Inertial KM-type extragradient scheme for solving a variational inequality and a hierarchical fixed point problems

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Abstract

We propose an inertial KM-type extragradient scheme to approximate a common solution of a variational inequality problem and a hierarchical fixed point problem for nonexpansive mappings. This scheme generalizes and unifies a number of known iterative schemes. Furthermore, we discuss the weak convergence for the proposed scheme. We also discuss an example to illustrate the main theorem.

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1 Introduction

Let \( C \) be a nonempty convex and closed set in a real Hilbert space \( \mathcal{H} \) and \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) denote the inner product and induced norm on \( \mathcal{H} \). A mapping \( U : C \to C \) is said to be nonexpansive if \( \| Uu - Uv \| \leq \| u - v \| \), \( \forall u, v \in C \). Note that if \( F(U) := \{ u \in C : Uu = u \} \neq \emptyset \) then set \( F(U) \) is convex and closed. Let \( F(U) \neq \emptyset \). The subdifferential of a proper function \( g : \mathcal{H} \to (-\infty, +\infty] \) is the set-valued operator \( \partial g : \mathcal{H} \to 2^\mathcal{H} \) defined by \( \partial g(u) = \{ w \in \mathcal{H} : \langle y - u, w \rangle + g(u) \leq g(y), \forall y \in \mathcal{H} \} \). Let \( u \in \mathcal{H} \). Then \( g \) is subdifferential at \( u \) if \( \partial g(u) \neq \emptyset \). The indicator function \( \psi_C : \mathcal{H} \to (-\infty, +\infty] \) is given by

\[
\partial \psi_C(u) = \begin{cases} 
0, & u \in C, \\
\infty, & \text{otherwise}.
\end{cases}
\]

Note that \( \psi_C \) is a convex function when \( C \) is a convex set.

In 2006, Moudafi et al. [1] discussed the convergence of a scheme for the following hierarchical fixed point problem (in short, H-FPP): Find \( \bar{u} \in F(U) \) such that

\[
\langle \bar{u} - V\bar{u}, \bar{u} - u \rangle \leq 0, \quad \forall u \in F(U),
\]

where the mappings \( U, V : C \to C \) are nonexpansive. Let \( \Phi \) denote the set of solutions of H-FPP(1.1). If \( \bar{u} \in F(U) \) then (1.1) \( \Leftrightarrow \langle -(I - V)\bar{u}, u - \bar{u} \rangle + \psi_{F(U)}(\bar{u}) \leq \psi_{F(U)}(u) \Leftrightarrow -(I - V)\bar{u} \in F(U) \).
∂ψ_{F(U)}(\bar{u}). Hence H-FPP(1.1) is equivalent to the variational inclusion: Find \bar{u} ∈ F(U) such that

$$0 ∈ (I - V)\bar{u} + N_{F(U)}(\bar{u}),$$

(1.2)

where the mapping I is identity on C and \( N_{F(U)}(\bar{u}) \) denotes the normal cone to F(U) at \( \bar{u} \) given by

$$N_{F(U)}(\bar{u}) = \partial \psi_{F(U)}(\bar{u}) = \begin{cases} \{ w ∈ H : \langle \mu - \bar{u}, w \rangle ≤ 0, \forall \mu ∈ F(U) \}, & \text{if } \bar{u} ∈ F(U), \\ \emptyset, & \text{otherwise.} \end{cases}$$

If we set V = I, then \( \Phi \) is just F(U). Furthermore, we mention that H-FPP(1.1) is worth to study because it includes as special cases, the important problems such as the variational inequality on fixed point sets and hierarchical minimization problems; see Moudafi [2].

In 2007, Moudafi [2] proposed the following Krasnoselski–Mann (KM)-type scheme for solving H-FPP(1.1): For given \( u_0 \in C \),

$$u_{k+1} = (1 - \alpha_k)u_k + \alpha_k \left( \sigma_k V u_k + (1 - \sigma_k)U u_k \right), \quad \forall n ≥ 0,$$

(1.3)

where \( \{ \alpha_k \} \subset (0,1) \) and \( \{ \sigma_k \} \subset (0,1) \). For further work related to scheme (1.3), see for example [1, 3–7].

In 2008, Mainge [8] introduced an inertial version of KM-type scheme by unifying the KM-type scheme and the inertial extrapolation, for approximating a fixed point of non-expansive mappings and discussed the weak convergence. Recently, Bot et al. [9] derived some the convergence results of the following inertial KM-type scheme to approximate a fixed point of nonexpansive mapping U on H which generalize the results of Mainge [8]:

$$\begin{align*}
  t_k &= u_k + \eta_k (u_k - u_{k-1}), \\
  u_{k+1} &= (1 - \alpha_k)t_k + \alpha_k U t_k,
\end{align*}$$

(1.4)

for each \( k ≥ 1 \), where \( \eta_k \) is a damping-type term and \( \alpha_k \) is a relaxation factor. Recently, the interest of studying inertial type algorithms has been increased due to their fast convergence. For further study of scheme (1.4) and its generalizations; see for example [10–13].

