Multiresolution Search of the Rigid Motion Space for Intensity Based Registration

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Abstract—We study the relation between the target functions of low-resolution and high-resolution intensity-based registration for the class of rigid transformations. Our results show that low resolution target values can tightly bound the high-resolution target function in natural images. This can help with analyzing and better understanding the process of multiresolution image registration. It also gives a guideline for designing multiresolution algorithms in which the search space in higher resolution registration is restricted given the fitness values for lower resolution image pairs. To demonstrate this, we incorporate our multiresolution technique into a Lipschitz global optimization framework. We show that using the multiresolution scheme can result in large gains in the efficiency of such algorithms. The method is evaluated by applying to 2D and 3D registration problems as well as the detection of reflective symmetry in 2D and 3D images.

I. INTRODUCTION

This paper investigates the implications of low-resolution image registration on the registration in higher resolutions. We focus on rigid intensity-based registration with correlation as fitness measure. We show that the fitness computed for lower resolution image pairs puts a bound on the fitness for higher resolution images. Our results can be exploited for the design and analysis of multiresolution registration techniques, especially the global optimization approaches.

Most of the approaches to globally optimally solving the registration problem deal with the alignment of point sets [1], [2], [3], [4], [5] or other geometric properties [6]. For example, Li and Hartley in [2] propose an algorithm to register two sets of points in a globally optimal way based on Lipschitz optimization [7]. Yang et al. [3] obtain a globally optimal solution by combing a branch and bound scheme with the Iterative Closest Point algorithm.

Such approaches have not been applied, as much, to intensity-based registration, due to the high computational cost. Nonetheless, in some applications it is worth to sacrifice efficiency to achieve more accurate results. A notable example is the construction of Shape Models [8] where the models are trained once and for all. In such a context, Cremers et al. [9] use Lipschitz optimization to search for optimal rotation and translation parameters for the alignment of 2D shape priors. After all, the computational burden is still the major limiting factor of such algorithms. It is well known that the multiresolution techniques can help with speeding up the registration algorithms [10], [11], [12]. However, to use a multiresolution scheme in a global optimization framework we need a theory which describes the relation among the target values of different resolutions. We provide such a theory in this paper, and based on that, propose an algorithm in which many candidate areas of the search space get rejected by examining the target value in lower resolutions.

This idea of early elimination has been well explored in the area of template matching, where a small template patch is searched in a target image. The successive elimination approach [13] rules out many candidate patch locations by examining a low-cost lower-bound on the cost function. The approach is extended in [14] to a multilevel algorithm, each level providing a tighter bound with higher computational cost. A multitude of approaches have been proposed based on the same idea of a multilevel succession of bounds [15], [16], [17], [18], [19], [20], [21]. In [22] some of the most successful algorithms of this class have been compared. Among them the work of Gharavi-Alkhansari [23] is very related to ours. In this approach each level corresponds to a different image resolution. By moving from low to high resolution we get tighter bounds requiring more computations. For most images many of the candidate solutions are eliminated by computing the match measure in lower resolutions. However, in this method, like all the approaches above, the search is only performed on a grid of translation vectors with 1 pixel intervals. Most of the work to extend this to more general transformations do not fully search the parameter space. We should mention, however, the work of Korman et al. [24] for affine pattern matching. It considers a grid of affine transformations and makes use of sublinear approximations to reduce the cost of examining each transformation. It runs a branch and bound algorithm in which a finer grid is used at each stage. Obtaining a solution close enough to the optimal target is guaranteed with high probability. The bounds, however, are in the form of asymptotic growth rates, and are learned empirically from a large data set. The algorithm, therefore, is not provably globally optimal in the exact mathematical sense.

Our work mainly deals with the alignment of two images rather than matching a small patch. Similar to [23] we take a multiresolution approach by providing bounds between different resolutions. However, since rotation is involved and also sub-pixel accuracy is required, the registration problem is analyzed in the continuous domain, by interpolating the discrete images. We incorporate the multiresolution paradigm into a Lipschitz optimization framework which searches the entire rigid motion space for a globally optimal solution. We show that the multiresolution approach can lead to huge efficiency gains. Another closely related problem is estimating reflective symmetry in 2D and 3D images [25]. We demonstrate that our algorithm can readily be applied to this problem as well. In short, the major contributions of this paper are...
Providing a framework to analyze the relation among registration targets at different resolutions,

- presenting inter-resolution bounds for several scenarios such as the use of different interpolation techniques or when only one image is decimated, and giving insights about how tighter bounds could be found,
- A novel algorithm to integrate the multiresolution paradigm into a Lipschitz global optimization framework resulting in considerable savings in computations.

We also introduce new effective bounds for the Lipschitz constants which are much smaller than what proposed in the literature. We should mention, however, that our main concern here is to demonstrate the efficiency gains of a multiresolution approach, and not optimizing the single-resolution algorithm itself. After providing a background in Sect. II, we introduce the inter-resolution bounds in Sect. III. A basic grid search algorithm is presented in Sect. IV, for better understanding the inter-resolution bounds in Sect. III. A basic grid search here is to demonstrate the efficiency gains of a multiresolution paradigm into a Lipschitz global optimization framework presenting inter-resolution bounds for several scenarios.

A. Discrete images

For discrete images we only have samples \( \{ f_i \} \) at discrete locations \( i \in \mathbb{Z}^d \) (\( d = 2 \) or \( d = 3 \)). One could think of \( \{ f_i \} \) as samples taken from a continuous image \( f \) according to \( f_i = f(Ti) \), where the scalar \( T \) is the sampling period. To get the continuous image from the discrete image we suppose that the continuous image is bandlimited to the corresponding Nyquist frequency \( \frac{1}{2T} \). This means that the continuous image can be obtained by sinc interpolation:

\[
f(x) = \sum_{i \in \mathbb{Z}^d} f_i \text{sinc}(\frac{x}{T} - i),
\]

where sinc(x) = \( \frac{\sin(\pi x)}{\pi x} \) for scalars, and

\[
sinc(x) = \prod_{i=1}^{d} \text{sinc}(x_i),
\]

for \( x \in \mathbb{R}^d \). Digital images are usually defined at a rectangular grid of pixels \( P \subseteq \mathbb{Z}^d \). For consistency with (3), we extend them by setting \( f_i = 0 \) outside the image boundaries where \( i \notin P \). Thus, the summation in (3) can be over \( P \) rather than \( \mathbb{Z}^d \). Notice that the continuous image created by (3) can be nonzero outside the image boundaries at non-integer coordinates.

The following proposition is useful for numerical integration in the frequency domain, and can be directly verified from the definition of the Discrete Fourier Transform (DFT).

**Proposition 1.** Consider a discrete image \( \{ f_i \}_{i \in P} \) defined over a square grid of pixels \( P \) with \( n \) pixels along each dimension. If the continuous image \( f \) is obtained using (3), the corresponding DFT values are equal to the samples of \( F(z) \), the Fourier transform of \( f(x) \), taken at \( \frac{1}{nT} \) intervals.

**Other interpolation techniques:** To accurately perform the sinc interpolation one needs to consider a large neighbourhood of pixels at any certain point. Therefore, in practice, kernels with a bounded support are used instead of the sinc function. A large class of interpolation techniques can be formulated as:

\[
f'(x) = \sum_{i \in \mathbb{Z}^d} f_i \cdot s(\frac{x}{T} - i),
\]

where \( s: \mathbb{R}^d \rightarrow \mathbb{R} \) is the interpolation kernel, which is typically a low-pass filter. To have a consistent interpolation, \( s(x) \) needs to be equal to 1 at the origin, and equal to zero at all other discrete sites with integer coordinates. Two examples of a bounded support kernel are:

**Nearest neighbor interpolation:** In this technique each point in the continuous domain gets the value of the closest discrete pixel. This can be obtained by using the box kernel

\[
s_n(x) = \begin{cases} 
1 & \|x\|_\infty \leq \frac{1}{2}, \\
0 & \text{elsewhere}.
\end{cases}
\]

**Bilinear interpolation:** In bilinear interpolation (trilinear for 3D) \( s \) is the product of triangular kernels in each dimension. The kernel has a bounded support, and can also be represented as the convolution of a box kernel with itself:

\[
s_t(x) = s_n(x) \ast s_n(x).
\]

\(^1\)Here, all integrals are over \( \mathbb{R}^d \) unless otherwise specified.

\(^2\)We have assumed that the discrete and continuous coordinates share the same origin. More generally, we can write \( f_i = f(Ti - x_0) \), where \( x_0 \in \mathbb{R}^d \) is the displacement of the origin.

