Joint estimation of conditional quantiles in multivariate linear regression models. An application to financial distress.

Lea Petrella\textsuperscript{a}, Valentina Raponi\textsuperscript{b,a},

\textsuperscript{a}MEMOTEF, Sapienza, Università di Roma, Rome, Italy
\textsuperscript{b}Imperial College Business School, Imperial College London, London, UK

Abstract

This paper proposes a maximum-likelihood approach to jointly estimate marginal conditional quantiles of multivariate response variables in a linear regression framework. We consider a slight reparameterization of the Multivariate Asymmetric Laplace distribution proposed by Kotz et al (2001) and exploit its location-scale mixture representation to implement a new EM algorithm for estimating model parameters. The idea is to extend the link between the Asymmetric Laplace distribution and the well-known univariate quantile regression model to a multivariate context, i.e. when a multivariate dependent variable is concerned. The approach accounts for association among multiple responses and study how the relationship between responses and explanatory variables can vary across different quantiles of the marginal conditional distribution of the responses. A penalized version of the EM algorithm is also presented to tackle the problem of variable selection. The validity of our approach is analyzed in a simulation study, where we also provide evidence on the efficiency gain of the proposed method compared to estimation obtained by separate univariate quantile regressions. A real data application is finally proposed to study the main determinants of financial distress in a sample of Italian firms.

Keywords: Multiple quantiles, Quantile Regression, Multivariate Asymmetric Laplace Distribution, EM algorithm, Maximum Likelihood, Multivariate response variables

1. Introduction

Quantile regression has become a widely used technique in many empirical applications, since the seminal work of Koenker and Basset (1978). It provides a way to model the conditional quantiles of a response variable with respect to a set of covariates in order to have a more complete picture of the entire conditional distribution than the ordinary least squares regression. This approach is quite suitable to be used in all the situations where specific features, like skewness, fat-tails, outliers,
truncation, censoring and heteroscedasticity arise. In fact, unlike standard linear regression models, which only consider the conditional mean of a response variable, quantile regression allows one to assume that the relationship between the response and explanatory variables can vary across the conditional distribution of the dependent variable. Many univariate quantile regression methods are now well consolidated in the literature and have been implemented in a wide range of different fields, like medicine (see e.g., Cole and Green (1992) and Royston and Altman (1994), Marino et al. (2016)), financial and economic research (Bassett and Chen (2001), Hendricks and Koenker (1992), Petrella et al. (2017)), and environmental modelling (see Pandey and Nguyen (1999) and Hendricks and Koenker (1992) for a discussion). Koenker (2005) provides an overview of the most used quantile regression techniques in a classical setting. In longitudinal studies, quantile regression models with random effects are also analysed, in order to account for the dependence between serial observations on the same subject. See, e.g Geraci and Bottai (2007), Koenker (2004), Koenker (2017), Marino and Farcomeni (2015) for references. Bayesian versions of quantile regression have also been extensively proposed (see Yu and Moyeed (2001), Kottas and Gelfand (2001), Kottas and Krnjajic (2009), Bernardi et al. (2015)).

It is well known in the literature that the univariate quantile regression approach has a direct link with the Asymmetric Laplace (AL) distribution. In fact, while frequentist quantile regression framework relies on the minimization of the asymmetric loss function introduced by Koenker and Basset (1978), the Bayesian approach introduces the Asymmetric Laplace (AL) distribution as inferential tool to estimate model parameters (see the seminal work Yu and Moyeed (2001)). The two approaches are both justified by the well-established relationship between the loss function and the AL density. That is, the loss function minimization problem is equivalent (in terms of parameter estimates) to the maximization of the likelihood associated with the AL density (see e.g Kozumi and Kobayashi (2011)). Therefore, the AL distribution could offer a convenient device to implement a likelihood-based inferential approach when dealing with quantile regression analysis.

Still in the context of univariate regression framework, part of the literature is concerned with estimation of multiple quantiles (say \( Q_{Y}(\tau_j) \), \( j=1,\ldots, p \)) of a given response variable \( Y \in \mathbb{R} \). In this case, joint estimation of these \( p \) quantiles provides a gain in efficiency compared to traditional sequential estimation of multiple quantiles (see Koenker and Basset, 1978). A recent extension to a longitudinal setting has been proposed by Cho et al. (2017).

When multivariate response variables are concerned, the existing literature on quantile regression is less extensive. The multivariate quantile problem has the goal to estimate the multivariate quantile, \( Q_{Y}(\tau) \), of a multivariate response variable \( Y \in \mathbb{R}^p \), with \( p > 1 \) and where the index \( \tau \) is a scalar. In this case, the main challenge concerns the definition of a multivariate quantile, given that there is no natural ordering in a \( p \)-dimensional space (Chakraborty (2003), Hallin et al. (2010), Kong and Mizera (2012)). Other attempts and extensions can be also found in Kong et al. (2015), Bernardi et al. (2018), Paindaveine and Siman (2012), Bocek and Siman (2017) and Alfó et al. (2016).
The main goal of the present paper is to extend the univariate linear quantile regression methodology to a multivariate context. In particular we want to generalize the inferential approach based on the AL distribution to a multivariate framework, by using the Multivariate Asymmetric Laplace (MAL) distribution introduced by Kotz et al. (2001). In this way we are not concerned to define a multivariate quantile. Instead, we are conducting a simultaneous inference on the marginal conditional quantiles of a multivariate response variable, taking also into account for the possible correlation among marginals. This need could arise in many situations where different responses might have similar distributions and/or might be affected by the same set of covariates at different parts of their distributions other than the mean. Hence, by jointly modeling them we can borrow information across responses and conduct joint inference at the marginal quantiles level, rather then defining a central point for their distributions.

Similar attempts in the literature have been proposed by Jun and Pinkse (2009), who introduce a seemingly unrelated quantile regression approach which entails a nonparametric estimation of a set of moment conditions for the conditional quantiles of interest. Waldmann and Kneib (2015) propose a bivariate quantile regression using a Bayesian approach and show how to estimate the conditional correlations between the two response variables. Their work, however, has not been generalized to the multivariate case.

Our paper offers a likelihood-based approach for modeling and estimating conditional marginal quantiles jointly, by using the MAL distribution as working likelihood in a linear regression framework. Specifically, we propose a slight reparametrization of the MAL distribution, subject to some specific constraints, which allows us to estimate regression coefficients via maximum likelihood (ML), accounting for the possible association among the responses. The inferential problem is solved by developing a suitable Expectation-Maximization (EM) algorithm, which exploits the mixture representation of the MAL distribution (see also Arslan, 2010).

Using simulation exercises, we assess the validity and the robustness of our approach, by considering different model distributional settings. We find that the estimation of the regression coefficients is not highly affected by the MAL distributional assumption. Moreover, the estimation efficiency of our multivariate approach is higher than the one obtained by running separate single quantile regression models on the marginals when estimating model parameters. That is, taking into account for the potential association among the response variables can significantly reduce the root mean square error of the estimated coefficients, hence improving the precision of the estimates.

When dealing with multivariate regression, the high dimensionality setting is an intrinsic part of the model building problem. In order to gain in parsimony and to conduct a variable selection procedure, we consider the penalized Least Absolute Shrinkage and Selecting Operator (LASSO) approach proposed by Tibshirani (1996). In particular, we propose a penalized version of the EM algorithm (PEM) accounting for an $l_1$ penalty term. We evaluate the estimation performance of the proposed approach through a simulation exercise, where we compute the bias and the root mean square error of the estimated parameters at different quantile levels.
The relevance of our approach is also shown empirically, contributing to the increasingly widespread literature that uses quantiles as measures of risk. In the recent years, due to the financial and economic crisis, a particular attention has been devoted to measuring and quantifying the level of financial risk and financial distress within a firm or investment portfolios. In this respect, many risk measures developed in literature are based on quantile values, like for example the Value at Risk (see Jorion (2007) and McNeil et al. (2005)). Moreover, quantile regression methods turn out to be very helpful to quantify either the magnitude and the causes of riskiness (see for example Engle and Manganelli (2004), Wong and Ting (2016)). In this paper, we implement the proposed quantile regression approach to investigate the main determinants of financial distress on a sample of 2,020 Italian firms. In particular, we use the definition of financial distress adopted in Bastos and Pindado (2013), which classify a firm as financial distressed if its earnings before interest and taxes depreciation and amortization (EBITDA) is under the first quartile of the sample or if its leverage is above the third quartile of the sample. Hence, starting from this definition, we apply our methodology to analyze the relationships between financial distress and firms’ characteristics and evaluate how it may vary when considering different (more extreme) quantiles of the distribution of leverage and EBITDA. In this way we are able to assess not only what are the main determinants for a firm’s risk of financial distress, but also how these factors matter as more serious levels of distress are considered.

The rest of the paper is organized as follows. In Section 2, we introduce the main notation and briefly revise the univariate quantile regression model. Section 3 introduces the joint quantile regression framework, while Section 4 proposes the EM-based Maximum Likelihood approach and the related Penalized EM (PEM) algorithm to estimate model parameters. In Section 5 we provide simulation results, while the empirical application is presented in Section 6. Section 7 summarizes our conclusions.

