Polynomial Triangles Revisited

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Abstract

A polynomial triangle is an array whose inputs are the coefficients in integral powers of a polynomial. Although polynomial coefficients have appeared in several works, there is no systematic treatise on this topic. In this paper we plan to fill this gap. We describe some aspects of these arrays, which generalize similar properties of the binomial coefficients. Some combinatorial models enumerated by polynomial coefficients, including lattice paths model, spin chain model and scores in a drawing game, are introduced. Several known binomial identities are then extended. In addition, we recursively calculate generating functions of column sequences. Interesting corollaries follow from these recurrence relations such as new formulae for the Fibonacci numbers and Hermite polynomials in terms of trinomial coefficients. Finally, we study some properties of the entropy density function which characterizes polynomial triangles in the thermodynamical limit.

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1 Introduction

The theme of our study is extremely simple. It consists in investigating several aspects of coefficients in integral powers of polynomials. These coefficients generate an array, called a polynomial triangle, whose k-th row consists of the coefficients of the powers of $t$ in $p(t)^k$, for a given polynomial $p(t)$. The table naturally resembles Pascal’s triangle and reduces to it when $p(t) = 1 + t$.

Historically, this very natural extension of the Pascal triangle may have been first discussed by Abraham De Moivre [6, 21] who found that the coefficient of $t^n$ of the polynomial

$$\left(1 + t + t^2 + \cdots + t^m\right)^k,$$

(1.1) arises in the solution of the following problem [34, p.389]:

“There are $k$ dice with $m+1$ faces marked from 1 to $m+1$; if these are thrown at random, what is the chance that the sum of the numbers exhibited shall be equal to $n$?”

A few decades later, Leonhard Euler [23, 24] published an analytical study of the coefficients of the polynomial (1.1). The elementary properties of the array generated by these coefficients closely mimic those of binomial ones. For example, each entry in the body of the (centered) triangle is the sum of the $m$ entries above it, extending a well-known property of Pascal’s triangle. This array, denoted in this paper by $T_m$, is termed the extended Pascal triangle [12], or Pascal-T triangle [52], or Pascal-De Moivre triangle [38]. In 1937, it was reintroduced by A. Tremblay [51] and in 1942 by P. Montel [40], and explicitly discussed by John Freund in 1956 [28], as arising in the solution of a restricted occupancy problem. In fact, the coefficient of $t^n$ in (1.1), denoted by George Andrews $\binom{k}{n_m}$ [5], is the number of distinct ways in which $n$ unlabeled objects can be distributed in $k$ labeled boxes allowing at most $m$ objects to fall in each box (see also the Riordan monograph [44, p.104]). In statistical physics, the Freund restricted occupancy model is known as the Gentile intermediate statistics, called after Giovanni Gentile Jr [29, 30]. This model interpolates between Fermi-Dirac statistics (binomial case : $m = 1$) and Bose-Einstein statistics ($m = \infty$). It will be referred here to as Gentile-Freund statistics (GFS).

The arrays $T_m$ have been extensively used in reliability and probability studies [6, 22, 39, 45]. Several results about the extended binomial coefficients, specially the trinomial ones ($m = 2$), are known [8, 10–14, 27]. Some generalizations have been discussed as well. For instance, Ollerton and Shannon [42] have investigated various properties and applications of a generalization of
\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 2 & 1 & 1 & 2 & 3 \\
1 & 3 & 6 & 7 & 6 & 3 & 1 & 2 \\
1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 \\
1 & 5 & 15 & 30 & 45 & 51 & 45 & 30 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

Figure 1: The centered triangle \( T_2 \)

the triangle \( T_m \) by extending the Freund occupancy model. We underline in passing that the entries of \( T_m \) have been \( q \)-generalized by George Andrews and Rodney J. Baxter [2] for \( m = 2 \) to solve the hard hexagon model in statistical mechanics and later by Warnaar [53] for arbitrary \( m \). This \( q \)-analog proved to have a deep connection with the Rogers-Ramanujan identities and the theory of partitions [3, 5].

The Pascal-De Moivre triangles \( T_2 \) (which begins as shown in Figure 1), \( T_3 \) and \( T_4 \) are recorded in Sloane’s Online Encyclopedia of Integer Sequences [47] as A027907, A008287 and A035343 respectively.

### 1.1 Preliminaries

Let us fix our terminology and notations.

**Definition 1.1.** let \( \mathbf{a} \) be a sequence of \( m + 1 \) numbers \((a_0, a_1, \ldots, a_m)\) and let \( p_{\mathbf{a}}(t) = \sum_{i=0}^{m} a_i t^i \) be its generating polynomial. The **polynomial coefficients** associated with the vector \( \mathbf{a} \) are defined by\(^1\)

\[
\binom{k}{n}_{\mathbf{a}} \overset{\text{def}}{=} \begin{cases} 
[t^n](p_{\mathbf{a}}(t))^k, & \text{if } 0 \leq n \leq mk \\
0, & \text{if } n < 0 \text{ or } n > mk 
\end{cases}
\quad (1.2)
\]

where we have used a vector-indexed binomial symbol mimicking the notation of Andrews. When \( a_i = 1 \) for all \( i \), the binomial symbol will be simply indexed by \( m \). The array of polynomial coefficients will be called \( \mathbf{a} \)-**polynomial triangle** or \( \mathbf{a} \)-**triangle** for short, and denoted by \( T(\mathbf{a}) \). Rows are indexed by \( k \) and columns by \( n \). The polynomial triangle \( T(\mathbf{a}) \) will be called **arithmetical** or **combinatorial** if the coefficients \( a_i \) are integers.

The term “polynomial coefficients” is inspired by the designation of Louis Comtet [19, p.78] for \( T_m \). Though apparently shallow, the map \( \mathbf{a} \mapsto T(\mathbf{a}) \) defined by (1.2) has quite nontrivial properties which relate to “deeper” mathematics. We stress that definition 1.1 appears in [36] as a consequence of “a generalization of Pascal’s triangle using powers of base numbers”. We quote also in this context the work of Noe [41] who studied in some detail the central coefficient in \((1 + bt + ct^2)^n\) for integers \( b, c \).

By application of the multinomial formula, we see that the polynomial coefficients are homogeneous polynomials of degree \( k \) in the numbers \( a_i \):

\[
\binom{k}{n}_{\mathbf{a}} = k! \sum_{k \in O(k,n)} \frac{\mathbf{a}^k}{k!},
\]

\( ^1 \text{We make use of the conventional notation for coefficients of entire series : } [t^n] \sum a_i t^i := a_n. \)
where the sum is over the set $O(k, n)$ of nonnegative integer vectors $k = (k_0, \ldots, k_m)$ subject to the constraints $k_0 + k_1 + \cdots + k_m = k$ and $k_1 + 2k_2 + \cdots + mk_m = n$, and where the following concise notations for powers and factorials of a vector are used

$$a^k = \prod_{i=0}^{m} a_i^{k_i}, \quad k! = \prod_{i=0}^{m} k_i!.$$ 

We note the following points about the polynomial coefficients. Eq. (1.3) can also be viewed as a “restricted” Bell polynomial in the indeterminates $a_i$. In particular cases, (1.3) can be expressed in terms of known orthogonal polynomials. For $m = 2$, it can be written in terms of the so-called two-variable one-parameter Gegenbauer polynomials defined by $(\alpha - 2xt + yt^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x, y; \alpha)t^n$. In particular, if $a_0a_2 > 0$, (1.2) is a value of the ordinary (one-variable) ultraspherical polynomials [1, p.783]:

$$\binom{k}{n}_a = a_0^{k-n/2}a_2^{n/2}C_{n}^{(-k)}\left(-\frac{a_1}{2\sqrt{a_0a_2}}\right). \quad (1.4)$$

A final point that needs to be addressed is that, with the exception of the binomial triangle, $T(a)$ is a Riordan array only if $p_a(0) = 0$, i.e., $a_0 = 0$, as can be seen from the bivariate generating function:

$$\sum_{k,n} \binom{k}{n}_a t^n u^k = \frac{1}{1- up_a(t)}.$$ 

For basic informations on Riordan arrays, the reader is referred to [46].

