On the index of a free abelian subgroup in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$

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Abstract

Let $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ denote the group of central units in the integral group ring $\mathbb{Z}[G]$ of a finite group $G$. A bound on the index of the subgroup generated by a virtual basis in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ is computed for a class of strongly monomial groups. The result is illustrated with application to groups of order $p^n$, $p$ prime, $n \leq 4$. The rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ and the Wedderburn decomposition of the rational group algebra of these $p$-groups have also been obtained explicitly in terms of $p$.

Keywords: integral group rings, unit group, central units, generalized Bass units, Wedderburn decomposition, strong Shoda pairs, strongly monomial groups, normally monomial groups.

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1 Introduction

Let $\mathcal{U}(\mathbb{Z}[G])$ denote the unit group of the integral group ring $\mathbb{Z}[G]$ of a finite group $G$. The centre of $\mathcal{U}(\mathbb{Z}[G])$ is denoted by $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$. It is well known that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm \mathcal{Z}(G) \times A$, where $A$ is a free abelian subgroup of $\mathcal{U}(\mathbb{Z}[G])$ of finite rank. In order to study $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$, a multiplicatively independent subset of such a subgroup $A$, i.e., a $\mathbb{Z}$-basis for such a free $\mathbb{Z}$-module $A$, is of importance, and is known only for a few groups ([11, 2, 15], see also [17], Examples 8.3.11 and 8.3.12). However, other papers deal with determining a virtual basis of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$, i.e., a multiplicatively independent subset of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ which generate subgroups of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ (see e.g. [6, 7, 8, 9, 10, 11, 12, 13, 14]).

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Analogous to well known cyclotomic units in cyclotomic fields, Bass [4] constructed units, so called Bass cyclic units, which generate a subgroup of finite index in $\mathcal{U}(\mathbb{Z}[G])$, when $G$ is cyclic. A virtual basis consisting of certain Bass cyclic units was also given by Bass. Generalizing the notion of Bass cyclic units, Jespers et al [11] defined generalized Bass units and have shown that these units generate a subgroup of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ for strongly monomial groups $G$. Recently, for a class of groups properly contained in finite strongly monomial groups, Jespers et al [13] provided a subset, denoted by $\mathcal{B}(G)$ (say), of the group generated by generalized Bass cyclic units, which forms a virtual basis of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$.

In this paper, we determine a bound on the index of the subgroup generated by $\mathcal{B}(G)$ in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ for the same class of groups as considered in [13] (Theorem 2). Our result is based on the ideas contained in [13] and Kummer’s work ([19], Theorem 8.2) on the index of cyclotomic units. Further in [13], Jespers et al have provided the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ for a strongly monomial group $G$, in terms of strong Shoda pairs of $G$. In section 4, we compute a complete and irredundant set of strong Shoda pairs of non abelian groups of order $p^n$, $p$ prime, $n \leq 4$, and provide, in terms of $p$, the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ of these $p$-groups along with the Wedderburn decomposition of their rational group algebras. We also derive a bound, in terms of $p$, on the index of subgroup generated by $\mathcal{B}(G)$ for these $p$-groups.

2 Preliminaries and Notation

We begin by recalling the definition of strong Shoda pair ([10], Definition 3.1). Let $K$ be a normal subgroup of a subgroup $H$ of $G$. Define $\hat{H} := \frac{1}{|H|} \sum_{h \in H} h$ and

$$\varepsilon(H, K) := \begin{cases} \hat{H}, & H = K; \\ \prod(\hat{K} - \hat{M}) = \hat{K} \prod(1 - \hat{M}), & \text{otherwise}, \end{cases}$$

where $M$ runs through the set of all minimal normal subgroups of $H$ containing $K$ properly. Set

$$e(G, H, K) := \text{the sum of distinct } G-\text{conjugates of } \varepsilon(H, K).$$

If the $G$-conjugates are orthogonal, then $e(G, H, K)$ is a central idempotent of $\mathbb{Q}[G]$. 

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A strong Shoda pair of $G$ is a pair $(H, K)$ of subgroups of $G$ with the properties that

(i) $K$ is normal in $H$ and $H$ is normal in the normalizer $N_G(K)$ of $K$ in $G$;

(ii) $H/K$ is cyclic and a maximal abelian subgroup of $N_G(K)/K$ and

(iii) the distinct $G$-conjugates of $\varepsilon(H, K)$ are mutually orthogonal.

If $(H, K)$ is a strong Shoda pair, then $e(G, H, K)$ is a primitive central idempotent of $\mathbb{Q}[G]$ ([16], Theorem 4.4). A group $G$ is called strongly monomial if every primitive central idempotent of $\mathbb{Q}[G]$ is of the form $e(G, H, K)$ for some strong Shoda pair $(H, K)$ of $G$.

Two strong Shoda pairs $(H_1, K_1)$ and $(H_2, K_2)$ are said to be equivalent if $e(G, H_1, K_1) = e(G, H_2, K_2)$. A complete set of representatives of distinct equivalence classes of strong Shoda pairs of $G$ is called a complete irredundant set of strong Shoda pairs of $G$. In order to know a complete irredundant set of primitive central idempotents of $\mathbb{Q}[G]$, $G$ strongly monomial, a complete irredundant set of strong Shoda pairs is of interest.

Recall that a group $G$ is called normally monomial if every complex irreducible character of $G$ is induced from a linear character of a normal subgroup of $G$. Theorem [1] as stated below, provides an algorithm to determine a complete irredundant set of strong Shoda pairs for a normally monomial group $G$ and also, in particular, yields that a normally monomial group is strongly monomial.

Let $\mathcal{N}$ be the set of all the distinct normal subgroups of a finite group $G$. For $N \in \mathcal{N}$, set

$A_N$ : a normal subgroup of $G$ containing $N$ such that $A_N/N$ is an abelian normal subgroup of maximal order in $G/N$.