2. Preliminaries on univariate quantile regression and the AL distribution

To better explain the link between the MAL distribution and the joint quantile regression, we briefly revise the univariate quantile regression model and its direct connection with the AL density. As argued in Yu and Moyeed (2001) and Kozumi and Kobayashi (2011), we say that a random variable \( Y \) is distributed as an AL with location parameter \( \mu \), scale parameter \( \delta > 0 \) and skewness parameter \( \tau \in (0, 1) \), i.e. \( AL(\mu, \delta, \tau) \), if its probability density function has the following representation

\[
f_{AL}(y; \mu, \delta, \tau) = \frac{\tau(1 - \tau)}{\delta} \exp \left\{ -\rho_{\tau} \left( \frac{y - \mu}{\delta} \right) \right\}
\]

where \( \rho_{\tau}(\cdot) \) denotes the so called loss (or check) function defined by \( \rho_{\tau}(x) = x(\tau - I\{x < 0\}) \), with \( I\{\cdot\} \) being the indicator function and where the quantity \( \rho_{\tau} \left( \frac{y - \mu}{\delta} \right) \) follows an exponential distribution with rate parameter equal to \( 1/\delta \). Kotz et al. (2001) show that the AL distribution in (1) admits a
Gaussian-mixture representation. In particular, if \( Y \sim AL(\mu, \delta, \tau) \), then \( Y \) can be also written as
\[
Y = \mu + \xi\tau U + \theta\tau \sqrt{\delta}UZ
\]
where \( U \) follows an Exponential distribution with rate parameter \( 1/\delta \) and \( Z \) is a standard Normal random variable. Moreover, in order to guarantee that the parameter \( \mu \) coincides with the quantile of \( Y \) at a chosen level \( \tau \), the following conditions on \( \xi\tau \) and \( \theta\tau \) must be satisfied
\[
\xi\tau = \frac{1 - 2\tau}{\tau(1 - \tau)}, \quad \text{and} \quad \theta\tau = \frac{2}{\tau(1 - \tau)}.
\]
Now, let \( y_i, i = 1, 2, ..., n \) be a response variable of interest and let \( x_i \) be a \( k \times 1 \) vector of covariates associated with the \( i \)-th observation. Let \( Q_{y_i}(\tau|x_i) \) denote the quantile regression function of \( y_i \) given \( x_i \) at a given level \( \tau \in (0, 1) \), and assume that the relationship between \( Q_{y_i}(\tau|x_i) \) and \( x_i \) can be modeled as
\[
Q_{y_i}(\tau|x_i) = x_i'\beta_{\tau}
\]
where \( \beta_{\tau} \) is a \( k \times 1 \) vector of regression coefficients. Notice that the relationship in (1) implies the following linear quantile regression model
\[
y_i = x_i'\beta_{\tau} + \epsilon_i, \quad i = 1, 2, ..., n
\]
where the error term \( \epsilon_i \) is such that its distribution is restricted to have the \( \tau \)-quantile equal to zero. If the distribution of the error term is left unspecified, then the parameter estimation proceeds by minimizing the following objective function
\[
\hat{\beta}_{\tau} = \arg\min_{\beta \in \mathbb{R}^k} \sum_{i=1}^{n} \rho_{\tau} \left( y_i - x_i'\beta_{\tau} \right).
\]
As the loss function \( \rho_{\tau}(\cdot) \) is not differentiable at zero, explicit solutions for \( \hat{\beta}_{\tau} \) cannot be derived and direct optimization is typically applied. As shown in Yu and Moyeed (2001) the AL distribution provides a direct connection between the minimization problem in (6) and maximum likelihood (ML) estimation. In fact, if we use the AL density as likelihood tool in (5), we have
\[
\mathcal{L}(\beta_{\tau}, \delta|y) = \frac{\tau^n(1 - \tau)^n}{\delta^n} \exp \left\{ -\sum_{i=1}^{n} \rho_{\tau} \left( \frac{y_i - x_i'\beta_{\tau}}{\delta} \right) \right\}.
\]
for a given \( \tau \), and with \( \delta > 0 \). It is easy to verify that the minimization of the objective function in (6) with respect to the parameter \( \beta_{\tau} \) is equivalent to the maximization of the likelihood in (7). Therefore, the AL distribution offers a valid tool to set up the quantile regression model in a likelihood framework.

In the next section we extend such link between the AL and the quantile regression to a multivariate framework.
3. Joint Quantile Regression and the MAL distribution

Extending the results of the previous section, we now show how to use the MAL distribution (Kotz et al. (2001)) for jointly modeling marginal conditional quantiles of a multivariate response variable. Let \( Y_i = [Y_{i1}, Y_{i2}, ..., Y_{ip}]' \) be a \( p \times 1 \) vector of error terms with univariate component-wise quantiles (at fixed levels \( \tau_1, ..., \tau_p \)) equal to zero. For the regression model in (9), consider now the following MAL distribution (Kotz et al., 2001) for jointly modeling marginal conditional quantiles of a multivariate response variable.

\[
\begin{align*}
Q_{Y_{i1}}(\tau_1|X_i) = \beta_{\tau_1}X_i, & \quad i = 1, 2, ..., n \\
Q_{Y_{i2}}(\tau_2|X_i), & \\
\vdots & \\
Q_{Y_{ip}}(\tau_p|X_i)
\end{align*}
\]

where \( Q_{Y_{ij}}(\tau_j|X_i) \) denotes the \( \tau_j \)-level quantile regression function of \( Y_{ij} \) given \( X_i \). Our objective is to provide joint estimation of the \( p \) marginal conditional quantiles of \( Y_i \in \mathbb{R}^p \). The representation in (8) implies the following multivariate linear regression model:

\[
Y_i = \beta X_i + \epsilon_i \quad i = 1, 2, ..., n
\]

where \( \epsilon_i \) denotes a \( p \times 1 \) vector of error terms with univariate component-wise quantiles (at fixed levels \( \tau_1, ..., \tau_p \), respectively) equal to zero. For the regression model in (9), consider now the following MAL distribution (Kotz et al., 2001)

\[
f_Y(y_i|\beta, X_i, D\tilde{\xi}, D\tilde{\Sigma}D) = \frac{2 \exp \left\{ (y_i - \beta X_i)'D^{-1}\tilde{\Sigma}^{-1}\tilde{\xi} \right\}}{(2\pi)^{p/2}|D\Sigma D|^{1/2}} \left( \frac{\tilde{m}}{2 + \tilde{d}} \right)^{\nu/2} K_\nu \left( \sqrt{(2 + \tilde{d})\tilde{m}} \right) \]

where \( \beta X_i \) is the location parameter vector, \( D\tilde{\xi} \in \mathbb{R}^p \) is the scale (or skew) parameter, with \( D = \text{diag}[\delta_1, \delta_2, ..., \delta_p] \), \( \delta_j > 0 \) and \( \tilde{\xi} = [\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_p]' \), having generic element \( \tilde{\xi}_j = \frac{1 - 2\tau_j}{\tau_j(1 - \tau_j)} \). \( \tilde{\Sigma} \) is a \( p \times p \) positive definite matrix such that \( \tilde{\Sigma} = \tilde{\Lambda}\Psi\tilde{\Lambda} \), with \( \Psi \) being a correlation matrix and \( \tilde{\Lambda} = \text{diag}[\tilde{\delta}_1, \tilde{\delta}_2, ..., \tilde{\delta}_p] \), with \( \tilde{\delta}_j^2 = \frac{2}{\tau_j(1 - \tau_j)} \). Moreover, \( \tilde{m} = (y_i - \beta X_i)'(D\tilde{\Sigma}D)^{-1}(y_i - \beta X_i) \), \( \tilde{d} = \tilde{\xi}'\tilde{\Sigma}\tilde{\xi} \), and \( K_\nu(\cdot) \) denotes the modified Bessel function of the third kind with index parameter \( \nu = (2 - p)/2 \). Using (9) and (10), and following Kotz et al. (2001), the MAL distribution can be written as a location-scale mixture, having the following representation:

\[
Y_i = \beta X_i + D\tilde{\xi}W + \sqrt{WDD\tilde{\Sigma}^{-1/2}}Z
\]
where $Z \sim N_p(0, I_p)$ denotes a $p$-variate standard Normal distribution and $W \sim \text{Exp}(1)$ has a standard Exponential distribution, with $Z$ being independent of $W$.

It is worth noticing that the constraints

$$\tilde{\xi}_j = \frac{1 - 2\tau_j}{\tau_j(1 - \tau_j)} \quad \text{and} \quad \tilde{\sigma}_j^2 = \frac{2}{\tau_j(1 - \tau_j)}, \quad j = 1, \ldots, p \quad (12)$$

represent necessary conditions to guarantee that the model in (8) holds, as shown in the proposition below.

**Proposition 1.** Let $Y \sim \text{MAL}_p(\mu, D\tilde{\xi}, D\tilde{\Sigma}D)$ as defined in (11), and let $\tau = [\tau_1, \tau_2, \ldots, \tau_p]'$ be a fixed $p \times 1$ vector, such that $\tau_j \in (0, 1)$, $j = 1, \ldots, p$, then,

$$P(Y_j < \mu_j) = \tau_j$$

if and only if

$$\tilde{\xi}_j = \frac{1 - 2\tau_j}{\tau_j(1 - \tau_j)} \quad \text{and} \quad \tilde{\sigma}_j^2 = \frac{2}{\tau_j(1 - \tau_j)}, \quad j = 1, \ldots, p \quad (13)$$

In addition, $Y_j \sim \text{AL}(\mu_j, \tau_j, \delta_j)$.

**Proof.** See Appendix.

Notice that, the representation in (10), under (12), is a suitable reparameterization of Kotz et al. (2001). Indeed the introduction of the diagonal matrix $D$ ensures that each $\delta_j$ represents the scale parameter of the marginal AL distribution of $Y_j$, for every $j = 1, 2, \ldots, p$ (see the proof of Proposition 1 in Appendix for further details).