### 1.2 Outline and Main results

Literature on the triangles $T_m$ remains quite sparse, and there is no systematic interest in their properties. In this research paper, we plan to fill this gap by presenting a unified approach to the subject. A more ambitious program is to extend all the known properties of the ubiquitous binomial coefficients as proposed by Comtet [19].

The following items give a sample of our results:

- Besides the weighted Gentile-Freund model, we give in Section 2 three combinatorial models enumerated by polynomial coefficients:
  
  1. we show that polynomial coefficients count the number of points a player can score in a game of drawing colored balls (Theorem 2.3). We discuss, by the way, a variant of the old problem of De Moivre and rediscover that polynomial coefficients provide a natural extension of the usual binomial probability distribution based on Bernoulli trials with more than two outcomes.
  
  2. we give, via a bijective proof, an interpretation of (1.2) as number of specified colored directed lattice paths (Theorem 2.4).
  
  3. we propose an interpretation in terms of spin chain systems (Proposition 2.5).

We discuss, also in this section, some important examples of arithmetical polynomial triangles related to restricted occupancy models.
• In Section 3, we give extensions of several binomial identities (Table 1, Identities 3.1, 3.2), and investigate some algebraic properties of the mapping \( a \mapsto T(a) \). The techniques are elementary and the proofs are straightforward, but the results seem interesting in their own right.

• In Section 4, we recursively calculate generating functions of the column sequences of \( T(a) \) (Proposition 4.1). Interesting corollaries follow from these recurrence relations such as new formulae for the Fibonacci numbers (Corollary 4.2) and Hermite polynomials (Corollary 4.3) in terms of trinomial coefficients.

• We introduce in Section 5 the notion of entropy density function in the thermodynamical limit (that is when \( k \) and \( n \) go to infinity and the ratio \( n/k \) is fixed) and study its properties (Theorem 5.3).

2 Combinatorial interpretations

Besides GFS, we give in this section three combinatorial models enumerated by arithmetical polynomial coefficients \( (a_i \in \mathbb{N}) \). Our proofs are mainly bijective. To do this, we begin by presenting our own approach to the GFS.

2.1 Restricted occupancy model

Consider a ball-in-box model in which \( n \) undistinguishable balls (particles) are distributed among \( k \) distinguishable boxes (states). Without restrictions on the boxes’s occupancies, this model is known as Bose-Einstein statistics. If one object at most is allowed to occupy a box, the model is called Fermi-Dirac statistics. In the more general problem, there is a maximum and minimum number of balls that any box can contain. Restricted occupancy models have many applications [16]. For instance, the Gentile-Freund model, where one allows at most \( m \) balls to fall in each box \((m \geq 1)\), has been applied to the analysis of socioeconomic and transport systems (see, e.g., [37]).

Assume a configuration in which \( k_i \) boxes among \( k \) ones are occupied by \( i \) balls, for each \( i = 1, \ldots m \). The number of vacant boxes is obviously \( k_0 = k - \sum_{i=1}^{m} k_i \) and the total number of balls is \( n = \sum_{i=1}^{m} ik_i \). Then there are \( \binom{k}{k_0, k_1, \ldots, k_m} = \frac{k!}{k_0! k_1! \cdots k_m!} \) ways to realize a configuration. The total number of ways to put the \( n \) balls in the boxes is obtained by summing over all \((m+1)\)-tuples of non-negative integers \((k_0, k_1, \cdots, k_m)\) subject to \( \sum_{i=0}^{m} k_i = k \) and \( \sum_{i=1}^{m} ik_i = n \). Then by using the multinomial formula, one easily finds that the ordinary generating function of this number is the \( k \)-th power of the polynomial \( 1 + t + t^2 + \cdots + t^m \). That is the polynomial coefficients associated with the vector \( a_i = [i \leq m]^2 \). These coefficients have well-known properties [10, 14, 28].

As is clear from (1.3), the polynomial coefficients associated with a general vector \( a \) count the total number of distributions of the balls in the above restricted occupancy model; but each configuration is now weighted by the monomials \( a^k \).

We note, by the way, a more elegant form of (1.3) in terms of weighted restricted partitions of \( n \):

\[ 2 \text{Throughout the paper, we use the Iverson bracket notation to indicate that } [P] = 1 \text{ if the proposition } P \text{ is true and } 0 \text{ otherwise.} \]
Lemma 2.1. The polynomial coefficient (1.2) can be written as

\[
\binom{k}{n} = \sum_{\lambda \vdash n, \ell(\lambda) \leq m} a_0^{k-l(\lambda)} h(\lambda) w_\lambda a_1^{k-l(\lambda)} \cdot \binom{k}{l(\lambda)}, \tag{2.1}
\]

where \( \lambda \vdash n \) indicates that the sum runs over all partitions of \( n : \lambda = (1^{k_1}2^{k_2} \cdots m^{k_m}) \) whose greatest part do not exceed \( m \), symbolized by \( \ell(\lambda') \leq m \), \( \lambda' \) being the conjugate partition of \( \lambda \); \( h \) is the function

\[
h(\lambda) = \binom{l(\lambda)}{k_1, k_2, \ldots, k_m},
\]

\( \ell(\lambda) = \sum_{i=1}^n k_i \) is the length of the partition \( \lambda \). \( w_\lambda \) is a function that assigns to \( \lambda \) the weight

\[
w_\lambda = \prod_{i=1}^m a_i^{k_i}.
\]

Proof. Identify a configuration in which \( k \) boxes among \( k \) ones are occupied by \( i \) balls, \( i = 0, \ldots, m \), with a restricted partition \( \lambda = (1^{k_1}2^{k_2} \cdots m^{k_m}) \) of the total number of balls \( n = \sum_{i=1}^m ik_i \). To such a partition, one can attach a Ferrers diagram where the number \( k_i \) represents the multiplicity of rows with \( i \) dots and the length of the first row is less than or equal to \( m \). Then the sum in (1.3) becomes over all restricted partitions of \( n \). The expression (2.1) follows by replacing \( k_0 \) by \( k - l(\lambda) \) and rewriting the multinomial coefficient as \( h(\lambda) \binom{k}{l(\lambda)} \).

