Scalar Resonant Relaxation of Stars around a Massive Black Hole

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Abstract

In nuclear star clusters, the potential is governed by the central massive black hole (MBH), so that stars move on nearly Keplerian orbits and the total potential is almost stationary in time. Yet, the deviations of the potential from the Keplerian one, due to the enclosed stellar mass and general relativity, will cause the stellar orbits to precess. Moreover, as a result of the finite number of stars, small deviations of the potential from spherical symmetry induce residual torques that can change the stars’ angular momentum faster than the standard two-body relaxation. The combination of these two effects drives a stochastic evolution of orbital angular momentum, a process named “resonant relaxation” (RR). Owing to recent developments in the description of the relaxation of self-gravitating systems, we can now fully describe scalar resonant relaxation (relaxation of the magnitude of the angular momentum) as a diffusion process. In this framework, the potential fluctuations due to the complex orbital motion of the stars are described by a random correlated noise with statistical properties that are fully characterized by the stars’ mean field motion. On long timescales, the cluster can be regarded as a diffusive system with diffusion coefficients that depend explicitly on the mean field stellar distribution through the properties of the noise. We show here, for the first time, how the diffusion coefficients of scalar RR, for a spherically symmetric system, can be fully calculated from first principles, without any free parameters. We also provide an open source code that evaluates these diffusion coefficients numerically.

Key words: black hole physics – galaxies: nuclei – gravitation – stars: kinematics and dynamics

1. Introduction

Nuclear star clusters with a central massive black hole (MBH) are dense environments where the interactions between stars play a crucial role. Although they are among the densest stellar environments in the universe, their gravitational potential is still dominated by the central MBH. As a result, stars move on nearly stationary Keplerian orbits. The gravitational potential from the stars themselves only leads to small potential perturbations that modify the purely Keplerian potential of the MBH. Nevertheless, these small perturbations, as well as the corrections from general relativity (GR), are the ones that drive the long-term evolution of the stellar cluster.

The evolution of a stellar system with a central MBH is a classical problem of stellar dynamics. It was first studied in the context of globular clusters with a central MBH by Peebles (1972) and Bahcall & Wolf (1976, 1977). These seminal works showed, under various simplifying approximations (Nelson & Tremaine 1999), that the two-body diffusion coefficients for a spherically symmetric and isotropic system can be calculated from first principles, where the only unknown is the Coulomb logarithm. This was subsequently generalized by Shapiro & Marchant (1978) and Cohn & Kulsrud (1978), who derived a two-dimensional diffusion equation (in energy and angular momentum) and calculated the associated diffusion coefficients (see, e.g., Bar-Or & Alexander 2016). Although the existence of central black holes in globular clusters is still unknown, many nuclear star clusters contain an MBH in their center (see Graham 2016 for a review).

In addition to the standard two-body relaxation driven by local scatterings, there exists in galactic nuclei a more efficient mechanism to change the angular momentum of stars. This process, named “resonant relaxation” (RR) by Rauch & Tremaine (1996), results from the coherent motion of the stars along their nearly fixed Keplerian orbits; a given test star will be subject to residual torques persisting on long timescales. RR can be separated into two different processes: scalar RR, which drives the evolution of the eccentricity, i.e., the magnitude of the angular momentum; and vector RR, which drives the orbital orientation. The residual torques associated with scalar RR are randomized by the in-plane orbital precession. The residual torques associated with vector RR persist on longer timescales, as they are randomized by the changes of the orbital orientations themselves. This implies that the orbital evolution by vector RR is much faster than the one by scalar RR, but can only affect the direction of the angular momentum vector (Rauch & Tremaine 1996; Hopman & Alexander 2006).

Extensive studies of vector RR were presented by Kocsis & Tremaine (2011, 2015) and Roupas et al. (2017). Here, our main focus is scalar RR.

Scalar RR, which can dominate the angular momentum’s evolution over standard two-body relaxation, did not have a formal self-consistent description for many years. Previous attempts at modeling this process were only qualitative, and many studies had to use ad hoc methods to include it (e.g., Hopman & Alexander 2006; Madigan et al. 2011; Merritt et al. 2011; Antonini & Merritt 2013; Merritt 2015). Recent advances in N-body simulations allowed for the study of RR numerically (Merritt et al. 2011; Hamers et al. 2014), but were lacking a fully self-consistent theory.

Only recently have several studies (Bar-Or & Alexander 2014; Sridhar & Touma 2016; Fouvy et al. 2017) put forward independently the foundation for a self-consistent kinetic theory of RR. Building upon these works, we show here that in the case...
of an isotropic spherical system, scalar RR can be described as a diffusion process, for which one can derive and calculate the diffusion coefficients from first principles.

This Letter is organized as follows. In Section 2 we write the orbit-averaged Hamiltonian of a test star as a Fourier sum over the orbital angles of both the test star and the field stars. Following the $\eta$-formalism (Bar-Or & Alexander 2014; Fouvry & Bar-Or 2017), these Fourier components are the random noise terms that drive the stochastic evolution of the test star’s orbital angular momentum, $J$. In Section 3 we follow this approach to write a closed expression for the diffusion coefficient of scalar RR. In Section 4 we briefly discuss the two-dimensional $(a, J/J_c)$ structure of the diffusion coefficients and compare it to two-body relaxation. Finally, we summarize our results and discuss future applications in Section 6.

