will consider only those input trajectories that satisfy $|r(t)| < c/2$ on the whole time interval $0 \leq t \leq \tau$. 
The corresponding trajectories of the process \((w, \sigma)\) cannot reach both left and right sides of the rectangle II during the same time interval. Other trajectories will be disregarded.

Thus, let us consider trajectories \((w(t), \sigma(t))\) corresponding to different realisations of the Brownian \(r(t)\) on the time interval \(0 \leq t \leq \tau\) and different initial data \((w(0), \sigma(0))\), restricting our attention to initial data from the right half of the rectangle II, i.e., with \(c/2 \leq w(0) \leq c\), \(0 \leq \sigma(0) \leq 2\) (Trajectories starting at the left half of II can be treated similarly). Since we assume that \(r(t) > -c/2\) for all \(0 \leq t \leq \tau\) (other inputs are disregarded due to \(\mathbf{(C4)}\)), a trajectory starting from the right half of II never reaches the left side of the rectangle. For such trajectories, the log-price increment \(\mathbf{(C5)}\) can be easily expressed in terms of the variables \(w(0), \sigma(0), r(\tau),\) and \(m(\tau) = \max_{0 \leq t \leq \tau} r(t)\), the maximum input value, where the probability density of the joint distribution for the Brownian motion and its running maximum is defined by the relation

\[
\rho_{br}(r, m) = \begin{cases} 
\frac{2(2m-r)}{\pi \sqrt{2\pi r}} e^{-\frac{(2m-r)^2}{2r}}, & m \geq 0, m \geq r, \\
0, & \text{otherwise.} 
\end{cases} 
\] (C7)

The two-dimensional random variable \((r(\tau), m(\tau))\) and the two-dimensional variable \((w(0), \sigma(0))\), which has the law \(\mathbf{(C2)}\), are independent. As the expression for \(\Delta r^*\) depends on relations between these variables, we classify trajectories into a few groups.

If \(\sigma(0) = 2\), then the trajectory remains on the upper boundary of the rectangle II all the time \((\sigma(t) = 2\) for all \(0 \leq t \leq \tau\)), hence the log-price increment \(\mathbf{(C5)}\) equals the increment of the input, \(\Delta r^* = r(\tau)\). Since \(r(\tau)\) is normally distributed, so is \(\Delta r^*\) for such trajectories:

\[
\rho(\Delta r^* = y, \sigma(0) = 2) = \frac{3c}{8(a - 2\kappa)} \cdot e^{-\frac{\pi^2 y^2}{2}}. 
\] (C8)

where \(P = 3c/(8(a - 2\kappa))\) is the total probability to find the point \((w(0), \sigma(0))\) on the right half of the upper side of the rectangle, see \(\mathbf{(C4)}\).

Another class consists of trajectories that start below the upper side of the rectangle II and never reach its right side during the day. This class is defined by the relations

\[
0 \leq \sigma(0) < 2; \quad c/2 \leq w(0) < c - m(\tau). 
\] (C9)

For such trajectories, \(\sigma(t) = \sigma(0)\) for all \(0 \leq t \leq \tau\) and hence \(\Delta r^* = r(\tau)\), as in the previous case. Integrating the product of the probability densities \(\rho_{st}(w, \sigma)\rho_{br}(r, m)\) over domain \(\mathbf{(C9)}\) with respect to the variables \(w(0) = w, \sigma(0) = \sigma\) and \(m(\tau) = m\), we obtain the probability density function of the log-price increment for this class of trajectories. After some manipulations, this probability density can be presented as the integral

\[
\rho(\Delta r^* = y, \sigma(0) < 2, w(0) < c - m(\tau)) = \frac{1}{2c(a - 2\kappa)} \int_{\max\{0, y\}}^{c/2} \rho_{br}(y, m) \left( \frac{c}{2} - m \right) \left( \frac{c}{2} + 4(m - m(\tau)) \right) \, dm, 
\] (C10)

which can be expressed explicitly in terms of the Gaussian and the error function.

