Uniform Inference in Nonparametric Predictive Regression, and a Unified Limit Theory for Spatial Density Estimation

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Abstract

Non-technical summary. A problem commonly encountered in time-series econometrics is that of having to relate series with different degrees of persistence, which arises in a particularly acute form in the empirical finance literature on the predictability of excess stock returns. This motivates the consideration of nonlinear predictive regression models, which unlike linear models are capable of relating series with different dependence properties.

However, a major obstacle to the estimation of these nonlinear models in practice has been the absence of the theoretical results needed to justify valid inference in these models. This paper fills this gap in the literature by showing that standard nonparametric tests (based on kernel regression) provide a means of conducting inference in nonlinear regression models that is completely robust to the degree of the persistence in the regressor. This is surprising in view of the known difficulties with the parametric estimation of the linear model in this same context.

Our results thus provide a sound theoretical basis for predictability tests that are robust both to the extent of the possible nonlinearity in the model, and to the dependence properties of the regressor.

Technical summary. A significant problem in predictive regression concerns the invalidity of standard OLS-based inferences when the regressor is highly persistent. Recent work on nonparametric methods has suggested that inference based on these may remain valid in this setting. However, existing results are insufficient to support the conclusion that standard nonparametric testing procedures have the correct asymptotic size, in the sense of controlling asymptotic null rejection probabilities uniformly in the parameters describing the persistence of the regressor. We provide a proof of precisely such a result, thereby establishing the posited validity of these methods. In the course of doing so, we develop novel technical results concerning additive functionals of autoregressive processes exhibiting moderate deviations from a unit root. This leads us to a unified theory for the behaviour of kernel density estimators within a class of processes that includes both stationary and integrated processes, and arrays formed from such processes.

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1 Introduction

Valid inference in a predictive regression – as distinct from a cross-sectional regression – faces two distinctive challenges. The first arises when the regressor is strongly serially dependent. As is now well known, in this case the limiting distribution of OLS is non-pivotal, being not only non-Gaussian but also depending on the unknown degree of persistence of the regressor. This renders conventional inferential procedures – such as referring OLS $t$-statistics to quantiles of the standard Gaussian distribution – invalid. The second difficulty concerns the possibility of a relationship between series with strikingly different dependence properties. For example, when testing for stock return predictability in finance, it is common to confront a series exhibiting martingale-difference-like behaviour, such as excess returns, with a candidate predictor that appears to be integrated (or nearly so), such as the dividend–price ratio. But parametric linear models, though widely used to test for such predictability, imply that both the regressor and the dependent variable should manifest a similar degree of persistence – unless, of course, they are entirely unrelated.

The first of these problems has been the subject of a substantial literature, which has sought to either: develop procedures capable of handling the non-standard limiting distribution of OLS; or to propose novel estimators that remain asymptotically Gaussian, regardless of the persistence of the regressor. However, since this work has all been carried out in a parametric linear regression setting, it goes no way toward addressing the second of the two problems noted above. The successful resolution of that problem seems to lie with nonlinear regression models, since the application of nonlinear transformations to dependent processes has been shown to produce new series with radically different memory properties (Marmer, 2008). The absence of any theoretical priors as to the functional form of these possible nonlinearities leads us naturally to consider nonparametric methods.

Some significant steps in this direction were taken in a recent paper by Kasparis, Andreou, and Phillips (2015, hereafter KAP), who studied the behaviour of kernel regression estimators – and associated $t$-statistic-based tests of non-predictability – within a certain class of strongly dependent regressor processes. Building on earlier work on local time density estimation by Wang and Phillips (2009a,b), the authors showed that, despite the assumed strong dependence of the regressor, nonparametric $t$-statistics enjoy standard Gaussian limits, exactly as they do when the regressors are weakly dependent. Their result holds out the prospect that nonparametric methods may be capable of simultaneously resolving both of the problems identified above: not only do they allow us to estimate models capable of relating series with differing degrees of persistence, but they also yield estimates whose limiting distributions (upon studentisation) are apparently unaffected by the persistence of the regressor.

One would like to be able to conclude that standard nonparametric tests retain their validity, in a predictive regression, regardless of the extent of the serial correlation affect-

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1See, amongst others, Cavanagh, Elliott, and Stock (1995), Campbell and Yogo (2006), Jansson and Moreira (2006), Magdalinos and Phillips (2009), Phillips and Lee (2013), and Elliott, Müller, and Watson (2015).
ing the regressor. Formally, what needs to be shown is that the asymptotic null rejection probabilities of these tests can be controlled uniformly in the parameters describing the persistence of the regressor – which in this paper will be summarised by an autoregressive coefficient $\rho$. But while KAP’s results – together with existing results for stationary (weakly dependent) regressors – are highly suggestive that such control is possible, they are insufficient to sustain any such claim. What we crucially require – and what is missing from the existing literature – are results concerning the asymptotics of kernel regression estimators when regressors are stationary but exhibit ‘moderate deviations from a unit root’ (Phillips and Magdalinos, 2007); we shall term such processes mildly integrated.

The uniformity sought in the present paper requires that we consider triangular arrays of regressor processes, which will be parametrised in terms of $\rho_n = \rho n$. Stationary processes are identified as those for which $\rho_n \to \rho < 1$, whereas local-to-unity processes (the class considered by KAP) have $\rho_n = 1 + O(n^{-1})$. Mildly integrated processes lie on exactly the bridge between these two classes, with $\rho_n \to 1$ but $n(1 - \rho_n) \to \infty$. They therefore inherit some of the properties of both stationary and local-to-unity processes, but are distinct from both, and their treatment requires the development of some genuinely novel limit theory.

The first contribution of the present paper is to show that nonparametric $t$-statistics remain asymptotically Gaussian when regressors are mildly integrated. This result – in conjunction with previous work – is sufficient to permit the conclusion that $t$-statistic-based tests and confidence intervals have the correct asymptotic size, in the sense that the relevant null rejection (or coverage) probabilities are controlled uniformly in the degree of persistence of the regressor (as described by $\rho$; see Section 2). In view of this, nonparametric inference may be conducted in a predictive regression entirely without regard for the possible temporal dependence of the regressor, without thereby endangering the validity of that inference.

Underpinning this finding are some new results concerning the asymptotics of kernel density estimators under mild integration (see Section 3). The proofs of these rely on a combination of arguments appropriate to the stationary and local-to-unity cases. Because the dependence of mildly integrated processes is sufficiently weak, kernel density estimators converge not to the local time of some limiting process, but to a (non-random) standard Gaussian probability density. In this respect, mildly integrated processes are more akin to stationary processes, except for the noted Gaussianity of the limiting density. On the other hand, they also share the diminished recurrence and slower rates of convergence characteristic of local-to-unity processes. In combination with previous work, the results of this paper yield a unified theory for the behaviour of kernel density estimators under all possible values – and drifting sequences – of the autoregressive parameter $\rho$. We refer to the possible limits of these estimators as spatial densities, since – depending on $\{\rho_n\}$ – these may alternately be (non-random) probability densities or (random) local time densities, but not both simultaneously. The technical results of this paper will undoubtedly prove useful for the analysis of other inferential problems, beyond the predictive regression setting considered in this paper.
Proofs of the main results appear in Appendices A–D. Proofs of results that are either straightforward, or closely related to those that have already appeared in the literature, are given in the Supplement, which also provides an index of notation.

Notation. All limits are taken as \( n \to \infty \) unless otherwise stated. For sequences \( \{a_n\}, \{b_n\} \): 

- \( a_n \asymp b_n \) denotes \( \lim_{n \to \infty} a_n/b_n = c \in \mathbb{R}\{0\} \), and \( a_n \sim b_n \) denotes \( \lim_{n \to \infty} a_n/b_n = 1 \). For positive sequences: \( a_n \preceq b_n \) denotes \( \limsup_{n \to \infty} a_n/b_n < \infty \) – equivalently, \( a_n = O(b_n) \). 
- \( a \preceq_C b \) denotes \( a \leq Cb \). For random sequences \( \{x_n\}, \{y_n\} \): \( x_n \preceq_p y_n \) denotes \( x_n = O_p(y_n) \). 
- \( \text{Lip} \) denotes the class of (real-valued) Lipschitz continuous functions on \( \mathbb{R} \), \( \text{BL} \) the class of bounded and Lipschitz functions on \( \mathbb{R} \), and \( \text{BIL} \) the subclass of \( \text{BL} \) functions that are Lebesgue integrable. \( L^p \) denotes the class of Lebesgue \( p \)-integrable functions on \( \mathbb{R} \). \( \rightsquigarrow \) denotes weak convergence in the sense of van der Vaart and Wellner (1996), and \( \rightsquigarrow_{fdd} \) the convergence of finite-dimensional distributions.

2 Uniform inference in nonparametric predictive regression

2.1 Data generating process (DGP)

The inferential problem that motivates our work concerns the nonlinear predictive regression model studied by KAP. The data generating process is described by

\[
y_t = m(x_{t-1}) + u_t \tag{2.1}
\]

where \( m \) and \( \{(x_{t-1}, u_t)\}_{t=1}^{n+1} \) are as per

Assumption DGP.

DGP1 \( m \in \mathcal{M} := \{m_0 : \mathbb{R} \to \mathbb{R} \mid \sup_{x \in \mathbb{R}} |m_0'(x)| < M \} \) for some \( M < \infty \).

DGP2 \( \{\varepsilon_t\}_{t=-\infty}^{\infty} \) is a scalar i.i.d. sequence; \( \varepsilon_0 \) has characteristic function \( \psi_{\varepsilon}(\lambda) := E e^{i\lambda \varepsilon_0} \) satisfying \( \psi_{\varepsilon} \in L^1 \) and a Lebesgue density \( f_{\varepsilon} \in \text{Lip} \) that is everywhere nonzero; 

- \( E\varepsilon_0 = 0 \) and \( E\varepsilon_0^2 = 1 \).

DGP3 \( \{x_t\}_{t=0}^{\infty} \) and \( \{v_t\}_{t=1}^{\infty} \) are generated according to

\[
x_t := \rho x_{t-1} + v_t \quad \quad v_t := \sum_{k=0}^{\infty} \phi_k \varepsilon_{t-k}, \tag{2.2}
\]

with \( x_0 = 0 \); \( \rho \in \mathbb{R} := [-1 + \delta, 1] \) for some \( \delta > 0 \); \( \phi_0 \neq 0 \); \( \sum_{k=0}^{\infty} |\phi_k| < \infty \); and 

- \( \phi := \sum_{k=0}^{\infty} \phi_k \neq 0 \).

DGP4 \( \{u_t\}_{t=1}^{\infty} \) is a martingale difference sequence with respect to \( \mathcal{G}_t := \sigma(\{x_s, u_s\}_{s \leq t}) \), with 

- \( E[u_t^2 \mid \mathcal{G}_{t-1}] = \sigma_u^2 \) a.s. and \( \sup_t E[u_t^2 \mid \mathcal{G}_{t-1}] \leq C < \infty \) a.s.

Remark 2.1. The assumption that \( f_{\varepsilon} \in \text{Lip} \) is used only in the stationary region, i.e. where \( \rho < 1 \). While this requirement could likely be dispensed with, we have retained it here.
so as to facilitate the direct application of results from Wu, Huang, and Huang (2010). 

$f_{\tau} \in \text{Lip}$ is implied, for example, if $\lambda \psi_{t}(\lambda) \in L^{1}$. The strict positivity of $f_{\tau}$ is also assumed merely for convenience, so as to ensure that the stationary solution to (2.2) has a density that is strictly positive at every $x \in \mathbb{R}$, thereby avoiding the possibility that we might (inadvertently) attempt to estimate $m(x)$ at points of zero density.

Remark 2.2. DGP3 is cognate with Assumptions 2.3 and 2.4 in KAP, with the key difference that we do not restrict $\{x_{t}\}$ to the local-to-unity region, in which $\rho = 1 + \frac{c}{n}$ for some fixed $c \in \mathbb{R}$, but instead allow the autoregressive parameter $\rho$ to range over the entirety of $R = [-1 + \delta, 1]$. Our main results easily extend to sequences of parameter spaces of the form $R = R_n := [-1 + \delta, 1 + \frac{c}{n}]$, for some fixed $\bar{c} < \infty$. Owing to the initialisation $x_0 = 0$, the regressor process is nonstationary, regardless of the value of $\rho$. However, when $\rho < 1$, (2.1) admits a stationary solution, which corresponds to the weak limit of $x_n$ as $n \to \infty$.

$\sum_{k=0}^{\infty} |\phi_k| < \infty$ implies that $\{v_t\}$ is a short-memory process, and so excludes the long-memory and anti-persistent cases that are also considered in KAP. It is likely that our results could also be extended to cover these, but we have refrained from considering these in order to keep the paper to a manageable length.

Remark 2.3. DGP4 implies that the regressor $x_{t-1}$ is exogenous, and ensures that $m$ is always identified from $m(x) = \mathbb{E}[y_t \mid x_{t-1} = x]$, regardless of the value of $\rho$. If the model (2.1) were reformulated with $x_t$ in place of $x_{t-1}$, then estimation of $m$ would remain possible when $\rho = 1$ (and, indeed, if $\rho = \rho_n \to 1$), despite the potential endogeneity of the regressor (see Wang and Phillips, 2009b). On the other hand, if $\rho < 1$, $m$ would no longer be identified, and any putative estimate of $m$ would be biased (even asymptotically).

The DGP is thus completely described by the unknown parameters $(m, \rho, \gamma)$, where $\gamma := (\psi, \{\phi_k\}, \sigma_u^2, \{F_{ut}\})$ and $F_{ut}$ denotes the conditional distribution $u_t \mid F_{t-1}$; let $\Gamma$ denote the set of possible values for $\gamma$. Here the regression function $m \in \mathcal{M}$ is the object of interest, whereas $(\rho, \gamma) \in R \times \Gamma$ are merely nuisance parameters. Let $x \in \mathbb{R}$ be given. For a hypothesis such as $H_0 : m(x) = \theta$, the subset of the parameter space consistent with $H_0$ is given by

$$H(\theta) := \{m \in \mathcal{M} \mid m(x) = \theta\} \times R \times \Gamma,$$

whence the size of a test of $H_0$ depends on its maximum rejection probability over all points in $H(\theta)$. In keeping with the literature on the parametric predictive regression problem, in which $\rho \in R$ is a particularly troublesome nuisance parameter – owing to the discontinuity in the (pointwise) limiting distribution of the OLS estimator at $\rho = 1$ – we shall only seek to control the (asymptotic) rejection probability of tests of $H_0$ on the smaller set

$$H^*(\theta) := \{m \in \mathcal{M} \mid m(x) = \theta\} \times R \times \{\gamma\}.$$

(In other words, our asymptotics shall hold $\gamma$ fixed as $n \to \infty$.) Our focus on $H^*(\theta)$, rather than $H(\theta)$, may be justified by the complications posed, even in the present setting, by controlling the (asymptotic) rejection probability of a test of $H_0$, uniformly in the
persistence parameter \( \rho \in \mathbb{R} \). The proof that standard nonparametric testing procedures indeed achieve such size control requires the development of some genuinely new asymptotic theory (see Section 3 below). On the other hand, the passage from \( H^*(\theta) \) to \( H(\theta) \) would merely call for relatively straightforward array extensions of existing results, along with those given in this paper.

2.2 Nonparametric estimation and inference

For our purposes, a suitable estimator of \( m \) in (2.1), at each \( x \in \mathbb{R} \), is provided by the local level (Nadaraya-Watson) kernel regression estimator,

\[
\hat{m}_n(x; h) := \frac{\sum_{t=1}^n K_h(x_t-x) y_{t+1}}{\sum_{t=1}^n K_h(x_t-x)},
\]

where \( K : \mathbb{R} \to \mathbb{R} \) is a smooth probability density, \( h > 0 \) denotes the bandwidth, and \( K_h(x) := h^{-1}K(h^{-1}u) \). We shall suppose \( h = h_n \), for \( \{h_n\} \) a shrinking bandwidth sequence as per

**Assumption SM** (smoothing).

**SM1** \( K \in \text{BIL} \) is positive and compactly supported, with \( \int_{\mathbb{R}} K = 1 \).

**SM2** \( h_n > 0 \) for all \( n \), \( h_n = o(1) \) and \( h_n^{-1} = o(n^{1/2}) \).

**Remark 2.4.** The persistence of \( x_t \), as summarised by \( \rho \), is intimately connected with the recurrence properties of \( x_t \), by which we mean the rate at which the local signal \( S_n(x; h) := \sum_{t=1}^n K_h(x_t-x) \) diverges, for each fixed \( x \in \mathbb{R} \). As is well known, when \( h \) is fixed, in the stationary region (\( \rho_n \to 0 < 1 \)) \( S_n \) diverges at rate \( n \) (probabilistically); whereas when in the local-to-unity region (\( \rho_n = 1 + O(n^{-1}) \)), this rate is reduced to \( n^{1/2} \). Mildly integrated processes are strictly intermediate between these cases, corresponding to a growth rate of \( n(1 - \rho_n^2)^{1/2} \) for \( S_n \). Insofar as \( \rho \) is unknown, the maximum rate at which \( h = h_n \) may shrink to zero, while still permitting the divergence of \( S_n \) – and hence, the consistency of \( \hat{m}_n \) – will thus be determined by the region in which that divergence rate is slowest, i.e. the local-to-unity region. This accounts for the requirement that \( h_n^{-1} = o(n^{1/2}) \) in SM2. This could be relaxed if \( h_n \) were chosen in such a way as to adapt to the (unknown) recurrence of \( \{x_t\} \), but a consideration of such procedures is beyond the scope of this paper.