Regrouping the terms with \( l(\lambda) = i \), we get the following useful form

\[
\binom{k}{n_a} = \sum_{i=\lfloor \frac{n}{m} \rfloor}^n a_0^{k-i} \alpha_{n,i} \binom{k}{i}, \tag{2.2}
\]

where

\[
\alpha_{n,i} = \sum_{\lambda \vdash n, \ell(\lambda) \leq m} h(\lambda) w_\lambda a_1^i n - i
\]

and \( a_1 = (a_1, \ldots, a_m) \).

We can interpret the summands in the formula (2.1) as follows: the binomial coefficient is the number of ways to collect \( l(\lambda) \) non-vacant boxes among \( k \) ones. A configuration \( \lambda \) being fixed, this number should be multiplied by the number \( h(\lambda) \) of ways to arrange \( k_1 \) boxes with 1 ball, \( k_2 \) boxes with 2 balls, \ldots, \( k_m \) boxes with \( m \) balls in a sequence of length \( l(\lambda) \). The result is finally weighted by the monomial \( w_\lambda \) which consists of the product of the weights of every row in the Ferrers diagram: each row with \( i \) dots has \( a_i \) colors, including the \( k_0 \) “empty” rows.

Remark 2.2. The number of permitted configurations is the number of partitions of \( n \) which fit inside a \( k \times m \) rectangle. It is given by the coefficient of \( q^n \) in the gaussian polynomial \( \left[ \begin{array}{c} k+m \\ k \end{array} \right]_q \).

To illustrate, let us consider 5 balls to be thrown into 4 boxes with maximal occupancy 3. There are \( [q^5]_4 \) 4 possible configurations: \( (23), (1^23), (1^32), (1^32). \) Summing the contributions of these configurations yields \( \binom{5}{8_a} = 12a_0^2a_2a_3 + 12a_0a_1^2a_3 + 12a_0a_1a_3^2 + 4a_1^3a_2. \)
For non-negative integers $a_i$, the interpretation of (1.2) suggests that $a_i = \binom{1}{i}$ can be regarded as the number of ways to throw $i$ balls in one box. To make this proposal plausible, one could consider that the interior of each box is discretized, i.e., balls live in cells whose occupation can be either single or multiple. This intrinsic “one-box structure” was proposed by Fang [25,26] who, in an essentially probabilistic approach to the Gentile-Freund model, considered the case where each box contains $m$ cells and the balls are assigned to the boxes in such a way that no cell can accommodate more than one ball. Obviously, according to this picture, if $a_j = 0$ for some index $j$, then $j$ balls are not allowed to lodge together in a same box. By reason of the above interpretation, the vector $a = (a_0, a_1, \ldots, a_m)$ will be called the color vector of the triangle $T(a)$.

### 2.2 Score in drawing colored balls

Let us consider the following game of chance: suppose that a box contains $N$ balls labeled by numbers from 0 to $m$ and assume we have $a_i$ balls with label (or color) $i$, $N = \sum_{i=0}^{m} a_i$. A ball is repeatedly drawn at random and put back in the box, with all balls having equal chances of being chosen at any time. Suppose that, in each trial, the capital of a player is increased by $j$ when ball number $j$ shows up with probability $a_j/N$. Let $g_i$ denote the gain in the $i$-th trial and let $G_k = g_1 + g_2 + \cdots + g_k$ be the partial gain at time $k$. The probability generating function of the random variable $G_k$ is

$$1 \frac{1}{N^k} \left( a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m \right)^k. $$

So, its probability mass function reads

$$P(G_k = n) = \frac{1}{N^k} \binom{k}{n} = \frac{1}{\sum_{i=0}^{mk} \binom{k}{i} a_i} \binom{k}{n}, \quad n = 0, 1, \ldots, mk. \quad (2.4)$$

where $a = (a_0, a_1, \ldots, a_m)$. This is the probability that the player shows $n$ points after $k$ draws. Note that the denominator in (2.4), i.e., the total number of possibilities, is the $k$-th row sum in the $a$-triangle. Since, moreover, the space of all possibilities is endowed with the uniform probability measure, we have

**Theorem 2.3.** Let $a = (a_0, a_1, \ldots, a_m) \in \mathbb{N}^{m+1}$. The polynomial coefficient associated with $a$ is the number of ways to record $n$ points after $k$ trials in the above described drawing game, the integer $a_j$ being the number of balls of color $j$.

Obviously, if $m = 1$, the variables $g_i$ are $(0,1)$ Bernoulli random variables and therefore $G_k$ has binomial distribution. In this sense, the probability mass function (2.4) is seen as a generalization of the binomial distribution based on trials with more than 2 outcomes, provided the variables $g_i$ are not decomposable to Bernoulli ones.

As noted in the introduction, the colorless version of the distribution (2.4) was first arrived at by De Moivre. In the middle of the twentieth century, it was restudied by Steyn as the limit of a generalization of the hypergeometric distribution [50]. It was also investigated by the authors of [43], where the distribution is termed “cluster binomial distribution” because of the multitude of outcomes for one trial. It was also studied in detail by the authors of [6].
2.3 Directed lattice paths

In probability theory, it is well-known that the evolution of sums of independent discrete random variables, like that of the last model, can be described by lattice paths. We plan to make use of this fact to propose the third model.

In this section, by a directed lattice path we mean a polygonal line of the discrete Cartesian half plane \( \mathbb{N} \times \mathbb{Z} \) whose “direction of increase” is the horizontal axis and the allowed steps are simple, i.e., of the form \((1, s)\) with \(s \in \mathbb{Z}\[7\].

We have the following combinatorial interpretation of the polynomial coefficients:

**Theorem 2.4.** Let \( S^{(a)}(k, n) \) denote the set of lattice paths of length \( k \) starting from the origin, ending in the point with coordinates \((k, n - mk/2)\) and using the steps \( s_i = (1, i - m/2)\), \( i = 0, \ldots, m \); step \( s_i \) coming in \( a_i \) colors. Then

\[
\binom{k}{n}_a = \# S^{(a)}(k, n).
\]

**Proof.** We will set up a bijection from the set of occupancy shapes of the GFS onto the set \( S^{(a)}(k, n) \). Consider \( k \) boxes numbered from 1 to \( k \) and arranged from left to right in ascending order as illustrated in Figure 2 for \( k = 4 \). Assume box \( n^0i \) is occupied by \( n_i \) balls. The occupancy shape of this configuration is the \( k \)-tuple \((n_1, \ldots, n_k)\); \( n_1 + \ldots + n_k = n \) and \( n_i \leq m \). An occupancy shape can then be regarded as a restricted composition of \( n \). We construct our bijection as follows. With the shape \((n_1, \ldots, n_k)\) we associate a lattice path of \( k \) steps starting from the origin in such a way that to box \( n^0i \) we assign bijectively a simple step \( s_i \) with slope \( n_i - m/2 \):

\[
(n_1, \ldots, n_k) \leftrightarrow (s_1, \ldots, s_k), \quad s_i = (1, n_i - m/2).
\]

Because box \( n^0i \) can have \( a_i \) colors, the corresponding step \( s_i \) can appear with so many incarnations. Moreover, since in a given configuration \( \lambda = (0^{k_0}1^{k_1} \cdots m^{k_m}) \), \( k_i \) boxes accommodate \( i \) balls, there are \( k_i \) steps \( s_i \) in the corresponding lattice path. Thus, the latter ends in the altitude \( \sum_{i=1}^{k}(n_i - m/2) = n - mk/2 \). This ends the proof.

