Linear Classification of data with Support Vector Machines and Generalized Support Vector Machines

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Abstract: In this paper, we study the support vector machine and introduced the notion of generalized support vector machine for classification of data. We show that the problem of generalized support vector machine is equivalent to the problem of generalized variational inequality and establish various results for the existence of solutions. Moreover, we provide various examples to support our results.

Keywords and Phrases: support vector machine, generalized support vector machine, control function.

2010 Mathematics Subject Classification: 62H30.

1 Support Vector Machine

Over the last decade, support vector machines (SVMs) [2, 3, 13, 14, 15] has been revealed as very powerful and important tools for pattern classification and regression. It has been used in various applications such as text classification [5], facial expression recognition [9], gene analysis [4] and many others [1, 6, 7, 8, 10, 11, 12, 17, 19, 20, 21, 22]. Recently, Wang et al. [15] presented SVM based fault classifier design for a water level control system. They also studied the SVM classifier based fault diagnosis for a water level process [16].

For the standard support vector classification (SVC), the basic idea is to find the optimal separating hyperplane between the positive and negative examples. The optimal hyperplane may be obtained by maximizing the margin between two parallel hyperplanes, which involves the minimization of a quadratic programming problem.
Support Vector Machines is based on the concept of decision planes that define decision boundaries. A decision plane is one that separates between a set of objects having different class memberships.

Support Vector Machines can be thought of as a method for constructing a special kind of rule, called a linear classifier, in a way that produces classifiers with theoretical guarantees of good predictive performance (the quality of classification on unseen data).

In this paper, we study the problems of support vector machine and define generalized support vector machine. We also show the sufficient conditions for the existence of solutions for problems of generalized support vector machine. We also support our results with various examples.

Thought this paper, by \( \mathbb{N}, \mathbb{R}, \mathbb{R}^n \) and \( \mathbb{R}_n^+ \) we denote the set of all natural numbers, the set of all real numbers, the set of all \( n \)-tuples real numbers, the set of all \( n \)-tuples of nonnegative real numbers, respectively.

Also, we consider \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) as Euclidean norm and usual inner product on \( \mathbb{R}^n \), respectively.

Furthermore, for two vectors \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \), we say that \( \mathbf{x} \leq \mathbf{y} \) if and only if \( x_i \leq y_i \) for all \( i \in \{1, 2, ..., n\} \), where \( x_i \) and \( y_i \) are the components of \( \mathbf{x} \) and \( \mathbf{y} \), respectively.

Linear Classifiers

Binary classification is frequently performed by using a function \( f : \mathbb{R}^n \to \mathbb{R} \) in the following way: the input \( \mathbf{x} = (x_1, ..., x_n) \) is assigned to the positive class if, \( f(\mathbf{x}) \geq 0 \) and otherwise to the negative class. We consider the case where \( f(\mathbf{x}) \) is a linear function of \( \mathbf{x} \), so that it can be written as

\[
f(\mathbf{x}) = \langle \mathbf{w} \cdot \mathbf{x} \rangle + b = \sum_{i=1}^{n} w_i x_i + b,
\]

where \( \mathbf{w} \in \mathbb{R}^n, b \in \mathbb{R} \) are the parameters that control the function and the decision rule is given by \( \text{sgn}(f(\mathbf{x})) \). The learning methodology implies that these parameters must be learned from the data.

**Definition 1.1.** We define the functional margin of an example \((\mathbf{x}_i, y_i)\) with respect to a hyperplane \((\mathbf{w}, b)\) to be the quantity

\[
\gamma_i = y_i (\langle \mathbf{w} \cdot \mathbf{x}_i \rangle + b),
\]

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where \( y_i \in \{-1, 1\} \). Note that \( \gamma_i > 0 \) implies correct classification of \((x_i, y_i)\).

If we replace functional margin by geometric margin we obtain the equivalent quantity for the normalized linear function \( \left( \frac{1}{\|w\|} w, \frac{1}{\|w\|} b \right) \), which therefore measures the Euclidean distances of the points from the decision boundary in the input space.

Actually geometric margin can be written as

\[
\tilde{\gamma} = \frac{1}{\|w\|} \gamma.
\]

To find the hyperplane which has maximal geometric margin for a training set \( S \) means to find maximal \( \tilde{\gamma} \). For convenience, we let \( \gamma = 1 \), the objective function can be written as

\[
\max \frac{1}{\|w\|}.
\]

Of course, there have some constraints for the optimization problem. According to the definition of margin, we have \( y_i (\langle w \cdot x_i \rangle + b) \geq 1, \, i = 1, \ldots, l \).

We rewrite the equivalent formation of the objective function with the constraints as

\[
\min \frac{1}{2} \|w\|^2 \quad \text{such that} \quad y_i (\langle w \cdot x_i \rangle + b) \geq 1, \, i = 1, \ldots, l.
\]

We denote this problem by SVM.

## 2 Generalized Support Vector Machines

We replace \( w, b \) by \( W, B \) respectively, the control function \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) defined as

\[
F(x) = Wx + B,
\]

where \( W \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^n \) are the parameters of control function.

Define

\[
\tilde{\gamma}_k^* = y_k (Wx_k + B) > 1 \quad \text{for} \quad k = 1, 2, \ldots, l,
\]

where \( y_k \in \{(-1, -1, \ldots, -1), (1, 1, \ldots, 1)\} \) is \( n \) dimensional vector.

