Hamiltonian Map to Conformal Modification of Spacetime Metric: Kaluza-Klein and TeVeS

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Abstract

It has been shown that the orbits of motion for a wide class of non-relativistic Hamiltonian systems can be described as geodesic flows on a manifold and an associated dual. This method can be applied to a four dimensional manifold of orbits in spacetime associated with a relativistic system. We show that a relativistic Hamiltonian which generates Einstein geodesics, with the addition of a world scalar field, can be put into correspondence with another Hamiltonian with conformally modified metric. Such a construction could account for part of the requirements of Bekenstein for achieving the MOND theory of Milgrom in the post-Newtonian limit. The constraints on the MOND theory imposed by the galactic rotation curves, through this correspondence, would then imply constraints on the structure of the world scalar field. We then use the fact that a Hamiltonian with vector gauge fields results, through such a conformal map, in a Kaluza-Klein type theory, and indicate how the TeVeS structure can be put into this framework.

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1. Introduction

The Hamiltonian[1]

\[ K = \frac{1}{2m} g_{\mu\nu} p^\mu p^\nu, \]  

(1)

with Hamilton equations (written in terms of derivatives with respect to an invariant world time \( \tau \) [2])

\[ \dot{x}_\mu = \frac{\partial K}{\partial p^\mu} = \frac{1}{m} g_{\mu\nu} p^\nu \]  

(2)

and

\[ \dot{p}^\mu = - \frac{\partial K}{\partial x_\mu} = -\frac{1}{2m} \frac{\partial g_{\mu\nu}}{\partial x_\mu} p^\mu p^\nu \]  

(3)

lead to the geodesic equation

\[ \ddot{x}_\rho = -\Gamma^\mu_{\rho\nu} \dot{x}_\nu \dot{x}_\mu, \]  

(4)

where what has appeared as a compatible connection form \( \Gamma_{\rho}^{\mu\nu} \) is given by

\[ \Gamma_{\rho}^{\mu\nu} = \frac{1}{2} g_{\rho\lambda} \left( \frac{\partial g^{\lambda\mu}}{\partial x_\nu} + \frac{\partial g^{\lambda\nu}}{\partial x_\mu} - \frac{\partial g^{\mu\nu}}{\partial x_\lambda} \right). \]  

(5)
These results are tensor relations over the usual diffeomorphisms admitted by the manifold $\{x_\mu\}$; writing the Hamiltonian in terms of (2), we see that the invariant interval on an orbit is proportional, through the constant Hamiltonian, to the square of the world time of evolution on the orbit, i.e.,

$$ds^2 = \frac{2}{m}Kd\tau^2.$$  

(6)

We shall first study, in the following, a generalization of (1) consisting of the addition of a scalar field $\Phi(x)$. The presence of such a scalar field can be considered as associated with the gauge covariant generalization of (1) in the Stückelberg-Schrödinger equation [3] in the absence of four-vector gauge fields, an energy distribution not directly associated with visible light. We then show that there is a corresponding Hamiltonian $\hat{K}$ with a conformally modified metric, and no explicit additive scalar field, which has the form of Bekenstein’s construction [4] for the realization of Milgrom’s MOND program (modified Newtonian dynamics) [5] for achieving the observed galactic rotation curves. This simple form of Bekenstein’s theory (called RAQUAL), which we discuss in detail in this work, for the sake of simplicity and clarity in the development of the mathematical method, does not properly account for causality and gravitational lensing; the theory has been further developed to include vector fields as well (TeVeS) [6] which has been relatively successful in accounting for these problems. We have shown previously that a gauge type Hamiltonian, with Minkowski metric and both vector and scalar fields [3] results, under a conformal map, in an effective Kaluza-Klein theory [7], and we shall indicate here (using a general Einstein metric) how the TeVeS structure can emerge, in terms of a Kaluza-Klein theory, in this way. More detailed analysis will be given in a subsequent publication.

In the case treated in detail here, known as RAQUAL, the correspondence between $K$ and $\hat{K}$ implies a relation between the conformal factor in $\hat{K}$ and the world scalar field $\Phi$, and thus a possible connection between the so-called dark matter problem and a dark energy distribution represented by $\Phi$. Application of the TeVeS theory can, furthermore, provide information on the Hamiltonian vector fields.

2. Addition of a scalar potential

The addition of a scalar potential to the Hamiltonian (1), in the form

$$K = \frac{1}{2m}g_{\mu\nu}p^\mu p^\nu + \Phi(x),$$

(7)

leads, according to the Hamilton equations, to the geodesic equation\(^1\)

$$\ddot{x}_\rho = -\Gamma^\rho_{\nu\mu} \dot{x}_\nu \dot{x}_\mu - \frac{1}{m}g_{\rho\nu} \frac{\partial \Phi}{\partial x_\nu}.$$  

(8)

Now, consider the Hamiltonian

$$\hat{K} = \frac{1}{2m}\hat{g}_{\mu\nu}p^\mu p^\nu,$$

(9)

\(^1\) Note that (8) does not admit an equivalence principle.
where
\[ \hat{g}_{\mu\nu} = \phi g_{\mu\nu}. \]  
(10)

This Hamiltonian can be put into correspondence with (7), as in the nonrelativistic case treated in [8], by defining
\[ \phi = \frac{k}{k - \Phi} \]  
(11)

with the constant mass shell constraint
\[ k = \hat{K} = K. \]  
(12)

As for (4), the Hamilton equations applied to (9) lead to the geodesic equation\(^2\)
\[ \ddot{x}_\rho = -\hat{\Gamma}^\rho_{\mu\nu} \dot{x}_\nu \dot{x}_\mu, \]  
(13)

where
\[ \hat{\Gamma}^\rho_{\mu\nu} = \frac{1}{2} \hat{g}^\lambda_{\rho} \left( \frac{\partial \hat{g}^{\lambda\mu}}{\partial x_\nu} + \frac{\partial \hat{g}^{\lambda\nu}}{\partial x_\mu} - \frac{\partial \hat{g}^{\mu\nu}}{\partial x_\lambda} \right). \]  
(14)

We remark that the construction based on Eqs. (9) and (10) admits the same family of diffeomorphisms as that of (7), since \( \phi \) is scalar. Under these diffeomorphisms, both \( g_{\mu\nu} \) and \( \hat{g}_{\mu\nu} \) are second rank tensors, and by construction of the connection forms, (4) and (13) are covariant relations. In the special coordinates for which (10) is taken explicitly, we have
\[ \frac{\partial \hat{g}^{\lambda\mu}}{\partial x_\nu} = \frac{\partial \phi}{\partial x_\nu} g^{\lambda\mu} + \phi \frac{\partial g^{\lambda\mu}}{\partial x_\nu}, \]  
(15)

so that
\[ \hat{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \frac{1}{2\phi} \left\{ \frac{\partial \phi}{\partial x_\nu} \delta^\rho_\mu + \frac{\partial \phi}{\partial x_\mu} \delta^\rho_\nu + g^{\mu\nu} g_{\rho\lambda} \frac{\partial \phi}{\partial x_\lambda} \right\}. \]  
(16)

Substituting (11) into (16), this becomes,
\[ \hat{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \frac{1}{2(k - \Phi)} \left\{ \frac{\partial \Phi}{\partial x_\nu} \delta^\rho_\mu + \frac{\partial \Phi}{\partial x_\mu} \delta^\rho_\nu + g^{\mu\nu} g_{\rho\lambda} \frac{\partial \Phi}{\partial x_\lambda} \right\}, \]  
(17)

and therefore the geodesic equation takes the form
\[ \ddot{x}_\rho = -\Gamma^\rho_{\mu\nu} \dot{x}_\nu \dot{x}_\mu - \frac{1}{m} g_{\rho\lambda} \frac{\partial \Phi}{\partial x_\lambda} + \frac{1}{k - \Phi} \frac{\partial \Phi}{\partial x_\nu} \dot{x}_\rho \dot{x}_\nu. \]  
(19)

\(^2\) Eq.(13) does admit an equivalence principle, since \( \hat{g}_{\mu\nu} \) and \( \hat{\Gamma}^\rho_{\mu\nu} \) are compatible.
This result (19) differs from the geodesic equation obtained from the Hamiltonian function \( K \) of Eq.(7). Let us, however, define a new velocity field, following the procedure used in [8],

\[
\dot{y}^\mu = \dot{g}^{\mu\nu} \dot{x}_\nu. \tag{20}
\]

Solving for \( \dot{x}_\nu \) and substituting into the general form (13), with the identity

\[
\dot{g}^{\mu\rho} \frac{\partial \dot{g}_{\mu\nu}}{\partial x_\lambda} \dot{y}^\nu = - \frac{\partial \dot{g}^{\rho\kappa}}{\partial x_\lambda}, \tag{21}
\]

we find the geodesic formula for the new velocity field

\[
\dddot{y}^\mu = - \dot{M}^\mu_{\nu\lambda} \dot{y}^\nu \dot{y}^\lambda, \tag{22}
\]

where

\[
\dot{M}^\mu_{\nu\lambda} = \frac{1}{2} \frac{\partial \dot{g}_{\nu\lambda}}{\partial x_\mu}. \tag{23}
\]

We assert that this result achieves a geometrical embedding of the motion generated by the Hamiltonian (7). Our method was to construct the Hamiltonian (9) which generates geodesic equations with a compatible connection, thus providing a geometric basis for the theory. By the same methods used to test stability of orbits as used in ref.[8], the geodesic deviation computed from the result (22) is effective in determining stability of the motion generated by the Hamiltonian (7). Applications of this type will be treated in a separate publication. To show that (22) is indeed a geometric embedding of the Hamiltonian (7), let us substitute the explicit form (10) for \( \dot{g}_{\mu\nu} \) into (23).

Using the definition (11),

\[
\frac{\partial \dot{g}_{\nu\lambda}}{\partial x_\mu} = \frac{\partial \dot{y}^\nu}{\partial x_\mu} + \frac{\partial \dot{y}^\lambda}{\partial x_\mu},
\]

and the fact that

\[
\frac{1}{2m} g_{\mu\nu} p_\mu p_\nu = k - \Phi,
\]

one obtains

\[
\dddot{y}^\mu = - \frac{1}{2} \frac{\partial \dot{g}_{\nu\lambda}}{\partial x_\mu} \dot{y}^\nu \dot{y}^\lambda - \frac{1}{m} \frac{\partial \Phi}{\partial x_\mu}. \tag{24}
\]

Now, considering our transformation of velocity fields (20) heuristically as a local change of variables\(^3\), so that

\[
\frac{\partial}{\partial x_\mu} = \hat{g}^{\rho\mu} \frac{\partial}{\partial y^\rho} = \phi^{-1} g^{\rho\mu} \frac{\partial}{\partial y^\rho}, \tag{25}
\]

\(^3\) The relation \( dy^\mu = g^{\mu\nu} dx_\nu \) is not integrable, and therefore does not uniquely define a set of coordinates \( \{ y^\mu \} \). For example, the “derivatives” \( \frac{\partial}{\partial y^\nu} = \dot{g}_{\nu\mu} \frac{\partial}{\partial x_\mu} \) are not commutative. The results of this identification have been, however, rigorously justified through a transformation of the affine parameter on the geodesic curves [9].
we obtain
\[ \ddot{y}^\mu = -\frac{1}{2} g^{\rho\nu} \frac{\partial g_{\nu\lambda}}{\partial y^\rho} \dot{y}^\nu \dot{y}^\lambda \frac{1}{m} g^{\mu\rho} \frac{\partial \Phi}{\partial y^\rho}. \] (26)

This result differs in its structure from (8) in that the connection form contains just one term, while the connection form in (8) has three terms. To complete the equivalence, we define a related velocity field within the framework of the geodesic motions \{\dot{y}^\mu\}. Let us define yet another velocity field
\[ \dot{z}_\nu = g_{\nu\mu} \dot{y}^\mu \] (27)

The derivative of \dot{y}^\mu then introduces an additional term, with indices symmetrized due to the bilinear form generated in the velocities. Using the relation (21) again (for derivatives of \(g^{\mu\nu}\)), and identifying heuristically, in the same way as done above,
\[ \frac{\partial}{\partial y^\rho} = g_{\rho\mu} \frac{\partial}{\partial z_\mu}, \] (28)

it follows from (26) that
\[ \ddot{z}_\rho = -\Gamma^\mu_{\rho\nu} \dot{z}_\nu \dot{z}_\mu - \frac{1}{m} g_{\rho\nu} \frac{\partial \Phi}{\partial z_\nu}, \] (29)

where \(\Gamma^\mu_{\rho\nu}\) is computed (in the same form) with all derivatives taken with respect to the variables \{\dot{z}_\mu\} in place of the \{\dot{x}_\mu\} in (5). Therefore, up to the transformation (27) within the family of velocity fields generated by the Hamilton equations from the conformally modified Hamiltonian (9), the geodesic equations (22) form a geometrical embedding of the original equations (8). Since \(\dot{\hat{g}}^{\mu\nu}\) and \(\hat{g}_{\mu\nu}\) are compatible, there is a local flat space on this manifold in which parallel transport can be defined, and the tensor properties carry the same class of diffeomorphisms as are implicit in (7) and (8).

