Linear Matrix Inequality Approaches to Koopman Operator Approximation

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1 Introduction

Koopman operator theory [1–4] provides a means to globally represent a nonlinear system as a linear system by transforming its states into an infinite-dimensional space of lifted states. The Koopman operator advances the current lifted state of the system to the next lifted state, much like the state transition matrix of a linear system. While originally proposed by B. O. Koopman in 1931 [1], modern computational resources, along with recent theoretical developments [2–4], have led to a resurgence of interest in using data-driven methods to approximate the Koopman operator. In general, the Koopman representation of a nonlinear system is infinite-dimensional. One way to approximate the Koopman operator in finite dimensions is to select a set of lifted states and use linear regression to find a matrix approximation of the Koopman operator, also called a Koopman matrix [5, 6]. The Koopman representation of a nonlinear system is particularly convenient for control systems design, as its linear representation of nonlinear systems is compatible with a wide variety of existing linear optimal control techniques [6–11].

The regression problem associated with finding an approximate Koopman operator is numerically challenging, requiring regularization techniques, such as Tikhonov regularization [12] or the lasso [13], to find a suitable solution. The novelty of this document is the reformulation of the Koopman matrix regression problem as a convex optimization problem with linear matrix inequality (LMI) constraints and the use of additional LMIs to, for instance, regularize the optimization problem. In particular, regularizers with LMI forms, such as the matrix two-norm or the $\mathcal{H}_\infty$ norm, can be added to the optimization problem in a modular fashion. Additional stability constraints can also be added in the same way. Although convex optimization and LMIs have previously been used to synthesize controllers for Koopman models [11], these tools have not yet been leveraged when solving the regression problem associated with finding the Koopman matrix. In [14], a related
optimization problem is posed where both the Koopman matrix and lifting functions are unknown. While the problem is NP-hard, a convex relaxation allows both to be found by solving two semidefinite programs.

A particular novelty of this document is solving the Koopman regression problem with a system norm regularizer. Although this document explores the use of the $H_\infty$ norm [15, §3.2] as a regularizer, any system norm can be used, such as the $H_2$ norm [15, §3.3] or a mixed $H_2$ norm [15, §3.5]. This systems perspective on the regression problem is a natural fit with the Koopman operator because the Koopman matrix describes the time evolution of the data associated with a dynamic system. While the $H_\infty$ norm of the Koopman operator has previously been considered in [16], it is in the form of a hard constraint on the system’s dissipativity. The use of a system norm to regularize an optimization problem enables a systems interpretation of the entire regularization procedure. For example, using a system norm as a regularizer enables the use of weighting functions that can explicitly penalize system gain in a particular frequency band.

This document focuses on the formulation of the Koopman matrix regression problem using convex optimization and LMIs, and demonstrates how LMIs can be leveraged to regularize or enforce additional constraints. This document does not present any numerical results, which are ongoing and will be continued in the future.

2 Koopman operator background

2.1 Koopman operator theory

Consider the discrete-time nonlinear process

$$x_{k+1} = f(x_k),$$

(1)

where $x_k \in \mathcal{M}$ evolves on a smooth manifold $\mathcal{M} \subseteq \mathbb{R}^{m \times 1}$, which is often just the entirety of $\mathbb{R}^{m \times 1}$.

Let $\psi : \mathcal{M} \to \mathbb{R}$ be a lifting function, where $\psi \in \mathcal{H}$. Any function of $x_k$ that returns a scalar is a lifting function. There are therefore infinitely many lifting functions, and they form a Hilbert space $\mathcal{H}$. The Koopman operator $U : \mathcal{H} \to \mathcal{H}$ is a linear operator that advances all scalar-valued lifting functions in time by one timestep. That is [5, §3.2],

$$(U \psi)(\cdot) = (\psi \circ f)(\cdot).$$

(2)

Using the Koopman operator, the dynamics of (1) may then be rewritten linearly in terms of $\psi$ as

$$\psi(x_{k+1}) = (U \psi)(x_k).$$

(3)

In finite dimensions, (3) is approximated by

$$\psi(x_{k+1}) = U \psi(x_k) + r_k,$$

(4)

where $\psi : \mathcal{M} \to \mathbb{R}^{p \times 1}$, $U \in \mathbb{R}^{p \times p}$, and $r_k$ is the residual error. Since each element of $\psi$ is a member of $\mathcal{H}$, $\psi$ is called a vector-valued lifting function. The Koopman matrix $U$ is a matrix approximation of the Koopman operator.
2.2 Koopman operator theory with inputs

If the discrete-time nonlinear process has exogenous inputs, the definitions of the lifting functions and Koopman operator must be adjusted. Consider
\[ x_{k+1} = f(x_k, u_k), \]
where \( x_k \in \mathcal{M} \subseteq \mathbb{R}^{m \times 1} \) and \( u_k \in \mathcal{N} \subseteq \mathbb{R}^{n \times 1} \).

