GENERATION GAPS AND ABELIANISED DEFECTS OF FREE PRODUCTS

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ABSTRACT. Let $G$ be a group of the form $G_1 \ast \cdots \ast G_n$, the free product of $n$ subgroups, and let $M$ be a $\mathbb{Z}G$-module of the form $\bigoplus_{i=1}^n M_i \otimes_{\mathbb{Z}G} \mathbb{Z}$. We shall give formulae in various situations for $d_{\mathbb{Z}G}(M)$, the minimum number of elements required to generate $M$. In particular if $C_1, C_2$ are nontrivial finite cyclic groups of coprime orders, $G = (C_1 \times \mathbb{Z}) \ast (C_2 \times \mathbb{Z})$ and $F/R \cong G$ is the free presentation obtained from the natural free presentations of the two factors, then the number of generators of the relation module, $d_{\mathbb{Z}G}(R/R')$, is three. It seems plausible that the minimum number of relators of $G$ should be 4, and this would give a finitely presented group with positive relation gap. However we cannot prove this last statement.

1. Introduction

Given a group $G$ and a $\mathbb{Z}G$-module $M$, let $d(G)$ denote the minimal number of generators of $G$, let $d_G(M)$ denote the minimal number of $\mathbb{Z}G$-module generators of $M$, and let $\Delta G$ denote the augmentation ideal of $G$, that is the ideal of $\mathbb{Z}G$ generated by $\{g^{-1} \mid g \in G\}$. Then the generation gap of the finitely generated group $G$ is the difference

$$\text{gap } G := d(G) - d_G(\Delta G).$$

If $G$ is finitely presentable and $R \hookrightarrow F \twoheadrightarrow G$ is a finite free presentation (meaning $F$ is a finitely generated free group and $R$ has a finite number of generators as a normal subgroup of $F$), then the abelianised defect $\text{adef}(F/R)$ of the free presentation is the difference $d_G(R/R') - d(F)$, where $R'$ is the commutator group of $R$ and the free abelian group $R/R'$ is viewed as a $\mathbb{Z}G$-module (the relation module of the presentation). A fair amount is known about the numerical functions gap and $\text{adef}$ for finite groups. Our purpose here is to make a start with their study for infinite groups. In particular, to indicate how results on finite groups can be used to discuss free products. In the next section, after reviewing some relevant facts of the finite theory, we state and discuss our main results. We mention just two striking consequences:

1.1. Proposition. If $G$ is the free product $G_1 \ast \cdots \ast G_n$ of the finite groups $G_1, \ldots, G_n$ of mutually coprime orders, then $\text{gap}(G) = 0$ if and only if there exists a positive integer $k$ such that $\text{gap}(G_k) = 0$ and $G_i$ is cyclic for all $i \neq k$. 

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1.2. Proposition. Let $G_1, G_2$ be nontrivial finite cyclic groups of coprime orders, let $C_1, C_2$ be infinite cyclic groups, and let $G = (G_1 \times C_1) \ast (G_2 \times C_2)$. If $F/R \cong G$ is the free presentation obtained from the natural 2-generator free presentations of the two factors, then $d_G(R/R') = 3$.

The second of these results could have an important consequence for the relation gap problem: given a finite free presentation $R \hookrightarrow F \twoheadrightarrow G$ of a group $G$, then the relation gap of the presentation is defined to be $d_F(R) - d_G(R/R')$, where $d_F(R)$ denotes the minimum number of elements needed to generate $R$ as a normal subgroup of $F$. No examples are known of positive relation gap. The group of Proposition 1.2 above needs at most 4 (obvious) relations. If 4 really equals $d_F(R)$, then $F/R$ would have positive relation gap. Proposition 1.2 was mentioned in the interesting survey article [10] by Jens Harlander.

A similar result was proved in [2, Proposition 1], where consequences to the “D(2) conjecture” [13] were discussed. We shall show how to obtain [2, Propositions 1,2] from Theorem 4.6 at the end of Section 6.

The first author Karl Gruenberg died on 10 October 2007. At that time this paper was more or less complete; mistakes that remain are entirely the fault of the second author, Peter Linnell. Karl was a close and generous friend to not only the second author, but many other mathematicians. His influence through his research and teaching will be greatly missed.

2. Review and Overview

Among finite groups, generation gap zero is the rule rather than the exception: all simple groups (indeed all 2-generator groups) and all soluble groups have generation gap zero. However there do exist groups of arbitrarily large generation gap. The smallest group with nonzero generation gap is known to be the direct product of 20 copies of the alternating group of degree 5 [5]. For more on these matters, see [7, §13].

Finite groups also behave well with respect to abelianised defect: on a given finite group $G$, the function $\text{adef}(F/R)$ as $F/R$ varies over all finite free presentations of $G$ is constant (cf. [7, §14]). Hence $\text{adef}(G)$ is an unambiguous notation for this integer. For finitely presentable groups $G$, the set of $\text{adef}(F/R)$ for varying finite free presentations $F/R$ is bounded below; so the lower bound may be written $\text{adef}(G)$. A free presentation $F/R \cong G$ is called minimal if $d(F) = d(G)$.

If $G = G_1 \ast \cdots \ast G_n$ and we are given free presentations $F_i/R_i \cong G_i$ for $i = 1, \ldots, n$, then $F = *F_i$ projects onto $G$ with kernel the normal closure of $R_1, \ldots, R_n$.

If $R$ is a relation group, we shall always abbreviate the resulting relation module $R/R'$ as $\overline{R}$. When the relation group is $R_i$, we shall use the streamlined notation $\overline{R_i}$ for $\overline{R_i}$.

2.1. Theorem. Let $G = G_1 \ast \cdots \ast G_n$ with all the $G_i$ finite groups of mutually coprime orders. Then

(a) $d_G(\Delta G) = \max_k \{d_{G_k}(\Delta G_k)\} + n - 1$

whence $\text{gap}(G) = \min_k \{\text{gap}(G_k) + \sum_{i \neq k} (d(G_i) - 1)\}$. 
(b) Let $F/R \cong G$ be the free presentation constructed from free presentations $F_i/R_i \cong G_i$ of the free factors. Then

$$d_G(\overline{R}) = \max_k \{d_{G_k}(\overline{R}_k) + \sum_{i \neq k} d(F_i)\},$$

whence $\text{adef}(F/R) = \max_k \{\text{adef}(G_k)\}$.

An immediate consequence of Theorem 2.1 a) is Proposition 1.1. Theorem 2.1 b) tells us, for example, that if each $G_i$ is an elementary abelian $p_i$-group of rank 2 and $d(F_i) = 2$, then $d_G(\overline{R}) = 3 + 2(n - 1) = 2n + 1$ because here (clearly) $d_{G_i}(\overline{R}_i) = 3$. One might expect $d_F(\overline{R})$ to be $3n$ since $\overline{R}$ is the normal closure of $R_1, \ldots, R_n$ and $d_{F_i}(\overline{R}_i) = 3$ for all $i$. However in the case $n = 2$, Hig-Haefl, Lustig and Metzler showed that $d_F(\overline{R}) = 5$. Addendum to Theorem 3 on p. 163] and hence the relation gap of $F/R$ is zero. Their results yield for general $n$ that $d_F(\overline{R}) \leq 3n - 1$. If there exists $n > 2$ such that $d_F(\overline{R}) = 3n - 1$, then the group involved will have positive relation gap, namely $3n - 1 - (2n + 1) = n - 2 > 0$.

Theorem 2.1 follows from the general Theorem 4.6 stated and proved in [1]. All our results on gap and adef are consequences of this. Theorem 4.6 is a join of two results, the first of which, Theorem 4.4, is our major contribution: it gives an upper bound for the minimal number of generators of a $\overline{R}$-submodule $M$, where $G$ has subgroups $G_1, \ldots, G_n$ and $M$ contains “good” $\mathbb{Z}G_i$-submodules $M_i$ that together $G$-generate $M$. When $G = G_1 \ast \cdots \ast G_n$ and $M = \bigoplus_{i=1}^n (M_i \otimes_{\mathbb{Z}G_i} \mathbb{Z}G)$, our upper bound for $d_G(M)$ becomes a lower bound by a remarkable theorem of George Bergman [1]. The restriction to what we call “good” modules (cf. Definition 4.3) is probably unavoidable. Fortunately it is no restriction at all for finite groups. The proof of this has no bearing on any of our other arguments and we therefore postpone it to an appendix.

The following two theorems look somewhat like Theorem 2.1 but here the factors are infinite groups and this leads to difficulties with the “good” property of modules.

2.2. Theorem. Let $H = H_1 \ast \cdots \ast H_n$, where each $H_i = G_i \times A_i$, $G_i$ is a finite nilpotent group, $A_i$ is a finitely generated nilpotent group with $A_i/A_i'$ torsion free and the orders of $G_1, \ldots, G_n$ are mutually coprime. Then

(a) $$d_H(\Delta H) = \max_k \{d(H_k) + \sum_{i \neq k} (d(A_i) + \delta_{A_i,1})\},$$

where $\delta_{A,1} = 1$ when $A_i = 1$ (equivalently $d(A_i) = 0$) and $\delta_{A,1} = 0$, when $A_i \neq 1$.

(b) $\text{gap}(H) = \min_k \{\sum_{i \neq k} (d(G_i) - \delta_{A_i,1})\}$.

We conclude that a free product of finitely generated abelian groups, with torsion groups of mutually coprime orders, has generation gap zero if, and only if, every factor but one is torsion free or finite cyclic. Theorem 2.2 is an immediate corollary of the more general Theorem 6.5 in [4].

Finally, we state the theorem of which Proposition 1.2 is a special case.

2.3. Theorem. Let $H = H_1 \ast \cdots \ast H_n$, where each $H_i = G_i \times C_i$ with $G_i$ finite cyclic and $C_i$ infinite cyclic. Assume that the orders of $G_1, \ldots, G_n$ are mutually coprime. Take natural minimal free presentations $F_i/R_i \cong H_i$ and let $F/R \cong H$ be the resulting free presentation of $H$. Then $d_H(\overline{R}) = n + 1$. 


