A SHARP IMPROVEMENT OF FIXED POINT RESULTS FOR QUASI-CONTRACTIONS IN $b$-METRIC SPACES

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Abstract. In this paper, a general fixed point theorem for quasi-contractions in $b$-metric spaces, which is a sharp improvement of Amini-Harandi’s result, Mitrovic and Hussain’s result, and is a generalization of many $b$-metric fixed point theorems in the literature, is proved. The technique overcomes some limits in $b$-metric fixed point theory compared to metric fixed point theory. The obtained results are also supported by examples.

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1. Introduction and Preliminaries

There have been many types of contraction conditions in metric spaces and generalized metric spaces [12], [19]. One of the most interesting types is the quasi-contraction [7]. Quasi-contractions have been studied and many nice results have been proved. In [4], Bessenyei studied nonlinear quasicontractions in complete metric spaces. In [5], Bessenyei studied weak $\varphi$-quasi-contractions and presented an elementary proof for known fixed point results in the literature. In [2], Amini-Harandi proved a fixed point theorem for quasi-contraction maps in $b$-metric spaces. Recently, Mitrović and Hussain [17] established fixed point results for weak $\varphi$-quasi-contractions involving comparison function in $b$-metric spaces.

Recall that the $b$-metric space is a generalization of a metric space. One of the main differences between a $b$-metric space and a metric space is that the modulus of concavity $\kappa \geq 1$ in the generalized triangle inequality, see Definition 1. (3) below. It implies that a $b$-metric is not necessarily continuous, see [3, Example 3.10] for example. It also implies that the contraction constants in certain $b$-metric fixed point theorems are in $[0; \frac{1}{\kappa^2})$ instead of $[0; 1)$, see [9, Remark 2.7] and [17, Corollary 3.5] for example. So, in many $b$-metric fixed point theorems, certain additional assumptions have been added to overcome the above difference such as the Fatou property in [2],

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the bounded orbit in [5] and [17]. For some recent improvements in \(b\)-metric fixed point theory, the reader may refer to [9], [18], [10], [14], [16], [15], [21].

In this paper, we are interested to improve the main results of [2] and [17]. By using a technical calculation that \(\lambda^n \in [0; \frac{1}{\kappa})\) for all \(\lambda \in [0; 1)\) and all \(n\) large enough, we prove a fixed point theorem for quasi-contractions in \(b\)-metric spaces which is a sharp improvement of the main results in [2] and [17], and is a generalization of many \(b\)-metric fixed point theorems in the literature. We also construct examples to support the obtained results.

Now we recall notions and results which will be useful in the next.

**Definition 1** ([8], page 263). Let \(X\) be a nonempty set, \(\kappa \geq 1\) and \(D : X \times X \rightarrow [0; \infty)\) be a function such that for all \(x, y, z \in X\),

1. \(D(x, y) = 0\) if and only if \(x = y\).
2. \(D(x, y) = D(y, x)\).
3. \(D(x, z) \leq \kappa[D(x, y) + D(y, z)]\).

Then

1. \(D\) is called a \(b\)-metric on \(X\) and \((X, D, \kappa)\) is called a \(b\)-metric space. Without loss of generality we may assume that \(\kappa\) is the smallest possible value, and it is called the modulus of concavity of the given \(b\)-metric.
2. The sequence \(\{x_n\}\) is called convergent to \(x\) if \(\lim_{n \to \infty} D(x_n, x) = 0\), written by \(\lim_{n \to \infty} x_n = x\).
3. The sequence \(\{x_n\}\) is called Cauchy if \(\lim_{n, m \to \infty} D(x_n, x_m) = 0\).
4. The \(b\)-metric space \((X, D, \kappa)\) is called compete if every Cauchy sequence is a convergent sequence.

**Definition 2** ([12], Definition 12.7). Let \(X\) be a nonempty set, \(\kappa \geq 1\) and \(D : X \times X \rightarrow [0, \infty)\) be a function such that for all \(x, y, z \in X\),

1. \(D(x, y) = 0\) if and only if \(x = y\).
2. \(D(x, y) = D(y, x)\).
3. \(D(x, z) \leq D(x, y) + \kappa D(y, z)\).

Then \(D\) is called a strong \(b\)-metric on \(X\) and \((X, D, \kappa)\) is called a strong \(b\)-metric space.

