Dynamics Compensation in Observation of Abstract Linear Systems*

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Abstract

This is the second part of four series papers, aiming at the problem of sensor dynamics compensation for abstract linear systems. Two major issues are addressed. The first one is about the sensor dynamics compensation in system observation and the second one is on the disturbance dynamics compensation in output regulation for linear system. Both of them can be described by the problem of state observation for an abstract cascade system. We consider these two apparently different problems from the same abstract linear system point of view. A new scheme of the observer design for the abstract cascade system is developed and the exponential convergence of the observation error is established. It is shown that the error based observer design in the problem of output regulation can be converted into a sensor dynamics compensation problem by the well known regulator equations. As a result, a tracking error based observer for output regulation problem is designed by exploiting the developed method. As applications, the ordinary differential equations (ODEs) with output time-delay and an unstable heat equation with ODE sensor dynamics are fully investigated to validate the theoretical results. The numerical simulations for the unstable heat system are carried out to validate the proposed method visually.

Keywords: Cascade system, observer, Sylvester equation, sensor dynamics, output regulation.

1 Introduction

When a sensor is installed on a control plant indirectly, the dynamics that connect the control plant and the sensor are referred to as sensor dynamics. In this case, one has to compensate the

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sensor dynamics in the observer design. The compensation of sensor dynamics dominated by partial differential equations (PDEs) has attracted much attention in recent years. The most commonly used example for infinite-dimensional sensor dynamics compensation is the output time-delay compensation in the context of the backstepping method ([9]), where the time-delay is regarded as the sensor dynamics dominated by a transport equation. Soon after [9], the sensor delay dynamics have been extended to the dynamics dominated by the first order hyperbolic equation [10, 21] and the wave equation [11]. All are by the PDE backstepping method ([12]).

The PDE backstepping method is powerful in the observer design for ODE-PDE cascade systems. However, it relies on a priori target system before the backstepping transformation can be performed. This means that the successful employment of the backstepping method requires a proper choice of target system. Although such choice seems natural in many situations like those in [9, 10, 11], it is more or less relying on intuition rather than strict analysis. This limits the applicability of the PDE backstepping method and can be seen from the systems described by Euler-Bernoulli beam equations or multi-dimensional PDEs for which the backstepping method is hardly applied. Roughly speaking, a sufficient condition that ensures the applicability of the PDE backstepping method is still lacking. Moreover, in the PDE backstepping method, the Lyapunov function has to be constructed in the stability analysis for the resulting closed-loop system, which gives rise to some challenges in particular for the treatment of PDEs with delays.

System observation through sensor dynamics can usually be modeled as a cascade observation system. Similarly, in output regulation for linear systems, the control plant driven by the disturbance generated from an exosystem can also be described by a cascade system. This implies that the sensor dynamics compensation is closely related to the observer design for estimation of the disturbance and the state simultaneously in the output regulation, although these two problems were investigated separately in literature. As will be seen, these two problems can be connected equivalently by an invertible transformation constructed by the well known regulator equations in the output regulation. As a consequence, an error based observer can be designed for the output regulation of abstract systems.

In this paper, we unify various types of sensor dynamics compensations from a general abstract framework point of view. We model the control system with indirect sensor configuration as a cascade system. Let $X_j$, $U_j$, and $Y_j$ be Hilbert spaces and let $A_j : X_j \to X_j$, $B_j : U_j \to X_j$, $C_1 : X_1 \to Y_1$ and $C_2 : X_2 \to U_1$ be related operators which are possibly unbounded, $j = 1, 2$. The considered cascade system is described by

$$
\begin{align*}
\dot{x}_1(t) &= A_1 x_1(t) + B_1 C_2 x_2(t), \\
\dot{x}_2(t) &= A_2 x_2(t) + B_2 u(t), \\
y(t) &= C_1 x_1(t),
\end{align*}
$$

where $y(t)$ is the measured output, $u(t)$ is the control and $B_1 C_2 : X_2 \to X_1$ represents the interconnection. The $x_2$-subsystem of (1.1) is the control plant but the sensor is installed indirectly on the $x_1$-subsystem.
The aim of this paper is to develop a systematic methodology for the observer design of system (1.1). The observer we shall design is the well known Luenberger-like observer. Owing to the generality of system (1.1), the sensor dynamics dominated by either ODE or PDE like the first order hyperbolic equation, heat equation, wave equation and the Euler-Bernoulli beam equation can be compensated effectively in a unified framework. When \( A_2 \) is a matrix, the existence of the observer gains can be characterized by the system (1.1) itself. More importantly, the more complicated problem that observation of the infinite-dimensional system through finite-dimensional sensor dynamics can still be addressed effectively. To demonstrate the effectiveness of the proposed approach, the observer design for an unstable heat equation with ODE sensor dynamics is considered.

We organize the paper as follows. In Section 2, we demonstrate the main idea of sensor dynamics compensation through a finite-dimensional cascade system. Some preliminary results on abstract systems are presented in Section 3. The well-posedness of the open-loop system is discussed in Section 4. The observability is discussed in Section 5. Section 6 is devoted to the Luenberger-like observer design for abstract linear systems, where the results in Section 2 for the finite-dimensional system are extended to the infinite-dimensional counterparts. The well-posedness and the exponential convergence of the observer are obtained. As an application, the disturbance and state observation in output regulation is investigated in Section 7. To validate the theoretical results, we consider the observations of ODEs with output delay in Section 8 and an unstable heat system with ODE sensor dynamics in Section 9. Some numerical simulations are presented in Section 10 to show the theory visually, followed up conclusions in Section 11. For the sake of readability, some results which are less relevant to the dynamics compensator design are arranged in the Appendix.

Throughout the paper, the inner product of the Hilbert space \( X \) is denoted by \( \langle \cdot, \cdot \rangle_X \), and the induced norm is denoted by \( \| \cdot \|_X \). Identity operator on the Hilbert space \( X_i \) is denoted by \( I_i, i = 1, 2 \). The space of bounded linear operators from \( X_1 \) to \( X_2 \) is denoted by \( L(X_1, X_2) \). If \( A \in L(X_1, X_2) \), we represent the kernel, domain, resolvent set and the spectrum of \( A \) as \( \text{Ker}(A), D(A), \rho(A) \) and \( \sigma(A) \), respectively.

### 2 Motivation

As mentioned, this section demonstrates the main idea of the observer design for system (1.1) in the setting of finite-dimensional state space. The Luenberger observer of system (1.1) is designed as

\[
\begin{cases}
\dot{x}_1(t) = A_1 x_1(t) + B_1 C_2 x_2(t) - F_1 y(t) - C_1 x_1(t), \\
\dot{x}_2(t) = A_2 x_2(t) + F_2 y(t) - C_1 x_1(t) + B_2 u(t),
\end{cases}
\]

(2.1)

where \( F_j \in L(Y_j, X_j), j = 1, 2 \) are the gain vectors to be determined. When system (1.1) is observable, \( F_1 \) and \( F_2 \) can be chosen easily by the pole assignment theorem of the linear systems. However, the problem becomes complicated if either \( A_1 \) or \( A_2 \) is an operator in an infinite-dimensional Hilbert space. We therefore need an alternative way to find \( F_1 \) and \( F_2 \) to be adaptive to the
infinite-dimensional setting. Let the observer errors be
\[
\tilde{x}_j(t) = x_j(t) - \hat{x}_j(t), \quad j = 1, 2,
\]
which are governed by
\[
\begin{aligned}
\dot{x}_1(t) &= (A_1 + F_1C_1)\tilde{x}_1(t) + B_1C_2\tilde{x}_1(t), \\
\dot{x}_2(t) &= A_2\tilde{x}_2(t) - F_2C_1\tilde{x}_1(t).
\end{aligned}
\]
(2.3)
If we select \( F_1 \) and \( F_2 \) properly such that system (2.3) is stable, then \((x_1(t), x_2(t))\) can be estimated in the sense that
\[
\|(x_1(t) - \tilde{x}_1(t), x_2(t) - \tilde{x}_2(t))\|_{X_1 \times X_2} \to 0 \quad \text{as} \quad t \to \infty.
\]
(2.4)
Inspired by the first part [4] of this series works, the \( F_1 \) and \( F_2 \) can be chosen easily by decoupling the system (2.3) as a cascade system. The corresponding transformation is
\[
\begin{pmatrix}
I_1 & S \\
0 & I_2
\end{pmatrix}
\begin{pmatrix}
A_1 + F_1C_1 & B_1C_2 \\
-F_2C_1 & A_2
\end{pmatrix}
\begin{pmatrix}
I_1 & S \\
0 & I_2
\end{pmatrix}^{-1}
\begin{pmatrix}
A_1 + (F_1 - SF_2)C_1 & SA_2 - [A_1 + (F_1 - SF_2)C_1]S + B_1C_2 \\
-F_2C_1 & A_2 + F_2C_1S
\end{pmatrix},
\]
(2.5)
where \( S \in \mathcal{L}(X_2, X_1) \) is to be determined. If we select \( S \) properly such that
\[
SA_2 - [A_1 + (F_1 - SF_2)C_1]S + B_1C_2 = 0,
\]
(2.6)
then the right side matrix of (2.5) is Hurwitz if and only if the matrices \( A_1 + (F_1 - SF_2)C_1 \) and \( A_2 + F_2C_1S \) are Hurwitz.

**Lemma 2.1.** Let \( X_1, X_2, U_1, U_2 \) and \( Y_1 \) be Euclidean spaces, and let \( A_j \in \mathcal{L}(X_j), B_j \in \mathcal{L}(U_j, X_j), \) \( j = 1, 2, C_2 \in \mathcal{L}(X_2, U_1) \) and \( C_1 \in \mathcal{L}(X_1, Y_1) \). Suppose that system (1.1) is observable. Then, there exist \( F_1 \in \mathcal{L}(Y_1, X_1) \) and \( F_2 \in \mathcal{L}(Y_1, X_2) \) such that the solution of the observer (2.1) satisfies:
\[
\|(x_1(t) - \tilde{x}_1(t), x_2(t) - \tilde{x}_2(t))\|_{X_1 \times X_2} \to 0 \quad \text{as} \quad t \to \infty.
\]
(2.7)
Moreover, \( F_1 \) and \( F_2 \) can be selected by the scheme: (a) Select \( F_0 \in \mathcal{L}(Y_1, X_1) \) such that \( A_1 + F_0C_1 \) is Hurwitz; (b) Solve the Sylvester equation \( (A_1 + F_0C_1)S - SA_2 = B_1C_2 \); (c) Select \( F_2 \in \mathcal{L}(Y_1, X_2) \) such that \( A_2 + F_2C_1S \) is Hurwitz; (d) Set \( F_1 = F_0 + SF_2 \).

**Proof.** Since system (1.1) is observable, its cascade structure implies that \((A_1, C_1)\) is observable as well. As a result, there exists an \( F_0 \in \mathcal{L}(Y_1, X_1) \) such that \( A_1 + F_0C_1 \) is Hurwitz, and at the same time,
\[
\sigma(A_1 + F_0C_1) \cap \sigma(A_2) = \emptyset.
\]
(2.8)
By [19], the Sylvester equation
\[
(A_1 + F_0C_1)S - SA_2 = B_1C_2
\]
(2.9)
admits a unique solution \( S \in \mathcal{L}(X_2, X_1) \). Hence, the invertible transformation in (2.5) satisfies

\[
\begin{pmatrix}
I_1 & S \\
0 & I_2
\end{pmatrix}
\begin{pmatrix}
A_1 + F_0 C_1 & B_1 C_2 \\
0 & A_2
\end{pmatrix}
\begin{pmatrix}
I_1 & S \\
0 & I_2
\end{pmatrix}^{-1} =
\begin{pmatrix}
A_1 + F_0 C_1 & 0 \\
0 & A_2
\end{pmatrix}
\]

(2.10)

and

\[
(C_1, 0)
\begin{pmatrix}
I_1 & S \\
0 & I_2
\end{pmatrix}^{-1} = (C_1, -C_1 S).
\]

(2.11)

Since the observability of system (1.1) keeps invariant under the output feedback, system (1.1) is observable if and only if the pair

\[
\begin{pmatrix}
A_1 + F_0 C_1 & 0 \\
0 & A_2
\end{pmatrix}, (C_1, -C_1 S)
\]

(2.12)

is observable. By (2.8), the observability of (2.12) implies that \((A_2, C_1 S)\) is observable [4, Lemma 10.2]. Therefore, there exists an \( F_2 \in \mathcal{L}(Y_1, X_2) \) such that \( A_2 + F_2 C_1 S \) is Hurwitz. Thanks to the scheme of choosing \( F_1 \), equation (2.9) becomes (2.6). As a result, (2.5) becomes

\[
\begin{pmatrix}
I_1 & S \\
0 & I_2
\end{pmatrix}
\begin{pmatrix}
A_1 + F_1 C_1 & B_1 C_2 \\
-F_2 C_1 & A_2
\end{pmatrix}
\begin{pmatrix}
I_1 & S \\
0 & I_2
\end{pmatrix}^{-1} =
\begin{pmatrix}
A_1 + F_0 C_1 & 0 \\
-F_2 C_1 & A_2 + F_2 C_1 S
\end{pmatrix},
\]

(2.13)

which is obviously Hurwitz. This shows that the error system (2.3) is stable and hence (2.7) holds true in terms of (2.2). This completes the proof of the lemma. \( \square \)

3 Preliminaries on abstract linear systems

In order to extend the finite-dimensional results in Section 2 to the infinite-dimensional systems, we introduce some necessary background on the infinite-dimensional linear systems, in particular for those systems with unbounded control and observation operators, which has been extensively discussed in [22].

