RELATIVE GL(V)-COMPLETE REDUCIBILITY

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Abstract. Let $K$ be a reductive subgroup of a reductive group $G$ over an algebraically closed field $k$. Using the notion of relative complete reducibility, in [3] a purely algebraic characterization of the closed $K$-orbits in $G^n$ was given, where $K$ acts by simultaneous conjugation on $n$-tuples of elements from $G$. This characterization generalizes work of Richardson and is also a natural generalization of Serre’s notion of $G$-complete reducibility. In this paper we revisit this idea, focusing on the particular case when the ambient group $G$ is a general linear group, giving a representation-theoretic characterization of relative complete reducibility. Along the way, we extend and generalize several results from [3].

1. Introduction

Let $G$ be a reductive linear algebraic group and let $n \in \mathbb{N}$. The group $G$ acts by simultaneous conjugation on $G^n$, the $n$-fold Cartesian product of $G$ with itself. In his seminal work [4, Thm. 16.4], Richardson characterized the closed $G$-orbits in $G^n$ in terms of the subgroup structure of $G$. In [2, Thm. 3.1] Richardson’s characterization was shown to be equivalent to a notion of Serre arising from representation theory, [5], and these ideas were further extended in [3] to give a characterization of the closed $K$-orbits in $G^n$ for an arbitrary reductive subgroup $K$ of $G$. This gave rise to the notion of relative complete reducibility, which we briefly recall now (see Section 2 for full definitions).

Let $H$ be a subgroup of $G$ and let $K$ be a reductive subgroup of $G$. Recall that (when $G$ is connected) the parabolic subgroups of $G$ have the form $P_{\lambda}$ where $\lambda$ is a cocharacter of $G$. Following [3], we say that $H$ is relatively $G$-completely reducible with respect to $K$ if for every cocharacter $\lambda$ of $K$ such that $H$ is contained in the subgroup $P_{\lambda}$ of $G$, there exists a cocharacter $\mu$ of $K$ such that $P_{\lambda} = P_{\mu}$ and $H \subseteq L_{\mu}$, a Levi subgroup of $P_{\lambda}$. For $K = G$, this definition coincides with the usual notion of $G$-complete reducibility due to Serre, cf. [2], [5].

The following algebraic characterization of the closed $K$-orbits in $G^n$ in terms of relative $G$-complete reducibility was given in [3, Thm. 1.1]:

Theorem 1.1. Let $K$ be a reductive subgroup of $G$. Let $H$ be the algebraic subgroup of $G$ generated by elements $x_1, \ldots, x_n \in G$. Then $K \cdot (x_1, \ldots, x_n)$ is closed in $G^n$ if and only if $H$ is relatively $G$-completely reducible with respect to $K$.

As is noted in [3, Cor. 3.6], if $G'$ is another reductive group with $G \subseteq G'$, then $H \subseteq G$ is relatively $G$-completely reducible with respect to $K$ if and only if $H$ is relatively $G'$-completely reducible with respect to $K$. Thus questions about relative complete reducibility can be reduced to questions about relative GL(V)-complete reducibility by choosing a suitable representation $V$. This is the focus of this paper: we study the case where $K$ is some reductive algebraic group, $V$ is a faithful representation of $K$ and $G = \text{GL}(V)$ is the general linear group.

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group of $V$, and we give “representation theoretic” characterizations of relative complete reducibility, extending earlier work from [3].

In order to state our main results, we need some further notation for subgroups of $GL(V)$. First recall that a parabolic subgroup of $GL(V)$ is the stabilizer $\text{Stab}_G(f)$ of a flag $f$ of subspaces in $V$. The poset of flags in $V$ is defined as the dual of the poset of parabolic subgroups in $GL(V)$, i.e., we set $f \leq f'$ provided $\text{Stab}_G(f) \supseteq \text{Stab}_G(f')$. For $K$ a reductive subgroup of $GL(V)$, we denote by $\mathcal{F}_K$ the set of flags in $V$ which stem from $K$, i.e., which correspond to parabolic subgroups $P_\lambda$ for $\lambda$ a cocharacter of $K$. A flag $f$ in $\mathcal{F}_K$ is called minimal in $\mathcal{F}_K$ provided $f' \leq f$ for $f'$ in $\mathcal{F}_K$ implies $f' = f$. Let $\mathcal{M}_K \subseteq \mathcal{F}_K$ be the set of minimal flags in $\mathcal{F}_K$. Note that members of $\mathcal{M}_K$ may have varying lengths, cf. Examples 3.1 and 3.2. Of course, $\mathcal{M}_{GL(V)}$ is the set of flags of length 1 in $V$ corresponding to the set of maximal parabolic subgroups in $GL(V)$.

Our first result characterizes relative $GL(V)$-complete reducibility with respect to $K$ in terms of the set of minimal flags $\mathcal{M}_K$.

**Theorem 1.2.** Let $H$ and $K$ be subgroups of $GL(V)$ with $K$ reductive. Then the following are equivalent:

(i) $H$ is relatively $GL(V)$-completely reducible with respect to $K$.

(ii) For every flag in $\mathcal{M}_K$ which is stabilized by $H$ there exists an opposite flag in $\mathcal{M}_K$ which is also stabilized by $H$.

