Multiplicative Quantum Cobordism Theory

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Abstract

We prove a twisting theorem for nodal classes in permutation-equivariant quantum $K$-theory, and combine it with existing theorems of Givental [5] to obtain a twisting theorem for general characteristic classes of the virtual tangent bundle. Using this result, we develop complex cobordism-valued Gromov-Witten invariants defined via $K$-theory, and relate those invariants to $K$-theoretic ones via the quantization of suitable symplectic transformations. This procedure is a $K$-theoretic analogue of the quantum cobordism theory developed by Givental and Coates in [6]. Using the universality of cobordism theory, we give an example of these results in the context of “Hirzebruch $K$-theory”, which is the cohomology theory determined by the Hirzebruch $\chi_y$-genus.

1 Introduction

In the complex cobordism theory $MU^*(\cdot)$, the Hirzebruch–Riemann–Roch formula

$$\pi_*(A) = \int [M] \cap \text{Td}(TM) \text{Ch}(A) \in \mathbb{Q} \otimes MU^*(pt)$$

expresses in cohomological terms the push-forward along the map $\pi : M \to pt$ to the point of a complex cobordism class $A \in MU^*(M)$ in a given (stably almost) complex manifold $M$. In this formula, Ch is the Chern-Dold character $MU^*(\cdot) \to H^\bullet(\cdot; \mathbb{Q} \otimes MU^*(pt))$, which is an isomorphism over $\mathbb{Q}$, while the “abstract Todd class”

$$\text{Td}(\cdot) = e^{\sum k > 0 s_k \text{ch}_k(\cdot)}$$

is the universal multiplicative characteristic class of complex vector bundles, where the arbitrary coefficients $s_1, s_2, \ldots$ form a certain set of free polynomial generators in the ring $\mathbb{Q} \otimes MU^*(pt)$. Consequently, one can interpret the cap-product $[M] \cap e^{\sum k > 0 s_k \text{ch}_k(TM)}$ as the cobordism-valued fundamental class of $M$. Henceforth we adopt the convention of [2] and denote the rational version of cobordism theory by $U^*$. If instead of a manifold, we begin with a complex orbifold $M$ with (virtual) tangent bundle $TM$, the right hand side of the equation still makes sense, and can be used to define cobordism-theoretic intersections. This leads to what is now known as fake cobordism-valued intersection theory. This point of view was adopted by Coates and Givental [3] in developing the theory of (albeit “fake” in our current terminology) cobordism-valued Gromov–Witten invariants, and expressing them in terms of cohomological ones (see [2, 13]). In that theory, $[M]$ is the virtual fundamental class of a moduli space $M$ of stable maps to a given Kähler target space $X$.

Complex cobordism theory reduces to complex K-theory when the abstract Todd class is specialized to its classical incarnation $\text{td}(\cdot) := \prod_{\text{Chern roots } x_i} x_i/(1 - e^{-x_i})$. Applied to a holomorphic orbibundle $V$ on a complex orbifold $M$, this leads to the fake holomorphic Euler characteristic $\chi^{\text{fake}}(M; V) := \int_M \text{ch}(V) \text{td}(TM)$. It is a rational number, which is only one summand (corresponding to $h = \text{id}$ in the orbifold’s isotropy groups) on the R.H.S. of the Kawasaki–Riemann–Roch formula

$$\chi(M; V) = \chi^{\text{fake}}\left(IM; \frac{\text{tr}_h V|_{N^h \wedge N^h}}{\text{str}_h N^h} \right).$$

The latter expresses the true (and integer) holomorphic Euler characteristic $\chi(M; V):= \text{sdim} H^\bullet(M; V)$ of the orbibundle in cohomological terms of the inertia orbifold $IM$. 

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Using this as a starting point, one can define true quantum K-theory (as opposed to the fake one), i.e. the theory of holomorphic Euler characteristics of holomorphic orbibundles on the moduli spaces of stable maps. It is based on the notion of the virtual structure sheaf introduced by Y.-P. Lee

For manifolds, there is a similar relation between cobordism theory and K-theory, analogous to the one between cobordism theory and cohomology, it works as follows: Given a compact complex manifold \(M\), to every integer polynomial \(P\) in \(\dim M\) variables one can associate the \(P\)-twisted (virtual) structure sheaf

\[
\mathcal{O}_P := \mathcal{O} \otimes P(\wedge^1 T_M, \wedge^2 T_M, \ldots, \wedge^{\dim M} T_M),
\]

and respectively define true holomorphic Euler characteristics

\[
\chi_P(M; V) := \chi(M; V \otimes \mathcal{O}_P)
\]

of vector bundles \(V \in K^0(M)\). In a similar manner to the cohomological case, these integers can be interpreted as cobordism-valued intersection numbers.

Indeed, taking in the role of \(P\) the Newton polynomials \(N_r\) expressed as polynomials of elementary symmetric functions, we obtain the Adams operations, \(\Psi^r(T_M)\). Over the rationals, the general multiplicative K-valued characteristic class of complex vector bundles has the form \(e^{\sum_{r>0} S_r \Psi^r(T_M)/r}\), where the arbitrary coefficients \(S_1, S_2, \ldots\) can be considered as certain independent elements in a completion of the coefficient ring of cobordism theory.

The analogue of the Chern-Dold character is denoted \(\text{Ch}_K\), which is an isomorphism:

\[
\text{Ch}_K : U^*(M) \xrightarrow{\text{Ch}_K} \text{Hev}(M; U^*(pt)) \xrightarrow{\text{ev}} K^0(M) \otimes U^*(pt)
\]

Thus, we can define

\[
U^*(M) \ni A \mapsto \pi_\ast(A) := \chi \left( M; \text{Ch}_K(A) \otimes e^{\sum_{r>0} S_r \Psi^r(T_M)/r} \right).
\]

The right hand side of this formalism makes sense in the context of orbifold K-theory as well. So we use K-theory to emulate a version of cobordism-theoretic intersection theory for an orbifold, by using the ring \(K^0(M) \otimes \text{hat} U^*(pt)\), with pushforward given by \(\pi_\ast \alpha = \chi(X; \alpha \otimes e^{\sum_{r>0} S_r \Psi^r(T_M)/r})\).

We call the brand of cobordism theory thus obtained multiplicative, and refer to \(\text{Ch}_K\) and \(Td_K\) as the multiplicative Chern-Dold character and multiplicative Todd class respectively. It is not genuinely “true” cobordism-valued intersection theory on \(M\), as we are unclear what that should mean, but being defined by means of the true (as opposed to fake) K-theory on \(M\), it is “less fake” than that of Coates–Givental.

Remark. The name “multiplicative” comes from the relationship between \(\text{Ch}_K\) and the formal group law determined by K-theory, which is that of the multiplicative group. This relationship will be explained in further detail in Section 4.1.

