Some symmetry group aspects of a perfect plane plasticity system

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Abstract

In this paper, all the known classical solutions of a plane perfect plasticity system under the Saint Venant–Tresca–von Mises yield criterion are associated with some group of point symmetries. The equations of slip-line families for all solutions are constructed, which allows one to explicitly determine the boundaries of the plastic areas. It is shown how one can determine the compatible velocity solution for known stresses by considering symmetries. Some invariant solutions of velocities for Prandtl stresses are constructed. The mechanical sense of the obtained velocity fields is discussed.

To the blessed memory of our teacher D D Ivlev

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1. Introduction

Let us consider the well known system of two-dimensional perfect plasticity [9]:

$$\begin{align*}
\frac{\partial \sigma}{\partial x} - 2k \left( \frac{\partial \theta_c}{\partial x} \cos 2\theta_c + \frac{\partial \theta_c}{\partial y} \sin 2\theta_c \right) &= 0, \\
\frac{\partial \sigma}{\partial y} - 2k \left( \frac{\partial \theta_c}{\partial x} \sin 2\theta_c - \frac{\partial \theta_c}{\partial y} \cos 2\theta_c \right) &= 0,
\end{align*}$$

(1)

which follows from equilibrium equations in the absence of body forces:

$$\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0, \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0
\end{align*}$$

(2)

and yields the Saint Venant–Tresca–von Mises criterion

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2$$

(3)
through the change of variables (due to Lévi):

\[
\sigma_x = \sigma - k \sin 2\theta^c, \quad \sigma_y = \sigma + k \sin 2\theta^c, \quad \tau_{xy} = k \cos 2\theta^c.
\] (4)

Here \(\sigma_x, \sigma_y\) are the normal components and \(\tau_{xy}\) the tangential component of stress in the rectangular coordinates \((x, y)\), \(\theta^c + \pi/4\) is a slope angle of the maximum principal stress relative to the axis \(\alpha_x\), \(\sigma\) is the mean compressive stress (or hydrostatic pressure), and \(k\) is a constant of the material.

To construct the velocity field, which is consistent with the solution of (1) for \(\sigma = \sigma_0(x, y), \theta^c = \theta^c_0(x, y)\) one needs to solve the following system of linear equations:

\[
\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \sin 2\theta_0 + \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \cos 2\theta_0 = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\] (5)

where \(u(x, y), v(x, y)\) are components of the velocity vector.

System (1) describes the stress state of the material which is being plastically deformed and has been studied over the past hundred years. The main contribution in determining closed-form solutions was made by Prandtl and Nadai at the beginning of the 20th century. These solutions are now known as classical ones and serve as good approximations for the real mechanical processes.

The symmetry method of solving differential equations was applied to system (1) for the first time in [1]. In [2] the main part of the symmetries for the system (2), (3) was calculated and some known solutions were associated with correspondent transformation groups. Finally, the complete Lie algebra of all admissible point symmetries of (1) was determined in [22] and all conservation laws as well as Lie–Backlund symmetries were constructed. Moreover, in the series of papers [23, 25, 27], point transformation groups were used to deform some known solutions and in [24] conservation laws were applied to solve the main boundary problems.

System (1) is a hyperbolic one. Two families of characteristic lines are defined by two equations:

\[
\frac{dy}{dx} = \tan \theta^c, \quad \frac{dy}{dx} = -\cot \theta^c,
\] (6)

and the corresponding Riemann invariants, which are constant along the characteristics, look like

\[
\frac{\sigma}{2k} - \theta^c = \xi, \quad \frac{\sigma}{2k} + \theta^c = \eta.
\] (7)

In the theory of plane perfect plasticity, characteristic lines coincide with the so-called slip-lines: curves whose directions at every point coincide with those of the maximum shear strain-rate. Equations (6) define the slip-line field. As indicated in [6], the field of slip-lines is the fundamental unknown element to be determined.

In practical applications it is necessary to find the solutions of system (1) which satisfy certain boundary conditions. There are three [9] fundamental kinds of boundary problems. The first one is the Cauchy problem: it is required to satisfy the prescribed values of \(\sigma\) and \(\theta^c\) on the given smooth arc \(C\), which nowhere coincides with a characteristic line and which intersects them only once. The second kind is the initial characteristic problem (the so-called Riemann or Goursat problem): the values of the functions \(\sigma, \theta\) are known on segments of slip-lines of different families emanating from the same point. The third boundary problem is a mixed one: the functions \(\sigma, \theta^c\) are known on a segment of the slip-line which joins a non-characteristic line, along which the angle \(\theta^c\) is given.

If the normal \(\sigma_n\) and tangential \(\tau_n\) components of stress are given on a boundary contour \(C\), then boundary values of \(\sigma, \theta^c\) are determined by the following relations

\[
\theta^c|_C = \varphi_n \pm \frac{1}{2} \arccos \frac{\tau_n}{k} + m\pi, \quad \sigma|_C = \sigma_n + k \sin 2(\theta^c - \varphi_n),
\]
where $\varphi_n$ is the angle between the normal to the contour and the $x$-axis, $|\tau_n| \leq k$, $m \in \mathbb{Z}$.

For example, along a free boundary $C$, $x = 0$ we have $\varphi_n = 0$, $\tau_n = 0$, $\sigma_n = 0$, then $\theta^c|_C = \pm \pi/4 + m\pi$, $\sigma|_C = \pm k$. The presence of two values for $\sigma$, $\theta^c$ is explained by the quadratic character of the yield criterion (3) and the sign is chosen from the mechanical formulation of the problem.

Let us note that a group analysis of the differential equations is a semi-inverse method. It means that firstly some solution should be found, then one can determine the corresponding boundary conditions.

It allows one to solve statically determined problems for (1), (5), i.e. when the boundary conditions involve only the stresses, which are sufficient to allow a determination of the plastic region and the state of stress without considering the velocities. The velocities can be calculated afterwards. An example of such problems are the plane and axisymmetric ones of the occurrence of plastic flow due to indentation by a rigid die to the semi-plane. There is a wide class of problems in which static boundary conditions are unknown, but can be determined by the known kinematic conditions. Such kinematically determined problems include the rolling of strips, wire drawing, etc [4].

Interest in the system of plane plasticity has recently been renewed. In [11] the known symmetries, admitted by (1), (5) were used to determine some solutions in the form of a propagation wave. In [12] the Lie algebra of symmetries for (1), (5) is calculated and the construction of some solution is discussed.

In [7], the plastic stress states of the round sphere are analyzed from the point of view of orthogonal equiareal patterns, and an interesting relation between the slip-line fields on the sphere and the sine-Gordon equation is shown.

The main goal of the present work is, on the one hand, to reconsider the relation between the known classical solutions and the group of point transformations (symmetries), because some results were previously omitted or were not published. On the other hand, we analyze velocities which are compatible with stresses, defined by (1) from the symmetry point of view.

The algorithm of the construction of invariant solutions is well known. The interested reader can find an extensive bibliography on this matter (see, for example, [3, 18, 19]). The main steps are: (1) for the given system define a Lie algebra of admissible operators, generating a group of point continuous transformations (symmetries); (2) determine the optimal system of non-similar subalgebras; (3) construct a set of invariants for the representatives of subalgebras and if possible, find out the form of the invariant solution; (4) by substituting the invariant solution form into the original system, the so-called factor-system is obtained, which in our case is the system of ordinary differential equations; (5) by solving a factor-system one can obtain the invariant solution.

The paper is organized as follows. In section 2 we consider some known solutions, obtained through heuristic methods and their mechanical interpretation. Section 3 is devoted to correspondence between symmetries and known classic closed-form solutions. Finally, the construction of the velocity field, related to the stress solution is discussed in section 4.

The following information will be useful. In polar coordinates $(r, \varphi)$, system (2), (3) has the form

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\varphi}}{\partial \varphi} + \frac{\sigma_r - \sigma_\varphi}{r} = 0,$$

$$\frac{\partial \tau_{r\varphi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\varphi}{\partial \varphi} + \frac{2}{r} \tau_{r\varphi} = 0,$$

$$(\sigma_r - \sigma_\varphi)^2 + 4\tau_{r\varphi}^2 = 4k^2,$$

where $\sigma_r, \sigma_\varphi$ are the radial and angular components and $\tau_{r\varphi}$ the tangential component of stress.
Introducing in a similar way
\[ \sigma_r = \sigma - k \sin 2\theta^p, \quad \sigma_\phi = \sigma + k \sin 2\theta^p, \quad \tau_{\psi} = k \cos 2\theta^p \]
into the above system, one can obtain:
\[ \frac{r \sigma'}{r} - 2k \left( r \frac{\partial \sigma'}{r} \cos 2\theta^p + \frac{\partial \sigma'}{\phi} \sin 2\theta^p \right) = 2k \sin 2\theta^p, \]
\[ \frac{\partial \sigma'}{\phi} - 2k \left( r \frac{\partial \sigma'}{r} \sin 2\theta^p - \frac{\partial \sigma'}{\phi} \cos 2\theta^p \right) = -2k \cos 2\theta^p, \tag{8} \]
where \( \sigma^p = \frac{\sigma_r + \sigma_\phi}{2} = \sigma \) and \( \theta^p \) is an angle between the radio and the slope of the slip-line.

