ALTERNATING RUNS OF PERMUTATIONS AND THE CENTRAL FACTORIAL NUMBERS

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Abstract. Let \( R(n, k) \) be the number of permutations of \( \{1, 2, \ldots, n\} \) with \( k \) alternating runs. In this paper, we establish the relationships between \( R(n, k) \) and the central factorial numbers of even indices as well as the number of signed permutations with a given number of alternating runs and the central factorial numbers of odd indices. The explicit formulas of the peak and left peak polynomials for permutations and the derivative polynomials of the tangent and secant functions are also established.

Keywords: Alternating runs; Eulerian polynomials; Peak polynomials; Derivative polynomials

1. Introduction

Let \( S_n \) be the symmetric group of all permutations of \( [n] \), where \( [n] = \{1, 2, \ldots, n\} \). Let \( \pi = \pi(1)\pi(2) \cdots \pi(n) \in S_n \). We say that \( \pi \) changes direction at position \( i \) if either \( \pi(i - 1) < \pi(i) > \pi(i + 1) \), or \( \pi(i - 1) > \pi(i) < \pi(i + 1) \), where \( i \in \{2, 3, \ldots, n - 1\} \). We say that \( \pi \) has \( k \) alternating runs if there are \( k - 1 \) indices \( i \) such that \( \pi \) changes direction at these positions. Let \( R(n, k) \) denote the number of permutations in \( S_n \) with \( k \) alternating runs. The enumeration of finite sequences according to the number of alternating runs has been studied extensively in the literature, starting with André [2] in 1884, restarting with Carlitz [7, 8, 9], and Comtet [11, p. 260-261] gave an exercise to this topic. In the past decades, several authors [5, 21, 28] have considered the explicit formulas of the numbers \( R(n, k) \) as well as their generating functions. The reader is referred to [4, 23, 29, 30] for the recent progress on this topic.

Following André [2], the numbers \( R(n, k) \) satisfy the recurrence relation

\[
R(n, k) = kR(n - 1, k) + 2R(n - 1, k - 1) + (n - k)R(n - 1, k - 2)
\]

for \( n, k \geq 1 \), where \( R(1, 0) = 1 \) and \( R(1, k) = 0 \) for \( k \geq 1 \). Let \( R_n(x) = \sum_{k=1}^{n-1} R(n, k)x^k \). It follows from (1) that

\[
R_{n+2}(x) = x(nx + 2)R_{n+1}(x) + x(1-x^2) \frac{d}{dx} R_{n+1}(x).
\]

Below are the polynomials \( R_n(x) \)’s for \( 2 \leq n \leq 5 \):

\[
R_2(x) = 2x, \\
R_3(x) = 2x + 4x^2, \\
R_4(x) = 2x + 12x^2 + 10x^3, \\
R_5(x) = 2x + 28x^2 + 58x^3 + 32x^4.
\]

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By the method of characteristics, Carlitz \[7\] proved that
\[
\sum_{n=0}^{\infty} (1 - x^2)^{-n/2} z^n n! \sum_{k=0}^{n} R(n+1,k)x^{n-k} = \frac{1-x}{1+x} \left( \frac{\sqrt{1-x^2} + \sin z}{x - \cos z} \right)^2.
\] (3)

Subsequently, Carlitz \[8, \text{ Eq. (1.3)}\] deduced the following explicit formula of \(R(n,k)\):
\[
\begin{cases}
R(2n - 1, 2n - s - 2) = \sum_{j=1}^{n} (-1)^{n-j} 2^{j+2} (2j - 1)! U(n, j) M(n, j, s), \\
R(2n, 2n - s - 1) = \sum_{j=1}^{n} (-1)^{n-j} 2^{j+1} (2j)! U(n, j) M(n, j, s).
\end{cases}
\] (4)

where
\[
U(n, j) = \frac{1}{(2j)!} \sum_{i=0}^{2j} (-1)^i \binom{2j}{i} (j-i)^{2n}, \quad M(n, j, s) = \sum_{t=0}^{n-j} (-1)^{t} \binom{n-j}{t} \binom{n-2}{s-t}.
\]