For a chosen spatial point \( x \in \mathbb{R} \), a test of \( H_0 : m(x) = \theta \) may be based on the nonparametric \( t \)-statistic

\[
\hat{t}_n(x; \theta, h) = s_n(x; h)^{-1} [\hat{m}_n(x; h) - \theta],
\]

where

\[
s_n^2(x; h) := \frac{\hat{\sigma}_n^2(x) \int_{\mathbb{R}} K^2}{h \sum_{t=1}^n K_h(x_t-x)} \quad \hat{\sigma}_n^2(x) := \frac{\sum_{t=1}^n K_{h_n}(x_t-x)y_{t+1} - \hat{m}_n(x)^2}{\sum_{t=1}^n K_{h_n}(x_t-x)}
\]

(2.5)
Test inversion leads to the familiar equal-tailed confidence interval for $m(x)$,

$$C_n(x; h) := \{ \theta \in \mathbb{R} \mid |\hat{i}_n(x; \theta, h)| \leq z_{1-\alpha/2} \} = [\hat{m}_n(x; h) - z_{1-\alpha/2}s_n(x; h), \hat{m}_n(x; h) + z_{1-\alpha/2}s_n(x; h)],$$

where $z_\tau$ denotes the $\tau$th quantile of the standard normal distribution. $C_n(x; h)$ is a ‘pointwise’ confidence interval, in the sense that it concerns the value of $m$ at a particular, fixed $x \in \mathbb{R}$, rather than over a collection of such points.

The (finite-sample) coverage probability of $C_n(x; h_n)$ is given by

$$CP_n(x; m, \rho) := \mathbb{P}_{m, \rho}(m(x) \in C_n(x; h_n)),$$

where $\mathbb{P}_{m, \rho}$ is indexed by the values of $m$ and $\rho$ generating the data. Of course, $\mathbb{P}_{m, \rho}$ also depends on $\gamma$: but in keeping the preceding discussion of the hypothesis testing problem, we shall regard $\gamma$ as fixed, and so suppress it from the notation here. We would like to control the asymptotic size of $C_n(x; h)$, where this is defined as

$$\text{AsySz}(x) := \liminf_{n \to \infty} \inf_{(m, \rho) \in \mathcal{M} \times \mathbb{R}} CP_n(x; m, \rho);$$

a related quantity is the asymptotic maximum coverage probability,

$$\text{AsyMaxCP}(x) := \limsup_{n \to \infty} \sup_{(m, \rho) \in \mathcal{M} \times \mathbb{R}} CP_n(x; m, \rho).$$

It is known from existing results that $\hat{i}_n(x; h_n) \sim N[0, 1]$ for every fixed $(m, \rho) \in \mathcal{M} \times \mathbb{R}$. However, while these results may be suggestive of $C_n$ as having the correct asymptotic size, they are insufficient to support this conclusion, which instead requires that this convergence hold along all drifting sequences $\{(m_n, \rho_n)\} \subset \mathcal{M} \times \mathbb{R}$.

Establishing the required uniformity with respect of $m \in \mathcal{M}$ poses no particular difficulty: due to the linearity of the local level estimator, $m$ affects only the bias of $\hat{m}_n$, and the uniform negligibility of this term follows from standard arguments. On the other hand, handling the nuisance parameter $\rho$ requires more care. Essentially, the problem is one of proving

$$v_n(x) := \frac{h_n^{1/2} \sum_{t=1}^{n} K_{h_n}(x_t - x) u_{t+1}}{\sigma_n \left[ \sum_{t=1}^{n} K_{h_n}(x_t - x) \int K^2 \right]^{1/2}} \sim N[0, 1]$$

along a sufficiently large class of drifting sequences $\{\rho_n\} \subset \mathbb{R}$. By adapting an argument from Andrews and Cheng (2012), it may be shown that, for the purposes of computing $\text{AsySz}(x)$ and $\text{AsyMaxCP}(x)$, it is sufficient to prove that (2.10) holds for the following classes of convergent sequences in $\mathbb{R}$ (see the proof of Proposition 2.1(ii) below):

- stationary (with parameter $\rho$): $\{\rho_n\} \in \mathcal{R}_{ST}^\rho$ if $\rho_n \to \rho \in [-1 + \delta, 1]$;
- mildly integrated: $\{\rho_n\} \in \mathcal{R}_{MI}$ if $\rho_n \to 1$ but $n(\rho_n - 1) \to -\infty$; and
• local to unity (with parameter $c$): $\{\rho_n\} \in \mathcal{R}_{LU}$ if $\rho_n \to 1$ and $n(\rho_n - 1) \to c \in \mathbb{R}$.

We may further restrict $\{\rho_n\} \in \mathcal{R}_{ST}$ so that $\rho_n \in [-1 + \delta, 1)$ for all $n$, and $\{\rho_n\} \in \mathcal{R}_{MI}$ so that $\rho_n \in (0, 1)$ for all $n$. Let $\mathcal{R}_{ST} := \bigcup_{\rho \in [-1 + \delta, 1)} \mathcal{R}_{ST}^\rho$, $\mathcal{R}_{LU} := \bigcup_{c \in \mathbb{R}} \mathcal{R}_{LU}^c$, and $\mathcal{R} := \mathcal{R}_{ST} \cup \mathcal{R}_{MI} \cup \mathcal{R}_{LU}$.

In all cases, the numerator of (2.10) is a martingale, and so is amenable to the application of existing martingale central limit theory. The principal difficulty is thus to show that the conditional variance 

$$\sigma^2_u \sum_{t=1}^{n} K^2_h (x_t - x)$$ 

upon standardisation, converges weakly to an a.s. nonzero limit. Results of this kind are available in the literature for the case when $\{\rho_n\} \in \mathcal{R}_{ST} \cup \mathcal{R}_{LU}$, but not when $\{\rho_n\} \in \mathcal{R}_{MI}$: this necessitates the theoretical work undertaken in Section 3.

2.3 Uniform validity of inferences

Our main result on the uniform validity of tests and pointwise confidence sets is the following. Let $\mathcal{X}$ denote a fixed, finite subset of $\mathbb{R}$. For a function $x \mapsto a(x)$, let $[a(x)]_{x \in \mathcal{X}}$ denote the vector $(a(x_1), \ldots, a(x_m))'$, for $\{x_1, \ldots, x_m\}$ an enumeration of $\mathcal{X}$.

**Proposition 2.1.** Suppose DGP and SM hold, and that additionally $h_n = o(n^{-1/3})$. Then

(i) for every finite $\mathcal{X} \subset \mathbb{R}$,

$$[\hat{t}_n(x; m_n(x), h_n)]_{x \in \mathcal{X}} \rightsquigarrow N[0, I_{\# \mathcal{X}}]$$

along every $\{m_n\} \subset \mathcal{M}$ and $\{\rho_n\} \in \mathcal{R}$; and

(ii) for each $x \in \mathbb{R}$, the confidence intervals $C_n(x; h_n)$ are asymptotically similar, i.e.

$$\text{AsySz}(x) = \text{AsyMaxCP}(x) = 1 - \alpha.$$ 

**Remark 2.5.** Part (i) establishes that any finite collection of $t$ statistics is jointly asymptotically normal and independent across the spatial points $x \in \mathcal{X}$, along all the drifting sequences that are relevant for the size calculation in part (ii). The latter further implies that the $t$-test of $H_0 : m(x) = \theta$ – the inversion of which was used to construct $C_n(x; h_n)$ – is also asymptotically similar.

**Remark 2.6.** $h_n = o(n^{-1/3})$ is required to undersmooth the bias. If DGP1 were strengthened such that the second derivatives of $m \in \mathcal{M}$ were assumed to be uniformly bounded, then it would be possible to relax this requirement to $h_n = o(n^{-1/6})$. Under the null of non-predictability considered below, $m$ is a constant function: in this case the bias of $\hat{m}_n$ vanishes, and the preceding condition on $h_n$ may be dropped entirely.

KAP are particularly concerned with testing the null that $x_{t-1}$ cannot predict $y_t$, which may be formally expressed as

$$H_0 : m(x) = \theta, \ \forall x \in \mathbb{R}.$$ 

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The authors base their tests of \(H_0\) on a vector of \(t\)-statistics, \([\hat{t}_n(x; \theta, h_n)]_{x \in \mathcal{X}}\). The resulting tests are perhaps more correctly regarded as tests of

\[H'_0: m(x) = \theta, \quad \forall x \in \mathcal{X}\]

rather than of \(H_0\), insofar as they only have power against alternatives to \(H'_0\).

Although \(\theta\) is unknown, it is consistently estimable at rate \(n^{-1/2}\) under \(H_0\), uniformly over \(\rho \in \mathbb{R}\), by \(\hat{\theta}_n := \frac{1}{n} \sum_{i=2}^{n+1} y_t\). Accordingly, \([\hat{t}_n(x; \hat{\theta}_n, h_n)]_{x \in \mathcal{X}}\) has the same limiting distribution as \([\hat{t}_n(x; \theta, h_n)]_{x \in \mathcal{X}}\). KAP consider the following test statistics,

\[
\hat{F}_{n,\text{sum}} := \sum_{x \in \mathcal{X}} \hat{t}_n^2(x; \hat{\theta}_n, h_n) \Rightarrow \zeta_{\text{sum}} \quad \hat{F}_{n,\text{max}} := \max_{x \in \mathcal{X}} \hat{t}_n^2(x; \hat{\theta}_n, h_n) \Rightarrow \zeta_{\text{max}}, \quad (2.11)
\]

where, by part (i) of Proposition 2.1, the stated convergence holds along all \(\{\rho_n\} \in \mathcal{R}\) (recall \(m(x) = \theta\) for all \(x \in \mathbb{R}\) under \(H_0\)), for \(\zeta_{\text{sum}}\) having a \(\chi^2(\#\mathcal{X})\) distribution, and \(\zeta_{\text{max}}\) the same distribution as the maximum of \(\#\mathcal{X}\) independent \(\chi^2(1)\) variates. For \(i \in \{\text{sum, max}\}\), let \(c_{\tau,i}\) denote the \(\tau\) quantile of \(\zeta_i\), so that an \(\alpha\)-level test of \(H_0\) based on \(\hat{F}_{n,i}\) rejects if and only if \(\hat{F}_{n,i} \geq c_{1-\alpha,i}\). Thus, by almost identical arguments as were used to prove part (ii) of Proposition 2.1, we have

**Proposition 2.2.** Suppose DGP and SM hold. Then for \(i \in \{\text{sum, max}\}\) and every finite \(\mathcal{X} \subset \mathbb{R}\), a test of \(H_0\) based on \(\hat{F}_{n,i}\) is asymptotically similar, i.e.

\[
\limsup_{n \to \infty} \sup_{\rho \in \mathcal{R}} \mathbb{P}_{0,\rho}\{\hat{F}_{n,i} \geq c_{1-\alpha,i}\} = \liminf_{n \to \infty} \inf_{\rho \in \mathcal{R}} \mathbb{P}_{0,\rho}\{\hat{F}_{n,i} \geq c_{1-\alpha,i}\} = \alpha. \quad (2.12)
\]

**Remark 2.7.** A more satisfying version of Proposition 2.1(i) would extend the result to any finite collection of - data-dependent - points \(\mathcal{X}_n \subset \mathbb{R}\) (with \(\#\mathcal{X}_n\) fixed as \(n \to \infty\)). Unfortunately, such a result is beyond the purview of the available martingale central limit theory, at least under DGP.

Proofs of Propositions 2.1 and 2.2 appear in Appendix A.

### 3 Spatial density estimation: a unified limit theory

#### 3.1 Preliminaries

The preceding section is underpinned by some new results concerning the limiting behaviour of the spatial density estimate \(\hat{c}_n^{-1} \sum_{t=1}^{n} K_{h_n}(x_t - x)\) when \(\{x_t\}\) is mildly integrated – that is, along drifting sequences \(\{\rho_n\} \in \mathcal{R}_{\text{MI}}\) that exhibit moderate deviations from a unit root; here \(e_n := e_n(\{\rho_n\})\) denotes a norming sequence. The proofs of these results in turn rely on the following extension of Theorem 2.1 in Wang and Phillips (2009a, hereafter WP).

We first restate their assumptions, some of which will also be needed here. Let \(\{\tilde{x}_{n,t}\}_{t=1}^{n}\) be a triangular array, \(\{\tilde{F}_{n,t}\}_{t=1}^{n}\) a collection of \(\sigma\)-fields such that each \(\tilde{x}_{n,t}\) is \(\tilde{F}_{n,t}\)-measurable, \(f: \mathbb{R} \to \mathbb{R}\), and define \(\Omega_n(\eta) := \{(l, k) \mid \eta n \leq k \leq (1-\eta)n, \; k + \eta n \leq l \leq n\}\) for some
\( \eta \in (0,1) \).

**Assumption wp.**

\( \text{WP}_1 \) \( f \in L^1 \cap L^2 \).

\( \text{WP}_2 \) There exists a stochastic process \( X(r) \) on \([0,1]\) having continuous local time \( \mathcal{L}_X(r,a) \) such that \( \tilde{x}_{n,[nr]} \rightsquigarrow X(r) \) in \( \ell_\infty([0,1]) \).

\( \text{WP}_3 \) For all \( 0 \leq s < t \leq n, n \geq 1 \), there exist constants \( d_{n,s,t} \) such that

(a) for some \( m_0 > 0 \) and \( C > 0 \), \( \inf_{(s,t) \in \Omega_n(\eta)} \eta^{m_0} / C \) as \( n \to \infty \), and

i. \( \lim_{\eta \to 0} \lim_{n \to \infty} \frac{1}{n} \sum_{t=(1-\eta)n}^{n} d_{n,0,t}^{-1} = 0 \),

ii. \( \lim_{\eta \to 0} \lim_{n \to \infty} \frac{1}{n} \max_{0 \leq s \leq (1-\eta)n} \sum_{t=s+1}^{s+\eta n} d_{n,s,t}^{-1} = 0 \),

iii. \( \limsup_{n \to \infty} \frac{1}{n} \max_{0 \leq s \leq n-1} \sum_{t=s+1}^{n} d_{n,s,t}^{-1} < \infty \);

(b) conditional on \( \tilde{F}_{n,s} \), \( (\tilde{x}_{n,t} - \tilde{x}_{n,s})/d_{n,s,t} \) has a density \( h_{n,s,t}(x) \) with is uniformly bounded by a constant \( K \), and

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{(s,t) \in \Omega_n(\delta^{1/2m_0})} \sup_{|u| \leq \delta} |h_{n,s,t}(u) - h_{n,s,t}(0)| = 0.
\]

Our extension of WP’s Theorem 2.1 consists of replacing \( \text{WP}_2 \) with the following

**Assumption wp (continued).**

\( \text{WP}_2' \) There exists a stochastic process \( \tilde{\mu} : [0,1] \times \mathbb{R} \to \mathbb{R}_+ \), which is continuous a.s. with \( \int_{\mathbb{R}} \tilde{\mu}(r,x) \, dx < \infty \) for all \( r \in [0,1] \), such that for every \( g \in BL \),

\[
\frac{1}{n} \sum_{t=1}^{[nr]} g(\tilde{x}_{n,t} - a) \rightsquigarrow_{fdd} \int_{\mathbb{R}} g(x-a) \tilde{\mu}(r,x) \, dx, \tag{3.1}
\]

over \( (r,a) \in [0,1] \times \mathbb{R} \).

**Proposition 3.1.** Suppose \( \text{WP}_1 \), \( \text{WP}_2' \) and \( \text{WP}_3 \) hold. Then for any \( c_n \to \infty \) and \( c_n/n \to 0 \)

\[
\frac{c_n}{n} \sum_{t=1}^{[nr]} f[c_n(\tilde{x}_{n,t} - a)] \rightsquigarrow_{fdd} \tilde{\mu}(r,a) \int_{\mathbb{R}} f \tag{3.2}
\]

over \( (r,a) \in [0,1] \times \mathbb{R} \).

**Remark 3.1.** While \( \text{WP}_2 \) is certainly sufficient for \( \text{WP}_2' \) with \( \tilde{\mu} = \mathcal{L}_X \), \( \text{WP}_2 \) is unnecessarily strong, being exclusive of certain processes for which (3.2) holds. Indeed, it is evident from Jeganathan (2004) that (3.2) may obtain even if the convergence in \( \text{WP}_2 \) holds only in the sense of the finite-dimensional convergence. The proof of his Lemma 8 implies that
WP2 holds whenever $\tilde{x}_{n,[nr]} \overset{\text{fdd}}{\rightarrow} X(r)$ and $\{\tilde{x}_{n,[nr]}\}$ is asymptotically equicontinuous in probability, in the sense that for every $\epsilon > 0$,
\begin{equation}
\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{|r_1 - r_2| \leq \delta} \mathbb{P}\{|\tilde{x}_{n,[nr_1]} - \tilde{x}_{n,[nr_2]}| > \epsilon\} = 0.
\end{equation}
However, as discussed in more detail in Remark 3.5 below, when $\{\tilde{x}_{n,t}\}$ is derived from a mildly integrated process, even such an apparently weak requirement as (3.3) fails to hold: though the finite-dimensional limit of $\tilde{x}_{n,[nr]}$ exists, it is not separable. For these processes, WP2 must therefore be verified by other means.

Remark 3.2. (3.1) extends straightforwardly, via a suitable choice of approximating BIL functions, to
\begin{equation}
F_n(a) := \frac{1}{n} \sum_{t=1}^{n} \mathbb{1}\{\tilde{x}_{n,t} \leq a\} \overset{\text{fdd}}{\rightarrow} \tilde{\mu}(1, x) \, \text{d}x =: F(a),
\end{equation}
where $F$ is itself a distribution function if $\int \tilde{\mu}(1, x) \, \text{d}x = 1$, as is generally the case. Insofar as (3.4) holds, $F$ may be identified as the spatial distribution associated to the finite-dimensional limit $X$ of $\tilde{x}_{n,[nr]}$. We shall accordingly refer to $x \mapsto \tilde{\mu}(1, x)$ as the spatial density associated to $X$. Some such unifying term is needed here, because depending on the process generating $\tilde{x}_{n,[nr]}$, $\tilde{\mu}(1, x)$ may correspond to either the (non-random) probability density of $X(1)$, or the local time density of $r \mapsto X(r)$, but not both.