Two families of characteristic lines are defined by the following equations:
\[ \frac{dr}{d\phi} = -r \tan \theta^p, \quad \frac{dr}{d\phi} = r \cot \theta^p. \tag{9} \]
From system (8), using (4) and \( \theta = \theta^c = \theta^p + \phi \), we have:
\[ \frac{\partial \sigma}{\partial r} - 2k \left( r \frac{\partial \theta_c}{r} \cos 2(\theta^c - \phi) + \frac{\partial \theta_c}{\phi} \sin 2(\theta^c - \phi) \right) = 0 \]
\[ \frac{\partial \sigma}{\partial \phi} - 2k \left( r \frac{\partial \theta_c}{r} \sin 2(\theta^c - \phi) - \frac{\partial \theta_c}{\phi} \cos 2(\theta^c - \phi) \right) = 0 \tag{10} \]

2. Notes about some known solutions

The description of classic solutions can be found in many textbooks on the mathematical theory of plasticity. Here we consider some of them, which were not analyzed in the series of works [2, 23, 25, 27].

2.1. The Revuzhenko solution

Revuzhenko in his paper [21] considered the so-called limiting equilibrium equations:
\[ \frac{1}{2k} \frac{\partial \sigma}{\partial \lambda_1} - \frac{\partial \phi}{\partial \lambda_1} = 0, \quad \frac{1}{2k} \frac{\partial \sigma}{\partial \lambda_2} + \frac{\partial \phi}{\partial \lambda_2} = 0, \tag{11} \]
\[ \frac{\partial y}{\partial \lambda_1} = \tan \left( \phi - \frac{\pi}{4} \right) \frac{\partial x}{\partial \lambda_1}, \quad \frac{\partial y}{\partial \lambda_2} = \tan \left( \phi + \frac{\pi}{4} \right) \frac{\partial x}{\partial \lambda_2}, \tag{12} \]
where \( \phi = \theta^c + \pi/4 \) and \( \lambda_1 = \eta, \lambda_2 = \xi \) are characteristic coordinates (7). Let us take \( 2k = 1 \), then the first two equations (11) give
\[ \sigma - \theta^c = F(\xi), \quad \sigma + \theta^c = G(\eta), \tag{13} \]
and equations (12) obtain the following form
\[ \frac{\partial y}{\partial \eta} = \tan \theta^c \frac{\partial x}{\partial \eta}, \quad \frac{\partial y}{\partial \xi} = -\cot \theta^c \frac{\partial x}{\partial \xi}, \tag{14} \]
where \( \theta^c = (G - F)/2, G(\eta) \) and \( F(\xi) \) are arbitrary functions and not identical constants.

Solving linear equations (14), closely related to (6), one can find the parametric equations \( x = x(\xi, \eta), y = y(\xi, \eta) \) of characteristic lines in Cartesian coordinates. If the slip-line field is known, then one can calculate functions \( \sigma(\xi, \eta), \theta(\xi, \eta) \) at the point \( (x, y) \) using (13).

In polar coordinates, system (14) takes the form
\[ \frac{1}{r} \frac{\partial r}{\partial \eta} = \cot(\theta^c - \phi) \frac{\partial \phi}{\partial \eta}, \quad \frac{1}{r} \frac{\partial r}{\partial \xi} = -\tan(\theta^c - \phi) \frac{\partial \phi}{\partial \xi} \tag{15} \]
and is similar to (9).
Eliminating $r$ from the above system, we obtain one equation for the function $u = \tan \theta^p$:
\[
\frac{\partial^2}{\partial \xi \partial \eta} \ln |u| - \frac{G}{2} \frac{\partial}{\partial \xi} \frac{1}{u} + \frac{F'}{2} \frac{\partial}{\partial \eta} u = 0.
\] (16)

Its solution can be determined by separating the variables and has the following form
\[
(a, b, c = \text{const}).
\]
\[
u^2 = \tan^2 \theta^p = aG(\eta) + bF'(\xi) + c.
\] (17)

Then, for simplicity, one can take
\[
\tan^2 \theta^p = \frac{\eta^2}{\xi^2}, \quad G = 2\eta^2 - \pi/4, \quad F = 2\xi^2 + \pi/4.
\] (18)

Regressing to variable $\varphi = \theta^c - \theta^p = (G - F)/2 - \theta^p$ we have:
\[
\varphi = \eta^2 - \xi^2 - \pi/4 \mp \arctan \frac{\eta}{\xi}.
\] (19)

The sign $\mp$ follows from the quadratic type of (17).

Taking $\tan \theta^p = \pm \eta/\xi$ and integrating (15), which take the form:
\[
\frac{1}{r} \frac{\partial r}{\partial \eta} = \pm \frac{\xi}{\eta} \frac{\partial \varphi}{\partial \eta}, \quad \frac{1}{r} \frac{\partial r}{\partial \xi} = \mp \frac{\eta}{\xi} \frac{\partial \varphi}{\partial \xi},
\] (20)

one can determine function $r$
\[
r = e^{\pm 2\pi / \xi^2 + \eta^2}.
\] (21)

The solution for $\sigma$, $\theta$ is given by functions $G$, $F$:
\[
\sigma = \frac{G + F}{2} = \xi^2 + \eta^2, \quad \theta^c = \frac{G - F}{2} = \eta^2 - \xi^2 - \pi/4.
\] (22)

The obtained solution can be used to describe the plastic state for a different angle area, hornlike area, etc. For more details of boundary conditions see [21].

2.2. The Nadai solution for two concentric circles

In [15] Nadai proposed the form of dependence $\tau_{\varphi \rho} = h(r) = k \cos 2\theta^p$, which from (8) gives:
\[
\cos 2\theta^p = C_1 r^{-2} + C_2, \quad \sigma = -2kC_2\varphi + f(r),
\] (23)

where $f(r)$ can be determined by the quadrature from equation
\[
f'' - 2k(\theta^p)' \cos 2\theta^p = 2kr^{-1} \sin 2\theta^p.
\] (24)

This solution is interpreted as stresses in the area, bounded by two concentric circles $r = a$, $r = b$, so that
\[
\tau_{\varphi \rho}|_{r=a} = -k, \quad \tau_{\varphi \rho}|_{r=b} = k,
\]

which gives
\[
C_1 = -2 \frac{a^2b^2}{b^2 - a^2}, \quad C_2 = \frac{a^2 + b^2}{b^2 - a^2}.
\] (25)

For simplicity, let us put $a = 1$, $b = \sqrt{2}$, then, taking into account that
\[
\tan \theta^p = \frac{1}{\sqrt{2}} \sqrt{\frac{2 - r^2}{r^2 - 1}}
\]
from the first equation for slip-lines (9) one can obtain the relation for the first family
\(1 \leq r \leq \sqrt{2}\)
\[
\frac{1}{\sqrt{2}} \arctan \frac{3r^2 - 4}{2\sqrt{2}\sqrt{(r^2 - 1)(2 - r^2)}} - \arcsin(2r^2 - 3) = \sqrt{2}\phi + K_1,
\]
which are parts of epicycloids.

The second equation gives the equation of the second family of slip-lines (hypocycloids):
\[
\sqrt{2} \arctan \frac{3r^2 - 4}{2\sqrt{2}\sqrt{(r^2 - 1)(2 - r^2)}} - \arcsin(2r^2 - 3) = 2\sqrt{2}\phi + K_2.
\]

Let us note that concentric circles are envelopes for corresponding families of epicycloids and hypocycloids (see figure 1).

In general, for all boundary problems with two envelopes as boundary lines, we have the following boundary conditions:
\[
\sigma_n|_{\Gamma_1} = \sigma|_{\Gamma_1}, \quad \tau_n|_{\Gamma_1} = -k, \quad \tau_n|_{\Gamma_2} = k, \quad (26)
\]
where \(\sigma_n, \tau_n\) are the normal and tangential components of stress along the contours \(\Gamma_1, \Gamma_2\) and \(\Gamma_1\) is an envelope of the first family of slip-lines and \(\Gamma_2\) is an envelope of the second family. Let us note that the first family of slip-lines forms the angle \(\pi/2\) with \(\phi_N(\Gamma_1)\) and forms the angle equal to \(\phi_N(\Gamma_2)\), where \(\phi_N(\Gamma)\) denotes the angle between the normal to the curve \(\Gamma\) and the \(\alpha\)-axis.
3. Group analysis and invariant solutions for stresses

It is known [22] that system (1) admits an infinite algebra of generalized (highest) symmetries. The basis of Lie algebra $L_{\sigma \theta}$ of point transformations is formed with the following operators (here we use $\partial_t = \partial / \partial t$ for simplicity):

\[
X_1 = x \partial_x + y \partial_y, \quad X_2 = -y \partial_x + x \partial_y, \quad X_3 = \partial_\sigma, \\
X_4 = \xi_1(x, y, \sigma, \theta) \partial_x + \xi_2(x, y, \sigma, \theta) \partial_y - 4k \theta \partial_\sigma - \frac{\sigma}{k} \partial_\theta, \\
X_5 = \xi(\sigma, \theta) \partial_\sigma + \eta(\sigma, \theta) \partial_\theta,
\]

where

\[
\xi_1 = x \cos 2\theta + y \sin 2\theta + \frac{\sigma}{k}, \quad \xi_2 = x \sin 2\theta - y \cos 2\theta - \frac{\sigma}{k},
\]

and $(\xi, \eta)$ is an arbitrary solution of the linear system

\[
\begin{align*}
\frac{\partial x}{\partial \theta} - 2k \left( \frac{\partial x}{\partial \sigma} \cos 2\theta + \frac{\partial y}{\partial \sigma} \sin 2\theta \right) &= 0, \\
\frac{\partial y}{\partial \theta} - 2k \left( \frac{\partial x}{\partial \sigma} \sin 2\theta - \frac{\partial y}{\partial \sigma} \cos 2\theta \right) &= 0,
\end{align*}
\]

obtained from (1) applying hodograph transformations of the form $x = x(\sigma, \theta), y = y(\sigma, \theta)$ and assuming that the Jacobian of the transformation is not equal to zero. Operator $X_5$ forms an infinite-dimensional subalgebra of $L_{\sigma \theta}$ and we consider it later (see subsection 3.6).

