The averaged tensors of the relative energy-momentum and angular momentum in general relativity

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Abstract

There exist at least a few different kind of averaging of the differences of the energy-momentum and angular momentum in normal coordinates $\text{NC}(\mathbf{P})$ which give tensorial quantities. The obtained averaged quantities are equivalent mathematically because they differ only by constant scalar dimensional factors. One of these averaging was used in our papers [1-8] giving the canonical superenergy and angular supermomentum tensors.

In this paper we present one other averaging of the energy-momentum and angular momentum differences which gives tensorial quantities with proper dimensions of the energy-momentum and angular momentum densities. But these averaged energy-momentum and angular momentum tensors, closely related to the canonical superenergy and angular supermomentum tensors, depend on some fundamental length $L$.

The averaged energy-momentum and angular momentum tensors of the gravitational field obtained in the paper can be applied, like the canonical superenergy and angular supermomentum tensors, to coordinate independent local (and also global) analysis of this field.

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I. THE AVERAGED ENERGY-MOMENTUM AND ANGULAR MOMENTUM TENSORS IN GENERAL RELATIVITY

In the papers [1-8] we have defined the canonical superenergy and angular supermomentum tensors, matter and gravitation, in general relativity (GR) and studied their properties and physical applications. In the case of the gravitational field these tensors gave us some substitutes of the non-existing gravitational energy-momentum and gravitational angular momentum tensors.

The canonical superenergy and angular supermomentum tensors were obtained pointwise as a result of some special averaging of the differences of the energy-momentum and angular momentum in normal coordinates NC(P). The dimensions of the components of these tensors can be written down as: [the dimensions of the components of an energy-momentum or angular momentum tensor (or pseudotensor)] × m⁻².

In this paper we propose a new averaging of the energy-momentum and angular momentum differences in NC(P) which is very like to the averaging used in [1-8] but which gives the averaged quantities with proper dimensionality of the energy-momentum and angular momentum densities.

Namely, we propose the following general definition of the averaged tensor (or pseudotensor) density \( \sqrt{|g|} T^b_a \)

\[
<T_a^b(P)> := \lim_{\varepsilon \to 0} \frac{\int_{\Omega} \left[ T^{(b)}_{(a)}(y) - T^{(b)}_{(a)}(P) \right] d\Omega}{\varepsilon^2/2 \int_{\Omega} d\Omega},
\]

where

\[
T^{(b)}_{(a)}(y) := \left( \sqrt{|g|} T^k_i(y) e^i_{(a)}(y) e_k^{(b)}(y) \right),
\]

\[
T^{(b)}_{(a)}(P) := \left( \sqrt{|g|} T^k_i(P) e^i_{(a)}(P) e_k^{(b)}(P) \right) = T^b_a(P)
\]

are the tetrad (or physical) components of a tensor or a pseudotensor density \( \sqrt{|g|} T^k_i(y) \) which describes an energy-momentum distribution, \( y \) is the collection of normal coordinates.
NC(P) at a given point P, \(e^i_{(a)}(y), \ e^b_k(y)\) denote an orthonormal tetrad field and its dual, respectively,

\[e^i_{(a)}(P) = \delta^i_a, \ e^a_k(P) = \delta^a_k, \ e^i_{(a)}(y)e^i_{(b)}(y) = \delta^b_a, \quad (4)\]

and they are parallelly propagated along geodesics through P.

For a sufficiently small domain \(\Omega\) which surrounds \(P\) we require

\[\int_\Omega y^i d\Omega = 0, \ \int_\Omega y^i y^k d\Omega = \delta^{ik} M, \quad (5)\]

where

\[M = \int_\Omega (y^0)^2 d\Omega = \int_\Omega (y^1)^2 d\Omega = \int_\Omega (y^2)^2 d\Omega = \int_\Omega (y^3)^2 d\Omega, \quad (6)\]

is a common value of the moments of inertia of the domain \(\Omega\) with respect to the subspaces \(y^i = 0, \ i = 0, 1, 2, 3\).

The procedure of averaging of an energy-momentum tensor or an energy-momentum pseudotensor given in (1) is a four-dimensional modification of the proposition by Mashhoon [9-12].

Let us choose \(\Omega\) as a small analytic ball defined by

\[(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 \leq R^2 = \varepsilon^2 L^2, \quad (7)\]

which can be described in a covariant way in terms of the auxiliary positive-definite metric \(h^{ik} := 2v^iv^k - g^{ik}\), where \(v^i\) are the components of the four-velocity of an observer \(O\) at rest at \(P\) (see, e.g., [1-8]). \(\varepsilon\) means a small parameter: \(\varepsilon \in (0; 1)\) and \(L\) is a fundamental length.

