Analysis of the Characteristics in the Meudon Constrained Evolution Scheme

I Cordero-Carrión¹, J Mª Ibáñez¹, J L Jaramillo², J L Jaramillo², J L Jaramillo², and E Gourgoulhon³, J L Jaramillo²

¹ Departamento de Astronomía y Astrofísica, Universidad de Valencia, Valencia, Spain
² Instituto de Astrofísica de Andalucía (IAA-CSIC), Apartado 3004 18080 Granada, Spain
³ Laboratoire de l’Univers et de ses Théories (LUTH), Observatoire de Paris, 92195 Meudon, France

Abstract. A first analysis of the characteristics associated with the evolving modes in the constraint evolution scheme proposed by the Meudon group in 2004 is presented. The system is written in a first-order hyperbolic form and a so-called generalized Dirac gauge is considered. Applications to inner boundary conditions in an excised approach to black hole evolutions are discussed.

1. Introduction

The work discussed in this communication corresponds to the second of two talks presented in the XXIX Spanish Relativity Meeting, having to do with the application of a constrained evolution scheme introduced in Ref. [1] for the resolution of Einstein equations. This evolution scheme gives rise to a coupled elliptic-hyperbolic system of partial differential equations. This talk focuses on the hyperbolic part, since the elliptic one was already discussed in the contribution by J.L. Jaramillo.

More concretely, our main goal here is to determine whether inner boundary conditions must be supplied to the radiative degrees of freedom or not, during the evolution of a black hole spacetime in an excised approach where the removed sphere coincides with the black hole horizon. This motivation configures the methodology we are going to follow; in particular, we will only aim at developing a preliminary analysis of the hyperbolicity of the system, so as to determine the characteristics on the horizon.

In the 3+1 formalism of general relativity space-time is foliated into spatial hypersurfaces, and Einstein equations are decomposed into a set of constraint equations and a set of dynamical equations [2]. In this framework the Meudon group has developed a so-called fully-constrained scheme [1]. In this fully-constrained scheme all equations and constraints are solved at each time step.

A methodological motivation for using this constrained evolution scheme follows from previous experience in solving numerically Einstein equations. More concretely, we will adapt the choice of the appropriate state-of-the-art numerical techniques to the specific structures of the considered
mathematical problems: on the one hand, spectral methods for elliptic systems and, on the other hand, modern high-resolution shock-capturing techniques for hyperbolic ones. Recently, a hybrid code with the above forementioned numerical strategy was built up [3]; we refer to it as MDM, from “Mariage des Maillages”. Accuracy and robustness of the MDM code, implementing the so-called Conformal Flatness Condition (CFC, see ref. [4, 5]) approach have been shown in [3]. Our middle-term goal is to extend this approach to the study of a generic space-time. In this sense, we present here a preliminary analysis of the structure of the system associated with the evolutionary part of Einstein equations in the Meudon formalism.

We introduce some notation (cf. contribution of J. L. Jaramillo). The 3-metric $\gamma_{ij}$ on the spacelike slice $\Sigma_t$ is written in terms of a conformal 3-metric $\tilde{\gamma}_{ij}$, namely $\tilde{\gamma}_{ij} = \psi^3 \gamma_{ij}$, where $\psi$ is the conformal factor. The difference between the inverse conformal metric, $\tilde{\gamma}^{ij}$, and the inverse of a background flat metric, $\gamma^{ij}$, is introduced: $\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$. The metric components with respect to a coordinate system $(x^\alpha)$ are expressed in terms of the lapse function, $N$, the shift vector, $\beta^i$, and the 3-metric, $\gamma_{ij}$, in the form $g_{\mu \nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt)$. The covariant derivative with respect to the flat metric is denoted by $D_i$. The Lie derivative is denoted by $L$. The extrinsic curvature, $K_{ij}$, is given by $K_{ij} = -(1/2) \nabla_n \gamma_{ij}$, where $n^\mu$ is the future directed unit normal to the slices $\Sigma_t$. Greek indices run from 0 to 3, and Latin indices from 1 to 3.