\(^3\)The equality might be up to a known global scaling factor depending on the convention used for defining DFT.
B. Computation of the target function

Assume that \{f_t\} and \{g_t\} are defined on a grid of pixels \(P\). By substituting \(f_t^{'\prime}\) (and \(g_t^{'\prime}\)) in (5) for \(f_t\) and \(g_t\) in (1), we can write the target function corresponding to the discrete image pair \{\(f_t\)\} and \{\(g_t\)\} as follows

\[
Q(R, t) = \sum_{i \in P} \sum_{j \in P} f_t(g_j \int s(\lambda t) \cdot s(\frac{R(x + t)}{T} - j) \, dx, \tag{8}
\]

\[
= T_d \sum_{i \in P} \sum_{j \in P} f_t(g_j \cdot W_h(R(i + \frac{t}{T}) - j)) \tag{9}
\]

where

\[
W_h(d) = \int s(Rx + d) \cdot s(x) \, dx. \tag{10}
\]

Notice that \(\frac{t}{T}\) in (9) is the translation in the pixel coordinates. Now look at \(R(i + \frac{t}{T}) - j\). This is equivalent to rigidly transforming \(i\) according to \(R\) and \(\frac{t}{T}\), and then taking the difference with the pixel position \(j\). The weight given to each pair of pixel values \(f_t\) and \(g_j\) is equal to \(W_h(R(i + \frac{t}{T}) - j)\). The value of \(W_h(d)\) is expected to decay as the vector \(d\) grows in size. If \(s\) has a bounded support like (6) or (7), then \(W_h(d)\) will have a bounded support too. In other cases \(W_h(d)\) is negligible for large enough vectors \(d\). Therefore, for each \(i\), we can sum over pixels \(j\) within a certain neighbourhood of \(i\). This means that computing the target (9) needs \(O(|P|)\) rather than \(O(|P|^2)\) computation, where \(|P|\) is the number of pixels.

An essential part in computing (9) is to find the weights \(W_h(R(i + \frac{t}{T}) - j)\). For the nearest neighbour kernel (6), the integral (10) is simply the intersection area of two squares. As for the sinc kernel, a formula for (10) can be calculated in the frequency domain using the Parseval’s theorem. Nevertheless, even if large neighbourhoods are avoided by using bounded support kernels, the computation of \(W_h(d)\) itself is still costly. One solution is to precompute \(W_h(d)\) on a grid of \(R\) and \(d\) values to look up when necessary. Most registration algorithms, however, consider a simple form for \(W_h(d)\) which does not depend on \(R\) once \(d\) is known. Basically, what they do is discretizing the correlation integral. Let us rewrite (8) as

\[
Q(R, t) = \sum_{k \in P} f_k s(\frac{R}{T} - k) \sum_{j \in P} g_j s(\frac{R(x + t)}{T} - j) \, dx. \tag{11}
\]

Now, we discretize the above integral at \(x = Tk\) for all \(k \in P\):

\[
Q(R, t) \approx T_d \sum_{k \in P} \sum_{j \in P} f_k s(i - k) \sum_{j \in P} g_j s(R(k + \frac{t}{T}) - j) \, dx,
\]

\[
= T_d \sum_{i \in P} \sum_{j \in P} f_i g_j s(R(i + \frac{t}{T}) - j) \, dx. \tag{12}
\]

Here, the weights are simply the kernel values \(s(R(i + \frac{t}{T}) - j)\) which do not depend on \(R\) if \(R(i + \frac{t}{T}) - j\) is known. Another observation is that (12) is independent of what kernel is used for interpolating \(\{f_t\}\) in the case where \(\{f_t\}\) and \(\{g_t\}\) are interpolated using two different kernels. For the nearest neighbour interpolation, \(\sum_{j \in P} g_j s(R(i + \frac{t}{T}) - j)\) is simply the intensity value of the closest pixel \(j\) to \(R(i + \frac{t}{T})\). For the sinc interpolation, this value may be approximated efficiently by looking up in an upsampled version of \(\{g_t\}\). In our experiments we observed that discretized computation of the correlation integral does not cause major problems, unless for extremely decimated images or when a very high accuracy is expected (e.g. less than .2 pixels for translation). However, one ideally needs to further consider the discretization error.

III. IMPLICATIONS OF LOW RESOLUTION REGISTRATION

Assume that the discrete images \(\{f_t\}\) and \(\{g_t\}\) are decimated to low-resolution images \(\{f_t^{'\prime}\}\) and \(\{g_t^{'\prime}\}\). One may ask the question “What can the fitness (9) computed for lower resolution images say about the fitness for higher resolution images?”. This is an important question since the target function can be computed much faster in low resolution. If the original images are decimated by a factor of \(m\), then computing the target function takes \(1/m^d\) less computations, where \(d = 2, 3\) for 2D and 3D images respectively. This fact is clear from (9), where it is shown that the amount of computations is proportional to the number of pixels.

Decimation of a discrete image \(\{f_t\}\) may be carried out by

1) low-pass filtering the corresponding continuous image \(f\) obtained from (3), and

2) sampling the filtered images at a period of \(mT\), where \(T\) is the sampling period of \(\{f_t\}\).

The low-pass filter handles the aliasing distortion caused by downsampling, and thus, must have a cutoff frequency of \(\frac{1}{2mT}\) or less along every direction. To simplify the derivations, we consider a radial filter with the ideal frequency response

\[
L(z) = \begin{cases} 
1 & \|z\| \leq \frac{1}{2mT}, \\
0 & \text{otherwise},
\end{cases} \tag{13}
\]

where \(\|z\|\) is the \(l^2\)-norm (length) of \(z\). The filtered image \(f^T\) can be obtained as \(f^T = \ast \cdot f\) where \(\ast\) is the impulse response of \(L\) and \(\ast \cdot \) is the convolution operator. This filter eliminates all the frequency components outside a ball of radius \(\frac{1}{2mT}\) around the origin. Therefore, nothing is lost by sampling the filtered image \(f^T\) at intervals \(mT\) to obtain \(\{f_t^{'\prime}\}\), the low resolution image. We can also define a complementary high-pass filter

\[
H(z) = \begin{cases} 
0 & \|z\| \geq \frac{1}{2mT}, \\
1 & \text{otherwise},
\end{cases} \tag{14}
\]

The values of \(H(z)\) for \(\|z\| > \frac{1}{2mT}\) do not matter, as \(H\) is applied to \(f, g\) which are bandlimited to \(\frac{1}{2mT}\) due to (3). One could filter \(f\) with \(H\) to obtain \(f^h\). Notice that \(f(x) = f^l(x) + f^h(x)\). To make a discrete image \(\{f_t^{'\prime}\}\) out of \(f^T\) we should sample at intervals \(T\). Therefore, the discrete image \(\{f_t^{'\prime}\}\) has the same size as \(\{f_t\}\) while \(\{f_t^{'\prime}\}\) has roughly \(1/m\) less samples in every direction. The corresponding continuous images of \(\{f_t^{'\prime}\}\) and \(\{f_t^{'\prime}\}\) computed from (3) are exactly equal to \(f^T\) and \(f^h\) respectively. Note that to use (3) on \(\{f_t^{'\prime}\}\) one must use \(mT\) instead of \(T\).

Now, let us have a closer look at the continuous images \(f^T, f^h, g^T\) and \(g^h\). An important observation is that \(f^T(x)\)

4In fact, the double summation in (9) implies that the amount of computations is proportional to the square of the number of pixels. But, in practice, we only consider \(j\)-s within a fixed neighbourhood of each \(i\).

5We do not sample outside the boundaries of \(f_t\) for creating \(f_t^{'\prime}\) and \(g_t^{'\prime}\), even though the samples might not be exactly zero after filtering. The loss is supposedly negligible given that the images have a dark (near zero) margin.
and \( g^h(R(x+t)) \) are orthogonal for any choice of \( R \) and \( t \). This is because \( f^l(x) \) has no frequency component outside the ball of radius \( \frac{1}{2\pi^d} \), while \( g^h(R(x+t)) \) has no frequency components inside this ball. One way to see this is to write the inner product (correlation) between these two functions in the frequency domain using the Parseval’s theorem:

\[
Q(R, t) = \int f^l(x) g^h(R(x+t)) \, dx = \int F^l(z) G^h(Rz) e^{2\pi i t^T z} \, dz = 0.
\]

Similarly, \( f^h(x) \) and \( g^l(R(x+t)) \) are orthogonal. This implies

\[
Q(R, t) = \int f(x) g(R(x+t)) \, dx = \int f^l(x) g^h(R(x+t)) \, dx + \int f^h(x) g^l(R(x+t)) \, dx = Q^l(R, t) + Q^h(R, t),
\]

where \( Q^l(R, t) \) is the target function computed for \( f^l \) and \( g^l \). Therefore, we have

\[
|Q(R, t) - Q^l(R, t)| = \left| \int f^h(x) g^h(R(x+t)) \, dx \right| \leq \sqrt{\int (f^h(x))^2 \, dx} \sqrt{\int (g^h(x))^2 \, dx} = \|f^h\| \|g^h\|,
\]

where \( \|f\| \) represents the \( L^2 \)-norm of \( f \). We obtain (18) using the Cauchy-Schwarz inequality followed by a change of variables \( x \leftarrow R(x+t) \) for the right square root. It follows

\[
Q^l(R, t) - E_{fg}^h \leq Q(R, t) \leq Q^l(R, t) + E_{fg}^h
\]

where \( E_{fg}^h = \|f^h\| \|g^h\| \). The idea here is that \( E_{fg}^h \) tends to be small as for natural images the energy is mostly concentrated in the lower frequency bands.

