Optimal bounds for the colorful fractional Helly theorem

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Abstract

The well known fractional Helly theorem and colorful Helly theorem can be merged into so called colorful fractional Helly theorem. It states: For every $\alpha \in (0,1]$ and every non-negative integer $d$, there is $\beta_{\text{col}} = \beta_{\text{col}}(\alpha,d) \in (0,1]$ with the following property. Let $F_1, \ldots, F_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$ of sizes $n_1, \ldots, n_{d+1}$ respectively. If at least $\alpha n_1 n_2 \cdots n_{d+1}$ of the colorful $(d+1)$-tuples have a nonempty intersection, then there is $i \in [d+1]$ such that $F_i$ contains a subfamily of size at least $\beta_{\text{col}} n_i$ with a nonempty intersection. (A colorful $(d+1)$-tuple is a $(d+1)$-tuple $(F_1, \ldots, F_{d+1})$ such that $F_i$ belongs to $F_i$ for every $i$.)

The colorful fractional Helly theorem was first stated and proved by Bárány, Fodor, Montejano, Oliveros, and Pór in 2014 with $\beta_{\text{col}} = \alpha/(d+1)$. In 2017 Kim proved the theorem with better function $\beta_{\text{col}}$, which in particular tends to 1 when $\alpha$ tends to 1. Kim also conjectured what is the optimal bound for $\beta_{\text{col}}(\alpha,d)$ and provided the upper bound example for the optimal bound. The conjectured bound coincides with the optimal bounds for the (non-colorful) fractional Helly theorem proved independently by Eckhoff and Kalai around 1984.

We prove Kim’s conjecture by extending Kalai’s approach to the colorful scenario. Moreover, we obtain optimal bounds also in more general setting when we allow several sets of the same color.

1 Introduction

The target of this paper is to provide optimal bounds for the colorful fractional Helly theorem first stated by Bárány, Fodor, Montejano, Oliveros, and Pór [BFM+14], and then improved by Kim [Kim17]. In order to explain the colorful fractional Helly theorem, let us briefly survey the preceding results.

The starting point, as usual in this context, is the Helly theorem:

\textbf{Theorem 1} (Helly’s theorem [Hel23]). Let $\mathcal{F}$ be a finite family of at least $d+1$ convex sets in $\mathbb{R}^d$. Assume that every subfamily of $\mathcal{F}$ with exactly $d+1$ members has a nonempty intersection. Then all sets in $\mathcal{F}$ have a nonempty intersection.

Helly’s theorem admits numerous extensions and two of them, important in our context, are the fractional Helly theorem and the colorful Helly theorem. The fractional Helly theorem of Katchalski and Liu covers the case when only some fraction of the $d+1$ tuples in $\mathcal{F}$ has a nonempty intersection:

\textbf{Theorem 2} (The fractional Helly theorem [KL79]). For every $\alpha \in (0,1]$ and every non-negative integer $d$, there is $\beta = \beta(\alpha,d) \in (0,1]$ with the following property. Given a finite family $\mathcal{F}$ of $n \geq d+1$ convex sets in $\mathbb{R}^d$ such that at least $\alpha \binom{n}{d+1}$ of the subfamilies of $\mathcal{F}$ with exactly $d+1$ members have a nonempty intersection. Then there is a subfamily of $\mathcal{F}$ with at least $\beta n$ members with a nonempty intersection. 

An interesting aspect of the fractional Helly theorem is not only to show the existence of $\beta(\alpha,d)$ but also to provide the largest value of $\beta(\alpha,d)$ with which the theorem is valid. This has been resolved independently by Kalai [Kal84] and by Eckhoff [Eck85] showing that the fractional Helly theorem holds with $\beta(\alpha,d) = 1 - (1 - \alpha)^{1/(d+1)}$. It is well known that this bound is sharp by considering a family $\mathcal{F}$

\footnote{D. B. is supported by GAČR grant no. 19-27871X. A. G. is supported by KAW-stipendiet 2015.0360 from the Knut and Alice Wallenberg Foundation. M. T. is supported by the GAČR grant 19-04113Y.}
consisting of $\approx (1 - (1 - \alpha)^{1/(d+1)})n$ copies of $\mathbb{R}^d$ and $\approx (1 - \alpha)^{1/(d+1)}n$ hyperplanes in general position; see, e.g., the introduction of [Kal84].

The colorful Helly theorem of Lovász covers the case where the sets are colored by $d + 1$ colors and only the ‘colorful’ $(d + 1)$-tuples of sets in $\mathcal{F}$ are considered. Given families $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ of sets in $\mathbb{R}^d$ a family of sets $\{\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}\}$ is a colorful $(d + 1)$-tuple if $\mathcal{F}_i \in \mathcal{F}_i$ for $i \in [d + 1]$, where for a non-negative integer $n \geq 1$ we use the notation $[n] := \{1, \ldots, n\}$. (The reader may think of $\mathcal{F}$ from preceding theorems decomposed into color classes $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$.)

**Theorem 3** (The colorful Helly theorem [Lov74, Bár82]). Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$. Let us assume that every colorful $(d + 1)$-tuple has a nonempty intersection. Then one of the families $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ has a nonempty intersection.

Both the colorful Helly theorem and the fractional Helly theorem with optimal bounds imply the Helly theorem. The colorful one by setting $\mathcal{F}_1 = \cdots = \mathcal{F}_{d+1} = \mathcal{F}$ and the fractional one by setting $\alpha = 1$ giving $\beta(1, d) = 1$.

The preceding two theorems can be merged into the following colorful fractional Helly theorem:

**Theorem 4** (The colorful fractional Helly theorem [BFM+14]). For every $\alpha \in (0, 1]$ and every non-negative integer $d$, there is $\beta_{\text{col}}(\alpha, d) \in (0, 1]$ with the following property. Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$ of sizes $n_1, \ldots, n_{d+1}$ respectively. If at least $\alpha n_1 \cdots n_{d+1}$ of the colorful $(d + 1)$-tuples have a nonempty intersection, then there is $i \in [d + 1]$ such that $\mathcal{F}_i$ contains a subfamily of size at least $\beta_{\text{col}} n_i$ with a nonempty intersection.

Bárány et al. proved the colorful fractional Helly theorem with the value $\beta_{\text{col}}(\alpha, d) = \frac{\alpha}{d+1}$ and they used it as a lemma [BFM+14, Lemma 3] in a proof of a colorful variant of a $(p, q)$-theorem. Despite this, the correct bound for $\beta_{\text{col}}$ seems to be of independent interest. In particular, the bound on $\beta_{\text{col}}$ has been subsequently improved by Kim [Kim17] who showed that the colorful fractional Helly theorem is true with $\beta_{\text{col}}(\alpha, d) = \max\left\{\frac{\alpha}{d+1}, 1 - (d + 1)(1 - \alpha)^{1/(d+1)}\right\}$. On the other hand, the value of $\beta_{\text{col}}(\alpha, d)$ cannot go beyond $1 - (1 - \alpha)^{1/(d+1)}$ because essentially the same example as for the standard fractional Helly theorem applies in this setting as well—it is sufficient to set $n_1 = n_2 = \cdots = n_{d+1}$ and take $\approx (1 - (1 - \alpha)^{1/(d+1)})n_i$ copies of $\mathbb{R}^d$ and $\approx (1 - (1 - \alpha)^{1/(d+1)})n_i$ hyperplanes in general position in each color class.1 (Kim [Kim17] provides a slightly different upper bound example showing the same bound.)

Coming back to the lower bound on $\beta_{\text{col}}(\alpha, d)$, Kim explicitly conjectured that $1 - (1 - \alpha)^{1/(d+1)}$ is also a lower bound, thereby an optimal bound for the colorful fractional Helly theorem. He also provides a more refined conjecture, that we discuss slightly later on (see Conjecture 8), which implies this lower bound. We prove the refined conjecture which therefore indeed gives the optimal bounds for the colorful fractional Helly theorem.