**Definition 2.1.** We define a map \( G : \mathbb{R}^n \rightarrow \mathbb{R}_+^n \) by

\[
G(w_i) = (\|w_i\|, \|w_i\|, \ldots, \|w_i\|) \quad \text{for} \quad i = 1, 2, \ldots, n,
\]

3
where \( w_i \) be the row of \( W_{n \times n} \) for \( i = 1, 2, \ldots, n \).

The problem is find \( w_i \in \mathbb{R}^n \) that satisfy

\[
\min_{w_i \in W} G(w_i) \text{ such that } \eta > 0,
\]

where \( \eta = y_k (W x_k + B) - 1 \).

We call this problem as the Generalized Support Vector Machine (GSVM).

The GSVM is equivalent to

\[
\text{find } w_i \in W : \langle G'(w_i), v - w_i \rangle \geq 0 \text{ for all } v \in \mathbb{R}^n \text{ with } \eta > 0,
\]

or more specifically

\[
\text{find } w_i \in W : \langle \eta G'(w_i), v - w_i \rangle \geq 0 \text{ for all } v \in \mathbb{R}^n.
\]

Hence the problem of GSVM becomes to the problem of generalized variational inequality.

**Example 2.2.** Let us take the group of points positive class \((1, 0), (0, 1)\) and negative class \((-1, 0), (0, -1)\).

First we use SVM to solve this problem to find the hyperplane \( < w, x > + b = 0 \) that separate this two kinds of points. Obviously, we know that the hyperplane is \( H \) which is shown in the Figure.
For two positive points, we have

\[
(w_1, w_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b = 1
\]

\[
(w_1, w_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + b = 1
\]

which implies

\[
w_1 + b = 1
\]

\[
w_2 + b = 1.
\]

For two negative points, we have

\[
(w_1, w_2) \begin{bmatrix} -1 \\ 0 \end{bmatrix} + b = -1
\]

\[
(w_1, w_2) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + b = -1
\]

implies that

\[
-w_1 + b = -1
\]

\[
-w_2 + b = -1.
\]

From the equations, we get \( w = (1, 1) \) and \( b = 0 \). The result is \( \|w\| = \sqrt{2} \).

Now we apply GSVM for this data.

For two positive points, we have

\[
\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

and

\[
\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

which gives

\[
\begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} w_{12} \\ w_{22} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

(2.6)
For two negative points, we have
\[
\begin{bmatrix}
  w_{11} & w_{12} \\
  w_{21} & w_{22}
\end{bmatrix}
\begin{bmatrix}
  -1 \\
  0
\end{bmatrix}
+ 
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix}
= 
\begin{bmatrix}
  -1 \\
  -1
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
  w_{11} & w_{12} \\
  w_{21} & w_{22}
\end{bmatrix}
\begin{bmatrix}
  0 \\
  -1
\end{bmatrix}
+ 
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix}
= 
\begin{bmatrix}
  -1 \\
  -1
\end{bmatrix},
\]
which provides
\[
\begin{bmatrix}
  -w_{11} \\
  -w_{21}
\end{bmatrix}
+ 
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix}
= 
\begin{bmatrix}
  -1 \\
  -1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
  -w_{12} \\
  -w_{22}
\end{bmatrix}
+ 
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix}
= 
\begin{bmatrix}
  -1 \\
  -1
\end{bmatrix}.
\] (2.7)

From (2.6) and (2.7), we get
\[
W = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
Thus
\[
\min G(w_i) = \min \{ G(w_1), G(w_2) \} = (\sqrt{2}, \sqrt{2}).
\]
Hence we get \( w = (1, 1) \) that minimize \( G(w_i) \) for \( i = 1, 2 \).

Conclusion: The above example shows that we get same result by applying any method SVM and GSVM.

In the next example, we consider the two distinct group of data, first solve both data for separate cases and then solve it for combine case for both methods SVM and GSVM.

Example 2.3. Let us consider the three categories of data:

Situation 1, suppose that, we have data \((1, 0), (0, 1)\) as positive class and data \((-1/2, 0), (0, -1/2)\) as negative class.
Using SVM to solve this problem, we have

\[
(w_1, w_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b = 1 \quad \text{and} \\
(w_1, w_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + b = 1,
\]

which implies

\[w_1 + b = 1 \quad \text{and} \quad w_2 + b = 1. \quad (2.8)\]

For two negative points, we have

\[
(w_1, w_2) \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} + b = -1, \quad \text{and} \\
(w_1, w_2) \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} + b = -1,
\]

which gives

\[-\frac{w_1}{2} + b = -1 \quad \text{and} \quad -\frac{w_2}{2} + b = -1. \quad (2.9)\]

From (2.8) and (2.9), we get \(w = \left(\frac{1}{3}, \frac{4}{3}\right)\) with \(b = \frac{1}{3}\), where \(\|w\| = \frac{\sqrt{17}}{3}\).
For **situation 2**, we consider the data \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\) as positive class, data \((-2, 0)\) and \((0, -2)\) as negative class.