Now, suppose that we want to find the maximum of the target function \( Q(R, t) \) over a grid \( \{R_k, t_k\} \) of registration parameters. Assume that we have computed the target function \( Q'(R, t) \) for the lower resolution images for all the grid values \( R_k, t_k \). Represent respectively by \( R^e, t^e \) and \( R^e, t^e \) the maximizers of \( Q^l(R, t) \) and \( Q(R, t) \) over the grid, where \( R^e, t^e \) are yet unknown. Using (20) we get

\[
Q(R^e, t^e) \geq Q(R^e, t^e) \geq Q^l(R^e, t^e) - E_{fg}^h.
\]

It means that we could rule out any \( R, t \) for which \( Q^l(R, t) < Q^l(R^e, t^e) - 2E_{fg}^h \), since in that case

\[
Q(R, t) \leq Q^l(R, t) + E_{fg}^h < Q^l(R^e, t^e) - E_{fg}^h \leq Q(R^e, t^e).
\]

Fig. 1(c) illustrates this idea. To further limit the search space one can compute the high resolution target function \( Q \) at \( R^e, t^e \) and discard all \( R, t \) with \( Q^l(R, t) < Q^l(R^e, t^e) - E_{fg}^h \), as shown in Fig. 1(d). The example provided in Fig. 1 shows the effectiveness of the proposed bound even when the images are decimated by a factor of 16 in each dimension.

### A. Bounded support interpolation

The results in the previous section hold when the sinc kernel (4) is used to compute the target function (20) to hold we need a large neighbourhood of pixels \( j \) around the location of each transformed pixel \( R(i + x_j / t) \), which requires a lot of computations. We now consider computing the target function (9) with two kernels of bounded support, namely the box kernel (6) and the triangular kernel (7) which respectively correspond to nearest neighbour and bilinear (or trilinear) interpolation methods. Many other interpolation techniques can be treated in a similar manner.

Here, we assume that the low resolution images are obtained, as before, by performing the ideal filter (13) on the continuous images \( f \) and \( g \) obtained by the sinc kernel using (3), and then resampling. However, for computing the target function using (9), we use a box or triangular kernel for both low-resolution and high-resolution images. Now, let us see what happens in the frequency domain. The interpolation formula (5) can be written as the following convolution

\[
f^l(x) = s\left(\frac{x}{T}\right) \ast \left( f(x) \cdot \sum_{i \in \mathbb{Z}^d} \delta(x - Ti) \right),
\]

where \( \delta \) is the Dirac delta function. The Fourier transform of the kernel \( s(x) \) is \( \text{sinc}^\alpha(z) \) with \( \alpha = 1 \) for the box kernel (6) and \( \alpha = 2 \) for the triangular kernel (7). The Fourier transform of (23) is then

\[
F^l(z) = \text{sinc}^\alpha(Tz) \cdot \sum_{i \in \mathbb{Z}^d} F(z - \frac{i}{T}).
\]

Notice that \( F(z) \) is bandlimited to \( \frac{1}{2T} \) in every dimension. Therefore, the term \( \sum_{i \in \mathbb{Z}^d} F(z - \frac{i}{T}) \) is just the periodic repetition of \( F(z) \) with a period of \( \frac{1}{T} \) along every dimension. In the same way, for the low-resolution image \( \{f^l_i\} \) we have

\[
F^l(z) = \text{sinc}^\alpha(mTz) \cdot \sum_{i \in \mathbb{Z}^d} F^l(z - \frac{i}{mT}).
\]

Notice that \( F^l(z) = F(z) \cdot L(z) \) is bandlimited to \( \frac{1}{2mT} \) along every dimension. Similarly, \( G^l(z) \) and \( G^dl(z) \) can be obtained.

Now, we like to find a bound on \( |Q'(R, t) - Q''l(R, t)| \), where \( Q'(R, t) \) is the target function (1) calculated for \( f^j \) and \( g^l \), and \( Q''l(R, t) \) is the target for \( f^{dl} \) and \( g^d \). Here, we present a simple way of doing this, and leave more elaborate bounds for future research. Define the energy of a function \( F \) within the frequency region \( S \) by

\[
E_S(F) = \int_S |F(z)|^2 \, dz,
\]

where \( \Omega \) the ball of radius \( \frac{1}{2\pi^d} \) around the origin in the frequency domain, and let \( \Omega' \) be its complement. Then we have the following proposition:

**Proposition 2.** The absolute difference on the target functions
where we have assumed as the reference image to perform affine warp to produce the reduced in the search space provided by such a low resolution registration is notable.

Notice that outside the frequency ball \( \Omega \) one expects \( F', G', F'' \) and \( G'' \) to have low energy and hence (29) is supposed to be small. As for (27) and (28), it is expected that within \( \Omega \), the energies \( E_{\Omega}(F' - F'') \) and \( E_{\Omega}(G' - G'') \) are small, as we will soon see. The proof is is given in Appendix A. Here, we give the value of the above energy terms calculated for function \( F \).

\[
\begin{align*}
Q'(\mathbf{r}, t) - Q''(\mathbf{r}, t) & \text{ is bounded from above by} \\
\sqrt{E_{\Omega}(F')} E_{\Omega}(G' - G'') + \sqrt{E_{\Omega}(G')} E_{\Omega}(F' - F'') & \quad (27) \\
+ \sqrt{E_{\Omega}(F' - F'')} E_{\Omega}(G' - G'') & \quad (28) \\
+ \sqrt{E_{\Omega}(F')} E_{\Omega}(G'') & \quad (29)
\end{align*}
\]

For the 1D case \((d = 1)\) it has the following formula

\[
\Phi_\alpha(z) = \int_{\Omega} \left( \Phi_\alpha(mTz) - \text{sinc}^2(\alpha(mTz)) \right) |F(z)|^2dz,
\]

where \( C \) is the \( l^\infty \)-ball of radius \( \frac{1}{M} \), that is \( \{z \mid |z| \leq \frac{1}{M} \} \), and the function \( \Phi_\alpha \) is defined on the \( l^\infty \)-ball of radius \( \frac{1}{M} \) as

\[
\Phi_\alpha(z) = \sum_{i \in \mathbb{Z}} \text{sinc}^2(\alpha(z + i)).
\]

For the 1D case \((d = 1)\) it has the following formula

\[
\Phi_\alpha(z) = \int_{\Omega} \left( \Phi_\alpha(mTz) - \text{sinc}^2(\alpha(mTz)) \right) |F(z)|^2dz,
\]

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\[
\Phi_\alpha(z) = \sum_{i \in \mathbb{Z}} \text{sinc}^2(\alpha(z + i)).
\]

It is simple to derive (30) and (31) from (24) and (25) considering the fact that \( \sum_{z \in \mathbb{Z}} F(z - \frac{1}{M}) = \sum_{z \in \mathbb{Z}} F'(z) = \frac{1}{MT} \) and \( \sum_{z \in \mathbb{Z}} F(z - \frac{1}{M}) \) are both equal to \( F(z) \) inside \( \Omega \). Notice that (31) tends to be relatively small for natural images as for small sized \( z \) the difference \( |\text{sinc}^2(Tz) - \text{sinc}^2(mTz)| \) is small and for larger \( z \) the frequency spectrum \( F(z) \) tends to be small. With a similar argument one can assert that the integrals (33) and (34) are small. The integral (32) is small as it is over \( C \setminus \Omega \). To obtain
\[
\int_{C \setminus \Omega} |F'(z)|^2 \, dz + \int_C |F'(z)|^2 \, dz \tag{37}
\]
As \( F'(z) = \sin^2(Tz)F(z) \) for all \( z \in C \), the integral over \( C \setminus \Omega \) is equal to (32). For the integral over \( C \), first for a vector \( \mathbf{v} \) and a set \( S \) define \( \mathbf{v} + S = \{ \mathbf{v} + z | z \in S \} \). Then, we have
\[
\sum_{t \in Z \setminus 0} \int_{C/T + C} |F'(z)|^2 \, dz = \sum_{t \in Z \setminus 0} \int_C |F'(z)|^2 \, dz = \sum_{t \in Z \setminus 0} \int_C |F(z)|^2 \sin^2(Tz + i) \, dz, \tag{38}
\]
which is equal to (33) as \( \sum_{t \in Z \setminus 0} \sin^2(Tz + i) = \Phi_\alpha(Tz) - \sin^2\alpha(Tz) \). In a similar way (34) can be obtained; only the period is \( \frac{1}{mT} \) instead of \( \frac{1}{T} \). This means that, instead of \( C \), we must consider the \( l^\infty \) ball \( C' \) of radius \( \frac{1}{mT} \). Notice that in this case \( \int_{C \setminus \Omega} |F'(z)|^2 \, dz \) is zero, as \( F'(z) = F(z) \cdot L(z) \) is zero outside \( \Omega \). All the integrals can be numerically computed using DFT in the light of Proposition 1. Fig. 2(a-c) illustrates the effect of the bound (27-29) for the example of Fig. 1.

