A Combinatorial $\tilde{O}(m^{3/2})$-time Algorithm for the Min-Cost Flow Problem

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February 19, 2014

Abstract

We present a combinatorial method for the min-cost flow problem and prove that its expected running time is bounded by $\tilde{O}(m^{3/2})$. This matches the best known bounds, which previously have only been achieved by numerical algorithms or for special cases. Our contribution contains three parts that might be interesting in their own right: (1) We provide a construction of an equivalent auxiliary network and interior primal and dual points with potential $P_0 = \tilde{O}(\sqrt{m})$ in linear time. (2) We present a combinatorial potential reduction algorithm that transforms initial solutions of potential $P_0$ to ones with duality gap below 1 in $\tilde{O}(P_0 \cdot \text{CEF}(n,m,\varepsilon))$ time, where $\varepsilon^{-1} = O(m^2)$ and $\text{CEF}(n, m, \varepsilon)$ denotes the running time of any combinatorial algorithm that computes an $\varepsilon$-approximate electrical flow. (3) We show that solutions with duality gap less than 1 suffice to compute optimal integral potentials in $O(m + n \log n)$ time with our novel crossover procedure. All in all, using a variant of a state-of-the-art $\varepsilon$-electrical flow solver, we obtain an algorithm for the min-cost flow problem running in $\tilde{O}(m^{3/2})$.

1 Introduction

The min-cost flow problem is one of the most well-studied problems in combinatorial optimization. Moreover, it represents an important special case of Linear Programming due to the integrality of the primal and dual polyhedra for arbitrary given integer costs $c$, capacities $u$, and demands $b$. That is, there are always integral primal and dual optimal solutions provided that the problem is feasible and finite. Since these solutions can be computed in polynomial time, min-cost flow algorithms are important building blocks in tackling many other problems. Combinatorial flow algorithms have dominated in past decades. However, interior point methods have been used more and more to solve several network flow problems, for example very successfully in the case of the max-flow problem, e.g. recently \cite{1}. By now, numerical methods lead the “horse-race” of the most efficient algorithms for various combinatorial problems. This is somewhat unsatisfactory. In particular for sparse graphs, the running time bounds of all combinatorial min-cost flow algorithms known from literature fail to break through the barrier of $n^2$, whereas Daitch and Spielman \cite{2} were the first to present an interior point method running in $\tilde{O}(m^{3/2})$ expected time.\cite{[1]} It is a dual central path following method. However, their algorithms are not combinatorial in any sense. In fact, they solve a more general problem and thus their method is more technical than necessary for the classical min-cost flow problem. It uses an efficient randomized solver for symmetric diagonally dominant (SDD) systems of linear equations based on the seminal work of Spielman and Teng \cite{3} and later by Koutis et al. \cite{4}. Only recently, Kelner et al. \cite{5} presented a simple, combinatorial, nearly-linear time algorithm for the electrical flow problem and thus also for finding approximate solutions to SDD system. It is combinatorial in

\textsuperscript{1}Throughout this paper, the $\tilde{O}$-notation is used to hide log-factors in $n$, $\|c\|_\infty$, $\|u\|_\infty$, $\|b\|_\infty$.\footnote{Throughout this paper, the $\tilde{O}$-notation is used to hide log-factors in $n$, $\|c\|_\infty$, $\|u\|_\infty$, $\|b\|_\infty$.}
the sense that it operates on the rationals and uses only the field operations (addition, subtraction, 
multiplication, division) as arithmetic operations. However, this alone is not sufficient to obtain an entirely 
combinatorial algorithm for the min-cost flow problem.

In this paper, we present a primal-dual potential reduction algorithm that uses a variant of the algorithm 
of Kelner et al. as a subroutine. Moreover, it is combinatorial in the same sense as their algorithm is. In 
particular, our method does not compute square roots or logarithms. The logarithms and square roots in 
this paper solely appear in the analysis, e.g. in the potential function that we use to guide our search. All 
running times in this paper are stated in terms of basic operations that also include comparisons in addition 
to the arithmetic operations.

After constructing an auxiliary network with the same optimum and primal and dual interior solutions 
of sufficient low potential for it, we update these interior points such that the potential function decreases 
by at least some constant in each step. The potential function serves us in two ways: (1) when it drops 
below 0, the duality gap is smaller than 1 and we may stop, (2) it keeps us away from the boundary. 
We thereby take a shortcut through the polyhedron instead of walking on the boundary as with most of 
the combinatorial methods, e.g. minimum-mean cycle canceling. We distinguish primal and dual steps 
and show that \( O(\sqrt{m}) \) steps are sufficient. The combinatorial interpretation is as follows. A primal step 
changes flow along cycles (which could be linear combinations of simple cycles). As mentioned above, 
the updates are guided by the potential function or more precisely by its gradient w.r.t. the primal variables 
at the current point. To this end, the gradient is projected onto the cycle-space. However, if the gradient 
is (nearly) orthogonal to the cycle-space, then we would not make sufficient progress. But in this case, the 
gradient is shallow w.r.t. the cut-space, which is the orthogonal complement of the cycle-space. Hence, we 
can do a dual step by modifying the dual variables corresponding to a cut. Computing the projection is 
equivalent to solving an electrical flow problem, where the resistances of the arcs are higher the smaller the 
corresponding values of the primal variables are. The gradient determines the current sources. Intuitively, 
approaching the boundary is impeded, because arcs with large resistance carry rather small quantities of 
electrical flow. We also give a novel method that takes the points of duality gap less than one and computes 
optimal integral potentials in near-linear time. Before we describe our contribution in more detail, we 
highlight other related work.

1.1 Other Related Work

We denote \( U := \|u\|_{\infty}, C := \|c\|_{\infty}, \) and \( \gamma := \max\{C,U\}. \) Edmonds and Karp [6] gave the first polynomial-
time min-cost flow algorithm in 1970. It can be implemented in \( O(m(m + n \log n) \log U) \) time [7]. Since 
then, there were many contributions on combinatorial flow algorithms. We mention some of the most 
important results such as the strongly polynomial time algorithm by Orlin [8] running in \( O(n^2 \log^2 n + 
mn \log n) \). Further scaling techniques like (generalized) cost-scaling were presented by Goldberg and Tar-
jan [9], and the double scaling technique by Ahuja et al. [10]. The latter yields a running time bound of 
\( O(nm \log \log U \log (nC)) \). As of yet, all combinatorial algorithms are at least quadratic in \( n \) even for sparse 
graphs. Only in the special case of small capacities, the algorithm of Gabow and Tarjan [11], achieves 
\( O(m^{3/2}), \) however its general bound of \( O((m^{3/2}U^{1/2} + \|u\|_1 \log \|u\|_1 \log (nC)) \) is only pseudo-polynomial.