Note that in the “absolute case” when $K = GL(V)$, we have $\mathcal{M}_K = \mathcal{M}_{GL(V)}$ and Theorem 1.2 reduces to the usual characterisation of complete reducibility in terms of submodules and complements. For the next result, which takes up this theme, we write $\mathcal{S}_K$ for the set of subspaces of $V$ which arise in flags from $\mathcal{F}_K$.

**Theorem 1.3.** Let $H$ and $K$ be subgroups of $GL(V)$ with $K$ reductive. Suppose that whenever $U \in \mathcal{S}_K$ is stabilized by $H$ there exists $W \in \mathcal{S}_K$ stabilized by $H$ so that $U \oplus W = V$. Then $H$ is relatively $GL(V)$-completely reducible with respect to $K$.

The converse of Theorem 1.3 is false in general, see Example 3.2. However, by requiring in addition that $\mathcal{M}_K \subseteq \mathcal{M}_{GL(V)}$, i.e., that every minimal flag in $\mathcal{M}_K$ corresponds to a maximal parabolic in $GL(V)$, we are able to obtain a converse to Theorem 1.3.

**Corollary 1.4.** Let $H, K$ be subgroups of $GL(V)$ with $K$ reductive. Suppose $\mathcal{M}_K \subseteq \mathcal{M}_{GL(V)}$. Then the following are equivalent:

(i) $H$ is relatively $GL(V)$-completely reducible with respect to $K$.

(ii) For each $U \in \mathcal{S}_K$ which is stabilized by $H$ there exists $W \in \mathcal{S}_K$ such that $H$ stabilizes $W$ and $V = U \oplus W$, as an $H$-module.

Corollary 1.4 readily follows from Theorem 1.2 and Corollary 2.4. Corollary 1.4 may be viewed as a generalization of [3, Prop. 5.1]. The latter gives a representation theoretic characterization of relative $GL(V)$-complete reducibility in case $K = GL(U)$ for a subspace $U$ of $V$ which is closely related to the condition in Corollary 1.4, see Lemma 3.4 and Corollary 2.4.

Particularly natural candidates for the subgroup $K$ in $GL(V)$ are the classical groups $SO(V)$ and $Sp(V)$. This is the theme of our next result which characterizes relative $GL(V)$-complete reducibility with respect to $SO(V)$ or $Sp(V)$ in terms of totally isotropic subspaces.
Corollary 1.5. Let $H$ be a subgroup of $\text{GL}(V)$ and let $K$ be $\text{SO}(V)$ or $\text{Sp}(V)$. Then the following are equivalent:

(i) $H$ is relatively $\text{GL}(V)$-completely reducible with respect to $K$.
(ii) Whenever $H$ stabilizes a totally isotropic subspace $U$ and its annihilator $U^\perp$, there exists a totally isotropic subspace $W \subseteq V$ such that $H$ stabilizes $W$ and $V = W \oplus U^\perp$, as an $H$-module.

Noting that in the setting of Corollary 1.5, flags in $\mathcal{F}_K$ have the form

$$U_1 \subseteq \ldots \subseteq U_r \subseteq U_r^\perp \subseteq \ldots \subseteq U_1^\perp \subseteq V,$$

so that minimal flags in $\mathcal{M}_K$ are of the form $U \subseteq U^\perp \subseteq V$ for $U$ a totally isotropic subspace, the result is immediate from Theorem 1.2.

Note that the condition in Corollary 1.5(ii) that $H$ must also stabilize the annihilator $U^\perp$ of $U$ cannot be relaxed in general, as $H$ does not need to leave the form on $V$ invariant, i.e., $H$ need not be a subgroup of $K$.

In the final Section 4, we briefly investigate the notion of relative $G$-complete reducibility over an arbitrary field, obtaining a rational version of Theorem 1.3, see Theorem 4.4, and noting that we also get rational counterparts of Theorem 1.2, and Corollaries 1.4 and 1.5.

2. Preliminaries

We work over an algebraically closed field $k$ with the exception of Section 4. Let $G$ be a reductive algebraic group defined over $k$ – we allow the possibility that $G$ is not connected. Let $H$ be a closed subgroup of $G$. We write $H^\circ$ for the identity component of $H$.

For the set of cocharacters (one-parameter subgroups) of $G$ we write $Y(G)$.

Suppose $G$ acts on a variety $X$ and let $x$ be in $X$. Then for each cocharacter $\lambda \in Y(G)$ we define a morphism of varieties $\overline{\phi}_{x,\lambda} : k^* \to X$ via $\phi_{x,\lambda}(a) = \lambda(a) \cdot x$. If this morphism extends to a morphism $\overline{\phi}_{x,\lambda} : k \to X$, then we say that the limit $\lim_{a \to 0} \lambda(a) \cdot x$ exists and set this limit equal to $\overline{\phi}_{x,\lambda}(0)$. For each cocharacter $\lambda \in Y(G)$, let $P_\lambda = \{ g \in G \mid \lim_{a \to 0} \lambda(a) g \lambda(a)^{-1} \text{ exists} \}$ and $L_\lambda = \{ g \in G \mid \lim_{a \to 0} \lambda(a) g \lambda(a)^{-1} = g \}$. Following [2, §6], we call $P_\lambda$ an $R$-parabolic subgroup of $G$ and $L_\lambda$ an $R$-Levi subgroup of $G$, noting that if $G$ is connected, then these $R$-parabolic subgroups and their $R$-Levi subgroups are precisely the parabolic subgroups and their Levi subgroups. If — as is often the case in this paper — $K$ is a reductive subgroup of $G$ and $\lambda \in Y(K)$, we always denote by $P_\lambda$ the $R$-parabolic subgroup of $G$ attached to $\lambda$; if we need to consider the corresponding $R$-parabolic subgroup of $K$ we write $P_\lambda(K)$ (and similarly for $R$-Levi subgroups).

