A PARTITION RELATION USING STRONGLY COMPACT CARDINALS

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Abstract. If $\kappa$ is strongly compact and $\lambda > \kappa$ and $\lambda$ is regular (or alternatively $\text{cf}(\lambda) \geq \kappa$), then $(2^{<\lambda})^+ \rightarrow (\lambda + \zeta)^2_\theta$ holds for $\zeta, \theta < \kappa$.

§0. Introduction

The aim of this paper is to prove the following theorem.

0.1 Theorem. If $\kappa$ is a strongly compact cardinal, $\lambda > \kappa$ is regular and $\zeta, \theta < \kappa$, then the partition relation $(2^{<\lambda})^+ \rightarrow (\lambda + \zeta)^2_\theta$ holds.

0.2 Theorem. Assume the conditions in Theorem 0.1 hold, with “$\lambda$ regular”. Then $\text{cf}(\lambda) > \kappa$ suffices.

We notice that our argument is valid in the case $\kappa = \omega$. As for the history of the problem we point out that Hajnal proved, in an unpublished work, that $(2^\omega)^+ \rightarrow (\omega_1 + n)^2_2$ holds for every $n < \omega$. Then it was shown in [Sh:20], §6, that for $\kappa > \omega$ regular and $2^{<\kappa} < \kappa$, the relation $(2^{<\kappa})^+ \rightarrow (\kappa + n)^2_2$ is true. More recently Baumgartner, Hajnal, and Todorcević in [BHT93] extended this to the case when the number of colors is arbitrarily finite. Earlier in [Sh:424], we have $(2^{<\lambda})^+ \rightarrow (\lambda \times m)^2_2$ for $n$ large enough (this was complimentary to the main result there that $\aleph_0 < \lambda = \lambda^{<\lambda} + 2^\lambda$ arbitrarily large does not imply $2^\lambda \rightarrow (\lambda \times \omega)^2_2$). Subsequently [BHT93] improves $n$. We hope that the way the strong compactness was used will be useful elsewhere; see [Sh:666] for a discussion of a possible consistency of failure. I also thank Peter Komjath for improving the presentation.

Notation. If $S$ is a set and $\kappa$ a cardinal, then $[S]^\kappa = \{a \subseteq S : |a| = \kappa\}$, $[S]^{<\kappa} = \{a \subseteq S : |a| < \kappa\}$. If $D$ is some filter over a set $S$, then $X \in D^+$ denotes that $S \setminus X \notin D$ and $X \subseteq S$. If $\kappa < \mu$ are regular cardinals, then $S^\kappa_\mu = \{\alpha < \mu : \text{cf}(\alpha) = \kappa\}$, a stationary set. The notation $A = \{x_\alpha : \alpha < \gamma\}^<_\gamma$, etc., means that $A$ is enumerated increasingly.

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\section{The case of \( \lambda \) regular}

1.1 Lemma. Assume \( \mu = \mu^\theta \). Assume that \( D \) is a normal filter on \( \mu^+ \) and \( A^* \in D^* \) satisfies \( \delta \in A^* \Rightarrow \text{cf}(\delta) > \theta \), and \( F' \) is a function with domain \([A^*]^2\) and range of cardinality \( \theta \). Then there are a normal filter \( D_0 \) on \( \mu^+ \) extending \( D, A_0 \in D_0 \) with \( A_0 \subseteq A^* \) and \( C_0 \subseteq \text{Rang}(F') \) satisfying \( \text{Rang}(F' \upharpoonright [A_0]^2) = C_0 \) such that, if \( X \in D_0^+ \), then \( \text{Rang}(F' \upharpoonright [X]^2) \supseteq C_0 \).

We first prove a claim.

1.2 Claim. Assume \( \mu = \mu^0 \) and \( F' : [S^*]^2 \to \theta, D \) is a normal filter on \( \mu^+, S^* \subseteq \mu^+ \) belongs to \( D^+ \) and \( \delta \in S^* \Rightarrow \text{cf}(\delta) > \theta \). There is a set \( A \subseteq D^* \) such that \( A \subseteq S^* \) and some \( C \subseteq \theta \) satisfying \( \text{Rang}(F' \upharpoonright [A]^2) = C \) and, if \( f : A \to \mu^+ \) is a regressive function, then for some \( \alpha < \mu^+ \) we have \( \text{Rang}(F' \upharpoonright [f^{-1}(\alpha)]^2) = C \) and \( f^{-1}(\alpha) \) is a subset of \( \mu^+ \) from \( D^+ \).