Theorem 2.3 is an immediate corollary of Theorem 6.13. It seems probable to us that Theorem 2.3 remains true (in an obviously modified enunciation) when each $C_i$ is allowed to be free abelian of finite rank and $G_i$ is a suitably restricted finite group. Just what may be needed to establish such a generalisation will become clear in §6.

3. Preliminaries

We explain some notation and collect a number of results, mostly well-known, that are needed later. Modules will be right modules and mappings will be written on the left. If $V$ is a finitely generated module over a ring $S$, we write $d_S(V)$ for the minimum number of elements needed to generate $V$ as an $S$-module. Of course if $X \subseteq V$, then $XS$ indicates the $S$-submodule of $V$ generated by $X$. In the case $S = kG$, the group ring of the group $G$ over the commutative ring $k$, we usually write $d_G(V)$ instead of $d_{kG}(V)$. If $G$ is finite, we let $\hat{G} = \sum_{g \in G} g \in kG$. If $K$ is a commutative ring containing $k$, then $KV$ means $K \otimes_k V$, so if $V$ is a $ZG$-module, $d_G(QV)$ means $d_{kG}(Q \otimes_2 V)$. For a prime $p$, we shall let $\mathfrak{p}_p$ indicate the field with $p$ elements. The commutator subgroup of $G$ will be denoted by $G'$ and we let $G^n = \langle g^n \mid g \in G \rangle$ for any positive integer $n$. If $H$ is a group, then $\delta_{G,H} \in \mathbb{Z}$ is the Kronecker $\delta$: so $\delta_{G,H} = 1$ if $G \cong H$ and $\delta_{G,H} = 0$ if $G \not\cong H$. Thus the $\delta$ in Theorem 2.2 is the Kronecker delta.

If $\pi$ is a set of primes, then $\pi'$ denotes the complementary set. Let $\pi(G)$ denote the set of primes dividing the orders of the finite subgroups of $G$. If $G$ is torsion free, set $\pi(G) = \emptyset$. Let $M$ be a finitely generated $ZG$-module. We call $M$ a Swan module if there exists a prime $p$ such that $d_G(M) = d_G(M/pM)$; in that case we call $p$ an $M$-prime. If $G$ is finite and $M$ is a $ZG$-lattice, then $M$-primes, if they exist, are divisors of $|G|$ (cf. 3.3). There is a well developed theory of Swan $ZG$-lattices when $G$ is finite: cf. [9, §7.1] for the basics. Augmentation ideals and relation modules of finite groups are Swan lattices. A relation prime of $G$ is an $\overline{\mathfrak{R}}$-prime, where $\overline{\mathfrak{R}}$ is the relation module coming from a finite free presentation of $G$. When $G$ is finite, this notion is independent of the chosen presentation: if $\overline{\mathfrak{R}}_1, \overline{\mathfrak{R}}_2$ are relation modules, then the $\overline{\mathfrak{R}}$-primes coincide with the $\overline{\mathfrak{R}}_2$-primes: cf. [9, Lemma 7.13].

3.1. Lemma. Let $H$ be a finitely generated group and let $G$ be a homomorphic image of $H$ such that $d(G) = d(H)$. If $\text{gap}(G) = 0$ and $\Delta G$ is a Swan module for $G$, then $\text{gap}(H) = 0$ and $\Delta H$ is a Swan module for $H$. Moreover every $\Delta G$-prime is a $\Delta H$-prime.

Proof. If $p$ is a $\Delta G$-prime, then $d(H) = d(G) = d_G(\Delta G) = d_G(\Delta G/p\Delta G) \leq d_H(\Delta H/p\Delta H) \leq d_H(\Delta H) \leq d(H)$. □

3.2. Corollary. If $H$ is a finitely generated nilpotent group, then $\text{gap}(H) = 0$ and $\Delta H$ is a Swan module. When $H/H'$ is torsion free, every prime is a $\Delta H$-prime; otherwise every $\Delta H$-prime belongs to $\pi(H/H')$.

Proof. There exists a prime $p$ such that $G := H/H'^p$ satisfies $d(G) = d(H)$. Since $G$ is an elementary abelian $p$-group, it has generation gap zero. When $H/H'$ is torsion free, any prime will do for $p$; but if $H/H'$ has torsion subgroup $T \neq 1$, then $p$ must divide $|T|$. □
3.3. Lemma. Let \( G \) be a finite group and let \( M \) be a finitely generated \( \mathbb{Z}G \)-module. Then \( d_G(\mathbb{Q}M) < d_G(M/pM) \) for every prime \( p \), while \( d_G(\mathbb{Q}M) = d_G(M/pM) \) for every \( p \notin \pi(G) \cup \pi(M) \).

Proof. First we show that \( d_G(\mathbb{Q}M) \leq d_G(M/Mp) \). Choose \( x_1, \ldots, x_n \in M \) to generate \( M \) as a \( \mathbb{Z}G \)-module modulo \( p \). Then the elements \( x_ig \) for \( 1 \leq i \leq n \) and \( g \in G \) generate \( M \) as a \( \mathbb{Z} \)-module modulo \( p \). Since \( M \) is finitely generated as a \( \mathbb{Z} \)-module, we see that the elements \( x_i g \) generate \( \mathbb{Q}M \) as a \( \mathbb{Q} \)-module. Hence the elements \( x_i \) generate \( \mathbb{Q}M \) as a \( \mathbb{Q} \)-module.

Next we show that \( d_G(M/Mp) \leq d_G(\mathbb{Q}M) \) when \( p \notin \pi(G) \cup \pi(M) \); this will complete the proof. Choose \( x_1, \ldots, x_n \in M \) to generate \( \mathbb{Q}M \) as a \( \mathbb{Q} \)-module. Write \( N = x_1\mathbb{Z}G + \cdots + x_n\mathbb{Z}G \). Then \( \mathbb{Q}N = \mathbb{Q}M \), consequently \( \mathbb{C} \otimes_{\mathbb{Z}} N \cong \mathbb{C} \otimes_{\mathbb{Z}} M \).

If \( M' \) is the torsion group of \( M \) and \( M = M/M' \), then \( M' \) is a \( \mathbb{Z} \)-lattice and \( M/pM \cong M/Mp \) since \( p \) is prime to \( |M'| \). Let \( k \) denote the algebraic closure of the field \( \mathbb{F}_p \). By Brauer character theory, we see by looking at the decomposition numbers that \( k \otimes_{\mathbb{Z}} M \) and \( k \otimes_{\mathbb{Z}} N \) have the same composition factors as \( kG \)-modules. Since \( p \) is prime to \( |G| \), all \( kG \)-modules are completely reducible and we deduce that \( k \otimes_{\mathbb{Z}} M \cong k \otimes_{\mathbb{Z}} N \). Therefore \( M/Mp \cong N/Np \) as \( \mathbb{F}_p G \)-modules by the Noether-Deuring theorem. Thus \( d_G(M/pM) \leq n \) as required. \( \square \)

4. The Main Technical Result

Given a finitely generated \( \mathbb{Z}G \)-module \( M \) and a finite set of primes \( \pi \), let \( (X_p)_{p \in \pi} \) be a family of subsets of \( M \) such that \( X_p \) is a \( \mathbb{Z}G \)-generating set of \( M \) modulo \( pM \) and \( X_p \supseteq X_q \) if, and only if, \( d_G(M/pM) \geq d_G(M/qM) \). We call \( (X_p)_{p \in \pi} \) a nested \( \pi \)-family in \( M \). Note that the nested structure ensures that if \( X_q \) is the minimal element of \( (X_p)_{p \in \pi} \), then for every \( p \in \pi \),

\[
(4.1) \quad d_G(M/(pM + X_q\mathbb{Z}G)) = d_G(M/pM) - d_G(M/qM).
\]

If \( \pi = \{p_1, \ldots, p_t\} \), we will write \( X_i = X_{p_i} \) for \( 1 \leq i \leq t \) and assume that \( X_1 \subseteq \cdots \subseteq X_t \).

4.2. Lemma. For every \( M \) and \( \pi \) there exists a nested \( \pi \)-family in \( M \).

Proof. Let \( \pi = \{p_1, \ldots, p_t\} \) be ordered so that \( d_G(M/p_iM) \geq d_G(M/p_jM) \) if, and only if, \( i \leq j \). Choose a subset \( Y_i \) of \( M \) so that its image in \( M/p_iM \) is a minimal \( \mathbb{Z}G \)-generating set of \( M/p_iM \). Then \( |Y_1| \geq |Y_2| \geq \cdots \geq |Y_t| \).

Let \( d = |Y_1| \) and enumerate the elements in each \( Y_i \) using zeros if necessary at the end: \( Y_i = \{y_{i,1}, y_{i,2}, \ldots, y_{i,d}\} \). Let \( p'_i = \prod_{j \neq i} p_j \). Then \( (p_i, p'_i) = 1 \) and \( p_im_i + p'_im'_i = 1 \) for suitable integers \( m_i, m'_i \). We define

\[
x_k = \sum_{i=1}^{t} p'_im'_i y_{i,k} \quad (1 \leq k \leq d).
\]

Thus \( x_k \equiv p'_im'_iy_{j,k} \equiv y_{j,k} \) modulo \( p_jM \), whence \( X_i = \{x_1, \ldots, x_{d_i}\} \) with \( d_i = |Y_i| \) gives the required nested \( \pi \)-family \( X_1 \supseteq X_2 \supseteq \cdots \supseteq X_t \). \( \square \)

4.3. Definition. Let \( G \) be a group and let \( M \) be a finitely generated \( \mathbb{Z}G \)-module. We say \( M \) is good for \( G \) if there exists a finite set of primes \( \pi(G, M) \) such that

\( (g1) \ d_G(M/pM) \) has constant value \( \delta_G(M) \) for all \( p \notin \pi(G, M) \), while \( \delta_G(M) \leq d_G(M/pM) \) for all \( p \in \pi(G, M) \);
(g2) for every finite set of primes \( \pi \) properly containing \( \pi(G, M) \) there exists a nested \( \pi \)-family whose minimal element \( X \) satisfies

\[
d_G(M/(pM + XZG)) \leq 1 \quad \text{for all } p \notin \pi,
\]

and \( M/XZG \) is a torsion group of finite exponent.