3.2 Finite-dimensional convergence

In applying Proposition 3.1 to the setting of DGP, we shall work with the scale-normalised array defined by
\begin{equation}
\tilde{x}_{n,t} := \text{var}(x_n)^{-1/2} x_t =: d_n^{-1} x_t,
\end{equation}
ensuring that the weak limit of $\tilde{x}_{n,n}$ has unit variance in all cases. Proposition 3.1 is broad enough to cover the class of regressor processes contemplated in DGP, even in cases where $\rho = \rho_n$ is allowed to depend on $n$. Indeed, it is the manner in which $\rho_n$ approaches unity (if at all) that determines the spatial density appearing in (3.1). In accordance with the division of the sequences $\{\rho_n\} \in \mathcal{R}$ given in Section 2.2 above, define
\begin{equation}
\mu(r, a; \{\rho_n\}) := \begin{cases} rv_\rho(a) & \text{if } \{\rho_n\} \in \mathcal{R}_{ST}^\rho \\ r\varphi(a) & \text{if } \{\rho_n\} \in \mathcal{R}_{MI} \\ \mathcal{L}_c(r, a) & \text{if } \{\rho_n\} \in \mathcal{R}_{LU}^c \end{cases}
\end{equation}
where $v_\rho$ is the density corresponding to the stationary solution to (2.2), normalised to have unit variance; $\varphi$ is the standard Gaussian density; and $\mathcal{L}_c(r, a)$ is the local time density (at
time \( r \in [0, 1] \) and point \( a \in \mathbb{R} \) associated to the normalised Ornstein–Uhlenbeck process,

\[
J_c(r) := \left( \int_0^1 e^{2(1-s)c} \, ds \right)^{-1/2} \int_0^r e^{(r-s)c} \, dW(s),
\]

(3.7)

for \( W \) a standard Brownian motion on \([0, 1]\).

For the purposes of the next result, \( \{h_n\} \) denotes a deterministic, nonzero bandwidth sequence; recall \( f_h(x) := h^{-1} f(h^{-1} x) \). The proof appears in Appendix B.

**Theorem 3.1.** Suppose DGP holds with \( \rho = \rho_n \) for some \( \{\rho_n\} \in \mathcal{R} \), and \( f \in L^1 \cap L^2 \). Then if \( h_n = o(d_n) \) and \( nd_n^{-1} h_n \to \infty \),

\[
\mu_n(r, a; f, h_n) := \frac{d_n}{n} \sum_{t=1}^{|nr|} f_{h_n}(x_t - d_n a) \sim_{\text{d.f.}} \mu(r, a; \{\rho_n\}) \int_{\mathbb{R}} f,
\]

(3.8)

over \((r, a) \in [0, 1] \times \mathbb{R}\).

**Remark 3.3.** In view of Remark 3.6 below, \( d_n \to \infty \) whenever \( \{\rho_n\} \in \mathcal{R}_{MI} \cup \mathcal{R}_{LU} \), and so the arguments given in the proof of Theorem 3.1 also imply that, in this case,

\[
\frac{d_n}{n} \sum_{t=1}^{|nr|} f_{h_n}(x_t - x) \sim \mu(r, 0; \{\rho_n\}) \int_{\mathbb{R}} f
\]

jointly with (3.8), for each \( x \in \mathbb{R} \).

**Remark 3.4.** The stationary (\( \{\rho_n\} \in \mathcal{R}_{ST} \)) and local-to-unity (\( \{\rho_n\} \in \mathcal{R}_{LU} \)) cases are covered by the results of Wu and Mielniczuk (2002), Wang and Phillips (2009b) and Wu, Huang, and Huang (2010). The proof under mild integration (\( \{\rho_n\} \in \mathcal{R}_{MI} \)) is new to the literature, and the arguments employed are a combination of those appropriate to the stationary and local-to-unity cases.

As in stationary case, one might envisage a ‘direct’ proof of (3.8), by proving the asymptotic negligibility of

\[
\mu_n - \mathbb{E} \mu_n = \frac{d_n}{n} \sum_{t=1}^n [f_{h_n}(x_t) - \mathbb{E} f_{h_n}(x_t)],
\]

and then demonstrating the convergence of \( \mathbb{E} \mu_n \) to the r.h.s. of (3.8) (here we have taken \( a = 0 \) and \( r = 1 \) for simplicity). However, the lesser recurrence of mildly integrated processes, as reflected in the reduced standardisation \( nd_n^{-1} \), significantly complicates the problem. Most notably, straightforward calculations show that the bound given in (13) in Wu, Huang, and Huang (2010) would here imply only that

\[
|\mu_n - \mathbb{E} \mu_n| \lesssim_{\rho} (nh_n)^{-1/2} d_n + n^{-1/2} d_n^3.
\]

(3.9)

Since \( d_n \asymp (1 - \rho_n^2)^{-1} \) under mild integration (see Remark 3.6 below), requiring negligibility of the r.h.s. would thus exclude those \( \{\rho_n\} \in \mathcal{R}_{MI} \) for which \( 1 - \rho_n \lesssim n^{-1/3} \).
The failure of the bound in (3.9) to be useful over the whole of the mildly integrated region necessitates the different proof strategy employed here, which is to use a kind of law of large numbers to establish (3.1) for the standardised series $\tilde{x}_{n,t} = d_n^{-1}x_t$ (see Proposition B.1 below), and then to invoke Proposition 3.1.

**Remark 3.5.** The tripartite classification in (3.6) is reflected in the different possible finite-dimensional limits $X(r; \{\rho_n\})$ of the standardised regressor process $X_n(r) := d_n^{-1}x_{\lfloor nr \rfloor}$. Under both stationarity and mild integration, the relatively weak dependence between $X_n(r_1)$ and $X_n(r_2)$ vanishes in the limit, and so $X$ has the property that $X(r_1)$ and $X(r_2)$ are independent for every $r_1 \neq r_2$. This explains why even such an apparently mild equicontinuity requirement as (3.3) is unavailing for the purposes of proving Theorem 3.1. Only under mild integration (where $d_n \to \infty$) does an invariance principle operate to ensure that the marginals of $X(r)$ are standard Gaussian; in the stationary case, these have density $\nu_\rho$.\(^2\) The limiting $X$ under mild integration thus corresponds to a continuous-time, standard Gaussian white noise process, which we shall denote by $G$.

The strong dependence between $X_n(r_1)$ and $X_n(r_2)$ that is a characteristic of local-to-unity processes ensures that, in this case, $X_n$ converges weakly to the diffusion $J_c$ (see (3.7) above). As $c \to -\infty$, the finite-dimensional distributions of $J_c$ converge to those of $G$, and in this sense there is continuity, in the limit, at the boundary demarcating the mildly integrated and local-to-unity regions.

**Remark 3.6.** When $\{\rho_n\} \in \mathcal{R}_{ST} \cup \mathcal{R}_{MI}$, it may be shown that $d_n^2 \sim n\omega^2_n(\rho_n)\phi^2$, where

$$\omega_n^2(\rho) := \int_0^1 e^{(1-r)n(\rho^2-1)} \, dr = \frac{1}{n(1-\rho^2)} \left[ 1 - e^{-n(1-\rho^2)} \right].$$

(See Section S.1 in the Supplement for the proof.) In particular, $d_n^2 \sim \phi^2(1 - \rho_n^2)^{-1}$ if $\{\rho_n\} \in \mathcal{R}_{MI}$, and $d_n^2 \sim n\phi^2 \int_0^1 e^{2(1-s)c} \, ds$ if $\{\rho_n\} \in \mathcal{R}_{LU}$.

### 3.3 Weak convergence of the spatial process

By adapting some of the arguments from Duffy (2015), and fixing $r = 1$, it is possible to strengthen the conclusion of Theorem 3.1 to weak convergence in $\ell_{ucr}(\mathbb{R})$, the space of bounded real-valued functions on $\mathbb{R}$, equipped with the topology of uniform convergence on compacta. We may also allow the bandwidth to be data-dependent, as well as to depend on the location $a \in \mathbb{R}$. Let $\mu(a; \{\rho_n\}) := \mu(1, a; \{\rho_n\})$.

**Assumption H.** $h_n : \mathbb{R} \to \mathbb{R}_+$ is continuous, with $h_n(a) \in \mathcal{H}_n := [h_n, \overline{h}_n]$ for all $a \in \mathbb{R}$ w.p.a.1., where $\overline{h}_n = o(h_n)$ and $h_n^{-1} = o(nd_n^{-1} \log^{-2} n)$.

**Theorem 3.2.** Suppose H and DGP hold, the latter with $\rho = \rho_n$ for some $\{\rho_n\} \in \mathcal{R}$. Then for any $f \in \text{BIL}$, with $\int_{\mathbb{R}} |f(x)x| \, dx < \infty$,

$$\mu_n(a; f, h_n) := \frac{d_n}{n} \sum_{t=1}^n f_{h_n(a)}(x_t - d_n a) \sim \mu(a; \{\rho_n\}) \int_{\mathbb{R}} f$$

\(^2\)Under mild integration, these assertions follow from arguments given in the proof of Proposition B.1(ii).
in $\ell_{ucc}(\mathbb{R})$.

Remark 3.7. The result may be extended to a broader class of functions than BIL, such as is allowed for by Theorem 3.1 in Duffy (2015), by means of a similar bracketing argument as is given in the proof of that result.

The proof of Theorem 3.2 appears in Appendix D.

4 Conclusion

This paper has established the validity of conventional nonparametric inferences in a predictive regression, where the degree of persistence of the regressor is unknown (and possibly very high). The opens the way for the systematic application of nonparametric methods to predictive regression, in which setting they enjoy the considerable advantage of being able to relate series with radically different memory properties. Our work on this problem has necessitated the development of some new limit theory for kernel density estimators, in the presence of mildly integrated processes. These new results fill an important gap in the existing literature, and allow for a unified treatment of these estimators in an autoregressive setting.

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A Proof of uniform validity of inferences

Here, as throughout the remainder of the Appendices (excepting Section B.1) and the Supplement, Assumptions DGP and SM are always maintained, even when not explicitly referenced. Let $e_n := e_n(\{\rho_n\}) := nd_n^{-1}$, where $d_n := \text{var}(x_n)^{1/2}$; we note that $e_n(\{\rho_n\}) \leq n$ for all $\{\rho_n\} \in R$.

Notation. For $p \in (1, \infty)$ and a function $f : \mathbb{R} \to \mathbb{R}$, $\|f\|_p := (\int |f(x)|^p \, dx)^{1/p}$ and $\|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)|$; for a random variable $X$, $\|X\|_p := (\mathbb{E}|X|^p)^{1/p}$, and $\|X\|_\infty$ denotes the essential supremum of $X$. BI denotes the class of bounded and Lebesgue-integrable functions on $\mathbb{R}$. $C$, $C_1$, etc., denote generic constants which may take on different values even at different places in the same proof. In keeping with the discussion of the nuisance parameters $\gamma$ in Section 2.1 above, any dependence of these constants on $\gamma$ is ignored throughout.
Proofs of the following auxiliary results appear in Section S.2 of the Supplement.

**Lemma A.1.** Suppose \( \{\rho_n\} \in \mathcal{R} \). Then there exists a \( C < \infty \) such that

\[
\frac{1}{e_n} \sum_{t=1}^{n} \mathbb{E}[|f(x_t)|] \leq C \|f\|_1
\]

**Lemma A.2.** For every \( x \in \mathbb{R} \), \( \sigma_u^2(x) = \sigma^2_u + o_p(1) \).

*Proof of Proposition 2.1(i).* Let \( x \in \mathbb{R} \), and \( \{m_n\} \) and \( \{\rho_n\} \) be as in the statement of the proposition. By Lemma A.2, straightforward calculations yield

\[
\hat{t}_n(x; m_n(x), h_n) = [v_n(x) + b_n(x)](1 + o_p(1))
\]

where \( v_n(x) \) is as in (2.10) above, and

\[
b_n(x) := \frac{h_n^{1/2} \sum_{t=1}^{n} K_{h_n}(x_t - x)[m_n(x_t) - m_n(x)]}{\sigma_u \left( \int_{\mathbb{R}} K^2 \sum_{t=1}^{n} K_{h_n}(x_t - x) \right)^{1/2}} = \frac{b_{n,1}(x)}{b_{n,2}(x)}.
\]

(A.1)

Under DGP1, a mean-value expansion gives

\[
|b_{n,1}(x)| \leq h_n^{3/2} \|m_n\|_\infty \sum_{t=1}^{n} K_{h_n}^{[1]}(x_t - x) \ll_p h_n^{3/2} e_n
\]

(A.2)

by Lemma A.1, since \( K_{h_n}^{[1]}(u) := K(u)|u| \in L^1 \). Also by Theorem 3.1,

\[
e_n^{-1/2} b_{n,2}(x) \prec \eta(x) := \sigma_u \left( \int_{\mathbb{R}} K^2 \right)^{1/2} \begin{cases} \nu_p(\sigma^{-1}_p x) & \text{if } \{\rho_n\} \in \mathcal{R}^p \setminus \mathcal{S} \\ \varphi(0) & \text{if } \{\rho_n\} \in \mathcal{R}_{\text{MI}} \\ \mathcal{L}_c(1,0) & \text{if } \{\rho_n\} \in \mathcal{R}_{\text{LU}} \end{cases}
\]

(A.3)

which is strictly positive and finite a.s. (see Remark 2.1 above); here \( \sigma_p^2 \) denotes the variance of the stationary solution to (2.2). Together (A.1)–(A.3) yield \( |b_n(x)| \ll_p h_n^{3/2} e_n^{1/2} \), which is \( o(1) \) since \( h_n = o(n^{-1/3}) \), and hence

\[
\hat{t}_n(x; m_n(x), h_n) = v_n(x)[1 + o_p(1)].
\]

The joint limiting distribution of \( [\hat{t}_n(x; m_n(x), h_n)]_{x \in \mathcal{X}} \) can thus be obtained via an application of an appropriate martingale CLT, and the Cramèr–Wold device. Consider

\[
M_n(x) := \left( \frac{h_n}{e_n} \right)^{1/2} \sum_{t=1}^{n} K_{h_n}(x_t - x) u_{t+1}.
\]

(A.4)

Under DGP4, \( M_n \) is a martingale with conditional variance

\[
\langle M_n(x) \rangle = \frac{\sigma_u^2}{e_n} \sum_{t=1}^{n} L_{h_n}(x_t - x) \prec \eta^2(x),
\]

(A.5)
where $L(u) := K^2(u)$. Furthermore, the (standardised) summands in (A.4) satisfy a conditional Lyapunov condition, since under DGP4

$$
\left( \frac{h_n}{e_n} \right)^2 \sum_{t=1}^n [K_{h_n}(x_t - x)]^4 \cdot \mathbb{E}[|u_{t+1}|^4 \mid G_t] \lesssim_p \frac{1}{e_n h_n} = o(1),
$$

using Lemma A.1. When $\{\rho_n\} \in \mathcal{R}_{ST}^\rho \cup \mathcal{R}_{MI}$, the r.h.s. of (A.5) is non-random, and so the asymptotic Gaussianity of (A.4) follows from Theorem 3.2 in Hall and Heyde (1980). When $\{\rho_n\} \in \mathcal{R}_{LU}$, we note further that

$$
\left( \frac{h_n}{n e_n} \right)^{1/2} \sum_{t=1}^n K_{h_n}(x_t - x)|\mathbb{E}[\varepsilon_{t+1} u_{t+1}]| \leq \sigma_u \left( \frac{h_n}{n e_n} \right)^{1/2} \sum_{t=1}^n K_{h_n}(x_t - x) = o(1)
$$

by the Cauchy-Schwarz inequality. Thus, an appeal to Theorem 2.1 of Wang (2014), together with the preceding, ensures that for all $\{\rho_n\} \in \mathcal{R}$,

$$
M_n(x) \rightsquigarrow \xi \eta(x),
$$

where $\xi = d N[0, 1]$, independent of $\eta$; the preceding holds jointly with (A.3).

Finally, regarding the joint limiting distribution of the vector $[M_n(x)]_{x \in \mathcal{X}}$, we note that, for any $x \neq x'$ and $\alpha_x, \alpha_{x'} \in \mathbb{R}$,

$$
\{\alpha_x M_n(x) + \alpha_{x'} M_n(x')\}
$$

$$
\begin{align*}
&= \frac{\sigma^2}{e_n} \sum_{t=1}^n \left( \alpha_x K_{h_n}(x_t - x) + \alpha_{x'} K_{h_n}(x_t - x') \right)^2 \\
&= \frac{\sigma^2}{e_n} \sum_{t=1}^n \left[ \alpha_x^2 L_{h_n}(x_t - x) + \alpha_{x'}^2 L_{h_n}(x_t - x') \right] + \alpha_x \alpha_{x'} \frac{\sigma^2}{e_n} \sum_{t=1}^n g_n(x_t),
\end{align*}
$$

where $L(u) := K^2(u)$ and $g_n(u) := h_n \cdot K_{h_n}(u - x) \cdot K_{h_n}(u - x')$. By Lemma A.1,

$$
\frac{1}{e_n} \sum_{t=1}^n |g_n(x_t)| \gtrsim_p \|g_n\|_1 = h_n \int_\mathcal{X} K_{h_n}(u - x) \cdot K_{h_n}(u - x') du = o(1),
$$

and so the Cramér–Wold device yields

$$
[M_n(x)]_{x \in \mathcal{X}} \rightsquigarrow [\xi(\cdot) \eta(\cdot)]_{x \in \mathcal{X}},
$$

where $[\xi(\cdot)]_{x \in \mathcal{X}} = d N[0, I_{\#\mathcal{X}}]$, independent of $[\eta(\cdot)]_{x \in \mathcal{X}}$. Since this occurs jointly with the convergence in (A.3), the result follows. \hfill \Box

**Proof of Proposition 2.1(ii).** Let $x \in \mathbb{R}$. We shall only give the proof that AsySz$(x) = 1 - \alpha$; the proof that AsyMaxCP$(x) = 1 - \alpha$ is analogous. Our arguments largely follow those given in the proof of Lemma 2.1 in Andrews and Cheng (2012). Let $\vartheta := (m, \rho) \in \mathcal{M} \times \mathbb{R} =: \Theta$. For a given $\vartheta_n = (m_n, \rho_n)$, we write $\{\vartheta_n\} \in \mathcal{R}$ to signify that $\{\rho_n\} \in \mathcal{R}$, and
similarly for the sets $\mathcal{R}_{ST}^\rho$, $\mathcal{R}_{MI}^\rho$, and $\mathcal{R}_{LU}^\rho$. We first note that for any $\{\vartheta^*_n\} \in \mathcal{R}$,

$$
\lim_{n \to \infty} \text{CP}_n(x; \vartheta^*_n) = \lim_{n \to \infty} \mathbb{P}_{m_n^*, \vartheta^*_n} \{m^*_n(x) \in C_n(x; h_n)\}
= \lim_{n \to \infty} \mathbb{P}_{m_n^*, \vartheta^*_n} \{|\hat{t}_n(x; m^*_n(x), h_n)| \leq z_{1-\alpha/2}\}
= 1 - \alpha,
$$

(\text{A.6})

where we have used (2.7), (2.6), and part (i) of the proposition, with this last ensuring that $\hat{t}_n(x; m^*_n(x), h_n) \sim N[0, 1]$ under $\{\vartheta^*_n\}$.

