Warped Geometry in Higher Dimensions with an Orbifold Extra Dimension

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Abstract

We solve the Einstein equations in higher dimensions with warped geometry where an extra dimension is assumed to have orbifold symmetry $S^1/Z_2$. The setup considered here is an extension of the five-dimensional Randall-Sundrum model to $5+D$ dimensions, and hidden and observable branes are fixed on the orbifold. It is assumed that the brane tension (self-energy) of each brane with $(4+D)$-dimensional spacetime is anisotropic and that the warped metric function of the four dimensions is generally different from that of the extra $D$ dimensions. We point out that the forms of the warped metric functions and the relations between the tensions of two branes depend on the integration constant appearing in the Einstein equations as well as on the sign of the bulk cosmological constant.

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1 Introduction

Motivated by the Hořava-Witten model in 11-dimensional theory ($M$ theory) compactified on the orbifold $S^1/Z_2$, many models have been proposed using the notion that there are two branes that represent the boundaries of higher dimensional spacetime [1]. Consequently, there has been growing interest among particle physicists and cosmologists in models with extra dimensions. Recent developments are based on the idea that ordinary matter fields could be confined to a three-brane world embedded in the higher dimensional space.

Adopting this idea further, there are several proposals that try to relate the Planck scale of the observable world to the higher dimensional Planck scale. In the model proposed by Arkani-Hamed Dimopoulos and Dvali [2], the fundamental scale $M_*$ can be related to the usual four-dimensional Planck scale $M_p$ via a volume factor, $M_p^2 = M_{*}^{n+2}R^n$, where $R^n$ is the volume of the compact space and $n$ is the number of extra dimensions. If $R$ is sufficiently large, $M_*$ can be as low as the 1 TeV scale; thus the model gives a possible solution to the gauge hierarchy problem. Furthermore, Randall and Sundrum [4, 5] have presented a static solution to the classical five-dimensional Einstein equations with negative bulk cosmological constant (AdS space). The warped metric (factor) in the model is an exponential scaling of the metric along the fifth dimension compactified on the $S^1/Z_2$ orbifold. This solution appeals to the possibility of an extra dimension limited by two three-branes with opposite tensions, and provides an alternative explanation for the hierarchy problem as due to the warped factor if the observable brane has negative brane tension. Both approaches assume that the standard model particles are confined to a three-brane embedded in higher dimensional spacetime and that gravity exists in the bulk.

An important question concerning these kinds of model is whether or not standard four-dimensional gravity is reproduced on the brane. In the Randall-Sundrum model, even if the fifth dimension is uncompactified, the usual gravity is shown to be recovered because of the existence of a massless graviton trapped in the brane [4]. Further, another problem is that the stabilization mechanism for the size of the extra dimensions is yet unknown. Introducing a bulk scalar field that interacts with the branes, several mechanisms have been proposed [10]. On the other hand, the existence of the extra dimensions allows much phenomenology, including the production of Kaluza-Klein excitations of gravitons at future colliders or their detection in high precision measurements at low energies.

The Randall-Sundrum static solution has been extended to time dependent solutions and their cosmological properties have been extensively studied [4]. In the framework of brane world cosmology, the serious problem emphasized recently is an unusual form of the Friedmann equations in the case of one extra dimension, which leads to a particular behavior of the Hubble parameter on the brane. In particular, the Hubble parameter $H$ is proportional to the energy density on the brane instead of the familiar dependence $H \sim \sqrt{\rho}$.

The purpose of this paper is to extend the five-dimensional Randall-Sundrum model to higher dimensional cases. Since the original Randall-Sundrum model was inspired by superstring theory or $M$ theory, the version in higher dimensions should be naturally motivated. In this case, we are interested in whether the tension of the higher dimensional brane is anisotropic or not, and in the relation between the brane tension and the bulk cosmological constant. We study the metric of the $(5 + D)$-dimensional Randall-Sundrum model, where the $5 + D$ dimensions are composed of the $(4 + D)$-dimensional spacetime and a dimension
compactified on the orbifold $S^1/Z_2$, $|y| \leq L$. The $(4 + D)$-dimensional world resides in the $(3 + D)$-brane, and two branes are fixed at $y = 0$ and $y = L$. The observable brane we live in is assumed to be the brane at $y = L$ and the hidden brane is at $y = 0$. The ways of taking the metric ansatz are various. In this paper, we consider the case that the four-dimensional warped metric function $a(y)$ is generally different from the extra $D$-dimensional warped metric function $c(y)$. Recently, the scenario in which $a(y)$ is equal to $c(y)$ has been discussed [11]. Furthermore, we assume that the brane tension of the $4 + D$ dimensions is anisotropic, namely, the brane tension of the four-dimensional spacetime is generally different from that of the extra $D$-dimensional space. Based on the above assumptions, we solve the $(5 + D)$-dimensional Einstein equation with the bulk cosmological constant and study the forms of $a(y)$ and $c(y)$ explicitly. Moreover, we derive the relations between the brane tension of the four dimensions and that of the extra $D$ dimensions and represent the behavior of each brane tension for the distance between two branes.

This paper is organized as follows. In section 2, the setup considered here is described and generalized Einstein equations with time dependence are explicitly expressed. In the simplest case of an isolated two-brane system, we give the higher dimensional Friedmann-type equation on the brane. In section 3, we solve the static $(5 + D)$-dimensional Randall-Sundrum model with the bulk cosmological constant $\Lambda$. For each case of $\Lambda < 0$, $\Lambda > 0$, and $\Lambda = 0$, the $(4 + D)$-dimensional metric functions can be obtained. We show that the forms of the warped metric functions and the relation between the brane tensions on the orbifold depend on the integration constant appearing in the Einstein equations as well as on the sign of the bulk cosmological constant. A summary and discussion are given in the final section. In an appendix, we review the Kasner solution of the $(4 + D)$-dimensional anisotropic cosmological model.

2 The Setup

We consider the higher dimensional spacetime with an orbifold extra dimension. This setup is an extension of the Randall-Sundrum model with five-dimensional warped metric. The two $3 + D$ branes with the $(4 + D)$-dimensional spacetime embedded in the $(5 + D)$-dimensional spacetime are located at $y = 0$ and at $y = L$, where the $y$ direction is compactified on the orbifold $S^1/Z_2$. This $(5 + D)$-dimensional model is described by the action

$$ S = \int_{-L}^{L} dy \int d^{4+D}x \sqrt{|g|} \left( \frac{1}{2\kappa^2} R - \Lambda \right) $$

in bulk, where $1/\kappa^2$ is the fundamental gravitational scale and $\Lambda$ is the bulk cosmological constant.

