NONSTANDARD BRAID RELATIONS AND CHEBYSHEV POLYNOMIALS

JONAH BLASIAK

Abstract. A fundamental open problem in algebraic combinatorics is to find a positive combinatorial formula for Kronecker coefficients, which are multiplicities of the decomposition of the tensor product of two $S_r$-irreducibles into irreducibles. Mulmuley and Sohoni attempt to solve this problem using canonical basis theory, by first constructing a nonstandard Hecke algebra $B_r$, which, though not a Hopf algebra, is a $u$-analogue of the Hopf algebra $\mathbb{C}S_r$ in some sense (where $u$ is the Hecke algebra parameter). For $r = 3$, we study this Hopf-like structure in detail. We define a nonstandard Hecke algebra $\mathcal{H}_3^{(k)} \subseteq \mathcal{H}_3^{\otimes k}$, determine its irreducible representations over $\mathbb{Q}(u)$, and show that it has a presentation with a nonstandard braid relation that involves Chebyshev polynomials evaluated at $\frac{1}{u+u^{-1}}$. We generalize this to Hecke algebras of dihedral groups. We go on to show that these nonstandard Hecke algebras have bases similar to the Kazhdan-Lusztig basis of $\mathcal{H}_3$ and are cellular algebras in the sense of Graham and Lehrer.

1. Introduction

The Kronecker coefficient $g_{\lambda\mu\nu}$ is the multiplicity of an irreducible $S_r$-module $M_{\nu}$ in the tensor product $M_{\lambda} \otimes M_{\mu}$. A fundamental and difficult open problem in algebraic combinatorics is to find a positive combinatorial formula for these coefficients. Although this problem has been studied since the early twentieth century, the general case still seems out of reach. In the last ten years this problem has seen a resurgence of effort, perhaps because of its recently discovered connections to quantum information theory [8] and complexity theory [15]. Much of the recent progress has been for Kronecker coefficients indexed by two two-row shapes, i.e., when $\lambda$ and $\mu$ have two rows: a positive formula was given by Remmel and Whitehead in [16] and further improvements were made by Rosas [18] and Briand-Orellana-Rosas [6]. Briand-Orellana-Rosas [6, 7] and Ballantine-Orellana [2] have also made progress on the special case of reduced Kronecker coefficients, sometimes called the stable limit, in which the first part of the partitions $\lambda, \mu, \nu$ is large.

In a series of recent papers, Mulmuley, in part with Sohoni and Narayanan, describes an approach to $P$ vs. $NP$ and related lower bound problems in complexity theory using algebraic geometry and representation theory, termed geometric complexity theory. Understanding Kronecker coefficients, particularly, having a good rule for when they are zero, is critical to their plan. In fact, Mulmuley gives a substantial informal argument claiming that if certain difficult separation conjectures like $P \neq NP$ are true, then

Key words and phrases. Hecke algebra, Chebyshev polynomials, cellular algebra, Kronecker problem, geometric complexity theory.

The author is currently an NSF postdoctoral fellow.
there is a \#P formula for Kronecker coefficients and a polynomial time algorithm that determines whether a Kronecker coefficient is nonzero \cite{13}. Thus from the complexity-theoretic perspective, there is hope that Kronecker coefficients will have nice formulae like those for Littlewood-Richardson coefficients, though experience suggests they will be much harder.

In \cite{15}, Mulmuley and Sohoni attempt to use canonical bases to understand Kronecker coefficients by constructing an algebra defined over \( \mathbb{Z}[u, u^{-1}] \) that carries some of the information of the Hopf algebra structure of the group algebra \( \mathbb{Z}S_r \) and specializes to it at \( u = 1 \). Specifically, they construct the nonstandard Hecke algebra \( \overline{H}_r \) (denoted \( B_r \) in \cite{15}), which is a subalgebra of the tensor square of the Hecke algebra \( H_r \) such that the inclusion \( \overline{\Delta} : \overline{H}_r \hookrightarrow \overline{H}_r \otimes \overline{H}_r \) is \( u \)-analogue of the coproduct \( \Delta \) of \( \mathbb{Z}S_r \) (see Definition 2.2). The goal is then to break up the Kronecker problem into two steps \cite{14}:

1. Determine the multiplicity \( n_{\lambda, \mu}^\alpha \) of an irreducible \( \overline{H}_r \)-module \( M_\alpha \) in the tensor product \( M_\lambda \otimes M_\mu \).
2. Determine the multiplicity \( m^\nu_\alpha \) of the \( S_r \)-irreducible \( M_\nu \) in \( M_\alpha |_{u = 1} \).

The resulting formula for Kronecker coefficients is

\[
g_{\lambda \mu \nu} = \sum_\alpha n_{\lambda, \mu}^\alpha m^\nu_\alpha. \tag{1}\]

Thus a positive combinatorial formula for \( n_{\lambda, \mu}^\alpha \) and \( m^\nu_\alpha \) would yield one for Kronecker coefficients.

However, this approach meets with serious difficulties. The defining relations of the algebras \( \overline{H}_r \) seem to be extremely complicated and remain mysterious even for \( r = 4 \). Problem (1) seems to be within reach, and, in the forthcoming paper \cite{5}, we solve it in the two-row case. For problem (2), the hope is to find a canonical basis of \( \overline{M}_\alpha \) that has a cellular decomposition into \( S_r \)-irreducibles at \( u = 1 \), however this seems to be extremely difficult.

In this paper we study a family of algebras \( \overline{H}_3^{(k)} \) that contains \( \overline{H}_3 \) and \( \overline{H}_3 \) as cases \( k = 1, 2 \). We discover a remarkable connection between the defining relations of these algebras and Chebyshev polynomials \( T_k(x) \). Specifically, we show that \( \overline{H}_3^{(k)} \) is generated by \( P_1^{(k)}, P_2^{(k)} \) and has a relation, which we call the nonstandard braid relation, that generalizes the braid relation for \( k = 1 \):

\[
P_1^{(k)}(P_{21}^{(k)} - ([2]^k a^{(1)})^2)(P_{21}^{(k)} - ([2]^k a^{(2)})^2) \ldots (P_{21}^{(k)} - ([2]^k a^{(k)})^2) =
P_2^{(k)}(P_{12}^{(k)} - ([2]^k a^{(1)})^2)(P_{12}^{(k)} - ([2]^k a^{(2)})^2) \ldots (P_{12}^{(k)} - ([2]^k a^{(k)})^2), \tag{2}\]

where \( [2] = u + u^{-1}, P_{ij}^{(k)} = P_{ji}^{(1)} P_{ij}^{(2)}, \) and the coefficient \( a^{(j)} \) is equal to \( T_j([2]) \).

Chebyshev polynomials come up in several places in nearby areas of algebra, however their appearance here seems to be genuinely new. For example, Chebyshev polynomials appear in the criterion for semisimplicity of Temperley-Lieb and Jones algebras \cite{10, 11} (also see \cite{14}). In \cite{14} they appear in three ways – as just mentioned, in giving the dimension of a centralizer algebra of a Temperley-Lieb algebra, and in calculating the decomposition of a Brauer algebra module into Temperley-Lieb algebra modules. In this paper, Chebyshev polynomials appear as the coefficients in the relations of an algebra.
In the remainder of this paper, we first define the nonstandard Hecke algebras $\mathcal{H}^{(k)}_W$ and establish some of their basic properties (§2). Section 3 contains our main results—a description of the irreducible representations of $\mathcal{H}^{(k)}_W$ (Theorem 3.4) and a presentation for $\mathcal{H}^{(k)}_3$ (Theorem 3.7). In §4 we generalize these results to nonstandard Hecke algebras $\mathcal{H}^{(k)}_W$ of dihedral groups. In this case, the nonstandard braid relation involves a multivariate version of Chebyshev polynomials and the $\mathcal{H}^{(k)}_W$-irreducibles are parameterized by signed compositions of $k$ (see Definition 4.2). We further show (§5) that the algebras $\mathcal{H}^{(k)}_W$ have bases generalizing the Kazhdan-Lusztig basis of $\mathcal{H}_3$ and are cellular algebras in the sense of Graham and Lehrer [11].

2. Nonstandard Hecke Algebras

After recalling the definition of the (standard) Hecke algebra $\mathcal{H}_W$ of a Coxeter group $W$, we introduce the nonstandard Hecke algebra $\mathcal{H}^{(k)}_W$ of [15]. Hecke algebras are not Hopf algebras in a natural way, and the nonstandard Hecke algebra $\mathcal{H}_W$ is in a sense the smallest deformation of $\mathcal{H}_W$ that also deforms the Hopf algebra structure of the group algebra $\mathbb{Z}W$. We show that the Hecke algebra $R\mathcal{H}_{S_2}$ is a Hopf algebra (for suitable rings $R$). We then define the sequence of algebras $\mathcal{H}^{(k)}_W$, $k \geq 1$, that begins with $\mathcal{H}_W$ ($k = 1$) and $\mathcal{H}_W$ ($k = 2$). We record some basic facts about the representation theory of these algebras and define anti-automorphisms that behave like the antipode of a Hopf algebra.

2.1. Let $(W,S)$ be a Coxeter group with length function $\ell$. We work over the ground ring $A = \mathbb{Z}[u,u^{-1}]$, the ring of Laurent polynomials in the indeterminate $u$.

**Definition 2.1.** The Hecke algebra $\mathcal{H}_W$ of a Coxeter group $(W,S)$ is the free $A$-module with basis $\{T_w : w \in W\}$ and relations generated by

$$T_u T_v = T_{uv}, \quad (T_s - u)(T_s + u^{-1}) = 0 \quad \text{if } \ell(uv) = \ell(u) + \ell(v)$$

$$\text{if } s \in S.$$

(3)

For any $J \subseteq S$, the parabolic subgroup $W_J$ is the subgroup of $W$ generated by $J$. We let $(\mathcal{H}_W)_J$ denote the subalgebra of $\mathcal{H}_W$ with $A$-basis $\{T_w : w \in W_J\}$, which is isomorphic to $\mathcal{H}_{W_J}$.

For any commutative ring $K$ and ring homomorphism $A \to K$, let $K\mathcal{H}_W = K \otimes_A \mathcal{H}_W$. We will often let $K = \mathbb{Q}(u)$, the quotient field of $A$. If the ring $K$ is understood and $A \to K$ is given by $u \mapsto z$, then we also write $\mathcal{H}_W|_{u=z}$ for $K\mathcal{H}_W$. The Hecke algebra $\mathcal{H}_W$ over $A$ is the generic Hecke algebra of $W$ and $K\mathcal{H}_W$ is a specialization of $\mathcal{H}_W$.

We are particularly interested in the type $A$ case in this paper, and in this case $(W,S) = (S_r,\{s_1,\ldots,s_{r-1}\})$ and we abbreviate $\mathcal{H}_S$ by $\mathcal{H}_r$.

2.2. The $u$-integers are $[k] := \frac{u^k - u^{-k}}{u - u^{-1}} \in A$. We also use the notation $[k]$ to denote the set $\{1,\ldots,k\}$, but these usages should be easy to distinguish from context. We also set $f = [2]^2$ because this constant appears particularly often.

Let $C_s = T_s + u$ and $C_s = T_s - u^{-1}$ for each $s \in S$. These are the simplest unsigned and signed Kazhdan-Lusztig basis elements. They are also proportional to the primitive central idempotents of $K(\mathcal{H}_W)_{\{s\}} \cong K\mathcal{H}_2$ when $K$ is a ring containing $\frac{1}{2}$: define $p_+$ =
Write $\epsilon_+, \epsilon_-$ for the one-dimensional trivial and sign representations of $\mathcal{H}_W$, which are defined by

$$
\epsilon_+: C_s' \mapsto [2], \quad \epsilon_-: C_s' \mapsto 0, \quad s \in S.
$$

We identify these algebra homomorphisms $\epsilon_+, \epsilon_-: \mathcal{H}_W \to A$ with left $\mathcal{H}_W$-modules in the usual way.

There is an $A$-algebra automorphism $\theta: \mathcal{H}_W \to \mathcal{H}_W$ defined by $\theta(T_s) = -T_s^{-1}$, $s \in S$. Let $1^\text{op}$ be the $A$-anti-automorphism of $\mathcal{H}_W$ given by $1^\text{op}(T_w) = T_w^{-1}$. Let $\theta^\text{op}$ be the $A$-anti-automorphism of $\mathcal{H}_W$ given by $\theta^\text{op} = \theta \circ 1^\text{op} = 1^\text{op} \circ \theta$. We will establish some basic properties of these anti-automorphisms in [2.5].

Let $\eta$ be the unique $A$-algebra homomorphism from $A$ to $\mathcal{H}_W$. At $u = 1$, the maps $\eta, \epsilon_+, 1^\text{op}$ specialize to the unit, counit, and antipode of the Hopf algebra $ZW$.

