ON p-ADIC EULER L-FUNCTIONS

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Abstract. In this paper, we define the p-adic Euler L-functions using the fermionic p-adic integral on $\mathbb{Z}_p$. By computing the values of the p-adic Euler L-functions at negative integers, we show that for Dirichlet characters with odd conductor, this definition is equivalent to the previous definition in [8] following Kubata-Leopoldt and Washington’s approach. We also study the behavior of p-adic Euler L-functions at positive integers. An interesting thing is that most of the results in Section 11.3.3 of Cohen’s book [3] are also established if we replace the generalized Bernoulli numbers with the generalized Euler numbers.

1. Introduction

Throughout this paper, we use the following notations.

- $\mathbb{C}$ – the field of complex numbers.
- $p$ – an odd rational prime number.
- $\mathbb{Z}_p$ – the ring of p-adic integers.
- $\mathbb{Q}_p$ – the field of fractions of $\mathbb{Z}_p$.
- $\mathbb{C}_p$ – the completion of a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$.
- $\mathbb{C}_\mathbb{Z}_p = \mathbb{Q}_p \setminus \mathbb{Z}_p$.
- $\nu_p$ – the p-adic valuation of $\mathbb{C}_p$ normalized so that $|p|_p = p^{-\nu_p(p)} = p^{-1}$.
- $\mathbb{Z}_p^\times$ – the group of p-adic units.

In 1964, Kubota-Leopoldt [14] defined the p-adic analogues of classical L-functions of Dirichlet which are closely related to the arithmetic of cyclotomic fields (see [5]). In [3, Chapter 11]), we can find another definition for the p-adic L-functions using Volkenborn integrals which is equivalent to Kubota-Leopoldt’s original definition. The Volkenborn integral was introduced by Volkenborn [19] and he also investigated many important properties of p-adic valued functions defined on the p-adic domain (see [19, 20]).

In 1749, Euler gave a paper to the Berlin Academy entitled Remarques sur un beau rapport entre les sérées des puissances tant directes que réciproques. In this paper, he studied

$$\phi(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$

2000 Mathematics Subject Classification. 11E95, 11B68, 11S80, 11M35.

Key words and phrases. Euler number and polynomial, p-adic integral, p-adic Euler L-function.
For $s \in \mathbb{C}$ and $\text{Re}(s) > 0$, the Euler zeta function and the Hurwitz-type Euler zeta function are defined by

$$
\zeta_E(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \quad \text{and} \quad \zeta_E(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + x)^s}
$$

respectively (see [1, 6, 11, 12, 15]). Notice that the Euler zeta functions can be analytically continued to the whole complex plane, and these zeta functions have the values of the Euler numbers or the Euler polynomials at negative integers.

Recently, we defined the $p$-adic Hurwitz-type Euler zeta functions using the fermionic $p$-adic integral on $\mathbb{Z}_p$ (see [9]). Notice that the fermionic $p$-adic integral on $\mathbb{Z}_p$ has been used by T. Kim [11] to derive useful formulas involving the Euler numbers and polynomials, and it has also been used by the second author to give a brief proof of Stein’s classical result on Euler numbers modulo power of two (see [7]). In [9], we first gave the definition for the $p$-adic Hurwitz-type Euler zeta function for $x \in C\mathbb{Z}_p$, then we gave the definition for the $p$-adic Hurwitz-type Euler zeta function for $x \in \mathbb{Z}_p$ using characters modulo $p^v$ and also gave a new definition for the $p$-adic Euler $\ell$-function for characters modulo $p^v$ as a special case. We showed that in this case the definition is equivalent to the second author’s previous definition in [8]. In [8], the second author defined the $p$-adic Euler $\ell$-functions for Dirichlet characters with odd conductor following Kubota-Leopoldt’s approach and Washington’s one and he also computed the derivative of $p$-adic Euler $\ell$-function at $s = 0$ and the values of $p$-adic Euler $\ell$-function at positive integers.

In this paper, we define the $p$-adic Euler $L$-functions using the fermionic $p$-adic integral on $\mathbb{Z}_p$. By computing the values of the $p$-adic Euler $L$-functions at negative integers, we show that for Dirichlet characters with odd conductor, this definition is equivalent to the second author’s previous definition in [8] following Kubota-Leopoldt and Washington’s approach (see Remark 4.6). We also study the behavior of $p$-adic Euler $L$-functions at positive integers. We show that most of the results in Section 11.3.3 of Cohen’s book [3] are also established if we replace the generalized Bernoulli numbers with the generalized Euler numbers.

2. Generalized Euler numbers

In this section, we recall some facts on the Euler numbers, the Euler polynomials and the generalized Euler numbers which will be used in the subsequent sections to study the properties for the $p$-adic Euler $L$-functions.

The definition of Euler polynomials is well known and can be found in many classical literatures (cf. see [17, p. 527-532]).

We consider the following generating function

$$
F(t, x) = \frac{2e^{xt}}{e^t + 1}.
$$
Expand $F(t, x)$ into a power series of $t$:

$$F(t, x) = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{2.2}$$

The coefficients $E_n(x), n \geq 0$, are called Euler polynomials. We note that there is a second kind Euler numbers, namely, from (2.1)

$$E_n = E_n(0), \tag{2.3}$$

For example

$$E_0 = 1, E_1 = -\frac{1}{2}, E_3 = \frac{1}{4}, E_5 = -\frac{1}{2}, E_7 = \frac{17}{8}, E_9 = -\frac{31}{2}, E_{11} = \frac{691}{4}, \ldots$$

and $E_{2k} = 0$ for $k = 1, 2, \ldots$. These numbers is a kind of ($\lambda$-)Euler numbers which defined by T. Kim in [11, p. 784]. This is different from the first kind Euler numbers named by Scherk in 1825 (see the first paragraph of [7, p. 2167]).

