Charged black holes in expanding Einstein–de Sitter universes

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Abstract

Inspired by a previous work by McClure and Dyer (2006) (*Class. Quantum Grav.* 23 1971), we analyze some solutions of the Einstein–Maxwell equations that were originally written to describe charged black holes in cosmological backgrounds. A detailed analysis of the electromagnetic sources for a sufficiently general metric is performed, and we then focus on deriving the electromagnetic four-current as well as the conserved electric charge of each metric. The charged McVittie solution is revisited, and a brief study of its causal structure is performed, showing that it may represent a charged black hole in an expanding universe, with the black hole horizon being formed at infinite late times. Charged versions of solutions originally put forward by Vaidya (Vd) and Sultana and Dyer (SD) are also analyzed. It is shown that the charged SD metric requires a global electric current, besides a central (spherically symmetric) electric charge. With the aim of comparing to the charged McVittie metric, new charged solutions of Vd and SD types are considered. In these cases, the original mass and charge parameters are replaced by particular functions of the cosmological time. In the new generalized charged Vaidya metric, the black hole horizon never forms, whereas in the new generalized SD case, both the Cauchy and the black hole horizons develop at infinite late times. A charged version of the Thakurta metric is also studied here. It is also a new solution. As in the charged SD case, the natural source of the electromagnetic field is a central electric charge with an additional global electric current. The global structure is briefly studied, and it is verified that the corresponding spacetime may represent a charged black hole in a cosmological background. All the solutions present initial singularities as found in the McVittie metric.

Keywords: black holes, cosmology, general relativity, Einstein–Maxwell equations
1. Introduction

1.1. Preliminary remarks

A major problem in general relativity and cosmology is related to the effects of the cosmological expansion on gravitating local systems, such as the solar system, accompanying the expansion of the Universe. This question has been the subject of studies since 1933, with McVittie [1]. Several models have emerged since then, and what is clear so far is that the way the local dynamics is affected strongly depends upon the choice of the specific metric to represent the system. Small-scale gravitating systems, small compared to the Hubble radius, participate in the expansion, but the effect of the expansion being so small is that it can be neglected. In such cases, the global dynamics is well described by the Friedman–Lamaitre–Robertson–Walker metric (FLRW), while the local dynamics is independent of the cosmological expansion and are described by an independent metric. For large-scale structures compared to the Hubble radius, the cosmic expansion becomes significant and cannot be neglected (see, e.g., [2]). In this situation, a fully dynamical metric, including the effects of the cosmic expansion, must be used. In the case of a local electromagnetic system, the analysis of [3] shows that a strongly connected system cannot be disturbed by the cosmic expansion. Based on this, we can say that in an expanding FLRW universe, only weakly bounded systems take part in the cosmic expansion [2]. However, it is still not clear what the effects of the cosmological expansion are on other, maybe more subtle, features of local gravitational systems such as the existence of trapping horizons in a black hole spacetime.

On the other hand, the description of black hole dynamics is not well settled; some points have been set aside as a first approximation to simplify the analysis. Although the black hole thermodynamics relies on the discovery of Hawking radiation [4, 5], to avoid further difficulties, the backreaction has been ignored in many studies about black hole dynamics in cosmic backgrounds. More complications arise when the black hole is immersed in a cosmic content other than a cosmological constant, such as dark matter and dark energy dynamic fields (see, e.g., [6] for an analysis of Hawking radiation from black holes asymptotic to the Einstein–de Sitter universe). In this way, if the observations indicate \( \omega < -1 \), the possibility of the Big Rip provides another motivation for the study of local dynamics of black holes, because the event horizon can follow the expansion and disappear, exposing the central singularity [7]. In general, the cosmic content tends to be incorporated into the black hole [8], and neglecting the accretion when modeling the black hole evolution results in unrealistic models.

Recently, observational evidence has accumulated, mainly by the observation of SNe Ia, that sets the equation of state parameter as \( \omega = -1.02^{+0.13}_{-0.19} \) [9], showing that the Universe goes through an accelerated expansion phase. This fact brought even more interest in the studies on the effects of the cosmic expansion on local gravitating systems. Therefore, it would be very useful to have exact solutions of the Einstein equations that could describe objects with a strong gravitational field embedded in an accelerated expanding Universe. Further study on the already known exact solutions to acquire a full understanding of the geometric and physical properties of such solutions is also an important task. This is one of the aims of the present work.

1.2. McVittie and Vaidya type solutions

The first known solution of Einstein equations that has been written as an attempt to describe a spherically symmetric pointlike particle in an expanding Universe is the McVittie metric [1].
In fact, such a metric may be used to investigate the effects of cosmic expansion on local systems. It is usually written in the form

\[ ds^2 = a^2(t) \left( 1 + \frac{m}{2ra(t)} \right)^{1/2} (dr^2 + r^2d\Omega^2) - \left( 1 - \frac{m}{2ra(t)} \right)^{1/2} \left( 1 + \frac{m}{2ra(t)} \right)^{-1/2} dt^2, \tag{1} \]

where \( t \) is the cosmological time, \( r \) is a radial comoving coordinate, and \( a(t) \) is to be interpreted as the cosmological scale factor. It was first thought that metric (1) could describe the Schwarzschild black hole geometry in a cosmological background. In this scenario, the mass \( m \) of the point mass (or the central black hole) is constant; there is no flow of energy towards the central object or escaping from it. This non-accretion condition also says that the size of the central object does not change during the expansion. However, it has been argued (see, e.g., [10–12]; see also [13, 14]) that the McVittie metric cannot describe a black hole evolving in an FLRW Universe due to the presence of a spacelike singularity at \( r = m/2a(t) \), which corresponds to an infinite pressure of the cosmic fluid surrounding the object.

According to these authors, the metric describes a singular central spherical object, but the event horizon never develops. A timelike limiting surface and an event horizon are formed, but both shrink due to expansion. On the other hand, it is known that in the Einstein–de Sitter Universe, in which \( a(t) \sim e^{H_0 t} \), with constant \( H_0 \), the McVittie metric reduces to the Schwarzschild–de Sitter metric, and the central object described by this metric is a black hole.

In fact, more recently it has been shown that if the cosmological expansion is such that for late times the Hubble parameter \( H(t) \equiv \dot{a}(t)/a(t) \) (with the over dot (\( \dot{} \)) standing for the total derivative) reaches a constant value, in which the spacetime asymptotes to the \( \Lambda \)CDM cosmology, an event horizon is formed and black hole and white hole regions are present in the McVittie solution [15–18].

Straightforwardly from the McVittie metric, one can build new black hole type solutions in cosmological backgrounds by replacing the mass parameter \( m \) by a function of the cosmic time \( m(\tau) \). For instance, by taking \( m = M \, a(\tau) \), with constant \( M \), and defining a new radial coordinate \( R \) by

\[ R = r \left( 1 + \frac{M}{2r} \right)^{1/2}, \]

in metric (1), we find

\[ ds^2 = a^2(t) \left( \frac{dr^2}{1 - \frac{2M}{R}} + \frac{R^2d\Omega^2}{1 - \frac{2M}{R}} \right) - \left( 1 - \frac{2M}{R} \right) dt^2, \tag{2} \]

which is a particular case of the Thakurta metric [19]. Of course, other choices of \( m(\tau) \) yield different geometries that may be interesting as candidates for representing black holes immersed in cosmological spacetimes (see below). Since the Einstein equations are not invariant under the change \( m \rightarrow m(\tau) \) one sees that such a procedure changes the sources. In the McVittie solution, the source is a perfect fluid with energy density \( \rho \) given by \( 8\pi\rho = 3H^2(t) \) and pressure, by \( 8\pi p = -3H^2(t) - 2\dot{H}(t)[1 + m/2a(t)]/[1 m/2a(t)] \). In turn, for the Thakurta metric (2) the simplest source is a fluid with density \( 8\pi\rho = \dot{H}(t)/\sqrt{(1 - 2M/R)} \), pressure \( 8\pi p = \left( 3H^2(t) + 2\dot{H}(t) \right)/\sqrt{(1 - 2M/R)} \), and with an energy flux \( Q \) given by \( 4\pi Q = M \, H(t)/R^2(1 - 2M/R)^{3/2} \) in the radial direction. Both spacetimes have singularities: McVittie at \( r = m/2a(t) \) and Thakurta at \( R = 2M \).

With the aim of describing black holes in expanding universes, other metrics have been written down. Vaidya [20] proposed a metric to describe a Kerr black hole immersed in a cosmological background, which, in the case with no rotation, can be written in the form
where \( m \) is a constant. Vaidya argued that the metric represents a black hole whose event horizon is located at \( r a(t) = 2m \). However, this seems a more subtle question than initially believed, since the surface \( r = 2m/a(t) \) does not look like an apparent horizon (see, e.g., [21]).

A similar (Vaidya type) metric as a possible representation of a Kerr black hole embedded in the Einstein–de Sitter Universe was examined by Thakurta [19]. As it was shown in [21], in the Vaidya and also in Thakurta solutions, some of the energy conditions are violated. Even though these are possible drawbacks, more studies on these metrics are necessary in order to understand the complete geodesic structure of the corresponding spacetimes.

Another class of solutions found by Sultana and Dyer (SD) [22] is also of interest for the present work. The SD metric,

\[
\text{\( d\bar{s}^2 = a^2(t) \left[ \text{d}r^2 + r^2 \text{d}\Omega^2 \right] + \frac{2m}{r} \left[ \frac{\text{d}t}{a(t)} + \text{d}r \right]^2 - \text{d}r^2, \)}
\]

with \( m \) being a constant, was proposed to represent a Schwarzschild black hole in an Einstein–de Sitter Universe. This solution is similar to the McVittie spacetime, but with the no-accretion condition being released; i.e., in the SD solution there is matter (energy) accretion onto (from) the central object. Another difference is that the McVittie solution can be sourced by a single perfect fluid, while SD needs a mixture of two non-interacting perfect fluids, a massive dust and a null (lightlike) dust fluid. In SD solution, both in radiation and matter dominated era, some of the energy conditions are violated.