Proof. Toward contradiction assume that no such sets \( A, C \) exist. We build a tree \( T \) as follows. Every node \( t \) of the tree will be of the form

\[
t = \langle \langle A_\alpha : \alpha \leq \varepsilon \rangle, \langle f_\alpha : \alpha < \varepsilon \rangle, \langle i_\alpha : \alpha < \varepsilon \rangle \rangle
\]

for some ordinal \( \varepsilon = \varepsilon(t) \) where \( \langle A_\alpha : \alpha \leq \varepsilon \rangle \) is a decreasing, continuous sequence of subsets of \( \mu^+ \); for every \( \alpha < \varepsilon, f_\alpha \) is a regressive function on \( A_\alpha \); and \( \langle i_\alpha : \alpha < \varepsilon \rangle \) is a sequence of distinct elements of \( \theta \). It will always be true that if \( t <_T t' \), then each of the three sequences of \( t' \) extend the corresponding one of \( t \).

To start, we make the node \( t \) with \( \varepsilon(t) = 0, A_0 = S^* \) the root of the tree.

At limit levels we extend (the obvious way) all cofinal branches to a node.

If we are given an element \( t = \langle \langle A_\alpha : \alpha \leq \varepsilon \rangle, \langle f_\alpha : \alpha < \varepsilon \rangle, \langle i_\alpha : \alpha < \varepsilon \rangle \rangle \) of the tree and the set \( A_\varepsilon = \emptyset \mod D \), then we leave \( t \) as a terminal node. Otherwise, let \( C = C_t = \text{Rang}(F' \upharpoonright [A_\varepsilon]^2) \) and notice that by hypothesis, toward contradiction, the pair \( A_\varepsilon, C_t \) cannot be as required in the Claim. There is, therefore, a regressive function \( f = f_t \) with domain \( A_\varepsilon \), such that for every \( x < \mu^+ \) the set \( \text{Rang}(F' \upharpoonright [f^{-1}(x)]^2) \) is a proper subset of \( C_t \) or \( f^{-1}(x) = \emptyset \mod D \) subset of \( \mu^+ \). We make as immediate extensions of \( t \) the sequences of the form \( t_x = \langle \langle A_\alpha : \alpha \leq x \rangle, \langle f_\alpha : \alpha < x \rangle, \langle i_\alpha : \alpha < x \rangle \rangle \) where \( A_{x+1} = f^{-1}(x), f_\alpha = f_t \) and \( i_\alpha \in C_t \) is some colour value such that if \( A_{x+1} \neq \emptyset \mod D \), then \( i_\alpha \) is not in the range of \( F' \upharpoonright [A_\varepsilon]^2 \).

Having constructed the tree, observe that every element \( x \in S^* \subseteq \mu^+ \) belongs to a set \( A^{(x)}_{\varepsilon(x)} \) for some (unique) terminal node \( t(x) \) of \( T \). Also, \( \varepsilon(x) < \theta^+(< \mu^+) \) holds by the selection of the \( \iota_{\alpha} \)'s as \( \langle \iota_{\alpha}^{(x)} : \alpha < \varepsilon(x) \rangle \) is a sequence of members of \( \theta \) with no repetitions while \( \theta \), the set of colours, has \( \theta \) members. For some set \( S \subseteq S^* \) of ordinals \( x < \mu^+ \) which belong to \( D^+ \) (by the normality of \( D \)), the value of \( \varepsilon(x) \) is the same, say \( \varepsilon \). For \( x \in S \) we let \( g_\alpha(x) = f^{(x)}_{\alpha^x} \) where \( f^{(x)}_{\alpha^x} \) is the \( \alpha \)-th regressive function in the node \( t(x) \in T \). Again, by \( \mu^0 = \mu \) and \( (\forall \alpha \in S)[\text{cf}(\alpha) > \theta] \) we have that \( (\forall x \in S')(\forall \alpha < \varepsilon)g_\alpha(x) = \beta_\alpha \) holds for some sequence \( \langle \beta_\alpha : \alpha < \varepsilon \rangle \) and subset \( S' \subseteq S \) from \( D^+ \). But then we get that the set \( S' \) satisfies \( x, y \in S' \Rightarrow (A^{(x)}_{\alpha^x}, f^{(x)}_{\alpha^x}, \iota^{(x)}_{\alpha^x}) = (A^{(y)}_{\alpha^y}, f^{(y)}_{\alpha^y}, \iota^{(y)}_{\alpha^y}) \) for every \( \alpha < \varepsilon \); we can prove this by induction on \( \alpha \). We can then prove that \( A^{(x)}_{\varepsilon(x)} = A^{(y)}_{\varepsilon(x)} \) for \( x, y \in S' \).
We can conclude that \( x, y \in S' \Rightarrow t(x) = t(y) \), so \( S' \subseteq A_{t(x)}^I \) for some terminal node \( t \), but this latter set is in \( D^+ \), a contradiction. \( \square_{1.2} \)

