Coherent states of accelerated relativistic quantum particles, vacuum radiation and the spontaneous breakdown of the conformal SU(2,2) symmetry

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Abstract
We give a quantum mechanical description of accelerated relativistic particles in the framework of coherent states (CSs) of the (3+1)-dimensional conformal group SU(2, 2), with the role of accelerations and ‘kinematical redshift’ played by special conformal transformations (SCTs) and with the role of (proper) time translations played by dilations. The accelerated ground state ˜ϕ₀ of first quantization is a CS of the conformal group. We compute the distribution function giving the occupation number of each energy level in ˜ϕ₀ and, with it, the partition function Z, mean energy E and entropy S, which resemble that of an ‘Einstein solid’. An effective temperature ˜T can be assigned to this ‘accelerated ensemble’ through the thermodynamic expression dE/dS, which leads to a (nonlinear) relation between acceleration and temperature different from Unruh’s (linear) formula. Then we construct the corresponding conformal-SU(2, 2)-invariant second-quantized theory and its spontaneous breakdown when selecting Poincaré-invariant degenerated θ-vacua (namely, CSs of conformal zero modes). SCTs (accelerations) destabilize the Poincaré vacuum and make it radiate.

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1. Introduction

The quantum analysis of accelerated frames of reference has been studied mainly in connection with quantum field theory (QFT) in curved spacetime. For example, the case of the quantization
of a Klein–Gordon field in Rindler coordinates \[1, 2\] entails a global mutilation of flat spacetime, with the appearance of event horizons, and leads to a quantization inequivalent to the standard Minkowski quantization. Physically one says that, whereas the Poincaré-invariant (Minkowskian) vacuum \(|0\rangle\) in QFT looks the same to any inertial observer (i.e. it is stable under Poincaré transformations), it converts into a thermal bath of radiation with temperature

\[T = \frac{\hbar a}{2\pi c k_B}\]

in passing to a uniformly accelerated frame \((a\) denotes the acceleration, \(c\) the speed of light and \(k_B\) the Boltzmann constant). This is called the Fulling–Davies–Unruh effect \[1, 3, 4\], which shares some features with the (black hole) Hawking \[5\] effect. Its explanation relies heavily upon Bogoliubov transformations, which find a natural explanation in the framework of coherent states (CSs) \[6–8\] and squeezed states \[9\].

In this paper, we also approach the quantum analysis of accelerated frames from a CS perspective but the scheme is rather different, although it shares some features with the standard approach commented on before. The situation will be similar in some respects to quantum many-body condensed matter systems describing, for example, superfluidity and superconductivity, where the ground state mimics the quantum vacuum in many respects and quasi-particles (particle-like excitations above the ground state) play the role of matter. We shall enlarge the Poincaré symmetry \(P\) to account for uniform accelerations and then spontaneously break it down back to Poincaré by selecting appropriate ‘non-empty vacua’\(^3\) stable under \(P\). Then the action of broken symmetry transformations (accelerations) will destabilize/excite the vacuum and make it radiate. The candidate for an enlargement of \(P\) will be the conformal group in \((3+1)\) dimensions \(SO(4, 2)\) incorporating dilations and special conformal transformations (SCTs)

\[x^\mu \rightarrow x'^\mu = \frac{x^\mu + a^\mu x^2}{1 + 2ax + a^2x^2},\]

which can be interpreted as transitions to systems of relativistic, uniformly accelerated observers with acceleration \((c = 1\) units) \(a = 2a\) (see e.g. \[10–12\] and later on equation \(8\)). From the conformal symmetry point of view, Poincaré-invariant vacua are regarded as a CS of conformal zero modes, which are undetectable (‘dark’) by inertial observers but unstable under SCTs.

A previous preliminary attempt to analyze quantum accelerated frames from a conformal group perspective was made in \[13\] (see also \[14\]), where a quite involved ‘second quantization formalism on a group \(G\)’ was developed and applied to the (finite part of the) conformal group in \((1+1)\) dimensions, \(SO(2, 2) \simeq SO(2, 1) \times SO(2, 1)\), which consists of two copies of the pseudo-orthogonal group \(SO(2, 1)\) (left- and right-moving modes, respectively). Here we shall use more conventional methods of quantization and work in realistic \((3+1)\) dimensions, using the (more involved) conformal group \(SO(4, 2) \simeq SU(2, 2)/\mathbb{Z}_4\). New consequences of this group-theoretical approach are obtained here, regarding a similitude between the accelerated ground state and the ‘Einstein solid’, the computation of entropies and a deviation from Unruh’s formula \(1\).

We would like to mention that (near-horizon two-dimensional) conformal symmetry has also played a fundamental role in the microscopic description of the Hawking effect. In fact, there is strong evidence that conformal field theories provide a universal (independent of the details of the particular quantum gravity model) description of low-energy black hole entropy, which is only fixed by symmetry arguments (see e.g. \[15, 16\]). Here, the Virasoro algebra turns

\(^3\) Actually, quantum vacua are not really empty to every observer, as the quantum vacuum is filled with zero-point fluctuations of quantum fields.
out to be the relevant subalgebra of surface deformations of the horizon of an arbitrary black hole and constitutes the general gauge (diffeomorphism) principle that governs the density of states. However, in (3+1) dimensions, conformal invariance is necessarily global (finite-(15)-dimensional). In this paper, we shall study zero-order effects that gravity has on quantum theory (uniform accelerations). To account for higher-order effects (like non-constant accelerations) in a group-theoretical framework, we should firstly promote the 3+1 conformal symmetry $SO(4,2)$ to a higher-(infinite)-dimensional symmetry. This is not a trivial task, although some steps have been done by the authors in this direction (see e.g. [14, 17–20]).

This paper is organized as follows. In section 2, we discuss the group-theoretical backdrop (conformal transformations, infinitesimal generators and commutation relations) and justify the interpretation of SCTs as transitions to relativistic uniform accelerated frames of reference and ‘kinematical redshift’. In section 3, we construct the Hilbert space and an orthonormal basis for our conformal particle in (3+1) dimensions, based on an holomorphic square-integrable irreducible representation of the conformal group on the eight-dimensional phase space $\mathbb{D}_8 = SO(4,2)/SO(4) \times SO(2)$ inside the complex Minkowski space $\mathbb{C}^4$. In section 4, we define conformal CSs, highlight the Poincaré invariance of the ground state (admissible/fiducial state as an Einstein solid, to obtain a deviation from Unruh’s formula (1) and to discuss the existence of a maximal acceleration. In section 5, we deal with the second-quantized (many-body) theory, where Poincaré-invariant (degenerated) pseudo-vacua are CSs of conformal zero modes. Selecting one of these Poincaré-invariant pseudo-vacua spontaneously breaks the existence of a maximal acceleration. In section 6, we discuss the outlook.

