A Dual CFT for Schwarzschild Black Hole

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Abstract

We consider horizon as a boundary and study the symmetries of the Schwarzschild metric in the near-horizon region. Appropriate boundary conditions result the boundary Killing vectors which obey a Virasoro algebra. The algebra of the corresponding charges has a central term with central charge $c = 12 \, r_H^3 = 96 \, M^3$. If we identify this central charge as the central charge of the dual CFT (defined at the horizon) and the Hawking temperature as the temperature of this CFT, Cardy formula reproduces the Bekenstein-Hawking entropy of the Schwarzschild black hole.
1 Introduction

Recently a new duality between four-dimensional extremal black holes and two-dimensional chiral CFTs was proposed [1, 2]. In fact, by making use of the Brown and Henneaux’s asymptotic symmetry method [3] and applying the Cardy formula, it was shown that the Bekenstein-Hawking entropy of the 4D extremal black holes is reproduced as the statistical entropy of the dual CFT.

Extremality is an important issue in this duality. It was shown in ref. [4] that the near-horizon geometry of the extremal Kerr-Newmann black holes has $SL(2, R) \times U(1)$ isometries. By imposing appropriate boundary conditions at the asymptotic boundary of the near-horizon geometry, the asymptotic Killing vectors obey a Virasoro algebra which is an enhancement of the $U(1)$ global symmetry. The $SL(2, R)$ part of the isometry does not appear in the asymptotic symmetry group. In fact it is argued in [4] that the $L_0$ of the $SL(2, R)$ measures the deviation from extremality and it is relevant for the entropy of the near-extremal black holes. Some success in improving such a CFT description for the near-extremal black holes has been achieved recently [5]-[7].

The aim of this paper is to study existence of a dual conformal field theory for the non-extreme cases. We note however that such a CFT description in the near horizon region of the non-extreme black holes was proposed earlier [8, 9] (see also [10]). Instead of taking near-horizon limit, the author of [9] formulates a boundary condition describes metrics with black hole horizon. The Diffeomorphisms that preserve this condition act non-trivially in the arbitrary small vicinity of the horizon and form a group of the conformal transformation in the $(t, r)$ space. The corresponding algebra is a copy of the Virasoro algebra with a central charge which is proportional to the Bekenstein-Hawking entropy of the black hole.

Indeed, we look at the symmetries of the horizon of the Schwarzschild black hole. To do so, we consider this black hole as a background solution and study the fluctuations which satisfy particular boundary conditions at the horizon. The requirement of preserving these boundary conditions leads to a set of diffeomorphisms which are generated by the so-called boundary Killing vectors. The boundary symmetry group contains the two-dimensional conformal group generated by

$$\zeta = T(t) \partial_t + 2r\dot{T}(t) \partial_r$$

where $T(t)$ is an arbitrary function. For $T(t) = t^{1+n}$, generators $\zeta_n$ obey a Virasoro algebra with no central term.

We use the covariant formalism introduced in [11, 12] to compute the corresponding charges. The algebra of the charges has a central extension and one can read its central charge:

$$c = 12 r_H^3 = 96 M^3.$$  

(1.2)

In light of the Brown and Henneaux’s method, it is plausible to identify this central charge as the central charge of a dual conformal field theory. The states of the dual CFT correspond to the Schwarzschild black hole and other metrics which have the same behavior.

\footnote{We note however that so far the covariant formalism has been used for the calculation of the charges in the boundaries located at the special infinity. It is assumed that the formalism is applicable for the horizons as well. The detailed analysis of this point is postponed to the future works.}
at the horizon. Moreover, if we identify the Hawking temperature of the black hole as the temperature of the CFT, the Cardy formula reproduces the Bekenstein-Hawking entropy. This is a good sign which make us defend from our proposal that the quantum gravity on the Schwarzschild background has a dual conformal description at the horizon.

The paper is organized as follows: In section 2 we briefly review Schwarzschild Black Hole. The boundary symmetry group of the horizon is discussed in section 3 and corresponding charges are calculated in section 4. We argue about existence of the dual CFT in section 5. Last section is devoted to discussion.