**Theorem 5** (The optimal colorful fractional Helly theorem). Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$ of sizes $n_1, \ldots, n_{d+1}$ respectively. If at least $\alpha n_1 \cdots n_{d+1}$ of the colorful $(d + 1)$-tuples have a nonempty intersection, for $\alpha \in (0, 1]$, then there is $i \in [d + 1]$ such that $\mathcal{F}_i$ contains a subfamily of size at least $(1 - (1 - \alpha)^{1/(d+1)})n_i$ with a nonempty intersection.

In the proof we follow the exterior algebra approach which has been used by Kalai [Kal84] in order to provide optimal bounds for the standard fractional Helly theorem. We have to upgrade Kalai’s proof to the colorful setting. This requires guessing the right generalization of several steps in Kalai’s proof. However, we note that after making these ‘guesses’ we follow Kalai’s proof quite straightforwardly.

Let us also compare one aspect of our proof with the previous proof of the weaker bound by Kim [Kim17]: Kim’s proof uses the colorful Helly theorem as a blackbox while our proof includes the proof of the colorful Helly theorem.

Last but not least, the exterior algebra approach actually allows to generalize Theorem 5 in several different directions. The extension to so called $d$-collapsible complexes is essentially mandatory for the well working proof while the other generalizations that we will present just follow from the method. We will discuss this in detail in forthcoming subsections of the introduction.

1We discuss this example in more detail at the end of Section 3 in a more general context. However, in this special case, it is perhaps much easier to check directly that $\beta_{\text{col}}$ cannot be improved due to this example.
1.1 $d$-representable and $d$-collapsible complexes

The nerve and $d$-representable complexes. The important information in Theorems 1, 2, 3, 4, and 5 is which subfamilies have a nonempty intersection. This information can be efficiently stored in a simplicial complex called the nerve that we will just define.

A (finite abstract) simplicial complex is a set system $K$ on a finite set of vertices $N$ such that whenever $A \in K$ and $B \subseteq A$, then $B \in K$. (The standard notation for the vertex set would be $V$ but this notation will be more useful later on when we will often use capital letters such as $R$ for some set and the corresponding lower case letters such as $r$ for its size.) The elements of $K$ are faces (a.k.a. simplices) of $K$. The dimension of a face $A \in K$ is defined as $\dim A = |A| - 1$; this corresponds to representing $A$ as an $|A|-1$-dimensional simplex. The dimension of $K$, denoted $\dim K$, is the maximum of the dimensions of faces in $K$. A face of dimension $k$ is a $k$-face in short. Vertices of $K$ are usually identified with 0-faces, that is, $v \in N$ is identified with $\{v\} \in K$. (We remark that though the definition of simplicial complex allows that $\{v\} \notin K$ for $v \in N$, in our applications we will always have $\{v\} \in K$ for $v \in N$.) Now given a family $\mathcal{F}$ of sets, the nerve of $\mathcal{F}$, is the simplicial complex whose set of vertices is $\mathcal{F}$ and whose faces are subfamilies with a nonempty intersection. A simplicial complex is $d$-representable if it is the nerve of a finite family of convex sets in $R^d$.

As a preparation for the $d$-collapsible setting, we now restate Theorem 5 in terms of $d$-representable complexes. For this we need two more notions. Given a simplicial complex $K$ and a subset $U$ of the vertex set $N$, the induced subcomplex $K[U]$ is defined as $K[U] := \{A \in K : A \subseteq U\}$. Now, let us assume that the vertex set $N$ is split into $d + 1$ pairwise disjoint subsets $N = N_1 \sqcup \cdots \sqcup N_{d + 1}$ (we can think of this partition as coloring each vertex of $N$ with one of the $d + 1$ possible colors). Then a colorful $d$-face is a $d$-face $A$, such that $|A \cap N_i| = 1$ for every $i \in [d + 1]$.

**Theorem 6** (Theorem 5 reformulated). Let $K$ be a $d$-representable simplicial complex with the set of vertices $N = N_1 \sqcup \cdots \sqcup N_{d + 1}$ divided into $d + 1$ disjoint subsets. Let $n_i := |N_i|$ for $i \in [d + 1]$ and assume that $K$ contains at least $\alpha n_1 \cdots n_{d + 1}$ colorful $d$-faces for some $\alpha \in (0, 1)$. Then there is $i \in [d + 1]$ such that $\dim K[N_i] \geq (1 - (1 - \alpha)^{1/(d + 1)}) n_i - 1$.

**Theorem 6** is indeed just a reformulation of **Theorem 5**: Considering $\mathcal{F}$ as disjoint union\(^2\) $\mathcal{F} = \mathcal{F}_1 \sqcup \cdots \sqcup \mathcal{F}_{d + 1}$, then $K$ corresponds to the nerve or $\mathcal{F}$, colorful $d$-faces correspond to colorful ($d + 1$)-tuples with nonempty intersection and the dimension of $K[V]$ corresponds to the size of largest subfamily of $\mathcal{F}$ with nonempty intersection minus 1. (The shift by minus 1 between size of a face and dimension of a face is a bit unpleasant; however, we want to follow the standard terminology.)

$d$-collapsible complexes. In [Weg75] Wegner introduced an important class of simplicial complexes, called $d$-collapsible complexes. They include all $d$-representable complexes, which is the main result of [Weg75], while they admit quite simple combinatorial description useful for induction.

Given a simplicial complex $K$, we say that a simplicial complex $K'$ arises from $K$ by an elementary $d$-collapse, if there are faces $L, M \in K$ with the following properties: (i) $\dim L \leq d - 1$; (ii) $M$ is the unique inclusion-wise maximal face which contains $L$; and (iii) $K' = K \setminus \{A \in K : L \subseteq A\}$. A simplicial complex $K$ is $d$-collapsible if there is a sequence of simplicial complexes $K_0, \ldots, K_\ell$ such that $K = K_\ell$; $K_i$ arises from $K_{i-1}$ by an elementary $d$-collapse for $i \in [\ell]$; and $K_\ell$ is the empty complex.

We will prove the following generalization of **Theorem 6** (equivalently of **Theorem 5**).

**Theorem 7** (The optimal colorful fractional Helly theorem for $d$-collapsible complexes). Let $K$ be a $d$-collapsible simplicial complex with the set of vertices $N = N_1 \sqcup \cdots \sqcup N_{d + 1}$ divided into $d + 1$ disjoint subsets. Let $n_i := |N_i|$ for $i \in [d + 1]$ and assume that $K$ contains at least $\alpha n_1 \cdots n_{d + 1}$ colorful $d$-faces for some $\alpha \in (0, 1)$. Then there is $i \in [d + 1]$ such that $\dim K[N_i] \geq (1 - (1 - \alpha)^{1/(d + 1)}) n_i - 1$.

1.2 Kim’s refined conjecture and further generalization

As a tool for a possible proof of **Theorem 5**, Kim [Kim17, Conjecture 4.2] suggested the following conjecture. (The notation $k_i$ in Kim’s statement of the conjecture is our $r_i + 1$.)

\(^2\)If there are any repetitions of sets in $\mathcal{F}$, which we generally allow for families of sets, then each repetition creates a new vertex in the nerve.
Conjecture 8 ([Kim17]). Let $n_i$ be positive and $r_i$ non-negative integers for $i \in [d+1]$ with $n_i \geq r_i + 1$. Let $F_1, \ldots, F_{d+1}$ be families of convex sets in $\mathbb{R}^d$ such that $|F_i| = n_i$ and there is no subfamily of $F_i$ of size $r_i + 1$ with non-empty intersection for every $i \in [d+1]$. Then the number of colorful $(d+1)$-tuples is at most

$$n_1 \cdots n_{d+1} = (n_1 - r_1) \cdots (n_{d+1} - r_{d+1}).$$

We explicitly prove this conjecture in the slightly more general setting of $d$-collapsible complexes. (Note that the condition ‘no subfamily of size $r_i + 1$’ translates as ‘no $r_i$-face’, that is, ‘the dimension is at most $r_i - 1$’.)

