Structural Results for Partially Observed Markov Decision Processes

A Tutorial

Vikram Krishnamurthy
University of British Columbia,
Vancouver, Canada. V6T 1Z4. Vancouver.
vikramk@ece.ubc.ca,
December 2015.
# Contents

1 **Introduction**  
1.1 Examples of Controlled (Active) Sensing  

2 **Partially Observed Markov Decision Processes (POMDPs)**  
2.1 Finite Horizon POMDP  
2.2 Belief State Formulation and Dynamic Programming  
2.3 Machine Replacement POMDP: Toy Example  
2.4 Finite Dimensional Controller for finite horizon POMDP  
2.5 Algorithms for Finite Horizon POMDPs with Finite Observation Space  
2.6 Discounted Infinite Horizon POMDPs  
2.7 Example: Optimal Search for a Markovian Moving Target  
2.8 Complements and Sources  

3 **Structural Results for Markov Decision Processes**  
3.1 Submodularity and Supermodularity  
3.2 First Order Stochastic Dominance  
3.3 Monotone Optimal Policies for MDPs  
3.4 How does the optimal cost depend on the transition matrix?  
3.5 Algorithms for Monotone Policies - Exploiting Sparsity  
3.6 Example: Transmission Scheduling over Wireless Channel  
3.7 Complements and Sources  

4 **Structural Results for Optimal Filters**  
4.1 Monotone Likelihood Ratio (MLR) Stochastic Order  
4.2 Total Positivity and Copositivity  
4.3 Monotone Properties of Optimal Filter  
4.4 Illustrative Example  
4.5 Discussion and Examples of Assumptions (F1)-(F4)  
4.6 Example: Reduced Complexity HMM Filtering with Stochastic Dominance Bounds  
4.7 Complements and Sources
## Contents

### Appendix to Chapter 4

**4.A Proofs** 54

### 5 Monotonicity of Value Function for POMDPs

**5.1 Model and Assumptions** 57

**5.2 Main Result: Monotone Value Function** 59

**5.3 Example 1: Monotone Policies for 2-state POMDPs** 60

**5.4 Example 2: POMDP Multi-armed Bandits Structural Results** 62

**5.5 Complements and Sources** 66

### 6 Structural Results for Stopping Time POMDPs

**6.1 Introduction** 68

**6.2 Stopping Time POMDP and Convexity of Stopping Set** 70

**6.3 Monotone Optimal Policy for Stopping Time POMDP** 73

**6.4 Characterization of Optimal Linear Decision Threshold for Stopping time POMDP** 76

**6.5 Example: Machine Replacement POMDP** 79

### Appendix to Chapter 6

**6.A Lattices and Submodularity** 80

**6.B MLR Dominance and Submodularity on Lines** 81

**6.C Proof of Theorem 6.3.4** 81

### 7 Quickest Change Detection

**7.1 Example 1: Quickest Detection with Phase-Distributed Change Time and Variance Penalty** 83

**7.2 Example 2: Risk Sensitive Quickest Detection with Exponential Delay Penalty** 88

**7.3 Example 3: Multi-agent Social Learning** 91

**7.4 Example 4: Quickest Detection with Controlled Sampling** 95

**7.5 Complements and Sources** 99

### 8 Myopic Policy Bounds for POMDPs and Sensitivity

**8.1 The Partially Observed Markov Decision Process** 100

**8.2 Myopic Policies using Copositive Dominance: Insight** 101

**8.3 Constructing Myopic Policy Bounds for Optimal Policy using Copositive Dominance** 103

**8.4 Optimizing the Myopic Policy Bounds to Match the Optimal Policy** 105

**8.5 Numerical Examples** 107

**8.6 Blackwell Dominance of Observation Distributions and Optimality of Myopic Policies** 110

**8.7 How does optimal POMDP cost vary with state and observation dynamics?** 113

**8.8 Complements and Sources** 114
Appendix to Chapter 8

8.A  POMDP Numerical Examples 115
References 118
This article provides an introductory tutorial on structural results in partially observed Markov decision processes (POMDPs). Typically, computing the optimal policy of a POMDP is computationally intractable. We use lattice programming methods to characterize the structure of the optimal policy of a POMDP without brute force computations. This article is a very short and somewhat incomplete treatment. Details, substantially more tutorial material, further examples and proofs can be found in the forthcoming book [57].

Contributions to POMDPs have been made by the several communities: operations research, robotics, machine learning, speech recognition, artificial intelligence, control systems theory, and economics. POMDPs have numerous examples in controlled sensing, wireless communications, machine learning, control systems, social learning and sequential detection.

We start with some terminology.

- A Markov Decision Process (MDP) is obtained by controlling the transition probabilities of a Markov chain as it evolves over time.
- A Hidden Markov Model (HMM) is a noisily observed Markov chain.
- A partially observed Markov decision process (POMDP) is obtained by controlling the transition probabilities and/or observation probabilities of an HMM.

These relationships are illustrated in Figure 1.1.

A POMDP specializes to a MDP if the observations are noiseless and equal to the state of the Markov chain. A POMDP specializes to an HMM if the control is removed. Finally, an HMM specializes to a Markov chain if the observations are noiseless and equal to the state of the Markov chain.

Figure 1.1 Terminology of HMMs, MDPs and POMDPs
Suppose a sensor provides noisy observations \( y_k \) of the evolving state \( x_k \) of a Markov stochastic system. The Markov system together with the noisy sensor constitute a partially observed Markov model (also called a stochastic state space model or Hidden Markov Model. The aim is to estimate the state \( x_k \) at each time instant \( k \) given the observations \( y_1, \ldots, y_k \).

In classical statistical signal processing, the optimal filter computes the posterior distribution \( \pi_k \) of the state at time \( k \) via the recursive algorithm

\[
\pi_k = T(\pi_{k-1}, y_k)
\]

where the operator \( T \) is essentially Bayes’ rule. Once the posterior \( \pi_k \) is evaluated, the optimal estimate (in the minimum mean square sense) of the state \( x_k \) given the noisy observations \( y_1, \ldots, y_k \) can be computed by integration.

Statistical signal processing deals with extracting signals from noisy measurements. Motivated by physical, communication and social constraints, we address the deeper issue of how to dynamically schedule and optimize signal processing resources to extract signals from noisy measurements. Such problems are formulated as POMDPs. Figure 1.2 displays the schematic setup.

As in the filtering problem, at each time \( k \), a decision maker has access to the noisy observations \( y_k \) of the state \( x_k \) of a Markov process. Given these noisy observations, the aim is to control the trajectory of the state and observation process by choosing actions \( u_k \) at each time \( k \). The decision maker knows ahead of time that if it chooses action \( u_k \) when the system is in state \( x_k \), then a cost \( c(x_k, u_k) \) will be incurred at time \( k \). (Of course the decision maker does not know state \( x_k \) at time \( k \) but can estimate the cost based on the observations \( y_k \).)

The goal of the decision maker is to choose the sequence of actions \( u_0, \ldots, u_{N-1} \) to minimize the expected cumulative cost \( E\{\sum_{k=0}^{N-1} c(x_k, u_k)\} \) where \( E \) denotes mathematical expectation.

The optimal choice of actions is determined by a policy (strategy) as \( u_k = \mu^*_k(\pi_k) \) where the optimal policy \( \mu^*_k \) satisfies Bellman’s stochastic dynamic program-
Introduction

\begin{equation}
\begin{aligned}
\mu_k^*(\pi) &= \arg\min_u Q_k(\pi, u), \\
J_k(\pi) &= \min_u Q_k(\pi, u), \\
Q_k(\pi, u) &= \sum_x c(x, u)\pi(x) + \sum_y J_{k+1}(T(\pi, y, u))\sigma(\pi, y, u).
\end{aligned}
\end{equation}

Here \( T \) is the optimal filter (1.1), and \( \sigma \) is a normalization term for the filter. Also \( \pi \) is the posterior computed via the optimal filter (1.1).

Chapter 2 starts our formal presentation of POMDPs. The POMDP model and stochastic dynamic programming recursion are formulated in terms of the belief state computed by the Bayesian filter. Several algorithms for solving POMDPs over a finite horizon are then presented. Optimal search theory for a moving target is used as an illustrative example of a POMDP.

In general, solving Bellman’s dynamic programming equation (1.2) for a POMDP is computationally intractable. The main aim of this article is to show that by introducing assumptions on the POMDP model, important structural properties of the optimal policy can be determined without brute-force computations. These structural properties can then be exploited to compute the optimal policy.

The main idea behind is to give conditions on the POMDP model so that the optimal policy \( \mu_k^*(\pi) \) is monotone\(^1\) in belief \( \pi \). In simple terms, \( \mu_k^*(\pi) \) is shown to be increasing in belief \( \pi \) by showing that \( Q_k(\pi, u) \) in Bellman’s equation (1.2) is submodular. The main result is:

\begin{equation}
\begin{aligned}
Q_k(\pi, u + 1) - Q_k(\pi, u) &\downarrow \pi \\
\text{submodular} \quad \Rightarrow \quad \mu_k^*(\pi) &\uparrow \pi.
\end{aligned}
\end{equation}

Obtaining conditions for \( Q_k(\pi, u) \) to be submodular involves powerful ideas in stochastic dominance and lattice programming.

Once the optimal policy of a POMDP is shown to be monotone, this structure can be exploited to devise efficient algorithms. Figure 1.3 illustrates an increasing optimal policy \( \mu_k^*(\pi) \) in the belief \( \pi \) with two actions \( u_k \in \{1, 2\} \). Note that any increasing function which takes on two possible values has to be a step function. So computing \( \mu_k^*(\pi) \) boils down to determining the single belief \( \pi_1^* \) at which the step function jumps. Computing (estimating) \( \pi_1^* \) can be substantially easier than directly solving Bellman’s equation (1.2) for \( \mu_k^*(\pi) \) for all beliefs \( \pi \), especially when \( \mu_k^*(\pi) \) has no special structure.

Chapter 3 gives sufficient conditions for a MDP to have a monotone (increasing) optimal policy. The explicit dependence of the MDPs optimal cumulative cost on transition probability is also discussed.

In order to give conditions for the optimal policy of a POMDP to be monotone, one first needs to show monotonicity of the underlying Hidden Markov

\(^1\)By monotone, we mean either increasing for all \( \pi \) or decreasing for all \( \pi \). “Increasing” is used here in the weak sense, it means “non-decreasing”. Similarly for decreasing.
Model filter. To this end, Chapter 4 discusses the monotonicity of Bayesian (Hidden Markov Model) filters. This monotonicity of the optimal filter is used to construct reduced complexity filtering algorithms that provably lower and upper bound the optimal filter.

Chapters 5 to 8 give conditions on the POMDP model for the dynamic programming recursion to have a monotone solution. Chapter 5 discusses conditions for the value function in dynamic programming to be monotone. This is used to characterize the structure of 2-state POMDPs and POMDP multi-armed bandits.

Chapter 6 gives conditions under which stopping time POMDPs have monotone optimal policies. As examples, Chapter 7 covers quickest change detection, controlled social learning and a variety of other applications. The structural results provide a unifying theme and insight to what might otherwise simply be a collection of examples.

Finally Chapter 8 gives conditions under which the optimal policy of a general POMDP can be lower and upper bounded by judiciously chosen myopic policies. Bounds on the sensitivity of the optimal cumulative cost of POMDPs to the parameters are also discussed.

This article is a butchered version (and incomplete version) of the forthcoming book [57] which contains a thorough treatment of structural results, dynamic programming algorithms for POMDPs, and reinforcement learning algorithms.

1.1 Examples of Controlled (Active) Sensing

This section outlines some applications of controlled sensing formulated as a POMDP. Controlled sensing also known as “sensor adaptive signal processing” or “active sensing” is a special case of a POMDP where the decision maker (controller) controls the observation noise distribution but not the dynamics of the stochastic system. The setup is as in Figure 1.2 with the link between the controller and stochastic system omitted.

In controlled sensing, the decision maker controls the observation noise distribution by switching between various sensors or sensing modes. An accurate sensor yields less noisy measurements but is expensive to use. An inaccu-
rate sensor yields more noisy measurements by is cheap to use. How should the decision maker decide at each time which sensor or sensing mode to use? Equivalently, how can a sensor be made “smart” to adapt its behavior to its environment in real time? Such an active sensor uses feedback control. As shown in Figure 1.2, the estimates of the signal are fed to a controller/scheduler that decides the sensor should adapt so as to obtain improved measurements; or alternatively minimize a measurement cost. Design and analysis of such closed loop systems which deploy stochastic control is non-trivial. The estimates from the signal processing algorithm are uncertain (they are posterior probability distribution functions). So controlled sensing requires decision making under uncertainty.

We now highlight some examples in controlled sensing.

**Example 1. Adaptive Radars**
Adaptive multifunction radars are capable of switching between various measurement modes, e.g., radar transmit waveforms, beam pointing directions, etc, so that the tracking system is able to tell the radar which mode to use at the next measurement epoch. Instead of the operator continually changing the radar from mode to mode depending on the environment, the aim is to construct feedback control algorithms that dynamically adapt where the radar radiates its pulses to achieve the command operator objectives. This results in radars that autonomously switch beams, transmitted waveforms, target dwell and revisit times.

**Example 2. Social Learning and Data Incest**
A social sensor (human-based sensor) denotes an agent that provides information about its environment (state of nature) to a social network. Examples of such social sensors include Twitter posts, Facebook status updates, and ratings on online reputation systems like Yelp and Tripadvisor. Social sensors present unique challenges from a statistical estimation point of view, since they interact with and influence other social sensors. Also, due to privacy concerns, they reveal their decisions (ratings, recommendations, votes) which can be viewed as a low resolution (quantized) function of their raw measurements.

**Example 3. Quickest Detection and Optimal Sampling**
Suppose a decision maker records measurements of a finite-state Markov chain corrupted by noise. The goal is to decide when the Markov chain hits a specific target state. The decision maker can choose from a finite set of sampling intervals to pick the next time to look at the Markov chain. The aim is to optimize an objective comprising of false alarm, delay cost and cumulative measurement sampling cost. Taking more frequent measurements yields accurate estimates but incurs a higher measurement cost. Making an erroneous decision too soon incurs a false alarm penalty. Waiting too long to declare the target state incurs a delay penalty. What is the optimal sequential strategy for the decision maker?
It is shown in §7.4 that the optimal sampling problem results in a POMDP that has a monotone optimal strategy in the belief state.

**Example 4. Interaction of Local and Global Decision Makers**

In a multi-agent network, how can agents use their noisy observations and decisions made by previous agents to estimate an underlying randomly evolving state? How do decisions made by previous agents affect decisions made by subsequent agents? In §7.3, these questions will be formulated as a multi-agent sequential detection problem involving social learning. Individual agents record noisy observations of an underlying state process, and perform social learning to estimate the underlying state. They make local decisions about whether a change has occurred that optimize their individual utilities. Agents then broadcast their local decisions to subsequent agents. As these local decisions accumulate over time, a global decision maker needs to decide (based on these local decisions) whether or not to declare a change has occurred. How can the global decision maker achieve such change detection to minimize a cost function comprised of false alarm rate and delay penalty? The local and global decision makers interact, since the local decisions determine the posterior distribution of subsequent agents which determines the global decision (stop or continue) which determines subsequent local decisions.

**Other Applications of POMDPs**

POMDP are used in numerous other domains. Some applications include:

- Optimal Search: see §2.7.
- Quickest Detection and other Sequential Detection Problems: see Chapter 6.
- Dialog Systems: see [133] and references therein.
- Robot navigation and planning: see [71] and references therein.
- Cognitive Radio dynamic spectrum sensing: see [134] and references therein.
Partially Observed Markov Decision Processes (POMDPs)

A POMDP is a controlled HMM. An HMM consists of an $X$-state Markov chain $\{x_k\}$ observed via a noisy observation process $\{y_k\}$. Figure 2.1 displays the schematic setup of a POMDP where the action $u_k$ affects the state and/or observation (sensing) process of the HMM. The HMM filter computes the posterior distribution $\pi_k$ of the state. The posterior $\pi_k$ is called the belief state. In a POMDP, the stochastic controller depicted in Figure 2.1 uses the belief state to choose the next action.

Figure 2.1 Partially Observed Markov Decision Process (POMDP) schematic setup. The Markov system together with noisy sensor constitute a Hidden Markov Model (HMM). The HMM filter computes the posterior (belief state) $\pi_k$ of the state of the Markov chain. The controller (decision maker) then chooses the action $u_k$ at time $k$ based on $\pi_k$.

This chapter is organized as follows. §2.1 describes the POMDP model. Then §2.2 gives the belief state formulation and the Bellman’s dynamic programming equation for the optimal policy of a POMDP. It is shown that a POMDP is equivalent to a continuous-state MDP where the states are belief states (posteriors). Bellman’s equation for continuous-state MDP was discussed in §2?. §2.3 gives a toy example of a POMDP. Despite being a continuous-state MDP, §2.4 shows that for finite horizon POMDPs, Bellman’s equation has a finite dimensional characterization. §2.5 discusses several algorithms that exploit this finite dimensional characterization to compute the optimal policy. §2.6 considers discounted cost infinite horizon POMDPs. As an example of a POMDP, optimal search of a moving target is discussed in §2.7.
2.1 Finite Horizon POMDP

A POMDP model with finite horizon \( N \) is a 7-tuple
\[
(X, U, Y, P(u), B(u), c(u), c_N).
\] (2.1)

1. \( X = \{1, 2, \ldots, X\} \) denotes the state space and \( x_k \in X \) denotes the state of a controlled Markov chain at time \( k = 0, 1, \ldots, N \).
2. \( U = \{1, 2, \ldots, U\} \) denotes the action space with \( u_k \in U \) denoting the action chosen at time \( k \) by the controller.
3. \( Y \) denotes the observation space which can either be finite or a subset of \( \mathbb{R} \).
   \( y_k \in Y \) denotes the observation recorded at each time \( k \in \{1, 2, \ldots, N\} \).
4. For each action \( u \in U \), \( P(u) \) denotes a \( X \times X \) transition probability matrix with elements
   \[
P_{ij}(u) = \mathbb{P}(x_{k+1} = j | x_k = i, u_k = u), \quad i, j \in X.
\] (2.2)
5. For each action \( u \in U \), \( B(u) \) denotes the observation distribution with
   \[
   B_{iy}(u) = \mathbb{P}(y_{k+1} = y | x_{k+1} = i, u_k = u), \quad i \in X, y \in Y.
\] (2.3)
6. For state \( x_k \) and action \( u_k \), the decision-maker incurs a cost \( c(x_k, u_k) \).
7. Finally, at terminal time \( N \), a terminal cost \( c_N(x_N) \) is incurred.

The POMDP model (2.1) is a partially observed model since the decision-maker does not observe the state \( x_k \). It only observes noisy observations \( y_k \) that depend on the action and the state specified by the probabilities in (2.3). Recall that an HMM is characterized by \( (X, Y, P, B) \); so a POMDP is a controlled HMM with the additional ingredients of action space \( U \), action dependent transition probabilities, action dependent observation probabilities and costs. In general, the transition matrix, observation distribution and cost can be explicit functions of time; however to simplify notation, we have omitted this time dependency.

Given the model (2.1), the dynamics of a POMDP proceed according to Algorithm 1. This involves at each time \( k \) choosing an action \( u_k \), accruing an instantaneous cost \( c(x_k, u_k) \), evolution of the state from \( x_k \) to \( x_{k+1} \), and observing \( x_{k+1} \) in noise as \( y_{k+1} \).

As depicted in (2.5), at each time \( k \), the decision maker uses all the information available until time \( k \) (namely, \( I_k \)) to choose action \( u_k = \mu_k(I_k) \) using policy \( \mu_k \). With the dynamics specified by Algorithm 1, denote the sequence of policies that the decision-maker uses from time \( 0 \) to \( N - 1 \) as \( \mu = (\mu_0, \mu_1, \ldots, \mu_{N-1}) \).

Objective

To specify a POMDP completely, in addition to the model (2.1), dynamics in Algorithm 1 and policy sequence\(^1 \mu \), we need to specify a performance criterion

\(^1\)§2.2 shows that a POMDP is equivalent to a continuous-state MDP. So it suffices to consider non-randomized policies to achieve the minimum in (2.7).
Algorithm 1 Dynamics of Partially Observed Markov Decision Process

At time $k = 0$, the state $x_0$ is simulated from initial distribution $\pi_0$.

For time $k = 0, 1, \ldots, N - 1$:

1. Based on available information
   \[ I_0 = \{\pi_0\}, I_k = \{\pi_0, u_0, y_1, \ldots, u_{k-1}, y_k\}, \] \hspace{1cm} (2.4)
   
   the decision-maker chooses action
   \[ u_k = \mu_k(I_k) \in U, \quad k = 0, 1, \ldots, N - 1. \] \hspace{1cm} (2.5)

   Here, $\mu_k$ denotes a policy that the decision maker uses at time $k$.

2. The decision-maker incurs a cost $c(x_k, u_k)$ for choosing action $u_k$.

3. The state evolves randomly with transition probability $P_{x_k x_{k+1}}(u_k)$ to the next state $x_{k+1}$ at time $k + 1$. Here
   \[ P_{ij}(u) = \Pr(x_{k+1} = j | x_k = i, u_k = u). \]

4. The decision-maker records a noisy observation $y_{k+1} \in \mathcal{Y}$ of the state $x_{k+1}$ according to
   \[ \Pr(y_{k+1} = y | x_{k+1} = i, u_k = u) = B_{iy}(u). \]

5. The decision-maker updates its available information as
   \[ I_{k+1} = I_k \cup \{u_k, y_{k+1}\}. \]

   If $k < N$, then set $k$ to $k + 1$ and go back to Step 1.
   If $k = N$, then the decision-maker pays a terminal cost $c_N(x_N)$ and the process terminates.

or objective function. This section considers the finite horizon objective

\[ J_{\mu}(\pi_0) = \mathbb{E}_{\mu} \left\{ \sum_{k=0}^{N-1} c(x_k, u_k) + c_N(x_N) \mid \pi_0 \right\}. \] \hspace{1cm} (2.6)

which is the expected cumulative cost incurred by the decision-maker when using policy $\mu$ up to time $N$ given the initial distribution $\pi_0$ of the Markov chain. Here, $\mathbb{E}_{\mu}$ denotes expectation with respect to the joint probability distribution of $(x_0, y_0, x_1, y_1, \ldots, x_{N-1}, y_{N-1}, x_N, y_N)$. The goal of the decision-maker is to determine the optimal policy sequence

\[ \mu^* = \arg \min_{\mu} J_{\mu}(\pi_0), \quad \text{for any initial prior } \pi_0 \] \hspace{1cm} (2.7)

that minimizes the expected cumulative cost. Of course, the optimal policy sequence $\mu^*$ may not be unique.

Remarks

1. The decision-maker does not observe the state $x_k$. It only observes noisy observations $y_k$ that depend on the action and the state via Step 4. Also, the
decision-maker knows the cost matrix $c(x, u)$ for all possible states and actions in $\mathcal{X}, \mathcal{U}$. But since the decision-maker does not know the state $x_k$ at time $k$, it does not know the cost accrued at time $k$ in Step 2 or terminal cost in Step 5. Of course, the decision-maker can estimate the cost by using the noisy observations of the state.

2. The term POMDP is usually reserved for the case when the observation space $\mathcal{Y}$ is finite. However, we consider both finite and continuous valued observations.

3. The action $u_k$ affects the evolution of the state (Step 3) and observation distribution (Step 4). In controlled sensing applications such as radars and sensor networks, the action only affects the observation distribution and not the evolution of the target.

4. More generally, the cost can be of the form $\bar{c}(x_k = i, x_{k+1} = j; y_k = y, y_{k+1} = \bar{y}, u_k = u)$. This is equivalent to the cost (see (2.13) below)

$$c(i, u) = \sum_{y \in \mathcal{Y}} \sum_{\bar{y} \in \mathcal{Y}} \sum_{j \in \mathcal{X}} \bar{c}(i, j, y, \bar{y}, u) P_{ijy}(u) B_{j\bar{y}}(u) B_{iy}(u).$$

(2.8)

### 2.2 Belief State Formulation and Dynamic Programming

This section details a crucial step in the formulation and solution of a POMDP, namely, the belief state formulation. In this formulation, a POMDP is equivalent to a continuous state MDP with states being the belief states. We then formulate the optimal policy as the solution to Bellman’s dynamic programming recursion written in terms of the belief state. Finally, we state and prove the main result: the solution of the dynamic programming recursion for a POMDP has an explicit piecewise linear and concave solution.

#### 2.2.1 Belief State Formulation of POMDP

Recall from §?? that for a fully observed MDP, the optimal policy is Markovian and the optimal action $u_k = \mu^*_k(x_k)$. In comparison, for a POMDP the optimal action chosen by the decision maker is in general

$$u_k = \mu^*_k(I_k), \quad \text{where} \quad I_k = (\pi_0, u_0, y_1, \ldots, u_k, y_k).$$

(2.9)

Since $I_k$ is increasing in dimension with $k$, to implement a controller, it is useful to obtain a sufficient statistic that does not grow in dimension. The posterior distribution $\pi_k$ computed via the HMM filter is a sufficient statistic for $I_k$. Define the posterior distribution of the Markov chain given $I_k$ as

$$\pi_k(i) = P(x_k = i|I_k), \quad i \in \mathcal{X}, \quad \text{where} \quad I_k = \{\pi_0, u_0, y_1, \ldots, u_{k-1}, y_k\}.$$  

(2.10)
2.2 Belief State Formulation and Dynamic Programming

We will call \( \pi_k \) as the belief state or information state at time \( k \). It is computed via the HMM filter namely \( \pi_k = T(\pi_{k-1}, y_k, u_{k-1}) \) where

\[
T(\pi, y, u) = \frac{B_y(u)P'(u)\pi}{\sigma(\pi, y, u)}, \quad \text{where } \sigma(\pi, y, u) = 1_X B_y(u)P'(u)\pi, \quad (2.11)
\]

\( B_y(u) = \text{diag}(B_{1y}(u), \cdots, B_{Xy}(u)) \).

The main point established below in Theorem 2.2.1 is that (2.9) is equivalent to

\[
u_k = \mu_k^+(\pi_k). \quad (2.12)
\]

In other words, the optimal controller operates on the belief state \( \pi_k \) (HMM filter posterior) to determine the action \( u_k \).

In light of (2.12), let us first define the space where \( \pi_k \) lives in. The beliefs \( \pi_k, k = 0, 1, \ldots \) defined in (2.10) are \( X \)-dimensional probability vectors. Therefore they lie in the \( X - 1 \) dimensional unit simplex denoted as

\[
\Pi(X) \overset{\text{defn}}{=} \{ \pi \in \mathbb{R}^X : 1^T \pi = 1, \quad 0 \leq \pi(i) \leq 1 \text{ for all } i \in \mathcal{X} = \{1, 2, \ldots, X\} \}.
\]

\( \Pi(X) \) is called the belief space. \( \Pi(2) \) is a one dimensional simplex (unit line segment), As shown in Figure 2.2, \( \Pi(3) \) is a two-dimensional simplex (equilateral triangle); \( \Pi(4) \) is a tetrahedron, etc. Note that the unit vector states \( e_1, e_2, \ldots, e_X \) of the underlying Markov chain \( x \) are the vertices of \( \Pi(X) \).

We now formulate the POMDP objective (2.6) in terms of the belief state. Consider the objective (2.6). Then

\[
J_{\mu}(\pi_0) = \mathbb{E}_{\mu} \left\{ \sum_{k=0}^{N-1} c(x_k, u_k) + c_N(x_N) \mid \pi_0 \right\}
\]

\[
\overset{(a)}{=} \mathbb{E}_{\mu} \left\{ \sum_{k=0}^{N-1} \mathbb{E}\{c(x_k, u_k) \mid \mathcal{I}_k\} + \mathbb{E}\{c_N(x_N) \mid \mathcal{I}_N\} \mid \pi_0 \right\}
\]
\[
= \mathbb{E}_\mu \left\{ \sum_{k=0}^{N-1} \sum_{i=1}^X c(i, u_k) \pi_k(i) + \sum_{i=1}^X c_N(i) \pi_N(i) \mid \pi_0 \right\} \\
= \mathbb{E}_\mu \left\{ \sum_{k=0}^{N-1} c'_u \pi_k + c'_N \pi_N \mid \pi_0 \right\}
\]

where (a) uses the smoothing property of conditional expectations. In (2.13), the \(X\)-dimensional cost vectors \(c_u(k)\) and terminal cost vector \(c_N\) are defined as

\[
c_u = [c(1, u) \cdots c(X, u)]', \quad c_N = [c_N(1) \cdots c_N(X)]'.
\]

**Summary:** The POMDP has been expressed as a continuous-state (fully observed) MDP with dynamics (2.11) given by the HMM filter and objective function (2.13). This continuous-state MDP has belief state \(\pi_k\) which lies in unit simplex belief space \(\Pi(X)\). Thus we have the following useful decomposition illustrated in Figure 2.1:

- An HMM filter uses the noisy observations \(y_k\) to compute the belief state \(\pi_k\)
- The POMDP controller then maps the belief state \(\pi_k\) to the action \(u_k\).

Determining the optimal policy for a POMDP is equivalent to partitioning \(\Pi(X)\) into regions where a particular action \(u \in \{1, 2, \ldots, U\}\) is optimal.

### 2.2.2 Stochastic Dynamic Programming for POMDP

Since a POMDP is a continuous-state MDP with state space being the unit simplex, we can straightforwardly write down the dynamic programming equation for the optimal policy as we did in §2.1 for continuous-state MDPs.

**Theorem 2.2.1** For a finite horizon POMDP with model (2.1) and dynamics given by Algorithm 1:

1. The minimum expected cumulative cost \(J_{\mu^*}(\pi)\) is achieved by deterministic policies

\[
\mu^* = (\mu_0^*, \mu_1^*, \ldots, \mu_{N-1}^*), \quad \text{where } u_k = \mu_k^*(\pi_k).
\]

2. The optimal policy \(\mu^* = (\mu_0, \mu_1, \ldots, \mu_{N-1})\) for a POMDP is the solution of the following Bellman’s dynamic programming backward recursion: Initialize \(J_N(\pi) = c'_N \pi\) and then for \(k = N - 1, \ldots, 0\)

\[
J_k(\pi) = \min_{u \in U} \left\{ c'_u \pi + \sum_{y \in Y} J_{k+1}(T(\pi, y, u)) \sigma(\pi, y, u) \right\}
\]

\[
\mu_k^*(\pi) = \arg\min_{u \in U} \left\{ c'_u \pi + \sum_{y \in Y} J_{k+1}(T(\pi, y, u)) \sigma(\pi, y, u) \right\}.
\]

(2.15)

The expected cumulative cost \(J_{\mu^*}(\pi)\) (2.13) of the optimal policy \(\mu^*\) is given by the value function \(J_0(\pi)\) for any initial belief \(\pi \in \Pi(X)\).

Since the belief space \(\Pi(X)\) is uncountable, the above dynamic programming recursion does not translate into practical solution methodologies. \(J_k(\pi)\) needs to be evaluated at each \(\pi \in \Pi(X)\), an uncountable set.
To illustrate the POMDP model and dynamic programming recursion described above, consider a toy example involving the machine replacement problem. Here we describe the 2-state version of the problem with noisy observations.

The state space is $X = \{1, 2\}$ where state 1 corresponds to a poorly performing machine while state 2 corresponds to a brand new machine. The action space is $U \in \{1, 2\}$ where action 2 denotes keep using the machine, while action 1 denotes replace the machine with a brand new one which starts in state 2. The transition probabilities of the machine state are

$$P(1) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad P(2) = \begin{bmatrix} 1 & 0 \\ \theta & 1 - \theta \end{bmatrix}. $$

where $\theta \in [0, 1]$ denotes the probability that machine deteriorates.

Assume that the state of the machine $x_k$ is indirectly observed via the quality of the product $y_k \in Y = \{1, 2\}$ generated by the machine. Let $p$ denote the probability that the machine operating in the good state produces a high quality product, and $q$ denote the probability that a deteriorated machine produces a poor quality product. Then the observation probability matrix is

$$B = \begin{bmatrix} p & 1 - p \\ 1 - q & q \end{bmatrix}. $$

Operating the machine in state $x$ incurs an operating cost $c(x, u = 2)$. On the other hand, replacing the machine at any state $x$, costs $R$, that is, $c(x, u = 1) = R$. The aim is to minimize the cumulative expected cumulative cost $E \mu \{ \sum_{k=0}^{N-1} c(x_k, u_k) | \pi_0 \}$ for some specified horizon $N$. Here $\pi_0$ denotes the initial distribution of the state of the machine at time 0.

Bellman’s equation (2.15) reads: Initialize $J_N(\pi) = 0$ (since there is no terminal cost) and for $k = N - 1, \ldots, 0$:

$$J_k(\pi) = \min \{ c'_1 \pi + J_{k+1}(e_1), \ c'_2 \pi + \sum_{y \in \{1,2\}} J_{k+1}(T(\pi, y, 2)) \sigma(\pi, y, 2) \} $$

where $T(\pi, y, 2) = \frac{B_y P'(2) \pi}{\sigma(\pi, y, 2)}$, $\sigma(\pi, y, 2) = 1' B_y P'(2) \pi$, $y \in \{1, 2\}$,

$$B_1 = \begin{bmatrix} p & 0 \\ 0 & 1 - q \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 - p & 0 \\ 0 & q \end{bmatrix}. $$

Since the number of states is $X = 2$, the belief space $\Pi(X)$ is a one dimensional simplex, namely the interval $[0, 1]$. So $J_k(\pi)$ can be expressed in terms of $\pi_2 \in [0, 1]$, because $\pi_1 = 1 - \pi_2$. Denote this as $J_k(\pi_2)$.

One can then implement the dynamic programming recursion numerically by discretizing $\pi_2$ in the interval $[0, 1]$ over a finite grid and running the Bellman’s equation over this finite grid. Although this numerical implementation is somewhat naive, the reader should do this to visualize the value function and optimal policy. The reader would notice that the value function $J_k(\pi_2)$ is
Partially Observed Markov Decision Processes (POMDPs)

piecewise linear and concave in $\pi_2$. The main result in the next section is that for a finite horizon POMDP, the value function is always piecewise linear and concave, and the value function and optimal policy can be determined exactly (therefore a grid approximation is not required).

2.4 Finite Dimensional Controller for finite horizon POMDP

Despite the belief space $\Pi(X)$ being continuum, the following remarkable result due to Sondik [115, 113] shows that Bellman’s equation (2.15) for a finite horizon POMDP has a finite dimensional characterization when the observation space $Y$ is finite.

**Theorem 2.4.1** Consider the POMDP model (2.1) with finite action space $U = \{1, 2, \ldots, U\}$ and finite observation space $Y = \{1, 2, \ldots, Y\}$. At each time $k$ the value function $J_k(\pi)$ of Bellman’s equation (2.15) and associated optimal policy $\mu^*_k(\pi)$ have the following finite dimensional characterization:

1. $J_k(\pi)$ is piecewise linear and concave with respect to $\pi \in \Pi(X)$. That is,

   $$J_k(\pi) = \min_{\gamma \in \Gamma_k} \gamma' \pi.$$  

   Here, $\Gamma_k$ at iteration $k$ is a finite set of $X$-dimensional vectors.

   Note $J_N(\pi) = c_N' \pi$ and $\Gamma_N = \{c_N\}$ where $c_N$ denotes the terminal cost vector.

2. The optimal policy $\mu^*_k(\pi)$ has the following finite dimensional characterization: The belief space $\Pi(X)$ can be partitioned into at most $|\Gamma_k|$ convex polytopes. In each such polytope $R_l = \{ \pi : J_k(\pi) = \gamma'_l \pi \}$, the optimal policy $\mu^*_k(\pi)$ is a constant corresponding to a single action. That is for belief $\pi \in R_l$ the optimal policy is

   $$\mu^*_k(\pi) = u(\arg\min_{\gamma \in \Gamma_k} \gamma'_l \pi)$$

   where the right hand side is the action associated with polytope $R_l$.

The above theorem says that at each time $k$, the belief space $\Pi(X)$ can be partitioned into at most $|\Gamma_k|$ convex polytopes and the optimal action within each such polytope is a constant. Also each vector $\gamma \in \Gamma_k$ will be called a gradient vector since if $J_k(\pi) = \gamma' \pi$, then $\gamma$ is the sub-gradient [13] of the concave function $J_k$ at belief state $\pi$.

Figure 2.3 illustrates the piecewise linear concave structure of the value function $J_k(\pi)$ for the case of a two-state Markov chain ($X = 2$). In this case, the belief state $\pi = \begin{bmatrix} 1 - \pi(2) \\ \pi(2) \end{bmatrix}$ is parametrized by the scalar $\pi(2) \in [0, 1]$ and the belief space is the one-dimensional simplex $\Pi(2) = [0, 1]$. Figure 2.3 also illustrates the finite dimensional structure of the optimal policy $\mu^*_k(\pi)$ asserted by the above theorem. In each region of belief space where $\gamma'_l \pi$ is active, the optimal policy takes on a single action.
2.4 Finite Dimensional Controller for finite horizon POMDP

Figure 2.3 Example of piecewise linear concave value function $J_k(\pi)$ of a POMDP with a 2-state underlying Markov chain ($X = 2$). Here $J_k(\pi) = \min\{\gamma'_1 \pi, \gamma'_2 \pi, \gamma'_3 \pi, \gamma'_4 \pi\}$ is depicted by solid lines. The figure also shows that the belief space can be partitioned into 4 regions. Each region where line segment $\gamma_l \pi$ is active (i.e., is equal to the solid line) corresponds to a single action, $u = 1$ or $u = 2$. Note that $\gamma_5$ is never active.

2.4.1 Proof of Theorem 2.4.1

The proof of Theorem 2.4.1 is important since it gives an explicit construction of the value function. Exact algorithms for solving POMDPs that will be described in §2.5 are based on this construction.

Theorem 2.4.1 is proved by backward induction for $k = N, \ldots, 0$. Obviously, $J_N(\pi) = c'_N \pi$ is linear in $\pi$. Next assume $J_{k+1}(\pi)$ is piecewise linear and concave in $\pi$: so $J_{k+1}(\pi) = \min_{\gamma' \in \Gamma_{k+1}} \gamma' \pi$. Substituting this in (2.15) yields

$$J_k(\pi) = \min_{u \in U} \left\{ c'_u \pi + \sum_{y \in Y} \min_{\gamma' \in \Gamma_{k+1}} \left\{ \frac{\gamma' B_y(u) P'(u) \pi}{\sigma(\pi, u, y)} \right\} \right\} = \min_{u \in U} \left\{ \sum_{y \in Y} \min_{\gamma' \in \Gamma_{k+1}} \left\{ \left[ \frac{c'_u + P(u) B_y(u) \gamma'}{Y} \right]' \pi \right\} \right\}. \quad (2.17)$$

The right hand side is the minimum (over $u$) of the sum (over $y$) of piecewise linear concave functions. Both these operations preserve the piecewise linear concave property. This implies $J_k(\pi)$ is piecewise linear and concave of the form

$$J_k(\pi) = \min_{\gamma' \in \Gamma_k} \gamma' \pi, \quad \text{where} \quad \Gamma_k = \cup_{u \in U} \oplus_{y \in Y} \left\{ \frac{c'_u + P(u) B_y(u) \gamma'}{Y} \mid \gamma' \in \Gamma_{k+1} \right\}. \quad (2.18)$$

Here $\oplus$ denotes the cross-sum operator: given two sets of vectors $A$ and $B$, $A \oplus B$ consists of all pairwise additions of vectors from these two sets. Recall $U =$
\{1, 2, \ldots, U\} and \(Y = \{1, 2, \ldots, Y\}\) are finite sets. A more detailed explanation of going from (2.17) to (2.18) is given in (2.19), (2.20).

2.5 Algorithms for Finite Horizon POMDPs with Finite Observation Space

This section discusses algorithms for solving a finite horizon POMDP when the observation set \(Y\) is finite. These algorithms exploit the finite dimensional characterization of the value function and optimal policy given in Theorem 2.4.1.

Consider the POMDP model (2.1) with finite action space \(U = \{1, 2, \ldots, U\}\) and finite observation set \(Y = \{1, 2, \ldots, Y\}\). Given the finite dimensional characterization in Theorem 2.4.1, the next step is to compute the set of gradients \(\Gamma_k\) that determine the piecewise linear segments of the value function \(J_k(\pi)\) at each time \(k\). Unfortunately, the number of piecewise linear segments can increase exponentially with the action space dimension \(U\) and double exponentially with time \(k\). This is seen from the fact that given the set of vectors \(\Gamma_k+1\) that characterizes the value function at time \(k + 1\), a single step of the dynamic programming recursion yields that the set of all vectors at time \(k\) are \(U|\Gamma_k+1|^Y\). (Of these it is possible that many vectors are never active such as \(\gamma_5\) in Figure 2.3.) Therefore, exact computation of the optimal policy is only computationally tractable for small state dimension \(X\), small action space dimension \(U\) and small observation space dimension \(Y\). Computational complexity theory gives worst case bounds for solving a problem. It is shown in [92] that solving a POMDP is a PSPACE complete problem. [72] gives examples of POMDPs that exhibit this worst case behavior.

2.5.1 Exact Algorithms: Incremental Pruning, Monahan and Witness

Exact algorithms for solving finite horizon POMDPs are based on the finite dimensional characterization of the value function provided by Theorem 2.4.1. The first exact algorithm for solving finite horizon POMDPs was proposed by Sondik [115]; see [80, 18, 19, 73] for several algorithms. Bellman’s dynamic programming recursion (2.15) can be expressed as the following three steps:

\[
Q_k(\pi, u, y) = \frac{c_{u,\pi}}{Y} + J_{k+1}(T(\pi, y, u)) \sigma(\pi, y, u)
\]

\[
Q_k(\pi, u) = \sum_{y \in Y} Q_k(\pi, u, y)
\]

\[
J_k(\pi) = \min_u Q_k(\pi, u).
\]

(2.19)

By “solving” we mean solving Bellman’s dynamic programming equation (2.15) for the optimal policy \(\mu_k^*(\pi), k = 0, \ldots, N-1\). Once the optimal policy is obtained, then the real time controller is implemented according to Algorithm 1.

Exact here means that there is no approximation involved in the dynamic programming algorithm. However, the algorithm is still subject to numerical round-off and finite precision effects.
Based on the above three steps, the set of vectors $\Gamma_k$ that form the piecewise linear value function in Theorem 2.4.1, can be constructed as

$$
\Gamma_k(u, y) = \left\{ \frac{c_u}{Y} + P(u)B(u)\gamma \ | \ \gamma \in \Gamma^{(k+1)} \right\}
$$

$$
\Gamma_k(u) = \oplus_y \Gamma_k(u, y)
$$

$$
\Gamma_k = \bigcup_{u \in U} \Gamma_k(u).
$$  \tag{2.20}

Here $\oplus$ denotes the cross-sum operator: given two sets of vectors $A$ and $B$, $A \oplus B$ consists of all pairwise additions of vectors from these two sets.

In general, the set $\Gamma_k$ constructed according to (2.20) may contain superfluous vectors (we call them “inactive vectors” below) that never arise in the value function $J_k(\pi) = \min_{\gamma \in \Gamma_k} \gamma'\pi$. The algorithms listed below seek to eliminate such useless vectors by pruning $\Gamma_k$ to maintain a parsimonious set of vectors.

**Incremental Pruning Algorithm:** We start with the incremental pruning algorithm described in Algorithm 2. The code is freely downloadable from [17].

**Algorithm 2** Incremental Pruning Algorithm for solving POMDP  

Given set $\Gamma_{k+1}$ generate $\Gamma_k$ as follows:  

* Initialize $\Gamma_k(u, y), \Gamma_k(u), \Gamma_k$ as empty sets  
* For each $u \in U$  
  * For each $y \in Y$  
    * $\Gamma_k(u, y) \leftarrow \text{prune} \left( \left\{ \frac{c_u}{Y} + P(u)B_y(u)\gamma \ | \ \gamma \in \Gamma^{(k+1)} \right\} \right)$  
    * $\Gamma_k(u) \leftarrow \text{prune} \left( \Gamma_k(u) \oplus \Gamma_k(u, y) \right)$  
  * $\Gamma_k \leftarrow \text{prune} \left( \Gamma_k \cup \Gamma_k(u) \right)$

Let us explain the “prune” function in Algorithm 2. Recall the piecewise linear concave characterization of the value function $J_k(\pi) = \min_{\gamma \in \Gamma_k} \gamma'\pi$ with set of vectors $\Gamma_k$. Suppose there is a vector $\gamma \in \Gamma_k$ such that for all $\pi \in \Pi(X)$, it holds that $\gamma'\pi \geq \bar{\gamma}'\pi$ for all vectors $\bar{\gamma} \in \Gamma_k - \{\gamma\}$. Then $\gamma$ dominates every other vector in $\Gamma_k$ and is never active. For example, in Figure 2.3, $\bar{\gamma}$ is never active. The prune function in Algorithm 2 eliminates such inactive vectors $\gamma$ and so reduces the computational cost of the algorithm.

Given a set of vectors $\Gamma$, how can an inactive vector be identified and therefore pruned (eliminated)? The following linear programming dominance test can be used to identify inactive vectors:

$$
\min \quad x
$$

subject to:  

$$
(\gamma - \bar{\gamma})'\pi \geq x, \quad \forall \bar{\gamma} \in \Gamma - \{\gamma\}
$$

$$
\pi(i) \geq 0, \quad i \in X, \quad 1'\pi = 1, \quad \text{i.e. } \pi \in \Pi(X).
$$  \tag{2.21}

Clearly, if the above linear program yields a solution $x \geq 0$, then $\gamma$ dominates all other vectors in $\Gamma - \{\gamma\}$. Then vector $\gamma$ is inactive and can be eliminated.
Monahan’s Algorithm: Mohahan [84] proposed an algorithm that is identical to Algorithm 2 except that the prune steps in computing $\Gamma_k(u, y)$ and $\Gamma_k(u)$ are omitted. So $\Gamma_k(u)$ comprises of $U|\Gamma_{k+1}|^Y$ vectors and these are then pruned according to the last step of Algorithm 2.

Witness Algorithm: The Witness algorithm [20], constructs $\Gamma_k(u)$ associated with $Q_k(\pi, u)$ (2.19) in polynomial time with respect to $X, U, Y$ and $|\Gamma_{k+1}|$. [21] shows that the incremental pruning Algorithm 2 has the same computational cost as the Witness algorithm and can outperform it by a constant factor.

2.5.2 Lovejoy’s Suboptimal Algorithm

Computing the value function and therefore optimal policy of a POMDP via the exact algorithms given above is intractable apart from small toy examples. Lovejoy [79] proposed an ingenious suboptimal algorithm that computes upper and lower bounds to the value function of a POMDP. The intuition behind this algorithm is depicted in Figure 2.4 and is as follows: Let $\bar{J}_k$ and $\underline{J}_k$, respectively, denote upper and lower bounds to $J_k$. It is obvious that by considering only a subset of the piecewise linear segments in $\Gamma_k$ and discarding the other segments, one gets an upper bound $\bar{J}_k$. That is, for any $\bar{\Gamma}_k \subset \Gamma_k$,

$$\bar{J}_k(\pi) = \min_{\gamma \in \bar{\Gamma}_k} \gamma'_\pi \geq \min_{\gamma \in \Gamma_k} \gamma'_\pi = J_k(\pi).$$

In Figure 2.4, $J_k$ is characterized by line segments in $\Gamma_k = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and the upper bound $\bar{J}_k$ is constructed from line segments in $\bar{\Gamma}_k = \{\gamma_2, \gamma_3\}$, i.e. discarding segments $\gamma_1$ and $\gamma_4$. This upper bound is displayed in dashed lines.