Suppose that \( X \) is a Hilbert space and \( A : D(A) \subset X \to X \) is a densely defined operator with \( \rho(A) \neq \emptyset \). The operator \( A \) can determine two Hilbert spaces: \((D(A), \| \cdot \|_1)\) and \((\mathcal{D}(A^*), \| \cdot \|_{-1})\), where \( \mathcal{D}(A^*) \) is the dual space of \( D(A) \) with respect to the pivot space \( X \), and the norms \( \| \cdot \|_1 \) and \( \| \cdot \|_{-1} \) are defined respectively by

\[
\begin{cases}
\| x \|_1 = \| (\beta - A) x \|_X, \quad \forall \ x \in D(A), \\
\| x \|_{-1} = \| (\beta - A)^{-1} x \|_X, \quad \forall \ x \in X,
\end{cases}
\]

\( \beta \in \rho(A) \). \hspace{1cm} (3.1)

These two spaces are independent of the choice of \( \beta \in \rho(A) \) because different choices of \( \beta \) lead to equivalent norms. For the sake of brevity, we denote the two spaces as \( D(A) \) and \( \mathcal{D}(A^*) \) thereafter. The adjoint of \( A^* \in \mathcal{L}(D(A^*), X) \), denoted by \( \bar{A} \), is defined as

\[
\langle \bar{A} x, y \rangle_{\mathcal{D}(A^*), D(A^*)} = \langle x, A^* y \rangle_X, \quad \forall \ x \in X, \ y \in D(A^*).
\]

(3.2)
Evidently, $\tilde{A}x = Ax$ for any $x \in D(A)$, which means that $\tilde{A} \in \mathcal{L}(X, [D(A^*)]'')$ is an extension of $A$. Since $A$ is densely defined, the extension is unique. By [22, Proposition 2.10.3], $(\beta - \tilde{A}) \in \mathcal{L}(X, [D(A^*)]'')$ and $(\beta - \tilde{A})^{-1} \in \mathcal{L}([D(A^*)]'', X)$, which imply that $\beta - \tilde{A}$ is an isomorphism from $X$ to $[D(A^*)]'$. If the operator $A$ generates a $C_0$-semigroup $e^{At}$ on $X$, then so is for its extension $\tilde{A}$ and $e^{\tilde{A}t} = (\beta - \tilde{A})e^{At}(\beta - \tilde{A})^{-1}$ for any $\beta \in \rho(A)$.

Suppose that $Y$ is an output Hilbert space and $C \in \mathcal{L}(D(A), Y)$. The $\Lambda$-extension of $C$ with respect to $A$ is defined by

\[
\left\{ \begin{array}{l}
C_\Lambda x = \lim_{\lambda \to +\infty} C\lambda(\lambda - A)^{-1}x, \ x \in D(C_\Lambda), \\
D(C_\Lambda) = \{ x \in X \mid \text{the above limit exists} \}.
\end{array} \right. \tag{3.3}
\]

Define the norm

\[
\|x\|_{D(C_\Lambda)} = \|x\|_X + \sup_{\lambda \geq \lambda_0} \|C\lambda(\lambda - A)^{-1}x\|_Y, \ \forall \ x \in D(C_\Lambda), \tag{3.4}
\]

where $\lambda_0 \in \mathbb{R}$ is any number so that $[\lambda_0, \infty) \subset \rho(A)$. By [25, Proposition 5.3], $D(C_\Lambda)$ with norm $\| \cdot \|_{D(C_\Lambda)}$ is a Banach space and $C_\Lambda \in \mathcal{L}(D(C_\Lambda), Y)$. Moreover, there exist continuous embeddings:

\[
D(A) \hookrightarrow D(C_\Lambda) \hookrightarrow X \hookrightarrow [D(A^*)]''. \tag{3.5}
\]

**Proposition 3.1.** ([27]) Let $X$, $U$ and $Y$ be the state space, input space and the output space, respectively. The system $(A, B, C)$ is said to be a regular linear system if and only if the following assertions hold:

(i) $A$ generates a $C_0$-semigroup $e^{At}$ on $X$;
(ii) $B \in \mathcal{L}(U, [D(A^*)]'')$ and $C \in \mathcal{L}(D(A), Y)$ are admissible for $e^{At}$;
(iii) $C_\Lambda(s - \tilde{A})^{-1}B$ exists for some (hence for every) $s \in \rho(A)$;
(iv) $s \to \|C_\Lambda(s - \tilde{A})^{-1}B\|$ is bounded on some right half-plane.

**Definition 1.** ([27]) Let $X$ and $Y$ be Hilbert spaces, let $A$ be the generator of $C_0$-semigroup $e^{At}$ on $X$ and let $C \in \mathcal{L}(D(A), Y)$. The operator $L \in \mathcal{L}(Y, [D(A^*)]'')$ is said to detect system $(A, C)$ exponentially if (a) $(A, L, C)$ is a regular linear system; (b) there exists an $s \in \rho(A)$ such that $I$ is an admissible feedback operator for $C_\Lambda(s - \tilde{A})^{-1}L$; (c) $A + LC_\Lambda$ is exponentially stable.

For other concepts of the admissibility for both control and observation operators, and regular linear systems, we refer to [23, 24, 25, 26].

**Definition 2.** [4] Suppose that $X$ is a Hilbert space and $A_j : D(A_j) \subset X \to X$ is a densely defined operator with $\rho(A_j) \neq \emptyset$, $j = 1, 2$. We say that the operators $A_1$ and $A_2$ are similar with the transformation $P$, denoted by $A_1 \sim_P A_2$, if the operator $P \in \mathcal{L}(X)$ is invertible and satisfies

\[
PA_1P^{-1} = A_2 \quad \text{and} \quad D(A_2) = PD(A_1). \tag{3.6}
\]
Lemma 3.1. Let $X$ and $Y$ be Hilbert spaces. Suppose that the operator $A_j : D(A_j) \subset X \to X$ generates a $C_0$-semigroup $e^{A_jt}$ on $X$ and $C_j \in \mathcal{L}(D(A_j), Y)$, $j = 1, 2$. If there is a $P \in \mathcal{L}(X)$ such that $A_1 \sim_P A_2$ and $C_1 = C_2P$, then, the following assertions hold:

(i) $C_1$ is admissible for $e^{A_1t}$ if and only if $C_2$ is admissible for $e^{A_2t}$;

(ii) $(A_1, C_1)$ is exactly (approximately) observable if and only if $(A_2, C_2)$ is exactly (approximately) observable.

Proof. Since $A_1 \sim_P A_2$, for any $x_2 \in D(A_2)$, there exists an $x_1 \in D(A_1)$ such that $Px_1 = x_2$. Consequently,

$$C_2e^{A_2t}x_2 = C_2Pe^{A_1t}P^{-1}Px_1 = C_1e^{A_1t}x_1. \quad (3.7)$$

Since $\|x_2\|_X \leq \|P\|\|x_1\|_X$ and $\|x_1\|_X \leq \|P^{-1}\|\|x_2\|_X$, the desired results can be concluded from the definitions of the admissibility and the exact (approximate) observability. \qed

4 Well-posedness of observation system

This section is devoted to the well-posedness of open-loop system (1.1). We shall show that the mapping from each initial data and control input to the state and the output is continuous. System (1.1) can be written as an abstract triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, where

$$\mathcal{A} = \begin{pmatrix} \tilde{A}_1 & B_1C_2A \\ 0 & \tilde{A}_2 \end{pmatrix}, \quad D(\mathcal{A}) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X_1 \times X_2 \mid \tilde{A}_1x_1 + B_1C_2A x_2 \in X_1, \tilde{A}_2x_2 \in X_2 \right\}, \quad (4.1)$$

and

$$\mathcal{B} = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}, \quad \mathcal{C} = (C_{1A}, 0), \quad D(\mathcal{C}) = D(C_{1A}) \times X_2. \quad (4.2)$$

Lemma 4.1. Suppose that the operator $A_j$ generates a $C_0$-semigroup $e^{A_jt}$ on $X_j$, $B_j \in \mathcal{L}(U_j, D(A_j^*))$ is admissible for $e^{A_jt}$ and $C_j \in \mathcal{L}(D(A_j), Y_j)$ is admissible for $e^{A_jt}$ with $Y_2 = U_1$, $j = 1, 2$. Then, the operator $\mathcal{A}$ defined by (4.1) generates a $C_0$-semigroup $e^{At}$ on $X_1 \times X_2$. Moreover, the following assertions hold:

(i) If system $(A_1, B_1, C_1)$ is regular, then $\mathcal{C}$ is admissible for $e^{At}$;

(ii) If system $(A_2, B_2, C_2)$ is regular, then $\mathcal{B}$ is admissible for $e^{At}$;

(iii) If both systems $(A_1, B_1, C_1)$ and $(A_2, B_2, C_2)$ are regular, then system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is also regular.

Proof. It has been proved in previous study [4] that the operator $\mathcal{A}$ generates a $C_0$-semigroup $e^{At}$ on $X_1 \times X_2$. We only need to prove (i), (ii) and (iii).

Proof of (i). Since $C_2$ is admissible for $e^{A_2t}$, for any $\tau > 0$, there exists a constant $c_\tau > 0$ such that

$$\int_0^\tau \left\| C_2e^{A_2s}x_2 \right\|_{Y_2}^2 ds \leq c_\tau \|x_2\|_{X_2}^2, \quad \forall \ x_2 \in D(A_2). \quad (4.3)$$
Since \((A_1, B_1, C_1)\) is a regular linear system, there exists a constant \(M_\tau > 0\) such that
\[
\int_0^\tau \left\| C_{1A} \int_0^t e^{A_1(t-s)} B_1 u_1(s) ds \right\|_{Y_1}^2 dt \leq M_\tau \int_0^\tau \| u_1(s) \|_{Y_2}^2 ds, \quad \forall u_1 \in L^2([0, \tau]; Y_2).
\] (4.4)

Combining (4.3) and (4.4), we arrive at
\[
\int_0^\tau \left\| C_{1A} \int_0^t e^{A_1(t-s)} B_1 C_{2A} e^{A_2 s} x_2 ds \right\|_{Y_1}^2 dt \leq M_\tau c_\tau \| x_2 \|_{X_2}^2, \quad \forall x_2 \in D(A_2).
\] (4.5)

A straightforward computation shows that, for any \((x_1, x_2)^T \in D(A)\),
\[
Ce^{At}(x_1, x_2)^T = C_{1A} e^{A_1 t} x_1 + C_{1A} \int_0^t e^{A_1(t-s)} B_1 C_{2A} e^{A_2 s} x_2 ds,
\] (4.6)

which, together with (4.5) and the admissibility of \(C_1\) for \(e^{A_1 t}\), leads to
\[
\int_0^\tau \left\| Ce^{At}(x_1, x_2)^T ds \right\|_{Y_1}^2 dt \leq L_\tau \| (x_1, x_2)^T \|_{X_1 \times X_2}, \quad \forall (x_1, x_2)^T \in D(A),
\] (4.7)

where \(L_\tau\) is a positive constant. Hence, \(C\) is admissible for \(e^{At}\).