To solve the Einstein equations, the metric ansatz can be written in the following form

$$ ds^2 = n^2(t, y) dt^2 - a^2(t, y) dx^2 - b^2(t, y) dy^2 - c^2(t, y) \left( dz_1^2 + \cdots + dz_D^2 \right) $$

$$ \equiv g_{AB} dx^A dx^B, $$(2)

where $A, B = 0, \ldots, 4 + D$. We shall use the notation $\{x^\mu\}$ with $\mu = 0, \ldots, 3$ for the coordinates on the four-dimensional spacetime $\{t, \vec{x}\}$, $x^4 = y$ for a coordinate on the orbifold.
compactification, and \( \{x^a\} \) with \( a = 5, \ldots, D+4 \) for coordinates on the extra D-dimensional space \( \{z_1, \ldots, z_D\} \). It is assumed that the distribution of the brane tension and the matter on the brane with \((4+D)\)-dimensional spacetime is anisotropic. The Einstein tensor corresponds to

\[
G_{AB} = \mathcal{R}_{AB} - \frac{1}{2} g_{AB} \mathcal{R},
\]

where \( \mathcal{R}_{AB} \) and \( \mathcal{R} \) represent the Ricci tensor and the scalar curvature, respectively. The Einstein equation is given by \( G_{AB} = \kappa^2 T_{AB} \), where \( T_{AB} \) is the energy-momentum tensor. It is assumed that there are contributions to \( T_{AB} \) from the bulk and the branes as

\[
T_{AB} = T_{AB}^{\text{bulk}} + T_{AB}^{\text{brane}}.
\]

From the bulk we have

\[
T_{AB}^{\text{bulk}} = g_{AB} \Lambda,
\]

where \( \Lambda \) is the cosmological constant in the bulk, and from the two branes

\[
T_{B}^{\text{brane}} = \frac{\delta(y)}{b} \text{diag}(V_1 + \rho_1, V_1 - p_1, V_1 - p_1, 0, V_1^* - p_1^*, \ldots, V_1^* - p_1^*)
\]

\[
+ \frac{\delta(y - L)}{b} \text{diag}(V_2 + \rho_2, V_2 - p_2, V_2 - p_2, V_2 - p_2, 0, V_2^* - p_2^*, \ldots, V_2^* - p_2^*)
\]

\[
(6)
\]

Here the indices 1 and 2 denote the brane at \( y = 0 \) and at \( y = L \), respectively. \( V, \rho, \) and \( p \) represent the brane tension, the density, and the pressure of the matter on each brane, respectively. The superscript \(^*\) corresponds to quantities in the extra \( D \)-dimensional space.

Using the metric ansatz Eq.(2), we can write the Einstein equation for each component. The \((0,0)\) component for the \( t \) direction is given by

\[
\frac{1}{n^2} \left[ 3 \left( \frac{\dot{a}}{a} \right)^2 + \frac{1}{2} D(D - 1) \left( \frac{\dot{c}}{c} \right)^2 + \frac{\dot{b}}{b} + D \frac{\dot{b}}{c} + 3 D \frac{\dot{a}}{a} \right]
\]

\[
- \frac{1}{b^2} \left[ 3 \left( \frac{a'}{a} \right)^2 + \frac{1}{2} D(D - 1) \left( \frac{c'}{c} \right)^2 + \frac{a''}{a} - 3 \frac{a' b'}{a b} + D \frac{c''}{c} + 3 D \frac{a' c'}{a c} - D \frac{b' c'}{b c} \right]
\]

\[
= \kappa^2 \Lambda + \kappa^2 \frac{V_1 + \rho_1}{b} \delta(y) + \kappa^2 \frac{V_2 + \rho_2}{b} \delta(y - L).
\]

The \((i,i)\) component for the three-dimensional space \((i = 1, 2, 3)\) is

\[
\frac{1}{n^2} \left[ -2 \frac{\dot{a}}{a} - \frac{\dot{b}}{b} - D \frac{\dot{c}}{c} - \left( \frac{\dot{a}}{a} \right)^2 - \frac{1}{2} D(D - 1) \left( \frac{\dot{c}}{c} \right)^2
\]

\[
- 2 \frac{\dot{a} b}{a b} - 2 D \frac{\dot{a} c}{a c} - D \frac{\dot{b} c}{b c} + 2 \frac{\dot{a} n}{a n} + \frac{\dot{b} n}{b n} + D \frac{\dot{c} n}{c n}\right]
\]

\[
(7)
\]
\[ + \frac{1}{b^2} \left[ 2 \frac{a''}{a} + \frac{n''}{n} + D \frac{c''}{c} + \left( \frac{a'}{a} \right)^2 + \frac{1}{2} D(D-1) \left( \frac{c'}{c} \right)^2 \right. \]
\[ \left. - 2 \frac{a'b'}{ab} + 2 \frac{a'n'}{an} - \frac{b'n'}{bn} + 2D \frac{a'c'}{ac} - D \frac{b'c'}{bc} + D \frac{c'c'}{cn} \right] \]
\[ = -\kappa^2 \Lambda - \kappa^2 V_1 - \frac{p_1}{b} \delta(y) - \kappa^2 V_2 - \frac{p_2}{b} \delta(y-L). \tag{8} \]

For the (4, 4) component for the y direction compactified on $S^1/Z_2$ we get
\[ + \frac{1}{n^2} \left[ -3 \frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - (D-1) \frac{\ddot{c}}{c} - 3 \left( \frac{\dot{a}}{a} \right)^2 - \frac{1}{2} D(D-1)(D-2) \left( \frac{\dot{c}}{c} \right)^2 \right. \]
\[ \left. - 3 \frac{\dot{a}b}{ab} - (D-1) \frac{\dot{b}c}{bc} - 3(D-1) \frac{\dot{a}c}{ac} + \frac{\dot{a}n}{an} + \frac{b\dot{n}}{bn} + (D-1) \frac{\dot{c}n}{cn} \right] \]
\[ + \frac{1}{b^2} \left[ 3 \frac{a''}{a} + \frac{n''}{n} + (D-1) \frac{c''}{c} + 3 \left( \frac{a'}{a} \right)^2 + \frac{1}{2} D(D-1)(D-2) \left( \frac{c'}{c} \right)^2 \right. \]
\[ \left. - 3 \frac{a'c'}{ac} + 3 \frac{a'n'}{an} - \frac{b'n'}{bn} + 3(D-1) \frac{a'c'}{ac} - (D-1) \frac{b'c'}{bc} + (D-1) \frac{n'c'}{nc} \right] \]
\[ = -\kappa^2 \Lambda - \kappa^2 V_1^* - \frac{p_1^*}{b} \delta(y) - \kappa^2 V_2^* - \frac{p_2^*}{b} \delta(y-L) \tag{9} \]