We generalize the above definition of $E_n$ and $E_n(x)$ as follows: let $\chi$ be a primitive Dirichlet character $\chi$ with an odd conductor $f = f_\chi$, and let

$$F_\chi(t) = 2 \sum_{a=1}^{f} \frac{(-1)^{\chi(a)} e^{at}}{e^{ft} + 1}, \quad |t| < \frac{\pi}{f}. \tag{2.4}$$

and

$$F_\chi(t, x) = F_\chi(t) e^{xt} = 2 \sum_{a=1}^{f} \frac{(-1)^{\chi(a)} e^{(a+x)t}}{e^{ft} + 1}, \quad |t| < \frac{\pi}{f}. \tag{2.5}$$

(see e.g. [11, p. 783]). Expanding thses into power series of $t$, let

$$F_\chi(t) = \sum_{n=0}^{\infty} \frac{E_{n, \chi} t^n}{n!} \tag{2.6}$$

and

$$F_\chi(t, x) = \sum_{n=0}^{\infty} E_{n, \chi}(x) \frac{t^n}{n!}. \tag{2.7}$$

Then $E_{n, \chi}(x) = \sum_{i=0}^{n} \binom{n}{i} E_i \chi x^{n-i}$ for $n \geq 0$. The coefficients $E_{n, \chi}$ and $E_{n, \chi}(x)$ for $n \geq 0$, are called generalized Euler numbers and polynomials, respectively. It is clear from (2.4) that $F_\chi(-t) = -\chi(-1) F_\chi(t)$, if $\chi \neq \chi^0$, the trivial character. Hence we obtain the result as follows:

**Proposition 2.1.** If $\chi \neq \chi^0$, the trivial character, then we have

$$(-1)^{n+1} E_{n, \chi} = \chi(-1) E_{n, \chi}, \quad n \geq 0.$$  

In particular we obtain

$$E_{n, \chi} = 0 \quad \text{if} \quad \chi \neq \chi^0, \quad n \not\equiv \delta_\chi \pmod{2},$$

where $\delta_\chi = 0$ if $\chi(-1) = -1$ and $\delta_\chi = 1$ if $\chi(-1) = 1$. 


Let $N$ be the least common multiple of $p$ and $f$. Then by (2.4), we have
\begin{equation}
F\chi(t) = 2 \sum_{m=0}^{\infty} (-1)^m \chi(m)e^{mt} = 2 \sum_{a=1}^{N} (-1)^a \chi(a) \frac{(e^{\frac{a}{N}})^{Nt}}{e^{Nt} + 1}.
\end{equation}
Therefore, by (2.1), (2.2), (2.4), (2.6) and (2.8), we obtain the result as follows:

**Proposition 2.2.** Let $N$ be the least common multiple of $p$ and $f$. Then
\begin{equation}
E_{n,\chi} = N^n \sum_{a=1}^{N} (-1)^a \chi(a) E_n \left( \frac{a}{N} \right).
\end{equation}

From (2.3), the generating function $F\chi(t, x)$ is given by
\begin{equation}
F\chi(t, x) = 2 \sum_{a=0}^{f} (-1)^a \chi(a) \sum_{k=0}^{\infty} (-1)^k e^{(a+x+fk)t}
= \sum_{n=0}^{\infty} \left( 2 \sum_{l=0}^{\infty} (-1)^l \chi(l)(l+x)^n \right) \frac{t^n}{n!}.
\end{equation}
Comparing coefficients of $t^n/n!$ on both sides of (2.7) and (2.9) gives
\begin{equation}
E_{n,\chi}(x) = 2 \sum_{l=0}^{\infty} (-1)^l \chi(l)(l+x)^n
\end{equation}
(see [11, p.784]). In particular, if $n \geq 0$ we have
\begin{equation}
E_{n,\chi}(x + N) = 2 \sum_{l=0}^{\infty} (-1)^l \chi(l)(l + x + N)^n.
\end{equation}
By (2.10) and (2.11), we obtain the results as follows:

**Proposition 2.3.** Let $N$ be the least common multiple of $f$ and $p$, we have
\begin{equation}
\sum_{r=0}^{N-1} (-1)^r \chi(r)(x + r)^n = \frac{E_{n,\chi}(x) - (-1)^N E_{n,\chi}(x + N)}{2}.
\end{equation}

### 3. The $p$-adic Fermionic Integral

In this section, we recall the $p$-adic fermionic integral and its relationship between the Euler numbers and polynomials.

Let $UD(\mathbb{Z}_p, \mathbb{C}_p)$ denote the space of all uniformly (or strictly) differentiable $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p$ (see [11]). The metric space $\mathbb{Q}_p$ has a basis of open sets consisting of all sets of the form $a + p^NZ_p = \{ x \in \mathbb{Q}_p \mid |x-a|_p \leq p^{-N} \} \subset \mathbb{Z}_p$ for $a \in \mathbb{Q}_p$ and $N \in \mathbb{Z}$. This means that any open subset of $\mathbb{Q}_p$ is a union of open subsets of this type. For $N \geq 1$, the $p$-adic fermionic distribution
\begin{equation}
\mu_{-1}(a + p^NZ_p) = (-1)^a
\end{equation}
is known as a measure on $\mathbb{Z}_p$ (see [11]). We shall write $d\mu_{-1}(x)$ to remind ourselves that $x$ is the variable of integration. It is well known that for
\[ f \in UD(Z_p, \mathbb{C}_p), \text{ its fermionic } p\text{-adic integral } I_{-1}(f) \text{ on } Z_p \text{ is defined to be} \]
\[ \lim_{N \to \infty} N^{-1} \sum_{a=0}^{p^N-1} f(a)(-1)^a \]
as \( N \to \infty \) (see [11, 7, 9]). Write down this integral as

