Approximate Kernel PCA Using Random Features: Computational vs. Statistical Trade-off

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Abstract

Kernel methods are powerful learning methodologies that provide a simple way to construct nonlinear algorithms from linear ones. Despite their popularity, they suffer from poor scalability in big data scenarios. Various approximation methods, including random feature approximation have been proposed to alleviate the problem. However, the statistical consistency of most of these approximate kernel methods are not well understood except for kernel ridge regression wherein it has been shown that the random feature approximation is not only computationally efficient but also statistically consistent with a minimax optimal rate of convergence. In this paper, we investigate the efficacy of random feature approximation in the context of kernel principal component analysis (KPCA) by studying the trade-off between computational and statistical behaviors of approximate KPCA. We show that the approximate KPCA is both computationally and statistically efficient compared to KPCA in terms of the error associated with reconstructing a kernel function based on its projection onto the corresponding eigenspaces. Depending on the eigenvalue decay behavior of the covariance operator, we show that only \( n^{2/3} \) features (polynomial decay) or \( \sqrt{n} \) features (exponential decay) are needed to match the statistical performance of KPCA. We also investigate their statistical behaviors in terms of the convergence of corresponding eigenspaces wherein we show that only \( \sqrt{n} \) features are required to match the performance of KPCA and if fewer than \( \sqrt{n} \) features are used, then approximate KPCA has a worse statistical behavior than that of KPCA.

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1 Introduction

Principal component analysis (PCA) (Jolliffe, 1986) is a popular statistical methodology for dimensionality reduction and feature extraction, wherein a low-dimensional representation that retains as much variance as possible of the original data is obtained. In fact, the low-dimensional representation is a projection of the original \( d \)-dimensional data onto the \( \ell \)-eigenspace, i.e., the span of eigenvectors associated with top \( \ell \) eigenvalues of the covariance matrix where \( \ell < d \), resulting in a \( \ell \)-dimensional representation. Using kernel trick, Schölkopf et al. (1998) extended the above idea

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to reproducing kernel Hilbert space (RKHS) (Aronszajn, 1950), resulting in kernel PCA (KPCA), which is a nonlinear form of PCA that better exploits the spatial structure of the data and also gives rise to nonlinear interpretation of dimensionality reduction of the original data. Due to this, KPCA is popular in applications such as image denoising (Mika et al., 1999), novelty detection (Hoffmann, 2007) and computer vision (see Lampert, 2009 and references therein), etc.

Despite the popularity of KPCA, one of its main drawbacks is the computational requirement of $O(n^3)$—to compute the eigenvectors of the Gram matrix—, where $n$ is the number of samples in the input space. In addition to KPCA, more generally, most of the kernel algorithms (see Schölkopf and Smola, 2002) has a space complexity requirement of $O(n^2)$ and time complexity requirement of $O(n^3)$ as in some sense, all of them involve an eigen decomposition of the Gram matrix. However, in big data scenarios where $n$ is large, the kernel methods including KPCA suffer from large space and time complexities. These space and computational complexity requirements arise as kernel methods solve learning problems in the dual, owing to the computational advantages associated with solving a finite dimensional optimization problem in contrast to an infinite dimensional problem in the primal (Schölkopf and Smola, 2002). To address these inherent computational issues, various approximation methods have been proposed and investigated during the last decade. Some of the popular approximation strategies include the incomplete Cholesky factorization (Fine and Scheinberg, 2001; Bach and Jordan, 2005), Nyström method (e.g., see Williams and Seeger, 2001; Drineas and Mahoney, 2005; Kumar et al., 2009), random features (e.g., see Rahimi and Recht, 2008; Kar and Karnick, 2012; Le et al., 2013), sketching (Yang et al., 2017), sparse greedy approximation (Smola and Schölkopf, 2000), etc. These methods can be grouped into two categories, wherein some employ an approximation to the dual problem while the other approximate the primal problem. Incomplete Cholesky factorization and sketching fall in the former category while the Nyström and random feature approximation belong to the latter category of approximating the primal problem. While it has been widely accepted that these approximate methods provide significant computational advantages and has been empirically shown to provide learning algorithms or solutions that do not suffer from significant deterioration in performance compared to those without approximation (Kumar et al., 2009; Rahimi and Recht, 2008; Yang et al., 2012, 2017), until recently, the statistical consistency of these approximate methods is not well understood. Most of the theoretical studies have dealt with the quality of kernel approximation (e.g., see Zhang et al., 2008; Drineas and Mahoney, 2005; Rahimi and Recht, 2008; Sriperumbudur and Szabó, 2015), which then have been used to study the statistical convergence of these approximate methods (e.g. Jin et al., 2013; Cortes et al., 2010). However, recently, sharper analysis on the statistical consistency of these approximate methods, particularly involving kernel ridge regression has been carried out (Bach, 2013; Alaoui and Mahoney, 2015; Rudi et al., 2015; Yang et al., 2017; Rudi and Rosasco, 2017), wherein it has been shown that Nyström, random feature and sketching based approximate kernel ridge regression are consistent and they also achieve minimax rates of convergence as achieved by the exact methods but using fewer features than the sample size. This means, these approximate kernel ridge regression algorithms are not only computationally efficient compared to their exact counterpart but also statistically efficient, i.e., achieve the best possible convergence rate.

On the other hand, the theoretical behavior of approximate kernel algorithms other than approximate kernel ridge regression is not well understood. The goal of this paper is to investigate the trade-off between computational and statistical efficiency of random feature based approximate KPCA (RF-KPCA). Before we discuss the problem and the related work, in the following, we briefly introduce the idea of random feature approximation, which involves computing a finite dimensional feature map that approximates the kernel function. Suppose say $k$ is a continuous
translation invariant kernel on $\mathbb{R}^d$, i.e., $k(x, y) = \psi(x - y), x, y \in \mathbb{R}^d$ where $\psi$ is a continuous positive definite function on $\mathbb{R}^d$. Bochner’s theorem (Wendland, 2005, Theorem 6.6) states that $\psi$ is the Fourier transform of a finite non-negative Borel measure $\Lambda$ on $\mathbb{R}$, positive definite function on $\mathbb{R}$.

$$k(x, y) = \psi(x - y) = \int_{\mathbb{R}^d} e^{-\sqrt{-1}(x,y)\omega} \Lambda(\omega)$$

where $\langle \cdot, \cdot \rangle_2$ denotes the usual Euclidean inner product and $(\ast)$ follows from the fact that $\psi$ is real-valued and symmetric. Since $\Lambda(\mathbb{R}^d) = \psi(0)$, we can write (1) as

$$k(x, y) = \psi(0) \int_{\mathbb{R}^d} \cos((x - y, \omega) \Lambda(\omega)$$

where $\Lambda(\psi(0))$ is a probability measure on $\mathbb{R}^d$. Therefore, without loss of generality, throughout the paper we assume that $\Lambda$ is a probability measure. Rahimi and Recht (2008) proposed a random approximation to $k$ by replacing the integral with Monte Carlo sums constructed from $(\omega_i)_{i=1}^m \sim \Lambda$, i.e.,

$$k_m(x, y) = \psi_m(x - y) = \frac{1}{m} \sum_{i=1}^m \cos(\langle x - y, \omega_i \rangle_2) = \langle \Phi_m(x), \Phi_m(y) \rangle_2, \quad (2)$$

where $\Phi_m(x) = \frac{1}{\sqrt{m}}(\cos(\langle x, \omega_1 \rangle_2), \ldots, \cos(\langle x, \omega_m \rangle_2), \sin(\langle x, \omega_1 \rangle_2), \ldots, \sin(\langle x, \omega_m \rangle_2))$ and $(\ast)$ holds based on the trigonometric identity: $\cos(a - b) = \cos a \cos b + \sin a \sin b$. This kind of random approximation to $k$ can be constructed for a more general class of kernels of the form

$$k(x, y) = \int_{\Theta} \varphi(x, \theta) \varphi(y, \theta) d\Lambda(\theta)$$

by using

$$k_m(x, y) = \frac{1}{m} \sum_{i=1}^m \varphi(x, \theta_i) \varphi(y, \theta_i) = \langle \Phi_m(x), \Phi_m(y) \rangle_2$$

where $\varphi(x, \cdot) \in L^2(\Theta, \Lambda)$—see Section 2 for notation—for all $x \in \mathcal{X}, (\theta_i)_{i=1}^m \sim \Lambda, \Phi_m(x) = \frac{1}{\sqrt{m}}(\varphi(x, \theta_1), \ldots, \varphi(x, \theta_m))$ with $\mathcal{X}$ and $\Theta$ being measurable spaces. As can be seen, the advantage of this approximation is that it yields a finite dimensional feature map $\Phi_m$ that allows to train kernel machines in the primal—for example, to obtain approximate kernel ridge regression, we simply apply linear regression on $\Phi_m(\cdot)$—and therefore has a computational complexity of $O(m^2 n^2)$ which is an improvement over $O(n^3)$ for $m < n$. As aforementioned, in the kernel ridge regression scenario, it has been shown that the approximate method is not only consistent but also efficient, i.e., achieves minimax rate of convergence for the excess error even if $m < n$. In this paper, we approximate KPCA by employing linear PCA on $\Phi_m(\cdot)$ which leads to solving an $m \times m$ eigen system and therefore has a computational complexity of $O(m^3)$. This is significantly better than $O(n^3)$ for $m < n$. The question therefore of interest is whether RF-KPCA consistent and how does $m$ depend on $n$ for RF-KPCA to achieve the same convergence rate as that of KPCA. The goal of this paper is to address these questions.

### 1.1 Contributions

The main contributions of the paper are as follows:
(i) First, in Section 4.1, we compare the performance of RF-KPCA with KPCA in terms of the reconstruction error of the associated ℓ-eigenspace, i.e., the error involved in reconstructing \( k(\cdot, x) \) based on its projections onto the corresponding ℓ-eigenspace. Since the ℓ-eigenspace associated with RF-KPCA is a subspace of \( \mathbb{R}^m \) in contrast to \( \mathcal{H} \) as is the case with KPCA, the notion of projecting \( k(\cdot, x) \in \mathcal{H} \) onto a subspace of \( \mathbb{R}^m \) is vacuous. To alleviate the problem, we define inclusion and approximation operators that embed both \( \mathcal{H} \) and \( \mathbb{R}^m \) as subspaces in \( L^2(\mathbb{P}) \), using which an appropriate notion of reconstruction error is defined to compare the behaviors of RF-KPCA and KPCA. We show that for \( m \) large enough but still smaller than \( n \), the reconstruction error associated with RF-KPCA has similar convergence rate to zero as that of KPCA (see Theorems 3 and 6). Precisely, if the covariance operator associated with KPCA has a polynomial eigenvalue decay, then only \( n^{2/3} \) random features are needed to attain similar behavior to that of KPCA, i.e., the computational complexity reduces from \( O(n^3) \) to \( O(n^2) \) (see Corollary 4 and 5). In addition, if the RKHS is more smooth, i.e., the eigenvalues decay at an exponential rate, then only \( \sqrt{n} \) random features are needed to match the statistical behavior of KPCA, thereby leading to \( O(n^{3/2}) \) computational requirement. These results show that the computational gain achieved by RF-KPCA is not at the expense of statistical efficiency. More subtly, there is an intimate connection between \( m, \ell \) and \( n \) and the smoothness of the RKHS, which affect the convergence rate of the reconstruction error. We explore this in detail in Theorems 3 and 6 and their associated corollaries.

(ii) Second, in Section 4.2, we investigate the statistical behavior of RF-KPCA and KPCA in terms of the convergence of ℓ-eigenspace, which is carried out by comparing the corresponding “projection operators” (see Theorem 9) on \( L^2(\mathbb{P}) \) in the operator norm. In Proposition 8, we show this notion of convergence to be stronger than converging in reconstruction error and then in Theorem 9 we show RF-KPCA to have similar convergence rates to that of KPCA if \( m \) is large enough (but still less than \( n \)). On the other hand, if \( m \) is not sufficiently large, then RF-KPCA has slower convergence rates than that of KPCA, which establishes the computational vs. statistical trade-off behavior. More precisely, we show \( m = \sqrt{n} \) to be a transition point wherein if \( m > \sqrt{n} \), then RF-KPCA and KPCA have similar convergence behavior while \( m \leq \sqrt{n} \) results in a statistical loss for RF-KPCA.

1.2 Related Work

To the best of our knowledge, not much investigation has been carried out on the statistical analysis of RF-KPCA. Recently, [Lopez-Paz et al. (2014)] studied the quality of approximation of the Gram matrix by the approximate Gram matrix (using random Fourier features) in operator norm and showed a convergence rate of \( n(\sqrt{\log n}/m + (\log n)/m) \). This approximation bound is too loose as we require \( m \) to grow faster than \( n \) to achieve convergence to zero, which defeats the purpose of random feature approximation. On the other hand, statistical behavior of KPCA is well understood. [Shawe-Taylor et al. (2005)] studied the statistical consistency of KPCA in terms of the reconstruction error of the estimated ℓ-eigenspace and obtained a convergence rate of \( n^{-1/2} \). Using localized arguments from M-estimation (e.g., see [Bartlett et al., 2003; Koltchinskii, 2006]), [Blanchard et al. (2007)] improved these rates by taking into account the decay rate of the eigenvalues of the covariance operator. However, unlike in this paper where the reconstruction error is defined in terms of convergence in \( L^2(\mathbb{P}) \), these works consider convergence in \( \mathcal{H} \). The question of convergence of ℓ-eigenspaces associated with KPCA was considered by [Zwald and Blanchard (2006)] as convergence of orthogonal projection operators on \( \mathcal{H} \) in Hilbert-Schmidt norm and obtained a convergence rate of \( n^{-1/2} \).
Various notations and definitions that are used throughout the paper are collected in Section 2. Preliminaries on KPCA and RF-KPCA are provided in Section 3. All the proofs are relegated to Section 6 with supplementary results collected in appendices.

2 Definitions & Notation

Define \( \|a\|_2 := \sqrt{\sum_{i=1}^{d}a_i^2} \) and \( \langle a, b \rangle_2 := \sum_{i=1}^{d}a_i b_i \), where \( a := (a_1, \ldots, a_d) \in \mathbb{R}^d \) and \( b := (b_1, \ldots, b_d) \in \mathbb{R}^d \). \( a \otimes_2 b := ab^\top \) denotes the tensor product of \( a \) and \( b \). \( I_n \) denotes an \( n \times n \) identity matrix. We define \( 1_n := (1, \ldots, 1)^\top \) and \( H_n := I_n - \frac{1}{n}1_n \otimes_2 1_n \). \( \delta_{ij} \) denotes the Kronecker delta. \( a \land b := \min(a, b) \) and \( a \lor b := \max(a, b) \). \( [n] := \{1, \ldots, n\} \) for \( n \in \mathbb{N} \). For constants \( a \) and \( b \), \( a \leq b \) (resp. \( a \geq b \)) denotes that there exists a positive constant \( c \) (resp. \( c' \)) such that \( a \leq cb \) (resp. \( a \geq c'b \)). For a random variable \( A \) with law \( P \) and a constant \( b \), \( A \leq_{\mu} b \) denotes that for any \( \delta > 0 \), there exists a positive constant \( c_{\delta} < \infty \) such that \( P(A \leq c_{\delta}b) \geq \delta \).

For a topological space \( X \), \( M^1_b(X) \) denotes the set of all finite non-negative Borel measures on \( X \). For \( \mu \in M^1_b(X) \), \( L^r(X, \mu) \) denotes the Banach space of \( r \)-power \((r \geq 1)\) \( \mu \)-integrable functions. For \( f \in L^r(X, \mu) \), \( \|f\|_{L^r(\mu)} := (\int_X |f|^r d\mu)^{1/r} \) denotes the \( L^r \)-norm of \( f \) for \( 1 \leq r < \infty \).

\( \mu^n := \mu \times \cdots \times \mu \) is the \( n \)-fold product measure. \( \mathcal{H} \) denotes a reproducing kernel Hilbert space with a reproducing kernel \( k : X \times X \rightarrow \mathbb{R} \).

Let \( H_1 \) and \( H_2 \) be abstract Hilbert spaces. For a bounded linear operator \( S : H_1 \rightarrow H_2 \), its \textit{operator norm} of \( S \) is defined as \( \|S\|_{\mathcal{L}(H_1, H_2)} := \sup\{|\langle Sx, x \rangle_{H_2} : x \in B_{H_1}\} \). Define \( \mathcal{L}(H_1, H_2) \) be the space of bounded linear operators from \( H_1 \) to \( H_2 \). For \( S \in \mathcal{L}(H_1, H_2) \), \( S^* \) denotes the \textit{adjoint} of \( S \). \( S \in \mathcal{L}(H) := \mathcal{L}(H, H) \) is called \textit{self-adjoint} if \( S^* = S \) and is called \textit{positive} if \( \langle Sx, x \rangle_{H} \geq 0 \) for all \( x \in H \). \( \alpha \in \mathbb{C} \) is an \textit{eigenvalue} of \( S \in \mathcal{L}(H) \) if there exists an \( x \neq 0 \) such that \( Sx = \alpha x \) and such an \( x \) is called the \textit{eigenvector/eigenfunction} of \( S \) and \( \alpha \). An eigenvalue is said to be \textit{simple} if it has multiplicity one. For compact, positive, self-adjoint \( S \in \mathcal{L}(H) \), \( S^r : H \rightarrow H, r \geq 0 \) is called a \textit{fractional power} of \( S \) and \( S^{1/2} \) is the \textit{square root} of \( S \), which we write as \( \sqrt{S} := S^{1/2} \).

An operator \( S \in \mathcal{L}(H_1, H_2) \) is \textit{Hilbert-Schmidt} if \( \|S\|_{\mathcal{L}^2(H_1, H_2)} := (\sum_{j \in J} \|Se_j\|_{H_2}^2)^{1/2} < \infty \) where \( (e_j)_{j \in J} \) is an arbitrary orthonormal basis of separable Hilbert space \( H_1 \). \( S \in \mathcal{L}(H_1, H_2) \) is said to be of \textit{trace class} if \( \|S\|_{\mathcal{L}^1(H_1, H_2)} := \sum_{j \in J} \|(S^* S)^{1/2} e_j\|_{H_1} < \infty \). If \( S \) is self-adjoint on \( H_1 \), then \( \|S\|_{\mathcal{L}^1(H_1, H_1), \mathcal{L}^2(H_1, H_1) \text{ and } \mathcal{L}^\infty(H_1, H_1)} \) are denoted as \( \|S\|_{\mathcal{L}^1(H_1, H_1)}, \|S\|_{\mathcal{L}^2(H_1, H_1)} \) and \( \|S\|_{\mathcal{L}^\infty(H_1, H_1)} \) respectively. For \( x, y \in H_1 \), \( x \otimes H_1 y \) is an element of the tensor product space \( H_1 \otimes H_1 \) which can also be seen as an operator from \( H_1 \) to \( H_1 \) as \( (x \otimes H_1, y)z = \langle x, y \rangle_{H_1} \) for any \( z \in H_1 \).

3 Variants of Kernel PCA: Population, Empirical and Approximate

In this section, we review kernel PCA (Schölkopf et al., 1998) in population and empirical settings and introduce approximate kernel PCA based on random features. This section not only provides preliminaries but also fixes some notation that will be used throughout the paper. To start with, we assume the following for the rest of the paper:

\((A_1)\) \( X \) is a separable topological space and \((H, k)\) is a separable RKHS of real-valued functions on \( X \) with a bounded continuous positive definite kernel \( k \) satisfying \( \sup_{x \in X} k(x, x) = : \kappa < \infty \).
3.1 PCA in Reproducing Kernel Hilbert Space

Classical PCA (Jolliffe, 1986) involves finding a direction \( \mathbf{a} \in \mathbb{R}^d \) such that \( \text{Var}(\langle \mathbf{a}, X \rangle) \) is maximized where \( X \) is a r.v. with law \( \mathbb{P} \) defined on \( \mathbb{R}^d \). By defining \( \Sigma \) as the covariance matrix of \( X \), the problem reduces to finding \( \mathbf{a} \) that solves \( \max \{ \langle \mathbf{a}, \Sigma \mathbf{a} \rangle : \| \mathbf{a} \|_2 = 1 \} \), which is nothing but the eigenvector of \( \Sigma \) associated with the largest eigenvalue of \( \Sigma \). Kernel PCA extends this idea in an RKHS by finding a function \( f \in \mathcal{H} \) such that \( \text{Var}[f(X)] \) is maximized, i.e.,

\[
\sup_{\| f \|_{\mathcal{H}} = 1} \text{Var}[f(X)] = \sup_{\| f \|_{\mathcal{H}} = 1} \mathbb{E} [ f(X) - \mathbb{E} [ f(X) ]]^2. \tag{3}
\]

Since \( f \in \mathcal{H} \), using the reproducing property \( f(X) = \langle f, k(\cdot, X) \rangle_{\mathcal{H}} \), we have \( \text{Var}[f(X)] = \mathbb{E} [ \langle f, k(\cdot, X) \rangle_{\mathcal{H}} - \langle f, m_{\mathbb{P}} \rangle_{\mathcal{H}} ]^2 \) where \( m_{\mathbb{P}} \in \mathcal{H} \) is the unique mean element of \( \mathbb{P} \), defined as

\[
\langle f, m_{\mathbb{P}} \rangle_{\mathcal{H}} = \mathbb{E} [ f(X) ] = \mathbb{E} [ \langle f, k(\cdot, X) \rangle_{\mathcal{H}} ] = \left\langle f, \int_X k(\cdot, x) \, d\mathbb{P}(x) \right\rangle_{\mathcal{H}} \quad (\forall f \in \mathcal{H}). \tag{4}
\]

The last equality in (4) holds based on the Riesz representation theorem (e.g., see Reed and Simon, 1980) and the fact that \( \int_X k(x, x) \, d\mathbb{P}(x) < \infty \), which ensures that \( k(\cdot, X) \) is \( \mathbb{P} \)-integrable in the Bochner sense (see Diestel and Uhl, 1977, Definition 1 and Theorem 2). Therefore,

\[
\text{Var}[f(X)] = \mathbb{E} [ (\langle f, k(\cdot, X) \rangle_{\mathcal{H}} - m_{\mathbb{P}} )^2 ] = \mathbb{E} [ (f( (k(\cdot, X) - m_{\mathbb{P}} ) \otimes_{\mathcal{H}} (k(\cdot, X) - m_{\mathbb{P}} ) ) f )_{\mathcal{H}} ] = \left\langle f, \frac{1}{2} \int_X (k(\cdot, x) - m_{\mathbb{P}} ) \otimes_{\mathcal{H}} (k(\cdot, x) - m_{\mathbb{P}} ) \, d\mathbb{P}(x) \right\rangle_{\mathcal{H}} = \langle f, \Sigma f \rangle_{\mathcal{H}}, \tag{5}
\]

where (\*) follows from the Riesz representation theorem and the fact that \( \int k(x, x) \, d\mathbb{P}(x) < \infty \), which ensures that \( k(\cdot, X) \otimes_{\mathcal{H}} k(\cdot, X) \) is \( \mathbb{P} \)-integrable in the Bochner sense. Here

\[
\Sigma := \int_X (k(\cdot, x) - m_{\mathbb{P}} ) \otimes_{\mathcal{H}} (k(\cdot, x) - m_{\mathbb{P}} ) \, d\mathbb{P}(x) = \int_X k(\cdot, x) \otimes_{\mathcal{H}} k(\cdot, x) \, d\mathbb{P}(x) - m_{\mathbb{P}} \otimes_{\mathcal{H}} m_{\mathbb{P}} \tag{6}
\]

is the covariance operator on \( \mathcal{H} \) whose action on \( f \in \mathcal{H} \) is defined as

\[
\Sigma f = \int_X k(\cdot, x) f(x) \, d\mathbb{P}(x) - m_{\mathbb{P}} \int_X f(x) \, d\mathbb{P}(x).
\]

Therefore, the kernel PCA problem exactly resembles classical PCA where the goal is to find \( f \in \mathcal{H} \) that solves

\[
\sup_{\| f \|_{\mathcal{H}} = 1} \langle f, \Sigma f \rangle_{\mathcal{H}}, \tag{7}
\]

with \( \Sigma \) being defined as in (6). Under the assumption that \( k \) is bounded, it can be shown that (see Proposition B.2 (iii)) that \( \Sigma \) is a trace-class operator and therefore Hilbert-Schmidt and compact. Also it is obvious that \( \Sigma \) is self-adjoint and positive and therefore by spectral theorem (Reed and Simon 1980, Theorems VI.16, VI.17), \( \Sigma \) can be written as

\[
\Sigma = \sum_{i \in I} \lambda_i \phi_i \otimes_{\mathcal{H}} \phi_i,
\]

where \( (\lambda_i)_{i \in I} \subset \mathbb{R}^+ \) are the eigenvalues and \( (\phi_i)_{i \in I} \) are the orthonormal system of eigenfunctions of \( \Sigma \) that span \( \mathcal{R}(\Sigma) \) with the index set \( I \) being either countable in which case \( \lambda_i \to 0 \) as \( i \to \infty \) or finite. It is therefore obvious that the solution to (7) is an eigenfunction of \( \Sigma \) corresponding to the largest eigenvalue. Here \( \mathcal{R}(\Sigma) \) denotes the range space of \( \Sigma \).
Note that the null space of $\Sigma$ is $\mathcal{N}(\Sigma) = \{ f \in \mathcal{H} : f(X) \text{ is constant $\mathbb{P}$-almost surely} \}$ since $\Sigma f = 0$ if and only if $\text{Var}[f(X)] = 0$, i.e., $f$ is constant $\mathbb{P}$-almost surely. Therefore if $\text{supp}(\mathbb{P}) = \mathcal{X}$ and $\mathcal{H}$ does not contain constant functions, then $\mathcal{N}(\Sigma) = \{0\}$ and $\Sigma$ is invertible. Assuming $k(\cdot, x) \in C_0(\mathcal{X})$ for all $x \in \mathcal{X}$ ensures that constant functions are not included in $\mathcal{H}$, where $C_0(\mathcal{X})$ is the space of continuous functions on $\mathcal{X}$ that decay to zero at infinity.

Throughout the paper, we assume that

(A$_2$) The eigenvalues $(\lambda_i)_{i \in I}$ of $\Sigma$ in (9) are simple, positive and without any loss of generality, they satisfy a decreasing rearrangement, i.e., $\lambda_1 > \lambda_2 > \cdots$.

