UNIFORM BOUNDS ON HARMONIC BELTRAMI DIFFERENTIALS AND WEIL-PETERSSON CURVATURES

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Abstract. In this article we show that for every finite area hyperbolic surface $X$ of type $(g,n)$ and any harmonic Beltrami differential $\mu$ on $X$, then the magnitude of $\mu$ at any point of small injectivity radius is uniform bounded from above by the ratio of the Weil-Petersson norm of $\mu$ over the square root of the systole of $X$ up to a uniform positive constant multiplication.

We apply the uniform bound above to show that the Weil-Petersson Ricci curvature, restricted at any hyperbolic surface of short systole in the moduli space, is uniformly bounded from below by the negative reciprocal of the systole up to a uniform positive constant multiplication. As an application, we show that the average total Weil-Petersson scalar curvature over the moduli space is uniformly comparable to $-g$ as the genus $g$ goes to infinity.

1. Introduction

In this paper, we derive uniform bounds on the curvature of the Weil-Petersson metric on $\mathcal{M}_g^n$ the moduli space of conformal structures on the surface of genus $g$ with $n$ punctures where $3g + n \geq 5$. We write $\mathcal{M}_g$ for $\mathcal{M}_g^0$ for simplicity. These bounds depend on new uniform bounds for the norm of harmonic Beltrami differentials in terms of injectivity radius.

Let $X \in \mathcal{M}_g^n$. Recall that the systole $\ell_{\text{sys}}(X)$ of $X$ is shortest length of closed geodesics in the hyperbolic surface $X$ and for $z \in X$, the injectivity radius $\text{inj}(z)$ is the maximum radius of an embedded ball centered at $z$. We denote the Margulis constant in dimension two by

$$\epsilon_2 = \sinh^{-1}(1).$$

By the Collar Lemma, for $r(z) \leq \epsilon_2$, then $z$ is either contained in a collar $C_\gamma$ about a closed geodesic $\gamma$ or $z$ is in a neighborhood $C_c$ about a cusp $c$. The tangent space $T_X \mathcal{M}_g^n$ of $\mathcal{M}_g^n$ at $X$ can be identified with the space of harmonic Beltrami differentials on $X$. Let $\mu \in T_X \mathcal{M}_g^n$. We denote by $||\mu||_{WP}$ the Weil-Petersson norm of $\mu$, which is also the $L^2$-norm of $\mu$ on $X$. One consequence of our analysis is the following Proposition.
Proposition 1.1. Let $X \in \mathcal{M}_g^n$ with $\ell_{\text{sys}}(X) \leq 2\epsilon_2$. Then for any $\mu \in T_X \mathcal{M}_g^n$ a harmonic Beltrami differential and $z \in X$ with injectivity radius $\text{inj}(z) \leq \epsilon_2$, \[
abla(\mu(z))^2 \leq \frac{||\mu||_{\text{WP}}^2}{\text{inj}(z)} \leq 2 \frac{||\mu||_{\text{WP}}^2}{\ell_{\text{sys}}(X)}.
\]

Remark 1.2. In [17, Corollary 11], Wolpert proved a similar bound when $\ell_{\text{sys}}(X)$ is smaller than a positive constant depending on $g$ and $n$. Our approach is similar to Wolpert’s, but using a detailed analysis of the thin parts, we are able to obtain the above uniform bounds independent of $g$ and $n$. Actually we will prove certain more precise uniform bounds which are Proposition 3.3 and Lemma 3.4. One may see Section 3 for more details.

Using Proposition 1.1, we derive uniform lower bounds on Weil-Petersson curvatures. More precisely, we prove

Theorem 1.3. For any $X \in \mathcal{M}_g^n$ with $\ell_{\text{sys}}(X) \leq 2\epsilon_2$, then

1. for any $\mu \in T_X \mathcal{M}_g^n$ with $||\mu||_{\text{WP}} = 1$, the Weil-Petersson Ricci curvature satisfies that
   \[\text{Ric}^{\text{WP}}(\mu) \geq -\frac{4}{\ell_{\text{sys}}(X)} ;\]

2. the Weil-Petersson scalar curvature at $X$ satisfies that
   \[\text{Sca}^{\text{WP}}(X) \geq -\frac{4}{\ell_{\text{sys}}(X)} \cdot (3g - 3 + n) .\]

Remark 1.4. In [10] Teo showed that for any $X \in \mathcal{M}_g$,

1. $\text{Ric}^{\text{WP}} \geq -2C(\frac{\ell_{\text{sys}}(X)}{2})^2$.
2. $\text{Sca}^{\text{WP}}(X) \geq -(6g - 6)C(\frac{\ell_{\text{sys}}(X)}{2})^2$.

Here the function $C(\cdot)$ is given by (3.1). As the systole $\ell_{\text{sys}}(X)$ of $X$ tends to zero, $C(\frac{\ell_{\text{sys}}(X)}{2})^2 = \frac{4}{\pi\ell_{\text{sys}}(X)^2} + O(\ell_{\text{sys}}(X)^2)$. Also $C(\frac{\ell_{\text{sys}}(X)}{2})^2$ tends to $\frac{3}{4\pi}$ as $\ell_{\text{sys}}(X)$ goes to infinity. Compared to Teo’s result, we obtain a better growth rate as $\ell_{\text{sys}}(X) \to 0$. Actually this growth rate $\frac{1}{\ell_{\text{sys}}(X)}$ is optimal: Wolpert in [17, Theorem 15] or [17, Corollary 16] computed the Weil-Petersson holomorphic sectional curvature along the gradient of certain geodesic length function and showed that it behaves as $\frac{3}{\pi\ell_{\text{sys}}(X)^2} + O(\ell_{\text{sys}}(X)^2)$ as $\ell_{\text{sys}}(X)$ goes to infinity. Compared to Teo’s result, we obtain a better growth rate as $\ell_{\text{sys}}(X) \to 0$. Actually this growth rate $\frac{1}{\ell_{\text{sys}}(X)}$ is optimal: Wolpert in [17, Theorem 15] or [17, Corollary 16] computed the Weil-Petersson holomorphic sectional curvature along the gradient of certain geodesic length function and showed that it behaves as $\frac{3}{\pi\ell_{\text{sys}}(X)^2} + O(\ell_{\text{sys}}(X)^2)$ as $\ell_{\text{sys}}(X) \to 0$, where $\alpha \subset X$ is a nontrivial loop. Part (1) of Teo’s results above in particular implies that the Weil-Petersson sectional curvature, restricted on any $\epsilon$-thick part of the moduli space, is uniformly bounded from below by a negative constant only depending on $\epsilon$. This was first obtained by Huang in [5]. One may also see [13] for more general statements.

Remark 1.5. The assumption $\ell_{\text{sys}}(X) \leq 2\epsilon_2$ in Theorem 1.3 can not be removed. One may see this in the following two different ways: (1). Tromba...
and Wolpert showed that for all \( X \in \mathcal{M}_g \),
\[
\text{Sca}^{\text{WP}}(X) \leq \frac{-3}{4\pi} \cdot (3g - 2).
\]

In particular for large enough \( g \), the uniform lower bound for scalar curvature in Theorem 1.3 does not hold for Buser-Sarnak surface \( X_g \) (see [2]) whose injectivity radius grows like \( \ln(g) \) as \( g \to \infty \). Similarly for (2). It was shown in [13, Theorem 1.1] that if \( \ell_{\text{sys}}(X) \) is large enough, then
\[
\min_{\text{span}(\mu, v) \subseteq T_X \mathcal{M}_g} K^{\text{WP}}(\mu, v) \leq -C < 0
\]
where \( C > 0 \) is a uniform constant independent of \( g \). In particular, the uniform lower bound for Ricci curvature in Theorem 1.3 does not hold for Buser-Sarnak surface \( X_g \) in [2] for large enough \( g \).

Let \( X \in \mathcal{M}_g^n \) with \( \ell_{\text{sys}}(X) \leq 2\epsilon_2 \), and let \( P(X) \subseteq T_X \mathcal{M}_g^n \) be the linear subspace generated by the gradient of short closed geodesic length functions and \( P(X)^\perp \) be its perpendicular. One may see (3.18) and (3.19) for the precise definitions. Our next result says that the Weil-Petersson curvature along any plane in \( T_X \mathcal{M}_g^n \) containing a \( \mu \in P(X)^\perp \) is uniformly bounded from below. More precisely,

**Theorem 1.6.** Let \( X \in \mathcal{M}_g^n \) with \( \ell_{\text{sys}}(X) \leq 2\epsilon_2 \), then for any \( \mu \neq 0 \in P(X)^\perp \) and \( v \in T_X \mathcal{M}_g^n \), the Weil-Petersson sectional curvature \( K^{\text{WP}}(\mu, v) \) along the plane spanned by \( \mu \) and \( v \) satisfies
\[
K^{\text{WP}}(\mu, v) \geq -4.
\]

It would be interesting to find upper bounds for \( K^{\text{WP}}(\mu, v) \) in terms of certain measurements of \( \mu \) and \( v \).

