Limiting Spectrum of Randomized Hadamard Transform and Optimal Iterative Sketching Methods

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Abstract

We provide an exact analysis of the limiting spectrum of matrices randomly projected either with the subsampled randomized Hadamard transform, or truncated Haar matrices. We characterize this limiting distribution through its Stieltjes transform, a classical object in random matrix theory, and compute the first and second inverse moments. We leverage the limiting spectrum and asymptotic freeness of random matrices to obtain an exact analysis of iterative sketching methods for solving least squares problems. Our results also yield optimal step-sizes and convergence rates in terms of simple closed-form expressions. Moreover, we show that the convergence rate for Haar and randomized Hadamard matrices are identical, and uniformly improve upon Gaussian random projections. The developed techniques and formulas can be applied to a plethora of randomized algorithms that employ fast randomized Hadamard dimension reduction.

Keywords: Randomized algorithms; Random projection; Random matrices; Subsampled randomized Hadamard transform; Haar matrices; Least-squares optimization.

1 Introduction

Random projections are a classical way of performing dimensionality reduction, and are widely used in many algorithmic and learning contexts [27, 16, 30, 9]. In this work, we compare the limiting spectral distributions of two classical subspace embeddings, that is, Gaussian embeddings and the subsampled randomized Hadamard transform (SRHT), and we study the performance of a randomized method, namely, the iterative Hessian sketch [23], in the context of (overdetermined) least-squares problems,

$$x^* := \arg\min_{x \in \mathbb{R}^d} \left\{ f(x) := \frac{1}{2} \|Ax - b\|^2 \right\},$$

where $A \in \mathbb{R}^{n \times d}$ is a given data matrix with $n \geq d$ and $b \in \mathbb{R}^n$ is a vector of observations. For simplicity of notations, we will assume throughout this work that $\text{rank}(A) = d$.

Many works have studied randomized algorithms [4, 24, 10, 22] for solving (1), based on sketching methods. The latter involve using a random matrix $S \in \mathbb{R}^{m \times n}$ to project the data $A$ and/or $b$
to a lower dimensional space \( m \ll n \), and then approximately solving the least-squares problem using the sketch \( SA \) and/or \( Sh \). The most classical sketch is a matrix \( S \in \mathbb{R}^{m \times n} \) with independent and identically distributed (i.i.d.) Gaussian entries \( N(0, m^{-1}) \), for which the matrix multiplication \( S \cdot A \) requires in general \( \mathcal{O}(mnd) \) basic operations (using classical matrix multiplication). This is larger than the cost \( \mathcal{O}(nd^2) \) of solving \( \mathbf{1} \) through standard matrix factorization methods, provided that \( m \gg d \). Another well-studied embedding is the (truncated) \( m \times n \) Haar matrix \( S \), whose rows are orthonormal and with range uniformly distributed among the subspaces of \( \mathbb{R}^n \) with dimension \( m \). However, it requires time \( \mathcal{O}(nm^2) \) to be formed, through a Gram-Schmidt procedure, which is also larger than \( \mathcal{O}(nd^2) \). An alternative embedding which verifies orthogonality properties is the SRHT \( \mathbf{1} \), which is based on the Walsh-Hadamard transform. Due to the recursive structure of the latter, the sketch \( SA \) can be formed in \( \mathcal{O}(nd \log m) \) time, so that the SRHT is often viewed as a standard reference point for comparing sketching algorithms.

It has been observed in several contexts that random projections with i.i.d. entries degrade the performance of the approximate solution compared to orthogonal projections \([16] [17] [9]\). Consequently, along with computational considerations, these empirical results suggest to consider the SRHT over Gaussian projections. On the other hand, in order to pick sharp algorithm’s parameters and obtain optimal performance, it is usually necessary to have a tight characterization of the eigenvalues of the matrix \( U^\top S^\top SU \), where \( U \) is the matrix of left singular vectors of \( A \). For Gaussian embeddings, this is the case thanks to standard tight Gaussian concentration bounds.

In this work, we aim to provide a tight characterization of the SRHT spectrum, in the asymptotic regime where the relevant dimensions and sample sizes go to infinity, and can have arbitrary aspect ratios. As for an application of interest, using the standard prediction (semi-)norm \( \|A(\hat{x} - x^*)\|^2 \) as the evaluation criterion for an estimator \( \hat{x} \), we aim to design an optimal version of the iterative Hessian sketch (IHS). Iterative methods (e.g., gradient descent or the conjugate gradient algorithm) have time complexity which usually scales proportionally to the condition number \( \kappa \) of the matrix \( A \) (or \( \sqrt{\kappa} \)), and this becomes prohibitively large when \( \kappa \gg 1 \). The IHS addresses this issue as follows. Given \( x_0 \in \mathbb{R}^d \), it uses a pre-conditioned Heavy-ball update with step sizes \( \{\mu_t\} \) and momentum parameters \( \{\beta_t\} \), given by

\[
x_{t+1} = x_t - \mu_t H_t^{-1} \nabla f(x_t) + \beta_t (x_t - x_{t-1}),
\]

where the Hessian \( H = A^\top A \) of the objective function \( f(x) \) is approximated, at each iteration, by \( H_t = A^\top S_t^\top S_t A \), and \( S_0, \ldots, S_t, \ldots \) are i.i.d. sketching matrices. From now on, we refer to the i.i.d. property of the sketching matrices as "refreshed" matrices. For Gaussian projections, \([14]\) provided an exact analysis of the error \( \|A(x_t - x^*)\|^2 \), which scales as \( (d/m)^t \). This rate makes intuitive sense, since the limiting spectral distribution of the Wishart matrix \( U^\top S^\top SU \) only depends on (the limit of) the aspect ratio \( (d/m) \). However, with SRHT embeddings, they observed that the predicted error – based on standard finite-sample bounds \([25]\) on the edge eigenvalues of \( U^\top S^\top SU \) – underestimates the practical performance of the IHS. Thus, by a refined (asymptotic) analysis of this spectrum, we aim to explain this theory-practice gap, and design an even better algorithm.

Beyond the IHS, there are many other efficient iterative methods which aim to address the aforementioned conditioning issue, based on an SRHT sketch of the data (or closely related sketches based on the Fourier transform). Randomized right pre-conditioning methods \([4] [24]\) compute first a matrix \( P \) – which itself depends on \( SA \) – such that the condition number of \( AP^{-1} \) is \( O(1) \), and then apply any standard iterative algorithm to the pre-conditioned least-squares objective \( \|AP^{-1}y - b\|^2 \). Besides least-squares, SRHT sketches are widely used for a wide range of applications across numerical linear algebra, statistics and convex optimization, such as low-rank matrix factorization \([12] [29]\), kernel regression \([31]\), random subspace optimization \([15]\), or, sketch and solve.
linear regression \[8\]. Hence, a refined analysis of the SRHT may also lead to better algorithms in these fields.

Our technical analysis is based on asymptotic random matrix theory \[5, 7\]. Classical results such as the Marchenko-Pastur law do not address well the case of the SRHT, and we leverage recent results on \textit{asymptotically liberating sequences} established by \[2\] (see also \[26\] for prior work). Further, we are inspired by the work of \[8\], who were the first, to our knowledge, to leverage these asymptotic freeness results in order to study the SRHT.

Throughout the paper, we will consistently use the following assumptions and notations for the aspect ratios, \( \gamma := \lim_{n,d \to \infty} \frac{\rho}{\xi} \in (0,1) \), \( \xi := \lim_{n,m \to \infty} \frac{m}{n} \in (\gamma, 1) \) and \( \rho_g := \frac{\gamma}{\xi} \in (0,1) \), and the subscript \( g \) (resp. \( h \)) will refer to Gaussian-related (resp. Haar-related) quantities.

We use the notations \( \|z\| \equiv \|z\|_2 \) for the Euclidean norm of a real vector \( z \), \( \|M\|_2 \) for the operator norm of a matrix \( M \), and \( \|M\|_F \) for its Frobenius norm. For a sequence of iterates \( \{x_t\} \), we denote the error vector \( \Delta_t := U^\top A(x_t - x^*) \), where \( U \) is the \( n \times d \) matrix of left singular vectors of \( A \). In particular, we have that \( \|\Delta_t\|^2 = \|A(x_t - x^*)\|^2 \).