It should be noted that \(U(n, j)\) are central factorial numbers of even indices. In \[28\], Stanley gave another explicit formula:
\[
R(n, k) = \sum_{i=0}^{k} \frac{1}{2^{k-i}} (-1)^{k-i} z_{k-i} \sum_{r+2m \equiv i, r \equiv i \mod 2} (-2)^m \binom{i-m}{(i+r)/2} \binom{n}{m} r^n,
\]
where \(z_0 = 2\) and \(z_n = 4\) for \(n \geq 1\). In \[5\], Canfield and Wilf pointed out that there is something wrong in \[4\], and showed that
\[
R(n, k) = \frac{1}{2^{k-2}} k^n - \frac{1}{2^{k-4}} (k-1)^n + \psi_2(n, k)(k-2)^n + \cdots + \psi_{k-1}(n, k) \quad \text{for} \ n \geq 2,
\]
in which each \(\psi_i(n, k)\) is a polynomial in \(n\) whose degree in \(n\) is \(\lfloor i/2 \rfloor\). By expressing \(R_n(x)\) in terms of the derivative polynomials of tangent function, Ma \[21\] found that
\[
R(n, s) = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} p(n, n - 2k + 1) E(n, k, s),
\]
where
\[
p(n, n - 2k + 1) = (-1)^{k} \sum_{i \geq 1} \frac{n!}{i!} (-2)^{n-i} \left\{ \binom{i}{n-i} - \binom{i}{n-2k+1} \right\},
\]
\[
E(n, k, s) = \sum_{j=0}^{\min(k, s)} (-1)^{k-j} \binom{n-k-1}{s-j} \binom{k}{j}.
\]

The peak polynomials for permutations and derivative polynomials of trigonometric functions appeared frequently in algebra, number theory, geometry and combinatorics, see \[17\ \[22\, \[25\, \[29\, \[30\\] and references therein. In the next section, we first present a revision of \[4\], and then we provide explicit formulas for the peak and left peak polynomials as well as the derivative polynomials of the tangent and secant functions. An explicit formula of the alternating run polynomials of signed permutations is also established. From the results of this paper, one can see that sometimes enumerative polynomials naturally occur in pairs, and it is often much better to tackle both at the same time.
The central factorial numbers of the second kind $T(n, k)$ are defined in Riordan’s book [26] p. 213-217 by

$$x^n = \sum_{k=0}^{n} T(n, k)x^k = \prod_{i=1}^{k-1} \left(x + \frac{k}{2} - i\right).$$

As usual, we denote by $U(n, k) = T(2n, 2k)$ and $V(n, k) = 4^{n-k}T(2n+1, 2k+1)$ for all $n, k \geq 0$. By definitions, these numbers satisfy the recurrence relations

$$U(n, k) = U(n - 1, k - 1) + k^2U(n - 1, k),$$
$$V(n, k) = V(n - 1, k - 1) + (2k + 1)^2V(n - 1, k),$$

with the initial conditions $U(1, 1) = 1$, $U(1, k) = 0$ if $k \neq 1$, $V(0, 0) = 1$ and $V(0, k) = 0$ if $k \neq 0$, see [14] [15] [18] for details. The central factorial numbers of even indices $U(n, k)$ first appeared in a paper of MacMahon [24, p. 106], and they can be defined by

$$x^n = \sum_{k=1}^{n} U(n, k)\prod_{i=1}^{k} (x - (i - 1)^2).$$

The numbers $U(n, k)$ count partitions of the set $\{1, -1, 2, -2, \ldots, n, -n\}$ with $k$ blocks $B_1, \ldots, B_k$ such that for each $j \in [k]$, if $i$ is the least integer such that $i$ or $-i$ belongs to $B_j$, then $\{i, -i\}$ is a subset of $B_j$, see [15] Section 2.1 for details. According to [26] p. 214, we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} V(n, k)x^{2k+1} \frac{z^{2n+1}}{(2n+1)!} = \sinh(x \sinh(z)) = xz + (x + x^3)\frac{z^3}{3!} + (x + 10x^3 + x^5)\frac{z^5}{5!} + \cdots.$$

