Quantum statistics effects and fluctuations of particle numbers near the critical point of nuclear matter

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Equation of state with quantum statistics corrections is derived for a multi-component gas of particles interacting through the repulsive and attractive van der Waals (vdW) forces up to first few orders over a small parameter \( \delta \approx \hbar^2 n (mT)^{-3/2} [g(1 - bn)]^{-1} \), where \( n \) and \( T \) are the particle number density and temperature, \( m \) and \( g \) the particle mass and degeneracy factor. The parameter \( b \) corresponds to the vdW excluded volume. For interacting system of Fermi nucleon and Bose \( \alpha \) particles, a small impurity of \( \alpha \) particles to the nucleon system at leading first order in both \( \alpha \) particle and nucleon small parameters \( \delta \) does not change much the basic results for the symmetric nuclear matter. The particle number fluctuations \( \omega \) determined by the isothermal in-compressibility \( K(n,T) \) can be obtained analytically at the same first order quantum-statistics approximation for symmetric nuclear matter. Our approximate analytical results appear to be in good agreement with the accurate numerical calculations.

I. INTRODUCTION

A study of hadron matter, first of all, an interacting system of protons and neutrons, has a long history; see, e.g., Refs. \[1-11\]. Realistic versions of the nuclear matter equation of state includes both the attractive and repulsive forces between particles. Thermodynamical behavior of this matter leads to the liquid-gas first-order phase transition which ends at the critical point. Experimentally, a presence of the liquid-gas phase transition in nuclear matter was reported and then analyzed in numerous papers (see, e.g., Refs. \[12-17\]). Critical points in different systems of hadrons were studied in Refs. \[18, 22, 23\], see also references therein.

Recently, the proposed van der Waals (vdW) equation of state accounting for the quantum statistics (QS) \[18, 22, 23\] was used to describe the properties of hadronic matter, also with many component extensions and applications to the fluctuation calculations for different thermodynamical averages \[24-31\]. The role and size of the effects of QS was studied analytically for nuclear matter, also for pure neutron and pure \( \alpha \)-particle matter in Ref. \[23\]. Particularly, we investigated a dependence of the critical point parameters on the particle mass \( m \), degeneracy factor \( g \), and the vdW parameters \( a \) and \( b \) which describe particle interactions for each of these systems. Our consideration was restricted to small temperatures, \( T \lesssim 30 \text{ MeV} \), and not too large particle densities. Within these restrictions, the number of nucleons becomes a conserved number, and the chemical potential of such systems regulates the number density of particles. An extension to the fully relativistic hadron resonances in a gas formulation with vdW interactions between baryons and between antibaryons was considered in Ref. \[32\]. An application of this extended model to net baryon number fluctuations in relativistic nucleus-nucleus collisions was developed in Ref. \[33\]. We do not include the Coulomb forces and make no differences between protons and neutrons (both these particles are named as nucleons). In addition, under these restrictions the non-relativistic treatment becomes very accurate and is adopted in our studies. In the present work we are going to apply the same analytical method as in Ref. \[23\] to the mixed two-component system of nucleons and \( \alpha \) particles. Another attractive subject of this work is to apply our analytical results to analysis of the particle number fluctuations near the critical points of the nuclear matter.

The paper is organized as the following. In Sec. II we recall some results of the ideal Bose and Fermi gases taking an exemplary case of the two-component \( N - \alpha \) system. In Sec. III the QS effects near the critical point are studied for the system of symmetric-nuclear and \( \alpha \)-particle matter. Our analytical results are used for nucleon number fluctuations in Sec. IV. These results are then discussed in Sec. V and summarized in Sec. VI.

II. IDEAL QUANTUM GASES

The pressure \( P_i(T,\mu) \) for the \( i \)-system of particles (e.g., \( i = \{N, \alpha\} \)) plays the role of the thermodynamical potential in the grand canonical ensemble (GCE) where temperature \( T \) and chemical potential \( \mu \) are independent variables. The particle number density \( n_i(T,\mu) \), entropy density \( s_i(T,\mu) \), and energy density \( \varepsilon_i(T,\mu) \) are given as

\[
n_i = \left( \frac{\partial P_i}{\partial \mu} \right)_T, \quad s_i = \left( \frac{\partial P_i}{\partial T} \right)_\mu, \quad \varepsilon_i = Ts_i + \mu n_i - P_i.
\]

In the thermodynamic limit \( V \rightarrow \infty \) considered in the present paper all intensive thermodynamical functions – \( P, n, s, \) and \( \varepsilon \) – depend on \( T \) and \( \mu \), rather than on the system volume \( V \), see for instance Ref. \[34\]. We start with the GCE expressions \( \sum_i P_i^{id}(T,\mu) \) for the pressure \( P_i^{id}(T,\mu) \) and particle number density \( n_i^{id}(T,\mu) = \sum_i n_i^{id}(T,\mu) \) for the ideal non-relativistic quantum gas \[24, 32\],

\[
P_i^{id} = \frac{1}{2} g_i \int \frac{dp}{(2\pi \hbar)^3} \frac{2}{m_i} \left[ \exp \left( \frac{p^2}{2m_i^2 \theta_i} - \frac{\mu}{\theta_i} \right) - 1 \right], \quad (2)
\]

\[
n_i^{id} = g_i \int \frac{dp}{(2\pi \hbar)^3} \left[ \exp \left( \frac{p^2}{2m_i^2 \theta_i} - \frac{\mu}{\theta_i} \right) - 1 \right], \quad (3)
\]

where \( m_i \) and \( g_i \) are, respectively, the particle mass and degeneracy factor of the \( i \) component. The value of \( \theta_i = -1 \) corresponds to the Fermi gas, \( \theta_i = 1 \) to the Bose gas, and \( \theta_i = 0 \) is the Boltzmann (classical) approximation.
when effects of the QS are neglected.

Equations (2) and (3) can be expressed in terms of the power series over fugacity, \( z \equiv \exp(\mu/T) \), as:

\[
P_i(T, z) \equiv \frac{g_i T}{\theta_i \Lambda_i} \operatorname{Li}_{3/2}(\theta_i z) = \frac{g_i T}{\theta_i \Lambda_i} \sum_{k=1}^{\infty} \frac{(\theta_i z)^k}{k^{3/2}},
\]

\[
n_i(T, z) \equiv \frac{\partial}{\partial \theta_i} \operatorname{Li}_{3/2}(\theta_i z) = \frac{g_i T}{\theta_i \Lambda_i} \sum_{k=1}^{\infty} \frac{\theta_i^k (\theta_i z)^k}{k^{3/2}}.
\]

Here,

\[
\Lambda_i \equiv \hbar \sqrt{\frac{2\pi}{m_i T}}
\]

is the de Broglie thermal wavelength \([22]\), and \( \operatorname{Li}_n \) is the polylogarithmic function \([36, 37]\). The values of \( \mu = 0 \) corresponds to an onset of the Bose-Einstein condensation in the system of bosons. For fermions, any values of \( \mu \) are possible, i.e., integrals (2) and (3) exist for \( \theta_i = -1 \) at all real values of \( \mu \). The power series (4) and (5) are obviously convergent at \( z < 1 \). For the Fermi statistics at \( z > 1 \), the integral representation of the corresponding polylogarithmic function can be used. Particularly, at \( z \to \infty \) one can use the asymptotic Sommerfeld expansion of the \( \operatorname{Li}_n(-z) \) functions over \( 1/\ln^2|z| \) \([38]\).