### 1.1 Overview of our results and contributions

We characterize the respective limiting spectral distributions of the matrix \( U^\top S^\top SU \), where \( U \) is an \( n \times d \) matrix with orthonormal columns and \( S \) is an \( m \times n \) Haar or SRHT matrix, and we show that these distributions are equal by finding explicitly their respective Stieltjes transforms. As an immediate consequence of our analysis, we provide the following new trace calculations

\[
\begin{align*}
\theta_{1,h} &= \left( \lim_{n \to \infty} \frac{1}{d} \operatorname{tr} \left( (U^\top S^\top SU)^{-1} \right) \right) = \frac{1-\gamma}{\xi-\gamma}, \\
\theta_{2,h} &= \left( \lim_{n \to \infty} \frac{1}{d} \operatorname{tr} \left( (U^\top S^\top SU)^{-2} \right) \right) = \frac{(1-\gamma)(\gamma^2 + \xi - 2\gamma\xi)}{(\xi-\gamma)^3}.
\end{align*}
\]

As an application of the above trace calculations, we characterize explicitly the optimal step sizes \( \mu_t \) and momentum parameters \( \beta_t \) of the IHS with Haar embeddings. That is, we find that at any time step \( t \geq 1 \),

\[
\lim_{n \to \infty} \frac{\mathbb{E}\|\Delta_t\|^2}{\|\Delta_0\|^2} = \rho_h^t,
\]

where the convergence rate \( \rho_h \) is given by \( \rho_h := \rho_g \cdot \frac{\xi(1-\xi)}{\xi^2 + \xi - 2\xi^2} \), and always satisfies \( \rho_h < \rho_g \). It implies that Haar embeddings have uniformly better performance, since the rate of the IHS with Gaussian embeddings is equal to \( \rho_g \) according to \[14\]. Further, we show that the optimal momentum parameters \( \beta_t \) are equal to 0, that is, Heavy-ball momentum does not accelerate the algorithm with refreshed Haar embeddings.

Importantly, we prove that, under the additional mild assumption on the initial error \( \Delta_0 \) that \( \mathbb{E}[\Delta_0 \Delta_0^\top] = d^{-1} I_d \), the IHS with SRHT embeddings also yields a sequence of iterates such that its rate of convergence is exactly equal to \( \rho_h \). Consequently, \textit{SRHT matrices outperform uniformly Gaussian embeddings}. Then, we confirm numerically the above theoretical statements.

We finally argue that our algorithm improves by a factor \( \log d \) the currently best known complexity \( C_c \) for solving \( \text{(1)} \) when the condition number is large. Precisely, given a fixed target error \( \|\Delta_t\|^2 \leq \varepsilon \) (such that \( \varepsilon \) is independent of the dimensions), we find that, with the sketch \( m \approx d \), our algorithm has complexity \( C_n \propto (nd \log d + d^3 + nd) \log(1/\varepsilon) \), whereas the current state-of-the-art algorithms for dense problems with their prescribed sketch size \( m \approx d \log d \) \[24, 6\] yield \( C_c \propto nd \log d + d^3 + nd \log(1/\varepsilon) \), so that, as \( d \to \infty \)

\[
C_n/C_c \propto 1/\log d.
\]
2 Technical Background

We introduce a few needed definitions, and we refer the reader to \[5, 21, 32\] for an extensive introduction to random matrix theory. Let \(\{M_n\}_n\) be a sequence of Hermitian random matrices, where each \(M_n\) has size \(n \times n\). For a fixed \(n\), the empirical spectral distribution (e.s.d.) of \(M_n\) is the (cumulative) distribution function of its eigenvalues \(\lambda_1, \ldots, \lambda_n\), i.e., \(F_{M_n}(x) := \frac{1}{n} \sum_{j=1}^{n} 1\{\lambda_j \leq x\}\) for \(x \in \mathbb{R}\), which has density \(f_{M_n}(x) = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j}(x)\) with \(\delta_{\lambda_j}\) the Dirac measure at \(\lambda_j\). Due to the randomness of the eigenvalues, \(F_{M_n}\) is random. The relevant aspect of some classes of large \(n \times n\) symmetric random matrices \(M_n\) is that, almost surely, the e.s.d. \(F_{M_n}\) converges weakly towards a non-random distribution \(F\), as \(n \to \infty\). This function \(F\), if it exists, will be called the limiting spectral distribution (l.s.d.) of \(M_n\).

A powerful tool in the analysis of random matrices is the Stieltjes transform. For \(\mu\) a probability measure supported on \([0, +\infty)\), its Stieltjes transform is defined over the complex space complementary to the support of \(\mu\) as

\[
m_\mu(z) := \int \frac{1}{x-z} \, d\mu(x) .
\]

It holds in particular that \(m_\mu\) is analytic over \(\mathbb{C} \setminus \mathbb{R}_+\), \(m_\mu(z) \in \mathbb{C}^+\) for \(z \in \mathbb{C}^+\), \(m_\mu(z) \in \mathbb{C}^-\) for \(z \in \mathbb{C}^-\) and \(\mu_\mu(z) > 0\) for \(z < 0\), where \(\mathbb{R}_+\) is the set of positive reals and \(\mathbb{C}^+\) is the set of complex numbers with positive imaginary part. Another useful transform for studying the product of random matrices is the S-transform, denoted \(S_\mu\). This is defined as the solution of the following equation, which is unique under certain conditions (see \[23\]),

\[
m_\mu\left(z + \frac{1}{z} S_\mu(z)\right) + z S_\mu(z) = 0.
\]

We introduce a few additional concepts from free probability that will be used in the proofs. We refer the reader to \[28, 13, 20, 3\] for an extensive introduction to this field. Consider the algebra \(A_n\) of \(n \times n\) random matrices. For \(X_n \in A_n\), we define the linear functional \(\tau_n(X_n) := \frac{1}{n} \mathbb{E}[\text{trace } X_n]\). Then, we say that a family \(\{X_{n,1}, \ldots, X_{n,I}\}\) of random matrices in \(A_n\) is asymptotically free if for every \(i \in \{1, \ldots, I\}\), \(X_{n,i}\) has a limiting spectral distribution, and if \(\tau\left(\prod_{j=1}^{m} P_{j}(X_{n,i_j} - \tau(P_j(X_{n,i_j})))\right) \to 0\) almost surely for any positive integer \(m\), any polynomials \(P_1, \ldots, P_m\) and any indices \(i_1, \ldots, i_m \in \{1, \ldots, I\}\) with \(i_1 \neq i_2, \ldots, i_{m-1} \neq i_m\). In particular, this definition implies that for two sequences of asymptotically free random matrices \(X_n, Y_n\), we have the trace decoupling relation

\[
\frac{1}{n} \mathbb{E}[\text{trace } X_n Y_n] - \frac{1}{n} \mathbb{E}[\text{trace } X_n] \frac{1}{n} \mathbb{E}[\text{trace } Y_n] \to 0 .
\]

Essential to our analysis is the following result. If two \(n \times n\) random matrices \(A_n\) and \(B_n\) are asymptotically free and have respective l.s.d. \(\mu_A\) and \(\mu_B\) with respective S-transforms \(S_A\) and \(S_B\), then the matrix product \(A_n B_n\) has l.s.d. \(\mu_{AB}\) whose S-transform is \(S_{AB}(z) = S_A(z) S_B(z)\). The distribution \(\mu\) is called the free multiplicative convolution of \(\mu_A\) and \(\mu_B\), and we denote \(\mu = \mu_A \boxtimes \mu_B\).

We will also make use of an alternative form of the Stieltjes transform: the \(\eta\)-transform is defined for \(z \in \mathbb{C} \setminus \mathbb{R}^-\) as

\[
\eta_\mu(z) := \int \frac{1}{1+z} \, d\mu(x) = \frac{1}{z} m_\mu\left(-\frac{1}{z}\right).
\]

There are standard examples of classes of random matrices and their limiting spectral behavior. We recall a classical result \[13\]. If \(S\) is an \(m \times d\) matrix with identically and independently distributed
entries \( \mathcal{N}(0,1/m) \), then, as \( m, d \to \infty \) with \( m/d \to \rho \in (0,1) \), the Marchenko-Pastur theorem (see \cite{05, 18}) states that the matrix \( S^T S \) has l.s.d. \( F_\rho \), whose Stieltjes transform is the unique solution of a certain fixed point equation, and whose density is explicitly given by

\[
\mu_\rho(x) = \frac{\sqrt{(b-x)_+(x-a)_+}}{2\pi \rho x},
\]

where \( y_+ = \max\{0,y\} \), \( a = (1-\sqrt{\rho})^2 \) and \( b = (1+\sqrt{\rho})^2 \). In our analysis of Haar and SRHT matrices, we will encounter similarly fixed-point equations satisfied by the Stieltjes (or \( \eta \)-) transform of their l.s.d.

### 3 Sketching with Haar matrices

Sketching matrices with i.i.d. entries are not ideal for sketching. Intuitively, i.i.d. projections distort the geometry of Euclidean space due to their non-orthogonality. Is it possible to overcome this using orthogonal random projections? In order to answer this question, we first aim to establish some properties of the l.s.d. of Haar matrices.

#### 3.1 Limiting spectral distribution of Haar matrices

Let \( S \) be a \( m \times n \) Haar matrix, and \( U \) be an \( n \times d \) deterministic matrix with orthonormal columns. Since we are interested in the limit \( n \to \infty \), we add the subscript \( n \) to matrices like \( S_n, U_n \) from now on. We set \( S_{1,n} := S_n U_n \). We next characterize the l.s.d. of \( S_{1,n}^T S_{1,n} \) through its Stieltjes transform.

