Generalized plane-fronted gravitational waves in any dimension

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Abstract

We study the gravitational waves in spacetimes of arbitrary dimension. They generalize the pp-waves and the Kundt waves, obtained earlier in four dimensions. Explicit solutions of the Einstein and Einstein-Maxwell equations are derived for an arbitrary cosmological constant.

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I. INTRODUCTION

Recently, there has been some progress in deriving the various generalizations of the plane wave solutions of the Maxwell and Einstein-Maxwell equations in higher-dimensional spacetimes [1–3]. The interest in such configurations is motivated, in particular, by applications that the plane wave solutions find in string theory, see e.g. [4,5] and the references therein. The gravitational plane-fronted waves in four dimensions represent a well known class of solutions which satisfy the so-called radiative conditions [6–10].

In addition, some attention has been paid recently to the generalizations of the plane wave solutions to metric-affine theories of gravity with torsion and nonmetricity waves along with the usual gravitational waves [11–14].

The aim of this work is to present exact gravitational wave solutions of the Einstein field equation with a cosmological term \( \Lambda \) for arbitrary dimensions \( 2 + N \).

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In Sec. II, the geometric ansatz for the metric (coframe) is formulated as an immediate generalization of the previous work done in four dimensions [15,16]. We demonstrate that the resulting curvature of spacetime satisfies the generalized radiation conditions. Using this ansatz, in Sec. III we derive a partial differential equation for the only unknown function. The explicit solutions are obtained in Sec. V. Our attention is not only confined to higher dimensions. The wave solutions in two and three spacetime dimensions represent a particular case which is analyzed separately in Sec. V. In the absence of the Maxwell radiative source, such spacetimes are isometric to the de Sitter (or anti-de Sitter) manifold. The four-dimensional case is only briefly mentioned. The higher-dimensional wave solutions are obtained under the assumption of the rotational symmetry. We study in detail the case of the positive cosmological constant $\Lambda > 0$. The solutions for the negative values $\Lambda < 0$ are briefly discussed in Sec. VI. In Sec. VII we summarize the results obtained and the Appendix contains some useful mathematical definitions and computations.

II. GEOMETRY OF THE ANSATZ

Let us denote the $2 + N$ local spacetime coordinates by $x^i = \{\sigma, \rho, z^1, \ldots, z^N\}$, with $i = 0, 1, \ldots, N + 1$. The upper case Latin indices, $A, B, \ldots = 0, 1$, label the first 2 spacetime dimensions which are relevant to a pp-wave. In particular, $x^A = \{\sigma, \rho\}$ are the wave coordinates with the wave fronts described by the surfaces of constant $\sigma$, and $\rho$ is an affine parameter along the wave vector of the null geodesic. The lower case Latin indices, $a, b, \ldots = 1, \ldots, N$, refer to an $N$-dimensional space of constant curvature. Greek indices, $\alpha, \beta, \ldots = 0, 1, \ldots, N + 1$, label the local anholonomic (co)frame components. We denote separate frame components by a circumflex over the corresponding index in order to distinguish them from coordinate components. The differential of an arbitrary function $f(x') = f(\sigma, \rho, z^a)$ with respect to the last $N$ coordinates is denoted $\text{df} := dz^a \partial_a f$.

The line element reads

$$ds^2 = g_{\alpha\beta} \partial^\alpha \otimes \partial^\beta,$$

(1)
with the half-null Lorentz metric

\[ g_{\alpha\beta} = \begin{pmatrix} g_{AB} & 0 \\ 0 & g_{ab} \end{pmatrix}, \quad g_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_{ab} = \delta_{ab}. \] (2)

With the functions \( p(z^a), q(\sigma, z^a), s(\sigma, \rho, z^a) \), the components of the coframe 1-form are given by

\[ \vartheta^0 = -d\sigma, \quad \vartheta^1 = \left( \frac{q}{p} \right)^2 (s \, d\sigma + d\rho), \quad \vartheta^a = \frac{1}{p} \, dz^a, \quad a = 1, \ldots, N. \] (3)

The dual frame basis (such that \( e^\alpha \vert \vartheta^\beta = \delta^\alpha_\beta \)) reads:

\[ e^0 = -\partial_\sigma + s \, \partial_\rho, \quad e^1 = \left( \frac{p}{q} \right)^2 \partial_\rho, \quad e^a = p \, \partial_a. \] (4)

We choose

\[ p = 1 + \frac{\Lambda}{4} z_\alpha z^\alpha, \quad q = q_0 \alpha + \beta_a z^a, \quad q_0 = 1 - \frac{\Lambda}{4} z_\alpha z^\alpha, \] (5)

\[ s = -\frac{\rho^2}{2} (\Lambda \alpha^2 + \beta_a \beta^a) + \rho \frac{\partial_\sigma q}{q} + \frac{p^{N/2}}{2} H(\sigma, z^a). \] (6)