Notice that the SD metric (4) can be obtained from the Vaidya metric (3) by replacing the mass parameter \( m \) by \( m a(t) \). In other words, the Vaidya and the SD metric differ from each other only by the rate of mass accretion onto the central body.

Finally we mention here the attempts made more recently as, for instance, the model proposed in [23] in which the mass of the central object changes due to accretion of a phantom fluid during the cosmic expansion considering backreaction. This metric differs from the solution of [8]; however, the results cannot be compared effectively since the latter does not consider backreaction.

1.3. Charged black holes in expanding universes

The charged black hole generalization of the McVittie metric was found by Vaidya and Shah [24, 25]. Later, a generalization of the Vaidya [20] solution including electric charge in the source was written [26]. More recently, a very interesting charged solution was put forward by Kastor and Traschen [27]. Starting with the Majumdar–Papapetrou multi-black holes static solution of [28], the authors found a solution of the Einstein–Maxwell equations which describes the dynamics of a system of charged multi-black holes in a cosmological background. Among other interesting aspects, the solution furnishes an analytical description of the dynamics of coalescing black holes. In the case of a single charged black hole, the solution corresponds to the extremal Reissner–Nordström black hole in a spatially flat Einstein–de Sitter Universe. The global structure of the Kastor–Traschen spacetime was explored in [29]. The construction of dynamical multi-centered black hole type solutions was generalized to an arbitrary FLRW Universe in a recent interesting work (see [30]).

Following a different path, and long before the works on multi-centered charged black holes [24, 25], a charged version of the McVittie metric was put forward, which is interpreted
as a charged Reissner–Nordström (RN) metric embedded in an FLRW cosmological background. The influence of the cosmological evolution on the size of the RN black hole was analyzed in [31]. Several properties of such a spacetime were investigated, but the global structure was not explored yet.

In [21] a set of metrics representing charged black hole type solutions in expanding backgrounds was displayed. The authors were interested mostly in verifying the energy conditions of the resulting spacetimes. They analyzed different charged metrics. The first one was a charged version of the McVittie metric, corresponding to a particular case of the solution presented in [25, 31]. A charged Vaidya metric was also considered. Other metrics studied there that interest us here were the charged Thakurta [19] and the charged SD [22] metrics. A new solution, similar to the McVittie spacetime, was also considered in [21]. The energy conditions were studied in all of the solutions, and it was shown that some of them may be of physical interest.

Some of the solutions considered in [21] may be obtained from the corresponding charged static solutions through conformal transformations, from which some of the properties of the resulting spacetime could be inferred. However, other cases such as the Vaidya-type metrics are not obtained in such a way, and the sources of the electromagnetic field, as well as the matter content of the background, must be analyzed with care. Our aim in this paper is first to investigate such electromagnetic sources in detail and explore the main properties of the corresponding energy–momentum tensor. Moreover, we also briefly investigate the global structure of the associated spacetimes.

1.4. This paper

The main aim of the present work is to investigate the electromagnetic sources of some charged black hole-type metrics in expanding universes. Inspired by the charged versions of the McVittie [1] and Vaidya [20] type metrics previously considered in the literature [21, 24, 25, 31], we write a sufficiently general metric with spherical symmetry and show that, under certain circumstances, there can be a global electric current in the radial direction. Then, some particular cases are studied. In particular, the charged Vaidya, SD, and Thakurta metrics reported in [21] are considered. The mass parameters of these solutions, as well as the charge parameters, are taken as a function of the cosmological time, and new charged solutions are then built. A brief analysis of each new metric is given.

The paper is structured as follows. In section 2 we write a sufficiently general metric and give the general form of the Faraday–Maxwell tensor and of the stress–energy (energy–momentum) tensor of the electromagnetic field related to such a metric. The definition of apparent horizon is also given in that section. Then, the charged McVittie metric is studied in section 3, where attention is paid to the electromagnetic aspects of the solution. Some aspects of the global structure of the corresponding spacetime are also considered. The studies reported in section 3 are not new, but the results are presented there for comparison to the other cases. Section 4 is dedicated to study the general properties of a charged metric written based on the original work by Vaidya [20]. Here we replace the mass parameter of the original metric by an appropriate function of the cosmic time. Again the main interest is on the electromagnetic quantities of the solution, and a brief analysis of the global structure of the resulting spacetime is given. The results on apparent horizons presented in this section are new. A charged version of a metric originally given by Sultana and Dyer [22] is analyzed in section 5. It is shown that the electromagnetic stress–energy tensor corresponds to a situation where there is a global electric current across the spacetime. Again, the mass and charge parameters of the original solutions are replaced by particular functions of the cosmic time.
and some aspects of the global structure, including the apparent horizons, of this new charged SD spacetime are also shown. Similarly, a charged version of the nonrotating Thakurta metric [19] is investigated in section 6. New mass and charge functions are considered, and some aspects of the global structure of the corresponding spacetime are also given. In section 7 we conclude.

2. A sufficiently general metric, the electromagnetic field, and the stress–energy tensor

2.1. The metric, Maxwell equations, and solutions

A sufficiently general metric for all cases we are interested in here is of the form

$$ds^2 = -f_0(r, t) \, dt^2 + a^2(t) \, f_1(r, t) \, dr^2 + 2 \, a(t) \, f_2(r, t) \, dt \, dr + a^2(t) \, r^2 \, f_3(r, t) \, d\Omega^2,$$

(5)

where $t$ is a timelike coordinate, $r$ is a spherical coordinate, $a(t)$ is the expansion factor, $d\Omega^2$ is the metric on the unit sphere, and functions $f_0(r, t)$, $f_1(r, t)$, $f_2(r, t)$, and $f_3(r, t)$ depend on the indicated coordinates. To simplify notation, we write $g_{ij} = f_0(r, t) \, x_1^2 + f_1(r, t) + f_2(r, t)$.

Let us stress that we are going to analyze a few particular cases of electrically charged, spherically symmetric black hole type solutions in expanding universes for which the functions $f_0, f_1, f_2,$ and $f_3$ are given. For instance, in the case of the charged Vaidya-type metrics considered in section 4, one has the simple form $f_3(r, t) = r^2$. The same holds in the case of the SD and Thakurta type metrics considered later. For these three particular metrics, $g_{ij}(r, t) = 1$ and the radial coordinate $r$ plays the role of a comoving radial coordinate, analogous to the comoving radial coordinate in FLRW cosmologies. With this in mind, we present in this section an analysis of the electromagnetic fields and sources for the given metrics by using the form given in equation (5) to represent any one of these cases.

The general form of the Faraday–Maxwell tensor $F^{\mu \nu}$ and the electromagnetic stress–energy tensor for the metric (5) are relevant to the present work. As a consequence of the spherical symmetry, the only nonzero components of $F^{\mu \nu}$ are $F^{tr} = -F^{rt} \equiv E(r, t)$. The contribution of the electromagnetic field to the energy–momentum tensor, i.e., the electromagnetic stress–energy tensor $\pi_{\mu \nu} = \pi_{\mu \sigma} \pi^{\sigma \nu} = \pi_{\mu \nu} = \pi_{rt} = \pi_{tr} = \pi_{rt}$, then is

$$\pi_{rt} = \pi_{tr} = -E^2(r, t) \, g_{rt}^2(r, t),$$

(6)

with all the other components of $\pi_{\mu \nu}$ being identically zero. The Maxwell equations give the two equations

$$\frac{\partial Q(r, t)}{\partial r} = 4\pi a^3(t) \, r^2 f_3(r, t) \, g_1(r, t) \, J^t(r, t),$$

(7)

$$\frac{\partial Q(r, t)}{\partial t} = 4\pi a^3(t) \, r^2 f_3(r, t) \, g_1(r, t) \, J^t(r, t),$$

(8)

where we defined

$$Q(r, t) = a^3(t) \, r^2 f_3(r, t) \, g_1(r, t) \, E(r, t),$$

(9)

with $J^t(r, t)$ and $J^r(r, t)$ being the only nonzero components of the electromagnetic current-density. The other components of the Maxwell equations are identically zero.
Equation (7) shows that the quantity \( Q(r, t) = 4\pi a^3(t) \int_0^r r^2 f_3(r, t) g_4(r, t) J'(r, t) \, dr \) may be interpreted as total electric charge inside a sphere of radius \( r \) at time \( t \). This means that, using equation (8), we may interpret \( \partial Q(r, t) / \partial t \) as the electric current across the spherical surface of radius \( r \) at time \( t \).

We are mostly interested in cosmological solutions with no net electric charge distribution across the whole spacetime. However, the location of the charged object cannot be determined without specifying the explicit form of the metric functions in equation (5), and, moreover, since there must be some electric charge somewhere in the spacetime, the non-stationary character of the metric implies some charge flux exists, i.e., a current-density which, of course, depends on the choice of the coordinates. If one takes \( J'(r, \ t) = 0 \), one has from equation \((7)\) \( Q(r, t) = Q(t) \), a function of time alone. The radial component of the current-density \( J'(r, \ t) \) may be nonzero. Therefore, there are at least two different solutions to equations (7) and (8) that are interesting for the present analysis. One holds in a region of the spacetime where there are no electromagnetic sources, \( J'(r, t) = 0 \) and \( J'(r, t) = 0 \), while the second is when electric current is allowed; i.e., for a special choice of coordinates, one may choose \( J'(r, t) = 0 \) but with \( J'(r, t) \neq 0 \). In the next sections, when studying charged versions of McVittie, Vaidya, SD and Thakurta metrics, we return to this subject.

We can determine the total electric charge inside a spherical surface at a fixed radial coordinate \( r \) for the spacetimes represented by metric (5). Let \( S \) represent the spherical surface \( \theta = \text{constant} \). The electric charge may be defined as

\[
Q(r, t) = \int_S F_{\mu n} t^\mu n^\nu \, dS,
\]

where \( t^\mu \) is the unit normal to surfaces of constant \( t \), \( n^\nu \) is the spacelike unit vector orthogonal to \( S \) pointing outwards, and \( dS \) is the area element on \( S \). Vectors \( t^\mu \) and \( n^\nu \) also satisfy the relations \( t^\mu n_\mu = -1 \), \( t^\nu n_\nu = 0 \), and \( n^\mu n_\mu = 1 \). In the case of metric (5), we may choose \( t^\mu = \delta^\mu_1 \left[ f_3(r, t) / g_4(r, t) - \delta^\mu_2 (r, t) / [a(t) g_4(r, t) \sqrt{f_3(r, t)}] \right] \) and \( n^\mu = \delta^\mu_1 / \sqrt{f_3(r, t)} \), and the resulting expression coincides with equation (9). Such a definition of electric charge is applied to some particular cases later.

