Formalization of dependent type theory: The example of $\text{CaTT}$

Thibaut Benjamin

1Université Paris-Saclay, CEA, List, F-91120, Palaiseau, France, thibaut.benjamin@cea.fr

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Abstract

We present the type theory $\text{CaTT}$, originally introduced by Finster and Mimram to describe globular weak $\omega$-categories, and we formalise this theory in the language of homotopy type theory. Most of the studies about this type theory assume that it is well-formed and satisfy the usual syntactic properties that dependent type theories enjoy, without being completely clear and thorough about what these properties are exactly. We use the formalisation that we provide to list and formally prove all of these meta-properties, thus filling a gap in the foundational aspect. We discuss the key aspects of the formalisation inherent to the theory $\text{CaTT}$, in particular that the absence of definitional equality greatly simplify the study, but also that specific side conditions are challenging to properly model. We present the formalisation in a way that not only handles the type theory $\text{CaTT}$ but also all the related type theories that share the same structure, and in particular we show that this formalisation provides a proper ground to the study of the theory $\text{MCaTT}$ which describes the globular monoidal weak $\omega$-categories. The article is accompanied by a development in the proof assistant Agda to actually check the formalisation that we present.

1 Introduction

This article aims at presenting and formalising the foundations of a class of dependent type theories; the theory $\text{CaTT}$ introduced by Finster and Mimram [10] being the motivating example. This dependent type theory is designed to encode a flavor of higher categorical structures called weak $\omega$-categories, and its semantics has been proved [6] to be equivalent to a definition of weak $\omega$-categories due to Maltsiniotis [15] based on an approach introduced by Grothendieck [11]. However, none of the aforementioned articles about the theory $\text{CaTT}$ provide a satisfying investigation of the type theoretic foundations, and simply assume that a lot of syntactic meta-properties are satisfied. This is not a shortcoming of those articles, but common practice in the type theory community since low-level descriptions are very lengthy and would make the articles essentially impossible to read. Thus it is usually accepted that such a description is possible and satisfies all the usually needed meta-theoretic properties can be proven by induction. Yet, there does not a general framework for presenting a dependent type theory, that would enforce the existence of low-level foundations.
together with the usual meta-theoretic properties. As a result, every new dependent type theory has to either be very specific about its foundations, or be vague enough and rely on the reader’s ability to infer the foundations and convince themselves that the meta-theoretic properties can be proven. This prevents a lot of results to be built from the ground up, and makes research about dependent type theory not readily available to experts.

The question of the choice of theoretical grounds to present a dependent type theory is also an interesting question in its own rights. It has been a long-standing goal to define type theory within type theory [8], and significant progress has been met in this direction, but there are still open problems when it comes to fully embracing proof-relevance. In particular, it has been theorised that homotopy type theory (HoTT) [16] should be completely definable within itself, but providing such a definition is still one of the main open problems of HoTT. Notably, this definition raises an important coherence problem that is reminiscent in its nature of the problems encountered in defining higher structures. Altenkirch and Kaposi [11] have made progress towards solving this issue by using quotient inductive inductive types, a type theoretic construct whose semantics is not completely understood.

In this article, we present a proof of concept for using the language of HoTT as a meta-theory for defining and studying dependent type theories. Our main objective is the theory CaTT, so our formalisation within HoTT also provide a much-needed foundation to this theory. Moreover this theory is particularly simple, since it does not have definitional equality, making it an ideal candidate to focus on one challenge posed by the use of HoTT. We first give a quick informal presentation of the theory CaTT, in order to assert our goal. This presentation relies on a simpler type theory called GSeTT which describes globular sets. We then discuss the formalisation of this theory, along with its meta-theoretic properties. Next, we move on to the formalisation of dependent type theories whose shapes are described by the theory GSeTT, that we call globular type theories, and whose CaTT is our primary example. Finally we present the theory CaTT and show how it can be formalised in the framework of globular type theories. We prove some of the interesting meta-properties of this theory with a particular emphasis on the ones that are important to understand its semantics and that are used in other articles without full proofs [10, 6]. All our definitions and proofs are accompanied by a formalisation in the theorem prover Agda (to be consistent with the language of HoTT, we also deactivated the use of the axiom K in Agda).

2 Introduction to the theory CaTT

We present here the type theory CaTT in order to motivate our formalisation work. The introduction we provide here focuses mostly on the syntactic aspects of the theory, but we also provide some intuition to the semantics, to help situate the theory in a broader picture. We refer the reader to existing articles [10, 6] for more in-depth discussions about the semantics of this type theory. The first presentation we provide here is informal, and serves as a guideline for the foundations that we are developing.

2.1 General setup for dependent type theories

All along this article, we only consider dependent type theories which support the contraction, exchange and weakening rules, so we simply refer to them as type theories and assume those rules

https://github.com/thibautbenjamin/catt-formalization
implicitly. Those theories are centred around four kinds of object, that we introduce here along with corresponding notations

contexts : \( \Gamma, \Delta, \ldots \)  
substitutions : \( \gamma, \delta, \ldots \)  
types : \( A, B, \ldots \)  
terms : \( t, u, \ldots \)  

Each of these object is associated to a well-formedness judgement

- \( \Gamma \) is a valid context : \( \Gamma \vdash A \)
- \( A \) is a valid type \( \Gamma : \Gamma \vdash A \)
- \( \gamma \) is a valid substitution from \( \Delta \) to \( \Gamma \) : \( \Delta \vdash \Gamma \)
- \( t \) is a term of type \( A \) in \( \Gamma : \Gamma \vdash t : A \)

The type theories we are interested in do not have definitional equalities, so these are the only judgements we do consider here.

### 2.2 The theory GSeTT

We first introduce the theory GSeTT which is simpler than the theory CaTT and serves as a basis on which this theory relies. In the theory GSeTT there are no term constructors, hence the only terms are the variables. There are two type constructors, that we denote \( \star \) and \( \to \). Those two constructors are subject to the following introduction rules

- \( \emptyset \vdash \) (EC)
- \( \Gamma \vdash \) (\( \star \)-INTRO)
- \( \Gamma, x : A \vdash \) (VAR)
- \( \Delta \vdash \) (\( \emptyset \)-INTRO)
- \( \Delta, x : A \vdash \) (ES)
- \( \Delta \vdash \gamma : \Gamma, x : A \vdash \Delta \vdash t : A[\gamma] \) (SE)
- \( \Gamma \vdash A \)
- \( \Gamma, x : A \vdash \Gamma \vdash t : A \) (CE)
- \( \Gamma \vdash u : A \)
- \( \Gamma \vdash t \to A \) (\( \to \)-INTRO)
- \( \Gamma \vdash t \to u \)

Where \( t[\gamma] \) denotes the application of the substitution \( \gamma \) to the term \( t \) and \( \text{Var()} \) denotes the set of variables needed to write a syntactic object. The contexts of the theory GSeTT can be represented by **globular sets**. Those are analogues to graph, except that they are allowed to have arrows in every dimension (often called **cells**). A cell of dimension \( n + 1 \) has for source and target a pair of cells of dimension \( n \) which are required to share the same source and target. In the type theory, this is imposed by rule (\( \to \)-INTRO) in which the terms \( t \) and \( u \) have to share the type \( A \).

**Example 1.** We illustrate the correspondence between contexts and finite globular sets with a few examples, using a diagrammatic representation of globular sets where we give the same name to a variable in a context and its corresponding cell in the corresponding globular set.

\[
\Gamma_c = (x : \star, y : \star, f : x \to y, z : \star, h : y \to z)
\]

\[
\Gamma_w = (x : \star, y : \star, f : x \to y, g : x \to y, \alpha : f \to g, z : \star, h : y \to z)
\]

\[
\Gamma_o = (x : \star, f : x \to x)
\]
This correspondence is not an actual bijection: Several contexts may correspond to the same globular set, if they only differ by reordering of the variables. One can account for this by considering the category of contexts with substitutions. This category is equivalent to the opposite of the category of finite globular sets [6, Th. 16]. A judgement $\Gamma \vdash x : A$ in the theory GSeTT corresponds to a well-defined cell $x$ in the globular set corresponding to $\Gamma$, and the type $A$ provides all the iterated sources and targets of $x$.