Here \( \Lambda \) is constant, the \( 1+N \) functions \( \alpha = \alpha(\sigma) \) and \( \beta_a = \beta_a(\sigma) \) depend on the coordinate \( \sigma \) only, and \( z_\alpha z^\alpha = \delta_{ab} z^a z^b, \beta_a \beta^a = \delta_{ab} \beta^a \beta^b; \) \( H(\sigma, z^a) \) is an unknown function to be determined by the field equation. Then, for arbitrary \( \alpha, \beta_a \), the curvature 2-form reads:

\[ R^{\alpha\beta} = -\Lambda \vartheta^\alpha \wedge \vartheta^\beta + 2 \gamma^{|\alpha} k^{|\beta} \wedge \vartheta^0. \] (7)

Here the null vector is \( k_\alpha = \delta^0_\alpha \), with \( k_\alpha k^\alpha = 0 \), and the components of the co-vector-valued 1-form \( \gamma_\alpha \) are given in terms of the derivatives of the function \( s \):

\[ \gamma^0 = 0, \quad \gamma^1 = 0, \quad \gamma_a = -D \left[ \left( \frac{q}{p} \right)^2 e^a \vert ds \right] + \left( \frac{q}{p} \right)^3 e^a \left[ d \left( \frac{p}{q} \right) \wedge ds \right]. \] (8)

The \( N \)-dimensional covariant derivative is denoted as \( D \) (the details of computations of the connection and the curvature are presented in the Appendix). The 1-form (8) “lives” on the \( N \)-submanifold and as a result, it has the evident properties

\[ \vartheta^a \wedge \gamma_\alpha = 0, \quad e_A \vert \gamma_\alpha = 0, \quad e_A \vert \gamma^a = e_a \vert \gamma^a. \] (9)
Substituting (5), (6) into (8), we find the trace of the 1-form explicitly:

\[ e_α \gamma^α = -\frac{p^{N/2}q}{2} \left( \partial^α \partial_α H + \frac{\Lambda N(N + 2)}{4p^2} H \right). \] (10)

In view of the obvious orthogonality relation \( \gamma^α k^α = 0 \), the tensor-valued 2-form \( S^{αβ} = R^{αβ} + \Lambda \vartheta^α \wedge \vartheta^β \) satisfies the so-called radiation conditions [16]:

\[ S^{αβ} k_β = 0, \quad S^{[αβ} k^{γ]} = 0. \] (11)

The same radiation conditions are also fulfilled by the Weyl 2-form which can be derived from (7):

\[ W^{αβ} = 2 \chi^{[α} k^{β]} \wedge \vartheta^0. \] (12)

This formula involves the traceless part \( \chi^{α} \) of the 1-form (8) which is defined by \( \chi^{\hat{α}} = \chi^{T} = 0 \) and by \( \chi^{α} := \gamma^α - 1/N (e_β | γ^β) \vartheta^α \). By definition, we have \( e_α | γ^α = e_α | γ^{\hat{α}} = 0 \).

An arbitrary function \( w = w(σ) \) defines the combined coordinate and coframe (Lorentz) transformation

\[
\sigma \rightarrow \int^σ w^2(σ)dσ, \quad \vartheta^A \rightarrow L^A_B \vartheta^B, \quad L^A_B = \begin{pmatrix} w^2 & 0 \\ 0 & w^{-2} \end{pmatrix},
\] (13)

which leaves invariant the form of the line element but rescales the three functions as follows:

\[
α \rightarrow α/w, \quad β_α \rightarrow β_α/w, \quad H \rightarrow H/w^2.
\] (14)

As a result, a nonzero \( α(σ) \) can always be rescaled to \( α = 1 \) with the help of (13), (14) by choosing \( w = α \).

III. GRAVITATIONAL EQUATION

The Ricci 1-form reads:

\[
eβ | R^{αβ} = \Lambda(N + 1) \vartheta^α - (e_α | γ^α) k^α \vartheta^0 = \left[ \Lambda(N + 1) \delta^α_β - (e_α | γ^α) k^α k_β \right] \vartheta^β.
\] (15)
Here we took into account that $\vartheta^0 = k_\alpha \vartheta^\alpha$.

In the tensor language, the Ricci tensor thus has the form:

$$Ric_{\alpha\beta} = \Lambda(N+1) g_{\alpha\beta} - (e_a | \gamma^a) k_\alpha k_\beta.$$  \hspace{1cm} (16)

Accordingly, the metric (1)-(6) is, by construction, a solution of the Einstein field equation with the cosmological term $\Lambda(N+1)$ and the "pure radiation" source. The latter is described by the null vector field $k_\alpha$ and the radiation energy density $-(e_a | \gamma^a)$.