The $(a, a)$ component for the $D$-dimensional space ($a = 5, \ldots, D + 4$) takes the form
\[ + \frac{1}{n^2} \left[ -3 \frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - (D-1) \frac{\ddot{c}}{c} - 3 \left( \frac{\dot{a}}{a} \right)^2 - \frac{1}{2} D(D-1)(D-2) \left( \frac{\dot{c}}{c} \right)^2 \right. \]
\[ \left. - 3 \frac{\dot{a}b}{ab} - (D-1) \frac{\dot{b}c}{bc} - 3(D-1) \frac{\dot{a}c}{ac} + \frac{\dot{a}n}{an} + \frac{b\dot{n}}{bn} + (D-1) \frac{\dot{c}n}{cn} \right] \]
\[ + \frac{1}{b^2} \left[ 3 \frac{a''}{a} + \frac{n''}{n} + (D-1) \frac{c''}{c} + 3 \left( \frac{a'}{a} \right)^2 + \frac{1}{2} D(D-1)(D-2) \left( \frac{c'}{c} \right)^2 \right. \]
\[ \left. - 3 \frac{a'c'}{ac} + 3 \frac{a'n'}{an} - \frac{b'n'}{bn} + 3(D-1) \frac{a'c'}{ac} - (D-1) \frac{b'c'}{bc} + (D-1) \frac{n'c'}{nc} \right] \]
\[ = -\kappa^2 \Lambda - \kappa^2 V_1^* - \frac{p_1^*}{b} \delta(y) - \kappa^2 V_2^* - \frac{p_2^*}{b} \delta(y-L) \tag{10} \]

and the non diagonal $(0, 4)$ component for the $t$ and $y$ directions is written as
\[ - 3 \frac{\dot{a}}{a} + 3 \frac{n\dot{a}}{na} + 3 \frac{\dot{b}}{b} + 3 \frac{n'\dot{a}}{na} + D \frac{n'\dot{c}}{nc} - \frac{\dot{c}}{c} + D \frac{b'c'}{bc} = 0. \tag{11} \]

Here the primes (overdots) denote the derivatives with respect to $y$ ($t$). Although the functions $a, n,$ and $c$ are continuous at the brane, their derivatives with respect to $y$ are discontinuous because of the presence of the brane. By matching the coefficients of the delta functions, the $(0, 0)$, $(i, i)$, and $(a, a)$ components of the Einstein equations are subject to jump conditions on the first derivatives of the functions. In order to derive jump conditions on $a, n,$ and $c$, we define the function $[f]_x$
\[ [f]_x = f(x+0) - f(x-0) \tag{12} \]
for an arbitrary function $f$. From Eqs. (7), (8), and (10), the integration over $y \in (-0, +0)$ yields
\[ - 3 \frac{[\dot{a}]}{a_0} - D \frac{[\dot{c}]}{c_0} = \kappa^2 b_0 (V_1 + p_1), \]

5
on the brane at \( y \)

Taking the average between \( y \)

conditions on \( y \)

evolution on the brane. This situation corresponds to the limit of \( L \)

pletely isolated from each other and give the equations of the higher dimensional cosmological expansion on the brane \([7]\). Below, we consider the simplest case where two branes are com-

Here we use the notation \([7]\)

conservation of the four-dimensional standard cosmology. We now define the average function \([7]\)

obtain the Friedmann-type equation on the brane at \( y \)

The jump conditions on \( n, a, \) and \( c \) are rewritten as

\[
\begin{align*}
\frac{[a']_0}{a_0} &= -\frac{\kappa^2 b_0}{D + 3} \left( V_1 + \rho_1 - D[V_1 - V_1^* - p_1 + p_1^*] \right), \\
\frac{[n']_0}{n_0} &= -\frac{\kappa^2 b_0}{D + 3} \left( V_1 - 2\rho_1 - 3p_1 - D[V_1 - V_1^* + \rho_1 + p_1^*] \right), \\
\frac{[c']_0}{c_0} &= -\frac{\kappa^2 b_0}{D + 3} \left( 4V_1 - 3V_1^* + \rho_1 - 3p_1 + 3p_1^* \right).
\end{align*}
\]  

(14)

It is noted that the above jump conditions at \( y = 0 \) depend on the tension, the density, and the pressure on the brane as well as on the number of extra dimensions. Similarly, the jump conditions at \( y = L \) can be derived. As mentioned later, these jump conditions are used to derive the relations between the brane tensions.

In the Randall-Sundrum model, it is important to study the equation for the cosmological expansion on the brane \([7]\). Below, we consider the simplest case where two branes are completely isolated from each other and give the equations of the higher dimensional cosmological evolution on the brane. This situation corresponds to the limit of \( L \rightarrow \infty \). Using the jump conditions in Eq.\((14)\), the difference between \( y = +0 \) and \( y = -0 \) in Eq.\((11)\) leads to energy conservation

\[
\begin{align*}
\hat{n} + \frac{9}{D + 3} \left( \rho_1 + p_1 + \frac{D}{9} [4V_1 - 3V_1^* + 4\rho_1 + 3p_1^*] \right) \frac{\dot{a}_0}{a_0} \\
-D \left\{ V_1 - 2\rho_1 - 3p_1 - D[V_1 - V_1^* + \rho_1 + p_1^*] \right\} \frac{\dot{c}_0}{c_0} = 0
\end{align*}
\]  

(15)

on the brane at \( y = 0 \). In the case \( D = 0 \), the above equation is reduced to the energy conservation of the four-dimensional standard cosmology. We now define the average function \([7]\)

\[
\{ f \} x = \frac{f(x + 0) + f(x - 0)}{2}.
\]  

(16)

Taking the average between \( y = +0 \) and \( y = -0 \) with respect to the \((4, 4)\) component, we can obtain the Friedmann-type equation on the brane at \( y = 0 \):

\[
\begin{align*}
\frac{1}{n_0^2} \left[ \frac{\ddot{a}_0}{a_0} + \left( \frac{\dot{a}_0}{a_0} \right)^2 + \frac{D}{3} \frac{\ddot{c}_0}{c_0} + \frac{1}{6} D(D - 1) \left( \frac{\dot{c}_0}{c_0} \right)^2 - \frac{\dot{n}_0 \dot{a}_0}{n_0 a_0} - \frac{D}{3} \frac{\dot{n}_0 \dot{c}_0}{n_0 c_0} + \frac{\dot{a}_0 \dot{c}_0}{a_0 c_0} \right] \\
= \frac{1}{3} \kappa^2 \Lambda \\
+ \frac{\kappa^4}{4(D + 3)^2} \left[ V_1 + \rho_1 - D(V_1 - V_1^* - p_1 + p_1^*) \right]
\end{align*}
\]  

(13)
Imposing the orbifold symmetry \( y \sim -y \), we have \( \{ f' \} = 0 \). Then we can drop all terms involving the average in Eq. (17). We have also fixed the time in such a way that \( n_0 = 1 \). This corresponds to the usual choice of time in conventional cosmology. We introduce the Hubble parameters \( H_a \equiv \dot{a}/a \) and \( H_c \equiv \dot{c}/c \) for the two scale factors on the brane at \( y = 0 \). Here it is assumed that after stabilizing the radion \( b_0 \) the matter on the brane is isotropic and that radiation dominates. This leads to \( p_1 = p_1^* \) and \( \rho_1 = (D + 3)p_1 \). Then the cosmological evolution equation on the brane at \( y = 0 \) becomes

\[
\dot{H}_a + 2H_a^2 + \frac{D}{3} H_c + \frac{1}{6} D(D + 1)H_c^2 + DH_a H_c = \frac{1}{3} \kappa^2 \Lambda + \frac{\kappa^4}{4(D + 3)^2} \left[ V_1 + \rho_1 - D(V_1 - V_1^*) \right] \times \left[ 2V_1 - \rho_1 \frac{6 + D}{3 + D} + D \left( 2V_1 - V_1^* - \rho_1 \frac{1}{3 + D} \right) \right] \\
+ \frac{D \kappa^4}{24(D + 3)^2} \left[ 4V_1 - 3V_1^* + \rho_1 \right] \times \left[ -2V_1 + 3V_1^* - \rho_1 \frac{21 + D}{3 + D} + D \left( 2V_1 - V_1^* - \rho_1 \frac{5 + D}{3 + D} \right) \right].
\]

The energy conservation equation and the cosmological evolution equation on the brane at \( y = L \) can be obtained by the same procedures as mentioned above.

