Cosmology in $f(Q)$ gravity: A unified dynamical system analysis at background and perturbation levels

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Motivated by the fact that cosmological models based on $f(Q)$ gravity are very efficient in fitting observational datasets at both background and perturbation levels, we perform a combined dynamical system analysis of both background and perturbation equations in order to examine the validity of this result through an independent method. We examine two studied $f(Q)$ models of the literature, namely the power-law and the exponential ones. For both cases, we obtain a matter-dominated saddle point characterized by the correct growth rate of matter perturbations, followed by the transition to a stable dark-energy dominated accelerated universe in which matter perturbations remain constant. Furthermore, analyzing the behavior of $f\sigma_8$, we find that the models fit observational data successfully, obtaining a behavior similar to that of $\Lambda$CDM scenario, although the exponential model does not possess the latter as a particular limit. Hence, through the independent approach of dynamical systems, we do verify the results of observational confrontation, namely that $f(Q)$ gravity can be considered as a very promising alternative to the $\Lambda$CDM concordance model.

I. INTRODUCTION

Einstein’s theory of General Relativity (GR) is the most successful theory to describe the gravitational interaction, and based on that, the Lambda Cold Dark Matter (LCDM) scenario is the concordance cosmological model. However, this standard gravitational and cosmological paradigm faces theoretical and observational problems such as the non-renormalizability of GR, the cosmological constant problem, the coincidence problem, the Hubble tension, the power-law and the exponential ones. For both cases, we obtain a matter-dominated saddle point characterized by the correct growth rate of matter perturbations, followed by the transition to a stable dark-energy dominated accelerated universe in which matter perturbations remain constant. Hence, through the independent approach of dynamical systems, we do verify the results of observational confrontation, namely that $f(Q)$ gravity can be considered as a very promising alternative to the $\Lambda$CDM concordance model.

The usual way to construct gravitational modifications is to add extra terms in the Einstein-Hilbert Lagrangian, resulting to $f(R)$ gravity [15–17], Gauss-Bonnet and $f(G)$ gravity [18–20], cubic and $f(P)$ gravity [21–23], Horndeski/Galileon scalar-tensor theories [24, 25], etc. Alternatively, one can add new terms to the equivalent formulation of gravity based on torsion, resulting to $f(T)$ gravity [11, 26], $f(T, T_G)$ gravity [27–29], $f(T, B)$ gravity [30, 31], scalar-torsion gravity [32], etc. Nevertheless, there is a third way to construct new classes of modified gravity, starting from the “symmetric teleparallel gravity”, which is based on the non-metricity scalar $Q$ [33], and extend it to a function $f(Q)$ in the Lagrangian.

The modified theory of $f(Q)$ gravity leads to interesting cosmological phenomenology at the background level [34–56]. Additionally, it has been successfully confronted with various background and perturbation observational data, such as the Cosmic Microwave Background (CMB), Supernovae type Ia (SNIa), Baryonic Acoustic Oscillations (BAO), Redshift Space Distortion (RSD), growth data, etc, [57–64], and this confrontation reveals that $f(Q)$ gravity may challenge the standard $\Lambda$CDM scenario. Finally, $f(Q)$ gravity comfortably passes the Big Bang Nucleosynthesis (BBN) constraints too [65].

Motivated by the exciting features of $f(Q)$ gravity, in this work, we employ the powerful mathematical tool of dynamical system analysis in order to investigate for the first time the cosmological dynamics of $f(Q)$ cosmology at both background and perturbation levels. Such investigation can be used to further confirm the results obtained from the observational analysis. We mention that usually, the dynamical system approach is applied at the background level [66–83], however relatively recently it was realized that the analysis can be applied at the perturbation level too [84–87]. Hence, with this combined analysis, we can determine both the background stable late-time solutions, as well as the growth of the structure formation, independent of the specific initial conditions. Moreover, we can examine how the matter perturbations back-react to the background solutions, too.