2.3 Ps-contexts

In the type theory CaTT there is an additional judgement on contexts, that recognises a special class of contexts that we call ps-contexts. We denote this judgement $\Gamma \vdash_{ps} x : A$. These two judgements are subject to the following derivation rules

$$
\frac{}{(x : \star) \vdash_{ps} x : \star} \quad \text{(PSS)}
$$

$$
\frac{\Gamma \vdash_{ps} f : x \to y \quad \Gamma, y : A, f : x \to y \vdash_{ps} f : x \to y}{\Gamma \vdash_{ps} y : A} \quad \text{(PSD)}
$$

The semantical intuition is that $\vdash_{ps}$ characterises the contexts corresponding pasting schemes [3, 15]. Those are the finite globular sets defining an essentially unique composition in weak $\omega$-categories. Finster and Mimram gave an alternate characterisation of the ps-contexts [10] using a relation on the variables of a context, denoted $x \prec y$. Each context $\Gamma$ defines this relation as the transitive closure of the relation generated by $x \prec y \prec z$ as soon as $\Gamma \vdash y : x \to z$ is derivable. The authors have proved that a context is isomorphic to a ps-context if and only if this relation is a linear order.

Additionally, a ps-context $\Gamma$ defines two subsets of its variables $\partial^- (\Gamma)$ and $\partial^+ (\Gamma)$ respectively called the source and the target set.

**Definition 2.** We define the $i$-source of ps-context as a list by induction

$$
\partial^- (i)(x : \star) = (x : \star) \quad \text{if dim } A \geq i
\partial^- (i)(\Gamma, y : A, f : x \to y) = \begin{cases} 
\partial^- (\Gamma) \\
\partial^- (\Gamma, y : A, f : x \to y)
\end{cases}
\text{otherwise}
$$

and its $i$-target by

$$
\partial^+ (i)(x : \star) = (x : \star) \quad \text{if dim } A > i
\partial^+ (i)(\Gamma, y : A, f : x \to y) = \begin{cases} 
\partial^+ (\Gamma) \\
\text{drop}(\partial^+ (\Gamma), y : A) \\
\partial^+ (\Gamma, y : A, f : x \to y)
\end{cases}
\text{otherwise}
$$

where drop is the list with its head removed. The source (resp. target) of a ps-context $\Gamma$ is the set of all variables in its dim $\Gamma$-source (resp. its dim $\Gamma$-target).

Semantically, these sets contain all the variables in the source and target of the result in the application of the essentially unique composition defined by the pasting scheme corresponding to $\Gamma$. 

4
Example 3. The contexts $\Gamma_c$ and $\Gamma_w$ defined in Example 1 are ps-contexts, while the context $\Gamma_\varnothing$ is not. We provide below a description of the relation $\triangleleft$ on those examples, as well as the source and target set variables for those that are ps-contexts.

\[
\begin{align*}
\Gamma_c &: \quad x \triangleleft f \triangleleft y \triangleleft h \triangleleft z \\
\partial^- (\Gamma_c) &= \{x\} \\
\partial^+ (\Gamma_c) &= \{z\}
\end{align*}
\]

\[
\begin{align*}
\Gamma_w &: \quad x \triangleleft f \triangleleft g \triangleleft y \triangleleft h \triangleleft z \\
\partial^- (\Gamma_w) &= \{x, f, y, h, z\} \\
\partial^+ (\Gamma_w) &= \{x, g, y, h, z\}
\end{align*}
\]

2.4 The type theory CaTT

The type theory CaTT is obtained from the type theory GSeTT by adding new term constructors that witness the operations of weak omega-categories. There are two of these constructors, $\text{op}$ and $\text{coh}$, in such a way that a term is either a variable or of the form $\text{op}_{\Gamma, A}[\gamma]$ or $\text{coh}_{\Gamma, A}[\gamma]$, where in both two last cases, $\Gamma$ is a ps-context, $A$ is a type and $\gamma$ is a substitution. These term constructors are subject to the following introduction rules.

\[
\begin{align*}
\frac{\Gamma \vdash ps \Gamma \vdash t : A \quad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \text{op}_{\Gamma, t}[\gamma] : t[\gamma] \rightarrow A[\gamma]} & \quad \frac{\Gamma \vdash ps \Gamma \vdash A \quad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \text{op}_{\Gamma, A}[\gamma] : A[\gamma]}\\
\end{align*}
\]

Both these rules apply only under extra side-conditions. We denote $(C_{\text{op}})$ the side condition of $(\text{op})$ and $(C_{\text{coh}})$ the one of $(\text{coh})$. Those side conditions are the following:

\[
(C_{\text{op}}) : \left\{ \begin{array}{l}
\text{Var}(t) \cup \text{Var}(A) = \partial^- (\Gamma) \\
\text{Var}(u) \cup \text{Var}(A) = \partial^+ (\Gamma)
\end{array} \right.
\]

\[
(C_{\text{coh}}) : \text{Var}(A) = \text{Var}(\Gamma)
\]

Recall that a ps-context is meant to represent an essentially unique composition in a weak $\omega$-category. The rules $(\text{op})$ and $(\text{coh})$ enforce this condition, in a weak sense analogue to requiring the type of composition to be contractible in homotopy type theory. More specifically, the rule $(\text{op})$ asserts that in a context $\Delta$, for every situation described by a ps-context $\Gamma$ (as witnessed by $\gamma$), there exists a term witnessing the existence of the composition of this situation. The rule $(\text{coh})$ imposes that any two such compositions are related by a higher cells. Indeed in this rule the type $A$ is necessarily of the form $a \rightarrow b$, where $a$ and $b$ represent two ways of composing the ps-context.

The role of the side condition is to prevent the composition to apply partially.

Example 4. Consider the context $\Gamma_c$ introduced in previous Example 1 we have established in Example 3 that it is a ps-context, with $\partial^- (\Gamma_c) = \{x\}$ and $\partial^+ (\Gamma_c) = \{z\}$. The type $x \rightarrow z$ thus satisfies the condition $(C_{\text{op}})$. This shows that for every context $\Delta$ with a substitution $\Delta \vdash \gamma : \Gamma_c$, we can define the term $\text{comp} \ \gamma$, and we have a derivation $\Delta \vdash \text{comp} \ \gamma : x[\gamma] \rightarrow z[\gamma]$. The substitution $\Delta \vdash \gamma : \Gamma_c$ picks out to composable arrows in $\Delta$, and the semantics is that the term $\text{comp} \ \gamma$ witnesses the composition of those arrows. Additional examples are presented in [10].

Since there is no definitional equality or computation rule in this theory, all the computational content is contained in the substitution calculus, i.e., the action of substitutions on terms and types, and the composition of substitutions.

3 Structural foundation of dependent type theory

In this section, we discuss the ways of presenting a dependent type theory and motivate our preferred choice in the light of our objective to formalise the theory CaTT and its properties. There are
essentially two ways of approaching this formalisation problem, each with its own strengths and weaknesses.

3.1 The typed syntax approach

An approach proposed by Dybjer [9] for internalising the semantics of dependent type theory is to define a typed syntax by induction induction. In this approach, we define the syntax and the judgements together, in such a way that every syntactic object intrinsically comes with its well-formedness judgement. For this reason we also refer to this approach as the *intrinsic type theory*. In this setting it makes no sense to even consider an ill-formed syntactic object, thus providing a very concise and consistent presentation of the theory. This approach captures very well the algebraic structure of dependent type theory. It also allows for a variable-free presentation, avoiding dealing with technical issues in managing the variable names (like accounting for \( \alpha \)-equivalence). For all these reasons, an intrinsic approach to dependent type theory is very valuable, the presentation is very direct and reflects the structure of the theory we are formalising.