In what follows, we develop multiplicative quantum cobordism theory\(^1\) (i.e. apply this construction to the moduli spaces of stable maps), and express thus defined cobordism-valued Gromov-Witten invariants in terms of K-theoretic ones.

The latter task reduces to computing K-theoretic Gromov-Witten invariants based on the twisted structure sheaves \(\mathcal{O}_P\) in terms of those with \(P = 1\).

For this, we need three types of “twisting” results of quantum K-theory. Two of them are already contained in \(^6\), and the third one is proved in Section 7 below. The proofs of these theorems rely on modifying the quantum adelic Hirzebruch-Riemann-Roch formula due to Givental \(^6\). This formula relies on the more general framework of permutation-equivariant quantum K-theory, and expresses true K-theoretic invariants in terms of fake ones.

The universal nature of cobordism theory means that the results of this paper can be specialized to other cohomology theories, as a particular example we consider “Hirzebruch K-theory”, the theory whose pushforward map is based on the Hirzebruch \(\chi_{-y}\) genus.

\(^1\)We are as yet unsure of the relationship between the quantum deformation of cobordism-theory obtained by incorporating our invariants into the product and the deformation recently introduced by Buchstaber-Veselov in \(^6\).
2 Permutation-Equivariant $K$-theoretic invariants

We recall the definition of permutation-equivariant $K$-theoretic Gromov-Witten invariants and the associated potentials, using the definition introduced in [6].

Given $h \in S_n$ with $\ell_r(h)$ cycles of length $r$, with $r$ ranging from 1 to $s$, $h$ acts on $X_{g,n,d}$ by permuting the marked points.

For each $r$, given inputs $w_{r1}, \ldots, w_{r\ell_r}$ each of the form $\sum \phi_m q^m$, for $\phi_m \in K^0(X) \otimes \Lambda$, associate to the input $w_{rk}$ the element $W_{rk} \in K^0(X_{g,n,d}) := \prod_{s=1}^{\ell_r} \sum_m \phi_m L_{r,s}^m$, where $\sigma_s$ are the marked points permuted by the $k$th cycle of length $r$, and $L_{r,s}$ are the corresponding cotangent line bundles on $X_{g,n,d}$.

Given a partition $\ell$, a genus $g$, and a degree $d$, $S_n$-equivariant correlators are defined as follows

$$\langle w_{11}, \ldots, w_{1\ell_1}, \ldots, w_{r1}, \ldots, w_{r\ell_r} \rangle_{g,d} := \prod_r r^{-\ell_r} \text{str}_k H^*(X_{g,n,d}, \mathcal{O}_{g,n,d}^\text{vir} \otimes \pi_1 \otimes_{j=1}^s \otimes_{i=1}^{\ell_i} W_{ij}).$$

The elements of $\Lambda$ act $\Psi$-linearly, i.e. scaling the $i$th input by $s \in \Lambda$ is equivalent to multiplication by $\Psi^s(r)$.

Define the genus $g$ potential function $\mathcal{F}_X^g$ and total descendant potential $\mathcal{D}_X$ are defined as follows:

$$\mathcal{F}_X^g := \sum_d Q^d \sum_{\ell} \frac{1}{\ell_1!} \ell_1^{(\ell_1 - 1)(\ell_1 - 2)} \cdots \ell_r^{(\ell_r - 1)(\ell_r - 2)} \cdots \ell_s^{(\ell_s - 1)(\ell_s - 2)} \langle \cdots, t_{i_1}, \cdots \rangle_{g,d}.$$  

$$\mathcal{D}_X := e^{\sum_{g \geq 0} H^0(S_n) \otimes \mathcal{F}_X^g(\mathcal{F}_X^g(\mathcal{F}_X^g(\cdots))))}.$$  

The variables $t_r$ are the same for all inputs coming from cycles of length $r$, the operator $R_r$ takes $F(t_1, t_2, \ldots)$ to $F(t_r, t_2, \ldots)$, and $\Psi^r(h) = h^r$.

After a dilaton shift of $1$ and $q$ in each input, $\mathcal{D}_X$ defines a quantum state in the symplectic loop space $\mathcal{K}^\infty$, which is given as a $K$-module by $\prod_{r \in \mathbb{Z}} \mathcal{K}_r$, equipped with the symplectic form $\Omega^\infty(f,g) = \bigoplus \frac{\delta f}{\delta g} \Omega(f,g_r)$.

The positive and negative spaces $\mathcal{K}_+^\infty$ and $\mathcal{K}_-^\infty$ are inherited from $\mathcal{K}$.

The ordinary genus-$g$ and descendant potentials $\mathcal{F}_X^{g,K}$ and $\mathcal{D}_X^K$ discussed in the introduction are recovered from the $S_n$-equivariant ones by letting $t_r = 0$ for $r > 1$. Concretely this means the following:

Ordinary $K$-theoretic correlators draw inputs from the algebra $K[\mathcal{M}]$, and are given by the formula

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{g,n,d} = \chi(X_{g,n,d}; \mathcal{O}_{g,n,d}^\text{vir} \otimes \mathcal{F}_X^g(\cdots) \otimes \mathcal{F}_X^g(\cdots) \otimes \mathcal{F}_X^g(\cdots)))$$

The genus-$g$ and total descendant potentials are defined by

$$\mathcal{F}_X^{g,K}(t) = \sum_{d \in H^2(X), n \geq 0} Q^d \sum_{r} \frac{1}{n} \ell_1^{(\ell_1 - 1)(\ell_1 - 2)} \cdots \ell_r^{(\ell_r - 1)(\ell_r - 2)} \cdots \ell_s^{(\ell_s - 1)(\ell_s - 2)} \langle t_{i_1}, t_{j_1}, \ldots \rangle_{g,d}.$$  

$$\mathcal{D}_X^K = e^{\sum_{g \geq 0} H^0(S_n) \otimes \mathcal{F}_X^{g,K}(\mathcal{F}_X^{g,K}(\mathcal{F}_X^{g,K}(\cdots))))}.$$  

2.1 Symplectic Loop Spaces and Quantization

We now introduce the spaces that will appear in our formalism: $\mathcal{K}_\infty$ consists of rational functions in $q$ with coefficients in $K$ and poles only at $0, \infty$ and roots of unity. The symplectic form is

$$\Omega^r(f,g) := -\text{Res}_{q=\infty}(f(q), g(q^{-1})^r) \frac{dq}{q}.$$  

Here $(,)^r$ denotes the $K$-theoretic Poincare pairing twisted by the operation $\Psi^r$.

The polarization is described as follows: $\mathcal{K}_+^\infty$ consists of Laurent polynomials in $q$ and represents inputs to $\mathcal{D}_X$ coming from cycles of length $r$, and $\mathcal{K}_-$ is $\{f : f(0) \neq \infty, f(\infty) = 0\}$. We also modify the quantization formulas by replacing $h$ with $h^r$.