Since the cosmology equation obtained here corresponds to a completely isolated brane system, it implies that the matter on one brane has nothing to do with the matter on another brane. However, in the case of finite \( L \), the two branes are closely related, so that the matter on one brane is constrained by the matter on the other brane. To study the cosmology equations constrained by two branes [8], the relation between the functions \( n, a, b, \) and \( c \) must obtained by integrating out Eq. (14). As for this point, we are going to provide the analysis in detail of the cosmological evolution in the setup presented in this paper [9].

### 3 Static solutions

We can obtain the static Randall-Sundrum-type solution by setting the density and the pressure of matter to zero. Note that the functions \( n, a, \) and \( c \) have time independence and
preserve Poincaré invariance in the (1 + 3)-dimensional metric
\[ n(y) = a(y), \quad b = 1, \]  
(19)
where \( b \) is normalized to be unity since it is assumed that the size in the \( y \) direction compactified on the orbifold is stabilized via some mechanism. We consider that the four-dimensional warped metric function \( a(y) \) is generally different from the extra \( D \)-dimensional \( c(y) \). Following from Eq. (7) to Eq. (10), the Einstein equations in the bulk are given by
\[
12 \left( \frac{a'}{a} \right)^2 + D(D - 1) \left( \frac{c'}{c} \right)^2 + 8D \frac{a'}{a} \frac{c'}{c} = -2\kappa^2 \Lambda, 
\]  
(20)
\[
6 \left( \frac{a'}{a} \right)^2 + D(D - 1) \left( \frac{c'}{c} \right)^2 + 6 \frac{a''}{a} + 2D \frac{c''}{c} + 6D \frac{a'}{a} \frac{c'}{c} = -2\kappa^2 \Lambda, 
\]  
(21)
\[
12 \left( \frac{a'}{a} \right)^2 + (D - 1)(D - 2) \left( \frac{c'}{c} \right)^2 + 8 \frac{a''}{a} + 2(D - 1) \frac{c''}{c} + 8(D - 1) \frac{a'}{a} \frac{c'}{c} = -2\kappa^2 \Lambda, 
\]  
(22)
where we used the fact that the \((0,0)\) component is equal to the \((a, a)\) component.

Here we can derive the solution of the five-dimensional classical Einstein equation to be the Randall-Sundrum model. Setting \( D = 0 \) and neglecting Eq. (22) coming from the \((a, a)\) component for the extra \( D \) dimensions, we have
\[
\left( \frac{a'}{a} \right)^2 = \frac{a''}{a} = -\frac{\kappa^2 \Lambda}{6}. 
\]  
(23)
For \( \Lambda < 0 \), the \( S^1/Z_2 \) orbifold symmetric solution is of the form
\[ a(y) = e^{\pm m_0 |y|}, \]  
(24)
where
\[ m_0 = \sqrt{-\frac{\kappa^2 \Lambda}{6}}. \]  
(25)
From Eq. (24), the jump conditions of \( a(y) \) at \( y = 0 \) and \( y = L \) lead to
\[ V_1 = -V_2 = \mp \sqrt{-\frac{\Lambda}{6\kappa^2}}, \]  
(26)
where upper and lower signs correspond to the signs in Eq. (24), respectively. Thus the brane tensions \( V_1, V_2 \) at \( y = 0 \) and \( y = L \) have opposite sign from each other. When \( V_2 \) is negative, the warped metric becomes
\[ ds^2 = e^{-2m_0|y|} g_{\mu\nu} dx^\mu dx^\nu - dy^2, \]  
(27)
where \( g_{\mu\nu} = \text{diag}(+, -, -, -) \). By using this warped metric, Randall and Sundrum proposed an alternative solution to the hierarchy problem. This solution appeals to the possibility of
an extra dimension limited by two branes with positive and negative tensions. Further, the resolution of the hierarchy problem is possible provided that the observable brane at \( y = L \) is the one with the negative tension. This model insists that the hierarchy has its origin in the geometry of the extra dimension.

We are interested in the solutions of the Randall-Sundrum model embedded in 5 + \( D \) dimensions with an orbifold compactification. The feature of this setup is that the warped metric function \( a(y) \) of the four-dimensional spacetime is generally different from the one \( c(y) \) of the extra \( D \)-dimensional space. After some algebra, we can rewrite the appropriate linear combination of Eqs.(20), (21), and (22) as

\[
12 \left( \frac{a'}{a} \right)^2 + D(D-1) \left( \frac{c'}{c} \right)^2 + 8D \frac{a'c'}{ac} = -2\kappa^2 \Lambda, \tag{28}
\]

\[
\frac{a''}{a} + 3 \left( \frac{a'}{a} \right)^2 + D \frac{a'c'}{ac} = -\frac{2\kappa^2}{D+3} \Lambda, \tag{29}
\]

\[
\frac{c''}{c} + (D-1) \left( \frac{c'}{c} \right)^2 + 4 \frac{a'c'}{ac} = -\frac{2\kappa^2}{D+3} \Lambda. \tag{30}
\]

To study the behavior of \( a(y) \) and \( c(y) \), let us consider the three cases of \( \Lambda < 0 \), \( \Lambda > 0 \), and \( \Lambda = 0 \) separately.

### 3.1 The solutions for \( \Lambda < 0 \)

In this case, in order to obtain the solution of the Einstein equations, we perform the changes of variables that allow the exact solution Eqs.(28)-(30). We define \( A(y) \) and \( C(y) \) by

\[
a(y) = e^{A(y)}, \quad c(y) = e^{C(y)}. \tag{31}
\]

Further, defining the parameter \( \omega \equiv 2\kappa^2 \), from Eqs.(28), (29) and (30), we obtain

\[
12(A')^2 + D(D-1)(C')^2 + 8DA'C' = -\omega \Lambda, \tag{32}
\]

\[
A'' + 4(A')^2 + DA'C' = -\frac{\omega \Lambda}{D+3}, \tag{33}
\]

\[
C'' + D(C')^2 + 4A'C' = -\frac{\omega \Lambda}{D+3}. \tag{34}
\]

We introduce the new variable \( Y \)

\[
dY = \mp e^{-4A-DC} dy. \tag{33}
\]