It is also worth adding a brief comment on parameters identifiability of the MAL distribution, as one could reasonably ask whether $D\tilde{\xi}$ and $D\tilde{\Sigma}D$ are uniquely identified. In the next proposition, we argue that the constraints (12) in (11) are necessary conditions for model identifiability, for any fixed quantile level $\tau_j, j = 1, \ldots, p$.

**Proposition 2.** Let $Y \in \mathbb{R}^p$ be distributed as a $\text{MAL}_p(\mu, D\tilde{\xi}, D\tilde{\Sigma}D)$, where $D = \text{diag}[\delta_1, \delta_2, \ldots, \delta_p]$, $\delta_j > 0$, $\tilde{\xi} = [\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_p]'$, with $\tilde{\xi}_j = \frac{1 - 2\tau_j}{\tau_j(1 - \tau_j)}$ being known for any fixed value of $\tau_j$. $\tilde{\Sigma}$ is a $p \times p$ positive definite matrix such that $\tilde{\Sigma} = \tilde{\Lambda}\Psi\tilde{\Lambda}$, with $\Psi$ being an unknown correlation matrix and $\tilde{\Lambda} = \text{diag}[\tilde{\sigma}_1, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_p]$, with fixed element $\tilde{\sigma}_j^2 = \frac{2}{\tau_j(1 - \tau_j)}$, for every $j = 1, \ldots, p$. Then, the parameters $D$ and $\Psi$ are uniquely identified.

**Proof.** See Appendix.

In other words, the constraints in (12) represent essential conditions not only to retrieve the joint quantile regression model in (8) (as discussed in Proposition 1), but also to ensure that the model does not suffer from identifiability problems (see Proposition 2).
4. Maximum Likelihood estimation

As shown in the previous sections, the MAL density represents a convenient tool to jointly model
marginal conditional quantiles of a multivariate response variable in a quantile regression framework.
In this section we introduce a maximum likelihood approach to estimate and make inference on model
parameters. We propose a suitable likelihood-based EM algorithm (Dempster et al, 1977), showing
that model parameters can be easily obtained in closed form, hence facilitating the computational
burden of the algorithm compared to the direct maximization approach. Moreover, as mentioned
in the Introduction, given the possible high dimensionality problem in multivariate settings, we also
propose a penalized version (PEM) of the EM algorithm by considering a LASSO regularization
approach (Tibshirani, 1996). The procedure essentially modifies the M-step of the EM algorithm by
introducing a penalty term and provides a data-driven procedure for variable selection.

4.1. The EM algorithm

In this section we propose a Maximum Likelihood-based approach to estimate the parameters of the
quantile regression models defined in (8). Specifically, we derive a new EM algorithm by exploiting the
Gaussian location-scale mixture representation (11) of the MAL distribution, under the constraints
in (12). The EM algorithm essentially alternates between performing an expectation (E) step, which
defines the expectation of the complete log-likelihood function evaluated using the current estimate for
the parameters, and a maximization (M) step, which computes parameter estimates by maximizing
the expected complete log-likelihood obtained in the E-step. The expected complete log-likelihood
function and the optimal parameter estimators are given below in the following two propositions. All
the proofs are collected in the Appendix.

For sake of clarity, in the following we introduce the notation $D(\delta)$ and $\tilde{\Sigma}(\Psi)$ to make clear that the
matrices $D$ and $\tilde{\Sigma}$ depend on the parameters $\delta = [\delta_1, ..., \delta_p]'$ and $\Psi$, respectively.

Proposition 3. For a given vector $\tau = [\tau_1, \tau_2, ..., \tau_p]'$, let $D(\delta) = \text{diag}(\delta)$, with $\delta = [\delta_1, ..., \delta_p]'$ and let
$\tilde{\Sigma}(\Psi) = \tilde{\Lambda} \Psi \tilde{\Lambda}$ with $\tilde{\Lambda}$ subject to the constrains in (13). Let $\hat{m}_i = (y_i - \beta_r X_i)' (D(\delta) \tilde{\Sigma}(\Psi) D(\delta))^{-1} (y_i - \beta_r X_i)$, for every $i = 1, 2, ..., n$, and define $\hat{d} = \tilde{\xi}' \tilde{\Sigma}(\Psi) \tilde{\xi}$. Then, the expected complete log-likelihood function (up to additive constants) is

$$E \left[ l_c(\beta_r, D(\delta), \tilde{\Sigma}(\Psi)|Y_i, \hat{m}_i, D(\delta), \tilde{\Sigma}(\Psi)) \right] = -\frac{n}{2} \log |D(\delta) \tilde{\Sigma}(\Psi) D(\delta)| + \sum_{i=1}^{n} (Y_i - \beta_r X_i)' D(\delta)^{-1} \tilde{\Sigma}(\Psi)^{-1} \tilde{\xi}' \tilde{\xi}$$

$$- \frac{1}{2} \sum_{i=1}^{n} z_i (Y_i - \beta_r X_i)' (D(\delta) \tilde{\Sigma}(\Psi) D(\delta))^{-1} (Y_i - \beta_r X_i)$$

$$- \frac{1}{2} \tilde{\xi}' \tilde{\Sigma}(\Psi) \tilde{\xi} \sum_{i=1}^{n} u_i.$$
where

\[ u_i = \left( \frac{\hat{m}_i}{2 + \hat{d}} \right)^{\frac{1}{2}} \frac{K_{\nu+1} \left( \sqrt{(2 + \hat{d})\hat{m}_i} \right)}{K_{\nu} \left( \sqrt{(2 + \hat{d})\hat{m}_i} \right)} \]

\[ z_i = \left( \frac{2 + \hat{d}}{\hat{m}_i} \right)^{\frac{1}{2}} \frac{K_{\nu+1} \left( \sqrt{(2 + \hat{d})\hat{m}_i} \right)}{K_{\nu} \left( \sqrt{(2 + \hat{d})\hat{m}_i} \right)} - \frac{2\nu}{\hat{m}_i} \] (17)

with

\[ \hat{m}_i = (y_i - \hat{\beta}_{\tau} X_i)' D_{(\hat{\delta})} \Sigma_{\Psi} D_{(\hat{\delta})}^{-1} (y_i - \hat{\beta}_{\tau} X_i) \] (18)

\[ \hat{d} = \xi' \Sigma_{(\Psi)} \xi. \] (19)

**Proof.** See Appendix.

For a given \( \tau \), maximizing the expectation of the complete data log-likelihood in (14)-(16) with respect to the parameters \( \beta_{\tau}, \Sigma_{\Psi}, \) and \( \delta \) leads to the following M-step updates.

**Proposition 4.** Given the vector \( \tau \), the values of \( \beta_{\tau}, \Sigma_{(\Psi)} \) and \( \delta \) maximizing (14)-(16) are

\[ \hat{\beta}_{\tau} = \left( \sum_{i=1}^{n} z_i X_i X_i' \right)^{-1} \left( \sum_{i=1}^{n} z_i X_i Y_i' - \sum_{i=1}^{n} X_i \xi' D_{(\hat{\delta})} \right) \] (20)

\[ \Sigma_{(\Psi)} = \frac{1}{n} \sum_{i=1}^{n} z_i D_{(\hat{\delta})}^{-1} (Y_i - \hat{\beta}_{\tau} X_i)(Y_i - \hat{\beta}_{\tau} X_i)' D_{(\hat{\delta})}^{-1} + \frac{1}{n} \sum_{i=1}^{n} u_i \xi' D_{(\hat{\delta})} - \frac{2}{n} D_{(\hat{\delta})}^{-1} \sum_{i=1}^{n} (Y_i - \hat{\beta}_{\tau} X_i)' \xi', \] (21)

while the estimation of \( \delta \) is obtained through a numerical optimization, by solving the following non-linear first order condition

\[ \sum_{i=1}^{n} (Y_i - \hat{\beta}_{\tau} X_i) \xi' \Sigma_{(\Psi)} + n D_{(\hat{\delta})} - \sum_{i=1}^{n} Z_i (Y_i - \hat{\beta}_{\tau} X_i)(Y_i - \hat{\beta}_{\tau} X_i)' D_{(\hat{\delta})}^{-1} \Sigma_{(\Psi)} = 0_{p \times p} \] (22)

**Proof.** See Appendix.

Therefore, the EM algorithm can be implemented as follows:

**E-step:** Set the iteration number \( h = 1 \). Fix the vector \( \tau \) at the chosen quantile levels \( \tau_1, \ldots, \tau_p \) of interest and initialize the parameter \( \theta = (\beta, \Sigma, \delta) \), deriving \( \Sigma_{(\Psi)} \) and \( D_{(\delta)} \). Then, given \( \theta = \theta^{(h)} \), calculate the weights...
Through the weights of

Notice that all the parameter estimates in (25)-(28) account for the multivariate structure of the data evaluated at two consecutive iterations is small enough

The procedure is iterated until convergence, that is when the difference between the likelihood function

where \( m_i^{(h)} = \left(Y_i - \beta^{(h)}_r X_i\right) \left(D_{(\delta)}^{(h)} \Sigma_{(\Psi)}^{(h)} D_{(\delta)}^{(h)}\right)^{-1} \left(Y_i - \beta^{(h)}_r X_i\right) \), and \( \tilde{d}^{(h)} = \tilde{\xi} \Sigma_{(\Psi)} \tilde{\xi} \), for \( i = 1, 2, \ldots, n \).

