D-branes and the conifold singularity

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Abstract

We analyze in detail the description of type IIB theory on a Calabi-Yau three-fold near a conifold singularity in terms of intersecting D-branes. In particular we study the singularity structure of higher derivative $F$-terms of the form $F_g W^{2g}$ where $W$ is the gravitational superfield. This singularity is expected to be due to a one-loop contribution from a charged soliton hypermultiplet becoming massless at the conifold point. In the intersecting D-brane description this soliton is described by an open string stretched between the two D-branes. After identifying the graviphoton vertex as a closed string operator we show that $F_g$'s have the expected singularity structure in the limit of vanishing soliton mass.

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1. Introduction

Solitonic objects play an important role in the understanding of dualities in field theory and string theory. In particular, in the case of $N = 2$ supersymmetric type II - heterotic string duality in four dimensions, an important role is played by the appearance of massless charged hypermultiplets of the gauge fields of the theory, arising from the zero-mass limit of solitonic objects in the theory. In the case of Calabi-Yau compactification of type IIB theories, the solitons are associated with Ramond-Ramond three-branes wrapped around the three-cycles of the Calabi-Yau, that shrink to zero size in the conifold limit \cite{I}. Following Polchinski\cite{II} these Ramond-Ramond solitons can be described as Dirichlet 3-branes. However it is difficult to use Polchinski’s prescription for these curved D-branes for a concrete string theory computation, especially where we would like to think of these D-branes as massless states, appearing for instance in the internal line of a corresponding field theory computation. An important step was taken by Bershadsky, Sadov and Vafa \cite{III} when they proposed that by using T-duality these solitons could be described as solitonic strings, or D-strings. These could then be transformed into fundamental strings using the SL(2,Z) duality of type IIB theory. In the end therefore the solitonic states would be accessible as perturbative string states, where the string propogates in a manifold dual to the original compactification, referred to as the D-manifold. Thus in this picture the effects of solitons interacting with other states in the theory would be perturbatively accessible.

In this note we examine one such application of the proposal of \cite{III}. We compute the topological amplitudes arising from the higher-derivative F-terms of the form $F_g(Z)W^{2g}$, where $W$ is the $N = 2$ gravitational superfield and $F_g$ depends on chiral vector superfields $Z$ (upto a holomorphic anomaly). These occur at genus $g$ respectively in the type II theory \cite{IV}. While in general the existence of such terms was shown, it is extremely difficult to compute exactly the coefficient $F_g$ at all orders. However, it was shown in \cite{III} that in the dual heterotic string theory these amplitudes occur at one-loop (with the excepection of $F_0$...
and $F_1$ which also receive tree-level contributions) and obey the same holomorphic anomaly equations as in the type II theory. Moreover the leading singularities of $F_g$ were studied in the conifold limit, and it was shown that they are universal poles of order $2g - 2$ with coefficients that are given by the Euler number of the moduli space of genus-$g$ Riemann surfaces, in line with the argument of [6] that the physics near the conifold singularity is governed by the $c = 1$ string theory at the self-dual point.

It was also pointed out in [5] that from the effective field theory point of view, the divergence comes from a one-loop graph with the massless charged hypermultiplet going through the loop. Hence in the type II theory it was expected that the leading singularities in $F_g$ should arise from a one-loop diagram involving the would-be massless charged black-hole of the conifold singularity in the internal line. The possibility of such a computation is precisely what is provided by the D-brane description of the neighbourhood of the conifold singularity. In particular, the leading singularity in $F_g$ should arise from a one-loop computation. However since the open string sector is what describes the soliton background, the one-loop refers to the open-string one loop, viz. the annulus. We show in this note that we reproduce the leading singularity behaviour of $F_g$'s as expected in the conifold limit.

In the next section we briefly review the D-manifold description of the type II theory near the conifold singularity and explicitly display the graviphoton operators that lie in the same space-time supersymmetry multiplet as the graviton. In section 3, we show the detailed computation of $F_g$ and show that it agrees with the previously calculated expressions in [5].