$D_N$ : the set of all subgroups $D$ of $A_N$ containing $N$ such that $\text{core}(D) = N$, $A_N/D$ is cyclic and is a maximal abelian subgroup of $N_G(D)/D$, where $\text{core}(D) = \bigcap_{x \in G} xDx^{-1}$, the largest normal subgroup of $G$ contained in $D$.

$T_N$ : a set of representatives of $D_N$ under the equivalence relation defined by conjugacy of subgroups in $G$.

$S_N$ : $\{(A_N, D) \mid D \in T_N\}$.

Note that if $N \in \mathcal{N}$ is such that $G/N$ is abelian, then, by ([1], Eq.(1)),

$$S_N = \begin{cases} \{(G, N)\}, & G/N \text{ cyclic;} \\ \emptyset, & \text{otherwise.} \end{cases} \quad (1)$$
Theorem 1 ([3], Theorem 1, Corollaries 1 and 2) The following statements are equivalent:

(i) $G$ is normally monomial;
(ii) $|G| = \sum_{N \in \mathcal{N}} \sum_{D \in \mathcal{D}_{N}} [G : A_{N}] \varphi([A_{N} : D])$, where $\varphi$ is the Euler phi function;
(iii) $S(G) := \bigcup_{N \in \mathcal{N}} S_{N}$ is a complete irredundant set of strong Shoda pairs of $G$.

Let $\zeta_n$ denote a complex root of unity of order $n$, $n > 1$ and $k$ an integer coprime to $n$. Then

$$\eta_k(\zeta_n) = \frac{1 - \zeta_n^k}{1 - \zeta_n} = 1 + \zeta_n + \zeta_n^2 + \ldots + \zeta_n^{k-1}$$

is a unit of $\mathbb{Z}[\zeta_n]$. The notation is extended by setting $\eta_k(1) = 1$. The units of the form $\eta_k(\zeta_j^n)$, with integers $j, k$ and $n$ such that $(k, n) = 1$ are called cyclotomic units of $\mathbb{Q}(\zeta_n)$, where $(k, n)$ denotes the greatest common divisor of $k$ and $n$.

Let $g$ be an element of $G$ of order $n$ and $k$ and $m$ positive integers such that $k^m \equiv 1 \pmod{n}$. Then,

$$u_{k,m}(g) = (1 + g + \ldots + g^{k-1})^m + \frac{1 - k^m}{n}(1 + g + \ldots + g^{n-1})$$

is a unit in the integral group ring $\mathbb{Z}[G]$. The units in $\mathbb{Z}[G]$ of this form are called Bass cyclic units (see [18], (10.3)).

Next, we recall the definition of generalized Bass unit of $\mathbb{Z}[G]$ defined by Jespers et al [14]. For a normal subgroup $M$ of $G$, $g \in G$ with parameters $k$ and $m$ as above,

$$u_{k,m}(1 - \hat{M} + g\hat{M}) = 1 - \hat{M} + u_{k,m}(g)\hat{M}$$

is an invertible element of $\mathbb{Z}[G](1 - \hat{M}) + \mathbb{Z}[G]\hat{M}$ and some power of it is invertible in $\mathbb{Z}[G]$. If $n_{G,M}$ is the minimal positive integer satisfying $u_{k,m}(1 - \hat{M} + g\hat{M})^{n_{G,M}} \in \mathcal{U}(\mathbb{Z}[G])$ for all $g \in G$, then the element

$$u_{k,m}(1 - \hat{M} + g\hat{M})^{n_{G,M}} = 1 - \hat{M} + u_{k,mn_{G,M}}(g)\hat{M}$$

is called a generalized Bass unit based on $G$ and $M$ with parameters $k$ and $m$. 


Let $G$ be a strongly monomial group with $\{(H_i, K_i) : 1 \leq i \leq m\}$, a complete and irredundant set of strong Shoda pairs such that $[H_i : K_i] = p_i^{n_i}$, $p_i$ prime, $n_i \geq 1$, where $[H_i : K_i]$ denotes the index of $K_i$ in $H_i$. Let $H_i/K_i = \langle g_iK_i \rangle$, and $L_j = \langle g_i^{p_i^{n_i-j}}, K_i \rangle$, $0 \leq j \leq n_i$. Let $k$ be a positive integer coprime with $p_i$ and let $r$ be an arbitrary integer. For every $0 \leq j \leq s \leq n_i$,

$$c_s^r(H_i, K_i, k, r) = 1,$$

and

$$c_j^i(H_i, K_i, k, r) = \left( \prod_{h \in L_j} u_{k, o_i n_i (k)}(g_i^{r p_i^{n_i-j}} h \hat{K}_i + 1 - \hat{K}_i) \right) \times \left( \prod_{l = j + 1}^{s-1} c_l^i(H_i, K_i, k, r) \right) \left( \prod_{l = 0}^{j-1} c_l^{i+l-j}(H_i, K_i, k, r)^{-1} \right),$$

where $o_i(k)$ denotes the multiplicative order of $k$ modulo $l$. The empty products equal 1. Let $T_i$ be a right transversal of $N_i = N_G(K_i)$ in $G$ and $I_i$ be a subset of $\{ k : 1 \leq k < \frac{p_i^{n_i}}{2}, (k, p_i) = 1 \}$, containing 1, which forms a set of representatives of $U(\mathbb{Z}/[H_i] : K_i[\mathbb{Z}])$ modulo $\langle N_i/H_i, -1 \rangle$. As explained in $[13]$, $N_i/H_i$ is identified as a subgroup of $Gal(\mathbb{Q}(\zeta_{p_i^n})/\mathbb{Q})$. Also, $N_i/H_i$ is either $\langle r_i \rangle$ or $\langle r_i \rangle \times \langle -1 \rangle$ for some $r_i \equiv 1 (mod 4)$. Call $d_i = 1$ if $-1 \in \langle r_i \rangle$ and 2 otherwise. Define

$$B(H_i, K_i) = \left\{ \prod_{x \in N_i/H_i} c_0^u(H_i, K_i, k, x) | k \in I_i \setminus \{1\} \right\}$$

and

$$B(H_i, K_i) = \left\{ \prod_{t \in T_i} u^t | u \in B(H_i, K_i) \right\},$$

where $u^t = t^{-1}ut$.