**Structure of the typed syntax.** The intrinsic formalisation of dependent type theory relies on four types defined in an inductive inductive way. They correspond to each of the four kinds of syntactic objects in the theory. We do not define those fully here, but still give their signature in Agda pseudo-code in order to discuss specific interesting points in following this route. We use the keyword `data` to introduce inductive or mutually inductive types.

```agda
data Ctx : Set 
data Ty : Ctx → Set 
data Sub : Ctx → Ctx 
data Tm : ∀ \( \Gamma \) → Ty \( \Gamma \) → Set 
```

In this framework the type `Ctx` is the type of well-formed contexts i.e., the type of all derivation trees of judgements of the form \( \Gamma \vdash \), and similarly for the three other types. This excludes ill-formed objects from the syntax. For this reason, the type of types `Ty` depends on the type `Ctx`: It does not make sense to require a type to be well-formed, without a reference to the context in which this well-formedness should be satisfied.

**The added value of proof-assistants.** Although it is not apparent just by looking at the signature, the constructors of the type `Ctx` actually depend on the type `Ty`, which itself depends on the type `Ctx` in its signature. Such hidden circular dependencies are everywhere in the definition of those four types. This means that checking the well-definedness of those types and of every construction that we define by induction on those types is non-trivial. As a result, using such an approach for a pen-and-paper formalisation is very challenging. Not only do the proofs have to be carried formally, but the proof that all the proofs are well-formed should also be written explicitly. However, proof assistants really can be a strong added-value in this situation, most of them have a termination checking algorithm that handles the well-formedness of each proof automatically for the user. The only downside is that since the halting problem is undecidable, this algorithm has to rely on heuristics, and forces the user to sometimes over-specify some of the functions for the termination checking algorithm to succeed.

**The dependency problem.** An intrinsic approach to formalising dependent type theory presents hard challenges to solve For instance, one of the first equality that we want to manipulate
is that the action of substitutions on types and terms respects the composition. This translates to
the following equalities
\[ A[\gamma][\delta] = A[\gamma \circ \delta] \quad t[\gamma][\delta] = t[\gamma \circ \delta] \]

Within the typed syntax, the second of these equations is not even a valid statement as is. Indeed,
suppose given the term \( t : Tm \Gamma A \) and two substitutions \( \gamma : Sub \Delta \Gamma \) and \( \delta : Sub \Theta \Delta \) then
the term \( t[\gamma \circ \delta] \) is of type \( Tm \Theta A[\gamma \circ \delta] \), whereas the term \( t[\gamma][\delta] \) is of type \( Tm \Theta A[\gamma][\delta] \).
Although these two types are equal by application of the two equalities, they are not definitionally
so. Thus to even state this equality, one has to transport one of the terms along the equality
between the types. Proving later properties then requires a proper handling of these transports to
show that they agree, and these new terms themselves have to be handled, and so on leading to
a coherence problem which would require the introduction of infinitely many axioms to describe
the theory as we want it. A solution to this issue was introduced by Chapman [8], by allowing
the equality to compare terms that live in different types. This idea was adapted by Lafont as a
heterogeneous predicate used to avoid transports. However this requires assuming the uniqueness
of identity proofs (UIP), at least for all types of the form \( Ty \Gamma \). This trivialises the coherence problem,
but breaks the computational behaviour of the theory: nothing reduce past the UIP axiom. Even
with this simplification, the aforementioned development is beyond the capabilities of Agda, and
requires deactivating its termination checker, which is one of the motivation to use a proof-assistant
in the first place.

**Typed syntax using quotient inductive inductive types.** Our overview of the intrinsic
approach to dependent type theory would not be complete without a discussion of the approach
developed by Altenkirch and Kaposi [1] with quotient inductive inductive types. The intuition is to
allow assumptions that are higher order constructors of higher inductive inductive types, and they
solve the coherence problem by leveraging the power of higher induction. They also truncate all the
higher constructors to level 1 (hence the name “quotient”). However, quotient inductive inductive
types are not well understood semantically, and most modern proof assistants do not support them
naively. As a result, formalising this approach in a proof assistant requires defining manually custom
recursors for the higher inductive types. Specifically in Agda, these recursors directly conflict with
the pattern-matching algorithm, and thus a formalisation in this setting has to carefully avoid the
pattern-matching mechanism when it is inconsistent, and rely on user defined methods. Those do
not scale up very well, and the lack of native facilities quickly become a practical obstruction to
any realistic non-trivial results about the theory.

### 3.2 The raw syntax approach

In this article, we take an alternate route to formalising dependent type theory, based on the
separation of the syntax and the rules of the theory. To highlight the difference we call **raw syntax**
the syntactic elements that are not tied to a derivation tree. Contrary to the typed syntax, raw
syntax may contain ill-formed entities that do not correspond to any entity of the theory. In this
approach, we delegate the computational duty to the raw syntax, which completely sidesteps the
coherence problem. However this approach does not give a direct description of the theory, it
requires combining two separate ingredients, the raw syntax and the judgements, to describe the
theory. For this reason, we call this approach an **extrinsic** formalisation of a dependent type theory.

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[8] https://github.com/amblafont/omegatt-agda/tree/2tt-fibrant
In this article we present a formalisation of the dependent type theory CaTT in an extrinsic way. Similar formalisations have been developed, notably by Finster and Rice, but with the intent of proving other kinds of theories. Moreover Lafont, Hirshowitz and Tabareau have also developed a formalisation of a type theory related to CaTT in a similar fashion in Coq with the goal of internalising the observation that types are weak ω-groupoids.

From now on we present our formalisation, and we use a lot of Agda pseudo-code. To simplify the notations, we use the convention that all the free variables are implicitly universally quantified. Moreover, we provide explanation for the unconventional syntax that we use, so that readers not familiar with Agda can read the article and the supplementary formalisation the same way.

**Variables management.** In order to compute operations on the raw syntax, we keep track of the variables to encode the information present in the typing but not in the raw syntax. From a semantics point of view, the names of the variables should be irrelevant and we prefer to always work up to α-equivalence. To avoid dealing with these issues, we develop a foundation in which the variables are natively normalised. Although each variable has a name, this name is uniquely determined by the rules of the theory. Thus, there is no possibility of renaming the variables and no quotient is needed. We achieve this with a variation on De Bruijn levels: We consider the type of variables to be the type of natural numbers and we require the contexts to enumerate their variables in increasing order. Since there is no variable binder in the theory CaTT, this specification alone suffices to completely determine all the variables in the theory.

**Structure of the raw syntax.** We now present the foundational structure of the dependent type theories that we are interested in. They are particular case of dependent type theories that support the contraction, weakening and exchange. In order to highlight the structure of the type theory itself and not get lost in technical difficulties tied to a specific dependent type theory like GSเยTT or CaTT, we first present the rule for the empty dependent type theory. It is the dependent type theory with no type constructors or term constructors. Semantically, this is a vacuous theory, but it still implements all the structure of dependent type theory. We first define the raw syntax: Contexts are supported by lists of pairs of the form (x, A) where x is a variable and A is a type, substitutions are supported by lists of pairs (x, t) where x is a variable and t is a term, since there are no term constructors, the only terms are variables and since there is no type constructors, there is no type.

**Definition 5.** We define the raw syntax of the empty dependent type theory as a collection of four types defined by mutual induction, representing respectively the (raw) contexts, substitutions, terms and types

\[
\begin{align*}
& \text{Pre-Ctx} = \text{list} \ (\mathbb{N} \times \text{Pre-Ty}) \\
& \text{Pre-Sub} = \text{list} \ (\mathbb{N} \times \text{Pre-Tm}) \\
& \text{data Pre-Tm where} \\
& \quad \text{Var} : \mathbb{N} \to \text{Pre-Tm} \\
& \text{data Pre-Ty where} \\
\end{align*}
\]

Here, the constructor Var produces an inhabitant of the type Pre-Tm from a variable (of type N), and there is no constructor for the type Pre-Ty since there is no type. Those types do not need to be mutually inductive here, because we have not added the constructors. When we add
constructors later, these types will become mutually inductive, and so we define them as such here for consistency.

