BEHAVIOR OF THE BERGMAN KERNEL
AND METRIC NEAR CONVEX BOUNDARY POINTS

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Abstract. The boundary behavior of the Bergman metric near a convex boundary point $z_0$ of a pseudoconvex domain $D \subset \mathbb{C}^n$ is studied. It turns out that the Bergman metric at points $z \in D$ in the direction of a fixed vector $X_0 \in \mathbb{C}^n$ tends to infinity, when $z$ is approaching $z_0$, if and only if the boundary of $D$ does not contain any analytic disc through $z_0$ in the direction of $X_0$.

For a domain $D \subset \mathbb{C}^n$ we denote by $L^2_0(D)$ the Hilbert space of all holomorphic functions $f$ that are square-integrable and by $\|f\|_D$ the $L^2$-norm of $f$. Let $K_D(z)$ be the restriction on the diagonal of the Bergman kernel function of $D$. It is well known (cf. [5]) that

$$K_D(z) = \sup\{|f(z)|^2 : f \in L^2_0(D), \|f\|_D \leq 1\}.$$ 

If $K_D(z) > 0$ for some point $z \in D$, then the Bergman metric $B_D(z; X)$, $X \in \mathbb{C}^n$, is well defined and can be given by the equality

$$B_D(z; X) = \frac{M_D(z; X)}{\sqrt{K_D(z)}},$$

where

$$M_D(z; X) = \sup\{|f'(z)X| : f \in L^2_0(D), \|f\|_D = 1, f(z) = 0\}.$$ 

We say that a boundary point $z_0$ of a domain $D \subset \mathbb{C}^n$ is convex if there is a neighborhood $U$ of this point such that $D \cap U$ is convex.

In [4], Herbort proved the following

Theorem 1. Let $z_0$ be a convex boundary point of a bounded pseudoconvex domain $D \subset \mathbb{C}^n$ whose boundary contains no nontrivial germ of an analytic curve near $z_0$. Then

$$\lim_{z \to z_0} B_D(z; X) = \infty$$

for any $X \in \mathbb{C}^n \setminus \{0\}$.

Herbort’s proof is mainly based on Ohsawa’s $\partial$-technique. The main purpose of this note is to generalize Theorem 1 using more elementary methods.

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For a convex boundary point \( z_0 \) of a domain \( D \subset \mathbb{C}^n \) we denote by \( L(z_0) \) the set of all \( X \in \mathbb{C}^n \) for which there exists a number \( \varepsilon_X > 0 \) such that \( z_0 + \lambda X \in \partial D \) for all complex numbers \( \lambda, |\lambda| \leq \varepsilon_X \). Note that \( L(z_0) \) is a complex linear space.

Then our result is the following one.

**Theorem 2.** Let \( z_0 \) be a convex boundary point of a bounded pseudoconvex domain \( D \subset \mathbb{C}^n \) and let \( X \in \mathbb{C}^n \). Then
(a) \( \liminf_{z \to z_0} K_D(z) \text{ dist}^2(z, \partial D) \in (0, \infty) \);
(b) \( \lim_{z \to z_0} B_D(z; X) = \infty \) if and only if \( X \notin L(z_0) \). Moreover, this limit is locally uniform in \( X \notin L(z_0) \);
(c) if \( L(z_0) = \{0\} \), then (a) and (b) are still true without the assumption that \( D \) is bounded.

**Proof of Theorem 2.** To prove (a) and (b) we will use the following localization theorem for the Bergman kernel and metric [2].

**Theorem 3.** Let \( D \subset \mathbb{C}^n \) be a bounded pseudoconvex domain and let \( V \subset \subset U \) be open neighborhoods of a point \( z_0 \in \partial D \). Then there exists a constant \( C \geq 1 \) such that
\[
\hat{C}^{-1} K_{D \cap U}(z) \leq K_D(z) \leq K_{D \cap U}(z),
\]
\[
\hat{C}^{-1} B_{D \cap U}(z; X) \leq B_D(z; X) \leq \hat{C} B_{D \cap U}(z; X)
\]
for any \( z \in D \cap V \) and any \( X \in \mathbb{C}^n \). (Here \( K_{D \cap U}(z) \) and \( B_{D \cap U}(z; \cdot) \) denote the Bergman kernel and metric of the connected component of \( D \cap U \) that contains \( z \).)