On the other hand, we consider the classical variational inequality (in short, VI): Find \( \bar{u} ∈ C \) such that

$$\langle h(\bar{u}), v - \bar{u} \rangle ≥ 0, \quad \forall v ∈ C,$$

(1.5)

introduced in [14] where \( h : H → H \). The set of solutions of VI(1.5) is denoted by \( \text{Sol}(\text{VI}(1.5)) \). Note that the projected gradient scheme for solving VI(1.5) is

$$u_{k+1} = P_C(u_k - \mu h(u_k)),$$

(1.6)

where \( \mu > 0 \) and \( P_C \) is the metric projection onto C. In order to converge, this scheme requires the restrictive condition that \( h \) is inverse strongly (or strongly) monotone. To
overcome this difficulty, Korpelevich [15] proposed an extragradient iterative scheme by
\[
v_k = \mathcal{P}_C(u_k - \mu h(u_k)), \\
u_{k+1} = \mathcal{P}_C(u_k - \mu h(v_k)),
\]
(1.7)
where \( \mu \in (0, \frac{1}{L}) \), where \( L > 0 \) is Lipschitz constant of \( h \). Since then many researchers improved scheme (1.7) in various directions; see, e.g. [16–24] and the references therein. Note that the calculation of two projections onto \( C \) might affect the efficiency of such scheme. Therefore, Dong et al. [25] proposed the following inertial KM-type extragradient scheme for VI(1.5):
\[
t_k = u_k + \eta_k(u_k - u_{k-1}), \\
v_k = \mathcal{P}_C(t_k - \mu h(t_k)), \\
u_{k+1} = (1 - \alpha_k)t_k + \alpha_k \mathcal{P}_C(t_k - \mu h(v_k)),
\]
(1.8)
where \( \{\eta_k\} \subset [0, \eta] \), \( \forall k \) is nondecreasing with \( \eta_1 = 0 \) and \( 0 \leq \eta_k \leq \eta < 1 \), for every \( k \geq 1 \) such that
\[
\delta > \frac{\eta[(1 + \mu L)^2 \eta(1 + \eta) + (1 - \mu^2 L^2) \eta \sigma + \sigma(1 + \mu L)^2]}{1 - \mu^2 L^2}
\]
and
\[
0 < \alpha \leq \frac{\delta(1 - \mu^2 L^2) - \eta[(1 + \mu L)^2 \eta(1 + \eta) + (1 - \mu^2 L^2) \eta \sigma + \sigma(1 + \mu L)^2]}{\delta[(1 + \mu L)^2 \eta(1 + \eta) + (1 - \mu^2 L^2) \eta \sigma + \sigma(1 + \mu L)^2]},
\]
where \( \alpha, \sigma, \delta > 0 \).

They proved the weak convergence theorem for scheme (1.8).

In this paper, we propose an inertial version of KM-type extragradient scheme by combining iterative schemes (1.3) and (1.8) to approximate a common solution of H-FPP(1.1) and VI(1.5). We prove a weak convergence theorem for the proposed scheme. Furthermore, we discuss an example to illustrate the main theorem. The theorems of the paper unify and generalize previously known corresponding theorems; see for example [2, 8, 9, 25–27].

2 Preliminaries

We give some definitions and results of convex and nonlinear analysis, which will be used in the proof of the weak convergence theorem.

A mapping \( \mathcal{P}_C \) is called the metric projection of \( \mathcal{H} \) onto \( C \) if for every point \( u \in \mathcal{H} \), there exists a unique point in \( C \) denoted by \( \mathcal{P}_C u \) such that
\[
\|u - \mathcal{P}_C u\| \leq \|u - v\|, \quad \forall v \in C.
\]
Note that \( \mathcal{P}_C \) is nonexpansive and satisfies
\[
\langle u - v, \mathcal{P}_C u - \mathcal{P}_C v \rangle \geq \|\mathcal{P}_C u - \mathcal{P}_C v\|^2, \quad \forall u \in \mathcal{H}.
\]
Moreover, $\mathcal{P}_C u$ is characterized by the fact $\mathcal{P}_C u \in \mathcal{C}$ and

$$\langle u - \mathcal{P}_C u, v - \mathcal{P}_C u \rangle \leq 0, \quad \forall v \in \mathcal{C},$$

which implies that

$$\|u - v\|^2 \geq \|u - \mathcal{P}_C u\|^2 + \|v - \mathcal{P}_C u\|^2, \quad \forall u \in \mathcal{H}, v \in \mathcal{C}.$$

**Definition 2.1** A mapping $h: \mathcal{H} \rightarrow \mathcal{H}$ is called:

(i) monotone, if for all $u, v \in \mathcal{H}$, we have

$$\langle hu - hv, u - v \rangle \geq 0;$$

(ii) $L$-Lipschitz continuous, if there exists a constant $L > 0$ such that, for all $u, v \in \mathcal{H}$, we have

$$\|hu - hv\| \leq L\|u - v\|.$$

**Lemma 2.1** If a mapping $U$ is nonexpansive on $\mathcal{H}$ then $I - U$ is maximal monotone [28] and demiclosed [29] on $\mathcal{H}$.

