First order Mean Field Games on networks

Claudio Marchi

University of Padova

work in progress with: Y. Achdou, P. Mannucci and N. Tchou
Outline

- A brief introduction to Mean Field Games
- Definition of networks
- A MFG problem on networks with control on the velocity
- Work in progress: control on the acceleration (with/without constraint on the control)
A brief introduction to Mean Field Games

The Mean Field Games (MFG) theory was proposed by Lasry-Lions, and independently by Huang-Malhamé-Caines, in 2006 for modelization of interactions among a very large ("infinite") number of agents when individual actions are related to mass behaviour and vice versa.

Applications: financial markets, fashion trends, pedestrian or vehicular traffic...

Distinctive features of the model:

- The agents are influenced only by the average behaviour of all other players (in analogy with Statistical Mechanics).
- The agents are rational: they choose a strategy so to minimize a cost.
- The agents are indistinguishable.
- The agents are individually neglectable: a single agent by itself cannot influence the collective behaviour.
Consider a game with $N$ players. The $i$-th player’s dynamics is

$$dX^i_s = \alpha^i_s ds + \sqrt{2\nu} dW^i_s, \quad X^i_t = x \in \mathbb{R}^n$$

where $\nu \geq 0$, $W^i$ are independent Brownian motions, while $\alpha^i$ is the control chosen so to minimize the cost functional

$$\mathbb{E} \left\{ \int_t^T \left[ |\alpha^i_s|^2 - \ell(X^i_s, s) + F \left[ \frac{1}{N-1} \sum_{j \neq i} \delta_{X^j_s} \right](X^i_s) \right] ds \right\}$$

$$+ G \left[ \frac{1}{N-1} \sum_{j \neq i} \delta_{X^j_T}(X^i_T) \right].$$

The Nash equilibria are characterized by a system of $2N$ equations. Nevertheless, as $N \to +\infty$, this system reduces to the following one:
MFG system

\[
\begin{align*}
-\partial_t u - \nu \Delta u + \frac{1}{2} |\nabla u|^2 + \ell(t, x) &= F[m(t)](x) \quad (t, x) \in (0, T) \times \mathbb{R}^n \\
\partial_t m - \nu \Delta m + \text{div}(m \nabla u) &= 0 \quad (t, x) \in (0, T) \times \mathbb{R}^n \\
u(T, x) &= G[m(T)](x) \quad x \in \mathbb{R}^n \\
m(0, x) &= m_0(x) \quad x \in \mathbb{R}^n
\end{align*}
\]

where \( m_0 \) is the initial distribution of players: \( m_0 \geq 0, \int_{\mathbb{R}^n} m_0 dx = 1 \).

- The first equation is a backward-in-time Hamilton-Jacobi(-Bellman) equation describing the expected value for a generic player.
- The second equation is a forward-in-time Fokker-Planck/continuity equation describing the density \( m \) of the players.
- Three couplings occur between the equations.
Variants / other applications

- the costs $F$ and $G$ may depend on $m$ in a local/nonlocal way;
- infinite horizon problem;
- dominant single player versus a population of small players;
- several populations of identical agents;
- cost depending on the velocity of other players and not on their positions;
- penalization of mass concentration;
- all players follow the same feedback law (*Mean Field Type Control*);
- the generic agent controls its acceleration (and $\nu = 0$);
- the agents’ positions are constrained in a closed subset of $\mathbb{R}^n$. 
First order case, i.e. $\nu = 0$

\[
\begin{align*}
\text{(HJ)} & \quad -\partial_t u + \frac{1}{2} |\nabla u|^2 + \ell(t, x) = F[m(t)](x) \quad (t, x) \in (0, T) \times \mathbb{R}^n \\
\text{(C)} & \quad \partial_t m + \text{div} \ (m \nabla u) = 0 \quad (t, x) \in (0, T) \times \mathbb{R}^n \\
& \quad u(T, x) = G[m(T)](x) \quad x \in \mathbb{R}^n \\
& \quad m(0, x) = m_0(x) \quad x \in \mathbb{R}^n
\end{align*}
\]

**Definition**

$(u, m) \in W^{1,\infty}_{\text{loc}}([0, T] \times \mathbb{R}^n) \times C([0, T]; \mathcal{P}_1(\mathbb{R}^n))$ is a solution if:

- (HJ)-equation is satisfied by $u$ in the viscosity sense
- (C)-equation is satisfied by $m$ in the sense of distributions.
Theorem (Cardaliaguet - PL Lions)