### Definition 2.2
The nonstandard Hecke algebra $\bar{\mathcal{H}}_W$ is the subalgebra of $\mathcal{H}_W \otimes \mathcal{H}_W$ generated by the elements

$$
P_s := C_s' \otimes C_s' + C_s \otimes C_s, \quad s \in S.
$$

We let $\bar{\Delta}: \bar{\mathcal{H}}_W \hookrightarrow \mathcal{H}_W \otimes \mathcal{H}_W$ denote the canonical inclusion, which we think of as a deformation of the coproduct $\Delta_{ZW}: ZW \to ZW \otimes ZW$, $w \mapsto w \otimes w$.

The nonstandard Hecke algebra is also the subalgebra of $\mathcal{H}_W \otimes \mathcal{H}_W$ generated by

$$
Q_s := f - P_s = -C_s' \otimes C_s - C_s \otimes C_s', \quad s \in S.
$$

Despite their simple definition, the nonstandard Hecke algebras seem to be extremely difficult to describe in terms of generators and relations. Indeed, the main purpose of this paper is to work out such a presentation for dihedral groups $W$. For the easiest case $W = S_2$, the story is quite nice.

### Proposition 2.3
Set $R = A[\frac{1}{2}]$ and $R_1 = Z[\frac{1}{2}]$. We have $R\bar{\mathcal{H}}_2 \cong R\mathcal{H}_2$ by $\frac{1}{f}P_1 \mapsto p_+$.

Then

(i) $R\bar{\mathcal{H}}_2$ is a Hopf algebra with coproduct $\Delta = \bar{\Delta}$, antipode $1^\text{op}$, counit $\epsilon_+$, and unit $\eta$.

(ii) the Hopf algebra $R\mathcal{H}_2|_{u=1}$, with Hopf algebra structure coming from (i), is isomorphic to the group algebra $R_1S_2$ with its usual Hopf algebra structure.

Moreover, the Hopf algebra structure of (i) is the unique way to make the algebra $R\mathcal{H}_2$ into a Hopf algebra so that (ii) is satisfied.

**Proof.** The isomorphism $R\bar{\mathcal{H}}_2 \cong R\mathcal{H}_2$ is immediate from the observation that $\Delta(\frac{1}{f}P_s) = p_+ \otimes p_+ + p_- \otimes p_-$ is an idempotent in $R\mathcal{H}_2 \otimes R\mathcal{H}_2$.

The axiom for the antipode is a special case of Proposition 2.4 to come (and also easy to check directly). We need to check the axioms

$$(\epsilon_+ \otimes id) \circ \Delta = id = (id \otimes \epsilon_+) \circ \Delta.$$
\[(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta,\]

which is easy. For example, for the second, observe that both sides applied to \(p_+\) yield

\[p_+ \otimes p_+ \otimes p_+ + p_+ \otimes p_- \otimes p_- + p_- \otimes p_+ \otimes p_- + p_- \otimes p_- \otimes p_+.
\]

It is straightforward to see that (ii) is satisfied. We check only that the coproduct commutes with specialization, and omit verifying that this is also so for the antipode, the counit, and the unit:

\[
\Delta|_{u=1}(\frac{1}{2}(1 + s_1)) = \frac{1}{2}(1 + s_1) \otimes \frac{1}{2}(1 + s_1) + \frac{1}{2}(1 - s_1) \otimes \frac{1}{2}(1 - s_1) = \frac{1}{2}(1 \otimes 1 + s_1 \otimes s_1) = \Delta_{R_1S_2}(\frac{1}{2}(1 + s_1)),
\]

where \(\Delta|_{u=1}\) is the specialization of \(\Delta\) and \(\Delta_{R_1S_2}\) is the usual coproduct on \(R_1S_2\).

For the uniqueness statement, we use that \(R \mathcal{H}_2 \otimes R \mathcal{H}_2\) is isomorphic to a product of matrix algebras. Explicitly,

\[R \mathcal{H}_2 \otimes R \mathcal{H}_2 \cong R(p_+ \otimes p_+) \oplus R(p_+ \otimes p_-) \oplus R(p_- \otimes p_+) \oplus R(p_- \otimes p_-).
\]

The map \(\Delta\) is determined by \(\Delta(p_+)\), and the image of \(\Delta\) is isomorphic to \(R \mathcal{H}_2\) if and only if \(\Delta(p_+)\) is an idempotent not equal to the identity. We also have that \(\Delta_{R_1S_2}(p_+) = p_+ \otimes p_+ + p_- \otimes p_-\). The only idempotent of \(R \mathcal{H}_2 \otimes R \mathcal{H}_2\) that specializes to \(\Delta_{R_1S_2}(p_+)\) at \(u = 1\) is \(p_+ \otimes p_+ + p_- \otimes p_-\), hence this must be \(\Delta(p_+)\) as desired. Additionally, the counit is determined uniquely by the comultiplication; the only anti-automorphisms of \(R \mathcal{H}_2\) are \(1^{op}\) and \(\theta^{op}\), and only \(1^{op}\) satisfies the required axiom. \(\square\)

For the next proposition, let \(\eta, \epsilon_+, \epsilon_-, 1^{op}, \theta^{op}\) be as in (2.7), and let \(\mu\) be the multiplication map for \(\mathcal{H}_W\).

**Proposition 2.4.** The involutions \(1^{op}\) and \(\theta^{op}\) are antipodes in the following sense:

\[
\mu \circ (1^{op} \otimes 1) \circ \bar{\Delta} = \eta \circ \bar{\epsilon}_+,
\]

\[\mu \circ (\theta^{op} \otimes 1) \circ \bar{\Delta} = \eta \circ \bar{\epsilon}_-, \tag{5}\]

where these are equalities of maps from \(\mathcal{H}_W\) to \(\mathcal{H}_W\).

**Proof.** First set \(S = 1^{op}\). The right-hand side of (4) is the algebra homomorphism defined by \(\eta \circ \bar{\epsilon}_+(Q_s) = 0, s \in S\). To see that the left-hand side defines the same thing, first observe that it takes \(Q_s\) to 0 and 1 to 1:

\[
\mu \circ (S \otimes 1)(Q_s) = \mu(Q_s) = -C_s'C_s - C_sC'_s = 0, s \in S.
\]

Next, let \(\bar{\Delta}(Q_s) = \sum_i a^{(i)} \otimes b^{(i)}\) and \(\bar{\Delta}(x) = \sum_j c^{(j)} \otimes d^{(j)}\) for some \(x \in \mathcal{H}_W\). Since \(\bar{\Delta}\) is an algebra homomorphism, there holds

\[
\mu \circ (S \otimes 1) \circ \bar{\Delta}(Q_s) = \mu \circ (S \otimes 1) \left( \sum_i (a^{(i)} \otimes b^{(i)}) \sum_j (c^{(j)} \otimes d^{(j)}) \right)
\]

\[= \sum_{i,j} S(c^{(j)})S(a^{(i)})b^{(i)}d^{(j)} = 0, \tag{7}\]

where the last equality is by (3).

We can similarly show that \(\mu \circ (S \otimes 1) \circ \bar{\Delta}(xQ_s) = 0\) for any \(x \in \mathcal{H}_W\). Thus the two-sided ideal generated by the \(Q_s\) is sent to zero, so the left and right-hand sides of
agree. Equation (4) is proved in a similar way by replacing \( Q_s \) with \( P_s \) and \( \epsilon_+ \) with \( \epsilon_- \) above. \( \square \)

2.4. By the above proposition, we can define \( \Delta^{(k)} : \mathcal{H}_2 \to \mathcal{H}_2 \otimes \mathcal{H}_2 \) inductively by \( \Delta^{(k)} := (\Delta^{(k-1)} \otimes 1) \circ \Delta = (1 \otimes \Delta^{(k-1)}) \circ \Delta \) and \( \Delta^{(2)} := \Delta \). Explicitly,

\[
\Delta^{(k)}(p_+) = \sum_{a \in \{+, -\}^k \setminus \{i : a_i = -\} \text{ is even}} p_{a_1} \otimes \cdots \otimes p_{a_k}.
\] (8)

It is now natural to generalize \( \mathcal{H}_W \) as follows.

**Definition 2.5.** The nonstandard Hecke algebra \( \mathcal{H}_W^{(k)} \) is the subalgebra of \( \mathcal{H}_W \otimes \mathcal{H}_W \) generated by the \( \mathcal{P}_s^{(k)} := [2]^k \Delta^{(k)}(p_+) \) for all \( s \in S \), where \( \Delta^{(k)} = \iota_\otimes \circ \Delta^{(k)} \) and \( \iota_\) is the inclusion \( \mathcal{H}_2 \cong (\mathcal{H}_W)_s \hookrightarrow \mathcal{H}_W \). Let \( \Delta^{(k)} : \mathcal{H}_W^{(k)} \hookrightarrow \mathcal{H}_W^{\otimes k} \) be the canonical inclusion.

Set \( Q_s^{(k)} = [2]^k - \mathcal{P}_s^{(k)} \). The set \( S \) of simple reflections of \( W \) will always be denoted \( \{s_1, \ldots, s_{r-1}\} \) in this paper, and we let \( \mathcal{P}_{s_1, \ldots, s_l}^{(k)} \) be shorthand for \( \mathcal{P}_s^{(k)} \mathcal{P}_s^{(k)} \cdots \mathcal{P}_s^{(k)} \). Note that \( \mathcal{H}_W^{(2)} = \mathcal{H}_W \), \( \mathcal{P}_s^{(2)} = \mathcal{P}_s \) and \( \mathcal{H}_W^{(1)} = \mathcal{H}_W \), \( \mathcal{P}_s^{(1)} = C'_s \), and we set \( \mathcal{H}_W^{(0)} = A \).

**Remark 2.6.** Proposition [2,3] supports the idea that the \( \mathcal{H}_W^{(k)} \) are “the smallest approximation to a Hopf algebra on \( \mathcal{H}_W \) deforming the Hopf algebra \( \mathbb{Z}W \).” An interesting problem is to make this precise by showing that \( \prod_{k \geq 0} \mathcal{H}_W^{(k)} \) (or a similar algebra) is a universal object in some categorical sense. This may be closely related to the problem of constructing a right adjoint to the forgetful functor from Hopf algebras to algebras, which was recently done in [1].

For any nonnegative integers \( k_l, k_r \) with \( k_l + k_r = k \), we have \( \Delta^{(k)} = (\Delta^{(k_l)} \otimes \Delta^{(k_r)}) \circ \Delta \); applying this to \([2]^k p_+\) yields

\[
\mathcal{P}_s^{(k)} = \mathcal{P}_s^{(k_l)} \otimes \mathcal{P}_s^{(k_r)} + Q_{s}^{(k_l)} \otimes Q_{s}^{(k_r)} = 2\mathcal{P}_s^{(k_l)} \otimes \mathcal{P}_s^{(k_r)} - [2]^{k_l} \otimes \mathcal{P}_s^{(k_r)} - \mathcal{P}_s^{(k_r)} \otimes [2]^{k_r} + [2]^k
\] (9)

for all \( s \in S \). As a consequence, there is a canonical inclusion

\[
\bar{\Delta}^{(k_l, k_r)} : \mathcal{H}_W^{(k_l)} \hookrightarrow \mathcal{H}_W^{(k_l)} \otimes \mathcal{H}_W^{(k_r)} , \quad \mathcal{P}_s^{(k_l)} \mapsto \mathcal{P}_s^{(k_l)} \otimes \mathcal{P}_s^{(k_r)} + Q_{s}^{(k_l)} \otimes Q_{s}^{(k_r)}.
\] (11)

Thus any pair \( \bar{M}_l, \bar{M}_r \), where \( \bar{M}_l \) is a \( \mathcal{H}_W^{(k_l)} \)-module and \( \bar{M}_r \) is a \( \mathcal{H}_W^{(k_r)} \)-module, gives rise to the \( \mathcal{H}_W^{(k)} \)-module \( \bar{M}_l \otimes \bar{M}_r \) by restriction.

We have the following commutativity property

\[
\text{Res}_W \mathcal{H}_W^{(k)} \bar{M}_l \otimes \bar{M}_r \cong \text{Res}_W \bar{M}_l \otimes \bar{M}_r ,
\] (12)

where the isomorphism is given by the map swapping tensor factors; this is an \( \mathcal{H}_W^{(k)} \)-module homomorphism by (9).

There are one-dimensional trivial and sign representations of \( \mathcal{H}_W^{(k)} \), which we denote by \( \bar{\epsilon}_+ \) and \( \bar{\epsilon}_- \):

\[
\bar{\epsilon}_+ : \mathcal{P}_s^{(k)} \mapsto [2]^k , \quad \bar{\epsilon}_- : \mathcal{P}_s^{(k)} \mapsto 0, \quad s \in S.
\]

For a ring homomorphism \( K \to A \), we have the specialization \( K \mathcal{H}_W^{(k)} := K \otimes_A \mathcal{H}_W^{(k)} \) of the nonstandard Hecke algebra.
2.5. Here we record some useful results about the anti-automorphisms $\theta^{op}$ and their corresponding anti-automorphisms $\lambda^{op}$ of $H_W$ and $\tilde{H}_W^{(k)}$. Many of the observations here are also made in [15, §10].