In this case, the lifting functions become \( \psi : \mathcal{M} \times \mathcal{N} \rightarrow \mathbb{R}, \psi \in \mathcal{H} \) and the Koopman operator \( \mathcal{U} : \mathcal{H} \rightarrow \mathcal{H} \) is instead defined so that
\[ (\mathcal{U}\psi)(x_k, u_k) = \psi(f(x_k, u_k), \star), \]
where \( \star = u_k \) if the input has state-dependent dynamics, or \( \star = 0 \) if the input has no dynamics [5, §6.5]. The input is state-dependent if it is computed by a controller.

Let the vector-valued lifting function \( \psi : \mathcal{M} \times \mathcal{N} \rightarrow \mathbb{R}^{p \times 1} \) be partitioned as
\[ \psi(x_k, u_k) = \begin{bmatrix} \vartheta(x_k) \\ \upsilon(x_k, u_k) \end{bmatrix}, \]
where \( \vartheta : \mathcal{M} \rightarrow \mathbb{R}^{p_{\vartheta} \times 1}, \upsilon : \mathcal{M} \times \mathcal{N} \rightarrow \mathbb{R}^{p_{\upsilon} \times 1} \), and \( p_{\vartheta} + p_{\upsilon} = p \). In the case where the input has no dynamics, (6) has the form [5, §6.5.1]
\[ \vartheta(x_{k+1}) = U \psi(x_k, u_k) + r_k, \]
where
\[ U = \begin{bmatrix} A & B \end{bmatrix}. \]
When expanded, this yields the familiar linear state space form,
\[ \vartheta(x_{k+1}) = A \vartheta(x_k) + B \upsilon(x_k, u_k) + r_k. \]

2.3 Approximating the Koopman operator

To approximate the Koopman matrix from data, consider a dataset \( \mathcal{D} = \{x_k, u_k\}_{k=0}^q \) and the corresponding lifted snapshot matrices
\[ \Psi = [\psi_0 \ \psi_1 \ \cdots \ \psi_{q-1}] \in \mathbb{R}^{p \times q}, \]
\[ \Theta_+ = [\vartheta_1 \ \vartheta_2 \ \cdots \ \vartheta_q] \in \mathbb{R}^{p_{\vartheta} \times q}, \]
where \( \psi_k = \psi(x_k, u_k) \) and \( \vartheta_k = \vartheta(x_k) \). Note that time-shifted input snapshots are not required.

The Koopman matrix that minimizes
\[ J(U) = \|\Theta_+ - U \Psi\|_F^2 \]
is therefore [5, §1.2.1]
\[ U = \Theta_+ \Psi^\dagger, \]
where \((\cdot)^\dagger\) denotes the Moore-Penrose pseudoinverse.
2.4 Extended DMD

Extended Dynamic Mode Decomposition (EDMD) [17] is a method to compute (14) that reduces the computational cost when the number of snapshots is much larger than the dimension of the lifted state (i.e., \( p \ll q \)) [5, §10.3.1]. Specifically, it reduces the size of the pseudoinverse required.

Consider the least-squares solution for the Koopman matrix,

\[
U = \Theta_+ \Psi^T \\
= \Theta_+ 1 \Psi^T \\
= \Theta_+ (\Psi^T \Psi^T) \Psi^T \\
= (\Theta_+ \Psi^T)(\Psi \Psi^T)^T \\
= GH^\dagger, 
\]

where

\[
G = \frac{1}{q} \Theta_+ \Psi^T \in \mathbb{R}^{p \times p}, \\
H = \frac{1}{q} \Psi \Psi^T \in \mathbb{R}^{p \times p}. 
\]

Since \( p \ll q \), EDMD greatly reduces the dimension of the pseudo-inverse operation required to compute \( U \) [5, §10.3.1]. To improve numerical conditioning, \( G \) and \( H \) are often scaled by the number of snapshots \( q \), as in (20) and (21).