Proof of (ii). When \((A_2, B_2, C_2)\) is a regular linear system, for any \(u_2 \in L^2_{\text{loc}}([0, \infty); U_2)\) and \(t \geq 0\), it follows that
\[
\int_0^t e^{A_2(t-s)} B_2 u_2(s) ds \in X_2 \quad \text{and} \quad C_{2A} \int_0^t e^{A_2(t-s)} B_2 u_2(s) ds \in L^2_{\text{loc}}([0, \infty); U_1).
\] (4.8)

These, together with the admissibility of \(B_1\) for \(e^{A_1 t}\), lead to
\[
\int_0^t e^{A(t-s)} B u_2(s) ds = \left( \int_0^t e^{A_1(t-s)} B_1 [C_{2A} \int_0^s e^{A_2(s-\alpha)} B_2 u_2(\alpha) d\alpha] ds \right) \in X_1 \times X_2.
\] (4.9)

Hence, \(B\) is admissible for \(e^{At}\).

Proof of (iii). Since both \((A_1, B_1, C_1)\) and \((A_2, B_2, C_2)\) are regular linear systems, for any \(\lambda \in \rho(A_1) \cap \rho(A_2) \subset \rho(A)\), we conclude that \(C_{jA}(\lambda - \tilde{A}_j)^{-1} B_j \in \mathcal{L}(U_j, Y_j)\) and \(\lambda \to \|C_{jA}(\lambda - \tilde{A}_j)^{-1} B_j\|\) is bounded on some right half-plane, \(j = 1, 2\). Moreover, a simple computation shows that
\[
C(\lambda - A)^{-1} B = C \begin{bmatrix} \lambda - (\tilde{A}_1 & B_1 C_{2A} \\ 0 & \tilde{A}_2 \end{bmatrix}^{-1} \begin{bmatrix} B \\ \end{bmatrix} \\
= (C_{1A}, 0) \begin{bmatrix} (\lambda - \tilde{A}_1)^{-1} & (\lambda - \tilde{A}_1)^{-1} B_1 C_{2A} (\lambda - \tilde{A}_2)^{-1} \\ 0 & (\lambda - \tilde{A}_2)^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ B_2 \end{bmatrix}
\] (4.10)

Consequently, \(C(\lambda - A)^{-1} B \in \mathcal{L}(U_2, Y_1)\) and \(\lambda \to \|C(\lambda - A)^{-1} B\|\) is bounded on some right half-plane. By Proposition 3.1, \((A, B, C)\) is a regular linear system.

As an immediate consequence of Lemma 4.1, we arrive at the following Theorem:
Theorem 4.1. Suppose that \((A_1, B_1, C_1)\) and \((A_2, B_2, C_2)\) are regular linear systems. Then, system (1.1) is well-posed: For any \((x_1(0), x_2(0))^{\top} \in X_1 \times X_2\) and \(u \in L^2_{\text{loc}}([0, \infty); U_2)\), there exists a unique solution \((x_1(t), x_2(t))^{\top} \in C([0, +\infty); X_1 \times X_2)\) to system (1.1) such that

\[
\|(x_1(t), x_2(t))^{\top}\|_{X_1 \times X_2} + \int_0^t \|y(s)\|_Y^2 \, ds \leq C_t \left[ \|(x_1(0), x_2(0))^{\top}\|_{X_1 \times X_2} + \int_0^t \|u(s)\|_{U_2}^2 \, ds \right] \tag{4.11}
\]

for any \(t > 0\), where \(C_t > 0\) is a constant that is independent of \((x_1(0), x_2(0))\) and \(u\).

5 Observability of system (1.1)

In this section, we consider the observability of system (1.1). We denote by \(\rho(\{A_1\})\) the connected component of \(\rho(A_1)\) which contains some right half-plane. This set has been used in [22, Proposition 2.4.3, p.34]. Obviously, there is only one such component. In particular, if \(\sigma(A_1)\) is countable, which is often the case in applications, then \(\rho(\{A_1\}) = \rho(A_1)\).

Theorem 5.1. Let \(X_2\) and \(Y_1\) be finite-dimensional Hilbert spaces and let \((A_1, B_1, C_1)\) be a regular linear system with the state space \(X_1\), input space \(U_1\) and the output space \(Y_1\). Suppose that \(A_2 \in \mathcal{L}(X_2)\), \(C_2 \in \mathcal{L}(X_2, U_1)\) and

\[
\sigma(A_2) \subset \rho(\{A_1\}). \tag{5.1}
\]

Then, system \((\mathcal{A}, C)\) is exactly (approximately) observable if and only if system \((A_1, C_1)\) is exactly (approximately) observable and

\[
\ker \left[ C_{1A}(\lambda - \tilde{A}_1)^{-1}B_1C_2 \right] \cap \ker(\lambda - A_2) = \{0\}, \quad \forall \lambda \in \sigma(A_2), \tag{5.2}
\]

where the operators \(\mathcal{A}\) and \(C\) are given by \((4.1)\) and \((4.2)\), respectively.

Proof. The assumption (5.1) implies that \(\sigma(A_1) \cap \sigma(A_2) = \emptyset\). It follows from [4, Lemma 4.2] that the Sylvester equation \(A_1P - PA_2 = B_1C_2\) admits a unique solution \(P \in \mathcal{L}(X_2, X_1)\) such that \(C_{1A}P \in \mathcal{L}(X_2, Y_1)\) and

\[
\tilde{A}_1Px_2 - PA_2x_2 = B_1C_2x_2, \quad \forall \ x_2 \in X_2. \tag{5.3}
\]

In terms of the solution \(P\), we introduce an invertible transformation \(\mathbb{P} \in \mathcal{L}(X_1 \times X_2)\) by

\[
\mathbb{P}(x_1, x_2)^\top = (x_1 + Px_2, \ x_2)^\top, \quad \forall \ (x_1, x_2)^\top \in X_1 \times X_2, \tag{5.4}
\]

whose inverse is given by

\[
\mathbb{P}^{-1}(x_1, x_2)^\top = (x_1 - Px_2, \ x_2)^\top, \quad \forall \ (x_1, x_2)^\top \in X_1 \times X_2. \tag{5.5}
\]

Define \(\mathcal{A}_\mathbb{P} = \text{diag}(A_1, A_2)\) with \(D(\mathcal{A}_\mathbb{P}) = D(A_1) \times X_2\). Since \(X_2\) is finite-dimensional, we have

\[
D(\mathcal{A}) = \left\{ (x_1, x_2)^\top \in X_1 \times X_2 \mid \tilde{A}_1x_1 + B_1C_2x_2 \in X_1 \right\}. \tag{5.6}
\]

For any \((x_1, x_2)^\top \in D(\mathcal{A}_\mathbb{P})\), it follows from (5.3) and the fact \(P \in \mathcal{L}(X_2, X_1)\) that

\[
\tilde{A}_1(x_1 - Px_2) + B_1C_2x_2 = \tilde{A}_1x_1 - PA_2x_2 \in X_1. \tag{5.7}
\]
which, together with (5.6) and (5.5), leads to that $P^{-1}(x_1, x_2) \in D(A)$. Hence, $D(A_F) \subset P D(A)$ due to the arbitrariness of $(x_1, x_2)$. On the other hand, for any $(x_1, x_2)^T \in D(A)$, by (5.3), (5.6) and the fact $P \in L(X_2, X_1)$, it follows that

$$\hat{A}_1(x_1 + Px_2) = \hat{A}_1x_1 + B_1C_2x_2 + \hat{A}_1Px_2 - B_1C_2x_2 = [\hat{A}_1x_1 + B_1C_2x_2] + PA_2x_2 \in X_1,$$

which implies that $P(x_1, x_2)^T \in D(A_F)$. Hence, $PD(A) \subset D(A_F)$. As a result, $PD(A) = D(A_F)$. Moreover, a simple computation shows that

$$A \sim_{F} A_F \text{ and } C_F^{-1} = (C_{1A}, -C_{1A}P).$$

By Lemma 3.1, $(A, C)$ is exactly (approximately) observable if and only if $(A_F, C_F^{-1})$ is exactly (approximately) observable. Now, it suffices to prove that $(A_F, C_F^{-1})$ is exactly (approximately) observable if and only if $(A_1, C_1)$ is exactly (approximately) observable and (5.2) holds.

Actually, for any $\lambda \in \sigma(A_2)$, since $\lambda \notin \sigma(A_1)$, it follows from (5.3) that

$$P = (\lambda - \hat{A}_1)^{-1}P(\lambda - A_2) - (\lambda - \hat{A}_1)^{-1}B_1C_2.$$

(5.10)

Suppose that $A_2x_2 = \lambda x_2, x_2 \in X_2$. Then, (5.10) yields

$$-C_{1A}Px_2 = C_{1A}(\lambda - \hat{A}_1)^{-1}B_1C_2x_2.$$

(5.11)

By [22, Remark 1.5.2, p.15], $(A_2, -C_{1A}P)$ is observable if and only if

$$\ker(C_{1A}P) \cap \ker(\lambda - A_2) = \{0\}, \forall \lambda \in \sigma(A_2).$$

(5.12)

Combining (5.11) and (5.12), we conclude that $(A_2, -C_{1A}P)$ is observable if and only if (5.2) holds.

When $(\hat{A}_1, C_{1A})$ is exactly (approximately) observable and (5.2) holds true, then $(A_2, -C_{1A}P)$ is observable. Furthermore, by (5.1) and [22, Theorem 6.4.2, p.190] ([22, Proposition 6.4.5, p.192]), $(A_F, C_F^{-1})$ is exactly (approximately) observable. Conversely, if $(A_F, C_F^{-1})$ is exactly (approximately) observable, by the block-diagonal structure of $A_F$ and (5.1), it is easily to obtain that $(A_1, C_1)$ is exactly (approximately) observable and $(A_2, -C_{1A}P)$ is observable (see, e.g., [4, Lemma 10.2]). In particular, (5.2) holds. The proof is complete.

Remark 5.1. When system $(A_1, B_1, C_1)$ is single-input-single-output and system $(A_2, C_2)$ is observable, the assumption (5.2) can be replaced by

$$C_{1A}(\lambda - \hat{A}_1)^{-1}B_1 \neq 0, \lambda \in \sigma(A_2).$$

(5.13)

Indeed, suppose that $x_2 \in \ker[C_{1A}(\lambda - \hat{A}_1)^{-1}B_1C_2] \cap \ker(\lambda - A_2)$ for some $\lambda \in \sigma(A_2)$. By (5.13) and the observability of $(A_2, C_2)$, we obtain $C_2x_2 = 0$ and hence $x_2 = 0$. This yields (5.2). The condition (5.13) implies that every point in $\sigma(A_2)$ is not the transmission zero of system $(A_1, B_1, C_1)$.
6 Luenberger-like observer for abstract linear system

In this section, we extend the results in Section 2 from finite-dimensional systems to infinite-dimensional ones. Inspired by the finite-dimensional observer (2.1), an infinite-dimensional Luenberger-like observer of (1.1) is designed as

\[
\begin{align*}
\dot{x}_1(t) &= A_1x_1(t) + B_1C_2x_2(t) - F_1[y(t) - C_1x_1(t)], \\
\dot{x}_2(t) &= A_2x_2(t) + F_2[y(t) - C_1x_1(t)] + B_2u(t),
\end{align*}
\]

where the tuning gain operators \( F_1 \) and \( F_2 \) are selected by the following scheme

- Find \( F_0 \in \mathcal{L}(Y_1, [D(A_1^*)]')' \) to detect system \((A_1, C_1)\) exponentially;
- Solve the following Sylvester operator equation:

\[
(A_1 + F_0C_{1A})S - SA_2 = B_1C_2;
\]

- Find \( F_2 \in \mathcal{L}(Y_1, [D(A_2^*)]')' \) to detect system \((A_2, C_{1A}S)\) exponentially;
- Set \( F_1 = F_0 + SF_2 \).