It is then automatic that \( |X| = \delta_G(M) \). Also if \( p \in \pi' \), then \( d_G(M/(pM + XZG)) = 1 \) if, and only if, \( p \in \pi(M/XZG) \) (because \( M \neq pM + XZG \) if, and only if, \( p \in \pi(M/XZG) \)).

4.4. Theorem. Let \( G \) be a group containing subgroups \( G_1, \ldots, G_n \), let \( M_i \) be a good \( \mathbb{Z}G_i \)-module and let \( M \) be a \( \mathbb{Z}G \)-module containing \( M_1, \ldots, M_n \) and generated by them as a \( \mathbb{Z}G \)-module. Assume that either all the \( M_i \) are Swan modules or that \( d_{G_i}(M_i/pM_i) > \delta_{G_i}(M_i) \) for some integer \( i \) and some prime \( p \). Then

\[
d_G(M) \leq \max_p \left\{ \sum_{i=1}^{n} d_{G_i}(M_i/pM_i) \right\}.
\]

Proof. Set \( \delta = \max_p \left\{ \sum_{i=1}^{n} d_{G_i}(M_i/pM_i) \right\}, e_i = \delta_{G_i}(M_i) \) and let \( \pi \) be a finite set of primes strictly containing \( \pi(G_1, M_1) \cup \cdots \cup \pi(G_n, M_n) \). Choose a nested \( \pi \)-family in \( M_1 \) as required for (g2) and let \( X_1 \) be the minimal element. Set \( N_1 = M_1/X_1\mathbb{Z}G_1 \). Now \( \pi(N_1) \) is finite and \( d_{G_1}(N_1/pN_1) \neq 0 \) if, and only if, \( p \in \pi(N_1) \). Define \( \pi_2 = \pi \cup \pi(N_1) \) and choose a nested \( \pi_2 \)-family in \( M_2 \) as required for (g2). Let its minimal element be \( X_2 \) and set \( N_2 = M_2/X_2\mathbb{Z}G_2 \). Continue in this manner, producing \( X_3, \ldots, X_n \), with \( N_i = M_i/X_i\mathbb{Z}G_i \) and \( \pi_i = \pi \cup \pi(N_1) \cup \cdots \cup \pi(N_{i-1}) \) for \( i = 2, \ldots, n \). This has ensured that if \( i < j \) and \( p \in \pi_j \), then \( d_{G_j}(N_j/pN_j) \) and \( d_{G_j}(N_j/pN_j) \) cannot both be non-zero. For suppose \( d_{G_j}(N_j/pN_j) \neq 0 \). Then \( p \in \pi(N_j) \), whence \( p \in \pi_j \) and then, since \( X_j \) is the minimal element in a nested \( \pi_j \)-family, \( d_{G_j}(N_j/pN_j) = d_{G_j}(M_j/pM_j) - \delta_{G_j}(M_j) \) by (4.1), where the right hand side is zero by (g1), because \( p \notin \pi(G_j, M_j) \). If \( d_{G_j}(N_j/pN_j) \neq 0 \), then \( p \notin \pi_j \) (otherwise \( d_{G_j}(N_j/pN_j) = 0 \) as we have just seen) and so \( p \in \pi(N_j)' \), which ensures \( d_{G_j}(N_j/pN_j) = 0 \). Thus we have

\[
(4.5) \quad \sum_{i=1}^{n} d_{G_i}(N_i/pN_i) \leq 1 \quad \text{for every } p \in \pi'.
\]

Suppose \( \delta = \sum_{i=1}^{n} e_i \). Since \( e_i \leq d_{G_i}(M_i/pM_i) \) for every \( p \) by (g1), we see that \( e_i \leq d_{G_i}(M_i/pM_i) \) for every \( p \) and every \( i \). In this case our hypothesis makes every \( M_i \) a Swan module, and so \( e_i = d_{G_i}(M_i) \) by using an \( M_i \)-prime \( p \). Thus \( \delta = \sum_{i=1}^{n} e_i, \) which gives the result. Therefore we may assume that \( \delta > \sum_{i=1}^{n} e_i \).

For each positive integer \( r \leq \delta - \sum_{i=1}^{n} e_i \), let \( \mathcal{P}_r \) denote the set of primes \( p \in \pi \) for which \( r \leq \sum_{i=1}^{n} d_{G_i}(N_i/pN_i) \). Thus \( \mathcal{P}_1 \neq \emptyset \) (because \( e_i < d_{G_i}(M_i/pM_i) \) for some \( i \) and some \( p \in \pi \)), \( \mathcal{P}_1 \supseteq \mathcal{P}_2 \supseteq \cdots \supseteq \mathcal{P}_s, \) where \( s = \delta - \sum_{i=1}^{n} e_i \) and \( \mathcal{P}_{s+1} = \emptyset \).

Recall \( N_i \neq pN_i \) for some \( p \in \pi' \) if and only if \( \pi(N_i) \cap \pi' \neq \emptyset \). Let \( \pi_0(N_i) = \pi(N_i) \cap \pi' \) and let \( I_0 \) be the set of all \( i \) such that \( \pi_0(N_i) \neq \emptyset \). By (4.3), the sets \( \pi_0(N_i), \) for \( i \in I_0, \) are disjoint. If \( i \in I_0, \) we know \( d_{G_i}(N_i/pN_i) = 1 \) for every \( p \in \pi(N_i) \) (property (g2)), whence we may choose a \( p \)-element \( u_{i,0}(p) \) which \( ZG_i \)-generates \( N_i \) modulo \( pN_i \). If \( i \notin I_0 \) set \( u_{i,0}(p) = 0 \) for all \( p \).

We now define inductively \( ZG_i \)-quotients \( N_{i,k} \) for \( N_i \) for \( 1 \leq k \leq s, \) starting with \( N_{i,1} = N_i, \) for \( 1 \leq i \leq n. \)
The set of primes $p \in \pi$ so that $N_i \neq pN_i$ for some $i$ is precisely $\mathcal{P}_1$. Let $\pi_1(N_i) = \pi(N_i) \cap \pi$ and let $I_1$ be all $i$ so that $\pi_1(N_i) \neq \emptyset$. With $i \in I_1$, choose a $p$-element $u_{i,1}(p) \in N_i$ so that $d_G(N_i/pN_i + u_{i,1}(p)\mathbb{Z}G_i) < d_G(N_i/pN_i)$ if $\pi_1(N_i) = \emptyset$, put $u_{i,1}(p) = 0$ for all $p$. Let $\pi(N_{i,1})$ be the union of the disjoint sets $\pi_0(N_i)$, $\pi_1(N_i)$ and define

$$K_{i,1} = \{u_{i,0}(p), u_{i,1}(p) \mid p \in \pi(N_{i,1})\} \mathbb{Z}G_i.$$  

Set $N_{i,2} = N_i/K_{i,1}$.  

If $N_{i,2} \neq pN_{i,2}$, then $p \in \mathcal{P}_2$ (because then $N_i \neq pN_i$, $p \in \pi_1(N_i)$ and $d_G(N_i/pN_i) \geq 2$). Thus $\pi(N_{i,2}) \subseteq \mathcal{P}_2$. Let $I_2$ be all $i$ for which $\pi(N_{i,2}) \neq \emptyset$. If $p \in \pi(N_{i,2})$, choose a $p$-element $u_{i,2}(p) \in N_i$ so that $d_G(N_{i,2}/pN_{i,2} + u_{i,2}(p)\mathbb{Z}G_i) < d_G(N_{i,2}/pN_{i,2})$, where $\tilde{u}_{i,2}(p) = u_{i,2}(p) + pN_{i,2}$. If $\pi(N_{i,2}) = \emptyset$, set $u_{i,2}(p) = 0$ for all $p$. Define

$$K_{i,2} = K_{i,1} + \{u_{i,2}(p) \mid p \in \pi(N_{i,2})\} \mathbb{Z}G_i$$

and $N_{i,3} = N_i/K_{i,2}$. Continue until we reach $N_i = K_{i,r}$, for some $r \leq s$ (because $\pi(N_{i,k}) \subseteq \mathcal{P}_k$ for all $k$).

By construction, for each $i$ and each $p$, the set of all non-zero elements $u_{i,k}(p)$ for $k \geq 0$ maps to a minimal set of $\mathbb{Z}_pG_i$-generators of $N_i/pN_i$. When $u_{i,0}(p) \neq 0$, the set consists just of this one element because $p \notin \pi$. The number of non-zero $u_{i,k}(p)$ for all $k \geq 0$ and all $i$ is $\sum_{i=1}^{n} d_G(N_i/pN_i)$. Thus

$$\max\{u_{i,k}(p) \mid k \geq 0 \text{ and all } i\} \leq s.$$  

Let $N = \bigoplus_{i=1}^{n} N_i \otimes_{\mathbb{Z}G_i} \mathbb{Z}G_i$. Now $A = \langle u_{i,k}(p) \mid \text{ all } i,k,p \rangle$ is a finite additive subgroup of $N$ and $N = A\mathbb{Z}G_i$ (because $N_i = K_{i,s}$ for all $i$). The subset $\{u_{i,k}(p) \mid \text{ all } i,k\}$ generates $A(p)$, the Sylow $p$-subgroup of $A$ and $d(A) = \max_p d(A(p))$. Hence $d_G(N) \leq \max_p d(A(p)) \leq s$.