Karmarkar [12] presented a polytime interior point method for solving linear programs in 1984. Af-
fter the ellipsoid method, this was the second type of method with polynomial running time. Karmarkar’s 
algorithm needs \( O(n^{3.5}L^2) \) time, where \( n \) is the number of variables and \( L \) the number of bits in the input. 
The work on interior point methods, and in particular on so-called potential reduction methods, was 
significantly advanced by Ye [13] in 1991. He presented an \( O(n^3 L) \)-time algorithm. Since interior point 
methods are known to be, asymptotically, the fastest methods for solving general linear optimization prob-
lems, there has been a huge interest in their application for solving network flow problems. To the best 
of our knowledge, the first attempt to analyze interior point methods, particularly for min-cost flow, was done 
by Vaidya [14] in 1989. He obtained a running time of \( O(n^2 \sqrt{m \log (n\gamma)}) \), which matched the best known 
bound then up to log-factors. Wallacher and Zimmermann [15] found a combinatorial interior point method 
in 1992, which they analyzed to run in \( O(am^2 L) \). Thus it could not keep up with the best combinatorial 
methods known at that time.
1.2 Our contribution

Our main contribution is a proof of the following theorem.

**Theorem 1.** There is a combinatorial algorithm for the min-cost flow problem terminating in

\[ O(m^{3/2}(\log \gamma + \log n) \log^3 n \log \log n) = \tilde{O}(m^{3/2}) \text{ time, with high probability.} \]

In contrast to previous results with this time bound our algorithm is combinatorial. Moreover, our analysis is less technical and very comprehensible though rigorous. It fits on about 12 pages. More precisely, our contribution contains the following parts that might be interesting on their own.

1. We show that it suffices to compute primal and dual points with duality gap below 1, since our novel crossover procedure finds optimal potentials in linear time then.

2. We give a combinatorial potential reduction method that, taking interior points of potential \( P_0 \), outputs interior points with duality gap below 1 in \( \tilde{O}(P_0 \cdot \text{CEF}(n, m, \varepsilon)) \), here CEF \((n, m, \varepsilon)\) is the complexity of an \( \varepsilon \)-electrical flow computation.

3. We give a method that, taking any min-cost flow problem as input, yields an auxiliary network with the same optimum and interior primal and dual points of potential \( P_0 = O(\sqrt{m}(\log \gamma + \log n)) \) in linear time.

Our crossover procedure takes solutions with duality gap less than one and efficiently rounds the potentials to integral values. Using one max-flow computation in the admissible network, one can also obtain primal optimal solutions. We remark that this max-flow computation is not needed if the input costs are randomly perturbed such that the optimal solution gets unique as it is for example done in [2]. In this case the admissible network is a tree and the corresponding tree solution can be obtained easily. For the combinatorial potential reduction method, we show how to use approximate electrical flow computations to reduce the duality gap of given primal and dual interior points of potential \( P_0 \) below any constant \( c \in \mathbb{R}_{>0} \), and thus its contribution is in \( O(\log n) \). In order to make this method combinatorial, we show how to normalize the cycle around which we are augmenting flow by the infinity norm, as opposed to previous approaches, where the normalization was done with the 2-norm. This would require computing square roots and thus is not allowed in our setting.

1.3 The Min-Cost Flow Problem and its Dual

In its most general form, the min-cost flow problem is stated as follows. Given a directed graph \( G = (V, A) \) with \(|V| = n\) and \(|A| = m\), node demands \( b \in \mathbb{Z}^n \) with \( 1^T b = 0 \), arc costs \( c \in \mathbb{Z}^m \) and arc capacities \( u \in (\mathbb{N} \cup \{\infty\})^m \), find a feasible flow \( x^* \in \mathbb{R}^m \), i.e. \( 0 \leq x^* \leq u \) and \( x^*(\delta^m(v)) - x^*(\delta^m(v)) = b_v \) for every \( v \in V \), such that \( c^T x^* \leq c^T x \) for all feasible flows \( x \) or assert that no such flow exists. However, it is well-known, see e.g. [16], that this problem can be reduced to a setting without capacity constraints and only non-negative costs. Furthermore, we assume w.l.o.g. that the problem is feasible as well as finite from now on. We will discuss how to reduce the general problem to the setting used here in Section 3. For the time being, we write the problem as primal-dual pair

\[ \min \{c^T x : Ax = b \text{ and } x \geq 0\} = \max \{b^T y : A^T y + s = c \text{ and } s \geq 0\}, \]

where \( A \in \{0, 1\}^{n \times m} \) is the node-arc incidence matrix of \( G \), i.e. \( A \) contains a column \( \alpha \) for every arc \((v, w)\) with \( \alpha_v = -1 \), \( \alpha_w = 1 \) and \( \alpha_i = 0 \) for all \( i \notin \{v, w\} \). The overloaded notation \( A \) for the set of arcs as well as for the node-arc incidence matrix is intended, because they are isomorphic.

\[ \text{We write } f(S) := \sum_{a \in S} f_a \text{ for } S \subseteq A \text{ for any vector } f \in \mathbb{R}^m. \]
2 Snapping to the Optimum

In this section, we show that solving the min-cost flow problem approximately, by the means of computing primal and dual solutions \( x^0, s^0 \) of duality gap less than 1, is sufficient, since optimal integral potentials can then be found in linear time. The main underlying idea of our new linear time rounding procedure is the following. We iteratively construct sets \( S^k \), starting with \( S^1 := \{s\} \) for an arbitrary vertex \( s \). During one iteration \( k \), we proceed as follows. Let us first assume \( b(S^k) < 0 \). Then, there has to be an outgoing arc from \( S^k \), otherwise the problem would be infeasible. We enlarge \( S^k \) by the vertex \( v^* \) such that \( a^k = (\hat{v}, v^*) \) for any \( \hat{v} \in S^k \) has minimal slack among all outgoing arcs from \( S^k \) and we increase the potentials \( y_{v^*} \) of all \( v \in V \setminus S^k \) by this minimal slack. It follows that the dual constraint of the arc \( a^k \) is satisfied with slack 0 and all other non-negativity constraints remain fulfilled. The objective value \( b^\tau y \) will be increased by this potential shift, since \( b(V \setminus S^k) > 0 \). In the case \( b(S^k) \geq 0 \), we decrease the potentials in \( V \setminus S^k \), analogously by the minimum slack of all incoming arcs. However to achieve a near-linear running time, these potential changes need to be performed in a lazy way. Using Fibonacci heaps, we can even reduce the running time to \( O(m + n \log n) \). We give the pseudo-code of this method in Algorithm 1 and show its correctness in Theorem 2.

