Limitations of the Invertible-Map Equivalences

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Abstract
This note draws conclusions that arise by combining two recent papers, by Anuj Dawar, Erich Grädel, and Wied Pakusa, published at ICALP 2019 and by Moritz Lichter, published at LICS 2021. In both papers, the main technical results rely on the combinatorial and algebraic analysis of the invertible-map equivalences $\equiv_{IM}^{k,Q}$ on certain variants of Cai-Fürer-Immerman structures (CFI-structures for short). These $\equiv_{IM}^{k,Q}$-equivalences, for a natural number $k$ and a set of primes $Q$, refine the well-known Weisfeiler-Leman equivalences used in algorithms for graph isomorphism. The intuition is that two graphs $G \equiv_{IM}^{k,Q} H$ cannot be distinguished by iterative refinements of equivalences on $k$-tuples defined via linear operators on vector spaces over fields of characteristic $p \in Q$.

In the first paper it has been shown, using considerable algebraic machinery, that for a prime $q \notin Q$, the $\equiv_{IM}^{k,Q}$ equivalences are not strong enough to distinguish between non-isomorphic CFI-structures over the field $\mathbb{F}_q$. In the second paper, a similar but not identical construction for CFI-structures over the rings $\mathbb{Z}_{2^t}$ has, again by rather involved combinatorial and algebraic arguments, been shown to be indistinguishable with respect to $\equiv_{IM}^{k,\mathbb{Z}_{2^t}}$. Together with earlier work on rank logic, this second result suffices to separate rank logic from polynomial time.

We show here that the two approaches can be unified to prove that CFI-structures over the rings $\mathbb{Z}_{2^t}$ are in fact indistinguishable with respect to $\equiv_{IM}^{k,P}$, for the set $P$ of all primes. In particular, this implies the following two results.

- There is no fixed $k$ such that the invertible-map equivalence $\equiv_{IM}^{k,P}$ coincides with isomorphism on all finite graphs.
- No extension of fixed-point logic by linear-algebraic operators over fields can capture polynomial time.

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1 Invertible-map equivalences and linear algebraic logics

Invertible-map equivalences are refinements of the Weisfeiler-Leman method, an important technique in the study of the graph isomorphism problem. For each positive integer $k$, the $k$-dimensional Weisfeiler-Leman method ($k$-WL method for short) defines an equivalence relation $\equiv_k$ which over-approximates isomorphism in the sense that if $G \equiv H$ for a pair of graphs $G$ and $H$, then $G \equiv_k H$ for any $k$. These equivalence relations get finer with increasing $k$ and approach isomorphism in the limit. Indeed, if $G$ and $H$ are $n$-vertex graphs then $G \equiv^k H$ if, and only if, $G \equiv H$ and, for each fixed $k$, the equivalence relation $\equiv^k$ is decidable in time $n^{O(k)}$. Thus, if there was a fixed $k$ such that $\equiv^k$ was the same as isomorphism, we would have a polynomial-time algorithm for graph isomorphism. However,
there is no such fixed $k$. Cai, Füredi, and Immerman [8] showed that there are pairs of non-isomorphic graphs $G$ and $H$ with $O(k)$ vertices such that $G \equiv^k H$. We call the construction of such graphs the CFI-construction. The Weisfeiler-Leman equivalences are also of central importance in descriptive complexity theory since they delimit the power of logics with counting operators, such as fixed-point logic with counting (FPC), which is a fundamental formalism in the quest for a logic for PTIME (see [14]).

The CFI-construction, in its original form, can be seen as a graph encoding of linear equation systems over the field $F_2$. Thus, while FPC is not strong enough to tell apart non-isomorphic CFI-structures, this can be done by stronger extensions of fixed-point logics that are powerful enough to solve such equation systems. A number of such extensions have been studied in [9]: the most influential one is rank logic (FPR), proposed in [7]. Rank logic extends fixed-point logic by operators for the rank of definable matrices over a given finite field $\mathbb{F}_p$. For a somewhat more powerful variant of rank logic $\text{FPR}^*$, studied in [10], it has until recently been open whether it defines all polynomial-time properties of finite structures.