We recall the notion of relative complete reducibility from [3].

Definition 2.1. Let $H$ and $K$ be subgroups of $G$ with $K$ reductive. We say that $H$ is relatively $G$-completely reducible with respect to $K$ if for every $\lambda \in Y(K)$ such that $H$ is contained in $P_\lambda$, there exists $\mu \in Y(K)$ such that $P_\lambda = P_\mu$ and $H$ is contained in $L_\mu$. We sometimes use the abbreviation relatively $G$-cr with respect to $K$.

Note that $H$ is relatively $G$-completely reducible with respect to $K$ if and only if $H$ is relatively $G$-completely reducible with respect to $K^\circ$. In the case when $K = G$, Definition 2.1 coincides with the usual definition of $G$-complete reducibility [2].

Let $K$ be a reductive subgroup of $\text{GL}(V)$. Recall that the parabolic subgroups of $\text{GL}(V)$ correspond to flags of subspaces in $V$, and that the partial order $\preceq$ on flags is the reverse of
the inclusion order on parabolic subgroups. When we need to specify the subspaces in a flag \( f \) we use the notation \( f = (U_1 \subseteq U_2 \subseteq \cdots \subseteq U_r \subseteq V) \), with the convention that \( U_1 \neq 0 \) and all the inclusions are proper. Such a flag is said to have length \( r \). We blur the distinction between subspaces \( U \) of \( V \) and flags \( (U \subseteq V) \) of length one. Recall from the Introduction that we denote by \( \mathcal{F}_K \) the set of flags in \( V \) stemming from \( K \), and by \( \mathcal{M}_K \) the set of minimal flags in \( \mathcal{F}_K \). The following observation is of interest in its own right.

**Lemma 2.2.** Let \( f \) be a flag in \( \mathcal{F}_K \) and let \( U \) be a subspace in \( f \). Then there is a flag \( f' \) in \( \mathcal{M}_K \) such that \( f' \leq f \) and \( U \) appears in \( f' \).

**Proof.** Let \( f = (U_1 \subseteq \cdots \subseteq U_r \subseteq V) \in \mathcal{F}_K \) of length \( r \) and \( U = U_i \) for some \( i \). We argue by induction on \( r \). If \( r = 1 \), there is nothing to prove. Suppose that \( r > 1 \) and that the statement is true for flags of length at most \( r - 1 \). If \( f \in \mathcal{M}_K \), there is nothing to show. Else there is an \( f' \in \mathcal{M}_K \) such that \( f' \leq f \). If \( U \) already appears in \( f' \), we are done. So, suppose that \( U \) does not appear in \( f' \).

Set \( G = \text{GL}(V) \). Let \( \lambda, \mu \in Y(K) \) such that \( P_\lambda = \text{Stab}_G(f) \subseteq P_\mu = \text{Stab}_G(f') \). There is a maximal torus \( T \) of \( G \) such that \( T \cap K \) is a maximal torus of \( K \), and there is a Borel subgroup of \( G \) so that \( T \subseteq B \subseteq P_\lambda \subseteq P_\mu \). Hence we may assume \( \lambda, \mu \in Y(T) \). Note that \( Y(T) \) is isomorphic to a subgroup of \( T \times T \), where \( n = \dim V \). Without loss we may assume that for an \( n \)-tuple \( (z_1, \ldots, z_n) \) in \( \mathbb{Z}^n \) corresponding to a parabolic subgroup containing \( B \), we have \( z_1 \geq \cdots \geq z_n \). There exists a basis \( v_1, \ldots, v_n \) of \( V \) such that \( \text{Stab}_G(\langle v_1 \rangle) \subseteq \langle v_1, v_2 \rangle \subseteq \cdots \subseteq V = B \). Let \((a_1, \ldots, a_n)\) and \((b_1, \ldots, b_n)\) be the \( n \)-tuples in \( \mathbb{Z}^n \) corresponding to \( P_\lambda \) and \( P_\mu \), respectively. Then we have

\[
P_\lambda = \text{Stab}_G(\langle v_1, \ldots, v_{\dim(U_i)} \rangle) \subseteq \cdots \subseteq \langle v_1, \ldots, v_{\dim(U_r)} \rangle \subseteq V.
\]

Hence

\[
a_1 = \cdots = a_{\dim(U_1)} > a_{\dim(U_1)+1} = \cdots = a_{\dim(U_2)} > \cdots > a_{\dim(U_r)+1} = \cdots = a_n.
\]

Choose \( i_0 \) so that the quotient \((b_i - b_{i+1})(a_i - a_{i+1})^{-1}\) is maximal, where we run over all \( i \) with \( a_i \neq a_{i+1} \). Set \( n_1 := b_{i_0} - b_{i_0+1} \) and \( n_2 := a_{i_0} - a_{i_0+1} \). Let \((c_1, \ldots, c_n)\) be the \( n \)-tuple in \( \mathbb{Z}^n \) corresponding to the cocharacter \( n_1 \lambda - n_2 \mu \in Y(T) \) and let \( \tilde{f} \) be the corresponding flag.

