THE DISCRETE HOMOTOPY PERTURBATION SUMUDU TRANSFORM METHOD FOR SOLVING PARTIAL DIFFERENCE EQUATIONS

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Abstract. In this paper, we introduce a combined form of the discrete Sumudu transform method with the discrete homotopy perturbation method to solve linear and nonlinear partial difference equations. This method is called the discrete homotopy perturbation Sumudu transform method (DHPSTM). The results reveal that the introduced method is very efficient, simple and can be applied to other linear and nonlinear difference equations. The nonlinear terms can be easily handled by use of He’s polynomials.

1. Introduction. Partial difference equations are types of difference equations that involve functions of two or more independent variables. Such equations arise in applications involving population dynamics with spatial migrations, chemical reactions, the approximation of solutions of partial differential equations by finite difference methods, random walk problems, the study of molecular orbits, dynamical systems, economics, biology and other fields.

In the early 1990’s, Watugala introduced the Sumudu transform and applied it to solve ordinary differential equations[30, 31, 32]. Belgacem and Karaballi introduced fundamental properties of Sumudu transform[9, 10]. There are several studies in this field[2, 3, 5, 6, 8, 11, 12, 14, 15, 16, 17, 26]. Recently, discrete Sumudu transform was defined in [23].

The homotopy perturbation method was first proposed by He in 1998[19]. Recently, the discrete homotopy perturbation method (DHPM) was used to obtain the numerical solution of Burgers’ equation and heat equation[33].

In this paper, we use discrete homotopy perturbation Sumudu transform method (DHPSTM) including discrete STM and DHPM in order to find solution of linear and nonlinear partial difference equations. Although the HPSTM has been widely applied to solve partial differential equations, to the best of our knowledge until

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now, the HPSTM was regarded only for the continuous equations. There is no discrete version HPSTM ever used to solve linear or nonlinear PDEs. This paper proposed the DHPSTM, and finds the DHPSTM has similar advantages of continuous HPSTM. To illustrate the method we apply the DHPSTM to partial difference equations.

2. Preliminaries and notations.

2.1. Discrete Homotopy Perturbation Method (DHPM). Consider the following general partial difference equation

\[ A(U_{m,n}) - f_{m,n} = 0, \quad m, n \in \mathbb{N}_0, \quad (1) \]

where \( A \) is a general difference operator. \( f_{m,n} \) is a given discretized function. \( U_{m,n} \) and \( f_{m,n} \) denote discrete approximations of \( U(x,t) \) and \( f(x,t) \) at the mesh point \((mh, n\tau)\), respectively. \( h \) is the space step in \( x \) direction, and \( \tau \) represents the increment in time. In this work we will take \( h = \tau = 1 \).

The operator \( A \) can be divided into two parts, which are \( L \) and \( N \), where \( L \) is a linear and \( N \) is non-linear operator. Therefore Eq(1) can be rewritten as follows:

\[ L(U_{m,n}) + N(U_{m,n}) - f_{m,n} = 0. \quad (2) \]

By constructing the homotopy technique to Eq(2), we have a homotopy in the form

\[ H(v_{m,n}(p), p) = (1 - p)[L(v_{m,n}(p)) - L(U_{m_0,n})] + p[A(v_{m,n}(p)) - f_{m,n}] = 0, \quad (3) \]

where \( p \in [0, 1] \) is an embedding parameter, \( U_{m_0,n} \) is an initial approximate solution of the original equation, which satisfies the boundary conditions. From Eq(3) we have

\[ H(v_{m,n}(0), 0) = L(v_{m,n}(0)) - L(U_{m_0,n}) = 0 \]
\[ H(v_{m,n}(1), 1) = A(v_{m,n}(1)) - f_{m,n} = 0 \]

changing the process of \( p \) from zero to unity is just change of \( v_{m,n}(p) \) from \( U_{m_0,n} \) to \( U_{m,n} \).

In topology, that is called homotopy. Assume that the solution of Eq(3) can be written as a series in \( p \):

\[ v_{m,n}(p) = v_{m_0,n} + pv_{m_1,n} + p^2v_{m_2,n} + \cdots \quad (4) \]

By setting \( p = 1 \), one can get an approximate solution of Eq(1)

\[ U_{m,n} = \lim_{p \to 1} v_{m,n}(p) = v_{m_0,n} + v_{m_1,n} + v_{m_2,n} + \cdots \quad (5) \]

2.2. Discrete Sumudu Transform. Here, we introduce the Discrete Sumudu Transform (DST).

**Definition 2.1** ([23]). If \( f : \mathbb{N}_0 \to \mathbb{C} \) is a function, then the discrete Sumudu transform is defined by

\[ S_d\{f(k)\}(u) = \frac{1}{u} \sum_{k=0}^{\infty} f(k) \left( \frac{u}{u+1} \right)^{k+1} \quad (6) \]

for all values of \( u \neq -1 \) such that the series converges.