**Proof of Lemma 1.1.** Apply Claim 1.2 with \( S^* = A^* \) to get corresponding \((C, A)\). Define the ideal \( I \) as follows. For \( X \subseteq \mu^+ \) we let \( X \in I \) iff there are a member \( E \) of \( D \) and a regressive function \( f : X \cap A \to \mu^+ \) such that every \( \text{Rang} \{ F' \mid [f^{-1}(\alpha)]^2 \} \) is a proper subset of \( C \) or \( f^{-1}(\alpha) \) is \( a = 0 \mod D \) subset of \( \mu^+ \).

Now:

1.3 Claim. \( I \) is a normal ideal on \( \mu^+ \) (and \( A^* = \mu^+ \mod I \)).

**Proof.** Straightforward.

Set \( D_0 \) to be the dual filter of \( I \), let \( A_0 = A \) and let \( C_0 = C \); by Claim 1.2 we are done. \( \square_{1.1} \)

1.4 Remark. 1) If Lemma 1.1 holds for some \( D_0, A_0, C_0 \), then it holds for \( D_1, A_1, C_1 \) when the normal filter \( D_1 \) extends \( D_0 \), and \( A_1 \in D_1 \) satisfies \( A_1 \subseteq A_0 \).

2) If \( D_0, A_0, C_0 \) satisfy Lemma 1.5, and \( X \in D_0^+ \), then \( X \) contains a homogeneous set of order type \( \lambda + 1 \) of color \( \xi \) for every \( \xi \in C_0 \).

3) Lemma 1.1 is closely related to the proof in \([Sh:26]\), i.e. 5.1 there.

**Proof of Theorem 0.1.** Let \( \mu = 2^{<\lambda}, \) and let \( F : [\mu^+]^2 \to \theta \) be a colouring. We apply Lemma 1.1 for \( A^* = S_{<\lambda}^{\mu^+} \), \( (F = F, \theta, \mu = \mu) \) and \( D \) the club filter. We shall write \( F(\alpha, \beta) \) for \( F(\{\alpha, \beta\}) \) and 0 for \( F(\alpha, \alpha) \).

We fix \( A_0, D_0, C_0 \) which we get by Lemma 1.1.

1.5 Lemma. Almost every \( \delta \in A_0 \) (i.e. for all but a set = \( \emptyset \mod D_0 \)) satisfies the following: if \( s \in [A_0 \cap \delta]^{<\lambda} \) and \( \{ z_\alpha : \alpha < \gamma \} \subseteq A_0 \cap [\delta, \mu^+] \) with \( \gamma < \kappa \), then there is \( \{ y_\alpha : \alpha < \gamma \} \subseteq A_0 \cap (\sup(s), \delta) \) such that:

(a) \( F(x, y_\alpha) = F(x, z_\alpha) \) (for \( x \in s, \alpha < \gamma \));
(b) \( F(y_\alpha, z_\beta) = F(z_\alpha, z_\beta) \) (for \( \alpha < \beta < \gamma \)).

**Proof.** By simple reflection (using the regularity of \( \lambda \)).

1.6 Lemma. There\(^1\) is \( A_0' \subseteq A_0 \) such that if \( \delta \in A_0', s \in [\delta]^{<\lambda} \) and \( \xi \in C_0 \), then there exists a \( \delta_1 \in A_0, \delta < \delta_1 \) such that:

(a) \( F(x, \delta) = F(x, \delta_1) \) (for \( x \in s \));
(b) \( F(\delta, \delta_1) = \xi \).