(A$_2$) ensures that $(\phi_i)_{i \in I}$ form an orthonormal basis and the eigenspace corresponding to $\lambda_i$ for any $i \in I$ is one-dimensional. This means, the orthogonal projection operator onto $\text{span}\{\phi_i\}_{i=1}^\ell$ is given by

$$P_\ell(\Sigma) = \sum_{i=1}^\ell \phi_i \otimes_\mathcal{H} \phi_i.$$  

3.2 Empirical Kernel PCA

In practice, $\mathbb{P}$ is unknown and the knowledge of $\mathbb{P}$ is available only through random samples $(X_i)_{i=1}^n$ drawn i.i.d. from it. The goal of empirical kernel PCA (EKPCA) is therefore to find $f \in \mathcal{H}$ such that

$$\widehat{\text{Var}}[f(X)] := \frac{1}{n} \sum_{i=1}^n f^2(X_i) - \left( \frac{1}{n} \sum_{i=1}^n f(X_i) \right)^2,$$

i.e., the empirical variance, is maximized. In the above, we considered an estimate of $\text{Var}[f(X)]$ based on the $V$-statistic representation although a $U$-statistic form can be equally used. Using the reproducing property, it is easy to show that $\widehat{\text{Var}}[f(X)] = \langle f, \widehat{\Sigma} f \rangle_{\mathcal{H}}$ where $\widehat{\Sigma} : \mathcal{H} \to \mathcal{H},$

$$\widehat{\Sigma} := \frac{1}{n} \sum_{i=1}^n k(\cdot, X_i) \otimes_\mathcal{H} k(\cdot, X_i) - \widehat{m} \otimes_\mathcal{H} \widehat{m}$$

is the empirical covariance operator and $\widehat{m} := \frac{1}{n} \sum_{i=1}^n k(\cdot, X_i)$ is an empirical estimator of the mean element, $m_\mathbb{P}$. While $\widehat{\Sigma}$ is a self-adjoint operator on (a possibly infinite dimensional) $\mathcal{H}$, it has rank of at most $n - 1$ and therefore is compact. It follows from the spectral theorem ([Reed and Simon, 1980, Theorems VI.16, VI.17]) that

$$\widehat{\Sigma} = \sum_{i=1}^n \widehat{\lambda}_i \widehat{\phi}_i \otimes_\mathcal{H} \widehat{\phi}_i,$$

where $(\widehat{\lambda}_i)_{i=1}^n \subset \mathbb{R}^+$ and $(\widehat{\phi}_i)_{i=1}^n$ are the eigenvalues and eigenfunctions of $\widehat{\Sigma}$. Similar to (A$_2$), we assume the following about $(\lambda_i)_{i=1}^n$.

(A$_3$) The eigenvalues $(\lambda_i)_{i=1}^n$ of $\widehat{\Sigma}$ in (9) are simple, rank$(\widehat{\Sigma}) = n - 1$ and without any loss of generality, they satisfy a decreasing rearrangement, i.e., $\lambda_1 > \lambda_2 > \cdots$.

Based on (A$_3$), a low-dimensional Euclidean representation of $X_i \in \mathcal{X}$ can be obtained as

$$\left( (k(\cdot, X_i), \widehat{\phi}_1)_H, \ldots, (k(\cdot, X_i), \widehat{\phi}_\ell)_H \right)^\top = \left( \widehat{\phi}_1(X_i), \ldots, \widehat{\phi}_\ell(X_i) \right)^\top \in \mathbb{R}^\ell,$$
where \( \ell < n \) and \( i \in [n] \). Clearly, the choice of \( k(\cdot, x) = \langle \cdot, x \rangle_2 \) for \( x \in \mathbb{R}^d \) in \((10)\) reduces to the usual low-dimensional representation using linear PCA. Under \((A_3)\), we denote the orthogonal projection operator onto \( \text{span}\{u_1, \ldots, u_m\} \) as \( P_m(\Sigma) \), which is given by

\[
P_m(\Sigma) = \sum_{i=1}^m \beta_i \Phi_i \phi_i.
\]

Note that the low-dimensional representation in \((10)\) requires the knowledge of \( (\phi_i)^{n-1}_{i=1} \), which are not obvious to compute even though \( \Sigma \) is finite rank, as they are solution to a possibly infinite dimensional linear system. The following result (proved in Section 6.1) shows that the eigensystem \((\lambda_i, \phi_i)^{n-1}_{i=1}\) of \( \Sigma \) can be obtained by solving a finite dimensional eigen problem of a matrix of size \( n \times n \). This means \( (\lambda_i, \phi_i)^{n-1}_{i=1} \) can computed in a time that scales as \( O(n^3) \).

**Proposition 1.** Let \((\lambda_i, \phi_i)_{i=1}^{n-1}\) be the eigensystem of \( \Sigma \) in \((9)\). Define \( K = [k(x_i, x_j)]_{i,j \in [n]} \). Then \((\lambda_i)_{i=1}^{n-1}\) are the eigenvalues of \( KH_n \) with

\[
\lambda_i = \frac{1}{\sqrt{n\lambda_i}} \sum_{j=1}^n \gamma_{i,j} k(\cdot, x_j),
\]

where \( \gamma_{i,j} = (\gamma_{i,1}, \ldots, \gamma_{i,n})^\top = \hat{\alpha}_i - \frac{1}{n}(1^\top \hat{\alpha}_i)1 \) with \( \hat{\alpha}_i \notin \mathcal{N}(H_n) \) and \((\hat{\alpha}_i)_{i=1}^{n-1}\) being the eigenvectors of \( KH_n \).

### 3.3 Approximate Kernel PCA using Random Features

In this section, we present approximate kernel PCA using random features, which we call as RF-KPCA. Throughout this section, we assume the following:

\((A_4)\) \( \mathcal{H} \) is a separable RKHS with reproducing kernel \( k \) of the form

\[
k(x,y) = \int:\varphi(x,\theta)\varphi(y,\theta)\,d\Lambda(\theta) = \langle \varphi(x,\cdot), \varphi(y,\cdot) \rangle_{L^2(\Lambda)},
\]

where \( \varphi: \mathcal{X} \times \Theta \to \mathbb{R}, \sup_{\theta \in \Theta, x \in \mathcal{X}} |\varphi(x, \theta)| \leq \sqrt{\kappa} \) and \( \Lambda \) is a probability measure on a separable topological space \( \Theta \).

The assumption of \( \Lambda \) being a probability measure on \( \mathcal{X} \) is not restrictive as any \( \Lambda \in M^b_+(\mathcal{X}) \) can be normalized to a probability measure. However, the uniform boundedness of \( \varphi \) over \( \mathcal{X} \times \Theta \) is somewhat restrictive as it is sufficient to assume \( \varphi(x, \cdot) \in L^2(\mathcal{X}, \Lambda), \forall x \in \mathcal{X} \) for \( k \) to be well-defined. However, the uniform boundedness of \( \varphi \) ensures that \( k \) is bounded, as assumed in \((A_1)\).

By sampling \((\theta_i)_{i=1}^m \overset{i.i.d.}{\sim} \Lambda \), an approximation to \( k \) can be constructed as

\[
k_m(x,y) = \frac{1}{m} \sum_{i=1}^m \varphi(x, \theta_i)\varphi(y, \theta_i) =: \sum_{i=1}^m \varphi_i(x)\varphi_i(y) = \langle \Phi_m(x), \Phi_m(y) \rangle_2,
\]

where \( \varphi_i := \frac{1}{\sqrt{m}}\varphi(\cdot, \theta_i) \) and \( \Phi_m(x) := (\varphi_1(x), \ldots, \varphi_m(x))^\top \in \mathbb{R}^m \) is the random feature map. It is easy to verify that \( k_m \) is the reproducing kernel of the RKHS

\[
\mathcal{H}_m = \left\{ f : f = \sum_{i=1}^m \beta_i \varphi_i, (\beta_i)_{i=1}^m \subset \mathbb{R} \right\}
\]
w.r.t. $\langle \cdot, \cdot \rangle_{\mathcal{H}_m}$ defined as $\langle f,g \rangle_{\mathcal{H}_m} := \sum_{i=1}^{m} \alpha_i \beta_i$ where $g = \sum_{i=1}^{m} \alpha_i \varphi_i$. Therefore $\mathcal{H}_m$ is isometrically isomorphic to $\mathbb{R}^m$.

Having obtained a random feature map, the idea of RF-KPCA is to perform linear PCA on $\Phi_m(X)$ where $X \sim \mathbb{P}$, i.e., RF-KPCA involves finding a direction $\beta \in \mathbb{R}^m$ such that $\text{Var}[\langle \beta, \Phi_m(X) \rangle_2]$ is maximized:

$$\sup_{\|\beta\|_2=1} \text{Var}[\langle \beta, \Phi_m(X) \rangle_2] = \sup_{\|\beta\|_2=1} \langle \beta, \Sigma_m \beta \rangle_2,$$

where

$$\Sigma_m := \text{Cov}[\Phi_m(X)] = \mathbb{E}[\langle (\Phi_m(X) - \mathbb{E}[\Phi_m(X)]) \otimes_2 (\Phi_m(X) - \mathbb{E}[\Phi_m(X)]) \rangle],$$

$$= \mathbb{E}[\Phi_m(X) \otimes_2 \Phi_m(X)] - \mathbb{E}[\Phi_m(X)] \otimes_2 \mathbb{E}[\Phi_m(X)]$$

(13) is a self-adjoint positive definite matrix. In fact, it is easy to verify that performing linear PCA on $\Phi_m(X)$ is same as performing KPCA in $\mathcal{H}_m$ since

$$\sup_{\|f\|_{\mathcal{H}_m}=1} \text{Var}[f(X)] = \sup_{\|\beta\|_2=1} \text{Var}[\langle \beta, \Phi_m(X) \rangle_2],$$

which follows from $\mathcal{H}_m$ being isometrically isomorphic to $\mathbb{R}^m$ and $f \in \mathcal{H}_m$ has the form $f = \langle \beta, \Phi_m(X) \rangle_2$. The empirical counterpart of RF-KPCA (we call it as RF-EKPCA) is obtained by solving

$$\sup_{\|\beta\|_2=1} \widetilde{\text{Var}}[\langle \beta, \Phi_m(X) \rangle_2] = \sup_{\|\beta\|_2=1} \langle \beta, \Sigma_m \beta \rangle_2$$

where

$$\widetilde{\Sigma}_m = \frac{1}{n} \sum_{i=1}^{n} \Phi_m(X_i) \otimes_2 \Phi_m(X_i) - \left( \frac{1}{n} \sum_{i=1}^{n} \Phi_m(X_i) \right) \otimes_2 \left( \frac{1}{n} \sum_{i=1}^{n} \Phi_m(X_i) \right)$$

(14) is a self-adjoint positive definite matrix. It is obvious that the solutions to the above mentioned optimization problems are the eigenvectors of $\Sigma_m$ and $\widetilde{\Sigma}_m$. Since $\Sigma_m$ and $\widetilde{\Sigma}_m$ are trace-class (see Proposition [3.4](iii)) and self-adjoint, spectral theorem (Reed and Simon, 1980, Theorems VI.16, VI.17) yields that

$$\Sigma_m = \sum_{i=1}^{m} \lambda_{m,i} \phi_{m,i} \otimes_2 \phi_{m,i} \quad \text{and} \quad \widetilde{\Sigma}_m = \sum_{i=1}^{m} \hat{\lambda}_{m,i} \hat{\phi}_{m,i} \otimes_2 \hat{\phi}_{m,i}$$

(15) where $(\lambda_{m,i})_{i=1}^{m} \in \mathbb{R}^+$ (resp. $(\hat{\lambda}_{m,i})_{i=1}^{m} \in \mathbb{R}^+$) and $(\phi_{m,i})_{i=1}^{m}$ (resp. $(\hat{\phi}_{m,i})_{i=1}^{m}$) are the eigenvalues and eigenvectors of $\Sigma_m$ (resp. $\widetilde{\Sigma}_m$). We will assume that

(A5) The eigenvalues $(\lambda_{m,i})_{i=1}^{m}$ (resp. $(\hat{\lambda}_{m,i})_{i=1}^{m}$) of $\Sigma_m$ (resp. $\widetilde{\Sigma}_m$) are simple, rank($\Sigma_m$) = $m$, rank($\widetilde{\Sigma}_m$) = $m-1$ and without any loss of generality, they satisfy a decreasing rearrangement, i.e., $\lambda_{m,1} > \lambda_{m,2} > \cdots$ (resp. $\hat{\lambda}_{m,1} > \hat{\lambda}_{m,2} > \cdots$).

Based on (A5), a low-dimensional representation of $X_i \in \mathcal{X}$ can be obtained as

$$\left( \langle \Phi_m(X_i), \phi_{m,1} \rangle_2, \ldots, \langle \Phi_m(X_i), \phi_{m,\ell} \rangle_2 \right)^\top \in \mathbb{R}^\ell$$

where $\ell \leq m$ and $i \in [n]$. Since $(\hat{\lambda}_i, \hat{\phi}_{m,i})_{i \in [m-1]}$ form the eigensystem of $m \times m$ matrix $\widetilde{\Sigma}_m$, the associated time complexity of finding this system scales as $O(m^3)$. This means, if $m < n$, RF-EKPCA is computationally cheaper than EKPCA as the latter involves solving an $n \times n$ eigensystem which scales as $O(n^3)$. 

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4 Computational vs. Statistical Trade-off: Main Results

The main goal of this paper is to investigate whether the above-mentioned computational saving achieved by RF-EKPCA is obtained at the cost of statistical “efficiency” or not. To pursue this investigation, we consider two different objectives to compare the statistical performance of RF-EKPCA to that of EKPCA. In Section 3, we investigate the statistical performance of EKPCA and RF-EKPCA in terms of the reconstruction error. To elaborate, in linear PCA, the quality of reconstruction after projecting a random variable $X \in \mathbb{R}^d$ onto the span of top $\ell$ eigenvectors of $\Sigma$ is captured by the reconstruction error, given by

$$
\mathbb{E}_{X \sim \mathbb{P}} \left\| (X - \mu) - \sum_{i=1}^{\ell} \langle (X - \mu), \phi_i \rangle \phi_i \right\|^2_f,
$$

where $(\phi_i)_{i \in [d]}$ are the eigenvectors of $\Sigma = \mathbb{E}[XX^\top] - \mu \mu^\top$ with $\mu := \mathbb{E}[X]$. Since $(\phi_i)_i$ form an orthonormal basis in $\mathbb{R}^d$, the above consideration makes sense and clearly, the choice of $\ell = d$ yields zero error. Since KPCA generalizes linear PCA—the choice of $k(x,y) = \langle x, y \rangle_2$ reduces kernel PCA to linear PCA—it is natural to consider the reconstruction error in KPCA and EKPCA to be

$$
\mathbb{E}_{X \sim \mathbb{P}} \left\| \overline{k}(\cdot, X) - \sum_{i=1}^{\ell} \langle \overline{k}(\cdot, X), \phi_i \rangle \mathcal{H} \phi_i \right\|^2_{\mathcal{H}} \quad \text{and} \quad \mathbb{E}_{X \sim \mathbb{P}} \left\| \overline{k}(\cdot, X) - \sum_{i=1}^{\ell} \langle \overline{k}(\cdot, X), \hat{\phi}_i \rangle \mathcal{H} \hat{\phi}_i \right\|^2_{\mathcal{H}}
$$

respectively, where $(\phi_i)_i$ and $(\hat{\phi}_i)_i$ are the orthonormal eigenfunctions of $\Sigma$ and $\hat{\Sigma}$ given in (4) and (5), corresponding to the eigenvalues $(\lambda_i)_i$ and $(\hat{\lambda}_i)_i$ satisfying $(A_2)$ and $(A_3)$ respectively. Here

$$
\overline{k}(\cdot, x) = k(\cdot, x) - \int_{\mathcal{X}} k(\cdot, x) \, d\mathbb{P}(x), \quad x \in \mathcal{X}.
$$

However, similar definition of reconstruction error for RF-KPCA and RF-EKPCA is not possible as the orthonormal eigenvectors of $\Sigma_m$ and $\hat{\Sigma}_m$ belong to $\mathbb{R}^m$ while $\overline{k}(\cdot, X)$ belongs to $\mathcal{H}$, which means the notion of projecting $\overline{k}(\cdot, X)$ onto $(\phi_{m,i})_i$ or $(\hat{\phi}_{m,i})_i$ is vacuous. In order to make the comparison between EKPCA and RF-EKPCA possible in terms of reconstruction error, we define certain operators below so that all the objects of interest are embedded into a common space, which we choose to be $L^2(\mathbb{P})$.

Define an inclusion operator (up to a constant)

$$
\mathcal{J} : \mathcal{H} \rightarrow L^2(\mathbb{P}), \quad f \mapsto f - f_{\mathbb{P}},
$$

where $f_{\mathbb{P}} := \int_{\mathcal{X}} f(x) \, d\mathbb{P}(x)$. It can be shown (see Proposition B.2) that

$$
\mathcal{J}^* : L^2(\mathbb{P}) \rightarrow \mathcal{H}, \quad f \mapsto \int_{\mathcal{X}} k(\cdot, x) f(x) \, d\mathbb{P}(x) - m_{\mathbb{P}} f_{\mathbb{P}} \quad \text{and} \quad \Sigma = \mathcal{J}^* \mathcal{J}.
$$

Usually, in kernel ridge regression (e.g., see Caponnetto and Vito, 2007; Smale and Zhou, 2007) the inclusion operator is defined as $\mathcal{J} : \mathcal{H} \rightarrow L^2(\mathbb{P}), f \mapsto f$ with adjoint $\mathcal{J}^* f = \int_{\mathcal{X}} k(\cdot, x) f(x) \, d\mathbb{P}(x)$ yielding an uncentered covariance operator $\Sigma = \mathcal{J}^* \mathcal{J} = \int_{\mathcal{X}} k(\cdot, x) \otimes \mathcal{H} k(\cdot, x) \, d\mathbb{P}(x)$. Since we work with centered covariance operator as defined in (4), we defined $\mathcal{J}$ appropriately as in (18).

Similarly, we define an approximation operator

$$
\mathfrak{A} : \mathbb{R}^m \rightarrow L^2(\mathbb{P}), \quad \beta \mapsto \sum_{i=1}^{m} \beta_i (\varphi_i - \varphi_{i,\mathbb{P}}),
$$

where $(\varphi_i)_{i \in [m]}$ denote the orthonormal basis in $\mathbb{R}^m$.
where
\[ \varphi_{i,P} := \int_X \varphi_i(x) \, d\mathbb{P}(x). \]

It can be shown (see Proposition B.4) that
\[ \mathfrak{A}^* : L^2(\mathbb{P}) \to \mathbb{R}^m, \quad f \mapsto (\langle f, \varphi_1 \rangle_{L^2(\mathbb{P})}, \ldots, \langle f, \varphi_m \rangle_{L^2(\mathbb{P})})^\top \] and \( \Sigma_m = \mathfrak{A}^* \mathfrak{A}. \)

Based on these operators, we redefine the reconstruction error for KPCA, EKPCA, RF-KPCA and RF-EKPCA in Section 4.1 and present convergence results comparing the statistical behavior of RF-EKPCA with that of EKPCA.

The second objective we consider to compare the behavior of EKPCA and RF-EKPCA is the convergence of eigenspaces of \( \hat{\Sigma} \) and \( \hat{\Sigma}_m \) to that of \( \Sigma \) in terms of the convergence of corresponding projection operators. While the convergence of eigenspace of \( \hat{\Sigma} \) to that of \( \Sigma \) is well-posed as both these eigenspaces are subspaces of \( \mathcal{H} \), the corresponding convergence of the eigenspace of \( \hat{\Sigma}_m \) to that of \( \Sigma \) is ill-posed. This is because the eigenspace of \( \hat{\Sigma}_m \) is a subspace in \( \mathbb{R}^m \) while that of \( \Sigma \) is a subspace in \( \mathcal{H} \). However, using the inclusion and approximation operators defined before in (18) and (19), we embed all these eigenspaces as subspaces in \( L^2(\mathbb{P}) \) and carry out the convergence study in Section 4.2.

### 4.1 Reconstruction Error: Convergence Analysis

Based on the discussion above, in this section, we redefine the reconstruction error associated with KPCA, EKPCA, RF-KPCA and RF-EKPCA as follows and then compare the convergence behavior of this error for EKPCA and RF-EKPCA. Define the reconstruction error associated with KPCA as
\[ R_{\Sigma,\ell} = \mathbb{E}_{X \sim \mathbb{P}} \left\| \mathcal{J} \hat{k}(:, X) - \sum_{i=1}^\ell \left\langle \mathcal{J} \hat{k}(:, X), \frac{\mathcal{J} \phi_i}{\sqrt{\lambda_i}} \right\rangle_{L^2(\mathbb{P})} \frac{\mathcal{J} \phi_i}{\sqrt{\lambda_i}} \right\|^2_{L^2(\mathbb{P})}. \] (20)

The above definition makes sense because as shown below, \( \left( \frac{\mathcal{J} \phi_i}{\sqrt{\lambda_i}} \right)_i \) forms an orthonormal system (ONS) in \( L^2(\mathbb{P}) \):
\[ \left\langle \frac{\mathcal{J} \phi_i}{\sqrt{\lambda_i}}, \frac{\mathcal{J} \phi_j}{\sqrt{\lambda_j}} \right\rangle_{L^2(\mathbb{P})} = \left\langle \mathcal{J}^* \mathcal{J} \phi_i, \phi_j \right\rangle_{\mathcal{H}} = \left\langle \mathcal{J}^* \mathcal{J} \phi_i, \phi_j \right\rangle_{\mathcal{H}} = \frac{\lambda_i}{\lambda_j} \left\langle \phi_i, \phi_j \right\rangle_{\mathcal{H}} = \delta_{ij}, \]
where we used Proposition B.2(iii) in (\( \ast \)). Similarly the reconstruction error associated with RF-KPCA can be defined as
\[ R_{\Sigma_m,\ell} = \mathbb{E}_{X \sim \mathbb{P}} \left\| \mathcal{J} \hat{k}(:, X) - \sum_{i=1}^\ell \left\langle \mathcal{J} \hat{k}(:, X), \frac{\mathcal{A} \phi_{m,i}}{\sqrt{\lambda_{m,i}}} \right\rangle_{L^2(\mathbb{P})} \frac{\mathcal{A} \phi_{m,i}}{\sqrt{\lambda_{m,i}}} \right\|^2_{L^2(\mathbb{P})}. \] (21)

which also makes sense as \( \left( \frac{\mathcal{A} \phi_{m,i}}{\sqrt{\lambda_{m,i}}} \right)_i \) forms an orthonormal system in \( L^2(\mathbb{P}) \) as
\[ \left\langle \frac{\mathcal{A} \phi_{m,i}}{\sqrt{\lambda_{m,i}}}, \frac{\mathcal{A} \phi_{m,j}}{\sqrt{\lambda_{m,j}}} \right\rangle_{L^2(\mathbb{P})} = \left\langle \mathfrak{A}^* \mathfrak{A} \phi_{m,i}, \phi_{m,j} \right\rangle_{\mathbb{R}^m} = \left\langle \mathfrak{A}^* \mathfrak{A} \phi_{m,i}, \phi_{m,j} \right\rangle_{\mathbb{R}^m} = \frac{\lambda_{m,i}}{\lambda_{m,j}} \left\langle \phi_{m,i}, \phi_{m,j} \right\rangle_{\mathbb{R}^m} = \delta_{ij}, \]
where we used Proposition B.4(iii) in (†). With these observations, we define the reconstruction error associated with EKPCA and RF-EKPCA as

$$R_{\Sigma, \ell} = \mathbb{E}_{X \sim \mathcal{P}} \left\| \mathcal{H}(\cdot, X) - \sum_{i=1}^{\ell} \left\langle \mathcal{H}(\cdot, X), \frac{\hat{\phi}_i}{\sqrt{\lambda_i}} \right\rangle_{L^2(\mathcal{P})} \right\|_{L^2(\mathcal{P})}^2$$

and

$$R_{\hat{\Sigma}, \ell} = \mathbb{E}_{X \sim \mathcal{P}} \left\| \mathcal{H}(\cdot, X) - \sum_{i=1}^{\ell} \left\langle \mathcal{H}(\cdot, X), \frac{\hat{\phi}_m,i}{\sqrt{\lambda_{m,i}}} \right\rangle_{L^2(\mathcal{P})} \right\|_{L^2(\mathcal{P})}^2$$

respectively. It is important to note that unlike with \((\mathcal{H}\phi_i/\sqrt{\lambda_i})_i\) and \((A\hat{\phi}_{m,i}/\sqrt{\lambda_{m,i}})_i\) which form ONS in \(L^2(\mathcal{P})\), \((\mathcal{H}\hat{\phi}_i/\sqrt{\lambda_i})_i\) and \((A\hat{\phi}_{m,i}/\sqrt{\lambda_{m,i}})_i\) do not form an ONS in \(L^2(\mathcal{P})\). Therefore, one may wonder whether the definitions of \(R_{\Sigma, \ell}\) and \(R_{\hat{\Sigma}, \ell}\) make sense. We show below that \(R_{\Sigma, \ell}\) and \(R_{\hat{\Sigma}, \ell}\) tend to zero as \(n \to \infty\), \(\ell \to \infty\) and \(m \to \infty\) with appropriate conditions on \(l, m, n\). This means, asymptotically \((\mathcal{H}\hat{\phi}_i/\sqrt{\lambda_i})_i\) and \((A\hat{\phi}_{m,i}/\sqrt{\lambda_{m,i}})_i\) capture all the information about \(k(\cdot, X)\) by forming an ONS in \(L^2(\mathcal{P})\).