A particular example of such a material source is represented by the energy-momentum of an electromagnetic wave. Taking the potential 1-form

$$A = \varphi(\sigma, z^a) \vartheta^0,$$  \hspace{1cm} (17)

we find the 2-form of the electromagnetic field $F = dA = \frac{1}{2} F_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta$ with the tensor components

$$F_{\alpha\beta} = 2n_{[\alpha} k_{\beta]}.$$  \hspace{1cm} (18)

Here $n^\alpha k_\alpha = 0$. In terms of the vector potential, the covector $n$ reads as follows:

$$n_0 = 0, \quad n_1 = 0, \quad n_a = e_a | d\varphi = p \partial_a \varphi.$$  \hspace{1cm} (19)

The unknown scalar function $\varphi$ is determined by the Maxwell equation $d^* F = 0$ that for the metric (1)-(3) reduces to the partial differential equation

$$\partial_a \left( p^{2-N} \partial^a \varphi \right) = 0.$$  \hspace{1cm} (20)

We can see from (18) that

$$F_{\alpha\gamma} F_{\beta}{}^\gamma = n_\gamma n^\gamma k_\alpha k_\beta, \quad F_{\alpha\beta} F^{\alpha\beta} = 0,$$  \hspace{1cm} (21)

and hence the energy-momentum reads

$$T_{\alpha\beta} = c \varepsilon_0 \left( F_{\alpha\gamma} F_{\beta}{}^\gamma - \frac{1}{4} g_{\alpha\beta} F_{\rho\sigma} F^{\rho\sigma} \right) = c \varepsilon_0 n_\gamma n^\gamma k_\alpha k_\beta.$$  

Combining (10), (16), and (21), we find that Einstein's equation

$$Ric_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \lambda g_{\alpha\beta} = \frac{8\pi G}{c^3} T_{\alpha\beta}$$  \hspace{1cm} (22)
with cosmological constant $\lambda = \Lambda (N+1)$ is satisfied provided the unknown function $H(\sigma, z^a)$ is a solution of

$$\partial^a \partial_a H + \frac{\Lambda N (N+2)}{4p^2} H = \frac{16\pi G\varepsilon_0 p^2 - N/2}{c^2 q}[\partial_a \varphi \partial^a \varphi].$$

(23)

Here $G$ is Newton’s gravitational constant and $\varepsilon_0$ the electric constant of the vacuum (vacuum permittivity).

**IV. VACUUM SOLUTION IN ANY DIMENSION**

In vacuum, when $\varphi = 0$, one can verify that the partial differential equation (23) has a solution:

$$H(\sigma, z^a) = f(\sigma) H_1(z^a), \quad H_1 = \frac{\tilde{q}}{p_N/2}.$$  

(24)

Here $f(\sigma)$ is an arbitrary function and

$$\tilde{q} = \left(1 - \frac{\Lambda}{4} z_a z^a\right) \tilde{\alpha} + \tilde{\beta}_a z^a.$$  

(25)

This has the same structure as the function $q$ from (5), but the arbitrary functions $\tilde{\alpha} = \tilde{\alpha}(\sigma)$ and $\tilde{\beta}_a = \tilde{\beta}_a(\sigma)$ are different from $\alpha, \beta_a$, in general.

Substituting this into (6), we find

$$s = -\frac{\rho^2}{2} (\Lambda \alpha^2 + \beta_a \beta^a) + \frac{\rho (\partial_{\sigma} q) + \tilde{q} f(\sigma)/2}{q}.$$  

(26)

As a result, we can verify that the 1-form (8) vanishes for such a solution, and the spacetime curvature (7) becomes constant: $R^{\alpha\beta} = -\Lambda \varphi^{\alpha} \wedge \varphi^\beta$. The resulting metric

$$ds^2 = -2 \left(\frac{q}{p}\right)^2 (s \, d\sigma^2 + d\sigma d\rho) + \frac{dz_a dz^a}{p^2}$$

(27)

with (26) and (25) thus represents the $(2 + N)$-dimensional de Sitter (or anti-de Sitter, depending on the sign of $\Lambda$) spacetime with a gravitational $pp$-wave propagating in it. In four dimensions ($N = 2$) this was demonstrated in [17].

The particular solution (24) satisfies the vacuum partial differential equation (23) for any dimension $N$. The complete solution depends significantly on the dimension, and it is instructive to analyze the different values of $N$ separately.
V. EXPLICIT SOLUTIONS FOR POSITIVE COSMOLOGICAL CONSTANT

A. \(N = 0\): Two-dimensional spacetime

In the two-dimensional spacetime, the \(z\) coordinates are absent. Hence the functions (6) reduce to \(p = 1\), \(q = \alpha(\sigma)\), and \(s = (-\Lambda \rho^2 \alpha^2 + H)/2\), with \(H = H(\sigma)\). We do not have an equation (23) in this case, and the resulting metric reads:

\[
d s^2 = -2\alpha^2 \left( \frac{-\Lambda \rho^2 \alpha^2 + H(\sigma)}{2} d\sigma^2 + d\sigma d\rho \right).
\]  

We can put \(\alpha = 1\) by means of the rescaling (13), (14).