Since at \(P\) the tetrad and normal components are equal, from now on we will write the components of any quantity at \(P\) without (tetrad) brackets, e.g., \(T^b_a(P)\) instead of \(T^b_a(P)\) \(T)\) and so on.

Let us now make the following expansions for the energy-momentum tensor of matter \(T^k_i(y)\) and for \(\sqrt{|g|}, e^i_{(a)}(y), e^b_k(y)\) [13]
\[ T^k_i(y) = \hat{T}^k_i + \nabla_i \hat{T}^k_i y^l + 1/2 \hat{T}^k_i,lm y^l y^m + R_3 \]

\[ = \hat{T}^k_i + \nabla_i \hat{T}^k_i y^l + 1/2 \left[ \nabla_l \nabla_m \hat{T}^k_i \right] y^l y^m + R_3, \quad (8) \]

\[ \sqrt{|g|} = 1 - 1/6 \hat{R}_{ab} y^a y^b + R_3 \quad (9) \]

\[ e^i_{(a)}(y) = \hat{e}^i_{(a)} + 1/6 \hat{R}^i_{lk} \hat{e}^k_{(a)} y^l y^m + R_3, \quad (10) \]

\[ e^k_{(b)}(y) = \hat{e}^k_{(b)} - 1/6 \hat{R}^k_{lp} \hat{e}^p_{(b)} y^l y^m + R_3, \quad (11) \]

which give (1) in the form

\[ <_m T^b_a(P) > = \lim_{\varepsilon \to 0} \int_{\Omega} \frac{\left[ \nabla_l \hat{T}^b_a y^l + 1/2 (\nabla_l \nabla_m \hat{T}^b_a - 2/3 \hat{R}_{lm} \hat{T}^b_a ) y^l y^m + THO \right]}{\varepsilon^2 / 2 \int_{\Omega} d\Omega} d\Omega \quad (12) \]

where \( THO \) means the terms of higher order in the expansion of the differences \( T^b_a(P) = T^b_a(P) - T^b_a(P), \) \( R_3 \) is the remainder of the third order and \( \nabla \) denotes covariant differentiation. Hat denotes the value of an object at \( P \).

The first and \( THO \) terms in the numerator of (12) do not contribute to \( <_m T^b_a(P) > \).

Hence, we finally get from (12)

\[ <_m T^b_a(P) > = \frac{1}{m} S_a^b(P) \frac{L^2}{6}, \quad (13) \]

where

\[ mS_a^b(P) := \delta^{mn} [\nabla_l \nabla_m \hat{T}^b_a - 2/3 \hat{R}_{lm} \hat{T}^b_a ] \quad (14) \]

is the canonical superenergy tensor of matter\(^1\) [1-8].

\(^1\)This tensor is something different than the tensor given in [1-8] because here we have averaged the tensor density \( \sqrt{|g|} T^k_i \); not the tensor \( T^k_i \) as in [1-8].
By introducing the four velocity $\hat{v}^l \hat{\delta}_l^l$, $v^kv_k = 1$ of an observer $O$ at rest at $P$ and the local metric $\hat{g}^{ab} \hat{\eta}^{ab}$, where $\eta^{ab}$ is the inverse Minkowski metric, one can write (14) in a covariant way as

$$m S^b_a(P; v^l) = (2\hat{v}^l \hat{v}^m - \hat{g}^{lm})[\nabla_i(\nabla_m \hat{T}_a^b - 2/3 \hat{R}_{lm} \hat{T}_a^b].$$

(15)

The sign $\hat{\delta}$ means that an equality is valid only in some special coordinates.

The matter superenergy tensor $m S^b_a(P; v^l)$ is symmetric. As a result of an averaging the tensor $m S^b_a(P; v^l)$, and in consequence the averaged tensor $< m T_a^b(P; v^l) >$, do not satisfy any local conservation laws in general relativity. However, these tensors satisfy trivial local conservation laws\(^2\) in special relativity (see, e.g., [1-8]).

