Linear Rank-Width of Distance-Hereditary Graphs

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Abstract We present a characterization of the linear rank-width of distance-hereditary graphs. Using the characterization, we show that the linear rank-width of every \( n \)-vertex distance-hereditary graph can be computed in time \( O(n^2 \cdot \log(n)) \), and a linear layout witnessing the linear rank-width can be computed with the same time complexity. For our characterization, we combine modifications of canonical split decompositions with an idea of [Megiddo, Hakimi, Garey, Johnson, Papadimitriou: The complexity of searching a graph. JACM 1988], used for computing the path-width of trees. We also provide a set of distance-hereditary graphs which contains the set of distance-hereditary vertex-minor obstructions for linear rank-width. The set given in [Jeong, Kwon, Oum: Excluded vertex-minors for graphs of linear rank-width at most k. STACS 2013: 221-232] is a subset of our obstruction set.

1 Introduction

Rank-width \cite{oum2006} is a graph parameter introduced by Oum and Seymour with the goal of efficient approximation of the clique-width \cite{bodlaender1998} of a graph. Linear rank-width can be seen as the linearized variant of rank-width, similar to path-width, which in turn can be seen as the linearized variant of tree-width. While path-width is a well-studied notion, much less is known about linear rank-width. Computing linear rank-width is NP-complete in general (this follows from \cite{garey1979}). Therefore it is natural to ask which graph classes allow for an efficient computation. Until now, the only (non-trivial) known such result is for forests \cite{adler2013}. A graph \( G \) is distance-hereditary, if for any two vertices \( u \) and \( v \) of \( G \), the distance between \( u \) and \( v \) in any connected, induced subgraph of \( G \) that contains both \( u \) and \( v \), is the same as the distance between \( u \) and \( v \) in \( G \). Distance-hereditary graphs are exactly the graphs of rank-width \( \leq 1 \) \cite{oum2006}. They include co-graphs (i.e. graphs of clique-width 2), complete (bipartite) graphs and forests.

We show that the linear rank-width of \( n \)-vertex distance-hereditary graphs can be computed in time \( O(n^2 \cdot \log(n)) \) (Theorem \textsuperscript{4}). Moreover, we show that a layout of the graph witnessing the linear rank-width can be computed with the same time complexity (Corollary \textsuperscript{3}). Given that computing the path-width of distance-hereditary graphs is NP-complete \cite{garey1979}, this is indeed surprising. We give a new characterization of linear rank-width of distance-hereditary graphs (Theorem \textsuperscript{4}).

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2 Preliminaries

For a set $A$, we denote the power set of $A$ by $2^A$. We let $A \setminus B := \{x \in A \mid x \notin B\}$ denote the difference of two sets $A$ and $B$. For a subset $X$ of a ground set $A$, let $\overline{X} := A \setminus X$.

In this paper, graphs are finite, simple and undirected, unless stated otherwise. Our graph terminology is standard, see for instance [8]. Let $G$ be a graph. We denote the vertex set of $G$ by $V(G)$ and the edge set by $E(G)$. An edge between $x$ and $y$ is written $xy$ (equivalently $yx$). If $X$ is a subset of the vertex set of $G$, we denote the subgraph of $G$ induced by $X$ by $G[X]$, and we let $G \setminus X := G[V(G) \setminus X]$. For a vertex $x \in V(G)$ we let $N_G(x) := \{y \in V(G) \mid x \neq y, xy \in E(G)\}$ denote the set of neighbors of $x$ (in $G$). The degree of $x$ (in $G$) is $\deg_G(x) := |N_G(x)|$. A partition of $V(G)$ into two sets $X$ and $Y$ is called a cut in $G$. We denote it by $(X,Y)$.

A tree is a connected, acyclic graph. A leaf of a tree is a vertex of degree one. A path is a tree where every vertex has degree at most two. The length of a path is the number of its edges. A rooted tree is a tree with a distinguished vertex $r$, called the root. A complete graph is the graph with all possible edges. A graph $G$ is called distance-hereditary (or DH for short) if for every two vertices $x$ and $y$ of $G$ the distance of $x$ and $y$ in $G$ equals the distance of $x$ and $y$ in any connected induced subgraph containing both $x$ and $y$ [3]. A star is a tree with a distinguished vertex, called its center, adjacent to all other vertices.

2.1 Linear Rank-Width and Vertex-Minors

Linear rank-width. For sets $R$ and $C$ an $(R,C)$-matrix is a matrix where the rows are indexed by elements in $R$ and columns indexed by elements in $C$. (Since we are only interested in the rank of matrices, it suffices to consider matrices up to permutations of rows and columns.) For an $(R,C)$-matrix $M$, if $X \subseteq R$ and $Y \subseteq C$,
we let $M[X,Y]$ be the submatrix of $M$ where the rows and the columns are indexed by $X$ and $Y$ respectively.

Let $A_G$ be the adjacency $(V(G),V(G))$-matrix of $G$ over the binary field. For a graph $G$, let $x_1,\ldots,x_n$ be a linear layout of $V(G)$. Every index $i \in \{1,\ldots,n\}$ induces a cut $(X_i,\overline{X_i})$, where $X_i := \{x_1,\ldots,x_i\}$ (and hence $\overline{X_i} = \{x_{i+1},\ldots,x_n\}$). The cutrank of the ordering $x_1,\ldots,x_n$ is defined as

$$\text{cutrk}_G(x_1,\ldots,x_n) := \max\{\text{rank}(A_G[X_i,\overline{X_i}]) \mid i \in \{1,\ldots,n\}\}.$$  

The linear rank-width of $G$ is defined as

$$\text{lwr}(G) := \min\{\text{cutrk}_G(x_1,\ldots,x_n) \mid x_1,\ldots,x_n \text{ is a linear layout of } V(G)\}.$$  

Disjoint unions of caterpillars have linear rank-width $\leq 1$. Ganian [11] gives an alternative characterization of the graphs of linear rank-width $\leq 1$ as thread graphs. It is proved in [2] that linear rank-width and path-width coincide on trees. It is easy to see that the linear rank-width of a graph is the maximum over the linear rank-widths of its connected components.

**Vertex-minors.** For a graph $G$ and a vertex $x$ of $G$, the local complementation at $x$ of $G$ consists in replacing the subgraph induced on the neighbors of $x$ by its complement. The resulting graph is denoted by $G \ast x$. If $H$ can be obtained from $G$ by a sequence of local complementations, then $G$ and $H$ are called locally equivalent. A graph $H$ is called a vertex-minor of a graph $G$ if $H$ is a graph obtained from $G$ by applying a sequence of local complementations and deletions of vertices.

For an edge $xy$ of $G$, let $W_1 := N_G(x) \cap N_G(y)$, $W_2 = N_G(x) \setminus N_G(y) \setminus \{y\}$, and $W_3 = N_G(y) \setminus N_G(x) \setminus \{x\}$. Pivoting on $xy$ of $G$, denoted by $G \triangle xy$, is the operation which consists in complementing the adjacencies between distinct sets $W_i$ and $W_j$, and swapping the vertices $x$ and $y$. It is known that $G \triangle xy = G \ast x \ast y \ast x = G \ast y \ast x \ast y$ [15].

**Lemma 1** ([15]). Let $G$ be a graph and let $x$ be a vertex of $G$. Then for every subset $X$ of $V(G)$, we have $\text{cutrk}_G(X) = \text{cutrk}_{G \ast x}(X)$. Therefore, every vertex-minor $H$ of $G$ satisfies $\text{lwr}(H) \leq \text{lwr}(G)$.

**Lemma 2** ([15]). Let $G$ be a graph and $xy, yz \in E(G)$. Then $G \triangle xy \triangle xz = G \triangle yz$.

### 2.2 Split Decompositions and Local Completions

**Split decompositions.** We will follow the definitions in [4]. Let $G$ be a connected graph. A split in $G$ is a cut $(X,Y)$ in $G$ such that $|X|,|Y| \geq 2$ and $\text{rank}(A_G[X,Y]) = 1$. In other words $(X,Y)$ is a split in $G$ if $|X|,|Y| \geq 2$ and there exist non-empty sets $X' \subseteq X$ and $Y' \subseteq Y$ such that $\{xy \in E(G) \mid x \in X, y \in Y\} = \{xy \in E(G) \mid x \in X', y \in Y'\}$. Notice that not all connected graphs have a split, and those that do not have a split are called prime graphs.

A marked graph $D$ is a connected graph $D$ with a distinguished set of edges $M(D)$, called marked edges, that form a matching, and such that every edge in $M(D)$ is an isthmus, i.e., its deletion increases the number of components. The ends of the marked edges are called marked vertices, and the components of $D \setminus M(D)$ are called bags of $D$. If $(X,Y)$ is a split in $G$, we construct a marked graph $D$ with vertex set $V(G) \cup \{x',y' \notin V(G)\}$ and edge set $E(G[X]) \cup E(G[Y]) \cup \{x'y'\} \cup E'$ where we define $x'y'$ as marked and

$$E' := \{x'x \mid x \in X \text{ and there exists } y \in Y \text{ such that } xy \in E(G)\} \cup \{y'y \mid y \in Y \text{ and there exists } x \in X \text{ such that } xy \in E(G)\}.$$
The marked graph $D$ is called a **simple decomposition**. A **decomposition** of a connected graph $G$ is a marked graph $D$ defined inductively to be either $G$ or a marked graph defined from a decomposition $D'$ by replacing a component $H$ of $D' \setminus M(D')$ by a simple decomposition of $H$. We call the transformation of $D'$ into $D$ a **refinement of $D'$**. Notice that in a decomposition of a connected graph $G$, the two ends of a marked edge do not have a common neighbor. For a marked edge $xy$ in a decomposition $D$, the **recomposition of $D$ along $xy$** is the decomposition $D' := D \setminus xy \setminus \{x, y\}$. For a decomposition $D$, we let $\hat{D}$ denote the connected graph obtained from $D$ by recomposing all marked edges. Since marked edges of a decomposition $D$ are isthmuses and form a matching, if we contract all the unmarked edges in $D$, we obtain a tree called the **decomposition tree of $G$ associated with $D$** and denoted by $T_D$. Obviously, the vertices of $T_D$ are in bijection with the bags of $D$, and we will also call them bags.

A decomposition $D$ of $G$ is called a **canonical split decomposition** if each bag of $D$ is either prime, or a star or a complete graph, and $D$ is not the refinement of a decomposition with the same property. Shortly, we call it a **canonical decomposition**. The following is due to Cunningham and Edmonds [6], and Dahlhaus [7].

**Theorem 1 ([6,7])**. Every connected graph $G$ has a unique canonical decomposition, up to isomorphism, that can be computed in time $O(|V(G)| + |E(G)|)$.

For a given connected graph $G$, by Theorem 1 we can talk about only one canonical decomposition of $G$ because all canonical decompositions of $G$ are isomorphic.

Let $D$ be a decomposition of $G$ with bags that are either primes, or complete graphs or stars (it is not necessarily a canonical decomposition). The **type of a bag** of $D$ is either $P$, or $K$ or $S$ depending on whether it is a prime, or a complete graph or a star. The **type of a marked edge $uv$** is $AB$ where $A$ and $B$ are the types of the bags containing $u$ and $v$ respectively. If $A = S$ or $B = S$, we can replace $S$ by $S_p$ or $S_c$ depending on whether the end of the marked edge is a leaf or the center of the star.

**Theorem 2 ([6])**. Let $D$ be a decomposition of a graph with bags of types $P$ or $K$ or $S$. Then $D$ is a canonical decomposition if and only if it has no marked edge of type $KK$ or $S_pS_c$.

We will use the following characterization of distance-hereditary graphs.

**Theorem 3 ([4])**. A connected graph is a distance-hereditary graph if and only if each bag of its canonical decomposition is of type $K$ or $S$.

**Local complementations in decompositions.** We now relate the decompositions of a graph and the ones of its locally equivalent graphs. Let $D$ be a decomposition. A vertex $v$ of $D$ **represents** an unmarked vertex $x$ (or is a representative of $x$) if $v = x$ or there is a path from $v$ to $x$ in $D$ starting with a marked edge such that marked edges and unmarked edges appear alternatively in the path. Two unmarked vertices $x$ and $y$ are **linked** in $D$ if there is a path from $x$ to $y$ in $D$ such that unmarked edges and marked edges appear alternatively in the path.

**Lemma 3.** Let $D$ be a decomposition of a graph. Let $v'$ and $w'$ be two marked vertices in a same bag of $D$, and let $v$ and $w$ be two unmarked vertices of $D$ represented by $v'$ and $w'$, respectively. Then $v$ and $w$ are linked in $D$ if and only if $vw \in E(D')$ if and only if $v'w' \in E(D')$.

**Proof.** It is easy to show that $v'$ and $w'$ are adjacent in $\hat{D}$ if and only if there is an alternating path from $v'$ to $w'$ in $D$. Now the proof follows from this and the definition of representativity.
A local complementation at an unmarked vertex $v$ in a decomposition $D$, denoted by $D \ast v$, is the operation which consists in replacing each bag $B$ containing a representative $w$ of $v$ with $B \ast w$. Observe that $M(D) = M(D \ast v)$.

**Lemma 4** ([H]). Let $D$ be a decomposition of a graph and let $v$ be an unmarked vertex of $D$. Then $D \ast v$ is a decomposition of $D \ast v$.