Gelineau and Zeng [15] Theorem 8 found that the central factorial numbers of odd indices $V(n, k)$ count partitions of $\{2n + 1\}$ into $2k + 1$ blocks of odd cardinality. Explicitly, one has

$$V(n, k) = \frac{1}{(2k)!4^k} \sum_{m=0}^{k} (-1)^{k-m} \frac{(2m + 1)^{2n+1}}{k + m + 1} \binom{2k}{k + m}.$$

We refer the reader to [27], A036969, A160562 for various results on central factorial numbers.

As pointed out in [18], even though the central factorial numbers are less known than Stirling numbers, they are as important as Stirling numbers. In this paper, we give some applications of the central factorial numbers in the study of the alternating run polynomials, peak and left peak polynomials as well as the derivative polynomials of tangent and secant functions.

2.1. Main results.

The first main result of this paper is given as follows.

**Theorem 1.** For any $n \geq 1$, one has

$$\frac{R_{2n-1}(x)}{(1 + x)^{n-2}} = \sum_{j=1}^{n} 2^{-j+1}(2j - 1)!U(n, j)x^j(1 - x)^{n-j},$$
$$\frac{R_{2n}(x)}{(1 + x)^{n-1}} = \sum_{j=1}^{n} 2^{-j+1}(2j)!U(n, j)x^j(1 - x)^{n-j}. \quad (6)$$
It is well known that the polynomials 

\[ R(2n, s)x^{2n-1-s} = \sum_{j=1}^{n} 2^{-j+2}(2j-1)!U(n, j)(x-1)^{n-j}(x+1)^{n-1}, \]

where \( R(n, s) \) are defined by

\[ R(n, s) = \frac{1}{2} \sum_{j=1}^{n} 2^{-j+2}(2j-1)!U(n, j)(x-1)^{n-j}(x+1)^{n-2}, \]

For any \( n \geq 2 \), we have

\[ \sum_{s=0}^{2n-2} R(2n-1, s)x^{2n-2-s} = \sum_{j=1}^{n} 2^{-j+2}(2j-1)!U(n, j)(x-1)^{n-j}(x+1)^{n-2}, \]

where \( R(n, s) \) are the coefficients of \( x^s \) in \((1+x)^{n-2}(1-x)^{n-j}\) and \( R(n, s) \) are the coefficients of \( x^s \) in \((1+x)^{n-1}(1-x)^{n-j}\).

The proof of Theorem 1 will be given in subsection 2.2. The following result is immediate.

**Corollary 2** (3). The polynomial \( R_n(x) \) is divisible by \((x+1)^{\lfloor n/2 \rfloor - 1}\) for any \( n \geq 2 \).

Recently, Bona [4] gave a proof of Corollary 2 by introducing a group action on permutations.

Let \( \pi = \pi(1)\pi(2)\cdots\pi(n) \in S_n \). An interior peak of \( \pi \) is an index \( i \in \{2, 3, \ldots, n-1\} \) such that \( \pi(i-1) < \pi(i) > \pi(i+1) \). Let \( \text{lpk}(\pi) \) denote the number of interior peaks of \( \pi \). The classical peak polynomials are defined by

\[ P_n(x) = \sum_{\pi \in S_n} x^{\text{lpk}(\pi)}, \]

which have been extensively studied in the past decades, see [22, 25, 30] and references therein. There is a close connection between \( R_n(x) \) and \( P_n(x) \) (see [22, Corollary 2]):

\[ R_n(x) = \frac{x(1+x)^{n-2}}{2n-2} P_n\left( \frac{2x}{1+x} \right) \quad \text{for} \quad n \geq 2. \]

Combining (6) and (8), we get the following result.