2. The graviphoton operator

We shall describe here only the D-manifold necessary for examining the region close to a single conifold singularity. We will not describe more general possibilities which include conifold singularities and we refer the interested reader to [3] and [7]. The region near the
confold singularity is described by two five-branes that intersect each other. The first we may take to be specified by Dirichlet boundary conditions on the directions $x_6$ to $x_9$ given by

$$ (x_6, x_7, x_8, x_9) = v $$

The rest of the directions will satisfy Neumann boundary conditions. The second D-brane is given by specifying the boundary condition

$$ (x_4, x_5, x_8, x_9) = (0, 0, 0, 0) $$

while all the other co-ordinates satisfy Neumann boundary conditions. When $v = 0$, it is clear that the two D-branes will intersect on a 3 + 1 world-volume. We can now examine the different sectors of the string whose end-points lie on the two D-branes, specifying the boundary conditions on both ends of the string. $x_8$ and $x_9$ are DD directions, while $x_4$, $x_5$, $x_6$, $x_7$ are ND directions and DN directions. $x_0$ to $x_3$ obey NN boundary conditions. The massless charged state of interest, which corresponds to a massless hypermultiplet comes from the ground state of the string between these two D-branes in the limit of the parameter $v$ going to 0. The U(1) gauge symmetry under which this state is charged arises from a linear combination of the two boundary U(1) gauge fields that are associated to the two D-branes.

As is well known, in each five-brane sector the correct space-time supersymmetry operator is given by a linear combination of the left and the right moving space-time supersymmetry generators of the form $Q + M\bar{Q}$, where M is made up of a product of gamma matrices. Since we are considering intersecting five-branes we have to find the common eigenvalues of the corresponding M-matrices [8]. This gives, as we shall see below, the effective space-time supersymmetry in the intersecting world-volume to be N=2. We will use this operator to act on the graviton operator and obtain the explicit expression for the anti-self-dual part of the gravi-photon vertex operator.
For the first five-brane the matrix $M$ appearing in the supersymmetry operator is:

$$M^{(1)} = \prod \gamma^6 \gamma^7 \gamma^8 \gamma^9 \gamma^{11} \quad (2.1)$$

Acting on a spinor in 10 dimensions, this matrix has two $+1$ and two $-1$ eigenvalues. For the second five-brane:

$$M^{(2)} = \prod \gamma^4 \gamma^5 \gamma^8 \gamma^9 \gamma^{11} \quad (2.2)$$

which has again two $+1$ and two $-1$ eigenvalues but in different planes. The surviving unbroken supersymmetry is given by the common eigenvalues of the two $M$ matrices which gives a $N = 2$ supersymmetry in the common 3 + 1 world-volume. The relevant chiral generators are

$$Q_1^\alpha + \tilde{Q}_1^\alpha; \quad \text{and} \quad Q_2^\alpha - \tilde{Q}_2^\alpha.$$

The operator $Q_i^\alpha$ is given in the general form

$$Q_i^\alpha \equiv \int e^{-\frac{1}{2} \phi} S^\alpha \Sigma_i$$

where $\phi$ is the bosonization of superghost. $S^\alpha$ and $\Sigma_i$ are the spin fields of the 4-dim. space-time and the internal parts respectively and they can be given explicitly in the bosonized form as follows:

$$S^\pm = e^{\pm \frac{i}{4} (\phi_1 + \phi_2)}$$

where $\phi_1$ and $\phi_2$ are the bosonization of the two complex fermions associated to 4-dim. space-time. Similarly bosonising the complex fermions associated to 4-5, 6-7 and 8-9 planes respectively via scalars $\phi_3$, $\phi_4$ and $\phi_5$ respectively, $\Sigma_k$ for $k = 1, 2$ can be expressed as:

$$\Sigma_k = e^{i\left(\frac{1}{2} - k\right)(\phi_3 + \phi_4) + i\frac{1}{2} \phi_5}$$