The work is organized as follows: In Sec. II, we present the field equations of $f(Q)$ gravity, from which one can arrive at the background and perturbed cosmological equations. Sec. III deals with the dynamical analysis of the combined system for the power-law and exponen-
II. $f(Q)$ COSMOLOGY

In this section we briefly review $f(Q)$ cosmology. The action of $f(Q)$ gravity is given by \cite{[33, 34]}

$$\mathcal{S} = \int \left[-\frac{1}{16\pi G}f(Q) + \mathcal{L}_m\right] \sqrt{-g} \, d^4x, \quad (1)$$

where $g$ is the determinant of the metric $g_{\mu\nu}$ and $\mathcal{L}_m$ is the matter Lagrangian density. $f(Q)$ is an arbitrary function of the non-metricity scalar \cite{[33]}

$$Q = -\frac{1}{4} Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} + \frac{1}{2} Q_{\alpha\beta} Q^{\gamma\beta\alpha} + \frac{1}{4} Q_{\alpha} Q^{\alpha} - \frac{1}{2} Q_{\alpha} \hat{Q}^{\alpha}, \quad (2)$$

where $Q_{\alpha} = Q_{\alpha}^{\mu}$ and $\hat{Q}^{\alpha} = Q^{\mu \alpha}$ are acquired from contractions of the non-metricity tensor $Q_{\alpha\mu\nu} = \nabla_{\alpha} g_{\mu\nu}$. Thus, Symmetric Teleparallel Equivalent of General Relativity, and therefore General Relativity, is recovered for $f(Q) = Q$.

Variation of action (1), and setting $8\pi G = 1$ for simplicity, leads to the field equations \cite{[34, 35]}

$$\frac{2}{\sqrt{-g}} \nabla_{\alpha} \left\{ \sqrt{-g} g_{\beta\nu} f_Q \left[ -\frac{1}{2} L^{\alpha\beta} + \frac{1}{4} g^{\mu\beta} \left( Q^{\alpha} - \hat{Q}^{\alpha} \right) - \frac{1}{8} \left( g^{\alpha\mu} Q^\beta + g^{\alpha\beta} Q^\mu \right) \right] - f_Q \left[ -\frac{1}{2} L^{\alpha\beta} + \frac{1}{8} \left( g^{\mu\alpha} Q^\beta + g^{\mu\beta} Q^\alpha \right) + \frac{1}{4} g^{\alpha\beta} \left( Q^{\mu} - \hat{Q}^{\mu} \right) \right] Q_{\alpha\beta} + \frac{1}{2} \delta^\nu_{\beta} f = T^\nu_{\alpha}, \quad (3)$$

where $L^{\alpha\beta} = \frac{1}{2} Q^{\alpha\mu} - Q^{\alpha}_{\mu \nu} v^\nu$ is the disformal tensor, $T^\nu_{\mu}$ is the matter energy-momentum tensor, and $f_Q \equiv \partial f/\partial Q$.

At the background level, we assume a homogeneous, isotropic, and spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, whose metric is of the form

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (4)$$

where $t$ is the cosmic time, $a(t)$ is the scale factor and $x, y, z$ are the Cartesian coordinates. Note that in FLRW metric, for the non-metricity scalar we obtain $Q = 6H^2$, where $H = \dot{a}/a$ is the Hubble function and the upper dot denotes derivative with respect to $t$. Imposing the splitting $f(Q) = Q + F(Q)$, and applying the FLRW metric, the corresponding Friedman equations are \cite{[33, 34]}

$$3H^2 = \rho + \frac{F}{2} - QF_Q, \quad (5)$$

$$(2QF_Q + F_Q + 1) \dot{H} + \frac{1}{4} \left( Q + 2QF_Q - F \right) = -2p, \quad (6)$$

with $F_Q \equiv \frac{dF}{dQ}$, $F_{QQ} \equiv \frac{d^2F}{dQ^2}$. In the above equations $\rho$ and $p$ are the energy density and pressure of the matter fluid, which for no interaction satisfy the conservation equation

$$\dot{\rho} + 3H(1 + w)\rho = 0, \quad (7)$$

with $w \equiv p/\rho$ the matter equation-of-state parameter.