**M-step:** Use \( u_i^{(h)} \) and \( z_i^{(h)} \) to maximize \( E[l_i(\theta|\theta^{(h)})] \) with respect to \( \theta \), and obtain the new parameter estimates as

\[
\hat{\beta}_r^{(h+1)} = \left(\sum_{i=1}^{n} z_i^{(h)} X_i X'_i\right)^{-1} \left(\sum_{i=1}^{n} z_i^{(h)} X_i Y'_i - \sum_{i=1}^{n} X_i \tilde{\xi} D_{(\delta)}^{(h)}\right) \tag{25}
\]

\[
\hat{\Sigma}_{(\Psi)}^{(h+1)} = \frac{1}{n} \sum_{i=1}^{n} z_i^{(h)} D_{(\delta)}^{-1(h)} \left(Y_i - \hat{\beta}_r^{(h+1)} X_i\right) \left(Y_i - \hat{\beta}_r^{(h+1)} X_i\right)' D_{(\delta)}^{-1(h)} + \frac{1}{n} \sum_{i=1}^{n} u_i^{(h)} \tilde{\xi} \tilde{\xi} - \frac{2}{n} D_{(\delta)}^{-1(h)} \sum_{i=1}^{n} Y_i - \hat{\beta}_r^{(h+1)} X_i)' \tilde{\xi}. \tag{27}
\]

while \( D_{(\delta)}^{(h+1)} \) is obtained as the solution of the following equation

\[
\sum_{i=1}^{n} \left(Y_i - \hat{\beta}_r^{(h+1)} X_i\right) \tilde{\xi} \sum_{(\Psi)}^{(h+1)} + n D_{(\delta)} - \sum_{i=1}^{n} z_i^{(h)} \left(Y_i - \hat{\beta}_r^{(h+1)} X_i\right) \left(Y_i - \hat{\beta}_r^{(h+1)} X_i\right)' D_{(\delta)}^{-1(h)} \sum_{(\Psi)}^{(h+1)} = 0_{p \times p} \tag{28}
\]

The procedure is iterated until convergence, that is when the difference between the likelihood function evaluated at two consecutive iterations is small enough.

Notice that all the parameter estimates in (25)-(28) account for the multivariate structure of the data through the weights \( u_i \) and \( z_i \) in (23) and (24), which depend on the index \( \nu \) (as a function of \( p \)). In the univariate case, \( \nu = \frac{p}{2} \) and the estimators reduce to the case of Sanchez et al. (2013).

---

1In our paper we set this convergence criterion equal to \( 10^{-6} \).
4.2. Variable selection and the Penalized EM (PEM) algorithm

As mentioned before, when dealing with high dimensional statistical problems, Lasso penalized procedures represent possible solutions to detect significant predictors from a large pool of candidate variables. Therefore, in this section we introduce a penalized version of the EM algorithm described in Section 4.1. The PEM algorithm was originally proposed by Green (1990) to allow for the maximization of a difficult-to-calculate penalized likelihood. Compared to the EM, the PEM algorithm leaves the E-step unchanged and modifies the M-step with a penalty function introduced to achieve shrinkage (e.g. ridge regression, Hoerl and Kennard, 1970), variable selection (e.g. Lasso, Tibshirani (1996)) or simultaneous shrinkage and variable selection (e.g elastic net, Zou and Hastie (2005)). For a chosen level \( \tau \), let us denote by \( \theta = (\beta_{\tau}, D(\delta), \Sigma(\psi)) \) the parameter set. Then, consider the following penalized-M (PM) step:

\[
\theta^{(h+1)} = \arg\max_{\theta} \left( Q(\theta|\theta^{(h)}) - \lambda \sum_{j=1}^{p} \sum_{s=1}^{k} |\beta_{js,\tau}| \right)
\]

where

\[
Q(\theta|\theta^{(h)}) = -\frac{n}{2} \log |D(\delta)\Sigma(\psi)D(\delta)| + \sum_{i=1}^{n} (Y_i - \beta_{\tau} X_i)' D^{-1}(\delta) \Sigma^{-1}(\psi) \xi
\]

\[
- \frac{1}{2} \sum_{i=1}^{n} z_i (Y_i - \beta_{\tau} X_i)' (D(\delta)\Sigma(\psi)D(\delta))^{-1} (Y_i - \beta_{\tau} X_i)
\]

\[
- \frac{1}{2} \xi' \Sigma(\psi) \xi \sum_{i=1}^{n} u_i.
\]

with all the quantities defined as in Section 4.1. In (29), the quantity \( \lambda \sum_{j=1}^{p} \sum_{s=1}^{k} |\beta_{js,\tau}| \) represents a convex penalty function, where \( \lambda \) is a tuning parameter that regulates the strength of the penalization assigned to the coefficients in the model. The choice of \( \lambda \) is made by performing cross-validation techniques which allows us to consider \( \lambda \) as a data-driven parameter. In particular, we compute the solutions for a decreasing sequence of values for \( \lambda \), starting from the smallest value \( \lambda_{\text{max}} \) for which the entire vector \( \hat{\beta}_k = 0 \). We then select a minimum value \( \lambda_{\text{min}} = \epsilon \lambda_{\text{max}} \) and construct a sequence of \( m \) values of \( \lambda \), decreasing from \( \lambda_{\text{max}} \) to \( \lambda_{\text{min}} \) on the log scale. This would lead to a more stable algorithm, see e.g. Friedman et al. (2010).

Notice that, even though the algorithm penalizes only the M-step, the expressions of the weights \( u_i \) and \( z_i \) in the E-step will be indirectly affected by the new (penalized) estimates in the M-step at each iteration.

In the next section we assess the performance of the EM and PEM algorithms using a simulation exercise.
5. Simulation study

In the following sections we conduct a simulation study to evaluate the small sample properties of the proposed methods. The idea of this exercise is to show that both the EM and the PEM algorithms represent valid procedures to estimate the quantile regression coefficients, regardless of the true data generation process.

5.1. Joint quantile regression

In this section we assess the performance of our estimation procedure by simulating a joint quantile regression model as described in Section 3. For this purpose, we consider a simple case of $n = 1000$ units, of dimension $p = 3$ and two explanatory variables. The observations are generated using the following data generating process:

$$Y_i = \beta_{\tau,0} + \beta_{\tau,1}X_{i2} + \beta_{\tau,2}X_{i3} + \epsilon_i \quad i = 1, 2, ..., n \quad (33)$$

The covariates are randomly drawn from a standard Normal distribution. The true value of $\beta_\tau$ is set equal to

$$\beta_\tau = \begin{bmatrix} -0.382 & -0.372 & 0.715 \\ 1.993 & 0.650 & 0.764 \\ 0.670 & 1.079 & 0.584 \end{bmatrix} \quad (34)$$

We analyze three different quantile vectors. In the first case, we assume $\tau = [0.50, 0.50, 0.50]'$, which implies that $\hat{\xi} = [0, 0, 0]'$ and $\hat{\Lambda} = \text{diag}(2.828, 2.828, 2.828)$. In the second scenario, we set $\tau = [0.25, 0.50, 0.75]'$ and, consequently, $\hat{\xi} = [2.667, 0, -2.667]'$ and $\hat{\Lambda} = \text{diag}(3.266, 2.828, 3.266)$. Finally, we consider a more extreme case with $\tau = [0.90, 0.50, 0.10]'$, $\hat{\xi} = [8.889, 0, -8.889]'$ and $\hat{\Lambda} = \text{diag}(4.714, 2.828, 4.714)$.

For each of the three cases, the parameter vector is represented by $\theta = [\beta_0, \beta_1, \beta_2, \delta_1, \delta_2, \delta_3, \rho_{12}, \rho_{13}, \rho_{23}]$, where the true values of $\beta_0$, $\beta_1$, $\beta_2$, are defined by the columns of $\beta_\tau$ in (34) and where we set $\delta_1 = 0.13$, $\delta_2 = 0.30$, $\delta_3 = 0.23$, with $\rho_{12} = 0.50$, $\rho_{13} = 0.30$ and $\rho_{23} = 0.40$.

Two different distributions for the error term generating process are considered in each simulation study: (a) a multivariate Normal random variable with zero mean and a variance-covariance matrix equal to $(\hat{D}\hat{\xi}\hat{\xi}'D + \hat{D}\hat{\Sigma}\hat{D})$, and (b) a multivariate Student $t$ distribution with 3 degrees of freedom, scale parameter $\hat{D}\hat{\Sigma}\hat{D}$ and non centrality parameter equal to $\hat{D}\hat{\xi}$. For each distribution of the error term, we carry out $B = 500$ Monte Carlo replications and report the relative bias and the root mean square error (RMSE), averaged across the 500 simulations, for each parameter value in $\theta$. The results are shown in Tables 1 and 2.

Table 1 analyzes the regression coefficient estimates for the three quantile levels described in the Panel A, Panel B and Panel C. For each of the three panels, the two error term distributions are considered.
and the corresponding point estimates and percentage bias are reported. As can be easily inferred, the bias effect is quite small when we analyze the median levels (see Panel A). As the quantile levels become more extreme (see Panels B and C), the bias slightly increases but it still remains reasonably small.

In Table 2 we report the Root Mean Square Error (RMSE) of the regression coefficients for the same \( \tau \)-levels under the same error distributions of Table 1 and compare the results obtained by running both the proposed joint quantile regression \((Joint \ QR \ RMSE \ in \ the \ table)\) and the univariate quantile regressions \((Univariate \ QR \ RMSE \ in \ the \ table)\) separately for each marginal \( Y_j \). In this way we want to highlight the added value of using the MAL distribution for multiple quantile regression purpose, which accounts for potential correlation among the responses. It is worth noting that in all the simulation situations, the effective gain of the proposed joint approach is reflected in the smaller RMSE of the estimates compared to those in the univariate case.