By construction, \( c_i = n_1 a_i - n_2 b_i \) for all \( i \).

If \( a_i > a_{i+1} \) and \( b_i = b_{i+1} \), then \( c_i > c_{i+1} \). If \( a_i = a_{i+1} \) and \( b_i = b_{i+1} \), then \( c_i = c_{i+1} \). The case \( a_i = a_{i+1} \) and \( b_i > b_{i+1} \) does not occur, since \( f' \leq f \). If \( a_i > a_{i+1} \) and \( b_i > b_{i+1} \), then our choice of \( n_1 \) and \( n_2 \) ensures that we have

\[
n_1(a_i - a_{i+1}) \geq n_2(b_i - b_{i+1}).
\]

Hence \( c_i \geq c_{i+1} \) in this case. So, whenever \( a_i = a_{i+1} \) we have \( c_i = c_{i+1} \) and whenever \( a_i > a_{i+1} \) we have \( c_i \geq c_{i+1} \). Thus \( \tilde{f} \leq f \).

Now, since \( U \) appears in \( f \), we have \( \dim(U) > \dim(U)+1 \), and since \( U \) does not appear in \( f' \), we have \( \dim(U) = \dim(U)+1 \). Therefore, \( c_{\dim(U)} > c_{\dim(U)+1} \), so the subspace \( U \) does not appear in \( f \). Finally, since we have got equality in (2.3) at least for \( i_0 \), so that \( c_{i_0} = c_{i_0+1} \) while \( a_{i_0} > a_{i_0+1} \), the length of \( \tilde{f} \) is strictly smaller than \( r \). Consequently, there exists a flag \( \hat{f} \in \mathcal{M}_K \) such that \( \hat{f} \leq f \) and \( U \) appears in \( \hat{f} \), by the induction hypothesis.

Recall that \( \mathcal{S}_K \) is the set of subspaces of \( V \) which appear in flags from \( \mathcal{F}_K \). We record an immediate consequence of Lemma 2.2.
Corollary 2.4. Let $K$ be a reductive subgroup of $GL(V)$. Then the following are equivalent:

(i) $S_K = \{ U \subseteq V \mid (U \subseteq V) \in F_K \}$.
(ii) $M_K \subseteq M_{GL(V)}$.

Next we recall [3, Lem. 3.3].

Lemma 2.5. Let $K$ be a reductive subgroup of $G$.

(i) Let $\lambda, \mu \in Y(K)$ such that $P_\lambda = P_\mu$ and $u \in R_u(P_\lambda(K))$ such that $uL_\lambda(K)u^{-1} = L_\mu(K)$. Then $uL_\lambda u^{-1} = L_\mu$.

(ii) Let $H$ be a subgroup of $G$. Then $H$ is relatively $G$-completely reducible with respect to $K$ if and only if for every $\lambda \in Y(K)$ such that $H \subseteq P_\lambda$ there exists $u \in R_u(P_\lambda(K))$ such that $H \subseteq L_u \lambda$.

In case both $G$ and $K$ are compatible products of reductive groups, our next result characterizes relative $G$-complete reducibility in terms of the factors of $K$.

Lemma 2.6. For $i = 1, 2$, let $K_i \subseteq G_i$ be reductive groups, $G := G_1 \times G_2$ and $K := K_1 \times K_2$. Let $H \subseteq G$ be a subgroup. Then $H$ is relatively $G$-completely reducible with respect to $K$ if and only if $H$ is relatively $G_i$-completely reducible with respect to $K_i$ for $i = 1, 2$.

Proof. Let $\lambda \in Y(K_1)$ such that $H \subseteq P_\lambda$. By the proof of [2, Lem. 2.12], the parabolic subgroups of $G$ stemming from $K$ have the form $P_{\lambda_i} \times P_{\lambda_2}$ with $\lambda_i \in Y(K_i)$ for $i = 1, 2$, since $G = G_1 \times G_2$ and $K = K_1 \times K_2$. Hence $P_\lambda = P_{\lambda_1}(G_1) \times P_{\lambda_2}(G_2)$. Since $H$ is relatively $G$-completely reducible with respect to $K$, there exists a $u = (u_1, u_2) \in R_u(P_{\lambda}(K))$ such that $H \subseteq L_{u, \lambda} = L_{u_1, \lambda}(G_1) \times G_2$, by Lemma 2.5(ii). Therefore, $u_1 \in R_u(P_{\lambda_1}(K_1))$. It follows that $H$ is relatively $G$-cr with respect to $K_1$, by Lemma 2.5(ii). The proof for $K_2$ is analogous.

For the reverse implication let $\lambda = (\lambda_1, \lambda_2) \in Y(K) = Y(K_1) \times Y(K_2)$ such that $H \subseteq P_{\lambda_1} \times P_{\lambda_2}$. Since $H$ is relatively $G$-completely reducible with respect to $K_i$ for $i = 1, 2$, there exits a $u_i \in R_u(P_{\lambda_i}(K_i))$ such that $H \subseteq L_{u_i, \lambda_i}(G_1) \times G_2$ resp. $H \subseteq G_1 \times L_{u_2, \lambda_2}(G_2)$. Therefore, we obtain

$$H \subseteq (L_{u_1, \lambda_1}(G_1) \times G_2) \cap (G_1 \times L_{u_2, \lambda_2}(G_2)) = L_{u, \lambda}$$

for $u = (u_1, u_2) \in R_u(P_{\lambda}(K))$. Once again, by Lemma 2.5(ii), $H$ is relatively $G$-cr with respect to $K$. \qed