The invertible-map equivalences have been defined in [8] as a tool to study the expressive power of rank logic. Like the $k$-WL equivalences, they are defined by iterated refinements of equivalences between $k$-tuples. However, the refinement process is not defined on the basis of counting, but on the basis of invertible maps between matrices obtained from the given tuples by appropriate substitutions. For a formal definition, we refer to [5, Sect. 3.1]. The equivalences $\equiv^k_{IM}$ properly refine the Weisfeiler-Leman equivalences in the sense that $G \equiv^k_{IM} H$ for sufficiently large $k'$ implies $G \equiv^k H$ for all graphs $G$ and $H$, but for the pairs $G, H$ obtained in the CFI-construction, $G \not\equiv^k_{IM} H$. As shown in [8] there is, for every formula $\varphi$ of rank logic FPR, a $k \in \mathbb{N}$ and a finite set $Q$ of primes such that the class of models of $\varphi$ is closed under $\equiv^k_{IM}$. But in fact, the invertible-map equivalences are potentially much finer than the equivalences under rank logic. They delimit the expressive power not just of rank logic, but of arbitrary extensions of fixed-point logic by linear-algebraic operators.

Intuitively, a linear-algebraic operator over a field $\mathbb{F}$ is any function $f$ that maps tuples $(M_1, \ldots, M_m)$ of $\mathbb{F}$-linear transformations on (subspaces of) an abstract vector space $\mathbb{R}^\mathbb{F}$ to some kind of linear-algebraic information $f(M_1, \ldots, M_m) \in \mathbb{N}$. We do not even require that the function $f$ is computable, but to define “linear-algebraic information” it has to be invariant under $\mathbb{F}$-vector space isomorphisms. This means that $f(M_1, \ldots, M_m) = f(N_1, \ldots, N_m)$ for any two sequences $(M_1, \ldots, M_m)$ and $(N_1, \ldots, N_m)$ that are simultaneously similar, in the sense that there is a $\mathbb{F}$-vector space isomorphism $S$ such that $N_i \cdot S = S \cdot M_i$ for all $i \leq m$.

The general linear-algebraic logics $\text{LA}_k(Q)$, defined in [6], are infinitary $k$-variable logics with generalized quantifiers for all linear-algebraic operators over finite vector spaces of characteristic $p \in Q$. For a detailed definition that is not needed here we refer to [5, Sect. 3.2].

Notice that the logics $\text{LA}_k(Q)$ and $\text{LA}_\omega(Q) = \bigcup_{k \in \omega} \text{LA}_k(Q)$ are non-effective, infinitary logics that are not intended for practical use. Their relevance stems from the fact that they encompass any extension of first-order logic or fixed-point logics by means of $Q$-linear-algebraic operators. Thus, inexpressibility results for $\text{LA}_k(Q)$ and $\text{LA}_\omega(Q)$ directly translate to inexpressibility results for all such logics, in particular for rank logic or logics with solvability operators for linear equation systems.

It has been shown in [6] that $\text{LA}_k(Q)$ is the logic for which the invertible-map equivalence $\equiv^k_{IM_{k,Q}}$ is the natural notion of elementary equivalence.

**Theorem 1.** Let $k \geq 2$ be a positive integer and $Q$ a set of prime numbers. For any finite structure $\mathfrak{A}$ and $a, b \in A^k$, the following are equivalent:

1. $(\mathfrak{A}, \bar{a}) \equiv^k_{IM_{k,Q}} (\mathfrak{A}, \bar{b})$; and
2. for every formula $\varphi$ of $\text{LA}_k(Q)$, $\mathfrak{A} \models \varphi[\bar{a}]$ if, and only if, $\mathfrak{A} \models \varphi[\bar{b}]$. 
2 Invertible-map equivalences for generalised CFI-structures

We next present a high-level exposition of the results in [6] and [13] on invertible-map equivalences of CFI-structures, and their consequences for graph isomorphism and descriptive complexity. We refer to the full versions of these papers, published on ArXiv [5, 14].

It is well-known that the CFI-construction can be adapted beyond the field $\mathbb{F}_2$ to many other algebraic structures. A general variant due to Holm [12] is based on arbitrary finite Abelian groups. In [10] a variant over prime fields $\mathbb{F}_p$ has been used to show that formulae of FPR that do not use a rank operator over the field $\mathbb{F}_p$ are no more expressive than formulae of FPC over these graphs. This separates the expressive power of FPR from that of FPR*, and proves that FPR does not capture PTIME. In [5], the same graph construction has been analysed with significantly deeper algebraic machinery, connecting it to invertible-map equivalences for primes $p \not\in \mathbb{Q}$.