This replacement of the variable is similar to the procedure of seeking the Kasner solution with time dependence in the higher dimensional cosmology in Ref.[3]. Thus we get

\[
12 \left( \frac{dA}{dY} \right)^2 + D(D-1) \left( \frac{dC}{dY} \right)^2 + 8D \frac{dA}{dY} \frac{dC}{dY} = -\omega \Lambda e^{8A+2DC}, \tag{34}
\]

\[
\frac{d^2A}{dY^2} = \frac{d^2C}{dY^2} = -\frac{\omega \Lambda}{D+3} e^{8A+2DC}. \tag{35}
\]
Note that the above equations are unchanged in either case when the lower or upper sign included in Eq.\((33)\) is taken. Hence, we can immediately write the integral of Eq.\((35)\):
\[
A - C = P_1 Y + P_2, \tag{36}
\]
where \(P_1\) and \(P_2\) are the integration constants. To solve the differential equation of Eq.\((34)\), we define a new variable \(Z\)
\[
Z = 8A + 2DC, \tag{37}
\]
Equation \((34)\) is translated as
\[
\frac{d^2 Z}{dY^2} = - \frac{2(D + 4)\omega \Lambda}{D + 3} e^Z. \tag{38}
\]
Integrating this equation, we obtain
\[
\left(\frac{dZ}{dY}\right)^2 = - \frac{4(D + 4)\omega \Lambda}{D + 3} e^Z + P_3, \tag{39}
\]
where \(P_3\) is the integration constant. However, \(P_3\) is a function of \(P_1\). This is because substitution of Eq.\((37)\) into Eq.\((39)\) leads to
\[
P_3 = \frac{16D}{D + 3} \left(\frac{dA}{dY} - \frac{dC}{dY}\right)^2 = \frac{16D}{D + 3} P_1^2. \tag{40}
\]
Therefore, we have
\[
\left(\frac{dZ}{dY}\right)^2 = - \frac{4(D + 4)\omega \Lambda}{D + 3} e^Z + \frac{16D}{D + 3} P_1^2. \tag{41}
\]
Since \(P_1\) and \(P_2\) are determined by the initial condition, the values are expected to be determined via some dynamics of the underlying physics. Below, we point out that different types of solution of Eq.\((41)\) are obtained depending on whether \(P_1\) is nonvanishing or vanishing.

First, we consider the case of \(P_1 \neq 0\) for the negative bulk cosmological constant. Equation \((41)\) can be simply solved as
\[
e^Z = \frac{4DP_1^2}{(D + 4)\omega |\Lambda| \sinh^2 \left(2\sqrt{\frac{D}{D+3}} P_1 Y\right)}. \tag{42}
\]
Using Eqs.\((36)\) and \((37)\), we get
\[
a = e^A = \left[ \frac{4DP_1^2}{(D + 4)\omega |\Lambda|} \frac{e^{2D(P_1 Y + P_2)}}{\sinh^2 \left(2\sqrt{\frac{D}{D+3}} P_1 Y\right)} \right]^{\frac{1}{2(D+4)}} ,
\]
\[
c = e^C = \left[ \frac{4DP_1^2}{(D + 4)\omega |\Lambda|} \frac{e^{-8(P_1 Y + P_2)}}{\sinh^2 \left(2\sqrt{\frac{D}{D+3}} P_1 Y\right)} \right]^{\frac{1}{2(D+4)}} . \tag{43}
\]
To write the above equations in terms of \(y\), we need to change the variable \(Y\) into \(y\). Equation (33) becomes

\[
dY = \mp \sqrt{\frac{D + 4}{4DP_2^2 \omega |\Lambda|}} \sinh \left( 2\sqrt{\frac{D}{D + 3}} P_1 Y \right) dy
\]

which leads to the relation between \(Y\) and \(y\):

\[
\exp \left[ 2\sqrt{\frac{D}{D + 3}} P_1 Y \right] = \coth \left( \frac{1}{2} \sqrt{\frac{D + 4}{D + 3}} \omega |\Lambda| (|y| + y_0) \right) ,
\]

where \(y_0\) is an integration constant and the positive sign in Eq.(44) is taken. After substitution of Eq.(45) into Eq.(43), the functions \(a\) and \(c\) are described in terms of \(y\):

\[
a(y) = \left( \frac{4DP_2^2}{(D + 4)\omega |\Lambda|} \right)^{\frac{1}{D+4}} \times \left[ \coth \sqrt{\frac{D+4}{D+3}} \left( \frac{1}{2} \sqrt{\frac{D + 4}{D + 3}} \omega |\Lambda| (|y| + y_0) \right) \sinh \left( \sqrt{\frac{D + 4}{D + 3}} \omega |\Lambda| (|y| + y_0) \right) \right]^{\frac{1}{D+4}},
\]

\[
c(y) = \left( \frac{4DP_2^2}{(D + 4)\omega |\Lambda|} \right)^{\frac{1}{D+4}} \times \left[ \tanh^2 \sqrt{\frac{D+4}{D+3}} \left( \frac{1}{2} \sqrt{\frac{D + 4}{D + 3}} \omega |\Lambda| (|y| + y_0) \right) \sinh \left( \sqrt{\frac{D + 4}{D + 3}} \omega |\Lambda| (|y| + y_0) \right) \right]^{\frac{1}{D+4}},
\]

where \(a\) and \(c\) respect the \(Z_2\) symmetry \(y \sim -y\). The constant \(P_2\) in Eq.(36) does not appear in the expression for the metric functions \(a(y)\) and \(c(y)\), namely, \(P_2\) could be set to zero since it can be absorbed into a redefinition of the extra \(D\)-dimensional coordinates. Moreover, the coefficient without \(P_2\) in \(a(y)\) and \(c(y)\) is considered to be physical irrelevant since we are interested only in the behavior with respect to \(y\). From Eq.(14), the jump conditions at \(y = 0\) lead to

\[
\frac{2\sqrt{|\lambda|}}{4 + D} \frac{2 \cosh \sqrt{|\lambda|} y_0 - \sqrt{D(D + 3)}}{2 \sinh \sqrt{|\lambda|} y_0} = -\kappa^2 \frac{4 + D}{D + 3} (V_1 - D[V_1 - V_1^*]),
\]

\[
\frac{2\sqrt{|\lambda|}}{4 + D} \frac{\cosh \sqrt{|\lambda|} y_0 + 2 \sqrt{\frac{D+3}{D}}}{\sinh \sqrt{|\lambda|} y_0} = -\kappa^2 \frac{4V_1 - 3V_1^*}{D + 3},
\]

where we define

\[
\lambda = \frac{D + 4}{D + 3} \omega \Lambda .
\]
Furthermore, the jump conditions at \( y = L \) lead to

\[
\frac{2 \sqrt{|\lambda|} \sinh \sqrt{|\lambda|} y_0 \left( 2 \cosh \sqrt{|\lambda|} y_0 - \sqrt{D(D + 3)} \cosh \sqrt{|\lambda|} L \right)}{4 + D \cosh 2 \sqrt{|\lambda|} y_0 - \cosh 2 \sqrt{D(D + 3)} \cosh \sqrt{|\lambda|} L} = \frac{\kappa^2}{D + 3} (V_2 - D[V_2 - V_2^*]) ,
\]

\[
\frac{2 \sqrt{|\lambda|} 2 \sinh \sqrt{|\lambda|} y_0 \left( \cosh \sqrt{|\lambda|} y_0 + 2 \sqrt{\frac{D + 3}{3}} \cosh \sqrt{|\lambda|} L \right)}{4 + D \cosh 2 \sqrt{|\lambda|} y_0 - \cosh 2 \sqrt{D(D + 3)} \cosh \sqrt{|\lambda|} L} = \frac{\kappa^2}{D + 3} (4V_2^* - 3V_2^*) .
\] (49)