By choosing $\bar{\Gamma}_k$ with small cardinality at each iteration $k$, one can reduce the computational cost of computing $\bar{J}_k$. This is the basis of Lovejoy’s [79] lower bound approximation. Lovejoy’s algorithm [79] operates as follows:

**Initialize:** $\bar{\Gamma}_N = \Gamma_N = \{c_N\}$. Recall $c_N$ is the terminal cost vector.

**Step 1:** Given a set of vectors $\Gamma_k$, construct the set $\bar{\Gamma}_k$ by pruning $\Gamma_k$ as follows: Pick any $R$ belief states $\pi_1, \pi_2, \ldots, \pi_R$ in the belief simplex $\Pi(X)$. (Typically, one often picks the $R$ points based on a uniform Freudenthal triangulization of $\Pi(X)$, see [79] for details). Then set

$$\bar{\Gamma}_k = \{\arg \min_{\gamma \in \Gamma_k} \gamma'_\pi_r, \quad r = 1, 2, \ldots, R\}.$$ 

**Step 2:** Given $\bar{\Gamma}_k$, compute the set of vectors $\Gamma_{k-1}$ using a standard POMDP algorithm.

**Step 3:** $k \rightarrow k - 1$ and go to Step 1.

Notice that $\bar{J}_k(\pi) = \min_{\gamma \in \bar{\Gamma}_k} \gamma'_\pi$ is represented completely by $R$ piecewise linear segments. Lovejoy [79] shows that for all $k$, $\bar{J}_k$ is an upper bound to the optimal value function $J_k$, Thus Lovejoy’s algorithm gives a suboptimal policy
2.5 Algorithms for Finite Horizon POMDPs with Finite Observation Space

Figure 2.4 Intuition behind Lovejoy’s suboptimal algorithm for solving a POMDP for $X = 2$. The piecewise linear concave value function $J_k$ is denoted by unbroken lines. Interpolation (dotted lines) yields a lower bound to the value function. Omitting any of the piecewise linear segments leads to an upper bound (dashed lines). The main point is that the dotted and dashed lines sandwich the value function (unbroken line).

that yields an upper bound to the value function at a computational cost of no more than $R$ evaluations per iteration $k$.

So far we have discussed Lovejoy’s upper bound. Lovejoy [79] also provides a constructive procedure for computing a lower bound to the optimal value function. The intuition behind the lower bound is displayed in Figure 2.4 and is as follows: a linear interpolation to a concave function lies below the concave function. Choose any $R$ belief states $\pi_1, \pi_2, \ldots, \pi_R$. Then construct $\underline{J}_k(\pi)$ depicted in dotted lines in Figure 2.4 as the linear interpolation between the points $(\pi_i, J_k(\pi_i)), i = 1, 2, \ldots, R$. Clearly due to concavity of $J_k$, it follows that $\underline{J}_k(\pi) \leq J_k(\pi)$ for all $\pi \in \Pi(X)$.

2.5.3 Point-Based Value Iteration Methods

Point-based value iteration methods seek to compute an approximation of the value function at special points in the belief space. The main idea is to compute solutions only for those belief states that have been visited by running the POMDP. This motivates the development of approximate solution techniques that use a sampled set of belief states on which the POMDP is solved [38, 109, 71].

As mentioned above, Lovejoy [79] uses Freudenthal triangulation to form a grid on the belief space and then computes the approximate policy at these belief states. Another possibility is to take all extreme points of the belief simplex or to use a random grid. Yet another option is to include belief states that are encountered when simulating the POMDP. Trajectories can be generated in the belief simplex by sampling random actions and observations at each time
More sophisticated schemes for belief sampling have been proposed in [109]. The SARSOP approach of [71] performs successive approximations of the reachable belief space from following the optimal policy.

2.5.4 Belief Compression POMDPs

An interesting class of suboptimal POMDP algorithms [106] involves reducing the dimension of the belief space \( \mathcal{B} \). The \( X \)-dimensional belief states \( \pi \) are projected to \( \bar{X} \)-dimensional belief states \( \bar{\pi} \) that live in the reduced dimension simplex \( \mathcal{B}(\bar{X}) \) where \( \bar{X} \ll X \). The principal component analysis (PCA) algorithm is used to achieve this belief compression as follows: Suppose a sequence of beliefs \( \pi_1, \ldots, \pi_N \) is generated; this is a \( X \times N \) matrix. Then perform a singular value decomposition \( \pi_1: N = U D V' \) and choose the largest \( \bar{X} \) singular values. Then the original belief \( \pi \) and compressed belief \( \bar{\pi} \) are related by \( \pi = U_{\bar{X}} \bar{\pi} \) or \( \bar{\pi} = U_{\bar{X}}' \pi \) where \( U_{\bar{X}} \) and \( D_{\bar{X}} \) denote the truncated matrices corresponding to the largest \( \bar{X} \) singular values. (PCA is suited to dimensionality reduction when the data lies near a linear manifold. However, POMDP belief manifolds are rarely linear and so [106] proposes an exponential family PCA.)

The next step is to quantize the low dimensional belief state space \( \mathcal{B}(\bar{X}) \) into a finite state space \( \bar{S} = \{ \bar{q}_1, \ldots, \bar{q}_L \} \). The corresponding full dimensional beliefs are \( S = \{ U_{\bar{X}} \bar{q}_1, \ldots, U_{\bar{X}} \bar{q}_L \} \). The reduced dimension dynamic programming recursion then is identical to that of a finite state MDP. It reads

\[
V_{k+1}(\bar{q}_i) = \min_{u \in \mathcal{U}} \{ \bar{c}((\bar{q}_i, u) + \sum_{j=1}^{L} \bar{T}(\bar{q}_i, \bar{q}_j, u)V_k(\bar{q}_j) \}
\]

where \( \bar{c}(\bar{q}_i, u) = c'(u)q_i \), and \( \bar{T}(\bar{q}_i, \bar{q}_j, u) = U_{\bar{X}} B_y(u)P'(u)q_i \).

Chapter 4 presents algorithms for approximating the belief with provable bounds. In [132, 131], we present stochastic gradient algorithms for estimating the underlying state of an HMM directly. Either of these algorithms can be used in the above reduced dimensional dynamic programming algorithm instead of PCA type compression.

2.6 Discounted Infinite Horizon POMDPs

So far we have considered finite horizon POMDPs. This section consider infinite horizon discounted cost POMDPs. The discounted POMDP model is a 7-tuple \( (\mathcal{X}, \mathcal{U}, \mathcal{Y}, P(u), B(u), c(u), \rho) \) where \( P(u), B(u) \) and \( c \) are no longer explicit functions of time and \( \rho \in [0, 1) \) in an economic discount factor. Also, compared to (2.1), there is no terminal cost \( c_N \).

Define a stationary policy sequence as \( \mu = (\mu, \mu, \mu, \cdots) \) where \( \mu \) is not an
explicit function of time \( k \). We will use \( \mu \) instead of \( \mu \) to simply notation. For stationary policy \( \mu : \Pi(\mathcal{X}) \to \mathcal{U}, \) initial belief \( \pi_0 \in \Pi(\mathcal{X}) \), discount factor \( \rho \in [0, 1) \), define the objective function as the discounted expected cost:

\[
J_{\mu}(\pi_0) = \mathbb{E}_{\mu} \left\{ \sum_{k=0}^{\infty} \rho^k c(x_k, u_k) \right\}, \text{ where } u_k = \mu(\pi_k)
\]

As in §2.2 we can re-express this objective in terms of the belief state as

\[
J_{\mu}(\pi_0) = \mathbb{E}_{\mu} \left\{ \sum_{k=0}^{\infty} \rho^k c'_{\mu(\pi_k)} \pi_k \right\}, \tag{2.22}
\]

where \( c_u = [c(1, u), \ldots, c(X, u)]' \), \( u \in \mathcal{U} \) is the cost vector for each action, and the belief state evolves according to the HMM filter \( \pi_k = T(\pi_{k-1}, y_k, u_{k-1}) \) (2.11).

The aim is to compute the optimal stationary policy \( \mu^* : \Pi(\mathcal{X}) \to \mathcal{U} \) such that \( J_{\mu^*}(\pi_0) \leq J_{\mu}(\pi_0) \) for all \( \pi_0 \in \Pi(\mathcal{X}) \). From the dynamic programming recursion, we have for any finite horizon \( N \) that

\[
J_k(\pi) = \min_{u \in \mathcal{U}} \left\{ \rho^k c'_{\pi} + \sum_{y \in \mathcal{Y}} J_{k+1}(T(\pi, y, u)) \sigma(\pi, y, u) \right\}
\]

initialized by \( J_N(\pi) = 0 \). For discounted cost problems, it is more convenient to work with a forward iteration of indices. Accordingly, define the following value function \( V_n(\pi) \):

\[
V_n(\pi) = \rho^{n-N} J_{N-n}(\pi), \quad 0 \leq n \leq N, \pi \in \Pi(\mathcal{X}).
\]

Then it is easily seen that \( V_n(\pi) \) satisfies the dynamic programming equation

\[
V_n(\pi) = c'_{\pi} + \rho \sum_{y \in \mathcal{Y}} V_{n-1}(T(\pi, y, u)) \sigma(\pi, y, u), \quad V_0(\pi) = 0. \tag{2.23}
\]

### 2.6.1 Bellman’s Equation for discounted infinite horizon POMDP

The main result for infinite horizon discounted cost POMDPs is as follows:

**Theorem 2.6.1** Consider an infinite horizon discounted cost POMDP with discount factor \( \rho \in [0, 1) \). Then

1. The optimal expected cumulative cost is achieved by a stationary deterministic Markovian policy \( \mu^* \).
2. The optimal policy \( \mu^*(\pi) \) and value function \( V(\pi) \) satisfy Bellman’s dynamic programming equation

\[
\mu^*(\pi) = \arg\min_{u \in \mathcal{U}} Q(\pi, u), \quad J_{\mu^*}(\pi_0) = V(\pi_0) \tag{2.24}
\]

\[
V(\pi) = \min_{u \in \mathcal{U}} Q(\pi, u), \quad Q(\pi, u) = c'_{\pi} + \rho \sum_{y \in \mathcal{Y}} V(T(\pi, y, u)) \sigma(\pi, y, u).
\]
where \( T(\pi, y, u) \) and \( \sigma(\pi, y, u) \) are the HMM filter and normalization (2.11).

The expected cumulative cost incurred by the optimal policy is \( J_{\mu^*}(\pi) = V(\pi) \).

3. The value function \( V(\pi) \) is continuous and concave in \( \pi \in \Pi(X) \).

2.6.2 Value Iteration Algorithm for discounted cost POMDPs

Let \( n = 1, 2, \ldots, N \) denote iteration number. The value iteration algorithm for a discounted cost POMDP is a successive approximation algorithm for computing the value function \( V(\pi) \) of Bellman’s equation (2.24) and proceeds as follows: Initialize \( V_0(\pi) = 0 \). For iterations \( n = 1, 2, \ldots, N \), evaluate

\[
V_n(\pi) = \min_{u \in U} Q_n(\pi, u), \quad \mu_n^*(\pi) = \arg\min_{u \in U} Q_n(\pi, u),
\]

\[
Q_n(\pi, u) = c'_u \pi + \rho \sum_{y \in Y} V_{n-1}(T(\pi, y, u)) \sigma(\pi, y, u).
\]  

(2.25)

Finally, the stationary policy \( \mu_N^* \) is used at each time instant \( k \) in the real time controller of Algorithm 1. The obvious advantage of the stationary policy is that only the policy \( \mu_N^*(\pi) \) needs to be stored for real time implementation of the controller in Algorithm 1.

Summary: The POMDP value iteration algorithm (2.25) is identical to the finite horizon dynamic programming recursion (2.15). So at each iteration \( n \), \( V_n(\pi) \) is piecewise linear and concave in \( \pi \) (by Theorem 2.2.1) and can be computed using any of the POMDP algorithms discussed in §2.5. The number of piecewise linear segments that characterize \( V_n(\pi) \) can grow exponentially with iteration \( n \). Therefore, except for small state, action and observation spaces, suboptimal algorithms (such as those discussed in §2.5) need to be used.

How are the number of iterations \( N \) chosen in the value iteration algorithm (2.25)? The value iteration algorithm (2.25) generates a sequence of value functions \( \{V_n\} \) that will converge uniformly (sup-norm metric) as \( N \to \infty \) to \( V(\pi) \), the optimal value function of Bellman’s equation. The number of iterations \( N \) in (2.25) can be chosen as follows: Let \( \epsilon > 0 \) denote a specified tolerance.

THEOREM 2.6.2 Consider the value iteration algorithm with discount factor \( \rho \) and \( N \) iterations. Then:

1. \( \sup_{\pi} |V_N(\pi) - V_{N-1}(\pi)| \leq \epsilon \) implies that \( \sup_{\pi} |V_N(\pi) - V(\pi)| \leq \frac{\epsilon \rho}{1 - \rho} \).

2. \( |V_N(\pi) - V(\pi)| \leq \frac{\rho^N}{1 - \rho} \max_{x,u} |c(x,u)| \).

Actually, similar to the value iteration algorithm, one can evaluate the expected discounted cumulative cost of an arbitrary stationary policy (not necessarily the optimal policy) as follows:

COROLLARY 2.6.3 (Policy Evaluation) For any stationary policy \( \mu \), the associated expected discounted cumulative cost \( J_\mu(\pi) \) defined in (2.22) for a POMDP satisfies

\[
J_\mu(\pi) = c'_\mu(\pi) \pi + \rho \sum_{y \in Y} J_\mu(T(\pi, y, \mu(\pi))) \sigma(\pi, y, \mu(\pi)).
\]  

(2.26)
Similar to the value iteration algorithm (2.25), \( J_\mu(\pi) \) can be obtained as 
\[
J_\mu(\pi) = \lim_{n \to \infty} V_{\mu,n}(\pi).
\]
Here \( V_{\mu,n}(\pi), n = 1, 2, \ldots \) satisfies the recursion
\[
V_{\mu,n}(\pi) = c_\mu(\pi) + \rho \sum_{y \in Y} V_{\mu,n-1}(T(\pi, y, \mu(\pi))) \sigma(\pi, y, \mu(\pi)), \quad V_{\mu,0}(\pi) = 0.
\]

Also Theorem 2.6.2 holds for \( J_\mu(\pi) \).

The proof is omitted since it is similar to the corresponding theorems for the optimal policy given above. Note that (2.26) can be written as
\[
J_\mu(\pi) = \gamma_\mu(\pi), \quad \text{where } \gamma_\mu(\pi) = c_u + \rho \sum_{y \in Y} \gamma_{\mu(T(\pi, y, u))} P(u) B_y(u), \quad (2.27)
\]
where \( u = \mu(\pi) \) on the right hand side. We will use this representation below.

### 2.7 Example: Optimal Search for a Markovian Moving Target

Optimal search of a moving target is a useful illustrative example of a POMDP. From an abstract point of view, many resource allocation problems involving controlled sensing and communication with noisy information can be formulated as an optimal search problem; see for example [46] where opportunistic transmission over a fading channel is formulated as an optimal search problem.

A target moves among \( X \) cells according to a Markov chain with transition matrix \( P \). At time instants \( k \in \{0, 1, 2, \ldots\} \), the searcher must choose an action from the action space \( \mathcal{U} \). The set \( \mathcal{U} \) contains actions that search a particular cell or a group of cells simultaneously. Assuming action \( u \) is selected by the searcher at time \( k \), it is executed with probability \( 1 - q(u) \). If the action cannot be executed, the searcher is said to be blocked for time \( k \). This blocking event with probability \( q(u) \) models the scenario when the search sensors are a shared resource and not enough resources are available to carry out the search at time \( k \). If the searcher is not blocked and action \( u \) searches the cell that the target is in, the target is detected with probability \( 1 - \beta(u) \); failure to detect the target when it is in the cell searched is called an overlook. So the overlook probability in cell \( u \) is \( \beta(u) \).

If the decision maker knows \( P, \beta, q \), in which order should it search the cells to find the moving target with minimum expected effort?

#### 2.7.1 Formulation of Finite-Horizon Search Problem

It is assumed here that the searcher has a total of \( N \) attempts to find the moving target. Given a target that moves between \( X \) cells, let \( \bar{X} = \{1, 2, \ldots, X, T\} \) denote the augmented state space. Here \( T \) corresponds to a fictitious terminal state that is added as a means of terminating search if the target is detected prior to exhausting the \( N \) search horizon.

Denote the observation space as \( \mathcal{Y} = \{F, \bar{F}, b\} \). Here \( F \) denotes “target found”, \( \bar{F} \) denotes “target not found” and \( b \) denotes “search blocked”.

An optimal search problem consists of the following ingredients:

1. **Markov State Dynamics**: The location of the target is modelled as a finite state Markov chain. The target moves amongst the \( X \) cells according to transition probability matrix \( P \). Let \( x_k \in X \) denote the state (location) of the target at the start of search epoch \( k \) where \( k = 0, 1, 2, \ldots, N - 1 \). To model termination of the search process after the target is found, we model the target dynamics by the **observation** dependent transition probability matrices \( P^y, y \in \mathcal{Y} = \{ F, \bar{F}, b \} \):

\[
P^F = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad P^\bar{F} = P^b = \begin{bmatrix} P & 0 \\ 0' & 1 \end{bmatrix}.
\]  

(2.28)

That is, \( \mathbb{P}(x_{k+1} = j|x_k = i, y_k = y) = P^y_{ij} \). As can be seen from the transition matrices above, the terminal state \( T \) is designed to be absorbing. A transition to \( T \) occurs only when the target is detected. The initial state of the target is sampled from \( \pi_0(i), i \in \{1, \ldots, X\} \).

2. **Action**: At each time \( k \), the decision maker chooses action \( u_k \) from the finite set of search actions \( \mathcal{U} \). In addition to searching the target in one of the \( X \) cells, \( \mathcal{U} \) may contain actions that specify the simultaneous search in a number of cells.

3. **Observation**: Let \( y_k \in \mathcal{Y} = \{ F, \bar{F}, b \} \) denote the observation received at time \( k \) upon choosing action \( u_k \). Here

\[
y_k = \begin{cases} 
F & \text{target is found,} \\
\bar{F} & \text{target is not found,} \\
b & \text{search action is blocked due to insufficient available resources.} 
\end{cases}
\]

Define the **blocking** probabilities \( q(u) \) and **overlook** probabilities \( \beta(u) \), \( u \in \mathcal{U} \) as:

\[
q(u) = \mathbb{P}(\text{insufficient resources to perform action } u \text{ at epoch } k),
\]

\[
\beta(u) = \mathbb{P}(\text{target not found} | \text{target is in the cell } u).
\]  

(2.29)

Then, the observation \( y_k \) received is characterized probabilistically as follows. For all \( u \in \mathcal{U} \) and \( j = 1, \ldots, X \),

\[
\mathbb{P}(y_k = F|x_k = j, u_k = u) = \begin{cases} 
1 - q(u)(1 - \beta(u)) & \text{if action } u \text{ searches cell } j, \\
0 & \text{otherwise,} 
\end{cases}
\]

\[
\mathbb{P}(y_k = \bar{F}|x_k = j, u_k = u) = \begin{cases} 
1 - q(u) & \text{if action } u \text{ does not search cell } j, \\
\beta(u)(1 - q(u)) & \text{otherwise,} 
\end{cases}
\]

\[
\mathbb{P}(y_k = b|x_k = j, u_k = u) = q(u).
\]  

(2.30)

Finally, for the fictitious terminal state \( T \), the observation \( F \) is always received regardless of the action taken, so that

\[
\mathbb{P}(y_k = F|x_k = T, u_k = u) = 1.
\]

4. **Cost**: Let \( c(x_k, u_k) \) denote the instantaneous cost for choosing action \( u_k \) when the target’s state is \( x_k \). Three types of instantaneous costs are of interest.
1. **Maximize Probability of Detection** [95, 30]. The instantaneous reward is the probability of detecting the target (obtaining observation $F$) for the current state and action. This constitutes a negative cost. So

$$
c(x_k = j, u_k = u) = -P(y_k = F|x_k = j, u_k = u) \quad \text{for } j = 1, ..., X,
$$

$$
c(x_k = T, u_k = u) = 0.
$$

(2.31)

2. **Minimize Search Delay** [95]. An instantaneous cost of 1 unit is accrued for every action taken until the target is found, i.e., until the target reaches the terminal state $T$:

$$
c(x_k = j, u_k = u) = 1 \quad \text{for } j = 1, ..., X,
$$

$$
c(x_k = T, u_k = u) = 0.
$$

(2.32)

3. **Minimize Search Cost**. The instantaneous cost depends only on the action taken. Let $c(u)$ denote the positive cost incurred for action $u$, then

$$
c(x_k = j, u_k = u) = c(u) \quad \text{for } j = 1, ..., X,
$$

$$
c(x_k = T, u_k = u) = 0.
$$

(2.33)

5. **Performance criterion**: Let $I_k$ denote the information (history) available at the start of search epoch $k$:

$$
I_0 = \{\pi_0\}, \quad I_k = \{\pi_0, u_0, y_0, \ldots, u_{k-1}, y_{k-1}\} \quad \text{for } k = 1, \ldots, N.
$$

(2.34)

$I_k$ contains the initial probability distribution $\pi_0$, the actions taken and observations received prior to search time $k$. A search policy $\mu$ is a sequence of decision rules $\mu = \{\mu_0, \ldots, \mu_{N-1}\}$ where each decision rule $\mu_k : I_k \rightarrow U$. The performance criterion considered is

$$
J_\mu(\pi_0) = E_\mu \left\{ \sum_{k=0}^{N-1} c(x_k, \mu_k(I_k)) | \pi_0 \right\}. \quad (2.35)
$$

This is the expected cost accrued after $N$ time points using search policy $\mu$ when the initial distribution of the target is $\pi_0$. The **optimal search problem** is to find the policy $\mu^*$ that minimizes (2.35) for all initial distributions, i.e.,

$$
\mu^* = \arg\min_{\mu \in \mathcal{U}} J_\mu(\pi_0), \quad \forall \pi_0 \in \Pi(X). \quad (2.36)
$$

Similar to (2.13), we can express the objective in terms of the belief state as

$$
J_\mu(\pi_0) = E_\mu \left\{ \sum_{k=0}^{N} c(x_k, \mu_k(I_k)) | \pi_0 \right\}, \quad
$$

$$
= E_\mu \left\{ \sum_{k=0}^{N} E\{c(x_k, \mu_k(I_k)) | I_k\} | \pi_0 \right\} = \sum_{k=0}^{N} E\{c_{u_k} \pi_k\} \quad (2.37)
$$

where belief state $\pi_k = [\pi_k(1), \ldots, \pi_k(X+1)]'$ is defined as $\pi_k(i) = P(x_k = i|I_k)$. 

The belief state is updated by the HMM predictor\textsuperscript{4} as follows:
\[
\pi_{k+1} = T(\pi_k, y_k, u_k) = \frac{P_{\nu u'} \hat{B}_y(u_k) \pi_k}{\sigma(\pi_k, y_k, u_k)}, \quad \sigma(\pi, y, u) = 1' \hat{B}_y(u) \pi \quad (2.38)
\]
\[
\hat{B}_y(u) = \text{diag}(\mathbb{P}(y_k = y|x_k = 1, u_k = u), \ldots, \mathbb{P}(y_k = y|x_k = X, u_k = u), \mathbb{P}(y_k = y|x_k = T, u_k = u)).
\]

Recall the observation dependent transition probabilities \( P_y \) are defined in (2.28).

The optimal policy \( \mu^* \) can be computed via the dynamic programming recursion (2.15) where \( T \) and \( \sigma \) are defined in (2.38).

\[\text{2.7.2 Formulation of Optimal Search as a POMDP}\]

In order to use POMDP software to solve the search problem (2.36) via dynamic programming, it is necessary to express the search problem as a POMDP. The search problem in §2.7.1 differs from a standard POMDP in two ways:

Timing of the events: In a POMDP, the observation \( y_{k+1} \) received upon adopting action \( u_k \) is with regards to the new state of the system, \( x_{k+1} \). In the search problem, the observation \( y_k \) received for action \( u_k \) conveys information about the current state of the target (prior to transition), \( x_k \).

Transition to the new state: In a POMDP the probability distribution that characterizes the system’s new state, \( x_{k+1} \), is a function of its current state \( x_k \) and the action \( u_k \) adopted. However, in the search problem, the distribution of the new state, \( x_{k+1} \), is a function of \( x_k \) and observation \( y_k \).

The search problem of §2.7.1 can be reformulated as a POMDP (2.1) as follows: Define the augmented state process \( s_{k+1} = (y_k, x_{k+1}) \).

Then consider the following POMDP with \( 2X + 1 \) underlying states.

State space: \( S = \{ (\bar{F}, 1), (\bar{F}, 2), \ldots, (\bar{F}, X), (b, 1), (b, 2), \ldots, (b, X), (F, T) \} \).

Action space: \( \mathcal{U} \) (same as search problem in §2.7.1)

Observation space: \( \mathcal{Y} = \{ F, \bar{F}, b \} \) (same as search problem in §2.7.1)

Transition probabilities: For each action \( u \in \mathcal{U} \), define
\[
P(u) = \begin{bmatrix}
B_{\bar{F}}(u) P & B_b(u) P & B_F(u) \mathbf{1} \\
B_{\bar{F}}(u) P & B_b(u) P & B_F(u) \mathbf{1} \\
\mathbf{0}' & \mathbf{0}' & 1
\end{bmatrix}
\quad (2.39)
\]

where \( P \) is the transition matrix of the moving target and for \( y \in \{ F, \bar{F}, b \} \)
\[
B_y(u) = \text{diag}(\mathbb{P}(y_k = y|x_k = 1, u_k = u), \ldots, \mathbb{P}(y_k = y|x_k = X, u_k = u)).
\quad (2.40)
\]

Recall these are computed in terms of the blocking and overlook probabilities using (2.30).

\textsuperscript{4}The reader should note the difference between the information pattern of a standard POMDP (2.9), namely, \( \mathcal{I}_k = (s_0, u_0, y_1, \ldots, u_{k-1}, y_k) \) and the information pattern \( \mathcal{I}_k \) for the search problem in (2.34). In the search problem, \( \mathcal{I}_k \) has observations until time \( k - 1 \), thereby requiring the HMM predictor (2.38) to evaluate the inner conditional expectation of the cost in (2.37).
Observation probabilities: For each action \( u \in \mathcal{U} \), define the observation probabilities \( R_{sy}(u) = \mathbb{P}(y_k = y| s_{k+1} = s, u_k = u) \) where for \( s = (\bar{y}, x) \),

\[
R_{\bar{y}x,y}(u) = \begin{cases} 
1 & \bar{y} = y, \\
0 & \text{otherwise}
\end{cases}, \quad y, \bar{y} \in \mathcal{Y}, \ x \in \{1, 2, \ldots, X\}.
\]

Costs: For each action \( u \in \mathcal{U} \), define the POMDP instantaneous costs as

\[
g(i, u) = \begin{cases} 
c(i, u) & \text{for } i = 1, \ldots, X, \\
c(i - X, u) & \text{for } i = X + 1, \ldots, 2X, \\
0 & \text{for } i = 2X + 1.
\end{cases}
\]

(2.41)

To summarize, optimal search over a finite horizon is equivalent to the finite horizon POMDP \((\mathcal{S}, \mathcal{U}, \mathcal{Y}, P(u), R(u), g(u))\).

2.8 Complements and Sources

The following websites are repositories of papers and software for solving POMDPs:
http://www.pomdp.org
http://bigbird.comp.nus.edu.sg/pmwiki/farm/appl/

The belief state formulation in partially observed stochastic control goes back to the early 1960s; see the seminal works of Stratonovich [119], Astrom [5] and Dynkin [29]. Sondik [115] first showed that Bellman’s equation for a finite horizon POMDP has a finite dimensional piecewise linear concave solution. This led to the influential papers [113, 116]; see [84, 80, 19, 110] for surveys. POMDPs have been applied in dynamic spectrum management for cognitive radio [39, 134], adaptive radars [85, 62, 60].

Optimal search theory is a well studied problem [118, 43]. Early papers in the area include [30, 95]. The paper [81] shows that in many cases optimal search for a target moving between two cells has a threshold type optimal policy - this verifies a conjecture made by Ross in [105]. The proof that the search problem for a moving target is a stochastic shortest path problem is given in [93, 112].

Radar scheduling using POMDPs has been studied in [52, 32, 53, 61, 62].
Structural Results for Markov Decision Processes

For finite state MDPs with large dimensional state spaces, computing the optimal policy by solving Bellman’s dynamic programming recursion or the associated linear programming problem can be prohibitively expensive. Structural results give sufficient conditions on the MDP model to ensure that the optimal policy \( \mu^*(x) \) is increasing (or decreasing) in the state \( x \). Such policies will be called monotone policies. To see why monotone policies are important, consider an MDP with two actions \( \mathcal{U} = \{1, 2\} \) and a large state space \( \mathcal{X} = \{1, 2, \ldots, X\} \). If the optimal policy \( \mu^*(x) \) is increasing in \( x \), then it has to be a step function of the form

\[
\mu^*(x) = \begin{cases} 
1 & x < x^* \\
2 & x \geq x^* 
\end{cases}
\]  

(3.1)

Here \( x^* \in \mathcal{X} \) is some fixed state at which the optimal policy switches from action 1 to action 2. A policy of the form (3.1) will be called a threshold policy and \( x^* \) will be called the threshold state. Figure 3.1 illustrates a threshold policy.

Note that \( x^* \) completely characterizes the threshold policy (3.1). Therefore, if one can prove that the optimal policy \( \mu^*(x) \) is increasing in \( x \), then one only needs to compute the threshold state \( x^* \). Computing (estimating) \( x^* \) is often more efficient (from a computational point of view) than solving Bellman’s equation when nothing is known about the structure of the optimal policy. Also real time implementation of a controller with monotone policy (3.1) is simple.

\(^1\)Throughout this book, increasing is used in the weak sense to mean increasing.

![Figure 3.1 Monotone increasing threshold policy \( \mu^*(x) \). Here, \( x^* \) is the threshold state at which the policy switches from 1 to 2.](image)
For a finite horizon MDP, Bellman’s equation for the optimal policy \( \mu^*_k(x) \) reads:

\[
Q_k(x, u) \overset{\text{def}}{=} c(x, u, k) + J'_{k+1}P_x(u) \tag{3.2}
\]

where \( J_{k+1} = [J_{k+1}(1), \ldots, J_{k+1}(X)]' \) denotes the value function. The key point is that the optimal policy is \( \mu^*_k(x) = \arg\min_{u \in U} Q_k(x, u) \).

What are sufficient conditions on the MDP model to ensure that the optimal policy \( \mu^*_k(x) \) is increasing in \( x \)? The answer to this question lies in the area of monotone comparative statics - which studies how the argmin or argmax of a function behaves as one of the variables changes. The main result of this chapter is to show that \( Q_k(x, u) \) in (3.2) being submodular in \((x, u)\) is a sufficient condition for \( \mu^*(x) \) to increase in \( x \). Since \( Q_k(x, u) \) is the conditional expectation of the cost to go given the current state, giving conditions on the MDP model to ensure that \( Q_k(x, u) \) is submodular requires characterizing how expectations vary as the state varies. For this we will use stochastic dominance.

In the next two sections we introduce these two important tools, namely, submodularity/supermodularity and stochastic dominance. They will be used to give conditions under which an MDP has monotone optimal policies.

### 3.1 Submodularity and Supermodularity

Throughout this chapter we assume that the state space \( \mathcal{X} = \{1, 2, \ldots, X\} \) and action space \( \mathcal{U} = \{1, 2, \ldots, U\} \) are finite.

#### 3.1.1 Definition and Examples

A real valued function \( \phi(x, u) \) is submodular in \((x, u)\) if

\[
\phi(x, u + 1) - \phi(x, u) \geq \phi(x + 1, u + 1) - \phi(x + 1, u). \tag{3.3}
\]

In other words, \( \phi(x, u + 1) - \phi(x, u) \) has decreasing differences with respect to \( x \). A function \( \phi(x, u) \) is supermodular if \(-\phi(x, u)\) is submodular.

Note that submodularity and supermodularity treat \( x \) and \( u \) symmetrically. That is, an equivalent definition is \( \phi(x, u) \) is submodular if \( \phi(x + 1, u) - \phi(x, u) \) has decreasing differences with respect to \( u \).

**Examples**: The following are submodular in \((x, u)\)

(i) \( \phi(x, u) = -xu \)

(ii) \( \phi(x, u) = \max(x, u) \)

(iii) Any function of one variable such as \( \phi(x) \) or \( \phi(u) \) is trivially submodular.

(iv) The sum of submodular functions is submodular.
30 Structural Results for Markov Decision Processes

Figure 3.2 Visual illustration of main idea of proof of Theorem 3.1.1. The solid (respectively, dotted) curve represents $\phi(x, u)$ (respectively $\phi(x + 1, u)$) plotted versus $u$. Also, $u^*(x + 1) = \arg\min_u \phi(x + 1, u)$. If for all values of $u$, $\phi(x, u)$ to the right of $u^*(x + 1)$ is larger than $\phi(x, u)$ to left of $u^*(x + 1)$, then clearly the $\arg\min_u \phi(x, u)$ must lie to the left of $u^*(x + 1)$. That is, $u^*(x) \leq u^*(x + 1)$.

3.1.2 Topkis’ Monotonicity Theorem

Let $u^*(x)$ denote the set of possible minimizers of $\phi(x, u)$ with respect to $u$:

$$u^*(x) = \{\arg\min_{u \in \mathcal{U}} \phi(x, u)\}, \quad x \in \mathcal{X}.$$  

In general there might not be a unique minimizer and then $u^*(x)$ has multiple elements. Let $\bar{u}^*(x)$ and $\underline{u}^*(x)$ denote the maximum and minimum elements of this set. We call these, respectively, the maximum and minimum selection of $u^*(x)$.

The key result is the following Topkis’ Monotonicity theorem.

**Theorem 3.1.1** Consider a function $\phi : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}$.

1. If $\phi(x, u)$ is submodular, then the maximum and minimal selections $\bar{u}^*(x)$ and $\underline{u}^*(x)$ are increasing in $x$.
2. If $\phi(x, u)$ is supermodular, then the maximum and minimal selections $\bar{u}^*(x)$ and $\underline{u}^*(x)$ are decreasing in $x$.

**Proof** We prove statement 1. The proof is illustrated visually in Figure 3.2. Fix $x$ and consider $\phi(x, u)$ as a function of $u$. Clearly, if $\phi(x, u)$ is larger than $\phi(x, \bar{u}^*(x + 1))$ for $u$ to the ‘right’ of $\bar{u}^*(x + 1)$, then $\arg\min_u \phi(x, u)$ must lie to the ‘left’ of $\bar{u}^*(x + 1)$ (see Figure 3.2). Therefore,

$$\phi(x, \bar{u}^*(x + 1)) \leq \phi(x, u) \text{ for } u \geq \bar{u}^*(x + 1) \implies u^*(x) \leq \bar{u}^*(x + 1).$$

Let us write this sufficient condition as

$$\phi(x, u) - \phi(x, \bar{u}^*(x + 1)) \geq 0 \text{ for } u \geq \bar{u}^*(x + 1).$$

Also by definition $\bar{u}^*(x + 1) \in \arg\min_u \phi(x + 1, u)$. So clearly

$$\phi(x + 1, u) - \phi(x + 1, \bar{u}^*(x + 1)) \geq 0 \text{ for all } u \in \mathcal{U}.$$  

2 Supermodularity for structural results in MDPs and game theory was introduced by Topkis in the seminal paper [123]. Chapter 6 gives a more general statement in terms of lattices.
From the above two inequalities, it is sufficient that
\[ \phi(x, u) - \phi(x, \bar{u}^*(x + 1)) \geq \phi(x + 1, u) - \phi(x + 1, \bar{u}^*(x + 1)) \text{ for } u \geq \bar{u}^*(x + 1). \]
A sufficient condition for this is that for any \( \bar{u} \in \mathcal{U} \),
\[ \phi(x, u) - \phi(x, \bar{u}) \geq \phi(x + 1, u) - \phi(x + 1, \bar{u}) \text{ for } u \geq \bar{u}. \]

\[ \square \]

3.2 First Order Stochastic Dominance

Stochastic dominance is the next tool that will be used to develop MDP structural results.

**Definition 3.2.1 (First order stochastic dominance)** Let \( \pi_1, \pi_2 \) denote two pmfs or pdfs with distribution functions \( F_1, F_2 \), respectively. Then \( \pi_1 \) is said to first order stochastically dominate \( \pi_2 \) (written as \( \pi_1 \geq_s \pi_2 \) or \( \pi_2 \leq_s \pi_1 \)) if
\[ 1 - F_1(x) \geq 1 - F_2(x), \quad \text{for all } x \in \mathbb{R}. \]
Equivalently, \( \pi_1 \geq_s \pi_2 \) if
\[ F_1(x) \leq F_2(x), \quad \text{for all } x \in \mathbb{R}. \]

In this chapter, we consider pmfs. For pmfs with support on \( X = \{1, 2, \ldots, X\} \), Definition 3.2.1 is equivalent to the following property of the tail sums:
\[ \pi_1 \geq_s \pi_2 \text{ if } \sum_{i=j}^{X} \pi_1(i) \geq \sum_{i=j}^{X} \pi_2(i), \quad j \in X. \]

For state space dimension \( X = 2 \), first order stochastic dominance is a complete order since \( \pi = [1 - \pi(2), \pi(2)] \) and so \( \pi_1 \geq_s \pi_2 \) if \( \pi_1(2) \geq \pi_2(2) \). Therefore, for \( X = 2 \), any two pmfs are first order stochastic orderable.

For \( X \geq 3 \), first order stochastic dominance is a partial order since it is not always possible to order any two belief pmfs \( \pi_1 \) and \( \pi_2 \).

The following is an equivalent characterization of first order dominance:

**Theorem 3.2.2 ([88])** Let \( V \) denote the set of all \( X \) dimensional vectors \( v \) with increasing components, i.e., \( v_1 \leq v_2 \leq \cdots \leq v_X \). Then \( \pi_1 \geq_s \pi_2 \) if and only if for all \( v \in V, v' \pi_1 \geq v' \pi_2 \). Similarly, pdf \( \pi_1 \geq_s \pi_2 \) if and only if \( \int \phi(x)\pi_1(x)dx \geq \int \phi(x)\pi_2(x)dx \) for any increasing function \( \phi(\cdot) \).

In other words, \( \pi_1 \geq_s \pi_2 \) if and only if, for any increasing function \( \phi(\cdot) \),
\[ E_{\pi_1}\{\phi(x)\} \geq E_{\pi_2}\{\phi(x)\}, \] where \( E_\pi \) denotes expectation with respect to the pmf

\[ ^3 \text{Recall the acronyms pmf (probability mass function) and pdf (probability density function).} \]
(pdf) π. As a trivial consequence, choosing φ(x) = x, it follows that π₁ ≥ s π₂ implies that the mean of pmf π₁ is larger than that of pmf π₂.

Finally, we need the following concept that combines supermodularity with stochastic dominance. We say that a transition probabilities Pᵢⱼ(u) are tail-sum supermodular in (i, u) if

$$\sum_{j=1}^{X} (P_{i,j}(u+1) - P_{i,j}(u))$$

is increasing in i, i ∈ X, u ∈ U. (3.4)

In terms of first order stochastic dominance, (3.4) can be re-written as

$$\frac{1}{2} (P_{i+1}(u+1) + P_i(u)) \geq \frac{1}{2} (P_{i+1}(u+1) + P_i(u+1)),$$

where Pᵢ(u) denotes the i-th row of the matrix P(u). Due to the term 1/2 both sides are valid probability mass functions. Thus we have the following result.

**Theorem 3.2.3** Let V denote the set of X dimensional vectors v with decreasing components, i.e., v₁ ≥ v₂ ≥ ⋯ ≥ vₓ. Then Pᵢⱼ(u) is tail-sum supermodular in (i, u) iff for all v ∈ V, v’Pᵢ(u) is submodular in (i, u), that is:

$$v' (P_{i+1}(u+1) - P_i(u)) \leq v' (P_{i+1}(u+1) - P_{i+1}(u)),$$

where Pᵢ(u) is the i-th row of the matrix P(u). Due to the term 1/2 both sides are valid probability mass functions. Thus we have the following result.

The proof follows immediately from Theorem 3.2.2 and (3.5).

### 3.3 Monotone Optimal Policies for MDPs

With the above two tools, we now give sufficient conditions for an MDP to have a monotone optimal policy.

For finite horizon MDPs, the model is the 5-tuple

$$(X, U, Pᵢⱼ(u,k), c(i,u,k), c_N(i)), \quad i, j \in X, u \in U.$$ (3.6)

Assume the MDP model satisfies the following 4 conditions:

**A1** Costs c(x, u, k) are decreasing in x.

The terminal cost c_N(x) is decreasing in x.

**A2** Pᵢ(u,k) ≤ s Pᵢ₊₁(u,k) for each i and u. Here Pᵢ(u,k) denotes the i-th row of the transition matrix for action u at time k.

**A3** c(x, u, k) is submodular in (x, u) at each time k. That is:

$$c(x, u + 1, k) - c(x, u, k)$$

is decreasing in x.

**A4** Pᵢⱼ(u, k) is tail-sum supermodular in (i, u) in the sense of (3.4). That is,

$$\sum_{j≥l} (P_{i,j}(u+1) - P_{i,j}(u))$$

is increasing in i.

For infinite horizon discounted cost and average cost MDPs, identical conditions will be used except that the instantaneous costs c(x, u) and transition matrix P(u) are time invariant, and there is no terminal cost.
Note that (A1) and (A2) deal with different states for a fixed action $u$, while (A3) and (A4) involve different actions and states.

The following is the main structural result for an MDP.

**Theorem 3.3.1**

1. Assume that a finite horizon MDP satisfies conditions (A1), (A2), (A3) and (A4). Then at each time $k = 0, 1, \ldots, N - 1$, there exists an optimal policy $\mu_k^*(x)$ that is increasing in state $x \in X$.

2. Assume that a discounted infinite horizon cost problem or unichain average cost problem satisfies (A1), (A2), (A3) and (A4). Then there exists an optimal stationary policy $\mu^*(x)$ that is increasing in state $x \in X$.

**Proof.**  

**Statement 1:** To prove statement 1, write Bellman’s equation as

$$
Q_k(i, u) \overset{\text{def}}{=} c(i, u, k) + J_{k+1}^P(u, k) \quad (3.7)
$$

where $J_{k+1} = [J_{k+1}(1), \ldots, J_{k+1}(X)]'$ denotes the value function. The proof proceeds in two steps.

**Step 1. Monotone Value Function.** Assuming (A1) and (A2), we show via mathematical induction that $Q_k(i, u)$ is decreasing in $i$ for each $u \in U$. So the value function $J_k(i)$ is decreasing in $i$ for $k = N, N - 1, \ldots, 0$.

Clearly $Q_N(i, u) = c_N(i)$ is decreasing in $i$ by (A1). Now for the induction step. Suppose $Q_{k+1}(i, u)$ is decreasing in $i \in X$ for each $u$. Then $J_{k+1}(i) = \min_u Q_{k+1}(i, u)$ is decreasing in $i$, since the minimum of decreasing functions is decreasing. So the $X$-dimensional vector $J_{k+1}$ has decreasing elements.

Next $P_i(u, k) \leq P_{i+1}(u, k)$ by (A2). Hence from Theorem 3.2.2, $J_{k+1}^P(u, k) \geq J_{k+1}^P(u, k)$. Finally since $c(i, u, k)$ is decreasing in $i$ by (A1), it follows that

$$c(i, u, k) + J_{k+1}^P(u, k) \geq c(i + 1, u, k) + J_{k+1}^P(u, k)$$

Therefore, $Q_k(i, u) \geq Q_k(i + 1, u)$ implying that $Q_k(i, u)$ is decreasing in $i$ for each $u \in U$. (This in turn implies that $J_k(i) = \min_u Q_k(i, u)$ is decreasing in $i$.) Hence the induction step is complete.

**Step 2. Monotone Policy.** Assuming (A3) and (A4) and using the fact that $J_k(i)$ is decreasing in $i$ (Step 1), we show that $Q_k(i, u)$ is submodular in $(i, u)$.

By (A3), $c(i, u, k)$ is submodular in $(i, u)$. By assumption (A4), since $J_{k+1}$ is a vector with decreasing elements (by Step 1), it follows from Theorem 3.2.3 that $J_{k+1}^P(u, k)$ is submodular in $(i, u)$. Since the sum of submodular functions is submodular, it follows that $Q_k(i, u) = c(i, u, k) + J_{k+1}^P(u, k)$ is submodular in $(i, u)$.

Since $Q_k(i, u)$ is submodular, it then follows from Theorem 3.1.1 that $\mu_k^*(i) = \min_{u \in U} Q_k(i, u)$ is increasing in $i$. (More precisely, if $\mu^*$ is not unique, then there exists a version of $\mu_k^*(i)$ that is increasing in $i$.)

$\square$
3.4 How does the optimal cost depend on the transition matrix?

How does the optimal expected cumulative cost $J_{\mu^*}$ of an MDP vary with transition matrix? Can the transition matrices be ordered so that the larger they are (with respect to some ordering), the larger the optimal cost? Such a result would allow us to compare the optimal performance of different MDPs, even though computing these via dynamic programming can be numerically expensive.

Consider two distinct MDP models with transition matrices $P(u)$ and $\bar{P}(u)$, $u \in \mathcal{U}$, respectively. Let $\mu^*(P)$ and $\mu^*(\bar{P})$ denote the optimal policies for these two different MDP models. Let $J_{\mu^*(P)}(x; P)$ and $J_{\mu^*(\bar{P})}(x; \bar{P})$ denote the optimal value functions corresponding to applying the respective optimal policies.

Introduce the following ordering on the transition matrices of the two MDPs.

(A5) Each row of $P(u)$ first order stochastic dominates the corresponding row of $\bar{P}(u)$ for $u \in \mathcal{U}$. That is, $P_i(u) \succeq \bar{P}_i(u)$ for $i \in \mathcal{X}, u \in \mathcal{U}$.

**Theorem 3.4.1** Consider two distinct MDPs with transition matrices $P(u)$ and $\bar{P}(u)$, $u \in \mathcal{U}$. If (A1), (A2), (A5) hold, then the expected cumulative costs incurred by the optimal policies satisfy $J_{\mu^*(P)}(x; P) \leq J_{\mu^*(\bar{P})}(x; \bar{P})$.

The theorem says that controlling an MDP with transition matrices $P(u)$, $u \in \mathcal{U}$ is always cheaper than an MDP with transition matrices $\bar{P}(u)$, $u \in \mathcal{U}$ if (A1), (A2) and (A5) hold. Note that Theorem 3.4.1 does need require numerical evaluation of the optimal policies or value functions.