Thus finally, $d_G(M) \leq d_G(N) + |X_1| + \cdots + |X_n| \leq s + \sum e_i = \delta$.  

All our results on free products are consequences of the following theorem which is an immediate consequence of Theorem 4.4 and a theorem of Bergman [1, Theorem 2.3]:

4.6. Theorem. Let $G = G_1 \ast \cdots \ast G_n$, let $M_i$ be a good $\mathbb{Z}G_i$-module for $1 \leq i \leq n$, and let $M = \bigoplus_{i=1}^{n} M_i \otimes_{\mathbb{Z}G_i} \mathbb{Z}G_i$. If either all the $M_i$ are Swan modules or $d_G(M_i/pM_i) > \delta_G(M_i)$ for some $i$ and some prime $p$, then

$$d_G(M) = \max_p \{\sum_{i=1}^{n} d_G(M_i/pM_i)\}.$$  

Proof. Theorem 4.4 gives the inequality $\leq$, while Bergman’s theorem tells us that, for each $p$, we have $\sum_{i=1}^{n} d_G(M_i/pM_i) = d_G(M/pM)$.  

Note that, by Bergman’s theorem, the conclusion of Theorem 4.6 is equivalent to the statement that $M$ is a Swan $\mathbb{Z}G$-module.

5. Finite Free Factors

If $G$ is a finite group, then every finitely generated $\mathbb{Z}G$-module $M$ is good for $G$ if we take $\pi(G,M) = \pi(G) \cup \pi(M)$. We already know that (g1) holds: this is Lemma 3.3 with $\delta_G(M) = d_G(\mathbb{Q}M)$. We postpone the proof of (g2) to the Appendix.

For the rest of this section we assume that $G_1, \ldots, G_n$ are finite groups and $G = G_1 \ast \cdots \ast G_n$. We consider various applications of Theorem 4.6 beginning with
augmentation ideals. Here $\bigoplus_{i=1}^{n} \Delta G_i \otimes \mathbb{Z}G = \mathbb{Z}G$ (e.g. [8, §8.6]), each $\Delta G_i$ is a Swan lattice and $d_{G_i}(\mathbb{Q}\Delta G_i) = 1$ (which makes $(g2)$ a triviality). Thus Theorem 4.6 gives

\begin{equation}
(5.1) \quad d_G(\Delta G) = \max_p \{ \sum_{i=1}^{n} d_{G_i}(\Delta G_i/p\mathbb{Z}G) \}.
\end{equation}

Hence the augmentation ideal of a free product of finite groups is a Swan module.

When $G_1, \ldots, G_n$ have mutually coprime orders, then (5.1) with Lemma 3.3 gives Theorem 2.1(a).

Next consider relation modules. Let $R_i \hookrightarrow F_i \to G_i$ be a free presentation and $R \to F \to G$ the resulting free presentation of $G$. Then $R = \bigoplus_{i=1}^{n} (R_i \otimes \mathbb{Z}G_i) \otimes \mathbb{Z}G$ and $d_{G_i}(\mathbb{Q}R_i) = d(G_i)$ (see e.g. [9, Theorem 2.7]). From Theorem 4.6 we obtain

\begin{equation}
(5.2) \quad d_G(R) = \max_p \{ \sum_{i=1}^{n} d_{G_i}(R_i/pR_i) \}.
\end{equation}

This has Theorem 2.1(b) as corollary. Furthermore, the Gruško-Neumann theorem tells us that a free presentation of the free product of the groups $G_i$ is of the form $*_{i}F_i \to *_{i}G_i$ with each $F_i$ a free group mapping onto $G_i$, consequently (5.2) shows that the relation module of a finite free presentation of a free product of finite groups is a Swan module.

The augmentation ideal of a group $G$ may be viewed as the first kernel of a $\mathbb{Z}G$-free resolution of $\mathbb{Z}$, while a relation module is the second kernel of such a resolution. We indicate how Theorem 2.1 may be generalised to arbitrary kernels.

Let $H$ be a finite group, let

\begin{equation}
(5.3) \quad \ldots \to (\mathbb{Z}H)^{\epsilon_2} \overset{\phi_2}{\to} (\mathbb{Z}H)^{\epsilon_1} \overset{\phi_1}{\to} \Delta H \overset{\phi_0}{\to} 0
\end{equation}

be a $\mathbb{Z}H$-free resolution of $\Delta H$ and write $K_i = \ker \phi_i$. We recall a criterion for a lattice to be a Swan module. This is originally due to Jacobinski [12].

Let $\mathbb{Z}(H)$ denote the semilocal subring $\{ a/b \mid a, b \in \mathbb{Z}, (b, |H|) = 1 \}$ of $\mathbb{Q}$ and take a minimal free presentation $K \hookrightarrow (\mathbb{Z}(H),H)^d \to (K_s(H))$, where $(K_s(H)) = K_s \otimes \mathbb{Z}(H)$. Then $K_s$ is a Swan lattice if $QK$ contains a copy of every non-trivial irreducible $QH$-module. Since $H$ is a finite group, every finitely generated $QH$-module is uniquely a direct sum of irreducible $QH$-modules and we have

$$QK - (d - e_s + e_{s-1} - \cdots + (-1)^{s}e_1)QH + (-1)^{s}Q\Delta H = 0$$

which shows that $Q\Delta H$ is a summand of $QK$ whenever $s$ is odd and also when $s$ is even provided $d - e_s + e_{s-1} - \cdots + (-1)^{s}e_1 \neq 1$; but if the sum is 1, then $K \cong \mathbb{Z}(H)$ and $\mathbb{Z}$ has projective period $\ell$ with $s + 2 \equiv 0 \mod \ell$.

Finally $QK_s = (QH)^s \oplus (-1)^sQ\Delta H$, where $e = e_s - e_{s-1} + \cdots + (-1)^{s-1}e_1$. Thus $QK_s$ needs $e + 1$ $QH$-module generators if $s$ is even and $e$ generators when $s$ is odd. If $K_s$ fails to be a Swan lattice, $\max_p \{ d_H(K_s/pK_s) \} = d = e + 1 = d_H(QK_s)$. Thus $d_H(K_s/pK_s)$ has constant value for all primes $p$.

Now return to $G = G_1 * \cdots * G_n$. Let

\begin{equation}
(5.4) \quad \ldots \to (\mathbb{Z}G_1)^{\ell_1,2} \overset{\theta_{1,2}}{\to} (\mathbb{Z}G_1)^{\ell_1,1} \overset{\theta_{1,1}}{\to} \Delta G_1 \overset{\theta_{1,0}}{\to} 0
\end{equation}

be a $G_i$-free resolution of $\Delta G_i$. Applying $\otimes_{ZG_i} ZG$ for each $i$, we obtain a $ZG$-free
resolution of $\Delta G_i \otimes_{ZG_i} ZG$. The sum of these resolutions yields a $G$-free resolution of $\Delta G$:
\[
\ldots \longrightarrow (ZG)^{f_2} \xrightarrow{\partial_2} (ZG)^{f_1} \xrightarrow{\partial_1} \Delta G \xrightarrow{\partial_0} 0,
\]
where $f_s = f_{1,s} + \cdots + f_{n,s}$ and $\ker \theta_s = \bigoplus_{i=1}^n \ker \theta_{i,s} \otimes_{ZG_i} ZG$.

5.6. Proposition. Let $G = G_1 \ast \cdots \ast G_n$ and take resolutions (5.4) and (5.5).
Suppose $s$ is a non-negative integer such that if $G_i$ has cohomological period $\ell_i$, then $s + 2 \not\equiv 0 \pmod{\ell_i}$. Then
\begin{itemize}
  \item[(a)] $d_G(\ker \theta_s) = \max \left\{ \sum_{i=1}^n d_{G_i}(\ker \theta_{i,s}/p \ker \theta_{i,s}) \right\}$;
  \item[(b)] if $G_1, \ldots, G_n$ have mutually coprime orders, then
    \[ d_G(\ker \theta_s) = \max_k \{ d_{G_k}(\ker \theta_{k,s}) + \sum_{i \neq k} d_{G_i}(\ker \theta_{i,s}) \} \]
    and $d_{G_i}(\ker \theta_{i,s}) = f_{i,s} - f_{i,s-1} + \cdots + (-1)^{s-1}f_{i,1} + \delta_s$, where $\delta_s = 1$ or 0 according as $s$ is even or odd.
\end{itemize}

Proof. By our discussion of $H$ above, $\ker \theta_{i,s}$ is a Swan lattice if $s$ is as stated. Therefore Theorem 4.6 yields (a); then (b) is a consequence of (a), again using facts proved about $H$.

Another application of Theorem 4.6 in the context of finite groups concerns finitely generated nilpotent groups. If $H$ is such a group, then there is a prime $p$ so that the elementary abelian image $H/H^p$ has the same minimal number of generators as does $H$. Let us call such a prime a generating prime for $H$.

5.7. Proposition. Let $H = H_1 \ast \cdots \ast H_n$, where each $H_i$ is a finitely generated nilpotent group. If every $H_i$, except possibly one, is cyclic or has $H_i/H_i^p$ torsion-free, then $\text{gap}(H) = 0$.

Proof. Suppose (without loss of generality) that $H_1$ is neither cyclic nor has $H_1/H_1^p$ torsion-free. Let $q$ be a generating prime for $H_1$ and $J(q)$ be the set of all $j$ such that $q$ is a generating prime for $H_j$. This absorbs all $H_j$ with $H_j/H_j^p$ torsion-free.

