Reference tracking of dynamic system using prefilter and integral extension

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Abstract. In this paper, a control approach for reference tracking in state-space that combines two existing methods for designing two degrees-of-freedom controllers for reference tracking is presented. One of these methods is the use of a prefilter $V$ to scale the reference input in order to compensate the existing DC-Gain of the closed-loop behavior. The second approach incorporates an integral extension in the state-space model by integrating the measured output and embedding it in the dynamic of the system to be controlled. Each of these approaches has advantages as well as disadvantages. By merging these two methods, the goal is to use the advantages of each method to overcome their drawbacks simultaneously.

1. Introduction
Linear control methods belong to the most commonly used control techniques for dynamic system. The biggest advantage of these methods is the simplicity of the design procedures. The first goal the controller is to feed the given dynamic system with an input, that stabilizes the system and make it reach the desired behaviour. However, the mathematical model does not exactly match the dynamic behaviour of the system. To ensure the stability of the system even by variation of some system characteristics, the controller should have a high performance.
Besides the stability of the dynamic system, another relevant aims of a controller is the reference tracking. There exist two well-known approaches for the reference tracking in state-space domain. The first approach divides the controller in two parts. The first part is responsible for stability and the second one ensures a DC-Gain of $k_\infty \neq 1$ between the reference and the output. This structure is not robust again process noises. The second approach used an integral extension of the state-space model. The stability and reference tracking are achieved with a simple state-feedback gain of the new system. This approach is robust against process-noises, but the reaction is slower due to the integrator incorporated in the system.
In this paper, we will first start with some mathematical preliminaries then we will present different approaches for reference tracking. lastly, the effect of each method will be tested on different dynamical systems.

2. Preliminaries
We consider the system dynamic (1) to be controlled. Many methods such as pole-placement and optimal control have been developed to design controllers that fulfil the main task of stability using the state-space representation. If we consider the autonomous part of the system (1) without noise, then the dynamic is given by the eigenvalues of the matrix $A$.

$$\begin{align*}
\dot{x} &= Ax + Bu + Ed \\
y &= Cx
\end{align*}$$

(1)
Theorem 2: Consider the autonomous dynamic system $\dot{x} = Ax$. If all the eigenvalues of $A$ are strictly negative, then this system is asymptotically stable.

The aim of the pole-placement procedure is to design a state feedback

$$u = -Kx \Rightarrow \dot{x} = (A - BK)x = \bar{Ax}$$

such that, the controlled system has the desired behaviour. [1]

However, all these techniques are based on linear approximations of the real system. To ensure the stability of the real system, the controller should have a high performance. Performance check is easily made in frequency-domain.

Using Laplace-transform, the dynamic of the closed-loop system (Figure 1) can be resumed in

$$Y(s) = G_w(s)W(s) + G_d(s)D(s).$$

To analyse the performance of a controlled system, the closed-loop transfer function for the reference $G_w(s)$ is used. The available tools make used of the open-loop transfer-function $L(s)$, which is used in the well known classical control-loop with negative output feedback (Figure 1).

$$G_w(s) = \frac{Y(s)}{W(s)} = \frac{L}{1+L}.$$ 

Theorem 3: Nyquist-criteria

An open-loop with the transfer-function $L(s)$ leads to an I/O-stable (input/output) loop if the Nyquist plot $L(j\omega)$ for $\omega = -\infty \cdots \infty$ encloses the point $-1 + j0$ of the complex plane $-n^+$-time clockwise. Where $n^+$ denotes the number of poles of $L(s)$ with positive real part. [2]

Some important frequency-domain measures are used to assess performance e.g. gain and phase margins [3]. Figure 2 shows gain and phase margins in Bode-diagram. The gain margin $GM$ defined as

$$GM = \frac{1}{|L(j\omega_c)|} \quad \text{with} \quad \angle L(j\omega_c) = -180^\circ. \quad [3]$$

The gain margin is the factor by which the loop gain $|L(j\omega)|$ can be increased before the closed-loop system becomes unstable. The gain margin is thus a direct safeguard against steady-state gain uncertainty [3]. The phase margin is defined as

$$PM = \angle L(j\omega_c) + 180^\circ, \quad \text{with} \quad |L(j\omega_c)| = 1. \quad [3]$$

The phase margin tells how much negative phase (phase lag) we can add to $L(s)$ at frequency $\omega_c$, before the phase at this frequency becomes $-180^\circ$ which corresponds to closed-loop instability. The phase margin is a direct safeguard against time delay uncertainty [3].