Operator $X_1$ gives the scaling transformation of $x$ and $y$, $X_2$ generates the rotation group and $X_3$ produces the $\sigma$-translation. The transformation group, corresponding to operator $X_4$ has been calculated and called the ‘quasi-scaling’ transformation in [25].

In polar coordinates the above operators (except $X_3$) take the following form:

\[
\begin{align*}
X'_1 &= r \partial_r, \\
X'_2 &= \partial_\varphi + \partial_\theta, \\
X'_4 &= r \cos 2(\theta - \varphi) \partial_r + \left( \sin 2(\theta - \varphi) - \frac{\sigma}{k} \right) \partial_\varphi - 4k \theta \partial_\sigma - \frac{\sigma}{k} \partial_\theta,
\end{align*}
\]

The non-zero commutators of (27) are as follows

\[
[X_2, X_4] = -4kX_3, \quad [X_3, X_4] = -X_2/k.
\]

From the symmetry point of view, it is necessary to construct the so-called optimal system of non-similar one-dimensional subalgebras [19] in order to define different invariant solutions. Let us note that taking the invariant solution $S$, corresponding to the subgroup $G$, and acting by transformations corresponding to the other (non-similar) subgroup $G'$ over $S$ one can obtain a family of ‘deformed’ solutions, depending on the group parameter. For system (1) some examples of this method were considered in [23, 25].

The optimal system of one-dimensional subalgebras for the finite part of $L_{\sigma \theta}$ is as follows $(\alpha \in \mathbb{R})$:

\[
\begin{align*}
\Theta_1 &= \langle X_4 + \alpha X_1 \rangle = \langle X_4 + \alpha r \partial_r \rangle, \\
\Theta_2 &= \langle X_3 + \alpha X_1 \rangle = \langle \partial_\sigma + \alpha r \partial_r \rangle, \\
\Theta_3 &= \langle X_2 + \alpha X_1 \rangle = \langle \partial_\varphi + \partial_\theta + \alpha r \partial_r \rangle, \\
\Theta_4^{(\pm)} &= \langle X_3 \pm X_2 + \alpha X_1 \rangle = \langle \partial_\sigma \pm \partial_\varphi + \partial_\theta + \alpha r \partial_r \rangle, \\
\Theta_5 &= \langle X_1 \rangle = \langle r \partial_r \rangle.
\end{align*}
\]

Non-similar subalgebras correspond to different values of $\alpha$.

Let us note that system (1) is invariant with respect to discrete symmetries:

\[
x \rightarrow -x, \quad \sigma \rightarrow -\sigma, \quad \theta \rightarrow -\theta; \quad y \rightarrow -y, \quad \sigma \rightarrow -\sigma, \quad \theta \rightarrow -\theta,
\]

\[7\]
and systems (8) and (10) admit transformations
\[ \varphi \rightarrow -\varphi, \sigma \rightarrow -\sigma, \theta^{p,c} \rightarrow -\theta^{p,c}; \]
\[ r \rightarrow -r \]
respectively.

Moreover, system (1) admits an equivalence transformation, acting on \( k \)
\[ \sigma' = \sigma \exp(a), \quad k' = k \exp(a), \quad a \in \mathbb{R}, \]
which allows one to choose the convenient value of \( k \) later on.

3.1. \( \Theta_1 \): quasi-scaling transformation

From the group analysis point of view, equation (16) admits the following symmetries:
\[ Y_1 = -\frac{F}{G} \partial_\xi + \frac{G}{F} \partial_\eta + \alpha \partial_\sigma, \quad Y_2 = \frac{2}{F} \partial_\xi, \quad Y_3 = \frac{2}{G} \partial_\eta, \]
and there is no extension of the group for any specific form of \( G \) and \( F \). For functions \( G, F \), taken as in (18), the above operators look like
\[ Y_1 = -\xi \partial_\xi + \eta \partial_\eta + 2u \partial_\sigma, \quad Y_2 = \xi^{-1} \partial_\xi, \quad Y_3 = \eta^{-1} \partial_\eta. \]

The invariant solution corresponding to operator \( Y_1 \) has the form \( u = \eta^2 f(z) \), where \( z = \xi \eta \). Substituting this form into (16) one can obtain its general solution in terms of Bessel functions. In particular, \( f = \pm 1/z \) is one of the solutions, therefore solution (18) \( u = \pm \eta^2/\xi \) is the particular one and is invariant with respect to \( Y_1 \).

Taking into account solutions (19), (21), (22), we can express operator \( Y_1 \) in terms of \( r, \varphi, \theta^{p,c}, \sigma \). Namely, for the higher sign solution \( \tan \theta^p = \eta^2/\xi \) we have:
\[ 2u \partial_\sigma = -2\xi \partial_\xi + 2\eta \partial_\eta, \]
\[ \partial_\xi = r \left( 2\eta \frac{\eta^2}{\xi (\xi^2 + \eta^2)} \right) \partial_\varphi + \left( -2\xi + \frac{\eta}{\xi^2 + \eta^2} \right) \partial_\varphi + 2\xi \partial_\sigma - 2\xi \partial_\sigma, \]
\[ \partial_\eta = r \left( 2\xi - \frac{\xi^2}{\eta (\xi^2 + \eta^2)} \right) \partial_\varphi + \left( 2\eta - \frac{\xi}{\xi^2 + \eta^2} \right) \partial_\varphi + 2\eta \partial_\sigma + 2\eta \partial_\sigma. \]
Taking into account that
\[ \cos 2\theta^p = \frac{1 - \tan^2 \theta^p}{1 + \tan^2 \theta^p} = \frac{\xi^2 - \eta^2}{\xi^2 + \eta^2}, \quad \sin 2\theta^p = \frac{2 \tan \theta^p}{1 + \tan^2 \theta^p} = \frac{2\xi \eta}{\xi^2 + \eta^2}, \]
we finally obtain
\[ \frac{1}{3} Y_1 = -r \cos 2\theta^p \partial_\xi + [2\sigma - \sin 2\theta^p] \partial_\varphi + 2\theta^p \partial_\varphi + 2\sigma \partial_\sigma + \frac{\pi}{2} \partial_\sigma = -X_\sigma^\theta + \frac{\pi}{2} X_\varphi^\theta. \]
For the solution with the lower sign, the operator \( Y_1 \) is transformed by analogy.

Subalgebra \( \{Y_1\} \) gives the Revuzhenko solution and is similar to subalgebra \( \Theta_1 \) when \( \alpha = 0 \). The case where \( \alpha \neq 0 \) will be considered in a future work.

3.2. \( \Theta_2 \): \( \sigma \)-translation and scaling

Let us take \( \alpha = -1/(2kc) \neq 0 \), then the invariants of \( \Theta_2 \) are: \( I_1 = \varphi, I_2 = \theta, I_3 = \sigma - 1/\alpha \ln r \) and the solution has the form
\[ \sigma = -2kc \ln r + f(\varphi), \quad \theta = \theta(\varphi). \]
Substituting the above form into (10), one can obtain the following system of ordinary differential equations:

\[ \theta' \sin 2(\theta - \varphi) = -c, \quad f' = 2kc \cot 2(\theta - \varphi). \]  

(33)

Integrating the first equation one can find the expression for \( \theta - \varphi \), then integrating the second equation, we obtain (\( c_i \in \mathbb{R} \))

\[ \sigma = -2kc \ln r - kc \ln [c + \sin 2(\theta - \varphi)] + \text{const}, \]  

(34)

\[ \theta = \varphi - \arctan \left( \frac{\sqrt{c^2 - 1}}{c} \tan \left[ \frac{\sqrt{c^2 - 1}}{c} (\theta + c_1) \right] - \frac{1}{c} \right), \quad c^2 > 1, \]  

(35)

\[ \theta = \varphi + \arctan \left( \frac{\sqrt{1 - c^2}}{c} \tanh \left[ \frac{\sqrt{1 - c^2}}{c} (\theta + c_2) \right] + \frac{1}{c} \right), \quad c^2 < 1. \]  

(36)

The slip-line equations are the following:

\[ r = k_1 \frac{e^{-\theta/c}}{\sqrt{c + \sin 2(\theta - \varphi)}}, \quad r = k_2 \frac{e^{\theta/c}}{\sqrt{c + \sin 2(\theta - \varphi)}}, \quad k_i \in \mathbb{R}, \]  

and \( \varphi = \varphi(\theta) \) is defined by (35) or (36).