Jespers et al ($[13]$, Theorem 3.5) proved that $B(G) = \bigcup_{i=1}^m B(H_i, K_i)$ is a virtual basis of $\mathcal{Z}(U(\mathbb{Z}[G]))$.

3 A bound on the index of $\langle B(G) \rangle$ in $\mathcal{Z}(U(\mathbb{Z}[G]))$

Throughout this section, we assume that $G$ is a strongly monomial group with $\{(H_i, K_i) : 1 \leq i \leq m\}$, a complete and irredundant set of strong Shoda pairs such that $[H_i : K_i] = p_i^{n_i}$, $p_i$ prime, $n_i \geq 1$. Without any specific mention, we continue to use the notation developed in Section 2. The following theorem provides a bound on the index $[\mathcal{Z}(U(\mathbb{Z}[G])) : \langle B(G) \rangle]$ of the subgroup generated by $B(G)$ in $\mathcal{Z}(U(\mathbb{Z}[G]))$:
Theorem 2 The index of the subgroup generated by $B(G)$ in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ is at most
$$\prod_{i=1}^{m} h_{p_i}^{*} \mathfrak{o}_i (p_i^{d-1}[N_i : H_i])^{l_i - 1},$$
where $\mathfrak{o}_i = \prod_{1 < k < p_i} (k)p_i^{n_i - 1}$ and $h_{p_i}^{*}$ denotes the class number of the maximal real subfield of $\mathbb{Q}(\zeta_{p_i^{n_i}})$.

We first prove the following lemma:

Lemma 1 Let $\mathcal{A}(H_i, K_i) = \mathcal{Z}(1 - e_i + \mathcal{U}(\mathbb{Z}[G]e_i))$ and $A(H_i, K_i) = \mathcal{Z}(1 - \varepsilon_i + \mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))$, where $e_i = e(G, H_i, K_i)$ and $\varepsilon_i = \varepsilon(H_i, K_i)$, $1 \leq i \leq m$. Then,
$$[\mathcal{A}(H_i, K_i) : \langle B(H_i, K_i) \rangle] = [A(H_i, K_i) : \langle B(H_i, K_i) \rangle].$$

Proof. From the proof of ([16], Proposition 3.4), we have isomorphism
$$\mathbb{Q}[G]e_i \overset{\theta_i}{\cong} M_{m_i}(\mathbb{Q}[N_i]\varepsilon_i),$$
where $m_i = [G : N_i]$, $N_i = N_G(K_i)$. Indeed, the above isomorphism is given by
$$\alpha e_i \overset{\theta_i}{\mapsto} (\alpha_{rs})_{m_i \times m_i},$$
where $\alpha_{rs} = \varepsilon_i t_r \alpha e_i t_s^{-1} \varepsilon_i$ and $\{t_j| 1 \leq j \leq m_i\}$ is a transversal of $N_i$ in $G$. This isomorphism in turn yields the group isomorphism
$$\mathcal{Z}(\mathcal{U}(\mathbb{Q}[G]e_i)) \overset{\theta_i}{\cong} \mathcal{Z}(\mathcal{U}(M_{m_i}(\mathbb{Q}[N_i]\varepsilon_i)))$$
given by
$$\alpha e_i \overset{\theta_i}{\mapsto} \varepsilon_i \alpha e_i \varepsilon_i I_{m_i \times m_i}.$$
As $\varepsilon_i \alpha e_i \varepsilon_i = \varepsilon_i \alpha \varepsilon_i$, the restriction of the above isomorphism $\theta_i$ to $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i))$ gives the isomorphism
$$\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i)) \overset{\theta_i}{\cong} \mathcal{Z}(\mathcal{U}(M_{m_i}(\mathbb{Z}[N_i]\varepsilon_i))).$$
However, $\mathcal{Z}(\mathcal{U}(M_{m_i}(\mathbb{Z}[N_i]\varepsilon_i)))$ being equal to $\{\beta \varepsilon_i I_{m_i \times m_i} : \beta \varepsilon_i \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))\}$, there is a natural group isomorphism $\mathcal{Z}(\mathcal{U}(M_{m_i}(\mathbb{Z}[N_i]\varepsilon_i))) \cong \mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))$ given by $\beta \varepsilon_i I_{m_i \times m_i} \mapsto \beta \varepsilon_i$. Consequently, we have the group isomorphism
$$\Theta_i : \mathcal{A}(H_i, K_i) \longrightarrow A(H_i, K_i).$$
by setting

\[1 - \varepsilon_i + \alpha \varepsilon_i \xrightarrow{\Theta_i} 1 - \varepsilon_i + \varepsilon_i \alpha \varepsilon_i.\]

We further see that if \(u = 1 - \varepsilon_i + \gamma \varepsilon_i \in B(H_i, K_i)\), \(\gamma \varepsilon_i \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i] \varepsilon_i))\), then \(\Theta_i(\prod_{t \in T} u_t) = \Theta_i(1 - \varepsilon_i + \sum_{t \in T} \gamma^t \varepsilon_i) = 1 - \varepsilon_i + \varepsilon_i (\sum_{t \in T} \gamma^t \varepsilon_i) = 1 - \varepsilon_i + \gamma \varepsilon_i = u\).

This yields \(\Theta_i(B(H_i, K_i)) = B(H_i, K_i)\) and consequently Lemma \ref{lemma} follows.