So, we may assume that \( D \) is convex.

To prove part (a) of Theorem 2, for any \( z \in D \) we choose a point \( \tilde{z} \in \partial D \) such that \( ||z - \tilde{z}|| = \text{dist}(z, \partial D) \). We denote by \( l \) the complex line through \( z \) and \( \tilde{z} \). By the Oshawa-Takegoshi extension theorem for \( L^2 \)-holomorphic functions [2], it follows that there exists a constant \( C_1 > 0 \) only depending on the diameter of \( D \) (not on \( l \)) such that
\[
K_D(z) \geq C_1 K_{D \cap l}(z).
\]
Since \( D \cap l \) is convex, it is contained in an open half-plane \( \Pi \) of the \( l \)-plane with \( \tilde{z} \in \partial \Pi \). Then
\[
K_{D \cap l}(z) \geq K_{\Pi}(z) = \frac{1}{4\pi \text{dist}^2(z, \partial \Pi)}.
\]
Now, part (a) of Theorem 2 follows from the inequalities (1), (2) and the fact that \( \text{dist}(z, \partial \Pi) \leq ||z - \tilde{z}|| = \text{dist}(z, \partial D) \).

To prove part (b) of Theorem 2, we denote by \( N(z_0) \) the complex affine space through \( z_0 \) that is orthogonal to \( L(z_0) \). Set \( E(z_0) = D \cap N(z_0) \). Note that \( E(z_0) \) is a nonempty convex set. So, part (b) of Theorem 2 will be a consequence of the following:

**Theorem 4.** Let \( z_0 \) be a boundary point of a bounded convex domain \( D \subset \mathbb{C}^n \). Then:
(i) \( \lim_{z \to z_0} B_D(z; X) = \infty \) locally uniformly in \( X \notin L(z_0) \);
(ii) for any compact set $K \subset E(z_0)$ there exists a constant $C > 0$ such that
$$B_D(z; X) \leq C||X||, \quad z \in K^0, \ X \in L(z_0),$$
where $K^0 := \{z_0 + tz : z \in K, \ 0 < t \leq 1\}$ is the cone generated by $K$.

Proof of Theorem 4. To prove (i) we will use the well-known fact that the Carathéodory metric $C_D(z; X)$ of $D$ does not exceed $B_D(z; X)$. On the other hand, we have the following simple geometric inequality [1]:
$$C_D(z; X) \geq \frac{1}{2d(z; X)},$$
where $d(z; X)$ denotes the distance from $z$ to the boundary of $D$ in the $X$-direction, i.e., $d(z; X) := \sup\{r : z + \lambda X \in D, \ \lambda \in \mathbb{C}, \ |\lambda| < r\}$. So, if we assume that (i) does not hold, then we may find a number $a > 0$ and sequences $D \supset (z_j)$, $z_j \to z_0$, $z_{j+1} \in X$ such that $d(z_j; X_j) \geq a$ which implies that for $|\lambda| \leq a$ the points $z_0 + \lambda X$ belong to $\hat{D}$ and, in view of convexity, they belong to $\partial D$. This means that $X \in L(z_0)$, a contradiction.

To prove part (ii) of Theorem 4, we may assume that $z_0 = 0$ and $L := L(0) = \{z \in \mathbb{C}^n : z_1 = \ldots = z_k = 0\}$ for some $k < n$. Then $N := N(0) = \{z \in \mathbb{C}^n : z_{k+1} = \ldots = z_n = 0\}$. From now on we will write any point $z \in \mathbb{C}^n$ in the form $z = (z', z'')$, $z' \in \mathbb{C}^k$, $z'' \in \mathbb{C}^{n-k}$. Note that $L \subseteq \partial D$ near 0, i.e., there exists a $c > 0$ such that
$$\{0'\} \times \Delta''_c \in \partial D,$$
where $\Delta''_c \subset \mathbb{C}^{n-k}$ is the polydisc with center at the origin and radius $c$. Since $K \subset E := E(0)$ and since $E$ is convex, there exists an $\alpha > 1$ such that $K \subset E_\alpha$, where $E_\alpha := \{z : az \in E\}$. Note that $K^0 \subset E_\alpha$. Using (3), the equality
$$(z', z'') = \frac{1}{\alpha}(\alpha z', 0') + (1 - \frac{1}{\alpha})(0', (1 - \frac{1}{\alpha})^{-1} z''),$$
and the convexity of $D$, it follows that
$$F_\alpha \times \Delta''_c \subset D,$$
where $\varepsilon := c(1 - \frac{1}{\alpha})$ and where $F_\alpha$ is the projection of $E_\alpha$ in $\mathbb{C}^k$ (we can identify $E_\alpha$ with $F_\alpha$). For $\delta := c(\alpha - 1)$ we obtain in the same way that
$$\hat{D} := D \cap (\mathbb{C}^k \times \Delta''_c) \subset F_\alpha \times \Delta''_\delta.$$