Let $v$ and $w$ be linked unmarked vertices in a decomposition $D$, and let $B_v$ and $B_w$ be the bags containing $v$ and $w$, respectively. Note that if $B$ is a bag of type S in the path from $B_v$ to $B_w$ in $T_D$, then the center of $B$ is a representative of either $v$ or $w$. Pivoting on $vw$ of $D$, denoted by $D \wedge vw$, is the decomposition obtained as follows: for each bag $B$ on the path from $B_v$ to $B_w$ in $T_D$, if $v', w' \in V(B)$ represent $v$ and $w$ in $D$, respectively, then we replace $B$ with $B \wedge v'w'$.

**Lemma 5.** Let $D$ be a decomposition of a distance-hereditary graph, and let $xy \in E(D)$. Then $D \wedge xy = D \ast x \ast y \ast x$.

**Proof.** Since $xy \in E(D)$, by Lemma 3, $x$ and $y$ are linked in $D$. It is easy to see that by the operation $D \ast x \ast y \ast x$, only the bags in the path from $x$ to $y$ are modified, and they are modified according to the definition of $D \wedge xy$. \qed

As a corollary of Lemmas 4 and 5 we get the following.

**Corollary 1.** Let $D$ be a decomposition of a distance-hereditary graph and $xy \in E(D)$. Then $D \wedge xy$ is a decomposition of $D \wedge xy$.

## 3 Limbs in Canonical Decompositions

In this section we define the notion of limb that is the key ingredient in our characterization. Intuitively, a limb in the canonical decomposition of a distance-hereditary graph $G$ is a subtree of the decomposition with the property that the linear rank-width of the graph obtained from the subtree by recomposing all marked edges is invariant under taking local complementations.

### 3.1 Definitions and Basic Properties

Let $D$ be the canonical decomposition of a distance-hereditary graph. We recall from Theorem 2 that each bag of $D$ is of type $K$ or $S$, and marked edges of types $KK$ or $S_pS_c$ do not occur. Given a bag $B$ of $D$, an unmarked vertex $y$ of $D$ represented by some marked vertex $w \in V(B)$, let $T$ be the component of $D \setminus V(B)$ containing $y$ and let $v \in V(T)$ be the neighbor of $w$ in $T$. We define the limb $\mathcal{L} := \mathcal{L}[D, B, y]$ as follows:

1. if $B$ is of type $K$, then $\mathcal{L} := T \ast v \setminus v$,
2. if $B$ is of type $S$ and $w$ is a leaf, then $\mathcal{L} := T \setminus v$,
3. if $B$ is of type $S$ and $w$ is the center, then $\mathcal{L} := T \wedge vy \setminus v$.

Note that in $T$, $v$ becomes an unmarked vertex, so a limb is well-defined. While $T$ is a canonical decomposition, $\mathcal{L}$ may not be a canonical decomposition at all, because deleting $v$ may create a bag of size 2. We let $\hat{\mathcal{L}} = \hat{\mathcal{L}}[D, B, y]$ denote the canonical decomposition obtained from $\mathcal{L}[D, B, y]$ by recomposing necessary marked edges to make it a canonical decomposition, and we let $\hat{\mathcal{L}} = \hat{\mathcal{L}}[D, B, y]$ denote the graph obtained from $\mathcal{L}[D, B, y]$ by recomposing all marked edges.

See Figure 1 for an example. If the original canonical decomposition $D$ is clear from the context, we remove $D$ in the notation $\mathcal{L}[D, B, y]$.

By the following lemma, all limbs are connected. We will use this fact implicitly in almost all the proofs.
Lemma 6. Let $D$ be the canonical decomposition of some distance-hereditary graph and let $B$ be a bag of $D$. If an unmarked vertex $y$ is represented by a marked vertex $v \in V(B)$, then $\mathcal{L}[B, y]$ is connected.

Proof. Let $T$ be the component of $D \setminus V(B)$ containing $y$. Let $B'$ be the bag of $T$ adjacent to $B$. Since local complementations maintain connectivity, it suffices to verify that $V(B')$ induces a connected subgraph in $\mathcal{L}[B, y]$. Using Theorem 1, this is not hard to see for each of the three cases. \hfill \Box

Figure 1. A distance-hereditary graph of linear rank-width 2 and its corresponding canonical decomposition $D$. If $B$ is the central bag, for every limb of $D \setminus V(B)$, $\mathcal{L}$ is isomorphic to an edge and hence has linear rank-width 1.

Lemma 7. Let $D$ be the canonical decomposition of a distance-hereditary graph and let $B$ be a bag of $D$. If two unmarked vertices $x$ and $y$ are represented by a marked vertex $w \in V(B)$, then $\mathcal{L}[B, x]$ is locally equivalent to $\mathcal{L}[B, y]$.

Proof. Since $x$ and $y$ are represented by the same vertex in $D$, they are contained in the same component of $D \setminus V(B)$, say $T$. If $B$ is a complete bag or a star bag having $w$ as a leaf, then by the definition, $\mathcal{L}[B, x] = \mathcal{L}[B, y]$. So, we may assume that $w$ is the center of the star bag $B$. Let $v$ be the marked vertex adjacent to $w$ in $D$. Since $v$ is linked to both $x$ and $y$ in $T$, by Lemma 2, $T \setminus vx \setminus xy = T \setminus vy$. So, we obtain that $(T \setminus vx \setminus v) \setminus xy = T \setminus vx \setminus vy \setminus v = T \setminus vy \setminus v$. Therefore $\mathcal{L}[B, x]$ is locally equivalent to $\mathcal{L}[B, y]$. \hfill \Box

For a bag $B$ in the canonical decomposition $D$ of a distance-hereditary graph and a component $T$ of $D \setminus V(B)$, we define $f(D, B, T)$ as the linear rank-width of $\mathcal{L}[D, B, y]$ for some unmarked vertex $y \in V(T)$. In fact, by Lemma 4, $f(D, B, T)$ does not depend on the choice of $y$. As in the notation $\mathcal{L}[D, B, x]$, if the canonical decomposition $D$ is clear from the context, we remove $D$ in the notation $f(D, B, T)$.

Proposition 1. Let $D$ be the canonical decomposition of a distance-hereditary graph and let $B$ be a bag of $D$. Let $x \in V(D)$ and let $y$ be an unmarked vertex represented in $D$ by $v \in B$. If $y'$ is represented by $v$ in $D \ast x$, then $\mathcal{L}[D, B, y]$ is locally equivalent to $\mathcal{L}[D \ast x, B, y']$. Therefore, $f(D, B, T) = f(D \ast x, B, T_x)$ where $T_x$ is the component of $(D \ast x) \setminus V(B)$ containing $y$.

Before proving the propositions, let us recall the following by Geelen and Oum.

Lemma 8 ([13]). Let $G$ be a graph and let $x, y$ be two distinct vertices in $G$. Let $xw \in E(G \ast y)$ and $xz \in E(G)$.

1. If $xy \notin E(G)$, then $(G \ast y) \setminus x$, $(G \ast y \ast x) \setminus x$, and $(G \ast y) \setminus xw \setminus x$ are locally equivalent to $G \setminus x$, $G \ast x \setminus x$, and $G \setminus xz \setminus x$, respectively.
2. If $xy \in E(G)$, then $(G * y) \setminus x$, $(G * y * x) \setminus x$, and $(G * y) \wedge xw \setminus x$ are locally equivalent to $G \setminus x$, $G \wedge xz \setminus x$, and $(G * x) \setminus x$, respectively.

Proof (of Proposition 7). We need only to prove the first statement because a local complementation preserves the linear rank-width of a graph. Let $T$ be a component of $D \setminus V(B)$ containing $y$ and let $T_x$ be the component of $(D * x) \setminus V(B)$ containing $y'$. Note that $V(T) = V(T_x)$. By Lemma 8, it is sufficient to prove that $\widehat{L}[D, B, y]$ is locally equivalent to $\widehat{L}[D * x, B, y']$ for some $y' \in V(T_x)$ represented by $v$ in $D * x$.

Let $u$ be the vertex of $T$ adjacent to $v$ in $D$. Note that $u \neq x$ and we can obtain $\widehat{L}[D, B, y]$ from $\widehat{T}$ by the three types in the definition of limbs.

First, suppose that $x \in V(T)$ and $x$ is not linked to $u$ in $T$. So $ux \notin E(\widehat{T})$ and $u$ is still linked to $y$ in $T * x$. In this case, we let $y' = y$. We observe the following cases.

**B is of type S and $v$ is a leaf of $B$.** Since $ux \notin E(\widehat{T})$, by Lemma 8, $\widehat{T} \setminus u$ is locally equivalent to $\widehat{T} * x \setminus u$.

**B is of type S and $v$ is the center of $B$.** Since $x$ is not linked to $u$ in $T$, after applying local complementation at $x$ in $T$, $y$ is still linked to $u$. Since $ux \notin E(\widehat{T})$, by Lemma 8, $\widehat{T} \setminus uy \setminus u$ is locally equivalent to $\widehat{T} * x \wedge uy \setminus u$.

**B is of type K.** Since $ux \notin E(\widehat{T})$, by Lemma 8, $\widehat{T} * u \setminus u$ is locally equivalent to $\widehat{T} * x * u \setminus u$.

Second, suppose that $x \in V(T)$ and $x$ is linked to $u$ in $T$. Because $x$ is still linked to $u$ in $T * x$, we let $y' = x$ in this case. Note that $uv \in E(\widehat{T})$.

**B is of type S and $v$ is a leaf of $B$.** Applying local complementation at $x$ does not change the type of the bag $B$. Since $ux \in E(\widehat{T})$, by Lemma 8, $\widehat{T} \setminus u$ is locally equivalent to $\widehat{T} * x \setminus u$.

**B is of type S and $v$ is the center of $B$.** Applying local complementation at $x$ changes the bag $B$ into a bag of type K, and the component $T$ into $T * x$. Since $ux \in E(\widehat{T})$, by Lemma 8, $\widehat{T} \setminus uy \setminus u$ is locally equivalent to $\widehat{T} * x \wedge uy \setminus u$.

**B is of type K.** Applying local complementation at $x$ changes the bag $B$ into a bag of type S such that the center of $B$ is adjacent to $u$. Since $ux \in E(\widehat{T})$, by Lemma 8, $\widehat{T} * u \setminus u$ is locally equivalent to $\widehat{T} * x \wedge ux \setminus u$.

Now suppose that $x \notin V(T)$. If $x$ has no representative in the bag $B$, then applying local complementation at $x$ does not change the bag $B$ and the component $T$. Therefore, we may assume that $x$ is represented by some vertex in $B$, necessarily adjacent to $v$. In this case, $u$ is still a representative of $y$ in $D * x$, so we let $y' = y$.

**B is of type S, $v$ is a leaf of $B$, and $x$ is represented by the center of $B$.**

Originally, $\mathcal{L}[D, B, y] = T \setminus u$ because $v$ is a leaf of $B$. Applying local complementation at $x$ changes $B$ into a complete bag, and the component $T$ into $T * u$. So the limb of $D * x$ corresponding $T$ is $(T * u) * u \setminus u = T \setminus u = \mathcal{L}[D, B, y]$.

**B is of type S and $v$ is the center of $B$.** In this case, $x$ is represented by a leaf of the bag $B$ and $\mathcal{L}[D, B, y] = T \wedge uy \setminus u$. Note that applying local complementation does not change the bag $B$, but it changes $T$ into $T * u$. So the limb of $D * x$ corresponding to $T$ is $(T * u) \wedge uy \setminus u$. Since $((T * u) \wedge uy \setminus u) * y = T * y * v * y \setminus u = T \wedge uy \setminus u$, they are locally equivalent.

**B is of type K.** The limb $\mathcal{L}[D, B, y] = T * u \setminus u$. After applying local complementation at $x$ in $D$, $B$ becomes a star such that a leaf of it is adjacent to $u$, and $T$ becomes $T * u$. Therefore, the limb of $D * x$ corresponding to $T$ is also $(T * u) \setminus u$. }
Proposition 2. Let $D$ be the canonical decomposition of a distance-hereditary graph and let $B_1$ and $B_2$ be two bags of $D$. Let $T_1$ be a component of $D \setminus V(B_1)$ such that $T_1$ does not contain the bag $B_2$, and let $T_2$ be the component of $D \setminus V(B_2)$ such that $T_2$ contains the bag $B_1$. Then $f(B_1, T_1) \leq f(B_2, T_2)$.

Proof. To prove Proposition 2 it is enough to prove the following lemma (Lemma 9) by Lemma 1.

Lemma 9. Let $D$ be the canonical decomposition of a distance-hereditary graph and let $B_1$ and $B_2$ be two bags of $D$. Let $T_1$ be a component of $D \setminus V(B_1)$ such that $T_1$ does not contain the bag $B_2$, and $T_2$ be the component of $D \setminus V(B_2)$ such that $T_2$ contains the bag $B_1$. Let $y_1$ and $y_2$ be two unmarked vertices in $T_1$ and $T_2$ which are represented by some vertices in the bags $B_1$ and $B_2$, respectively. Then $\hat{\mathcal{L}}[B_1, y_1]$ is a vertex-minor of $\hat{\mathcal{L}}[B_2, y_2]$.

Proof. We use induction on the length of the path from $B_1$ to $B_2$ in $T_D$. We first assume that $B_1$ and $B_2$ are adjacent in $T_D$. Let $u_1$ and $u_2$ be the vertices of $B_1$ adjacent to $T_1$ and $T_2$, respectively, and let $u_3$ be a vertex of $B_1$ which is adjacent to neither $B_2$ nor $T_1$.