If \( mk \) is even then there exist a path that ends in the \( x \)-axis, i.e., a bridge if we use the terminology of [7] (cf. Figure 2). This case concerns an occupancy model with half-filling, counted by the central polynomial coefficient \( \binom{k}{m/2}_a \). For further lattice paths interpretations of central trinomial coefficients, see the study of David Callan [15].
2.4 A spin chain model

Consider a chain of \( k \) sites; each site is occupied by a particle with spin \( \frac{m}{2} \). The \( m + 1 \) components of the spin runs over the set \( \{-\frac{m}{2}, -\frac{m}{2} + 1, \ldots, \frac{m}{2}\} \). As for the Ising model, define the “magnetization” of a spin configuration of the system as the sum of spin projections divided by \( k \):

\[
\frac{\text{sum of up spins} \uparrow - |\text{sum of down spins} \downarrow|}{k}.
\]

Identifying the slope \( n_i - \frac{m}{2} \) of the \( i \)-th step in the lattice path model with the spin projection \( n_i - \frac{m}{2} \), as illustrated in Figure 2, we have the following interpretation:

**Proposition 2.5.** The polynomial coefficient associated with the vector \( a \) is the number of spin configurations with magnetization \( \frac{n}{k} - \frac{m}{2} \); spin projection \( n_i - \frac{m}{2} \) comes in \( a_i \) colors.

The half-filling occupation discussed in the end of the last subsection concerns now the spin configuration with vanishing magnetization.

2.5 Examples of combinatorial polynomial triangles

Let us now discuss some instances of arithmetical polynomial triangles associated with specific color vectors.

**Example 2.6 (Polynomial triangle associated with binomial coefficients).** Let \( a_i = \binom{m}{i} \). Here, the polynomial coefficients reduce to the binomial coefficients \( \binom{m}{n} \), i.e., the \( n \) balls are distributed into \( m \) copies of \( k \) boxes according to Fermi-Dirac statistics. In this case, equation (1.3) leads to the binomial formula

\[
\binom{mk}{n} = \sum_{k_1+k_2+\ldots+k_m=n, l=0}^{m} \prod_{i=1}^{m} \binom{k_i}{l} \left( \frac{m}{l} \right)^{k_i}.
\]

**Example 2.7.** Let \( a_i = a_i[p \leq i \leq m], a_p \neq 0 \). In this case we have weighted GFS where no boxes occupied by less than \( p \) balls are permitted. Here the sum (2.1) reduces to a sum over partitions such that \( p \leq l(\lambda') \leq m \). The number of possible ways is readily found to be

\[
\binom{k}{n}_a = \binom{k}{n-kp}_{a_p} \quad \text{if} \quad kp \leq n \quad \text{and} \quad \binom{k}{n}_a = 0 \quad \text{if} \quad kp > n.
\]

where \( a_p = (a_p, \ldots, a_m) \). In particular, the number of ways in which \( n \) unlabeled objects can be distributed in \( k \) uncolored labeled boxes allowing at most \( m \) objects and at least \( p \) objects to fall in each box, is \( \binom{k}{n-kp}_{a-p} \). This is understandable, since we have to fill each box with \( p \) objects to guarantee the minimum occupancy level and distribute the remaining \( n - kp \) ones among \( k \) boxes allowing at most \( m - p \) balls per box. We notice that for \( p = 1 \), \( \binom{k}{n-k-1} \) is also the number of compositions of \( n \) with exactly \( k \) parts, each less than or equal to \( m \) [4, p.55].

**Example 2.8 (Restricted occupancy model with distinguishable balls).** Let \( a_i = 1/i! [0 \leq i \leq m] \). This case is the exponential version of the GFS:

\[
\binom{k}{n}_a = [t^n] (e_m(t))^k,
\]
where \( e_m(t) \) is simply the \( m \)th section of the exponential series. When \( m = \infty \) the coefficient \( n! \binom{k}{n} = k^n \) is the number of ways in which \( n \) distinguishable balls can be thrown in \( k \) distinguishable boxes (The so-called Maxwell-Boltzmann statistics). For finite \( m \), the integer \( n! \binom{k}{n} \) enumerates the same occupancy model but with the restriction that no more than \( m \) labeled balls can lodge in the same box. For \( m = 2 \), the triangle is recorded as A141765.

If \( a_i = 1/i! \) \([1 \leq i \leq m]\), the integer

\[
n! \binom{k}{n} = n! \binom{e_m(t) - 1}{k}, \quad (k \leq n \leq mk)
\]

is the statistical weight of the above restricted Maxwell-Boltzmann model allowing that no box is left unoccupied. For \( m = \infty \), we have that

\[
n! \binom{k}{n} \frac{k}{k!} \binom{k}{n_A} = \binom{n}{k},
\]

\( \binom{n}{k} \) being the Stirling numbers of the second kind. Therefore, for finite \( m \), the integer \( n!/k! \binom{k}{n} \) is the number of ways of partitioning a set of \( n \) elements into \( k \) nonempty subsets with the restriction that no subset can contain more than \( m \) elements. In the particular case \( m = 2 \), we have

\[
n! \binom{k}{n} \frac{k}{k!} \binom{k}{n_A} = \frac{n!}{2^{n-k}(2k-n)!((n-k))!},
\]

which are the coefficients of the so-called Bessel polynomials:

\[
\sum_{n=k}^{2k} n! \binom{k}{n} \frac{k}{k!} \binom{k}{n_A} t^n = \sum_{i=0}^{k} \frac{(k+i)!}{(k-i)!i!} \left( \frac{t}{2} \right)^i = y_k(t).
\]

These numbers have been studied by Choi and Smith [17, 18].

### 3 Polynomial coefficient Identities

Binomial coefficients satisfy an amazing plethora of dazzling identities. It is natural to seek their extensions to the polynomial case. In this section we demonstrate generalizations of some of the binomial coefficient identities.

#### 3.1 Extension of Top Ten binomial identities

In Table 1 we propose polynomial extensions of some of the top ten binomial identities displayed in table 174 of the “concrete Mathematics” by Graham, Knuth, and Patashnik [33]. The proof of these generalizations is straightforward.