**Theorem 3.1.** The matrix \( S_{1,n}^T S_{1,n} \) admits a l.s.d. whose Stieltjes transform \( m_h \) is given by

\[
m_h(z) = \frac{z(2\gamma - 1) + \xi - \gamma - \sqrt{(\gamma + \xi - 2 + z)^2 + 4(z - 1)(1 - \gamma)(1 - \xi)}}{2\gamma z(1 - z)},
\]

for any \( z \in \mathbb{C} \setminus \mathbb{R}_+ \).

One might wonder how the l.s.d. of Haar matrices and that of Gaussian embeddings – the Marchenko-Pastur law \( \mu_\rho_g \) – differ. Consider the re-scaled matrix \( \frac{1}{m} S_{1,n}^T S_{1,n} \), whose expectation is equal to the identity. Crucially, the l.s.d. \( \mu_\rho_g \) does not depend on the sample size \( n \) but only on the limit ratio between \( d \) and \( m \), whereas the distribution \( F_h \) involves the ratios \( \gamma \) and \( \xi \). Numerically, we observe in Figure 1 that, for fixed \( \gamma = 0.2 \), as \( \xi \) increases, the empirical Haar density departs from the Marchenko-Pastur density \( \mu_\rho_g \), and concentrates more and more relatively to \( \mu_\rho_g \). Importantly, we see that the support of \( F_h \) is included within the support of \( \mu_\rho_g \), and thus, more concentrated around 1.

In the rest of Section 3.1, we present the main steps for proving Theorem 3.1. First, observe that since both \( S_n \) and \( U_n \) are rectangular orthogonal matrices, we can embed them into full orthogonal matrices as \( S_n = \begin{pmatrix} S_n & \mathbf{0} \end{pmatrix} \) and \( U_n = \begin{pmatrix} U_n & U_n^\perp \end{pmatrix} \). Then, we can write

\[
S_{1,n} = \begin{pmatrix} I_m & 0 \end{pmatrix} S_n U_n \begin{pmatrix} I_d \\ 0 \end{pmatrix}.
\]

(13)
so that \( \gamma \) fixed-point equation. We defer details of the proof to Appendix A.1.

The above expression (14) of the matrix \( C_n \) admits a l.s.d. \( F_C \), whose Stieltjes transform \( m_C \) is given by

\[
m_C(z) = \frac{z + \gamma + \xi - 2 - \sqrt{(\gamma + \xi - 2 + z)^2 + 4(z - 1)(1 - \gamma)(1 - \xi)}}{2z(1 - z)},
\]
for any \( z \in \mathbb{C} \setminus \mathbb{R}_+ \).

**Proof.** The above expression (14) of the matrix \( C_n \) has the required form to apply Theorem 4.11 by [\text{[14]}], and hence characterize the e.s.d. of \( C_n \) through its \( \eta \)-transform which has to satisfy a fixed-point equation. We defer details of the proof to Appendix A.1.

Now, we use the fact that the matrices \( S_{1,n}^\top S_{1,n} \) and \( C_n \) have the same non-zero eigenvalues. Almost surely, there are exactly \( d \) of them, which we denote \( \lambda_1, \ldots, \lambda_d \). Then, the e.s.d. \( F_{C_n} \) of \( C_n \) can be decomposed as

\[
F_{C_n}(x) = \left(1 - \frac{d}{n}\right) 1_{\{x \geq 0\}} + \frac{1}{n} \sum_{i=1}^{d} 1_{\{x \geq \lambda_i\}} = \left(1 - \frac{d}{n}\right) 1_{\{x \geq 0\}} + \frac{d}{n} F_{h,n}(x),
\]
where \( F_{h,n} \) is the e.s.d. of \( S_{1,n}^\top S_{1,n} \). Taking the limit \( n \to \infty \), we find that \( F_{1,n} \) converges weakly almost surely to

\[
F_h(x) = \frac{1}{\gamma} \left( F_C(x) - (1 - \gamma) 1_{\{x \geq 0\}} \right).
\]

By definition of \( m_h \) and using (17), it follows that for \( z \in \mathbb{C} \setminus \mathbb{R}_+ \)

\[
m_h(z) = \int \frac{1}{x - z} dF_h(x) = \frac{1}{\gamma} \int \frac{1}{x - z} dF_C(x) - \frac{1 - \gamma}{\gamma} \int \frac{1}{x - z} \delta_0(x) dx = \frac{1}{\gamma} m_C(z) + \frac{1 - \gamma}{\gamma z}.
\]

Plugging-in the expression of \( m_C \), we obtain the claimed formula (12) for \( m_h \).
3.2 First and second inverse moments, and application to least-squares optimization

**Theorem 3.2.** Suppose that $S$ is an $m \times n$ Haar matrix, and let $U$ be an $n \times d$ deterministic matrix with orthonormal columns. It holds that

\[
\theta_{1,\theta} := \lim_{n \to \infty} \frac{1}{d} \text{trace } E\left[(U^T S^T SU)^{-1}\right] = \frac{1 - \gamma}{\xi - \gamma},
\]

\[
\theta_{2,\theta} := \lim_{n \to \infty} \frac{1}{d} \text{trace } E\left[(U^T S^T SU)^{-2}\right] = \frac{(1 - \gamma)(\gamma^2 + \xi - 2\gamma\xi)}{(\xi - \gamma)^3}.
\]

**Proof.** The proof essentially relies on relating the traces of the inverse moments to limits and derivatives of $m_h(z)$ evaluated at $z = 0$. We defer calculations to Appendix A.2.

As an application of the formulas for $\theta_{1,\theta}$ and $\theta_{2,\theta}$, we consider the IHS (2) with refreshed Haar matrices $\{S_t\}$. For $S$ an $m \times n$ Haar matrix, we have the equality in distribution, $U^T S^T SU \overset{d}{=} \Omega^T U^T S^T SU\Omega$, for $\Omega$ a $d \times d$ Haar matrix independent of $S$. Consequently, the matrix of eigenvectors of $U^T S^T SU$ is itself Haar-distributed. Along with the trace calculations $\theta_{1,\theta}$ and $\theta_{2,\theta}$, it implies the following result whose proof is deferred to Appendix A.3.

**Theorem 3.3.** With refreshed Haar matrices $\{S_t\}$, step sizes $\mu_t = \theta_{1,\theta}/\theta_{2,\theta}$ and momentum parameters $\beta_t = 0$, the sequence of error vectors $\{\Delta_t\}$ satisfies

\[
\rho_h := \left(\lim_{n \to \infty} \frac{E\|\Delta_t\|^2}{\|\Delta_0\|^2}\right)^{1/t} = \rho_g \cdot \frac{\xi(1 - \xi)}{\gamma^2 + \xi - 2\xi\gamma}.
\]

Further, for any sequence of step sizes $\{\mu_t\}$ and momentum parameters $\{\beta_t\}$, we have that, for the resulting sequence of error vectors $\{\Delta_t\}$,

\[
\rho_h \leq \liminf_{t \to \infty} \left(\lim_{n \to \infty} \frac{E\|\Delta_t\|^2}{\|\Delta_0\|^2}\right)^{1/t},
\]

that is, $\rho_h$ is the optimal rate one may achieve using Haar embeddings.

Hence, orthogonal projections are uniformly better than Gaussian i.i.d. projections. Indeed, the ratio between the convergence rates $\rho_h$ and $\rho_g$ is equal to $\xi(1 - \xi)/(\gamma^2 + \xi - 2\gamma\xi)$, and is always strictly smaller than 1. To see this, note that $\xi(1 - \xi)/(\gamma^2 + \xi - 2\gamma\xi) < 1$ if and only if $\xi(1 - \xi) < \gamma^2 + \xi - 2\gamma\xi$, and after simplification, we obtain the condition $(\xi - \gamma)^2 > 0$. In the small sketch size regime $d \leq m \ll n$, we have $\rho_h/\rho_g \approx 1$. As the sketch size $m$ increases relatively to $n$, the convergence rates’ ratio scales as $\rho_h/\rho_g \approx (1 - \xi)$, and one can improve on the number of iterations – and thus, data passes – with Haar embeddings by making $1 - \xi$ bounded away from 1. Further, observe that if we do not reduce the size of the original matrix, so that $m = n$ and $\xi = 1$, then the algorithm converges in one iteration. This means that we do not lose any information in the linear model. In contrast, Gaussian projections introduce more distortions than rotation, even though the rows of a Gaussian matrix are almost orthogonal to each other in the high-dimensional setting.

Interestingly, momentum does not accelerate the refreshed sketch. Leveraging past information through the Heavy-ball update (2) does not provide any benefit, possibly due to the independence between the sketching matrices $\{S_t\}$. On the other hand, it remains an open question – both for Gaussian and Haar embeddings – whether there exists a first-order method which uses past iterates along with refreshed matrices, and provide acceleration over gradient descent updates.
However, the time complexity of generating an $m \times n$ Haar matrix using the Gram-Schmidt procedure is $O(nm^2)$, which is, for instance, larger than the classical cost $O(nd^2)$ for solving the least-squares problem \[[1]\], and we now turn to the analysis of another orthogonal matrix, the SRHT, which contains less randomness, but is more structured and faster to generate.

