An extension of the linear theory of isothermal sound propagation in an aerosol

Nils T. Basse
Taftehøj 23, Høruphav, 6470 Sydals, Denmark

May 17, 2019

Abstract
The existing linear theory of isothermal sound propagation in an aerosol considers Stokes drag and treats particles which are infinitely viscous. We extend the theory by applying the Coriolis flowmeter ”bubble theory”. Here, the drag force is a function of the Stokes number and the particle-to-fluid ratio of the dynamic viscosity [S.-M.Yang and L.G.Leal, A note on the memory-integral contributions to the force on an accelerating spherical drop at low Reynolds number, Phys. Fluids A 3, 1822-1824 (1991)]. Aerosol examples are presented and differences between the original and extended theories are discussed.

Keywords: Linear theory extension, Isothermal sound propagation in an aerosol, Drag force, Coriolis flowmeter ”bubble theory”

1. Introduction

Sound is damped by propagation through an aerosol; this can have important practical implications, e.g. for jet engines and rocket motors [1]. An aerosol is defined to be a suspension of solid particles or liquid droplets in air or another gas.

The linear theory of isothermal sound propagation in an aerosol has been presented in [2, 3] using Stokes drag and infinitely viscous particles. We will name this theory the ”sound propagation theory”.

Another linear theory of two-phase flow considers the reaction force on an oscillating fluid-filled container due to entrained particles [4]. This theory was motivated by the need to model two-phase flow in Coriolis flowmeters

\footnote{nils.basse@npb.dk}
and is known as the ”bubble theory”. The bubble theory has been used to model both (i) measurement errors \[5\] and (ii) damping \[6\] experienced by Coriolis flowmetering of two-phase flow. The analogy between the bias flow aperture theory \[7\] and the bubble theory has been explored in \[8\].

In terms of included physics in the two theories, the main difference is that another drag force \[9\] than Stokes drag is included in the bubble theory; this drag force is a function of the Stokes number and of the particle-to-fluid ratio of the dynamic viscosity. Thus, the sound propagation theory is a limiting case of the more general bubble theory.

The paper is organised as follows: In Section 2, we summarize the sound propagation theory, followed by a corresponding overview of the bubble theory in Section 3. We use mixture examples to compare the two theories in Section 4 and conclude in Section 5.

2. Isothermal sound propagation in an aerosol

Here, we use the nomenclature from \[3\]. The theory is derived for a dilute suspension, i.e. \( \phi_v \ll 1 \), where \( \phi_v \) is the particle volume concentration.

The result of the theory is the dispersion relation for an acoustic wave passing through the aerosol:

\[
A \equiv \left( c^2_{TF} \frac{k^2}{\omega^2} - 1 \right) / \eta_{m0} = \frac{1}{1 - i\omega \tau_d},
\]

(1)

where \( c_{TF} \) is the isothermal sound speed in the fluid, \( k \) is the complex wavenumber of the acoustic wave and \( \omega \), the angular frequency of the acoustic wave, is assumed to be real and positive. \( \eta_{m0} \) is the mass loading and \( \tau_d \) is the dynamic relaxation time of the particle:

\[
\tau_d = \frac{m_p}{6\pi \mu_f a^2} = \frac{2a^2}{9\nu_f \delta},
\]

(2)

where \( m_p \) is the particle (p) mass, \( \mu_f \) is the dynamic viscosity of the fluid (f), \( a \) is the particle radius, \( \nu_f \) is the kinematic viscosity of the fluid and \( \delta = \rho_f / \rho_p \). Here, \( \rho_f \) is the fluid density and \( \rho_p \) is the particle density.

The real (imaginary) part of \( A \) is the phase velocity (attenuation) of the acoustic wave, respectively.

3. The Coriolis flowmeter bubble theory

Note: This entire section is taken (almost) verbatim from \[8\].

The Coriolis flowmeter bubble theory was first presented in \[4\]. It is a linear theory for an incompressible, low Reynolds number flow. The force on a
fluid-filled, oscillating container due to entrained particles is calculated. The particles can either be solid or consist of a fluid. The motion of the container leads to decoupled motion of the fluid and the particles, which leads to both (i) measurement errors and (ii) damping of Coriolis flowmeters. These effects have been studied in [5] and [6], respectively. The entrained particles mean that a two-phase flow is considered by the theory. The force on the container is given by:

$$F_{f,z} = (\rho_f - \rho_p)V_p a_c F,$$

where $V_p$ is the particle volume, $a_c$ is the container acceleration, $z$ is the acceleration direction and $F$ is the reaction force coefficient:

$$F = 1 + \frac{4(1 - \tau)}{4\tau - (9iG/\beta^2)}$$

The real part of $F$ is a virtual mass loss and the imaginary part of $F$ represents damping which acts against the vibrating force.

The density ratio is

$$\tau = 1/\delta = \frac{\rho_p}{\rho_f}$$

The Stokes number is

$$\beta = \frac{a}{\delta} = a \sqrt{\frac{\omega \rho_f}{2 \mu_f}},$$

where $\omega$ is the oscillation frequency of the container.