We choose a canonical decomposition $D'$ which is locally equivalent to $D$ such that the bag $B_3$ is the star having $u_3$ as the center. Let $T'_1 = D'[V(T_1)]$ and let $y'_1$ and $y'_2$ be two unmarked vertices in $T'_1$ and $T'_2$ which are represented by some vertices in the bags $B_1$ and $B_2$, respectively. Then, by Proposition 2, each $\hat{\mathcal{L}}[D, B_1, y_1]$ is locally equivalent to $\hat{\mathcal{L}}[D', B_1, y'_1]$, and therefore, it is sufficient to show that $\hat{\mathcal{L}}[D', B_1, y'_1]$ is a vertex-minor of $\hat{\mathcal{L}}[D', B_2, y'_2]$.

Since $B_3$ is the star having $u_3$ as the center, $\hat{\mathcal{L}}[D', B_2, y'_2]$ is a limb of type 1 or 2 in the definition. Since $u_1$ is not adjacent to $u_2$, $T_1$ is an induced subgraph of $\hat{\mathcal{L}}[D', B_2, y'_2]$, and therefore, $\hat{\mathcal{L}}[D', B_1, y'_1]$ is a vertex-minor of $\hat{\mathcal{L}}[D', B_2, y'_2]$, as required.

Now suppose that $B_1$ is not adjacent to $B_2$ in $T_2$. We choose a bag $B_3$, which is adjacent to $B_2$, on the path from $B_1$ to $B_2$ in $T_D$. Let $T_3$ be the component of $D \setminus V(B_3)$ such that $T_3$ contains the bag $B_1$ and let $y_3$ be an unmarked vertex in $T_3$ which is represented by some vertex in the bag $B_3$. By induction hypothesis, $\hat{\mathcal{L}}[B_1, y_1]$ is a vertex-minor of $\hat{\mathcal{L}}[B_3, y_3]$ and $\hat{\mathcal{L}}[B_3, y_3]$ is a vertex-minor of $\hat{\mathcal{L}}[B_2, y_2]$. Therefore, $\hat{\mathcal{L}}[B_1, y_1]$ is a vertex-minor of $\hat{\mathcal{L}}[B_2, y_2]$.

4 Characterizing the Linear Rank-Width of DH Graphs

In this section, we prove the main theorem of the paper, which characterizes distance-hereditary graphs of linear rank-width $k$ (cf. Figure 1).

Theorem 4 (Main Theorem). Let $k$ be a positive integer and let $D$ be the canonical decomposition of a distance-hereditary graph. Then $\text{lrw}(D) \leq k$ if and only if for each bag $B$ of $D$, $D$ has at most two components $T$ of $D \setminus V(B)$ such that $f(B, T) = k$, and for all the other components $T'$ of $D \setminus V(B)$, $f(B, T') \leq k - 1$.

Proposition 3. Let $k$ be a positive integer. Let $D$ be the canonical decomposition of a distance-hereditary graph and let $B$ be a bag of $D$. If $D \setminus V(B)$ has at least three components $T$ such that $f(B, T) = k$, then $\text{lrw}(D) \geq k + 1$.

Proof. We may assume that $D \setminus V(B)$ has exactly three components $T_1$, $T_2$ and $T_3$, where each component $T_i$ satisfies $f(B, T_i) = k$. For each $1 \leq i \leq 3$,

- let $w_i \in V(T_i)$ be the marked vertex of $D$ adjacent to $B$,
- let $N_i$ be the set of the unmarked vertices in $T_i$ linked to $w_i$,
- let $u_i \in N_i$ and $T_i' = L[D, B, u_i]$.

We will show that $\text{lrw}(\hat{D}) \geq k + 1$. Since removing a vertex from a graph does not increase the linear rank-width, we may assume that $B$ consists of exactly three marked vertices which are adjacent to one of $T_1$, $T_2$ and $T_3$. Now, every unmarked vertex of $D$ is contained in one of $T_1$, $T_2$ and $T_3$.

Note that for each $T_i$, $f(B, T_i)$ does not change when applying local complementations by Proposition 1. Moreover, by Lemmas 1 and 2 for any canonical decomposition $D'$ obtained from $D$ by applying local complementations, we have $\text{lrw}(\hat{D}) = \text{lrw}(\hat{D'})$. So we may assume that $B$ is a complete bag.

We first claim that $L[D, B, u_2] = (D \ast u_1)[V(T_2) \setminus w_2]$. Since the bag $B$ is complete, by definition, $L[D, B, u_2] = T_2 \ast w_2 \setminus w_2$. Since $u_1$ is linked to $w_1$ in $T_1$ and there is an alternating path from $w_1$ to $w_2$ in $D$, by concatenating alternating paths it is easy to see that

$$(D \ast u_1)[V(T_2) \setminus w_2] = T_2 \ast w_2 \setminus w_2 = L[D, B, u_2],$$

as claimed. See Figure 2.

Towards a contradiction, suppose that $\hat{D}$ has a linear layout $L$ of width $k$. Let $a$ and $b$ be the first vertex and the last vertex of $L$, respectively. Since $B$ has no unmarked vertices, without loss of generality, we may assume that $a, b \in V(\hat{T}_1) \cup V(\hat{T}_3)$. With this assumption, we will prove that $\hat{T}_2$ has linear rank-width at most $k - 1$.

Let $v \in V(\hat{T}_2)$ and $S_v := \{x \in V(\hat{D}) \mid x \leq_L v\}$ and $T_v := V(\hat{D}) \setminus S_v$. Since $v$ is arbitrary, it is sufficient to show that

$$\text{cutrk}_{\hat{T}_2}(S_v \cap V(\hat{T}_2)) \leq k - 1.$$  

We have four cases. We recall that $N_i$ is the set of vertices in $V(\hat{T}_i)$ which has a neighbor in $V(\hat{D}) \setminus V(\hat{T}_i)$.

1. There exist $x_1, x_2 \in N_1 \cup N_2$ such that $x_1 \in S_v$ and $x_2 \in T_v$.
2. $N_1 \cup N_2 \subseteq S_v$.
3. $N_1 \cup N_3 \subseteq T_v$.

For Case 1, we assume that there exist $x_1, x_2 \in N_1 \cup N_3$ such that $x_1 \in S_v$ and $x_2 \in T_v$. We claim that

$$\text{cutrk}_{\hat{T}_2}(S_v \cap V(\hat{T}_2)) = \text{cutrk}_{\hat{D}[V(\hat{T}_2) \cup \{x_1, x_2\}]}((S_v \cap V(\hat{T}_2)) \cup \{x_1\}) - 1.$$  

Because

$$\text{cutrk}_{\hat{D}[V(\hat{T}_2) \cup \{x_1, x_2\}]}((S_v \cap V(\hat{T}_2)) \cup \{x_1\}) \leq \text{cutrk}_{\hat{D}}(S_v) \leq k,$$

the claim will imply that $\text{cutrk}_{\hat{T}_2}(S_v \cap V(\hat{T}_2)) \leq k - 1$. Note that $x_1$ and $x_2$ have the same neighbors on $V(\hat{T}_2)$ in $D$, they also have the same neighbors in $\hat{D}[V(\hat{T}_2) \cup \{x_1, x_2\}]$. Since $x_1$ is adjacent to $x_2$ in $\hat{D}[V(\hat{T}_2) \cup \{x_1, x_2\}]$, $x_2$ become a leaf in $\hat{D}[V(\hat{T}_2) \cup \{x_1, x_2\}] \ast x_1$ having exactly one neighbor, $x_1$. We already observed that

$$(D \ast x_1)[V(T_2) \setminus w_2] = L[D, B, u_2],$$

and therefore, we have

$$\hat{D}[V(\hat{T}_2) \cup \{x_1, x_2\}] \ast x_1 \setminus x_1 \setminus x_2 = (\hat{D} \ast x_1)[V(\hat{T}_2)] = \hat{T}_2.$$
Therefore,
\[
\text{cutrk}_{\mathcal{D}}(V(T_2)) + 1 = \text{cutrk}_{\mathcal{D}}(V(T_2)) + 1,
\]
as claimed.

Now let us assume that \(N_1 \cup N_3 \subseteq S_v\) or \(N_1 \cup N_3 \subseteq T_v\). In both cases, the
sub-matrix of \(A_G[S_v, T_v]\) induced by the vertices in \(V(T_1') \cup V(T_3')\) will contribute
at least 1 to \(\text{cutrk}_{\mathcal{D}}(S_v)\). Indeed, this follows easily using the facts that \(a, b \in V(T_1') \cup V(T_3')\), that the graph \(\mathcal{D}[V(T_1') \cup V(T_3')]\) is connected, and that \(|V(T_1')| \geq 2\)
and \(|V(T_3')| \geq 2\). Therefore, we will have \(\text{cutrk}_{\mathcal{D}}(S_v) \leq k - 1\). Thus, \(T_2\)
has linear rank-width at most \(k - 1\), which is a contradiction. \(\square\)

![Figure 2](image_url)

**Figure 2.** Realize a limb without removing the bag in Proposition 3. Since \(B\) is a complete
bag, the limb \(\mathcal{L}[D, B, w_2] = (D * u_1)[V(T_2) \setminus w_2]\).

To prove the converse direction, we use the following technical lemmas.

**Lemma 10.** Let \(k\) be a positive integer. Let \(D\) be the canonical decomposition of a
distance-hereditary graph and let \(B\) be a bag of \(D\) of type \(S\) such that the center \(x\)
of \(B\) and a leaf \(y\) of \(B\) are unmarked vertices of \(D\).

If for every component \(T\) of \(D \setminus V(B)\), \(f(B, T) \leq k - 1\), then the graph \(\mathcal{D}\)
has a linear layout of width at most \(k\) such that the first vertex and the last vertex of it
are \(x\) and \(y\), respectively.

**Proof.** Let \(T_1, T_2, \ldots, T_{\ell}\) be the components of \(D \setminus V(B)\) and for each \(1 \leq i \leq \ell\), let \(w_i\) be the marked vertex of \(T_i\) adjacent to a vertex of \(B\). Since each \(w_i\) is adjacent
to a leaf of \(B\), \(T_i \setminus w_i\) is the limb of \(D\) with respect to \(B\) and \(T_i\).
Suppose that for every component $T$ of $D \setminus V(B)$, $f(B, T) \leq k - 1$. We may assume without loss of generality that $B$ has only two unmarked vertices $x$ and $y$. For each $1 \leq i \leq \ell$, let $L_i$ be a linear layout of $T_i \setminus w_i$ of width $k - 1$. We claim that $L := (x) \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_{\ell} \oplus (y)$ is a linear layout of $\widehat{D}$ of width $k$. It is sufficient to prove that for all $w \in V(\widehat{D}) \setminus \{x, y\}$, $\text{cutrk}_{\widehat{D}}(\{v \mid v \leq_L w\}) \leq k$.

Let $w \in V(\widehat{D}) \setminus \{x, y\}$, and let $S_w := \{v : v \leq_L w\}$ and $T_w := V(\widehat{D}) \setminus S_w$. We assume that $w \in L_j$ for some $1 \leq j \leq \ell$. Note that

$$\text{cutrk}_{\widehat{D}}(S_w) = \text{rank} \left( \begin{array}{c|c|c|c} x & y T_w \cap V(\widehat{T}_j) & T_w \setminus \{y\} \setminus V(\widehat{T}_j) \\ \hline S_w \cap V(\widehat{T}_j) & 1 & * & 0 \\ S_w \setminus \{x\} \setminus V(\widehat{T}_j) & 0 & 0 & * \\ \end{array} \right)$$

$$= \text{rank} \left( \begin{array}{c|c|c} x & y T_w \cap V(\widehat{T}_j) & T_w \setminus \{y\} \setminus V(\widehat{T}_j) \\ \hline S_w \cap V(\widehat{T}_j) & 1 & 0 & 0 \\ S_w \setminus \{x\} \setminus V(\widehat{T}_j) & 0 & * & 0 \\ \end{array} \right)$$

$$= \text{cutrk}_{T_j \setminus w_j}(S_w \cap V(\widehat{T}_j)) + 1 \leq (k - 1) + 1 = k.$$ 

Therefore, $L$ is a linear layout of $\widehat{D}$ of width $k$ such that the first vertex of it is $x$ and the last vertex is $y$.

**Lemma 11.** Let $k$ be a positive integer. Let $D$ be the canonical decomposition of a distance-hereditary graph such that for each bag $B$ of $D$, there are at most two components $T$ of $D \setminus V(B)$ satisfying $f(B, T) = k$ and for all the other components $T'$ of $D \setminus V(B)$, $f(B, T') \leq k - 1$. If $P$ is the set of bags $B$ in $D$ such that exactly two components $T$ of $D \setminus V(B)$ satisfy $f(B, T) = k$, then either $P = \emptyset$ or $T_D[P]$ is a path.