2.5 Extended DMD with Tikhonov regularization

Tikhonov regularization [12, 18], which penalizes the Frobenius norm of the unknown matrix in a linear regression problem, can be used to improve the condition number of \( H \) in (21). Consider the regularized EDMD cost function,

\[
\begin{align*}
J(U; \alpha) &= \frac{1}{q} \| \Theta_+ - U \Psi \|_F^2 + \frac{\alpha}{q} \| U \|_F^2 \\
&= \frac{1}{q} \text{tr} \left( (\Theta_+ - U \Psi)(\Theta_+ - U \Psi)^T \right) + \frac{\alpha}{q} \text{tr}(U U^T) \\
&= \text{tr} \left( \frac{1}{q} \Theta_+ \Theta_+^T - G U^T - U G^T + U \left( H + \frac{\alpha}{q} 1 \right) U^T \right).
\end{align*}
\]

Comparing (24) with the unregularized cost function, given by

\[
J(U) = \text{tr} \left( \frac{1}{q} \Theta_+ \Theta_+^T - G U^T - U G^T + U H U^T \right),
\]

demonstrates that

\[
U = GH_{\alpha}^\dagger 
\]

minimizes the Tikhonov-regularized cost function, where

\[
H_{\alpha} = H + \frac{\alpha}{q} 1.
\]
3 Reformulating EDMD using LMIs

3.1 Reformulating the regression problem

To add other regularizers in a modular fashion, the Koopman operator regression problem is reformulated as a convex optimization problem with LMI constraints. Recall that the Koopman matrix $U$ minimizes

$$J(U) = \|\Theta_+ - U\Psi\|^2_F.$$  

This cost function can be rewritten as a convex optimization problem with linear matrix inequality (LMI) constraints. Specifically, consider

$$J(U) = \frac{1}{q}\|\Theta_+ - U\Psi\|^2_F,$$

$$= \frac{1}{q}\text{tr}\left( (\Theta_+ - U\Psi)(\Theta_+ - U\Psi)^T \right),$$

$$= \frac{1}{q}\text{tr}(\Theta_+\Theta_+^T) - \frac{1}{q}\text{tr}(\Theta_+\Psi^TU^T + U\Psi\Theta_+^T) + \frac{1}{q}\text{tr}(U\Psi\Psi^TU^T),$$

$$= \frac{1}{q}\text{tr}(\Theta_+\Theta_+^T) - 2\text{tr}\left( U \left( \frac{1}{q}\Psi\Theta_+^T \right) \right) + \text{tr}\left( U \left( \frac{1}{q}\Psi\Psi^T \right) \right)^T.$$

$$= c - 2\text{tr}(UG^T) + \text{tr}(UHU^T),$$

where $c$ is a scalar constant, $G$ is defined in (20), and $H$ is defined in (21).

The minimization of (33) is equivalent to the minimization of

$$J(U, \nu, W) = c - 2\text{tr}(UG^T) + \nu$$

subject to

$$\text{tr}(W) < \nu,$$

$$W > 0,$$

$$UHU^T < W,$$

where $\nu$ and $W$ are slack variables that allow the cost function to be rewritten using LMIs [15, §2.15.1]. Using the Schur complement [15, §2.3.1], the quadratic term (37) can be rewritten as

$$W - UHU^T > 0 \iff \begin{bmatrix} W & U \\ U^T & H^{-1} \end{bmatrix} > 0, \ H > 0.$$  

Note that $H = H^T > 0$ if the columns of $\Psi$ are linearly independent. Minimizing (13) is therefore equivalent to

$$\min J(U, \nu, W) = c - 2\text{tr}(UG^T) + \nu$$

s.t. $\text{tr}(W) < \nu,$

$$W > 0,$$

$$\begin{bmatrix} W & U \\ U^T & H^{-1} \end{bmatrix} > 0.$$

Both the objective function and constraints are convex, and $U$ appears linearly in all of them.
3.2 Introducing Tikhonov regularization

Recall the Tikhonov-regularized cost function,

\[ J(U; \alpha) = \| \Theta - U \Psi \|_F^2 + \alpha \| U \|_F^2. \]  

(43)

Its equivalent LMI form is

\[
\begin{aligned}
\min J(U, \nu, W; \alpha) &= c - 2 \text{tr}(UG^T) + \nu \\
\text{s.t.} \ & \ \text{tr}(W) < \nu, \\
W > 0, \\
\begin{bmatrix}
W & U \\
U^T & H^-1
\end{bmatrix} > 0,
\end{aligned}
\]

(47)

where, \( H \) has simply been replaced with \( H_\alpha \) in (47).