The value of the constant \( c \) is important for the mechanical interpretation of stresses.

In the case \( c > 1 \), the families of slip-lines have the envelopes \( \varphi = \pm \alpha \). Here we have the well known Nadai solution [6, 14], which describes the flow of plastic material through the wedge-shaped converging channel (total angle \( 2\alpha \))

\[ \alpha + \frac{\pi}{4} = \frac{c}{\sqrt{c^2 - 1}} \arctan \frac{c + 1}{c - 1}, \quad \alpha \in (0, \frac{\pi}{2}). \]  

Two straight lines \( \varphi = \pm \alpha \) are boundaries of the channel. For more details on this solution see [27].

In the case \( c^2 < 1 \), there is no envelope for slip-line families. However, solution (34), (36) can still be used to describe a plastic state of the channel with the shear stress \( \tau_p = k \cos 2(\theta_{(1,2)}^{(1,2)} - \varphi_{(1,2)}^{(1,2)}) \) along straight-line borders \( \varphi = \varphi_{(1,2)}^{(1,2)} = \text{const}. \), because \( \theta^p \) is constant along the straight line \( \varphi = \text{const}. \) The relation between \( \theta_{(1,2)}^{(1,2)} \) and \( \varphi_{(1,2)}^{(1,2)} \) is given by (36).

Unfortunately, a new partially invariant solution for the angle \( \theta \) announced in [12] coincides with the above well known Nadai solution for the channel. A compatible velocity solution can be found, for example, in [6].

In the case \( c = 1 \), from (33) and after applying (31), we have a singular solution of the form

\[ \theta = \varphi + \frac{\pi}{4}, \quad \sigma = 2k \ln \frac{r}{R} + k - p, \]  

(37)

which is a well known Nadai solution describing the plastic state around a circular cavity of radius \( R \), situated in an infinite medium loaded by uniformly distributed pressure \( p \), with the tangential stress equal to zero. For more details on this solution see [25]. Let us note that the solution obtained in [12] as a partially invariant solution, corresponding to the subalgebra generated by operator \( K \sim X_4 \), coincides with (37). As for the velocity solution of (5), compatible with the Nadai solution see, for example, [26].

The non-singular solution of (33), when \( c = 1 \) is

\[ \sigma = -2k \ln r - k \ln [1 + \sin 2(\theta - \varphi)] + \text{const}. \]

\[ \varphi = \theta + \arctan \left( 1 + \frac{1}{\theta - A} \right), \quad A \in \mathbb{R}. \]
The case \( \alpha = 0 \) does not produce any invariant solution, because the necessary condition of existence of the invariant solution [19] is not satisfied.

### 3.3. \( \Theta_3 \): rotation and scaling

(a) The case \( \alpha = 0 \) corresponds to the rotation subgroup \( \langle \partial_{\varphi} + \partial_{\theta} \rangle \) and has the invariants

\[
I_1 = r, \quad I_2 = \sigma, \quad I_3 = \theta - \varphi,
\]

then the invariant solution is as follows:

\[
\sigma = f(r), \quad \theta = \varphi + g(r).
\]

After substituting into (10) we have:

\[
rf' = \frac{2k}{\sin 2g}, \quad rg' = \cot 2g.
\]

(38)

The second equation gives two solutions

\[
\cot g = \pm \sqrt{C_2^2 r^4 - 1}.
\]

Taking the boundary condition at \( r = R \) as \( \theta(R) = \varphi - \pi/2 \) we obtain \( C_2^2 = 1/R^4 \) and \( \cos 2g = -R^2/r^2 \), so \( C = -1/R^2 \). Integrating the first equation of (38) and taking the boundary condition \( \sigma(R) = -p = \text{const} \).

\[
\sigma = \pm k \ln \left( \frac{r^2}{R^2} + \sqrt{\frac{r^4}{R^4} - 1} \right) - p.
\]

For function \( \theta \)

\[
\theta = \varphi + \frac{1}{2} \arccos \left( -\frac{R^2}{r^2} \right) = \varphi + \frac{\pi}{2} - \frac{1}{2} \arccos \frac{R^2}{r^2}.
\]

(39)

Finally, taking \( \tan g = \left( \frac{r^2}{R^2} + 1 \right) / \sqrt{r^2/R^4 - 1} \) and applying reflection (31) to (39) we obtain the well known vortex flow Nadai solution [16]:

\[
\sigma = -k \ln \tan \left( \frac{1}{2} \arccos \frac{R^2}{r^2} + \frac{\pi}{4} \right) - p, \quad \theta = \varphi - \frac{\pi}{2} + \frac{1}{2} \arccos \frac{R^2}{r^2}.
\]

(40)

The boundary condition for the above solution is \( \tau_{\varphi} \big|_{r=R} = k \cos 2\theta_p = k \cos 2g(R) = -k, \sigma \big|_{r=R} = -p \). The homotopy of the above solution with the Prandtl solution (54) was analyzed in [27]. The generalization of the Nadai solution was obtained by Mikhlin [13] for \( \tau_{\varphi} = q, \ |q| < k \), when \( r = R \). For the corresponding velocity solution see [26].

(b) In the case \( \alpha \neq 0 \), the invariants of the operator \( \langle \partial_{\varphi} + \partial_{\theta} + \alpha r \partial_r \rangle \) are as follows:

\[
I_1 = z = re^{-\alpha \varphi}, \quad I_2 = \sigma, \quad I_3 = -\theta + \frac{\varphi}{2} + \frac{1}{2\alpha} \ln r
\]

and the form of the invariant solution is

\[
\sigma = f(z), \quad \theta^r = \frac{\varphi}{2} + \frac{1}{2\alpha} \ln r = \varphi + \frac{1}{2\alpha} \ln z \sim \theta^p = \frac{1}{2\alpha} \ln z.
\]

Substituting this form into system (10) gives a compatibility condition \( \alpha^2 = -1 \). This means that \( z = x \pm iy \) and there is no real solution.
3.4. $\Theta_4^{(+)}$: $\sigma$-translation, rotation and scaling

(a) Let $\alpha = 0$, $\Theta_4^{-} = (\partial_x - \partial_y - \partial_z)$. The form of the invariant solution is

$$\sigma = -A \varphi + f(r), \quad \theta^c = \varphi + g(r), \quad A > 0,$$

which coincides with (23), taking into account that $g(r) = \theta^p(r)$. The Nadai solution for two concentric circles is an invariant one with respect to the operator of $\Theta_4^{(-)}$.

(b) If $\alpha = 0$ and $\Theta_4^{(+)} = (\partial_x + \partial_y + \partial_z)$ is taken, the form of the invariant solution is

$$\sigma = A \varphi + f(r), \quad \theta^c = \varphi + g(r), \quad A > 0,$$

which is dual to solution (23)–(25), in the sense that families of the characteristics change between themselves, and there is no new solution.

(c) $\alpha \neq 0$. In such a case, the invariant, relating independent variable is $\lambda = re^{-\alpha \varphi}$. Taking into account that $A \partial_\sigma$ is a symmetry, let us consider the following form of the invariant solution

$$\sigma = A \varphi + f(\lambda), \quad \theta^c = \varphi + g(\lambda), \quad (41)$$

where $A = \text{const.} \neq 0$ and $g = \theta^p$. Let us note that if $A = \pm 2k$ there is a simple wave solution, considered later (see subsection 3.5).

Substituting (41) into (10) gives two conditions on functions $f$ and $g$:

$$f' = \frac{A}{\lambda} \frac{2k/A + \cos 2\varphi - \alpha \sin 2\varphi}{2\alpha \cos 2\varphi + (1 - \alpha^2) \sin 2\varphi},$$

$$g' = \frac{1}{\lambda} \frac{A/(2k) + \cos 2\varphi - \alpha \sin 2\varphi}{2\alpha \cos 2\varphi + (1 - \alpha^2) \sin 2\varphi},$$

which are two separable equations with a closed-form solution.

For simplicity let us take $\alpha = 1$. Then function $g$ is defined implicitly by relation

$$\ln \lambda = \int \frac{2 \cos 2\varphi}{A/(2k) + \cos 2\varphi - \sin 2\varphi} \, dg = I_2(g) + \text{const.}$$

Relations (9) along characteristic lines take the form:

$$\frac{d \ln r}{d \varphi} = -\tan g, \quad \frac{d \ln r}{d \varphi} = \cot g.$$

But $d \ln r/d\varphi = d \ln \lambda/d\varphi + 1$, so for the first family we have

$$r = e^{\varphi + I_2(g)}, \quad \varphi = I_1(g) + C_1, \quad (42)$$

and the second family is defined by

$$r = e^{\varphi + I_2(g)}, \quad \varphi = \tilde{I}_1(g) + C_2, \quad (43)$$

where $C_i$ are the constants of the characteristic lines, and functions $I_1$ and $\tilde{I}_1$ are as follows:

$$I_1(g) = -\int \frac{2 \cos 2\varphi}{A/(2k) + \cos 2\varphi - \sin 2\varphi \tan g + 1} \, dg,$$

$$\tilde{I}_1(g) = \int \frac{2 \cos 2\varphi}{A/(2k) + \cos 2\varphi - \sin 2\varphi \cot g + 1} \, dg.$$
Figure 2. Slip-line field for the spiral solution which describes the plastic state in the curvilinear channel, bounded by a lower curve (44) and an upper curve (45). The first family of slip-lines, having (45) as the envelope, is given by parametric relations (42) with $C_1 = -0.2, -0.4, \ldots$. The second family of slip-lines is determined by relations (43) with $C_2 = 0, -0.2, \ldots$. The boundary conditions are: $\sigma|_{\Gamma_1} = \sqrt{2}\varphi + f(\lambda_1)$, $\tau_n|_{\Gamma_1} = -1/2$ and $\sigma|_{\Gamma_2} = \sqrt{2}\varphi + f(\lambda_2)$, $\tau_n|_{\Gamma_2} = 1/2$.