For nucleon gas we take \( m_N \approx 938 \text{ MeV} \) neglecting a small difference between proton and neutron masses. The degeneracy factor is then \( g_N = 4 \) which takes into account two spin and two isospin states of nucleon. For ideal Bose gas of \( \alpha \)-nuclei, one has \( g_\alpha = 1 \) and \( m_\alpha \approx 3727 \text{ MeV} \).

At \( z \ll 1 \), only one term \( k = 1 \) is enough in Eqs. (4) and (5) which leads to the classical ideal gas relation

\[
P = n T.
\]

Note that the result (7) follows automatically from Eqs. (2) and (3) at \( \theta_i = 0 \). The classical Boltzmann approximation at \( z \ll 1 \) is valid for large \( T \) and/or small \( n \) region of the \( n-T \) plane. In fact, at very small \( n \), one observes \( z < 1 \) at small \( T \) too.

Inverting the \( z^k \) power series in Eq. (5), one transforms the power expansion of \( z(\epsilon_i) \) to the parameter \( \epsilon_i \) (see, e.g., Ref. [29]),

\[
\epsilon_i = -\frac{\theta_i n \Lambda_i^3}{4\sqrt{2} g_i} \equiv -\theta_i \epsilon_i,
\]

where

\[
\epsilon_i = \frac{\hbar^3 \pi^{3/2} n_i}{2 g_i (m_i T)^{3/2}}.
\]

Taking a given component \( i \), e.g., for nucleon matter (\( \theta_i = -1 \)), for simplicity, we will omit subscript \( i \)

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1 The units with Boltzmann constant \( k_B = 1 \) are used. We keep the Plank constant in the formulae to illustrate the effects of QS, but put \( \hbar = \hbar/2\pi = 1 \) in all numerical calculations. For simplicity, we omitted here and below the subscript \( i \) for the ideal gas everywhere where it will not lead to a misunderstanding.

In discussions of Fig. 1. The exact fugacity \( z(\epsilon) \) can be obtained by multiplying equation (5) by the factor \( \Lambda^3/(4\sqrt{2} g) \) to get \( \epsilon = \epsilon(z) \) and, then, inverting this equation with respect to \( z \). Different other curves in Fig. 1 present the maximal power \( k_{\text{max}} \) of the sum of Eq. (5) over \( k \) after the same multiplying and cut-off the series for the polylogarithmic function \( \operatorname{Li}(z) \) in powers of \( z \) (at the order \( k_{\text{max}} \)). As seen from this figure, one has the asymptotic convergence over \( k_{\text{max}} \) - the better the smaller \( \epsilon \). Even the first-order correction is leading and good in the region of \( \epsilon \approx \epsilon_c = 0.1 - 0.2 \) where \( z \approx 1 \). The second \( (k_{\text{max}} = 2) \) correction improves the convergence such that the cut-off sum for \( \operatorname{Li} \) at the power \( k_{\text{max}} \) practically coincides with the exact result (Fig. 1). For larger \( \epsilon \), say, \( \epsilon > 1 \), where the fugacity \( z \) is much larger than 1 (e.g., in the small temperature limit), we need more and more terms and one has a divergence of the series in \( k_{\text{max}} \). In this region the series for \( \operatorname{Li} \) fails, and one has to use another asymptotic expansion, for instance, over \( 1/z^2 \) as suggested by Sommerfeld \([38]\).

Fig. 2 shows the contour graphics in the \( n - T \) plane where black lines mean \( z(n, T) = \text{const} \) on left, and \( \epsilon(n, T) = \text{const} \) on right with the values written in white squares. As seen from these plots, all values of \( z \leq 1 \) correspond to \( \epsilon \ll 1 \) above blue regions, and therefore, together with Fig. 1 this explains reasons for using the expansion in small parameter \( \epsilon \), even when the fugacity is of the order of 1 and somewhat larger. In particular, the critical points obtained in Ref. [22] belong to such a region.

The expansion of \( z(\epsilon) \) in powers of \( \epsilon \) is inserted then into Eq. (1). At small \( \epsilon_i < 1 \) the expansion of the pressure over the powers of \( \epsilon_i \) is rapidly convergent asymptotically, i.e., converges to the exact (polylogarithmic) function result (4) and (5), the faster the smaller \( \epsilon_i \), such that a few first terms give already a good approximation of the QS effects. Notice that the fugacity values of \( z \) can be larger 1, however, for small \( \epsilon \) and, similarly, for other corrections of a maximal power \( k_{\text{max}} \) in the \( \operatorname{Li}(z) \) polynomials. Taking the two terms, \( k = 1 \), and 2, in Eqs. (4) and (5), one obtains a classical gas result (7) plus the leading first few-order corrections due to the effects of QS:

\[
P_i(T, n_i) = n_i T \left[ 1 + \epsilon_i - c_2 \epsilon_i^2 - c_3 \epsilon_i^3 + O(\epsilon_i^4) \right],
\]

where \( c_2 = 4[16/(9\sqrt{3}) - 1] \approx 0.106 \), \( c_3 = 4(15 + 9\sqrt{2} - 16\sqrt{3})/3 \approx 0.0201 \), and so on. For brevity, we call the linear and quadratic \( \epsilon_i \)-terms in Eq. (10) as the first and second (order) quantum corrections.

Equation (10) demonstrates explicitly a deviation of the quantum ideal gas pressure from the classical ideal-gas value (7): the Fermi statistics leads to an increasing of the classical pressure, while the Bose statistics to its decreasing. This is often interpreted \([22]\) as the effective Fermi ‘repulsion’ and Bose ‘attraction’ between QS particles.
Figure 1. Fugacity $z$ as function of the quantum statistics parameter $\epsilon$ for small values where one finds the critical points ($\epsilon_c = 0.1 - 0.2$ in nuclear matter). Solid black curve shows the exact fugacity $z(\epsilon)$, and $k_{\text{max}}$ is the maximal power of cut-off series for polylogarithm $Li$. 

III. VDW MODEL WITH QUANTUM-STATISTICS CORRECTIONS

For the infinite system of a mixture of different particles, e.g., Fermi and Bose particles – nucleons and $\alpha$ particles, one can present the pressure function of the

$$P(T, n) = P_{\text{id}}^N(T, \mu_N^*) + P_{\text{id}}^\alpha(T, \mu_\alpha^*) - a_{NN}n_N^2 - 2a_{N\alpha}n_Nn_\alpha - a_{\alpha\alpha}n_\alpha^2,$$

(11)
where

\[ P_{id}^N(T, \mu_N^*) = \frac{4g_N}{3\sqrt{\pi \hbar N_{\alpha}}} \int_0^\infty d\eta \frac{n^{3/2}}{\exp\left(\eta - \frac{\mu_N^*}{T}\right) + 1}, \]

\[ P_{id}^\alpha(T, \mu_\alpha^*) = \frac{4g_0}{3\sqrt{\pi \hbar N_{\alpha}}} \int_0^\infty d\eta \frac{n^{1/2}}{\exp\left(\eta - \frac{\mu_\alpha^*}{T}\right) + 1}, \]

(12)