Recall that the boundary \( \partial \mathcal{M}_g \) of \( \mathcal{M}_g \) consists of nodal surfaces. As \( X \) goes to \( \partial \mathcal{M}_g \), the Weil-Petersson scalar curvature \( \text{Sca}^{\text{WP}}(X) \) always blows up to \(-\infty\) because the Weil-Petersson sectional curvature at \( X \) along certain direction goes to \(-\infty\) (e.g., see [9] or [17, Corollary 16]). It was not known whether the total scalar curvature \( \int_{\mathcal{M}_g} \text{Sca}^{\text{WP}}(X) dX \) is finite. We will show it is truly finite. Moreover, combining Theorem 1.3 and a result of Mirzakhani in [8] we will determine the asymptotic behavior of \( \int_{\mathcal{M}_g} \text{Sca}^{\text{WP}}(X) dX \) as \( g \to \infty \). More precisely, we prove

**Theorem 1.7.** As \( g \to \infty \),
\[
\frac{\int_{\mathcal{M}_g} \text{Sca}^{\text{WP}}(X) dX}{\text{Vol}_{\text{WP}}(\mathcal{M}_g)} \asymp -g.
\]

**Notation.** In this paper, we say two functions
\[
f_1(g) \asymp f_2(g)
\]
if there exists a universal constant $C \geq 1$, independent of $g$, such that

$$\frac{f_2(g)}{C} \leq f_1(g) \leq C f_2(g).$$

**Plan of the paper.** Section 2 provides some necessary background and the basic properties on Teichmüller theory and the Weil-Petersson metric. Refined results of Proposition 1.1 are proved in Section 3. We prove several results on uniform lower bounds for Weil-Petersson curvatures including Theorem 1.3 and 1.6. Theorem 1.7 is proved in Section 5.

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## 2. Preliminaries

In this section, we set our notation and review the relevant background material on Teichmüller space and Weil-Petersson curvature.

### 2.1. Teichmüller space.

We denote by $S^n_g$ an oriented surface of genus $g$ with $n$ punctures where $3g + n \geq 5$. Then the Uniformization theorem implies that the surface $S^n_g$ admits hyperbolic metrics of constant curvature $-1$. We let $T^n_g$ be the Teichmüller space of surfaces of genus $g$ with $n$ punctures, which we consider as the equivalence classes under the action of the group $\text{Diff}_0(S^n_g)$ of diffeomorphisms isotopic to the identity of the space of hyperbolic surfaces $X = (S^n_g, \sigma(z)|dz|^2)$. The tangent space $T_X T^n_g$ at a point $X = (S^n_g, \sigma(z)|dz|^2)$ is identified with the space of finite area harmonic Beltrami differentials on $X$, i.e. forms on $X$ expressible as $\mu = \overline{\psi}/\sigma$ where $\psi \in Q(X)$ is a holomorphic quadratic differential on $X$. Let $z = x + iy$ and $dA = \sigma(z)dx\,dy$ be the volume form. The **Weil-Petersson metric** is the Hermitian metric on $T^n_g$ arising from the the **Petersson scalar product**

$$\langle \varphi, \psi \rangle = \int_X \frac{\varphi \cdot \overline{\psi}}{\sigma^2} \, dA$$

via duality. We will concern ourselves primarily with its Riemannian part $g_{WP}$. Throughout this paper we denote by $\text{Teich}(S^n_g)$ the Teichmüller space endowed with the Weil-Petersson metric. By definition it is easy to see that the mapping class group $\text{Mod}^n_g := \text{Diff}^+(S^n_g)/\text{Diff}^0(S^n_g)$ acts on $\text{Teich}(S^n_g)$ as isometries. Thus, the Weil-Petersson metric descends to a metric, also called the Weil-Petersson metric, on the moduli space of Riemann surfaces $\mathcal{M}^n_g$ which is defined as $T^n_g/\text{Mod}^n_g$. Throughout this paper we also denote by $\mathcal{M}^n_g$ the moduli space endowed with the Weil-Petersson metric and write $\mathcal{M}_g = \mathcal{M}^0_g$ for simplicity. One may refer to [16] for recent developments on Weil-Petersson geometry.
2.2. Weil-Petersson curvatures. The Weil-Petersson metric is Kähler. The curvature tensor of the Weil-Petersson metric is given as follows. Let $\mu_i, \mu_j$ be two elements in the tangent space $T_X \mathcal{M}_g^n$ at $X$, so that the metric tensor written in local coordinates is

$$ g_{ij} = \int_X \mu_i \cdot \overline{\mu_j} dA. $$

For the inverse of $(g_{ij})$, we use the convention

$$ g^{ij} g_{kj} = \delta_{ik}. $$

Then the curvature tensor is given by

$$ R_{ijkl} = \frac{\partial^2}{\partial t^k \partial t^l} g_{ij} - g^{st} \frac{\partial}{\partial t^k} g_{it} \frac{\partial}{\partial t^l} g_{sj}. $$

We now describe the curvature formula of Tromba [11] and Wolpert [14] which gives the curvature in terms of the Beltrami-Laplace operator $\Delta$. It has been applied to study various curvature properties of the Weil-Petersson metric. Tromba [11] and Wolpert [14] showed that $\mathcal{M}_g^n$ has negative sectional curvature. In [9] Schumacher showed that $\mathcal{M}_g^n$ has strongly negative curvature in the sense of Siu. Liu-Sun-Yau in [7] showed that $\mathcal{M}_g^n$ has dual Nakano negative curvature, which says that the complex curvature operator on the dual tangent bundle is positive in some sense. The third named author in [18] showed that the $\mathcal{M}_g^n$ has non-positive definite Riemannian curvature operator. One can also see [4, 5, 10, 11, 17, 19] for other aspects of the curvature of $\mathcal{M}_g^n$.

Set $D = -2(\Delta - 2)^{-1}$ where $\Delta$ is the Beltrami-Laplace operator on $X = (S, \sigma |dz|^2) \in \mathcal{M}_g^n$. The operator $D$ is positive and self-adjoint.

**Theorem 2.1** (Tromba [11], Wolpert [14]). The curvature tensor satisfies

$$ R_{ijkl} = \int_X D(\mu_i \mu_j) \cdot (\mu_k \mu_l) dA + \int_X D(\mu_i \mu_l) \cdot (\mu_k \mu_j) dA. $$

2.2.1. Weil-Petersson holomorphic sectional curvatures. Recall that a holomorphic sectional curvature is a sectional curvature along a holomorphic line. Let $\mu \in T_X \mathcal{M}_g^n$ be a harmonic Beltrami differential. By Theorem 2.1 the holomorphic sectional curvature $\text{HolK}^{\text{WP}}(\mu)$ along the holomorphic line spanned by $\mu$ is

$$ \text{HolK}^{\text{WP}}(\mu) = \frac{-2 \cdot \int_X D(|\mu|^2) \cdot (|\mu|^2) dA}{||\mu||^4_{WP}}. $$

Assume that $||\mu||_{WP} = 1$. From [13 Proposition 2.7], which relies on an estimation of Wolf in [12], we know that

$$ -2 \int_X |\mu|^4 dA \leq \text{HolK}^{\text{WP}}(\mu) \leq -\frac{2}{3} \int_X |\mu|^4 dA. $$
2.2.3. \textit{Weil-Petersson Ricci curvatures.} Let $\{\mu_i\}_{i=1}^{3g-3+n}$ be a holomorphic orthonormal basis of $T_{X,\mathcal{M}_g^n}$. Then the Ricci curvature $\text{Ric}^{WP}(\mu_i)$ at $X$ in the direction $\mu_i$ is given by

$$\text{Ric}^{WP}(\mu_i) = -\sum_{j=1}^{3g-3+n} R_{i\mu_j\mu_j},$$

$$= -\sum_{j=1}^{3g-3+n} \left( \int_X D(\mu_i \mu_j^\ast) \cdot (\mu_j \mu_j^\ast) dA + \int_X D(|\mu_i|^2) \cdot (|\mu_j|^2) dA \right).$$

Since $\int_X D(f) \cdot f dA \geq 0$ for any function $f$ on $X$, by applying the argument in the proof of \eqref{2.2} we have

$$-2 \leq \frac{\text{Ric}^{WP}(\mu_i)}{\sum_{j=1}^{3g-3+n} \int_X D(|\mu_i|^2) \cdot (|\mu_j|^2) dA} \leq -1.$$
It is known from [13, Proposition 2.5] that $-\text{Sca}_{WP}(X)$ is uniformly comparable to the quantity $||\sum_{i=1}^{3g-2n+3} |\mu_i|^2||^2_{WP}$. More precisely,

$$(2.5) \quad -2 \int_X \left( \sum_{i=1}^{3g-2n+3} |\mu_i|^2 \right)^2 dA \leq \text{Sca}_{WP}(X) \leq -\frac{1}{3} \int_X \left( \sum_{i=1}^{3g-2n+3} |\mu_i|^2 \right)^2 dA.$$ 

3. Bounding the pointwise norm by the $L^2$ norm

In this section we will bound the pointwise norm of a harmonic Beltrami differential $\mu = \bar{\phi}/\sigma$ in terms of its Weil-Petersson norm and the injectivity radius function. Our results will improve on prior work of Teo [10] and Wolpert [17], giving the optimal asymptotics of Wolpert with its uniformity of Teo. As in Wolpert [17, Proposition 7], our approach will be to first decompose $\phi$ in the thin part of the surface into the leading and non-leading parts of its Laurent expansion. Then by a detailed analysis, we describe the leading term and give an explicit exponentially decaying upper bound on the non-leading term.