### 2.2. The scale factor \( a(t) \)

For the charged McVittie, Vaidya, SD, and Thakurta type metrics that are considered in the present work, the function \( a(t) \) in equation (5) corresponds to the cosmological expansion factor. In general, this function is defined by the matter content of the universe after solving the full Einstein–Maxwell system of equations. However, regarding the kind of study being performed here, we satisfy ourselves by taking two particularly simple cases, defined next.

The first choice is, for comparison, the expansion factor used in [16, 18]

\[
a(t) = \left[ \sinh \left( \frac{3k t}{2} \right) \right]^{2/3},
\]

with \( k \) being a constant to be chosen appropriately. This choice has an initial power-law expansion with \( a(t) \sim t^{2/3} \) and a final de Sitter accelerated phase. From now on, this choice is called the case (a).

The second choice is a power-law expansion,

\[
a(t) = t^\alpha,
\]

\[\]
with constant $\alpha$, which corresponds to a Universe filled with a perfect fluid whose equation of state is of the form $p = \omega \rho$, with $\omega = (2 - 3 \alpha)/2$. Our choice shall be $\alpha = 2/3$, and so $\omega = 0$, a cold dark matter-dominated model. This is called the case (b).

### 2.3. Singularities and horizons

Besides the study on the electromagnetic sources, we investigate the curvature singularities and horizons of the chosen metrics.

The interest in curvature singularities in the cosmological scenario is because they are physical, i.e., true spacetime singularities, meaning that physical quantities such as the energy density or pressure of the cosmological fluid diverge at the singularity. It is then necessary to calculate the usual curvature invariants, which are built from the Riemann tensor $R_{\mu\nu\rho\sigma}$. As argued in [15], for a metric of the form given by equation (5), the Ricci ($\mathcal{R}$) and the Kretschmann ($\mathcal{K}$) scalars are necessary to identify the spacetime singularities. These are calculated for each metric investigated in the present work (see later).

In order to analyze the global properties of a spacetime whose metric has the form of equation (5), it is useful to perform a coordinate transformation, which brings in a new radial coordinate $R$ defined by

$$R = a(t) r \sqrt{f_3(r, t)}. \quad (13)$$

With this, metric (5) assumes the form

$$ds^2 = \left[ h(R, t) - \frac{C^2(R, t)}{A(R, t)} \right] dt^2 + A(R, t) dR^2 - 2C(R, t) dR \, dt + R^2 d\Omega^2, \quad (14)$$

where the functions $A(R, t), h(R, t),$ and $C(R, t)$ are given in terms of the original functions $f_0(r, t), f_1(r, t), f_2(r, t),$ and $f_3(r, t)$, but we do not write such relations here.

At this point it is interesting to find the equations for the expansions of the outgoing (+) and ingoing (−) congruences of null geodesics, $\theta_{\pm} = \nabla_\nu k^\nu_{\pm}$, where $k^\nu_{\pm}$ is the tangent to the radial null geodesics. Using this definition and the geodesic equations for the metric (14), it follows

$$\theta_{\pm} = \frac{2}{R} \left[ \frac{C(R, t)}{A(r, t)} \pm \sqrt{\frac{h(R, t)}{A(R, t)}} \right] \left( \frac{dr}{dp} \right)_{\pm}, \quad (15)$$

where $\left( \frac{dr}{dp} \right)_{\pm} = k^r_{\pm}$, with $p$ being an affine parameter along the geodesic curve. For well-behaved $k^r_{\pm}$, one then sees that the zeros of $\theta_{\pm}$ are given by the zeros of the function between brackets in equation (15). This means that surfaces for which the product $\theta_+ \theta_- = 0$, if they exist, are located at regions of the spacetime satisfying the equation

$$F_H(R, t) = \frac{h(R, t)}{A(R, t)} \left( \frac{C^2(R, t)}{h(R, t) A(R, t)} - 1 \right) = 0. \quad (16)$$

According to Nolan [11], which follows [32], for a spherically symmetric metric, the zeros of $F_H(R, t)$, which are in fact the zeros of the function $\chi(R, t) = \left( V_p R \right) \sqrt{V^p V R}$, define a trapping boundary. For the kind of metrics we are interested in here, these are also trapping horizons. On the other hand, other authors (see, e.g., [7, 13, 15]) call such surfaces generically apparent horizons. We follow the nomenclature of [13, 15].

We can also verify that the zeros of the expansions $\theta_{\pm}$ coincide with stationary points (regions) of the radial null geodesics of the metric (14). In fact, the radial null geodesics are
given by solutions of the equation
\[
\left( \frac{dR}{dz} \right)_z = \frac{1}{A(R, t)} \left( C(R, t) \pm \sqrt{h(R, t) A(R, t)} \right),
\]  
(17)
where the plus (minus) sign indicates outgoing (ingoing) geodesics. We see also that the functions \( \frac{dR}{dz} \) change sign at the zeros of the function \( F_H(R, t) \), meaning that the zeros may be apparent horizons separating the spacetime into trapped and non-trapped (regular) regions.

It is worth mentioning that for the metric of equation (14), one has \( F_H(R, t) = h(R, t)g^{RR} \), with \( g^{RR} \) being the component of the inverse metric. Indeed, this procedure of looking for the zeros of the function \( g^{RR} \) to find the possible apparent horizons of black hole type metrics in expanding spacetimes was followed by many authors (see, e.g., [11, 15, 16, 18] for McVittie spacetime, [29] for the Kastor–Traschen solution [27], and [13] for a Thakurta-type metric).

2.4. About the next sections

In the following we investigate a few particular charged metrics written to describe charged black holes in expanding universes. The main aim is to understand the physical nature of the electromagnetic sources of the corresponding spacetimes. Another interest is to look for apparent horizons in each case; the ultimate goal is to determine if black hole horizons are really present, besides finding the spacetime singularities in order to identify the physical nature of the central object in each case. In order to do that, the complete geodesic structure of the given spacetime must be investigated. However, this study is lengthy and, moreover, all the metrics we consider here deserve a separate detailed analysis because of the interest in the subject in recent literature. Hence, to avoid a very long paper, in this work we investigate only the form of the apparent horizons and the curvature singularities for each one of the studied metrics.

3. Charged McVittie metric

The generalized version of the McVittie metric, including the electric charge of the central body, was put forward in [25] (see also [31]). Here we review the electromagnetic sources of such a metric and present a short analysis of curvature singularities and apparent horizons.

3.1. The solution

In the case of an asymptotically spatially flat FLRW spacetime, the charged McVittie metric may be obtained from the general form (5) by setting \( f_0(r, t) = f^2(r, t)g^2(r, t) \), \( f_1(r, t) = f_2(r, t) = g^2(r, t) \), and \( f_3(r, t) = 0 \), which gives \( g_1(r, t) = f(r, t) \), where the functions \( f(r, t) \) and \( g(r, t) \) are defined, respectively, by
\[
f(r, t) = 1 - \frac{m^2}{4a^2(t) \, r^2} + \frac{q^2}{4a^2(t) \, r^2},
\]  
(18)
\[
g(r, t) = \left( 1 + \frac{m}{2a(t) \, r} \right)^2 - \frac{q^2}{4a^2(t) \, r^2}.
\]  
(19)
with $m$ and $q$ being constants related, respectively, to the mass and charge of the central body. The resulting metric is

$$dx^2 = -\frac{f^2(r, t)}{g^2(r, t)} dt^2 + a^2(t) g^2(r, t) \left( dr^2 + r^2 d\Omega^2 \right).$$

(20)

The function $a(t)$ is interpreted as the expansion factor, with $t$ being a cosmological time. The asymptotic metric ($r \to \infty$) results in

$$\Omega = - + + \left( \frac{1}{a^2(t)} r^2 \right) \left( dx^2 + x^2 d\Omega^2 \right),$$

which is the FLRW metric in the case of flat three space. This metric suffers from the same illness as the uncharged McVittie metric, but at least when the Hubble parameter $H(t) = \dot{a}(t)/a$ asymptotes a constant at $t \to \infty$, such a solution has an event horizon and represents a black hole in an expanding Universe (see, e.g., [16, 18]).

Notice that a further generalization of the charged McVittie metric (20) is obtained simply by replacing the constant parameters $m$ and $q$ with a function of the cosmological time $m(t)$ and $q(t)$. As discussed in the literature (see, e.g., [10, 11, 13]), this implies the violation of the non-accretion hypothesis of McVittie, and there must be some kind of energy flux in the radial direction throughout the spacetime. In respect to the electric charge, it means a non-constant electric charge of the source, and the presence of a radial electric current throughout the spacetime (see also the next section).

### 3.2. Faraday and stress–energy tensors, conserved electric charge, and current-density

The electromagnetic source and the stress–energy tensor for the metric (20) were analyzed in [25, 31] and we do not need to reproduce the results here. However, further comments with respect to the Maxwell equations are in order. According to our analysis in section 2, the non-zero components of the Faraday–Maxwell strength tensor are $F^\tau = -F^\tau \equiv E(r, t)$, and for the charged McVittie metric, one finds (see also [25, 31]) $E(r, t) = \frac{\dot{q}_0 h(t)}{a^2(t)} \frac{r^2 f(r, t)}{g^2(r, t)}$.

The solution presented in [25] assumes $h(t) = 1$, leading to $Q(r, t) = q_0 = \text{constant}$, and $q_0$ is a conserved quantity identified with the total electric charge of the source, so that $q_0 = q$, with $q$ being the charge parameter of the metric. This result means that comoving observers in the charged McVittie spacetime (20) attribute a constant electric charge to the central object. Furthermore, with such a choice, the Einstein–Maxwell equations in the presence of a cosmic perfect fluid are satisfied, resulting in Friedmann type equations for the energy-density and pressure. No additional terms in the density or pressure arise because of the presence of the electric charge.