In view of (2.8), there is a sequence $\{\vartheta_n\} \subset \Theta$ and a subsequence $\{p_n\}$ such that

$$
\text{AsySz}(x) = \liminf_{n \to \infty} \text{CP}_n(x; \vartheta_n) = \lim_{n \to \infty} \text{CP}_{p_n}(x; \vartheta_{p_n}).
$$

(\text{A.7})

Since $\mathcal{R}$ is compact, we may choose $\{p_n\}$ such that $\{\rho_{p_n}\}$ also converges to some $\rho \in \mathcal{R}$. Let $c_n := n(\rho_n - 1)$, and let $\{w_n\} \subset \{p_n\}$ be a further subsequence, to be chosen below. Now either:

(i) $\rho < 1$: we may choose $\{w_n\}$ such that $\rho_{w_n} < 1$ for all $n$; or
(ii) $\rho = 1$: then either:

(a) $c_{p_n}$ is bounded: choose $\{w_n\}$ such that $c_{w_n} \to c \in \mathbb{R}$; or
(b) $c_{p_n}$ is unbounded: choose $\{w_n\}$ such that $c_{w_n} \to -\infty$, and $c_{w_n} < 0$ for all $n$.

Consistent with the preceding division, consider a new sequence $\{\vartheta^*_n\}$ satisfying either:

(i) $\vartheta^*_n = \vartheta_{w_{k-1}}$ for $w_{k-1} < n \leq w_k$; or
(ii) $\vartheta^*_n = (m_{w_k}, 1 + n^{-1}c_{w_k})$ for $w_{k-1} < n \leq w_k$;

as appropriate. Then $\vartheta^*_{w_n} = \vartheta_{w_n}$ for all $n$ by construction, and either: (i) $\{\vartheta^*_n\} \in \mathcal{R}_{ST}^\rho$; (ii)(a) $\{\vartheta^*_n\} \in \mathcal{R}_{LU}^\rho$; or (ii)(b) $\{\vartheta^*_n\} \in \mathcal{R}_{MI}$. Thus $\{\vartheta^*_n\} \in \mathcal{R}$ in all cases, and so

$$
1 - \alpha = \lim_{n \to \infty} \text{CP}_n(x; \vartheta^*_n) = \lim_{n \to \infty} \text{CP}_{w_n}(x; \vartheta^*_{w_n})
= \lim_{n \to \infty} \text{CP}_{w_n}(x; \vartheta_{w_n}) = \lim_{n \to \infty} \text{CP}_{p_n}(x; \vartheta_{p_n}) = \text{AsySz}(x).
$$

by (A.6) and then (A.7).

\[ \square \]

**Proof of Proposition 2.2.** Under $H_0$, $\mathbb{E}(\hat{t}_n - \theta)^2 = n^{-1}\sigma^2_u$, and so $\hat{t}_n$ is $n^{-1/2}$-consistent. Therefore, the same arguments as were employed in the proof of Proposition 2.1(i) now give

$$
[\hat{t}_n(x; \hat{\theta}_n, h_n)]_{x \in \mathcal{X}} = [\hat{t}_n(x; \theta, h_n)]_{x \in \mathcal{X}} + o_p(1) \sim N[0, I_{*\mathcal{X}}],
$$

(\text{A.8})

along every $\{\rho_n\} \in \mathcal{R}$. (Note that since $m = 0$ under the null, the nonparametric estimator $\hat{m}_n$ has no bias, and so $h_n = o(n^{-1/3})$ is not needed to prove (A.8).) Hence, by the
uniform inference in nonparametric predictive regression

\[ \limsup_{n \to \infty} \mathbb{P}_{0,\rho_n} \{ \hat{F}_{n,i} \geq c_{1-\alpha,i} \} = \liminf_{n \to \infty} \mathbb{P}_{0,\rho_n} \{ \hat{F}_{n,i} \geq c_{1-\alpha,i} \} = \alpha \quad (A.9) \]

for \( i \in \{ \text{sum, max} \} \), along every \( \{ \rho_n \} \in \mathcal{R} \). The passage from (A.9) to (2.12) now follows exactly the same lines as the proof of Proposition 2.1(ii). \( \square \)

B Proof of finite-dimensional convergence

B.1 Proof of Proposition 3.1

Similarly to the proof of Theorem 2.1 in Wang and Phillips (2009a), define

\[ L_n(r, a) := \frac{c_n}{n} \sum_{k=1}^{\lfloor nr \rfloor} f(c_n(\tilde{x}_{k,n} - a)) \]
\[ L_n,\epsilon(r, a) := \frac{c_n}{n} \sum_{k=1}^{\lfloor nr \rfloor} \int_{\mathbb{R}} f(c_n(\tilde{x}_{k,n} - a + z\epsilon)) \varphi(z) \, dz, \]

and set \( \varphi_\epsilon(x) := e^{-1} \varphi(e^{-1} x) \). It follows from Lemma 7 in Jeganathan (2004) that, for each \( \epsilon > 0 \) fixed, there is a non-random \( \delta_n = o(1) \) such that

\[ \left| L_n,\epsilon(r, a) - \frac{T_n}{n} \sum_{k=1}^{\lfloor nr \rfloor} \varphi_\epsilon(\tilde{x}_{k,n} - a) \int_{\mathbb{R}} f \right| \leq \delta_n \to 0. \]

Furthermore, the arguments used by Wang and Phillips (2009a) to prove that

\[ \lim_{\epsilon \to 0} \lim_{n \to \infty} \mathbb{E}|L_n(r, a) - L_n,\epsilon(r, a)| = 0, \]

for each \( a \in \mathbb{R} \), which corresponds to (5.1) in that paper, require only their Assumptions 2.1 and 2.3, both of which are maintained here (as WP1 and WP3 respectively). Finally, by WP2',

\[ \frac{1}{n} \sum_{k=1}^{\lfloor nr \rfloor} \varphi_\epsilon(\tilde{x}_{k,n} - a) \sim_{fdd} \int_{\mathbb{R}} \varphi_\epsilon(x - a) \mu(r, x) \, dx \]
\[ = \int_{\mathbb{R}} \varphi(x) \mu(r, \epsilon x + a) \, dx = \mu(r, a) + o_p(1) \]

over \( (r, a) \in [0, 1] \times \mathbb{R} \) as \( n \to \infty \) and then \( \epsilon \to 0 \), since \( \mu \) is continuous a.s. \( \square \)

B.2 Proof of Theorem 3.1

\( \{ \rho_n \} \in \mathcal{R}_{LU} \). Proposition 7.1 in Wang and Phillips (2009b), together with the arguments used to prove their Proposition 7.2, establish that \( \{ \tilde{x}_{n,t} \} \) satisfies WP2 and WP3. (Technically, the authors only consider sequences of the form \( \rho_n = 1 + c/n \) for fixed \( c \in \mathbb{R} \),
but their arguments clearly carry over to the slightly more general situation in which $ho_n = 1 + \frac{c}{n}$ for $c_n \to c \in \mathbb{R}$, as permitted by $\mathcal{R}_{LU}$. Thus, in this case, the result follows by Proposition 3.1.

$\{\rho_n\} \in \mathcal{R}_{MI}$. In this case, we shall need the following two results, the proofs of which are given in Appendix C. Recall the definition of $\tilde{x}_{n,t}$ given in (3.5) above.

**Proposition B.1.** Suppose $g \in BL$, $\{\rho_n\} \in \mathcal{R}_{MI}$. Then

(i) $\frac{1}{n} \sum_{t=1}^{[nr]} g(\tilde{x}_{n,t}) = \frac{1}{n} \sum_{t=1}^{[nr]} \mathbb{E}g(\tilde{x}_{n,t}) + o_p(1)$; and

(ii) $\frac{1}{n} \sum_{t=1}^{[nr]} \mathbb{E}g(\tilde{x}_{n,t}) \to r \int_{\mathbb{R}} g(x)\varphi(x) \, dx$.

**Proposition B.2.** Suppose $\{\rho_n\} \in \mathcal{R}_{MI}$. Then $\tilde{x}_{n,t}$ satisfies wp3 with $\tilde{F}_{n,t} := \sigma(\{\varepsilon_s\}_{s \leq t})$.

It follows immediately from Proposition B.1 that for every $g \in BL$,

$$
\frac{1}{n} \sum_{t=1}^{[nr]} g(\tilde{x}_{n,t} - a) = \frac{1}{n} \sum_{t=1}^{[nr]} \mathbb{E}g(\tilde{x}_{n,t} - a) + o_p(1) \xrightarrow{r} r \int_{\mathbb{R}} g(x-a)\varphi(x) \, dx
$$

for each $(r, a) \in [0,1] \times \mathbb{R}$. Thus wp2' holds with $\tilde{\mu}(r,a) = r\varphi(a)$. By Proposition B.2, $\{\tilde{x}_{n,t}\}$ satisfies wp3, whence the result follows by Proposition 3.1.

$\{\rho_n\} \in \mathcal{R}_{ST}$. Since $d_n \preceq 1$ in this case, it follows from Theorem 1 in Wu, Huang, and Huang (2010), with minor modifications, that

$$
\mu_n(r,a;f,h_n) = \frac{d_n}{n} \sum_{t=1}^{[nr]} \mathbb{E}f_{h_n}(x_t - d_n a) + o_p(1).
$$

Let $p_{\rho,t}$ and $\psi_{\rho,t}$ respectively denote the Lebesgue density and characteristic function of $x_t$, and $p_\rho$ and $\psi_\rho$ those of the stationary solution to (2.2), for $\rho < 1$. The sequence $\{\psi_{\rho,t}\}_{t=1}^{\infty}$ is uniformly integrable, since $|\psi_{\rho,t}(\lambda)| \leq |\psi_{\bar{t}}(\lambda)|$ for all $t$, and $\psi_{\bar{t}} \in L^1$.

Let $t_n \in \{1, \ldots, n\}$ with $t_n \to \infty$. Since $\rho_n$ is bounded away from unity, $\psi_{\rho_n,t_n}(\lambda) - \psi_{\rho_n}(\lambda) \to 0$ for each $\lambda \in \mathbb{R}$, and thus

$$
\|p_{\rho_n,t_n} - p_{\rho_n}\|_\infty \leq \int_{\{\lambda \leq A\}} |\psi_{\rho_n,t_n}(\lambda) - \psi_{\rho_n}(\lambda)| \, d\lambda + \int_{\{\lambda > A\}} [||\psi_{\rho_n,t_n}(\lambda)| + |\psi_{\rho_n}(\lambda)|] \, d\lambda
$$

$$
\to 0,
$$

as $n \to \infty$ and then $A \to \infty$; a similar argument yields $\|p_{\rho_n} - p_\rho\|_\infty \to 0$, as $\rho_n \to \rho < 1$. Thus

$$
\mathbb{E}f_{h_n}(x_{t_n} - d_n a) = \int_{\mathbb{R}} f(x)p_{\rho_n,t_n}(d_n a + h_n x) \, dx
$$

$$
\quad = \int_{\mathbb{R}} f(x)p_\rho(d_n a + h_n x) \, dx + o(1) \to p_\rho(\sigma_n a) \int_{\mathbb{R}} f,
$$

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where we have used the fact that \( d_n \to \sigma_\rho \), the standard deviation of the stationary solution to (2.2). Thus
\[
\frac{d_n}{n} \sum_{t=1}^{[nr]} \mathbb{E} f_{h_n}(x_t - d_n a) \to r \sigma_\rho \rho_a(\sigma_\rho a) \int_{\mathbb{R}} f = r \nu_\rho(a) \int_{\mathbb{R}} f.
\]

\[\square\]

C Proofs of auxiliary results for mild integration

C.1 Preliminaries

Under DGP, we may regard \( x_t \) as a nonstationary linear process, i.e. a weighted sum of the underlying i.i.d. \( \epsilon_t \)'s. Direct calculations show that \( x_t = \sum_{k=0}^{\infty} a_{t,t-k} \epsilon_{t-k}, \) where
\[
a_{t,t-k} := a_{t,t-k}(\rho) := \sum_{l=0}^{k \wedge (t-1)} \rho^l \phi_{t-l}.
\]

Observe that this quantity does not depend on \( t \) for \( k \leq t - 1 \). Accordingly, we define
\[
a_k := a_k(\rho) := \sum_{l=0}^{k} \rho^l \phi_{t-l} = a_{t,t-k}(\rho)
\]
for \( 0 \leq k \leq t - 1 \).

We shall make frequent use, throughout the following, of the decomposition
\[
x_t = \sum_{k=0}^{\infty} a_{t,t-k} \epsilon_{t-k} = \sum_{k=t-s+1}^{\infty} a_{t,t-k} \epsilon_{t-k} + \sum_{k=0}^{t-s} a_k \epsilon_{t-k} =: x_{-\infty,s-1,t} + x_{s,t,t},\tag{C.3}
\]
for \( s \in \{1, \ldots, t\} \); here \( x_{-\infty,s-1,t} \) and \( x_{s,t,t} \) are independent, being \( \mathcal{F}_{-\infty}^{s-1} \) and \( \mathcal{F}_s \)-measurable respectively, for \( \mathcal{F}_s := \sigma(\{\epsilon_r \}_{r=s}) \). For \( r \in \{s+1, \ldots, t-1\} \), \( x_{s,t,t} \) further decomposes as
\[
x_{s,t,t} = \sum_{k=s}^{t} a_{t-k} \epsilon_k = \sum_{k=s}^{t-1} a_{t-k} \epsilon_k + \sum_{k=r+1}^{t} a_{t-k} \epsilon_k = x_{s,r,t} + x_{r+1,t,t},\tag{C.4}
\]
where \( x_{s,r,t} \) and \( x_{r+1,t,t} \) are respectively \( \mathcal{F}_s \)- and \( \mathcal{F}_{r+1} \)-measurable. Taking \( s = 1 \) in (C.3), we have by independence that
\[
\mathbb{E} x_t^2 = \sum_{k=0}^{\infty} a_{t,t-k}^2 = \sum_{k=0}^{t-1} \left( \sum_{l=0}^{k} \rho^{k-l} \phi_l \right)^2 + \sum_{k=t}^{\infty} \left( \sum_{l=0}^{t-1} \rho^l \phi_{t-l} \right)^2 =: V_{1,t}(\rho) + V_{2,t}(\rho),\tag{C.5}
\]
where \( V_{1,t} \) can be rewritten as
\[
V_{1,t}(\rho) = \sum_{i=0}^{t-1} \phi_i \sum_{k=0}^{t-i-1} \rho^{2k} + 2 \sum_{i=0}^{t-1} \sum_{j=i+1}^{t-1} \phi_i \phi_j \sum_{k=0}^{t-j-1} \rho^{2k}\tag{C.6}
\]
(see Section S.3 of the Supplement for details).

Define $k_n := k_n(\{\rho_n\})$ to be the largest integer for which

\[ k_n(\{\rho_n\}) \leq \begin{cases} 
((1 - \rho_n)^{-1} \land n)/2 & \text{if } \{\rho_n\} \in \mathcal{R}_{MI} \\
n/2 & \text{if } \{\rho_n\} \in \mathcal{R}_{LU},
\end{cases} \tag{C.7} \]

for each $n$ sufficiently large. Observe that $k_n \asymp d_n^2$ in both cases. Proofs of the following elementary results are given in Section S.3 of the Supplement.

**Lemma C.1.** Suppose $\{\rho_n\} \in \mathcal{R}_{MI} \cup \mathcal{R}_{LU}$. Then

(i) there exist $0 < a < \pi < \infty$ and $k_0, n_0 \in \mathbb{N}$ with $k_0$ even, such that

\[ a \leq \min_{k_0/2 \leq k \leq 2k_0} |a_k(\rho_n)| \leq \max_{0 \leq k \leq n} |a_k(\rho_n)| \leq \pi \tag{C.8} \]

for all $n \geq n_0$; and

(ii) there exists a $\gamma \in (0, \infty)$ such that

\[ \sup_{n \geq n_0} \max_{k_0/2 \leq k \leq 2k_0} |\psi_k[a_k(\rho_n)]\lambda| \leq \begin{cases} 
e^{-\gamma\lambda^2} & \text{if } |\lambda| \leq 1, \\
e^{-\gamma} & \text{if } |\lambda| \geq 1.
\end{cases} \]

**Lemma C.2.** Suppose $\{\rho_n\} \in \mathcal{R}_{MI}$. Then

(i) $\rho_n^{\text{ref}} \to 0$ for any $\epsilon > 0$;

(ii) $(1 - \rho_n^{2}) \sim 2(1 - \rho_n)$.