Using SVM to solve this problem, we have

\[
\begin{align*}
(w_1, w_2) \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + b &= 1 \text{ and } \\
(w_1, w_2) \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} + b &= 1,
\end{align*}
\]

which implies

\[
\frac{1}{2}w_1 + b = 1 \text{ and } \frac{1}{2}w_2 + b = 1. \tag{2.10}
\]

From the negative points, we have

\[
\begin{align*}
(w_1, w_2) \begin{bmatrix} -2 \\ 0 \end{bmatrix} + b &= -1 \text{ and } \\
(w_1, w_2) \begin{bmatrix} 0 \\ -2 \end{bmatrix} + b &= -1,
\end{align*}
\]

implies that

\[
-2w_1 + b = -1 \text{ and } -2w_2 + b = -1. \tag{2.11}
\]

From (2.10) and (2.11), we get \(w = \left(\frac{4}{5}, \frac{4}{5}\right)\) and \(b = \frac{3}{5}\) with \(\|w\| = \frac{\sqrt{32}}{5}\).
In the next situation 3, we combine of this two groups of data. Now, we have data $(1/2, 0)$, $(0, 1/2)$, $(1, 0)$, $(0, 1)$ as positive class and $(-1/2, 0)$, $(0, -1/2)$, $(-2, 0)$, $(0, -2)$ as negative class.

Using SVM to solve this problem, we have

$$
(w_1, w_2) \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + b = 1 \text{ and }
$$

$$
(w_1, w_2) \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} + b = 1,
$$

which implies

$$w_1/2 + b = 1 \text{ and } w_2/2 + b = 1. \quad (2.12)
$$

For two negative points, we have

$$
(w_1, w_2) \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} + b = -1 \text{ and }
$$

$$
(w_1, w_2) \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} + b = -1,
$$

implies that

$$
-\frac{1}{2}w_1 + b = -1 \text{ and } -\frac{1}{2}w_2 + b = -1. \quad (2.13)
$$

From (2.12) and (2.13), we obtain $w = (2, 2)$ and $b = 0$, where $\|w\| = 2\sqrt{2}$. 

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Now we solve the same problem for all three situations by using GSVM.

For two positive points of situation 1, we have
\[
\begin{bmatrix}
w_{11} & w_{12} \\
w_{21} & w_{22}
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
= \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
w_{11} & w_{12} \\
w_{21} & w_{22}
\end{bmatrix}
\begin{bmatrix}
0 \\
1
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
= \begin{bmatrix}
1 \\
1
\end{bmatrix},
\]
which implies
\[
\begin{bmatrix}
w_{11} \\
w_{21}
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
= \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
w_{12} \\
w_{22}
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
= \begin{bmatrix}
1 \\
1
\end{bmatrix}. \quad (2.14)
\]

Again, for the negative points, we have
\[
\begin{bmatrix}
w_{11} & w_{12} \\
w_{21} & w_{22}
\end{bmatrix}
\begin{bmatrix}
-1/2 \\
0
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
= \begin{bmatrix}
-1 \\
-1
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
w_{11} & w_{12} \\
w_{21} & w_{22}
\end{bmatrix}
\begin{bmatrix}
0 \\
-1/2
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
= \begin{bmatrix}
-1 \\
-1
\end{bmatrix},
\]
which gives
\[
\begin{bmatrix}
-\frac{1}{3}w_{11} \\
-\frac{1}{3}w_{21}
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
= \begin{bmatrix}
-1 \\
-1
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
-\frac{1}{3}w_{12} \\
-\frac{1}{3}w_{22}
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
= \begin{bmatrix}
-1 \\
-1
\end{bmatrix}. \quad (2.15)
\]

From (2.14) and (2.15), we get
\[
W = \begin{bmatrix}
\frac{4}{3} & \frac{4}{3} \\
\frac{4}{3} & \frac{4}{3}
\end{bmatrix}
\quad \text{and} \quad B = \begin{bmatrix}
-\frac{1}{3} \\
-\frac{1}{3}
\end{bmatrix}.
\]

Thus we get
\[
\min_{w_i \in W} G(w_i) = \left(\frac{4\sqrt{2}}{3}, \frac{4\sqrt{2}}{3}\right).
\]

Hence we get \( w = \left(\frac{4}{3}, \frac{4}{3}\right) \) that minimize \( G(w_i) \) for \( i = 1, 2 \).

Now, for positive points of situation 2, we have
\[
\begin{bmatrix}
w_{11} & w_{12} \\
w_{21} & w_{22}
\end{bmatrix}
\begin{bmatrix}
1/2 \\
0
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
= \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
  w_{11} & w_{12} \\
  w_{21} & w_{22}
\end{bmatrix}
\begin{bmatrix}
  0 \\
  1/2
\end{bmatrix} +
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  1 \\
  1
\end{bmatrix},
\]
which gives
\[
\begin{bmatrix}
  \frac{1}{2}w_{11} \\
  \frac{1}{2}w_{21}
\end{bmatrix} +
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  1 \\
  1
\end{bmatrix} \text{ and } \begin{bmatrix}
  \frac{1}{2}w_{12} \\
  \frac{1}{2}w_{22}
\end{bmatrix} +
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  1 \\
  1
\end{bmatrix}.
\]