**Proof.** Suppose that $P \neq \emptyset$. If $B_1$ and $B_2$ are in $P$, then there exists a component $T_1$ of $D \setminus V(B_1)$ not containing $V(B_2)$ such that $f(B_1, T_1) = k$, and there exists a component $T_2$ of $D \setminus V(B_2)$ not containing $V(B_1)$ such that $f(B_2, T_2) = k$. So by Proposition 2, for all $B$ on the path from $B_1$ to $B_2$ in $T_D$, $B$ must be contained in $P$. So $P$ forms a connected subtree in $T_D$. Suppose now that $P$ has a vertex $B$ of degree at least three, and let $B_1, B_2$, and $B_3$ be the neighbors of $B$ in $P$. Then, again by Proposition 2, $D$ must have three components $T$ of $D \setminus V(B)$ such that $f(B, T) = k$, which contradicts the assumption. Therefore, $P$ forms a path in $T_D$. \hfill \Box

**Proposition 4.** Let $k$ be a positive integer. Let $D$ be the canonical decomposition of a distance-hereditary graph and let $B$ be a bag of $D$ with two unmarked vertices $x, y$. If for every component $T$ of $D \setminus V(B)$, $f(B, T) \leq k - 1$, then the graph $\widehat{D}$ has a linear layout of width at most $k$ such that the first vertex and the last vertex of it are $x$ and $y$, respectively.

**Proof.** Choose a canonical decomposition $D'$ which is locally equivalent to $D$ such that the bag $B$ is a star with the center $x$. By Proposition 11, for each component $T$ of $D \setminus V(B)$, $f(D, B, T) = f(D', B, T')$ where $T'$ is the component of $D'[\setminus V(D') \setminus V(B)]$ corresponding to $T$. Since $D'$ is locally equivalent to $D$, by Lemma 10, we can easily check the statement. \hfill \Box

**Lemma 12.** Let $k$ be a positive integer. Let $D$ be the canonical decomposition of a distance-hereditary graph such that for each bag $B$ of $D$, there are at most two
components $T$ of $D \setminus V(B)$ satisfying $f(B, T) = k$ and for all the other components $T'$ of $D \setminus V(B)$, $f(B, T') \leq k - 1$. Then $T_D$ has a path $P$ such that for each bag $B$ in $P$ and a component $T$ of $D \setminus V(B)$ not containing a bag of $P$, $f(B, T) \leq k - 1$.

Proof. Let $P'$ be the set of bags $B$ in $D$ such that exactly two components $T$ of $D \setminus V(B)$ satisfy $f(B, T) = k$. By Lemma 11, either $P' = \emptyset$ or $T_D[P']$ is a path.

We first assume that $P' \neq \emptyset$. Let $T_D[P'] = B_1 - B_2 - \cdots - B_n$. By the definition, there exists a component $T_1$ of $D \setminus V(B_1)$ such that $T_1$ does not contain a bag of $P'$ and $f(B_1, T_1) = k$. Let $B_0$ be the bag of $T_1$ which is adjacent to $B_1$ in $D$. Similarly, there exists a component $T_n$ of $D \setminus V(B_n)$ such that $T_n$ does not contain a bag of $P'$ and $f(B_n, T_n) = k$. Let $B_{n+1}$ be the bag of $T_n$ which is adjacent to $B_n$ in $D$. Then $P := B_0 - B_1 - B_2 - \cdots - B_n - B_{n+1}$ is the required path.

Now we assume that $P' = \emptyset$. We choose a bag $B_0$ in $D$. If $D$ has no component $T$ of $D \setminus V(B_0)$ such that $f(B_0, T) = k$, then $P := B_0$ satisfies the condition. If not, we take a maximal path $P := B_0 - B_1 - \cdots - B_{n+1}$ in $T_D$ such that

1. for each $0 \leq i \leq n$, $D \setminus V(B_i)$ has one component $T_i$ such that $f(B_i, T_i) = k$, and $B_{i+1}$ is the bag of $T_i$ adjacent to $B_i$ in $D$,
2. every component $T$ of $D \setminus V(B_{n+1})$ not containing a bag of $P$ is such that $f(B_{n+1}, T) \leq k - 1$.

By the maximality, $P$ is a path in $T_D$ such that for each bag $B$ in $P$ and a component $T$ of $D \setminus V(B)$ not containing a bag of $P$, $f(B, T) \leq k - 1$.

We are now ready to prove the converse direction of the proof of Theorem 4.

**Proposition 5.** Let $k$ be a positive integer. Let $D$ be the canonical decomposition of a distance-hereditary graph such that for each bag $B$ of $D$, $D$ has at most two components $T$ of $D \setminus V(B)$ satisfying $f(B, T) = k$ and for all the other components $T'$ of $D \setminus V(B)$, $f(B, T') \leq k - 1$. Then $\text{lrw}(\hat{D}) \leq k$.

Proof. Let $P := B_0 - B_1 - \cdots - B_n - B_{n+1}$ be the path in $T_D$ such that for each bag $B$ in $P$ and a component $T$ of $D \setminus V(B)$ not containing a bag of $P$, $f(B, T) \leq k - 1$ (such a path exists by Lemma 12). If $B_0$ does not have an unmarked vertex, then we add one unmarked vertex to $B_0$ and we call it $a_0$. Similarly for $B_{n+1}$, but the added unmarked vertex is called $b_{n+1}$.

Now for each $0 \leq i \leq n$, let $b_i$ be the marked vertex of $B_i$ adjacent to $B_{i+1}$ and let $a_{i+1}$ be the marked vertex of $B_{i+1}$ adjacent to $b_i$. And for each $0 \leq i \leq n + 1$, let $D_i$ be the subdecomposition of $D$ induced on the bag $B_i$ and the components of $D \setminus V(B_i)$ which do not contain a vertex of $P$. Notice that the vertices $a_i$ and $b_i$ are unmarked vertices in $D_i$. Since every component $T$ of $D_i \setminus V(B_i)$ is such that $f(D_i, B_i, T) \leq k - 1$, by Proposition 4 $D_i$ has a linear layout $L_i'$ of width $k$ such that the first vertex of it is $a_i$ and the last vertex of it is $b_i$. For each $1 \leq i \leq n$, let $L_i$ be the linear layout obtained from $L_i'$ by removing $a_i$ and $b_i$. Let $L_1$ and $L_{n+1}$ be obtained from $L_1'$ and $L_{n+1}'$ by removing $b_0$ and $a_{n+1}$, respectively, and also the vertices $a_0$ and $b_{n+1}$, respectively, if they were added. Then we can easily check that $L := L_0 \oplus L_1 \oplus \cdots \oplus L_{n+1}$ is a linear layout of $\hat{D}$ having width at most $k$. Therefore $\text{lrw}(\hat{D}) \leq k$.

**5 Computing the Linear Rank-Width of DH Graphs**

In this section, we describe an algorithm to compute the linear rank-width of distance-hereditary graphs. Since the linear rank-width of a graph is the maximum linear rank-width over all its connected components, we will focus on connected distance-hereditary graphs.
Whenever a merging operation on two bags $B$ have to merge two bags to be able to turn a limb into a canonical decomposition, if they are incomparable, then we regard it as a new one.

Let $G$ be the rooted canonical decomposition of a distance-hereditary graph with the root $R$. Let $B$ be a non-root bag of $D$ and let $B'$ be the parent of $B$. We introduce the following notations.

1. Let $T_1(D, B)$ be the component of $D \setminus V(B')$ which contains the bag $B$, and $F_1(D, B) := f(D, B', T_1(D, B))$.
2. Let $U_1(D, B)$ be the set of unmarked vertices in $T_1(D, B)$ represented by a vertex of $B'$ in $D$, and for $v \in U_1(D, B)$, let $SL_1(D, B, v) := \tilde{C}[D, B', v]$.
3. Let $T_2(D, B)$ be the component of $D \setminus V(B)$ which contains the bag $B'$, and $F_2(D, B) := f(D, B, T_2(D, B))$.
4. Let $U_2(D, B)$ be the set of unmarked vertices in $T_2(D, B)$ represented by a vertex of $B$ in $D$, and for $v \in U_2(D, B)$, let $SL_2(D, B, v) := \tilde{C}[D, B, v]$.

Below we will compute $SL_1(D, B, v)$ or $SL_2(D, B, v)$, and we will sometimes have to merge two bags to be able to turn a limb into a canonical decomposition. Whenever a merging operation on two bags $B_1$ and $B_2$ appears, if $B_2$ is a descendant of $B_1$ (or $B_1$ is a descendant of $B_2$), then we regard the merged bag as $B_1$ (or $B_2$), and if they are incomparable, then we regard it as a new one.

We define the root $R'$ of $SL_1(D, B, v)$ or $SL_2(D, B, v)$ as follows. If the root $R$ of $D$ exists in $SL_1(D, B, v)$ or $SL_2(D, B, v)$, then let $R' := R$. Assume the root $R$ does not exist in $SL_1(D, B, v)$ or $SL_2(D, B, v)$. If $R$ was a bag, then $R'$ is removed and either two children of $R$ are merged or they are linked by a marked edge. If $R$ was a marked edge, then one of the bags incident with $R$ is removed and either two children of it are merged or linked by a marked edge. In both cases, if two children of the merged bag are merged, then let $R'$ be the merged bag, and if otherwise, let $R'$ be the marked edge between them. From the definition, we have the following.

**Remark 1.** Let $D$ be the rooted canonical decomposition of a distance-hereditary graph and let $B$ be a non-root bag of $D$. If $B'$ is a non-root bag of $SL_i(D, B, v)$, then $B'$ is a non-root bag of $D$ (for $i = 1, 2$).

Our algorithm uses methods from the algorithm for the vertex separation of trees in [9]. Let $D$ be the rooted canonical decomposition of a distance-hereditary graph $G$. Our algorithm works bottom-up on $D$, and computes $F_i(D, B)$ for all bags $B$ in $D$ using a dynamic programming. Let $B$ be a bag of $D$, and let $B_1, B_2, \ldots, B_m$ be the children of $B$ in $D$. Let $k := \max_{1 \leq i \leq m} F_1(D, B_i)$. We can easily observe that $k \leq F_1(D, B) \leq k + 1$. We discuss now how to determine $F_1(D, B)$. A bag $B$ of $D$ is called $k$-critical if $F_1(D, B) = k$ and $B$ has two children $B_1$ and $B_2$ such that $F_1(D, B_1) = F_1(D, B_2) = k$. We first observe the following which can be derived from Theorem 4 and Proposition 2.

**Proposition 6.** Let $D$ be the rooted canonical decomposition of a distance-hereditary graph $G$ such that $k = \max \{ F_1(D, B) \mid B \text{ is a non-root bag of } D \}$. Assume that $D$
has neither a bag $B$ having at least three children $B'$ such that $\mathcal{F}_1(D, B') = k$ nor it has two incomparable bags $B_1$ and $B_2$ with $B_1$ a $k$-critical bag and $\mathcal{F}_1(D, B_2) = k$. Let $B$ be a $k$-critical bag of $D$. Then $B$ is the unique $k$-critical bag of $D$. Moreover, $\text{lrw}(G) = k + 1$ if and only if $\mathcal{F}_2(D, B) = k$.

Proof. We first show that $B$ is the unique $k$-critical bag of $D$. Let $B'$ be a $k$-critical bag of $D$ which is distinct from $B$. If two bags $B$ and $B'$ are comparable in $D$, then without loss of generality, we may assume that $B$ is a descendant of $B'$ in $D$. Then by the definition of $k$-criticality, $B'$ has a child $B_1'$ such that $\mathcal{F}_1(D, B_1') = k$ and $B$ is not a descendant of $B_1'$ in $D$. Thus, $B$ is incomparable with $B_1'$ in $D$ and $\mathcal{F}_1(D, B_1') = k$, which contradicts the assumption.

Let $B$ be the unique $k$-critical bag of $D$. Now we claim that $\text{lrw}(G) = k + 1$ if and only if $\mathcal{F}_2(D, B) = k$. Note that by the assumption on $k$, $\text{lrw}(G) \leq k + 1$. So, the converse direction is easy. For the forward direction, suppose that $\text{lwr}(G) = k + 1$. Since $D$ has no bag $B$ having at least three children $B_1, B_2$ and $B_3$ such that $\mathcal{F}_1(D, B_1) = \mathcal{F}_1(D, B_2) = \mathcal{F}_1(D, B_3) = k$, by Theorem 4 there should exist a $k$-critical bag $B'$ of $D$ such that $\mathcal{F}_2(D, B') \geq k$. If $\mathcal{F}_2(D, B) = k + 1$, then there must be a bag $B'$ in $D$, which is incomparable to $B$, such that $\mathcal{F}_1(D, B') = k$. Thus, $\mathcal{F}_2(D, B) = k$. Since $B$ is the unique $k$-critical bag of $D$, $\mathcal{F}_2(D, B) = \mathcal{F}_2(D, B') = k$, as required.

By Proposition 6, the computation of $\mathcal{F}_1(D, B)$ is reduced to the computation of $\mathcal{F}_2(\mathcal{SL}_1(D, B, v), B_c)$ when $\mathcal{SL}_1(D, B, v)$ has the unique $k$-critical bag $B_c$. In order to compute $\mathcal{F}_2(\mathcal{SL}_1(D, B, v), B_c)$, we can call recursively the algorithm computing the linear rank-width in $\mathcal{SL}_2(\mathcal{SL}_1(D, B, v), B_c)$. However, we will prove that these recursive calls are not needed if we compute more than the linear rank-width, and it is the key for the $O(n^2 \cdot \log(n))$ time algorithm.

| $j$ | $PD(B, j)$ | $LD(B, j)$ | Status |
|-----|-------------|-------------|--------|
| 10  | 8           | 9           | $D' \in D(B, 10)$ has no 10-critical bags. |
| 9   | 8           | 9           | $D' \in D(B, 9)$ has no 9-critical bags. |
| 8   | 8           | 9           | $D' \in D(B, 8)$ has the unique 8-critical bag $B_c$ and the maximum $\mathcal{F}_1$ value over all bags $B'$ except the root in $\mathcal{SL}_1(D', B_c, v)$ is 7. |
| 7   | 7           | 8           | $D' \in D(B, 7)$ has a bag having three children $B'$ such that $\mathcal{F}_1(D', B') = 7$. Thus, $LD(B, 7) = 8$. |
| 6   | -           | -           | Once we have $LD(B, \ell) = \ell + 1$, it is unnecessary to compute $D(B, j)$ where $j < \ell$. |

Table 1. Examples of $PD(B, j)$ and $LD(B, j)$.