More precisely, this variant of the CFI-construction associates with every connected, 3-regular, ordered, and simple base graph $G = (V, E, \leq)$, every prime field $\mathbb{F}_p$, and every function $\lambda : V \rightarrow \mathbb{F}_p$ a CFI-structure $\text{CFI}[G, \mathbb{F}_p, \lambda]$, with the following properties:

- The automorphism group of $\text{CFI}[G, \mathbb{F}_p, \lambda]$ is an elementary Abelian $p$-group.
- Two CFI-structures $\text{CFI}[G, \mathbb{F}_p, \lambda]$ and $\text{CFI}[G, \mathbb{F}_p, \sigma]$ over the same base graph $G$ are isomorphic if, and only if, $\sum \lambda = \sum_{v \in V} \lambda(v) = \sum_{v \in V} \sigma(v) = \sum \sigma$.

The CFI-problem (over a class $\mathcal{F}$ of base graphs and a field $\mathbb{F}_p$) is to decide, given a structure $\text{CFI}[G, \mathbb{F}_p, \lambda]$ with $G \in \mathcal{F}$, whether $\sum \lambda = 0$. The CFI-problem is solvable in polynomial time, for instance by Gaussian elimination.

For proving logical inexpresibility results, the full power of the CFI-construction is unfolded when the graphs in the underlying class $\mathcal{F}$ are highly connected. The class used in [5] is a family $\mathcal{F} = \{G_n : n \in \mathbb{N}\}$ of 3-regular, connected expander graphs where $G_n$ has $\Theta(n)$ vertices. By the Cai-Führer-Immerman Theorem [3] and its well-known generalisations to other algebraic structures than $\mathbb{F}_2$, we have the following property:

- For every $G_n \in \mathcal{F}$ and all $\lambda, \sigma : V \rightarrow \mathbb{F}_p$ we have that $\text{CFI}[G_n, \mathbb{F}_p, \lambda] \equiv^{\mathcal{O}(n)} \text{CFI}[G_n, \mathbb{F}_p, \sigma]$.

A final important fact about these CFI-structures is a homogeneity property: Despite the fact that counting logic cannot determine the full isomorphism type of a CFI-structure, it can, with $\Theta(k)$ many variables, distinguish between those pairs of $k$-tuples which are not related via an automorphism of the CFI-structure.

- For all $k$-tuples $\bar{a}, \bar{b}$ in a CFI-structure $\mathfrak{A} = \text{CFI}[G, \mathbb{F}_p, \lambda]$ with $G \in \mathcal{F}$, we have that $(\mathfrak{A}, \bar{a}) \equiv^{3k} (\mathfrak{A}, \bar{b})$ if, and only if, $f(\bar{a}) = \bar{b}$ for some automorphism $f$ of $\mathfrak{A}$.

Based on these properties, and on methods from the representation theory of finite groups, such as Maschke’s Theorem, the main technical result of [5] says the following: on CFI-structures for $\mathcal{F}$ and the field $\mathbb{F}_p$, the distinguishing power of $\equiv^{M}_{k, Q}$, where $p \not\in \mathbb{Q}$, is no greater than the counting equivalence $\equiv^\ell$ for some fixed $\ell$.

\begin{itemize}
  \item \textbf{Theorem 2.} Let $p \not\in \mathbb{Q}$. For every $k$ there is an $n$ such that for every $G_m \in \mathcal{F}$ satisfying $m \geq n$ and all $\lambda, \sigma$ we have that $\text{CFI}[G_m, \mathbb{F}_p, \lambda] \equiv^{M}_{k, Q} \text{CFI}[G_m, \mathbb{F}_p, \sigma]$.
  \item \textbf{Corollary 3.} If $Q \not\in \mathbb{P}$, there is no fixed $k$ such that $\equiv^{M}_{k, Q}$ coincides with isomorphism on all finite structures.
\end{itemize}

The interesting question left open by this result is, of course, the case when $Q = \mathbb{P}$. Since the CFI-problem, for arbitrary base graphs, is solvable in polynomial time by solving systems
of linear equations, we get the following limitations for the expressive power of the logics $\text{LA}^\omega(Q)$.

▶ Corollary 4. If $Q \neq \mathbb{P}$, there is a class of finite structures that is decidable in polynomial time, but not definable in $\text{LA}^\omega(Q)$.

Since $\text{LA}^\omega(Q)$ subsumes FPC, no extension of fixed-point logic by $Q$-linear algebraic operators can capture PTIME, unless it includes such operators for all prime characteristics.