B. \(N = 1\): Three-dimensional spacetime

In three spacetime dimensions, we have \(N = 1\) and a single \(z\)-coordinate, so that the partial derivatives in (23) reduce to the ordinary ones. Then (24), (25) evidently describes a general solution of the resulting homogeneous equation, with \(\tilde{\alpha}\) and \(\tilde{\beta}\) being the two arbitrary integration functions. We rewrite this general solution of the homogeneous equation as

\[
H(\sigma, z) = f_1(\sigma) H_1(z) + f_2(\sigma) H_2(z).
\]  

Here \(f_1\) and \(f_2\) are the two arbitrary functions (replacing \(\tilde{\alpha}\) and \(\tilde{\beta}\)) and

\[
H_1(z) = \frac{q_0}{\sqrt{p}}, \quad H_2(z) = \frac{\sqrt{\Lambda} z}{2\sqrt{p}}.
\]  

In accordance with our analysis above, the vacuum solution (29) yields a vanishing 1-form (8), and the curvature of the 3-spacetime again reduces to a de Sitter one \(R^{\alpha\beta} = -\Lambda \bar{\psi}^\alpha \wedge \bar{\psi}^\beta\).

It is straightforward to take the electromagnetic source into account. Integration of (20) yields the scalar potential

\[
\varphi = \nu(\sigma) \varphi_0(z), \quad \varphi_0(z) = \arctan(\sqrt{\Lambda} z/2),
\]  

with an arbitrary function \(\nu(\sigma)\). The particular solution of the inhomogeneous equation (23) then reads:
\[ H_i(\sigma, z) = \frac{4\pi \varepsilon_0 \nu^2 \Lambda}{c^2(\Lambda \alpha^2 + \beta^2)\sqrt{p}} \left[ 2\sqrt{\Lambda}(\alpha z - \beta q_0/\Lambda) \arctan(\sqrt{\Lambda}z/2) - q \ln(p/|q|) \right]. \] (32)

Correspondingly, for \( N = 1 \), we find the general solution of (23) as

\[ H(\sigma, z) = f_1(\sigma) H_1(z) + f_2(\sigma) H_2(z) + H_i(\sigma, z). \] (33)

**C. \( N = 2 \): Four-dimensional spacetime**

For \( N = 2 \) it is convenient to combine the two real coordinates \( z^a, a = 1, 2 \), into a complex variable \( \zeta = z^1 + iz^2 \). Then (23) can be recast into

\[ \partial_{\zeta \bar{\zeta}} H + \frac{\Lambda}{2p^2} H = \frac{16\pi \varepsilon_0 p}{c^2 q} \partial_{\zeta} \varphi \partial_{\bar{\zeta}} \varphi. \] (34)

We now have \( p = 1 + \Lambda \zeta \bar{\zeta}/4 \) and \( q = (1 - \Lambda \zeta \bar{\zeta}/4) \alpha + (\zeta \bar{\beta} + \bar{\zeta} \beta)/2 \) with \( \beta = \beta_1 + i\beta_2 \). The overbar denotes complex conjugate quantities. The integration of the equation (34) is based on the following differential identity which holds for an arbitrary function \( f(\sigma, \zeta, \bar{\zeta}) \):

\[ \partial^2_{\zeta \bar{\zeta}} \left[ p^2 \partial_\zeta \left( \frac{f}{p^2} \right) \right] + \frac{\Lambda}{2p^2} \left[ p^2 \partial_\zeta \left( \frac{f}{p^2} \right) \right] \equiv \partial_\zeta \left[ p^2 \partial_\zeta \left( \frac{\partial_\zeta f}{p^2} \right) \right]. \] (35)

As a result, we have the general solution of (34) in the form,

\[ H(\sigma, \zeta, \bar{\zeta}) = p^2 \left[ \partial_\zeta \left( \frac{f}{p^2} \right) + \partial_{\bar{\zeta}} \left( \frac{f}{p^2} \right) \right], \quad f = f_0(\sigma, \zeta) + f_1(\sigma, \zeta, \bar{\zeta}), \] (36)

where \( f_0(\sigma, \zeta) \) is an arbitrary holomorphic function of \( \zeta \) and

\[ f_1(\sigma, \zeta, \bar{\zeta}) = \frac{8\pi \varepsilon_0}{c^2} \int^\zeta d\zeta' p^2 \int^{\zeta'} d\zeta'' p^{-2} \int^{\zeta''} d\zeta''' (p/q) \partial_{\zeta'''} \varphi \partial_{\zeta''} \varphi. \] (37)

Equation (20) reduces to \( \partial^2_{\zeta \bar{\zeta}} \varphi = 0 \) which means that the electromagnetic potential can be expressed as \( \varphi = \varphi_0 + \bar{\varphi}_0 \) in terms of an arbitrary holomorphic function \( \varphi_0(\sigma, \zeta) \).