For each bag $B$ of $D$ and $0 \leq j \leq \lfloor \log |V(G)| \rfloor$, we recursively define the set $D(B, j)$ of canonical decompositions and the positive integers $PD(B, j)$ and $LD(B, j)$. The integer $j$ will not be larger than the linear rank-width of the distance-hereditary graph, and the inequality $j \leq \lfloor \log |V(G)| \rfloor$ came from the following fact.

**Lemma 13.** For a distance-hereditary graph $G$, $\text{lwr}(G) \leq \log |V(G)|$.

**Proof.** We can easily modify the proof of [17, Theorem 4.2] to show it.

Let $D(D, |\log |V(G)||) := \{\mathcal{SL}_1(D, B, v) : v \in U_{\ell}(D, B)\}$. Roughly, $D(B, j)$ consists of all canonical decompositions obtained from canonical decompositions in $D(B, |\log |V(G)||)$ by recursively removing the unique $k$-critical bag, for $k \geq j + 1$. 


For each set $D(B,j)$ and $D' \in D(B,j)$, let $PD(B,j)$ be the maximum $F_1(D',B')$ over all non-root bags $B'$ in $D'$, and let $LD(B,j) := \text{lrw}(D')$. The essential cases are when $PD(B,j) = j$, and in these cases, we want to determine whether $LD(B,j) = j$ or $j + 1$. Now we define $D(B,j)$ precisely.

1. Let $D(B, \lfloor \log|V(G)| \rfloor) := \{SL_1(D,B,v) : v \in U_1(D,B)\}$.
2. For all $1 \leq j \leq \lfloor \log|V(G)| \rfloor$, if $PD(B,j) \neq j$, let $D(B,j) := D(B,j)$. If $PD(B,j) = j$, then for $D' \in D(B,j)$,
   (a) if $(D')$ has a bag with at least 3 children $B_1$ such that $LD(B_1,j) = j$ or $LD(B_2,j) = j$ or $(D')$ has no $j$-critical bags, then let $D(B,j - 1) := D(B,j)$.
   (b) if $D'$ has the unique $j$-critical bag $B_c$, then let $D(B,j - 1) := \{SL_2(D',B_c,v) : D' \in D(B,j), v \in U_2(D',B_c)\}$.

Note that all decompositions in $D(B,j)$ are locally equivalent. So, $PD(B,j)$ and $LD(B,j)$ are well-defined. We prove the following.

**Proposition 7.** Let $D$ be the rooted canonical decomposition of a distance-hereditary graph $G$ and let $B$ be a non-root bag of $D$. Let $i$ be an integer such that $0 \leq i \leq \lfloor \log|V(G)| \rfloor$ and $PD(B,i) \leq i$. Let $D' \in D(B,i)$ and let $B'$ be a non-root bag of $D'$. Then $B'$ is also a non-root bag of $D$ and $PD(B',i) \leq i$. Moreover, for $v \in U_i(D',B')$ and $D'' \in D(B',i)$, $SL_1(D',B',v)$ is locally equivalent to $D''$. Therefore, $F_1(D',B') = LD(B',i)$.

To prove Proposition 7, we need the following technical lemma.

**Lemma 14.** Let $D$ be the canonical decomposition of a distance-hereditary graph. Let $B_1$ and $B_2$ be two bags of $D$ and let $T_i$ be the component of $D \setminus V(B_i)$ containing the bag in $\{B_1,B_2\} \setminus \{B_i\}$ such that $V(T_1) \cap V(T_2)$ has at least one unmarked vertex in $D$, and let $v_i$ be the marked vertex of $B_i$ adjacent to $T_i$.

Then there exists a canonical decomposition $D'$ locally equivalent to $D$ such that for each $i \in \{1,2\}$, $B_i$ is a star in $D'$ and $v_i$ is a leaf of $B_i$.

**Proof.** For one bag $B_i$, it is easy to make $B_i$ into a bag having $v_i$ as a leaf by applying local complementations. Without loss of generality, we may assume that $v_1$ is a leaf of $B_1$. If $v_2$ is a leaf of $B_2$, then we are done. If $B_2$ is a complete bag, then by applying one local complementation, we may change $B_2$ into a bag having $v_2$ as a leaf. This local complementation does not change the bag $B_1$, because already $v_1$ is a leaf of $B_1$ and $B_1 \ast v_1 = B_1$. So, we conclude the result. Therefore, we may assume that $v_2$ is the center of the star bag $B_2$.

Let $T = D[V(T_1) \cap V(T_2)]$ and let $w_2 \in V(T_2)$ be the marked vertex adjacent to $v_2$ in $D$. By the definition of a canonical decomposition, $w_2$ is not a leaf of a star bag in $D$. Therefore, there exists an unmarked vertex $y \in V(T)$ of $D$ such that $y$ is linked to $w_2$ in $T$. Let $y'$ be an unmarked vertex of $D$ represented by $w_2$ in $D$. Note that $y$ is linked to $y'$ in $D$ and the paths from $y$ to $y'$ in $D$ pass through $B_2$ but not $B_1$. Thus, each $v_i$ is a leaf of $B_i$ in $D \setminus yy'$, as required.

**Proposition 8.** Let $D$ be the canonical decomposition of a distance-hereditary graph. Let $B_1$ and $B_2$ be two bags of $D$ and let $T_i$ be the component of $D \setminus V(B_i)$ containing the bag in $\{B_1,B_2\} \setminus \{B_i\}$ such that $V(T_1) \cap V(T_2)$ has at least two unmarked vertices in $D$. For $i = 1,2$, let $v_i$ be the marked vertex of $B_i$ adjacent to $T_i$, let $y_i$ be an unmarked vertex represented in $D$ by $v_i$ and let $y'_i$ be an unmarked vertex represented in $\hat{L}[D,B_i,y_i]$ by $\{v_1,v_2\} \setminus \{v_i\}$.

Then $\hat{L}[\hat{L}[D,B_1,y_1],B_2,y'_1]$ is locally equivalent to $\hat{L}[\hat{L}[D,B_2,y_2],B_1,y'_2]$. 

Proof. For each $i = 1, 2$, let $v_i$ be the marked vertex adjacent to $v_i$ in $D$, and let $T = D[V(T_1) \cap V(T_2)]$.

We first assume that each $v_i$ is a leaf of $B_i$ in $D$. In this case, we can obtain both $\bar{L}[^{\bar{L}}[D, B_1, y_1], B_2, y_2]$ and $\bar{L}[\bar{D}, B_1, y_1]$ from the decomposition $T \setminus w_1 \setminus w_2$ by making it into a canonical decomposition. Thus $\bar{L}[\bar{L}[^{\bar{L}}[D, B_1, y_1], B_2, y_2]] = \bar{L}[^{\bar{L}}[D, B_2, y_2], B_1, y_2]$.

Now, we consider other cases. By Lemma [1], there exists a canonical decomposition $\bar{D}'$ locally equivalent to $D$ such that for each $i \in \{1, 2\}$, $v_i$ is a leaf of $B_i$ in $\bar{D}'$. Let $z_i$ be an unmarked vertex represented in $\bar{D}'$ by $v_i$ and let $z_i'$ be an unmarked vertex represented in $\bar{L}[\bar{D}', B_i, z_i]$ by $\{v_1, v_2\} \setminus \{v_1\}$.

Since $D$ is locally equivalent to $\bar{D}'$, by Proposition [1], $\bar{L}[\bar{D}, B_1, y_1]$ is locally equivalent to $\bar{L}[\bar{D}', B_1, z_1]$. Again, since $\bar{L}[\bar{D}, B_1, y_1]$ is locally equivalent to $\bar{L}[\bar{D}', B_1, z_1]$, by Proposition [1],

$$\bar{L}[\bar{L}[^{\bar{L}}[D, B_1, y_1], B_2, y_2]] = \bar{L}[\bar{L}[\bar{D}', B_1, z_1], B_2, z_2].$$

Similarly, we obtain that

$$\bar{L}[\bar{L}[\bar{D}, B_2, y_2], B_1, y_2'] = \bar{L}[\bar{L}[\bar{D}', B_2, z_2], B_1, z_2].$$

Since each $v_i$ is a leaf of $B_i$ in $\bar{D}'$, from the earlier case we analyzed,

$$\bar{L}[\bar{L}[\bar{D}', B_1, z_1], B_2, z_2'] = \bar{L}[\bar{L}[\bar{D}', B_2, z_2], B_1, z_2].$$

Therefore,

$$\bar{L}[\bar{L}[^{\bar{L}}[D, B_1, y_1], B_2, y_2]] = \bar{L}[\bar{L}[\bar{D}', B_2, y_2], B_1, y_2'],$$

as required. \qed

Proposition 9. Let $D$ be the canonical decomposition of a distance-hereditary graph. Let $B_1$ and $B_2$ be two bags of $D$. Let $T_1$ be a component of $D \setminus V(B_1)$ which does not contain $B_2$ and let $T_2$ be the component of $D \setminus V(B_2)$ containing the bag $B_1$. For $i = 1, 2$, let $v_i$ be the marked vertex of $B_i$ adjacent to $T_i$, and let $y_i$ be an unmarked vertex represented by $v_i$ in $D$. If $B_1$ is a bag of $\bar{L}[\bar{L}[\bar{D}, B_2, y_2], B_1, y_2]$, then $\bar{L}[\bar{L}[\bar{D}, B_1, y_1]]$ is locally equivalent to $\bar{L}[\bar{L}[\bar{D}'[D, B_2, y_2], B_1, y_2]$, where $y_2'$ is an unmarked vertex represented in $\bar{L}[\bar{D}'[D, B_2, y_2], B_1, y_2]$.

Proof. Suppose $B_1$ is a bag of $\bar{L}[\bar{L}[\bar{D}, B_2, y_2], B_1, y_2']$ and $y_2'$ is an unmarked vertex represented in $\bar{L}[\bar{L}[\bar{D}, B_2, y_2]]$ by $v_1$. If $y_2$ is a leaf of a star bag $B_2$, it is easy to show that $\bar{L}[\bar{L}[\bar{D}, B_1, y_1]] = \bar{L}[\bar{L}[\bar{D}, B_2, y_2], B_1, y_2']$ because $B_1$ and $T_1$ in $D$ are not different with $B_1$ and $T_1$ in $\bar{L}[\bar{L}[\bar{D}, B_2, y_2]]$, respectively. Now, we consider other cases. Note that there exists a canonical decomposition $\bar{D}'$ locally equivalent to $D$ such that $v_2$ is a leaf of $B_2$ in $\bar{D}'$. Let $z_i$ be an unmarked vertex represented by $v_i$ in $\bar{D}'$ and let $z_i'$ be an unmarked vertex represented in $\bar{L}[\bar{D}', B_2, z_2]$ by $v_1$.

Since $D$ is locally equivalent to $\bar{D}'$, by Proposition [1], $\bar{L}[\bar{L}[\bar{D}, B_1, y_1]]$ is locally equivalent to $\bar{L}[\bar{L}[\bar{D}', B_1, z_1]]$. Similarly, we obtain that $\bar{L}[\bar{L}[\bar{D}, B_2, y_2]]$ is locally equivalent to $\bar{L}[\bar{L}[\bar{D}', B_2, z_2]]$. Since $\bar{L}[\bar{L}[\bar{D}, B_2, y_2]]$ is locally equivalent to $\bar{L}[\bar{L}[\bar{D}', B_2, z_2]]$, by Proposition [1],

$$\bar{L}[\bar{L}[\bar{L}[\bar{D}, B_2, y_2], B_1, y_2']] = \bar{L}[\bar{L}[\bar{L}[\bar{D}', B_2, z_2], B_1, z_2']].$$

Since $v_2$ is a leaf of $B_2$ in $\bar{D}'$, from the earlier case we analyzed, $\bar{L}[\bar{L}[\bar{L}[\bar{D}, B_2, y_2], B_1, y_2']] = \bar{L}[\bar{L}[\bar{L}[\bar{D}', B_2, z_2], B_1, z_2']$, and therefore,

$$\bar{L}[\bar{L}[\bar{L}[\bar{D}, B_1, y_1]]] = \bar{L}[\bar{L}[\bar{L}[\bar{D}', B_2, y_2], B_1, y_2']],$$

as required. \qed
The following are needed to prove Proposition [7].

**Lemma 15.** Let $D$ be the rooted canonical decomposition of a distance-hereditary graph $G$ and let $B$ be a non-root bag of $D$. Let $i$ be an integer such that $0 \leq i < \lfloor \log |V(G)| \rfloor$.

If $PD(B, i) \leq i$, then $PD(B, i + 1) \leq i + 1$.