Note that every strong \(b\)-metric is continuous, and the convergence and completeness in strong \(b\)-metric spaces are defined as in \(b\)-metric spaces.

**Definition 3** ([2], Definition 2.4). A \(b\)-metric space \((X, D, \kappa)\) is called to have Fatou property if for all \(x, y \in X\) and \(\lim_{n \to \infty} x_n = x\) we have

\[D(x, y) \leq \liminf_{n \to \infty} D(x_n, y)\].
Theorem 1 ([2], Theorem 2.8). Let \((X, D, \kappa)\) be a complete b-metric space having Fatou property and \(f : X \to X\) be a map such that for some \(\lambda \in [0; \frac{1}{\kappa})\) and all \(x, y \in X\),
\[
D(f(x), f(y)) \leq \lambda \max \{D(x, y), D(x, f(x)), D(y, f(y)), D(x, f(y)), D(y, f(x))\}.
\] (1.1)
Then \(f\) has a unique fixed point \(x^*\) and \(\lim_{n \to \infty} f^n(x) = x^*\) for all \(x \in X\).

Theorem 2 ([5], Theorem on page 289). Assume that

1. \((X, D, \kappa)\) is a complete metric space and \(f : X \to X\) is a map such that for all \(x, y \in X\),
\[
D(f(x), f(y)) \leq \varphi(\text{diam}O(x, y))
\] (1.2)
where \(\varphi : [0; \infty) \to [0; \infty)\) is an increasing, upper semicontinuous function,
\(\varphi(0) = 0\) and \(\varphi(t) < t\) for all \(t > 0\), and
\[
O(x, y) = \{f^n(x), f^n(y) : n \in \mathbb{N} \cup \{0\}\}.
\]
2. Each orbit of \(f\) is bounded.
Then \(f\) has unique fixed point \(x^*\) and \(\lim_{n \to \infty} f^n(x) = x^*\) for all \(x \in X\).

Theorem 3 ([17], Theorem 3.3). Assume that

1. \((X, D, \kappa)\) is a complete b-metric space and \(f : X \to X\) is a map such that for some \(\lambda \geq 0\) and all \(x, y \in X\),
\[
D(f(x), f(y)) \leq \lambda \text{diam}O(x, y).
\] (1.3)
2. \(\lambda \in [0; 1)\) and each orbit of \(f\) is bounded.
Then we have

1. There exists \(x^* \in X\) such that \(\lim_{n \to \infty} f^n(x) = x^*\) for all \(x \in X\).
2. \(f\) has unique fixed point \(x^*\) if one of the following holds
   (a) \(f\) is continuous at \(x^*\).
   (b) \(D\) is continuous.

Theorem 4 ([17], Corollary 3.5). Let \((X, D, \kappa)\) be a complete b-metric space and \(f : X \to X\) be a map such that for some \(\lambda \in [0; \frac{1}{\kappa})\) and all \(x, y \in X\),
\[
D(f(x), f(y)) \leq \lambda \max \{D(x, y), D(x, f(x)), D(y, f(y)), \frac{D(x, f(y))}{2\kappa}, \frac{D(y, f(x))}{2\kappa}\}.
\] (1.4)
Then \(f\) has a unique fixed point.

Theorem 5 ([17], Corollary 3.6). Let \((X, D, \kappa)\) be a complete strong b-metric space and \(f : X \to X\) be a map such that for some \(\lambda \in [0; 1)\) and all \(x, y \in X\),
\[
D(f(x), f(y)) \leq \lambda \max \{D(x, y), D(x, f(x)), D(y, f(y)), D(x, f(y)), D(y, f(x))\}.
\]
Then \(f\) has a unique fixed point.
The main result is Theorem 6 below. Note that