**Proof** Suppose

\[ Q_i(i, u) = c(i, u, k) + J_{k+1}' P_i(u), \quad \bar{Q}_i(i, u) = c(i, u, k) + J_{k+1}' \bar{P}_i(u) \]

The proof is by induction. Clearly $J_N(i) = \bar{J}_N(i) = c_N(i)$ for all $i \in \mathcal{X}$. Now for the inductive step. Suppose $J_{k+1}(i) \leq \bar{J}_{k+1}(i)$ for all $i \in \mathcal{X}$. Therefore $J_{k+1}' P_i(u) \leq \bar{J}_{k+1}' \bar{P}_i(u)$. By (A1), (A2), $\bar{J}_{k+1}(i)$ is decreasing in $i$. By (A5), $P_i \succeq \bar{P}_i$. Therefore $J_{k+1}' P_i \leq \bar{J}_{k+1}' \bar{P}_i$. So $c(i, u, k) + J_{k+1}' P_i(u) \leq c(i, u, k) + J_{k+1}' \bar{P}_i(u)$ or equivalently, $Q_k(i, u) \leq \bar{Q}_k(i, u)$. Thus $\min_u Q_k(i, u) \leq \min_u \bar{Q}_k(i, u)$ or equivalently, $J_k(i) \leq \bar{J}_k(i)$ thereby completing the induction step.

\[ \square \]

3.5 Algorithms for Monotone Policies - Exploiting Sparsity

Consider an average cost MDP. Assume that the costs and transition matrices satisfy (A1)-(A4). Then by Theorem 3.3.1 the optimal stationary policy $\mu^*(x)$ is increasing in $x$. How can this monotonicity property be exploited to compute (estimate) the optimal policy? This section discusses several approaches. (These approaches also apply to discounted cost MDPs.)
3.5 Algorithms for Monotone Policies - Exploiting Sparsity

3.5.1 Policy Search and Q-learning with Submodular Constraints

Suppose, for example, $\mathcal{U} = \{1, 2\}$ so that the monotone optimal stationary policy is a step function of the form (3.1) (see Figure 3.1) and is completely defined by the threshold state $x^*$. We need an algorithm to search for $x^*$ over the finite state space $\mathcal{X}$. In [132], discrete-valued stochastic optimization algorithms for implementing this. Another possibility is to solve a continuous-valued relaxation as follows: Define the parametrized policy $\mu_\psi(x)$ where $\psi \in \mathbb{R}_+^2$ denotes the parameter vector. Also define the sample path cumulative cost estimate $\hat{C}_N(\psi)$ over some fixed time horizon $N$ as

$$\mu_\psi(x) = \begin{cases} 1 & \frac{1}{1+\exp(-\psi_1(x-\psi_2))} < 0.5, \\ 2 & \text{otherwise} \end{cases}, \quad \hat{C}_N(\psi) = \frac{1}{N+1} \sum_{k=0}^N c(x_k, \mu_\psi(x_k)).$$

Note $\mu_\psi$ is a sigmoidal approximation to the step function (3.1). Consider the stochastic optimization problem: Compute $\psi^* = \arg \min_{\psi} \mathbb{E}\{\hat{C}_N(\psi)\}$. This can be solved readily via simulation based gradient algorithms such as the SPSA Algorithm.

Alternatively, instead of exploiting the monotone structure in policy space, the submodular structure of the value function can be exploited. From Theorem 3.3.1, the Q-function $Q(x, u)$ in (3.7) is submodular. This submodularity can be exploited in Q-learning algorithms as described in [27, 26].

The above methods operate without requiring explicit knowledge of transition matrices. Such methods are useful in transmission scheduling in wireless communication where often by modeling assumptions (A1)-(A4) hold, but the actual values of the transition matrices are not known.

3.5.2 Sparsity Exploiting Linear Programming

Here we describe how the linear programming formulation for the optimal policy can exploit the monotone structure of the optimal policy. Suppose the number of actions $U$ is small but the number of states $X$ is large. Then the monotone optimal policy $\mu^*(x)$ is sparse in the sense that it is a piecewise constant function of the state $x$ that jumps upwards at most at $U - 1$ values (where by assumption $U$ is small). In other words, $\mu^*(x+1) - \mu^*(x)$ is non-zero for at most $U - 1$ values of $x$. In comparison, an unstructured policy can jump between states at arbitrary values and is therefore not sparse.

How can this sparsity property of a monotone optimal policy be exploited to compute the optimal policy? A convenient way of parametrizing sparsity in a monotone policy is in terms of the conditional probabilities $\theta_{x,u}$ defined in (??). Indeed $\theta_{x,u} - \theta_{x-1,u}$ as a function of $x$ for fixed $u$ is non-zero for up to only two values of $x \in \mathcal{X}$.
sensing [37], is to add a Lagrangian sum-of-norms term
\[ \lambda \sum_{x \geq 2} \| \theta_x - \theta_{x-1} \|_2, \quad \lambda \geq 0, \quad (3.8) \]
to a cost function whose minimum yields the optimal policy. Here \( \theta_x = [\theta_{x,1}, \ldots, \theta_{x,U}]' \).

The term (3.8) is a variant of the fused lasso\(^5\) or total variation penalty, and can be interpreted as a convex relaxation of a penalty on the number of changes of conditional probability \( \theta \) (as a function of state \( x \)). We refer to [68] for details.

### 3.6 Example: Transmission Scheduling over Wireless Channel

The conditions given in §3.3 are sufficient for the optimal policy to have a monotone structure. We conclude this chapter by describing an example where the sufficient conditions in §3.3 do not hold. However, a somewhat more sophisticated proof shows that the optimal policy is monotone. The formulation below generalizes the classical result of [25],[105] to the case of Markovian dynamics.

Consider the transmission of time sensitive video (multimedia) packets in a wireless communication system with the use of an ARQ protocol for retransmission. Suppose \( L \) such packets stored in a buffer need to transmitted over \( N \geq L \) time slots. At each time slot, assuming the channel state is known, the transmission controller decides whether to attempt a transmission. The quality of the wireless channel (which evolves due to fading) is represented abstractly by a finite state Markov chain. The channel quality affects the error probability of successfully transmitting a packet. If a transmission is attempted, the result (an ACK or NACK of whether successful transmission was achieved) is received. If a packet is transmitted but not successfully received, it remains in the buffer and may be retransmitted. At the end of all \( N \) time slots, no more transmission is allowed and a penalty cost is incurred for packets that remain in the buffer.

How should a transmission controller decide at which time slots to transmit the packets? It is shown below that the optimal transmission scheduling policy is a monotone (threshold) function of time and buffer size. The framework is applicable to any delay-sensitive real time packet transmission system.

#### 3.6.1 MDP Model for Transmission Control

We formulate the above transmission scheduling problem as a finite horizon MDP with a penalty terminal cost. The wireless fading channel is modeled as a finite state Markov chain. Let \( s_k \in S = \{ \gamma_1, \ldots, \gamma_K \} \) denote the channel state

\(^4\)Each term in the summation is a \( l_2 \) norm, the overall expression is the sum of norms. This is similar to the \( l_1 \) norm which is the sum of absolute values.

\(^5\)The lasso (least absolute shrinkage and selection operator) estimator was originally proposed in [121]. This is one of the most influential papers in statistics since the 1990s. It seeks to determine \( \theta^* = \arg\max_\theta \| y - A\theta \|_2^2 + \lambda \| \theta \|_1 \) given an observation vector \( y \), input matrix \( A \) and scalar \( \lambda > 0 \).
at time slot $k$. Assume $s_k$ evolves as a Markov chain according to transition probability matrix $P = \{P_{ss'} : s, s' = 1, 2, \ldots, K\}$, where $P_{ss'} = P(s_{k+1} = \gamma_{ss'} | s_k = s)$. Here the higher the state $s$, the better the quality of the channel.

Let $U = \{u_0 = 0 \text{ (do not transmit)}, u_1 = 1 \text{ (transmit)}\}$ denote the action space. In a time slot, if action $u$ is selected, a cost $c(u)$ is accrued, where $c(\cdot)$ is an increasing function. The probability that a transmission is successful is an increasing function of the action $u$ and channel state $s$:

$$\gamma(u, s) = \begin{cases} 0 & \text{if } u = 0 \\ 1 - P_e(s) & \text{if } u = 1. \end{cases}$$

(3.9)

Here $P_e(s)$ denotes the error probability for channel state $s$ and is a decreasing function of $s$.

Let $n$ denotes the residual transmission time: $n = N, N - 1, \ldots, 0$. At the end of all $N$ time slots, i.e. when $n = 0$, a terminal penalty cost $c_N(i)$ is paid if $i$ untransmitted packets remain in the buffer. It is assumed that $c_N(i)$ is increasing in $i$ and $c_N(0) = 0$.

The optimal scheduling policy $\mu_n^*(i, s)$ is the solution of Bellman’s equation:6

$$V_n(i, s) = \min_{u \in U} Q_n(i, s, u), \quad \mu_n^*(i, s) = \arg \min_{u \in U} Q_n(i, s, u), \quad (3.10)$$

$$Q_n(i, s, u) = \left\{ c(u) + \sum_{s' \in S} P_{ss'} \left[ \gamma(u, s)V_{n-1}(i-1, s') + (1 - \gamma(u, s))V_{n-1}(i, s') \right] \right\}$$

initialized with $V_n(0, s) = 0, V_0(i, s) = c_N(i)$. A larger terminal cost $c_N(i)$ emphasizes delay sensitivity while a larger action cost $c(u)$ emphasizes energy consumption. If $P_{ss} = 1$ then the problem reduces to that considered in [25, 105].

### 3.6.2 Monotone Structure of Optimal Transmission Policy

**Theorem 3.6.1** The optimal transmission policy $\mu_n^*(i, s)$ in (3.10) has the following monotone structure:

1. If the terminal cost $c_N(i)$ is increasing in the buffer state $i$, then $\mu_n^*(i, s)$ is decreasing in the number of transmission time slots remaining $n$.
2. If $c_N(i)$ is increasing in the buffer state $i$ and is integer convex, i.e.,

$$c_N(i + 2) - c_N(i + 1) \geq c_N(i + 1) - c_N(i) \quad \forall i \geq 0,$$

(3.11)

then $\mu_n^*(i, s)$ is a threshold policy of the form:

$$\mu_n^*(i, s) = \begin{cases} 0 & \text{if } i < i_{n,s}^* \\ 1 & \text{if } i \geq i_{n,s}^* \end{cases}$$

Here the threshold buffer state $i_{n,s}^*$ depends on $n$ (time remaining) and $s$ (channel state). Furthermore, the threshold $i_{n,s}^*$ is increasing in $n$.

6It is notationally convenient here to use Bellman’s equation with forward indices. So we use $V_n = J_{N-n}$ for the value function. This notation was used previously for MDPs in (??).
The theorem says that the optimal transmission policy is aggressive since it is optimal to transmit more often when the residual transmission time is less or the buffer occupancy is larger. The threshold structure of the optimal transmission scheduling policy can be used to reduce the computational cost in solving the dynamic programming problem or the memory required to store the solutions. For example, the total number of transmission policies given \(L\) packets, \(N\) time slots and \(K\) channel states is \(2^{NLK}\). In comparison, the number of transmission policies that are monotone in the number of transmission time slots remaining and the buffer state is \(NLK\), which can be substantially smaller. \[40\] gives several algorithms (e.g., (modified) value iteration, policy iteration) that exploit monotone results to efficiently compute the optimal policies.

To prove Theorem 3.6.1, the following results can be established.

**Lemma 3.6.2** The value function \(V_n(i, s)\) defined by (3.10) is increasing in the number of remaining packets \(i\) and decreasing in the number of remaining time slots \(n\).

**Lemma 3.6.3** If \(c_N(\cdot)\) is an increasing function (of the terminal buffer state) then the value function \(V_n(i, s)\) satisfies the following submodularity condition:

\[
V_n(i + 1, s) - V_n(i, s) \geq V_{n+1}(i + 1, s) - V_{n+1}(i, s),
\]

for all \(i \geq 0, s \in S\). Hence, \(Q_n(i, s, u)\) in (3.10) is supermodular in \((n, u)\). Furthermore, if the penalty cost \(c_N(\cdot)\) is an increasing function and satisfies (3.11) then \(V_n(i, s)\) has increasing differences in the number of remaining packets:

\[
V_n(i + 2, s) - V_n(i + 1, s) \geq V_n(i + 1, s) - V_n(i, s),
\]

for all \(i \geq 0, s \in S\). Hence \(Q_n(i, s, u)\) is submodular in \((i, u)\).

With the above two lemmas, the proof of Theorem 3.6.1 is as follows:

First statement: If \(c_N(i)\) is increasing in \(i\) then \(V_n(i, s)\) satisfies (3.12) and \(Q_n(i, s, u)\) given by (3.10) is supermodular in \((u, n)\) (provided that \(\gamma(u, s)\) defined in (3.9) is an increasing function of the action \(u\)). Therefore, \(\mu_n^*(i, s)\) is decreasing in \(n\).

Second statement: Due to Lemma 3.6.3, if \(c_N(i)\) is increasing in \(i\) and satisfies (3.11) then \(V_n(i, h)\) satisfies (3.13) and \(Q_n(i, h, u)\) is submodular in \((u, i)\) (provided that \(\gamma(u, h)\) increases in \(u\)). The submodularity of \(Q_n(i, h, u)\) in \((u, i)\) implies that \(\mu_n^*(i, h)\) is increasing in \(i\). The result that \(i_n^*\) is increasing in \(n\) follows since \(\mu_n^*(i, h)\) is decreasing in \(n\) (first statement).

### 3.7 Complements and Sources

The use of supermodularity for structural results in MDPs and game theory was pioneered by Topkis in the seminal paper \[123\] culminating in the book \[124\]. We refer the reader to \[4\] for a tutorial description of supermodularity with
applications in economics. [40, Chapter 8] has an insightful treatment of sub-
modularity in MDPs. Excellent books in stochastic dominance include [88, 108].
[114] covers several cases of monotone MDPs. The paper [83] is highly influen-
tial in the area of monotone comparative statics (determining how the argmax
or argmin behaves as a parameter varies) and discusses the single crossing con-
dition. [99] has some recent results on conditions where the single crossing
property is closed under addition. A more general version of Theorem 3.4.1
is proved in [87]. The example in §3.6 is expanded in [90] with detailed numerical
examples. Also [91] considers the average cost version of the transmission
scheduling problem on a countable state space (to model an infinite buffer). [42]
gives structural results for Markov decision games. [130] studies supermodu-
larity and monotone policies in discrete event systems. [3] covers deeper results
in multimodularity, supermodularity and extensions of convexity to discrete
spaces for discrete event systems. In [1], gradient based stochastic approxima-
tion algorithms are presented to estimate the optimal policy of a constrained
MDP. It would be of interest to generalize these results by adding constraints
for a monotone policy.
4 Structural Results for Optimal Filters

This chapter and the following four chapters develop structural results for the optimal policy of a POMDP. In Chapter 3, we used first order stochastic dominance to characterize the structure of optimal policies for finite state MDPs. However, first order stochastic dominance is not preserved under Bayesian rule for the belief state update. So a stronger stochastic order is required to obtain structural results for POMDPs. We will use the monotone likelihood ratio (MLR) stochastic order to order belief states. This chapter develops important structural results for the HMM filter using the MLR order. These results form a crucial step in formulating the structural results for POMDPs.

Outline of this Chapter

Recall from Chapter 2 that for a POMDP, with controlled transition matrix $P(u)$ and observation probabilities $B_{xy}(u) = p(y|x,u)$, given the observation $y_{k+1}$, the HMM filter recursion (2.11) for the belief state $\pi_{k+1}$ in terms of $\pi_k$ reads

$$
\pi_{k+1} = T(\pi_k, y_{k+1}, u_k) = \frac{B_{yk+1}(u_k)P'(u_k)\pi_k}{\sigma(\pi_k, y_{k+1}, u_k)}, \sigma(\pi, y, u) = 1'B_y(u)P'(u)\pi, \\
B_y(u) = \text{diag}(B_{1y}(u), \ldots, B_{Xy}(u)), \quad u \in U = \{1, 2, \ldots, U\}, y \in Y. \quad (4.1)
$$

The two main questions addressed in this chapter are:

1. How can beliefs (posterior distributions) $\pi$ computed by the HMM filter be ordered within the belief space (unit simplex) $\Pi(X)$?
2. Under what conditions does the HMM filter $T(\pi, y, u)$ increase with belief $\pi$, observation $y$ and action $u$?

Answering the first question is crucial to define what it means for a POMDP to have an optimal policy $\mu^*(\pi)$ increasing with $\pi$. Recall from Chapter 3 that for the fully observed MDP case, we gave conditions under which the optimal policy is increasing with scalar state $x$ – ordering the states was trivial since they were scalars. However, for a POMDP, we need to order the belief states $\pi$ which are probability mass functions (vectors) in the unit simplex $\Pi(X)$.

The first question above will be answered by using the monotone likelihood ratio (MLR) stochastic order to order belief states. The MLR is a partial order that is ideally suited for Bayesian estimation (optimal filtering) since it is preserved under conditional expectations. §4.1 discusses the MLR order.
The answer to the second question is essential for giving conditions on a POMDP so that the optimal policy \( \mu^*(\pi) \) increases in \( \pi \). Recall that Bellman’s equation (2.24) for a POMDP involves the term \( \sum_y V(T(\pi, y, u)) \sigma(\pi, y, u) \). So to show that the optimal policy is monotone, it is necessary to characterize the behavior of the HMM filter \( T(\pi, y, u) \) and normalization term \( \sigma(\pi, y, u) \). §4.2 to §4.5 discuss structural properties of the HMM filter.

Besides their importance in establishing structural results for POMDPs, the structural results for HMM filters developed in this chapter are also useful for constructing reduced complexity HMM filtering algorithms that provably upper and lower bound the optimal posterior (with respect to the MLR order). We shall describe the construction of such reduced complexity HMM filters in §4.6. Sparse rank transition matrices for the low complexity filters will be constructed via nuclear norm minimization (convex optimization).

### 4.1 Monotone Likelihood Ratio (MLR) Stochastic Order

For dealing with POMDPs, the MLR stochastic order is the main concept that will be used to order belief states in the unit \( X-1 \) dimensional unit simplex

\[
\Pi(X) = \{ \pi \in \mathbb{R}^X : 1'\pi = 1, \ 0 \leq \pi(i) \leq 1, \ i \in X = \{1, 2, \ldots, X\} \}.
\]

#### 4.1.1 Definition

**Definition 4.1.1 (Monotone Likelihood Ratio (MLR) ordering)** Let \( \pi_1, \pi_2 \in \Pi(X) \) denote two belief state vectors. Then \( \pi_1 \) dominates \( \pi_2 \) with respect to the MLR order, denoted as \( \pi_1 \succeq_\text{MLR} \pi_2 \), if

\[
\pi_1(i) \pi_2(j) \leq \pi_2(i) \pi_1(j), \quad i < j, i, j \in \{1, \ldots, X\}, \tag{4.2}
\]

Similarly \( \pi_1 \preceq_\text{MLR} \pi_2 \) if in (4.2) is replaced by \( \geq \).

Equivalently, \( \pi_1 \succeq_\text{MLR} \pi_2 \) if the likelihood ratio \( \pi_1(i)/\pi_2(i) \) is increasing. Similarly, for the case of pmfs, define \( \pi_1 \succeq_\text{MLR} \pi_2 \) if their ratio \( \pi_1(x)/\pi_2(x) \) is increasing in \( x \in \mathbb{R} \).

**Definition 4.1.2** A function \( \phi : \Pi(X) \to \mathbb{R} \) is said to be MLR increasing if \( \pi_1 \succeq_\text{MLR} \pi_2 \) implies \( \phi(\pi_1) \geq \phi(\pi_2) \). Also \( \phi \) is MLR decreasing if \( -\phi \) is MLR increasing.

Recall the definition of first order stochastic dominance (Definition 3.2.1 in Chapter 3). MLR dominance is a stronger condition than first order dominance.

**Theorem 4.1.3** For pmfs or pdfs \( \pi_1 \) and \( \pi_2 \), \( \pi_1 \succeq_\text{MLR} \pi_2 \) implies \( \pi_1 \succeq_\text{fod} \pi_2 \).

**Proof** \( \pi_1 \succeq_\text{MLR} \pi_2 \) implies \( \pi_1(x)/\pi_2(x) \) is increasing in \( x \). Denote the corresponding cdfs as \( F_1, F_2 \). Define \( t = \{ \sup x : \pi_1(x) \leq \pi_2(x) \} \). Then \( \pi_1 \succeq_\text{MLR} \pi_2 \) implies that for \( x \leq t, \pi_1(x) \leq \pi_2(x) \) and for \( x \geq t, \pi_1(x) \geq \pi_2(x) \). So for \( x \leq t, F_1(x) \leq F_2(x) \). Also for \( x > t, \pi_1(x) \geq \pi_2(x) \) implies \( 1 - \int_x^\infty \pi_1(x) \, dx \leq 1 - \int_x^\infty \pi_2(x) \, dx \) or equivalently, \( F_1(x) \leq F_2(x) \). Therefore \( \pi_1 \succeq_\text{fod} \pi_2 \). \( \square \)
Structural Results for Optimal Filters

For state space dimension $X = 2$, MLR is a complete order and coincides with first order stochastic dominance. The reason is that for $X = 2$, since $\pi(1) + \pi(2) = 1$, it suffices to choose the second component $\pi(2)$ to order $\pi$. Indeed, for

$$X = 2, \quad \pi_1 \geq_r \pi_2 \iff \pi_1 \geq_s \pi_2 \iff \pi_1(2) \geq \pi_2(2).$$

For state space dimension $X \geq 3$, MLR and first order dominance are partial orders on the belief space $\Pi(X)$. Indeed, $[\Pi(X), \geq_r]$ is a partially ordered set (poset) since it is not always possible to order any two belief states in $\Pi(X)$.

Example (i): $[0.2, 0.3, 0.5]' \geq_r [0.4, 0.5, 0.1]'$

Example (ii): $[0.3, 0.2, 0.5]'$ and $[0.4, 0.5, 0.1]'$ are not MLR comparable.

Figure ?? gives a geometric interpretation of first order and MLR dominance for $X = 3$.

4.1.2 Why MLR ordering?

The MLR stochastic order is useful in filtering and POMDPs since it is preserved under conditional expectations (or more naively, application of Bayes rule).

**Theorem 4.1.4** MLR dominance is preserved under Bayes rule: For continuous or discrete-valued observation $y$ with observation likelihoods $B_y = \text{diag}(B_{1y}, \ldots, B_{Xy})$, $B_{xy} = p(y|x)$, given two beliefs $\pi_1, \pi_2 \in \Pi(X)$, then

$$\pi_1 \geq_r \pi_2 \iff \frac{B_y \pi_1}{1'B_y \pi_1} \geq_r \frac{B_y \pi_2}{1'B_y \pi_2}$$

providing $1'B_y \pi_1$ and $1'B_y \pi_2$ are non-zero.\(^1\)

*Proof* By definition of MLR dominance, the right hand side is

$$B_{y\pi_1} 1'B_{1\pi_1} (i) \pi_2 (i+1) \leq B_{y\pi_1} 1'B_{1\pi_1} (i+1) \pi_1 (i+1) \pi_2 (i)$$

which is equivalent to $\pi_1 \geq_r \pi_2$. \qed

Notice that in the fully observed MDP of Chapter 3, we used first order stochastic dominance. However, first order stochastic dominance is not preserved under conditional expectations and so is not useful for POMDPs.

4.1.3 Examples

1. **First order stochastic dominance is not closed under Bayes rule**: Consider beliefs $\pi_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})'$, $\pi_2 = (0, \frac{2}{3}, \frac{1}{3})'$. Then clearly $\pi_1 \leq_s \pi_2$. Suppose $P = I$ and the observation likelihoods $B_{xy}$ have values: $P(y|x = 1) = 0$, $P(y|x = 2) = 0.5$, $P(y|x = 3) = 0.5$. Then the filtered updates are $T(\pi_1, y, u) = (0, \frac{1}{2}, \frac{1}{2})'$ and $T(\pi_1, y, u) = (0, \frac{1}{2}, \frac{1}{2})'$.

\(^1\)A notationally elegant way of saying this is: Given two random variables $X$ and $Y$, then $X \preceq_r Y$ iff $X[A \leq_r Y] Y \in A$ for all events $A$ providing $P(X \in A) > 0$ and $P(Y \in A) > 0$. Requiring $1'B_y \pi > 0$ avoids pathological cases such as $\pi = [1, 0]'$ and $B_y = \text{diag}(0, 1)$, i.e., prior says state 1 with certainty, while observation says state 2 with certainty.
4.2 Total Positivity and Copositivity

This section defines the concepts of total positivity and copositivity. These are crucial concepts in obtaining monotone properties of the optimal filter with respect to the MLR order.

**Definition 4.2.1 (Totally Positive of Order 2 (TP2))** A stochastic matrix $M$ is TP2 if all its second order minors are non-negative. That is, determinants

$$
\begin{vmatrix}
M_{i,i,j_1} & M_{i,j_2} \\
M_{i_2,j_1} & M_{i_2,j_2}
\end{vmatrix} \geq 0 \text{ for } i_2 \geq i_1, j_2 \geq j_1.
$$

(4.3)

Equivalently, a transition or observation kernel\(^2\) denoted $M$ is TP2 if the $i+1$-th row MLR dominates the $i$-th row: that is, $M_{i,:} \succeq_r M_{j,:}$ for every $i > j$.

Next, we define a copositive ordering. Recall from §?? that an HMM is parameterized by the transition and observation probabilities $(P, B)$. Start with the following notation. Given an HMM $(P(u), B(u))$ and another HMM\(^3\) $(P(u +$

\(^2\)We use the term “kernel” to allow for continuous and discrete valued observation spaces $\mathcal{Y}$. If $\mathcal{Y}$ is discrete, then $B(u)$ is a $X \times Y$ stochastic matrix. If $\mathcal{Y} \subseteq \mathbb{R}$, then $B_{xy}(u) = p(y|x, u)$ is a probability density function.

\(^3\)The notation $u$ and $u+1$ is used to distinguish between the two HMMs. Recall that in POMDPs, $u$ denotes the actions taken by the controller.
Structural Results for Optimal Filters

1), B(u + 1)), define the sequence of \( X \times X \) dimensional symmetric matrices \( \Gamma^{j,u,y}, j = 1, \ldots, X - 1, y \in \mathcal{Y} \) as

\[
\Gamma^{j,u,y} = \frac{1}{2} \left[ \gamma^{j,u,y}_{mn} + \gamma^{j,u,y}_{nm} \right]_{X \times X}, \quad \text{where}
\]

\[
\gamma^{j,u,y}_{mn} = B_{j,y}(u)B_{j+1,y}(u+1)P_{m,j}(u)P_{n,j+1}(u+1) - B_{j+1,y}(u)B_{j,y}(u+1)P_{m,j+1}(u+1)P_{n,j}(u+1).
\]

**Definition 4.2.2 (Copositive Ordering \( \preceq \) of Transition and Observations Probabilities)**

Given \((P(u), B(u))\) and \((P(u+1), B(u+1))\), we say that \((P(u), B(u)) \preceq (P(u+1), B(u+1))\) if the sequence of \( X \times X \) matrices \( \Gamma^{j,u,y}, j = 1, \ldots, X - 1, y \in \mathcal{Y} \) are copositive.\(^4\)

That is,

\[
\pi' \Gamma^{j,u,y} \pi \geq 0, \forall \pi \in \Pi(X), \text{ for each } j, y.
\]

The above notation \((P(u), B(u)) \preceq (P(u+1), B(u+1))\) is intuitive since it will be shown below that the copositive condition (4.5) is a necessary and sufficient condition for the HMM filter update to satisfy \( T(\pi, y, u) \leq_r T(\pi, y, u+1) \) for any posterior \( \pi \in \Pi(X) \) and observation \( y \in \mathcal{Y} \). This will be denoted as Assumption (F3) below.

We are also interested in the special case of the optimal HMM predictors instead of optimal filters. Recall that optimal prediction is a special case of filtering obtained by choosing non-informative observation probabilities, i.e., all elements of \( B_{x,y} \) versus \( x \) are identical. In analogy to Definition 4.2.2 we make the following definition.

**Definition 4.2.3 (Copositive Ordering of Transition Matrices)**

Given \( P(u) \) and \( P(u+1) \), we say that \( P(u) \preceq P(u+1) \) if the sequence of \( X \times X \) matrices \( \Gamma^{j,u}, j = 1, \ldots, X - 1 \) are copositive, i.e.,

\[
\pi' \Gamma^{j,u} \pi \geq 0, \forall \pi \in \Pi(X), \text{ for each } j, \text{ where}
\]

\[
\Gamma^{j,u} = \frac{1}{2} \left[ \gamma^{j,u}_{mn} + \gamma^{j,u}_{nm} \right]_{X \times X}, \quad \gamma^{j,u}_{mn} = P_{m,j}(u)P_{n,j+1}(u+1) - P_{m,j+1}(u+1)P_{n,j}(u+1).
\]

### 4.3 Monotone Properties of Optimal Filter

With the above definitions, we can now give sufficient conditions for the optimal filtering recursion \( T(\pi, y, u) \) to be monotone with respect to the MLR order. The following are the main assumptions.

\(^4\)A symmetric matrix \( M \) is positive semidefinite if \( x' M x \geq 0 \) for any vector \( x \). In comparison, \( M \) is copositive if \( \pi' M \pi \geq 0 \) for any probability vector \( \pi \). Clearly if a symmetric matrix is positive definite then it is copositive. Thus copositivity is a weaker condition than positive definiteness.
4.3 Monotone Properties of Optimal Filter

(F1) $B(u)$ with elements $B_{x,y}(u)$ is TP2 for each $u \in U$ (see Definition 4.2.1).

(F2) $P(u)$ is TP2 for each action $u \in U$.

(F3) $(P(u), B(u)) \preceq (P(u + 1), B(u + 1))$ (copositivity condition in Definition 4.2.2).

(F3') All elements of the matrices $Γ^{j,u,y}$ are non-negative. (This is sufficient for (F3).)

(F3) $P(u) \preceq P(u + 1)$ (copositivity condition in Definition 4.2.3).

(F3') All the elements of $Γ^{j,u}$ are non-negative. (This is sufficient for (F3)).

(F4) $\sum_{y \in Y} \sum_{j \in X} [P_{i,j}(u)B_{j,y}(u) - P_{i,j}(u + 1)B_{j,y}(u + 1)] \leq 0$ for all $i \in X$ and $\bar{y} \in \mathcal{Y}$.

Assumptions (F1), (F2) deal with the transition and observation probabilities for a fixed action $u$. In comparison, (F3), (F3'), (F3), (F4) are conditions on the transition and observation probabilities of two different HMMs corresponding to the actions $u$ and $u + 1$.

Main Result: The following theorem is the main result of this chapter. The theorem characterizes how the HMM filter $T(\pi, y, u)$ and normalization measure $\sigma(\pi, y, u)$ behave with increasing $\pi, y$ and $u$. The theorem forms the basis of all the structural results for POMDPs presented in subsequent chapters.

THEOREM 4.3.1 (Structural result for filtering) Consider the HMM filter $T(\pi, y, u)$ and normalization measure $\sigma(\pi, y, u)$ defined as

$$T(\pi, y, u) = \frac{B_y(u)P'(u)\pi}{\sigma(\pi, y, u)} , \quad \sigma(\pi, y, u) = 1'B_y(u)P'(u)\pi, \quad \text{where}$$

$$B_y(u) = \text{diag}(B_{1y}(u), \ldots, B_{Xy}(u)), \quad u \in U = \{1, 2, \ldots, U\}, y \in \mathcal{Y}. \quad (4.7)$$

Suppose $\pi_1, \pi_2 \in \Pi(X)$ are arbitrary belief states. Then

1. (a) For $\pi_1 \succeq_r \pi_2$, the HMM predictor satisfies $P'(u)\pi_1 \succeq_r P'(u)\pi_2$ iff (F2) holds.
   (b) Therefore, for $\pi_1 \succeq_r \pi_2$, the HMM filter satisfies $T(\pi_1, y, u) \succeq_r T(\pi_2, y, u)$ for any observation $y$ iff (F2) holds (since MLR dominance is preserved by Bayes rule, Theorem 4.1.4).

2. Under (F1), (F2), $\pi_1 \succeq_r \pi_2$ implies $\sigma(\pi_1, u) \succeq_s \sigma(\pi_2, u)$ where

$$\sigma(\pi, u) \equiv [\sigma(\pi, 1, u), \ldots, \sigma(\pi, Y, u)].$$

3. For $y, \bar{y} \in \mathcal{Y}$, $y > \bar{y}$ implies $T(\pi_1, y, u) \succeq_r T(\pi_1, \bar{y}, u)$ iff (F1) holds.

4. Consider two HMMs $(P(u), B(u))$ and $(P(u + 1), B(u + 1))$. Then
   (a) $T(\pi, y, u + 1) \succeq_r T(\pi, y, u)$ iff (F3) holds.
   (b) Under (F3'), $T(\pi, y, u + 1) \succeq_r T(\pi, y, u)$.

5. Consider two HMMs $(P(u), B)$ and $(P(u + 1), B)$. Then
   (a) (F3) is necessary and sufficient for $P'(u + 1)\pi \succeq_r P'(u)\pi$.
   (b) (F3') is sufficient for $P'(u + 1)\pi \succeq_r P'(u)\pi$.

5 Any square matrix $M$ with non-negative elements is copositive since $\pi'M\pi$ is always non-negative for any belief $\pi$. So a sufficient condition for the copositivity (4.5) is that the individual elements $\frac{1}{2} [\pi_{\min}^{j,u,y} + \pi_{\max}^{j,u,y}] \geq 0$. 


(c) Either (F3) or (F3') are sufficient for the optimal filter to satisfy $T(\pi, y, u+1) \geq_r T(\pi, y, u)$ for any $y \in \mathcal{Y}$.

6. Under (F4), $\sigma(\pi, u+1) \geq_s \sigma(\pi, u)$.
7. Statement 4 holds for discrete valued observation space $\mathcal{Y}$. All the other statements hold for discrete and continuous-valued $\mathcal{Y}$.

The proof is the appendix §4.A.1.

Statement 1a asserts that a TP2 transition matrix is sufficient for a one-step ahead HMM predictor to preserve MLR stochastic dominance with respect to $\pi$. As a consequence Statement 1b holds since applying Bayes rule to $P'(u)\pi_1$ and $P'(u)\pi_2$, respectively, yields the filtered updates, and Bayes rule preserves MLR dominance (recall Theorem 4.1.4).

Statement 2 asserts that the normalization measure of the HMM filter $\sigma(\pi, y, u)$ is monotone increasing in $\pi$ (with respect to first order dominance) if the observation kernel is TP2 (F1) and transition matrix is TP2 (F2).

Statement 3 asserts that the HMM filter $T(\pi, y, u)$ is monotone increasing in the observation $y$ iff (F1) holds. That is, a larger observation yields a larger belief if and only if the observation kernel is TP2.

Finally Statements 4, 5 and 6 compare the filter update and normalization measures for two different HMMs indexed by actions $u$ and $u+1$. Statements 4 and 6 say that if $(P(u), B(u))$ and $(P(u+1), B(u+1))$ satisfy the specified conditions, then the belief update and normalization term with parameters $(P(u+1), B(u+1))$ dominate those with parameters $(P(u), B(u))$. Statement 5 gives a similar result for predictors and HMMs with identical observation probabilities.

### 4.4 Illustrative Example

This section gives simple examples to illustrate Theorem 4.3.1. Suppose

$$P(1) = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}, \quad P(2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \pi_1 = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix}, \quad \pi_2 = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.5 \end{bmatrix}.$$

It can be checked that the transition matrix $P(1)$ is TP2 (and so (F2) holds). $P(2)$ is not TP2 since the second order minor comprised of the (1,1), (1,2) (2,1) (2,2) elements is $-1$. Finally, $\pi_1 \geq_r \pi_2$ since the ratio of their elements $[2/3, 1, 6/5]$ is increasing.

**Example (i):** Statement 1a says that $P'(1)\pi_1 \geq_r P'(1)\pi_2$ which can be verified since the ratio of elements $[0.8148, 1.1.1282]'$ is increasing. On the other hand since $P(2)$ is not TP2, $P'(2)\pi_1$ is not MLR smaller than $P'(2)\pi_2$. The ratio of elements of $P'(2)\pi_1$ with $P'(2)\pi_2$ is $[1, 0.6667, 1.2]$ implying that they are not MLR orderable (since the ratio is neither increasing or decreasing).

Statement 1b (MLR order is preserved by Bayes rule) was illustrated numerically in §4.1.3.
Example (ii): To illustrate Statement 2, suppose $B(1) = P(1)$ so that (F1), (F2) hold. Then

$$
\sigma(\pi_1, 1) = [0.2440, 0.3680, 0.3880]',
\sigma(\pi_2, 1) = [0.2690, 0.3680, 0.3630]'.
$$

Clearly $\sigma(\pi_1, 1) \geq \sigma(\pi_2, 1)$.

Example (iii): Regarding Statement 3, if $B = P(1)$ then $B_1 = \text{diag}(0.6, 0.2, 0.1)$, $B_2 = \text{diag}(0.3, 0.5, 0.3)$. Then writing $T(\pi, y, u)$ as $T(\pi, y)$,

$$
T(\pi_1, y = 1) = [0.5410, 0.2787, 0.1803]',
T(\pi_1, y = 2) = [0.1793, 0.4620, 0.3587]'.
$$

implying that $T(\pi_1, y = 1) \leq_r T(\pi_1, y = 2)$.

Example (iv): Consider a cost vector $c = [c(x = 1), c(x = 2), c(x = 3)]' = [3, 2, 1]'$.

Suppose a random variable $x$ has a prior $\pi$ and is observed via noisy observations $y$ with observation matrix $B(1)$ above. Then the expected cost after observing $y$ is $c'T(\pi, y)$. It is intuitive that a larger observation $y$ corresponds to a larger state and therefore a smaller expected cost (since the cost vector is decreasing in the state). From Theorem 4.3.1(3) this indeed is the case since $T(\pi, y) \geq_r T(\pi, \bar{y})$ for $y > \bar{y}$ which implies that $T(\pi, y) \geq_r T(\pi, \bar{y})$ and therefore $c'T(\pi, y)$ is decreasing in $y$.

### 4.5 Discussion and Examples of Assumptions (F1)-(F4)

Since Assumptions (F1)-(F4) will be used a lot in subsequent chapters, we now discuss their motivation with examples.

**Assumption (F1)**

(F1) is required for preserving the MLR ordering with respect to observation $y$ of the Bayesian filter update. (F1) is satisfied by numerous continuous and discrete distributions, see any classical detection theory book such as [98]. Since (F1) is equivalent to each row of $B$ being MLR dominated by subsequent rows, any of the examples in §4.1.3 yield TP2 observation kernels. For example, if the $i$-th row of $B$ is $\mathcal{N}(y - x; \mu_i, \sigma^2)$ with $\mu_i < \mu_{i+1}$ then $B$ is TP2. The same logic applies to Exponential, Binomial, Poisson, Geometric, etc. For a discrete distribution example, suppose each sensor obtains measurements $y$ of the state $x$ in quantized Gaussian noise. Define

$$
\mathbb{P}(y|x = i) = \frac{\bar{b}_{iy}}{\sum_{y=1}^{Y} \bar{b}_{iy}} \text{ where } \bar{b}_{iy} = \frac{1}{\sqrt{2\pi\Sigma}} \exp\left(-\frac{1}{2} \frac{(y - g_i)^2}{\Sigma}\right)
$$

(4.8)

Assume the state levels $g_i$ are increasing in $i$. Also, $\Sigma \geq 0$ denotes the noise variance and reflects the quality of the measurements. It is easily verified that (A2) holds. As another example, consider equal dimensional observation and
state spaces \((X = Y)\) and suppose \(P(y = i|x = i) = p_{ii}, P(y = i-1|x = i) = P(y = i+1|x = i) = (1-p_{ii})/2.\) Then for \(1/(\sqrt{2} + 1) \leq p_{ii} \leq 1\), \(A2\) holds.

**Assumption (F2)**

\((F2)\) is essential for the Bayesian update \(T(\pi, y, u)\) preserving monotonicity with respect to \(\pi\). TP2 stochastic orders and kernels have been studied in great detail in [48]. \((F2)\) is satisfied by several classes of transition matrices; see [51, 50].

The left-to-right Bakis HMM used in speech recognition [100] has an upper triangular transition matrix which has a TP2 structure under mild conditions, e.g., if the upper triangular elements in row \(i\) are \((1-P_{ii})/(X-i)\) then \(P\) is TP2 if \(P_{ii} \leq 1/(X-i)\).

As another example, consider a tridiagonal transition probability matrix \(P\) with \(P_{ij} = 0\) for \(j \geq i + 2\) and \(j \leq i - 2\). As shown in [34, pp.99–100], a necessary and sufficient condition for tridiagonal \(P\) to be TP2 is that \(P_{ii}P_{i+1,i+1} \geq P_{i,i+1}P_{i+1,i}\).

Karlin’s classic book [49, pp.154] shows that the matrix exponential of any tridiagonal generator matrix is TP2. That is, \(P = \exp(Qt)\) is TP2 if \(Q\) is a tridiagonal generator matrix (nonnegative off-diagonal entries and each row adds to 0) and \(t > 0\).

The following lemmas give useful properties of TP2 transition matrices.

**Lemma 4.5.1** If \(P\) is TP2, i.e., \((F2)\) holds, then \(P_{11} \geq P_{21} \geq P_{31} \geq \cdots \geq P_{X1}\).

**Proof** We prove the contrapositive, that is, \(P_{11} < P_{1+1,1}\) implies \(P\) is not TP2. Recall from \((A3-Ex1)\), TP2 means that \(P_{11}P_{i+1,j} \geq P_{i+1,1}P_{ij}\) for all \(j\). So assuming \(P_{11} < P_{1+1,1}\), to show that \(P\) is not TP2, we need to show that there is at least one \(j\) such that \(P_{i+1,j} < P_{ij}\). But \(P_{11} < P_{i+1,1}\) implies \(\sum_{k \neq 1} P_{i+1,k} < \sum_{k \neq 1} P_{ik}\), which in turn implies that at least for one \(j\), \(P_{i+1,j} < P_{ij}\). \(\square\)

**Lemma 4.5.2** The product of two TP2 matrices is TP2.

Lemma 4.5.2 lets us construct TP2 matrices by multiplying other TP2 matrices. The lemma is also used in the proof of the main Theorem 4.3.1.

**Assumption (F3), (F3’) and Optimal Prediction**

Assumption \((F3)’\) is sufficient condition for the belief due to action \(u+1\) to MLR dominate the belief due to action \(u\), i.e., in the terminology of [82], \(u+1\) yields a more “favorable outcome” than \(u\). In general, the problem of verifying copositivity of a matrix is NP-complete [15]. Assumption \((F3)’\) is a simpler but more restrictive sufficient condition than \((F3)\) to ensure that \(P_{j,u,y}^{u+1}\) in (4.5) is copositive. Here is an example of \(\langle P(1), B(1) \rangle \preceq \langle P(2), B(2) \rangle\) which satisfies \((F3)’\):

\[
P(1) = \begin{bmatrix} 0.8000 & 0.1000 & 0.1000 \\ 0.2823 & 0.1804 & 0.5373 \\ 0.1256 & 0.1968 & 0.6776 \end{bmatrix}, \quad B(1) = \begin{bmatrix} 0.8000 & 0.1000 & 0.1000 \\ 0.0341 & 0.3665 & 0.5994 \\ 0.0101 & 0.2841 & 0.7058 \end{bmatrix}
\]
4.6 Example: Reduced Complexity HMM Filtering with Stochastic Dominance Bounds

\[ P(2) = \begin{bmatrix} 0.0188 & 0.1981 & 0.7831 \\ 0.0051 & 0.1102 & 0.8847 \\ 0.0016 & 0.0626 & 0.9358 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 0.0041 & 0.1777 & 0.8182 \\ 0.0025 & 0.1750 & 0.8225 \\ 0.0008 & 0.1290 & 0.8701 \end{bmatrix}. \]

(F3) is necessary and sufficient for the optimal predictor with transition matrix \( P(u+1) \) to MLR dominate the optimal predictor with transition matrix \( P(u) \). (F3') is a sufficient condition for (F3) since it requires all the elements of the matrix to be non-negative which trivially implies copositivity. We require MLR dominance of the predictor since then by Theorem 4.1.4 MLR dominance of the filter is assured for any observation distribution. (First order dominance is not closed under Bayesian updates).

A straightforward sufficient condition for (4.6) to hold is if all rows of \( P(u+1) \) MLR dominate the last row of \( P(u) \).

Assumption (F4)
This ensures that the normalized measure \( \sigma(\pi, u + 1) \) first order stochastically dominates \( \sigma(\pi, u) \).

Assumptions (F3') and (F4) are relaxed versions of Assumptions (c), (e), (f) of [78, Proposition 2] and Assumption (i) of [102, Theorem 5.6] in the stochastic control literature. The assumptions (c), (e), (f) of [78] require that \( P(u + 1) \geq_{\text{TP2}} P(u) \) and \( B(u + 1) \geq_{\text{TP2}} B(u) \) (where \( \geq_{\text{TP2}} \) denotes TP2 stochastic order) which is impossible for stochastic matrices, unless \( P(u) = P(u+1) \), \( B(u) = B(u+1) \) or the matrices \( P(u), B(u) \) are rank 1 for all \( u \) meaning that the observations are non-informative.

4.6 Example: Reduced Complexity HMM Filtering with Stochastic Dominance Bounds

The main result Theorem 4.3.1 can be exploited to design reduced complexity HMM filtering algorithms with provable sample path bounds. In this section we derive such reduced-complexity algorithms by using Assumptions (F2) and (F3') with statement 5 of Theorem 4.3.1 for the transition matrix.

4.6.1 Upper and Lower Sample Path Bounds for Optimal Filter
Consider an HMM with \( X \times X \) transition matrix \( P \) and observation matrix \( B \) with elements \( B_{xy} = p(y_k = y | x_k = x) \). The observation space \( Y \) can be discrete or continuous-valued; so that \( B \) is either a pdf or a pmf. The HMM filter computes the posterior

\[ \pi_{k+1} = T(\pi_k, y_{k+1}; P), \quad \text{where} \quad T(\pi, y; P) = \frac{B_y P^\top \pi}{1'B_y P^\top \pi}, \quad B_y = \text{diag}(B_{1y}, \ldots, B_{Xy}). \]
The above notation explicitly shows the dependence on the transition matrix $P$. Due to the matrix-vector multiplication $P'\pi$, the HMM filter involves $O(X^2)$ multiplications and can be excessive for large $X$.

The main idea of this section is to construct low rank transition matrices $\bar{P}$ and $\hat{P}$ such that the above filtering recursion using these matrices form lower and upper bounds to $\pi_k$ in the MLR stochastic dominance sense. Since $\bar{P}$ and $\hat{P}$ are low rank (say $r$), the cost involved in computing these lower and upper bounds to $\pi_k$ at each time $k$ will be $O(Xr)$ where $r \ll X$.

Since that plan is to compute filtered estimates using $\bar{P}$ and $\hat{P}$ instead of the original transition matrix $P$, we need additional notation to distinguish between the posteriors and estimates computed using $P$, $\bar{P}$ and $\hat{P}$. Let

$$\bar{\pi}_{k+1} = T(\bar{\pi}_k, y_{k+1}; \hat{P}), \quad \bar{\pi}_{k+1} = T(\bar{\pi}_k, y_{k+1}; P), \quad \bar{\pi}_{k+1} = T(\bar{\pi}_k, y_{k+1}; \bar{P})$$

denote the posterior updated using the optimal filter (4.9) with transition matrices $P$, $\bar{P}$ and $\hat{P}$, respectively. Assuming that the state levels of the Markov chain are $g = (1, 2, \ldots, X)'$, the conditional mean estimates of the underlying state computed using $P$, $\bar{P}$ and $\hat{P}$, respectively, will be denoted as

$$\hat{x}_k = \mathbb{E}\{x_k|y_{0:k}; P\} = g'\bar{\pi}_k, \quad \bar{x}_k = \mathbb{E}\{x_k|y_{0:k}; P\} = g'\bar{\pi}_k,$$

Also denote the maximum a posteriori (MAP) state estimates computed using $P$ and $\hat{P}$ as

$$\hat{x}_{k\text{MAP}} = \arg\max_i \bar{\pi}_k(i), \quad \bar{x}_{k\text{MAP}} = \arg\max_i \bar{\pi}_k(i), \quad \hat{x}_{k\text{MAP}} = \arg\max_i \bar{\pi}_k(i).$$

The following is the main result of this section (proof in [57]).