**Lemma 2.2** ([30]) Let $\{\psi_k\}, \{\delta_k\}$ and $\{\eta_k\}$ be the sequences in $[0, \infty)$ such that $\psi_{k+1} \leq \psi_k + \eta_k(\psi_k - \psi_{k-1}) + \gamma_k, \forall k \geq 1$, $\sum_{k=1}^{\infty} \gamma_k < +\infty$ and there is a number $\eta$ with $0 \leq \eta_k \leq \eta < 1, \forall k \geq 1$. Then the following hold:

(a) $\sum_{k=1}^{\infty} |\psi_k - \psi_{k-1}| < +\infty$, where $[r] := \max\{r, 0\};$

(b) there is a $\psi^* \in [0, \infty)$ such that $\lim_{k \rightarrow \infty} \psi_k = \psi^*$.

**Lemma 2.3** ([31]) Let $C$ be a nonempty subset of $\mathcal{H}$ and the sequence $\{u_k\}$ in $\mathcal{H}$ satisfy the conditions:

(a) $\lim_{k \rightarrow \infty} \|u_k - u\|$ exists for every $u \in C$;

(b) any weak cluster point of $\{u_k\}$ is in $C$.

Then $\{u_k\}$ is weak convergent to a point in $C$.

### 3 Weak convergence theorem

We propose the following inertial KM-type extragradient scheme for solving H-FPP (1.1) and VI (1.5).

**Scheme** Choose initial values $u_0, u_1 \in \mathcal{H}$ arbitrarily. The sequence $\{u_k\}$ be generated by the scheme:

\[
\begin{align*}
t_k &= u_k + \eta_k(u_k - u_{k-1}), \\
v_k &= \mathcal{P}_C(t_k - \mu h(t_k)), \\
w_k &= \mathcal{P}_C(t_k - \mu h(v_k)), \\
u_{k+1} &= (1 - \alpha_k)u_k + \alpha_k(\sigma_k Vw_k + (1 - \sigma_u)Uw_k),
\end{align*}
\]

(3.1)
where \( \{\eta_k\} \subset [0, \eta], \forall k, \) is nondecreasing with \( \eta_1 = 0 \) and \( 0 \leq \eta_k \leq \eta < 1 \), \( \{\sigma_k\} \subset [c, d], \) \( c, d \in (0, 1) \), \( \mu \in (0, \frac{1}{L}) \), \( L > 0 \) and \( \{\alpha_k\} \) is a real sequence with conditions:

\[
\delta > \frac{\eta^2(1 + \eta) + \eta \sigma}{1 - \eta^2} \quad \text{and} \quad 0 < \alpha \leq \alpha_k \leq \frac{\delta - \eta(\eta(1 + \eta) + \eta \delta + \sigma)}{\delta(1 + \eta(1 + \eta) + \eta \delta + \sigma)}, \quad \text{where} \quad \alpha, \sigma, \delta > 0.
\]

Now, we discuss the weak convergence for scheme (3.1).

**Theorem 3.1** Let \( \mathcal{H} \) be a real Hilbert space and \( C \subset \mathcal{H} \) be a nonempty, convex and closed set; let the mappings \( U, V : \mathcal{C} \to \mathcal{C} \) be nonexpansive and \( h : \mathcal{H} \to \mathcal{H} \) be \( L \)-Lipschitz continuous and monotone. Assume that \( \Gamma = \text{Sol}(\mathcal{V}(1.5)) \cap \Phi \cap F(V) \neq \emptyset \). Let the sequence \( \{u_k\} \) be defined by scheme (3.1). Then the following results hold:

(a) \( \sum_{k=1}^{\infty} \|u_{k+1} - u_k\|^2 < +\infty; \)

(b) \( \{u_k\} \) converges weakly to \( \bar{u} \in \Gamma \).

**Proof (a).** Let for any \( q \in \Gamma \). Since \( h \) is \( L \)-Lipschitz continuous and monotone then we can easily get

\[
\|w_k - q\|^2 \leq \|t_k - q\|^2 - (1 - \mu^2L^2)\|t_k - v_k\|^2;
\]  
(3.2)

see [3]. From the nonexpansivity of \( P_C \) and Lipschitz continuity of \( h \), it follows that

\[
\|v_k - w_k\| = \|P_C(t_k - \mu h(t_k)) - P_C(t_k - \mu h(v_k))\| \leq \mu \|h(t_k) - h(v_k)\| \leq \mu L \|t_k - v_k\|, \tag{3.3}
\]

which yields

\[
\|t_k - w_k\| \leq \|t_k - v_k\| + \|v_k - w_k\| \leq (1 + \mu L)\|t_k - v_k\|. \tag{3.4}
\]