More recently, a somewhat different CFI-construction over the rings $\mathbb{Z}_{2^i}$ has been used by Lichter [14] to separate rank logic from PTIME. His construction of CFI-structures $\text{CFI}[G, \mathbb{Z}_{2^i}, \lambda]$ is not based on 3-regular graphs, but on highly connected regular graphs of large degree and girth. Further, but this is a minor point, the last component is not a function on vertices, but a function $\lambda : E \rightarrow \mathbb{Z}_{2^i}$ defining the values by which edges are twisted. Analogous properties as above apply. In particular,

- The automorphism group of $\text{CFI}[G, \mathbb{Z}_{2^i}, \lambda]$ is an Abelian $2$-group.
- Two CFI-structures $\text{CFI}[G, \mathbb{Z}_{2^i}, \lambda]$ and $\text{CFI}[G, \mathbb{Z}_{2^i}, \sigma]$ are isomorphic if, and only if, $\sum_{e \in E} \lambda(e) = \sum_{e \in E} \sigma(e) = \sum \sigma$.

The analysis of these CFI-structures is done in terms of the game-theoretic description of the invertible-map equivalences, the so-called invertible-map game introduced in [8], using combinatorial objects called blurers. The main technical result of [14] shows that these CFI-structures cannot be told apart by invertible-map equivalences for the prime 2.

▶ Theorem 5. For each $k$ there exists a graph $G = (V, E, \leq)$, a number $i$, and two functions $\lambda, \sigma : E \rightarrow \mathbb{Z}_{2^i}$ such that $\sum \sigma = \sum \lambda + 2^{i-1}$ and $\text{CFI}[G, \mathbb{Z}_{2^i}, \lambda] \equiv_{\text{IM}_k[2]} \text{CFI}[G, \mathbb{Z}_{2^i}, \sigma]$.

Further, Lichter refines an argument from [10] to show that on the CFI-structures over $\mathbb{Z}_{2^i}$, every formula of FPR$^*$ is equivalent to an FPR formula with rank operators only over the field $\mathbb{F}_2$. But these cannot tell apart $\equiv_{\text{IM}_k[2]}$-equivalent structures. Thus, there exists a variant of the CFI-problem that is not definable in rank logic.

▶ Corollary 6. FPR$^*$ does not capture PTIME.

3 Combining the constructions

To combine the results of [5] and [14] we want to show that the CFI-structures $\text{CFI}[G, \mathbb{Z}_{2^i}, \lambda]$ are not just $\equiv_{\text{IM}_k[2]}$-equivalent but in fact $\equiv_{\text{IM}_k[2]}$-equivalent for the set of all primes $\mathbb{P}$. For this, we have to show that the differences in the two CFI-constructions do not really matter.

Both CFI-structures are based on the well-known CFI-gadgets. These gadgets originally consist of inner and outer vertices. Every outer vertex is adjacent to some inner vertices in the gadget. Two gadgets are connected by connecting their corresponding outer vertices. For $d$-regular graphs, the inner vertices can be replaced by $d$-ary relations, which is done in [5]. Alternatively, [14] leaves out the outer vertices and directly connects the inner vertices, which is important to yield structures of the same signature for different degrees of the base graph. When using only one sort of vertices (so either only inner or only outer ones) fewer case distinctions are needed.

For a simple and connected base graph $G = (V, E, \leq)$ and a function $\lambda : E \rightarrow \mathbb{Z}_{2^i}$, we define the two constructions $\text{CFI}_0[G, \mathbb{Z}_{2^i}, \lambda]$ using only outer vertices and $\text{CFI}_1[G, \mathbb{Z}_{2^i}, \lambda]$ using only inner vertices, respectively.
The relation $A$ vertices $u$ used. This results in isomorphic structures. In [14] more relations apart from slightly from the ones in [5] and [14]. In [5] functions $\lambda: V \to F_p$ instead of $\lambda: E \to \mathbb{Z}_2^r$ are used. This results in isomorphic structures. In [14] more relations apart from $I$ are added.