\[ I_{-1}(f) = \int_{Z_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} N^{-1} \sum_{a=0}^{p^N-1} f(a)(-1)^a \in \mathbb{C}_p \]
(see [11] for more details). By (3.2), it is well known that the Euler numbers \( E_n \) and the Euler polynomials \( E_n(x) \) are connected with \( I_{-1} \)-integrals as follows. For any \( n \geq 0 \),

\[ I_{-1}(x^n) = \int_{Z_p} x^n d\mu_{-1}(x) = E_n \]
and

\[ I_{-1}((x+y)^n) = \int_{Z_p} (x+y)^n d\mu_{-1}(y) = E_n(x) \]
(see [11, 7]).

We can now obtain the generalized Euler numbers and polynomials on \( Z_p \). Let \( \chi \) be a primitive Dirichlet character \( \chi \) with an odd conductor \( f = f_{\chi} \), and let \( N \) be the least common multiple of \( p \) and \( f \). Then by (3.2), (3.4) and Proposition 2.2, we see that

\[ \int_{Z_p} \chi(x)x^n d\mu_{-1}(x) = \lim_{r \to \infty} Np^{-1} \sum_{a=0}^{Np^r-1} (-1)^a \chi(a)a^n \]

\[ = \lim_{r \to \infty} N^{-1} \sum_{k=0}^{N-1} \sum_{a=0}^{p^r-1} (-1)^{Na+k} \chi(Na+k)(Na+k)^n \]

\[ = N^n \sum_{k=0}^{N-1} (-1)^k \chi(k) E_n \left( \frac{k}{N} \right) \]
is equivalent to

\[ I_{-1}(\chi(x)x^n) = \int_{Z_p} \chi(x)x^n d\mu_{-1}(x) = E_{n,\chi}. \]

Similarly, by (3.5) we have

\[ I_{-1}(\chi(y)(x+y)^n) = \int_{Z_p} \chi(y)(x+y)^n d\mu_{-1}(y) = E_{n,\chi}(x). \]

Therefore we have the results as follows:
**Proposition 3.1.** Let $\chi$ be a primitive Dirichlet character $\chi$ with an odd conductor. Then for $n \geq 0$ we have

1. $I_{-1}(x^n) = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n$.
2. $I_{-1}((x+y)^n) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x)$.
3. $I_{-1}(\chi(x)x^n) = \int_{\mathbb{Z}_p} \chi(x)x^n d\mu_{-1}(x) = E_{n,\chi}$.
4. $I_{-1}(\chi(y)(x+y)^n) = \int_{\mathbb{Z}_p} \chi(y)(x+y)^n d\mu_{-1}(y) = E_{n,\chi}(x)$.

**4. $p$-adic Euler $L$-functions**

In this section, we define the $p$-adic Euler $L$-functions and compute its value at negative integers.

Given $x \in \mathbb{Z}_p$, $p \nmid x$ and $p > 2$, there exists a unique $(p-1)$-th root of unity $\omega(x) \in \mathbb{Z}_p$ such that

$$x \equiv \omega(x) \pmod{p},$$

where $\omega$ is the Teichmüller character. Let $\langle x \rangle = \omega^{-1}(x)x$, so $\langle x \rangle \equiv 1 \pmod{p}$. We extend the notation $\langle \rangle$ to $\mathbb{Q}_p^\times$ by setting

$$(4.1) \quad \langle x \rangle = \left( \frac{x}{p^{\nu_p(x)}} \right).$$

If $x \in \mathbb{Q}_p^\times$, we define $\omega_v(x)$ by

$$(4.2) \quad \omega_v(x) = \langle x \rangle = p^{\nu_p(x)} \omega \left( \frac{x}{p^{\nu_p(x)}} \right)$$

(see [3] p. 280, Definition 11.2.2 for more details). Now we give a definition for the $p$-adic Euler $L$-functions.

**Definition 4.1.** Let $\chi$ be a primitive character of conductor $f$ and $m$ be the least common multiple of $f$ and $p$. For $s \in \mathbb{C}_p$ such that $|s| < R_p = p^{(p-2)/(p-1)}$, we have

$$L_{p,E}(\chi, s) = \langle m \rangle^{1-s} \sum_{a=0}^{f-1} \chi(a) \zeta_{p,E} \left( s, \frac{a}{f} \right) (-1)^a,$$

where the $p$-adic Hurwitz-type Euler zeta function $\zeta_{p,E}(s, x)$ is defined in Definition 3.2 and 4.1 of [9] for $x \in CZ_p$ and $x \in \mathbb{Z}_p$, respectively. We define $L_{p,E}(\chi, 1) = \lim_{s \to 1} L_{p,E}(\chi, s)$, when the limit exist. In particular if $\chi$ is the trivial character $\chi_0$, we set $\zeta_{p,E}(s) = L_{p,E}(\chi_0, s) = \zeta_{p,E}(s, 0)$ (see Theorem 4.10 (3) in [9]).

**Definition 4.2** (see [3] p. 302). (1) Let $m \in \mathbb{Z}_{>0}$. We define $\chi_{0,m}$ to be the trivial character modulo 1 when $p \nmid m$, and to be the trivial character modulo $p$ when $p \mid m$. In other words, $\chi_{0,m}(a) = 1$ when $p \nmid a$ or when $p \mid a$ but $p \nmid m$ and $\chi_{0,m}(a) = 0$ when $p \mid a$ and $p \mid m$.