**Proof.** Otherwise, there is some \( X \subseteq A_0, X \in D_0^+ \) such that for every \( \delta \in X \) there are \( s(\delta) \in [\delta]^{<\lambda} \) and \( \xi(\delta) \in C_0 \) such that there is no \( \delta_1 > \delta \) satisfying (a) and (b). By normality and \( \mu = \mu^{<\lambda} \) we can assume that \( s(\delta) = s \) and \( \xi(\delta) = \xi \) holds for \( \delta \in X \). By Lemma 1.1, that is, the choice of \( (A_0, D_0, C_0) \), there must exist \( \delta < \delta_1 \) in \( X \) with \( F(\delta, \delta_1) = \xi \), and this is a contradiction. \( \square_{1.6} \)

**Continuation of the proof of Theorem 0.1.** Let \( A_0' \subseteq A_0 \) satisfy Lemmas 1.1 and 1.6 and pick some \( \delta_1 \in A_0' \). Then let \( T = A_0' \setminus (\delta_1 + 1) \).

\(^1\)In fact, if \( A_1^* \in D_0^+ \), then for some \( A_0' \subseteq A_1 \cap A_0, A_1 \setminus A_0' = \emptyset \mod D_0 \) and the conclusion holds for every \( \delta \in A_0' \).
1.7 Lemma. There exists a function $G : T \times T \to C_0$ such that if $s \in [\delta_1]^{< \lambda}, \gamma < \kappa$, and $Z = \{ x_\alpha : \alpha < \gamma \} \subseteq T$, then there is $\{ y_\alpha : \alpha < \gamma \} \subseteq (\sup(s), \delta_1)$ such that:

(a) $F(x, y_\alpha) = F(x, z_\alpha)$ (for $x \in s, \alpha < \gamma$);
(b) $F(y_\alpha, y_\beta) = F(z_\alpha, z_\beta)$ (for $\alpha < \beta < \gamma$);
(c) $F(y_\alpha, z_\beta) = G(z_\alpha, z_\beta)$ (for $\alpha < \beta < \gamma$).

Proof. As $\kappa$ is strongly compact, it suffices to show that for every $Z \in [T]^{< \kappa}$ there exists a function $G : Z \times Z \to \theta$ as required. Clauses (a) and (b) are obvious by Lemma 1.5, and it is clear that, if we fix $Z$, then for every $s \in [\delta_1]^{< \lambda}$ there is an appropriate $G : Z \times Z \to \theta$. We show that there is some $G : Z \times Z \to \theta$ that works for every $s$. Assume otherwise, that is, for every $G : Z \times Z \to \theta$ there is some $s_G \in [\delta_1]^{< \lambda}$ such that $G$ is not appropriate for $s_G$. Notice that the number of these functions $G$ is less than $\kappa$. Then no $G$ could be right for $s = \bigcup \{ s_G : G \text{ a function from } Z \times Z \text{ to } \theta \} \in [\delta_1]^{< \lambda}$, a contradiction. $\square_{1.7}$

Continuation of the proof of Theorem 0.1. We now apply Lemma 1.1 to the colouring $G(x, y) = G(x, y) = \langle F(x, y), G(x, y) \rangle$ for $x < y$ in $T$ and otherwise, and the filter $D_0$ and the set $T$ to get the normal filter $D_1 \supseteq D_0$, the set $A_1 \subseteq T \subseteq A_2$ such that $A_1 \subseteq D_1$ and the colour set $C_1 \subseteq \theta \times \theta$. Notice that actually $C_1 \subseteq C_0 \times C_0$. We can also apply Lemmas 1.5 and 1.6 to get some set $A_1' \subseteq A_1$.

1.8 Lemma. There is a set $a \in [A_1']^{< \kappa}$ such that for every decomposition $a = \bigcup \{ a_\xi : \xi \in C_1 \}$ there is some $\xi \in C_1$ such that:

(\alpha) for every $\xi \in C_1$ there is an $\varepsilon$-homogeneous subset for the colouring $G$ of order type $\zeta$ in $a_\varepsilon$;
(\beta) similarly for every $\varepsilon \in C_0$ and $F$.

Proof. This follows from the strong compactness of $\kappa$, as $A_1'$ itself has this partition property (see Claim 2.8 for more details). $\square_{1.8}$

Continuation of the proof of Theorem 0.1. Fix a set $a$ as in Lemma 1.8.

