ASYMPTOTIC SPECTRAL DISTRIBUTIONS OF DISTANCE $k$-GRAPHS OF STAR PRODUCT GRAPHS.

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Abstract. Let $G$ be a finite connected graph and let $G^{[N,k]}$ be the distance $k$-graph of the $N$-fold star power of $G$. For a fixed $k \geq 1$, we show that the large $N$ limit of the spectral distribution of $G^{[N,k]}$ converges to a centered Bernoulli distribution, $1/2\delta_{-1} + 1/2\delta_1$. The proof is based in a fourth moment lemma for convergence to a centered Bernoulli distribution.

1. Introduction and Statement of Results

The interest in asymptotic aspects of growing combinatorial objects has increased in recent years. In particular, the asymptotic spectral distribution of graphs has been studied from the quantum probabilistic point of view, see Hora [10] and Hora and Obata [10]. Moreover, as observed in Accardi, A. Ben Ghorbal and Obata [3], Obata [16] and Accardi, Lenczewski and Salapata [2], the cartesian, star, rooted and free products of graphs correspond to natural independences in non-commutative probability, see [5, 14, 18]. This has lead to state central limit theorems for these product of graphs by reinterpreting the classical, free [6, 20], Boolean [19] and monotone [15] central limit theorems.

More recently, in a series of papers [8, 9, 11, 12, 13, 17] the asymptotic spectral distribution of the distance $k$-graph of the $N$-fold power of the cartesian product was studied. These investigations, finally lead to the following theorem which generalizes the central limit theorem for cartesian products of graphs.

Theorem 1.1 (Hibino, Lee and Obata [9]). Let $G = (V, E)$ be a finite connected graph with $|V| \geq 2$. For $N \geq 1$ and $k \geq 1$ let $G^{[N,k]}$ be the distance $k$-graph of $G^N = G \times \cdots \times G$ (N-fold Cartesian power) and $A^{[N,k]}$ its adjacency matrix. Then, for a fixed $k \geq 1$, the eigenvalue distribution of $N^{-k/2}A^{[N,k]}$ converges in moments as $N \to \infty$ to the probability distribution of

\begin{equation}
\left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!} \tilde{H}_k(g),
\end{equation}

where $\tilde{H}_k$ is the monic Hermite polynomial of degree $k$ and $g$ is a random variable obeying the standard normal distribution $N(0,1)$.

In this note we study the distribution (with respect to the vacuum state) of the star product of graphs. That is, we prove the analog of Theorem 1.1 by changing the cartesian product by the star product.

Theorem 1.2. Let $G = (V, E, e)$ be a locally finite connected graph and let $k \in \mathbb{N}$ be such that $G^{[k]}$ is not trivial. For $N \geq 1$ and $k \geq 1$ let $G^{[\star N,k]}$ be the distance
$k$-graph of $G^\star N = G \star \cdots \star G$ ($N$-fold star power) and $A^{[\star N,k]}$ its adjacency matrix. Furthermore, let $\sigma = V^k_1$ be the number of neighbours of $e$ in the distance $k$-graph of $G$, then the distribution with respect to the vacuum state of $(N\sigma)^{-1/2}A^{[\star N,k]}$ converges in distribution as $N \to \infty$ to a centered Bernoulli distribution. That is,

$$
\frac{A^{[\star N,k]}}{\sqrt{N\sigma}} \xrightarrow{\text{weakly}} \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1,
$$

The limit distribution above is universal in the sense that it is independent of the details of a factor $G$, but also in this case the limit does not depend on $k$. The proof of Theorem 1.2 is based in a fourth moment lemma for convergence to a centered Bernoulli distribution.

Apart from the introduction, this note is organized as follows. Section 2 is devoted to preliminaries. In Section 3 we prove a fourth moment lemma for convergence to a centered Bernoulli distribution. Finally in Section 4 we prove Theorem 1.2.

2. Preliminaries

In this section we give very basic preliminaries on graphs the Cauchy transform, Jacobi parameters and non-commutative probability. The reader familiar with these objects may skip this section.