**Proof of Theorem** \ref{theorem} Since \(\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))\) is a subgroup of \(\prod_{i=1}^m A(H_i, K_i)\), we have

\[
[\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) : \langle B(G) \rangle] \leq \prod_{i=1}^m A(H_i, K_i) : \langle B(G) \rangle
= \prod_{i=1}^m A(H_i, K_i) : \langle \cup_{i=1}^m B(H_i, K_i) \rangle
= \prod_{i=1}^m [A(H_i, K_i) : \langle B(H_i, K_i) \rangle].
\]

In view of Lemma \ref{lemma} it is enough to prove that

\[
[A(H_i, K_i) : \langle B(H_i, K_i) \rangle] \leq h_{\nu_i}^+ p_i^\nu_i a_i (p_i^d - 1[N_i : H_i])|H_i|^{-1}. \tag{3}
\]

The center of \(\mathbb{Q}(1 - \varepsilon_i) + \mathbb{Q}[N_i] \varepsilon_i\), which is equal to \(\mathbb{Q}(1 - \varepsilon_i) + (\mathbb{Q}[H_i] \varepsilon_i)^{N_i/H_i}\), is embedded inside the algebra \(\mathbb{Q}[H_i] \hat{K}_i \oplus \mathbb{Q}(1 - \hat{K}_i)\), via the embedding

\[
r(1 - \varepsilon_i) + u \varepsilon_i \mapsto (r(1 - \varepsilon_i) + u \varepsilon_i) \hat{K}_i + r(1 - \hat{K}_i), \tag{4}
\]

where \(r \in \mathbb{Q}, u \varepsilon_i \in (\mathbb{Q}[H_i] \varepsilon_i)^{N_i/H_i}\). Here, \(N_i/H_i\) acts on \(\mathbb{Q}[H_i] \varepsilon_i\) by the action \((\alpha \varepsilon_i)^{n_i} = n_i^{-1} \alpha n_i \varepsilon_i, \alpha \varepsilon_i \in \mathbb{Q}[H_i] \varepsilon_i, n_i H_i \subset N_i/H_i\) and \((\mathbb{Q}[H_i] \varepsilon_i)^{N_i/H_i}\) are the elements kept fixed under this action.

Let \(\pi\) denote the projection of \(\mathbb{Q}[H_i] \hat{K}_i \oplus \mathbb{Q}(1 - \hat{K}_i)\) onto \(\mathbb{Q}(\zeta_{p_i^{n_i}})\) under the isomorphism \(\mathbb{Q}[H_i] \hat{K}_i \oplus \mathbb{Q}(1 - \hat{K}_i) \cong \oplus_{k=0}^\tau \mathbb{Q}(\zeta_{p_i^{n_i}}) \oplus \mathbb{Q}(1 - \hat{K}_i)\) given by

\[
x \hat{K}_i + a(1 - \hat{K}_i) \mapsto (\sum_{j=0}^{\nu_i - 1} x_j, \sum_{j=0}^{\nu_i - 1} x_j \zeta_{p_i^j}, \ldots, \sum_{j=0}^{\nu_i - 1} x_j \zeta_{p_i^{\nu_i - 1}}, a(1 - \hat{K}_i)), \tag{5}
\]

where \(x \hat{K}_i = \sum_{j=0}^{\nu_i - 1} x_j g_j^i \hat{K}_i, x_j \in \mathbb{Q}\). Since \(\pi \circ \tau \circ \iota\) is injective on \(\mathcal{Z}(\mathbb{Q}(1 - \varepsilon_i) + \mathbb{Q}[N_i] \varepsilon_i)\) and \(\pi \circ \tau \circ \iota(A(H_i, K_i)) \subseteq \mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i})\), we have

\[
[A(H_i, K_i) : \langle B(H_i, K_i) \rangle] = \left[\pi \circ \tau \circ \iota(A(H_i, K_i)) : \pi \circ \tau \circ \iota(\langle B(H_i, K_i) \rangle)\right]
\leq \left[\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i}) : \pi \circ \tau \circ \iota(\langle B(H_i, K_i) \rangle)\right]. \tag{6}
\]
Set
\[ N_{(H, K_i)} = \langle \pi_{N_i/H_i}(\eta_k(\zeta_{p_{ni}}^{n_i})^{\alpha_{p_{ni}}(k)p_{ni}^{n_i-1}a_i}) : k \in I_i \setminus \{1\} \rangle, \]
\[ F_{(H, K_i)} = \langle \eta_k(\zeta_{p_{ni}}^{n_i})^{\alpha_{p_{ni}}(k)p_{ni}^{n_i-1}a_i} : 1 < k < \frac{p_{ni}}{2}, (k, p_i) = 1 \rangle, \]
\[ O_{(H, K_i)} = F_{(H, K_i)} \times \langle \zeta_{p_{ni}}^{n_i-1} \rangle, \]
\[ P_{(H, K_i)} = \langle \eta_k(\zeta_{p_{ni}}^{n_i}) : k \in \mathcal{U}(\mathbb{Z}/p_{ni}\mathbb{Z}) \rangle = \langle \eta_k(\zeta_{p_{ni}}^{n_i}) : 1 < k < \frac{p_{ni}}{2}, (k, p_i) = 1 \rangle \times \langle \zeta_{p_{ni}}^{n_i} \rangle, \]
\[ Q_{(H, K_i)} = \mathcal{U}(\mathbb{Z}[\zeta_{p_{ni}}^{N_i/H_i}]) \cap O_{(H, K_i)}. \]

where \( \pi_{N_i/H_i}(u) = \prod_{\sigma \in N_i/H_i} \sigma(u), \ u \in \mathcal{Q}(\zeta_{p_{ni}}^{n_i}). \)

By (13), Proposition 3.4,
\[ \pi \circ \tau \circ \iota \langle B(H_i, K_i) \rangle = N_{(H, K_i)}. \]  (7)