The 4-dimensional problem was studied in detail in [16,12,13,17], and the above solution was derived together with the explicit computation of the integral (37).
D. Arbitrary $N$: Rotationally symmetric solutions

For $N > 2$, the complete integration of the partial differential equation (23) becomes rather nontrivial. Here we will not consider this problem in its full generality, but from now on we confine ourselves to the rotationally symmetric solutions which are described by the functions $H(\sigma, \xi)$ with the radial variable $\xi = \sqrt{z_a z^a} = \sqrt{\delta_{ab} z^a z^b}$. Recalling (24), we readily have one such solution in vacuum which is given by

$$H_1(\sigma, \xi) = \frac{q_0}{p^{N/2}}. \quad (38)$$

Note that $p = p(\xi) = 1 + \Lambda \xi^2/4$ and $q_0 = q_0(\xi) = 1 - \Lambda \xi^2/4$.

The second independent vacuum solution $H_2$ can be obtained by using the Liouville formula after rewriting the $N$-dimensional Laplacian of (23) in hyperspherical coordinates:

$$\partial_a \partial^a H = \xi^{1-N} \partial_{\xi}(\xi^{N-1} \partial_{\xi} H) + \xi^{-2} \Delta_{S_{N-1}} H \quad \text{(with $\Delta_{S_{N-1}}$ the Laplacian on a $(N-1)$-hypersphere).}$$

Denoting $x = \sqrt{\Lambda \xi/2}$, we then obtain

$$H_2 = \frac{q_0}{p^{N/2}} \int dx \frac{(1 + x^2)^N}{(1 - x^2)^2 x^{N-1}}. \quad (39)$$

The final explicit result depends essentially on whether $N$ is an odd or an even number.

For the odd $N = 2n + 1, n = 0, 1, \ldots$, we have the second rotationally symmetric vacuum solution:

$$H_2 = 2^{N-2} \frac{q_0}{p^{N/2}} \sum_{k=0}^{n} \frac{(n)}{2k-1} \left( \frac{-q_0}{\sqrt{\Lambda \xi}} \right)^{2k-1}. \quad (40)$$

For the even $N = 2n, n = 1, 2, \ldots$, we find a more complicated result:

$$H_2 = \frac{1}{4 p^{N/2}} \left\{ \binom{2n}{n} \left[ p + q_0 n \ln (\Lambda \xi^2/4) \right] + \sum_{k=0}^{n} \left[ p + q_0 n/k \right] \left( \binom{2n}{n-k} \Lambda \xi^2/4 \right)^k \right\}. \quad (41)$$

As usual, $\binom{n}{k}$ denotes the binomial coefficient. It is instructive to write down several examples for the lower dimensions. For $N = 1$ (three-dimensional spacetime), Eq. (40) yields $H_2 = (\sqrt{\Lambda \xi}/2)/\sqrt{p}$, thus recovering (30). For $N = 2$ (four spacetime dimensions), we have from (41): $H_2 = 1 + (q_0/p) \ln \left( \sqrt{\Lambda \xi}/2 \right)$. In a five-dimensional spacetime (for $N = 3$), the equation
(40) yields $H_2 = 2(p^2 - 2q_0^2)/(\sqrt{\Lambda}\xi q_0 p^{3/2})$. In six dimensions (for $N = 4$), we read off from (41): $H_2 = (4/p) \left[ 1 - q_0^2/(2\Lambda\xi^2) + (3q_0/2p) \ln \left( \sqrt{\Lambda}\xi/2 \right) \right]$.

The general rotationally symmetric solution in vacuum reads:

$$H(\sigma, z^a) = f_1(\sigma) H_1(z^a) + f_2(\sigma) H_2(z^a),$$

(42) with $H_{1,2}$ given by (38), (40), (41), and the two arbitrary functions $f_{1,2}(\sigma)$. It is important to notice that whereas the function $H$ is rotationally symmetric, the spacetime metric is not. The frame and the line element involve the functions $q$ and $s$, given by (5) and (6), which are not rotationally symmetric for nontrivial $\beta_a$.

In order to find solutions for the nontrivial matter source, we have to analyze the electromagnetic field equations. Since we are interested in the rotationally symmetric configurations, we also impose this symmetry condition on the electromagnetic field. It is then straightforward to verify that a general rotationally symmetric solution of (20) reads (putting $x = \sqrt{\Lambda}\xi/2$):

$$\varphi = \nu(\sigma) \varphi_0(\xi), \quad \varphi_0(\xi) = \int dx \frac{(1 + x^2)^{N-2}}{x^{N-1}}.$$  
(43)

The value of this integral essentially depends on $N$. For $N = 1$, we find (31), whereas for higher odd dimensions $N = 2n + 1$ with $n = 1, 2, \ldots$, we obtain

$$\varphi_0 = 2^{N-2} \sum_{k=0}^{n-1} \frac{(n-1)}{2k+1} \left( \frac{-q_0}{\sqrt{\Lambda}\xi} \right)^{2k+1}.$$  
(44)