We now describe the construction of the required homogeneous subset. Let $\delta_2 \in A_1'$ be some element with $\delta_2 > \sup(a)$. For $\xi = (\xi_1, \xi_2) \in C_1 \subseteq \theta \times \theta$ let $a_\xi$ be the following set:

$$a_\xi = \{ x \in a : \bar{G}(x, \delta_2) = \xi \}.$$ 

By Lemma 1.8, there is some $\xi = (\xi_1, \xi_2) \in C_1$ for which the statement in that lemma is true and necessarily (as $a \cup \{ \delta_2 \} \subseteq A_1' \subseteq A_0$ and $a_\xi \neq \emptyset$) we have $\xi_1, \xi_2 \in C_0$. Select some $b \subseteq a_\xi$, otp$(b) = \zeta$ such that $F$ is constantly $\xi_2$ on $b$; this is possible by clause (\beta) of Lemma 1.8. This set $b$ will be the $\zeta$ part of our homogeneous set of ordinals of order type $\lambda + \zeta$, so we will have to construct a set of order type $\lambda$ below $b$. By induction on $\alpha$ we will choose $x_\alpha$ such that the set $\{ x_\alpha : \alpha < \lambda \} \subseteq \delta_1$ satisfies the following conditions:

$\ast_1 F(x_\beta, x_\alpha) = \xi_2$ (for $\beta < \alpha$),
$\ast_2 F(x_\alpha, b \cup \{ \delta_2 \}) = \xi_2$, i.e. $F(x_\alpha, y) = \xi_2$ when $y \in b \cup \{ \delta_2 \}$.

Assume that we have reached step $\alpha$, that is, we are given the set of ordinals with $\{ x_\beta : \beta < \alpha \} \subseteq \delta_1$ and call this set $s$. Applying Lemma 1.6 for $A_1, A_1', \delta_2$ and $s \cup b$ and the colouring $\bar{G}$ here standing for $A_0, A_0', \delta, s$ and the colouring $F$ there (that is, the choice of $A_1'$) we get that there exists some $\delta_3 > \delta_2$ (standing for $\delta_1$ there)
such that:

(i) $\delta_3 \in A_1$;
(ii) $G(x, \delta_3) = \bar{G}(x, \delta_2)$ for $x \in s \cup b$;
(iii) $G(\delta_2, \delta_3) = (\xi_1, \xi_2)$.

Hence

\[ (*_3) \quad F(x_\beta, \delta_3) = \xi_2 \quad \text{(for $\beta < \alpha$).} \]

[Why? As $F(x_\beta, \delta_3) = F(x_\beta, \delta_2)$ by (ii) and the choice of $\bar{G}$ and $F(x_\beta, \delta_2) = \xi_2$ by $(*_2)$ from the induction hypothesis.]

\[ (*_4) \quad G(b \cup \{ \delta_2 \}, \delta_3) = \xi_2, \text{ i.e. } G(y, \delta_3) = \xi_2 \text{ when } y \in b \cup \{ \delta_2 \}. \]

[Why? If $y \in b$, then by (ii) and the definition of $\bar{G}$ we have $G(y, \delta_3) = G(y, \delta_2)$, but $b \subseteq a_\xi$ so by the choice of $a_\xi$ we have $G(y, \delta_2) = \xi_2$. For $y = \delta_2$ use clause (iii), that is, $(\xi_1, \xi_2) = G(\delta_2, \delta_3) = (F(\delta_2, \delta_3), G(\delta_2, \delta_3))$.]

By the choice of $G$ this implies that there is some $x_\alpha$ as required; that is, by the choice of $G$ (see Lemma 1.7) applied to $Z = \{ z_i : i < \gamma \}$, enumerating the set $b \cup \{ \delta_2, \delta_3 \}$ and $s$ as above, we get $\{ y_i : i < \gamma \}$, now necessarily $\delta_3 = z_{\gamma - 1}$, and we can choose $y_{\gamma - 1}$ as $x_\alpha$. $\square_{1.1}$

§2. THE CASE OF $\lambda$ SINGULAR

We prove version 0.2 of the main theorem.

Proof of Theorem 0.2. Let $\sigma = \text{cf}(\lambda)$. Let $\lambda = \sum_{\xi \in \sigma} \lambda_\xi$ with $\lambda_\xi > \sigma \geq \kappa > \theta$ strictly increasing. Let $\mu_\xi = 2^{\lambda_\xi}$ and $\mu = \Sigma \{ \mu_\xi : \xi < \sigma \} = 2^{<\lambda}$. We also fix $F : [\mu^+]^2 \rightarrow \theta$.