If \( R = \limsup \frac{|f(k)|}{k^2} \), then we have one of the following three cases:

(i) If \( 0 < R < \infty \), then the series (6) converges for \( \frac{|u+1|}{u} > R \) and diverges elsewhere;

(ii) If \( R = 0 \), then the series (6) converges for all \( u \) except possibly when \( u = -1 \);

(iii) If \( R = \infty \), the series (6) diverges everywhere.
Below there are some properties of the discrete Sumudu transform:

**Lemma 2.2 ([23]).** If $S_d\{f(k)\}(u) = F(u)$ for $|\frac{u+1}{u}| > A \geq R$, then for the same values of $u$

$$S_d\{f(k + 1)\}(u) = \frac{u + 1}{u} F(u) - \frac{f(0)}{u}$$

and in general

$$S_d\{f(k + m)\}(u) = \left(\frac{u + 1}{u}\right)^m F(u) - \sum_{n=0}^{m-1} \frac{\left(u + 1\right)^{m-n-1}}{u^{m-n}} f(n).$$

**Lemma 2.3 ([23]).** If $S_d\{f(k)\}(u) = F(u)$ for $|\frac{u+1}{u}| > A$, then for the same values of $u$

$$S_d\{\Delta f(k)\}(u) = \frac{1}{u} \left[F(u) - f(0)\right]$$

and in general

$$S_d\{\Delta^m f(k)\}(u) = u^{-n} \left[F(u) - \sum_{i=0}^{n-1} u^i |\Delta^i f(k)|_{k=0}\right],$$

where $\Delta f(k) = f(k + 1) - f(k)$ and $\Delta^0 f(k) = f(k)$.

### 2.3. Discrete Homotopy Perturbation Sumudu Transform Method (DHPSTM)

To illustrate this method we consider the partial difference equation of the form

$$\Delta_n U_{m,n} - \phi_{m,n} \Delta^2_m U_{m,n} = g_{m,n}, \quad m, n \in \mathbb{N}_0,$$

with subject to initial condition

$$U_{m,0} = f_m,$$

where $\phi_{m,n}$ and $f_m$ are given discrete functions. $g_{m,n}$ is the source term. The forward partial differences $\Delta_m$ and $\Delta_n$ are defined as usual, i.e., $\Delta_m U_{m,n} = U_{m+1,n} - U_{m,n}$ and $\Delta_n U_{m,n} = U_{m,n+1} - U_{m,n}$. Second order partial difference is defined by $\Delta^2_m U_{m,n} = \Delta_m (\Delta_m U_{m,n})$ and $\Delta^0_n U_{m,n} = U_{m,n}$. $\Delta^0_n U_{m,n} = U_{m,n}$.

By applying discrete Sumudu transform on both sides of Eq(7) with respect to $n$, we get

$$S_d\{\Delta_n U_{m,n}\} = S_d\{\phi_{m,n} \Delta^2_m U_{m,n} + g_{m,n}\}$$

$$\frac{1}{u} \left[S_d\{U_{m,n}\} - f_m\right] = S_d\{\phi_{m,n} \Delta^2_m U_{m,n} + g_{m,n}\}$$

Taking the inverse discrete Sumudu transform to Eq(9) we have

$$U_{m,n} = G_{m,n} + S_d^{-1}\{u S_d\{\phi_{m,n} \Delta^2_m U_{m,n}\}\},$$

where $G_{m,n}$ represents the term arising from the source term and the prescribed initial conditions.