4 Sketching with SRHT matrices

We have seen in the previous section that Haar random projections have a better performance than Gaussian i.i.d. random projections. However, they are still slow to generate and apply. Can we get the same good statistical performance as Haar projections with faster methods? Here we consider the SRHT. This is faster as it relies on the well-structured Walsh-Hadamard transform, which is defined as follows. For an integer $n = 2^p$ with $p \geq 1$, the Walsh-Hadamard transform is defined recursively as $H_n = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n/2} & H_{n/2} \\ H_{n/2} & -H_{n/2} \end{bmatrix}$ with $H_1 = 1$. We consider a version of the SRHT which is slightly different from the classical SRHT \[[1]\]. Our transform $A \mapsto SA$ first randomly permutes the rows of $A$, before applying the classical transform. This has negligible cost $O(n)$ compared to the cost $O(nd \log m)$ of the matrix multiplication $A \mapsto SA$, and breaks the non-uniformity in the data. That is, we define the $n \times n$ subsampled randomized Hadamard matrix as $S = BH_n DP$, where $B$ is an $n \times n$ diagonal sampling matrix of i.i.d. Bernoulli random variables with success probability $m/n$, $H_n$ is the $n \times n$ Walsh-Hadamard matrix, $D$ is an $n \times n$ diagonal matrix of i.i.d. sign random variables, equal to $\pm 1$ with equal probability, and $P \in \mathbb{R}^{n \times n}$ is a uniformly distributed permutation matrix. At the last step, we discard the zero rows of $S$, so that it becomes an $\tilde{m} \times n$ orthogonal matrix with $\tilde{m} \sim \text{Binomial}(m/n, n)$, and the ratio $\tilde{m}/n$ concentrates fast around $\xi$ while $n \to \infty$. Although the dimension $\tilde{m}$ is random, we refer to $S$ as an $m \times n$ SRHT matrix.

4.1 Limiting spectral distribution of the SRHT

Similarly to an $m \times n$ Haar matrix, the rows of an SRHT are subsampled rows of an orthogonal matrix. Thus, in spite of the reduced amount of randomness, we can still expect the SRHT to be similar to the uniform orthogonal projection.

**Theorem 4.1.** Let $S$ be an $m \times n$ SRHT matrix, $S_h$ be an $m \times n$ Haar matrix, and $U$ an $n \times d$ deterministic matrix with orthonormal columns. Then, the matrices $U^T S^T SU$ and $U^T S_h^T S_h U$ have the same limiting spectral distribution. Consequently, it holds that

$$\lim_{n \to \infty} \frac{1}{d} \text{trace} \mathbb{E} \left( (U^T S^T SU)^{-1} \right) = \theta_{1,h}, \quad (24)$$

$$\lim_{n \to \infty} \frac{1}{d} \text{trace} \mathbb{E} \left( (U^T S^T SU)^{-2} \right) = \theta_{2,h}. \quad (25)$$

The rest of Section 4.1 is devoted to the proof of Theorem 4.1. Our analysis proceeds in a way similar to the analysis of the Haar case, and we describe in this paragraph the main steps. Denote by $F_S$ the l.s.d. of $U^T S^T SU$ and by $F_{S,n}$ its e.s.d. As we did for the Haar case with the matrix $C_n$, we introduce here an auxiliary matrix $G_n$ whose e.s.d. is related to $F_{S,n}$. Then, we characterize the $\eta$-transform $\eta_C$ of its l.s.d. $F_G$. Our analysis for $\eta_C$ uses recent results on asymptotically liberating sequences from free probability \[[2]\], which is a key technical innovation of our work. Finally, we show that $\eta_C$ is equal to the $\eta$-transform $\eta_C$ of $F_C$, and we conclude that $F_S = F_h$.

Let $S = BH_n DP$ be the $n \times n$ SRHT matrix (before discarding the rows) as defined above, and $U$ be an $n \times d$ deterministic matrix with orthonormal columns. Note that whether we consider the
zero rows or not in the matrix $S$, the matrix $U^T S^T SU$ remains the same, and so does its l.s.d. The matrices $B, H_n$ and $D$ are all symmetric matrices, and they respectively satisfy $B^2 = B, H_n^2 = I_n$ and $D^2 = I_n$, and $P$ is also an orthogonal matrix. Then, we have that $S^T S = P^T D H_n B H_n D P$, and further,

$$ (S^T S)^2 = P^T D H_n B H_n D P P^T D H_n B H_n D P = P^T D H_n B H_n D P = S^T S. $$

We first have the following observation, whose proof is deferred to Appendix [1,2.

**Lemma 4.1.** For $P, B, D, H_n$ and $U$ defined as above, we have the following equality in distribution

$$ U^T (P^T D H_n) B (H D P) U \overset{d}{=} U^T (P^T D H_n D P) B (P^T D H_n D P) U. \quad (26) $$

We now proceed with asymptotic statements, and we introduce the subscript $n$ to all matrices. We set $W_n := P_n D H_n D_n P_n$. It holds that the matrix $U_n^T W_n B_n W_n U_n$ has the same nonzero eigenvalues as $G_n := B_n W_n U_n^T W_n B_n$, so that we first find the l.s.d. of the matrix $G_n$. The reader may notice that $G_n$ plays a similar role in the analysis of the SRHT case, to that of the matrix $C_n$ in the analysis of the Haar case.

The following result states the asymptotic freeness of the matrices $B_n$ and $W_n U_n U_n^T W_n$. Its proof follows directly from Corollaries 3.5 and 3.7 by [2].

**Lemma 4.2.** Let $B_n, W_n, U_n$ be defined as above. Then, the matrices \{B_n, W_n U_n U_n^T W_n\} are asymptotically free in the limit of the non-commutative probability spaces of random matrices. Consequently, the e.s.d. of the matrix $G_n = B_n W_n U_n U_n^T W_n B_n$ converges to the freely multiplicative convolution of the l.s.d. $F_B$ of $B_n$ and the l.s.d. $F_U$ of $U_n U_n^T$, that is, $G_n$ has l.s.d. given by $F_G = F_B \boxtimes F_U$.

Since the density of the l.s.d. $F_B$ is $f_B = \xi \delta_1 + (1 - \xi) \delta_0$ and and the density of $F_U$ is $f_U = \gamma \delta_1 + (1 - \gamma) \delta_0$, we have that the $S$-transforms $S_B$ of $F_B$ and $S_U$ of $F_U$ are respectively equal to $S_B(y) = \frac{y+1}{y+\xi}$ and $S_U(y) = \frac{y+1}{y+\gamma}$. From Lemma 4.2, it follows that the $S$-transform $S_G$ of $F_G$ is the product of $S_B$ and $S_U$, i.e.,

$$ S_G(y) = S_U(y) S_B(y) = \frac{(y+1)^2}{(y+\xi)(y+\gamma)}. \quad (27) $$

First, note that using their respective definitions, the $S$-transform of $F_G$ and its $\eta$-transform $\eta_G$ are related by the equation $\eta_G\left(-\frac{y}{y+1} S_G(y)\right) = y + 1$. Plugging-in the expression (27) of $S_G(y)$ into the latter equation, we obtain that

$$ \eta_G\left(-\frac{y(y+1)}{(y+\gamma)(y+\xi)}\right) = y + 1. $$

Letting $z = -\frac{(y+\gamma)(y+\xi)}{(y(y+1)}$ and using the relationship [10] between the Stieltjes and $\eta$-transforms, we find that the Stieltjes transform $m_G$ of $G$ is equal to

$$ m_G(z) = \frac{z + \gamma + \xi - 2 - \sqrt{g(z)}}{2z(1-z)}, $$

where $g(z) = (\gamma + \xi - 2 + z)^2 + 4(z-1)(1-\gamma)(1-\xi)$. Hence, we get that $m_G(z) = m_C(z)$, that is, $F_G = F_C$. 