We remark that the sequence of transformations (20) and (27) consists of
\[ \dot{z}_\nu = g_{\nu\mu} \dot{y}^\mu = g_{\nu\mu} \hat{g}^{\mu\lambda} \dot{x}_\lambda = \phi^{-1} \dot{x}_\nu, \] (30)

independently of the coordinate system, since any Jacobians applied to these tensors will cancel. However, it is Eq. (22) that constitutes a nontrivial embedding of the orbits generated by (7). Our interest in this Section has been in relating the Hamiltonian (9) to the simplest Bekenstein-Milgrom form of MOND, without concern in the development of this simplified case for lensing or causal effects, for which a TeVeS type theory would be required, and with this, to be able to state restrictions on the form of the scalar field \(\Phi\). In the next Section, we indicate how a TeVeS can be generated in this framework, i.e., as a result of a conformal map.

3. TeVeS and Kaluza-Klein Theory

In this section, we show that the TeVeS theory can be cast into the form of a Kaluza-Klein construction. There has recently been a discussion[7], from the point of view of conformal correspondence, of a relativistic Hamiltonian with gauge invariant form
\[ K = \frac{1}{2m} \eta_{\mu\nu} (p^\mu - ea^\mu)(p^\nu - ea^\nu) - ea^5, \] (31)
where the \(\{a^\mu\}\), as fields, may depend on the affine parameter \(\tau\) as well as \(x^\mu\), and the \(a^5\) field is necessary for the gauge invariance of the \(\tau\) derivative in the quantum mechanical Stueckelberg-Schrödinger equation. Here \(\eta_{\mu\nu}\) is the Minkowski metric \((-1,+1,+1,+1)\). As remarked in this work, Wesson[10] and Liko[11], as well as previous work on this struc-
ure[3], have associated the \(a^5\) field with mass density. It was shown[7] that a Hamiltonian of the form

\[
\hat{K} = \frac{1}{2m}\hat{g}_{\mu\nu}(p^\mu - ea^\mu)(p^\nu - ea^\nu) \tag{32}
\]

can be put into correspondence with \(K\) by taking \(\hat{g}_{\mu\nu}\) to have the conformal form

\[
\hat{g}_{\mu\nu} = \eta_{\mu\nu}\frac{k}{k + ea^5}, \tag{33}
\]

where \(k\) is the common (constant) value of \(K\) and \(\hat{K}\). In this correspondence, the equations of motion generated by \(\hat{K}\) through the Hamilton equations, have extra terms beyond those provided by the connection form associated with \(\hat{g}_{\mu\nu}\), due to the presence of the gauge fields. Calculating the geodesic deviation, one could identify a curvature form associated with an effective five dimensional metric, consistent with the connection form in what then becomes the geodesic equation for the motion of a particle generated by the Hamilton equations obtained from \(\hat{K}\). This five dimensional effective metric is that of a Kaluza-Klein theory.

We may apply the same procedure to the Hamiltonian

\[
K = \frac{1}{2m}g_{\mu\nu}(p^\mu - U^\mu)(p^\nu - U^\nu) + \Phi, \tag{34}
\]

where \(g_{\mu\nu}\) is the Einstein metric, and \(\Phi\) is a world scalar field, and \(U^\mu\) are gauge-like vector fields, as in Eq. (31). We shall give a more complete discussion of the dynamical properties of the equivalence in a subsequent paper, but it suffices for our purpose here to define, as in Eq. (33), the conformally modified metric

\[
\hat{g}_{\mu\nu} = g_{\mu\nu}\frac{k}{k - \Phi} \equiv e^{-2\phi}g_{\mu\nu}; \tag{35}
\]

a Kaluza-Klein effective metric then emerges from the Hamilton equations applied to the “equivalent” Hamiltonian

\[
\hat{K} = \frac{1}{2m}\hat{g}_{\mu\nu}(p^\mu - U^\mu)(p^\nu - U^\nu), \tag{36}
\]

as in ref.[7]4.

Consider the Hamiltonian

\[
K_K = \frac{1}{2m}\tilde{g}_{\mu\nu}p^\mu p^\nu, \tag{37}
\]

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4 One can choose \(\hat{K}\) (as in (32)) to be \(m/2\), which results, according to the Hamilton equations, in \(d\tau\) as the invariant interval.
where

$$\tilde{g}_{\mu\nu} = e^{-2\phi}(g_{\mu\nu} + U_\mu U_\nu) - e^{2\phi} U_\mu U_\nu$$

(38)

This Hamiltonian then has the form

$$K_K = e^{-2\phi} g_{\mu\nu} p^\mu p^\nu - 2 \sinh 2\phi (U_\mu p^\mu)^2,$$

(39)

where $U_\mu = g_{\mu\nu} U^\nu$, i.e. with the same tensor properties as the fields appearing in Eq. (34).

Let us now define a Kaluza-Klein type metric (of the form obtained in [7])

$$g_{AB} = \begin{pmatrix} \hat{g}_{\mu\nu} & U_\nu \\ U_\mu & g_{55} \end{pmatrix}.$$  

(40)

Contraction to a bilinear form with the (5D) vectors $p_A = \{p^\lambda, p^5\}$, with indices $\lambda = \nu$ on the right and $\lambda = \mu$ on the left, one finds

$$g_{AB} p^A p^B = \hat{g}_{\mu\nu} p^\mu p^\nu + 2 p^5 (p^\mu U_\mu) + (p^5)^2 g_{55}.$$  

(41)

If we take

$$p^5 = - \sinh 2\phi (p^\mu U_\mu)$$  

(42)

and $g_{55} = 0$ (the null choice of the constant assumed in ref.[7]), one sees that the Hamiltonian (31) can be represented in terms of this Kaluza-Klein metric as

$$K_K = \frac{1}{2m} g_{AB} p^A p^B.$$  

(43)

Note that with the constraint that the fields $U^\mu$ are timelike unit vectors[6], enforced by using a Lagrange parameter, the product $(p^\mu U_\mu)$ corresponds, in an appropriate local frame, to the energy of the particle, close to its mass in the case of a nonrelativistic particle, or to the frequency in the case of on-shell photons. It clearly remains to understand more deeply the apparently ad hoc choice of $p^5$ in (42) in terms of a 5D canonical dynamics, along with the structure of the 5D Einstein equations for $g_{AB}$ that follow from the geometry associated with (43).

4. Conclusions

A map of the type discussed in refs. [7],[8], of a Hamiltonian containing an Einstein metric, generating the connection form of general relativity, and a world scalar field, representing a distribution of energy on the spacetime manifold, into a corresponding Hamiltonian with a conformal metric (and compatible connection form), can account for the structure of the RAQUAL theory of Bekenstein and Milgrom[4]. Furthermore, applying this correspondence to a Hamiltonian with gauge-type structure, we have shown that one obtains a non-compact Kaluza-Klein effective metric which can account for the TeVeS structure of Bekenstein, Sanders and Milgrom[6]. This method can be applied to the Brans-Dicke theory or other scalar-tensor theories as well.
The phenomenological constraints placed on the $TeVeS$ variables in its astrophysical applications and by its MOND limit would, in principle, place constraints on the vector and scalar fields appearing in the corresponding Hamiltonian model.

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Abstract
It has been shown that the orbits of motion for a wide class of non-relativistic Hamiltonian systems can be described as geodesic flows on a manifold and an associated dual by means of a conformal map. This method can be applied to a four dimensional manifold of orbits in spacetime associated with a relativistic system. We show that a relativistic Hamiltonian which generates Einstein geodesics, with the addition of a world scalar field, can be put into correspondence in this way with another Hamiltonian with conformally modified metric. Such a construction could account for part of the requirements of Bekenstein for achieving the MOND theory of Milgrom in the post-Newtonian limit. The constraints on the MOND theory imposed by the galactic rotation curves, through this correspondence, would then imply constraints on the structure of the world scalar field. We then use the fact that a Hamiltonian with vector gauge fields results, through such a conformal map, in a Kaluza-Klein type theory, and indicate how the TeVeS structure of Bekenstein and Saunders can be put into this framework. We exhibit a class of infinitesimal gauge transformations on the gauge fields $U_\mu(x)$ which preserve the Bekenstein-Sanders condition $U_\mu U^\mu = -1$. The underlying quantum structure giving rise to these gauge fields is a Hilbert bundle, and the gauge transformations induce a non-commutative behavior to the fields, i.e. they become of Yang-Mills type. Working in the infinitesimal gauge neighborhood of the initial Abelian theory we show that in the Abelian limit the Yang-Mills field equations provide residual nonlinear terms which may avoid the caustic singularity found by Contaldi et al.

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1. Introduction

The Hamiltonian [1]

$$K = \frac{1}{2m} g^{\mu\nu} p_\mu p_\nu,$$

(1)

with Hamilton equations (written in terms of derivatives with respect to an invariant world time $\tau$ [2])

$$\dot{x}_\mu = \frac{\partial K}{\partial p_\mu} = \frac{1}{m} g^{\mu\nu} p_\nu$$

(2)

and

$$\dot{p}_\mu = -\frac{\partial K}{\partial x_\mu} = -\frac{1}{2m} \frac{\partial g^{\lambda\gamma}}{\partial x_\mu} p_\lambda p_\gamma$$

(3)
lead to the geodesic equation
\[ \ddot{x}^\rho = -\Gamma^\rho_{\mu\nu} \dot{x}^\nu \dot{x}^\mu, \] (4)
where what has appeared as a compatible connection form \( \Gamma^\rho_{\mu\nu} \) is given by
\[ \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left( \frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right). \] (5)

These results are tensor relations over the usual diffeomorphisms admitted by the manifold \( \{x^\mu\} \); writing the Hamiltonian in terms of (2), we see that the invariant interval on an orbit is proportional, through the constant Hamiltonian, to the square of the world time of evolution on the orbit, \( \text{i.e.} \)
\[ ds^2 = \frac{2}{m} K d\tau^2. \] (6)

We shall first study, in the following, a generalization of (1) consisting of the addition of a scalar field \( \Phi(x) \). The presence of such a scalar field may be considered as a gauge compensation field for the \( \tau \) derivative in the evolution term of the covariant generalization of (1) in the Stückelberg-Schrödinger equation [3], an energy distribution not directly associated with electromagnetic radiation in the usual sense. We then follow the method of ref.[4] to show that there is a corresponding Hamiltonian \( \hat{K} \) with a conformally modified metric, and no explicit additive scalar field, which has the form of the construction of Bekenstein and Milgrom [5] for the realization of Milgrom’s MOND program (modified Newtonian dynamics) [6] for achieving the observed galactic rotation curves. This simple form of Bekenstein’s theory (called RAQUAL), which we discuss in some detail below for the sake of simplicity and clarity in the development of the mathematical method, does not properly account for causality and gravitational lensing; the theory has been further developed to include vector fields (which we shall call Bekenstein-Sanders fields) as well ([TeVeS]) [7], which has been relatively successful in accounting for these problems. It has been shown [8], moreover, that a gauge type Hamiltonian, with Minkowski metric and both vector and scalar fields results, under a conformal map, in an effective Kaluza-Klein theory. We shall indicate here (using a general Einstein metric) how the \( \text{TeVeS} \) structure can emerge in terms of a Kaluza-Klein theory in this way. It is essential in this construction that the Bekenstein-Sanders fields be considered as gauge fields. As a realization of this possibility, we exhibit an infinitesimal gauge transformation on the underlying quantum theory for which the vector fields, which we shall call \( \mathcal{U}^\mu(x) \), emerge as gauge compensation fields, such that, as required by the \( \text{TeVeS} \) theory the property \( \mathcal{U}^\mu \mathcal{U}_\mu = -1 \) [7] is preserved under such gauge transformations. The corresponding quantum theory then has the form of a Hilbert bundle and, in this framework, the gauge fields are of (generalized) Yang-Mills type [9]. Working in the infinitesimal neighborhood of a gauge in which the fields are Abelian, we show that in the limit the contributions from the nonabelian sector provide nonlinear terms in the field equations which may avoid the caustic singularity found by Contaldi \textit{et al} [10]. Further investigation of this structure will be given in a subsequent publication.

For both the RAQUAL and the \( \text{TeVeS} \) theories, the correspondence between \( K \) and \( \hat{K} \) implies a relation between the conformal factor in \( \hat{K} \) and the world scalar field \( \Phi, \) and
thus a possible connection between the so-called dark matter problem and a dark energy distribution represented by $\Phi$.