As follows from (3.2), (3.4) and \( \mu L \in (0, 1) \), we have

\[
\|w_k - q\|^2 \leq \|t_k - q\|^2 - \frac{1 - \mu^2L^2}{(1 + \mu L)^2}\|t_k - w_k\|^2. \tag{3.5}
\]

Let for any \( q \in \Gamma \) and \( T_{\sigma_k} := \sigma_k V + (1 - \sigma_k)U \). Now, by using (3.5), we estimate

\[
\|u_{k+1} - q\|^2 = \|(1 - \alpha_k)t_k + \alpha_k T_{\sigma_k}w_k - q\|^2 \\
\leq (1 - \alpha_k)\|t_k - q\|^2 + \alpha_k \|T_{\sigma_k}w_k - q\|^2 - \alpha_k(1 - \alpha_k)\|T_{\sigma_k}w_k - t_k\|^2 \\
\leq (1 - \alpha_k)\|t_k - q\|^2 + \alpha_k \|Vw_k - q\|^2 + (1 - \sigma_k)\|Uw_k - q\|^2 \\
- \sigma_k(1 - \sigma_k)\|Vw_k - Uw_k\|^2 - \alpha_k(1 - \alpha_k)\|T_{\sigma_k}w_k - t_k\|^2 \\
\leq \|t_k - q\|^2 - \alpha_k \sigma_k(1 - \sigma_k)\|Vw_k - Uw_k\|^2 - \frac{1 - \mu^2L^2}{(1 + \mu L)^2}\|t_k - v_k\|^2 \\
- \alpha_k(1 - \alpha_k)\|T_{\sigma_k}w_k - t_k\|^2 \leq \|t_k - q\|^2 - \alpha_k(1 - \alpha_k)\|T_{\sigma_k}w_k - t_k\|^2. \tag{3.6}
\]

(3.6)
Next, we estimate

$$
\|u_k - q\|^2 = \|u_k + \eta_k (u_k - u_{k-1}) - q\|^2 \\
= (1 + \eta_k) \|u_k - q\|^2 - \eta_k \|u_{k-1} - q\|^2 \\
+ \eta_k (1 + \eta_k) \|u_k - u_{k-1}\|^2. \quad (3.8)
$$

From (3.7) and (3.8), we have

$$
\|u_{k+1} - q\|^2 - (1 + \eta_k) \|u_k - q\|^2 + \eta_k \|u_{k-1} - q\|^2 \leq -\alpha_k (1 - \alpha_k) \|T_{\sigma_k} u_k - \ell_k\|^2 \\
+ \eta_k (1 + \eta_k) \|u_k - u_{k-1}\|^2. \quad (3.9)
$$

Furthermore, from scheme (3.1), we have

$$
\|T_{\sigma_k} w_k - \ell_k\|^2 = \left\| \frac{1}{\alpha_k} (u_{k+1} - u_k) + \frac{\eta_k}{\alpha_k} (u_{k-1} - u_k) \right\|^2 \\
\geq \frac{1}{\alpha_k^2} \|u_{k+1} - u_k\|^2 + \frac{\eta_k^2}{\alpha_k^2} \|u_k - u_{k-1}\|^2 \\
+ \frac{\eta_k}{\alpha_k^2} \left( -\rho_k \|u_{k+1} - u_k\|^2 - \frac{1}{\rho_k} \|u_k - u_{k-1}\|^2 \right), \quad (3.10)
$$

where $\rho_k := \frac{1}{\eta_k + \delta \alpha_k}$. Thus, it follows from (3.9) and (3.10) that

$$
\|u_{k+1} - q\|^2 - (1 + \eta_k) \|u_k - q\|^2 + \eta_k \|u_{k-1} - q\|^2 \leq \frac{(1 - \alpha_k) (\eta_k \rho_k - 1)}{\alpha_k} \|u_{k+1} - u_k\|^2 \\
+ \gamma_k \|u_k - u_{k-1}\|^2, \quad (3.11)
$$

where

$$
\gamma_k := \eta_k (1 + \eta_k) + \eta_k (1 - \alpha_k) \frac{(1 - \eta_k \rho_k)}{\alpha_k \rho_k} > 0, \quad (3.12)
$$

since $\eta_k \rho_k < 1$ and $\alpha_k \in (0, 1)$. It follows from $\delta = \frac{(1 - \eta_k \rho_k)}{\alpha_k \rho_k}$ and (3.12) that

$$
\gamma_k := \eta_k (1 + \eta_k) + \eta_k (1 - \alpha_k) \delta \leq \eta (1 + \eta) + \eta \delta, \quad \forall k \geq 1. \quad (3.13)
$$

Next, we define the sequences $\{\phi_k\}$ and $\{\psi_k\}$ by

$$
\phi_k := \|x_k - q\|^2, \quad \psi_k := \phi_k - \eta_k \phi_{k-1} + \gamma_k \|u_k - u_{k-1}\|^2, \quad \forall k \geq 1. \quad (3.14)
$$