5.2. Penalized joint quantile regression

In this section, a simulation study is proposed to evaluate the performance of the penalized joint quantile regression, which uses the PEM algorithm proposed in Section 4.2. We analyze a simple case with \( n = 1000, p = 3 \) and a set of four explanatory variables using the same data generating process of the previous section. For a fixed \( \tau \)-level, the matrix of regression coefficients \( \beta_x \) contains 15 elements, where we set six of them (namely, \( \beta_{12}, \beta_{14}, \beta_{23}, \beta_{24}, \beta_{32}, \beta_{33} \)) equal to zero. As in the previous section, we analyze three different quantile vectors, i.e. \( \tau = [0.50, 0.50, 0.50]' \), \( \tau = [0.25, 0.50, 0.75]' \), and \( \tau = [0.10, 0.50, 0.90]' \). For each of the three cases we perform 100 Monte Carlo simulations, under either the \( N_3(0, D\tilde{\xi}D + D\tilde{\Sigma}D) \) and the \( t_3(D\tilde{\Sigma}D, D\tilde{\xi}) \) distributions as possible data generating process. Then, for each case, we estimate the model parameters using the penalized objective function defined in (29). The estimation of the tuning parameter \( \lambda \) is obtained using a 10-fold cross validation method, where the initial grid of the possible values for \( \lambda \in [\lambda_{min}, \lambda_{max}] \) has been described in Section 4.2.

Table 3 reports the true positive rate (TPR) for each of the true coefficients initially set equal to zero. The TPR gives a measure of how sensitive a given method is at discovering non-zero entries and we calculate it as the ratio between the number of simulations that correctly identify the parameter as a zero value, over the total number of simulations (i.e. the number of true zeros for each coefficients). The results show that the PEM method performs quite well, with an average TPR of more than the 80% across the three simulation scenarios and regardless of the distributional assumption of the errors.

6. Empirical application

As stated in the Introduction, recent financial and economic crises have put the need for a thorough analysis of the causes and the effects of the financial distress. In this context, quantile regression
has turned out to be an effective framework to study and evaluate financial stability of systems. In line with the recent literature that links quantile regression with measures of financial risks (see for example Adrian and Brunnermeier (2016), Covas et al. (2014), Engle and Manganelli (2004) and Xiao et al. (2015)), in this section we use our methodology to identify the main determinants for a risk of financial distress. We use data on 2,020 private limited non-financial Italian firms from the Amadeus Bureau van Dijk dataset, with reference year 2015. Following Pindado et al. (2008) and Bastos and Pindado (2013) among others, we adopt a definition of financial distress that evaluates the firm’s capacity to satisfy its financial obligations. Specifically, they classify a company as financially distressed not only when it files for bankruptcy but also when the two following events occur: (1) its earnings before interest and taxes depreciation and amortization (EBITDA) is under the first quartile of the sample or (2) the firm’s leverage is above the third quartile of the sample.

The idea of this real data application is to use our joint estimation approach to study the entire distribution of both firm’s leverage and EBITDA and assess how the impact of firm’s characteristics (such as profitability, financial expenses and earnings) varies with different quantile levels. This allows us to identify not only the main determinants of firm’s risk of financial distress, but also how the effect of these factors may vary depending on the severity of the distress a firm is facing.

As a measure of firm’s leverage (leverage) we use the ratio between firm’s total asset and equity. As explanatory variables, we consider indexes of firm’s profitability (profit), financial expenses (finexp), and retained earnings (earnings), as suggested in Pindado et al. (2008). We also consider the impact of short-term debt (current debt) and the ratio between firm’s cashflow and total assets (cashflow), and control for other firm’s specific characteristics such as firms’ total fixed assets over total asset ratio (fixassets, which can be interpreted as an indicator of firm’s collaterals), firm’s net income scaled by total assets (netincome, as a proxy of the activity level of the firm), and size, measured by the number of employees in the firm (employees).

The results for the two cases when \( \tau = [0.75, 0.25] \) and \( \tau = [0.90, 0.10] \) are reported in Table 4, Panel A and Panel B, respectively. In the table, parameter estimates are displayed in boldface when significant at the standard 5% level. Standard errors of the estimates are computed using non-parametric bootstrap (see e.g., Geraci and Bottai (2007)) and are reported in brackets.

We find a negative relationship between profitability and financial distress, as the coefficient of profitability (profit) shows a significant and negative impact on both EBITDA and leverage. This is in line with the results in Pindado et al. (2008) and Campbell et al. (2008), who argue that firms that face financial distress are most likely unable to fulfill their financial obligations. This effect remains constant also at more extreme quantile levels of the distribution, as shown in Panel B of Table 4.

We also find a positive effect of financial expenses, whose magnitude is amplified especially for the leverage component. This effect is still consistent with the findings in Pindado et al. (2008) and confirms the expectations that the risk of financial distress increases as the firm’s risk of not being able to comply with its financial obligations rises. This effect captures the firm’s financial vulnerability, which increases when considering more extreme (riskier) values of leverage.
In analyzing the effect of retained earnings on financial distress, we find evidence of a positive impact on both EBITDA and leverage, even though it decreases as riskier quantiles are analyzed. This is in contrast with the expected results in the literature, where a negative relationship with financial distress likelihood is typically postulated, as a firm should have a lower capacity of self-financing during periods of higher financial stress. However, several papers have documented the relationship between financial distress and earnings management practices in economy like Italy, and find evidence that private companies experiencing financial distress tend to manipulate their earnings to portray better financial performance and obtain bank financing (see e.g Bisogno and De Luca (2015), among others). In this case, earnings should not be considered a very informative determinant of firm’s financial distress.

Another important factor in explaining financial distress is the ratio between firm’s fixed asset over total asset. The claim is typically that tangible assets tend to reduce the financial distress costs because of the liquidation possibility in case of default (see e.g Charalambakis and Psychoyios (2012)). Hence, the higher is the risk of a financial distress, the higher is the level of tangible assets a firm will have on its balance sheet. Our results confirm this evidence, showing a positive effect of fixasset on both EBITDA and leverage.

Firm’s cash flow over total asset ratio is also a significant predictor of financial distress. It’s impact on leverage is highly negative, whilst it is found to be positive on the EBITDA component, with a sensible decrease when moving towards more extreme quantile. This is could be due to the fact that firms with low cash flow are less likely to make leverage adjustments, especially during period of financial stress. Predictive power is also shown by the firm’s short-term debt. Finally, no significant effect is found in terms of size (measured by the number of employees) and activity level (netincome).

In Table 5 we report the LASSO estimates computed on the same firms’ sample and on the same quantile levels, which essentially confirm the above findings.

7. Conclusions

This paper proposes a new likelihood-based method to jointly estimate marginal conditional quantiles of a multivariate response variable in a linear regression framework. We use a suitable reparameterization of the Multivariate Asymmetric Laplace distribution of Kotz et al. (2001), whose mixture representation allows us to implement a new EM algorithm. Using this procedure, we show that the regression parameters can be easily estimated in closed form, hence avoiding direct maximization procedures. A penalized version of the algorithm is also proposed as an automatic data-driven procedure to perform variable selection. The good performance of the two methods is evaluated using a simulation exercise, where extreme quantiles are also considered as possible simulation scenarios. An empirical application to financial distress analysis on a sample of Italian firms is finally presented.

As the approach of quantile regression is widely used in the literature, several extensions of the results obtained in this paper can be analyzed in future research. Longitudinal settings and/or random
effects models are of particular interest in this context, with the goal of characterizing the change in the response variables over time and accounting also for the dependence between serial observations on the same subject. Other forms of penalized algorithm could be also promising, as the simultaneous regularization of the parameters could perform better when a large set of (possibly correlated) explanatory variables is used in the application.
Table 1: Parameter estimates at different quantile levels: point estimates and relative bias.

|                  | $\hat{\beta}_{01}$ | $\hat{\beta}_{02}$ | $\hat{\beta}_{03}$ | $\hat{\beta}_{11}$ | $\hat{\beta}_{12}$ | $\hat{\beta}_{13}$ | $\hat{\beta}_{21}$ | $\hat{\beta}_{22}$ | $\hat{\beta}_{23}$ |
|------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| **Panel A:** $\tau = [0.50, 0.50, 0.50]$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| $\epsilon_i \sim \mathcal{N}_3(0, D\hat{\xi}D + D\hat{\Sigma}D)$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| Estimate         | -0.382              | 1.984               | 0.667               | -0.374              | 0.653               | 1.083               | 0.713               | 0.771               | 0.586               |
| Bias (%)         | -0.063              | -0.423              | -0.325              | 0.560               | 0.443               | 0.423               | -0.199              | 0.871               | 0.305               |
| $\epsilon_i \sim \mathcal{t}_3(D\hat{\Sigma}D, D\hat{\xi})$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| Estimate         | -0.379              | 1.980               | 0.666               | -0.369              | 0.646               | 1.072               | 0.706               | 0.756               | 0.508               |
| Bias (%)         | 0.632               | 0.105               | 0.698               | 0.749               | 0.367               | 0.452               | 0.778               | 0.056               | 0.898               |
| **Panel B:** $\tau = [0.25, 0.50, 0.75]$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| $\epsilon_i \sim \mathcal{N}_3(0, D\hat{\xi}D + D\hat{\Sigma}D)$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| Estimate         | -0.394              | 1.992               | 0.697               | -0.371              | 0.647               | 1.083               | 0.714               | 0.768               | 0.590               |
| Bias (%)         | 3.141               | -0.030              | 4.030               | -0.166              | -0.478              | 0.405               | -0.175              | 0.609               | 1.189               |
| $\epsilon_i \sim \mathcal{t}_3(D\hat{\Sigma}D, D\hat{\xi})$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| Estimate         | -0.0418             | 2.023               | 0.709               | -0.373              | 0.655               | 1.086               | 0.714               | 0.764               | 0.589               |
| Bias (%)         | 7.515               | 1.501               | 4.776               | 0.337               | 0.817               | 0.664               | 0.510               | 0.448               | 0.956               |
| **Panel C:** $\tau = [0.10, 0.50, 0.90]$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| $\epsilon_i \sim \mathcal{N}_3(0, D\hat{\xi}D + D\hat{\Sigma}D)$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| Estimate         | -0.431              | 2.006               | 0.731               | -0.363              | 0.651               | 1.065               | 0.771               | 0.764               | 0.576               |
| Bias (%)         | 12.821              | 0.669               | 9.104               | -1.413              | 0.181               | -1.218              | -0.539              | 0.051               | -1.271              |
| $\epsilon_i \sim \mathcal{t}_3(D\hat{\Sigma}D, D\hat{\xi})$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| Estimate         | -0.436              | 2.012               | 0.755               | -0.386              | 0.659               | 1.100               | 0.717               | 0.764               | 0.590               |
| Bias (%)         | 14.222              | 0.974               | 12.694              | 2.748               | 1.407               | 1.987               | 1.013               | 0.435               | 0.987               |
Table 2: Joint quantile estimation versus univariate quantile regressions: RMSEs.

