ON THE EQUIVALENCE OF CERTAIN COSET CONFORMAL FIELD THEORIES

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ABSTRACT. We demonstrate the equivalence of Kazama-Suzuki cosets \(G(m, n, k)\) and \(G(k, n, m)\) based on complex Grassmannians by proving that the corresponding conformal precosheaves are isomorphic. We also determine all the irreducible representations of the conformal precosheaves.

§1. INTRODUCTION

One of the largest two-dimensional conformal field theories (CFT) arises from coset construction. This construction is examined from the algebraic quantum field theory or Local Quantum Physics (LQF) (cf. [H]) point of view in [X1-3], and many mathematical results are obtained which have resisted other attempts. Kazama and Suzuki showed in [KS] that the superconformal algebra based on coset \(G/H\) possesses an extended \(N=2\) superconformal symmetry if, for rank \(G=\)rank \(H\), the coset \(G/H\) is a Kähler manifold. In this paper, we focus on the class of Kazama-Suzuki models based on the complex Grassmannian manifold \(SU(m+n)/[SU(m) \times SU(n) \times U(1)]\). It will be written as coset

\[
G(m, n, k) := \frac{SU(m+n)_k \times Spin(2mn)}{SU(m)_{n+k} \times SU(n)_{m+k} \times U(1)_{mn(m+n)(m+n+k)}}.
\]

The numerical subscripts are the levels of the representations (cf. [PS]). The invariance of the central charge (cf. [KW]) of the coset \(G(m, n, k)\)

\[
c^{m,n,k} = \frac{3mnk}{m+n+k}
\]

under any permutation of \(m, n, k\) suggests that the models themselves may be invariant [KS]. The invariance of the cosets under the exchange of \(m\) and \(n\) is manifest from their definition, but the symmetry under the the exchange of \(m\) and \(k\) is unexpected, as \(m\) and \(k\) play rather different roles. In [NS], strong evidence for the symmetry is provided including the identification of chiral quantities such as conformal weights, modular transformation matrices and fusion rules under certain
conditions. The goal of this paper is to study this symmetry and related questions in the same spirit of [X1-3].

According to the basic idea of LQF, all the chiral quantities should be obtained by studying the representations the conformal precosheaf of the underlying CFT. We will recall the definition of conformal precosheaf in §2. Denote by $\mathcal{A}(G(m, n, k))$ the conformal precosheaf associated with the coset $G(m, n, k)$. Hence to show that coset conformal field theory based on $G(m, n, k)$ is equivalent to the one based on $G(k, n, m)$, we just have to show that

$$\mathcal{A}(G(m, n, k)) \simeq \mathcal{A}(G(k, n, m)) \quad (1.1)$$

where the isomorphism $\simeq$ between two conformal precosheaves is naturally defined in §2.1.

(1.1) is proved in §3 (cf. Th. 3.7) by representing the two conformal precosheaves on a larger Fock space and use a version of level-rank duality (cf. Prop. 10.6.4 of [PS]). An immediate corollary (cf. Cor. 3.8) is the existence of a one to one map between the irreducible representations (primary fields) of the two cosets and identification of all chiral quantities including braiding and fusion matrices. However, it may be tedious to write down explicitly this map in general. Under certain conditions, such a map is given explicitly in [NS] which we believe to be the right one.

Our second goal in this paper is to determine all the irreducible representations (primary fields) of $\mathcal{A}(G(m, n, k))$. We first determine all the Vacuum Pairs (VPs) of the coset, a concept introduced in [K] which we recall in §2. VPs play an important role in fixed point resolutions and identifications of representations (cf. §4). It is usually easy to come up with VPs based on simple symmetry considerations, but it is in general a nontrivial question to determine all VPs. A list of VPs for $G(m, n, k)$ is given in [LVW] and [NS] based on Dynkin diagram symmetries, but it is known ([DJ]) that there may be VPs which are not related to Dynkin diagram symmetries. We show that the list of VPs for $G(m, n, k)$ given in [LVW] and [NS] is indeed all there is (cf. Th. 4.4). The proof is a mixture of solving VP equation (2.4) for simple cases and using the ring structure of sectors (cf. Lemma 2.7). Using Th. 4.4 and [X3], we determine all the irreducible representations in Th. 4.7.

This paper is organized as follows: In §2.1 we give the definition of coset conformal precosheaves and their properties. In §2.2 we recall some basic results from [X1] in Th. 2.2 and Prop. 2.3 to set up notations, and in Th. 2.4 we show that the coset $G(m, n, k)$ has various expected properties, a result which is implicitly contained in [X2] and [X3]. In §2.3 we describe the notion of Vacuum Pairs of [K] in our setting. While Lemma 2.5 follows directly from definitions, Lemma 2.7 depends on Prop. 2.3. Lemma 2.7 plays an important in §4.

In §3, after recalling some basic facts about the representations of loop groups in Prop. 3.1 from [PS] and [W], we prove Lemmas 3.1-3.6. Th. 3.7 follows from these lemmas, and Cor. 3.8 follows from Th. 3.7. In §4 we first recall simple selection rules about the representations of $\mathcal{A}(G(m, n, k))$ in §4.1. In §4.2 we determine all
the VPs of $A(G(m,n,k))$ in Th. 4.4. Th. 4.4 is proved by using Lemmas 4.1-4.3. Cor. 4.5 follows from Th. 4.4. Lemma 4.6 shows that the conditions of Lemma 2.1 of [X2] are satisfied, and so one can apply Lemma 2.1 of [X2] in the proof of Th. 4.7.

The ideas of this paper apply to KS models based on other Grassmannians as in [FS]. We hope to discuss those cases in the future publication.

In the end of this introduction we describe in more details of the inclusion in the coset $G(m,n,k)$. The inclusion is given by $H_1 \subset G_1$ with $H_1 = SU(m)_n \times SU(n)_m \times U(1)_{mn(m+n)(m+n+k)}$ and $G_1 = SU(m+n)_k \times Spin(2mn)_1$. We will use $H_2$ and $G_2$ to denote $H_1$ and $G_1$ respectively under the exchange of $m$ and $k$. The inclusion $H_1 \subset G_1$ is constructed by the composition of two inclusions:

$$H_1 \subset SU(m)_n \times SU(m)_k \times SU(n)_m \times SU(n)_k \times U(1)_{mn(m+n)(m+n+k)} \times U(1)_{mn(m+n)(k)}$$

(1.2)

and

$$(SU(m)_n \times SU(n)_m \times U(1)_{mn(m+n)(m+n)}) \times (SU(m)_k \times SU(n)_k \times U(1)_{mn(m+n)(k)}) \subset Spin(2mn)_1 \times SU(m+n)_k.$$ (1.3)

The inclusion in (1.2) is diagonal. To describe the inclusion in (1.3), note that the tangent space of the Grassmanian

$$\frac{SU(m+n)}{SU(m) \times SU(n) \times U(1)}$$

at the point corresponding to the identity of $SU(m+n)$ is isomorphic to $\mathbb{C}^m \otimes \mathbb{C}^n$, which is a fundamental representation of $Spin(2mn)$. The natural action of $SU(m) \times SU(n) \times U(1)$ on the tangent space gives the conformal inclusion (cf. §4.2 of [KW])

$$SU(m)_n \times SU(n)_m \times U(1)_{mn(m+n)(m+n)} \subset Spin(2mn)_1.$$  

The inclusion

$$SU(m)_k \times SU(n)_k \times U(1)_{mn(m+n)(k)} \subset SU(m+n)_k$$

comes from the conformal inclusion (cf. Prop. 4.2 of [KW])

$$SU(m)_1 \times SU(n)_1 \times U(1)_{mn(m+n)} \subset SU(m+n)_1.$$
§2. Preliminaries

§2.1 Coset conformal precosheaf. In this section we recall the basic properties enjoyed by the family of the von Neumann algebras associated with a conformal Quantum Field Theory on $S^1$ (cf. [GL1]). This is an adaption of DHR analysis (cf. [H]) to chiral CFT which is most suitable for our purposes.

By an interval we shall always mean an open connected subset $I$ of $S^1$ such that $I$ and the interior $I'$ of its complement are non-empty. We shall denote by $\mathcal{I}$ the set of intervals in $S^1$.

An irreducible conformal precosheaf $\mathcal{A}$ of von Neumann algebras on the intervals of $S^1$ is a map

$$ I \to \mathcal{A}(I) $$

from $\mathcal{I}$ to the von Neumann algebras on a Hilbert space $\mathcal{H}$ that verifies the following property:

**A. Isotony.** If $I_1, I_2$ are intervals and $I_1 \subset I_2$, then

$$ \mathcal{A}(I_1) \subset \mathcal{A}(I_2). $$

**B. Conformal invariance.** There is a nontrival unitary representation $U$ of $G$ (the universal covering group of $PSL(2, \mathbb{R})$) on $\mathcal{H}$ such that

$$ U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in G, \quad I \in \mathcal{I}. $$

The group $PSL(2, \mathbb{R})$ is identified with the Möbius group of $S^1$, i.e. the group of conformal transformations on the complex plane that preserve the orientation and leave the unit circle globally invariant. Therefore $G$ has a natural action on $S^1$.

**C. Positivity of the energy.** The generator of the rotation subgroup $U(R)(\cdot)$ is positive.

Here $R(\vartheta)$ denotes the (lifting to $G$ of the) rotation by an angle $\vartheta$.

**D. Locality.** If $I_0, I$ are disjoint intervals then $\mathcal{A}(I_0)$ and $\mathcal{A}(I)$ commute.

The lattice symbol $\lor$ will denote ‘the von Neumann algebra generated by’.

**E. Existence of the vacuum.** There exists a unit vector $\Omega$ (vacuum vector) which is $U(G)$-invariant and cyclic for $\forall I \in \mathcal{I} \mathcal{A}(I)$.

**F. Uniqueness of the vacuum (or irreducibility).** The only $U(G)$-invariant vectors are the scalar multiples of $\Omega$.

Assume $\mathcal{A}$ is as defined in above. A covariant representation $\pi$ of $\mathcal{A}$ is a family of representations $\pi_I$ of the von Neumann algebras $\mathcal{A}(I), I \in \mathcal{I}$, on a Hilbert space $\mathcal{H}_\pi$ and a unitary representation $U_\pi$ of the covering group $G$ of $PSL(2, \mathbb{R})$, with
positive energy, i.e. the generator of the rotation unitary subgroup has positive generator, such that the following properties hold:

\[
I \supset \bar{I} \Rightarrow \pi_{\bar{I}} = \pi_I \quad \text{(isotony)}
\]

\[
\text{ad}U_\pi(g) \cdot \pi_I = \pi_{gI} \cdot \text{ad}U(g) \quad \text{(covariance)}.
\]

A unitary equivalence class of representations of \( \mathcal{A} \) is called superselection sector.

The composition of two superselection sectors are known as Connes’s fusion \([W]\). The composition is manifestly unitary and associative, and this is one of the most important virtues of the above formulation. The main question is to study all superselection sectors of \( \mathcal{A} \) and their compositions.