According to DHPM, we construct a homotopy in the as following

$$(1 - p)[v_{m,n}(p) - U_{m,0,n}] + p[v_{m,n}(p) - G_{m,n} - S_d^{-1}\{u S_d\{\phi_{m,n} \Delta^2_m v_{m,n}(p)\}\}] = 0$$

or equivalently

$$v_{m,n}(p) = U_{m,0,n} - pU_{m,0,n} + pG_{m,n} + pS_d^{-1}\{u S_d\{\phi_{m,n} \Delta^2_m v_{m,n}(p)\}\} = 0.$$
Let
\[ v_{m,n}(p) = \sum_{i=0}^{\infty} p^i v_{m_i,n}, \quad U_{m,0} = U_{m_0,n}. \] (12)

Substituting (12) into (11) and comparing the coefficients of the term with identical powers of \( p \), lead to
\[ p^0 : v_{m_0,n} = U_{m_0,n} = U_{m,0} = f_m \]
\[ p^1 : v_{m_1,n} = -U_{m_0,n} + G_{m,n} + S_d^{-1}\{uS_d\{\phi_{m,n}\Delta_m^2 v_{m_0,n}\}\} \]
\[ p^2 : v_{m_2,n} = S_d^{-1}\{uS_d\{\phi_{m,n}\Delta_m^2 v_{m_1,n}\}\} \]
\[ p^3 : v_{m_3,n} = S_d^{-1}\{uS_d\{\phi_{m,n}\Delta_m^2 v_{m_2,n}\}\} \]
\[ \vdots \]

When the limit get for \( p \to 1 \), the solutions obtain as following:
\[ U_{m,n} = \lim_{p \to 1} \left( \sum_{i=0}^{\infty} p^i v_{m_i,n} \right) \]
\[ = v_{m_0,n} + v_{m_1,n} + v_{m_2,n} + \cdots \]
\[ = \sum_{i=0}^{\infty} v_{m_i,n}. \] (13)

3. Applications of DHPSTM. In this section, we shall examine some applications of our newly developed method through the following examples.

**Example 1.** We consider the following partial difference equation
\[ \Delta_2 U_{m,n} = \Delta_2^2 U_{m,n}, \quad m,n \in \mathbb{N}_0, \] (14)
with the initial condition
\[ U_{m,0} = 5^m. \] (15)

We construct the discrete Sumudu transform with respect to \( n \) for Eq (14) as follows:
\[ \frac{1}{u} \left[ S_d\{U_{m,n}\} - U_{m,0} \right] = S_d\{\Delta_2^m U_{m,n}\} \]
\[ S_d\{U_{m,n}\} = 5^m + uS_d\{U_{m+2,n} - 2U_{m+1,n} + U_{m,n}\} \]
\[ U_{m,n} = S_d^{-1}\{5^m + uS_d\{U_{m+2,n} - 2U_{m+1,n} + U_{m,n}\}\}. \]

To solve initial value problem (14)-(15) by DHPSTM, we construct the following homotopy:
\[ (1-p)[v_{m,n}(p) - U_{m_0,n}] + p[v_{m,n}(p) - 5^m - S_d^{-1}\{uS_d\{v_{m+2,n}(p) - 2v_{m+1,n}(p) + v_{m,n}(p)\}\}] = 0. \] (16)

Substituting (12) into (16) and comparing the coefficients of the term with identical powers of \( p \), lead to
\[ p^0 : v_{m_0,n} - U_{m_0,n} = 0 \]
\[ p^1 : v_{m_1,n} + U_{m_0,n} - 5^m - S_d^{-1}\{uS_d\{v_{m+2,n} - 2v_{m+1,n} + v_{m,n}\}\} = 0 \]
\[ p^2 : v_{m_2,n} - S_d^{-1}\{uS_d\{v_{m+2,n} - 2v_{m+1,n} + v_{m,n}\}\} = 0 \]
\[ p^3 : v_{m3,n} - S_d^{-1}\{uS_d\{v_{m2+2,n} - 2v_{m2+1,n} + v_{m2,n}\}\} = 0 \]
\[ \vdots \]
\[ p^l : v_{ml,n} - S_d^{-1}\{uS_d\{v_{m_{l-1}+2,n} - 2v_{m_{l-1}+1,n} + v_{m_{l-1},n}\}\} = 0. \]

When the initial value is
\[ v_{m0,n} = U_{m,0} = 5^m, \]
then the following recurrence results are obtained
\[ v_{m1,n} = 4^2 5^m n \]
\[ v_{m2,n} = \frac{4^4 5^m n^{(2)}}{2!} \]
\[ v_{m3,n} = \frac{4^6 5^m n^{(3)}}{3!} \]
\[ \vdots \]
\[ v_{ml,n} = \frac{4^{2l} 5^m n^{(l)}}{l!}, \]

where \( n^{(l)} = n(n-1)(n-2) \cdots (n-l+1) \).