In figure 2 one can observe two families of slip-lines and their corresponding envelopes:

$$\Gamma_1 : r = \lambda_1 e^{\varphi}, \quad \lambda_1 = \exp\left(\frac{\pi - \sqrt{2}}{4} - \frac{1}{2(2 - \sqrt{2})} + \frac{1}{4} \ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right), \quad (44)$$

$$\Gamma_2 : r = \lambda_2 e^{\varphi}, \quad \lambda_2 = \exp\left(-\frac{\pi}{8} - \frac{3}{4} \ln 2\right), \quad (45)$$

which are two logarithmic spirals. To simplify the above integrals we take $A = 2\sqrt{2}k$, then functions $I_1$, $\bar{I}_1$ and $I_2$ can be expressed in terms of simple functions.

It seems that the spiral solution was first mentioned by Hartmann [5]. Unfortunately, this work was not published, but one can find some results in [17]. In works [1] and [2] the stresses were described. Here we give the corresponding slip-line field.

3.5. $\Theta_3$: scaling transformation

The form of the invariant solution is

$$\sigma = \sigma(\varphi), \quad \theta = \theta(\varphi).$$

From the first equation of (10) one can obtain $\theta - \varphi = \pi n/2$, $n \in \mathbb{Z}$. The second equation gives

$$\sigma = (-1)^{n+1}2k\varphi + \text{const.} = (-1)^{n+1}2k\theta^c + \text{const.} \quad (46)$$

In other words, there are simple wave solutions, which are well known [9] and which are called simple stress states. Let us note that in Cartesian coordinates, these solutions correspond to the so-called similarity (or dimensionless) ones, i.e. they depend on the $y/x$ variable only. Unfortunately, this was not seen in [12]. The solution obtained by the symmetry reduction for the representative subalgebra of symmetry $B_1 = -v\partial_x + u\partial_\varphi$ has the form (46).
For example, in the case $\sigma = 2k\theta + \text{const.}$ from (1), function $\theta(x, y)$ is defined by the following relation:

$$x \cos \theta + y \sin \theta = \Phi(\theta),$$

(47)

where $\Phi(\theta)$ is an arbitrary function. Of course, one can take different forms of $\Phi$ (as in [11, 12]) to define some solution, but all these solutions correspond to one characteristic family, which is a family of straight lines $\theta = C_1$

$$x \cos C_1 + y \sin C_1 = \Phi(C_1),$$

(48)

with the relation (14) along the characteristic $\frac{dy}{dx} = -\cot C_1$. From (47), taking into account the second relations along the characteristic $\frac{dy}{dx} = \tan \theta$, one can obtain the linear equation

$$\frac{dx}{d\theta} - \cot \theta \left(x + f'(\theta) \sin^2 \theta\right) = 0,$$

where $f(\theta) = \Phi(\theta)/\sin \theta$, with the solution

$$x = \sin \theta \left(\int f' \cos \theta \, d\theta + \tau\right), \quad \tau = \text{const.}$$

(49)

The family of characteristics (47), (49) is orthogonal to the family (48). Equation

$$x + f'(\theta) \sin^2 \theta = 0$$

(50)

defines the envelope of straight lines. All the above relations are due to [10].

The velocity fields for simple stress states have some trivial properties [26]. To construct these fields let us use components $U$ and $V$ of the velocity along characteristic directions (6):

$$u = U \cos \theta - V \sin \theta, \quad v = U \sin \theta + V \cos \theta.$$

The velocity component along each straight line is constant. Thus, in the case of solution (47) $U = \text{const.}$ along the characteristic line and for $V$ from (5) we obtain the equation

$$\frac{\partial V}{\partial x} \sin \theta - \frac{\partial V}{\partial y} \cos \theta = 0,$$

with the general solution $V(x, y) = V(J_1)$, where $V$ is an arbitrary function of $J_1 = x \cos \theta + y \sin \theta$. Finally,

$$u = U(\theta) \cos \theta - V(J_1) \sin \theta, \quad v = U(\theta) \sin \theta + V(J_1) \cos \theta.$$  

If the second family is a family of straight lines, then by analogy, we have

$$u = U(J_2) \cos \theta - V(\theta) \sin \theta, \quad v = U(J_2) \sin \theta + V(\theta) \cos \theta,$$

where $J_2 = x \sin \theta - y \cos \theta$, and $U, V$ are arbitrary functions of their arguments.

Let us note that the form of the solution, corresponding to simple wave, can be quite complicated. For example, one can verify that

$$\sigma = 2k\theta^c + \text{const.}, \quad r \cos(\theta^c - \varphi) = Ce^{\theta^c}$$

(51)

is a solution of (10). Then, $\Phi(\theta) = C \exp(\theta^c)$ and from (50) one can obtain a logarithmic spiral $\Gamma: r = C \sqrt{2} e^{\pi/4}$ as the envelope for the family of straight lines $r \cos(C_1 - \varphi) = Ce^{\theta^c}$. The second family of characteristics in the form of spirals is defined by the following parametric relations:

$$r \cos \varphi = Ce^{\theta^c} (\sin \theta^c + \cos \theta^c) + \tau \sin \theta^c,$$

$$r \sin \varphi = Ce^{\theta^c} (\sin \theta^c - \cos \theta^c) - \tau \cos \theta^c,$$

(52)

where $\tau$ is the number of the spiral and $\theta^c$ is the parameter.

In figure 3 one can observe the corresponding slip-line field, describing the plastic state near the contour $\Gamma$, loaded by constant tangential stress $\tau_n = -k$ and the normal stress has the form $\sigma_n|\Gamma = 2k\varphi$. Let us note that $\theta|\Gamma = -\pi/4$. 

13
3.6. Infinite part of the symmetries

The classic Prandtl solution [20] for system (2), (3) has the form

\[
\sigma_x \frac{k}{k} = -c - \frac{m x}{h} + 2 \sqrt{1 - \frac{m^2 y^2}{h^2}},
\]
\[
\sigma_y \frac{k}{k} = -c - \frac{m x}{h},
\]
\[
\tau_{xy} \frac{k}{k} = \frac{m y}{h},
\]

where \( h > 0, 0 \leq m \leq 1 \) and \( c \) are constants. This solution describes the stress state of a thin plastic block of height \( 2h \), compressed between rough parallel plates and it satisfies the following boundary conditions:

\[
\sigma / k |_{y=\pm h} = -c - \frac{m x}{h} + \sqrt{1 - m^2}, \quad \tau_{xy} |_{y=\pm h} = \pm mk. \tag{53}
\]

Let us take \( c = 0, 2k = 1, h = 1 \) and \( m = 1 \) (corresponding to perfectly rough plates). Then, in terms of functions \( \sigma, \theta \) we have:

\[
2\sigma = -x + \sqrt{1 - y^2}, \quad y = \cos 2\theta. \tag{54}
\]

This solution is invariant with respect to the subalgebra of the form \( \{\partial_x - 2\partial_y\} \), where operator \( \partial_x \) is of \( X_5 \) form.

Equations for the slip-line families (\( \theta \) is the parameter) appear as follows:

\[
x = \mp 2\theta + \sqrt{1 - y^2} + \text{const.}, \quad y = \cos 2\theta,
\]

and represent two orthogonal families of cycloids.

Let us consider the solution, proposed in [11] in a form of propagation wave

\[
\sigma = f(\xi), \quad \theta = g(\xi), \quad \xi = a_1 x + a_2 y,
\]

where \( a_i \) are constants. It is easy to see that this solution is invariant with respect to the operator \( X = a_2 \partial_x - a_1 \partial_y \), which is of \( X_5 \) form. The Lie algebra \( L_5 \) with the basis \( \{X_1, X_2, X_3, \partial_x, \partial_y\} \), considered in [2], has the non-similar subalgebra \( \Theta_0 = \{\partial_x + \gamma X_1\}, \gamma \in \mathbb{R} \). If \( a_1 \neq 0 \), then by
the rotation transformation, \( a_1 \to 0 \) and \( X \sim \partial_x \) \( \in \Theta_0|_{y=0} \). That is why one can take \( a_1 = 0 \), \( a_2 = 1 \), and the form of the invariant solution is \( \sigma = f(y), \theta = g(y) \), which gives the trivial constant solution.