Now, let $(z, X) \in K^0 \times L$. Note that $z = (z', 0')$ and $X = (0', X'')$. Then, using (4) and the product properties of the Bergman kernel and metric, we have
$$M_D(z; X) \leq M_{F_\alpha \times \Delta''_c}(z; X) \leq M_{\Delta''_\delta}(0' ; X'') \sqrt{K_{F_\alpha}(z')} \leq C_1||X|| \sqrt{K_{F_\alpha}(z')}$$
for some constant $C_1 > 0$. On the other hand, since $K^0 \subset \subset \mathbb{C}^k \times \Delta''_\delta$, by virtue of Theorem 3 there exists a constant $\tilde{C} > 1$ such that
$$K_D(z) \geq \tilde{C}^{-1} K_{\hat{D}}(z).$$
Moreover, in view of (5), we have
$$K_{\hat{D}}(z) \geq K_{F_\alpha \times \Delta''_c}(z) K_{\Delta''_\delta}(0')$$
and hence
\begin{equation}
K_D(z) \geq (C_2)^2 K_{F_1}(z')
\end{equation}
for some constant $C_2 > 0$. Now, by (6) and (7), it follows that
\begin{equation}
B_D(z; X) = \frac{M_D(z; X)}{\sqrt{K_D(z)}} \leq \frac{C_1}{C_2} ||X|| \sqrt{\frac{K_{F_1}(z')}{K_{F_2}(z')}}.
\end{equation}

Note that $z' \to \alpha^{-2}z'$ is a biholomorphic mapping from $F_1$ onto $F_2$ and, therefore,
\begin{equation}
K_{F_1}(z') = \alpha^{-4k} K_{F_2}(\alpha^{-2}z').
\end{equation}

In view of (8) and (9), in order to finish (ii) we have to find a constant $C_3 > 0$ such that
\begin{equation}
K_{F_2}(z') \leq C_3 K_{F_2}(\alpha^{-2}z')
\end{equation}
for any $z' \in H^0$ with $H^0 := \{tz': z' \in H, 0 < t \leq 1\}$, where $H$ is the projection of $K$ into $\mathbb{C}^k$ (we can identify $K$ with $H$).

To do this, note first that $\gamma := \text{dist}(H, \partial F_0) > 0$ since $K \subset E_{\alpha}$. Fix $\tau \in (0, 1]$ and $z' \in H^0$, and denote by $T_{\tau, z'}$ the translation that maps the origin in the point $\tau z'$. It is easy to check that
\begin{equation}
T_{\tau, z'}(\bar{F}_\alpha \cap B_\gamma) \subset F_\alpha,
\end{equation}
where $B_\gamma$ is the ball in $\mathbb{C}^k$ with center at the origin and radius $\gamma$. To prove (10), we will consider the following two cases:

Case I. $z' \in H^0 \setminus B_2$: Then
\begin{equation}
K_{F_\alpha}(z') \leq \frac{m_1}{m_2} K_{F_\alpha}(\alpha^{-2}z'),
\end{equation}
where $m_1 := \sup_{H^0 \setminus B_2} K_{F_\alpha}$ and $m_2 := \inf K_{F_\alpha}$.