Note that the spin field part coming from the mixed ND and DN directions $\phi_3$ and $\phi_4$ appear with the same sign as in a $T^4/Z_2$ orbifold model and, together with the part coming from the DD directions, it has the structure of the internal spin field of the heterotic string.
on $T^2 \times T^4/Z_2$, having $N = 2$ spacetime supersymmetry in 4 dimensions. Following the notation of [3], we write down the graviton vertex operator in the 0-ghost picture,

$$V_{gr}(p_1^+) = (\partial Z_2^\pm + i p_1^+ \psi_1^\pm \psi_2^\pm)(\bar{\partial} \bar{Z}_2^\pm + i p_1^+ \bar{\psi}_1^\pm \bar{\psi}_2^\pm) e^{ip_1^+ Z_1^\pm}.$$ 

The graviphoton is obtained by the action of $(Q_1^a + \bar{Q}_1^a)(Q_2^a - \bar{Q}_2^a)$ on the graviton operator given above. This gives rise to two types of terms; one from the action of $(Q_1^a Q_2^a - \bar{Q}_1^a \bar{Q}_2^a)$ and the other from the action of $(\bar{Q}_1^a Q_2^a - Q_1^a \bar{Q}_2^a)$. The first term is of the NS-NS type:

$$[(e^{-\phi}\psi_5^\pm)(\bar{\partial} \bar{Z}_2^\pm + i p_1^+ \bar{\psi}_1^\pm \bar{\psi}_2^\pm) - (\partial Z_2^\pm + i p_1^- \psi_1^\pm \psi_2^\pm)(e^{-\bar{\phi}} \bar{\psi}_5^\pm)] e^{ip_1^+ Z_1^\pm}.$$ 

while the second one is of the R-R type:

$$i p_1^+ e^{-1/2(\phi + \bar{\phi})} S^+ \bar{S}^+ \Sigma_i \bar{\Sigma}_j \epsilon_{ij}$$

It is clear that the graviphoton is the sum of both these terms; while the second is similar in form to the original type IIB graviphoton operator (compactified on Calabi-Yau), the first is similar in structure to the gravi-photon in the heterotic string (compactified on $T^4/Z_2 \times T^2$). Note also that the first term is actually the anti-symmetric tensor component $B_{\mu I} (I = 8, 9)$ instead of the metric component $G_{\mu I}$. In fact the latter is actually not gauge invariant: indeed one can check that its longitudinal mode, although a total derivative, in general contributes to amplitudes involving world sheets with boundaries. More precisely the boundary operator that it gives rise to is the scalar associated to the overall translation of all the 5-branes. This is similar to what happens with the NS $B_{\mu \nu}$ field whose longitudinal mode is the $U(1)$ gauge field partner of the above scalar. As a result these fields become massive due to Cremmer-Scherk mechanism.

3. Computation of $F_g$

The leading singularity in $F_g$ would appear in effective field theory as a one loop effect involving the would-be massless solitonic state propagating in the loop. Since this solitonic
state is mapped to an open string stretched between two intersecting $D$-branes the relevant world-sheet is an annulus with the two boundaries on the two intersecting $D$-branes respectively. In order to calculate $F_g$ it turns out to be more convenient to consider amplitudes involving $2g$ graviphoton field strengths that appear as the lowest component of $W^{2g}$ and two matter $U(1)$ gauge field strengths appearing in the highest component from the expansion of $F_g$. Since the leading singularity in $F_g$ is expected to be a constant times $\mu^{2-2g}$, where $\mu$ is the mass of the would-be massless soliton, the relevant $U(1)$ gauge field strengths are the ones in the vector multiplet corresponding to the modulus $\mu$. This amplitude therefore will compute $\partial_{\mu}^2 F_g$. As mentioned earlier the vertex for this $U(1)$ gauge field is just the difference of the boundary operator:

