A direct calculation of critical exponents of two-dimensional anisotropic Ising model

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Using an exact solution of the one-dimensional (1D) quantum transverse-field Ising model (TFIM), we calculate the critical exponents of the two-dimensional (2D) anisotropic classical Ising model (IM). We verify that the exponents are the same as those of isotropic classical IM. Our approach provides an alternative means of obtaining and verifying these well-known results.

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It is well known that the two-dimensional (2D) classical Ising model (IM)\textsuperscript{1} has a thermodynamic continuous phase transition (CPT) from (anti-)ferromagnetic phase to paramagnetic phase at a critical temperature. Later it was also known\textsuperscript{2,3} that a one-dimensional (1D) quantum transverse-field Ising model (TFIM) can be mapped into the 2D anisotropic classical IM under certain limit. Thus, a connection between the ground state of 1D quantum TFIM and thermodinamical equilibrium state of a 2D classical IM was established. So far, our knowledge about the thermodynamical properties of 2D IM, especially those about critical phenomena, were mainly obtained by some sophisticate techniques such as high-temperature and low-temperature expansions, renormalization group calculations,\textsuperscript{4,5} Onsager's exact matrix-calculation and its variations\textsuperscript{6,7}. Normally, these techniques are computational involving, and it is often not easy to be understood by the beginners and people outside of the field. In this short paper, we would like to introduce a direct way to obtain the quantities like critical exponents of 2D IM. The method is based on the exact solution of the ground state of 1D TFIMs\textsuperscript{8}, which, in turn, can be obtained by using a simple Jordan-Winger transformation\textsuperscript{9}.

The Hamiltonian of a 2D classical IM is

$$H = -J_x \sum_{x,y} \sigma_{x,y} \sigma_{x+1,y} - J_y \sum_{x,y} \sigma_{x,y} \sigma_{x,y+1}$$  \hspace{1cm} (1)

where $\sigma_{x,y}$ is the classical Ising spin on site $(x,y)$ with two possible values $\pm 1$, and $J_x$ and $J_y$ are spin-spin couplings along $x$-axis(row) and $y$-axis(column) directions, respectively. It is well known that this model has a thermodynamic CPT at a critical temperature $T = T_c$. Later it was shown\textsuperscript{10} that with the relations and limits

$$\frac{J_y}{k_B T} \rightarrow +\infty$$
$$\frac{J_x}{k_B T} = \frac{J_y}{k_B T} \approx \frac{1}{\tau} \rightarrow 0$$

the transfer matrix between two neighboring columns

$$\hat{T} \approx \hat{1} - \tau \hat{H}$$  \hspace{1cm} (3)

where $\hat{1}$ is identity operator and

$$\hat{H} = -J \sum_n \hat{\sigma}_n^x \hat{\sigma}_{n+1}^x - h \sum_n \hat{\sigma}_n^z$$  \hspace{1cm} (4)

is the Hamiltonian operator of a 1D quantum TFIM where $\hat{\sigma}_n^x$ and $\hat{\sigma}_n^z$ are standard Pauli operators on site $n$. Thus the 2D anisotropic classical IM is mapped into a 1D quantum TFIM with the following correspondences between the two models\textsuperscript{8,9}:

1. the equilibrium state of 2D IM vs. the ground state of 1D TFIM;
2. the free energy of 2D IM vs. the ground-state energy of 1D TFIM;
3. the ensemble averages of physical quantities of 2D IM vs. the ground-state expectation values of time-ordered operators of 1D TFIM;
4. the temperature $T$ of 2D IM vs. the transverse field $h$ of 1D TFIM.

With the so-called hard-core boson transformation

$$\hat{\sigma}_n^x = \hat{b}_n^\dagger + \hat{b}_n$$
$$\hat{\sigma}_n^y = i(\hat{b}_n^\dagger - \hat{b}_n)$$
$$\hat{\sigma}_n^z = 2\hat{b}_n^\dagger \hat{b}_n - 1,$$  \hspace{1cm} (5)

the Jordan-Wigner transformation\textsuperscript{9}

$$\hat{b}_n = \hat{c}_n e^{-i\pi \sum_{\ell=1}^{n-1} \hat{c}_\ell^\dagger \hat{c}_\ell}$$
$$\hat{b}_n = \hat{c}_n^\dagger e^{i\pi \sum_{\ell=1}^{n-1} \hat{c}_\ell^\dagger \hat{c}_\ell}$$  \hspace{1cm} (6)

and the Fourier transformation

$$\hat{c}_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} dk \hat{c}_k e^{-ikn}$$
$$\hat{c}_n^\dagger = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} dk \hat{c}_k^\dagger e^{ikn}$$  \hspace{1cm} (7)

the Hamiltonian of 1D TFIM is rewritten into

$$\hat{H} = N h + \int_0^\pi dk \hat{H}_k$$
$$\hat{H}_k = 2i J \sin k (\hat{c}_k^\dagger \hat{c}_{-k} + \hat{c}_{-k}^\dagger \hat{c}_k)$$
$$-2 (\hat{h} + J \cos k) (\hat{c}_k^\dagger \hat{c}_k + \hat{c}_{-k}^\dagger \hat{c}_{-k})$$  \hspace{1cm} (8)