Case II. $z' \in H^0 \cap B_2$: By Theorem 3 there exists a constant $\tilde{C}_3 \geq 1$ such that
\begin{equation}
\tilde{C}_3 K_{F_\alpha} \geq K_{F_\alpha \cap B_\gamma} \text{ on } F_\alpha \cap B_2.
\end{equation}
In particular,
\begin{equation}
\tilde{C}_3 K_{F_\alpha}(\alpha^{-2}z') \geq K_{F_\alpha \cap B_\gamma}(\alpha^{-2}z').
\end{equation}

On the other hand, by (11) with data $T := T_{1-\alpha^{-2}, z'}$ it follows that
\begin{equation}
K_{F_\alpha \cap B_\gamma}(\alpha^{-2}z') = K_{T(F_\alpha \cap B_\gamma)}(z') \geq K_{F_\alpha}(z').
\end{equation}

Now, (12), (13), and (14) imply that (10) holds for $C_3 := \max\{\frac{m_1}{m_2}, \tilde{C}_3\}$. This completes the proofs of Theorem 4 and part (b) of Theorem 2. \hfill \Box

Remark. The approximation (5) of the domain $D \cap (\mathbb{C}^k \times \Delta^0_N)$ by the domain $E_1 \times \Delta^0_N$ can be replaced by using the Oshawa-Takegoshi theorem \[7\] with the data $D$ and $N$. 

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Finally, part (c) of Theorem 2 will be a consequence of the following two theorems.

**Theorem 5 [8].** Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain and let $U$ be an open neighborhood of a local (holomorphic) peak point $z_0 \in \partial D$. Then
\[
\lim_{z \to z_0} \frac{K_D(z)}{K_{D \cap U}(z)} = 1
\]
and
\[
\lim_{z \to z_0} \frac{B_D(z; X)}{B_{D \cap U}(z; X)} = 1
\]
locally uniformly in $X \subset \mathbb{C}^n \setminus \{0\}$.

**Theorem 6.** Let $z_0$ be a boundary point of a bounded convex domain $D \subset \mathbb{C}^n$. Then the following conditions are equivalent:

1. $z_0$ is a (holomorphic) peak point;
2. $z_0$ is the unique analytic curve in $\bar{D}$ containing $z_0$;
3. $L(z_0) = \{0\}$.

Note that the only nontrivial implication is (3) $\implies$ (1). It is contained in [8]. Now, part (c) of Theorem 2 is a consequence of this implication, Theorem 5, and part (b) of Theorem 2.

Proof of Theorem 6. The implication (2) $\implies$ (3) is trivial.

The implication (1) $\implies$ (2) easily follows by the maximum principle and the fact that there are a neighborhood $U$ of $z_0$ and a vector $X \subset \mathbb{C}^n$ such that $(\bar{D} \cap U) + (0, 1]X \subset D$ (cf. (11)).

Denote by $A^0(D)$ the algebra of holomorphic functions on $D$ which are continuous on $\bar{D}$. Now, following [8] we shall prove the implication (3) $\implies$ (1); namely, (3) implies that $z_0$ is a peak point with respect to $A^0(D)$. This is equivalent to the fact (cf. [3]) that the point mass at $z_0$ is the unique element of the set $A(z_0)$ of all representing measures for $z_0$ with respect to $A^0(D)$, i.e. $\text{supp } \mu = \{z_0\}$ for any $\mu \in A(z_0)$.

Let $\mu \in A(z_0)$. Since $D$ is convex, we may assume that $z_0 = 0$ and $D \subset \{z \in \mathbb{C}^n : \text{Re}(z_1) < 0\}$. Note that if $a$ is a positive number such that $a \inf_{z \in D} \text{Re}(z_1) > -1$ ($D$ is bounded), then the function $f_1(z) = \exp(z_1 + az_1^2)$ belongs to $A^0(D)$ and $|f_1(z)| < 1$ for $z \in \bar{D} \setminus \{z_1 = 0\}$. This easily implies (cf. [3]) that $\text{supp } \mu \subset D_1 := \partial D \cap \{z_1 = 0\}$. Since $L(0) = 0$, the origin is a boundary point of the compact convex set $D_1$. As above, we may assume that $D_1 \subset \{z \in \mathbb{C}^n : \text{Re}(z_2) \leq 0\}$ ($z_2$ is independent of $z_1$) and then construct a function $f_2 \in A^0(D)$ such that $|f_2(z)| < 1$ for $z \in D_1 \setminus \{z_2 = 0\}$. This implies that $\text{supp } \mu \subset D_1 \cap \{z_2 = 0\}$. Repeating this argument we conclude that $\text{supp } \mu = \{0\}$, which completes the proofs of Theorems 6 and 2. \hfill $\Box$

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