For the even dimensions $N = 2n$, with $n = 1, 2, \ldots$, we find

$$\varphi_0 = \frac{1}{2} \left[ \left( \frac{2n-2}{n-1} \right) \ln (\Lambda\xi^2/4) + \sum_{k=-n+1}^{+n-1} \frac{(2n-2)}{k} \left( \frac{\Lambda\xi^2}{4} \right)^{k} \right].$$  
(45)

Now we are in position to find the particular solution of the inhomogeneous equation (23). However, we evidently have a problem: the right-hand side of (23) contains $q$ which makes the rotationally symmetric ansatz inconsistent unless we assume that $\beta_a = 0$. Accordingly, we now put $\beta_a = 0$ and $\alpha = 1$ (which is always possible by the rescaling (13), (14)). Then
\( q = q_0 \) and the direct computation then yields the rotationally symmetric solution of the inhomogeneous equation (23):

\[
H_i(\sigma, \xi) = \frac{16\pi G \varepsilon_0 \nu^2}{c^2} \left[ \varphi_0(\xi) H_2(\xi) - \psi(\xi) H_1(\xi) \right],
\]  

where (again using the variable \( x = \sqrt{\Lambda} \xi/2 \))

\[
\psi(\xi) = \int dx \frac{(1 + x^2)^{N-2}}{x^{N-1}} \int x' \frac{(1 + x'^2)^N}{(1 - x'^2)^2 x^{N-1}} \]

\[
= \frac{N - 1}{2} \left( \int dx \frac{(1 + x^2)^{N-2}}{x^{N-1}} \right)^2 - \int dx \frac{(1 + x^2)^{2N-3}}{(1 - x^2)x^{2N-3}}. \tag{47}
\]

The last integrals essentially depend on \( N \). For \( N = 1 \), we get

\[
\psi(\xi) = \frac{1}{4} \ln(p/|q_0|), \tag{48}
\]

and then (46) reduces to a particular case of (32) after taking into account (30) and (31).

For higher dimensions, \( N > 1 \), we finally find:

\[
\psi(\xi) = \frac{N - 1}{2} \varphi_0^2(\xi) - 2^{N-4} \left[ \ln \left( \frac{|q_0|}{\sqrt{\Lambda} \xi} \right) + \sum_{k=1}^{N-2} \frac{1}{2k} \left( \frac{p^2}{\Lambda \xi^2} \right)^k \right]. \tag{49}
\]

The general rotationally symmetric solution of (23) for an arbitrary \( N \) with the electromagnetic wave source thus reads:

\[
H(\sigma, \xi) = f_1(\sigma) H_1(\xi) + f_2(\sigma) H_2(\xi) + H_i(\sigma, \xi). \tag{50}
\]

\textbf{VI. NEGATIVE COSMOLOGICAL CONSTANT}

Above, we have considered the exact solutions for the case of the positive cosmological constant, \( \Lambda > 0 \). For completeness, let us give the explicit results which corresponds to the case of the negative cosmological constant, \( \Lambda < 0 \). It will be convenient to consider the first lowest values of \( N \) separately, following the scheme of the previous section.
A. $N = 0$

Here we have again the geometry (28) of the two-dimensional spacetime of the (now, negative) constant curvature with an arbitrary function $H(\sigma)$ and the trivial functions $p = 1$ and $q = \alpha(\sigma)$.

B. $N = 1$

Instead of (30) we now find the two independent solutions of the homogeneous equation (23) to be $H_2(z) = q_0/\sqrt{|p|}$ and $H_2(z) = \sqrt{|\Lambda|} z/2 \sqrt{|p|}$ with $q_0 = 1 + |\Lambda| z^2/4$, and the solution of the Maxwell equations now reads

$$\varphi = \nu(\sigma) \varphi_0(z), \quad \varphi_0(z) = \ln \left( \frac{1 + \sqrt{|\Lambda|} z/2}{1 - \sqrt{|\Lambda|} z/2} \right).$$

(51)

This replaces (31). Then the general solution is again (33) with the particular solution (32) of the inhomogeneous equation replaced by

$$H_i(\sigma, z) = \frac{4\pi G \varepsilon_0 \nu^2 \Lambda}{c^2 (\Lambda \alpha^2 + \beta^2) \sqrt{|p|}} \left[ \sqrt{|\Lambda|}(\alpha z - \beta q_0/\Lambda) \ln \left( \frac{1 - \sqrt{|\Lambda|} z/2}{1 + \sqrt{|\Lambda|} z/2} \right) - q \ln(|p/q|) \right].$$

(52)

C. $N = 2$

In the four-dimensional spacetime, the main formulas (35)-(37) remain valid for any sign of $\Lambda$, and the corresponding solutions with negative cosmological constant are obtained by analytic continuation from the solutions with positive cosmological constant. See the refs. [16,12,13] where both cases are studied in detail and [17] where the complete classification of the solutions is given.