Given two irreducible conformal precosheaves \( \mathcal{A}_1, \mathcal{A}_2 \) on Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) with vacuum vectors \( \Omega_1 \) and \( \Omega_2 \) respectively. One can define naturally that \( \mathcal{A}_1 \) is isomorphic to \( \mathcal{A}_2 \) if there is a unitary map \( U : \mathcal{H}_1 \to \mathcal{H}_2 \) such that:

\[
U^* \mathcal{A}_2(I)U = \mathcal{A}_1(I), \forall I \in \mathcal{I}; U\Omega_1 = \Omega_2.
\]

Note that by \([GL2]\), \( U \) as defined above intertwines the representation of the conformal group \( G \).

We have the following (cf. Prop. 1.1 of \([GL1]\)):

**2.1 Proposition.** Let \( \mathcal{A} \) be an irreducible conformal precosheaf. The following hold:

(a) **Reeh-Schlieder theorem:** \( \Omega \) is cyclic and separating for each von Neumann algebra \( \mathcal{A}(I), I \in \mathcal{I} \).

(b) **Bisognano-Wichmann property:** \( U \) extends to an (anti-)unitary representation of \( G \times_{\sigma_r} \mathbb{Z}_2 \) such that, for any \( I \in \mathcal{I} \),

\[
U(\Lambda_I(2\pi t)) = \Delta_I^{it}
\]

\[
U(r_I) = J_I
\]

where \( \Delta_I, J_I \) are the modular operator and the modular conjugation associated with \( (\mathcal{A}(I), \Omega) \). For each \( g \in G \times_{\sigma_r} \mathbb{Z}_2 \)

\[
U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI).
\]

(c) **Additivity:** if a family of intervals \( I_i \) covers the interval \( I \), then

\[
\mathcal{A}(I) \subset \vee_i \mathcal{A}(I_i).
\]

(d) **Haag duality:** \( \mathcal{A}(I)' = \mathcal{A}(I') \).
Let us give some examples of conformal precosheaves.

Let $G$ be a compact Lie group. Denote by $LG$ the group of smooth maps $f : S^1 \mapsto G$ under pointwise multiplication. The diffeomorphism group of the circle $\text{Diff} S^1$ is naturally a subgroup of Aut($LG$) with the action given by reparametrization. In particular $G$ acts on $LG$. We will be interested in the projective unitary representations (cf. Chap. 9 of [PS]) $\pi$ of $LG$ that are both irreducible and have positive energy. This implies that $\pi$ should extend to $LG \ltimes \text{Rot}$ so that the generator of the rotation group Rot is positive. It follows from Chap. 9 of [PS] that for a fixed level there are only finite number of such irreducible projective representations.

Now let $G$ be a connected compact Lie group and let $H \subset G$ be a Lie subgroup. Let $\pi^i$ be an irreducible representation of $LG$ with positive energy at level $k^1$ on Hilbert space $H^i$. Suppose when restricting to $LH$, $H^i$ decomposes as:

$$H^i = \sum_{\alpha} H_{i,\alpha} \otimes H_{\alpha},$$

and $\pi_{\alpha}$ are irreducible representations of $LH$ on Hilbert space $H_{\alpha}$. The set of $(i, \alpha)$ which appears in the above decompositions will be denoted by $\text{exp}$.

We shall use $\pi^1$ (resp. $\pi_1$) to denote the vacuum representation of $LG$ (resp. $LH$) on $H^1$ (resp. $H_1$). Let $\Omega$ (resp. $\Omega_0$) be the vacuum vector in $\pi^1$ (resp. $\pi_1$) and assume

$$\Omega = \Omega_{0,0} \otimes \Omega_0$$

with $\Omega_{0,0} \in H_{1,1}$.

We shall assume that $H \subset G$ is not a conformal inclusion (cf. [KW]) to avoid triviality.

For each interval $I \subset S^1$, we define:

$$A(I) := P \pi^1(L_I H)^{\prime} \cap \pi^1(L_I G)^{\prime\prime} P,$$

where $P$ is the projection from $H^1$ to a closed subspace spanned by

$$\bigvee_{J \in I} \pi^1(L_J H)^{\prime} \cap \pi^1(L_J G) \Omega.$$

Here $\pi^1(L_I G)^{\prime\prime}$ denotes the von Neumann algebra generated by

$$\pi^1(a), a \in LG, \text{Supp} a \subset I.$$

It follows from [X1] that $A(I)$ is an irreducible conformal precosheaf on the closed space. We define this to be irreducible conformal precosheaf of coset ($H \subset G$) CFT and denote it by $A_{G/H}$. Note the similarity of this definition to the vertex operator algebraic definition (cf. [FZ]). Note also that $\pi_{(i, \alpha)}$ above naturally gives rise to the covariant representations of $A_{G/H}$. $A_{G/H}$ corresponds to coset construction of CFT.

For the inclusion $H_1 \subset G_1$ at the end of §1, we will also denote $A_{G_1/H_1}$ by $A(G(m, n, k))$.

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1When $G$ is the direct product of simple groups, $k$ is a multi-index, i.e., $k = (k_1, ..., k_n)$, where $k_i \in \mathbb{N}$ corresponding to the level of the $i$-th simple group. The level of $LH$ is determined by the Dynkin indices of $H \subset G$. To save some writing we write the coset simply as $H \subset G_k$ or $H \subset G$ when the levels are clear from the context.
§2.2 Some results from [X1]. We recall some results from [X1] which will be used in the following. We refer the reader to [X1] for more details.

Let $M$ be a properly infinite factor and $\text{End}(M)$ the semigroup of unit preserving endomorphisms of $M$. In this paper $M$ will always be a type $III_1$ factor. Let $\text{Sect}(M)$ denote the quotient of $\text{End}(M)$ modulo unitary equivalence in $M$. It follows from [L3] and [L4] that $\text{Sect}(M)$ is endowed with a natural involution $\theta \mapsto \bar{\theta}$, and $\text{Sect}(M)$ is a semiring: i.e., there are two operations $+, \times$ on $\text{Sect}(M)$ which verifies the usual axioms. The multiplication of sectors is simply the composition of sectors. Hence if $\theta_1, \theta_2$ are two sectors, we shall write $\theta_1 \times \theta_2$ as $\theta_1 \theta_2$. In [X4], the image of $\theta \in \text{End}(M)$ in $\text{Sect}(M)$ is denoted by $[\theta]$. However, since we will be mainly concerned with the ring structure of sectors in this paper, we will denote $[\theta]$ simply by $\theta$ if no confusion arises.

Assume $\theta \in \text{End}(M)$, and there exists a normal faithful conditional expectation $\epsilon : M \to \theta(M)$. We define a number $d_\epsilon$ (possibly $\infty$) by:

$$d_\epsilon^{-2} := \text{Max}\{\lambda \in [0, +\infty)|\epsilon(m_+ - m_+) \geq \lambda m_+, \forall m_+ \in M_+\}$$

(cf. [PP]).

If $d_\epsilon < \infty$ for some $\epsilon$, we say $\theta$ has finite index or statistical dimension. In this case we define

$$d_\theta = \text{Min}_\epsilon\{d_\epsilon|d_\epsilon < \infty\}.$$ 

$d_\theta$ is called the statistical dimension of $\theta$. $d_\theta^2$ is called the minimal index of $\theta$. In fact in this case there exists a unique $\epsilon_\theta$ such that $d_{\epsilon_\theta} = d_\theta$. $\epsilon_\theta$ is called the minimal conditional expectation. It is clear from the definition that the statistical dimension of $\theta$ depends only on the unitary equivalence classes of $\theta$. When $N \subset M$ with $N \simeq M$, we choose $\theta \in \text{End}(M)$ such that $\theta(M) = N$. The statistical dimension (resp. minimal index) of the inclusion $N \subset M$ is defined to be the statistical dimension (resp. minimal index) of $\theta$.

Let $\theta_1, \theta_2 \in \text{Sect}(M)$. By Th. 5.5 of [L3], $d_{\theta_1+\theta_2} = d_{\theta_1} + d_{\theta_2}$, and by Cor. 2.2 of [L5], $d_{\theta_1\theta_2} = d_{\theta_1}d_{\theta_2}$. These two properties are usually referred to as the additivity and multiplicativity of statistical dimensions. Also note by Prop. 4.12 of [L4] $d_\theta = d_{\bar{\theta}}$. If a sector does not have finite statistical dimension in any of the above three equations, then the equation is understood as the statement that both sides of the equation are $\infty$.

Assume $\lambda, \mu$, and $\nu \in \text{End}(M)$ have finite statistical dimensions. Let $\text{Hom}(\lambda, \mu)$ denote the space of intertwiners from $\lambda$ to $\mu$, i.e. $a \in \text{Hom}(\lambda, \mu)$ iff $a\lambda(p) = \mu(p)a$ for any $p \in M$. $\text{Hom}(\lambda, \mu)$ is a finite dimensional vector space and we use $\langle \lambda, \mu \rangle$ to denote the dimension of this space. Note that $\langle \lambda, \mu \rangle$ depends only on $[\lambda]$ and $[\mu]$. Moreover we have $\langle \nu\lambda, \mu \rangle = \langle \lambda, \nu\bar{\mu} \rangle$, $\langle \nu\lambda, \mu \rangle = \langle \nu, \mu\bar{\lambda} \rangle$ which follows from Frobenius duality (See [L2]). We will also use the following notation: if $\mu$ is a subsector of $\lambda$, we will write as $\mu < \lambda$ or $\lambda \triangleright \mu$. A sector is said to be irreducible if it has only one subsector.

Let $\theta_i, i = 1, \ldots, n$ be a set of irreducible sectors with finite index. The ring generated by $\theta_i, i = 1, \ldots, n$ under compositions is defined to be a vector space
(possibly infinite dimensional) over \( \mathbb{C} \) with a basis \( \{ \xi_j, j \geq 1 \} \), such that \( \xi_j \) are irreducible sectors, \( \xi_j \neq \xi_{j'} \) if \( j \neq j' \), and the set \( \{ \xi_j, j \geq 1 \} \) is a list of all irreducible sectors which appear as subsectors of finite products of \( \theta_i, i = 1, \ldots, n \). The ring multiplication on the vector space is obtained naturally from that of \( \text{Sect}(M) \).

Let \( M(J), J \in \mathcal{I} \) be an irreducible conformal precosheaf on Hilbert space \( \mathcal{H}^1 \). Suppose \( N(J), J \in \mathcal{I} \) is an irreducible conformal precosheaf and \( \pi^i \) is a covariant representation of \( N(J) \) on \( \mathcal{H}^1 \) such that \( \pi^1(N(J)) \subset M(J) \) is a directed standard net as defined in Definition 3.1 of [LR] for any directed set of intervals. Fix an interval \( I \) and denote by \( N := N(I), M := M(I) \). For any covariant representation \( \pi_\lambda \) (resp. \( \pi^i \)) of the irreducible conformal precosheaf \( N(J), J \in \mathcal{I} \) (resp. \( M(J), J \in \mathcal{I} \)), let \( \lambda \) (resp. \( i \)) be the corresponding endomorphism of \( N \) (resp. \( M \)) as defined in §2.1 of [GL1]. These endomorphisms are obtained by localization in §2.1 of [GL1] and will be referred to as localized endomorphisms for convenience. The corresponding sectors will be called localized sectors.