2. Hamiltonian

Let us consider a star of mass $M_*$ moving on a nearly Keplerian orbit of semimajor axis (SMA) $a$ around an MBH of mass $M$, embedded in a spherically symmetric and isotropic in velocity star cluster of density $\rho(r)$. The orbits of the stars in the potential of the MBH can be described in angle-action variables. In this case, it is convenient to use the Delaunay variables (e.g., Binney & Tremaine 2008), where the three actions are: $J_z = \sqrt{GM/2a}$, the maximal (circular) angular momentum for a given $a$; $J = \sqrt{1 - e^2}J_z$, the specific orbital angular momentum with $e$ the eccentricity; and $J_z = J \cos(\theta)$, the $z$ component of the angular momentum with $\theta$ the inclination angle w.r.t. an inertial reference frame. The corresponding angles are: the mean anomaly $M$, the argument of pericenter $\omega$, and the longitude of the ascending node $\Omega$.

Following Bar-Or & Alexander (2014), we use the addition theorem for spherical harmonics to write the secular (orbit-averaged) specific Hamiltonian of the test star as a multipole expansion

$$ H = H_0(a, J) + \sum_{m, n = -\infty}^{\infty} e^{i(m\Omega + n\omega)} \eta_{mn}(a, J, J_z, t). \quad (1) $$

In this equation, the first term is the mean field potential, while the second term describes the potential fluctuations around the mean field due to the intricate motion of the finite number of field stars. The mean field potential reads

$$ H_0(a, J) = \Phi_{MBH}(a) + \Phi_{GR}(a, J) + \Phi_s(a, J). \quad (2) $$

It is composed of the Keplerian potential of the central MBH,

$$ \Phi_{MBH}(a) = -\frac{1}{2} \nu_t(a) J_z, \quad (3) $$

where $\nu_t(a) = \sqrt{GM/2a^3}$ is the fast orbital frequency imposed by the central MBH, an effective correction to the Keplerian potential,

$$ \Phi_{GR}(a, J) = -\frac{3}{a} \nu_t(a) J_z, \quad (4) $$

which reproduces the orbit-averaged Schwarzschild (in-plane) orbital precession, where $r_s = GM/c^2$, and $\Phi_s(a, J)$ the mean field potential due to the stellar cluster around the MBH.

The last two terms of $H_0$ induce, respectively, a prograde and retrograde in-plane orbital precession, and the combined precession is

$$ \dot{\omega} = \nu_p(a, J) = \frac{\partial H_0}{\partial J} = \nu_{GR}(a, J) + \nu_M(a, J), \quad (5) $$

where

$$ \nu_{GR}(a, J) = \frac{\partial \Phi_{GR}}{\partial J} = \frac{3}{a} \frac{r_s}{J} \nu_t(a), \quad (6) $$

is the precession induced by GR, and

$$ \nu_M(a, J) = \frac{\partial \Phi_s}{\partial J} = \frac{\nu_t(a)}{\pi M_c} \int_0^\infty dM_{\text{enc}}(r[f]) \cos f \quad (7) $$

is the mass-induced precession (Kocsis & Tremaine 2015), where $M_{\text{enc}}(r)$ is the stellar mass enclosed within a radius $r$, and $f$ is the true anomaly.

The last term in Equation (1) is due to the discrete nature of the stellar potential and describes the fluctuations around the mean field potential due to the motion of the field stars. Note that this is the only time-dependent term in the Hamiltonian, and therefore this term drives the orbital diffusion of the test star. These terms take the form

$$ \eta_{mn}(a, J, J_z, t) = \sum_{k=1}^N \sum_{n' = -\infty}^\infty e^{-i(m\Omega k + n\omega k)} \psi_{mn}(J, J_z(t)), \quad (8) $$

where the first sum is over the $N$ field stars and $M_k$ is the mass of the $k$th field star. In the large $N$ limit, $\eta_{mn}$ can be considered as random Gaussian noise terms, with $\langle \eta_{mn} \rangle = 0$. In Equation (8), we also introduced the angular Fourier components of the pairwise orbit-averaged interaction potential given by

$$ \psi_{mn}(J, J_z) = -\sum_{\ell = \ell_{\text{min}}}^{\ell_{\text{max}}} G^\ell_{mn}(\theta, \theta') K^\ell_{mn}(a, J, a', J'), \quad (9) $$

with

$$ K^\ell_{mn}(a, J, a', J') = \langle K_2(r, r') e^{i(n\theta - n'\theta')} \rangle \quad (10) $$

$$ = \langle K_2(r, r') \cos(n\theta') \cos(n'\theta') \rangle, $$

where $\ell_{\text{max}} = \text{max} \{1, |n|, |n'|\}$, $I = (J_z, J, J_c)$ stands for the action vector, $\langle \cdot \rangle$ denotes the orbit-average, and $K_2(r, r') = \min(r, r')/\max(r, r')^{1/2}$ is the usual min-max term from the Legendre expansion of the Keplerian potential. Here, we note that the component $\ell = 0$ does not contribute to the diffusion. Finally, $G^\ell_{mn}(\theta, \theta')$ is the geometrical factor

$$ G^\ell_{mn}(\theta, \theta') = \frac{4\pi \gamma^m \gamma^{*n} \gamma^{*n}}{2\ell + 1} d^\ell_{\text{min}}(\theta) d^\ell_{\text{max}}(\theta'), \quad (11) $$

where $d^\ell_{\text{min}}(\theta)$, related to the Wigner’s rotation matrices $D^\ell_{mn}(\alpha, \beta, \gamma) = e^{-i\alpha n^*} d^\ell_{mn}(\beta) e^{-i\gamma m^*}$ (e.g., Rose 1995), satisfies $\langle |d^\ell_{mn} |^2 \rangle = 2/(2\ell + 1)$, and $\nu^2 = Y^2_{\ell}(\pi/2, \pi/2)$ satisfies $4\pi \gamma^m \gamma^{*n} / (2\ell + 1) = [(k_+ - 1)!!(k_+ - 1)!!][/(k_+ + k_+)]!$ and is non-zero only if $k_+ = k - n$ are even.