Proposition 9. Let $n_i$ be positive and $r_i$ non-negative integers for $i \in [d+1]$ with $n_i \geq r_i + 1$. Let $K$ be a $d$-collapsible simplicial complex with the set of vertices $N = N_1 \cup \cdots \cup N_{d+1}$ divided into $d+1$ disjoint subsets. Assume that $|N_i| = n_i$ and that $\dim K[N_i] \leq r_i - 1$ for every $i \in [d+1]$. Then $K$ contains at most

$$n_1 \cdots n_{d+1} = (n_1 - r_1) \cdots (n_{d+1} - r_{d+1}).$$

colorful $d$-faces.

Our main technical result. Now, let us present our main technical tool for a proof of Proposition 9 and therefore for a proof of Theorem 7 as well.

Let us use the notation $N$ for the set of positive integers whereas $N_0$ is the set of non-negative integers. Let us consider the vectors $k = (k_1, \ldots, k_{d+1}), r = (r_1, \ldots, r_{d+1}) \in N_0^{d+1}$ and $n = (n_1, \ldots, n_{d+1}) \in N_0^{d+1}$ such that $k, r + 1 \leq n$. (Here the notation $a \leq b$ means that $a$ is less or equal to $b$ in every coordinate and $1 = (1, \ldots, 1) \in N_0^{d+1}$.) We will also use the notation $k := k_1 + \cdots + k_{d+1}$, $n := n_1 + \cdots + n_{d+1}$, and $r = r_1 + \cdots + r_{d+1}$. Let $N$ be a set with $n$ elements partitioned as usual $N = N_1 \cup \cdots \cup N_{d+1}$ where $|N_i| = n_i$ for $i \in [d+1]$. By $\binom{N}{k}$ we denote the set of all subsets $A$ of $N$ such that $|A \cap N_i| = k_i$ for every $i \in [d+1]$. Note that $\binom{N}{k} \subseteq \binom{n}{k}$ where $\binom{n}{k}$ denotes the set of all subsets of $N$ of size $k$.

Now let $K$ be a simplicial complex with the vertex set $N$ as above. We say that a face $A$ of $K$ is $k$-colorful if $A \in \binom{N}{k}$, that is, $|A \cap N_i| = k_i$ for every $i \in [d+1]$. The earlier notion of colorful face corresponds to setting $k = 1$. By $f_k$ we denote the $k$-colorful $f$-vector of $K$, that is, the number of $k$-colorful faces in $K$.

Now let us further assume that we are also given sets $R_i \subseteq N_i$ with $|R_i| = r_i$ for every $i \in [d+1]$ and let $R = R_1 \cup \cdots \cup R_{d+1}$ and $\bar{R} = N \setminus R$. Then we define the set system

$$P_k(n, d, r) = \left\{ S \in \binom{N}{k} : |S \cap \bar{R}| \leq d \right\}.$$

We remark that $P_k(n, d, r)$ is not a simplicial complex as it contains only sets in $\binom{n}{k}$. However, this set system is useful for estimating the number of $k$-colorful faces in a $d$-collapsible complex. In sequel, we will use notation $p_k(n, d, r)$ for the size of $P_k(n, d, r)$, that is, $p_k(n, d, r) = |P_k(n, d, r)|$.

Theorem 10. For an integer $d \geq 1$, let $K$ be a $d$-collapsible simplicial complex with vertex partition $N = N_1 \cup \cdots \cup N_{d+1}$ and let $n = (n_1, \ldots, n_{d+1}) \in N_0^{d+1}$ be the vector with $n_i = |N_i|$. For $r = (r_1, \ldots, r_{d+1}) \in N_0^{d+1}$ such that $\dim K[N_i] \leq r_i - 1$ for $i \in [d+1]$ and $k \in N_0^{d+1}$ such that $k \leq n$ it follows that

$$f_k(K) \leq p_k(n, d, r).$$

Theorem 10 is proved in Section 2. Here we show the implications Theorem 10 $\Rightarrow$ Proposition 9 and Proposition 9 $\Rightarrow$ Theorem 7. In addition, we advertise that Theorem 10 yields further generalizations of Theorem 7. We explain in Section 3.

Proof of Proposition 9 modulo Theorem 10. We use Theorem 10 with $k = 1$. Then it is just sufficient to compute $p_1(n, d, r)$. The size of $\binom{n}{1}$ is $n_1 \cdots n_{d+1}$. Furthermore $A$ belongs to $\binom{n}{1} \setminus P_1(n, d, r)$ if and only if $|A \cap (N_i \setminus R_i)| = 1$ for every $i \in [d+1]$, that is, the number of such $A$ is $(n_1 - r_1) \cdots (n_{d+1} - r_{d+1})$.

This gives the required formula

$$p_1(n, d, r) = n_1 \cdots n_{d+1} - (n_1 - r_1) \cdots (n_{d+1} - r_{d+1}).$$
Proof of Theorem 7 modulo Proposition 9. For contradiction, let us assume that for every $i \in [d+1]$ we get $\dim K[N_i] < (1 - (1 - \alpha)^{1/(d+1)}) n_i - 1$. Let us set $r_i := \dim K[N_i] + 1 < (1 - (1 - \alpha)^{1/(d+1)}) n_i$. Then Proposition 9 gives that the number of colorful $d$-faces is at most

$$n_1 \cdots n_{d+1} - (n_1 - r_1) \cdots (n_{d+1} - r_{d+1}) < n_1 \cdots n_{d+1} - n_1 \cdots n_{d+1}(1 - (1 - \alpha)^{1/(d+1)})^{d+1}$$

which is a contradiction due to the strict inequality on the first line. \qed

2 Exterior algebra

In this section we prove Theorem 10. First we overview the required tools from exterior algebra—here we follow \cite[Section 2]{Kal84} very closely.

Let $N$ be a finite set with $n$ elements and let $V = \mathbb{R}^N$ be the $n$-dimensional real vector space with standard basis vectors $e_i$ for $i \in N$. Let $\Lambda V$ be the $2^n$ dimensional exterior algebra over $V$ with basis vectors $e_S$ for $S \subseteq N$. The exterior product \wedge on this algebra is defined so that it satisfies (i) $e_\emptyset$ is a neutral element, that is $e_\emptyset \wedge e_S = e_S \wedge e_\emptyset = e_S$; (ii) $e_S = e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $S = \{i_1, \ldots, i_k\} \subseteq N$ where $i_1 < \cdots < i_k$ and we identify $e_i$ with $e_{\{i\}}$ for $i \in N$; (iii) $e_i \wedge e_j = -e_j \wedge e_i$ for $i, j \in N$. By $\Lambda^k V$ we denote the subspace of $\Lambda V$ generated by $(e_S)_{S \subseteq \binom{N}{k}}$ where $0 \leq k \leq n$. We consider the standard inner product on both $V$ and $\Lambda V$ so that $(e_i)_{i \in N}$ and $(e_S)_{S \subseteq \binom{N}{k}}$ are their orthonormal bases respectively. Then $(e_S)_{S \subseteq \binom{N}{k}}$ is also an orthonormal basis of $\Lambda^k V$.

Given another basis $(g_i)_{i \in N}$, let $A = (a_{ij})_{i,j \in N}$ be the $N \times N$ transition matrix\footnote{Here we index rows and columns of a matrix by elements from some set, not necessarily integers. That is by $N \times N$ matrix we mean the matrix where both rows and columns are indexed by elements of $N$.} from $(e_i)_{i \in N}$ to $(g_i)_{i \in N}$, that is, $g_i = \sum_{j \in N} a_{ij} e_j$ for any $i \in N$. The basis $(g_i)_{i \in N}$ induces a basis of $\Lambda V$ given by $g_S = g_{i_1} \wedge \cdots \wedge g_{i_k}$ for $S = \{i_1, \ldots, i_k\} \subseteq N$. Transition from the standard basis $(e_S)_{S \subseteq \binom{N}{k}}$ of $\Lambda^k V$ to $(g_S)_{S \subseteq \binom{N}{k}}$ is given by

$$g_S = \sum_{T \subseteq \binom{N}{k}} \text{det} A_{S \setminus T} e_T$$

where $A_{S \setminus T} = (a_{ij})_{i \in S, j \in T}$ for $S, T \subseteq N$.