2. Evolution equations for $h^{ij}$

Using this evolution scheme, maximal slicing ($K \equiv K^{ij}_i = 0$) and the generalized Dirac gauge ($D_i h^{ij}_i = 0$), a set of elliptic equations for the lapse function, the shift vector and the conformal factor is deduced. Moreover, coupled with this elliptic system, there is a set of six second order partial differential equations for the evolution of the deviations with respect to the flat metric, $h^{ij}$. Our work here focuses on this last set of equations. More specifically, we aim at performing a preliminary analysis of the hyperbolic part of the fully-constrained evolution system. Rather than studying the whole coupled elliptic-hyperbolic system, following a systematic line like the one developed in [6], we consider the fields in the elliptic part, that is $\psi$, $N$ and $\beta^i$, as given fixed functions acting as “parameters” for the hyperbolic system. This is therefore a first approach to the study of the whole coupled system.

2.1. Equations for $h^{ij}$ as a first-order evolution system.

In addition to considering $\psi$, $N$ and $\beta^i$ as fixed functions, we construct the differential operator governing the evolution in terms of the connection associated with the flat metric $D_i$, instead of the one corresponding to the conformal metric $\tilde{D}_i$. The main motivation for this simplification is the numerical scheme proposed in Ref. [1] for the resolution of the evolution problem, which heavily relies on the use of spherical coordinates. The correction terms linking the flat connection with the conformal one are passed to the sources. In this work we do not consider the effects of the non-linear sources on the features of the hyperbolic system. Instead, an analysis of these sources will be carried out elsewhere.

Let us write the equations for $h^{ij}$ as a first-order system. For this let us define the following auxiliary variables: the derivative of $h^{ij}$ respect to the evolution vector is defined by $u^{ij}$, and the covariant derivative of $h^{ij}$ with respect to the flat metric is defined by $w_k^{ij}$, that is:
With these new variables the system for $h^{ij}$ can be cast into:

$$\frac{\partial w^i}{\partial t} = D^k_h w^i, \quad \text{(2)}$$

where $\phi$ is a source term which does not contain partial derivatives of $u$ or $w$. From definition (2) and $\partial_i f^{ij} = 0$, it is satisfied:

$$\frac{\partial w^i}{\partial t} = D^k_h u^i. \quad \text{(4)}$$

Then the system of equations (1, 3, 4) can be written as:

$$\frac{\partial \varphi}{\partial t} + A^l D_l \varphi = g\left(\beta^k, N, \varphi, \hat{\varphi}_\mu, \hat{\varphi}_\mu, \hat{\varphi}_\mu, h^{ij}, u^{ij}, w^i_k\right), \quad \text{(5)}$$

where the vectors $\varphi$ and $g$, corresponding respectively to the variables of the first-order system and the sources, are defined as:

$$\varphi = \begin{pmatrix} h^{ij} \\ u^{ij} \\ w^i_j \end{pmatrix}, \quad g\left(\beta^k, N, \varphi, \hat{\varphi}_\mu, \hat{\varphi}_\mu, \hat{\varphi}_\mu, h^{ij}, u^{ij}, w^i_k\right) = \begin{pmatrix} u^i \\ \phi \\ 0 \end{pmatrix}.$$

The dimension of $\varphi$ is 30. This follows from the fact that $h^{ij}$ is a symmetric tensor, and therefore also $u^{ij}$ and $w^i_j$. Indices run as $1 \leq i, j \leq 3$, $1 \leq k \leq 3$. In order to write the matrices of the system in a simple way, some quantities are defined:

$$q_1 = (\beta^1)^2 - N^2 \psi^4 \gamma^{11}, \quad q_2 = (\beta^1, \beta^2)^2 - N^2 \psi^4 \gamma^{12}, \quad q_3 = (\beta^1, \beta^2, \beta^3)^2 - N^2 \psi^4 \gamma^{13}, \quad q_4 = (\beta^2)^2 - N^2 \psi^4 \gamma^{22}, \quad q_5 = (\beta^2, \beta^3)^2 - N^2 \psi^4 - \gamma^{23}, \quad q_6 = (\beta^3)^2 - N^2 \psi^4 \gamma^{33}.$$

Then, the matrices $A^l$, for each value of $l = 1, 2, 3$, have the form:
2.2. A first hyperbolicity analysis: characteristics

Once we dispose of the first-order evolution scheme, we study some basic hyperbolicity issues, mainly attempting to obtain explicit expressions of the characteristics in terms of the functions $\psi$, $N$, $\beta^i$ and $\tilde{\gamma}^{ij}$, with the practical goal of determining the freedom to impose boundary conditions for $h^{ij}$ on the inner excised boundary.

In particular, our analysis is far from being exhaustive and we do not investigate issues like existence, uniqueness or well-posedness (see for instance Ref. [7]).