### C.2 Proofs of Propositions B.1–B.2

We first state and prove the following auxiliary lemma, which which is the key ingredient in the proof of the first part of Proposition B.1. For a function $g \in BL$, let $\|g\|_{\text{Lip}} := \sup_{x \neq y}|g(x) - g(y)|/|x - y|$.

**Lemma C.3.** For any $g \in BL$,

\[ \mathbb{E} \left[ \sum_{t=1}^{n} |g(x_t) - \mathbb{E}g(x_t)| \right] \leq \|g\|_{\text{Lip}} \sum_{k=0}^{\infty} \left( \sum_{l=1}^{n} \phi_{l,k}^2 \right)^{1/2} \leq \|g\|_{\text{Lip}} n^{1/2} \sum_{k=0}^{\infty} |\phi_k| / (1 - |\rho|), \tag{C.9} \]

where the second inequality holds if $|\rho| < 1$.

**Proof.** Let $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_{t-1}]$. Decompose

\[ Q_n := \sum_{t=1}^{n} [g(x_t) - \mathbb{E}g(x_t)] = \sum_{k=0}^{\infty} \sum_{t=1}^{n} [\mathbb{E}_{t-k} g(x_t) - \mathbb{E}_{(t-1)-k} g(x_t)] = \sum_{k=0}^{\infty} M_{nk}. \tag{C.10} \]
Clearly, by the orthogonality of martingale differences,

\[ \mathbb{E}M_{nk}^2 = \sum_{t=1}^{n} \mathbb{E}[E_{t-k}g(x_t) - E_{(t-1)-k}g(x_t)]^2. \]  

(C.11)

Now by the decomposition (C.3),

\[ x_t = \sum_{s=0}^{k-1} a_{t,t-s} \varepsilon_{t-s} + a_{t,t-k} \varepsilon_{t-k} + \sum_{s=k+1}^{\infty} a_{t,t-s} \varepsilon_{t-s} \]
\[ = d \sum_{s=0}^{k-1} a_{t,t-s} \varepsilon_{t-s} + a_{t,t-k} \varepsilon^* + \sum_{s=k+1}^{\infty} a_{t,t-s} \varepsilon_{t-s} =: x_t^*, \]

where \( \varepsilon^* = d \varepsilon_0 \) is independent of \( \{ \varepsilon_t \} \), and hence also of \( \mathcal{F}_{t-k} \). Thus \( E_{(t-1)-k}g(x_t) = E_{t-k}g(x_t^*), \) whence

\[ |E_{t-k}g(x_t) - E_{(t-1)-k}g(x_t)| = |E_{t-k}[g(x_t) - g(x_t^*)]| \leq \|g\|_{\text{Lip}} a_{t,t-k} E_{t-k} |\varepsilon_{t-k} - \varepsilon^*|. \]

Hence, by (C.11) and Jensen’s inequality, and recalling that \( \sigma_{\varepsilon}^2 = 1, \)

\[ \mathbb{E}M_{nk}^2 \leq 2\|g\|_{\text{Lip}}^2 a_{t,t-k}^2, \]

which together with (C.10) yields the first inequality in (C.9).

For the second inequality, we recall from (C.1) that

\[ |a_{t,t-k}| \leq \sum_{l=0}^{n-1} |\rho|^l |\phi_{k-l}| =: b_{n,k} \]

for \( 1 \leq t \leq n \), with the convention that \( \phi_{-l} := 0 \) for \( l > 0 \). Hence if \( |\rho| < 1, \)

\[ \sum_{k=0}^{\infty} \left( \sum_{l=1}^{n} a_{t,l-k}^2 \right)^{1/2} \leq n^{1/2} \sum_{k=0}^{\infty} b_{n,k} = n^{1/2} \sum_{l=0}^{n-1} |\rho|^l \sum_{k=0}^{\infty} |\phi_{k-l}| \leq n^{1/2} \sum_{k=0}^{\infty} |\phi_k| \frac{1}{1 - |\rho|}. \]

Proof of Proposition B.1(i). We take \( r = 1 \) for simplicity; the proof for fixed \( r \in (0,1) \) is analogous. When \( \rho \in (0,1) \), applying Lemma C.3 to the unstandardised process \( \{ x_t \} \) gives the bound

\[ \mathbb{E} \left| \sum_{t=1}^{n} [g(x_t) - \mathbb{E}g(x_t)] \right| \leq \|g\|_{\text{Lip}} n^{1/2} \sum_{k=0}^{\infty} |\phi_k| \frac{1}{1 - \rho}. \]  

(C.12)

It follows that replacing \( x_t \) by the rescaled process \( \tilde{x}_{n,t} = d_n^{-1} x_t \) in (C.12) gives

\[ \mathbb{E} \left| \sum_{t=1}^{n} [g(\tilde{x}_{n,t}) - \mathbb{E}g(\tilde{x}_{n,t})] \right| \leq \frac{1}{d_n} \frac{n^{1/2}}{1 - \rho_n} \]
\[ \times \frac{1}{n^{1/2}} \frac{(1 - \rho_n^2)^{1/2}}{1 - \rho_n} \times \frac{1}{[n(1 - \rho_n)]^{1/2}} = o(1), \]

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where we have used Lemma C.2(ii) and the fact that \( d_n^2 \asymp (1 - \rho_n^2)^{-1} \) (see Remark 3.6).

**Proof of Proposition B.1(ii).** Let \( \epsilon > 0 \). It is proved below that along every sequence \( \{t_n\} \subseteq [n\epsilon, n] \),

\[
\tilde{x}_{n,t_n} \rightsquigarrow N[0,1],
\]

whence \( \mathbb{E}g(\tilde{x}_{n,t_n}) \to \tau(g) := \int \mathbb{E}g(x)\varphi(x)\,dx \), since \( g \) is bounded. Also by boundedness,

\[
\left| \frac{1}{n}\sum_{t=1}^{[nr]}[\mathbb{E}g(\tilde{x}_{n,t}) - \tau(g)] \right| \leq \epsilon\|g\|_{\infty} + \sup_{t\in[n\epsilon,n]}|\mathbb{E}g(\tilde{x}_{n,t}) - \tau(g)| \to \epsilon\|g\|_{\infty}.
\]

Since \( \epsilon \) was arbitrary, the result follows.

It remains to prove (C.13). To that end, decompose

\[
x_t = \sum_{k=0}^{t-1} a_{t-k}e_{t-k} + \sum_{k=t}^{\infty} a_{t-k}e_{t-k} = x_{1,t,t} + x_{-\infty,0,t} =: x_t^{(1)} + x_t^{(2)}.
\]

Note that \( x_t^{(1)} \) and \( x_t^{(2)} \) are independent, with variances respectively given by \( V_{1,t} \) and \( V_{2,t} \) in (C.5) above. We shall prove below that for \( t = t_n \in [n\epsilon, n] \), \( \rho = \rho_n \) and \( r_n := (1 - \rho_n^2)^{-1} \),

\[
r_n^{-1}\text{var}(x_{n,t}) = r_n^{-1}V_{1,t_n}(\rho_n) + o(1) \to \phi^2.
\]

Thus \( \text{var}(x_{n,t}) \sim r_n\phi^2 \sim n\omega_n^2(\rho_n)\phi^2 \), where \( \omega_n^2 \) is defined in (3.10) above. Further, we may write \( r_n^{-1/2}x_t^{(1)} = \sum_{k=-\infty}^{n} \delta_{n,k}e_k \), where

\[
\delta_{n,k} = \begin{cases} r_n^{-1/2}a_{t_n,k} & \text{if } 1 \leq k \leq t_n \\ 0 & \text{otherwise.} \end{cases}
\]

Since \( \rho_n \to 1 \) it follows that for all \( n \) sufficiently large

\[
\max_{k \leq n} |\delta_{n,k}| \leq r_n^{-1/2} \max_{1 \leq k \leq t_n} |a_{t_n,k}| \leq (1 - \rho_n^2)^{1/2} \sum_{i=0}^{\infty} |\phi_i| = o(1).
\]

Together, (C.14) and (C.15) permit the application of Lemma 2.1(i) in Abadir, Distaso, Giraitis, and Koul (2014), yielding

\[
\tilde{x}_{n,t_n} = \text{var}(x_{n,t})^{-1/2}x_{t_n} = \phi^{-1}r_n^{-1/2}x_t^{(1)} + o(1) \rightsquigarrow N[0,1].
\]

We turn lastly to the proof of (C.14). From (C.6), and the fact that \( \rho_n \in (0,1) \), we have

\[
(1 - \rho_n^2)V_{1,t_n}(\rho_n) = \sum_{i=0}^{t_n-1} \phi_i^2(1 - \rho_n^{2(t_n-i)}) + \sum_{i=0}^{t_n-1} \sum_{j=i+1}^{t_n-1} \phi_i\phi_j(1 - \rho_n^{2(t_n-j)}).
\]
Also by Lemma C.2, $|\rho_n^{2(t_n-i)}| \leq 1$ for all $0 \leq i \leq t_n - 1$, and $\rho_n^{2(|\nu|)} \leq \rho_n^{2(\lceil \nu \rceil - i)} \to 0$ for each fixed $i \geq 0$. Thus, in view of $\sum_{i=0}^{\infty} |\phi_i| < \infty$,

$$(1 - \rho_n^2)V_{1,t_n}(\rho_n) = \sum_{i=0}^{t_n-1} \phi_i^2 + 2 \sum_{i=0}^{t_n-1} \sum_{j=i+1}^{t_n-1} \phi_i \phi_j + o(1) \to \varphi^2.$$ 

Regarding $V_{2,t_n}$, we note from (C.5) that

$$V_{2,t_n}(\rho_n) \lesssim \sum_{k=t_n}^{\infty} \sum_{l=0}^{t_n-1} \rho_n^l |\phi_{k-l}| \leq \sum_{l=0}^{t_n-1} \rho_n^l \tilde{\phi}_{t_n-l},$$

where $\tilde{\phi}_j := \sum_{i=j}^{\infty} |\phi_i|$. Finally,

$$\sum_{l=0}^{t_n-1} \rho_n^l \tilde{\phi}_{t_n-l} = \left( \sum_{l=0}^{[t_n/2]-1} + \sum_{l=[t_n/2]}^{t_n-1} \rho_n^l \tilde{\phi}_{t_n-l} \right) \lesssim \left( \tilde{\phi}_{[t_n/2]} + \rho_n^{[t_n/2]} \right) \sum_{l=0}^{[t_n/2]} \rho_n^l = o(r_n),$$

using that $\tilde{\phi}_{[t_n/2]} \to 0$ and $\rho_n^{[t_n/2]} \to 0$ (by Lemma C.2(i)), and

$$\sum_{l=0}^{[t_n/2]} \rho_n^l \leq (1 - \rho_n)^{-1} \asymp (1 - \rho_n^2)^{-1} = r_n$$

by Lemma C.2(ii), whence $V_{2,t_n}(\rho_n) = o(r_n)$. \hspace{1cm} \square

**Proof of Proposition B.2.** We take $d_{n,s,t} = 1$ for all $n$, $s$, and $t$. Then part (a) of WP3 is trivially satisfied. For part (b), recall the decomposition in (C.3),

$$x_t = \sum_{k=0}^{t-s-1} a_{t,t-k} \xi_{t-k} + \sum_{k=t-s}^{\infty} a_{t,t-s} \xi_{t-s}$$

$$= \sum_{k=0}^{t-s-1} a_k \xi_{t-k} + \sum_{k=t-s}^{\infty} a_{t,t-s} \xi_{t-s} := x_{s+1,t,t} + x_{-\infty,s,t},$$

for $1 \leq s < t \leq n$, so that

$$x_t - x_s = x_{s+1,t,t} + (x_{-\infty,s,t} - x_s) =: x_{s+1,t,t} + y_{s,t},$$

where $x_{s+1,t,t}$ is independent of $y_{s,t}$. Noting $x_{s+1,t,t} = d x_{1,t-s,t-s}$ and taking $r := t - s$, we thus have

$$\tilde{x}_{n,t} - \tilde{x}_{n,s} = d_n^{-1} x_{1,r,r} + y^*_n,$$

where $y^*_{n,s,t} = d_n^{-1} y_{n,s,t}$ is independent of $x_{1,r,r}$.

In view of the definition of $\Omega_n(\eta)$, part (b) of WP3 only concerns $s$ and $t$ for which $(1 - \delta)n \geq t - s = r = r_n \geq n\delta$ for some $\delta \in (0, 1)$. For such $r_n$, it follows from arguments given in the proof of Proposition B.1(ii) that $z_n := d_n^{-1} x_{1,r_n,r_n} \rightsquigarrow N(0,1)$. Letting $f_n$
denote the characteristic function of $z_n$, arguments given in the proof of Corollary 2.2 in Wang and Phillips (2009a) then imply that part (b) holds if the sequence $\{f_n\}$ is uniformly integrable. In order to prove this, we first observe that

$$|f_n(\lambda)| = |\mathbb{E}\exp(i\lambda z_n)| \leq \prod_{s=k_0}^{k_n} |\psi_{\varepsilon}[\lambda d_n^{-1} a_k(\rho_n)]|$$

whenever $n$ is sufficiently large that $t_n \geq n\delta \geq k_n$. Hence for any $A > 0$,

$$\int_{\{\lambda \geq A\}} |f_n(\lambda)| \, d\lambda \leq \int_{\{\lambda \geq A\}} \prod_{s=k_0}^{k_n} |\psi_{\varepsilon}[\lambda d_n^{-1} a_k(\rho_n)]| \, d\lambda,$$

$$= d_n \int_{\{\lambda \geq A d_n^{-1}\}} \prod_{s=k_0}^{k_n} |\psi_{\varepsilon}[\lambda a_k(\rho_n)]| \, d\lambda$$

$$\leq C \, d_n \int_{\{\lambda \geq A d_n^{-1}\}} e^{-\gamma \lambda^2 k_n} \, d\lambda + d_n e^{-\gamma k_n/2} \int_{\{\lambda \geq 1\}} |\psi_{\varepsilon}[\lambda a_k(\rho_n)]| \, d\lambda$$

$$\leq C_1 \int_{\{\lambda \geq A\}} e^{-C_2 \lambda^2} \, d\lambda + d_n e^{-C_2 d_n^2} \int_{\mathbb{R}} |\psi_{\varepsilon}(\lambda)| \, d\lambda$$

$$\to 0$$

as $n \to \infty$ and then $A \to 0$, where we have used Lemma C.1 and the fact that $k_n \approx d_n^2$. Thus $\{f_n\}$ is uniformly integrable.

\[\Box\]

D Proof of weak convergence (Theorem 3.2)

$\{\rho_n\} \in \mathcal{R}_{\text{M1}} \cup \mathcal{R}_{\text{LU}}$. In this case, the proof of Theorem 3.2 closely follows the proof of Theorem 3.1(i) in Duffy (2015), as outlined in Section 4 of that paper. In particular, appropriate analogues of two key intermediate results – Propositions 4.1 and 4.2 in Duffy (2015), reproduced below as Proposition D.1 – are proved below by means of a martingale decomposition similar to that developed in Section 7.1 of that paper. To this end, let $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_{-\infty}^t]$, and consider

$$f(x_t) = \sum_{k=1}^{t \wedge k_n} [\mathbb{E}_{t-k+1} f(x_t) - \mathbb{E}_{t-k} f(x_t)] + \mathbb{E}_{t-k_n+} f(x_t),$$

where $[a]_+ := a \vee 0$: note that unlike Duffy (2015, Sec. 7.1), the decomposition here is truncated at $[t - k_n]_+$ rather than at 0. Defining

$$\xi_{k,t} f := \mathbb{E}_t f(x_{t+k}) - \mathbb{E}_{t-1} f(x_{t+k})$$

(D.1)

we have
$S_n f := \sum_{t=1}^n f(x_t) = \sum_{t=1}^n E_{[t-k_n]} f(x_t) + \sum_{k=0}^{k_n-1} \sum_{t=k+1}^n [E_{t-k} f(x_t) - E_{t-k-1} f(x_t)]$

$$= N_n f + \sum_{k=0}^{k_n-1} \sum_{t=1}^n [E_{t-k} f(x_{t+k}) - E_{t-1} f(x_{t+k})] = N_n f + \sum_{k=0}^{k_n-1} M_{n,k} f$$  \hspace{5mm} (D.2)

where $N_n f := \sum_{t=1}^n E_{[t-k_n]} f(x_t)$ and $M_{n,k} f := \sum_{t=1}^{n-k} \xi_{k,t} f$. \{\xi_{k,t}, F^t_{\infty}\}_{t=1}^{n-k}$ forms a martingale difference sequence for each $k$ by construction, and so control over each $M_{n,k} f$ will be deduced from control over

$$U_{n,k} f := [M_{n,k} f] = \sum_{t=1}^{n-k} \xi_{k,t} f \hspace{5mm} V_{n,k} f := \langle M_{n,k} f \rangle = \sum_{t=1}^{n-k} \xi_{k,t}^2 f. \hspace{5mm} (D.3)$$

To state our bounds on the foregoing, we first define the norm

$$\| f \|_{[\beta]} := \inf \{c \in \mathbb{R}_+ \mid \| f (\lambda) \| \leq c |\lambda|^\beta, \forall \lambda \in \mathbb{R} \}, \hspace{5mm} (D.4)$$

for $\beta \in (0,1]$, where $\hat{f}(\lambda) := \int e^{i\lambda x} f(x) \, dx$ denotes the Fourier transform of $f$. (See Section 4.2 and Lemma 9.1 in Duffy (2015) for more details on $\| f \|_{[\beta]}$.) Let $BI_{[\beta]} := \{ f \in BI \mid \| f \|_{[\beta]} < \infty \}$,

$$\varsigma_n(\beta, f) := \| f \|_{[\beta]} + e_n d_n^{-\beta} (\| f \|_1 + \| f \|_{[\beta]})$$

and

$$\sigma_{n,k}^2(\beta, f) := \begin{cases} \| f \|_\infty^2 + e_n \| f \|_2^2 & \text{if } k \in \{0, \ldots, k_0 - 1\}, \\ e_n \left[ k^{-(3+2\beta)/2} \| f \|_{[\beta]}^2 + e^{-\gamma_1 k} \| f \|_1^2 \right] & \text{if } k \in \{k_0, \ldots, k_n - 1\}; \end{cases}$$

and for $\mathcal{F} \subseteq BI_{[\beta]}$,

$$\delta_n(\beta, \mathcal{F}) := \| \mathcal{F} \|_{[\beta]} + e_n^{1/2} \| \mathcal{F} \|_2 + e_n d_n^{-\beta} (\| \mathcal{F} \|_1 + \| \mathcal{F} \|_{[\beta]}), \hspace{5mm} (D.5)$$

where $\| \mathcal{F} \| := \sup_{f \in \mathcal{F}} \| f \|$. Define

$$\overline{\beta} := \overline{\beta} (\{ \rho_n \}) := \sup \{ \beta \in (0,1] \mid e_n^{-1/2} d_n^{\beta} = o(1) \}.$$

Since $d_n \lesssim n^{1/2} \lesssim e_n$, $\overline{\beta} (\{ \rho_n \}) \geq \frac{1}{2}$ for all $\{ \rho_n \} \in R_{\text{MI}} \cup R_{\text{LU}}$. Proofs of the following results appear in Section S.4 of the Supplement.