Now, using the monotonicity of $\{\eta_k\}$ and the fact that $\phi_k \geq 0$ for all $k \in \mathbb{N}$, we have

$$
\psi_{k+1} - \psi_k \leq \phi_{k+1} - (1 + \eta_k) \phi_k + \eta_k \phi_{k-1} + \gamma_{k+1} \|u_{k+1} - u_k\|^2 - \gamma_k \|u_k - u_{k-1}\|^2. \quad (3.15)
$$
Hence, it follows from \((3.11)\) and \((3.15)\) that
\[
\psi_{k+1} - \psi_k \leq \frac{(1 - \alpha_k)(\eta_k \rho_k - 1)}{\alpha_k} \|u_{k+1} - u_k\|^2 + \gamma_{k+1} \|u_{k+1} - u_k\|^2
= \left(\frac{(1 - \alpha_k)(\eta_k \rho_k - 1)}{\alpha_k} + \gamma_{k+1}\right) \|u_{k+1} - u_k\|^2.
\] (3.16)

Now, we note that
\[
\frac{(1 - \alpha_k)(\eta_k \rho_k - 1)}{\alpha_k} + \gamma_{k+1} \leq -\sigma, \quad \forall k \geq 1;
\] (3.17)
see [9].

Therefore, it follows from \((3.16)\) and \((3.17)\) that
\[
\psi_{k+1} - \psi_k \leq -\sigma \|u_{k+1} - u_k\|^2.
\] (3.18)

Since \(\eta_1 = 0\), it follows from \((3.14)\) that \(\psi_1 = \phi_1 \geq 0\) and hence \((3.18)\) shows that \(\{\psi_k\}\) is bounded. Furthermore, \((3.14)\) and the boundedness of \(\{\eta_k\}\) yield
\[
-\eta \phi_{k-1} \leq \phi_k - \eta \phi_{k-1} \leq \psi_k \leq \psi_1.
\] (3.19)

Thus, we obtain
\[
\phi_k \leq \eta^k \phi_0 + \psi_1 \sum_{j=1}^{k-1} \eta^j \leq \eta^k \phi_0 + \frac{1}{1 - \eta} \psi_1.
\] (3.20)

Now, it follows from \((3.18), (3.19), (3.20)\) and the boundedness of \(\{\psi_k\}\) that
\[
\sigma \sum_{j=1}^{k} \|u_{j+1} - u_j\|^2 \leq \psi_1 - \psi_{k+1} \leq \psi_1 + \eta \phi_k \leq \psi_1 + \eta^k \phi_0 + \frac{1}{1 - \eta} \psi_1,
\] (3.21)
which implies that \(\sum_{k=1}^{\infty} \|u_{k+1} - u_k\|^2 < +\infty\).

**Proof of (b).** Since \(\eta_k \rho_k < 1\), it follows from \((3.11), (3.13)\), \(\sum_{k=1}^{\infty} \|u_{k+1} - u_k\|^2 < +\infty\), and Lemma 2.2 that
\[
\lim_{k \to \infty} \|u_k - q\| \quad \text{exists and finite},
\] (3.22)
and hence \(\{u_k\}\) is bounded. It follows furthermore from \(\sum_{k=1}^{\infty} \|u_{k+1} - u_k\|^2 < +\infty\) that
\[
\lim_{k \to \infty} \|u_{k+1} - u_k\| = 0.
\] (3.23)