Any anti-automorphism $S$ of an $A$-algebra $H$ allows us to define duals of $H$-modules: let $\langle , \rangle : M \otimes M^* \rightarrow A$ be the canonical pairing, where $M^*$ is the $A$-module $\text{Hom}_A(M, A)$. Then the $H$-module structure on $M^*$ is defined by

$$\langle m, hm' \rangle = \langle S(h)m, m' \rangle$$

for any $h \in H$, $m \in M$, $m' \in M^*$. We write $M^\circ$ (resp. $M^\#$) for the $H_W$-module dual to $M$ corresponding to the anti-automorphism $\theta^{op}$ (resp. $\lambda^{op}$).

We note that for $W = S$, dualization $M \mapsto M^\#$ corresponds to transposing partitions.

**Proposition 2.7** ([5] (see also [12, Exercises 2.7, 3.14])). Let $M_\lambda$ be the Specht module of $H_r$ of shape $\lambda$. Then

$$M_\lambda^\circ \cong M_{\lambda'} \quad \text{and} \quad M_\lambda^\# \cong M_{\lambda'},$$

where $\lambda'$ is the transpose of the partition $\lambda$.

Let $\mathcal{A}_\theta$ be the subgroup of automorphisms of $H_W^{\otimes k}$ generated by

$$\theta_i = 1 \otimes \cdots \otimes 1 \otimes \theta \otimes 1 \otimes \cdots \otimes 1,$$

where the $\theta$ appears in the $i$-th tensor factor. Let $\mathcal{A}_\theta^0$ be the subgroup of $\mathcal{A}_\theta$ generated by $\theta_i \theta_j, i, j \in [k]$. This is a subgroup of index 2 and consists of the involutions involving an even number of $\theta$'s and the rest 1's.

**Proposition 2.8.** The elements of $\mathcal{A}_\theta$ restrict nicely to $H_W^{(k)}$:

(i) $\alpha(\mathcal{P}_s^{(k)}) = \begin{cases} \mathcal{P}_s^{(k)} & \text{if } \alpha \in \mathcal{A}_\theta^0 \\ \mathcal{Q}_s^{(k)} & \text{if } \alpha \in \theta_1 \mathcal{A}_\theta^0 \end{cases}$ for all $s \in S$.

(ii) $F_W^{(k)}$ is left stable by the elements of $\mathcal{A}_\theta$.

(iii) The restriction of an element of $\mathcal{A}_\theta$ to $F_W^{(k)}$ corresponds to the map $\mathcal{A}_\theta \rightarrow \text{Aut}(F_W^{(k)}), \theta_i \mapsto \theta^{(k)}$, where $\theta^{(k)} : H_W^{(k)} \rightarrow H_W^{(k)}$ is the $A$-algebra homomorphism determined by $\theta^{(k)}(\mathcal{P}_s^{(k)}) = \mathcal{Q}_s^{(k)}$ for all $s \in S$.

(iv) The homomorphism $\theta^{(k)}$ is an involution of $F_W^{(k)}$.

(v) The inclusion $\Delta^{(k)}_{(k_1, k_2)}$ of (11) induces a map $\text{Aut}(F_W^{(k_1)} \otimes F_W^{(k_2)}) \rightarrow \text{Aut}(F_W^{(k)}), \text{which sends } \theta^{(k_1)} \otimes 1 \text{ and } 1 \otimes \theta^{(k_2)} \text{ to } \theta^{(k)}$.

**Proof.** It is enough to check (i) for $W = S_2$, and this can be seen directly from the observation $\theta(p_+) = p_-$ and by comparing

$$\frac{1}{2^k} \mathcal{Q}_1^{(k)} = \Delta^{(k)}(p_-) = \sum_{\{a_i \in \{+,-\}^k \mid \{i : a_i = -\} \text{ is odd}} p_{a_1} \otimes \ldots \otimes p_{a_k}$$

with the similar expression for $\Delta^{(k)}(p_+)$ in [8]. The remaining statements follow easily. \(\square\)
There are also anti-automorphisms \(1^\circ := 1^\circ \otimes \ldots \otimes 1^\circ\) and \((\theta(j))^\circ := 1^\circ \circ \theta(j)\) of \(\mathcal{H}_W^{(k)}\), where \(\theta(j)\) is defined in Proposition \(2.8\) (iii). We also write \(M^o\) (resp. \(M^\#\)) for the \(\mathcal{H}_W^{(k)}\)-module dual to \(M\) corresponding to the anti-automorphism \(1^\circ\) (resp. \((\theta(j))^\circ\)). The next proposition is immediate from Proposition \(2.8\) (v) and definitions.

**Proposition 2.9.** Let \(\tilde{M}_l\) be an \(\mathcal{H}_W^{(k)}\)-module and \(\tilde{M}_r\) an \(\mathcal{H}_W^{(k)}\)-module and assume that these are free and finite-dimensional as \(A\)-modules. Then

\[
(\tilde{M}_l \otimes \tilde{M}_r)^o \cong \tilde{M}_l^o \otimes \tilde{M}_r^o \cong \tilde{M}_l^\# \otimes \tilde{M}_r^\#;
(\tilde{M}_l \otimes \tilde{M}_r)^\# \cong \tilde{M}_l^o \otimes \tilde{M}_r^\# \cong \tilde{M}_l^\# \otimes \tilde{M}_r^o.
\]

### 3. The nonstandard braid relation

Here we determine the irreducible representations of the nonstandard Hecke algebra \(\mathcal{H}_{3}^{(k)}\) and find that it has a two-dimensional irreducible with defining constant \([2]^kT_j(\frac{1}{[2]^3})\) for each \(j \in [k]\). We deduce from this the nonstandard braid relation \([2]\) for \(\mathcal{H}_{3}^{(k)}\).

3.1. It is convenient to prove our results for \(\mathcal{H}_{3}^{(k)}\) in the following general setup. All of the two-dimensional irreducible representations of \(K\mathcal{H}_W^{(k)}\) (for \(W\) any dihedral group, not just \(W = S_3\), and suitable field \(K\)) are of the form \(\tilde{X}^{(k)}(c) \cong K^2\), for some constant \(c \in K\), defined by the following matrices giving the action of \(P_i^{(k)}\) on \(\tilde{X}^{(k)}(c)\):

\[
P_1^{(k)} \mapsto \begin{pmatrix} [2]^k & c \\ 0 & 0 \end{pmatrix},\; P_2^{(k)} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & [2]^k \end{pmatrix}.
\]

(13) Here we have specified a basis \((x_1, x_2)\) for \(\tilde{X}^{(k)}(c)\) and are thinking of matrices as acting on the left on column vectors, so that the \(j\)-th column of these matrices gives the coefficients of \(P_i^{(k)}x_j\) in the basis \((x_1, x_2)\).

Recall that a representation of an algebra over a field \(K\) is absolutely irreducible if it is irreducible over any field extension of \(K\) (the appendix of [12] is a good quick reference for this and other basic definitions and results about finite dimensional algebras over a field). Observe that \(\tilde{X}^{(k)}(c)\) has a one-dimensional submodule if and only if \(
\begin{pmatrix} 0 \\ 1 \end{pmatrix}\) or \(
\begin{pmatrix} [2]^k \\ -1 \end{pmatrix}\) spans a submodule. Since \(
\begin{pmatrix} 0 \\ 1 \end{pmatrix}\) spans a submodule if and only if \(c = 0\) and \(
\begin{pmatrix} [2]^k \\ -1 \end{pmatrix}\) spans a submodule if and only if \(c = f^k\), we conclude that

\[
\tilde{X}^{(k)}(c)\text{ is absolutely irreducible } \iff c \notin \{0, f^k\}.
\]

(14) \((\tilde{X}^{(k)}(c)\) can always be considered as a module for the subalgebra of \(\text{End}_K(K^2)\) generated by the matrices above, even if it is not a module for the algebras we are interested in.)

One also checks easily that

\[
X^{(k)}(c) \cong X^{(k)}(c') \iff c = c'.
\]

(15)
Proposition 3.1. Let $k_l, k_r$ be positive integers with $k_l + k_r = k$. Suppose that $\bar{X}_l = \bar{X}^{(k_l)}(c_l), \bar{X}_r = \bar{X}^{(k_r)}(c_r)$ are irreducible $K\mathcal{H}_{W}^{(k_l)}, K\mathcal{H}_{W}^{(k_r)}$-modules, respectively with constants $c_l, c_r \in K$. Put $a_l = \sqrt{c_l}, a_r = \sqrt{c_r}$ and define
\begin{equation}
    a_\pm = a_la_r \pm \sqrt{(f^{k_l} - c_l)(f^{k_r} - c_r)}.
\end{equation}
Assume that $a_la_r, a_\pm \in K$ and that $K$ is an integral domain in which $2 \neq 0$. Then $\bar{X}_l \otimes \bar{X}_r$ is a $K\mathcal{H}_{W}^{(k)}$-module via $\Delta^{(k_l,k_r)}$ (see (11)) with the following decomposition into irreducibles
\begin{equation}
    \bar{X}_l \otimes \bar{X}_r \cong \begin{cases}
    \bar{X}^{(k)}(a^2_+) \oplus \bar{X}^{(k)}(a^2_-) & a^2_+ \neq f^k \text{ and } a_- \neq 0, \\
    \bar{e}_+ + \bar{e}_- & a^2_+ = f^k \text{ and } a_- \neq 0, \\
    \bar{e}_1 + \bar{e}_2 & a^2_+ \neq f^k \text{ and } a_- = 0, \\
    \bar{e}_+ \bar{e}_- + \bar{e}_1 \bar{e}_2 & a^2_+ = f^k \text{ and } a_- = 0,
    \end{cases}
\end{equation}
where $\bar{e}_1$ and $\bar{e}_2$ are one-dimensional representations given by
\begin{align*}
    \bar{e}_1 : & \quad P_1^{(k)} \mapsto [2]^k, \quad P_2^{(k)} \mapsto 0, \\
    \bar{e}_2 : & \quad P_1^{(k)} \mapsto 0, \quad P_2^{(k)} \mapsto [2]^k.
\end{align*}

Remark 3.2. A direct calculation shows that, with the appropriate convention for square roots,
\begin{align*}
    a^2_+ = f^k & \iff f^{k_1}c_l = f^{k_2}c_r, \\
    a_- = 0 & \iff f^k = f^{k_1}c_r + f^{k_2}c_l.
\end{align*}
Note that by (14) and (16), our assumptions on $K$ and that $\bar{X}_l$ and $\bar{X}_r$ are irreducible imply $a^2_+ \neq a^2_-$. Thus the last three cases of (17) are the only degenerate cases that can occur.

Proof. Let $(x_{l1}, x_{l2})$ and $(x_{r1}, x_{r2})$ be bases for $\bar{X}_l$ and $\bar{X}_r$, respectively, corresponding to the matrices in (13). Using (10) we compute that the matrices for $\bar{X}_l \otimes \bar{X}_r$ in the basis $(x_{l1} \otimes x_{r2}, x_{l2} \otimes x_{r1}, x_{l2} \otimes x_{r2})$ are given by
\begin{equation}
    P_1^{(k)} \mapsto \begin{pmatrix}
    [2]^k & [2]^k c_r & [2]^k c_l \\
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0
    \end{pmatrix}, \quad P_2^{(k)} \mapsto \begin{pmatrix}
    [2]^k & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    2 & [2]^k c_l & [2]^k c_r & [2]^k
    \end{pmatrix}
\end{equation}
Now set
\begin{align}
    z_{1+} = \begin{pmatrix}
    a_la_r & a_+ & -[2]^k c_l \\
    -[2]^k c_r & a_+ & [2]^k
    \end{pmatrix}, \quad z_{2+} = \begin{pmatrix}
    [2]^k a_la_r a_+ \\
    -[2]^k a_la_r a_+ \\
    -[2]^k a_la_r a_+
    \end{pmatrix}, \quad z_{1-} = \begin{pmatrix}
    a_la_r a_- & -[2]^k c_l \\
    -[2]^k c_r & -[2]^k a_la_r a_-
    \end{pmatrix}, \quad z_{2-} = \begin{pmatrix}
    [2]^k a_la_r a_- \\
    -[2]^k a_la_r a_- \\
    -[2]^k a_la_r a_-
    \end{pmatrix},
\end{align}
The vectors $z_{1+}, z_{2+}, z_{1-}, z_{2-}$ were found using the form of the matrices in (18) to ensure that $z_{i+}$ and $z_{i-}$ span the $\bar{e}_+$-isotypic component of $\text{Res}_{(k)}^{(k)} \bar{X}_l \otimes \bar{X}_r$ for $i = 1, 2$. A direct computation shows that $K\{z_{1+}, z_{2+}\}$ (resp. $K\{z_{1-}, z_{2-}\}$) is a submodule of $\bar{X}_l \otimes \bar{X}_r$ provided $a_+$ (resp. $a_-$) is a solution to the quadratic equation
\begin{equation}
    y^2 - 2a_la_r y - f^k + f^{k_r}c_l + f^{k_l}c_r = 0.
\end{equation}
The constants $a_\pm$ are defined (16) so that this is so.