The following result (proved in Section 6.2) presents an alternate and simple representation of \((20)-(23)\), which will be useful to obtain convergence rates on the above defined reconstruction errors. The result involves truncation operators which are heavily used in the proofs and therefore we define them separately as follows:

$$\Sigma_{\ell} = \sum_{i=1}^{\ell} \lambda_i \phi_i \otimes \mathcal{H} \phi_i, \quad \Sigma_{\ell}^{-1} = \sum_{i=1}^{\ell} \frac{1}{\lambda_i} \phi_i \otimes \mathcal{H} \phi_i, \quad \hat{\Sigma}_{\ell}^{-1} = \sum_{i=1}^{\ell} \frac{1}{\lambda_i} \hat{\phi}_i \otimes \mathcal{H} \hat{\phi}_i,$$

$$\Sigma_{m,i}^{-1} = \sum_{i=1}^{\ell} \frac{1}{\lambda_{m,i}} \phi_{m,i} \otimes \mathcal{H} \phi_{m,i} \quad \text{and} \quad \hat{\Sigma}_{m,i}^{-1} = \sum_{i=1}^{\ell} \frac{1}{\lambda_{m,i}} \hat{\phi}_{m,i} \otimes \mathcal{H} \hat{\phi}_{m,i}.$$

**Proposition 2.** The following hold:

(i) \(R_{\Sigma, \ell} = \left\| \Sigma - \Sigma_{\ell} \right\|_{L^2(\mathcal{H})}^2;\)

(ii) \(R_{\hat{\Sigma}, \ell} = \left\| \Sigma^{1/2} \left( I - \hat{\Sigma}_{\ell}^{-1} \Sigma \right) \Sigma^{1/2} \right\|_{L^2(\mathcal{H})}^2;\)

(iii) \(R_{\Sigma, \ell} = \left\| (I - A\Sigma_{m,i}^{-1} A^*) \mathcal{J} \mathcal{H}^* \right\|_{L^2(\mathcal{P})}^2;\)

(iv) \(R_{\hat{\Sigma}, \ell} = \left\| (I - A\hat{\Sigma}_{m,i}^{-1} A^*) \mathcal{J} \mathcal{H}^* \right\|_{L^2(\mathcal{P})}^2.\)

The above result shows that the reconstruction errors are the Hilbert-Schmidt norms of certain operators which approximate \(\Sigma = \mathcal{H}^* \mathcal{J}\) and \(\mathcal{J} \mathcal{H}^*\). Based on the representation provided above, the following result (proved in Section 6.3) provides a finite-sample bound on the reconstruction error, using which convergence rates can be obtained.
Theorem 3. Suppose \((A_1) - (A_4)\) hold. Define \(N_{\Sigma}(\ell) = \sum_{i > \ell} \lambda_i^2\). Then the following hold:

(i) \(R_{\Sigma,\ell} = N_{\Sigma}(\ell)\);

(ii) Let \(\varepsilon > 0\) and \(n \geq 1 \vee 8\varepsilon\). Suppose there exists \(a > 1\) such that \(\lambda_\ell \geq 7a\kappa\sqrt{\frac{2\varepsilon}{n}}\). Then

\[
\mathbb{P}^n \left\{ (X_i)_{i=1}^n : R_{\Sigma,\ell} \leq N_{\Sigma}(\ell) + C_1 \left( \frac{\varepsilon^{3/2}}{\lambda_\ell^2 n^{3/2}} + \frac{\sqrt{\varepsilon}}{\sqrt{n}} + \frac{\varepsilon}{n \lambda_\ell} + \frac{\varepsilon}{n} \right) \right\} \geq 1 - 4e^{-\varepsilon},
\]

where \(C_1\) is a constant that depends only on \(a\) and \(\kappa\) and not on \(n, \ell\) and \(\varepsilon\).

(iii) Let \(\varepsilon > 0\) and \(m \geq 1 \vee 8\varepsilon\). Then

\[
\Lambda^m \left\{ (\theta_j)_{j=1}^m : R_{\Sigma,m,\ell} \leq N_{\Sigma}(\ell) + C_2 \frac{\varepsilon}{m} \right\} \geq 1 - 2e^{-\varepsilon},
\]

where \(C_2\) is a constant that depends only on \(\kappa\) and not on \(m, \ell\) and \(\varepsilon\).

(iv) Let \(\varepsilon > 0\) and \(n \wedge m \geq 1 \vee 8\varepsilon\). Suppose there exists \(b > 1\) such that \(\lambda_\ell \geq 15 b\kappa \sqrt{\frac{2\varepsilon}{n\wedge m}}\). Then

\[
\mathbb{P}^n \times \Lambda^m \left\{ (X_i)_{i=1}^n, (\theta_j)_{j=1}^m : R_{\Sigma,m,\ell} \leq N_{\Sigma}(\ell) + C_3 \left( \frac{\varepsilon}{n\lambda_\ell^2 (m \wedge \sqrt{n})} + \frac{\sqrt{\varepsilon}}{\sqrt{n}} + \frac{\varepsilon}{n \lambda_\ell} + \frac{\varepsilon}{n \wedge m} \right) \right\} \geq 1 - 6e^{-\varepsilon},
\]

where \(C_3\) is a constant that depends only on \(b\) and \(\kappa\) and not on \(n, \ell\) and \(\varepsilon\).

Since \(\Sigma\) is trace class, it is obvious that \(\lambda_\ell \to 0\) and \(N_{\Sigma}(\ell) \to 0\) as \(\ell \to \infty\). By assuming a decay condition on \((\lambda_i)_i\), a convergence rate for \(N_{\Sigma}(\ell)\) can be obtained. From Theorem\(3\)(ii), it therefore follows that \(R_{\Sigma,\ell} \to 0\) as \(\ell, n \to \infty\) and \(\lambda_\ell \sqrt{n} \to \infty\) with the same convergence rate as that of \(R_{\Sigma,\ell}\) if \(\ell\) is chosen such that \(N_{\Sigma}(\ell)\) dominates \(\frac{\sqrt{\varepsilon}}{\sqrt{n}}\). Similarly, in Theorem\(3\)(iv), if \(m \geq c\sqrt{n}\) (where \(c\) is a constant independent of \(n\) and \(\ell\) and \(\lambda_\ell \sqrt{m \wedge n} \to \infty\) then the bound on \(R_{\Sigma,m,\ell}\) behaves similarly to that of \(R_{\Sigma,\ell}\) and therefore achieves the same convergence behavior as that of \(R_{\Sigma,\ell}\). This means, as long as \(m \geq c\sqrt{n}\) and \(\ell\) does not go to infinity too fast, the statistical behavior of RF-EKPCA is same as that of EKPCA, i.e., without losing any statistical efficiency, RF-EKPCA is computationally efficient compared to that of EKPCA. We observe similar behavior for the population versions, i.e., KPCA and RF-KPCA where if \(m\) is large enough, then RF-KPCA has similar reconstruction error to that of KPCA while RF-KPCA only requires solving a finite dimensional eigenvalue problem compared to the infinite dimensional eigenvalue problem for KPCA. The following corollaries to Theorem\(3\) investigate the statistical behavior of EKPCA and RF-EKPCA in detail under the polynomial and exponential decay condition on the eigenvalues of \(\Sigma\).

Corollary 4. Suppose \(A^{-\alpha} \leq \lambda_i \leq \overline{A}^{-\alpha}\) for \(\alpha > 1\) and \(\overline{A}, A \in (0, \infty)\). Let \(\ell = n^{\frac{\alpha}{2}}\), \(\theta > 0\). Then the following hold:

(i) \(n^{-2\theta(1 - \frac{1}{2\alpha})} \leq R_{\Sigma,\ell} \leq n^{-2\theta(1 - \frac{1}{2\alpha})}\);

(ii) There exists \(\tilde{n} \in \mathbb{N}\) such that for all \(n > \tilde{n}\),

\[
R_{\Sigma,\ell} \lesssim_p \begin{cases} n^{-2\theta(1 - \frac{1}{2\alpha})}, & 0 < \theta \leq \frac{\alpha}{2\alpha - 1} \\ n^{-\left(1 - \frac{\theta}{\alpha\alpha - 1}\right)}, & \frac{\alpha}{2\alpha - 1} < \theta < \frac{1}{2} \end{cases};
\]
(iii) 
\[ R_{Σ_m,ℓ} ≲_p \left\{ \begin{array}{ll}
 n^{-2θ(1−\frac{1}{2α})}, & \tau ≥ 2θ \left( 1 - \frac{1}{2α} \right) \\
 n^{−1}, & \tau ≤ 2θ \left( 1 - \frac{1}{2α} \right).
\end{array} \right. \]

(iv) There exists \( \tilde{n} \in \mathbb{N} \) such that for all \( n > \tilde{n} \),

\[ R_{\hat{Σ}_m,ℓ} ≲_p \left\{ \begin{array}{ll}
 n^{-2θ(1−\frac{1}{2α})}, & 0 < θ ≤ \frac{α}{4α−1} \\
 n^{-\left( \frac{1}{2} - \frac{α}{2α−1} \right)}, & \frac{α}{4α−1} ≤ θ < \frac{1}{2}
\end{array} \right. \]

for \( τ > 2θ \) with \( m = n^τ \).

Remark 1. (i) Since \( Σ \) is a trace class operator (see Proposition B.1(iii)), the condition of \( \alpha ≥ 1 \) is required. Comparing the behavior of \( R_{\hat{Σ}_ℓ} \) (i.e., the reconstruction error associated with EKPCA) to that of \( R_{Σ,ℓ} \) (i.e., the reconstruction error associated with KPCA), it follows that if \( ℓ \) grows to infinity slower than \( n^{1−1/2α} \), then KPCA and EKPCA have similar asymptotic behavior in reconstructing the kernel function in the mean squared sense. On the other hand, if \( ℓ \) grows to infinity faster than \( n^{1−1/2α} \), then EKPCA has a slower asymptotic convergence to zero than that of KPCA. This is precisely the bias-variance trade-off wherein choosing \( ℓ \) to grow slower than \( n^{1−1/2α} \) yields smaller variance and larger bias, i.e., the bias dominates the variance and \( R_{Σ,ℓ} \) is precisely of the same order as the bias. On the other hand, if \( ℓ \) grows faster than \( n^{1−1/2α} \), then variance dominates the bias and so EKPCA has slower convergence rates compared to KPCA.

(ii) Comparing \( R_{Σ_m,ℓ} \) (i.e., the reconstruction error associated with RF-EKPCA) with \( R_{\hat{Σ}_ℓ} \), it can be noted that RF-EKPCA has similar asymptotic behavior to that of EKPCA as long as the number of features \( m \) grow sufficiently fast to infinity, which is determined by the rate of growth of \( ℓ \). To obtain similar asymptotic behavior for RF-EKPCA and EKPCA, it is required that the error due to random feature approximation is of smaller order than the bias and variance terms that are controlled by \( ℓ \). Since bias dominates at smaller \( ℓ \) and variance at larger \( ℓ \), the number of random features to approximate EKPCA depends on the rate of growth of \( ℓ \).

(iii) Note that the choice of \( θ = \frac{α}{4α−1} \) yields the best rate of convergence for EKPCA and RF-EKPCA, which is given by \( n^{-1/2} \). Since \( \frac{α}{4α−1} \) is a decreasing function of \( α \) and \( \frac{2α−1}{4α−1} \) is an increasing function of \( α \), it is obvious that a faster rate of convergence with fewer random features is obtained for a smoother RKHS (i.e., large \( α \)) compared to that of a rougher RKHS. This means \( n^{−1/2} \) is the best convergence rate possible for EKPCA and RF-EKPCA with the minimal number of features being \( \sqrt{m} \), which are obtained as \( α → ∞ \). In addition, Corollary B(iv) yields that RF-EKPCA only requires \( n^{2/3} \) number of random features for any \( α > 1 \) to have performance similar to that of KPCA as \( θ = \frac{2}{4α−1} < \frac{1}{5} \).

(iv) Comparing \( R_{Σ_m,ℓ} \) to \( R_{Σ,ℓ} \), it follows that both have similar behavior if enough features are used, i.e., \( m \) is large enough depending on \( ℓ \). If \( m \) is not large enough, then the approximation error dominates resulting in a slower convergence rate for \( R_{Σ_m,ℓ} \). Also note that compared to RF-EKPCA, RF-KPCA requires slightly fewer random features to have a behavior similar to that of KPCA and this requirement of slightly more features is to counter the effect of sampling from \( P \).

Corollary 5. Suppose \( \underline{A}e^{−γi} ≤ λ_i ≤ \overline{A}e^{−γi} \) where \( \underline{A}, \overline{A} ∈ (0,∞) \) and \( γ > 0 \). Let \( ℓ = \frac{1}{γ} \log n^θ \), \( θ > 0 \). Then the following hold:

(i) \( n^{-2θ} ≲ R_{Σ,ℓ} ≲ n^{-2θ} \);
There exists \( \tilde{n} \in \mathbb{N} \) such that for all \( n > \tilde{n} \),

\[
R_{\Sigma, \ell} \lesssim_p \left\{ \begin{array}{ll}
\frac{n^{-2\theta}}{\log \frac{n}{n}}, & 0 < \theta < \frac{1}{4} \\
\frac{1}{4} \leq \theta < \frac{3}{2}
\end{array} \right.
\]

(iii) \[
R_{\Sigma_m, \ell} \lesssim_p \left\{ \begin{array}{ll}
\frac{n^{-2\theta}}{\log \frac{n}{n}}, & \tau \geq 2\theta \\
\frac{1}{\tau} \leq 2\theta
\end{array} \right.
\]

(iv) There exists \( \tilde{n} \in \mathbb{N} \) such that for all \( n > \tilde{n} \),

\[
R_{\Sigma_m, \ell} \lesssim_p \left\{ \begin{array}{ll}
\frac{n^{-2\theta}}{\log \frac{n}{n}}, & 0 < \theta < \frac{1}{4} \\
\frac{1}{4} \leq \theta < \frac{1}{2}
\end{array} \right.
\]

for \( \tau > 2\theta \) with \( m = n^\tau \).

While Corollary 5 also enjoys most of the observations raised in Remark 1, it can be seen that under the exponential decay assumption on the eigenvalues, all the methods (KPCA, EKPCA, RF-KPCA and RF-EKPCA) achieve the best convergence rate of \( n^{-\frac{1}{2}} \) up to \( \log n \) terms—such a rate is obtained in Corollary 4 as \( \alpha \to \infty \)—when \( \ell = \frac{1}{\gamma} \log \frac{n}{n} \) and \( m > \sqrt{n} \). In other words, since the RKHS considered in Corollary 5 is smoother than that of in Corollary 4, fast convergence rates for the reconstruction error is achieved while requiring fewer random features. Corollary 4 and 5 show that EKPCA and RF-EKPCA achieve optimal reconstruction error rates (with reference to KPCA) for an appropriate range of \( \ell \) and \( m \), which means RF-EKPCA does not suffer any statistical loss to achieve computational gain over EKPCA.

In the following theorem, we improve the bound in Theorem 3 (ii, iv) by making an additional assumption on the decay behavior of the eigen gaps. The proof (given in Section 6.5) involves bounding the weighted distance between certain projection operators and is of independent interest in the theory of perturbation of linear operators (see Theorem 6 (ii)). We specialize Theorem 6 to exponential decay behavior of the eigenvalues in Corollary 7, wherein we demonstrate that this improved bound provides optimal convergence rates but with weaker requirements on \( \ell \) and \( m \).

Theorem 6. Suppose \((A_1) - (A_4)\) hold. Define \( \delta_i = \frac{1}{2}(\lambda_i - \lambda_{i+1}) \), \( i \in \mathbb{N} \). The following hold:

(i) Let \( \varepsilon > 0 \) and \( n \geq 1 \lor 8\varepsilon \). Suppose \( \delta_\ell \geq 14\kappa \sqrt{\frac{2\varepsilon}{n} \log n} \) and there exists \( a > 1 \) such that \( \lambda_\ell \geq 7a\kappa \sqrt{\frac{2\varepsilon}{n}} \). Then

\[
\mathbb{P}^n \left\{ (X_i)_{i=1}^n : R_{\Sigma, \ell} \leq N_\Sigma(\ell) + C_4 \left( \frac{\varepsilon^{3/2}}{\lambda_\ell^2 n^{3/2}} + \frac{\varepsilon}{\lambda_\ell n} + \frac{\ell^2 \lambda_\ell^2 \varepsilon}{\delta_\ell^2 n} \right) \right\} \geq 1 - 4e^{-\varepsilon},
\]

where \( C_4 \) is a constant that depends only on \( a \) and \( \kappa \) and not on \( n, \ell \) and \( \varepsilon \).

(ii) Let \( \varepsilon > 0 \) and \( n \land m \geq 1 \lor 8\varepsilon \). Suppose there exists \( b > 1 \) and \( c > 1 \) such that \( \lambda_\ell \geq 15b\kappa \sqrt{\frac{2\varepsilon}{n\land m}} \) and \( \delta_\ell \geq 22ck \sqrt{\frac{2\varepsilon}{n\land m}} \). Then
\[ \mathbb{P}^n \times \Lambda^m \left\{ ((X_i)_{i=1}^n, (\theta_j)_{j=1}^m) : R_{\Sigma_{m,\ell}}^n \leq N_\Sigma(n) + C_5 \left( \frac{\varepsilon (\sqrt{\varepsilon} \vee \varepsilon)}{n \lambda_1^2(m \wedge \sqrt{n})} + \frac{\varepsilon}{\lambda m} + \frac{\ell^2 \lambda_1^2 \varepsilon}{\delta^2 n} + \frac{\varepsilon}{n \wedge m} \right) \right\} \leq 1 - 6e^{-\varepsilon}, \]

where \( C_5 \) is a constant that depends only on \( b, c \) and \( \kappa \) and not on \( n, \ell \) and \( \varepsilon \).

Comparing Theorem \( \text{(iii)} \) with Theorem \( \text{(i)} \), it is clear that if \( \frac{\ell^2 \lambda_1^2}{\delta^2 n} \) is of lower order than \( \frac{1}{\sqrt{n}} \), then the bound in the latter is sharper than that in the former. Similar result also holds while comparing the bound in Theorem \( \text{(ii)} \) to that of Theorem \( \text{(i)} \). The following corollary explores and compares the above result to Theorem \( \text{Corollary 7.} \)

Suppose \( A e^{-\gamma i} \leq \lambda_1 \leq A e^{-\gamma i} \) where \( A, A \in (0, \infty) \) and \( \gamma > \log \frac{A}{A} \). Let \( \ell = \frac{1}{\gamma} \log n^\theta \), \( \theta > 0 \). Then the following hold:

(i) There exists \( \tilde{n} \in \mathbb{N} \) such that for all \( n > \tilde{n} \),

\[ R_{\Sigma,\ell}^n \lesssim_p \begin{cases} n^{-2\theta}, & 0 < \theta \leq \frac{3}{4} \\ n^{-(1-\theta)}, & \frac{3}{4} \leq \theta < 1 \end{cases} \]

(ii) There exists \( \tilde{n} \in \mathbb{N} \) such that for all \( n > \tilde{n} \),

\[ R_{\Sigma_{m,\ell}}^n \lesssim_p \begin{cases} n^{-2\theta}, & 0 < \theta \leq \frac{3}{4} \\ n^{-(1-\theta)}, & \frac{3}{4} \leq \theta \leq 1 \end{cases} \]

for \( \tau > 2\theta \) with \( m = n^\tau \).

Remark 2. (i) Comparing Corollary \( \text{Corollary 7} \) to Corollary \( \text{Corollary 5} \), it is clear that the former provides the best rate of \( n^{-2/3} \) with \( \tau > \frac{2}{3} \) compared to the best rate of \( \sqrt{\log n/n} \) for the latter with \( \tau > \frac{1}{2} \). In fact in Corollary \( \text{Corollary 5} \), when \( \theta = \frac{1}{3} \), which corresponds to \( \tau > \frac{2}{3} \), the reconstruction error rate is only \( \sqrt{\log n/n} \) in contrast to \( n^{-2/3} \) in Corollary \( \text{Corollary 7} \). Also, when \( \theta = \frac{1}{4} \), we obtain the error rate of \( n^{-1/2} \) in Corollary \( \text{Corollary 7} \) compared to \( \sqrt{\log n/n} \) in Corollary \( \text{Corollary 5} \). Therefore, Corollary \( \text{Corollary 7} \) improves upon Corollary \( \text{Corollary 5} \) by exploiting the behavior of eigengaps.

(ii) Under the assumptions on the eigenvalue decay in Corollary \( \text{Corollary 7} \) we used the following lower bound on the eigengap in Theorem \( \text{Theorem 6} \) to obtain the rates in Corollary \( \text{Corollary 7} \) \( \delta_i \geq \frac{1}{2} e^{-\gamma i} (A - A e^{-\gamma}) \gtrsim e^{-\gamma i} \), where the assumption \( \gamma > \log \frac{A}{A} \) is exploited. Note that no specific assumption is made on the decay rate of eigengap as the decay rate of eigenvalues completely determine the decay behavior of the eigengap. In particular we have \( e^{-\gamma i} \lesssim \delta_i \lesssim e^{-\gamma i} \). On the other hand, the counterpart to Corollary \( \text{Corollary 4} \) under polynomial decay condition of eigenvalues for the setting of Theorem \( \text{Theorem 6} \) requires a lower decay condition on \( \delta_i \), i.e., \( \delta_i \geq B i^{-\beta} \) where \( \beta \geq \alpha \) and \( B \in (0, \infty) \). The assumption of \( \beta \geq \alpha \) is required as \( \delta_i \leq \frac{1}{2} i^{-\alpha} \left( 1 - A \left( \frac{i}{1 + i} \right)^\alpha \right) \lesssim i^{-\alpha} \). Therefore, in contrast to Theorem \( \text{Theorem 8} \), both \( \beta \) and \( \alpha \) control the reconstruction error rates, in Theorem \( \text{Theorem 6} \) both \( \beta \) and \( \alpha \) control the reconstruction error rates and the rate at which \( \ell \) and \( m \) can grow with \( n \). It can shown that the bound of Theorem \( \text{Theorem 8} \) is better than that of Theorem \( \text{Theorem 6} \) for a certain range of \( \beta \) and if \( \beta \) is too large, i.e., the eigengaps are too small, then \( \frac{\ell^2 \lambda_1^2}{\delta^2 n} \) will dominate the rate, in which case EKPCA and RF-EKPCA will have worse error rate behavior than that of KPCA.
4.2 Convergence Analysis of Eigenspaces

Apart from reconstruction error, another natural performance measure for EKPCA and RF-EKPCA is the convergence of corresponding “eigenspaces” to the eigenspace of KPCA. As mentioned in the last paragraph before Section 4.1, it is not possible to directly compare the eigenspaces corresponding to $\Sigma$, $\hat{\Sigma}$ and $\Sigma_m$ as the eigenspace corresponding to $\Sigma_m$ is a subspace of $\mathbb{R}^m$ whereas those of the former are subspaces in $\mathcal{H}$. Using the inclusion and approximation operators, we can define suitable embeddings of these eigenspaces to $L^2(\mathbb{P})$ which can therefore be compared. The following result (proved in Section 6.6) defines the suitable “projection” operators and establishes the relation between these operators and the reconstruction error. In fact, it shows the notion of eigenspace convergence (in terms of the convergence of these “projection” operators) to be stronger than that of the reconstruction error and therefore it is of interest to understand the performance of RF-EKPCA and EKPCA w.r.t. this notion of eigenspace convergence.

**Proposition 8.** Define

\[ P_\ell(\mathfrak{J}^\ast) := \sum_{i=1}^\ell \frac{\mathfrak{J}_i}{\sqrt{\lambda_i}} \otimes L^2(\mathbb{P}), \quad P_\ell(\mathfrak{A}^\ast) := \sum_{i=1}^\ell \frac{\mathfrak{A}_{\phi_{m,i}}}{\sqrt{\lambda_{m,i}}} \otimes L^2(\mathbb{P}), \]

\[ \hat{P}_\ell := \sum_{i=1}^\ell \frac{\hat{\lambda}_i}{\sqrt{\lambda_i}} \otimes L^2(\mathbb{P}) \quad \text{and} \quad \hat{P}_{m,\ell} := \sum_{i=1}^\ell \frac{\hat{\lambda}_{m,i}}{\sqrt{\lambda_{m,i}}} \otimes L^2(\mathbb{P}). \]

Then

(i) \[ \left| \sqrt{R_{\Sigma,\ell}} - \sqrt{R_{\hat{\Sigma},\ell}} \right| \leq \| \Sigma \|_{L^2(\mathcal{H})} \left\| P_\ell(\mathfrak{J}^\ast) - \hat{P}_\ell \right\|_{L^\infty(L^2(\mathbb{P}))}; \]

(ii) \[ \left| \sqrt{R_{\Sigma,\ell}} - \sqrt{R_{\hat{\Sigma},\ell}} \right| \leq \| \Sigma \|_{L^2(\mathcal{H})} \left\| P_\ell(\mathfrak{J}^\ast) - P_\ell(\mathfrak{A}^\ast) \right\|_{L^\infty(L^2(\mathbb{P}))}; \]

(iii) \[ \left| \sqrt{R_{\Sigma,\ell}} - \sqrt{R_{\hat{\Sigma},\ell}} \right| \leq \| \Sigma \|_{L^2(\mathcal{H})} \left\| P_\ell(\mathfrak{J}^\ast) - \hat{P}_{m,\ell} \right\|_{L^\infty(L^2(\mathbb{P}))}. \]

Note that $P_\ell(\mathfrak{J}^\ast)$ and $P_\ell(\mathfrak{A}^\ast)$ are orthogonal projection operators onto the span of $(\mathfrak{J}_i/\sqrt{\lambda_i})_{i=1}^\ell$ and $(\mathfrak{A}_{\phi_{m,i}}/\sqrt{\lambda_{m,i}})_{i=1}^\ell$, which are the eigenfunctions corresponding to the top $\ell$ eigenvalues of $\mathfrak{J}^\ast$ and $\mathfrak{A}^\ast$ respectively. While $\hat{P}_\ell$ and $\hat{P}_{m,\ell}$ are not projection operators onto the span of $(\hat{\mathfrak{J}}_i/\sqrt{\hat{\lambda}_i})_{i=1}^\ell$ and $(\hat{\mathfrak{A}}_{\phi_{m,i}}/\sqrt{\hat{\lambda}_{m,i}})_{i=1}^\ell$ respectively, they can be thought of as approximations to $P_\ell(\mathfrak{J}^\ast)$ and therefore the study of convergence behavior of the r.h.s. in Proposition 8(i,iii) is interesting. The following result (proved in Section 6.7) provides a finite sample bound on the convergence of “projection” operators, using which convergence rates are derived in Corollary 10 and 11.