Further, the matrix $G_n$ has the same non-zero eigenvalues as the matrix $U_n^TW_nB_nW_nU_n$ which, according to Lemma 4.1, is equal in distribution to $U_n^T\Sigma_n\Sigma_nU_n$. Denote by $\lambda_1, \ldots, \lambda_d$ the non-zero eigenvalues of $U_n^T\Sigma_n\Sigma_nU_n$, where $\hat{d}$ is itself a random number due to the randomness of non-zero rows $\tilde{m}$. Hence, the e.s.d $F_{G,n}$ of $G_n$ and the e.s.d. $F_{S,n}$ of $U_n^T\Sigma_n\Sigma_nU_n$ satisfy (see Appendix B.3)

$$F_{G,n}(x) \equiv \left(1 - \frac{d}{n}\right)1_{\{x \geq 0\}} + \frac{d}{n}F_{S,n}(x).$$

Thus, we obtain that $F_{S,n}$ converges weakly almost surely to the distribution

$$F_S(x) := \frac{1}{\gamma}(F_C(x) - (1 - \gamma)1_{\{x \geq 0\}}) = \frac{1}{\gamma}(F_C(x) - (1 - \gamma)1_{\{x \geq 0\}}).$$

The latter expression is equal to $F_h(x)$ according to (28), so that $F_S(x) = F_h(x)$. The analysis of the traces of the expected first and second inverse moments only involves the limiting distribution (we refer the reader to the proof of the expressions of $\theta_{1,h}$ and $\theta_{2,h}$, in Section A.2). Due to the equality $F_h = F_S$, they remain the same with SRHT matrices, which concludes the proof of Theorem 4.1.

In Figure 2 we verify that the empirical densities with Haar and SRHT matrices are indeed very close.

![Figure 2: Empirical densities of the matrices $\frac{1}{m}U^TS^TU$ for $S$ an $m \times n$ Haar matrix and SRHT matrix, versus Marchenko-Pastur density with shape parameter $d/m$. We use $n = 4096$, $d = 820$ and $m \in \{860, 1640, 2450\}$, so that $\gamma \approx 0.2$ and $\xi \in \{0.21, 0.4, 0.6\}$.

4.2 Application to least-squares optimization

Although the SRHT has much less randomness, its spectrum behaves asymptotically as that of a Haar matrix according to Theorem 4.1. Hence, we should expect the iterative Hessian sketch to have the same rate of convergence. However, it is a challenging problem to characterize the asymptotic behavior of the eigenvectors of the matrix $U^TS^TU$, which is beyond the scope of this paper.

For SRHT matrices, we first rely on an additional mild assumption on the initialization of the least-squares problem (1) which allow us to exactly characterize the convergence of the IHS with refreshed SRHT embeddings. We then provide a tight-upper bound without relying on the initialization assumption. The proofs are deferred to Appendix A.4.

**Theorem 4.2.** Suppose that the initial (random) error vector $\Delta_0 = x_0 - x^* \in \mathbb{R}^d$ satisfies the condition $\mathbb{E}[\|\Delta_0\|^2] = d^{-1}I_d$. Then, with refreshed SRHT matrices $\{S_t\}$ and step sizes $\mu_t = \theta_1^h/\theta_2^h$, the sequence of error vectors $\{\Delta_t\}$ satisfy

$$\rho_s := \left(\lim_{n \to \infty} \mathbb{E}\|\Delta_t\|^2\right)^{1/t} = \rho_g \cdot \frac{\xi(1 - \xi)}{\gamma^2 + \xi - 2\xi\gamma} = \rho_h.$$  

(30)
Remark 4.1. We note that the initialization condition $E \left[ \Delta_0 \Delta_0^\top \right] = d^{-1} I_d$ can be achieved by picking $x_0$ uniformly random on the unit $d$-sphere $S^{d-1}$, followed by a uniformly random signed permutation and scaling to the columns of $A$ to obtain $E x^* x^T \propto I_d$.

We also present the following upper-bound on the error, which holds for any deterministic or random initialization $x_0$ and exhibits an identical convergence rate.

Theorem 4.3. For any initialization, with refreshed SRHT matrices $\{S_t\}$ and step sizes $\mu_t = \theta_{1,h}/\theta_{2,h}$, the sequence of error vectors $\{\Delta_t\}$ satisfy

$$\lim_{n \to \infty} \frac{1}{d} \mathbb{E} \frac{\|\Delta_t\|^2}{\|\Delta_0\|^2} \leq \rho_h.$$  

While providing significant computational benefits for forming the sketch $SA$, SRHT embeddings are still able to match the convergence rate of Haar matrices, and thus, also improves on Gaussian sketches.

4.2.1 Numerical Simulations

We now verify Theorems 3.3 and 4.2 numerically, and we compare the performance of refreshed Haar/SRHT sketches to refreshed Gaussian sketches. For SRHT embeddings, we use the optimal step sizes $\mu_t = \theta_{1,h}/\theta_{2,h}$ and momentum parameters $\beta_t = 0$, where we replace $\xi$ and $\gamma$ by their finite sample approximations. For Gaussian embeddings, we use the optimal parameters $\mu_t = \theta_{1,g}/\theta_{2,g}$ and $\beta_t = 0$, which were derived by [14], and where $\theta_{p,g} = \mathbb{E}[(U^T S^T S U)^{-p}]$ for $p \in \{1, 2\}$. The expressions of $\theta_{p,g}$ can also be found in [11]. We set $n = 8192$, $d = 800$ and we vary the sketch size $m$. As $m$ increases, then Haar/SRHT embeddings are increasingly better, and the empirical curves match our theoretical predictions.

![Figure 3: Error $\|\Delta_t\|^2/\|\Delta_0\|^2$ versus number of iterations for the iterative Hessian sketch: (a) $m = 980$, (b) $m = 2450$ and (c) $m = 4100$ (error averaged over 10 independent trials).](image)

4.2.2 Complexity Analysis

Let us now turn to a complexity analysis of the IHS with SRHT embeddings, and compare it to the currently best known complexity for solving (1). The latter is achieved, among others, by the pre-conditioned conjugate gradient algorithm [24]. As described in Section 1, this algorithm uses a sketch $SA$ to compute a pre-conditioning matrix $P$, such that $A^{-1}$ has a small condition number, and then it solves the least-squares problem $\min_y \|AP^{-1} y - b\|^2$, using the conjugate-gradient method. As for the IHS, it can be decomposed into three parts: sketching, factoring (computing $P$ and $AP^{-1}$) versus computing $H_t^{-1}$, and then the iterations.
The pre-conditioned conjugate gradient prescribes at least the sketch size \( m \approx d \log d \) in order to converge, with high-probability guarantees. This theoretical prescription is based on the finite-sample bounds on the extremal eigenvalues of the matrix \( U^T S^T SU \) derived by [25]. Then, given a fixed (and independent of the dimensions) error \( \varepsilon > 0 \) and with the prescribed \( m \approx d \log d \), the resulting complexity to achieve \( \|\Delta_t\|^2 \leq \varepsilon \) scales as
\[
C_c \asymp nd \log d + d^3 \log d + nd \log(1/\varepsilon),
\] (32)
where \( nd \log d \) is the cost of forming the sketch \( SA \), \( d^3 \log d \) the pre-conditioning cost, and \( nd \log(1/\varepsilon) \) is the per-iteration cost times the number of iterations. Following our new analysis, we obtain similarly that the IHS with the SRHT can use \( m \approx d \), with resulting complexity
\[
C_n \asymp (nd \log d + d^3 + nd) \log(1/\varepsilon).
\] (33)
Note that the number of iterations multiplies the sum of the sketching, factor and per-iteration costs, and this is due to refreshing the sketches. Then, treating the term \( \log(1/\varepsilon) \) as a constant independent of the dimensions, we find that
\[
C_n/C_c \asymp 1/\log d, \quad d \to \infty.
\] (34)
Hence, with a smaller sketch size, the resulting complexity improves by a factor \( \log d \) over the current state-of-the-art in randomized preconditioning for dense problems (e.g., see [6, 19]). We also note that the \( O(d^3) \) term can be improved to \( O(d^{\omega}) \), where \( \omega \) is the exponent of matrix multiplication.

**Acknowledgements**

This work was partially supported by the National Science Foundation under grant IIS-1838179.

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A Sketching with Haar matrices – Proofs of main and auxiliary results

A.1 Proof of Lemma 3.1

Recall the definition (14) of the matrix $C_n$,

\[
C_n = \left( \begin{array}{ccc} I_m & 0 \\ 0 & 0 & 0 \end{array} \right) W_n \left( \begin{array}{ccc} I_d & 0 \\ 0 & 0 & 0 \end{array} \right) W_n^\top \left( \begin{array}{ccc} I_m & 0 \\ 0 & 0 & 0 \end{array} \right).
\]

We leverage Theorem 4.11 from [7], which we recall for the sake of completeness.