Finally, let us consider the case where the discretized integration (12) is used for computing the target function. As discussed earlier, this approximation does not depend on how \( \{f_1\} \) is interpolated. Therefore, we may assume that \( \{f_1\} \) is interpolated with a sinc kernel, that is \( F' = F \) and \( F'' = F \). Under these assumptions, one can show that \(|Q'(R, t) - Q''(R, t)|\) is bounded from above by
\[
\sqrt{E_{\Omega}(F) E_{\Omega}(G' - G'')} + \sqrt{E_{\Omega}(F') E_{\Omega}(G' - G'')} \tag{39}
\]
where \( \Omega' \) is the ball of radius \( \frac{\sqrt{2}}{T} \) (see Appendix B). The effect of using this bound is depicted in Fig. 2(d,e,f). The bound is generally smaller than (27-29). However, here the discretization error for computing the target integral has been neglected. This is evident from the slight fluctuations in the bounds in Fig. 2(d,e) for very low resolutions.

\subsection*{B. Low-resolution to high-resolution registration}

In this section we examine the situation where only one of the images \( \{f_1\} \) is decimated. We study how this affects the amount of computations and the quality of the bounds. We also consider the case where the other image \( \{g_1\} \) is upsampled. Assume that \( \{f_1\} \) is decimated by a factor of \( m \) to create the low-resolution image \( \{f_1^m\} \). Using (5) a corresponding continuous image can be obtained as
\[
f^m(x) = \sum_{i \in P^m} f_1^m s\left(\frac{x}{mT} - 1\right), \tag{40}
\]
where \( P^m \) is the grid of low-resolution pixels. For the image \( \{g_1\} \) we do not change the resolution, and thus, we have \( g(x) = \sum_{i \in P} g_i s\left(\frac{x}{T} - 1\right) \). Now, there are two ways to compute the correlation target between \( f^m(x) \) and \( g(R(x + t)) \): the exact way like (9) and the approximate way by discretizing the integral like (12). We leave it to the reader to check that for computing the exact integral similar to (9), lowering the resolution of just one image does not significantly reduce the computations. This is because while the number of pixels in the first image is reduced by a factor of \( m^d \), the size of the neighbourhood around each transformed pixel required for the computation of \( W_{\delta}(d) \) is increased by the same factor.

However, if we discretize the integral, like in (12) we get
\[
Q'''(R, t) = \int f^m(x) g'(R(x + t)) \, dx \approx (mT)^d \sum_{i \in P^m} \sum_{j \in P} g_j s\left(R(mi + \frac{t}{T}) - j\right) \, dx. \tag{41}
\]
The above shows that if \( s \) has a compact support then for every \( i \) we only need to consider those pixels \( j \) which are in the corresponding neighbourhood of \( R(mi + \frac{t}{T}) \). The size of this neighbourhood is the same as that of (12). Since the first sum is over the low-resolution pixel grid \( P^m \) which has about \( m^d \) times less pixels than \( P \), we see that the amount of required computation when lowering the resolution of just one image is the same as when both images are decimated. This, of course, comes at the cost of having an approximate integral by discretization.

Now, let us see what happens to the bounds. If the interpolation is done using the sinc kernel \( (s = \text{sinc}, \text{and thus } f'' = f' \text{ and } g'' = g) \), then the target function (41) is equal to \( Q'(R, t) = \int f'(x) g'_1(R(x + t)) \, dx \), and hence the bound (20) does not change. This is due to the fact that \( f'(x) \) and \( g'_1(R(x + t)) \) are orthogonal, and thus, the correlation between \( f'(x) \) and \( g(R(x + t)) = g'(R(x + t)) + g''(R(x + t)) \) is the same as the correlation between \( f'(x) \) and \( g'(R(x + t)) \). However, for other interpolation techniques the bounds can be further tightened. In a similar way to Proposition 2 we can obtain
\[
|Q'(R, t) - Q'''(R, t)| \leq \sqrt{E_{\Omega}(G') E_{\Omega}(F'' - F')} + \sqrt{E_{\Omega}(F') E_{\Omega}(G' - G')} \tag{42}
\]
which is generally smaller than the bound in Proposition 2. If the discretized integral is used, similar to the way (39) is obtained, one can obtain the following bound
\[
\sqrt{E_{\Omega}(F) E_{\Omega}(G')} \tag{43}
\]
which is obtained by replacing \( G''' \) with \( G' \) in (39). Fig. 3(a) shows an example of applying this bound.

Now, let us see what happens if the second image \( \{g_1\} \) is upsampled by a factor of \( p \geq 2 \) to obtain \( \{g_2^p\} \). We assume that the new samples are obtained by the natural sinc interpolation. Therefore, the corresponding continuous function remains the same, which means \( g''(x) = g(x) \). Similarly to the way we obtained (24), if \( \{g_2^p\} \) is interpolated by the nearest neighbour \( (a=1) \) or the bilinear \( (a=2) \) method to obtain the interpolated image \( g'^p \), the corresponding Fourier transform will be
\[
G'^p(z) = \frac{\sin^p\left(\frac{Tz}{p}\right)}{\sin\left(\frac{Tz}{p}\right)} \sum_{i \in Z \setminus 0} G(z - \frac{pi}{T}). \tag{45}
\]
Notice that \( \sum_{i \in Z \setminus 0} G(z - \frac{pi}{T}) \) is the periodic repetition of \( G(z) \) with period \( p/T \). Since \( G(z) \) is bandlimited to \( T/2 \) in every
corresponding target functions are different. This, however, this does not make it a better bound, since the search space reduction approach is the same as that of Fig. 1(d). The bounds are not as effective as that of the sinc interpolation, but still are useful. (d,e,f) the same figures, but this time discretized integration is used instead of exact integration to compute the target function, and (39) is used for the bounds. Notice that the discretized integration has resulted in fluctuations in the bounds which increase as the decimation rate \( m \) gets larger.

Fig. 2. The bounds obtained for the nearest neighbour interpolation for the registration problem given in Fig. 1. Images are decimated by a factor of (a) 16, (b) 8 and (c) 4. The search space reduction approach is the same as that of Fig. 1(d). The bounds are not as effective as that of the sinc interpolation,

\[ |Q(R,t) - Q'(R,t)| = \left| \int_{\Omega} F(z) G(Rz(x)) e^{2\pi it^T x} dz \right| \]

\[ \leq \sum_{k=1}^{P} \sqrt{\int_{\Omega_k} |F(z)|^2 dz} \sqrt{\int_{\Omega_k} |G(z)|^2 dz} \]

where \( \Omega \) is the area outside the ball of radius \( \frac{1}{2mT} \) around the origin. The above is obtained using the Parseval’s theorem given the fact that \( F^h(z) = H(z) F(z) \) and \( G^h(z) = H(z) G(z) \) where the high-pass filter \( H \) was defined in (14). Now, we partition \( \Omega \) to radial bands \( \Omega_1, \Omega_2, \ldots, \Omega_p \) as illustrated in Fig. 4. More precisely, \( \Omega_i = \{ z \mid r_i \leq |z| < r_{i+1} \} \) for radii \( r_1 < r_2 < \cdots < r_{p+1} \). Then we have

The reader might have notice that the bound (46) is actually smaller than (19). This, however, this does not make it a better bound, since the corresponding target functions are different.
the registration problem of Fig. 1. In all cases, the image \( f \) is decimated by a factor of \( m = 16 \), and \( g \) has been upsampled by a factor of (a) \( p = 1 \), (b) \( p = 2 \) and (c) \( p = 4 \). Changing \( p \) from 2 to 4 has not made a noticeable difference.

IV. A BASIC ALGORITHM

We showed that the search space for high-resolution images can be limited given the low-resolution target values. This inspires a multiresolution search strategy whereby the target values at each resolution level further restrict the search space for the next (higher) level of resolution, as formalized in Algorithm 1. This is a very basic algorithm which searches a grid of the registration parameters \( G \). Here, we use \( l \) as an index to represent the resolution level, as opposed to the previous section, where \( l \) was a symbol indicating low-resolution. The resolution levels are \( l = 0, 1, \ldots, l_{\text{max}} \), where \( l=0 \) and \( l_{\text{max}} \) respectively correspond to highest and lowest resolutions. The target function for each resolution is represented by \( Q^l \). Therefore, \( Q^0(R, t) = Q(R, t) \). At each resolution level, the algorithm finds the maximizer \( (R^*, t^*) \) of the target function \( Q^l \) over \( G \) (line 3), updates \( Q^* \), the so-far best target value (line 4), and discards some elements of \( G \) based on \( Q^* \) and the inter-resolution bounds \( B^l_{\text{res}} \) (line 5). Here, \( B^l_{\text{res}} \) can be any of the bounds introduced in the previous section. The algorithm might not seem interesting from a practical perspective as the target function has to be evaluated at every element of the grid, at least at the lowest resolution. Nonetheless, it demonstrates a strategy purely based on our inter-resolution bounds, and gives insights about how to reduce computations in globally optimal registration algorithms where the whole parameter space is searched, which is the subject of the next section.