2. The conformal group and its generators

The conformal group in (3+1) dimensions, $SO(4,2)$, is composed of Poincaré $\mathcal{P} = SO(3,1) \otimes \mathbb{R}^4$ (a semidirect product of spacetime translations $b^\mu \in \mathbb{R}^4$ times Lorentz $\Lambda^\mu_\nu \in SO(3,1)$) transformations augmented by dilations ($\epsilon^\mu \in \mathbb{R}_+$) and relativistic uniform accelerations (SCTs, $a^\mu \in \mathbb{R}^4$) which, in Minkowski spacetime, have the following realization:

$$x'^\mu = x^\mu + b^\mu, \quad x'^\mu = \Lambda^\mu_\nu(x^\nu), \quad x'^\mu = e^\mu x^\mu, \quad x'^\mu = \frac{x^\mu + a^\mu x^2}{1 + 2ax + a^2x^2},$$

respectively. The infinitesimal generators (vector fields) of transformations (3) are easily deduced,

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad M_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}, \quad D = x^\mu \frac{\partial}{\partial x^\mu}, \quad K_\mu = -2x_\mu x^2 \frac{\partial}{\partial x^\mu} + x^2 \frac{\partial}{\partial x^\mu},$$

and they close into the conformal Lie algebra:

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho},$$
$$[P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho, \quad [P_\mu, P_\nu] = 0,$$
$$[K_\mu, M_{\rho\sigma}] = \eta_{\mu\rho}K_\sigma - \eta_{\mu\sigma}K_\rho, \quad [K_\mu, K_\nu] = 0,$$
$$[D, P_\mu] = -P_\mu, \quad [D, K_\mu] = K_\mu, \quad [D, M_{\mu\nu}] = 0,$$
$$[K_\mu, P_\nu] = 2(\eta_{\mu\nu}D + M_{\mu\nu}).$$
The conformal quadratic Casimir operator

\[ C_2 = D^2 - \frac{1}{2} M_{\mu\nu} M^{\mu\nu} + \frac{1}{2} \left( P_{\mu} K^{\mu} + K_{\mu} P^{\mu} \right) \]  

(6)

generalizes the Poincaré Casimir \( P^2 = P_{\mu} P^{\mu} \) which, for scalar fields \( \phi \), leads to the Klein–Gordon equation \( P^2 \phi = m_0^2 \phi \), with \( m_0^2 \) the squared rest mass. The fact that \([D, P^2] = -2P^2\) implies that conformal fields must either be massless or have a continuous mass spectrum (see e.g. the classical references [22, 26]). Actually, just like the Poincaré-invariant mass \( m_0 \) comprises a continuum of ‘Galilean’ masses \( m \), a conformally invariant mass \( m_{00} \) can be defined by the Casimir (6), which comprises a continuum of Poincaré masses \( m_0 \). The eigenvalue equation \( C_2 \phi = m_{00}^2 \phi \) can be seen as a generalized Klein–Gordon equation, where \( D \) replaces \( P_0 \) as the (proper) time evolution generator (see [21] for more information) and \( m_{00} \) replaces \( m_0 \) (see [22] for the formulation of other conformally invariant massive field equations of motion in generalized Minkowski space).

In this paper, we shall deal with discrete series representations of the conformal group having a continuous mass spectrum and the corresponding wavefunctions having support on the whole four-dimensional Minkowski spacetime, with the dilation parameter \( \tau \) playing the role of proper time. We shall report on this model of conformal quantum particles later on in section 3. The reader can also consult our recent reference [21] for the underlying gauge-invariant Lagrangian approach (of nonlinear sigma-model type) behind our quantum model of conformal particles, which is built upon a generalized Dirac method for the quantization of constrained systems which resembles in some aspects the standard approach to quantizing coadjoint orbits of a group \( G \) (see e.g. classical references [23–25]).

### 2.1. Special conformal transformations as transitions to uniform relativistic accelerated frames and kinematical redshift

The interpretation of SCTs (2) as transitions from inertial reference frames to systems of relativistic, uniformly accelerated observers was identified many years ago by the authors of [10–12]. More precisely, denoting by \( u^\mu = \frac{dx^\mu}{d\tau} \) and \( a^\mu = \frac{du^\mu}{d\tau} \) the 4-velocity and 4-acceleration of a point particle, respectively, the relativistic motion with constant acceleration is characterized by the usual condition [27] \( a_{\mu} a^{\mu} = -g^2 \), where \( g \) is the magnitude of the acceleration in the instantaneous rest system. Then, from \( u_\mu u^\mu = 1 \) (in \( c = 1 \) units), we can derive the differential equation to be satisfied for all systems with constant relative acceleration:

\[ \frac{da^\mu}{d\tau} = g^2 u^\mu. \]  

(7)

In 1945, Hill [10] proved that the kinematical invariance group of (7) is precisely the conformal group \( SO(4,2) \) (see also [11, 12]). Here we shall provide a simple explanation of this fact. For simplicity, let us take an SCT along the ’\( z \)' axis, \( a^\mu = (0, 0, 0, a) \), and the temporal path \( x^\mu = (t, 0, 0, 0) \). Then the transformation (2) reads

\[ \begin{align*}
  t' &= \frac{t}{1 - a^2 t^2}, \\
  z' &= \frac{aw^2}{1 - a^2 t^2}.
\end{align*} \]  

(8)

As a curiosity, this formula turns out to be equivalent to the vanishing of the von Laue 4-vector \( E^\mu = \frac{1}{2} e^2 (\dot{\omega} a^\mu + a \dot{a} u^\mu) \) of an accelerated point charge; that is, a compensation between the Schott term \( \frac{1}{2} e^2 \frac{\dot{\omega}}{c} \) and the Abraham–Lorentz–Dirac radiation reaction force \( \frac{1}{2} e^2 a_{\mu} a^\mu u^\mu \) (minus the rate at which energy and momentum are carried away from the charge by radiation).
Writing \( z' \) in terms of \( t' \) gives the usual formula for the relativistic uniform accelerated (hyperbolic) motion:

\[
z' = \frac{1}{g} \left( \sqrt{1 + g^2 t'^2} - 1 \right)
\]

(9)

with \( g = 2a \). In the same year, Hill [28] also noticed a very interesting relation coming from the time (\( \mu = 0 \)) component of SCTs (2) generated by \( K_0 \). Taking now \( a^\mu = (a^0, 0, 0, 0) \) and denoting \( \bar{u} = \frac{dx}{dt} \) and \( \bar{u}' = \frac{dx'}{dt'} \) the velocities in both reference frames, equation (2) leads to the velocity formula

\[
\bar{u}' = \bar{u} - 2a^0 (1 - \bar{u}^2)x + O((a^0)^2),
\]

(10)

which, to first order of approximation, resembles Hubble’s law of redshift when identifying \( H_0 = -2a^0 \) (the Hubble constant). Indeed, the added term is a simple radial velocity with magnitude proportional to the distance \( \bar{x} \) from the observer. Note that the previous derivation is purely kinematical and does not appeal to the relativistic theory of gravitation. The physical implications of this formula (and we think of SCTs in general) have been overlooked for nearly 65 years. It could lead to an ambiguity in current interpretations of stellar redshifts. Recently, Wulfman [29] has proposed several experiments, based on an analysis of the anomalous frequency shifts uncovered in the Pioneer 10 and 11 spacecraft studies, pursuing the determination of the value of the group parameter \( a^0 \) and thereby removing the possible ambiguity in Hubble’s formula.