**Theorem 2.** Let Algorithm 1 be initialized with primal and dual solutions \( x^0, s^0 \) with \( x^\tau s^0 < 1 \). The algorithm outputs optimal integral potentials \( y \) in \( O(m + n \log n) \).

**Proof.** We assume w.l.o.g. that the vertices are labeled \( 1, \ldots, n \) in the order in which they are added to \( S \). We show, by induction, that the potentials

\[ y^k_v = \begin{cases} y^0_v + \Delta^{k-1} & k \leq v \\ y^{k-1}_v & k > v \end{cases} \]

are feasible, i.e.

\[ s^k_a := c_a + y^k_v - y^k_w \geq 0 \text{ for all } a = (v, w). \]

For the induction base, note that \( y^1_v \) is just \( y^0_v \) shifted by \( \Delta^0 = -y^0_s \) and hence it constitutes valid potentials. For the inductive step let us consider iteration \( k > 1 \) and let \( a = (v, w) \) be an arbitrary arc. Let \( i := \min\{v, w\} \) and \( j := \max\{v, w\} \). With the convention \( c(i, j) = -c(i, j) \) and thus \( s^k_{(i, j)} = -s^k_{(i, j)} \), we obtain

\[ s^k_{(i, j)} = c(i, j) + y^k_i - y^k_j = c(i, j) + \begin{cases} y^0_i - y^0_j, & k \leq i \\ y^{k-1}_i - (y^0_j + \Delta^{k-1}), & i < k \leq j \\ y^{k-1}_j - y^{k-1}_i, & j < k \end{cases} \]

For the first and third case, we apply the induction hypothesis and obtain \( s^k_{(i, j)} \geq 0 \). For the second case, we
first note that

\[ \Delta^{k-1} = \sigma \cdot c_{a^{k-1}} + y_{j^{k-1}} - y_{j^{k}} \]

where \( \sigma = \begin{cases} 1 & \text{if } b(S^{k-1}) < 0 \text{ or } \delta^m(S^{k-1}) = \emptyset \\ -1 & \text{otherwise} \end{cases} \)

\[ = \sigma \cdot c_{a^{k-1}} + (y_{j^{k-1}} - \Delta^{k-2}) - y_{j^{k}} + \Delta^{k-2} = \sigma \cdot \Delta^{k-1} + \Delta^{k-2} \]

Since \( i < k \leq j \) and thus \((i, j) \in \delta^m(S^{k-1})\), this yields

\[ s_{i, j}^k = c_{(i, j)} + y_{j}^{k-1} - (y_{i}^{k} + \Delta^{k-2}) - \sigma \cdot \Delta^{k-1} = \sigma \cdot \Delta^{k-1} \]

Independent of \( a \) being \((i, j)\) or \((j, i)\), we get \( s_{a}^{k} = s_{a}^{k-1} + \Delta s_{a}^{k-1} \geq 0 \) since \( \Delta s_{a}^{k-1} \) is a minimizer and by the non-negativity of the slacks due to the induction hypothesis. Hence, the output potentials are feasible. In addition, we construct one tight constraint in each iteration, since \( s_{a}^{k} = 0 \) if \( a = a^{k-1} \). Since \( y_{j} = 0 \) and \( c \in \mathbb{Z}^m \), we conclude that after termination \( y \) is integral. Note that the optimum objective value is integer and thus \([b^Ty]_0 < 1\). We have

\[ b^Ty - b^Ty_q = \sum_{v \in V} b_{v}y_{v} - \sum_{v \in V} b_{v}y_{v} - \sum_{v \geq k} b_{v}(y_{v}^{0} + \Delta^{k-1}) - \sum_{v \geq k} b_{v}y_{v}^{k-1} \]

\[ = \sum_{v \geq k} b_{v}y_{v}^{0} + \sum_{v \geq k} b_{v} \Delta^{k-1} - \sum_{v \geq k} b_{v}y_{v}^{0} - \sum_{v \geq k} b_{v} \Delta^{k-2} = -\sigma \cdot \Delta^{k-1} \cdot b(S^{k-1}) \geq 0 \]

because \( \delta^m(S^{k-1}) = \emptyset \) implies that \( b(S^{k-1}) \leq 0 \) or that the instance is infeasible. Since \( b, y \in \mathbb{Z}^m \) and \( b^Ty - [b^Ty_q] < 1 \) we have that \( y \) is optimal. A similar implementation as used for Dijkstra’s or Prim’s algorithms but with two Fibonacci Heaps, one for the nodes adjacent to \( S \) through \( \delta^m(S) \) and \( \delta^m(S^k) \) each, yields the run time of \( O(m + n \log n) \).

### 3 Combinatorial Potential Reduction Algorithm

We will now describe our Combinatorial Potential Reduction Algorithm, it maintains a primal solution \( x \) and dual slacks \( s \). We evaluate such a pair by the potential function

\[ P(x, s) := q \ln(x^Ts) - \sum_{a \in A} \ln(s_as_a) - m \ln m \]

for some scalar \( q = m + p \in \mathbb{Q} \) to be chosen later. Note that the duality gap \( x^Ts - b^Ty = c^Tx \) serves as measure for the distance to optimality of \( x \) and \( s \). An equivalent formulation of the potential function yields

\[ P(x, s) = p \ln(x^Ts) + m \ln \left( \frac{1}{m} \sum_{a \in A} x_as_a \right) - m \ln \left( \sqrt{\prod_{a \in A} s_as_a} \right) \geq p \ln(x^Ts), \quad (1) \]

because the arithmetic mean is bounded by the geometric mean from below. Thus, \( P(x, s) < 0 \) implies \( x^Ts < 1 \). As we have shown in Section\[\text{[3]}\] solutions satisfying \( x^Ts < 1 \) can be efficiently rounded to integral optimal solutions. Thus, we follow the strategy to minimize the potential function by a combinatorial gradient descent until the duality gap drops below 1. \[\square\]

To this end, we shall project the gradient \( g := \nabla_x P = \frac{1}{x} x - X^{-1} 1, \text{where } X := \text{diag}(x), \text{on the cycle space of the network. However, we do not use the standard scalar product for the projection but a skewed one as it is common in the literature on interior point methods. This skewed scalar product may also be considered as the standard one in a scaled space where } x  \text{ is mapped to } X^{-1} x = 1. \text{ By setting } s':= Xs, \text{ the duality gap } x^Ts' = 1^Ts' \text{ and the potential function } P(x, s') = P(1, s') \text{ remain unchanged. Accordingly, we define } \hat{A} := AX \text{ and } g' := \nabla_{s'} P|_{x=1, s=x} = Xg. \]

We start with given initial primal and dual solutions \( x, s \) or rather with their analogs \( 1, s \) in the scaled space, which may be found for example with our initialization method described in Section\[\text{[4]}\]. Now, it would be desirable to move \( x \) in the direction of \( -g' \), the direction of steepest descent of the potential.