2. If $I \subset \mathbb{Z}$, we set

$$\sum_{a \in I}^{(p)} g(a) = \sum_{a \in I \atop p|a} g(a)$$
and similarly
\[ \prod_{a \in I}^{(p)} g(a) = \prod_{a \in I} g(a). \]

In particular, if \( p \mid m \) we have
\[ \sum_{0 \leq a < m}^{(p)} g(a) = \sum_{a = 0}^{m-1} \chi_{0,m}(a) g(a). \]

**Lemma 4.3** (see [3, p. 302]). Let \( \chi \) be a nontrivial primitive character of conductor \( f \) and \( m \) be a common multiple of \( f \) and \( p \). Then
\[ \sum_{0 \leq a < m}^{(p)} \chi(a) = 0. \]

**Proposition 4.4.** Let \( \chi \) be a primitive character of an odd conductor \( f \) and \( m \) be the least common multiple of \( f \) and \( p \). Let \( s \in \mathbb{C}_p \) be such that \( |s| < R_p \).

1. We have
   \[ L_{p,E}(\chi, s) = \langle m \rangle^{1-s} \sum_{a=0}^{m-1} \chi_{0,m}(a) \chi(a) \zeta_{p,E} \left( s, \frac{a}{m} \right) (-1)^a. \]

2. If, in addition, \( p \mid m \), we have
   \[ L_{p,E}(\chi, s) = \langle m \rangle^{1-s} \sum_{0 \leq a < m}^{(p)} \chi(a)(-1)^a \sum_{i=0}^{\infty} \left( \frac{1-s}{i} \right) \frac{m^i}{a^i} E_i. \]

3. If \( \chi \neq \chi_0 \), then
   \[ L_{p,E}(\chi, 1) = \sum_{0 \leq a < m}^{(p)} \chi(a)(-1)^a. \]

**Proof.** (1) Writing \( a = kf + r \), we have
\[ \langle m \rangle^{1-s} \sum_{a=0}^{m-1} \chi_{0,m}(a) \chi(a) \zeta_{p,E} \left( s, \frac{a}{m} \right) (-1)^a. \]
(4.3)
\[ = \langle m \rangle^{1-s} \sum_{r=0}^{f-1} \chi(r)(-1)^r \sum_{k=0}^{m-1} \chi_{0,m}(kf + r) \zeta_{p,E} \left( s, \frac{m}{m} \right) (-1)^k. \]

Case (I): If \( p \nmid m \), then \( \chi_{0,m} \) to be a trivial character modulo 1 by Definition 4.2. Using Theorem 3.10 (3) in [3], we have
\[ \sum_{k=0}^{m-1} \chi_{0,m}(kf + r) \zeta_{p,E} \left( s, \frac{m}{m} \right) (-1)^k = \sum_{k=0}^{m-1} \zeta_{p,E} \left( s, \frac{r + \frac{k}{f}}{m} \right) (-1)^k \]
\[ = \zeta_{p,E} \left( s, \frac{r}{f} \right). \]
Case (II): If \( p \mid m \), then \( \chi_{0,m} \) to be a trivial character modulo \( p \) by Definition 4.2. Thus we have
\[
\sum_{k=0}^{m-1} \chi_{0,m}(kf + r) \zeta_{p,E} \left( s, \frac{r}{m} + \frac{k}{m} \right) (-1)^k = \sum_{k=0}^{m-1} \zeta_{p,E} \left( s, \frac{r}{m} + \frac{k}{m} \right) (-1)^k.
\]

Using Corollary 4.5 in [9], we obtain the following:

(a) Suppose \( p \nmid f \). Then \( r/f \in \mathbb{Z}_p \) and
\[
p \nmid (kf + r) \quad \text{if and only if} \quad p \nmid (r/f + k).
\]
Thus
\[
(4.4) \quad \sum_{k=0}^{m-1} \zeta_{p,E} \left( s, \frac{r}{m} + \frac{k}{m} \right) (-1)^k = \zeta_{p,E} \left( s, \frac{r}{f} \right).
\]

(b) If \( p \mid f \), then \( \chi_{0,m}(kf + r) = \chi_{0,m}(r) \). We have
\[
\sum_{k=0}^{m-1} \chi_{0,m}(kf + r) \zeta_{p,E} \left( s, \frac{r}{m} + \frac{k}{m} \right) (-1)^k = \chi_{0,m}(r) \sum_{k=0}^{m-1} \zeta_{p,E} \left( s, \frac{r}{m} + \frac{k}{m} \right) (-1)^k.
\]
If \( p \mid r \), then
\[
(4.5) \quad \sum_{k=0}^{m-1} \chi_{0,m}(kf + r) \zeta_{p,E} \left( s, \frac{r}{m} + \frac{k}{m} \right) (-1)^k = 0.
\]
If \( p \nmid r \), then \( r/f \in \mathbb{C}\mathbb{Z}_p \). Thus by Theorem 3.10 (3) in [9] and the definition of \( \chi_{0,m}(r) \), we have
\[
(4.6) \quad \chi_{0,m}(r) \sum_{k=0}^{m-1} \zeta_{p,E} \left( s, \frac{r}{m} + \frac{k}{m} \right) (-1)^k = \chi_{0,m}(r) \zeta_{p,E} \left( s, \frac{r}{f} \right).
\]
Thus if \( p \nmid f \), substitute \((4.4)\) to \((4.3)\), we have
\[
\langle m \rangle^{1-s} \sum_{a=0}^{m-1} \chi_{0,m}(a) \chi(a) \zeta_{p,E} \left( s, \frac{a}{m} \right) (-1)^a
\]
\[
= \langle m \rangle^{1-s} \sum_{r=0}^{f-1} \chi(r)(-1)^r \sum_{k=0}^{m-1} \chi_{0,m}(kf + r) \zeta_{p,E} \left( s, \frac{r}{m} + \frac{k}{m} \right) (-1)^k
\]
\[
= \langle m \rangle^{1-s} \sum_{r=0}^{f-1} \chi(r)(-1)^r \zeta_{p,E} \left( s, \frac{r}{f} \right)
\]
\[
= L_{p,E}(\chi, s).
\]
If \( p \mid f \), substitute (4.5) and (4.6) to (4.3), we have