In this paper, if we use \( 1 \) to denote a sector or a covariant representation, it should be understood as the identity sector or vacuum representation.

We will use \( d_\lambda \) and \( d_i \) to denote the statistical dimensions of \( \lambda \) and \( i \) respectively. \( d_\lambda \) and \( d_i \) are also called the statistical dimensions of \( \pi_\lambda \) and \( \pi^i \) respectively, and they are independent of the choice of \( I \) (cf. Prop. 2.1 of [GL1]).

Let \( \pi^i \) be a covariant representation of \( M(J), J \in \mathcal{I} \) which decomposes as:

\[
\pi^i = \sum_\lambda b_{i\lambda} \pi_\lambda
\]

when restricted to \( N(J), J \in \mathcal{I} \), where the sum is finite and \( b_{i\lambda} \in \mathbb{N} \). Let \( \gamma_i := \sum_\lambda b_{i\lambda} \lambda \) be the corresponding sector of \( N \). It is shown (cf. (1) of Prop. 2.8 in [X1]) that there are sectors \( \rho, \sigma_i \in \text{Sect}(N) \) such that:

\[
\rho \sigma_i \bar{\rho} = \gamma_i.
\]

Notice that \( \sigma_i \) are in one-to-one correspondence with covariant representations \( \pi^i \), and in fact the map \( i \rightarrow \sigma_i \) is an isomorphism of the ring generated by \( i \) and the ring generated by \( \sigma_i \). The subfactor \( \bar{\rho}(N) \subset N \) is conjugate to \( \pi^1(N(I)) \subset M(I) \) (cf. (2) of Prop. 2.6 in [X4]).

Now we assume \( \pi^1(N(I)) \subset M(I) \) has finite index. Then for each localized sector \( \lambda \) of \( N \) there exists a sector denoted by \( a_\lambda \) of \( N \) such that the following theorem is true (cf. [X4]):

**Theorem 2.2.** (1) The map \( \lambda \rightarrow a_\lambda \) is a ring homomorphism;

(2) \( \rho a_\lambda = \lambda \rho, a_\lambda \bar{\rho} = \bar{\rho} a_\lambda, d_\lambda = d_{a_\lambda} \);

(3) \( \langle \rho a_\lambda, \rho a_\mu \rangle = \langle a_\lambda, a_\mu \rangle = \langle a_\lambda \bar{\rho}, a_\mu \bar{\rho} \rangle \);

(4) \( \langle \rho a_\lambda, \rho \sigma_i \rangle = \langle a_\lambda, \sigma_i \rangle = \langle a_\lambda \bar{\rho}, \sigma_i \bar{\rho} \rangle \);

(5) (3) (resp. (4)) remains valid if \( a_\lambda, a_\mu \) (resp. \( a_\lambda \)) is replaced by any of its subsectors;
(6) \( a_{\lambda} \sigma_i = \sigma_i a_{\lambda} \).

We will apply the results of Th. 2.2 to the case when \( N(I) = \mathcal{A}_{G/H}(I) \otimes \pi_1(L_1H)^\prime \) and \( M(I) = \pi^1(L_1G)^\prime \) under the assumption that \( H \subset G \) is cofinite, i.e., \( \pi^1(N(I)) \subset M(I) \) has finite index, where \( \mathcal{A}_{G/H}(I) \) is defined in §2.1 (cf. \([X1]\)). By Th. 2.2, for every localized endomorphisms \( \lambda \) of \( N(I) \) we have a map \( a : \lambda \rightarrow a_{\lambda} \) which verifies (1) to (6) in Th. 2.2.

Tensor Notation. Let \( \theta \in \text{End}(\mathcal{A}_{G/H}(I) \otimes \pi_1(L_1H)^\prime) \). We will denote \( \theta \) by \( \rho_1 \otimes \rho_2 \) if

\[
\theta(p \otimes 1) = \rho_1(p) \otimes 1, \forall p \in \mathcal{A}_{G/H}(I), \theta(1 \otimes p') = 1 \otimes \rho_2(p'), \forall p' \in \pi_1(L_1H)^\prime,
\]

where \( \rho_1 \in \text{End}(\mathcal{A}_{G/H}(I)), \rho_2 \in \text{End}(\pi_1(L_1H)^\prime) \).

Recall from §2.1 \( \pi_{i,\alpha} \) of \( \mathcal{A}_{G/H}(I) \) are obtained in the decompositions of \( \pi^1 \) of \( LG \) with respect to subgroup \( LH \), and we denote the set of such \( (i, \alpha) \) by \( \exp \). We will denote the sector corresponding to \( \pi_{(i, \alpha)} \) simply by \( (i, \alpha) \). Under the conditions that \( (i, \alpha), (j, \beta) \) have finite indices, we have that \( (i, \alpha) \) is an irreducible sector if and only if \( \pi_{i,\alpha} \) is an irreducible covariant representation, and \( (i, \alpha) \succ (j, \beta) \) if and only of \( \pi_{j,\beta} \) appears as a direct summand of \( \pi_{i,\alpha} \), and \( (i, \alpha) \) is equal to \( (j, \beta) \) as sectors if and only \( \pi_{i,\alpha} \) is unitarily equivalent to \( \pi_{j,\beta} \) (cf. \([GL1]\)).

Given \( (i, \alpha) \in \text{End}(\mathcal{A}_{G/H}(I)) \) as above, we define \( (i, \alpha) \otimes 1 \in \text{End}(N(I)) \) so that:

\[
(i, \alpha) \otimes 1(p \otimes p') = (i, \alpha)(p) \otimes p', \forall p \in \mathcal{A}_{G/H}(I), p' \in \pi_1(L_1H)^\prime.
\]

It is easy to see that \( (i, \alpha) \otimes 1 \) corresponds to the covariant representation \( \pi_{i,\alpha} \otimes \pi_1 \) of \( N(I) \). Note that this notation agrees with our tensor notation above. Also note that for any covariant representation \( \pi_x \) of \( \mathcal{A}_{G/H}(I) \), we can define a localized sector \( x \otimes 1 \) of \( N(I) \) in the same way as in the case when \( \pi_x = \pi_{i,\alpha} \).

Each covariant representation \( \pi^i \) of \( LG \) gives rise to an endomorphism \( \sigma_i \in \text{End}(N(I)) \) and

\[
\rho \sigma_i \varrho = \gamma_i = \sum_{\alpha} (i, \alpha) \otimes (\alpha)
\]

(2.1)

where the summation is over those \( \alpha \) such that \( (i, \alpha) \in \exp \). The following is Prop. 4.2 of \([X1]\):

**Proposition 2.3.** Assume \( H \subset G \) is cofinite. We have:

1. Let \( x, y \) be localized sectors of \( \mathcal{A}_{G/H}(I) \) with finite index. Then

\[
\langle x, y \rangle = \langle a_x \otimes 1, a_y \otimes 1 \rangle;
\]

2. If \( (i, \alpha) \in \exp \), then \( a_{(i, \alpha) \otimes 1} \prec a_{1 \otimes \alpha} \sigma_i \).
Denote by $d_{(i,\alpha)}$ the statistical dimension of $(i,\alpha)$. Then $d_{(i,\alpha)} \leq d_i d_\alpha$, where $d_i$ (resp. $d_\alpha$) is the statistical dimension of $i$ (resp. $\alpha$).

Let us denote by $S_{ij}$ (resp. $S_{\alpha\beta}$) the $S$ matrices of $LG$ (resp. $LH$) at level $k$ (resp. certain level of $LH$ determined by the inclusion $H \subset G_k$) as defined on P. 264 of [Kac]. Define

$$b(i,\alpha) = \sum_{(j,\beta)} S_{ij} \overline{S_{\alpha\beta}} \langle (j,\beta), (1,1) \rangle$$

(2.2)

Note the above summation is effectively over those $(j,\beta)$ such that $(j,\beta) \in \exp$. Note that by [KW], if $b(i,\alpha) > 0$, then $(i,\alpha) \in \exp$. The Kac-Wakimoto Conjecture (KWC) states that if $(i,\alpha) \in \exp$, then $b(i,\alpha) > 0$. Under certain conditions, a stronger result than KWC is obtained in [X3], and the results of [X3] apply to the coset $Gr(m,n,k)$. More precisely we have:

**Theorem 2.4.** (1): The coset $Gr(m,n,k)$ is cofinite (cf. [X1] or definition after Th. 2.2);

(2): There are only a finite number of irreducible representations of $A(G(m,n,k))$,

and each irreducible representation appears as a direct summand of some $(i,\alpha) \in \exp$;

(3): The statistical dimension $d_{(i,\alpha)}$ of the coset sector $(i,\alpha)$ is given by

$$d_{(i,\alpha)} = \frac{b(i,\alpha)}{b(1,1)}$$

where $b(i,\alpha)$ is defined in (2.2);

(4): The irreducible representations of $A(G(m,n,k))$ generate a unitary modular category as defined in [Tu].

**Proof.** (1) is proved at the end of §3.2 of [X2]. (2) and (3) follows from (1), Cor. 3.2 and Th. 3.4 of [X3]. We note that it is assumed in [X3] that all the groups involved are type $A$ groups so one can use the results of [W] and [X6]. But it is easy to show that these results hold for $U(1)$ (cf. P. 58 of [X5]) since all sectors involved are automorphisms, and in fact it is already implicitly contained in §4 of [X6]. Hence all results of [X3] apply to $U(1)$ too. (4) follows from Prop. 2.4 of [X3]. We note that (4) also follows from (1) and [L1].

\[ \square \]

---

2Our $(j,\beta)$ corresponds to $(M,\mu)$ on P.186 of [KW], and $\langle (j,\beta), (1,1) \rangle$ is equal to $\text{mult}_M(\mu,\nu)$ which appears in 2.5.4 of [KW] by (2.5). So our formula (2.2) is identical to 2.5.4 of [KW].

3In fact using (1) and [L1] one can obtain a stronger result, i.e., $A(G(m,n,k))$ is completely rational (cf. [KLM]).
§2.3 Vacuum Pairs.