### 2. Addition of a scalar potential and conformal equivalence

The addition of a scalar potential to the Hamiltonian (1), in the form

$$K = \frac{1}{2m} g^{\mu\nu} p_\mu p_\nu + \Phi(x), \quad (7)$$

leads, according to the Hamilton equations, to the geodesic equation\(^1\)

$$\ddot{x}^\rho = -\Gamma^\rho_{\mu\nu} \dot{x}^\nu \dot{x}^\mu - \frac{1}{m} g_{\rho\nu} \frac{\partial \Phi}{\partial x^\nu}. \quad (8)$$

Now, consider the Hamiltonian (we carry out the calculations explicitly here since we shall have need of some of the intermediate results)

$$\hat{K} = \frac{1}{2m} \hat{g}^{\mu\nu} (y)p_\mu p_\nu. \quad (9)$$

It follows from the Hamilton equations that

$$\dot{y}^\mu = \frac{\partial \hat{K}}{\partial \hat{p}_\mu} = \frac{1}{m} \hat{g}^{\mu\nu} p_\nu, \quad (10)$$

so that

$$p_\nu = m \hat{g}_{\mu\nu} \dot{y}^\mu$$

and

$$\dot{p}_\mu = -\frac{\partial \hat{K}}{\partial y^\mu} = -\frac{1}{2m} \frac{\partial \hat{g}^{\lambda\gamma}}{\partial y^\mu} p_\lambda p_\gamma. \quad (11)$$

As in (4), it then follows that

$$\ddot{y}^\mu = -\hat{\Gamma}^\mu_{\lambda\sigma} \dot{y}^\lambda \dot{y}^\sigma, \quad (11)$$

where, as for (4),

$$\hat{\Gamma}^\mu_{\lambda\sigma} = \frac{1}{2} \hat{g}^{\mu\nu} \left\{ \frac{\partial \hat{g}_{\nu\sigma}}{\partial y^\lambda} + \frac{\partial \hat{g}_{\nu\lambda}}{\partial y^\sigma} - \frac{\partial \hat{g}_{\lambda\sigma}}{\partial y^\nu} \right\}. \quad (12)$$

We now establish an equivalence between the Hamiltonians (7) and (9) by assuming the momenta $p_\mu$ equal at every moment $\tau$ in the two descriptions. With the constraint

$$\hat{K} = K = k, \quad (13)$$

if we assume the conformal form

$$\hat{g}^{\nu\sigma}(y) = \phi(y) g^{\nu\sigma}(x), \quad (14)$$

\(^1\) Note that (8) does not admit an equivalence principle, but (11), arising from (9) does.
it follows that

$$\phi(y)(k - \Phi(x)) = k. \quad (15)$$

The relation (15) is not sufficient to construct \(y\) as a function of \(x\), but if we impose the relation

$$\delta x^\mu = \phi^{-1}(y)\delta y^\mu \quad (16)$$

between variations generated on position in the two coordinate systems, it is sufficient to evaluate derivatives of \(\phi(y)\) in terms of derivatives with respect to \(x\) of the scalar field \(\Phi(x)\).[11](see also [12]). We review this construction briefly below.

We remark that the construction based on Eqs. (9) and (14) admits the same family of diffeomorphisms as that of (7), since \(\phi\) is scalar. Under these diffeomorphisms, both \(g_{\mu\nu}\) and \(\hat{g}_{\mu\nu}\) are second rank tensors, and by construction of the connection forms, (4) and (11) are covariant relations.

To see how these derivatives are constructed on the constraint hypersurface determined by (15), let us, for brevity, define

$$F(x) \equiv \frac{k}{k - \Phi(x)}, \quad (17)$$

so that the constraint relation (15) reads

$$\phi(y) = F(x). \quad (18)$$

Then, since variations in \(x\) and \(y\) are related by (16),

$$\phi(y + \delta y) = F(x + \delta x) \equiv F(x) + \delta x^\mu \frac{\partial F(x)}{\partial x^\mu}. \quad (19)$$

To first order in Taylor’s series on the left, we obtain the relation

$$\frac{\partial \phi(y)}{\partial y^\mu} = \phi^{-1}(y) \frac{\partial F(x)}{\partial x^\mu}. \quad (20)$$

We may therefore define a derivative, restricted to the constraint hypersurface

$$\frac{\tilde{\partial} F(x)}{\partial y^\mu} = \phi^{-1}(y) \frac{\partial F(x)}{\partial x^\mu}. \quad (21)$$

The Leibniz relation follows easily for the product of functions, it e.g., for \(\phi(y)g^{\mu\nu}(x)\).

In a similar way, the second derivative can be obtained from (19) by recognizing that the variation in \(x\) is to be computed at the point \(y + \delta y\). Keeping terms of second order in the expansion of both sides, one can define the second derivative restricted to the constraint hypersurface defined by (18); although it appears that a second derivative defined by (21) would not be symmetric, both the derivative of (21) and the second derivative computed from (19) on the constraint hypersurface agree and are symmetric [11], i.e.,

$$\frac{\tilde{\partial}^2 F(x)}{\partial y^\mu \partial y^\nu} = \frac{\tilde{\partial}^2 F(x)}{\partial y^\nu \partial y^\mu}. \quad (22)$$
This implies that the restricted derivative defined by (21) behaves as a *bona fide* derivative on the constraint hypersurface, admitting the consistent coexistence of the coordinates \( x \) and \( y \) related by (15). We will not have further use of (22) here, primarily relevant for the calculation of stability criteria through geodesic deviation (the second derivative occurs in the curvature tensor).

In the following, we complete our argument of equivalence by reconstructing the equations of motion following from the Hamilton equations applied to (7), i.e., Eq. (8).

We begin our reconstruction, in analogy with the procedure used in the nonrelativistic problem[8], by defining the new variable \( z_\mu \) such that

\[
\dot{z}_\mu = \hat{g}_{\mu\nu}(y) \dot{y}^\nu. \tag{23}
\]

Substituting \( \dot{y}^\nu = \hat{g}^{\mu\nu}(y) \dot{z}_\mu \) into (11), the \( \tau \) derivatives of \( \hat{g}^{\mu\nu}(y) \) generate terms that cancel two of the terms in \( \hat{\Gamma}^{\mu}_{\lambda\sigma}, \) leaving

\[
\ddot{z}_\nu = \frac{1}{2} \frac{\partial \hat{g}^{\lambda\sigma}}{\partial y^\nu} \dot{y}^\lambda \dot{y}^\sigma. \tag{24}
\]

Now, substituting for \( \dot{y}^\lambda \) from (23), and using the identity

\[
\hat{g}^{\gamma\lambda} \frac{\partial \hat{g}^{\lambda\sigma}}{\partial y^\nu} \hat{g}^{\sigma\rho} = -\frac{\partial \hat{g}^{\gamma\rho}}{\partial y^\nu}, \tag{25}
\]

we find

\[
\ddot{z}_\nu = -\frac{1}{2} \frac{\partial \hat{g}^{\gamma\rho}}{\partial y^\nu} \dot{z}_\gamma \dot{z}_\rho. \tag{26}
\]

Finally, from the variational type argument we used above,

\[
\hat{g}^{\rho\gamma}(y + \delta y) - \hat{g}^{\rho\gamma}(y) = \frac{\partial \hat{g}^{\gamma\rho}}{\partial y^\nu} \delta y^\nu
\]

\[
= \frac{\partial \hat{g}^{\rho\gamma}}{\partial y^\nu} \dot{y}^\nu \delta z_\lambda, \tag{27}
\]

so that

\[
\frac{\partial \hat{g}^{\rho\gamma}}{\partial y^\nu} \dot{y}^\nu = \frac{\partial \hat{g}^{\rho\gamma}}{\partial z_\lambda} \dot{z}_\lambda.
\]

Thus

\[
\frac{\partial \hat{g}^{\rho\gamma}}{\partial y^\nu} = \hat{g}^{\rho\gamma}_{\nu\lambda} \frac{\partial \hat{g}^{\rho\gamma}}{\partial z_\lambda} \tag{28}
\]

We therefore have the alternative form

\[
\ddot{z}_\nu = -\frac{1}{2} \hat{g}^{\rho\gamma}_{\nu\lambda} \frac{\partial \hat{g}^{\rho\gamma}}{\partial z_\lambda} \dot{z}_\rho \dot{z}_\gamma. \tag{29}
\]
This result constitutes a “geometric” embedding of the Hamiltonian motion induced by (7) in the same way as for the nonrelativistic case. Substituting the explicit form of $\hat{g}^{\rho\gamma}$ in terms of the original Einstein metric from (14), one obtains

$$\ddot{z}_\nu = -\frac{1}{2}g_{\nu\lambda} \frac{\partial g^{\rho\gamma}}{\partial z_\lambda} \dot{z}_\rho \dot{z}_\gamma - \frac{1}{2}\phi^{-1}g_{\nu\lambda} \frac{\partial \phi}{\partial z_\lambda} g^{\rho\gamma} \dot{z}_\gamma \dot{z}_\rho$$

(30)

The second term contains the potential field, as in the Hamilton equations, but the first term does not contain the full connection form. We may finally, however, define a “decontraction” of the connection in (30) using the Einstein metric. In fact, since according to (16), $\dot{y}^\nu = \phi \dot{x}^\nu$, and by (23),

$$\dot{z}_\mu = \hat{g}_{\mu\nu} \dot{y}^\nu = \phi^{-1}g_{\mu\nu} \dot{y}^\nu,$$

(31)

it follows that

$$\dot{z}_\mu = g_{\mu\nu} \dot{x}^\nu.$$

(32)

Making this substitution in (30) leads explicitly, taking into account the $k$ shell constraint (13) and the form of (7), to the equation (8). We have thus completed our demonstration of the equivalence between the purely metric form of the Hamiltonian (9) and the Hamilton (7), for which the relation (29) corresponds to a dynamics generated by a compatible connection form, and constitute a “geometric” embedding of the original Hamiltonian motion.

Our interest in this section has been in relating the Hamiltonian (7) to the simplest Bekenstein-Milgrom form of MOND, without concern in the development of this simplified case for lensing or causal effects, for which a TeVeS type theory would be required. In the next Section, we indicate how a TeVeS theory can be generated in this framework, i.e., as a result of a conformal map.

3. TeVeS and Kaluza-Klein Theory

In this section, we show that the TeVeS theory can be cast into the form of a Kaluza-Klein construction. There has recently been a discussion [8], from the point of view of conformal correspondence, of the equivalence of a relativistic Hamiltonian with an electromagnetic type gauge invariant form [3] (here $\eta^{\mu\nu}$ is the Minkowski metric ($-1, +1, +1, +1$))

$$K = \frac{1}{2m} \eta^{\mu\nu} (p_\mu - ea_\mu)(p_\nu - ea_\nu) - ea_5,$$

(33)

where the $\{a_\mu\}$, as fields, may depend on the affine parameter $\tau$ as well as $x^\mu$, and the $a^5$ field is necessary for the gauge invariance of the $\tau$ derivative in the quantum mechanical Stueckelberg-Schrödinger equation, with a Kaluza-Klein theory. As remarked in this work, Wesson [13] and Liko [14], as well as previous work on this structure[3], have associated the source of the $a_5$ field with mass density. A Hamiltonian of the form

$$\tilde{K} = \frac{1}{2m} \hat{g}^{\mu\nu} (p_\mu - ea_\mu)(p_\nu - ea_\nu)$$

(34)
can be put into correspondence with $K$ by taking $\hat{g}^{\mu\nu}$ to be

$$\hat{g}^{\mu\nu} = \eta^{\mu\nu} \frac{k}{k + ea_5},$$

(35)

where $k$ is the common (constant) value of $K$ and $\hat{K}$. In this correspondence, the equations of motion generated by $\hat{K}$ through the Hamilton equations, have extra terms, beyond those provided by the connection form associated with $\hat{g}^{\mu\nu}$, due to the presence of the gauge fields. These additional terms can be identified as belonging to a connection form associated with a five dimensional metric, that of a Kaluza-Klein theory.

We may apply the same procedure to the Hamiltonian

$$K = \frac{1}{2m} g^{\mu\nu} (p_\mu - \epsilon U_\mu)(p_\nu - \epsilon U_\nu) + \Phi,$$

(36)

where $g^{\mu\nu}$ is an Einstein metric, $\Phi$ is a world scalar field, and $U_\mu$ are to be identified with the Bekenstein-Sanders fields for which $[6] U_\nu U^\nu = -1$, with $U^\mu = g^{\mu\nu} U_\nu$.

We discuss in Section 4 a class of gauge transformations on the wave functions of the underlying quantum theory for which the $U_\mu$ arise as gauge compensation fields.

Let us define, as in Eq. (35), the conformally modified metric

$$\hat{g}^{\mu\nu} = g^{\mu\nu} \frac{k}{k - \Phi} \equiv e^{-2\phi} g^{\mu\nu}. $$

(37)

The "equivalent" Hamiltonian

$$\hat{K} = \frac{1}{2m} \hat{g}^{\mu\nu} (p_\mu - \epsilon U_\mu)(p_\nu - \epsilon U_\nu)$$

(38)

then generates, through the Hamilton equations, an equation of motion which corresponds to the geodesic equation for an effective Kaluza-Klein metric, as in ref.[8].

Now, consider the Hamiltonian

$$K_K = \frac{1}{2m} \tilde{g}^{\mu\nu} p_\mu p_\nu,$$

(39)

with the Bekenstein-Sanders metric[7]

$$\tilde{g}^{\mu\nu} = e^{-2\phi} (g^{\mu\nu} + U_\mu U^\nu) - e^{2\phi} U^\mu U^\nu$$

(40)

The Hamiltonian $K_K$ then has the form

$$K_K = e^{-2\phi} g^{\mu\nu} p_\mu p_\nu - 2 \sinh 2\phi (U^\mu p_\mu)^2,$$

(41)

Let us now define a Kaluza-Klein type metric of the form obtained in [7], arising from the equations of motion generated by (38),

$$g^{AB} = \begin{pmatrix} \hat{g}^{\mu\nu} & U^\nu \\ U_\mu & \tilde{g}^{55} \end{pmatrix}.$$
Contraction to a bilinear form with the (5D) vectors \( p_A = \{ p_\lambda, p_5 \} \), with indices \( \lambda = \nu \) on the right and \( \lambda = \mu \) on the left, one finds

\[
g^{AB} p_A p_B = \hat{g}^{\mu\nu} p_\mu p_\nu + 2p_5(p_\mu U^\mu) + (p_5)^2 g^{55}.
\] (43)

If we take

\[
p_5 = -\frac{(p_\mu U^\mu)}{g^{55}} \left(1 \pm \sqrt{1 - 2g^{55} \sinh 2\phi}\right),
\] (44)

then the Kaluza-Klein theory coincides with (41), i.e.,

\[
K_K = \frac{1}{2m} g^{AB} p_A p_B.
\] (45)

As discussed by Wesson [13], Kaluza [15] chose \( g^{55} = \text{const.} \) for consistency with electromagnetism, while Wesson [13] makes the more general choice of a world scalar field. In particular, the value \( g^{55} = 0 \) is well defined (as in [8]).