**Lemma 6.1.** Suppose that \( X_1, X_2, U_1, Y_1 \) are Hilbert spaces and \( A_j : D(A_j) \subset X_j \to X_j \) is a densely defined operator with \( \rho(A_j) \neq \emptyset, \ j = 1, 2 \). Suppose that \( B_1 \in \mathcal{L}(U_1, [D(A_1^*)]')', C_1 \in \mathcal{L}(D(A_1), Y_1), C_2 \in \mathcal{L}(D(A_2), U_1), F_0 \in \mathcal{L}(Y_1, [D(A_1^*)]'), F_2 \in \mathcal{L}(Y_1, [D(A_2^*)]') \) and \( S \in \mathcal{L}(X_2, X_1) \) solves the Sylvester equation (6.2) in the sense that

\[
(\tilde{A}_1 + F_0C_{1A})Sx_2 - SA_2x_2 = B_1C_2x_2, \ \forall \ x_2 \in D(A_2).
\]

Then, the following assertions hold true:

(i) If \((A_1, B_1, C_1)\) is a regular linear system, then \( C_{1A}S \in \mathcal{L}(D(A_2), Y_1) \);

(ii) If the systems \((A_2, F_2, C_2)\) and \((A_2, F_2, C_{1A}S)\) are regular, then there exists an extension of \( S \), still denoted by \( S \), such that \( SF_2 \in \mathcal{L}(Y_1, [D(A_1^*)]')' \) and

\[
(\tilde{A}_2 + F_2C_{1A})Sx_2 - S\tilde{A}_2x_2 = B_1C_{2A}x_2, \ \forall \ x_2 \in X_2F_2,
\]

where

\[
X_2F_2 = D(A_2) + (\beta - \tilde{A}_2)^{-1}F_2U_2, \ \beta \in \rho(A_2).
\]

**Proof.** (i) For any \( x_2 \in D(A_2) \), the assumption (6.3) yields \( Sx_2 \in D(C_{1A}) \) directly, provided \( F_0 \neq 0 \). When \( F_0 = 0 \), it follows from (6.3) that \( \tilde{A}_1Sx_2 - B_1C_2x_2 = SA_2x_2 \in X_1 \), which implies that \( Sx_2 \in D(A_1^*) + (\alpha - \tilde{A}_1)^{-1}B_1U_1 \) with \( \alpha \in \rho(A_1^*) \). Since \((A_1, B_1, C_1)\) is a regular linear system, \( D(A_1^*) + (\alpha - \tilde{A}_1)^{-1}B_1U_1 \subset D(C_{1A}) \). Therefore, \( Sx_2 \in D(C_{1A}) \) and \( S(D(A_2)) \subset D(C_{1A}) \). Since \( S \in \mathcal{L}(D(A_2), X_1) \) and \( D(C_{1A}) \) is continuously embedded in \( X_1 \), the inclusion \( S(D(A_2)) \subset D(C_{1A}) \) implies that \( S \) is a closed operator from \( D(A_2) \) to \( D(C_{1A}) \). By the closed graph theorem, \( S \in \mathcal{L}(D(A_2), D(C_{1A})) \) and hence \( C_{1A}S \in \mathcal{L}(D(A_2), Y_1) \).
(ii) In terms of the solution $S ∈ \mathcal{L}(X_2, X_1)$ of (6.3), we define the operator $\tilde{S}$ by
\[
\tilde{S} = B_1C_2A(β - \tilde{A}_2)^{-1} + (β - \tilde{A}_1)S(β - \tilde{A}_2)^{-1} - F_0C_1AS(β - \tilde{A}_2)^{-1}, \quad β ∈ ρ(A_2).
\] (6.6)
For any $x_2 ∈ X_2$, since $(β - \tilde{A}_2)^{-1}x_2 ∈ D(A_2)$, it follows from (6.6) and (6.3) that
\[
\tilde{S}x_2 = B_1C_2A(β - \tilde{A}_2)^{-1}x_2 + (β - \tilde{A}_1)S(β - \tilde{A}_2)^{-1}x_2 - F_0C_1AS(β - \tilde{A}_2)^{-1}x_2
\]
\[
= -S\tilde{A}_2(β - \tilde{A}_2)^{-1}x_2 + Sβ(β - \tilde{A}_2)^{-1}x_2
\]
\[
= S(β - \tilde{A}_2)^{-1}x_2 = Sx_2,
\]
which implies that $\tilde{S}$ is an extension of $S$. On the other hand, by (6.6) and the regularity of $(A_2, F_2, C_2)$ and $(A_2, F_2, C_1S)$, we can conclude that
\[
\tilde{S}F_2 = B_1C_2A(β - \tilde{A}_2)^{-1}F_2 + (β - \tilde{A}_1)S(β - \tilde{A}_2)^{-1}F_2 - F_0C_1AS(β - \tilde{A}_2)^{-1}F_2,
\] (6.8)
which implies that $\tilde{S}F_2 ∈ \mathcal{L}(Y_1, [D(A_1^*)]'')$. Moreover, for any $y_1 ∈ Y_1$, if we let $x_β = (β - \tilde{A}_2)^{-1}F_2y_1$, then, it follows from (6.8) and (6.7) that
\[
(\tilde{A}_1 + F_0C_1A)\tilde{S}x_β - B_1C_2Ax_β = β\tilde{S}x_β - \tilde{S}F_2y_1 = β\tilde{S}x_β - \tilde{S}(β - \tilde{A}_2)x_β = \tilde{S}\tilde{A}_2x_β,
\] (6.9)
which implies that $\tilde{S}$ solves the Sylvester equation (6.2) on $(β - \tilde{A}_2)^{-1}F_2Y_1$. Since $\tilde{S}|x_2 = S$, (6.3) and (6.5), we can obtain (6.4) easily with the replacement of $S$ by $\tilde{S}$.

**Theorem 6.1.** Let $(A_j, B_j, C_j)$ be a regular linear system with the state space $X_j$, input space $U_j$ and the output space $Y_j$, $j = 1, 2$. Suppose that $U_1 = Y_2$, $F_0 ∈ \mathcal{L}(Y_1, [D(A_1^*)]'')$ detects system $(A_1, C_1)$ exponentially, $F_2 ∈ \mathcal{L}(Y_1, [D(A_2^*)]'')$ detects system $(A_2, C_1AS)$ exponentially, $(A_2, F_2, C_2)$ is a regular linear system and $F_1 = F_0 + SF_2$, where $S ∈ \mathcal{L}(X_2, X_1)$ is the solution of Sylvester equation (6.2) in the sense of (6.3). Then, the observer (6.1) of system (1.1) is well-posed: For any $(\tilde{x}_1(0), \tilde{x}_2(0))^T ∈ X_1 × X_2$ and $u ∈ L^2_{loc}([0, ∞); U_2)$, the observer (6.1) admits a unique solution $(\hat{x}_1, \hat{x}_2)^T ∈ C([0, ∞); X_1 × X_2)$ such that
\[
e^{ωt}\|x_1(t) - \hat{x}_1(t), x_2(t) - \hat{x}_2(t)\|^T_{X_1 × X_2} → 0 \text{ as } t → ∞,
\] (6.10)
where $ω$ is a positive constant that is independent of $t$.

**Proof.** By Theorem 4.1, for any $(x_1(0), x_2(0))^T ∈ X_1 × X_2$ and $u ∈ L^2_{loc}([0, ∞); U_2)$, system (1.1) admits a unique solution $(x_1, x_2)^T ∈ C([0, ∞); X_1 × X_2)$ such that $y = C_1Ax_1 ∈ L^2_{loc}([0, ∞); Y_1)$. Set the errors
\[
\tilde{x}_i(t) = x_i(t) - \hat{x}_i(t), \quad i = 1, 2,
\] (6.11)
which are governed by
\[
\begin{cases}
\dot{\tilde{x}}_1(t) = (A_1 + F_1C_1)\tilde{x}_1(t) + B_1C_2\tilde{x}_2(t), \\
\dot{\tilde{x}}_2(t) = A_2\tilde{x}_2(t) - F_2C_1\tilde{x}_1(t).
\end{cases}
\] (6.12)

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System (6.12) can be written as
\[
\frac{d}{dt}(\tilde{x}_1(t), \tilde{x}_2(t)) = \mathcal{A}(\tilde{x}_1(t), \tilde{x}_2(t))^{\top},
\] (6.13)
where
\[
\mathcal{A} = \begin{pmatrix}
\tilde{A}_1 + F_1 C_{1A} & B_1 C_{2A} \\
-F_2 C_{1A} & \tilde{A}_2
\end{pmatrix},
\]
\[
D(\mathcal{A}) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X_1 \times X_2 \mid (\tilde{A}_1 + F_1 C_{1A})x_1 + B_1 C_{2A}x_2 \in X_1, \frac{\tilde{A}_2}{2} x_2 - F_2 C_{1A}x_1 \in X_2 \right\}.
\] (6.14)

In terms of the solution \(S\) of the Sylvester equation (6.2), we introduce a bounded invertible transformation \(S \in \mathcal{L}(X_1 \times X_2)\) by
\[
S(x_1, x_2)^\top = (x_1 + Sx_2, x_2)^\top, \quad \forall (x_1, x_2)^\top \in X_1 \times X_2,
\] (6.15)
whose inverse is given by
\[
S^{-1}(x_1, x_2)^\top = (x_1 - Sx_2, x_2)^\top, \quad \forall (x_1, x_2)^\top \in X_1 \times X_2.
\] (6.16)

By virtue of Lemma 6.1, \(C_{1A}S \in \mathcal{L}(D(A_2), Y_1)\), \(SF_2 \in \mathcal{L}(Y_1, [D(A_1^*)]^*)\) and (6.4) holds. Since system \((A_2, F_2, C_{1A}S)\) is regular, it follows that \(C_{1A}S(\beta - \tilde{A}_2)^{-1}F_2 \in \mathcal{L}(Y_1)\) for any \(\beta \in \rho(A_2)\), which, together with (6.5) and the fact \(C_{1A}S \in \mathcal{L}(D(A_2), Y_1)\), implies that \(C_{1A}S(X_{2F_2}) \subset Y_1\).

Define
\[
\mathcal{A}_S = \begin{pmatrix}
\tilde{A}_1 + F_0 C_{1A} & 0 \\
-F_2 C_{1A} & \tilde{A}_2 + F_2 C_{1A}S
\end{pmatrix}
\] (6.17)
with
\[
D(\mathcal{A}_S) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X_1 \times X_2 \mid (\tilde{A}_1 + F_0 C_{1A})x_1 \in X_1, (\tilde{A}_2 + F_2 C_{1A}S)x_2 - F_2 C_{1A}x_1 \in X_2 \right\}.
\] (6.18)

We claim that \(\mathcal{A} \sim_S \mathcal{A}_S\). Indeed, for any \((x_1, x_2)^\top \in D(\mathcal{A})\), since \(X_{2F_2}\) defined by (6.5) can be characterized as ([25, Remark 7.3]):
\[
X_{2F_2} = \left\{ x_2 \in X_2 \mid \tilde{A}_2 x_2 + F_2 y_1 \in X_2, \ y_1 \in Y_1 \right\},
\] (6.19)
\((x_1, x_2)^\top \in D(\mathcal{A})\) implies that \(x_2 \in X_{2F_2}\) and hence \(C_{1A}Sx_2 \in Y_1\). By (6.14), we have
\[
(\tilde{A}_2 + F_2 C_{1A}S)x_2 - F_2 C_{1A}(x_1 + Sx_2) = \tilde{A}_2 x_2 - F_2 C_{1A}x_1 \in X_2.
\] (6.20)

Noting that \(F_1 = F_0 + SF_2\), it follows from (6.14) that
\[
(\tilde{A}_1 + F_0 C_{1A})x_1 + SF_2 C_{1A}x_1 + B_1 C_{2A}x_2 = (\tilde{A}_1 + F_1 C_{1A})x_1 + B_1 C_{2A}x_2 \in X_1,
\] (6.21)
which, together with (6.4), (6.20), (6.21) and \(S \in \mathcal{L}(X_2, X_1)\), yields
\[
(\tilde{A}_1 + F_0 C_{1A})(x_1 + Sx_2) = (\tilde{A}_1 + F_0 C_{1A})x_1 + B_1 C_{2A}x_2 + S\tilde{A}_2 x_2
\] (6.22)
\[
= [(\tilde{A}_1 + F_0 C_{1A})x_1 + SF_2 C_{1A}x_1 + B_1 C_{2A}x_2] + S(\tilde{A}_2 x_2 - F_2 C_{1A}x_1) \in X_1.
\]
From (6.18), (6.20) and (6.22), we can conclude that \( S(x_1, x_2)^\top \in D(\mathcal{A}_2) \) and thus \( S(D(\mathcal{A})) \subset D(\mathcal{A}_2) \).