### Construction using outer vertices
This construction requires that $G$ is $d$-regular. For each vertex $u \in V$ with neighbourhood $N_G(u) = \{v_1, \ldots, v_d\}$ we define a gadget consisting of vertices $A_u := \mathbb{Z}_2^d \times N_G(u)$ and two relations:

\[
R_u := \{(a_1, v_1), \ldots, (a_d, v_d) \in A_u^d : \sum_{i=1}^d a_i = 0\}, \quad u \in V,
\]

\[
C_u := \{(a, v), (a+1, v) \in A_u^2 : a \in \mathbb{Z}_2^d, v \in N_G(u)\}, \quad u \in V.
\]

The CFI-relation $R_u$ connects all $d$-tuples of vertices for each neighbour with sum 0 (in $\mathbb{Z}_2^d$) and the cycle relation realizes the automorphism group $\mathbb{Z}_2^d$ on the vertices of each neighbour of $u$. We obtain the CFI-structure $\text{CFI}_0 [G, \mathbb{Z}_2^r, \lambda] := (A, R, C, I, \preceq)$ as follows: The universe $A$ is given by the disjoint union of the $A_u$ for all $u \in V$, and likewise $R := \bigcup_{u \in V} R_u$ and $C := \bigcup_{u \in V} C_u$. The inverse relation pairs additive inverses for each edge (shifted by $\lambda$):

\[
I := \{(a, v), (b, u) \in A_u \times A_v : \{u, v\} \in E, a + b = \lambda(\{u, v\})\}.
\]

Finally, the preorder $\preceq$ is just the extension of $\leq$ to the gadgets: for $(a, u') \in A_u$ and $(b, v') \in A_v$, we have $(a, u') \preceq (b, v')$ if $(u, u')$ is lexicographically smaller than $(v, v')$.

### Construction using inner vertices
To define $\text{CFI}_1 [G, \mathbb{Z}_2^r, \lambda]$ we replace the $d$-ary relation $R$ with vertices and thus can omit the restriction to a fixed degree. For each vertex $u \in V$ we define a gadget consisting of vertices $B_u$ and two families of relations:

\[
B_u := \{\bar{a} \in \mathbb{Z}_2^{N_G(u)} : \sum_{i} a_i = 0\}, \quad u \in V,
\]

\[
N_{u,v} := \{(\bar{a}, b) \in B_u^2 : \bar{a}(v) = b(v)\}, \quad u \in V, v \in N_G(u),
\]

\[
C_{u,v} := \{(\bar{a}, b) \in B_u^2 : \bar{a}(v) + 1 = b(v)\}, \quad u \in V, v \in N_G(u).
\]

The relation $N_{u,v}$ identifies a set of vertices in $B_u$ corresponding to the vertex $(a, v) \in A_u$. For every $v \in N_G(u)$ and $\bar{a} \in \mathbb{Z}_2^d$, a clique is added between the vertices in the set corresponding to $(a, v)$. These cliques are a partition of $B_u$ for a fixed $v$. The other relation $C_{u,v}$ represents the relation $C$ by adding directed complete bipartite graphs between subsequent cliques. We need different relations for every neighbour $v \in N_G(u)$ because the relations overlap.

We obtain the CFI-structure $\text{CFI}_1 [G, \mathbb{Z}_2^r, \lambda] := (B, N, C, \lambda, \preceq)$ as follows: the universe $B$ is given by the disjoint union of all $B_u$ for all $u \in V$. The relations $N$ and $C$ are 4-ary equivalence relations on pairs, such that the $N_{u,v}$ respectively $C_{u,v}$ are given as union of equivalence classes:

\[
N := \{(\bar{a}, b, \bar{a}', \bar{b}') : \{u, v\} \in N_{u,v}) \leq \{(u', v') : (\bar{a}', \bar{b}') \in N_{u', v'}\}\}.
\]

Here we extended $\leq$ to sets of pairs of vertices in the base graph. The relation $C$ is defined similarly. The preorder $\preceq$ is again the preorder obtained as the lexicographical extension of $\leq$ to the vertices in $B_u$. Now connecting gadgets becomes similar to the case of $\text{CFI}_0 [G, \mathbb{Z}_2^r, \lambda]$. Instead of adding an edge between two vertices in $A_u \times A_v$, we add complete bipartite graphs between the corresponding vertices in $B_u \times B_v$:

\[
I := \{(\bar{a}, b) \in B_u \times B_v : \{u, v\} \in E, \bar{a}(v) + \bar{b}(u) = \lambda(\{u, v\})\}.
\]

For easier presentation, the two structures $\text{CFI}_1 [G, \mathbb{Z}_2^r, \lambda]$ and $\text{CFI}_0 [G, \mathbb{Z}_2^r, \lambda]$ still differ slightly from the ones in [5] and [14]. In [5] functions $\lambda: V \to F_p$ instead of $\lambda: E \to \mathbb{Z}_2^r$ are used. This results in isomorphic structures. In [14] more relations apart from $I$ are added.
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to make local isomorphism types more informative. Nevertheless, these structures have the same automorphisms and in fact the additional relations are definable in 3-variable logic using the relation \( I \).