Therefore,
\[ \mathcal{U}(\mathbb{Z}[\zeta_{p_{ni}}^{N_i/H_i}]) \cap \langle B(H_i, K_i) \rangle \]
\[ = \mathcal{U}(\mathbb{Z}[\zeta_{p_{ni}}^{N_i/H_i}]) \cap (B(H_i, K_i)) \]
\[ = \mathcal{U}(\mathbb{Z}[\zeta_{p_{ni}}^{N_i/H_i}]) \cap O_{(H, K_i)}[Q_{(H, K_i)} : N_{(H, K_i)}] \]
\[ \leq \mathcal{U}(\mathbb{Z}[\zeta_{p_{ni}}^{N_i/H_i}]) \cap Q_{(H, K_i)}[Q_{(H, K_i)} : N_{(H, K_i)}]. \]  (8)

Further,
\[ \mathcal{U}(\mathbb{Z}[\zeta_{p_{ni}}^{N_i/H_i}]) \cap O_{(H, K_i)} \]
\[ = \mathcal{U}(\mathbb{Z}[\zeta_{p_{ni}}^{N_i/H_i}]) \cap P_{(H, K_i)}[O_{(H, K_i)} : P_{(H, K_i)}] \]  (9)

Clearly,
\[ P_{(H, K_i)} : O_{(H, K_i)} = p_{i}^{n_i-1} \prod_{1 < k < \frac{p_{ni}}{2}, (k, p_i) = 1} o_{p_{ni}}(k)p_{ni}^{n_i-1}a_i = p_{i}^{n_i-1}o_i. \]  (10)

Also, by (19), Theorem 8.2,
\[ \mathcal{U}(\mathbb{Z}[\zeta_{p_{ni}}^{N_i/H_i}]) : P_{(H, K_i)} = h_{p_{ni}}^{N_i/H_i}. \]  (11)

Next, observe that \( Q_{(H, K_i)} \cap F_{(H, K_i)} \) is a free abelian group, and by (13), Lemma 3.2, it has rank at most \( |I_i| - 1 \). Furthermore, any element of \( Q_{(H, K_i)} \cap F_{(H, K_i)} \) is of order at most \( p_{i}^{n_i-1}|N_i/H_i| \). To see this, let \( u \in Q_{(H, K_i)} \cap F_{(H, K_i)} \) and write \( u = \prod_{1 < k < \frac{p_{ni}}{2}, (k, p_i) = 1} (\eta_k(\zeta_{p_{ni}}^{n_i})^{\alpha_{p_{ni}}(k)p_{ni}^{n_i-1}a_k})^{\alpha_k}, \alpha_k \geq 0 \). Since \( \pi_{N_i/H_i}(\eta_k(\zeta_{p_{ni}}^{n_i})) = 1 \) and
\[ \pi_{N_i/H_i}(\eta_k(\zeta_{p_{ni}}^{n_i})) = \pi_{N_i/H_i}(-\zeta_{p_{ni}}^{n_i})^{\pi_{N_i/H_i}(\eta_k(\zeta_{p_{ni}}^{n_i}))}, \]
\( t \geq 0, j \in I_i, \) it turns out that \( u^{N_i/H_i}|p_{ni}^{n_i-1} = (\pi_{N_i/H_i}(u))^{p_{ni}^{n_i-1}} \in N_{(H, K_i)} \cap F_{(H, K_i)} \).
Consequently,

\[ [Q(H_i, K_i) \cap F(H_i, K_i) : N(H_i, K_i) \cap F(H_i, K_i)] \leq (p_i^{d_i-1}|N_i/H_i|)^{|I_i|-1} \]  \hspace{1cm} (12)

and therefore,

\[ [Q(H_i, K_i) : N(H_i, K_i)] \leq [Q(H_i, K_i) : Q(H_i, K_i) \cap F(H_i, K_i)] [Q(H_i, K_i) : N(H_i, K_i) \cap F(H_i, K_i)] \leq p_i (p_i^{d_i-1}|N_i/H_i|)^{|I_i|-1}. \]  \hspace{1cm} (13)

Finally, Eqs. (12)-(13) yield the claim, i.e., Eq. (13), which in view of Eq. (2) and Lemma 1 completes the proof. □

The above theorem, in particular, for abelian \( p \) groups yields the following:

**Corollary 1** Let \( G \) be an abelian \( p \) group, \( p \) prime, and let \( K_i, 1 \leq i \leq m \), be all the cyclic subgroups of \( G \). If \( |K_i| = p^{n_i}, 1 \leq i \leq m \), then the index of the subgroup generated by \( B(G) \) in \( U(Z[G]) \) is at most

\[ \prod_{i=1}^{m} h_{p_i}^{+} p^{n_i} \left( \prod_{1 < k < p_i^{n_i}} o_{p_i}(k) p^{n_i} n_i \right). \]

## 4 Non Abelian groups of order \( p^n \), \( n \leq 4 \)

### 4.1 Non Abelian groups of order \( p^3 \)

If \( p \) is an odd prime, then up to isomorphism, the only two non-abelian groups of order \( p^3 \) are:

\[ \langle a, b \mid a^p = b^p = 1, \ ba = a^{p+1} b \rangle \]

and

\[ \langle a, b, c \mid a^p = b^p = c^p = 1, ab = bac, ac = ca, bc = cb \rangle. \]

In (3), Theorems 3, 4), a complete and irredundant set of strong Shoda pairs for these groups has been found. Applying Theorem 2 and (13), Theorem 3.1), we obtain that for any of these two non-abelian groups \( G \) of order \( p^3 \), \( p \) odd prime,

\[ \text{Rank of } Z(U(Z[G])) = \frac{(p - 3)(p + 2)}{2} \]

and

\[ [Z(U(Z[G])) : \langle B(G) \rangle] \leq \left( p^{\frac{p-1}{2}} h_{p}^{+} \prod_{1 < k < \frac{p}{2}} o_{p}(k) \right)^{p+2}. \]
If \( p = 2 \), then \( G \) is either isomorphic to \( D_4 \), the dihedral group of order 8 or is isomorphic to \( Q_8 \), the group of quaternions. Both these groups satisfy the hypothesis of ([18], Theorem 6.1). Therefore, we already know that the group of central units in integral group rings of these groups consist of only trivial units.

