Mirzakhani’s Curve Counting
Research Announcement

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Abstract. Mirzakhani wrote two papers on counting curves of given type on a surface: one for simple curves [11], and one for arbitrary ones [12]. We give a complete argument deriving Mirzakhani’s result for general curves from the one about simple ones. We then sketch an argument to give a new proof of both results – full details and other related matters will appear in [7].

1. Counting curves

Let \( S \) be an orientable surface of negative Euler characteristics and of genus \( g \), with \( k \) ends, and with \( \partial S = \emptyset \), and suppose for the sake of concreteness that \( S \) is not a thrice punctured sphere. We will also ignore the difference between essential closed curves and their homotopy classes. Under a multiciuvre we understand a finite formal sum \( \gamma = \sum a_i \gamma_i \) of pairwise non-homotopic, primitive closed curves \( \gamma_i \), with coefficients \( a_i \geq 0 \). The length \( \ell_X(\gamma) \) of \( \gamma \) with respect to a hyperbolic structure \( X \) on \( S \) is the weighted sum \( \sum a_i \ell_X(\gamma_i) \) of the lengths of the associated geodesics. The multicurve \( \gamma \) is simple if all the curves \( \gamma_i \) are simple and disjoint. The mapping class group \( \text{Map}(S) = \text{Homeo}(S)/\text{Homeo}_0(S) \) of \( S \) acts on the set of (homotopy classes of) curves, and hence on the set of multicurves. Two multicurves \( \gamma \) and \( \gamma' \) are of the same type if they belong to the same mapping class group orbit, meaning that there \( \phi \in \text{Map}(S) \) with \( \gamma' = \phi(\gamma) \).

In this note we are interested in two results of Mirzakhani describing the asymptotic behavior of the number of multicurves of given type and length bounded by \( L \), where length is measured with respect to a, for instance, hyperbolic metric on \( S \). In [11], Mirzakhani proved the following result:

**Theorem 1.1** (Mirzakhani). For any essential simple multicurve \( \gamma \) in \( S \), and for any complete, finite volume, hyperbolic structure \( X \) on \( S \) we have

\[
\lim_{L \to \infty} \frac{\# \{ \gamma' \text{ of type } \gamma \text{ and with } \ell_X(\gamma') \leq L \}}{L^{6g-6+2k}} = \frac{B(X) \cdot c(\gamma)}{m_{g,k}},
\]

where \( B(X) = \mu_{\text{Thu}}(\{ \lambda \in \mathcal{ML} \text{ with } \ell_X(\lambda) \leq 1 \}) \) is the Thurston measure of the set of measured laminations \( \lambda \in \mathcal{ML}(S) \) with \( \ell_X(\lambda) \leq 1 \), where \( m_{g,k} = \int B(X) \) is the Weil-Petersson integral of \( B(X) \) over the moduli space, and where \( c(\gamma) \) is a rational only depending on \( \gamma \).

The constant \( c(\gamma) \) in Theorem 1.1 is related to the Weil-Petersson volumes of the set of hyperbolic surfaces where \( \gamma \) has given length. She gives a recursive formula to compute these volumes using a generalization [10] of McShane’s identity [9], and it is from there that she derives that \( c(\gamma) \) is rational. It should be already clear from this brief description that simplicity of \( \gamma \) is used all over [11]. However, Mirzakhani later came up with another argument [12] that allowed her to treat the non-simple case as well:
Theorem 1.2 (Mirzakhani). For any essential multicurve $\gamma$ in $S$, and for any complete, finite volume, hyperbolic structure $X$ on $S$ we have

$$\lim_{L \to \infty} \frac{\# \{ \gamma' \text{ of type } \gamma \text{ and with } \ell_X(\gamma') \leq L \}}{L^{6g-6+2k}} = \frac{B(X) \cdot c(\gamma)}{m_{g,k}}$$

where $B(X)$ and $m_{g,k}$ are as in Theorem 1.1, and where $c(\gamma)$ is again rational.

As we already mentioned in the abstract, the goals of this note are

(1) give a complete proof of Theorem 1.2 by combining Theorem 1.1 and results from our earlier paper [6], and

(2) sketch how the same strategy can be used to give a proof of Theorem 1.1 in the first place.