3. Reference Tracking

Reference tracking is a very important task of a controller. In this part, we will present two different approaches for reference tracking in state-space domain. Then we will merge the two approaches.

3.1. Reference Tracking with prefilter

Consider the dynamic System (1) to be controlled with a simple state-feedback matrix and the reference such that

$$u = -Kx + Vw$$

Figure 1. Classical control-loop

Figure 2. Gain and phase margin in Bode-diagram [3]
With the Laplace-transform following relations is derived for reference and noise transfer-functions:

\[ G_w(s) = \frac{Y(s)}{W(s)} = C(sI - A + BK)^{-1}BV, \quad G_d(s) = \frac{Y(s)}{D(s)} = C(sI - A + BK)^{-1}E. \]  (8)

For reference tracking, we consider the behaviour of the transfer function for low frequencies \((\omega \to 0)\). It holds

\[ \lim_{\omega \to 0} G_w(j\omega) = 0, \quad \lim_{\omega \to 0} G_d(j\omega) = -C(A - BK)^{-1}BV \equiv 1 \Rightarrow V = -(C(A - BK)^{-1}B)^{-1}. \]  (9)

The behaviour of the noise transfer function \(G_d(s)\) should also be analysed to see the effect of process noise on the system.

\[ \lim_{\omega \to 0} G_d(j\omega) = -C(A - BK)^{-1}E, \quad \lim_{\omega \to \infty} G_w(j\omega) = 0 \]  (10)

The relation (10) shows that high-frequency noises will be damped, but it will not be the case for low-frequency noises. For instance, step noise will be transferred to the system with the DC-Gain

\[ k_{d,\infty} = -(C(A - BK)^{-1}E). \]

To overcome this problem, another approach is used.

### 3.2. Integral extension for reference tracking

Figure 4 show the idea of the integral extension. In this procedure, the integral of the error \(e = y - w\) is included in the state-space model and the new representation is used to design a feedback gain. Consider \(z\) as new state-space representation. An equivalent dynamic system can be derived

\[ z = \int (y - w) \, dt \Rightarrow [\dddot{x}] = \begin{bmatrix} 0 & C \\ A & x \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u + \begin{bmatrix} 1 \\ E \end{bmatrix} w + \begin{bmatrix} 0 \\ E_0 \end{bmatrix} \]

\[ y = \begin{bmatrix} 0 & C \end{bmatrix} \begin{bmatrix} \dddot{x} \\ x \end{bmatrix}. \]  (11)

Using a state feedback

\[ u = -\bar{K}\ddot{x} = -[K_f \quad K_p] \begin{bmatrix} \dddot{x} \\ x \end{bmatrix} \]  (13)

to control the system and applying the Laplace-transform lead to the transfer functions

\[ G_w(s) = \frac{Y(s)}{W(s)} = \bar{C}(sI - \bar{A} + \bar{BK})^{-1}E = \frac{1}{s} C \left( sI - A + BK_p + \frac{1}{s} BK_f C \right)^{-1} BK_f, \]  (14)

\[ G_d(s) = \frac{Y(s)}{D(s)} = \bar{C}(sI - \bar{A} + \bar{BK})^{-1}E_0 = C \left( sI - A + BK_p + \frac{1}{s} BK_f C \right)^{-1} E. \]  (15)

For \(\omega \to 0\) and \(\omega \to \infty\), it holds

\[ \lim_{\omega \to 0} G_w(j\omega) = C(BK_f C)^{-1}BK_f \xrightarrow{(*)} 1, \quad \lim_{\omega \to 0} G_w(j\omega) = 0, \]  (16)

\[ \lim_{\omega \to \infty} G_d(j\omega) = 0, \quad \lim_{\omega \to \infty} G_d(j\omega) = 0. \]  (17)

\( (*) \): if \(BK_f C\) non-singular.

For slow varying reference, the tracking occurs successfully. High-frequency noises are damped as well as low-frequency noise. This was not the case with the prefilter. However, due to the integrator embedded in the new dynamic for controller design, the response of the system may be slower.