By comparing the first and last components of $z_{i+}, z_{i-}$, one checks that if $a_+^2 \neq f_k$ (resp. $a_- \neq 0$), then $K\{z_{i+}, z_{2+}\}$ (resp. $K\{z_{i-}, z_{2-}\}$) is two-dimensional. Moreover, by the remark before the proof, $a_+^2 \neq f_k$ (resp. $a_- \neq 0$) implies $\bar{X}(a_+^2)$ (resp. $\bar{X}(a_-^2)$) is irreducible. Then with the assumptions of the first case of (17), $(z_{1+}, z_{2+}, z_{1-}, z_{2-})$ is a basis of $\bar{X}_l \otimes \bar{X}_r$ and the action of $P_i^{(k)}$ in this basis is given by

$$
P_1^{(k)} \mapsto \begin{pmatrix} [2] & a_+^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & [2] & a_-^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_2^{(k)} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & [2] & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & [2] \end{pmatrix},$$

which verifies (17) in this case.

Next suppose $a_+^2 = f_k$ and $a_- \neq 0$, and define $z_{2+}'$ by

$$z_{2+}' := \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$ 

Then $(z_{1+}, z_{2+}', z_{1-}, z_{2-}')$ is a basis and the action of $P_i^{(k)}$ in this basis is given by

$$
P_1^{(k)} \mapsto \begin{pmatrix} [2] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & [2] & a_-^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_2^{(k)} \mapsto \begin{pmatrix} [2] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & [2] \end{pmatrix}.$$ 

The second case of (17) follows. The third and fourth cases of (17) are similar: if $a_+^2 \neq f_k$ and $a_- = 0$, then $(z_{1+}, z_{2+}, z_{1-}, z_{2-}')$ is a basis, where

$$z_{2-}' := \begin{pmatrix} [2] & a_la_r \\ -[2] & a_la_r \\ -[2] & a_la_r \\ a_- \end{pmatrix}.$$ 

If $a_+^2 = f_k$ and $a_- = 0$, then $(z_{1+}, z_{2+}', z_{1-}, z_{2-}')$ is a basis of $\bar{X}_l \otimes \bar{X}_r$. □

3.2. The $k$-th Chebyshev polynomial $T_k(x)$ of the first kind is the polynomial expressing $\cos(k\theta)$ in terms of $x = \cos(\theta)$. Chebyshev polynomials appear in many areas of mathematics including numerical analysis, special functions, approximation theory, and ergodic theory. Explicit formulas, recurrences, and generating functions are known for Chebyshev polynomials (17), though in this paper all we need are simple trigonometric identities. Recall from the introduction the constants

$$a^{(k)} = T_k\left(\frac{1}{[2]}\right), \quad k \geq 1.$$ 

(23)
We will see that \([2]^k a(j), j \in [k]\) are the defining constants of the two-dimensional irreducible representations of \(\mathcal{H}_3^{(k)}\). The first few coefficients \([2]^k a^{(k)}\) are

\[
\begin{align*}
[2]^1 a^{(1)} &= 1 = 1 \\
[2]^2 a^{(2)} &= -f + 2 = -u^2 - u^{-2} \\
[2]^3 a^{(3)} &= -3f + 4 = -3u^2 - 2 - 3u^{-2} \\
[2]^4 a^{(4)} &= f^2 - 8f + 8 = u^4 - 4u^2 - 2 - 4u^{-2} + u^{-4}.
\end{align*}
\] (24)

Though Chebyshev polynomials are usually defined for \(k \geq 0\), it is convenient to define them for all integers \(k\). Note that the definition above still makes sense and we have \(T_k(x) = T_{-k}(x)\). Also let \(T_k^1(x)\) be the element of \(\sqrt{1-x^2}\mathbb{Z}[x]\) obtained by expressing \(\sin(kx)\) in terms of \(x = \cos(\theta)\); more precisely, we should write \(T_k^1(x) \in y\mathbb{Z}[x] \subseteq \mathbb{Z}[x,y]/(x^2 + y^2 - 1)\). The \(k\)-th Chebyshev polynomial \(U_k(x)\) of the second kind is the polynomial expressing \(\sin((k+1)\theta)/\sin(\theta)\) in terms of \(x = \cos(\theta)\). Then we have \(T_k^1(x) = \sqrt{1-x^2}U_{k-1}(x)\) for \(k \geq 1\). Again, we may allow \(k\) to be any integer and there holds \(T_{-k}(x) = -T_k^1(x)\).

The calculation decomposing the tensor products \(\bar{X}_l \otimes \bar{X}_r\) (Proposition \ref{prop:tensor_product_decomposition}) and the following identity for Chebyshev polynomials are all we need to determine the irreducibles of \(\mathcal{H}_3^{(k)}\).

**Lemma 3.3.** For non-negative integers \(k_l, k_r\), there holds the following identity for Chebyshev polynomials (omitting the dependence of \(T_k(x)\) on \(x\))

\[
T_{k_l \mp k_r} = T_{k_l} T_{k_r} \pm T_{k_l}^1 T_{k_r}^1.
\]

**Proof.** This is immediate from the trigonometric identity for the cosine of a sum of angles:

\[
T_{k_l} T_{k_r} \pm T_{k_l}^1 T_{k_r}^1 = \cos(k_l\theta) \cos(k_r\theta) \pm \sin(k_l\theta) \sin(k_r\theta) = \cos((k_l \mp k_r)\theta) = T_{k_l \mp k_r}.
\]

**Theorem 3.4.** Define \(\bar{M}^{(k)} = \bar{X}^{(k)}(([2]^k a^{(k)})^2)\). For \(K = \mathbb{Q}(u)\), the irreducible representations of \(K \mathcal{H}_3^{(k)}\) consist of the trivial and sign representations

\[
\bar{\epsilon}_+, \bar{\epsilon}_-,
\]

and the \(k\) two-dimensional representations

\[
\epsilon_+^{\otimes k-1} \otimes \bar{M}^{(1)}, \epsilon_+^{\otimes k-2} \otimes \bar{M}^{(2)}, \ldots, \bar{M}^{(k)}.
\]

**Proof.** The proof is by induction on \(k\). The result is well-known for \(k = 1\), so assume \(k > 1\).

Since \(K \mathcal{H}_3^{(k)}\) is a subalgebra of \(K \mathcal{H}_3^{(k-1)} \otimes K \mathcal{H}_3^{(k-1)}\), every irreducible representation of \(K \mathcal{H}_3^{(k)}\) belongs to the composition series of \(X_l \otimes X_r\) for some irreducible \(K \mathcal{H}_3^{(k)}\)-module \(X_l\) and irreducible \(K \mathcal{H}_3^{(k-1)}\)-module \(X_r\). From the case \(X_l = \epsilon_+\), we conclude by induction that \(\bar{\epsilon}_+, \bar{\epsilon}_-\), and \(\epsilon_+^{\otimes j} \otimes \bar{M}^{(k-j)}, j \in [k-1]\) are distinct irreducibles of \(K \mathcal{H}_3^{(k)}\). To obtain the complete list of irreducibles, by commutativity \[\otimes\], it remains to decompose \(\bar{M}^{(1)} \otimes \bar{M}^{(k-1)}\) and \(\epsilon_- \otimes \bar{M}^{(k-1)}\) into irreducibles.

Decomposing \(\bar{M}^{(1)} \otimes \bar{M}^{(k-1)}\) into irreducibles is the crux of the proof, and this is a special case of Proposition \ref{prop:tensor_product_decomposition}. Note that \(\epsilon_+^{\otimes 2} \otimes \bar{M}^{(k-2)} = \bar{X}^{(k)}(([2]^k a^{(k-2)})^2)\). Then by
Proposition 3.1 and Lemma 3.3 with $k_1 = 1, k_r = k - 1$, $\bar{M}^{(1)} \otimes \bar{M}^{(k-1)}$ decomposes into irreducibles as

$$\bar{M}^{(1)} \otimes \bar{M}^{(k-1)} \cong (\epsilon_+^{\otimes 2} \otimes \bar{M}^{(k-2)}) \oplus \bar{M}^{(k)} \quad \text{if } k > 2, \quad (25)$$

$$\bar{M}^{(1)} \otimes \bar{M}^{(1)} \cong \bar{\epsilon}_+ \oplus \bar{\epsilon}_- \oplus \bar{M}^{(2)} \quad \text{if } k = 2. \quad (26)$$

Finally, $\epsilon_- \otimes \bar{M}^{(k-1)} \cong \epsilon_- \otimes (\bar{M}^{(k-1)})^\circ \cong \epsilon_+ \otimes (\bar{M}^{(k-1)})^\#$, where the second isomorphism is by Proposition 2.9 and the first is by $\bar{\epsilon}$ follows by induction using (25) and (26); the base case $\bar{M}^{(1)} \cong (\bar{M}^{(1)})^\circ \cong (\bar{M}^{(1)})^\#$ is the $r = 3$ case of Proposition 2.7. Thus the irreducible constituent of $\epsilon_- \otimes \bar{M}^{(k-1)}$ is already in our list.

It remains to check that $\bar{M}^{(k)}$ is distinct from $\epsilon_+^{\otimes j} \otimes \bar{M}^{(k-j)}$, $j \in [k - 1]$. This holds by (15) and the fact that $T_k(x)$ is a degree $k$ polynomial in $x$, implying $[2]^k a(j)$ is a polynomial in $[2]$ whose constant coefficient is nonzero if and only if $j = k$.

**Corollary 3.5.** The algebra $K\mathcal{H}_3^{(k)}$ is split semisimple for $K = \mathbb{Q}(u)$.

**Proof.** Proceed by induction on $k$. We can view the inclusion $\bar{\Delta}^{(1,k-1)}: \mathcal{H}_3^{(k)} \to \mathcal{H}_3 \otimes \mathcal{H}_3^{(k-1)}$ of (11) as one of left $\mathcal{H}_3^{(k)}$-modules. Since localizations are flat, $K$ is a flat $A$-module; thus, $K\bar{\mathcal{H}}_3^{(k)} \to K\mathcal{H}_3 \otimes K\mathcal{H}_3^{(k-1)}$ is also an inclusion (this fails for the specialization $A \to \mathbb{Z}$, $u \mapsto 1$). Hence to show that $K\bar{\mathcal{H}}_3^{(k)}$ is semisimple it suffices to show that $X_i \otimes \bar{X}_r$ is a direct sum of irreducible $K\bar{\mathcal{H}}_3^{(k)}$-modules for any irreducible $K\mathcal{H}_3$-module $X_i$ and irreducible $K\mathcal{H}_3^{(k-1)}$-module $\bar{X}_r$. The proof above gives an explicit decomposition of $X_i \otimes \bar{X}_r$ into irreducibles.

The algebra $K\bar{\mathcal{H}}_3^{(k)}$ is split because the irreducibles $\epsilon^{\otimes k-j} \otimes \bar{M}^{(j)} \cong \bar{X}^{(k)}(([2]^k a(j))^2), j \in [k]$ are absolutely irreducible by (14).

**Remark 3.6.** It was claimed in [15] that $\mathbb{R}\mathcal{H}_r$ is semisimple for any specialization $A \to \mathbb{R}$, however there is a mistake in the proof. In fact, the specialization $u \mapsto 1$ is not semisimple if $r > 2$. I have been convinced by the authors of [15] that the proof can be easily fixed to show that $\mathbb{R}(u)\mathcal{H}_r$ is semisimple. We strongly suspect that all the algebras $\mathbb{Q}(u)\mathcal{H}_W^{(k)}$ are semisimple for finite Weyl groups $W$, but we do not yet have a proof.

3.3. Now we can determine the nonstandard braid relation for $\bar{\mathcal{H}}_3^{(k)}$. Define

$$F_k(y) = (y - ([2]^k a(1))^2)(y - ([2]^k a(2))^2) \ldots (y - ([2]^k a(k))^2),$$

a polynomial in an indeterminate $y$ with coefficients in $A$.