\[
\langle m \rangle^{1-s} \sum_{a=0}^{m-1} \chi_{0,m}(a) \chi(a) \zeta_{p,E} \left( s, \frac{a}{m} \right) (-1)^a = \langle m \rangle^{1-s} \sum_{r=0}^{f-1} \chi(r)(-1)^r \sum_{k=0}^{m-1} \chi_{0,m}(kf + r) \zeta_{p,E} \left( s, \frac{r}{f} + \frac{k}{m} \right) (-1)^k
\]

\[
= \langle m \rangle^{1-s} \sum_{r=0}^{f-1} \chi(r)(-1)^r \sum_{k=0}^{m-1} \chi_{0,m}(kf + r) \zeta_{p,E} \left( s, \frac{r}{m} + \frac{k}{m} \right) (-1)^k
\]

\[
= \langle m \rangle^{1-s} \sum_{r=0}^{f-1} \chi(r)(-1)^r \zeta_{p,E} \left( s, \frac{r}{m} \right)
\]

\[
= L_{p,E}(\chi, s),
\]

since \( p \mid f \) and \( p \mid r \), we have \((f,r) \neq 1\), so \( \chi(r) = 0 \), thus we get the last equality.

(2) If \( p \mid m \), from (1), we have

\[
L_{p,E}(\chi, s) = \langle m \rangle^{1-s} \sum_{a=0}^{m-1} \chi_{0,m}(a) \chi(a) \zeta_{p,E} \left( s, \frac{a}{m} \right) (-1)^a.
\]

Since \( p \mid m \), by Definition 4.2 we have

\[
L_{p,E}(\chi, s) = \langle m \rangle^{1-s} \sum_{a=0}^{m-1} \chi_{0,m}(a) \chi(a) \zeta_{p,E} \left( s, \frac{a}{m} \right) (-1)^a
\]

\[
= \langle m \rangle^{1-s} \sum_{0 \leq a < m} (p) \chi(a) \zeta_{p,E} \left( s, \frac{a}{m} \right) (-1)^a.
\]

Also since in the above equality, \( p \nmid a \) and \( p \mid m \), we obtain

\[
\frac{a}{m} \in \mathbb{Q}_p.
\]

By Theorem 3.5 of [9], we have

\[
\zeta_{p,E} \left( s, \frac{a}{m} \right) = \left\langle \frac{a}{m} \right\rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} E_i \frac{m^i}{a^i},
\]

where \( E_i \) are Euler numbers. Substitute (4.8) to (4.7), we have

\[
L_{p,E}(\chi, s) = \sum_{0 \leq a < m} (p) \chi(a)(-1)^a \left\langle \frac{a}{m} \right\rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} \frac{m^i}{a^i} E_i.
\]

(3) Since \( p \mid m \), by (2), we have

\[
L_{p,E}(\chi, 1) = \sum_{0 \leq a < m} (p) \chi(a)(-1)^a
\]

and the result is established. \( \square \)
Next we compute the values of $p$-adic Euler $L$-functions at negative integers.

**Proposition 4.5.** Keep the above assumptions.

1. The function $L_{p,E}(\chi, s)$ is a $p$-adic analytic function for $|s| < R_p$.
2. For $k \in \mathbb{Z}_{\geq 1}$, we have
   \[ L_{p,E}(\chi, 1 - k) = (1 - p^k \chi_k(p)) E_{k, \chi_k}, \]
   where $\chi_k = \chi \omega^{-k}$ and $\chi \neq \chi_0$, the trivial character.
3. If $\chi$ is an even character the function $L_{p,E}(\chi, s)$ is identically equal to zero.

**Remark 4.6.** By comparing Proposition 4.5 (2) with equalities (3.2) and (3.3) in [8, p. 6], from Lemma 1 in [5, p. 19] we conclude that for Dirichlet characters with odd conductor the definition of $p$-adic Euler $L$-functions in this paper is equivalent to the second author’s previous definition in [8] following Kubata-Leopoldt’s approach.

**Proof.** (1) By definition of $L_{p,E}(\chi, s)$, Theorem 3.5 and Proposition 4.7 in [9], the result follows.

(2) Let $m$ be the least common multiple of $p$ and $f$. Then by Proposition 4.4 (1), we have

\[ L_{p,E}(\chi, 1 - k) = \langle m \rangle^{k-1} \sum_{a=0}^{m-1} \chi_0, m(a) \chi(a) \zeta_{p,E} \left( 1 - k, \frac{a}{m} \right) (-1)^a. \]

Since $p | m$, by Definition 4.2 (2), we have

\[ L_{p,E}(\chi, 1 - k) = \langle m \rangle^{k} \sum_{0 \leq a < m}^{(p)} \chi(a) \zeta_{p,E} \left( 1 - k, \frac{a}{m} \right) (-1)^a. \]