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\[\text{This method is similar to Ye’s primal-dual algorithm \[\text{[13]}\]. We mostly follow the notation and proof strategy from lecture notes of Michel Goemans on Linear Programming.}\]
function. However, \(g'\) may not be a feasible direction, since \(\bar{A}g' \neq 0\) in general. Thus, we wish to find a direction \(d'\) in the kernel of \(A\) that is closest to \(g'\). Computing \(d'\) amounts to solve the optimization problem

\[
\min\{\|g' - d'\|^2 : \bar{A}d' = 0\} = \min\{\|f\|^2 : Af = \chi\},
\]

where we set \(f = X(g' - d')\), \(R = X^{-2}\) and \(\chi = \bar{A}g'\). The latter is actually an electrical flow problem. We briefly review electrical flows, for more details, see for example [5].

### 3.1 Electrical Flows

Let \(\chi \in \mathbb{Q}^n\) be a current source vector with \(1^T \chi = 0\) and let \(r \in \mathbb{Q}_{>0}^m\) be a resistance vector on the arcs, denote \(R = \text{diag}(r)\) and \(\|v\|_R := \sqrt{v^TRv}\) for \(v \in \mathbb{R}^m\).

**Definition 1 (Electrical Flow).** Let \(\chi \in \mathbb{Q}^n\) with \(1^T \chi = 0\).

1. The unique flow \(f^* \in \mathbb{Q}^m\) with \(\|f^*\|^2_R = \min\{\|f\|^2_R : Af = \chi\}\) is the electrical flow.
2. Let \(\varepsilon \geq 0\) and \(f \in \mathbb{R}^m\) with \(Af = \chi\) and \(\|f\|^2_R \leq (1 + \varepsilon)\|f^*\|^2_R\), then \(f\) is called an \(\varepsilon\)-electrical flow.
3. Let \(s\) be a fixed node, \(T\) a spanning tree, \(P(s,v)\) the unique path in \(T\) from \(s\) to \(v\) and \(f \in \mathbb{R}^m\). The tree induced voltages \(\pi \in \mathbb{R}^n\) are defined by \(\pi(v) := \sum_{a \in P(s,v)} ra\).
4. For any \(a = (v,w) \in A\setminus T\), we define \(C_a := \{a\} \cup P(v,w)\) and \(r(C_a) := \sum_{b \in C_a} rb\). We write \(\tau(T) := \sum_{a \in A \setminus T} r(C_a)/ra\) for the tree condition number of \(T\).

The dual of the electrical flow problem is \(\max\{2\pi^T \chi - \pi^T A^{-1}A^T \pi : \pi \in \mathbb{R}^n\}\), where \(\pi\) are called voltages. We conclude that an optimal solution \(\pi^*\) satisfies \(AR^{-1}A^T \pi^* = \chi\).

**Definition 2 (Certifying \(\varepsilon\)-Electrical Flow Algorithm).** Let \(\varepsilon > 0\). A certifying \(\varepsilon\)-electrical flow algorithm is an algorithm that computes an \(\varepsilon\)-electrical flow \(f\) and voltages \(\pi \in \mathbb{Q}^n\) such that

\[
\|\pi - \pi^*\|^2_{AR^{-1}A^T} \leq \varepsilon\|\pi^*\|^2_{AR^{-1}A^T},
\]

where \(\pi^*\) is an optimal dual solution. We define \(\text{CEF}(n,m,\varepsilon)\) to be a bound on the running time of a certifying \(\varepsilon\)-electrical flow algorithm for directed graphs with \(n\) nodes and \(m\) arcs.

Kelner et al. [5] present a combinatorial \(\varepsilon\)-electrical flow algorithm with expected approximation guarantee. However, we transform their algorithm to one with an exact approximation guarantee and linear running time with high probability. Similarly to them, we compute a low-stretch spanning tree \(T\) (w.r.t. the resistances), which has tree condition number \(\tau(T) = O(m\log n\log\log n)\) using the method of Abraham and Neiman [17] that runs in \(O(m\log n\log\log n)\). We then sample non-tree edges \(a\) according to the same probability distribution \(p_a := \frac{1}{(\tau(T))^2} \frac{r(C_a)}{ra}\) and push flow along the cycle \(C_a\) until the gap between primal and dual objective value becomes less than \(\varepsilon\). The running time of this approach is \(O(m\log^2 n\log(1/\varepsilon)\log\log n) = \tilde{O}(m)\) for \(\varepsilon^{-1} = O(\text{poly}(n))\) with high probability as we show in Theorem 3. Note that it suffices for our purpose to mimic their \texttt{SimpleSolver}, which scales with \(\log(n/\varepsilon)\) instead of \(\log(1/\varepsilon)\) as their improved version does. We remark that, as in their solver, the flow updates should be performed using a special tree data structure [5 Section 5], which allows updating the flow in \(O(\log n)\). Moreover, gap should only be computed every \(m\) iterations, which results in \(O(1)\) amortized time per iteration for the update of gap.

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4 The kernel of \(\bar{A}\), up to the scaling with \(X\), corresponds to the cycle space of the graph.
Algorithm 2: Combinatorial Potential Reduction Algorithm

Input: Feasible flow $x > 0$ and feasible dual variables $y$ and $s > 0$, parameter $\delta$
Output: Feasible flow $x > 0$ and feasible dual variables $y$ and $s > 0$ s.t. $x^T s < 1$.

while $x^T s \geq 1$ do
  $g' := \frac{2}{X} X s - I, \chi := \tilde{A} g', r := X^{-2} I$
  /* $\varepsilon$-electrical flow computation, similar to SimpleSolver in [5] */
  $T :=$ low-stretch spanning tree w.r.t. $r$, $\tau(T) := \sum_{a \in A \setminus T} \frac{r(C_a)}{r_a}, P_a := \frac{r(C_a)}{\tau(T) r_a}$
  $f :=$ tree solution with $A f = \chi$ for $T$, $\pi :=$ tree induced voltages of $f$
  gap := $f^T R f - 2 \pi^T \chi + \pi^T \tilde{A}^T \pi$
  repeat
    Randomly sample $a \in A \setminus T$ with probability $p_a$
    Update $f$ by pushing $\sum_{b \in C_a} r_b f_b / r(C_a)$ flow through $C_a$ in the direction of $a$
    Occasionally compute tree-induced voltages $\pi$ and gap
  until gap $\leq \delta$
  /* Move in primal or dual direction. */
  Set $\tilde{X} := g' - X^{-1} f, \tilde{\chi} = \tilde{A}^T \pi$ and $\tilde{z}' = g' - \tilde{\chi}$.
  if $\| \tilde{z}' \|_2 \geq 1/4$ then
    Do a primal step, i.e. $X := I - \lambda \frac{\tilde{g'}}{\max \{ 1, \| \tilde{z}' \|_2 \}}$, where $\lambda = 1/4$.
  else
    Do a dual step, i.e. $s' := s' - \mu \tilde{z}'$ and $y := y + \mu \pi$, where $\mu = \frac{\| f \|_2}{\| X \|_2}$.
  return $x$ and $y, s$