On the other hand, for any \((x_1, x_2)^\top \in D(\mathcal{A}_2), \) we have
\[
\tilde{A}_2 x_2 - F_2 C_1 A (x_1 - S x_2) \in X_2,
\] which implies that \( x_2 \in X_2 F_2 \). By \( F_1 = F_0 + S F_2, \) (6.4), (6.18) and \( S \in L(X_2, X_1) \),
\[
(\tilde{A}_1 + F_1 C_1 A)(x_1 - S x_2) + B_1 C_2 A x_2 = (\tilde{A}_1 + F_0 C_1 A)x_1 - S[\tilde{A}_2 x_2 - F_2 C_1 A (x_1 - S x_2)] \in X_1.
\] (6.24)
It follows from (6.23) and (6.24) that \( S^{-1}(x_1, x_2)^\top \in D(\mathcal{A}) \) and thus \( D(\mathcal{A}_2) \subset S(D(\mathcal{A})) \). We therefore arrive at \( D(\mathcal{A}_2) = S(D(\mathcal{A})) \). By exploiting (6.4), a straightforward computation shows that \( S(\mathcal{A} S^{-1}(x_1, x_2)^\top) = \mathcal{A}_2 (x_1, x_2)^\top \) for any \((x_1, x_2)^\top \in D(\mathcal{A}_2) \). This proves that \( \mathcal{A}_2 \) and \( \mathcal{A} \) are similar each other.

Since \( F_0 \) and \( F_2 \) detect exponentially systems \((A_1, C_1)\) and \((A_2, C_1 S)\), respectively, \( e^{(\tilde{A}_1 + F_0 C_1 A)t} \) and \( e^{(\tilde{A}_2 + F_2 C_1 S)t} \) are exponentially stable in \( X_1 \) and \( X_2 \), respectively. Moreover, \( C_1 \) is admissible for \( e^{(\tilde{A}_1 + F_0 C_1 A)t} \) and \( F_2 \) is admissible for \( e^{(\tilde{A}_2 + F_2 C_1 S)t} \). By [4, Lemma 3.3], \( \mathcal{A}_2 \) generates an exponentially stable \( C_0 \)-semigroup \( e^{\mathcal{A}_2 t} \) on \( X_1 \times X_2 \). From the similarity of \( \mathcal{A}_2 \) and \( \mathcal{A} \), the operator \( \mathcal{A} \) generates an exponentially stable \( C_0 \)-semigroup \( e^{\mathcal{A} t} \) on \( X_1 \times X_2 \) as well. As a result, system (6.12) with the initial state \((\tilde{x}_1(0), \tilde{x}_2(0)) = (x_1(0) - \tilde{x}_1(0), x_2(0) - \tilde{x}_2(0)) \) admits a unique solution \((\tilde{x}_1, \tilde{x}_2)^\top \in C([0, \infty); X_1 \times X_2) \). Let \((\hat{x}_1(t), \hat{x}_2(t)) = (x_1(t) - \tilde{x}_1(t), x_2(t) - \tilde{x}_2(t)) \) be the solution of the observer (6.1) and satisfies (6.10). The uniqueness of the solution can be obtained easily by the linearity of the observer. The proof is complete.

\[\text{Remark 6.1.}\] When a system is given, one needs to solve the Sylvester equation (6.2) to get the tuning gain operators \( F_1 \) and \( F_2 \) of the observer (6.1). This is usually not a trivial task, particularly for the case that both \( A_1 \) and \( A_2 \) are unbounded. Consequently, Theorem 6.1 does not mean that we can always design an observer for the given system. However, under some reasonable additional assumptions, we still can solve the Sylvester equation analytically or numerically even for the cascade system involving a multi-dimensional PDE (see, e.g., [13] and [17]). In particular, the situation will become easier provided one of \( A_1 \) and \( A_2 \) is bounded. In this case, the solution of the Sylvester equation (6.2) always exists provided (see, e.g.,[4])
\[
\sigma(A_1 + F_0 C_1 A) \cap \sigma(A_2) = \emptyset.
\] (6.25)

When the cascade system consists of an ODEs and a one-dimensional PDE, the problem becomes quite easy. In this case, an implementable way to solve the Sylvester equation is given at the end of Section 5 of the first paper [4] of this series works. To show the effectiveness, this method will be applied to observer design for ODEs with output delay and unstable heat equation with ODE sensor dynamics in Sections 8 and 9, respectively.
When $X_2$ is finite-dimensional, we can characterize the existence of the tuning gains $F_1$ and $F_2$ through the system (1.1) itself.

**Corollary 6.1.** Let $A_2 \in \mathcal{L}(X_2)$ be a matrix with $\sigma(A_2) \subset \{ s \mid \text{Res} \geq 0 \}$, and let $(A_1, B_1, C_1)$ be a regular linear system. Suppose that system $(A_1, C_1)$ is exponentially detectable and $(A, C)$ is approximately observable, where $A$ and $C$ are given by (4.1) and (4.2), respectively. Then, there exist $F_1 \in \mathcal{L}(Y_1, [D(A_1^*)]^\prime)$ and $F_2 \in \mathcal{L}(Y_1, X_2)$ such that the observer (6.1) of system (1.1) is well-posed: For any $(\hat{x}_1(0), \hat{x}_2(0))^\top \in X_1 \times X_2$ and $u \in L_{\text{loc}}^2([0, \infty); U_2)$, the observer (6.1) admits a unique solution $(\hat{x}_1, \hat{x}_2)^\top \in C([0, \infty); X_1 \times X_2)$ such that (6.10) holds for some positive constant $\omega$.

**Proof.** Since system $(A_1, C_1)$ is exponentially detectable, there exists an $F_0 \in \mathcal{L}(Y_1, [D(A_1^*)]^\prime)$ to detect system $(A_1, C_1)$ exponentially. In particular, $A_1 + F_0C_1A$ generates an exponentially stable $C_0$-semigroup $e^{(A_1 + F_0C_1A)t}$ on $X_1$. Noting that the matrix $A_2$ satisfies $\sigma(A_2) \subset \{ s \mid \text{Res} \geq 0 \}$, we have (6.25). It follows from [13] that the Sylvester equation (6.2) admits a solution $S \in \mathcal{L}(X_2, X_1)$ in the sense that

$$
(\hat{A}_1 + F_0C_1A)Sx_2 - SA_2x_2 = B_1C_2x_2, \quad x_2 \in X_2.
$$

(6.26)

By Lemma 6.1, we have $C_{1A}S \in \mathcal{L}(X_2, Y_1)$. Define

$$
A_{F_0} = \begin{pmatrix}
\hat{A}_1 + F_0C_1A & B_1C_2A \\
0 & A_2
\end{pmatrix}
$$

(6.27)

with

$$
D(A_{F_0}) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X_1 \times X_2 \mid \begin{array}{l}
(\hat{A}_1 + F_0C_1A)x_1 + B_1C_2Ax_2 \in X_1 \\
A_2x_2 \in X_2
\end{array} \right\}.
$$

(6.28)

As in the proof of Theorem 5.1, it follows from (6.26) that

$$
A_{F_0} \sim_S A_S \quad \text{and} \quad C_S = C_S^{-1} = (C_{1A}^\top, -C_{1A}S),
$$

(6.29)

where the operator $S$ is given by (6.15) and $A_S = \text{diag}(\hat{A}_1 + F_0C_1A, A_2)$ with

$$
D(A_S) = \left\{ (x_1, x_2)^\top \in X_1 \times X_2 \mid (\hat{A}_1 + F_0C_1A)x_1 \in X_1, \quad A_2x_2 \in X_2 \right\}.
$$

(6.30)

By a simple computation, the operator $A_{F_0}$ can be written as $A_{F_0} = A + F_0C$, where $F_0 = (F_0, 0)^\top$. Moreover, for any $u \in L_{\text{loc}}^2([0, \infty); Y_1)$, it follows that

$$
\int_0^t e^{A(t-s)}F_0u(s)ds = \begin{pmatrix}
\int_0^t e^{\hat{A}_1(t-s)}F_0u(s)ds \\
0
\end{pmatrix} \in X_1 \times X_2,
$$

(6.31)

which implies that $F_0$ is admissible for $e^{At}$. By Lemma 4.1, $C$ is admissible for $e^{At}$ as well. Noting that $F_0 \in \mathcal{L}(Y_1, [D(A_1^*)]^\prime)$ detects system $(A_1, C_1)$ exponentially and

$$
C(s - A)^{-1}F_0 = C_{1A}(s - \hat{A}_1)^{-1}F_0, \quad \forall \ s \in \rho(A),
$$

(6.32)

we conclude that $(A, F_0, C)$ is a regular linear system and $I$ is an admissible feedback operator for $C(s - A)^{-1}F_0$. Since $(A, C)$ is approximately observable, it follows from [25, Remark 6.5] that...
system \((A + F_0C, C) = (A_{F_0}, C)\) is approximately observable. By Lemma 3.1 and the similarity (6.29), system \((A_S, C_S)\) is approximately observable as well. Thanks to [22, Remark 6.1.8, p.175], we have

\[
\text{Ker}(\lambda - A_S) \cap \text{Ker}(C_S^{-1}) = \{0\}, \quad \forall \lambda \in \sigma(A_S) \subset \sigma(\tilde{A}_1 + F_0C_{1A}) \cup \sigma(A_2). \tag{6.33}
\]

For any \(h_2 \in \text{Ker}(\lambda - A_2) \cap \text{Ker}(C_{1A}S)\), it follows that

\[
(0, h_2)^\top \in \text{Ker}(\lambda - A_S) \cap \text{Ker}(C_S^{-1}), \quad \lambda \in \sigma(A_2), \tag{6.34}
\]

which, together with (6.33), implies that \(\text{Ker}(\lambda - A_2) \cap \text{Ker}(C_{1A}S) = \{0\}\). Furthermore, by [22, Remark 1.5.2, p.15], \((A_2, -C_{1A}S)\) is observable. From the pole assignment theorem of the linear systems, there exists an \(F_2 \in \mathcal{L}(Y_1, X_2)\) such that \(A_2 + F_2C_{1A}\) is Hurwitz. Let \(F_1 = F_0 + SF_2\).

The proof is then accomplished by Theorem 6.1.

The Corollary 6.1 covers a fair amount of dynamics compensations for the finite-dimensional systems with infinite-dimensional sensor dynamics dominated by the first order hyperbolic equation [9], heat equation [10, 21] as well as the one-dimensional wave equation [11]. In particular, the sensor delay for ODEs in [12] can also be compensated by Corollary 6.1.

### 7 Disturbance compensation in output regulation

Thanks to the abstract setting (1.1), it is found that the sensor dynamics compensation of linear systems is closely related to the disturbance and state estimation in the output regulation, which were usually considered as two different problems in existing literatures. We refer the results about sensor dynamics compensation to [9, 10, 11, 12] and the results on output regulation to [13, 16, 17, 18].

Consider the following output regulation problem in the state space \(X_1\), input space \(U_1\) and the output space \(Y_1\):

\[
\begin{aligned}
\dot{z}_1(t) &= A_1 z_1(t) + B_d d(t) + B_1 u(t), \\
y(t) &= C_1 z_1(t) + r(t),
\end{aligned} \tag{7.1}
\]

where \(A_1 : X_1 \to X_1, B_1 : U_1 \to X_1, C_1 : X_1 \to Y_1\) are the system operator, control operator and the observation operator, respectively, \(u(t)\) is the control, \(d(t)\) is the disturbance in a Hilbert space \(U_d, B_d : U_d \to X_1, y(t)\) is the tracking error, and \(r(t)\) is the reference signal.

As in the conventional output regulation problem discussed in [17], we suppose that the disturbance and reference signal are generated from the following exosystem in Hilbert space \(X_2\):

\[
\begin{aligned}
\dot{z}_2(t) &= A_2 z_2(t), \\
d(t) &= C_d z_2(t), \\
r(t) &= C_2 z_2(t),
\end{aligned} \tag{7.2}
\]

where \(A_2 \in \mathcal{L}(X_2), C_d : X_2 \to U_d\) and \(C_2 : X_2 \to Y_1\). In this section, we assume that

\[
X_2 = \mathbb{C}^n, \quad C_d \in \mathcal{L}(X_2, U_d), C_2 \in \mathcal{L}(X_2, Y), \quad \sigma(A_2) \subset \{\lambda \mid \text{Re}\lambda \geq 0\} \tag{7.3}
\]
and $A_1$ generates an exponentially stable $C_0$-semigroup $e^{A_1 t}$ on $X_1$. Combining system (7.1) and exosystem (7.2), we obtain the following coupled system:

$$
\begin{aligned}
\dot{z}_1(t) &= A_1 z_1(t) + B_2 C_d z_2(t) + B_1 u(t), \\
\dot{z}_2(t) &= A_2 z_2(t), \\
y(t) &= C_1 z_1(t) + C_2 z_2(t).
\end{aligned}
$$

(7.4)

The main objective of the output regulation is to design a tracking error based feedback control to regulate the tracking error to zero as $t \to \infty$. By [13], there exists a full state feedback law $u(t) = -Q z_2(t)$ that solves the regulation problem (7.4) if and only if the following regulator equations

$$
\begin{aligned}
A_1 \Pi - \Pi A_2 &= -B_d C_d + B_1 Q, \\
C_1 A \Pi + C_2 &= 0
\end{aligned}
$$

(7.5)

admits a solution $\Pi \in \mathcal{L}(X_2, X_1)$ and $Q \in \mathcal{L}(X_2, U_1)$. In order to improve the results in [13] where only the full state feedback is considered, we consider, in this section, an observer based error feedback design. By the separation principle of the linear systems, an error based feedback can be designed easily provided that we have an observer for system (7.4). The only measurement for the observer design is the tracking error $y(t)$.

