Vibrational dressing in kinetically constrained spin systems

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Quantum spin systems with kinetic constraints have become paradigmatic for exploring collective dynamical behaviour in many-body systems. Here we discuss a facilitated spin system which is inspired by recent progress in the realization of Rydberg quantum simulators. This platform allows to control and investigate the interplay between facilitation dynamics and the coupling of spin degrees of freedom to lattice vibrations. Developing a minimal model, we show that this leads to the formation of polaronic quasiparticle excitations which are formed by many-body spin states dressed by phonons. We investigate in detail the properties of these quasiparticles, such as their dispersion relation, effective mass and the quasiparticle weight. Rydberg lattice quantum simulators are particularly suited for studying this phonon-dressed kinetically constrained dynamics as their exaggerated length scales permit the site-resolved monitoring of spin and phonon degrees of freedom.

Introduction.— The precise control and manipulation of quantum systems is of utmost importance both in fundamental physics and for applications in quantum technologies. The last decade has seen an immense effort in the improvement of experimental techniques which enable the exploration of quantum many-body systems [1]. Rydberg atoms are notably suitable for this scope due to their versatility in simulating many-body models [2]. In particular, they provide an ideal platform for the realization of spin systems, with applications ranging from quantum information processing [3] to the exploration of fundamental questions concerning thermalization in quantum mechanics [4–6].

Recently, there has been a growing interest in the study of quantum systems in the presence of kinetic constraints, that impose restrictions on the connectivity between many-body configurations. In particular, it has been observed how constraints, that prevent the system from fully exploring the Hilbert space, can lead to peculiar dynamics and an unexpected lack of thermalization even in systems without explicit symmetries [7–20]. Facilitation is a specific instance of a constrained dynamics. The concept was introduced by Fredrickson and Andersen [21] in the study of kinetic aspects of the glass transition using spin models [22]. Here the excitation of one spin enhances the excitation probability of a neighboring spin. In Rydberg gases such dynamical behavior occurs naturally in the so-called anti-blockade regime [23–27], and the emerging many-body effects have been investigated in detail in many recent works [28, 29]. Among the studied phenomena are nucleation and growth [30–34], non-equilibrium phase transitions [35–39] as well as Anderson [28, 40] and many-body localization [41].

In this work we are interested in exploring the interplay between facilitated spin excitations and vibrational degrees of freedom. Such a scenario naturally occurs in Rydberg lattice quantum simulators [2], where individual atoms are held in oscillator potentials [see Fig. 1(a)], and coupling between spin and vibrations is caused by
state-dependent mechanical forces [42, 43]. We develop a minimal model that describes the emerging complex many-body dynamics and permits a perturbative expansion in the spin-phonon coupling strength. The dressing of the spin dynamics through lattice vibrations leads to the formation of a polaronic quasiparticle [44] for which we analyse the dispersion relation, the effective mass and the Z-factor, determining the quasiparticle weight. The perturbative results are compared with numerical simulations. Using Rydberg quantum simulators for exploring this physics is particularly appealing as these platforms allow the probing of spin and vibrational degrees of freedom. Thus, using side-band spectroscopy [45], the phonon cloud that dresses the spin excitation should be directly observable in experiments.

**Facilitated Rydberg lattice.** – We consider a chain of $N$ traps (e.g. optical tweezers) [46, 47] each loaded with a single Rydberg atom (see Fig. 1). The Rydberg atoms can be effectively described as a two-level system in which $|\downarrow_i\rangle$, $|\uparrow_j\rangle$ are indices that label the lattice sites, $\Omega$ is the Rabi frequency, and $\Delta$ is the detuning of the Rydberg excitation laser from the single atom resonance. The vibrational dressing. Through Eq. (3) it is evident that bosons, that correspond to the trap vibrations, interact only with states where $s = 2$, i.e., with states in which there are two adjacent excited spins, as shown in Fig. 1(c). In order to simplify the description we rewrite the Hamiltonian (3) by introducing the Fouriertransformed bosonic modes $a_j = \frac{1}{\sqrt{N}} \sum_{\pi=−N/2}^{N/2} e^{i\frac{2\pi}{N}jp} A_p$, which yields