**Theorem 9.** Suppose $(A_1) - (A_4)$ hold. Define $\delta_i = \frac{1}{2} (\lambda_i - \lambda_{i+1})$, $i \in \mathbb{N}$. The following hold:

(i) Let $\varepsilon > 0$ and $n \geq 1 \lor 8\varepsilon$ . Suppose $\delta_\ell \geq 14\kappa \sqrt{\frac{2\varepsilon}{n}}$ and there exists $a > 1$ such that $\lambda_\ell \geq 7ak \frac{\sqrt{2\varepsilon}}{n}$. Then

\[ \mathbb{P}^n \left\{ (X_i)_{i=1}^n : \left\| P_\ell(\mathfrak{J}^\ast) - \hat{P}_\ell \right\|_{L^\infty(L^2(\mathbb{P}))} \leq C_6 \left( \sqrt{\frac{\varepsilon}{\delta_\ell \sqrt{n}}} + \sqrt{\frac{\varepsilon}{\lambda_\ell \sqrt{n}}} + \frac{\varepsilon^{1/4}}{\sqrt{\lambda_\ell n^{1/4}}} \right) \right\} \geq 1 - 4e^{-\varepsilon}, \]

where $C_6$ is a constant that depends only on $a$ and $\kappa$ and not on $n, \ell$ and $\varepsilon$. 

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(ii) Let $\varepsilon > 0$ and $m \geq 1 \lor 8\varepsilon$. Then

$$\Lambda^m \left\{ (\theta_j)_{j=1}^m : \| P_\ell(\mathcal{H}^*) - P_\ell(\mathfrak{A}^*) \|_{L^\infty(L^2(\mathbb{P}))} \leq C_7 \frac{\sqrt{\varepsilon}}{\delta \ell \sqrt{m}} \right\} \geq 1 - 2e^{-\varepsilon},$$

where $C_7$ is a constant that does not depend on $m$, $\ell$ and $\varepsilon$.

(iii) Let $\varepsilon > 0$ and $n \land m \geq 1 \lor 8\varepsilon$. Suppose there exists $b > 1$ and $c > 1$ such that $\lambda_\ell \geq 15bk\sqrt{\frac{2\varepsilon}{n \land m}}$ and $\delta_\ell \geq 30ck\sqrt{\frac{2\varepsilon}{n \land m}}$. Then

$$\mathbb{P}^n \times \Lambda^m \left\{ \left( (X_i)_{i=1}^n, (\theta_j)_{j=1}^m \right) : \| P_\ell(\mathcal{H}^*) - \widehat{P}_m,\ell \|_{L^\infty(L^2(\mathbb{P}))} \leq C_8 \left( \frac{\sqrt{\varepsilon}}{\delta_\ell \sqrt{m}} + \frac{\sqrt{\varepsilon}}{\lambda_\ell \sqrt{n}} + \varepsilon^{1/4} \right) \right\} \geq 1 - 6e^{-\varepsilon},$$

where $C_8$ is a constant that depends only on $b$, $c$ and $\kappa$ and not on $n$, $\ell$ and $\varepsilon$.

Comparing the bounds in Theorem 9, it is clear that for $m \geq c' \sqrt{n}$, the eigenspaces corresponding to EKPCA and RF-EKPCA have similar statistical convergence behavior to $P_\ell(\mathcal{H}^*)$. The following corollaries specialize Theorem 9 to the polynomial and exponential decay behaviors of the eigenvalues.

**Corollary 10.** Suppose $\underline{A} \alpha^{-\alpha} \leq \lambda_i \leq \overline{A} \alpha^{-\alpha}$, $\delta \ell \geq B i^{-\beta}$ for $\alpha > 1$, $\beta \geq \alpha$ and $\overline{A}, \overline{A}, B \in (0, \infty)$. Let $\ell = n^{\frac{2}{\alpha}}$, $\theta \geq 0$ and $m = n^{\tau}$, $\tau \in (0, 1]$. Then the following hold:

(i) There exists $\tilde{n} \in \mathbb{N}$ such that for all $n > \tilde{n}$,

$$\| P_\ell(\mathcal{H}^*) - \widehat{P}_n \|_{L^\infty(L^2(\mathbb{P}))} \lesssim_p n^{-\left( \frac{1}{2} - \frac{\theta}{\alpha} \right) \lor \left( \frac{1}{2} - \frac{\theta}{2(2\beta - \alpha)} \right)} \lor \left( \frac{1}{2} - \frac{\theta}{\alpha} \right)};$$

(ii) $\| P_\ell(\mathcal{H}^*) - P_\ell(\mathcal{A}^*) \|_{L^\infty(L^2(\mathbb{P}))} \lesssim_p n^{-\left( \frac{1}{2} - \frac{\theta}{\alpha} \right)}$;

(iii) There exists $\tilde{n} \in \mathbb{N}$ such that for all $n > \tilde{n}$,

$$\| P_\ell(\mathcal{H}^*) - \widehat{P}_m,\ell \|_{L^\infty(L^2(\mathbb{P}))} \lesssim_p \begin{cases} n^{-\left( \frac{1}{2} - \frac{\theta}{2(2\beta - \alpha)} \right)}, & 0 \leq \theta < \frac{\theta(2\beta - \alpha)}{2}, \tau > \frac{1}{2} + \frac{\theta(2\beta - \alpha)}{\alpha} \\ n^{-\left( \frac{1}{2} - \frac{\theta}{\alpha} \right)}, & 0 \leq \theta < \frac{\theta(2\beta - \alpha)}{2}, \tau \leq \frac{1}{2} + \frac{\theta(2\beta - \alpha)}{\alpha} \lor \frac{\theta(2\beta - \alpha)}{\alpha} \land \theta \geq \frac{2\beta}{\alpha} \lor \frac{\theta(2\beta - \alpha)}{\alpha} \land \theta \geq \frac{2\beta}{\alpha} \\ n^{-\left( \frac{1}{2} - \frac{\theta}{\alpha} \right)}, & \frac{\theta}{2(2\beta - \alpha)} \leq \theta < \frac{\theta}{2}, \tau > \frac{2\beta}{\alpha} \lor \frac{\theta}{2} \land \theta \geq \frac{2\beta}{\alpha} \lor \frac{\theta}{2} \land \theta \geq \frac{2\beta}{\alpha} \end{cases}.$$
(iii) There exists $\tilde{n} \in \mathbb{N}$ such that for all $n > \tilde{n}$,
\[
\|P_{\ell} (\mathfrak{M}^*) - \hat{P}_{m, \ell}\|_{L^\infty (L^2 (\mathbb{P}))} \leq_{\mathcal{P}} \begin{cases} 
 n^{-\left(\frac{1}{2} - \frac{\theta}{2}\right)}, & 0 \leq \theta < \frac{1}{2}, \tau \geq \frac{1}{2} + \theta \\
 n^{-\left(\frac{\tau}{2} - \theta\right)}, & 0 \leq \theta < \frac{1}{2}, \tau \leq \frac{1}{2} + \theta
\end{cases}.
\]

Remark 3. (i) It is clear from Corollary 10 and 11 that the best rate of convergence for EKPCA and RF-EKPCA is $n^{-1/4}$ and is obtained at $\theta = 0$, i.e., for a constant $\ell$ that does not grow with $n$, and the rate becomes slower with increase in $\ell$. EKPCA and RF-EKPCA have similar statistical behavior not only for fixed $\ell$ with $m > \sqrt{n}$, but also for $\ell$ growing with $n$ as long as $\ell$ does not grow too fast (in the polynomial case) with $n$ and enough random features ($m$) are used. On the other hand, if $\ell$ grows too fast with $n$ (in the polynomial case) and fewer features are used, the behavior of RF-EKPCA is different with slower convergence rates than that of EKPCA, which means, RF-EKPCA achieves computational advantage at the cost of statistical efficiency.

(ii) $\beta$ along with $\alpha$ controls the rate at which $\ell$ can grow with $n$. If the eigenvalue gaps are arbitrarily small, i.e., $\beta$ arbitrarily large, then one can only work with fixed $\ell$, i.e., $\ell$ cannot grow with $n$.

(iii) Using Corollary 10 and 11 rates of convergence for $R_{\Sigma, \ell}$ and $R_{\Sigma_m, \ell}$ can be obtained by employing the bounds in Proposition 8, resulting in weaker rates with stronger requirements on $\theta$ and $\tau$ than that obtained in Corollary 4 and 5.

5 Summary & Discussion

To summarize, we investigated the computational vs. statistical trade-off in the problem of approximating kernel PCA using random features. While it is obvious that approximate kernel PCA using $m$ random features has lower computational complexity than kernel PCA when $m < n$ with $n$ being the number of samples, it is not obvious that this computational gain is not achieved at the cost of statistical efficiency. Through inclusion and approximation operators, we defined two appropriate statistical notions to study the statistical behavior of kernel PCA and its approximate version: (i) error in reconstructing a kernel function using $\ell$ eigenfunctions and (ii) error in the “projection” operators that correspond to the eigenspace spanned by $\ell$ eigenfunctions. In both these settings, we showed that kernel PCA and its approximate version have similar statistical behavior as long as $m$ is large enough (but still $m < n$) with $m$ depending on the number of eigenfunctions $\ell$ being considered. If $\ell$ is large, then more features are needed to maintain the statistical behavior, which increases the computational cost. In particular, we showed that for smooth RKHS—exponential decay in the eigenvalues of the covariance operator—, only $\sqrt{n}$ random features are needed to achieve optimal convergence rates in the reconstruction error.

An open question to be answered is whether the requirement on $m$ optimal? The current requirement of $m > n^{2\theta}$ where $\theta = \alpha \log \ell n$ arises from the requirement that $\lambda_\ell \sqrt{m} \geq c$ where $\lambda_\ell$ is the $\ell^{th}$ largest eigenvalue of the covariance operator $\Sigma$ and $c$ is some constant. Note that $\alpha$ denotes the index in the polynomial decay of the eigenvalues of $\Sigma$. This requirement of $\lambda_\ell \sqrt{m} \geq c$ appears in the proof because of the following trivial bound we employed: $|\lambda_\ell (A) - \hat{\lambda}_\ell (B)| \leq \|(\lambda_i (A) - \hat{\lambda}_i (B))\|_{L^2} \leq \|A - B\|_{L^2 (H)}$ for $A = \Sigma, \Sigma_m$ and its empirical versions $B = \tilde{\Sigma}, \tilde{\Sigma}_m$. Here $A$ and $B$ denote self-adjoint, positive, Hilbert-Schmidt operators on a separable Hilbert space, $H$. Since $\lambda_\ell (A) \to 0$ and $\lambda_\ell (B) \to 0$ as $\ell \to \infty$, it is reasonable to expect that $|\lambda_\ell (A) - \hat{\lambda}_\ell (B)|$ gets small with large $\ell$ and therefore the bound of $\|A - B\|_{L^2 (H)}$ for all $\ell$ is trivial. It will be interesting
to see whether relative bounds of the type $|\lambda_{\ell}(A) - \lambda_{\ell}(B)| \leq \lambda_{\ell}(A)\|A - B\|_{L^2(H)}$ can be obtained, which will in turn weaken the requirement on $m$.

6 Proofs

In this section we present the proofs of the results in Sections 3 and 4.

6.1 Proof of Proposition 1

Define the sampling operator

$$ S : H \to \mathbb{R}^n, \quad f \mapsto \frac{1}{\sqrt{n}}(f(X_1), \ldots, f(X_n))^\top $$

whose adjoint, called the reconstruction operator can be shown (see Proposition B.1 (i)) to be

$$ S^* : \mathbb{R}^n \to H, \quad \alpha \mapsto \frac{1}{\sqrt{n}} \sum_{i=1}^n \alpha_i k(\cdot, X_i), $$

where $\alpha := (\alpha_1, \ldots, \alpha_n)^\top$. It follows from Proposition B.1 (ii) that $\hat{\Sigma} = S^* H_n S$, which implies

$$ (\hat{\phi}_i) \text{ satisfy } S^* H_n S \hat{\phi}_i = \hat{\lambda}_i \hat{\phi}_i, $$

where $\hat{\lambda}_i \geq 0$. Multiplying both sides of (24) on the left by $S$, we obtain that $(\hat{\alpha}_i), \hat{\alpha}_i := S \hat{\phi}_i, i \in [n]$ are eigenvectors of $S S^* H_n = \frac{1}{n} K H_n$, i.e., they satisfy the finite dimensional linear system

$$ K H_n \hat{\alpha}_i = n \hat{\lambda}_i \hat{\alpha}_i, $$

(25)

where $K$ is the Gram matrix, i.e., $(K)_{ij} = k(X_i, X_j), i, j \in [n]$ and the fact that $K = n S S^*$ follows from Proposition B.1 (iii). It is important to note that $(\hat{\alpha}_i)_i$ do not form an orthogonal system in the usual Euclidean inner product but in the weighted inner product where the weighting matrix is $H_n$. Indeed, it is easy to verify that

$$ \langle \hat{\alpha}_i, H_n \hat{\alpha}_j \rangle_2 = \langle S \hat{\phi}_i, H_n S \hat{\phi}_j \rangle_2 = \langle \hat{\phi}_i, \hat{\Sigma} \hat{\phi}_j \rangle_H = \hat{\lambda}_j \langle \hat{\phi}_i, \hat{\phi}_j \rangle_H = \hat{\lambda}_j \delta_{ij}, $$

where $\delta_{ij}$ is the Kronecker delta. Having obtained $(\hat{\alpha}_i)_i$ from (25), the eigenfunctions of $\hat{\Sigma}$ are obtained from (24) as

$$ \hat{\phi}_i = \frac{1}{\hat{\lambda}_i} S^* H_n \hat{\alpha}_i. $$

(26)

The result therefore follows by applying $S^*$ to $H_n \hat{\alpha}_i = \hat{\alpha}_i - \frac{1}{n}(1^\top \hat{\alpha}_i)1$.

Remark 4. A result similar to (25) and (26) is usually obtained through representer theorem [Kimeldorf and Wahba, 1971; Schölkopf et al., 2001] which yields that $\hat{\phi}_i = \sum_{j=1}^n \beta_{i,j} k(\cdot, X_j)$ where $\beta_{i,j} = K^{-1/2} \delta_{ij}$ and $(\delta_{ij})_i$ are eigenvectors of $K^{1/2} H_n K^{1/2}$ assuming $K$ is invertible. In the above proof, we do not require the invertibility of $K$.

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6.2 Proof of Proposition 2

Before presenting the proof, let us consider the following simple lemma.

**Lemma 12.** Let $H$ and $G$ be Hilbert spaces. For any $A, B : H \to G$ and $f, g \in H$,
\[
\langle Af, Bg \rangle_GBg = B(g \otimes_H g)B^*Af.
\]

**Proof.** $\langle Af, Bg \rangle_GBg = B(B^*Af, g)hg = B(g \otimes_H g)B^*Af.$ \qed

(i) Based on Lemma 12 it follows that
\[
R_{\Sigma, \ell} = E \| J K(\cdot, X) - J \Sigma^{-1} \Sigma K(\cdot, X) \|^2_{L^2(\mathbb{P})}^{(*)} = E \| J K(\cdot, X) - J \Sigma^{-1} \Sigma K(\cdot, X) \|^2_{L^2(\mathbb{P})}
\]

\[
= E \langle J (K(\cdot, X) - \Sigma^{-1} \Sigma K(\cdot, X)) , J (k(\cdot, X) - \Sigma^{-1} \Sigma K(\cdot, X)) \rangle_{L^2(\mathbb{P})}
\]

\[
= E \langle \Sigma (k(\cdot, X) - \Sigma^{-1} \Sigma K(\cdot, X)) , (k(\cdot, X) - \Sigma^{-1} \Sigma K(\cdot, X)) \rangle_{\mathcal{H}}
\]

\[
= E \langle \Sigma (I - \Sigma^{-1} \Sigma) \bar{k}(\cdot, X) , (I - \Sigma^{-1} \Sigma) \bar{k}(\cdot, X) \rangle_{\mathcal{H}}
\]

\[
= E \langle \Sigma (I - \Sigma^{-1} \Sigma) \bar{k}(\cdot, X) \otimes_{\mathcal{H}} k(\cdot, X) \rangle_{L^2(\mathcal{H})}.
\]

where we used Proposition 12(iii) in $(*)$. By Lemma 12, since $E\|k(\cdot, X) \otimes_{\mathcal{H}} k(\cdot, X)\|^2_{L^2(\mathcal{H})} = E\|k(\cdot, X)\|^2_{L^2(\mathcal{H})} < \infty$ as $k$ is bounded, $k(\cdot, X) \otimes_{\mathcal{H}} k(\cdot, X)$ is $\mathbb{P}$-integrable in the Bochner sense (see Diestel and Uhl 1977, Definition 1 and Theorem 2), and therefore it follows from Diestel and Uhl 1977, Theorem 6) that

\[
R_{\Sigma, \ell} = \langle \langle I - \Sigma^{-1} \Sigma \rangle^* \Sigma (I - \Sigma^{-1} \Sigma) , E \bar{k}(\cdot, X) \otimes_{\mathcal{H}} k(\cdot, X) \rangle \rangle_{L^2(\mathcal{H})}.
\]

Note that $\Sigma \Sigma^{-1} \Sigma = \Sigma^{-1} \Sigma \Sigma = \Sigma$ and the result follows.

(ii) Based on Lemma 12 it follows that $R_{\Sigma, \ell} = E \| J K(\cdot, X) - J \Sigma^{-1} \Sigma K(\cdot, X) \|^2_{L^2(\mathbb{P})}$. Carrying out the calculations verbatim as in (i), we obtain

\[
R_{\Sigma, \ell} = \langle \Sigma (I - \Sigma^{-1} \Sigma) , (I - \Sigma^{-1} \Sigma) \Sigma \rangle_{L^2(\mathcal{H})},
\]

which is exactly the r.h.s. of (27) but with $\Sigma^{-1} \Sigma$ replaced by $\Sigma^{-1} \Sigma$. Therefore,

\[
R_{\Sigma, \ell} = \langle \Sigma (I - \Sigma^{-1} \Sigma) , (I - \Sigma^{-1} \Sigma) \Sigma \rangle_{L^2(\mathcal{H})} = \text{Tr} \left[ (I - \Sigma^{-1} \Sigma)^* \Sigma (I - \Sigma^{-1} \Sigma) \right]
\]

\[
= \text{Tr} \left[ \Sigma^{1/2} (I - \Sigma^{-1} \Sigma)^* \Sigma (I - \Sigma^{-1} \Sigma) \Sigma^{1/2} \right] = \left\| \Sigma^{1/2} (I - \Sigma^{-1} \Sigma) \Sigma^{1/2} \right\|^2_{L^2(\mathcal{H})}.
\]

(iii) Again based on Lemma 12 we obtain

\[
R_{\Sigma, \ell} = E \| J K(\cdot, X) - \Sigma \Sigma^{-1} \Sigma K(\cdot, X) \|^2_{L^2(\mathbb{P})} = E \left\langle \left( I - \Sigma \Sigma^{-1} \Sigma \right) J K(\cdot, X) \right\rangle_{L^2(\mathbb{P})}
\]

\[
= E \left\langle \left( I - \Sigma \Sigma^{-1} \Sigma \right) J K(\cdot, X) , \left( I - \Sigma \Sigma^{-1} \Sigma \right) J K(\cdot, X) \right\rangle_{L^2(\mathbb{P})}
\]

\[
= E \left\langle J \left( I - \Sigma \Sigma^{-1} \Sigma \right) J K(\cdot, X) , \left( I - \Sigma \Sigma^{-1} \Sigma \right) J K(\cdot, X) \right\rangle_{\mathcal{H}}.
\]
and the result follows. Here in (†), we used the Bochner integrability of $k(\cdot, X) \otimes \mathcal{H} k(\cdot, X)$ as in (i) along with Proposition B.2(iii).

(iv) The proof is exactly the same as (iii) but with $\Sigma_{m,t}$ replaced by $\tilde{\Sigma}_{m,t}$.

6.3 Proof of Theorem 3

(i) The result follows by noting that

$$R_{\Sigma, t} = \|\Sigma - \Sigma_t\|_{L^2(\mathcal{H})}^2 = \left\| \sum_{i > t} \lambda_i \phi_i \otimes \mathcal{H} \phi_i \right\|_{L^2(\mathcal{H})}^2 = \sum_{i > t} \lambda_i^2.$$

(ii) Consider

$$\Sigma^{1/2} \left( I - \tilde{\Sigma}^{-1} \Sigma \right) \Sigma^{1/2} = \Sigma - \Sigma^{1/2} \tilde{\Sigma}_{t-1}^{-1} \Sigma^{3/2}$$

$$= \Sigma - \left( \Sigma^{1/2} - \tilde{\Sigma}_{t-1}^{1/2} + \tilde{\Sigma}_{t-1}^{1/2} \right) \tilde{\Sigma}_{t-1}^{-1} \left( \Sigma^{3/2} - \tilde{\Sigma}_{t-1}^{3/2} + \tilde{\Sigma}_{t-1}^{3/2} \right)$$

$$= \Sigma - \left( \Sigma^{1/2} - \tilde{\Sigma}_{t-1}^{1/2} \right) \tilde{\Sigma}_{t-1}^{-1} \left( \Sigma^{3/2} - \tilde{\Sigma}_{t-1}^{3/2} \right) - \tilde{\Sigma}_{t-1}^{1/2} \tilde{\Sigma}_{t-1}^{-1} \left( \Sigma^{3/2} - \tilde{\Sigma}_{t-1}^{3/2} \right)$$

Therefore

$$\sqrt{R_{\Sigma, t}} = \left\| \Sigma^{1/2} \left( I - \tilde{\Sigma}^{-1} \Sigma \right) \Sigma^{1/2} \right\|_{L^2(\mathcal{H})} \leq (1) + (2) + (3) + (4). \quad (28)$$

We now bound (1) – (4) as follows:

$$1 := \left\| \Sigma - \tilde{\Sigma}^{1/2} \tilde{\Sigma}_{t-1}^{-1} \Sigma^{3/2} \right\|_{L^2(\mathcal{H})} = \left\| \Sigma - \tilde{\Sigma}_{t-1} \right\|_{L^2(\mathcal{H})}$$

$$\leq \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} + \left\| \tilde{\Sigma} - \tilde{\Sigma}_{t-1} \right\|_{L^2(\mathcal{H})} = \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} + \left\| \lambda_i > t \right\|_{L^2}$$

$$\leq \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} + \left\| (\lambda_i > t) \right\|_{L^2} + \left\| (\lambda_i - \tilde{\lambda}_i) \right\|_{L^2}$$

$$\leq \sqrt{N(\Sigma)} + 2 \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})}, \quad \text{where (†) follows from Theorem B.2(i).}$$

Let us assume that there exists $a > 1$ such that

$$\lambda_t \geq a \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})}. \quad (29)$$

$$2 := \left\| \left( \Sigma^{1/2} - \tilde{\Sigma}_{t-1}^{1/2} \right) \tilde{\Sigma}_{t-1}^{-1} \left( \Sigma^{3/2} - \tilde{\Sigma}_{t-1}^{3/2} \right) \right\|_{L^2(\mathcal{H})}$$

$$\leq \left\| \Sigma^{1/2} - \tilde{\Sigma}_{t-1}^{1/2} \right\|_{L^\infty(\mathcal{H})} \left\| \tilde{\Sigma}_{t-1}^{-1} \right\|_{L^\infty(\mathcal{H})} \left\| \Sigma^{3/2} - \tilde{\Sigma}_{t-1}^{3/2} \right\|_{L^2(\mathcal{H})}$$

$$\leq \frac{3}{2\lambda_t} \left( \left\| \Sigma \right\|_{L^\infty(\mathcal{H})}^{1/2} \left\| \Sigma \right\|_{L^\infty(\mathcal{H})}^{1/2} \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} \right) \leq \frac{3}{2\lambda_t} \left( \left\| \Sigma \right\|_{L^\infty(\mathcal{H})} + \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} \right) \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})}^{3/2}$$

$$\leq \left( \frac{3}{2\lambda_t} \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} \right) \left( \left\| \Sigma \right\|_{L^\infty(\mathcal{H})} + \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} \right) \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})}^{3/2}$$

$$\leq \left( \frac{3}{2\lambda_t} \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} \right) \left( \left\| \Sigma \right\|_{L^\infty(\mathcal{H})} + \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} \right) \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})}^{3/2}$$

$$\leq \left( \frac{3}{2\lambda_t} \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} \right) \left( \left\| \Sigma \right\|_{L^\infty(\mathcal{H})} + \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} \right) \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})}^{3/2}$$

$$\leq \left( \frac{3}{2\lambda_t} \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} \right) \left( \left\| \Sigma \right\|_{L^\infty(\mathcal{H})} + \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} \right) \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})}^{3/2}$$

$$\leq \left( \frac{3}{2\lambda_t} \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} \right) \left( \left\| \Sigma \right\|_{L^\infty(\mathcal{H})} + \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})} \right) \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})}^{3/2}$$
where (**) follows from Theorem D.2 (ii, iii) and (†) follows from Theorem D.2 (i).

\[ \leq \frac{3}{2\sqrt{\lambda_1}} \left( \|\Sigma\|^{1/2}_{L^\infty(H)} \vee \|\hat{\Sigma}\|^{1/2}_{L^\infty(H)} \right) \|\Sigma - \hat{\Sigma}\|^{1/2}_{L^2(H)} \]

and

\[ \leq \frac{3}{2\lambda_2 - \|\Sigma - \hat{\Sigma}\|_{L^2(H)}^2} \left( \|\Sigma\|^{1/2}_{L^\infty(H)} + \|\Sigma - \hat{\Sigma}\|^{1/2}_{L^2(H)} \right) \|\Sigma - \hat{\Sigma}\|_{L^2(H)} \]

where the bounds for (3) and (4) use results from Theorem D.2 similar to that of in (ii). Also The result therefore follows by combining (1) – (4) in (29), noting that \( \lambda_1 \leq \|\Sigma\|_{L^2(H)} \leq 2\kappa \), using Lemma C.1 to bound \( \|\Sigma - \hat{\Sigma}\|_{L^2(H)} \) and verifying that (29) holds under the assumption that \( \lambda_\ell \geq 7\tau_\kappa \sqrt{\frac{2\kappa}{n}} \).