For two negative points for this case, we have
\[
\begin{bmatrix}
  w_{11} & w_{12} \\
  w_{21} & w_{22}
\end{bmatrix}
\begin{bmatrix}
  -2 \\
  0
\end{bmatrix} +
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  -1 \\
  -1
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
  w_{11} & w_{12} \\
  w_{21} & w_{22}
\end{bmatrix}
\begin{bmatrix}
  0 \\
  -2
\end{bmatrix} +
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  -1 \\
  -1
\end{bmatrix},
\]
which gives
\[
\begin{bmatrix}
  -2w_{11} \\
  -2w_{21}
\end{bmatrix} +
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  -1 \\
  -1
\end{bmatrix} \text{ and } \begin{bmatrix}
  -2w_{12} \\
  -2w_{22}
\end{bmatrix} +
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  -1 \\
  -1
\end{bmatrix}.
\]

Thus, we obtain that
\[
W = \begin{bmatrix}
  \frac{4}{5} & \frac{4}{5} \\
  \frac{0}{5} & \frac{4}{5}
\end{bmatrix} \text{ and } B = \begin{bmatrix}
  \frac{3}{5} \\
  \frac{3}{5}
\end{bmatrix}.
\]

Thus we get
\[
\min_{i \in \{1,2\}} G(w_i) = \left( \frac{4\sqrt{2}}{5}, \frac{4\sqrt{2}}{5} \right).
\]

Hence we get \( w = \left( \frac{4}{5}, \frac{4}{5} \right) \) that minimize \( G(w_i) \) for \( i = 1, 2 \).

For the positive points of the combination of situation 3, we have
\[
\begin{bmatrix}
  w_{11} & w_{12} \\
  w_{21} & w_{22}
\end{bmatrix}
\begin{bmatrix}
  1/2 \\
  0
\end{bmatrix} +
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  1 \\
  1
\end{bmatrix},
\]
and
\[
\begin{bmatrix}
  w_{11} & w_{12} \\
  w_{21} & w_{22}
\end{bmatrix}
\begin{bmatrix}
  0 \\
  1/2
\end{bmatrix} +
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  1 \\
  1
\end{bmatrix},
\]
which gives
\[
\begin{bmatrix}
  \frac{1}{2}w_{11} \\
  \frac{1}{2}w_{21}
\end{bmatrix} +
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  1 \\
  1
\end{bmatrix} \text{ and } \begin{bmatrix}
  \frac{1}{2}w_{12} \\
  \frac{1}{2}w_{22}
\end{bmatrix} +
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  1 \\
  1
\end{bmatrix}.
\]
For two negative points for this case, we have
\[
\begin{bmatrix}
w_{11} & w_{12} \\
w_{21} & w_{22}
\end{bmatrix}
\begin{bmatrix}
-\frac{1}{2} \\
0
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
= \begin{bmatrix}
-1 \\
-1
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
w_{11} & w_{12} \\
w_{21} & w_{22}
\end{bmatrix}
\begin{bmatrix}
0 \\
-\frac{1}{2}
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
= \begin{bmatrix}
-1 \\
-1
\end{bmatrix},
\]
which gives
\[
\begin{bmatrix}
-\frac{1}{2}w_{11} \\
-\frac{1}{2}w_{21}
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
= \begin{bmatrix}
-1 \\
-1
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
-\frac{1}{2}w_{12} \\
-\frac{1}{2}w_{22}
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
= \begin{bmatrix}
-1 \\
-1
\end{bmatrix}.
\]
From this, we obtain that
\[
W = \begin{bmatrix}
2 & 2 \\
2 & 2
\end{bmatrix}
\]
and
\[
B = \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]
Thus we get
\[
\min_{i \in \{1, 2\}} G(w_i) = (2\sqrt{2}, 2\sqrt{2}).
\]
Hence we get \(w = (2, 2)\) that minimize \(G(w_i)\) for \(i = 1, 2\). □

**Proposition 2.4.** Let \(G : \mathbb{R}^n \rightarrow \mathbb{R}^n_+\) be a differentiable operator. An element \(w^* \in \mathbb{R}^n\) minimize \(G\) if and only if \(G'(w^*) = 0\), that is, \(w^* \in \mathbb{R}^n\) solves GSVM if and only if \(G'(w^*) = 0\).

**Proof.** Let \(G'(w^*) = 0\), then for all \(v \in \mathbb{R}^n\),
\[
< \eta G'(w^*), v - w^* > = < 0, v - w^* > = 0.
\]
Consequently, the inequality
\[
< \eta G'(w^*), v - w^* > \geq 0
\]
holds for all \(v \in \mathbb{R}^n\). Hence \(w^* \in \mathbb{R}^n\) solves problem of GSVM. Conversely, assume that \(w^* \in \mathbb{R}^n\) satisfies
\[
< \eta G'(w^*), v - w^* > \geq 0 \quad \forall \ v \in \mathbb{R}^n.
\]
Take \(v = w^* - G'(w^*)\) in the above inequality implies that
\[
< \eta G'(w^*), -G'(w^*) > \geq 0,
\]
which further implies
\[-\eta \|G'(w^*)\|^2 \geq 0.\]

Since \( \eta > 0 \), we get \( G'(w^*) = 0 \). \( \square \)