$$H = \Omega \sum_{\alpha} \langle \alpha | (e^{i\frac{2\pi}{N}jp} − 1) e^{-i\frac{2\pi}{N}a \hat{A}_p} + h.c. \rangle \hat{a}_p A_p.$$ (4)

where $\hat{a}_p = \sum_{\alpha} \alpha \langle \alpha | \hat{a} \rangle \hat{a} \hat{\alpha}$ denotes the lattice position operator. In the next step we decouple the lattice from the bosonic modes using the Lee-Low-Pines transformation [48]

$$U = \exp \left[ −i \alpha \sum_p \frac{2\pi p}{N} A_p A_p \right].$$ (5)
and by introducing the Fourier modes of the quasiparticles, $|\alpha\rangle = \frac{1}{\sqrt{N}} \sum_{q=-N/2}^{N/2} e^{i \frac{2\pi q}{N}} |q\rangle$. The transformed Hamiltonian reads $U^\dagger H U = \sum_q (q| H_q |q\rangle$, with

$$H_q = \Omega \left[ \mu^+ \left( 1 + e^{-i \frac{2\pi q}{N} \left( \sum_p p A_p^+ A_p + q \right) } \right) + \text{h.c.} \right] + \frac{\kappa (\mu^2 - 1)}{\sqrt{2N}} \sum_p \left[ (e^{-i \frac{2\pi q}{N} p - 1}) A_p^+ + \text{h.c.} \right] + \omega \sum_p A_p^+ A_p.$$  

By virtue of the canonical transformation the quasiparticle momentum $q$ is now a conserved quantum number, which simplifies tremendously the subsequent analysis. Further manipulations, which are detailed in the Supplemental Material, allow us to finally obtain

$$H_q = \mu \sum_p A_p^+ A_p + \Omega \cos \left( \frac{\pi}{N} \left( \sum_p p A_p^+ A_p + q \right) \right) \mu^z + H_{\text{int}}^q + \text{H.c.}$$  

with $H_{\text{int}}^q = -\frac{\kappa^2}{\omega} (1 - \mu^z)$ and the displaced bosonic operators $A_p = A_p + \frac{\mu}{\omega N} \left( e^{-i \frac{2\pi q}{N} p - 1} \right)$. An explicit expression for the term $H_{\text{int}}^q$ is given in the Supplemental Material. Note, that despite the achieved simplification, the Hamiltonian (7) is highly non-trivial and now describes many-body spin states coupled to a bath of interacting phonons.

To investigate the vibrational dressing of the facilitation dynamics we first consider the decoupling limit $\kappa = 0$. In this case the spectrum of Hamiltonian (7) is given by bands that appear in pairs with positive and negative curvature [see Fig. 2(a)]. There are infinitely many pairs, forming a ladder with a spacing given by the trap frequency $\omega$. The ground state band has the tight-binding dispersion relation

$$E_{\text{GS}} = -\Omega \cos \left( \frac{\pi}{N} q \right).$$  

Note that in the limit of $N \gg 1$, the argument of the cosine becomes a continuous variable $-\pi \leq \frac{2\pi q}{N} \leq \pi$. In the following we assume for simplicity that the trap (phonon) frequency is larger than the twice the laser Rabi frequency, $\omega > 2\Omega$. In this case the ground state band is well separated from the remaining ones. Crucially, this regime is within reach of current technology from an experimental point of view. In fact, in order to be able to observe coherent dynamics, we must have $\omega > 2\Omega \gg \gamma$ with $\gamma$ being the decay rate of the Rydberg atoms. Typically, $\gamma \approx 10^4$ Hz and frequencies larger than $10^5$ Hz can be achieved experimentally, for both $\Omega$ [40] and $\omega$ [45].