**Corollary 3.** For any \( n \geq 1 \), we have

\[ xP_{2n-1}(x) = \sum_{j=1}^{n} 2^{2n-2j}(2j-1)!U(n, j)x^j(1-x)^{n-j}, \]

\[ xP_{2n}(x) = \sum_{j=1}^{n} 2^{2n-2j}(2j-1)!U(n, j)x^j(1-x)^{n-j}. \]

Let \( \pi = \pi(1)\pi(2)\cdots\pi(n) \in S_n \). A left peak of \( \pi \) is an index \( i \in [n-1] \) such that \( \pi(i-1) < \pi(i) > \pi(i+1) \), where we take \( \pi(0) = 0 \). Let \( \text{lpk}(\pi) \) denote the number of left peaks of \( \pi \). The left peak polynomials are defined by

\[ \hat{P}_n(x) = \sum_{\pi \in S_n} x^{\text{lpk}(\pi)}. \]

It is well known that the polynomials \( \hat{P}_n(x) \) satisfy the following recurrence relation

\[ \hat{P}_{n+1}(x) = (nx + 1)\hat{P}_n(x) + 2x(1-x)\frac{d}{dx}\hat{P}_n(x), \]
with initial conditions $\hat{P}_1(x) = 1$ and $\hat{P}_2(x) = 1 + x$, see [27] A008971] and references therein. As a dual of Corollary 3, we can now present the second main result of this paper.

**Theorem 4.** For any $n \geq 1$, we have

$$\hat{P}_{2n}(x) = \sum_{j=0}^{n} (2j)!V(n,j)x^j(1-x)^{n-j},$$

$$\hat{P}_{2n+1}(x) = \sum_{j=0}^{n} (2j+1)!V(n,j)x^j(1-x)^{n-j}. \quad (11)$$

*Proof.* Note that

$$\hat{P}_2(x) = 1 + x = (1 - x) + 2x, \quad \hat{P}_3(x) = 1 + 5x = (1 - x) + 6x,$$

$$\hat{P}_4(x) = 1 + 18x + 5x^2 = (1 - x)^2 + 20x(1-x) + 24x^2,$$

$$\hat{P}_5(x) = 1 + 58x + 61x^2 = (1 - x)^2 + 60x(1-x) + 120x^2.$$

Thus the two expressions in (11) hold for $n = 1, 2$. We proceed by induction. For $m \geq 2$, assume that the result holds for $n = m$. It follows from (10) that

$$\hat{P}_{2m+2}(x) = (2mx + x + 1)\hat{P}_{2m+1}(x) + 2x(1-x)\frac{d}{dx}\hat{P}_{2m+1}(x)$$

$$= ((2mx + 1) + x)\sum_{j=0}^{m} (2j+1)!V(m,j)x^j(1-x)^{m-j} +$$

$$2\sum_{j=0}^{m} (2j+1)!V(m,j)x^j(1-x)^{m-j+1} - 2\sum_{j=0}^{m} (2j+1)!(m-j)V(m,j)x^j(1-x)^{m-j}$$

$$= \sum_{j=0}^{m} (2j+1)!V(m,j)x^j(1-x)^{m-j} + \sum_{j=0}^{m} (2j+1)!V(m,j)x^j(1-x)^{m-j}.$$

where the last equality follows from the fact that $2mx + 1 + 2j(1-x) = 2m(1-x) = 2j + 1$. Using $1 = (1 - x) + x$, we obtain

$$\hat{P}_{2m+2}(x) = \sum_{j=0}^{m} (2j+2)!V(m,j)x^{j+1}(1-x)^{m-j} + \sum_{j=0}^{m} (2j+1)!V(m,j)x^j(1-x)^{m-j+1}.$$