The \( k \)-orbits of a CFI-structure \( \text{CFI}_k [G, \mathbb{Z}_2^i, \lambda] \) over a \((k + 3)\)-connected base graph \( G = (V, E, \leq) \) can be defined in \((k + 2)\)-variable counting logic. The proof is analogous to the one in [10] for the case of \( E_p \) instead of \( \mathbb{Z}_2^i \).

**Combining results.** Our ultimate goal is to prove the following theorem:

**Theorem 7.** For each \( k \) there exists a graph \( G = (V, E, \leq) \), a number \( i \), and two functions \( \lambda, \sigma : E \to \mathbb{Z}_2^i \) such that \( \sum \sigma = \sum \lambda + 2^i - 1 \) and \( \text{CFI}_k [G, \mathbb{Z}_2^i, \lambda] \equiv_{\forall \exists} \text{CFI}_k [G, \mathbb{Z}_2^i, \lambda] \).

This theorem is proved by combining the proofs of Theorem 2 and Theorem 5. Specifically, we examine the properties of the CFI-structures used in that proof and argue that they are (sufficiently) satisfied by the alternative CFI-structures used to prove Theorem 2 in [5]. Specifically, we take a close look at the base graphs used to prove Theorem 5 in [14] immediately gives the following.

**Lemma 8.** For each \( k \) there exist \( c, d, g, \) and \( i \) such that for every regular base graph \( G = (V, E, \leq) \) of degree at least \( d \), vertex-connectivity at least \( c \), and girth at least \( g \) there are functions \( \lambda, \sigma : E \to \mathbb{Z}_2^i \) such that \( \sum \sigma = \sum \lambda + 2^i - 1 \) and \( \text{CFI}_k [G, \mathbb{Z}_2^i, \lambda] \equiv_{\forall \exists} \text{CFI}_k [G, \mathbb{Z}_2^i, \lambda] \).

Our aim is to argue that the case of primes other than 2 can be covered by the methods used to prove Theorem 2 in [5]. Specifically, we examine the properties of the CFI-structures used in that proof and argue that they are (sufficiently) satisfied by the alternative CFI-structures defined here. The proofs in Section 8 in [5] depend on the following properties of CFI-structures:

- **Homogeneity:** A structure is called \( \ell \)-homogeneous, if for every \( t \) the \( \ell \)-orbits of the structure can be defined in counting logic with \( \ell \cdot t \) variables. The proof in [5] relies on the fact (proved in [9]) that if the base graph is a 3-regular expander, then the resulting CFI-structures are \( \ell \)-homogeneous for some fixed value of \( \ell \). However, the construction we are using here uses base graphs that are \( d \)-regular (for increasing values of \( d \)) and not necessarily expanders. The proof of Lemma 8 relies on a weaker connectivity assumption: that the graphs are \( c \)-connected. With this, we cannot prove that the structure \( \text{CFI}_k [G, \mathbb{Z}_2^i, \lambda] \) is homogeneous. However, we can show that the \( \ell \)-orbits for \( t \leq c \) are definable in counting logic with no more than \( \ell \cdot t \) variables for some constant \( \ell \). Homogeneity is used in the proof of Theorem 8.2 in [5] to construct a formula of counting logic ordering the \( \ell \)-orbits of CFI-structures. It is clear from the proof of the theorem that we need this only for values of \( t \) not exceeding \( k \), the number of variables for which we aim to establish equivalence in Theorem 2. Thus, the full strength of homogeneity is not necessary. We can choose, for any \( k \), base graphs with sufficiently large values of \( d, c \), and \( g \) as in Lemma 8, and in these, \( \ell \)-orbits for all values of \( t \) do not need to be ordered in counting logic.

- **Structure of automorphism groups:** It is pointed out in [5] that the automorphism groups of the CFI-structures \( \text{CFI}_k [G, \mathbb{F}_p, \lambda] \) constructed there are elementary Abelian \( p \)-groups. For our structures \( \text{CFI}_k [G, \mathbb{Z}_2^i, \lambda] \), the automorphism groups are Abelian 2-groups but not necessarily elementary. However, the proof of Theorem 2 does not use the assumption of elementariness anywhere. The fact that it is an Abelian \( p \)-group is sufficient.