Now let us take the gravitational field and make the expansion

$$\sqrt{|g|} e^{k}\iota(y) = \frac{\alpha}{9} \left[ \hat{B}^k_{ilm} + \hat{P}^k_{ilm} ight. \\
- \frac{\delta^k_i}{2} \hat{R}^{abc}_{l}(\hat{\check{R}}_{abcd} + \hat{\check{R}}_{abcd}) + 2\beta^2 \delta^k_i \hat{E}_{(i)g}^{(g)} |^{l}_{g} + \\
\left. - 3\beta^2 \hat{E}_{i(l)}^{k} |^{m}_{l} + 2\beta \hat{R}^k_{(g)(l)} |^{g}_{m} \right] y^l y^m + R_3.$$  

(16)

Here

$$\alpha = \frac{c^4}{2G} = \frac{1}{2\beta}, \quad E^k_i := T^k_i - 1/2\delta^k_i T.$$  

(17)

The expansion (16) with the help of (9),(10) and (11) gives the following averaged gravitational energy-momentum tensor

$$< g t^b_a(P; v^l) >= g S^b_a(P; v^l) \frac{L^2}{6},$$  

(18)

where the tensor $g S^b_a(P; v^l)$ is the canonical superenergy tensor for the gravitational field [1-8].

\(^2\)Trivial local conservation laws because the integral superenergetic quantities or, equivalently, integral averaged energy-momentum calculated from them for aclosed system in special relativity vanish.
We have [1-8]
\[ g S_a^b (P; v^l) = \frac{2\alpha}{9} (2\hat{v}^l \hat{v}^m - \hat{g}^{lm}) \left[ \hat{B}_{alm}^b + \hat{P}_{alm}^b ight] - 1/2\delta_a^b \hat{R}^{ijk}\;_m \hat{R}_{ijkl} + 2\beta^2 \delta_a^b \hat{E} \hat{E}_m^g \\
- 3\beta^2 \hat{E}_a \hat{E}_m^g + 2\beta \hat{R}_b \hat{E}_l^g \\] (19)

where

\[ B_{alm}^b := 2\hat{R}^{bik} \;_{(l} \hat{R}^{aik}\;_{m)} - 1/2\delta_a^b \hat{R}^{ijk}\;_l \hat{R}_{ijkl}. \] (20)

is the Bel-Robinson tensor, while

\[ P_{alm}^b := 2\hat{R}^{bik} \;_{(l} \hat{R}^{aik}\;_{m)} - 1/2\delta_a^b \hat{R}^{ijk}\;_l \hat{R}_{ijkl}. \] (21)

In vacuum the tensor \( g S_a^b (P; v^l) \) reduces to the simpler form

\[ g S_a^b (P; v^l) = \frac{8\alpha}{9} (2\hat{v}^l \hat{v}^m - \hat{g}^{lm}) \left[ \hat{R}^{b(ik)} \;_{(l} \hat{R}^{aik}\;_{m)} - 1/2\delta_a^b \hat{R}^{ijk}\;_{(l} \hat{R}_{ikp}\;_{m)} \right], \] (22)

which is symmetric and the quadratic form \( g S_{ab} (P; v^l) \hat{v}^a \hat{v}^b \) is positive-definite.

In vacuum we also have the local conservation laws

\[ \nabla_b g S_a^b = 0. \] (23)

and the analogical laws satisfied by the averaged tensor \( g T_a^b (P; v^l) \).

The averaged energy-momentum tensors \( <_g T_a^b (P; v^l) > \) and \( <_g t_a^b (P; v^l) > \) can be considered as the averaged tensors of the relative energy-momentum. They can also be interpreted as the fluxes of the appropriate canonical superenergy. It is easily seen from the formulas (13) and (18).

Now let us consider the averaged angular momentum tensors in GR. The constructive definition of these tensors, in analogy to the definition of the averaged energy-momentum tensors, is as follows.

In normal coordinates NC(P) we define
< M^{(a)(b)(c)}(P) >= M^{abc}(P) := \lim_{\varepsilon \to 0} \frac{\int_{\Omega} [M^{(a)(b)(c)}(y) - M^{(a)(b)(c)}(P)] d\Omega}{\varepsilon^2/2 \int_{\Omega} d\Omega}, \quad (24)

where

\[ M^{(a)(b)(c)}(y) := M^{ijkl}(y) e_i^{(a)}(y) e_k^{(b)}(y) e_l^{(c)}(y), \quad (25) \]

\[ M^{(a)(b)(c)}(P) := M^{ijkl}(P) e_i^{(a)}(P) e_k^{(b)}(P) e_l^{(c)}(P) = M^{ijkl}(P) \delta_i^a \delta_k^b \delta_l^c = M^{abc}(P), \quad (26) \]

are the physical (or tetrad) components of the field \( M^{ijkl}(y) = (-)M^{kil}(y) \) which describes the angular momentum densities.\(^3\) As in (2) and (3), \( e_i^{(a)}(y) \), \( e_k^{(b)}(y) \) denote mutually dual orthonormal tetrads parallelly propagated along geodesics through \( P \) such that \( e_i^{(a)}(P) = \delta_i^a \), \( e_k^{(b)}(P) = \delta_k^b \). \( \Omega \) is a sufficiently small four-dimensional ball with centre at \( P \) and with the radius \( R = \varepsilon L \).