Now, given an $m$-element set $M$ and $M \times N$-matrix $A$ and $k \leq m, n$, let $C_k(A)$ be the matrix $(\text{det} A_{S \setminus T})_{S \subseteq \binom{N}{k}, T \subseteq \binom{M}{k}}$.

The following lemma is implicitly contained in \cite{Kal84}.

Lemma 11. If the columns of $A$ are linearly independent, then the columns of $C_k(A)$ are linearly independent as well.

Proof. If columns of $A$ are linearly independent, then $n \leq m$. Consider an arbitrary square submatrix $B$ of rank $n$. Considering $B$ as a transition matrix from $(e_i)_{i \in N}$ to $(g_i)_{i \in N}$, we get that $C_k(B)$ is a transition matrix from $(e_S)_{S \subseteq \binom{N}{k}}$ to $(g_S)_{S \subseteq \binom{N}{k}}$, thus $C_k(B)$ has full rank. However, $C_k(B)$ is also a submatrix of $C_k(A)$ with all $\binom{n}{k}$ columns. \qed

Now, let us in addition assume that $(g_i)_{i \in N}$ is an orthonormal basis of $V$. As pointed out by Kalai, it follows from the Cauchy-Binet formula that $(g_S)_{S \subseteq \binom{N}{k}}$ is also an orthonormal basis of $\Lambda^k V$.

We also define the left interior product on $\Lambda V$ so that for $f, g \in \Lambda V$ we let $g T \cdot f$ be the unique element of $\Lambda V$ which satisfies $(u \cdot g, f) = (u \cdot g, f)$. It turns out that $g T \cdot g S$ is non-zero only if $T \subseteq S$, in which case $g T \cdot g S = \pm g_{S \setminus T}$.

Colored exterior algebra. Now we extend the previous tools to colored setting. From now on, let us assume that $N$ is an $n$-element set decomposed into $(d+1)$-color classes, $N = N_1 \sqcup \cdots \sqcup N_{d+1}$, and $N = \bigcup_{i,j \neq N} N_i N_j$. We pick an $N \times N$-matrix $A$ so that it is a block-diagonal matrix with blocks corresponding to individual $N_i$. That is, $A_{N_i N_j}$ is a zero matrix whenever $i \neq j$. On the other hand, as shown by Kalai \cite[Section 2]{Kal84},

\begin{thebibliography}{9}
\bibitem{Kal84} Kalai, G., 1984. \textit{On the number of colorful faces in a...}
\end{thebibliography}
Section 2], it is possible to pick each \( A_{N_i|N_i} \), so that \((g_j)_{j \in N_i}\) is an orthogonal basis of the subspace of \( V \) generated by \((e_j)_{j \in N_i}\), and moreover each square submatrix of \( A_{N_i|N_i} \) has full rank. Therefore, from now on, we assume that we picked \( A \) and the vectors \( g_j \) this way.

Similarly as in the introduction, let us set \( n = (n_1, \ldots, n_{d+1}) \) so that \( n_i = |N_i| \) for \( i \in [d+1] \); for simplicity, let us assume that each \( N_i \) is nonempty—that is, \( n \) is a \((d+1)\)-tuple of positive integers. Let us also consider another \((d+1)\)-tuple \( k = (k_1, \ldots, k_{d+1}) \) of non-negative integers such that \( k \leq n \) and we set \( k = k_1 + \cdots + k_{d+1} \). Then by \( \bigwedge^k V \) we mean the subspace of \( \bigwedge V \) generated by \((e_s)_{s \in \binom{N}{k}}\); recall that \( \binom{N}{k} \) is the set of all subsets \( A \) of \( N \) such that \( |A \cap N_i| = k_i \) and that \( \binom{N}{k} \subseteq \binom{N}{k} \). Thus we also get that \( \bigwedge^k V \) is a subspace of \( \bigwedge^1 V \). In addition, due to our choice of \((g_j)_{j \in N_i}\) we get that \( g_s \in \bigwedge^k V \) if \( S \in \binom{N}{k} \). In addition let \( A_{S|T} = 0 \) if \( T \in \binom{N}{k} \backslash \binom{N}{k} \) because \( A_{S|T} \) is in this case a block matrix such that some of the blocks is not a square. Thus the formula (1) simplifies to

\[
g_s = \sum_{T \in \binom{N}{k}} \det A_{S|T} e_T.
\]

Proof of Theorem 10. For \( k \in \mathbb{N}^{d+1} \) such that \( k \leq d \) we have that \( P_k(n,d,r) = \binom{N}{k} \), thus the theorem follows trivially. On the other hand, if \( k > r \), then \( k_i > r_i \) for some \( i \) and consequently \( f_k(K) = 0 \) due to our assumption \( \dim K[N_i] \leq r_i - 1 \); therefore the theorem again follows trivially. In sequel, we assume \( d < k \leq r \).

Let us define the subspaces of \( \bigwedge^k V \)

\[
A_k = \left\{ m \in \bigwedge^k V : \left( \forall T \in \binom{R}{k-d} \right) g_T \cdot m = 0 \right\},
\]

and

\[
W_k = \text{span} \left\{ e_S \in \bigwedge^k V : S \in \binom{N}{k} \text{ and } S \subseteq K \right\},
\]

from the definition it follows that the colorful \( f \)-vector and the dimension of \( W_k \) coincide, i.e. \( f_k = \dim(W_k) \).

We claim that

\[
\dim(A_k) \geq \left| \binom{N}{k} \right| - p_k(n,d,r).
\]

If \( S \in \binom{N}{k} \) such that \( S \notin P_k(n,d,r) \), then \( |S \cap R| > d \). As \( S \subseteq R \cup \bar{R} = N \) and \( |S| = k \) we have that \( |S \cap \bar{R}| < k - d \). If \( T \in \binom{R}{k-d} \) we have that \( S \notin T \); therefore \( g_T \cdot g_S = 0 \). From this it follows that \( g_S \in A_k \) and finally the claim because \( g_S \in \bigwedge^k V \).

The core of the proof is to show \( A_k \cap W_k = \{0\} \). Once we have this, we get \( f_k(K) = \dim(W_k) \leq \dim \bigwedge^k V - \dim A_k \geq p_k(n,d,r) \) which proves the theorem.

For contradiction, let \( m \in A_k \cap W_k \) be a non-zero element. Because \( m \in W_k \), it can be written as \( m = \sum_{S \in \binom{N}{k}} \alpha_S e_S \) where the sum is over all \( S \in \binom{N}{k} \) such that \( S \subseteq K \). Let \( K_0, \ldots, K_r \) be a sequence of simplicial complexes showing \( d \)-collapsibility of \( K \) as in the definition of \( d \)-collapsible complex. In addition, due to [Kal84, Lemma 3.2], it is possible to assume that \( K_j \) arises from \( K_{j-1} \) by so called special elementary \( d \)-collapse which is either a removal of a maximal face of dimension at most \( d - 1 \) or the minimal face (the face \( L \) in the definition) has dimension exactly \( d - 1 \). Now let us consider the first step from \( K_{1-1} \) to \( K_1 \) such that a face \( U \in \binom{N}{k} \) with non-zero \( \alpha_U \) is eliminated. Denote \( L \) and \( M \) the faces determining the collapse as in the definition. We have \( L \subseteq U \subseteq M \), \( |M| \geq |U| = k-d \) and therefore \( |L| = d \) (equivalently, \( \dim L = d-1 \)), because the collapse is special. For \( T \in \binom{R}{k-d} \) let \( \mathbf{t} = (t_1, \ldots, t_{d+1}) \in \mathbb{N}^{d+1} \) be such that \( t_i = |T \cap N_i| \). Then \( g_T = \sum_{P \in \binom{N}{k}} \det(A_{T|P})e_P \) via (2). We also need to simplify the expression \( \langle e_L, g_T \cdot e_S \rangle \) for \( S \in \binom{N}{k} \). We obtain

\[
\langle e_L, g_T \cdot e_S \rangle = \langle e_L \wedge g_T, e_S \rangle = \sum_{P \in \binom{N}{k}} \det(A_{T|P})(e_L \wedge e_P, e_S) \]