We test Algorithm 1 for the registration of the image pairs in Fig. 1. First, we consider only the rotation parameter using a grid of rotation angles between -180 to 180 degrees with 0.1 degree intervals. Six levels of resolution has been used with decimation factors 32, 16, 8, 4, 2 and 1. For further implementation details we refer the reader to Sect. VI-A. It takes about 0.2 seconds for the algorithm to find the best grid element. The second column of Table I shows the percentage of grid points ruled out at each resolution level by the bounds introduced in the previous section. The algorithm finds the maximizer

\[
\begin{align*}
\Omega \ni (R, t) &\quad \text{so-far best target value} \\
\text{for} \ l &\quad \text{low to high resolution} \\
\text{end for} \\
\text{return} \ R^*, t^*
\end{align*}
\]

Algorithm 1 A basic multiresolution rigid motion space search algorithm for intensity-based registration.

**Input:**
\[ \{f^l_i\}, \{g^l_i\}: \text{registration image pairs for each resolution } l = 0, 1, \ldots, l_{\text{max}}; \]
\[ B^l_{\text{res}}: \text{inter-resolution bounds for each } l, \]
\[ G: \text{a search grid of parameters } (R, t), \]
\[ m_0 < m_1 < \cdots < m_{\text{max}}: \text{A set of decimation rates } (m_0=1), \]
\[ R_0, t_0: \text{an initial solution (optional).} \]

1: \( Q^* \leftarrow Q(R_0, t_0) \) or \(-\infty\) \>
2: for \( l \leftarrow l_{\text{max}} \) downto 0 do \>
3: \( R^*, t^* \leftarrow \arg \max_{R, t \in G} Q^l(R, t) \)
4: \( Q^* \leftarrow \max(Q^*, Q^l(R^*, t^*)) \)
5: \( G \leftarrow G \setminus \{R, t \in G | Q^l(R, t) < Q^* - B^l_{\text{res}}\} \)
6: end for
7: return \( R^*, t^* \)
maxima exhibit sharp peaks in the target function. Depending on these factors, the multiresolution scheme might lead to different efficiency gains in different images.

| decimation | ruled out (rot.) | ruled out (rot. + trans.) |
|------------|----------------|-------------------------|
| $m = 32$   | 99.25 %        | 99.97 %                 |
| $m = 16$   | 0.00 %         | 0.0220 %                |
| $m = 8$    | 0.139 %        | 0.0055 %                |
| $m = 4$    | 0.028 %        | 7.0e-6 %                |
| $m = 2$    | 0.056 %        | 1.8e-6 %                |
| $m = 1$    | 0 %            | 0 %                     |

TABLE I

The percentage of grid elements ruled out at each level of resolution for a search grid of rotation (column 2) and rotation plus translation (column 3) parameters.

V. EMBEDDING INTO A GLOBAL OPTIMIZATION FRAMEWORK

In order to demonstrate the practical value of our results, we present an example of how the multiresolution technique can be integrated into a globally optimal search algorithm. Here, we consider the Lipschitz optimization framework [7], a branch and bound approach which exploits the concept of Lipschitz continuity to obtain upper bounds within each sub-region. A function $h: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is called Lipschitz continuous if there exists a constant $L$ such that for any $x_1, x_2 \in D$

$$|h(x_2) - h(x_1)| \leq L \|x_2 - x_1\|.$$  

(50)

The smallest such $L$ is called the Lipschitz constant. For differentiable $h$ the Lipschitz constant is the supremum of the gradient size within $D$. If $h(x, y)$ has Lipschitz constants $L_x$ and $L_y$ with respect to $x$ and $y$ respectively, then

$$|h(x_2, y_2) - h(x_1, y_1)| \leq L_x \|x_2 - x_1\| + L_y \|y_2 - y_1\|.$$  

(51)

In Lipschitz optimization, the target function is computed at the centre of each sub-domain, and then, using (50) or (51) an upper-bound is found on the target values within the sub-domain. The sub-domain is either rejected or split based on whether or not the upper bound is smaller than the currently best target value.

A. Lipschitz bounds

The efficiency of the algorithm highly depends on the quality of the Lipschitz bounds. An example of Lipschitz bounds for intensity-based registration is given in [9] in the context of shape models. Here, we present better bounds by looking at the target function in the frequency domain. According to our experiments, our bounds are usually smaller by at least one order of magnitude. Let $F(z) = \text{real}(F(z)), \text{imag}(F(z)) \in \mathbb{R}^2$, and similarly define $G(z)$. Then using the fact that $F(-z) = \overline{F(z)}$ and $G(-z) = G(z)$ for real-valued $f$ and $g$, we can rewrite (2) as

$$Q(R, t) = \int F(z)^T \Gamma(2\pi t^T z) G(Rz) \, dz.$$  

(52)

where $\Gamma(\phi)$ is the 2D rotation matrix of angle $\phi$, that is

$$\Gamma(\phi) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}.$$  

(53)

Now, the gradient of the target with respect to $t$ is

$$\frac{d}{dt} Q(R, t) = 2\pi \int z F(z)^T \Gamma'(2\pi t^T z) G(Rz) \, dz.$$  

(54)

where $\Gamma'(\phi) = \frac{d}{d\phi} \Gamma(\phi)$, which is also a rotation matrix. To obtain a fairly good upper bound on the magnitude of (54) we first divide the frequency domain into radial bands $\Omega_0, \Omega_1, \ldots, \Omega_P$, where $\Omega_i = \{z : r_i \leq \|z\| < r_{i+1}\}$ for radii $0 = r_0 < r_1 < \ldots < r_{P+1}$. This is similar to the approach taken in Sect. III-C1, only instead of partitioning the area outside $\Omega$, we partition the entire frequency domain (look at Fig. (4) and replace $\Omega$ by $\Omega_0$). Now, from (54) we can write

$$\left\| \frac{d}{dt} Q(R, t) \right\| \leq 2\pi \sum_{i=0}^P \int_{\Omega_i} \|z\| \left| F(z)^T \Gamma'(2\pi t^T z) G(Rz) \right| \, dz$$

\leq 2\pi \sum_{i=0}^P \left( \inf_{\Omega_i} \|z\| \right) \int_{\Omega_i} \|F(z)^T\| \|G(Rz)\| \, dz$$

\leq 2\pi \sum_{i=0}^P \int_{\Omega_i} \|F(z)^T\| \, dz \int_{\Omega_i} \|G(Rz)\| \, dz.$$  

(55)

In the third line of the above we used the fact that $\Gamma'(2\pi t^T z)$ is a rotation matrix. Also, notice that $\|F(z)^T\|$ is simply equal to the magnitude of the complex quantity $F(z)$. The integrals can be computed similarly to Sect. III-C1. What makes the above a good bound is that large values of $r_{i+1}$ are multiplied by high-frequency components of $F$ and $G$, which are typically small in natural images. This has been possible due to the division of the frequency domain into radial bands $\Omega_i$.

The parameterization of the rotation matrix can be quite different in 2D and 3D. Assume that $\rho \in \mathbb{R}$ is a single parameter representing one degree of freedom in the rotation space. Then

$$\frac{d}{d\rho} Q(R, t) = \int F(z)^T \Gamma(2\pi t^T z) J_G(Rz) \frac{dR}{d\rho} \, dz,$$  

(56)

where $J_G$ is the Jacobian matrix of $G$. It follows

$$\left| \frac{d}{d\rho} Q(R, t) \right| \leq \sum_{i=0}^P \int_{\Omega_i} \|F(z)^T\| \|G(Rz)\| \frac{dR}{d\rho} \, dz$$

\leq \sum_{i=0}^P \sqrt{\int_{\Omega_i} \|F(z)^T\|^2 \, dz} \sqrt{\int_{\Omega_i} \|G(Rz)\|^2 \, dz}.$$  

(57)

For 2D the only parameter is the angle of rotation $\rho = \theta$. Notice that $\frac{d\theta}{d\rho} R^{-1}$ is equivalent to a 90 degrees rotation counterclockwise. Thus, we have

$$J_G(z) \frac{dR}{d\rho} R^{-1} z = J_G(z) z^\perp,$$  

(58)

where $z^\perp = [-z_2, z_1]^T$ is the vector $z$ rotated by 90 degrees.