Next, by the definition of \(t_k\) in \((3.1)\) and \(\eta_k \leq \eta, \forall k\), we have
\[
\|t_k - u_k\| = \eta_k \|u_k - u_{k-1}\| \leq \eta \|u_k - u_{k-1}\|,
\]
which implies that
\[
\lim_{k \to \infty} \| t_k - u_k \| = 0, \tag{3.24}
\]
and hence \( \{ t_k \} \) is bounded. Since
\[
\| t_k - u_{k+1} \| \leq \| t_k - u_k \| + \| u_k - u_{k+1} \|, \tag{3.25}
\]
it follows from (3.23), (3.24) and (3.25) that
\[
\lim_{k \to \infty} \| t_k - u_{k+1} \| = 0. \tag{3.26}
\]
From (3.6) and (3.26), and \( \{ \alpha_k \} \subseteq (0, 1), \{ \sigma_k \} \subseteq [c, d], c, d \in (0, 1) \), we have
\[
\begin{align*}
\alpha_k \sigma_k (1 - \sigma_k) \| Vw_k - Uw_k \|^2 & = \| t_k - q \|^2 - \| u_{k+1} - q \|^2 \\
& \leq \| t_k - u_{k+1} \| (\| t_k - q \| + \| u_{k+1} - q \|) \\
& = \| t_k - u_{k+1} \| M_1,
\end{align*}
\]
where \( M_1 := \sup_k (\| t_k - q \| + \| u_{k+1} - q \|) \). Hence, it follows
\[
\lim_{k \to \infty} \| Vw_k - Uw_k \| = 0. \tag{3.27}
\]
From (3.6) and (3.26), and \( \mu L \in (0, 1) \), we have
\[
\begin{align*}
\frac{1 - \mu^2 L^2}{(1 + \mu^2 L^2)^2} \| t_k - w_k \|^2 & \leq \| t_k - q \|^2 - \| u_{k+1} - q \|^2 \\
& = \| t_k - u_{k+1} \| M_1,
\end{align*}
\]
it follows that
\[
\lim_{k \to \infty} \| t_k - w_k \| = 0. \tag{3.28}
\]
It follows from (3.26) and (3.28) that
\[
\lim_{k \to \infty} \| t_k - u_{k+1} - \alpha_k (t_k - w_k) \| = 0. \tag{3.29}
\]
Furthermore, we have
\[
\begin{align*}
\alpha_k \| Uw_k - w_k \| & \leq \| u_{k+1} - t_k \| + \alpha_k \| t_k - w_k \| + \alpha_k \sigma_k \| Uw_k - Vw_k \|, \\
\| Uw_k - w_k \| & \leq \frac{1}{\alpha_k} \| u_{k+1} - t_k \| + \| t_k - w_k \| + \sigma_k \| Uw_k - Vw_k \|. \tag{3.30}
\end{align*}
\]
Since \( \alpha_k > 0, \forall k \), it follows from (3.26), (3.27), (3.28) and (3.30) that
\[
\lim_{k \to \infty} \| Uw_k - w_k \| = 0. \tag{3.31}
\]
From (3.27) and (3.31), we have
\[
\lim_{k \to \infty} \| V w_k - w_k \| = 0. \tag{3.32}
\]

Now, let $\widetilde{u}$ be a sequential weak cluster point of $\{u_k\}$, that is, there exists a subsequence $\{u_{k_i}\}$ of $\{u_k\}$ such that $\{u_{k_i}\}$ converges weakly to $\widetilde{u}$, say, in $H$. Furthermore, (3.24) and (3.28) imply that $\{u_{k_i}\}$, $\{t_{k_i}\}$ and $\{w_{k_i}\}$ all have the same asymptotic behavior and hence there exist subsequences $\{t_{k_i}\}$ of $\{t_k\}$ and $\{w_{k_i}\}$ of $\{w_k\}$ such that $t_{k_i}$ and $w_{k_i}$ both converge weakly to $\widetilde{u}$. Now, Lemma 2.1, (3.31) and (3.32) imply that $\widetilde{u} \in F(U)$ and $\widetilde{u} \in F(V)$.

Next, we prove that $\widetilde{u} \in \Phi_1$. Since
\[
u_{k+1} - t_k = \alpha_k \left( w_k - t_k \right) + \alpha_k \left( \sigma_k (Vw_k - w_k) + (1 - \sigma_k) (Uw_k - w_k) \right), \tag{3.33}
\]
and hence
\[
\frac{1}{\alpha_k \sigma_k} \left( t_k - u_{k+1} - \alpha_k (t_k - w_k) \right) = (I - V) w_k + \frac{1 - \sigma_k}{\sigma_k} (I - U) w_k, \tag{3.34}
\]
and therefore for all $z \in F(U)$ and by making use of the monotonicity of $I - V$, we have
\[
\frac{1}{\alpha_k \sigma_k} \left( t_k - u_{k+1} - \alpha_k (t_k - w_k) \right), w_k - z \right) = \left( (I - V) w_k - (I - V) z, w_k - z \right)
+ \left( (I - V) z, w_k - z \right)
+ \frac{1 - \sigma_k}{\sigma_k} (w_k - Uw_k, w_k - z)
\geq \left( (I - V) z, w_k - z \right)
+ \frac{1 - \sigma_k}{\sigma_k} (w_k - Uw_k, w_k - z). \tag{3.35}
\]

Hence,
\[
\frac{1}{\alpha_k \sigma_k} \left( t_k - u_{k+1} - \alpha_k (t_k - w_k) \right), w_k - z \right) \geq \left( (I - V) z, w_k - z \right)
+ \frac{1 - \sigma_k}{\sigma_k} (w_k - Uw_k, w_k - z). \tag{3.36}
\]

Using (3.29), (3.31), and the conditions on the parameters $\alpha_k$ and $\sigma_k$ in (3.36), we have
\[
\limsup_{i \to \infty} \langle z - V z, w_{k_i} - z \rangle \leq 0 \quad \forall z \in F(U). \tag{3.37}
\]

Since $w_{k_i}$ converges weakly to $\widetilde{u}$, we get
\[
\langle (I - V) z, \widetilde{u} - z \rangle \leq 0, \quad \forall z \in F(U). \tag{3.38}
\]