From Eqs. (47) and (49), the brane tensions at \( y = 0 \) and \( y = L \) are expressed as

\[
V_1 = -\frac{2 \sqrt{|\lambda|}}{\kappa^2(4 + D)} \frac{2(D + 3) \cosh \sqrt{|\lambda|} y_0 + \sqrt{D(D + 3)}}{2 \sinh \sqrt{|\lambda|} y_0},
\]

\[
V_1^* = -\frac{2 \sqrt{|\lambda|}}{\kappa^2(4 + D)} \frac{(D + 3) \cosh \sqrt{|\lambda|} y_0 - 2 \sqrt{\frac{D + 3}{D}}}{\sinh \sqrt{|\lambda|} y_0},
\]

\[
V_2 = \frac{2 \sqrt{|\lambda|}}{\kappa^2(4 + D)} \frac{2 \left( (D + 3) \cosh \sqrt{|\lambda|} y_0 + \sqrt{D(D + 3)} \cosh \sqrt{|\lambda|} L \right) \sinh \sqrt{|\lambda|} y_0}{\cosh 2 \sqrt{|\lambda|} y_0 - \cosh 2 \sqrt{|\lambda|} L},
\]

\[
V_2^* = \frac{2 \sqrt{|\lambda|}}{\kappa^2(4 + D)} \frac{2 \left( (D + 3) \cosh \sqrt{|\lambda|} y_0 - 2 \sqrt{\frac{D + 3}{D}} \cosh \sqrt{|\lambda|} L \right) \sinh \sqrt{|\lambda|} y_0}{\cosh 2 \sqrt{|\lambda|} y_0 - \cosh 2 \sqrt{|\lambda|} L}.
\] (50)

From the above equations, the integration constant \( y_0 \) is expressed as

\[
y_0 = \frac{1}{\sqrt{|\lambda|}} \arcsinh \left[ \frac{1}{\kappa(V_1^* - V_1)} \right].
\] (51)

The sign of \( y_0 \) depends on the sign of the difference between \( V_1 \) and \( V_1^* \). If \( V_1 > V_1^* \), \( y_0 \) becomes negative, and Eq. (46) leads to the conclusion that \( a(y) \) has a singular point as long as \( D \neq 1 \). To avoid this a singular point over \( |y| \leq L \), \( -y_0 > L \) is required.

From Eq. (50), the brane tensions \( V_2, V_2^* \) of the observable brane at \( y = L \) can be described in terms of \( V_1, V_1^* \) of the hidden brane at \( y = 0 \). The ratios \( V_1/V_1^* \) and \( V_2/V_2^* \) cannot be unity as long as \( D \) is a positive integer. Thus, each brane tension becomes anisotropic in this setup, and each brane tension is closely related to the other because of the presence of two branes. Taking the limit \( L \gg y_0 \) in the infinite orbifold extra dimension where the observable brane is fixed far away from the origin \( \mathbb{R}^3 \), the ratio \( V_2 \) to \( V_2^* \) of the observable brane becomes

\[
\frac{V_2}{V_2^*} = -\frac{1}{4} D.
\] (52)

\( V_2 \) and \( V_2^* \) have opposite signs to each other and the magnitude of the ratio depends on the number of extra \( D \) dimensions.

Next, let us consider the case \( P_1 = 0 \). There exists a solution of Eq. (11) with a negative bulk cosmological constant. Integrating it we obtain

\[
e^z = \frac{D + 3}{(D + 4)\omega|\Lambda|Y^2}.
\] (53)
Using Eqs. (36) and (37), we have

\[ e^A = \left[ \frac{D + 3}{(D + 4)\omega|\Lambda|} \right] \left[ \frac{e^{2DP_2}Y^2}{Y^2} \right]^{\frac{1}{2(4+D)}} \],

\[ e^C = \left[ \frac{D + 3}{(D + 4)\omega|\Lambda|} \right] \left[ \frac{e^{-8P_2}Y^2}{Y^2} \right]^{\frac{1}{2(4+D)}}. \] (54)

Equation (33) leads to the relation between \( Y \) and \( y \):

\[ Y = Y_0 \exp \left[ \mp \sqrt{\frac{D + 4}{D + 3} \omega|\Lambda| y} \right], \] (55)

where \( Y_0 \) is an integration constant. After substitution of Eq. (55) into Eq. (54), \( a \) and \( c \) are described in terms of \( y \):

\[ a(y) = c(y) = \exp \left[ \pm \sqrt{\frac{\omega|\Lambda|}{(D + 3)(D + 4)}} |y| \right]. \] (56)

Hence the upper and lower signs correspond to the signs in Eq. (55) and both \( a \) and \( c \) are normalized to be unity at \( y = 0 \). The jump conditions of \( y = 0 \) and \( y = L \) lead to

\[ \pm 2 \sqrt{\frac{\omega|\Lambda|}{(D + 3)(D + 4)}} = - \frac{\kappa^2}{D + 3} \left( V_1 - D[V_1 - V_1^*] \right) = - \frac{\kappa^2}{D + 3} \left( 4V_1 - 3V_1^* \right) \]

\[ = \frac{\kappa^2}{D + 3} \left( V_2 - D[V_2 - V_2^*] \right) = \frac{\kappa^2}{D + 3} \left( 4V_2 - 3V_2^* \right), \] (57)

then the brane tensions are given by

\[ V_1 = V_1^* = -V_2 = -V_2^* = \mp \frac{2}{\kappa} \sqrt{\frac{D + 3}{D + 4}|\Lambda|}. \] (58)

In this case, we find that \( a(y) \) is equal to \( c(y) \) and the brane tension of each brane is automatically guaranteed to be isotropic. The lower sign in Eq. (58) corresponds to the case that the brane tension of the observable brane at \( y = L \) is negative. Consequently, the warped metric function becomes the exponential damping factor since the lower sign (negative) in Eq. (56) is selected. This situation is similar to the five-dimensional Randall-Sundrum solution. As in the Randall-Sundrum scenario, the hierarchy between the physical mass scale \( m_{\text{hid}} \) on the hidden brane at \( y = 0 \) and \( m_{\text{obs}} \) on the observable brane at \( y = L \) can be generated from the warped metric.