1. *The MFG system has a solution* \((u, m)\);
2. \(m(x, s) = \Phi(x, 0, s)\# m_0(x)\), *where* \(\Phi\) *is the flow of the dynamics*
   \[
   x'(s) = -\nabla u(x(s), s), \quad x(0) = x.
   \]

**Ingredients of the proof**

i) for a.e. \(x\), optimal trajectories may bifurcate only at initial time;

ii) the optimal controls are bounded uniformly w.r.t. \(x\);

iii) the value function is Lipschitz continuous and semiconcave;

iv) for a.e. \(x\), system (1) describes the (unique) optimal trajectory of the optimal control problem;

v) Schauder fixed point theorem.
Literature for 1\textsuperscript{st} order MFG

- **MFG on Euclidean spaces**
  - \textit{classical approach}
    - P.L. Lions’ lectures at Collège de France 2012 - Cardaliaguet “Notes on Mean Field Games”,
    - Cardaliaguet, DGA 2013
    - Gomes-Pimentel-Voskanyan, SpringerBrief 2016
  - \textit{Lagrangian approach}
    - Benamou-Carlier-Santambrogio, Springer 2016
    - Cannarsa-Capuani, Springer-Indam 28, 2018
    - Mazanti-Santambrogio, M3AS 2019

- **MFG on discrete sets**
  - Gomes-Mohr-Souza, JMPA 2010
  - Gomes-Mohr-Souza, AMO 2013
  - Guéant, AMO 2015

- **MFG on networks (all for 2\textsuperscript{nd} order case)**
  - Camilli-M., SIAM JCO 2016
  - Achdou-Dao-Ley-Tchou, NHM 2019
  - Achdou-Dao-Ley-Tchou, CVPDE 2020
A network is a connected, embedded in $\mathbb{R}^n$, set $\mathcal{N}$ and it is formed by a set of vertices $V := \{v_i\}_{i \in I}$ and a set of regular edges $E := \{e_j\}_{j \in J}$. We assume that the network is compact and without boundary.

(a) An example of network
Notations

- \( \text{Inc}_i := \{ j \in J : e_j \text{ incident to } v_i \in V \} \).
- Any edge \( e_j \) is parametrized by a smooth function \( \pi_j : [0, l_j] \to \mathbb{R}^n \).

For a function \( u : \mathcal{N} \to \mathbb{R} \) we denote by \( u_j : [0, l_j] \to \mathbb{R} \) its restriction to \( e_j \), i.e. \( u(x) = u_j(y) \) for \( x \in e_j \), \( y = \pi_j^{-1}(x) \).

- The derivative is considered w.r.t. the parametrization.
- In \( v_i \in V \), the oriented derivative of \( u \) is

\[
\partial_j u(v_i) := \begin{cases} 
\lim_{h \to 0^+} [u_j(h) - u_j(0)]/h, & \text{if } v_i = \pi_j(0) \\
\lim_{h \to 0^+} [u_j(l_j - h) - u_j(l_j)]/h, & \text{if } v_i = \pi_j(l_j)
\end{cases}
\]

and \( D_x u(v_i) := (\partial_j u(v_i))_{j \in \text{Inc}_i} \).
Dynamics of a generic player

The state of a generic player is constrained in the network and, when it is inside an edge \( e_j \), it obeys to

\[
x'(t) = \alpha(t)
\]

where \( \alpha \) is the control.

Cost for the generic player

The generic player aims at choosing \( \alpha \in L^2 \) so to minimize the cost

\[
J(x, t, \alpha) = \int_t^T \left[ \frac{|\alpha(s)|^2}{2} - \ell(x(s), s) + F[m(s)](x(s)) \right] ds
\]

\[
+ G[m(T)](x(T))
\]

where \( m(s) \) is the distribution of the whole population at time \( s \).
Notations

- $\Gamma = AC(0, T; \mathcal{N})$
- $\Gamma[x] = \{\gamma \in \Gamma : \gamma(0) = x\}$
- $\mathcal{P}(\Gamma) = \{\text{Borel probability measures on } \Gamma\}$
- $\forall t \in [0, T]$, the evaluation map is $e_t : \Gamma \to \mathcal{N}$ with $e_t(\gamma) = \gamma(t)$
- $\mathcal{P}_{m_0}(\Gamma) = \{\eta \in \mathcal{P}(\Gamma) : e_0\#\eta = m_0\}$
- for each $\eta \in \mathcal{P}_{m_0}(\Gamma)$, we set

$$J^n(t, x, \alpha) = \int^T_t \left[ \frac{|\alpha(s)|^2}{2} - \ell(\gamma(s), s) + F[e_s\#\eta](\gamma(s)) \right] ds + G[e_T\#\eta](\gamma(T))$$

where $\gamma(t) = x$ and $\gamma' = \alpha$ and

$$\Gamma^n[x] = \{\gamma \in \Gamma[x] : J^n(0, x, \gamma') \leq J^n(0, x, \tilde{\gamma}') \quad \forall \tilde{\gamma} \in \Gamma[x]\}.$$
Definition