### 4.2 Non Abelian groups of order \( p^4 \)

We first assume that \( p \) is an odd prime. The following is the complete list of non abelian groups of order \( p^4 \), \( p \) an odd prime (see [5], §117):

1. \( G_1 = \langle a, b : a^{p^3} = b^p = 1, ba = a^{1+p^2}b \rangle \);
2. \( G_2 = \langle a, b, c : a^{p^2} = b^p = c^p = 1, cb = a^pbc, ab = ba, ac = ca \rangle \);
3. \( G_3 = \langle a, b : a^{p^2} = b^{p^2} = 1, ba = a^{1+p}b \rangle \);
4. \( G_4 = \langle a, b, c : a^{p^2} = b^p = c^p = 1, ca = a^{1+p}c, ba = ab, cb = bc \rangle \);
5. \( G_5 = \langle a, b, c : a^{p^2} = b^p = c^p = 1, ca = abc, ab = ba, bc = cb \rangle \);
6. \( G_6 = \langle a, b, c : a^{p^2} = b^p = c^p = 1, ba = a^{1+p}b, ca = abc, cb = bc \rangle \);
7. \( G_7 = \begin{cases} \langle a, b, c : a^{p^2} = b^p = 1, c^p = a^p, ab = ba^{1+p}, ac = cab^{-1}, cb = bc \rangle, & \text{if } p = 3; \\
\langle a, b, c : a^{p^2} = b^p = c^p = 1, ba = a^{1+p}b, ca = a^{1+p}bc, cb = a^pbc \rangle, & \text{if } p > 3; \end{cases} \)
8. \( G_8 = \begin{cases} \langle a, b, c : a^{p^2} = b^p = 1, c^p = a^{-p}, ab = ba^{1+p}, ac = cab^{-1}, cb = bc \rangle, & \text{if } p = 3; \\
\langle a, b, c : a^{p^2} = b^p = c^p = 1, ba = a^{1+p}b, ca = a^{1+dp}bc, cb = a^{dp}bc \rangle, & \text{if } p > 3 \\
d \not\equiv 0, 1(\mod p) \end{cases} \)
9. \( G_9 = \langle a, b, c, d : a^p = b^p = c^p = d^p = 1, dc = acd, bd = db, ad = da, bc = cb, ac = ca, ab = ba \rangle \);
10. \( G_{10} = \begin{cases} \langle a, b, c : a^{p^2} = b^p = c^p = 1, ab = ba, ac = cab, bc = ca^{-p}b \rangle, & \text{if } p = 3, \\
\langle a, b, c, d : a^p = b^p = c^p = d^p = 1, dc = bcd, db = abd, ad = da, \\
bc = cb, ac = ca, ab = ba \rangle, & \text{if } p > 3. \end{cases} \)
Theorem 3 A complete and irredundant set $S(G_i)$ of strong Shoda pairs for each $G_i$, $1 \leq i \leq 10$, is as follows:

(i) $S(G_1) = \{(a, \langle 1 \rangle), (G_1, \langle a \rangle), (G_1, G_1)\} \cup \{(G_1, \langle a^p, a^p b \rangle), (G_1, \langle a^p a b \rangle) \mid 0 \leq i \leq p - 1\};$

(ii) $S(G_2) = \{(a, b), (G_2, \langle a, b \rangle), (G_2, G_2)\} \cup \{(G_2, \langle a b, c \rangle), (G_2, \langle b a^i, c a^j \rangle) \mid 0 \leq i, j \leq p - 1\};$

(iii) $S(G_3) = \{(a, b^p), (G_3, \langle a \rangle), (G_3, G_3)\} \cup \{(a, b^p, \langle a^{p^i} b^p \rangle), (G_3, \langle a^p, b a^i \rangle) \mid 0 \leq i \leq p - 1\} \cup \{(G_3, \langle a^p, a^k b^p \rangle) \mid 1 \leq k \leq p - 1\};$

(iv) $S(G_4) = \{(a, b), (G_4, G_4)\} \cup \{(a, \langle a^p \rangle), (G_4, \langle a, b \rangle), (G_4, \langle b a^i, c a^j \rangle) \mid 0 \leq i, j \leq p - 1\};$

(v) $S(G_5) = \{(a, \langle a \rangle), (G_5, \langle a^p, b, c \rangle), (G_5, \langle a, b \rangle), (G_5, G_5)\} \cup \{(G_5, \langle b, c a^p \rangle) \mid 0 \leq i \leq p - 1\} \cup \{(a, \langle a b^k \rangle), (G_5, \langle b, c a^k \rangle) \mid 1 \leq k \leq p - 1\};$

(vi) $S(G_6) = \{(a^p, b, c), (a, b), (G_6, \langle a^p, b, c \rangle), (G_6, G_6)\} \cup \{(a^p, b, c, \langle b, c a^p \rangle) \mid 0 \leq i \leq p - 1\} \cup \{(G_6, \langle b, c a^k \rangle) \mid 1 \leq k \leq p - 1\};$

(vii) $S(G_7) = \{(b, c), (b, c), \langle c \rangle, (G_7, \langle a, b \rangle), (G_7, G_7)\} \cup \{(G_7, \langle b, c a^i \rangle) \mid 0 \leq i \leq p - 1\};$

(viii) $S(G_7) = \{(b, \langle c \rangle), (b, \langle c \rangle), \langle c \rangle, (G_7, \langle a, b \rangle), (G_7, G_7)\} \cup \{(G_7, \langle b, c a^i \rangle) \mid 0 \leq i \leq p - 1\};$

(ix) $S(G_8) = \{(b, \langle c \rangle), (b, \langle c \rangle), \langle c \rangle, (G_7, \langle a, b \rangle), (G_7, G_7)\} \cup \{(G_8, \langle b, c a^i \rangle) \mid 0 \leq i \leq p - 1\};$