3. Proofs of Theorems 1.3 and 1.2

Let $K$ be a reductive subgroup of $G = GL(V)$. The following explicit characterization of relative $GL(V)$-complete reducibility in terms of opposite flags is used in the sequel without further reference. Let $H$ be a subgroup of $G$. Then $H$ is relatively $G$-completely reducible with respect to $K$ if, and only if, for each flag $(U_1 \subseteq \ldots \subseteq U_m \subseteq V)$ in $F_K$ which is stabilised by $H$ there exists an opposite flag $(W_1 \subseteq \ldots \subseteq W_m \subseteq V)$ in $F_K$ stabilised by $H$, i.e. such that $V = U_i \oplus W_{m+1-i}$ as $H$-modules, for each $i = 1, \ldots, m$. With this in hand, we first attend to Theorem 1.3.

Proof of Theorem 1.3. Let $\lambda \in Y(K)$ such that $H \subseteq P_{\lambda}$. We wish to show that $H \subseteq L_\mu$ for some $\mu \in Y(K)$ with $P_\mu = P_{\lambda}$. Let $(U_1 \subseteq \ldots \subseteq U_r \subseteq V)$ be the flag corresponding to $P_{\lambda}$. Arguing by induction on $i$ we first prove that there exist subspaces $W_{r+1-i}$ such that for

$$1 \leq i \leq r, U_i \oplus W_{r+1-i} = V$$

as an $H$-module and $W_1 \subseteq \ldots \subseteq W_r$ with $W_r \in S_K$. 

Since $H$ stabilizes $U_1$, there exists $W_r \in \mathcal{S}_K$ such that $U_1 \oplus W_r = V$ and $H$ stabilizes $W_r$, by hypothesis, which gives the base case for our induction and the final claimed condition. So now suppose that we have found the subspaces $W_r, \ldots, W_{r+1-i}$. Set $\tilde{U}_{i+1} := U_{i+1} \cap W_{r+1-i}$ and $W_{r-i} := \langle W_{r+1-i} \setminus \tilde{U}_{i+1} \rangle$. By induction hypothesis, $U_i \oplus W_{r+1-i} = V$ and $W_{r+1-i}$ is stabilized by $H$. Since $H$ stabilizes $U_{i+1}$ and $W_{r+1-i}$, $H$ stabilizes $\tilde{U}_{i+1}$. Hence $W_{r-i}$ is stabilized by $H$ as well. Since $V = U_i \oplus W_{r+1-i}$ and $U_i \subseteq U_{i+1}$, we have $U_{i+1} = U_i \oplus \tilde{U}_{i+1}$.

Since $W_{r-i} = \langle W_{r+1-i} \setminus \tilde{U}_{i+1} \rangle$, we have $W_{r+1-i} = \tilde{U}_{i+1} \oplus W_{r-i}$ (to see this direct sum, write down a basis for $\tilde{U}_{i+1}$ and then extend to a basis for $W_{r+1-i}$; the vectors added will form a basis for $W_{r-i}$). Therefore,

$$V = U_i \oplus W_{r+1-i} = U_i \oplus \tilde{U}_{i+1} \oplus W_{r-i} = U_{i+1} \oplus W_{r-i}.$$ 

This completes the induction step, so we may assume that we have found subspaces $W_1 \subseteq \ldots \subseteq W_r$, with $W_r \in \mathcal{S}_K$ and $U_i \oplus W_{r+1-i} = V$ as an $H$-module for $1 \leq i \leq r$.

Let $P_\mu$ be the stabilizer of the flag $(W_1 \subseteq \ldots \subseteq W_r \subseteq V)$ from the previous paragraph and note that $P_\mu$ is opposite to $P_\lambda$ and by construction $H \subseteq P_\mu$. Since $W_r$ is from $\mathcal{S}_K$, there is some flag $(W_1' \subseteq \ldots \subseteq W_m' \subseteq V) \in \mathcal{F}_K$ with $W_i' = W_r$ for some $i$. Let $P_{\lambda'}$ be the stabilizer of this flag with $\lambda' \in Y(K)$.

Without loss we can assume that $\lambda, \lambda' \in Y(S) \subseteq Y(T)$ for a maximal torus $S$ of $K$ and a maximal torus $T$ of $G$. We thus have

$$S \subseteq T \subseteq P_\lambda \cap P_{\lambda'} \subseteq P_\lambda \cap \text{Stab}_G(W_1' \subseteq \ldots \subseteq W_m' \subseteq V)$$

$$\subseteq \text{Stab}_G(U_1 \subseteq \ldots \subseteq U_r \subseteq V) \cap \text{Stab}_G(W_r)$$

$$= \text{Stab}_G(U_1 \subseteq \ldots \subseteq U_r \subseteq V) \cap \text{Stab}_G(W_1 \subseteq \ldots \subseteq W_r \subseteq V)$$

$$= P_\lambda \cap P_\mu.$$

In particular, $T$ belongs to the opposite parabolic $P_\mu$ of $P_\lambda$. However, since $P_{-\lambda}$ is also opposite to $P_\lambda$ and contains $T$, by the uniqueness of the parabolic subgroup opposite to $P_\lambda$ containing $T$, it follows that $P_\mu = P_{-\lambda}$. Therefore, $H \subseteq P_\lambda \cap P_\mu = P_\lambda \cap P_{-\lambda} = L_\lambda$. Thus $H$ is relatively $G$-completely reducible with respect to $K$, as claimed. \hfill $\square$

Next we illustrate Theorem 1.3 with an explicit example when $V$ is a faithful irreducible representation of a simple algebraic group of exceptional type.