Given $X \in M^g_n$ a hyperbolic surface of finite volume, for $z \in X$ we will let $r(z) = \text{inj}(z)$ be the injectivity radius at $z$. We will refer several times to the a function $C(r)$ introduced by Teo in [10] which is given by

$$(3.1) \quad C(r) = \left( \frac{4\pi}{3} \left( 1 - \text{sech}^6 \left( \frac{r}{2} \right) \right) \right)^{-\frac{1}{2}}$$

$$= \left( \frac{4\pi}{3} \left( 1 - \left( \frac{4e^r}{(1+e^r)^2} \right)^3 \right) \right)^{-\frac{1}{2}}.$$ 

It follows that $C(r)$ is decreasing with respect to $r$ and as $r$ tends to zero we have

$$C(r) = \frac{1}{\sqrt{\pi r}} + O(1).$$ 

Furthermore $C(r)$ tends to $\sqrt{\frac{2}{4\pi}}$ as $r$ tends to infinity.

Let $X = (\mathcal{S}^n_g, \sigma(z)|dz|^2) \in \mathcal{M}_g^n$ and $\phi \in Q(X)$ where $Q(X)$ is the space of holomorphic quadratic differentials on $X$. We set

$$(3.2) \quad ||\phi(z)|| := \frac{||\phi(z)||}{\sigma(z)} \quad \text{for all } z \in X,$$

and

$$(3.3) \quad ||\phi||_2 := \left( \int_X ||\phi(z)||^2 \cdot \sigma(z)|dz|^2 \right)^{\frac{1}{2}}.$$ 

We have the following result of Teo.

Lemma 3.1. (Teo, [10, Proposition 3.1]) Let $\phi \in Q(X)$ be a holomorphic quadratic differential on a hyperbolic surface $X \in \mathcal{M}_g^n$, and $r : X \rightarrow \mathbb{R}_+$ be
the injectivity radius function. Then
\[ \|\phi(z)\| \leq C(r(z)) \cdot \|\phi\|_2 = \frac{\|\phi\|_2}{\sqrt{\pi} \cdot r(z)} (1 + o(r(z))) \]
where the constant \( C(\cdot) \) is given by (3.1).

In [17], Wolpert gave the following asymptotically optimal bound.

**Lemma 3.2.** (Wolpert, [17, Corollary 11]) Let \( S \) be a surface of genus \( g \) with \( n \) punctures, and \( X \in \mathcal{M}_g^n \) be any hyperbolic surface. Then for any \( \epsilon > 0 \) there exists a \( \delta(\epsilon, S) > 0 \) such that if \( \ell_{\text{sys}}(X) \leq \delta(\epsilon, S) \) then for any \( \phi \in Q(X) \) and \( z \in X \)
\[ \|\phi(z)\| \leq (1 + \epsilon) \sqrt{\frac{2}{\pi}} \|\phi\|_2 \sqrt{\ell_{\text{sys}}(X)}. \]

We will now derive a uniform bound that gives the asymptotics of Wolpert’s bound above.

### 3.1. Collar Neighborhoods

We let \( \phi \in Q(X) \) be a holomorphic quadratic differential on a Riemann surface \( X \in \mathcal{M}_g^n \) and \( \gamma \) be a simple closed geodesic of length \( L \) in \( X \). We lift \( \phi \) to \( \tilde{\phi} \) on the annulus \( A = \{ z | e^{-\frac{\pi^2}{2}} < |z| < e^{\frac{\pi^2}{2}} \} \). Then \( \tilde{\phi}(z) = \frac{f(z)}{z^2} dz^2 \) where \( f \) is holomorphic on \( A \). Therefore we have the Laurent series
\[ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n. \]

We define
\[ f_-(z) = \sum_{n<0} a_n z^n, \quad f_0(z) = a_0, \quad f_+(z) = \sum_{n>0} a_n z^n. \]

We therefore have the decomposition
\[ \tilde{\phi}(z) = (f_-(z) + f_0(z) + f_+(z)) \frac{dz^2}{z^2} = \phi_-(z) + \phi_0(z) + \phi_+(z) \]

Let \( \gamma \subset X \in \mathcal{M}_g^n \) be a closed geodesic of length \( L \leq 2\epsilon_2 \). By the Collar lemma (see [3, Chapter 4]) there is an embedded collar \( C_\gamma \) of \( \gamma \) in \( X \) as follows.
\[ (3.4) \quad C_\gamma := \{ z \in X | d(z, \gamma) \leq \arcsinh\left( \frac{1}{\sinh\left( \frac{L}{2} \right)} \right) \}. \]

We set
\[ (3.5) \quad \|\phi|_{C_\gamma}\|_2 := \left( \int_{C_\gamma} \|\phi(z)\|^2 \cdot \sigma(z) |dz|^2 \right)^{\frac{1}{2}}. \]
As $C_\gamma$ embeds in $A$, we have that the injectivity radius function $r$ on $A$ coincides with the injectivity radius function on $C_\gamma \subseteq X$. Also if $z \in A$ has distance $d(z, \gamma)$ from the core closed geodesic then
\[
\sinh(r(z)) = \sinh(L/2) \cosh(d(z, \gamma))
\]
Therefore it follows that
\[
C_\gamma = \{ z \in A \mid r(z) \leq \sinh^{-1}(\cosh(L/2)) \}.
\]
For $0 < t \leq \sinh^{-1}(\cosh(L/2))$ we then define
\[
C_t = \{ z \in A \mid r(z) \leq t \}.
\]
In part of the following Proposition we will need to restrict to a sub-collar of the standard collar $C_\gamma$. For this we define the constant
\[
\tau_2 = \log(3) = \sinh^{-1} \left( \frac{1}{\sqrt{3}} \right).
\]
We prove the following

**Proposition 3.3.** Let $\phi \in Q(X)$ and $C_\gamma$ be the collar about a closed geodesic $\gamma$ of length $L \leq 2\epsilon_2$. Then

1. For any $z \in C_\gamma$
   \[
   ||\phi_0(z)|| \leq \frac{1}{\sqrt{Lc_0(L)}} \frac{\sinh^2(L/2)}{\sinh^2(r(z))} ||\phi|_{C_\gamma}||_2
   \]
   where
   \[
c_0(L) = \cos^{-1}(\tanh(L/2)) + \frac{1}{2} \sin \left( 2 \cos^{-1}(\tanh(L/2)) \right) = \frac{\pi}{2} - \frac{L^3}{12} + O(L^5).
   \]
2. On $C_t$, $||\phi_\pm(z)||$ attains its maximum on $\partial C_t$.
3. For $z \in C_\gamma$ in the sub-collar $C_{\tau_2} = \{ z \in A \mid r(z) \leq \tau_2 \}$
   \[
   ||\phi_\pm(z)|| \leq F(r(z)) ||\phi|_{C_\gamma}||_2
   \]
   where
   \[
   F(r(z)) = \frac{e^{\pi \sqrt{3} C(\tau_2)}}{3 \sinh^2(r(z))} \leq C(\tau_2).
   \]
4. For $z \in C_\gamma$ in the sub-collar $C_{\tau_2}$ with $r(z) \leq \tau_2$
   \[
   ||\phi(z)|| \leq G(r(z)) ||\phi|_{C_\gamma}||_2
   \]
   where
   \[
   G(r) = \frac{1}{\sqrt{2rc_0(2r)}} + \frac{2e^{\pi \sqrt{3} C(\tau_2)}e^{-\sinh(r(z))}}{3 \sinh^2(r)} = \frac{1}{\sqrt{r}} \left( 1 + \frac{2r^3}{3\pi} + O(r^5) \right).
   \]
5. For $z \in C_\gamma$ with $r(z) \leq \epsilon_2$ then
   \[
   ||\phi(z)|| \leq \frac{||\phi||_2}{\sqrt{r(z)}}.
   \]
Proof. Let $S = \{ z = x + iy \mid |y| < \pi/2 \}$ be the strip, then the hyperbolic metric on $S$ is $\rho_S(z) = |dz|/\cos(y)$. By the Collar Lemma [3, Theorem 4.1.6] the injectivity radius function on $S$ satisfies