Also since $p \nmid a$ and $p | m$, we have $a/m \in C\mathbb{Z}_p$. Using Theorem 3.9 (2) in [9] we have

\[ \zeta_{p,E} \left( 1 - k, \frac{a}{m} \right) = \frac{1}{\omega_v^k \left( \frac{a}{m} \right)} E_k \left( \frac{a}{m} \right). \]

where $E_k(x)$ are the Euler polynomials. Substitute (4.10) to (4.9), we have

\[ L_{p,E}(\chi, 1 - k) = \langle m \rangle^{k} \sum_{0 \leq a < m}^{(p)} \chi(a) (-1)^a \frac{1}{\omega_v^k \left( \frac{a}{m} \right)} E_k \left( \frac{a}{m} \right). \]

Since $p \nmid a$, we have

\[ \omega_v(a) = p^{v(a)} \omega \left( \frac{a}{p^{v(a)}} \right) = \omega(a). \]
Thus if \( \chi \neq \chi_0 \), by Proposition 2.2, we have

\[
L_{p,E}(\chi, 1 - k) = \langle m \rangle^k \omega_v^k(m) \sum_{0 \leq a < m}^{(p)} (-1)^a \chi \omega^{-k}(a) E_k \left( \frac{a}{m} \right)
\]

\[
= \langle m \rangle^k \omega_v^k(m) \left( \sum_{0 \leq a < m}^{m-1} (-1)^a \chi \omega^{-k}(a) E_k \left( \frac{a}{m} \right) \right)
\]

\[
= \left( \sum_{a=0}^{m-1} (-1)^a \chi \omega^{-k}(a) E_k \left( \frac{a}{m} \right) \right)
\]

\[
= \left( \sum_{a=0}^{m-1} (-1)^a \chi \omega^{-k}(pa) E_k \left( \frac{pa}{m} \right) \right)
\]

\[
= \left( \sum_{a=0}^{m-1} (-1)^a \chi \omega^{-k}(a) E_k \left( \frac{a}{m} \right) \right)
\]

\[
= (1 - \chi_k(p)) E_k \chi_k,
\]

where \( \chi_k = \chi \omega^{-k} \).

(3) From Definition 4.1, we have

\[
L_{p,E}(\chi, s) = \langle m \rangle^{1-s} \sum_{a=0}^{f-1} \chi(a) \zeta_{p,E} \left( s, \frac{a}{f} \right) (-1)^a.
\]

Let \( b = f - 1 - a \), and let \( \chi \) be an even character. Then by Corollary 3.6 and Theorem 4.10 in [9], we have

\[
L_{p,E}(\chi, s) = \langle m \rangle^{1-s} \sum_{b=0}^{f-1} \chi(f - 1 - b) \zeta_{p,E} \left( s, \frac{f - 1 - b}{f} \right) (-1)^{f-1-b}
\]

\[
= \langle m \rangle^{1-s} \sum_{b=0}^{f-1} \chi(1 + b) \zeta_{p,E} \left( s, \frac{1 + b}{f} \right) (-1)^b
\]

\[
= \langle m \rangle^{1-s} \sum_{b'=1}^{f} \chi(b') \zeta_{p,E} \left( s, \frac{b'}{f} \right) (-1)^{b'} (-1),
\]

so \( L_{p,E}(\chi, s) = -L_{p,E}(\chi, s) \). Therefore \( L_{p,E}(\chi, s) = 0 \) if \( \chi \) is an even character. \( \square \)
5. \( p \)-adic Euler L-function at positive integers

In this section we study the behavior of \( p \)-adic Euler \( L \)-functions at positive integers following Cohen’s approach in Section 11.3.3 of [3]. We show that most of the results in Section 11.3.3 of Cohen’s book are also established if we replace the generalized Bernoulli numbers with the generalized Euler numbers.

**Proposition 5.1.** Let \( f \) be an odd integer and \( \chi \) be a primitive character modulo \( f \) and \( m \) be the least common multiple of \( f \) and \( p \).

1. For \( k \in \mathbb{Z} \setminus \{0\} \), we have
   \[
   L_{p,E}(\chi, k + 1) = \lim_{N \to \infty} \sum_{0 \leq n < mp^N}^{(p)} \chi_k(n) \frac{(-1)^n}{n^k}.
   \]

2. \[
   L_{p,E}(\chi, 1) = E_{0,\chi},
   \]
   where \( E_{0,\chi} \) defined in (2.6).

**Proof.** (1) By Theorem 3.9 (1) in [9], for \( p \nmid a \), we have

\[
\zeta_{p,E}(k + 1, \frac{a}{m}) = \omega_v^k \left( \frac{a}{m} \right) \int_{\mathbb{Z}_p} \frac{1}{(a + j)k} d\mu_{-1}(j)
\]

\[
= \omega_v^k \left( \frac{a}{m} \right) \lim_{N \to \infty} \sum_{j=0}^{p^N - 1} \frac{(-1)^j}{(a + j)k}.
\]

(5.1)

\[
= \omega_v^k \left( \frac{a}{m} \right) m^k \lim_{N \to \infty} \sum_{j=0}^{p^N - 1} \frac{(-1)^j}{(a + mj)k}.
\]

\[
= \omega_v^k(a) \frac{m^k}{\omega_k(m)} \lim_{N \to \infty} \sum_{j=0}^{p^N - 1} \frac{(-1)^j}{(a + mj)k}.
\]

So that, by Definition 4.2 (2) and Proposition 4.4 (1), we have

\[
L_{p,E}(\chi, k + 1) = (m)^{-k} \sum_{a=0}^{m-1} \chi_{0,m}(a) \chi(a) \zeta_{p,E} \left( k + 1, \frac{a}{m} \right) (-1)^a
\]

(5.2)

\[
= (m)^{-k} \sum_{0 \leq a < m}^{(p)} \chi(a) \zeta_{p,E} \left( k + 1, \frac{a}{m} \right) (-1)^a.
\]