Since the fields \( U^\mu \) are timelike unit vectors [7], \( (p^\mu U_\mu) \) corresponds, in an appropriate local frame, to the energy of the particle, close to its mass in the case of a nonrelativistic particle, or to the frequency in the case of on-shell photons. It clearly remains to understand more deeply the apparently \textit{ad hoc} choice of \( p^5 \) in (44) in terms of a 5D canonical dynamics, along with the structure of the 5D Einstein equations for \( g_{AB} \) that follow from the geometry associated with (45). We shall study these questions in a succeeding paper.

4 The Bekenstein-Sanders Vector Field as a Gauge Field

Essential features of the Bekenstein-Sanders field [7] of the \textit{TeVeS} theory are that it be a local field, i.e., \( U_\mu(x) \), and there is a normalization constraint

\[
U^\mu U_\mu = -1,
\] (46)

so that the vector is timelike. To preserve the normalization condition (46) under gauge transformation, we shall study the construction of a class of gauge transformations which essentially moves the \( U(x) \) field on a hyperbola with a Lorentz transformation (at the point \( x \)).

If we think of our underlying quantum structure, which generates the gauge field, as a fiber bundle with base \( x^\mu \), then we must think of the transformation acting in such a way that the absolute square (norm) of the wave function attached to the base point \( x^\mu \) preserves its value [9].

An analogy can be drawn to the usual Yang-Mills gauge [9] on \( SU(2) \), where there is a two-valued index for the wave function \( \psi_\alpha(x) \). The gauge transformation in this case is a two by two matrix function of \( x \), and acts only on the indices \( \alpha \). The condition of invariant absolute square (probability) is

\[
\sum_\alpha |\sum_\beta U_{\alpha\beta} \psi_\beta|^2 = \sum |\psi_\alpha|^2
\] (47)
Generalizing this structure, one can take the indices $\alpha$ to be continuous, so that (47) becomes
\[ \int (dU) | \int (dU') U(U,U') \psi(U',x)|^2 = \int (dU) |\psi(U,x)|^2, \]
(48)
implying that $U(U,U')$ is a unitary operator on a Hilbert space $L^2(dU)$. Since we are assuming that $U_\mu$ lies on an orbit determined by (48), the measure is
\[ (dU) = \frac{d^3 U}{U^0}, \]
(49)
i.e., a three dimensional Lorentz invariant integration measure.

Moreover, the Lorentz transformation on $U_\mu$ is generated by a non-commutative operator, and therefore the gauge transformation is non-Abelian. We demonstrate the resulting noncommutativity of the operator valued fields, $U'$, after an infinitesimal gauge transformation of this type, explicitly below.

This construction is somewhat similar to the treatment of the electromagnetic potential vector and its time derivative as oscillator variables in the process of second quantization of the radiation field (the energy density of the field is given by these variables in the form of an oscillator).

We now examine the gauge condition:
\[ (p_\mu - \epsilon U'_\mu) U \psi = U (p_\mu - \epsilon U_\mu) \psi \]
(50)
Identifying $p_\mu$ with $-i \partial / \partial x^\mu$, and cancelling the terms $U p_\mu \psi$ on both sides, we obtain
\[ U'_\mu = U U_\mu U^{-1} - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} U^{-1}, \]
(51)
in the same form as the Yang-Mills theory [9]. It is evident in the Yang-Mills theory, that due to the matrix nature of the second term, the field will be algebra-valued, resulting in the usual structure of the Yang-Mills non-Abelian gauge theory. Here, if the transformation $U$ is a Lorentz transformation, the numerical valued field $U_\mu$ would be carried, in the first term, to a new value on a hyperbola. However, the second term may well be operator valued on $L^2(dU)$, and thus, as in the Yang-Mills theory, $U'^\mu$ would become nonabelian.

It follows from (51) that the field strengths
\[ f_{\mu \nu} = \frac{\partial U_\mu}{\partial x^\nu} - \frac{\partial U_\nu}{\partial x^\mu} + i \epsilon [U_\mu, U_\nu] \]
(52)
are related to the field strengths in the transformed form
\[ f'_{\mu \nu} = \frac{\partial U'_\mu}{\partial x^\nu} - \frac{\partial U'_\nu}{\partial x^\mu} + i \epsilon [U'_\mu, U'_\nu] \]
(53)
according to
\[ f'_{\mu \nu}(x) = U f_{\mu \nu}(x) U^{-1}, \]
(54)
just as in the finite dimensional Yang-Mills theories.

This result follows from writing out, from (51),

\[
\frac{\partial U'_{\mu}}{\partial x^\nu} = \frac{\partial U}{\partial x^\nu} U^{-1}_{\mu} + U \frac{\partial U'_{\mu}}{\partial x^\nu} U^{-1} + U U_{\mu} \frac{\partial U^{-1}}{\partial x^\nu} - \frac{i}{\epsilon} \frac{\partial^2 U}{\partial x^\mu \partial x^\nu} U^{-1} - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} \frac{\partial U^{-1}}{\partial x^\nu},
\]

and subtracting the same expression with \(\mu, \nu\) reversed. Then add the result to

\[
\begin{align*}
\iota e[ U'_{\mu}, U'_{\nu}] &= \iota e U[ U_{\mu}, U_{\nu}] U^{-1} + [U U_{\mu} U^{-1}, \frac{\partial U}{\partial x^\nu} U^{-1}] \\
&+ [\frac{\partial U}{\partial x^\mu} U^{-1}, U U_{\nu} U^{-1}] - \frac{i}{\epsilon} [\frac{\partial U}{\partial x^\mu} U^{-1}, \frac{\partial U}{\partial x^\nu} U^{-1}]
\end{align*}
\]

Whenever the combination

\[
U^{-1} \frac{\partial U}{\partial x^\mu} U^{-1}
\]

appears, it should be replaced by

\[- \frac{\partial U^{-1}}{\partial x^\mu}.
\]

The result (54) then follows after a little manipulation.

Now, consider the possibility that this finite gauge transformation leaves \( U_{\mu} U^{\mu} = -1 \). We write out

\[
(U U_{\mu} U^{-1} - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} U^{-1})(U U^{\mu} U^{-1} - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} U^{-1}) = -1 - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} U^{\mu} U^{-1} - \frac{i}{\epsilon} U U_{\mu} U^{-1} \frac{\partial U}{\partial x^\mu} U^{-1} - \frac{1}{\epsilon^2} \frac{\partial U}{\partial x^\mu} U^{-1} \frac{\partial U}{\partial x^\mu} U^{-1}
\]

\[
= -1 - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} U^{\mu} U^{-1} + \frac{i}{\epsilon} U U_{\mu} \frac{\partial U^{-1}}{\partial x^\mu} + \frac{1}{\epsilon^2} \frac{\partial U}{\partial x^\mu} \frac{\partial U^{-1}}{\partial x^\mu}.
\]

It may be possible that \( U \) can be chosen to make all but the first term in (57) vanish, but in the case of finite gauge transformations, it is not so easy to see how to construct examples. For the infinitesimal case, it is, however, easy to construct a gauge function with the required properties. For

\[
U = 1 + i G,
\]

where \( G \) is infinitesimal, (51) becomes

\[
U'_{\mu} = U_{\mu} + i [G, U_{\mu}] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\mu} + O(G^2),
\]

(59)
Then,
\[
U'_\mu U^\mu \cong U_\mu n_\mu + i(U_\mu [G, U^\mu] + [G, U_\mu]U^\mu) \\
+ \frac{1}{\epsilon} \left( \frac{\partial G}{\partial x^\mu} U^\mu + U_\mu \frac{\partial G}{\partial x^\mu} \right).
\]

Let us take
\[
G = -i\epsilon \frac{\sum \{ \omega_{\lambda\gamma}(U, x), (U^\lambda \frac{\partial}{\partial U_{\gamma}} - U^\gamma \frac{\partial}{\partial U_{\lambda}}) \}}{2} \equiv \frac{\epsilon}{2} \sum \{ \omega_{\lambda\gamma}(U, x), N^{\lambda\gamma} \}
\]
where symmetrization is required since \( \omega_{\lambda\gamma} \) is a function of \( U \) as well as \( x \), and
\[
N^{\lambda\gamma} = -i(U^\lambda \frac{\partial}{\partial U_{\gamma}} - U^\gamma \frac{\partial}{\partial U_{\lambda}}).
\]

This construction is valid in the initially special gauge, which we shall call the “special abelian gauge”, in which the components of \( U^\mu \) commute. The appearance of \( U^\mu \) in the gauge functions is then admissible since this quantity acts on the wave functions \( \langle U, x | \psi \rangle = \psi(U, x) \) at the point \( x \), in the representation in which the operator \( U^\mu \) on \( L^2(dU) \) is diagonal.

Our investigation in the following will be concerned with a study of the infinitesimal gauge neighborhood of this limit, where the components of \( U^\mu \) do not commute, and therefore constitute a Yang Mills type field. We shall show in the limit that the corresponding field equations acquire nonlinear terms, and may therefore suppress the caustic singularities found by Contaldi et al [10]. They found that nonlinear terms associated with a non-Maxwellian type action, such as \((\partial_{\mu} U^\mu)^2\), could avoid this caustic singularity, so that the nonlinear terms we find as a residue of the Yang-Mills structure induced by our gauge transformation might achieve this effect in a natural way.

The second term of (60), which is the commutator of \( G \) with \( U_\mu U^\mu \) vanishes, since this product is Lorentz invariant (the symmetrization in \( G \) does not affect this result).

We now consider the third term in (60).
\[
\frac{1}{\epsilon} \left( \frac{\partial G}{\partial x^\mu} U^\mu + U_\mu \frac{\partial G}{\partial x^\mu} \right) = \frac{1}{2} \{ \omega_{\lambda\gamma} \, N^{\lambda\gamma} \} U^\mu + U^\mu \left\{ \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu}, N^{\lambda\gamma} \right\} \\
= \frac{1}{2} \left\{ N^{\lambda\gamma} \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} U^\mu + \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} N^{\lambda\gamma} U^\mu \right\} \\
+ U^\mu N^{\lambda\gamma} \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} + U^\mu \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} N^{\lambda\gamma}
\]

There are two terms proportional to
\[
\frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} U^\mu.
\]
If we take (locally)
\[
\omega_{\lambda\gamma}(U, x) = \omega_{\lambda\gamma}(k_\nu x^\nu),
\]

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where \( k_\nu U^\nu = 0 \), then
\[
\frac{\partial \omega^{\lambda\gamma}}{\partial x^\mu} U_\mu = k_\mu U^\mu \omega^{\lambda\gamma}_\mu = 0. \tag{65}
\]
For the remaining two terms,
\[
U^\mu N^{\lambda\gamma} \frac{\partial \omega^{\lambda\gamma}}{\partial x^\mu} + \frac{\partial \omega^{\lambda\gamma}}{\partial x^\mu} N^{\lambda\gamma} U^\mu =
= N^{\lambda\gamma} U^\mu \frac{\partial \omega^{\lambda\gamma}}{\partial x^\mu}
+ [U^\mu, N^{\lambda\gamma}] \frac{\partial \omega^{\lambda\gamma}}{\partial x^\mu} + \frac{\partial \omega^{\lambda\gamma}}{\partial x^\mu} U^\mu N^{\lambda\gamma}
+ \frac{\partial \omega^{\lambda\gamma}}{\partial x^\mu} [N^{\lambda\gamma}, U^\mu]. \tag{66}
\]
Since the commutators contain only terms linear in \( U_\mu \) and they have opposite sign, they cancel. The remaining terms are zero by the argument (65). The condition \( U_\mu U^\mu = -1 \) is therefore invariant under this gauge transformation, involving the coefficient \( \omega^{\lambda\gamma}_\mu \) which is a function of the projection of \( x^\mu \) onto a hyperplane orthogonal to \( U_\mu \), i.e., a function of \( k_\mu x^\mu \), where \( k_\mu U^\mu = 0 \). The vector \( k_\mu \), of course, depends on \( U_\mu \) (for example, \( k_\mu = U_\mu(U \cdot b) + b_\mu \), for some \( b_\mu \neq 0 \)).