**The action of substitutions.** The computational content of the theory is completely determined by the action of substitutions and the composition of substitution. We define these operations on the raw syntax levels, and they are the reason why we need to introduce variable names. The coherence problem does not appear at this level since none of the types constituting the raw syntax are dependent. The composition of substitutions can be seen as an action of substitutions on substitutions.

**Definition 6.** We define the action of substitutions on types, terms and the composition of substitutions as the following mutually inductive operations

\[
\begin{align*}
\_ \_T : \text{Pre-Ty} & \to \text{Pre-Sub} \to \text{Pre-Ty} \\
\_ \_t : \text{Pre-Tm} & \to \text{Pre-Sub} \to \text{Pre-Tm} \\
\_ \circ \_ : \text{Pre-Sub} & \to \text{Pre-Sub} \to \text{Pre-Sub}
\end{align*}
\]

\[
\begin{align*}
\text{(} & \gamma \text{)} \_ \_T \\
\text{Var} \ x \ [\text{nil}] \_ \_T & = \text{Var} \ x \\
\text{Var} \ x \ [\gamma :: (v, t)] \_ \_T & = \text{if} \ x \equiv v \ \text{then} \ t \text{ else} \ ((\text{Var} \ x) \ [\gamma] \_ \_T)
\end{align*}
\]

\[
\begin{align*}
\text{nil} \circ \gamma & = \text{nil} \\
(\gamma :: (x, t)) \circ \delta & = (\gamma \circ \delta) :: (x, (t \ [\delta] \_ \_T))
\end{align*}
\]

Here the case \((\gamma \_\_T)\) represents the empty case, since there are no constructors for the type \text{Pre-Ty}. Moreover, \text{nil} denotes the empty list, and \text{\_::\_} is the cons operation on lists. Again there is no need strictly speaking to use mutually inductive types here, but we do for the sake of consistency with other dependent type theories.

**Judgements and inference rules.** In order to recover the actual dependent type theory from the raw syntax, we need to eliminate the ill-formed syntactic entities and for this we introduce the judgements together with their inference rules. We define those as mutually inductive types that depend on the raw syntax of the theory with the following signature

\[
\begin{align*}
data \_ \_\vdash \text{-C} : \text{Pre-Ctx} & \to \text{Set} \\
data \_ \_\vdash \text{-T} : \text{Pre-Ctx} & \to \text{Pre-Ty} \to \text{Set} \\
data \_ \_\vdash \_\_\_\# : \text{Pre-Ctx} & \to \text{Pre-Tm} \to \text{Pre-Ty} \to \text{Set} \\
data \_ \_\vdash \_\_\_\_\_ : \text{Pre-Ctx} & \to \text{Pre-Sub} \to \text{Pre-Ctx} \to \text{Set}
\end{align*}
\]

The type constructors of these mutually inductive types are exactly the inference rules of the theory. The type \(\Gamma \vdash \text{C}\), for instance, is meant to represent the judgement \(\Gamma \vdash\), an inhabitant of this type is built out of the type constructors, corresponding to inference rules. Hence an element of this type can be thought of as a derivation tree. Reasoning by induction on derivation trees, a common technique to prove meta-theoretic properties of dependent type theory, just translates to reasoning by induction on those four types. This discussion holds because there is computation rules (i.e., rules postulating definitional equalities) in our theories: In the presence of such rules, one would need to consider higher inductive types, and the computation rules would be higher order constructors.
Definition 7. We define the following inference rules for constructing contexts, substitutions and variable terms, which are the inference rules of the empty type theory (again, the type \( \vdash T \) has no constructor, since it is empty).

```
data \( \vdash C \) where
  ec : \( \text{nil} \vdash C \)
  cc : \( \Gamma \vdash C \rightarrow \Gamma \vdash A \rightarrow x == \text{length} \rightarrow \Gamma : (x, A)) \vdash C \)
```

```
data \( \vdash T \) where
  data \( \vdash T \) where
  var : \( \Gamma \vdash C \rightarrow (x, A) \in \Gamma \rightarrow \Gamma \vdash \text{Var} x \# A \)
```

```
data \( \vdash S \) where
  es : \( \Delta \vdash C \rightarrow \Delta \vdash S \text{nil > nil} \)
  sc : \( \Delta \vdash S \gamma > \Gamma \rightarrow (\Gamma : (x, A)) \vdash C \rightarrow (\Delta \vdash t \# (A \gamma \gamma \text{Pre-Ty})) \rightarrow x == y \rightarrow \Delta \vdash S (\gamma \gamma : (y, t)) > (\Gamma : (x, A)) \)
```

For our presentation, we rely on the correspondence between the term constructors of those types and the rules to name the constructors accordingly to the names we have given for the rules. In the rules cc and sc, we consider free variables and require equalities between them, instead of considering syntactically equal terms so that we are able to eliminate on this equality only when needed. This lets us avoid the use of axiom K. In the rule cc, the condition \( x == \text{length} \rightarrow \Gamma \) is the one enforcing our condition that context enumerate their variables in order.

4 Formalisation and properties of the theory GSeTT

From now on, we consider the type theoretic foundations that we have presented of an extrinsic formalisation of dependent type theory. We present our first formalisation of an actual dependent type theory with the theory GSeTT. To this end, we extend our definition of the empty type theory by adding two constructors for the type \( \text{Pre-Ty} \), corresponding to the two type constructors of the type theory GSeTT. We then show some interesting meta-properties that the theory GSeTT enjoys.

4.1 Formal presentation of the theory GSeTT

Recall that the type theory GSeTT has only two type constructors \( \star \) and \( \rightarrow \). We thus introduce two constructors to the type \( \text{Pre-Ty} \). Note that this addition changes the entire raw syntax, since the other types \( \text{Pre-Ctx}, \text{Pre-Tm} \) and \( \text{Pre-Sub} \) are all mutually inductively defined with \( \text{Pre-Ty} \). For simplicity purposes, we only present here the parts whose definition changes, but these changes actually propagate to the entire raw syntax.

Definition 8. The raw syntax of the type theory GSeTT is defined the same way as the raw syntax of the empty type theory, replacing the type \( \text{Pre-Ty} \) with the type

```
data \( \text{Pre-Ty} \) where
  \( \star : \text{Pre-Ty} \)
  \( \Rightarrow : \text{Pre-Ty} \rightarrow \text{Pre-Tm} \rightarrow \text{Pre-Tm} \rightarrow \text{Pre-Ty} \)
```

Additionally, we introduce the action of substitution to be defined the same way on raw terms and substitutions, and defined on raw types by
* [ γ ]Pre-Ty = *
⇒ A t u [ γ ]Pre-Ty = ⇒ (A [ γ ]Pre-Ty) (t [ γ ]Pre-Tm) (u [ γ ]Pre-Tm)

We also specify the judgements of the theory, again those are defined in the same way for the empty type theory, except for the type judgement \( \vdash \). Since these judgements are mutually inductive, a change here again propagates to all the judgements of the theory.

**Definition 9.** The type theory \( \text{GSeTT} \) is obtained, from its raw syntax by adding the judgements defined in the same way as for the empty type theory, except for the following data \( \vdash \).

\[
\begin{align*}
\text{data } \vdash \text{ where} \\
\text{ob} : \forall \{\Gamma\} \to \Gamma \vdash C \to \Gamma \vdash T \\
\text{ar} : \forall \{\Gamma A t u\} \to \Gamma \vdash t A \# A \to \Gamma \vdash u A \# A \to \Gamma \vdash T \Rightarrow A t u
\end{align*}
\]

With this formalisation of the theory \( \text{GSeTT} \), we can prove a lot of interesting meta-properties that are required to study its semantics. All of these properties are proved by induction on the derivation trees. A lot of these proofs are straightforward and we just give a discussion on the non-trivial proof techniques that come up. We refer the reader to the Agda implementation provided as supplementary material for the complete proofs of all of these properties.

### 4.2 Structure of the dependent type theory

First we show a few properties about the theory \( \text{GSeTT} \), ensuring that the raw syntax together with a typing rule describe a dependent type theory with all the expected structure. Although these results are used a lot to study the semantics of the theory [10, 6], they are generally admitted and proving them requires as much specificity on the foundational aspects as we have provided here.