Let us recall the definition of vacuum pairs according to P. 236 of [K] (or [KW]) in our notations. As in §2.1 let $\pi^i$ be an irreducible representation of $LG$ with positive energy at level $k$ on Hilbert space $H^i$. Suppose when restricting to $LH$, $H^i$ decomposes as:

$$H^i = \sum_\alpha H_{i,\alpha} \otimes H_\alpha,$$

and $\pi_\alpha$ are irreducible representations of $LH$ on Hilbert space $H_\alpha$. By [GKO], the generator $L_G(0)$ of the rotation group for $LG$ act on $H^i$ as

$$L_G(0) = L_{G/H}(0) \otimes id + 1 \otimes L_H(0)$$

(2.3)

The eigenvalues of $L_G(0)$ on $H^i$ are given by $h_i + m, m \in \mathbb{Z}_{\geq 0}$, where $h_i$ is the conformal dimension or trace anomaly defined in (1.4.1) of [KW]. Let $\Omega_{i,\alpha} \otimes \Omega_\alpha$ be a unit vector with weight $i' := i - r$ of $LG$ where $\Omega_\alpha$ is the highest weight vector of $LH$ on $H_\alpha$, and $r$ is a sum of positive roots of $LG$. By (3.2.6) of [KW],

$$L_G(0)(\Omega_{i,\alpha} \otimes \Omega_\alpha) = (h_i + m)\Omega_{i,\alpha} \otimes \Omega_\alpha$$

where $m$ is a nonnegative integer determined by $i'$. But we also have

$$L_G(0)(\Omega_{i,\alpha} \otimes \Omega_\alpha) = L_{G/H}(0)(\Omega_{i,\alpha}) \otimes \Omega_\alpha + \Omega_{i,\alpha} \otimes L_H(0)(\Omega_\alpha)$$

$$= L_{G/H}(0)(\Omega_{i,\alpha}) \otimes \Omega_\alpha + h_\alpha \Omega_{i,\alpha} \otimes \Omega_\alpha,$$

and since the eigenvalues of $L_{G/H}(0)$ are non-negative (cf. §3 of [KW]), we must have

$$h_i + m \geq h_\alpha$$

According to [K], we will say that $\{i, \alpha\}$ is a Vacuum Pair if

$$h_i + m = h_\alpha$$

(2.4)

As noted above $m$ is determined by $i' := i - r$ and $\alpha$ is obtained by restriction from weight $i'$ of $LG$ to $LH$. Note that since there are only finitely many $i, \alpha$, (2.4) has only a finite number of solutions. We will denote the finite set of of VPs simply as $VPS$. However it is in general nontrival to determine $VPS$.

From the equations before (2.4) we must have that (2.4) hold iff

$$L_{G/H}(0)(\Omega_{i,\alpha}) = 0,$$

i.e., $\Omega_{i,\alpha}$ is a vacuum vector, and it follows immediately that $H_{(1,1)}$ is a direct summand of $H_{(i,\alpha)}$. Hence if the sector $\{i, \alpha\}$ has finite index, then $\{i, \alpha\}$ is a VP iff $(i, \alpha) \in \text{exp}$ and

$$\langle (i, \alpha), (1,1) \rangle > 0$$

(2.5)
One can see the importance of such VPs in calculating (2.2).

For the rest of this section, we will assume that all sectors or representations considered have finite indices.

Assume that \( H_1 \subset H_2 \subset G \). For simplicity we will use \( \pi_x, \pi_y, \pi_z \) to denote the irreducible representations of \( LH_1, LH_2 \) and \( LG \) respectively, and \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) to denote the conformal precosheaves of cosets \( H_1 \subset H_2, H_2 \subset G, H_1 \subset G \) respectively. Note we have natural inclusions \( \mathcal{A}(I) \otimes \mathcal{B}(I) \subset \mathcal{C}(I) \), corresponding to the natural inclusions

\[
(\pi(L_1 H_1)' \cap \pi(L_1 H_2)'') \otimes (\pi(L_1 H_2)' \cap \pi(L_1 G)'') \subset \pi(L_1 H_1)' \cap \pi(L_1 G)'',
\]

where \( I \) is a proper open interval of \( S^1 \). From the decompositions:

\[
\pi_z \simeq \sum_y \pi(z,y) \otimes \pi_y \simeq \sum_{y,x} \pi(z,y) \otimes \pi(y,x) \otimes \pi_x \simeq \sum_x \pi(z,x) \otimes \pi_x
\]

we conclude that

\[
\pi(z,x) \simeq \sum_y \pi(z,y) \otimes \pi(y,x)
\]

which is understood as the decomposition of representation \( \pi(z,x) \) of \( \mathcal{C} \) when restricted to \( \mathcal{A} \otimes \mathcal{B} \subset \mathcal{C} \). The following lemma follows immediately from (2.3) and (2.4):

**Lemma 2.5.** \( \{z, x\} \) is a VP for \( H_1 \subset G \) iff there exists a \( y \) such that \( \{z, y\} \) and \( \{y, x\} \) are VPs for \( H_1 \subset H_2 \) and \( H_2 \subset G \) respectively.

One can usually find VPs by examining the symmetry of Dynkin diagrams (cf. 2.7.12 of [KW]). This motivates the following:

**Definition 2.6 (SVP).** \( \{i, \alpha\} \) is called a simple vacuum pair if \( d_\alpha = 1 \).

We will denote the set of all SVPs as \( \text{SVPS} \).

**Lemma 2.7.** \( \text{SVPS} \) is an abelian group under the compositions of sectors.

**Proof.** Let \( \{i, \alpha\} \in \text{SVPS} \). By (2.5) \( \langle (i, \alpha), (1, 1) \rangle > 0 \). A useful property which follows from Th. 2.2 and (2.1) is

\[
\langle \sigma_i, a_{x \otimes \alpha} \rangle = \langle (i, \alpha), x \rangle.
\]

Set \( x = (1, 1) \) we get \( \langle \sigma_i, a_{1 \otimes \alpha} \rangle > 0 \), and so \( \sigma_i \prec a_{1 \otimes \alpha} \) since \( \sigma_i \) is irreducible. Since \( d_{a_1 \otimes \alpha} = d_\alpha = 1 \), \( a_{1 \otimes \alpha} \) must be irreducible and \( \sigma_i = a_{1 \otimes \alpha} \). In particular \( d_i = 1 \). So

\[
a_{(i, \alpha) \otimes 1} = a_{(1, 1) \otimes 1} = \sigma_i a_{1 \otimes \alpha}.
\]

It follows from

\[
a_{(i, \alpha) \otimes 1} = \sigma_i a_{1 \otimes \alpha} = a_{(1, 1) \otimes 1}
\]
that \{\bar{i}, \alpha\} \in SVPS. Now let \{i, \alpha\}, \{j, \beta\} \in SVPS. We must have \(ij = k, \alpha \beta = \delta\) for some \{\beta, \delta\} since all sectors have statistical dimension 1. To show that SVPS is an abelian group, we just have to show that \{k, \delta\} \in VPS. Note that

\[ a(k, \delta)^{\otimes 1} = \sigma_k a_1 \otimes \bar{\delta} = \sigma_i \sigma_j a_1 \otimes \alpha a_\beta = a(1, 1)^{\otimes 1} \]

and this shows that \{k, \delta\} \in VPS by (1) of Prop. 2.3.

\[ \square \]

§3. \(A(G(m, n, k)) \simeq A(G(k, n, m))\)

We will first recall some facts from [PS]. The reader is referred to [PS] for more details.

Let \(H\) denote the Hilbert space \(L^2(S^1; \mathbb{C})\) of square-summable \(\mathbb{C}\)-valued functions on the circle. The group \(LU(N)\) of smooth maps \(S^1 \to U(N)\) acts on \(H\) multiplication operators.

Let us decompose \(H = H_+ \oplus H_-\), where

\[ H_+ = \{ \text{functions whose negative Fourier coefficients vanish} \} \]

We denote by \(P\) the projection from \(H\) onto \(H_+\).

Denote by \(U_{res}(H)\) the group consisting of unitary operator \(A\) on \(H\) such that the commutator \([P, A]\) is a Hilbert-Schmidt operator. There exists a central extension \(U_{res}^\sim\) of \(U_{res}(H)\) as defined in §6.6 of [PS]. The central extension \(L^U(N)\) of \(LU(N)\) induced by \(U_{res}^\sim\) is called the basic extension.

The basic representation \(\pi\) of \(LU(N)\) is the representation on Fermionic Fock space \(F(\mathbb{C}^N) := \Lambda(PH) \otimes \Lambda((1 - p)H)^*\) as defined in §10.6 of [PS]. Note that if \(\mathbb{C}^N = \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2}\), then \(F(\mathbb{C}^N)\) is canonically isomorphic to \(F(\mathbb{C}^{N_1}) \otimes F(\mathbb{C}^{N_2})\).

Let \(I = \bigcup_{i=1}^{\alpha} I_i\) be a proper subset of \(S^1\), where \(I_i\) are intervals of \(S^1\). Denote by \(M(I, \mathbb{C}^N)\) the von Neumann algebra generated by \(c(\xi)'s\), with \(\xi \in L^2(I, \mathbb{C}^N)\). Here \(c(\xi) = a(\xi) + a(\xi)^*\) and \(a(\xi)\) is the creation operator defined as in Chapter 1 of [W2]. Let \(K : F(\mathbb{C}^N) \to F(\mathbb{C}^N)\) be the Klein transformation given by multiplication by 1 on even forms and by \(i\) on odd forms. We will denote the set of even forms as \(F(\mathbb{C}^N)^{ev}\). Note that the vacuum vector \(\Omega \in F(\mathbb{C}^N)^{ev}\). An operator on \(F(\mathbb{C}^N)\) is called even if it commutes with \(K\).

\(F(\mathbb{C}^N)\) supports a projective representation of \(LSpin(2N)\) at level 1 (also denoted by \(\pi\) ), and in fact \(F(\mathbb{C}^N)^{ev}\) is the vacuum representation of \(LSpin(2N)\) (cf. P. 246-7 of [PS]).

**Proposition 3.1.** (1): The vacuum vector \(\Omega\) is cyclic and separating for \(M(I, \mathbb{C}^N)\) and \(M(I, \mathbb{C}^N)' = K^{-1} M(I', \mathbb{C}^N) K;\)

(2): \(M(I, \mathbb{C}^N) = \pi(L_I U(N))''\);

(3): \(\pi^{ev}(L_I U(N))'' = \pi(L_{I Spin(2N)})''\) where \(\pi^{ev}(L_I U(N))''\) denotes the even elements of \(\pi(L_I U(N))''\);
(4): If $\mathbb{C}^N = \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2}$ and let $\pi(LU(N_1))$ (resp. $\pi(LU(N_2))$) be the representation induced from the map $U(N_1) \to U(N_1) \oplus id_{N_2}$ (resp. $U(N_2) \to id_{N_1} \oplus U(N_2)$), then

$$\pi(L_1U(N))'' = \pi(L_1U(N_1))'' \vee \pi(L_1U(N_2))''.$$ 

**Proof.** (1) is proved in §15 of [W]. (2) is implied in §15 of [W], also cf. Lemma 3.1 of [X6]. To prove (3), note that by (2)

$$\pi(L_1Spin(2N))'' \subset M^{ev}(I, \mathbb{C}^N) = \pi^{ev}(L_1U(N))''.$$

Note that both sides are invariant under the action of the modular group (cf. [W]), by [T], it is sufficient to show that

$$\pi(L_1Spin(2N))'' \Omega \supset M^{ev}(I, \mathbb{C}^N)\Omega.$$

By Reeh-Schlieder theorem

$$\pi(L_1Spin(2N))''\Omega = \pi(LSpin(2N))''\Omega,$$

and

$$\pi(LSpin(2N))''\Omega = F(\mathbb{C}^N)^{ev}$$

by P. 246-7 of [PS]. Since

$$M^{ev}(I, \mathbb{C}^N)\Omega \subset F(\mathbb{C}^N)^{ev},$$

the proof of (3) is complete.