2.1 Claim. For some $\bar{C}$ we have:

\begin{itemize}
  \item[(a)] $\bar{C} = \langle C_\alpha : \alpha \in S \rangle$;
  \item[(b)] $S \subseteq \mu^+, \bar{C}_\delta \subseteq \delta$;
  \item[(c)] $\text{otp}(\bar{C}_\delta) \leq \sigma$;
  \item[(d)] $S^* = \{ \delta < \lambda : \text{otp}(\bar{C}_\delta) = \sigma \}$ is stationary;
  \item[(e)] $C_\delta$ unbounded in $\delta$ if $\text{otp}(\bar{C}_\delta) = \sigma$;
  \item[(f)] $\alpha \in C_\delta \Rightarrow \alpha \in S$ and $C_\alpha = C_\delta \cap \alpha$.
\end{itemize}

$\square_{2.1}$

Proof. By [Sh:420] §1 as $\sigma^+ < \mu^+, \sigma = \text{cf}(\sigma)$.

Continuation of the proof of Theorem 0.2. Let $D_0, A_0, C_0$ be as given by Lemma 1.1 with the club filter of $\mu^+, S^*$ (from clause (d) of Claim 2.1 above) here standing for $D, A^*$ there, so $A_0 \subseteq S^*$.

Notation. $\varepsilon(\alpha) = \text{otp}(C_\alpha)$.

2.2 Claim. Let $\chi > 2^{\mu}, <^{\chi} \text{ a well ordering of } \mathcal{H}(\chi)$. For any $x \in \mathcal{H}(\chi)$ we can find $\mathfrak{B} = \langle B_\alpha : \alpha < \lambda \rangle$ such that:

\begin{itemize}
  \item[(a)] $\mathfrak{B}_\alpha < (\mathcal{H}(\chi), \in, <^{\chi})$;
  \item[(b)] $\lambda, \mu, F, \langle \lambda_\xi : \xi < \sigma \rangle, \bar{C}, A_0, C_0, D_0$ belong to $\mathfrak{B}_\alpha$;
  \item[(c)] $\mathfrak{B}_\beta : \beta < \alpha \in B_\alpha$ if $\alpha \notin S^*$;
  \item[(d)] $|B_\beta| = \mu_{\varepsilon(\beta)}$ and $|B_\beta|^{\lambda_{\varepsilon(\beta)}} \subseteq B_\beta$ and $\mu_{\varepsilon(\beta)} + 1 \subseteq B_\beta$ (actually follows);
  \item[(e)] $B_\alpha = \bigcup\{ B_\beta : \beta \in C_\alpha \}$ if $\alpha \in S^*$.
\end{itemize}

Proof. Straightforward.
2.3 Observation. 1) We have ε(α) < ε(β), and ℶ_α ⊆ ℶ_β and ℶ_α < ℶ_β if α ∈ ℶ_β.

2.4 Claim. There is a set \( A'_0 \subseteq A_0 \) such that:

(a) \( A'_0 \subseteq D_0 \) and \( α < δ \in A'_0 \implies \sup(ℏ_α ∩ μ^+) < δ \);
(b) if \( ξ \in C_0 \) and \( δ \in A'_0 \) and \( s \in \bigcup \{[δ ∩ ℏ_α]^{≤ λ_κ(α)} : α ∈ ℶ_δ \} \), then there is \( δ_1 ∈ A_0 \) such that \( δ < δ_1 \) and
   (a) \( F(x, δ) = F(x, δ_1) \) for \( x ∈ s \),
   (b) \( F(δ, δ_1) = ξ \).

Proof. Requirement (a) holds for all but a nonstationary set of \( δ ∈ A_0 \). Requirement (b) is proved as in Lemma 1.6. □

2.5 Claim. There is a function \( G_ε : T × T → C_0 \) such that:

- if \( s ∈ [δ ∩ ℏ_α]^{≤ λ_κ(α)} \) and \( ε = ε(α) \) and \( α ∈ ℶ_δ \) and \( γ < κ \) and \( Z = \{ζ_β : β < γ\} \subseteq T \), then there is \( \{ζ_β : β < γ\} ⊆ [δ ∩ ℏ_α]^{μ^+} \cap ℏ_α, y_0 > \sup(s) \) such that:
  (a) \( F(x, y_β) = F(x, ζ_β) \) for \( x ∈ s, β < δ \);
  (b) \( F(ζ_β, y_β) = G(y_β, ζ_β) \);
  (c) \( F(ζ_1, y_2) = F(y_1, y_2) \) for \( β_1 < β_2 < γ \).