Now, we will try to combines the two techniques in order to have the fast reaction of the prefilter and the noise rejection of integral extension.
3.3. Combination of prefilter and integral extension for reference tracking

Now, we consider the extended plant (21) with the control input

\[ u = -\bar{K}\bar{x} + Vw = -[K_I \; K_P]Z_x + Vw, \quad \text{with} \quad V = -(CA - BK)^{-1}B^{-1} \quad (18) \]

(Figure 5). The resulting transfer functions are

\[ G_w(s) = C \left( sI - A + BK_P + \frac{1}{s}BK_I C \right)^{-1} B \left( \frac{1}{s}K_I + V \right), \quad (19) \]
\[ G_d(s) = C \left( sI - A + BK_P + \frac{1}{s}BK_I C \right)^{-1} E. \quad (20) \]

The noise transfer function is the same as from the extended plant. Thus, it is also robust against low and high-frequency noises. For the transfer function \( G_w(s) \) we have

\[ \lim_{\omega \to 0} G_w(\omega) = C(BK_I C)^{-1}BK \to 1 \quad (21) \]
\[ \lim_{\omega \to \infty} G_w(\omega) = 0. \quad (22) \]

We could not derive some observations about the reference transfer function, but using some examples we want to illustrate it in the next section.

4. Example

In this section, we want to present the result of each methods of reference tracking control on closed-loop behaviour of a dynamic system. We have tested the three approaches on five different dynamic systems. In this section, one of them will be presented.

The chosen system has following dynamics

\[ \dot{x} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ -2 & -3 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d, \]
\[ y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x. \]

For the design of \( K \) and \( V \) we used

\[ p_1 = -2, \; p_2 = -2 \] and \( p_3 = -2.5 \)

as poles, and for \( \bar{K} = [K_I \; K_P] \) we used

\[ p_1 = -2, \; p_2 = -2, \; p_3 = 2.5 \] and \( p_4 = -1.5 \).

For each method, we got following transfer functions

\[ G_w^{pr}(s) = \frac{-5s^2 + 30s + 30}{6s^3 + 33s^2 + 57s + 30} \]
\[ G_w^{in}(s) = \frac{4.5s^4 + 31.53s^3 + 79.94s^2 + 150}{5.62s^2 + 33.78s + 33.78} \]
\[ G_w^{co}(s) = \frac{1.35s^4 + 9.46s^3 + 23.98s^2 + 26.01s + 10.13}{-1.13s^3 + 5.07s^2 + 16.89s + 10.13} \]

\( pr \) for prefilter approach, \( in \) for integral extension, \( co \) for combination

for the reference and

\[ G_d^{pr}(s) = \frac{16s^2 + 221s + 178}{16s^3 + 88s^2 + 152s + 80} \]
\[ G_d^{in}(s) = \frac{4.5s^4 + 31.53s^3 + 79.94s^2 + 86.69s + 33.78}{4.5s^3 + 81.49s^2 + 68.26s} \]
\[ G_d^{co}(s) = \frac{4.5s^4 + 31.53s^3 + 79.94s^2 + 86.69s + 33.78}{4.5s^3 + 81.49s^2 + 68.26s} \]

for the noise. Figure 6 shows the Bode diagram of the reference transfer functions for each approach. For low frequencies, they all have the same magnitude behaviour.
Nevertheless, the prefilter has a higher phase than the integral extension; therefore, it should have a faster reaction and a higher phase margin than the integral extension.

Figure 7 shows the Bode diagram of the noise transfer functions for each method. It shows that the prefilter will transmit low-frequency process noises with an amplification of 6.95 dB while the integral extension will reject them with a higher phase i.e. faster reaction. Both figures show, that from the combination of both approaches applied on the system should result the fast reaction to changes in reference of prefilter and the good noise rejection behaviour of integral extended plant.

The chosen plant was simulated for each controller with the initial state

\[ x_0 = [-1 \quad -1 \quad 2]^T \]

and step reference introduced at \( t = 5s \) with amplitude 10. A step noise with amplitude 3 was introduced at \( t = 10s \). Figure 8 shows on the first graph the output of the system with the prefilter, on the second graph the output of the system with integral extension and on the third graph the output of the system with the combined approach. As expected, the noise was not damped by the prefilter while the integral extension had a slower response. The mixture of prefilter and integral extension has given in this case a faster dynamic than the integral extension and the step noise was successfully rejected.

5. Conclusion

In this paper, we presented two well-known approaches for reference tracking of a control system in time domain using state-space representation and we showed some of their characteristics using the frequency-domain. We first presented the prefilter to compensate the DC-gain of the reference transfer function. The prefilter was not able to reject low-frequency noises, though. Then we presented the integral extension of the state-space model. The integral extension was able to reject the noise, but its reaction for reference tracking is slower. Mixing both approaches eventually yields to a good noise rejection behaviour while increasing the reaction speed of the system. This was tested on five different systems. They all have similar results and one of those examples was presented in Section 4.

6. References

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