$$V_F = \int (\partial_{\tau}X^\mu + ip\psi\gamma_{\tau}\psi^\mu)e^{ip\cdot X}$$  \hspace{1cm} (3.1)$$
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on the two boundaries of the annulus under consideration. We shall take a kinematic configuration so that $g$ of the graviphotons are of the form:

$$V_T(p^-_1) = \frac{1}{p_1^-}(Q^-_1 + \tilde{Q}^-_1)(Q^-_2 - \tilde{Q}^-_2)V_{gr}(p^-_1)$$  \hspace{1cm} (3.2)$$

and similarly $g$ graviphotons with momenta $p^+_1$ which are obtained by exchanging the signs in the superscripts.

These graviphoton vertex operators are in $(-1)$ ghost picture and therefore one needs to insert $2g$ picture changing operators on the annulus. Since each of the graviphoton vertex carries charge 1 (left plus right charge) along the DD plane, the only non-vanishing contribution comes from the part $(e^{\phi}e^{-\phi_5}\partial Z^+_5 + \text{right moving part})$ of the picture changing operator. Here $Z^+_5$ is the complex scalar associated to the DD plane. $\partial Z^+_5$ (and $\bar{\partial} Z^+_5$) are necessarily replaced by zero modes as they cannot contract among themselves. The zero mode contribution can be most easily seen when one represents the annulus by a rectangle defined by $0 \leq \sigma \leq 1$ and $0 \leq \tau \leq t$, where $t$ is the modulus of the world sheet. The various bosonic fields have periodic boundary conditions along $\tau$ direction while along $\sigma$
direction they have the appropriate Neumann or Dirichlet boundary conditions. $Z^+_5$ then has the following mode expansion:

$$Z^+_5 = \bar{\mu}\sigma + \text{oscillators}$$

where $\bar{\mu} = v_8 + iv_9$. Thus each picture changing operator gives a factor of $\bar{\mu}$ so that altogether we get a factor $\bar{\mu}^2g$.

Now we construct the correlation functions between all the vertex operators. First thing to note is that in each of the operators appearing in the correlation function the ratio of the superghost charge and the charge along the DD plane (i.e. the coefficients of $\phi$ and $\phi_5$) is $-1$. This, together with the fact that the spin-structures of the superghost and the DD fermions are the same, implies that the correlation function of the superghost system cancels exactly with that of DD fermion system. Finally we are left with the correlation functions of the remaining 4 complex fermion systems whose charges span a 4-dimensional lattice that is closely related to $SO(8)$ weight lattices. More precisely since two of the complex fermions come with mixed boundary conditions and therefore behave like $Z_2$ twisted fermions the $SO(8)$ weight lattices are shifted by a spinor of the $SO(4)$. In any case, summing over the spin structures using Riemann theta identity is equivalent to using the triality relations of the underlying $SO(8)$ weight lattices. If we denote by $a_i$ the charges of $\phi_i$ (for $i=1,\ldots,4$) then one can show that summing over spin-structure is equivalent to the following triality map:

$$a_1 \rightarrow (a_1 + a_2 + a_3 + a_4)/2$$
$$a_2 \rightarrow (a_1 + a_2 - a_3 - a_4)/2$$
$$a_3 \rightarrow (a_1 - a_2 + a_3 - a_4)/2$$
$$a_4 \rightarrow (-a_1 + a_2 + a_3 - a_4)/2$$