**Definition 2.5.** Let \( K \) be a closed and convex subset of \( \mathbb{R}^n \). Then, for every point \( x \in \mathbb{R}^n \), there exists a unique nearest point in \( K \), denoted by \( P_K(x) \), such that \( \|x - P_K(x)\| \leq \|x - y\| \) for all \( y \in K \) and also note that \( P_K(x) = x \) if \( x \in K \). \( P_K \) is called the metric projection of \( \mathbb{R}^n \) onto \( K \). It is well known that \( P_K: \mathbb{R}^n \to K \) is characterized by the properties:

(i) \( P_K(x) = z \) for \( x \in \mathbb{R}^n \) if and only if \( < z, y - z > \geq < x, y - z > \) for all \( y \in \mathbb{R}^n \);

(ii) For every \( x, y \in \mathbb{R}^n \), \( \|P_K(x) - P_K(y)\|^2 \leq < x - y, P_K(x) - P_K(y) > \);

(iii) \( \|P_K(x) - P_K(y)\| \leq \|x - y\| \), for every \( x, y \in \mathbb{R}^n \), that is, \( P_K \) is non-expansive map.

**Proposition 2.6.** Let \( G: \mathbb{R}^n \to \mathbb{R}^n_+ \) be a differentiable operator. An element \( w^* \in \mathbb{R}^n \) minimize mapping \( G \) defined in (2.3) if and only if \( w^* \) is the fixed point of map

\[
P_{\mathbb{R}^n_+} (I - \rho G') : \mathbb{R}^n \to \mathbb{R}^n_+ \quad \text{for any } \rho > 0.
\]

that is,

\[
w^* = P_{\mathbb{R}^n_+} (I - \rho G') (w^*) = P_{\mathbb{R}^n_+} (w^* - \rho G' (w^*)),
\]

where \( P_{\mathbb{R}^n_+} \) is a projection map from \( \mathbb{R}^n \) to \( \mathbb{R}^n_+ \).

**Proof.** Suppose \( w^* \in \mathbb{R}^n_+ \) is solution of \( GSVM \) then for \( \eta > 0 \), we have

\[
< \eta G'(w^*) , w - w^* > \geq 0 \quad \text{for all } w \in \mathbb{R}^n.
\]

Adding \( < w^*, w - w^* > \) on both sides, we get

\[
< w^*, w - w^* > + < \eta G'(w^*) , w - w^* > \geq < w^*, w - w^* > \quad \text{for all } w \in \mathbb{R}^n,
\]

which further implies that

\[
< w^*, w - w^* > \geq < w^* - \eta G'(w^*) , w - w^* > \quad \text{for all } w \in \mathbb{R}^n,
\]

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which is possible only if \( w^* = P_{\mathbb{R}_+^n} (w^* - \rho G' (w^*)) \), that is, \( w^* \) is the fixed point of \( G' \).

Conversely, let \( w^* = P_{\mathbb{R}_+^n} (w^* - \rho G' (w^*)) \), then we have
\[
< w^*, w - w^* > \geq < w^* - \eta G' (w^*), w - w^* > \quad \text{for all } w \in \mathbb{R}^n,
\]
which implies
\[
< w^*, w - w^* > - < w^* - \eta G' (w^*), w - w^* > \geq 0 \quad \text{for all } w \in \mathbb{R}^n,
\]
and so
\[
< \eta G' (w^*), w - w^* > \geq 0 \quad \text{for all } w \in \mathbb{R}^n.
\]
Thus \( w^* \in \mathbb{R}_+^n \) is the solution of GSVM. \( \square \)

**Definition 2.7.** A map \( G : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is said to be

(I) \( L \)-Lipschitz if for every \( L > 0 \),
\[
\| G (x) - G (y) \| \leq L \| x - y \| \quad \text{for all } x, y \in \mathbb{R}^n.
\]

(II) monotone if
\[
< G (x) - G (y), x - y > \geq 0 \quad \text{for all } x, y \in \mathbb{R}^n.
\]

(III) strictly monotone if
\[
< G (x) - G (y), x - y > > 0 \quad \text{for all } x, y \in \mathbb{R}^n \text{ with } x \neq y.
\]

(IV) \( \alpha \)-strongly monotone if
\[
< G (x) - G (y), x - y > \geq \alpha \| x - y \|^2 \quad \text{for all } x, y \in \mathbb{R}^n.
\]

Note that, every \( \alpha \)-strongly monotone map \( G : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is strictly monotone and every strictly monotone map is monotone.

**Example 2.8.** Let \( G : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a mapping defined as
\[
G (w_i) = \alpha w_i + \beta,
\]
where \( \alpha \) is any non negative scalar and \( \beta \) is any real number. Then \( G \) is Lipschitz continuous with Lipschitz constant \( L = \alpha \).
Also, for any \( x, y \in \mathbb{R}^n \),
\[
< G'(x) - G'(y), x - y > = \alpha \| x - y \|^2
\]
which show that \( G \) is \( \alpha \)-strongly monotone. \( \square \)

**Theorem 2.9.** Let \( K \subseteq \mathbb{R}^n \) be closed and convex and \( G' : \mathbb{R}^n \rightarrow K \) is strictly monotone. If there exists a \( w^* \in K \) which is the solution of \( GSVM \), then \( w^* \) is unique in \( K \).