Pick a generating prime $p_i$ for each $i$, but take $p_i = q$ whenever $i \in J(q)$. Set $G_i = H_i/H_i^p$. Then $G = G_1 \ast \cdots \ast G_n$ is a homomorphic image of $H$ and $d(G) = d(H)$. By reordering if necessary we may assume $J(q) = \{ 1, \ldots, r \}$. Then
$G_i$ is cyclic for all $i > r$ and $p_i \neq q$. Hence
\[
\max_p \left\{ \sum_{i=1}^n d_{G_i}(\Delta G_i/p\Delta G_i) \right\} = \sum_{i=1}^r d_{G_i}(\Delta G_i/q\Delta G_i) + n - r = d(G)
\]
and so $\text{gap}(G) = 0$ (by Theorem 4.6) whence $\text{gap}(H) = 0$ by Lemma 5.4.

It would be interesting to have a converse of Proposition 5.7. The closest we come in this direction is Theorem 2.2 (cf. Theorem 6.5).

6. Infinite Free Factors

In order to use Theorem 4.6 for infinite groups, we must find good Swan modules. We begin with augmentation modules.
6.1. Proposition. Let $A$ be a non-trivial finitely generated nilpotent group with $A/A'$ torsion-free and let $G$ be a finite group such that

(a) $d(G) = d(G/G')$ and  
(b) $d_G(\Delta G/p\Delta G) = d(G/G'G^p)$ for all $p \in \pi(G)$.

If $H = G \times A$, then $\text{gap}(H) = 0$, $\Delta H$ is both a Swan lattice and a good $\mathbb{Z}H$-module for $\pi(H, \Delta H) = \pi(G)$. Furthermore $d(H) = d(G) + d(A)$ and $\delta_H(\Delta H) = d(A)$.

For the proof of Proposition 6.1 we require the following result.

6.2. Lemma. Let $G$ be a finite group, let $B$ be a finite nilpotent group and let $E = G \times B$. For any prime $p$,

$$d_E(\Delta E/p\Delta E) = \max\{d_G(\Delta G/p\Delta G), \dim G/G'G^p + \dim B/B'B^p\}.$$

Proof. If $B(p)$ is the Sylow $p$-subgroup of $B$, $d_B(\Delta B/p\Delta B) = d(B(p))$ (Lemma 7.17(i)) and $d(B(p)) = d(B/B'B^p)$. The result now follows from Theorem 2p. □

Lemma 6.2 shows that every finite nilpotent group $G$ has property (b) of Proposition 6.1. There are plenty of non-nilpotent finite groups having properties (a) and (b). Let $G = P \times B$, where $P = P'$, $B' = 1$, $\pi(P) \subseteq \pi(B)$ and $d(B/B^p) \geq d_P(\Delta P)$ for all $p \in \pi(G)$. For any $k \geq 1$, the direct power $G^{(k)}$ of $G$ satisfies $d_G^{(k)}(\Delta G^{(k)}/p\Delta G^{(k)}) = k d(B/B^p)$ for all $p \in \pi(G)$ (again use Theorem 2p). Thus $G^{(k)}$ has property (b) for all $k \geq 1$. If $k$ is large enough, then by a theorem of Wiegold (e.g. Theorem 6.13) $G^{(k)} = kd(G/G') = d(G^{(k)}/G^{(k)'}),$ whence $G^{(k)}$ has property (a).

Proof of Proposition 6.1. When $G = 1$ the result follows from Corollary 3.2, so henceforth we assume $G \neq 1$. Set $\bar{A} = A/A^{[G]}$ and $\bar{H} = G \times A$. Now $\text{gap}(G/G') = 0$ implies $\text{gap}(G) = 0$, by 3.1, and this implies $\text{gap}(\bar{H}) = 0$ (use (2.3)) and note that what is called in that paper the presentation rank of $E$ of the finite group $E$ coincides with $\text{gap}(E)$: [7, §2] and also that $\Delta H$ is a Swan module, provided that $d(H) = d(\bar{H})$.

If $n \geq 2$ is a positive integer, $C$ is a finite abelian group such that $nC = 0$, and $D$ is a finitely generated free abelian group, then $d(C \times D/nD) = d(C) + d(D)$. Applying this with $n = |G|$, $C = G/G'$ and $D = A/A'$, we find that

$$d(H) \leq d(G) + d(A) = d(G/G') + d(A) = d((G/G') \times (A/A')) \leq d(\bar{H}) \leq d(H).$$

Thus $d(H) = d(\bar{H})$ and also $d(H) = d(G) + d(A)$.

Consider condition (g1). Let $a_1, \ldots, a_e$ be free generators of $A$. By Corollary 3.2, $\{a_1 - 1, \ldots, a_e - 1\}$ is a minimal set of $\mathbb{Z}A$-module generators of $\Delta A$ modulo $p\Delta A$ for every prime $p$. Choose $p \notin \pi(G)$. Since $d_G(\Delta G/p\Delta G) = 1$ (by Lemma 3.3) there exists $x \in \Delta G$ so that $x\mathbb{Z}G + p\Delta G = \Delta G$. Let

$$X = \{x + (a_1 - 1)\bar{G}, a_2 - 1, \ldots, a_e - 1\}$$

(6.3)

(where $\bar{G}$ is defined in the first paragraph of [3]). We claim $X \mathbb{Z}H + p\Delta H = \Delta H$. For if $z \mapsto \bar{z}$, then $\Delta H \mapsto \Delta H/p\Delta H$ denotes the natural epimorphism and $W = X \mathbb{Z}H$, then

$$(x + \bar{G}(a_1 - 1))(|G| - \bar{G}) = |G| x$$

(6.4)
shows \( \bar{x} \in W \), whence \( \hat{G}(a_1 - 1) \in W \). Also \( \bar{g} - 1 \in W \) for all \( g \in G \) since \( xZH \) contains \((\Delta G + p\Delta H)/p\Delta H\). Hence \( |G| - \hat{G} + p\Delta H \in W \), which gives
\[
|G|(a_1 - 1) = (|G| - \hat{G} + \hat{G})(a_1 - 1) \in W,
\]
and so \((a_1 - 1) \in W \). Thus \( W = \Delta H/p\Delta H \). We conclude \( d_H(\Delta H/p\Delta H) \leq e \) and hence equality because \( \Delta H \to \Delta A \). Thus \( d_H(\Delta H) = e \) and of course \( d_H(\Delta H/p\Delta H) \geq e \) for every prime \( p \), whence property (g1).

Finally property (g2). Choose a finite set of primes \( \pi \) properly containing \( \pi(G) \) and find a nested \( \pi \)-family \( Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_i \) in \( \Delta G \) that satisfies (g2). We may assume that there is a positive integer \( s \leq t \) such that \( p_i \in \pi(G) \) for \( i \leq s \) and \( p_i \notin \pi(G) \) for \( i > s \). Let \( Y_i = \{y_1, \ldots, y_{d_i}\} \) (so \( Y_t = \{y_1\} \)). Define
\[
X_i = \{y_1 + (a_1 - 1)\hat{G}, a_2 - 1, \ldots, a_e - 1, a_1 - 1, y_2, \ldots, y_{d_i}\}.
\]
Then \( X_1 \supseteq \cdots \supseteq X_t \) and we claim this is a nested \( \pi \)-family in \( \Delta H \). First consider \( i < s \). Since \( X_t \) generates the same \( \mathbb{Z}H \)-submodule as \( S := \{y_1, y_2, \ldots, y_{d_i}, a_1 - 1, a_2 - 1, \ldots, a_e - 1\} \) and clearly \( SZH + p_i\Delta H = \Delta H \), we need to show that \( d_H(\Delta H/p_i\Delta H) = d_i + e \). With \( \hat{H} = \hat{G} \times \hat{A} \) as above, \( H \to \hat{H} \) induces a bijection \( S \simeq \hat{S} \) (since the kernel of \( \Delta \) is \( \hat{G} \)). Thus we are reduced to proving
\[
d_H(\Delta H/p_i\Delta H) = d_i + e.
\]
This is true by property (b) on \( G \) and Lemma 6.2.

Next consider the case \( i \geq s \). Since \( X_s \) is the set \( X_i \) of \( x = y_1 \) and \( p = p_i \), we see that \( X_s \) minimally generates \( \Delta H \) modulo \( p_i\Delta H \). Simplify the notation now by setting \( X = X_s(= X_t) \) and \( x = y_1 \). Let \( q \) be a prime not in \( \pi \). Then (by (g2) for \( \Delta \)), \( d_G(\Delta G/(q\Delta G + xZG)) \leq 1 \), whence there exists \( z \in \Delta G \) such that \( \{x, z\}ZG + q\Delta G = \Delta G \). Then \( X \cup \{z\} \) generates \( \Delta H \) modulo \( q\Delta H \).

To finish the proof, we must show that \( \Delta H/XZH \) has finite exponent. Suppose \( m \) is the exponent of the finite module \( \Delta G/xZG \). Then \( m \) is prime to \( p (= p_i) \) and \( \Delta \) shows that \( |G|[x] \in XZH \). So \( |G|m\Delta G \) is contained in \( XZH \), which gives \( |G|m(|G| - \hat{G}) \in XZH \). Also
\[
|G|(a_1 - 1) = |G|(x - \hat{G}(a_1 - 1)) - |G|x \in XZH,
\]
whence \( |G|^2m(a_1 - 1) = |G|m(|G| - \hat{G} + \hat{G})(a_1 - 1) \in XZH \). Therefore \( |G|^2m\Delta H \subseteq XZH \), which completes the verification of (g2) and hence the proof of Proposition 6.1. \( \square \)

The following result is an immediate consequence of Theorem 6.6 and Proposition 6.1.

6.5. **Theorem.** Let \( H = H_1 \ast \cdots \ast H_n \), where each \( H_i \) is of the form \( G_i \times A_i \), \( A_i \) is a finitely generated nilpotent group with \( A_i/A_i' \) torsion-free and \( G_i \) is a finite group such that
(a) \( d(G_i) = d(G_i/G_i') \) and
(b) \( d_{G_i}(\Delta G_i/p\Delta G_i) = d(G_i/G_i'G_i^p) \) for every \( p \in \pi(G_i) \). Then
\[
d_H(\Delta H) = \max \{\sum_{i=1}^{n} d_{H_i}(\Delta H_i/p\Delta H_i)\}.
\]
Theorem 6.6 is an immediate corollary of Theorem 6.5 provided we note that
\[ \delta_H((\Delta H)) = (1 - \delta_{A,1})d(A_1) + \delta_{A,1}. \]

We now turn to relation modules. These are harder to deal with. Let \( G \) be a non-trivial finite group, let \( C \) be an infinite cyclic group, and let \( H = G \times C \). Choose a finitely generated free presentation \( R \to F \to G \) for \( G \) and form the finitely generated free presentation \( S \to F * C \to H \) for \( H \).