Define $\mathcal{K}^\infty := \prod_{r>0} \mathcal{K}^r$. 

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, with symplectic form $\Omega(f, g) = \sum_r \Omega^r(f_r, g_r)$, and polarizations inherited from $K^r$.

$D_X$ is a function on $K_X^\infty$, and thus defines a quantum state $\langle D_X \rangle$. To make certain equations homogeneous, we impose that this construction is done after applying the dilaton shift, a translation replacing $t_r$ with $t_r - (1 - q)$. Since specializing to ordinary $K$-theoretic invariants is equivalent to setting $t_r = 0$ for $r > 1$, $D_X^K$ naturally defines a quantum state on $K := K^1$.

3 Twisting theorems

We can define twisted $K$-theoretic invariants by tensoring $O^{uvr}$ with other classes from $K^0(X_{g,n,d})$.

These classes take the form $E = e \sum_k \frac{\delta_k}{1 + \psi^k}$. Where the $E_k$s must all be of the following 3 types:

- **Type I:** $E_k = ft_{\psi} ev_{n+1}^* V_k$, where $V_k \in K^0(X)$.
- **Type II:** $E_k = ft_{\psi} ev_{n+1}^* F_k(L_{n+1})$, where $L_{n+1}$ is the universal cotangent line and $F_k$ is a Laurent polynomial with coefficients in $K^0(X)$, with $F_k(1) = 0$. If $F_k = \sum q_k g^k$ then ev_{n+1}^* F_k(L_{n+1}) is shorthand for $\sum_k ev_{n+1}^* q_k L_{n+1}$, these are $K$-theoretic versions of the $\kappa$-classes introduced by Kabanov and Kimura in [8].
- **Type III:** $E_k = ft_{\psi} i_e ev_{n+1}^* F_k(L_+, L_-)$, where $i : Z \to U_{g,n,d}$ is the inclusion of the codimension-2 locus of nodes, and $F_k$ is a symmetric Laurent polynomial in two variables with coefficients as above. $L_+$ and $L_-$ denote cotangent line bundles to the branches at the node. Note we could equivalently write $E_k = ft_{\psi} (ev_{n+1}^* F_k(L_+, L_-) \otimes O_Z)$.

Henceforth we will omit the subscript $n + 1$ from the evaluation map when there is no ambiguity. The construction of the twisted total descendant potential $D_X^E$ is identical to its untwisted counterpart, except that the operator $R_{\psi}$ also replaces $E_k$ with $E_{\psi} k$.

The effect of twistings of any type on $\langle D_X \rangle$ is as follows: $\langle D_X^E \rangle = \nabla \langle D_X \rangle$, where $\nabla$ is some operator on Fock space. The quantization formulas are also adjusted slightly, being replaced by $\hat{h}$ with $\hat{h}^r$ in the $r$th component. Twistings can also be taken on top of each other, if $E$ splits as $E_1 E_0$, where $E_1$ is a twisting of a particular type, and $E_0$ is made up of other types, we also have $\langle D_X^{E_1} \rangle = \nabla \langle D_X^{E_0} \rangle$.

The following theorems, proven in [5], describes $\nabla$ for twistings of type 1 and 2 in terms of the symplectic geometry of $K^\infty$.

**Theorem 3.1.** For a twisting of type I, $\nabla$ is the quantization of the multiplication operator $f_r \to \Phi_r f_r$ in the $r$th component, where $\Phi_r$ is the Euler-Maclaurin asymptotics of $\sum_{s=0}^{\frac{1}{\Psi} F_k r k} \frac{\sum_{s=0}^{\frac{1}{\Psi} F_k r k} \chi(a \otimes b \otimes e^{\sum_{s=0}^{\frac{1}{\Psi} F_k r k} \Phi_h})}{K^{\frac{1}{\Psi} F_k r k}}$. Here $\Phi$ is regarded as a $\Psi$-linear symplectomorphism from $K^\infty$ with symplectic form governed by the twisted Poincare pairing $(a, b)^{\Psi, r} = \chi(a \otimes b \otimes e^{\sum_{s=0}^{\frac{1}{\Psi} F_k r k} \Phi_h})$ to $K^\infty$ with the standard symplectic form.

**Theorem 3.2.** For a twisting of type II, the operator $\nabla$ is the translation on Fock space that changes the dilaton shift from $v_r = 1 - q$ to $v_r = (1 - q) e^{\sum_{s=0}^{\frac{1}{\Psi} F_k r k} \Phi_h} (1 + q)$. This can be interpreted as leaving the quantum states the same, but changing how they are obtained from the actual potentials.

In this work, we prove an analogous theorem for twistings of type III:

**Theorem 3.3.** For a twisting of type III, the operator $\nabla$ is of the form $e^\Psi \sum \Delta_r$, where $\Delta_r$ is an order-2 differential operator on determined by insertion of the symmetric tensor

$$\sum \frac{k}{1 - q} (F_{r+1}(L_+, L_-)) \phi_{\Psi} \otimes \phi_{\Psi} = \frac{1}{1 - L_+ L_-} \in K_+ \otimes K_+$$

into $t_r$.

The result is equivalent to changing the negative space of the polarization on $K^\infty$.

We take a moment to describe concretely the change of polarization in Theorem 3.3. The operator is the quantization of the time-1 flow of the quadratic Hamiltonian given in standard Darboux coordinates by $(p, Sp)$, which results in the symplectomorphism given in Darboux coordinates as $(p, q) \to (p, q + Sp)$.  

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Phrased invariantly, $S$ is the map $K_- \to K_+$ given in the $r$th coordinate by dualizing the symmetric tensor $\sum \frac{1}{1-L_L-L_-} \phi_{rk}(1-L_L^k L^-_k) \phi^\alpha \phi_{\alpha}$ with respect to the symplectic form. It induces a change of negative space from $q = 0$ to $q = Sp$, but leaves $K_+$ constant. The twisted potential represents the same quantum state as the untwisted one, but in the Fock space on $K^\infty$ constructed with respect to a different negative polarization.

If one interprets $\frac{1}{1-L_L-L_-}$ as a tensor in $K^\infty \otimes K^\infty$, then dualizing gives the identity map from $K^\infty$ to $K^\infty$. Renaming $L_+$ to $q$ and $L_-$ to $x$, this map is given by $f(q) \mapsto -\text{Res}_{q=0} f(q) \frac{d}{dq}$, provided $x$ is interpreted as being close to infinity.

Adding this to the map $K^\infty \to K^\infty$ determined from the operator gives the map $K^\infty \to K^\infty$ sending $(p, 0)$ to $(p, Sp)$. In summary, the new polarization is determined in the $r$th coordinate by the image of $K_-$ under the map obtained by dualizing the expression

$$\frac{1}{1-L_L-L_-} \phi_{rk}(1-L_L^k L^-_k) \phi^\alpha \phi_{\alpha}$$

We will henceforth use expressions of this kind to label polarizations.