On the other hand, if one chooses $h(t) \neq 1$, the Maxwell equations are satisfied with the inclusion of a radial current-density given by $J^\tau = \frac{q_0 h(t)}{4\pi r^2 a^3(t)} g^2(r, t)$, and the charge of the central object is not constant. In addition, in such a case there would be additional contributions to the energy density and pressure of the cosmic fluid coming from the electromagnetic field. For instance, the energy density has a term of the form $q^2 h^2(t) r^4 - q_0^2 h^2(t) r^4$, which for $h(t) \neq 1$ and constant $q$, implies a violation of the energy conditions. This solution has not been investigated in the literature, since it is regarded as unphysical. It is worth mentioning that if we replace the parameter $q$ by a function of time and take $q(t) = q_0 h(t)$, the Maxwell equations are satisfied and the spurious term in the energy (and pressure) of the perfect fluid vanishes. However, in order to satisfy the Einstein equations, additional matter components must be added to the background. A detailed study of such a case is not the purpose of this work.
3.3. Singularities and horizons

As already mentioned, we do not investigate the complete geodesic structure of the charged McVittie metric here, since a complete analysis is lengthy and out of the scope of this paper. However, the results on the singularities and horizons are of interest. We consider here just the case of constant mass and charge parameters \( m \) and \( q \), and show the singularities and apparent horizons of metric (20) for comparison to the other solutions investigated in the next sections. More details about the apparent horizons and singularities in the case of constant parameters \( m \) and \( q \) can be found in [33].

To say something about curvature singularities of the charged McVittie metric (20), we calculate the Ricci and the Kretschmann scalars (see the appendix). To simplify expressions, it is convenient to use a new radial coordinate \( R \) defined by \( R = a(t) r g(r, t) \) (see [31]). The Ricci scalar is then \( R = 12H^2(t) + 6H(t)/\sqrt{1 - 2M/R + Q^2/R^2} \), where we have written \( M = m \) and \( Q = q \). The Kretschmann scalar is written in appendix A. We immediately see possible singularities when \( H(t) \) diverges, and at the region of the spacetime where \( h(R) = 1 - 2M/R + Q^2/R^2 = 0 \), i.e., for \( R = M \pm \sqrt{M^2 - Q^2} \), or, in terms of the original coordinate \( r \), at \( r a(t) = \pm \sqrt{M^2 - Q^2}/2 \). In the overcharged case \( M^2 < Q^2 \), the function \( h(R) \) has no real roots and the singularity is at \( R = 0 \) (at \( r a(t) = -(M \pm Q)/2 \)). We consider here just the undercharged case \( (M^2 > Q^2) \).

The apparent horizons of the McVittie metric are given by the roots of the equation (see equation (16))

\[
F_M(R, t) = -H^2(t) R^2 + \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right) = 0, \tag{21}
\]

which is a fourth-order polynomial equation for \( R \), given the solutions in terms of the time \( t \). Note also that the zeros of \( F_M(R, t) \) coincide with the zeros of the radial coordinate velocity \( dR/dt \) of ingoing lightlike geodesics, i.e., the zeros of

\[
\frac{dR}{dt} = H(t) R \sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}} - \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right),
\]

for which \( 1 - 2M/R + Q^2/R^2 \neq 0 \).

For the undercharged McVittie metric \( (M^2 > Q^2) \), and for the chosen form and parameters of the scale factor, for which \( k \times M \) is sufficiently small (case (a)), and \( \alpha = 2/3 \) (case (b)), function \( F_M(R, t) \) has just one real root at small times, while it has three real roots \( R_1(t), R_2(t), \) and \( R_3(t) \) for sufficiently large times (see figure 1). The root \( R_1(t) \) is always inside the singularity radius \( R_s \) and \( R_0 \) which are the asymptotic values of the functions \( R_2(t) \) and \( R_3(t) \) at \( t \rightarrow \infty \), respectively. According to the study of [16], the black hole horizon \( H \) is defined by \( R_2; t = \infty \), while \( R_4 \) defined by \( (R_3, t \rightarrow \infty) \) gives a de Sitter horizon (case (a)) or the Hubble horizon (case (b)). Our \( R_0 \) here corresponds to the asymptotic limit of \( R_a \) of [16]. The apparent horizon \( R_1(t) \) is inside the singularity, but it has an interesting interpretation, since it asymptotes a charged black hole inner (Cauchy) horizon at \( t \rightarrow \infty \), i.e., the region \( (R_a, t = \infty) \) is similar to the Reissner–Nordström–de Sitter inner horizon; it is a Cauchy horizon. In case (b), in which \( \lim_{t \rightarrow \infty} H(t) = 0 \), the conformal structure of the boundary of the charged McVittie spacetime

\[
\text{Class. Quantum Grav. 32 (2015) 115004 M G Rodrigues and V T Zanchin}
\]
is similar to that of a Reissner–Nordström black hole, apart from the naked initial singularities. In this case, the asymptotic form of the metric in $R t$ coordinates is time independent. In both cases (a) and (b), the asymptotic metric at $t \to \infty$ can be put into stationary forms so that both admit Killing vectors associated to time translations. It is interesting to explore the geodesic structure of the charged McVittie spacetime in detail, including the extreme and overcharged cases. However, we are not going to report the results of such a study here.

4. Charged Vaidya-type metrics

A charged version of the Vaidya metric [20] was reported in [21]. Here we review the electromagnetic sources of a modified version of that metric and present a brief analysis of curvature singularities and apparent horizons. The modifications of the metric are done by considering also time-varying mass and charge parameters when compared to the solution given in [21].

4.1. The solution

According to our notation of section 2, the charged Vaidya-type solution given in [21] is obtained by choosing $f_0(r, t) = 1 - g(r)/a^2(t)$, $f_1(r, t) = 1 + g(r)/a^2(t)$, $f_2(r, t) = g(r)/a^2(t)$, and $f_3(r, t) = 1$, with $g(r) = 2M/r - q^2/r^2$, which gives $g_3(r, t) = 1$ (see equation (5)). Here we consider a generalized version of the metric given in [21] by taking the mass and charge parameters as a function of time, i.e., $m = m(t)$ and $q = q(t)$, and so we write
\[ ds^2 = a^2(t) \left[ dr^2 + r^2 d\Omega^2 \right] + g(r, t) \left[ \frac{dr}{a(t)} + dr \right]^2 - dr^2, \]  

where

\[ g(r, t) = \frac{2m(t)}{r} - \frac{q^2(t)}{r^2}. \]  

As promptly seen from \((22)\), for \(a(t) = 1\) and constant \(m\) and \(q\), it results in the Reissner–Nordström black hole spacetime, and far from the central body, \(r \to \infty\), it results in the Einstein–de Sitter cosmological spacetime.

The non-zero components of the Einstein tensor corresponding to metric \((22)\) are the diagonal ones \(G^{\theta\theta}, G^{\phi\phi}\), besides the ones that mix the radial and time coordinates, namely, \(G^{\theta\phi}\) and \(G^{\phi\theta}\). We do not write the full expressions of these components here, but we observe that there are changes in the sources in comparison to the results given in \([21]\) because of the time dependence of \(m(t)\) and \(q(t)\). To be explicit, we take, for instance, the new terms that arise in the component \(G^{\theta\theta}\), which are

\[ \frac{-2H(t)(m(t)a(t)r - q(t)q(t)/a^2(t)r^2).} \]  

As in the original Vaidya metric \([20]\), these terms are singular at \(r = 0\). Their contributions to the energy density depend on the explicit form of the functions \(m(t)\) and \(q(t)\), and in the case \(m(t) = M/a(t), q(t) = Q/a(t)\), considered in the next subsection, the problem of negative energy densities (at early times \(t\) and large \(r\)) of the original Vaidya solution persists (see, e.g., \([21]\), see also appendix A). We need to point out explicitly the linear terms on \(q^2(t)\) that do not contain derivatives of \(a(t)\) because they are directly related to the electromagnetic stress–energy tensor, which is one of our main points of interest. The contribution of these terms to the Einstein tensor is as follows

\[ G^{\theta\theta} = G^{\phi\phi} = -\frac{q^2(t)}{a^2(t)r^4}. \]  

From the analysis of section 2, it is expected that terms of the form \(q^2(t)/a^4(t) r^4\) would be exactly canceled by the corresponding components of the electromagnetic stress–energy tensor \(E^\mu_\nu\). This fact is confirmed in the next subsection.

### 4.2. Faraday and stress–energy tensors, conserved electric charge and current-density

Following the analysis of section 2, one gets the Faraday–Maxwell tensor field as \(E(r, t) = q_0 h(t)a^3(t)r^2\), with \(q_0\) being a constant, which yields \(Q(r, t) = q_0 h(t)\) (see equation \((9)\)), and the electromagnetic stress–energy tensor then is

\[ E^\theta_\theta = E^\phi_\phi = -\frac{q^2 h^2(t)}{8\pi a^4(t) r^4}. \]  

with the other components being identically zero.

It is seen that by choosing \(h(t) = 1\), i.e., \(E(r, t) = q_0/a^3(t)r^2\), with constant \(q_0\), the Maxwell equations are satisfied with zero current-density, \(J^i(r, t) = 0\), which implies that \(q(t) = q_0\) is the total conserved electric charge of the source, as expected. In fact, from Maxwell equations \((7)\) and \((8)\), we see that the electric four-current is not well defined just in the region of the spacetime for which \(a^3(t) r^2 \to 0\). Everywhere else in the spacetime, the electromagnetic sources vanish, and then it is natural to associate the loci where \(a^3(t) r^2 \to 0\) as the actual location of the source. Furthermore, comparing equations \((25)\) with equations \((24)\), it is seen that the components of the electromagnetic stress–energy tensor
cancel the corresponding terms of the Einstein tensor through Einstein equations 
\( G_{\mu\nu} = 8\pi \, E_{\mu\nu} \).