In the case of a negative bulk cosmological constant, whether the integration constant \( P_1 \) is nonzero or zero determines the form of the warped metric function. If \( P_1 \neq 0 \), the warped metric functions \( a(y) \) and \( c(y) \) have the forms of different hyperbolic functions and the brane tension is anisotropic. If \( P_1 = 0 \) and the observable brane has negative brane tension, both the warped metric functions have the same form of exponential damping factor and the brane tension is isotropic. Thus, whether the brane tension on the orbifold is isotropic or anisotropic depends on the value of \( P_1 \), namely, the integration constant \( P_1 \) controls the solution of the Einstein equation in bulk and it is expected that \( P_1 \) is determined by the initial configuration of the brane world in this setup. Hence it is assumed that the dynamical mechanism for fixing the value of \( P_1 \) is unknown.
3.2 The solutions for $\Lambda > 0$

When the bulk cosmological constant $\Lambda$ is positive, Eq. (41) implies that $P_1$ should be nonzero. Solving this equation, we obtain

$$e^Z = \frac{4DP_1^2}{(D+4)\omega\Lambda \cosh^2 \left( 2\sqrt{\frac{D}{D+3}} P_1 Y \right)}.$$  

(59)

Furthermore, we have

$$e^A = \left[ \frac{4DP_1^2}{(D+4)\omega\Lambda \cosh^2 \left( 2\sqrt{\frac{D}{D+3}} P_1 Y \right)} \right]^{\frac{1}{2(4+D)}},$$

$$e^C = \left[ \frac{4DP_1^2}{(D+4)\omega\Lambda \cosh^2 \left( 2\sqrt{\frac{D}{D+3}} P_1 Y \right)} \right]^{\frac{1}{2(4+D)}}.$$  

(60)

Following Eq. (33),

$$\exp \left[ 2\sqrt{\frac{D}{D+3}} P_1 Y \right] = \mp \tan \left( \frac{1}{2} \sqrt{\frac{D+4}{D+3}} \omega\Lambda \left( y + y_1 \right) \right),$$

(61)

where $y_1$ is an integration constant. By imposing the orbifold symmetry, we can rewrite $a$ and $c$ in terms of $y$

$$a(y) = \left( \frac{4DP_1^2}{(D+4)\omega\Lambda} \right)^{\frac{1}{2(D+4)}} \left[ \tan \frac{\sqrt{D(D+3)}}{2} \left( |y| + y_1 \right) \sin \left( \sqrt{\lambda} \left( |y| + y_1 \right) \right) \right]^{\frac{1}{2}},$$

$$c(y) = \left( \frac{4DP_1^2}{(D+4)\omega\Lambda} \right)^{\frac{1}{2(D+4)}} \left[ \cot \frac{\sqrt{D(D+3)}}{2} \left( |y| + y_1 \right) \sin \left( \sqrt{\lambda} \left( |y| + y_1 \right) \right) \right]^{\frac{1}{2}},$$

(62)

where $\lambda$ is defined in Eq. (48) and $P_2$ can be set to zero as mentioned previously. The jump conditions lead to brane tensions as follows:

$$V_1 = -\frac{2\sqrt{\lambda}}{k^2(4+D)} \frac{2(D+3) \cos \sqrt{\lambda}y_1 - \sqrt{D(D+3)}}{2 \sin \sqrt{\lambda}y_1},$$

$$V_1^* = -\frac{2\sqrt{\lambda}}{k^2(4+D)} \frac{(D+3) \cos \sqrt{\lambda}y_1 + 2\sqrt{D+3}}{\sin \sqrt{\lambda}y_1},$$

$$V_2 = \frac{2\sqrt{\lambda}}{k^2(4+D)} \frac{\sinh \sqrt{\lambda}y_1 \left( 2(D+3) \cosh \sqrt{\lambda}y_1 + \sqrt{D(D+3)} \cosh \lambda L \right)}{\cosh 2\sqrt{\lambda}y_1 - \cosh 2\sqrt{\lambda}L},$$

$$V_2^* = \frac{2\sqrt{\lambda}}{k^2(4+D)} \frac{2\sinh \sqrt{\lambda}y_1 \left( (D+3) \cosh \sqrt{\lambda}y_1 - 2\sqrt{D+3} \cosh \sqrt{\lambda}L \right)}{\cosh 2\sqrt{\lambda}y_1 - \cosh 2\sqrt{\lambda}L}.$$  

(63)
Using the above equations, the integration constant $y_1$ is expressed as

$$y_1 = \frac{1}{\sqrt{\lambda}} \arcsin \left[ \frac{1}{\kappa(V_1 - V_1^*)} \sqrt{\frac{2(D + 4)}{D}\Lambda} \right].$$ \hspace{1cm} (64)

The sign of $y_1$ depends on the sign of the difference between $V_1$ and $V_1^*$. If $V_1^* > V_1$, $y_1$ becomes negative and it is found that $c(y)$ has a singularity. Moreover, Eq.(64) leads to the constraint on the location of $y = L$:

$$L < \pi \sqrt{\frac{D + 3}{(D + 4)\omega A}} - y_1.$$ \hspace{1cm} (65)

As in the case of negative bulk cosmological constant, the brane tension becomes anisotropic. The warped metric functions obtained here have the forms of different trigonometric functions.

### 3.3 The solutions for $\Lambda = 0$

In the case of zero bulk cosmological constant, the metric functions are quite similar to the Kasner solution in higher dimensional cosmology given in the appendix. We take the power-law form by taking account of the orbifold symmetry,

$$a(y) = \left(\frac{|y|}{y_2} + 1\right)^k,$$

$$c(y) = \left(\frac{|y|}{y_2} + 1\right)^l,$$ \hspace{1cm} (66)

where $a$ and $c$ are normalized to be unity at $y = 0$ and $y_2$ is a constant to be determined later. Substituting the above equations into Eqs.(20)-(22), we obtain two equations for the exponents

$$4k + Dl = 1,$$

$$4k^2 + Dl^2 = 1.$$ \hspace{1cm} (67)

Solving the above equations, $k$ and $l$ are given by

$$k = \frac{2 \pm \sqrt{D(D + 3)}}{2(D + 4)},$$

$$l = \frac{D \pm 2 \sqrt{D(D + 3)}}{D(D + 4)}.$$ \hspace{1cm} (68)

Here, we cannot determine whether the sign included in Eq.(68) is the lower or upper sign at this stage.

From Eq.(14), the jump conditions on $a$ and $c$ at $y = 0$ yield

$$\frac{2k}{y_2} = -\frac{\kappa^2}{D + 3} (V_1 - D[V_1 - V_1^*]),$$

$$\frac{2l}{y_2} = -\frac{\kappa^2}{D + 3} (4V_1 - 3V_1^*),$$ \hspace{1cm} (69)
furthermore, the jump conditions at \( y = L \) lead to
\[
\begin{align*}
\frac{2ky_2}{L^2 - y_2^2} &= -\frac{\kappa^2}{D+3} (V_2 - D[V_2 - V_2^*]), \\
\frac{2ly_2}{L^2 - y_2^2} &= -\frac{\kappa^2}{D+3} (4V_2 - 3V_2^*). 
\end{align*}
\tag{70}
\]
Using the above equations, the brane tensions are given by
\[
\begin{align*}
V_1 &= -\frac{2}{\kappa^2 y_2} (1 - k), \\
V_1^* &= -\frac{2}{\kappa^2 y_2} (1 - l), \\
V_2 &= -\frac{2y_2}{\kappa^2 (L^2 - y_2^2)} (1 - k), \\
V_2^* &= -\frac{2y_2}{\kappa^2 (L^2 - y_2^2)} (1 - l). 
\end{align*}
\tag{71}
\]
Following the constraint on the exponents in Eq.(67), the constant \( y_2 \) becomes
\[
y_2 = -\frac{2(D + 3)}{\kappa^2 (4V_1 + DV_1^*)}. \tag{72}
\]
Consequently, the ratios \( V_1/V_1^* \) and \( V_2/V_2^* \) of each brane cannot be unity for arbitrary positive \( D \), namely, the brane tension becomes anisotropic. From Eq.\((71)\), we have \( V_1/V_1^* = V_2/V_2^* = (1 - k)/(1 - l) \), which means that the ratio for each brane is the same. Furthermore, since \( k \) and \( l \) cannot be beyond unity, the sign of both \( V_1 \) and \( V_1^* \) is always negative and the relative size between \( L \) and \( y_2 \) determines the sign of both \( V_2 \) and \( V_2^* \). Taking the limit \( L \gg y_2 \) to be the infinitely fixed observable brane \([4]\), both \( V_2 \) and \( V_2^* \) approach zero. Moreover, if the extra \( D \)-dimensional space has infinite dimension, taking the limit of \( D \to \infty, k \to \pm1/2 \) and \( l \to 0 \), and the ratio becomes
\[
\frac{V_1}{V_1^*} = \frac{V_2}{V_2^*} = \frac{1}{2} \cdot \frac{3}{2}. \tag{73}
\]
Thus, \( V_1 \) and \( V_1^* \) have the same sign and \( V_2 \) and \( V_2^* \) do also.