Thus, this geometrical factor satisfies

$$ \langle |G^\ell_{mn}(\theta, \theta') |^2 \rangle_{\theta, \theta'} = \frac{16\pi^2 \gamma^m \gamma^{*n} \gamma^{*n}}{(2\ell + 1)^4}, \quad (12) $$
To obtain \( \bar{C}_n(J, n\nu_p(a, J)) \) is the Fourier transform, 
\[
\tilde{C}_n(a, J, t - t') = \int_{-\infty}^{\infty} df(t) e^{i\omega t}, \text{ of the correlation function } 
C_n(a, J, t - t') = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{df_J}{2\pi} \langle \eta_{hm}(a, J, J, t') \eta_{hm}(a, J, J, t) \rangle, 
\]
and we used the fact that \( \bar{C}_n(a, J, n\nu_p(a, J)) \) is invariant under \( n \to -n \), to sum only over positive \( n \), which introduces a factor 2 w.r.t. Fouvy & Bar-Or (2017) and Equation (35).

The correlation function in Equation (15) depends upon time through the motion of the field stars. As the Keplerian orbits of the field stars evolve, the cluster’s potential changes, and on long timescales, the system is reshuffled and the potential fluctuations become uncorrelated. Here, the main source of orbital evolution is the apsidal precession of the orbits due to the enclosed stellar mass, \( M_{\text{tot}}(r) \), as well as the relativistic in-plane precessions. Assuming that the field stars are moving on Keplerian orbits precessing in-plane because of the mean field Hamiltonian \( H_0 \) (Equation (2)) and (ignoring collective effects (see Fouvy & Bar-Or 2017)), the diffusion coefficient from Equation (14) can be written explicitly as (see Appendix B)

\[
D_{JJ}^{RR}(a, J) = 4\pi G^2 \sum_{i} \sum_{i'=-\infty}^{\infty} n_i^2 \int d\alpha' n_i(a') J dJ'|J'| \left| A_{nn}(a, J, a', J') \right|^2 [\nu_p(a, J) - n'\nu_p(a', J')] \left| \psi_{\text{max}}(J, J') \right|^2, 
\]

where we considered a mass spectrum \( M_i \) of field stars and defined a susceptibility coefficient which is averaged over both \( J \) of the test star and the field star

\[
A_{nn}(a, J, a', J') \left| \psi_{\text{max}}(J, J') \right|^2 
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{i=0}^{\infty} \int_{J_{\text{max}}}^{J_{\text{min}}} dJ \int_{J_{\text{max}}}^{J_{\text{min}}} dJ' \int_{J_{\text{max}}}^{J_{\text{min}}} dJ'' \left| \psi_{\text{max}}(J, J') \right|^2 K_{mn}(a, J, a', J'), 
\]

where \( \ell_{mn} = \max \{|i|, |m|, |n|\} \).

By solving the resonant condition in Equation (16), we can carry out the integral over \( J' \) to obtain

\[
D_{JJ}^{RR}(a, J) = 4\pi G^2 \sum_{i} \sum_{i'=-\infty}^{\infty} n_i^2 \int d\alpha' n_i(a') J dJ'|J'| \left| A_{nn}(a, J, a', J') \right|^2 
\]

\[
\times \sum_{j'} \left| \frac{f_{J,J'}(J', J)}{\partial\alpha'} \right| \left| \bar{C}_n(a, J, n\nu_p(a, J)) \right|, 
\]

where the sum on \( J' \) runs over the solutions, \( J_1, \ldots, J_{\ell_{mn}} \), of the resonant conditions \( \nu_p(a', J_\ell) = (n/n') \nu_p(a, J) \), which depend on \( a', a, J \) and on the ratio, \( n/n' \), of the resonant numbers.

In addition to diffusion, the interaction of a test star with the field stars also induces a drift by polarization. This drifts originates from the “wake” created by the test star as it moves through the field stars (Binney & Tremaine 2008). Here we use a simple method to calculate this term. As \( D_{JJ}^{RR} \) can be obtained directly from the Balescu–Lenard equation (Heyvaerts 2010;
Chavanis 2012), the associated friction by polarization is obtained by the replacements \( M_i^2 \rightarrow m_i^2M_i/2 \) and \( f(J, E) \rightarrow \partial f(J, E) \) or equivalently \( f(J) \rightarrow \partial f(J) = f_J(J) - f_J(J) \), where \( M_i \) is the mass of the test particle (e.g., Fouvy & Bar-O 2017). Therefore, the friction force by polarization acting on a given test particle is given by

\[
D_{J, \text{pol}}(a, J) = 4\pi G^2 M_i \sum_i M_i \sum_{n,u=1}^{\infty} n \int da' \mathcal{N}_i(a') \times \sum_{J'} J' \partial f(J'; a')/J' \times |A_{nu}(a, J, a', J')|^2/J' \partial J' \nu_T(a', J') \mathcal{I}_c(a', J') \mathcal{I}_c(a', J').
\]

The information about the underlying cluster is contained in the angular momentum distribution function \( f_{J,i}(J; a) \), in the mass-weighted SMA distribution \( \sum_n M_i^2 \mathcal{N}_i(a) \), and in the stellar contribution to the precession which enters the resonant condition \( \delta (n\nu_T(a, J) - n'\nu_T(a', J')) \) in Equation (16), while scaling with mass as \( \sim \sum_n (M_i/M) \mathcal{N}_i(<a)\nu_T(a) \) (see Equation (7)). In a multi-mass population, the system will undergo a strong mass segregation, where heavier masses will develop a steeper density slope than the lighter ones (Alexander & Hopman 2009). This means that at small SMAs the heavy stars will dominate the diffusion, which in turn increases the diffusion rate by the heavy to light mass ratio. For simplicity, in the upcoming applications, we limit ourselves to a single-mass population.