**Lemma D.1.** Suppose $\{ \rho_n \} \in R_{\text{MI}} \cup R_{\text{LU}}$ and $\beta \in (0,\overline{\beta})$. Then there exists a $C < \infty$ such that

$$\| N_n f \|_{[\beta]} \leq C \varsigma_n(\beta, f) \hspace{5mm} (D.6)$$

and

$$\| U_{n,k} f \|_{[\beta]} \vee \| V_{n,k} f \|_{[\beta]} \leq C \sigma_{n,k}^2(\beta, f) \hspace{5mm} (D.7)$$
for all \( n \geq n_0, 0 \leq k \leq k_n - 1 \) and \( f \in \text{BI} \).

**Lemma D.2.** Suppose \( \{\rho_n\} \in \mathcal{R}_{\text{MI}} \cup \mathcal{R}_{\text{LU}} \) and \( \beta \in (0, \beta) \). Then there exists a \( C < \infty \) such that

\[
\sup_{f \in \mathcal{G}} \zeta_n(\beta, f) + \sum_{k=0}^{k_n-1} \sup_{f \in \mathcal{G}} \sigma_{n,k}(\beta, f) \leq C \delta_n(\beta, \mathcal{G})
\]

for all \( \mathcal{G} \subset \text{BI}[\beta] \).

With the preceding lemmas taking the places of Lemmas 7.3 and 7.5 in Duffy (2015), the next result follows from almost exactly the same arguments as are used to prove Propositions 4.1 and 4.2 in that paper. The very minor modifications that are required merely reflect the slight differences between \( \zeta_n, \sigma_{n,k} \) and \( \delta_n \) as they appear above, and the corresponding quantities in Duffy (2015); the reader is accordingly referred to that paper for the details of the proof. Let \( \kappa(x) := (1 - |x|)1\{x \in [-1, 1]\} \) denote the triangular kernel function, and \( \mu_n(a; f) := \mu_n(1, a; f, 1) \). As in Duffy (2015, Sec. 4.2), \( \|\cdot\|_{\tau_3/2} \) denotes the Orlicz norm associated to the convex and increasing function

\[
\tau_{3/2}(x) := \begin{cases} 
 x(e - 1) & \text{if } x \in [0, 1], \\
 e^{x^{2/3}} - 1 & \text{if } x \in (1, \infty).
\end{cases}
\]

**Proposition D.1.** Suppose \( \{\rho_n\} \in \mathcal{R}_{\text{MI}} \cup \mathcal{R}_{\text{LU}} \) and \( \beta \in (0, \beta) \). Then

(i) there exists a \( C < \infty \) such that

\[
\sup_{a_1, a_2 \in \mathbb{R}} \|\mu_n(a_1; \kappa) - \mu_n(a_2; \kappa)\|_{\tau_{3/2}} \leq C|a_1 - a_2|^{\beta};
\]

(ii) if \( \mathcal{F}_n \subset \text{BI}[\beta] \) with \( \#\mathcal{F}_n \lesssim n^C \), then \( \max_{f \in \mathcal{F}_n} |S_n f| \lesssim_p \delta_n(\beta, \mathcal{F}_n) \log n \), whence

\[
e_n^{-1} \max_{f \in \mathcal{F}_n} |S_n f| = o(1)
\]

if \( \|\mathcal{F}_n\|_1 \lesssim 1, \|\mathcal{F}_n\|_\beta = o(d_n^2) \) and \( \|\mathcal{F}_n\|_\infty = o(e_n \log^{-2} n) \).

The proof of Theorem 3.2 may now proceed almost exactly along the lines of the proof of Theorem 3.1(i) in Duffy (2015), with Proposition D.1 here playing the role of Propositions 4.1 and 4.2 there. Let \( M < \infty \); the desired convergence in \( \ell_{\text{ucc}}(\mathbb{R}) \) will follow from convergence in \( \ell_\infty([-M, M]) \), the space of bounded functions on \([-M, M] \), equipped with the topology of uniform convergence. As per the argument in Section 6 of Duffy (2015), it follows immediately from part (i) of Proposition D.1 that \( \mu_n(a; \kappa) \) is tight in \( \ell_\infty([-M, M]) \), whence \( \mu_n(a; \kappa) \sim \mu(a) \) in \( \ell_\infty([-M, M]) \). Further, for any \( f \) as in the
statement of Theorem 3.2,

\[
\sup_{a \in [-M,M]} \left| \mu_n(a; f, h_n(a)) - \mu(a; \kappa) \int_{\mathbb{R}} f \right| \\
\leq \sup_{(a,h) \in [-M,M] \times \mathbb{R}_n} \left| \mu_n(a; f, h) - \mu(a; \kappa) \int_{\mathbb{R}} f \right| = o_p(1)
\]

where the inequality holds w.p.a.1 under \( H \), while the equality follows from part (ii) of Proposition D.1, together with (6.2)–(6.3) in Duffy (2015) and the subsequent arguments. Thus \( \mu_n(a; f, h_n(a)) \rightharpoonup \mu(a) \) in \( \ell_\infty([-M,M]) \).

\( \{\rho_n\} \in \mathcal{R}_{ST} \). In this case, the result follows from essentially the same arguments as are used to prove Theorem 2 in Wu, Huang, and Huang (2010).
Supplementary material

S.1 Verification of Remark 3.6

The claim to be proved is that, when \( \{\rho_n\} \in R_{\alpha} \cup R_{\beta} \), \( d_{\alpha}^2 := \text{var}(x_n) \sim n\omega_n^2(\rho_n)\phi^2 \), for \( \omega_n \) as in (3.10). When \( \{\rho_n\} \in R_{\alpha} \), this follows from (C.14) with \( t_n = n \). We therefore suppose that \( \{\rho_n\} \in R_{\beta} \). In this case, \( c_n := n(\rho_n - 1) \rightarrow c \in \mathbb{R} \) and \( \omega_n^2(\rho_n) \rightarrow \int_0^1 e^{2(1-s)} c \, ds \).

Define

\[
g_k(\rho) := \frac{1 - \rho 2^k}{1 - \rho^2} = \sum_{l=0}^{k-1} \rho^{2^l}
\]

for \( k \in \{1, \ldots, n\} \), with the final equality holding by continuity when \( \rho = 1 \). Recall from (C.5)–(C.6) above that \( E x_n^2 = V_{1,n}(\rho) + V_{2,n}(\rho) \), where

\[
V_{1,n}(\rho) = \sum_{i=0}^{n-1} \phi_i^2 g_{n-i}(\rho) + 2 \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} \phi_i \phi_j g_{n-j}(\rho). \tag{S.2}
\]

If we can show that

(i) \( \frac{1}{n} g_{n-i}(\rho_n) \rightarrow \int_0^1 e^{2(1-s)} c \, ds \) as \( n \rightarrow \infty \), for each fixed \( i \geq 0 \); and

(ii) \( \max_{1 \leq k \leq n} \frac{1}{n} |g_k(\rho_n)| \) is uniformly bounded;

then in view of \( \sum_{i=0}^{\infty} |\phi_i| < \infty \) and (S.2), it will follow immediately that

\[
n^{-1}V_{1,n}(\rho_n) \rightarrow \phi^2 \int_0^1 e^{2(1-s)} c \, ds
\]

as \( n \rightarrow \infty \), whence \( V_{1,n}(\rho_n) \sim n\omega_n^2(\rho_n)\phi^2 \).

For (i), we first suppose that \( c_n \rightarrow c \neq 0 \). Then

\[
\frac{1}{n} g_{n-i}(\rho) = \left( 1 + \frac{\omega_n}{n(\rho_n^2 - 1)} \right)^{2(n-i)} - 1 \rightarrow e^{2c} - 1 = \frac{e^c}{2c} = \int_0^1 e^{2(1-s)} c \, ds.
\]

To handle the case where \( c_n \rightarrow 0 \), we note first that \( y^x = 1 + x + o(x) \) as \( (y, x) \rightarrow (e, 0) \). Hence

\[
\frac{1}{n} g_{n-i}(\rho) = \frac{2c_n(1 + o(1))}{2c_n(1 + o(1))} \rightarrow 1 = \int_0^1 e^{2(1-s)} c \, ds \bigg|_{c=0}.
\]

For (ii), we note from (S.1) that \( |g_k(\rho)| \leq k - 1 \) whenever \( \rho \leq 1 \), while if \( \rho > 1 \), \( |g_k(\rho)| \) is maximised by taking \( k = n \), and so boundedness follows from (i) with \( i = 0 \).

It remains to show that \( V_{2,n}(\rho_n) = o(n) \). Taking \( n \) sufficiently large, \( \rho_n > 0 \) and so \( \rho_n^k \in [\rho_n^{-n}, \rho_n^n] \) for any \( 0 \leq k \leq n \). Since \( (\rho_n^{-n}, \rho_n^n) \rightarrow (e^{-c}, e^c) \) and \( \sum_{i=0}^{\infty} |\phi_i| < \infty \),

\[
V_{2,n}(\rho_n) = \sum_{k=n}^{\infty} \left( \sum_{l=0}^{n-1} \rho_n^l \phi_{k-l} \right)^2 \lesssim \sum_{k=n}^{\infty} \left( \sum_{l=0}^{n-1} |\phi_{k-l}| \right)^2 \lesssim \sum_{k=n}^{\infty} \sum_{l=0}^{n-1} |\phi_{k-l}|.
\]
Finally, \[
\sum_{k=n}^{\infty} \sum_{l=0}^{n-1} |\phi_{k-l}| = \left( \sum_{k=n}^{2n} + \sum_{k=2n+1}^{\infty} \right) \sum_{l=0}^{n-1} |\phi_{k-l}| \leq \sum_{k=0}^{n} \sum_{l=k}^{\infty} |\phi_l| + n \sum_{k=n}^{\infty} |\phi_k| = o(n). \]

S.2 Proofs of Lemmas A.1–A.2

Proof of Lemma A.1. Suppose \( \{\rho_n\} \in \mathcal{R}_{ST} \). Then Fourier inversion and the decomposition \( x_t = \phi_0 \varepsilon_t + x_{-\infty} \) gives, for any positive-valued \( f \in L^1 \),

\[
\mathbb{E} f(x_t) \leq C \int |\hat{f}(\lambda)| |\psi_\varepsilon(-\phi_0 \lambda)| d\lambda \leq \phi_0^{-1} \|\psi_\varepsilon\|_1 \|f\|_1 \leq C \|f\|_1 \tag{S.3}
\]

since \( \phi_0 \neq 0 \). This, together with the fact that \( e_n \simeq n \), yields the result in this case.

Now suppose \( \{\rho_n\} \in \mathcal{R}_{MI} \cup \mathcal{R}_{LU} \). In this case, (S.3) continues to hold, and an appeal to Lemma S.2(i) below yields

\[
\frac{1}{e_n} \sum_{t=1}^{n} \mathbb{E} f(x_t) \leq C \frac{1}{e_n} \left( k_0 + \sum_{t=k_0}^{k_n} k^{-1/2} + (n - k_n)k_n^{-1/2} \right) \|f\|_1.
\]

The bracketed term on the r.h.s. has the same order as

\[
e_n^{-1}(k_n^{1/2} + nk_n^{-1/2}) = e_n^{-1}d_n + 1 \lesssim 1
\]

since, in particular, \( nk_n^{-1/2} \simeq nd_n^{-1} = e_n \). \( \square \)

Proof of Lemma A.2. The proof follows the same lines as the proof of Theorem 3.2 in Wang and Phillips (2009b). We first note that

\[
\sum_{t=1}^{n} K_{h_n}(x_t - x) [y_{t+1} - \hat{m}_n(x)]^2
\]

\[
= \sum_{t=1}^{n} K_{h_n}(x_t - x) u_{t+1}^2 + \sum_{t=1}^{n} K_{h_n}(x_t - x) [m_n(x_t) - \hat{m}_n(x)]u_{t+1}
\]

\[
+ \sum_{t=1}^{n} K_{h_n}(x_t - x) [m_n(x_t) - \hat{m}_n(x)]^2
\]

\[
=: A_{n,1} + A_{n,2} + A_{n,3}.
\]

Now, letting \( \xi_t := u_{t+1}^2 - \sigma_n^2 \), we claim that

\[
\frac{A_{n,1}}{e_n} = \frac{\sigma_n^2}{e_n} \sum_{t=1}^{n} K_{h_n}(x_t - x) + \frac{1}{e_n} \sum_{t=1}^{n} K_{h_n}(x_t - x) \xi_{t+1}
\]

\[
\sim \sigma_n^2 \eta(x)
\]

for \( \eta(x) \) as defined in (A.3) above. The convergence of the first r.h.s. term in (S.4) follows
from Theorem 3.1. For the second term, we note that

\[
\mathbb{E}\left( \frac{1}{e_n} \sum_{t=1}^{n} K_{\nu_n}(x_t - x)\zeta_{t+1} \right)^2 \leq \frac{1}{e_n^2 h_n} \sum_{t=1}^{n} \mathbb{E} \left[ L_{\nu_n}(x_t - x) \cdot \mathbb{E} \left[ \zeta_{t+1}^2 \mid G_t \right] \right] \leq \frac{C}{e_n h_n} ||L||_1 = o(1)
\]

by Lemma A.1 and the a.s. boundedness of \( \sup_t \mathbb{E}[\zeta_{t+1}^2 \mid G_t] \), where \( L(u) := K^2(u) \).

Next, note that for any \( x \in \mathbb{R} \), \( |\hat{m}_n(x) - m_n(x)| = o_p(1) \) follows from arguments similar to those used in the proof of Proposition 2.1(i). Thus

\[
\frac{A_{n,3}}{e_n} \leq C \frac{1}{e_n} \sum_{t=1}^{n} K_{\nu_n}(x_t - x) \{ |m_n(x_t) - m_n(x)|^2 + |\hat{m}_n(x_t) - m_n(x)|^2 \}
\leq ||m'||_{\infty}^2 \frac{h_n^2}{e_n} \sum_{t=1}^{n} K_{\nu_n}^{[2]}(x_t - x) + o_p(1)
= o_p(1),
\]

by a mean-value expansion and Lemma A.1; here \( K^{[2]}(u) := K(u)u^2 \). Finally, by the Cauchy-Schwarz inequality,

\( A_{n,2} \leq (A_{n,1})^{1/2}(A_{n,3})^{1/2} \),

and so by Theorem 3.1 and the preceding,

\[
\sigma_u^2(x) = \frac{A_{n,1} + A_{n,2} + A_{n,3}}{\sum_{t=1}^{n} K_{\nu_n}(x_t - x)} \rightarrow \frac{\sigma_u^2 \eta(x)}{\eta(x)} = \sigma_u^2. \tag{\ref*{A.1.1}} \]

S.3 Proofs of auxiliary results from Appendix C

Proof of (C.6). Letting \( m := t - 1 \), we have that \( V_{t,t}(\rho) \) is equal to

\[
\sum_{k=0}^{m} \sum_{i=0}^{k} \sum_{j=0}^{k} \rho^{2k-i-j} \phi_i \phi_j = \sum_{i=0}^{m} \sum_{j=0}^{m} \phi_i \phi_j \sum_{k=1}^{m} \rho^{2k-i-j}
= \sum_{i=0}^{m} \phi_i^2 \sum_{k=1}^{m} \rho^{2(k-i)} + 2 \sum_{i=0}^{m} \phi_i \phi_j \sum_{k=j}^{m} \rho^{2(k-j)}
= \sum_{i=0}^{m} \phi_i^2 \rho^{2k} + 2 \sum_{i=0}^{m} \phi_i \phi_j \sum_{k=0}^{m-j} \rho^{2k}. \tag{\ref*{A.1.2}} \]

Proof of Lemma C.1(i). When \( \{\rho_n\} \in \mathcal{R}_{LU} \), the result follows essentially from arguments given in Wang and Phillips (2009b): see their (7.14), in particular. We therefore turn to the case where \( \{\rho_n\} \in \mathcal{R}_{MT} \). Then \( \rho_n \in (0,1) \), and the upper bound in (C.8) follows trivially from \(|a_k(\rho_n)| \leq \sum_{i=0}^{\infty} |\phi_i| \). Further, for any \( 0 \leq k \leq 2k_n \),

\[
\rho_n^{2k_n} \leq \rho_n^k \leq \rho_n^{-k} \leq \rho_n^{-2k_n}.
\]
Noting that $\rho^{(1-\rho)^{-1}} \to e^{-1}$ as $\rho \to 1$, and $2k_n \sim (1-\rho_n)^{-1}$, it follows that $(\rho_n^{2k_n}, \rho_n^{-2k_n}) \to (e^{-1}, e)$ as $n \to \infty$. Thus there exists an $n_0$ and $C_1, C_2 \in (0, \infty)$ such that $\rho_n^{k_n}, \rho_n^{-k_n} \in [C_1, C_2]$ for all $n \geq n_0$ and $0 \leq k \leq 2k_n$.