To conclude this section, let us also say that at least two alternative meanings of SCTs have also been proposed [30, 31]. One is related to Weyl’s idea of different lengths at different points of spacetime [30]: ‘the rule for measuring distances changes at different positions’. The other is Kastrup’s interpretation of SCTs as geometrical gauge transformations of Minkowski space [31].

3. A model of conformal quantum particles

In this section we report on a model for quantum particles with conformal symmetry. The reader can find more details in [21], where we formulate a gauge-invariant nonlinear sigma model on the conformal group and quantize it according to a generalized Dirac method for constrained systems.

3.1. Compactified Minkowski space and the isomorphism \( SO(4, 2) = SU(2, 2)/\mathbb{Z}_4 \)

In [21] it is shown how Minkowski space arises as the support of constrained wavefunctions on the conformal group. Actually, compactified Minkowski space \( \mathbb{M}_4 = S^3 \times \mathbb{Z}_2 \) naturally lives inside the conformal group \( SO(4, 2) \) as the coset \( \mathbb{M}_4 = SO(4, 2)/W \), where \( W \) denotes the Weyl subgroup generated by \( K_\mu, M_{\mu\nu} \), and \( D \) (i.e. a Poincaré subgroup \( P \) augmented by the dilations \( R^+ \)). The Weyl group \( W \) is the stability subgroup (the little group in physical usage) of \( x^\mu = 0 \). The conformal group acts transitively on \( \mathbb{M}_4 \) and is free from singularities. Instead of \( SO(4, 2) \), we shall work for convenience with its four covering group:

\[
SU(2, 2) = \left\{ g = \begin{pmatrix} AB \\ CD \end{pmatrix} \in \text{Mat}_{4 \times 4}(\mathbb{C}) : g^\dagger \Gamma g = \Gamma, \quad \det(g) = 1 \right\},
\]

(11)

where \( \Gamma \) denotes a Hermitian form of signature \(+ + --\). The conformal Lie algebra (5) can also be realized in terms of \( 4 \times 4 \) gamma matrices in, for instance, the Weyl basis:

\[
\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = 4y^0 y^1 y^2 y^3 = \begin{pmatrix} -\sigma^0 \\ 0 \\ 0 \end{pmatrix}.
\]

(12)
where $\tilde{\sigma}^\mu = \sigma^\mu$ (we are using the convention $\eta = \text{diag}(1, -1, -1, -1)$) and $\sigma^\mu$ are the standard Pauli matrices. Indeed, the choice

$$D = \frac{\gamma^5}{2}, \quad M^{\mu\nu} = \frac{[\gamma^\mu, \gamma^\nu]}{4} = \frac{1}{4} \begin{pmatrix} \sigma^\mu \tilde{\sigma}^\nu - \sigma^\nu \tilde{\sigma}^\mu & 0 \\ 0 & \tilde{\sigma}^\mu \sigma^\nu - \tilde{\sigma}^\nu \sigma^\mu \end{pmatrix},$$

fulfills the commutation relations (5). These are the Lie algebra generators of the fundamental representation of $SU(2, 2)$. The group $SU(2, 2)$ acts transitively on compactified Minkowski space $\mathbb{M}_4$, which can be identified with the set of Hermitian $2 \times 2$ matrices $X = x_\mu \sigma^\mu$, as follows:

$$X \mapsto X' = (AX + B)(CX + D)^{-1}. \quad (14)$$

With this identification, the transformations (3) can be recovered from (14) as follows.

(i) Standard Lorentz transformations, $\chi^\mu = \Lambda^\mu_\nu (\omega) \chi^\nu$, correspond to $B = C = 0$ and $A = D^{\dagger} = \text{SL}(2, \mathbb{C})$, where we are making use of the homomorphism (spinor map) between $SO^+(3, 1)$ and $\text{SL}(2, \mathbb{C})$ and writing $X' = AXA^\dagger, A \in \text{SL}(2, \mathbb{C})$ instead of $X'^\mu = \Lambda^\mu_\nu (\omega) X^\nu.$

(ii) Dilations correspond to $B = C = 0$ and $A = D^{\dagger} = e^{t^2/2}I$.

(iii) Spacetime translations are $A = D = I, C = 0$ and $B = b_\mu \sigma^\mu$.

(iv) SCTs correspond to $A = D = I$ and $C = a_\mu \sigma^\mu, B = 0$ by noting that $\text{det}(CX + I) = 1 + 2ax + a^2x^2$:

$$X' = X(CX + I)^{-1} \Leftrightarrow \chi'^\mu = \frac{\chi^\mu + a^\mu x^2}{1 + 2ax + a^2x^2}.$$

### 3.2. Unirreps of the conformal group: discrete series

We shall consider the complex extension of the compactified Minkowski space $\mathbb{M}_4 = U(2)$ to the eight-dimensional conformal (phase) space:

$$\mathbb{D}_4 = U(2, 2)/U(2)^2 = \{ Z \in \text{Mat}_{2 \times 2}(\mathbb{C}) : I - ZZ^\dagger > 0 \}, \quad (15)$$

of which $\mathbb{M}_4 = \{ Z \in \text{Mat}_{2 \times 2}(\mathbb{C}) : I - ZZ^\dagger = 0 \}$ is the Shilov boundary. It can be proved (see e.g. [21, 32]) that the action

$$[U_z(\phi)(Z) = |CZ + D|^{-1/2} \phi(Z'), \quad Z' = (AZ + B)(CX + D)^{-1} \quad (16)$$

constitutes a unitary irreducible representation of $SU(2, 2)$ on the Hilbert space $\mathcal{H}_z(\mathbb{D}_4)$ of square-integrable holomorphic functions $\phi$ with invariant integration measure

$$d\mu_\lambda(Z, Z') = \pi^{-4}(\lambda - 1)(\lambda - 2)^2(\lambda - 3) \det(I - ZZ^\dagger)^{-1/2} |dZ|,$$