It is maybe fair to say that already (1) has some interest. In fact, all our admiration and appreciation for Mirzakhani’s work notwithstanding, the paper [12] is very hard to read, containing some rather arcane parts. We are however mostly interested in the fact that by combining (1) and (2), one gets a proof of both of Mirzakhani’s results, really different in spirit to the original ones. This should be clear to the reader if she continues skimming through this note, but let us for example stress that (i) the Weil-Petersson metric does not play any role at all, (ii) the only form of “equidistribution” needed is Masur’s theorem [8] asserting that the Thurston measure is ergodic, and (iii) that moduli space only plays a role because it is needed in the proof of Masur’s theorem.

Remark. We have stressed that our approach is different from Mirzakhani’s, and this is really true. Still, we suspect that she might have been aware of many, possibly most, of the things we say here. We still hope that she might have been amused by it.

Anyways, we believe that it is worth having a coherent, basically self-contained write-up of these and some related matters. This is why we use this text to also announce that we are writing [7], a not very long book serving this purpose.

Acknowledgements. Our arguments are to a large extent elementary. Often it seems that we are playing Three-card Monte with different kinds of limits. We have only been slowly understanding how much mileage we can get out of that game, and discussing with other people has often helped enormously. This is particularly true of Kasra Rafi, a true expert. In fact, we are highly influenced by [13].

Simplifying assumptions. We will only discuss here the case that $S$ is a closed surface of genus $g \geq 3$. The assumption that $S$ is closed arises from the fact that we will work with currents, and when working with open surfaces one has to be slightly more careful, using a few more words. The assumption that the genus is at least 3 comes from the fact that we want to avoid keeping track of the kernel of the action of the mapping class group on the space of measured laminations. The reader should have no difficulties if she continues skimming through this note, but let us for example stress that (i) the Weil-Petersson metric does not play any role at all, (ii) the only form of “equidistribution” needed is Masur’s theorem [8] asserting that the Thurston measure is ergodic, and (iii) that moduli space only plays a role because it is needed in the proof of Masur’s theorem.

2. Proof of Theorem 1.2

In this section we deduce Theorem 1.2 from Theorem 1.1 using results we proved earlier in [6]. We refer to [12] for basic facts and definitions about geodesic currents. Denoting by $C = C(S)$
the space of geodesic currents, we consider for a given (possibly non-simple) multicurve \( \sigma \subset S \) the measure
\[
\nu^L_{\sigma} = \frac{1}{L^{6g-6}} \sum_{\sigma' \text{ of type } \sigma} \delta_{T_{\sigma'}}
\]
where \( \delta_x \) stands for the Dirac measure on \( C \) centred at \( x \). In \([6]\) we studied the behaviour of \( \nu^L_{\sigma} \) when \( L \to \infty \) and proved:

**Proposition 2.1.** \([6, \text{Prop. 4.1}]\) Any sequence \((L_n)_n\) of positive numbers with \( L_n \to \infty \) has a subsequence \((L_{n_i})_i\) such that the measures \((\nu^L_{\sigma})_i\) converge in the weak-*-topology to the measure \( c \cdot \mu_{\text{Thu}} \) on \( \mathcal{MC} \subset C \) for some \( c > 0 \).

**Remark.** Masur’s theorem asserting the ergodicity of the Thurston measure \([8]\) plays a crucial role in the proof of Proposition 2.1. This is the only time that ergodicity is used in our arguments. Also, and just to assuage the possible concern that what we are doing here is circular, we would like to stress that Proposition 2.1 is independent of Mirzakhani’s work.

Suppose now that \( \sigma \) is a filling multicurve and that it is generic in the sense that for all \( \phi \in \text{Map}(S) \setminus \{\text{Id}\} \) we have
\[
\mu_{\text{Thu}}(\{\lambda \in \mathcal{MC} \text{ with } \iota(\lambda, \sigma) = \iota(\phi(\lambda), \sigma)\}) = 0.
\]

It is not hard to construct generic multicurves – for example \( \sigma \) can be taken to be a union of simple curves. Anyways, note that genericity implies that the stabiliser \( \text{Stab}(\sigma) \subset \text{Map}(S) \) of \( \sigma \) in the mapping class group is trivial, and hence that we can identify the mapping class group with the \( \text{Map}(S) \)-orbit of \( \sigma \).

Applying Proposition 2.1 to the measures \((\nu^L_{\sigma})_i\) we get a sequence \((L_n)_n\) tending to infinity and such that
\[
\lim_{n \to \infty} \nu^L_{\sigma} = C \cdot \mu_{\text{Thu}}
\]
for some \( C > 0 \). Our aim is to compute \( C \), proving that it is independent of the sequence \((L_n)_n\).