We can now introduce the effective, total, energy density $\rho_{\text{eff}}$ and pressure $p_{\text{eff}}$, respectively as

$$\rho_{\text{eff}} \equiv \rho + \frac{F}{2} - QF_Q, \quad \rho_{\text{eff}} \equiv \frac{\rho(1 + w)}{2QF_Q + F_Q + 1} - \frac{Q}{2}, \quad (8)$$

and thus the corresponding total equation of state $w_{\text{eff}}$ is given by

$$w_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = -1 + \frac{\Omega_m(1 + w)}{2QF_Q + F_Q + 1}. \quad (9)$$

We mention that for an accelerated Universe one requires $w_{\text{eff}} < -\frac{1}{3}$. Finally, it proves convenient to introduce the energy density parameters as

$$\Omega_m = \frac{\rho}{3H^2}, \quad \Omega_Q = \frac{F}{3H^2}, \quad (10)$$

and thus the first Friedmann equation (5) becomes simply

$$\Omega_m + \Omega_Q = 1. \quad (11)$$

Let us proceed to the investigation of the linear perturbation level, focusing on the matter density contrast $\delta = \frac{\rho}{\rho}\Delta$, where $\rho$ is the perturbation of the matter energy density. In particular, the evolution equation of the matter overdensity at the quasi-static limit is given by \cite{[34, 61]}

$$\ddot{\delta} + 2H\dot{\delta} = \frac{\rho\delta}{2(1 + F_Q)}, \quad (12)$$

where the denominator of the last term accounts for the appearance of an effective Newton’s constant. We mention that in the quasi-static limit the terms involving time derivatives in the perturbed equations are neglected, and only spatial derivative terms remain. It is worth mentioning that such an approximation is sufficient at small scales, well within the cosmic horizon \cite{[88]}.

III. DYNAMICAL SYSTEM ANALYSIS

In this section we construct the dynamical system of the background and perturbed equations, for a general function $F(Q)$. In this regard, we transform the equations (5)-(7) and (13) into a first-order autonomous system, by considering the following dynamical variables:

$$x = \frac{F}{6H^2}, \quad y = -2F_Q, \quad u = \frac{d(\ln\delta)}{d(\ln a)}. \quad (14)$$
Hence, while the variables $x, y$ are associated with the background evolution of the Universe, the variable $u$ quantifies the growth of matter perturbations. Therefore, $u > 0$ signifies the growth and $u < 0$ indicates the decay of matter perturbations, whenever the matter density contrast $\delta$ is positive.

The background cosmological parameters $\Omega_m, \Omega_Q$ and $w_{\text{eff}}$ can expressed as

$$\Omega_m = 1 - x - y,$$

$$\Omega_Q = x + y,$$

$$w_{\text{eff}} = -1 + \frac{(1 - x - y)(1 + w)}{2QF_{QQ} - \frac{y}{2} + 1}. \quad (15)$$

Now, in terms of variables (14), the cosmological equations can be written as the following dynamical system:

$$x' = -\frac{\dot{H}}{H^2}(y + 2x), \quad (16)$$

$$y' = 2y\frac{\dot{H}QF_{QQ}}{F_Q}, \quad (17)$$

$$u' = -u(u + 2) + \frac{3(1 - x - y)}{(2 - y)} - \frac{\dot{H}}{H^2}u, \quad (18)$$

where a prime stands for differentiation with respect to $\ln a$, and

$$\frac{\dot{H}}{H^2} = -\frac{3 - 3(x + y)}{4QF_{QQ} - y + 2}. \quad (19)$$

The physical system is a product space of the background phase space $\mathcal{B}$, which includes the variables $x, y,$ and the perturbed space $\mathcal{P}$ which consists of the variable $u$. Under the physical condition $0 \leq \Omega_m \leq 1$, the phase space of the combined system is

$$\Psi = \mathcal{B} \times \mathcal{P} = \{(x, y, u) \in \mathbb{R}^2 \times \mathbb{R} : 0 \leq x + y \leq 1\}. \quad (20)$$

It is worth mentioning that the projection of orbits of the product space $\Psi$ on space $\mathcal{B}$ reduces to the corresponding background orbits.

As a next step we shall determine the dynamical evolution of the system by extracting its critical points and examining their stability. Physically, a stable point with $u > 0$ indicates an indefinite growth of matter perturbations and hence the system is not stable with respect to matter perturbations. However, a stable point with $u < 0$ indicates the decay of matter perturbations and therefore the system is asymptotically stable with respect to perturbations. Finally, when $u = 0$ at a stable point implies that matter perturbations remain constant. In summary, for a viable model one desires to have an unstable or saddle point with $u > 0$, required for the description of the matter epoch of the universe, in which the matter perturbations grow but which does not hold eternally, followed by a stable late-time attractor corresponding to acceleration but with $u = 0$.