1. Theorem 6 is an improvement of Theorem 1 in the sense that the assumption of Fatou property is omitted, and the contraction constant \( \lambda \in [0; 1) \).
2. Theorem 6 is an improvement of Theorem 4 in the sense that the right side of (2.1) is greater than that of (1.4), and the contraction constant \( \lambda \in [0; 1) \).
3. Theorem 6 is an improvement of Theorem 5 in the sense that the strong \( b \)-metric is replaced by a continuous \( b \)-metric.
4. Theorem 6 is a generalization of many \( b \)-metric fixed point theorems in the literature such as [1, Theorem 2.1], [1, Theorem 3.1], [11, Corollary 3.12], [20, Corollary 2.6].
5. Recently, an analogue of Reich contraction in \( b \)-metric spaces was proved [13, Theorem 3.1]. In the proof on [13, page 85], the author claimed \( \lim_{n \to \infty} d(x_{n+1}, Tx^*) = d(x^*, Tx^*) \) provided that \( \lim_{n \to \infty} x_n = x^* \). Unfortunately, this claim does not hold since the \( b \)-metric \( d \) is not necessarily continuous. In fact, the conclusion in [13, Theorem 3.1] does not hold which was proved in [9, Remark 2.7].

**Theorem 6.** Assume that

1. \((X, D, \kappa)\) is a complete \( b \)-metric space and \( f : X \to X \) is a map such that for some \( \lambda \geq 0 \) and all \( x, y \in X \),
\[
D(f(x), f(y)) \leq \lambda \max \{ D(x, y), D(x, f(x)), D(y, f(y)), D(x, y), D(y, x) \}.
\]
(2.1)

2. One of the following holds
   (a) \( D \) is continuous and \( \lambda \in [0; 1) \).
   (b) \( \lambda \in [0; \frac{1}{\kappa}) \).

Then \( f \) has a unique fixed point \( x^* \) and \( \lim_{n \to \infty} f^n(x) = x^* \) for all \( x \in X \).

**Proof.** For \( m \leq i \leq n - 1 \) and \( m \leq j \leq n \), from (2.1) we find that
\[
D(f^i(x), f^j(x)) = D(f^{i-1}(x), f^{j-1}(x)) \leq \lambda \max \{ D(f^{i-1}(x), f^{j-1}(x)), D(f^{i-1}(x), f^{j-1}(x)), D(f^{i-1}(x), f^{j-1}(x)), D(f^{i-1}(x), f^{j-1}(x)) \},
\]
(2.2)

For \( i, j \leq n \), from (2.2), we get
\[
\max \{ D(f^i(x), f^j(x)) : m \leq i, j \leq n \}
\]
for all $n$

By (2.3) we have

$$\leq \lambda \max\{D(f^i(x), f^j(x)) : m - 1 \leq i, j \leq n\}$$

$$\leq \ldots$$

$$\leq \lambda^m \max\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq n\}.$$  \hfil (2.3)

It implies that

$$\max\{D(f^i(x), f^j(x)) : 1 \leq i, j \leq n\} \leq \lambda \max\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq n\}.$$  

Since $0 \leq \lambda < 1$, we see that

$$\max\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq n\} = \max\{D(x, f^i(x)) : 1 \leq i \leq n\}.$$  

So there exists $1 \leq k_n(x) \leq n$ such that

$$D(x, f^{k_n(x)}(x)) = \max\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq n\}.$$  

Since $\lambda \in [0; 1)$, there exists $n_0$ such that $\lambda^{n_0} < \frac{1}{\kappa}$. If $k_n(x) \leq n_0$, then

$$D(x, f^{k_n(x)}(x)) \leq \max\{D(x, f^i(x)) : 0 \leq i \leq n_0\}.$$  \hfil (2.4)

If $k_n(x) > n_0$, then using (2.3) we find that

$$D(x, f^{k_n(x)}(x))$$

$$\leq \kappa[D(x, f^{m_0}(x)) + D(f^{m_0}(x), f^{k_n(x)}(x))]$$

$$\leq \kappa[D(x, f^{m_0}(x)) + \lambda^{m_0} \max\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq k_n(x)\}]$$

$$\leq \kappa[D(x, f^{m_0}(x)) + \lambda^{m_0} \max\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq n\}]$$

$$= \kappa[D(x, f^{m_0}(x)) + \lambda^{m_0} D(x, f^{k_n(x)}(x))].$$

Note that $\lambda^{m_0} < \frac{1}{\kappa}$. So we get

$$D(x, f^{k_n(x)}(x)) \leq \frac{\kappa}{1 - \kappa \lambda^{m_0}} D(x, f^{m_0}(x)).$$  \hfil (2.5)