where the label $\lambda \in \mathbb{Z}, \lambda \geq 4$ is the conformal, scale or mass dimension ($|dZ|$ denotes the Lebesgue measure in $\mathbb{C}^4$). The factor $\pi^{-4}(\lambda - 1)(\lambda - 2)^2(\lambda - 3)$ in $d\mu_\lambda(Z, Z')$ is chosen so that the constant function $\phi(Z) = 1$ has unit norm. Besides the conformal dimension $\lambda$, the discrete series representations of $SU(2, 2)$ have two extra spin labels $s_1, s_2 \in \mathbb{N}/2$ associated with the (stability) subgroup $SU(2) \times SU(2)$. Here we shall restrict ourselves to scalar fields ($s_1 = s_2 = 0$) for the sake of simplicity (see e.g. [21] for the spinning unirreps of $SU(2, 2)$). The reduction of this representation into unitary irreducible representations of the Poincaré subgroup indicates that we are dealing with fields with a continuous mass spectrum extending from zero to infinity [33].
3.3. The Hilbert space of our conformal particle

It has been proved in [32] that the infinite set of homogeneous polynomials

\[ \psi^{j,m}_{q_1,q_2}(Z) = \sqrt{\frac{2j+1}{\lambda-1}} \frac{(m+\lambda-2)!(m+2j+\lambda-1)!}{(j+q_1)!j!(j+q_2)!p!(p+q_1-q_2)!} \left( \frac{1}{p} \frac{j-q_2}{p-q_1-q_2} \right) \]

with

\[ D^{j}_{q_1,q_2}(Z) = \sqrt{\frac{2j+1}{\lambda-1}} \frac{(m+\lambda-2)!(m+2j+\lambda-1)!}{(j+q_1)!j!(j+q_2)!p!(p+q_1-q_2)!} \left( \frac{1}{p} \frac{j-q_2}{p-q_1-q_2} \right) \]

the standard Wigner’s D-matrices \( j \in \mathbb{N}/2 \), verifies the following closure relation (the reproducing Bergman kernel or \( \lambda \)-extended MacMahon–Schwinger’s master formula):

\[ \sum_{(j)_{\mathbb{N}/2}} \sum_{(m)_{\mathbb{N}_0}} \sum_{j} \psi^{j,m}_{q_1,q_2}(Z) \psi^{j,m}_{q_1,q_2}(Z') = \frac{1}{\det(I-Z'Z)^{2}} \]

and constitutes an orthonormal basis of \( \mathcal{H}_{\lambda}(\mathbb{D}_4) \) (the sum on \( j \) accounts for all non-negative half-integer numbers). The identity (19) will be useful for us in the following.

3.4. Hamiltonian and energy spectrum

In [21] we have argued that the dilation operator \( D \) plays the role of the Hamiltonian of our conformal quantum theory. Actually, the replacement of time translations by dilations as kinematical equations of motion has already been considered in the literature (see e.g. [34, 35]), when quantizing field theories on space-like Lorentz-invariant hypersurfaces \( x^{2} = x^{\mu}x_{\mu} = r^{2} = \text{constant} \). In other words, if one wishes to proceed from one surface at \( x^{2} = \tau^{2} \) to another at \( x^{2} = \tau^{2} \), this is done by scale transformations; that is, \( D = \frac{d}{d\tau} \) is the evolution operator in a proper time \( \tau \). We must say that other possibilities exist for choosing a conformal Hamiltonian, namely the combination \( \mathcal{P}_0 = (\mathcal{P}_{0} + K_{0})/2 \), which has been used in [26].

From the general expression (16), we can compute the finite left action of dilations \( (B = 0 = C \text{ and } A = e^{\tau/2}\sigma^{0} = D^{-1} \Rightarrow g = e^{\tau/2}\text{diag}(1,1,-1,-1)) \) on wavefunctions:

\[ [U_{\tau}(g)\phi](Z) = e^{\frac{\tau}{2}}\phi(e^{\tau}Z). \]

The infinitesimal generator of this transformation is the Hamiltonian operator:

\[ H = \lambda + \sum_{i,j=1}^{2} Z_{ij} \frac{\partial}{\partial Z_{ij}} = \lambda + z_{\mu} \frac{\partial}{\partial z_{\mu}}, \]

where we have set \( Z = z_{\mu}\sigma^{\mu} \) in the last equality. This Hamiltonian has the form of that of a four-dimensional (relativistic) harmonic oscillator in the Bargmann representation. The set of functions (17) constitutes a basis of Hamiltonian eigenfunctions (homogeneous polynomials) with energy eigenvalues \( E_{n}^{l} \) (the homogeneity degree) given by

\[ H\psi^{j,m}_{q_1,q_2} = E_{n}^{l}\psi^{j,m}_{q_1,q_2}, \quad E_{n}^{l} = \lambda + n, \quad n = 2j + 2m. \]

Actually, each energy level \( E_{n}^{l} \) is \( (n+1)(n+2)(n+3)/6 \) times degenerated (just like a four-dimensional harmonic oscillator). This degeneracy coincides with the number of linearly independent polynomials \( \prod_{i,j=1}^{2} Z_{ij}^{n_{ij}} \) of fixed degree of homogeneity \( n = \sum_{i,j=1}^{2} n_{ij} \). This also proves that the set of polynomials (17) is a basis for analytic functions \( \phi \in \mathcal{H}_{\lambda}(\mathbb{D}_4) \). The spectrum is equispaced and bounded from below, with ground state \( \psi^{0,0}_{0,0} = 1 \) and zero-point energy \( E_{0}^{l} = \lambda \) (the conformal, scale or mass dimension).
4. Coherent states of accelerated relativistic particles, distribution functions and mean values

In the following two subsections, we shall compute the distribution function and mean values for CSs of accelerated relativistic quantum particles based on the unirrep of the conformal group previously mentioned.

4.1. Conformal CS and the accelerated ground state

Among the infinite set \( \{ \phi_{0,0}^0(Z) \} \) of homogeneous polynomials (17), we shall choose the ground state \( \phi_{0,0}^0(Z) = 1 \) (of zero degree/energy) as an admissible vector (see [32] for a proof of admissibility). The set of CSs in the orbit of \( \phi_{0,0}^0 \) under the action (16) is

\[
\phi_{0,0}^0(Z) = \left[ U_{\phi}(g) \phi_{0,0}^0 \right](Z) = \det(CZ + D)^{-\lambda}.
\]

(23)

Note that Poincaré transformations (zero acceleration \( C = 0 \) and \( \det(D) = 1 \)) leave the ground state invariant, that is, \( \phi_{0,0}^0 \) looks the same to every inertial observer. We shall call \( \phi_{0,0}^0 \) the ‘accelerated’ ground state. For arbitrary accelerations, \( C = \alpha \sigma^\mu \neq 0 \), we can decompose \( \phi_{0,0}^0 \) using the Bergman kernel expansion (19) as

\[
\phi_{0,0}^0(Z) = \det(D)^{-\lambda} \sum_{m=0}^{\infty} \sum_{j} \phi_{q_1,q_2}^{m,j}(-C) \phi_{q_1,q_2}^{m,j}(Z),
\]

(24)

where \( C \equiv D^{-1}C \) is a ‘rescaled acceleration matrix’. From (24), we interpret the coefficient \( \phi_{q_1,q_2}^{m,j}(-C) \) as the probability amplitude of finding the accelerated ground state in the excited level \( \phi_{q_1,q_2}^{m,j} \) of energy \( E_n^\lambda = \lambda + 2j + 2m = \lambda + n \) (up to a global normalizing factor \( \det(D)^{-\lambda} \)).