**Proof.** (sketch) The polynomial symmetry relation is readily established by writing that \( p_A(t) \) is the reciprocal polynomial of \( p_A(t) \):

\[
p_A(t) = \sum_{i=0}^{m} a_{m-i} t^i = t^m p_A(t^{-1})
\]
| Identity                        | Binomial                                                                 | Polynomial                                                                 |
|-------------------------------|--------------------------------------------------------------------------|---------------------------------------------------------------------------|
| Factorial expansion           | $\binom{k}{n} = \frac{k!}{n!(k-n)!}$                                     | Equation (1.3)                                                            |
| Symmetry†                     | $\binom{k}{n} = \binom{k}{k-n}$                                          | $\binom{k}{n}_a = \binom{k}{mk-n}_a$                                     |
| Absorption/Extraction         | $\binom{k}{n} = \frac{k}{n} \binom{k-1}{n-1}$                          | $\binom{k}{n}_a = \frac{k}{n} \sum_{i=1}^{m} i a_i \binom{k-1}{n-i}_a$   |
| Vandermonde convolution       | $\sum_{i+j=n} \binom{r}{i} \binom{s}{j} = \binom{r+s}{n}$              | $\sum_{i+j=n} \binom{r}{i}_a \binom{s}{j}_a = \binom{r+s}{n}_a$         |
| Addition/Induction            | $\binom{k}{n} = \binom{k-1}{n} + \binom{k-1}{n-1}$                     | $\binom{k}{n}_a = \sum_{i=0}^{m} a_i \binom{k-1}{n-i}_a$                |
| Binomial theorem              | $\sum_{n=0}^{k} \binom{k}{n} x^n y^{k-n} = (x+y)^k$                      | $\sum_{n=0}^{m} \binom{k}{n}_a x^n y^{mk-n} = \left( \sum_{i=0}^{m} a_i x^i y^{m-i} \right)^k$ |
| Upper summation‡              | $\sum_{0 \leq k \leq p} \binom{k}{n} = \binom{p+1}{n+1}$              | $\sum_{0 \leq k \leq p} \frac{1}{\alpha_0} \binom{k}{n}_a = \sum_{i=1}^{m} \frac{1}{\alpha_0} \alpha_{n,i} \binom{p+1}{i+1}$ |
| Upper negation                | $\binom{k}{n} = (-1)^n \binom{n-k-1}{n}$                               | $\binom{k}{n}_a = \sum_{i=0}^{n} \alpha_0^{-i} (-1)^i \alpha_{n,i} \binom{i-k-1}{i}$ |
| A recurrence relation with respect to $n \geq 1$ | $\binom{k}{n} = \frac{k+1-n}{n} \binom{k}{n-1}$                         | $\binom{k}{n}_a = \frac{1}{\alpha_0} \sum_{l=1}^{m} (k+1) I - n \alpha_l \binom{k}{n-I}_a$ |

Table 1: Extensions of eight of the top ten binomial coefficients identities.

† $J$ is the $(m+1) \times (m+1)$ backward identity matrix, that is matrix with 1’s on the anti-diagonal and 0’s elsewhere.

‡ The coefficients $\alpha_{n,i}$ are defined in (2.3).
and
\[
\binom{k}{mk-n} = (t^{mk-n})^t \binom{mk-n}{n} = \binom{k}{n}.
\]

The Absorption/Extraction property follows by taking the derivative of both sides of \( p_a(t) = \sum_{n=0}^{mk} \binom{k}{n} t^n \) with respect to \( t \), and equating the coefficients of \( t^n \). The Vandermonde convolution is usually obtained by equating coefficients on both sides of \( p^{k+s}_a(t) = p^s_a(t) p^s_a(t) \). The Addition/Induction relation is a particular case of Vandermonde convolution with \( r = 1 \) and \( s = k - 1 \). The generalized binomial theorem is an obvious consequence of definition 1.1. To prove the generalized Upper summation and Upper negation identities, we apply the binomial upper summation and upper negation to the binomial coefficients in right-hand-side of (2.2). Finally, the recurrence with respect to \( n \geq 1 \) is a particular case of a recurrence for powers of Taylor series (see [32] and references therein). To prove it take logarithms of the equation \( (p_a(t))^k = \sum_{n=0}^{mk} \binom{k}{n} t^n \) and differentiate with respect to \( t \) and equate the coefficients of \( t^n \) on both sides of the obtained equation [32].

**Remarks on the polynomial symmetry.** Two points about the polynomial symmetry are worthy of note:

- If \( a \) is a palindrome, i.e., \( a_i = a_{m-i} \forall i = 0, \ldots, m \), then \( a = Ja \) and thus, through the generalized symmetry relation, the centered triangle \( T(a) \) is mirror symmetric across the median column (cf. Figure 1).

- In physics literature, the term “holes” is used to designate missing occupancies in ball-in-box models and the notion of particle-hole duality implies that instead of studying particles, one can get similar information by studying the holes. We infer that the generalized symmetry relation provides a particle-hole duality. To see this, consider the restricted occupancy model discussed in subsection 2.1 and assume, following Fang [25, 26], that each box contains \( m \) cells; no more than 1 ball can lodge in the same cell. For our purpose, a particle is simply an occupied cell while a hole is identified with an empty one. Therefore, it is clear that if \( n \) balls (particles) are distributed among \( k \) boxes (states), there are \( mk - n \) holes. According to this picture, we learn from the symmetry relation that a system of \( n \) particles governed by GFS associated with the color vector \( a \) can equivalently be described by \( mk - n \) missing particles obeying GFS associated with the color vector \( Ja \). This equivalence is just the particle-hole duality. Particularly, if \( a \) is palindromic, (in this case the polynomial \( p_a \) is self-reciprocal) particles and holes obey the same statistics. In the point of view of lattice paths, this duality acts as a simple reflection about the \( x \)-axis.

### 3.2 More identities

**A pretty identity.**

\[
\sum_l \binom{r}{p+l} \binom{s}{n+l} = \binom{r+s}{mr-p+n} = \binom{r+s}{ms+p-n}.
\]

**Proof.** Follows from the symmetry relation and the application of Vandermonde convolution.
If \( p = n = 0 \), the identity (3.1) can be cast into the following matrix form

\[
T(Ja) \cdot T(a)^t = S(a) = S(Ja)^t,
\]

where \( S(a) \) is the array whose \((r, s)\)-entry is \( \binom{r+s}{mr}_a \). Obviously, the matrix \( S(a) \) is symmetric if \( a \) is a palindrome, and generalizes the familiar Pascal symmetric matrix. For instance, \( S((1,1,1)) \) begins as

\[
S((1,1,1)) = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & \ldots \\
1 & 3 & 6 & 10 & 15 & 21 & \ldots \\
1 & 6 & 19 & 45 & 90 & 161 & \ldots \\
1 & 10 & 45 & 141 & 357 & 784 & \ldots \\
1 & 15 & 90 & 357 & 1107 & 2907 & \ldots \\
1 & 21 & 161 & 784 & 2907 & 8953 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Putting in (3.1) \( r = s = k \) and \( p = n = 0 \), we get the identity:

**Sum of squares.** If \( a \) is a palindrome then

\[
\sum_{n=0}^{mk} \binom{k}{n}_a^2 = \binom{2k}{mk}_a.
\]

(3.2)

This identity generalizes the well-known formula \( \sum_{n=0}^{k} \binom{k}{n}^2 = \binom{2k}{k} \), [9, p.78].

### 3.3 Some algebraic facts about the mapping \( a \mapsto T(a) \)

Now we prove some general algebraic properties.