The result of the spin structure therefore is correlation function of the transformed operators in the odd spin structure. Since the charges of all the operators appearing in our amplitude
satisfy \( a_1 = a_2 \) and \( a_3 = a_4 \), we note that the transformed operators do not carry any \( \phi_3 \) and \( \phi_4 \) charges. Thus the contribution of the \( \phi_3 \) and \( \phi_4 \) system exactly cancels the partition function of the bosons along DN and ND directions. Furthermore the bosonic partition function of the DD plane cancels that of the bosonic ghost system and we are just left with the correlation function of the space-time bosons and fermions. The above triality map results in the following transformations of the various operators appearing in the amplitude:

\[
Q_1^+ \rightarrow \int dz \psi_1^+
\]

\[
Q_1^- \rightarrow \int dz \psi_2^-
\]

\[
Q_2^+ \rightarrow \int dz \psi_2^+
\]

\[
Q_2^- \rightarrow \int dz \psi_1^-
\]

and similarly for the right-movers with \( dz \) replaced by \( d\bar{z} \), upto possible relative signs which can be absorbed in the definitions of \( \psi \) and \( \tilde{\psi} \)'s. \( V_{gr} \) and \( V_F \) under the triality map go to themselves. Note that the supersymmetry generators involve the combinations \( Q_1^\pm + \tilde{Q}_1^\pm \) and \( Q_2^\pm - \tilde{Q}_2^\pm \). Under the triality map therefore they go over to the combinations \((\psi_1^\pm + \tilde{\psi}_1^\pm)\) and \((\psi_2^\pm + \tilde{\psi}_2^\pm)\) respectively, where we have made a definite choice of relative signs between the left and the right movers. This choice together with the requirement of the closure of the contours involved in the supersymmetry charges then implies the following boundary conditions for \( \psi_1^\pm \) and \( \psi_2^\pm \):

\[
\tilde{\psi}_1^\pm (z, \bar{z}) = \psi_1^\pm (-\bar{z}, -z)
\]

\[
\tilde{\psi}_2^\pm (z, \bar{z}) = \psi_2^\pm (-\bar{z}, -z)
\]

By applying the supersymmetry generators on the graviton vertices one finds that the graviphoton operators are transformed by triality map to:

\[
V_T(p^\pm) \rightarrow (\partial_\tau Z_2^\pm + ip^\mp_1(\psi_1^\pm - \tilde{\psi}_1^\pm)(\psi_2^\pm - \tilde{\psi}_2^\pm))e^{ip^\mp_1 Z_1^\pm}
\]
The correlation function of the bosonic part of the above vertex operator \( \partial_\tau Z^\pm_2 \) gives a total derivative which upon partial integration brings down one power of momentum \( p^\pm_1 \) together with \( Z^\pm_1 \) in each of the graviphoton vertex. One can now take the zero momentum limit after extracting one momentum from each vertex and then \( F_g \) is related to a correlation function involving \( 2g \) of the operators \( (Z^\pm_1 \partial_\tau Z^\pm_2 + (\psi^\pm_1 - \tilde{\psi}^\pm_1)(\psi^\pm_2 - \tilde{\psi}^\pm_2)) \). One can now write down a generating function for \( F_g \)'s following [5] as:

\[
G(\lambda) = \sum_g \frac{\lambda^{2g}}{(g!)^2} \partial^2 F_g
\]

which amounts to perturbing the sigma model action for space-time bosons and fermions by \( \lambda \bar{\mu} \) times the integral of the above operators and computing the correlation function of the two matter \( U(1) \) gauge fields in the presence of this perturbation.

The important point to note here is that the perturbed sigma model still admits four fermion zero modes given by \( \tilde{\psi}^\pm_1 = \psi^\pm_1 = \text{constant} \) and \( \tilde{\psi}^\pm_2 = \psi^\pm_2 = \text{constant} \). Thus the partition function vanishes. This is to be expected because the partition function computes an amplitude involving \( 2g \) graviphotons which is the lowest component of the corresponding \( F \)-term. We would like to emphasize here that for this vanishing of the partition function it is crucial that the full graviphoton vertex is used in the computation including both the NS-NS part as well as R-R part. In order to obtain a non-zero amplitude one must insert two \( U(1) \) gauge fields as explained earlier. The two \( U(1) \) gauge fields in our case are the vector partners of the modulus field \( \mu \) and are given as boundary operators. The four fermion zero modes are then soaked by the fermion bilinear pieces in the two vertices. Thus the generating function for the \( F_g \)'s (or more precisely \( \partial^2 F_g \)'s) is obtained by computing the non-zero mode determinants of bosons and fermion in the perturbed action.