In the second-quantized (many-particles) theory, the squared modulus \( |\phi_{q_1,q_2}^{m,j}(-C)|^2 \) gives us the occupation number of the corresponding state (see section 5).

4.2. The accelerated ground state as a statistical ensemble: ‘the Einstein solid’

For canonical ensembles, the (discrete) energy levels \( E_n \) of a quantum system in contact with a thermal bath at temperature \( T \) are ‘populated’ according to the Boltzmann distribution function \( f_n(T) \sim e^{-E_n/k_B T} \). For other external reservoirs or interactions (like, for instance, electric and magnetic fields acting on a charged particle) one could also compute (in principle) the distribution function giving the population of each energy level. Actually, if one were able to unitarily implement the external interaction in the original quantum system, then one could deduce the distribution function for the population of each energy level from first quantum mechanical principles. This is precisely what we have done with uniform accelerations of Poincaré-invariant relativistic quantum particles, where the unitary transformation (24) gives the population of each energy level \( E_n^\lambda \) in the accelerated ground state \( \phi_{0,0}^0 \). Let us consider then the CS (23) itself as a statistical (‘accelerated’) ensemble. Using (19) we can explicitly compute the partition function as

\[
Z(C) = \sum_{j} \sum_{m=0}^{\infty} \sum_{q_1,q_2=-j}^{j} |\phi_{q_1,q_2}^{m,j}(C)|^2 = \frac{1}{\det(I - C^T C)^{\lambda}} = \frac{1}{(1 - \text{tr}(C^T C) + \det(C^T C))^{\lambda}}.
\]

(25)

Using this result, the fact that \( \phi_{q_1,q_2}^{m,j}(C) \) are homogeneous polynomials of degree \( 2j + 2m \) in \( C \) (recall equation (22), with the Hamiltonian operator given by (21)) and that \( \text{tr}(C^T C) \)
and \( \det(C^T C) \) are homogeneous polynomials of degrees 1 and 2 in \( C \), respectively, the (dimensionless) mean energy in the accelerated ground state (23) can be calculated as

\[
\mathcal{E}(C) = \frac{\sum_{j\in \mathbb{N}/2} \sum_{m=0}^{\infty} \sum_{q_1, q_2 = -j} E_0^j |\psi_{q_1, q_2}^m(C)|^2}{\sum_{j\in \mathbb{N}/2} \sum_{m=0}^{\infty} \sum_{q_1, q_2 = -j} |\psi_{q_1, q_2}^m(C)|^2} = \frac{1 - \det(C^T C)}{\det(I - C^T C)} = \lambda + \frac{-\det(C^T C)}{\det(I - C^T C)} = E_0 + E_B(C),
\]

where we have detached the zero-point (‘dark’ energy) contribution \( E_0 = \lambda \) from the rest (‘bright’ energy) \( E_B(C) \) for convenience. For the particular case of an acceleration \( \alpha \) along the ‘\( \varepsilon \)’ axis, \( C = \alpha \sigma^3 \), expressions (25) and (26) acquire the simpler form:

\[
Z(\alpha) = (1 - \alpha^2)^{-2\lambda}, \quad \mathcal{E}(\alpha) = \lambda + 2\lambda \frac{\alpha^2}{1 - \alpha^2}.
\]

Note that the mean energy \( \mathcal{E}(\alpha) \) is of Planckian type for the identification:

\[
\alpha^2(T) \equiv e^{-\frac{\pi^2}{k_B T}}.
\]

where we have introduced \( \varepsilon \) (the quantum of energy of our four-dimensional harmonic oscillator). At this stage, the identification (28) is an \textit{ad hoc} assignment but, eventually, we shall justify it from first thermodynamical principles (see the following subsection).

Note also that, for the identification (28), the partition function \( Z(\alpha) \) matches that of an Einstein solid with \( 2\lambda \) degrees of freedom and Einstein temperature \( T_\varepsilon = \varepsilon/k_B \) (see e.g. [36]). We remind the reader that an Einstein solid consists of \( N \) independent (non-coupled) three-dimensional harmonic oscillators in a lattice (i.e. \( \phi = 3N \) degrees of freedom).

Let us pursue this curious analogy a bit further. The total number of ways to distribute \( n \) quanta of energy among \( \phi \) one-dimensional harmonic oscillators is given in general by the binomial coefficient \( W_\phi(n) = \binom{n+\phi-1}{\phi-1} \). For example, for \( \phi = 4 \) we recover the degeneracy \( W_4(n) = (n+1)(n+2)(n+3)/6 \) of each energy level \( E^j_n \) of our four-dimensional ‘conformal oscillator’ given after (22). Let us see how \( W_\phi(n) \), for \( \phi = 2\lambda \), arises from the distribution function \( |\psi_{q_1, q_2}^m(C)|^2 \). Indeed, for \( C = \alpha \sigma^3 \), \( |\psi_{q_1, q_2}^m(C)|^2 \) can be cast as

\[
|\psi_{q_1, q_2}^m(\alpha)\rangle^2 = \frac{2j+1}{\lambda-1} \left( \frac{m + \lambda - 2}{\lambda - 2} \right) \left( \frac{m + 2j + \lambda - 1}{\lambda - 2} \right) \left( 2\lambda \right)^{2m} |D_{q_1, q_2}^j(\alpha \sigma^3)|^2
\]