2.1. Graphs. A directed graph or digraph is a pair $G = (V,E)$ such that $V$ is a non-empty set and $E \subseteq V \times V$. The elements of $V$ and $E$ are called the vertices and the edges of the digraph $G$, respectively. Two vertices $x, y \in V$ are adjacent, or neighbours if $(x,y) \in E$.

We call loop an edge of the form $(v,v)$ and we say that a graph is simple if has no loops. A digraph is called undirected if $(v,w) \in E$ implies $(w,v) \in E$.

We will work with simple undirected digraphs and use the word graph for a simple undirected digraph without any further reference.

The adjacency matrix of $G$ is the matrix indexed by the vertex set $V$, where $A_{xy} = 1$ when $(x,y) \in E$ and $A_{xy} = 0$ otherwise.

A path is a graph $P = (V,E)$ with vertex set $V = \{v_1, \ldots, v_k\}$ and edges $E = \{(v_1,v_2), \ldots, (v_{k-1},v_k)\}$. A walk is a path that can repeat edges. We say that a graph is connected if every pair of distinct vertices $x, y \in V$ are connected by a walk (or equivalently by a path).

In this note we focus on specific types of graphs coming from the distance $k$-graphs of the star product of finite rooted graph.

For a given graph $G = (V,E)$ and a positive integer $k$ the distance $k$-graph is defined to be a graph $G^k = (V,E^k)$ with

$$E^k = \{(x,y) : x, y \in V, \partial_G (x,y) = k\},$$

where $\partial_G (x,y)$ is the graph distance. Figure 1 shows the distance 2-graph induced by the 3 dimensionial cube.

A rooted graph is a graph with a labeled vertex $o \in V$. Finally we define the star product of rooted graphs. For $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two graph with distinguished vertices $o_1 \in V_1$ and $o_2 \in V_2$, the star product graph of $G_1$ with
Figure 1. 3-Cube and its distance 2-graph

$G_2$ is the graph $G_1 \star G_2 = (V_1 \times V_2, E)$ such that for $(v_1, w_1), (v_2, w_2) \in V_1 \times V_2$ the edge $e = (v_1, w_1) \sim (v_2, w_2) \in E$ if and only if one of the following holds:

1. $v_1 = v_2 = o_1$ and $w_1 \sim w_2$
2. $v_1 \sim v_2$ and $w_1 = w_2 = o_2$.

As we can see, the star product is a graph obtained by gluing two graphs at their distinguished vertices $o_1$ and $o_2$.

Figure 2. Star product of two cycles

2.2. The Cauchy Transform. We denote by $\mathcal{M}$ the set of Borel probability measures on $\mathbb{R}$. The upper half-plane and the lower half-plane are respectively denoted as $\mathbb{C}^+$ and $\mathbb{C}^-$.

For a measure $\mu \in \mathcal{M}$, the Cauchy transform $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ is defined by the integral

$$G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(dx)}{z - x}, \quad z \in \mathbb{C}^+$$

The Cauchy transform is an important tool in non-commutative probability. For us, the following relation between weak convergence and the Cauchy Transform will be important.

**Proposition 2.1.** Let $\mu_1$ and $\mu_2$ be two probability measures on $\mathbb{R}$ and

$$d(\mu_1, \mu_2) = \sup \{ |G_{\mu_1}(z) - G_{\mu_2}(z)| ; \Im(z) \geq 1 \}.$$  \hfill (2.1)

Then $d$ is a distance which defines a metric for the weak topology of probability measures. Moreover, $|G_\mu(z)|$ is bounded in $\{ z : \Im(z) \geq 1 \}$ by 1.
In other words, a sequence of probability measures \( \{\mu_n\}_{n \geq 1} \) on \( \mathbb{R} \) converges weakly to a probability measure \( \mu \) on \( \mathbb{R} \) if and only if for all \( z \) with \( \Im(z) \geq 1 \) we have

\[
\lim_{n \to \infty} G_{\mu_n}(z) = G_\mu(z).
\]