\begin{equation}
\sinh(r(z)) = \frac{\sinh(L/2)}{\cos(y)}.
\end{equation}

We have the $Z$ cover $\pi : S \to A$ given by $\pi(z) = e^{\frac{2\pi i z}{h}}$. Therefore the hyperbolic metric on $A$ is given by

$$
\rho(z) = \frac{L}{2\pi |z| \cos \left( \frac{L}{2\pi} \log |z| \right)}.
$$

It follows that $C_\gamma$ lifts to the strip $S_\gamma = \{ w = x + iy \mid |y| < h(L) \}$ where

$$
h(L) = \cosh^{-1}(\tanh(L/2)).
$$

Therefore $C_\gamma = \{ z \in A \mid e^{-s(L)} < |z| < e^{s(L)} \}$ where

$$
s(L) = 2\pi \cdot \frac{h(L)}{L}.
$$

We first show that $\phi_-, \phi_0, \phi_+$ are all orthogonal on $C_\gamma$. We have

$$
\|\phi|_{C_\gamma}\|_2^2 = \int_{C_\gamma} |\phi(z)|^2 \rho^2(z) = \sum_{n,m} \int_{e^{-s(L)}}^{e^{s(L)}} \int_0^{2\pi} a_n b_m z^n \bar{z}^m |z|^4 \rho^2(r) r drd\theta.
$$

$$
= \sum_{n,m} \left( \int_{e^{-s(L)}}^{e^{s(L)}} \frac{a_n b_m r^{n+m-3}}{\rho^2(r)} dr \right) \left( \int_0^{2\pi} e^{i(n-m)\theta} d\theta \right)
$$

$$
= 2\pi \sum_n \int_{e^{-s(L)}}^{e^{s(L)}} \frac{|a_n|^2 r^{2n-3}}{\rho^2(r)} dr.
$$

Therefore

\begin{equation}
\|\phi|_{C_\gamma}\|_2^2 = \|\phi_-|_{C_\gamma}\|_2^2 + \|\phi_0|_{C_\gamma}\|_2^2 + \|\phi_+|_{C_\gamma}\|_2^2.
\end{equation}

This gives the bound

$$
\|\phi|_{C_\gamma}\|_2^2 \geq \|\phi_0|_{C_\gamma}\|_2^2 = 2\pi |a_0|^2 \frac{16\pi^4}{L^3} \int_{e^{-s(L)}}^{e^{s(L)}} \cos^2 \left( \frac{L}{2\pi} \log r \right) dr.
$$

We let $t = \frac{L}{2\pi} \log r$ giving $dt = \frac{L}{2\pi} dr$ and

$$
\|\phi_0|_{C_\gamma}\|_2^2 = |a_0|^2 \frac{16\pi^4}{L^3} \int_{-h(L)}^{h(L)} \cos^2(t) dt.
$$

We define

$$
c_0(L) = \int_{-h(L)}^{h(L)} \cos^2(t) dt = h(L) + \frac{1}{2} \sin(2h(L)) = \frac{\pi}{2} - \frac{L^3}{12} + O(L^5).
$$

Then

$$
\|\phi_0|_{C_\gamma}\|_2^2 = \frac{16\pi^4}{L^3} c_0(L).
$$
For $z \in C_\gamma$, we have

$$\|\phi_0(z)\| = \frac{4\pi^2|a_0|}{L^2} \cos^2\left(\frac{L}{2\pi} \log |z|\right) = \frac{1}{\sqrt{Lc_0(L)}} \sinh^2(L/2) \|\phi_0|_{C_\gamma}\|_2$$

where in the last equality we apply the following version of formula (3.6)

$$\cos \left(\frac{L}{2\pi} \log |z|\right) = \frac{\sinh(L/2)}{\sinh(r(z))}.$$ 

Thus

$$\|\phi_0(z)\| \leq \frac{1}{\sqrt{Lc_0(L)}} \sinh^2(L/2) \|\phi|_{C_\gamma}\|_2$$

giving (1).

We consider $\phi_+(z) = f_+(z)dz^2/z^2$. We have that $f_+(z)$ is holomorphic on the disk $D_+ = \{ z \mid |z| < e^{s_2} \}$. Furthermore $f_+(z)/z$ extends holomorphically to $D_+$. By the maximum principle the maximum modulus of $f_+(z)/z$ on $B(s) = \{ z \mid |z| \leq s \}$ is on the boundary. Therefore the maximum modulus of $f_+(z)/z$ on $B(s)$ is at some $z_s \in \partial B(s)$ with $M_s = |f(z_s)|/|z_s|$. We have for $z \in B(s)$

$$\|\phi_+(z)\| = \frac{|f_+(z)|}{|z|^2} \cdot \frac{4\pi^2}{L^2} |z|^2 \cos^2\left(\frac{L}{2\pi} \log |z|\right) \leq M_s \frac{4\pi^2}{L^2} |z| \cos^2\left(\frac{L}{2\pi} \log |z|\right).$$

Recall that

$$\|\phi_+(z_s)\| = M_s \frac{4\pi^2}{L^2} s \cos^2\left(\frac{L}{2\pi} \log s\right).$$

Therefore

$$\|\phi_+(z_s)\| \leq \frac{\|\phi_+(z_s)\|}{s \cos^2\left(\frac{L}{2\pi} \log s\right)} \left(|z| \cos^2\left(\frac{L}{2\pi} \log |z|\right)\right).$$

We observe that $x \cos^2\left(\frac{L}{2\pi} \log x\right)$ is monotonically increasing on $[1, e^{s(L)}]$. To see this, we consider equivalently the function $u(t) = e^{2\pi t/L} \cos^2(t)$ on $[-h(L), h(L)]$. Differentiating it we get

$$u'(t) = 2e^{2\pi t/L} \cos(t) \left(\frac{\pi}{L} \cos(t) - \sin(t)\right).$$

Thus $u$ is monotonic for $\tan(t) \leq \frac{\pi}{2}$. As $t \leq h(L) = \cos^{-1}(\tanh(L/2))$ we have

$$\tan(t) \leq \tan(h(L)) = \frac{1}{\sinh(L/2)} \leq \frac{2}{L} \leq \frac{\pi}{L}.$$

Thus $u$ is monotonic on $[1, \frac{L}{2\pi} \cdot s(L)]$. Therefore $\|\phi_+(z)\|$ has maximum modulus in $C_t$ on the boundary. Similarly one may prove that $\|\phi_-(z)\|$ has maximum modulus in $C_t$ on the boundary by using $\frac{1}{z}$ as a variable. This proves (2).

To prove (3) we use Teo’s bound from Lemma [3.1]. By Teo

$$\|\phi_+(z_s)\| \leq C(r(z_s)) \cdot \|\phi_+|_{B(z_s,r(z_s))}\|_2$$
where \( B(z, r) \) is the hyperbolic ball about \( z \) of radius \( r \). We choose \( z_a \) in the collar such that \( B(z_a, r(z_a)) \subseteq \mathcal{C}_\gamma \). By the Collar Lemma \( [3, \text{Theorem 4.1.6}] \), a point of injectivity radius \( r \) is a distance \( d \) from the boundary of the collar where

\[
\sinh(r) = \cosh\left(\frac{L}{2}\right) \cos d - \sinh d.
\]

We note that solving \( d = r \) gives

\[
r = \tanh^{-1}\left(\frac{\cosh\left(\frac{L}{2}\right)}{2}\right) \geq \tanh^{-1}(1/2).
\]

Therefore we choose \( z_a \) such that \( r(z_a) = \tanh^{-1}(1/2) = \tau_2 \). Then by Lemma \( 3.1 \) and \( 3.7 \)

\[
\|\phi_+(z_a)\| \leq C(\tau_2) \cdot \|\phi_+|c\|_2 \leq C(\tau_2) \cdot \|\phi|c\|_2.
\]

This together with \( 3.9 \) implies that

\[
\|\phi_+(z)\| \leq \frac{C(\tau_2)}{s \cos^2\left(\frac{L}{2\pi} \log s\right)} \cdot \|\phi|c\|_2 \left( |z| \cos^2\left(\frac{L}{2\pi} \log |z|\right) \right).
\]

Recall that \( 3.6 \) gives

\[
\cos\left(\frac{L}{2\pi} \log |z|\right) = \frac{\sinh(L/2)}{\sinh(r(z))}.
\]

Therefore

\[
|z| = e^{\pm \frac{2\pi}{L} \left( \cos^{-1}\left(\frac{\sinh(L/2)}{\sinh(r(z))}\right) \right)}
\]

where the sign depends on which side of the core closed geodesic you are on. We rewrite the bound in terms of injectivity radius. Recall that \( s > 1 \).