(3)

If \( S \notin L \) then \( \langle e_L \wedge e_P, e_S \rangle = 0 \) for all \( P \), and therefore \( \langle e_L, g_T \cdot e_S \rangle = 0 \). If \( S \subseteq L \) then \( \langle e_L \wedge e_P, e_S \rangle = 0 \) unless \( P = S \setminus L \) and therefore \( \langle e_L, g_T \cdot e_S \rangle = \langle e_L \wedge e_{S \setminus L}, e_S \rangle \det(A_{T|S \setminus L}) \).
Since \( m \in A_k \), for arbitrary \( T \in \binom{R}{k-d} \) we get

\[
0 = \langle e_L, g_{T \setminus L} m \rangle = \sum_{S \in \binom{L}{k} : S \subseteq T} a_S \langle e_L, g_{T \setminus L} e_S \rangle = \sum_{S \in \binom{L}{k} : S \subseteq K_{i-1}} a_S \langle e_L, g_{T \setminus L} e_S \rangle = \sum_{S \in \binom{L}{k} : S \supseteq L} a_S \langle e_L, g_{T \setminus L} e_S \rangle
\]

where the third equality follows from the fact that \( a_S = 0 \) for \( S \in K \setminus K_{i-1} \) due to our choice of \( K_{i-1} \) and the last two equalities follow from our earlier simplification of \( \langle e_L, g_{T \setminus L} e_S \rangle \). (We also use that the expressions \( S \supseteq L \) and \( M \supseteq S \supseteq L \) are equivalent as \( M \) is the unique maximal face containing \( L \).

We also have \( U = \binom{L}{k} \) with \( M \supseteq U \supseteq L \) for which \( \langle e_U, \xi_{U \setminus L}, e_U \rangle \) is nonzero (the latter one equals \( \pm 1 \)). Therefore the expression above is a linear dependence of the columns of \( C_{k-d}(A_{R|M:L}) \) however, we will also show that columns of \( C_{k-d}(A_{R|M:L}) \) are linearly independent. Because \( A \) is a block-matrix with blocks \( A_{N_{i-1} \setminus L} \), we get that \( A_{R|M:L} \) is a block matrix with blocks \( A_{R_i \mid M:L \cap N_i} \). Thus it is sufficient to check that the columns are independent in each block. But this follows from our assumptions how we picked \( A \) in each block, using that \( |R_i| = r_i \geq |(M \setminus L) \cap N_i| \) as \( |M \cap N_i| \leq r_i \) due to our assumption \( \dim K[V_i] \leq r_i - 1 \).

### 3 k-colorful fractional Helly theorem

Theorem 10 allows to generalize Theorem 7 in two more directions.

The first generalization of Theorem 7 is already touched in the introduction. We can deduce analogy of Theorem 7 for k-colorful faces (instead of just colorful \( d \)-faces) where \( k = (k_1, \ldots, k_{d+1}) \in N_0^{d+1} \) is some vector. For example, if \( d = 2 \), \( k = (2, 1, 1) \) and we understand the partition of \( N = N_1 \cup N_2 \cup N_3 \) as coloring the vertices of \( K \) red, green, or blue. Then we seek for number of faces that contain two red vertices, one green vertex and one blue vertex.

For the second generalization, let us first observe that in the conclusion of Theorem 7 there is the same coefficient \( 1 - (1 - \alpha)^{1/(d+1)} \) independently of \( i \). However, in the notation of Theorem 7, we may also seek for \( i \) such that \( \dim K[N_i] \geq \beta_i n_i + 1 \) where \( \beta = (\beta_1, \ldots, \beta_{d+1}) \in (0, 1]^{d+1} \) is some fixed vector. Then for given \( \beta \), we want to find the lowest \( \alpha \in (0, 1] \) with which we reach the conclusion analogous as in Theorem 7. This is a natural analog of various Ramsey type statements: For example, if the edges of a complete graph \( G \) with at least 9 vertices are colored blue or red, then the graph contains either a blue copy of the complete graph on 3 vertices or a red copy of the complete graph on 4 vertices.

For purpose of stating the generalization, let us set

\[
L_k(d) := \{ \ell = (\ell_1, \ldots, \ell_{d+1}) \in N_0^{d+1} : \ell_1 + \cdots + \ell_{d+1} \leq d \text{ and } \ell_i \leq k_i \text{ for } i \in [d+1] \}
\]

and

\[
a_k(d, \beta) := \sum_{\ell = (\ell_1, \ldots, \ell_{d+1}) \in L_k(d)} \prod_{i=1}^{d+1} \binom{k_i}{\ell_i} (1 - \beta_i)^{\ell_i} \beta_i^{k_i - \ell_i}.
\]

**Theorem 12.** Let \( d \geq 1 \) and \( k = (k_1, \ldots, k_{d+1}) \in N_0^{d+1} \) be such that \( k := k_1 + \cdots + k_{d+1} \geq d + 1 \). Let \( K \) be a \( d \)-collapsible simplicial complex with the set of vertices \( N = N_1 \cup \cdots \cup N_{d+1} \) divided into \( d \) disjoint subsets. Let \( n_i := |N_i| \) for \( i \in [d+1] \) and assume that \( K \) contains at least \( a_k(d, \beta) \binom{N}{k} \) k-colorful faces for some \( \beta = (\beta_1, \ldots, \beta_{d+1}) \in (0, 1]^{d+1} \). Then there is \( i \in [d+1] \) such that \( \dim K[N_i] \geq \beta_i n_i - 1 \).

The formula (5) for \( a_k(d, \beta) \) in Theorem 12 is a bit complicated. However, this is the optimal value for \( \alpha \) in the theorem. We first prove Theorem 12 and then we will provide an example showing that for every \( d \), \( k \) and \( \beta \) as in the theorem, the value for \( \alpha \) cannot be improved. The remark below is a probabilistic interpretation of (5). (This, for example, easily reveals that \( a_k(d, \beta) \in (0, 1] \) for given parameters and will help us with checking monotonicity in \( \beta \).)

**Remark 13.** Consider a random experiment when we gradually for each \( i \) pick \( k_i \) numbers \( x_{i1}, \ldots, x_{ki} \) in the interval \( [0, 1] \) independently at random (with uniform distribution). Let \( \ell_i \) be the number of \( x_{ij} \)
which are greater than \( \beta_i \) and let us consider the event \( A_k(d, \beta) \) expressing that \( \ell_1 + \cdots + \ell_{d+1} \leq d \). Then \( \alpha_k(d, \beta) \) is the probability \( \mathbb{P}[A_k(d, \beta)] \).

Indeed, the probability that the number of \( x_j^i \) which are greater than \( \beta_i \) is exactly \( \ell_i \) is given by the expression beyond the sum in (3). Therefore, we need to sum this over all options giving \( \ell_1 + \cdots + \ell_{d+1} \leq d \) and \( \ell_i \leq k_i \).

In the proof of Theorem 12 we will need the following slightly modified proposition. We relax ‘at least’ to ‘more than’ while we aim at strict inequality in the conclusion—this innocent change will be a significant advantage in the proof. On the other hand, after this change we can drop the assumption \( k \geq d + 1 \). But this is only a cosmetic change, because the proposition below is vacuous if \( \alpha_k(d, \beta) = 1 \) which in particular happens if \( k < d + 1 \).