It follows from (5) that the coefficient of $x^j(1-x)^{m-j+1}$ on the right-hand side is $(2j)!V(m+1,j)$. By the recurrence relation (10), we have

$$\hat{P}_{2m+3}(x) = (2mx + 2x + 1)\hat{P}_{2m+2}(x) + 2x(1-x)\frac{d}{dx}\hat{P}_{2m+2}(x)$$

$$= (2mx + 2x + 1)\sum_{j=0}^{m+1} (2j)!V(m+1,j)x^j(1-x)^{m+1-j} +$$

$$2(1-x)\sum_{j=0}^{m+1} (2j)!V(m+1,j)x^j(1-x)^{m+1-j} -$$

$$2x\sum_{j=0}^{m+1} (2j)!(m+1-j)V(m+1,j)x^j(1-x)^{m+1-j}.$$
The coefficient of \(x^j(1-x)^{m+1-j}\) on the right-hand side is \((2j+1)!V(m+1,j)\), since
\[
2mx + 2x + 1 + 2j(1-x) - 2x(m+1-j) = 2j+1.
\]
Therefore, the two expressions in (11) hold for \(n = m+1\). This completes the proof. \(\Box\)

Denote by \(B_n\) the hyperoctahedral group of rank \(n\). Elements \(\pi\) of \(B_n\) are signed permutations of the set \(\pm[n]\) such that \(\pi(-i) = -\pi(i)\) for all \(i\), where \(\pm[n] = \{\pm1, \pm2, \ldots, \pm n\}\). As usual, we identify a signed permutation \(\pi = \pi(1)\cdots\pi(n)\) with the word \(\pi(0)\pi(1)\cdots\pi(n)\), where \(\pi(0) = 0\). A run of a signed permutation \(\pi\) is defined as a maximal interval of consecutive elements on which the elements of \(\pi\) are monotonic in the order
\[
\ldots < \underbrace{2} < \underbrace{1} < 0 < 1 < 2 < \cdots.
\]

The up signed permutations are signed permutations with \(\pi(1) > 0\). Let \(\hat{R}(n,k)\) denote the number of up signed permutations in \(B_n\) with \(k\) alternating runs and let \(\hat{R}_n(x) = \sum_{k=1}^n \hat{R}(n,k)x^k\). Following [10, Theorem 4], we have
\[
\hat{R}_n(x) = x(1+x)^{n-1}\hat{P}_n\left(\frac{2x}{1+x}\right)
\]
for \(n \geq 1\). Combining this with Theorem [11] we obtain the following result.

**Corollary 5.** For \(n \geq 1\), we have
\[
\hat{R}_{2n}(x) = x(1+x)^{n-1}\sum_{j=0}^n 2^j(2j)!V(n,j)x^j(1-x)^{n-j},
\]
\[
\hat{R}_{2n+1}(x) = x(1+x)^n\sum_{j=0}^n 2^j(2j+1)!V(n,j)x^j(1-x)^{n-j}.
\]

In the sequel, we discuss derivative polynomials of the tangent and secant functions. In 1879, André [11] observed that
\[
\sum_{n=0}^{\infty} \frac{E_n z^n}{n!} = \tan z + \sec z = 1 + z + \frac{z^2}{2!} + 2\frac{z^3}{3!} + 5\frac{z^4}{4!} + 16\frac{z^5}{5!} + \cdots.
\]

Note that
\[
\sum_{n=0}^{\infty} \frac{E_{2n+1} z^{2n+1}}{(2n+1)!} = \tan z, \quad \sum_{n=0}^{\infty} \frac{E_{2n} z^{2n}}{(2n)!} = \sec z.
\]

For this reason the numbers \(E_{2n+1}\) are sometimes called tangent numbers and the numbers \(E_{2n}\) are secant numbers. The derivative polynomials of tangent and secant functions are respectively defined as follows:
\[
\frac{d^n}{d\theta^n} \tan \theta = Q_n(\tan \theta), \quad \frac{d^n}{d\theta^n} \sec \theta = \sec \theta \cdot \hat{Q}_n(\tan \theta).
\]

The study of these polynomials was initiated by Knuth and Buckholtz [19]. They noted that \(Q_{2n-1}(0) = E_{2n-1}\) and \(\hat{Q}_{2n}(0) = E_{2n}\). It is well known that (see [7] [16]):
\[
Q(x; z) = \sum_{n=0}^{\infty} Q_n(x) \frac{z^n}{n!} = \frac{x + \tan z}{1 - x \tan z}.
\]
\[
\tilde{Q}(x; z) = \sum_{n=0}^{\infty} \tilde{Q}_n(x) \frac{z^n}{n!} = \frac{\sec z}{1 - x \tan z}.
\]