(x) $S(G_8) = \{(b, \langle c a^d \rangle), (b, \langle c a^d \rangle), \langle c a^d \rangle, (G_7, \langle a, b \rangle), (G_7, G_7)\} \cup \{(G_8, \langle b, c a^i \rangle) \mid 0 \leq i \leq p - 1\};$

(xi) $S(G_9) = \{(G_9, \langle a, b, d \rangle), (G_9, G_9)\} \cup \{(a, b, d), \langle d, b a^i \rangle, \langle d, b a^i \rangle, \langle d, c b^i, d b^i \rangle \mid 0 \leq i, j \leq p - 1\};$

(xii) $S(G_{10}) = \{(a, b), (b, a), (G_{10}, \langle a, b \rangle), (G_{10}, G_{10})\} \cup \{(G_{10}, \langle b, c a^d \rangle) \mid 0 \leq i \leq p - 1\};$

(xiii) $S(G_{10}) = \{(a, b, c), (a, c), (G_{10}, \langle a, b, d \rangle)\} \cup \{(a, b, c), \langle c a^b, b \rangle, (G_{10}, \langle a, b, c a^d \rangle), (G_{10}, G_{10}) \mid 0 \leq i \leq p - 1\};$

Proof (i) Define $N_0 := \langle 1 \rangle$, $N_1 := \langle a^p \rangle$, $N_2 := \langle a^p \rangle$, $N_3 := \langle a \rangle$, $H_i := \langle a^p, a^p b \rangle$, $K_j := \langle a^p, a^j b \rangle$ where $0 \leq i, j \leq p - 1$. Observe that these subgroups are normal in $G_1$. Using Eq. [3] we have $S_{N_1} = S_{N_2} = \emptyset$, $S_{N_3} = \{(G_1, N_3)\}$, $S_{H_i} = \{(G_1, H_i)\}$, $S_{K_j} = \{(G_1, K_j)\}$. In order to find $S_{N_0}$, we see that $\langle a \rangle$ is a maximal abelian subgroup of $G_1$. Further, the only subgroup $D$ of $\langle a \rangle$ which is corefree in $G_1$ is
$D = \langle 1 \rangle$. This gives $S_{N_0} = \{ \langle a \rangle, \langle 1 \rangle \}$. Define

$$N_1 = \{ \langle 1 \rangle, \langle a^p \rangle, \langle a \rangle, \langle a, b \rangle \} \cup \{ \langle a^p, a^{p^j} b \rangle, \langle a^p, a^j b \rangle | 0 \leq i, j \leq p - 1 \}.$$  

Observe that $\sum_{N \in N_1} \sum_{D \in D_N} [G : A_N]|\varphi([A_N : D]) = p^4$. Now, if $\mathcal{N}$ is the set of all normal subgroups of $G_1$, then

$$p^4 = |G_1| = \sum_{N \in N} \sum_{D \in D_N} [G : A_N]|\varphi([A_N : D]) \quad \text{(by Theorem 1)}$$

$$\geq \sum_{N \in N_1} \sum_{D \in D_N} [G : A_N]|\varphi([A_N : D]) \quad \text{(as $N_1 \subseteq \mathcal{N}$)}$$

$$= p^4.$$

This yields $S_N = \phi$ if $N \not\in N_1$ and consequently, by Theorem 1, $\bigcup_{N \in N_1} S_N$ is a complete irredundant set of strong Shoda pairs of $G_1$.

(ii)-(xiii) For $2 \leq i \leq 10$, consider the following set $N_i$ of normal subgroups of $G_i$:

$$N_2 = \{ \langle 1 \rangle, \langle a^p \rangle, \langle a^p, b \rangle, \langle a, b \rangle, \langle a, b, c \rangle \} \cup \{ \langle a^p, b^i c \rangle, \langle a, b^i c \rangle, \langle a b^i c, a c^i \rangle | 0 \leq i, j \leq p - 1 \};$$

$$N_3 = \{ \langle 1 \rangle, \langle a^p \rangle, \langle a \rangle, \langle b^p \rangle, \langle a^p b^p \rangle, \langle a, b \rangle \} \cup \{ \langle a^{p^i} b^j \rangle, \langle a^{p^i} b^j \rangle | 0 \leq i \leq p - 1 \};$$

$$N_4 = \{ \langle 1 \rangle, \langle a^p \rangle, \langle a^p b \rangle, \langle a, b, c \rangle \} \cup \{ \langle a^p, b^i c \rangle, \langle a, b^i c \rangle, \langle a b^i c, a c^i \rangle | 0 \leq i, j \leq p - 1 \};$$

$$N_5 = \{ \langle 1 \rangle, \langle b \rangle, \langle a^p b \rangle, \langle a^p, b, c \rangle, \langle a, b, c \rangle \} \cup \{ \langle b^i a^p \rangle | 0 \leq i \leq p - 1 \};$$

$$N_6 = \{ \langle 1 \rangle, \langle a^p \rangle, \langle a^p b \rangle, \langle a, b, c \rangle \} \cup \{ \langle b, ca^k \rangle | 1 \leq k \leq p - 1 \};$$

$$N_7 = \{ \langle 1 \rangle, \langle a^p \rangle, \langle a^p b \rangle, \langle a, b, c \rangle \} \cup \{ \langle b, ca^i \rangle | 0 \leq i \leq p - 1 \};$$

$$N_8 = \{ \langle 1 \rangle, \langle a^p \rangle, \langle a^p b \rangle, \langle a, b, c \rangle \} \cup \{ \langle b, ca^i \rangle | 0 \leq i \leq p - 1 \};$$

$$N_9 = \{ \langle 1 \rangle, \langle a \rangle, \langle a, b, d \rangle, \langle a, d \rangle, \langle a, b, c, d \rangle \} \cup \{ \langle b^i a^k \rangle, \langle a, b^i c^k \rangle, \langle a, c^k d \rangle, \langle a, b, c, d \rangle | 0 \leq i, j \leq p - 1 \};$$

$$N_{10} = \{ \langle 1 \rangle, \langle a^3 \rangle, \langle b^3 \rangle, \langle a, b, c \rangle, \langle b, a, c \rangle, \langle b, c a^2 \rangle, \langle a, b \rangle, \langle a, b, c \rangle \} \cup \{ \langle a, b, c d \rangle | 0 \leq i \leq p - 1 \}, \text{if } p = 3;$$

$$\{ \langle 1 \rangle, \langle a \rangle, \langle a, b \rangle, \langle a, b, d \rangle, \langle a, b, c, d \rangle \} \cup \{ \langle a, b, c d \rangle | 0 \leq i \leq p - 1 \}, \text{if } p > 3.$$  