In order to proceed to specific analysis, we need to specify the function $F$ and hence determine the term $QF_{QQ}/F_Q$. In the following subsections we will consider two specific models, which are known to lead to interesting cosmological phenomenology.

### A. Model I: $F(Q) = \alpha \left(\frac{Q}{Q_0}\right)^n$

We start our analysis by considering a power-law model with [34, 58, 60]

$$F(Q) = \alpha \left(\frac{Q}{Q_0}\right)^n, \quad (21)$$

with $\alpha$ and $n$ two parameters and where $Q_0 = 6H_0^2$ is the present value of $Q$ (note that applying the first Friedmann equation at present, $\alpha$ can be eliminated in terms of $n$ and the present value $\Omega_{m0}$). This model can describe the late-time universe acceleration and it is also compatible with BBN constraints [65]. We mention that for $n = 0$ this model is equivalent to the concordance $\Lambda$CDM scenario, while for $n = 1$ the model reduces to the symmetric teleparallel equivalent of general relativity [34, 58, 60]. In this case, we have $QF_{QQ} = \frac{(1-n)y}{4}$ and hence the system (16)-(18) closes.

The corresponding dynamical system contains the following four critical points:

- **Point $A_1 (0, 0, 1)$**: This point corresponds to a matter-dominated critical solution with the background parameters $\Omega_m = 1$ and $w_{\text{eff}} = 0$. At the perturbation level we have $u = 1$, which implies that the matter overdensity $\delta$ varies as the scale factor $a$ and hence increases with the universe expansion. The corresponding Jacobian matrix has the eigenvalues $-\frac{5}{2}, 3$ and $\frac{3}{2}(1 - n)$, therefore point $A_1$ for any value of $n$ is always a saddle one. Hence, the trajectories pass through this point and leave it as they are attracted by a late-time stable point. Thus, we conclude that this point could be the main candidate for describing the structure formation during the matter domination era at both the background and perturbation levels.

- **Point $B_1 (0, 0, -\frac{3}{2})$**: At the background level this point corresponds to matter domination, with $\Omega_m = 1$ and $w_{\text{eff}} = 0$. However, this point could not describe the formation of structures at the perturbation level, since $u = -\frac{3}{2}$, and hence the matter overdensity $\delta$ varies as $a^{-\frac{3}{2}}$. The eigenvalues of the Jacobian matrix are $\frac{3}{2}, 3$ and $\frac{3}{2}(1 - n)$, and therefore this point is unstable for $n < 1$ and saddle for $n > 1$.

- **The curve of critical points $C_1 (1 - y, y, 0)$**: Each point on this curve corresponds to a solution dominated by the effective dark-energy component, i.e. $\Omega_Q = 1$, in which the universe is accelerated with a cosmological-constant-like behavior, namely with $w_{\text{eff}} = -1$. Furthermore, at the perturbation level
we have \( u = 0 \), which implies that the matter perturbation remains constant. The corresponding eigenvalues are \(-2, -3\) and 0, and since the curve is one-dimensional with one vanishing eigenvalue, it is normally hyperbolic [67], and one can determine its stability by examining the signature of the remaining non-vanishing eigenvalues [67]. Therefore, we conclude that it is always stable. In summary, the curve \( C_1 \) describes the late time dark-energy dominated Universe, at both background and perturbation levels.

- **Curve of critical points \( D_1 \) \((1 - y, y, -2)\):** Similarly to \( C_1 \), this curve of critical points also corresponds to a cosmological-constant-like solution, i.e. with \( w_{\text{eff}} = -1 \), dominated by the effective dark-energy component. Additionally, it is characterized by the decay of matter perturbations, since \( u = -2 \). However, it is saddle with eigenvalues 2, \(-3\) and 0. Hence, unlike \( C_1 \), curve \( D_1 \) cannot describe a late-time dark-energy dominated Universe at the perturbation level.

Our analysis reveals that different critical points describe different modes of matter perturbations. Additionally, we mention that identical background critical points behave differently at the perturbation level. For instance, we showed that points \( A_1 \) and \( B_1 \) describe the decelerated matter-dominated era at the background level, but only point \( A_1 \) has the correct growth of matter structure. Interestingly, point \( A_1 \) is saddle and thus it provides the natural exit towards a late-time accelerated epoch. On the other hand, at late times the curves of critical points \( C_1 \) and \( D_1 \) are identical at the background level, describing the accelerated dark-energy dominated epoch. However, only curve \( C_1 \) is physically and observationally interesting at the perturbation level, since it is stable and exhibits constant matter perturbations. Lastly, examining for completeness whether there are critical points at infinity, we find that no such physical points exist.