Since $F(U)$ is convex, $\beta z + (1 - \beta) \widetilde{u} \in F(U)$ for $\beta \in (0, 1)$ and hence
\[
\langle (I - V) (\beta z + (1 - \beta) \widetilde{u}), \widetilde{u} - (\beta z + (1 - \beta) \widetilde{u}) \rangle \tag{3.39}
\]
\[ = \beta \langle (I - V)(\beta z + (1 - \beta)\bar{u}), \bar{u} - z \rangle \]

\[ \leq 0, \quad \forall z \in F(U), \] (3.40)

which implies

\[ \langle (I - V)(\beta z + (1 - \beta)\bar{u}), \bar{u} - z \rangle \leq 0, \quad \forall z \in F(U). \] (3.41)

On taking the limit \( \beta \to 0^+ \), we have

\[ \langle (I - V)\bar{u}, \bar{u} - z \rangle \leq 0, \quad \forall z \in F(U), \] (3.42)

which implies \( \bar{u} \in \Phi \).

Now, we show that \( \bar{u} \in \text{Sol}(VI(1.5)) \). Since \( \lim_{k \to \infty} \| v_k - t_k \| = 0 \) and \( \lim_{k \to \infty} \| t_k - u_k \| = 0 \), there exist subsequences \( \{ t_{ki} \} \) of \( \{ t_k \} \) and \( \{ v_{ki} \} \) of \( \{ v_k \} \), respectively, such that \( \{ t_{ki} \}, \{ v_{ki} \} \) both converge weakly to \( \bar{u} \). Let

\[ G_v = \begin{cases} 
  hv + N_C(v), & \text{if } v \in C; \\
  \emptyset, & \text{if } v \notin C,
\end{cases} \]

then the monotone mapping \( G \) is maximal [32] and hence \( 0 \in G_v \) if and only if \( v \in \text{Sol}(VI(1.5)) \) [33]. Let \( (v, w) \in \text{graph}(G) \), then \( w \in G_v = hv + N_C(v) \) and hence \( w - hv \in N_C(v) \), i.e., \( \langle v - u, w - hv \rangle \geq 0 \), for all \( u \in C \).

On the other hand, from \( v_k = P_C((I - \mu h)t_k) \) and \( v \in C \), we get

\[ \langle (I - \mu h)t_k - v_k, v_k - v \rangle \geq 0. \]

This implies that

\[ \left\{ v^* - v_k, \frac{v_k - t_k}{\mu} + ht_k \right\} \geq 0. \]

Since \( \langle v - u, w - hv \rangle \geq 0 \), for all \( u \in C \) and \( v_{ki} \in C \), using the monotonicity of \( h \), we have

\[ \langle v - v_{ki}, w \rangle \geq \langle v - v_{ki}, hv \rangle \]

\[ \geq \langle v - v_{ki}, hv - hv_{ki} \rangle + \langle v - v_{ki}, hv_{ki} - h t_{ki} \rangle - \left\{ v - y_{ki}, \frac{v_{ki} - t_{ki}}{\mu} \right\} \]

\[ \geq \langle v - v_{ki}, hv_{ki} - h t_{ki} \rangle - \left\{ v - v_{ki}, \frac{v_{ki} - t_{ki}}{\mu} \right\}. \]

Since \( h \) is continuous, on taking the limit \( i \to \infty \) we have \( \langle v - \bar{u}, w \rangle \geq 0 \). Since \( G \) is maximal monotone, we have \( \bar{u} \in G^{-1}0 \) and hence \( \bar{u} \in \text{Sol}(VI(1.5)) \) and thus \( \bar{u} \in \Gamma \).
Finally, it follows from (3.22) and Lemma 2.3 that the sequence \( \{u_k\} \) converges weakly to \( \bar{u} \in \Gamma \).

**Remark 3.2** One can derive a number of schemes from scheme (3.1); some special cases are as follows:

(i) Setting \( \eta_k = 0, \forall k \) then scheme (3.1) reduces to extragradient scheme for solving VI(1.5) and H-FPP(1.1).

(ii) Setting \( \sigma_k = 0, \forall k, \) and \( V = I, U = I \) then scheme (3.1) reduces to scheme (1.8) for solving VI(1.5) and hence we recover Theorem 3.1 [25].

(iii) Setting \( V = I, \sigma_k = 0, U = J_{\delta_k}^B := (I + \lambda_kB)^{-1} (\text{where } B : \mathcal{H} \to 2^{\mathcal{H}} \text{ is maximal monotone and } \lambda_k \in (0, \infty)), \) and \( \alpha_k = \alpha \ \forall k, \) scheme (3.1) takes the following form:

\[
\begin{aligned}
t_k &= u_k + \eta_k(u_k - u_{k-1}), \\
v_k &= \mathcal{P}_C(t_k - \mu h(t_k)), \\
w_k &= \mathcal{P}_C(t_k - \mu h(v_k)), \\
u_{k+1} &= (1 - \alpha)t_k + \alpha_j^k w_k,
\end{aligned}
\]

which was considered with an additional error tolerance strategy in [34].