At \( P \) the tetrad and normal components of an object are equal. We apply this again and omit tetrad brackets for the indices of any quantity attached to the point \( P \); for example, we write \( M^{abc}(P) \) instead of \( M^{(a)(b)(c)}(P) \) and so on.

For matter as \( M^{ijkl}(y) \) we take

\[ m M^{ijkl}(y) = \sqrt{|g|} [y^j T^{kl}(y) - y^k T^{jl}(y)], \quad (27) \]

where \( T^{ik}(y) = T^{ki}(y) \) are the components of a symmetric energy-momentum tensor of matter and \( y^i \) denote the normal coordinates \( \text{NC}(P) \).

The formula (27) gives the total angular momentum densities, orbital and spinorial, because the dynamical energy-momentum tensor of matter \( T^{ik} = T^{ki} \) comes from the canonical one by using the Belinfante-Rosenfeld symmetrization procedure and, therefore, includes the canonical spin of matter [14].

Note that the normal coordinates \( y^i \) form the components of the local radius-vector \( \vec{y} \) with respect to the origin \( P \). Consequently, the components \( m M^{ijkl}(y) \) form a local tensor density.

\(^3\)Of course, \( M^{abc}(P) = 0 \), but we leave \( M^{abc}(P) \) in our formulas.
For the gravitational field we take the gravitational angular momentum pseudotensor proposed by Bergmann and Thomson [14,17] as

\[ g M^{ikl}(y) = F U^{[kl]}(y) - F U^{[id]}(y) + \sqrt{|g|} (y^i_{BT} t^{kl} - y^k_{BT} t^{il}) , \tag{28} \]

where

\[ F U^{[kl]} := g^{im} F U_{m}^{[kl]} = \alpha g^{im} \frac{g_{ma}}{\sqrt{|g|}} \left[ (-g)(g^{ka} g^{lb} - g^{la} g^{kb}) \right]_{,b} \tag{29} \]

are Freud’s superpotentials with the first index raised and

\[ BT t^{kl} := g^{ki} E t^{l}_{i} + \frac{g^{mk}}{\sqrt{|g|}} F U_{m}^{[lp]} \tag{30} \]

are the components of the Bergmann-Thomson gravitational energy-momentum pseudotensor [14,17].

\[ E t^{k}_{i} = \alpha \left\{ \delta^{k}_{i} g^{ms} (\Gamma^l_{mr} \Gamma^r_{sl} - \Gamma^r_{ms} \Gamma^l_{rl}) \\
+ g^{ms} \Gamma^r_{ms} - 1/2 (\Gamma^k_{tp} g^{tp} - \Gamma^k_{it} g^{kt}) g_{ms} \\
- 1/2 (\delta^{k}_{r} \Gamma^l_{ml} + \delta^{k}_{m} \Gamma^l_{rl}) \right\} \tag{31} \]

is the Einstein canonical gravitational energy-momentum pseudotensor of the gravitational field.

The Bergmann-Thomson gravitational angular pseudotensor is most closely related to the Einstein canonical energy-momentum complex and it has better physical and transformational properties than the famous gravitational angular momentum pseudotensor proposed by Landau and Lifschitz [15,16,17]. This is why we apply it here.

One can interpret the Bergmann-Thomson gravitational angular momentum pseudotensor as the sum of the spinorial part

\[ S^{ikl} := F U^{i[kl]} - F U^{k[id]} \tag{32} \]

and the orbital part

\[ O^{ikl} := \sqrt{|g|} (y^i_{BT} t^{kl} - y^k_{BT} t^{il}) \tag{33} \]
of the gravitational angular momentum “densities”.

Substitution of (27) and (28) (expanded up to third order) and (9),(10),(11) into (24) gives the following averaged angular momentum tensors for matter and gravitation respectively

\[
\left<\text{m} M^{abc}(P; v^l)\right> = \text{m} S^{abc}(P; v^l) \frac{L^2}{6},
\]

\[
\left<g M^{abc}(P; v^l)\right> = g S^{abc}(P; v^l) \frac{L^2}{6}.
\]