Fixing \( n = 2j + 2m \), the (unnormalized) probability of finding \( \phi_{0,0}^0 \) in the energy level \( E^j_n \) is

\[
j^j_n(\alpha) \equiv \sum_{j = \frac{n}{2}}^{n/2} \sum_{\lambda = \{0, 1/2\}}^{\phi=\lambda+j} |\psi_{q_1, q_2}^m(\alpha)|^2
\]

\[
= \frac{n/2}{\lambda-1} \left( \frac{2j + 1}{\lambda - 1} \left( \frac{n - j + \lambda - 2}{\lambda - 2} \right) \left( \frac{n - j + \lambda - 1}{\lambda - 2} \right) \right)^{2\lambda^n} = \frac{\lambda^n}{2\lambda - 1} \sum_{j=0}^{\lambda} W_{2\lambda}(\alpha \sigma^3)n^{2\lambda^n},
\]

where \([0, 1/2]\) is 0 for \( n \) even and 1/2 for \( n \) odd (in this summation, the \( j \) steps are unity). Here, \( W_{2\lambda}(n) \) plays the role of an ‘effective’ degeneracy and \( \alpha^2 \) a Boltzmann-like factor. In fact, the partition function in (27) can be obtained again as

\[
Z(\alpha) = \sum_{n=0}^{\infty} j^j_n(\alpha) = \sum_{n=0}^{\infty} W_{2\lambda}(\alpha \sigma^3)n^{2\lambda^n} = \left( \sum_{n=0}^{\infty} \alpha^{2n} \right)^{2\lambda^n} = (1 - \alpha^2)^{-2\lambda},
\]
Figure 1. Probability $\pi_\lambda^n(\alpha)$ for fixed $\alpha = 0.8$ and different values of $\lambda$.

Figure 2. Probability $\pi_\lambda^n(\alpha)$ for fixed $\lambda = 4$ and different values of $\alpha$.

where we have identified the Maclaurin series expansion of $(1 - \alpha^2)^{-2\lambda}$ and the geometric series sum $z(\alpha) \equiv \sum_{n=0}^{\infty} \alpha^{2n} = 1/(1 - \alpha^2)$ with ratio $\alpha^2$. The fact that $Z(\alpha) = (z(\alpha))^{2\lambda}$ (the product of $2\lambda$ partition functions $z(\alpha)$) reinforces the analogy between our accelerated ground state and the Einstein solid with $2\lambda$ degrees of freedom (see the following section for the computation of the entropy).

Note that the distribution function $\pi_\lambda^n(j) \equiv f_\lambda^n(\alpha)/Z(\alpha)$ has a maximum for a given $n = n_0(\alpha, \lambda)$, with $n_0(\alpha, \lambda)$ increasing in $\lambda$ (see figure 1) and in $\alpha$ (see figure 2).

Furthermore, inside each energy level $E_\lambda^n$, the allowed angular momenta $j = [0, 1/2], \ldots, n/2$ appear with different (unnormalized) probabilities:

$$f_{\lambda,n,j}(\alpha) \equiv \frac{(2j + 1)^2}{\lambda - 1} \left( \frac{\frac{n}{2} - j + \lambda - 2}{\lambda - 2} \right) \left( \frac{\frac{n}{2} + j + \lambda - 1}{\lambda - 2} \right) \alpha^{2n}.$$  \hspace{1cm} (32)

Actually, the distribution function $\pi_\lambda^n(j) \equiv f_{\lambda,n,j}(\alpha)/f_{\lambda,n}(\alpha)$, which is independent of $\alpha$, has a maximum for a given $j = j_0(n, \lambda)$, with $j_0(n, \lambda)$ an increasing sequence of $n$ and decreasing on $\lambda$ (see figure 3).

4.3. Entropy, temperature and 'maximal acceleration'

Note that deriving the partition function $Z(\alpha)$ and mean energy $E(\alpha)$ from the distribution function (29), (30) does not involve any thermal (but just pure quantum mechanical) input.
the same way, we can also compute the entropy as a logarithmic measure of the density of states. In fact, denoting by $p_n(\alpha) = \alpha^{2n} Z(\alpha)$ the probability of finding our ‘Einstein solid’ in the energy level $n$ with degeneracy $W_\lambda(n)$, the entropy can be calculated as

$$S(\alpha) = -\sum_{n=0}^{\infty} W_\lambda(n) p_n(\alpha) \ln p_n(\alpha)$$

$$= -\sum_{n=0}^{\infty} \frac{(2\lambda + n - 1)}{n} \alpha^{2n} \ln((1 - \alpha^2)^{2\lambda} \alpha^{2n})$$

$$= -(1 - \alpha^2)^{2\lambda} \left( \sum_{n=0}^{\infty} \frac{(2\lambda + n - 1)}{n} \alpha^{2n} \ln((1 - \alpha^2)^{2\lambda}) + \sum_{n=0}^{\infty} \frac{(2\lambda + n - 1)}{n} \alpha^{2n} \ln(\alpha^{2n}) \right)$$

$$= -(1 - \alpha^2)^{2\lambda} \left( 2\lambda \ln(1 - \alpha^2) \sum_{n=0}^{\infty} \frac{(2\lambda + n - 1)}{n} \alpha^{2n} + 2 \ln(\alpha) \sum_{n=1}^{\infty} \frac{(2\lambda + n - 1)}{n} \alpha^{2n} \right)$$

$$= -2\lambda \left( \frac{\alpha^2 \ln(\alpha^2)}{1 - \alpha^2} + \ln(1 - \alpha^2) \right), \quad (33)$$

where we have identified the partition function $Z(\alpha)$ and its derivative $\alpha^2 \frac{d}{d\alpha} Z(\alpha)$ in the last two summations. Again, there is no thermal input up to now. If we wanted to assign an ‘effective’ temperature $T$ to our ‘accelerated ensemble’, we could use the universal thermodynamic expression (derivative of the energy with respect to the entropy):

$$T = \frac{d\mathcal{E}(\alpha)}{dS(\alpha)} = -\frac{1}{\ln(\alpha^2)}, \quad (34)$$

given in units of the Einstein temperature $T_E = \epsilon/k_B$ (i.e. $T = T/TE^2$). Equality (34) can be inverted to formula (28), giving the announced derivation of the assignment (28) from first thermodynamic principles. One could still check consistency (if desired) with other classical formulas relating mean energy and entropy to the partition function, namely,

$$\mathcal{E}(\alpha) = -\frac{d \ln Z(\alpha)}{d\beta}, \quad S(\alpha) = \frac{d}{dT} (T \ln Z(\alpha)), \quad \beta \equiv 1/T, \quad (35)$$