Then for \( |z| \geq 1 \), i.e., \( |z| = e^{\frac{2\pi}{L} \left( \cos^{-1}\left(\frac{\sinh(L/2)}{\sinh(r(z))}\right) \right)} \),

\[
\|\phi_+(z)\| \leq \frac{C(\tau_2) \sinh^2(\tau_2)e^{\frac{2\pi}{L} \left( \cos^{-1}\left(\frac{\sinh(L/2)}{\sinh(r(z))}\right) - \cos^{-1}\left(\frac{\sinh(L/2)}{\sinh(\tau_2)}\right) \right)}}{\sinh^2(r(z))} \cdot \|\phi|c\|_2.
\]

Note that \( \sinh(\tau_2) = 1/\sqrt{3} \). Also for \( 0 < x < y \leq \pi \) then \( x - y \leq \cos(y) - \cos(x) \) giving

\[
\|\phi_+(z)\| \leq \frac{C(\tau_2)e^{\frac{2\pi}{L} \left( \frac{1}{\sinh(r(z))} - \sqrt{3} \right)}}{3 \sinh^2(r(z))} \cdot \|\phi|c\|_2.
\]

As \( \sinh(x) \geq x \) we have for \( |z| \geq 1 \),

\[
(3.10) \|\phi_+(z)\| \leq \frac{C(\tau_2)e^{-\pi \left( \frac{1}{\sinh(r(z))} - \sqrt{3} \right)}}{3 \sinh^2(r(z))} \cdot \|\phi|c\|_2 = F(r(z)) \cdot \|\phi|c\|_2.
\]

We note that \( r(z) = r(1/z) \). Also by the above, the maximum of \( \|\phi_+(z)\| \) on \( \{ z \mid 1/c \leq |z| \leq c \} \) is on the boundary \( |z| = c \) where \( 1 < c \leq e^{a(L)} \).

Therefore for \( |z| = 1/c < 1 \) we have

\[
\|\phi_+(z)\| \leq \max_{|w|=1/c} \|\phi_+(w)\| \leq \max_{|w|=c} \|\phi_+(w)\| \leq F(r(z)) \cdot \|\phi|c\|_2.
\]
Thus for \( r(z) \leq \tau_2 \)

\[(3.11) \quad ||\phi_+(z)|| \leq \left( \frac{C(\tau_2)e^{\pi \sqrt{3}}e^{-\frac{\pi}{\sinh^2(r(z))}}}{3 \sinh^2(r(z))} \right) ||\phi||_{C_2} \leq C(\tau_2)||\phi||_{C_2} \]

where in the last inequality we apply that \( e^{-\frac{\pi}{\sinh^2(r(z))}} \) is increasing. Similar as in the proof of Part (2) if we consider \( \frac{1}{r} \) as a variable, one may also get the same bound for \( ||\phi_-(z)|| \). This proves (3).

For proving (4), we combine the bounds above using

\[ ||\phi(z)|| \leq ||\phi_-(z)|| + ||\phi_0(z)|| + ||\phi_+(z)||. \]

First observe that both \( \frac{\sinh(L/2)}{\sqrt{L}} \) and \( \frac{\sinh(L/2)}{\sqrt{c_0(L)}} \) are increasing. Since \( 2r(z) \geq L \), for any \( z \in C_\gamma \) we have

\[ ||\phi_0(z)|| \leq \frac{1}{\sqrt{2r(z)c_0(2r(z))}} ||\phi||_{C_\gamma}. \]

Therefore for \( z \in C_\gamma \) with \( r(z) \leq \tau_2 \),

\[(3.12) \quad ||\phi(z)|| \leq G(r(z))||\phi||_{C_\gamma} \]

where

\[ G(r) = \frac{1}{\sqrt{2rc_0(2r)}} + \frac{2e^{\pi \sqrt{3}}C(\tau_2)e^{-\frac{\pi}{\sinh^2(r)}}}{3 \sinh^2(r)}. \]

This proves (4).

To prove (5) we combine the above bound for \( r(z) \leq \tau_2 \) with Teo’s bound for \( r(z) \leq \epsilon_2 \). If \( r(z) \geq \tau_2 \) by Lemma 3.1 we have that

\[ ||\phi(z)|| \leq C(r(z)) \cdot ||\phi||_2 \leq \sqrt{r(z)}C(r(z)) \cdot ||\phi||_2 \frac{1}{\sqrt{r(z)}}. \]

As \( C(x)\sqrt{x} \) is monotonically decreasing with \( C(\tau_2)\sqrt{\tau_2} = .8091 \) we have

\[ ||\phi(z)|| \leq \sqrt{\tau_2}C(\tau_2) \cdot ||\phi||_2 \frac{1}{\sqrt{r(z)}} = .8091 \frac{||\phi||_2}{\sqrt{r(z)}}. \]

We now consider \( r(z) \leq \tau_2 \). We have that \( H(r) = G(r) \cdot \sqrt{r} \) is monotonically decreasing. Therefore Part (4) above together with Lemma 3.1 imply that

\[ ||\phi(z)|| \leq \min \left( H(r(z)), \sqrt{r(z)}C(r(z)) \right) \cdot \frac{||\phi||_2}{\sqrt{r(z)}}. \]

Considering \( m(r) = \min(H(r), \sqrt{r}C(r)) \) on \( (0, \tau_2] \) we have by computation that \( m(r) \leq m_0 = .9137 \) (see figure 1).

Therefore for \( r(z) \leq \epsilon_2 \)

\[(3.13) \quad ||\phi(z)|| \leq \max\{ .8091, .9137 \} \cdot \frac{||\phi||_2}{\sqrt{r(z)}} \leq \frac{||\phi||_2}{\sqrt{r(z)}} \]

which completes the proof. \( \Box \)
3.2. Cusp neighborhoods. We now consider the cusp neighborhoods of \( X \in \mathcal{M}_g^n \). Then each cusp \( c \) gives a cover \( \pi : \Delta^* \to X \) where \( \Delta^* = \{ z \mid 0 < |z| < 1 \} \). The hyperbolic metric on \( \Delta^* \) is \( \rho(z) = -1/|z| \log |z| \).

By the Collar Lemma (see [3, Chapter 4]), \( c \) has a collar \( C_c \) which lifts to \( A_c := \{ z \mid 0 < |z| < e^{-\pi} \} \) with \( \pi \) injective on \( A_c \). Furthermore as \( C_c \) is embedded, the injectivity radius function \( r \) on \( X \) lifts to the injectivity radius function on \( A_c \) with \( A_c := \{ z \in A \mid r(z) < \epsilon_2 \} \). We have

**Lemma 3.4.** Let \( X \in \mathcal{M}_g^m \) and \( \phi \in Q(X) \). If \( z \in C_c \), then

\[
\|\phi(z)\| \leq K(r(z))\|\phi\|_2 \leq C(\epsilon_2)\|\phi\|_2
\]

where

\[
K(r) = \left( \frac{C(\epsilon_2)e^{\pi}e^{-\frac{\pi}{\sinh(r)}}}{\sinh^2(r)} \right)
\]

and \( C(\epsilon_2) = .7439 \).

**Proof.** As before we have \( \phi = \phi_- + \phi_0 + \phi_+ \). We have the hyperbolic metric on \( A_c \) is \( \rho(z) = -1/|z| \log |z| \). The lemma is trivially true if \( \|\phi\|_2 = \infty \). Therefore we consider \( \|\phi\|_2 < \infty \). It follows that \( \phi_0 = \phi_- = 0 \). We now bound \( \|\phi(z)\| \) as above. If \( \phi(z) = f(z)dz^2/z^2 \) then \( f(z)/z \) extends to \( B(s) = \{ z \mid |z| < s \} \) and has maximum modulus at \( z_s \) with \( |z_s| = s \). Therefore

\[
\|\phi(z)\| \leq \|\phi(z_s)\| / s(\log |s|)^2 \cdot |z|(|z|)^2.
\]

It can easily be checked that \( |z|(|z|)^2 \) is monotonic on \( A_c \). By the Collar Lemma, the the injectivity radius on \( A_c \) satisfies \( \sinh(r(z)) = -\pi/\log |z| \). Therefore by letting \( s = e^{-\pi} \) (the maximal cusp) and using Lemma 3.1 we obtain that for \( r(z) \leq \epsilon_2 \),

\[
\|\phi(z)\| \leq \left( \frac{C(\epsilon_2)e^{\pi}e^{-\frac{\pi}{\sinh(r(z))}}}{\sinh^2(r(z))} \right) \|\phi\|_2 = K(r(z))\|\phi\|_2.
\]
The function \( e^{-\sinh(r)} \sinh^2(r) \) is monotonically increasing on \([0, \epsilon_2]\). Recall that \( \sinh(\epsilon_2) = 1 \). So we have
\[
||\phi(z)|| \leq C(\epsilon_2)||\phi||_2.
\]
Which completes the proof. \( \square \)

### 3.3 Uniform upper bounds for \( ||\phi|| \)

In this subsection we discuss several applications of Proposition 3.3 and Lemma 3.4. The first one is to show Proposition 1.1.