A measure \( \eta \in \mathcal{P}_{m_0}(\Gamma) \) is a MFG equilibrium for \( m_0 \) if

\[
\text{supp}(\eta) \subset \bigcup_{x \in \text{supp}(m_0)} \Gamma^\eta[x].
\]

Theorem

Assume

- \( m_0 \in \mathcal{P}(\mathcal{N}) \)
- \( \ell \in C^0(\mathcal{N}) \)
- \( F[\cdot], G[\cdot] : \mathcal{P}(\mathcal{N}) \rightarrow C^0(\mathcal{N}) \) are bounded and continuous.

Then, there exists a MFG equilibrium \( \eta \) for \( m_0 \).
Proof (sketch)
Following the Lagrangian approach of [Cannarsa-Capuani, '18], we introduce the multivalued map

\[ E : \mathcal{P}_{m_0}(\Gamma) \to \mathcal{P}_{m_0}(\Gamma) \]

\[ E(\eta) = \left\{ \hat{\eta} \in \mathcal{P}_{m_0}(\Gamma) : \text{supp}(\hat{\eta}) \subset \bigcup_{x \in \text{supp}(m_0)} \Gamma^\eta[x] \right\} \]

and we apply Kakutani fixed point Theorem to obtain a MFG equilibrium. Indeed, there holds

a) \( \forall \eta \in \mathcal{P}_{m_0}(\Gamma), E(\eta) \) is a nonempty set
b) \( \forall \eta \in \mathcal{P}_{m_0}(\Gamma), E(\eta) \) is a convex set
c) the map \( E \) fulfills the closed graph property.
**Definition**

A couple \((u, m)\) is a **mild solution** to the MFG if there exists a MFG equilibrium \(\eta \in \mathcal{P}_{m_0}(\Gamma)\) such that

- \(m(t) = e_t \# \eta\) \quad \forall t \in [0, T] \)
- \(u\) is the the **value function** associated to \(\eta\):

\[
u(t, x) = \inf_{\alpha \text{ adm.}} J^\eta(t, x, \alpha).\]

**Corollary**

There exists a mild solution \((u, m)\) to the MFG.
Hamilton-Jacobi problem for $u$

\[
\begin{cases}
-\partial_t u + \frac{1}{2} |\partial_j u|^2 + \ell = F[m(t)] & (t, x) \in (0, T) \times e_j \\
-\partial_t u + \max_{j \in \text{Inc}_i} \left\{ \frac{1}{2} ((\partial_j u)_-) \right\} + \ell = F[m(t)] & (t, v_i) \in (0, T) \times V \\
u(T, x) = G[m(T)](x) & x \in \mathcal{N}.
\end{cases}
\]

**Definition (viscosity solution)**

$u$ is a **subsolution** (resp., a **supersolution**) if: for all $\varphi \in C^1((0, T) \times \mathcal{N})$ s.t. $u - \varphi$ has a maximum (resp., a minimum) at $(t, x)$, there holds

\[
-\partial_t \varphi(t, x) + \frac{|\partial_j \varphi(t, x)|^2}{2} + \ell(t, x) \leq (\geq) F[m(t)](x)
\]

if $x \in e_j$

\[
-\partial_t \varphi(t, x) + \max_{j \in \text{Inc}_i} \left\{ \frac{((\partial_j \varphi(t, x))_-)^2}{2} \right\} + \ell(t, x) \leq (\geq) F[m(t)](x)
\]

if $x \in V$.

$u$ is a **solution** when it is both a sub- and a supersolution.

**Proposition**

$u$ is the viscosity solution to the Hamilton-Jacobi problem.
Dynamics of a generic player

Inside an edge $e_j$, the state of a player obeys to

$$x'(t) = v(t), \quad v'(t) = \alpha(t)$$

where the control $\alpha$ is chosen either. Two cases:

- $\alpha$ is chosen in $\mathbb{R}$
- $\alpha$ is chosen in $[-1, 1]$.

Cost for the generic player

$$J(x, v, t, \alpha) = \int_t^T \left[ \frac{|\alpha(s)|^2}{2} - \ell(x(s), v(s), s) + F[m(s)](x(s), v(s)) \right] ds + G[m(T)](x(T), v(T)).$$

Difficulties. Inertia of dynamics, viability set, ...
Thank You!