Now $a_k(\rho_n) = \rho_n^k \sum_{l=0}^{k} \rho_n^{-l} \phi_l$, and for any $m \leq k \leq 2k_n$,

$$\sum_{l=0}^{k} \rho_n^{-l} \phi_l = \sum_{l=0}^{m} \phi_l - \sum_{l=0}^{m} (1 - \rho_n^{-l}) \phi_l + \sum_{l=m+1}^{k} \rho_n^{-l} \phi_l.$$ 

Therefore, since $|\rho_n^k| \leq 1$,

$$|a_k(\rho_n) - \rho_n^k \sum_{l=0}^{m} \phi_l| \leq \sum_{l=0}^{m} |1 - \rho_n^{-l}| ||\phi_l|| + \sum_{l=m+1}^{k} ||\phi_l||$$

Let $m_0$ be chosen such that both

$$\rho_n^k \sum_{l=0}^{m_0} \phi_l \geq C_1 \sum_{l=0}^{m_0} \phi_l \geq \frac{C_1}{2} ||\phi|| =: 3a$$

for all $n \geq n_0$, and $\sum_{l=m_0+1}^{\infty} ||\phi_l|| \leq a$. Since $\rho_n^{-l} \to 1$ for each $l$, there exists an $n_1 \geq n_0$ such that

$$|a_k(\rho_n)| \geq \rho_n^k \sum_{l=0}^{m_0} \phi_l - \sum_{l=0}^{m_0} |1 - \rho_n^{-l}| ||\phi_l|| - \sum_{l=m_0+1}^{k} ||\phi_l|| \geq a$$

for all $n \geq n_1$. Taking $k_0 := 2m_0$ and re-designating $n_1$ as $n_0$ completes the proof. \qed

Proof of Lemma C.1(ii). Since $\psi \in L^1$, $\varepsilon_0$ has a bounded continuous density. Thus by the Riemann-Lebesgue lemma (Feller, 1971, Lem. XV.3.3) $\limsup_{|\lambda| \to \infty} |\psi(\lambda)| = 0$. Further, $\psi \in L^1$ cannot be periodic, and so $|\psi(\lambda)| < 1$ for all $\lambda \neq 0$ (Feller, 1971, Lem. XV.1.4).

Since $\psi$ is necessarily continuous (Feller, 1971, Lem. XV.1.1), it follows from part (i) of the lemma that

$$\sup_{n \geq n_0} \sup_{k_0 \leq k \leq n} \sup_{|\lambda| \geq 1} |\psi(\lambda)| \leq e^{-\gamma_0}$$

for some $\gamma_0 \in (0, \infty)$. By the moments theorem for characteristic functions (Feller, 1971, Lem. XV.4.2), we have the expansion $\psi(\lambda) = 1 - \frac{1}{2} \lambda^2 (1 + r(\lambda))$ where $r(\lambda) \to 0$ as $\lambda \to 0$. In view of part (i) of the lemma, there thus exists a $\gamma_1 \in (0, \infty)$ such that

$$\sup_{n \geq n_0} \sup_{k_0 \leq k \leq n} |\psi(\lambda)| \leq e^{-\gamma_1 \lambda^2}$$

for all $|\lambda| \leq 1$. The result now follows by taking $\gamma := \gamma_0 \wedge \gamma_1$. \qed

Proof of Lemma C.2. Letting $c_n := n(\rho_n - 1) \to -\infty$, we note that for every $M < \infty$, we
may take \( n \) sufficiently large such that \( e_n < -M \), whence 
\[
\rho_n^{ne} = \left(1 + \frac{c_n}{n}\right)^{ne} \leq \left(1 - \frac{M}{n}\right)^{ne} \to e^{-Me} \to 0
\]
as \( n \to \infty \) and then \( M \to \infty \). Thus (i) holds. (ii) follows from 
\[
\frac{1 - \rho_n^2}{1 - \rho_n} = 1 + \rho_n \to 2.
\]

### S.4 Proofs of Lemmas D.1–D.2

The proof of Lemma D.1 requires the following two results, which here play the role of 
Lemmas 7.4 and 9.3 in Duffy (2015); see Section S.5 for the proofs.

**Lemma S.1.** Suppose \( \beta \in (0, \beta) \). Then there exists a \( C < \infty \) such that 
\[
\|\xi_{k,t}^2 f\|_\infty + \sum_{s=1}^{n-k-t} \|\mathbb{E}_t \xi_{k,t+s}^2 f\|_\infty \leq C \sigma_{n,k}^2 (\beta, f) \tag{S.5}
\]
when \( k \in \{0, \ldots, k_0 - 1\} \), and 
\[
\|\xi_{k,t}^2 f\|_\infty + \sum_{s=1}^{(n-k-t)\wedge k_n} \|\mathbb{E}_t \xi_{k,t+s}^2 f\|_\infty \leq C n^{-1} k_n \sigma_{n,k}^2 (\beta, f) \tag{S.6}
\]
when \( k \in \{k_0, \ldots, k_n - 1\} \), for all \( n \geq n_0, 1 \leq t \leq n - k \) and \( f \in \text{BI}_{[\beta]} \).

**Lemma S.2.** Suppose \( f \in \text{BI} \). There exists a \( C < \infty \), not depending on \( f \), such that 
(i) for every \( t \geq 0 \) and \( k_0 \leq k \leq n - t \), 
\[
\mathbb{E}_t |f(x_{t+k})| \leq_C (k \wedge k_n)^{-1/2} \|f\|_1;
\]
(ii) if in addition \( f \in \text{BI}_{[\beta]} \) for some \( \beta \in (0, 1) \), then for every \( t \geq 0 \) and \( k_0 \leq k \leq n - t \), 
\[
|\mathbb{E}_t f(x_{t+k})| \leq_C (k \wedge k_n)^{-(1+\beta)/2} \|f\|_{[\beta]} + e^{-\gamma_1 (k \wedge k_n)} \|f\|_1.
\]

**Proof of Lemma D.1.** By Lemma S.2(ii), 
\[
|\mathcal{N}_n f| \leq \left(\sum_{t=1}^{k_0-1} + \sum_{t=k_0}^{k_n} + \sum_{t=k_n+1}^{n} \right) |\mathbb{E}_{t-k_n} f(x_t)| \\
\lesssim \|f\|_\infty + ne^{-\gamma_1 k_n} \|f\|_1 + nk_n^{-(1+\beta)/2} \|f\|_{[\beta]}
\]
whence (D.6), noting that \( ne^{-\gamma_1 k_n} \lesssim nk_n^{-(1+\beta)/2} \approx nd_n^{-(1+\beta)} = e_n d_n^{-\beta} \). Regarding (D.7), 
in view of Lemma 7.2 in Duffy (2015) it suffices to show 
\[
\|U_{n,k} f\|_p \vee \|V_{n,k} f\|_p \leq C p^{1/p} \sigma_{n,k}^2 (\beta, f), \tag{S.7}
\]
for every \( p \in \mathbb{N} \). To prove (S.7), consider decomposing \( V_{n,k} f \) into \( L \) blocks, as per

\[
V_{n,k} f = \sum_{l=1}^{L} \sum_{t=\tau_{l-1}}^{\tau_l} \mathbb{E} \xi_{k,t}^2 f =: \sum_{l=1}^{L} V_{n,k,l} f
\]

for endpoints \( 0 \leq \tau_0 \leq \cdots \leq \tau_L \leq n - k \). For the \( l \)th block, repeated application of the law of iterated expectations yields

\[
\mathbb{E} |V_{n,k,l} f|^p \leq p! \cdot \sum_{t_1 = \tau_{l-1} + 1}^{\tau_l} \cdots \sum_{t_{p-1} = t_{p-2}}^{\tau_{l-1}} \mathbb{E} \left[ \mathbb{E}_{t_1-1}(\xi_{k,t_1}^2 f) \cdots \mathbb{E}_{t_{p-1}-1}(\xi_{k,t_{p-1}}^2 f) \right] 
\cdot \left( \|\xi_{k,t_{p-1}}^2 f\|_\infty + \sum_{s=1}^{\tau_{l-1} - t_{p-1}} \|\mathbb{E}_{t_{p-1} - 1} \xi_{k,t_{p-1} + s}^2\|_\infty \right) ; \quad (S.8)
\]

Since \( \|\mathbb{E}_{t_{p-1} - 1} \xi_{k,t_{p-1} + s}^2\|_\infty \leq \|\mathbb{E}_{t_{p-1} - 1} \xi_{k,t_{p-1} + s}^2\|_\infty \), suitable bounds for the final term (in parentheses) are provided by Lemma S.1. In particular, when \( k \in \{0, \ldots, k_{0}\} \), we may take \( \tau_0 = 0 \) and \( \tau_1 = n - k \), so that (S.5) immediately yields

\[
\mathbb{E} |V_{n,k} f|^p \leq p! \cdot C_p \sigma_{n,k}^{2p}(\beta, f).
\]

When \( k \in \{k_{0}, \ldots, k_{n} - 1\} \), we set \( \tau_l := k_n l \wedge (n - k) \), with \( L = L_n \) chosen to be the smallest integer such that \( k_n L_n \geq n - k \). Then applying (S.6) to (S.8) gives

\[
\mathbb{E} |V_{n,k,l} f|^p \leq p! \cdot C_p (n^{-1} k_n)^p \sigma_{n,k}^{2p}(\beta, f)
\]

for \( l \in \{1, \ldots, L\} \), and

\[
\|V_{n,k} f\|_p \leq \sum_{l=1}^{L} \|V_{n,k,l} f\|_p \leq C_p l^{1/p} \sigma_{n,k}^{2p}(\beta, f)
\]

since \( L_n \lesssim n k^{-1} \). Thus \( V_{n,k} f \) satisfies (S.7); an analogous argument establishes that this is also true of \( U_{n,k} f \).

Proof of Lemma D.2. This follows exactly as per the proof of Lemma 7.5 in Duffy (2015): we need only to note that, in the present case,

\[
e_n^{1/2} \sum_{k=0}^{k_n} k^{-(3+2\beta)/4} \lesssim e_n^{1/2} k_n^{1/4} \sum_{k=0}^{k_n} k^{-1-\beta/2} \lesssim e_n^{1/2} d_n^{1/2-\beta} \lesssim e_n d_n^{1/2-\beta},
\]

since \( k_n^{1/4} \lesssim d_n^{1/2} \lesssim e_n^{1/2} \).
S.5 Proofs of Lemmas S.1–S.2

We first recall the following useful inequality, from Lemma 9.1(i) in Duffy (2015):

\[ |\hat{f}(\lambda)| \leq (|\lambda|^\beta \|f\|_{|\beta|}) \wedge \|f\|_1 \]  

(S.9)

for every \( \beta \in (0, 1] \) and \( f \in B\ell_{|\beta|} \); recall \( \hat{f} \) denotes the Fourier transform of \( f \). We shall also need the following results, whose proofs appear at the end of this section. Let

\[ \vartheta(z_1, z_2) := E[e^{-iz_1\epsilon_0} - Ee^{-iz_1\epsilon_0}][e^{-iz_2\epsilon_0} - Ee^{-iz_2\epsilon_0}] \].

**Lemma S.3.** There exists a \( C < \infty \) such that for every \( z_1, z_2 \in \mathbb{R} \),

\[ |\vartheta(z_1, z_2)| \leq C \|z_1\|^2 \wedge 1 \|z_2\|^2 \wedge 1 \].

**Lemma S.4.** There exists a \( \gamma_1 > 0 \) and a \( C < \infty \) such that, for every \( p \in [0, 5] \), \( z_1, z_2 \in \mathbb{R}_+ \), and \( k_0 \leq k \leq 2k_n \),

\[ \int_{\mathbb{R}} (z_1|\lambda|^p \wedge z_2) \prod_{k \in K} |\psi(a_i(\rho_n)\lambda)| \, d\lambda \leq C \, z_1 k^{-(p+1)/2} + z_2 e^{-\gamma_1 k} \]

uniformly over all \( K \subseteq \{[k/2], \ldots, k\} \) with \( \#K \geq k/4 \).

**Corollary S.1.** There exists a \( \gamma_1 > 0 \) and a \( C < \infty \) such that

(i) for every \( p \in [0, 5] \), \( z_1, z_2 \in \mathbb{R}_+ \), \( k_0 \leq k \leq k_n \) and \( 1 \leq t \leq n - k \),

\[ \int_{\mathbb{R}} (z_1|\lambda|^p \wedge z_2) \|Ee^{-i\lambda x_{t+k}} - E_0e^{-i\lambda x_{t+k}}\| \, d\lambda \leq C \, z_1 (k \wedge k_n)^{-(p+1)/2} + z_2 e^{-\gamma_1 (k \wedge k_n)} \];

(ii) and additionally, for every \( 2 \leq s \leq t \),

\[ \int_{\mathbb{R}} \|E e^{-i\lambda x_{s+t-1,k+1}} - E_0 e^{-i\lambda x_{s+t-1,k+1}}\| \, d\lambda \leq C \, (s \wedge k_n)^{-1/2} \].

**Proof of Lemma S.1.** The argument is similar to that used to prove Lemma 7.4 in Duffy (2015). We first suppose that \( k \in \{0, \ldots, k_0 - 1\} \). Trivially, \( \|\xi_{k,t}^2 f\|_{\infty} \lessapprox C_1 \|f\|_{\infty}^2 \), while by Jensen’s inequality and Lemma S.2(i),

\[ \|E_t \xi_{k,t+s}^2 \| \lessapprox C \|E_t f^2(x_{t+s+k})\| \leq C_1 \begin{cases} \|f\|_{\infty}^2 & \text{if } 1 \leq s \leq k_0 - 1, \\ (s \wedge k_n)^{-1/2} \|f\|_{\infty}^2 & \text{if } s \geq k_0. \end{cases} \]

Hence, noting that \( \sum_{s=1}^{k_0} s^{-1/2} \lessapprox k_0^{1/2} \lessapprox n k_0^{1/2} \) and \( n k_0^{-1/2} \lessapprox n d_{k_0}^{-1} = e_n \),

\[ \|\xi_{k,t}^2 f\|_{\infty} + \sum_{s=1}^{n-k-t} \|E_t \xi_{k,t+s}^2 \| \lessapprox C_1 \|f\|_{\infty}^2 + n k_0^{-1/2} \|f\|_{\infty}^2 \lessapprox C_1 \|f\|_{\infty} + e_n \|f\|_{\infty}^2 \].
as required for (S.5).

It remains to consider the case where \( k \in \{k_0, \ldots, k_n - 1\} \). We shall obtain a bound for \( \mathbb{E}_{t-s} \xi_{k,t}^2 f \) – for \( t \in \{2, \ldots, n - k\} \) and \( s \in \{1, \ldots, k_n \wedge (t - 1)\} \) – which depends on \( k \) and \( s \) but not \( t \), thus permitting us to deduce the required bound for \( \mathbb{E}_{t} \xi_{k,t+s}^2 f \) in (S.6). As per (C.3) and (C.4) above, decompose

\[
x_{t+k} = x_{-\infty,0,t+k} + x_{1,t-1,t+k} + \theta k \varepsilon_t + x_{t+1,t+k,t+k},
\]

so that by Fourier inversion

\[
\xi_{k,t} f = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\lambda) e^{-i\lambda x_{-\infty,0,t+k}} e^{-i\lambda x_{1,t-1,t+k}} \left[ e^{-i\lambda \theta} - \mathbb{E} e^{-i\lambda \theta} \right] e^{-i\lambda x_{t+1,t+k}} d\lambda,
\]

whence

\[
\xi_{k,t}^2 f = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\lambda_1) \hat{f}(\lambda_2) e^{-i(\lambda_1 + \lambda_2) x_{-\infty,0,t+k}} e^{-i(\lambda_1 + \lambda_2) x_{1,t-1,t+k}}
\]

\[
\cdot \left[ e^{-i\lambda_1 \theta} - \mathbb{E} e^{-i\lambda_1 \theta} \right] e^{-i\lambda_2 \theta} e^{-i\lambda_2 \theta} e^{-i\lambda x_{t+1,t+k}} d\lambda_1 d\lambda_2.
\]