4 Multiplicative quantum cobordism theory

4.1 Complex-oriented cohomology theories and formal group laws

A complex-oriented cohomology theory is a generalized cohomology theory which admits Chern classes for complex vector bundles. For a theory $A^*$, a complex orientation is determined by the image of the tautological orientation $\gamma$ determined from the operator gives the map $K^\infty \to K^\infty$ sending $(p, 0)$ to $(p, Sp)$. In summary, the new polarization is determined in the $r$th coordinate by the image of $K_-$ under the map obtained by dualizing the expression

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The logarithm of the formal group law of cobordism theory is given by Mischenko’s formula as
\[ z(u) = u + \sum_{n \geq 1} [\mathcal{C}P^n] u^{n+1} / n + 1. \]

One can thus explicitly compute \( u(1 - q^{-1}) \) as the series inverse of \( 1 - e^{-z(u)} \). There are some generators \( b_k \) of \( U^*(pt) \) such that: \( u(1 - q^{-1}) = 1 - q^{-1} + \sum_{k \geq 1} b_k (1 - q^{-1})^{k+1}. \)

Similarly: \( \ln\left( \frac{1 - q^{-1}}{u(1 - q^{-1})} \right) = \sum_{k \geq 1} a_k (1 - q^{-1})^k \) for a different set of generators \( a_k \). After completing with respect to this grading, \( \sum_{k \geq 1} a_k (1 - q^{-1})^k \) can be rewritten as a series in \( q^{-k} \), denoted \( s(q) = \sum_{k \geq 0} c_k q^{-k} \). The \( c_k \) are independent in the completion of \( U^*(pt) \), but \( c_0 \) is determined by the requirement \( s(1) = 0 \).

Since \( Td_K \) is multiplicative and Adams operations are additive, the formula for the multiplicative Todd class of a general bundle is:
\[ Td_K() = e^{\sum_{k \geq 0} \frac{\Psi^k(\cdot)}{k}}. \]

Here \( \Psi^0 \) is the rank operator. This is the universal \( K \)-theoretic characteristic class mentioned in the introduction, with the additional requirement of stability.

The stability requirement can be relaxed in the following way. Given a characteristic class \( C \) with \( C(1) = t \) for \( t \) some unit, we can regard it as coming from a series \( \frac{1 - q^{-t}}{u(1 - q^{-t})} \), where \( u(1 - q^t) \) is a homomorphism the multiplicative group with orientation given by \( (1 - q^t) \) instead of \( (1 - q^{-1}) \). This scales the logarithm \( z(u) \) by a factor of \( \frac{1}{t} \).

Using \( C \) and \( u(1 - q^{-1}) \) define new versions of \( Ch_K \) and \( Td_K \), however, the resulting pushforwards have the same value as if we used the normalized version of \( C \) instead. To see this, apply the ordinary Riemann-Roch formula to rewrite \( \chi(X; Ch_K(\alpha) Td_K(T\xi)) \) as an integral over \( X \). The \( \frac{1}{t} \) coefficients appearing from the expansion of \( t^d(Td_K(T\xi)) \) and \( ch(Ch_K(\alpha)) \) cancel in the top degree. The same result is true in the orbifold setting, which can be shown by applying Kawasaki-Riemann-Roch and then considering top-degree terms on each stratum.

Keeping this in mind, the \( c_0 \) term in the exponential expression for \( Td_K \) can be ignored, provided we apply the above modifications consistently.

### 4.2 Cobordism-valued Gromov-Witten invariants

Define the algebra \( U \) to be \( \hat{U}^*(X) \), where the hat denotes completion by the grading introduced in the previous section, and further completion to ensure \( u(1 - q^{-1}) \) is a Laurent polynomial in \( q \) (the latter may involve adding an additional variable).

Define \( q(u) \) to be \( e^{z(u)} \). The inputs to cobordism-theoretic correlators are drawn from \( U[q(u)^\pm] \), regarded as a subalgebra of \( U(u) \). This algebra contains \( u \) as well as \( u^* := u(1 - q(u)) \), which represents the first \( U^* \)-theoretic Chern class of the dual to the universal line bundle.

For \( \alpha_i \in U[q(u)^\pm] \), the cobordism theoretic correlators are defined via the right hand side of the Hirzebruch-Riemann-Roch formula, i.e.
\[ \langle \alpha_1, \ldots, \alpha_n \rangle^U_{g,n,d} = \chi(X_{g,n,d}; \mathcal{O}_{\mathcal{X}}^{vir} \cdot Td_K(T^{vir}) \prod_{i=1}^n ev_i^* Ch_K \alpha_i(L_i)). \]

The genus \( g \) and total descendant potentials \( \mathcal{F}_X^U \) and \( D_X^U \) are defined in the same way as for \( K \)-theory.

### 4.3 The loop space \( \mathcal{U} \)

We construct the space \( \mathcal{U} \) in a similar manner to \( K \). As a \( U \)-module, \( \mathcal{U} \) is defined as \( U[q(u)^\pm] \) localized at \( 1 - q(u)^m \) for each \( m \in \mathbb{Z}_{\neq 0} \). The symplectic form is
\[ \Omega^U(f, g) := Res_{q(u)=0, \infty} (f(u), g(u^*))^U dz(u). \]

\((\cdot)^U \) denotes the cobordism-theoretic Poincare pairing.

As with \( K \), \( \mathcal{U} \) is \( U[q(u)^\pm] \), however the negative space is not the natural analogue of \( K_{-} \), consisting of functions holomorphic at 0 and vanishing at \( \infty \). Rather, it is obtained from that space by dualizing the symmetric tensor \( \frac{1}{2}(\frac{1}{2}, \frac{1}{2}) \), and taking the image under the resulting linear map. With the above data, \( D_X^U \) defines a quantum state \( \langle D_X^U \rangle \) of \( \mathcal{U} \) after a dilaton shift of \( u^* \).
5 Formula for $\mathcal{D}_X^U$

We are now in the position to state the formula relating $\mathcal{D}_X^K$ and $\mathcal{D}_X^K$:

The quantum multiplicative Chern character $q\text{Ch}_K$, defined by extending $\text{Ch}_K$ by $u \mapsto u(1-q^{-1})$, (equivalently $q(u) \mapsto q$), is a linear isomorphism from $\mathcal{U}$ to $\mathcal{K}$ (provided $\Lambda$ is chosen to be $u^*(pt)$ completed appropriately). $q\text{Ch}_K$ is not a symplectomorphism, since it transforms the cobordism-theoretic Poincare pairing into the $K$-theoretic pairing with the insertion of $t \cdot \text{Td}_K(TX)$. Furthermore, it does not identify dilaton shifts nor polarizations. Roughly, after correcting these discrepancies, $q\text{Ch}_K$ identifies the quantum states. More precisely, the following formula holds:

**Theorem 5.1.**

$$q\text{Ch}_K(\mathcal{D}_X^U) = \nabla(\mathcal{D}_X^K)$$

Where $\nabla$ consists of 3 operators:

- The quantization of the scalar multiplication by the asymptotic expansion of $\text{Td}_K(TX-1)$, which is regarded as a symplectomorphism from $\mathcal{K}$ with symplectic structure twisted by $t \cdot \text{Td}_K(TX) \cdot \text{K}$ to $\mathcal{K}$ with its original symplectic structure. Thus viewed, the quantization acts in the opposite direction.