**Proof.** Suppose that $PD(B, i + 1) \geq i + 2$. By the definition of $D(B, i)$, $D(B, i) = D(B, i + 1)$ and therefore, $PD(B, i) \geq i + 2$, which is contradiction. \(\square\)

**Lemma 16.** Let $D$ be the rooted canonical decomposition of a distance-hereditary graph $G$ and let $B$ be a non-root bag of $D$. Let $D' \in D(B, \lfloor \log |V(G)| \rfloor)$, and let $B'$ be a non-root bag of $D'$. Then $B'$ is a non-root bag of $D$. Moreover, for $v \in U_1(D', B')$ and $D'' \in D'(D', \lfloor \log |V(G)| \rfloor)$, $\mathcal{SL}_1(D', B', v)$ is locally equivalent to $D''$.

**Proof.** By Remark [1] $B'$ is a non-root bag of $D$. Since $B'$ is a bag of $D'$, by Proposition [7], $\mathcal{SL}_1(D', B', v)$ is locally equivalent to $D''$ for all $D'' \in D'(D', \lfloor \log |V(G)| \rfloor)$.

**Proof (of Proposition [7]).** By Remark [1] $B'$ is a non-root bag of each $D' \in D(B, j)$ where $i \leq j \leq \lfloor \log |V(G)| \rfloor$ and $B'$ is also a non-root bag of $D$. Also, by Lemma [15], $PD(B, j) \leq j$ for all $i \leq j \leq \lfloor \log |V(G)| \rfloor$. Clearly $PD(B', i) \leq i$, otherwise, $PD(B, i) \geq i + 1$.

We show, by induction on $\lfloor \log |V(G)| \rfloor - i$, that for $v \in U_1(D', B')$ and $D'' \in D'(D', B', v)$ is locally equivalent to $D''$. If $i = \lfloor \log |V(G)| \rfloor$, then by Lemma [16] $\mathcal{SL}_1(D', B', v)$ is locally equivalent to $D''$.

Suppose that $i < \lfloor \log |V(G)| \rfloor$. Let $v \in U_1(D', B')$, $D'' \in D'(B', i + 1)$ and let $D_1 \in D(B, i + 1)$, $D_2 \in D(B', i + 1)$. Let $z \in U_1(D_1, B')$. By the induction hypothesis, $\mathcal{SL}_1(D_1, B', z)$ is locally equivalent to $D_2$. Note that $PD(B, i + 1) \leq i + 1$.

If $PD(B, i + 1) \leq i$, then it is easy to see that $PD(B', i + 1) \leq i$. Therefore, $D(B, i) = D(B, i + 1)$ and $D(B', i) = D(B', i + 1)$. Thus, $\mathcal{SL}_1(D', B', v)$ is locally equivalent to $\mathcal{SL}_1(D_1, B', z)$ and $D''$ is locally equivalent to $D_2$. By the induction hypothesis, we conclude that $\mathcal{SL}_1(D', B', v)$ is locally equivalent to $D''$.

Now we may assume that $PD(B, i + 1) = i + 1$. Since $PD(B, i + 1) = i + 1$ and $PD(B, i) \leq i$, by the definition of $D(B, i)$, neither $D_1$ has a bag having at least three children $B_1$ such that $F_1(D_1, B_1) = i + 1$, nor $D_1$ has two incomparable bags $B_1$ and $B_2$ with an $(i + 1)$-critical bag $B_1$ and $LD(B_2, i + 1) = i + 1$. So, either $D_1$ has no $(i + 1)$-critical bag, or $D_1$ has the unique $(i + 1)$-critical bag. If $D_1$ has no $(i + 1)$-critical bag, then $D_2$ also has no $(i + 1)$-critical bag. Thus, $D(B, i) = D(B, i + 1)$ and $D(B', i) = D(B', i + 1)$, and by the induction hypothesis, $\mathcal{SL}_1(D', B', v)$ is locally equivalent to $D''$, as required.

Suppose that $D_1$ has the unique $(i + 1)$-critical bag $B_c$. Let $w \in U_2(D_1, B_c)$ and let $w' \in U_1(\mathcal{SL}_2(D_1, B_c, w), B')$. From the definition of $D(B, i)$, $D''$ is locally equivalent to $\mathcal{SL}_2(D_1, B_c, w)$. Thus, $\mathcal{SL}_2(D_1, B_c, w)$ contains the bag $B'$, and so, $B'$ is not a descendant of $B_c$ in $D_1$. Therefore, there are two cases:

1. $B_c$ is incomparable to $B'$ in $D_1$.
2. $B_c$ is a descendant of $B'$ in $D_1$.

First assume that $B_c$ is incomparable to $B'$ in $D_1$. Then there is no $(i + 1)$-critical bag in $\mathcal{SL}_1(D_1, B', z)$. So, $D_2$ has no $(i + 1)$-critical bag and therefore, $D(B', i) = D(B', i + 1)$. Therefore, $D_2$ is locally equivalent to $D''$. Moreover, by Proposition [9]

$\mathcal{SL}_1(D_1, B', z)$ is locally equivalent to $\mathcal{SL}_1(\mathcal{SL}_2(D_1, B_c, w), B', w')$. 


Case 1: \( B_c \) is incomparable to \( B' \) in \( D_1 \).

\[
\begin{align*}
\mathcal{S}\mathcal{L}_1(D_1, B', z) & \quad \mathcal{S}\mathcal{L}_1(D_1, B', z) & \quad D_2 & \quad D'' \\
\sim & \quad \Rightarrow & \quad \Rightarrow & \quad \Rightarrow & \quad \Rightarrow & \quad \sim \\
\mathcal{S}\mathcal{L}_1(\mathcal{S}\mathcal{L}_2(D_1, B_c, w), B', w') & \quad \mathcal{S}\mathcal{L}_1(D', B', v) & \quad \mathcal{S}\mathcal{L}_2(D', B', v) & \quad \mathcal{S}\mathcal{L}_2(D', B', v) & \quad \mathcal{S}\mathcal{L}_1(D', B', v)
\end{align*}
\]

Case 2: \( B_c \) is a descendant of \( B' \) in \( D_1 \).

\[
\begin{align*}
\mathcal{S}\mathcal{L}_2(\mathcal{S}\mathcal{L}_1(D_1, B', z), B_c, z') & \quad \mathcal{S}\mathcal{L}_2(\mathcal{S}\mathcal{L}_1(D_1, B', z), B_c, z') & \quad \mathcal{S}\mathcal{L}_2(D_2, B_c, z') & \quad D'' \\
\sim & \quad \Rightarrow & \quad \Rightarrow & \quad \Rightarrow & \quad \sim & \quad \sim \\
\mathcal{S}\mathcal{L}_1(\mathcal{S}\mathcal{L}_2(D_1, B_c, w), B', w') & \quad \mathcal{S}\mathcal{L}_1(D', B', v) & \quad \mathcal{S}\mathcal{L}_1(D', B', v) & \quad \mathcal{S}\mathcal{L}_1(D', B', v) & \quad \mathcal{S}\mathcal{L}_1(D', B', v)
\end{align*}
\]

Figure 3. The two cases in the proof of Proposition 7.

Since \( D' \) is locally equivalent to \( \mathcal{S}\mathcal{L}_2(D_1, B_c, w) \),

\[
\mathcal{S}\mathcal{L}_1(D_1, B', z) \text{ is locally equivalent to } \mathcal{S}\mathcal{L}_1(D', B', v).
\]

Also, by the induction hypothesis, \( D_2 \) is locally equivalent to \( \mathcal{S}\mathcal{L}_1(D_1, B', z) \), and therefore, \( D_2 \) is locally equivalent to \( \mathcal{S}\mathcal{L}_1(D', B', v) \). Thus, we conclude that \( D'' \) is locally equivalent to \( \mathcal{S}\mathcal{L}_1(D', B', v) \), as required. See Figure 3.

Now suppose that \( B_c \) is a descendant of \( B' \) in \( D_1 \). Let \( T = D_1[V(T_1(D_1, B')) \cap V(T_2(D_1, B_c))] \). If \( V(T) \) has only one unmarked vertex in \( D_1 \), then \( B' \) is the parent of \( B_c \) and \( |B'| = 3 \). However, in this case, \( B' \) does not exist in \( \mathcal{S}\mathcal{L}_2(D_1, B_c, w) \in D(B, t) \), which contradicts the assumption. Therefore, we may assume that \( V(T) \) has at least two unmarked vertices in \( D_1 \).

Let \( z' \in U_2(\mathcal{S}\mathcal{L}_1(D_1, B', z), B_c) \). Since \( V(T) \) has at least two unmarked vertices in \( D_1 \), by Proposition 5

\[
\mathcal{S}\mathcal{L}_1(\mathcal{S}\mathcal{L}_2(D_1, B_c, w), B', w') \text{ is locally equivalent to } \mathcal{S}\mathcal{L}_2(\mathcal{S}\mathcal{L}_1(D_1, B', z), B_c, z').
\]

Since \( D' \) is locally equivalent to \( \mathcal{S}\mathcal{L}_2(D_1, B_c, w) \),

\[
\mathcal{S}\mathcal{L}_1(D', B', v) \text{ is locally equivalent to } \mathcal{S}\mathcal{L}_2(\mathcal{S}\mathcal{L}_1(D_1, B', z), B_c, z').
\]

Also, since \( \mathcal{S}\mathcal{L}_1(D_1, B', z) \) is locally equivalent to \( D_2 \)

\[
\mathcal{S}\mathcal{L}_1(D', B', v) \text{ is locally equivalent to } \mathcal{S}\mathcal{L}_2(D_2, B_c, z').
\]

At last, \( \mathcal{S}\mathcal{L}_2(D_2, B_c, z') \) is locally equivalent to \( D'' \), and therefore, we conclude that \( \mathcal{S}\mathcal{L}_1(D', B', v) \) is locally equivalent to \( D'' \). See Figure 3.

Now we describe the algorithm explicitly. To ease the understanding, we modify the given canonical decomposition as follows. For the canonical decomposition \( D' \) of a distance-hereditary graph \( G \), we modify \( D' \) into a canonical decomposition \( D \) by adding a bag \( R \) adjacent to a bag \( R' \) in \( D \) so that \( f(D, R, D') = \text{lrw}(G) \). So, if we regard \( R \) as the root bag of \( D \), then \( \mathcal{F}_1(D, R') = \text{lrw}(G) \), and therefore it is sufficient to compute \( LD(R', |\log|V(G)||) = \mathcal{F}_1(D, R') \). The basic strategy is to compute \( LD(B, i) \) for all non-root bags \( B \) of \( D \) and integers \( i \) such that \( PD(B, i) \leq i \). If \( B \) is a non-root leaf bag of \( D \), then clearly \( \mathcal{F}_1(D, B) = 1 \), so let \( LD(B, i) = 1 \) for all \( 0 \leq i \leq |\log|V(G)|| \). For convenience, let \( t = |\log|V(G)|| \).

1. Compute the canonical decomposition \( D' \) of \( G \), and obtain a canonical decomposition \( D \) from \( D' \) by adding a root bag \( R \) adjacent to a bag \( R' \) in \( D \) so that \( \text{lrw}(G) = LD(R', t) \).
2. For all non-root leaf bags $B$ in $D$, set $LD(B, j) := 1$ for all $0 \leq j \leq t$.
3. While ($D$ has a non-root bag $B$ such that $LD(B, t)$ is not computed.)
   (a) Choose a non-root bag $B$ in $D$ such that for every child $B'$ of $B$, $LD(B', t)$
   is computed.
   (b) Choose some $v \in \mathcal{U}_t(D, B)$ and compute $D_i := SL_1(D, B, v) \in D(B, t)$.
   (c) Compute $k := PD(B, t)$ and set $D_k := D_i$ and $i := k$.
   (d) Let $S$ be a stack.
   (e) While (true) do.
      i. If either ($D_i$ has a bag with at least 3 children $B_1$ such that $LD(B_1, i) =$
         $1$ or ($D_i$ has two incomparable bags $B_1$ and $B_2$ with $B_1$ an $i$-critical
         bag and $LD(B_2, i) = i$) or ($D_i$ has no $i$-critical bags), then stop this
         loop.
      ii. Find the unique $i$-critical bag in $D_i$.
      iii. Compute $D_{i-1} \in D(B, i - 1)$ and push($S, i$).
      iv. Set $j := i - 1$ and $i := PD(B, i - 1)$ and $D_i := D_j$.
   (f) If either ($D_i$ has a bag with at least 3 children $B_1$ such that $LD(B_1, i) =$
      $i$ or ($D_i$ has two incomparable bags $B_1$ and $B_2$ with $B_1$ an $i$-critical
      bag and $LD(B_2, i) = i$), then set $LD(B, i) := i + 1$, else, $LD(B, i) := i$.
   (g) While ($S \neq \emptyset$) do.
      i. Set $j := pull(S)$.
      ii. If $LD(B, i) = j$, then $LD(B, j) := j + 1$, else $LD(B, j) := j$.
      iii. For $\ell = i + 1$ to $j - 1$, set $LD(B, \ell) := LD(B, i)$.
      iv. Set $i := j$.
   (h) Set $LD(B, j) := LD(B, k)$ for all $k < j \leq t$.
4. Return $LD(R', t)$.

**Proof (of Theorem 5).** We will show that for each bag $B$ and each $0 \leq j \leq t =
\lfloor \log(n) \rfloor$ such that $PD(B, j) \leq j$, the algorithm computes $LD(B, j)$ correctly. For
each non-root leaf bag $B$, by Lemma 6, Step 2 correctly puts the values $LD(B, j)$.
(The limb $SL_1(D, B, v)$ is isomorphic to either a complete graph or a star and it has
at least two vertices.) Now we assume that $B$ is not a leaf and for all its children $B'$
and integers $0 \leq \ell \leq t = \lfloor \log(n) \rfloor$ such that $PD(B', \ell) \leq \ell$, $LD(B', \ell)$ is computed.