Contrarily, if one takes \( h(t) \neq 1 \), so that \( E(r, t) = q_0 h(t) a^3(t) r^2 \), the radial electric 

current-density is nonzero, \( J_r(r, t) = -q_0 \frac{h(t)}{4\pi r^2 a^3(t)} \), and the total charge of the source is not a constant. In this case, there may arise contributions to the energy–momentum tensor from the electromagnetic field that violate the energy conditions. In fact, similarly to the uncharged case (as investigated, e.g., in [20]), the charged Vaidya metric (22) can be sourced by a global 

perfect cosmic fluid in addition to a heat flux, besides the spherical charged object. If \( h(t) \neq 1 \), so that \( E(t) = q_0 h^2(t) \), the radial electric 

current-density is nonzero, \( J_r(t) = -q_0 \frac{h^2(t)}{r^4 a^4(t)} \). This is similar to what happens in the charged McVittie metric we investigated in section 3.2; if \( q_0^2 h^2(t) \), the energy conditions may be violated and the corresponding solutions are disregarded as unphysical.

4.3. Singularities and horizons

Let us comment here on the curvature singularities and possible apparent horizons of the spacetime defined by the charged Vaidya type metric (22).

If we choose constant \( m \) and \( q \), the Ricci scalar is given by 
\[ R = 2H^2(t) \left( 6 + g(r)/a^2(t) \right) + 2H(t) \left( 3 + 2g(r)/a^2(t) \right) - 4mH(t)/a^3(t)r^2, \]
while the Kretschmann scalar is given in appendix A. We can note that both scalars are singular when \( H(t) \to \infty \) and at \( a(t) r \to 0 \).

For the non-constant \( m(t) \) and \( q(t) \) case, there are, of course, many interesting choices. To keep the metric similar to the McVittie one, and to compare with the SD and Thakurta type metrics in the following, we take the case \( m(t) = M/a(t) \) and \( q(t) = Q/a(t) \) with constant \( M \) and \( Q \). A long but straightforward calculation shows that the curvature scalars are singular at the same regions as in the case with constant \( m \) and \( q \), where \( a(t) r = 0 \). As commented in appendix A, the curvature scalars show that the singularities are the same as for constant \( m \) and \( q \).

The trapping horizons of a spacetime whose metric has the form of equation (22) may be found by analyzing the roots of the equation (see equation (16)),

\[ F_\gamma (R, t) \equiv H^2(t) \, R^2 - \left[ H(t) \, R - 1 \right]^2 \frac{g(R)}{a^2(t)} - 1 = 0, \]  

where now \( R = ra(t) \), \( g(R) = 2M/R - Q^2/R^2 \), and we defined \( M = m \, a(t) \) and \( Q = q \, a(t) \). Here we consider the case of constant \( M \) and \( Q \) only.

The zeros of the function \( F_\gamma (R, t) \) tell us where the expansion rate of radial null geodesics changes signs, which is a necessary condition to find apparent horizons. The short analysis we give here is by comparison to the McVittie metric studied in [16, 18]. The solutions of equation (26) depend explicitly on \( a(t) \) and hence we need to define such a function \textit{a priori}. In the present analysis we use the two cases mentioned in section 2.2.

Let us first mention that equation (26) has at least one real positive root for all times given by \( H(t) R = 1 \). This solution is shown by the curves \( R_3(t) \) in figure 2; it is the Hubble radius, it tends asymptotically to a cosmological de Sitter horizon in case (a), with \( R = 1/H(\infty) = \) constant, and is exactly \( 3t/2 \) in case (b). Moreover, since in an expanding cosmological scenario the scale factor becomes arbitrarily large for infinite time, the second term in equation (26) vanishes and \( R_3(t) \) is the only root at late times. For simplicity we
consider here the undercharged case only, \( M^2 > Q^2 \). The roots \( R_1(t) \), \( R_2(t) \), and \( R_3(t) \) as function of the time \( t \) are plotted in figure 2 for the case with \( M = 2.0 \) and \( Q = 1.0 \), and for the scale factor \( a(t) \) labeled as cases (a) and (b); see section 2.2. Since the asymptotic form (for infinite time) of the roots of the function \( F_V(R, t) \) dictates the structure of the boundaries of the given spacetime, we see that a black hole horizon analogous to \( R = \infty \) of the charged McVittie metric never forms in the charged Vaidya metric. The black hole type apparent horizon \( R_2(t) \) is formed after the initial singularity (at the same time as the inner apparent horizon \( R_1(t) \)), expands faster than \( R_1(t) \) at early times but starts to shrink after a given time, while \( R_1(t) \) keeps growing slowly with time. Finally, after a finite time interval, both apparent horizons coalesce and disappear. The singularity at \( R = 0 \) is naked for all later times. Assuming, for instance, a matter-dominated universe at late times, then \( \lim_{t \to \infty} a(t) H(t) = 0 \) (case (b)) and the only real positive root increases indefinitely with time, corresponding to the Hubble radius. When the asymptotic form of the scale factor \( a(t) \) corresponds to an accelerated expanding universe (case (a)), where \( \lim_{t \to \infty} H(t) = H_0 = \text{constant} \), the only real positive root tends to a constant, which corresponds to a de Sitter cosmological horizon. Even though these partial results have been found, a more detailed study is necessary to determine the global structure of the spacetime generated by metric \( (22) \).

5. Charged Sultana–Dyer type metrics

5.1. The solution

A charged version of the SD metric, which is obtained through a conformal transformation of the Reissner–Nordström metric written in Eddington–Finkelstein coordinates, was given in [21]. For the present analysis we take the following generalized SD type metric:

\[ ... \]
\[\text{d} s^2 = a^2(t) \left[ \text{d} r^2 + r^2 \text{d} \Omega^2 \right] + g(r, t) \left[ \text{d} r + a(t) \text{d} t \right]^2 - \text{d} t^2, \quad (27)\]

where again we have put \(g(r, t) = 2m(t)/r - q^2(t)/r^2\) (see equation (23)). As noted earlier, the factor \(a(t)\) is interpreted as the cosmological scale factor, with \(t\) being the cosmological time. This charged SD metric is obtained from our general metric (5) through the identifications \(f_0(r, t) = 1 - 2m(t)/r + q^2(t)/r^2; \quad f_1(r, t) = 1 + 2m(t)/r - q^2(t)/r^2; \quad f_2(r, t) = g(r, t) = 2m(t)/r - q^2(t)/r^2; \quad f_3(r, t) = 1, \) from what follows \(\dot{g}(r, t) = 1\). One can see that the SD metric (27) follows from the Vaidya metric by substituting \(m(t)\) and \(q(t)\), respectively, by \(m(t) a^2(t)\) and \(q(t) a(t)\) into the metric (22).

The uncharged version of metric (27) has been considered in the literature as a good candidate to represent a Schwarzschild-like black hole immersed in an expanding cosmological background (see, e.g., [13, 22]). On the other hand, this charged version (27) presents an additional issue that deserves to be considered carefully. For future reference, let us note that metric (27) yields an Einstein tensor whose relevant components presents terms like additional issue that deserves to be considered carefully. For future reference, let us note that metric (27) yields an Einstein tensor whose relevant components presents terms like

\[G^t_t = G^\theta_\theta = -G^\varphi_\varphi = -\frac{q^2(t)}{a^2(t) \ r^4}. \quad (28)\]

Compared to the Vaidya metric case, equation (24), the difference is just by a factor of \(a^2(t)\). In the following subsection we give the Faraday–Maxwell and the electromagnetic stress–energy tensors for the charged SD metric and compare them to the Einstein tensor (28) to analyze the source of the electromagnetic field.

5.2. Faraday and stress–energy tensors, conserved electric charge, and current-density

According to the study of section 2, the Faraday–Maxwell tensor field for the SD metric (4) is of the form \(F^\mu_\nu = E(t, r) = q_0 h(t)a^2(t)r^2\), where \(q_0\) is a constant and \(h(t)\) is an arbitrary function of time, and then the electric charge obtained from equation (9) is \(Q(r, t) = q_0 h(t)\).

The nonzero components of the electromagnetic stress–energy tensor are

\[E^t_t = E^\theta_\theta = -E^\varphi_\varphi = \frac{-q_0^2 h^2(t)}{8\pi a^2(t) \ r^4}. \quad (29)\]

Comparing this with the Einstein tensor components (equation (28)), we get terms of the form \(G^t_t - 8\pi E^t_t = \left( q_0^2 h^2(t) - q^2(t) a^2(t) \right)/\left(a^4(t) r^4\right)\). The exact cancellation of such terms is expected, and then we must have \(q^2(t) = q_0^2 h^2(t)/a^2(t)\).

If one takes \(q(t) = q_0 = \text{constant}\), as the solution given in [21], the expected cancellation happens only if \(h(t) = a(t)\). However, in that case the constant \(q_0 = q\) is not the conserved electric charge. In fact, using the charge definition (10) we get \(Q(r, t) = q_0 a(t)\). If so, there would be a non-zero electric current in the radial direction \(J^t(r, t) = -q_0 H(t)(4\pi a^2(t) r^2)\), implying that the total charge of the central object varies with time. This is the solution considered in [21], but the presence of the radial electric current was not noticed there.

On the other hand, the natural choice is to take \(Q(t) = q_0 = \text{constant}\), where the total charge of the central body is constant (cf equation (10)), with \(h(t) = 1\), which corresponds to the case of zero four-current density throughout the spacetime, \(J^\mu = 0\). In such a case, one must have \(q^2(t) = q_0^2 /a^2(t)\), giving rise to a new metric of SD type. To see how this choice affects the sources, let us take the full expression for \(G^t_t - 8\pi E^t_t\) in the case with \(m(t) = M/a(t)\) and \(q(t) = Q/a(t)\), where \(M\) and \(Q\) are constant parameters. This gives

\[G^t_t - 8\pi E^t_t = -H^2(t) \left( 3 + \frac{4M}{a(t) t} - \frac{Q^2}{a^2(t) r^2} \right) + 4MH(t)(4\pi a^2(t) r^2). \quad \text{In the original charged SD}\]
metric of [21] one has \( G^i_j - 8\pi E^i_j = -3H^2(t)\left(1 + \frac{2m}{r} - \frac{q^2}{r^2}\right) + \frac{4mH(t)}{a(t)r^2} \). Hence, apart from the time dependence of the mass and charge functions, this component of the Einstein equations has no significant differences in comparison with the solution given in [21]. Similar changes occur in the other components of the Einstein tensor, but we do not analyze them here.