2.3. The Jacobi Parameters. Let \( \mu \) be a probability measure with all moments, that is \( m_n(\mu) := \int_{\mathbb{R}} x^n \mu(dx) < \infty \). The Jacobi parameters \( \gamma_m = \gamma_m(\mu) \geq 0, \beta_m = \beta_m(\mu) \in \mathbb{R} \), are defined by the recursion

\[
xP_m(x) = P_{m+1}(x) + \beta_m P_m(x) + \gamma_{m-1} P_{m-1}(x),
\]

where the polynomials \( P_{-1}(x) = 0, P_0(x) = 1 \) and \( (P_m)_{m \geq 0} \) is a sequence of orthogonal monic polynomials with respect to \( \mu \), that is,

\[
\int_{\mathbb{R}} P_m(x) P_n(x) \mu(dx) = 0 \text{ si } m \neq n.
\]

A measure \( \mu \) is supported on \( m \) points iff \( \gamma_{m-1} = 0 \) and \( \gamma_n > 0 \) for \( n = 0, \ldots, m - 2 \).

The Cauchy transform may be expressed as a continued fraction in terms of the Jacobi parameters, as follows.

\[
G_{\mu}(z) = \int_{-\infty}^{\infty} \frac{1}{z - t} \mu(dt) = \frac{1}{z - \beta_0 - \frac{\gamma_0}{z - \beta_1 - \frac{\gamma_1}{z - \beta_2 - \cdots}}}
\]

An important example for this paper is the Bernoulli distribution \( b = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_1 \) for which \( \beta_0 = 0, \gamma_0 = 1, \) and \( \beta_n = \gamma_n = 0 \) for \( n \geq 1 \). Thus, the Cauchy transform is given by

\[
G_{b}(z) = \frac{1}{z - 1/z}.
\]

In the case when \( \mu \) has \( 2n + 2 \)-moments we can still make an orthogonalization procedure until the level \( n \). In this case the Cauchy transform has the form

\[
G_{\mu}(z) = \frac{1}{z - \beta_0 - \frac{\gamma_0}{z - \beta_1 - \frac{\gamma_1}{z - \beta_2 - \cdots}}}
\]

where \( \nu \) is a probability measure.

2.4. Non-Commutative Probability Spaces. A \( C^*-\)probability space is a pair \((A, \varphi)\), where \( A \) is a unital \( C^*\)-algebra and \( \varphi : A \to \mathbb{C} \) is a positive unital linear functional. The elements of \( A \) are called (non-commutative) random variables. An element \( a \in A \) such that \( a = a^* \) is called self-adjoint.

The functional \( \varphi \) should be understood as the expectation in classical probability.

For \( a_1, \ldots, a_k \in A \), we will refer to the values of \( \varphi(a_{i_1} \cdots a_{i_n}) \), \( 1 \leq i_1, \ldots, i_n \leq k, n \geq 1 \), as the mixed moments of \( a_1, \ldots, a_k \).

For any self-adjoint element \( a \in A \) there exists a unique probability measure \( \mu_a \) (its spectral distribution) with the same moments as \( a \), that is,

\[
\int_{\mathbb{R}} x^k \mu_a(dx) = \varphi(a^k), \quad \forall k \in \mathbb{N}.
\]
We say that a sequence \( a_n \in \mathcal{A}_n \) converges in distribution to \( a \in \mathcal{A} \) if \( \mu_{a_n} \) converges in distribution to \( \mu_a \).

In this note we will only consider the \( C^* \)-probability spaces \((\mathcal{M}_n, \phi_1)\), where \( \mathcal{M}_n \) is the set of matrices of size \( n \times n \) and for a matrix \( M \in \mathcal{M}_n \) the functional \( \phi_1 \) evaluated in \( M \) is given by

\[
\phi_1(M) = M_{11}.
\]

Let \( G = (V, E, 1) \) be a finite rooted graph with vertex set \( \{1, ..., n\} \) and let \( A_G \) be the adjacency matrix. We denote by \( A(G) \subset \mathcal{M}_n \) be the adjacency algebra, i.e., the \( * \)-algebra generated by \( A_G \).

It is easy to see that the \( k \)-th moment of \( A \) with respect to the \( \phi_1 \) is given the number of walks in \( G \) of size \( k \) starting and ending at the vertex 1. That is,

\[
\phi_1(A^k) = |\{(v_1, ..., v_k) : v_1 = v_k = 1 \text{ and } (v_i, v_{i+1}) \in E\}|.
\]

Thus one can get combinatorial information of \( G \) from the values of \( \phi_1 \) in elements of \( A(G) \) and vice versa.