- **Automorphisms as ordered objects:** Part (3) of the proof of Theorem 8.2 in [5] exploits the fact that automorphisms of CFI-structures can be represented as ordered objects. This works for \( \text{CFI}_k [G, \mathbb{Z}_2^i, \lambda] \) exactly as for \( \text{CFI}_0 [G, \mathbb{Z}_2^i, \lambda] \) by exploiting the total order on the vertices (and hence of the edges) of \( G \): for every edge it is stored by which amount an automorphism twists the edge.
Thus, we have seen that both CFI-constructions satisfy the same crucial properties, which permits us to establish the following lemma.

**Lemma 9.** For every $k$ there exists a number $c$ such that for every $c$-connected base graph $G = (V, E, \leq)$ and every $\lambda, \sigma : E \to \mathbb{Z}_{2^k}$ it holds that \( \text{CFI}_1 [G, \mathbb{Z}_{2^k}, \lambda] \equiv_{k, \lambda} \text{CFI}_1 [G, \mathbb{Z}_{2^k}, \sigma] \).

So we know that for every $k$ there is a pair $\mathcal{A}, \mathcal{B}$ of CFI-structures satisfying $\mathcal{A} \equiv_{k, \lambda} \mathcal{B}$ and $\mathcal{A} \equiv_{k, \lambda} \mathcal{B}$. To combine these results, we show in general that if $\mathcal{A} \equiv_{k, \lambda} \mathcal{B}$ and $\mathcal{A} \equiv_{k, \lambda} \mathcal{B}$, then $\mathcal{A} \equiv_{k, \lambda} \mathcal{B}$. This is not immediate, because it is not clear whether nesting linear-algebraic operators of characteristics in $P$ and $Q$ increases the distinguishing power.

**Lemma 10.** Let $P, Q$ be two sets of primes, $k, i \in \mathbb{N}$, $G = (V, E, \leq)$ be a $(k + 3)$-connected base graph, $\lambda, \sigma : E \to \mathbb{Z}_{2^k}$, $\mathcal{A} = \text{CFI}_1 [G, \mathbb{Z}_{2^k}, \lambda]$, and $\mathcal{B} = \text{CFI}_1 [G, \mathbb{Z}_{2^k}, \sigma]$. If $\mathcal{A} \equiv_{k, \lambda} \mathcal{B}$ and $\mathcal{A} \equiv_{k, \lambda} \mathcal{B}$, then $\mathcal{A} \equiv_{k, \lambda} \mathcal{B}$.

**Proof.** We say that two tuples $\bar{a}, \bar{b} \in A^n$ and $\bar{c}, \bar{d} \in B^n$ (for some $n \leq k$) have the same type, if the same $(k + 2)$-variable counting logic formula defines the orbit of $\bar{a}$ and $\bar{b}$. We show by induction on formulae that for every $\text{LA}^k(P \cup Q)$ formula $\varphi$ and every $\bar{a} \in A^k$ and $\bar{b} \in B^k$ that have the same type it holds that $\mathfrak{A} \models \varphi[\bar{a}]$, if, and only if, $\mathfrak{B} \models \varphi[\bar{b}]$.

The only interesting case is the one of a linear-algebraic operator $f$ of characteristic $p \in P \cup Q$. Assume without loss of generality that $p \in P$. For simplicity, we denote the generalized quantifier corresponding to $f$ by $\psi = Q^{m, p}_f (\varphi_1, \ldots, \varphi_t)$ for $t \in \mathbb{N}$ and $2m \leq k$.

Here $\varphi_1, \ldots, \varphi_t$ are $\text{LA}^k(P \cup Q)$ formulae, where $2m$ variables are bound by the quantifier. These formulae correspond to $0/1 A^m \times A^m$ matrices $M_1, \ldots, M_t$ and likewise to $B^m \times B^m$ matrices $N_1, \ldots, N_t$ (for details we refer to [5]). The generalized quantifier has $n \leq 2m$ free variables and is satisfied if $f(M_1, \ldots, M_t) = t$.

Let $\bar{a} \in A^n$ and $\bar{b} \in B^n$ have the same type. By induction hypothesis $\mathfrak{A} \models \varphi_i[\bar{a}^i]$ if, and only if, $\mathfrak{B} \models \varphi_i[\bar{b}^i]$ for every $i \in \ell$, $\bar{a}^i \in A^{2m}$, and $\bar{b}^i \in B^{2m}$ such that $\bar{a}^i$ and $\bar{b}^i$ have the same type. That is, there are $(k + 2)$-variable counting logic formulae $\varphi_1', \ldots, \varphi_t'$ equivalent to the $\varphi_i$ on $\mathfrak{A}$ and $\mathfrak{B}$ (namely the disjunction of all orbit-defining formulae for all orbits satisfying $\varphi_i$). With an additional free variable we can simulate counting and obtain equivalent $\varphi_1'', \ldots, \varphi_t''$ $\text{LA}^{k+3}(P)$ formulae.