3.3 Avoiding matrix inversion

Computing the inverse of \( H_\alpha \) in (47) is numerically problematic and can be avoided using a matrix decomposition to split \( H_\alpha \) into

\[ H_\alpha = L_\alpha L_\alpha^T. \]

(48)

The matrix \( L_\alpha \) can be found using a Cholesky decomposition or eigendecomposition of \( H_\alpha \), or a singular value decomposition of \( \Psi \). Assuming this decomposition has been performed, the quadratic term in the optimization problem becomes

\[
W - UH_\alpha U^T = W - UL_\alpha L_\alpha^T U^T \\
= W - (UL_\alpha) 1 (UL_\alpha)^T.
\]

(49)

(50)

Applying the Schur complement [15, §2.3.1] once again yields a new form of (47),

\[
\begin{bmatrix}
W & UL_\alpha \\
L_\alpha^T U & 1
\end{bmatrix} > 0.
\]

(51)

This form trades off a matrix inverse for a matrix decomposition. The new optimization problem without matrix inversion is

\[
\begin{aligned}
\min J(U, \nu, W; \alpha) &= c - 2 \text{tr}(UG^T) + \nu \\
\text{s.t.} \ & \ \text{tr}(W) < \nu, \\
W > 0, \\
\begin{bmatrix}
W & UL_\alpha \\
L_\alpha^T U & 1
\end{bmatrix} > 0,
\end{aligned}
\]

(52)

(53)

(54)

(55)

where \( H_\alpha = L_\alpha L_\alpha^T \). This formulation of the optimization problem is almost always preferable to the formulation that requires inverting \( H_\alpha \).
Two possible methods to compute $L_\alpha$ are presented, but any suitable matrix decomposition can be used. Using the eigendecomposition

$$H_\alpha = V\Lambda V^T$$

$$= V\sqrt{\Lambda} \sqrt{\Lambda} V^T,$$  \hspace{1cm} (56)

it follows that

$$L_\alpha = V\sqrt{\Lambda}.$$  \hspace{1cm} (58)

Another option is to leverage the singular value decomposition of $\Psi$, 

$$\Psi = Q\Sigma Z^T.$$  \hspace{1cm} (59)

Recalling that $H = \frac{1}{q} \Psi \Psi^T$, the matrix $H_\alpha$ then becomes

$$H_\alpha = \frac{1}{q} Q \Sigma^2 Q^T + \frac{\alpha}{q} I$$

$$= \frac{1}{q} Q \Sigma^2 Q^T + Q \left( \frac{\alpha}{q} I \right) Q^T$$

$$= Q \left( \frac{1}{q} \Sigma^2 + \frac{\alpha}{q} I \right) Q^T.$$ \hspace{1cm} (61)

The matrix $L_\alpha$ is therefore

$$L_\alpha = Q \sqrt{\frac{1}{q} \Sigma^2 + \frac{\alpha}{q} I}.$$ \hspace{1cm} (63)

Note that the matrix square root in (63) is easy to compute, as its radicand is diagonal.

Tikhonov regularization can be added to any cost function in this document by substituting $H$ and $L$ for $H_\alpha$ for $L_\alpha$. This allows for mixed regularization in the style of the elastic net [19]. However, for the sake of brevity, this is not shown.

4 Matrix norm regularization

4.1 Matrix two-norm regularization

Matrix two-norm regularization has an LMI form that can be easily incorporated into the optimization problem. The regularized cost function is

$$J(U; \beta) = \| \Theta_+ - U\Psi \|_F^2 + \beta \| U \|_2,$$ \hspace{1cm} (64)

where $\beta$ is the regularization coefficient. The matrix two-norm of a matrix is its maximum singular value. That is,

$$\| U \|_2 = \sqrt{\lambda(U^T U)}$$

$$= \sigma(U),$$ \hspace{1cm} (65) \hspace{1cm} (66)
where $\bar{\lambda}(\cdot)$ is the maximum eigenvalue and $\bar{\sigma}(\cdot)$ is the maximum singular value.