**Weakens and judgement preservation.** We have claimed that the type theories we formalise implement the weakenings, contractions and exchange rules. Our naming convention for the contexts actually prevent two variables to have the same name, so all our contexts are reduced with respect to contraction and the contraction rule trivial. Exchange is a bit difficult to express due to the dependency, and is not useful for future applications, so we do not prove it here. The following proposition shows, among other things the weakenings.

**Proposition 10.** The theory \( \text{GSeTT} \) supports weakenings for types, terms and substitutions.

\[
\begin{align*}
\text{wkT} & : \Gamma \vdash T A \to (\Gamma :: (y , B)) \vdash C \to (\Gamma :: (y , B)) \vdash T A \\
\text{wkT} & : \Gamma \vdash t A \# A \to (\Gamma :: (y , B)) \vdash C \to (\Gamma :: (y , B)) \vdash T A \\
\text{wkS} & : \Delta \vdash S \gamma > \Gamma \to (\Delta :: (y , B)) \vdash C \to (\Delta :: (y , B)) \vdash S \gamma > \Gamma
\end{align*}
\]

Moreover, any sub-term of a derivable term is itself derivable. More precisely, we have the following (c.f. admitted results [10, Lemma 6] and [6, Lemma 6])

\[
\begin{align*}
\Gamma \vdash A \to \Gamma : \Gamma \vdash T A \to \Gamma \vdash C \\
\Gamma \vdash t : A \to \Gamma : \Gamma \vdash t A \# A \to \Gamma \vdash C \\
\Delta \vdash \gamma : \Gamma \to \Delta : \Delta \vdash S \gamma > \Gamma \to \Gamma \vdash C \\
\Delta \vdash \gamma : \Gamma \to \Delta : \Delta \vdash S \gamma > \Delta \to \Delta \vdash C \\
\Gamma \vdash \text{src} : \Gamma \vdash T \Rightarrow A t u \to \Gamma \vdash t A \\
\Gamma \vdash \text{tgt} : \Gamma \vdash T \Rightarrow A t u \to \Gamma \vdash u A
\end{align*}
\]
Structure of category with families. Categories with families \[\text{(CwF)}\] are a categorical framework designed to describe the structure of a dependent type theory. They are ubiquitous in the study of the semantics of dependent type theory. With the foundations that we have given, we can prove by induction on the derivation trees that the type theory \(\text{GSeTT}\) defines a CwF. However, developing category theory within HoTT can prove quite challenging, and to avoid doing so, we prove the separately all the required ingredients that constitute a CwF.

**Proposition 11.** The action of a derivable substitution on a derivable type (resp. term) is again a derivable type (resp. term)

\[
\begin{align*}
[] T : & \Gamma \vdash T A \to \Delta \vdash S \gamma > \Gamma \to \Delta \vdash T (A [\gamma ]\text{Pre-Ty}) \\
[] t : & \Gamma \vdash t # A \to \Delta \vdash S \gamma > \Gamma \to \Delta \vdash t (t [\gamma ]\text{Pre-Tm}) # (A [\gamma ]\text{Pre-Ty})
\end{align*}
\]

**Proposition 12.** In the theory \(\text{GSeTT}\), there is an identity substitution defined by

\[
\begin{align*}
\text{Pre-id} : & \forall (\Gamma : \text{Pre-Ctx}) \to \text{Pre-Sub} \\
\text{Pre-id} \text{ nil} = & \text{ nil} \\
\text{Pre-id} (\Gamma :: (x, A)) = & (\text{Pre-id} \Gamma) :: (x, \text{Var} x)
\end{align*}
\]

It acts trivially on types, terms and substitutions on a raw syntax level and it is derivable.

\[
\begin{align*}
[id] T : & (A [\text{Pre-id} \Gamma] \text{Pre-Ty}) == A \\
[id] t : & (t [\text{Pre-id} \Gamma] \text{Pre-Tm}) == t
\end{align*}
\]

**Proposition 13.** The action of substitution is compatible with the composition of substitution

\[
\begin{align*}
[] T : & \Gamma \vdash T A \to \Delta \vdash S \gamma > \Gamma \to \Theta \vdash S \delta > \Delta \to \\
& ((A [\gamma ]\text{Pre-Ty}) [\delta ]\text{Pre-Ty}) == (A [\gamma \circ \delta ]\text{Pre-Ty}) \\
[] t : & \Gamma \vdash t # A \to \Delta \vdash S \gamma > \Gamma \to \Theta \vdash S \delta > \Delta \to \\
& ((t [\gamma ]\text{Pre-Tm}) [\delta ]\text{Pre-Tm}) == (t [\gamma \circ \delta ]\text{Pre-Tm})
\end{align*}
\]

**Proposition 14.** The composition of substitutions preserves the derivability of substitutions. Moreover, it is associative, and the identity is the left unit of the composition.

\[
\begin{align*}
\circ - \text{adm} : & \Delta \vdash S \gamma > \Gamma \to \Theta \vdash S \delta > \Delta \to \Theta \vdash S (\gamma \circ \delta ) > \Gamma \\
\circ - \text{left-unit} : & \Delta \vdash S \gamma > \Gamma \to (\text{Pre-id} \Gamma \circ \gamma ) == \gamma \\
a : & \Delta \vdash S \gamma > \Gamma \to \Theta \vdash S \delta > \Delta \to \Xi \vdash S \theta > \Theta \to (\gamma \circ \delta ) \circ \theta == (\gamma \circ (\delta \circ \theta )
\end{align*}
\]

The result in Proposition 13 and 14 may not hold at the raw syntax level, and they do require derivability hypothesis to hold. All these propositions together show a large part of the structure of CwF of the theory \(\text{GSeTT}\). We have presented them in an unusual order which highlights their dependency structure: Each of the later cited result uses the previously cited ones. These results together correspond to \[6\] Propositions 8, 9 and 10], where they are used without proof.

### 4.3 Proof-theoretic considerations

In our foundational framework, we can also show meta-properties of a proof-theoretic nature. From now on we will use the language of HoTT as a convenience, even though we do not need the univalence axiom, and bare Martin-Löf type theory without any axiom is sufficient.
Decidability of type checking. In Martin-Löf type theory, we express that a type \(A\) is decidable, by exhibiting an inhabitant of the type \(\text{dec } A = A + \neg A\).

**Theorem 15.** Type checking in the theory \(\text{GSeTT}\) is a decidable problem

\[
\begin{align*}
\text{dec} \vdash C & : \forall \Gamma \rightarrow \text{dec } (\Gamma \vdash C) \\
\text{dec} \vdash S & : \forall \Delta \Gamma \gamma \rightarrow \text{dec } (\Delta \vdash S \gamma > \Gamma) \\
\text{dec} \vdash T & : \forall \Gamma A \rightarrow \text{dec } (\Gamma \vdash T A) \\
\text{dec} \vdash t & : \forall \Gamma A t \rightarrow \text{dec } (\Gamma \vdash t t # A)
\end{align*}
\]

The proof of this theorem is by mutual induction of the derivation tree, however the structure of the induction is quite complicated and showing that it is well-founded is a hard problem. This is where the use of a proof assistant like Agda becomes extremely useful, since its termination checker is able to verify this automatically. Proving this theorem in Agda amounts to implementing a certified type checker for the theory \(\text{GSeTT}\).

Uniqueness of derivation tree. In order to express uniqueness, we use the language of HoTT, and we define the types

\[
\begin{align*}
\text{is-contr } A & = \Sigma A (\lambda x \rightarrow ((y : A) \rightarrow x == y)) \\
\text{is-prop } A & = \forall (x y : A) \rightarrow \text{is-contr } (x == y)
\end{align*}
\]

witnessing respectively that a type \(A\) has a unique inhabitant, and that \(A\) is either empty or has a unique inhabitant.