By the chain rule, we get
\[
Q_{n+1}(x) = (1 + x^2) \frac{d}{dx} Q_n(x), \quad \tilde{Q}_{n+1}(x) = (1 + x^2) \frac{d}{dx} \tilde{Q}_n(x) + x \tilde{Q}_n(x),
\]
with \(Q_0(x) = x\) and \(\tilde{Q}_0(x) = 1\). According to [20, Theorem 2], for \(n \geq 1\), we have
\[
Q_n(x) = (x^{n-1} + x^{n+1})P_n(1 + x^{-2}), \quad \tilde{Q}_n(x) = x^n \tilde{P}_n(1 + x^{-2}).
\]
Combining this with Corollary 3 and Theorem 4 we obtain the following result.

**Theorem 6.** For \(n \geq 1\), we have
\[
\begin{align*}
Q_{2n-1}(x) &= \sum_{j=1}^{n} (-4)^{n-j}(2j - 1)!U(n, j)(1 + x^2)^j, \\
Q_{2n}(x) &= x \sum_{j=1}^{n} (-4)^{n-j}(2j)!U(n, j)(1 + x^2)^j, \\
\tilde{Q}_{2n}(x) &= \sum_{j=0}^{n} (-1)^{n-j}(2j)!V(n, j)(1 + x^2)^j, \\
\tilde{Q}_{2n+1}(x) &= x \sum_{j=0}^{n} (-1)^{n-j}(2j + 1)!V(n, j)(1 + x^2)^j.
\end{align*}
\]

There has been much work on special values and explicit formulas of the derivative polynomials of trigonometric functions, see [12, 17] for instance. In [17], Hoffman noted that \(\tilde{Q}_n(1)\) are the Springer numbers of root systems of type \(B_n\), which also count snakes of type \(B_n\). A snake of type \(B_n\) is a signed permutation \(\pi(1)\pi(2) \cdots \pi(n)\) of \(B_n\) such that \(0 < \pi(1) > \pi(2) < \cdots \pi(n)\). Setting \(s_n = \tilde{Q}_n(1)\), then we have
\[
\sum_{n=0}^{\infty} s_n \frac{z^n}{n!} = \frac{1}{\cos z - \sin z}.
\]

From the above discussion, we get the following result.

**Corollary 7.** For \(n \geq 1\), we have
\[
\begin{align*}
E_{2n-1} &= \sum_{j=1}^{n} (-4)^{n-j}(2j - 1)!U(n, j), \quad E_{2n} = \sum_{j=0}^{n} (-1)^{n-j}(2j)!V(n, j), \\
S_{2n} &= \sum_{j=0}^{n} (-1)^{n-j}(2j)!2^jV(n, j), \quad S_{2n+1} = \sum_{j=0}^{n} (-1)^{n-j}(2j + 1)!2^jV(n, j).
\end{align*}
\]

### 2.2. The proof of Theorem 1
Along the same lines as in [17], setting \(x = \cos 2\theta\) and replacing \(z\) by \(2z\) in (3), we get
\[
\sum_{n=0}^{\infty} (\sin 2\theta)^{-n} 2^n \frac{z^n}{n!} \sum_{k=0}^{n} R(n + 1, k) \cos^{n-k} 2\theta = \tan^2 \theta \cot^2 (\theta - z).
\]
Thus by replacing \( z \) by \( -z \), we obtain
\[
\sum_{n=0}^{\infty} (\sin 2\theta)^{-n}2^n(1)n^2/n! \sum_{k=0}^{n} R(n+1,k)\cos^{n-k} 2\theta
= \tan^2 \theta \cot^2 (\theta + z)
= \tan^2 \theta \csc^2 (\theta + z) - \tan^2 \theta.
\]

By the Taylor's theorem, it is easy to derive that
\[
(-1)^n 2^n (\sin 2\theta)^{-n} \sum_{k=0}^{n} R(n+1,k) \cos^{n-k} 2\theta = \tan^2 \theta \frac{d^n}{d\theta^n} \csc^2 \theta
\]  
(12)

Carlitz deduced the following result by a simple induction.