**Fact 3.1.** Product of two polynomial triangles. As infinite matrices, the product of two polynomial triangles is a polynomial triangle, i.e.,

\[
T(a) \cdot T(b) = T(a \circ b)
\]

where for \( a = (a_0, \ldots, a_m) \), and \( b = (b_0, \ldots, b_p) \), \( a \circ b \) is the \((mp+1)\)-vector whose \( i \)-th component is given by

\[
(a \circ b)_i = \sum_{j=0}^{m} a_j \binom{j}{i}_b, \quad i = 0, \ldots, mp.
\]

(3.4)

**Proof.** The generating polynomial of \( a \circ b \) is

\[
p_{a \circ b}(t) = \sum_{i=0}^{mp} \left( \sum_{j=0}^{m} a_j \binom{j}{i}_b \right) t^i = \sum_{j=0}^{m} a_j \sum_{i=0}^{jp} \binom{j}{i}_b t^i = \sum_{j=0}^{m} a_j p_b(t)^j = p_a(p_b(t)).
\]

Hence the \((k, n)\)-entry of \( T(a \circ b) \) can be written as

\[
\binom{k}{n}_{a \circ b} = [t^n] (p_a(p_b(t)))^k = \sum_{l=0}^{mk} \binom{k}{l}_a [t^l] p_b(t)^l = \sum_{l=0}^{mk} \binom{k}{l}_a \binom{l}{n}_b.
\]

The last sum is exactly the \((k, n)\)-entry of the array \( T(a) \cdot T(b) \). \( \square \)
The formula (3.3) generalizes the identity \( \sum_{l \geq 0} \binom{k}{l} \binom{l}{n} = 2^{k-n} \binom{k}{n} \) [9, p.78].

The set of all vectors equipped with the operation (3.4) is a monoid with identity element given by \( \mathbf{e} = (0, 1, 0, \ldots) \); the triangle \( T(\mathbf{e}) \) is the infinite identity matrix. The facts 3.2, 3.3, 3.4, 3.5 below are straightforward features of the binary operation (3.4). We leave their proofs as easy exercises for the reader.

**Fact 3.2.** The operation \( \circ \), which can be written in matrix form as

\[ a \circ b = T(b)^{t} a, \]

is associative and obviously linear with respect to the first argument, but not with respect to the second. However, one has the property

\[ a \circ (\lambda b) = (I_{\lambda} a) \circ b, \]

where, for every real number \( \lambda \), \( I_{\lambda} = \text{diag}(1, \lambda, \lambda^2, \ldots) \).

**Fact 3.3.** Let \( \lambda \) be an arbitrary real number. Then

\[ T(\lambda a) = I_{\lambda} \cdot T(a). \]
\[ T(I_{\lambda} a) = T(a) \cdot I_{\lambda}. \]

A particularly interesting case is that of binomial triangles \( (m = 1) \) for which the map \( T(\cdot) \) is a group isomorphism:

**Fact 3.4.** The triangle \( T((a_0, a_1)) \) is a Riordan array [46]:

\[ T((a_0, a_1)) = \left( \frac{1}{1-a_0 t}, \frac{a_1 t}{1-a_0 t} \right), \]

and the set of these arrays:

\[ \mathcal{T}_2 := \{ T(a) | a \in \mathbb{R} \times \mathbb{R}^{*} \} \]

is a subgroup of the Riordan group isomorphic to \( (\mathbb{R} \times \mathbb{R}^{*}, \circ) \).

**Fact 3.5.** The group \( (\mathcal{T}_2, \cdot) \) is isomorphic to the “ax + b” group generated by affine transformations of the real line.

Recall that the “ax + b” group, generated by all dilations and translations of the real line, is isomorphic to the multiplicative group of all (real) \( 2 \times 2 \) matrices of the form [49, p. 35]:

\[ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}; \quad a \neq 0. \]

4 Generating functions

In this section, we prove several properties of the generating functions of columns of \( T(a) \).

Let \( \mathcal{F}_n(u) \) and \( \mathcal{E}_n(u) \) be respectively the ordinary and the exponential generating functions for the \( n \)-th column of the (left-justified) triangle \( T(a) \). Then we have
Proposition 4.1. The generating functions \( \mathcal{F}_n \) and \( \mathcal{E}_n \) take the forms

\[
\mathcal{F}_n(u) = \frac{P_n^{(a)}(u)}{(1 - a_0 u)^{n+1}} \quad \text{and} \quad \mathcal{E}_n(u) = e^{a_0 u} R_n^{(a)}(u),
\]

(4.1)

where

\[
P_n^{(a)}(u) = \sum_{i=\left[\frac{n}{a}\right]}^{n} \alpha_{n,i} u^i (1 - a_0 u)^{n-i} \quad \text{and} \quad R_n^{(a)}(u) = \sum_{i=\left[\frac{n}{a}\right]}^{n} \alpha_{n,i} \frac{u^i}{i!};
\]

\(\alpha_{n,i}\) is defined by (2.3). Moreover, the polynomials \(P_n^{(a)}(u)\) and \(R_n^{(a)}(u)\) are subject to the recursive equations

\[
P_n^{(a)}(u) = u \sum_{i=1}^{m} a_i (1 - a_0 u)^{i-1} P_{n-i}^{(a)}(u)
\]

(4.2)

\[
\frac{\partial R_n^{(a)}(u)}{\partial u} = \sum_{i=1}^{m} a_i R_{n-i}^{(a)}(u),
\]

(4.3)

with the initial conditions \(P_0^{(a)}(u) = R_0^{(a)}(u) = 1\) and \(P_n^{(a)}(u) = R_n^{(a)}(u) = 0\) for \(n < 0\).

Proof. Using the form (2.2), we have

\[
\mathcal{F}_n(u) = \sum_{k=0}^{\infty} \binom{k}{\frac{n}{a}} u^k = \sum_{i=\left[\frac{n}{a}\right]}^{n} a_0^{-i} \alpha_{n,i} \sum_{k=0}^{\infty} \binom{k}{i} (a_0 u)^k.
\]

Employing the generating function of binomial coefficients : \(\sum_{k=0}^{\infty} \binom{k}{n} u^k = u^n/(1 - u)^{n+1}\), we find that \(\mathcal{F}_n(u)\) can be displayed in the form (4.1) with

\[
P_n^{(a)}(u) = \sum_{i=\left[\frac{n}{a}\right]}^{n} \alpha_{n,i} u^i (1 - a_0 u)^{n-i}.
\]

As for the expressions of \(\mathcal{E}_n(u)\) and \(R_n^{(a)}(u)\), it results in the same way, using the exponential generating function \(\sum_{k=0}^{\infty} \binom{k}{n} u^k/k! = e^{a_0 u^n}/n!\).

To prove the recursion relations (4.2) and (4.3) we use the Addition /Induction relation of Table 1. It is immediate that for \(n > 0\) (recall \(\binom{0}{n}_a = 0\) if \(n > 0\))

\[
\mathcal{F}_n(u) = \sum_{k=0}^{\infty} \binom{k}{\frac{n}{a}} u^k = \sum_{k=1}^{\infty} \sum_{i=0}^{m} a_i \binom{k-1}{n-i}_a u^k = u \sum_{i=0}^{m} a_i \mathcal{F}_{n-i}(u).
\]

i.e.,

\[
(1 - a_0 u) \mathcal{F}_n(u) = u \sum_{i=1}^{m} a_i \mathcal{F}_{n-i}(u),
\]

which yields the desired recurrence. On the other hand, employing also the Addition/Induction relation, we find for \(n > 0\)

\[
\mathcal{E}_n(u) = \sum_{k=1}^{\infty} \binom{k}{\frac{n}{a}} u^k = \sum_{i=0}^{m} a_i \left( \sum_{l=0}^{\infty} \binom{l}{n-i}_a \frac{u^{l+1}}{(l+1)!} \right).
\]
Taking the derivative of both sides, we find
\[ \frac{\partial E_n}{\partial u}(u) = \sum_{i=0}^{m} a_i E_{n-i}(u), \]
from which equation (4.3) results straightforwardly.