Equation (18) is the main result of this work. It shows that for a spherically symmetric and isotropic stellar distribution, the diffusion coefficients associated with RR can be derived and calculated from first principles. Carrying out the integral in Equation (18) is conceptually straightforward but can be technically challenging. It requires solving the resonant condition and integrating over \( a' \) and the two true anomalies \( f \) and \( f' \) in \( |A_{nu}|^2 \) (see Equation (10)). We provide a code, SCRPRY,2 in which the integration is carried out using the Vegas Monte-Carlo integration scheme (Lepage 1978).

The Fokker-Planck Equation (13) can be rewritten in the more traditional form

\[
\frac{\partial P(J, t)}{\partial t} = -\frac{\partial}{\partial J} D_J P(J, t) + \frac{1}{2} \frac{\partial}{\partial J^2} D_{J^2} P(J, t),
\]

where the two-diffusion coefficients satisfy the fluctuation–dissipation relation

\[
D_J = \frac{1}{2J} \frac{\partial (J D_{J^2})}{\partial J} + D_{J, \text{pol}},
\]

with \( D_J = D_{JJ}^{\text{RR}} \) and \( D_{J, \text{pol}} = D_{J, \text{pol}}^{\text{RR}} \).

Assuming an isotropic J distribution and a single-mass population, Equation (18) can be written as

\[
D_{JJ}^{\text{RR}}(a, J) = 2\pi^2 \mathcal{I}_c(a) d_{\mathcal{RR}}(a, J),
\]

where \( \mathcal{T}(a) = (M_i/M) \sqrt{N(<2a)\nu_T(a)} \) is the typical strength of the residual torque (Gürkan & Hopman 2007), \( T_c(a) \sim 1/\nu_M(a) \) is the typical coherence time (see below), \( N(<2a) \) is the number of stars with SMAs smaller than \( 2a \), and

\[
d_{\mathcal{RR}}(a, J) = \sum_{n,u=1}^{\infty} n \int da' \mathcal{N}_i(a') \times \sum_{J'} 4\pi J'^2 |A_{nu}(a, J, a', J')|^2/J' \partial J' \mathcal{I}_c(a', J') \mathcal{I}_c(a', J') \mathcal{I}_c(a', J'),
\]

is dimensionless.

Because we assumed that the cluster is isotropic in velocities, i.e., \( \partial f(J, E)/\partial J = 0 \), or \( J\partial f_J(J) = f_J(J) \), there is no dynamical friction,3 and no amplification through collective effects (Fouvy & Bar-O 2017). In the absence of a loss-cone, the zero-flux steady-state solution reads \( f(J) = J/J_c^2 \). In practice, \( f(J) \) is logarithmically suppressed near the loss-cone, \( J_e \), and therefore deviates from the isotropic \( f(J) \propto J \) distribution. As a result, both dynamical friction and collective effects can become important near the loss-cone. However, as RR is quenched near the loss-cone (see Figure 1), these effects are expected to be of no practical importance for the overall relaxation (which also includes two-body relaxation).

In Figure 1 we show the RR diffusion coefficient for the normalized angular momentum \( j = J/J_c \), with the notation \( D_{JJ} = D_{JJ}^{\text{RR}} / J_c^2 \), as a function of \( j \) and compare it with two-body relaxation (Bar-O & Alexander 2016, Equation (125)) at \( a \sim 6 \) Mpc. Here, we assume a stellar population with a Bahcall–Wolf (BW) cusp density profile \( \rho(r) \propto r^{-7/4} \) and stars of one solar mass \( M_s = M_* \) around an MBH of mass \( M_h = 4 \times 10^6 M_\odot \), with a total stellar mass \( M_* \) within the radius of influence \( r_h = 2 \) pc. The various bumps appearing in Figure 1 are associated with the respective contributions from

\footnote{Available at https://github.com/benbaror/scrpry.}

Figure 1. Normalized angular momentum \( j = J/J_c \) diffusion coefficients as a function of \( j \) for two-body (dotted line) and resonant relaxation (RR, dots). The RR diffusion coefficient is calculated using Equation (18), while the two-body relaxation time is calculated from Equation (125) in Bar-O & Alexander (2016). The sharp drop in the RR diffusion coefficient occurs where the precession frequency of the test star \( \nu_T(j) \) is comparable to the coherence frequency of the system 1/\( T_c(a) \) (see Figure 2). We also show the convergence w.r.t. the maximal harmonic number \( f \) in black lines, from \( f = 1 \) (thickest line) to \( f = 5 \) (thinnest line with dots). In this figure, the diffusion coefficients are evaluated at \( a \simeq 6 \) Mpc for an isotropic Bahcall–Wolf cusp \( \rho(J) \propto J^{7/4} \) of solar mass stars around an MBH of \( M_h = 4 \times 10^6 M_\odot \) and a total stellar mass \( M_* \) within the radius of influence \( r_h = 2 \) pc, using SCRPRY.

5 Here we define dynamical friction as the drift term proportional to the mass of the subject star, as opposed to the parametric drift proportional to the mass of the field stars (Chavanis 2012).
different resonance pairings \((n, n')\). Figure 1 also demonstrates the fast convergence of \(D_{jj}^{\text{RR}}\) w.r.t. the harmonic number \(f\). Unlike two-body relaxation which has a logarithmic divergence (manifested in the Coulomb logarithm), RR has no divergences.