**Theorem 4.6.1 (Stochastic Dominance Sample-Path Bounds)** Consider the HMM filtering updates $T(\pi, y; P)$, $T(\pi, y; \hat{P})$ and $T(\pi, y; \bar{P})$ where $T(\cdot)$ is defined in (4.9) and $P$ denotes the transition matrix of the HMM.

1. For any transition matrix $P$, there exist transition matrices $\bar{P}$ and $\hat{P}$ such that $\bar{P} \preceq P \preceq \hat{P}$ (recall $\preceq$ is the copositive ordering defined in Definition 4.2.3).

2. Suppose transition matrices $\bar{P}$ and $P$ are constructed such that $\bar{P} \preceq P \preceq \hat{P}$. Then for any $y$ and $\pi \in \Pi(X)$, the filtering updates satisfy the sandwich result

$$T(\pi, y; P) \leq_r T(\pi, y; \bar{P}) \leq_r T(\pi, y; \hat{P}).$$

3. Suppose $P$ is TP2 (Assumption (F2)). Assume the filters $T(\pi, y; P)$, $T(\pi, y; \bar{P})$ and $T(\pi, y; \hat{P})$ are initialized with common prior $\pi_0$. Then the posteriors satisfy

$$\bar{\pi}_k \leq_r \pi_k \leq_r \bar{\pi}_k, \quad \text{for all time } k = 1, 2, \ldots$$

As a consequence for all time $k = 1, 2, \ldots$.

(a) The conditional mean state estimates defined in (4.10) satisfy $\bar{x}_k \leq \hat{x}_k \leq \bar{\pi}_k$.
(b) The MAP state estimates defined in (4.11) satisfy $\hat{x}_{k}^{\text{MAP}} \leq \bar{x}_{k}^{\text{MAP}} \leq \bar{x}_{k}^{\text{MAP}}$. 

Statement 1 says that for any transition matrix $P$, there always exist transition matrices $\underline{P}$ and $\bar{P}$ such that $\underline{P} \preceq P \preceq \bar{P}$ (copositivity dominance). Actually if $P$ is TP2, then one can trivially construct the tightest rank 1 bounds $\underline{P}$ and $\bar{P}$ as shown below.

Given existence of $\underline{P}$ and $\bar{P}$, the next step is to optimize the choice of $\underline{P}$ and $\bar{P}$. This is discussed in §4.6.2 where nuclear norm minimization is used to construct sparse eigenvalue matrices $\underline{P}$ and $\bar{P}$.

Statement 2 says that for any prior $\pi$ and observation $y$, the one step filtering updates using $\underline{P}$ and $\bar{P}$ constitute lower and upper bounds to the original filtering problem. This is simply a consequence of (F3') and Statement 5 of Theorem 4.3.1.

Statement 3 globalizes Statement 2 and asserts that with the additional assumption that the transition matrix $P$ of the original filtering problem is TP2, then the upper and lower bounds hold for all time. Since MLR dominance implies first order stochastic dominance (see Theorem 4.1.3), the conditional mean estimates satisfy $\underline{x}_{k} \leq \hat{x}_{k} \leq \bar{x}_{k}$.

4.6.2 Convex Optimization to Compute Low Rank Transition Matrices

It only remains to give algorithms for constructing low rank transition matrices $\underline{P}$ and $\bar{P}$ that yield the lower and upper bounds $\underline{\pi}_{k}$ and $\bar{\pi}_{k}$ for the optimal filter posterior $\pi_{k}$. These involve convex optimization [33] for minimizing the nuclear norm. The computation of $\underline{P}$ and $\bar{P}$ is independent of the observation sample path and so the associated computational cost is irrelevant to the real time filtering. Recall that the motivation is as follows: If $\underline{P}$ and $\bar{P}$ have rank $r$, then the computational cost of the filtering recursion is $O(rX)$ instead of $O(X^2)$ at each time $k$.

Construction of $\underline{P}$, $\bar{P}$ without rank constraint

Given a TP2 matrix $P$, the transition matrices $\underline{P}$ and $\bar{P}$ such that $\underline{P} \preceq P \preceq \bar{P}$ can be constructed straightforwardly via an LP solver. With $\underline{P}_{1}, \underline{P}_{2}, \ldots, \underline{P}_{X}$ denoting the rows of $\underline{P}$, a sufficient condition for $\underline{P} \preceq P$ is that $\underline{P}_{i} \preceq \bar{P}_{i}$ for any row $i$. Hence, the rows $\underline{P}_{i}$ satisfy linear constraints with respect to $\bar{P}_{i}$ and can be straightforwardly constructed via an LP solver. A similar construction holds for the upper bound $\bar{P}$, where it is sufficient to construct $\bar{P}_{i} \succeq \underline{P}_{i}$.

Rank 1 bounds: If $P$ is TP2, an obvious construction is to construct $\underline{P}$ and $\bar{P}$ as follows: Choose rows $\underline{P}_{i} = P_{i}$ and $\bar{P}_{i} = P_{X}$ for $i = 1, 2, \ldots, X$. These yield rank 1 matrices $\underline{P}$ and $\bar{P}$. It is clear from Theorem 4.6.1 that $\underline{P}$ and $\bar{P}$ constructed in this manner are the tightest rank 1 lower and upper bounds.
Nuclear Norm Minimization Algorithms to Compute Low Rank Transition Matrices \(P, \bar{P}\)

This subsection constructs \(P\) and \(\bar{P}\) as low rank transition matrices subject to the condition \(P \preceq P \preceq \bar{P}\). To save space we consider the lower bound transition matrix \(P\); construction of \(\bar{P}\) is similar. Consider the following optimization problem for \(P\):

\[
\text{Minimize rank of } X \times X \text{ matrix } P \quad (4.12)
\]

subject to the constraints \(\text{Cons}(\Pi(X), P, m)\) for \(m = 1, 2, \ldots, X - 1\), where for \(\epsilon > 0\),

\[
\text{Cons}(\Pi(X), P, m) \equiv \begin{cases} 
\Gamma^{(m)} \text{ is copositive on } \Pi(X) \\
\|P'\pi - P'\bar{\pi}\|_1 \leq \epsilon \text{ for all } \pi \in \Pi(X) \\
P \geq 0, \quad P1 = 1.
\end{cases} \tag{4.13a, b, c}
\]

Recall \(\Gamma\) is defined in (4.6) and (4.13a) is equivalent to \(P \preceq P \preceq \bar{P}\). The constraints \(\text{Cons}(\Pi(X), P, m)\) are convex in matrix \(P\), since (4.13a) and (4.13c) are linear in the elements of \(P\) and (4.13b) is convex (because norms are convex). The constraints (4.13a), (4.13c) are exactly the conditions of Theorem 4.6.1, namely that \(P\) is a stochastic matrix satisfying \(P \preceq P \preceq \bar{P}\).

The convex constraint (4.13b) is equivalent to \(\|P - P\|_1 \leq \epsilon\), where \(\|\cdot\|_1\) denotes the induced 1-norm for matrices.\(^6\)

To solve the above problem, we proceed in two steps:

1. The objective (4.12) is replaced with the reweighted nuclear norm (see §4.6.2 below).
2. Optimization over the copositive cone (4.13a) is achieved via a sequence of simplicial decompositions (see remark at end of §4.6.2).

**Reweighted Nuclear Norm**

Since the rank is a non-convex function of a matrix, direct minimization of the rank (4.12) is computationally intractable. Instead, we follow the approach developed by Boyd and coworkers [33] to minimize the iteratively reweighted nuclear norm. Inspired by Candès and Tao [16], there has been much recent interest in minimizing nuclear norms for constructing matrices with sparse eigenvalue sets or equivalently low rank. Here we compute \(P, \bar{P}\) by minimizing their nuclear norms subject to copositivity conditions that ensure \(P \preceq P \preceq \bar{P}\).

Let \(\|\cdot\|_*\) denote the nuclear norm, which corresponds to the sum of the singular values of a matrix. The re-weighted nuclear norm minimization proceeds as a sequence of convex optimization problems indexed by \(n = 0, 1, \ldots\). Initialize \(P^{(0)} = I\). For \(n = 0, 1, \ldots\), compute \(X \times X\) matrix

\[
P^{(n+1)} = \text{argmin}_P \|W^{(n)} P W^{(n)}\|_ \quad (4.14)
\]

\(^6\)The three statements \(\|P'\pi - P'\bar{\pi}\|_1 \leq \epsilon, \|P - P\|_1 \leq \epsilon\) and \(\sum_{i=1}^X \| (P' - P')_i \|_1 \pi(i) \leq \epsilon\) are all equivalent since \(\|\pi\|_1 = 1\) because \(\pi\) is a probability vector (pmf)
subject to: constraints $\text{Cons}(\Pi(X), \underline{P}, m)$, $m = 1, \ldots, X - 1$

namely, (4.13a), (4.13b), (4.13c).

Notice that at iteration $n + 1$, the previous estimate, $\underline{P}^{(n)}$, appears in the cost function of (4.14) in terms of weighting matrices $W_1^{(n)}$, $W_2^{(n)}$. These weighting matrices are evaluated iteratively as

$$
W_1^{(n+1)} = (W_1^{(n)} - U\Sigma U^T W_1^{(n)} + \delta I)^{-1/2},
$$

$$
W_2^{(n+1)} = (W_2^{(n)} - V\Sigma V^T W_2^{(n)} + \delta I)^{-1/2}.
$$

(4.15)

Here $W_1^{(n)} \underline{P}^{(n)} W_2^{(n)} = U\Sigma V^T$ is a reduced singular value decomposition, starting with $W_1^{(0)} = W_2^{(0)} = I$ and $\underline{P}^{(0)} = P$. Also $\delta$ is a small positive constant in the regularization term $\delta I$. In numerical examples of §??, we used YALMIP with MOSEK and CVX to solve the above convex optimization problem.

The intuition behind the reweighting iterations is that as the estimates $\underline{P}^{(n)}$ converge to the limit $\underline{P}^{(\infty)}$, the cost function becomes approximately equal to the rank of $\underline{P}^{(\infty)}$.

Remark: Problem (4.14) is a convex optimization problem in $\underline{P}$. However, one additional issue needs to be resolved: the constraints (4.13a) involve a copositive cone and cannot be solved directly by standard interior point methods. To deal with the copositive constraints (4.13a), one can use the state-of-the-art simplicial decomposition method detailed in [15], see [67] for details.

4.6.3 Discussion. Reduced Complexity Predictors

If one were interested in constructing reduced complexity HMM predictors (instead of filters), the results in this section are straightforwardly relaxed using first order dominance $\leq_s$ instead of MLR dominance $\leq_r$ as follows: Construct $\underline{P}$ by nuclear norm minimization as in (4.14), where (4.13a) is replaced by the linear constraints $P_i \leq_s P_i$, on the rows $i = 1, \ldots, X$, and (4.13b), (4.13c) hold. Thus the construction of $\underline{P}$ is a standard convex optimization problem and the bound $P' \pi \leq_s P' \pi$ holds for the optimal predictor for all $\pi \in \Pi(X)$.

Further, if $\underline{P}$ is chosen so that its rows satisfy the linear constraints $P_i \leq_s P_{i+1}$, $i = 1, \ldots, X - 1$, then the following global bound holds for the optimal predictor: $(P' k \pi \leq_s (P' k \pi$ for all time $k$ and $\pi \in \Pi(X)$. A similar result holds for the upper bounds in terms of $\bar{P}$.

It is instructive to compare this with the filtering case, where we imposed a TP2 condition on $P$ for the global bounds of Theorem 4.6.1(3) to hold wrt $\leq_r$. We could have equivalently imposed a TP2 constraint on $\underline{P}$ and allow $P$ to be arbitrary for the global filtering bounds to hold, however the TP2 constraint is non-convex and so it is difficult to then optimize $\underline{P}$.

Finally, note that the predictor bounds in terms of $\leq_s$ do not hold if a filtering update is performed since $\leq_s$ is not closed w.r.t. conditional expectations.
4.7 Complements and Sources

The books [88, 108] give comprehensive accounts of stochastic dominance. [126] discusses the TP2 stochastic order which is a multivariate generalization of the MLR order. Karlin’s book [47] is a classic on totally positive matrices. The classic paper [48] studies multivariate TP2 orders; see also [102].

The material in §4.6 is based on [67]; where additional numerical results are given. Also it is shown in [67] how the reduced complexity bounds on the posterior can be exploited by using a Monte-Carlo importance sampling filter. The approach in §4.6 of optimizing the nuclear norm as a surrogate for rank has been studied as a convex optimization problem in several papers [75]. Inspired by the seminal work of Candès and Tao [16], there has been much recent interest in minimizing nuclear norms in the context of sparse matrix completion problems. Algorithms for testing for copositive matrices and copositive programming have been studied recently in [14, 15].

There has been extensive work in signal processing on posterior Cramér-Rao bounds for nonlinear filtering [122]; see also [104] for a textbook treatment. These yield lower bounds to the achievable variance of the conditional mean estimate of the optimal filter. However, such posterior Cramér-Rao bounds do not give constructive algorithms for computing upper and lower bounds for the sample path of the filtered distribution. The sample path bounds proposed in this chapter have the attractive feature that they are guaranteed to yield lower and upper bounds to both hard and soft estimates of the optimal filter. It would be of interest to develop similar results for jump Markov linear systems, in particular to use such constraints for particle filtering algorithms [28].

Appendix 4.A Proofs

4.A.1 Proof of Theorem 4.3.1

First recall from Definition 4.2.1 that $\pi_1 \geq_r \pi_2$ is equivalent to saying that the matrix $\begin{bmatrix} \pi_2' \\ \pi_1' \end{bmatrix}$ is TP2. This TP2 notation is more convenient for proofs.

**Statement 1a** If (F2) holds, then $\pi_1 \geq_r \pi_2$ implies $P'\pi_1 \geq_r P'\pi_2$: Showing that $P'\pi_1 \geq_r P'\pi_2$ is equivalent to showing that $\begin{bmatrix} \pi_2' P \\ \pi_1' P \end{bmatrix}$ is TP2. But $\begin{bmatrix} \pi_2' P \\ \pi_1' P \end{bmatrix} = \begin{bmatrix} \pi_2' \\ \pi_1' \end{bmatrix} P$. Also since $\pi_1 \geq_r \pi_2$, the matrix $\begin{bmatrix} \pi_2' \\ \pi_1' \end{bmatrix}$ is TP2. By (F2), $P$ is TP2. Since the product of TP2 matrices is TP2 (see Lemma 4.5.2), the result holds.

If $\pi_1 \geq_r \pi_2$ implies $P'\pi_1 \geq_r P'\pi_2$ then (F2) holds: Choose $\pi_1 = e_j$ and $\pi_2 = e_i$ where $j > i$, and as usual $e_i$ is the unit vector with 1 in the $i$-th position. Clearly then $\pi_1 \geq_r \pi_2$. Also $P'e_j$ is the $i$-th row of $P$. So $P'e_j \geq_r P'e_i$ implies the $j$-th
row is MLR larger than the i-th row of P. This implies that P is TP2 by definition 4.2.1.

Statement 1b follows by applying Theorem 4.1.4 to 1a.

Statement 2. Since MLR dominance implies first order dominance, by (F1), \( \sum_{y \geq y} B_{x,y}(u) \) is increasing in \( x \). By (F2), \( (P_{i,1}, \ldots, P_{i,x}) \leq_s (P_{j,1}, \ldots, P_{j,x}) \) for \( i \leq j \). Therefore \( \sum_j P_{ij}(u) \sum_{y \geq y} B_{j,y}(u) \) is increasing in \( i \in X \). Therefore \( \pi_1 \geq \pi_2 \) implies \( \sigma(\pi_1, u) \geq \sigma(\pi_2, u) \).

Statement 3. Denote \( P'(u)\pi_1 = \bar{\pi} \). Then \( T(\pi_1, y, u) \geq_r T(\pi_1, \bar{y}, u) \) is equivalent to

\[
(B_{i,y}B_{i+1,y} - B_{i+1,y}B_{i,y})\bar{\pi}(i)\bar{\pi}(i+1) \leq 0, \quad y > \bar{y}.
\]

This is equivalent to \( B \) being TP2, namely condition (F1).

Statement 4a. By definition of MLR dominance, \( T(\pi, y, u) \leq_r T(\pi, y, u+1) \) is equivalent to

\[
\sum_m \sum_n B_{j,y}(u+1)B_{j+1,y}(u)P_{nj}(u+1)\pi_n \pi_m \leq \sum_m \sum_n B_{j,y}(u)B_{j+1,y}(u+1)P_{nj}(u)P_{nj+1}(u+1)\pi_n \pi_m
\]

and also

\[
\sum_m \sum_n B_{j,y}(u+1)B_{j+1,y}(u)P_{nj}(u+1)\pi_n \pi_m \leq \sum_m \sum_n B_{j,y}(u)B_{j+1,y}(u+1)P_{nj}(u)P_{nj+1}(u+1)\pi_n \pi_m
\]

This is equivalent to (F3').

Statement 4b follows since (F3') is sufficient for (F3).

The proofs of Statement 5 and 6 are very similar to Statement 4 and omitted.
This chapter gives sufficient conditions on the POMDP model so that the value function in dynamic programming is decreasing with respect to the monotone likelihood ratio (MLR) stochastic order. That is, $\pi_1 \geq_r \pi_2$ (in terms MLR dominance) implies $V(\pi_1) \leq V(\pi_2)$. To prove this result, we will use the structural properties of the optimal filter established in Chapter 4.

Giving conditions for a POMDP to have a monotone value function is useful for several reasons: It serves as an essential step in establishing sufficient conditions for a stopping time POMDPs to have a monotone optimal policy – this is discussed in Chapter 6. For more general POMDPs (discussed in Chapter 8), it allows us to upper and lower bound the optimal policy by judiciously constructed myopic policies. Please see Figure ?? for the sequence of chapters on POMDP structural results.

After giving sufficient conditions for a monotone value function, this chapter also gives two examples of POMDPs to illustrate the usefulness of this result:

- **Example 1. Monotone Optimal Policy for 2-state POMDP**: §5.3 gives sufficient conditions for a 2 state POMDP to have a monotone optimal policy. The optimal policy is characterized by at most $U - 1$ threshold belief states (where $U$ denotes the number of possible actions). One only needs to compute (estimate) these $U - 1$ threshold belief states in order to determine the optimal policy. This is easier than solving Bellman’s equation when nothing is known about the structure of the optimal policy. Also real time implementation of a controller with a monotone policy is simple; only the threshold belief states need to be stored in a lookup table. Figure 5.1 illustrates a monotone policy for a two state POMDP with $U = 3$.

- **Example 2. POMDP Multi-armed Bandits and Opportunistic Scheduling**: §5.4 discusses how monotone value functions can be used to solve POMDP multi-armed bandit problems efficiently. It is shown that for such problems, the
5.1 Model and Assumptions

Figure 5.1 §5.3 gives sufficient conditions for a 2-state POMDP to have a monotone optimal policy. The figure illustrates such a monotone policy for \( U = 3 \). The optimal policy is completely determined by the threshold belief states \( \pi^*_1 \) and \( \pi^*_2 \).

optimal strategy is “opportunistic”: choose the bandit with the largest belief state in terms of MLR order.

5.1 Model and Assumptions

Consider a discrete time, infinite horizon discounted cost POMDP which was formulated in §2.6. The state space for the underlying Markov chain is \( \mathcal{X} = \{1, 2, \ldots, X\} \), the action space is \( \mathcal{U} = \{1, 2, \ldots, U\} \) and the belief space is the unit \( X - 1 \) dimensional unit simplex

\[
\Pi(X) = \{ \pi \in \mathbb{R}^X : 1^t \pi = 1, \quad 0 \leq \pi(i) \leq 1, \quad i \in \mathcal{X} = \{1, 2, \ldots, X\} \}.
\]

For stationary policy \( \mu : \Pi(X) \to \mathcal{U} \), initial belief \( \pi_0 \in \Pi(X) \), discount factor \( \rho \in [0, 1) \), the discounted cost is

\[
J_\mu(\pi_0) = \mathbb{E}_\mu \left\{ \sum_{k=1}^{\infty} \rho^{k-1} C(\pi_k, \mu(\pi_k)) \right\}.
\] (5.1)

Here \( C(\pi, u) \) is the cost accrued at each stage and is not necessarily linear in \( \pi \). The belief evolves according to the HMM filter \( \pi_k = T(\pi_{k-1}, y_k, u_k) \) where

\[
T(\pi, y, u) = \frac{B_y(u) P'(u) \pi}{\sigma(\pi, y, u)}, \quad \sigma(\pi, y, u) = 1'_X B_y(u) P'(u) \pi,
\]

\[
B_y(u) = \text{diag}(B_{1y}(u), \ldots, B_{Xy}(u)), \quad \text{where } B_{xy}(u) = p(y|x, u).
\] (5.2)

Throughout this chapter \( y \in \mathcal{Y} \) can be discrete-valued in which case \( p \) in (5.2) is a pmf or continuous-valued in which case \( p \) is a pdf.

This chapter considers discounted cost POMDPs for notational convenience to avoid denoting the time dependencies of parameters and policies. The main result of this chapter, namely Theorem 5.2.1 also holds for finite horizon POMDPs providing conditions (C), (F1) and (F2) hold at each time instant for the time dependent cost, observation matrix and transition matrix.
Monotonicity of Value Function for POMDPs

The optimal stationary policy \( \mu^* : \Pi(X) \to \mathcal{U} \) such that \( J_{\mu^*}(\pi_0) \leq J_{\mu}(\pi_0) \) for all \( \pi_0 \in \Pi(X) \) satisfies Bellman’s dynamic programming equation (2.24)

\[
\mu^*(\pi) = \arg\min_{u \in \mathcal{U}} Q(\pi, u), \quad J_{\mu^*}(\pi_0) = V(\pi_0)
\]

\[
V(\pi) = \min_{u \in \mathcal{U}} Q(\pi, u), \quad Q(\pi, u) = C(\pi, u) + \rho \sum_{y \in Y} V(T(\pi, y, u)) \sigma(\pi, y, u).
\]

Assumptions

(C) The cost \( C(\pi, u) \) is first order stochastically decreasing with respect to \( \pi \) for each action \( u \in \{1, 2, \ldots, U\} \). That is \( \pi_1 \succeq_s \pi_2 \) implies \( C(\pi_1, u) \leq C(\pi_2, u) \).

For linear costs \( C(\pi, u) = c_u^T \pi \), (C) is equivalent to the condition:

The instantaneous cost \( c(x, u) \) is decreasing in \( x \) for each \( u \).

(F1) The observation probability kernel \( B(u) \) is TP2 for each action \( u \in \{1, 2, \ldots, U\} \).

(F2) The transition matrix \( \mathcal{P}(u) \) is TP2 for each action \( u \in \{1, 2, \ldots, U\} \).

Recall that assumptions (F1) and (F2) were discussed in Chapter 4; see (F1), (F2) on page 45. (F1) and (F2) are required for the Bayesian filter \( T(\pi, y, u) \) to be monotone increasing with observation \( y \) and \( \pi \) with respect to the MLR order. This is a key step in showing \( V(\pi) \) is MLR decreasing in \( \pi \).

Sufficient Conditions for (C)

We pause briefly to discuss Assumption (C), particularly in the context of non-linear costs that arise in controlled sensing (discussed in Chapter ??).

For linear costs \( C(\pi, u) = c_u^T \pi \), obviously the elements of \( c_u \) decreasing is necessary and sufficient for \( C(\pi, u) \) to be decreasing with respect to \( \succeq_s \).

For nonlinear costs, we can give the following sufficient condition for \( C(\pi, u) \) to be decreasing in \( \pi \) with respect to first order stochastic dominance, Consider the subset of \( \mathbb{R}_+^X \) defined as \( \Delta = \{ \delta : 1 = \delta(1) \geq \delta(2) \cdots \geq \delta(X) \} \). Define the \( X \times X \) matrix

\[
\Psi = \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
0 & 1 & -1 & \cdots & 0 \\
& & & \ddots & \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}.
\]

Clearly every \( \pi \in \Pi(X) \) can be expressed as \( \pi = \Psi \delta \) where \( \delta \in \Delta \). Consider two beliefs \( \pi_1 = \Psi \delta_1 \) and \( \pi_2 = \Psi \delta_2 \) such that \( \pi_1 \succeq_s \pi_2 \). The equivalent partial order induced on \( \delta_1 \) and \( \delta_2 \) is: \( \delta_1 \succeq \delta_2 \) where \( \succeq \) is the componentwise partial order on \( \mathbb{R}^X \).

**Lemma 5.1.1** Consider a nonlinear cost \( C(\pi_2, u) \) that is differentiable in \( \pi \).

1. For \( \pi_1 \succeq_s \pi_2 \), a sufficient condition for \( C(\pi_1, u) \leq C(\pi_2, u) \) is \( \frac{d}{d\pi} C(\Psi \delta) \leq 0 \) element wise.

2. Consider the special case of a quadratic cost \( C(\pi, u) = \phi_u^T \pi - \alpha(h^T \pi)^2 \) where \( \alpha \) is a non-negative constant, \( \phi_u, h \in \mathbb{R}_+^X \) with elements \( \phi_{iu}, h_i, i = 1, \ldots, X \). Assume \( h \)}
5.2 Main Result: Monotone Value Function

The following is the main result of this chapter.

**Theorem 5.2.1** Consider an infinite horizon discounted cost POMDP with continuous or discrete-valued observations. Then under (C), (F1), (F2), $Q(\pi, u)$ is MLR decreasing in $\pi$. As a result, the value function $V(\pi)$ in Bellman's equation (5.3) is MLR decreasing in $\pi$. That is, $\pi_1 \geq \pi_2$ implies that $V(\pi_1) \leq V(\pi_2)$.

**Proof** The proof is by mathematical induction on the value iteration algorithm and makes extensive use of the structural properties of the HMM filter developed in Theorem 4.3.1. Recall from (2.25) that the value iteration algorithm proceeds as follows: Initialize $V_0(\pi) = 0$ and for iterations $n = 1, 2, \ldots$, 

$$V_n(\pi) = \min_{u \in U} Q_n(\pi, u), \quad Q_n(\pi, u) = C(\pi, u) + \rho \sum_{y \in Y} V_{n-1}(T(\pi, y, u)) \sigma(\pi, y, u).$$

Assume that $V_{n-1}(\pi)$ is MLR decreasing in $\pi$ by the induction hypothesis. Under (F1), Theorem 4.3.1(3) says that $T(\pi, y, u)$ is MLR increasing in $y$. As a result, $V_{n-1}(T(\pi, y, u))$ is decreasing in $y$. Under (F1), (F2), Theorem 4.3.1(2) says 

$$\pi \geq \bar{\pi} \implies \sigma(\pi, y, u) \geq \sigma(\bar{\pi}, y, u). \quad (5.7)$$

$V_{n-1}(T(\pi, y, u))$ decreasing in $y$ and the first order dominance (5.7) implies 

2 Note that $C(\pi, u)$ first order increasing in $\pi$ implies that $C(\pi, u)$ is MLR increasing in $\pi$, since MLR dominance implies first order dominance.
using Theorem 3.2.2 that
\[
\pi \geq \bar{\pi} \implies \sum_y V_{n-1}(T(\pi, y, u)) \sigma(\pi, y, u) \leq \sum_y V_{n-1}(T(\bar{\pi}, y, u)) \sigma(\bar{\pi}, y, u)
\] (5.8)

Next, from Theorem 4.3.1(1), it follows that under (F2),
\[
\pi \geq \bar{\pi} \implies T(\pi, y, u) \geq T(\bar{\pi}, y, u)
\]

Using the induction hypothesis that \(V_{n-1}(\pi)\) is MLR decreasing in \(\pi\) implies
\[
\pi \geq \bar{\pi} \implies V_{n-1}(T(\pi, y, u)) \leq V_{n-1}(T(\bar{\pi}, y, u))
\]

which in turn implies
\[
\pi \geq \bar{\pi} \implies \sum_y V_{n-1}(T(\pi, y, u)) \sigma(\pi, y, u) \leq \sum_y V_{n-1}(T(\bar{\pi}, y, u)) \sigma(\bar{\pi}, y, u).
\] (5.9)

Combining (5.8), (5.9), it follows that
\[
\pi \geq \bar{\pi} \implies \sum_y V_{n-1}(T(\pi, y, u)) \sigma(\pi, y, u) \leq \sum_y V_{n-1}(T(\bar{\pi}, y, u)) \sigma(\bar{\pi}, y, u).
\] (5.10)

Finally, under (C), \(C(\pi, u)\) is MLR decreasing (see Footnote 2)
\[
\pi \geq \bar{\pi} \implies C(\pi, u) \leq C(\bar{\pi}, u).
\] (5.11)

Since the sum of decreasing functions is decreasing, it follows that
\[
\pi \geq \bar{\pi} \implies C(\pi, u) + \sum_y V_{n-1}(T(\pi, y, u)) \sigma(\pi, y, u) \leq C(\bar{\pi}, u) + \sum_y V_{n-1}(T(\bar{\pi}, y, u)) \sigma(\bar{\pi}, y, u)
\]

which is equivalent to \(Q_n(\pi, u) \leq Q_n(\bar{\pi}, u)\). Therefore \(Q_n(\pi, u)\) is MLR decreasing in \(\pi\). Since the minimum of decreasing functions is decreasing, \(V_n(\pi) = \min_u Q_n(\pi, u)\) is MLR decreasing in \(\pi\). Finally, since \(V_n\) converges uniformly to \(V\), it follows that \(V(\pi)\) is also MLR decreasing.

To summarize, although value iteration is not useful from a computational point of view for POMDPs, we have exploited its structure of prove the monotonicity of the value function. In the next two chapters, several examples will be given that exploit the monotone structure of the value function of a POMDP.

### 5.3 Example 1: Monotone Policies for 2-state POMDPs

This section gives sufficient conditions for the optimal policy \(\mu^*(\pi)\) to be monotone increasing in \(\pi\) when the underlying Markov chain has \(X = 2\) states (see Figure 5.1). For \(X = 2\), since \(\pi\) is a two-dimensional probability mass function
with \( \pi(1) + \pi(2) = 1 \), it suffices to order the beliefs in terms of the second component \( \pi(2) \) which lies in the interval \([0, 1]\).

Consider a discounted cost POMDP \( (X, U, Y, P(u), B(u), c(u), \rho) \) where state space \( X = \{1, 2\} \), action space \( U = \{1, 2, \ldots, U\} \), observation space \( Y \) can be continuous or discrete, and \( \rho \in [0, 1) \). The main assumptions are as follows:

(C) \( c(x, u) \) is decreasing in \( x \in \{1, 2\} \) for each \( u \in U \).

(F1) \( B \) is totally positive of order 2 (TP2).

(F2) \( P(u) \) is totally positive of order 2 (TP2).

(F3) \( P_{12}(u + 1) - P_{12}(u) \leq P_{22}(u + 1) - P_{22}(u) \) (tail-sum supermodularity).

(S) The costs are submodular: \( c(1, u + 1) - c(1, u) \geq c(2, u + 1) - c(2, u) \).

Recall (C) on page 58 and (F1), (F2) on page 45. The main additional assumption above is the submodularity assumption (S). Apart from (F1), the above conditions are identical to the fully observed MDP case considered in Theorem 3.3.1 on page 33. Indeed (A2) and (A4) in Theorem 3.3.1 are equivalent to (F2) and (F3), respectively, for \( X = 2 \).

**Theorem 5.3.1** Consider a POMDP with an underlying \( X = 2 \) state Markov chain. Under (C), (F1), (F2), (F3), (S), the optimal policy \( \mu^*(\pi) \) is increasing in \( \pi \). Thus \( \mu^*(\pi(2)) \) has the following finite dimensional characterization: There exist \( U + 1 \) thresholds (real numbers) \( 0 = \pi_0^* \leq \pi_1^* \leq \cdots \leq \pi_U^* \leq 1 \) such that

\[
\mu^*(\pi) = \sum_{u \in U} u I (\pi(2) \in (\pi_{u-1}^*, \pi_u^*]) .
\]

The proof is in [57]. It exploits the fact that the value function \( V(\pi) \) is decreasing in \( \pi \) (Theorem 5.2.1 on page 59) and is concave to show that \( Q(\pi, u) \) is submodular. That is

\[
Q(\pi, u) - Q(\pi, \bar{u}) - Q(\bar{\pi}, u) + Q(\bar{\pi}, \bar{u}) \leq 0, \quad u \geq \bar{u}, \quad \pi \geq \bar{\pi}.
\]  \( \tag{5.12} \)

where \( Q(\pi, u) \) is defined in Bellman’s equation (5.3). Recall that for \( X = 2, \geq_s \) (first order dominance) and \( \geq_r \) (MLR dominance) coincide, implying that \( \pi \geq \bar{\pi} \iff \pi \geq_s \bar{\pi} \iff \pi(2) \geq \bar{\pi}(2) \). As a result, for \( X = 2 \), submodularity of \( Q(\pi, u) \) needs to be established with respect to \((\pi(2), u)\) where \( \pi(2) \) is a scalar in the interval \([0, 1]\). Hence the same simplified definition of submodularity used for a fully observed MDP (Theorem 3.1.1 on page 30) can be used.

**Summary:** We have given sufficient conditions for a 2 state POMDP to have a monotone (threshold) optimal policy as illustrated in Figure 5.1. The threshold values can then be estimated via simulation based policy gradient algorithm such as the SPSA Algorithm, see also §6.4.2. Theorem 5.3.1 only holds for \( X = 2 \) and does not generalize to \( X \geq 3 \). For \( X \geq 3 \), determining sufficient conditions for submodularity (5.12) to hold is an open problem. In Chapter 8, we will instead construct judicious myopic bounds for \( X \geq 3 \).
5.4 Example 2: POMDP Multi-armed Bandits Structural Results

In this section, we show how the monotone value function result of Theorem 5.2.1 facilitates solving POMDP multi-armed bandit problems efficiently.

The multi-armed bandit problem is a dynamic stochastic scheduling problem for optimizing in a sequential manner the allocation effort between a number of competing projects. Numerous applications of finite state Markov chain multi-armed bandit problems appear in the operations research and stochastic control literature, see [35], [127] for examples in job scheduling and resource allocation for manufacturing systems. The reason why multi-armed bandit problems are interesting is because their structure implies that the optimal policy can be found by a so-called Gittins index rule [35, 105]: At each time instant, the optimal action is to choose the process with the highest Gittins index, where the Gittins index of each process is a function of the state of that process. So the problem decouples into solving individual control problems for each process.

This section considers multi-armed bandit problems where the finite state Markov chain is not directly observed – instead the observations noisy measurements of the unobserved Markov chain. Such POMDP multi-armed bandits are a useful model in stochastic scheduling.

5.4.1 POMDP Multi-armed Bandit Model

The POMDP multi-armed bandit has the following model: Consider $L$ independent projects $l = 1, \ldots, L$. Assume for convenience each project $l$ has the same finite state space $X = \{1, 2, \ldots, X\}$. Let $x^{(l)}_k$ denote the state of project $l$ at discrete time $k = 0, 1, \ldots$. At each time instant $k$ only one of these projects can be worked on. The setup is as follows:

- If project $l$ is worked on at time $k$:
  1. An instantaneous non-negative reward $\rho^k r(x^{(l)}_k)$ is accrued where $0 \leq \rho < 1$ denotes the discount factor.
  2. The state $x^{(l)}_k$ evolves according to an $X$-state homogeneous Markov chain with transition probability matrix $P$.
  3. The state of the active project $l$ is observed via noisy measurements $y^{(l)}_{k+1} \in Y = \{1, 2, \ldots, Y\}$ of the active project state $x^{(l)}_{k+1}$ with observation probability $B_{xy} = \mathbb{P}(y^{(l)} = y|x^{(l)} = x)$.
- The states of all the other $(L - 1)$ idle projects are unaffected, i.e., $x^{(l)}_{k+1} = x^{(l)}_k$, if project $l$ is idle at time $k$. No observations are obtained for idle projects.

For notational convenience we assume all the projects have the same reward functions, transition and observation probabilities and state spaces. So the reward $r(x^{(l)}_k, l)$ is denoted as $r(x^{(l)}_k)$, etc. All projects are initialized with $x^{(l)}_0 \sim \pi^{(l)}_0$ where $\pi^{(l)}_0$ are specified initial distributions for $l = 1, \ldots, L$. Denote $\pi_0 = (\pi^{(1)}_0, \ldots, \pi^{(L)}_0)$. 
Let \( u_k \in \{1, \ldots, L\} \) denote which project is worked on at time \( k \). So \( x^{(u_k)}_{k+1} \) is the state of the active project at time \( k+1 \). Denote the history at time \( k \) as
\[
I_0 = \pi_0, \quad I_k = \{\pi_0, y_1^{(u_0)}, \ldots, y_k^{(u_{k-1})}, u_0, \ldots, u_{k-1}\}.
\]
Then the project at time \( k \) is chosen according to \( u_k = \mu(I_k) \), where the policy denoted as \( \mu \) belongs to the class of stationary policies. The cumulative expected discounted reward over an infinite time horizon is given by
\[
J_\mu(\pi) = \mathbb{E}_\mu \left\{ \sum_{k=0}^{\infty} \rho^k r \left( x_k^{(u_k)} \right) \right\} | \pi_0 = \pi, \ u_k = \mu(I_k). \quad (5.13)
\]
The aim is to determine the optimal stationary policy \( \mu^*(\pi) = \arg \max_\mu J_\mu(\pi) \) which yields the maximum reward in (5.13).

Note that we have formulated the problem in terms of rewards rather than costs since typically the formulation involves maximizing rewards of active projects. Of course the formulation is equivalent to minimizing a cost.

At first sight (5.13) seems intractable since the equivalent state space dimension is \( X^L \). The multi-armed bandit structure yields a remarkable simplification - the problem can be solved by considering \( L \) individual POMDPs each of dimension \( X \). Actually with the structural result below, one only needs to evaluate the belief state for the \( L \) individual HMMs and choose the largest belief at each time (with respect to the MLR order).

### 5.4.2 Belief State Formulation

We now formulate the POMDP multi-armed bandit in terms of the belief state. For each project \( l \), denote by \( \pi_k^{(l)} \) the belief at time \( k \) where
\[
\pi_k^{(l)}(i) = \mathbb{P}(x_k^{(l)} = i | I_k)
\]
The POMDP multi-armed bandit problem can be viewed as the following scheduling problem: Consider \( P \) parallel HMM state estimation filters, one for each project. The project \( l \) is active, an observation \( y_k^{(l)} \) is obtained and the belief \( \pi_k^{(l)} \) is computed recursively by the HMM state filter
\[
\pi_{k+1}^{(l)} = T(\pi_k^{(l)}, y_k^{(l)}) \quad \text{if project } l \text{ is worked on at time } k \quad (5.14)
\]
where
\[
T(\pi^{(l)}, y^{(l)}) = \frac{B_{y^{(l)}} P^{(l)} \pi^{(l)}}{\sigma(\pi^{(l)}, y^{(l)})}, \quad \sigma(\pi^{(l)}, y^{(l)}) = 1' B_{y^{(l)}} P^{(l)} \pi^{(l)}
\]
In (5.14) \( B_{y^{(l)}} = \text{diag}(\mathbb{P}(y^{(l)}|x^{(l)} = 1), \ldots, \mathbb{P}(y^{(l)}|x^{(l)} = X)) \).

The beliefs of the other \( L - 1 \) projects remain unaffected, i.e.
\[
\pi_{k+1}^{(q)} = \pi_k^{(q)} \quad \text{if project } q \text{ is not worked on, } \ q \in \{1, \ldots, L\}, \ q \neq l \quad (5.15)
\]
Note that each belief \( \pi^{(l)} \) lives in the unit simplex \( \Pi(X) \).
Let $r$ denote the $X$ dimensional reward vector $[r(x^{(l)}_k = 1), \ldots, r(x^{(l)}_k = X)]'$. In terms of the belief state, the reward functional (5.13) can be re-written as

$$J_\mu(\pi) = \mathbb{E}\{ \sum_{k=0}^{\infty} \rho^k r'(u_k) | (\pi_0^{(1)}, \ldots, \pi_0^{(L)}) = \pi \}, \quad u_k = \mu(\pi_1^{(k)}, \ldots, \pi_L^{(k)}).$$

The aim is to compute the optimal policy $\mu^*(\pi) = \arg \max_\mu J_\mu(\pi)$.

### 5.4.3 Gittins Index Rule

Define $\bar{M} \overset{\text{def}}{=} \max_i r(i)/(1 - \rho)$ and let $M$ denote a scalar in the interval $[0, \bar{M}]$.

It is known that the optimal policy of a multi-armed bandit has an indexable rule [127]. Translated to the POMDP multi-armed bandit the result reads:

**Theorem 5.4.1 (Gittins index)** Consider the POMDP multi-armed bandit problem comprising $L$ projects. For each project $l$ there is a function $\gamma(\pi^{(l)}_k)$ called the Gittins index, which is only a function of the parameters of project $l$ and the information state $\pi^{(l)}_k$, whereby the optimal scheduling policy at time $k$ is to work on the project with the largest Gittins index:

$$\mu^*(\pi^{(1)}_k, \pi^{(2)}_k, \ldots, \pi^{(L)}_k) = \max_{l \in \{1, \ldots, L\}} \{ \gamma(\pi^{(l)}_k) \}$$

(5.17)

The Gittins index of project $l$ with belief $\pi^{(l)}_k$ is

$$\gamma(\pi^{(l)}_k) = \min \{M : V(\pi^{(l)}_k, M) = M\}$$

(5.18)

where $V(\pi^{(l)}_k, M)$ satisfies Bellman’s equation

$$V(\pi^{(l)}_k, M) = \max \left\{ r'\pi^{(l)}_k + \rho \sum_{y=1}^Y V(T(\pi^{(l)}_k, y), M) \sigma(\pi^{(l)}_k, y), M \right\}$$

(5.19)

Theorem 5.4.1 says that the optimal policy is “greedy”: at each time choose the project with the largest Gittins index; and the Gittins index for each project can be obtained by solving a dynamic programming equation for that project. Theorem 5.4.1 is well known in the multi-armed bandit literature [35] and will not be proved here.

Bellman’s equation (5.19) can be approximated over any finite horizon $N$ via the value iteration algorithm. Since as described in Chapter 2, the value function of a POMDP at each iteration has a finite dimensional characterization, the Gittins index can be computed explicitly for any finite horizon using any of the exact POMDP algorithms in Chapter 2. Moreover, the error bounds for value iteration for horizon $N$ (compared to infinite horizon) of Theorem 2.6.2 directly translate to error bounds in determining the Gittins index. However, for large dimensions, solving individual POMDP to compute the Gittins index is computationally intractable.
5.4 Example 2: POMDP Multi-armed Bandits Structural Results

5.4.4 Structural Result: Characterization of Monotone Gittins Index

Our focus below is to show how the monotone value function result of Theorem 5.2.1 facilitates solving (5.17), (5.18) efficiently. We show that under reasonable conditions on the rewards, transition matrix and observation probabilities, the Gittins index is monotone increasing in the belief (with respect to the MLR order). This means that if the information states of the \( L \) processes at a given time instant are MLR comparable, the optimal policy is to pick the process with the largest belief. This is straightforward to implement and makes the solution practically useful.

Since we are dealing rewards rather than costs, we say that assumption (C) holds if \( r(i) \) is increasing in \( i \in X \). (This corresponds to the cost decreasing in \( i \)).

**Theorem 5.4.2** Consider the POMDP multi-armed bandit where all the \( L \) projects have identical transition and observation matrices and reward vectors. Suppose assumptions (C), (F1) and (F2) on page 45 hold for each project. Then the Gittins index \( \gamma(\pi) \) is MLR increasing in \( \pi \). Therefore, if the beliefs \( \pi_{k}^{(l)} \) of the \( L \) projects are MLR comparable, then the optimal policy \( \mu^{*} \) defined in (5.17) is opportunistic:

\[
\mu^{*} = \arg\max_{l \in \{1, \ldots, L\}} \pi_{k}^{(l)}
\]  

(5.20)

**Proof** First using exactly the same proof as Theorem 5.2.1, it follows that \( V(\pi, M) \) is MLR increasing in \( \pi \).

Given that the value function \( V(\pi, M) \) is MLR increasing in \( \pi \), we can now characterize the Gittins index. Recall from (5.18) that \( \gamma(\pi) = \min\{M : V(\pi, M) - M = 0\} \). Suppose \( \pi^{(1)} \geq \pi^{(2)} \). This implies \( V(\pi^{(1)}, M) \geq V(\pi^{(2)}, M) \) for all \( M \). So \( V(\pi^{(1)}, \gamma(\pi^{(2)})) - \gamma(\pi^{(2)}) \geq V(\pi^{(2)}, \gamma(\pi^{(2)})) - \gamma(\pi^{(2)}) = 0 \). Since \( V(\pi, M) - M \) is decreasing in \( M \) (this is seen by subtracting \( M \) from both sides of (5.19)), it follows from the previous inequality that the point \( \min\{M : V(\pi^{(1)}, M) - M = 0\} > \min\{M : V(\pi^{(2)}, M) - M = 0\} \). So \( \gamma(\pi^{(1)}) \geq \gamma(\pi^{(2)}) \).

**Discussion**

It is instructive to compare (5.17) with (5.20). Theorem 5.4.2 says that instead of choosing the project with the largest Gittins index, it suffices to choose the project with the largest MLR belief (providing the beliefs of the \( L \) projects are MLR comparable). In other words, the optimal policy is opportunistic (other terms used are “greedy” or “myopic”) with respect to the beliefs ranked by MLR order. The resulting optimal policy is trivial to implement and makes the solution practically useful. There is no need to compute the Gittins index.

The following examples yield trajectories of belief states which are MLR comparable across the \( L \) projects. As a result, under (C), (F1), (F2), the optimal policy is opportunistic and completely specified by Theorem 5.4.2.

**Example 1** If \( X = 2 \), then all beliefs are MLR comparable.

**Example 2** Suppose \( B \) is a bi-diagonal matrix. Then if \( \pi_{0}^{(l)} \) is a unit indicator
vector, then all subsequent beliefs are MLR comparable (since all beliefs comprise of two consecutive non-zero elements and the rest are zero elements).

Example 3: Suppose \( B_{iy} = 1/Y \) for all \( i, y \). Suppose all processes have same initial belief \( \pi_0 \) and pick \( P \) such that either \( P'\pi_0 \geq_r \pi_0 \) or \( P'\pi_0 \leq_r \pi_0 \). Then from Theorem 4.3.11a, if \( P \) is TP2, all beliefs are MLR comparable.