**Proposition 14.** Let \( d \geq 1 \) and \( k = (k_1, \ldots, k_{d+1}) \in \mathbb{N}_0^{d+1} \). Let \( K \) be a \( d \)-collapsible simplicial complex with the set of vertices \( N = N_1 \sqcup \cdots \sqcup N_{d+1} \) divided into \( d + 1 \) disjoint subsets. Let \( n_i := |N_i| \) for \( i \in [d+1] \) and assume that \( K \) contains more than \( \alpha_k(d, \beta)(\binom{m}{k}) \) \( k \)-colorful faces for some \( \beta = (\beta_1, \ldots, \beta_{d+1}) \in (0,1]^{d+1} \). Then there is \( i \in [d+1] \) such that \( \dim K[N_i] > \beta_i n_i - 1 \).

First we show how Theorem 12 follows from Proposition 14 by a limit transition. Then we prove Proposition 14.

**Proof of Theorem 12 modulo Proposition 14.** Let us consider \( \epsilon > 0 \) such that \( \beta - \epsilon \in (0,1]^{d+1} \) for \( \epsilon = (\epsilon, \ldots, \epsilon) \in (0,1]^{d+1} \).

First, we need to check \( \alpha_k(d, \beta) > \alpha_k(d, \beta - \epsilon) \). For this we will use Remark 13 and we also use \( k \geq d + 1 \). It is easy to check \( A_k(d, \beta) \geq A_k(d, \beta - \epsilon) \) which gives \( \alpha_k(d, \beta) \geq \alpha_k(d, \beta - \epsilon) \). In order to show the strict inequality, it remains to show that \( A_k(d, \beta) \setminus A_k(d, \beta - \epsilon) \) has positive probability. Consider the output of the experiment when each \( x_j^{i*} \in (\beta_i - \epsilon, \beta_i) \). This output has positive probability \( \epsilon^k \). In addition, this output belongs to \( A_k(d, \beta) \) whereas it does not belong to \( A_k(d, \beta - \epsilon) \) as required.

This means, that we can apply Proposition 14 with \( \alpha_k(d, \beta - \epsilon) \) as we know that \( K \) has at least \( \alpha_k(d, \beta - \epsilon)(\binom{m}{k}) \) \( k \)-colorful faces by assumptions of Theorem 12 which is more than \( \alpha_k(d, \beta - \epsilon)(\binom{m}{k}) \). We obtain \( \dim K[N_i] > (\beta_i - \epsilon)n_i - 1 \). By letting \( \epsilon \) tend to 0, we obtain the required \( \dim K[N_i] \geq \beta_i n_i - 1 \). \( \square \)

**Boosting the complex.** In the proof of Proposition 14, we will need the following procedure for boosting the complex. For a given complex \( K \) with vertex set \( N = N_1 \sqcup \cdots \sqcup N_{d+1} \) partitioned as usual, and a non-negative integer \( m \) we define the complex \( K(m) \) as a complex with the vertex set \( N \times [m] = N_1 \times [m] \sqcup \cdots \sqcup N_{d+1} \times [m] \) whose maximal faces are of the form \( S \times [n] \), where \( S \) is a maximal face of \( K \). We will also use the notation \( \delta_k(K) := f_k(K)/\binom{m}{k} \) for the density of \( k \)-colorful faces of \( K \).

**Lemma 15.** Let \( K \) be a simplicial complex with vertex partition \( N = N_1 \sqcup \cdots \sqcup N_{d+1} \) and \( k = (k_1, \ldots, k_{d+1}) \in \mathbb{N}_0^{d+1} \), then

(i) \( \delta_k(K(m)) \geq \delta_k(K) \); and

(ii) if \( K \) is \( d \)-collapsible, then \( K(m) \) is \( d \)-collapsible as well.

**Proof.** Let us start with the proof of (i). If \( \delta_k(K) = 0 \) there is nothing to prove. Thus we may assume that \( \delta_k(K) > 0 \) (equivalently \( f_k(K) > 0 \)) and consequently we have that \( |N_i| \geq k_i \). Let us interpret \( \delta_k(K) \) as the probability that a random \( k \)-tuple of vertices in \( N \) is a simplex of \( K \), and we interpret \( \delta_k(K(m)) \) analogously. Let \( \pi: N \times [m] \to N \) be the projection to the first coordinate. Now, let \( U \) be a \( k \)-tuple of vertices in \( N \times [m] \) taken uniformly at random. Considering the set \( \pi(U) \subseteq N \), it need not be a \( k \)-tuple (this happens exactly when two points in \( U \) have the same image under \( \pi \)) but it can be extended to a \( k \)-tuple \( W \) using that \( |N_i| \geq k_i \) for every \( i \). In sequel \( W \) is an extension of \( \pi(U) \) to a \( k \)-tuple, taken uniformly at random among all possible choices. Because of the choices we made, \( W \) is in fact a \( k \)-tuple of vertices in \( N \) taken uniformly at random. (Note that the choices done in each \( N_i \) or \( N_i \times [m] \) are independent of each other.) Altogether, using \( \mathbb{P} \) for probability, we get

\[
\delta_k(K(m)) = \mathbb{P}[U \in K(m)] = \mathbb{P}[\pi(U) \in K] \geq \mathbb{P}[W \in K] = \delta_k(K).
\]

This shows (i).
For (ii), we follow the idea of splitting a vertex from [AKMM02, Proposition 14(i)] which proves a similar statement for d-Leray complexes. For a complex $K$ and a vertex $v \in K$ let $K^v_{\rightarrow v_1, v_2}$ be a complex obtained from $K$ by splitting the vertex $v$ into two newly introduced vertices $v_1$ and $v_2$. That is, if $V$ is the set of vertices of $K$, then the set of vertices of $K^v_{\rightarrow v_1, v_2}$ is $(V \cup \{v_1, v_2\}) \setminus \{v\}$ assuming $v_1, v_2 \notin V$. The maximal simplices of $K^v_{\rightarrow v_1, v_2}$ are obtained from maximal simplices $S$ of $K$ by replacing $v$ with $v_1$ and $v_2$, if $S$ contains $v$ (otherwise $S$ is kept as it is). Our aim is to show that if $K$ is $d$-collapsible, then $K^v_{\rightarrow v_1, v_2}$ is $d$-collapsible as well. This will prove (ii) because $K_{(m)}$ can be obtained from $K$ by repeatedly splitting some vertex. For the proof, we extend the notation $K^v_{\rightarrow v_1, v_2}$ to setting $K^v_{\rightarrow v_1, v_2} = K$ if $v$ does not belong to $K$.

Let $K_0 = K, K_1, \ldots, K_\ell = \emptyset$ be a sequence such that $K_i$ arises from $K_{i-1}$ by an elementary $d$-collapse. Our task is to show that $K^v_{\rightarrow v_1, v_2} K_{i-1}$-collapses to $K^v_{\rightarrow v_1, v_2} K_i$ for $i \in [\ell]$. This will show the claim as $K^v_{\rightarrow v_1, v_2} = \emptyset$. For simplicity of the notation, we will treat only the elementary $d$-collapse from $K$ to $K_1$ as other steps are analogous. We will assume $v \in K$ as there is nothing to do, if $v \notin K$.

Let $L$ and $M$ be the faces from the definition of the elementary $d$-collapse. That is, $\dim L \leq d - 1$; $M$ is the unique maximal face in $K$ which contains $L$ and $K_1$ is obtained from $K$ by removing all faces that contain $L$, including $L$. We will distinguish three cases according to whether $v \notin L$ or $v \in M$.

If $v \notin M$ (which implies $v \notin L$), then $M$ is the unique maximal face containing $L$ in $K^v_{\rightarrow v_1, v_2}$ and the elementary $d$-collapse removing $L$ and all its superfaces yields $K^v_{\rightarrow v_1, v_2}$.