**Theorem 3.7.** The algebra $\bar{\mathcal{H}}_3^{(k)}$ is the associative $A$-algebra generated by $\mathcal{P}_s^{(k)}, s \in S = \{s_1, s_2\}$, with quadratic relations

$$(\mathcal{P}_s^{(k)})^2 = [2]^k \mathcal{P}_s^{(k)}, \quad s \in S, \quad (27)$$

and nonstandard braid relation

$$\mathcal{P}_1^{(k)} F_k(\mathcal{P}_2^{(k)}) = \mathcal{P}_2^{(k)} F_k(\mathcal{P}_1^{(k)}), \quad (28)$$
proof. The quadratic relations follow from the fact that $\mathcal{H}_2$ is a Hopf algebra. Set
\[ h := \mathcal{P}_1^{(k)} F_k(\mathcal{P}_{21}^{(k)}) - \mathcal{P}_2^{(k)} F_k(\mathcal{P}_{12}^{(k)}) \in \mathcal{H}_3^{(k)}. \]
We next show that the nonstandard braid relation holds in $K\mathcal{H}_3^{(k)}$, for $K = \mathbb{Q}(u)$, i.e. $1 \otimes h = 0$ in $K\mathcal{H}_3^{(k)}$. To see this, one computes easily using (13) that $\mathcal{P}_2^{(k)} (\mathcal{P}_{12}^{(k)} - (2^k a^{(j)})^2)$ and $\mathcal{P}_1^{(k)} (\mathcal{P}_{21}^{(k)} - (2^k a^{(j)})^2)$ act on $\varepsilon_+^{\otimes k-j} \otimes M^{(j)}$ by 0. We also have that $\mathcal{P}_i^{(k)}$ acts by 0 on $\varepsilon_-$. Further noting that both sides of (28) act on $\varepsilon_+$ by the constant $[2]^k F_k(\mathbb{2}^{2k})$, we conclude using Theorem 3.4 that $1 \otimes h$ acts by 0 on all irreducible representations of $K\mathcal{H}_3^{(k)}$. The semisimplicity of $K\mathcal{H}_3^{(k)}$ then implies $1 \otimes h = 0$.

We next claim that $\mathcal{H}_3^{(k)} \to K\mathcal{H}_3^{(k)}$ is injective, which would imply $h = 0$, i.e. the nonstandard braid relation holds. The claim holds because $K\mathcal{H}_3^{(k)}$ is the localization of $\mathcal{H}_3^{(k)}$ at the multiplicative set $U = \mathbb{A} \setminus \{0\}$ and $U$ contains no zero divisors (the facts about localization needed for this would be standard in the commutative setting, and it is not difficult to show that they carry over to the case that the multiplicative set lies in the center of the ring). The multiplicative set $U$ contains no zero divisors because $\mathcal{H}_3^{(k)}$ is a subalgebra of the free $\mathbb{A}$-module $\mathcal{H}_W^{(k)}$.

To see that no other relations hold in $\mathcal{H}_3^{(k)}$, let $H$ be the $\mathbb{A}$-algebra generated by the $\mathcal{P}_i^{(k)}$ with relations (27) and (28). The algebra $H$ is a free $\mathbb{A}$-module with basis given by the $4k + 2$ monomials
\[ 1, \ (\mathcal{P}_{12}^{(k)})^j, \ (\mathcal{P}_{21}^{(k)})^j, \ \mathcal{P}_1^{(k)} (\mathcal{P}_{21}^{(k)})^{j-1}, \ \mathcal{P}_2^{(k)} (\mathcal{P}_{12}^{(k)})^{j-1}, \ \mathcal{P}_1^{(k)} (\mathcal{P}_{21}^{(k)})^{k}, \ j \in [k]. \]
By the split semisimplicity of $\mathbb{Q}(u)\mathcal{H}_3^{(k)}$, $\dim_{\mathbb{Q}(u)} \mathbb{Q}(u)\mathcal{H}_3^{(k)} = 4k + 2 = \dim_{\mathbb{Q}(u)} \mathbb{Q}(u)H$. An additional relation holding in $\mathcal{H}_3^{(k)}$ and not in $H$ would force $\dim_{\mathbb{Q}(u)} \mathbb{Q}(u)\mathcal{H}_3^{(k)} < \dim_{\mathbb{Q}(u)} \mathbb{Q}(u)H$, hence $H \cong \mathcal{H}_3^{(k)}$. \qed

In the proof above we have shown that

**Corollary 3.8.** The algebra $\mathcal{H}_3^{(k)}$ is free as an $\mathbb{A}$-module.

4. **Generalizations to dihedral groups**

Let $W$ be the dihedral group of order $2m$ with simple reflections $S = \{s_1, s_2\}$ and relations
\[ s^2 = id, \quad s \in S \]
\[ (s_1s_2)^m = id. \quad (29) \]
We will generalize the results of the previous section to the algebras $\mathcal{H}_W^{(k)}$. The two-dimensional irreducibles of $\mathcal{H}_W^{(k)}$ are indexed by signed compositions of $k$ (Definition 4.2) and their defining constants are given by the evaluation of a multivariate generalization of Chebyshev polynomials. This generalization does not seem to agree with previously studied generalizations of Chebyshev polynomials (for instance, it does not agree with those studied in [3]).
4.1. We first recall the description of the irreducible representations of $\mathcal{H}_W$.

**Theorem 4.1** (Kilmoyer-Solomon (see, for instance, [9, Theorem 8.3.1])). Let $\zeta$ be a primitive $m$-th root of unity and define $w_j = 2 + \zeta^j + \zeta^{-j}$. Set $n = \frac{m-1}{2}$ if $m$ is even and $n = \frac{m}{2}$ if $m$ is odd. Suppose $K$ is a subfield of $\mathbb{C}(u)$ containing the $w_j$. The irreducible representations of $K\mathcal{H}_W$ consist of

(a) the trivial and sign representations $\epsilon_+,$ $\epsilon_-,$
(b) $n$ two-dimensional representations denoted $M_j,$ $j \in [n],$ where $M_j = X^{(1)}(w_j),$ (c) if $m$ is even, two one-dimensional representations $\epsilon_1$ and $\epsilon_2$ determined by

\[ \epsilon_1 : \quad C'_1 \mapsto [2], \quad C'_2 \mapsto 0, \]
\[ \epsilon_2 : \quad C'_1 \mapsto 0, \quad C'_2 \mapsto [2]. \]

4.2. Here we introduce the combinatorics and multivariate generalization of Chebyshev polynomials needed to describe the irreducibles of $\mathcal{H}_W^{(k)}$.

**Definition 4.2.** A signed $n$-composition of $k$ is an $n$-vector $k = (k_1, \ldots, k_n)$ of integers such that $\sum_{i=1}^n |k_i| = k$. If $k$ is a signed composition of $k$, we also write $|k| = k$. Two signed $n$-compositions $k, k'$ are equivalent if $k = k'$ or $k = -k'$.

Let $\text{SComp}_{n,k}$ be the set of equivalence classes of signed $n$-compositions of $k$. Write $\text{SComp}_{n,\leq k} = \bigcup_{1 \leq k' \leq k} \text{SComp}_{n,k'}$ and $\text{SComp}_{n,<k} = \bigcup_{1 \leq k' < k} \text{SComp}_{n,k'}$.

**Example 4.3.** The signed 2-compositions of 3, with equivalent compositions in the same column, are

\[
\begin{align*}
(3,0) & \quad (2,1) & \quad (2,-1) & \quad (1,2) & \quad (1,-2) & \quad (0,3) \\
(-3,0) & \quad (-2,-1) & \quad (-2,1) & \quad (-1,-2) & \quad (-1,2) & \quad (0,-3).
\end{align*}
\]

Since the number of $i$-compositions of $k$ is $\binom{k-1}{i-1}$, we have the enumerative result

\[ |\text{SComp}_{n,k}| = \sum_{i=1}^{n} 2^{i-1} \binom{n}{i} \binom{k-1}{i-1}. \]

**Definition 4.4.** Let $k$ be a signed $n$-composition of $k$. The multivariate Chebyshev polynomial $T_k(x_1, \ldots, x_n, y_1, \ldots, y_n) \in R$ is the polynomial expressing $\cos(k_1\theta_1 + k_2\theta_2 + \cdots + k_n\theta_n)$ in terms of $x_j = \cos(\theta_j), y_j = \sin(\theta_j)$, where

\[ R = \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]/\bigoplus_j (x_j^2 + y_j^2 - 1). \]

Similarly, $T^1_k(x_1, \ldots, x_n, y_1, \ldots, y_n) \in R$ is the polynomial expressing $\sin(k_1\theta_1 + k_2\theta_2 + \cdots + k_n\theta_n)$ in terms of $x_j = \cos(\theta_j), y_j = \sin(\theta_j)$.

The polynomials $T_k, T^1_k$ can be expressed in terms of the univariate polynomials $T^0_k(x) := T_k(x,y)$ and $T^1_k(x,y) = yU_{k-1}(x)$ (as defined in [3,2] explicitly as follows. Let $\mathbb{F}_2$ be the finite field of order 2. If $k$ is an $n$-vector, then a vector $\alpha \in \mathbb{F}_2^n$ is
the irreducible representations of $A_4$. We give names to the two-dimensional irreducibles of $\overline{\text{A}}$.

Definition 4.5. Define $A^* = A[\sqrt{w_1}, \ldots, \sqrt{w_n}, \sqrt{f-w_1}, \ldots, \sqrt{f-w_n}]$. For each signed $n$-composition $k$ of $k$, define the following constants, which after being multiplied by $[2]^k$, belong to $A^*$:

- $a_k = T_k \left( \frac{\sqrt{w_1}}{2}, \ldots, \frac{\sqrt{w_n}}{2}, \frac{1}{2} \sqrt{f-w_1}, \ldots, \frac{1}{2} \sqrt{f-w_n} \right)$,
- $a_k^1 = T_k^1 \left( \frac{\sqrt{w_1}}{2}, \ldots, \frac{\sqrt{w_n}}{2}, \frac{1}{2} \sqrt{f-w_1}, \ldots, \frac{1}{2} \sqrt{f-w_n} \right)$.

To better understand these coefficients, it is helpful to compute the specialization

$$\left( a_k \right)_{u=1} = \cos \left( \frac{\pi}{m} \sum_{j=1}^{n} j k_j \right), \quad \left( a_k^1 \right)_{u=1} = \sin \left( \frac{\pi}{m} \sum_{j=1}^{n} j k_j \right).$$

4.3. We give names to the two-dimensional irreducibles of $\mathcal{H}_W^{(k)}$:

$$N_{k'}^{(k)} := \tilde{X}^{(k)}(([2]^k a_{k'})^2), \quad k' \in \text{SComp}_{n \leq k},$$

$$N_{k'}^{1(k)} := \tilde{X}^{(k)}(([2]^k a_{k'}^1)^2) \cong \epsilon_1 \otimes N_{k'}^{1(k-1)}, \quad k' \in \text{SComp}_{n < k}.$$

Note that these definitions make sense because $([2]^k a_{k'})^2$ and $([2]^k a_{k'}^1)^2$ are independent of the equivalence class of $k'$. The isomorphism in the second line will be seen in the course of the proof below. We also remark that $N_{k'}^{(k)} \cong \epsilon_{+}^{\otimes k-k'} \otimes N_{k'}^{(k)}$. In the special case that $k$ has only one nonzero component $k_j = k$, $N_{k}^{(k)} \cong \tilde{X}^{(k)}(([2]^k T_k(\sqrt{\frac{f}{2}}))^2)$ is similar to the representation $\tilde{M}^{(k)}$ in the $W = S_3$ case.

Theorem 4.6. Let $W, m$, and $n$ be as in Theorem 4.1. For $K$ the quotient field of $A^*$, the irreducible representations of $K \mathcal{H}_W^{(k)}$ consist of

- the trivial and sign representations $\epsilon_+, \epsilon_-$,
- for each $k' \in \text{SComp}_{n \leq k}$, a two-dimensional representation $N_{k'}^{(k)}$,
- if $m$ is even, two one-dimensional representations $\tilde{\epsilon}_1 \cong \epsilon_1 \otimes \epsilon_{+}^{\otimes k-1}$ and $\tilde{\epsilon}_2 \cong \epsilon_2 \otimes \epsilon_{+}^{\otimes k-1}$ given by
  $$\tilde{\epsilon}_1 : \mathcal{P}_1^{(k)} \mapsto [2]^k, \quad \mathcal{P}_2^{(k)} \mapsto 0,$$
  $$\tilde{\epsilon}_2 : \mathcal{P}_1^{(k)} \mapsto 0, \quad \mathcal{P}_2^{(k)} \mapsto [2]^k,$$
- if $m$ is even, for each $k' \in \text{SComp}_{n < k}$, a two-dimensional representation $N_{k'}^{1(k)}$. 
Proof. The proof is by induction on \( k \) and similar to that of Theorem 3.4. The base case \( k = 1 \) is Theorem 4.1.

