The natural frequency ratio between Timoshenko and Euler effect by variation of parameter

Xie Lei*

Civil Engineering Department, Yanbian University, Yanji City, Jilin Province, 133002, China
*Author’s e-mail: xielei@ybu.edu.cn

Abstract. Applied energy principle analysis natural frequency equations of Euler beam and Timoshenko respectively. The expression of natural frequency is calculated by the separation of variables. Consider the natural frequency ratio between the Euler beam and Timoshenko one to analyze the relationship between geometric parameters and frequency ratio that effect by parameters of frequency. Calculations show that the relationship between frequency ratio and geometric parameters present non-linear parabola characteristic. When the increment of geometric parameters is identical, the frequency value of the beam was lowered by both transverse shear effect and rotary inertia whereby non-linear characteristics are mainly determined by the transverse shear effect.

1. Introduction
As an important component in engineering, the beam has an irreplaceable role in the structural stress system[1-3]. Many structural forms can be simplified as beams or beam systems under certain conditions. For many years, scholars have done lots of research on the mechanical properties of beams under loads, including theoretical studies[4-5] and experimental studies[6]. As one of the key computational parameters, the natural frequency plays an important role in the theoretical calculation and experimental research of the beam. The classical beam analysis theory is divided into Euler beam theory and Timoshenko beam theory, in which the latter considers the shear effect and rotary inertia effect. The influence of these two effects on the value of frequency parameters needs to be further discussed. In this paper, the frequency expressions of Euler and Timoshenko beams are analyzed through the energy principle. The effect of shear effect and rotational inertia effect on the frequency is studied by analyzing the relation between the section geometry parameters and the frequency ratio.

2. Derivation of Euler beam dynamic model
According to the basic assumptions of Bernoulli-Euler beams and the one-dimensional beam theory, the displacement field is[7].

\[ u_1 = -z \psi(x,t) \quad u_2 = 0 \quad u_3 = w(x,t) \] (1)

Where, \( u_1, u_2, u_3 \) are the displacement in the x, y, and z directions, respectively; \( \psi(x) \) is the rotation angle of the axis of the beam

\[ \psi = \frac{\partial w}{\partial x} \] (2)

Strain

\[ \varepsilon_{xx} = -z \frac{\partial^2 w(x,t)}{\partial x^2} \quad \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0 \] (3)
The sagging moment of the beam is

\[ M = \int_{-h/2}^{h/2} \sigma_{xx} zbdz = -EI \frac{\partial \psi}{\partial x} \]  

(4)

On the one hand, the kinetic energy of the beam is

\[ T = \frac{1}{2} \int_{-h/2}^{h/2} \left[ \rho \dot{u} \ddot{u} dx \right] dz = \frac{1}{2} \int_{-h/2}^{h/2} \left[ \rho I \left( \ddot{w}/\ddot{t} \right)^2 + \rho A \left( \ddot{w}/\ddot{t} \right)^2 \right] dx \]  

(5)

On the other hand, the sum of the work done by the external force and the strain energy is

\[ U + V = \int_{0}^{L} \left[ EI \left( \ddot{w}/\ddot{t} \right)^2 / 2 - qw \right] dx \]  

(6)

According to Hamilton principle

\[ \delta \int_{h}^{t} L dt = \delta \int_{h}^{t} \left[ \rho I \left( \ddot{w}/\ddot{t} \right)^2 + \rho A \left( \ddot{w}/\ddot{t} \right)^2 \right] \left[ /2 - EI \left( \ddot{w}/\ddot{t} \right)^2 / 2 + qw \right] dx dt = 0 \]  

(7)

Finally, Eq.(7) is thus

\[ \int_{h}^{t} \int_{0}^{L} \left[ (\rho - E) I w'''' - \rho A \ddot{w} + q \right] \delta w dx dt + \int_{h}^{t} \left( \rho - E \right) I w'' \delta w_{0}'' \right] dt 

\[- \int_{h}^{t} \left( \rho - E \right) I w'' \delta w_{0}'' \right] dt + \int_{0}^{L} \rho A \ddot{w} \delta w_{0}'' \right] dx = 0 \]  

(8)

The governing equation of Euler beam, we get

\[ (\rho - E) I w'''' - \rho A \ddot{w} + q = 0 \]  

(9)

The solution of Eq.(9) is thus

\[ w(x,t) = W(x) \cos(\omega t) \]  

(10)

Where,

\[ W(x) = A_{n} \sin(n\pi x/L) \]

When q=0, the frequency of Euler is thus

\[ \omega_{Euler} = \sqrt{(E-\rho)/(1/A)} / (n\pi/L)^2 \]  

(11)