**Lemma 8** ([8] Eqs (2.11), (2.12)). For \( n \geq 1 \), one has
\[
\frac{d^{2n-2}}{d\theta^{2n-2}} \csc^2 \theta = \sum_{j=1}^{n} (-1)^{n-j} 2^{2n-2j}(2j-1)!U(n,j) \csc^{2j} \theta,
\]
\[
\frac{d^{2n-1}}{d\theta^{2n-1}} \csc^2 \theta = \sum_{j=1}^{n} (-1)^{n-j+1} 2^{2n-2j}(2j)!U(n,j) \csc^{2j} \theta \cot \theta,
\]

**A proof** Theorem [11]. The proof of the expression of \( \sum_{s=0}^{2n-2} R(2n-1,s)x^{2n-2-s} \) is the same as in [8]. However, we give a proof of it for completeness. Replacing \( n \) by \( 2n-2 \), then (12) becomes
\[
2^{2n-2}(\sin 2\theta)^{-2n+2} \sum_{s=0}^{2n-2} R(2n-1,s) \cos^{2n-2-s} 2\theta = \tan^2 \theta \frac{d^{2n-2}}{d\theta^{2n-2}} \csc^2 \theta.
\]

Combining Lemma [8] and the double angle formula \( \sin 2\theta = 2 \sin \theta \cos \theta \), we get
\[
\sum_{s=0}^{2n-2} R(2n-1,s) \cos^{2n-2-s} 2\theta
= 2^{2-2n}(\sin 2\theta)^{2n-2} \tan^2 \theta \frac{d^{2n-2}}{d\theta^{2n}} \csc^2 \theta
= \sum_{j=1}^{n} (-1)^{n-j} 2^{2n-2j}(2j-1)!U(n,j) \sin^{2n-2j} \theta \cos^{2n-4} \theta.
\]

Since \( \sin^2 \theta = \frac{1-\cos 2\theta}{2} \), \( \cos^2 \theta = \frac{1+\cos 2\theta}{2} \), we obtain
\[
\sum_{s=0}^{2n-2} R(2n-1,s) \cos^{2n-2-s} 2\theta
= \sum_{j=1}^{n} (-1)^{n-j} 2^{2-j}(2j-1)!U(n,j)(1-\cos 2\theta)^{n-j}(1+\cos 2\theta)^{n-2-j}.
\]

Setting \( x = \cos 2\theta \), we get the expression of \( \sum_{s=0}^{2n-2} R(2n-1,s)x^{2n-2-s} \).

Similarly, replacing \( n \) by \( 2n-1 \), the identity (12) becomes
\[
\sum_{s=0}^{2n-1} R(2n,s) \cos^{2n-1-s} 2\theta = (-1)^{21-2n}(\sin 2\theta)^{2n-1} \tan^2 \theta \frac{d^{2n-1}}{d\theta^{2n-1}} \csc^2 \theta.
\]
Then combining Lemma 8 and double angle formulas, we get

\[
\sum_{s=0}^{2n-1} R(2n, s) \cos^{2n-1-s} 2\theta
\]

\[
= \sum_{j=1}^{n} (-1)^{n-j} 2^{2n-2j} (2j)! U(n, j) \sin^{2n-2j} \theta \cos^{2n-2} \theta
\]

\[
= \sum_{j=1}^{n} (-1)^{n-j} 2^{1-j} (2j)! U(n, j) (1 - \cos 2\theta)^{n-j} (1 + \cos 2\theta)^{n-1}
\]

Setting \( x = \cos 2\theta \), we get the expression of \( \sum_{s=0}^{2n-1} R(2n, s)x^{2n-1-s} \). This completes the proof. □

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