As in the proof of Theorem 3.4, it suffices to decompose \( X_l \otimes \bar{X}_h \) into irreducibles for every irreducible \( K \mathcal{H}_W \)-module \( X_l \) and irreducible \( K \mathcal{H}_W^{(k-1)} \)-module \( \bar{X}_h \). The calculation we need is a special case of the following: suppose that \( k_l, k_r \) are positive integers such that \( k = k_l + k_r \) and \( k_l, k_r \) are signed \( n \)-compositions of \( k_l \), \( k_r \) respectively. Then by Proposition 3.1 and Lemma 4.8 (below), \( \bar{N}^{(k)}_{k_l} \otimes \bar{N}^{(k)}_{k_r} \) decomposes into irreducibles as (assume without loss of generality that if \( k_l = 1 \) is Theorem 4.1.

The proof is by induction on \( k \).

In particular, if we are given some signed \( n \)-composition \( k \) of \( k \), then we may certainly choose \( k_l, k_r \) such that \( k_l + k_r = k \). Thus the first summand of the right-hand side of (34) is \( \bar{N}^{(k)}_{k_l} \), so this is a representation of \( K \mathcal{H}_W^{(k)} \). And certainly \( |k_l - k_r|, |k_l + k_r| \leq k \), so the irreducibles on the right-hand side of (34) and (35) are in our list.

Next, we consider the case \( X_l = \epsilon_\cdot \). We have
\[
\epsilon_\cdot \otimes \bar{N}^{(k-1)}_{k_r} = \epsilon_\cdot \otimes \bar{N}^{(k-1)}_{k_r},
\]
which is in our list of irreducibles. The first isomorphism of (36) is \( \bar{N}^{(k-1)}_{k_r} \), which can be checked directly from (13), and the second is by Proposition 2.9. Also, if \( m \) is even, \( \epsilon_\cdot \otimes \bar{N}^{(k-1)}_{1} \), hence \( \epsilon_\cdot \otimes \bar{N}^{(k-1)}_{k_r} = \bar{N}^{(k-1)}_{k_r} \).

Finally, we consider the case that \( m \) is even and \( X_l = \epsilon_1 \). The action of \( P^{(k)}_{1} \) on \( \epsilon_1 \otimes \bar{X}^{(k-1)}(c) \) is given by
\[
P^{(k)}_{1} \mapsto \left( \begin{array}{cc} [k] \cdot & [c] \cdot \\ 0 & 0 \end{array} \right), \quad P^{(k)}_{2} \mapsto \left( \begin{array}{cc} [k] \cdot & 0 \\ -[c] \cdot & 0 \end{array} \right).
\]
(37)

Changing to the basis
\[
\left( \begin{array}{cc} 1 \\ 0 \end{array} \right), \quad \left( \begin{array}{cc} [k] \cdot \\ [c] \cdot \end{array} \right)
\]
shows that \( \epsilon_1 \otimes \bar{X}^{(k-1)}(c) \cong \bar{X}^{(k)}(f^k - cf) \), hence \( \epsilon_1 \otimes \bar{N}^{(k-1)}_{1} \cong \bar{N}^{(k)}_{1} \). A similar computation shows that \( \epsilon_2 \otimes \bar{N}^{(k-1)}_{1} \cong \bar{N}^{(k)}_{1} \).

It remains to prove that the list of irreducibles is distinct, and this is Proposition 4.9 (below).

Corollary 4.7. The algebra \( K \mathcal{H}_W^{(k)} \) is split semisimple for \( K \) the quotient field of \( A^* \), where \( A^* \) is as in Definition 4.5.

Proof. This follows from the proof above, similar to the \( W = S_3 \) case. The base case \( k = 1 \) requires the split semisimplicity of \( K \mathcal{H}_W \). A proof of this is given in [9, Corollary 8.3.2].

Lemma 4.8. For \( k_l, k_r \) signed \( n \)-compositions of \( k_l \), \( k_r \) respectively, there holds
\[
T_{k_l+1}^{k_l} = T_{k_l} T_{k_r} \pm T_{k_l}^{1} T_{k_r}^{1}.
\]
Proof. This is immediate from the trigonometric identity for the cosine of a sum of angles:

\[ T_k T_k^r \pm T_k^r T_k = \cos(k_{11} \theta_1 + \cdots + k_{1n} \theta_n) \cos(k_{r1} \theta_1 + \cdots + k_{rn} \theta_n) \pm \sin(k_{11} \theta_1 + \cdots + k_{1n} \theta_n) \sin(k_{r1} \theta_1 + \cdots + k_{rn} \theta_n) \]

(38)

Proposition 4.9. The coefficients \([2]^k a'_k, k' \in S\text{Comp}_{n \leq k},\) together with, if \(m\) is even, \([2]^k a'_k, k' \in S\text{Comp}_{n < k},\) are distinct elements of \(A^*\) (see Definition 4.5).

Proof. It suffices to consider the specializations \(\sigma : A \to \mathbb{R}, \sigma(u) \in \mathbb{R}_{>0}\). Set \(x = \frac{1}{\sigma(u) + \sigma(u^{-1})}\); the image of the map \(\mathbb{R}_{>0} \to \mathbb{R}, \sigma(u) \mapsto x\) is \((0, \frac{1}{2})\). If \(m\) is odd, it suffices to show that for any signed \(n\)-compositions \(k', k''\),

\[ \cos^2(k'_1 \arccos(\sqrt{w_1} x) + \cdots + k'_n \arccos(\sqrt{w_n} x)) = \cos^2(k''_1 \arccos(\sqrt{w_1} x) + \cdots + k''_n \arccos(\sqrt{w_n} x)) \]

(an equality of real-valued functions on \((0, \frac{1}{2})\)) implies \(k'\) is equivalent to \(k''\). Observe that \(\cos^2(a) = \cos^2(b)\), \(a, b \in \mathbb{R}\), if and only if

\[ a + b \in 2\pi \mathbb{Z} \cup (\pi + 2\pi \mathbb{Z}) \quad \text{or} \quad a - b \in 2\pi \mathbb{Z} \cup (\pi + 2\pi \mathbb{Z}). \]

Define the functions \(\alpha_\pm : (0, \frac{1}{2}) \to \mathbb{R}\) by

\[ x \mapsto (k'_1 \pm k''_1) \arccos(\sqrt{w_1} x) + \cdots + (k'_n \pm k''_n) \arccos(\sqrt{w_n} x) \]

If there exists \(x_0 \in (0, \frac{1}{2})\) such that \(\alpha_+(x_0) \notin 2\pi \mathbb{Z} \cup (\pi + 2\pi \mathbb{Z})\), then \(\alpha_-(x)\) is constant in some neighborhood \(N'\) of \(x_0\). Otherwise, \(\alpha_+\) is constant on \((0, \frac{1}{2})\). Since we can choose square roots such that \(\sqrt{w_j} = 2 \cos(\frac{\pi j}{m}) \in (0, 2)\), the result follows from the lemma below with \(z_j = 2 \cos(\frac{\pi j}{m})\). The additional arguments needed in the case \(m\) is even are similar. \(\square\)

We are grateful to Sergei Ivanov for the proof of the following lemma.

Lemma 4.10. Suppose \(g(x) = \sum_{i=0}^{\infty} a_i x^i\) is a non-polynomial real analytic function, convergent to this series on a neighborhood \(N\) of the origin. If \(z_1, \ldots, z_n\) are distinct positive real numbers, then the functions \(1, g(z_1 x), \ldots, g(z_n x),\) with domains restricted to \(N'\) for some \(N' \subseteq N\) having a limit point in \(N\), are linearly independent over \(\mathbb{R}\).

Proof. Suppose for a contradiction that \(c = c_1 g(z_1 x) + \cdots + c_n g(z_n x),\) \(c, c_j \in \mathbb{R}\) with some \(c_j\) nonzero. Assume without loss of generality that all of the \(c_j\) are nonzero and that \(z_1\) is the largest of the \(z_j\). Then

\[ c = \sum_{i=0}^{\infty} \left( \sum_{j=1}^{n} c_j z_j^i \right) a_i x^i \]

holds for all \(x \in N'\) implies \(\sum_j c_j z_j^i = 0\) for all \(i > 0\) such that \(a_i \neq 0\). For large \(i\), the term \(c_1 z_1^i\) dominates this sum, hence \(c_1 = 0\), contradiction. \(\square\)
4.4. Now we can determine the nonstandard braid relation for $\mathcal{H}_W^{(k)}$. Define

$$G_{m,k}(y) = \begin{cases} 
\prod_{k' \in S_{\text{Comp}_{n}, \leq k}} (y - ([2]^k a_{k'})^2) & \text{if } m \text{ is odd}, \\
y \prod_{k' \in S_{\text{Comp}_{n}, \leq k}} (y - ([2]^k a_{k'})^2) \prod_{k' \in S_{\text{Comp}_{n}, \leq k}} (y - ([2]^k a_{1}^1)^2) & \text{if } m \text{ is even}.
\end{cases}$$

(39)

Theorem 4.11. Let $W, m, n$ be as in Theorem 4.7. The algebra $A^* \mathcal{H}_W^{(k)}$ is the associative $A^*$-algebra generated by $P_s^{(k)}$, $s \in S = \{s_1, s_2\}$, with quadratic relations

$$(P_s^{(k)})^2 = [2]^k P_s^{(k)}, \quad s \in S,$$

and nonstandard braid relation

$$P_1^{(k)} G_{m,k}(P_{21}^{(k)}) = P_2^{(k)} G_{m,k}(P_{12}^{(k)}) \quad \text{if } m \text{ is odd},$$

(41)

$$G_{m,k}(P_{21}^{(k)}) = G_{m,k}(P_{12}^{(k)}) \quad \text{if } m \text{ is even}. $$

(42)

Proof. The proof is similar to that of Theorem 3.7. In the case $m$ is even, we need to check that both sides of (42) act on $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ by 0. This is clear since $P_1^{(k)}$ acts on these representations by 0.

In fact, we can deduce a stronger statement, which does not seem to be easy to prove directly.

Corollary 4.12. The polynomial $G_{m,k}(y)$ belongs to $A[y]$. Therefore Theorem 4.11 holds with $A$ in place of $A^*$.

Proof. Let $K$ (resp. $K^*$) be the field of fractions of $A$ (resp. $A^*$). Let $F$ be the $A$-algebra generated by $P_s^{(k)}$, $s \in S$, with quadratic relations (40). The ideal of relations $I$ is defined by the exact sequence

$$0 \to I \to F \to \mathcal{H}_W^{(k)} \to 0.$$ 

(43)

Since localizations are flat and free modules are flat, the sequence remains exact after tensoring with $K$ and $K^*$. Thus Theorem 4.11 says that

$$h := P_1^{(k)} G_{m,k}(P_{21}^{(k)}) - P_2^{(k)} G_{m,k}(P_{12}^{(k)}) \in K^* F$$

generates $K^* I$ if $m$ is odd (the even case is similar).

Now choose a graded lexicographic term order on monomials in $F$ and let $\{g_1, g_2, \ldots\}$ be a Groebner basis for $KI$. There exists an $i$ such that $g_i$ and $h$ have the same leading monomial. We must then have $g_i = ch$, $c \in K^*$, because if not we could cancel the leading terms of $g_i$ and $h$, contradicting that $\{h\}$ is a Groebner basis of $K^* I$. Since $g_i \in KF$, and the leading coefficient of $h$ is 1, we must have $c \in K$. It follows that $h \in KF$ and $KI = \langle h \rangle$. The desired conclusion $I = \langle h \rangle$ then follows by repeating the argument from the proof of Theorem 3.7.

We may also conclude, as in the $W = S_3$ case,

Corollary 4.13. The algebra $\mathcal{H}_W^{(k)}$ is free as an $A$-module.
5. A cellular basis for $\mathcal{H}_W^{(k)}$

Graham and Lehrer’s theory of cellular algebras [11] formalizes the notion of an algebra with a basis well-suited for studying representations of the algebra. The theory is modeled after the Kazhdan-Lusztig basis of $\mathcal{H}$ in which a basis element $C'_w$ is naturally labeled by the insertion and recording tableaux of $w$. We briefly introduce this theory (following some of the conventions in [12]) and show that $R\mathcal{H}_W^{(k)}$ (for $R$ a suitable localization of $A^*$) is a cellular algebra with a cellular basis generalizing the Kazhdan-Lusztig basis of $\mathcal{H}_3$ and the basis of $\mathcal{H}_3$ given in [13].