To compute the non-zero mode determinants let us make the mode expansion for the non-zero mode parts of fermions and bosons:

\[
\psi^\pm_k = \sum_{(m,n) \neq (0,0)} \eta^{\pm(m,n)}_k e^{im\pi\sigma} e^{im\frac{2\mu\tau}{\tau}}
\]
\[ \psi_k^\pm = \sum_{(m,n) \neq (0,0)} \eta_k^{(m,n)} e^{-in\pi \sigma} e^{im\frac{2\pi}{\tau}} \]
\[ Z_k^\pm = \sum_{(m,n) \neq (0,0)} \alpha_k^{(m,n)} \cos(n\pi \sigma) e^{im\frac{2\pi}{\tau}} \]

By plugging in these expansions one can easily see that for \( m \neq 0 \), the boson and fermion determinants cancel. On the other hand for \( m = 0 \), the \( \lambda \)-dependent term in the fermion part of the action drops out while it remains in the bosonic part of the action. The ratio of the fermion and boson determinant then gives

\[ \prod_{n=1}^{\infty} \left[ 1 - \left( \frac{\lambda \mu t}{n\pi} \right)^2 \right]^{-2} = \left| \frac{\lambda \mu t}{\sin(\lambda \mu t)} \right|^2 \]

Putting together the zero mode part of the bosons \( Z_k^\pm \) as well as the boson \( Z_5^\pm \) along DD directions one finds the following result for the generating function:

\[ G(\lambda) \equiv \sum_{g=1}^{\infty} \lambda^{2g} \partial_\mu^2 F_g = \int_0^\infty \frac{dt}{t} \left[ \frac{\lambda \mu t}{\sin(\lambda \mu t)} \right]^2 e^{-|\mu|^2t} \]
\[ = \sum_{g=1}^{\infty} \frac{(2g - 1)}{2g} B_{2g} \mu^{-2g} \tag{3.3} \]

where \( B_{2g} \) are the Bernoulli numbers. Integrating this expression with respect to \( \mu \), one finds the leading singularities in \( F_g \) for \( g \geq 2 \):

\[ F_g \to \chi_g \mu^{2-2g} \]

where \( \chi_g \) is the Euler character of the moduli space of genus \( g \) Riemann surfaces. Similarly by integrating the equation for \( G(\lambda) \) for \( g = 1 \) we find:

\[ F_1 \to -\frac{1}{12} \log \mu \]

The behaviour of the prepotential \( F_0 \) near the conifold can be easily calculated by inserting two \( U(1) \) gauge fields corresponding to the modulus \( \mu \) and the modulus field \( \mu \) itself (the latter is inserted just to get non-vanishing on-shell amplitude). One can then verify that the prepotential has the expected behaviour near the conifold:

\[ F_0 \to \frac{1}{2} \mu^2 \log \mu \]
The above results are in agreement with the singularity structure one expects for $F_g$’s near the conifold as due to the appearance of a massless hypermultiplet. Our calculations here show that the description of the physics near conifold by intersecting D-branes correctly reproduces the singularity structure. Note that in the computation for $F_g$’s all non-trivial dependence on the world-sheet modulus $t$ drops out. This is to be expected, because only BPS-states contribute to $F_g$’s [9] and the only BPS state running through the loop in the above calculation is the ground state of the string stretched between the two intersecting D-branes. Our computation also identifies the graviphoton vertex as a closed string operator involving both NS-NS and R-R parts. On the other hand the sum over all the $U(1)$’s associated to each $D$-brane would not reproduce this singularity structure as can be seen by using the method of [10], and therefore cannot be identified with the graviphoton. In fact this is consistent with the observation that this sum over all $U(1)$’s is eaten up by the NS-NS $B_{\mu\nu}$ via Cremmer-Scherk mechanism.

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