5 The semisimple character of the group $SU(2,2)$ allows us to express all kinematic magnitudes by pure numbers. From a ‘Galilean’ viewpoint, we could say that in conformal kinematics there is a characteristic length, a characteristic time and a characteristic speed which may be used as natural units, and then lengths, times and speeds are dimensionless (see [37, 38] for a thorough study on kinematic groups and dimensional analysis).
A hurried analysis of the relation $a^2 = e^{-1/T}$ would lead us to think of the existence of a ‘maximal acceleration’ $a^2 = 1$ (in dimensionless units). Actually, in the process toward the calculation of thermodynamical quantities, we have made use of a rescaling of the original acceleration $C = a_0 \sigma^\mu$, in the expression (24), to $C' = D^{-\lambda} C = \alpha \sigma^\mu$. We can find the relation between $a_0$ and $\alpha$ as follows. Taking into account that $\varphi^{00}$ is normalized and the representation (16) is unitary (see appendix C and proposition 5.2 of [32]), we know that the accelerated ground state (24) is also normalized. This means that the normalizing global factor $\det(D)^{-\lambda}$ in (24) is related to the partition function $Z(C)$ in (25) by

$$\det(DD^\dagger)^{-\lambda} = 1/Z(C) = \det(I - C'C)^\lambda \Rightarrow \det(DD^\dagger) = \frac{1}{\det(I - C'C)}. \tag{36}$$

Therefore, for $C = \alpha \sigma^3$ and $C' = \alpha \sigma^3$, the relation $C'C = C'(DD^\dagger)^{-1}C$ reads

$$a^2 = \frac{\alpha^2}{1 + \alpha^2} \Rightarrow a^2 = \frac{\alpha^2}{1 - \alpha^2}. \tag{37}$$

With this identification, the mean energy $E = \lambda + 2\lambda \alpha^2 1 - \alpha^2 = \lambda + 2\lambda a^2$ turns out to be a quadratic function of the acceleration ‘$a$’. The dependence of ‘$a$’ with the effective temperature $T$ is then

$$a = \sqrt{\frac{e^{-1/T}}{1 - e^{-1/T}}} = \sqrt{T} + O\left(\frac{1}{\sqrt{T}}\right) \quad \text{for} \quad T \gg 1. \tag{39}$$

This behavior departs from Unruh’s formula (1) even in the limit of high temperatures. However, at high temperatures, it is in accordance with the equipartition theorem for an Einstein solid with $2k$ degrees of freedom since the energy (38) becomes $E(T) = \varepsilon E = E_0 + 2\lambda k_B T + O(1/\sqrt{T})$, with $E_0 = \varepsilon \lambda$ and $T = T_0^2 T$.

We have seen that the fact that $\alpha$ is bounded is just due to a rescaling of ‘$a$’, so that there is not a maximal acceleration in our model as such. Nevertheless, we would like to comment on other arguments in the literature supporting the existence of a bound $a_{\max}$ for proper accelerations. One was given some time ago in [39] in connection with conformal kinematics; there the authors analyzed the physical interpretation of the singularities, $1 + 2\alpha x + \alpha^2 x^2 = 0$, of the SCT (2). When applying the transformation to an extended object of size $\ell$, an upper limit to the proper acceleration, $a_{\max} \simeq 1/\ell$ (in $c = 1$ units), is shown to be necessary in order for the tenets of special relativity not to be violated (see [39] for more details). Before, Caianiello [40] derived the existence and physical consequences of a maximal acceleration connected with Born’s reciprocity principle (BRP) [41, 42]. Indeed, one can deduce the existence of a maximal acceleration from the positivity of Born’s line element

$$d\tau^2 = dx^\mu dx^\nu + \frac{r^4}{l^2} dp^\mu dp^\nu = d\tau \sqrt{1 - \frac{a^2}{a_{\max}^2}}, \tag{40}$$

where $d\tau^2 = dx^\mu dx^\nu$ and $dp^\mu/d\tau = md^2 x^\mu/d\tau^2 = ma$, as usual. An adaptation of the BRP to the conformal relativity has been put forward by some of us in [21], where a conformal analogue of the line element (40) in the phase space $\mathbb{D}_4$ has been considered. However, the existence of a maximal acceleration inside the conformal group does not seem to be apparent from this conformal adaptation of the BRP either.

In the past few years, many papers have been published (see e.g. [43] and references therein), each one introducing the maximal acceleration starting from different motivations and from different theoretical schemes. Among the large list of physical applications of...
Caianiello’s model we would like to point out the one in cosmology which avoids an initial singularity while preserving inflation. Also, a maximal-acceleration relativity principle leads to a variable fine structure ‘constant’ [43], according to which it could have been extremely small (zero) in the early Universe and then all matter in the Universe could have emerged via the Unruh effect. Moreover, in a non-commutative geometry setting [44], the non-vanishing commutators among the four components of $P_\mu = (P_\mu + K_\mu)/2$ can be seen as a sign of the granularity (non-commutativity) of spacetime in conformal-invariant theories, along with the existence of a minimal length $\ell_{\text{min}}$ or, equivalently, a maximal acceleration $a_{\text{max}} = 1/\ell_{\text{min}}$ (in $c = 1$ units).

5. Second-quantized theory, conformal zero modes and Poincaré $\theta$-vacua

We have discussed the effect of relativistic accelerations in first quantization. However, the proper setting to analyze radiation effects is in the second-quantized theory. Let us denote (for space-saving notation) by $n = \{j, m, q_1, q_2\}$ the multi-index of the one-particle basis wavefunctions $\psi_n$ in (17) and by $\hat{a}_n$ (resp. $\hat{a}_n^\dagger$) operators annihilating (resp. creating) a particle in the state $|n\rangle$. An orthonormal basis for the Hilbert space of the second-quantized theory is constructed by taking the orbit through the conformal vacuum (0) of the creation operators $\hat{a}_n^\dagger$,

$$|q(n_1), \ldots, q(n_p)\rangle \equiv \frac{(\hat{a}_n^\dagger)^{q(n_1)} \cdots (\hat{a}_n^\dagger)^{q(n_p)}}{(q(n_1)! \cdots q(n_p)!)^{1/2}} |0\rangle,$$

where $q(n) \in \mathbb{N}$ denotes the occupation number of the state $n$ with energy $2j + 2m$.

The fact that the ground state of the first quantization, $\psi_0$, is invariant under Poincaré transformations (remember the discussion after (23)) implies that the annihilation operator $\hat{a}_0$ of zero (‘dark’-) energy modes commutes with all Poincaré' generators. It also commutes with all annihilation operators and creation operators of particles with positive (‘bright’) energy,

$$[\hat{a}_0, \hat{a}_n^\dagger] = 0, \quad n \neq 0. \quad (42)$$

Therefore, by Schur’s lemma, $\hat{a}_0$ must behave as a multiple of the identity when conformal symmetry is broken/restricted to Poincaré' symmetry. This means that we can choose Poincaré'-invariant vacua $|\theta\rangle$ as being eigenstates of $\hat{a}_0$, namely,

$$\hat{a}_0 |\theta\rangle = \theta |\theta\rangle \Rightarrow |\theta\rangle = e^{\theta \hat{a}_0^\dagger - \hat{a}_0} |0\rangle,$$