D. Arbitrary $N$

For the rotationally symmetric solutions with $\Lambda < 0$ and $N > 1$, we straightforwardly find the following results. We have now $p = 1 - |\Lambda| \xi^2/4$ and $q_0 = 1 + |\Lambda| \xi^2/4$. Furthermore, as
before, the first independent solution of the homogeneous equation (23) is (38), whereas the particular solution of the inhomogeneous equation is again given by (46), for \( \alpha = 1, \beta_a = 0 \).

However, the previous explicit result (49) for the function \( \psi(\xi) \) should be replaced by

\[
\psi(\xi) = \frac{N-1}{2} \varphi_0^2(\xi) - 2^{2N-4}(-1)^N \left[ \ln \left( \frac{q_0}{\sqrt{|\Lambda|\xi}} \right) + \sum_{k=1}^{N-2} \frac{1}{2k} \left( -p^2 \right)^k \right].
\] (53)

Here we assume \( N > 1 \) (the lower dimensional cases are reported in the previous subsections).

The form of the second solution of the homogeneous equation \( H_2 \) and the solution of the Maxwell equation \( \varphi_0 \) depends significantly on whether \( N \) is an odd or an even number.

For the odd \( N = 2n + 1, n = 1, 2, \ldots \), we obtain the second independent homogeneous solution \( H_2(\xi) \) and the electromagnetic potential \( \varphi_0(\xi) \), respectively:

\[
H_2 = 2^{N-2} \frac{(-1)^n q_0}{p^{N/2}} \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{2k-1} \left( \frac{q_0}{\sqrt{|\Lambda|\xi}} \right)^{2k-1},
\] (54)

\[
\varphi_0 = 2^{N-2} (-1)^{n-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{n-1}{k}}{2k+1} \left( \frac{q_0}{\sqrt{|\Lambda|\xi}} \right)^{2k+1}.
\] (55)

For the even dimensions \( N = 2n \), with \( n = 1, 2, \ldots \), we find more complicated expressions:

\[
H_2 = \frac{(-1)^n}{4 p^{N/2}} \left\{ \binom{2n}{n} \left[ p + q_0 n \ln \left( |\Lambda|\xi^2/4 \right) \right] + \sum_{k=0}^{n} \frac{\left( -1 \right)^{n-k} \binom{2n}{2n-k} \left( -|\Lambda|\xi^2/4 \right)^k}{\binom{n-k}{k} \binom{n+1}{k}} \right\},
\] (56)

\[
\varphi_0 = \frac{(-1)^n}{2} \left[ \binom{2n-2}{n-1} \ln \left( |\Lambda|\xi^2/4 \right) + \sum_{k=0}^{n-1} \frac{\left( -1 \right)^{n-k} \binom{2n-2}{n-1-k} \left( -|\Lambda|\xi^2/4 \right)^k}{\binom{n-1-k}{k}} \right].
\] (57)

The general rotationally symmetric solution of (23) for an arbitrary \( N > 1 \) with the electromagnetic wave source then again is given by (50), provided we use (53)-(57) in it.

VII. DISCUSSION AND CONCLUSION

In this paper, we report on the gravitational wave solutions in any spacetime dimension (lower and higher than 4) with an arbitrary cosmological constant \( \Lambda \). The wave fronts, obtained as the leaves of constant \( \sigma \), are then the \( N \)-dimensional surfaces of constant (positive
or negative, depending on the sign of $\Lambda$) curvature. In four dimensions (for $N = 2$), such solutions have been recently [17] interpreted as the radiative spacetimes with a non-expanding shear-free and twist-free null congruence. When $N \neq 2$, neither the Newman-Penrose formalism nor Petrov classification is available for the characterization of the new generalized wave solutions. However, we can formally distinguish several classes of solutions according to the values of $\Lambda$ and the combination $\Lambda \alpha^2 + \beta_a \beta^a$. Following [17], we have the generalized Kundt waves for $\Lambda > 0$, but for $\Lambda < 0$ we find the generalized $pp$-waves, the Kundt waves, or the “Lobatchevski planes waves” when $\Lambda \alpha^2 + \beta_a \beta^a$ is negative, positive, or zero, respectively.

In this paper we have derived the $N$-dimensional generalizations of the pure gravitational waves of all classes mentioned above, and an extension of the Kundt waves for the nontrivial electromagnetic radiation source.