where $N$ is site number, $\hat{c}_n^\dagger, \hat{c}_n, \hat{c}_k, \hat{c}_k^\dagger$ are fermion operators. Denote $|0\rangle$ as fermions’ vacuum state, the ground
state and ground-state energy of $\hat{H}$ are obtained as:

$$|\Phi_G\rangle = \prod_{k=0}^{\pi} (u_k\hat{c}_k^\dagger\hat{c}_{-k} + v_k)|0\rangle$$

$$E_G = N\hbar + \int_0^\pi dk e_G(k)$$

where

$$u_k = \frac{-e_G(k)}{[e_G^2(k) + 4J^2\sin^2 k]^{1/2}}$$

$$v_k = \frac{2iJ\sin k}{[e_G^2(k) + 4J^2\sin^2 k]^{1/2}}$$

$$e_G(k) = -2\left(h + J\cos k + \sqrt{\hbar^2 + 2Jh\cos k + J^2}\right).$$

Denote

$$t = \frac{h}{|J|},$$

the excitation gap of $\hat{H}$ is

$$\Delta_E = 2|J||t - 1|.$$ (12)

Thus the critical point of the ground-state CPT of the 1D TFIM is at $t = t_c = 1$ where $\Delta_E$ vanishes.

Now that the ground state of 1D quantum TFIM is obtained, critical behaviors of physical quantities of the ground state can be calculated directly. Let us first consider the ‘heat capacitance’. According to the correspondences between 2D IM and 1D TFIM, the ‘heat capacitance’ in 1D TFIM is

$$C_{TFIM} = -\hbar^2 E_G \frac{dE_G}{dh^2}. $$

Put in the ground-state energy obtained in Eq. 10 and 11, we have

$$C_{TFIM} = t^{-1/2} \int_0^\pi \frac{\sin^2 k dk}{[g(t) + 4\cos^2 k]^{3/2}}$$

where

$$g(t) \equiv (t - t_c)/\sqrt{t}. $$

Near the critical point, i.e., $t \approx t_c = 1$, the denominator in the integral can be expanded as

$$\left[ g^2(t) + 4\cos^2 k \right]^{3/2} \approx 8\cos^3 k/2 + 3g^2(t)\cos k/2.$$ (16)

After some calculations one can obtain

$$C_{TFIM} \approx \left( \frac{1}{2} \ln \frac{32}{3} - 1 \right) - \ln |t - t_c|. $$

Therefore, the critical behavior of the heat capacitance is the same as that of 2D isotropic classical IM.

The magnetization along the spin-spin interaction direction in 1D TFIM is defined as

$$M_x = \frac{1}{N}\sum_{n=1}^{N} \langle \Phi_G|\hat{\sigma}^x_n|\Phi_G\rangle = \langle \Phi_G|\hat{\sigma}^x_n|\Phi_G\rangle.$$ (18)

However, there are two facts which make it unable to obtain the correct value of $M_x$ directly. One is that the 1D TFIM in Eq. 10 has no magnetic field component along the spin-spin interaction direction so that $M_x$ defined in Eq. 18 is always zero. The other is that $\hat{\sigma}^x_n$ in fermion representation is a non-local operator so that it is very difficult to calculate its expectation value even when the ground state of 1D TFIM with non-zero magnetic field component along the spin-spin interaction direction is obtained. Fortunately, the magnetization can be extracted from the ground-state spin-spin correlation function defined as

$$G_{xx}(r) = \langle \Phi_G|\hat{\sigma}^x_r\hat{\sigma}^x_{r+n}|\Phi_G\rangle.$$ (19)

This correlation contains two parts of contributions. The first part is a long-range correlation of constant value due to the long-range magnetic order of the ground state, and the amplitude of this part is the square of the magnetization. The second part is the correlation of quantum fluctuations, and the amplitude of this part goes to zero in the limit $r \to \infty$. Thus the magnetization equals to the square root of the amplitude of the correlation function in Eq. 19 in the limit $r \to \infty$. The spin-spin correlation function of 1D TFIM was evaluated extensively in Barouch and McCoy’s paper, where the correlation function was expressed in the form of a Töplitz determinant and its asymptotic behaviors at large yet finite distance $r$ were calculated with the use of a theorem of Szego. One exact result is

$$G_{xx}(r \to \infty) = [\text{sign}(J)]^r(1 - t^2)^{1/4} $$

where $\text{sign}(J)$ is the sign of $J$. Here we provide a direct way to obtain this result. Let us consider a finite chain of $N$ spins with free boundary condition and calculate the correlation function of the first spin and the $N$-th spin, i.e., the ground-state expectation value of $\hat{\sigma}^x_1\hat{\sigma}^x_N$ in the limit $N \to \infty$. With Eq. 14, Eq. 15 and Eq. 16, it is easy to show