Now proceeding as in (i), we get the required complete and irredundant set of strong Shoda pairs of $G_i$, $2 \leq i \leq 10$. □
Theorem 3 along with ([6], Theorem 3.6) and ([13], Theorem 3.1) also yield the following:

**Corollary 2** The Wedderburn decomposition of $\mathbb{Q}[G_i]$ and the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G_i]))$, $1 \leq i \leq 10$, are as follows:

| $G$  | $\mathbb{Q}[G]$                                                                                     | Rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ |
|------|-----------------------------------------------------------------------------------------------------|--------------------------------------------------|
| $G_1$ | $\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus \mathbb{Q}(\zeta_p^2)^{(p)} \oplus M_p(\mathbb{Q}(\zeta_p^3))$ | $(p+1)(p^2-5)$                                   |
| $G_2$ | $\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p+p^2)} \oplus M_p(\mathbb{Q}(\zeta_p^2))$              | $\frac{p^3-p^2-3p-5}{2}$                         |
| $G_3$ | $\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus \mathbb{Q}(\zeta_p^2)^{(p)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(p)}$ | $\frac{p^3+p^2-7p-3}{2}$                         |
| $G_4$ | $\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p+p^2)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(p)}$          | $\frac{(p-3)(p+1)^2}{2}$                         |
| $G_5$ | $\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus \mathbb{Q}(\zeta_p^2)^{(p)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(p)}$ | $\frac{p^3+p^2-7p-3}{2}$                         |
| $G_6$ | $\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(1+p)}$           | $(p-3)(p+1)$                                     |
| $G_7$ | $\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus M_p(\mathbb{Q}(\zeta_p)) \oplus M_p(\mathbb{Q}(\zeta_p^2))$ | $p^2-p-4$                                       |
| $G_8$ | $\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus M_p(\mathbb{Q}(\zeta_p)) \oplus M_p(\mathbb{Q}(\zeta_p^2))$ | $p^2-p-4$                                       |
| $G_9$ | $\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p+p^2)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(p)}$          | $\frac{(p-3)(p+1)^2}{2}$                         |
| $G_{10}$ ($p=3$) | $\mathbb{Q} \oplus \mathbb{Q}(\zeta_3)^{(4)} \oplus M_3(\mathbb{Q}(\zeta_3)) \oplus M_3(\mathbb{Q}(\zeta_9))$ | 2                                              |
| $G_{10}$ ($p>3$) | $\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(1+p)}$           | $(p-3)(p+1)$                                     |

Theorems 2 and 3 immediately yield the following:

**Corollary 3** If $I_{G_i}$ denotes the index of $\langle B(G_i) \rangle$ in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G_i]))$, $1 \leq i \leq 10$, then

(i) $I_{G_1} \leq 2^{\frac{p^3-p^2-2p}{2}} \cdot 3^{\frac{p^3-p^2-2}{2}} \cdot p^\frac{4p^3-p^2-2p-3}{2} \cdot b_1^{1+p}b_2^pb_3^p$;

(ii) $I_{G_2} \leq 2^{\frac{p^3-p^2-2p}{2}} \cdot p^{\frac{p^3+p^2-2p-1}{2}} \cdot b_1^{1+p}b_2^p$;

(iii) $I_{G_3} \leq 2^{p(p^2-p-2)}p^\frac{2p^3+p-5}{2} \cdot b_1^{1+p}b_2^{2p}$. 

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and the other non abelian groups of order $2^4$ phic groups of order $2^4$.

Theorem 6.1). Hence, if $G$ has a pair of the rank of $H$.

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We now take the case, when $p = 2$. Upto isomorphism, there are 9 non isomorphic groups of order $2^4$ as listed in ([1], §118). Except the following two groups:

$$H_1 = \langle a, b : a^8 = b^2 = 1, ba = a^7b \rangle$$

and

$$H_2 = \langle a, b : a^8 = b^4 = 1, ba = a^7b, a^4 = b^2 \rangle,$$

the other non abelian groups of order $2^4$ again satisfy the hypothesis of ([11], Theorem 6.1). Hence, if $G$ is a non abelian group of order $2^4$ other than $H_1$ and $H_2$, then $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm \mathcal{Z}(G)$.

For the groups $H_1$ and $H_2$, we obtain using Theorem [1] that $\{(\langle a \rangle, \{1\}), (\langle a \rangle, \langle a^4 \rangle), (H_1, \langle a \rangle), (H_1, \langle a^2, b \rangle), (H_1, \langle a^2, ab \rangle), (H_1, H_1)\}$ and $\{(\langle a \rangle, \{1\}), (\langle a \rangle, \langle b^2 \rangle), (H_2, \langle a \rangle), (H_2, \langle a^2, b \rangle), (H_2, \langle a^2, ab \rangle), (H_2, H_2)\}$ are complete irredundant sets of strong Shoda pairs of $H_1$ and $H_2$ respectively. Theorem [2] and ([13], Theorem 3.1) now yield that the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = 1$ and $[\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) : \langle \mathcal{B}(G) \rangle] \leq 3.2^{14}$, $G = H_1$ or $H_2$.

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