(iii) Note that

\[ \sqrt{R_{\Sigma_m,\ell}} = \left\| (I - \mathfrak{A} \Sigma_m^{-1} \mathfrak{A}^*) \mathfrak{J}^* \right\|_{L^2(L^2(\mathbb{P}))} \leq 5 + 6, \]

where 5 := \( \left\| (I - \mathfrak{A} \Sigma_m^{-1} \mathfrak{A}^*) \mathfrak{J}^* \right\|_{L^2(L^2(\mathbb{P}))} \) and 6 := \( \left\| (I - \mathfrak{A} \Sigma_m^{-1} \mathfrak{A}^*) (\mathfrak{J}^* - \mathfrak{A} \mathfrak{A}^*) \right\|_{L^2(L^2(\mathbb{P}))} \). We now bound (5) and (6) as follows.

\[ \leq \text{Tr}^{1/2} \left[ \mathfrak{A} \mathfrak{A}^* (I - \mathfrak{A} \Sigma_m^{-1} \mathfrak{A}^*) \mathfrak{J}^* \mathfrak{J}^* \right] \]

\[ \leq \text{Tr}^{1/2} \left[ \mathfrak{S}_m (\mathfrak{J}^* - \mathfrak{S}_m \Sigma_m^\perp \mathfrak{A}^*) (\mathfrak{A} - \mathfrak{A} \Sigma_m^{-1} \mathfrak{A} \mathfrak{A}^*) \right] \]

\[ \leq \text{Tr}^{1/2} \left[ \mathfrak{S}_m (I - \Sigma_m \Sigma_m^\perp \Sigma_m^\perp \Sigma_m) \right] \]

\[ \leq \left\| \mathfrak{J}^* - \mathfrak{A} \mathfrak{A}^* \right\|_{L^2(L^2(\mathbb{P}))} + \sqrt{\mathbb{N}\Sigma(\ell)}, \]
where we used Proposition B.4(iii) in (\(*) and Theorem D.2(i) in (\(†)).

\[ 6 = \| (I - \mathfrak{A}_m^{-1} \mathfrak{A}^*) (\mathfrak{A}^* - \mathfrak{A} \mathfrak{A}^*) \|_{L^2(P)} \leq \| \mathfrak{A}^* - \mathfrak{A} \mathfrak{A}^* \|_{L^2(P)} \| I - \mathfrak{A}_m^{-1} \mathfrak{A}^* \|_{L^\infty(P)} \]  

where

\[ \| I - \mathfrak{A}_m^{-1} \mathfrak{A}^* \|_{L^\infty(P)} \leq 1 + \| \mathfrak{A}_m^{-1} \mathfrak{A}^* \|_{L^\infty(P)} = 2, \]  

since \( \mathfrak{A}_m^{-1} \mathfrak{A}^* = \sum_{i=1}^\ell \frac{\mathfrak{A} \phi_{m,i}}{\sqrt{\lambda_{m,i}}} \otimes L^2(P) \frac{\mathfrak{A} \phi_{m,i}}{\sqrt{\lambda_{m,i}}} \) is an orthogonal projection operator onto the span of \( (\mathfrak{A} \phi_{m,i}/\sqrt{\lambda_{m,i}})_{i=1}^\ell \). The result therefore follows by combining the bounds on \( 5 \) and \( 6 \) in \( 30 \).

(iv) Note that

\[ \sqrt{R_{\Sigma_m,\ell}} = \| (I - \mathfrak{A}_m^{-1} \mathfrak{A}^*) \mathfrak{A} \|_{L^2(P)} \]  

where \( 7 \) := \( \| (I - \mathfrak{A}_m^{-1} \mathfrak{A}^*) \mathfrak{A} \|_{L^2(P)} \) and \( 8 \) := \( \| (I - \mathfrak{A}_m^{-1} \mathfrak{A}^*) (\mathfrak{A}^* - \mathfrak{A} \mathfrak{A}^*) \|_{L^2(P)} \). We now bound \( 7 \) and \( 8 \) as follows. Carrying out the exact calculations as in \( 5 \), we obtain from \( 31 \) that

\[ 7 = \| \Sigma_m^{1/2} (I - \hat{\Sigma}_m \hat{\Sigma}_m) \Sigma_m^{1/2} \|_{L^2(R^m)}. \]  

Clearly the r.h.s. of \( 35 \) is the same as \( \sqrt{R_{\Sigma,\ell}} \) in \( 28 \) but for \( \Sigma \) replaced by \( \Sigma_m \) and \( \hat{\Sigma} \) replaced by \( \hat{\Sigma}_m \). Therefore we decompose \( 7 \) as

\[ 7 \leq 7.1 + 7.2 + 7.3 + 7.4, \]  

where

\[ 7.1 \leq \| (\lambda_{m,i})_{i>\ell} \|_{L^2} + 2 \| \Sigma_m - \hat{\Sigma}_m \|_{L^2(R^m)} \leq \| (\lambda_{i})_{i>\ell} \|_{L^2} + \| (\lambda_{m,i} - \lambda_{i})_{i} \|_{L^2} + 2 \| \Sigma_m - \hat{\Sigma}_m \|_{L^2(R^m)} \]  

\[ \leq \sqrt{N_{\Sigma}(\ell)} + \| \mathfrak{A}^* - \mathfrak{A} \mathfrak{A}^* \|_{L^2(P)} + 2 \| \Sigma_m - \hat{\Sigma}_m \|_{L^2(R^m)}, \]  

\[ 7.2 \leq \frac{3}{2 \left( \lambda_{m,\ell} - \| \Sigma_m - \hat{\Sigma}_m \|_{L^2(R^m)} \right)} \left( \| \Sigma_m \|_{L^\infty(R^m)}^{1/2} + \| \Sigma_m - \hat{\Sigma}_m \|_{L^2(R^m)}^{1/2} \right) + \| \Sigma_m - \hat{\Sigma}_m \|_{L^2(R^m)}^{3/2}, \]  

where we used the observation that \( (\lambda_{m,i})_{i} \) and \( (\lambda_{i})_{i} \), which are eigenvalues of \( \Sigma_m \) and \( \Sigma \) respectively, are also the eigenvalues of \( \mathfrak{A} \mathfrak{A}^* \) and \( \mathfrak{A} \mathfrak{A}^* \) respectively and then employed Theorem D.2(i) in \( \ast \). \( \ast \ast \) follows from noting that \( \| \Sigma_m \|_{L^\infty(R^m)} \leq 2\kappa \) and assuming there exists \( b > 1 \) such that

\[ \lambda_{\ell} \geq b \| \mathfrak{A} \mathfrak{A}^* \|_{L^2(P)} + b \| \Sigma_m - \hat{\Sigma}_m \|_{L^2(R^m)}. \]  

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\[(7.3) \leq \frac{3}{2 \sqrt{\lambda_m \ell} - \| \Sigma_m - \tilde{\Sigma}_m \|_{L^2(\mathbb{R}^m)}} \left( \| \Sigma_m \|_{L^\infty(\mathbb{R}^m)}^{1/2} + \| \Sigma_m - \tilde{\Sigma}_m \|_{L^2(\mathbb{R}^m)}^{1/2} \right) \| \Sigma_m - \tilde{\Sigma}_m \|_{L^2(\mathbb{R}^m)} \]

\[\leq \frac{3}{2 \sqrt{\lambda_1 - \| \mathcal{J}_\pi - \mathcal{A}_\pi^\ast \|_{L^2(\mathbb{P})}}) - \| \Sigma_m - \tilde{\Sigma}_m \|_{L^2(\mathbb{R}^m)} \]

\[(7.4) \leq \ell^{1/4} \left( \| \Sigma_m \|_{L^\infty(\mathbb{R}^m)}^{1/2} + \| \Sigma_m - \tilde{\Sigma}_m \|_{L^2(\mathbb{R}^m)}^{1/2} \right) \| \Sigma_m - \tilde{\Sigma}_m \|_{L^\infty(\mathbb{R}^m)}^{1/2} \]

where we used Theorem \[D2(i)\] in \[(7.3)\] and \[(7.4)\] as in \[(7.2)\].

\[8 = \| (I - \mathcal{A}_m^\ast \mathcal{A}^\ast)(\mathcal{J}_\pi^\ast - \mathcal{A}^\ast) \|_{L^2(\mathbb{P})} \leq \| \mathcal{J}_\pi^\ast - \mathcal{A}^\ast \|_{L^2(\mathbb{P})} \| I - \mathcal{A}_m^\ast \mathcal{A}^\ast \|_{L^\infty(\mathbb{P})} \]

where

\[\| I - \mathcal{A}_m^\ast \mathcal{A}^\ast \|_{L^\infty(\mathbb{P})} \leq 1 + \| \mathcal{A}_m^\ast \mathcal{A}^\ast \|_{L^\infty(\mathbb{P})} \]

Note that

\[\| \mathcal{A}_m^\ast \mathcal{A}^\ast \|_{L^\infty(\mathbb{P})} = \sup_{f \in L^2(\mathbb{P})} \frac{\| \mathcal{A}_m^\ast \mathcal{A}^\ast f \|_{L^2(\mathbb{P})}}{\| f \|_{L^2(\mathbb{P})}} = \sup_{f \in L^2(\mathbb{P})} \| \mathcal{A}_m^\ast \mathcal{A}^\ast f \|_{L^2(\mathbb{P})} \]

\[= \sup_{f \in L^2(\mathbb{P})} \| \mathcal{A}_m^\ast \mathcal{A}^\ast f \|_{\mathbb{R}^m} = \| \mathcal{A}_m^\ast \mathcal{A}^\ast \|_{L^\infty(\mathbb{P}, \mathbb{R}^m)} \]

\[\leq \| \mathcal{A}_m^\ast \|_{L^\infty(\mathbb{P})} \| \mathcal{A}^\ast \|_{L^\infty(\mathbb{P}, \mathbb{R}^m)} \]

where

\[\| \mathcal{A}_m^\ast \|_{L^\infty(\mathbb{P})} \leq \| \mathcal{A}_m^\ast \|_{L^2(\mathbb{P})} \leq \frac{1}{\sqrt{\lambda_m \ell}} \| \Sigma_m - \tilde{\Sigma}_m \|_{L^2(\mathbb{R}^m)}^{1/2} + 1 \]

\[\| \mathcal{A}^\ast \|_{L^\infty(\mathbb{P}, \mathbb{R}^m)} \leq \frac{1}{\sqrt{\lambda_1 - \| \mathcal{J}_\pi^\ast - \mathcal{A}^\ast \|_{L^2(\mathbb{P})}}) - \| \Sigma_m - \tilde{\Sigma}_m \|_{L^2(\mathbb{R}^m)} + 1 \]

\[\| \mathcal{J}_\pi^\ast - \mathcal{A}^\ast \|_{L^2(\mathbb{P})} \leq \| \Sigma_m - \tilde{\Sigma}_m \|_{L^2(\mathbb{R}^m)}^{1/2} \]
\[
\frac{\sqrt{\mathcal{B}} \left\| \Sigma_m - \hat{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)}^{1/2}}{\sqrt{(b-1)\sqrt{\lambda_\ell}}} + 1
\]  
(41)

and
\[
\left\| \hat{\Sigma}_{m,\ell}^{-1/2} \mathbf{q}^* \right\|_{L^\infty(L^2(\mathbb{P}),\mathbb{R}^m)} = \sup_{f \in L^2(\mathbb{P})} \frac{\left\| \hat{\Sigma}_{m,\ell}^{-1/2} \mathbf{q}^* f \right\|_{\mathbb{R}^m}}{\|f\|_{L^2(\mathbb{P})}} = \sup_{f \in L^2(\mathbb{P})} \frac{\left\langle \hat{\Sigma}_{m,\ell}^{-1/2} \mathbf{q}^* f, \hat{\Sigma}_{m,\ell}^{-1/2} \mathbf{q}^* f \right\rangle_{\mathbb{R}^m}^{1/2}}{\|f\|_{L^2(\mathbb{P})}} \leq \left( \frac{\sqrt{b} \left\| \Sigma_m - \hat{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)}^{1/2}}{\sqrt{(b-1)\sqrt{\lambda_\ell}}} + 1 \right)^{1/2} \left\| \hat{\Sigma}_{m,\ell}^{-1/2} \mathbf{q}^* \right\|_{L^\infty(L^2(\mathbb{P}))}^{1/2},
\]  
(42)

with Theorem \textbf{D.2}(i,ii) being used in (**) and (*) respectively. Combining (41) and (42) in (40), we obtain
\[
\left\| \hat{\Sigma}_{m,\ell}^{-1} \mathbf{q}^* \right\|_{L^\infty(L^2(\mathbb{P}))} \leq \left( \frac{\sqrt{b} \left\| \Sigma_m - \hat{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)}^{1/2}}{\sqrt{(b-1)\sqrt{\lambda_\ell}}} + 1 \right)^2 \leq 2 + \frac{2b \left\| \Sigma_m - \hat{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)}}{(b-1)\lambda_\ell},
\]  
(43)

Combining (43) and (39) and using the result in (38), we obtain
\[
(8) \leq \left( 3 + \frac{2b \left\| \Sigma_m - \hat{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)}}{(b-1)\lambda_\ell} \right) \left\| \mathcal{J} \mathbf{q}^* - \mathbf{A} \mathbf{q}^* \right\|_{L^2(L^2(\mathbb{P}))}.
\]  
(44)

Combining the bounds on (7.1) - (7.4) in (38) and combining the result with the bound on (8) (see (44)) in (34) completes the proof through an application of Lemma \textbf{C.1} for \( \left\| \Sigma_m - \hat{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)} \) conditioned on \( (\theta_i)_{i=1}^m \) and then through an application of Proposition \textbf{B.5} for \( \left\| \mathcal{J} \mathbf{q}^* - \mathbf{A} \mathbf{q}^* \right\|_{L^2(L^2(\mathbb{P}))} \) while noting that (37) holds under the assumed condition of \( \lambda_\ell \geq 15b\sqrt{\frac{2\epsilon}{n\lambda/m}} \).

6.4 Proof of Corollary 4

(i) Under the assumption on \( (\lambda_i) \) and \( \alpha > 1 \), it follows that
\[
\mathcal{N}_\Sigma(\ell) = \sum_{i > \ell} \lambda_i^2 \leq A^2 \sum_{i > \ell} i^{-2\alpha} \leq A^2 \int_{\ell}^\infty x^{-2\alpha} \, dx \leq \ell^{-2\alpha+1}
\]
and
\[
\mathcal{N}_\Sigma(\ell) = \sum_{i > \ell} \lambda_i^2 \geq A^2 \sum_{i > \ell} i^{-2\alpha} \geq A^2 \int_{\ell+1}^\infty x^{-2\alpha} \, dx \geq (\ell + 1)^{-2\alpha+1}.
\]
Using $\ell = n^{\frac{\alpha}{2}}$, the upper bound follows trivially. For the lower bound, observe that $(\ell + 1)^{-2\alpha + 1} = n^{-2\theta(1 - \frac{1}{\ell})} \left(1 + n^{-\frac{\alpha}{4}}\right)^{-2\alpha + 1} \gtrsim n^{-2\theta(1 - \frac{1}{\ell^2})}$ since $n \geq 1$.

(ii) Since $0 < \theta < \frac{1}{2}$, it is guaranteed that $\lambda_\ell \sqrt{n} \rightarrow \infty$ which implies there exists $\tilde{n} \in \mathbb{N}$ such that for all $n > \tilde{n}$, the condition on $\lambda_\ell$ is satisfied. Therefore, it is easy to verify that $R_{\tilde{X},\ell} = O_p\left(N_\Sigma(\ell) + \frac{\sqrt{\ell}}{\sqrt{n}}\right)$ since \( \frac{1}{\sqrt{m} \sqrt{n}} = o_p\left(\frac{\sqrt{\ell}}{\sqrt{n}}\right) \) and \( \frac{1}{\sqrt{m} \lambda_\ell} = o_p\left(\frac{\sqrt{\ell}}{\sqrt{n}}\right) \) as $\ell, n, \lambda_\ell \sqrt{n} \rightarrow \infty$.

This means, $R_{\tilde{X},\ell} \lesssim_p n^{-2\theta(1 - \frac{1}{\ell^2})} + n^{-\left(1 - \frac{\theta}{2\ell}\right)}$ and the result follows by noting that the first term dominates the second if $0 < \theta \leq \frac{\alpha}{4\alpha - 1}$ and the second term dominates the first, otherwise.

(iii) The result follows by noting that $R_{\tilde{X}_{m},\ell} \lesssim_p n^{-2\theta(1 - \frac{1}{\ell^2})} + n^{-\tau}$.

(iv) Since $\tau > 2\theta$ and $0 < \theta < \frac{1}{2}$, we have $\lambda_\ell \sqrt{m \wedge n} \rightarrow \infty$, which means there exists $\tilde{n} \in \mathbb{N}$ such that for all $n > \tilde{n}$, the condition on $\lambda_\ell$ is satisfied. Note that $\frac{\sqrt{\ell}}{\sqrt{n}}$ dominates $\frac{1}{\sqrt{m \wedge n}}$ and $\frac{1}{\sqrt{m \lambda_\ell}}$ as $\ell, n, m, \lambda_\ell \sqrt{m \wedge n} \rightarrow \infty$ and $n^{-2\theta(1 - \frac{1}{\ell^2})}$ dominates $\frac{1}{m^\gamma n}$ as $m, n \rightarrow \infty$. Therefore $R_{\tilde{X}_{m},\ell} \lesssim_p n^{-2\theta(1 - \frac{1}{\ell^2})} + n^{-\left(1 - \frac{\theta}{2\ell}\right)}$ and the result follows.

6.5 Proof of Theorem 6

(i) Define $P_\ell(\Sigma) := \sum_{i=1}^\ell \phi_i \otimes \mathcal{H} \phi_i$ and $P_\ell(\tilde{\Sigma}) := \sum_{i=1}^\ell \tilde{\phi}_i \otimes \mathcal{H} \tilde{\phi}_i$. Consider

$$\Sigma^{1/2}\left(I - \tilde{\Sigma}_\ell^{-1}\Sigma\right)\Sigma^{1/2} = \Sigma^{1/2}\left(I - P_\ell(\Sigma) + P_\ell(\tilde{\Sigma}) - P_\ell(\Sigma) + P_\ell(\tilde{\Sigma})\right)\Sigma^{1/2}$$

$$= \Sigma^{1/2}\left(I - P_\ell(\Sigma)\right)\Sigma^{1/2} + \Sigma^{1/2}\left(P_\ell(\tilde{\Sigma}) - P_\ell(\Sigma)\right)\Sigma^{1/2}$$

$$+ \Sigma^{1/2}\left(P_\ell(\tilde{\Sigma}) - \tilde{\Sigma}_\ell^{-1}\Sigma\right)\Sigma^{1/2}$$

$$(\ast) \Sigma = \Sigma - \Sigma_\ell + \Sigma^{1/2}\left(P_\ell(\Sigma) - P_\ell(\tilde{\Sigma})\right)\Sigma^{1/2} + \Sigma^{1/2}\tilde{\Sigma}_\ell^{-1}\left(\tilde{\Sigma}_\ell - \Sigma\right)\Sigma^{1/2}.$$  

Define $\tilde{X}_{i,\ell} := \sum_{i=1}^\ell \tilde{\phi}_i \otimes \mathcal{H} \tilde{\phi}_i$. The third term in the r.h.s. of $(\ast)$ can be written as

$$\Sigma^{1/2}\tilde{\Sigma}_\ell^{-1}\left(\tilde{\Sigma}_\ell - \Sigma\right)\Sigma^{1/2} = \Sigma^{1/2}\tilde{\Sigma}_\ell^{-1}\left(\tilde{\Sigma}_\ell - \tilde{\Sigma}_{i,\ell} - \Sigma\right)\Sigma^{1/2} = \Sigma^{1/2}\tilde{\Sigma}_\ell^{-1}\left(\tilde{\Sigma}_\ell - \Sigma\right)\Sigma^{1/2}$$

since $\Sigma^{1/2}\tilde{\Sigma}_\ell^{-1}\tilde{\Sigma}_{i,\ell}\Sigma^{1/2} = 0$ as $\langle \tilde{\phi}_j, \tilde{\phi}_i \rangle = 0$ for all $i \leq \ell$ and $j > \ell$. Therefore,

$$\Sigma^{1/2}\left(I - \tilde{\Sigma}_\ell^{-1}\Sigma\right)\Sigma^{1/2} = \Sigma - \Sigma_\ell + \Sigma^{1/2}\left(P_\ell(\Sigma) - P_\ell(\tilde{\Sigma})\right)\Sigma^{1/2} + \Sigma^{1/2}\tilde{\Sigma}_\ell^{-1}\left(\tilde{\Sigma}_\ell - \Sigma\right)\Sigma^{1/2},$$

which implies

$$\sqrt{R_{\tilde{X},\ell}} = \left\|\Sigma^{1/2}\left(I - \tilde{\Sigma}_\ell^{-1}\Sigma\right)\Sigma^{1/2}\right\|_{L^2(\mathcal{H})} \leq (\mathbf{A}) + (\mathbf{B}) + (\mathbf{C}).$$

(45)

We now bound $(\mathbf{A}) - (\mathbf{C})$ as follows:

$(\mathbf{A}) = \|\Sigma - \Sigma_\ell\|_{L^2(\mathcal{H})} = \sqrt{N_\Sigma(\ell)}.$

It follows from Theorem 6.1 that

$(\mathbf{B}) = \left\|\Sigma^{1/2}\left(P_\ell(\Sigma) - P_\ell(\tilde{\Sigma})\right)\Sigma^{1/2}\right\|_{L^2(\mathcal{H})} \leq \frac{\ell \lambda_\ell}{\delta_\ell} \|\Sigma - \tilde{\Sigma}\|_{L^2(\mathcal{H})}$

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assuming
\[ \delta_\ell \geq 2 \left\| \Sigma - \hat{\Sigma} \right\|_{L^2(H)} . \] (46)

\[ C = \left\| \Sigma^{1/2} \hat{\Sigma}^{-1} \left( \Sigma - \Sigma \right) \Sigma^{1/2} \right\|_{L^2(H)} \]
\[ \leq \left\| \left( \Sigma^{1/2} - \hat{\Sigma}^{1/2} \right) \hat{\Sigma}^{-1} \left( \Sigma - \Sigma \right) \Sigma^{1/2} \right\|_{L^2(H)} + \left\| \hat{\Sigma}^{-1/2} \left( \Sigma - \Sigma \right) \Sigma^{1/2} \right\|_{L^2(H)} \]
\[ \leq \left\| \Sigma^{1/2} - \hat{\Sigma}^{1/2} \right\|_{L^\infty(H)} \left\| \hat{\Sigma}^{-1} \right\|_{L^\infty(H)} \left\| \Sigma - \hat{\Sigma} \right\|_{L^2(H)} \left\| \Sigma^{1/2} \right\|_{L^\infty(H)} \]
\[ + \left\| \hat{\Sigma}^{-1/2} \right\|_{L^\infty(H)} \left\| \Sigma - \hat{\Sigma} \right\|_{L^2(H)} \left\| \Sigma^{1/2} \right\|_{L^\infty(H)} \]
\[ \leq \frac{\sqrt{\lambda_1}}{\lambda_\ell - \left\| \Sigma - \hat{\Sigma} \right\|_{L^2(H)}} \left\| \Sigma - \hat{\Sigma} \right\|_{L^2(H)}^{3/2} + \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_\ell - \left\| \Sigma - \hat{\Sigma} \right\|_{L^2(H)}}} \left\| \Sigma - \hat{\Sigma} \right\|_{L^2(H)} , \]
\[ \leq \frac{a\sqrt{\lambda_1}}{(a-1)\lambda_\ell} \left\| \Sigma - \hat{\Sigma} \right\|_{L^2(H)}^{3/2} + \frac{a\lambda_1}{(a-1)\lambda_\ell} \left\| \Sigma - \hat{\Sigma} \right\|_{L^2(H)} , \]

where we used Theorem D.2(i, ii) in (\textit{d}). Combining $A - C$ in (45) and using Lemma C.1 yields the result by noting that (29) and (46) hold under the conditions that $\lambda_\ell \geq 7a\kappa \sqrt{\frac{2e}{n}}$ and $\delta_\ell \geq 14\kappa \sqrt{\frac{2e}{n}}$.