We now demonstrate explicitly the nonabelian nature of the gauge fields after infinitesimal gauge transformation. With (59), the commutator term in (53) is
\[
[U'_\mu, U'_\nu] = (U_\mu + i [G, U_\mu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\mu})(U_\nu + i [G, U_\nu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\nu})
- (U_\nu + i [G, U_\nu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\nu})(U_\mu + i [G, U_\mu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\mu})
= \frac{1}{\epsilon} \{ [U_\mu, \frac{\partial G}{\partial x^\nu}] - [U_\nu, \frac{\partial G}{\partial x^\mu}] \}
+ i [U_\mu, [G, U_\nu]] - i [U_\nu, [G, U_\mu]], \tag{67}
\]
where the remaining terms have identically cancelled. Note that this expression does not contain any noncommutative quantities. Now,
\[
[G, U_\nu] = 2 i \epsilon \omega^{\nu}_\gamma U_\gamma \tag{68}
\]
and
\[
[U_\mu, \frac{\partial G}{\partial x^\nu}] = 2 i \epsilon U_\lambda \frac{\partial \omega^{\lambda}_\mu}{\partial x^\nu}. \tag{69}
\]

The terms involving \([G, U_\nu]\) and \([G, U_\mu]\) therefore cancel, so that
\[
[U'_\mu, U'_\nu] = 2 i U_\lambda \left( \frac{\partial \omega^{\lambda}_\mu}{\partial x^\nu} - \frac{\partial \omega^{\lambda}_\nu}{\partial x^\mu} \right) \tag{70}
\]
We have taken \( \omega^{\lambda}_\mu = \omega^{\lambda}_\mu(k_{\sigma} x^\sigma) \), so that
\[
\frac{\partial \omega^{\lambda}_\mu}{\partial x^\nu} = k_\nu \omega^{\lambda}_\mu, \tag{71}
\]
We now demonstrate explicitly the nonabelian nature of the gauge fields after infinitesimal gauge transformation. With (59), the commutator term in (53) is
\[
[U'_\mu, U'_\nu] = (U_\mu + i [G, U_\mu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\mu})(U_\nu + i [G, U_\nu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\nu})
- (U_\nu + i [G, U_\nu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\nu})(U_\mu + i [G, U_\mu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\mu})
= \frac{1}{\epsilon} \{ [U_\mu, \frac{\partial G}{\partial x^\nu}] - [U_\nu, \frac{\partial G}{\partial x^\mu}] \}
+ i [U_\mu, [G, U_\nu]] - i [U_\nu, [G, U_\mu]], \tag{67}
\]
where the remaining terms have identically cancelled. Note that this expression does not contain any noncommutative quantities. Now,
\[
[G, U_\nu] = 2 i \epsilon \omega^{\nu}_\gamma U_\gamma \tag{68}
\]
and
\[
[U_\mu, \frac{\partial G}{\partial x^\nu}] = 2 i \epsilon U_\lambda \frac{\partial \omega^{\lambda}_\mu}{\partial x^\nu}. \tag{69}
\]

The terms involving \([G, U_\nu]\) and \([G, U_\mu]\) therefore cancel, so that
\[
[U'_\mu, U'_\nu] = 2 i U_\lambda \left( \frac{\partial \omega^{\lambda}_\mu}{\partial x^\nu} - \frac{\partial \omega^{\lambda}_\nu}{\partial x^\mu} \right) \tag{70}
\]
We have taken \( \omega^{\lambda}_\mu = \omega^{\lambda}_\mu(k_{\sigma} x^\sigma) \), so that
\[
\frac{\partial \omega^{\lambda}_\mu}{\partial x^\nu} = k_\nu \omega^{\lambda}_\mu, \tag{71}
\]
and therefore

$$[U'_\mu, U'_\nu] = 2i(k_\nu \omega'^{\lambda}_\mu - k_\mu \omega'^{\lambda}_\nu)U_\lambda,$$  \hspace{1cm} (72)

generally not zero. This demonstrates the nonabelian character of the fields. In the Abelian limit, we may take $\omega' \to 0$, but as we shall a residual nonlinearity, which depends on $\omega''$ may remain in the field equations.

We now consider the derivation of field equations from a Lagrangian constructed with the $\psi$'s and $f^{\mu\nu}f_{\mu\nu}$. We take the Lagrangian to be of the form (the indices are raised and lowered with $g_{\mu\nu}$)

$$L = L_f + L_m,$$  \hspace{1cm} (73)

where

$$L_f = -\frac{1}{4} f^{\mu\nu}f_{\mu\nu},$$  \hspace{1cm} (74)

and

$$L_m = \psi^* \left( i \frac{\partial}{\partial t} - \frac{1}{2M} (p_\mu - \epsilon U_\mu) g^{\mu\nu} (p_\nu - \epsilon U_\nu) - \Phi \right) \psi + \text{c.c.}$$  \hspace{1cm} (75)

We shall be working in the infinitesimal neighborhood of the special gauge for Abelian $U_\mu$, for which it has the form given in (59) for infinitesimal $G$. It is therefore not Abelian to first order, but we take its variation $\delta U$ to be a c-number function, carrying the variation, to lowest order, by variation of the first term in (59), and not varying the part of $U$ introduced by the infinitesimal gauge transformation (evaluated on the original value of $U$).

In carrying out the variation of $L_m$, the contributions of varying the $\psi$'s with respect to $U$ vanish due to the field equations (Stueckelberg-Schrödinger equation) obtained by varying $\psi^*$ (or $\psi$), and therefore in the variation with respect to $U$, only the explicit presence of $U$ in (75) need be taken into account.

Note that for the general case of $U$ generally operator valued, we can write

$$\psi^* (p_\mu - \epsilon U_\mu) g^{\mu\nu} (p_\nu - \epsilon U_\nu) \psi = g^{\mu\nu} ((p_\mu - \epsilon U_\mu) \psi^*) (p_\nu - \epsilon U_\nu) \psi,$$  \hspace{1cm} (76)

since the Lagrangian density (75) contains an integration over $(dU')(dU'')$ (considered in lowest order) as well as an integration over $(dx)$ in the action and the operators $U$ are Hermitian. In the limit in which $U$ is evaluated in the special Abelian gauge (real valued), and noting that $p_\mu$ is represented by an imaginary differential operator, we can write this as

$$g^{\mu\nu} \psi^* (p_\mu - \epsilon U_\mu) (p_\nu - \epsilon U_\nu) \psi = -g^{\mu\nu} (p_\mu + \epsilon U_\mu) \psi^* (p_\nu - \epsilon U_\nu) \psi,$$  \hspace{1cm} (77)

i.e., replacing explicitly $p_\mu$ by $-i(\partial/\partial x^\mu) \equiv -i \partial_\mu$, we have

$$\delta U L_m = -i \frac{\epsilon}{2M} \{ \psi^* (\partial_\mu - i \epsilon U_\mu) \psi - (\partial_\mu + i \epsilon U_\mu) \psi^* \} \delta U^\mu,$$  \hspace{1cm} (78)

where we have called $g^{\mu\nu} \delta U_\nu = \delta U_\mu$, or

$$\delta U L_m = j_\mu (U, x) \delta U^\mu,$$  \hspace{1cm} (79)
where \( j_\mu \) has the usual form of a gauge invariant current.

For the calculation of the variation of \( \mathcal{L}_f \) we note that the commutator term in (52) is, in lowest order, a c-number function, as given in (72).

Calling

\[
\omega' \lambda_{\mu} U_{\lambda} \equiv v_\mu,
\]

we compute the variation of

\[
[U'_{\mu}, U'_{\nu}] = 2i(k_\nu v_\mu - k_\mu v_\nu).
\]  

Then, for

\[
\delta_{\mu} [U'_{\mu}, U'_{\nu}] = \delta_{\mu} \frac{\partial}{\partial U_{\gamma}} [U'_{\mu}, U'_{\nu}],
\]

we compute

\[
\frac{\partial}{\partial U_{\gamma}} [U'_{\mu}, U'_{\nu}] = 2i(\frac{\partial k_\nu}{\partial U_{\gamma}} v_\mu + k_\nu \frac{\partial v_\mu}{\partial U_{\gamma}}) - (\mu \leftrightarrow \nu).
\]  

With our choice of \( k_\nu = U_\nu (U \cdot b) + b_\nu \),

\[
\frac{\partial k_\nu}{\partial U_{\gamma}} = \delta_\nu^{\gamma} (U \cdot b) + U_\nu b_{\gamma},
\]

so that

\[
\frac{\partial}{\partial U_{\gamma}} [U'_{\mu}, U'_{\nu}] = 2i(\delta_\nu^{\gamma} (U \cdot b) + U_\nu b_{\gamma}) v^\mu
\]

\[
+ k_\nu \frac{\partial v_\mu}{\partial U_{\gamma}} - (\mu \leftrightarrow \nu)
\]  

\[
\equiv \mathcal{O}_\nu^\mu,
\]

i.e.

\[
\delta_{\mu} [U'_{\mu}, U'_{\nu}] = \mathcal{O}_\nu^\mu \delta U_{\gamma}.
\]  

The quantity \( v_\mu \) is proportional to the derivative of \( \omega_\lambda^\mu \). In the limit that \( \omega, \omega' \to 0 \) (cf. (81)), the second derivative, \( \omega'' \) which appears in \( \mathcal{O}_{\nu}^{\mu \nu} \) may not vanish (somewhat analogous to the case in gravitational theory when the connection form vanishes but the curvature does not), so that this term can contribute in limit to the special Abelian gauge.

Returning to the variation of \( \mathcal{L}_f \) in (74), we see that

\[
\delta \mathcal{L}_f = -\partial^\nu f_{\mu \nu} \delta U^\mu + 2i f_{\mu \nu} \delta [U_{\mu}, U_{\nu}],
\]

where we have taken into account the fact that \([U_{\mu}, U_{\nu}]\) is a commuting function, and integrated by parts the derivatives of \( \delta U \). With (86) we obtain

\[
\delta \mathcal{L}_f = -\partial^\nu f_{\mu \nu} \delta U^\mu + 2i \epsilon f_{\lambda \sigma} \mathcal{O}_{\nu}^{\lambda \sigma} \mu \delta U^\mu.
\]

Since the coefficient of \( \delta U^\mu \) must vanish, we obtain, with (79), the Yang-Mills equations for the fields given the source currents

\[
\partial^\nu f_{\mu \nu} = j_\mu - 2i \epsilon f_{\lambda \sigma} \mathcal{O}_{\nu}^{\lambda \sigma} \mu,
\]
which is nonlinear in the fields $U_\mu$, as we have seen, even in the Abelian limit, where, from (78) and (79),

$$j_\mu = -i \frac{\epsilon}{2M} \{ \psi^*(\partial_\mu - i e U_\mu) \psi - ((\partial_\mu + i e U_\mu) \psi^*) \psi \}. \quad (90)$$

We point out that this current corresponds to a flow of the matter field; the absolute square of the wave functions corresponds to an event density. The coupling $\epsilon$ is not necessarily charge, and the fields $U$ are not necessarily electromagnetic even in the Abelian limit. However, the Hamiltonian (36) leads directly to a Lorentz type force, similar in form to that generated by the Hilbert-Einstein action. The dynamics of this system will be investigated in a forthcoming paper.

5. Conclusions

A map of the type discussed in ref.[8], of a Hamiltonian containing an Einstein metric, generating the connection form of general relativity, and a world scalar field, representing a distribution of energy on the spacetime manifold, into a corresponding Hamiltonian with a conformal metric (and compatible connection form), can account for the structure of the RAQUAL theory of Bekenstein and Milgrom[5]. Furthermore, applying this correspondence to a Hamiltonian with gauge-type structure, we have shown that one obtains a non-compact Kaluza-Klein effective metric which can account for the $TeVeS$ structure of Bekenstein, Sanders and Milgrom[7].

In order to maintain the constraint condition $U_\mu U^\mu = -1$ for the Bekenstein-Sanders fields, under local gauge transformations, we have introduced a class of gauge of gauge transformations on the underlying quantum theory which acts on the Hilbert bundle, quite analogous to that arising in the second quantization of the electromagnetic field (where the vector potentials and their time derivatives are considered as quantum oscillator variables) associated with the values of the gauge fields. The action of this class of gauges induces a nonabelian structure on the fields, which therefore satisfy Yang-Mills type field equations with source currents associated with matter flow. In the Abelian limit, these equations contain residual non-linear terms which may avoid the caustic singularities found by Contaldi et al for an electromagnetic type gauge field.

The phenomenological constraints placed on the $TeVeS$ variables in its astrophysical applications and on its MOND limit[16] would, in principle, place constraints on the vector and scalar fields appearing in the corresponding Hamiltonian model, for which the additive world scalar field corresponds to an energy distribution not associated with electromagnetic radiation.

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Hamiltonian Map to Conformal Modification of Spacetime Metric: Kaluza-Klein and TeVeS

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Abstract

It has been shown that the orbits of motion for a wide class of non-relativistic Hamiltonian systems can be described as geodesic flows on a manifold and an associated dual by means of a conformal map. This method can be applied to a four dimensional manifold of orbits in spacetime associated with a relativistic system. We show that a relativistic Hamiltonian which generates Einstein geodesics, with the addition of a world scalar field, can be put into correspondence in this way with another Hamiltonian with conformally modified metric. Such a construction could account for part of the requirements of Bekenstein for achieving the MOND theory of Milgrom in the post-Newtonian limit. The constraints on the MOND theory imposed by the galactic rotation curves, through this correspondence, would then imply constraints on the structure of the world scalar field. We then use the fact that a Hamiltonian with vector gauge fields results, through such a conformal map, in a Kaluza-Klein type theory, and indicate how the TeVeS structure of Bekenstein and Saunders can be put into this framework. We exhibit a class of infinitesimal gauge transformations on the gauge fields $U_{\mu}(x)$ which preserve the Bekenstein-Sanders condition $U_{\mu}U^{\mu} = -1$. The underlying quantum structure giving rise to these gauge fields is a Hilbert bundle, and the gauge transformations induce a non-commutative behavior to the fields, i.e. they become of Yang-Mills type. Working in the infinitesimal gauge neighborhood of the initial Abelian theory we show that in the Abelian limit the Yang-Mills field equations provide residual nonlinear terms which may avoid the caustic singularity found by Contaldi et al.