The case of vanishing $\Lambda = 0$ is straightforwardly obtained in our general framework. We have then $p = 1$ and $q = \alpha + \beta_a z^a$. Both, the Maxwell equation (20) and the Einstein equation (23), reduce to the Laplace and the Poisson equation, respectively, in an $N$-dimensional Euclidean space. When $\beta_a \neq 0$, the particular solution reads $H = (16 \pi G \varepsilon_0 / c^2)(\phi^a \phi_a / \beta_b \beta^b) q (\ln q - 1)$ and $\varphi(\sigma, z^a) = \phi_a(\sigma) z^a$. A different particular $pp$-wave solution is recovered for $\beta_a = 0$: $H(\sigma, z^a) = h_{ab}(\sigma) z^a z^b$ with $h^a_a = (8 \pi G \varepsilon_0 / c^2) \phi^a \phi_a$.

Besides the possible applications of the higher-dimensional solutions in string-theoretic and brane models [5], the new lower-dimensional wave solutions appear to be helpful in discussing the problem of the gravitational collapse.

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**VIII. APPENDIX**

Here we collect some details about the connection and the curvature. The connection 1-form is determined by the first structure equation $d\vartheta^\alpha + \Gamma^\alpha_{\beta \gamma} \wedge \vartheta^\beta = 0$. The explicit solution
\[ \Gamma_{\alpha}^\beta = \frac{1}{2} \left[ e_{\alpha} \right] d\vartheta^\beta - e^\beta \left[ d\vartheta \right] e_{\alpha} - \left( e_{\alpha} \right] e^\beta \left] d\vartheta \right) \vartheta^\gamma. \quad (58) \]

Substituting (3), we find the different components of the connection as follows:

\[ \Gamma_{a}^b = \vartheta_a \partial_b p - \vartheta^b \partial_a p, \quad (59) \]

\[ \Gamma_{a}^B = -q \partial_a (p/q) \vartheta^B - \frac{q^2}{p} (\partial_a s) \vartheta^0 k^B, \quad (60) \]

\[ \Gamma_{A}^B = \left[ (q/p) \vartheta (p/q) - \vartheta^0 \left( \partial_a s - 2q^{-1} \partial_a q \right) \right] \eta^B_A. \quad (61) \]

Here \( k^A = \delta^A_1 \) is the null vector introduced above. The indices \( A, B, \ldots \) and \( a, b, \ldots \) are raised and lowered by means of the metrics (2). The 2-dimensional Levi-Civita tensor \( \eta_{AB} = -\eta_{BA} \) has the components: \( \eta^0_0 = -1, \eta^1_1 = 1 \) and \( \eta^0_1 = \eta^1_0 = 0. \)

The curvature 2-form is constructed from the connection according to \( R_{\alpha}^\beta = d\Gamma_{\alpha}^\beta + \Gamma_{\gamma}^\beta \wedge \Gamma_{\alpha}^\gamma. \) Substituting here (59)-(61), we find explicitly:

\[ R_{a}^b = p'(-2p/\xi + p') \partial_a \wedge \vartheta^b = -\Lambda \partial_a \wedge \vartheta^b, \quad (62) \]

\[ R_{a}^B = -L_{a} \wedge \vartheta^B + k^B \gamma_a \wedge \vartheta^0, \quad (63) \]

\[ R_{A}^B = -K \partial_A \wedge \vartheta^B. \quad (64) \]

Here \( \xi = \sqrt{z^a z^a} = \sqrt{\delta_{ab} z^a z^b} \) and the prime \( ' \) denotes the derivative with respect to this variable (note that \( p = p(\xi) = 1 + \Lambda \xi^2/4 \)). Furthermore,

\[ L_{a} = D[q \partial_a (p/q)] - (q^2/p) \partial_a (p/q) d(p/q), \quad (65) \]

\[ K = -(p/q)^2 \left[ \partial^2_{\rho \rho} s + (e_{\alpha}) d(q/p))^2 \right], \quad (66) \]

whereas the components of the 1-form \( \gamma_a \) are given in (8). The covariant derivative \( D \) is defined here by means of the connection (59) on an \( N \)-dimensional space of constant curvature. For example, for a covector \( v_a \), we have \( Dv_a = dv_a - \Gamma_{a}^b v_b \). Finally, taking into account (5) and (6), we can straightforwardly verify that \( L_{a} = \Lambda \partial_a \) and \( K = \Lambda \) for arbitrary \( \alpha, \beta_a. \)
In compact form, the components of the curvature 2-form (62)-(64) are then given by the equation (7). The Riemann tensor represents the components of the curvature 2-form expanded with respect to the coframe: $R_{\alpha \beta} = \frac{1}{2} R_{\mu \nu \alpha \beta} \vartheta^\mu \wedge \vartheta^\nu$. Correspondingly, the Ricci tensor $Ric_{\alpha \beta} = R_{\mu \alpha \beta}^\mu$ represents the components of the 1-form $e_\beta \lrcorner R_{\alpha \beta} = Ric_{\alpha \beta} \vartheta^\beta$. The curvature scalar is, as usual, $R = e_\alpha \lrcorner e_\beta \lrcorner R^{\alpha \beta} = g^{\alpha \beta} Ric_{\alpha \beta}$. 
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