Consider the modified optimization problem

$$
\min J(U, \gamma; \beta) = \frac{1}{q} \|\Theta_+ - U\Psi\|_F^2 + \frac{\beta}{q} \gamma \\
\text{s.t. } \bar{\sigma}(U) < \gamma.
$$

(67)

(68)

The constraint (68) can be rewritten as [15, §2.11.1]

$$
\begin{bmatrix}
\gamma 1 & U \\
U^T & \gamma 1
\end{bmatrix} > 0.
$$

(69)

It follows that the optimization problem

$$
\min J(U, \nu, W, \gamma; \beta) = c - 2\text{tr}(UG^T) + \nu + \frac{\beta}{q} \gamma \\
\text{s.t. } \text{tr}(W) < \nu, \\
W > 0, \\
\begin{bmatrix}
W & UL \\
L^T U^T & 1
\end{bmatrix} > 0, \\
\begin{bmatrix}
\gamma 1 & U \\
U^T & \gamma 1
\end{bmatrix} > 0,
$$

(70)

(71)

(72)

(73)

(74)

where $H = LL^T$, is equivalent to minimizing (64).

### 4.2 Nuclear norm regularization

Nuclear norm regularization [20, 21] can be incorporated to favour low-rank Koopman operators. The regularized cost function is

$$
J(U; \beta) = \|\Theta_+ - U\Psi\|_F^2 + \beta \|U\|_*,
$$

(75)

where $\beta$ is the regularization coefficient. The nuclear norm of a matrix is defined as

$$
\|U\|_* = \text{tr} \left( \sqrt{U^T U} \right) \\
= \sum_{i=0}^{p \times p - 1} \sigma_i(U),
$$

(76)

(77)

where $\sigma_i(\cdot)$ is the $i$th singular value. Recall that $U \in \mathbb{R}^{p \times p}$. The solution to the optimization problem

$$
\min \|U\|_*
$$

(78)

is equivalent to the solution to the optimization problem [20] [15, §2.11.6]

$$
\min \frac{1}{2} (\text{tr}(V_1) + \text{tr}(V_2)) \\
\text{s.t. } \begin{bmatrix}
V_1 & U \\
U^T & V_2
\end{bmatrix} \geq 0,
$$

(79)

(80)
where $V_1 = V_1^T$ and $V_2 = V_2^T$. It follows that the optimization problem

$$\begin{align*}
\min J(U, \nu, W, \gamma, V_1, V_2; \beta) &= c - 2\text{tr}(UG^T) + \nu + \frac{\beta}{q} \gamma \\
\text{s.t.} \quad &\text{tr}(W) < \nu, \\
& W > 0, \\
& \begin{bmatrix} W & UL \\
L^T U & 0 \end{bmatrix} > 0, \\
& \text{tr}(V_1) + \text{tr}(V_2) \leq 2\gamma, \\
& \begin{bmatrix} V_1 & U \\
U^T & V_2 \end{bmatrix} \geq 0, \\
& H = LL^T,
\end{align*}$$

where $H = LL^T$, is equivalent to minimizing (75).

## 5 Asymptotic stability constraint

To ensure that all eigenvalues associated with the matrix $A$, where $U = [A \ B]$, have magnitude strictly less than one, thus ensuring asymptotic stability, a modified Lyapunov constraint [22, §1.4.4]

$$\begin{align*}
P > 0, \\
A^T PA - \rho^2 P &< 0, \\
\end{align*}$$

can be added to ensure that the magnitude of the largest eigenvalue of $A$ is no larger than $0 < \bar{\rho} < 1$. Applying the Schur complement to (88) yields

$$\begin{align*}
A^T PA - \rho^2 P &< 0 \iff (A^T P)^{-1} (A^T P)^T - \rho^2 P < 0 \\
&\iff (A^T P)^{-1} (A^T P)^T - \rho P < 0 \\
&\iff \begin{bmatrix} -\rho P & A^T P \\
P^T A & -\rho P \end{bmatrix} < 0, -\rho P < 0 \\
&\iff \begin{bmatrix} \rho P & A^T P \\
P^T A & \rho P \end{bmatrix} > 0, P > 0.
\end{align*}$$

The full optimization problem with asymptotic stability constraint is therefore

$$\begin{align*}
\min J(U, \nu, W, P; \bar{\rho}) &= c - 2\text{tr}(UG^T) + \nu \\
\text{s.t.} \quad &\text{tr}(W) < \nu, \\
&W > 0, \\
& \begin{bmatrix} W & UL \\
L^T U & 0 \end{bmatrix} > 0, \\
&P > 0, \\
& \begin{bmatrix} \rho P & A^T P \\
P^T A & \rho P \end{bmatrix} > 0,
\end{align*}$$
where $H = LL^T$ and $U = \begin{bmatrix} A & B \end{bmatrix}$.