|                  | $\hat{\beta}_{01}$ | $\hat{\beta}_{02}$ | $\hat{\beta}_{03}$ | $\hat{\beta}_{11}$ | $\hat{\beta}_{12}$ | $\hat{\beta}_{13}$ | $\hat{\beta}_{21}$ | $\hat{\beta}_{22}$ | $\hat{\beta}_{23}$ |
|------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| **Panel A: $\tau = [0.50, 0.50, 0.50]$** |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| $\epsilon_i \sim N_3(0, \tilde{D}\tilde{\xi}'D + \tilde{D}\tilde{\Sigma}D)$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| Joint QR RMSE    | 0.022 0.046 0.038   | 0.024 0.049 0.040   | 0.020 0.048 0.032   |                     |                     |                     |                     |                     |                     |
| Univariate QR RMSE | 0.023 0.049 0.039   | 0.029 0.053 0.041   | 0.020 0.049 0.034   |                     |                     |                     |                     |                     |                     |
| $\epsilon_i \sim t_3(\tilde{D}\tilde{\Sigma}D, \tilde{D}\tilde{\xi})$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| Joint QR RMSE    | 0.028 0.060 0.050   | 0.027 0.067 0.049   | 0.030 0.051 0.034   |                     |                     |                     |                     |                     |                     |
| Univariate QR RMSE | 0.034 0.074 0.058   | 0.030 0.077 0.044   | 0.027 0.067 0.059   |                     |                     |                     |                     |                     |                     |
| **Panel B: $\tau = [0.25, 0.50, 0.75]$** |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| $\epsilon_i \sim N_3(0, \tilde{D}\tilde{\xi}'D + \tilde{D}\tilde{\Sigma}D)$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| Joint QR RMSE    | 0.467 0.049 0.283   | 0.029 0.049 0.049   | 0.031 0.044 0.049   |                     |                     |                     |                     |                     |                     |
| Univariate QR RMSE | 0.470 0.055 0.324   | 0.029 0.049 0.051   | 0.038 0.048 0.051   |                     |                     |                     |                     |                     |                     |
| $\epsilon_i \sim t_3(\tilde{D}\tilde{\Sigma}D, \tilde{D}\tilde{\xi})$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| Joint QR RMSE    | 0.428 0.059 0.292   | 0.053 0.049 0.096   | 0.056 0.106 0.041   |                     |                     |                     |                     |                     |                     |
| Univariate QR RMSE | 0.491 0.061 0.572   | 0.056 0.049 0.103   | 0.061 0.196 0.049   |                     |                     |                     |                     |                     |                     |
| **Panel C: $\tau = [0.10, 0.50, 0.90]$** |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| $\epsilon_i \sim N_3(0, \tilde{D}\tilde{\xi}'D + \tilde{D}\tilde{\Sigma}D)$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| Joint QR RMSE    | 0.981 0.055 0.835   | 0.063 0.039 0.130   | 0.062 0.045 0.131   |                     |                     |                     |                     |                     |                     |
| Univariate QR RMSE | 1.063 0.05 3.274   | 0.087 0.048 0.140   | 0.075 0.047 0.142   |                     |                     |                     |                     |                     |                     |
| $\epsilon_i \sim t_3(\tilde{D}\tilde{\Sigma}D, \tilde{D}\tilde{\xi})$ |                     |                     |                     |                     |                     |                     |                     |                     |                     |
| Joint QR RMSE    | 0.997 0.101 0.936   | 0.093 0.241 0.165   | 0.065 0.048 0.152   |                     |                     |                     |                     |                     |                     |
| Univariate QR RMSE | 0.692 0.135 1.228   | 0.097 0.243 0.182   | 0.071 0.053 0.169   |                     |                     |                     |                     |                     |                     |
Table 3: The performance of the PEM algorithm: True Positive Rate (TPR).

|                  | $\hat{\beta}_{12}$ | $\hat{\beta}_{14}$ | $\hat{\beta}_{23}$ | $\hat{\beta}_{24}$ | $\hat{\beta}_{32}$ | $\hat{\beta}_{33}$ |
|------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|
| $\mathbf{e}_1 \sim N_3(0, D\tilde{\xi}^T D + D\tilde{\Sigma}D)$ |                    |                    |                    |                    |                    |                    |
| $\tau = [0.50, 0.50, 0.50]'$ | 89.122             | 92.475             | 84.341             | 89.342             | 91.753             | 89.134             |
| $\tau = [0.25, 0.50, 0.75]'$ | 87.221             | 83.453             | 79.978             | 86.397             | 83.641             | 87.550             |
| $\tau = [0.10, 0.50, 0.90]'$ | 83.341             | 82.512             | 80.123             | 86.101             | 82.308             | 86.761             |
| $\mathbf{e}_1 \sim t_3(D\tilde{\Sigma}D, D\tilde{\xi})$ |                    |                    |                    |                    |                    |                    |
| $\tau = [0.50, 0.50, 0.50]'$ | 88.324             | 89.546             | 84.209             | 86.008             | 88.331             | 88.198             |
| $\tau = [0.25, 0.50, 0.75]'$ | 86.130             | 83.346             | 80.001             | 87.121             | 82.453             | 85.345             |
| $\tau = [0.10, 0.50, 0.90]'$ | 83.007             | 81.978             | 81.321             | 86.004             | 80.423             | 84.121             |
Table 4: Estimated coefficients at different quantile levels and standard errors.

|                  | Leverage | EBITDA | Leverage | EBITDA |
|------------------|----------|--------|----------|--------|
|                  | Panel A: $\tau = [0.75, 0.25]$ | Panel B: $\tau = [0.90, 0.10]$ |
| Constant         | -4.472 (0.501) | -0.321 (0.105) | -4.359 (0.557) | -0.274 (0.087) |
| profit           | -0.022 (0.003) | -0.024 (0.001) | -0.022 (0.003) | -0.024 (0.005) |
| finexp           | 13.598 (0.538) | 0.451 (0.107) | 15.849 (0.567) | 0.312 (0.091) |
| earnings         | 1.118 (0.509)  | 0.748 (0.110) | 0.984 (0.571)  | 0.621 (0.092)  |
| employee         | -0.006 (0.005) | -0.001 (0.001) | -0.003 (0.005) | -0.002 (0.001) |
| fixasset         | 2.445 (0.132)  | 0.187 (0.027) | 2.907 (0.119)  | 0.151 (0.020)  |
| netincome        | -0.371 (1.654) | 0.347 (0.337) | -3.816 (1.658) | 0.345 (0.245)  |
| current debt     | 0.107 (0.008)  | -0.009 (0.002) | 0.122 (0.008)  | -0.007 (0.001) |
| cashflow         | -4.089 (1.376) | 6.251 (0.251) | -4.704 (1.256) | 4.407 (0.212)  |
| $\delta_j$      | 0.952 (0.018)  | 0.190 (0.003) | 0.800 (0.012)  | 0.138 (0.002)  |
| $\rho_{12}$     | -0.131 (0.027) | -0.005 (0.032) |                  |                  |
| n                | 2,020 | 2,020 | 2,020 | 2,020 |
Table 5. **LASSO parameter estimates.**

| EBITDA | Leverage | EBITDA | Leverage |
|--------|----------|--------|----------|
| Panel A: $\tau = [0.75, 0.25]'$ | Panel B: $\tau = [0.90, 0.10]'$ |
| Constant | -3.085 | -0.408 | -4.566 | -0.364 |
| profit | -0.012 | -0.343 | -0.014 | -0.038 |
| finexp | 12.624 | 0.019 | 16.222 | 0.014 |
| earnings | 0.478 | 1.422 | 1.376 | 0.389 |
| employee | – | – | – | – |
| fixasset | 2.038 | 0.307 | 1.067 | 0.235 |
| netincome | -0.676 | – | -1.267 | – |
| current debt | 0.721 | -0.330 | 0.220 | -0.107 |
| cashflow | -3.035 | 5.606 | -2.072 | 5.236 |
| $\delta_i$ | 0.937 | 0.831 | 0.827 | 0.135 |
| $\rho_{12}$ | -0.114 | -0.006 | 2.259 | 5.612 |
| $n$ | 2,020 | 2,020 | 2,020 | 2,020 |
Appendix. Proofs of Propositions

Proof of Proposition 1 Using the result in (11), where we denote $\beta \tau X_i = \mu$ without loss of generality, for each component $Y_j$ the following holds

$$Y_j = \mu_j + \delta_j \xi_j W + \delta_j \sqrt{W} \sigma_j Z_j, \quad \forall j = 1, 2, \ldots, p$$  \ \ (35)

where $Z_j \sim N(0, 1)$ represents the $j$-th component of $Z$. Then, defining $V = \delta_j W \sim \text{Exp}(\frac{1}{\delta_j})$, the representation in (35) can be also written as

$$Y_j = \mu_j + \xi_j V + \sigma_j \sqrt{\delta_j V} Z_j.$$  \ \ (36)

Following Kotz et al. (2001) and Kozumi and Kobayashi (2011), and imposing (13), the result follows since (36) represents a univariate AL distribution with location, skewness and scale parameter equal to $\mu_j$, $\tau_j$ and $\delta_j$, respectively.