We then conclude that the simplest case with \( q(t) = q_0 = \) constant and equal to the electric charge of the spherical source is physically not interesting. The Einstein–Maxwell equations are satisfied only with the presence of additional energy–momentum terms that may violate the energy conditions. The energy density acquires an extra term of the form \( q_0^2\left(1 - a^4(t)\right)/a^4(t)r^4 \). This term changes sign at time \( t \) such that \( a^2(t) = 1 \), which is an arbitrary time, since the scale factor \( a(t) \) can be normalized in such a way that \( a^2(t) = 1 \) at any given time. A similar situation happens in the case of the charged Thakurta spacetime, which we investigate in more detail later.

### 5.3. Singularities and horizons

Here we comment briefly on the causal structure of the charged SD spacetime given by the metric (27). The Ricci and the Kretschmann scalars for this metric are written in appendix A, first for constant \( m \) and \( q \) and afterwards for \( m(t) = M/a(t) \) and \( q(t) = Q/a(t) \), with constant \( M \) and \( Q \). As in the case of charged Vaidya solution, both scalars are singular when \( H(t) \to \infty \) and at \( a(t) = r = 0 \). The singularities are the same when the mass \( m \) and the charge \( q \) depend on time, as can be seen by calculating the Ricci and the Kretschmann scalars for such a case. Again, we consider here the case where \( m(t) = M/a(t) \) and \( q(t) = Q/a(t) \). As is seen later, this choice implies that metric (4) has a geodesic structure similar to the charged McVittie metric.

Looking for singularities in metric (27), we see that the locus where \( g_{rr} = g^{rr} = 0 \) coincides with the horizons of the Reissner–Nordström metric, \( r = \ell_\pm(t) = m(t) \pm \sqrt{m^2(t) - q^2(t)} \), which are roots of the function \( f(r, t) = 1 - 2m(t)/r + q^2(t)r^2 \). However, these are not spacetime singularities nor apparent horizons. To find the locus of an apparent horizon, we proceed as in the case of the preceding section and transform to a new radial coordinate given by \( R = ra(t) \). Again, apparent horizons can be found by analyzing the roots of the equation \( g^{\delta\kappa}(R, t) = 0 \). In the case of SD metric (27), this gives

\[
F_{SD}(R, t) \equiv H^2(t)R^2 + [1 - H(t)R]^2g(R, t) - 1 = 0,
\]

where now \( g(R, t) = 2M/R - Q^2/R^2 \). With \( M = m(t) \) \( a(t) \) and \( Q = q(t) \( a(t) \). The resulting equation to be solved is a fourth-order polynomial equation whose zeros depend on \( H(t) \), \( M \), and \( Q \). For simplicity, we assume here the condition \( M^2 > Q^2 \), with \( Q \neq 0 \) and consider the two cases for \( a(t) \) as earlier, i.e., \( a(t) \) given by equations (11) and (12). For a given range of the parameters \( M \) and \( Q \), and for the two given choices of \( a(t) \), the real roots of equation (30), \( R_i(t) (i = 1, 2, \ldots) \), result in the curves drawn in figure 3. At early times, there is one real positive root \( R_1(t) \), which is a cosmological apparent horizon (or the Hubble radius) and then the initial singularity at \( R = ra(t) = 0 \) is not hidden by any kind of horizon. Indeed, the root \( R_3(t) \) is exactly the Hubble time, i.e., the inverse of the Hubble parameter, \( R_3(t) = 1/H(t) \). At infinitely large times \( R_3(t) \) tends to a constant \( (R_3(t) \to 1/k) \) and will form the de Sitter cosmological horizon in case (a), i.e., \( (R_3, t = \infty) \) is a de Sitter horizon, or it will be the Hubble surface \( (R_3(t) = 3t/2) \) for late times in case (b). For sufficiently large times, two other
solutions $R_1(t)$ and $R_2(t)$ become real and positive. The three roots exist for all later times and, interestingly, their asymptotic form suggests the formation of charged black hole and white hole regions, with a cosmological horizon. For instance, if $\lim_{t \to \infty} H(t) = H_0 = \text{constant}$, case (a), the three roots tend to fixed values $R_1(t) \to R_{c1}$, $R_2(t) \to R_{c2}$, and $R_3(t) \to R_{c3} = 1/k$. We claim that these values correspond respectively to a Cauchy horizon, a charged black hole horizon, and a de Sitter (cosmological) horizon. The de Sitter horizon is defined by $(R_{c3}, t = \infty)$, the black hole horizon $H$ is given by $(R_{c2}, t = \infty)$, and the inner horizon is defined by $(R_{c1}, t = \infty)$. In the case of a matter-dominated Universe, case (b), $\lim_{t \to \infty} H(t) = 0$, $R_1(t)$, and $R_2(t)$ also tend to constant values with properties of Cauchy and black hole horizons, respectively, while $R_3(t)$ tends to $3t/2$, the Hubble radius. In both cases, the charged SD metric (27) seems to represent a charged black hole in an expanding universe.

6. Charged Thakurta-type metrics

6.1. The solution

The original Thakurta [19] solution is conformal to the Kerr metric and was put forward to represent a rotating pointlike mass in an expanding universe. For the non-rotating case, the Thakurta metric is conformal to the Schwarzschild metric and is also a candidate to describe the gravitational field of a point mass in an expanding Universe. The resulting metric in this case was analyzed in [13], even though the name used there was the SD metric. A charged version of the non-rotating Thakurta solution was given in [21]. Such a metric is of the form

$$ ds^2 = -f(r, t)dt^2 + a^2(t)\left[f^{-1}(r, t)dr^2 + r^2d\Omega^2\right]. $$

(31)
where
\[ f(r, t) = 1 - \frac{2m(t)}{r} + \frac{q^2(t)}{r^2}, \]  
(32)

with \( m(t) \) and \( q(t) \) being related, respectively, to the mass and electric charge of the central body and are fixed parameters in the original solution. Metric (31) may be obtained from our general metric (5) by choosing
\[ f_0(r, t) = f_1^{-1}(r, t) = 1 - 2m(t)/r + q^2(t)/r^2, \]
and \( f_2(r, t) = 0 \), so that \( g_t(r, t) = 1 \). As mentioned earlier, if \( m \) and \( q \) are both constant parameters, metric (31) is conformal to the Reissner–Nordström metric, as seen by redefining the time coordinate by writing \( dt = a(t) \, dt \). Despite having been briefly analyzed in [13, 21] (for constant \( m \) and constant \( q \)), several aspects of the resulting metric remain to be investigated. In particular, the sources of the electromagnetic field deserve further analysis in this case.

Since we are mostly interested in the electromagnetic sources, we write here only the metric contributions to the Einstein tensor which do not depend on \( H(t) \),
\[ G'_t = G'_r = -G'^{\theta\theta} = -G'^{\phi\phi} = -\frac{q^2(t)}{a^2(t) \, r^4}. \]  
(33)
Note that these are exactly the same as for the charged SD metric (cf equation (28)). These terms should be canceled by the relative components of the electromagnetic stress–energy tensor (see next).

### 6.2. Faraday and stress–energy tensors, conserved electric charge, and current-density

Applying the analysis of section 2 to the Thakurta metric (31), we find the same Faraday–Maxwell tensor as for the SD case, \( F' = -F' = E(r, t) = q_0 \, h(t)/(r^2 a^3(t)) \), where, as pointed out earlier, \( q_0 \) is a constant and \( h(t) \) is an arbitrary function of the cosmological time \( t \) alone, and the only non-zero component of the current-density is
\[ J_t(r, t) = -q_0 \, h(t)/(4 \pi a^3(t) r^2). \]  
(34)
Interestingly, the electromagnetic field and the related stress–energy tensor of the charged Thakurta metric (31) are quite the same as for the charged SD geometry (27).

The components (34) should cancel exactly the associated terms in the Einstein tensor (33). Hence, we must choose \( q_0^2 h^2(t) = q^2(t) a^2(t) \). As in the case of the SD metric (see section 5), the choice of constant \( q(t) = q_0 \) as done in [21] implies \( h(t) = a(t) \) and leads to the presence of a global electric flow in the radial direction with \( J'(r, t) = -q_0 H(t)/(4 \pi a^3(t) r^2) \), and the total charge of the source varies with time, \( Q(t) = q_0 a(t) \). On the other hand, the choice \( Q = \) constant (\( h(t) = 1 \)) has the required physical property that the total electric charge of the source is constant with the cosmological time. This corresponds to \( q(t) = Q/a(t) \), resulting in a different metric from the solution presented in [21].

In conclusion, as in the SD case, there are two interesting cases to consider. The first case is the one with \( h(t) = a(t) \), \( q(t) = q_0 = \) constant, in which the total charge of the source is \( Q(t) = q_0 a(t) \) and there is a non-zero radial electric current \( J'(r, t) \). As a matter of fact, this is the charged Thakurta solution investigated in [21], but the presence of the non-zero radial electric current was not reported in that study. The second case is the one with \( h(t) = 1 \),
\(Q(t) = q_0 = \text{constant, and } q(t) = q_0/a(t),\) which leads to a new metric of the Thakurta type, which is briefly investigated later.