Since both $A$ and $P$ are unknown, this optimization problem is bilinear, and can be solved iteratively by holding either $A$ or $P$ fixed while solving for the other. Iteration must be performed until the cost function stops changing significantly.

## 6 System norm regularization

A system norm like the $H_\infty$ norm, the $H_2$ norm, or a mixed $H_2$ norm can be used to regularize the Koopman regression problem when it is posed as in Section 3. The use of the $H_\infty$ norm as a regularizer when finding the Koopman matrix via regression is explored next. Minimizing the $H_\infty$ norm guarantees that the resulting LTI system will be asymptotically stable, and allows the regularization problem to be tuned in the frequency domain with weighting functions.

The Koopman representation of a nonlinear ODE can be thought of as a discrete-time LTI system $\mathcal{G} : \ell_2 \rightarrow \ell_2$, where $\ell_2$ is the extended inner product sequence space [23], $U = \begin{bmatrix} A & B \end{bmatrix}$, $C = 1$, and $D = 0$. That is,

$$\mathcal{G} \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad (99)$$

where $\sim$ denotes a minimal state space realization [24, §16.9.16]. The $H_\infty$ norm of $\mathcal{G}$ is the worst-case gain from $\|u\|_2$ to $\|G u\|_2$. That is [15, §3.2.2],

$$\|G\|_\infty = \sup_{u \in \ell_2, u \neq 0} \frac{\|G u\|_2}{\|u\|_2}. \quad (100)$$

With $H_\infty$ norm regularization, the cost function associated with the regression problem is

$$J(U; \beta) = \|\Theta_+ - U \Psi\|_F^2 + \beta \|G\|_\infty, \quad (101)$$

where $\beta$ is the regularization coefficient.

The $H_\infty$ norm has an LMI formulation. The inequality $\|G\|_\infty < \gamma$ holds if and only if [15, §3.2.2]

$$P > 0,$$

$$\begin{bmatrix} P & A & P B & 0 \\ P A^T & P & 0 & P C^T \\ B^T & 0 & \gamma I & D^T \\ 0 & C P & D & \gamma I \end{bmatrix} > 0. \quad (103)$$
The full optimization problem with $\mathcal{H}_\infty$ regularization is

\[
\min J(U, \nu, W, \gamma, P; \beta) = c - 2\text{tr}(UG^T) + \nu + \frac{\beta}{q}\gamma
\]  

subject to

\[
\text{tr}(W) < \nu,
\]

\[
W > 0,
\]

\[
\begin{bmatrix}
W & U \\
L^TU & 1
\end{bmatrix} > 0,
\]

\[
P > 0,
\]

\[
\begin{bmatrix}
P & AP & B & 0 \\
PA^T & P & 0 & PC^T \\
B^T & 0 & \gamma 1 & D^T \\
0 & CP & D & \gamma 1
\end{bmatrix} > 0,
\]

where $H = LL^T$ and $U = \begin{bmatrix}A & B\end{bmatrix}$.

This is a bilinear optimization problem as both $P$ and $A$ are unknown. It must be solved iteratively by holding either $P$ or $A$ fixed while solving for the other. Iteration must be performed until the cost function stops changing significantly.

### 7 Reproducible research

The methods presented in this document are implemented in pykoop, the authors’ open source Koopman operator identification library [25].

### 8 Conclusion

Regression is one way to approximate a Koopman matrix from data. The presented LMI-based methods to regularize and constrain the Koopman matrix regression problem are part of a modular approach that can be readily adjusted for the problem at hand. The proposed method of regularizing the Koopman matrix regression problem with the $\mathcal{H}_\infty$ norm provides a systems perspective to the problem and allows the regularization to be tuned in the frequency domain using weighting functions. Other system norms, like the $\mathcal{H}_2$ norm, the generalized $\mathcal{H}_2$ norm, the peak-to-peak norm [15, §3] can also be used as regularizers. The unique properties of these system norms may prove useful in the identification of approximate Koopman operators from data. Further exploration of these properties, along with investigation into improved methods to solve the bilinear matrix inequalities arising in these problems is the subject of future research.

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