**Proof.** Suppose that \( w^*_1, w^*_2 \in K \) with \( w^*_1 \neq w^*_2 \) be the two solutions of \( GSVM \), then we have
\[
< \eta G'(w^*_1), w - w^*_1 > \geq 0 \text{ for all } w \in \mathbb{R}^n \quad (2.16)
\]
and
\[
< \eta G'(w^*_2), w - w^*_2 > \geq 0 \text{ for all } w \in \mathbb{R}^n, \quad (2.17)
\]
where \( \eta > 0 \). Putting \( w = w^*_2 \) in (2.16) and \( w = w^*_1 \) in (2.17), we get
\[
< \eta G'(w^*_1), w^*_2 - w^*_1 > \geq 0 \quad (2.18)
\]
and
\[
< \eta G'(w^*_2), w^*_1 - w^*_2 > \geq 0. \quad (2.19)
\]
Eq. (2.18) can be further write as
\[
< -\eta G'(w^*_1), w^*_1 - w^*_2 > \geq 0. \quad (2.20)
\]
Adding (2.19) and (2.20) implies that
\[
< \eta G'(w^*_2) - \eta G'(w^*_1), w^*_1 - w^*_2 > \geq 0
\]
which implies
\[
\eta < G'(w^*_1) - G'(w^*_2), w^*_1 - w^*_2 > \leq 0
\]
or
\[
< G'(w^*_1) - G'(w^*_2), w^*_1 - w^*_2 > \leq 0. \quad (2.21)
\]
Since \( G' \) is strictly monotone, so we must have
\[
< G'(w^*_1) - G'(w^*_2), w^*_1 - w^*_2 > > 0,
\]
which contradicts (2.21). Thus \( w_1^* = w_2^* \). □

**Theorem 2.10.** Let \( K \subseteq \mathbb{R}^n \) be closed and convex. If the map \( G' : \mathbb{R}^n \to K \) is \( L \)-Lipchitz and \( \alpha \)-strongly monotone then there exists a unique \( w^* \in K \) which is the solution of \( GSVM \).

**Proof. Uniqueness:**
Suppose that \( w_1^*, w_2^* \in K \) be the two solutions of \( GSVM \), then for \( \eta > 0 \), we have
\[
< \eta G'(w_1^*), w - w_1^* > \geq 0 \text{ for all } w \in \mathbb{R}^n \tag{2.22}
\]
and
\[
< \eta G'(w_2^*), w - w_2^* > \geq 0 \text{ for all } w \in \mathbb{R}^n. \tag{2.23}
\]
Putting \( w = w_2^* \) in (2.22) and \( w = w_1^* \) in (2.23), we get
\[
< \eta G'(w_1^*), w_2^* - w_1^* > \geq 0 \tag{2.24}
\]
and
\[
< \eta G'(w_2^*), w_2^* - w_1^* > \geq 0. \tag{2.25}
\]
Eq. (2.24) can be further write as
\[
< -\eta G'(w_1^*), w_2^* - w_1^* > \geq 0. \tag{2.26}
\]
Adding (2.25) and (2.26) implies that
\[
< \eta G'(w_2^*) - \eta G'(w_1^*), w_1^* - w_2^* > \geq 0
\]
which implies
\[
\eta < G'(w_1^*) - G'(w_2^*), w_1^* - w_2^* > \leq 0. \tag{2.27}
\]
Since \( G' \) is \( \alpha \)-strongly monotone, so we have
\[
\alpha \eta \|w_1^* - w_2^*\|^2 \leq \eta < G'(w_1^*) - G'(w_2^*), w_1^* - w_2^* > \leq 0,
\]
which implies that
\[
\alpha \eta \|w_1^* - w_2^*\|^2 \leq 0.
\]
Since \( \alpha \eta > 0 \), so we must have \( \|w_1^* - w_2^*\| = 0 \) and hence \( w_1^* = w_2^* \).

**Existence:**
As we know that if $w^* \in \mathbb{R}_+^n$ is solution of GSVM then for $\eta > 0$, we have

$$< \eta G'(w^*), w - w^* > \geq 0 \text{ for all } w \in \mathbb{R}^n$$

if and only if

$$w^* = P_{\mathbb{R}_+^n}(w^* - \rho G'(w^*)) = F(w^*) \text{ (say).}$$

Now for any $w_1^*, w_2^* \in \mathbb{R}_+^n$, we have

$$\|F(w_1^*) - F(w_2^*)\|^2 = \|P_{\mathbb{R}_+^n}(w_1^* - \rho G'(w_1^*)) - P_{\mathbb{R}_+^n}(w_2^* - \rho G'(w_2^*))\|^2 \leq \|(w_1^* - \rho G'(w_1^*)) - (w_2^* - \rho G'(w_2^*))\|^2$$

(as $P_{\mathbb{R}_+^n}$ is nonexpansive)

$$= \|(w_1^* - w_2^*) - \rho[G'(w_1^*) - G'(w_2^*)]\|^2 \leq \langle (w_1^* - \rho G'(w_1^*)) - (w_2^* - \rho G'(w_2^*)), (w_1^* - w_2^*) - \rho[G'(w_1^*) - G'(w_2^*)]\rangle > 0.$$}