Since system (7.4) without control is a cascade of the control plant and the exosystem, the observer design of system (7.4) is closely related to the problem of sensor dynamics compensation. Actually, after an invertible transformation, we can regard the control plant as the sensor dynamics of the exosystem. Suppose that $(z_1, z_2)^\top \in C([0, +\infty); X_1 \times X_2)$ is a solution of system (7.4). If we define, in terms of the solution of regulator equations (7.5), the transformation

$$
\begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix} =
\begin{pmatrix}
I_1 & -\Pi \\
0 & I_2
\end{pmatrix}
\begin{pmatrix}
z_1(t) \\
z_2(t)
\end{pmatrix},
$$

(7.6)

then system (7.4) is transferred into

$$
\begin{aligned}
\dot{x}_1(t) &= A_1 x_1(t) + B_1 Q x_2(t) + B_1 u(t), \\
\dot{x}_2(t) &= A_2 x_2(t), \\
y(t) &= C_1 x_1(t),
\end{aligned}
$$

(7.7)

which takes the same form as system (1.1). As a result, an observer of system (7.4) can be designed by combining the transformation (7.6) and Theorem 6.1, which takes the form:

$$
\begin{aligned}
\dot{\hat{z}}_1(t) &= A_1 \hat{z}_1(t) + B_d C_d \hat{z}_2(t) - K_1 [y(t) - C_1 \hat{z}_1(t) - C_2 \hat{z}_2(t)] + B_1 u(t), \\
\dot{\hat{z}}_2(t) &= A_2 \hat{z}_2(t) + K_2 [y(t) - C_1 \hat{z}_1(t) - C_2 \hat{z}_2(t)],
\end{aligned}
$$

(7.8)

where the tuning gains $K_1 \in \mathcal{L}(Y_1, X_1)$ and $K_2 \in \mathcal{L}(Y_1, X_2)$ can be selected by the following scheme:
• Solve the following Sylvester equation on $X_2$:
\[ A_1 \Gamma - \Gamma A_2 = B_d C_d; \quad (7.9) \]

• Find $K_2 \in L(Y_1, X_2)$ to detect system $(A_2, C_1 \Lambda - C_2)$ exponentially;

• Set $K_1 = \Gamma K_2 \in L(Y_1, X_1)$.

**Theorem 7.1.** Let $(A_1, B_1, C_1)$ be a regular linear system with the state space $X_1$, input space $U_1$ and the output space $Y_1$. Suppose that the exosystem satisfies (7.3), system (7.4) is approximately observable, $B_d \in L(U_d, [D(A^*_1)])$ and $A_1$ generates an exponentially stable $C_0$-semigroup $e^{A_1t}$ on $X_1$. If the regulation problem (7.4) is solvable, i.e., the regulator equations (7.5) admits a solution $K \in L(X_2, X_1)$ and $K_2 \in L(Y_1, X_2)$ such that the observer (7.8) of system (7.4) is well-posed: For any $(\hat{z}_1(0), \hat{z}_2(0))^T \in X_1 \times X_2$ and $u \in L^2_{loc}([0, \infty); U)$, the observer (7.8) admits a unique solution $(\hat{z}_1, \hat{z}_2)^T \in C([0, \infty); X_1 \times X_2)$ such that
\[ e^{\omega t}(|z_1(t) - \hat{z}_1(t), z_2(t) - \hat{z}_2(t)|^T)_{X_1 \times X_2} \to 0 \quad \text{as} \quad t \to \infty, \quad (7.10) \]
where $\omega$ is a positive constant that is independent of $t$.

**Proof.** Since the semigroup $e^{A_1t}$ is exponentially stable in $X_1$ and $\sigma(A_2) \subset \{ \lambda \mid \text{Re} \lambda \geq 0 \}$, we have $\sigma(A_1) \cap \sigma(A_2) = \emptyset$. By [13], the Sylvester equation (7.9) admits a unique solution $\Gamma \in L(X_2, X_1)$. In terms of the solution $\Pi \in L(X_2, X_1)$, $Q \in L(X_2, U_1)$ of the regulator equations (7.5), we can define the transformation (7.6) to convert system (7.4) into system (7.7). Moreover, a simple computation shows that
\[ \dot{A}_1 S x_2 - S A_2 x_2 = B_1 Q x_2, \quad S = \Gamma + \Pi \in L(X_2, X_1), \quad \forall x_2 \in X_2. \quad (7.11) \]
In view of the observer (6.1), the observer of system (7.7) is
\[
\begin{cases}
\dot{x}_1(t) = A_1 \dot{x}_1(t) + B_1 Q \dot{x}_2(t) - F_1[y(t) - C_1 \dot{x}_1(t)] + B_1 u(t), \\
\dot{x}_2(t) = A_2 \dot{x}_2(t) + F_2[y(t) - C_1 \dot{x}_1(t)],
\end{cases} \quad (7.12)
\]
where the tuning gains $F_1$ and $F_2$ satisfy: $F_2 \in L(Y_1, X_2)$ detects system $(A_2, C_1 \Lambda S)$ and $F_1 = SF_2 \in L(Y_1, X_1)$. By exploiting Corollary 6.1, for any initial state $(\dot{x}_1(0), \dot{x}_2(0))^T \in X_1 \times X_2$, the observer (7.12) admits a unique solution $(\dot{x}_1, \dot{x}_2)^T \in C([0, \infty); X_1 \times X_2)$ such that (6.10) holds for some positive constant $\omega$. Let
\[ \begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} I_1 & \Pi \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix}, \quad K_2 = F_2 \quad \text{and} \quad K_1 = \Gamma K_2. \quad (7.13) \]
Then, it is easy to see that such a defined $(\dot{z}_1, \dot{z}_2)^T \in C([0, \infty); X_1 \times X_2)$ is a solution of the observer (7.8) and moreover, the convergence (7.10) holds.

Finally, we show that both $K_1$ and $K_2$ are always bounded. Indeed, $K_2 \in L(Y_1, X_2)$ is trivial. Since $\Gamma \in L(X_2, X_1)$ and $K_1 = \Gamma K_2$, we have $K_1 \in L(Y_1, X_1)$. This completes the proof of the theorem. \[ \square \]
By (7.8), Theorem 7.1 and [13], an observer based error feedback control for system (7.4) can be designed naturally as
\[ u(t) = -Q\hat{z}_2(t), \]
where \( \hat{z}_2(t) \) comes from the observer (7.8). The exponential stability of the resulting closed-loop system can be obtained easily by the separation principle of the linear systems.

**Remark 7.1.** Although the regulator equations (7.5) have been used in the process of observer design, it is interesting that the observer (7.8) itself is free from the regulator equations, in other words, we can design an observer for system (7.4) without solving the regulator equations (7.5).

** Remark 7.2.** Generally speaking, we need to consider the “robustness” of the feedback (7.14) in robust output regulation. This problem can be investigated by the internal model principle [18]. Since the robustness is beyond the topic of this paper, we do not touch it here. However, the problem is almost trivial provided \( Y_1 = \mathbb{C} \). Indeed, a simple computation shows that the feedback (7.14) contains 1-copy internal model of the exosystem and hence is conditionally robust [16, 17]. Consequently, the controller (7.14) is robust to all unknown \( C_2 \) and \( C_d \) provided \( Y_1 = \mathbb{C} \). In other words, both \( C_2 \) and \( C_d \) can be selected specially as in [7].

**Remark 7.3.** When \( B_1 = B_d \) in system (7.4), the disturbance \( C_dz_2(t) \) is in the control channel. System (7.4) can be regarded as a stabilization problem with the input disturbance \( C_dz_2(t) \) and the output disturbance \( C_2z_2(t) \). When the disturbance is estimated, it can be compensated directly by its estimation. This is the main idea of the active disturbance rejection control (ADRC).

## 8 ODEs with output delay

In this section, we suppose that \( X_2 = \mathbb{R}^n \), \( A_2 \in \mathcal{L}(X_2) \) and \( U \) is a Hilbert space. We will validate the theoretical results through the output delay compensation for ODEs. Consider the following single output system:
\[
\dot{x}_2(t) = A_2x_2(t) + B_2u(t), \quad y(t) = C_2x_2(t - \tau), \tag{8.1}
\]
where \( C_2 \in \mathcal{L}(X_2, \mathbb{R}) \) is the observation operator, \( B_2 \in \mathcal{L}(U, X_2) \) is the control operator, \( u(t) \) is the control and \( y(t) \) is the measurement with delay \( \tau > 0 \). Let \( w(x, t) = C_2x_2(t - x) \) for \( x \in [0, \tau] \) and \( t \geq \tau \). Then, system (8.1) can be written as
\[
\begin{cases}
\dot{x}_2(t) = A_2x_2(t) + B_2u(t), \\
w_t(x, t) + w_x(x, t) = 0, \quad x \in (0, \tau), \\
w(0, t) = C_2x_2(t), \\
y(t) = w(\tau, t).
\end{cases} \tag{8.2}
\]

Define \( A_1 : D(A_1) \subset L^2[0, \tau] \to L^2[0, \tau] \) by
\[
A_1f = -f', \quad \forall \ f \in D(A_1) = \{ f \in H^1[0, \tau] \mid f(0) = 0 \}, \tag{8.3}
\]
the operator $B_1 : \mathbb{R} \to [D(A_1^\dagger)]'$ by $B_1 q = q \delta(\cdot)$ for any $q \in \mathbb{R}$, where $\delta(\cdot)$ is the Dirac distribution, and the operator $C_1 : D(A_1) \subset L^2[0, \tau] \to \mathbb{R}$ by
\[
C_1 f = f(\tau), \quad \forall f \in D(A_1).
\] (8.4)

Then, system (8.2) is put into an abstract form:
\[
\begin{aligned}
&\begin{cases}
  w_t(\cdot, t) = A_1 w(\cdot, t) + B_1 C_2 x_2(t), \\
  \dot{x}_2(t) = A_2 x_2(t) + B_2 u(t), \\
  y(t) = C_1 w(\cdot, t).
\end{cases}
\end{aligned}
\] (8.5)

Since $A_1$ generates an exponentially stable $C_0$-semigroup $e^{A_1 t}$ on $L^2[0, \tau]$, we need to solve the Sylvester equation $A_1 S - S A_2 = B_1 C_2$ for the observer design.