Here \( n \) is the baryon number density, \( n = n_N + 4n_\alpha \), \( P_{id}^N \) is given by Eq. (2), and \( \mu_N^* \) are the solutions of the transcendental equations:

\[ n_N^* = n_N^{id}(T, \mu_N^*) = \frac{2g_N}{\sqrt{\pi \hbar N}} \int_0^\infty d\eta \frac{\eta^{3/2}}{\exp\left(\eta - \frac{\mu_N^*}{T}\right) + 1}, \]

\[ n_\alpha^* = n_\alpha^{id}(T, \mu_\alpha^*) = \frac{2g_0}{\sqrt{\pi \hbar N}} \int_0^\infty d\eta \frac{\eta^{1/2}}{\exp\left(\eta - \frac{\mu_\alpha^*}{T}\right) + 1}, \]

(13)

and \( n_i^* \) is defined by Eq. (3), see more details in Refs. [18, 24]. The relationship between the densities \( n_i \) of Eq. (11) and auxiliary ones \( n_i^* \) can be written in the following form \[18, 24, 25\]:

\[ n_N = \frac{n_N^*[1 + (b_{\alpha\alpha} + b_{NN}^* - b_{N\alpha})n_{\alpha}]^2}{1 + b_{NN}n_N + b_{\alpha\alpha}n_{\alpha}^2 + (b_{NN}^* - b_{\alpha\alpha})n_{\alpha}n_N^*}, \]

(14)

\[ n_\alpha = \frac{n_\alpha^*[1 + (b_{\alpha\alpha} - b_{NN}^*)n_{\alpha}]^2}{1 + b_{NN}n_N + b_{\alpha\alpha}n_{\alpha}^2 + (b_{NN}^* - b_{\alpha\alpha})n_{\alpha}n_N^*}, \]

(15)

where \( b_{ij} \) are the vdWM exclusion volume constants \[18\]:

\[ b_{NN} = 3.35 \text{ fm}^3, \quad b_{\alpha\alpha} = 16.76 \text{ fm}^3, \]

\[ b_{\alpha N} = 13.95 \text{ fm}^3, \quad b_{NN}^* = 2.85 \text{ fm}^3. \]

(16)

Notice that for Bose particles, the restriction \( \mu_\alpha^* \leq 0 \) for the non-relativistic chemical potential should be satisfied. These restrictions correspond to those \( \mu_\alpha \leq 0 \) in the non-relativistic case in the ideal Bose gas. In Eq. (11), the constants \( a_{ij} > 0 \) and \( b_{ij} > 0 \) are responsible for respectively attractive and repulsive interactions between particles.

In the Boltzmann approximation, i.e. at \( \theta = 0 \) in Eqs. (2) and (3), the QvdWM is reduced to the classical vdWM \[25\]:

\[ P_i = \sum_j \left[ \frac{n_i T}{1 - n_j b_{ij}} - a_{ij} n_i n_j \right]. \]

(17)

Note that the classical vdWM \[17\] is further reduced to the ideal classical gas \[7\] at \( a_{ij} = 0 \) and \( b_{ij} = 0 \). At \( a_{ij} = 0 \) and \( b_{ij} = 0 \) the QvdWM turns into the quantum ideal gas Eqs. (2) and (3).

Following Ref. [18], one can fix the model parameters \( a_{ij} \) and \( b_{ij} \) using the ground state properties of the corresponding system components, (see, e.g., Ref. [39]) by

\[ a_{NN} = 329.8 \text{ MeV} \cdot \text{fm}^3, \quad a_{N\alpha} = a_{\alpha N} = a_{\alpha\alpha} = 0. \]

(18)

These values are very close to those found in Refs. [18, 24]. Other constants are taken from Ref. [18] [see Eq. (16)]. Small differences appear because of the non-relativistic formulation used in the present studies. Notice that the system of \( N + \alpha \) was studied in Ref. [19] in the Skyrme model, and the QvdW approach is criticized because the Bose condensation cannot be described in the QvdW model.

In what follows, a few first quantum corrections of the QvdWM will be considered. Expanding \( P_{id}^N(T, \mu_N^*) \), Eq. (13) used in Eq. (14), over small parameters \( \epsilon_i^* \) (Eq. (8) with \( i = N, n_i = n_N^* \) or \( i = \alpha, n_i = n_\alpha^* \), and superscript in \( \epsilon_i^* \) corresponds to that of \( n_i^* \)), one obtains

\[ P_{id}^N(T, n_N^*) = n_N^* T \left[ 1 + \epsilon_N^* \right] \]

(19)

and

\[ P_{id}^\alpha(T, n_\alpha^*) = n_\alpha^* T \left[ 1 - \epsilon_\alpha^* \right], \]

(20)

where \( \epsilon_i^* \) is given by Eq. (9) with replacing \( n_i \) by \( n_i^* \). These expressions are similar to those of Eq. (10) at the first order in \( \epsilon_i \). We proved that at small \( \epsilon_i^* \) the expansion of the pressure over powers of \( \epsilon_i^* \) becomes rapidly convergent to the exact results, and a few first terms give already a good approximation. Our Eq. (19), in contrast to Eq. (10) discussed in Refs. [25, 26], takes into account the particle interaction effects (cf. with the previous section [11]). A new point of our consideration is the analytical estimates of the QS effects in a mixed system of interacting fermions and bosons. Similarly to the ideal gases, the quantum corrections in Eq. (19) increases with the particle number density \( n_i \) and decreases with the system temperature \( T \), particle mass \( m_i \), and degeneracy factor \( g_i \).

As in Ref. [18], we introduce now the impurity contribution of the \( \alpha \)-particles in the symmetric nuclear matter as the ratio of the number of nucleons in the \( \alpha \) particle impurity referred to the total number of nucleons,

\[ X_\alpha = \frac{4n_\alpha}{n_N + 4n_\alpha} = \frac{4n_\alpha}{n}. \]

(21)

where \( n \) is the baryon number density defined already above (below Eq. (12)). According to the numerical solutions in Ref. [18], for the parameters of Eq. (18), the value of \( X_\alpha \) has been approximately obtained, \( X_\alpha \approx 0.013 \). We will use below this value in our calculations. Taking this estimate for a simple exemplary case, one can find \( n_N^* \) and \( n_\alpha^* \) from equations (14) and (15). Then, using Eqs. (24) and (16), one can present them in the following approximate form:

\[ n_N^* \approx \frac{r_1 n}{1 - b_N n}, \quad n_\alpha^* \approx \frac{r_2 n}{1 - b_\alpha n}, \]

(22)

where

\[ r_1 = (1 - X_\alpha) = 0.987, \quad r_2 = \frac{X_\alpha}{4} = 0.0033. \]

(23)

Here, \( b_i \) are coefficients related approximately to the interaction constants \( b_{ij} \), Eq. (16).

\[ b_N \approx 3.29 \text{ fm}^3, \quad b_\alpha \approx 2.81 \text{ fm}^3. \]

(24)

For another interaction parameter \( a_{1} \), one can use

\[ a_{1} = r_1^2 a_{NN} \approx 321.3 \text{ MeV} \cdot \text{fm}^3. \]

(25)
Using also Eq. (11) with Eqs. (19) and (20), for the parameter values of the order of mentioned above, one arrives at

\[ P(T, n) = T r_1 n \frac{1 + \delta_N}{1 - b_N n} + T r_2 n \frac{1 - \delta_\alpha}{1 - b_\alpha n} - a_1 n^2, \]  

where

\[ \delta_i = \frac{\epsilon_i}{1 - b_i n}, \quad n_N = r_1 n, \quad n_\alpha = r_2 n, \]  

and \( i = N, \alpha, r_1 \) and \( r_2 \) are given by Eq. (23). Note that the expression for the pressure, Eq. (26), in a case of \( r_2 = 0 \) and \( r_1 = 1 \), exactly the same as for a pure nuclear matter in Ref. [23]. A new feature of the quantum effects in the system of particle with the vdW interactions is the additional factors \((1 - b_i n)^{-1}\) in the quantum correction \(\delta_i\), i.e., the QS effects becomes stronger due to the repulsive interactions between particles.