When the trajectories of beliefs for the individual bandit processes are not MLR comparable, they can be projected to MLR comparable beliefs, and a suboptimal policy implemented as follows:

Assume at time instant \( k \), the beliefs of all \( L \) processes are MLR comparable. Let \( \sigma(1), \ldots, \sigma(L) \) denote the permutation of \( (1, \ldots, L) \) so that

\[
\pi_{\sigma(1)} \geq_r \pi_{\sigma(2)} \geq_r \ldots \geq_r \pi_{\sigma(L)}.
\]

From Theorem 5.4.2, the optimal action is \( u_k = \sigma(1) \). But the updated belief \( \pi_{\sigma(1)}^{k+1} \) may not be MLR comparable with the other \( L - 1 \) information states. So we project \( \pi_{\sigma(1)}^{k+1} \) to the nearest belief denoted \( \tilde{\pi} \) in the simplex \( \Pi(X) \) that is MLR comparable with the other \( L - 1 \) information states. That is, at time \( k + 1 \) solve the following \( L \) optimization problems: Compute the projection distances

\[
\mathcal{P}(\tilde{\pi}(1)) = \min_{\bar{\pi} \in \Pi(X)} \| \bar{\pi} - \pi_{\sigma(1)}^{k+1} \| \text{ subject to } \pi_{\sigma(1)}^{k+1} \geq_r \pi_{\sigma(2)}^{k+1}, \pi_{\sigma(1)}^{k+1} \geq_r \ldots \geq_r \pi_{\sigma(L)}^{k+1}
\]

\[
\mathcal{P}(\tilde{\pi}(p)) = \min_{\bar{\pi} \in \Pi(X)} \| \bar{\pi} - \pi_{\sigma(p+1)}^{k+1} \| \text{ subject to } \pi_{\sigma(p+1)}^{k+1} \geq_r \pi_{\sigma(p+2)}^{k+1}, \ldots \geq_r \pi_{\sigma(L)}^{k+1}, p = 2, \ldots, L - 1
\]

\[
\mathcal{P}(\tilde{\pi}(L)) = \min_{\bar{\pi} \in \Pi(X)} \| \bar{\pi} - \pi_{\sigma(L)}^{k+1} \| \text{ subject to } \pi_{\sigma(L)}^{k+1} \geq_r \ldots \geq_r \pi_{\sigma(1)}^{k+1}
\]

Here \( \| \cdot \| \) denotes some norm, and \( \mathcal{P}, \tilde{\pi}^p \) denote, respectively, the minimizing value and minimizing solution of each of the problems. Finally set \( \pi_{\sigma(1)}^{k+1} = \arg\min_{\tilde{\pi}} \mathcal{P}(\tilde{\pi}) \). The above \( L \) problems are convex optimization problems and can be solved efficiently in real time. Thus all the beliefs at time \( k + 1 \) are MLR comparable, the action \( u_{k+1} \) is chosen as the index of the largest belief.

**Summary:** For POMDP multi-armed bandits that satisfy the conditions of Theorem 5.4.2, the optimal policy is to choose the project with the largest belief.

### 5.5 Complements and Sources

The proof that under suitable conditions the POMDP value function is monotone with respect to the MLR order goes back to Lovejoy [78]. The MLR order is the natural setting for POMDPs due to the Bayesian nature of the problem. More generally, a similar proof holds for multivariate observation distributions - in this case the TP2 stochastic order (which is a multivariate version of the MLR order) is used - to establish sufficient conditions for monotone value function for a multivariate POMDP; see [102].

The result in §5.3 that establishes a threshold policy for 2-state POMDPs is from [2]. However, the proof does not work for action dependent (controlled)
observation probabilities. In §8.6 we will use Blackwell ordering to deal with action dependent observation probabilities. For optimal search problems with 2 states, [81] proves the optimality of threshold policies under certain conditions. Other types of structural result for 2-state POMDPs are in [36, 11].

The POMDP multi-armed bandit structural result in §5.4 is from [69] where several numerical examples are presented; see also [53] for applications in radar. [127] and [35] are classic works in Bayesian multi-armed bandits. More generally, [74] establishes the optimality of indexable policies for restless bandit POMDPs with 2 states. (In a restless bandit, the state of idle projects also evolves.) §?? gives a short discussion on non-Bayesian bandits.
6 Structural Results for Stopping Time POMDPs

6.1 Introduction

The previous chapter established conditions under which the value function of a POMDP is monotone with respect to the MLR order. Also conditions were given for the optimal policy for a two-state POMDP to be monotone (threshold). This and the next chapter develop structural results for the optimal policy of multi-state POMDPs. To establish the structural results, we will use submodularity, and stochastic dominance on the lattice$^1$ of belief states to analyze Bellman’s dynamic programming equation – such analysis falls under the area of “Lattice Programming” [40]. Lattice programming and “monotone comparative statics” pioneered by Topkis [124] (see also [4, 6]) provide a general set of sufficient conditions for the existence of monotone strategies. Once a POMDP is shown to have a monotone policy, then gradient based algorithms that exploit this structure can be designed to estimate this policy. This and the next two chapters rely heavily on the structural results for filtering (Chapter 4) and monotone value function (Chapter 5). Please see Figure ?? on page ?? for the context of this chapter.

6.1.1 Main Results

This chapter deals with structural results for stopping time POMDPs. Stopping time POMDPs have action space $\mathcal{U} = \{1 \text{ (stop)}, 2 \text{ (continue) }\}$. They arise in sequential detection such as quickest change detection and machine replacement. Establishing structural results for stopping time POMDPs are easier than that for general POMDPs (which is considered in the next chapter). The main structural results in this chapter regarding stopping time POMDPs are:

1. Convexity of stopping region: §6.2 shows that the set of beliefs where it is optimal to apply action 1 (stop) is a convex subset of the belief space. This result unifies several well known results about the convexity of the stopping set for sequential detection problems.

$^1$A lattice is a partially ordered set (in our case belief space $\Pi(X)$) in which every two elements have a supremum and infimum (in our case with respect to the monotone likelihood ratio ordering). The appendix gives definitions of supermodularity on lattices.
2. **Monotonicity of the optimal policy:** §6.3 gives conditions under which the optimal policy of a stopping time POMDP is monotone with respect to the monotone likelihood ratio (MLR) order. The MLR order is naturally suited for POMDPs since it is preserved under conditional expectations.

![Figure 6.1](image)

Figure 6.1 Illustration of the two main structural results for stopping time POMDP established in this chapter. Theorem 6.2.1 shows that the stopping set (where action \( u = 1 \) is optimal) is convex. Theorem 6.3.4 shows that the optimal policy is increasing on any line from \( e_1 \) to \( (e_2, e_3) \) and decreasing on any line \( e_3 \) to \( (e_1, e_2) \). Therefore, the stopping set includes state \( 1 \) \( (e_1) \). Also the boundary of the stopping set \( \Gamma \) intersects any line from \( e_1 \) to \( (e_2, e_3) \) at most once; similarly for any line from \( e_3 \) to \( (e_1, e_2) \). Thus the set of beliefs where \( u = 2 \) is optimal is a connected set. Figure 6.1 shows several types of stopping sets that are excluded by Theorem 6.3.4.

Figure 6.1 displays these structural results. For \( X = 2 \), we will show that stopping set is the interval \([\pi^*, 1]\) and the optimal policy \( \mu^*(\pi) \) is a step function; see Figure 6.1(a)). So it is only necessary to compute the threshold state \( \pi^* \).

Most of this chapter is devoted to characterizing the optimal policy for stopping time POMDPs when \( X \geq 3 \). The main result shown is that under suitable conditions, the optimal policy for a stopping time POMDP is MLR increasing and therefore has a threshold switching “curve” (denoted by \( \Gamma \) in Figure 6.1(b)). So one only needs to estimate this curve \( \Gamma \), rather than solve a dynamic programming equation. We will show that the threshold curve \( \Gamma \) has useful properties: it can intersect any line from \( e_1 \) to the edge \( (e_2, e_3) \) only once. Similarly, it can intersect any line from \( e_3 \) to the edge \( (e_1, e_2) \) only once (These are the dashed lines in Figure 6.1(b)). It will be shown that the optimal MLR linear threshold policy, which approximates the curve \( \Gamma \), can then be estimated via simulation based stochastic approximation algorithms. Such a linear threshold policy is straightforward to implement in a real time POMDP controller.

§7? discusses structural results for POMDPs with multivariate observations. The multivariate TP2 stochastic order is used.

The structural results presented in this chapter provide a unifying theme and insight into what might otherwise simply be a collection of techniques and results in sequential detection. In Chapter 7 we will present several examples of stopping time POMDPs in sequential quickest change detection, multi-agent social learning and controlled measurement sampling.
6.2 Stopping Time POMDP and Convexity of Stopping Set

A stopping time POMDP has action space \( \mathcal{U} = \{1 \text{ (stop)}, 2 \text{ (continue)} \} \).

For continue action \( u = 2 \), the state \( x \in \mathcal{X} = \{1, 2, \ldots, X\} \) evolves with transition matrix \( P \) and is observed via observations \( y \) with observation probabilities \( B_{xy} = \mathbb{P}(y_k = y|x_k = x) \). An instantaneous cost \( c(x, u = 2) \) is incurred. Thus for \( u = 2 \), the belief state evolves according to the HMM filter \( \pi_k = T(\pi_{k-1}, y_k) \) defined in (2.11). Since action 1 is a stop action and has no dynamics, to simplify notation, we write \( T(\pi, y, 2) \) as \( T(\pi, y) \) and \( \sigma(\pi, y, 2) \) as \( \sigma(\pi, y) \) in this chapter.

The action 1 incurs a terminal cost of \( c(x, u = 1) \) and the problem terminates.

We consider the class of stationary policies

\[
\mu_k = \mu(\pi_k) \in \mathcal{U} = \{1 \text{ (stop)}, 2 \text{ (continue)} \}. \tag{6.1}
\]

Let \( \tau \) denote a stopping time adapted to \( \{\pi_k\}, k \geq 0 \). That is, with \( u_k \) determined by decision policy (6.1),

\[
\tau = \{ \inf k : u_k = 1 \}. \tag{6.2}
\]

Let \( \Pi(\pi) = \{\pi \in \mathbb{R}^\mathcal{X} : 1^t\pi = 1, \quad 0 \leq \pi(i) \leq 1 \text{ for all } i \in \mathcal{X}\} \) denote the belief space. For stationary policy \( \mu : \Pi(\pi) \to \mathcal{U} \), initial belief \( \pi_0 \in \Pi(\mathcal{X}) \), discount factor\(^2 \rho \in [0, 1] \), the discounted cost objective is

\[
J_{\mu}(\pi_0) = \mathbb{E}_\mu \left\{ \sum_{k=0}^{\tau-1} \rho^k c(x_k, 2) + \rho^\tau c(x_\tau, 1) \right\} = \mathbb{E}_\mu \left\{ \sum_{k=0}^{\tau-1} \rho^k c_k^2 \pi_k + \rho^\tau c^2_1 \pi_\tau \right\}. \tag{6.3}
\]

where \( c_k = [c(1, u), \ldots, c(X, u)]^t \). The aim is to determine the optimal stationary policy \( \mu^* : \Pi(\mathcal{X}) \to \mathcal{U} \) such that \( J_{\mu^*}(\pi_0) \leq J_{\mu}(\pi_0) \) for all \( \pi_0 \in \Pi(\mathcal{X}) \).

For the above stopping time POMDP, \( \mu^* \) is the solution of Bellman’s equation which is of the form\(^3 \) (where \( V(\pi) \) below denotes the value function):

\[
\mu^*(\pi) = \arg\min_{u \in \mathcal{U}} Q(\pi, u), \quad V(\pi) = \min_{u \in \mathcal{U}} Q(\pi, u), \tag{6.4}
\]

\[
Q(\pi, 1) = c^1_1 \pi, \quad Q(\pi, 2) = c^2_2 \pi + \rho \sum_{y \in \mathcal{Y}} V(T(\pi, y)) \sigma(\pi, y). \quad s
\]

where \( T(\pi, y) \) and \( \sigma(\pi, y) \) are the HMM filter and normalization (5.2).

6.2.1 Convexity of Stopping Region

We now present the first structural result for stopping time POMDPs: the stopping region for the optimal policy is convex. Define the stopping set \( \mathcal{R}_1 \) as the

\(^2\) In stopping time POMDPs we allow for \( \rho = 1 \) as well.

\(^3\) The stopping time POMDP can be expressed as an infinite horizon POMDP. Augment \( \Pi(\pi) \) to include the fictitious stopping state \( e_{X+1} \) which is cost free, i.e., \( c(e_{X+1}, u) = 0 \) for all \( u \in \mathcal{U} \). When decision \( u_k = 1 \) is chosen, the belief state \( \pi_{k+1} \) transitions to \( e_{X+1} \) and remains there indefinitely. Then (6.3) is equivalent to \( J_{\mu}(\pi) = \mathbb{E}_\mu \left\{ \sum_{k=0}^{\tau-1} \rho^k c^2_2 \pi_k + \rho^\tau c^2_1 \pi + \sum_{k=\tau+1}^{\infty} \rho^k c(e_{X+1}, u_k) \right\} \), where the last summation is zero.
set of belief states for which stopping \((u = 1)\) is the optimal action. Define \(R_2\) as the set of belief states for which continuing \((u = 2)\) is the optimal action. That is
\[
R_1 = \{ \pi : \mu^*(\pi) = 1 \text{ (stop)} \}, \quad R_2 = \{ \pi : \mu^*(\pi) = 2 \} = \Pi(X) - R_1. \quad (6.5)
\]

The theorem below shows that the stopping set \(R_1\) is convex (and therefore a connected set). Recall that the value function \(V(\pi)\) is concave on \(\Pi(X)\). (This essential property of POMDPs was proved in Theorem 2.4.1.)

**Theorem 6.2.1 ([76])** Consider the stopping-time POMDP with value function given by (6.4). Then the stopping set \(R_1\) is a convex subset of the belief space \(\Pi(X)\).

**Proof** Pick any two belief states \(\pi_1, \pi_2 \in R_1\). To demonstrate convexity of \(R_1\), we need to show for any \(\lambda \in [0, 1]\),
\[
V(\lambda \pi_1 + (1 - \lambda) \pi_2) \geq \lambda V(\pi_1) + (1 - \lambda) V(\pi_2)
\]
\[
= \lambda Q(\pi_1, 1) + (1 - \lambda) Q(\pi_2, 1) \quad \text{(since } \pi_1, \pi_2 \in R_1)\]
\[
= Q(\lambda \pi_1 + (1 - \lambda) \pi_2, 1) \quad \text{(since } Q_1(\pi, 1) \text{ is linear in } \pi)\]
\[
\geq V(\lambda \pi_1 + (1 - \lambda) \pi_2) \quad \text{(since } V(\pi) \text{ is the optimal value function)}
\]
Thus all the inequalities above are equalities, and \(\lambda \pi_1 + (1 - \lambda) \pi_2 \in R_1\). \(\square\)

Note that the theorem says nothing about the “continue” region \(R_2\). In Theorem 6.3.4 below we will characterize both \(R_1\) and \(R_2\). Figure 6.1 illustrates the assertion of Theorem 6.2.1 for \(X = 2\) and \(X = 3\).

**6.2.2 Example 1. Classical Quickest Change Detection**

Quickest detection is a useful example of a stopping time POMDP that has applications in biomedical signal processing, machine monitoring and finance [97, 9]. The classical Bayesian quickest detection problem is as follows: An underlying discrete-time state process \(x\) jumps changes at a geometrically distributed random time \(\tau^0\). Consider a sequence of random measurements \(\{y_k, k \geq 1\}\), such that conditioned on the event \(\{\tau^0 = t\}\), \(y_k, \{k \leq t\}\) are i.i.d. random variables with distribution \(B_{1t}\) and \(\{y_k, k > t\}\) are i.i.d. random variables with distribution \(B_{2y}\). The quickest detection problem involves detecting the change time \(\tau^0\) with minimal cost. That is, at each time \(k = 1, 2, \ldots, \), a decision \(u_k \in \{\text{continue, stop and announce change}\}\) needs to be made to optimize a tradeoff between false alarm frequency and linear delay penalty.\(^4\)

A geometrically distributed change time \(\tau^0\) is realized by a two state \((X = 2)\)

\(^4\)There are two general formulations for quickest time detection. In the first formulation, the change point \(\tau^0\) is an unknown deterministic time, and the goal is to determine a stopping rule such that a worst case delay penalty is minimized subject to a constraint on the false alarm frequency (see, e.g., [86, 96, 129, 97]). The second formulation, which is the formulation considered in this book (this chapter and also Chapter 7), is the Bayesian approach where the change time \(\tau^0\) is specified by a prior distribution.
Markov chain with absorbing transition matrix $P$ and prior $\pi_0$ as follows:

\[
P = \begin{bmatrix}
1 & 0 \\
1 - P_{22} & P_{22}
\end{bmatrix}, \quad \pi_0 = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad \tau^0 = \inf\{k : x_k = 1\}.
\] (6.6)

The system starts in state 2 and then jumps to the absorbing state 1 at time $\tau^0$. Clearly $\tau^0$ is geometrically distributed with mean $1/(1 - P_{22})$.

The cost criterion in classical quickest detection is the Kolmogorov–Shiryayev criterion for detection of disorder [111]

\[
J_\mu(\pi) = d \mathbb{E}_\mu \{(\tau - \tau^0)^+\} + \mathbb{P}_\mu(\tau < \tau^0), \quad \pi_0 = \pi.
\] (6.7)

where $\mu$ denotes the decision policy. The first term is the delay penalty in making a decision at time $\tau > \tau^0$ and $d$ is a positive real number. The second term is the false alarm penalty incurred in announcing a change at time $\tau < \tau^0$.

**Stopping time POMDP:** The quickest detection problem with penalty (6.7) is a stopping time POMDP with $U = \{1 \text{ (announce change and stop)}, 2 \text{ (continue)}\}$, $X = \{1, 2\}$, transition matrix in (6.6), arbitrary observation probabilities $B_{xy}$, cost vectors $c_1 = [0, 1]'$, $c_2 = [d, 0]'$ and discount factor $\rho = 1$.

Theorem 6.2.1 then implies the following structural result.

**COROLLARY 6.2.2**  The optimal policy $\mu^*$ for classical quickest detection has a threshold structure: There exists a threshold point $\pi^* \in [0, 1]$ such that

\[
u_k = \mu^*(\pi_k) = \begin{cases}
2 \quad \text{(continue)} & \text{if } \pi_k(2) \in [\pi^*, 1] \\
1 \quad \text{(stop and announce change)} & \text{if } \pi_k(2) \in [0, \pi^*].
\end{cases}
\] (6.8)

**Proof**  Since $X = 2$, $\Pi(X)$ is the interval $[0, 1]$, and $\pi(2) \in [0, 1]$ is the belief state. Theorem 6.2.1 implies that the stopping set $\mathcal{R}_1$ is convex. In one dimension this implies that $\mathcal{R}_1$ is an interval of the form $[a^*, \pi^*]$ for $0 \leq a < \pi^* \leq 1$. Since state 1 is absorbing, Bellman’s equation (6.4) with $\rho = 1$ applied at $\pi = \epsilon_1$ implies

\[
\mu^*(\epsilon_1) = \arg\min_u \{c(1, u = 1), \quad d(1 - \pi(2)) + V(\epsilon_1)\} = 1.
\]

So $\epsilon_1$ or equivalently $\pi(2) = 0$ belongs to $\mathcal{R}_1$. Therefore, $\mathcal{R}_1$ is an interval of the form $[0, \pi^*)$. Hence the optimal policy is of the form (6.8).

Theorem 6.2.1 says that for quickest detection of a multi-state Markov chain, the stopping set $\mathcal{R}_1$ is convex. (Recall $\mathcal{R}_1$ is the set of beliefs where $u = 1$ = stop is optimal.) What about the continuing set $\mathcal{R}_2 = \Pi(X) - \mathcal{R}_1$ where action $u = 2$ is optimal? For $X = 2$, using Corollary 6.2.2, $\mathcal{R}_2 = [\pi^*, 1]$ and is therefore convex. However, for $X > 2$, Theorem 6.2.1 does not say anything about the structure of $\mathcal{R}_2$; indeed, $\mathcal{R}_2$ could be a disconnected set. In §6.3, we will use more powerful POMDP structural results to give sufficient conditions for both $\mathcal{R}_1$ and $\mathcal{R}_2$ to be connected sets.
6.3 Monotone Optimal Policy for Stopping Time POMDP

We now consider the next major structural result: sufficient conditions to ensure that a stopping time POMDP has a monotone optimal policy.

Consider a stopping time POMDP with state space and action space

\[ \mathcal{X} = \{1, \ldots, X\}, \quad \mathcal{U} = \{1\text{ (stop)}, 2\text{ (continue)}\} \]

Action 2 implies continue with transition matrix \( P \), observation distribution \( B \) and cost \( C(\pi, 2) \), while action 1 denotes stop with stopping cost \( C(\pi, 1) \). So the model is almost identical to the previous section except that the costs \( C(\pi, u) \) are in general nonlinear functions of the belief. Recall such nonlinear costs were motivated by controlled sensing applications in Chapter ??.

In terms of the belief state \( \pi \), Bellman’s equation reads

\[
Q(\pi, u = 1) = C(\pi, 1), \quad Q(\pi, u = 2) = C(\pi, 2) + \rho \sum_y V(T(\pi, y, 2)) \sigma(\pi, y, 2),
\]

\[
V(\pi) = \min_{u \in \{1, 2\}} Q(\pi, u), \quad \mu^*(\pi) = \arg\min_{u \in \{1, 2\}} Q(\pi, u). \tag{6.9}
\]

6.3.1 Objective

One possible objective would be to give sufficient conditions on a stopping time POMDP so that the optimal policy \( \mu^*(\pi) \) is MLR increasing on \( \Pi(X) \). That is,

\[
\pi_1, \pi_2 \in \Pi(X), \quad \pi_1 \succeq_r \pi_2 \implies \mu^*(\pi_1) \geq \mu^*(\pi_2). \tag{6.10}
\]

However, because \( \Pi(X) \) is only partially orderable with respect to the MLR order, it is difficult to exploit (6.10) for devising useful algorithms. Instead, in this section, our aim is to give (less restrictive) conditions that lead to

\[
\pi_1, \pi_2 \in \mathcal{L}(e_i, \pi), \quad \pi_1 \succeq_r \pi_2 \implies \mu^*(\pi_1) \geq \mu^*(\pi_2), \quad i \in \{1, X\}. \tag{6.11}
\]

Here \( \mathcal{L}(e_i, \pi) \) denotes any line segment in \( \Pi(X) \) which starts at \( e_1 \) and ends at any belief \( \pi \) in the subsimplex \( \{e_2, \ldots, e_X\} \), or any line segment which starts at \( e_X \) and ends at any belief \( \pi \) in the subsimplex \( \{e_1, \ldots, e_{X-1}\} \). (These line segments are the dashed lines in Figure 6.1(b).) So instead of proving \( \mu^*(\pi) \) is MLR increasing for any two beliefs in the belief space, we will prove that \( \mu^*(\pi) \) is MLR increasing for any two beliefs on these special line segments \( \mathcal{L}(e_i, \pi) \).

The main reason is that the MLR order is a total order on such lines (not just a partial order) meaning that any two beliefs on \( \mathcal{L}(e_i, \pi) \) are MLR orderable.

Proving (6.11) yields in turn two very useful results:

1. The optimal policy \( \mu^*(\pi) \) of a stopping time POMDP is characterized by switching curve \( \Gamma \); see Theorem 6.3.4. This is illustrated in Figure 6.1(b).
2. The optimal linear approximation to switching curve \( \Gamma \) that preserves (6.11) can be estimated via a simulation based stochastic approximation algorithm thereby facilitating a simple controller; see §6.4.
6.3.2 MLR Dominance on Lines

Since our plan is to prove (6.11) on line segments in the belief space, we formally define these line segments. Define the sub-simplices, $\mathcal{H}_1$ and $\mathcal{H}_X$:

$$
\mathcal{H}_1 = \{ \pi \in \Pi(X) : \pi(1) = 0 \}, \quad \mathcal{H}_X = \{ \pi \in \Pi(X) : \pi(X) = 0 \}. 
$$ (6.12)

Denote a generic belief state that lies in either $\mathcal{H}_1$ or $\mathcal{H}_X$ by $\bar{\pi}$. For each such $\bar{\pi} \in \mathcal{H}_i$, $i \in \{1, X\}$, construct the line segment $\mathcal{L}(e_i, \bar{\pi})$ that connects $\bar{\pi}$ to $e_i$. Thus each line segment $\mathcal{L}(e_i, \bar{\pi})$ comprises of belief states $\pi$ of the form:

$$
\mathcal{L}(e_i, \bar{\pi}) = \{ \pi \in \Pi(X) : \pi = (1 - \epsilon)\bar{\pi} + \epsilon e_i, \; 0 \leq \epsilon \leq 1 \}, \; \bar{\pi} \in \mathcal{H}_i. 
$$ (6.13)

To visualize (6.12) and (6.13), Figure 6.2 illustrates the setup for $X = 3$. The sub-simplex $\mathcal{H}_1$ is simply the line segment $(e_2, e_3)$; and $\mathcal{H}_3$ is the line segment $\{e_1, e_2\}$. Also shown are examples of line segments $\mathcal{L}(e_1, \bar{\pi})$ and $\mathcal{L}(e_3, \bar{\pi})$ for arbitrary points $\bar{\pi}_1$ and $\bar{\pi}_2$ in $\mathcal{H}_1$ and $\mathcal{H}_3$.

We now define the MLR order on such line segments. Recall the definition of MLR order $\geq_r$ on the belief space $\Pi(X)$ in Definition 4.1.1 on page 41.

**Definition 6.3.1 (MLR ordering $\geq_{L_i}$ on lines)** $\pi_1$ is greater than $\pi_2$ with respect to the MLR ordering on the line $\mathcal{L}(e_i, \bar{\pi})$, $i \in \{1, X\}$ — denoted as $\pi_1 \geq_{L_i} \pi_2$, if $\pi_1, \pi_2 \in \mathcal{L}(e_i, \bar{\pi})$ for some $\bar{\pi} \in \mathcal{H}_i$, and $\pi_1 \geq_r \pi_2$.

Appendix 6.B shows that the partially ordered sets $[\mathcal{L}(e_1, \bar{\pi}), \geq_{L_X}]$ and $[\mathcal{L}(e_X, \bar{\pi}), \geq_{L_1}]$ are chains, i.e., totally ordered sets. All elements $\pi_1, \pi_2 \in \mathcal{L}(e_X, \bar{\pi})$ are comparable, i.e., either $\pi_1 \geq_{L_X} \pi_2$ or $\pi_2 \geq_{L_X} \pi_1$ (and similarly for $\mathcal{L}(e_1, \bar{\pi})$). The largest element (supremum) of $[\mathcal{L}(e_1, \bar{\pi}), \geq_{L_X}]$ is $\bar{\pi}$ and the smallest element (infimum) is $e_1$. 

---

**Figure 6.2** Examples of sub-simplices $\mathcal{H}_1$ and $\mathcal{H}_3$ and points $\bar{\pi}_1 \in \mathcal{H}_1$, $\bar{\pi}_2 \in \mathcal{H}_3$. Also shown are the lines $\mathcal{L}(e_1, \bar{\pi}_1)$ and $\mathcal{L}(e_3, \bar{\pi}_2)$ that connect these points to the vertices $e_1$ and $e_3$. 

---
6.3.3 Submodularity with MLR order

To prove the structural result (6.11), we will show that $Q(\pi, u)$ in (6.9) is a submodular function on the chains $[\mathcal{L}(e_1, \bar{\pi}), \geq L_1]$ and $[\mathcal{L}(e_1, \bar{\pi}), \geq L_X]$. This requires less restrictive conditions than submodularity on the entire simplex $\Pi(X)$.

**Definition 6.3.2 (Submodular function)** Suppose $i = 1$ or $X$. Then $f : \mathcal{L}(e_i, \bar{\pi}) \times \mathcal{U} \rightarrow \mathbb{R}$ is submodular if $f(\pi, u) - f(\bar{\pi}, u) \leq f(\bar{\pi}, u) - f(\bar{\pi}, \bar{u})$, for $u \leq \bar{u}, \pi \geq L_i, \bar{\pi}$.

A more general definition of submodularity on a lattice is given in Appendix 6.A on page 80. Also Appendix 6.B contains additional properties that will be used in proving the main theorem below.

The following key result says that for a submodular function $Q(\pi, u)$, there exists a version of the optimal policy $\mu^*(\pi) = \arg\min_u Q(\pi, u)$ that is MLR increasing on lines.

**Theorem 6.3.3 (Topkis Theorem)** Suppose $i = 1$ or $X$. If $f : \mathcal{L}(e_i, \bar{\pi}) \times \mathcal{U} \rightarrow \mathbb{R}$ is submodular, then there exists a $\mu^*(\pi) = \arg\min_{u \in \mathcal{U}} f(\pi, u)$, that is increasing on $[\mathcal{L}(e_i, \bar{\pi}), \geq L_i]$, i.e., $\pi^0 \geq L_i, \pi \Rightarrow \mu^*(\pi) \leq \mu^*(\pi^0)$.

6.3.4 Assumptions and Main Result

For convenience, we repeat (C) on page 58 and (F1), (F2) on page 45. The main additional assumption below is the submodularity assumption (S).

(C) $\pi_1 \geq_s \pi_2$ implies $C(\pi_1, u) \leq C(\pi_2, u)$ for each $u$.

For linear costs, the condition is: $c(x, u)$ is decreasing in $x$ for each $u$.

(F1) $B$ is totally positive of order 2 (TP2).

(F2) $P$ is totally positive of order 2 (TP2).

(S) $C(\pi, u)$ is submodular on $[\mathcal{L}(e_X, \bar{\pi}), \geq L_X]$ and $[\mathcal{L}(e_1, \bar{\pi}), \geq L_1]$.

For linear costs the condition is $c(x, 2) - c(x, 1) \geq c(X, 2) - c(X, 1)$ and $c(1, 2) - c(1, 1) \geq c(x, 2) - c(x, 1)$.

**Theorem 6.3.4 (Switching Curve Optimal Policy)** Assume (C), (F1), (F2) and (S) hold for a stopping time POMDP. Then:

1. There exists an optimal policy $\mu^*(\pi)$ that is $\geq L_X$ increasing on lines $\mathcal{L}(e_X, \bar{\pi})$ and $\geq L_1$ increasing on lines $\mathcal{L}(e_1, \bar{\pi})$.

2. Hence there exists a threshold switching curve $\Gamma$ that partitions belief space $\Pi(X)$ into two individually connected regions $\mathcal{R}_1, \mathcal{R}_2$, such that the optimal policy is

$$
\mu^*(\pi) = \begin{cases} 
\text{continue} = 2 & \text{if } \pi \in \mathcal{R}_2 \\
\text{stop} = 1 & \text{if } \pi \in \mathcal{R}_1
\end{cases}
$$

(6.14)

The threshold curve $\Gamma$ intersects each line $\mathcal{L}(e_X, \bar{\pi})$ and $\mathcal{L}(e_1, \bar{\pi})$ at most once.

3. There exists $i^* \in \{0, \ldots, X\}$, such that $e_1, \ldots, e_{i^*} \in \mathcal{R}_1$ and $e_{i^*+1}, \ldots, e_X \in \mathcal{R}_2$.

4. For the case $X = 2$, there exists a unique threshold point $\pi^*(2)$.

\[A set is connected if it cannot be expressed as the union of disjoint nonempty closed sets [107].]
Let us explain the intuition behind the proof of the theorem. As shown in Theorem 5.2.1 on page 59, (C), (F1) and (F2) are sufficient conditions for the value function $V(\pi)$ to be MLR decreasing in $\pi$.

(S) is sufficient for the costs $c'_{u}\pi$ to be submodular on lines $\mathcal{L}(e_{X}, \bar{\pi})$ and $\mathcal{L}(e_{1}, \bar{\pi})$. Finally (C),(F1) and (S) are sufficient for $Q(\pi, u)$ to be submodular on lines $\mathcal{L}(e_{X}, \bar{\pi})$ and $\mathcal{L}(e_{1}, \bar{\pi})$.

As a result, Topkis Theorem 6.3.3 implies that the optimal policy is monotone on each chain $[\mathcal{L}(e_{X}, \bar{\pi}), \geq L_{X}]$. So there exists a threshold belief state on each line $\mathcal{L}(e_{X}, \bar{\pi})$ where the optimal policy switches from 1 to 2. (A similar argument holds for lines $[\mathcal{L}(e_{1}, \bar{\pi}), \geq L_{1}]$).

The entire simplex $\Pi(X)$ can be covered by the union of lines $\mathcal{L}(e_{X}, \bar{\pi})$. The union of the resulting threshold belief states yields the switching curve $\Gamma(\bar{\pi})$. This is illustrated in Figure 6.3.

### 6.4 Characterization of Optimal Linear Decision Threshold for Stopping time POMDP

In this section, we assume (C), (F1), (F2) and (S) hold. Therefore Theorem 6.3.4 applies and the optimal policy $\mu^*(\pi)$ is characterized by a switching curve $\Gamma$ as illustrated in Figure 6.3.

How can the switching curve $\Gamma$ be estimated (computed)? In general, any
user-defined basis function approximation can be used to parametrize this curve. However, such parametrized policy needs to capture the essential feature of Theorem 6.3.4: it needs to be MLR increasing on lines.

6.4.1 Linear Threshold Policies

We derive the optimal linear approximation to the switching curve $\Gamma$ on simplex $\Pi(X)$. Such a linear decision threshold has two attractive properties: (i) Estimating it is computationally efficient. (ii) We give conditions on the coefficients of the linear threshold that are necessary and sufficient for the resulting policy to be MLR increasing on lines. Due to the necessity and sufficiency of the condition, optimizing over the space of linear thresholds on $\Pi(X)$ yields the "optimal" linear approximation to switching curve $\Gamma$.

Since $\Pi(X)$ is a subset of $\mathbb{R}^{X-1}$, a linear hyperplane on $\Pi(X)$ is parametrized by $X-1$ coefficients. So, on $\Pi(X)$, define the linear threshold policy $\mu_\theta(\pi)$ as

$$
\mu_\theta(\pi) = \begin{cases} 
\text{stop} = 1 & \text{if } \left[\begin{array}{c} 0 \\ 1 \\ \theta \end{array}\right]^T \begin{bmatrix} \pi \\ -1 \end{bmatrix} < 0 \\
\text{continue} = 2 & \text{otherwise}
\end{cases} \quad \pi \in \Pi(X). \tag{6.15}
$$

Here $\theta = (\theta(1), \ldots, \theta(X-1))^T \in \mathbb{R}^{X-1}$ denotes the parameter vector of the linear threshold policy.

Theorem 6.4.1 below characterizes the optimal linear decision threshold approximation to the threshold switching curve on $\Pi(X)$. Assume conditions (C), (F1) and (S) hold so that Theorem 6.3.4 holds. Also assuming that the stopping region $R_1$ is non empty, then $\epsilon_1$ lies in the stopping set. This implies that the $(X-1)$-th component of $\theta$ satisfies $\theta(X-1) > 0$.

**Theorem 6.4.1 (Optimal Linear Threshold Policy)** For belief states $\pi \in \Pi(X)$, the linear threshold policy $\mu_\theta(\pi)$ defined in (6.15) is
(i) MLR increasing on lines $\mathcal{L}(e_X, \bar{\pi})$ iff $\theta(X - 2) \geq 1$ and $\theta(i) \leq \theta(X - 2)$ for $i < X - 2$.
(ii) MLR increasing on lines $\mathcal{L}(e_1, \bar{\pi})$ iff $\theta(i) \geq 0$, for $i < X - 2$. \hfill $\square$

**Proof** Given any $\pi_1, \pi_2 \in \mathcal{L}(e_X, \bar{\pi})$ with $\pi_2 \geq_{L_X} \pi_1$, we need to prove: $\mu_\theta(\pi_1) \leq \mu_\theta(\pi_2)$ iff $\theta(X - 2) \geq 1$, $\theta(i) \leq \theta(X - 2)$ for $i < X - 2$. But from the structure of (6.15), obviously $\mu_\theta(\pi_1) \leq \mu_\theta(\pi_2)$ is equivalent to $\left[\begin{array}{c} 0 \\ 1 \\ \theta \end{array}\right]^T \begin{bmatrix} \pi_1 \\ -1 \end{bmatrix} \leq \left[\begin{array}{c} 0 \\ 1 \\ \theta \end{array}\right]^T \begin{bmatrix} \pi_2 \\ -1 \end{bmatrix}$, or equivalently,

$$
\left[\begin{array}{c} 0 \\ 1 \\ \theta(1) \end{array}\right] \left[\begin{array}{c} \pi_1 \\ -1 \end{array}\right] + \left[\begin{array}{c} 1 \\ \theta(2) \end{array}\right] \left[\begin{array}{c} \pi_2 \\ -1 \end{array}\right] \leq 0.
$$

Now from Lemma 6.6.3(i), $\pi_2 \geq_{L_X} \pi_1$ implies that $\pi_1 = \epsilon_1 e_X + (1 - \epsilon_1)\bar{\pi}$, $\pi_2 = \epsilon_2 e_X + (1 - \epsilon_2)\bar{\pi}$ and $\epsilon_1 \leq \epsilon_2$. Substituting these into the above expression,
we need to prove
\[(\epsilon_1 - \epsilon_2)(\theta(X - 2) - [0 1 \theta(1) \cdots \theta(X - 2)]\mathcal{H}_X), \quad \forall \mathcal{H}_X \]
iff \(\theta(X - 2) \geq 1, \theta(i) \leq \theta(X - 2), i < X - 2\). This is obviously true.

A similar proof shows that on lines \(L(e_1, \mathcal{H}_X)\) the linear threshold policy satisfies \(\mu_\theta(\pi_1) \leq \mu_\theta(\pi_2)\) iff \(\theta(i) \geq 0\) for \(i < X - 2\).

As a consequence of Theorem 6.4.1, the optimal linear threshold approximation to switching curve \(\Gamma\) of Theorem 6.3.4 is the solution of the following constrained optimization problem:

\[\theta^* = \arg \min_{\theta \in \mathbb{R}^X} J_{\mu_\theta}(\pi), \quad \text{subject to } 0 \leq \theta(i) \leq \theta(X - 2), \theta(X - 2) \geq 1 \text{ and } \theta(X - 1) > 0\]

where the cumulative cost \(J_{\mu_\theta}(\pi)\) is obtained as in (5.1) by applying threshold policy \(\mu_\theta\) in (6.15).

**Remark:** The constraints in (6.16) are necessary and sufficient for the linear threshold policy (6.15) to be MLR increasing on lines \(L(e_X, \mathcal{H}_X)\) and \(L(e_1, \mathcal{H}_X)\). That is, (6.16) defines the set of all MLR increasing linear threshold policies — it does not leave out any MLR increasing polices; nor does it include any non MLR increasing policies. Therefore optimizing over the space of MLR increasing linear threshold policies yields the optimal linear approximation to threshold curve \(\Gamma\).

### 6.4.2 Algorithm to compute the optimal linear threshold policy

In this section a stochastic approximation algorithm is presented to estimate the threshold vector \(\theta^*\) in (6.16). Because the cumulative cost \(J_{\mu_\theta}(\pi)\) in (6.16) of the linear threshold policy \(\mu_\theta\) cannot be computed in closed form, we resort to simulation based stochastic optimization. Let \(n = 1, 2, \ldots\), denote iterations of the algorithm. The aim is to solve the following linearly constrained stochastic optimization problem:

\[\text{Compute } \theta^* = \arg \min_{\theta \in \mathbb{R}^X} \mathbb{E}\{J_n(\mu_\theta)\}\]

subject to \(0 \leq \theta(i) \leq \theta(X - 2), \theta(X - 2) \geq 1 \text{ and } \theta(X - 1) > 0\).  

Here the sample path cumulative cost \(J_n(\mu_\theta)\) is evaluated as

\[J_n(\mu_\theta) = \sum_{k=0}^{\infty} \rho^k C(\pi_k, u_k), \quad \text{where } u_k = \mu_\theta(\pi_k) \text{ is computed via (6.15)}\]

with prior \(\pi_0\) sampled uniformly from \(\Pi(X)\). A convenient way of sampling uniformly from \(\Pi(X)\) is to use the Dirichlet distribution (i.e., \(\pi_0(i) = x_i / \sum_i x_i\), where \(x_i \sim \text{unit exponential distribution}\).

The above constrained stochastic optimization problem can be solved by a variety of methods. One method is to convert it into an equivalent unconstrained
problem via the following parametrization: Let \( \phi = (\phi(1), \ldots, \phi(X - 1))' \in \mathbb{R}^{X-1} \) and parametrize \( \theta \) as \( \theta^\phi = [\theta^\phi(1), \ldots, \theta^\phi(X - 1)]' \) where

\[
\theta^\phi(i) = \begin{cases} 
\phi^2(X - 1) & \text{if } i = X - 1 \\
1 + \phi^2(X - 2) & \text{if } i = X - 2 \\
(1 + \phi^2(X - 2)) \sin^2(\phi(i)) & \text{if } i = 1, \ldots, X - 3
\end{cases}
\]

(6.19)

Then \( \theta^\phi \) trivially satisfies constraints in (6.17). So (6.17) is equivalent to the following unconstrained stochastic optimization problem:

Compute \( \mu^\phi^*(\pi) \) where \( \phi^* = \arg \min_{\phi \in \mathbb{R}^{X-1}} \mathbb{E}\{J_n(\phi)\} \) and \( J_n(\phi) \) is computed using (6.20) with policy \( \mu^\phi^*(\pi) \) in (6.19).

(6.20)

Algorithm 3 uses the SPSA Algorithm to generate a sequence of estimates \( \hat{\phi}_n \), \( n = 1, 2, \ldots \), that converges to a local minimum of the optimal linear threshold \( \phi^* \) with policy \( \mu^\phi^*(\pi) \).

**Algorithm 3** Policy Gradient SPSA Algorithm for computing optimal linear threshold policy

Assume (C), (F1), (F2), (S) hold so that the optimal policy is characterized by a switching curve in Theorem 6.3.4.

Step 1: Choose initial threshold coefficients \( \hat{\phi}_0 \) and linear threshold policy \( \mu^\phi_0 \).

Step 2: For iterations \( n = 0, 1, 2, \ldots \)

- Evaluate sample cumulative cost \( J_n(\hat{\phi}_n) \) using (6.20).

- Update threshold coefficients \( \hat{\phi}_n \) via SPSA Algorithm as

\[
\hat{\phi}_{n+1} = \hat{\phi}_n - \epsilon_{n+1} \nabla_{\phi} J_n(\hat{\phi}_n)
\]

(6.21)

The stochastic gradient algorithm (6.21) converges to a local optimum. So it is necessary to try several initial conditions \( \hat{\phi}_0 \). The computational cost at each iteration is linear in the dimension of \( \hat{\phi} \).

6.5 Example. Machine Replacement POMDP

We continue here with the machine replacement example discussed in §?? and §2.3. We show that the conditions of Theorem 6.3.4 hold and so the optimal policy for machine replacement is characterized by a threshold switching curve.

Recall the state space is \( \mathcal{X} = \{1, 2, \ldots, X\} \) where state \( X \) denotes the best state (brand new machine) while state 1 denotes the worst state, and the action space is \( \mathcal{U} = \{1 \text{ (replace)}, 2 \text{ (continue)}\} \). Consider an infinite horizon discounted cost
version of the problem. Bellman’s equation reads
\[ \mu^*(\pi) = \arg\min_{u \in \mathcal{U}} Q(\pi, u), \quad V(\pi) = \min_{u \in \mathcal{U}} Q(\pi, u) \]
\[ Q(\pi, 1) = R + \rho V(e_X), \quad Q(\pi, 2) = c_2^\pi + \rho \sum_{y \in \mathcal{Y}} V(T(\pi, y)) \sigma(\pi, y). \] (6.22)

Since every time action \( u = 1 \) (replace) is chosen, the belief state switches to \( e_X \), Bellman’s equation (6.22) is similar to that of a stopping POMDP. The cost of operating the machine \( c(x, u = 2) \) is decreasing in state \( x \) since the smaller \( x \) is, the higher the cost incurred due to loss of productivity. So (C) holds. The transition matrix \( P(2) \) defined in (??) satisfies (F2). Assume that the observation matrix \( B \) satisfies (F1). Finally, since \( c(x, 2) \) is decreasing in \( x \) and \( c(x, 1) = R \) is independent of \( x \), it follows that \( c(x, u) \) is submodular and so (S) holds. Then from Theorem 6.3.4, the optimal policy is MLR increasing and characterized by a threshold switching curve. Also from Theorem 6.2.1, the set of beliefs \( \mathcal{R}_1 \) where it is optimal to replace the machine is convex. Since the optimal policy is MLR increasing, if \( \mathcal{R}_1 \) is non-empty, then \( e_1 \in \mathcal{R}_1 \). Algorithm 3 can be used to estimate the optimal linear parametrized policy that is MLR increasing.

**Appendix 6.A  Lattices and Submodularity**

Definition 6.3.2 on page 75 on submodularity suffices for our treatment of POMDPs. Here we outline a more abstract definition; see [124] for details.

(i) **Poset:** Let \( \Omega \) denote a nonempty set and \( \preceq \) denote a binary relation. Then \((\Omega, \preceq)\) is a partially ordered set (poset) if for any elements \( a, b, c \in \Omega \), the following hold:
1. \( a \preceq a \) (reflexivity)
2. if \( a \preceq b \) and \( b \preceq a \), then \( a = b \) (anti-symmetry)
3. if \( a \preceq b \) and \( b \preceq c \), then \( a \preceq c \) (transitivity).

For a POMDP, clearly \((\Pi(X), \leq_r)\) is a poset, where
\[ \Pi(X) = \{ \pi \in \mathbb{R}^X : 1')\pi = 1, \quad 0 \leq \pi(i) \leq 1 \text{ for all } i \in \mathcal{X} \} \]
is the belief space and \( \leq_r \) is the MLR order defined in (4.2).

(ii) **Lattice:** A poset \((\Omega, \preceq)\) is called a lattice if the following property holds:
\( a, b \in \Omega \), then \( a \lor b \overset{\text{defn}}{=} \max\{a, b\} \in \Omega \) and \( a \land b \overset{\text{defn}}{=} \min\{a, b\} \in \Omega \). (Here \( \min \) and \( \max \) are with respect to partial order \( \preceq \).)

Clearly, \((\Pi(X), \leq_r)\) is a lattice. Indeed if two beliefs \( \pi_1, \pi_2 \in \Pi(X) \), then if \( \pi_1 \leq_r \pi_2 \), obviously \( \pi_1 \lor \pi_2 = \pi_2 \) and \( \pi_1 \land \pi_2 = \pi_1 \) belong to \( \Pi(X) \). If \( \pi_1 \) and \( \pi_2 \) are not MLR comparable, then \( \pi_1 \lor \pi_2 = e_X \) and \( \pi_1 \land \pi_2 = e_1 \), where \( e_i \) is the unit vector with 1 in the \( i \)-th position.

Note that \( \Omega = \{ e_1, e_2, e_3 \} \) is not a lattice if one uses the natural element wise ordering. Clearly, \( e_1 \lor e_2 = (1, 1, 0) \notin \Omega \) and \( e_1 \land e_2 = (0, 0, 0) \notin \Omega \).
Finally, \( \Pi(X) \times \{1, 2, \ldots, U\} \) is also a lattice. This is what we use in our POMDP structural results.

(iii) **Submodular function:** Let \((\Omega, \leq)\) be a lattice and \(f: \Omega \to \mathbb{R}\). Then \(f\) is submodular if for all \(a, b \in \Omega\),
\[
 f(a) + f(b) \geq f(a \lor b) + f(a \land b). \tag{6.23}
\]

For the two component case, namely \(a = (\pi_1, u + 1), b = (\pi_2, u)\), with \(\pi_1 \leq_r \pi_2\), clearly Definition (6.23) is equivalent to Definition 6.3.2 on page 75; and this suffices for our purposes. When each of \(a\) and \(b\) consist of more than two components, the proof needs more work; see [124].

### Appendix 6.B  MLR Dominance and Submodularity on Lines

Recall the definition of \(\mathcal{L}(e_i, \bar{\pi})\) in (6.13).

**Lemma 6.B.1** The following properties hold on \([\Pi(X), \geq_r], [\mathcal{L}(e_i, \bar{\pi}), \geq_{L_X}]\).

(i) On \([\Pi(X), \geq_r], e_1\) is the least and \(e_X\) is the greatest element. On \([\mathcal{L}(e_i, \bar{\pi}), \geq_{L_X}]\), \(\bar{\pi}\) is the least and \(e_X\) is the greatest element and all points are MLR orderable.

(ii) Convex combinations of MLR comparable belief states form a chain. For any \(\gamma \in [0, 1], \pi \leq_r \pi^0 \Longrightarrow \pi \leq_r \gamma \pi + (1 - \gamma) \pi^0 \leq_r \pi^0\).