If $v \in M$ while $v \notin L$, then $(M \cup \{v_1, v_2\}) \setminus \{v\}$ is the unique maximal face containing $L$ in $K^v_{\rightarrow v_1, v_2}$ and the elementary $d$-collapse removing $L$ and all its superfaces yields $K^v_{\rightarrow v_1, v_2}$.

Finally, if $v \in M$ and $v \in L$, then we need to perform the $d$-collapse from $K^v_{\rightarrow v_1, v_2}$ to $K^v_{\rightarrow v_1, v_2}$ by two elementary steps; see Figure 1. First we realize that $(M \cup \{v_1, v_2\}) \setminus \{v\}$ is the unique maximal face containing $(L \cup \{v_1\}) \setminus \{v\}$ in $K^v_{\rightarrow v_1, v_2}$. Because $\dim (L \cup \{v_1\}) \setminus \{v\} = \dim L$, we can perform an elementary $d$-collapse removing $(L \cup \{v_1\}) \setminus \{v\}$ and all its superfaces obtaining a complex $K'$. In $K'$ we have that $(M \cup \{v_2\}) \setminus \{v\}$ is the unique maximal face containing $(L \cup \{v_2\}) \setminus \{v\}$. After removing $(L \cup \{v_2\}) \setminus \{v\}$ and all its superfaces, we get desired $K^v_{\rightarrow v_1, v_2}$ (note that in this case $K^v_{\rightarrow v_1, v_2}$ is indeed obtained from $K^v_{\rightarrow v_1, v_2}$ by removing $(L \cup \{v_1\}) \setminus \{v\}$, $(L \cup \{v_2\}) \setminus \{v\}$ and all their superfaces).

**Density of $P_k(n, d, r)$.** Now, we will provide a formula for density of $P_k(n, d, r)$. In the following computations, we also set $\delta_k(n, d, r) = p_k(n, d, r) / |[N_k]|$ using the notation from the definition of $P_k(n, d, r)$. We get
\[ p_k(n, d, r) = \left\lfloor \frac{N!}{k!} : |S \cap R| \leq d \right\rfloor \]

Then, using \((x)_m := x \cdot (x - 1) \cdots (x - (m - 1))\), the density is given by

\[ \delta_k(n, d, r) = \frac{p_k(n, d, r)}{\prod_{i=1}^{d+1} (n_i)} = \frac{\sum_{\ell=(\ell_1, \ldots, \ell_{d+1}) \in L_d(n)} \prod_{i=1}^{d+1} \left( n_i - r_i \right) r_i (r_i - 1)}{\prod_{i=1}^{d+1} (n_i)}. \] (6)

**Proof of Proposition 14.** For contradiction, let us assume that for every \(i \in [d + 1]\) we have that \(\dim(K[V_i]) \leq \beta_i n_i - 1\). Let us set \(r_i := \dim(K[V_i]) + 1 \leq \beta_i n_i\). Note that the conclusion of Theorem 10 can be restated as \(\delta_k(K) \leq \delta_k(n, d, r)\).

Now we get

\[ \delta_k(K) \leq \liminf_{m \to \infty} \delta_k(K_{(m)}) \] by Lemma 15(i)

\[ \leq \liminf_{m \to \infty} \delta_k(mn, d, mr) \] by Theorem 10 using Lemma 15(ii)

\[ \leq \liminf_{m \to \infty} \delta_k(mn, d, [mn_i]) \] using \(r_i \leq \beta_i n_i\) and monotonicity of \(p_k(n, d, r)\) in \(r\)

\[ = \liminf_{m \to \infty} \sum_{\ell=(\ell_1, \ldots, \ell_{d+1}) \in L_d(n)} \prod_{i=1}^{d+1} \left( n_i - r_i \right) r_i (r_i - 1) \] by (6)

\[ = \sum_{\ell=(\ell_1, \ldots, \ell_{d+1}) \in L_d(n)} \prod_{i=1}^{d+1} \left( n_i - r_i \right) r_i (r_i - 1) \] (using the notation of Theorem 12)

which is a contradiction with assumptions. \(\square\)

**Remark 16.** It would be much more natural to try to avoid boosting and to show directly \(\delta_k(K) \leq \delta_k(n, d, r) \leq \alpha_k(d, \beta)\) in the proof of Proposition 14. The former inequality follows from Theorem 10. However, the latter inequality turned out to be somewhat problematic for us when we attempted to show it directly from the definition of \(\alpha_k(d, \beta)\) and from (6). Thus, in our computations, we take an advantage of the fact that the computations in the limit are easier.

**Tightness of Theorem 12.** We conclude this section by showing that the bound given in Theorem 12 is tight.

Let us fix \(d \in \mathbb{N}\), \(k = (k_1, \ldots, k_{d+1}) \in \mathbb{N}_{0}^{d+1}\) with \(k := k_1 + \cdots + k_{d+1} \geq d+1\) and \(\beta = (\beta_1, \ldots, \beta_{d+1}) \in (0, 1)^{d+1}\) as in the statement of Theorem 12. Let \(0 \leq \alpha' < \alpha_k(d, \beta)\). We will find a complex \(K\) which contains at least \(\alpha' \left( \frac{N!}{k!} \right)\) \(k\)-colorful faces while \(\dim(K[N_i]) \leq \beta_i n_i - 1\) for every \(i \in [d + 1]\) (using the notation of the statement of Theorem 12).

Similarly as in the proof of Theorem 12 let us consider \(\varepsilon > 0\) such that \(\beta - \varepsilon \in (0, 1)^{d+1}\) for \(\varepsilon = (\varepsilon, \ldots, \varepsilon) \in (0, 1)^{d+1}\). In addition, because \(\alpha_k(d, \beta)\) is continuous in \(\beta\) due to its definition (5), we may pick \(\varepsilon\) such that \(\alpha' < \alpha_k(d, \beta - \varepsilon)\). For simplicity of notation, let \(\beta' = (\beta'_1, \ldots, \beta'_{d+1}) := \beta - \varepsilon\).

Now we pick a positive integer \(m\) and set \(n = (m, \ldots, m) \in \mathbb{N}^{d+1}\), that is, \(n_1 = \cdots = n_{d+1} = m\) and \(n = m(d + 1)\) in our standard notation. We also set \(r = (r_1, \ldots, r_{d+1})\) so that \(r_i := \lfloor \beta'_i/m \rfloor\). We assume

\[ \text{This choice of } n \text{ will yield a counterexample where each color class has equal size. It would be also possible to vary the sizes.} \]
that \( m \) is large enough so that \( r_i \geq k_i \) for each \( i \in [d + 1] \). We define families \( N_i \) of convex sets in \( \mathbb{R}^d \) so that each \( N_i \) contains \( r_i \) copies of \( \mathbb{R}^d \) and \( m \) \( - \) \( r_i \) hyperplanes in general position. We also assume that the collection of all hyperplanes in \( N_1, \ldots, N_{d+1} \) is in general position. We set \( K \) to be the nerve of the family \( N = N_1 \sqcup \cdots \sqcup N_{d+1} \).

First, we check \( \dim K[N_i] < \beta d - m - 1 \) provided that \( m \) is large enough. A subfamily of \( N_i \) with nonempty intersection contains at most \( d \) hyperplanes from \( N_i \). Therefore \( \dim K[N_i] < r_i + d = [\beta d] + d < \beta d - m - 1 \) for \( m \) large enough.