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1. Introduction

The Hamiltonian [1]

$$K = \frac{1}{2m} g^{\mu\nu} p_{\mu} p_{\nu},$$

with Hamilton equations (written in terms of derivatives with respect to an invariant world time $\tau$ [2])

$$i^{\mu} = \frac{\partial K}{\partial p_{\mu}} = \frac{1}{m} g^{\mu\nu} p_{\nu},$$

and

$$\dot{p}_{\mu} = -\frac{\partial K}{\partial x^{\mu}} = \frac{1}{2m} \frac{\partial g^{\lambda\gamma}}{\partial x^{\mu}} p_{\lambda} p_{\gamma},$$
lead to the geodesic equation
\[ \ddot{x}^\rho = -\Gamma^\rho_{\mu\nu} \dot{x}^\nu \dot{x}^\mu, \]  
where what has appeared as a compatible connection form \( \Gamma^\rho_{\mu\nu} \) is given by
\[ \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left( \frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right). \]  

These results are tensor relations over the usual diffeomorphisms admitted by the manifold \( \{x^\mu\} \); writing the Hamiltonian in terms of (2), we see that the invariant interval on an orbit is proportional, through the constant Hamiltonian, to the square of the world time of evolution on the orbit, i.e.,
\[ ds^2 = \frac{2}{m} K d\tau^2. \]  

We shall first study, in the following, a generalization of (1) consisting of the addition of a scalar field \( \Phi(x) \). The presence of such a scalar field may be considered as a gauge compensation field for the \( \tau \) derivative in the evolution term of the covariant generalization of (1) in the Stückelberg-Schrödinger equation [3], an energy distribution not directly associated with electromagnetic radiation in the usual sense. We then follow the method of ref.[4] to show that there is a corresponding Hamiltonian \( \hat{K} \) with a conformally modified metric, and no explicit additive scalar field, which has the form of the construction of Bekenstein and Milgrom [5] for the realization of Milgrom’s MOND program (modified Newtonian dynamics) [6] for achieving the observed galactic rotation curves. This simple form of Bekenstein’s theory (called RAQUAL), which we discuss in some detail below for the sake of simplicity and clarity in the development of the mathematical method, does not properly account for causality and gravitational lensing; the theory has been further developed to include vector fields (which we shall call Bekenstein-Sanders fields) as well (TeVeS) [7], which has been relatively successful in accounting for these problems. It has been shown [8], moreover, that a gauge type Hamiltonian, with Minkowski metric and both vector and scalar fields results, under a conformal map, in an effective Kaluza-Klein theory. We shall indicate here (using a general Einstein metric) how the TeVeS structure can emerge in terms of a Kaluza-Klein theory in this way. It is essential in this construction that the Bekenstein-Sanders fields be considered as gauge fields. As a realization of this possibility, we exhibit an infinitesimal gauge transformation on the underlying quantum theory for which the vector fields, which we shall call \( U^\mu(x) \), emerge as gauge compensation fields, such that, as required by the TeVeS theory the property \( U^\mu U_\mu = -1 \) [7] is preserved under such gauge transformations. The corresponding quantum theory then has the form of a Hilbert bundle and, in this framework, the gauge fields are of (generalized) Yang-Mills type [9]. Working in the infinitesimal neighborhood of a gauge in which the fields are Abelian, we show that in the limit the contributions from the nonabelian sector provide nonlinear terms in the field equations which may avoid the caustic singularity found by Contaldi et al [10]. Further investigation of this structure will be given in a subsequent publication.

For both the RAQUAL and the TeVeS theories, the correspondence between \( K \) and \( \hat{K} \) implies a relation between the conformal factor in \( \hat{K} \) and the world scalar field \( \Phi \), and
thus a possible connection between the so-called dark matter problem and a dark energy distribution represented by Φ.

2. Addition of a scalar potential and conformal equivalence

The addition of a scalar potential to the Hamiltonian (1), in the form

\[ K = \frac{1}{2m} g^{\mu\nu} p_\mu p_\nu + \Phi(x), \]  

leads, according to the Hamilton equations, to the geodesic equation\(^1\)

\[ \ddot{x}^\rho = - \Gamma^\rho_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \frac{1}{m} g_{\rho\nu} \frac{\partial \Phi}{\partial x^\nu}. \]  

Now, consider the Hamiltonian (we carry out the calculations explicitly here since we shall have need of some of the intermediate results)

\[ \hat{K} = \frac{1}{2m} \hat{g}^{\mu\nu}(y) p_\mu p_\nu. \]  

It follows from the Hamilton equations that

\[ \dot{y}^\mu = \frac{\partial \hat{K}}{\partial p_\mu} = \frac{1}{m} \hat{g}^{\mu\nu} p_\nu, \]  

so that

\[ p_\nu = m \hat{g}_{\mu\nu} \dot{y}^\mu \]  

and

\[ \dot{p}_\mu = - \frac{\partial \hat{K}}{\partial y^\mu} = - \frac{1}{2m} \frac{\partial \hat{g}^{\lambda\gamma}}{\partial y^\mu} p_\lambda p_\gamma. \]  

As in (4), it then follows that

\[ \ddot{y}^\mu = - \hat{\Gamma}^\mu_{\lambda\sigma} \dot{y}^\lambda \dot{y}^\sigma, \]  

where, as for (4),

\[ \hat{\Gamma}^\mu_{\lambda\sigma} = \frac{1}{2} \hat{g}^{\mu\nu} \left\{ \frac{\partial \hat{g}_{\nu\sigma}}{\partial y^\lambda} + \frac{\partial \hat{g}_{\nu\lambda}}{\partial y^\sigma} - \frac{\partial \hat{g}_{\lambda\sigma}}{\partial y^\nu} \right\}. \]  

We now establish an equivalence between the Hamiltonians (7) and (9) by assuming the momenta \( p_\mu \) equal at every moment \( \tau \) in the two descriptions. With the constraint

\[ \hat{K} = K = k, \]  

if we assume the conformal form

\[ \hat{g}^{\mu\nu}(y) = \phi(y) g^{\mu\nu}(x), \]  

\(^1\) Note that (8) does not admit an equivalence principle, but (11), arising from (9) does.
it follows that
\[ \phi(y)(k - \Phi(x)) = k. \] (15)

The relation (15) is not sufficient to construct \( y \) as a function of \( x \), but if we impose the relation
\[ \delta x^\mu = \phi^{-1}(y)\delta y^\mu \] (16)
between variations generated on position in the two coordinate systems, it is sufficient to evaluate derivatives of \( \phi(y) \) in terms of derivatives with respect to \( x \) of the scalar field \( \Phi(x) \) [11] (see also [12]). We review this construction briefly below.

We remark that the construction based on Eqs. (9) and (14) admits the same family of diffeomorphisms as that of (7), since \( \phi \) is scalar. Under these diffeomorphisms, both \( g_{\mu\nu} \) and \( \hat{g}_{\mu\nu} \) are second rank tensors, and by construction of the connection forms, (4) and (11) are covariant relations.

To see how these derivatives are constructed on the constraint hypersurface determined by (15), let us, for brevity, define
\[ F(x) \equiv \frac{k}{k - \Phi(x)}, \] (17)
so that the constraint relation (15) reads
\[ \phi(y) = F(x). \] (18)

Then, since variations in \( x \) and \( y \) are related by (16),
\[ \phi(y + \delta y) = F(x + \delta x) \equiv F(x) + \delta x^\mu \frac{\partial F(x)}{\partial x^\mu}. \] (19)

To first order in Taylor’s series on the left, we obtain the relation
\[ \frac{\partial \phi(y)}{\partial y^\mu} = \phi^{-1}(y)\frac{\partial F(x)}{\partial x^\mu}. \] (20)

We may therefore define a derivative, restricted to the constraint hypersurface
\[ \frac{\partial F(x)}{\partial y^\mu} = \phi^{-1}(y)\frac{\partial F(x)}{\partial x^\mu}. \] (21)

The Leibniz relation follows easily for the product of functions, it e.g., for \( \phi(y)g^{\mu\nu}(x) \).

In a similar way, the second derivative can be obtained from (19) by recognizing that the variation in \( x \) is to be computed at the point \( y + \delta y \). Keeping terms of second order in the expansion of both sides, one can define the second derivative restricted to the constraint hypersurface defined by (18); although it appears that a second derivative defined by (21) would not be symmetric, both the derivative of (21) and the second derivative computed from (19) on the constraint hypersurface agree and are symmetric [11], i.e.,
\[ \frac{\partial^2 F(x)}{\partial y^\mu \partial y^\nu} = \frac{\hat{\partial}^2 F(x)}{\partial y^\nu \partial y^\mu} \] (22)
This implies that the restricted derivative defined by (21) behaves as a \textit{bona fide} derivative on the constraint hypersurface, admitting the consistent coexistence of the coordinates $x$ and $y$ related by (15). We will not have further use of (22) here, primarily relevant for the calculation of stability criteria through geodesic deviation (the second derivative occurs in the curvature tensor).

In the following, we complete our argument of equivalence by reconstructing the equations of motion following from the Hamilton equations applied to (7), \textit{i.e.}, Eq. (8).

We begin our reconstruction, in analogy with the procedure used in the nonrelativistic problem[8], by defining the new variable $z_\mu$ such that

$$\dot{z}_\mu = \hat{g}_{\mu\nu}(y)\dot{y}^\nu. \quad (23)$$

Substituting $\dot{y}^\nu = \hat{g}^{\mu\nu}(y)\dot{z}_\mu$ into (11), the $\tau$ derivatives of $\hat{g}^{\mu\nu}(y)$ generate terms that cancel two of the terms in $\hat{\Gamma}_{\lambda\sigma}^{\mu}$, leaving

$$\ddot{z}_\nu = \frac{1}{2} \frac{\partial \hat{g}_{\lambda\sigma}}{\partial y^\nu} \dot{y}^\lambda \dot{y}^\sigma. \quad (24)$$

Now, substituting for $\dot{y}^\lambda$ from (23), and using the identity

$$\hat{g}^{\gamma\lambda} \frac{\partial \hat{g}_{\lambda\sigma}}{\partial y^\nu} \hat{g}^{\sigma\rho} = - \frac{\partial \hat{g}^{\gamma\rho}}{\partial y^\nu}, \quad (25)$$

we find

$$\ddot{z}_\nu = - \frac{1}{2} \frac{\partial \hat{g}^{\gamma\rho}}{\partial y^\nu} \dot{z}_\gamma \dot{z}_\rho. \quad (26)$$

Finally, from the variational type argument we used above,

$$\hat{g}^{\rho\gamma}(y + \delta y) - \hat{g}^{\rho\gamma}(y) = \frac{\partial \hat{g}^{\gamma\rho}}{\partial y^\nu} \delta y^\nu$$

so that

$$\frac{\partial \hat{g}^{\rho\gamma}}{\partial y^\nu} \hat{g}_{\nu\lambda} = \frac{\partial \hat{g}^{\rho\gamma}}{\partial z_\lambda} \quad \text{or} \quad \frac{\partial \hat{g}^{\rho\gamma}}{\partial y^\nu} = \hat{g}_{\nu\lambda} \frac{\partial \hat{g}^{\rho\gamma}}{\partial z_\lambda} \quad (28)$$

We therefore have the alternative form

$$\ddot{z}_\nu = - \frac{1}{2} \hat{g}_{\nu\lambda} \frac{\partial \hat{g}^{\rho\gamma}}{\partial z_\lambda} \dot{z}_\rho \dot{z}_\gamma. \quad (29)$$
This result constitutes a “geometric” embedding of the Hamiltonian motion induced by (7) in the same way as for the nonrelativistic case. Substituting the explicit form of $\hat{g}^{\rho\gamma}$ in terms of the original Einstein metric from (14), one obtains

$$\ddot{z}_\nu = -\frac{1}{2}g_{\nu\lambda}\frac{\partial g^{\rho\gamma}}{\partial z_\lambda}\ddot{z}_\rho - \frac{1}{2}\phi^{-1}g_{\nu\lambda}\frac{\partial \phi}{\partial z_\lambda}g^{\rho\gamma}\dot{z}_\gamma\dot{z}_\rho$$

(30)

The second term contains the potential field, as in the Hamilton equations, but the first term does not contain the full connection form. We may finally, however, define a “decontraction” of the connection in (30) using the Einstein metric. In fact, since according to (16), $\dot{y}^\nu = \phi \dot{x}^\nu$, and by (23),

$$\dot{z}_\mu = \hat{g}_{\mu\nu}\dot{y}^\nu = \phi^{-1}g_{\mu\nu}\dot{y}^\nu,$$

(31)

it follows that

$$\dot{z}_\mu = g_{\mu\nu}\dot{x}^\nu.$$  

(32)

Making this substitution in (30) leads explicitly, taking into account the $k$ shell constraint (13) and the form of (7), to the equation (8). We have thus completed our demonstration of the equivalence between the purely metric form of the Hamiltonian (9) and the Hamilton (7), for which the relation (29) corresponds to a dynamics generated by a compatible connection form, and constitute a “geometric” embedding of the original Hamiltonian motion.

Our interest in this section has been in relating the Hamiltonian (7) to the simplest Bekenstein-Milgrom form of MOND, without concern in the development of this simplified case for lensing or causal effects, for which a TeVeS type theory would be required. In the next Section, we indicate how a TeVeS theory can be generated in this framework, i.e., as a result of a conformal map.