In the presence of interactions between the propagating Rydberg excitation and the phonons, i.e. for $\kappa \geq 0$, the energy bands, defining the spectrum of Eq. (7), are modified. In particular, we observe the lifting of the degeneracy of the ground state and the first excited band at the band edges together with a flattening of the band structure. The latter decrease of the band curvature, shown in Fig. 2(b), is a consequence of the phonon-dressing of the spin excitation which leads to the formation of a polaron quasiparticle which is characterized by a correspondingly increased effective band mass.

In order to obtain a qualitative understanding of the observed renormalization of the band structure we adopt a perturbative approach to the solution of Eq. (7). The only interaction term that at first order in perturbation theory yields a non-zero energy correction is $H_{\text{int}}^q$. This term couples states with quasiparticle momentum $q$ of the ground state band, $|\Psi_q^{(0)}\rangle$, to the first excited band. At first order in perturbation theory, the calculation reduces to the solution of a two-level eigenvalue problem for each $q$. This yields the dressed value for the ground state band, $E_{\text{GS}}^{(1)}(q)$, and the first excited band, $E_2^{(1)}(q)$:

$$E_{\text{GS}}^{(1)} = -\sqrt{\Omega^2 \cos^2 \left( \frac{\pi}{N} q \right) + \frac{\kappa^4}{\omega^2} - \frac{\kappa^2}{\omega}},$$  

and $E_2^{(1)} = -E_{\text{GS}}^{(1)} - 2\frac{\kappa^2}{\omega}$. As we can be seen in Fig. 2, there is good agreement between the analytical result and
the numerics.

**Dressed facilitation dynamics.**—The interaction between the Rydberg atoms and the phonons that leads to the phonon dressing and corresponding band flattening, results in a slowdown of propagating facilitated Rydberg excitations. This effect is shown in Fig. 3(a), where we display the real-time dynamics of both Rydberg excitations and phonons. For the simulations we performed exact diagonalization on a system of size $N = 10$ and we truncated the local bosonic Hilbert space allowing a maximum number of three bosons per site. The initial state contains a single Rydberg excitation at the left edge of the lattice and no bosons, i.e. $|\uparrow\downarrow\downarrow\ldots\rangle\otimes|0,0,0\ldots\rangle$. Consequently, such wave packet states of the form $|\psi_m\rangle = |\ldots\uparrow\downarrow\downarrow\ldots\rangle$ in real space correspond to superpositions of momentum states that live on the first two excited bands due to a mixing between the states introduced in the diagonalization of the Hamiltonian (6).

The data in Fig. 3(a) shows that the stronger the coupling $\kappa$ the more pronounced becomes the phonon trail that is carried and left behind by the propagating Rydberg excitation. In Rydberg quantum simulator experiments it is standard to measure the Rydberg density [2]. It is, however, also possible to determine the local phonon density by side-band spectroscopy, as demonstrated in Ref. [45]. Remarkably, this makes it possible to use Rydberg quantum simulators to directly detect and map out the phonon cloud in-situ and in real-time, which remains elusive in solid state systems and most ultracold atom platforms.

The magnitude of the phonon dressing can be quantified by the $Z$-factor which is defined by the overlap of the dressed polaron state $|\tilde{\psi}_q\rangle$ with its non-interacting counterpart $|\psi_q^{(0)}\rangle$, $Z_q = |\langle \psi_q^{(0)} | \tilde{\psi}_q \rangle|^2$ [44, 49, 50]. The calculation of the $Z$-factor from exact diagonalization, see Fig. 3(b) shows that, although the phonon dressing is strong, still a well-defined polaron quasiparticle exists.