- A translation operator on Fock space which changes the dilaton shift to from $1-q$ to $q\text{Ch}_K(u^*) = u(1-q)$.

- The quantization of a symplectomorphism of the form $(p,q) \mapsto (p,q + Sp)$, which leaves $\mathcal{K}$ unchanged and changes $\mathcal{K}$ into $q\text{Ch}_K(\mathcal{U})$.

$q\text{Ch}_K$ identifies the potentials $\mathcal{D}_X^K$ and $\mathcal{D}_X^{K,tw}$, where the twisting class is $\text{Td}_K(T^\mathcal{vir})$. We recall the decomposition of $T^\mathcal{vir}$ in $K^0(X_{g,n,d})$ proved in [2]:

$$T^\mathcal{vir} = -\mathfrak{f}_q(L^{-1} - 1) + \mathfrak{f}_q(\mathfrak{e}_q(TX - 1)) - \mathfrak{f}_q i_* \mathcal{O}_Z^*$.

Thus twisting by $\text{Td}_K(T^\mathcal{vir})$ induces one twisting of each type.

By theorems 2.1, 2.2, and 2.3, $\langle \mathcal{D}_X^K \rangle = \nabla'\langle \mathcal{D}_X^K \rangle$, where $\nabla'$ is an operator that encodes a change of symplectic form, dilaton shift, and polarization.

So the formula is equivalent to showing that $\nabla = \nabla'$, and that $q\text{Ch}_K$ is a symplectomorphism, which respects dilaton shift and polarization, provided that the symplectic structure on $\mathcal{K}$ is the one determined by $\nabla$.

Twisting by $\text{Td}_K(T^\mathcal{vir}) = e^{\sum_{k < 0} \frac{q}{q(k)}(s_k T^\mathcal{vir})}$, for some particular choices of $s_k$, results in three twistings, each of type:

- Type I: $e^{\sum_{k < 0} \frac{q}{q(k)}(s_k \mathfrak{f}_q(1-L^{-1}))}$
- Type II: $e^{\sum_{k < 0} \frac{q}{q(k)}(s_k \mathfrak{f}_q e_q(TX - 1))}$
- Type III: $e^{\sum_{k < 0} \frac{q}{q(k)}(s_k (-\mathfrak{f}_q i_* \mathcal{O}_Z)^*)}$

These result in the following changes:

- Multiplication operator and symplectic pairing: Since the twisting of type I is $\text{Td}_K(\mathfrak{f}_q e_q(TX - 1))$, the resulting multiplication operator is equivalent to changing the Poincare pairing into the following:

$$\frac{1}{\text{Td}_K(1)} \text{Res}_{q=0, \infty} \chi(X; f(q)g(q^{-1}) \text{Td}_K(TX)) \frac{dq}{q}.$$

The operator itself is the asymptotic expansion of $\prod_{m>0} \text{Td}_K^{(m)}(\mathfrak{f}_q q^m)$. The residue operations on $\mathcal{U}$ and $\mathcal{K}$ themselves coincide since $q\text{Ch}_K(dz(u)) = \frac{1}{\text{Td}_K(1)} \frac{dq}{q}$.

- Dilaton shift: The dilaton shift changes to $(1-q)e^{\sum_{k < 0} \frac{q}{q(k)}(s_k (1-q))}$, which is the asymptotic expansion of $(1-q)\text{Td}_K(q^{-1}) = u(1-q) = q\text{Ch}_K(u^*)$.  


• Change in polarization: The twisting of type III is by $\text{Td}_K(-f t_s i, O_Z^*) = 1/ \text{Td}_K(t_s f i, O_Z)$. Here $\text{Td}_K^r(V)$ denotes $\text{Td}_K(V^r)$. So the expression determining the new polarization is:

$$\frac{1}{(\text{Td}_K^r(1 - L_r L_*)(1 - L_r L_*)(1 - L_r L_*) - 1)} = \frac{\text{Td}_K^r(L_r L_*(1 - L_r L_*)(1 - L_r L_*)}{\text{Td}_K^r(1)(1 - L_r L_*)(1 - L_r L_*)} = q \text{Ch}_K(c_r^*(L_r L_*)^*)$$

6 Specialization and examples

6.1 Other cohomology theories

Over $\mathbb{Q}$, the universality of cobordism theory also holds for cohomology rings. Given a cohomology theory $A$, the specialization map $\phi : U^*(pt) \to A^*(pt)$ is given by sending $[CP^n]$ to the pushforward to the point of the class $1 \in A^*(CP^n)$. One can recover $A^*(X)$ by restriction of scalars from $U^*(X)$, which is exact over $\mathbb{Q}$.

In this way, one can in principle specialize the constructions of the previous section to any complex oriented cohomology theory, and thus define Gromov-Witten invariants valued in that theory. However, since we use a completed version of $U^*$, the map $q \text{Ch}_K$ and the class $\text{Td}_K^r(T^*\text{vir})$ will only be well-defined if $\phi$ factors through the completion, i.e. if $u_A(1 - q^{-1})$ is actually a Laurent polynomial in $q$.

We can also use the same framework to define invariants for algebraically-oriented theories. Levine and Morel’s theory of algebraic cobordism outlined in [10] has the same universality properties among algebraic theories as $MU^*$ does for complex oriented ones. The necessary Riemann-Roch theorems are due to Smirnov ([11]). So the formalism we have constructed works equally well in this context.

We can also in principle extend multiplicative cobordism theory can also to permutation-equivariant invariants by replacing the holomorphic Euler characteristics used to define the correlators with supertraces, and constructing the resulting potentials analogously to the $\tau$-genus. Hence $\text{Td}_K^r(T^*\text{vir})$ is a line bundle then $K$-theory of Grassmanians, as in [7]. Note that this also means these invariants extend naturally to the permutation-equivariant case.

Using the formula from section 5, the transition to Hirzebruch $K$-theory has the following effects on the symplectic loop space:

• The multiplication operator from the type I part of the twisting changes the Poincare pairing to $$(a, b) = \chi(X; a \cdot b \cdot \text{Td}_Y(TX))$$, and scales the symplectic form by $\text{Td}_Y(1) = \frac{1}{1 - y}.$
• The dilaton shift becomes \(u(1 - q) = \frac{1 - q}{1 - q^q}\), demonstrating formal group inversion.