3. **TeVeS and Kaluza-Klein Theory**

In this section, we show that the TeVeS theory can be cast into the form of a Kaluza-Klein construction. There has recently been a discussion [8], from the point of view of conformal correspondence, of the equivalence of a relativistic Hamiltonian with an electromagnetic type gauge invariant form [3] (here $\eta^{\mu\nu}$ is the Minkowski metric $(-1, +1, +1, +1)$)

$$K = \frac{1}{2m}\eta^{\mu\nu}(p_\mu - ea_\mu)(p_\nu - ea_\nu) - ea_5,$$

(33)

where the $\{a_\mu\}$, as fields, may depend on the affine parameter $\tau$ as well as $x^\mu$, and the $a^5$ field is necessary for the gauge invariance of the $\tau$ derivative in the quantum mechanical Stueckelberg-Schrödinger equation, with a Kaluza-Klein theory. As remarked in this work, Wesson [13] and Liko [14], as well as previous work on this structure[3], have associated the source of the $a_5$ field with mass density. A Hamiltonian of the form

$$\dot{K} = \frac{1}{2m}\hat{g}^{\mu\nu}(p_\mu - ea_\mu)(p_\nu - ea_\nu)$$

(34)
can be put into correspondence with \( K \) by taking \( \hat{g}^{\mu\nu} \) to be

\[
\hat{g}^{\mu\nu} = \eta^{\mu\nu} \frac{k}{k + ea_5},
\]

(35)

where \( k \) is the common (constant) value of \( K \) and \( \hat{K} \). In this correspondence, the equations of motion generated by \( \hat{K} \) through the Hamilton equations, have extra terms, beyond those provided by the connection form associated with \( \hat{g}^{\mu\nu} \), due to the presence of the gauge fields. These additional terms can be identified as belonging to a connection form associated with a five dimensional metric, that of a Kaluza-Klein theory.

We may apply the same procedure to the Hamiltonian

\[
K = \frac{1}{2m} g^{\mu\nu} (p_\mu - eU_\mu)(p_\nu - eU_\nu) + \Phi,
\]

(36)

where \( g^{\mu\nu} \) is an Einstein metric, \( \Phi \) is a world scalar field, and \( U_\mu \) are to be identified with the Bekenstein-Sanders fields for which \( U_\nu U^\nu = -1 \), with \( U^\mu = g^{\mu\nu} U_\nu \).

We discuss in Section 4 a class of gauge transformations on the wave functions of the underlying quantum theory for which the \( U_\mu \) arise as gauge compensation fields.

Let us define, as in Eq. (35), the conformally modified metric

\[
\hat{g}^{\mu\nu} = g^{\mu\nu} \frac{k}{k - \Phi} = e^{-2\phi} g^{\mu\nu}.
\]

(37)

The “equivalent” Hamiltonian

\[
\hat{K} = \frac{1}{2m} \hat{g}^{\mu\nu} (p_\mu - eU_\mu)(p_\nu - eU_\nu)
\]

(38)

then generates, through the Hamilton equations, an equation of motion which corresponds to the geodesic equation for an effective Kaluza-Klein metric, as in ref.[8].

Now, consider the Hamiltonian

\[
K_K = \frac{1}{2m} \tilde{g}^{\mu\nu} p_\mu p_\nu,
\]

(39)

with the Bekenstein-Sanders metric[7]

\[
\tilde{g}^{\mu\nu} = e^{-2\phi} (g^{\mu\nu} + U^\mu U_\nu) - e^{2\phi} U^\mu U^\nu
\]

(40)

The Hamiltonian \( K_K \) then has the form

\[
K_K = e^{-2\phi} g^{\mu\nu} p_\mu p_\nu - 2 \sinh 2\phi (U^\mu p_\mu)^2,
\]

(41)

Let us now define a Kaluza-Klein type metric of the form obtained in [7], arising from the equations of motion generated by (38),

\[
g^{AB} = \begin{pmatrix} \hat{g}^{\mu\nu} & U^\nu \\ U_\mu & g^{55} \end{pmatrix}.
\]

(42)
Contraction to a bilinear form with the (5D) vectors \( p_A = \{ p_\lambda, p_5 \} \), with indices \( \lambda = \nu \) on the right and \( \lambda = \mu \) on the left, one finds

\[
g^{AB} p_A p_B = \hat{g}^{\mu\nu} p_\mu p_\nu + 2p_5 (p_\mu U^\mu) + (p_5)^2 g^{55}. \tag{43}
\]

If we take

\[
p_5 = -\frac{(p_\mu U^\mu)}{g^{55}} (1 \pm \sqrt{1 - 2g^{55}\sinh 2\phi}), \tag{44}
\]

then the Kaluza-Klein theory coincides with (41), i.e.,

\[
K_K = \frac{1}{2m} g^{AB} p_A p_B. \tag{45}
\]

As discussed by Wesson [13], Kaluza [15] chose \( g_{55} = \text{const.} \) for consistency with electromagnetism, while Wesson [13] makes the more general choice of a world scalar field. In particular, the value \( g^{55} = 0 \) is well defined (as in [8]).

Since the fields \( U^\mu \) are timelike unit vectors[7], \( (p_\mu U^\mu) \) corresponds, in an appropriate local frame, to the energy of the particle, close to its mass in the case of a nonrelativistic particle, or to the frequency in the case of on-shell photons. It clearly remains to understand more deeply the apparently \textit{ad hoc} choice of \( p_5 \) in (44) in terms of a 5D canonical dynamics, along with the structure of the 5D Einstein equations for \( g_{AB} \) that follow from the geometry associated with (45). We shall study these questions in a succeeding paper.

## 4 The Bekenstein-Sanders Vector Field as a Gauge Field

Essential features of the Bekenstein-Sanders field [7] of the \textit{TeVeS} theory are that it be a local field, \textit{i.e.}, \( U_\mu(x) \), and there is a normalization constraint

\[
U^\mu U_\mu = -1, \tag{46}
\]

so that the vector is timelike. To preserve the normalization condition (46) under gauge transformation, we shall study the construction of a class of gauge transformations which essentially moves the \( U(x) \) field on a hyperbola with a Lorentz transformation (at the point \( x \)).

If we think of our underlying quantum structure, which generates the gauge field, as a fiber bundle with base \( x^\mu \), then we must think of the transformation acting in such a way that the absolute square (norm) of the wave function attached to the base point \( x^\mu \) preserves its value [9].

An analogy can be drawn to the usual Yang-Mills gauge [9] on \( SU(2) \), where there is a two-valued index for the wave function \( \psi_\alpha(x) \). The gauge transformation in this case is a two by two matrix function of \( x \), and acts only on the indices \( \alpha \). The condition of invariant absolute square (probability) is

\[
\sum_\alpha \sum_\beta |U_{\alpha\beta}\psi_\beta|^2 = \sum |\psi_\alpha|^2 \tag{47}
\]
Generalizing this structure, one can take the indices $\alpha$ to be continuous, so that (47) becomes
\[
\int (dU)|\int (dU')U(U',x)|^2 = \int (dU)|\psi(U, x)|^2,
\]
implying that $U(U, U')$ is a unitary operator on a Hilbert space $L^2(dU)$. Since we are assuming that $U_\mu$ lies on an orbit determined by (48), the measure is
\[
(dU) = \frac{d^3U}{U^0},
\]
i.e., a three dimensional Lorentz invariant integration measure.

Moreover, the Lorentz transformation on $U_\mu$ is generated by a non-commutative operator, and therefore the gauge transformation is non-Abelian. We demonstrate the resulting noncommutativity of the operator valued fields, $U'$, after an infinitesimal gauge transformation of this type, explicitly below.

This construction is somewhat similar to the treatment of the electromagnetic potential vector and its time derivative as oscillator variables in the process of second quantization of the radiation field (the energy density of the field is given by these variables in the form of an oscillator).

We now examine the gauge condition:
\[
(p_\mu - eU'_\mu)U\psi = U(p_\mu - eU_\mu)\psi
\]
Identifying $p_\mu$ with $-i\partial/\partial x^\mu$, and cancelling the terms $Up_\mu\psi$ on both sides, we obtain
\[
U'_\mu = UU_\mu U^{-1} - \frac{i}{e} \frac{\partial U}{\partial x^\mu} U^{-1},
\]
in the same form as the Yang-Mills theory [9]. It is evident in the Yang-Mills theory, that due to the matrix nature of the second term, the field will be algebra-valued, resulting in the usual structure of the Yang-Mills non-Abelian gauge theory. Here, if the transformation $U$ is a Lorentz transformation, the numerical valued field $U_\mu$ would be carried, in the first term, to a new value on a hyperbola. However, the second term may well be operator valued on $L^2(dU)$, and thus, as in the Yang-Mills theory, $U'^\mu$ would become nonabelian.

It follows from (51) that the field strengths
\[
f_{\mu\nu} = \frac{\partial U_\mu}{\partial x^\nu} - \frac{\partial U_\nu}{\partial x^\mu} + ie[U_\mu, U_\nu]
\]
are related to the the field strengths in the transformed form
\[
f'_{\mu\nu} = \frac{\partial U'_\mu}{\partial x^\nu} - \frac{\partial U'_\nu}{\partial x^\mu} + ie[U'_\mu, U'_\nu]
\]
according to
\[
f'_{\mu\nu}(x) = U f_{\mu\nu}(x)U^{-1},
\]
just as in the finite dimensional Yang-Mills theories. This result follows from writing out, from (51),

\[
\frac{\partial U_\mu'}{\partial x^\nu} = \frac{\partial U}{\partial x^\nu} U_\mu U^{-1} + U \frac{\partial U_\mu}{\partial x^\nu} U^{-1} + U \frac{\partial U_\mu}{\partial x^\nu} U^{-1} - \frac{i}{\epsilon} \frac{\partial^2 U}{\partial x^\mu \partial x^\nu} U^{-1} - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} \frac{\partial U}{\partial x^\nu} U^{-1} + \frac{1}{\epsilon} \frac{\partial U}{\partial x^\mu} \frac{\partial U}{\partial x^\nu} U^{-1}
\]

(55)

and subtracting the same expression with \( \mu, \nu \) reversed. Then add the result to

\[
i \epsilon [U_\mu', U_\nu'] = i \epsilon U [U_\mu, U_\nu] U^{-1} + [U U_\mu U^{-1}, \frac{\partial U}{\partial x^\nu} U^{-1}] + [\frac{\partial U}{\partial x^\nu} U^{-1}, U U_\nu U^{-1}] - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} U^{-1}, \frac{\partial U}{\partial x^\nu} U^{-1}
\]

(56)

Whenever the combination

\[U^{-1} \frac{\partial U}{\partial x^\mu} U^{-1}\]

appears, it should be replaced by

\[-\frac{\partial U^{-1}}{\partial x^\mu} .\]

The result (54) then follows after a little manipulation.

Now, consider the possibility that this finite gauge transformation leaves \( U_\mu U^\mu = -1 \). We write out

\[
(U U_\mu U^{-1} - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} U^{-1})(U U^\mu U^{-1} - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} U^{-1}) = -1 - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} U^\mu U^{-1} - \frac{i}{\epsilon} U U_\mu U^{-1} \frac{\partial U}{\partial x^\mu} U^{-1} - \frac{1}{\epsilon^2} \frac{\partial U}{\partial x^\mu} U^{-1} \frac{\partial U}{\partial x^\mu} U^{-1} - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} U^\mu U^{-1}
\]

(57)

It may be possible that \( U \) can be chosen to make all but the first term in (57) vanish, but in the case of finite gauge transformations, it is not so easy to see how to construct examples. For the infinitesimal case, it is, however, easy to construct a gauge function with the required properties. For

\[U \simeq 1 + i G ,\]

(58)

where \( G \) is infinitesimal, (51) becomes

\[U_\mu' = U_\mu + i [G, U_\mu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\mu} + O(G^2).
\]

(59)
Then,\[ U'_\mu U^\mu \cong U_\mu n_\mu + i(U_\mu [G, U^\mu] + [G, U_\mu]U^\mu) \]
\[ + \frac{1}{\epsilon} \left( \frac{\partial G}{\partial x_\mu} U^\mu + U_\mu \frac{\partial G}{\partial x_\mu} \right). \] (60)

Let us take\[ G = -i\epsilon \sum \{ \omega_{\lambda\gamma}(U, x), (U_\lambda \frac{\partial}{\partial U_\gamma} - U_\gamma \frac{\partial}{\partial U_\lambda}) \} \]
\[ \equiv \frac{\epsilon}{2} \sum \{ \omega_{\lambda\gamma}(U, x), N^{\lambda\gamma} \} \] (61)
where symmetrization is required since \( \omega_{\lambda\gamma} \) is a function of \( U \) as well as \( x \), and
\[ N^{\lambda\gamma} = -i(U_\lambda \frac{\partial}{\partial U_\gamma} - U_\gamma \frac{\partial}{\partial U_\lambda}). \] (62)

This construction is valid in the initially special gauge, which we shall call the “special abelian gauge”, in which the components of \( U^\mu \) commute. The appearance of \( U^\mu \) in the gauge functions is then admissible since this quantity acts on the wave functions \( <U, x|\psi> = \psi(U, x) \) at the point \( x \), in the representation in which the operator \( U^\mu \) on \( L^2(dU) \) is diagonal.

Our investigation in the following will be concerned with a study of the infinitesimal gauge neighborhood of this limit, where the components of \( U^\mu \) do not commute, and therefore constitute a Yang Mills type field. We shall show in the limit that the corresponding field equations acquire nonlinear terms, and may therefore suppress the caustic singularities found by Contaldi et al [10]. They found that nonlinear terms associated with a non-Maxwellian type action, such as \((\partial_\mu U^\mu)^2\), could avoid this caustic singularity, so that the nonlinear terms we find as a residue of the Yang-Mills structure induced by our gauge transformation might achieve this effect in a natural way.