Next we check that \( K \) contains at least \( \alpha' ![\binom{N}{d}] \) \( k \)-colorful faces provided that \( m \) is large enough. Partitioning \( N_i \) so that \( R_i \) is the subfamily of the copies of \( \mathbb{R}^d \) and \( R_i \) is the subfamily of hyperplanes, we get

\[ f_k(K) = p_k(n, d, r) \]

from the definition of \( p_k(n, d, r) \). Therefore (6) gives

\[ \delta_k(K) = \sum_{\ell = (\ell_1, \ldots, \ell_{d+1}) \in L_k(d)} \prod_{i=1}^{d+1} \binom{k_i}{\ell_i} (m - [\beta d] \ell_i, ([\beta d] \ell_i)_{k_i} - \ell_i) \prod_{i=1}^{d+1} (m)_{k_i} \]

Passing to the limit (considering the dependency of \( K \) on \( m \)), we get

\[ \lim_{m \to \infty} \delta_k(K) = \sum_{\ell = (\ell_1, \ldots, \ell_{d+1}) \in L_k(d)} \prod_{i=1}^{d+1} \binom{k_i}{\ell_i} (1 - \beta d) \ell_i, ([\beta d] \ell_i)_{k_i} - \ell_i, \alpha_k(d, \beta') \]

Therefore, for \( m \) large enough \( K \) contains at least \( \alpha' ![\binom{N}{d}] \) \( k \)-colorful as \( \alpha' < \alpha_k(d, \beta') \).

### 4 A topological version?

A simplicial complex \( K \) is \( d \)-Leray if the ith reduced homology group \( H_i(L) \) (over \( \mathbb{Q} \)) vanishes for every induced subcomplex \( L \leq K \) and every \( i \geq d \). As we already know, every \( d \)-representable complex is \( d \)-collapsible, and in addition every \( d \)-collapsible complex is \( d \)-Leray [Weg75]. Helly-type theorems usually extend to \( d \)-Leray complexes and such extensions are interesting because they allow topological versions of Helly-type when collections of convex sets are replaced with good covers. We refer to several concrete examples [Hel30, KM05, AKMM02] or to the survey [Tan13].

We believe that it is possible to extend Theorem 7 to \( d \)-Leray complexes:

**Conjecture 17** (The optimal colorful fractional Helly theorem for \( d \)-Leray complexes). Let \( K \) be a \( d \)-Leray simplicial complex with the set of vertices \( N = N_1 \sqcup \cdots \sqcup N_{d+1} \) divided into \( d + 1 \) disjoint subsets. Let \( n_i := |N_i| \) for \( i \in [d + 1] \) and assume that \( K \) contains at least \( \alpha n_1 \cdots n_{d+1} \) \( \alpha \)-colorful \( d \)-faces for some \( \alpha \in (0, 1) \). Then there is \( i \in [d + 1] \) such that \( \dim K[N_i] \geq (1 - (1 - \alpha)^{1/(d+1)}) n_i - 1 \).

In fact, our original approach how to prove Theorem 7 was to prove directly Conjecture 17. Indications that this could be possible are that both the optimal fractional Helly theorem [AKMM02, Kal02] and the colorful Helly theorem [KM05] hold for \( d \)-Leray complexes. In addition, there is a powerful tool, algebraic shifting, developed by Kalai [Kal02], which turned out to be very useful in attacking similar problems.

In the remainder of this section we briefly survey a possible approach towards Conjecture 17 but also the difficulty that we encountered. Because we do not really prove any new result in this section, our description is only sketchy.

Our starting point is the proof of the optimal fractional Helly theorem for \( d \)-Leray complexes. The key ingredient is the following theorem of Kalai [AKMM02, Theorem 13].

**Theorem 18.** Let \( K \) be a \( d \)-Leray complex and \( f_0(K) = n \). Then \( f_d(K) > \binom{n}{d+1} - \binom{n-r}{d+1} \) implies \( f_{d+r}(K) > 0 \) (where \( f(K) \) denotes the \( f \)-vector of \( K \)).

As far as we can judge, the only proof of Theorem 18 in the literature follows from the first and the third sentence in the following remark in [Kal02]:
“It is not hard to see (although it has been overlooked for a long time) that the class of $d$-Leray complexes (for some $d$) with complete $(d - 1)$-dimensional skeletons is precisely the Alexander dual of the class of Cohen-Macaulay complexes. This observation implies that the fact that shifting preserves the Leray property easily follows from the fact that shifting preserves the Cohen-Macaulay property. Moreover, it shows that the characterization of face numbers of $d$-Leray complexes follows from the corresponding characterization for Cohen-Macaulay complexes.”

For completeness we add that the characterization of face numbers of Cohen-Macaulay complexes has been done by Stanley [Sta75]. Given a simplicial complex $K$ on vertex set $V$ its Alexander dual is a simplicial complex defined as $K^*:=\{\sigma \subseteq V: V \setminus \sigma \notin K\}$. We skip the definition of Cohen-Macaulay complex because we will only use it implicitly but we refer, for example, to [Kal02, §4] for more details.

A simplicial complex $K$ on vertex set $[n]$ is called shifted if for all integers $i$ and $j$ with $1 \leq i < j \leq n$ and all faces $A$ of $K$ such that $j \in A$ and $i \notin A$, the set $(A \setminus \{j\}) \cup \{i\}$ is a face of $K$. Exterior algebraic shifting is a function that associates to a simplicial complex $K$ a shifted complex $K^*$, while preserving many interesting invariants of $K$. Below we list some properties of exterior algebraic shifting that we will use. A simplicial complex is pure if all its inclusion-maximal faces have the same dimension.

**Theorem 19.**

(i) [Kal02][Theorem 2.1] Exterior algebraic shifting preserves the $f$-vector.

(ii) [Kal02][Theorem 4.1] If $K$ is Cohen-Macaulay, then $K^*$ is Cohen-Macaulay, in particular, pure.

(iii) [Kal02][3.5.6] Exterior algebraic shifting and Alexander duality commute.

The next lemma is a possible replacement of the third sentence in Kalai’s remark how to prove Theorem 18. We prove it as motivation for the tools we would need in the colorful scenario.

**Lemma 20.** Let $K$ be a $d$-Leray complex on $[n]$ with complete $(d - 1)$-skeleton and let $D = \dim(K) + 1$. Then $K^* \subseteq \Delta_{D-d-1} \ast \Delta_{n-D+d-1}$.

**Proof.** By the first sentence of Kalai’s remark, the Alexander dual $K^*$ of $K$ is a Cohen-Macaulay complex. By the definition of Alexander dual, it has dimension $n - d - 2$ and contains complete $(n - D - 2)$-skeleton. Hence, properties (i) and (ii) of Theorem 19 imply that the exterior algebraic shifting $(K^*)^e$ of $K^*$ is a pure shifted complex of dimension $n - d - 2$ with complete $(n - D - 2)$-skeleton. If we take any subset $A$ of size $n - D - 1$ in $(K^*)^e$, then $A$ is a face and by purity there must be a face of size $n - d - 1$ that contains $A$. Now, since $(K^*)^e$ is shifted we have that $\{1, 2, \ldots, D - d\} \cup A \in (K^*)^e$. This implies that $\Delta_{D-d-1} \ast \Delta_{n-D+d-1} \subseteq (K^*)^e = (K^*)^*$, by Theorem 19(iii). Taking Alexander dual from both sides proves the first part of the statement. (Using that $(L^*)^* = L^*; L_1 \subseteq L_2 \Rightarrow L_1^* \subseteq L_2^*; \text{ and } (\Delta_{D-d-1} \ast \Delta_{n-D+d-1})^* = \Delta_{D-d-1} \ast \Delta_{n-D+d-1}^*$).

For completeness, Theorem 18 quickly follows from Lemma 20. Indeed, if $K$ is $d$-Leray such that $f_{d+r}(K) = 0$, then $D := \dim(K) + 1 \leq d + r$. In addition, we can assume without loss of generality that $K$ contains complete $(d - 1)$-skeleton. Consequently, Lemma 20 gives $f_d(K) = f_d(K^*) \leq f_d(\Delta_{D-d-1} \ast \Delta_{n-D+d-1}) = (n \choose d+1) - (n-d+1) \leq (n \choose d+1) - (n-d+1)$.

Now, in order to attack Conjecture 17, we would like to do something similar in colorful setting. In particular, we need to preserve the colorful $f$-vector. Babson and Novik [BN06] give a definition of colorful algebraic shifting which preserves the colorful $f$-vector. Nevertheless, the conjecture does not follow immediately from their result as the Alexander dual of a $d$-Leray complex is not in general balanced.

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