**Theorem A.1** (Theorem 4.11 in [7]). Let $D_n \in \mathbb{R}^{n \times n}$ and $T_n \in \mathbb{R}^{n \times n}$ be diagonal non-negative matrices, and $W_n \in \mathbb{R}^{n \times n}$ be a Haar matrix. Denote $F_D$ and $F_T$ the respective l.s.d. of $D_n$ and $T_n$. Denote $C_n$ the matrix $C_n := D_n^\frac{1}{2} W_n T_n W_n^\top D_n^\frac{1}{2}$. Then, as $n$ tends to infinity, the e.s.d. of $C_n$ converges to $F$ whose $\eta$-transform $\eta_F$ satisfies

\[
\eta_F(z) = \int \frac{1}{z \gamma(z) x + 1} dF_D(x),
\]

\[
\gamma(z) = \int \frac{x}{\eta_F(z) + z \delta(z) x} dF_T(x),
\]

\[
\delta(z) = \int \frac{x}{z \gamma(z) x + 1} dF_D(x).
\]
The e.s.d. of \( \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} \) converges to the distribution \( F_\gamma \) with density \( \gamma \delta_1 + (1 - \gamma) \delta_0 \), and the e.s.d. of \( \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \) converges to the distribution \( F_\xi \) with density \( \xi \delta_1 + (1 - \xi) \delta_0 \). Then, according to Theorem A.1, the e.s.d. of \( C_n \) converges to a distribution \( F_C \), whose \( \eta \)-transform \( \eta_C \) is solution of the following system of equations,

\[
\eta_C(z) = \int \frac{1}{z \gamma(z)x + 1} \, dF_\xi(x), \tag{35}
\]

\[
\gamma(z) = \int \frac{x}{\eta_C(z) + z \delta(z)x} \, dF_\gamma(x), \tag{36}
\]

\[
\delta(z) = \int \frac{x}{z \gamma(z)x + 1} \, dF_\xi(x). \tag{37}
\]

Plugging the above expressions of \( F_\xi \) and \( F_\gamma \) into the above equations, and after simplification, we obtain that \( \eta_C \) is solution of the following second-order equation

\[
\eta_C(z) = (1 - \gamma) + \frac{\gamma}{1 + z \left(1 + \frac{\xi - 1}{\eta_C(z)}\right)}, \tag{38}
\]

Plugging the relationship \([10]\) between the Stieltjes and \( \eta \)-transforms into \([38]\), we find that

\[
m_C(z) = \frac{z + \gamma + \xi - 2 - \sqrt{g(z)}}{2z(1 - z)}, \tag{39}
\]

where \( g(z) = (\gamma + \xi - 2 + z)^2 + 4(z - 1)(1 - \gamma)(1 - \xi) \), and we choose the branch of the square-root such that \( m_C(z) \in \mathbb{C}^+ \) for \( z \in \mathbb{C}^+ \), \( m_C(z) \in \mathbb{C}^- \) for \( z \in \mathbb{C}^- \) and \( m_C(z) > 0 \) for \( z < 0 \).

### A.2 Proof of the trace calculations \( \theta_{1,h} \) and \( \theta_{2,h} \)

We will need the following result regarding the support of \( F_h \), which is proved in Appendix B.1.

**Lemma A.1.** The support of \( F_h \) satisfies

\[
\inf \, \text{supp}(F_h) \geq \frac{(1 - \sqrt{\rho_g})^2}{(1 + \frac{1}{\sqrt{\xi}})^2}. \tag{40}
\]

Thus, the support of \( F_h \) is bounded away from 0, so is the intersection of the support of \( F_C \) and \( \mathbb{R}^* \). Further, the distribution \( F_C \) has a point mass at 0 equal to \( 1 - \gamma \). We now turn to the trace calculations.

#### A.2.1 Computing \( \theta_{1,h} \)

Using the facts that \( F_C \) has support within \([0, +\infty)\) and a point mass equal to \((1 - \gamma)\) at 0, its \( \eta \)-transform \( \eta_C \) is well-defined on \( \{ z \in \mathbb{R} \mid z > 0 \} \), and, for \( z > 0 \), it can be decomposed as

\[
\eta_C(z) = 1 - \gamma + \int_{x \neq 0} \frac{1}{1 + zx} \, dF_C(x). \tag{41}
\]
The function \( \frac{1}{x} \) is integrable on the set \( \{ x > 0 \} \) with respect to \( F_C \), since the support of \( F_C \) on \( \mathbb{R}^* \) is bounded away from 0. Since \( \left| \frac{z}{1 + x z} \right| < \frac{1}{x} \) when \( z > 0, x > 0 \), it follows by the dominated convergence theorem that

\[
\lim_{z \to \infty} \int_{x \neq 0} \frac{z}{1 + x z} dF_C(x) = \int_{x \neq 0} \lim_{z \to \infty} \frac{z}{1 + x z} dF_C(x) = \int_{x \neq 0} \frac{1}{x} dF_C(x). \tag{42}
\]

Using (41), it follows that

\[
\lim_{z \to \infty} z \left( \eta_C(z) - (1 - \gamma) \right) = \int_{x \neq 0} \frac{1}{x} dF_C(x), \tag{43}
\]

On the other hand, we have that

\[
\lim_{z \to \infty} \eta_C(z) = (1 - \gamma) + \lim_{z \to \infty} \int_{x \neq 0} \frac{1}{1 + x z} dF_C(x) \tag{44}
\]

\[
= (1 - \gamma) + \int_{x \neq 0} \lim_{z \to \infty} \frac{1}{1 + x z} dF_C(x) \tag{45}
\]

\[
= 1 - \gamma. \tag{46}
\]

where the second equality is again justified by the dominated convergence theorem. Subtracting \( 1 - \gamma \) from both sides of (38), multiplying by \( z \left( 1 + \xi - \frac{1}{\eta_C(z)} \right) \) and letting \( z \to \infty \), we obtain

\[
\lim_{z \to \infty} z \left( 1 + \frac{\xi - 1}{\eta_C(z)} \right) \left( \eta_C(z) - (1 - \gamma) \right) = \lim_{z \to \infty} z \left( 1 + \frac{\xi - 1}{\eta_C(z)} \right) \left( \frac{\gamma}{1 + z(1 + \frac{\xi - 1}{\eta_C(z)})} \right).
\]

Note that the right-hand side of the above equation is equal to \( \gamma \), and the left-hand side satisfies

\[
\lim_{z \to \infty} z \left( 1 + \frac{\xi - 1}{\eta_C(z)} \right) \left( \eta_C(z) - (1 - \gamma) \right) = \lim_{z \to \infty} z \left( \eta_C(z) - (1 - \gamma) \right) \left( 1 + \frac{\xi - 1}{1 - \gamma} \right)
\]

\[
= \frac{\xi - \gamma}{1 - \gamma} \cdot \int_{x \neq 0} \frac{1}{x} dF_C(x),
\]

where we used (43) and (46). This shows that \( \gamma = \frac{\xi - \gamma}{1 - \gamma} \int_{x \neq 0} \frac{1}{x} dF_C(x) \). We conclude by observing that

\[
\theta_{1,h} = \lim_{n \to \infty} \frac{1}{d} \text{trace} \mathbb{E} \left[ (S_{1,n}^T S_{1,n})^{-1} \right] = \frac{1}{\gamma} \cdot \lim_{n \to \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{d} \frac{1}{\lambda_i} \right] = \frac{1}{\gamma} \int_{x \neq 0} \frac{1}{x} dF_C(x),
\]

and consequently, \( \theta_{1,h} = \frac{1 - \gamma}{\xi - \gamma} \), which is the claimed result.

A.2.2 Computing \( \theta_{2,h} \)

Unrolling its definition, we have that

\[
\theta_{2,h} = \lim_{n \to \infty} \frac{1}{d} \text{trace} \mathbb{E} \left[ (S_{1,n}^T S_{1,n})^{-2} \right] = \frac{1}{\gamma} \cdot \lim_{n \to \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{d} \frac{1}{\lambda_i^2} \right] = \frac{1}{\gamma} \int_{x \neq 0} \frac{1}{x^2} dF_C(x),
\]

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where the limit in the third equation holds and is finite since $F_C$ has support bounded away from 0 on $\mathbb{R}^n$. By definition of $m_C$ and using the fact that $F_C$ has point mass 1 at 0, we get that
\[
\frac{d m_C(z)}{dz} = \int \frac{1}{(x-z)^2} \, dF_C(x) = \frac{1 - \gamma}{z^2} + \int_{\{x \neq 0\}} \frac{1}{(x-z)^2} \, dF_C(x).
\]

Using again the fact that $F_C$ has support bounded away from 0 on $\mathbb{R}^n$ and the dominated convergence theorem, we have that $\gamma \theta_{2,h} = \lim_{z \to 0} \int_{x \neq 0} \frac{1}{(x-z)^2} \, dF_C(x)$, and thus,
\[
\gamma \theta_{2,h} = \lim_{z \to 0} \left\{ \frac{d m_C(z)}{dz} - \frac{1 - \gamma}{z^2} \right\}.
\]

We denote
\[
\begin{align*}
\Delta &= (\gamma + \xi - 2 + z)^2 + 4(z-1)(1-\gamma)(1-\xi), \\
\Delta' &= \frac{d\Delta}{dz} = 2(z + \gamma + \xi - 2) + 4(1-\gamma)(1-\xi).
\end{align*}
\]