**Lemma 6.B.2** (i) For \(i \in \{1, X\}, \pi_i \geq_{L_i} \pi_2\) is equivalent to \(\pi_j = (1 - e_j)\bar{\pi} + e_j \pi_X\) and \(e_1 \geq e_2\) for \(\bar{\pi} \in \mathcal{H}_i\), where \(\mathcal{H}_i\) is defined in (6.12).

(ii) So submodularity on \(\mathcal{L}(e_i, \bar{\pi}), i \in \{1, X\}\), is equivalent to showing
\[
\pi^\epsilon = (1 - \epsilon)\bar{\pi} + \epsilon e_i \implies C(\pi^\epsilon, 2) - C(\pi^\epsilon, 1) \text{ decreasing w.r.t. } \epsilon \tag{6.24}
\]

The proof of Lemma 6.B.2 follows from Lemma 6.B.1 and is omitted.

As an example motivated by controlled sensing, consider costs that are quadratic in the belief. Suppose \(C(\pi, 2) - C(\pi, 1)\) is of the form \(\phi'\pi + \alpha (h' \pi)^2\). Then from (6.24), sufficient conditions for submodularity on \(\mathcal{L}(e_X, \bar{\pi})\) and \(\mathcal{L}(e_i, \bar{\pi})\) are for \(\bar{\pi} \in \mathcal{H}_X\) and \(\mathcal{H}_i\), respectively,
\[
\phi_X - \phi' \bar{\pi} + 2\alpha h' \pi^\epsilon (h_X - h' \bar{\pi}) \leq 0, \quad \phi_1 - \phi' \bar{\pi} + 2\alpha h' \pi^\epsilon (h_1 - h' \bar{\pi}) \geq 0 \tag{6.25}
\]
If \(h_i \geq 0\) and increasing or decreasing in \(i\), then (6.25) is equivalent to
\[
\phi_X - \phi' \bar{\pi} + 2\alpha h_X (h_X - h' \bar{\pi}) \leq 0, \quad \phi_1 - \phi' \bar{\pi} + 2\alpha h_X (h_1 - h' \bar{\pi}) \geq 0 \tag{6.26}
\]
where \(\bar{\pi} \in \mathcal{H}_X\) and \(\bar{\pi} \in \mathcal{H}_i\), respectively.

### Appendix 6.C  Proof of Theorem 6.3.4

**Part 1:** Establishing that \(Q(\pi, u)\) is submodular, requires showing that \(Q(\pi, 1) - Q(\pi, 2)\) is \(\geq_r\), on lines \(\mathcal{L}(e_X, \bar{\pi})\) for \(i = 1\) and \(X\). Theorem 5.2.1 shows by induction that for each \(k\), \(V_k(\pi)\) is \(\geq_r\) decreasing on \(\Pi(X)\) if (C), (F1), (F2) hold. This
implies that $V_k(\pi)$ is $\geq L_1$, decreasing on lines $\mathcal{L}(e_X, \bar{\pi})$ and $\mathcal{L}(e_1, \bar{\pi})$. So to prove $Q_k(\pi, u)$ in (6.9) is submodular, we only need to show that $C(\pi, 1) - C(\pi, 2)$ is $\geq L_1$, decreasing on $\mathcal{L}(e_i, \bar{\pi})$, $i = 1, X$. But this is implied by (S). Since submodularity is closed under pointwise limits [124, Lemma 2.6.1 and Corollary 2.6.1], it follows that $Q(\pi, u)$ is submodular on $\geq L_1$, $i = 1, X$ Having established $Q(\pi, u)$ is submodular on $\geq L_1$, $i = 1, X$, Theorem 6.3.3 implies that the optimal policy $\mu^*(\pi)$ is $\geq L_1$, and $\geq L_X$ increasing on lines.

**Part 2(a) Characterization of Switching Curve $\Gamma$.** For each $\bar{\pi} \in \mathcal{H}_X$ (6.13), construct the line segment $\mathcal{L}(e_X, \bar{\pi})$ connecting $\mathcal{H}_X$ to $e_X$ as in (6.13). By Lemma 6.B.1, on the line segment connecting $(1 - \epsilon)\bar{\pi} + \epsilon e_X$, all belief states are MLR orderable. Since $\mu^*(\pi)$ is monotone increasing for $\pi \in \mathcal{L}(e_X, \bar{\pi})$, moving along this line segment towards $e_X$, pick the largest $\epsilon$ for which $\mu^*(\pi) = 1$. (Since $\mu^*(e_X) = 1$, such an $\epsilon$ always exists). The belief state corresponding to this $\epsilon$ is the threshold belief state. Denote it by $\Gamma(\bar{\pi}) = \pi^{\epsilon, \bar{\pi}} \in \mathcal{L}(e_X, \bar{\pi})$ where $\epsilon^* = \sup\{\epsilon \in [0, 1] : \mu^*(\pi^{\epsilon, \bar{\pi}}) = 1\}$.

The above construction implies that on $\mathcal{L}(e_X, \bar{\pi})$, there is a unique threshold point $\Gamma(\bar{\pi})$. Note that the entire simplex can be covered by considering all pairs of lines $\mathcal{L}(e_X, \bar{\pi})$, for $\bar{\pi} \in \mathcal{H}_X$, i.e., $\Pi(X) = \bigcup_{\bar{\pi} \in \mathcal{H}_X} \mathcal{L}(e_X, \bar{\pi})$. Combining all points $\Gamma(\bar{\pi})$ for all pairs of lines $\mathcal{L}(e_X, \bar{\pi})$, $\bar{\pi} \in \mathcal{H}_X$, yields a unique threshold curve in $\Pi(X)$ denoted $\Gamma = \bigcup_{\bar{\pi} \in \mathcal{H}_X} \Gamma(\bar{\pi})$.

**Part 2(b) Connectedness of $R_1$:** Since $e_1 \in R_1$, call $R_{1a}$ the subset of $R_1$ that contains $e_1$. Suppose $R_{1b}$ was a subset of $R_1$ that was disconnected from $R_{1a}$. Recall that every point in $\Pi(X)$ lies on a line segment $\mathcal{L}(e_1, \bar{\pi})$ for some $\bar{\pi}$. Then such a line segment starting from $e_1 \in R_{1a}$ would leave the region $R_{1a}$, pass through a region where action 2 was optimal, and then intersect the region $R_{1b}$ where action 1 is optimal. But this violates the requirement that $\mu(\pi)$ is increasing on $\mathcal{L}(e_1, \bar{\pi})$. Hence $R_{1a}$ and $R_{1b}$ have to be connected.

**Connectedness of $R_2$:** Assume $e_X \in R_2$, otherwise $R_2 = \emptyset$ and there is nothing to prove. Call the region $R_2$ that contains $e_X$ as $R_{2a}$. Suppose $R_{2b} \subset R_2$ is disconnected from $R_{2a}$. Since every point in $\Pi(X)$ can be joined by the line segment $\mathcal{L}(e_X, \bar{\pi})$ to $e_X$. Then such a line segment starting from $e_X \in R_{2a}$ would leave the region $R_{2a}$, pass through a region where action 1 was optimal, and then intersect the region $R_{2b}$ (where action 2 is optimal). This violates the property that $\mu(\pi)$ is increasing on $\mathcal{L}(e_X, \bar{\pi})$. Hence $R_{2a}$ and $R_{2b}$ are connected.

**Part 3:** Suppose $e_i \in R_1$. Then considering lines $\mathcal{L}(e_i, \bar{\pi})$ and ordering $\geq L_1$, it follows that $e_{i-1} \in R_1$. Similarly if $e_i \in R_2$, then considering lines $\mathcal{L}(e_{i+1}, \bar{\pi})$ and ordering $\geq L_1$, it follows that $e_{i+1} \in R_2$.

**Part 4** follows trivially since for $X = 2$, $\Pi(X)$ is a one dimensional simplex.
Chapter 6 presented three structural results for stopping time POMDPs: convexity of the stopping region (for linear costs), the existence of a threshold switching curve for the optimal policy (under suitable conditions) and characterization of the optimal linear threshold policy. This chapter discusses several examples of stopping time POMDPs in quickest change detection. We will show that for these examples, convexity of the stopping set and threshold optimal policies arise naturally. Therefore, the structural results of Chapter 6 serve as a unifying theme and give insight into what might otherwise be considered as a collection of sequential detection methods.

7.1 Example 1. Quickest Detection with Phase-Distributed Change Time and Variance Penalty

Here we formulate quickest detection of a phase-distributed change time as a stopping time POMDP and analyze the structure of the optimal policy. The reader should review §6.2.2 where the classical quickest detection problem with geometric change times was discussed. We will consider two generalizations of the classical quickest detection problem: phase-type (PH) distributed change times and variance stopping penalty.

PH-distributions include geometric distributions as a special case and are used widely in modelling discrete event systems [89]. The optimal detection of a PH-distributed change point is useful since the family of PH-distributions forms a dense subset for the set of all distributions. As described in [89], a PH-distributed change time can be modelled as a multi-state Markov chain with an absorbing state. So the space of public belief states \( \pi \) now is a multidimensional simplex of probability mass functions. We will formulate the problem as a stopping time POMDP and characterize the optimal decision policy.

The second generalization we consider is a stopping penalty comprising of the false alarm and a variance penalty. The variance penalty is essential in stopping problems where one is interested in ultimately estimating the state \( x \). It penalizes stopping too soon if the uncertainty of the state estimate is large.\(^1\)

\(^1\)In [8], a continuous time stochastic control problem is formulated with a quadratic stopping cost, and the existence of the solution to the resulting quasi-variational inequality is proved.
The variance penalty results in a stopping cost that is quadratic in the belief state $\pi$.

### 7.1.1 Formulation of Quickest Detection as a Stopping Time POMDP

Below we formulate the quickest detection problem with PH-distributed change time and variance penalty as a stopping time POMDP. We can then use the structural results of Chapter 6 to characterize the optimal policy.

#### Transition Matrix for PH distributed change time

The change point $\tau^0$ is modeled by a phase type (PH) distribution. The family of all PH-distributions forms a dense subset for the set of all distributions [89] i.e., for any given distribution function $F$ such that $F(0) = 0$, one can find a sequence of PH-distributions $\{F_n, n \geq 1\}$ to approximate $F$ uniformly over $[0, \infty)$. Thus PH-distributions can be used to approximate change points with an arbitrary distribution. This is done by constructing a multi-state Markov chain as follows:

- Assume state ‘1’ (corresponding to belief $e_1$) is an absorbing state and denotes the state after the jump change.
- The states $2, \ldots, X$ (corresponding to beliefs $e_2, \ldots, e_X$) can be viewed as a single composite state that $x$ resides in before the jump.
- To avoid trivialities, assume that the change occurs after at least one measurement. So the initial distribution $\pi_0$ satisfies $\pi_0(1) = 0$. The transition probability matrix is of the form

$$P = \begin{bmatrix} 1 & 0 \\ \tilde{P}(X-1) \times 1 & \tilde{P}(X-1) \times (X-1) \end{bmatrix}. \quad (7.1)$$

The “change time” $\tau^0$ denotes the time at which $x_k$ enters the absorbing state 1:

$$\tau^0 = \min\{k : x_k = 1\}. \quad (7.2)$$

The distribution of $\tau^0$ is determined by choosing the transition probabilities $\tilde{P}, \tilde{P}$ in (7.1). To ensure that $\tau^0$ is finite, assume states 2, 3, ..., $X$ are transient. This is equivalent to $\tilde{P}$ satisfying $\sum_{n=1}^{\infty} \tilde{P}_{ii} < \infty$ for $i = 1, \ldots, X - 1$ (where $\tilde{P}_{ii}$ denotes the $(i, i)$ element of the $n$-th power of matrix $\tilde{P}$). The distribution of the absorption time to state 1 is

$$\nu_0 = \pi_0(1), \quad \nu_k = \tilde{\pi}_0^k \tilde{P}^{k-1} \pi_k, \quad k \geq 1, \quad (7.3)$$

where $\tilde{\pi}_0 = [\pi_0(2), \ldots, \pi_0(X)]'$. The key idea is that by appropriately choosing the pair $(\pi_0, P)$ and the associated state space dimension $X$, one can approximate any given discrete distribution on $[0, \infty)$ by the distribution $\{\nu_k, k \geq 0\}$; see [89, pp.240-243]. The event $\{x_k = 1\}$ means the change point has occurred at time $k$ according to PH-distribution (7.3). In the special case when $x$ is a 2-state Markov chain, the change time $\tau^0$ is geometrically distributed.
7.1 Example 1. Quickest Detection with Phase-Distributed Change Time and Variance Penalty

**Observations**
The observation \( y_k \in \mathcal{Y} \) given state \( x_k \) has conditional probability pdf or pmf
\[
B_{xy} = p(y_k = y|x_k = x), \quad x \in \mathcal{X}, y \in \mathcal{Y}.
\]
where \( \mathcal{Y} \subset \mathbb{R} \) (in which case \( B_{xy} \) is a pdf) or \( \mathcal{Y} = \{1, 2, \ldots, Y\} \) (in which case \( B_{xy} \) is a pmf). In quickest detection, states 2, 3, \ldots, \( X \) are fictitious and are defined to generate the PH-distributed change time \( \tau^0 \) in (7.2). So states 2, 3, \ldots, \( X \) are indistinguishable in terms of the observation \( y \). That is, the observation probabilities \( B \) in (7.4) satisfy
\[
B_{2y} = B_{3y} = \cdots = B_{Xy} \text{ for all } y \in \mathcal{Y}.
\]

**Actions**
At each time \( k \), a decision \( u_k \) is taken where
\[
u_k = \mu(\pi_k) \in \mathcal{U} = \{1 \text{ (announce change and stop)}, 2 \text{ (continue)} \}.
\]
In (7.6), the policy \( \mu \) belongs to the class of stationary decision policies.

**Stopping Cost**
If decision \( u_k = 1 \) is chosen, then the decision maker announces that a change has occurred and the problem terminates. If \( u_k = 1 \) is chosen before the change point \( \tau^0 \), then a false alarm and variance penalty is paid. If \( u_k = 1 \) is chosen at or after the change point \( \tau^0 \), then only a variance penalty is paid. Below these costs are formulated.

Let \( h = (h_1, \ldots, h_X)' \) specify the physical state levels associated with states 1, 2, \ldots, \( X \) of the Markov chain \( x \in \{e_1, e_2, \ldots, e_X\} \). The variance penalty is
\[
\mathbb{E}\{\|x_k - \pi_k\|_2^2 \mid \mathcal{I}_k\} = H'\pi_k(i) - (h'\pi_k)^2,
\]
where \( H_i = h_i^2 \) and \( H = (H_1, H_2, \ldots, H_X) \),
\[
\mathcal{I}_k = (y_1, \ldots, y_k, u_0, \ldots, u_{k-1}).
\]
This conditional variance penalizes choosing the stop action if the uncertainty in the state estimate is large. Recall we discussed POMDP with nonlinear costs in §2.

Next, the false alarm event \( \cup_{i \geq 2} \{x_k = e_i\} \cap \{u_k = 1\} = \{x_k \neq e_1\} \cap \{u_k = 1\} \) represents the event that a change is announced before the change happens at time \( \tau^0 \). To evaluate the false alarm penalty, let \( f_i I(x_k = e_i, u_k = 1) \) denote the cost of a false alarm in state \( e_i, i \in \mathcal{X} \), where \( f_i \geq 0 \). Of course, \( f_1 = 0 \) since a false alarm is only incurred if the stop action is picked in states 2, \ldots, \( X \). The expected false alarm penalty is
\[
\sum_{i \in \mathcal{X}} f_i \mathbb{E}\{I(x_k = e_i, u_k = 1)\mid \mathcal{I}_k\} = f'\pi_k I(u_k = 1),
\]
where \( f = (f_1, \ldots, f_X)' \), \( f_1 = 0 \). (7.8)
The false alarm vector $f$ is chosen with increasing elements so that states further from state 1 incur larger penalties.

Then with $\alpha, \beta$ denoting non-negative constants that weight the relative importance of these costs, the stopping cost expressed in terms of the belief state at time $k$ is

$$\bar{C}(\pi_k, u_k = 1) = \alpha (H' \pi_k - (h' \pi_k)^2) + \beta f' \pi_k.$$  \hfill (7.9)

One can view $\alpha$ as a Lagrange multiplier in a stopping time problem that seeks to minimize a cumulative cost subject to a variance stopping constraint.

**Delay cost of continuing**

We allow two possible choices for the delay costs for action $u_k = 2$:

(a) Predicted Delay: If action $u_k = 2$ is taken then $\{x_{k+1} = e_1, u_k = 2\}$ is the event that no change is declared at time $k$ even though the state has changed at time $k+1$. So with $d$ denoting a non-negative constant, $d I(x_{k+1} = e_1, u_k = 2)$ depicts a delay cost. The expected delay cost for decision $u_k = 2$ is

$$\bar{C}(\pi_k, u_k = 2) = d E\{I(x_{k+1} = e_1, u_k = 2)|F_k\} = de'_1 P' \pi_k.$$ \hfill (7.10)

The above cost is motivated by applications (e.g., sensor networks) where if the decision maker chooses $u_k = 2$, then it needs to gather observation $y_{k+1}$ thereby incurring an additional operational cost denoted as $C_o$. Strictly speaking, $\bar{C}(\pi, 2) = de'_1 P' \pi + C_o$. Without loss of generality set the constant $C_o$ to zero, as it does not affect our structural results. The penalty $d I(x_{k+1} = e_1, u_k = 2)$ gives incentive for the decision maker to predict the state $x_{k+1}$ accurately.

(b) Classical Delay: Instead of (7.10), a more ‘classical’ formulation is that a delay cost is incurred when the event $\{x_k = e_1, u_k = 2\}$ occurs. The expected delay cost is

$$\bar{C}(\pi_k, u_k = 2) = d E\{I(x_k = e_1, u_k = 2)|I_k\} = de'_1 \pi_k.$$ \hfill (7.11)

Remark: Due to the variance penalty, the cost $\bar{C}(\pi, 1)$ in (7.9) is quadratic in the belief state $\pi$. Therefore, the formulation cannot be reduced to a standard stopping problem with linear costs in the belief state.

Summary: It is clear from the above formulation that quickest detection is simply a stopping-time POMDP of the form (6.3) with transition matrix (7.1) and costs (7.8), (7.10) or (7.11). In the special case of geometric change time ($X = 2$), and delay cost (7.11), the cost function assumes the classical Kolmogorov–Shiryayev criterion for detection of disorder (6.7).

### 7.1.2 Main Result. Threshold Optimal Policy for Quickest Detection

Note first that for $\alpha = 0$ in (7.9), the stopping cost is linear. Then Theorem 6.2.1 applies implying that the stopping set $\mathcal{R}_1$ is convex. Below we focus on establishing Theorem 6.3.4 to show the existence of a threshold curve for the
7.1 Example 1. Quickest Detection with Phase-Distributed Change Time and Variance Penalty

optimal policy. As discussed at the end of §7.2, such a result goes well beyond establishing convexity of the stopping region.

We consider the predicted cost and delay cost cases separately below:

**Quickest Detection with Predicted Delay Penalty**

First consider the quickest detection problem with the predicted delay cost (7.10). For the stopping cost $\bar{C}(\pi, 1)$ in (7.9), choose $f = [0, 1, \ldots, 1]' = X - e_1$.

This weighs the states $2, \ldots, X$ equally in the false alarm penalty. With assumption (7.5), the variance penalty (7.7) becomes $\alpha(e_1' \pi - (e_1' \pi)^2)$. The delay cost $\bar{C}(\pi, 2)$ is chosen as (7.10). To summarize

$$\bar{C}(\pi, 1) = \alpha(e_1' \pi - (e_1' \pi)^2) + \beta(1 - e_1' \pi), \quad \bar{C}(\pi, 2) = de_1' P' \pi.$$  \hspace{1cm} (7.12)

Theorem 6.3.4 is now illustrated for the costs (7.12). Before proceeding there is one issue we need to fix. Even for linear costs in (7.12) ($\alpha = 0$), denoting $\bar{C}(\pi, u) = c' \pi$, it is seen that the elements of $c_1$ are increasing while the elements of $c_2$ are decreasing. So at first sight, it is not possible to apply Theorem 6.3.4 since assumption (C) of Theorem 6.3.4 requires that the elements of the cost vectors are decreasing. But the following novel transformation can be applied, which is nicely described in [40, pp.389–390] (for fully observed MDPs).

Define

$$V(\pi) = \bar{V}(\pi) - (\alpha + \beta)f' \pi, \quad C(\pi, 1) = \alpha(H' \pi - (h' \pi)^2) - \alpha f' \pi$$

$$C(\pi, 2) = \bar{C}(\pi, 2) - (\alpha + \beta)f' \pi + \rho(\alpha + \beta)f' P' \pi.$$  \hspace{1cm} (7.13)

Then clearly $V(\pi)$ satisfies Bellman’s dynamic programming equation

$$\mu^*(\pi) = \arg \min_{u \in U} Q(\pi, u), \quad J_{\mu^*}(\pi) = V(\pi) = \min_{u \in \{1, 2\}} Q(\pi, u) \quad \text{(7.14)}$$

where $Q(\pi, 2) = C(\pi, 2) + \rho \sum_{y \in Y} V(T(\pi, y)) \sigma(\pi, y)$, $Q(\pi, 1) = C(\pi, 1)$

Even though the value function is now changed, the optimal policy $\mu^*(\pi)$ and hence stopping set $R_1$ remain unchanged with this coordinate transformation. The nice thing is that the new costs $C(\pi, u)$ in (7.13) can be chosen to be decreasing under suitable conditions. For example, if $\alpha = 0$, then clearly $C(\pi, 1) = 0$ (and so is decreasing by definition) and it is easily checked that if $d \geq \rho \beta$ then $C(\pi, 2)$ is also decreasing.

With the above transformation, we are now ready to apply Theorem 6.3.4. Assumptions (C) and (S) of Theorem 6.3.4 specialize to the following assumptions on the transformed costs in (7.13):

\begin{enumerate}
  \item[(C-Ex1)] $d \geq \rho(\alpha + \beta)$
  \item[(S-Ex1)] $(d - \rho(\alpha + \beta))(1 - P_{21}) \geq \alpha - \beta$
\end{enumerate}

**Theorem 7.1.1** Under (C-Ex1), (F1), (F2) and (S-Ex1), Theorem 6.3.4 holds implying that the optimal policy for quickest detection with PH-distributed change times has a threshold structure. Thus Algorithm 3 estimates the optimal linear threshold.
Proof The proof follows from Theorem 6.3.4. We only need to show that (C-Ex1) and (S-Ex1) hold. These are specialized versions of conditions (C) and (S) arising in Theorem 6.3.4.

First consider (C-Ex1): Consider Lemma 5.1.1 with the choice of \( \phi = 2\alpha e_1, \ h = e_1 \) in (5.5). This yields \( 2\alpha \geq 0 \) and \( 2\alpha \geq 2\alpha \) which always hold. So \( C(\pi, 1) \geq r_{decreasing} \) for any non-negative \( \alpha \). It is easily verified that (C-Ex1) is sufficient for the linear cost \( C(\pi, 2) \) to be decreasing.

Next consider (S-Ex1). Set \( \phi_i = (d - \rho(\alpha + \beta)) P' e_i + (\beta - \alpha) e_1, \ h = e_1 \) in (6.26). The first inequality is equivalent to: (i) \( (d - \rho(\alpha + \beta))(P_X - P_{i1}) \leq 0 \) for \( i \geq 2 \) and (ii) \( (d - \rho(\alpha + \beta))(1 - P_{i1}) \geq \alpha - \beta \). Note that (i) holds if \( d \geq \rho(\alpha + \beta) \).

The second inequality in (6.26) is equivalent to \( (d - \rho(\alpha + \beta))(1 - P_{i1}) \geq \alpha - \beta \). Since \( P \) is TP2, from Lemma 4.5.1 it follows that (S-Ex1) is sufficient for these inequalities to hold. \( \square \)

Quickest Detection with Classical Delay Penalty
Finally, consider the ‘classical’ delay cost \( \tilde{C}(\pi, 2) \) in (7.11) and stopping cost \( \tilde{C}(\pi, 1) \) in (7.9) with \( h \) in (7.5). Then

\[
\tilde{C}(\pi, 1) = \alpha \left( e'_1 \pi - \left( e'_1 \pi \right)^2 \right) + \beta f' \pi_k, \quad \tilde{C}(\pi, 2) = de'_1 \pi. \tag{7.15}
\]

Assume that the decision maker designs the false alarm vector \( f \) to satisfy the following linear constraints:

\[\begin{align*}
\text{(AS-Ex1)} & \quad (i) \ f_i \geq \max \{ 1, \rho \frac{\alpha + \beta}{\beta} f' P' e_i + \frac{\alpha - d}{\beta} \}, \ i \geq 2. \\
 & \quad (ii) \ f_j - f_i \geq \rho f' P'(e_j - e_i), \ j \geq i, \ i \in \{ 2, \ldots, X - 2 \} \\
 & \quad (iii) \ f_X - f_i \geq \rho \frac{\alpha + \beta}{\beta} f' P'(e_X - e_i), \ i \in \{ 2, \ldots, X - 1 \}.
\end{align*}\]

Feasible choices of \( f \) are easily obtained by a linear programming solver.

Then Theorem 6.3.4 continues to hold, under conditions (AS-Ex1), (F1),(F2).

Summary: We modeled quickest detection with PH-distributed change time as a multi-state POMDP. We then gave sufficient conditions for the optimal policy to have a threshold structure. The optimal linear parametrized policy can be estimated via the policy gradient Algorithm 6.4.2.

7.2 Example 2: Risk Sensitive Quickest Detection with Exponential Delay Penalty

In this example, we generalize the results of [96], which deals with exponential delay penalty and geometric change times. We consider exponential delay penalty with PH-distributed change time. Our formulation involves risk sensitive partially observed stochastic control. We first show that the exponential penalty cost function in [96] is a special case of risk-sensitive stochastic control cost function when the state space dimension \( X = 2 \). We then use the risk-sensitive stochastic control formulation to derive structural results for PH-distributed change time. In particular, the main result below (Theorem 7.2.1)
shows that the threshold switching curve still characterizes the optimal stopping region $R_1$. The assumptions and main results are conceptually similar to Theorem 6.3.4.

Risk sensitive stochastic control with exponential cost has been studied extensively [24, 45, 10]. In simple terms, quickest time detection seeks to optimize the objective $E\{J^0\}$ where $J^0$ is the accumulated sample path cost until some stopping time $\tau$. In risk sensitive control, one seeks to optimize $J = E\{exp(\epsilon J^0)\}$. For $\epsilon > 0$, $J$ penalizes heavily large sample path costs due to the presence of second order moments. This is termed a risk-averse control. Risk sensitive control provides a nice formalization of the exponential penalty delay cost and allows us to generalize the results in [96] to phase-distributed change times.

Below, we will use $c(e_i, u = 1)$ to denote false alarm costs and $c(e_i, u = 2)$ to denote delay costs, where $i \in \mathcal{X}$. Risk sensitive control [12] considers the exponential cumulative cost function

$$J_\mu(\pi) = \mathbb{E}_\mu \left\{ \exp \left( \epsilon \sum_{k=0}^{\tau-1} c(x_k, u_k = 2) + \epsilon c(x_\tau, u_\tau = 1) \right) \right\}$$

(7.16)

where $\epsilon > 0$ is the risk sensitive parameter.

Let us first show that the exponential penalty cost in [96] is a special case of (7.16) for consider the case $X = 2$ (geometric distributed change time). For the state $x \in \{e_1, e_2\}$, choose $c(x, u = 1) = \beta I(x \neq e_1, u = 1) = \beta(1 - e'_1 x)$ (false alarm cost), $c(x, u = 2) = d I(x = e_1, u = 2) = de'_1 x$ (delay cost). Then it is easily seen that $\sum_{k=0}^{\tau-1} c(x_k, u_k = 2) + c(x_\tau, u_\tau = 1) = d |\tau - \tau_0|^+ + \beta I(\tau < \tau_0)$.

Therefore (recall $\tau_0$ is defined in (7.2) and $\tau$ is defined in (6.2)),

$$J_\mu(\pi) = \mathbb{E}_\mu \left\{ \exp \left( \epsilon d |\tau - \tau_0|^+ + \epsilon \beta I(\tau < \tau_0) \right) \right\} \left[ I(\tau < \tau_0) + I(\tau = \tau_0) + I(\tau > \tau_0) \right]$$

$$= \mathbb{E}_\mu \left\{ \exp(\epsilon \beta) I(\tau < \tau_0) + \exp(\epsilon d |\tau - \tau_0|^+) I(\tau > \tau_0) + 1 \right\}$$

$$= \mathbb{E}_\mu \{ (e^{\epsilon \beta} - 1) I(\tau < \tau_0^0) + e^{\epsilon d |\tau - \tau_0|^+} \}$$

$$= (e^{\epsilon \beta} - 1) \mathbb{P}_\mu(\tau < \tau_0) + \mathbb{E}_\mu \{ e^{\epsilon d |\tau - \tau_0|^+} \}$$

(7.17)

which is identical to exponential delay cost function in [96, Eq.40]. Thus the Bayesian quickest time detection with exponential delay penalty in [96] is a special case of a risk sensitive stochastic control problem.

We consider the delay cost as in (7.10); so for state $x \in \{e_1, \ldots, e_X\}$, $c(x, u_k = 2) = de'_k P^k x$. To get an intuitive feel for this modified delay cost function, for the case $X = 2$,

$$\sum_{k=0}^{\tau-1} c(x_k, u_k = 2) + c(x_\tau, u_\tau = 1) = d |\tau - \tau_0|^+ + \beta I(\tau < \tau_0) + d P_{21}(\tau_0 - 1) I(\tau_0 < \tau)$$

Therefore, for $X = 2$, the exponential delay cumulative cost function is

$$J_\mu(\pi) = (e^{\epsilon \beta} - 1) \mathbb{P}_\mu(\tau < \tau_0) + \mathbb{E}_\mu \{ e^{\epsilon d |\tau - \tau_0|^+} + P_{21}(\tau_0 - 1) I(\tau_0 < \tau) \}.$$  

(7.18)
This is similar to (7.17) except for the additional term \( P_{21}(\tau_0 - 1)I(\tau_0 < \tau) \) in the exponential.

With the above motivation, we consider risk sensitive quickest detection for PH-distributed change time, i.e. \( X \geq 2 \). Let \( \pi \) denote the risk sensitive belief state, see [31, 45] for extensive descriptions of the risk sensitive belief state and verification theorems for dynamic programming in risk sensitive control. Bellman’s equation reads

\[
\tilde{V}(\pi) = \min \{ \tilde{C}(\pi, 1), \sum_{y \in Y} \tilde{V}(T(\pi, y))\sigma(\pi, y) \} \quad \text{where} \quad (7.19)
\]

\[
\tilde{C}(\pi, 1) = R_i' \pi, \quad T(\pi, y) = \frac{B_y P' \text{diag}(R_2) \pi}{\sigma(\pi, y)}, \quad \sigma(\pi, y) = 1'B_y P' \text{diag}(R_2) \pi \\
R_1 = (1, e^{\beta}, \ldots, e^{\beta^X})', \quad R_2 = (e^{\beta d}, e^{\beta d P_{21}}, \ldots, e^{\beta d P_{X1}})', \quad B_y = \text{diag}(B_{1y}, \ldots, B_{Xy}).
\]

Similar to the transformation used in (7.13), define \( V(\pi) = \tilde{V}(\pi) - \tilde{C}(\pi, 1) \). Then \( V(\pi) \) satisfies Bellman’s equation (7.14) with

\[
C(\pi, 1) = 0, \quad C(\pi, 2) = R_i' (P' \text{diag}(R_2) - I) \pi. \quad (7.20)
\]

Assume the following condition holds

(C-Ex3) The elements of \( R_i' (P' \text{diag}(R_2) - I) \) are decreasing w.r.t. \( i = 1, 2, \ldots, X \).

Evaluating \( C(\pi, 2) = R_i' (P' \text{diag}(R_2) - I) \pi \), then (C-Ex3) is equivalent to

\[
e^{\beta} - 1 \geq e^{\beta d P_{21}}(P_{21} + e^{\beta (1 - P_{21})}) - e^{\beta} \quad \text{and} \quad e^{\beta d P_{11}}(P_{11} + e^{\beta (1 - P_{11})})
\]

decreasing in \( i \in \{2, \ldots, X \} \). For example, if \( d = \epsilon = 1 \), then for \( \beta \geq 1 \), the following are verified by elementary calculus:

(i) (C-Ex3) always holds for \( \beta \geq 1 \) when \( X = 2 \) (geometric distributed change time).

(ii) For PH-distributed change time, if (F2) holds, then (C-Ex3) always holds providing \( P_{21} < 1/(e^{\beta} - 1) \).

**Theorem 7.2.1** The stopping region \( R_1 \) is a convex subset of \( \Pi(X) \). Under (C-Ex3), (F1), (F2), Theorem 6.3.4 holds. Thus Algorithm 3 estimates the optimal linear threshold.

**Proof** The only difference compared to a standard stopping time POMDP is the update of the belief state (7.19) which now includes the term \( \text{diag}(R_a) \). The elements of \( R_a \) are non-negative and functionally independent of the observation \( y \). Therefore the three main requirements that \( T(\pi, y) \) is MLR increasing in \( \pi, T(\pi, y) \) is MLR increasing in \( Y \), and \( \sigma(\pi, \cdot) \) is \( \geq_r \) increasing in \( \pi \) continue to hold. Then the rest of the proof is identical to Theorem 6.3.4.

**Remarks**: (i) Delay Formulation in [96]: Consider the formulation in [96] which is equivalent to (7.17). Then for the geometric distributed case \( X = 2 \), the convexity of \( R_1 \) holds using a similar proof to above. Since \( \Pi(X) \) is a 1-dimensional simplex and \( e_1 \in R_1 \), convexity implies there exists (a possible degenerate)
threshold point $\pi^*$ that characterizes $\mathcal{R}_1$ such that the optimal policy is of the form (6.8). As a sanity check, the analogous condition to (C-Ex3) reads $e^{cd} - 1 > P_{21}(1 - e^{cd})$. This always holds for $\epsilon \geq 0$. Therefore, assuming (F1) holds, the above theorem holds for the exponential delay penalty case under (F1). (Recall (F2) holds trivially when $X = 2$). Finally, for $X > 2$, using a similar proof, the conclusions of Theorem 7.2.1 hold.

7.3 Example 3: Multi-agent Social Learning

This section deals with a multi-agent Bayesian stopping time problem where agents perform greedy social learning and reveal their actions to subsequent agents. Given such a protocol of local decisions, how can the multi-agent system make a global decision when to stop? We show that the optimal decision policy of the stopping time problem has multiple thresholds. The motivation for such problems arise in automated decision systems (e.g., sensor networks) where agents make local decisions and reveal these local decisions to subsequent agents. The multiple threshold behavior of the optimal global decision shows that making global decisions based on local decision involves non-monotone policies.

7.3.1 Motivation: Social Learning amongst myopic agents

We refer the reader to [22, 66, 63] for details of social learning. Consider a multi-agent system with agents indexed as $k = 1, 2, \ldots$ performing social learning to estimate an underlying random state $x$ with prior $\pi_0$. (We assume in this section that $x$ is a random variable and not a Markov chain.) Let $y_k \in \mathcal{Y} = \{1, 2, \ldots, Y\}$ denote the private observation of agent $k$ and $a_k \in \mathcal{A} = \{1, 2, \ldots, A\}$ denote the local action agent $k$ takes. Define:

$$\mathcal{H}_k = (a_1, \ldots, a_{k-1}, y_k), \quad \mathcal{G}_k = (a_1, \ldots, a_{k-1}, a_k).$$

Let us highlight the key Bayesian update equations in the social learning protocol; see [22, 66, 64]:

A time $k$, based on its private observation $y_k$ and public belief $\pi_{k-1}$, agent $k$:

1. Updates its private belief $\eta_k = \mathbb{E}\{x|\mathcal{H}_k\}$ as

$$\eta_k = \frac{B_{y_k} \pi_{k-1}}{V B_{y_k} \pi_{k-1}}$$

2. Takes local myopic action $a_k = \arg \min_{a \in \mathcal{A}} \{c'_a \eta_k\}$ where $\mathcal{A} = \{1, 2, \ldots, A\}$ denotes the set of local actions.

3. Based on $a_k$, the public belief $\pi_k = \mathbb{E}\{x_k|\mathcal{G}_k\}$ is updated (by subsequent agents) via the social learning filter (initialized with $\pi_0$)

$$\pi_k = T(\pi_{k-1}, a_k), \text{ where } T(\pi, a) = \frac{R^*_a \pi}{\sigma(\pi, a)}, \quad \sigma(\pi, a) = 1' X R^*_a \pi$$
In (7.23), $R^\alpha = \text{diag}(P(a|x = e_i, \pi), i \in \mathcal{X})$ with elements

$$
P(a_k = a|x = e_i, \pi_{k-1} = \pi) = \sum_{y \in \mathcal{Y}} P(a_k = a|y, \pi)P(y|x = e_i) = \sum_{y \in \mathcal{Y}} \prod_{a \in \mathcal{A} - \{a\}} I(c'_a B_y \pi < c'_a B_y \pi) P(y|x = e_i)
$$

(7.24)

Here $I(\cdot)$ is the indicator function and $B_y = \text{diag}(P(y|x = e_1), \ldots, P(y|x = e_X))$.

The procedure then repeats at time $k + 1$ and so on.

Recall that in classical social learning after some finite time $\bar{k}$, all agents choose the same action and the public belief freezes resulting in an information cascade; see [22, 66] and [58, 59] for a financial application.

### 7.3.2 Example 3: Stopping time POMDP with Social Learning: Interaction of Local and Global Decision Makers

Suppose a multi-agent system makes local decisions and performs social learning as above. How can the multi-agent system make a global decision when to stop? Such problems are motivated in decision systems where a global decision needs to be made based on local decisions of agents. Figure ?? on page ?? shows the setup with interacting local and global decision makers.

We consider a Bayesian sequential detection problem for state $x = e_1$. Our goal below is to derive structural results for the optimal stopping policy. The main result below (Theorem 7.3.1) is that the global decision of when to stop is a multi-threshold function of the belief state.

Consider $\mathcal{X} = \mathcal{Y} = \{1, 2\}$ and $\mathcal{A} = \{1, 2\}$ and the social learning model of §7.3.1, where the costs $c(e_i, a)$ satisfy

$$
c(e_1, 1) < c(e_1, 2), \quad c(e_2, 2) < c(e_2, 1).
$$

(7.25)

Otherwise one action will always dominate the other action and the problem is un-interesting.

Let $\tau$ denote a stopping time adapted to $\mathcal{G}_k$, $k \geq 1$ (see (7.21)). In words, each agent has only the public belief obtained via social learning to make the global decision of whether to continue or stop. The goal is to solve the following stopping time POMDP to detect state $e_1$: Choose stopping time $\tau$ to minimize

$$
J_\mu(\pi) = \mathbb{E}_\mu\left\{\sum_{k=0}^{\tau-1} \rho^k \mathbb{E}\left\{dI(x = e_1)|\mathcal{G}_k\right\} + \rho^\tau \beta \mathbb{E}\{I(x \neq e_1)|\mathcal{G}_{\tau}\}\right\}
$$

(7.26)

The first term is the delay cost and penalizes the decision of choosing $u_k = 2$ (continue) when the state is $e_1$ by the non-negative constant $d$. The second term is the stopping cost incurred by choosing $u_\tau = 1$ (stop and declare state 1) at time $k = \tau$. It is the error probability of declaring state $e_1$ when the actual state
is $e_2$. $\beta$ is a positive scaling constant. In terms of the public belief, (7.26) is

$$J_\mu(\pi) = \mathbb{E}_\mu\left\{ \sum_{k=0}^{\tau-1} \rho^k \bar{C}(\pi_{k-1}, u_k = 2) + \rho^\tau \bar{C}(\pi_{\tau-1}, u_\tau = 1) \right\}$$

(7.27)

$$\bar{C}(\pi, 2) = de_1\pi$$, \quad $\bar{C}(\pi, 1) = \beta e_2\pi$.

The global decision $u_k = \mu(\pi_{k-1}) \in \{1 \text{ (stop) }, 2 \text{ (continue)}\}$ is a function of the public belief $\pi_{k-1}$ updated according to the social learning protocol (7.22), (7.23). The optimal policy $\mu^*(\pi)$ and value function $V(\pi)$ satisfy Bellman’s equation (7.14) with

$$Q(\pi, 2) = C(\pi, 2) + \rho \sum_{a \in \mathcal{A}} V(T(\pi, a)) \sigma(\pi, a) \text{ where } (7.28)$$

$C(\pi, 2) = \bar{C}(\pi, 2) - (1 - \rho)\bar{C}(\pi, 1)$, \quad $Q(\pi, 1) = C(\pi, 1) = 0$.

Here $T(\pi, a)$ and $\sigma(\pi, a)$ are obtained from the social learning filter (7.23). The above stopping time problem can be viewed as a macro-manager that operates on the public belief generated by micro-manager decisions. Clearly the micro and macro-managers interact – the local decisions $a_k$ taken by the micro-manager determine $\pi_k$ and hence determines decision $u_{k+1}$ of the macro-manager.

Since $\mathcal{X} = \{1, 2\}$, the public belief state $\pi = [1 - \pi(2), \pi(2)]$ is parametrized by the scalar $\pi(2) \in [0, 1]$, and the belief space is the interval $[0, 1]$. Define the following intervals which form a partition of the interval $[0,1]$:

$$\mathcal{P}_l = \{ \pi(2) : \kappa_l < \pi(2) \leq \kappa_{l-1}\}, \quad l = 1, \ldots, 4 \text{ where } \kappa_0 = 1, \quad \kappa_1 = \frac{(c(e_1, 2) - c(e_1, 1))B_{11}}{(c(e_1, 2) - c(e_1, 1))B_{11} + (c(e_2, 1) - c(e_2, 2))B_{21}}$$

$$\kappa_2 = \frac{(c(e_1, 2) - c(e_1, 1))B_{12}}{(c(e_1, 2) - c(e_1, 1))B_{12} + (c(e_2, 1) - c(e_2, 2))B_{22}}, \quad \kappa_3 = 0.$$ (7.29)

$\kappa_0$ corresponds to belief state $e_1$, and $\kappa_4$ corresponds to belief state $e_2$. (See discussion at the end of this section for more intuition about the intervals $\mathcal{P}_l$).

It is readily verified that if the observation matrix $B$ is TP2, then $\kappa_3 \leq \kappa_2 \leq \kappa_1$. The following is the main result.

**Theorem 7.3.1** Consider the stopping time problem (7.27) where agents perform social learning using the social learning Bayesian filter (7.23). Assume (7.25) and $B$ is symmetric and satisfies (F1). Then the optimal stopping policy $\mu^*(\pi)$ has the following structure: The stopping set $\mathcal{R}_1$ is the union of at most three intervals. That is $\mathcal{R}_1 = \mathcal{R}_1^a \cup \mathcal{R}_1^b \cup \mathcal{R}_1^c$ where $\mathcal{R}_1^a, \mathcal{R}_1^b, \mathcal{R}_1^c$ are possibly empty intervals. Here
1. The stopping interval $\mathcal{R}_1^a \subseteq \mathcal{P}_1 \cup \mathcal{P}_4$ and is characterized by a threshold point. That is, if $\mathcal{P}_1$ has a threshold point $\pi^*$, then $\mu^*(\pi) = 1$ for all $\pi(2) \in \mathcal{P}_4$ and

$$\mu^*(\pi) = \begin{cases} 2 & \text{if } \pi(2) \geq \pi^* \\ 1 & \text{otherwise} \end{cases}, \quad \pi(2) \in \mathcal{P}_1.$$ (7.30)
Similarly, if $\mathcal{P}_4$ has a threshold point $\pi_4^*$, then $\mu^*(\pi) = 2$ for all $\pi(2) \in \mathcal{P}_1$.

(ii) The stopping intervals $R_{12} \subseteq \mathcal{P}_2$ and $R_{14} \subseteq \mathcal{P}_3$.

(iii) The intervals $\mathcal{P}_1$ and $\mathcal{P}_4$ are regions of information cascades. That is, if $\pi_k \in \mathcal{P}_1 \cup \mathcal{P}_4$, then social learning ceases and $\pi_{k+1} = \pi_k$ (see Theorem ?? for definition of information cascade).

The proof of Theorem 7.3.1 is in [57]. The proof depends on properties of the social learning filter and these are summarized in Lemma ?? in Appendix ??.

The proof is more complex than that of Theorem 6.2.1 since now $V(\pi)$ in is not necessarily concave over $\Pi(X)$, since $T(\cdot)$ and $\sigma(\cdot)$ are functions of $R_\pi$ (7.24) which itself is an explicit (and in general non-concave) function of $\pi$.

Example: To illustrate the multiple threshold structure of the above theorem, consider the stopping time problem (7.27) with the following parameters:

$$\rho = 0.9, \quad d = 1.8, \quad B = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}, \quad c(e_1, a) = \begin{bmatrix} 4.57 & 5.57 \\ 2.57 & 0 \end{bmatrix}, \quad \beta = 2. \quad (7.31)$$

Figure 7.1(a) and (b) show the optimal policy and value function. These were computed by constructing a grid of 500 values for $\Pi = [0, 1]$. The double threshold behavior of the stopping time problem when agents perform social learning is due to the discontinuous dynamics of the social learning filter (7.23).

Discussion

The multiple threshold behavior (nonconvex stopping set $R_1$) of Theorem 7.3.1 is unusual. One would have thought that if it was optimal to ‘continue’ for a particular belief $\pi^*(2)$, then it should be optimal to continue for all beliefs $\pi(2)$ larger than $\pi^*(2)$. The multiple threshold optimal policy shows that this is not true. Figure 7.1(a) shows that as the public belief $\pi(2)$ of state 2 decreases, the optimal decision switches from ‘continue’ to ‘stop’ to ‘continue’ and finally ‘stop’. Thus the global decision (stop or continue) is a non-monotone function of public beliefs obtained from local decisions.
The main reason for this unusual behavior is the dependence of the action likelihood $R^\pi_a$ on the belief state $\pi$. This causes the social learning Bayesian filter to have a discontinuous update. The value function is no longer concave on $\Pi(X)$ and the optimal policy is not necessarily monotone. As shown in the proof of Theorem 7.3.1, the value function $V(\pi)$ is concave on each of the intervals $\mathcal{P}_l$, $l = 1, \ldots, 4$.

### 7.4 Example 4: Quickest Detection with Controlled Sampling

This section discusses quickest change detection when the decision maker controls how often it observes (samples) a noisy Markov chain. The aim is to detect when a nosily observed Markov chain hits a target state by minimizing a combination of false alarm, delay cost and measurement sampling cost. There is an inherent trade-off between these costs: Taking more frequent measurements yields accurate estimates but incurs a higher measurement cost. Making an erroneous decision too soon incurs a false alarm penalty. Waiting too long to declare the target state incurs a delay penalty. Since there are multiple “continue” actions, the problem is not a standard stopping time POMDP. For the 2-state case, we show that under reasonable conditions, the optimal policy has the following intuitive structure: if the belief is away from the target state, look less frequently; if the belief is close to the target state, look more frequently.

#### 7.4.1 Controlled Sampling Problem

Let $t = 0, 1, \ldots$ denote discrete time and $\{x_t\}$ denote a Markov chain on the finite state space $\mathcal{X} = \{e_1, \ldots, e_X\}$ with transition matrix $P$.