From (13) we have
\[ U_{m,n} = \sum_{j=0}^{\infty} v_{mj,n} \]
\[ = 4^2 5^m n + \frac{4^4 5^m n^{(2)}}{2!} + \frac{4^6 5^m n^{(3)}}{3!} + \cdots \]
\[ = \sum_{j=0}^{\infty} \frac{4^{2j} 5^m n^{(j)}}{j!}. \]

Figure 1 shows the DHPSTM solution of initial value problem (14)-(15) with side view.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Numerical illustration of solution \( U_{m,n} \) by DHPSTM}
\end{figure}
Example 2. Consider the discrete diffusion equation
\[ \Delta_m U_{m,n} = \Delta^2_m U_{m,n} + m \Delta_m U_{m,n} + U_{m,n}, \quad m, n \in \mathbb{N}_0, \] (17)
with subject to initial condition
\[ U_{m,0} = m. \] (18)

We construct the following discrete Sumudu transform with respect to \( n \) for Eq(17):
\[
\frac{1}{\ell!} \left[ S_d \{ U_{m,n} \} - U_{m,0} \right] = S_d \{ \Delta^2_m U_{m,n} + m \Delta_m U_{m,n} + U_{m,n} \}
\]
\[
S_d \{ U_{m,n} \} = m + u S_d \{ U_{m+2,n} + (m-2)U_{m+1,n} + (2-m)U_{m,n} \}
\]
\[
U_{m,n} = S^{-1}_d \{ m + u S_d \{ U_{m+2,n} + (m-2)U_{m+1,n} + (2-m)U_{m,n} \} \}
\]
\[
U_{m,n} = m + S^{-1}_d \{ u S_d \{ U_{m+2,n} + (m-2)U_{m+1,n} + (2-m)U_{m,n} \} \}.
\]

To solve initial value problem (17)-(18) by DHPSTM, we construct the following homotopy:
\[
(1 - p) [v_{m,n}(p) - U_{m,n}] + p [v_{m,n}(p) - m - S^{-1}_d \{ u S_d \{ v_{m+2,n}(p) + (m-2)v_{m+1,n}(p) + (2-m)v_{m,n}(p) \} \} = 0. \] (19)

Substituting (12) into (19) and comparing the coefficients of the term with identical powers of \( p \), lead to
\[
p^0 : v_{m_0,n} - U_{m_0,n} = 0
\]
\[
p^1 : v_{m_1,n} + U_{m_0,n} - m - S^{-1}_d \{ u S_d \{ v_{m_0+2,n} + (m-2)v_{m_0+1,n} + (2-m)v_{m_0,n} \} \} = 0
\]
\[
p^2 : v_{m_2,n} - S^{-1}_d \{ u S_d \{ v_{m_1+2,n} + (m-2)v_{m_1+1,n} + (2-m)v_{m_1,n} \} \} = 0
\]
\[
p^3 : v_{m_3,n} - S^{-1}_d \{ u S_d \{ v_{m_2+2,n} + (m-2)v_{m_2+1,n} + (2-m)v_{m_2,n} \} \} = 0
\]
\[
\vdots
\]
\[
p^\ell : v_{m_\ell,n} - S^{-1}_d \{ u S_d \{ v_{m_{\ell-1}+2,n} + (m-2)v_{m_{\ell-1}+1,n} + (2-m)v_{m_{\ell-1},n} \} \} = 0.
\]

When the initial value is
\[ v_{m_0,n} = U_{m_0,n} = U_{m,0} = m, \]
then the following recurrence results are obtained
\[ v_{m_1,n} = 2mn \]
\[ v_{m_2,n} = \frac{2^2 m(n)n}{2!} \]
\[ v_{m_3,n} = \frac{2^3 m(n)n(n)}{3!} \]
\[ \vdots \]
\[ v_{m_\ell,n} = \frac{2^\ell m(n)n(\ell)}{\ell!} \]

From (13) we have
\[ U_{m,n} = \sum_{j=0}^{\infty} \frac{2^j m(n)n(j)}{j!}. \]

Figure 2 shows the DHPSTM solution of initial value problem (17)-(18) with side view.
Example 3. Consider the discrete nonlinear Burgers’ equation

$$\Delta_n U_{m,n} + U_{m,n} \Delta_m U_{m,n} = \Delta_m^2 U_{m,n}, \quad m, n \in \mathbb{N}_0,$$

with subject to initial condition

$$U_{m,0} = 2m.$$  \hfill (21)