Inspired by [4], we suppose that $S : X_2 \to L^2[0, \tau]$ takes the form:
\[
S v = \sum_{j=1}^n v_j s_j := \langle s, v \rangle_{X_2}, \quad \forall v = (v_1, v_2, \cdots, v_n)^T \in X_2,
\] (8.6)

where $s : [0, \tau] \to \mathbb{R}^n$ is a vector-valued function given by $s(x) = (s_1(x), s_2(x), \cdots, s_n(x))^T$ for any $x \in [0, \tau]$, $s_j \in L^2[0, \tau]$, $j = 1, 2, \cdots, n$. Inserting (8.6) into the corresponding Sylvester equation will lead to a vector-valued ODE:
\[
s'(x) + A_2^* s(x) = 0, \quad s(0) = -C_2^*.
\] (8.7)

We solve (8.7) to get $s(x) = -e^{-A_2^* x}C_2^*$, $x \in [0, \tau]$. Hence, the solution of the Sylvester equation $A_1 S - S A_2 = B_1 C_2$ is found to be
\[
(S v)(x) = -C_2 e^{-A_2^* x} v, \quad \forall v \in X_2, \quad x \in [0, \tau].
\] (8.8)

Moreover, $C_1 S x_2 = -C_2 e^{-A_2^* x_2}$, $\forall x_2 \in X_2$. According to the scheme of gain operators choice in Section 6, if $F_2$ detects $(A_2, C_1 S)$, we can choose $F_1 = S F_2 = -C_2 e^{-A_2^*} F_2$. In view of (6.1), the observer of system (8.5) is
\[
\begin{aligned}
&\begin{cases}
  \dot{w}_t(x, t) + \dot{w}_x(x, t) = C_2 e^{-A_2^*} F_2[w(\tau, t) - \dot{w}(\tau, t)], \quad x \in (0, \tau), \\
  \dot{x}_2(t) = A_2 \dot{x}_2(t) + F_2[w(\tau, t) - \dot{w}(\tau, t)] + B_2 u(t), \\
  \dot{w}(0, t) = C_2 \ddot{x}_2(t).
\end{cases}
\end{aligned}
\] (8.9)

In particular, if we choose $F \in L(\mathbb{R}, X_2)$ such that $A_2 + F C_2$ is Hurwitz, then the operator $A_2 + e^{A_2^*} F C_2 e^{-A_2^*} = A_2 - e^{A_2^*} F C_1 S$ is also Hurwitz due to the invertibility of $e^{A_2^*}$. Then, we can choose $F_2 = -e^{A_2^*} F$ and $F_1 = C_2 e^{A_2^*} F$. The observer of system (8.5) turns to be
\[
\begin{aligned}
&\begin{cases}
  \dot{w}_t(x, t) + \dot{w}_x(x, t) = -C_2 e^{A_2^*} F[w(\tau, t) - \dot{w}(\tau, t)], \quad x \in (0, \tau), \\
  \dot{x}_2(t) = A_2 \dot{x}_2(t) - e^{A_2^*} F[w(\tau, t) - \dot{w}(\tau, t)] + B_2 u(t), \\
  \dot{w}(0, t) = C_2 \ddot{x}_2(t),
\end{cases}
\end{aligned}
\] (8.10)
According to Theorem 6.1, to design the observer for system (9.4), we should first choose \( A \) such that \( A \) is Hurwitz and then solve the following Sylvester equation on \( D(A_2) \):

\[
(A_1 + F_0 C_1) S - S A_2 = B_1 C_2.
\]  

9 Unstable heat with ODE dynamics

In this section, we consider a more complicated problem to show the effectiveness of the proposed approach. We will observe an unstable heat equation through ODE sensor dynamics. Comparing with the ODE system with PDE sensor dynamics, the PDE system with ODE sensor dynamics is more complicated. The intuitive reason behind this is that we need to observe an infinite-dimensional plant through a finite-dimensional system.

Consider the following cascade system of ODEs and an unstable heat equation:

\[
\begin{aligned}
\dot{t}(t) &= A_1 v(t) + B_1 w(1, t), \\
\dot{w}(x, t) &= w_{xx}(x, t) + \mu w(x, t), \quad x \in (0, 1), \quad \mu > 0, \\
w(0, t) &= 0, \quad w_x(1, t) = u(t), \\
y(t) &= C_1 v(t),
\end{aligned}
\]  

(9.1)

where \( A_1 \in \mathbb{R}^{m \times m} \), \( B_1 \in \mathcal{L}(\mathbb{R}, \mathbb{R}^m) \), \( C_1 \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) \), \( u(t) \) is the control and \( y(t) \) is the measured output. The heat subsystem is the control plant and the ODE system serves as the sensor dynamics. This makes the observation of (9.1) very different from the existing results in [10] and [12]. Define the operator \( A_2 : D(A_2) \subset L^2[0, 1] \rightarrow L^2[0, 1] \) by

\[
A_2 f = f'' + \mu f, \quad \forall f \in D(A_2) = \left\{ f \in H^2[0, 1] \mid f(0) = f'(1) = 0 \right\},
\]  

(9.2)

the operator \( B_2 \in \mathcal{L}(\mathbb{R}, [D(A_2)]') \) by \( B_2 c := \delta(\cdot - 1)c \) for any \( c \in \mathbb{R} \) and \( C_2 \in \mathcal{L}(D(A_2), \mathbb{R}) \) by

\[
C_2 f = f(1), \quad \forall f \in D(A_2),
\]  

(9.3)

where \( \delta(\cdot) \) is the Dirac distribution. With these operators at hand, system (9.1) can be written as an abstract form in the state space \( \mathbb{R}^m \times L^2[0, 1] \):

\[
\begin{aligned}
\dot{t}(t) &= A_1 v(t) + B_1 C_2 w(\cdot, t), \\
\dot{w}(\cdot, t) &= A_2 w(\cdot, t) + B_2 u(t), \\
y(t) &= C_1 v(t).
\end{aligned}
\]  

(9.4)

According to Theorem 6.1, to design the observer for system (9.4), we should first choose \( F_0 \in \mathbb{R}^m \) such that \( A_1 + F_0 C_1 \) is Hurwitz and then solve the following Sylvester equation on \( D(A_2) \):

\[
(A_1 + F_0 C_1) S - S A_2 = B_1 C_2.
\]  

(9.5)
Inspired by [4], we define the vector-valued function \( s : [0, 1] \to \mathbb{R}^m \) by \( s(x) = (s_1(x), s_2(x), \ldots, s_m(x))^\top \) for any \( x \in [0, 1] \), where \( s_j \in L^2[0, 1] \) will be determined later, \( j = 1, 2, \ldots, m \). Suppose that the solution of Sylvester equation (9.5) takes the form
\[
Sf = \begin{pmatrix}
\langle f, s_1 \rangle_{L^2[0, 1]} \\
\langle f, s_2 \rangle_{L^2[0, 1]} \\
\vdots \\
\langle f, s_m \rangle_{L^2[0, 1]}
\end{pmatrix} := \langle f, s \rangle_{L^2[0, 1]}, \quad \forall f \in L^2[0, 1].
\tag{9.6}
\]
Then, inserting (9.6) into (9.5), we obtain
\[
s''(x) + \mu s(x) = (A_1 + F_0 C_1) s(x), \quad s(0) = 0, \quad s'(1) = B_1,
\tag{9.7}
\]
which is a vector-valued ODE with respect to the variable \( x \). Solve equation (9.7) to obtain
\[
s(x) = x G(xG)(\cosh G)^{-1} B_1, \quad G^2 = A_1 + F_0 C_1 - \mu,
\tag{9.8}
\]
where
\[
G(s) = \begin{cases}
\sinh s, & s \neq 0, s \in \mathbb{C}, \\
1, & s = 0.
\end{cases}
\tag{9.9}
\]
By (9.8) and (9.6), \( S \in \mathcal{L}(L^2[0, 1], \mathbb{R}^m) \) satisfies
\[
Sf = \int_0^1 f(x)s(x)dx, \quad \forall f \in L^2[0, 1],
\tag{9.10}
\]
and hence \( C_1 S \in \mathcal{L}(L^2[0, 1], \mathbb{R}) \) given by
\[
C_1 Sf = \int_0^1 f(x)C_1 s(x)dx = \langle f, C_1 s \rangle_{L^2[0, 1]}, \quad \forall f \in L^2[0, 1].
\tag{9.11}
\]
According to the developed scheme, we need to design \( F_2 \in \mathcal{L}(\mathbb{R}, [D(A_2^*)]') \) to detect system \((A_2, C_1 S)\) exponentially, i.e., design \( F_2 \) such that the following system is exponentially stable
\[
\begin{aligned}
z_t(x, t) &= z_{xx}(x, t) + \mu z(x, t) + F_2 \int_0^1 C_1 s(x)z(x, t)dx, \\
z(0, t) &= z_x(1, t) = 0.
\end{aligned}
\tag{9.12}
\]
We treat this problem by the modal decomposition approach which has been used in [2, 14, 20].

By a simple computation, system (9.12) can be written as
\[
z_t(\cdot, t) = (A_2 + F_2 C_1 S)z(\cdot, t).
\tag{9.13}
\]
Let
\[
\phi_n(x) = \sqrt{2} \sin \sqrt{\lambda_n}x, \quad \lambda_n = \left(n - \frac{1}{2}\right)^2 \pi^2, \quad x \in [0, 1], \quad n = 1, 2, \ldots.
\tag{9.14}
\]
Then, \( \{\phi_n(\cdot)\}_{n=1}^\infty \) forms an orthonormal basis for \( L^2[0, 1] \), which satisfies
\[
\phi_n''(x) = -\lambda_n \phi_n(x), \quad \phi_n(0) = \phi_n(1) = 0, \quad n = 1, 2, \ldots.
\tag{9.15}
\]
In view of (6.1), (9.21) and (9.22), the observer of system (9.1) is designed as
\[
\dot{C}(t) = \sum_{n=1}^{\infty} z_n(t) \phi_n(t), \quad C_1 s(\cdot) = \sum_{n=1}^{\infty} \gamma_n \phi_n(\cdot),
\]
where \(z_n(t)\) and \(\gamma_n\) are the Fourier coefficients, given by
\[
z_n(t) = \int_0^1 z(x,t) \phi_n(x) dx, \quad \gamma_n = \int_0^1 C_1 s(x) \phi_n(x) dx, \quad n = 1, 2, \cdots.
\]
Finding the derivative of \(z_n(t)\) along the system (9.12), we obtain
\[
\dot{z}_n(t) = \int_0^1 z_t(x,t) \phi_n(x) dx
= \int_0^1 \left[ z_{xx}(x,t) + \mu z(x,t) + F_2 \int_0^1 C_1 s(x) z(x,t) dx \right] \phi_n(x) dx
= (\lambda_n + \mu) z_n(t) + \left( F_2 \sum_{j=1}^{\infty} \gamma_j \dot{z}_j(t), \phi_n \right)_{[D(A_2)]', D(A_2)}
= (\lambda_n + \mu) z_n(t) + \left[ \sum_{j=1}^{\infty} \gamma_j \dot{z}_j(t) \right] F_2^* \phi_n.
\]
Since \(\lambda_n \to +\infty (n \to \infty)\), there exists a positive integer \(N\) so that
\[
(\lambda_n + \mu) < 0, \quad \forall \ n > N.
\]
Suppose that there exists an \(L_N = [l_1, l_2, \cdots, l_N]^\top \in \mathbb{R}^N\) such that \(\Lambda_N + L_N \Gamma_N\) is Hurwitz, where
\[
\Lambda_N = \text{diag}\{-\lambda_1 + \mu, \cdots, -\lambda_N + \mu\}, \quad \Gamma_N = [\gamma_1, \gamma_2, \cdots, \gamma_N] \in \mathbb{R}^{1 \times N}.
\]
If we choose \(F_2 \in L^2(\mathbb{R}, L^2[0, 1])\) by
\[
F_2 c = c \sum_{n=1}^{N} l_n \phi_n(\cdot), \quad \forall \ c \in \mathbb{R},
\]
then, it follows from Lemma 12.1 in Appendix that the operator \(A_2 + F_2 C_1 S\) generates an exponentially stable \(C_0\)-semigroup \(e^{(A_2 + F_2 C_1 S)t}\) on \(L^2[0, 1]\). As a result, \(F_2 \in L(\mathbb{R}, L^2[0, 1])\) detects system \((A_2, C_1 S)\) exponentially.

Let \(F_1 = F_0 + SF_2\). It follows from (9.21) and (9.10) that
\[
F_1 c = \left( F_0 + \int_0^1 s(x) \sum_{n=1}^{N} l_n \phi_n(x) dx \right) c, \quad \forall \ c \in \mathbb{R}.
\]
In view of (6.1), (9.21) and (9.22), the observer of system (9.1) is designed as
\[
\begin{cases}
\dot{v}(t) = A_1 \dot{v}(t) + B_1 \dot{w}(1,t) - \left[ F_0 + \int_0^1 s(x) \sum_{n=1}^{N} l_n \phi_n(x) dx \right] [C_1 v(t) - C_1 \dot{v}(t)], \\
\dot{w}(t) = \dot{w}_{xx}(x,t) + \mu \dot{w}(x,t) + \sum_{n=1}^{N} l_n \phi_n(x) [C_1 v(t) - C_1 \dot{v}(t)], \\
\dot{v}(0,t) = 0, \quad \dot{w}_x(1,t) = u(t).
\end{cases}
\]
Theorem 9.1. Let $\phi_n(\cdot)$ and $\lambda_n$ be given by (9.14) and let $s(\cdot)$ be given by (9.8). Suppose that system (9.1) is approximately observable and the inequality (9.19) holds with the positive integer $N$. Then, there exists an $F_0 \in \mathbb{R}^m$ and $L_N = [l_1, l_2, \ldots, l_N] \in \mathbb{R}^N$ such that observer (9.23) of system (9.1) is well-posed: For any $(\hat{v}(0), \hat{w}(0)) \in \mathbb{R}^m \times L^2[0, 1]$ and $u \in L^2_{loc}[0, \infty)$, the observer (9.23) admits a unique solution $(\hat{v}, \hat{w}) \in C([0, \infty); \mathbb{R}^m \times L^2[0, 1])$ such that
\[
e^{\omega t} \| (v(t) - \hat{v}(t), w(\cdot, t) - \hat{w}(\cdot, t)) \|_{\mathbb{R}^m \times L^2[0, 1]} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \tag{9.24}\]
where $\omega$ is a positive constant that is independent of $t$.