**Example 3.1.** Suppose $\text{char}(k) \neq 2$. Let $K$ be the simple group of type $G_2$ over $k$ and let $V$ be the seven-dimensional irreducible representation of $K$. Let $e_1, \ldots, e_7$ be the canonical basis for $V$, in which the corresponding maximal torus $S$ of $K$ has the form

$$S = \{ \text{diag}(t, s, st^{-1}, 1, s^{-1}t, t^{-1}, s^{-1}) \mid s, t \in k^* \}$$

Let $V_5$ be the span of the first five of these basis vectors, and let $H$ be the image of $\text{GL}(V_5)$ embedded in $\text{GL}(V)$ via

$$A \mapsto \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The flags in $\mathcal{F}_K$ have subspaces of dimension $(1, 3, 4, 6, 7), (2, 5, 7)$ and $(1, 2, 3, 4, 5, 6, 7)$. Suppose that $(U_1 \subseteq \ldots \subseteq U_m \subseteq V)$ belongs to $\mathcal{F}_K$. Note that in the proof of Theorem 1.3 the
hypothesis that whenever $U$ in $S_K$ is stabilized by $H$ there exists a $W$ in $S_K$ such that $H$ stabilizes $W$ and $U \oplus W = V$ is only used for $U_1$. Thus in our case it sufficient to check that whenever $H$ stabilizes a one-dimensional or two-dimensional subspace from $S_K$, then $H$ stabilizes a complement in $S_K$. If $H$ stabilizes a one- or two-dimensional subspace $U$, then $U \subseteq \langle e_6, e_7 \rangle$. Note that $H$ stabilizes each of the subspaces $\langle e_1, \ldots, e_6 \rangle, \langle e_1, \ldots, e_5, e_7 \rangle, \langle e_1, \ldots, e_5 \rangle$ in $S_K$. If $U$ is one-dimensional, then $\langle e_1, \ldots, e_6 \rangle$ or $\langle e_1, \ldots, e_5, e_7 \rangle$ is a complement to $U$. If $U$ is two-dimensional then $\langle e_1, \ldots, e_5 \rangle$ is a complement to $U$. By Theorem 1.3, $H$ is relatively $G$-cr with respect to $K$. 

The converse of Theorem 1.3 is false in general, as the following example illustrates.

**Example 3.2.** Let $G = \text{GL}_4(k)$ and let $K$ be the subgroup of diagonal matrices of the form $\text{diag}(t, s, s^{-1}, t^{-1})$ with $s, t \in k^*$. Let $e_1, \ldots, e_4$ be the standard basis of $k^4$ and $U = \langle e_1, e_2, e_3 \rangle$. Suppose that $H$ is the parabolic subgroup of $G$ corresponding to the flag $U \subseteq V$. Since the flags from $F_K$ have subspaces of dimension $(2, 4), (1, 3, 4)$ and $(1, 2, 3, 4)$, the group $H$ is not contained in $P_{\lambda}$ for any $\lambda \in Y(K) \setminus \{1\}$. Hence trivially, $H$ is relatively $G$-cr with respect to $K$. Note that $U \in S_K$ and $H$ stabilizes $U$. One checks that the complement to $U$ in the set $S_K$ is $W = \langle e_4 \rangle$. But $H$ does not stabilize $W$.

Armed with Lemma 2.2 and the arguments from the proof of Theorem 1.3, we are in a position to address Theorem 1.2.

**Proof of Theorem 1.2.** Assume (ii). Suppose that $(U_1 \subseteq \ldots \subseteq U_r \subseteq V) \in F_K$ is stabilized by $H$. Let $\lambda \in Y(K)$ be the corresponding cocharacter. Then there exists $f \in M_K$ such that $H$ stabilizes $f$ and $U_1$ is one of the subspaces of $f$, by Lemma 2.2. Hence there exists a flag $\hat{f} \in M_K$ opposite to $f$ such that $H$ stabilizes $\hat{f}$. Then there exists a subspace $W$ in $\hat{f}$ such that $V = U_1 \oplus W$. Note that $H$ stabilizes $W$ and $W$ belongs to $S_K$. In the proof of Theorem 1.3 the condition that whenever $U$ in $S_K$ is stabilized by $H$ there exists $W$ in $S_K$ stabilized by $H$ so that $U \oplus W = V$ is only used for the first subspace in the flag. So we can apply the proof from Theorem 1.3 to conclude that there exists $\mu \in Y(K)$ such that $H \subseteq I_\mu$ and $P_{\lambda} = P_\mu$. Hence (i) follows, by Theorem 1.3.