We also compute the phonon occupation number in momentum space in the ground state $|\psi_q^{(0)}\rangle$, i.e. $n_{ph}(p,q)$, with $p$ and $q$ being the phonon and quasi-particle momentum, respectively:

$$n_{ph}(p,q) = \langle \tilde{\psi}_q | A_p^d A_p | \tilde{\psi}_q \rangle,$$  \hfill (10)

While this quantity cannot be computed exactly analytically, at first non-zero order one finds:

$$n_{ph}^{(1)}(q,p) = \frac{\kappa^2}{\omega^2 N} |\psi_q^{(0)}| \left( e^{i\frac{2\pi}{N} p} - 1 \right) \left( e^{-i\frac{2\pi}{N} p} - 1 \right) |\psi_q^{(0)}\rangle,$$

$$= \frac{2 \kappa^2}{\omega^2 N} \left[ 1 - \cos \left( \frac{2\pi}{N} p \right) \right],$$  \hfill (11)

where $-\pi \leq \frac{2\pi}{N} p \leq \pi$. Note, that this result does not depend on the quasiparticle momentum $q$. Such a dependence enters at higher order in perturbation theory and leads to a $q$-dependent coefficient to Eq. (11). In fact our numerical calculations confirm a dependence of the form

$$n_{ph}(p,q) = 2 \frac{\kappa^2}{\omega^2 N} C^\kappa(q) \left[ 1 - \cos \left( \frac{2\pi}{N} p \right) \right],$$  \hfill (12)

**Figure 3:** Polaron dynamics: (a) Density plot of the Rydberg and phonon excitation for different values of the interaction strength $\kappa/\omega$. The first (second) row shows the Rydberg (phonon) density. For $\kappa = 0$, i.e. in absence of interactions, a ballistic spreading of the Rydberg excitation is observed and no phonons are generated (orange dashed lines are a guide to the eye). At $\kappa = 0.3 \omega$ ($\kappa = 1.5 \omega$ and $\omega = 5\Omega$) the propagation of the Rydberg excitation is slowed down until it almost comes to a halt (on the timescale shown) when $\kappa = 0.5 \omega$. (b) Momentum-dependent $Z$-factor for different values of the coupling strength. (c) Occupation number of the phonon modes in momentum space (labelled by $p$) of the ground state for quasi-particle momentum $\frac{2\pi}{N} q = \pm \pi$ and $\kappa = 0.3 \Omega$. We compare the analytical result of Eq. (10) with numerical data obtained using exact diagonalization on a system up to $N = 12$ sites.
with a numerically determined coefficient \( C^*(q) \). In Fig. 3(c) which we show the phonon occupation number \( \nu_{ph}(p, q) \) at the edges of the ground state band, \( \frac{2\pi}{N} q = \pm \pi \), at \( \kappa = 0.3\Omega \). The agreement between numerical and analytical results from Eq. (11) is excellent.

Conclusions. – We have shown how the non-equilibrium dynamics of a facilitated Rydberg atoms chain is dramatically affected by interactions with trap vibrations. This coupling leads to a dressing of the propagating excitations and shows the emergence of a slow-dynamics induced by a flattening of the quasi-particle bands. The latter can be interpreted as an polaronic effect that leads to an increase of the effective mass. The phonon dressing, as discussed here, might have links to other timely research questions: it was recently pointed out, that lattice Hamiltonians coupled to bosons can offer a possible setup for the observation of fractons [51], which are currently much studied in the context of ergodicity breaking in quantum systems.

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Supplemental Material: Vibrational dressing in kinetically constrained spin systems

In this supplemental material we show step-by-step how the Hamiltonian (3) in the main text can be rewritten as in Eq. (7). We will also write explicitly all the interaction terms.

HAMILTONIAN IN THE EFFECTIVE SPACE

Let us start by considering the Hamiltonian describing Rydberg atoms in the effective “constrained” Hilbert space. This reads (see Eq.(3) in the main text):

\[ H = \Omega \left( \sum_\alpha |\alpha\rangle \langle \alpha | \mu^+ + \mu^- |\alpha + 1\rangle \langle \alpha | + \text{h.c.} \right) + \kappa \sum_\alpha \frac{\mu^2 - 1}{2} |\alpha\rangle \langle \alpha | (a_{\alpha+1}^+ + a_{\alpha+1} - a_{\alpha}^+ - a_{\alpha}) + \omega \sum_\alpha a_{\alpha}^+ a_{\alpha}, \] (S1)

where the \( \mu \)-operators are the ones defined in the main text. The first step is to move to the Fourier space for the bosonic modes of the harmonic traps. This is achieved by defining

\[ a_m = \frac{1}{\sqrt{N}} \sum_{p=\pm N/2} A_p e^{i 2\pi mp/N}. \] (S2)