(4) follows immediately from (2).

□

We will consider $N = mn + mk + nk$ and

$$\mathbb{C}^N = \mathbb{C}^m \otimes \mathbb{C}^n \oplus \mathbb{C}^m \otimes \mathbb{C}^k \oplus \mathbb{C}^m \otimes \mathbb{C}^k$$

in the following.

Denote by $\pi$ the representation of $LG_1, LG_2$ on $F(\mathbb{C}^N)$ induced by the natural inclusions of $G_1 \subset U(N), G_2 \subset U(N)$. Note that the levels of representations match.

The $U(1)$ factor of $H_1$ is mapped into $U(N)$ as

$$a \to a^{m+n}id_n \otimes id_m \oplus a^n id_m \otimes id_k \oplus a^{-m}id_n \otimes id_k.$$ 

This gives a map $P_1 : LU(1) \to LU(N)$. Denote by $P_2 : LU(1) \to LU(N)$ the map induced by

$$a \to a^{m+n}id_m \otimes (id_n \oplus id_k) \oplus id_n \otimes id_k.$$ 

The representations $\pi(P_1(LU(1)))$ and $\pi(P_2(LU(1)))$ have levels $mn(m+n)(m+n+k)$ and $(m+n)^2m(n+k)$ respectively. We will denote them by

$$\pi(LU(1)_{mn(m+n)(m+n+k)})$$

and $\pi(LU(1)_{(m+n)^2m(n+k)})$ respectively. We first state a simple result about representations of $LU(1)$:
Lemma 3.2. If \( \pi \) is a positive energy representation of \( LU(1) \), then it is strongly additive (cf. [L1]), i.e., if \( I_1, I_2 \) are intervals obtained by removing an interior point of interval \( I \), the
\[
\pi(L_1U(1))'' = \pi(L_{I_1}U(1))'' \lor \pi(L_{I_2}U(2))''
\]

Proof: The representation of the connected component \( LU(1)^0 \) of \( LU(1) \) is strongly additive by [TL]. Note that \( L_1U(1) \) is generated by \( L_1U(1)^0 \) and any loop of winding number 1 with support on \( I \), and we can choose this loop to have support on \( I_1 \). This shows that
\[
\pi(L_1U(1))'' \subset \pi(L_{I_1}U(1))'' \lor \pi(L_{I_2}U(2))''
\]
and completes the proof.

\[\square\]

Lemma 3.3.
\[
\pi(L_1U(1)^{mn(m+n)(m+n+k)})' \cap \pi(L_1G_I)' = \pi(L_1U(1)^{m+n})^{m(n+k)}' \cap \pi(L_1G_I)''.
\]

Proof: It is sufficient to show that
\[
\pi(L_1U(1)^{mn(m+n)(m+n+k)})'' \lor \pi(L_1G_I)' = \pi(L_1U(1)^{m+n})^{m(n+k)}'' \lor \pi(L_1G_I)'.
\]
Note that for any \( \beta \in L_1U(1) \), \( P_1(\beta) = P_2(\beta)P_3(\beta) \), where \( P_3(\beta): P_2(\beta)^{-1}P_1(\beta) \in LU(k)^{m+n} \). Also \( \pi(LU(k)^{m+n})'' \subset \pi(L_1G_I)' \). Hence \( \pi(P_1(\beta)) \in \pi(P_1(\beta)) \lor \pi(L_1G_I)' \). This shows \( \subset \) in the lemma. The other inclusion is similar.

\[\square\]

Lemma 3.4. (1).
\[
\pi(L_1SU(m)^{k+n})' \cap K\pi(L_1U(mn + mk))''K^{-1} = K\pi(L_1U(k + n)m)''K^{-1}.
\]
(2).
\[
\pi(L_1SU(m)^{k+n})' \cap \pi(L_1U(1)^{m+n})^{m(n+k)}' \cap K\pi(L_1U(mn + mk))''K^{-1} = \pi(L_1SU(k + n)m)''K^{-1}.
\]

Proof: Ad (1): Since elements of \( \pi(L_1SU(m)^{k+n})'' \) commute with \( K \), it is sufficient to show that:
\[
\pi(L_1SU(m)^{k+n})' \cap \pi(L_1U(mn + mk))'' = \pi(L_1U(k + n)m)''.
\]
By local equivalence (cf. Th. B of [W]), it is sufficient to show the above equality for the restriction \( \pi_1 \) of \( \pi \) to \( \mathcal{F}(C^{mn+mk}) \). Note that
\[
\pi_1(L_1SU(m)^{k+n})' \cap \pi_1(L_1U(mn + mk))'' \supset \pi_1(L_1U(n + k)m)''
\]
and both sides are invariant under the action of modular group. By [T], it is sufficient to show that

\[ \pi_1(LSU(m)_{k+n})' \cap \pi_1(LU(mn + mk))'' \Omega \subset \pi(LU(n + k)_m)'' \Omega \]

By the decomposition of \( F(\mathbb{C}^{mn + mk}) \) with respect to \( L_SU(m)_{k+n} \times LU(n + k)_m \) given in Prop. 10.6.4 of [PS], \( \Omega = \Omega_1 \otimes \Omega_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2 \), where \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are vacuum representations of \( LSU(m)_{k+n} \) and \( LU(n + k)_m \), and \( \Omega_1, \Omega_2 \) are vacuum vectors. By Reeh-Schlieder theorem, \( \pi(LU(n + k)_m)'' \Omega = \Omega_1 \otimes \mathcal{H}_2 \). Now let \( x \in \pi_1(LSU(m)_{k+n})' \cap \pi_1(LU(mn + mk))'' \), then \( x \in \pi_1(LSU(m)_{k+n})' \vee \pi_1(LSU(m)_{k+n})' = \pi_1(LSU(m)_{k+n})' \) by strong additivity (cf. [TL]), and so \( x \Omega \in \Omega_1 \otimes \mathcal{H}_2 \), and the proof is complete.

Ad (2): Note that the right hand side is contained in the left hand side. By (1) it is sufficient to show that

\[ \pi(LU(1)_{(m+n)^2m(n+k)})' \cap K \pi(LU(k + n)_m)'' K^{-1} \subset \pi(LSU(k + n)_m)'' \]

Note that both sides are invariant under the action of modular group. Let \( a \) be an element of the left hand side. Then \( a \in \pi(LSU(m)_{k+n})' \cap \pi(LU(1)_{(m+n)^2m(n+k)})' \) by strong additivity of \( LSU(m) \) (cf. [TL]) and \( LU(1) \) (cf. Lemma 3.2). Now the proof is similar to that of (1). By using Reeh-Schlieder theorem and decompositions given in Prop. 10.6.2 and 10.6.4 of [PS], we have that \( a \Omega \subset \pi(LSU(k + n)_m)'' \Omega \).

By [T], this shows (2).

\[ \square \]

**Lemma 3.5.**

\[ \pi(LH_1)' \cap \pi(LG_1)'' \subset \pi(LG_2)''. \]

**Proof:** By Lemma 3.3

\[ \pi(LH_1)' \cap \pi(LG_1)'' \subset \pi(LSU(m)'_{n+k} \cap \pi(LU(1)_{(m+n)^2m(n+k)})' \cap \pi(LG_1)''. \]

Note that by (4) of Prop. 3.1

\[ \pi(LG_1)'' \subset K \pi(LU(mn + mk + nk))'' K^{-1} = K \pi(LU(mn + mk))'' K^{-1} \vee K \pi(LU(nk))'' K^{-1}. \]

By Lemma 3.4

\[ \pi(LH_1)' \cap \pi(LG_1)'' \subset K \pi(LSU(m)'_{n+k} \cap \pi(LU(1)_{(m+n)^2m(n+k)})' \cap K \pi(LU(mn + mk))'' K^{-1} \vee K \pi(LU(nk))'' K^{-1} \]

\[ = \pi(LSU(n + k)_m)'' \cap K \pi(LU(nk))'' K^{-1}. \]
Note that by (3) of Prop. 3.1 \( \pi(L_1 \text{Spin}(2nk))'' \) are the even elements of 
\[ \pi(L_1 U(nk))'', \]
and the elements of \( \pi(L_1 H_1)' \cap \pi(L_1 G_1)'' \) and \( \pi(L_1 SU(n+k)''_m \) are even, it follows that
\[ \pi(L_1 H_1)' \cap \pi(L_1 G_1)'' \subset \pi(L_1 SU(n+k)''_m \lor \pi(L_1 \text{Spin}(2nk))'' = \pi(L_1 G_2)''. \]

\[ \square \]

**Lemma 3.6.**
\[ \pi(L_1 H_1)' \cap \pi(L_1 G_1)'' \subset \pi(L_1 H_2)' . \]

**Proof:** By definitions it is enough to show that
\[ \pi(L_1 H_1)' \cap \pi(L_1 G_1)'' \subset \pi(L_1 U(1)_{kn(k+n)(k+n+m)})' \]
or equivalently
\[ \pi(L_1 H_1)'' \lor \pi(L_1 G_1)' \supset \pi(L_1 U(1)_{kn(k+n)(k+n+m)})'' . \]

Note that \( \pi(L_1 U(1)_{kn(k+n)(k+n+m)}) \) is actually \( \pi(P(\alpha)) \), where \( P : LU(1) \rightarrow LU(N) \) is given by
\[ \alpha \rightarrow \alpha^{k+n}id_n \otimes id_k \oplus \alpha^n id_k \otimes id_m \oplus \alpha^{-k}id_n \otimes id_m . \]

So
\[ P(\alpha) = [\alpha^{k+n}(id_n \oplus id_m) \otimes id_k \oplus id_n \times id_m] \]
\[ \times [\alpha^{-k}id_m \otimes (id_k \oplus id_n) \oplus id_k \times id_n] \]

Denote by
\[ \alpha_1 : = \alpha^{k+n}(id_n \oplus id_m) \otimes id_k \oplus id_n \otimes id_m, \]
\[ \alpha_2 : = \alpha^{-k}id_m \otimes (id_k \oplus id_n) \oplus id_k \times id_n . \]

Then \( \pi(P(\alpha)) \) is equal to \( \pi(\alpha_1)\pi(\alpha_2) \) up to a scalar. Note that
\[ \pi(\alpha_1) \in \pi(LU(k)_{n+m}'') \]
and \( \pi(\alpha_2) \in \pi(LU(m)_{k+n}'') \), so
\[ \pi(L_1 U(1)_{kn(k+n)(k+n+m)})'' \subset \pi(LU(k)_{n+m}'') \lor \pi(LU(m)_{k+n}'') . \]

By definition, \( \pi(L_1 G_1)' \supset \pi(LU(k)_{n+m}'') \), and by Lemma 3.5,
\[ \pi(L_1 H_1)'' \lor \pi(L_1 G_1)' \supset \pi(L_1 G_2)' \supset \pi(LU(m)_{k+n}'') . \]

It follows that
\[ \pi(L_1 H_1)'' \lor \pi(L_1 G_1)' \supset \pi(L_1 U(1)_{kn(k+n)(k+n+m)})'' . \]

\[ \square \]
Theorem 3.7. The conformal precosheaves \( A(G(m, n, k)) \) and \( A(G(k, n, m)) \) are isomorphic.