Then, using the expression (15) of $m_C$ and taking the derivative, it follows that
\[
\frac{d m_C(z)}{dz} - \frac{1 - \gamma}{z^2} = \frac{1 - \frac{1}{\sqrt{\Delta}}(2(z + \gamma + \xi - 2) + 4(1-\gamma)(1-\xi))}{2z(1-z)} + \frac{(z + \gamma + \xi - 2 - \sqrt{\Delta})(2z - 1)}{2z^2(z-1)^2} + \frac{\gamma - 1}{z^2}
\]
\[
= \frac{1}{2z^2(z-1)^2} [\Delta_1 + (2\gamma \xi - \gamma - \xi)\Delta_2 - \Delta_3 + \Delta_4],
\]
where
\[
\begin{align*}
\Delta_1 &= z^2(z-1) \\
\Delta_2 &= \frac{z(z-1)}{\sqrt{\Delta}} \\
\Delta_3 &= (z-1)\sqrt{\Delta} \\
\Delta_4 &= z(1-z) + (z + \gamma + \xi - 2)(2z - 1) + 2(\gamma - 1)(z - 1)^2.
\end{align*}
\]

According to L’Hospital rule,
\[
\gamma \theta_{2,h} = \lim_{z \to 0} \frac{\Delta_1'' + (2\gamma \xi - \gamma - \xi)\Delta_2'' - \Delta_3'' + \Delta_4''}{2(12z^2 - 12z + 2)} = \lim_{z \to 0} \frac{\Delta_1'' + (2\gamma \xi - \gamma - \xi)\Delta_2'' - \Delta_3'' + \Delta_4''}{4}
\]
\[
= \frac{\Delta_1''}{4},
\]
where $\Delta_i''$ denotes the second derivative of $\Delta_i$ with respect to $z$. After some calculations, we find that
\[
\Delta_1''|_{z=0} = -\frac{2}{\xi - \gamma},
\]
\[
\Delta_2''|_{z=0} = \frac{2}{\xi - \gamma} + \frac{4\gamma\xi - 2\gamma - 2\xi}{(\xi - \gamma)^3},
\]
\[
\Delta_3''|_{z=0} = \frac{4(2\gamma\xi - \gamma - \xi) - 1}{\xi - \gamma} + \frac{(2\gamma\xi - \gamma - \xi)^2}{(\xi - \gamma)^3},
\]
\[
\Delta_4''|_{z=0} = 2(2\gamma - 1).
\]

Using (50), it follows that
\[
\gamma \theta_{2,h} = \frac{1}{4} \left( -\frac{(2\gamma - 1)^2}{\xi - \gamma} + \frac{(2\gamma\xi - \gamma - \xi)^2}{(\xi - \gamma)^3} \right) = \frac{\gamma(1-\gamma)(\gamma^2 + \xi - 2\gamma\xi)}{(\xi - \gamma)^3},
\]
and finally, we obtain the claimed expression, that is, $\theta_{2,h} = \frac{1 - \gamma(\gamma^2 + \xi - 2\gamma\xi)}{(\xi - \gamma)^3}$. 

A.3 Proof of Theorem 3.3

Let \( \{S_t\} \) be a sequence of independent \( m \times n \) Haar matrices, and let \( \{x_t\} \) be the sequence of iterates generated by the update (2) with \( \mu_t = \theta_{1,h}/\theta_{2,b} \) and \( \beta_t = 0 \). Recall that we denote \( \Delta_t = U^T A(x_t - x^*) \), where \( A = USV^T \) is a thin singular value decomposition of \( A \). For \( t \geq 0 \), we have that

\[
A \left( A^T S^T S A \right)^{-1} A^T = U \Sigma V^T \left( V \Sigma U^T S^T S U \Sigma V^T \right)^{-1} V \Sigma U^T = U \Sigma V^T V \Sigma^{-1} (U^T S^T S U)^{-1} \Sigma^{-1} V V^T \Sigma U^T = U (U^T S^T S U)^{-1} U^T
\]

Multiplying both sides of the update formula (2) by \( A \), subtracting \( A x^* \) and using the normal equation \( A^T A x^* = A^T b \), we find that

\[
A (x_{t+1} - x^*) = \left( I_n - \mu_t U (U^T S_t^T S_t U)^{-1} U^T \right) A (x_t - x^*). \tag{51}
\]

Multiplying both sides of (51) by \( U^T \), using the definition of \( \Delta_t \) and the fact that \( U^T U = I_d \), it follows that

\[
\Delta_{t+1} = U^T \left( I_n - \mu_t U (U^T S_t^T S_t U)^{-1} U^T \right) A (x_t - x^*) \]

\[
= \left( U^T - \mu_t U (U^T S_t^T S_t U)^{-1} U^T \right) (Ax_t - x^*) \]

\[
= \left( I_d - \mu_t (U^T S_t^T S_t U)^{-1} \right) \Delta_t,
\]

and then, taking the squared norm,

\[
\|\Delta_{t+1}\|^2 = \Delta_t^T \left( I_d - \mu_t (U^T S_t^T S_t U)^{-1} \right)^2 \Delta_t.
\]

Taking the expectation with respect to \( S_t \) and using the independence of \( S_t \) with respect to \( S_0, \ldots, S_{t-1} \), we obtain that

\[
\mathbb{E}_{S_t} [\|\Delta_{t+1}\|^2] = \Delta_t^T \mathbb{E} \left[ \left( I_d - \mu_t (U^T S_t^T S_t U)^{-1} \right)^2 \right] \Delta_t \tag{52}
\]

\[
= \Delta_t^T \left( I_d - 2 \mu_t \mathbb{E} \left[ (U^T S_t^T S_t U)^{-1} \right] + \mu_t^2 \mathbb{E} \left[ (U^T S_t^T S_t U)^{-2} \right] \right) \Delta_t. \tag{53}
\]

We write the spectral decomposition \( U^T S_t^T S_t U = V \Sigma V^T \) where \( \Sigma \) is diagonal with positive entries \( \lambda_1, \ldots, \lambda_d \) and \( V_t = [v_1, \ldots, v_d] \) is a \( d \times d \) orthogonal matrix. The matrix \( S_t U \) is distributed as the \( m \times d \) upper-left block of an \( n \times n \) Haar matrix. Therefore, \( S_t U \) is right rotationally invariant, and so is the matrix \( V \). It follows that \( \lambda_i v_{ik} v_{i\ell} \overset{d}{=} -\lambda_i v_{ik} v_{i\ell} \) for any index \( i \) and any indices \( k \neq \ell \). Then, for any \( p \in \{1, 2\} \) and any \( k \neq \ell \), we have

\[
\mathbb{E} \left[ \left( (U^T S^T S U)^{-p} \right)_{k\ell} \right] = \sum_{i=1}^d \mathbb{E} \left[ \lambda_i^{-p} v_{ik} v_{i\ell} \right] = - \sum_{i=1}^d \mathbb{E} \left[ \lambda_i^{-p} v_{ik} v_{i\ell} \right],
\]

which implies that the off-diagonal term \( \mathbb{E} \left[ \left( (U^T S^T S U)^{-p} \right)_{k\ell} \right] \) is equal to 0. Further, by permutation invariance of the matrix \( V \), we get that for any \( k \),

\[
\mathbb{E} \left[ \left( (U^T S^T S U)^{-p} \right)_{kk} \right] = \frac{1}{d} \text{trace} \mathbb{E} \left[ (U^T S^T S U)^{-p} \right],
\]

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or equivalently, \( \mathbb{E} \left[ (U^T S^T SU)^{-p} \right] = \theta_{p,n} I_d \) where \( \theta_{p,n} := d^{-1} \text{trace} \left[ (U^T S^T SU)^{-p} \right] \). Then, using (53), it follows that

\[
\mathbb{E}_{S_t} \left[ \| \Delta_{t+1} \|^2 \right] = \mathbb{E}_{S_t} \left[ \Delta_t^T \left( I_d - 2\mu_t \theta_{1,n} I_d + \mu_t^2 \theta_{2,n} I_d \right) \Delta_t \right]
\]

\[
= \left( 1 - 2\mu_t \theta_{1,n} + \mu_t^2 \theta_{2,n} \right) \| \Delta_t \|^2
\]

By induction, we further obtain

\[
\mathbb{E}_{S_t} \left[ \| \Delta_0 \|^2 \right] = \prod_{j=0}^{t-1} \left( 1 - \frac{\theta_{1,n}^2}{\theta_{2,n}} + \left( \frac{\theta_{1,n}}{\sqrt{\theta_{2,n}}} - \mu_j \sqrt{\theta_{2,n}} \right)^2 \right) \cdot \| \Delta_t \|^2.
\]

Taking the limit \( n \to \infty \) and using the definition \( \theta_{h,p} = \lim_{n \to \infty} \theta_{p,n} \) for \( p \in \{1, 2\} \), we find that

\[
\lim_{n \to \infty} \mathbb{E}_{S_t} \left[ \| \Delta_0 \|^2 \right] = \prod_{j=0}^{t-1} \left( 1 - \frac{\theta_{1,h}^2}{\theta_{2,h}} + \left( \frac{\theta_{1,h}}{\sqrt{\theta_{2,h}}} - \mu_j \sqrt{\theta_{2,h}} \right)^2 \right) \cdot \left( 1 - \frac{\theta_{1,h}^2}{\theta_{2,h}} \right)^t.
\]

The above right-hand side is minimized at \( \mu_j = \theta_{1,h}/\theta_{2,h} \) for all times steps \( j \geq 0 \), which yields the error formula

\[
\lim_{n \to \infty} \mathbb{E}_{S_t} \left[ \| \Delta_0 \|^2 \right] = \left( 1 - \frac{\theta_{1,h}^2}{\theta_{2,h}} \right)^t.
\]

Plugging-in the expressions of \( \theta_{1,h} \) and \( \theta_{2,h} \), we obtain the claimed convergence rate \( \rho_h \).