It follows from (2.4) and (2.5) that

$$\max\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq n\} \leq \frac{\kappa}{1 - \kappa \lambda^{m_0}} \max\{D(x, f^i(x)) : 0 \leq i \leq n_0\}$$

for all $n$. So

$$\sup\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq n\} \leq M < \infty$$

where

$$M = \frac{\kappa}{1 - \kappa \lambda^{m_0}} \max\{D(x, f^i(x)) : 0 \leq i \leq n_0\}.$$  

By (2.3) we have

$$\sup\{D(f^i(x), f^j(x)) : m \leq i, j \leq \infty\} \leq \lambda \sup\{D(f^i(x), f^j(x)) : m - 1 \leq i, j \leq \infty\}$$

$$\leq \ldots \leq \lambda^m \sup\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq \infty\} \leq \lambda^m M.$$  

Then

$$\lim_{m \to \infty} \sup\{D(f^i(x), f^j(x)) : m \leq i, j \leq \infty\} = 0.$$
Therefore the sequence \( \{f^n(x)\} \) is a Cauchy sequence. Since \( X \) is complete, there exists \( x^* \in X \) such that
\[
\lim_{n \to \infty} f^n(x) = x^*. \tag{2.6}
\]
By (2.1) we get
\[
D(f^{n+1}(x), f(x^*)) = D(f^n(x), f(x^*)) \tag{2.7}
\]
\[
\leq \lambda \max \{ D(f^n(x), x^*), D(f^n(x), f^{n+1}(x)), D(x^*, f(x^*)), D(f^n(x), f(x^*)), D(x^*, f^{n+1}(x)) \}. \tag{2.8}
\]
We consider two following cases.

**Case 1.** \( D \) is continuous and \( \lambda \in [0; 1) \).

Using (2.7) and the continuity of \( D \), we obtain
\[
D(x^*, f(x^*)) \leq \lambda \max \{ 0, 0, D(x^*, f(x^*)), D(x^*, f(x^*)), 0 \} = \lambda D(x^*, f(x^*)). \]
Since \( \lambda \in [0; 1) \), we have \( D(x^*, f(x^*)) = 0 \). So \( x^* \) is a fixed point of \( f \).

**Case 2.** \( \lambda \in [0; \frac{1}{\kappa}) \).

It follows from (2.7) that
\[
\liminf_{n \to \infty} D(f^{n+1}(x), f(x^*)) \leq \lambda \max \{ 0, 0, D(x^*, f(x^*)), \liminf_{n \to \infty} D(f^n(x), f(x^*)), 0 \} = \lambda \max \{ D(x^*, f(x^*)), \liminf_{n \to \infty} D(f^n(x), f(x^*)) \}. \tag{2.8}
\]
By (2.8), we consider two following subcases.

**Subcase 2.1.** \( \liminf_{n \to \infty} D(f^{n+1}(x), f(x^*)) \leq \lambda D(x^*, f(x^*)) \)

We find that
\[
D(x^*, f(x^*)) \leq \kappa[D(x^*, f^{n+1}(x)) + D(f^{n+1}(x), f(x^*)]]. \tag{2.9}
\]
From (2.6) and (2.9) we deduce that
\[
\liminf_{n \to \infty} D(f^{n+1}(x), f(x^*)) \geq \frac{1}{\kappa} D(x^*, f(x^*)). \tag{2.10}
\]
On the contrary, suppose that \( x^* \neq f(x^*) \). Note that \( 0 \leq \lambda < \frac{1}{\kappa} \). Then
\[
\liminf_{n \to \infty} D(f^{n+1}(x), f(x^*)) \leq \lambda D(x^*, f(x^*)) < \frac{1}{\kappa} D(x^*, f(x^*)).
\]
This is a contradiction with (2.10). Therefore \( x^* = f(x^*) \).

**Subcase 2.2.** \( \liminf_{n \to \infty} D(f^{n+1}(x), f(x^*)) \leq \lambda \liminf_{n \to \infty} D(f^n(x), f(x^*)) \).
From \( \lim\inf_{n \to \infty} D(f^n(x), f(x^*)) = \lim\inf_{n \to \infty} D(f^{n+1}(x), f(x^*)) \) and \( 0 \leq \lambda < \frac{1}{\kappa} \) we have \( \lim\inf_{n \to \infty} D(f^n(x), f(x^*)) = 0 \). So there exists a subsequence \( \{ f^{k_n}(x) \} \) of \( \{ f^n(x) \} \) such that

\[
\lim_{n \to \infty} f^{k_n}(x) = f(x^*). \tag{2.11}
\]

Note that

\[
D(x^*, f(x^*)) \leq \kappa[D(x^*, f^{k_n}(x)) + D(f^{k_n}(x), f(x^*))], \tag{2.12}
\]

Letting \( n \to \infty \) in (2.12) and using (2.11), (2.6) we obtain \( D(x^*, f(x^*)) = 0 \). Then \( x^* = f(x^*) \).