(ii) The result follows by bounding (7) (see (35)) using the decomposition involving projection operators considered in (i) but applied to $\Sigma_m$ and $\hat{\Sigma}_m$; and combining it with the bound on (8) obtained in (11). To this end, define $P_\ell(\Sigma_m) := \sum_{i=1}^\ell \phi_{m,i} \otimes 2 \phi_{m,i}$ and $P_\ell(\hat{\Sigma}_m) := \sum_{i=1}^\ell \hat{\phi}_{m,i} \otimes 2 \hat{\phi}_{m,i}$. Following the decomposition in (i) verbatim for $\Sigma_m$ and $\hat{\Sigma}_m$, we obtain
\[ (7) \leq (7.5) + (7.6) + (7.7) , \] (47)

where
\[ (7.5) = \left\| \Sigma_m - \Sigma_m,\ell \right\|_{L^2(\mathbb{R}^m)} = \left\| (\lambda_{i,\ell})_{i > \ell} \right\|_{\ell^2} \leq \left\| (\lambda_i)_{i > \ell} \right\|_{\ell^2} + \left\| (\lambda_m,\ell - \lambda_i)_i \right\|_{\ell^2} \]
\[ \leq \sqrt{N\Sigma(\ell)} + \left\| \mathcal{F}^* - \mathcal{A}^* \right\|_{L^2(L^2(\mathbb{R}))} , \]

with Theorem D.2(ii) being invoked in (\textit{*}). It follows from Theorem A.1 that
\[ (7.6) = \left\| \Sigma_m^{1/2} \left( P_\ell(\Sigma_m) - P_\ell(\hat{\Sigma}_m) \right) \Sigma_m^{1/2} \right\|_{L^2(\mathbb{R}^m)} \leq \frac{\ell \lambda_m,\ell}{\delta_{m,\ell}} \left\| \Sigma_m - \hat{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)} \]
assumed
\[ \delta_{m,\ell} \geq 2 \left\| \Sigma_m - \hat{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)} . \] (48)

Note that
\[ |\delta_{m,\ell} - \delta_\ell| \leq \frac{1}{2} |\lambda_{m,\ell} - \lambda_\ell| + \frac{1}{2} |\lambda_{m,\ell+1} - \lambda_{\ell+1}| \leq \left\| (\lambda_{m,i} - \lambda_i)_i \right\|_{\ell^2} \leq \left\| \mathcal{A}^* - \mathcal{F}^* \right\|_{L^2(L^2(\mathbb{R}))} , \] (50)
where the last inequality follows from Theorem D.2(i). Using (50) in (48), we obtain

\[
\ell \left( \lambda_\ell + \| \mathbf{X}^* - \mathcal{F}^* \|_{L^2(L^2(\mathbb{P}))} \right) \leq \delta_\ell - \| \mathbf{X}^* - \mathcal{F}^* \|_{L^2(L^2(\mathbb{P}))} \left\| \Sigma_m - \tilde{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)}
\]

assuming

\[
\delta_\ell \geq \| \mathbf{X}^* - \mathcal{F}^* \|_{L^2(L^2(\mathbb{P}))} + 2 \left\| \Sigma_m - \tilde{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)},
\]

which ensures that (49) holds. Let us assume that there exists \( c > 1 \) such that

\[
\delta_\ell \geq c \| \mathbf{X}^* - \mathcal{F}^* \|_{L^2(L^2(\mathbb{P}))} + 2c \left\| \Sigma_m - \tilde{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)},
\]

which ensures that (52) is satisfied. Using (53) in (51), we obtain

\[
\ell \leq \frac{(b + 1)\ell \lambda_\ell}{b(c - 1)\delta_\ell} \left\| \Sigma_m - \tilde{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)},
\]

Carrying out the bounding as in (C), we obtain that

\[
\text{(7.7)} = \left\| \Sigma_m^{1/2} \tilde{\Sigma}_m^{1/2} \right\|_{L^2(\mathbb{R}^m)} \left( \Sigma_m - \tilde{\Sigma}_m \right) \left\| \Sigma_m^{1/2} \right\|_{L^2(\mathbb{R}^m)} \leq \frac{\sqrt{\lambda_m,1}}{\lambda_m,1} \left\| \Sigma_m - \tilde{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)}^{3/2}
\]

\[
\leq \frac{\sqrt{\lambda_m,1}}{\lambda_m,1} \left\| \Sigma_m - \tilde{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)}^{3/2} + \frac{\sqrt{\lambda_m,1}}{\lambda_m,1} \left\| \Sigma_m - \tilde{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)}^{3/2}
\]

\[
\leq \frac{b \sqrt{2k}}{(b - 1)\lambda_\ell} \left\| \Sigma_m - \tilde{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)}^{3/2} + \frac{2b \kappa}{(b - 1)\lambda_\ell} \left\| \Sigma_m - \tilde{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)}^{3/2}.
\]

Combining the bounds on (7.5) and (7.7) with the bound on (7.6) (see (54)) in (47) and then combining the result with the bound on (8) (see (44)) completes the proof through an application of Lemma C.1 for \( \left\| \Sigma_m - \tilde{\Sigma}_m \right\|_{L^2(\mathbb{R}^m)} \) conditioned on \( (\theta_i)_{i=1}^m \) and then through an application of Proposition B.5 for \( \| \mathcal{F}^* - \mathbf{X}^* \|_{L^2(L^2(\mathbb{P}))} \) while noting that (37) and (53) hold under the assumed conditions of \( \lambda_\ell \geq 15bk \sqrt{\frac{2k}{n \times m}} \) and \( \delta_\ell \geq 22c\kappa \sqrt{\frac{2k}{n \times m}} \).
6.6 Proof of Proposition 8

(i) Note that \( R_{\Sigma,\ell} = \|(I - \mathcal{J}\Sigma^{-1}_\ell)^* \mathcal{J}\|^2_{L^2(P)} \) since
\[
\|(I - \mathcal{J}\Sigma^{-1}_\ell)^* \mathcal{J}\|^2_{L^2(P)} = \|\mathcal{J} (I - \Sigma^{-1}_\ell) \mathcal{J}\|^2_{L^2(P)} = \text{Tr} \left[ (I - \Sigma^{-1}_\ell) \Sigma (I - \Sigma^{-1}_\ell) \Sigma \right] = \|\Sigma - \Sigma\ell\|_{L^2(H)}^2.
\]
Similarly, it is easy to verify that \( R_{\bar{\Sigma},\ell} = \|(I - \mathcal{J}\bar{\Sigma}^{-1}_\ell)^* \mathcal{J}\|^2_{L^2(P)} \). Therefore,
\[
\left| \sqrt{R_{\Sigma,\ell}} - \sqrt{R_{\bar{\Sigma},\ell}} \right| = \left| \left\| (I - \mathcal{J}\Sigma^{-1}_\ell)^* \mathcal{J}\|_{L^2(P)} - \left\| (I - \mathcal{J}\bar{\Sigma}^{-1}_\ell)^* \mathcal{J}\|_{L^2(P)} \right. \right|
\leq \left\| \left( \mathcal{J}\Sigma^{-1}_\ell - \mathcal{J}\bar{\Sigma}^{-1}_\ell \right) \mathcal{J}\|_{L^2(P)}
\leq \|\mathcal{J}\|_{L^2(P)} \left\| \Sigma^{-1}_\ell - \Sigma^{-1}\bar{\ell} \right\|_{L^2(H)}
\]
and the result follows by noting that \( \|\mathcal{J}\|_{L^2(P)} = \|\Sigma\|_{L^2(H)} \), \( P_\ell(\mathcal{J}\) = \( \mathcal{J}\Sigma^{-1}_\ell \mathcal{J}\) and \( \bar{P}_\ell = \mathcal{J}\bar{\Sigma}^{-1}_\ell \mathcal{J}\).

(ii) From (i) above and Proposition 2(iii), we have
\[
\left| \sqrt{R_{\Sigma,\ell}} - \sqrt{R_{m,\ell}} \right| = \left| \left\| (I - \mathcal{J}\Sigma^{-1}_\ell)^* \mathcal{J}\|_{L^2(P)} - \left\| (I - \mathcal{J}\Sigma^{-1}_m\ell)^* \mathcal{J}\|_{L^2(P)} \right. \right|
\leq \left\| \left( \mathcal{J}\Sigma^{-1}_\ell - \mathcal{J}\Sigma^{-1}_m\ell \right) \mathcal{J}\|_{L^2(P)}
\leq \left\| \mathcal{J}\Sigma^{-1}_\ell - \mathcal{J}\Sigma^{-1}_m\ell \right\|_{L^2(H)} \|\Sigma\|_{L^2(H)}
\]
and the result follows by observing that \( P_\ell(\mathcal{J}\Sigma^{-1}_\ell) = \mathcal{J}\Sigma^{-1}_m\ell \mathcal{J}\).

(iii) The result is verbatim (ii) with \( \Sigma_{m,\ell} \) replaced by \( \bar{\Sigma}_{m,\ell} \).

6.7 Proof of Theorem 9

(i) Since \( P_\ell(\mathcal{J}\Sigma^{-1}_\ell) = \mathcal{J}\Sigma^{-1}_\ell \mathcal{J}\) and \( \bar{P}_\ell = \mathcal{J}\bar{\Sigma}^{-1}_\ell \mathcal{J}\), we have
\[
\left\| P_\ell(\mathcal{J}\Sigma^{-1}_\ell) - \bar{P}_\ell \right\|_{L^\infty(L^2(P), H)} = \left\| \mathcal{J} \left( \Sigma^{-1}_\ell - \bar{\Sigma}^{-1}_\ell \right) \mathcal{J} \right\|_{L^\infty(L^2(P), H)} = \left\| \Sigma^{1/2} \left( \Sigma^{-1}_\ell - \bar{\Sigma}^{-1}_\ell \right) \Sigma^{1/2} \right\|_{L^\infty(L^2(P), H)}
\leq \left\| \Sigma^{1/2} \left( \Sigma^{-1}_\ell - \bar{\Sigma}^{-1}_\ell \right) \Sigma^{1/2} \right\|_{L^\infty(H)}
\leq \left\| \Sigma^{1/2} \left( \Sigma^{-1}_\ell - \bar{\Sigma}^{-1}_\ell \right) \Sigma^{1/2} \right\|_{L^\infty(H)} \left( \Sigma^{-1/2} \right),
\]
where in the last line, we used the fact that
\[
\left\| \Sigma^{-1/2} \right\|_{L^\infty(L^2(P), H)} = \left\| \Sigma^{-1/2} \right\|_{L^\infty(L^2(P), H)} = \left\| P_\ell(\mathcal{J}\Sigma^{-1}_\ell) \right\|_{L^\infty(L^2(P), H)} = 1.
\]
Hence, bounding the r.h.s. of (55), we have

$$\left\| \Sigma^{1/2} \left( \Sigma^{-1} - \tilde{\Sigma}^{-1} \right) \Sigma^{1/2} \right\|_{L^\infty(\mathcal{H})} = \left\| P_\ell(\Sigma) - \Sigma^{1/2} \tilde{\Sigma}^{-1} \Sigma^{1/2} \right\|_{L^\infty(\mathcal{H})}$$

where $P_\ell(\Sigma) := \sum_{i=1}^\ell \phi_i \otimes_\mathcal{H} \phi_i$ is an orthogonal projection operator onto the span of $(\phi_i)^\ell_{i=1}$. Using the following decomposition,

$$P_\ell(\Sigma) - \Sigma^{1/2} \tilde{\Sigma}^{-1} \Sigma^{1/2} = P_\ell(\Sigma) - P_\ell(\tilde{\Sigma}) - \left( \Sigma^{1/2} - \tilde{\Sigma}^{1/2} \right) \tilde{\Sigma}^{-1} \left( \Sigma^{1/2} - \tilde{\Sigma}^{1/2} \right)$$

$$- \left( \Sigma^{1/2} - \tilde{\Sigma}^{1/2} \right) \tilde{\Sigma}^{-2} \left( \Sigma^{1/2} - \tilde{\Sigma}^{1/2} \right)$$

we have

$$\left\| \Sigma^{1/2} \left( \Sigma^{-1} - \tilde{\Sigma}^{-1} \right) \Sigma^{1/2} \right\|_{L^\infty(\mathcal{H})} \leq \overline{D} + \overline{E} + \overline{F}$$

where

$$\overline{D} := \left\| P_\ell(\Sigma) - P_\ell(\tilde{\Sigma}) \right\|_{L^\infty(\mathcal{H})} \overset{(\dagger)}{\le} \frac{1}{\delta_\ell} \left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})},$$

$$\overline{E} := \left\| \left( \Sigma^{1/2} - \tilde{\Sigma}^{1/2} \right) \tilde{\Sigma}^{-1} \left( \Sigma^{1/2} - \tilde{\Sigma}^{1/2} \right) \right\|_{L^\infty(\mathcal{H})} \leq \frac{1}{\lambda_\ell} \left\| \Sigma^{1/2} - \tilde{\Sigma}^{1/2} \right\|_{L^\infty(\mathcal{H})},$$

$$\overline{F} := \left\| \left( \Sigma^{1/2} - \tilde{\Sigma}^{1/2} \right) \tilde{\Sigma}^{-1/2} + \tilde{\Sigma}^{-1} \left( \Sigma^{1/2} - \tilde{\Sigma}^{1/2} \right) \right\|_{L^\infty(\mathcal{H})} \leq \frac{2}{\sqrt{\lambda_\ell}} \left\| \Sigma^{1/2} - \tilde{\Sigma}^{1/2} \right\|_{L^\infty(\mathcal{H})},$$

assuming (46) holds and we used Theorem A.1(i) in (\dagger).

and

where we used Theorem D.2(ii) in (\dagger) and Theorem D.2(i) in (\dagger). Combining the bounds on $\overline{D} - \overline{F}$ in (56) and then in (55), invoking Lemma C.1 to bound $\left\| \Sigma - \tilde{\Sigma} \right\|_{L^2(\mathcal{H})}$ and verifying that (29) and (46) hold under the assumptions $\lambda_\ell \geq 7\kappa \sqrt{\frac{2\delta_\ell}{n}}$ and $\delta_\ell \geq 14\kappa \sqrt{\frac{2\delta_\ell}{n}}$ yields the result.

(ii) The result is a straightforward application of Theorem A.1(i) and Lemma C.1

(iii) Since $\hat{P}_{m,\ell} = A \tilde{\Sigma}^{-1}_{m,\ell} A^*$, we have

$$\left\| P_\ell(\mathcal{I}^*) - \hat{P}_{m,\ell} \right\|_{L^\infty(L^2(\mathcal{F}))} = \left\| \gamma \Sigma^{-1} \mathcal{I}^* - A \tilde{\Sigma}^{-1}_{m,\ell} A^* \right\|_{L^\infty(L^2(\mathcal{F}))} \leq \overline{G} + \overline{H}$$

(57)
Carrying out the bounding as in (55) for 

\[ \mathbf{G} := \left\| \mathcal{J}_{\ell}^{-1} \mathcal{I} - \mathcal{A} \mathcal{J}_{\ell}^{-1} \mathcal{A}^* \right\|_{L^\infty(L^2(\mathbb{P}))} \quad \text{and} \quad \mathbf{H} := \left\| \mathcal{A} \mathcal{J}_{\ell}^{-1} \mathcal{I} - \mathcal{A} \mathcal{J}_{\ell}^{-1} \mathcal{A}^* \right\|_{L^\infty(L^2(\mathbb{P}))}. \]

Clearly, 

\[ \mathbf{G} = \| P_{\ell}(\mathcal{I}^*) - P_{\ell}(\mathcal{A}^*) \|_{L^\infty(L^2(\mathbb{P}))} \leq \frac{1}{\delta_{\ell}} \| \mathcal{J}^* - \mathcal{A}^* \|_{L^2(L^2(\mathbb{P}))}, \] 

(58)

where the inequality follows from Theorem A.1(i) under the assumption that 

\[ \delta_{\ell} \geq 2 \| \mathcal{J}^* - \mathcal{A}^* \|_{L^2(L^2(\mathbb{P}))}. \] 

(59)

Carrying out the bounding as in (55) for \( \mathbf{H} \), we obtain 

\[ \mathbf{H} \leq \left\| \Sigma_{m,\ell}^{1/2} \left( \Sigma_{m,\ell}^{-1} - \hat{\Sigma}_{m,\ell}^{-1} \right) \Sigma_{m,\ell}^{1/2} \right\|_{L^\infty(\mathbb{R}^m)}. \]

The result follows by using the decomposition in (56) for the r.h.s. of the above inequality under conditions (37) and (53), applying Lemma C.1 on \( \Sigma_{m,\ell} - \hat{\Sigma}_{m,\ell} \) conditioned on \( (\theta_i)_i^{m} \), applying Proposition 13.5 for \( \| \mathcal{J}^* - \mathcal{A}^* \|_{L^\infty(L^2(\mathbb{P}))} \), and verifying that (37) and (53) hold under the assumed conditions of \( \lambda_{\ell} \geq 15bn\sqrt{\frac{2}{m+n}} \) and \( \delta_{\ell} \geq 30\varepsilon\sqrt{\frac{2}{m+n}}. \)

6.8 Proof of Corollary 10

(i) It is easy to verify that 

\[ \| P_{\ell}(\mathcal{J}^*) - \hat{P}_{\ell} \|_{L^\infty(L^2(\mathbb{P}))} = O_P \left( \frac{1}{\delta_{\ell}\sqrt{n}} + \frac{1}{\sqrt{\lambda_{n,\ell}}} \right) \]

since \( \frac{1}{\lambda_{n,\ell}} = o_P \left( \frac{1}{\delta_{\ell}\sqrt{n}} \right) \) as \( \ell, n, \delta_{\ell}\sqrt{n} \to \infty \). Since \( \delta_{\ell}\sqrt{n} \to \infty \) for \( \| P_{\ell}(\mathcal{J}^*) - \hat{P}_{\ell} \|_{L^\infty(L^2(\mathbb{P}))} \to 0 \), it is clear that there exists \( \bar{n} \in \mathbb{N} \) such that for all \( n > \bar{n} \), the conditions on \( \delta_{\ell} \) and \( \lambda_{\ell} \) are satisfied if \( \theta < \frac{\alpha}{2\beta} \). Therefore, 

\[ \| P_{\ell}(\mathcal{J}^*) - \hat{P}_{\ell} \|_{L^\infty(L^2(\mathbb{P}))} \lesssim_p n^{-\left(\frac{1}{2} - \frac{\alpha}{2\beta} \right)} \]

and the result follows by noting that the first term dominates the second if \( 0 < \theta \leq \frac{\alpha}{2(2\beta - \alpha)} \) and the second term dominates the first, otherwise.

(iii) Arguing as above, it can be verified that 

\[ \| P_{\ell}(\mathcal{J}^*) - \hat{P}_{m,\ell} \|_{L^\infty(L^2(\mathbb{P}))} = O_P \left( \frac{1}{\delta_{\ell}\sqrt{m}} + \frac{1}{\sqrt{\lambda_{n,\ell}}} \right) \]

as \( \ell, n, \delta_{\ell}\sqrt{m} \to \infty \). This yields the constraint \( \tau > \frac{2\alpha\beta}{\alpha} \) and also ensures that there exists \( \bar{n} \in \mathbb{N} \) such that for all \( n > \bar{n} \), the conditions on \( \lambda_{\ell} \) and \( \delta_{\ell} \) are satisfied. Therefore, 

\[ \| P_{\ell}(\mathcal{J}^*) - \hat{P}_{m,\ell} \|_{L^\infty(L^2(\mathbb{P}))} \lesssim_p \]

\[ n^{-\left(\frac{1}{2} - \frac{\alpha}{2\beta} \right)} + n^{-\left(\frac{\alpha}{2\beta} - \frac{\alpha}{2(2\beta - \alpha)} \right)}. \]

Note that the first term dominates the rate if \( \tau > \frac{1}{2} + \frac{\theta(2\beta - \alpha)}{\alpha} \) which requires that \( 0 < \theta < \frac{\alpha}{2(2\beta - \alpha)} \) since \( \tau \in (0, 1] \). Similarly, the second term dominates the rate if 

\[ \frac{2\alpha\beta}{\alpha} < \tau \leq \frac{1}{2} + \frac{\theta(2\beta - \alpha)}{\alpha} \]

and the result follows depending on \( \theta < \frac{\alpha}{2(\beta + \alpha)} \) or \( \frac{\alpha}{2(2\beta - \alpha)} < \theta < \frac{\alpha}{2\beta} \).

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A  A Perturbation Result for Projection Operators

The first part of the following perturbation result for orthogonal projection operators is quoted from [Zwald and Blanchard (2000), Theorem 3], while the second part is new.

**Theorem A.1.** Let $A$ be a self-adjoint positive Hilbert-Schmidt operator on the Hilbert space $H$ with simple nonzero eigenvalues $\lambda_1(A) > \lambda_2(A) > \cdots$. Let $D > 1$ be an integer such that $\lambda_D(A) > 0$. Define $\delta_D(A) = \frac{1}{2}(\lambda_D(A) - \lambda_{D+1}(A))$. Let $B \in H$ be a self-adjoint operator such that $A + B$ is positive and $\|B\|_{L^2(H)} \leq \frac{\delta_D(A)}{2}$. Let $P^D(A)$ (resp. $P^D(A + B)$) denote the orthogonal projector onto the subspace spanned by the first $D$ eigenvectors of $A$ (resp. $A + B$). Then

\[
\begin{align*}
& (i) \quad \|P^D(A) - P^D(A + B)\|_{L^2(H)} \leq \frac{\|B\|_{L^2(H)}}{\delta_D(A)^2}; \\
& (ii) \quad \|A^{1/2} (P^D(A) - P^D(A + B)) A^{1/2}\|_{L^2(H)} \leq \frac{\|B\|_{L^2(H)}}{\delta_D(A)^2} \cdot D\lambda_D(A) \\
\end{align*}
\]

**Proof.** (ii) The resolvent of a symmetric operator $A$ is defined as $R_A(z) := (A - zI)^{-1}, z \in \mathbb{C}$. It is a well-known result (e.g., see [Kato (1980)] that a projection operator onto the $D$-eigenspace (i.e., span of $D$ eigenfunctions corresponding to the top $D$ eigenvalues) of $A$ can be written in terms of the resolvent of $A$ as

\[
P^D(A) = \frac{-1}{2\pi i} \oint_{\gamma_A} R_A(z) \, dz,
\]

where $\gamma_A$ is a simple closed curve in $\mathbb{C}$ enclosing exactly the first $D$ eigenvalues of $A$. Therefore,

\[
P^D(A) - P^D(A + B) = \frac{-1}{2\pi i} \oint_{\gamma} (R_A(z) - R_{A+B}(z)) \, dz
\]

\[
= \frac{-1}{2\pi i} \oint_{\gamma} R_A(z) BR_{A+B}(z) \, dz, \quad (A.1)
\]

where $\gamma$ is a simple closed curve in $\mathbb{C}$ enclosing exactly the first $D$ eigenvalues of $A$ and $A + B$. In (A.1), the last equality follows from the fact that:

\[
R_A(z) - R_{A+B}(z) = (A - zI)^{-1} - (A + B - zI)^{-1}
\]

\[
= (A - zI)^{-1} ((A + B - zI) - (A - zI)) (A + B - zI)^{-1}
\]

\[
= R_A(z) BR_{A+B}(z).
\]

Note that $R_{A+B}(z) = (I + R_A(z)B)^{-1} R_A(z) = \sum_{j \geq 0} (-1)^j (R_A(z)B)^j R_A(z)$, where the infinite series is the von-Neumann representation for $(I + R_A(z)B)^{-1}$ in the sense that

\[
\sup_{z \in \mathbb{C}} ||(I + R_A(z)B)^{-1} - S_N(z)||_{L^\infty(H)} \xrightarrow{N \to \infty} 0,
\]

where $S_N(z) := \sum_{j=1}^{N} (-1)^j (R_A(z)B)^j$. Therefore, $R_A(z)BR_{A+B}(z) = R_A(z)BS_{\infty}(z)R_A(z)$, where $S_{\infty}(z) := \sum_{j \geq 0} (-1)^j (R_A(z)B)^j$. Using this in (A.1), we have

\[
\begin{align*}
\|A^{1/2} (P^D(A) - P^D(A + B)) A^{1/2}\|_{L^2(H)} &= \frac{1}{2\pi} \left\| \left( \oint_{\gamma} R_A(z) BS_{\infty}(z) R_A(z) \, dz \right) A^{1/2} \right\|_{L^2(H)} \\
& \leq \frac{1}{2\pi} \int_{\gamma} \left\| (A^{1/2} R_A(z) BS_{\infty}(z) R_A(z) A^{1/2}) \, dz \right\|_{L^2(H)} \\
& \leq \frac{1}{2\pi} \int_{\gamma} \left\| A^{1/2} R_A(z) BS_{\infty}(z) R_A(z) A^{1/2} \right\|_{L^2(H)} \, dz
\end{align*}
\]
\[
\leq \frac{1}{2\pi} \oint_{\gamma} \left\| A^{1/2} R_A(z) \right\|_{L^\infty(H)}^2 \| B \|_{L^2(H)} \| S_\infty(z) \|_{L^\infty(H)} \, dz
\]

\[
\leq \frac{\| B \|_{L^2(H)}}{\pi} \oint_{\gamma} \left\| A^{1/2} R_A(z) \right\|_{L^\infty(H)}^2 \, dz,
\]

where (*) holds if

\[
\sup_{z \in \gamma} \| R_A(z) \|_{L^\infty(H)} \| B \|_{L^2(H)} \leq \frac{1}{2},
\]

since

\[
\| S_\infty(z) \|_{L^\infty(H)} \leq \sum_{j \geq 0} \| R_A(z) B \|_{L^\infty(H)}^j \leq 2, \quad \forall \ z \in \gamma.
\]

Since \( \| B \|_{L^2(H)} \leq \frac{\delta_D}{2} \), (A.3) holds if

\[
\delta_D \leq \sup_{z \in \gamma} \| R_A(z) \|_{L^\infty(H)} = \frac{1}{\sup_{z \in \gamma} \sup_i |\lambda_i(A) - z|^{-1}} = \inf \inf |\lambda_i - z|,
\]

where \( \lambda_i := \lambda_i(A) \) and \( \delta_D := \delta_D(A) \). From Theorem D.2(i), we have \( |\lambda_i(A) - \lambda_i(A + B)| \leq \delta_D \) for all \( i \). Under this constraint, in the following we choose a \( \gamma \) that satisfies (A.4) and then obtain a bound on (A.2).