We first observe the computation of a canonical decomposition in $D(B, j)$. Note
that by Lemma 15, $PD(B, i) \leq i$ for all $j \leq i \leq t$. Also, since 2(a) in the definition
of $D(B, j)$ occurs for some $j$, we have $PD(B, \ell) = j$ for all $\ell \leq j - 1$. In other words,
a canonical decomposition in $D(B, j)$ is obtained from a canonical decomposition
in $D_i \in D(B, t)$ by a sequence of operations which are either

1. $D_{i-1} = D_i$ when $PD(B, i) \leq i - 1$, or
2. $D_{i-1} = SL_1(D_i, B_i, v)$ where $B_i$ is the unique $i$-critical bag of $D_i$ and $v \in
\mathcal{U}_t(D_i, B_i)$ when $PD(B, i) = i$.

To skip the procedures taking $D_{i-1} = D_i$, it is sufficient to deal with $D_j$ directly
instead of $D_{i-1}$ if $j = PD(B, i - 1)$ for some $i$. Therefore, Step 3(d) correctly
compute $D_j$ for all $j$ such that $PD(B, j) = j$.

Now we verify the procedure of computing $LD(B, j)$. Let $0 \leq \ell \leq t$ be the
minimum integer such that $D_{i} = D_i$ is computed. If $\ell = 0$, then the linear rank-width of
$D_0$ must be 1 because $D_0$ must have at least two vertices. If $\ell \geq 1$, then since $D_{\ell-1}$
is not computed, by the definition,

1. $LD(B, \ell) = \ell + 1$ if either $D_{\ell}$ has a bag with at least 3 children $B'$ such that
   $LD(B', \ell) = \ell$ or $D_{\ell}$ has two incomparable bags $B_1$ and $B_2$ with $B_1$ an $i$-critical
   bag and $LD(B_2, i) = i$,
2. $LD(B, \ell) = \ell$ if otherwise.
So, Step 3(f) correctly computes it.

Note that by Proposition 6, we can compute \( LD(B,j) \) for all \( \ell + 1 \leq j \leq t \). By the definition, if \( D_j \) has the unique \( j \)-critical bag and \( LD(B,j-1) = j \), then \( LD(B,j) = j + 1 \), and if either \( PD(B,j) \leq j - 1 \) or \( LD(B,j-1) \leq j - 1 \), then \( LD(B,j) = j \). In the loop 3(e) in the algorithm, we use a stack to pile up the integers \( j \) such that \( D_j \) has the unique \( j \)-critical bag. So, from the lower value in the stack we compute \( LD(B,j) \) recursively. Therefore, Steps 3(g-h) compute all \( LD(B,j) \) correctly where \( PD(B,j) \leq j \).

Now we analyze the time complexity. By Propositions 6 and 7 the steps of the algorithm outlined above computes the linear rank-width of every connected distance-hereditary graph \( G \). Let us now analyze its running time. Let \( n \) and \( m \) be the number of vertices and edges of \( G \). Its canonical decomposition \( D' \) can be computed in time \( O(n + m) \) by Theorem 1 and one can of course add a new bag to obtain a new canonical decomposition \( D \) and root it in constant time. The number of bags in \( D \) is bounded by \( O(n) \) (see [12, Lemma 2.2]). For each bag \( B, LD(B,j) \) for all \( 0 \leq j \leq t \) can be computed in time \( O(n \cdot \log(n)) \). In fact, Steps 3(a-c) can be done in time \( O(n) \). The loop in 3(e) runs \( \log(n) \) times since \( k \leq \log(n) \), and all the steps in 3(e) can be implemented in time \( O(n) \). Since Steps 3(f-h) can be done in time \( O(n) \), we conclude that this algorithm runs in time \( O(n^2 \cdot \log(n)) \).

**Corollary 2.** For every connected distance-hereditary graph \( G \), we can compute in time \( O(n^2 \cdot \log(n)) \) a layout of the vertices of \( G \) witnessing \( \text{lrw}(G) \).

**Proof.** We follow the same proof as in [9] and establish a linear layout witnessing \( \text{lrw}(G) = k \). We first run the algorithm computing \( \text{lrw}(G) \). At the end, each bag \( B \) has a label \( \lambda_B = (a_1,a_2,\ldots,a_k) \) corresponding to the computed values \( LD(B,j) \). Then we search for the path depicted in Lemma 13 and this can be done in linear time. Now for all the subtrees pending on that path, the linear rank-width of the corresponding limbs are at most \( k - 1 \). So we apply recursively the same algorithm on each of them. We can therefore output an ordering witnessing \( \text{lrw}(G) = k \). Since the depth of the recursive calls is bounded by \( k \) and in each call, the path is found in \( O(n) \), we can compute optimal layout in time \( O(n \log(n)) \), once \( \lambda_B \) are computed, which can be done in \( O(n^2 \cdot \log(n)) \).

### 6 Obstructions

A graph \( H \) is a vertex-minor obstruction for (linear) rank-width \( k \) if it has (linear) rank-width \( k + 1 \) and every proper vertex-minor of \( H \) has (linear) rank-width at most \( k \). The set of pairwise locally non-equivalent vertex-minor obstructions for (linear) rank-width \( k \) is not known, but for rank-width \( k \) a bound on their size is known [15], which is not the case for linear rank-width \( k \). For \( k = 1 \), Adler, Farley, and Proskurowski [1] characterized the distance-hereditary vertex-minor obstructions for linear rank-width at most 1 by two pairwise locally non-equivalent graphs. For general \( k \), Jeong, Kwon, and Oum recently provided a \( 2^{O(3^k)} \) lower bound on the number of pairwise locally non-equivalent distance-hereditary vertex-minor obstructions for linear rank-width at most \( k \) [16]. Using our characterization, we generalize the construction in [16] and conjecture a subset of the given set to be the set of distance-hereditary vertex-minor obstructions.

We will use the notion of one-vertex extension introduced in [14]. We call a graph \( G' \) an one-vertex extension of a distance-hereditary graph \( G \) if \( G' \) is a graph obtained from \( G \) by adding a new vertex \( v \) with some edges and \( G' \) is again distance-hereditary. For convenience, if \( D \) and \( D' \) are canonical decompositions of \( G \) and \( G' \), respectively, then \( D' \) is also called an one-vertex extension of \( D \). For examples, one might see that an one-vertex extension of \( K_2 \) is isomorphic to either \( K_3 \) or \( K_{1,2} \).
For each non-negative integer $k$, we construct the sets $\Psi_k$ and $\Psi_k'$ of canonical decompositions as follows.

1. $\Psi_0$ consists of the canonical decomposition of the graph $K_2$. (It is isomorphic to $K_2$.)
2. For $k \geq 0$, let $\Psi_k'$ be the union of $\Psi_k$ and the set all the one-vertex extensions of canonical decompositions in $\Psi_k$.
3. For $k \geq 1$, let $\Psi_k$ be the set of all canonical decompositions $D$ defined as follows. Choose three canonical decompositions $D_1, D_2, D_3$ in $\Psi_{k-1}$ and take one-vertex extensions $D_i'$ of $D_i$ with new vertices $v_i$ for each $i$. We introduce a new bag $B$ of type $K$ or $S$ having three vertices $v_1, v_2, v_3$ and
   - (a) if $v_i$ is in a complete bag, then $D_i'' = D_i' * v_i$,
   - (b) if $v_i$ is the center of a star bag, then $D_i'' = D_i' \land w_i z_i$ for some $z_i$ linked to $w_i$ in $D_i'$,
   - (c) if $v_i$ is a leaf of a star bag, then $D_i'' = D_i'$.
We define $D$ as the canonical decomposition obtained by the disjoint union of $D_1', D_2', D_3'$ and $B$ by adding the marked edges $v_1 v_2, v_2 v_3, v_3 v_1$.

It is worth noticing that if $k \geq 1$, then $D \in \Psi_k$ if and only if there exists a bag $B$ in $D$ such that the three limbs corresponding to $B$ are contained in $\Psi_{k-1}$. We prove the following.

**Theorem 6.** Let $k \geq 0$ and let $G$ be a distance-hereditary graph such that $\text{lrw}(G) \geq k + 1$. Then there exists a canonical decomposition $D$ in $\Psi_k$ such that $G$ contains a vertex-minor isomorphic to $\hat{D}$.

Instead of showing Theorem 6 we will prove the following which implies it clearly since $\Psi_k \subseteq \Psi_k'$.

**Theorem 7.** Let $k \geq 0$ and let $G$ be a distance-hereditary graph such that $\text{lrw}(G) \geq k + 1$. Then there exists a canonical decomposition $D$ in $\Psi_k'$ such that $G$ contains a vertex-minor isomorphic to $\hat{D}$.

**Lemma 17.** Let $D$ be the canonical decomposition of a distance-hereditary graph. Let $B_1$ and $B_2$ be two distinct bags of $D$ and let $y_1$ be the vertex of $B_1$ such that the distance between $y_1$ to $B_2$ is minimum, and similarly we define $y_2$ such that

- $y_1$ is not a center of a star bag $B_1$, and
- $B_2$ is a star bag and $y_2$ is a leaf of $B_2$.

Let $L_1$ be the set of all bags $B$ in $D$ such that every path from $B$ to $B_2$ in $T_D$ contains $B_1$, and similarly we define $L_2$. Then $D$ has a vertex-minor isomorphic to $\hat{D}'$ where $D'$ is a canonical decomposition such that

1. $D[\bigcup_{B \in L_1} V(B)] = D'[\bigcup_{B \in L_1} V(B)]$,
2. $D[\bigcup_{B \in L_2} V(B)] = D'[\bigcup_{B \in L_2} V(B)]$, and
3. either $D'$ has no bags between $B_1$ and $B_2$, or $D'$ has only one another bag $B$ which is contained in neither $L_1$ nor $L_2$, and $|V(B)| = 3$, $B$ is star, and two leaves are adjacent to $y_1$ and $y_2$ in $D'$.

**Proof.** If there is no bag between $B_1$ and $B_2$ in $D$, then there is no problem. Let $P = p_1 p_2 \ldots p_{\ell}$ be the shortest path from $y_1 = p_1$ to $y_2 = p_{\ell}$ in $D$ and we assume $\ell \geq 3$.

For all bags $C$ in $D$ which contains only two vertices $p_i, p_{i+1}$ of $P$, we remove $C$ and add a marked edge $p_{i-1} p_{i+2}$. This corresponds to removing all vertices from $\hat{D}$ which are represented by some vertices of $C$ except $p_i$ and $p_{i+1}$ in $D$. By this procedure, we may assume that all bags except $B_1$ and $B_2$ contain three or no
vertices of $P$. Note that $y_2$ is a leaf of $B_2$ but $y_1$ is not a center of $B_1$. So, if there is no bag between $B_1$ and $B_2$ after modification, by Theorem 2 the resulting decomposition is again a canonical decomposition. If there is a bag between $B_1$ and $B_2$, then all marked edges on the path from $B_1$ to $B_2$ in the modified decomposition tree are not types of $S_pS_c$ or $KK$, and therefore by Theorem 2 it is again a canonical decomposition.

If there exist two adjacent bags $C_1$ and $C_2$ in $D$ such that $p_i, p_{i+1}, p_{i+2} \in V(C_1)$ and $p_{i+3}, p_{i+4}, p_{i+5} \in V(C_2)$. Then clearly, $p_{i+1}$ and $p_{i+4}$ are the centers of star bags $C_1$ and $C_2$, respectively. By pivoting two vertices represented by $p_{i+1}$ and $p_{i+4}$ in $D$, we can modify two bags $C_1$ and $C_2$ so that $p_ip_{i+2}p_{i+3}p_{i+5}$ become a path. By Lemma 3, this pivoting does not affect on any bag in $L_1$ and $L_2$, so we remove $C_1$ and $C_2$ from $D$, and add a marked edge $p_{i-1}p_{i+6}$. That is, we can reduce two such bags simultaneously. At the end, we have no bags between $B_1$ and $B_2$ or only one star bag whose two leaves are adjacent to $y_1$ and $y_2$, and we conclude the result. \[
\]

The next proposition says how we can replace limbs having linear rank-width $\geq k = 1$ into one-vertex extensions of canonical decompositions in $\Psi^*_D$ using Lemma 17.

**Proposition 10.** Let $D$ and $A$ be canonical decompositions of distance-hereditary graphs and let $B$ be a star bag of $D$. Let $T$ be a component of $D \setminus V(B)$ such that a leaf $v$ of $B$ is adjacent to $T$, and let $w$ be an unmarked vertex of $D$ represented by $v$. If $\mathcal{L}[D, B, w]$ has a vertex-minor isomorphic to either $\hat{A}$ or an one-vertex extension of $\hat{A}$, then there exists a canonical decomposition $D'$ such that

1. $\hat{D}$ has a vertex-minor isomorphic to $\hat{D}'$,
2. either $D'[V(D) \setminus V(T)] = D[V(D) \setminus V(T)]$ or $D'[V(D) \setminus V(T)] = D[V(D) \setminus V(T)] \ast v$, and
3. the limb of $D'$ with respect to the component of $D' \setminus V(B)$ which is adjacent to $v$ is either $A$ or an one-vertex extension of $A$.