We construct the discrete Sumudu transform with respect to $n$ for Eq(20) as follow

$$\frac{1}{u} \left[ S_d\{U_{m,n}\} - U_{m,0}\right] = S_d\{\Delta_m^2 U_{m,n} - U_{m,n} \Delta_m U_{m,n}\} = 2m + u S_d\{U_{m+2,n} - 2U_{m+1,n} + U_{m,n} - U_{m,n}U_{m+1,n} + U_{m,n}^2\}$$

$$U_{m,n} = S_d^{-1}\{2m + u S_d\{U_{m+2,n} - 2U_{m+1,n} + U_{m,n} - U_{m,n}U_{m+1,n} + U_{m,n}^2\}\}$$

To solve initial value problem(20)-(21)by DHPSTM, we construct the following homotopy:

$$(1 - p)[v_{m,n}(p) - U_{m,0,n}] + p[v_{m,n}(p) - 2m - S_d^{-1}\{u S_d\{v_{m+2,n}(p) - 2v_{m+1,n}(p) + v_{m,n}(p) - v_{m,n}(p)v_{m+1,n}(p) + v_{m,n}^2(p)\}\} = 0.$$  \hfill (22)

Substituting (12) into (22) and comparing the coefficients of the term with identical powers of $p$, lead to

$$p^0 : v_{m_0,n} - U_{m_0,n} = 0$$

$$p^1 : v_{m_1,n} + U_{m_0,n} - 2m - S_d^{-1}\{u S_d\{v_{m_0+2,n} - 2v_{m_0+1,n} + v_{m_0,n} - v_{m_0,n}v_{m+1,n} + v_{m_0,n}^2\}\} = 0$$

$$p^2 : v_{m_2,n} - S_d^{-1}\{u S_d\{v_{m_1+2,n} - 2v_{m_1+1,n} + v_{m_1,n} - v_{m_0,n}v_{m+1,n} + v_{m_1,n}v_{m+1,n} + 2v_{m_0,n}v_{m+1,n}\}\} = 0$$
\[ p^3 : v_{m_3,n} - S^{-1}_d \{ uS_d \{ v_{m_2+2,n} - 2v_{m_2+1,n} + v_{m_2,n} - v_{m_0,n}v_{m_2+1,n} \\
- v_{m_1,n}v_{m_1+1,n} - v_{m_2,n}v_{m_0+1,n} + \frac{v_{m_1,n}^2}{2} + 2v_{m_0,n}v_{m_2,n} \} \} = 0 \]

\[ : \]

\[ p^\ell : v_{m_\ell,n} - S^{-1}_d \{ uS_d \{ \Delta^2_m v_{m_{\ell-1},n} - \sum_{k=0}^{\ell-1} \left( v_{m_k,n}\Delta_m v_{m_{\ell-1-k},n} \right) \} \} = 0. \]

When the initial value is

\[ v_{m_0,n} = U_{m_0,n} = U_{m,0} = 2m, \]

then the following recurrence results are obtained

\[ v_{m_1,n} = -4mn \]
\[ v_{m_2,n} = 8mn^{(2)} \]
\[ v_{m_3,n} = -16mn^{(3)} - 8mn^{(2)} \]
\[ v_{m_4,n} = 32mn^{(4)} + \frac{160}{3}mn^{(3)} \]

\[ : \]

From (13) we have

\[ U_{m,n} = v_{m_0,n} + v_{m_1,n} + v_{m_2,n} + v_{m_3,n} + v_{m_4,n} \]
\[ = 2m - 4mn + 8mn^{(2)} - 16mn^{(3)} - 8mn^{(2)} + 32mn^{(4)} + \frac{160}{3}mn^{(3)} + \ldots \]

Figure 3 shows the DHPSTM approximate solution of \( U_{m,n} \) with side view.

Figure 3. Numerical illustration of approximate solution \( U_{m,n} \) by DHPSTM

4. Discussions and conclusions. Discrete homotopy perturbation Sumudu transform method is applied successfully for finding exact solutions for linear difference equations and approximate solutions for nonlinear difference equations. The efficiency and accuracy of the proposed method is demonstrated test problems. The results showed that the DHPSTM is extremely simple and easy to handle the nonlinear terms. The basic ideas described, and the general guidelines expanded in this
paper is expected to be further employed to solve other similar linear and nonlinear partial difference equations.

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