**Proof of Proposition 1.1.** Let \( z \in X \) with \( \text{inj}(z) \leq \epsilon_2 \). Then \( z \) is in either a collar or a cusp. If \( z \) is in a collar, the claim follows by Part (5) of Proposition 3.3. If \( z \) is in a cusp, the claim follows by Lemma 3.4. \( \square \)

We define \( \ell_{\text{sys}}^+(X) = \min(2\epsilon_2, \ell_{\text{sys}}(X)) \). Then we have

**Corollary 3.5.** Let \( X \in \mathcal{M}_g^n \) and \( \phi \in Q(X) \). Then
\[
||\phi||_\infty \leq \sqrt{\frac{2}{\ell_{\text{sys}}^+(X)}} ||\phi||_2.
\]

**Proof.** If \( r(z) \geq \epsilon_2 \) or \( z \) is in a cusp neighborhood then as \( \ell_{\text{sys}}^+(X) \leq 2\epsilon_2 \), it follows by Lemma 3.1 and Lemma 3.4 that
\[
||\phi(z)|| \leq C(\epsilon_2)||\phi||_2 \leq \sqrt{2\epsilon_2} C(\epsilon_2) \frac{||\phi||_2}{\ell_{\text{sys}}^+(X)}.
\]
We have \( \sqrt{2\epsilon_2} C(\epsilon_2) = .9877 < \sqrt{2} \). So the claim follows for these two cases.

If \( z \) is in a collar neighborhood with \( r(z) \leq \epsilon_2 \), it follows by (3.13) that
\[
(3.14) \quad ||\phi(z)|| \leq m_0 \frac{||\phi||_2}{\sqrt{r(z)}} \leq \sqrt{2} \cdot m_0 \cdot \frac{||\phi||_2}{\ell_{\text{sys}}(X)}.
\]
The claim also follows as \( m_0 < .9137 \). \( \square \)

**Remark 3.6.** We note that we can use Proposition 3.3 to give a bound for Wolpert’s Lemma 3.7 which is independent of topology. We let \( H(r) = G(r) \cdot \sqrt{r} \). Then \( H(r) \) is monotonically increasing with
\[
\lim_{r \to 0} H(r) = \frac{1}{\sqrt{\pi}}.
\]
We note for from Part (4) of Proposition 3.3 that for \( r(z) \leq \epsilon_2 \)
\[
||\phi(z)|| \leq \frac{||\phi||_2}{\sqrt{r(z)}}.
\]
Thus for \( \pi/2 \cdot \ell_{\text{sys}}(X) \leq r(z) \leq \epsilon_2 \) we have
\[
(3.15) \quad ||\phi(z)|| \leq \frac{||\phi||_2}{\sqrt{r(z)}} \leq \sqrt{\frac{2}{\pi}} \cdot \frac{||\phi||_2}{\ell_{\text{sys}}(X)}.
\]
We choose \( \delta_1 \) such
\[
\delta_1 = \frac{2}{\pi} H^{-1} \left( \frac{1 + \epsilon}{\sqrt{\pi}} \right).
\]
Then it follows by Part (4) of Proposition 3.3 that for \( \ell_{\text{sys}}(X) < \delta_1 \) and
\[ r(z) \leq \min\{ \frac{\pi}{2} \cdot \ell_{\text{sys}}(X), \tau_2 \} \leq \min\{ H^{-1} \left( \frac{1 + \epsilon}{\sqrt{\pi}} \right), \tau_2 \} \]
(3.16)
\[
\| \phi(z) \| \leq (1 + \epsilon) \sqrt{\frac{2}{\pi}} \frac{\| \phi \|}{\sqrt{\ell_{\text{sys}}(X)}}.
\]
Now for \( r(z) \geq \tau_2 \) as \( C(\tau_2) = 1.09 < 2 \)
\[
\| \phi(z) \| \leq C(\tau_2) \| \phi \| \leq 2 \| \phi \|.
\]
Thus for \( \ell_{\text{sys}}(X) < \frac{1}{2\pi} \) and \( r(z) \geq \tau_2 \) we have
(3.17)
\[
\| \phi(z) \| \leq \sqrt{\frac{2}{\pi}} \frac{\| \phi \|}{\sqrt{\ell_{\text{sys}}(X)}}.
\]
We therefore choose \( \delta = \min(\delta_1, \frac{1}{2\pi}) \) to get the following result.

**Theorem 3.7.** Let \( X \in \mathcal{M}^n_g \) be any hyperbolic surface. Then for any \( \epsilon > 0 \) there exists a constant \( \delta(\epsilon) > 0 \) only depending on \( \epsilon \) such that if \( \ell_{\text{sys}}(X) \leq \delta(\epsilon) \) then for any \( \phi \in Q(X) \) and \( z \in X \),
\[
\| \phi(z) \| \leq (1 + \epsilon) \sqrt{\frac{2}{\pi}} \frac{\| \phi \|}{\sqrt{\ell_{\text{sys}}(X)}}.
\]

We note by the expansion of \( G \) we have for \( \epsilon \) small,
\[
\delta(\epsilon) = \frac{2}{\pi} H^{-1} \left( \frac{1 + \epsilon}{\sqrt{\pi}} \right) \approx \left( \frac{12\epsilon}{\pi^2} \right)^{1/3}.
\]

### 3.4. Fixing the length of short curves

Let \( X \in \mathcal{M}^n_g \) and for \( \alpha \) a closed curve, we let \( l_\alpha \) be the geodesic length function on \( \mathcal{M}^n_g \). Then we let \( dL_\alpha \in T^*(\mathcal{M}^n_g) \) be the complex one-form such that \( \operatorname{Re} dL_\alpha = dl_\alpha \). We define
(3.18)
\[
P(X) \subseteq T^*_X(\mathcal{M}_g) = \operatorname{span}\{(dL_\alpha)_X \mid l_\alpha(X) \leq \epsilon_2\}
\]
and
(3.19)
\[
P(X)^\perp = \{ \mu \mid \langle \phi, \mu \rangle = 0, \forall \phi \in P(X) \} \subseteq T_X(\mathcal{M}^n_g).
\]
The plane \( P(X)^\perp \) is the set of directions that fix the length of short curves. We have the following immediate consequence of Proposition 3.3.

**Lemma 3.8.** Let \( \mu \in P(X)^\perp \) then
\[
\| \mu(z) \| \leq \sqrt{2} \cdot \| \mu \|.
\]
Furthermore for \( r(z) \leq \tau_2 \)
\[
\| \mu(z) \| \leq 2 \cdot F(r(z)) \cdot \| \mu \|.
\]
Where \( F(r(z)) \) is defined in Proposition 3.3.
Proof. Let \( \mu = \frac{\phi}{\rho^2} \in P(X)^\perp \). Recall that \( C(\tau_2) = 1.0917 \). If \( r(z) \geq \tau_2 \), then by Lemma 3.1
\[
\|\mu(z)\| \leq C(\tau_2) \cdot \|\mu\|_2 \leq \sqrt{2} \|\mu\|_2.
\]
Similarly if \( z \) is in a cusp neighborhood, then
\[
\|\mu(z)\| \leq K(r(z)) \leq C(\epsilon_2) \cdot \|\mu\|_2 \leq C(\tau_2) \|\mu\|_2.
\]
Now we consider the remaining case. That is, \( r(z) \leq \epsilon_2 \) and \( z \in C_\alpha \) where \( \alpha \subset X \) is a closed geodesic with \( l_\alpha(X) \leq 2\tau_2 \). We lift \( \phi \) to \( \hat{\phi} \) on the annulus \( A \) and have as before \( \hat{\phi}(z) = \phi_- + \phi_0 + \phi_+ \) with \( \phi_0(z) = a \frac{dz^2}{z^2} \) for \( a \in \mathbb{C} \). By the Gardiner formula [6] we have
\[
0 = \langle dL_\alpha, \mu \rangle = \frac{2}{\pi} \int_A \frac{\hat{\phi}(z)}{\rho(z)^2} \frac{dz^2}{z^2}
= \frac{2}{\pi} \int_A \frac{\phi_0(z)}{\rho(z)^2} \frac{dz^2}{z^2} = \frac{2a}{\pi} \int_A \frac{dx dy}{r^4 \rho^2(r)} = a l_\alpha(X).
\]
Therefore \( a = 0 \) and \( \phi_0 = 0 \) (see also [12, Proposition 8.5]). Then it follows from Part (3) of Proposition 3.3 that
\[
\|\phi(z)\| \leq \|\phi_-(z)\| + \|\phi_+(z)\| \leq 2 \cdot F(r(z)) \|\phi\|_2
\]
where
\[
F(r) = \frac{e^{\pi \sqrt{3}} C(\tau_2) e^{-\frac{\pi}{3 \sinh(r)}}}{3 \sinh^2(r)}.
\]
Together with Lemma 3.1 by letting \( m'(r) = \min(2F(r), C(r)) \) we have
\[
\|\phi(z)\| \leq m'(r(z)) \|\phi\|_2.
\]
On \((0, \tau_2]\) by computation we have \( m'(r) \leq 1.2333 \) (see figure 2).

\[\text{Figure 2. Plot of } 2F(r) \text{ and } C(r) \text{ on } (0, \tau_2]\]

Therefore
\[
\|\phi(z)\| \leq \sqrt{2} \cdot \|\phi\|_2
\]
and proving the first inequality.
We note that $K(r) \leq 2F(r)$ on $(0, \tau_2]$ where $K(r)$ is defined in Lemma 3.4. Then it follows by Lemma 3.4 and (3.22) for all $r(z) \leq \tau_2$

$$\|\mu(z)\| \leq 2 \cdot F(r(z)) \cdot \|\mu\|_2$$

which completes the proof. \hfill \Box

4. Uniform lower bounds for Weil-Petersson curvatures

The following bounds is essentially due to Teo [10]. As we need a slightly modified version, we give the following version due to Ken Bromberg.