For the more complex 3D case we use the upper bound:

$$\left\| J_G(z) \frac{dR}{d\rho} R^{-1} z \right\| \leq \|J_G(z)\| \left\| \frac{dR}{d\rho} R^{-1} z \right\|,$$  

(59)

where $\|J_G(z)\|$ is the spectral norm of the matrix $J_G(z)$. We use the axis-angle representation of the rotation, with the unit...
vector \( \omega \) representing the axis of rotation and \( \theta \) representing the rotation angle. The rotation can be formulated as

\[
Rz = \cos(\theta) (1 - \omega \omega^T) z + \sin(\theta) (\omega \times z) + \omega \omega^T z
\]

where \( \times \) denotes the cross product. Similarly, \( R^{-1} z \) is obtained by replacing \( \theta \) in the above with \(-\theta\). Simple calculations give

\[
\frac{dR}{d\theta} R^{-1} z = \omega \times z \leq \|z\|.
\]

Now, we parameterize \( \omega \) in the spherical coordinates using two angles \( \phi \) and \( \psi \), as \( \omega = \left[ \cos(\phi) \cos(\psi), \sin(\phi) \cos(\psi), \sin(\psi) \right]^T \).

Let \( \tau = [-\sin(\phi), \cos(\phi), 0]^T \) and \( \nu = [-\cos(\phi) \sin(\psi), -\sin(\phi) \sin(\psi), \cos(\psi)]^T \), and notice that \( \omega, \tau \) and \( \nu \) form an orthonormal basis with \( \omega \times \tau = \nu \). Further, we have

\[
\frac{d\omega}{d\phi} = \cos(\psi) \tau, \quad \frac{d\tau}{d\phi} = \nu.
\]

By taking \( s = \sin(\theta/2) \) and \( c = \cos(\theta/2) \), and using \( \nu \times z = (\tau \omega^T - \omega \tau^T) z \) we get

\[
\frac{dR}{d\psi} z = 2s^2 (\omega \nu^T + \nu \omega^T) z + 2c^2 (\nu \times z) = 2s \text{USU}^T z, \quad (63)
\]

where \( \text{USU} \) and \( S = \begin{bmatrix} 0 & -c & s \\ c & 0 & 0 \\ s & 0 & 0 \end{bmatrix} \).

As \( \text{USU} \) is orthogonal and \( S \) has spectral norm 1, we have

\[
\left\| \frac{dR}{d\psi} R^{-1} z \right\| = \|2s \text{USU}^T R^{-1} z\| \leq 2 \sin(\theta/2) \|z\|. \quad (65)
\]

In a similar way, for \( \phi \) one can obtain

\[
\left\| \frac{dR}{d\phi} R^{-1} z \right\| \leq 2 \sin(\theta/2) \|\cos(\psi)\| \|z\|. \quad (66)
\]

Lipschitz bounds may be obtained from (65) and (66) by simply maximizing \( \sin(\theta/2) \) and \( \cos(\psi) \) within each subregion. By substituting into (57), one realizes that the second square root linearly grows with the size of \( z \). However, this is compensated by the fact that larger vectors \( z \) lie inside higher frequency bands \( \Omega \), and thus, are multiplied by high-frequency components of \( F \) and \( J_G \) which are typically small. Another important observation is that the Lipschitz bounds get smaller as images are more decimated. This can be easily seen by looking at (55) and (57), and noticing that the spectrum of a low-resolution image is obtained by zeroing out the high-resolution spectrum outside a certain ball.

**B. A multiresolution branch and bound algorithm**

We divide the registration parameter space into a number of hypercubes. In Lipschitz optimization, each hypercube either gets rejected or gets split depending on whether or not the corresponding upper bound on the target value is smaller than the largest target value found so far. The idea behind our multiresolution algorithm is to reduce the computations by using lower resolution images, as much as possible, to make the reject/split decision. To see how this is done, consider a certain cube \( C \) whose centre corresponds to parameters \( R_0, t_0 \). Like Sect. IV, we have a set of resolution levels \( l = 0, 1, \ldots, l_{\text{max}} \). For each resolution level \( l \) there are two different bounds: the inter-resolution bound \( B_{\text{res}}^l \) and the Lipschitz bound \( B_{\text{Lip}}^l \). The Lipschitz bound can be obtained from (50) or (51), and bounds the variation of \( Q'(R, t) \) from \( Q'(R_0, t) \) within the cube \( C \), that is \( Q'(R_0, t) - Q'(R_0, t_c) \leq B_{\text{Lip}}^l \) for all \( (R, t) \in C \). The inter-resolution bound gives \( |Q'(R_0, t) - Q'(R_0, t_c)| \leq B_{\text{res}}^l \).

Thus, we can define a total bound \( B_{\text{total}}^l = B_{\text{res}}^l + B_{\text{Lip}}^l \) giving

\[
|Q(R, t) - Q'(R_0, t_c)| \leq B_{\text{total}}^l \quad \text{for all } (R, t) \in C. \quad (67)
\]

Therefore, if \( Q'(R_0, t_c) + B_{\text{total}}^l \) is smaller than \( Q^* \), the currently best target value, we can safely discard \( C \). Otherwise, we test the above for the next (higher) resolution level \( l+1 \).

More precisely, our strategy is as follows. First, for any cube \( C \) we define \( l_{\text{min}} \) as the minimum total bound \( B_{\text{total}}^l \) for each cube, we start from \( l = l_{\text{max}} \) (the lowest resolution) and examine the upper bounds for \( l = l_{\text{max}} - 1, \ldots, l_{\text{min}} \) in order. If at any resolution \( l \) it happens that \( Q'(R_0, t_c) + B_{\text{total}}^l \leq Q^* \), then we discard \( C \). Otherwise, we split the cube. We do not go further below \( l_{\text{min}} \) as otherwise \( B_{\text{total}}^l \) would become larger and there would be a high change of \( Q'(R_0, t_c) + B_{\text{total}}^l \) falling above \( Q^* \) again.

Notice that, \( B_{\text{res}}^l \) is equal or proportional to the residual of the energy after low-pass filtering. Thus, we have \( 0 = B^0_{\text{res}} \leq B^1_{\text{res}} \leq \cdots \leq B^n_{\text{res}} \). On the contrary, we observed in the previous section that \( B^0_{\text{Lip}} \geq B^1_{\text{Lip}} \geq \cdots \geq B^n_{\text{Lip}} \). On the first iterations of the algorithm, the cubes are large, and therefore, \( B^n_{\text{Lip}} \)-s dominate \( B^n_{\text{res}}\)-s. Consequently, \( l_{\text{min}} \) becomes equal or close to \( l_{\text{max}} \), and only lower resolution images are used for making the reject/split decision. As the algorithm goes on, the cubes get divided into smaller cubes. Thus, \( B^n_{\text{Lip}} \) becomes smaller compared to \( B^n_{\text{res}} \), and \( l_{\text{min}} \) moves towards zero. Therefore, higher resolution images are examined. Due to this strategy, a large portion of the search space is explored in the lower-resolution domains.

Algorithm 2 describes our approach. It uses the breadth-first search strategy, using a queue data structure. We use a special element called level-delimiter to separate different levels of the search tree in the queue. After searching all cubes of each level, we have a candidate optimal parameter \( (R^n, t^n) \), which is the maximizer of \( Q_{\text{min}}(R^n, t^n) - B_{\text{res}}^n \) among all cubes in this level (see line 21 of the algorithm). We examine the original cost function \( Q = Q^0 \) on this candidate parameter set. If \( Q(R^n, t^n) \) > \( Q^* \), then \( Q^* \), \( R^n \) and \( t^n \) are updated (line 8).

It is not always necessary to split across every dimension of the cube, when a split is needed. Notice that the bound \( B_{\text{Lip}}^n \) is the sum of the bounds for each parameter, as shown in (51) for two parameters. We sort the bounds descendingly, start from the parameter with the biggest bound, and keep splitting across the parameters, until the sum of the rest of the bounds are smaller than \( Q^* - B_{\text{res}}^n \), which is the margin required for rejecting the cube at resolution \( l_{\text{min}} \).

The algorithm continues until \text{FinishCondition()} \) is satisfied. Different criteria can be used as the finishing condition, such as when all the remaining cubes are small enough, or if the target function does not change much between two consecutive search levels. However, what is usually used in the branch and bound algorithms is to ensure that the currently best target \( Q^* \) is within a certain distance \( \varepsilon \) of the optimal
Algorithm 2 A multiresolution Lipschitz optimization algorithm for rigid intensity-based registration.

Input: \( \{ f^l \}, \{ g^l \} \): registration image pairs for each resolution \( l \), \( B^l_{\text{res}} \): inter-resolution bounds for each \( l \), \( L^l \): Lipschitz constant estimates for all parameters, for each \( l \), \( C_0 \): the initial hypercube containing the parameter space, \( R_0, t_0 \): an initial solution (optional).

