This paper is dedicated to Professor Tsuyoshi Ando

Abstract. We extend Jordan’s notion of principal angles to work for two subspaces of quaternionic space, and so have a method to analyze two orthogonal projections in $M_n(\mathbb{A})$ for $\mathbb{A}$ the real, complex or quaternionic field (or skew field). From this we derive an algorithm to turn almost commuting projections into commuting projections that minimizes the sum of the displacements of the two projections. We quickly prove what we need using the universal real $C^*$-algebra generated by two projections.

1. Two projections, the three-fold way

The general form of two projections on complex Hilbert space is well known, going back to at least Dixmier [3]. The real case is older, being implicit in the work of Jordan [8]. Restricted to the finite-dimensional case, we can think of these as theorems about two projections in certain finite-dimensional real $C^*$-algebras. One would therefore expect the same result to hold in all finite-dimensional real $C^*$-algebras such as $M_n(\mathbb{H})$ where $\mathbb{H}$ is the skew field of quaternions.

Notation we will use gives us the needed supply of pairs of small projections. For $0 \leq \theta \leq \pi/2$ define

$$P_\theta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_\theta = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}.$$

By the dimension of a projection, we mean its trace, or the dimension over $\mathbb{A}$ of its range.

Theorem 1.1. Suppose $\mathbb{A}$ equals $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Suppose $P$ and $Q$ are projections in $M_n(\mathbb{A})$. If $\dim(P) \leq \dim Q$ then there is a unitary $U$ in $M_n(\mathbb{A})$ so that

$$P = U \begin{bmatrix} P_{\theta_1} & & & \\ & P_{\theta_2} & & \\ & & \ddots & \\ & & & P_{\theta_j} \end{bmatrix} U^* \begin{bmatrix} 0I_r & \\ & 0I_s \end{bmatrix}.$$

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1 Corresponding author.

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and
\[
Q = U \begin{bmatrix}
P_{\theta_1} & P_{\theta_2} & \cdots & P_{\theta_J} & I_r & 0I_s \\
0I_r & 0I_s & & & & \\
\end{bmatrix}
\]

where \( r = \dim(P) - \dim(Q) \) and \( s = n - r \). Moreover, \( \theta_1, \ldots, \theta_J \) and \( r \) and \( s \) are uniquely determined, the \( \theta_j \) up to order.

The real and complex cases are known. There is a short proof for the quaternionic case involving universal real \( C^* \)-algebras. We present this proof in the next section.

**Definition 1.2.** The principal angles between the range of \( P \) and the range of \( Q \) in \( \mathbb{H}^n \) are \( \theta_1, \ldots, \theta_J \).

We can make sense of “principal vectors” if we consider a subspace of \( \mathbb{H}^n \) as a subspace of \( \mathbb{C}^{2n} \) that is closed under the anti-unitary symmetry
\[
T \left( \begin{bmatrix} v \\ w \end{bmatrix} \right) = \begin{bmatrix} -\overline{w} \\ \overline{v} \end{bmatrix}.
\]

In terms of Dyson’s three-fold way [4], we are discussing class AII.

**Corollary 1.3.** Suppose \( M \) and \( N \) are subspaces of \( \mathbb{C}^{2n} \) with \( T(M) = M \) and \( T(N) = N \). Then the principal angles (see [5]) have even multiplicity and the principal vectors can be selected to be pairs of the form \( v, T v \).

**Proof.** If we replace these subspaces by \( UM \) and \( UN \) for some unitary \( U \) then \( U \) will send principal vectors to principal vectors. If \( U \) is symplectic then it will commute with \( T \) (see [10]) and so the conditions involving \( T \) will be preserved. Thus we can assume, by Theorem 1.1, that \( M \) and \( N \) are the ranges of \( A \) and \( B \) where
\[
A = \begin{bmatrix} P & P \\ P & P \end{bmatrix}, \quad Q = \begin{bmatrix} Q & Q \\ Q & Q \end{bmatrix}
\]

and
\[
P = \begin{bmatrix}
P_{\theta_1} & P_{\theta_2} & \cdots & P_{\theta_J} & I_r & 0I_s \\
P_{\theta_2} & P_{\theta_1} & & & & \\
\vdots & \vdots & & & & \\
P_{\theta_J} & 0I_r & & & & \\
0I_r & 0I_s & & & & \\
\end{bmatrix}, \quad Q = \begin{bmatrix}
P_{\theta_1} & P_{\theta_2} & \cdots & P_{\theta_J} & I_r & 0I_s \\
P_{\theta_2} & P_{\theta_1} & & & & \\
\vdots & \vdots & & & & \\
P_{\theta_J} & 0I_r & & & & \\
0I_r & 0I_s & & & & \\
\end{bmatrix}.
\]