3.2 The Method

Using any certifying $\varepsilon$-electrical flow algorithm, we can compute an approximation of $d'$ by solving problem (2) and obtain an $\varepsilon$-electrical flow $f$. In the electrical flow problem the resistances $R$ are given by $X^{-2}$ and the current sources $\chi$ by $\tilde{A} g'$. We compute a cycle $\tilde{X} = g' - X^{-1} f$ from the flow as well as a cut $\tilde{\chi} = \tilde{A}^T \pi$ from the voltages $\pi$. The idea is to push flow around the cycle $\tilde{X}$ in a primal step, whereas, in a dual step, we modify the slacks along the cut $\tilde{\chi}$. In Ye’s algorithm the decision whether to make a primal or dual step is made dependent on $\| d' \|_2$. In our setting, however, we do not know the exact projection $d'$ of $g'$. Nevertheless, we can show that the 2-norm of $\tilde{X} = g' - \tilde{\chi}$ does not differ too much from $\| d' \|_2$, so deciding dependent on $\| \tilde{X} \|_2$ is possible. We note that another crucial difference between Ye’s algorithm and our Combinatorial Potential Reduction Algorithm is that we normalize by $\max \{ 1, \| \tilde{X} \|_\infty \}$ in the primal step, whereas in Ye’s algorithm the normalization is done with $\| \tilde{X} \|_2$, which requires taking square roots and could thus yield irrational numbers.

We remark that our method works with any certifying $\varepsilon$-electrical flow algorithm. However, we merge the version of the SimpleSolver of Kelner et al. [5] as described above in our pseudocode implementation of Algorithm 2 to be more self-contained.

3.3 Analysis

It is not hard to see that the primal and dual steps in the algorithm are in fact feasible moves.

Lemma 1. The new iterates $\tilde{x} = X(1 - \lambda \frac{\tilde{g'}}{\max \{ 1, \| \tilde{X} \|_\infty \}})$, $\tilde{y} = y + \mu \pi$ and $\tilde{s} = X^{-1} (s' - \mu \tilde{z}')$ are feasible.

Proof. Clearly, $A \tilde{x} = A \tilde{X} = \tilde{A} (g' - X^{-1} f) = \chi - A f = 0$. Note that $X \tilde{x} > 0$ if and only if $\tilde{x}' > 0$. It holds that

$$\tilde{X}' = 1 - \lambda \frac{\tilde{g'}}{\max \{ 1, \| \tilde{X} \|_\infty \}} \geq 1 - \lambda \frac{\| \tilde{X} \|_\infty}{\max \{ 1, \| \tilde{X} \|_\infty \}} \geq 1 - \lambda = 3/4 \quad \text{for every } a \in A.$$
For the dual variables, we have \( A^T(y + \mu \pi) + X^{-1}(s' - \mu \delta') = A^T y + \mu A^T \pi + s - \mu X^{-1} A^T \pi = c \). In addition, we obtain
\[
\hat{s}'_a = s'_a - \mu \delta'_a = x_a s_a - \frac{x^T s}{q} (g'_a - \delta'_a) = \frac{x^T s}{q} (1 + \delta'_a) \geq \frac{1}{2q} > 0,
\]
since \(|\delta'_a| \leq \|\delta'\|_\infty \leq \|\delta'\|_2 \leq 1/2 \) and \( x^T s \geq 1 \).

The following lemma shows that the potential is reduced by a constant amount in each step. We remark that although the proof for the dual step is essentially similar to the proof for Ye’s algorithm, the normalization with the \( \infty \)-norm requires non-trivial changes in the proof for the primal step.

**Lemma 2.** If \( \delta \leq 1/8 \) and \( p^2 \geq m \geq 4 \), the potential reduction is constant in each step.

\[
P(1, s') - P(1 - \lambda \nu, s') \geq 1/64 \quad \text{and} \quad P(1, s') - P(1, s' - \mu \delta') \geq \frac{1}{12}.
\]

**Proof.** 1. We first show the estimate for the primal step. Let \( v \) be any vector with \( \|v\|_\infty \leq 1 \), then
\[
P(1, s') - P(1 - \lambda \nu, s') = -q \ln \left( 1 - \lambda \frac{v^T s'}{\max\{1, \|s'\|_\infty\}} \right) + \sum_{a \in A} \ln (1 - \lambda v_a)
\]
\[
\geq q \lambda \frac{v^T s'}{\|s'\|_\infty} - \lambda \sum_{a \in A} v_a - \frac{\lambda^2}{2(1 - \lambda)} \sum_{a \in A} v_a^2 = \lambda g^T v - \frac{\lambda^2}{2(1 - \lambda)} \|v\|_2^2.
\]
where the inequality follows because \( \ln (1 + \gamma) \geq \gamma - \gamma^2 / (2(1 - |\gamma|)) \) for any \( \gamma \in (-1, 1) \). The variable gap from Algorithm\[\text{II}\] can be written as
\[
gap := f^T R f - 2\pi T x + \pi T A^T \pi = \|g' - \delta'\|_2^2 - 2g^T (g' - \delta') + \|g' - \delta'\|_2^2
\]
\[
= \|g' - \delta'\|_2^2 - \|g'\|_2^2 + \|\delta'\|_2^2 = -2g^T \delta' + \|\delta'\|_2^2 + \|\delta'\|_2^2,
\]
which for the primal step, where \( \|\delta'\|_2 \geq 1/4 \), yields the estimate
\[
2g^T \delta' = \|\delta'\|_2^2 + \|\delta'\|_2^2 \geq \|\delta'\|_2^2 + 1/8, \quad \text{since} \ (\text{gap} < \delta \leq 1/8).
\]

**Case** \( \max\{1, \|\delta'\|_\infty\} = 1 \): Then, from (3) with \( v = \delta' \) and (5) we obtain
\[
P(1, s') - P(1 - \lambda \delta', s') \geq \frac{\lambda (\|\delta'\|_2^2 + 1/8)}{2} - \frac{\lambda^2}{2(1 - \lambda)} \|\delta'\|_2^2
\]
\[
= \frac{1}{2} \left( (\lambda - \lambda^2) \|\delta'\|_2^2 + \frac{\lambda}{8} \right) \geq \frac{1}{64} \quad \text{for} \ \lambda = 1/4.
\]