1: \( Q^* \leftarrow Q(R_0, t_0) \) or \(-\infty\) \( \triangleright \) so-far best target value
2: \( Q^l_{\triangleright} \leftarrow -\infty \) \( \triangleright \) best low-resolution lower-bound on target
3: QUEUE.ENQUEUE\((C_0)\)
4: QUEUE.ENQUEUE(level-delimiter)
5: repeat
6: \( C \leftarrow \) QUEUE.DEQUEUE()
7: if \( C = \) level-delimiter then \( \triangleright \) level completed
8: \( Q^*, R^*, t^* \leftarrow \text{AMAX}(Q^*, R^*, t^*, Q(R^*, 0, t^*), R^*, t^*) \)
9: QUEUE.ENQUEUE(level-delimiter)
10: go to line 26 \( \triangleright \) next iteration
end if
12: \( R_c, t_c \leftarrow \text{GETCENTRE}(Q) \)
13: \( B^l_{\text{Lip}} \leftarrow \text{LIPSCHITZBOUND}(L^l, C) \) for all \( l \)
14: \( B^l_{\text{total}} \leftarrow B^l_{\text{res}} + B^l_{\text{Lip}} \) for all \( l \)
15: \( l_{\text{min}} \leftarrow \text{argmin}_l B^l_{\text{total}} \)
16: for \( l \leftarrow l_{\text{max down to}} l_{\text{min}} \) do
17: if \( Q^l(R_c, t_c) + B^l_{\text{total}} \leq Q^* \) then
18: go to line 26 \( \triangleright \) reject \( C \), next iteration
end if
19: end for
20: \( Q^l_{\text{oversamp}}, R^l_{\text{oversamp}}, t^l_{\text{oversamp}} \leftarrow \text{AMAX}(Q^l_{\text{oversamp}}, R^l_{\text{oversamp}}, t^l_{\text{oversamp}}) - B^l_{\text{min (res, c)}} \)
21: \( Q^*, R^*, t^* \leftarrow \text{AMAX}(Q^*, R^*, t^*, Q^l_{\text{oversamp}}, R^l_{\text{oversamp}}, t^l_{\text{oversamp}}) \)
22: for \( C' \) in \text{SPLIT}(C) do
23: QUEUE.ENQUEUE\((C')\)
24: end for
25: end for
26: until FINISHCONDITION()
27: return \( R^*, t^* \)

procedure \text{AMAX}(Q_1, R_1, t_1, Q_2, R_2, t_2)
28: if \( Q_1 \geq Q_2 \) then return \( Q_1, R_1, t_1 \)
29: else return \( Q_2, R_2, t_2 \)
30: end procedure

VI. EXPERIMENTS

The only globally optimal approach we found on rigid intensity-based registration is \cite{9}, running a single-resolution Lipschitz optimization for 2D images with larger Lipschitz bounds than ours. Their proposed estimation of Lipschitz constant is more than 20 times larger than ours for rotation and more than 10 times larger for translation for the image pair of Fig. 1. Therefore, here we compare the single-resolution and multiresolution versions of our own algorithm and report the speed gain offered by the multiresolution technique.

A. Implementation

Algorithm setup: We use the low-resolution to high-resolution registration of Sect. III-B with discretized integration and the bound (46), where \( g \) is upsampled by a factor of 2, and bilinear (or trilinear) interpolation is used. The highest decimation rate is set to the largest power of 2 where \( E^l_{\text{LS}} = \| f^l \| \| g^l \| \) in (20) is less than 0.05 of \( \| f \| \| g \| \). For the example of Fig. 1 this gives a maximum decimation rate of 32. However, for images with a less compact spectrum this can be smaller. We have found experimentally that this is a safe threshold for avoiding errors caused by discretized integration. We set the convergence threshold \( \varepsilon \) to 0.01 \( \| f \| \| g \| \) (see Sect. V-B). For the starfish example of Fig. 1 this provides a solution with an accuracy of about 0.2 pixels for translation and 0.04 degrees for rotation.

Platform: The main algorithm is implemented under C++, while constructing the multiresolution pyramid and the bounds are done under Matlab. The machine has Intel Core i7-5500U 2.4GHz CPU and 8GB of RAM.

Low-pass filtering and downsampling: To implement the ideal filter (13) we zero out the DFT coefficients of the discrete image outside the corresponding ball around the origin, and then take inverse DFT. This is not in general equivalent to applying the ideal filter (13) to the corresponding continuous image. But, we can get closer to the ideal case by using higher-resolution DFTs. Another potential source of error is due to downsampling. After taking inverse DFT, we downsample the resulting discrete image by starting at the top-left pixel \((1, 1)\) and then sampling every \( m \)-th pixel. This means that the corresponding continuous image is sampled only within the boundary of the discrete image. However, there is no guarantee that the continuous image will be zero at the sampling locations outside the image boundary after low-pass filtering. One may solve this by sampling outside the boundaries up to a certain margin. Both these sources of error are negligible for images with a fairly large dark (near zero) margin. We can obtain such an image by zero padding, or reversing the intensity values for images with a bright background.

Upsampling: To upsample, zero-valued higher-frequency components are added to the DFT of the discrete image, and inverse DFT is taken. Again, this is not equal to upsampling with sinc interpolation, but is accurate enough for images with a dark margin, or when higher-resolution DFT is used.

Image intensity: All images are converted to grayscale prior to registration. For images with a white or light background the intensity values are inverted.

B. 2D registration

First we try the starfish image image of Fig. 1(a). The main image is \( 782 \times 782 \). The second image is obtained by applying a translation of \((-50, 20)\) pixels and a rotation of 45 degrees to the first image. The multiresolution Lipschitz optimization algorithm takes 44 seconds to finish, while the
single resolution version takes 3 hours and 13 minutes. This means a large speedup of order 263.

Now, we apply our algorithm to register two different slices of the same brain MRI volume being 10 slices away from each other (Fig. 5). We rotate the second slice by 160 degrees and then translate it by (-10,14) pixels. The images are 192×156. The single-resolution algorithm takes 534 seconds to perform the registration. The multiresolution algorithm does it in 119 seconds. This time we get a 4.5 times speedup. This shows that the efficiency gain can vary based on how concentrated the image is in the frequency domain and how the target value of the true solution stands out in the registration parameter space.

Fig. 5. Registering two different slices of a brain MRI image. (a) the first slice, (b) the second slice rotated and translated, and (c) the first slice registered to the second slice. The data is obtained from [26].

C. 3D registration

Our current branch and bound algorithm is not yet able to handle the 6 degrees of freedom of a full 3D registration in a reasonable time. What is done here is to align the centres of masses of the two images and then perform a rotation-only search around the centre of mass. We consider the binary volumes of the same vertebra in two different subjects (Fig. 6). Both volumes are 81×90×24 and are obtained by manually labeling CT images. The accurate alignment of such images is usually required as the first step of constructing a shape model. The registration results can be seen in Fig. 6. The single-resolution algorithm performs the registration in 19 minutes and 47 seconds. For the multiresolution approach it takes 29.8 seconds, which is a 40-fold speedup.

Fig. 6. 3D registration of binary vertebra shapes, (a,b) the registration image pair, (c) the first image registered to the second image.

D. Reflective symmetry detection

A closely related problem to rigid registration is detecting the axis or plane of symmetry in 2D and 3D images. Most natural images are not perfectly symmetric. One, however, can estimate the best choice by maximizing the correlation of an image with its mirror reflection across a line (2D) or a plane (3D). The problem formulation and other details are given in Appendix C. Here, we demonstrate the application of the multiresolution Lipschitz algorithm.

Fig. 7 shows three examples of symmetry detection in 2D images, comparing the execution time of single-resolution and multiresolution algorithms. In all cases the multiresolution algorithm is faster by a factor of more than 4. The reader may have noticed that the butterfly image takes more time than the face image despite being smaller. This is due to sharper edges (larger high-frequency components) in the butterfly image, leading to larger Lipschitz bounds.

| resolution   | 300×300  | 340×320  | 218×182  |
|--------------|----------|----------|----------|
| multi-res. time | 14.7s  | 3.9s  | 0.92s  |
| single-res. time | 64.3s  | 16.5s  | 4.4s  |
| speedup      | 4.4      | 4.2      | 4.7      |

Fig. 7. Detecting axis of symmetry for images of a butterfly, a face and a brain MRI slice. The butterfly image is from http://hdwallpapersrocks.com. The brain MRI image was obtained from http://www.brain-map.org.

For estimating plane of symmetry, we use the binary volume of a human vertebra (Fig. 8). We use the same data as for 3D registration. Reflective symmetry detection for binary volumes is specially useful in shape analysis and modeling. The 3D image is 114×106×44. The single-resolution algorithm takes 2 hours and 51 minutes to detect the optimal plane of symmetry. The multiresolution algorithm does this in 29.2 seconds. This is about 350 times faster, which is a considerable speedup.