Select the obvious real principal vectors for \( P \) and \( Q \) and then double each such \( v \) as
\[
\begin{bmatrix} v \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v \end{bmatrix} = T \begin{bmatrix} v \\ 0 \end{bmatrix}.
\]

\( \square \)
Remark 1.4. To unify things, we can regard a subspace of \( \mathbb{R}^n \) as a subspace of \( \mathbb{C}^n \) that is closed under conjugation, \( T_+ (v) = \overline{v} \). For two such subspaces of \( \mathbb{C}^n \) we can select principal vectors so that each such \( v \) satisfies \( T_+ (v) = v \). So in class AI we do not see the "Kramers pairs" effect that we see in class AII, but in both cases principal vectors can be selected to respect the relevant antiunitary symmetry.

It is no doubt possible to prove Corollary 1.3 directly, and then derive Theorem 1.1. However, the universal real \( C^* \)-algebra does much more than this. It can be used to prove technical results relevant to real \( K \)-theory, or, as we shall see, illuminate an algorithm for dealing with three relatively easy classes of almost commuting matrices.

2. A universal real \( C^* \)-algebra

In this article we interpret "real \( C^* \)-algebra" to mean specifically an \( \mathbb{R}^* \)-algebra. A real Banach algebra \( A \) with involution is an \( \mathbb{R}^* \)-algebra so long as its norm extends to the complexification \( A_\mathbb{C} \) to make that a \( \mathbb{C}^* \)-algebra. One can see [13] for a precise definition of what is allowed when doing relations on \( \mathbb{R}^* \)-algebras, but it certainly is allowed to say that a generator \( p \) satisfies \( p^2 = p^* = p \). For simplicity, we consider only the case of \( x_1, \ldots, x_n \) as generators. We say \( U \), along with \( \iota \) mapping \( \{x_1, \ldots, x_n\} \) into \( U \), is the universal \( \mathbb{R}^* \)-algebra for a set of relations if the following is true. Given any \( \mathbb{R}^* \)-algebra \( A \) with \( y_n, \ldots, y_n \) satisfying those relations, there is a unique *-homomorphism \( \phi : U \to A \) so that \( \phi (\iota (x_j)) = y_j \). Colloquially speaking, there is always exactly one extension of the mapping \( x_j \mapsto y_j \) to a *-homomorphism.

The following is very easy, given the machinery in developed by Sørensen in [13]. We call it a Theorem only because so much follows from it that is not so obvious.

**Theorem 2.1.** The universal \( \mathbb{R}^* \)-algebra generated by two elements \( p \) and \( q \) subject to the relations \( p^2 = p^* = p \) and \( q^2 = q^* = q \) is

\[
B = \left\{ f \in C \left([0, \frac{\pi}{2}], M_2(\mathbb{R})\right) \middle| f(0) \in \begin{bmatrix} \mathbb{R} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } f(1) \in \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{bmatrix} \right\}
\]

and the universal generators are \( p_0 \) and \( q_0 \) where

\[
p_0(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad q_0(t) = \begin{bmatrix} \cos^2(t) & \sin(t) \cos(t) \\ \sin(t) \cos(t) & \sin^2(t) \end{bmatrix}.
\]

**Proof.** The complexification of \( B \) is clearly

\[
A = \left\{ f \in C \left([0, 1], M_2(\mathbb{C})\right) \middle| f(0) \in \begin{bmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } f(1) \in \begin{bmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{bmatrix} \right\}
\]

and this is known to be the universal complex \( C^* \)-algebra for the relations of being two orthogonal projections. For example, see [11, §3]. By [13, Theorem 5.2.6.], the universal \( \mathbb{R}^* \)-algebra for these relations is the closed real *-algebra in \( A \) generated by \( \{p_0, q_0\} \), which is \( B \). \( \square \)
Proof of Theorem 1.1. Every finite-dimensional quotient of \( B \) is of the form
\[
\mathcal{C} = M_2(\mathbb{R}) \oplus \cdots \oplus M_2(\mathbb{R}) \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}
\]
with any number of the \( M_2(\mathbb{R}) \) and up to two of the \( \mathbb{R} \), with the surjection from \( B \) being evaluation at various \( t \) in \([0, 1)\) and also
\[
f \mapsto f(1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad f \mapsto f(1) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]
The \(*\)-homomorphisms between finite-dimensional \( R^* \)-algebras are known, say by [6]. Up to unitary equivalence, the only embedding of \( \mathcal{C} \) into \( M_n(\mathbb{H}) \) is found by the obvious embedding into
\[
\mathcal{D} = M_2(\mathbb{H}) \oplus \cdots \oplus M_2(\mathbb{H}) \oplus \mathbb{H} \oplus \cdots \oplus \mathbb{H}
\]
followed by an embedding that puts the \( M_k(\mathbb{H}) \) down the diagonal, perhaps with multiplicity in each summand.