By the conclusions of Case 1 and Case 2, we find that \( f \) has a fixed point \( x^* \) and by (2.6), \( \lim_{n \to \infty} f^n(x) = x^* \).

Finally, we prove the uniqueness of the fixed point of \( f \). Indeed, let \( x^*, y^* \) be two fixed points of \( f \). From (2.1) we have

\[
D(x^*, y^*) = D(f(x^*), f(y^*)) \leq \lambda \max \{ D(x^*, y^*), D(x^*, f(x^*)), D(y^*, f(y^*)), D(x^*, f(y^*)), D(y^*, f(x^*)) \} = \lambda D(x^*, y^*).
\]

Since \( \lambda \in [0; 1) \), we obtain \( D(x^*, y^*) = 0 \), that is, \( x^* = y^* \). Then the fixed point of \( f \) is unique. \( \square \)

Next we present some examples to illustrate the obtained result. The following example shows there exists the map \( f : X \to X \) so that Theorem 6 is applicable but Theorem 1, Theorem 4 and Theorem 5 are not.

**Example 1.** Let \( X = \mathbb{R} \), and \( D(x, y) = |x - y|^2 \) for all \( x, y \in X \), and the map \( f : X \to X \) be defined by \( f(x) = \frac{3}{4} x \) for all \( x \in X \). Then

1. \( (X, D, \kappa) \) is a complete \( b \)-metric space with the modulus of concavity \( \kappa = 2 \), \( D \) is continuous, and the condition (2.1) holds for all \( \lambda \in [\frac{3}{4}, 1) \). Then Theorem 6 is applicable to \( f \).

2. The conditions (1.1) and (1.4) do not hold for all \( \lambda \in [0, \frac{1}{\kappa}) \). Then Theorem 1 and Theorem 4 are not applicable to \( f \).

3. \( D \) is not a strong \( b \)-metric. Then Theorem 5 is not applicable to \( f \).

**Proof.** (1). It is easy to check that \( (X, D, \kappa) \) is a complete \( b \)-metric space with the modulus of concavity \( \kappa = 2 \), \( D \) is continuous, and the condition (2.1) holds for all \( \lambda \in [\frac{3}{4}, 1) \). Then Theorem 6 is applicable to \( f \).

(2). For \( x = 0, y = 1 \) and \( \lambda \in [0, \frac{1}{\kappa}) = [0, \frac{1}{2}) \), we find that

\[
D(f(0), f(1)) = \frac{9}{16}
\]
\[ \geq \lambda = \lambda. \max \{ D(0, 1), D(0, f(0)), D(1, f(1)), D(0, f(1)), D(1, f(0)) \} \]

This proves that conditions (1.1) and (1.4) do not hold for all \( \lambda \in [0, \frac{1}{K}) \). Then Theorem 1 and Theorem 4 are not applicable to \( f \).

(3) On the contrary, suppose that \( D \) is a strong \( b \)-metric. Then there exists \( K \geq 1 \) such that for all \( x, y, z \in X \),

\[ |x - y|^2 \leq |x - z|^2 + K|z - y|^2. \] (2.13)

For \( n \in \mathbb{N} \), and \( x_0 = \frac{1}{n}, y_0 = 1 + \frac{1}{n}, z_0 = 1 \) we have

\[ |x_0 - y_0|^2 = 1 \]

\[ |x_0 - z_0|^2 + K|z_0 - y_0|^2 = \left( \frac{1}{n} - 1 \right)^2 + \frac{K}{n^2} = 1 + \frac{1 - 2n + K}{n^2}. \]

So for \( n > K \) we have

\[ |x_0 - z_0|^2 + K|z_0 - y_0|^2 < 1 = |x_0 - y_0|^2. \]

It is a contradiction to (2.13). Then \( D \) is not a strong \( b \)-metric, and Theorem 5 is not applicable to \( f \).