To that end consider a second multicurve \( \alpha \subset S \), which for now is arbitrary but later will be assumed to be simple. Now, the mapping class group acts properly discontinuously on the subset of \( C \) consisting of filling currents \([5]\). In particular, \( \text{Stab}(\alpha) \) also acts properly discontinuously on the set
\[
\mathcal{C}_\alpha = \{\lambda \in \mathcal{C} \text{ such that } \lambda + \alpha \text{ is filling}\}.
\]

An (open) fundamental domain for this action is easy to describe:
\[
\mathcal{D}_\alpha = \{\lambda \in \mathcal{C}_\alpha \text{ with } \iota(\lambda, \sigma) < \iota(\lambda, \phi(\sigma)) \text{ for all } \phi \in \text{Stab}(\alpha) \setminus \{\text{Id}\}\}.
\]
We note a few simple properties of \( \mathcal{D}_\alpha \):

- Its closure is the set \( \bar{\mathcal{D}}_\alpha = \{\lambda \in \mathcal{C}_\alpha \text{ with } \iota(\lambda, \sigma) \leq \iota(\lambda, \phi(\sigma)) \text{ for all } \phi \in \text{Stab}(\alpha)\}\).
- For every \( \lambda \in \mathcal{C}_\alpha \) there is at most one \( \phi \in \text{Stab}(\alpha) \) with \( \phi(\lambda) \in \mathcal{D}_\alpha \).
- For every \( \lambda \in \mathcal{C}_\alpha \) there is at least one \( \phi \in \text{Stab}(\alpha) \) with \( \phi(\lambda) \in \bar{\mathcal{D}}_\alpha \).

It follows that there is a set \( \Theta \subset \text{Map}(S) \) of representatives of the classes \( \text{Map}(S)/\text{Stab}(\alpha) \) such that
\[
\{\phi \in \text{Map}(S) \text{ with } \phi^{-1}(\sigma) \in \mathcal{D}_\alpha \} \subset \Theta \subset \{\phi \in \text{Map}(S) \text{ with } \phi^{-1}(\sigma) \in \bar{\mathcal{D}}_\alpha \}.
\]
The first inclusion in (2.2) yields:

\[ \#\{\alpha' \text{ of type } \alpha \text{ with } \iota(\alpha', \sigma) \leq L_n\} = \# \{\phi(\alpha) \text{ where } \phi \in \text{Map}(S) \text{ and } \iota(\phi(\alpha), \sigma) \leq L_n\} \]

\[ = \# \{\phi \in \Theta \text{ with } \iota(\phi(\alpha), \sigma) \leq L_n\} \]

\[ \geq \# \{\phi \in \text{Map}(S) \text{ with } \phi^{-1}(\sigma) \in D_\alpha \text{ and } \iota(\alpha, \phi^{-1}(\sigma)) \leq L_n\} \]

\[ = \# \left\{ \phi \in \text{Map}(S) \text{ with } \phi(\sigma) \in D_\alpha \text{ and } \iota \left( \alpha, \frac{\phi(\sigma)}{L_n} \right) \leq 1 \right\} \]

\[ = \left( \sum_{\phi \in \text{Map}(S)} \delta_{\frac{1}{L_n} \phi(\sigma)} \right) \left( \{\lambda \in D_\alpha, \iota(\alpha, \lambda) \leq 1\} \right) \]

\[ = \nu^L_\sigma \left( \{\lambda \in D_\alpha, \iota(\alpha, \lambda) \leq 1\} \right). \]

In light of (2.1) we thus get that

\[ \lim_{n \to \infty} \frac{\#\{\alpha' \text{ of type } \alpha \text{ with } \iota(\alpha', \sigma) \leq L_n\}}{L_n^{6g-6}} \geq \lim_{n \to \infty} \nu^L_\sigma \left( \{\lambda \in D_\alpha, \iota(\alpha, \lambda) \leq 1\} \right) \]

\[ \geq C \cdot \mu_{\text{Thu}} \left( \{\lambda \in D_\alpha, \iota(\alpha, \lambda) \leq 1\} \right) \]

An analogue computation using the right hand side of (2.2) implies

\[ \lim_{n \to \infty} \frac{\#\{\alpha' \text{ of type } \alpha \text{ with } \iota(\alpha', \sigma) \leq L_n\}}{L_n^{6g-6}} \leq C \cdot \mu_{\text{Thu}} \left( \{\lambda \in D_\alpha, \iota(\alpha, \lambda) \leq 1\} \right) \]