Proof: By Lemmas 3.5-3.6, for every interval \( I \),
\[
\pi(L_I H_1)' \cap \pi(L_I G_1)'' \subset \pi(L_I H_2)' \cap \pi(L_I G_2)''.
\]
Exchanging \( m \) and \( k \) in Lemmas 3.5-3.6, we get
\[
\pi(L_I H_1)' \cap \pi(L_I G_1)'' \supset \pi(L_I H_2)' \cap \pi(L_I G_2)'',
\]
and so
\[
\pi(L_I H_1)' \cap \pi(L_I G_1)'' = \pi(L_I H_2)' \cap \pi(L_I G_2)''.
\]
Let \( \mathcal{H} \) be the closure of \( \pi(L_I H_1)' \cap \pi(L_I G_1)'' \Omega \), and let \( P_0 \) be the projection onto \( \mathcal{H} \). Let \( A \) be the conformal precosheaf given by
\[
A := \pi(L_I H_1)' \cap \pi(L_I G_1)'' P_0 = \pi(L_I H_2)' \cap \pi(L_I G_2)'' P_0
\]
on \( \mathcal{H} \). It follows by definitions that \( A(G(m, n, k)) \) and \( A(G(k, n, m)) \) are both isomorphic to \( A \).
\( \square \)

Note that by Th. 2.4 \( A(G(m, n, k)) \) has only finitely number of irreducible representations, and they generate a unitary modular category. Denote this modular category by \( MC(G(m, n, k)) \). Th. 3.7 implies that

Corollary 3.8. There exists a one to one correspondence between the irreducible representations of \( A(G(m, n, k)) \) and \( A(G(k, n, m)) \) such that the three manifold invariants (including colored ones, cf. [Tu]) calculated from \( MC(G(m, n, k)) \) are identical to that from \( MC(G(k, n, m)) \).

In particular the corollary shows the existence of identifications between all chiral quantities of \( A(G(m, n, k)) \) and \( A(G(k, n, m)) \). By using Th. 4.7 and [X5], one can write down a formula for the closed three manifold invariants from \( MC(G(m, n, k)) \). We will omit the formula, but we note that the symmetry under the exchange of \( m \) and \( k \) agrees with §3 of [X5].

§4. Representations of \( A(G(m, n, k)) \)

By (2) of Th. 2.4, every irreducible representation of \( A(G(m, n, k)) \) occurs in \((i, \alpha)\) for some \((i, \alpha) \in \text{exp}\). So we need to determine \( \text{exp} \). It is also known that there may be field identifications, i.e., there may be \((j, \beta)\) with \( i \neq j \) or \( \beta \neq \alpha \) but \((j, \beta)\) is equivalent to \((i, \alpha)\) as representations. There are also issues of fixed point resolutions, i.e., as a representation \((i, \alpha)\) may not be irreducible, and we need to decompose \((i, \alpha)\) into irreducible pieces. To answer these questions, it turns out one needs to determine all VPs for \( A(G(m, n, k)) \). Let us first introduce some notations.
Note that the inclusion $H_1 \subset G_1$ is a composition of two inclusions as described at the end of §1:

$$H_1 \subset SU(m)_n \times SU(m)_k \times SU(n)_m \times SU(n)_k$$

$$\times U(1)_{mn(m+n)(m+n)} \times U(1)_{mn(m+n)(k)}$$

and

$$(SU(m)_n \times SU(n)_m \times U(1)_{mn(m+n)(m+n)}) \times (SU(m)_k \times SU(n)_k$$

$$\times U(1)_{mn(m+n)(k)}) \subset G_1 := \text{Spin}(2mn)_1 \times SU(m+n)_k.$$ 

We note that the inclusion

$$(SU(m)_n \times SU(n)_m \times U(1)_{mn(m+n)(m+n)}) \subset \text{Spin}(2mn)_1$$

is a conformal inclusion, which is in fact a composition of two conformal inclusions

$$SU(m)_n \times SU(n)_m \subset SU(mn)_1$$

and

$$SU(mn)_1 \times U(1)_{mn(m+n)(m+n)} \subset \text{Spin}(2mn)_1.$$ 

We will use $\pi_0, \lambda_0, \hat{\lambda}_1, \hat{\lambda}_2$ and $\hat{q}$ to denote the representations of

$$\text{Spin}(2mn)_1, SU(m+n)_k, SU(m)_{n+k}, SU(n)_{m+k}$$

and $U(1)_{mn(m+n)(m+n+k)}$ respectively. So the general coset labels $(i, \alpha)$ in §2.2 can be identified in the case of $H_1 \subset G_1$ as $i = \{(\pi_0, \lambda_0)\}$ and $\alpha = \{\hat{\lambda}_1, \hat{\lambda}_2, \hat{q}\}$.

We will use $\lambda_1, \lambda_2, \hat{\lambda}_1, \hat{\lambda}_2, \hat{q}$ and $q$ to denote the representations of

$$SU(m)_k, SU(n)_k, SU(m)_n, SU(n)_m, U(1)_{mn(m+n)(m+n)}$$

and $U(1)_{mnk(m+n)}$ respectively.

We use $\tau$ to denote the generator of some symmetries of the extended Dynkin diagram of the Kac-Moody algebra, and it is defined as follows:

Acting on an $SU(K)_M$ representation $\lambda$, $\tau$ rotates the extended Dynkin indices, i.e., $a_i(\tau(\lambda)) = a_{i+1}(\lambda)$, where $a_{i+K} = a_i$. Acting on the representations of $\text{Spin}(2L)_1$, $\tau$ exchanges the vacuum and vector representations, and exchanges the two spinor representations.

In accordance with the conventions of §2.1, if 1 is used to denote a representation of $SU(K)_M$ or $\text{Spin}(2L)_1$, it will always be the vacuum representation. We will however use 0 to label the vacuum representation of $U(1)_{2M}$.

Denote by $H_3 := SU(m)_k \times SU(n)_k \times U(1)_{mn(m+n)(k)}$ and $G_3 := SU(m+n)_k$. 

§4.1 Selection Rules. Let \((\lambda_0; \lambda_1, \lambda_2, q)\) be in the \(\exp\) of \(H_3 \subset G_3\). By looking at the actions of the centers of \(H_3, G_3\), we can get constraints on the conjugacy classes of the representations. These are known as selection rules. For a representation \(\lambda\) of \(SU(K)_M\), we denote by \(r_\lambda\) the number of boxes in the Young tableau corresponding to \(\lambda\).

First we have

\[
[e^{\frac{2\pi i}{m}} id_m \oplus id_n] \times [e^{\frac{2\pi i}{m(n+n)}} id_m \oplus e^{\frac{2\pi i}{m+n}} id_n] = e^{\frac{2\pi i}{m+n}} (id_m \oplus id_n).
\]

Note that \(e^{\frac{2\pi i}{m}} id_m \oplus id_n\) and \(e^{\frac{2\pi i}{m+n}} (id_m \oplus id_n)\) are in the centers of \(SU(m)\) and \(SU(m+n)\) respectively. By considering the actions of these elements on the space labeled by \((\lambda_0; \lambda_1, \lambda_2, q)\), we get:

\[
e^{\frac{2\pi i}{m+n}} r_\lambda \mu = e^{\frac{2\pi i}{m}} r_\lambda \mu e^{\frac{-2\pi i}{m+n}} q,
\]

and so

\[
q = -mr_\lambda \mu + (m + n)r_\lambda \mu \mod m(m + n) \quad (4.1)
\]

Similarly by considering the center of \(SU(n)\) we get

\[
q = nr_\mu - (m + n)r_\lambda \mu \mod n(m + n) \quad (4.2)
\]

Now let \((\lambda_0, \pi_0; \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{q})\) be in the set \(\exp\) of \(H_1 \subset G_1\). Similarly by looking at the action of the centers of \(G_1\) and \(H_1\) as above, we get the following constraint on the conjugacy classes of the representations:

\[
\dot{q} = -mr_\lambda \mu + (m + n)r_\lambda \mu + \frac{1}{2}nm(m + n)\epsilon \mod m(m + n) \quad (4.3)
\]

\[
\dot{q} = nr_\mu - (m + n)r_\lambda \mu + \frac{1}{2}nm(m + n)\epsilon \mod n(m + n) \quad (4.4)
\]

where \(\epsilon = 1 or 1 if \pi_0\) is a spin representation or otherwise.

§4.2 Vacuum Pairs for \(G(m, n, k)\). Now we are ready to determine the VPs for \(H_1 \subset G_1\). By Lemma 2.5, \((\lambda_0, \pi_0; \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{q})\) is a vacuum pair iff there exist \(\lambda_1, \tilde{\lambda}_1, \lambda_2, \tilde{\lambda}_2, q, \tilde{q}\) such that \((\pi_0; \tilde{\lambda}_1, \tilde{\lambda}_2), (\lambda_0; \lambda_1, \lambda_2), (\lambda_1, \lambda_1; \lambda_1), (\lambda_2, \tilde{\lambda}_2; \tilde{\lambda}_2)\) and \((\tilde{q}, q; \tilde{q})\) are VPs. By 2.7.12 of [KW], \((\lambda_1, \lambda_2; \lambda_1), (\lambda_1, \lambda_2; \lambda_2)\) are VPs iff

\[
(\lambda_1, \tilde{\lambda}_1; \lambda_1) = (\tau^j(1), \tau^j(1); \tau^j(1)), (\lambda_2, \tilde{\lambda}_2; \lambda_2) = (\tau^i(1), \tau^i(1); \tau^i(1))
\]

for some \(0 \leq j \leq m - 1, 0 \leq i \leq n - 1\). So we should determine VPs of the form \((\tilde{q}, q; \tilde{q})\) and \((\lambda_0; \tau^j(1), \tau^i(1))\). The following lemma solves the first question:
Lemma 4.1. The VPs for the diagonal inclusion

\[ U(1)_{2a+2b} \subset U(1)_{2a} \times U(1)_{2b} \]

are given by \((\frac{a}{(a,b)}i, \frac{b}{(a,b)}i; \frac{a+b}{(a,b)}i)\) where \(0 \leq i \leq 2(a, b) - 1\), and \((a, b)\) is the greatest common divisor of \(a\) and \(b\).

Proof: We use \(0 \leq x \leq 2a - 1, 0 \leq y \leq 2b - 1, 0 \leq z \leq 2(a + b) - 1\) to label the representations. Using

\[ h_x = \frac{x^2}{4a}, h_y = \frac{y^2}{4b}, h_z = \frac{z^2}{4(a + b)} \]

one checks easily from (2.4) that the list in the lemma are indeed VPs. We want to show that the list is complete. This is an easy exercise and we will prove it by calculating (2.2) in §2. Note that \((x, y; z) \in \exp\) if and only if \(x + y - z\) is divisible by \(2(a, b)\). Note that all the sectors in this coset have statistical dimensions equal to 1. By lemma 2.2 and (2) of Prop. 3.1 (set \(i = 1, \alpha = 1, z = 1\)) in [X3], we get

\[ \frac{1}{b(1,1)^2} \times 2(a + b) = 2a \times 2b \times \left(\frac{a + b}{(a, b)}\right)^2. \]

It follows that

\[ b(1,1) = \frac{(a, b)}{\sqrt{2ab(a + b)}}. \]

Note by definition

\[ b(1,1) = \sum_{(x,y,z) \in VPS} \frac{1}{\sqrt{8ab(a + b)}}, \]

and by comparing with the value of \(b(1,1)\) we conclude that the number of VPs must not exceed \(2(a, b)\). Thus the list of VPs in the lemma is complete.