Computing principal vectors. The standard for computing principal angles and vectors is an algorithm by Björck and Golub [1]. Let us assume our subspaces are given as the ranges of projections \( P \) and \( Q \). Their algorithm first obtains partial isometries \( E \) and \( F \) so that \( EE^* = P \) and \( FF^* = Q \). Then a singular value decomposition \( U \Omega V^* \) of \( E^*F \) is computed, and the principal vectors are found by pairing each column from \( EU \) with a column from \( FV \).

We describe here a different algorithm. We have no particular application in mind, so do not explore speed or accuracy issues. Moreover, the algorithm is simpler if it is restricted to the case \( \|P - Q\| \leq 1/\sqrt{2} \). We use always the operator norm, so \( \|X\| \) is the largest singular value of \( X \). See [10] for details regarding the norm in the case of a matrix of quaternions.

Following an idea from [12], we let \( U \) be the unitary in the polar decomposition of \( X = QP + (I - Q)(I - P) \). We take an orthonormal basis of eigenvectors for \( PQP \), and for each \( v \) in that basis coming from an eigenspace at or above \( \frac{1}{2} \) we find that \( (v, Uv) \) is a pair of principal vectors. Assuming the eigen-decomposition is done with the appropriate symmetry respected, the result will have the correct symmetry.

This algorithm can be validated, in exact arithmetic, from Theorem 1.1. Notice that the condition \( \|P - Q\| \leq 1/\sqrt{2} \) causes the \( \theta_j \) to be at most \( \frac{\pi}{2} \). For each \( P = P_\theta \) and \( Q = Q_\theta \) we note that
\[
X = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 \\ 0 & \cos(\theta) \end{bmatrix}
\]
so
\[
U = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}
\]
and the eigenvector for
\[
PQP = \begin{bmatrix} \cos^2(\theta) & 0 \\ 0 & 0 \end{bmatrix}
\]
for an eigenvalue above one-half will be
\[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]
and this will get paired with
\[
\begin{bmatrix}
\cos(\theta) \\
\sin(\theta)
\end{bmatrix}.
\]

Notice that $X$ will be invertible, and indeed have $\|X\| \leq 1$ and $\|X^{-1}\| \leq \sqrt{2}$. Thus $U$ can be quickly and accurately computed by Newton’s method \[7\]. Here is the algorithm in Matlab, assuming that $p$ and $q$ are the projection matrices.

```matlab
u = q*p + (eye(n)-q)*(eye(n)-p);
for iteration = 1:5
    u = (1/2)*(u + inv(u'));
end
central = p*q*p;
central = 0.5*(central + central');
[v,D] = eigs(central, dim);
a = v;
b = u*v;
```
The pairs of principal vectors are in the columns of $a$ and $b$. Code that tests this algorithm is available as an auxiliary file to the arxiv.org preliminary version of this paper.

3. Almost commuting projections

Almost commuting projections are much easier to understand than almost commuting hermitian contractions. Indeed, Lin’s theorem \[9\] is sufficiently difficult that there are no algorithms implementing it. An algorithm for a related problem might be helpful.

We can easily impose on our universal real $C^*$-algebra a relation that bounds the commutator.

**Corollary 3.1.** Suppose $0 \leq \delta < \frac{1}{2}$. Let $C = \frac{1}{2} \arcsin(2\delta)$. The universal $R^*$-algebra generated by two elements $p$ and $q$ subject to the relations $p^2 = p^* = p$ and $q^2 = q^* = q$ and $\|pq - qp\| \leq \delta$

is

\[ B_\delta = \left\{ f \in C(I_C, M_2(\mathbb{R})) \mid f(0) \in \begin{bmatrix} \mathbb{R} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } f(1) \in \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{bmatrix} \right\} \]

where $I_C = [0, C] \cup \left[ \frac{\pi}{2} - C, \frac{\pi}{2} \right]$ and the universal generators are $p_0$ and $q_0$ as in equation \[2.7\].