Fig. 8. Estimating the plane of symmetry for the binary volume of a vertebra

VII. CONCLUSION AND FUTURE WORK

We showed that low-resolution target values can tightly bound the high-resolution target function for rigid intensity based image registration. This led to a multiresolution search scheme in which the search at each resolution limits the search space for the next higher resolution level. By embedding into the Lipschitz optimization framework, we showed that this strategy can significantly speed up the globally optimal registration algorithms. This paper was mostly focused on providing a working example of such an embedding and demonstrating the efficiency gains it provides. Optimizing the efficiency of the single-resolution algorithm itself can be the subject of future research. This can be done by finding better Lipschitz bounds or employing more effective search
strategies instead of the breadth first method. An extension of this work could be to find bounds for more general transformations (similarity, affine, nonlinear, etc.) and other types of target functions (robust, information theoretic, etc.). The effectiveness of our multiresolution approach comes from the energy compaction of natural images in the Fourier basis. An important extension is to examine the application of similar ideas for other bases, and obtaining optimized basis functions tailored for each image pair.

APPENDIX A

BOUNDED SUPPORT INTERPOLATION

Proof of Proposition 2: The difference \(|Q'(R,t) − Q''(R,t)|\) can be written in the frequency domain using (2):

\[
\left| \int \bar{F}(z) G'(Rz) e^{2\pi it^Tz} \, dz - \int \bar{F}(z) G''(Rz) e^{2\pi it^Tz} \, dz \right|. 
\] (68)

Using the triangle inequality, we break the above into integrations over \(Ω\) and its complement \(\bar{Ω}\):

\[
\left| \int D(z) \, dz \right| \leq \left| \int D(z) \, dz \right| + \left| \int D(z) \, dz \right|, \tag{69}
\]

where \(D(z) = e^{2\pi it^Tz} \left( \int \bar{F}(z) G'(Rz) - \int \bar{F}(z) G''(Rz) \right)\). Let us first deal with the integral inside \(Ω\). Since \(F' z = F z\) for all \(z \in Ω\), we expect \(F'(x)\) and \(F''(x)\) to be close. A similar argument goes for \(G'(x)\) and \(G''(x)\). Now using the equality

\[
ab - a'b' = a(b - b') + b(a - a') - (a - a')(b - b') \tag{70}
\]

for any four complex numbers \(a, b, a'\) and \(b'\) we can write

\[
\begin{align*}
\left| \int_{Ω^c} \left( \int \bar{F}(z) G'(Rz) - \int \bar{F}(z) G''(Rz) \right) e^{2\pi it^Tz} \, dz \right| & \leq \left| \int_{Ω^c} \left( \int \bar{F}(z) (G'(Rz) - G''(Rz)) e^{2\pi it^Tz} \, dz \right) \right| \\
& + \left| \int_{Ω^c} \left( \int \bar{F}(z) (\bar{F}(z) - \bar{F}'(z)) (G'(Rz) - G''(Rz)) e^{2\pi it^Tz} \, dz \right) \right| \\
& + \left| \int_{Ω^c} \left( \int \bar{F}'(z) - \bar{F}(z) \right) (G'(Rz) - G''(Rz)) e^{2\pi it^Tz} \, dz \right|.
\end{align*}
\]

This is equal to (29).

APPENDIX B

To obtain (39), we treat the frequency areas \(Ω\) and \(\bar{Ω}\) separately, like in (69). We use the same approach from which (27) and (28) are obtained. However, since in this case \(F' = F''\) and \(F'' = \bar{F}'\), and also \(F(z) = F(z)\) inside \(Ω\), then we have \(E_{Ω}(F' - F'') = E_{Ω}(F' - F'') = 0\). Therefore, (27) and (28) reduce to \(\sqrt{E_{Ω}(F)} E_{\bar{Ω}}(G'' - G''')\), which is the first term in (39). Now, outside \(Ω\) we have \(F'(z) = 0\), and therefore

\[
\int_{Ω^c} D(z) \, dz = \int_{Ω^c} \bar{F}(z) G'(Rz) e^{2\pi it^Tz} \, dz \tag{73}
\]

where \(D(z) = \left( \int \bar{F}(z) G'(Rz) - \int \bar{F}(z) G''(Rz) \right) e^{2\pi it^Tz}\) and \(Ω^c\) is the ball of radius \(\frac{1}{2T}\). This is equal to the second term in (39). The equality (73) holds since \(F'(z)\) is bandlimited to \(\frac{1}{2T}\) in every dimension, and hence the integrand is zero outside \(Ω^c\). The inequality (74) is obtained by Cauchy-Schwarz followed by a change of variables \(z \leftarrow \bar{Rz}\) in the second integral. The change of variables is possible since \(Rz \in Ω\) if and only if \(z \in Ω^c\). Last line uses the fact that \(E_{Ω^c\setminus Ω}(F) = E_{Ω^c}(F)\).

APPENDIX C

AXIS AND PLANE OF SYMMETRY

We represent a plane by its normal \(u\) and its distance \(α\) from the origin. The reflection of a point \(x\) with respect to the plane can be formulated as \(x' = (1 - 2uu^T)(x - 2αu)\). The correlation-based target function then becomes

\[
Q(u, α) = \int f(x) f((1 - 2uu^T)(x - 2αu)) \, dx. \tag{76}
\]

All the inter-resolution bounds in the registration problem works here by simply replacing \(g\) with \(f\). One can verify this by using \(f, (1 - 2uu^T)\) and \(-2μu\), respectively, instead of \(g, R\), and \(t\) in the derivations. It is only left to find Lipschitz bounds for the new parameters \(α\) and \(u\). Following a similar approach as Sect. V-A, we can write \(Q(u, α)\) as

\[
Q(u, α) = \int F(z)^T Γ(−4παuu^T)z F((1 - 2uu^T)z) \, dz. \tag{77}
\]
To be concise, let $\beta = -4\pi \alpha u^T z$. A bound for $\alpha$ can be

$$\left| \frac{d}{d\alpha} Q(u, \alpha) \right| = 4\pi \int u^T z F(z)^T \Gamma(\beta) F(\frac{1}{2} uu^T) z \, dz$$

$$\leq 4\pi \sum_{i=0}^{P} \int \Omega_i \|F(z)\| \|F(\frac{1}{2} uu^T) z\| \, dz$$

$$\leq 4\pi \sum_{i=0}^{P} \int \Omega_i \|F(z)\|^2 \, dz. \quad (78)$$

To bound the derivative with respect to $u$ we represent it in polar/spherical coordinates. Assume $\gamma$ is a parameter by which $u$ is parameterised, and let $u' = \frac{d}{d\gamma} u$. Then we have

$$\frac{d}{d\gamma} Q(u, \alpha) = -4\pi \alpha \int u'^T z F(z)^T \Gamma(\beta) F(\frac{1}{2} uu^T) z \, dz$$

$$- 2\int F(z)^T \Gamma(\beta) J_F(\frac{1}{2} uu^T) z (u' u + u' u^T) z \, dz. \quad (79)$$

As $u'$ is always perpendicular to $u$ (see polar/spherical parameterization of $\omega$ in Sect. V-A), we have

$$\frac{d}{d\gamma} Q(u, \alpha) \leq 4\pi \alpha \|u'\| \int \Omega_i \|F(z)\|^2 \, dz.$$ For the 2D case it equals $\|\cos(\phi), \sin(\phi)\|^T$. Then, we have

$$\left| \frac{d}{d\phi} Q(u, \alpha) \right| = 4\pi \alpha \int \Omega_i \|F(z)\|^2 \, dz$$

$$+ 2 \sum_{i=0}^{P} \int E_{\Omega_i}(F) \int \Omega_i \|J_F(z)\|^2 \|z\|^2 \, dz, \quad (80)$$

where $z^\perp$ was defined in (58). For 3D, we use spherical coordinates: $u = [\cos(\phi), \cos(\phi), \sin(\phi), \sin(\phi)]^T$. Thus, we have

$$\left| \frac{d}{d\phi} Q(u, \alpha) \right| \leq 4\pi \alpha \int \Omega_i \|F(z)\|^2 \, dz$$

$$+ 2 \|J_F(z)\| \sum_{i=0}^{P} \int E_{\Omega_i}(F) \int \Omega_i \|J_F(z)\|^2 \|z\|^2 \, dz, \quad (81)$$

where $\|J_F(z)\|$ denotes the spectral norm of $J_F(z)$, and

$$\left| \frac{d}{d\psi} Q(u, \alpha) \right| \leq 4\pi \alpha \sum_{i=0}^{P} \int \Omega_i \|F(z)\|^2 \|z\|^2 \, dz$$

$$+ 2 \sum_{i=0}^{P} \int E_{\Omega_i}(F) \int \Omega_i \|J_F(z)\|^2 \|z\|^2 \, dz, \quad (82)$$

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