The second term of (60), which is the commutator of \( G \) with \( U^\mu U_\mu \) vanishes, since this product is Lorentz invariant (the symmetrization in \( G \) does not affect this result).

We now consider the third term in (60).
\[ \frac{\epsilon}{2} \left( \frac{\partial G}{\partial x_\mu} U^\mu + U_\mu \frac{\partial G}{\partial x_\mu} \right) = \frac{1}{2} \{ \frac{\partial \omega_{\lambda\gamma}}{\partial x_\mu}, N^{\lambda\gamma} \} U^\mu + U^m \{ \frac{\partial \omega_{\lambda\gamma}}{\partial x_\mu}, N^{\lambda\gamma} \} \]
\[ = \frac{1}{2} \left\{ N^{\lambda\gamma} \frac{\partial \omega_{\lambda\gamma}}{\partial x_\mu} U^\mu + \frac{\partial \omega_{\lambda\gamma}}{\partial x_\mu} N^{\lambda\gamma} U^\mu \right\} \]
\[ + U^\mu N^{\lambda\gamma} \frac{\partial \omega_{\lambda\gamma}}{\partial x_\mu} + U^\mu \frac{\partial \omega_{\lambda\gamma}}{\partial x_\mu} N^{\lambda\gamma} \} \] (63)

There are two terms proportional to
\[ \frac{\partial \omega_{\lambda\gamma}}{\partial x_\mu} U^\mu. \]

If we take (locally)
\[ \omega_{\lambda\gamma}(U, x) = \omega_{\lambda\gamma}(k_\nu x_\nu), \] (64)
where \( k_\nu U_\nu = 0 \), then
\[
\frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} U_\mu = k_\mu U_\mu \omega'_{\lambda\gamma} = 0.
\] (65)

For the remaining two terms,
\[
U_\mu N^{\lambda\gamma} \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} + \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} N^{\lambda\gamma} U_\mu
= N^{\lambda\gamma} U_\mu \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} + [U_\mu, N^{\lambda\gamma}] \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} + \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} U_\mu N^{\lambda\gamma}
+ \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} [N^{\lambda\gamma}, U_\mu].
\] (66)

Since the commutators contain only terms linear in \( U_\mu \) and they have opposite sign, they cancel. The remaining terms are zero by the argument (65). The condition \( U_\mu U_\mu = -1 \) is therefore invariant under this gauge transformation, involving the coefficient \( \omega_{\lambda\gamma} \) which is a function of the projection of \( x^\mu \) onto a hyperplane orthogonal to \( U_\mu \), i.e., a function of \( k_\mu x_\mu \), where \( k_\mu U_\mu = 0 \). The vector \( k_\mu \), of course, depends on \( U_\mu \) (for example, \( k_\mu = U_\mu (U \cdot b) + b_\mu \), for some \( b_\mu \neq 0 \)).

We now demonstrate explicitly the nonabelian nature of the gauge fields after infinitesimal gauge transformation. With (59), the commutator term in (53) is
\[
[U'_\mu, U'_\nu] = (U_\mu + i[G, U_\mu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\mu})(U_\nu + i[G, U_\nu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\nu})
- (U_\nu + i[G, U_\nu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\nu})(U_\mu + i[G, U_\mu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\mu})
= \frac{1}{\epsilon} \{ [U_\mu, \frac{\partial G}{\partial x^\nu}] - [U_\nu, \frac{\partial G}{\partial x^\mu}] \}
+ i[U_\mu, [G, U_\nu]] - i[U_\nu, [G, U_\mu]],
\] (67)

where the remaining terms have identically cancelled. Note that this expression does not contain any noncommutative quantities. Now,
\[
[G, U_\nu] = 2i\epsilon \omega_\nu^\gamma U_\gamma
\] (68)
and
\[
[U_\mu, \frac{\partial G}{\partial x^\nu}] = 2i\epsilon U_\lambda \frac{\partial \omega^\lambda_{\mu}}{\partial x^\nu}.
\] (69)

The terms involving \([G, U_\nu]\) and \([G, U_\mu]\) therefore cancel, so that
\[
[U'_\mu, U'_\nu] = 2i\epsilon U_\lambda \left( \frac{\partial \omega^\lambda_{\mu}}{\partial x^\nu} - \frac{\partial \omega^\lambda_{\nu}}{\partial x^\mu} \right)
\] (70)

We have taken \( \omega^\lambda_{\mu} = \omega^\lambda_{\mu}(k_\sigma x^\sigma) \), so that
\[
\frac{\partial \omega^\lambda_{\mu}}{\partial x^\nu} = k_\nu \omega^\lambda_{\mu},
\] (71)
and therefore
\[ [\mathcal{U}_\mu', \mathcal{U}_\nu'] = 2i(k_{\lambda\mu} - k_{\mu\lambda})U_{\lambda}, \] (72)
generally not zero. This demonstrates the nonabelian character of the fields. In the Abelian limit, we may take \( \omega' \to 0 \), but as we shall a residual nonlinearity, which depends on \( \omega'' \) may remain in the field equations.

We now consider the derivation of field equations from a Lagrangian constructed with the \( \psi \)'s and \( f^{\mu\nu} f_{\mu\nu} \). We take the Lagrangian to be of the form (the indices are raised and lowered with \( g_{\mu\nu} \))
\[ \mathcal{L} = \mathcal{L}_f + \mathcal{L}_m, \] (73)
where
\[ \mathcal{L}_f = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu} \] (74)
and
\[ \mathcal{L}_m = \psi^* \left( i \frac{\partial}{\partial \tau} - \frac{1}{2M} (p_\mu - eU_\mu) g^{\mu\nu} (p_\nu - eU_\nu) - \Phi \right) \psi + \text{c.c.} \] (75)

We shall be working in the infinitesimal neighborhood of the special gauge for Abelian \( U_\mu \), for which it has the form given in (59) for infinitesimal \( G \). It is therefore not Abelian to first order, but we take its variation \( \delta U_\mu \) to be a c-number function, carrying the variation, to lowest order, by variation of the first term in (59), and not varying the part of \( U \) introduced by the infinitesimal gauge transformation (evaluated on the original value of \( U \)).

In carrying out the variation of \( \mathcal{L}_m \), the contributions of varying the \( \psi \)'s with respect to \( U \) vanish due to the field equations (Stueckelberg-Schrödinger equation) obtained by varying \( \psi^* \) (or \( \psi \)), and therefore in the variation with respect to \( U \), only the explicit presence of \( U \) in (75) need be taken into account.

Note that for the general case of \( U \) generally operator valued, we can write
\[ \psi^* (p_\mu - eU_\mu) g^{\mu\nu} (p_\nu - eU_\nu) \psi = g^{\mu\nu} ( (p_\mu - eU_\mu) \psi^* ) (p_\nu - eU_\nu) \psi, \] (76)
since the Lagrangian density (75) contains an integration over \((dU')(dU'')\) (considered in lowest order) as well as an integration over \((dx)\) in the action and the operators \( U \) are Hermitian. In the limit in which \( U \) is evaluated in the special Abelian gauge (real valued), and noting that \( p_\mu \) is represented by an imaginary differential operator, we can write this as
\[ g^{\mu\nu} \psi^* (p_\mu - eU_\mu) (p_\nu - eU_\nu) \psi = -g^{\mu\nu} (p_\mu + eU_\mu) \psi^* (p_\nu - eU_\nu) \psi, \] (77)
i.e., replacing explicitly \( p_\mu \) by \(-i(\partial/\partial x^\mu) \equiv -i\partial_\mu \), we have
\[ \delta_U \mathcal{L}_m = -i \frac{\epsilon}{2M} \left\{ \psi^* (\partial_\mu - i\epsilon U_\mu) \psi - ((\partial_\mu + i\epsilon U_\mu) \psi^* ) \psi \right\} \delta U^\mu; \] (78)
where we have called \( g^{\mu\nu} \delta U_\nu = \delta U_\mu \), or,
\[ \delta_U \mathcal{L}_m = j_\mu (U, x) \delta U^\mu, \] (79)
where $j_\mu$ has the usual form of a gauge invariant current.

For the calculation of the variation of $L_f$ we note that the commutator term in (52) is, in lowest order, a c-number function, as given in (72).

Calling

$$\omega^\lambda_\mu U_\lambda \equiv v_\mu, \quad (80)$$

we compute the variation of

$$[U'_\mu, U'_\nu] = 2i(k_\nu v_\mu - k_\mu v_\nu) \quad (81)$$

Then, for

$$\delta U[U'_\mu, U'_\nu] = \delta U, \frac{\partial}{\partial U_\gamma} [U'_\mu, U'_\nu], \quad (82)$$

we compute

$$\frac{\partial}{\partial U_\gamma} [U'_\mu, U'_\nu] = 2i(\frac{\partial k_\nu}{\partial U_\gamma} v_\mu + k_\nu \frac{\partial v_\mu}{\partial U_\gamma} - (\mu \leftrightarrow \nu)). \quad (83)$$

With our choice of $k_\nu = U_\nu (U \cdot b) + b_\nu$,

$$\frac{\partial k_\nu}{\partial U_\gamma} = \delta_\nu^\gamma (U \cdot b) + U_\nu b^\gamma, \quad (84)$$

so that

$$\frac{\partial}{\partial U_\gamma} [U'_\mu, U'_\nu] = 2i(\delta_\nu^\gamma (U \cdot b) + U_\nu b^\gamma) v^\mu + k_\nu \frac{\partial v_\mu}{\partial U_\gamma} - (\mu \leftrightarrow \nu)) \equiv O^\gamma_{\mu \nu}, \quad (85)$$

i.e.

$$\delta U[U'_\mu, U'_\nu] = O^\gamma_{\mu \nu} \delta U_\gamma \quad (86)$$

The quantity $v_\mu$ is proportional to the derivative of $\omega^\lambda_\mu$. In the limit that $\omega, \omega' \rightarrow 0$ (cf. (81)), the second derivative, $\omega''$ which appears in $O^\gamma_{\mu \nu}$ may not vanish (somewhat analogous to the case in gravitational theory when the connection form vanishes but the curvature does not), so that this term can contribute in limit to the special Abelian gauge.

Returning to the variation of $L_f$ in (74), we see that

$$\delta L_f = -\partial^\nu f_{\mu \nu} \delta U^\mu + 2i f_{\mu \nu} \delta [U_\mu, U_\nu], \quad (87)$$

where we have taken into account the fact that $[U_\mu, U_\nu]$ is a commuting function, and integrated by parts the derivatives of $\delta U$. With (86) we obtain

$$\delta L_f = -\partial^\nu f_{\mu \nu} \delta U^\mu + 2i \epsilon f_{\lambda \sigma} O^{\lambda \sigma}_{\mu \nu} \delta U^\mu \quad (88)$$

Since the coefficient of $\delta U^\mu$ must vanish, we obtain, with (79), the Yang-Mills equations for the fields given the source currents

$$\partial^\nu f_{\mu \nu} = j_\mu - 2i \epsilon f_{\lambda \sigma} O^{\lambda \sigma}_{\mu \nu}, \quad (89)$$
which is nonlinear in the fields $U_\mu$, as we have seen, even in the Abelian limit, where, from (78) and (79),
\[
j_\mu = -\frac{\epsilon}{2M} \{ \psi^* (\partial_\mu - i e U_\mu) \psi - (\partial_\mu + i e U_\mu) \psi^* \}.
\]

(90)

We point out that this current corresponds to a flow of the matter field; the absolute square of the wave functions corresponds to an event density. The coupling $\epsilon$ is not necessarily charge, and the fields $U$ are not necessarily electromagnetic even in the Abelian limit. However, the Hamiltonian (36) leads directly to a Lorentz type force, similar in form to that generated by the Hilbert-Einstein action. The dynamics of this system will be investigated in a forthcoming paper.

5. Conclusions

A map of the type discussed in ref.[8], of a Hamiltonian containing an Einstein metric, generating the connection form of general relativity, and a world scalar field, representing a distribution of energy on the spacetime manifold, into a corresponding Hamiltonian with a conformal metric (and compatible connection form), can account for the structure of the RAQUAL theory of Bekenstein and Milgrom[5]. Furthermore, applying this correspondence to a Hamiltonian with gauge-type structure, we have shown that one obtains a non-compact Kaluza-Klein effective metric which can account for the TeVeS structure of Bekenstein, Sanders and Milgrom[7].

In order to maintain the constraint condition $U_\mu U^\mu = -1$ for the Bekenstein-Sanders fields, under local gauge transformations, we have introduced a class of gauge of gauge transformations on the underlying quantum theory which acts on the Hilbert bundle, quite analogous to that arising in the second quantization of the electromagnetic field (where the vector potentials and their time derivatives are considered as quantum oscillator variables) associated with the values of the gauge fields. The action of this class of gauges induces a nonabelian structure on the fields, which therefore satisfy Yang-Mills type field equations with source currents associated with matter flow. In the Abelian limit, these equations contain residual non-linear terms which may avoid the caustic singularities found by Contaldi et al for an electromagnetic type gauge field.

The phenomenological constraints placed on the TeVeS variables in its astrophysical applications and on its MOND limit[16] would, in principle, place constraints on the vector and scalar fields appearing in the corresponding Hamiltonian model, for which the additive world scalar field corresponds to an energy distribution not associated with electromagnetic radiation.

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