**Proposition 4.1.** Fix $z \in X$ and let $U \subset T_X\mathcal{M}_g^0$ be a subspace and $K_z > 0$ a constant such that for all harmonic Beltrami differentials $\mu \in U$ we have

$$\|\mu(z)\| \leq K_z \|\mu\|_2.$$

Then if $\mu_1, \ldots, \mu_k$ is an orthonormal family in $U$ we have

$$\sum_{i=1}^k \|\mu_i(z)\|^2 \leq K_z^2.$$

**Proof.** Pick constants $c_1, \ldots, c_k$ such that $|c_i| = \|\mu_i(z)\|$ and the directions of maximal and minimal stretch of the Beltrami differentials $c_i\mu_i$ all agree at $z$.\footnote{For example if we choose a chart near $z$, in the chart the $\mu_i$ are realized by functions and we can let $c_i = \mu_i(z)$. Then, in this chart, the directions of maximal and minimal stretch at $z$ of each $c_i\mu_i$ are the real and imaginary axis.} We then let

$$\mu_z = \sum_{i=1}^k c_i\mu_i$$

and observe that our conditions on the directions of maximal and minimal stretch give that

$$\|\mu_z(z)\| = \sum_{i=1}^k |c_i\mu_i(z)| = \sum_{i=1}^k \|\mu_i(z)\|^2.$$

As the $\mu_i$ are orthonormal we also have

$$\|\mu_z\|^2 = \sum_{i=1}^k |c_i|^2 = \sum_{i=1}^k \|\mu_i(z)\|^2.$$

As $\mu_z$ is a linear combination of harmonic Beltrami differentials it is also a harmonic Beltrami differential so

$$\|\mu_z(z)\| \leq K_z \|\mu_z\|$$

and therefore

$$\|\mu_z(z)\|^2 \leq K_z^2 \|\mu_z\|^2 = K_z^2 \|\mu_z(z)\|.$$  
Dividing by $\|\mu_z(z)\| = \sum_{i=1}^k \|\mu_i(z)\|^2$ gives the result. \hfill \Box
In this section we prove Theorem 1.3. Before proving it, we provide a uniform upper bound for any holomorphic orthonormal frame at \( X \in \mathcal{M}_g^n \).

First we make a thick-thin decomposition of \( X \in \mathcal{M}_g^n \) into three pieces as follows. Let \( \epsilon \) be the Margulis constant as in previous sections. We set

\[
X_1 := \{ q \in X : \text{inj}(q) \geq \epsilon \}, \\
X_2 := \{ q \in \text{cusps} : \text{inj}(q) < \epsilon \}, \\
X_3 := \{ q \in \text{collars} : \text{inj}(q) < \epsilon \}.
\]

So \( X = \bigcup_{i=1}^{3} X_i \). We note that the set \( X_2 \) and \( X_3 \) may be empty. Actually Buser and Sarnak \[2\] showed that \( \sup_{X \in \mathcal{M}_g^n} \text{inj}(X) \asymp \ln(g) \) for all \( g \geq 2 \).

Let \( \{ \mu_i \}_{i=1}^{3g-3+n} \) be a holomorphic orthonormal basis of \( T_X \mathcal{M}_g^n \). Our aim is to bound

\[
\sum_{i=1}^{3g-3+n} |\mu_i|^2(z) \leq C(\epsilon^2)^2 = .5533.
\]

This bound is an easy application of Lemma 3.1 and Proposition 4.1.

Next we consider the case on \( X_2 \). Recall that Lemma 3.4 says that for any \( x \in X_2 \), \( ||\phi(z)|| \leq C(\epsilon_2)||\phi||_2 \). Therefore it follows by Proposition 4.1 that

\[
\sup_{z \in X_2} \sum_{i=1}^{3g-3+n} |\mu_i|^2(z) \leq C(\epsilon)^2 = .5533
\]

Now we deal with the case on \( X_3 \). Considering \( (3.14) \) we let \( K_0 = 2 \times (.9137)^2 = 1.6697 \). Then by Proposition 4.1 we have

\[
\sup_{z \in X_3} \sum_{i=1}^{3g-3+n} |\mu_i|^2(z) \leq \frac{K_0}{\ell_{\text{sys}}(X)} = \frac{1.6697}{\ell_{\text{sys}}(X)}.
\]

On the thick part of the moduli space \( \mathcal{M}_g^n \), the Weil-Petersson curvature has been well studied in \[5, 10, 13\]. Now we study the Weil-Petersson curvatures on Riemann surfaces with short systoles. Our first result in this section is as follows.

**Theorem 4.2** (\( = \text{Theorem 1.3} \)). For any \( X \in \mathcal{M}_g^n \) with \( \ell_{\text{sys}}(X) \leq 2\epsilon_2 \), then

1. for any \( \mu \in T_X \mathcal{M}_g^n \) with \( ||\mu||_{\text{WP}} = 1 \), the Ricci curvature satisfies

\[
\text{Ric}_{\text{WP}}(\mu) \geq -\frac{4}{\ell_{\text{sys}}(X)}.
\]

2. The scalar curvature at \( X \) satisfies

\[
\text{Sc}_{\text{WP}}(X) \geq -\frac{4}{\ell_{\text{sys}}(X)} \cdot (3g - 3 + n).
\]
Proof. We first show Part (1). Let $\mu \in T_X \mathcal{M}_g^n$ with $||\mu||_{WP} = 1$ and one may choose a holomorphic orthonormal basis $\{\mu_i\}_{i=1}^{3g-3+n}$ of $T_X \mathcal{M}_g^n$ such that $\mu = \mu_1$. Now we split the lower bound in (2.3) into three parts. Since $X_1, X_2$ and $X_3$ are mutually disjoint,

$$\text{Ric}^{WP}(\mu) \geq -2 \sum_{j=1}^{3g-3+n} \int_X D(|\mu|^2) \cdot (|\mu_j|^2) dA$$

$$= -2 \int_{X_1} D(|\mu|^2) \cdot (\sum_{j=1}^{3g-3+n} |\mu_j|^2) dA$$

$$- 2 \int_{X_2} D(|\mu|^2) \cdot (\sum_{j=1}^{3g-3+n} |\mu_j|^2) dA$$

$$- 2 \int_{X_3} D(|\mu|^2) \cdot (\sum_{j=1}^{3g-3+n} |\mu_j|^2) dA.$$

Since $D$ is a positive operator (see [14]), $D(|\mu|^2) \geq 0$. Then it follows by (4.1), (4.2) and (4.3) that

$$\text{Ric}^{WP}(\mu) \geq -3 \cdot 3.3394 \cdot \frac{\ell_{sys}(X)}{\ell_{sys}(X)} \cdot (3g-3+n).$$

where in the last inequality we note that $2K_0 = 4(0.9137^2) = 3.394$ and $C(\epsilon_2) = 0.7438$. Recall that the operator $D$ is self-adjoint and $D(1) = 1$. So

$$\int_X D(|\mu|^2) dA = \int_X |\mu|^2 \cdot D(1) dA = ||\mu||_{WP}^2 = 1.$$ Therefore

$$\text{Ric}^{WP}(\mu) \geq -\frac{3.3394}{\ell_{sys}(X)} \cdot (3g-3+n).$$

(4.4)

Part (2) follows by Part (1) as

$$\text{Sca}^{WP}(X) = \sum_{i=1}^{3g-3+n} \text{Ric}^{WP}(\mu_i) \geq -\frac{4}{\ell_{sys}(X)} \cdot (3g-3+n).$$

(4.5)

The proof is complete. \hfill \Box

Remark 4.3. For $\mathcal{M}_g = \mathcal{M}_g^0$, the lower bound in Part (2) of Theorem 4.2 can be extended to $-\frac{11}{\ell_{sys}(X)} \cdot (g-1)$ because (4.4) implies that

$$\text{Sca}^{WP}(X) = \sum_{i=1}^{3g-3} \text{Ric}^{WP}(\mu_i) \geq -\frac{3 \times 3.3394}{\ell_{sys}(X)} \cdot (g-1)$$

$$\geq -\frac{11}{\ell_{sys}(X)} \cdot (g-1).$$
Since the Weil-Petersson sectional curvature is negative \([11,14]\), we have that for any \(X \in \mathcal{M}_g^n\) and \(\mu, v \in T_X \mathcal{M}_g^n\),
\[
\max \{\text{Ric}^{WP}(\mu), \text{Ric}^{WP}(v)\} < K^{WP}(\mu, v).
\]
The following result is a direct consequence of Theorem 4.4.