From (4.1), the generating functions for the polynomials \( P_n^{(a)}(u) \) and \( R_n^{(a)}(u) \) are
\[ \sum_{n=0}^{\infty} P_n^{(a)}(u) z^n = \frac{1 - a_0 u}{1 - u p_a ((1 - a_0 u) z)}; \quad (4.4) \]
\[ \sum_{n=0}^{\infty} R_n^{(a)}(u) z^n = \exp(u (p_a(z) - a_0)). \quad (4.5) \]

4.1 The special case \( m = 2 \)

If \( m = 2 \), the two-term recurrence (4.2) can be explicitly solved by standard techniques. For the colorless case, this yields
\[ P_n^{(2)}(u) = \frac{(u + \sqrt{u(4 - 3u)})^{n+1} - (u - \sqrt{u(4 - 3u)})^{n+1}}{2^{n+1} \sqrt{u(4 - 3u)}}. \quad (4.6) \]
The first few polynomials are
\[
\begin{array}{c|c}
 n & u^{-\lceil n/2 \rceil} P_n^{(2)}(u) \\
\hline
 0 & 1 \\
 1 & 1 \\
 2 & 1 \\
 3 & 2 - u \\
 4 & 1 + u - u^2 \\
 5 & 3 - 2u \\
 6 & 1 + 3u - 4u^2 + u^3 \\
 7 & 4 - 2u - 2u^2 + u^3.
\end{array}
\]

From (2.3), we derive \( \alpha_n = \binom{i}{\frac{n}{2}} \). Since \( \sum_i \binom{i}{\frac{n}{2}} = F_{n+1} \), where \( F_n \) is the \( n \)-th Fibonacci number, we have \( 2^n P_n^{(2)}(1/2) = F_{n+1} \). Actually, we find the following appealing connection between Fibonacci numbers and trinomial coefficients:

**Corollary 4.2.** For \( n \geq 1 \)
\[ \sum_{k=\lceil (n-1)/2 \rceil}^{\infty} \binom{k}{n-1} \frac{1}{2^{k+1}} = F_n. \]

Moreover, from the generating function (4.5), we see that \( R_n^{(2)}(u) \) can be expressed as:
\[ R_n^{(2)}(u) = (-1)^n \left( \frac{\sqrt{-u}}{n!} \right)^n H_n \left( \frac{\sqrt{-u}}{2} \right), \]
where \( H_n \) is the \( n \)-th Hermite polynomial [1]. As an interesting by-product of this connection, we find an expression of the Hermite polynomials in terms of colorless trinomial coefficients:
Corollary 4.3. For all \( n \), we have the following representation of Hermite polynomials:

\[
H_n(x) = \frac{(-1)^n n!}{2^n} e^{2x^2} \sum_{k=[n/2]}^{\infty} \frac{k!}{n!} \frac{(-4)^{k/2k-n}}{k!}.
\]

4.2 On the zeros of \( P_n^{(m)} \) - A conjecture

The rational form of \( F_n(u) \) in Proposition 4.1 is characteristic of the generating functions of polynomials. The polynomials \( P_n^{(m)} \) play the role of Eulerian polynomials appearing in the numerator of the generating function \( \sum_{k} k^n u^k = A_n(u)/(1-u)^{n+1} \) [48, p.209]. The Eulerian polynomials \( A_n(u) \) are known to have all zeros real [35]. It is quite normal to see if this property is also valid for the polynomials \( P_n^{(m)} \).

If we take a look at the polynomial (4.6), we observe that a non-trivial zero of it (i.e. \( u \neq 0 \)) must be such that \( (u - \sqrt{u(4-3u)})/(u + \sqrt{u(4-3u)}) \) is an \((n+1)\)-th root of unity. If \( u_p \) denote such zeros then (\( i = \sqrt{-1} \))

\[
u_p = \frac{(1 + e^{i \frac{2\pi p}{n+1}})^2}{e^{i \frac{4\pi p}{n+1}} + e^{i \frac{2\pi p}{n+1}} + 1} = \frac{1 + \cos \left( \frac{2\pi p}{n+1} \right)}{1 + 2 \cos \left( \frac{2\pi p}{n+1} \right)}
\]

with \( p \not\in \{(n+1)/3, 2(n+1)/3\} \) whenever \( n+1 \) is a multiple of 3. Thus the polynomials (4.6) has real zeros only. Similar investigations for the case \( m = 3 \) leads to the same conclusion. Actually, several numerical experimentations suggest forcibly the truth of

Conjecture 4.4. For all \( m \geq 1 \), the colorless polynomials \( P_n^{(m)} \) have real zeros only.

5 Asymptotics: the entropy density function

This section is devoted to the study of a function that characterizes the polynomial triangles in the limit where the row index \( k \) tends to infinity and the column index \( n \) increases proportionally, namely the asymptotic entropy density function. But before defining this notion which originates from statistical mechanics and information theory, we recall a general formula that we shall rely upon:

Theorem. (Daniels [20, p.646], Good [31, p.868]) For a power series or a polynomial \( f(t) \) with non-negative real coefficients and a strictly positive radius of convergence, define

\[
\Delta f(t) = t f'(t) / f(t) ; \quad \delta f(t) = f''(t) / f(t) - \left( f'(t) / f(t) \right)^2 + f'(t) / tf(t).
\]

Assume that the function \( f(t) \) is aperiodic, i.e., gcd\{i, [t^i]f(t) > 0\} = 1, and suppose that the equation \( \Delta f(t) = n/k \) has a real positive solution \( x \) smaller than the radius of convergence of \( f \). Then, for \( n, k \to +\infty \) and \( n/k \) finite,

\[
[t^n] (f(t))^k = \frac{f^k(x)}{x^{n+1} \sqrt{2\pi k \delta f(x)}} (1 + o(1)), \tag{5.1}
\]

uniformly as \( k \to \infty \).
The ratio \( n/k \), denote it \( \rho \), is the mean number of balls in one box. As a function of the saddle point \( x \), \( \rho \) is strictly increasing given that \( x \partial_x \rho(x) = x^2 \delta f(x) \) is the variance of the (non-degenerate) random variable taking a value \( i \in \{0, \ldots, m\} \) with probability \( a_i x^i / f(x) \). The function \( \rho(x) = \Delta f(x) \) itself being the expectation value of this variable.

**Remark 5.1.** The Daniels-Good theorem leads to the known asymptotic of the central trinomial coefficient, i.e, \( n/k = 1 \text{ (A002426)} \). For \( a = (a_0, a_1, a_2) ; a_i > 0 \), a little calculation gives

\[
\binom{k}{k} a \sim \frac{(a_1 + 2 \sqrt{a_0 a_2})^{k+1/2}}{2 \sqrt{a_0 a_2} \sqrt{\pi k}} \quad \text{as} \quad k \to \infty.
\]

Choosing \( a = (1, 2, 1) \), we recover also the asymptotic of central binomial coefficient \( \binom{2k}{k} \sim 4^{k}/\sqrt{\pi k} \) as \( k \to \infty \).