Let $\tau_0, \tau_1, \ldots, \tau_k$ denote discrete time instants at which measurement samples have been taken, where by convention $\tau_0 = 0$. Let $\tau_k$ denote the current time-instant at which a measurement is taken. The measurement sampling protocol proceeds according to the following steps:

**Step 1. Observation:** A noisy measurement $y_k \in \mathcal{Y}$ at time $t = \tau_k$ of the Markov chain is obtained with conditional pdf or pmf $B_{xy} = p(y_k = y|x_{\tau_k} = x)$.

**Step 2. Sequential Decision Making:** Let $I_k = \{y_1, \ldots, y_k, u_0, u_1, \ldots, u_{k-1}\}$ denote the history of past decisions and available observations. At times $\tau_k$, an action $u_k$ is chosen according to the stationary policy $\mu$, where

$$u_k = \mu(I_k) \in \mathcal{U} = \{0 \text{ (announce change)}, 1, 2, \ldots, L\}. \quad (7.32)$$

Here, $u_k = l$ denotes take the next measurement after $D_l$ time points, $l \in \{1, 2, \ldots, L\}$. The initial decision at time $\tau_0 = 0$ is $u_0 = \mu(\pi_0)$ where $\pi_0$ is the initial distribution. Also, $D_1 < D_2 < \cdots < D_L$ are $L$ distinct positive integers that denote the set of possible sampling time intervals. Thus the decision $u_k$ specifies the next time $\tau_{k+1}$ to make a measurement as follows:

$$\tau_{k+1} = \tau_k + D_{u_k}, \quad u_k \in \{1, 2, \ldots, L\}, \quad \tau_0 = 0. \quad (7.33)$$
Quickest Change Detection

Step 3. Costs: If decision \( u_k \in \{1, 2, \ldots, L\} \) is chosen, a decision cost \( c(x_t, u_k) \) is incurred by the decision-maker at each time \( t \in [\tau_k, \ldots, \tau_{k+1} - 1] \) until the next measurement is taken at time \( \tau_{k+1} \). Also at each time \( \tau_k, k = 0, 1, \ldots, k^* - 1 \), the decision maker pays a non-negative measurement cost \( \tilde{m}(x_{\tau_k}, x_{\tau_{k+1}}, y_{k+1}, u_k) \) to observe the noisy Markov chain at time \( \tau_{k+1} = \tau_k + D_{u_k} \). In terms of \( \mathcal{I}_k \), this is equivalent to choosing the measurement cost as (see (2.8))

\[
    r(x_{\tau_k} = e_i, u_k) = \sum_j P^{D_{u_k}}_{ij} B_{jy} \tilde{m}(x_{\tau_k} = e_i, x_{\tau_{k+1}} = e_j, y_{k+1} = y, u_k) \tag{7.34}
\]

where \( P^{D_{u}}_{ij} \) denotes the \((i, j)\) element of matrix \( P^{D_{u}} \).

Step 4: If at time \( t = \tau_{k^*} \) the decision \( u_{k^*} = 0 \) is chosen, then a terminal cost \( c(x_{\tau_{k^*}}, 0) \) is incurred and the problem terminates.

If decision \( u_k \in \{1, 2, \ldots, L\} \), set \( k \) to \( k + 1 \) and go to Step 1.

In terms of the belief state, the objective to be minimized can be expressed as

\[
    J_\mu(\pi) = E_\mu \left\{ \sum_{k=0}^{k^* - 1} C(\pi_k, u_k) + C(\pi_{k^*}, u_{k^*} = 0) \right\} \tag{7.35}
\]

where \( C(\pi, u) = C^*_u \pi \) for \( u \in \mathcal{U} \)

\[
    C_u = \begin{cases} 
        r_u + (I + P + \ldots + P^{D_{u} - 1})c_u & u \in \{1, 2, \ldots, L\} \\
        c_0 & u = 0 
    \end{cases}, 
    c_u = [c(e_1, u), \ldots, c(e_X, u)]', 
    r_u = [r(e_1, u), \ldots, r(e_X, u)]'. \tag{7.36}
\]

Define the stopping set \( \mathcal{R}_1 \) as

\[
    \mathcal{R}_1 = \{\pi \in \Pi(X) : \mu^*(\pi) = 0\} = \{\pi \in \Pi(X) : Q(\pi, 0) \leq Q(\pi, u), u \in \{1, 2, \ldots, L\}\}. \tag{7.37}
\]

Bellman’s dynamic programming equation reads

\[
    \mu^*(\pi) = \arg \min_{u \in \mathcal{U}} Q(\pi, u), \quad J_{\mu^*}(\pi) = V(\pi) = \min_{u \in \mathcal{U}} Q(\pi, u), 
    Q(\pi, u) = C(\pi, u) + \sum_{y \in \mathcal{Y}} V(T(\pi, y, u)) \sigma(\pi, y, u), 
    u = 1, \ldots, L, 
    Q(\pi, 0) = C(\pi, 0). \tag{7.38}
\]

7.4.2 Example: Quickest Change Detection with Optimal Sampling

We now formulate the quickest detection problem with optimal sampling which serves as an example to illustrate the above model. Recall that decisions (whether to stop, or continue and take next observation sample after \( D_l \) time points) are made at times \( \tau_1, \tau_2, \ldots \). In contrast, the state of the Markov chain (which models the change we want to detect) can change at any time \( t \). We need to construct the delay and false alarm penalties to take this into account.
1. **Phase-Distributed (PH) Change time**: As in Example 1, in quickest detection, the target state (labelled as state 1) is absorbing. The transition matrix $P$ is specified in (7.1). Denote the time at which the Markov chain hits the target state as
\[ \tau^0 = \min\{t : x_t = 1\}. \]  
(7.39)

2. **Observations**: As in (7.5) of Example 1, $B_{2y} = B_{3y} = \cdots = B_{Xy}$.

3. **Costs**: Associated with the quickest detection problem are the following costs.

   (i) **False Alarm**: Let $\tau_k$ denote the time at which decision $u_k = 0$ (stop and announce target state) is chosen, so that the problem terminates. If the decision to stop is made before the Markov chain reaches the target state 1, i.e., $\tau_k < \tau^0$, then a unit false alarm penalty is paid. Choosing $j = 1 - e_1$ in (7.8), in terms of the belief state, the false alarm penalty at epoch $k = k^*$ is
   \[ \sum_{i \neq 1} \mathbb{E}\{I(x_{\tau_k} = e_1, u_k = 0) | \mathcal{I}_k\} = (1 - e_1)\pi_k I(u_k = 0). \]  
(7.40)

   (ii) **Delay cost of continuing**: Suppose decision $u_k \in \{1, 2, \ldots, L\}$ is taken at time $\tau_k$. So the next sampling time is $\tau_{k+1} = \tau_k + D_{u_k}$. Then for any time $t \in [\tau_k, \tau_{k+1} - 1]$, the event $\{x_t = e_1, u_k\}$ signifies that a change has occurred but not been announced by the decision maker. Since the decision maker can make the next decision (to stop or continue) at $\tau_{k+1}$, the delay cost incurred in the time interval $[\tau_k, \tau_{k+1} - 1]$ is $d \sum_{I=\tau_k}^{\tau_{k+1}-1} I(x_t = e_1, u_k)$ where $d$ is a non-negative constant. For $u_k \in \{1, 2, \ldots, L\}$, the expected delay cost in interval $[\tau_k, \tau_{k+1} - 1] = [\tau_k, \tau_k + D_{u_k} - 1]$ is
   \[ d \sum_{I=\tau_k}^{\tau_{k+1}-1} \mathbb{E}\{I(x_t = e_1, u_k) | \mathcal{F}_k\} = d e_1' (I + P + \cdots + P^{D_{u_k}-1})' \pi_k. \]

   (iii) **Measurement Sampling Cost**: Suppose decision $u_k \in \{1, 2, \ldots, L\}$ is taken at time $\tau_k$. As in (7.36) let $r_{u_k} = (r(x_{\tau_k} = e_i, u_k), i \in \mathcal{X})$ denote the non-negative measurement cost vector for choosing to take a measurement. Next, since in quickest detection, states $2, \ldots, X$ are fictitious states that are indistinguishable in terms of cost, choose $r(e_2, u) = \ldots = r(e_X, u)$. Choosing a constant measurement cost at each time (i.e., $r(e_i, u)$ independent of state $i$ and action $u$), still results in non-trivial global costs for the decision maker. This is because choosing a smaller sampling interval will result in more measurements until the final decision to stop, thereby incurring a higher total measurement cost for the global decision maker.

**Remarks**: (i) **Quickest State Estimation**: The setup is identical to above, except that unlike (7.1), the transition matrix $P$ no longer has an absorbing target state. Therefore the Markov chain can jump in and out of the target state. To avoid pathological cases, we assume $P$ is irreducible. Also there is no requirement for the observation probabilities to satisfy $B_{2y} = B_{3y} = \cdots = B_{Xy}$. 

**Example 4**: Quickest Detection with Controlled Sampling 

---

7.4 Example 4: Quickest Detection with Controlled Sampling

---
(ii) **Summary:** In the notation of (7.36), the costs for quickest detection/estimation optimal sampling are \( C(\pi, u) = C'_u \pi \) where \( C_0 = c_0 = 1 - e_1 \) and
\[
C_u = r_u + (I + P + \cdots + P^{D_u-1})c_u, \quad c_u = de_u, \quad u \in \{1, 2, \ldots, L\}. \quad (7.41)
\]

(iii) **Structural Results:** As mentioned earlier, since there are multiple “continue” actions \( u \in \{1, 2, \ldots, L\} \), the problem is not a standard stopping time POMDP. Of course, Theorem 6.2.1 applies and so the optimal policy for the “stop” action, i.e., stopping set \( R_1 \) (7.37), is a convex set. Characterizing the structure of the policy for the actions \( \{1, 2, \ldots, L\} \) is more difficult. For the 2-state case, we obtain structural results in §7.4.3. For the multi-state case, we will develop results in Chapter 8.

7.4.3 Threshold Optimal Policy for Quickest Detection with Sampling

Consider quickest detection with optimal sampling for geometric distributed change time. The transition matrix is \( P = \begin{bmatrix} 1 & 0 \\ 1 - P_{22} & P_{22} \end{bmatrix} \) and expected change time is \( \mathbb{E}\{\tau^0\} = \frac{1}{1 - P_{22}} \) where \( \tau^0 \) is defined in (7.39). For a 2-state Markov chain since \( \pi(1) + \pi(2) = 1 \), it suffices to represent \( \pi \) by its second element \( \pi(2) \in [0, 1] \). That is, the belief space \( \Pi(X) \) is the interval \([0, 1]\).

**Theorem 7.4.1** Consider the quickest detection optimal sampling problem of §7.4.2 with geometric-distributed change time and costs (7.41). Assume the measurement cost \( r(e_i, u) \) satisfies (C), (S) and the observation distribution satisfies (F2). Then there exists an optimal policy \( \mu^*(\pi) \) with the following monotone structure: There exist up to \( L \) thresholds denoted \( \pi^*_1, \ldots, \pi^*_L \) with \( 0 = \pi^*_0 \leq \pi^*_1 \leq \ldots \leq \pi^*_L \leq \pi^*_{L+1} = 1 \) such that, for \( \pi(2) \in [0, 1] \),
\[
\mu^*(\pi) = l \text{ if } \pi(2) \in [\pi^*_l, \pi^*_{l+1}), \quad l = 0, 1, \ldots, L. \quad (7.42)
\]

Here the sampling intervals are ordered as \( D_1 < D_2 < \ldots < D_L \). So the optimal sampling policy (7.42) makes measurements less frequently when the posterior \( \pi(2) \) is away from the target state and more frequently when closer to the target state. (Recall the target state is \( \pi(2) = 0 \).)

The proof follows from that of Theorem 5.3.1 on page 61. There are two main conclusions regarding Theorem 7.4.1. First, for constant measurement cost, (C) and (S) hold trivially. For the general measurement cost \( \tilde{m}(x_{k+1}, y_{k+1}, u_k) \) (see (7.34)) that depends on the state at epoch \( k + 1 \), then \( r(e_i, u) \) in (7.34) automatically satisfies (S) if \( P \) satisfies (F1) and \( \tilde{m} \) is decreasing in \( j \). Second, the optimal policy \( \mu^*(\pi) \) is monotone in posterior \( \pi(1) \) and therefore has a finite dimensional characterization. To determine the optimal policy, one only needs to compute the values of the \( L \) thresholds \( \pi^*_1, \ldots, \pi^*_L \). These can be estimated via a simulation-based stochastic optimization algorithm.
7.5 Complements and Sources

The book [97] is devoted to quickest detection problems and contains numerous references. PH-distributed change times are used widely to model discrete event systems [89] and are a natural candidate for modeling arrival/demand processes for services that have an expiration date [23]. It would be useful to do a performance analysis of the various optimal detectors proposed in this chapter – see [120, 101] and references therein. [7] considers a measurement control problem for geometric-distributed change times (2-state Markov chain with an absorbing state). [56] considers joint change detection and measurement control POMDPs with more than 2 states.

In [54, 55] similar structural results are developed for one-shot Bayesian games to characterize the Nash equilibrium.
Chapter 6 discussed stopping time POMDPs and gave sufficient conditions for the optimal policy to have a monotone structure. In this chapter we consider more general POMDPs (not necessarily with a stopping action) and present the following structural results:

1. **Upper and Lower Myopic Policy Bounds using Copositivity Dominance**: For general POMDPs it is difficult to provide sufficient conditions for monotone policies. Instead, we provide sufficient conditions so that the optimal policy can be upper and lower bounded by judiciously chosen myopic policies. These sufficient conditions involve the copositive ordering described in Chapter 4. The myopic policy bounds are constructed to maximize the volume of belief states where they coincide with the optimal policy. Numerical examples illustrate these myopic policies for continuous and discrete valued observations.

2. **Upper Myopic Policy Bounds using Blackwell Dominance**: Suppose the observation probabilities for actions 1 and 2 can be related via the following factorization: $B(1) = B(2) R$ where $R$ is a stochastic matrix. We then say that $B(2)$ Blackwell dominates $B(1)$. If this Blackwell dominance holds, we will show that a myopic policy coincides with the optimal policy for all belief states where choosing action 2 yields a smaller instantaneous cost than choosing action 1. Thus, the myopic policy forms an upper bound to the optimal policy. We provide two examples: scheduling an optimal filter versus an optimal predictor, and scheduling with ultrametric observation matrices.

3. **Sensitivity to POMDP parameters**: The final result considered in this chapter is: How does the optimal cumulative cost of POMDP depend on the transition and observation probabilities? The ordinal results use the copositive ordering of transition matrices and Blackwell dominance of observation matrices that yield an ordering of the achievable optimal costs of a POMDP.

### 8.1 The Partially Observed Markov Decision Process

Throughout this chapter we will consider discounted cost infinite horizon POMDPs discussed in §2.6. Let us briefly review this model. A discrete time Markov chain evolves on the state space $X = \{e_1, e_2, \ldots, e_X\}$ where $e_i$ denotes the unit $X$-dimensional vector with 1 in the $i$-th position. Denote the action space as
The aim is to compute the optimal stationary policy \( \mu^* : \Pi(X) \to \mathcal{U} \) such that \( J_{\mu^*}(\pi_0) \leq J_{\mu}(\pi_0) \) for all \( \pi_0 \in \Pi(X) \). Obtaining the optimal policy \( \mu^* \) is equivalent to solving Bellman’s dynamic programming equation: \( \mu^*(\pi) = \arg\min_{u \in \mathcal{U}} Q(\pi, u) \), where

\[
V(\pi) = \min_{u \in \mathcal{U}} Q(\pi, u), \quad Q(\pi, u) = c_u\pi + \rho \sum_{y \in Y} V(T(\pi, y, u)) \sigma(\pi, y, u). \tag{8.3}
\]

Since \( \Pi(X) \) is continuum, Bellman’s equation (8.3) does not translate into practical solution methodologies. This motivates the construction of judicious myopic policies that upper and lower bound \( \mu^*(\pi) \).

### 8.2 Myopic Policies using Copositive Dominance: Insight

For stopping time POMDPs, in Chapter 6 we gave sufficient conditions for \( Q(\pi, u) \) in Bellman’s equation to be submodular, i.e., \( Q(\pi, u + 1) - Q(\pi, u) \) is decreasing in \( \pi \) with respect to the monotone likelihood ratio order. This implied that the optimal policy \( \mu^*(\pi) \) was MLR increasing in belief \( \pi \) and had a threshold structure.

Unfortunately, for a general POMDP, giving sufficient conditions for \( Q(\pi, u) \) to be submodular is still an open problem.\(^1\) Instead of showing submodularity, in this chapter we will give sufficient conditions for \( Q(\pi, u) \) to satisfy

\[
Q(\pi, u + 1) - Q(\pi, u) \leq C(\pi, u + 1) - C(\pi, u) \tag{8.4}
\]

\(^1\)For the two-state case, conditions for submodularity are given in §5.3, but these do not generalize to more than two states.
where $C(\pi, u)$ is a cleverly chosen instantaneous cost in terms of the belief state.

A nice consequence of (8.4) is the following: Let $\overline{\pi}(\pi) = \text{argmin}_u C(\pi, u)$ denote the myopic policy that minimizes the instantaneous cost. Then (8.4) implies that the optimal policy $\mu^*(\pi)$ satisfies

$$\mu^*(\pi) \leq \overline{\pi}(\pi), \quad \text{for all } \pi \in \Pi(X).$$

In words: if (8.4) holds, then the myopic policy $\mu(\pi)$ is provably an upper bound to the optimal policy $\mu^*(\pi)$. Since the myopic policy is trivially computed, this is a useful result. But there is more! As will be described below, for discounted cost POMDPs, the optimal policy remains unchanged for a family of costs $C(\pi, u)$. So by judiciously choosing these costs we can also construct myopic policies $\mu(\pi)$ that lower bound the optimal policy. To summarize, for any belief state $\pi$, we will present sufficient conditions under which the optimal policy $\mu^*(\pi)$ of a POMDP can be upper and lower bounded by myopic policies denoted by $\overline{\pi}(\pi)$ and $\mu(\pi)$, respectively, i.e., (see Figure 8.1 for a visual display)

$$\mu(\pi) \leq \mu^*(\pi) \leq \overline{\pi}(\pi) \quad \text{for all } \pi \in \Pi(X). \quad (8.5)$$

Clearly, for belief states $\pi$ where $\mu(\pi) = \mu(\pi)$, the optimal policy $\mu^*(\pi)$ is completely determined.

![Figure 8.1](image)

**Figure 8.1** Illustration of main result of this chapter. The aim is to construct an upper bound $\overline{\mu}$ (dashed line) and lower bound $\mu$ (dotted line), to the optimal policy $\mu^*$ (solid line) such that (8.5) holds for each belief state $\pi$. Thus the optimal policy is sandwiched between the judiciously chosen myopic policies $\mu$ and $\overline{\mu}$ over the entire belief space $\Pi(X)$. Note for $\pi$ where $\overline{\pi}(\pi) = \mu(\pi)$, they coincide with the optimal policy $\mu^*(\pi)$. Maximizing the volume of beliefs where $\mu(\pi) = \overline{\pi}(\pi)$ is achieved by solving a linear programming problem as described in §8.4.

Interestingly, these judiciously constructed myopic policies are independent of the actual values of the observation probabilities (providing they satisfy a sufficient condition) which makes the structural results applicable to both discrete and continuous observations. Finally, we will construct the myopic policies, $\overline{\pi}(\pi)$ and $\mu(\pi)$, to maximize the volume of the belief space where they coincide with the optimal policy $\mu^*(\pi)$.

As an extension of the above results, motivated by examples in controlled sensing [61, 128, 8], one can show that similar myopic bounds hold for POMDPs with quadratic costs in the belief state.
Numerical examples are presented to illustrate the performance of these myopic policies. To quantify how well the myopic policies perform we use two parameters: the volume of the belief space where the myopic policies coincide with the optimal policy, and an upper bound to the average percentage loss in optimality due to following this optimized myopic policy.

**Context.**
The papers [78, 102, 103] give sufficient conditions for (8.4) so that the optimal policy of a POMDP can be upper bounded\(^2\) by a myopic policy. Unfortunately, despite the enormous usefulness of such a result, the sufficient conditions given in [78] and [102] are not useful - it is impossible to generate non-trivial examples that satisfy the conditions (c), (e), (f) of [78, Proposition 2] and condition (i) of [102, Theorem 5.6]. In this chapter, we provide a fix to these sufficient conditions so that the results of [78, 102] hold for constructing a myopic policy that upper bounds the optimal policy. It turns out that Assumptions (F3') and (F4) described in Chapter 4 are precisely the fix we need. We also show how this idea of constructing a upper bound myopic policy can be extended to constructing a lower bound myopic policy.

### 8.3 Constructing Myopic Policy Bounds for Optimal Policy using Copositive Dominance

With the above motivation, we are now ready to construct myopic policies that provably sandwich the optimal policy for a POMDP.

**Assumptions**

(F3’) and (F4) below are the main copositivity assumptions.

(C1) There exists a vector \( g \in \mathbb{R}^{X} \) such that the \( X \)-dimensional vector \( C_u \equiv c_u + (I - \rho P(u)) g \) is strictly increasing elementwise for each action \( u \in U \).

(C2) There exists a vector \( f \in \mathbb{R}^{X} \) such that the \( X \)-dimensional vector \( C_u \equiv c_u + (I - \rho P(u)) f \) is strictly decreasing elementwise for each action \( u \in U \).

(F1) \( B(u), u \in U \) is totally positive of order 2 (TP2). That is, all second-order minors and nonnegative.

(F2) \( P(u), u \in U \) is totally positive of order 2 (TP2).

(F3') \( \gamma_{j,u,y}^{m,n} + \gamma_{j,u,y}^{m,n} \geq 0 \) \( \forall m, n, j, u, y \) where

\[
\gamma_{j,u,y}^{m,n} = B_{j,y}^{1}(u)B_{j+1,y}^{1}(u+1)P_{m,j}(u)P_{n,j+1}(u+1) - B_{j+1,y}^{1}(u)B_{j,y}^{1}(u+1)P_{m,j+1}(u)P_{n,j}(u+1)
\]

\(^2\)Since [78] deals with maximization rather than minimization, the myopic policy constructed in [78] forms a lower bound.
(F4) $\sum_{y \leq \bar{y}} \sum_{j=1}^{X} [P_{i,j}(u)B_{j,y}(u) - P_{i,j}(u+1)B_{j,y}(u+1)] \leq 0$ for $i \in \{1, 2, \ldots, X\}$ and $\bar{y} \in \mathcal{Y}$.

**Discussion**

Recall (F1), (F2), (F3'), (F4) were discussed in Chapter 4. As described in Chapter 4, (F3') and (F4) are a relaxed version of Assumptions (c), (e), (f) of [78, Proposition 2] and Assumption (i) of [102, Theorem 5.6]. In particular, the assumptions (c), (e), (f) of [78] require that $P(u+1) \geq \rho P(u)$ and $B(u+1) \geq \rho B(u)$, where $\rho$ (TP2 stochastic ordering) is defined in [88], which is impossible for stochastic matrices, unless $P(u) = P(u+1)$, $B(u) = B(u+1)$ or the matrices $P(u)$, $B(u)$ are rank 1 for all $u$ meaning that the observations are non-informative.

Let us now discuss (C1) and (C2). If the elements of $c_u$ are strictly increasing then (C1) holds trivially. Similarly, if the elements of $c_u$ are strictly decreasing then (C2) holds; indeed then (C2) is equivalent to (C) on page 58.

(C1) and (C2) are easily verified by checking the feasibility of the following linear programs:

\[
\begin{align*}
LP_1 : \min_{g \in S_g} & \ 1' \chi g, \\
LP_2 : \min_{f \in S_f} & \ 1' \chi f.
\end{align*}
\]

where $\chi = \{g : c_u e_i \leq c_u e_{i+1} \ \forall u, i \in \{1, 2, \ldots, X\}\}$ and $\chi = \{f : c_u e_i \geq c_u e_{i+1} \ \forall u, i \in \{1, 2, \ldots, X\}\}$.

**8.3.1 Construction of Myopic Upper and Lower Bounds**

We are interested in myopic policies of the form $\arg\min_{u \in \mathcal{U}} C'_u \pi$ where cost vectors $C_u$ are constructed so that when applied to Bellman’s equation (8.3), they leave the optimal policy $\mu^*(\pi)$ unchanged. This is for several reasons: First, similar to [78], [102] it allows us to construct useful myopic policies that provide provable upper and lower bounds to the optimal policy. Second, these myopic policies can be straightforwardly extended to 2-stage or multi-stage myopic costs. Third, such a choice precludes choosing useless myopic bounds such as $\overline{\mu}(\pi) = \mathcal{U}$ for all $\pi \in \Pi(X)$.

Accordingly, for any two vectors $g$ and $f \in \mathbb{R}^X$, define the myopic policies associated with the transformed costs $\overline{C}_u$ and $\underline{C}_u$ as follows:

\[
\overline{\mu}(\pi) \equiv \arg\min_{u \in \mathcal{U}} \overline{C}_u \pi, \quad \text{where} \quad \overline{C}_u = c_u + (I - \rho P(u)) g \tag{8.10}
\]

\[
\underline{\mu}(\pi) \equiv \arg\min_{u \in \mathcal{U}} \underline{C}_u \pi, \quad \text{where} \quad \underline{C}_u = c_u + (I - \rho P(u)) f. \tag{8.11}
\]

It is easily seen that Bellman’s equation (8.3) applied to optimize the objective (8.1) with transformed costs $\overline{C}_u$ and $\underline{C}_u$ yields the same optimal strategy $\mu^*(\pi)$.
8.4 Optimizing the Myopic Policy Bounds to Match the Optimal Policy

as the Bellman’s equation with original costs $c_u$. The corresponding value functions are $\mathbf{V}(\pi) \equiv V(\pi) + g'(\pi)$ and $\underline{\mathbf{V}}(\pi) \equiv V(\pi) + f'(\pi)$.

**Theorem 8.3.1** Consider a POMDP $(X, U, Y, P(u), B(u), c, \rho)$ and assume (C1), (F1), (F2), (F3'), (F4) holds. Then the myopic policies, $\overline{\mu}(\pi)$ and $\underline{\mu}(\pi)$, defined in (8.10), (8.11) satisfy: $\underline{\mu}(\pi) \leq \mu^*(\pi) \leq \overline{\mu}(\pi)$ for all $\pi \in \Pi(X)$.

The above result where the optimal policy $\mu^*(\pi)$ is sandwiched between $\underline{\mu}(\pi)$ and $\overline{\mu}(\pi)$ is illustrated in Figure 8.1 for $X = 2$.

**Proof** We show that under (C1),(F1), (F2), (F3') and (F4), $\underline{\mu}(\pi) \leq \mu^*(\pi) \leq \overline{\mu}(\pi)$ for all $\pi \in \Pi(X)$.

Let $\mathbf{V}$ and $\overline{\mathbf{Q}}$ denote the variables in Bellman’s equation (8.3) when using costs $C_u$ defined in (8.10). Then from Theorem 5.2.1 in Chapter 6, $\mathbf{V}(T(\pi, y, u))$ is increasing in $y$. From Theorem 4.3.1(5), under (F4), $\sigma(\pi, u + 1) \geq \sigma(\pi, u)$.

Therefore,

$$
\sum_{y \in Y} \mathbf{V}(T(\pi, y, u)) \sigma(\pi, y, u) \leq \sum_{y \in Y} \mathbf{V}(T(\pi, y, u + 1)) \sigma(\pi, y, u + 1)
$$

Inequality (b) holds since from Theorem 4.3.1(4) and Theorem 5.2.1,

$$
\mathbf{V}(T(\pi, y, u + 1)) \geq \mathbf{V}(T(\pi, y, u)) \quad \forall y \in Y.
$$

Equation (8.12) implies that $\sum_{y \in Y} \mathbf{V}(T(\pi, y, u)) \sigma(\pi, y, u)$ is increasing w.r.t. $u$ or equivalently,

$$
\overline{\mathbf{Q}}(\pi, u) - C_u \pi \leq \overline{\mathbf{Q}}(\pi, u + 1) - C_{u+1} \pi.
$$

It therefore follows that

$$
\{ \pi : C_u \pi \geq C_u \pi \} \subseteq \{ \pi : \overline{\mathbf{Q}}(\pi, u') \geq \overline{\mathbf{Q}}(\pi, u) \}, \quad u' > u
$$

which implies that $\overline{\mu}(\pi) \leq u \implies \mu^*(\pi) \leq u$. The proof that $\mu^*(\pi) \geq \mu(\pi)$ is similar and omitted. (See [77, Lemma 1] for a more general statement.)

8.4 Optimizing the Myopic Policy Bounds to Match the Optimal Policy

The aim of this section is to determine the vectors $g$ and $f$, in (8.8) and (8.9), that maximize the volume of the simplex where the myopic upper and lower policy bounds, specified by (8.10) and (8.11), coincide with the optimal policy. That is, we wish to maximize the volume of the ‘overlapping region’

$$
\text{vol}(\Pi_O), \text{ where } \Pi_O \equiv \{ \pi : \overline{\mu}(\pi) = \mu(\pi) = \mu^*(\pi) \}.
$$

Notice that the myopic policies $\overline{\mu}$ and $\mu$ defined in (8.10), (8.11) do not depend on the observation probabilities $B_u$ and so neither does $\text{vol}(\Pi_O)$. So $\overline{\mu}$ and $\mu$
can be chosen to maximize $\text{vol}(\Pi_O)$ independent of $B(u)$ and therefore work for discrete and continuous observation spaces. Of course, the proof of Theorem 8.3.1 requires conditions on $B(u)$.

Optimized Myopic Policy for Two Actions

For a two action POMDP, obviously for a belief $\pi$, if $\underline{\pi}(\pi) = 1$ then $\mu^*(\pi) = 1$. Similarly, if $\overline{\mu}(\pi) = 2$, then $\mu^*(\pi) = 2$. Denote the set of beliefs (convex polytopes) where $\underline{\pi}(\pi) = \mu^*(\pi) = 1$ and $\overline{\mu}(\pi) = \mu^*(\pi) = 2$ as

$$
\Pi(X)_1 = \{ \pi : \underline{\pi} \leq \overline{\pi} \}, \quad \Pi(X)_2 = \{ \pi : \underline{\pi} \leq \overline{\pi} \}.
$$

Similarly, if $\overline{\Pi}_O = \Pi(X)_1 \cup \Pi(X)_2$. Our goal is to find $g^* \in S_g$ and $f^* \in S_f$ such that $\text{vol}(\Pi_O)$ is maximized.

**THEOREM 8.4.1** Assume that there exists two fixed $X$-dimensional vectors $g^*$ and $f^*$ such that

$$
(P(2) - P(1))g^* \leq (P(2) - P(1))g, \forall g \in S_g
$$

$$
(P(1) - P(2))f^* \leq (P(1) - P(2))f, \forall f \in S_f,
$$

(8.16)

where for $X$-dimensional vectors $a$ and $b$, $a \preceq b \Rightarrow [a_1 \leq b_1, \ldots, a_X \leq b_X]$. If the myopic policies $\overline{\pi}$ and $\underline{\pi}$ are constructed using $g^*$ and $f^*$, then $\text{vol}(\Pi_O)$ is maximized.

**Proof** The sufficient conditions in (8.16) ensure that $\Pi(X)_1 \supseteq \Pi(X)_2 \forall g \in S_g$ and $\Pi(X)_2 \supseteq \Pi(X)_1 \forall f \in S_f$. Indeed, to establish that $\text{vol}(\Pi(X)_1) \geq \text{vol}(\Pi(X)_2) \forall g \in S_g$, we have

$$
(P(1) - P(2))g^* \geq (P(1) - P(2))g, \forall g \in S_g
$$

(8.17)

$$
\Rightarrow c_1 - c_2 - \rho (P(1) - P(2))g^* \geq c_1 - c_2 - \rho (P(1) - P(2))g, \forall g \in S_g
$$

$$
\Rightarrow \Pi(X)_1 \supseteq \Pi(X)_2 \forall g \in S_g \Rightarrow \text{vol}(\Pi(X)_1) \geq \text{vol}(\Pi(X)_2) \forall g \in S_g
$$

So $\text{vol}(\Pi(X)_1) \geq \text{vol}(\Pi(X)_2) \forall g \in S_g$ and $\text{vol}(\Pi(X)_2) \geq \text{vol}(\Pi(X)_1) \forall g \in S_g$. Since $\Pi_O = \Pi(X)_1 \cup \Pi(X)_2$, the proof is complete. \qed

Theorem 8.4.1 asserts that myopic policies $\overline{\pi}$ and $\underline{\pi}$ characterized by two fixed vectors $g^*$ and $f^*$ maximize $\text{vol}(\Pi_O)$ over the entire belief space $\Pi(X)$. The existence and computation of these policies characterized by $g^* \in S_g$ and $f^* \in S_f$ are determined by Algorithm 4. Algorithm 4 solves $X$ linear programs to obtain $g^*$. If no $g^* \in S_g$ satisfying (8.16) exists, then Algorithm 4 will terminate with no solution. The procedure for computing $f^*$ is similar.
Algorithm 4 Compute $g^*$

1: for all $i \in X$ do
2: \[ \alpha_i \leftarrow \min_{g \in S} e_i^i P(2) - P(1) \] g
3: end for
4: \[ g^* \in S^\alpha, S^\alpha \equiv \{ g^* : g^* \in S_g, e_i^i P(2) - P(1) g^* = \alpha_i, i = 1, \ldots, X \} \]
5: \[ \bar{\pi}(\pi) = \arg\min_{u \in \{1, 2\}} \pi' C^*_u \forall \pi \in \Pi(X), \text{where } C_u = c_u + (I - \rho P(u)) g^* \]
6: \[ \bar{\pi}(\pi) = \mu^*(\pi) = 1, \forall \pi \in \Pi(X)_1^\alpha \]

8.5 Numerical Examples

Optimizing Myopic Policies for more than 2 actions

Unlike Theorem 8.4.1, for the case $U > 2$, we are unable to show that a single fixed choice of $\bar{\pi}$ and $\mu$ maximizes $\text{vol}(\Pi_O)$. Instead at each time $k$, $\bar{\pi}$ and $\mu$ are optimized depending on the belief state $\pi_k$. Suppose at time $k$, given observation $y_k$, the belief state, $\pi_k$, is computed by using (8.2). For this belief state $\pi_k$, the aim is to compute $g^* \in S_g$ (8.8) and $f^* \in S_f$ (8.9) such that the difference between myopic policy bounds, $\bar{\pi}(\pi_k) - \mu(\pi_k)$, is minimized. That is,

\[
(g^*, f^*) = \arg\min_{g \in S_g, f \in S_f} \bar{\pi}(\pi_k) - \mu(\pi_k). \tag{8.18}
\]

(8.18) can be decomposed into following two optimization problems,

\[
g^* = \arg\min_{g \in S_g} \bar{\pi}(\pi_k), f^* = \arg\max_{f \in S_f} \mu(\pi_k). \tag{8.19}
\]

If assumptions (C1) and (C2) hold, then the optimizations in (8.19) are feasible. Then $\bar{\pi}(\pi_k)$ in (8.10) and $g^*$, in (8.18) can be computed as follows: Starting with $\bar{\pi}(\pi_k) = 1$, successively solve a maximum of $U$ feasibility LPs, where the $i$th LP searches for a feasible $g \in S_g$ in (8.8) so that the myopic upper bound yields action $i$, i.e. $\bar{\pi}(\pi_k) = i$. The $i$th feasibility LP can be written as

\[
\min_{g \in S_g} 1^i \pi g \\
\text{s.t.}, \quad C_i^i \pi_k \leq C_u^i \pi_k \forall u \in U, u \neq i \tag{8.20}
\]

The smallest $i$, for which (8.20) is feasible, yields the solution $(g^*, \bar{\pi}(\pi_k) = i)$ of the optimization in (8.19). The above procedure is straightforwardly modified to obtain $f^*$ and the lower bound $\mu(\pi_k)$ (8.11).

8.5 Numerical Examples

Recall that on the set $\Pi_O$ (8.14), the upper and lower myopic bounds coincide with the optimal policy $\mu^*(\pi)$. What is the performance loss outside the set $\Pi_O$?
To quantify this, define the policy

$$\tilde{\mu}(\pi) = \begin{cases} 
\mu^*(\pi) & \forall \pi \in \Pi_O \\
\text{arbitrary action (e.g. 1)} & \forall \pi \notin \Pi_O
\end{cases}$$

Let $J_{\tilde{\mu}}(\pi_0)$ denote the discounted cost associated with $\tilde{\mu}(\pi_0)$. Also denote

$$\tilde{J}_{\mu^*}(\pi_0) = E\left\{ \sum_{k=0}^{\infty} \rho^k \tilde{c}_{\mu^*}(\pi_k) \pi_k \right\},$$

where

$$\tilde{c}_{\mu^*}(\pi) = \begin{cases} 
c_{\mu^*}(\pi) & \pi \in \Pi_O \\
\min_{u \in U} c(1, u), \cdots, \min_{u \in U} c(X, u) & \pi \notin \Pi_O
\end{cases}$$

Clearly an upper bound for the percentage loss in optimality due to using policy $\tilde{\mu}$ instead of optimal policy $\mu^*$ is

$$\epsilon = \frac{J_{\tilde{\mu}}(\pi_0) - \tilde{J}_{\mu^*}(\pi_0)}{\tilde{J}_{\mu^*}(\pi_0)}$$  \hspace{1cm} (8.21)

In the numerical examples below, to evaluate $\epsilon$, 1000 Monte-Carlo simulations were run to estimate the discounted costs $J_{\tilde{\mu}}(\pi_0)$ and $\tilde{J}_{\mu^*}(\pi_0)$ over a horizon of 100 time units. The parameters $\epsilon$ and $\text{vol}(\Pi_O)$ are used to evaluate the performance of the optimized myopic policy bounds constructed according to §8.4. Note that $\epsilon$ depends on the choice of observation distribution $B$, unlike $\text{vol}(\Pi_O)$, see discussion below (8.14) and also Example 2 below.

Example 1. Sampling and Measurement Control with Two Actions: In this problem discussed in §7.4, at every decision epoch, the decision maker has the option of either recording a noisy observation (of a Markov chain) instantly (action $u = 2$) or waiting for one time unit and then recording an observation using a better sensor (action $u = 1$). Should one record observations more frequently and less accurately or more accurately but less frequently?

We chose $X = 3$, $U = 2$ and $Y = 3$. Both transition and observation probabilities are action dependent (parameters specified in the Appendix). The percentage loss in optimality is evaluated by simulation for different values of the discount factor $\rho$. Table 8.1(a) displays $\text{vol}(\Pi_O)$, $\epsilon_1$ and $\epsilon_2$. For each $\rho$, $\epsilon_1$ is obtained by assuming $\pi_0 = e_3$ (myopic bounds overlap at $e_3$) and $\epsilon_2$ is obtained by uniformly sampling $\pi_0 \notin \Pi_O$. Observe that $\text{vol}(\Pi_O)$ is large and $\epsilon_1$, $\epsilon_2$ are small, which indicates the usefulness of the proposed myopic policies.

Example 2. 10-state POMDP: Consider a POMDP with $X = 10$, $U = 2$. Consider two sub-examples: the first with discrete observations $Y = 10$ (parameters in Appendix), the second with continuous observations obtained using the additive Gaussian noise model, i.e. $y_k = x_k + n_k$ where $n_k \sim N(0, 1)$. The percentage loss in optimality is evaluated by simulation for these two sub-examples.
8.5 Numerical Examples

Table 8.1 Performance of optimized myopic policies versus discount factor $\rho$ for five numerical examples. The performance metrics $\text{vol}(\Pi_O)$ and $\epsilon$ are defined in (8.14) and (8.21).

(a) Example 1

| $\rho$ | $\text{vol}(\Pi_O)$ | $\epsilon_1$ | $\epsilon_2$ |
|-------|---------------------|-------------|-------------|
| 0.4   | 95.3%               | 0.30%       | 16.6%       |
| 0.5   | 94.2%               | 0.61%       | 13.9%       |
| 0.6   | 92.4%               | 1.56%       | 11.8%       |
| 0.7   | 90.2%               | 1.63%       | 9.1%        |
| 0.8   | 87.4%               | 1.44%       | 6.3%        |
| 0.9   | 84.1%               | 1.00%       | 3.2%        |

(b) Example 2

| $\text{vol}(\Pi_O)$ | $\epsilon_1^{d}$ | $\epsilon_2^{d}$ | $\epsilon_1^{c}$ | $\epsilon_2^{c}$ |
|---------------------|-------------------|-------------------|-------------------|-------------------|
| 64.27%              | 7.73%             | 12.88%            | 6.92%             | 454.31%           |
| 55.27%              | 8.58%             | 12.36%            | 8.99%             | 298.51%           |
| 46.97%              | 9.73%             | 11.91%            | 12.4%             | 205.50%           |
| 39.87%              | 9.38%             | 11.44%            | 14.4%             | 136.31%           |
| 34.51%              | 9.84%             | 12.49%            | 12.4%             | 99.71%            |
| 29.62%              | 11.2%             | 12.24%            | 20.5%             | 52.16%            |

(c) Example 3

| $\text{vol}(\Pi_O)$ | $\epsilon_1^{d}$ | $\epsilon_2^{d}$ | $\epsilon_1^{c}$ | $\epsilon_2^{c}$ |
|---------------------|-------------------|-------------------|-------------------|-------------------|
| 61.4%               | 2.5%              | 10.1%             |                   |                   |
| 56.2%               | 2.3%              | 6.9%              |                   |                   |
| 47.8%               | 1.7%              | 4.9%              |                   |                   |
| 40.7%               | 1.4%              | 3.5%              |                   |                   |
| 34.7%               | 1.1%              | 2.3%              |                   |                   |
| 31.8%               | 0.7%              | 1.4%              |                   |                   |

(d) Example 4

| $\text{vol}(\Pi_O)$ | $\text{vol}(\Pi_O)$ | $\tau_1$ | $\epsilon_1$ | $\epsilon_2$ |
|---------------------|---------------------|----------|--------------|--------------|
| 98.9%               | 84.5%               | 0.10%    | 6.17%        | 1.45%        |
| 98.6%               | 80.0%               | 0.18%    | 7.75%        | 1.22%        |
| 98.4%               | 75.0%               | 0.23%    | 11.62%       | 1.00%        |
| 98.1%               | 68.9%               | 0.26%    | 14.82%       | 0.75%        |
| 97.8%               | 61.5%               | 0.27%    | 19.74%       | 0.51%        |
| 97.6%               | 52.8%               | 0.25%    | 24.08%       | 0.26%        |

and denoted by $\epsilon_1^{d}$, $\epsilon_2^{d}$ (discrete observations) and $\epsilon_1^{c}$, $\epsilon_2^{c}$ (Gaussian observations) in Table 8.1(b).

$\epsilon_1^{d}$ and $\epsilon_1^{c}$ are obtained by assuming $\pi_0 = e_5$ (myopic bounds overlap at $e_5$). $\epsilon_2^{d}$ and $\epsilon_2^{c}$ are obtained by sampling $\pi_0 \notin \Pi_O$. Observe from Table 8.1(b) that $\text{vol}(\Pi_O)$ decreases with $\rho$.

Example 3. 8-state and 8-action POMDP: Consider a POMDP with $X = 8, U = 8$ and $Y = 8$ (parameters in Appendix). Table 8.1(c) displays $\text{vol}(\Pi_O)$, $\epsilon_1$ and $\epsilon_2$. For each $\rho$, $\epsilon_1$ is obtained by assuming $\pi_0 = e_1$ (myopic bounds overlap at $e_1$) and $\epsilon_2$ is obtained by uniformly sampling $\pi_0 \notin \Pi_O$. The results indicate that the myopic policy bounds are still useful for some values of $\rho$.

Example 4. Myopic Bounds versus Transition Matrix: The aim here is to illustrate the performance of the optimized myopic bounds over a range of transition probabilities. Consider a POMDP with $X = 3, U = 2$, additive Gaussian noise
model of Example 2, and transition matrices

\[
P(2) = \begin{bmatrix}
1 & 0 & 0 \\
1 - 2\theta_1 & \theta_1 & \theta_1 \\
1 - 2\theta_2 & \theta_2 & \theta_2
\end{bmatrix}, \quad P(1) = P^2(2)
\]

It is straightforward to show that \(\forall \theta_1, \theta_2\) such that \(\theta_1 + \theta_2 \leq 1, \theta_2 \geq \theta_1\), \(P(1)\) and \(P(2)\) satisfy (F2) and (F3'). The costs are \(c_1 = [1, 1, 1, 2]\)' and \(c_2 = [1, 2, 1, 1]'\).

Table 8.1(d) displays the worst case and best case values for performance metrics \((\text{vol}(\Pi_O), \epsilon_1, \epsilon_2)\) versus discount factor \(\rho\) by sweeping over the entire range of \((\theta_1, \theta_2)\). The worst case performance is denoted by \(\text{vol}(\Pi_O), \epsilon_1, \epsilon_2\) and the best case by \(\text{vol}(\Pi_O), \overline{\epsilon}_1, \overline{\epsilon}_2\).

8.6 Blackwell Dominance of Observation Distributions and Optimality of Myopic Policies

In previous sections of this chapter, we used copositive dominance to construct upper and lower myopic bounds to the optimal policy of a POMDP. In this section we will use another concept, called Blackwell dominance, to construct lower myopic bounds to the optimal policy for a POMDP.

8.6.1 Myopic Policy Bound to Optimal Decision Policy

Motivated by active sensing applications, consider the following POMDPs where based on the current belief state \(\pi_{k-1}\), agent \(k\) chooses sensing mode

\[u_k \in \{1 \text{ (low resolution sensor)}, 2 \text{ (high resolution sensor)}\}.
\]

Depending on its mode \(u_k\), the sensor views the world according to this mode – that is, it obtains observation from a distribution that depends on \(u_k\). Assume that for mode \(u \in \{1, 2\}\), the observation \(y(u) \in Y(u) = \{1, \ldots, Y(u)\}\) is obtained from the matrix of conditional probabilities

\[B(u) = (B_{iy(u)}(u), i \in \{1, 2, \ldots, X\}, y(u) \in Y(u))\]

where \(B_{iy(u)}(u) = P(y(u)|x = c_i, u)\).

The notation \(Y(u)\) allows for mode dependent observation spaces. In sensor scheduling [52], the tradeoff is as follows: Mode \(u = 2\) yields more accurate observations of the state than mode \(u = 1\), but the cost of choosing mode \(u = 2\) is higher than mode \(u = 1\). Thus there is an tradeoff between the cost of acquiring information and the value of the information.

The assumption that mode \(u = 2\) yields more accurate observations than mode \(u = 1\) is modeled as follows: We say mode 2 Blackwell dominates mode 1, denoted as

\[B(2) \succeq_B B(1) \quad \text{if} \quad B(1) = B(2) R. \quad (8.22)
\]
Here $R$ is a $Y^{(2)} \times Y^{(1)}$ stochastic matrix. $R$ can be viewed as a confusion matrix that maps $Y^{(2)}$ probabilistically to $Y^{(1)}$. (In a communications context, one can view $R$ as a noisy discrete memoryless channel with input $y^{(2)}$ and output $y^{(1)}$). Intuitively (8.22) means that $B^{(2)}$ is more accurate than $B^{(1)}$.

The goal is to compute the optimal policy $\mu^*(\pi) \in \{1, 2\}$ to minimize the expected cumulative cost incurred by all the agents

$$J_\mu(\pi) = \mathbb{E}_\mu\left\{ \sum_{k=0}^{\infty} \rho^k C(\pi_k, u_k) \right\}. \quad (8.23)$$

where $\rho \in [0, 1)$ is the discount factor. Even though solving the above POMDP is computationally intractable in general, using Blackwell dominance, we show below that a myopic policy forms a lower bound for the optimal policy.