Thus if we set

\[n = mj + a, \text{ where } 0 \leq j \leq p^N - 1, \ 0 \leq a \leq m - 1, \ p \nmid a.\]

Then

\[0 \leq n \leq mp^N - 1, \text{ and } p \nmid m.\]
Substitute (5.1) to (5.2), since $f$ is an odd integer and $m$ is the least common multiple of $f$ and $p$, we have

$$L_{p,E}(\chi, k + 1) = \sum_{0 \leq a < m}^{(p)} \chi(a) \omega^k(a)(-1)^a \lim_{N \to \infty} \sum_{j=0}^{pN-1} \frac{(-1)^j}{(a + mj)^k}$$

$$= \sum_{0 \leq a < m}^{(p)} \chi(a) \omega^k(a) \lim_{N \to \infty} \sum_{j=0}^{pN-1} \frac{(-1)^{j+a}}{(a + mj)^k}$$

$$= \sum_{0 \leq a < m}^{(p)} \chi \omega^k(a) \lim_{N \to \infty} \sum_{j=0}^{pN-1} \frac{(-1)^{ma+j}}{(ma + j)^k}$$

$$= \lim_{N \to \infty} \sum_{0 \leq a < mpN-1}^{(p)} \chi \omega^k(a) \frac{(-1)^n}{n^k}.$$  

(2) By Definition of $L_{p,E}(\chi, s)$, Theorem 3.8 and Corollary 4.4 in [9], we have

$$L_{p,E}(\chi, k + 1) = \sum_{a=0}^{(f-1)} \chi(a) \zeta_{p,E} \left(1, \frac{a}{f}\right) (-1)^a$$

$$= \sum_{a=0}^{(f-1)} \chi(a)(-1)^a$$

$$= E_{0,\chi}.$$  

**Corollary 5.2.** Let $k \in \mathbb{Z}\{0\}$. If $\chi$ is a primitive character modulo a power of $p$, then we have

$$L_{p,E}(\chi, k + 1) = \int_{\mathbb{Z}_p^\times} \frac{\chi \omega^k(x)}{x^k} d\mu_{-1}(x).$$

In particular,

$$L_{p,E}(\omega^{-k}, k + 1) = \int_{\mathbb{Z}_p^\times} \frac{1}{x^k} d\mu_{-1}(x).$$

**Proof.** If $f$ is a power of $p$, that is $f = p^v$, then we have

$$L_{p,E}(\chi, s) = \sum_{a=0}^{(p^v-1)} \chi(a) \zeta_{p,E} \left(s, \frac{a}{p^v}\right) (-1)^a$$

$$= \zeta_{p,E}(\chi, s, 0)$$

$$= \ell_{p,E}(\chi, s).$$
by Corollary 4.3 and Remark 4.11 in [9]. Thus
\[
L_{p,E}(\chi, s) = \ell_{p,E}(x, k + 1)
= \zeta(\chi, s, 0)
= \int_{\mathbb{Z}_p} \chi(x) x^{-k} d\mu_{-1}(x)
= \int_{\mathbb{Z}_p} \chi(x) x^{-k} d\mu_{-1}(x)
= \int_{\mathbb{Z}_p} \chi(x) x^{-k} d\mu_{-1}(x)
= \int_{\mathbb{Z}_p} \chi(x) x^{-k} d\mu_{-1}(x).
\]

□

**Proposition 5.3.** Let \(\chi\) be a primitive character modulo \(f\) and \(\Phi\) be the Euler-phi function. Then for all \(k \in \mathbb{Z}\), we have
\[
L_{p,E}(\chi, k + 1) = \lim_{r \to \infty} E_{\Phi(p^r) - k, \chi \omega^k}.
\]
In particular,
\[
\lim_{r \to \infty} E_{\Phi(p^r) - k} = L_{p,E}(\omega^{-k}, k + 1),
\]
where \(E_m\) are the Euler numbers.

**Proof.** Denote \(\chi_k = \chi \omega^{-k}\). Since \(L_{p,E}(\chi, s)\) is a continuous function of \(s\), for all \(k \in \mathbb{Z}\), we have
\[
L_{p,E}(\chi, k + 1) = \lim_{r \to \infty} L_{p,E}(\chi, k + 1 - \Phi(p^r))
= \lim_{r \to \infty} L_{p,E}(\chi, 1 - (\Phi(p^r) - k))
= \lim_{r \to \infty} (1 - p^{\Phi(p^r) - k} \chi \Phi(p^r) - k(p)) E_{\Phi(p^r) - k, \chi \omega^{k - \Phi(p^r)}}
\]
using Proposition 4.3. Since
\[
\omega^{\Phi(p^r)} = \omega^{p^{r-1}(p-1)} = 1 \quad \text{and} \quad \chi \Phi(p^r) - k = \chi \omega^{k - \Phi(p^r)} = \chi \omega^k,
\]
thus
\[
L_{p,E}(\chi, k + 1) = \lim_{r \to \infty} E_{\Phi(p^r) - k, \chi \omega^k}.
\]
This completes the proof of the Theorem. \(\square\)

**Definition 5.4.** For \(k \in \mathbb{Z}\), we define the \(p\)-adic \(\chi\)-Euler numbers by
\[
E_{k,p,\chi} = \lim_{r \to \infty} E_{\Phi(p^r) + k, \chi} = L_{p,E}(\chi \omega^k, 1 - k).
\]