### 5.1 Entropy density function

We define the **entropy density function** as follows

**Definition 5.2.** When \( k \) goes to infinity and \( \rho \) is fixed (the so-called thermodynamical limit), we define the entropy density function (or entropy per box) as the limit

\[
\lim_{k \to \infty} \frac{1}{k} \ln \binom{k}{k} a_{\rho} \overset{\text{def}}{=} h(a)(\rho), \quad 0 \leq \rho \leq m
\]

The existence of the limit is guaranteed by the Daniels-Good theorem.

In what follows, we assume that the \( a_i \)'s are non-negative and the polynomial \( p_a \) is aperiodic (in other words, there exists no integer \( r \) such that \( p_a(t) = \sum_i a_i t^i \)). Thus, specializing (5.1) for the polynomial coefficients, we get for a given \( \rho \in [0, m] \)

\[
h(a)(\rho) = \ln p_a(x(\rho)) - \rho \ln x(\rho), \quad (5.2)
\]

where \( x(\rho) \) is the (unique) real positive zero of the polynomial \( \sum_{i=0}^m (i - \rho) a_i t^i \). Explicit expressions of the entropy density function can be found for \( m \leq 4 \). For instance

\[
h(a_0, a_1)(\rho) = (1 - \rho) \ln a_0 + \rho \ln a_1 - \rho \ln \rho - (1 - \rho) \ln(1 - \rho),
\]

which coincide, in the uncolored case, with the entropy function for the Bernoulli trial with parameter \( \rho \) as probability of success.

Obviously, the function (5.2) is continuous and differentiable for \( 0 < \rho < m \). We now prove the main claim of this section.

**Theorem 5.3.** The density function \( h(a) \) fulfils the following properties

(i) \( h(a) \) is strictly concave;

(ii) \( h(a) \) is unimodal and reaches its peak at the point \( \mu = \sum_i i a_i / \sum_i a_i \) and

\[
\max_{0 \leq \rho \leq m} h(a)(\rho) = \ln \left( \sum_{i=0}^m a_i \right); \quad (5.3)
\]
Figure 3: Entropy density function vs $\rho = n/k$ for the un-weighted quadrinomial coefficients.

(iii) $h^{(a)}(\rho) \geq 0$ for all $\rho$, whenever $a_0 \geq 1$ and $a_m \geq 1$;

(iv) Particle-Hole duality

\[ h^{(a)}(\rho) = h^{(1a)}(m - \rho); \]

(v) As a function of $a$:

\[ \Theta_a h^{(a)}(\rho) = 1, \quad \forall \rho \]

where $\Theta = \sum_{i=0}^{m} a_i \partial_{a_i}$ is the Theta (or homogeneity) operator.

Proof.

(i) Consider the random variable $\xi$ whose probability mass function is given by $P(\xi = i) = a_i x^i / p_a(x)$, for $i = 0, \ldots, m$. As noted above, the variance of $\xi$ is given by

\[ \nabla(\xi) = x \partial_x \rho(x) = x^2 \delta p_a(x). \]

Moreover, from (5.2), we derive

\[ \partial_\rho h^{(a)}(x) = -\ln x(\rho). \]

Whence

\[ \partial_\rho^2 h^{(a)}(x) = -\frac{1}{x(\rho)} \partial_\rho x(\rho) = -\frac{1}{\nabla(\xi)} < 0. \]

Thus, the entropy density function is strictly concave.

(ii) Since $h^{(a)}$ is concave, its maximum is attained when $x(\rho) = 1$ as we can read from (5.6), i.e., when $h^{(a)}(\rho(x = 1)) = \ln p_a(1)$ according to (5.2). To show that $h^{(a)}$ is monotonically increasing for $0 < \rho < \mu$ and monotonically decreasing for $\mu < \rho < m$, recall that

\[ \rho(x) = x \left. \frac{p_a(t)}{p_a(x)} \right|_{t=x}, \]

and remark that $\partial_\rho h^{(a)}(x) > 0$ if $0 < x < 1$, i.e., $0 = \rho(0) < \rho(x) < \rho(1) = \mu$, because the function $\rho(x)$ is strictly increasing as noted above, and $\partial_\rho h^{(a)}(x) < 0$ if $x > 1$, i.e., $\rho(x) > \mu$. 

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(iii) We see from (5.7) that if \( \rho \) goes to 0, then \( x \to 0 \), since the zeros of \( p'_{a}(t)|_{t=x} \) are essentially negative and if \( \rho \to m^+ \) then \( x \to +\infty \). Moreover, from (5.2) we derive that

\[
\lim_{\rho \to 0} h^{(a)}(\rho) = \lim_{x \to 0} h^{(a)}(\rho(x)) = \ln a_0,
\]

and

\[
\lim_{\rho \to m} h^{(a)}(\rho) = \lim_{x \to +\infty} h^{(a)}(\rho(x)) = \ln a_m.
\]

These limits together with the strict concavity implies that the entropy density is non-negative if \( a_0 \geq 1 \) and \( a_m \geq 1 \).

(iv) The Particle-Hole duality is obvious from the polynomial symmetry (Table 1).

(v) In fact the differential equation is valid for all \( k \) and \( n \), in particular in the thermodynamical limit:

\[
\frac{1}{T_a} \ln \binom{k}{n} = \frac{1}{k} \binom{k}{n}^{-1} \Theta_a \binom{k}{n} = 1,
\]

where we have used the homogeneity of (1.2) as a polynomial in \( a \): \( \Theta_a \binom{k}{n} = k \binom{k}{n} \), which follows readily from the trivial formula \( \Theta_a p_a^k(t) = k p_a^k(t) \).

6 Concluding remarks

In his paper, Richard C. Bollinger [12], concludes with the hopes that,

“like \( T_1 \) has certainly been a rich source of interesting and useful mathematics, its extended relatives (i.e, \( T_m \)) potentially may serve as equally fruitful objects of study.”

Our study concretizes in some extent the Bollinger’s suggestion. We too believe that deeper aspects still be discovered for the polynomial triangles. The following items give a sample of our observations that make interesting exercises:

1. Through the expression (1.4), several recurrences of Gegenbauer polynomials can be rewritten in terms of trinomial coefficients and, possibly, could be extended to general polynomial ones. For instance, we find [54]

\[
(2k - n - 1)(2k - n) \binom{k}{n} = k(7k - 3n - 5) \binom{k-1}{n} - 3(k - 1)k \binom{k-2}{n},
\]

\[
2(k + 1) \binom{k}{n} = (2k - n + 2) \binom{k+1}{n} - (k + 1) \binom{k}{n-1},
\]

\[
n(2k - n) \binom{k}{n} = k(2k - 1) \binom{k-1}{n-1} + 3(k - 1)k \binom{k-2}{n-2}.
\]

Can we find recurrences of the same type for the coefficients (1.2)?
2. The exponential generating function of row sequences of $T(a)$:

$$\sum_n \binom{k}{n} \frac{t^n}{n!}$$

provides a natural extension of Laguerre polynomials: $L_k(t) = \sum_n \binom{k}{n}(-t)^n/n!$. Can orthogonality and other properties of Laguerre polynomials be generalized?

3. It may also be of interest to extend the present work to multivariate polynomials.

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