**Case** \( \max\{1, \|\delta'\|_\infty\} = \|\delta'\|_\infty \): We use (3) with \( v = \delta' / \|\delta'\|_\infty \) and (5). Then we conclude
\[
P(1, s') - P(1, s') \geq \frac{\lambda (\|\delta'\|_\infty^2 + 1/8)}{2\|\delta'\|_\infty^2} - \frac{\lambda^2}{2(1 - \lambda)} \|\delta'\|_\infty^2
\]
\[
\geq \frac{1}{2} \left( (\|\delta'\|_\infty^2 + 1/8) \lambda - \frac{\lambda^2}{1 - \lambda} \right) \|\delta'\|_\infty^2 \geq \frac{1}{12} \quad \text{for} \ \lambda = 1/4.
\]

2. For the dual step, observe that \( s' := s' - \mu \delta' = \frac{\lambda' s'}{\|s'\|_\infty} (1 + \delta') \). We obtain
\[
P(1, s') - P(1, s') = -q \ln \left( \frac{m + \|T s'\|}{q} \right) - \sum_{a \in A} \ln s'_a + \sum_{a \in A} \ln s'_a
\]
\[
\geq -q \ln \left( \frac{m + \|T s'\|}{q} \right) - m \ln \left( \frac{\|T s'\|}{m} \right) + \sum_{a \in A} \ln \left( \frac{\|T s'\|}{q} (1 + z'_a) \right)
\]
\[
= -p \ln \left( 1 + \frac{\|T s'\| - p}{q} \right) - m \ln \left( 1 + \frac{\|T s'\|}{m} \right) + \sum_{a \in A} \ln (1 + z'_a),
\]
In the dual step we have \( P(1, s') - P(1, s) \geq \frac{p^2 - p\|\zeta\|^2}{q} - \|\zeta\|^2 + \frac{\|\zeta\|^2}{2(1 - \|\zeta\|_\infty)} \geq \frac{p^2 - p\sqrt{m} \|\zeta\|_2}{p + m} - \frac{\|\zeta\|_2^2}{2(1 - \|\zeta\|_2)} \geq \frac{p^2 - p^2/2}{p + p^2} \geq \frac{1/4}{2(1 - 1/2)} = \frac{p}{2(p + 1)} Q \geq \frac{1}{12}

using \( p^2 \geq m \geq 4 ] and \( \|\zeta\|_2 < 1/2 \).\)

We already remarked that \( P(x, s) < 0 \) implies \( x^Ts < 1 \), hence \( P(x, s) \geq 0 \) holds throughout the algorithm. With Lemma 3.2 we can also keep running the algorithm until \( 16 \) iterations is bounded by \( O(m \log^3(m) \log \log m) \).

Theorem 3. Given primal and dual interior points with potential \( P_0 \) as input, Algorithm 3.2 outputs interior primal and dual solutions \( x \) and \( y \) with \( x^Ts < 1 \) after \( O(P_0) \) iterations. It can be implemented such that it terminates after \( O(P_0 \cdot m \log^3(m) \log \log m) \) time with probability at least \( 1 - \exp(-m \log^3(m) \log \log m) \).

Proof. Kelner et al. give the following convergence result [5, Theorem 4.1]

\[ E[f_j^T Rf_j - f^*^T Rf^*] \leq \left(1 - \frac{1}{\tau}\right) (f_0^T Rf_0 - f^*^T Rf^*), \]  

(6)

here \( f^* \) denotes an optimal electrical flow and \( f_j \) the flow computed in the \( j \)th iteration. Let \( \text{gap}_j \) denote the value of gap in the \( j \)th iteration and let \( X \) denote a random variable counting the number of iterations of the Repeat-Until loop in Algorithm 3.2. It follows that

\[ \Pr[X > i] = \prod_{j=1}^i \Pr[f_j^T Rf_j - 2\pi_j^T X + \pi_j^T \tilde{A}^T \pi_j \geq \delta] \leq \prod_{j=1}^i \Pr[f_j^T Rf_j - f^*^T Rf^* \geq \frac{\delta}{\tau}], \]

since \( f_j^T Rf_j - f^*^T Rf^* \geq \text{gap}_j / \tau \), see [5, Lem. 6.2]. Using Markov’s bound, equation (6) and the bound on the initial energy \( f_0^T Rf_0 \leq sl(T) f^*^T Rf^* \), see [5, Lem. 6.1], yields

\[ \Pr[X > i] \leq \prod_{j=1}^i \frac{E[f_j^T Rf_j - f^*^T Rf^*]}{\delta/\tau} \leq \left(\frac{\tau}{\delta}(f_0^T Rf_0 - f^*^T Rf^*)\right)^i \left(1 - \frac{1}{\tau}\right)^{i^{\left\lfloor \frac{\delta}{\tau} \right\rfloor}} \leq \left(\frac{\tau}{\delta}(sl(T) - 1)\|g^* - d^*\|_2\right)^i \left(1 - \frac{1}{\tau}\right)^{i^{\left\lfloor \frac{\delta}{\tau} \right\rfloor}} \leq \exp(-m \log^3(m) \log \log m) \]

with \( i = O(m \log^2(m) \log \log m) \), and the guarantee on the low-stretch spanning tree of Abraham and Neiman [17], which yields \( \tau = O(sl(T)) = O(m \log(m) \log \log m) \). Hence, the number of times the Repeat-Until loop is executed during one of the \( O(P_0) \) iterations is bounded by \( O(m \log^2(m) \log \log m) \) with exponentially high probability. We remark that the updates of the flow \( f \) should not be done in the naive way but using a simple data structure exactly as it is also described in Kelner et al [5] for their SimpleSolver. One iteration takes \( O(\log n) \) time then. We can compute gap in every \( m \)th iteration in \( O(m) \) time, which yields that we make at most \( m \) steps to much and need amortized constant run-time for the update of gap in each iteration. This yields the bound.\)

We remark that we can also keep running the algorithm until \( x^Ts < c \) for any \( c \in \mathbb{R}^\geq 0 \) without affecting the running time. In addition, we get the following more general result. To prove it, it remains to show that a \( 1/(16q^2) \)-electrical flow fulfills gap \( \leq 1/8 \).
Theorem 4. Given primal and dual interior points with a potential of $P_0$ as input, there is a combinatorial algorithm that outputs interior primal and dual solutions $x$ and $y$ with $x^T s < 1$ and needs

$$O(P_0 \cdot \text{CEF}(n, m, 1/(16q^2)))$$

time, where $q = m + \min\{k \in \mathbb{Z} : k^2 \geq m\}$ and CEF($n, m, \epsilon$) is the running time of a certifying $\epsilon$-electrical flow algorithm.