**Proposition 5.5.** Assume that the conductor of \(\chi\) is a power of \(p\). Then for \(k \in \mathbb{Z}\), we have
\[
E_{k,p,\chi} = \lim_{r \to \infty} \sum_{0 \leq n < p^r}^{(p)} \chi(n)n^k(-1)^n = \int_{\mathbb{Z}_p} \chi(x)x^k d\mu_{-1}(x).
\]
Proof. We have
\[ E_{k,p,\chi} = L_{p,E}(\chi \omega^k, 1 - k) \]
\[ = \int_{\mathbb{Z}_p^*} \chi(x)x^k d\mu_{-1}(x) \]
\[ = \lim_{r \to \infty} \sum_{0 \leq n < p^r}^{(p)} \chi(n)n^k(-1)^n. \]
using Corollary 5.2. Thus
\[ E_{k,p,\chi} = \lim_{r \to \infty} \sum_{0 \leq n < p^r}^{(p)} \chi(n)n^k(-1)^n = \int_{\mathbb{Z}_p} \chi(x)x^k d\mu_{-1}(x). \]

Proposition 5.6.
(1) If \( \chi(-1) = (-1)^k \), then we have
\[ E_{k,p,\chi} = 0. \]
(2) If \( k \geq 1 \), then we have
\[ E_{k,p,\chi} = (1 - p^k \chi(p)) E_{k,\chi}. \]
(3) Let \( m \) be the least common multiple of \( f \) and \( p \), and set
\[ H_n(x) = \sum_{0 \leq a < m}^{(p)} \frac{\chi(a)}{a^n} (-1)^a. \]
If \( k \geq 1 \) and \( \chi(-1) = (-1)^{k-1} \), we have
\[ E_{-k,p,\chi} = \sum_{i=0}^{m} (-1)^i \binom{k+i-1}{k-1} m^{i} E_i H_{k+i}(x), \]
where \( E_i \) is the Euler numbers.
(4) For all \( k \), we have \( v_p(E_{k,p,\chi}) \geq 0 \).

Proof. (1) If \( \chi(-1) = 1 \) and \( k \equiv 0 \) (mod 2), then \( \Phi(p^r) + k \equiv 0 \) (mod 2), we have \( E_{\Phi(p^r)+k,\chi} = 0 \) by Proposition 2.1, thus
\[ E_{k,p,\chi} = \lim_{r \to \infty} E_{\Phi(p^r)+k,\chi} = 0. \]
If \( \chi(-1) = -1 \) and \( k \equiv 1 \) (modulo 2), then \( \Phi(p^r) + k \equiv 1 \) (mod 2), we have \( E_{\Phi(p^r)+k,\chi} = 0 \) by Proposition 2.1, thus
\[ E_{k,p,\chi} = \lim_{r \to \infty} E_{\Phi(p^r)+k,\chi} = 0. \]

(2) By Proposition 4.5 (2) and Definition 5.4, we have
\[ E_{k,p,\chi} = L_{p,E}(\chi \omega^k, 1 - k) = (1 - p^k \chi(p)) E_{k,\chi}. \]
(3) By Proposition 4.4, we have

\[ L_{p,E}(\chi\omega^{-k}, k + 1) = \langle m \rangle^{-k} \sum_{0 \leq a < m}^{(p)} \chi\omega^{-k}(a)(-1)^{a} \left\langle \frac{a}{m} \right\rangle^{-k} \sum_{i=0}^{\infty} \left( -\frac{k}{i} \right) \frac{m^{i}}{a^{i}} E_{i} \]

\[ = \sum_{0 \leq a < M}^{(p)} \chi_{k}(a)(-1)^{a} \left\langle \frac{a}{m} \right\rangle^{-k} \sum_{i=0}^{\infty} \left( -\frac{k}{i} \right) \frac{m^{i}}{a^{i}} E_{i} \]

\[ = \sum_{i=0}^{\infty} \left( -\frac{k}{i} \right) m^{i} E_{i} \sum_{0 \leq a < m}^{(p)} \chi_{k}(a)(-1)^{a} \left\langle \frac{a}{m} \right\rangle^{-k} \frac{1}{a^{i}} \]

\[ = \sum_{i=0}^{\infty} \left( -\frac{k}{i} \right) m^{i} E_{i} \sum_{0 \leq a < m}^{(p)} \chi(k)(a)(-1)^{a} \omega^{k}(a) \frac{1}{a^{i+k}} \]

By Definition 5.4 (1), we have

\[ E_{-k,p,\chi} = L_{p,E}(\chi\omega^{-k}, k + 1) \]

\[ = \sum_{i=0}^{\infty} \left( -\frac{k}{i} \right) m^{i} E_{i} H_{k+i}(x) \]

\[ = \sum_{i=0}^{\infty} (-1)^{i} \left\langle \frac{k+i-1}{k-1} \right\rangle m^{i} E_{i} H_{k+i}(x). \]

(4) We have

\[ v_{p}(E_{k,\chi}) = v_{p}\left( N^{k} \sum_{a=1}^{N} (-1)^{a} \chi(a) E_{k}\left( \frac{a}{N} \right) \right) \]

\[ \geq \min_{1 \leq a \leq N} v_{p}\left( N^{k}(-1)^{a} \chi(a) E_{k}\left( \frac{a}{N} \right) \right) \]

\[ = \min_{1 \leq a \leq N} v_{p}\left( N^{k}(-1)^{a} \chi(a) \sum_{j=0}^{k} \binom{k}{j} \left( \frac{a}{N} \right)^{j} E_{k-j} \right) \]

\[ \geq \min_{1 \leq a \leq N} v_{p}\left( N^{k}(-1)^{a} \chi(a) \binom{k}{j} \left( \frac{a}{N} \right)^{j} E_{k-j} \right) \]

proving (4). \qed

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