• The subsequent polarization changes to the one determined by \(\frac{1 - q^q}{1 - q^q\bar{q}}\). The map \(f \mapsto -\text{Res}_{0,\infty} \frac{1 - q^q}{1 - q^q\bar{q}} f(q(1-q^{-1}x)) \frac{d}{q}\) sends \(f \in \mathcal{K}^\infty\) to \(f + \frac{u}{1-q} f(0)\), so the new negative space is \(\{f : f(\infty) = yf(0) \neq \infty\}\).

For genuinely smooth orbifolds, when \(y = 1\), this version of the \(\chi_{-y}\)-genus becomes the ordinary topological Euler characteristic. So in cases where \(X_{q,n,d}\) is genuinely smooth, which includes cases where \(X\) is homogeneous, in particular \(M_{g,n}\). Applying the corresponding approach using the cohomologically defined invariants of \(3\) instead yields the orbifold Euler characteristic, which is a rational number given by a weighted count of simplices. This illustrates the general principle that multiplicative cobordism-theoretic invariants will have different relationships to the orbifold structure of \(X_{q,n,d}\) than “fake” ones.

However, the symplectic formalism degenerates in this limit, so any computations must be done for a general \(y\), and then specialized. We postpone a detailed discussion of the kinds of invariants that thus occur to another work.

7 Proof of theorem 3.3

7.1 Adelic Formula for \(\mathcal{D}_X\)

We recall the adelic formula for \(\mathcal{D}_X\), which recasts the \(K\)-theoretic potential into purely cohomological terms. The proof of theorem 2.2 will rely heavily on this formula.

We define the adelic symplectic loop space \(\mathcal{K}_{\infty} = \bigoplus_{M \in \mathbb{Z}} \mathcal{K}^\text{fake}(X \times BZ_M)\), where \(\mathcal{K}^\text{fake}(X \times BZ_M)\) denotes the loop space of the fake quantum \(K\)-theory of the orbifold \(X \times BZ_M\). Each summand splits as a direct sum of \(M\) sectors \(\mathcal{K}_M^\infty\) labelled by roots of unity \(\zeta\), each isomorphic to \(K((q - 1))\).

The symplectic structure on \(\mathcal{K}^\text{fake}(X \times BZ_M)\) comes from an additional twisting of fake quantum \(K\)-theory which we outline later, and is described as follows: The symplectic form \(\Omega_{\text{tw}}\) pairs \(\mathcal{K}_M^\infty\) with \(\mathcal{K}_M^{-1}\) by \(\Omega_{\text{tw}}(f, g) = \frac{1}{\zeta} \langle f(q), g(q^{-1}) \rangle^{(r)}\), where \((\Psi^a, \Psi^b)^{(r)} = r \Psi^a(b, a)\), for \((a, b)\) the usual Poincare pairing, and \(r\) is the index of \((\zeta)\) in \(\mathbb{Z}_M\). Let \(m(\zeta) = \frac{M}{\tilde{r}(\zeta)}\) denote the primitive order of \(\zeta\).

Define the adelic potential \(\mathcal{D}_X\) to be \(\bigotimes_{M} \mathcal{D}_X^{\text{tw}} X \times BZ_M\).

We can resum the component spaces according to \(r\), to describe the adelic space as:

\[
\bigoplus_{\zeta} \bigoplus_{r} \mathcal{K}_M^\infty
\]

Where the first sum is taken over all roots of unity.

After resumming, the symplectic form becomes

\[
\Omega_{\infty}(f, g) = \sum_{\zeta} \frac{1}{m(\zeta)} \sum_{r} \text{Res}_{q=1} \langle f_{\zeta}^r(q^{-1}), g_{\zeta}^{-1}(q) \rangle^{(r)} dq.
\]

The adelic map \(\Phi : (f_1, \ldots) \mapsto \Psi^r(f_r(\frac{q^r}{\zeta}))\) defines a symplectic \(\Psi\)-linear transformation between \(\mathcal{K}^\infty\) and \(\mathcal{K}_{\infty}\), which respects positive, but not negative polarizations, since an element of \(\mathcal{K}_{\infty}\) will not be polar at every root of unity.

A result of \(\ref{5}\) is that \(\langle \mathcal{D}_X \rangle = \Phi^* e^{\hbar/2} \sum_{r, \zeta, \eta} \nabla_{r, \zeta, \eta} \mathcal{D}_X\), where \(exp(h/2 \sum_{r, \zeta, \eta, r \neq 1} \sum_{\zeta, \eta})\) is the quantization of the rotation changing the standard polarization on \(\mathcal{K}^\infty\) to the uniform polarization, which is determined by the image of \(\mathcal{K}_{\infty}\) under \(\Phi\).

This formula has the form of Wick’s summation over graphs, and arises from the application of the Lefschetz-Kawasaki-Riemann-Roch theorem to \(\mathcal{D}_X\). The theorem states that for \(\mathcal{X}\) a orbifold, \(V\) and orbibundle, and \(h\) a discrete automorphism of \(\mathcal{X}\) that lifts to \(V\):

\[
\text{str}_h(\mathcal{X}; V) = \chi^\text{fake}(\mathcal{X}^h; \frac{\text{tr}_h(V)}{\text{str}_h N_{\mathcal{X}/\mathcal{X}}})
\]
Here $\tilde{h}$ some lifting of $h$ on each component of $\mathcal{I}X^h$, and $\chi^{fake}(A;V)$ is defined to be $\int_A \text{ch}(V) \text{td}(TA)$, i.e. the pushforward in fake $K$-theory. This theorem is consequence of the usual Kawasaki-Riemann-Roch theorem, which was shown by Tonita in [12] to hold for virtually smooth orbifolds.

We recall from [6] the following description of $\mathcal{I}X_{g,n,d}^h$:

The total space itself corresponds to a moduli space of stable maps from curves $C$ with a symmetry $\tilde{h}$ accomplishing the permutation $h$ of marked points.

A connected component (henceforth referred to as a Kawasaki stratum) of this space is described by certain combinatorial data:

- A graph $G$ dual to the quotient of the curve by the cyclic group generated by $\tilde{h}$.
- A positive integer $M_v$ for each vertex $v$. $M_v$ representing the order of $\tilde{h}$ on the vertex $v$.
- The discrete characteristics (genus, degree) of the map on each irreducible component.
- A labelling of the vertices of $G$ with eigenvalues of $\tilde{h}$ on the tangent lines to the branches at the ramification points of order $r$. These eigenvalues will be primitive $m$th roots of unity for $m = \frac{M_v}{r}$.
- A labeling of the edges of $G$ (corresponding to nodes) with pairs of eigenvalues of $\tilde{h}$ on each branch to the node. We require that these eigenvalues not be inverse to each other (i.e. the node is unbalanced), so the node cannot be smoothed within the stratum.