We thus see that the difference between the phonon creation operators appearing in the interaction term can be rewritten as

\[ a_{m+1}^+ - a_m^+ = \frac{1}{\sqrt{N}} \sum_p \left[ (e^{-i \frac{2\pi m+1}{N}p} - e^{-i \frac{2\pi m}{N}p}) A_p^\dagger \right]. \] (S3)

As we showed in the main text (see Eq. (4)) this leads to the Hamiltonian

\[ H = \Omega \sum_\alpha |\alpha\rangle \langle \alpha | \mu^+ + \Omega \sum_\alpha (\mu^- |\alpha + 1\rangle \langle \alpha | + \text{h.c.}) + \kappa \sum_\alpha \frac{\mu^2 - 1}{2} |\alpha\rangle \langle \alpha | (e^{-i 2\pi p N/2} - 1) e^{-i 2\pi \hat{a}} A_p^\dagger + \text{h.c.} \] (S4)

in which \( \hat{a} = \sum_\alpha \alpha |\alpha\rangle \langle \alpha |. \) At this point we can apply the Lee-Low-Pines transformation, which is defined as

\[ U = \exp \left[ -i \hat{a} \sum_p \frac{2\pi p}{N} A_p A_p^\dagger \right] \] (S5)
\[ U^\dagger = \exp \left[ i \hat{a} \sum_p \frac{2\pi p}{N} A_p A_p^\dagger \right]. \] (S6)

We stress, again, that this transformation is important because it decouples the lattice degrees of freedoms from the phonons. Applying the transformation (S6) to the operators in Eq. (S4) we have:

\[ U^\dagger A_p U = \exp \left\{ -\frac{2\pi p}{N} \hat{\alpha} \right\} A_p, \] (S7)

and

\[ U^\dagger |m+1\rangle |m\rangle U = e^{i \sum_p A_p^\dagger A_p \frac{2\pi p(m+1)}{N}} |m+1\rangle |m\rangle e^{-i \sum_p A_p^\dagger A_p \frac{2\pi pm}{N}} = |m+1\rangle |m\rangle e^{-i \frac{2\pi}{N} \hat{a} \sum_p A_p^\dagger A_p}. \] (S8)
Therefore, Hamiltonian (S4) can be rewritten as

\[ U^\dagger H U = \Omega \sum_\alpha [\langle \alpha | \mu^x | \alpha + 1 \rangle | e^{-i\frac{2\pi}{N} \sum_{p} A_p^\dagger A_p} \mu^- + |\alpha + 1\rangle | e^{i\frac{2\pi}{N} \sum_{p} A_p^\dagger A_p} \mu^+ ] \]

\[ + \frac{\kappa (\mu^x - 1)}{2\sqrt{N}} \sum_x \left[ \left( e^{-i\frac{2\pi}{N} p} - 1 \right) e^{-i\frac{2\pi}{N} \alpha} A_p^\dagger + \mathrm{h.c.} \right] + \omega \sum_p A_p^\dagger A_p. \]  

(S9)

In order to get the rid of the lattice labels \( \alpha \) we move to the Fourier space for the quasi-particles:

\[ |\alpha\rangle = \frac{1}{\sqrt{N}} \sum_{q=-N/2}^{N/2} e^{i\frac{2\pi}{N} q} |q\rangle \]  

(S10)

We then obtain

\[ \hat{H} = \Omega \sum_q |q\rangle \langle q| \left[ \mu^x + \mu^- e^{-i\frac{2\pi}{N} \sum_{p} A_p^\dagger A_p + q} + \mu^+ e^{i\frac{2\pi}{N} \sum_{p} A_p^\dagger A_p + q} \right] \]

\[ + \frac{\kappa (\mu^x - 1)}{2\sqrt{N}} \sum_x \left[ \left( e^{-i\frac{2\pi}{N} p} - 1 \right) A_p^\dagger + \mathrm{h.c.} \right] + \omega \sum_p A_p^\dagger A_p. \]  

(S11)

Note, that Hamiltonian (S11) is diagonal in the quasi-particles momentum \( q \). Hence, we can diagonalize for every \( q \) the free part of it, i.e. the Hamiltonian corresponding to \( \kappa = 0 \).