2 Schwarzschild Black Hole

Schwarzschild black hole is a solution of the Einstein-Hilbert action

\[ I = \frac{1}{16\pi} \int d^4 x \sqrt{-g} R, \quad (2.1) \]

which describes a neutral static black hole with mass \( M \). The corresponding metric is

\[ ds^2 = -\frac{(\hat{r} - r_H)}{\hat{r}} dt^2 + \frac{\hat{r}}{(\hat{r} - r_H)} dr^2 + r^2 d\Omega^2, \quad (2.2) \]

where \( r_H \) is the radius of the horizon and

\[ r_H = 2M. \quad (2.3) \]

The Bekenstein-Hawking entropy and Hawking temperature are

\[ S = \pi r_H^2, \quad T_H = \frac{1}{4\pi r_H}. \quad (2.4) \]

It is useful to make a coordinate transformation and define a new radial coordinate

\[ r = \frac{1}{\hat{r} - r_H}. \quad (2.5) \]

The metric of the black hole in this new coordinate is

\[ ds^2 = -\frac{1}{1 + rr_H} dt^2 + \frac{1 + rr_H}{r^4} dr^2 + \frac{(1 + rr_H)^2}{r^2} d\Omega^2. \quad (2.6) \]

The horizon is now at \( r = \infty \) and asymptotic infinity is mapped to \( r = 0 \). This geometry has \( U(1) \times SO(3) \) isometries where \( U(1) \) is generated by \( \partial_t \) Killing vector.

3 Boundary Symmetry Group

We consider horizon as a boundary and follow the Brown and Henneaux’s method [3]. To do so, excitations around (2.6) must be studied at the horizon. These excitations are restricted by imposing some boundary conditions. This leads to the concept of the boundary symmetry
which includes diffeomorphisms ζ preserving form of the metric at the boundary up to the perturbations satisfy boundary conditions.

We choose the boundary conditions as follows:²

\[ h_{\mu\nu} = O \left( \begin{array}{cccc} 1/r^2 & 1/r^2 & 1/r^2 & 1/r \\ 1/r^3 & 1/r^2 & 1/r^3 & 1/r \\ 1/r & 1 & 1 & 1 \\ \end{array} \right) \]  

where basis are \((t, r, \theta, \phi)\) and

\[ h_{\mu\nu} \equiv \delta g_{\mu\nu} = \mathcal{L}_\zeta g_{\mu\nu}. \]  

Here, \(g_{\mu\nu}\) is the metric \((2.6)\).

The most general diffeomorphisms which preserve these boundary conditions are

\[ \zeta = [T(t) + O(1/r)] \partial_t + \left[ 2r \dot{T}(t) + O(1) \right] \partial_r + \left[-P'(\phi) + O(1/r^2) \right] \partial_\theta \]

\[ + \left[W(\theta) + \cot(\theta)P(\phi) + O(1/r^2) \right] \partial_\phi, \]  

where \(T(t), W(\theta)\) and \(P(\phi)\) are arbitrary functions. For \(T(t) = t^{1+n}\) and \(W(\theta) = P(\theta) = 0\), we have

\[ \zeta_n = t^n \left( t \partial_t + 2(n+1)r \partial_r \right), \]  

satisfying

\[ [\zeta_n, \zeta_m] = (m-n)\zeta_{m+n}. \]  

Note that \(\zeta_{-1} = \partial_t\) is the \(U(1)\) global isometry of the geometry, hence the \(U(1)\) isometry of the Schwarzschild metric is enhanced to a Virasoro algebra in the vicinity of the horizon.

The rest of the diffeomorphisms generated by \(\partial_\theta\) and \(\partial_\phi\) using the \(P(\phi)\) and \(W(\theta)\) functions are an enhancement of \(SO(3)\) isometry.

We should emphasize that all of the diffeomorphisms characterized by \((3.3)\) do not result non-zero charges, i.e some of them are trivial. Hence boundary symmetry group of the horizon is the group of allowed diffeomorphisms \((3.3)\) modulo these trivial transformations.

### 4 Calculation of Charges

In this section we want to compute the charges associated with the boundary symmetries of the horizon.

We use the covariant formalism of ref. [11, 12]. The infinitesimal charge difference between geometries \(g_{\mu\nu}\) and \(g_{\mu\nu} + h_{\mu\nu}\), associated with the transformation \(\zeta\), is defined by

\[ \delta Q_\zeta = \frac{1}{8\pi} \int_{\partial \Sigma} k_\zeta[h; g] \]  

²We have checked consistency of these boundary conditions by making use of the linearized equations of motion.
where
\[ k_\zeta = -\frac{1}{4} \epsilon_{\alpha\beta\mu\nu} [\zeta^\nu D^\mu h - \zeta^\nu D_{\sigma} h^{\mu\sigma} + \zeta_\sigma D^\nu h_{\mu\sigma} + \frac{1}{2} h D^\nu \zeta_\mu \\
- h^{\nu\sigma} D_\sigma \zeta_\mu + \frac{1}{2} h^{\sigma\nu} (D^\mu \zeta_\sigma + D_\sigma \zeta_\mu) ] dx^\alpha \wedge dx^\beta, \]
(4.2)
with \( h \equiv g^{\mu\nu} h_{\mu\nu} \).