Conversely, suppose (i). Let $f \in M_K \subseteq F_K$ such that $H$ stabilizes $f$. Then there exists $\hat{f} \in F_K$ stabilized by $H$ such that $f$ and $\hat{f}$ are opposite. Suppose that $\hat{f} \not\in M_K$. Then there exists a flag $\check{f} \in M_K$ such that $\check{f} \prec \hat{f}$. Let $T$ be a maximal torus of $K$ which is contained in $\text{Stab}_G(f)$ and $\text{Stab}_G(\check{f})$. Hence there exists $\lambda \in Y(T)$ such that $P_{\lambda} = \text{Stab}_G(f)$ and $P_{-\lambda} = \text{Stab}_G(\check{f})$. Since $T \subseteq P_{-\lambda} \subseteq \text{Stab}_G(\hat{f})$, there exists a $\mu \in Y(T)$ such that $P_\mu = \text{Stab}_G(\hat{f})$. Suppose that $f' \in F_K$ is the flag corresponding to $-\mu$. Since $P_{\lambda} \subseteq P_{-\mu}$, we have $f' \preceq \check{f}$ and $f'$ is stabilized by $H$. Since the length of $\check{f}$ is smaller than that of $\hat{f}$, the length of $\hat{f}$ is smaller than that of $f$. But this is a contradiction to the minimality of $f$. Therefore, we conclude that $\hat{f} \in M_K$ and thus (ii) holds, as claimed. □

**Remark 3.3.** Whenever $M_K \not\subseteq M_G$, there exists a subgroup $H$ of $G$ such that $H$ is relatively $G$-cr with respect to $K$ and $H$ stabilizes a subspace $U' \in S_K$ but does not stabilize any complement to $U'$. To see this, note that since $S_K \neq \{U \subseteq V : (U \subseteq V) \in F_K\}$, by Corollary 2.4, there exists a $U'$ in $S_K$ such that $(U' \subseteq V) \not\in F_K$. Set $H := \text{Stab}_G(U' \subseteq V)$. Then $H$ is not contained in $P_{\lambda}$ for any $\lambda \in Y(K) \setminus \{1\}$. Trivially, $H$ is relatively $G$-cr with respect to $K$. Note that $H$ stabilizes $U'$ in $S_K$ but does not stabilize any complement to $U'$, since $H$ is a maximal parabolic subgroup of $G$. 


For a fixed subspace $U \subseteq V$, we show in the following lemma that $K = \text{GL}(U)$ satisfies the condition in Corollary 2.4(i). A maximal torus of $G$ also satisfies the condition. So Corollary 1.4 applies in these instances, thanks to Corollary 2.4.

**Lemma 3.4.** Let $U$ be a subspace of $V$. Fix a complement $\tilde{U}$ to $U$ in $V$. Let $K = \text{GL}(U) \subseteq G$, embedded via the decomposition $V = U \oplus \tilde{U}$. Then $\mathcal{S}_K = \{W \subseteq V \mid (W \subseteq V) \in \mathcal{F}_K\}$.

**Proof.** Let $(W_1 \subseteq \ldots \subseteq W_m \subseteq V)$ be in $\mathcal{F}_K$. One sees by inspection that for each $1 \leq i \leq m$ we have $W_i \subseteq U$ or $\tilde{U} \subseteq W_i$. On the other hand, suppose that $W$ is a subspace contained in $U$. Then we can find a complement $W'$ to $W$ containing $\tilde{U}$ and the cocharacter which acts with weight 1 on $W$ and weight 0 on $W'$ lies in $Y(K)$ and gives the flag $(W \subseteq V) \in \mathcal{F}_K$. Similarly, all flags $(W \subseteq V)$ with $\tilde{U} \subseteq W$ are in $\mathcal{F}_K$. Hence

$$\mathcal{F}_K = \{(W_1 \subseteq \ldots \subseteq W_m \subseteq V) \in \mathcal{F}_K \mid W_i \subseteq U \text{ or } \tilde{U} \subseteq W_i \text{ for } 1 \leq i \leq m, \text{ for some } m\},$$

and so

$$(3.5) \quad \mathcal{S}_K = \{W' \subseteq V \mid W' \subseteq U \text{ or } \tilde{U} \subseteq W'\} = \{W \subseteq V \mid (W \subseteq V) \in \mathcal{F}_K\},$$

as claimed. $\square$

In view of (3.5) and Corollary 2.4, Corollary 1.4 and Lemma 3.4 imply [3, Prop. 5.1]. So Corollary 1.4 may be viewed as a generalization of the special case treated in [3, Prop. 5.1].

Corollary 1.5 and Example 3.1 consider situations when $K$ acts irreducibly on $V$. We close this section with a characterization of relative $G$-complete reducibility in case $V$ decomposes as a direct sum of $K$-modules, where $K$ is a direct product of reductive subgroups, the result is immediate from Lemma 2.6 and [3, Cor. 3.6].

**Corollary 3.6.** Let $K$ be a direct product of reductive subgroups $K_1$ and $K_2$ of $G$. Then $H$ is relatively $G$-completely reducible with respect to $K$ if and only if it is relatively $G$-completely reducible with respect to $K_i$ for $i = 1, 2$.

Both implications in Corollary 3.6 fail in general without the assumption that $K$ is a direct product, as illustrated by our final example.

**Example 3.7.** Let $G = \text{GL}(k^4)$, $K = \{\text{diag}(t, s, t^{-1}, s^{-1}) \mid t, s \in k^*\}$, and $\{e_1, e_2, e_3, e_4\}$ is the canonical basis for $k^4$. Set $V_1 = \langle e_1, e_2 \rangle$ and $V_2 = \langle e_3, e_4 \rangle$. Let $K_i$ be the image of the projection from $K$ to $\text{GL}(V_i)$ for $i = 1, 2$.