In the picture above, choosing \( L \) such that \( L > \delta_D \) ensures that (A.4) holds and \( \gamma \) encloses only the top \( D \) eigenvalues of \( A \) and \( A + B \) as required. Therefore

\[
\oint_{\gamma} \left\| A^{1/2} R_A(z) \right\|_{L^\infty(H)}^2 \, dz = \oint_{\gamma} \sup_i \frac{\lambda_i}{|\lambda_i - z|^2} \, dz = \int \sup_i \frac{\lambda_i}{|\lambda_i - \gamma(t)|^2} |\gamma'(t)| \, dt
\]

\[
= \int_0^1 \sup_i \frac{\lambda_i}{|\lambda_i - \gamma_1(t)|^2} |\gamma_1'(t)| \, dt
\]

\[
+ \int_{-1}^0 \sup_i \frac{\lambda_i}{|\lambda_i - \gamma_2(t)|^2} |\gamma_2'(t)| \, dt
\]
where $\gamma_1(t) = \lambda_1(1 - t) + (\lambda_D - \delta_D)t - iL, t \in [0, 1]$; $\gamma_2(t) = \lambda_D - \delta_D + iLt, t \in [-1, 1]$; $\gamma_3(t) = (\lambda_D - \delta_D)(1 - t) + \lambda_1 t + iL, t \in [0, 1]$ and $\gamma_4(t) = \lambda_1 - L \cos t + iL \sin t, t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. So

\[
\int_{0}^{1} \sup_i \frac{\lambda_i}{|\lambda_i - \gamma_1(t)|^2} |\gamma'_1(t)| \, dt \leq \sup_i \lambda_i \int_{0}^{1} \frac{\lambda_1 - \lambda_D + \delta_D}{L^2} \, dt = \|A\|_{\mathcal{L}^\infty(H)} \frac{\lambda_1 - \lambda_D + \delta_D}{L^2}, \quad (A.5)
\]

and

\[
\int_{0}^{1} \sup_i \frac{\lambda_i}{|\lambda_i - \gamma_3(t)|^2} |\gamma'_3(t)| \, dt \leq \sup_i \lambda_i \int_{0}^{1} \frac{\lambda_1 - \lambda_D + \delta_D}{L^2} \, dt = \|A\|_{\mathcal{L}^\infty(H)} \frac{\lambda_1 - \lambda_D + \delta_D}{L^2}, \quad (A.7)
\]

In the following, we will obtain an estimate for $\int_{-1}^{1} \sup_i \frac{\lambda_i}{|\lambda_i - \gamma_2(t)|} |\gamma'_2(t)| \, dt$. Note that

\[
\int_{-1}^{1} \sup_i \frac{\lambda_i}{|\lambda_i - \gamma_2(t)|^2} |\gamma'_2(t)| \, dt = 2L \int_{0}^{1} \sup_i \frac{\lambda_i}{(\lambda_1 - \lambda_D + \delta_D)^2 + L^2 t^2} \, dt = 2L \int_{0}^{1} f(\lambda_i, t) \, dt,
\]

where

\[
f(\lambda_i, t) := \frac{\lambda_i}{(\lambda_i - a)^2 + L^2 t^2}, \quad \text{and} \quad a := \lambda_D - \delta_D.
\]

We first argue that $\forall i \geq D + 1$, $t \in [0, 1]$, $f(\lambda_i, t) \leq f(\lambda_D, t)$. For $i \geq D + 1$, it is clear that $\lambda_D > \lambda_i$ and for all $j \geq 1$, $(\lambda_j - a)^2$ is minimized at $j = D$, $D + 1$, i.e., $(\lambda_i - a)^2 \geq (\lambda_D - a)^2$. Therefore, $\sup_i f(\lambda_i, t) = \sup_{i \in [D]} f(\lambda_i, t)$ and

\[
\int_{-1}^{1} \sup_i \frac{\lambda_i}{|\lambda_i - \gamma_2(t)|^2} |\gamma'_2(t)| \, dt = 2L \int_{0}^{1} \sup_{i \in [D]} f(\lambda_i, t) \, dt. \quad (A.9)
\]

Suppose $1 \leq j < i \leq D$. It is easy to verify that $f(\lambda_j, t) \leq f(\lambda_i, t), \forall t \in [0, t_{ij}]$ and $f(\lambda_j, t) > f(\lambda_i, t), \forall t \in (t_{ij}, 1]$ where

\[
t_{i,j} := \sqrt{\frac{\lambda_i \lambda_j - a^2}{L^2}}.
\]

Since $t_{i+1,i}^2 = \frac{\lambda_i \lambda_{i+1} - a^2}{L^2} < \frac{\lambda_i - \lambda_{i-1} - a^2}{L^2} = t_{i,i-1}^2$, we may partition the interval $[0, 1]$ as

\[
[0, 1] = [0, t_{D,D-1}] \cup (t_{D,D-1}, t_{D-1,D-2}] \cup \cdots \cup (t_{2,1}, 1],
\]

wherein it is easy to see that

\[
f(\lambda_D, t) \geq f(\lambda_i, t), \forall t \in [0, t_{D,D-1}], i < D, \quad f(\lambda_D, t) < f(\lambda_{D-1}, t), \forall t > t_{D,D-1},
\]

\[
f(\lambda_{D-1}, t) \geq f(\lambda_i, t), \forall t \in [0, t_{D-1,D-2}], i < D - 1, \quad f(\lambda_{D-1}, t) < f(\lambda_{D-2}, t), \forall t > t_{D-1,D-2},
\]

\[
\vdots
\]

\[
f(\lambda_2, t) \geq f(\lambda_1, t), \forall t \in [0, t_2, 1], \quad f(\lambda_2, t) < f(\lambda_1, t), \forall t > t_2, 1.
\]
Therefore,

\[
\sup_{1 \leq i \leq D} f(\lambda_i, t) = \begin{cases} 
  f(\lambda_D, t), & \forall t \in [0, t_{D,D-1}] \\
  f(\lambda_{D-1}, t), & \forall t \in (t_{D,D-1}, t_{D-1,D-2}] \\
  \vdots \\
  f(\lambda_2, t), & \forall t \in (t_{3,2}, t_{2,1}] \\
  f(\lambda_1, t), & \forall t \in (t_{2,1}, 1] 
\end{cases}.
\]  

(A.10)

Using (A.10) in the r.h.s. of (A.9), we have

\[
\int_0^1 \sup_{i \in [D]} f(\lambda_i, t) \, dt = \int_0^{t_{D,D-1}} f(\lambda_D, t) \, dt + \int_{t_{D,D-1}}^{t_{D-1,D-2}} f(\lambda_{D-1}, t) \, dt + \cdots \\
+ \int_{t_{3,2}}^{t_{2,1}} f(\lambda_2, t) \, dt + \int_{t_{2,1}}^1 f(\lambda_1, t) \, dt.
\]  

(A.11)

Note that for \(2 \leq i \leq D - 1\),

\[
\int_{t_{i+1,i}}^{t_{i,i-1}} f(\lambda_i, t) \, dt = \int_{t_{i+1,i}}^{t_{i,i-1}} \frac{\lambda_i}{(\lambda_i-a)^2 + \lambda_i^2} \, dt \\
= \frac{\lambda_i}{(\lambda_i-a)L} \left( \tan^{-1} \left( \frac{L \lambda_{i-1}}{\lambda_i-a} \right) - \tan^{-1} \left( \frac{L \lambda_i}{\lambda_i-a} \right) \right) \\
= \frac{\lambda_i}{(\lambda_i-a)L} \left( \tan^{-1} \left( \frac{\sqrt{\lambda_i \lambda_{i-1} - a^2}}{\lambda_i-a} \right) - \tan^{-1} \left( \frac{\sqrt{\lambda_i \lambda_{i-1} - a^2}}{\lambda_i-a} \right) \right) \\
\leq \frac{\pi \lambda_i}{2(\lambda_i-a)L},
\]  

(A.12)

where in (*) we used the fact that \(\frac{\sqrt{\lambda_i \lambda_{i-1} - a^2}}{\lambda_i-a} > 0\),\(\frac{\sqrt{\lambda_i \lambda_{i-1} - a^2}}{\lambda_i-a} > 0\) and \(\tan^{-1}\) with principal value has the range \((-\frac{\pi}{2}, \frac{\pi}{2})\). Similarly,

\[
\int_0^{t_{D,D-1}} f(\lambda_D, t) \, dt = \frac{\lambda_D}{\delta_D L} \tan^{-1} \left( \frac{L t_{D,D-1}}{\delta_D} \right) = \frac{\lambda_D}{\delta_D L} \tan^{-1} \left( \frac{\sqrt{\lambda_D \lambda_{D-1} - a^2}}{\delta_D} \right) \leq \frac{\pi \lambda_D}{2 \delta_D L} \]  

(A.13)

and

\[
\int_{t_{2,1}}^1 f(\lambda_1, t) \, dt = \frac{\lambda_1}{(\lambda_1-a)L} \left( \tan^{-1} \left( \frac{L}{\lambda_1-a} \right) - \tan^{-1} \left( \frac{L t_{2,1}}{\lambda_1-a} \right) \right) \\
= \frac{\lambda_1}{(\lambda_1-a)L} \left( \tan^{-1} \left( \frac{L}{\lambda_1-a} \right) - \tan^{-1} \left( \frac{\sqrt{\lambda_1 \lambda_1 - a^2}}{\lambda_1-a} \right) \right) \\
\leq \frac{\lambda_1}{(\lambda_1-a)L} \tan^{-1} \left( \frac{L}{\lambda_1-a} \right).
\]  

(A.14)

Substituting (A.12)–(A.14) in (A.11) and using the result in (A.9), we obtain

\[
\int_0^1 \frac{\lambda_i}{|\lambda_i - \gamma_2(t)|^2} |\gamma_2(t)| \, dt \leq \pi \sum_{i=2}^{D} \frac{\lambda_i}{\lambda_i-a} + \frac{2 \lambda_1}{\lambda_1-a} \tan^{-1} \left( \frac{L}{\lambda_1-a} \right) \\
\leq \frac{\pi (D-1) \lambda_D}{\delta_D} + \frac{2 \lambda_D}{\delta_D} \tan^{-1} \left( \frac{L}{\lambda_1-a} \right),
\]  

(A.15)
where in (†), we use the fact that $\frac{\lambda_i}{\lambda_{D-a}} \leq \frac{\lambda_0}{\lambda_{D-a}} = \frac{\lambda_0}{\delta_D}$ for all $1 \leq i \leq D$. Substituting (A.6)–(A.8) and (A.15) in (A.3), and using the result in (A.2) by letting $L \rightarrow \infty$ yields the result. \hfill \Box

Remark A.1. Note that using Theorem A.1(i), the following trivial bound can be obtained:

$$
\left\| A^{1/2} (P^D(A) - P^D(A + B)) A^{1/2} \right\|_{L^2(H)} \leq \|A\|_{L^\infty(H)} \left\| P^D(A) - P^D(A + B) \right\|_{L^2(H)} \\
\leq \frac{\|A\|_{L^\infty(H)} \|B\|_{L^2(H)}}{\delta_D(A)},
$$

which is significantly improved in Theorem A.1(ii).

B Sampling, Inclusion and Approximation Operators

In this appendix, we present some technical results related to the properties of sampling, inclusion and approximation operators.

B.1 Properties of the sampling operator

The following result presents the properties of the sampling operator, $S$ and its adjoint. While these results are known in the literature (e.g., see Smale and Zhou [2007]), we present it here for completeness.

**Proposition B.1.** Let $H$ be an RKHS of real-valued functions on a non-empty set $X$ with $k$ as the reproducing kernel. Define $S : H \rightarrow \mathbb{R}^n$, $f \mapsto \frac{1}{\sqrt{n}} (f(X_1), \ldots, f(X_n))^T$ where $(X_i)_i \subset X$. Then the following hold:

(i) $S^* : \mathbb{R}^n \rightarrow H$, $\alpha \mapsto \frac{1}{\sqrt{n}} \sum_{i=1}^n \alpha_i k(\cdot, X_i)$;

(ii) $\hat{\Sigma} = S^* H_n S$ where $\hat{\Sigma}$ is defined in (9);

(iii) $K = n S S^*$.

**Proof.** (i) For any $g \in H$ and $\alpha \in \mathbb{R}^n$, we have

$$
\langle S^* \alpha, g \rangle_H = \langle \alpha, S g \rangle_2 = \frac{1}{\sqrt{n}} \sum_{i=1}^n \alpha_i g(X_i) = \left\langle \frac{1}{\sqrt{n}} \sum_{i=1}^n \alpha_i k(\cdot, X_i), g \right\rangle_H,
$$

where the last equality follows from the reproducing property.

(ii) For any $f \in H$,

$$
\langle f, \hat{\Sigma} f \rangle_H = \frac{1}{n} \sum_{i=1}^n f^2(X_i) - \left( \frac{1}{n} \sum_{i=1}^n f(X_i) \right)^2 = \langle S f, S f \rangle_2 - \frac{1}{n} \langle 1_n, S f \rangle_2^2 \\
= \langle f, S^* S f \rangle_H - \frac{1}{n} \langle S^* 1_n, f \rangle_H^2 = \langle f, S^* S f \rangle_H - \frac{1}{n} \langle f, S^* (1_n \otimes 1_n) S f \rangle_H = \langle f, S^* H_n S f \rangle_H.
$$

(iii) For any $\alpha \in \mathbb{R}^n$,

$$
S S^* \alpha = S \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \alpha_i k(\cdot, X_i) \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \alpha_i S k(\cdot, X_i) = \frac{1}{n} K \alpha,
$$

where in the second equality, we used the fact $S$ is a linear operator. \hfill \Box
B.2 Properties of the inclusion operator

The following result captures the properties of the inclusion operator \( \mathcal{I} \). A variation of the result is known in the literature (e.g., see Steinwart and Christmann 2008, Theorem 4.26).

**Proposition B.2.** Suppose \((A_1)\) holds. Define \( \mathcal{I} : \mathcal{H} \rightarrow L^2(\mathbb{P}) \), \( f \mapsto f - f_\mathbb{P} \), where \( f_\mathbb{P} := \int f(x) \, d\mathbb{P}(x) \). Then the following hold:

(i) \( \mathcal{I}^*: L^2(\mathbb{P}) \rightarrow \mathcal{H} \), \( f \mapsto \int_X k(\cdot, x) f(x) \, d\mathbb{P}(x) - m_\mathbb{P} f_\mathbb{P} \) where \( m_\mathbb{P} := \int_X k(\cdot, x) \, d\mathbb{P}(x) \).

(ii) \( \mathcal{I} \) and \( \mathcal{I}^* \) are Hilbert-Schmidt.

(iii) \( \Sigma = \mathcal{I}^* \mathcal{I} \) is trace-class, where \( \Sigma \) is defined in (3).

(iv) \( \mathcal{I} \mathcal{I}^* = \mathcal{Y} - (1 \otimes L^2(\mathbb{P})) 1 = \mathcal{Y} - \mathcal{Y}(1 \otimes L^2(\mathbb{P})) 1 + (1 \otimes L^2(\mathbb{P})) \mathcal{Y}(1 \otimes L^2(\mathbb{P})) 1 \) is trace-class where \( \mathcal{Y} : L^2(\mathbb{P}) \rightarrow L^2(\mathbb{P}) \), \( f \mapsto \int_X k(\cdot, x) f(x) \, d\mathbb{P}(x) \).

**Proof.** (i) For any \( f \in L^2(\mathbb{P}) \) and \( g \in \mathcal{H} \),

\[
\langle \mathcal{I}^* f, g \rangle_{\mathcal{H}} = \langle f, \mathcal{I} g \rangle_{L^2(\mathbb{P})} = \int_X f(x) \langle \mathcal{I} g(x) \rangle_{\mathcal{H}} \, d\mathbb{P}(x) = \int_X f(x) \langle g(x) - g_\mathbb{P} \rangle \, d\mathbb{P}(x)
\]

\[
= \int_X f(x) \langle k(\cdot, x), g \rangle_{\mathcal{H}} \, d\mathbb{P}(x) - \langle m_\mathbb{P}, g \rangle_{\mathcal{H}} f_\mathbb{P} = \left( \int_X k(\cdot, x) f(x) \, d\mathbb{P}(x), g \right) - \langle m_\mathbb{P} f_\mathbb{P}, g \rangle_{\mathcal{H}}.
\]

Clearly \( f_\mathbb{P} \) is well defined as for any \( f \in L^2(\mathbb{P}) \), \( f_\mathbb{P} \leq \int |f(x)| \, d\mathbb{P}(x) \leq \|f\|_{L^2(\mathbb{P})} < \infty \) and for \( f \in \mathcal{H} \), \( f_\mathbb{P} = \langle f, m_\mathbb{P} \rangle_{\mathcal{H}} \leq \|f\|_{\mathcal{H}} \sqrt{\int \langle k(x, x) \rangle \, d\mathbb{P}(x)} < \infty \) and the result therefore follows.

(ii) For any orthonormal basis \( \{e_j\}_j \) in \( \mathcal{H} \),

\[
\|\mathcal{I}\|_{L^2(\mathcal{H}, L^2(\mathbb{P}))}^2 = \sum_j \|\mathcal{I} e_j\|_{L^2(\mathbb{P})}^2 = \sum_j \|e_j - e_j, f_\mathbb{P}\|_{L^2(\mathbb{P})}^2 = \sum_j \|e_j\|_{L^2(\mathbb{P})}^2 - e_{j, f_\mathbb{P}}^2 \leq \sum_j \|e_j\|_{L^2(\mathbb{P})}^2
\]

\[
= \sum_j \int_X \langle e_j, k(\cdot, x) \rangle_{\mathcal{H}}^2 \, d\mathbb{P}(x) \overset{(*)}{=} \int_X \sum_j \langle e_j, k(\cdot, x) \rangle_{\mathcal{H}}^2 \, d\mathbb{P}(x) = \int_X k(x, x) \, d\mathbb{P}(x) < \infty,
\]

where \((*)\) follows from monotone convergence theorem. Since \( \|\mathcal{I}\|_{L^2(\mathcal{H}, L^2(\mathbb{P}))} = \|\mathcal{I}^*\|_{L^2(\mathcal{H})} \), the result follows.

(iii) For any \( f \in \mathcal{H} \), \( \mathcal{I} \mathcal{I}^* f = \mathcal{I}^* (f - f_\mathbb{P}) = \mathcal{I}^* f - \mathcal{I}^* f_\mathbb{P} = \mathcal{I}^* f \), where we use the fact that \( \mathcal{I}^* f_\mathbb{P} = 0 \) since \( f_\mathbb{P} \) is a constant function. By using the reproducing property,

\[
\mathcal{I} \mathcal{I}^* f = \mathcal{I}^* f = \int_X f(x) k(x, \cdot) \, d\mathbb{P}(x) - m_\mathbb{P} f_\mathbb{P} = \int_X \langle k(\cdot, x), \langle k(\cdot, x), f \rangle_{\mathcal{H}} d\mathbb{P} - m_\mathbb{P} \langle m_\mathbb{P}, f \rangle_{\mathcal{H}}
\]

\[
= \int_X (k(\cdot, x) \otimes m_\mathbb{P}) k(\cdot, x) f \, d\mathbb{P}(x) - \langle m_\mathbb{P} \otimes m_\mathbb{P} m_\mathbb{P} \rangle f = \Sigma f
\]

and the result follows. Since \( \|\mathcal{I}\|_{L^2(\mathcal{H}, L^2(\mathbb{P}))}^2 = \|\mathcal{I}^*\|_{L^1(\mathcal{H})} \), \( \Sigma \) is trace-class.

(iv) For any \( f \in L^2(\mathbb{P}) \),

\[
(\mathcal{I} \mathcal{I}^*) f = \mathcal{I} (\mathcal{I}^* f) = \mathcal{I} \left( \int_X k(\cdot, x) f(x) \, d\mathbb{P}(x) - m_\mathbb{P} f_\mathbb{P} \right)
\]

\[
= \int_X k(\cdot, x) f(x) \, d\mathbb{P}(x) - m_\mathbb{P} f_\mathbb{P} - \int_X \int_X k(y, x) f(x) \, d\mathbb{P}(x) \, d\mathbb{P}(y)
\]

\[
+ f_\mathbb{P} \int_X \int_X k(y, x) \, d\mathbb{P}(x) \, d\mathbb{P}(y)
\]

\[
= \mathcal{Y} f - \mathcal{Y} 1_{f} \langle f \rangle_{L^2(\mathbb{P})} - \langle \mathcal{Y} 1, f \rangle_{L^2(\mathbb{P})} + 1_{\mathcal{Y} 1} \langle 1, f \rangle_{L^2(\mathbb{P})} 1_{f} \langle f \rangle_{L^2(\mathbb{P})}
\]

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and the result follows, where in the last line we use the fact that $\Upsilon$ is self-adjoint, which follows from [Steinwart and Christmann, 2008, Theorem 4.27]. Since $\|\Upsilon^*\|_{L^2(\mathcal{H})} = \|\Upsilon\|_{L^2(\mathbb{P})}$, it follows that $\Upsilon^*$ is trace-class. 

The following result presents a representation for $\Upsilon$ if $k$ satisfies $(A_4)$.

**Lemma B.3.** Suppose $(A_4)$ holds. Then $\Upsilon = \int_{\Theta} \varphi(\cdot, \theta) \otimes L^2(\mathbb{P}) \varphi(\cdot, \theta) d\Lambda(\theta)$.

**Proof.** Since $k(x, y) = \int_{\Theta} \varphi(x, \theta) \varphi(y, \theta) d\Lambda(\theta)$, for any $f \in L^2(\mathbb{P})$,

$$\Upsilon f = \int_{\mathcal{X}} k(\cdot, f) d\mathbb{P}(x) = \int_{\mathcal{X}} \int_{\Theta} \varphi(\cdot, \theta) \varphi(x, \theta) d\Lambda(\theta) f(x) d\mathbb{P}(x)$$

$$= \int_{\Theta} \varphi(\cdot, \theta) \left( \int_{\mathcal{X}} \varphi(x, \theta) f(x) d\mathbb{P}(x) \right) d\Lambda(\theta) = \int_{\Theta} \varphi(\cdot, \theta) \langle \varphi(\cdot, \theta), f \rangle_{L^2(\mathbb{P})} d\Lambda(\theta)$$

$$= \int_{\Theta} (\varphi(\cdot, \theta) \otimes L^2(\mathbb{P}) \varphi(\cdot, \theta)) f d\Lambda(\theta) = \left( \int_{\Theta} \varphi(\cdot, \theta) \otimes L^2(\mathbb{P}) \varphi(\cdot, \theta) d\Lambda(\theta) \right) f,$$

where Fubini’s theorem is applied in $(*)$. 

---

### B.3 Properties of the approximation operator

The following result presents the properties of the approximation operator, $\mathfrak{A}$.

**Proposition B.4.** Define $\mathfrak{A} : \mathbb{R}^m \rightarrow L^2(\mathbb{P})$, $\beta \mapsto \sum_{i=1}^m \beta_i (\varphi_i - \varphi_i \mathbb{P})$ where $\varphi_i \mathbb{P} := \int_{\mathcal{X}} \varphi_i(x) d\mathbb{P}(x)$ and $\sup_{x \in \mathcal{X}} |\varphi_i(x)| \leq \sqrt{\frac{m}{m}}$ for all $i \in [m]$ with $k < \infty$. Then the following hold:

(i) $\mathfrak{A}^* : L^2(\mathbb{P}) \rightarrow \mathbb{R}^m$, $f \mapsto (\langle f, \varphi_1 \rangle_{L^2(\mathbb{P})} - f(\varphi_1, \mathbb{P}), \ldots, (\langle f, \varphi_m \rangle_{L^2(\mathbb{P})} - f(\varphi_m, \mathbb{P}))^T$;

(ii) $\mathfrak{A}$ and $\mathfrak{A}^*$ are Hilbert-Schmidt;

(iii) $\sum_{m} = \mathfrak{A}^* \mathfrak{A}$ is trace-class;

(iv) $\mathfrak{A} \mathfrak{A}^* = \Pi - (1 \otimes L^2(\mathbb{P}) 1) \Pi - (1 \otimes L^2(\mathbb{P}) 1) \Pi (1 \otimes L^2(\mathbb{P}) 1)$ is trace-class where $\Pi := \sum_{i=1}^m \varphi_i \otimes L^2(\mathbb{P}) \varphi_i : L^2(\mathbb{P}) \rightarrow L^2(\mathbb{P})$.

**Proof.** The proof is similar to that of Proposition B.2.

(i) For any $\beta \in \mathbb{R}^m$ and $f \in L^2(\mathbb{P})$,

$$\langle \mathfrak{A} \beta, f \rangle_2 = \langle f, \mathfrak{A} \beta \rangle_{L^2(\mathbb{P})} = \sum_{i=1}^m \beta_i \int_{\mathcal{X}} f(x)(\varphi_i(x) - \varphi_i \mathbb{P}) d\mathbb{P}(x) = \sum_{i=1}^m \beta_i \langle f, \varphi_i \rangle_{L^2(\mathbb{P})} - f(\varphi_i, \mathbb{P})$$

and the result follows.

(ii) For any orthonormal basis $(e_j)_{j=L^2(\mathbb{P})}$,

$$\|\mathfrak{A}^*\|_{L^2(\mathbb{P}), \mathbb{R}^m}^2 = \sum_{j} \|\mathfrak{A}^* e_j\|_2^2 = \sum_{j} \sum_{i=1}^m (\langle e_j, \varphi_i \rangle_{L^2(\mathbb{P})} - e_j \mathbb{P} \varphi_i \mathbb{P})^2$$

$$= \sum_{j} \sum_{i=1}^m (e_j, \varphi_i)_{L^2(\mathbb{P})}^2 + e_j \mathbb{P} \varphi_i \mathbb{P} - 2\epsilon_j \mathbb{P} \varphi_i \mathbb{P} \langle e_j, \varphi_i \rangle_{L^2(\mathbb{P})}$$

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\[
\begin{align*}
&= \sum_{i=1}^{m} \|\varphi_i\|_{L^2(P)}^2 + \sum_{i=1}^{m} \varphi_{i,P}^2 \sum_{j} \langle e_j, 1 \rangle_{L^2(P)}^2 - \sum_{i=1}^{m} \varphi_{i,P} \sum_{j} \langle e_j, (\varphi_i \otimes L^2(P) 1 + 1 \otimes L^2(P) \varphi_i) e_j \rangle_{L^2(P)} \\
&= \sum_{i=1}^{m} \|\varphi_i\|_{L^2(P)}^2 + \sum_{i=1}^{m} \varphi_{i,P}^2 - 2 \sum_{i=1}^{m} \varphi_{i,P} \langle \varphi_i, 1 \rangle_{L^2(P)} \leq \sum_{i=1}^{m} \|\varphi_i\|_{L^2(P)}^2 \leq \kappa < \infty,
\end{align*}
\]

and so \(\mathfrak{A}\) and \(\mathfrak{A}^*\) are Hilbert-Schmidt.

(iii) For any \(\beta \in \mathbb{R}^m\),

\[
\mathfrak{A}^* \beta = \mathfrak{A}^* \left( \sum_{i=1}^{m} \beta_i (\varphi_i - \varphi_{1,P}) \right) = \sum_{i=1}^{m} \beta_i \mathfrak{A}^* (\varphi_i - \varphi_{1,P})
\]

\[
= \sum_{i=1}^{m} \beta_i (\langle \varphi_i - \varphi_{1,P}, \varphi_1 \rangle_{L^2(P)}, \ldots, \langle \varphi_i - \varphi_{1,P}, \varphi_m \rangle_{L^2(P)})^T
\]

\[
= \sum_{i=1}^{m} \beta_i (\langle \varphi_i, \varphi_1 \rangle_{L^2(P)} - \varphi_{1,P} \varphi_1, \ldots, \langle \varphi_i, \varphi_m \rangle_{L^2(P)} - \varphi_{1,P} \varphi_m, \ldots, \langle \varphi_{1,P}, \varphi_{1,P} \varphi_1, \ldots, \langle \varphi_{1,P}, \varphi_{1,P} \varphi_m \rangle_{L^2(P)} - \varphi_{1,P} \varphi_m, \ldots, \langle \varphi_{1,P}, \varphi_{1,P} \varphi_{1,P} \varphi_1, \ldots, \langle \varphi_{1,P}, \varphi_{1,P} \varphi_{1,P} \varphi_m \rangle_{L^2(P)} - \varphi_{1,P} \varphi_m, \ldots, \langle \varphi_{1,P}, \varphi_{1,P} \varphi_{1,P} ..., \langle \varphi_{1,P}, \varphi_{1,P} \varphi_{1,P} \varphi_{1,P} \varphi_1, \ldots, \langle \varphi_{1,P}, \varphi_{1,P} \varphi_{1,P} \varphi_{1,P} \varphi_m \rangle_{L^2(P)} - \varphi_{1,P} \varphi_m, \ldots, \langle \varphi_{1,P}, \varphi_{1,P} \varphi_{1,P} ...) = \Gamma \beta,
\]

where \(\Gamma_{ij} = \langle \varphi_i, \varphi_j \rangle_{L^2(P)} - \varphi_{i,P} \varphi_{j,P} = \text{Cov}(\varphi_i(X), \varphi_j(X)), i, j \in [m]\) with \(X \sim P\). Clearly \(\Sigma = \Sigma_m\) and the result follows. \(\Sigma_m\) is trace-class since \(\mathfrak{A}\) is Hilbert-Schmidt.