5.1. Let $H$ be an algebra over a commutative ring $R$.

Definition 5.1. Suppose that $(\Lambda, \geq)$ is a (finite) poset and that for each $\lambda \in \Lambda$ there is a finite indexing set $\mathcal{T}(\lambda)$ and distinct elements $C^\lambda_{ST} \in H$ for all $S, T \in \mathcal{T}(\lambda)$ such that

$$C = \{C^\lambda_{ST} : \lambda \in \Lambda \text{ and } S, T \in \mathcal{T}(\lambda)\}$$

is a (free) $R$-basis of $H$. For each $\Lambda' \subseteq \Lambda$ let $H_{\Lambda'}$ be the $R$-submodule of $H$ with basis $\{C^\mu_{ST} : \mu \in \Lambda' \text{ and } S, T \in \mathcal{T}(\mu)\}$; write $H_{\lambda}, H_{\lambda \leq \lambda}$ in place of $H_{\{\lambda\}}, H_{\{\mu \in \Lambda : \mu \leq \lambda\}}$.

The triple $(C, \Lambda, \mathcal{T})$ is a cellular basis of $H$ if

(i) the $R$-linear map $* : H \to H$ determined by $(C^\lambda_{ST})^* = C^\lambda_{TS}$, for all $\lambda \in \Lambda$ and all $S$ and $T$ in $\mathcal{T}(\lambda)$, is an algebra anti-isomorphism of $H$,

(ii) for any $\lambda \in \Lambda$ and $h \in H$ there exist $r_{S', S} \in R$, for $S', S \in \mathcal{T}(\lambda)$, such that for all $T \in \mathcal{T}(\lambda)$

$$hC^\lambda_{ST} \equiv \sum_{S' \in \mathcal{T}(\lambda)} r_{S', S}C^\lambda_{S'T} \mod H_{<\lambda}.$$

If $H$ has a cellular basis then we say that $H$ is a cellular algebra.

The cellular basis for $R\mathcal{H}_W^{(k)}$ ($R$ to be specified) is similar to the “banal” example of [11], which we now recall.

Example 5.2. For each element $\lambda \in \Lambda$, we are given an element $\sigma_\lambda$ of $R$. Let $H = R[y]/g(y)$ where $g(y) = \prod_{\lambda \in \Lambda} (y - \sigma_\lambda)$. Choose a partial order on $\Lambda$ such that for each pair $\mu, \lambda \in \Lambda$, we have $\mu \leq \lambda, \lambda \leq \mu$ or $\sigma_\mu - \sigma_\lambda$ is invertible in $R$ (for example, $\leq$ can be a total order). For $\lambda \in \Lambda$, let $\mathcal{T}(\lambda) = \{\lambda\}$ and set

$$C^\lambda_{\lambda, \lambda} = C^\lambda = \prod_{\mu \not\in \lambda} (y - \sigma_\mu).$$

The triple $(\{C^\lambda\}_{\lambda \in \Lambda}, \Lambda, \mathcal{T})$ is a cellular basis of $H$: one checks that $\{C^\lambda\}_{\lambda \in \Lambda}$ is an $R$-basis by evaluating a linear relation $\sum_{\lambda \in \Lambda} a_\lambda C^\lambda = 0$ at $x = \sigma_\mu$ for $\mu$ a maximal element of $\Lambda$; one concludes that $a_\mu = 0$ and shows by induction that the other $a_\lambda$’s are 0. Similar considerations show that for any $\lambda \in \Lambda$, $\{C^\mu : \mu \leq \lambda\}$ is an $R$-basis of the ideal of $H$ generated by $\prod_{\mu \not\in \lambda} (y - \sigma_\mu)$. We conclude that for $h(y) \in R[y]$, there holds $h(y)C^\lambda \equiv h(\sigma_\lambda)C^\lambda \mod H_{<\lambda}$.

The data for the cellular basis of $R\mathcal{H}_W^{(k)}$ is as follows:

- $\Lambda = \Lambda_1 \cup \Lambda_2$, where
The set $\text{SComp}_{n,k}^1$ is equal to $\text{SComp}_{n,k}$ and is decorated with a superscript 1 to distinguish its elements from those of $\text{SComp}_{n,k}$. It is convenient to define $\sigma_k$ for all $k \in \Lambda_2$ by

$$
\sigma_k = [2]^k a_k, \quad k \in \text{SComp}_{n,k},
\sigma_k = [2]^k a_1^k, \quad k \in \text{SComp}_{n,k}^1 \text{ and } m \text{ even},
$$

where $a_k, a_1^k$ are as in Definition 4.5.

Choose a partial order on $\Lambda_2$ such that for each $k, k' \in \Lambda_2$, either $k \leq k'$, $k' \leq k$, or $\sigma_k - \sigma_{k'}$ is invertible in $R$. Then let the poset on $\Lambda$ be that of $\Lambda_2$ with the elements of $\Lambda_1$ added as in the following diagrams.

\begin{equation}
\begin{array}{c}
\bar{\epsilon}_+ \\
\Lambda_2 \\
\bar{\epsilon}_-
\end{array}
\begin{array}{c}
\bar{\epsilon}_+ \\
\Lambda_2 \\
\bar{\epsilon}_-
\end{array}
\end{equation}

(44)

The cellular basis $C$ of $R \mathcal{H}_W^{(k)}$ consists of, if $m$ is odd,

$$
\begin{align*}
\epsilon^- & := 1, \\
C^{\epsilon^-}_{11} & := \sigma_k \mathcal{P}^{(k)}_1 \prod_{k' \leq k} (\mathcal{P}_1^{(21)} - \sigma_{k'}^2), \\
C^{\epsilon^-}_{21} & := \mathcal{P}^{(21)}_1 \prod_{k' \leq k} (\mathcal{P}_1^{(21)} - \sigma_{k'}^2), \\
C^{\epsilon^+}_{11} & := \sigma_k \mathcal{P}^{(k)}_2 \prod_{k' \leq k} (\mathcal{P}_2^{(12)} - \sigma_{k'}^2), \\
C^{\epsilon^+}_{22} & := \mathcal{P}^{(k)}_2 \prod_{k' \leq k} (\mathcal{P}_2^{(12)} - \sigma_{k'}^2), \\
C^{\epsilon^-}_{12} & := \mathcal{P}^{(12)}_1 \prod_{k' \leq \bar{\epsilon}^+} (\mathcal{P}_1^{(21)} - \sigma_{k'}^2), \\
C^{\epsilon^+}_{21} & := \mathcal{P}^{(21)}_1 \prod_{k' \leq \bar{\epsilon}^+} (\mathcal{P}_1^{(21)} - \sigma_{k'}^2),
\end{align*}
$$

for all $k \in \Lambda_2$ and the products are over all $k' \in \Lambda_2$ satisfying the stated conditions.

Note that the quantity defining $C^{\epsilon^+}$ above is exactly $\mathcal{P}^{(k)}_1 G_{m,k} (\mathcal{P}_2^{(21)})$, equal to $\mathcal{P}^{(k)}_2 G_{m,k} (\mathcal{P}_2^{(12)})$ by the nonstandard braid relation (41).
If \( m \) is even, then the cellular basis \( \mathcal{C} \) consists of

\[
\begin{align*}
C^{\varepsilon -} & := 1, \\
C^k_{11} & := \sigma_k \mathcal{P}_1^{(k)} \prod_{k' \not< k} (\mathcal{P}_{21}^{(k)} - \sigma_{k'}^2), \\
C^k_{21} & := \mathcal{P}_{21}^{(k)} \prod_{k' \not< k} (\mathcal{P}_{21}^{(k)} - \sigma_{k'}^2), \\
C^k_{22} & := \sigma_k \mathcal{P}_2^{(k)} \prod_{k' \not< k} (\mathcal{P}_{12}^{(k)} - \sigma_{k'}^2), \\
C^k_{12} & := \mathcal{P}_{12}^{(k)} \prod_{k' \not< \varepsilon_1} (\mathcal{P}_{21}^{(k)} - \sigma_{k'}^2), \\
C^\varepsilon_1 & := \mathcal{P}_1^{(k)} \prod_{k' \not< \varepsilon_1} (\mathcal{P}_{21}^{(k)} - \sigma_{k'}^2), \\
C^\varepsilon_2 & := \mathcal{P}_2^{(k)} \prod_{k' \not< \varepsilon_2} (\mathcal{P}_{12}^{(k)} - \sigma_{k'}^2), \\
C^\varepsilon_+ & := \mathcal{P}_{12}^{(k)} \prod_{k' \not< \varepsilon_+} (\mathcal{P}_{12}^{(k)} - \sigma_{k'}^2),
\end{align*}
\]

(46)

where the products are over all \( k' \in \Lambda_2 \) satisfying the stated conditions. The quantity defining \( C^{\varepsilon +} \) above is exactly \( G_{m,k}(\mathcal{P}_{12}^{(k)}) \), equal to \( G_{m,k}(\mathcal{P}_{21}^{(k)}) \) by the nonstandard braid relation (12).

**Proposition 5.3.** Let \( \Lambda, \mathcal{T}, \mathcal{C} \) and \( \sigma_k \) be as above and let \( R \) be a ring containing \( \mathbb{A}^* \) such that \( \sigma_k \) is invertible in \( R \) for all \( k \in \Lambda_2 \). The algebra \( H = R \hat{H}^{(k)}_W \) is a cellular algebra with cellular basis (\( \mathcal{C}, \hat{\Lambda}, \mathcal{T} \)). The anti-automorphism \( * \) of Definition 5.7 (i) is equal to \( 1^{op} \) of Definition 2.5.

**Proof.** We must show that \( \mathcal{C} \) is an \( R \)-basis of \( H \). As an \( H \times H^{op} \)-bimodule, the \( \bar{\varepsilon}_+ \times \bar{\varepsilon}_+ \)-isotypic component of the restriction of \( H \) to a \( H_{\{s_1\}} \times H_{\{s_1\}}^{op} \)-bimodule has an \( R \)-basis consisting of monomials of the form \( \mathcal{P}_1^{(k)}(\mathcal{P}_{21}^{(k)})^j \). The set \( \{C^k_{ij} : k \in \Lambda_2\} \cup \{\bar{\varepsilon}_i\} \) (united with \( \{\bar{\varepsilon}_1\} \) if \( m \) is even) is also an \( R \)-basis for this space by the same argument as in Example 5.2. Similar considerations show that for any \( \lambda \in \Lambda \) and \( i, j \in [2] \), the set

\[
\{C^{k'}_{ij} : k' < \lambda\} \cup \begin{cases} \{\bar{\varepsilon}_+, \bar{\varepsilon}_1\} & \text{if } m \text{ is even and } i = j, \\
\{\bar{\varepsilon}_+\} & \text{otherwise,}
\end{cases}
\]

is an \( R \)-basis for the \( \bar{\varepsilon}_+ \times \bar{\varepsilon}_+ \)-isotypic component of the \( H_{\{s_1\}} \times H_{\{s_1\}}^{op} \)-restriction of \( I_\lambda \), where \( I_\lambda \) is the two-sided ideal of \( H \) generated by \( \prod_{k' < \lambda} (\mathcal{P}_{12}^{(k)} - \sigma_{k'}^2) \). That \( \mathcal{C} \) is an \( R \)-basis of \( H \) follows by applying this to \( \lambda = \bar{\varepsilon}_- \). We may also deduce that

\[
\mathcal{P}_1^{(k)} C^k_{21} = \mathcal{P}_1^{(k)} \prod_{k' < k} (\mathcal{P}_{21}^{(k)} - \sigma_{k'}^2) = \mathcal{P}_1^{(k)} (\mathcal{P}_{21}^{(k)} - \sigma_k^2 + \sigma_{k'}^2) \prod_{k' < k} (\mathcal{P}_{21}^{(k)} - \sigma_{k'}^2) = \mathcal{P}_1^{(k)} \prod_{k' < k} (\mathcal{P}_{21}^{(k)} - \sigma_{k'}^2) + \sigma_k C^k_{11} \equiv \sigma_k C^k_{11} \mod H_{<k}. \tag{47}
\]

Similarly, \( \mathcal{P}_2^{(k)} C^k_{12} \equiv \sigma_k C^k_{22} \mod H_{<k} \). Thus the left action of \( H \) on \( H_{\leq k}/H_{<k} \) in the basis \( \{C^k_{11}, C^k_{21}, C^k_{22}, C^k_{12}\} \) is given by

\[
\mathcal{P}_1^{(k)} \mapsto \begin{pmatrix} [2]^k \sigma_k & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \sigma_k & [2]^k \end{pmatrix}, \quad \mathcal{P}_2^{(k)} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\
\sigma_k & [2]^k & 0 & 0 \\
0 & 0 & [2]^k & \sigma_k \\
0 & 0 & 0 & 0 \end{pmatrix}. \tag{48}
\]

If \( m \) is odd, this verifies condition (ii) of Definition 5.1.
If $m$ is even, we also need the following, immediate from the definition of $C$ in \((46)\),
\[
\begin{align*}
\mathcal{P}_1^{(k)} \varepsilon_1 &= [2]^{k} \varepsilon_1, \\
\mathcal{P}_2^{(k)} \varepsilon_1 &= \varepsilon_1 \equiv 0 \mod H_{< \varepsilon_1}, \\
\mathcal{P}_1^{(k)} \varepsilon_2 &= \varepsilon_+ \equiv 0 \mod H_{< \varepsilon_2}, \\
\mathcal{P}_2^{(k)} \varepsilon_2 &= [2]^{k} \varepsilon_2.
\end{align*}
\]

The claim that $* = 1^{op}$ is straightforward.  

\[\square\]

5.2. Cellular algebras are well-suited for studying specializations. For this, we need some additional definitions from \([11]\). Let $M_\lambda$ be the left $H$-module that is the submodule of $H_{<\lambda}/H_{<\lambda}$ with $R$-basis \(\{C^\lambda_{ST} : S \in \mathcal{T}(\lambda)\}\) for some $T \in \mathcal{T}(\lambda)$; this basis is independent of $T$ and we denote its elements by $C^\lambda_S = C^\lambda_{ST}, S \in \mathcal{T}(\lambda)$.