which implies that Poincaré' ‘$\theta$-vacua’ $|\theta\rangle$ are (canonical) CSs of conformal zero modes. Unlike the conformal vacuum $|0\rangle$, which is invariant under the whole conformal group, Poincaré' $|\theta\rangle$ are not stable under SCTs (accelerations). In fact, the second-quantized version of (24), for an acceleration $C = \sigma \sigma^3$ along the third axis, is given by the transformation of annihilation (resp. creation) operators:

$$\tilde{\hat{a}}_0 = \sum_{n=0}^{\infty} \psi_n(\sigma) \hat{a}_n. \quad (44)$$

We shall assume that $\sum_n |\psi_n(\sigma)|^2 = 1$ (normalized probabilities) so that this transformation preserves the original commutation relations $[\hat{a}_0, \hat{a}_n^\dagger] = 1$. Therefore, accelerated Poincaré' $\theta$-vacua are

$$|\tilde{\theta}\rangle = e^{\theta \hat{a}_0^\dagger - \hat{a}_0} |0\rangle = e^{\theta \sum_{n=0}^{\infty} \psi_n(\sigma) a_n^\dagger |\theta\rangle.} \quad (45)$$

We can think of conformal zero modes as ‘virtual particles’ without ‘bright’ energy and undetectable by inertial observers. However, from an accelerated frame, they become ‘visible’
to a Poincaré observer. The average number of particles with energy $E_n$ in the accelerated vacuum (45) is then given by

$$N_n(\alpha) = \langle \hat{\theta} | \hat{a}_n^\dagger \hat{a}_n | \hat{\theta} \rangle = |\theta|^2 |\varphi_n(\alpha)|^2,$$

(46)

where $|\theta|^2$ is the total average number of particles in $|\theta\rangle$ and $|\varphi_n(\alpha)|^2$ is the occupation number of the energy level $E_n$ of the accelerated vacuum $|\hat{\theta}\rangle$. The situation resembles that in many condensed-matter systems (like Bose–Einstein condensates, superconductors, etc), where one also finds non-empty, coherent ground states. In the same way, the probability $P_n(q, \alpha)$ of observing $q$ particles with energy $E_n$ in $|\hat{\theta}\rangle$ can be calculated as

$$P_n(q, \alpha) = |\langle q(n) | \hat{\theta} \rangle|^2 = \frac{e^{-|\theta|^2}}{q!} |\theta|^2 |\varphi_n(\alpha)|^2 q = \frac{e^{-|\theta|^2}}{q!} N_n^{q}(\alpha).$$

(47)

Therefore, the relative probability of observing a state with total energy $E$ in the excited vacuum $|\hat{\theta}\rangle$ is

$$P(E) = \sum_{\sum_{n=0}^{\infty} q_n = E \text{ in } E} \prod_{n=0}^{k} P_n(q_n, \alpha).$$

(48)

For the case studied in this paper, this distribution function can be factorized as $P(E) = \Omega(E) e^{-E/T}$, where $\Omega(E)$ is a relative weight proportional to the number of states with energy $E$ and the factor $e^{-E/T}$ fits this weight properly to a temperature $T$.

One can also compute the total mean energy

$$E(\alpha) = \langle \hat{\theta} | \sum_{n=1}^{\infty} E_n n^2 \varphi_n^\dagger \varphi_n | \hat{\theta} \rangle = |\theta|^2 \sum_{n=1}^{\infty} |\varphi_n(\alpha)|^2 E_n = |\theta|^2 \mathcal{E}(\alpha),$$

(49)

which, as expected, is the product of $\mathcal{E}(\alpha)$ in (27) times the average number of particles $|\theta|^2$ in $|\theta\rangle$. The free parameter $|\theta|^2$ is also linked to a vacuum (‘dark’) energy $E_0 = |\theta|^2 \mathcal{E}_0 = |\theta|^2 \lambda$, just like, for example, the ‘cosmological constant’. Like other non-zero vacuum expectation values, zero-point energy leads to observable consequences such as, for instance, the Casimir effect and influences the behavior of the Universe at cosmological scales, where the vacuum (dark) energy is expected to contribute to the cosmological constant, which affects the expansion of the Universe (see e.g. [45] for a nice review). Actually, dark energy is the most popular way to explain recent observations that the Universe appears to be expanding at an accelerating rate.

6. Comments and outlook

As already commented in the introduction, conformal field theories also seem to provide a universal description of low-energy black hole thermodynamics, which is only fixed by symmetry arguments (see [15, 16] and references therein). Actually, Unruh’s temperature (1) coincides with Hawking’s temperature

$$T = \frac{\hbar c^3}{8\pi M k_B G} = \frac{2\pi GM}{\Sigmack_B}$$

(50)

($\Sigma = 4\pi r_s^2 = 8\pi G^2 M^2/c^4$ stands for the surface of the event horizon) when the acceleration is that of a free-falling observer on the surface $\Sigma$, i.e. $a = c^4/(4GM) = GM/r_s^2$. Here, the Virasoro algebra proves to be a physically important subalgebra of the gauge algebra of surface deformations that leave the horizon fixed for an arbitrary black hole. Thus, the fields on the surface must transform according to irreducible representations of the Virasoro algebra, which is the general symmetry principle that governs the density of microscopic states. Therefore, in the Hawking effect, the calculation of thermodynamical quantities, linked
to the statistical mechanical problem of counting microscopic states, is reduced to the study of the representation theory of the conformal group.

Although our approach to the quantum analysis of accelerated frames shares with the previous description of black hole thermodynamics the existence of an underlying conformal invariance, we should not confuse the two schemes. Conformal invariance in the Hawking effect manifests itself as an infinite-dimensional gauge algebra of (two-dimensional) surface deformations. However, the infinite-dimensional character of conformal symmetry seems to be an exclusive patrimony of two-dimensional physics, and conformal invariance in (3+1) dimensions is finite-(15)-dimensional, thus accounting for transitions to uniformly accelerated frames only. To account for higher-order effects of gravity on QFT from a group-theoretical point of view, one should consider more general diffeomorphism (Lie) algebras. Higher-dimensional analogies of the infinite two-dimensional conformal symmetry have been proposed by us in [17, 18, 14, 19, 20]. We think that these infinite $W$-like symmetries can play some fundamental role in quantum gravity models, as a gauge guiding principle.

To conclude, we would also like to mention that the same spontaneous $SU(2,2)$-symmetry breaking mechanism explained in this paper applies to general $SU(N,M)$-invariant quantum theories, where an interesting connection between ‘curvature and statistics’ has emerged [46, 47]. We hope that many more interesting physical phenomena remain to be unraveled inside conformal-invariant quantum (field) theory. As stated long time ago by Hill [28], “a more complete analysis of the physical interpretation of the full conformal group of transformations will be required before all of its implications can be appreciated”.

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