### 3. The Fourth Moment Lemma

The following lemma which shows that the first, second and fourth moments are enough to ensure convergence to a Bernoulli distribution was observed in [3]. We give a new proof in terms of Jacobi parameters for the convenience of the reader.

**Lemma 3.1.** Let \( \{X_n\}_{n \geq 1} \subset (\mathcal{A}, \varphi) \) be a sequence of self-adjoint random variables in some non-commutative probability space, such that \( \varphi(X_n) = 0 \) and \( \varphi(X_n^2) = 1 \). If \( \varphi(X_n^4) \to 1 \), as \( n \to \infty \), then \( \mu_{X_n} \) converges in distribution to a symmetric Bernoulli random variable \( b \).

**Proof.** Let \( \{(\gamma_i(\mu_{X_n})) , (\beta_i(\mu_{X_n}))\} \) be the Jacobi parameters of the measures \( \mu_{X_n} \). The first moments \( \{m_n\}_{n \geq 1} \) are given in terms of the Jacobi Parameters as follows, see [4].

\[
\begin{align*}
m_1 &= \beta_0 \\
m_2 &= \beta_0^2 + \gamma_0 \\
m_3 &= \beta_0^3 + 2\beta_0 \gamma_0 + \beta_1 \gamma_0 \\
m_4 &= \beta_0^4 + 3\beta_0^2 \beta_1 + 2\beta_0 \beta_1 \gamma_0 + \beta_1^2 \gamma_0 + \gamma_0^2 + \gamma_0 \gamma_1.
\end{align*}
\]

Since \( m_1(\mu_{X_n}) = 0 \) and \( m_2(\mu_{X_n}) = 1 \) we have

\[
\beta_0(\mu_{X_n}) = 0 \quad \text{and} \quad \gamma_0(\mu_{X_n}) = 1 \quad \forall n \geq 1,
\]

Hence,

\[
m_4(\mu_{X_n}) = \beta_1^2(\mu_{X_n}) + 1 + \gamma_1(\mu_{X_n}).
\]

Now, since \( m_4(\mu_{X_n}) \) \( 1 \) and \( \gamma_1 \geq 0 \) we have the convergence

\[
\beta_1(\mu_{X_n}) \to 0 \quad \text{and} \quad \gamma_1(\mu_{X_n}) \to 0.
\]

Let \( G_{\mu_n} \) be the Cauchy transform of \( \mu_n \). By [2.2] we can expand \( G_{\mu} \) as a continued fraction as follows

\[
G_{\mu_n}(z) = \frac{1}{z - \frac{1}{z - \beta_1 - \gamma_1 G_{\nu_n}(z)}}
\]
where $\nu_n$ is some probability measure. Now, recall that $|G_{\nu_n}(z)|$ is bounded by 1 in the set $\{z; \Im(z) \geq 1\}$ and thus, since $\gamma_1 \to 0$ and $\beta_1 \to 0$ we see that $\gamma_n G_{\nu_n}(z) \to 0$. This implies the point-wise convergence

$$ G_{\mu_n}(z) \to \frac{1}{z - \frac{1}{z}} $$

in the set $\{z; \Im(z) \geq 1\}$, which then implies the weakly convergence $\mu_n \to b$. \hfill \Box

From the proof of the previous lemma one can give a quantitative version in terms of the distance given in eq (2.1).

**Proposition 3.2.** Let $\mu$ be a probability measure such that $m_4 := m_4(\mu)$ is finite. Then

$$ d(\mu, \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}) \leq 4\sqrt{m_4 - 1}. $$

**Proof.** If $m_4 - 1 > 1/16$ then the statement is trivial since $d(\mu, 1/2\delta_1 + 1/2\delta_{-1}) \leq 1$ for any measure $\mu$. Thus we may assume that $(m_4 - 1) \leq 1/16$.