For zero bulk cosmological constant, the warped metric functions \( a(y) \) and \( c(y) \) have the forms of different power laws whose exponents are similar to the constraints appearing in the Kasner solution of higher dimensional cosmology. Furthermore, in the case of an infinite orbifold dimension the brane tension of the observable brane becomes zero.

4 Summary and Discussion

We study the warped metric in the \((5 + D)\)-dimensional Einstein equation with an extra dimension compactified on the orbifold \( S^1/Z_2 \), where two \((3 + D)\)-branes are fixed on the \( y \) direction in orbifold compactification, the hidden brane at \( y = 0 \) and the observable brane at
$y = L$. It is assumed that the energy-momentum tensor on the brane has anisotropic brane tension, anisotropic density, and anisotropic pressure. With the ansatz metric in this paper, the warped metric function $a(y)$ of four dimensions is generally different from that $c(y)$ of $D$ dimensions. We solved the Einstein equations in this setup.

For a negative bulk cosmological constant, whether the integration constant $P_1$ in the differential equation coming from the Einstein equation is non zero or zero controls the forms of $a(y)$ and $c(y)$. If $P_1 \neq 0$, $a(y)$ and $c(y)$ have the forms of different hyperbolic functions, and we pointed out that the brane tension becomes anisotropic. If $P_1 = 0$, $a(y)$ and $c(y)$ have the same form of exponential factor and the brane tension becomes isotropic. Furthermore, if the observable brane has negative brane tension, the warped metric function is the exponential damping factor and this case is similar to five-dimensional Randall-Sundrum scenario. For positive bulk cosmological constant, the integration constant $P_1$ is required to be nonzero in order for the solution of the differential equation to exist. $a(y)$ and $c(y)$ have the forms of different trigonometric functions and the brane tension becomes anisotropic. On the other hand, a zero bulk cosmological constant causes $a(y)$ and $c(y)$ to have the forms of different power laws whose exponents are constrained and thus the brane tension becomes anisotropic.

As mentioned above, the dynamics of the differential equation depend on the integration constant $P_1$ which is determined by the initial condition. The mechanism for fixing the value of $P_1$ is unknown; however, it is expected that this is determined via the dynamics of the underlying physics, namely, the initial configuration of the brane world. In section 2, we derived the cosmological evolution equation in the isolated two-brane system embedded in $5 + D$ dimensions with warped metric. We are going to study the cosmology constrained by two branes in this setup. Moreover, it is necessary to explore the massless gravitational fluctuations about our classical solution obtained here and to study the stabilization mechanism for compactification. Finally, we expect that this setup may be connected to the $D$-brane configuration in the framework of superstring theory.

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Appendix: Review of the Kasner Solution

In this appendix, we review the Kasner solution in higher dimensional cosmology \[8, 12\]. The original Kasner cosmology is famous as an example of the anisotropic four-dimensional cosmological model, and the metric of Kasner form is

$$ds^2 = dt^2 - t^{2p} dx_1^2 - t^{2q} dx_2^2 - t^{2r} dx_3^2,$$ \hspace{1cm} (74)

where $p$, $q$, and $r$ are parameters. The Kasner cosmology corresponds to the vacuum (empty) cosmological model where the numbers $p$, $q$, and $r$ satisfy the constraints

$$p + q + r = 1, \quad p^2 + q^2 + r^2 = 1.$$ \hspace{1cm} (75)
The above equations are determined via the Einstein equations. The space becomes anisotropic if at least two of the three \( p, q, \) and \( r \) are different.

Next, we describe an extension of the four-dimensional Kasner cosmology to \( 4 + D \) dimensions, and the metric is given by \([3, 12]\)
\[
ds^2 = dt^2 - a(t)^2 d\vec{x}^2 - c(t)^2 \left( dz_1^2 + \cdots + dz_D^2 \right). \tag{76}
\]
Here \( a(t) \) and \( c(t) \) represent the scale factor of the three-space and that of the extra \( D \)-space, respectively. This metric corresponds to \( n \equiv 1, b \equiv 0, \) and \( y \) independence of \( a \) and \( c \) in Eq.(2). Moreover, there are no contributions of the bulk and the brane due to the emptiness. We can rewrite the Einstein equations by performing the appropriate linear combinations of Eqs.(7), (8), and (10)
\[
\begin{align*}
3\ddot{a} + D\ddot{c} &= 0, \\
\frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 + D\frac{\dot{a} \dot{c}}{ac} &= 0, \\
\frac{\ddot{c}}{c} + (D-1) \left( \frac{\dot{c}}{c} \right)^2 + 3\frac{\dot{a} \dot{c}}{ac} &= 0. \tag{77}
\end{align*}
\]
We take the power-law form (so-called Kasner solution) as
\[
\begin{align*}
a(t) &= a_0 \left( \frac{t}{t_0} \right)^p, \\
c(t) &= c_0 \left( \frac{t}{t_0} \right)^q, \tag{78}
\end{align*}
\]
where \( a_0, c_0, \) and \( t_0 \) are constants and the scale factors are normalized to be zero at \( t = 0. \) The exponents \( p \) and \( q \) are subject to the constraints
\[
\begin{align*}
3p + Dq &= 1, \\
3p^2 + Dq^2 &= 1. \tag{79}
\end{align*}
\]
The above equations can be simply checked by the substitution of Eq.(78) into Eq.(77). Solving this, we have
\[
\begin{align*}
p &= \frac{3 \pm \sqrt{3D(D+2)}}{3(D+3)}, \\
q &= \frac{D \pm \sqrt{3D(D+2)}}{D(D+3)}. \tag{80}
\end{align*}
\]
Taking the upper sign in Eq.(80), these solutions describe the case where the scale factor \( a(t) \) of three-dimensional space expands while \( c(t) \) of the extra \( D \)-dimensional space shrinks.

In the case of zero bulk cosmological constant, the form of the metric functions with \( y \) dependence in Eq.(66) resembles the Kasner solutions with \( t \) dependence in Eq.(78). Moreover, the Kasner solutions with radion potential in the framework of large extra dimensions are discussed in Ref. [3].
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