**Proof.** Since $\mathcal{L}[D, B, w]$ has a vertex-minor isomorphic to either $\hat{A}$ or an one-vertex extension of $\hat{A}$, there exists a sequence $x_1, x_2, \ldots, x_m$ of vertices of $\mathcal{L}[D, B, w]$ and $S \subseteq V(\mathcal{L}[D, B, w])$ such that $\mathcal{L}[D, B, w] \ast x_1 \ast x_2 \ast \ldots \ast x_m \ast S$ is either $\hat{A}$ or an one-vertex extension of $\hat{A}$. Note that since $v$ is a leaf of $B$, $\mathcal{L}[D, B, w]$ is a limb of type 1 in the definition. Thus, $D \ast x_1 \ast x_2 \ast \ldots \ast x_m$ also has a decomposition of either $A$ or an one-vertex extension of $A$ as an induced subgraph.

When we apply a local complementation at each $x_j$ on $D$, the decomposition induced on $V(D) \setminus V(T)$ either is affected by a local complementation at $y$ or is not changed. Therefore, $(D \ast x_1 \ast x_2 \ast \ldots \ast x_m)[V(D) \setminus V(T)]$ will be either $D[V(D) \setminus V(T)]$ or $D[V(D) \setminus V(T)] \ast y$. Note that if we remove $S$ and unnecessary vertices from $D \ast x_1 \ast x_2 \ast \ldots \ast x_m$, then it could be disconnected between the vertices of the part $A$ and the bag $B$, so we should be careful.

We choose a bag $B'$ in $D \ast x_1 \ast x_2 \ast \ldots \ast x_m$ such that $B'$ has at least two vertices represent unmarked vertices of $A$ and the distance from $B'$ to $B$ in $D \ast x_1 \ast x_2 \ast \ldots \ast x_m$ is minimum. Among the vertices of $B'$, let $y$ be the vertex such that the distance between $y$ and $B$ is minimum. By the definition, $y$ does not represent any unmarked vertices of $A$ in $D \ast x_1 \ast x_2 \ast \ldots \ast x_m$. We remark one important fact that since $A$ is connected and at least two vertices of $B'$ represent the vertices of $A$, $y$ is not the center of a star bag.

Let $L$ be the set of all bags $C$ in $D \ast x_1 \ast x_2 \ast \ldots \ast x_m$ such that every path from $C$ to $B$ in $T_{D \ast x_1 \ast x_2 \ast \ldots \ast x_m}$ contains $B'$. Applying Lemma 17, the underlying graph of $D \ast x_1 \ast x_2 \ast \ldots \ast x_m$ has a vertex-minor isomorphic to $\hat{D}'$ such that

1. $(D \ast x_1 \ast x_2 \ast \ldots \ast x_m)[\bigcup_{B \in L} V(B)] = D'[\bigcup_{B \in L} V(B)]$, and
2. $(D \ast x_1 \ast x_2 \ast \ldots \ast x_m)[V(D) \setminus V(T)] = D'[V(D) \setminus V(T)]$, and

3. either $D'$ has no bags between $B$ and $B'$, or $D'$ has only one another bag $C$ which is contained in neither $\bigcup_{B \in L} V(B)$ nor $V(D) \setminus V(T)$, and $|V(C)| = 3$, $C$ is star, and two leaves of $C$ are adjacent to $y$ and $v$ in $D'$.

If $D'$ has no bags between $B$ and $B'$, then the limb corresponding to $B$ in $D'$ is exactly $(D \ast x_1 \ast x_2 \ast \ldots \ast x_m) \setminus y$ whose underlying graph has $A$ as an induced subgraph.

So we may assume that $D'$ has one bag $C$ between $B$ and $B'$ where $|V(C)| = 3$, $C$ is star, and two leaves of $C$ are adjacent to $y$ and $v$ in $D'$. Let $c$ be the center of $C$. If $\mathcal{L}[D, B, w]$ has a vertex-minor isomorphic to $\hat{A}$, then we can regard $A$ with the vertex $c$ as an one-vertex extension of $A$. Therefore, we may assume that $\mathcal{L}[D, B, w]$ has a vertex-minor isomorphic to an one-vertex extension of $A$ with a new vertex $a$. In this case, we first remove the remaining vertices of $S$ except $c$ and unnecessary vertices from $D'$, and say $D''$.

If $B'$ is star and the center of it is an unmarked vertex, then by applying local complementation at $c$ and removing $c$, we can attach the part on $\bigcup_{B \in L} V(B)$ to $V(D) \setminus V(T)$ with applying one local complementation at $v$ on $D''[V(D) \setminus V(T)]$. And if $B'$ is complete and one vertex $y'$ of it is an unmarked vertex, then by pivoting $y'c$ on $D''$ and removing $c$, we can attach the part on $\bigcup_{B \in L} V(B)$ to $V(D) \setminus V(T)$ without changing anything.

Thus, we may assume that all vertices of $B'$ are marked vertices in $D''$. In this case, we can observe that $D'' \setminus a$ is still connected because $A \setminus a$ is connected and $c$ is linked to at least two unmarked vertices of $D''$ contained in $V(A)$. Since the underlying graph of the limb of $D'' \setminus a$ corresponding to the component having $c$ is $\hat{D}''[V(A) \setminus \{a\} \cup \{c\}]$, which is still a distance-hereditary graph, we prove the claim.

\begin{proof}[Proof of Theorem 7] If $k = 0$, then $\text{lrw}(G) \geq 1$ and $G$ has an edge, so it has a vertex-minor isomorphic to $K_2$. Therefore, we may assume that $k \geq 1$.

Let $D$ be the canonical decomposition of $G$. Since $G$ has linear rank-width at least $k + 1$, by Theorem 4 there exists a bag $B$ in $D$ with three components $T_1, T_2, T_3$ of $D$ corresponding to $B$ such that for corresponding limbs $D_i, \hat{D}_i$ of $D$ has linear rank-width at least $k$. Let $v_i$ be the vertex in $B$ adjacent to $T_i$.

By Proposition 1, we may assume that $B$ is a star with the center $v_3$. Then by Proposition 10 $D$ has a vertex-minor isomorphic to a graph $\hat{D}'$ where $D'$ is obtained from $D \setminus V(T_1) \setminus V(T_2)$ by adding one-vertex extensions of two decompositions $T'_1, T'_2 \in \Psi_{k-1}$ with new vertices $t_1, t_2$, respectively, and adding marked edges $t_1v_1, t_2v_2$. Note that $T_3$ can be changed into either $T_3$ or $T_3 \ast w$ in $D'$ where $wv_3$ is a marked edge in $D$, when apply Proposition 10. Let $T_3'$ be the component of $D' \setminus V(B)$ corresponding to $T_3$.

For each $i$, choose an unmarked vertex $z_i$ in $V(T'_i)$ such that $z_i$ is represented by $v_i$ in $D'$, and let $w_i$ be the vertex in $T'_i$ adjacent to $v_i$. Note that if we apply local complementation at $z_3$ and $z_2$ subsequently in $D'$, then

1. $B$ is changed to a star with the center $v_2$,  
2. $T'_1$ is the same as before,  
3. $T'_2$ is changed to $T'_2 \ast w_2 \ast z_2$,  
4. $T'_3$ is changed to $T'_3 \ast z_3 \ast w_3$.

Now, we apply Proposition 10 to the component $T'_3 \ast z_3 \ast w_3$, Then we have that the underlying graph of $D' \ast z_3 \ast z_2$ has a vertex-minor isomorphic to $\hat{D}''$ where $D''$ is obtained from $D' \ast z_3 \ast z_2 \setminus V(T'_3)$ by adding an one-vertex extension of a decomposition $T''_3 \in \Psi_{k-1}$ with a new vertex $t_3$ and adding a marked edge $t_3v_3$.

We have two cases. We first assume that the outside of $T''_3$ is not changed when we apply Proposition 10. Then $D''$ consists of
1. $B$ is a star with the center $v_2$.
2. and three components of $D'' \setminus V(B)$ are $T'_1$, $T'_2 \ast w_2 \ast z_2$, and $T''_3$.

In this case, $D'' \ast z_2 \in \Psi_k$ because $D'' \ast z_2 \in \Psi_k$ consists of

1. $B$ is complete,
2. and three components of $D'' \setminus V(B)$ are $T'_1$, $T'_2 \ast w_2$, and $T''_3 \ast w_3$.

and therefore, each limb of $D''$ with respect to $B$ are contained in $\Psi'_{k-1}$.

Now we assume that the outside of $T''_3$ is affected by one local complementation. Then $B$ is not changed, but $T''_2$ is affected by a local complementation. Then $D''$ consists of

1. $B$ is a star with the center $v_2$,
2. and three components of $D'' \setminus V(B)$ are $T'_1$, $T'_2 \ast w_2 \ast z_2 \ast w_2$, and $T''_3$.

We can see that $D'' \in \Psi_k$ because each limb with respect to $B$ are contained in $\Psi'_{k-1}$. Therefore, $G$ has a vertex-minor isomorphic to $D''$ where $D'' \in \Psi_k \subseteq \Psi'_{k}$, as required.

In order to prove that $\Psi_k$ is the set of canonical decompositions of distance-hereditary vertex-minor obstructions for linear rank-width at most $k$, we need to prove that for every $D \in \Psi_k$, $D$ has linear rank-width $k+1$ and every of its proper vertex-minors has linear rank-width $\leq k$. However, we were not able to prove it. We will now identify a subset which satisfies this desired property. For each non-negative integer $k$, we define the set $\Phi_k$ of canonical decompositions as follows.

1. $\Phi_0 := \Psi_0$.
2. For $k \geq 1$, let $\Phi_k$ be the set of all canonical decompositions $D$ defined as follows.

Choose three canonical decompositions $D_1, D_2, D_3$ in $\Phi_{k-1}$ and take one-vertex extensions $D'_i$ of $D_i$ with new vertices $w_i$ for each $i$. We introduce a new bag $B$ of type $K$ or $S$ having three vertices $v_1, v_2, v_3$ and (a) if $v_i$ is in a complete bag, then $D''_i = D'_i \ast w_i$, (b) if $v_i$ is the center of a star bag, then $D''_i = D'_i \setminus w_i$ for some $z_i$ linked to $w_i$ in $D'$, (c) if $v_i$ is a leaf of a star bag, then $D''_i = D'_i$.

We define $D$ as the canonical decomposition obtained by the disjoint union of $D'_1, D'_2, D'_3$ and $B$ by adding the marked edges $v_1 w_1, v_2 w_2, v_3 w_3$.

The set $\Phi_k$ is clearly a subset of $\Psi_k$. One also observes that the obstructions constructed in [11] and [16] are contained in $\Phi_k$ for all $k \geq 1$. We have moreover the following.

**Proposition 11.** Let $k \geq 0$ and let $D \in \Phi_k$. Then $\text{lrw}(\hat{D}) = k+1$ and every proper vertex-minor of $\hat{D}$ has linear rank-width at most $k$.

To prove Proposition 11, we need some more lemmas.

**Lemma 18.** Let $D \in \Phi_k$ and let $v$ be an unmarked vertex in $D$. Then $D \ast v \in \Phi_k$.

**Proof.** We proceed by induction on $k$. If $k = 0$, then $D$ is the canonical decomposition of a graph isomorphic to $K_2$, clearly, $D \ast v \in \Phi_k$ for any unmarked vertex $v$ in $D$. We assume that $k \geq 1$. By the construction, there exists a bag $B$ of $D$ such that the three limbs $D_1, D_2, D_3$ in $D$ corresponding to the bag $B$ are contained in $\Phi_{k-1}$.

Let $D'_1, D'_2, D'_3$ be the three limbs of $D \ast v$ corresponding to the bag $B$ such that $D'_i$ and $D_i$ came from the same component of $D \setminus V(B)$. Then by Proposition 11, $D'_i$ is locally equivalent to $D_i$. So by the induction hypothesis, $D'_i \in \Phi_{k-1}$. And $D \ast v$ is the canonical decomposition obtained from $D'_i$ following the construction of $\Phi_k$. Therefore, $D \ast v \in \Phi_k$. \qed
Lemma 19 (Bouchet [4]). Let $G$ be a graph, $v$ be a vertex of $G$ and $w$ an arbitrary neighbor of $v$. Then every elementary vertex-minor obtained from $G$ by deleting $v$ is locally equivalent to either $G \setminus v$, $G * v \setminus v$, or $G \land vw \setminus v$.

Proof. By Lemma 18 and Lemma 19, it is sufficient to show that if $D \in \Phi_k$ and $v$ is an unmarked vertex of $D$, then $D \setminus v$ has linear rank-width at most $k$. We use induction on $k$. We may assume that $k \geq 1$. Let $B$ be the bag of $D$ such that $D \setminus V(B)$ has exactly three limbs whose underlying graphs are contained in $\Phi_{k-1}$. Clearly there is no other bag having the same property. Since $B$ has no unmarked vertices, $v$ is contained in one of the limbs $D'$, and by induction hypothesis, $D' \setminus v$ has linear rank-width at most $k - 1$. Therefore, by Theorem 18, $D \setminus v$ has linear rank-width at most $k$. $\square$

We leave open the question to identify a set $\Phi_k \subset \Theta_k \subset \Psi_k$ that forms the set of canonical decompositions of distance-hereditary vertex-minor obstructions for linear rank-width $k$.

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