Now, both bounds agree because \( \sigma \) was chosen to be generic. We have proved that

\[ (2.3) \quad \lim_{n \to \infty} \frac{\#\{\alpha' \text{ of type } \alpha \text{ with } \iota(\alpha', \sigma) \leq L_n\}}{L_n^{6g-6}} = C \cdot \mu_{\text{Thu}} \left( \{\lambda \in D_\alpha, \iota(\alpha, \lambda) \leq 1\} \right). \]

Recall now another result from [6], an almost direct consequence of Proposition 2.1 above:

**Theorem 2.2.** [6 Corollary 4.4] If \( \sigma_1, \sigma_2 \in \mathcal{C} \) are filling currents, then we have

\[ \lim_{L \to \infty} \frac{\#\{\gamma' \text{ of type } \gamma \text{ with } \iota(\gamma', \sigma_1) \leq L\}}{\#\{\gamma' \text{ of type } \gamma \text{ with } \iota(\gamma', \sigma_2) \leq L\}} = \frac{B(\sigma_1)}{B(\sigma_2)} \]

for every multicurve in \( S \).

In Theorem 2.2 we are using the notation

\[ B(\sigma) = \mu_{\text{Thu}} \left( \{\lambda \in \mathcal{ML}, \iota(\lambda, \sigma) \leq 1\} \right) \]

for a filling current \( \sigma \). Note that this is consistent with the notation used in Theorem 1.1.

Combining (2.3) and Theorem 2.2, and noting that the length function \( \ell_X \) is given by taking intersections with a Liouville current, we get that, for any multicurve \( \alpha \)

\[ (2.4) \quad \lim_{n \to \infty} \frac{\#\{\alpha' \text{ of type } \alpha \text{ with } \ell_X(\alpha') \leq L_n\}}{L_n^{6g-6}} = C \cdot \mu_{\text{Thu}} \left( \{\lambda \in D_\alpha, \iota(\alpha, \lambda) \leq 1\} \right) \cdot \frac{B(X)}{B(\sigma)}. \]

Now comes the moment of invoking Mirzakhani’s theorem for simple curves. Indeed, assuming now that \( \alpha \) is simple we get from Theorem 1.1 that

\[ (2.5) \quad \lim_{L \to \infty} \frac{\#\{\alpha' \text{ of type } \alpha \text{ with } \ell_X(\alpha') \leq L\}}{L^{6g-6}} = \frac{B(X) \cdot c(\alpha)}{m_g}. \]

Setting the right-hand sides of equations (2.4) and (2.5) equal and solving for \( C \) we get

\[ (2.6) \quad C = \frac{1}{m_g} \cdot \frac{c(\alpha) \cdot B(\sigma)}{\mu_{\text{Thu}} \left( \{\lambda \in D_\alpha, \iota(\alpha, \lambda) \leq 1\} \right)}. \]

Noting now that this equality for \( C \) is completely independent of the sequence \( (L_n) \) we have proved:
Fact 1. We have
\[
\lim_{L \to \infty} \nu^L_\sigma = \frac{1}{m_g} \cdot \frac{c(\alpha)}{\mu_{\text{Thu}}(\{\lambda \in \mathcal{D}_\alpha, \iota(\alpha, \lambda) \leq 1\})} \cdot B(\sigma) \cdot \mu_{\text{Thu}}
\]
for any filling current \(\sigma\) and any (auxiliary) simple curve \(\alpha\).

Note then that by Fact 1 we can just write everywhere \(L \to \infty\) instead of \(L_n \to \infty\).

We are now ready to conclude the proof of Theorem 1.2. Still letting \(\sigma\) and \(\alpha\) be the previously fixed filling and simple curve, respectively, suppose now that \(\gamma\) is any multicurve, simple or not, filling or not. Applying (2.4) to \(\gamma\), and putting in the value for \(C\) given in (2.6) we get:
\[
\lim_{L \to \infty} \frac{\#\{\gamma' \text{ of type } \gamma \text{ with } \ell_X(\gamma') \leq L\}}{L^{6g-6}} = \frac{C \cdot \mu_{\text{Thu}}(\{\lambda \in \mathcal{D}_\gamma, \iota(\gamma, \lambda) \leq 1\}) \cdot B(X)}{B(\sigma)} = \frac{c(\alpha) \cdot \mu_{\text{Thu}}(\{\lambda \in \mathcal{D}_\alpha, \iota(\alpha, \lambda) \leq 1\}) \cdot B(X)}{m_g \cdot \mu_{\text{Thu}}(\{\lambda \in \mathcal{D}_\alpha, \iota(\alpha, \lambda) \leq 1\})}
\]
where \(c(\alpha)\) is Mirzakhani’s constant from Theorem 1.1. We have proved Theorem 1.2. \(\square\)