The vdW, both in its classical form (17) and in its QvWM extension (11) and (26), describes the first order liquid-gas phase transition. As the value of \(X_\alpha\), used in our derivations, is very small, the approximate critical points in the considered approach will be determined by the following equations:

\[ \left( \frac{\partial P(T, n)}{\partial n} \right)_T = 0, \quad \left( \frac{\partial^2 P(T, n)}{\partial n^2} \right)_T = 0. \]  

Using Eq. (26) in the first approximation in \(\delta_i\), one derives from Eq. (28) the system of two equations for the CP parameters \(n_c\) and \(T_c\) at the same first order:

\[ 2 a_1 = \frac{T r_1 (1 + 2 \delta_N)}{(1 - b_N n)} + \frac{T r_2 (1 - 2 \delta_\alpha)}{(1 - b_\alpha n)}, \]

\[ a_1 = \frac{T r_1 b_N}{(1 - b_N n)} \left[ 1 + \delta_N \frac{(1 + 2 \delta_N)}{b_N n} \right] + \frac{T r_2 b_\alpha}{(1 - b_\alpha n)} \left[ 1 - \delta_\alpha \frac{(1 + 2 \delta_\alpha)}{b_\alpha n} \right]. \]  

Note that the equations (29) and Eq. (30) for the CP in the case of \(r_2 = 0\) exactly the same as for a pure nucleon matter in Ref. [23].

For the CP parameters of the classical vdWM, which are found from Eq. (28) for the equation (17), one has

\[ T_c^{(0)} = \frac{\hbar}{2\pi a_1^{\frac{1}{2}}} \approx 29.2 \text{ MeV}, \quad n_c^{(0)} = \frac{1}{a_1} \approx 0.100 \text{ fm}^{-3}, \]

\[ P_c^{(0)} = \frac{\hbar}{2\pi a_1^{\frac{1}{2}}} \approx 1.09 \text{ MeV} \cdot \text{fm}^{-3}. \]  

The numerical calculations within the full QvWM (11), (12), (15) and (14) give (see also Refs. [18, 22, 23])

\[ T_c \approx 19.9 \text{ MeV}, \quad n_c \approx 0.0733 \text{ fm}^{-3}, \]

\[ P_c \approx 0.562 \text{ MeV} \cdot \text{fm}^{-3}. \]  

These our results (32) appear to be essentially the same as those obtained in Ref. [18].

A summary of the results for the CP parameters is presented in Tables I (with Figs. 2 and 3) and II. For symmetrical nuclear matter \((X_\alpha = 0)\), Fig. 3 shows the isotherms of the pressure \(P\) as function of the reduced volume \(v\) (left) and the particle number density \(n\) (right panel) with the first (and second) order corrections. For the same case, a difference of the results for the classical vdWM (31) and QvWM (32) demonstrates a role of the effects of Fermi and Bose statistics at the CP of the symmetric nuclear particle matter. The size of these effects appears to be rather significant for the case of impurity contributions \(X_\alpha \approx 1\) of the \(\alpha\)-particles into the nucleon matter. On the other hand, it is remarkable that the first order correction (Table I) reproduces these QS effects with a high accuracy. The contribution of high order corrections in \(\delta_i\), - second, third and fourth order is much smaller than the first-order correction that shows a fast convergence in \(\delta_i\) by first-order terms. Therefore, high-order corrections due to the QS effects can be neglected for evaluations of the critical points values.

Table I shows that for the case of the mixed \(N - \alpha\) system with \(X_\alpha\), Eq. (24), even the first order corrections are in good agreement with exact numerical QvWM results (32), see Refs. [18, 22, 23]. As seen from Table I the QS effects of the \(\alpha\)-particle impurity can be neglected because, first of all, of too small relative concentration \(X_\alpha\) of this impurity, according to Eq. (24) as suggested in Ref. [18]. By this reason, one can simplify our calculations of the particle number fluctuations in the next section IV taking a pure symmetric nuclear matter.

Many other examples were recently considered in Ref. [40]. All models investigated in that paper have rather different high-order virial-expansion coefficients. However, if the parameters of these different models are fixed by a requirement to reproduce properties of the ground state, the obtained values of \(T_c\) and \(n_c\) appear to be quite similar. For example, different \(T_c\) values come to the narrow region \(T_c = 18 \pm 2 \text{ MeV}\) for the symmetric nuclear matter \((X_\alpha = 0)\). The effects of Fermi statistics leads to much stronger changes of the \(T_c\) values: about 10 MeV in the nucleon matter.

### IV. PARTICLE NUMBER FLUCTUATIONS

From the Gibbs probability distribution for a gas of classical particles interacting through the repulsive and attractive forces at large temperatures \(T\) and small enough particle number density average \((n)\), one can use the vdW equation of state (17) [see sec. (11) and Ref. [23]]. In what follows, to simplify notations, we will omit angle brackets for statistical averages if it does not lead to misunderstanding. Following, e.g., Ref. [31], the fluctuations of the particle number \(\omega\) as the dispersion of the

| Critical points | vdW \((k_{\text{max}} = 0)\) | 1 | 2 | QvWM |
|----------------|-----------------|---|---|------|
| \(T_c \ [\text{MeV}]\) | 29.2 | 19.0 | 19.67 | 19.7 |
| \(n_c \ [\text{fm}^{-3}]\) | 0.100 | 0.065 | 0.072 | 0.072 |
| \(P_c \ [\text{MeV} \cdot \text{fm}^{-3}]\) | 1.09 | 0.48 | 0.52 | 0.52 |
Figure 3. Pressures $P$ as functions of the reduced volume $v$ (left) and particle number density $n$ (right panel) at different temperatures $T$ (in units of the critical value $T_c$) for the simplest case of the symmetric nucleon matter. The critical point is shown by the close circle found from the exact solution of equations (23). The dotted line shows the second order approximation [Ref. [23] and Eq. (26) ($r_1 = 1, r_2 = 0$) employing for nucleon matter]. The horizontal lines are plotted by using the Maxwell area law in left and correspondingly in right panels. The unstable and metastable parts of the isothermal lines are presented by dashed and dash-dotted lines, respectively. Other closed dots show schematically a binodal boundary for the two phase coexistence curve in the transition from two- to one-phase range [23].

Table II. Results for the CP parameters of the vdWM (2nd column), the symmetric nuclear (N) (g = 4, m = 938 MeV, 3rd and 4th columns) and the mixed symmetric-nuclear and α-particle (g = 1, m = 3737 MeV) matter (N + α, 5th and 6th columns).

| Critical points | Eq. (31) | N 1st-order | N QvdWM | N + α 1st-order | N + α QvdWM |
|-----------------|---------|-------------|---------|-----------------|-------------|
| $T_c$ [MeV]     | 29.2    | 19.0        | 19.7    | 19.4            | 19.9        |
| $n_c$ [fm$^{-3}$]| 0.100   | 0.065       | 0.072   | 0.072           | 0.073       |
| $P_c$ [MeV fm$^{-3}$]| 1.09   | 0.48        | 0.52    | 0.51            | 0.56        |

GCE Gibbs distribution function integrated over the excitation energy can be expressed in terms of the derivative of particle number-density average.