The following example shows that the continuity of \( D \) in Theorem 6. (2a) and the condition \( \lambda \in [0, \frac{1}{K}) \) in Theorem 6. (2b) are essential.

Example 2. Let \( X = \{ 0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots \} \), and

\[ D(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \in \{ 0, 1 \} \\ |x - y| & \text{if } x \neq y \in \{ 0 \} \cup \{ \frac{1}{2n} : n = 1, 2, \ldots \} \\ \frac{1}{4} & \text{otherwise,} \end{cases} \]

and let \( f : X \rightarrow X \) be defined by

\[ f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{10^n} & \text{if } x = \frac{1}{n}, n = 1, 2, \ldots \end{cases} \]

Then

(1) \( (X, D, \kappa) \) is a complete \( b \)-metric space with the modulus of concavity \( \kappa = 4 \).

(2) There exist \( \lambda \geq 0 \) such that the contraction condition (2.1) holds for all \( x, y \in X \).

(3) \( D \) is not continuous and \( \lambda \in [\frac{1}{4}, 1) \).

(4) \( f \) is fixed point free.

Proof. By [9, Example 2.6], \( (X, D, \kappa) \) is a complete metric-type space with the modulus of concavity \( \kappa = 4 \). Then \( (X, D, \kappa) \) is also a complete \( b \)-metric on \( X \) with the modulus of concavity \( \kappa = 4 \). The remaining conclusions were proved in [9, Example 2.6] and [9, Remark 2.7].
The following example shows that the assumption of bounded orbit in Theorem 2 and Theorem 3 is essential. Moreover, for the case of unbounded orbit, the value \( \text{diam}O(x,y) \) cannot be replaced by \( \max\{d(x,f(y)),d(y,f(x))\} \). However, the value \( \text{diam}O(x,y) \) can be replaced by \( d(x,y) \) in the class of complete regular semimetric spaces, which is a generalization of the class of complete \( b \)-metric spaces, see [6, Theorem 1].

**Example 3.** Let \( X = \{1,2,3,\ldots\}, d(x,y) = |x-y| \) for all \( x,y \in X \), \( f(x) = x+2 \) for all \( x \in X \) and \( \varphi : [0;\infty) \to [0;\infty) \) be defined by

\[
\varphi(t) = \begin{cases} 
\frac{1}{2} t & \text{if } t \in [0;3) \\
 t - 1 & \text{if } t \geq 3.
\end{cases}
\]

Then we have

1. \((X,d)\) is a complete metric space, and \( \varphi \) is an increasing, upper semicontinuous function, \( \varphi(0) = 0 \) and \( \varphi(t) < t \) for all \( t > 0 \). In particular, \((X,d)\) is also a complete \( b \)-metric space.
2. Every orbit of \( f \) is unbounded. So \( \text{diam}O(x,y) = \infty \) and then (1.3) holds for all \( x,y \in X \) and all \( \lambda \in (0;1) \).
3. \( d(f(x),f(y)) \leq \varphi(\max\{d(x,f(y)),d(y,f(x))\}) \) for all \( x,y \in X \). Then (1.2) holds for all \( x,y \in X \).
4. \( f \) is fixed point free.

**Proof.** (1). It is clear that \((X,d)\) is a complete metric space, and \( \varphi \) is an increasing, upper semicontinuous function, \( \varphi(0) = 0 \) and \( \varphi(t) < t \) for all \( t > 0 \).

(2). For all \( x \in X \) we have \( O(x) = \{x,x+2,x+4,\ldots\} \) which is an unbounded orbit of \( f \).

(3). Let \( x,y \in X \). We may assume that \( x < y \). The we have

\[
d(f(x),f(y)) = |x-y| = y-x
\]

and

\[
d(x,f(y)) = y-x+2 \geq 3.
\]

Then we have

\[
\varphi(\max\{d(x,f(y)),d(y,f(x))\}) = \varphi(d(x,f(y))) = y-x+1 > d(f(x),f(y)).
\]

This proves that (1.2) holds for all \( x,y \in X \).

(4). Since \( f(x) = x+2 \) for all \( x \in X \), \( f \) is fixed point free. \( \square \)

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