Denoting by $f(z) = \beta_1 - \gamma_1 G_{\nu_n}(z)$ we have

$$ |G_\mu(z) - G_b(z)| = \left| \frac{1}{z - \frac{1}{z}} - \frac{1}{z - \frac{1}{z} - f(z)} \right| = \left| \frac{f(z)}{(z^2 - 1)(z^2 - 1 - f(z)z)} \right|. $$

From (3.1) we get the inequalities $\sqrt{m_4 - 1} \geq |\beta_1|$ and $\sqrt{m_4 - 1} \geq m_4 - 1 \geq \gamma_1$. Since, for $\Im(z) > 1$, we have that, $|G_\nu(z)| < 1$ we see that $|f(z)| = |\beta_1 - \gamma_1 G_\nu(z)| \leq 2\sqrt{m_4 - 1} \leq 1/2$, from where we can easily obtain the bound $\frac{1}{(z^2 - 1 - f(z)z)} \leq 2$. Also, for $\Im(z) > 0$ we have the bound $\frac{1}{(z^2 - 1)} < 1$. Thus we have

\begin{align*}
(3.2) \quad |G_\mu(z) - G_b(z)| &= \frac{f(z)}{(z^2 - 1)(z^2 - 1 - f(z)z)} \\
(3.3) &= |f(z)| \left| \frac{1}{(z^2 - 1)} \right| \left| \frac{1}{z^2 - 1 - f(z)z} \right| \\
(3.4) &\leq 2|f(z)| \leq 4\sqrt{m_4 - 1}.
\end{align*}

as desired. \hfill \Box

### 4. Proof of Theorem 1.2

Before proving Theorem 1.2 we will prove a lemma about the structure of distance $k$-graph of the iterated star product of a graph.

**Lemma 4.1.** Let $G = (V, E, e)$ be a connected finite graph with root $e$ and $k$ such that $G^{[k]}$ is a non-trivial graph. Let $G^{*N[k]}$ be the distance $k$-graph of the $N$-th star product of $G$, then $G^{*N[k]}$ admits a decomposition of the form

$$ G^{*N[k]} = (G^{[k]})^{*N} \cup \hat{G}. $$

where $\partial G(z, e) < k$, for all $z \in \hat{G}$.
Proof. Let \( G_1, G_2, \ldots, G_N \) be the \( N \) copies of \( G \), that form the star product graph \( G^{*N} \) by gluing them at \( e \). For \( x, y \in G_i \), the distance between \( x \) and \( y \) is given by
\[
\partial_{G^{*N}}(x, y) = \partial_{G_i}(x, y) = \partial_{G}(x, y),
\]
hence
\[
(x, y) \in E\left(G^{[k]}_i \right) \text{ if and only if } (x, y) \in E\left((G^{*N})^{[k]} \right),
\]
therefore we have \( (G^{[k]}_i)^{*N} \subseteq (G^{*N})^{[k]} \).

Now, if \( x \in G_i \) and \( y \in G_j \) with \( j \neq i \), by definition all the paths in \( G^{*N} \) from \( x \) to \( y \) must pass throw \( e \), then we have
\[
\partial_{G^{*N}}(x, y) = \partial_{G_i}(x, e) + \partial_{G_j}(y, e),
\]
thus
\[
(x, y) \in E\left((G^{*N})^{[k]} \right) \text{ if and only if } \partial_{G_i}(x, e) + \partial_{G_j}(y, e) = k.
\]
Since \( \partial_{G_i}(x, e), \partial_{G_j}(y, e) > 0 \), we obtain the desired result. \( \square \)

Now, we are in position to prove the main theorem of the paper.

Proof of Theorem 1.2. Consider the non-commutative probability space \((A, \phi_1)\) with \( \phi_1(M) = M_{11} \), for \( M \in A \) (see Section 2). Then, recall that, if \( A \) is an adjacency matrix, \( \phi_1(A^k) \) equals the number of walks of size \( k \) starting and ending at the vertex 1.