Another invariant, considered in [11] is \( \tau = \sigma + a_1 x + a_2 y \), which corresponds to the operator \( X = 2a_1 \partial_x - \partial_y - a_1/a_2 \partial_y \). In the same way we obtain \( X \sim 2\partial_x - \partial_y \in \Theta_0|_{y=0} \), which produces the Prandtl solution (54), so there is no new invariant solution.

4. The velocity field

The Lie algebra of operators, admitted by (1), (5) is known (see [2, 12]). Substituting solution (4.1), (4.5) is another velocity solution for Nadai stresses.

However, there is another way of obtaining invariant solutions for velocities, based on the fact that \( u = -y \), \( v = x \) is the solution of (5). Substituting \( x = v \), \( y = -u \) to a stress solution \( \sigma = \sigma_0(x, y), \theta = \theta_0(x, y) \) of system (1), one obtains relations of the form

\[
F_1(\sigma, \theta, u, v) = 0, \quad F_2(\sigma, \theta, u, v) = 0.
\]

Changing in the above equations \( \sigma = \sigma_1(x, y), \theta = \theta_1(x, y) \), \( \sigma_1 \neq \sigma_0, \theta_1 \neq \theta_0 \) is the invariant stress solution, we obtain the invariant solution for \( u \) and \( v \).

For the polar coordinates, let us take a solution for stresses: \( \sigma = \sigma_0(r, \varphi), \theta = \theta_0(r, \varphi), \psi = \psi_0(\sigma, \theta) \) (or take them in implicit form) and put \( \rho = r_0(\sigma_1(x, y), \theta_1(x, y)), \psi = -\psi_0(\sigma_1(x, y), \theta_1(x, y)) \) in order to obtain the corresponding invariant velocity solution

\[
u = \rho \sin \psi, \quad v = \rho \cos \psi.
\]

The above method does not require integration. Only algebraic operations for functions are used. To illustrate it, let us consider the Nadai solution (37) as \( (\sigma_0, \theta_0) \). It has the classic velocities: \( v_r = r^{-1}, v_\varphi = 0 \), where \( (v_r, v_\varphi) \) are radial and angular components, related to \( u, v \) in the following, well known way:

\[
u = v_r \cos \varphi - v_\varphi \sin \varphi, \quad v = v_r \sin \varphi + v_\varphi \cos \varphi.
\]

Let us take the Prandtl solution (54) as \( (\sigma_1, \theta_1) \). Then

\[
2\sigma_1 = -r \cos \varphi + \sqrt{1 - r^2} \sin^2 \varphi = -v + \sqrt{1 - u^2} \to 2\sigma_0 = 4k \ln r,
\]

\[
\theta_1 = -\frac{1}{2} \arccos(r \sin \varphi) = \frac{1}{2} \arccos u - \frac{\pi}{2} \to \theta_0 = \varphi + \frac{\pi}{4},
\]

then

\[
u = \sin 2\varphi = \frac{2xy}{x^2 + y^2}, \quad v = -2k \ln r^2 - \cos 2\varphi = -\ln(x^2 + y^2) - \frac{x^2 - y^2}{x^2 + y^2}
\]

is another velocity solution for Nadai stresses.

Any invariant stress solution of the finite part of algebra \( L_{xy} \) produces an invariant velocity solution for the Prandtl solution, constructed on the base of the \( X_5 \) operator.

From (54) we have

\[
x = -2\sigma - \sin 2\theta, \quad y = \cos 2\theta.
\]
Taking into account that \( \partial_y = -2\partial_x, \partial_x = -2y\partial_y + 2\sqrt{1 - y^2}\partial_x \), we have the following Lie algebra \( L_{uv} \) of point symmetries:

\[
Z_1 = u\partial_u + v\partial_v, \quad Z_2 = -2y\partial_y + 2\sqrt{1 - y^2}\partial_y - v\partial_u + u\partial_v, \quad Z_3 = \partial_x,
\]

\[
Z_4 = 2[\arccos y + y(x - \sqrt{1 - y^2})]\partial_x + 2(-x + \sqrt{1 - y^2})\sqrt{1 - y^2}\partial_y - (xy + yv)\partial_v
\]

\[
+ [yu + v(x - 2\sqrt{1 - y^2})]\partial_u,
\]

\[
Z_5 = y\partial_y - xu\partial_x, \quad Z_6 = u_0(x, y)\partial_u + v_0(x, y)\partial_v,
\]

where \( u_0, v_0 \) is an arbitrary solution of (5). As in the case of the Lie algebra \( L_{xy} \), algebra \( L_{uv} \) contains the infinite-dimensional operator \( Z_6 \), which corresponds to the principle of linear superposition of solutions of linear equations. Symmetry \( Z_2 \) is an analogue of the rotation group and \( Z_4 \) is an analogue of quasi-scaling transformation.

It is clear that Lie algebra \( L_{xy} \), \( X_1, ..., X_3 \) of the symmetries of (1), gives symmetries of the whole system (1), (5) under change \( x \rightarrow v, y \rightarrow -u \). Thus, operator \( B_1 = -v\partial_v + u\partial_u \), named as a new one in [12], represents operator \( -Z_5 = -y\partial_y + xu\partial_x \) of the form \( Z_6 \). Finally, we can say that symmetry analysis of the velocity system (5) is secondary with respect to the analysis of the stress system (1).

The non-zero commutators for the finite part of Lie algebra \( L_{uv} \) are as follows:

\[
[Z_2, Z_4] = -2Z_3, \quad [Z_3, Z_4] = -Z_2,
\]

and for the infinite part we have

\[
[Z_1, Z_3] = -Z_5, \quad [Z_3, Z_5] = -\partial_x,
\]

\[
[Z_2, Z_5] = (-x + 2\sqrt{1 - y^2})\partial_u + y\partial_v,
\]

\[
[Z_4, Z_5] = (x^2 - 3y^2 - 4x\sqrt{1 - y^2} + 2)\partial_u + (-2\arccos y - 2xy + 2y\sqrt{1 - y^2})\partial_v.
\]

### 4.1. Infinite subalgebra

It is interesting to note that relation (59) produces the well known Nadai solution for velocities [6]:

\[
u = -y,
\]

and from (60) we have \((C_i \in \mathbb{R})\):

\[
u = 2y\sqrt{1 - y^2} - 2\arccos y - 2xy + C_2.
\]

Using the scaling transformation \( \tilde{u} = au, \tilde{v} = av \), corresponding to the operator \( Z_1 \), one can multiply the above relation by the same factor \( a \neq 0 \). And with the help of translations \( \partial_u, \partial_v \in Z_6 \) we can add any constant to the velocity components.

In general, any combination of commutators of the form \([Z_{2,4}, [Z_{2,4}, ..., [Z_{2,4}, Z_5] ...]\) produces solutions for velocities. Thus, commutator \([Z_2, [Z_4, Z_5]]\) gives

\[
u = 2xy - 2\arccos y - 2y\sqrt{1 - y^2} + C_1,
\]

which coincides with Ivlev–Senashov [2] solution of the form \((c_i \in \mathbb{R})\)

\[
u = xy + \arcsin y - y\sqrt{1 - y^2} + c_1, \quad v = -(x^2 + y^2)/2 + c_2.
\]

Taking into account that \( \arcsin y = \pi/2 - \arccos y \). Let us note that the above solution is invariant with respect to the operator \((Z_3 + Z_5)\), and the Nadai solution is the invariant one for the operator \((Z_3 + \partial_u)\).
Let us give a mechanical interpretation. The Nadai solution (59) satisfies linear boundary conditions for \( u \) on the edges \( y = \pm 1 \) of the plates, while \( v \) is a constant:

\[
\begin{align*}
  u(x, \pm 1) &= x, \\
  v &= \mp 1.
\end{align*}
\]

(64)

If \( C_1 = 3, C_2 = \pi \), then solution (62) satisfies the simple symmetric boundary conditions on the plates \( y = \pm 1 \)

\[
\begin{align*}
  u(x, \pm 1) &= x^2, \\
  v(x, \pm 1) &= \mp (2x - \pi),
\end{align*}
\]

(65)

in other words, solution (62), in the sense of boundary conditions, is a generalization of the Nadai solution, because now for the velocity \( u \) we have quadratic dependence on \( x \), and for \( v \) the dependence is linear.

One of the main properties of plastic deformation is indicated in [9]: the dissipation must be positive in the plastic zone, because a plastic deformation is accompanied by irreversible energy consumption. This is a condition of the compatibility of stress and velocity fields. The following condition ensures the non-negativity of the plastic dissipation of energy:

\[
\begin{align*}
  \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \sigma_x - \sigma_y &= 0, \\
  \sigma_x = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \\
  \sigma_y = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}, \\
  \tau_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.
\end{align*}
\]

(66)

If the above inequality is not satisfied, then boundary conditions are wrongly formulated from the mechanical point of view. Thus, for the Nadai solution condition (66) is satisfied for any \( x : 2/\sqrt{1 - y^2} > 0 \). For solution (62) the plastic dissipation is non-negative if

\[
x \geq 2\sqrt{1 - y^2},
\]

(67)

i.e. when Nadai solution component \( u \geq 0 \).