The next step is therefore the study the hyperbolicity of the system of first order with respect to the evolution vector $\partial_t$, whose components are $\xi^a = (1,0,0,0)$. A generic spacelike covector of components $\xi^a = (0,\xi^i)$ is chosen. The associated eigenvalue problem we must solve is (see e.g. Ref. [8]):

$$\left( A^t \xi^i \right)\lambda \xi = 0$$

where $\lambda$ denotes the eigenvalue and $X_{\lambda}$ the corresponding eigenvector.

First of all, we note that if the Dirac gauge is not imposed, i.e. $D_k h^{ij}$ is not set to zero, then the structure of the $A^t$ matrices change and imaginary parts appear in the eigenvalues. This is simply a
reflect that, as it is well known, Einstein equations by themselves have not a definite type, without the specification of a gauge. When imposing \( D_k h^{ji} = 0 \), the eigenvalues of the linear combination \( A^i \xi_l \) are real. So the first conclusion is that our system is hyperbolic. This underlines the importance of the choice of the gauge. Even though this is a crucial result for our analysis, this was also expected from the general structure of equation (3) and the properties of gauges analogous to the Dirac one, like the Coulomb gauge in electromagnetism.

The main goal of the work here is to obtain explicit expressions for the characteristics of the hyperbolic system. It is easy to see that the first six vectors are eigenvectors of the eigenvalue 0 and they are orthogonal to the rest. Therefore, the rest can be studied independently and the eigenvalues associated with the chosen spatial vector are:

\[
\lambda_0 = 0,
\]

\[
\lambda_\pm^{(s)} = -\beta^\mu \xi_\mu \pm N \left( \xi^\mu \xi_\mu \right)^{1/2} = -\beta^\mu \xi_\mu \pm N \left( \xi^\mu \xi_\mu \right)^{1/2},
\]

where \( \lambda_0 \) has multiplicity 12, and \( \lambda_\pm^{(s)} \) has each one multiplicity 6.

3. Application to a particular case: inner boundary conditions

The previous analysis of the system can be applied to the inner boundary conditions problem in an excised approach to black hole evolutions (see talk by J. L. Jaramillo). A crucial point in such an approach to black hole evolution is to assess the freedom to impose boundary conditions on the excised surface. For this we must determine the characteristics on the excised surface.

We now specify the generic vector \( \xi^i \) as the unitary outgoing radial vector on the horizon, \( s^j \). By definition \( \beta^\perp = \beta^\mu s_\mu \), so the expressions of the eigenvalues which are not identical to zero are just \( \lambda_\pm^{(s)} = -\beta^\perp \pm N \). Because of the boundary conditions of the elliptic equations, we have \( -\beta^\perp + N \leq 0 \) on the excised surface (which coincides with the black hole horizon). And since the lapse function is positive, \( -\beta^\perp - N < 0 \). This shows that the characteristics on the excised surface associated with the vector \( s^j \) points outwards with respect to the integration domain. There is a key interplay between the elliptic boundary conditions and the properties of the hyperbolic degrees of freedom. All the equations, elliptic and hyperbolic ones, must be considered as an ensemble.

The fact that all the characteristics curves are ingoing into the black hole is something we could expect from the physical intuition. In consequence, there is no residual freedom in the prescription of the inner boundary conditions of \( h^{ij} \) during the evolution. The system is fully determined and we must only guarantee that the values of \( h^{ij} \) are consistent with the choice of the Dirac gauge (something that can actually be tricky in the numerical implementation).

4. Conclusions

In this study we have written as a first-order system the evolution part of the fully-constrained scheme [1] for the resolution of Einstein equations. We have shown that this system is hyperbolic under the adoption of a Dirac gauge and explicit expressions for the characteristics of the system have been given. In the particular case of an excised black hole evolution, we have established that no additional boundary conditions must be enforced on the evolution degrees of freedom, once the excised surface has been adapted to the black hole by means of the boundary conditions for the elliptic part. The next step consists in implementing the system numerically. An important issue to address in such an implementation will be the manner of guaranteeing that the Dirac gauge is actually satisfied during the evolution: this will be governed by the resolution of the elliptic equation for the shift \( \beta^\perp \) (following from the momentum constraint and \( D_k h^{ji} = 0 \)). Consequently, even though we have shown that no freedom is left for the updating of the \( h^{ij} \) data for on the excised surface during the evolution, the
numerical implementation must guarantee that inner values of $h^{ij}$ are consistent with the chosen boundary conditions for $\beta^i$.

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