In order to give the above information in a more transparent way, we display the phase portrait of the system (16)-(18) in Fig. 1. As we see, the system follows the orbit \( B_1 \to A_1 \to C_1 \). Furthermore, in Fig. 2 we present the evolution of the background parameters and the matter growth rate variable \( u \), in terms of the redshift \( z = \frac{a_0}{a} - 1 \) with \( a_0 = 1 \) the current scale factor. As we see, the model describes the transition from matter domination towards an accelerated dark-energy dominated epoch.

![Phase portrait of the system (16)-(18)](image1)

**FIG. 1.** The phase portrait of the system (16)-(18), for the power-law model I of (21) with \( n = 0.5 \). This particular example exhibits the evolution \( B_1 \to A_1 \to C_1 \).

![Evolution of density parameters](image2)

**FIG. 2.** Upper graph: Evolution of the density parameters of matter \( \Omega_m \) and of non-metricity (dark-energy) \( \Omega_Q \), as well as of the total, effective equation-of-state parameter \( w_{\text{eff}} \), as functions of the redshift, for the power-law model I of (21) with \( n = 0.5 \). Lower graph: Evolution of the perturbation variable \( u = \frac{d(ln\delta)}{d ln a} \).

In summary, the present power-law model can describe the desired thermal history of the Universe, both at the background and perturbation levels. Our analysis indicates that in principle the above hold for any value of \( n \). Nevertheless, we should mention that a tuning of initial
conditions is required in order to have a sufficiently long matter-dominated era.

Finally, in order to test the predictions on the matter growth with observational data, in Fig. 3 we provide the evolution of \( f\sigma_8 \). This quantity is defined as the product of the growth rate factor, \( f = u = \frac{d\ln \delta}{d\ln a} \), and the root-mean-square normalization of the matter power spectrum \( \sigma_8 \). The value of \( \sigma_8 \) usually depends on the model, however here we have taken \( \sigma_8 = 0.8 \) which could alleviate the present \( \sigma_8 \) tension between the RSD and Planck data [2]. We mention here that we have checked that for \( n > 0 \) the evolution of \( f\sigma_8 \) coincides with that of \( \Lambda \)CDM scenario, namely with the case \( n = 0 \). From Fig. 3, we observe that models with \( n < 0 \) have smaller value of \( f\sigma_8 \), however, the data prefer comparatively larger values. Hence, models with \( n < 0 \) are not favored by the data. Additionally, it is worth noting that observational data still favour \( n < 1 \) [58–60, 63]. Thus, from our analysis we can conclude that the condition \( 0 < n < 1 \) is required in order to acquire consistency with observations.

In this subsection we consider the exponential model [61]
\[
F(Q) = Q e^{\frac{\beta Q}{Q_0}} - Q,
\]
(22)
with \( \beta \) the only dimensionless parameter. For \( \beta = 0 \) the model is equivalent to GR without a cosmological constant, however the interesting feature is that for \( \beta \neq 0 \) this model can fit observations in a very satisfactory way, although it does not include a cosmological constant [61]. Note that applying the first Friedmann equation at present, \( \beta \) can be early times \( Q \gg Q_0 \), the model tends to GR limit and hence it trivially passes the BBN constraints [61, 65].

In this case we have \( Q F_{QQ} = \left( \frac{x+1}{x-1} \right)^2 + x(y-2) + \frac{y^2}{2} - 1 \), and therefore the dynamical system (16)-(18) has four curves of critical points. In what follows, we shall describe the properties of each curve.

- **Curve of critical points** \( A_2 \left( \frac{y}{2}, y, 1 \right) \): This curve corresponds to a matter scaling solution with \( \Omega_m = 1 - \frac{y}{2} \) and \( w_{\text{eff}} = 0 \). The corresponding Jacobian matrix has the eigenvalues \(-\frac{5}{2}, 3, 0\), and thus the corresponding points are always saddle. Furthermore, since \( u = 1 \), it is implied that the matter perturbations grow, and hence the solution is of interest from the structure formation point of view.