3. Sketch of proof of Theorem 1.1

What might be somewhat curious is that one can actually use the same kind of arguments to prove Theorem 1.1 as well. The idea is to replace, in the argument above, the existence of (2.5) by the use of the limit
\[
(3.1) \quad B(\sigma) = \lim_{L \to \infty} \frac{\#\{\alpha \in \mathcal{M}_L \text{ with } \iota(\alpha, \sigma) \leq L\}}{L^{6g-6}}.
\]
Here \(\mathcal{M}_L\) denotes the set of simple multicurves with integral weights. The existence of (3.1) follows immediately from (or maybe “is”) the construction of the Thurston measure \(\mu_{\text{Thu}}\) as a scaling limit. Anyways, ignoring momentarily the issue of exchanging limits in (\(\ast\)) below, we have for any \((L_n)\) as in (2.1) that
\[
B(\sigma) = \lim_{n \to \infty} \frac{\#\{\alpha \in \mathcal{M}_L \text{ with } \iota(\alpha, \sigma) \leq L\}}{L_n^{6g-6}} = \sum_{\gamma \in \mathcal{M}_L/\text{Map}(S)} \frac{\#\{\gamma' \text{ of type } \gamma \text{ with } \iota(\gamma', \sigma) \leq L_n\}}{L_n^{6g-6}}
\]
\[
(\ast) \quad \sum_{\gamma \in \mathcal{M}_L/\text{Map}(S)} \lim_{L \to \infty} \frac{\#\{\gamma' \text{ of type } \gamma \text{ with } \iota(\gamma', \sigma) \leq L_n\}}{L_n^{6g-6}}
\]
\[
= \sum_{\gamma \in \mathcal{M}_L/\text{Map}(S)} \mu_{\text{Thu}}(\{\lambda \in \mathcal{D}_\gamma, \iota(\gamma, \lambda) \leq 1\}) = \sum_{\gamma \in \mathcal{M}_L/\text{Map}(S)} \mu_{\text{Thu}}(\{\lambda \in \mathcal{D}_\gamma, \iota(\gamma, \lambda) \leq 1\}),
\]
where the fourth equality follows from (2.3). We can now use (3.2) to compute \(C\), getting:
\[
C = \frac{B(\sigma)}{\sum_{\gamma \in \mathcal{M}_L/\text{Map}(S)} \mu_{\text{Thu}}(\{\lambda \in \mathcal{D}_\gamma, \iota(\gamma, \lambda) \leq 1\})}.
\]
In particular, \(C\) is independent of the sequence \((L_n)\). Hence, by setting
\[
\kappa = \sum_{\gamma \in \mathcal{M}_L/\text{Map}(S)} \mu_{\text{Thu}}(\{\lambda \in \mathcal{D}_\gamma, \iota(\gamma, \lambda) \leq 1\})
\]
we get the following new “version” of Fact 1.
Fact 2. For any filling multicurve \( \sigma \) we have
\[
\lim_{L\to\infty} \nu_L^\sigma = \frac{1}{\kappa} \cdot B(\sigma) \cdot \mu_{\text{Thu}}
\]
where \( \kappa \) is as in (3.3). 

Now, Fact 2 and (2.3) together imply that
\[
\lim_{n\to\infty} \frac{\{\alpha' \text{ of type } \alpha \text{ with } \ell_X(\alpha') \leq L_n\}}{L_n^{6g-6}} = \frac{B(X)}{\kappa} \cdot c(\alpha)
\]
for any multicurve \( \alpha \), simple or not. Invoking Theorem 2.2 again we get:

**Theorem 3.1.** For any multicurve \( \alpha \), simple or not, we have
\[
\lim_{n\to\infty} \frac{\{\alpha' \text{ of type } \alpha \text{ with } \ell_X(\alpha') \leq L_n\}}{L_n^{6g-6}} = \frac{B(X)}{\kappa} \cdot c(\alpha)
\]
where \( c(\alpha) = \mu_{\text{Thu}}(\{\lambda \in D_{\alpha}, \nu(\alpha, \lambda) \leq 1\}) \) and \( \kappa \) is as in (3.3).