6.6. Proposition. \( \overline{s} \) satisfies (g1) with \( \delta_H(\overline{s}) = d(F) \).

Proof. Let \( \alpha: F \ast C \to G \ast C \) denote the natural epimorphism induced by the surjection of \( F \) onto \( G \) and the identity on \( C \), let \( \beta: G \ast C \to G \times C \) denote the natural epimorphism induced by the identity on \( G \) and \( C \), and let \( \gamma: F \ast C \to G \times C \) denote the natural epimorphism induced by the surjection of \( F \) onto \( G \) and the identity on \( C \). Set \( R_0 = \ker \alpha, S = \ker \gamma \). Then we have a commutative diagram with exact rows

\[
\begin{array}{ccccccc}
1 & \to & R_0 & \to & F \ast C & \to & G \ast C & \to & 1 \\
& | & & | & & | & & |
\downarrow & & \downarrow & & \downarrow & & |
1 & \to & S & \to & F \ast C & \to & G \times C & \to & 1
\end{array}
\]

where the map \( R_0 \to S \) is the natural inclusion. Let \( E = \ker \beta \) and write \( C = \langle c \rangle \). By the snake lemma (or the map \( \beta \) can be identified with the natural map \( (F \ast C)/R_0 \to (F \ast C)/S \)), we see that \( S/R_0 \cong E \). Also \( E \) is the free group on \( \{ [g, c^i] \mid 1 \neq g \in G, \ 0 \neq i \in \mathbb{Z} \} \), so \( S = R_0 \times E \), which gives the extension of \( \mathbb{Z}H \)-modules

\[ R_0/[R_0, S] \to \overline{s} \to E/E'. \]

Choose an inverse \( t: G \to F \) to the free presentation \( F \to G \) (so \( t(G) \) is a transversal for \( R \) in \( F \)) and assume that \( t(1) = 1 \). Then \( R_0 = * w(w^{-1} Rw) \) (the free product of the subgroups \( w^{-1} Rw \)) where \( w \) runs through all those elements of \( F \ast C \) that can be expressed as a word in \( t(G) \) and \( C \), and start with an element of \( C \). Hence \( R_0 = \overline{R} \otimes_{\mathbb{Z}} G \ast C \) and then, because \( R_0 \to S \to E \) is exact, \( R_0/[R_0, S] = (R \otimes_{\mathbb{Z}} G \ast C) / \overline{R} \otimes_{\mathbb{Z}} G \ast C \otimes_{\mathbb{Z}} E \otimes_{\mathbb{Z}} \mathbb{Z}H \cong \overline{R} \otimes_{\mathbb{Z}} G \ast C \otimes_{\mathbb{Z}} E \otimes_{\mathbb{Z}} \mathbb{Z}H \); also \( E/E' \cong \Delta(G) / \mathbb{Z}C \) as \( \mathbb{Z}H \)-modules via \( [g, c^i] \mapsto (g - 1) \otimes (c^i - 1) \) and \( \Delta(C) \cong \mathbb{Z}C \) via \( c - 1 \mapsto 1 \). Hence (6.7) becomes

\[ \overline{R} \otimes_{\mathbb{Z}} \mathbb{Z}C \to \overline{s} \to \Delta(G) \otimes_{\mathbb{Z}} \mathbb{Z}C. \]

Now \( \tau: \Delta G \otimes_{\mathbb{Z}} \mathbb{Z}C \to \overline{s} \) defined by \( (g - 1) \otimes c^i \mapsto c^{-1} [t(g), c^i] = [t(g), c^i] \) is a \( \mathbb{Z}C \)-homomorphism, whence \( \tau \overline{G} \) is a \( \mathbb{Z}H \)-homomorphism. One checks that \( (\tau \overline{G})c = |G| \text{id} \).

So if \( q \) is prime to \(|G|\), then (6.8) taken mod \( q \) is a split extension. The free presentation \( R \to F \to G \) determines the relation sequence \( \overline{R} \to \mathbb{Z}G \to \Delta G \) and this splits modulo \( q \). Thus \( \overline{R} \otimes_{\mathbb{Z}} \mathbb{Z}C / q \Delta G \cong (\mathbb{Z}G / q \mathbb{Z}G)^{d(F)} \) and therefore

\[ \overline{s} / q \overline{s} \cong (\mathbb{Z}G / q \mathbb{Z}G)^{d(F)} \otimes_{\mathbb{Z}} \mathbb{Z}C / q \mathbb{Z}C \cong (\mathbb{Z}H / q \mathbb{Z}H)^{d(F)}, \]

which gives \( \delta_H(\overline{s}/p \overline{s}) = d(F) \).

Finally, let \( p \in \pi(G) \) and write \( E = F \ast C \). Then \( \overline{s} \to SE'/E' \) is an \( H \)-homomorphism, \( SE'/E' \) is \( H \)-trivial and free abelian of rank \( d(F) \), whence \( \overline{s}/p \overline{s} \) has an \( H \)-trivial \( H \)-image of \( \mathbb{F}_p \)-dimension \( d(F) \). Thus \( d_H(\overline{s}/p \overline{s}) \geq d(F) \). \( \square \)
Continuing with the notation of the last proof, take $C$-coinvariants of (6.8), giving

\[ R \to S \otimes_{ZC} Z \to \Delta G, \]

which is exact because $H_1(C, ZC) = 0$. The cocycle determining the extension (6.8) is $\psi : h \mapsto \tau h - \tau$. If $h = g e^d$, then $\psi(h) = \psi(g)$ because $\tau$ is a $C$-homomorphism. The cocycle determining (6.9) is $\psi_C := \psi \otimes_{ZC} 1_Z$.

Let $t(a)(b) = t(ab)\rho(a, b)$ for $a, b \in G$. Then $\psi(g) : (a-1) \otimes 1 \mapsto ((a g^{-1} - g^{-1}) \otimes 1) \tau g - ((a-1) \otimes 1) \tau = \tilde{\rho}(a, g^{-1}) g(1-c) \in S \Delta C$, whence $\psi_C(g) = 0$. We conclude that (6.9) splits.

6.10. Lemma. Assume that $G$ is nilpotent and that there exists a relation prime of $G$ that is also a $\Delta G$-prime. Then $\overline{S}$ is a Swan module and every $p$ that is both $\Delta G$-prime and $\overline{R}$-prime is an $\overline{S}$-prime.

Proof. Let $p$ be an augmentation prime and also a relation prime for $G$. Then $d_G(\Delta G) = d(G/G'G^p) [9]$ Lemma 7.17(i)] and $d_G(\overline{R}) = d(R/[R, F]R^p) [9]$ Lemma 7.17(ii)]. Hence

\[ d_G(\overline{R}) + d_G(\Delta G) = d(R/[R, F]R^p) + d(G/G'G^p). \]

Since (6.9) splits, there is an epimorphism $(S \otimes_{ZC} Z)/p(S \otimes_{ZC} Z) \to R/[R, F]R^p \oplus G/G'G^p$, which shows

\[ d(R/[R, F]R^p) + d(G/G'G^p) \leq d_G(\overline{S} \otimes_{ZC} Z)/p(\overline{S} \otimes_{ZC} Z) \leq d_H(\overline{S}). \]

But $d_H(\overline{S}) \leq d_G(\overline{R}) + d_G(\Delta G)$, by (6.8), so that we have achieved $d_H(\overline{S}) = d_H(\overline{S}/p\overline{S})$.

Unfortunately the nilpotence of $G$ is not enough to ensure that the remaining hypothesis in Lemma 6.10 is true: cf. [9] Proposition 7.19]. But if $G$ is cyclic (as it is from here on), then the set of all relation primes of $G$ coincides with the set of all $\Delta G$-primes and this set is $\pi(G)$.

6.11. Lemma. Continue with the previous notation. If $G$ is cyclic, then $\overline{S}$ satisfies (g2) for $\pi(G \times C, \overline{S}) = \pi(G)$.

Proof. Let $n = |G|$, write $G = \langle a \rangle$, let $F = \langle x \rangle$ (so $F$ is infinite cyclic), let $R = \langle x^n \rangle$, and let $x \mapsto a$ in the presentation $R \to F \to G$. Then $S$ is the normal closure in $E = F * C$ of $\{x^n, [x, c] \}$, whence $[S, E]$ is generated mod $E_3$ (the third term $[[E, E], E]$ of the lower central series of $E$) by $[x^n, c] \equiv [x, c]^n \mod E_3$. Thus $[S, E] \neq E'$, since $E'/E_3$ is infinite cyclic on $[x, c]E_3$. Therefore $d_H(\overline{S}) = 2$. Hence $d_H(\overline{S}/p\overline{S}) = 2$ for every $p \in \pi(G)$ (by 6.10).