Since \( 1 \leq s \leq t - 1 \), making the further decomposition

\[
x_{1,t-1,t+k} = x_{1,t-s,t+k} + x_{t-s+1,t-1,t+k}
\]

and taking conditional expectations on both sides of (S.10) gives

\[
\mathbb{E}_{t-s} \xi_{k,t}^2 f = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\lambda_1) \hat{f}(\lambda_2) e^{-i(\lambda_1 + \lambda_2) x_{-\infty,0,t+k}} e^{-i(\lambda_1 + \lambda_2) x_{1,t-s,t+k}}
\]

\[
\cdot \mathbb{E} e^{-i(\lambda_1 + \lambda_2) x_{t-s+1,t-1,t+k}} \cdot \mathbb{E} e^{-i\lambda_1 \theta} e^{-i\lambda_2 \theta} d\lambda_1 d\lambda_2,
\]

where \( \mathbb{E}[e^{-iz_{t,0}} - \mathbb{E} e^{-iz_{t,0}}] [e^{-iz_{t,0}} - \mathbb{E} e^{-iz_{t,0}}] \). Thus by (S.9) and Lemmas C.1(i) and S.3, there exist \( C, C_1 < \infty \) such that

\[
\mathbb{E}_{t-s} \xi_{k,t}^2 f \leq C \int_{\mathbb{R}^2} |\hat{f}(\lambda_1) \hat{f}(\lambda_2)| \left[ |\lambda_1|^2 \wedge 1 \right] \left[ |\lambda_2|^2 \wedge 1 \right]^{1/2}
\]

\[
\cdot \left| \mathbb{E} e^{-i(\lambda_1 + \lambda_2) x_{t-s+1,t-1,t+k}} \right|
\]

\[
\cdot \left| \mathbb{E} e^{-i\lambda_1 x_{t+1,t+k,t+k}} \right| \left| \mathbb{E} e^{-i\lambda_2 x_{t+1,t+k,t+k}} \right| d\lambda_1 d\lambda_2
\]

\[
\leq C_1 \int_{\mathbb{R}} |\hat{f}(\lambda_1)|^2 \left[ |\lambda_1|^2 \wedge 1 \right] \left[ \mathbb{E} e^{-i\lambda_1 x_{t+1,t+k,t+k}} \right]
\]

\[
\cdot \left[ \mathbb{E} e^{-i(\lambda_1 + \lambda_2) x_{t-s+1,t-1,t+k}} \right] \left| \mathbb{E} e^{-i\lambda x_{t+1,t+k,t+k}} \right| d\lambda_2 d\lambda_1.
\]
where we have used $|ab| \leq |a|^2 + |b|^2$, and appealed to symmetry (in $\lambda_1$ and $\lambda_2$) to obtain the final bound. Now by Corollary S.1(ii), and recalling that $s \leq k_n$,

$$
\int_{\mathbb{R}} |\mathbb{E}e^{-i(\lambda_1+\lambda_2)x_{t-s+t+1,t+k}}| \, d\lambda_2 = \int_{\mathbb{R}} |\mathbb{E}e^{-i\lambda x_{t-s+t+1,t+k}}| \, d\lambda \leq C (s \wedge k_n)^{-1/2} = s^{-1/2},
$$

(S.12)

while (S.9) and Corollary S.1(i) give

$$
\int_{\mathbb{R}} |\hat{f}(\lambda)|^2 (|\lambda|^2 \wedge 1) |\mathbb{E}e^{-i\lambda x_{t+k,t+k}}| \, d\lambda \\
\leq \int_{\mathbb{R}} [((|\lambda|^{2(1+\beta)}/|f|)^2)^{1/2}] \wedge \|f\|_1^2 |\mathbb{E}e^{-i\lambda x_{t+k,t+k}}| \, d\lambda \\
\leq C k^{-3(1+\beta)/2} \|f\|_1^2 + e^{-\gamma_1 k} \|f\|_1^2.
$$

(S.13)

Together, (S.11)–(S.13) yield

$$
\mathbb{E}_{t-k} \xi_{k,t}^2 f \leq C s^{-1/2} \left(k^{-3(1+\beta)/2} \|f\|_1^2 + e^{-\gamma_1 k} \|f\|_1^2\right),
$$

which does not depend on $t$, and thus applies also to $\|\mathbb{E}_{t-k} \xi_{k,t+s}^2 f\|_\infty$. Hence

$$
\sum_{s=1}^{(n-k-t)/2} \|\mathbb{E}_{t-k} \xi_{k,t+s}^2 f\|_\infty \leq C k_n^{1/2} \left(k^{-3(1+\beta)/2} \|f\|_1^2 + e^{-\gamma_1 k} \|f\|_1^2\right).
$$

(S.14)

We come finally to $\|\xi_{k,t}^2 f\|_\infty$. Returning to (S.10), we have by (S.9) and Corollary S.1(i) that

$$
\|\xi_{k,t}^2 f\|_\infty \leq C \left(\int_{\mathbb{R}} |\hat{f}(\lambda)| |\mathbb{E}e^{-i\lambda x_{t+1,t+k}}| \, d\lambda\right)^2 \\
\leq C_1 \left(\int_{\mathbb{R}} [(|\lambda|^{2(1+\beta)}/|f|)^2)] \wedge \|f\|_1^2 |\mathbb{E}e^{-i\lambda x_{t+1,t+k}}| \, d\lambda\right)^2 \\
\leq C_2 k^{-3(1+\beta)/2} \|f\|_1^2 + e^{-\gamma_1 k} \|f\|_1^2 \\
\leq C_3 k_n^{1/2} \left(k^{-3(1+\beta)/2} \|f\|_1^2 + e^{-\gamma_1 k} \|f\|_1^2\right),
$$

(S.15)

where the final bound follows because $k \leq k_n$. The result now follows from (S.14) and (S.15), and the fact that

$$
k_n^{1/2} = (n^{-1} k_n) n k_n^{-1/2} \asymp (n^{-1} k_n) e_n.
$$

Proof of Lemma S.2. Exactly as in the proof of Lemma 9.3 in Duffy (2015), for $f \in BI$

$$
\mathbb{E}t |f(x_{t+k})| \leq C \|f\|_1 \int_{\mathbb{R}} |\mathbb{E}e^{-i\lambda x_{t+1,t+k}}| \, d\lambda,
$$
while for \( f \in \mathcal{B}I[\beta] \),
\[
|E_t f(x_{t+k})| \leq C \int_{\mathbb{R}} [||f||_{\beta} |\lambda|^\beta] \lambda E e^{-i\lambda x_{t-1,t+k}} \, d\lambda,
\]
whereupon both parts follow by Corollary S.1(i).

**Proof of Lemma S.3.** Exactly as in the proof of Lemma 9.4 in Duffy (2015),
\[
E|e^{-i\lambda \varepsilon_0} - E e^{-i\lambda \varepsilon_0}|^2 \leq C E[|\lambda \varepsilon_0|^2 \wedge 1] \leq C_1 |\lambda|^2
\]
since \( E\varepsilon_0^2 < \infty \). The result now follows by noting that the l.h.s. is also bounded by 4, and applying the Cauchy-Schwarz inequality.

**Proof of Lemma S.4.** Let \( h(\lambda) := z_1 |\lambda|^p \wedge z_2 \) and \( K := \# K \). By Hölder’s inequality,
\[
\int_{\mathbb{R}} h(\lambda) \prod_{l \in K} |\psi_l(a_l(\rho_n) \lambda)| \, d\lambda \leq \prod_{l \in K} \left( \int_{\mathbb{R}} h(\lambda)|\psi_l(a_l(\rho_n) \lambda)|^K \, d\lambda \right)^{1/K}
\]
\[
\leq \max_{l \in K} \int_{\mathbb{R}} h(\lambda)|\psi_l(a_l(\rho_n) \lambda)|^K \, d\lambda
\]
\[
\leq \int_{\mathbb{R}} h(\lambda) \max_{k_0/2 \leq k \leq 2k_n} |\psi_l(a_l(\rho_n) \lambda)|^K \, d\lambda.
\]
Further, by Lemma C.1(ii), the preceding is bounded by
\[
z_1 \int_{\mathbb{R}} |\lambda|^p e^{-\lambda \gamma^2 K} \, d\lambda + z_2 e^{-\gamma K} \|\psi_\varepsilon\|_1 \leq C \, z_1 K^{-(p+1)/2} + z_2 e^{-\gamma K}.
\]
Since \( K \geq k/4 \), the result follows.

**Proof of Corollary S.1.** Since
\[
x_{t+1,t+k,t+k} = \sum_{l=0}^{k-1} a_l \xi_{t+k-l} \quad x_{t-s+1,t-1,t+k} = \sum_{l=k+1}^{k+s-1} a_l \xi_{t+k-l},
\]
we have
\[
|E e^{-i\lambda x_{t+1,t+k,t+k}}| \leq \prod_{l=[(k \wedge k_n)/2] + 1}^{k \wedge k_n \!} |\psi_\varepsilon(a_l(\rho_n) \lambda)|
\]
and so part (i) follows immediately from Lemma S.4. For part (ii), we note that
\[
|E e^{-i\lambda x_{t-s+1,t-1,t+k}}| \leq \prod_{l=k+1}^{k-1+s \wedge k_n} |\psi_\varepsilon(a_l(\rho_n) \lambda)|,
\]
and so, when \( s \geq k_0 \), the required bound also follows from Lemma S.4. When \( s < k_0 \), the crude bound \( |E e^{-i\lambda x_{t-s+1,t-1,t+k}}| \leq |\psi_\varepsilon(a_{k+1}(\rho_n) \lambda)| \) suffices, in view of \( \psi_\varepsilon \in L^1 \) and Lemma C.1(i).
## S.6 Index of notation

**Greek and Roman symbols**

Listed in (Roman) alphabetical order. Greek symbols are listed according to their English names: thus \( \omega \), as ‘omega’, appears before \( \theta \), as ‘theta’.

| Symbol       | Description                                                                 | Page |
|--------------|------------------------------------------------------------------------------|------|
| \( a_{t,t-k} \) | coefficient sequence                                                        | (C.1) |
| \( a_k \)    | equals \( a_{t,t-k} \) for \( 0 \leq k \leq t - 1 \)                        | (C.2) |
| AsySz        | asymptotic size                                                              | (2.8) |
| AsyMaxCP     | asymptotic maximum coverage probability                                       | (2.9) |
| BI           | bounded and integrable functions on \( \mathbb{R} \)                        | App. A |
| BI[\( \beta \)] | \( f \in BI \) with \( \| f \|_{[\beta]} < \infty \)                     | (D.4) |
| BL           | bounded and Lipschitz functions on \( \mathbb{R} \)                        | Sec. 1 |
| BIL          | bounded, integrable and Lipschitz functions on \( \mathbb{R} \)            | Sec. 1 |
| \( C, C_1 \) | generic constants                                                            | App. A |
| \( C_n \)   | confidence set                                                               | (2.6) |
| CP\(_n\)     | coverage probability                                                         | (2.7) |
| \( d_n \)   | equals \( \text{var}(x_n)^{1/2} \)                                          | (3.5) |
| \( \delta_n(\beta, \mathcal{F}) \) | appears in Prop. D.1(ii)                                                 | (D.5) |
| \( \varepsilon_t \) | innovation sequence                                                          | DGP2 |
| \( e_n \)   | norming sequence, equals \( nd_n^{-1} \)                                    | App. A |
| \( \mathbb{E}_t \) | expectation conditional on \( \mathcal{F}^t_{\infty} \)                    | App. C.2 |
| \( \eta \)  | mixing variate in limiting variance                                          | (A.3) |
| \( \hat{f} \) | Fourier transform of \( f \)                                                 | App. D |
| \( \hat{F}_{n,i} \) | non-predictability test statistic                                           | (2.11) |
| \( \mathcal{F}_s^t \) | \( \sigma(\{ \varepsilon_r \}_{r=s}^t) \)                                 | App. C.1 |
| \( \mathcal{G}_t \) | \( \sigma(\{ x_s, u_s \}_{s \leq t}) \)                                    | DGP4 |
| \( \gamma \) | nuisance parameters \((\psi_\varepsilon, \{ \phi_k \}, \sigma_u^2, \{ F_{at} \})\) | Sec. 2.1 |
| \( \Gamma \) | parameter space for \( \gamma \)                                            | Sec. 2.1 |
| \( h, h_n \) | bandwidth                                                                  | (2.3) |
| \( \underline{h}_n, \overline{h}_n \) | upper and lower bounds defining \( \mathcal{H}_n \)                      | H    |
| \( \mathcal{H}_n \) | set of allowable bandwidths                                                  | H    |
| \( k_n \)   | real sequence related to \( \rho_n \)                                       | (C.7) |
| \( K, K_h \) | smoothing kernel, \( K_h(x) := h^{-1}K(h^{-1}u) \)                         | (2.3) |
| \( J_e \)   | standardised OU process                                                     | (3.7) |
| \( \ell_{ucc}(\mathbb{R}) \) | bounded functions with ucc topology                                         | Sec. 3.3 |
| \( L^p \)   | Lebesgue \( p \)-integrable functions on \( \mathbb{R} \)                  | Sec. 1 |
| \( L_c \)   | local time of \( J_c \)                                                     | (3.6) |
| Symbol | Description |
|--------|-------------|
| Lip    | Lipschitz continuous functions on $\mathbb{R}$ | Sec. 1 |
| $m$    | regression function | (2.1) |
| $\tilde{m}_n$ | local level estimate of $m$ | (2.3) |
| $\mathcal{M}$ | class of allowable regression functions | DGP1 |
| $\mathcal{M}_{n,k}f$ | martingale components in decomposition of $\mathcal{S}_nf$ | (D.2) |
| $\mu$ | limiting spatial density under $\mathcal{R}$ | (3.6) |
| $\tilde{\mu}$ | generic limiting spatial density | (3.1) |
| $\mu_n$ | spatial density estimate | (3.8) |
| $N_{n,f}$ | remainder from decomposition of $\mathcal{S}_nf$ | (D.2) |
| $\nu_\rho$ | limiting stationary density | (3.6) |
| $\omega_n^2$ | scaling factor | (3.10) |
| $\phi$ | sum of the $\phi_k$'s | DGP3 |
| $\phi_k$ | coefficients defining the linear process $v_t$ | (2.2) |
| $\varphi$ | standard Gaussian density | (3.6) |
| $\psi_{\varepsilon}$ | characteristic function of $\varepsilon_0$ | DGP2 |
| $\rho$ | autoregressive parameter | (2.2) |
| $R$ | parameter space for $\rho$ | DGP3 |
| $\mathcal{R}$ | $\mathcal{R}_{\text{ST}} \cup \mathcal{R}_{\text{MI}} \cup \mathcal{R}_{\text{LU}}$ | Sec. 2.2 |
| $\mathcal{R}_{\text{ST}}$ | stationary sequences in $\mathbb{R}$ | Sec. 2.2 |
| $\mathcal{R}_{\text{MI}}$ | mildly integrated sequences in $\mathbb{R}$ | Sec. 2.2 |
| $\mathcal{R}_{\text{LU}}$ | local-to-unity sequences in $\mathbb{R}$ | Sec. 2.2 |
| $s_n^2$ | asymptotic variance estimator | (2.5) |
| $\mathcal{S}_n$ | summation operator, $\mathcal{S}_n f := \sum_{t=1}^n f(x_t)$ | (D.2) |
| $\sigma_\rho^2$ | stationary variance at $\rho < 1$ | App. B.2 |
| $\sigma_u^2$ | conditional variance of $u_t$ | DGP4 |
| $\hat{\sigma}_u^2$ | estimate of $\sigma_u^2$ | (2.5) |
| $\hat{t}_n$ | $t$-statistic | (2.4) |
| $\theta$ | hypothesised value of $m(x)$ | Sec. 2.1 |
| $\hat{\theta}_n$ | sample mean of $y_t$ | Sec. 2.3 |
| $u_t$ | regression disturbance | (2.1) |
| $\mathcal{U}_{n,k}f$ | squared variation of $\mathcal{M}_{n,k}f$ | (D.3) |
| $v_n$ | martingale component of $\hat{t}_n$ | (2.10) |
| $v_t$ | linear process built from $\{\varepsilon_t\}$ | (2.2) |
| $V_{i,t}$ | variance components | (C.5) |
| $\mathcal{V}_{n,k}f$ | conditional variance of $\mathcal{M}_{n,k}f$ | (D.3) |
| $W$ | standard Brownian motion | (3.7) |
\( x_t \) regressor, partial sum of \( \{v_t\} \) ........................................  (2.2)
\( \tilde{x}_{n,t} \) standardised regressor ........................................ (3.5)
\( x_{s,r,t} \) component of \( x_t \) .................................................. (C.4)
\( X_n \) standardised regressor process .................................. Rem. 3.5
\( X \) finite-dimensional limit of \( X_n \) ...................................... Rem. 3.5
\( \mathcal{X} \) subset of \( \mathbb{R} \) ......................................................... Sec. 2.3
\( \xi_{kt} f \) martingale difference components of \( \mathcal{M}_{nk} f \) .......... (D.1)
\( y_t \) dependent variable in regression ................................. (2.1)

Symbols not connected to Greek or Roman letters

Ordered alphabetically by their description.

\( =_d \) both sides have the same distribution
\( \xrightarrow{p} \) converges in probability to
\( \xrightarrow{\text{fdd}} \) finite-dimensional convergence .......................... Sec. 1
\( \lfloor \cdot \rfloor \) floor function (integer part) .............................. Sec. 1
\( \|f\|_{[\beta]} \) Fourier norm .................................................. (D.4)
\( \lesssim_p \) l.h.s. bounded in probability by the r.h.s. ............... Sec. 1
\( \lesssim_C \) l.h.s. less than \( C \) times the r.h.s. .......................... Sec. 1
\( \lesssim \) l.h.s of no greater order than the r.h.s. ....................... Sec. 1
\( \|f\|_{\text{Lip}} \) Lipschitz norm .......................................... App. C.2
\( \|f\|_p \) \( L^p \) norm, \( (\int |f(x)|^p)^{1/p} \), for function \( f \) .......... App. A
\( \|X\|_p \) \( L^p \) norm, \( (\mathbb{E}|X|^p)^{1/p} \), for random variable \( X \) .......... App. A
\( \sim \) strong asymptotic equivalence ................................. Sec. 1
\( \|\mathcal{F}\| \) supremum of norm \( \|\cdot\| \) over \( \mathcal{F} \): \( \sup_{f \in \mathcal{F}} \|f\| \) .......... App. D
\( [a(x)]_{x \in \mathcal{X}} \) vector \( (a(x_1), \ldots, a(x_m))^\prime \), for \( \{x_1, \ldots, x_m\} = \mathcal{X} \) ........ Sec. 2.3
\( \asymp \) weak asymptotic equivalence .................................. Sec. 1
\( \asymp \) weak convergence (van der Vaart and Wellner, 1996) ...... Sec. 1