To complete this section, let us comment about the changes in the energy–momentum tensor implied by the replacement of constant \(m\) and \(q\) by the functions \(m(t) = M/a(t)\) and \(q(t) = Q/a(t).\) The time–time component of the Einstein equations yields

\[
G_{tt} = 8\pi E_{tt} = -\frac{3H^2(t)}{f(r, t)} + \frac{2H^2(t)}{f^2(r, t)} \left( M \frac{1}{a(t)r} - \frac{Q^2}{a^2(t)r^2} \right),
\]

where \(f(r, t) = 1 - 2M/a(t)r + Q^2/a^2(t)r^2.\) The first term has the same form as in the original Thakurta metric, but now with \(f(r, t)\) replacing \(f(r)\) of the original solution. Considering the source as a fluid with heat flow, we find that the energy density of the fluid acquires extra contributions, which changes the functional form of the energy density. The original energy density is positive in all regions of the spacetime where \(f(r) > 0\), while the new energy density becomes negative for values of \(R = a(t)r\) very close to the largest zero of the function \(f(r, t)\). Similar changes occur in the pressure and heat flow, but we do not show such functions here. This negative energy density may be interpreted as a kind of phantom matter. On the other hand, we can avoid such a situation by considering, for instance, a mixture of fluids instead of a single fluid with heat flow. Another interesting way out of this problem is to take different functions for \(m(t)\) while keeping \(q(t) = Q/a(t)\), which is the appropriate electromagnetic source. To simplify the analysis and for the sake of comparison with the McVittie metric, we treat here the case \(m(t) = M/a(t)\) only.

6.3. Singularities and horizons of the charged Thakurta metric

As earlier, for simplicity, we comment only on the undercharged case.

For constant \(m\) and constant \(q\), the Ricci scalar for the metric (31) may be written as

\[
R = 12\left(\frac{H^2(t)}{f(r, t)} + 6\frac{H(t)}{f(r, t)}\right) + \text{Kretschmann scalar (in appendix A).}
\]

The other curvature scalars present the same singularities as those found from Ricci (A.9) and Kretschmann (A.10).

As seen from the Ricci scalar, there are singularities at regions of the spacetime where at least one of the following conditions is satisfied: when \(H(t)\) diverges, when \(a(t)\) vanishes, and where \(f(r) = 0\). Thus, besides a possible big-bang singularity when \(a(t) \to 0\) (\(H(t) \to \infty\)), the singularities are defined by the zeros of function \(f(r) = 1 - 2M/(r + q^2/r^2),\) i.e., at \(r = r_\pm = m \pm \sqrt{m^2 + q^2}.\) Of course, the existence of such singularities depends upon the asymptotic behavior of the Hubble parameter \(H(t)\) for \(t \to \infty\). For instance, if \(\lim_{t \to \infty} H(t) = 0\) and \(\lim_{t \to \infty} H(t) = 0,\) then the singularity at \(f(r) = 0\) that exists for finite \(t\) disappears in the infinite future of time \(t\).

Furthermore, from the Kretschmann scalar (and from other curvature scalars such as \(R_{\mu\nu}R^{\mu\nu}\)), it follows that \(r = 0\) is also a singularity. Hence, the singularity is at \(r = 0\) in the case \(f(r)\) has no real roots, i.e., in the overcharged case, where the central object has electric charge \(q\) larger than the mass parameter \(m, m^2 < q^2.\)

The same singularities are present for non-constant \(m(t)\) and \(q(t)\). The only difference is that the singularities defined by the zeros of the function \(f(r, t) = 1 - 2m(t)/(r + q^2(t)/r^2)\) now depend on time. The curvature scalars for this case are not written here since their expressions are cumbersome (see the appendix).

As in the previously analyzed spacetimes, it is useful to define a physical radial coordinate by \(R = ra(t)\) so that metric (31) assumes the form
\[ ds^2 = - \left( f(R) - \frac{H^2(t) R^2}{f(R)} \right) dt^2 - \frac{2H(t) R}{f(R)} dr dR + \frac{dR^2}{f(R)} + R^2 d\Omega^2, \]  
(35)

where \( f(R) = 1 - 2M/R + Q^2/R^2 \) and the new parameters \( M = m(t)a(t) \) and \( Q = q(t)a(t) \) are defined. As earlier, the search for apparent horizons is performed just in the case \( m(t) = M/a(t) \) and \( q(t) = Q/a(t) \), with constant \( M \) and \( Q \).

In terms of the new parameters \( M \) and \( Q \), for \( M^2 \geq Q^2 \) the curvature singularities are located at the loci of the spacetime where \( f(R) = 1 - 2M/R + Q^2/R^2 = 0 \), i.e., \( R_\pm = M \pm \sqrt{M^2 - Q^2} \), where \( R_\pm = a(t) r_\pm = a(t) \left( m \pm \sqrt{m^2 - q^2} \right) \). In the overcharged case, \( M^2 < Q^2 \), there is a curvature singularity at \( R = 0 \).

Since there are singularities in the generalized Thakurta spacetime given by metric (35), it is interesting to verify whether there are horizons hiding the singularity to external observers, so we look for apparent horizons. In terms of the metric (35), apparent horizons, if they exist, are given by the roots of the equation \( H^2(t) R^2 - f^2(R) = 0 \) or

\[ F_t(R, t) = H^2(t) R^2 - \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^2 = 0, \]  
(36)

which results in a third-order polynomial equation for \( R \).

The analytical expressions for the three roots of the previous polynomial equation (36) for \( R (R_1(t), R_2(t), \text{and } R_3(t)) \) are not written explicitly here. The solutions depend explicitly on the Hubble parameter \( H(t) \). Restricting the analysis to expanding universes \( H(t) \geq 0 \), so that \( H(t) R > 0 \), the nature of such roots depends on the charge to mass relation. A deeper analysis shows that for \( M^2 \geq Q^2 \) one of the roots, \( R_1(t) \), say, is smaller than \( R_+ = M - \sqrt{M^2 - Q^2} \) for all times, while the other two are larger than \( R_- = M + \sqrt{M^2 - Q^2} \). On the other hand, for \( M^2 < Q^2 \) there is only one real positive root of equation (36), but as mentioned earlier, we report here mostly on the undercharged case \((M^2 > Q^2)\). The apparent horizons of the extremely charged Thakurta metric \((M^2 = Q^2)\) have a similar time behavior as for the undercharged case.

If \( M^2 \geq Q^2 \) there is a curvature singularity at \( R = R_+ \), with finite \( t \), and the three roots may be real and positive. According to the big-bang cosmological models, \( H(t) \) is large at early times and then, for small times, equation (36) has only one real positive root \((R_1(t))\), the other two roots being complex conjugates to each other. As time goes on, \( H(t) \) decreases and there is a specific time \( t_b \) for which \( 27(1 + 6MH(t_b) - Q^2H^2(t_b)) = 2(1 - 6MH(t_b))^3/2 \) when these roots become real and equal to each other, \( R_2(t_b) = R_3(t_b) \). Thereafter, for all later times, the three horizons develop independently. The situation is depicted in figure 4, where we plot \( R_+ \) and the three solutions of equation (36) as a function of the coordinate time \( t \) for two cases of \( a(t) \). At early times, the singularity \( R = R_\pm (t \text{ finite}) \) is completely naked, and after some intermediate time \( t = t_b \) an apparent horizon is formed at \( R = R_2(t_b) = R_3(t_b) \); it immediately bifurcates into two branches that evolves independently.

The solution \( R_2(t) = R_3(t) \), which is satisfied at a particular time, bifurcates and after that \( R_2(t) \) and \( R_3(t) \) evolve, forming a cosmological apparent horizon (or Hubble-type surface) \( R_2(t) \) and a black hole-type apparent horizon \( R_3(t) \). Asymptotically (for \( t \to \infty \)), in the case (a) where \( \lim_{t \to \infty} H(t) = H_0 > 0 \), \( R_3(t) \) tends to a de Sitter-type cosmological horizon, \( \lim_{t \to \infty} R_3(t) = R_\infty = \text{constant in time} \) (\( R_\infty \) is the largest real root of the equation \( 1 - 2M/R + Q^2/R^2 - H_0 R = 0 \)). In the case (b), where \( \lim_{t \to \infty} H(t) = 0 \), \( R_3(t) \) tends to the Hubble radius, \( R_3(t) \sim 1/t \). On the other hand, in both cases (a) and (b), \( R_2(t) \) tends to a black hole horizon. In the case (a) \( \lim_{t \to \infty} R_2(t) = R_\infty = \text{constant} \) (which is the intermediate
solution of the equation $1 - 2M/R + Q^2/R^2 - H_0 R = 0$, while in case (b) $R_2(t)$ tends to the Reissner–Nordström black hole horizon $R_+ = M + \sqrt{M^2 - Q^2}$. Following [16] (see also [18]), we see that the region $(R_2, t = \infty)$ is a black hole horizon $H$. Notice also that in the limit $t \to \infty$, for case (b) the Ricci scalar tends to zero and the regions of the spacetime $R_+$ and $R_-$ are no longer curvature singularities. For this particular case, the true singularity is at $R = 0$, as it follows from the Kretschmann scalar (A.10). With this we may infer that the boundaries of the resulting spacetime are similar to the Reissner–Nordström case.

Let’s stress also that if the Universe has a phase dominated by the cosmological constant (at late times), the Hubble parameter $H(t)$ tends to a constant, and $R_2(t)$ tends to a constant value larger than $R_+$, i.e., $\lim_{t \to \infty} R_2(t) = R_2 = \text{constant}$, and thus a black hole horizon is formed. The cosmological-like horizon $R_1(t)$ tends to a de Sitter horizon (see [16]). In this case we may infer that the causal structure of the region $R > R_+$ of the charged Thakurta-type spacetime (35) is similar to the McVittie metric [16, 18].

7. Further comments and conclusion

The McVittie metric was proposed in 1933 and has been studied by many authors. However, we can say that a good comprehension of the global structure of the corresponding spacetime has been achieved only recently. It is then clear that a lot of effort is still needed to reach a good level of understanding of the other metrics that have been written to represent black holes in cosmological backgrounds. This is so for the Vaidya metric [20] and also for the SD [22] and Thakurta [19] metrics. What we mean is that not even the original versions of such metrics were investigated in detail, and so any progress in the charged cases is of high interest.