$$\langle \Phi_G|\hat{\sigma}^x_1\hat{\sigma}^x_N|\Phi_G\rangle = \langle \Phi_G|\hat{b}_1^\dagger\hat{b}_N + \hat{b}_N^\dagger\hat{b}_1|\Phi_G\rangle = \langle \Phi_G|\sum_{l=1}^{N}\hat{c}_l^\dagger\hat{c}_l - \hat{c}_N^\dagger\hat{c}_N|\Phi_G\rangle = \langle \Phi_G|\sum_{l=1}^{N}\hat{c}_l^\dagger\hat{c}_l + \hat{c}_N^\dagger\hat{c}_N|\Phi_G\rangle = \langle \Phi_G|\sum_{l=1}^{N}\hat{c}_l^\dagger\hat{c}_l + \hat{c}_N^\dagger\hat{c}_N - \hat{c}_N^\dagger\hat{c}_N|\Phi_G\rangle = \langle \Phi_G|\sum_{l=1}^{N}\hat{c}_l^\dagger\hat{c}_l + \hat{c}_N^\dagger\hat{c}_N - \hat{c}_N^\dagger\hat{c}_N|\Phi_G\rangle $$

with the use of

$$\exp[i\pi(\hat{c}_k^\dagger\hat{c}_k + \hat{c}_{-k}^\dagger\hat{c}_{-k})](u_k\hat{c}_k^\dagger\hat{c}_{-k} + v_k)|0\rangle \equiv (u_k\hat{c}_k^\dagger\hat{c}_{-k} + v_k)|0\rangle.$$ (21)

Thus the rest work is to transform $(\hat{c}_1^\dagger + \hat{c}_N^\dagger)(\hat{c}_N - \hat{c}_N)$ into $k$–space representation by Eq. 4 and calculate its
ground-state expectation value. From the form of \( |\Phi_G> \) in Eq.(19), it is obvious that only the following four kinds of terms have non-zero expectation values

\[
<\Phi_G|\hat{c}_k\hat{c}_k^\dagger|\Phi_G> = 1 - u_k^2 \\
<\Phi_G|\hat{c}_k^\dagger\hat{c}_k|\Phi_G> = u_k^2 \\
<\Phi_G|\hat{c}_{-k}\hat{c}_k^\dagger|\Phi_G> = u_k v_k \\
<\Phi_G|\hat{c}_{-k}\hat{c}_k|\Phi_G> = u_k v_k^* 
\]  

(23)

After doing some integrals, one finally obtains that in the thermodynamic limit \( N \to \infty \)

\[
<\Phi_G|\hat{\sigma}_x^z\hat{\sigma}_y^z|\Phi_G> = [\text{sign}(J)]^N(1 - t^2)^{1/4} 
\]  

(24)

for \( t < t_c = 1 \) and is zero for \( t > t_c \), which is the same as Eq.(20). From this result, one can see that

\[
M_x = (1 - t^2)^{1/8} \approx (t_c - t)^{1/8} 
\]  

(25)

for \( t_c - t \to +0 \) and is zero for \( t > t_c \), which is exactly the same as the critical behavior of the magnetization of 2D isotropic IM. Thus we have found that the critical behaviors of the heat capacitance and the magnetization of 1D TFIM and 2D classical isotropic IM are the same. According to the scaling laws of critical phenomena, we can conclude that all the six critical exponents of the two models are equal. Thus our approach serves as an alternative way to obtain and verify the well-known results of 2D isotropic IM. In fact, another exact result obtained by Barouch and McCoy, the value at the transition point \( t = t_c \) which decays as a power-law function of \( r \),

\[
G_{xx}(t = t_c, r) = [\text{sign}(J)]^r r^{-1/4} e^{1/4} 2^{1/12} A^{-3} \propto r^{-1/4} 
\]  

(26)

where \( A = 1.282427130 \) is Glaisher’s constant, was shown to be the same as that of 2D isotropic classical IM. The exponential decay of the quantum-fluctuation-induced spin-spin correlation around the transition point was also shown to be equivalent to that in 2D isotropic IM. It should be noted that it is unable to obtain the critical behaviors of zero-field susceptibility \( \chi \) and the magnetization at the critical point under vanishing field because the 1D TFIM in Eq.(1) has no magnetic field component along the spin-spin interaction direction.

We would like to make some discussions about the above results. From the mapping between 2D classical IM and 1D TFIM, one can see that the approximation in Eq.(5) is valid only when the couplings along one direction, say, \( J_y \), goes to positive infinity while the couplings along the other direction, say, \( J_x \), goes to zero. This means that a 1D TFIM with anti-ferromagnetic coupling, i.e., \( J < 0 \), should correspond to a 2D classical IM with infinitely large ferromagnetic coupling along one direction and vanishing anti-ferromagnetic coupling along the other direction. Thus the equivalence of the critical behaviors of 1D TFIM and 2D classical isotropic IM shows that the critical behaviors of 2D classical IM is very robust.

In summary, with the use of the exact solution of the 1D TFIM, we calculate the critical exponents of the 2D anisotropic classical IM and verify that the exponents are the same as those of 2D isotropic classical IM. This shows that the universality class of critical behaviors in 2D classical IM is quite robust, since the 2D classical IM corresponding to a 1D TFIM is extremely anisotropic.

Our approach provides an alternative means to obtain and verify those well-known results.

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