Proof. As in Claim 1.7. □

2.6 Claim. There is a function \( G : T × T → C_0 \) such that if \( s ∈ [T]^{<κ} \), then for arbitrarily large \( ε < σ \) we have \( G | (s × s) = G_ε | (s × s) \).

Proof. Let \( D^* \) be a uniform κ-complete ultrafilter on \( σ \) and define \( G \) by \( G(α, β) = G_ε(α, β) = ξ \) ∈ \( D^* \). □

Continuation of the proof of Theorem 0.2. Now we apply Lemma 1.1 to the colouring \( G_ε \) where \( G(x, y) = (F(x, y), G_ε(x, y) = (F(x, y), G(x, y)) \) for \( x < y \) in \( T \) and zero otherwise and to the filter \( D_0 \) and the set \( T \). We get a normal filter \( D_1 \) and a set \( A_1 \subseteq T \subseteq A'_0 \) and a set of colours \( C_1 \). As \( A_1 \subseteq A_0 \) necessarily \( C_1 ⊆ C_0 × C_0 \).

2.7 Claim. There is \( A'_1 \subseteq A_1 \) such that:

(a) \( A_1 \setminus A'_1 = \emptyset \) mod \( D_1 \);
(b) if \( δ ∈ A'_1, α ∈ ℶ_δ \) and \( s ∈ [δ ∩ ℏ_α]^{≤ λ_κ(α)} \) and \( ℶ_δ ∈ C_1 \), then for some \( δ_κ \) we have \( δ < δ_κ ∈ A_1 \) and
   (a) \( G(x, δ) = G(x, δ_1) \) for every \( x ∈ s \),
   (b) \( G(δ_1, δ_κ) = ξ \).

Proof. As in the proof of Lemma 1.6. □

2.8 Claim. There is a set \( a ∈ [A'_1]^{<κ} \) such that:

- for every decomposition of \( a \) as \( ∪ \{a_ξ : ξ ∈ C_1\} \) there is \( ξ ∈ C_1 \) such that:
  (a) for every \( ε ∈ C_1 \) there is \( b ⊆ a_ξ \) of order type \( ζ \) such that \( G | [b]^2 \) is constantly \( ε \);
  (b) for every \( ε ∈ C_1 \) there is \( b ⊆ a_ξ \) of order type \( ζ \) such that \( F | [b]^2 \) is constantly \( ε \).
Proof. The claim holds since $A'_1$ has this property and $\kappa$ is strongly compact. If $A'_1 = \bigcup \{ a_\xi : \xi \in C_1 \}$ for some $\xi, a_\xi \in D_1^+$ hence clause (a) holds by the choice of $D_1, C_1$; and clause (b) holds as $D_1^+ \subseteq D_0^+$ (as $D_0 \subseteq D_1$) and the choice of $D_0, C_0$. □

Continuation of the proof of Theorem 0.2. Now choose $\delta_2 \in A'_1$ such that $\delta_2 > \sup(a)$ and for $\xi = (\xi_1, \xi_2) \in C_1 \subseteq \theta \times \theta$ define $a_\xi$ as

$$\bar{a}_\xi = \{ x \in a : \bar{G}(x, \delta_2) = \bar{\xi} \}.$$ 

Clearly $\langle a_\xi : \bar{\xi} \in C_1 \rangle$ is a decomposition of $a$ and so there is $\bar{\xi} = (\xi_1, \xi_2) \in C_1$ as guaranteed by $\square$ of Claim 2.8. In particular, there is $b \subseteq a_\xi$ of order type $\zeta$ such that $F \upharpoonright |b|^2$ is constantly $\xi_2$ (note that $(\xi_1, \xi_2) \in C_1 \subseteq C_0 \times C_0$ so $\xi_2 \in C_0$).