A. Fluctuations and susceptibility in the GCE

For calculations of classical fluctuations of the particle numbers, $\omega$, within the grand canonical ensemble (GCE) one can start with the particle number average [23, 31]

$$\langle N \rangle = \int \rho_N(q, p; \mu, T, V) d\Gamma ,$$

(33)

where $\rho_N(q, p; \mu, T, V)$ is the GCE distribution function of the phase space variables $q, p$, $d\Gamma = dq dp$ (normalized as usually for a classical system), $\mu$ is the chemical potential, $T$ the temperature, and $V$ the volume of the classical system. The Gibbs probability distribution can be written as

$$\rho_N(q, p; \mu, T, V) = Z^{-1} \exp \left[ - (H_N(q, p) - \mu N) / T \right] .$$

(34)

Here $H_N(q, p)$ is the classical Hamiltonian, $Z$ the normalization factor which is the partition function,

$$Z = \sum_N \int d\Gamma \exp \left[ - (H_N(q, p) - \mu N) / T \right] .$$

(35)

Taking variations of both sides of Eq. (33) over $\mu$ with the help of Eqs. (31) and (32) and changing the order of the integral over the phase space $\Gamma$ and derivative over the chemical potential $\mu$, at first order variations, i.e., the second order in fluctuations one obtains (see Refs. [23, 31])

$$\omega(n, T) = \frac{\langle \Delta N \rangle^2}{\langle N \rangle} = T \frac{\partial^2 \langle n \rangle / \partial \mu}{n} ,$$

(36)

where $\Delta N = N - \langle N \rangle$ is the fluctuation of $N$ around its average $\langle N \rangle$, $n = n(\mu, T)$ is the particle number-density average in the GCE. In Eq. (37), the variational derivative,

$$\chi = \frac{\partial \langle n \rangle / \partial \mu}{T} ,$$

(37)

is the isothermal susceptibility. For the linear (first order) variations,

$$\chi_1 = \frac{\partial \langle n \rangle / \partial \mu}{T} ,$$

(38)

one has explicitly,

$$\omega_1(n, T) \equiv S_2 = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} = T \frac{\partial^2 \langle n \rangle / \partial \mu}{n} .$$

(39)

The particle number density $n(\mu, T)$, entropy density $s(\mu, T)$, and energy density $\varepsilon(T, \mu)$ in the GCE are given by Eq. (1).
Figure 4. Contour plots for the vdW [zeroth, upper] and first (lower plots) orders in the QS expansion over a small parameter $\delta$ for the particle number fluctuations as functions of the density $n$ and temperature $T$ (in units of $n_c$ and $T_c$) with full in-compressibility $K_T(n, T)$ (left) and the main derivative approximation (MDA) (right panels).

Let us consider variations of the relationship (33) over the chemical potential $\mu$ taking into account high order variations, for instance, second-order ones. For simplicity, we shall still take these variations at constant temperature, i.e. consider non-linear (second-order) isothermal susceptibility. Eq. (38) is correct for any order of the variational derivative (non-linear susceptibility, Eq. (37)) but now we can specify it at the 2nd order. Taking immediately the variations over $\mu$ up to the second order at $T = \text{const}$ in Eq. (38), one obtains the next (2nd) order corrections to Eqs. (39) and (38), which were considered at first order. These corrections are proportional to the so called kurtosis, defined in Ref. [22] in a slightly different way. Fluctuations accounting for the third cumulant moment of the Gibbs distribution, take the form:

$$\frac{T}{\langle N \rangle} \delta_N \langle N \rangle = S_2 (\delta\mu)^2 + \frac{1}{2T} S_3 (\delta\mu)^2 + \ldots , \quad (40)$$

where $S_3$ is the kurtosis which can be normalized by the average $\langle N \rangle$ as $S_2$, Eq. (39),

$$S_3 = \frac{\langle N^3 \rangle - \langle N \rangle^3}{\langle N \rangle} . \quad (41)$$

Similarly, one can obtain the 4th order moment (or 4th-order cumulant moment) of the Gibbs distribution, i.e., from the third order variations of the average $\langle N \rangle$ over chemical potential $\mu$, and so on. This allows us to go beyond the restrictions of the 2nd order cumulant moment fluctuations $\omega_1$, shown explicitly in Eq. (39), i.e. beyond the first variational derivative for the susceptibility $\chi$, i.e. linear susceptibility $\chi_1$, Eq. (38).

The expression (35) for the fluctuation $\omega$ of the particle number is more general though it is still singular exactly at the CP where the linear susceptibility $\chi_1$, Eq. (38) is $\infty$ in the sum (40).

The integral traces of cumulants as given by

$$C_N = \int dqdp \exp \{-\beta [H_N(q, p) - \mu N]\} , \quad (42)$$

can be calculated by the saddle point method (SDM).
We may try to introduce the entropy

\[ S = - \sum_N P_N \log P_N , \tag{43} \]

where \( P_N = \rho_N(q,p,\mu,T,V) \) is the probability distribution \[34\] and

\[ P_N = \exp\left\{ -\beta [H_N(q,p) - E] - \mu N \right\} , \tag{44} \]

Writing the SPM condition \( \delta S = 0 \), i.e.,

\[ \left( \frac{\partial S}{\partial q} \right)^* = 0, \quad \left( \frac{\partial S}{\partial p} \right)^* = 0, \tag{45} \]

one obtains the classical trajectories \( q^*(t), p^*(t) \) from these Hamiltonian equations \[43\]. We have also to specify the mean field in the Hamilton function, \( H_N(q,p) = \sum_r \left( \frac{p_r^2}{2m} + V_K(q_r) \right) \). If we are far from the bifurcations (CPs), one can use the standard SPM, and the non-zero 2nd order terms of the entropy expansion. Such derivations lead to the results of the standard thermodynamics but near the CP. Near the bifurcation, where the second order terms of the entropy expansion is zero, we may employ the improved SPM (ISPM) \[41, 42\] transforming the ISPM for the action phase integral of the POT \[40\] to the real exponent argument – the entropy \( S \), Eq. \[43\]. The simplest ISPM is the second order expansion of the entropy \[43\], but with finite integration limits. Now, one can take the path integral analytically in terms of the erf functions of the real argument. Here is the place where we can apply for the catastrophe theory of Fedoryuk \[45, 48\] by expanding the entropy to the third order terms. In this way, we arrive at the Airy-kind integrals with the finite contributions of the two SPM points which turn into one bifurcation point at the limit to the CP. Note that in order to remove singularity of the fluctuations \( \omega \) near at the critical point with generalization to the QS description, one can calculate \( \omega \) as the moments of the statistical level density \( \rho(E,A) \) \[49\], also with using the ISPM.

Thus, for the first simplest classical dynamic case, the integral for the fluctuation \[50\] can be presented as the Feynman path integral over the formal trajectories \( q(t) \) and \( p(t) \) with the SDM condition \[44\] for the main contributions from the classical trajectories at large excitation energy with the system temperature, \( T = (\partial S/\partial E)^* \), where the standard thermodynamics but with critical points is working well.