If $P$ and $Q$ almost commute, and we have candidates $P'$ and $Q'$ that are commuting projections, we can hope to have minimized either

\[
\|P' - P\| + \|Q' - Q\|
\]
or
\[ \max (\|P' - P\|, \|Q' - Q\|). \]
In the first case, we can just let \( P' = P \) and set \( Q' \) to be the spectral projection for \( \left[ \frac{1}{2}, \infty \right) \) of
\[ PQP + (I - P)Q(I - P). \]
This leads to the well-known result for the sum of displacements, namely
\[ \|P' - P\| + \|Q' - Q\| = \sin \left( \frac{1}{2} \arcsin(2x) \right). \]

Controlling the max of the displacements does not seem to have been considered before.

We observe that for \( 0 \leq \theta \leq \pi/4 \),
\[ \left\| P_\theta - Q_{\frac{\theta}{2}} \right\| = \left\| Q_\theta - Q_{\frac{\theta}{2}} \right\| = \sin \left( \frac{\theta}{2} \right) \]
while for \( \pi/4 \leq \theta \leq \pi/2 \), we let \( \theta' = \frac{\theta}{2} + \frac{\pi}{4} \) and observe
\[ \left\| P_\theta - (I - Q_{\theta'}) \right\| = \left\| Q_\theta - Q_{\theta'} \right\| = \sin \left( \frac{\theta}{2} \right). \]
For all \( \theta \) we find
\[ \left\| P_\theta Q_\theta - Q_\theta P_\theta \right\| = \frac{1}{2} \sin(2\theta). \]
Finally, when we start with 0 and 1 or 0 and 0 we just leave those alone.

**Theorem 3.2.** Suppose \( \mathbb{A} \) equals \( \mathbb{R} \), \( \mathbb{C} \) or \( \mathbb{H} \). If \( P \) and \( Q \) are projections in \( M_n(\mathbb{A}) \) then there are projections \( P' \) and \( Q' \) in \( M_n(\mathbb{A}) \) that commute and so that
\[ \|P - P'\| = \|Q - Q'\| = \sin \left( \frac{1}{4} \arcsin(2\delta) \right) \]
where
\[ \delta = \|PQ - QP\|. \]
The choice of \( P' \) and \( Q' \) can be made so that it is continuous in \( P \) and \( Q \).

**Proof.** We can simply work in \( \mathcal{B}_\delta \) and use the well-know fact that naturality in \( C^* \)-algebra constructions leads to continuity. \( \square \)

**Theorem 3.3.** For \( \delta = \|PQ - QP\| < \frac{1}{2} \), the commuting projections \( P' \) and \( Q' \) of Theorem 3.2 can be computed by the following formulas: let
\[
R = \frac{1}{2} (PQP + QPQ) \\
S = \frac{1}{2} ((I - P) Q (I - P) + Q (I - P) Q) \\
T = PQP + (I - P) (I - Q) (I - P)
\]
Figure 1. The two eigenvalues of $r(x)$ for various scalar values of $x$.

and then let $E_R$ and $E_S$ be the spectral projections for $R$ and $S$ corresponding to the set $\left[\frac{1}{5}, \infty\right)$ and $E_T$ the spectral projections for $T$ corresponding to the set $\left[\frac{1}{2}, \infty\right)$, and finally

$$P' = E_T E_R E_T + (I - E_T)(I - E_S)(I - E_T)$$
$$Q' = E_T E_R E_T + (I - E_T)E_S(I - E_T)$$

Proof. We notice that for

$$p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} \cos^2(x) & \sin(x) \cos(x) \\ \sin(x) \cos(x) & \sin^2(x) \end{bmatrix},$$

and with $0 \leq x \leq \frac{\pi}{4}$, if we set

$$r = \frac{1}{2} (pq + qp)$$

then

$$r = \begin{bmatrix} \cos \left(\frac{x}{2}\right) & \sin \left(\frac{x}{2}\right) \\ \sin \left(\frac{x}{2}\right) & -\cos \left(\frac{x}{2}\right) \end{bmatrix} \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix} \begin{bmatrix} \cos \left(\frac{x}{2}\right) & \sin \left(\frac{x}{2}\right) \\ \sin \left(\frac{x}{2}\right) & -\cos \left(\frac{x}{2}\right) \end{bmatrix}$$