Proof of Proposition 2 Remember that $D = \text{diag}[\delta_1, \delta_2, \ldots, \delta_j]$ is an unknown matrix with each $\delta_j > 0$, $j = 1, \ldots, p$. Moreover $\tilde{\Sigma} = \tilde{\Lambda} \Psi \tilde{\Lambda}$, where $\tilde{\Lambda}$ is a known $p \times p$ diagonal matrix with $(j, j)$-th element equal to $\tilde{\sigma}_j = \sqrt{\frac{2}{\tau_j(1-\tau_j)}}$, and where $\Psi$ is an unknown correlation matrix to be estimated.

Suppose we have two sets of parameters (say, $D$ and $D^*$, and $\Psi$ and $\Psi^*$), such that

$$D \tilde{\Lambda} \Psi \tilde{\Lambda} D = D^* \tilde{\Lambda} \Psi^* \tilde{\Lambda} D^*.$$  \ \ (37)

Then, identifiability analysis asks whether it is possible to identify both $D$ and $\Psi$ uniquely. That is, if (37) holds for some $D \neq D^*$ and some $\Psi \neq \Psi^*$, then the model is not identifiable, as two different sets of parameters give the same MAL distribution. Now, notice that (37) implies that

$$\text{diag} \left( D \tilde{\Lambda} \Psi \tilde{\Lambda} D \right) = \text{diag} \left( D^* \tilde{\Lambda} \Psi^* \tilde{\Lambda} D^* \right).$$  \ \ (38)

Since both $\Psi$ and $\Psi^*$ are correlation matrices then, for each $j$-th diagonal element we have $\Psi_{jj} = \Psi^*_{jj} = 1, \forall j = 1, \ldots, p$. This implies that (38) can be rewritten as

$$(\delta_j)^2 \sigma_j^2 = (\delta_j^*)^2 \sigma_j^2, \quad j = 1, \ldots, p.$$  \ \ (39)

But then, since by assumption each $\delta_j$ (and $\delta_j^*$) are greater than zero, the relationship in (39) is satisfied if and only if $\delta_j = \delta_j^*, \forall j = 1, \ldots, p$, or equivalently $D = D^*$. Given this, the relationship in (37) reduces to

$$D \tilde{\Lambda} \Psi \tilde{\Lambda} D = D \tilde{\Lambda} \Psi^* \tilde{\Lambda} D.$$
Finally, since both $D$ and $\hat{\Lambda}$ are squared diagonal matrices (hence invertible), we have

$$(D\hat{\Lambda})^{-1}D\hat{\Lambda}\Psi\hat{\Lambda}D(D\hat{\Lambda})^{-1} = (D\hat{\Lambda})^{-1}D\hat{\Lambda}\Psi^*\hat{\Lambda}D(D\hat{\Lambda})^{-1},$$

which implies that $\Psi = \Psi^*$. Therefore, both $D$ and $\Psi$ are just identified, as (37) holds if and only if $D = D^*$ and $\Psi = \Psi^*$.

\[\square\]

**Proof of Proposition 3** Notice that, under the constraints in (13), the representation in (11) implies that

$$Y_i|W_i = w_i \sim N_p(\beta_\tau X_i + D\hat{\xi}w_i, w_i D\hat{\Sigma}D)$$

$$W_i \sim Exp(1)$$

This implies that the joint density function of $Y$ and $W$ is

$$f_{Y,W}(y,w) = \frac{\exp\left\{(y - \mu)'D^{-1}\Sigma^{-1}\xi\right\}}{(2\pi)^{p/2}|\Sigma|^{1/2}} \left(w^{-p/2}\exp \left\{-\frac{1}{2}w \left( -1 + \frac{1}{2}w(d + 2) \right) \right\} \right).$$

Then, using (42), the complete log-likelihood function (up to additive constant terms) can be written as follows:

$$l_c(\beta_\tau, D(\delta), \Sigma(\Psi)|Y_i, W_i) = \sum_{i=1}^n (Y_i - \beta_\tau X_i)'D^{-1}(\delta)^{-1}\Sigma^{-1}(\Psi)\hat{\xi} - \frac{n}{2}\log |D(\delta)\hat{\Sigma}(\Psi)D(\delta)|$$

$$- \frac{1}{2} \sum_{i=1}^n \frac{1}{W_i}(Y_i - \beta_\tau X_i)'(D(\delta)^{-1}\Sigma(\Psi)D(\delta))^{-1}(Y_i - \beta_\tau X_i)$$

$$- \frac{1}{2} \hat{\xi}'\hat{\Sigma}(\Psi)\hat{\xi} \sum_{i=1}^n W_i.$$  

where we use the notation $D(\delta)$ and $\hat{\Sigma}(\Psi)$ to express the matrices $D$ and $\hat{\Sigma}$ as a function of their parameters, $\delta$ and $\Psi$, respectively. In practice, $W_i$ is a latent variable and, hence, not observable. For this reason, the E-step of the EM algorithm considers the conditional expectation of the complete log-likelihood function, given the observed data $Y_i$ and the parameter estimates $(\hat{\beta}_\tau, D(\delta), \hat{\Sigma}(\Psi))$. That is,
Using the joint distribution of \( Y \) which corresponds to a Generalized Inverse Gaussian (GIG) distribution with parameters 

\[ \nu, \tilde{a}, \tilde{b} \]

Let \( \tilde{m}_i \) given in (10), we have that \( \tilde{m}_i = (y_i - \beta_r X_i)'(D_{i} \tilde{\Sigma}_{i} \tilde{\Sigma}_{i})^{-1}(y_i - \beta_r X_i) \), for every \( i = 1, 2, ..., n \), and define \( \tilde{d} = \tilde{\xi} \tilde{\Sigma}_{i} \tilde{\xi} \).

Using the joint distribution of \( Y \) and \( W \) derived in (42) and the pdf of \( Y \) given in (10), we have that

\[
f_{W|Y}(W_i|Y_i = y_i) = \frac{f_{W,Y}(w_i, y_i)}{f_Y(y_i)} = \frac{w_i^{-p/2} \left( \frac{x + d}{m_i} \right)^{\nu/2} \exp \left\{ -\frac{\tilde{m}_i}{2w_i} - \frac{w_i(2 + \tilde{d})}{x} \right\}}{2K_{\nu} \left( \sqrt{(2 + \tilde{d})\tilde{m}_i} \right)}
\]

which corresponds to a Generalized Inverse Gaussian (GIG) distribution with parameters \( \nu, 2 + \tilde{d}, \tilde{m}_i \), i.e.\(^2\)

\[
f_{W|Y}(W_i|Y_i = y_i) \sim \text{GIG} \left( \nu, 2 + \tilde{d}, \tilde{m}_i \right).
\]

It follows that

\[
E[W_i|Y_i, \tilde{Y}_i, D_{i}, \tilde{\Sigma}_{i}] = \left( \frac{\tilde{m}_i}{2 + \tilde{d}} \right)^{\frac{\nu}{2}} K_{\nu+1} \frac{\sqrt{(2 + \tilde{d})\tilde{m}_i}}{K_{\nu} \left( \sqrt{(2 + \tilde{d})\tilde{m}_i} \right)}
\]

and

\[
E[W_i^{-1}|Y_i, \tilde{Y}_i, D_{i}, \tilde{\Sigma}_{i}] = \left( \frac{2 + \tilde{d}}{\tilde{m}_i} \right)^{\frac{\nu}{2}} K_{\nu+1} \frac{\sqrt{(2 + \tilde{d})\tilde{m}_i}}{K_{\nu} \left( \sqrt{(2 + \tilde{d})\tilde{m}_i} \right)} - \frac{2 \nu}{\tilde{m}_i}
\]

\(^2\)The pdf of a GIG(\( p, a, b \)) distribution is defined as 

\[ f_{GIG}(x; p, a, b) = \frac{\left( \frac{a}{x} \right)^{p/2} e^{-x} e^{-\frac{b}{(ax+bx^{-1})}}}{2K_{p}(\sqrt{ab})} \], \text{ with } a > 0, b > 0 \text{ and } p \in \mathbb{R}.\]
where

\[
\hat{m}_i = (y_i - \hat{\beta}_\tau X_i)'(D_{(\hat{\delta})} \hat{\Sigma}_{\hat{\Psi}} D_{(\hat{\delta})})^{-1}(y_i - \hat{\beta}_\tau X_i)
\] (53)

\[
\hat{d} = \hat{\xi}' \hat{\Sigma}_{\hat{\Psi}} \hat{\xi}.
\] (54)

Denoting the two conditional expectations in (51) and (52) by \(u_i\) and \(z_i\) respectively, conclude the proof.

\[\Box\]

**Proof of Proposition 4**. Imposing the first order conditions on (14) - (16) with respect to \(\beta_\tau\) and \(\hat{\Sigma}_{\hat{\Psi}}\) gives the parameter estimates in (20), (21) and (22).
References

[1] Adrian, T., Brunnermeier M. K., (2016). CoVaR. *American Economic Review*, Vol. 106, pp. 1705–41.