- **Curve of critical points** \( B_2 \left( -\frac{y}{2}, y, -\frac{3}{2} \right) \): Similarly to \( A_2 \), this curve corresponds to a matter scaling solution. However, since \( u < 0 \), we deduce that the matter perturbations decay and therefore it cannot describe the growth of structures during the matter-dominated epoch. It corresponds to an unstable node with eigenvalues \( \frac{5}{2}, 3, 0 \).

In this case we have \( Q F_{QQ} = (x+1)^2 + x(y-2) + \frac{y^2}{2} - 1 \), and therefore the dynamical system (16)-(18) has four curves of critical points. In what follows, we shall describe the properties of each curve.

- **Curves of critical points** \( C_2 \left( 1 - y, y, 0 \right) \) and \( D_2 \left( 1 - y, y, -2 \right) \): Both these curves correspond to accelerated solutions, dominated by the non-metricity component. The stability and cosmological properties of curves \( C_2 \) and \( D_2 \) are exactly the same with the curves \( C_1 \) and \( D_1 \). Finally, we find that only \( C_2 \) is interesting for the late-time universe at the perturbation level.
Similarly to Model I of the previous subsection, we see that the inclusion of perturbations distinguishes critical points that are equivalent at the background level. Hence, from the combined background and perturbation analysis we find that only curve $A_2$ is physically interesting to describe the matter-dominated epoch, where matter perturbations are generated. On the other hand, curve $C_2$ corresponds to late-time dark-energy domination, with a fixed evolution of matter perturbations, as observations require. Lastly, for this model we also find no critical points at infinity.

In Fig. 4 we present the phase-space evolution, describing the transition $B_2 \rightarrow A_2 \rightarrow C_2$. Furthermore, in Fig. 5 we depict the evolution of the background cosmological parameters and the growth rate $u$, where we observe the transition from matter domination towards a late-time dark-energy dominated epoch. As we mentioned above, it is interesting that even though the present model does not possess a $\Lambda$CDM limit for any parameter choice, the corresponding dynamics is qualitatively similar with that of $\Lambda$CDM (see Fig. 5). Hence, since the model is free from the cosmological constant problem, it may be considered as slightly preferred over the $\Lambda$CDM scenario, constituting an interesting alternative.

Finally, in Fig. 6 we investigate the evolution of $f\sigma_8$, using $\sigma_8 = 0.7$. As we observe, the behavior is comparable with that of $\Lambda$CDM paradigm. In summary, the unified dynamical system analysis confirms the observational investigation at background and perturbation levels performed in [61, 63].

Motivated by the fact that cosmological models based on $f(Q)$ gravity are very efficient in fitting observational datasets at both background and perturbation levels [61, 63], in the present work we performed a combined dynamical system analysis of both background and perturbation equations in order to examine the validity of this result through an independent method.

After transforming the background and perturbation equations into an autonomous system, we focused on two studied $f(Q)$ models of the literature, namely the power-law and the exponential ones. Due to the extra variable related to matter perturbations, each background critical point split into two points, characterized by different behavior of matter perturbations and stability.

Concerning the power-law model, we obtained a matter-dominated saddle point characterized by the correct growth rate of matter perturbations, followed by the transition to a stable dark-energy dominated accelerated
universe in which matter perturbations remain constant. Furthermore, we studied the growth of matter perturbations by analyzing the behavior of \( f_{\sigma_8} \), and we found that the model fits observational data successfully if the exponent lies within \( 0 < n < 1 \), in which case we obtained a behavior similar to that of ΛCDM scenario.

Concerning the exponential model, we also found curves of points corresponding to matter domination and matter perturbation growth, and the fact that they are saddle provides a successful transition to the stable late-time dark-energy dominated solution with constant matter perturbations. The interesting feature of this model is that this desired behavior is obtained although the model does not possess ΛCDM scenario as a particular limit, namely, it arises solely from the non-metricity structure. Additionally, we found that while the power-law model resembles ΛCDM cosmology for \( n < 1 \), the exponential model resembles the latter for any choice of the model parameter.

In summary, the combined dynamical analysis at the background and perturbation levels do verify the results of observational confrontation, showing through an independent way that \( f(Q) \) gravity, and specifically, the exponential model can be considered as a very promising alternative to ΛCDM concordance model.

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