After normalizing at the unbalanced nodes, each vertex represents a component of a Chen-Ruan moduli space of stable maps to the orbifold $X \times B\mathbb{Z}_M$. After doing this, the eigenvalue at a marked point or node also determines the sector of $\mathcal{I}(X \times B\mathbb{Z}_M)$ in which the evaluation map at that marked point lands.

Thus the KRR formula relates a correlator to some fake $K$-theoretic correlators of $X \times B\mathbb{Z}_M$, which are additionally twisted by the denominator terms. These account for the twistings of fake $K$-theory that appear in the adelic space formalism.

Marrying the vertices at edges involves the application of a propagator operator for each edge, which will involve comparing the respective edge operators.

### 7.2 Twisted potentials

The exact same argument applies essentially verbatim to twisted potentials, with two differences. The vertex potentials are further twisted by the restriction of the twisting class (we label the resulting potentials $\mathcal{D}^\text{tw,E}_X$). And, only in the case of type III twistings, the edge operators are modified as well.

Our strategy will thus be to begin with the twisted potential $\mathcal{D}^\text{tw,E}_X$, where $E$ denotes a twisting of type III, we pass to the adelic potential $\mathcal{D}^\text{E}_X$, and analyze the vertex contributions coming from $E$ to relate $\mathcal{D}^E_X$ and $\mathcal{D}_X$. Then we use the adelic formula to convert that to a relationship between $\mathcal{D}^E_X$ and $\mathcal{D}_X$, which will involve comparing the respective edge operators.

Rather than beginning with $\mathcal{D}_X$, we could take $E$ to be the composition of $E_0$, a twisting of type I and II, and $E_1$, a twisting of type III. The resulting argument would give a relationship between $\mathcal{D}^E_X$ and $\mathcal{D}^{E_0}_X$, and is identical to the case where $E_0$ is trivial, so we just work in the latter setting to minimize notation.

#### 7.2.1 Vertex Contributions

Let $\hat{\mathcal{M}}$ be a Kawasaki stratum with ambient moduli space $X_{g,n,d}$ (from which the twisting classes are inherited). Let $C$ be the universal curve, and $\hat{C} = C$ be the universal quotient curve by $h$. Let $ft$, $ev$, $i$ denote the structure maps of $C$ (the unitalized such maps denote the ones coming from the ambient space $X_{g,n,d}$). Let the vertex and edge nodes of $C$ be labelled $Z_v$, respectively, and label the cotangent branches by $L_{\pm}$. Any hatted version of the previously introduced notation refers to the corresponding construction on $\hat{C}$.

We have

$$ft_*i_*ev^*V|_{\hat{\mathcal{M}}} = ft_*i_*O_{Z_v} F_k(L_+, L_-)Eu(N),$$

for $N$ some excess normal bundle bundle, and $Eu$ the $K$-theoretic Euler class.
N_{Z_e} is trivial, since all vertex nodes can be smoothed within the stratum. This allows us to recast the nodal twisting restricted to 1-vertex strata solely in terms of the nodal loci of those strata. Let \( \hat{M} \) now denote a stratum with one vertex and no edges.

If we denote a stratum with one vertex and no edges.

Lemma 7.1. Let \( \{\}_{i=1}^{m} \lambda_{i}^{-r_{i}} \zeta^{(b-a)_{i}} \) denote a stratum with one vertex and no edges. Let \( L_{+}^{m} \) be a generic bundle on the covering curve, \( \hat{M} \) extracts the contribution of the locus of nodes with \( r \) copies on the covering curve, which we refer to as \( Z_r \).

Differentiating the twisting class in \( E_k \) brings down the factor
\[
\hat{M} L_{+}^{m} \otimes \mathbb{C}_{\lambda^{-1}}
\]

Since \( \hat{M} L_{+}^{m} \) is invariant under the \( h^{r} \)-action, the Todd class is invariant under \( h^{r} \).

So the pushforward is equal to
\[
\hat{M} \sum_{i=1}^{m} \lambda_{i}^{-r_{i}} \zeta^{(b-a)_{i}} \times \hat{M} L_{+}^{m} \otimes \mathbb{C}_{\lambda^{-1}}
\]

Since \( \sum_{i=1}^{m} \lambda_{i}^{-r_{i}} \zeta^{(b-a)_{i}} = 0 \) unless \( \lambda^{r_{i}} \zeta^{(b-a)_{i}} = 1 \), in which case the result is:
\[
\hat{M} \sum_{i=1}^{m} \lambda_{i}^{-r_{i}} \zeta^{(b-a)_{i}} \times \hat{M} L_{+}^{m} \otimes \mathbb{C}_{\lambda^{-1}}
\]

The factor \( m \) from the \( m \) copies of the node, labelled by elements of the automorphism group of the node, labelled by powers of \( h^{r} \), which acts on \( L_{\pm} \) by \( \zeta^{r+1} \). Since \( L_{+} \) is invariant under the \( h^{r} \)-action, the Todd class is invariant under \( h^{r} \).

So the pushforward is equal to
\[
\hat{M} \sum_{i=1}^{m} \lambda_{i}^{-r_{i}} \zeta^{(b-a)_{i}} \times \hat{M} L_{+}^{m} \otimes \mathbb{C}_{\lambda^{-1}}
\]

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\[
\hat{M} \sum_{i=1}^{m} \lambda_{i}^{-r_{i}} \zeta^{(b-a)_{i}} \times \hat{M} L_{+}^{m} \otimes \mathbb{C}_{\lambda^{-1}}
\]

The factor \( m \) from the \( m \) copies of the node, labelled by elements of the automorphism group of the node.

Since the Chern character intertwines Adams operations and cohomological power operations (denoted here \( P^{h} \), the contribution of the term \( \hat{M} \zeta^{(b-a)_{i}} \times \hat{M} L_{+}^{m} \otimes \mathbb{C}_{\lambda^{-1}} \) to \( \hat{M} \zeta^{(b-a)_{i}} \times \hat{M} L_{+}^{m} \otimes \mathbb{C}_{\lambda^{-1}} \) can be described as the following cohomological pushforward:
\[
\sum_{\lambda'^{r_{i}} = 1, \lambda'^{b-a} = \zeta^{b-a}} \lambda'^{P_{k}} \times \hat{M} \zeta^{(b-a)_{i}} \times \hat{M} L_{+}^{m} \otimes \mathbb{C}_{\lambda^{-1}}
\]