Now since $G'$ is $L$-Lipchitz and $\alpha$-strongly monotone, so we get

$$\|F(w_1^*) - F(w_2^*)\|^2 \leq \|w_1^* - w_2^*\|^2 - 2\rho \|w_1^* - w_2^*\|^2$$

$$\quad + \rho^2 L^2 \|w_1^* - w_2^*\|^2$$

$$= (1 + \rho^2 L^2 - 2\rho \alpha) \|w_1^* - w_2^*\|^2,$$

that is,

$$\|F(w_1^*) - F(w_2^*)\| \leq \theta \|w_1^* - w_2^*\|, \quad (2.30)$$

where $\theta = \sqrt{1 + \rho^2 L^2 - 2\rho \alpha}$. Since $\rho > 0$, so that when $\rho \in (0, \frac{2\alpha}{L^2})$, then we get $\theta \in [0, 1)$. Now, by using Principle of Banach contraction, we obtain the fixed point of map $F$, that is, there exists a unique $w^* \in \mathbb{R}_+^n$ such that

$$F(w^*) = P_{\mathbb{R}_+^n}(w^* - \rho G'(w^*)) = w^*.$$

Hence $w^* \in \mathbb{R}_+^n$ is the solution of GSVM.

**Example 2.11.** Let us take the group of data of positive class $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, 0),$ $(\alpha_1, \alpha_2, \ldots, \alpha_{n-2}, 0, \alpha_n),$ ..., $(0, \alpha_2, \alpha_3, \ldots, \alpha_n)$ and negative class $(k\alpha_1, k\alpha_2, \ldots, k\alpha_{n-1}, 0),$ $(k\alpha_1, k\alpha_2, \ldots, k\alpha_{n-2}, 0, k\alpha_n),$ ..., $(0, k\alpha_2, k\alpha_3, \ldots, k\alpha_n)$ for $n \geq 2$, where each $\alpha_i \neq 0$ for $i \in \mathbb{N}$ and $k \neq 1$. 

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A map \( G : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be given as
\[
G(w_i) = (\| w_i \|, \| w_i \|, ..., \| w_i \|) \quad \text{for} \quad i = 1, 2, ..., n,
\]
where \( w_i \) are the row of \( W_{n \times n} \) for \( i = 1, 2, ..., n \). Then we have
\[
G'(w_i) = \frac{1}{\| w_i \|} w_i \quad \text{for} \quad i = 1, 2, ..., n.
\]

Now from the given data, we get
\[
W = \frac{2}{(n - 1)(1 - k)} \begin{bmatrix}
\frac{1}{\alpha_1} & \frac{1}{\alpha_2} & \cdots & \frac{1}{\alpha_n} \\
\frac{1}{\alpha_1} & \frac{1}{\alpha_2} & \cdots & \frac{1}{\alpha_n} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{\alpha_1} & \frac{1}{\alpha_2} & \cdots & \frac{1}{\alpha_n}
\end{bmatrix}
\]
and so we have
\[
G(w_i) = \frac{2}{(n - 1)(1 - k)} \sqrt{\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \cdots + \frac{1}{\alpha_n^2}(1, 1, ..., 1)} \quad \text{for} \quad i = 1, 2, ..., n
\]
and
\[
G'(w_i) = \frac{1}{\sqrt{\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \cdots + \frac{1}{\alpha_n^2}}} \left( \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, ..., \frac{1}{\alpha_n} \right) \quad \text{for} \quad i = 1, 2, ..., n.
\]

Note that, for any \( w_1, w_2 \in W \),
\[
\| G'(w_1) - G'(w_2) \| = 0 = L \| w_1 - w_2 \|
\]
is satisfied where \( L \) is any nonnegative real number. Also
\[
< G'(w_1) - G'(w_2), w_1 - w_2 > \geq 0
\]
is satisfied which show that \( G' \) is monotone operator. Moreover, \( w = \frac{2}{(n - 1)(1 - k)} \left( \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, ..., \frac{1}{\alpha_n} \right) \) is the solution of GSVM with \( \| w \| = \frac{2}{(n - 1)(1 - k)} \sqrt{\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \cdots + \frac{1}{\alpha_n^2}} \).

\[\square\]

**Example 2.12.** Let us take the group of data of positive class \((\alpha_1, \alpha_2, ..., \alpha_m, 0, 0, ..., 0)\), \((0, \alpha_2, \alpha_3, ..., \alpha_{m+1}, 0, 0, ..., 0)\), ..., \((\alpha_1, \alpha_2, ..., \alpha_{m-1}, 0, 0, ..., 0, \alpha_n)\) and negative
class \((k\alpha_1, k\alpha_2, \ldots, k\alpha_m, 0, 0, 0), (0, k\alpha_2, k\alpha_3, \ldots, k\alpha_{m+1}, 0, 0, 0), \ldots, (k\alpha_1, k\alpha_2, \ldots, k\alpha_{m-1}, 0, 0, 0, \ldots)\) for \(n > m \geq 1\), where each \(\alpha_i \neq 0\) for \(i \in \mathbb{N}\) and \(k \neq 1\).