As shown in Figure 2, near \(J = J_c\) (i.e., \(e \ll 1\)), \(D_{jj}^{\text{RR}}\) is well approximated by \(D_{jj}^{\text{RR}} \approx 2T_c \langle \tau^2 \rangle\), where \(\langle \tau^2 \rangle / \tau^2 \approx 0.07(1 - j)\) is the averaged residual torque in the direction of the angular momentum (Bar-Or & Alexander 2016), and \(T_c\) is the typical correlation time of the system. Here, we also find that the ansatz \(T_c \approx \sqrt{\pi / 2} \nu_p^{-1}(2a, 1/\sqrt{2})\) from Bar-Or & Alexander (2016), for which the correlation time is proportional to the median precession time evaluated at \(2a\) (the median eccentricity is \(e = j = 1/\sqrt{2}\)), provides a good approximation of \(D_{jj}^{\text{RR}}\), as long as \(\nu_p(2a, 1/\sqrt{2})\) is dominated by mass precession; i.e., most of the field stars at this SMA are non-relativistic.

This implies that for non-relativistic orbits, RR scales as \(D_{jj}^{\text{RR}} / J_c^2 \sim (M/M_\bullet)\nu_p(a)\), which is independent of the number of stars (Rauch & Tremaine 1996; Hopman & Alexander 2006), while two-body relaxation scales like \(D_{jj}^{\text{NR}} \sim N(<a) (M^2/M_\bullet^2)\nu_p(a)\log A\). Therefore, because \(N(<a) \ll M/M_\bullet\), RR can be significantly faster than two-body relaxation in some regions of orbital space (e.g., Bar-Or & Alexander 2016).

When the precession frequency of \(\nu_p(a, j)\), the test star, approaches \(1/T_c\), the diffusion coefficient, \(D_{jj}^{\text{RR}}\), sharply drops as it enters the relativistic regime where the precession frequency of the test star is higher than the precession of the bulk of the field stars, and \(J\) becomes an adiabatic invariant. In Figure 2 we show that this suppression of RR occurs at \(J_0\) where \(\nu_{\text{GR}}(a, j_0) \approx 0.45/T_c\).

4. Discussion

In this section, we briefly investigate the phase-space structure of \(D_{jj}^{\text{RR}}\) and compare it to the standard two-body relaxation diffusion coefficients, and comment about its contribution to various physical phenomena in the vicinity of the MBH. We use SCRRPY to calculate \(D_{jj}^{\text{RR}}\) and for simplicity, we consider as previously a stellar cluster composed of a single-mass population with a BW power-law density cusp \(\rho(r) \propto r^{-7/4}\).

As shown in Figure 3, diffusion by RR can be faster than two-body relaxation in a limited region of phase space. Interestingly, the orbits of the young stellar population cluster in the Milky Way Galactic center (the S-stars cluster, Ghez et al. 2003; Schödel et al. 2003; Gillessen et al. 2009, 2017) are within this region.

At low \(j = J/J_c\) and low \(a\), RR is quenched by adiabatic invariance. This is because the relativistic precession \(\nu_{\text{GR}}\) increases as \(1/\sqrt{(aj^2)}\) (see Equation (5)), and when \(\nu_{\text{GR}}(j)\) is larger than the coherence frequency \(1/T_c\), the diffusion coefficient decays rapidly, as demonstrated in Figure 1. In Figure 3, this translates to a line in \((a, J/J_c)\) phase space where RR is quenched and two-body relaxation takes over. This line is associated with the so-called “Schwarzschild barrier” that is observed in \(N\)-body simulations (Merritt et al. 2011). At large SMA, the mass precession time becomes comparable to the orbital time and two-body relaxation wins over RR.

Generally, event rates associated with loss-cone dynamics like tidal disruption events (TDEs) and binary disruptions will be governed either by the dynamics near the boundary between the full and empty loss-cone or by the dynamics near the radius of influence \(r_h\), depending upon which radius is smaller. Typically, for an MBH with a mass of \(10^5-10^7 M_\odot\), these are of the same order of magnitude (e.g., Alexander & Bar-Or 2017).

Near the radius of influence, the precession time is comparable to the orbital period, and thus the dynamics is governed by two-body relaxation. Therefore, RR is not expected to have a significant effect on these rates.

As shown in Figure 3, RR can have an effect on the dynamics of stars deep in the cusp. Hopman & Alexander (2006) suggested that RR can significantly increase the rates of extreme mass ratio inspirals (EMRIs). Additionally, \(N\)-body simulations (Merritt et al. 2011; Brem et al. 2014), which inherently contain both RR and two-body relaxation, showed that in practice the rates of EMRIs are comparable to the ones obtained by considering only two-body relaxation. Bar-Or & Alexander (2016) showed that as long as the region where the RR diffusion dominates over two-body relaxation is far from the region where gravitational wave (GW) emission dominates the orbital evolution (see dotted contour in Figure 3), RR will not contribute significantly to the EMRIs event rate. This is indeed the case for the cusp considered in Figure 3.

Both RR and two-body relaxation will drive the stellar distribution toward an isotropic distribution in angular momentum, i.e., \(f(J) = 2J/J_c^2\), or \(f(e) = 2e\) in eccentricity, when neglecting loss-cone effects. This will happen over the relaxation timescale, which is of order of the diffusion timescale \(T_r(a) = 1/D_{jj}^{\text{NR}}(a)\), where \(D_{jj}^{\text{NR}}(a) = \int \langle \delta J \rangle \delta J / DJ\) is the isotropic averaged diffusion coefficient.