Let $H$ be the stabilizer of $U := \langle e_2, e_4 \rangle$ in $G$ and note that $(U \subseteq k^4)$ belongs to $\mathcal{F}_K$. So $H$ is a maximal parabolic subgroup of $G$ stemming from $K$, and as such it is not relatively $G$-cr with respect to $K$. However, $H$ does not stem from $K_i$ and so is not contained in any parabolic subgroup of $G$ stemming from $K_i$, for $i = 1, 2$. Hence $H$ is relatively $G$-irreducible with respect to $K_i$, so it is relatively $G$-cr with respect to $K_i$, for $i = 1, 2$.

Now let $\tilde{H}$ be the stabilizer of $\tilde{U} := \langle e_1 \rangle$ in $G$. Note that $\tilde{H}$ is a maximal parabolic subgroup in $G$ stemming from $K_1$, thus it is not relatively $G$-cr with respect to $K_1$. However, since $\tilde{H}$ does not stem from $K$, it is relatively $G$-irreducible with respect to $K$ so in particular, is relatively $G$-cr with respect to $K$.
4. Rationality Questions

In this section, $k$ denotes an arbitrary field and we assume that $G$ is a reductive $k$-defined group and $K$ is a reductive $k$-defined subgroup of $G$ and let $G(k)$ and $K(k)$ denote the groups of $k$-points of $G$ and $K$, respectively. First, we recall the definition of relative $G$-complete reducibility over $k$ from [3, Def. 4.1].

**Definition 4.1.** Let $H$ be a subgroup of $G$. We say that $H$ is relatively $G$-completely reducible over $k$ with respect to $K$ if for every $\lambda \in Y(K)$ such that $P_\lambda$ is $k$-defined and $H$ is contained in $P_\lambda$, there exists $\mu \in Y(K)$ such that $P_\lambda = P_\mu$, $H$ is contained in $L_\mu$ and $L_\mu$ is $k$-defined.

The set of $k$-defined cocharacters of $G$ is denoted by $Y_k(G)$. Let $X$ be an affine variety over $k$ on which $G$ acts. We recall the definition of a cocharacter-closed $G(k)$-orbit in $X$ from [1, Def. 1.1].

**Definition 4.2.** Let $x \in X$. The orbit $G(k) \cdot x$ is cocharacter-closed over $k$ provided for all $\lambda \in Y_k(G)$ if $x' := \lim_{a \to 0} \lambda(a) \cdot x$ exists, then $x' \in G(k) \cdot x$.

Analogous to Theorem 1.1, we have the following “geometric” characterization of relative $G$-complete reducibility over $k$ in terms of cocharacter-closure, thanks to [3, Thm. 4.12(iii)].

**Theorem 4.3.** Let $K$ be a reductive subgroup of $G$. Let $H$ be the algebraic subgroup of $G$ generated by elements $x_1, \ldots, x_n \in G$. Then $K(k) \cdot (x_1, \ldots, x_n)$ is cocharacter-closed over $k$ if and only if $H$ is relatively $G$-completely reducible with respect to $K$ over $k$.

Now let $G = \text{GL}(V)$ and recall the notation from the Introduction. The $k$-defined parabolic subgroups of $G$ are precisely the stabilizers of rational flags in $V$ i.e., flags consisting of $k$-defined subspaces of $V$. Let $\mathcal{F}_K(k)$ be the set of rational flags in $\mathcal{F}_K$, and let $\mathcal{S}_K(k)$ denote the subset of $\mathcal{S}_K$ consisting of $k$-defined subspaces. We write $Y_k(K)$ for the set of $k$-defined cocharacters of $K$.

The following result is the rational analogue of Theorem 1.3.

**Theorem 4.4.** Suppose that $H$ is a subgroup of $G$. Suppose that whenever $U \in \mathcal{S}_K(k)$ is stabilized by $H$ there exists $W \in \mathcal{S}_K(k)$ stabilized by $H$ so that $U \oplus W = V$. Then $H$ is relatively $G$-completely reducible over $k$ with respect to $K$.

**Proof.** Let $\lambda \in Y_k(K)$ such that $H \subseteq P_\lambda$. Note that the flag corresponding to $P_\lambda$ is $k$-defined. Let $P_\mu = \text{Stab}_G(W_1 \subseteq \ldots \subseteq W_r)$ be the opposite parabolic subgroup to $P_\lambda$ from Theorem 1.3. By [6, Lem. 16.1.2], $P_\lambda \cap \text{Stab}_G(W_r)$ is $k$-defined, since $\text{Stab}_G(W_r)$ is a $k$-defined parabolic subgroup of $G$. Thanks to the proof of Theorem 1.3, we have $P_\lambda \cap \text{Stab}_G(W_r) = P_\lambda \cap P_\mu$. Then $H$ is contained in the $k$-defined Levi-subgroup $P_\lambda \cap P_\mu$ stemming from $K$. Hence $H$ is relatively $G$-completely reducible over $k$ with respect to $K$. \[\square\]

We note that the rational analogues of Theorem 1.2, Lemma 2.2 and Corollaries 1.4, 1.5 readily follow from the results above, by replacing $\mathcal{F}_K$, $\mathcal{M}_K$ resp. $\mathcal{S}_K$ by $\mathcal{F}_K(k)$, $\mathcal{M}_K(k)$ resp. $\mathcal{S}_K(k)$ and using the rational version of Theorem 1.3, Theorem 4.4.
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