The value function $V(\pi)$ and optimal policy $\mu^*(\pi)$ satisfy Bellman’s equation

$$V(\pi) = \min_{u \in U} Q(\pi, u), \quad \mu^*(\pi) = \arg \min_{u \in U} Q(\pi, u), \quad J_{\mu^*}(\pi) = V(\pi)$$

$$Q(\pi, u) = C(\pi, u) + \rho \sum_{y^{(u)} \in Y^{(u)}} V(T(\pi, y, u)) \sigma(\pi, y, u),$$

$$T(\pi, y, u) = \frac{B_y(y^{(u)}) P'_\pi}{\sigma(\pi, y, u)}, \quad \sigma(\pi, y, u) = 1_X B_y(y^{(u)}) P'_\pi. \quad (8.24)$$

We now present the structural result. Let $\Pi^s \subset \Pi$ denote the set of belief states for which $C(\pi, 2) < C(\pi, 1)$. Define the myopic policy

$$\mu(\pi) = \begin{cases} 
2 & \pi \in \Pi^s \\
1 & \text{otherwise}
\end{cases}$$

**Theorem 8.6.1** Assume that $C(\pi, u)$ is concave with respect to $\pi \in \Pi(X)$ for each action $u$. Suppose $B(2) \succeq_B B(1)$, i.e., $B(1) = B(2)R$ holds where $R$ is a stochastic matrix. Then the myopic policy $\mu(\pi)$ is a lower bound to the optimal policy $\mu^*(\pi)$, i.e., $\mu^*(\pi) \geq \mu(\pi)$ for all $\pi \in \Pi$. In particular, for $\pi \in \Pi^s$, $\mu^*(\pi) = \mu(\pi)$, i.e., it is optimal to choose action 2 when the belief is in $\Pi^s$. \qed

**Remark:** If $B(1) \succeq_B B(2)$, then the myopic policy constitutes an upper bound to the optimal policy.

Theorem 8.6.1 is proved below. The proof exploits the fact that the value function is concave and uses Jensen’s inequality. The usefulness of Theorem 8.6.1 stems from the fact that $\mu(\pi)$ is trivial to compute. It forms a provable lower bound to the computationally intractable optimal policy $\mu^*(\pi)$. Since $\mu$ is sub-optimal, it incurs a higher cumulative cost. This cumulative cost can be evaluated via simulation and is an upper bound to the achievable optimal cost.

Theorem 8.6.1 is non-trivial. The instantaneous costs satisfying $C(\pi, 2) < C(\pi, 1)$, does not trivially imply that the myopic policy $\mu(\pi)$ coincides with the optimal policy $\mu^*(\pi)$, since the optimal policy applies to a cumulative cost function involving an infinite horizon trajectory of the dynamical system.

It is instructive to compare Theorem 8.6.1 with Theorem 8.3.1 on page 105.
Theorem 8.3.1 used copositive dominance (with several assumptions on the POMDP model) to construct both upper and lower bounds to the optimal policy. In comparison, (Theorem 8.6.1) needs no assumptions on the POMDP model apart from the Blackwell dominance condition \( B(1) = B(2) \) \( R \) and concavity of costs with respect to the belief; but only yields an upper bound.

8.6.2 Example. Optimal Filter vs Predictor Scheduling

Suppose \( u = 2 \) is an active sensor (filter) which obtains measurements of the underlying Markov chain and uses the optimal HMM filter on these measurements to compute the belief and therefore the state estimate. So the usage cost of sensor 2 is high (since obtaining observations is expensive and can also result in increased threat of being discovered), but its performance cost is low (performance quality is high).

Suppose sensor \( u = 1 \) is a predictor which needs no measurement. So its usage cost is low (no measurement is required). However its performance cost is high since it is more inaccurate compared to sensor 2.

Since the predictor has non-informative observation probabilities, its observation probability matrix is \( B(1) = 1_{X \times Y} \). So clearly \( B(1) = B(2) \) meaning that the filter (sensor 2) Blackwell dominates the predictor (sensor 1) Theorem 8.6.1 then says that if the current belief is \( \pi_k \), then if \( C(\pi_k, 2) < C(\pi_k, 1) \), it is always optimal to deploy the filter (sensor 2).

8.6.3 Proof of Theorem 8.6.1

\( C(\pi, u) \) concave implies that \( V(\pi) \) is concave on \( II(X) \). We then use the Blackwell dominance condition (8.22). In particular,

\[
T(\pi, y(1), 1) = \sum_{y(2) \in Y(2)} T(\pi, y(2), 2) \frac{\sigma(\pi, y(2), 2)}{\sigma(\pi, y(1), 1)} P(y(1)|y(2))
\]

\[
\sigma(\pi, y(1), 1) = \sum_{y(2) \in Y(2)} \sigma(\pi, y(2), 2) P(y(1)|y(2)).
\]

Therefore \( \frac{\sigma(\pi, y(2), 2)}{\sigma(\pi, y(1), 1)} P(y(1)|y(2)) \) is a probability measure w.r.t. \( y(2) \) (since the denominator is the sum of the numerator over all \( y(2) \)). Since \( V(\cdot) \) is concave, using Jensen’s inequality it follows that

\[
V(T(\pi, y(1), 1)) = V\left( \sum_{y(2) \in Y(2)} T(\pi, y(2), 2) \frac{\sigma(\pi, y(2), 2)}{\sigma(\pi, y(1), 1)} P(y(1)|y(2)) \right)
\]

\[
\geq \sum_{y(2) \in Y(2)} V(T(\pi, y(2), 2)) \frac{\sigma(\pi, y(2), 2)}{\sigma(\pi, y(1), 1)} P(y(1)|y(2))
\]

\[
\Rightarrow \sum_{y(1)} V(T(\pi, y(1), 1)) \sigma(\pi, y(1), 1) \geq \sum_{y(2)} V(T(\pi, y(2), 2)) \sigma(\pi, y(2), 2). \quad (8.25)
\]

Therefore for \( \pi \in \Pi^s \),
\[
C(\pi, 2) + \rho \sum_{y^{(2)}} V(T(\pi, y^{(2)}, 2) \sigma(\pi, y^{(2)}, 2)) \leq C(\pi, 1) + \rho \sum_{y^{(1)}} V(T(\pi, y^{(1)}, 1) \sigma(\pi, y^{(1)}, 1)).
\]
So for \( \pi \in \Pi^s \), the optimal policy \( \mu^*(\pi) = \arg\min_{u \in U} Q(\pi, u) = 2 \). So \( \mu(\pi) = \mu^*(\pi) = 2 \) for \( \pi \in \Pi^s \) and \( \bar{\mu}(\pi) = 1 \) otherwise, implying that \( \bar{\mu}(\pi) \) is a lower bound for \( \mu^*(\pi) \).

### 8.7 How does optimal POMDP cost vary with state and observation dynamics?

This and the next section focus on achievable costs attained by the optimal policy. This section presents gives bounds on the achievable performance of the optimal policies by the decision maker. This is done by introducing a partial ordering of the transition and observation probabilities – the larger these parameters with respect to this order, the larger the optimal cumulative cost incurred.

How does the optimal expected cumulative cost \( J_{\mu^*} \) of a POMDP vary with transition matrix \( P \) and observation distribution \( B \)? Can the transition matrices and observation distributions be ordered so that the larger they are, the larger the optimal cumulative cost? Such a result is very useful – it allows us to compare the optimal performance of different POMDP models, even though computing these is intractable. Recall that the transition matrix specifies the mobility of the state and the observation matrix specifies the noise distribution; so understanding how these affect the achievable optimal cost is important.

Consider two distinct POMDPs with transition matrices \( \theta = P \) and \( \bar{\theta} = \bar{P} \), respectively. Alternatively, consider two distinct POMDPs with observation distributions \( \theta = B \) and \( \bar{\theta} = \bar{B} \), respectively. Assume that the instantaneous costs \( C(\pi, u) \) and discount factors \( \rho \) for both POMDPs are identical.

Let \( \mu^*(\theta) \) and \( \mu^*(\bar{\theta}) \) denote, respectively, the optimal policies for the two POMDPs. Let \( J_{\mu^*(\theta)}(\pi; \theta) = V(\pi; \theta) \) and \( J_{\mu^*(\bar{\theta})}(\pi; \bar{\theta}) = V(\pi; \bar{\theta}) \) denote the optimal value functions corresponding to applying the respective optimal policies.

Consider two arbitrary transition matrices \( P \) and \( \bar{P} \). Recalling Definition 4.2.3 for \( \succeq \) and (F3'), assume the copositive ordering
\[
P \succeq \bar{P}. \tag{8.26}
\]
Recall the Blackwell dominance of observation distributions: \( \bar{B} \) Blackwell dominates \( B \) denoted as
\[
\bar{B} \succeq_B B \text{ if } B = \bar{B} R \tag{8.27}
\]
where \( R = (R_{lm}) \) is a stochastic kernel, i.e., \( \sum_{l} R_{lm} = 1 \).

The question we pose is: How does the optimal cumulative cost \( J_{\mu^*(\theta)}(\pi; \theta) \) vary with transition matrix \( P \) or observation distribution \( B \)? For example, in quickest change detection, do certain phase-type distributions for the change
time result in larger optimal cumulative cost compared to other phase-type distributions? In controlled sensing, do certain noise distributions incur a larger optimal cumulative cost than other noise distributions?

**Theorem 8.7.1** 1. Consider two distinct POMDP problems with transition matrices \( P \) and \( \bar{P} \), respectively, where \( P \succeq \bar{P} \) with respect to copositive ordering (8.26). If (C), (F1), (F2) hold, then the optimal cumulative costs satisfy

\[
J_{\mu^*}(P)(\pi; P) \leq J_{\mu^*}(\bar{P})(\pi; \bar{P}).
\]

2. Consider two distinct POMDP problems with observation distributions \( B \) and \( \bar{B} \), respectively, where \( \bar{B} \succeq_B B \) with respect to Blackwell ordering (8.27). Then

\[
J_{\mu^*}(B)(\pi; B) \geq J_{\mu^*}(\bar{B})(\pi; \bar{B}).
\]

The proof is in Appendix [57]. Computing the optimal policy and associated cumulative cost of a POMDP is intractable. Yet, the above theorem facilitates comparison of these optimal costs for different transition and observation probabilities.

It is instructive to compare Theorem 8.7.1(1) with Theorem 3.4.1 of §3.4, which dealt with the optimal costs of two fully observed MDPs. Comparing the assumptions of Theorem 3.4.1 with Theorem 8.7.1(1), we see that the assumption on the costs (A1) is identical to (C). Assumption (A2) in Theorem 3.4.1 is replaced by (F1), (F2) which are conditions on the transition and observation probabilities. The first order dominance condition (A2) in Theorem 3.4.1 is a weaker condition than the TP2 condition (F1). In particular, (F1) implies (A2). Finally, (A5) in Theorem 3.4.1 is replaced by the stronger assumption of copositivity (F3'). Indeed, (F3') implies (A5).

Remark: An obvious consequence of Theorem 8.7.1(1) is that a Markov chain with transition probabilities \( P_{iX} = 1 \) for each state \( i \) incurs the lowest cumulative cost. After one transition such a Markov chain always remains in state \( X \). Since the instantaneous costs are decreasing with state (C), clearly, such a transition matrix incurs the lowest cumulative cost. Similarly if \( P_{i1} = 1 \) for each state \( i \), then the highest cumulative cost is incurred. A consequence of Theorem 8.7.1(2) is that the optimal cumulative cost incurred with perfect measurements is smaller than that with noisy measurements.

### 8.8 Complements and Sources

This chapter is based on [65] and extends the structural results of [78, 102]. Constructing myopic policies using Blackwell dominance goes back to [125]. [102] uses the multivariate TP2 order for POMDPs with multivariate observations. [41] shows the elegant result that the \( p \)-th root of a stochastic matrix \( P \) is a stochastic matrix providing \( P^{-1} \) is an M-matrix. The structural results can also be developed for stochastic control of continuous time HMMs. By using a robust formulation of the continuous-time HMM filter [44], time discretization
yields exactly the HMM filter. One can then consider the resulting discretized Hamilton Jacobi-Bellman equation and obtain structural results.

Although, not discussed in this article, the parameters of the underlying HMM can be estimated recursively via online estimators such as those in [70] or offline via EM type algorithms.

Appendix 8.A  POMDP Numerical Examples

Parameters of Example 1: For the first example the parameters are defined as,

\[
c = \begin{pmatrix} 1.0000 & 1.5045 & 1.8341 \\ 1.5002 & 1.0000 & 1.0000 \end{pmatrix}, \quad P(2) = \begin{pmatrix} 1.0000 & 0.0000 & 0.0000 \\ 0.4677 & 0.4149 & 0.1174 \\ 0.3302 & 0.5220 & 0.1478 \end{pmatrix}, \quad P(1) = P^2(2)
\]

\[
B(1) = \begin{pmatrix} 0.6373 & 0.3405 & 0.0222 \\ 0.3118 & 0.6399 & 0.0483 \\ 0.0422 & 0.8844 & 0.0734 \end{pmatrix}, \quad B(2) = \begin{pmatrix} 0.5927 & 0.3829 & 0.0244 \\ 0.4986 & 0.4625 & 0.0389 \\ 0.1395 & 0.79 & 0.0705 \end{pmatrix}
\]

Parameters of Example 2: For discrete observations \(B(u) = B \forall u \in \mathcal{U} \),

\[
P(1) = \begin{pmatrix} 0.9496 & 0.0056 & 0.0056 & 0.0056 & 0.0056 & 0.0056 & 0.0056 & 0.0056 & 0.0056 & 0.0056 \\ 0.9023 & 0.0081 & 0.0112 & 0.0112 & 0.0112 & 0.0112 & 0.0112 & 0.0112 & 0.0112 \end{pmatrix}
\]

\[
P(2) = \begin{pmatrix} 0.5688 & 0.0143 & 0.0521 & 0.0521 & 0.0521 & 0.0521 & 0.0521 & 0.0521 & 0.0521 & 0.0522 \\ 0.5400 & 0.0144 & 0.0557 & 0.0557 & 0.0557 & 0.0557 & 0.0557 & 0.0557 & 0.0557 & 0.0557 \\ 0.5133 & 0.0145 & 0.0590 & 0.0590 & 0.0590 & 0.0590 & 0.0590 & 0.0590 & 0.0590 & 0.0592 \\ 0.4877 & 0.0145 & 0.0622 & 0.0622 & 0.0622 & 0.0622 & 0.0622 & 0.0622 & 0.0622 & 0.0624 \\ 0.4631 & 0.0145 & 0.0653 & 0.0653 & 0.0653 & 0.0653 & 0.0653 & 0.0653 & 0.0653 & 0.0653 \\ 0.4400 & 0.0144 & 0.0682 & 0.0682 & 0.0682 & 0.0682 & 0.0682 & 0.0682 & 0.0682 & 0.0682 \\ 0.4181 & 0.0144 & 0.0709 & 0.0709 & 0.0709 & 0.0709 & 0.0709 & 0.0709 & 0.0709 & 0.0712 \\ 0.3969 & 0.0143 & 0.0736 & 0.0736 & 0.0736 & 0.0736 & 0.0736 & 0.0736 & 0.0736 & 0.0736 \\ 0.3771 & 0.0141 & 0.0761 & 0.0761 & 0.0761 & 0.0761 & 0.0761 & 0.0761 & 0.0761 & 0.0761 \\ 0.3585 & 0.0140 & 0.0784 & 0.0784 & 0.0784 & 0.0784 & 0.0784 & 0.0784 & 0.0784 & 0.0787 \end{pmatrix}
\]
Myopic Policy Bounds for POMDPs and Sensitivity

$c = \begin{pmatrix}
0.5986 & 0.5810 & 0.6116 & 0.6762 & 0.5664 & 0.6188 & 0.7107 & 0.4520 & 0.5986 & 0.7714 \\
0.6986 & 0.6727 & 0.7017 & 0.7649 & 0.6536 & 0.6005 & 0.6924 & 0.4324 & 0.5790 & 0.6714
\end{pmatrix}$

Parameters of Example 3: $B(u) = T_\omega \forall u \in \mathcal{U}$, where $T_\omega$ is a tridiagonal matrix defined as

$T_\omega = [\xi_{ij}]_{X \times X}, \xi_{ij} = \begin{cases} 
\varepsilon & i = j \\
1 - \varepsilon & (i, j) = (1, 2), (X - 1, X) \\
1 - \varepsilon & (i, j) = (i, i + 1), (i, i - 1), i \neq 1, X \\
\frac{1}{2} & \text{otherwise}
\end{cases}$

$P(1) = \begin{pmatrix}
0.1851 & 0.1692 & 0.1630 & 0.1546 & 0.1324 & 0.0889 & 0.0546 & 0.0522 \\
0.1538 & 0.1531 & 0.1601 & 0.1580 & 0.1395 & 0.0994 & 0.0667 & 0.0694 \\
0.1307 & 0.1378 & 0.1489 & 0.1595 & 0.1372 & 0.1143 & 0.0799 & 0.0847 \\
0.1157 & 0.1307 & 0.1437 & 0.1591 & 0.1496 & 0.1199 & 0.0840 & 0.0973 \\
0.1053 & 0.1196 & 0.1388 & 0.1579 & 0.1320 & 0.1248 & 0.0888 & 0.1128 \\
0.0850 & 0.1056 & 0.1326 & 0.1618 & 0.1585 & 0.1348 & 0.0977 & 0.1240 \\
0.0707 & 0.0966 & 0.1217 & 0.1578 & 0.1629 & 0.1447 & 0.1078 & 0.1438 \\
0.0549 & 0.0757 & 0.1095 & 0.1502 & 0.1666 & 0.1576 & 0.1189 & 0.1666
\end{pmatrix}$

$P(2) = \begin{pmatrix}
0.0488 & 0.0696 & 0.1016 & 0.1413 & 0.1599 & 0.1614 & 0.1270 & 0.1904 \\
0.0413 & 0.0604 & 0.0882 & 0.1292 & 0.1503 & 0.1661 & 0.1425 & 0.2220 \\
0.0329 & 0.0482 & 0.0752 & 0.1195 & 0.1525 & 0.1694 & 0.1519 & 0.2504 \\
0.0248 & 0.0368 & 0.0649 & 0.1097 & 0.1503 & 0.1732 & 0.1643 & 0.2740 \\
0.0196 & 0.0309 & 0.0566 & 0.0985 & 0.1429 & 0.1805 & 0.1745 & 0.2965 \\
0.0158 & 0.0258 & 0.0517 & 0.0934 & 0.1392 & 0.1785 & 0.1794 & 0.3162 \\
0.0134 & 0.0221 & 0.0463 & 0.0844 & 0.1335 & 0.1714 & 0.1822 & 0.3467 \\
0.0110 & 0.0186 & 0.0406 & 0.0783 & 0.1246 & 0.1679 & 0.1899 & 0.3691
\end{pmatrix}$

$P(3) = \begin{pmatrix}
0.0077 & 0.0140 & 0.0337 & 0.0704 & 0.1178 & 0.1632 & 0.1983 & 0.3949 \\
0.0058 & 0.0117 & 0.0297 & 0.0659 & 0.1122 & 0.1568 & 0.1954 & 0.4225 \\
0.0041 & 0.0090 & 0.0244 & 0.0581 & 0.1011 & 0.1494 & 0.2013 & 0.4526 \\
0.0022 & 0.0055 & 0.0165 & 0.0439 & 0.0865 & 0.1328 & 0.2006 & 0.5120 \\
0.0017 & 0.0044 & 0.0132 & 0.0362 & 0.0751 & 0.1264 & 0.2046 & 0.5384 \\
0.0012 & 0.0033 & 0.0106 & 0.0317 & 0.0702 & 0.1211 & 0.1977 & 0.5642 \\
0.0009 & 0.0025 & 0.0091 & 0.0273 & 0.0638 & 0.1134 & 0.2004 & 0.5826
\end{pmatrix}$

$P(4) = \begin{pmatrix}
0.0007 & 0.0020 & 0.0075 & 0.0244 & 0.0609 & 0.1104 & 0.2013 & 0.5928 \\
0.0005 & 0.0016 & 0.0063 & 0.0208 & 0.0527 & 0.1001 & 0.1991 & 0.6189 \\
0.0004 & 0.0013 & 0.0049 & 0.0177 & 0.0468 & 0.0923 & 0.1981 & 0.6385 \\
0.0003 & 0.0009 & 0.0038 & 0.0149 & 0.0407 & 0.0854 & 0.2010 & 0.6530 \\
0.0002 & 0.0007 & 0.0031 & 0.0123 & 0.0346 & 0.0781 & 0.2022 & 0.6688 \\
0.0001 & 0.0005 & 0.0023 & 0.0100 & 0.0303 & 0.0713 & 0.1980 & 0.6875 \\
0.0001 & 0.0004 & 0.0019 & 0.0083 & 0.0266 & 0.0683 & 0.1935 & 0.7009 \\
0.0001 & 0.0003 & 0.0014 & 0.0069 & 0.0240 & 0.0651 & 0.1878 & 0.7144
\end{pmatrix}$
8.A POMDP Numerical Examples

\[ P(5) = \begin{pmatrix}
0.0000 & 0.0002 & 0.0010 & 0.0054 & 0.0204 & 0.0590 & 0.1772 & 0.7368 \\
0.0000 & 0.0001 & 0.0008 & 0.0041 & 0.0168 & 0.0515 & 0.1663 & 0.7604 \\
0.0000 & 0.0001 & 0.0006 & 0.0038 & 0.0156 & 0.0480 & 0.1596 & 0.7723 \\
0.0000 & 0.0001 & 0.0005 & 0.0032 & 0.0139 & 0.0450 & 0.1603 & 0.777 \\
0.0000 & 0.0001 & 0.0004 & 0.0028 & 0.0124 & 0.0418 & 0.1590 & 0.7835 \\
0.0000 & 0.0001 & 0.0003 & 0.0023 & 0.0106 & 0.0389 & 0.1547 & 0.7931 \\
0.0000 & 0.0000 & 0.0002 & 0.0015 & 0.0080 & 0.0325 & 0.1386 & 0.8192
\end{pmatrix} \]

\[ P(6) = \begin{pmatrix}
0.0000 & 0.0000 & 0.0001 & 0.0012 & 0.0067 & 0.0296 & 0.1331 & 0.8293 \\
0.0000 & 0.0000 & 0.0001 & 0.0010 & 0.0059 & 0.0275 & 0.1238 & 0.8417 \\
0.0000 & 0.0000 & 0.0001 & 0.0009 & 0.0056 & 0.0272 & 0.1238 & 0.8424 \\
0.0000 & 0.0000 & 0.0001 & 0.0009 & 0.0053 & 0.0269 & 0.1234 & 0.8434 \\
0.0000 & 0.0000 & 0.0001 & 0.0006 & 0.0043 & 0.0237 & 0.1189 & 0.8524 \\
0.0000 & 0.0000 & 0.0001 & 0.0005 & 0.0038 & 0.0215 & 0.1129 & 0.8612 \\
0.0000 & 0.0000 & 0.0000 & 0.0004 & 0.0032 & 0.0191 & 0.1094 & 0.8679 \\
0.0000 & 0.0000 & 0.0000 & 0.0003 & 0.0025 & 0.0161 & 0.1011 & 0.8800
\end{pmatrix} \]

\[ P(7) = \begin{pmatrix}
0.0000 & 0.0000 & 0.0000 & 0.0003 & 0.0022 & 0.0143 & 0.0938 & 0.8894 \\
0.0000 & 0.0000 & 0.0000 & 0.0002 & 0.0019 & 0.0136 & 0.0901 & 0.8942 \\
0.0000 & 0.0000 & 0.0000 & 0.0002 & 0.0017 & 0.0126 & 0.0849 & 0.9006 \\
0.0000 & 0.0000 & 0.0000 & 0.0002 & 0.0015 & 0.0118 & 0.0819 & 0.9046 \\
0.0000 & 0.0000 & 0.0000 & 0.0001 & 0.0013 & 0.0108 & 0.0754 & 0.9124 \\
0.0000 & 0.0000 & 0.0000 & 0.0001 & 0.0011 & 0.0098 & 0.0714 & 0.9176 \\
0.0000 & 0.0000 & 0.0000 & 0.0001 & 0.0010 & 0.0090 & 0.0713 & 0.9186 \\
0.0000 & 0.0000 & 0.0000 & 0.0001 & 0.0009 & 0.0084 & 0.0675 & 0.9231
\end{pmatrix} \]

\[ P(8) = \begin{pmatrix}
0.0000 & 0.0000 & 0.0000 & 0.0001 & 0.0008 & 0.0078 & 0.0665 & 0.9248 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0007 & 0.0068 & 0.0626 & 0.9299 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0006 & 0.0061 & 0.0581 & 0.9352 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0005 & 0.0057 & 0.0561 & 0.9377 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0005 & 0.0053 & 0.0558 & 0.9384 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0004 & 0.0051 & 0.0558 & 0.9387 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0004 & 0.0045 & 0.0522 & 0.9429 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0003 & 0.0040 & 0.0505 & 0.9452
\end{pmatrix} \]

\[
c = \begin{pmatrix}
1.0000 & 2.2486 & 4.1862 & 6.9509 & 11.2709 & 15.9589 & 21.4617 & 27.6965 \\
31.3230 & 8.8185 & 9.6669 & 11.4094 & 14.2352 & 17.8532 & 22.3155 & 27.5353 \\
50.0039 & 26.3162 & 14.6326 & 15.3534 & 17.1427 & 19.7455 & 23.1064 & 27.3025 \\
65.0359 & 40.2025 & 27.5380 & 19.5840 & 20.3017 & 21.8682 & 24.2022 & 27.4108 \\
79.1544 & 53.1922 & 39.5408 & 30.5670 & 23.3697 & 23.9185 & 25.1941 & 27.4021 \\
90.7494 & 63.6983 & 48.6593 & 38.6848 & 30.4868 & 25.7601 & 26.0012 & 27.1867 \\
99.1985 & 71.1173 & 55.0183 & 44.0069 & 34.7860 & 29.0205 & 26.9721 & 27.1546 \\
106.3851 & 77.2019 & 60.0885 & 47.8917 & 37.6330 & 30.8279 & 27.7274 & 26.4338
\end{pmatrix}
\]
References

[1] F. Vazquez Abad and V. Krishnamurthy. Constrained stochastic approximation algorithms for adaptive control of constrained Markov decision processes. In 42nd IEEE Conference on Decision and Control, pages 2823–2828, 2003.

[2] S.C. Albright. Structural results for partially observed Markov decision processes. Operations Research, 27(5):1041–1053, Sept.-Oct. 1979.

[3] E. Altman, B. Gaujal, and A. Hordijk. Discrete-Event Control of Stochastic Networks: Multimodularity and Regularity. Springer-Verlag, 2004.

[4] R. Amir. Supermodularity and complementarity in economics: An elementary survey. Southern Economic Journal, 71(3):636–660, 2005.

[5] K. J. Åström. Optimal control of Markov processes with incomplete state information. Journal of Mathematical Analysis and Applications, 10(1):174–205, 1965.

[6] S. Athey. Monotone comparative statics under uncertainty. The Quarterly Journal of Economics, 117(1):187–223, 2002.

[7] T. Banerjee and V. Veeravalli. Data-efficient quickest change detection with on-off observation control. Sequential Analysis, 31:40–77, 2012.

[8] J.S. Baras and A. Bensoussan. Optimal sensor scheduling in nonlinear filtering of diffusion processes. SIAM Journal Control and Optimization, 27(4):786–813, July 1989.

[9] M. Basseville and I.V. Nikiforov. Detection of Abrupt Changes — Theory and Applications. Information and System Sciences Series. Prentice Hall, New Jersey, USA, 1993.

[10] N. Bäuerle and U. Rieder. More risk-sensitive Markov decision processes. Mathematics of Operations Research, 39(1):105–120, 2013.

[11] T. Ben-Zvi and A. Grosfeld-Nir. Partially observed Markov decision processes with binomial observations. Operations Research Letters, 41(2):201–206, 2013.

[12] A. Bensoussan. Stochastic Control of Partially Observable Systems. Cambridge University Press, Cambridge, UK, 1992.

[13] D. P. Bertsekas. Nonlinear Programming. Athena Scientific, Belmont, MA., 2000.

[14] S. Bundfuss and M. Dür. Algorithmic copositivity detection by simplicial partition. Linear Algebra and its Applications, 428(7):1511–1523, 2008.

[15] S. Bundfuss and M. Dür. An adaptive linear approximation algorithm for copositive programs. SIAM Journal on Optimization, 20(1):30–53, 2009.

[16] E. J. Candès and T. Tao. The power of convex relaxation: Near-optimal matrix completion. IEEE Transactions on Information Theory, 56(5):2053–2080, May 2009.

[17] A. R. Cassandra. Tony’s POMDP page. http://www.cs.brown.edu/research/ai/pomdp/index.html.
[18] A. R. Cassandra. Exact and Approximate Algorithms for Partially Observed Markov Decision Process. PhD thesis, Dept. Computer Science, Brown University, 1998.
[19] A. R. Cassandra. A survey of POMDP applications. In Working Notes of AAAI 1998 Fall Symposium on Planning with Partially Observable Markov Decision Processes, pages 17–24, 1998.
[20] A. R. Cassandra, L. Kaelbling, and M. L. Littman. Acting optimally in partially observable stochastic domains. In AAAI, volume 94, pages 1023–1028, 1994.
[21] A. R. Cassandra, M. L. Littman, and N. L. Zhang. Incremental pruning: A simple fast exact method for partially observed Markov decision processes. In Proceedings of the 13th Annual Conference on Uncertainty in Artificial Intelligence (UAI-97), Providence, Rhode Island, 1997.
[22] C. Chamley. Rational herds: Economic Models of Social Learning. Cambridge University Press, 2004.
[23] S. Dayanik and C. Goulding. Detection and identification of an unobservable change in the distribution of a Markov-modulated random sequence. IEEE Transactions on Information Theory, 55(7):3323–3345, 2009.
[24] E. Denardo and U. Rothblum. Optimal stopping, exponential utility, and linear programming. Mathematical Programming, 16(1):228–244, 1979.
[25] C. Derman, G. J. Lieberman, and S. M. Ross. Optimal system allocations with penalty cost. Management Science, 23(4):399–403, December 1976.
[26] D. Djonin and V. Krishnamurthy. MIMO transmission control in fading channels - a constrained Markov decision process formulation with monotone randomized policies. IEEE Transactions on Signal Processing, 55(10):5069–5083, 2007.
[27] D. Djonin and V. Krishnamurthy. Q-learning algorithms for constrained Markov decision processes with randomized monotone policies: Applications in transmission control. IEEE Transactions on Signal Processing, 55(5):2170–2181, 2007.
[28] A. Doucet, N. Gordon, and V. Krishnamurthy. Particle filters for state estimation of jump Markov linear systems. IEEE Transactions on Signal Processing, 49:613–624, 2001.
[29] E. Dynkin. Controlled random sequences. Theory of Probability & Its Applications, 10(1):1–14, 1965.
[30] J. N. Eagle. The optimal search for a moving target when the search path is constrained. Operations Research, 32:1107–1115, 1984.
[31] R. J. Elliott, L. Aggoun, and J. B. Moore. Hidden Markov Models – Estimation and Control. Springer-Verlag, New York, 1995.
[32] R. Evans, V. Krishnamurthy, and G. Nair. Networked sensor management and data rate control for tracking maneuvering targets. IEEE Transactions on Signal Processing, 53(6):1979–1991, June 2005.
[33] M. Fazel, H. Hindi, and S. P. Boyd. Log-det heuristic for matrix rank minimization with applications to Hankel and Euclidean distance matrices. In Proceedings of the 2003 American Control Conference, 2003.
[34] F. R. Gantmacher. Matrix Theory, volume 2. Chelsea Publishing Company, New York, 1960.
[35] J. C. Gittins. Multi–armed Bandit Allocation Indices. Wiley, 1989.
[36] A. Grosfeld-Nir. Control limits for two-state partially observable Markov decision processes. European journal of operational research, 182(1):300–304, 2007.
[37] T. Hastie, R. Tibshirani, and J. Friedman. *The elements of statistical learning*. Springer-Verlag, 2009.

[38] M. Hauskrecht. Value-function approximations for partially observable Markov decision processes. *Journal of Artificial Intelligence Research*, 13(1):33–94, 2000.

[39] S. Haykin. Cognitive radio: Brain-empowered wireless communications. *IEEE Journal on Selected Areas Communications*, 23(2):201–220, Feb. 2005.

[40] D. P. Heyman and M. J. Sobel. *Stochastic Models in Operations Research*, volume 2. McGraw-Hill, 1984.

[41] N. Higham and L. Lin. On pth roots of stochastic matrices. *Linear Algebra and its Applications*, 435(3):448–463, 2011.

[42] J. Huang and V. Krishnamurthy. Transmission control in cognitive radio as a markovian dynamic game: Structural result on randomized threshold policies. *IEEE Transactions on Communications*, 58(1):301–310, 2010.

[43] K. Iida. *Studies on the Optimal Search Plan*, volume 70 of Lecture Notes in Statistics. Springer-Verlag, 1990.

[44] M. R. James, V. Krishnamurthy, and F. LeGland. Time discretization of continuous-time filters and smoothers for HMM parameter estimation. *IEEE Transactions on Information Theory*, 42(2):593–605, March 1996.

[45] M.R. James, J.S. Baras, and R.J. Elliott. Risk-sensitive control and dynamic games for partially observed discrete-time nonlinear systems. *IEEE Transactions on Automatic Control*, 39(4):780–792, April 1994.

[46] L. Johnston and V. Krishnamurthy. Opportunistic file transfer over a fading channel - a POMDP search theory formulation with optimal threshold policies. *IEEE Transactions on Wireless Commun.*, 5(2):394–405, Feb. 2006.

[47] S. Karlin. *Total Positivity*, volume 1. Stanford Univ., 1968.

[48] S. Karlin and Y. Rinott. Classes of orderings of measures and related correlation inequalities. I. Multivariate totally positive distributions. *Journal of Multivariate Analysis*, 10(4):467–498, December 1980.

[49] S. Karlin and H. M. Taylor. *A Second Course in Stochastic Processes*. Academic Press, 1981.

[50] J. Keilson and A. Kester. Monotone matrices and monotone Markov processes. *Stochastic Processes and their Applications*, 5(3):231–241, 1977.

[51] M. Kijima. *Markov Processes for Stochastic Modelling*. Chapman and Hall, 1997.

[52] V. Krishnamurthy. Algorithms for optimal scheduling and management of hidden Markov model sensors. *IEEE Transactions on Signal Processing*, 50(6):1382–1397, June 2002.

[53] V. Krishnamurthy. Emission management for low probability intercept sensors in network centric warfare. *IEEE Transactions on Aerospace and Electronic Systems*, 41(1):133–152, Jan. 2005.

[54] V. Krishnamurthy. Decentralized activation in dense sensor networks via global games. *IEEE Transactions on Signal Processing*, 56(10):4936–4950, 2008.

[55] V. Krishnamurthy. Decentralized spectrum access amongst cognitive radios-an interacting multivariate global game-theoretic approach. *IEEE Transactions on Signal Processing*, 57(10):3999–4013, Oct. 2009.

[56] V. Krishnamurthy. How to schedule measurements of a noisy Markov chain in decision making? *IEEE Transactions on Information Theory*, 59(9):4440–4461, July 2013.
[57] V. Krishnamurthy. Partially Observed Markov Decision Processes. From Filtering to Controlled Sensing. Cambridge University Press, 2016.

[58] V. Krishnamurthy and A. Aryan. Quickest detection of market shocks in agent based models of the order book. In Proceedings of the 51st IEEE Conference on Decision and Control, Maui, Hawaii, Dec. 2012.

[59] V. Krishnamurthy and A. Aryan. Detecting asset value dislocations in multi-agent models for market microstructure. In ICASSP 2013, May 2013.

[60] V. Krishnamurthy, R. Bitmead, M. Gevers, and E. Miehling. Sequential detection with mutual information stopping cost: Application in GMTI radar. IEEE Transactions on Signal Processing, 60(2):700–714, 2012.

[61] V. Krishnamurthy and D. Djonin. Structured threshold policies for dynamic sensor scheduling—a partially observed Markov decision process approach. IEEE Transactions on Signal Processing, 55(10):4938–4957, Oct. 2007.

[62] V. Krishnamurthy and D.V. Djonin. Optimal threshold policies for multivariate POMDPs in radar resource management. IEEE Transactions on Signal Processing, 57(10), 2009.

[63] V. Krishnamurthy, O. Namvar Ghareshshiran, and M. Hamdi. Interactive sensing and decision making in social networks. Foundations and Trends® in Signal Processing, 7(1-2):1–196, 2014.

[64] V. Krishnamurthy and W. Hoiles. Online reputation and polling systems: Data incest, social learning and revealed preferences. IEEE Transactions Computational Social Systems, 1(3):164–179, Jan. 2015.

[65] V. Krishnamurthy and U. Pareek. Myopic bounds for optimal policy of POMDPs: An extension of Lovejoy’s structural results. Operations Research, 62(2):428–434, 2015.

[66] V. Krishnamurthy and H. V. Poor. A tutorial on interactive sensing in social networks. IEEE Transactions on Computational Social Systems, 1(1):3–21, March 2014.

[67] V. Krishnamurthy and C. Rojas. Reduced complexity HMM filtering with stochastic dominance bounds: A convex optimization approach. IEEE Transactions on Signal Processing, 62(23):6309–6322, 2014.

[68] V. Krishnamurthy, C. Rojas, and B. Wahlberg. Computing monotone policies for Markov decision processes by exploiting sparsity. In 3rd Australian Control Conference (AUCC), pages 1–6. IEEE, 2013.

[69] V. Krishnamurthy and B. Wahlberg. POMDP multiarmed bandits – structural results. Mathematics of Operations Research, 34(2):287–302, May 2009.

[70] V. Krishnamurthy and G. Yin. Recursive algorithms for estimation of hidden Markov models and autoregressive models with Markov regime. IEEE Transactions on Information Theory, 48(2):458–476, February 2002.

[71] H. Kurniawati, D. Hsu, and W. S. Lee. Sarsop: Efficient point-based POMDP planning by approximating optimally reachable belief spaces. In Robotics: Science and Systems, volume 2008, 2008.

[72] M. L. Littman. Algorithms for sequential decision making. PhD thesis, Brown University, 1996.

[73] M. L. Littman. A tutorial on partially observable Markov decision processes. Journal of Mathematical Psychology, 53(3):119–125, 2009.
[74] K. Liu and Q. Zhao. Indexability of restless bandit problems and optimality of Whittle index for dynamic multichannel access. *IEEE Transactions on Information Theory*, 56(11):5547–5567, 2010.

[75] Z. Liu and L. Vandenberghe. Interior-point method for nuclear norm approximation with application to system identification. *SIAM Journal on Matrix Analysis and Applications*, 31(3):1235–1256, 2009.

[76] W. S. Lovejoy. On the convexity of policy regions in partially observed systems. *Operations Research*, 35(4):619–621, July-August 1987.

[77] W. S. Lovejoy. Ordered solutions for dynamic programs. *Mathematics of Operations Research*, 12(2):269–276, 1987.

[78] W. S. Lovejoy. Some monotonicity results for partially observed Markov decision processes. *Operations Research*, 35(5):736–743, Sept.-Oct. 1987.

[79] W. S. Lovejoy. Computationally feasible bounds for partially observed Markov decision processes. *Operations Research*, 39(1):162–175, January–February 1991.

[80] W. S. Lovejoy. A survey of algorithmic methods for partially observed Markov decision processes. *Annals of Operations Research*, 28:47–66, 1991.

[81] I. MacPhee and B. Jordan. Optimal search for a moving target. *Probability in the Engineering and Information Sciences*, 9:159–182, 1995.

[82] P. Milgrom. Good news and bad news: Representation theorems and applications. *Bell Journal of Economics*, 12(2):380–391, 1981.

[83] P. Milgrom and C. Shannon. Monotone comparative statics. *Econometrica*, pages 157–180, 1994.

[84] G. E. Monahan. A survey of partially observable Markov decision processes: theory, models and algorithms. *Management Science*, 28(1), January 1982.

[85] W. Moran, S. Suvorova, and S. Howard. Application of sensor scheduling concepts to radar. In A. Hero, D. Castanon, D. Cochran, and K. Kastella, editors, *Foundations and Applications for Sensor Management*, pages 221–256. Springer-Verlag, 2006.

[86] G.B. Moustakides. Optimal stopping times for detecting changes in distributions. *Annals of Statistics*, 14:1379–1387, 1986.

[87] A. Muller. How does the value function of a Markov decision process depend on the transition probabilities? *Mathematics of Operations Research*, 22:872–885, 1997.

[88] A. Muller and D. Stoyan. *Comparison Methods for Stochastic Models and Risk*. Wiley, 2002.

[89] M. F. Neuts. *Structured stochastic matrices of M/G/1 type and their applications*. Marcel Dekker, N.Y., 1989.

[90] M. H. Ngo and V. Krishnamurthy. Optimality of threshold policies for transmission scheduling in correlated fading channels. *IEEE Transactions on Communications*, 57(8):2474–2483, 2009.

[91] M. H. Ngo and V. Krishnamurthy. Monotonicity of constrained optimal transmission policies in correlated fading channels with ARQ. *IEEE Transactions on Signal Processing*, 58(1):438–451, 2010.

[92] C. H. Papadimitriou and J.N. Tsitsiklis. The complexity of Markov decision processes. *Mathematics of Operations Research*, 12(3):441–450, 1987.

[93] S. Patek. On partially observed stochastic shortest path problems. In *Proceedings of 40th IEEE Conference on Decision and Control*, pages 5050–5055, Orlando, Florida, 2001.
[94] J. Pineau, G. Gordon, and T. Sebastian. Point-based value iteration: An anytime algorithm for POMDPs. In IJCAI, volume 3, pages 1025–1032, 2003.
[95] S.M. Pollock. A simple model of search for a moving target. Operations Research, 18:893–903, 1970.
[96] H. V. Poor. Quickest detection with exponential penalty for delay. Annals of Statistics, 26(6):2179–2205, 1998.
[97] H. V. Poor and O. Hadjiliadis. Quickest Detection. Cambridge University Press, 2008.
[98] H.V. Poor. An Introduction to Signal Detection and Estimation. Springer-Verlag, New York, 2 edition, 1993.
[99] J. Quah and B. Strulovici. Aggregating the single crossing property. Econometrica, 80(5):2333–2348, 2012.
[100] L.R. Rabiner. A tutorial on hidden Markov models and selected applications in speech recognition. Proceedings of the IEEE, 77(2):257–285, 1989.
[101] V. Raghavan and V. Veeravalli. Bayesian quickest change process detection. In ISIT, pages 644–648, Seoul, 2009.
[102] U. Rieder. Structural results for partially observed control models. Methods and Models of Operations Research, 35(6):473–490, 1991.
[103] U. Rieder and R. Zagst. Monotonicity and bounds for convex stochastic control models. Mathematical Methods of Operations Research, 39(2):187–207, June 1994.
[104] B. Ristic, S. Arulampalam, and N. Gordon. Beyond the Kalman Filter: Particle Filters for Tracking Applications. Artech, 2004.
[105] S. Ross. Introduction to Stochastic Dynamic Programming. Academic Press, San Diego, California., 1983.
[106] N. Roy, G. Gordon, and S. Thrun. Finding approximate POMDP solutions through belief compression. Journal of Artificial Intelligence Research, 23:1–40, 2005.
[107] W. Rudin. Principles of Mathematical Analysis. McGraw-Hill, 1976.
[108] M. Shaked and J. G. Shanthikumar. Stochastic orders. Springer-Verlag, 2007.
[109] G. Shani, R. Brafman, and S. Shimony. Forward search value iteration for POMDPs. In IJCAI, pages 2619–2624, 2007.
[110] G. Shani, J. Pineau, and R. Kaplow. A survey of point-based POMDP solvers. Autonomous Agents and Multi-Agent Systems, 27(1):1–51, 2013.
[111] A..N. Shiryaev. On optimum methods in quickest detection problems. Theory of Probability and its Applications, 8(1):22–46, 1963.
[112] S. Singh and V. Krishnamurthy. The optimal search for a Markovian target when the search path is constrained: the infinite horizon case. IEEE Transactions on Automatic Control, 48(3):487–492, March 2003.
[113] R. D. Smallwood and E. J. Sondik. Optimal control of partially observable Markov processes over a finite horizon. Operations Research, 21:1071–1088, 1973.
[114] J. E. Smith and K. F. McCardle. Structural properties of stochastic dynamic programs. Operations Research, 50(5):796–809, 2002.
[115] E. J. Sondik. The optimal control of partially observed Markov processes. PhD thesis, Electrical Engineering, Stanford University, 1971.
[116] E. J. Sondik. The optimal control of partially observable Markov processes over the infinite horizon: discounted costs. Operations Research, 26(2):282–304, March-April 1978.
[117] M. Spaan and N. Vlassis. Perseus: Randomized point-based value iteration for POMDPs. *J. Artif. Intell. Res. (JAIR)*, 24:195–220, 2005.

[118] L. Stone. What’s happened in search theory since the 1975 Lanchester prize. *Operations Research*, 37(3):501–506, May–June 1989.

[119] R. L. Stratonovich. Conditional Markov processes. *Theory of Probability and its Applications*, 5(2):156–178, 1960.

[120] A. G. Tartakovsky and V. V. Veeravalli. General asymptotic Bayesian theory of quickest change detection. *Theory of Probability and its Applications*, 49(3):458–497, 2005.

[121] R. Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 267–288, 1996.

[122] P. Tichavsky, C. H. Muravchik, and A. Nehorai. Posterior Cramér-Rao bounds for discrete-time nonlinear filtering. *IEEE Transactions on Signal Processing*, 46(5):1386–1396, May 1998.

[123] D. M. Topkis. Minimizing a submodular function on a lattice. *Operations Research*, 26:305–321, 1978.

[124] D. M. Topkis. *Supermodularity and Complementarity*. Princeton University Press, 1998.

[125] C. C. White and D. P. Harrington. Application of Jensen’s inequality to adaptive suboptimal design. *Journal of Optimization Theory and Applications*, 32(1):89–99, 1980.

[126] W. Whitt. Multivariate monotone likelihood ratio and uniform conditional stochastic order. *Journal Applied Probability*, 19:695–701, 1982.

[127] P. Whittle. Multi-armed bandits and the Gittins index. *J. R. Statist. Soc. B*, 42(2):143–149, 1980.

[128] J. Williams, J. Fisher, , and A. Willsky. Approximate dynamic programming for communication-constrained sensor network management. *IEEE Transactions on Signal Processing*, 55(8):4300–4311, 2007.

[129] B. Yakir, A. M. Krieger, and M. Pollak. Detecting a change in regression: First-order optimality. *Annals of Statistics*, 27(6):1896–1913, 1999.

[130] D. Yao and P. Glasserman. *Monotone Structure in Discete-Event Systems*. Wiley, 1st edition, 1994.

[131] G. Yin, C. Ion, and V. Krishnamurthy. How does a stochastic optimization/approximation algorithm adapt to a randomly evolving optimum/root with jump Markov sample paths. *Mathematical programming*, 120(1):67–99, 2009.

[132] G. Yin, V. Krishnamurthy, and C. Ion. Regime switching stochastic approximation algorithms with application to adaptive discrete stochastic optimization. *SIAM Journal on Optimization*, 14(4):117–1215, 2004.

[133] S. Young, M. Gasic, B. Thomson, and J. Williams. POMDP-based statistical spoken dialog systems: A review. *Proceedings of the IEEE*, 101(5):1160–1179, 2013.

[134] Q. Zhao, L. Tong, A. Swami, and Y. Chen. Decentralized cognitive MAC for opportunistic spectrum access in ad hoc networks: A POMDP framework. *IEEE Journal on Selected Areas Communications*, pages 589–600, 2007.