A map \(G : \mathbb{R}^n \to \mathbb{R}^n_+\) be given as

\[
G(w_i) = (\|w_i\|, \|w_i\|, \ldots, \|w_i\|) \quad \text{for} \quad i = 1, 2, \ldots, n,
\]

where \(w_i\) are the row of \(W_{n \times n}\) for \(i = 1, 2, \ldots, n\). Then we have

\[
G'(w_i) = \frac{1}{\|w_i\|}w_i \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

Now from the given data, we get

\[
W = \frac{2}{m \cdot (1 - k)} \begin{bmatrix} \frac{1}{\alpha_1} & \frac{1}{\alpha_2} & \cdots & \frac{1}{\alpha_n} \\ \frac{1}{\alpha_1} & \frac{1}{\alpha_2} & \cdots & \frac{1}{\alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\alpha_1} & \frac{1}{\alpha_2} & \cdots & \frac{1}{\alpha_n} \end{bmatrix}
\]

and so we have

\[
G(w_i) = \frac{2}{(n - 1) \cdot (1 - k)} \sqrt{\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \cdots + \frac{1}{\alpha_n^2}} (1, 1, \ldots, 1) \quad \text{for} \quad i = 1, 2, \ldots, n
\]

and

\[
G'(w_i) = \frac{1}{\sqrt{\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \cdots + \frac{1}{\alpha_n^2}}} \left( \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n} \right) \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

It is easy to verify that \(G'\) is monotone and Lipchitz continuous operator. The vector \(w = \frac{2}{m(1-k)}(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n})\) is the solution of GSVM with \(\|w\| = \frac{2}{m(1-k)} \sqrt{\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \cdots + \frac{1}{\alpha_n^2}}. \quad \square
\]

**Example 2.13.** Consider \((\alpha_1, 0, 0), (0, \alpha_2, 0), (0, 0, \alpha_3), (\beta_1, 0, 0), (0, \beta_2, 0), (0, 0, \beta_3)\) as data of positive class and \((k\alpha_1, 0, 0), (0, k\alpha_2, 0), (0, 0, k\alpha_3), (k\beta_1, 0, 0), (0, k\beta_2, 0), (0, 0, k\beta_3)\) as negative class of data, where \(\alpha_i, \beta_i\) and \(k\) are positive real numbers with each \(\alpha_i \leq \beta_i\) for \(i = 1, 2, 3\) and \(k \neq 1\).

The map \(G : \mathbb{R}^n \to \mathbb{R}^n_+\) is given as

\[
G(w_i) = (\|w_i\|, \|w_i\|, \ldots, \|w_i\|) \quad \text{for} \quad i = 1, 2, 3,
\]
where \( w_i \) are the row of \( W_{3 \times 3} \) for \( i = 1, 2, ..., n \). Then we have

\[
G'(w_i) = \frac{1}{\|w_i\|} w_i \quad \text{for} \quad i = 1, 2, 3.
\]

Now from the given data, we get

\[
W = \frac{2}{(1 - k)} \begin{bmatrix}
\frac{1}{\alpha_1} & \frac{1}{\alpha_2} & \frac{1}{\alpha_3} \\
\frac{1}{\alpha_1} & \frac{1}{\alpha_2} & \frac{1}{\alpha_3} \\
\frac{1}{\alpha_1} & \frac{1}{\alpha_2} & \frac{1}{\alpha_3}
\end{bmatrix}
\]

and so we have

\[
G(w_i) = \frac{2}{(1 - k)} \left( \sqrt{\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2}}, \sqrt{\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2}}, \sqrt{\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2}} \right)
\]

and

\[
G'(w_i) = \frac{1}{\sqrt{\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2}}} \left( \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3} \right).
\]

Note that, for any \( w_1, w_2 \in W \),

\[
\|G'(w_1) - G'(w_2)\| = 0 = L \|w_1 - w_2\|
\]

is satisfied for \( L > 0 \). Also

\[
< G'(w_1) - G'(w_2), w_1 - w_2 > \geq 0
\]

is satisfied which show that \( G' \) is monotone operator. Moreover, \( w = \frac{2}{(1 - k)} \left( \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3} \right) \) is the solution of GSVM with \( \|w\| = \frac{2}{(1 - k)} \sqrt{\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2}} \). \( \square \)

**Example 2.14.** Let us take the group of data of positive class \((1, 0, 0), (1, 1, 0), (0, 1, 1)\) and negative class \((-\frac{1}{2}, 0, 0), (-\frac{1}{2}, -\frac{1}{2}, 0), (0, -\frac{1}{2}, -\frac{1}{2})\).

Now from the given data, we have

\[
W = \begin{bmatrix}
\frac{4}{3} & 0 & \frac{4}{3} \\
\frac{4}{3} & 0 & \frac{4}{3} \\
\frac{4}{3} & 0 & \frac{4}{3}
\end{bmatrix}
\]

with

\[
G(w_i) = \frac{4}{3} (\sqrt{2}, \sqrt{2}, \sqrt{2}) \quad \text{for} \quad i = 1, 2, 3
\]
and

\[ G'(w_i) = \frac{1}{\sqrt{2}}(1, 0, 1) \text{ for } i = 1, 2, 3. \]

It is easy to verify that \( G' \) is monotone operator and Lipschitz continuous. Moreover, \( w = (\frac{4}{3}, 0, \frac{4}{3}) \) is the solution of GSVM with \( \|w\| = \frac{4}{3}\sqrt{2}. \)

Conclusion. Recently many results appeared in the literature giving the problems related to the support vector machine and its applications. In this paper, initiate the study of generalized support vector machine and present linear classification of data by using support vector machine and generalized support vector machine. We also provide sufficient conditions under which the solution of generalized support vector machine exist. Various examples are also present to show the validity of these results.

\end{conclusion}

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