The next set of conditions,

\[
m(\tau) + w(0) > c; \quad \frac{m(\tau) + w(0) - c}{\kappa_\kappa - \kappa} < 2 - \sigma(0), 
\] (C11)

ensures that a trajectory reaches the right side, but not the upper side of the rectangle II. For such trajectories,

\[
\Delta r^* = r(\tau) + \frac{\kappa}{\kappa_\kappa - \kappa}(m(\tau) + w(0) - c). 
\] (C12)

Hence, we obtain the probability density function \(\rho(\Delta r^* = y)\) of the log-price increment for this class by integrating the product \(\rho_{st}(w, \sigma)\rho_{br}(r, m)\) where \(r = r(\tau)\) is related to the variables \(w = w(0), \sigma = \sigma(0), m = m(\tau)\) by formula \(\mathbf{(C12)}\) with \(\Delta r^* = y\) kept fixed; relations \(\mathbf{(C11)}\) define the domain of integration in the product of the domain II of the pair \((w, \sigma)\) and the line \(m\). The resulting triple integral can be reduced to the sum of the following two terms:

\[
\rho(\Delta r^* = y, 0 < \sigma(0) < \sigma(\tau) < 2) = c_0 \int_{\max\{0, (y-c)/\kappa_\kappa\}}^{c/2} (2 - p) \, dp \int_{\max\{0, y-k\kappa, q + (\kappa_\kappa - \kappa)p\}}^{c/2} \rho_{br}(y - kp, q + (\kappa_\kappa - \kappa)p) \, dq. 
\] (C13)
\[ \rho(\Delta r^* = y, 0 = \sigma(0) < \sigma(\tau) < 2) = \frac{c_0}{\kappa_c - \kappa} \int_{\max\{0, (y-c/2)/\kappa_c\}}^{c/2} dp \int_{\max\{0, y, p\}}^{c/2} q \rho_{\sigma r}(y - \kappa p, q + (\kappa_c - \kappa)p) dq, \quad (C14) \]

where \( c_0 = (\kappa_c - \kappa)^2 / (c(a - 2\kappa)) \).

Finally, there are trajectories starting below the upper side of the rectangle that reach the upper side during the day. This class is defined by the conditions

\[ 0 < 2 - \sigma(0) < \frac{m(\tau) + w(0) - c}{\kappa_c - \kappa} \quad (C15) \]

and the corresponding log-price increment equals \( \Delta r^* = \Delta r + \kappa(2 - \sigma(0)) \). For the subcritical parameter set we consider, the probability of having such trajectories is small and their contribution has almost no effect on the profile of the probability density plot. Hence, we have discarded a correction to the probability density function of \( \Delta r^* \) due to such trajectories.

Thus, denoting the sum of contributions \((C8), (C10), (C13), (C14)\) from different classes of trajectories starting in the right half of \( \Pi \) by \( \rho_r(y) \), the symmetrized sum

\[ \rho(\Delta r^* = y) = \rho_r(y) + \rho_r(-y) \quad (C16) \]

provides an analytic approximation to the probability density function of the log-price daily increments, see the theoretical curve in Fig. 4(c). The term \( \rho_r(-y) \) accounts for trajectories starting in the left half of \( \Pi \).

We now look at the critical value \( \kappa = \kappa_c \). In the critical case, each trajectory that reaches the right side of the rectangle, immediately jumps to its upper side. Hence, \( \rho_r(y) \) is the sum of expressions \((C8)\) and \((C10)\) only (with no terms of the form \((C13), (C14)\)). The symmetrized sum \((C16)\) describes the main central mode of the probability density distribution shown in Fig. 4(d). One small side mode appears due to trajectories that start on the lower side of the rectangle and reach (jump to) the upper side, that is trajectories that have been disregarded in the subcritical case. The profile of the side modes is described by the left and right shifts \( \rho_{\text{side}}(\pm(y + 2\kappa)) \) of the function

\[ \rho_{\text{side}}(y) = \frac{1}{2c^2} \int_{\max\{0, y, 0\}}^{c/2} m^2 \rho_{\sigma r}(y, m) dm. \quad (C17) \]

Hence, the central mode and side modes can be explicitly expressed as a combination of the Gaussian and the error function.

In the supercritical case \( \kappa > \kappa_c \), the central mode is the same as in the critical case, while the side modes have the same shape as in the critical case but shift further to left and right.

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