In Figure 4 we show the relaxation times for RR and for two-body relaxation. While the two-body diffusion timescales as \(T_{\text{RR}}(a) \sim (M_\bullet/M_\odot)^2 P(a)/(N(<a)\log A)\), where \(P(a)\) is the orbital period, the RR diffusion timescales as \(T_{\text{RR}}(a) \sim (M_\bullet/M_\odot) P(a)\) in the region where the precession is dominated by mass precession. As shown in Figure 4, RR can be significantly faster than two-body relaxation deep in the cusp until it is quenched by the relativistic precession. Although faster than two-body relaxation, the RR diffusion timescale is longer than the ages of some of the young stars.
observed in our Galactic center (Habibi et al. 2017). This suggests that these stars did not have the time to relax by RR to the current nearly thermal \( f(e) \propto e \) distribution observed today (Gillessen et al. 2017). Let us note, however, that in a multi-mass system, one has \( T_{j,RR}^\text{iso} (a) \sim \langle M_j \rangle / \langle M_\ast \rangle \, P(a) \) and the diffusion time can be shorter by an order of magnitude in regions where stellar black holes dominate the total enclosed mass, as expected from strong mass segregation (Alexander & Hopman 2009).

An additional relaxation mechanism that will randomize the orbital orientation is the so-called “vector resonant relaxation” (Rauch & Tremaine 1996; Hopman & Alexander 2006; Kocsis & Tremaine 2015). As shown in Figure 4, this process can randomize the orientations of the orbits (but not their eccentricities) on a shorter timescale \( T_{j,VR}^\text{VR} \simeq (\langle M_j \rangle / \langle M_\ast \rangle) (P(a) / \sqrt{N(<a)}) \) (Kocsis & Tremaine 2015).

5. Comparing to Numerical Simulations

In this section we compare our analytical diffusion coefficients to the ones measured in Hamers et al. (2014) test particles’ simulations. Hamers et al. (2014) investigated the diffusion of test particles interacting with field stars that are moving on precessing Keplerian orbits and are not interacting between themselves. The precession of these field stars is driven by both the mass and the GR precessions.

Here we repeat the measurements of Hamers et al. (2014) by computing the diffusion coefficients directly from the changes in \( J \) along the trajectory of each test star. In addition, we slightly modify the procedure used by Hamers et al. (2014) and perform these measurements only in non-overlapping time intervals. That is, for a given test star’s trajectory, we split the time series into equal non-overlapping sequences of duration \( \Delta \tau \), and measure the changes in angular momentum \( \Delta J \) in each sequence. For each \( \Delta J \), we assign \( a \) and \( j = J / J_c \) by their values at the beginning of the sequence, and bin the data into linear bins in \( a \) and logarithmic bins in \( j \). Close encounters make the distribution of \( \Delta J \) heavy-tailed decaying like \( 1/|\Delta J|^3 \) at large \( \Delta J \) (Weissbein & Sari 2017). This is a two-body effect that we suppress by using the fractional moment \( \langle (\Delta J)^{2k} \rangle = \lim_{\Delta \rightarrow 0} (\langle \Delta J \rangle)^{2k} / g(\delta) \), where \( g(\delta) = 2[\Gamma((1 + \delta) / 2)/\pi]^{2/k} \), and \( \Gamma \) is the Gamma function (see Bar-Or et al. 2013).

![Figure 3](image1.png)

**Figure 3.** Phase-space structure of \( D_{\text{RR}}^\text{iso}/J_c^3 \), shown as a color map on a logarithmic scale. Diffusion by RR is faster than two-body or non-resonant relaxation (NR) in a limited region of phase space (solid black contours), which is far from the region where gravitational wave emission dominates the orbital evolution (dashed contour). The orbits of S-stars observed in the Galactic center (Gillessen et al. 2017; red circles, lie in the RR-dominated region. Orbits beyond the relativistic loss-cone, \( J_c = 4GM_\ast c \) (solid red line), are short-lived. Solar mass stars will tidally disrupt if their orbital pericenter distance is smaller than \( \epsilon_c = (\langle M_j \rangle / \langle M_\ast \rangle)^{1/2} R_c \) (Alexander & Bar-Or 2017; dashed red line). The cusp’s parameters are the same as in Figure 1.

![Figure 4](image2.png)

**Figure 4.** Angular momentum relaxation timescales \( T_j = 1/D_{\text{RR}}^\text{iso} \) by RR (solid line), and by two-body relaxation (NR; dashed line), as a function of semimajor axis. Relaxation by RR is faster than two-body relaxation in a limited range of semimajor axis. For comparison, we show the main sequence ages of some of the S-stars estimated recently by Habibi et al. (2017; red circles with error bars) and the vector RR timescale (dotted line). The cusp’s parameters are the same as in Figure 1.

\[\log_{10}(a/\text{time}) = -0.5 \times \log_{10}(J/J_c) + 1.5\]
In order to compare the numerically measured diffusion coefficients with our analytical ones, one first has to account for the fact that $\Delta J$ are computed for a finite time $\Delta T$. Over a finite time $\Delta T$, following Equations (14) and (31), the change in $J$ over time $\Delta T$ is characterized by

$$\frac{\langle (\Delta J) \rangle}{\Delta T} = 2 \sum_{n=1}^{\infty} d\nu \frac{C}{n^2} (a, J, \nu) K_{\Delta T}(\nu - n\nu_n(a, J))$$

(24)

where $K_{\Delta T}(\nu) \equiv (1 - \cos(\Delta T \nu))/2 \pi \nu^2 \Delta T$ is a window function associated the finite time measurement. For small $\Delta T$, the change in angular momentum is linear with time and $\langle (\Delta J) \rangle/\Delta T = \langle \tau_0 \rangle \Delta T$. In the limit $\Delta T \to \infty$, $K_{\Delta T}(\nu)$ becomes a Dirac delta and $\langle (\Delta J) \rangle/\Delta T \to D_{RR}^L$, but on finite times $\langle (\Delta J) \rangle/\Delta T$ will differ from the true underlying diffusion coefficients.