To analyze the complete solution of (1), (5), one needs to know both the slip-line field and the velocity field. To construct streamlines for the known components of the velocity vector field, we can solve the equation

\[
\frac{dx}{u} = \frac{dy}{v}.
\]

(68)

Thus, for the Nadai solution the equation of the streamlines is

\[
x = y\sqrt{1 - y^2} + \arcsin y + \text{const},
\]

(69)

and for solution (62) it appears as follows

\[
2x \arccos y + xy^2 - 2xy\sqrt{1 - y^2} - \pi x - y^3 + 3y = \text{const}.
\]

(70)

The theory of plane plastic strain does not involve the time, and calculated stresses do not depend on the rate of strain. The progress of the deformation can be expressed by monotonically varying quantity: a load, an angle or, as in case of the Prandtl solution, one can take the height of the block (see [26]). Streamlines coincide with the path of material particles and help to visualize the flow. One can observe vector fields and streamlines for the Nadai solution and for solution (62) in figures 4 and 5, respectively.

Due to linearity of the velocity equations there is an infinite number of their solutions. The following condition can be useful for selecting the appropriate one. It is known [8], that for the real velocity field, due to the modified Mises criterion, the dissipation

\[
D = \sigma_x e_x + \sigma_y e_y + 2\tau_{xy} \gamma_{xy},
\]

(71)

where \( e_x = \partial u/\partial x, e_y = \partial v/\partial y \) and \( 2\gamma_{xy} = \partial v/\partial x + \partial u/\partial y \) are corresponding strains, should take a maximum value: \( D_{\text{real}} \geq D_{\text{possible}} \).
Figure 4. The Nadai velocity solution for Prandtl stresses with boundary conditions (53), (64) with $\epsilon = 0$, $2k = 1$, $h = 1$ and $m = 1$. Solid lines represent the streamlines (69).

Figure 5. Velocity solution (62) for Prandtl stresses with boundary conditions (53), (65). Solid lines are streamlines (70).

Thus, for the Nadai solution $D = 1/\sqrt{1-y^2}$ and for solution (62) $D = 2(x-2\sqrt{1-y^2})/\sqrt{1-y^2}$, therefore, when $x > 1/2 + 2\sqrt{1-y^2}$, the dissipation of solution (62) is greater than dissipation of the Nadai solution.

Let us note that the Prandtl solution is a good approximation for experimental data only at sufficiently great distance from the center [6]. The same is true for the velocity field.

4.2. Invariant solutions

Let us consider the finite part of Lie algebra $L_{\text{sym}}$. Noting that commutators (58) coincide with (30), when $k = 1$, one can conclude that the optimal systems are similar:

$$
\Theta_1 = \langle Z_4 + \alpha Z_1 \rangle, \quad \Theta_2 = \langle Z_3 + \alpha Z_1 \rangle, \quad \Theta_3 = \langle Z_2 + \alpha Z_1 \rangle,
\Theta_4^{(b)} = \langle Z_3 \pm Z_2 + \alpha Z_1 \rangle, \quad \Theta_5 = \langle Z_1 \rangle.
$$

This is obvious, because we have made a non-singular change of variables (57), which does not change the structure of the Lie algebra. If we take (55), then operators $Z_i$ are equivalent to (29) with the change $\rho \rightarrow r$, $\psi \rightarrow -\varphi$ and taking into account (54).
4.2.1. $\Theta_1$: rotation symmetry. Let us consider $\alpha = 0$ (only rotation operator), then operator $Z_2$ takes the form

$$\tilde{Z}_2 = -2y\partial_x + 2\sqrt{1 - y^2}\partial_y - \partial_\psi.$$  (72)

Its invariants are $I_1 = z = x - \sqrt{1 - y^2}$, $I_2 = \psi + 1/2 \arcsin y$, $I_3 = \rho$ and the form of the invariant solution is

$$\rho = f(z), \quad \psi = g(z) - \frac{1}{2} \arcsin y.$$  

Substituting the above relations to (5) and simplifying, we obtain the system

$$g' + \frac{1}{2} \sin 2g = 0, \quad f' - \frac{1}{2} \cos 2g = 0.$$  

Putting $g = 1/2 \arccos h(z)$ the above system takes the form

$$h' = 1 - h^2, \quad f' = \frac{1}{2} h f,$$

so

$$h = \tanh(z + c_1), \quad f = c_2 \sqrt{\cosh(z + c_1)}, \quad c_i = \text{const.}$$

and

$$u = c_2 \sqrt{\cosh(z + c_1)} \sin \left(\frac{1}{2} \arccos \tanh(z + c_1) - \frac{1}{2} \arccos y\right),$$

$$v = c_2 \sqrt{\cosh(z + c_1)} \cos \left(\frac{1}{2} \arccos \tanh(z + c_1) - \frac{1}{2} \arccos y\right).$$

After simplification we obtain

$$u^2 = \frac{c_2^2}{2} (\cosh(z + c_1) - y \sinh(z + c_1) - \sqrt{1 - y^2}),$$

$$v^2 = \frac{c_2^2}{2} (\cosh(z + c_1) + y \sinh(z + c_1) + \sqrt{1 - y^2}).$$  (73)

Let us apply the algorithm with no integration. Let us take the vortex flow Nadai solution (40), invariant with respect to the rotation operator. Then:

$$\rho^2 = r^2 = R^2 \cosh(\sigma + p), \quad \psi = -\psi = -\theta + \arctan e^{-2(\sigma + p)} - \frac{3\pi}{4},$$

and

$$u^2 = \frac{R^2}{2} (\cosh(\sigma + p) + \sqrt{1 - y^2} \sinh(\sigma + p) + y),$$

$$v^2 = \frac{R^2}{2} (\cosh(\sigma + p) - \sqrt{1 - y^2} \sinh(\sigma + p) - y).$$  (74)

Let us note that solution (73) goes to solution (74), under the change $\psi \rightarrow \psi + \pi/4$, $\theta \rightarrow \theta - \pi/4$ (corresponding to the rotation transformation) and taking into account that $z = -2\sigma$.

4.2.2. $\Theta_4$: x-translation and rotation. Let us consider the case of $\Theta_4^{(+)}$, where $\alpha = 0$. Then the symmetry operator in terms of variables (55) is slightly different from (72) and appears as follows

$$(1 - 2y)\partial_x + 2\sqrt{1 - y^2}\partial_y - \partial_\psi.$$  

The form of the invariant solution is

$$\rho = f(z), \quad \psi = -\frac{1}{2} \arcsin y + g(z), \quad z = x - \sqrt{1 - y^2} - \frac{1}{2} \arcsin y.$$
To determine functions $f$, $g$ by analogy with the above section we come to the system
\[
\left(\frac{1}{2} - \sin 2g\right) f' - f g' \cos 2g = 0, \quad f' \cos 2g - \left(\frac{1}{2} + \sin 2g\right) f g' = \frac{f}{2},
\]
with the solution in quadratures
\[g = \arctan \left[ 2 - \sqrt{3} \tanh \left( \frac{\sqrt{3}}{3} z + \text{const.} \right) \right], \quad \ln f = \frac{2}{3} \int \cos 2g(z) \, dz.
\]

In the case where $\alpha \neq 0$, one can use the algorithm and spiral stress solution in form (41) to obtain the invariant velocity solution.

4.2.3. $\Theta_2$: $x$-translation and scaling. The case $\alpha = 0$ gives a trivial solution $u, v = \text{const.}$ Let us consider $\alpha \neq 0$, then the invariant solution appears as follows:
\[u = e^{\alpha y} f(y), \quad v = e^{\alpha y} g(y).
\]
This form was proposed in [2], but the solution was not determined.

For functions $f$, $g$ there is the system:
\[\sqrt{1 - y^2} \left( f' + \alpha g \right) + y \left( g' - \alpha f \right) = 0, \quad \alpha f + g' = 0.
\]
Eliminating $f$, we obtain the linear equation for $g$
\[g'' - 2\alpha \frac{y}{\sqrt{1 - y^2}} g' - \alpha^2 g = 0 \]
with the particular solution (when $\alpha = -1/2$) of the form
\[g_1(y) = e^{\frac{y}{\sqrt{1 - y^2}}} \left( 1 - \sqrt{1 - y^2} \right)^{\frac{1}{2}}.
\]
The second particular solution can be found in a well known way and is as follows:
\[g_2(y) = \frac{g_1(y)}{y} \left( 1 + \sqrt{1 - y^2} + y \arcsin y \right).
\]
Finally, the solution in the form (75) is
\[u(x, y) = y \frac{e^{\frac{y}{\sqrt{1 - y^2}}} \left[ c_1 + \frac{c_2}{y} \left( -1 + \sqrt{1 - y^2} + y \arcsin y \right) \right]}{(1 - \sqrt{1 - y^2})^{\frac{3}{2}}},
\]
\[v(x, y) = \frac{e^{\frac{y}{\sqrt{1 - y^2}}} \left[ c_1 + \frac{c_2}{y} \left( 1 + \sqrt{1 - y^2} + y \arcsin y \right) \right]}{(1 - \sqrt{1 - y^2})^{\frac{3}{2}}},
\]
In the case where $c_1 \neq 0, c_2 = 0$, from equation (68) we obtain a family of streamlines
\[x + \text{const.} = \sqrt{1 - y^2} + \ln(1 - \sqrt{1 - y^2})
\]
and boundary conditions $u(x, \pm 1) = \pm e^{-s/2}, v(x, \pm 1) = e^{-s/2}$.
Let us note that this case corresponds to the singular solution (37), for which
\[\rho = C e^{\psi r}, \quad \psi = -\varphi = \frac{\pi}{4} - \theta_p,
\]
and the corresponding velocities are quite simple:
\[u = \exp \left( -\frac{x}{2} + \frac{\sqrt{1 - y^2}}{2} \right) \left( \sqrt{1 + y} + \sqrt{1 - y} \right),
\]
\[v = \exp \left( -\frac{x}{2} + \frac{\sqrt{1 - y^2}}{2} \right) \left( \sqrt{1 + y} - \sqrt{1 - y} \right).
\]
However, the case $c_1 = 0$, $c_2 < 0$ seems to be more interesting from a mechanical point of view, because in such a case the boundary conditions are:

$$u(x, \pm 1) = c_2 (\pi/2 - 1) e^{-x/2}, \quad v(x, \pm 1) = \pm c_2 (1 + \pi/2) e^{-x/2},$$

(76)

i.e. the plates are coming close to one another with the same velocity.