### 4 Numerical example

We discuss an example to illustrate Theorem 3.1.

**Example 4.1** Let \( \mathcal{H} = \mathbb{R} \). Let \( C = (-\infty, +\infty) \), the mappings \( h : \mathcal{H} \to \mathcal{H} \) be defined by \( h(u) = 3u - 2, \forall u \in C \); and \( U, V : C \to C \) be defined by \( Uu = \frac{u + 1}{3}, \forall u \in C \), respectively. Setting \( \{\alpha_k\} = 0.8, \{\eta_k\} = 0.4 \) and \( \{\sigma_k\} = \{0.1, 0.2\}, \forall k \geq 1 \). Then there are unique sequences \( \{u_k\}, \{v_k\} \) and \( \{w_k\} \) obtained by scheme (3.1) converging to \( \bar{u} = \frac{2}{3} \in \Gamma \).

**Proof** Since \( h \) is Lipschitz continuous with \( L = 3 \) and monotone and hence \( \mu \in (0, \frac{1}{4}) \), we take \( \mu = \frac{1}{4} \). Observe that the mappings \( U, V \) are nonexpansive with \( F(U) = \{\frac{1}{4}\}, F(V) = \{\frac{1}{3}\} \), and hence \( \Phi = \text{Sol}(H-(FPP)) = \{\frac{1}{3}\} \). One can also obtain \( \text{Sol}(VI(1.5)) \neq \emptyset \). Furthermore, scheme (3.1) reduces to the following scheme: Given initial values \( u_0, u_1 \),

\[
\begin{aligned}
t_k &= u_k + \eta_k(u_k - u_{k-1}), \\
v_k &= \mathcal{P}_C(t_k - \mu h(t_k)) = \left\{ \begin{array}{ll}
0, & \text{if } u < 0, \\
1, & \text{if } u > 1, \\
\frac{1}{2}t_k + \frac{1}{2}, & \text{otherwise,}
\end{array} \right.
\\w_k &= \mathcal{P}_C(t_k - \mu h(v_k)) = \left\{ \begin{array}{ll}
t_k + \frac{1}{2}, & \text{if } u < 0, \\
t_k + \frac{1}{3}, & \text{if } u > 1, \\
t_k - \frac{1}{3}(3y_k - 2), & \text{otherwise,}
\end{array} \right.
\\u_{k+1} &= (1 - \alpha_k)t_k + \alpha_k((\gamma_k + 6) + (1 - \alpha_k)\frac{1}{2}).
\end{aligned}
\]

Finally, using MATLAB, we have Fig. 1 and Table 1, which show that \( \{u_k\}, \{v_k\} \) and \( \{w_k\} \) converge to \( \bar{u} = \frac{2}{3} \) as \( k \to +\infty \). □
Figure 1: Convergence of \( \{u_k\} \), \( \{v_k\} \) and \( \{w_k\} \) when 
\( u_0 = 1, u_1 = 2 \)

| No. of iterations | \( u_k \) \((u_0 = 1, u_1 = 2)\) | \( v_k \) | \( w_k \) |
|-------------------|--------------------------------|---------|---------|
| 1                 | 0.824408                       | 0.650000| 0.612500|
| 2                 | 0.738007                       | 0.688543| 0.737764|
| 3                 | 0.721799                       | 0.682885| 0.718937|
| 4                 | 0.700135                       | 0.676815| 0.699649|
| 5                 | 0.687021                       | 0.673869| 0.686825|
| 6                 | 0.679051                       | 0.670444| 0.678942|
| 7                 | 0.674203                       | 0.668966| 0.674313|
| 8                 | 0.671253                       | 0.668066| 0.671214|
| 9                 | 0.669458                       | 0.667518| 0.669434|
| 10                | 0.668366                       | 0.667185| 0.668351|
| 11                | 0.667701                       | 0.666982| 0.667692|
| 12                | 0.667296                       | 0.666859| 0.667291|
| 13                | 0.667050                       | 0.666784| 0.667047|
| 14                | 0.666900                       | 0.666738| 0.666982|
| 15                | 0.666809                       | 0.666710| 0.666807|
| 20                | 0.666679                       | 0.666670| 0.666678|
| 25                | 0.666668                       | 0.666667| 0.666668|
| 29                | 0.666667                       | 0.666667| 0.666667|
| 30                | 0.666667                       | 0.666667| 0.666667|

Concluding remark 4.1 In this paper, we considered a variational inequality problem (VI) and a hierarchical fixed point problem (H-FPP) in Hilbert space. We proposed an inertial version of Krasnoselskii–Mann (KM)-type extragradient scheme (3.1) by combining the KM-type scheme (1.3) and an inertial version of the extragradient scheme (1.8) to approximate a common solution of H-FPP (1.1) and VI (1.5). Furthermore, we proved a weak convergence theorem for the proposed scheme (3.1). Finally, we discussed an example to illustrate Theorem 3.1.

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