**Theorem 16.** In the theory \(\text{GSeTT}\), every derivable judgement has a unique derivation (stated without proof \([6, \text{Lemma 7}]\))

\[
\begin{align*}
\text{is-prop} \vdash C & : \text{is-prop } (\Gamma \vdash C) \\
\text{is-prop} \vdash S & : \text{is-prop } (\Delta \vdash S \gamma > \Gamma) \\
\text{is-prop} \vdash T & : \text{is-prop } (\Gamma \vdash T A) \\
\text{is-prop} \vdash t & : \text{is-prop } (\Gamma \vdash t t # A)
\end{align*}
\]

We again prove by mutual induction, and the proof is fairly straightforward. The key ingredient is that the theory does not have definitional equality. We can recover the typed syntax from the raw syntax and the judgements, by considering dependent pairs of a an element of the raw syntax together with its judgement as follows. For instance for contexts, we define the type \(\text{Ctx} = \Sigma \text{Pre-Ctx} (\lambda \Gamma \rightarrow \Gamma \vdash C)\), and similarly for types, terms and substitutions, we define the types \(\text{Ty } \Gamma\), \(\text{Tm } \Gamma A\) and \(\text{Sub } \Delta \Gamma\).

### 4.4 Familial representability of types

We now use our foundational framework to define the disks and sphere contexts, which are families of contexts that play an important in the understanding of the semantics of the theory \([6]\).

**Definition 17.** For every number \(n\), we define a type \(\Rightarrow_{u} n\) (\(u\) stands for “universal”) and two contexts \(\text{Pre-}S n\) and \(\text{Pre-}D n\) by mutual induction in the raw syntax as follows (where \(\ell\) is the length of the list)

\[
\begin{align*}
\Rightarrow_{u} 0 & = * \\
\Rightarrow_{u} (S n) & = \Rightarrow_{u} n (\text{Var } (2 n)) (\text{Var } (2 n + 1)) \\
S 0 & = \text{nil} \\
S (S n) & = (D n) :: (\ell (D n), \Rightarrow_{u} n) \\
D n & = (S n) :: (\ell (S n), \Rightarrow_{u} n)
\end{align*}
\]

13
Proposition 18. The disk and sphere contexts are valid contexts in the theory $GSeTT$, and the type $\Rightarrow u$ is derivable in the sphere context

$S\vdash : \forall n \to S \ n \vdash C$

The sphere contexts play a particular role in the theory since they classify the types in a context: types in a context are equivalent to substitution from that context to a disk context. This is a result that we call familial representability of types [6], and that we formally prove in our foundational framework, using the definition of equivalence $is-equiv$ usual to HoTT.

Theorem 19. For every context $\Gamma$, and any derivable substitution from $\Gamma$ to a sphere, we define a derivable type in $\Gamma$ by applying the substitution on the type $\Rightarrow u \_$. The resulting map defines an equivalence

$Ty-n : \forall \Gamma \to \Sigma N (\lambda n \to Sub \Gamma (S n)) \to Ty \Gamma$

$Ty-n \Gamma (n, (\gamma, (I\vdash\gamma:Sn)) = ((\Rightarrow u n)[ \gamma ]Pre-Ty), ([]T (S\vdash n) I\vdash\gamma:Sn)$

$Ty-classifier : \forall \Gamma \to is-equiv (Ty-n \Gamma)$

This result is substantially harder to prove formally than the previously mentioned ones, and relies on the uniqueness of derivation trees.

5 Formalisation and properties of the theory $CaTT$

In this section we adapt our foundational framework to define and study the properties of the theory $CaTT$. This requires the introduction of the two term constructors $op$ and $coh$. We break down this work in two steps: First, we define a notion of globular type theory - a dependent type theory with the same type constructors as $GSeTT$ and generically indexed term constructors - and we extend the meta-theoretic results of the previous section all these theories. Then we define the type theory $CaTT$ as a particular instance of a globular type theory, by specifying the index for the term constructors.

5.1 Globular type theories

Globular type theories are dependent type theories that have the same type structure as the theory $GSeTT$, but have term constructors. In order to describe not only the type theory $CaTT$, but also other dependent type theories, we define these term constructors generically. To this end, we assume a type $I$, which serves as an index to all the term constructors.

Definition 20. The raw syntax of a globular type theory is defined by the four mutually inductive types.

```
data Pre-Ty where
  * : Pre-Ty
  \Rightarrow : Pre-Ty \to Pre-Tm \to Pre-Tm \to Pre-Ty
```

data Pre-Tm where

```
Var : N \to Pre-Tm
Tm-c : \forall (i : index) \to Pre-Sub \to Pre-Tm
```
data Pre-Sub where
   <> : Pre-Sub
   <_,_↦→> : Pre-Sub → N → Pre-Tm → Pre-Sub

data Pre-Ctx where
   ∅ : Pre-Ctx
   _--#_ : Pre-Ctx → N → Pre-Ty → Pre-Ctx

   Due to the mutually inductive definition, raw contexts and substitutions, cannot be defined as mere lists. However, they behave the exact same way as lists, and for all intents and purposes, we treat them as normal lists of pairs in the rest of this article.

Definition 21. The action of substitution on the raw syntax is computed the same way as the action of substitutions on the raw syntax of the theory \textit{GSeTT}, except on terms where it is defined by

\[
\text{Var } x \ [ < > ] \text{Pre-Tm} = \text{Var } x \\
\text{Var } x \ [ < \delta , v \mapsto t > ] \text{Pre-Tm} = \text{if } x \equiv v \text{ then } t \text{ else } ((\text{Var } x) [ \delta ] \text{Pre-Tm}) \\
\text{Tm-c } i \gamma [ \delta ] \text{Pre-Tm} = \text{Tm-c } i (\gamma \circ \delta)
\]

Due to the added dependency, the composition of substitution is now mutually defined with the action on types and terms, whereas in the theory \textit{GSeTT}, it was separately defined.

Inference rules of globular type theories. We give a presentation of a generic form for the introduction of the indexed term constructors in globular type theories. To achieve this, we parameterise the rules, in such a way that every term constructor corresponds to its own introduction rule. We allow to have term constructors in the pre-syntax that do not correspond to any derivable term, if the rule is inapplicable. From now on, we assume that the type I has decidable equality, that is, we have a term

\[
eq \text{decI} : \forall (x y : I) \rightarrow \text{dec} (x == y)
\]

In order to parameterise the rules, we suppose that for every inhabitant i of the type I, there exists a context Ci i in the raw syntax of \textit{GSeTT} and a type Ti i in the raw syntax of the globular type theory. Moreover, we assume that the context Ci i is derivable in the theory \textit{GSeTT}.

Definition 22. A globular type theory is a theory obtained from its syntax by imposing the same judgement rules as in the theory \textit{GSeTT} for contexts, types and substitutions, and imposing for terms

\[
\text{data } _\vdash-t_\#_ where \\
\text{var} : \Gamma \vdash C \rightarrow (x , A) \in \Gamma \rightarrow \Gamma \vdash t (\text{Var } x) \# A \\
\text{tm} : \text{Ci i } \vdash T \text{ Ti i } \rightarrow \Delta \vdash S \gamma > \text{Ci i } \rightarrow \Delta \vdash t \text{ Tm-c } i \gamma \# (\text{Ti i } [ \gamma ] \text{Pre-Ty})
\]

Note that again, the judgements of the theory are defined mutually inductively, and this change propagates to the other types. In practice, we have introduced a type A freely along with the hypothesis A == Ti i [ \gamma ]Pre-Ty in order to eliminate on this equality.
Properties of globular type theories. Most of the meta-theoretic properties extend from the theory GSeTT to any globular type theory, but there can be some difficulties in doing so.

**Proposition 23.** Every globular type theory satisfy all the results presented in Propositions 10, 11, 12, 13 and 14 in this case, these results are a bit more involved to prove, because of the added dependency of terms on substitutions. Many results that could be proven separately in the case of GSeTT now depend on each other and have to be proven by mutual induction. Again, termination checking is not trivial, this is one instance where using Agda is a strong benefit.