Since \( G \) is a simple graph, it has no loops and then \( G^{*N} \) is also a simple graph. Thus,
\[
\phi_1\left(\frac{A^{[N,k]}}{\sqrt{N|V_e^{[k]}|}}\right) = 0.
\]

Now, observe that since the graph \( G^{*N} \) has no loops, the only walks in \( G \) of size 2 which start in \( e \) and end in \( e \) are of the form \((exe)\), where \( x \) is a neighbor of \( e \) in \((G^{*N})^{[k]} \). The number of neighbors of \( e \) is exactly \( N|V_e^{[k]}| \), thus
\[
\phi_1\left(\frac{A^{[N,k]}}{\sqrt{N|V_e^{[k]}|}}\right)^2 = \frac{1}{N|V_e^{[k]}|} \phi_1\left(\left(\frac{A^{[N,k]}}{\sqrt{N|V_e^{[k]}|}}\right)^2\right) = \frac{1}{N|V_e^{[k]}|} N|V_e^{[k]}| = 1.
\]

Thus we have seen that \( \phi(A_N) = 0 \) and \( \phi(A_N^2) = 1 \). Hence, it remains to show that \( \phi(A_N^4) \rightarrow 1 \) as \( N \rightarrow \infty \).

We are interested in counting the number of walk of size 4 that start and finish at \( e \) in \((G^{*N})^{[k]} \). We will divide this walks in two types.

Type 1. The first type of walk is of the form \( exe \). That is, the walk starts at \( e \), then visits a neighbor \( x \) of \( e \) to then come back to \( e \), this can be done in \( N|V_e^{[k]}| \) ways. After this, he again visits a a neighbour \( y \) (which could be again \( x \)) of \( e \) to
finally come back to \( e \). Again, this second step can be done in \( N|V_e^{[k]}| \) different ways, so there is \( \left( N|V_e^{[k]}| \right)^2 \) walks of this type. Thus

\[
1 = \frac{\left( N|V_e^{[k]}| \right)^2}{\left( N|V_e^{[k]}| \right)^2} \leq \phi_1 \left( \left( \frac{A^{[\ast,N,k]}}{\sqrt{N|V_e^{[k]}|}} \right)^4 \right).
\]

**Type 2.** Let \( G_x^{[k]} \) be the copy of \( G^{[k]} \) in the distance-\( k \) graph of the star product \((G^{[k]})^{\ast N}\) which contains \( x \). The second type of walks is as follows. From \( e \) is goes to some \( x \in V_e^{[k]} \) (which can be chosen in \( N|V_e^{[k]}| \) different ways), and then from \( x \) then he goes to some \( y \in V_e^{[k]} \) (which can be chosen in \( N|V_e^{[k]}| \) different ways), and then from \( y \) then he goes to some \( y' \in V_e^{[k]} \). This \( y' \) should belong to \( G_x^{[k]} \). Indeed, since \( \delta_{(G^{\ast N})}\left( e, x \right) = k \), if \( y' \) would be in another copy of \( G^{[k]} \) the distance \( \delta_{(G^{\ast N})}\left( y, x \right) \) between \( y \) and \( x \) would be bigger than \( k \). The number of ways of choosing \( y \) is bounded by the number of neighbours of \( x \).

For the next step of the walk, from \( y \) we can only go to a neighbor of \( e \), say \( z \in V_e^{[k]} \) (since in the last step it must come back to \( e \)). This \( z \) indeed must also belong to \( G_x^{[k]} \). If this wouldn’t be the case and \( z \notin G_x^{[k]} \), then we would have that \( \delta_{(G^{\ast N})}\left( e, z \right) \neq k \), which is a contradiction because of Lemma \ref{lem:4.1}.

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**Figure 3.** Types of walks of size 4

**Figure 4.** Obstructions
Finally, let $M = \max_{x \in V} |V_x^{[k]}|$. Then, from the above considerations we see that the number of walks of Type 2 is bounded by $M \left( N |V_x^{[k]}| \right)^2 \left( |V_x^{[k]}| \right)^2 \leq \left( \frac{N |V_x^{[k]}|}{N |V_x^{[k]}|} \right)^4 + \frac{N |V_x^{[k]}| M |V_x^{[k]}|}{N |V_x^{[k]}|} = 1 + \frac{M}{N} \rightarrow_{N \to \infty} 1,$

since $M$ does not depend on $N$. Thanks to Lemma 3.1 we obtain the desired result.

□

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