Proof. It is easy to verify that $(A_2, B_2, C_2)$ is a regular linear system, where $A_2, B_2, C_2$ are defined by (9.2)-(9.3) (see, e.g. [1]). By [8, Definition 1.2, p.3], the matrix-valued function $G(x)G^\top$ and $\cosh G$ are well-defined. Since system (9.1) is approximately observable, it follows from [22, Remark 6.1.8, p.175] and the cascaded structure of system (9.1) that system $(A_1, C_1)$ is observable as well. As a result, there exists an $F_0 \in \mathcal{L}(Y_1, X_1)$ such that $A_1 + F_0C_1$ is Hurwitz, and at the same time, $\sigma(A_1 + F_0C_1) \cap \sigma(A_2) = \emptyset$. By a simple computation, it follows that
\[
\sigma(A_2 - \mu) = \left\{ -\left( n - \frac{1}{2} \right)^2 \pi^2 \mid n \in \mathbb{N} \right\}, \tag{9.25}\]
which, together with (9.8), leads to $\sigma(G^2) \cap \left\{ -\left( n - \frac{1}{2} \right)^2 \pi^2 \mid n \in \mathbb{N} \right\} = \emptyset$. Hence, for any $\lambda \in \sigma(G)$, we have $\lambda^2 \notin \left\{ -\left( n - \frac{1}{2} \right)^2 \pi^2 \mid n \in \mathbb{N} \right\}$, which implies that $\lambda \notin \left\{ i \left( n - \frac{1}{2} \right) \pi \mid n \in \mathbb{Z} \right\}$ and hence $\cosh \lambda \neq 0$. Consequently, $\cosh G$ is invertible.

By a simple computation, the operator $S$, given by (9.10) and (9.8), solves the Sylvester equation (9.5) and moreover, $C_1S$ given by (9.11) satisfies $C_1S \in \mathcal{L}(L^2[0, 1], \mathbb{R})$. Define $A_{F_0} = \begin{pmatrix} A_1 + F_0C_1 & B_1C_{2A} \\ 0 & A_2 \end{pmatrix}$ and $C_1 = (C_1, 0) \in \mathcal{L}(\mathbb{R}^m \times L^2[0, 1], \mathbb{R})$. As the proof of Corollary 6.1, it follows from [25, Remark 6.5] that $(A_{F_0}, C_1)$ is approximately observable. By Lemma 3.1, the pair
\[
(SA_{F_0}S^{-1}, C_1S^{-1}) = \left( \begin{pmatrix} A_1 + F_0C_1 & 0 \\ 0 & A_2 \end{pmatrix}, (C_1, -C_1S) \right) \tag{9.26}\]
is approximately observable as well where the invertible transformation $S$ is given by
\[
S(v, f)^\top = (v + Sf, f)^\top, \quad \forall (v, f)^\top \in \mathbb{R}^m \times L^2[0, 1].
\]
Thanks to the block-diagonal structure of $SA_{F_0}S^{-1}$ and [4, Lemma 10.2], system $(A_2, C_1S)$ is approximately observable. Since $\{\phi_n(\cdot)\}_{n=1}^{\infty}$ defined by (9.14) forms an orthonormal basis for $L^2[0, 1]$, we conclude from (9.11) and the approximate observability of $(A_2, C_1S)$ that
\[
\gamma_n = \int_0^1 C_1S(x)\phi_n(x)dx \neq 0, \quad n = 1, 2, \ldots \tag{9.27}\]
As a result, the finite-dimensional system $(\Lambda_N, \Gamma_N)$ is observable, where $\Lambda_N$ and $\Gamma_N$ are given by (9.20). Hence, there exists an $L_N = [l_1, l_2, \ldots, l_N] \in \mathbb{R}^N$ such that $\Lambda_N + L_N\Gamma_N$ is Hurwitz. By Lemma 12.1, $F_2 \in \mathcal{L}(\mathbb{R}, L^2[0, 1])$ given by (9.21) detects system $(A_2, C_1S)$ exponentially. Noting that $F_1 = F_0 + SF_2$ satisfies (9.22), the well-posedness of the observer (9.23) can be obtained by Theorem 6.1 directly. \hfill \square
10 Numerical simulations

In this section, we carry out some numerical simulations for systems (9.1) and (9.23) to validate the theoretical results. The finite difference scheme is adopted in discretization. The numerical results are programmed in Matlab. The time step and the space step are taken as $4 \times 10^{-5}$ and $10^{-2}$, respectively. We choose

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C_1 = (1, 1), \quad \mu = 4.$$ (10.1)

With this setting, it is easy to check that the assumptions in Theorem 9.1 are fulfilled with $N = 1$. The initial states of systems (9.1) and (9.23) are chosen as

$$v(0) = (1, 1)^\top, \quad w(x, 0) = \sin \pi x, \quad \hat{w}(x, 0) \equiv 0, \quad \hat{v}(0) = 0, \quad x \in [0, 1].$$ (10.2)

We assign the poles to get the gains

$$F_0 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad F_1 = \begin{pmatrix} -1.5847 \\ -3.9479 \end{pmatrix}, \quad F_2 = 5.0978 \times \sqrt{2} \sin \frac{\pi}{2} x,$$ (10.3)

which lead to $\sigma(\Lambda_N + L_N \Gamma_N) = \{-2\}$. The error between the state $w(\cdot, t)$ and $\hat{w}(\cdot, t)$ is plotted in Figure 1(a) and the error between the state $v(t) = (v_1(t), v_2(t))^\top$ and $\hat{v}(t) = (\hat{v}_1(t), \hat{v}_2(t))^\top$ is plotted in Figure 1(b). Both of them show that the convergence is effective and smooth, which validates numerically the effectiveness of the proposed method.

![Figure 1: Observation error.](image)

11 Conclusions

This paper attributes the observer design for abstract cascade systems. Two important issues are addressed. The first one is the sensor dynamics compensation for a linear system and the
second one is on the error based observer design in the problem of output regulation. It is found that these two different problems can be dealt with in a united way from the abstract linear cascade systems framework point of view. As a result, the observer design approaches in sensor dynamics compensation and the disturbance estimation approaches in output regulation can be used interactively. We propose a new scheme for the observer design by seeking tuning gain operators of the Luenberger-like observer. Both the well-posedness and the exponential convergence of the observer are established. The proposed method gives an alternative method for observer design of the PDE cascade systems, which avoids the target system seeking and the Lyapunov function constructing in the popular PDE backstepping method.

It should be pointed out that the proposed approach in Theorem 6.1 is not limited to the examples considered in Sections 8 and 9. Actually, the approach opens up a new road leading to the observer design of cascade systems particularly for those systems which consist of ODE and multi-dimensional PDE. More importantly, it can also be applied to the delay compensation for general infinite-dimensional systems. This will be considered in the third paper [5] of this series works. Moreover, as pointed in Remark 7.3, the approach gives rise to a new idea of active disturbance rejection control. In the last paper [6] of this series works, we will give a new observer, i.e., extended dynamics observer, in the framework of the active disturbance rejection control.

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12 Appendix A

Lemma 12.1. Let the operator $A_2$ be given by (9.2), let $\phi_n(\cdot)$ and $\lambda_n$ be given by (9.14) and let $N$ be a positive integer such that (9.19) holds with $\mu > 0$. Suppose that there exists an $L_N = [l_1, l_2, \cdots, l_N]^T \in \mathbb{R}^N$ such that $\Lambda_N + L_N \Gamma_N$ is Hurwitz, where $\Lambda_N$ and $\Gamma_N$ are given by (9.20). Suppose that $F_2 : \mathbb{R} \to L^2[0, 1]$ is defined by (9.21) and $C_1S$ is given by (9.11). Then, the operator $A_2 + F_2C_1S$ generates an exponentially stable $C_0$-semigroup $e^{(A_2 + F_2C_1S)t}$ on $L^2[0, 1]$.

Proof. By (9.21) and (9.11), the operator $F_2C_1S$ is bounded on $L^2[0, 1]$. Since $A_2$ generates an analytic semigroup on $L^2[0, 1]$, so is for the operator $A_2 + F_2C_1S$ due to the boundedness of $F_2C_1S$ ([15, Corollary 2.3, p.81]). The proof will be accomplished if we can show that $\sigma(A_2 + F_2C_1S) \subset \{s \mid \text{Res} < 0\}$. For any $\lambda \in \sigma(A_2 + F_2C_1S)$, we consider characteristic equation $(A_2 + F_2C_1S)f = \lambda f$ with $f \neq 0$.

When $f \in \text{Span}\{\phi_1, \phi_2, \cdots, \phi_N\}$, assume that $f = \sum_{j=1}^{N} f_j \phi_j$. The characteristic equation becomes

$$\sum_{j=1}^{N} f_j A_2 \phi_j + \sum_{j=1}^{N} f_j F_2 C_1 S \phi_j = \sum_{j=1}^{N} \lambda f_j \phi_j.$$  \hspace{1cm} (12.1)

By (9.21) and (9.17), we have

$$F_2 C_1 S \phi_j = F_2 \int_{0}^{1} C_1 s(x) \phi_j(x) dx = F_2 \gamma_j = \gamma_j \sum_{n=1}^{N} l_n \phi_n, \hspace{0.5cm} j = 1, 2, \cdots, N.$$ \hspace{1cm} (12.2)

Since $A_2 \phi_j = (-\lambda_j + \mu) \phi_j$, equation (12.1) takes the form

$$\sum_{j=1}^{N} f_j (-\lambda_j + \mu) \phi_j + \sum_{j=1}^{N} f_j \gamma_j \cdot \sum_{n=1}^{N} l_n \phi_n = \sum_{j=1}^{N} \lambda f_j \phi_j.$$ \hspace{1cm} (12.3)

Take the inner product with $\phi_k$, $k = 1, 2, \cdots, N$ on equation (12.3) to get

$$(-\lambda_k + \mu) f_k + l_k \sum_{j=1}^{N} f_j \gamma_j = \lambda f_k, \hspace{0.5cm} k = 1, 2, \cdots, N.$$ \hspace{1cm} (12.4)
which, together with (9.20), leads to

\[
(\lambda - \Lambda_N - L_N \Gamma_N) \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix} = 0. \tag{12.5}
\]

Since \((f_1, f_2, \cdots, f_N) \neq 0\), we have

\[
\text{Det}(\lambda - \Lambda_N - L_N \Gamma_N) = 0, \tag{12.6}
\]

which shows that \(\lambda \in \sigma(\Lambda_N + L_N \Gamma_N) \subset \{s \mid \text{Re } s < 0\}\), since \(\Lambda_N + L_N \Gamma_N\) is Hurwitz.

When \(f \notin \text{Span}\{\phi_1, \phi_2, \cdots, \phi_N\}\), there must exist \(j_0 > N\) so that \(\int_0^1 f(x) \phi_{j_0}(x)dx \neq 0\). Take the inner product with \(\phi_{j_0}\) on equation \((A_2 + F_2C_1S)f = \lambda f\) to get

\[
(-\lambda_{j_0} + \mu) \int_0^1 f(x) \phi_{j_0}(x)dx = \lambda \int_0^1 f(x) \phi_{j_0}(x)dx. \tag{12.7}
\]

As a result, \(\lambda = -\lambda_{j_0} + \mu < 0\). This shows also \(\lambda \in \sigma(A_2 + F_2C_1S) \subset \{s \mid \text{Re } s < 0\}\). The proof is complete. \(\square\)