**DIAGONALIZATION OF THE FREE PART**

Let us rewrite Eq. (S11) in matrix form, i.e. writing explicitly the matrices \( \mu^x, \mu^\pm \) and completing the squares for the bosonic part

\[ \hat{H}_q = \Omega \begin{pmatrix} 0 & e^{i\frac{2\pi}{N} \sum_{p} A_p^\dagger A_p + q} \\ e^{-i\frac{2\pi}{N} \sum_{p} A_p^\dagger A_p + q} & 1 \end{pmatrix} \]

\[ + \frac{\kappa}{\omega \sqrt{N}} \left( e^{i\frac{2\pi}{N} p} - 1 \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) + \omega \sum_p A_p \left( \begin{array}{cc} 1 & \frac{\kappa}{\omega \sqrt{N}} (e^{i\frac{2\pi}{N} p} - 1) \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \\ \frac{\kappa}{\omega \sqrt{N}} \sum_p \left[ 1 - \cos \left( \frac{2\pi}{N} p \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \right] \right). \]  

(S12)

Defining a displacement operator for the bosons, i.e.

\[ \hat{D} = \exp \left[ -\frac{\kappa}{\sqrt{N} \omega} \sum_p (e^{-i\frac{2\pi}{N} p} - 1) A_p^\dagger - \mathrm{h.c.} \right] \]  

(S13)

such that \( \hat{D}^\dagger \hat{A}_p \hat{D} = A_p \), with \( \hat{A}_p = A_p + \frac{\kappa}{\omega \sqrt{N}} (e^{-i\frac{2\pi}{N} p} - 1) \), we can cast Eq. (S12) in the following form:

\[ \hat{D} \hat{H}_q \hat{D}^\dagger = \omega \sum_p \hat{A}_p^\dagger \hat{A}_p + \Omega \begin{pmatrix} 0 & e^{i\frac{2\pi}{N} \sum_{p} \hat{A}_p^\dagger \hat{A}_p + q} \\ e^{-i\frac{2\pi}{N} \sum_{p} \hat{A}_p^\dagger \hat{A}_p + q} & 1 \end{pmatrix} + \frac{\kappa^2}{\omega} \hat{n} + \Omega \mu^+ (\hat{D} - 1) \left( e^{i\frac{2\pi}{N} \sum_{p} \hat{A}_p^\dagger \hat{A}_p + q} + 1 \right) + \mathrm{h.c.} \]  

(S14)

Note, that the effect of the interaction between the lattice and the phonons is only in the argument of the displacement operator. We can now diagonalize the off-diagonal matrix appearing in (S14). Abbreviating \( \theta = \sum_p p \hat{A}_p^\dagger \hat{A}_p + q \), the matrix we want to diagonalize has therefore the form

\[ \begin{pmatrix} 0 & e^{i2\theta \pi/N} + 1 \\ e^{-2\theta \pi/N} + 1 & 0 \end{pmatrix}. \]  

(S15)
Its eigenvectors are \((-e^{i\theta\pi/N}, 1)\) and \((e^{i\theta\pi/N}, 1)\), therefore the unitary matrix \(S\) which implements the diagonalization is
\[
S = \begin{pmatrix}
-e^{i\theta\pi/N} & e^{i\theta\pi/N} \\
1 & 1
\end{pmatrix}.
\] (S16)