3. Derivation of Timoshenko beam dynamic model

According to the one-dimensional stress field hypothesis, the strain Eq.(12) of the Timoshenko beam is thus

\[ \varepsilon_{xx} = -z \frac{\partial \psi}{\partial x} \quad \varepsilon_{zz} = \frac{\partial w}{\partial x} - \psi \]  

\[ \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = 0 \]  

(12)

The bending moment of the beam is written as

\[ M = \int_{-h/2}^{h/2} \sigma_{xx} zbdz = -EI \frac{\partial \psi}{\partial x} \]  

(13)

![Figure 1. Cross-section of the beam.](image-url)
According to Hooke’s law and \( \tau_{xz} = 2\varepsilon_{xx}G \)

\[
V_s = \int_{-h/2}^{h/2} \tau_{xz} bdz = kGA(\partial w/\partial x - \psi)
\]  

(14)

Where \( k \) is shear constant;

According to the Hamilton principle, kinetic energy is thus

\[
T = \frac{1}{2} \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} \rho \dddot{u} \dddot{u} dxdydz = \frac{1}{2} \int_0^L \left[ \rho I (\partial^2 \psi / \partial t^2)^2 + \rho A (\partial w / \partial t)^2 \right] dx
\]

(15)

The sum of strain energy and potential energy is thus

\[
U + V = \int_0^L \left[ EI (\partial^2 \psi / \partial x^2)^2 / 2 + kGA (\partial w / \partial x - \psi)^2 / 2 - qw \right] dx
\]

(16)

we get

\[
\delta \int_0^{L_1} \delta Ldt = \delta \int_0^{L_1} \left[ \frac{1}{2} \rho I (\partial^2 \psi / \partial t^2)^2 + \rho A (\partial w / \partial t)^2 \right] dx
\]

(17)

Notice at the time \( t_1 \) and \( t_2 \), \( \partial \psi = \partial w = 0 \)

Then Eq.(17) is written as

\[
\int_0^{L_1} \left[ -\partial (\rho I \dddot{u}) / \partial t + \partial (EI \dddot{u} / \partial x) / \partial x + kGA (\partial w / \partial x - \psi) \right] \partial \psi + \int_0^{L_1} EI (\dddot{u} / \partial t) \partial \psi \bigg|_{t_1}^{t_2} dt - \int_0^{L_1} kGA (\partial w / \partial x - \psi) \partial w \bigg|_{t_1}^{t_2} dt = 0
\]

(18)

The equation of Euler-Lagrange is thus

\[
-\partial (\rho A \dddot{w}) / \partial t + \partial kGA (w_x - \psi) / \partial x + q = 0
\]

(19)

\[
-\partial (\rho I \dddot{u}) / \partial t + \partial (EI \dddot{u} / \partial x) / \partial x + kGA (w_x - \psi) = 0
\]

(20)

From Eq.(14), we get

\[
\partial V_s / \partial x = \rho A \dddot{w} - q
\]

(21)

\[
V_s - \partial M / \partial x = \rho I \dddot{u}
\]

(22)

By decoupling, the transverse displacement equation is thus

\[
EI \dddot{w} / \partial x^4 + \rho A \dddot{w} / \partial t^2 - \rho I (1 + E / kG) \dddot{w} / \partial t^2 \partial x^2 + \rho^2 I / kG \left( \dddot{w} / \partial t^4 \right)
\]

(23)

Separable variables are used for the solution of Eq.(23)

\[
w(x,t) = W(x) \cos(\omega t)
\]

(24)

\[
W(x) = A_n \sin(n\pi x / L)
\]

(25)

Take Eq.(24), (25) into Eq.(23), then

\[
EI (n\pi / L)^4 - \rho A \omega_n^2 - \rho I (1 + E / kG) \omega_n^2 (n\pi / L)^2 + \rho^2 I \omega_n^4 / kG = 0
\]

(26)

3.1. Frequency governing equation of Timoshenko beam

Eq.(26) divide by \( -EI / L^2 \), then

\[
-\rho^2 L^2 \omega_n^4 / E kG + \left[ \rho AL^2 / EI + \rho L^2 (1 + E / kG)(n\pi)^2 / E \right] \omega_n^2 - (n\pi)^4 = 0
\]

(27)
Then the solution of Eq.(27) is thus

$$
\omega_{\text{bend}} = \sqrt{\frac{L^2/r^2 + \left[ L^2 (1 + E/kG) \right] (n\pi)^2 - \sqrt{\left[ L^2/r^2 + \left[ L^2 (1 + E/kG) \right] (n\pi)^2 \right]^2 - 4EL^4 (n\pi)^4 / kG}}{2\rho L^4 / kG}} \tag{28}
$$