where

$$\lambda_1(x) = \cos^2(x) \left(\frac{1}{2} \cos(x) + \frac{1}{2}\right)$$

and

$$\lambda_2(x) = -\cos^2(x) \left(\frac{1}{2} \cos(x) - \frac{1}{2}\right).$$

Suppose $e_r$ is the spectral projection of $r$ for $\left[\frac{1}{5}, \infty\right)$. Since

$$\lambda_2(x) \leq \lambda_2\left(\frac{\pi}{4}\right) = \frac{2 - \sqrt{2}}{8} \leq \frac{1}{5} \leq \frac{2 + \sqrt{2}}{8} \leq \lambda_2\left(\frac{\pi}{4}\right) \leq \lambda_2(x)$$

we find that
Figure 2. Distance to computed commuting projections by the Formulas in implemented in Matlab. There were 100 pairs of 200-by-200 real projections of distance at most 0.49 apart. The blue line is the exact answer of \( \sin(\arcsin(2x)/4) \).

\[ e_x = \begin{bmatrix} 
    \cos^2\left(\frac{x}{2}\right) & \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) \\
    \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) & \sin^2\left(\frac{x}{2}\right) 
\end{bmatrix} \]

which is on the midpoint of the canonical path between \( p \) and \( q \). (See [2].) By symmetry, set

\[ s = \frac{1}{2} ((1 - p)q(1 - p) + q(1 - p)q) \]

and find that the spectral projection \( e_s \) of \( s \) for \([\frac{1}{4}, \infty)\) satisfies

\[ e_s = \begin{bmatrix} 
    \cos^2\left(\frac{x}{2} - \frac{\pi}{2}\right) & \sin\left(\frac{x}{2} - \frac{\pi}{2}\right) \cos\left(\frac{x}{2} - \frac{\pi}{2}\right) \\
    \sin\left(\frac{x}{2} - \frac{\pi}{2}\right) \cos\left(\frac{x}{2} - \frac{\pi}{2}\right) & \sin^2\left(\frac{x}{2} - \frac{\pi}{2}\right) 
\end{bmatrix}, \]

and so for \( \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \), and this is the “midpoint” between \( 1 - p \) and \( q \). These projections do not become zero when \( x \) is in the opposite subinterval, as indicated by Figure 1.

For all \( x \) we use

\[ t = pqp + (1 - p)(1 - q)(1 - p) \]

which is

\[ t = \begin{bmatrix} 
    \cos^2(x) & 0 \\
    0 & \sin^2(x) 
\end{bmatrix}. \]

Thus the spectral projection \( e_t \) for \( t \) corresponding to \([\frac{1}{2}, \infty)\) is

\[ \begin{bmatrix} 
    1 & 0 \\
    0 & 0 
\end{bmatrix} \]

for \( x \) less than \( \frac{\pi}{2} \) and

\[ \begin{bmatrix} 
    0 & 0 \\
    0 & 1 
\end{bmatrix} \]
Figure 3. The errors with the same test matrices as in Figure 2.
(a) The sum of errors, in operator norm, in the relations $P^2 = P, Q^2 = Q, P^* = P, Q^* = Q$ and $P'Q' = Q'P'$. (b) The errors from the optimal in $\max(\|P - P\|, \|Q - Q\|)$.

for $x > \pi/2$.

If we define $r$ and $s$ and $t$ by the above formulas and set

$$p_1 = e_t e_r e_t + (1 - e_t)(1 - e_s)(1 - e_t)$$
$$q_1 = e_t e_r e_t + (1 - e_t)s(1 - e_t)$$

and find these are exactly commuting projections and

$$\max(\|p - p_1\|, \|q - q_1\|) = \sin\left(\frac{x}{2}\right).$$
Since 
\[ \delta = \|pq - qp\| = \sin(x) \cos(x) = \frac{1}{2} \sin(2x) \]
we have 
\[ \max(\|p - p_1\|, \|q - q_1\|) = \sin \left( \frac{1}{4} \arcsin(2\delta) \right). \]

□

Some applications of the half-angle formula give us
\[ \sin \left( \frac{1}{4} \arcsin(2\delta) \right) = \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{1 - 4\delta^2}}} \]
should someone think this is an improvement.

This is readily programmable for complex matrices, and can be done so real
and quaternionic matrices lead to real and quaternionic matrices during the calculation. Code that tests this is available as an auxiliary file to the arxiv.org
preliminary version of this paper. The results on real matrices is shown in Figure 2 with numerical errors shown in Figure 3. The data as shown were created with testCommute(200,100) using the code in the auxiliary file testCommute.m.

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Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, USA.

E-mail address: loring@math.unm.edu