Proof. It remains to show that a $1/(16q^2)$-electrical flow fulfills gap $\leq 1/8$. The approximation guarantee from the certifying $\epsilon$-electrical flow algorithm for the primal and dual solution yield

$$
\|g' - \bar{c}\|_2^2 \leq (1 + \epsilon)\|g' - d'\|_2^2 \quad \text{and} \quad \|\pi - \pi^*\|_{\bar{A}^T}^2 \leq \epsilon\|\pi^*\|_{\bar{A}^T}^2,
$$

the second guarantee equivalently writes as

$$
\epsilon\|g' - d'\|_2^2 \geq \|\bar{c}' - d'\|_2^2 = \|\bar{c}'\|_2^2 - 2\bar{d}^T (g' - \bar{A}^T \pi) + \|d'\|_2^2 = \|\bar{c}'\|_2^2 - \|d'\|_2^2.
$$

Together with (4) and (7), we obtain

$$
gap = \|g' - \bar{c}\|_2^2 - \|g'\|_2^2 + \|\bar{c}'\|_2^2 \leq (1 + \epsilon)\|g' - d'\|_2^2 + \epsilon\|g' - d'\|_2^2 + \|d'\|_2^2 \leq 2\epsilon\|g' - d'\|_2^2 \leq 2\epsilon q^2 \leq \frac{1}{8}.
$$

4 Initialization

In this section, we describe how to find initial points with $P_0 = \hat{O}(\sqrt{m})$ that we can use to initialize Algorithm\footnote{We assume w.l.o.g. that the given min-cost flow instance is finite, that the capacities are finite and that the costs are non-negative. In order to be self-contained, we also justify these assumptions.}

We first describe how one recognizes unbounded instances. Consider the graph $G_\infty = (V,A_\infty)$, where $A_\infty := \{a \in A_0 : u_a = \infty\}$ denotes the set of arcs with infinite capacity. By running a shortest path algorithm for graphs with possibly negative arc length, as for example the one presented by Goldberg in [18], we can detect whether $G_\infty$ contains a negative cycle in $O(\sqrt{m}m \log C)$ time. If there is such a uncapacitated negative cycle, the problem is unbounded and the solution is $-\infty$, otherwise the solution is finite. Now, since we know the problem is finite, provided that it is feasible as well, there will always be an optimal basic solution. Hence, the maximum flow on any arc in this solution will be bounded by $\|b\|_1/2$. Hence, we set the capacity of every uncapacitated arc to $u_a = \|b\|_1/2$. There is also a well-known technique to remove the negative costs: Saturate the arcs with negative cost and consider the residual network, this gives an equivalent problem with $c \geq 0$. Note that the increase in $\|b\|_1$ due to this construction is only polynomial. We remark that we do not need to check feasibility, since the crossover procedure presented above enables us to recognize infeasibility. This is described at the end of this section.

4.1 Removing Capacity Constraints

Using a standard reduction, we modify the network in order to get rid of the upper bound constraints $x \leq u$. We briefly review the construction since we will later extend it to obtain the auxiliary network flow problem with interior primal and dual points of low potential. Let $G_0 = (V_0,A_0)$ denote the original input graph. For an edge $a = (v,w) \in A_0$, we proceed as follows, see from left to middle in Figure 1. Remove $a$, insert a node $vw$, insert arcs $\hat{a} = (v,vw)$ and $\bar{a} = (w,vw)$ with $\hat{c}_a = c_a$ and $\bar{c}_a = 0$, respectively\footnote{The accents reflect the direction in which the arc is drawn in Figure 1}. Moreover, set $\hat{b}_{vw} = u_a$ and subtract $u_a$ from $b_{vw}$.

4.2 Finding the Initial Flow

Recall the equivalent form of the potential function in \[1\]. It illustrates that the potential becomes small if the ratio between the arithmetic and the geometric mean does. This in turn is the case if the variance of
Figure 1: The transition from the left to the middle, which is done for each arc, removes the capacity constraint. From the middle to the right: In order to balance the \( x_a s_a \), we introduce the arc \( \hat{a} = (v, w) \) with high cost \( c_{\hat{a}} \) and reroute flow along it. The direction of \( \hat{a} \) depends on a tree solution \( z \) in \( G_0 \). It is flipped if \( z_a \geq u_a / 2 \).

The \( x_a s_a \) is low over all \( a \in A \). This observation is crucial for our Algorithm\textsuperscript{3} that finds an initial flow with low potential. Our aim is to balance the flows on \((v, vw)\) and \((w, vw)\), by introducing the arc \((v, w)\) or \((w, v)\), see Figure\textsuperscript{1} from middle to right. Since we perform the two transitions of Figure\textsuperscript{1} together, we will refer to the resulting graph as \( G_1 = (V_1, A_1) \) with \(|V_1| = n_1\) and \(|A_1| = m_1 = 3w\). For the sake of presentation, we assume w.l.o.g. that all capacities \( u_a \) of the original graph \( G \) were odd such that \( z_a - u_a / 2 \neq 0 \) for all integers \( z_a \).

Algorithm 3: Balance Arcs

\begin{itemize}
  \item \textbf{Input}: \( G_0 = (V_0, A_0) \), parameter \( t \).
  \item \textbf{Output}: Graph \( G_1 \), primal and dual solutions \( x, y, s \), such that \( x_a s_a \in [t, t + CU / 2] \).
\end{itemize}

\begin{algorithm}
  \textbf{Compute a tree solution in} \( G_0 \) \textbf{and obtain an integral (not necessarily feasible) flow} \( z \).
  \for \textbf{for every arc} \( a \in A_0 \)
  \begin{algorithmic}[1]
    \State Insert node \( vw \), arcs \( \hat{a} = (v, vw) \), \( \hat{a} = (w, vw) \) with \( c_{\hat{a}} = c_a \), \( c_{\hat{a}} = 0 \), set \( x_{\hat{a}} = x_{\hat{a}} = u_a / 2 \)
    \If {\( z_a > u_a / 2 \)}
      \State Replace a by \( \hat{a} = (v, w) \)
    \Else
      \State Replace a by \( \hat{a} = (w, v) \)
    \EndIf
    \State \( c_{\hat{a}} = \lceil t / (z_a - u_a / 2) \rceil \), \( x_{\hat{a}} = \lceil z_a - u_a / 2 \rceil \), \( y_{vw} := -2t / u_a \) and \( y_v, y_w := 0 \)
  \end{algorithmic}
  \textbf{return} the resulting graph \( G_1 = (V_1, A_1) \) and \( x, y \) with corresponding slacks \( s \).
\end{algorithm}

Theorem 5. Let \( G_1 = (V_1, A_1) \), \( x, y, s \) be output by Algorithm\textsuperscript{3} and \( \Gamma := \max \{ C, U ; \| b \|_1 / 2 \} \).