The map $t : G \to F$ is here $t(a^i) = x^i$ for $0 \leq i < n$ and $\tau$ maps $(a-1) \otimes 1$ to $[x, c]$. The calculation of $\psi(g)$ gives

\[ \tau \hat{G} : (a-1) \otimes 1 \mapsto n[\overline{x}, \overline{c}] + \sum_{g \in G} \tilde{\rho}(a, g^{-1}) g(1-c). \]

The term $\sum_{g \in G} \tilde{\rho}(a, g^{-1}) g = \overline{x^n}$. Set $z = n[\overline{x}, \overline{c}] - \overline{x^n}c = ((a-1) \otimes 1) \tau \hat{G} - \overline{x^n}$. Since $\overline{S} = \{\overline{x^n}, [x, c]\}ZH$, we see that $\overline{S} = \{z, [x, c]\}ZH$. Now

\[ z(n - \hat{G}c) = n^2[\overline{x}, \overline{c}] - n\overline{x^n}c - ((a-1) \otimes 1) \tau \hat{G} - \overline{x^n}) \hat{G}c = n^2[\overline{x}, \overline{c}], \]
which shows that \( \overline{S} = z\mathbb{Z}H + q\overline{S} \) for any \( q \) prime to \( n \); and \( \overline{S}/z\mathbb{Z}H \) has exponent dividing \( n^2 \).

If \( \pi \) is a finite set of primes properly containing \( \pi(G) \), then \( \{ z, [x, q] \} \geq \{ z \} \) is a nested \( \pi \)-family whose minimal element has the required properties for \((g2)\). \(\square\)

Combining Proposition 6.10, Lemma 6.10 and Lemma 6.11 gives

6.12. Proposition. If \( F/R \cong G \) is a minimal free presentation of the finite cyclic group \( G \), \( C \) is an infinite cyclic group and \( (F \ast C)/S \cong G \times C \), then \( \overline{S} \) is a good Swan module for \( G \times C \) with \( \pi(G \times C, \overline{S}) = \pi(G) \).

We call the free presentation in Proposition 6.12 the natural minimal free presentation of \( G \times C \). Proposition 6.12 and Theorem 4.6 combine to give

6.13. Theorem. Let \( H = H_1 \ast \cdots \ast H_n \), where each \( H_i = G_i \times C_i \), \( G_i \) is finite cyclic and \( C_i \) is infinite cyclic. Take natural minimal free presentations \( F_i/R_i \cong H_i \) and let \( F/R \cong H \) be the resulting free presentation of \( H \). Then

\[
d_H(\overline{R}) = \max_p \left( \sum_{i=1}^n d_{H_i}(\overline{R_i}/p\overline{R_i}) \right).
\]

Finally in this section, we discuss [2, Proposition 2] (which is a generalisation of [2, Proposition 1]). Let \( n \) be a positive integer, let \( \rho_n = \rho_n(x, t) \) be the word defined by \( \rho_n(x, t) = (txt^{-1})x(txt^{-1})^{-1}x^{-n-1} \), and let \( Q_n = \langle x, t \mid \rho_n = x^n = 1 \rangle \). Then [2, Lemma 3] shows that \( Q_n \) is isomorphic to the HNN-extension \( (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z})_\phi \), where \( \phi \) maps the first factor isomorphically to the second. Let \( R \hookrightarrow F \twoheadrightarrow Q_n \) be the presentation determined by the generators and relators for \( Q_n \) from above. Then, of course, the relation module \( \overline{R} \) is generated as a \( \mathbb{Z}Q_n \)-module by the images of \( \rho_n, x^n \) in \( \overline{R} \). Define \( q_n = (n + 1)^n - 1 \) and \( c_n = nq_n \). Let \( \overline{p}_n \) indicate the image of \( \rho_n \) in \( \overline{R} \). Now [2, Lemma 4] proves that \( [(txt^{-1})^n, x^n] = x^n \) in \( \langle x, t \mid \rho_n \rangle \) (here \( [a, b] \) denotes the commutator \( aba^{-1}b^{-1} \)). Furthermore since \( [(txt^{-1})^n, x^n] \in R' \) and \( \overline{R}/\overline{p}_n \mathbb{Z}Q_n \) is generated as a \( \mathbb{Z}Q_n \)-module by the image of \( x^n \) in \( \overline{R}/\overline{p}_n \mathbb{Z}Q_n \), we see that \( \overline{R}/\overline{p}_n \mathbb{Z}Q_n \) is a group of exponent dividing \( q_n \). We deduce that if \( p \) is a prime that does not divide \( q_n \), then \( \overline{R} = \overline{p}R + \overline{p}Q_n \). We conclude that \( \overline{R} \) is a good \( \mathbb{Z}Q_n \)-module with \( \delta_{Q_n}(\overline{R}) = 1 \) and \( \pi(Q_n, \overline{R}) = \pi(q_n) \), where \( \pi(q_n) \) indicates the set of primes dividing \( q_n \). Also if \( n \geq 2 \) and \( p \) is prime which divides \( n \), it is not difficult to see that \( d_{Q_n}(\overline{R}/\overline{p}R) = 2 \) and hence \( \overline{R} \) is a Swan module. We can now apply Theorem 4.6 to obtain the following result.

6.14. Proposition. Suppose \( r \) is a positive integer and \( (q_{m_1}, q_{m_2}) = 1 \) for \( 1 \leq i < j \leq r \). Assume that \( m_i \geq 2 \) for all \( i \). If \( \overline{R} \) is the relation module for the group \( \Gamma := Q_{m_1} \ast \cdots \ast Q_{m_r} \) and presentation

\[
\langle x_{m_1}, t_{m_1}, \ldots, x_{m_r}, t_{m_r} \mid \rho_{m_1}(x_{m_1}, t_{m_1}), \ldots, \rho_{m_r}(x_{m_r}, t_{m_r}) \rangle,
\]

then \( d_{\Gamma}(\overline{R}) = r + 1 \).

This result is essentially that of [2, Proposition 2], though there explicit generators for the relation module are given. Of course, one could obtain further results by combining the previous paragraph with Proposition 6.12 and Theorem 4.6.
7. Appendix

Throughout this section $G$ is a finite group. Let $M$ be a finitely generated $ZG$-module. As usual, $M(p)$ denotes the localization of $M$ at the prime $p$. To establish that $M$ is a good $ZG$-module with $\pi(G,M) = \pi(G) \cup \pi(M)$ we must prove

7.1. Theorem. Let $\pi$ be a finite set of primes properly containing $\pi(G,M)$. Then there exists a nested $\pi$-family in $M$ whose minimal element $X$ satisfies

$$d_G(M/(XZG + pM)) \leq 1$$

for every prime $p \notin \pi$.

We know that $M$ has property (g1) (by Lemma 3.3) and $M/XZG$ is a finitely generated abelian group with $p(M/XZG) = M/XZG$ for $p \in \pi \setminus \{\pi(G) \cup \pi(M)\}$, whence $M/XZG$ is finite. Thus Theorem 7.1 supplies what is missing of (g2).

We recall some relevant facts. If $R$ is a subring of $\mathbb{Q}$ containing $1/|G|$, then $RG$ is a maximal order in $\mathbb{Q}G$, every central idempotent of $\mathbb{Q}G$ belongs to $RG$ and every $RG$-lattice is $RG$-projective [4] respectively Proposition 27.1, Theorem 26.20, Theorem 26.12]. If $U$ is an irreducible $ZG$-module such that $pU = 0$ for some $p \in \pi(G)'$, then $U$ can be viewed as a $Z(p)$-module, whence there exists a unique primitive central idempotent $e$ such that $Ue \neq 0$.

7.2. Lemma. If $V,W$ are projective $Z(p)G$-lattices such that $QV \cong QW$, then $V \cong W$ [4] Theorem 32.1].

7.3. Lemma. Let $M$ be a finite $ZG$-module such that $M(p) = 0$ for all $p \in \pi(G)$. Then the composition factors of $M$ can be arranged in any given order.

This follows from [14] Lemma 9.3]. Here is another proof, shown to us by Steve Donkin.

Proof. It suffices to prove the result when $M$ has prime-power order, so assume that $|M| = p^n$. Then $p \notin \pi(G)$ and $M$ is a module over $Z(p)G$.

Let $M = M_0 \supset M_1 \supset \cdots \supset M_d = 0$ be a composition series and let $(U_1, \ldots, U_d)$ be a (possibly new) sequence of the composition factors. We show by an induction on $d$ that there exists a composition series of $M$ producing factors in this order.

When $d = 1$ there is nothing to prove. Suppose $d = 2$. If $pM = 0$, then $M$ is semi-simple and so we are done. If $pM \neq 0$, then $pM$ must be simple and so is $M/pM$. Then $M \twoheadrightarrow pM$ has kernel $pM$, whence $M/pM \cong pM$ and again we are done.

Let $d > 2$ and $M/M_1 \cong U_r$. If $r = 1$, then induction gives a new series on $M_1$ that produces $(U_2, \ldots, U_d)$, completing the proof. Suppose $r > 1$ and $U_1 \cong M_k/M_{k+1}$. Then $k > 0$ and by induction we can find a composition series of $M_1$ with $U_1$ at the top. By the case $d = 2$, there exists a composition series $M = M_0 \supset M'_1 \supset M'_2$ with factors $(U_1, U_r)$. Finally, use induction on $M'_1$ to obtain a composition series of $M'_1$ with factors $(U_2, \ldots, U_d)$. \hfill \Box

7.4. Lemma. Let $M$ be a finitely generated $ZG$-module, let $p$ be a prime not in $\pi(G,M)$ and let $U$ be an irreducible $ZG$-module such that $pU = 0$. Let $e$ be the primitive central idempotent of $QG$ such that $Ue = U$ and assume that $(QM)e$ is not irreducible. If $\theta, \phi : M \twoheadrightarrow U$ are $ZG$-epimorphisms and $q$ is an integer prime to $p$, then there exists a $ZG$-automorphism $\mu$ of $M$ such that $\theta = \phi \mu$ and $\mu \equiv \text{id}$ mod $qM$. 
Proof. Let $M'$ be the torsion submodule of $M$. Since $p \nmid |M'|$, we see that $M(p)$ is a $\mathbb{Z}(p)$-module and $M(p)\neq 0$. If $(\mathbb{Q}M)e = V_1 \oplus V_2$ with $V_i \neq 0$, set $M_i = V_i \cap M(p)e$ and then $M_1 \oplus M_2$ is a $\mathbb{Z}(p)$-module of finite index in $M(p)e$. Therefore $M(p)e = M_1 \oplus M_2$ by Lemma [7.2] and there exist epimorphisms $\sigma_1: M_1 \rightarrow U$, $\sigma_2: M_2 \rightarrow U$.