[2] Alfó, M., Marino, M.F., Ranalli, M.G., Salvati, N. (2016). M-quantile regression for multivariate longitudinal data. *48th scientific meeting of the Italian Statistical Society*, ISBN:9788861970618.

[3] Arslan, O., (2010). An alternative multivariate skew Laplace distribution: properties and estimation, *Statistical Papers*, Vol. 51, pp. 865–887.

[4] Bassett Jr., G., Chen, HL, (2001). Portfolio style: Return-based attribution using quantile regression. *Empirical Economics*, Vol. 26, pp 293–305.

[5] Bastos, R., Pindado, J., (2013). Trade credit during a financial crisis: A panel data analysis. *Journal of Business Research*, Vol 6, pp. 614–620.

[6] Bernardi, M., Gayraud, G., Petrella, L., (2015). Bayesian tail risk interdependence using quantile regression. *Bayesian Analysis*, Vol. 10, pp. 553–603.

[7] Bernardi, M., Petrella, L., Stolfi, P., (2018). The sparse method of simulated quantiles: an application to portfolio optimisation. *Forthcoming Statistica Neerlandica*.

[8] Bisogno, M., De Luca, R., (2015). Financial Distress and Earnings Manipulation: Evidence from Italian SMEs. *Journal of Accounting and Finance*, Vol. 4, pp. 42–51.

[9] Bocek, P., Siman, M., (2017). On weighted and locally polynomial directional quantile regression. *Computational Statistics*, Vol. 32, pp. 929–946.

[10] Campbell, J., Hilscher, J., Szilagyi, J., (2008). In search of distress risk. *The Journal of Finance*, Vol. 63, pp. 2899–2939.

[11] Chakraborty, B., (2003). On multivariate quantile regression. *Journal of Statistical Planning and Inference*, Vol. 110, pp. 109–132.

[12] Charalambakis, E.C., Psychoyios, D., (2012). What do we know about capital structure? Revisiting the impact of debt ratios on some firm-specific factors. *Applied Financial Economics*, Vol. 22, pp. 1727–1742.

[13] Cho, H., Kim, S., Kim, M., (2017). Multiple quantile regression analysis of longitudinal data: Heteroscedasticity and efficient estimation. *Journal of Multivariate Analysis*, Vol. 155, pp. 334–343.
[14] Cole, T. J., Green, P. J. (1992). Smoothing reference centile curves: The lms method and penalized likelihood. *Statistics in Medicine*. Vol 11. pp 1305–1319.

[15] Covas, F.B., Rump, B., Zakrajsek, E., (2014). Stress-testing US bank holding companies: A dynamic panel quantile regression approach. *International Journal of Forecasting*, Vol. 30, pp. 691–713.

[16] Dempster, A. P., Laird, N. M., Rubin, D. B., (1977). Maximum Likelihood from Incomplete Data via the EM Algorithm. *Journal of the Royal Statistical Society. Series B*. Vol. 39, pp. 1–38.

[17] Friedman, J., Hastie, T., Tibshirani, R., (2010). Regularization Paths for Generalized Linear Models via Coordinate Descent. *Journal of Statistical Software*, Vol.33 (1), pp. 1–22.

[18] Engle, R.F., Manganelli, S., (2004). CAViaR: Conditional Autoregressive Value at Risk by Regression Quantiles. *Journal of Business & Economic Statistics*, Vol. 22, pp. 367–381.

[19] Geraci, M., Bottai, M., (2007). Quantile regression for longitudinal data using the asymmetric Laplace distribution. *Biostatistics*, Vol. 8, pp. 140–154.

[20] Green, P.J. (1990). On use of the EM algorithm for penalized likelihood estimation. *Journal of Royal Statistical Society Series B*, Vol. 52, pp. 443–452.

[21] Hallin, M., Paindaveine, D., Siman, M., (2010). Multivariate quantiles and multiple-output regression quantiles: from L1 optimization to half space depth. *The Annals of Statistics*, Vol. 38, pp. 635–669.

[22] Hendricks, W., Koenker, R. (1992). Hierarchical spline models for conditional quantiles and the demand for electricity. *Journal of the American Statistical Association*, Vol. 87, pp 58–68.

[23] Hoerl, A., Kennard, R. (1970) Ridge regression. In *Encyclopedia of Statistical Sciences*, Vol. 8, pp. 129–136. New York: Wiley.

[24] Jorion, P., (2007). Value at Risk: The New Benchmark for Managing Financial Risk, 3rd Edition

[25] Jun, S.J., and Pinkse, J., (2009). Efficient Semiparametric Seemingly Unrelated Quantile Regression Estimation. *Econometric Theory*, Vol. 25, pp 1392–1414.

[26] Koenker, R. (2004). Quantile Regression for Longitudinal Data, *Journal of Multivariate Analysis*, Vol. 91, pp 74–89.

[27] Koenker, R. (2005): Quantile Regression. *Cambridge University Press.*

[28] Koenker, R. (2017): Quantile Regression: 40 Years On. *Annual Review of Economics*. Vol. 9, pp 155–176.
[29] Koenker, R., Bassett, G. (1978) Regression quantiles. *Econometrica*. Vol. 46, pp. 33–50.

[30] Koenker, R., Geling, O., (2001). Reappraising Medfly Longevity: A Quantile Regression Survival Analysis. *Journal of the American Statistical Association*. Vol. 96, pp. 458–468.

[31] Kong, L., I. Mizera (2012). Quantile tomography: Using quantiles with multivariate data. *Statistica Sinica*. Vol 22, pp. 1589–1610.

[32] Kong, L., Shu, H., Heo, G., He, QC., (2015). Estimation for bivariate quantile varying coefficient model. *arXiv preprint arXiv:1511.02552*.

[33] Kottas, A., Gelfand, A.E., (2001). Bayesian semiparametric median regression modeling. *Journal of the American Statistical Association*, Vol 96, pp. 1458–1468.

[34] Kottas, A., Krnjajic, M., (2009). Bayesian semiparametric modelling in quantile regression. *Scandinavian Journal of Statistics*, Vol 36, pp. 297–319.

[35] Kotz, S., Kozabowski, T. J., Podgorski, K. (2001). The Laplace Distribution and Generalizations: A Revisit with Applications to Communications, Economics, Engineering, and Finance. Boston: Birkhauser.

[36] Kozumi, H, Kobayashi, G., (2011). Gibbs sampling methods for Bayesian quantile regression. *Journal of Statistical Computation and Simulation*, Vol. 81, pp. 1565–1578.

[37] Marino, M.F., Farcomeni, A., (2015). Linear quantile regression models for longitudinal experiments: an overview. *Metron*, Vol. 73, pp. 229–247.

[38] Marino, M.F., Tzavidis, N., Alfó, M., (2016). Mixed hidden Markov quantile regression models for longitudinal data with possibly incomplete sequences. *Statistical Methods in Medical Research*, ISSN:0962-2802 DOI.

[39] McNeil, A., Frey, R., Embrechts, P. (2005). Quantitative Risk Management: Concepts, Techniques and Tools. Princeton University Press, Princeton.

[40] Paindaveine, D., Siman, M., (2012). Computing multiple-output regression quantile regions from projection quantiles. *Computational Statistics*, Vol. 27, pp. 29–49.

[41] Pandey G. R., Nguyen V. T., (1999). A comparative study of regression based methods in regional flood frequency analysis, *Journal of Hydrology*, Vol. 225, pp. 92–101.

[42] Petrella L., La Porta, A., Merlo, L., (2017). Cross-country assessment of systemic risk in the European stock market: evidence from a CoVar analysis. *Working Paper*.

[43] Pindado, J., Rodrigues L., De la Torre, C., (2008). Estimating financial distress likelihood. *Journal of Business Research*, Vol. 61, pp. 995–1003.
[44] Royston, P., Altman, D.G., (1994). Regression using fractional polynomials of continuous covariates: parsimonious parametric modelling. *Journal of Applied Statistics* 43, 429–467.

[45] Sanchez, L.B., Lachos, V., Labra, F.V., (2013). Likelihood Based Inference for Quantile Regression Using the Asymmetric Laplace Distribution. Universidade Estadual de Campinas, Technical Report 15.

[46] Tibshirani, R., (1996). Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society. Series B.*, Vol. 58, pp. 267–288.

[47] Yu, K. and Moyeed, R.A., (2001), Bayesian quantile regression. *Statistics and Probability Letters*, Vol. 54, pp. 437–447.

[48] Waldmann, E., Kneib, T., (2015). Bayesian bivariate quantile regression. *Statistical Modelling*. Vol. 15, pp. 326–344.

[49] Wong, C.M., Ting, L.L.O., (2016). A Quantile Regression Approach to the Multiple Period Value at Risk Estimation. *Journal of Economics and Management*, Vol. 12, pp. 1–35.

[50] Xiao, Z., Guo H., Lam M.S. (2015). Quantile Regression and Value at Risk. *Handbook of Financial Econometrics and Statistics. Springer, New York, NY*.

[51] Zou, H., Hastie, T., (2005). Regularization and variable selection via the elastic net. *Journal of the Royal Statistical Society. Series B.*, Vol. 67, pp. 301–320.