If \( \lambda^{r} = \zeta^{b-a} \) and \( \lambda^{r} \) is an \( M \)th root of unity, we necessarily have that \( r | M \). So we can relabel \( k \) as \( r l_{0} \). Collecting the \( r \) terms corresponding to the eigenvalues with \( \lambda^{r} = \zeta^{b-a} \) terms yields the above expression is equal to:
\[
\zeta^{l_{0}(b-a)_{i}} \times \hat{M} \zeta^{(b-a)_{i}} \times \hat{M} L_{+}^{m} \otimes \mathbb{C}_{\lambda^{-1}}
\]

Since orbifold Gromov-Witten theory uses the cotangent lines \( \hat{L}_{\pm} \) on the quotient curve, we rewrite \( L_{\pm} \) as \( \hat{L}_{\pm} \), which is valid in fake K-theory even though such a bundle may not exist genuinely. Pulling back \( \Delta_{r l_{0}} \) to \( \hat{Z}_{r} \), renaming \( \hat{M} \) (\( F_{r} \)) to \( S_{r} \), and reverting to the notation of fake K-theory yields:
\[
\hat{Z}_{r} \hat{f}_{r} \Delta_{r l_{0}} = \hat{F} \hat{E}_{r} S_{r} \zeta^{-1} \hat{L}_{+}^{\hat{L}_{+}} \zeta \hat{L}_{-}^{\hat{L}_{-}} (1 - \hat{L}_{+}^{\hat{L}_{+}} \hat{L}_{-}^{\hat{L}_{-}})
\]

The factor \( (1 - \hat{L}_{+}^{m} \hat{L}_{-}^{m}) \) occurs from pulling back \( \hat{f}_{r} \hat{f}_{r} \hat{O}_{\hat{Z}_{r}} \), and is the K-theoretic Euler class of the normal bundle of \( \hat{Z}_{r} \) in \( \hat{M} \).

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To compute the correlator as an integral on $\hat{Z}_r$, we use the general formula for a morphism $Y \to X$:

$$\chi^{fake}(X; V) = \chi^{fake}(Y; f^*(V)/\text{Eu}(N_f))$$

If we label the twisting class, contributions from the KRR denominators, and correlator inputs together as $B$, we thus have:

$$\chi^{fake}(\hat{\mathcal{M}}; \Delta_k \cdot B \cdot \mathcal{O}^{tw}_{X \times BZ_{2\mathbb{M}}}) =$$

$$\chi^{fake}(\hat{Z}_r; \Psi^e(S_k(-1)^{1/m}(1 - L_{+}^{(m)/1/m})(1 - L_{-}^{(m)/1/m})) \cdot \hat{\imath}^* \hat{i}^*(B \cdot \mathcal{O}^{tw}_{X \times BZ_{2\mathbb{M}}}))$$

Ungluing the nodes and integrating over the moduli spaces of component curves yields an order-2 recurrence relation on the correlators, in which the tensor structure of the components to match $\mathcal{O}^{tw}$, and accounts for the fact that deformations of a node on the quotient curve correspond to coherent deformations of the $r$ preimages on the covering curve, whereas in general they can be deformed independently.

A more detailed account of how ungluing the nodes interacts with cohomological nodal twisting classes is given in [13] (see Proposition 3.9) for the case where $F_k$ are constants, the addition of nonconstant terms does not alter the argument.

The differential operator determined from this recurrence adds a factor of $h^r/2$, due to the symmetry between $L_+$ and $L_-$, and the genus reduction (one node on the quotient curve corresponds to $r$ nodes on the covering curve). So the potential $D^{tw, E}_{X \times BZ_{2\mathbb{M}}}$ satisfies the same differential equation as $\sum D^{tw}_{X \times BZ_{2\mathbb{M}}}$, where $\sum$ corresponds to changing the polarization in the sectors of order $r$ and eigenvalue $\zeta$ using the expression $\Psi^{r}(\sum \phi_{i} F_{k} (\zeta^{-1} L_{+}^{1/m} \zeta L_{-}^{1/m}))$.

### 7.2.2 Edge contributions

Recall that an edge in the graph of a Kawasaki stratum corresponds to an unbalanced node in the quotient curve corresponding to $r$ nodes on the cover curve where $\hat{h}^r$ acts on the tangent branches with eigenvalues $\nu_+, \nu_-$, which are respectively primitive $m_+, m_-$, roots of unity, let $\mathcal{M}$ be the order of $h$ on the stratum, and let $m = \frac{m_+ m_-}{m_+ + m_-}$.

Fixing a particular edge $e_0$, we perform the same procedure as the vertices to compute the contribution of the nodal locus $Z_{e_0}$. The Euler factor $\text{Eu}(N)$ in the previous section becomes $1 - L_+ L_-$, since smoothing the edge node is normal to $\hat{\mathcal{M}}$.

Differentiating in $E_k$ as before brings out the term

$$\text{ch}(\Delta_k^e) = \chi^{tr} \frac{\psi_k}{k} \chi(f \circ \hat{\imath}^* e_v^* F_k (L_+ L_-)(1 - L_+ L_-)) = \chi(\sum_{\lambda^{\nu_+} = 1} \lambda^{\nu_+} \chi^k(\hat{\imath}^* F_k (L_+ L_-)(1 - L_+ L_-)))$$

The map $\hat{f} \circ \hat{\imath}$ is an isomorphism on coarse spaces, since every point in $\hat{\mathcal{M}}$ has a node corresponding to the edge. At the level of stacks, the automorphism group of the node is contracted to the identity, thus the (genuine) $K$-theoretic pushforward only extracts $h^r$ invariants. The term $\hat{e}_v^* L_+^{1/m} L_{-}^{1/m} \otimes C_{\lambda^{\nu_+}}$ only has a nonzero contribution when $\lambda^r = \mu^{-1} \nu^{-3}$.

Thus if $k = r l_0$, then

$$\hat{\imath}^* \hat{f}^* \Delta_k = \Psi^r(\hat{e}_v^* S^k(L_+^{1/m} L_+^{1/m} \nu^{-1})(1 - \mu^{-1} \nu^{1} L_+ L_-))$$

This means that ungluing the edge nodes is done by applying the operator: $e_{\sum_{e_0 \neq e} r^e \Psi^e(h/2)}$, where

$$\sum_{e_0 \neq e} = \frac{e_{\sum_{e} \Psi^e(F_k(L_+^{1/m} L_{-}^{1/m} \nu^{-1})(\phi^e \otimes \phi_{e_0})).}{1 - \mu^{-1} \nu^{1} L_+^{1/m} L_{-}^{1/m}}$$

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The other ingredients here are the same as the ones calculated in [6]: The denominator is the contribution of the normal bundle of \( \hat{M} \) in the denominator of Kawasaki-Riemann-Roch formula, \( \phi^\alpha \), \( \phi_\alpha \) constitute a Poincaré-dual basis of \( K_0(X) \), which unglues the diagonal constraint at the nodes.

The resulting change of polarization on the adelic map pulls back to the one described in the theorem statement.

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