The algebra of the boundary charges is
\[ \{ Q_\zeta, Q_\eta \}_{DB} = \delta_\eta Q_\zeta = Q_\zeta [Q_\zeta, Q_\eta] + \frac{1}{8\pi} \int_{\partial\Sigma} k_\zeta [\mathcal{L}_\eta \tilde{g}, \tilde{g}] \]
(4.3)
where \( \tilde{g} \) denotes the metric of the background (metric (2.6) for our case).

Since the asymptotic Killing vectors (3.4) are time dependent, according to the definition (4.1) the corresponding charges depend on time too. Following ref. [5] we consider the analytic continuation of \( t \) and define the generators as
\[ L_n = \frac{1}{2\pi i} \int dt Q_\zeta, \]
(4.4)
which results following algebra:
\[ \{ L_m, L_n \}_{DB} = (m - n)L_{m+n} + \frac{c}{12} (m^3 - Bm)\delta_{m+n,0} \]
(4.5)
where \( c \) is the central charge and \( B \) is a constant that can be absorbed by a shift in \( L_0 \).

Using (4.2) and (4.3) one finds
\[ \frac{1}{2\pi i} \lim_{r \to \infty} \int dt \frac{1}{8\pi} \int k_{\zeta_m} [\mathcal{L}_{\zeta_n} \tilde{g}, \tilde{g}] = r_H^3 (m^3 - m)\delta_{m+n,0}. \]
(4.6)
Hence we have
\[ c = 12r_H^3 = 96 M^3. \]
(4.7)

5 Dual CFT of the Horizon

In light of the Brown and Henneaux’s method, we can use the results of the previous sections to propose a dual conformal field theory for the quantum gravity in the background of the Schwarzschild black hole. This CFT is at the horizon and there is a correspondence between its Hilbert space and the geometries similar to the Schwarzschild. This similarity is at the horizon and they may have different behavior in the bulk. This proposal will be firm if we find a consistency between the entropy of the Schwarzschild black hole and the degeneracy of the corresponding states in the CFT. The interesting point is that if we identify the temperature of the CFT as the Hawking temperature and its central charge as (4.7), the Cardy formula results
\[ S = \frac{\pi^2}{3} c T = \frac{\pi^2}{3} (12 r_H^3) \left( \frac{1}{4\pi r_H} \right) = \pi r_H^2, \]
(5.1)
which is the Bekenstein-Hawking entropy of the black hole. This is a good evidence that the
dual CFT of the gravity in the Schwarzschild background is a chiral CFT at the horizon.

Note that the final result for the central charge in ref. [9] is $c = 3 \pi q^2 r_H^2$, where $q$ is an
arbitrary constant. It is consistent with the our result if

$$q^2 = \frac{4 r_H}{\pi}. \quad (5.2)$$

Moreover, $L_0$ in [9] depends on $q$ too. For the value of $q$ given by (5.2) we have

$$L_0 = \frac{M}{4}. \quad (5.3)$$

We also note that the central charge in [8] is given by

$$c = \frac{3 A \beta}{2 \pi T} \quad (5.4)$$

where $A$ is the area of the event horizon, $\beta$ is the inverse of the Hawking temperature and
$T$ is an arbitrary period. Consistency of (5.4) with our result $c = 12 r_H^3$ determines period
as $T = 2\pi$.

6 Conclusion

We propose that the quantum gravity in the background of the Schwarzschild black hole has
a dual chiral CFT. This CFT is defined at the horizon and its central charge and temperature are

$$c = 12 r_H^3, \quad T = \frac{1}{4 \pi r_H}. \quad (6.1)$$

This proposal is based on the observation that in the vicinity of the horizon appropriate
boundary conditions result a boundary symmetry group which is an infinite dimensional
conformal group in the $(t, r)$ space. Moreover, the Cardy formula produces the same entropy
as the Bekenstein-Hawking area law.

It seems that this results can be generalized and there exist a dual CFT at the hori-
zon of the generic four-dimensional black holes. If we follow lessons of the Schwarzschild
black hole, the dual of the non-extreme Kerr-Newmann has the following central charge and
temperature:

$$c = \frac{12(r_+^2 + a^2)^2}{r_+ - r_-^3}, \quad T = \frac{r_+ - r_-}{4 \pi (r_+^2 + a^2)}, \quad (6.2)$$

where $r_+$ and $r_-$ are the radius of the outer and inner horizons. It is clear that the temper-
ature is the same as the Hawking temperature.

For the extremal case $r_+ = r_-$, the central charge blows up while the temperature goes
to zero and the Cardy formula still gives the correct answer. We postpone investigation of
this CFT to the future works.
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