1. It holds that \( x_a s_a \in [t, t + \Gamma] \) for all \( a \in A_1 \).

2. Setting \( t = m \Gamma^3 \) and \( p = \min \{ k \in \mathbb{Z} : k^2 \geq m_1 \} \) yields \( P(x, s) = O(\sqrt{m} \log(n \gamma)) \).

Proof.

1. Let \( a \in A_0 \) be any arc in \( G_0 \). We have \( x_a = x_{\hat{a}} = u_a / 2 \). It holds that,

\[ x_{\hat{a}} s_{\hat{a}} = \frac{u_a}{2} \left( c_{\hat{a}} + \frac{2t}{u_a} \right) = t + \frac{u_a c_{\hat{a}}}{2} \leq t + \Gamma^2 , \quad x_{\hat{a}} s_{\hat{a}} = \frac{u_a}{2} \left( c_{\hat{a}} + \frac{2t}{u_a} \right) = t \quad \text{and} \]

\[ x_{\hat{a}} s_{\hat{a}} - \left| \frac{u_a}{2} \right| t = t \quad \text{and} \quad x_{\hat{a}} s_{\hat{a}} - \left| \frac{u_a}{2} \right| t \left( \frac{t}{z_a - u_a / 2} \right) = t + \Gamma . \]

\footnote{This is justified by the following argument: If the capacity of an arc is even, then we add a parallel arc of capacity 1 and reduce the capacity of the original arc by 1.}
2. We consider the potential function with $q = m_1 + p$, we will fix $p$ below. We have
\[
P(x, s) = q \ln(x^T s) - \sum_{a \in A_1} \ln(x_a s_a) - m_1 \ln m_1 \leq q \ln (m_1 t + 2m^2) - m_1 \ln(m_1 t)
\]
\[
\leq q \ln \left(1 + \frac{m_1 t^2}{m_1 t}\right) + p \ln(m_1 t) \leq \frac{m_1 + p}{m_1 t} \ln(m_1 t) + p \ln m_1 t^2, \text{since } t = m^2 3.
\]

For $p = \min\{z \in \mathbb{Z} : z^2 \geq m_1\}$, we get $P(x, s) = \mathcal{O}(\sqrt{m \log m \Gamma}) = \mathcal{O}(\sqrt{m \log n} \gamma)$.

Algorithm\[\text{3}\] can be implemented in $\mathcal{O}(m)$ time. We remark that, due to the high costs of the arcs in $A_1 \setminus A_0$, there will never be flow on them in an optimal solution. In particular, these arcs are more expensive than any path in the original network because $c_\hat{a} = |t/|z_a - u_a/2| | \geq mCU$. Therefore, the optimum of the problem is not changed by the introduction of the arcs $\hat{a}$. Note that the resulting network is always feasible. This is why we can assume feasibility in Section\[\text{2}\].

5 Summary

We first run Algorithm\[\text{3}\] on the input graph $G_0$ to construct the auxiliary network $G_1$. We then initialize Algorithm\[\text{2}\] with the obtained interior points. If $|b^T y| > mCU$, the problem in $G_0$ was infeasible, since any solution in $G_0$ is bounded by $mCU$. Otherwise, we apply Algorithm\[\text{1}\] and obtain optimal integral potentials $y$ in $G_1$. Let $H_1$ be the admissible network, i.e. the graph $G_1$ with all arcs with dual slack 0. Notice that the resulting network is always feasible.

We consider the potential function with $q = m_1 + p$, we will fix $p$ below. We have
\[
P(x, s) = q \ln(x^T s) - \sum_{a \in A_1} \ln(x_a s_a) - m_1 \ln m_1 \leq q \ln(m_1 t + 2m^2) - m_1 \ln m_1 t
\]
\[
\leq q \ln \left(1 + \frac{m_1 t^2}{m_1 t}\right) + p \ln(m_1 t) \leq \frac{m_1 + p}{m_1 t} \ln(m_1 t) + p \ln m_1 t^2, \text{since } t = m^2 3.
\]

For $p = \min\{z \in \mathbb{Z} : z^2 \geq m_1\}$, we get $P(x, s) = \mathcal{O}(\sqrt{m \log m \Gamma}) = \mathcal{O}(\sqrt{m \log n} \gamma)$.

Algorithm\[\text{3}\] can be implemented in $\mathcal{O}(m)$ time. We remark that, due to the high costs of the arcs in $A_1 \setminus A_0$, there will never be flow on them in an optimal solution. In particular, these arcs are more expensive than any path in the original network because $c_\hat{a} = |t/|z_a - u_a/2| | \geq mCU$. Therefore, the optimum of the problem is not changed by the introduction of the arcs $\hat{a}$. Note that the resulting network is always feasible. This is why we can assume feasibility in Section\[\text{2}\].

We first run Algorithm\[\text{3}\] on the input graph $G_0$ to construct the auxiliary network $G_1$. We then initialize Algorithm\[\text{2}\] with the obtained interior points. If $|b^T y| > mCU$, the problem in $G_0$ was infeasible, since any solution in $G_0$ is bounded by $mCU$. Otherwise, we apply Algorithm\[\text{1}\] and obtain optimal integral potentials $y$ in $G_1$. Let $H_1$ be the admissible network, i.e. the graph $G_1$ with all arcs with dual slack 0. Consider $H_0$, the graph resulting by removing all arcs $\hat{a}$ from $H_1$ that were introduced by Algorithm\[\text{3}\]. By a max-flow computation we compute a feasible solution $x$ in $H_0$, which is optimal in $G_0$ by complementary slackness. If $H_0$ is however infeasible, there is a set $S$ with $b(S) \leq -1$ and $\delta_{H_0}^{\text{out}}(S) = \emptyset$ \[\text{7, Corollary 11.2}]. Since $y$ is optimal in $G_1$, there is an arc $a \in \delta_{G_1}^{\text{in}}(S)$ with $x_a = 0$, thus $a \in \delta_{H_0}^{\text{in}}(S)$. It follows that there is always a feasible and integral solution $z$ in $H_1$ with $z_\hat{a} \geq 1$ that is optimal in $G_1$. With $c_\hat{a} \geq mCU$, we conclude that the cost of $z$ is larger than $mCU$, which contradicts $|b^T y| \leq mCU$. Since the max-flow computation requires $\mathcal{O}(m^{3/2} \log (n^2/m) \log U)$ if it is carried out with the algorithm of Goldberg and Rao \[\text{19}\], this concludes the proof of Theorem\[\text{1}\].

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