Our proof follows that of [14, Proposition 9.5] but with some refinements. Given $\mathbb{Z}(p)$-maps $\alpha, \alpha_2$ of $M_1, M_2$ to $U$, we shall write $\alpha_1 \oplus \alpha_2$ for the map $(x_1, x_2) \mapsto \alpha_1(x_1) + \alpha_2(x_2)$. Let $\theta_i$ be the map on $M_i$ induced by $\theta$. Clearly one of $\theta_1, \theta_2$ is an epimorphism. Let us suppose it is $\theta_2$.

Since $M_1$ is $\mathbb{Z}(p)$-projective, there exists $\psi: M_1 \rightarrow M_2$ so that $\theta_2 \psi = \theta_1 - \sigma_1$. Define $\hat{\psi}: M_1 \oplus M_2 \rightarrow M_1 \oplus M_2$ by $(x_1, x_2) \mapsto (0, \psi(x_1))$. Then $\hat{\psi}^2 = 0$. Since $M(p)e$ is a direct summand of $M(p)$, any endomorphism of $M(p)e$ can be considered as one of $M(p)$ by defining it to be zero on $M(p)(1-e)$. Then $\text{End}_{\mathbb{Z}(p)}G(M(p)) \cong \mathbb{Z}(p) \otimes \text{End}_{\mathbb{Z}G}(M)$ shows $\hat{\psi} = (1/r) \otimes \tau$ for a suitable integer $r$ prime to $p$ and $\tau \in \text{End}_{\mathbb{Z}G}(M)$.

As $|M'|, r, q$ are all prime to $p$, we may choose $s$ so that $qr| |M'|/s \equiv 1 \mod p$; then $t := |M'|/s$ satisfies $tM' = 0$. Now $tM \cap M' = 0$ shows $tM$ is a $\mathbb{Z}G$-module and $\tau$ restricts to an endomorphism of $tM$. Since $(tM)(p) \cong M(p)$ and $\hat{\psi}^2 = 0$, we see that $\tau^2(p) = 0$, which gives $\tau^2(tM) = 0$. Define $\alpha = \text{id}_M + (qt)\tau$. This is a $\mathbb{Z}G$-module of $M$ and we readily check that

$$(\sigma_1 \oplus \theta_2)\alpha(p) = \theta_1 \oplus \theta_2.$$  

A similar procedure leads to $\psi': M_2 \rightarrow M_1$ such that $\sigma_1 \psi' = \sigma_2 \theta_2$, $\tilde{\psi}': (x_1, x_2) \mapsto (\psi'(x_2), 0)$, $r' \tilde{\psi}' = r'(p)$ for some $r'$ coprime to $p$ and some $\tau' \in \text{End}_{\mathbb{Z}G}M$. Then the $\mathbb{Z}G$-module $\alpha' = \text{id}_M + (qt)\tau'$ of $M$ satisfies

$$(\sigma_1 \oplus \theta_2)\alpha'(p) = \sigma_1 \oplus \sigma_2.$$  

We conclude that the $\mathbb{Z}G$-module $\lambda = \alpha'^{-1} \alpha'$ of $M$ satisfies $(\theta_1 \oplus \theta_2)\lambda(p) = \sigma_1 \oplus \sigma_2$. Similarly we can find an automorphism $\nu$ such that $(\phi_1 \oplus \phi_2)\nu(p) = \sigma_1 \oplus \sigma_2$. Then $\nu = \nu^{-1}$ is an automorphism as required: for $\phi(p)\mu(p) = \theta(p)$ implies $\phi \mu = \theta$ mod $p$ (because $|M'| = 1$) whence $\phi \mu = \theta$. 

7.5. Lemma. Let $L$ be a submodule of the $\mathbb{Z}G$-lattice $M$ such that $M/L$ is finite of order prime to $|G|$. Then there exists a submodule $N$ of $M$ such that $M/N$ is a finite cyclic $\mathbb{Z}G$-module with the same composition factors as $M/L$.

Proof. Since $M/L$ is the direct sum of its primary parts, we may assume that $M/L$ has $p$-power order for some prime $p$, so we may view $M/L$ as a $\mathbb{Z}(p)$-module. Let $V_1, \ldots, V_n$ be the irreducible $\mathbb{Q}G$-modules which appear in $QM$. Then we may choose a $\mathbb{Q}G$-submodule $V$ of $QM$ which is isomorphic to $\bigoplus_{i=1}^n V_i$. Set $A = V \cap M(p)$. Since $p$ is prime to $|G|$, all $\mathbb{Z}(p)$-G-lattices are projective. Therefore we may write $M(p) = A \oplus B$ for a suitable $\mathbb{Z}(p)$-G-submodule $B$ of $QM$, because $M(p)/A$ is a $\mathbb{Z}(p)$-G-module. Also we see from Lemma [7.2] that every submodule of $A$ is cyclic, and if $A_0$ is a $\mathbb{Z}(p)$-submodule of $A$ such that $A/A_0$ is finite, then every composition factor of $M/L$ is a $\mathbb{Z}G$-homomorphic image of $A$. Thus by induction on the length of the composition series of $M/L$, we see that there is a $\mathbb{Z}G$-submodule $C$ of $A$ such that $A/C$ has the same composition factors as $M/L$. Clearly $A/C$ is a cyclic $\mathbb{Z}G$-module. Now set $N = (C \oplus B) \cap M$. Then $M/N \cong A/C$ as $\mathbb{Z}G$-modules and the result follows. 

7.6. **Lemma.** Let $L, N$ be submodules of the finitely generated $\mathbb{Z}G$-module $M$ such that $M/L, M/N$ are finite with the same composition factors and $\pi(M/L) \cap \pi(G, M) = \emptyset$. Assume that if $U$ is a composition factor of $M/L$ with $pU = 0$ and attached to the primitive central idempotent $e$, then $(QM)e$ is not irreducible as $QG$-module. Let $q$ be any integer prime to $|M/L|$. Then there exists an isomorphism $\theta : L \cong N$ such that $\theta(x) \equiv x \mod qM$ for all $x \in L$.

**Proof.** By Lemma 7.1 we may arrange the composition factors of $M/L$ and $M/N$ so that they appear in the same order; in other words, there exist chains of submodules $M = L_0 \supset L_1 \supset \cdots \supset L_d = L$ and $M = N_0 \supset N_1 \supset \cdots \supset N_d = N$ for some $d \geq 1$ such that, for each $i = 0, 1, \ldots, d - 1, L_i/L_{i+1}$ is $\mathbb{Z}G$-irreducible and $L_i/L_{i+1} \cong N_i/N_{i+1}$. We prove the result by induction on $d$.

By Lemma 7.4 there exists a $\mathbb{Z}G$-automorphism $\mu$ of $M$ such that $\mu$ induces $L_1 \cong N_1$ and $\mu(x) \equiv x \mod qM$ for all $x \in M$. By induction there exists an isomorphism $\nu : \mu L \cong N$ such that $\nu(y) \equiv y \mod qN_1$ for all $y \in \mu L$. Then $\nu \mu : L \cong N$ and $\nu \mu(x) \equiv x \mod qM$ for all $x \in L$. \hfill $\square$

**Proof of Theorem 7.4.** Choose a nested $\pi$-family $X$ in $M$ and let $X$ be the minimal element. Let $e_1, \ldots, e_n$ be the primitive central idempotents in $QG$ and define

$$I = \{i \mid (QM)e_i \text{ is not irreducible}\},$$

$$J = \{i \mid (QM)e_i \text{ is irreducible}\}.$$

Set $e = \sum_{i \in J} e_i$ and let $M_1/XZG$ be the $\pi'$-primary component of $M/XZG$. Define the $\mathbb{Z}G$-submodule $M_2$ of $M_1$ containing $XZG$ by $M_2/XZG = (M_1/XZG)e$. Then each composition factor of $M_2/XZG$ is attached to a unique $e_i$ with $i \in I$. The kernel of $M_2 \rightarrow M_2/XZG$ contains $M_2'$, the torsion group of $M_2$, because $\pi \cap \pi(G) = \emptyset$.

By Lemma 7.5 there exists $N \supseteq M'_2$ such that $M_2/N$ has the same composition factors as $M_2/XZG$ and $M_2/N$ is a cyclic $\mathbb{Z}G$-module. Let $q$ be an integer prime to $|M_2/XZG|$. By Lemma 7.4, there exists an isomorphism $\theta : XZG \cong N$ such that $x \equiv \theta(x) \mod qM$ for all $x \in X$. Thus

$$X \equiv \theta(X) \mod pM$$

for all $p \in \pi$. So if we replace the subset $X$ of each member of the nested $\pi$-family $X$ by $\theta(X) = Y$, then we obtain a new nested $\pi$-family whose minimal element is $Y$. Observe $N = YZG$ and $M/YZG$ is finite because $M/M_2$ is finite. Also $(M_1/M_2)e = 0$, which makes $M_2/YZG = (M_1/YZG)e$.

We claim $M_1/YZG$ is cyclic. Since we already know $(M_1/YZG)e$ is cyclic, it remains to check $(M_1/YZG)(1 - e)$ is also cyclic. It suffices to do this for each primary component. If $p$ occurs in the primary decomposition, then $(M_1)_{(p)}(1 - e) = \sum_{i \in J} (M_1)_{(p)} e_i$, where $(QM)e_i$ for $i \in J$ are distinct irreducible modules, whence $(M_1)_{(p)}(1 - e)$ is cyclic. Therefore so is $(M_1/YZG)(1 - e)$. Finally, if $p \in \pi'$, then $M/(pM + YZG) = M_1/(pM + YZG)$, which gives our result. \hfill $\square$

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