In the next sections, we will study a more popular formula (see Appendix A, Ref. \[25, 26, 30\]) used for calculations of the fluctuations \( \omega \), which is expressed in terms of the isothermal in-compressibility \( K_T \), and compare the results obtained by different approximations. Our purpose of the next sections is to find the ranges of good agreement between the approximate expansion near the critical point and accurate analytical result for the vdWM to check a validness of both expressions through the non-linear susceptibility \( \chi \) and non-linear in-compressibility \( K_T \).

\section{B. Fluctuations and in-compressibility}

For the relative fluctuations of the particle numbers \( \omega \), one has \[24, 31\]

\[ \omega(n,T) = \frac{T}{K_T} , \tag{46} \]

where \( K_T \) is the isothermal in-compressibility,

\[ K_T = \left( \frac{\delta P}{\delta n} \right)_T , \tag{47} \]

and \( P \) is given by equation of state which is given in the one-component QvdW (symmetric nucleon matter) by Eq. \[26\] (with \( r_1 = 1, r_2 = 0, b = b_{NN} \)). The in-compressibility \( K_T \), Eq. \[17\], in Eq. \[16\] as function of the density \( n \) and temperature \( T \), can be expanded in power series near the critical point \( n_c, T_c \) over both variables \( n \) and \( T \) but taking derivatives at the current point \( n, T \),

\[ K_T = (\frac{\partial P}{\partial n})_T + (\frac{\partial^2 P}{\partial n^2})_T (n - n_c) + (\frac{\partial^3 P}{\partial n^3})_T (n - n_c)^2 + \ldots . \tag{48} \]

Using approximately the definition \[28\] valid at the critical point \(4 \), and assuming that the linear in temperature and quadratic in density variations are dominating above other high order variations, one can define the main derivative approximation (MDA):

\[ K_{T\text{MDA}} \approx (\frac{\partial^2 P}{\partial n \partial T})_T (T - T_c) + \frac{1}{2} (\frac{\partial^3 P}{\partial n^3})_T (n - n_c)^2 . \tag{49} \]

We may compare their contributions \( K_{T\text{MDA}} \) into the full expansion \[45\] with the definition \( K_T(n,T) \), Eq. \[17\] in terms of the first order derivative in the expansion \[48\] as function of \( n, T \),

\[ K_T \approx K^{(1)}_T = \left( \frac{\partial P}{\partial n} \right)_T . \tag{50} \]

Notice that Eq. \[16\] can be derived from Eq. \[30\] by using linear variations for the chemical potential \( \mu \) as function of the particle number density \( n \) (see, e.g., Appendix A).

Approximating Eq. \[17\] for in-compressibility \( K_T(n,T) \) by the first derivative of the pressure, Eq. \[50\], at the first order in a small quantum-statistics parameter \( \delta \), Eq. \[27\] \((r_1 = 1, r_2 = 0, b = b_{NN} = b_0\)), one obtains

\[ \omega(n,T) = \frac{T}{T[1 + 2\delta](1 - nb)^2 - 2na} . \tag{51} \]

Studying now a behavior of \( \omega(T,n) \), Eq. \[31\], near the critical point \( n_c, T_c \), see 3rd column in Table \[4\] within

\[ \text{2 The CP is assumed to be of the simplest second order, in contrast to a high order CP when high order derivatives become also zero.} \]
for nucleon system as function of the particle number density \( n \) (in units of \( n_c \)) at the critical value of the temperature \( T = T_c \) with zeroth (vdW), \( k_{\text{max}} = 0 \), and first corrections of the QS expansion, \( k_{\text{max}} = 1 \). Solids show Eq. (50) for the full (without expansion near the CP) fluctuations \( \omega \) and dashed curves present the tested main-derivative approximation (MDA), Eq. (46) with (49).

Similarly, using Eq. (54), for the fluctuations \( \omega(n, T) \), Eq. (51), at the constant \( T = T_c \) one finds

\[
\delta((1 + \nu)n_c, T_c) \approx \frac{k^3 \pi^{3/2} n}{2 g (mT_c)^{3/2} (1 - \tilde{n})} \left( 1 - \frac{3}{2} \tau \right).
\]

Using also Eq. (51), one finds

\[
\omega(n_c, (1 + \tau)T_c) \approx \frac{(1 - \delta_0)^2}{1 - \delta_0} \tau^{-1}
\]

Taking \( b \) from Eq. (10), one finally obtains \( G_\tau \approx 0.29 \), that is only slightly different from the value \( G_\tau \approx 0.26 \) of Ref. [21]. For the case of the classical vdWM, one respectively arrives at \( G_\tau = 1/6 \).

C. Discussion of the results

Fig. 4 shows the particle number fluctuations \( \omega(n, T) \) in units of the critical values \( n_c \) and \( T_c \) for symmetric nuclear matter at the zeroth [vdW, upper] and first (lower panels) order in the quantum statistics expansion. Left
and right contour plots of Fig. 4 present the calculations using respectively the standard in-compressibility $\kappa_T^{(1)}(n, T)$, Eq. (50), and its MDA, Eq. (49). For the MDA calculations we assume the dominance of the derivative contributions of Eq. (48) above high order variations in the in-compressibility, see Eq. (49), and neglect first- and second-derivative terms by using approximately Eq. (28). As seen from Fig. 4 (cf. lower with upper plots), the quantum statistics effects is significant for the fluctuations $\omega$ even after exclusion of a large shift of the critical point by choosing the scaling units to a lower critical values due to the quantum statistics effect, in agreement with the accurate numerical result [Eq. (52)] (see also Ref. 18). Contour plots for fluctuations $\omega$ at a few next high orders (e.g., $k_{\text{max}} = 2 - 4$) are almost the same as for the first order and, therefore, is not shown in Fig. 4. It is clearly seen that a convergence of the MDA fluctuations $\omega$, Eqs. (46) and (49), with those calculated through the the equation (39) for in-compressibility $\kappa_T$ takes place, except for small white ranges near the CP. (see Fig. 4).

Fig. 5 presents more details in the comparison between fluctuations $\omega$ with the in-compressibility $\kappa_T$, Eq. (50), using the pressure $p_{\omega}$ ($r_1 = 1, r_2 = 0$) for nucleons at the zeroth and first orders of the QS expansion and their MDA calculations by Eq. (49) for $\kappa$, at $T = T_c$. The derivatives of the MDA are calculated analytically at the $(n, T)$ point on a small but finite distance from the critical point $(n_c, T_c)$ by assuming that the third derivative term over the density $n$ is leading at the temperature $T = T_c$ (Eq. (49)) for variations of the pressure of equation of state, which is determined by Eq. (26) for symmetric nuclear matter. As shown in Fig. 5 the fluctuations calculated through the in-compressibility, Eq. (50) (solids), and the MDA, Eq. (49) (dashed lines), within a given order $k_{\text{max}} = 0$ or 1 of the QS expansion shows a huge bump in the density dependence, largely in agreement with the approximate simple analytical asymptotic expression (35), and more accurate analytical formula, Eq. (41), also with numerical calculations.

Concerning the MDA, one finds a good agreement with the expression (49) for the vdW ($k_{\text{max}} = 0$), and first ("1") approximations in the QS expansion over $\delta$ near the critical point up to a small distance from the CP. As this distance decreases, one can see a divergence of the MDA as for the vdW approach. For larger distances from the CP in the range $n \gtrsim 1.5$ and $n \lesssim 0.5$, the discrepancy between these approximations are due to the fact that the MDA, Eq. (49), becomes worse because these MDA high-order derivative contributions are not already dominating above the first two derivatives terms of Eq. (49). Note that for the calculations of fluctuations $\omega$ with QS corrections, it was convenient to use the expression (49) for the fluctuation $\omega$ in terms of the susceptibility using the fugacity variable $z$ instead of the particle density variable $n$. Similarly, one can consider the fluctuations $\omega(T, n)$ at $n = n_c$ with analogous properties.