In Figure 5 we illustrate how $\langle (\Delta J)^2 \rangle/\langle J^2 \rangle \Delta T$ converges to the diffusion coefficient, $D_{RR}^L$, as $\Delta T \to \infty$, and that the numerically measured $\langle (\Delta J)^2 \rangle/\Delta T$ is consistent with the analytical prediction from Equation (24).

6. Summary

Relaxation processes in dense stellar systems around an MBH are a classical problem of stellar dynamics. Understanding these processes is crucial for the long-term steady-state stellar distribution of nuclear clusters and mass segregation therein; short-term transient phenomena such as tidal disruptions, GW emissions, and hypervelocity stars; and the distribution of unique source populations such as young stars, X-ray binaries, and radio pulsars, to name a few (Alexander 2017).

All of these phenomena depend upon both the relaxation in energy and in angular momentum. The relaxation in energy is well described by two-body relaxation, where the diffusion coefficients can be calculated from first principles for an isotropic distribution function. The only poorly determined quantity is the Coulomb logarithm, which has only a small effect on the diffusion.

Despite the approximations made in the derivation of these diffusion coefficients (e.g., Nelson & Tremaine 1999), they are in a good agreement with the ones measured in direct $N$-body simulations (Bar-Or et al. 2013). However, relaxation in angular momentum can be dominated by RR (Rauk & Tremaine 1996; Hopman & Alexander 2006). While this was demonstrated by Eilon et al. (2009) and especially by Merritt et al. (2011) using direct $N$-body simulations, a complete and self-consistent theory of RR was still lacking. The foundation for a concrete kinetic theory of RR was put forward independently by Bar-Or & Alexander (2014), Sridhar & Touma (2016), and Fouvry et al. (2017). In Fouvry & Bar-Or (2017), we generalized the method of Bar-Or & Alexander (2014) to a general stochastic Hamiltonian with integrable mean field and showed it to be equivalent to the (degenerate) Balescu–Lenard and Landau Equations (Heyvaerts 2010; Chavanis 2012, 2013). This means that the different approaches of Bar-Or & Alexander (2014), Sridhar & Touma (2016) and Fouvry et al. (2017), although different in details, are essentially equivalent.

Building upon Bar-Or & Alexander (2014) and Fouvry & Bar-Or (2017), we presented here, for the first time, a calculation of the scalar RR diffusion coefficients from first principles and without any free parameters. This brings to a close the long journey, started by Rauk & Tremaine (1996), of bringing the kinetic theory of RR to the same level of completeness as the standard two-body relaxation one. Although this treatment is limited to the diffusion of the angular momentum magnitude in a spherical and isotropic background distribution, for which collective effects can be ignored (e.g., Nelson & Tremaine 1999), the same limitations also apply to standard two-body relaxation (see Vasiliev 2015 for applications to non-spherically symmetric systems). Here, we also assumed that the MBH dominates the potential. This assumption breaks down close to the radius of influence, where the contribution of the underlying stellar population is comparable to that of the MBH. This is not a significant limitation, as RR is negligible compared to two-body relaxation at this point (see Figure 4). Some of these limitations could be mitigated in future studies, and the entire kinetic theory could be tested against future $N$-body simulations that are already approaching a realistic number of stars in galactic nuclei (Panamarev et al. 2018).

The ability to calculate RR diffusion coefficients provides us with the opportunity to make more realistic estimates on the effects of RR on astrophysical phenomena in galactic nuclei. As shown in Figure 4, RR can dramatically reduce the relaxation time in angular momentum. As a result, even short-lived populations (like the young S-star cluster) can be relaxed to a thermal eccentricity distribution. As RR can efficiently drive the angular momentum evolution, it may contribute to the supply rate of stellar objects into the loss-cone. This contribution is significant only if the loss-cone is close to the region where RR dominates the diffusion over two-body relaxation, and will depend upon the underlying stellar distribution and the specific loss-cone scenario. We show that for a standard stellar population following a BW cusp around a $4 \times 10^6 M_\odot$ MBH, the contribution of RR to the EMRs and TDEs rates is negligible.
We dedicate this paper to the memory of Tal Alexander, who passed away in May 2018. He will be greatly missed. We are grateful to Adrian Hamers for sharing with us the data from his Test Particle Integrator runs used in Hamers et al. (2014). We thank Tal Alexander, Adrian Hamers, Kirill Lezhnin, John Magorrian, Christophe Pichon and Scott Tremaine for fruitful discussions. B.B. acknowledges support from the Schmidt Fellowship. J.B.F. acknowledges support from Program number NAS5-02655. This research is carried out in part within the framework of the Spin(e) collaboration (ANR-13-BS05-0005, http://cosmicorigin.org).

Appendix A
Detailed Derivation of the Diffusion Coefficient

In this appendix, we follow Binney & Lacey (1988) to recover Equation (14), which expresses the diffusion coefficients as a function of the power spectrum of the noise. Let us therefore consider a test particle whose dynamics is governed by the single particle Hamiltonian

\[ H(\theta, J, t) = H_0(J) + \sum_k e^{i k \theta} \eta_k(J, t), \]  

(25)

where \( \theta \) and \( J \) are the angle and actions of the mean field Hamiltonian \( H_0 \), and \( \eta_k \) is the Fourier transform in angle of the noise. The equations of motions of the test particle are given by Hamilton’s equations and read

\[ \ddot{\theta} = \Omega(J) + \sum_k e^{ik\theta} \frac{\partial}{\partial J} \eta_k(J, t); \quad \dot{J} = -ik \sum_k e^{ik\theta} \eta_k(J, t), \]  