Now let $E = \{ \varepsilon < \sigma : G_\varepsilon(\alpha, \delta_2) = G(\alpha, \delta_2) \}$ for every $\alpha \in b \}$. By the definition of $G$ this is an unbounded subset of $\sigma$ and clearly

\[ (*) \quad \text{if } \varepsilon \in E \text{ and } \alpha \in b, \text{ then } G_\varepsilon(\alpha, \delta_2) = G(\alpha, \delta_2) = (\xi_1, \xi_2). \]

For $\alpha < \lambda$ let $\Upsilon(\alpha) = \min \{ \varepsilon \in E : \alpha < \lambda \}$ and let $C_{\delta_3} = \{ \Upsilon(\gamma) : \Upsilon < \sigma \}$. Now we try to choose by induction on $\alpha < \lambda$ a element $x_\alpha$ satisfying

\[ (*)_0 \quad \text{if } \varepsilon < \sigma \text{ and } \alpha \in b, \text{ then } G_{\varepsilon}(\alpha, \delta_2) = G(\alpha, \delta_2), \]

\[ (*)_1 \quad \text{if } \varepsilon < \sigma \text{ and } \alpha \in b, \text{ then } G_{\varepsilon}(\alpha, \delta_2) = G(\alpha, \delta_2), \]

\[ (*)_2 \quad \text{if } \varepsilon < \sigma \text{ and } \alpha \in b, \text{ then } G_{\varepsilon}(\alpha, \delta_2) = G(\alpha, \delta_2). \]

At step $\alpha$, by Claim 2.7, that is, by the choice of $A'_1$ applying clause (b) there with $\{ x_{\beta} : \beta < \alpha \} \cup \delta_2, \bar{\xi}$ here standing for $s, \delta, \xi$ there, we can find $\delta_3$ satisfying the requirement there on $\delta_1$, so

1. $\delta_2 < \delta_3 \in A_1$,
2. $G(\delta_2, \delta_3) = G(x_\alpha, \delta_2)$ for $x \in s \cup b$,
3. $G(\delta_2, \delta_3) = (\xi_1, \xi_2)$.

Now

\[ (*)_3 \quad G_{\varepsilon}(\delta_2, \delta_3) = \varepsilon \text{ for } \beta < \alpha. \]

[Why? By (ii) we have $G(\delta_2, \delta_3) = G(\delta_2, \delta_2)$, hence $F(\delta_2, \delta_3) = F(\delta_2, \delta_2)$, but the latter by $(*)_2$ is equal to $\varepsilon_2$.]

\[ (*)_4 \quad G(\beta, \delta_3) = \varepsilon \text{ for } \beta \in b. \]

[Why? By (ii) and as $\beta \in b \Rightarrow G(\beta, \delta_2) = (\xi_1, \xi_2) \Rightarrow G(\beta, \delta_2) = \varepsilon_2$.]

\[ (*)_5 \quad G(\delta_2, \delta_3) = \varepsilon. \]

[Why? By clause (iii).]

\[ (*)_6 \quad \{ x_{\beta} : \beta < \alpha \} \text{ is a subset of } \delta_3 \cap \mathcal{B}_{\Upsilon(\alpha)}. \]

Let $\langle y_i : i < \zeta + 2 \rangle$ list $b \cup \{ \delta_2, \delta_3 \}$ increasing order.

Now we use the choice of $G_{\Upsilon(\alpha)}$ to choose an increasing sequence $\langle z_i : i < \zeta + 2 \rangle$ in $\delta_3 \cap \mathcal{B}_{\Upsilon(\alpha)}$, $z_0 > x_{\beta}$ for $\beta < \alpha$ such that $F(z_i, y_j) = G(y_i, y_j)$ for $i, j < \zeta + 2$ and $F(x_{\beta}, z_1) = F(x_{\beta}, y_1)$ for $i < \zeta + 2$. Let $x_{\alpha} = z_{\zeta + 1}$ so $x_{\alpha} = \delta_1 \cap \mathcal{B}_{\Upsilon(\alpha)}$ is $> x_{\beta}$ for $\beta < \alpha$.

Also $x_{\alpha}$ satisfies $(*)_0$ of the recursive definition. Now $\beta < \alpha \Rightarrow F(x_{\beta}, x_{\alpha}) = F(x_{\beta}, z_{\zeta + 1}) = F(x_{\beta}, y_{\zeta + 1}) = F(x_{\beta}, \delta_3)$ which is $\xi_2$ by $(*)_3$ above, so for our choice of $x_{\alpha}$, $(*)_1$ holds. Next if $\beta \in b \cup \{ \delta_2 \}$, then $F(x_{\alpha}, x_{\beta}) = F(x_{\beta}, z_{\zeta + 1}) = G(x_{\beta}, \delta_3)$ which is $\xi_2$ by $(*)_4$ or $(*)_5$. So $x_{\alpha}$ is as required. □
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