In the argument above, the only point we have not explained in detail is equality (*) in (3.2). This is indeed not completely obvious, but follows easily from the following result:

**Proposition 3.2.** For every \( \epsilon > 0 \) there exist a constant \( L_0 \) and a finite set \( N \subset ML\mathbb{Z}/\text{Map}(S) \) of types of simple multicurves with
\[
\sum_{\gamma \in ML\mathbb{Z}/\text{Map}(S), \gamma \notin N} \#\{\gamma' \text{ of type } \gamma \text{ with } \ell_X(\gamma') \leq L_n\} \leq \epsilon \cdot L_n^{6g-6}
\]
for all \( L \geq L_0 \).

The proof of Proposition 3.2 uses arguments similar to those used in [6] and full details will appear in [7].

4. **Forthcoming pamphlet**

What else will also appear in [7]? Well, for one, we will use the kind of arguments given here to get to the other theorems proved by Mirzakhani in [11, 12]. For example, what she refers to as studying the statistics of pants decompositions – see Theorem 1.2 in [12]. We will also explain how to use Theorem 3.1 to derive a lattice counting result due to Rafi and the second author of this note [13], including the finite volume case that was not treated earlier. Also, results closely related to another forthcoming paper of Kasra Rafi and the second author [13] will also enter into [7]. In [14] we give in principle non-recursive numerical expressions for the constant \( c(\alpha) \) in Theorem 3.1.

Among other things, we recover Mirzakhani computations [11] of relative frequencies of different kinds of curves (it seems to be well understood that some of her computations are off by a factor of the form \( 2^k \) – our numbers are thus different from hers). Some of these computations will also be explained in [7]. It is indeed important to make clear that these constants are computable because they are related, via other results of Mirzakhani, to intersections of Chern classes in moduli space, and through that to all sorts of important areas of fundamental science that we ignore completely. Anyways, Mirzakhani’s constants were closely linked to the Weil-Petersson volume of certain moduli spaces, which she was able to compute recursively using her version [10] of the McShane identity. As we mentioned above, our computations are in principle non-recursive, but this might be kind of misleading because they involve knowing the number of triangulations of a surface with this or that property. In simple situations one can get hold of those constants by hand, but probably the best one can actually do to get exact numbers is to give a recursive formula...

In any case, we intend to write a more or less self-contained book, giving complete proofs of all the results mentioned here and of the needed background. Other than a certain familiarity with
hyperbolic surfaces, laminations, and train tracks, we will only be assuming Masur’s ergodicity theorem.

References

[1] J. Aramayona and C. Leininger, Hyperbolic structures on surfaces and geodesic currents, in Algorithms and geometric topics around automorphisms of free groups, Advanced Courses CRM-Barcelona, Birkhäuser.
[2] F. Bonahon, Bouts des variétés hyperboliques de dimension 3, Ann. of Math. 124 (1986).
[3] F. Bonahon, The geometry of Teichmüller space via geodesic currents, Invent. Math. 92 (1988).
[4] F. Bonahon, Geodesic currents on negatively curved groups, in Arboreal group theory MSRI Publ., 19, Springer, 1991.
[5] V. Erlandsson and G. Mondello, Ergodic invariant measures on the space of geodesic currents, arXiv:1807.02144.
[6] V. Erlandsson and J. Souto, Counting Curves in Hyperbolic Surfaces, GAFA 26, (2016).
[7] V. Erlandsson and J. Souto, booklet in preparation
[8] H. Masur, Ergodic actions of the mapping class group, Proc. Amer. Math. Soc. 94, (1985).
[9] G. McShane, Simple geodesics and a series constant over Teichmüller space, Invent. Math. 132 (1998).
[10] M. Mirzakhani, Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces, Invent. Math. 167 (2007).
[11] M. Mirzakhani, Growth of the number of simple closed geodesics on hyperbolic surfaces, Ann. of Math.(2), 168(1) (2008).
[12] M. Mirzakhani, Counting mapping class group orbits on hyperbolic surfaces, arXiv:1601.03342.
[13] K. Rafi and J. Souto, Geodesics currents and counting problems, to appear in GAFA.
[14] K. Rafi and J. Souto, Statistics of simple curves on surfaces, revisited, in preparation.

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