(iv) For any \(f \in L^2(P)\),

\[
\mathfrak{A}^* f = \mathfrak{A}((f, \varphi_1)_{L^2(P)} - f_{\varphi_1,P}, \ldots, (f, \varphi_m)_{L^2(P)} - f_{\varphi_m,P})^T
\]

\[
= \sum_{i=1}^{m} (\langle f, \varphi_i \rangle_{L^2(P)} - f_{\varphi_i,P})(\varphi_i - \varphi_{1,P})
\]

\[
= \sum_{i=1}^{m} (\langle f, \varphi_i \rangle_{L^2(P)} - \langle f, 1 \rangle_{L^2(P)} \langle \varphi_i, 1 \rangle_{L^2(P)})(\varphi_i - \varphi_{1,P})
\]

\[
= \Pi f - \langle \Pi 1, f \rangle_{L^2(P)} - \Pi(1 \otimes L^2(P) 1)f + \langle (1 \otimes L^2(P) 1)\Pi 1, f \rangle_{L^2(P)}
\]

\[
= \Pi f - (1 \otimes L^2(P) 1)\Pi f - \Pi(1 \otimes L^2(P) 1)f + (1 \otimes L^2(P) 1)\Pi(1 \otimes L^2(P) 1)f
\]

and the result follows. \(\mathfrak{A}^*\) is trace-class since \(\mathfrak{A}^*\) is Hilbert-Schmidt.

B.4 A concentration inequality for inclusion and approximation operators

The following result provides a bound on \(\|\mathfrak{A}^* - \mathfrak{A}^*\|_{L^2(L^2(P))}\):

**Proposition B.5.** Suppose \((A_1)\) holds. Then for any \(\tau > 0\) and \(m \geq 8\tau\),

\[
\Lambda^m \left( (\theta_i)_{i=1}^{m} : \|\mathfrak{A}^* - \mathfrak{J}^*\|_{L^2(L^2(P))} \leq 8 \kappa \sqrt{\frac{2\tau}{m}} \right) \geq 1 - 2e^{-\tau}.
\]

**Proof.** From Proposition [B.2](iv), Lemma [B.3] and Proposition [B.3](iv), we have

\[
\mathfrak{J}^* = \mathcal{Y} - (1 \otimes L^2(P) 1)\mathcal{Y} - \mathcal{Y}(1 \otimes L^2(P) 1) + (1 \otimes L^2(P) 1)\mathcal{Y}(1 \otimes L^2(P) 1)
\]

and

\[
\mathfrak{A}^* = \Pi - (1 \otimes L^2(P) 1)\Pi - \Pi(1 \otimes L^2(P) 1) + (1 \otimes L^2(P) 1)\Pi(1 \otimes L^2(P) 1)
\]
where
\[ \Upsilon := \int_\varnothing \varphi(\cdot, \theta) \otimes L^2(\mathcal{F}) \varphi(\cdot, \theta) d\Lambda(\theta) \]
and
\[ \Pi := \sum_{i=1}^m \varphi_i \otimes L^2(\mathcal{F}) \varphi_i = \frac{1}{m} \sum_{i=1}^m \varphi(\cdot, \theta_i) \otimes L^2(\mathcal{F}) \varphi(\cdot, \theta_i). \]

Therefore
\[ \| \mathfrak{A} \# \mathfrak{B} - \mathfrak{C} \|_{L^2(L^2(\mathcal{F}))} \leq \| \Pi - \Upsilon \|_{L^2(L^2(\mathcal{F}))} + \| (1 \otimes L^2(\mathcal{F}) 1)(\Pi - \Upsilon) \|_{L^2(L^2(\mathcal{F}))} \]
\[ + \| (\Pi - \Upsilon)(1 \otimes L^2(\mathcal{F}) 1) \|_{L^2(L^2(\mathcal{F}))} \]
\[ \leq \| \Pi - \Upsilon \|_{L^2(L^2(\mathcal{F}))} + \| 1 \otimes L^2(\mathcal{F}) 1 \|_{L^\infty(L^2(\mathcal{F}))} \| \Pi - \Upsilon \|_{L^2(L^2(\mathcal{F}))} \]
\[ + \| 1 \otimes L^2(\mathcal{F}) 1 \|_{L^\infty(L^2(\mathcal{F}))} \| \Pi - \Upsilon \|_{L^2(L^2(\mathcal{F}))} \]
\[ = 4 \| \Pi - \Upsilon \|_{L^2(L^2(\mathcal{F}))}, \]  \hspace{1cm} (B.1)

where we note that \( \| 1 \otimes L^2(\mathcal{F}) 1 \|_{L^\infty(L^2(\mathcal{F}))} = \| 1 \|_{L^2(\mathcal{F})} = 1 \) which follows from Lemma C.3. The result follows by combining (B.1) with (C.2) (see Lemma C.1).

C Technical Results

The following result presents a concentration inequality for the centered covariance operator. This result is well-known in the literature (e.g., see Caponnetto and Vito, 2007) for the uncentered covariance operator \( \Sigma_c := \int_X k(\cdot, x) \otimes \mathcal{H} k(\cdot, x) d\mathbb{P}(x) \) on RKHS \( \mathcal{H} \). While the result for centered covariance operator does not pose any technical hurdles as it follows from a straightforward application of Bernstein’s inequality (Theorem D.1), we present here a general result for completeness.

**Lemma C.1.** Let \( H \) be a separable Hilbert space, \( \mathcal{X} \) be a separable topological space and \( \nu : \mathcal{X} \to H \) be measurable. Define

\[ \mathcal{C} = \int_\mathcal{X} \nu(x) \otimes_H \nu(x) d\mathbb{P}(x) - t \otimes_H t \]
and
\[ \mathcal{C} = \frac{1}{s} \sum_{i=1}^s \nu(X_i) \otimes_H \nu(X_i) - \hat{t} \otimes_H \hat{t}, \]

where \( t := \int_\mathcal{X} \nu(x) d\mathbb{P}(x) \), \( \hat{t} := \frac{1}{s} \sum_{i=1}^s \nu(X_i) \) with \( (X_i)_{i=1}^s \) i.i.d. \( \sim \mathbb{P} \). Assume \( \sup_{x \in \mathcal{X}} \| \nu(x) \|_H^2 \leq \varepsilon \). Then for any \( \tau > 0 \) and \( s \geq 8\tau \), the following holds:

\[ \mathbb{P}^s \left\{ (X_i)_{i=1}^s : \| \mathcal{C} - \mathcal{C} \|_{L^2(H)} \leq 7\varepsilon \sqrt{\frac{2\tau}{s}} \right\} \geq 1 - 4e^{-\tau}. \]  \hspace{1cm} (C.1)

If \( t = \hat{t} = 0 \), then for any \( \tau > 0 \) and \( s \geq 8\tau \),

\[ \mathbb{P}^s \left\{ (X_i)_{i=1}^s : \| \mathcal{C} - \mathcal{C} \|_{L^2(H)} \leq 2\varepsilon \sqrt{\frac{2\tau}{s}} \right\} \geq 1 - 2e^{-\tau}. \]  \hspace{1cm} (C.2)

**Proof.** Define \( \mathcal{C} = \mathcal{C}_c - t \otimes_H t \) where \( \mathcal{C}_c := \int_X \nu(x) \otimes_H \nu(x) d\mathbb{P}(x) \). Similarly, define \( \mathcal{C} := \mathcal{C}_c - \hat{t} \otimes_H \hat{t} \) where \( \mathcal{C}_c := \frac{1}{s} \sum_{i=1}^s \nu(X_i) \otimes_H \nu(X_i) \). Therefore

\[ \| \mathcal{C} - \mathcal{C} \|_{L^2(H)} \leq \| \mathcal{C}_c - \mathcal{C}_c \|_{L^2(H)} + \| \hat{t} \otimes_H \hat{t} - t \otimes_H t \|_{L^2(H)} \]

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\[ \sup_{x \in \mathcal{X}} \|\nu(x) \otimes H \nu(x) - \mathcal{C}_c\|_{L^2(H)} \leq \sup_{x \in \mathcal{X}} \|\nu(x) \otimes H \nu(x)\|_{L^2(H)} + \|\mathcal{C}_c\|_{L^2(H)} \leq \sup_{x \in \mathcal{X}} \|\nu(x)\|^2_H + \int_{\mathcal{X}} \|\nu(x) \otimes H \nu(x)\|_{L^2(H)} \, dp(x) \leq 2\varepsilon \]

and \(\mathbb{E}\|\xi_i\|_{L^2(H)}^2 \leq \mathbb{E}\|\nu(X_i) \otimes H \nu(X_i)\|_{L^2(H)}^2 \leq \varepsilon\). This means, \((\xi_i)_{i=1}^s\) satisfy (D.1) with \(\theta = \varepsilon\) and \(B = 2\varepsilon\). Therefore by Bernstein’s inequality (Theorem D.1), for any \(\tau > 0\), with probability at least \(1 - 2e^{-\tau}\) over the choice of \((X_i)_{i=1}^s\), we obtain

\[ \|\widehat{\mathcal{C}}_c - \mathcal{C}_c\|_{L^2(H)} \leq \frac{4\varepsilon\tau}{s} + \sqrt{\frac{2\varepsilon^2 \tau}{s}}. \quad \text{(C.4)} \]

We now bound \(\|\widehat{\tau} - t\|_H\). Define \(\eta_i := \nu(X_i) - t\). It is easy to verify that \((\eta_i)_{i=1}^s\) are i.i.d. random elements in \(H\) with \(\mathbb{E}\eta_i = 0\) for all \(i \in [s]\) and \(\frac{1}{s} \sum_{i=1}^s \eta_i = \widehat{\tau} - t\). For \(r > 2\) and any \(i \in [s]\),

\[ \mathbb{E}\|\eta_i\|_{H}^r \leq \mathbb{E}\|\eta_i\|_{H}^p \sup_{x \in \mathcal{X}} \|\nu(x) - t\|_{H}^{p-2} \]

where \(\sup_{x \in \mathcal{X}} \|\nu(x) - t\|_{H} \leq 2\sqrt{\varepsilon}\) and \(\mathbb{E}\|\eta_i\|_{H}^p \leq \mathbb{E}\|\nu(X_i)\|_{H}^p \leq \varepsilon\). This means, \((\eta_i)_{i=1}^s\) satisfy (D.1) with \(\theta = \sqrt{\varepsilon}\) and \(B = 2\sqrt{\varepsilon}\). Therefore by Bernstein’s inequality (Theorem D.1), for any \(\tau > 0\), with probability at least \(1 - 2e^{-\tau}\) over the choice of \((X_i)_{i=1}^s\), we obtain

\[ \|\widehat{\tau} - t\|_H \leq \frac{4\tau \sqrt{\varepsilon}}{s} + \sqrt{\frac{2\varepsilon \tau}{s}}. \quad \text{(C.5)} \]

The result in (C.1) follows by combining (C.4) and (C.5) in (C.3) under the assumption that \(s \geq 8\tau\). (C.2) follows from (C.4) under the assumption that \(s \geq 8\tau\). \(\square\)

The following result provides a bound on \(\|f \otimes H f - g \otimes H g\|_{L^2(H)}\) in terms of \(\|f - g\|_H\).

**Lemma C.2.** Let \(H\) be a Hilbert space with \(f, g \in H\). Then

\[ \|f \otimes H f - g \otimes H g\|_{L^2(H)} \leq \|f - g\|_H \sqrt{\|f\|_H^2 + \|g\|_H^2 + 4\|f\|_H\|g\|_H}. \]

**Proof.** Note that

\[ \|f \otimes H f - g \otimes H g\|_{L^2(H)}^2 = \|f\|_H^4 + \|g\|_H^4 - 2\langle f, g \rangle_H^2 = \|f\|_H^2 - \|g\|_H^2 - 2\|f\|_H^2\|g\|_H^2 + 2\|f\|_H^2\|g\|_H^2 - \langle f, g \rangle_H^2 \]

\[ = \|f\|_H^2\|g\|_H - \langle f, g \rangle_H^2 + \|f\|_H\|g\|_H^2 + \langle f, g \rangle_H^2 \]

\[ + \|f\|_H^2 - \|g\|_H^2 \]

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\[
\begin{align*}
&= \left( \| f - g \|_H^2 - \| f \|_H - \| g \|_H^2 \right) \left( \| f \|_H \| g \|_H + \langle f, g \rangle_H \right) + \| f \|_H^2 - \| g \|_H^2 \| f \|_H^2 - \| g \|_H^2 \right) \\
&\leq \| f \|_H^2 - \| g \|_H^2 + \| f - g \|_H^2 \left( \| f \|_H \| g \|_H + \langle f, g \rangle_H \right).
\end{align*}
\]

Since
\[
\| f \|_H^2 - \| g \|_H^2 = \| f \|_H^2 - \| g \|_H^2 \| f \|_H + \| g \|_H \| f \|_H + \| g \|_H^2 \leq \| f - g \|_H^2 \| f \|_H + \| g \|_H^2
\]
and
\[
\| f - g \|_H^2 \left( \| f \|_H \| g \|_H + \langle f, g \rangle_H \right) \leq 2 \| f \|_H \| g \|_H \| f - g \|_H,
\]
the result follows. \hfill \square

The following result computes the operator, Hilbert-Schmidt and trace norms of a rank one operator.

**Lemma C.3.** Define \( B = f \otimes_H f \) where \( H \) is a separable Hilbert space and \( f \in H \). Then \( \| B \|_{\mathcal{L}_{\infty}(H)} = \| B \|_{\mathcal{L}^2(H)} = \| B \|_{\mathcal{L}^1(H)} = \| f \|_H^2 \).

**Proof.** Since \( B \) is self-adjoint, \( \| B \|_{\mathcal{L}_{\infty}(H)} = \lambda_1(B) = \sup_{\| g \|_H = 1} \langle g, B g \rangle_H = \sup_{\| g \|_H = 1} \langle f, g \rangle_H^2 = \| f \|_H^2 \). Note that \( \| B \|_{\mathcal{L}^1(H)} = \sum_j \langle e_j, (f \otimes_H f) e_j \rangle_H = \sum_j \langle f, e_j \rangle_H^2 = \| f \|_H^2 \) for any orthonormal basis \( (e_j)_j \) in \( H \). \hfill \square

## D Supplementary Results

In this appendix, we collect some standard results that are used to prove the results of this paper. The first result is Bernstein’s inequality in separable Hilbert spaces which is quoted from Yurinsky (1995, Theorem 3.3.4). The second result is a collection of results from operator theory.

**Theorem D.1** (Bernstein’s inequality). Let \( (\Omega, A, \mathbb{P}) \) be a probability space, \( H \) be a separable Hilbert space, \( B > 0 \) and \( \theta > 0 \). Furthermore, let \( \xi_1, \ldots, \xi_n : \Omega \to H \) be zero mean i.i.d. random variables satisfying
\[
\mathbb{E}\| \xi_1 \|_H^r \leq \frac{r!}{2} \theta^2 B^r, \quad \forall \ r > 2.
\]

Then for any \( \tau > 0 \),
\[
\mathbb{P}^n \left\{ (\xi_1, \ldots, \xi_n) : \left\| \frac{1}{n} \sum_{i=1}^{n} \xi_i \right\|_H \geq \frac{2B\tau}{n} + \sqrt{\frac{2\theta^2\tau}{n}} \right\} \leq 2e^{-\tau}.
\]

**Theorem D.2.** Let \( A \) and \( B \) be self-adjoint positive Hilbert-Schmidt operators on a Hilbert space, \( H \) with eigenvalues \( (\lambda_i)_i \) and \( (\tau_i)_i \) respectively. Then
\[
(i) \quad |\lambda_i - \tau_i| \leq \| (\lambda_i - \tau_i)_i \|_{\ell_2} = \| A - B \|_{\mathcal{L}^2(H)}, \quad \forall \ i;
\]
\[
(ii) \quad \| A^t - B^t \|_{\mathcal{L}_{\infty}(H)} \leq \| A - B \|_{\mathcal{L}_{\infty}(H)}, \quad 0 \leq t \leq 1;
\]
\[
(iii) \quad \| A^t - B^t \|_{\mathcal{L}_{\infty}(H)} \leq t \left( \| A \|_{\mathcal{L}_{\infty}(H)}^{-1} \lor \| B \|_{\mathcal{L}_{\infty}(H)}^{-1} \right) \| A - B \|_{\mathcal{L}_2(H)}^t, \quad t \geq 1.
\]

**Proof.** (i) follows from Theorem II in [Kato, 1987] which is an infinite dimensional extension of Hoffman-Wielandt inequality (see Kato, 1980, Theorem II-6.11). (ii) follows from Theorem X.1.1 in [Bhatia, 1997] for the operator monotone function \( x \mapsto x^t \) on \( (0, \infty) \). (iii) follows from Lemma 7 in [De Vito et al., 2014] for the Lipschitz function \( x \mapsto x^t \) on \( (0, \| A \|_{\mathcal{L}_{\infty}(H)} \lor \| B \|_{\mathcal{L}_{\infty}(H)}) \). \hfill \square
References

A. Alaoui and M. Mahoney. Fast randomized kernel ridge regression with statistical guarantees. In C. Cortes, N. D. Lawrence, D. D. Lee, M. Sugiyama, and R. Garnett, editors, Advances in Neural Information Processing Systems 28, pages 775–783. Curran Associates, Inc., 2015.

N. Aronszajn. Theory of reproducing kernels. Trans. Amer. Math. Soc., 68:337–404, 1950.

F. Bach. Sharp analysis of low-rank kernel matrix approximations. In S. Shalev-Shwartz and I. Steinwart, editors, Proc. of the 26th Annual Conference on Learning Theory, volume 30 of Proceedings of Machine Learning Research, pages 185–209. PMLR, 2013.

F. Bach and M. I. Jordan. Predictive low-rank decomposition for kernel methods. In L. D. Raedt and S. Wrobel, editors, Proc. of the 22nd International Conference on Machine Learning, pages 33–40, 2005.

P. Bartlett, O. Bousquet, and S. Mendelson. Local Rademacher complexities. Annals of Statistics, 33(4):1497–1537, 2005.

R. Bhatia. Matrix Analysis. Springer-Verlag, New York, 1997.

G. Blanchard, O. Bousquet, and L. Zwald. Statistical properties of kernel principal component analysis. Machine Learning, 66(2):259–294, 2007.

A. Caponnetto and E. De Vito. Optimal rates for regularized least-squares algorithm. Foundations of Computational Mathematics, 7:331–368, 2007.

C. Cortes, M. Mohri, and A. Talwalkar. On the impact of kernel approximation on learning accuracy. In Y. W. Teh and M. Titterington, editors, Proc. of the 13th International Conference on Artificial Intelligence and Statistics, volume 9 of Proceedings of Machine Learning Research, pages 113–120. PMLR, 2010.

E. De Vito, L. Rosasco, and A. Toigo. Learning sets with separating kernels. Applied and Computational Harmonic Analysis, 37(2):185–217, 2014.

J. Diestel and J. J. Uhl. Vector Measures. American Mathematical Society, Providence, 1977.

P. Drineas and M. W. Mahoney. On the Nyström method for approximating a Gram matrix for improved kernel-based learning. Journal of Machine Learning Research, 6:2153–2175, December 2005.

S. Fine and K. Scheinberg. Efficient SVM training using low-rank kernel representations. Journal of Machine Learning Research, 2:243–264, 2001.

H. Hoffmann. Kernel PCA for novelty detection. Pattern Recognition, 40:863–874, 2007.

R. Jin, T. Yang, M. Mahdavi, Y-F. Li, and Z-H. Zhou. Improved bounds for the Nyström method with application to kernel classification. IEEE Transactions on Information Theory, 59(10):6939–6949, 2013.

I. Jolliffe. Principal Component Analysis. Springer-Verlag, New York, USA, 1986.
P. Kar and H. Karnick. Random feature maps for dot product kernels. In N. D. Lawrence and M. Girolami, editors, Proc. of the 15th International Conference on Artificial Intelligence and Statistics, volume 22 of Proceedings of Machine Learning Research, pages 583–591. PMLR, 2012.

T. Kato. Perturbation Theory for Linear Operators. Springer-Verlag, New York, USA, 1980.

T. Kato. Variation of discrete spectra. Communications in Mathematical Physics, 111:501–504, 1987.

G. S. Kimeldorf and G. Wahba. Some results on Tchebycheffian spline functions. Journal of Mathematical Analysis and Applications, 33:82–95, 1971.

V. Koltchinskii. Local Rademacher complexities and oracle inequalities in risk minimization. Annals of Statistics, 34(6):2593–2656, 2006.

S. Kumar, M. Mohri, and A. Talwalkar. Ensemble Nyström method. In Y. Bengio, D. Schuurmans, J. D. Lafferty, C. K. I. Williams, and A. Culotta, editors, Advances in Neural Information Processing Systems 22, pages 1060–1068. Curran Associates, Inc., 2009.

C. Lampert. Kernel methods in computer vision. Foundations and Trends in Computer Graphics and Vision, 4(3):193–285, 2009.

Q. Le, T. Sarlós, and A. J. Smola. Fast food – Computing Hilbert space expansions in loglinear time. In S. Dasgupta and D. McAllester, editors, Proc. of the 30th International Conference on Machine Learning, volume 28 of Proceedings of Machine Learning Research, pages 244–252. PMLR, 2013.

D. Lopez-Paz, S. Sra, A. Smola, Z. Ghahramani, and B. Schölkopf. Randomized nonlinear component analysis. In E. P. Xing and T. Jebara, editors, Proceedings of the 31st International Conference on Machine Learning, volume 32 of Proceedings of Machine Learning Research, pages 1359–1367. PMLR, 2014.

S. Mika, B. Schölkopf, A. J. Smola, K-R. Müller, M. Scholz, and G. Rätsch. Kernel PCA and de-noising in feature spaces. In M. J. Kearns, S. A. Solla, and D. A. Cohn, editors, Advances in Neural Information Processing Systems 11, pages 536–542. MIT Press, 1999.

A. Rahimi and B. Recht. Random features for large-scale kernel machines. In J. C. Platt, D. Koller, Y. Singer, and S. T. Roweis, editors, Advances in Neural Information Processing Systems 20, pages 1177–1184. Curran Associates, Inc., 2008.

M. Reed and B. Simon. Methods of Modern Mathematical Physics: Functional Analysis I. Academic Press, New York, 1980.

A. Rudi and L. Rosasco. Generalization properties of learning with random features. In I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, Advances in Neural Information Processing Systems 30, pages 3215–3225. Curran Associates, Inc., 2017.

A. Rudi, R. Camoriano, and L. Rosasco. Less is more: Nyström computational regularization. In C. Cortes, N. D. Lawrence, D. D. Lee, M. Sugiyama, and R. Garnett, editors, Advances in Neural Information Processing Systems 28, pages 1657–1665. Curran Associates, Inc., 2015.

B. Schölkopf and A. J. Smola. Learning with Kernels. MIT Press, Cambridge, MA, 2002.
B. Schölkopf, A. Smola, and K.-R. Müller. Nonlinear component analysis as a kernel eigenvalue problem. *Neural Computation*, 10:1299–1319, 1998.

B. Schölkopf, R. Herbrich, and A. Smola. A generalized representer theorem. In *Proc. of the 14th Annual Conference on Computational Learning Theory and 5th European Conference on Computational Learning Theory*, pages 416–426, London, UK, 2001. Springer-Verlag.

J. Shawe-Taylor, C. Williams, N. Christianini, and J. Kandola. On the eigenspectrum of the Gram matrix and the generalisation error of kernel PCA. *IEEE Transactions on Information Theory*, 51(7):2510–2522, 2005.

S. Smale and D.-X. Zhou. Learning theory estimates via integral operators and their approximations. *Constructive Approximation*, 26:153–172, 2007.

A. J. Smola and B. Schölkopf. Sparse greedy matrix approximation for machine learning. In *Proc. 17th International Conference on Machine Learning*, pages 911–918. Morgan Kaufmann, San Francisco, CA, 2000.

B. K. Sriperumbudur and Z. Szabó. Optimal rates for random Fourier features. In C. Cortes, N. D. Lawrence, D. D. Lee, M. Sugiyama, and R. Garnett, editors, *Advances in Neural Information Processing Systems 28*, pages 1144–1152. Curran Associates, Inc., 2015.

I. Steinwart and A. Christmann. *Support Vector Machines*. Springer, New York, 2008.

H. Wendland. *Scattered Data Approximation*. Cambridge University Press, Cambridge, UK, 2005.

C.K.I. Williams and M. Seeger. Using the Nyström method to speed up kernel machines. In V. Tresp T. K. Leen, T. G. Diettrich, editor, *Advances in Neural Information Processing Systems 13*, pages 682–688, Cambridge, MA, 2001. MIT Press.

T. Yang, Y. Li, M. Mahdavi, R. Jin, and Z-H. Zhou. Nyström method vs random Fourier features: A theoretical and empirical comparison. In F. Pereira, C. J. C. Burges, L. Bottou, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems 25*, pages 476–484. Curran Associates, Inc., 2012.

Y. Yang, M. Pilanci, and M. J. Wainwright. Randomized sketches for kernels: Fast and optimal non-parametric regression. *Annals of Statistics*, 45(3):991–1023, 2017.

V. Yurinsky. *Sums and Gaussian Vectors*, volume 1617 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.

K. Zhang, I. W. Tsang, and J. T. Kwok. Improved Nyström low-rank approximation and error analysis. In *Proceedings of the 25th international conference on Machine learning*, pages 1232–1239. ACM, 2008.

L. Zwald and G. Blanchard. On the convergence of eigenspaces in kernel principal component analysis. In Y. Weiss, B. Schölkopf, and J. C. Platt, editors, *Advances in Neural Information Processing Systems 18*, pages 1649–1656. MIT Press, 2006.