Then $\psi' := Q^{m, p}_f (\varphi_1', \ldots, \varphi_t')$ is an $\text{LA}^{k+3}(P)$ formula equivalent to $\psi$ on $\mathfrak{A}$ and $\mathfrak{B}$. For the sake of contradiction, assume without loss of generality that $\mathfrak{A} \models \varphi'[\bar{a}]$ but $\mathfrak{B} \not\models \varphi'[\bar{b}]$.

Let $\chi[\bar{x}]$ be the $(k + 2)$-variable logic formula defining the orbits of $\bar{a}$ and $\bar{b}$ and $\chi'[\bar{x}]$ be the equivalent $\text{LA}^{k+3}(P)$ formula. Then the $\text{LA}^{k+3}(P)$ sentence $\forall \bar{x}, \chi[\bar{x}] \Rightarrow \varphi'[\bar{x}]$ distinguishes $\mathfrak{A}$ and $\mathfrak{B}$. But by assumption and Theorem 7 such a sentence does not exist.

We believe that with a more careful analysis the decrease of the number of variables from $k + 3$ for $P$ and $Q$ to $k$ for $P \cup Q$ in Lemma 10 is not needed. Finally, we are ready to prove Theorem 7.

**Proof of Theorem 7.** For every $d \geq 2$ and $g \geq 3$ there exists a $d$-regular graph of girth $g$ [15], in particular there exists a $(d, g)$-cage (a graph with minimal order for the parameters $d$ and $g$). Every $(d, g)$-cage for an odd $d \geq 7$ is $\lceil \frac{d}{2} \rceil$-connected [2].

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[1] Formally, $\text{LA}^k(P \cup Q)$ uses an interpretation instead of $\ell$ many plain formulae, but the argument remains the same. For details, we refer to [5].
8 Limitations of the Invertible-Map Equivalences

Fix $k$ and let $c_1$, $d$, $g$, and $i$ be the constants given by Lemma 8 for $k + 3$. Furthermore, let $c_2$ be the constant given by Lemma 9 for $k + 3$. We set $c := \max\{c_1, c_2, k + 3\}$ and assume that $d$ is odd (otherwise we increase $d$ by one). Let $G = (V, E, \leq)$ be a $(\max\{d, 2c + 1\}, g)$-cage (for an arbitrary order $\leq$). By Lemma 8 there are functions $\lambda, \sigma : E \to \mathbb{Z}_2^i$ such that $\sum\sigma = \sum\lambda + 2^{i - 1}$ and $\text{CFI}_I[G, \mathbb{Z}_{2^i}, \lambda] \equiv_{IM}^{k + 3, \{2\}} \text{CFI}_I[G, \mathbb{Z}_{2^i}, \sigma]$. By Lemma 9, it holds that $\text{CFI}_I[G, \mathbb{Z}_{2^i}, \lambda] \equiv_{IM}^{k + 3, P, \{2\}} \text{CFI}_I[G, \mathbb{Z}_{2^i}, \sigma]$. The claim follows with Lemma 10.

4 Conclusion

There are two important conclusions that can be drawn from Theorem 7. The first is the immediate one that there is no constant $k$ for which the $k$-invertible-map test yields a complete isomorphism test.

▶ Corollary 11. There is no fixed $k$ such that $\equiv_{IM}^{k, P}$ coincides with isomorphism on finite structures.

The CFI-structures constructed in the proof of Theorem 5 are large: their size is super-exponential in $k$. In particular, we get only a weak lower bound on $k$ in terms of the size of the CFI-structures needed to distinguish them with the invertible-map equivalence $\equiv_{IM}^{k, P}$. The bound is super-constant but sub-logarithmic. This should be contrasted with the linear lower bound for the dimension of the Weisfeiler-Leman method needed to distinguish the CFI-structures. It is an interesting question whether the bound for the invertible-map equivalence can be strengthened.

The second consequence is that no linear-algebraic logic captures PTIME. Indeed, the problem of determining, for a structure $\text{CFI}_I[G, \mathbb{Z}_{2^i}, \lambda]$, whether $\sum\lambda = 0$ is decidable in polynomial time [14].

▶ Corollary 12. No extension of fixed-point logic by linear-algebraic operators over fields captures PTIME.

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