A validity of the expressions (46) and (49) for fluctuations $\omega$ and their MDAs can be evaluated from these calculations (Figs. 4 and 5) by the ranges where one finds good agreement between the approximations (49) and (50) for the in-compressibility $\kappa(n, T)$. Their rough relative estimates, about 1.5%, are found approximately for both the cases, $T = T_c$ (Fig. 5) in the dependence on density $n$ and in dependence on temperature at $n = n_c$ (see also Fig. 4). The fluctuation based on the MDA Eq. (49), converges to that with the standard Eq. (50) for the in-compressibility of the expression (49) (or Eq. (46)) on much smaller relative distances, 0.05% for $T = T_c$ and from -0.02% to 0.005% for $n = n_c$, with respect to the critical values of Table I.

Notice that it is difficult (impossible) to realize practically the conditions for the application of the MDA in the limit to the CP, in particular, if we introduce the restrictions $T = T_c$ or $n = n_c$. In the way to the CP, one has to stop at small but finite distance from the CP when a huge bump appear : the MDA variations fail because it becomes smaller or of the order of next derivatives contributions in expansion (49) of the in-compressibility $\kappa_T$ in the denominator of the fluctuations $\omega$, Eq. (49), see Refs. 27, 28. The derivations of Eqs. (34) and (36) become invalid on enough small but finite distances from the CP because, probably, we use the mean field approach (in particular, the vdWM) as the basis of the QS perturbation expansion. As shown in Ref. 22, in this case the correlation length of the correlation function, or the two-body amplitude of scattering in the quasi-particle Landau theory (51), infinitely diverges by increasing relatively, in the considered limit to the CP, with respect to the mean distance between particles. In this case, the arguments of validness for the derivations of Eqs. (34) and (36) for the fluctuations $\omega$ through the derivatives of the thermodynamic averages (pressure or particle number density) contradict 27, 28 with the background of the statistical physics for which we should have an opposite tendency such that the considered relative fluctuations must be small; see, e.g., Refs. 22, 27, 28, 51.

V. SUMMARY

The QvdWM equation of state has been derived analytically and used to study the quantum statistics effects in a vicinity of the critical point of two-component system of nucleon and $\alpha$-particle matter. The expressions for the pressure were obtained by using the quantum statistics expansion over the small parameters $\delta_i$ ($i = \{N, \alpha\}$) near the vdW approach. A simple and explicit dependence on the system parameters, such as the particle mass $m_i$ and degeneracy factor $g_i$, is demonstrated at the first order of this expansion. Such a dependence is absent within the classical vdWM. The quantum corrections to the CP parameters of the symmetric-nuclear and $\alpha$-particle matter appear to be quite significant. For example, the value of $T_c^{(1)} = 29.2$ MeV in the classical vdW model decreases dramatically to the value $T_c^{(1)} = 19.4$ MeV. On the other hand, this approximate analytical result within the first-order quantum correction is already close to the accurate numerical value of...
\( T_\text{c} = 19.9 \text{ MeV} \) obtained by the numerical calculations within the full QvdWM. The trend of the critical-value changes because of inclusion of the \( \alpha \) particles into the nucleon system occurs in the correct direction, namely the CPs are somewhat increased in the critical point as compared to those for pure nucleon system, and these analytical results are in good agreement with more exact numerical calculations.

The particle number fluctuations for symmetric nucleon matter have been derived within the same analytical QvdW approach near the critical point. Their behavior near the critical point in standard calculations through the in-compressibility is in good agreement with more exact numerical calculations. Main features, as a huge bump near the CP, for the same QvdWM equation of state was found as similar to the approximate analytical and full numerical results obtained with and without using the expansion of the in-compressibility near the CP at zero (vdW) and first order over a small parameter of quantum statistics. The convergence of the main derivative approximation for the isothermal in-compressibility near the CP by accounting for contributions of the mixed second density-temperature and third density derivative terms to the corresponding full expansion of the in-compressibility was studied and the rough estimates for ranges of validness of these QvdW approximations was obtained.

As perspectives, we will study the fluctuations near the critical point by using the improved saddle point method similarly as applied for the oscillating components of the single-particle density of states within the semiclassical periodic orbit theory of critical points (bifurcations) \cite{14,12,15} and in terms of the moments of the statistical level density. Note also that our consideration made for the QvdWM can be straightforwardly extended to other types of inter-particle interactions.

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**Appendix A: Derivations of the classical particle-number fluctuations**

Within the canonical ensemble (CE), one can use the free energy \( F(V,T) \) as a characteristic thermodynamic function of the volume \( V \) and temperature \( T \) for a fixed particle number \( N \). Assuming the thermodynamic limit condition for our infinite system, one can express \( F \) in terms of that per particle \[ \text{[22]}, \]

\[
F(V,T) = N f(\tilde{v}, T),
\]

where

\[
\tilde{v} = \frac{1}{n}, \quad n = N/V.
\]

For the pressure \( P \) and chemical potential \( \mu \), one has

\[
P = -\left( \frac{\partial F}{\partial V} \right)_T = -\left( \frac{\partial f}{\partial \tilde{v}} \right)_T,
\]

and

\[
\mu = \left( \frac{\partial F}{\partial V} \right)_T = f - \frac{1}{n} \left( \frac{\partial f}{\partial \tilde{v}} \right)_T,
\]

where the volume per particle \( \tilde{v} \) is given by Eq. \[ \text{[A2]} \]

Taking the first variation of Eq. \[ \text{[A4]} \] over particle number density \( n \) through the relationship \[ \text{[A2]} \], one obtains

\[
\delta \mu = \frac{1}{n^3} \left( \frac{\partial^2 f}{\partial \tilde{v}^2} \right)_T \delta n.
\]

Therefore, one finds

\[
\left( \frac{\partial n}{\partial \mu} \right)_T = \frac{n^3}{\left( \partial^2 f/\partial \tilde{v}^2 \right)_T}.
\]

According to Eq. \[ \text{[69]} \] and Eqs. \[ \text{[66], [A3], and [A2]} \], one arrives at Eq. \[ \text{[46]} \].

Note that the same result can be obtained much shortly by using the Jacobian transformations within the GCE \[ \text{[23]} \],

\[
\left( \frac{\partial n}{\partial \mu} \right)_T = \frac{D(n,T)}{D(\mu,T)} = \frac{1}{D(\mu,T)/D(n,T)}
\]

and

\[
n = \left( \frac{\partial P}{\partial \mu} \right)_T = \frac{D(P,T)}{D(\mu,T)}
\]

see Eq. \[ \text{[41]} \]. Therefore, substituting these equations \[ \text{[A7]} \] and \[ \text{[A8]} \] into Eq. \[ \text{[59]} \] for the particle number fluctuations \( \omega \), one can do cancellation in ratios of the denominator by using the Jacobian properties. Finally, one obtains Eq. \[ \text{[40]} \].