Take $r^2 = I/A$, frequency comparison between the Euler beam and Timoshenko beam, we get

$$
\frac{\omega_{\text{Euler}}}{\omega_{\text{bend}}} = \sqrt{\frac{2(E - \rho)/kGr (n\pi)^2}{L^4/r^2 + \left[ L^2 (1 + E/kG) \right] (n\pi)^2 - \sqrt{\left[ L^2/r^2 + \left[ L^2 (1 + E/kG) \right] (n\pi)^2 \right]^2 - 4EL^4 (n\pi)^4 / kG}} \tag{29}
$$

### 3.1.1. An example.

A steel beam, elastic modulus $E=200$GPa=$2g/(cm\cdot\mu s^2)$, modulus of shear deformation $G=80$GPa=$0.8g/(cm\cdot\mu s^2)$, rectangular cross-section, length 200cm, $k=5/6$, density $0.078g/cm^3$. By calculation, the relationship between the frequency ratio and the parameters is in Figure 2.

The ratio of the Euler beam to the Timoshenko beam presents a nonlinear parabolic relationship with the increase of parameter $r$ as shown in Figure 2. It shows that the frequency values of the Euler beam are higher than those of the Timoshenko beam. When $n$ value increases, the nonlinear characteristic of the frequency ratio decreases with the increase of $r$. But for $r$ value with the same increment, the increased amplitude of frequency ratio increases continuously. When both transverse shear effect and rotational inertia are considered in the beam frequency, both of them affect the beam frequency.
3.2. Frequency governing equation of Timoshenko beam containing only the moment of rotation inertia

The term containing the constant K in the formula is the transverse shear term. So when we just include the moment of inertia of rotation, we set that to be zero. That is, the Eq.(30) is zero.

\[-E(L/r)^4/kG \quad \text{and} \quad (L/r)^2 E/kG\]

Then Eq.(27) turns into

\[\left(\rho AL^4/EI + \rho L^2 (n\pi)^2/E\right)\omega_n^2 - (n\pi)^4 = 0\]  \hspace{1cm} (31)

The solution of Eq.(31) is thus

\[\omega_n^2 = (EA/\rho I)\left((n\pi)^4/(L/r)^2 + (L/r)^2 + (n\pi)^2\right)\] \hspace{1cm} (32)

In the meantime, parameter r is shown in Section 3.1.

\[\omega_{Euler} / \omega_{Timoshenko} = \sqrt{1 - \rho/E} \left[1 + (r/L)^2 (n\pi)^2\right]\] \hspace{1cm} (33)

When the example in Section 3.1.1 is used to calculate, the relationship between the frequency ratio in Eq.(33) and the parameter r as shown in Figure 3.

| n = 1 | n = 2 | n = 3 |
|-------|-------|-------|
| 0.8   | 1.2   | 1.8   |
| 1.0   | 1.4   | 1.6   |
| 1.0   | 1.4   | 1.8   |

4. Conclusion

In this paper, the formula derivation of the Euler beam and Timoshenko beam frequency is discussed by Hamilton respectively. The relation between geometric parameters and frequency ratio is discussed. Meanwhile, the influence of shear lateral effect and rotational inertia on frequency is also considered. The relationship between the geometric characteristic parameters and the ratio of frequency presents a nonlinear parabolic feature. In the meantime, as n increases, the increase in frequency ratio gradually increases. The transverse shear effect and rotational inertia both affect the beam frequency, and the transverse shear effect plays a decisive role in the nonlinear characteristics of the curve.
References
[1] Nguyen, D.T., Christoph, B., Nguyen, V.T. (2020) Analysis of the crack development and shear transfer mechanisms of reinforced concrete beams with low amounts of shear reinforcement. J. Engineering Structures. 222:1-20.
[2] Liu, B., Zhang, J.R., Wang L., Yang, R.H. (2011) The review of the shear capacity calculation of reinforced concrete beams (part1). J. China & foreign highway. 31: 150-158.
[3] Liu, B., Zhang, J.R., Wang L., Yang, R.H. (2011) The review of the shear capacity calculation of reinforced concrete beams (part2). J. China & foreign highway. 31: 109-114.
[4] Fang, Q., Wu, P.A. (2003) Main factors affecting failure modes of blast loaded RC beams. J. Computational mechanics. 20:39-42.
[5] Fang, Q., Chen, G.L., Chen, L. (2013) The linear dynamic responses of columns subjected to blast loads. J. Engineering mechanics. 30:112-119.
[6] Zhang, D., Yao, S.J., Lu, F.Y., Chen, X.G., Lin, Y.L. (2013) Experimental study on scaling of RC beams under close-in blast loading. J. Engineering Failure Analysis. 33:497-504.
[7] Clive, L.D., Irving H.S. (2013) Solid Mechanics A Variational Approach. Springer, New York.