(26)

where \( \Omega(J) = \partial H_0(J)/\partial J \) are the mean field frequencies. Starting from \( \theta_0 \) and \( J_0 \) at time \( t = t_0 \), the changes in \( \theta \) and \( J \) after time \( t \) can be written as \( \Delta \theta(t) = \Omega(J_0) t + \Delta \theta(t) \) and \( \Delta J(t) = \Delta J_0(t) \), where

\[ \Delta \theta(t) = \int_0^t ds \int dJ \delta(J - J_0 - \Delta J_0(s)) \left[ \Omega(J) - \Omega(J_0) \right] 
+ \sum_k e^{i k \theta} e^{i k \Omega(J_0) s} e^{ik \Delta \theta_0(s)} \frac{\partial}{\partial J} \eta_k(J, s). \]  

(27)

and

\[ \Delta J_0(t) = -ik \int_0^t ds \sum_k e^{i k \theta} e^{i k \Omega(J_0) s} 
\times e^{ik \Delta \theta_0(s)} \eta_k(J(s), s). \]  

(28)

One may then expand Equations (27) and (28) in orders of \( \eta \). To second order in \( \eta \), the change in \( J \) is given by

\[ \Delta J = -i \sum_k k \int_0^t ds \left[ e^{i k \theta} e^{i k \Omega(J_0) s} \eta_k(J, s) \right] 
+ \int_0^t ds \int_0^t ds' \sum_{k, k'} k e^{ik'k'} e^{i(k-k')\theta} e^{i(k-k')\Omega(J)} 
\times \left[ \eta_k(J, s) \eta_{k'}(J, s') (s - s') k' \cdot \frac{\partial}{\partial J} \eta_k(J, s') \right] 
+ \eta_k(J, s) k \cdot \frac{\partial}{\partial J} \eta_k(J, s') + \eta_k(J, s') k' \cdot \frac{\partial}{\partial J} \eta_k(J, s) \]  

(29)

Taking the ensemble average of \( \Delta J \) and \( (\Delta J)^2 \), and dividing by the time interval \( \Delta T \), one obtains the effective diffusion coefficients

\[ D_{\Delta T}[\Delta J_i] = \int_0^{\Delta T} ds \int_0^s ds' \sum_k k_i k_j e^{i k_i \Omega(J_0) s} C_k(J, s - s') \]  

\[ = \frac{1}{2} \sum_k k_i k_j \int d\omega K_{\Delta T}(\omega - k \cdot \Omega(J)) \tilde{C}_k(J, \omega), \]  

(30)

and

\[ D_{\Delta T}[\Delta J_i, \Delta J_j] = \int_0^{\Delta T} ds \int_0^{\Delta T} ds' \sum_k k_i k_j e^{i k_i \Omega(J_0) s} C_k(J, s - s') \]  

\[ = \sum_k k_i k_j \int d\omega K_{\Delta T}(\omega - k \cdot \Omega(J)) \tilde{C}_k(J, \omega), \]  

(31)

where \( C_k(s - s') = \langle \eta_k(s) \eta_k^*(s') \rangle \) is the correlation function and

\[ K_{\Delta T}(\omega) = \frac{1 - \cos(T \omega)}{\pi T \omega^2}, \]  

(32)

is the time-dependent kernel with \( \int d\omega K_{\Delta T}(\omega) = 1 \), and \( \lim_{T \to \infty} K_{\Delta T}(\omega) = \delta(\omega) \).

As expected, these diffusion coefficients satisfy the fluctuation–dissipation relation

\[ D_{\Delta T}[\Delta J_i] = \frac{1}{2} \frac{\partial}{\partial J_i} D_{\Delta T}[\Delta J_i, \Delta J_j], \]  

(33)

In the limit \( t \gg T_c \), we can approximate \( K_{\Delta T}(\omega) \) by a Dirac delta, and the diffusion coefficients becomes the standard ones

\[ D[\Delta J_i] = \sum_{k > 0} k_i k_j \tilde{C}_k(J, k \cdot \Omega(J)) \]  

(34)

and

\[ D[\Delta J_i, \Delta J_j] = \sum_k k_i k_j \tilde{C}_k(J, k \cdot \Omega(J)). \]  

(35)
Finally, using the fluctuation–dissipation relation, the Fokker–Planck equation takes the form
\[
\frac{df}{dt} = \frac{1}{2} \frac{\partial}{\partial p} \left\{ D \left[ \Delta J \Delta J' \right] \frac{\partial f}{\partial J} \right\}.
\] (36)

Appendix B
Detailed Derivation of the Correlation Function

In this appendix we briefly illustrate how one can obtain Equation (16) from Equations (8) and (14). We start by assuming that the field stars are moving on (in-plane) precessing orbits, and we neglect their self-induced potential fluctuations. The noise term from Equation (8) then takes the form
\[
\eta_{nm}(a, J, J', t) = \sum_{k=1}^{N} GM_k \sum_{n'=-\infty}^{\infty} e^{-i\Omega t_{k}(0)} - i\nu_{k}(a, J) \nu_{n'nm'}(J, J').
\] (37)

As defined in Equation (15), the correlation function is given by
\[
C_n(a, J, t) = G^2M^2 \sum_{m=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \int da' \int dJ' |\psi_{nm'}(J, J')|^2
\times \int_{J}^{J'} \frac{dJ}{2J} \int_{J'}^{J''} \frac{dJ''}{2J''} e^{-i\nu_{n'}(a', J')} |\psi_{nm}(J, J')|^2,
\] (38)
and its Fourier transform is
\[
\tilde{C}_n(a, J, \nu) = 2\pi G^2 M^2 \sum_{n'=-\infty}^{\infty} \int da' \int dJ' |\psi_{nm}(a, J', J')|^2 \times \delta(\nu - n'\nu_{q}(a', J')) |\psi_{nm}(a, J, a', J')|^2.
\] (39)

When injected in Equation (14) while considering a multi-mass spectrum, one recovers Equation (16).