Note that these derivations based on the first derivative transformations fail near the critical point because of the divergence of fluctuations due to zeros in the denominators and, therefore, strictly speaking, cannot be used in enough a small vicinity of the critical point, see Eq. \[ \text{[28]} \], in contrast to the fluctuation formula as the Gibbs distribution dispersion (sec. \[ \text{[III]} \]) and Eq. \[ \text{[66]} \] in terms of the susceptibility \[ \text{[51]} \].

As stated in the paper, our analysis can be applied beyond the vdW approach. In fact, similar estimates of the quantum statistic effects can be straightforwardly done also for the mean-field models. Concerning these models see, e.g., \[ \text{[51]} \] and references therein.
[1] B. K. Jennings, S. Das Gupta, and N. Mobed, Phys. Rev. C 25, 278 (1982).
[2] G. Röpke, L. Münchow, and H. Schulz, Nucl. Phys. A 379, 536 (1982).
[3] G. Fái and J. Randrup, Nucl. Phys. A 381, 557 (1982).
[4] T. Biro, H. W. Barz, B. Lukač, and J. Zimanyi, Phys. Rev. C 27, 2695 (1983).
[5] L. P. Csernai, H. Stöcker, P. R. Subramanian, G. Buchwald, G. Graebner, A. Rosenhauer, J. A. Maruhn, and W. Greiner, Phys. Rev. C 28, 2001 (1983).
[6] L. P. Csernai and J. I. Kapusta, Phys. Rept. 131, 223 (1986).
[7] B. D. Serot and J. D. Walecka, Adv. Nucl. Phys. 16, 1 (1986).
[8] J. Zimanyi and S.A. Moszkowski, Phys. Rev. C 42, 1416 (1990).
[9] R. Brockmann and R. Machleidt, Phys. Rev. C 42, 1965 (1990).
[10] H. Mueller and B. D. Serot, Nucl. Phys. A 606, 508 (1996).
[11] M. Bender, P. H. Heenen and P. G. Reinhard, Rev. Mod. Phys. 75, 121 (2003).
[12] J. E. Finn et al., Phys. Rev. Lett. 49, 1321 (1982).
[13] R. W. Minich et al., Phys. Lett. B 118, 458 (1982).
[14] A. S. Hirsch et al., Phys. Rev. C 29, 508 (1984).
[15] J. Fochodzalla et al., Phys. Rev. Lett. 75, 1040 (1995).
[16] J. B. Natowitz, K. Hägel, Y. Ma, M. Murray, L. Qin, R. Wada, and J. Wang, Phys. Rev. Lett. 89, 212701 (2002).
[17] V. A. Karnaukhov et al., Phys. Rev. C 67, 011601 (2003).
[18] V. Vovchenko, A. Motornenko, P. Alba, M.I. Gorenstein, L.M. Satarov, and H. Stoecker, Phys. Rev. C 96, 015206 (2017).
[19] L.M. Satarov, I.N. Mishustin, A. Motornenko, V. Vovchenko, M.I. Gorenstein, and H. Stocker, Phys. Rev. C 99, 024909 (2019).
[20] R. V. Poberezhnyuk, V. Vovchenko, M. I. Gorenstein, and H. Stoecker, Phys. Rev. C 99, 024907 (2019).
[21] R.V. Poberezhnyuk, V. Vovchenko, D.V. Anisimov, M.I. Gorenstein, J. Mod. Phys. E, 26, 1750061 (2017).
[22] V. Vovchenko, D. Anchishkin, and M. Gorenstein, Phys. Rev. C, 91, 064314 (2015).
[23] S.N. Fedotkin, A.G. Magner, and M.I. Gorenstein, Phys. Rev. C, 100, 054334 (2019).
[24] M Anisimov and V. Sychev, Thermodynamics of critical state for individual sustances, (Energoatomizdat, Moscow, 1990)(in Russian).
[25] L.D. Landau and E.M. Lifshitz, Statistical Physics, Course of Theoretical Physics, (Pergamon, Oxford, UK, 1975), Vol. 5.
[26] R. Balescu, Equilibrium and nonequilibrium statistical mechanics (Wiley, New York, 1975), Vol. 1.
[27] R.C. Tolman, The principles of statistical mechanics (Oxford at the Clarendon Press, 1938).
[28] J.S. Rowlinson, The properties of real gases, Encyclopedia of Physics, Vol. 3/12 (Springer-Verlag, Academic Edition, Berlin, 1958), ISBN: 978-3-642-45894-1.
[29] K.B. Tolpygo, Thermodynamics and Statistical Physics, (Kiev University, Kiev, 1966) (in Russian).
[30] A. Ishihara, “Statistical Physics” (Academic Press, New York, 1971).
[31] D. Zubarev, V. Morozov, and G. Röpke, Statistical Mechanics of Nonequilibrium Processes, Vol. I (Moscow, Fizmatlit, 2002)(in Russian).
[32] V. Vovchenko, M. I. Gorenstein, and H. Stoecker, Phys. Rev. Lett. 118, 182301 (2017).
[33] V. Vovchenko, L. Jiang, M. I. Gorenstein, and H. Stoecker, Phys. Rev. C98, 024910 (2018).
[34] V.V. Begun and M.I. Gorenstein, Phys. Rev. C 77, 064903 (2008).
[35] V. Vovchenko, L. Neise, and H. Stöcker, Thermodynamics and Statistical Mechanics, 1995 Springer-Verlag New York, Inc.
[36] I.S. Gradstein and I.M. Ryzhik, Tables of Integrals, Series and Products (Moscow, Fizmatlit, 4th edition, 1963).
[37] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, Integrals and Series, (Moscow, Nauka, 1986).
[38] M. Brack, C. Guet, and H-B. Hakanson, Phys. Rep. 123, 275 (1985).
[39] H. A. Bethe, Ann. Rev. Nucl. Part. Sci. 21, 93 (1971).
[40] V. Vovchenko, Phys. Rev. C 96, 015206 (2017).
[41] A.G. Magner, K. Arita, S. N. Fedotkin, and K. Matsuyanagi, Prog. Theor. Phys. 108 (2002) 853.
[42] A.G. Magner, Y.S. Yatsysyhn, K. Arita, and M. Brack, Phys. At. Nucl. 74, 1445 (2011).
[43] M. C. Gutzwiller, Chaos in Classical and Quantum Mechanics (Springer-Verlag, New York, 1990).
[44] M. C. Gutzwiller, Chaos in Classical and Quantum Mechanics (Springer-Verlag, New York, 1990).
[45] V. M. Strutinsky, Nukleonika, 20 (1975) 679; V. M. Strutinsky and A. G. Magner, Sov. Phys. Part. Nucl., 7 (1977) 138.
[46] M. Brack and R.K. Bhaduri, Semiclassical Physics, Frontiers in Physics No. 96, 2nd ed. (Westview Press, Boulder, CO, 2003).
[47] M. V. Fedoryuk, Sov. J. of Comp. Math. and Math. Phys.,4 (1964) 671; ibid. 10 (1970) 286.
[48] A. G. Magner, K. Arita, S. N. Fedotkin, Progr. Theor. Phys., 115 (2006) 523.
[49] A.G. Magner, A.I. Sanzhur, S.N. Fedotkin, A.I. Levon, and S. Shlomo, [arXiv:2006.03868v2 [nucl-th] 2020, submitted to the Phys. Lett. B, 2020.
[50] E.M. Lifshitz and Pitajevsky, Vol. 9 (Moscow, Fizmatlit, 2004) (Russian).
[51] D. Anchishkin and V. Vovchenko, J. Phys. G 42, 105102 (2015).