The quantities below are defined in [9]:

$$G = 1 + \lambda + \frac{\lambda^2}{9} - \frac{(1 + \lambda)^2 f(\lambda)}{\kappa[\lambda^3 - \lambda^2 \tanh \lambda - 2 f(\lambda)] + (\lambda + 3)f(\lambda)},$$

where

$$\lambda = (1 + i)\beta$$

and

$$f(\lambda) = \lambda^2 \tanh \lambda - 3\lambda + 3 \tanh \lambda$$

The viscosity ratio is

$$\kappa = \frac{\mu_p}{\mu_f}$$
G is “proportional to the drag force on a spherical particle undergoing harmonic motion in a surrounding (stagnant) liquid” \(4\) (fluid):

\[
F_D = -u_p(6\pi\mu_f aG),
\]

(11)

where \(u_p\) is the particle velocity. Note that if \(G = 1\), the drag force reduces to the Stokes drag:

\[
F_{D,\text{Stokes}} = -u_p(6\pi\mu_f a)
\]

(12)

4. Comparison of the two theories

The angular frequency \(\omega\) has a different physical meaning for the two theories: For the acoustic propagation theory, it is the acoustic wave frequency and for the bubble theory, it is the frequency of the container oscillation. However, mathematically they are completely equivalent.

4.1. The acoustic propagation theory limit of the bubble theory

We will now derive that under certain assumptions, the acoustic propagation theory is a limit of the bubble theory.

First we use Eq. (4) with \(\tau \gg 1\):

\[
F \approx -\frac{i9G/\beta^2}{4\tau - i9G/\beta^2}
\]

(13)

To bring this into a form comparable to the acoustic propagation theory, we multiply Eq. (13) by \(i\beta^2/9G\):

\[
F = \frac{1}{1 + i\left(\frac{4\tau}{9G}\right) \beta^2}
\]

(14)

Further, for small \(\beta\) and \(\kappa \gg 1\), \(G \approx 1\) \[8\]:

\[
F = \frac{1}{1 + i\left(\frac{4\tau}{9G}\right) \beta^2} \approx \frac{1}{1 + i\omega \tau_d}
\]

(15)

We find that Eq. (15) for \(F\) is equal to Eq. (1) for \(A\), except for the sign of the imaginary part in the denominator. However, this sign is a convention, i.e. whether one associates a negative or positive sign of the imaginary part with damping.
4.2. Mixture examples

For previous work using the bubble theory [5, 6, 8], we considered two-phase mixtures where water was the fluid and particles consisted of air, oil and sand. In this paper, we have air as the fluid and water, oil and sand as the particles.

The real part of $F$ (Eq. (4)) and $A$ (Eq. (1)) is compared in Fig. 1. The results match quite closely.

The imaginary part of $F$ (Eq. (4)) and $A$ (Eq. (1)) is compared in Fig. 2. $\beta$ for maximum damping agrees quite well for the two theories, but there is some deviation in the amplitude: In general, the damping amplitude for the bubble theory is 5% lower than for the sound propagation theory.

The changes in the damping amplitude are due to differences in $G$; the main effect is the increase of $G$ with $\beta$, a smaller effect is the particle-to-fluid ratio of the dynamic viscosity, see Fig. 3. For increasing $\beta$, the real part of $G$ increases from 1 and the imaginary part of $G$ increases from 0; the combined effect is the reduction of the peak damping amplitude compared to the sound
propagation theory.

As we have seen, the differences between the two theories are small; to make them more visible, we overlay results for the water-air mixture (water droplets in air) in Fig. 4.

5. Conclusions

We have extended the linear theory of isothermal sound propagation in an aerosol by applying the Coriolis flowmeter “bubble theory”: Here, the drag force is a function of the Stokes number and the particle-to-fluid ratio of the dynamic viscosity.

Aerosol examples are presented with air as the fluid - the most important modification is a reduction of the damping peak magnitude by around 5% for the bubble theory.
Acknowledgements

The author is grateful to Dr. John Hemp for creating, providing and explaining/discussing the Coriolis flowmeter bubble theory [4].

References

[1] M.S. Howe, Acoustics of Fluid-Structure Interactions, Cambridge University Press, 1998.

[2] S. Temkin and R.A. Dobbins, Attenuation and dispersion of sound by particulate-relaxation processes, J. Acoust. Soc. Am. 40, 317-324 (1966).

[3] S. Temkin, Suspension Acoustics, An Introduction to the Physics of Suspensions, Cambridge University Press, 2005.

[4] J. Hemp, Reaction force of a bubble (or droplet) in a liquid undergoing simple harmonic motion, Unpublished, 1-13 (2003).

[5] N.T. Basse, A review of the theory of Coriolis flowmeter measurement errors due to entrained particles, Flow. Meas. Instrum. 37, 107-118 (2014).

[6] N.T. Basse, Coriolis flowmeter damping for two-phase flow due to decoupling, Flow. Meas. Instrum. 52, 40-52 (2016).

[7] M.S. Howe, On the theory of unsteady high Reynolds number flow through a circular aperture, Proc. R. Soc. Lond. A. 366, 205-223 (1979).

[8] N.T. Basse, On the analogy between the bias flow aperture theory and the Coriolis flowmeter ”bubble theory”, https://arxiv.org/abs/1901.04930

[9] S.-M. Yang and L.G. Leal, A note on the memory-integral contributions to the force on an accelerating spherical drop at low Reynolds number, Phys. Fluids A 3, 1822-1824 (1991).