□

In the next few lemmas we determine VPs for \(H_3 \subset G_3\) of the form

\( (\lambda_0; \tau^j(1), \tau^l(1), q) \).

Note that the sectors \(\tau^j(1), \tau^l(1), q\) have statistical dimensions equal to 1, and by the argument of Lemma 2.7, such VPs form an abelian group with group law being the composition of sectors. We denote this abelian group by \(S\). Also the statistical dimension of \(\lambda_0\) is equal to 1, so \(\lambda_0\) must be irreducible for any sector \(x\) of \(SU(m + n)_k\). Choose \(x\) corresponding to the fundamental representation of \(SU(m + n)\) and using the well known fusion rules (cf. [W]), we conclude that \(\lambda_0 = \tau^l(1)\) for some \(0 \leq l \leq m + n - 1\). We will choose the roots \(\alpha_1, \ldots, \alpha_{m+n-1}\) of \(SU(m + n)\) such that \(\alpha_1, \ldots, \alpha_{m-1}\) and \(\alpha_{m+1}, \ldots, \alpha_{m+n-1}\) are roots of \(SU(m)\) and \(SU(n)\) respectively. We will denote the fundamental weights of \(LSU(m + n), LSU(m)\) and \(LSU(n)\) by \(\Lambda_j, \Lambda_{j'}\) and \(\Lambda_{j''}\) respectively, where \(0 \leq j \leq m+n-1, 0 \leq j' \leq m-1, 0 \leq j'' \leq n-1\).
Lemma 4.2. \( (1; \tau^j(1), \tau^i(1), 0) \in S \) iff \( j \equiv 0 \mod m, i \equiv 0 \mod n \).

Proof. Assume that \( j \neq 0, i \neq 0 \) and that the coset vacuum vector in \( H_{(1; \tau^j(1), \tau^i(1), 0)} \) appear in the weight space of \( LH_3 \) with weight \( k\Lambda_0 - r \), where

\[
r = \sum_{0 \leq s \leq m+n-1} y_s \alpha_s, y_s \geq 0, 0 \leq s \leq m+n-1
\]

Note that \( \alpha_0 = \delta - \sum_{1 \leq s \leq m+n-1} \alpha_s \) (cf. § 1 of [KW]). By the equation (2.4), we get:

\[
(y_0 - y_1)\alpha_1 + \ldots + (y_0 - y_{m-1})\alpha_{m-1} + (y_m - y_0)\tilde{\Lambda}_{m-1} = k\tilde{\Lambda}_j
\]

\[
(y_0 - y_{m+1})\alpha_{m+1} + \ldots + (y_0 - y_{m+n-1})\alpha_{m+n-1} + (y_m - y_0)\tilde{\Lambda}_{m+1} = k\tilde{\Lambda}_i
\]

\[
y_0 - y_m = 0, y_0 = h_k\tilde{\Lambda}_j + h_k\tilde{\Lambda}_i
\]

Solving these equations, we get in particular

\[
y_0 - y_j = \frac{k(j - m)}{m}, y_0 - y_{m+i} = \frac{ki(n - j)}{n}, y_0 = \frac{k(j - m)}{2m} + \frac{ki(n - j)}{2n}.
\]

Note that \( y_j \geq 0, y_{m+i} \geq 0 \) and so \( y_j = y_{m+i} = 0, y_0 = y_m > 0 \). It follows that the weight \( k\Lambda_0 - r \) is degenerate (cf. P. 190 of [K]) with respect to \( k\Lambda_0 \), contradicting Lemma 11.2 of [K].

\( \square \)

Lemma 4.3. If \( (\tau^l(1); \tau^j(1), \tau^i(1), q) \in S \), then \( l = j + i \mod m + n \) and \( q = (nj - mi)k \mod mn(m + n)k \).

Proof.: One checks easily using definitions that

\[
(\tau(1); \tau(1), 1, nk) \in S
\]

and

\[
(\tau(1); 1, \tau(1), -mk) \in S.
\]

Since \( S \) is an abelian group, it follows that all \( (\tau^{j+i}(1); \tau^j(1), \tau^i(1), nj - mi) \) form a subgroup \( S' \) of \( S \). The lemma is equivalent to \( S' = S \). Without loss of generality let us assume that \( n \leq m \). Let \( (\tau^l(1); \tau^j(1), \tau^i(1), q) \in S \), to show that \( (\tau^l(1); \tau^j(1), \tau^i(1), q) \in S' \), by multiplying elements of \( S' \) if necessary, we just have to consider the case \( l = 0, i = 0 \), and we denote by \( S'' \) the abelian group generated by such elements.

Note that \( (1; \tau^n(1), 1, n(n + m)k) \in S''. \) Let \( (1; \tau^j(1), 1, q) \in S'' \) be an element such that \( q \) is the least positive integer. By Lemma 4.2, \( S'' \) is a cyclic group generated by \( (1; \tau^j(1), 1, q) \). So there exists a positive integer \( k_1 \) such that

\[
n(n + m)k = qk_1, n = jk_1 \mod m.
\]
To complete the proof we just have to show that $k_1 = 1$.

As in the proof of Lemma 4.2, we have the following equation for $(1; \tau^i(1), 1, q)$ by (2.4):

\[
(y_0 - y_1)\alpha_1 + \ldots + (y_0 - y_{m-1})\alpha_{m-1} + (y_m - y_0)\hat{\lambda}_{m-1} = k\hat{\lambda}_j
\]
\[
(y_0 - y_{m+1})\alpha_{m+1} + \ldots + (y_0 - y_{m+n-1})\alpha_{m+n-1} + (y_m - y_0)\hat{\lambda}_{m+1} = 0
\]
\[
y_0 - y_m = \frac{q}{m+n}, y_0 = h_{k\hat{\lambda}_j} + \frac{q^2}{2mn(m+n)k}.
\]

By solving the equations, we find in particular that
\[
y_0 = \frac{kj(m-j)}{2m} + \frac{q^2}{2(m+n)mnk}, y_0 - y_j = \frac{kj(m-j)}{m} + \frac{qj}{m+n}.
\]

Since $y_j \geq 0$, we have the following inequality:
\[
q^2 \geq n(m+n)(m-j)k^2 + 2nkjq,
\]
and so
\[
q \geq k(nj + \sqrt{njm(m+n-j)}).
\]

Using $n(n+m)k = qk_1$, we get
\[
\frac{n(m+n)}{k_1} \geq nj + \sqrt{njm(m+n-j)}.
\]

Solving this equality for $0 \leq j \leq m-1$, we get inequality
\[
j \leq \frac{n}{k_1} + \frac{m}{2} - \sqrt{\frac{m^2}{4} + \frac{nm(k_1 - 1)}{k_1^2}},
\]
and so
\[
k_1j \leq n
\]
with equality iff $k_1 = 1$. Since $k_1j = n \mod m$ and $n \leq m$, we conclude that $k_1 = 1$.

\[
\square
\]

Now we are ready to prove the following theorem:

**Theorem 4.4.** All the VPs of $\mathcal{A}(G(m, n, k))$ are given by

\[
(\tau^{j+i}(1), \tau^{jm+im}(1); \tau^j(1), \tau^i(1), (nj-mi)(m+n+k))
\]

where $j, i$ are integers.

**Proof.** By Lemma 2.5, $(\lambda_0, \pi_0; \hat{\lambda}_1, \hat{\lambda}_2, \hat{q})$ is a vacuum pair iff there exist
\[
\lambda_1, \hat{\lambda}_1, \lambda_2, \hat{\lambda}_2, q, \hat{q}
\]
such that \((\pi_0; \tilde{\lambda}_1, \tilde{\lambda}_2), (\lambda_0; \lambda_1, \lambda_2), (\lambda_1, \tilde{\lambda}_1; \tilde{\lambda}_1), (\lambda_2, \tilde{\lambda}_2; \tilde{\lambda}_2)\) and \((\tilde{q}, q; \tilde{q})\) are VPs. By 2.7.12 of [KW], \((\lambda_1, \tilde{\lambda}_1; \tilde{\lambda}_1), (\lambda_2, \tilde{\lambda}_2; \tilde{\lambda}_2)\) are VPs iff

\[
(\lambda_1, \tilde{\lambda}_1; \tilde{\lambda}_1) = (\tau^j(1), \tau^j(1); \tau^j(1)), (\lambda_2, \tilde{\lambda}_2; \tilde{\lambda}_2) = (\tau^i(1), \tau^i(1); \tau^i(1))
\]

for some \(0 \leq j \leq m - 1, 0 \leq i \leq n - 1\). By Lemma 4.1 and Lemma 4.3, we have \(\lambda_0 = \tau^{j+i}(1), \lambda_1 = \tau^j(1), \lambda_2 = \tau^i(1)\) and \(q = (nj - mi)(m + n + k)\). Since \((\pi_0; \tilde{\lambda}_1, \tilde{\lambda}_2)\) is the VP associated to a regular conformal inclusion, it is determined by Prop. 4.2 of [KW], and one checks easily that \(\pi_0\) takes the form stated in the theorem.

\(\Box\)

**Corollary 4.5.** (1): Assume that \((\lambda_0, \pi_0; \hat{\lambda}_1, \hat{\lambda}_2, \hat{q})\) verifies selection rules (4.3) and (4.4). Then

\[
b(\lambda_0, \pi_0; \hat{\lambda}_1, \hat{\lambda}_2, \hat{q}) = d_{\lambda_0}d_{\hat{\lambda}_1}d_{\hat{\lambda}_2}b(1, 1; 1, 1, 0);
\]

(2): \((\lambda_0, \pi_0; \hat{\lambda}_1, \hat{\lambda}_2, \hat{q}) \in \exp\) if and only if it verifies selection rules (4.3) and (4.4).

(3) The statistical dimension of \((\lambda_0, \pi_0; \hat{\lambda}_1, \hat{\lambda}_2, \hat{q})\) is \(d_{\lambda_0}d_{\hat{\lambda}_1}d_{\hat{\lambda}_2}\).

**Proof.** Ad (1): To save some writing denote by \(i := \{\lambda_0, \pi_0\}, \alpha := \{\hat{\lambda}_1, \hat{\lambda}_2, \hat{q}\}\). By definition

\[
b(\lambda_0, \pi_0; \hat{\lambda}_1, \hat{\lambda}_2, \hat{q}) = \sum_{w \in VPS} S_{iw(1)} S_{\alpha w(1)}.
\]

Using Th. 4.4, the assumption and symmetry properties of \(S\) matrices (cf. §2 of [NS]), we conclude that

\[
b(\lambda_0, \pi_0; \hat{\lambda}_1, \hat{\lambda}_2, \hat{q}) = d_{\lambda_0}d_{\hat{\lambda}_1}d_{\hat{\lambda}_2}b(1, 1; 1, 1, 0).
\]

Ad (2): This follows immediately from (1) and Th. B of [KW].