Using it to transform \(\hat{H}_q\), we have
\[
S^\dagger \hat{H}_q S = \omega \sum_p \hat{A}_p^\dagger \hat{A}_p - \Omega \cos \left[ \frac{\pi}{N} \left( \sum_p p \hat{A}_p^\dagger \hat{A}_p + q \right) \right] \mu^z - \frac{\kappa^2}{\omega} (1 - \mu^z) + \Omega S^\dagger \mu^+ S (S^\dagger \hat{D} S - 1) \left( e^{i\frac{2\pi}{N}} + 1 \right) + \text{h.c.},
\] (S17)
i.e. the diagonalization induces a mixing between the states \(|q, \mu^z = 1\rangle\) and \(|q, \mu^z = 2\rangle\). It is straightforward to see that
\[
S^\dagger \mu^+ S = \frac{1}{2} e^{-i\frac{\pi}{N}\theta} (-i\mu^y - \mu^z),
\] (S18)
and therefore
\[
S^\dagger \mu^+ \hat{D} S = -\frac{1}{2} e^{-i\pi/\theta} \hat{D}(\mu^z + i\mu^y).
\] (S19)

Using these expressions we obtain
\[
\Omega [S^\dagger \mu^+ \hat{D} S - S^\dagger \mu^+ S] \left( e^{2i\pi\theta/N} + 1 \right) =
\]
\[
= -\frac{\Omega}{2} e^{-i\pi/\theta\mu^z(\hat{D} - 1)} e^{i\pi/\theta} e^{-i\pi/\theta} \left( e^{2i\pi\theta/N} + 1 \right) [\mu^z + i\mu^y] =
\]
\[
= -\Omega (\hat{D} - 1) \cos \left( \frac{2\pi}{N} \theta \right) [\mu^z + i\mu^y],
\] (S20)
where we defined
\[
\hat{D} = e^{-i\frac{\pi}{N} \sum_p p \hat{A}_p^\dagger \hat{A}_p} e^{-\frac{\pi}{N} \sum_p [(e^{-i2\pi p/N} - 1) \hat{A}_p^\dagger - (e^{i2\pi p/N} - 1) \hat{A}_p] e^{i\frac{\pi}{N} \sum_p p \hat{A}_p^\dagger \hat{A}_p}. \quad (S21)
\]

It is possible to rewrite \(\hat{D}\) by noting that
\[
\left( 1 - i \frac{\pi}{N} p \hat{A}_p^\dagger \hat{A}_p \right) \hat{A}_p \left( 1 + i \frac{\pi}{N} p \hat{A}_p^\dagger \hat{A}_p \right) + \cdots =
\]
\[
= \hat{A}_p - i \frac{\pi}{N} p [\hat{A}_p^\dagger \hat{A}_p, \hat{A}_p] + \cdots =
\]
\[
= \hat{A}_p + i \frac{\pi}{N} p \hat{A}_p + \cdots = e^{i\frac{\pi}{N} p \hat{A}_p}, \quad (S22)
\]
therefore we have
\[
\hat{D} = \prod_{p=-N/2}^{N/2} \hat{D}_p \quad \text{with} \quad \hat{D}_p = \hat{D} \left( -\frac{\kappa}{\omega \sqrt{N}} e^{-i(p/\pi) \mu^z (e^{-i(2\pi/\pi)p} - 1)} \right).
\] (S23)
The complete Hamiltonian is therefore
\[
\hat{H}_q = \omega \sum_p \hat{A}_p^\dagger \hat{A}_p - \Omega \cos \left[ \frac{\pi}{N} \left( \sum_p p \hat{A}_p^\dagger \hat{A}_p + q \right) \right] \mu^z + H_{\text{int}}^1 + H_{\text{int}}^{1\dagger},
\] (S24)
with
\[
H_{\text{int}}^1 = -\frac{\kappa^2}{\omega} (1 - \mu^z)
\] (S25)
and
\[
H_{\text{int}}^{1\dagger} = -\Omega (\hat{D} - 1) \cos \left[ \frac{\pi}{N} \left( \sum_p p \hat{A}_p^\dagger \hat{A}_p + q \right) \right] (\mu^z + i\mu^y) + \text{h.c.}.
\] (S26)
For the leading order correction to the energy of the ground state band \(E_{\text{GS}}(q)\) the only term which gives a non-zero contribution is \(H_{\text{int}}^1\) (see discussion in main text).