**Theorem 24** (c.f. Theorem 16). In any globular type theory, every derivable judgement has a single derivation tree.

Definition 17 of the disks and sphere contexts also makes sense in any globular type theory. We also call those the disk and sphere contexts in the raw syntax of the globular theory.

**Theorem 25** (cf Proposition 18 and Theorem 19). The disk and sphere context define valid contexts in any globular type theory, and the sphere contexts classify the types: There is an equivalence between the derivable types in a context and the substitutions from that context to a sphere context.

**Decidability of type checking.** The decidability of type checking is a result that does not generalise as well to any globular type theory, because the generic form we have given for the rules is too permissive. Trying to reproduce the proof of GSeTT yields a proof whose termination cannot be checked by Agda: There is not a variant that decreases along the rules. And indeed, it is possible to devise a globular type theory for which type checking is not decidable. However, we can restrict our attention a little further, and consider theories that satisfies an extra hypothesis

\[ \forall i \rightarrow Ci i \vdash Ti i \rightarrow \text{dim} (Ci i) \leq \text{dim} (Ti i) \]

where \( \text{dim} \) is the dimension of a type (i.e., the number of iterated arrows it is built with) and \( \text{dimC} \) is the dimension of a context (i.e., the maximal dimension among the types it contains). The theory CaTT satisfies this hypothesis.

**Theorem 26** (cf Theorem 15). For every globular type theory satisfying the hypothesis \( \text{wfI} \), the type checking is decidable.

Proving this by induction is fairly straightforward in principle, but ensuring that the induction is well-formed is quite involved. Indeed, there is no obvious decreasing variant and the proof relies on keeping track of both the dimension and the number of nested term constructors in a precise way to exhibit one. This argument is really non-trivial and for this result the use of a termination checker such as Agda’s is extremely valuable.

### 5.2 Ps-contexts and the theory CaTT

We leverage the definition of globular type theory to formalise and prove some meta-theoretic properties of the theory CaTT. To this end, we define a particular type \( J \) to index the term constructors, as well as the contexts \( Ci j \) and the types \( Ti j \) to define the inference rules.
**Ps-contexts.** In our formalism, there is no specific difference between the term constructors op and coh, both of them are term constructors of the form \( \text{Tm-c} \). If anything, formally, op and coh correspond to families of term constructors and not term constructors. One of the ingredients to index these families are the ps-contexts that we formally define here.

**Definition 27.** We define the judgements \( \_ \vdash \text{ps} \) and \( \_ \vdash \#_\_ \) over the raw syntax of the type theory \( \text{GSeTT} \) as the following inductive types (where we denote \( l \) for \( \ell \) \( \Gamma \))

\[
data \_ \vdash \text{ps}_\#_\_ : \text{Pre-Ctx} \rightarrow \mathbb{N} \rightarrow \text{Pre-Ty} \rightarrow \text{Set}
\]

where

\[
\begin{align*}
\text{pss} : & (\text{nil} :: (0 \ , \ *)) \vdash \text{ps} \ 0 \ # \ * \\
\text{psd} : & \Gamma \vdash \text{ps} \ f \ # \ (\Rightarrow \ A \ (\text{Var} \ x) \ (\text{Var} \ y)) \rightarrow \Gamma \vdash \text{ps} \ y \ # \ A \\
\text{pse} : & \Gamma \vdash \text{ps} \ x \ # \ A \rightarrow ((\Gamma \ :: (1 \ , \ A)) :: (S \ 1 \ , \ \Rightarrow \ A \ (\text{Var} \ x) \ (\text{Var} \ 1))) \vdash \text{ps} \ S \ 1 \ # \ \Rightarrow \ A \ (\text{Var} \ x) \ (\text{Var} \ 1)
\end{align*}
\]

**Proposition 28.** The ps-contexts are valid contexts of the theory \( \text{GSeTT} \).

\( \Gamma \vdash \text{ps} \rightarrow \Gamma^- : \Gamma \vdash \text{ps} \rightarrow \Gamma \vdash \)

**The relation \( \triangleleft \).** To work with ps-contexts formally, we define the relation \( \triangleleft \) introduced by Finster and Mimram [10]. The main purpose of this relation is to perform inductive reasoning.

**Definition 29.** Given a contest \( \Gamma \), we define a generating relation \( \Gamma , \_ \triangleleft_0 \_ \), together with its transitive closure \( \Gamma , \_ \triangleleft \_ \) as follows

\[
data \_ , \_ \triangleleft_0 \_ \Gamma \ x \ y : \text{Set}
\]

where

\[
\begin{align*}
\triangleleft_0^- : & \Gamma \vdash t \ (\text{Var} \ y) \ # \ (\Rightarrow \ A \ (\text{Var} \ x) \ (\text{Var} \ z)) \rightarrow \Gamma \ , \ x \triangleleft_0 \ y \\
\triangleleft_0^+ : & \Gamma \vdash t \ (\text{Var} \ x) \ # \ (\Rightarrow \ A \ (\text{Var} \ z) \ (\text{Var} \ y)) \rightarrow \Gamma \ , \ x \triangleleft_0 \ y
\end{align*}
\]

**Proposition 30.** The ps-contexts are linear for the relation \( \_ , \_ \triangleleft \_ \), i.e., whenever \( \Gamma \) is a ps-context, the relation \( \Gamma , \_ \triangleleft \_ \) defines a linear order on the variables of \( \Gamma \).

**Theorem 31.** The judgement \( \_ \vdash \text{ps} \) is decidable, and any two derivation of the same judgement of this form are equal.

\[
is\text{-prop} \vdash \text{ps} : \forall \ \Gamma \rightarrow \text{is-prop} \ (\Gamma \vdash \text{ps})
\]

\[
dec \vdash \text{ps} : \forall \ \Gamma \rightarrow \text{dec} \ (\Gamma \vdash \text{ps})
\]
These results are proven by induction on the derivation trees, but they are not straightforward. Indeed, for instance in the case of the uniqueness, a derivation of $\Gamma \vdash ps$ necessarily comes from a derivation of the form $\Gamma \vdash ps \ x \ # \ *$ and by induction this derivation is necessarily unique. However the hard part is to prove that there can only be a unique $x$ such that we have $\Gamma \vdash ps \ x \ # \ *$. Using the $\lhd$-linearity, we can prove a more general lemma: if we have a derivation of $\Gamma \vdash ps \ x \ # \ A$ and of $\Gamma \vdash ps \ y \ # \ A$ with $A$ and $B$ two types of the same dimension, then $x = y$. The decidability also presents a difficulty: The rule $psd$ contains variables in its premises that are not bound in its conclusion. However, these variables have to belong to the context, so we can solve this issue by enumeration of the variables and $\lhd$-linearity.

**Index of term constructors.** With the $ps$-contexts, we can define the index type for the term constructors in the theory $CaTT$. Recall that the term constructors in this theory are defined by $op_{\Gamma, A}$ and $coh_{\Gamma, A}$, where $\Gamma$ is a $ps$-context and $A$ is a type. In our informal presentation, we also required the side conditions $(C_{op})$ and $(C_{coh})$ in the derivation rules. For convenience, we integrate these side conditions in the index type in our formalisation, and define $J$ to be the type of pairs of the form $(\Gamma, A)$, where $\Gamma$ is a $ps$-context and $A$ is a type satisfying either $(C_{op})$ or $(C_{coh})$.

The conditions $(C_{op})$ and $(C_{coh})$ are not straightforward to formalise in HoTT, because the same variable may appear several times in the same term, so there may be several witnesses that a term contains all the desired variables. However, the intended semantics is that of a proposition. To solve this issue, we work with the type $set$ of sets of numbers, for which the membership relation is a proposition. We define the type $A \subset B$ of witnesses that a set $A$ is included in a set $B$, as well as the type $A \circ B = (A \subset B) \times (B \subset A)$ of set equality. We can show that these two types are proposition since we are only manipulating finite subsets of $\mathbb{N}$, and thus we recover the intended semantics.