It remains to prove that \( \rho_h \) is the best rate one may achieve with the update (2) along with Haar embeddings. It is actually an immediate consequence of Theorem 2 by [14], whose assumptions are satisfied by Haar embeddings.

### A.4 Proof of Theorem 4.2 and 4.3

Let \( \{S_t\} \) be a sequence of independent \( m \times n \) SRHT matrices, and let \( \{x_t\} \) be the sequence of iterates generated by the update (2) with \( \mu_t = \theta_{1,h}/\theta_{2,h} \) and \( \beta_t = 0 \). Denote \( \Delta_t = U^T S_t (x_t - x^*) \) the sequence of error vectors. The proof follows exactly the same lines as for Theorem 4.2 up to the relationship (53), which we recall here,

\[
\mathbb{E}_{S_t} \left[ \| \Delta_{t+1} \|^2 \right] = \mathbb{E}_{S_t} \left[ \Delta_t^T \left( I_d - \mu_t (U^T S_t S_t U)^{-1} \right)^2 \Delta_t \right].
\]

Denote \( Q_t = (I_d - \mu_t (U^T S_t S_t U)^{-1})^2 \). It holds that \( \Delta_{t+1} = Q_t \Delta_t \) as previously shown. Hence, by induction, we obtain that

\[
\mathbb{E} \left[ \| \Delta_t \|^2 \right] = \text{trace} \mathbb{E} \left[ Q_0 \ldots Q_{t-1} Q_{t-1} \ldots Q_0 \Delta_0 \right].
\]

Using the independence of \( \Delta_0 \) and the \( Q_t \), and the assumption \( \mathbb{E} [\Delta_0 \Delta_0^\top] = I_d/d \), it follows that

\[
\mathbb{E} \left[ \| \Delta_t \|^2 \right] = \frac{1}{d} \text{trace} \mathbb{E} \left[ Q_1 \ldots Q_{t-1} Q_{t-1} \ldots Q_0^2 \right].
\]
It holds that the matrix $Q_0^2$ is asymptotically free from $Q_{t-1}\ldots Q_1$. Therefore, using the trace decoupling relation (4), we have that

$$
\lim_{n \to \infty} \mathbb{E} [\|\Delta_t\|^2] = \lim_{n \to \infty} \frac{1}{d} \text{trace} \mathbb{E} [Q_1 \ldots Q_{t-1}Q_t \ldots Q_0^2] = \lim_{n \to \infty} \frac{1}{d} \text{trace} \mathbb{E} [Q_0^2] \cdot \lim_{n \to \infty} \frac{1}{d} \text{trace} \mathbb{E} [Q_2 \ldots Q_{t-1}Q_t \ldots Q_1^2].
$$

Note that $\lim_{n \to \infty} \frac{1}{d} \text{trace} \mathbb{E} [Q_0^2] = (1 - 2\mu_0 \theta_{1,h} + \mu_2^2 \theta_{2,h})$. Repeating the same asymptotic freeness argument between $Q_1^2$ and $Q_{t-1} \ldots Q_2$ and plugging-in $\mu_j = \theta_{1,h}/\theta_{2,h}$, we finally obtain the claimed result,

$$
\lim_{n \to \infty} \mathbb{E} [\|\Delta_{t+1}\|^2] = \prod_{j=0}^{t-1} (1 - \mu_j \theta_{1,h} + \mu_j^2 \theta_{2,h}) = \left(1 - \frac{\theta_{1,h}^2}{\theta_{2,h}}\right)^t.
$$

The proof of Theorem 4.3 immediately follows from an alternative upper-bound on the expression (55) for the norm of the error. In particular, we note that

$$
\mathbb{E} [\|\Delta_t\|^2] = \text{trace} \mathbb{E} [Q_0 \ldots Q_{t-1}Q_t \ldots Q_0 \Delta_0 \Delta_0^\top] \leq \|\Delta_0 \Delta_0^\top\|_2 \text{trace} \mathbb{E} [Q_0 \ldots Q_{t-1}Q_t \ldots Q_0] = d\|\Delta_0\|_2^2 \frac{1}{d} \text{trace} \mathbb{E} [Q_0 \ldots Q_{t-1}Q_t \ldots Q_0].
$$

We then combine the earlier expression (56) with the above upper-bound and complete the proof.

## B Proofs of the auxiliary results

### B.1 Proof of the bounds on the support of $F_h$

We show that the support of $F_h$ satisfies

$$\inf \text{supp}(F_h) \geq \frac{(1 - \sqrt{p_g})^2}{(1 + \frac{1}{\sqrt{\xi}})^2}.$$

Let $S$ be an $m \times n$ Haar matrix, $U$ an $n \times d$ deterministic matrix with orthonormal columns, and $S_g$ be an $m \times n$ matrix independent of $S$, with i.i.d. entries $\mathcal{N}(0, 1/m)$. Write $S_g = \Omega_\ell \Sigma \Omega_r$ a singular value decomposition of $S_g$. It holds that $\Omega_\ell$ is an $m \times m$ Haar matrix, independent of the $m \times m$ diagonal matrix of singular values $\Sigma$, and $\Omega_r \stackrel{d}{=} S$, so that $\Omega_\ell \Sigma S \stackrel{d}{=} S_g$. Further, the operator norm of $\Sigma$ satisfies $\lim_{n \to \infty} \|\Sigma\|_2 = \left(1 + \frac{1}{\sqrt{\xi}}\right)$ almost surely. Then,

$$\sigma_{\min}(SU) = \min_{\|x\|=1} \|SUx\| \geq \min_{\|x\|=1} \frac{\|\Sigma SUx\|}{\|\Sigma\|_2} = \frac{1}{\|\Sigma\|_2} \cdot \min_{\|x\|=1} \|\Omega_\ell \Sigma SUx\|.
$$

Almost surely, $\min_{\|x\|=1} \|\Omega_\ell \Sigma Sx\| \to (1 - \sqrt{p_g})$ as $n \to \infty$. Thus, almost surely, $\liminf_{n \to \infty} \sigma_{\min}(SU) \geq \frac{(1 - \sqrt{p_g})}{(1 + \frac{1}{\sqrt{\xi}})^2}$, which yields the claimed lower bound on the support of $F_h$. 

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B.2 Proof of Lemma 4.1

Note that both $B$ and $D$ are diagonal matrices whose diagonal entries are i.i.d. random variables, and $P$ is a permutation matrix. Define $\tilde{B} = PBP^T$ and $\tilde{D} = P^TDP$, then we have

$$\tilde{B}^d = B, \quad \tilde{D}^d = D$$

and

$$DP = P\tilde{D}, \quad P^TD = \tilde{D}P^T. \quad (57)$$

It follows that

$$U^T P^T DH_n DPBP^T DH_n DPU = U^T P^T DH_n P\tilde{D}B\tilde{D}P^T H_n DPU$$

$$= U^T P^T DH_n PB\tilde{D}^2 P^T H_n DPU$$

$$= U^T P^T DH_n PBP^T H_n DPU$$

$$= U^T P^T DH_n \tilde{B}H_n DPU$$

$$= U^T P^T DH_n \tilde{B}H_n DPU, \quad \text{d}$$

where the first equation follows from (57), the second equation holds because $\tilde{D}$ and $B$ are diagonal so they commute, while the third equation holds because $\tilde{D}^2 = I_n$.

B.3 Proof of the identity (28)

We note that

$$F_{G_n}(x) \overset{d}{=} \left(1 - \frac{\tilde{d}}{n}\right) 1_{\{x \geq 0\}} + \frac{1}{n} \sum_{j=1}^{\tilde{d}} 1_{\{x \geq \lambda_j\}}$$

$$= \left(1 - \frac{\tilde{d}}{n}\right) 1_{\{x \geq 0\}} + \frac{d}{n} \cdot \frac{1}{d} \sum_{j=1}^{\tilde{d}} 1_{\{x \geq \lambda_j\}}$$

$$= \left(1 - \frac{\tilde{d}}{n}\right) 1_{\{x \geq 0\}} + \frac{d}{n} \left(F_{S,n}(x) - \left(\frac{d - \tilde{d}}{d}\right) 1_{\{x > 0\}}\right)$$

$$= \left(1 - \frac{d}{n}\right) 1_{\{x \geq 0\}} + \frac{d}{n} F_{S,n}(x),$$

which proves (28).