**Theorem 4.4.** For any \(X \in \mathcal{M}_g^n\) with \(\ell_{\text{sys}}(X) \leq 2\epsilon_2\), then for any \(\mu, v \in T_X \mathcal{M}_g^n\), the Weil-Petersson sectional curvature satisfies that
\[
K^{WP}(\mu, v) \geq -\frac{4}{\ell_{\text{sys}}(X)}.
\]

**Remark 4.5.** Huang in \([4]\) showed that \(K^{WP}(\mu, v) \geq -c(g)\ell_{\text{sys}}(X)\) on \(\mathcal{M}_g\) where \(c(g) > 0\) is a constant depending on \(g\).

**Remark 4.6.** The upper bound \(2\epsilon_2\) for \(\ell_{\text{sys}}(X)\) in Theorem 4.4 may not be optimal. However, the upper bound for \(\ell_{\text{sys}}(X)\) can not be removed: actually it was shown in \([13]\, \text{Theorem 1.1}\) that if \(\ell_{\text{sys}}(X)\) is large enough, then
\[
\min_{\text{span}(\mu,v) \subset T_X \mathcal{M}_g} K^{WP}(\mu, v) \leq -C < 0
\]
where \(C > 0\) is a uniform constant independent of \(g\). In particular, (4.8) does not hold for Buser-Sarnak surface \(X_g\) in \([2]\) whose injectivity radius grows like \(\ln (g)\) as \(g \to \infty\).

We close this subsection by proving Theorem 1.6.

**Theorem 4.7** (=Theorem 1.6). For any \(X \in \mathcal{M}_g^n\) with \(\ell_{\text{sys}}(X) \leq 2\epsilon_2\), then for any \(\mu \neq 0 \in P(X)^\perp\) and \(v \in T_X \mathcal{M}_g^n\), the Weil-Petersson sectional curvature \(K^{WP}(\mu, v)\) along the plane spanned by \(\mu\) and \(v\) satisfies that
\[
K^{WP}(\mu, v) \geq -4.
\]
**Proof.** Since \(\mu \in P(X)^\perp\), by Lemma 3.8 we have
\[
\sup_{z \in X} |\mu|(z) \leq \sqrt{2}||\mu||_{WP}.
\]
By taking a rescaling one may assume ||\mu||_{WP} = 1. We normalize \(v\) such that ||\(v||_{WP} = 1\). Then it follows by (2.2) that
\[
K^{WP}(\mu, v) \geq -2 \int_X D(|v|^2) |\mu|^2 dA
\geq -4 \int_X D(|v|^2) \cdot 1 dA
= -4 \int_X |v|^2 dA = -4
\]
which completes the proof. \(\square\)
5. Total scalar curvature for large genus

It is known \[9, 17\] that the Weil-Petersson scalar curvature always tends to negative infinity as the surface goes to the boundary of the moduli space. In this section we focus on \( \mathcal{M}_g \) and study the total Weil-Petersson scalar curvature

\[
\int_{\mathcal{M}_g} \text{Sc}^{\text{WP}} (X) dX
\]

over the moduli space \( \mathcal{M}_g \), where \( dX \) is the Weil-Petersson measure induced by the Weil-Petersson metric on \( \mathcal{M}_g \).

For any \( \epsilon > 0 \), the \( \epsilon \)-thick part \( \mathcal{M}_g^{\geq \epsilon} \) is the subset defined as

\[
\mathcal{M}_g^{\geq \epsilon} := \{ X \in \mathcal{M}_g : \ell_{\text{sys}}(X) \geq \epsilon \}.
\]

The complement \( \mathcal{M}_g^{< \epsilon} := \mathcal{M}_g \setminus \mathcal{M}_g^{\geq \epsilon} \) is called the \( \epsilon \)-thin part of the moduli space. We first recall the following result of Mirzakhani which we will apply.

**Theorem 5.1.** (Mirzakhani, \[8, Corollary 4.3\]) As \( g \to \infty \),

\[
\int_{\mathcal{M}_g} \frac{1}{\ell_{\text{sys}}(X)} dX \asymp \text{Vol}_{\text{WP}}(\mathcal{M}_g).
\]

Now we are ready to state our result in this section.

**Theorem 5.2** (\( = \)Theorem 1.7). As \( g \to \infty \),

\[
\frac{\int_{\mathcal{M}_g} \text{Sc}^{\text{WP}} (X) dX}{\text{Vol}_{\text{WP}}(\mathcal{M}_g)} \asymp -g.
\]

**Proof.** First by Wolpert \[14\] or Tromba \[11\] we know that for all \( X \in \mathcal{M}_g \),

\[
\text{Sc}^{\text{WP}} (X) \leq -\frac{3}{4\pi} \cdot (3g - 2).
\]

Thus,

\[
(5.1) \quad \frac{\int_{\mathcal{M}_g} \text{Sc}^{\text{WP}} (X) dX}{\text{Vol}_{\text{WP}}(\mathcal{M}_g)} \leq -C_1 \cdot g
\]

where \( C_1 > 0 \) is a uniform constant independent of \( g \).

Next we prove the other direction. That is to show that

\[
(5.2) \quad \int_{\mathcal{M}_g} \text{Sc}^{\text{WP}} (X) dX \geq -C'_1 \cdot g \cdot \text{Vol}_{\text{WP}}(\mathcal{M}_g)
\]

where \( C'_1 > 0 \) is a uniform constant independent of \( g \). We split the total scalar curvature into two parts. More precisely we let \( \epsilon_2 = \sinh^{-1}(1) > 0 \),

\[
(5.3) \quad \int_{\mathcal{M}_g} \text{Sc}^{\text{WP}} (X) dX = \int_{\mathcal{M}_g^{\geq \epsilon_2}} \text{Sc}^{\text{WP}} (X) dX + \int_{\mathcal{M}_g^{< \epsilon_2}} \text{Sc}^{\text{WP}} (X) dX.
\]

On \( \mathcal{M}_g^{\geq \epsilon_2} \) it follows by Lemma 3.1 of Teo that

\[
\text{Sc}^{\text{WP}} (X) \geq -(6g - 6) \cdot C^2(\epsilon_2).
\]
Thus, we have

\[\int_{\mathcal{M}_g^{\leq \epsilon^2}} \text{Sca}^{\text{WP}}(X) dX \geq -(6g - 6) \cdot C^2(\epsilon^2) \cdot \text{Vol}_{\text{WP}}(\mathcal{M}_g^{\leq \epsilon^2})\]

\[\geq -(6g - 6) \cdot C^2(\epsilon^2) \cdot \text{Vol}_{\text{WP}}(\mathcal{M}_g)\]

\[\geq -C_2 \cdot g \cdot \text{Vol}_{\text{WP}}(\mathcal{M}_g)\]

where \(C_2 > 0\) is a uniform constant independent of \(g\).

On \(\mathcal{M}_g^{< \epsilon^2}\) it follows by Theorem 4.2 that

\[\text{Sca}^{\text{WP}}(X) \geq -\frac{11}{\ell_{\text{sys}}(X)} \cdot (g - 1).\]

Thus, we have

\[\int_{\mathcal{M}_g^{< \epsilon^2}} \text{Sca}^{\text{WP}}(X) dX \geq -11(g - 1) \cdot \int_{\mathcal{M}_g^{< \epsilon^2}} \frac{1}{\ell_{\text{sys}}(X)} dX\]

\[\geq -11(g - 1) \cdot \int_{\mathcal{M}_g} \frac{1}{\ell_{\text{sys}}(X)} dX.\]

By Theorem 5.1 of Mirzakhani we have

\[\int_{\mathcal{M}_g^{\leq \epsilon^2}} \text{Sca}^{\text{WP}}(X) dX \geq -C_3 \cdot g \cdot \text{Vol}_{\text{WP}}(\mathcal{M}_g)\]

where \(C_3 > 0\) is a uniform constant independent of \(g\).

Then the claim (5.2) follows by (5.3), (5.4) and (5.5). \(\square\)

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