**Definition 32.** We define the set of source variables and the set of target variables of a $ps$-context. We proceed by induction and first define the $i$-sources and $i$-targets by induction on the judgement $\Gamma \vdash ps \ x \ # \ A$ (we denote $l$ for $\ell \Gamma$)

$$
\begin{align*}
src_{i\text{-var}} & (i \ pss) = \text{if } i \equiv 0 \text{ then nil else (nil :: 0)} \\
src_{i\text{-var}} & (i \ psd) = src_{i\text{-var}} \Gamma \vdash psx \\
src_{i\text{-var}} & (i \ pse \{\Gamma = \Gamma \} \{A = A\} \Gamma \vdash psx \text{ with } \text{dec-} \leq i \ (S \ (\text{dim } A)) \\
\quad \text{... | inl i} \leq \text{SdimA} = src_{i\text{-var}} \Gamma \vdash psx \\
\quad \text{... | inr SdimA} < i = (src_{i\text{-var}} \Gamma \vdash psx :: 1) :: (S 1) \\
tgt_{i\text{-var}} & (i \ pss) = \text{if } i \equiv 0 \text{ then nil else (nil :: 0)} \\
tgt_{i\text{-var}} & (i \ psd) = tgt_{i\text{-var}} \Gamma \vdash psx \\
tgt_{i\text{-var}} & (i \ pse \{\Gamma = \Gamma \} \{A = A\} \Gamma \vdash psx \text{ with } \text{dec-} \leq i \ (S \ (\text{dim } A)) \\
\quad \text{... | inl i} \leq \text{SdimA} = \text{if } i \equiv S \ (\text{dim } A) \text{ then drop(tgt}_{i\text{-var}} \Gamma \vdash psx) :: 1 \\
\quad \text{else tgt}_{i\text{-var}} \Gamma \vdash psx \\
\quad \text{... | inr SdimA} < i = (tgt_{i\text{-var}} \Gamma \vdash psx :: 1) :: (S 1) 
\end{align*}
$$

Here the with construction matches on the decidability of the order in $\mathbb{N}$. For instances matching on $\text{dec-} \leq i \ n$ produces two cases of type $i \leq n$ and $n < i$. Moreover $\text{drop}$ takes a list and removes its head. The source and target sets of a $ps$-context are the sets corresponding to the source and target lists in dimensions maximal.

$$
src \ (\Gamma, \ ps \ \Gamma \vdash psx) = \text{set-of-list (src}_{\text{-var}} \ (\text{dimC } \Gamma) \ \Gamma \vdash psx)
$$
tgt-var (Γ, ps ⊢ psx) = set-of-list (tgt-var (dimC Γ) ⊢ psx)

**Definition 33.** We define the type \_is-full-in_, witnessing whether either the condition (C\_op) or (C\_coh) is satisfied, as follows

\[
\text{data } \_\text{is-full-in}_\_ \text{ where}
\]
\[
\text{Cop} : (\text{src-var } Γ) \cong ((\text{varT } A) \cup \text{-set (vart } t)) \to
\text{tgt-var } Γ \cong ((\text{varT } A) \cup \text{-set (vart } u)) \to
\Rightarrow A \to \text{is-full-in } Γ
\]
\[
\text{Ccoh} : (\text{varC (fst } Γ)) \cong (\text{varT } A) \to A \text{ is-full-in } Γ
\]

where \text{varT} (resp. \text{vart}, \text{varC}) is the set of variables associated to a type (resp. to a term, to a context).

**Definition 34.** The type \( J = \Sigma (\text{ps-ctx } \times \text{Ty}) \lambda \{ (Γ, A) \to A \text{ is-full-in } Γ \} \) to be the index type for the term constructors of the theory \( \text{CaTT} \). It is the types of pairs \((Γ, A)\) where \( Γ \) is a ps-context and \( A \) is a raw type satisfying \((C\_op)\) or \((C\_coh)\).

The raw syntax of the type theory \( \text{CaTT} \) is the raw syntax of the globular type theory with index type \( J \).

**The type theory \( \text{CaTT} \)** We give a definition of the dependent type theory \( \text{CaTT} \), by adding the rules to the raw syntax. For this it suffices to define a context \( C_i \) a type \( T_i \) for every inhabitant \( i \) of the type \( J \).

**Definition 35.** Considering a term \(((\Gamma, Γ \vdash ps), A), A\text{-full})\) of type \( J \), we pose

\[
C_i (((Γ, Γ \vdash ps), A), A\text{-full}) = (Γ, Γ \vdash ps \to Γ \vdash Γ \vdash ps)
\]
\[
T_i (((Γ, Γ \vdash ps), A), A\text{-full}) = A
\]

The theory \( \text{CaTT} \) is the globular type theory obtained from these assignments.

**Proposition 36.** The type \( J \) has decidable equality and satisfies the technical condition of well-foundedness

\[
eq\text{dec}_J : \forall (x \ y : J) \to \text{dec } (x = y)
\]
\[
wfJ : \forall j \to C_i j \vdash T_i j \to \text{dimC } (C_i j) \leq \text{dim } (T_i j)
\]

Since this definition realises \( \text{CaTT} \) as a particular case of a globular type theory, it enjoys all the properties that we have already proved for them. In particular we have already proved

**Theorem 37.** In the theory \( \text{CaTT} \) the following statements hold.

- The theory supports weakening and derivability is preserved by the inference rules.
- The theory \( \text{CaTT} \) defines a category with families.
- Every derivable judgement in \( \text{CaTT} \) has a unique derivation tree.
- The sphere contexts in \( \text{CaTT} \) classify the types.
- Type checking is decidable in the theory \( \text{CaTT} \).
6 Conclusion and further work

We have presented a full formalisation of the foundational aspects of the dependent type theory CaTT. Although this dependent type theory is quite simple, in that it does not have any definitional equality, proving formally all the relevant aspects that we expected turned out to be a substantial amount of work with highly non-trivial challenges to solve. In particular for some of the aspects such as the decidability of type checking, the use of a proof-assistant such as Agda appears almost mandatory given the subtlety of the arguments. Ideally, such a foundational work should be carried only once and made accessible for future work to rely on. Unfortunately, there is no unifying framework for every dependent type theory, that could allow us to obtain these properties for free for all dependent type theory, and so in practice one has to redevelop this entire construction for every dependent type theory. The notion of globular type theory is very limited attempt at such a framework. A more promising approach could be to follow the work of Gylterud with the Myott project\footnote{https://git.app.uib.no/Hakon.Gylterud/myott}, and to formalise the interesting properties for a large class of dependent type theories.

The formalisation that we have presented, and in particular the proof for the decidability of type checking constitutes a verified implementation of a type checker for the theory CaTT. Besides, we have developed regular implementation of such a type checker\footnote{https://github.com/thibautbenjamin/catt} in OCaml. However, to improve the user experience, we have defined some meta-operations (called suspension and functorialization) on the syntax of the theory, that we proved correct manually\footnote{\cite{1,7}}. However, to avoid relying on the correctness of our implementation, the software computes the result of these operations and checks them like any user inputs. This leads to inefficiency in the implementation, and is not very satisfying. A better practice would be to define and prove formally those meta-operations, and then export the results to executable code in order to have a natively certified implementation of these meta-operations.

Finally, we have also defined a dependent type theory to describe monoidal weak $\omega$-categories, that we call MCaTT. This dependent type theory is also a globular type theory, and as such it would be valuable to formalise it as well in the framework we have presented (there are actually two slightly different but equivalent formulations for this theory \cite{5} and \cite{4}, and only the second one corresponds to a globular type theory as we have defined). Moreover, our understanding of the semantics relies strongly on translations back and forth between the theories CaTT and MCaTT. Such translations are defined by induction on the syntax and are tedious to carry, defining them formally and proving their correctness constitutes a relevant problem to tackle in further works.

In general, giving a formulation of higher categorical results in terms of a syntax and a dependent type theory allows to perform this kind of reasoning that is well understood formally by proof-assistant. We believe that it constitutes a strong asset for higher category theory, where the complexity of the theory itself quickly becomes a meaningful obstacle for any non-trivial exploration by hand of the theories.

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