We started our study reviewing the charged metrics given in [21], and soon verified that the electromagnetic field of those solutions deserved further examination, since in some cases
the correct sources of the Maxwell electromagnetic field were not given. In particular, in the charged versions of the SD and Thakurta metrics, the presence of a radial current density across the spacetime was not observed. Then, for the sake of comparison, we started by investigating this issue in the charged McVittie solution given in [25], and also considered a charged Vaidya-type metric. In both cases, Maxwell equations were solved by choosing the Faraday–Maxwell field in such a way that the four-current density vanishes everywhere except at singular points (regions) of the spacetime. The total charge of the central object is a constant, and the stress–energy tensor emerging from such a Faraday–Maxwell gauge field exactly cancels the corresponding terms in the Einstein tensor produced by the given metric.

On the other hand, in the charged versions of the SD and Thakurta type metrics, the situation is a little more interesting. The Faraday–Maxwell tensor field related to zero four-current density outside the source contributes to a stress–energy tensor that does not cancel the corresponding terms of the Einstein tensor coming from the metrics given in [21]. This gives rise to additional energy and pressure terms that have no direct physical interpretation. On the other hand, if one chooses the correct Faraday–Maxwell field so to get rid of the undesirable energy terms, then there is a global (radial) electric current throughout the spacetime, and the electric charge of the source varies with the cosmological time.

As mentioned earlier, further studies are necessary to complete the analysis of the metrics considered here. As we have shown, some of them present properties that may represent charged black holes in expanding universes. It is worth mentioning once more that the geodesic structure of most of those spacetimes was not investigated in the literature, not even of the uncharged original metrics. We are now investigating the global structure of the charged SD and Thakurta metrics. The complete analysis, including the conformal diagrams, will be published in separate papers.

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Appendix. Curvature scalars

A.1. Charged McVittie metric

The charged McVittie metric gets a simpler form by using a Schwarszchild-like radial coordinate $R$ given by

$$R = a(t) \left[ g(r, t) = a(t) \left( 1 + \frac{m}{2a(t)r} \right)^3 - \frac{q^2}{4a(t)r} \right].$$

(A.1)

In terms of this new radial coordinate, and taking constant mass and charge parameters, $m = M$ and $q = Q$, the Ricci and the Kretschmann scalars, $\mathcal{R}$ and $K$, respectively, for the charged McVittie metric (20) may be written as

$$\mathcal{R} = 12H^2(t) + \frac{6H(t)}{\sqrt{\mathcal{R}(R)}},$$

(A.2)
\[ \mathcal{K} = 48 \left( \frac{M}{R^3} - \frac{Q^2}{R^4} \right)^2 + \frac{8Q^4}{R^8} + 24H^4(t) + \frac{4H(t)}{\sqrt{f(R)}} \left( 3\dot{H}(t) + 6H^2(t) - \frac{2Q^2}{R^4} \right). \] 

(A.3)

where we defined \( f(R) = 1 - 2M/R + Q^2/R^2 \). These curvature scalars are used in the study performed in section 3.

### A.2. Charged Vaidya-type metrics

The Ricci and the Kretschmann scalars, \( \mathcal{R} \) and \( \mathcal{K} \), respectively, for the Vaidya metric (22), considering \( m = \) constant and \( q = \) constant, are given by

\[
\mathcal{R} = 2H^2(t) \left( 6 + \frac{g(r)}{a^2(t)} \right) + 2\dot{H}(t) \left( 3 + \frac{2g(r)}{a^2(t)} \right) - \frac{4mH(t)}{a^3(t)r^2}, \]

(A.4)

\[
\mathcal{K} = 12 \left( 2H^4(t) + 2H^2(t)H(t) + H^2(t) \right) + \frac{8g(r)}{a^2(t)}
\times \left( H^4(t) + 2H^2(t)H(t) + 2H^2(t) \right) + \frac{g^2(r)}{a^2(t)}
\times \left( 8\dot{H}^2(t) + H^4(t) - \frac{16H(t)}{a^6(t)r^4} \left( q^2 - mr \right) \right) \left( 1 - a(t)rH(t) \right)
\]

\[
- \frac{8\dot{H}(t)q^2 + 16mH(t)}{a^4(t)r^4} + \frac{8H^2(t)}{a^6(t)r^2} \left( \frac{7q^4}{r^4} - \frac{20mq^2}{r^3} + \frac{16m^2}{r^2} \right)
\]

\[
- \frac{16H(t)}{a^2(t)r^3} \left( \frac{6q^4}{r^4} - \frac{13mq^2}{r^3} + \frac{8m^2}{r^2} \right) + \frac{8}{a^8(t)r^2} \left( \frac{7q^4}{r^4} - \frac{12mq^2}{r^3} + \frac{6m^2}{r^2} \right), \]

(A.5)

where \( g(r) = \frac{2m}{r} - \frac{q^2}{r^2} \). These scalars are used in the analysis of the singularities of the charged Vaidya spacetime in section 4.

Now, taking time-dependent mass and charge parameters, i.e., with \( m = m(t) \) and \( q = q(t) \), besides the terms showed by equations (A.4) and (A.5) with \( m \) and \( q \) replaced by \( m(t) \) and \( q(t) \), the resulting Ricci and Kretschmann scalar acquire a large number of additional terms; thus, to avoid cumbersome expressions in the paper, we do not write them here. Such extra terms contain negative powers of \( a(t) \), positive powers \( H(t) \) and \( \dot{H}(t) \), and negative powers of \( r \). These extra terms contain also positive powers of \( m(t) \), \( q(t) \), \( \dot{m}(t) \), and \( \dot{q}(t) \).

If we take \( m(t) = M/a(t) \) and \( q(t) = Q/a(t) \), with constant \( M \) and \( Q \), as considered in the text, the Ricci scalar becomes

\[
\mathcal{R} = 12H^2(t) + 3\dot{H}(t) \left( 2 + \frac{g(r, t)}{a^2(t)} \right) + \frac{\dot{H}(t)Q^2}{a^4(t)r^2}, \]

(A.6)

where \( g(r, t) = \frac{2M}{a(t)r} - \frac{Q^2}{a^2(t)r^2} \). The Kretschmann scalar results in too long an expression to be written here, but it has the same kind of terms as in the case of constant \( m \) and \( q \). Hence, the singularities are the same as in the case with constant \( m \) and \( q \). Namely, a big-bang type singularity when \( a(t) = 0 \) or, equivalently, when \( H(t) \) and \( \dot{H}(t) \) become infinitely large, and a singularity at the region \( r = 0, t \) finite).
A.3. Charged SD-type metrics

For constant $m$ and $q$, the Ricci and the Kretschmann scalars obtained from metric (27) are, respectively,

$$ R = 6 \left( 2H^2(t) + \dot{H}(t) \right) \left[ 1 + g(r) \right] - \frac{12m H(t)}{a(t)r^2}, \quad (A.7) $$

$$ K = 12(1 + g(r))^2 \left( 2H^4(t) + 2H^2(t) \dot{H}(t) + H^2(t) \right) + \frac{16q^2 H(t)}{a^3(t)r^3} \left( 2q^2 - 3mr \right) $$

$$ - \frac{16 \dot{H}(t)H(t)}{a(t)r^3} \left( q^2 g(r) + mr \right) - \frac{8 \dot{H}(t)}{a^2(t)r^3} (1 + g(r)) - \frac{48m H^3(t)}{a^3(t)r^2} (1 + g(r)) $$

$$ + \frac{32H^2(t)}{a^4(t)r^6} \left( q^4 - 3mq^2r + 3m^2r^2 \right) + \frac{8}{a^4(t)r^8} \left( 7q^4 - 12mq^2r + 6m^2r^2 \right), \quad (A.8) $$

where $g(r) = \frac{2m}{r} - \frac{q^2}{r^2}$.

The modifications of these scalars in the case $m = Ma(t)$ and $q = Qa(t)$, with constant $M$ and $Q$, correspond to the presence of new terms with the same functional dependence on $a(t)$, $H(t)$, and $r$, as the previous terms. For instance, the Ricci scalar changes to

$$ R = 6H^2(t) \left( 2 + \frac{2M}{a(t)r} - \frac{Q^2}{a^2(t)r^2} \right) + 2H(t) \left[ 3 + \frac{5M}{a(t)r} - \frac{Q^2}{a^2(t)r^2} \right] - \frac{8MH(t)}{a(t)r^2}. $$

We do not write the resulting expression for the Kretschmann scalar because it is too cumbersome. These scalars are used in the analysis of singularities of the charged SD spacetime in section 5.

A.4. Charged Thakurta-type metrics

Taking constant parameters $m$ and $q$, the Ricci and the Kretschmann scalars of the charged Thakurta metric (31) may be written, respectively, as

$$ R = \frac{12H^2(t)}{f(r)} + \frac{6H(t)}{f(r)}, \quad (A.9) $$

$$ K = \frac{12}{f^2(r)} \left( H(t) + H^2(t) \right)^2 - \frac{8q^2 H(t)}{a^2(t) r^4 f(r)} + \frac{12H^4(t)}{f^2(r)} $$

$$ + \left( \frac{m}{r^3} - \frac{q^2}{r^3} \right) \left( \frac{48}{a^4(t)} - \frac{16H^2(t)}{a^2(t) f^2(r)} \right) + \frac{8q^4}{a^4(t) r^8}. \quad (A.10) $$

These scalars are used in the analysis of section 6.

As in the SD metric, replacing $m$ and $q$ by functions of time $m(t)$ and $q(t)$ does not affect the big-bang singularity $a(t) = 0$, or $H(t) = \infty$ and $\dot{H}(t) = \infty$. The singularities at the zeros of the function $f(r, t) = 1 - 2m(t)r + q^2(t)r^2$ are also present, but now depend on the cosmic time $t$. We do not write here the expressions for the Ricci and Kretschmann scalars for the general case because they are too large. In the particular case where $m(t) = Ma(t)$ and $q(t) = Qa(t)$, the Ricci scalar becomes

$$ R = 12H^2(t)f(r, t) - 2H(t)(6f(r, t) + Ma(t)r - Q^2/a^3(t)r^2) f^2(r, t) + 2H^2(t) \left[ 4(M/a(t)r - Q^2/a^3(t)r^2)^2 - f(r, t)(6M/a(t)r - 5Q^2/a^3(t)r^2) \right] f^2(r, t), $$

and we avoid to write the huge expression for the Kretschmann scalar.
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