Ad (3): This follows from (1) and (3) of Th. 2.4.

\(\Box\)

Assume that \((\lambda_0, \pi_0; \hat{\lambda}_1, \hat{\lambda}_2, \hat{q})\) verifies selection rules (3) and (4). By Cor. 4.5, \((\lambda_0, \pi_0; \hat{\lambda}_1, \hat{\lambda}_2, \hat{q}) \in \exp\). We will determine the irreducible components of representation

\[(\lambda_0, \pi_0; \hat{\lambda}_1, \hat{\lambda}_2, \hat{q}) .\]

To save some writing denote by \(i := \{\lambda_0, \pi_0\}, \alpha := \{\hat{\lambda}_1, \hat{\lambda}_2, \hat{q}\}\). By (2) of Prop. 2.3, \(a_{(i, \alpha) \otimes 1} \preceq a_{1 \otimes \sigma_i}\), but by (3) of Cor. 4.5, \(a_{(i, \alpha)} = d_i d_{\alpha}\), it follows that

\[
a_{(i, \alpha) \otimes 1} = a_{1 \otimes \sigma_i} .\]
So by the same argument as in the derivation of (**) in [X1] and use Th. 4.4 we get

$$\langle (i, \alpha), (i', \alpha') \rangle = \langle a_1 \otimes \bar{\alpha}_\sigma i, a_1 \otimes \bar{\alpha}_\sigma i' \rangle = \sum_{w \in VPS} \delta_{w(i), i'} \delta_{w(\alpha), \alpha'}$$

(4.5)

By setting $i = i'$, $\alpha = \alpha'$ in (4.5), we get

$$\langle (i, \alpha), (i, \alpha) \rangle = t$$

where $t$ is the number of elements in the set

$$F(i, \alpha) := \{ w \in VPS, w(i) = i, w(\alpha) = (\alpha) \}$$

(4.6)

Lemma 4.6. $F(i, \alpha)$ is a cyclic group of order $t$. Moreover, let \{j, $\beta$\} be the generator. Then $\sigma_j = a_1 \otimes \beta$ has order $t$, i.e., $t$ is the least positive integer such that $\sigma_j^t = 1$.

Proof: Let $w \in F(i, \alpha)$. By Th. 4.4 and definitions one checks easily the following property:

If \{1, $\pi_0; \hat{\lambda}_1, \hat{\lambda}_2, \hat{q}\} \in F(i, \alpha)$, then $\pi_0 = 1, \hat{\lambda}_1 = 1, \hat{\lambda}_2 = 1, \hat{q} = 0$;

It follows from that the projection of $w \in F(i, \alpha)$ onto its first component in $\mathbb{Z}_{m+n}$ is an embedding, and so $F(i, \alpha)$ must be a cyclic group of order $t$ which is a divisor of $m+n$.

Now let \{j, $\beta$\} be the order $t$ generator of $F(i, \alpha)$. So $j^t = 1, \beta^t = 1$ and $t$ is the least positive integer with this property. Let $t_1, t_2$ be the orders of $j, \beta$ respectively. Then $t$ is the least common multiple of $t_1$ and $t_2$. Since \{j, $\beta$\} $\in VPS$, $\sigma_j = a_1 \otimes \beta$, and it follows that $t_1$ is a divisor of $t_2$ since $j \rightarrow \sigma_j$ is an embedding. Note that

$$a_1 \otimes \beta^t_1 = \sigma_j^t_1 = 1,$$

and so \{1, $\beta^t_1$\} $\in VPS$ and also fix $(i, \alpha)$, by the property above we must have $\beta^t_1 = 1$, so $t_2$ is also a divisor of $t_1$. It follows that $t_1 = t_2 = t$.

□

By the formula before (4.5) the map

$$(i, \alpha) \rightarrow a_1 \otimes \bar{\alpha}_\sigma i$$

is a ring isomorphism. By definitions, $\sigma_i \sigma_j = \sigma_i$ and

$$a_1 \otimes \bar{\alpha}_\sigma j = a_1 \otimes \bar{\alpha}_\sigma a_1 \otimes \bar{\beta} = a_1 \otimes \bar{\alpha}$$

where \{j, $\beta$\} is as in Lemma 4.6. Moreover, by (4.5)

$$\langle a_1 \otimes \bar{\alpha}_\sigma i, a_1 \otimes \bar{\alpha}_\sigma i \rangle = t$$

and by Lemma 4.6 $\sigma_j$ has order $t$. Applying Lemma 2.1 of [X2] in the present case with $a = \sigma_i, b = a_1 \otimes \bar{\alpha}$ and $\tau = \sigma_j$, we conclude that the representation $(i, \alpha)$ decomposes into $t$ distinct irreducible pieces and each irreducible piece has equal statistical dimension. We record this result in the following:
Theorem 4.7. Let \((\lambda_0, \pi_0; \hat{\lambda}_1, \hat{\lambda}_2, \hat{q}) \in \exp\) be a representation of \(A(G(m,n,k))\). Then the representation decomposes into \(t\) distinct irreducible representations where \(t\) is the number of elements in the set (4.6), and each such irreducible representation has equal statistical dimension.

We note that by (2) of Th. 2.4, Th. 4.7 and formula (4.5) above, we can give a list of all the irreducible representations of \(A(G(m,n,k))\) as follows:

First we write down all \((\lambda_0, \pi_0; \hat{\lambda}_1, \hat{\lambda}_2, \hat{q})\) which verifies (4.3) and (4.4). Denote such a set by \(\exp\). \(\exp\) admits a natural action of \(VPS\) given in Th. 4.4. Suppose that \(\exp\) is the union of \(l\) orbits \(\exp_1, \ldots, \exp_l\). Let \((i_p, \alpha_p) \in \exp_p, 1 \leq p \leq l\) be representatives of the orbits. We note that two different representative of the same orbit are unitarily equivalent representations of \(A(G(m,n,k))\) by (4.5) and Th. 4.7. Let \(t_p\) be the order of \(F(i_p, \alpha_p), 1 \leq p \leq l\) as defined in (4.6). Then each representation \((i_p, \alpha_p)\) of \(A(G(m,n,k))\) decomposes into \(t_p\) distinct irreducible pieces, and hence the number of irreducible representations of \(A(G(m,n,k))\) is given by \(\sum_{1 \leq p \leq l} t_p\).

References

[DJ] D. Dunbar and K. Joshi, *Characters for coset conformal field theories and Maverick examples*, Inter. J. Mod. Phys. A, Vol.8, No. 23 (1993), 4103-4121.

[FZ] I. Frenkel and Y. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras*, Duke Math. Journal (1992), Vol. 66, No. 1, 123-168.

[FS] J. Fuchs and C. Schweigert, *Level-rank duality of WZW theories and isomorphisms of \(N=2\) coset models*, Ann. Physics 234 (1994), no. 1, 102–140.

[GL1] D. Guido and R. Longo, *The Conformal Spin and Statistics Theorem*, Comm.Math.Phys., 181, 11-35 (1996).

[GL2] D. Guido and R. Longo, *Relativistic invariance and charge conjugation in quantum field theory*, Comm.Math.Phys., 148, 521-551 (1992).

[GKO] P. Goddard and D. Olive, eds., *Kac-Moody and Virasoro algebras*, Advanced Series in Math. Phys., Vol 3, World Scientific 1988.

[H] R. Haag, *Local Quantum Physics*, Springer-Verlag 1992.

[K] V. G. Kac, *Infinite Dimensional Lie Algebras*, 3rd Edition, Cambridge University Press, 1990.

[KW] V. G. Kac and M. Wakimoto, *Modular and conformal invariance constraints in representation theory of affine algebras*, Advances in Math., 70, 156-234 (1988).

[KLM] Y. Kawahigashi, R. Longo and M. Müger, *Multi-interval Subfactors and Modularity of Representations in Conformal Field theory*, to appear in Comm.Math.Phys., also see [math.OA/9903104](http://arxiv.org/abs/math.OA/9903104).

[KS] Y. Kazama and H. Suzuki, *New \(N=2\) superconformal field theories and superstring compactification*, Nuclear Phys. B 321 (1989), no. 1, 232–268.
[LVW] W. Lerche, C. Vafa and N. P. Warner, Nucl. Phys. B324 (1989) 427.

[L1] R. Longo, *Conformal Subnets and Intermediate Subfactors*, math.OA/0102196.

[L2] R. Longo, *Duality for Hopf algebras and for subfactors*, I, Comm. Math. Phys., 159, 133-150 (1994).

[L3] R. Longo, *Index of subfactors and statistics of quantum fields*, I, Comm. Math. Phys., 126, 217-247 (1989).

[L4] R. Longo, *Index of subfactors and statistics of quantum fields*, II, Comm. Math. Phys., 130, 285-309 (1990).

[L5] R. Longo, *Minimal index and braided subfactors*, J.Funct.Analysis 109 (1992), 98-112.

[LR] R. Longo and K.-H. Rehren, *Nets of subfactors*, Rev. Math. Phys., 7, 567-597 (1995).

[NS] S. Naculich and H. Schnitzer, *Superconformal coset equivalence from level-rank duality*, Nuclear Phys. B 505 (1997), no. 3, 727–748.

[PP] M.Pimsner and S.Popa, *Entropy and index for subfactors*, Ann. Sci.Éc.Norm.Sup. 19, 57-106 (1986).

[PS] A. Pressley and G. Segal, *Loop Groups*, O.U.P. 1986.

[T] M. Takesaki, *Conditional expectation in von Neumann algebra*, J. Funct. Analysis 9 (1972), 306-321.

[TL] V. Toledano Laredo, *Fusion of Positive Energy Representations of LSpin_{2n}*. Ph.D. dissertation, University of Cambridge, 1997

[Tu] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, Walter de Gruyter, Berlin, New York 1994.

[W] A. Wasserman, *Operator algebras and Conformal field theories III*, Invent. Math. Vol. 133, 467-539 (1998)

[X1] F.Xu, *Algebraic coset conformal field theories*, Comm. Math. Phys. 211 (2000) 1-43.

[X2] F.Xu, *Algebraic coset conformal field theories II*, Publ. RIMS, vol. 35 (1999), 795-824.

[X3] F.Xu, *On a conjecture of Kac-Wakimoto*, Publ. RIMS, vol. 37 (2001), 165-190.

[X4] F.Xu, *New braided endomorphisms from conformal inclusions*, Comm.Math.Phys. 192 (1998) 349-403.

[X5] F.Xu, *3-manifold invariants from cosets*, math.GT/9907077.

[X6] F.Xu, *Jones-Wassermann subfactors for Disconnected Intervals*, Comm. Contemp. Math. Vol. 2, No. 3 (2000) 307-347.