Moreover, the dissipation function (71) has the form

$$D = -c_2 \sqrt{1 - y^2} - 1 + y \arcsin y + \frac{1}{2} \sqrt{1 - y^2} e^{i(-x + \sqrt{1 - y^2})}$$

(77)

and is non-negative for any $y \in [-1, 1]$, so $D \geq 0$ along the whole plastic block.

The velocity field and a family of streamlines are shown in figure 6. Let us note that streamlines appear as in the experimental data, indicated in [16].

4.2.4. $\Theta_1$: quasi-scaling symmetry. In the case where $\alpha = 0$, the operator $Z_4$ in terms of the variables $\rho, \psi$ (55), $\theta = -1/2 \arccos y$ and $-2\sigma = x + \sin 2\theta$ takes the form

$$Z_4 = 2\theta \partial \rho + 2\sigma \partial \rho - \cos(2\theta + 2\psi) \rho \partial \rho + [\sin(2\theta + 2\psi) - 2\sigma] \partial \psi \sim X_{p}^\rho.$$

The invariant for independent variables is $z = \sigma^2 - \theta^2$. However, there are some difficulties in directly finding invariants for the dependent variables $\rho, \psi$. Let us apply the algorithm exposed at the beginning of subsection 4.2. Considering the Revuzhenko solution (19), (21), (22) and taking

$$\xi^2 = \frac{\sigma - \theta}{2} - \frac{\pi}{8} = \frac{1}{4} (-x + \sqrt{1 - y^2} - \arcsin y),$$

$$\eta^2 = \frac{\sigma + \theta}{2} + \frac{\pi}{8} = \frac{1}{4} (-x + \sqrt{1 - y^2} + \arcsin y),$$

we have

$$\rho(x, y) = e^{i22\pi} (\xi^2 + \eta^2)^{1/2}, \quad \psi(x, y) = \xi^2 - \eta^2 + \frac{\pi}{4} \pm \arctan \frac{\eta}{\xi},$$

and from (55) we obtain the invariant velocities $u, v$ for Prandtl stresses, associated with the quasi-scaling point transformation.

Subalgebra $\Theta_2$ does not produce any form of invariant solution.
5. Conclusions

For the first time, the Revuzhenko solution is interpreted as an invariant one with respect to the quasi-scaling transformation. The direct construction of the mechanically significant solution, invariant with respect to the quasi-scaling transformation is quite difficult, because the general solution of the corresponding factor-system of ordinary differential equations is expressed in terms of Bessel functions. Moreover, the symmetry group is pointed out, which gives the Nadai solution for two concentric circles.

All of the known classical solutions of the plane perfect plasticity system are invariant with respect to some group of point symmetries. One can observe that for different values of the parameters, involved in one non-similar subalgebra, there are different solutions from a mechanical point of view. The equations of slip-line families for all solutions are constructed, which allows one to explicitly determine the boundaries of plastic areas.

It is shown how for known stresses, one can determine the compatible velocity solution, considering symmetries. It seems there are no advantages in looking for invariant solutions using symmetries of a complete system (stresses and velocities) in comparison to the traditional method exposed in this paper: firstly to solve the system for stresses, substitute its known solution into the system for velocities and determine the invariant velocity solutions. Moreover, the algorithm to construct invariant velocity solutions without integration is proposed. Thus, for Prandtl stresses, it is possible to obtain all invariant velocity solutions.

As one can see, the streamlines of the velocity field even for the simplest Prandtl solution can be different. However, the slip-line field is unique and it defines the region of the plastic state, bounded by envelopes of slip-line families (or sometimes by a characteristic line). That is why, when one gives a mechanical interpretation of the obtained solution it is necessary to consider both fields simultaneously.

For the Prandtl solution the well known Nadai velocity solution is compared to some other solutions. A new velocity field in the form of the exponential function is determined. It is shown that the Nadai velocity solution is not preferable in the whole plastic area. There are other velocity fields where the dissipation is greater. Due to the infinite number of solutions, the problem of the construction of real velocities for Prandtl solutions is still open.

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References

[1] Annin B D 1978 A new exact solution of equations of the plane problem of ideal plasticity with von Mises condition Euromech III: Symp. on Constitutive Modelling in Inelasticity (Marianske Lazne, Czechoslovakia, 26–28 Sept. 1978) pp 6–8
[2] Annin B D, Bytev V O and Senashov S I 1985 Group Properties of the Equations of Elasticity and Plasticity (Novosibirsk: Nauka) (in Russian)
[3] Cantwell B J 2002 Introduction to Symmetry Analysis (Cambridge: Cambridge University Press)
[4] Druyanov B A and Nepershin R I 1994 Problems of Technological Plasticity (Amsterdam: Elsevier)
[5] Hartmann W 1925 Über die Integration der Differential Gleichungen des ebene Gleichgewichtszustandes für den allgemein-plastischen Körper Thesis University of Gottingen, Germany
[6] Hill R 1950 The Mathematical Theory of Plasticity (Oxford: Oxford University Press)
[7] Hlaváč A and Marvan M 2013 Another integrable case in two-dimensional plasticity J. Phys. A: Math. Theor. 46 045203
[8] Ivlev D D 1966 Theory of Ideal Plasticity (Moscow: Nauka) (in Russian)
[9] Kachanov L M 2004 *Fundamentals of the Theory of Plasticity* (Dover Books on Engineering) (New York: Dover)

[10] Khristianovich S A 1936 The plane problem of mathematical plasticity theory for the closed contour, loaded by external forces *Sh. Math.* 1 511–34 (in Russian)

[11] Lamothe V 2012 Symmetry group analysis of an ideal plastic flow *J. Math. Phys.* 53 033704

[12] Lamothe V 2012 Group analysis of an ideal plasticity model *J. Phys. A: Math. Theor.* 45 285203

[13] Mikhlin S G 1945 *The Mathematical Theory of Plasticity* (Providence, RI: Applied Mathematics Group, Brown University)

[14] Nadai A 1924 Über die Gleit- und Verzweigungsflächen einiger Gleichgewichtszustände bildsamer Massen und die Nachspannungen bleibend verzerrter Körper *Z. Phys.* 30 106–38

[15] Nadai A 1928 Plastizität und Erddruck *Handbuch der Physik Bd. VI* (Berlin: Springer)

[16] Nadai A 1950 *Theory of Flow and Fracture of Solids* vol 1 (New York: McGraw-Hill)

[17] Nadai A 1963 *Theory of Flow and Fracture of Solids* vol 2 (New York: McGraw-Hill)

[18] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (New York: Springer)

[19] Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)

[20] Prandtl L 1923 Anwendung beispiele zu einem Henckyschen Satz über das plastische Gleichgewicht *Z. Angew. Math. Mech.* 3 401–6

[21] Revuzhenko A F 1975 A class of exact solutions of ideal plasticity equations *J. Appl. Mech. Tech. Phys.* 16 235–8

[22] Senashov S I and Vinogradov A M 1988 Symmetries and conservation laws of 2-dimensional ideal plasticity *Proc. Edinburgh Math. Soc.* 31 415–39

[23] Senashov S I and Yakhno A 2007 Reproduction of solutions of bidimensional ideal plasticity *Int. J. Nonlinear Mech.* 42 500–3

[24] Senashov S I and Yakhno A 2012 Conservation laws, hodograph transformation and boundary value problems of plane plasticity *SIGMA* 8 071 (arXiv:1210.3673)

[25] Senashov S I, Yakhno A and Yakhno L 2009 Deformation of characteristic curves of the plane ideal plasticity equations by point symmetries *Nonlinear Anal. Theor. Methods Appl.* 71 e1274–84

[26] Sokolovskii V V 1950 *Teoriya Plastičnosti* (Moscow–Leningrad: Gosudarstv. Izdat. Tehn.-Teor. Lit.) (in Russian)

[27] Yakhno A and Yakhlo L 2010 ‘Homotopy’ of Prandtl and Nadai solution *Int. J. Nonlinear Mech.* 45 793–9