**Definition 5.4.** For $\lambda \in \Lambda$, the bilinear form $\phi_\lambda : M_\lambda \times M_\lambda \to R$ is defined in the basis \(\{C^\lambda_S : S \in \mathcal{T}(\lambda)\}\) as follows. For $S, T \in \mathcal{T}(\lambda)$ let $\phi_\lambda(C^\lambda_S, C^\lambda_T)$ be the element of $R$ determined by
\[
C^\lambda_{UV}C^\lambda_{TV} \equiv \phi_\lambda(C^\lambda_S, C^\lambda_T)C^\lambda_{UV} \mod H_{<\lambda},
\]
where $U$ and $V$ are any elements of $\mathcal{T}(\lambda)$.

**Proposition 5.5.** The bilinear forms $\phi_k, k \in \Lambda_2$, for the cellular algebra $R\mathcal{K}_W^{(k)}$ of Proposition 5.3 are given in the basis $(C^k_1, C^k_2) \times (C^k_1, C^k_2)$ by
\[
\left( \prod
\begin{array}{c}
\sigma_{k'}^2 - \sigma_{k'}^{2}\n\end{array}
\right)
\left(
\begin{array}{cc}
[2]^k & \sigma_k \\
\sigma_k & [2]^k
\end{array}
\right),
\]
\[\text{(49)}\]

**Proof.** By \((48)\), the action of $\mathcal{P}_2^{(k)}(\mathcal{P}_1^{(k)})^j$ on $M_k$ in the basis $(C^k_1, C^k_2)$ is given by
\[
\left(
\begin{array}{cc}
0 & 0 \\
\sigma_k^{2j+1} & [2]^k \sigma_k^{2j}
\end{array}
\right).
\]

It follows that $C^k_{22}C^k_{21} = [2]^k \prod_{k' \leq k} (\sigma_{k'}^2 - \sigma_{k'}^{2} / C^k_{21}) \mod H_{<k}$, which accounts for the $(2, 2)$ entry of the matrix in \((49)\). The other entries are computed similarly.  
\[\square\]

Now suppose that $R$ is a field. Let $\text{rad}(M_\lambda) = \{x \in M_\lambda : \phi_\lambda(x, y) = 0 \text{ for all } y \in M_\lambda\}$. Define $\Lambda' = \{\lambda \in \Lambda : \phi_\lambda \neq 0\}$.

**Proposition 5.6** (Graham-Lehrer \([11]\)). Maintain the notation of the general setup of Definition 5.4. Let $\lambda \in \Lambda$. Then
(i) $\text{rad}(M_\lambda)$ is an $H$-submodule of $M_\lambda$.
(ii) If $\phi_\lambda \neq 0$, the quotient $L_\lambda := M_\lambda / \text{rad}(M_\lambda)$ is absolutely irreducible.
(iii) If $\phi_\lambda \neq 0$, $\text{rad}(M_\lambda)$ is the minimal submodule of $M_\lambda$ with semisimple quotient.
(iv) The set $\{L_\lambda : \lambda \in \Lambda'\}$ is a complete set of absolutely irreducible $H$-modules.

**Definition 5.7.** The decomposition matrix of a cellular algebra $H$ is the matrix $(d_{\lambda \mu})_{\lambda \in \Lambda, \mu \in \Lambda'}$, where $d_{\lambda \mu}$ is the multiplicity of $L_\mu$ in $M_\lambda$.

We next compute the decomposition matrix of the specialization $\mathcal{K}_W^{(k)}|_{u=1}$. For $k \in \Lambda_2$, the residue of $k$, denoted $r(k)$, is the unique integer in
\[
\left\{
\begin{array}{ll}
tm + \alpha \sum_{j=1}^n jk_j : t \in \mathbb{Z}, \alpha \in \{1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor \} & \text{if } k \in \text{SComp}_{n, \leq k}, \\
\frac{tm}{2} + \alpha \sum_{j=1}^n jk_j : t \in \mathbb{Z}, \alpha \in \{1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor \} & \text{if } k^1 \in \text{SComp}_{n, < k}.
\end{array}
\right.
\]
It is convenient to define \( r(\bar{e}_+) = r(\bar{e}_-) = 0 \) and \( r(\bar{e}_2) = \frac{m}{2} \).

**Proposition 5.8.** Maintain the notation of Proposition 5.3. Suppose that \( m \) is odd, \( \Lambda_2 \) is totally ordered, and \( R = \mathbb{C} \) with \( A^* \rightarrow R \) given by \( u \mapsto 1 \). Then for any \( \lambda \in \Lambda \),

\[
\phi_\lambda \text{ has rank } \begin{cases} 2 & \text{if } \lambda \text{ is the maximal element of } \Lambda \text{ with its residue and } r(\lambda) \neq 0, \\ 1 & \text{if } \lambda \text{ is the maximal element of } \Lambda_2 \text{ with residue } 0, \\ 1 & \lambda = \bar{e}_-, \\ 0 & \text{otherwise.} \end{cases}
\]

The decomposition matrix of \( \mathcal{H}_W^{(k)} |_{u=1} \) is given by

\[
d_{\lambda \mu} = \begin{cases} 1 & \text{if } r(\lambda) = r(\mu) \text{ and } \{\lambda, \mu\} \neq \{\bar{e}_+, \bar{e}_-\}, \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** This follows from the computation of \( \alpha_k |_{u=1}, a_k^1 |_{u=1} \) in (32), Proposition 5.5, (48), and (15). In the case \( \phi_k \) has rank 1 (\( k \in \Lambda_2 \)), a direct computation shows that \( M_k \) has \( \bar{e}_- \) as an irreducible submodule with quotient \( L_k = \bar{e}_+ \).

**Remark 5.9.** Proposition 5.3 does not apply to the \( u = 1 \) specialization if \( m \) is even because \( \sigma_k \) can be 0. It is not difficult to modify the cellular basis to obtain a result similar to Proposition 5.8 which we do below. This will show that the conclusions of cellular algebra theory do indeed not hold in this case, so the assumptions of Proposition 5.3 are necessary, not an artifact of our choice of basis.

**Proposition 5.10.** Suppose that \( m \) is even and otherwise maintain the setup of Proposition 5.8. Let \( \mathcal{C}' \) be the same as \( \mathcal{C} \) in (46) with the following modifications

\[
C'^{k}_{11} := P^{(k)}_1 \prod_{k' \neq k} (P^{(k)}_{21} - \sigma_{k'2}^2), \\
C'^{k}_{22} := P^{(k)}_2 \prod_{k' \neq k} (P^{(k)}_{12} - \sigma_{k'2}^2).
\]

Then \( \mathcal{C}' \) is a free \( R \)-basis of \( H = R \mathcal{H}_W^{(k)} \). Let \( M_j = \bar{X}^{(1)}(w_j) \) as in Theorem 4.4. Then the left-representation afforded by \( \{C'^{k}_{11}, C'^{k}_{21}\} \)

- is isomorphic to \( M_{r(k)} |_{u=1} \) if \( r(k) \in [n] \),
- has a submodule isomorphic to \( \bar{e}_- \) with quotient \( \bar{e}_+ \) if \( r(k) = 0 \),
- has a submodule isomorphic to \( \bar{e}_2 \) with quotient \( \bar{e}_1 \) if \( r(k) = \frac{m}{2} \).

While the left-representation afforded by \( \{C'^{k}_{22}, C'^{k}_{12}\} \)

- is isomorphic to \( M_{r(k)} |_{u=1} \) if \( r(k) \in [n] \),
- has a submodule isomorphic to \( \bar{e}_- \) with quotient \( \bar{e}_+ \) if \( r(k) = 0 \),
- has a submodule isomorphic to \( \bar{e}_1 \) with quotient \( \bar{e}_2 \) if \( r(k) = \frac{m}{2} \).

**Proof.** The proof is similar to that of Propositions 5.3 and 5.8.

**Corollary 5.11.** The maximal semisimple quotient \( \mathcal{H}_W^{(k)} |_{u=1} / \text{rad}(\mathcal{H}_W^{(k)} |_{u=1}) \) is isomorphic to \( \mathbb{C}W \).
ACKNOWLEDGMENTS

I am extremely grateful to Anton Geraschenko for discussions that helped clarify many of the ideas in this paper. I thank Sergei Ivanov for his nice proof of Lemma 4.10 and math overflow for coordinating this help. I am grateful to Monica Vazirani and Ketan Mulmuley for helpful discussions and to James Courtois and John Wood for help typing and typesetting figures.

REFERENCES

[1] A.L. Agore. Categorical constructions for Hopf algebras. Preprint, arXiv:0905.2613v3, 2010.
[2] C. M. Ballantine and R. C. Orellana. On the Kronecker product $s_{(n-p,p)} \ast s_\lambda$. Electron. J. Combin., 12:Research Paper 28, 26 pp. (electronic), 2005.
[3] R. J. Beerends. Chebyshev polynomials in several variables and the radial part of the Laplace-Beltrami operator. Trans. Amer. Math. Soc., 328(2):779–814, 1991.
[4] Georgia Benkart and Dongho Moon. Tensor product representations of Temperley-Lieb algebras and Chebyshev polynomials. In Representations of algebras and related topics, volume 45 of Fields Inst. Commun., pages 57–80. Amer. Math. Soc., Providence, RI, 2005.
[5] Jonah Blasiak. Representation theory of the nonstandard Hecke algebra. In preparation, 2010.
[6] Emmanuel Briand, Rosa Orellana, and Mercedes Rosas. Quasipolynomial formulas for the kronecker coefficients indexed by two two-row shapes (extended abstract), 2008.
[7] Emmanuel Briand, Rosa Orellana, and Mercedes Rosas. Reduced Kronecker coefficients and counter-examples to Mulmuley’s strong saturation conjecture $SH$: with an appendix by Ketan Mulmuley. Comput. Complex., 18(4):577–600, 2009.
[8] A. Brown, S. van Willigenburg, and M. Zabrocki. Expressions for Catalan Kronecker Products. ArXiv e-prints, September 2008.
[9] Meinolf Geck and G"{o}tz Pfeiffer. Characters of finite Coxeter groups and Iwahori-Hecke algebras, volume 21 of London Mathematical Society Monographs. New Series. The Clarendon Press Oxford University Press, New York, 2000.
[10] Frederick M. Goodman, Pierre de la Harpe, and Vaughan F. R. Jones. Coxeter graphs and towers of algebras, volume 14 of Mathematical Sciences Research Institute Publications. Springer-Verlag, New York, 1989.
[11] J. J. Graham and G. I. Lehrer. Cellular algebras. Invent. Math., 123(1):1–34, 1996.
[12] Andrew Mathas. Iwahori-Hecke algebras and Schur algebras of the symmetric group, volume 15 of University Lecture Series. American Mathematical Society, Providence, RI, 1999.
[13] Ketan Mulmuley. Geometric complexity theory VI: the flip via saturated and positive integer programming in representation theory and algebraic geometry. CoRR, abs/0704.0229, 2007.
[14] Ketan Mulmuley. Geometric complexity theory VII: Nonstandard quantum group for the plethysm problem. CoRR, abs/0709.0749, 2007.
[15] Ketan Mulmuley and Milind A. Sohoni. Geometric complexity theory IV. Quantum group for the Kronecker problem. CoRR, abs/cs/0703110, 2007.
[16] Jeffrey B. Remmel and Tamsen Whitehead. On the Kronecker product of Schur functions of two row shapes. Bull. Belg. Math. Soc. Simon Stevin, 1(5):649–683, 1994.
[17] Theodore J. Rivlin. The Chebyshev polynomials. Wiley-Interscience [John Wiley & Sons], New York, 1974. Pure and Applied Mathematics.
[18] Mercedes H. Rosas. The kronecker product of schur functions indexed by two-row shapes or hook shapes. J. Algebraic Comb., 14(2):153–173, 2001.

The University of Michigan, Department of Mathematics, 530 Church Street, Ann Arbor, MI 48109-1043, USA

E-mail address: jblasiak@gmail.com