Here

\[
m S^{abc}(P; v^l) = 2[(\hat{v}^p \hat{v}^p - \hat{g}^{ap}) \nabla_p \hat{T}^{bc} - (\hat{v}^b \hat{v}^p - \hat{g}^{bp}) \nabla_p \hat{T}^{ac}],
\]

and

\[
g S^{abc}(P; v^l) = \alpha (\hat{v}^p \hat{v}^l - \hat{g}^{pl}) \left[ \beta (\hat{g}^{ac} \hat{g}^{br} - \hat{g}^{bc} \hat{g}^{ar}) \nabla_p \hat{E}_{pr} \right.
\]
\[
+ 2 \hat{g}^{ar} \nabla_p \hat{R}^{(b \, c)}_{\, (a \, p \, r)} - 2 \hat{g}^{br} \nabla_p \hat{R}^{(a \, c)}_{\, (b \, p \, r)}
\]
\[
+ 2/3 \hat{g}^{bc} (\nabla_r \hat{R}^{(a \, b)}_{\, (r \, p \, c)} - \beta \nabla_p \hat{E}_a) - 2/3 \hat{g}^{ac} (\nabla_r \hat{R}^{(b \, a)}_{\, (r \, p \, c)} - \beta \nabla_p \hat{E}_b) \left].
\]

are the components of the canonical angular supermomentum tensors for matter and gravitation, respectively \[4,6,8\].

In special relativity the averaged tensors \(<g M^{abc}(P; v^l)\>) , <m M^{abc}(P; v^l)\>) , like as the canonical angular supermomentum tensors, satisfy trivial conservation laws [1-8]. In the framework of the GR only the tensors \(g S^{abc}(P; v^l)\) and \(<g M^{abc}(P; v^l)\>) satisfy local conservation laws in vacuum.

In vacuum, when \(T_{ik} = 0 \iff E_{ik} := T_{ik} - 1/2 g_{ik} T = 0\), the canonical gravitational angular supermomentum tensor \(g S^{abc}(P; v^l)\) given by (37) simplifies to

\[
g S^{abc}(P; v^l) = 2\alpha (\hat{v}^p \hat{v}^l - \hat{g}^{pl}) \left[ \hat{g}^{ar} \nabla_p \hat{R}^{(b \, c)}_{\, (r \, t \, i)} - \hat{g}^{br} \nabla_p \hat{R}^{(a \, c)}_{\, (b \, t \, r)} \right].
\]

Some remarks are in order:
1. The orbital part \( O^{ikl} = \sqrt{|g|}(y^{ikl}_{HT} - y^{kl}_{HT}) \) of the \( gM^{ikl} \) does not contribute to the tensor \( gS^{abc}(P; v^l) \) and to the tensor \( <_g M^{abc}(P; v^l) > \). Only the spinorial part \( S^{ikl} = F^{ikl}U^{[ik]} - F^{ikl}U^{[kl]} \) gives nonzero contribution to these tensors.

2. The averaged angular momentum tensors \( <_g M^{abc}(P; v^l) >, <_m M^{abc}(P; v^l) > \), like as the canonical angular supermomentum tensors, do not need any radius-vector for existing.

The averaged tensors \( <_m M^{abc}(P; v^l) >, <_g M^{abc}(P; v^l) >, \) likely as the averaged energy-momentum tensors, can be interpreted as the *averaged tensors of the relative angular momentum*\(^1\) and also as the *fluxes* of the appropriate angular supermomentum.

The formulas \( (13),(18),(34),(35) \) give the direct link between the canonical superenergy and angular supermomentum tensors

\[
g^{abc}(P; v^l), m^{abc}(P; v^l), gS^{abc}(P; v^l), mS^{abc}(P; v^l) \tag{39}
\]

and the averaged energy-momentum and angular momentum tensors

\[
<_g t^{a b}(P; v^l), <_m t^{a b}(P; v^l), <_g M^{abc}(P; v^l), <_m S^{abc}(P; v^l) >. \tag{40}
\]

It is easily seen that the averaged energy-momentum and angular momentum tensors *differ* from the canonical superenergy and angular supermomentum tensors *only* by the constant scalar multiplicator \( \frac{L^2}{6} \), where \( L \) means some fundamental length. Thus, from the mathematical point of view, these two kind of tensors are equivalent. Physically they *are not* because their components have different dimension. Moreover the averaged energy-momentum and angular momentum tensors depend on a fundamental length \( L \). Owing to the last fact and the formulas \( (13),(18), (34), (35) \) it seems that the canonical superenergy and angular supermomentum tensors are *more fundamental* than the averaged energy-momentum and angular momentum tensors. But one should emphasize that the averaged energy-momentum and

\(^1\)The angular momentum is, of course, always relative quantity, by definition.
angular momentum tensors have an important superiority over the canonical superenergy and angular supermomentum tensors: their components possess proper dimensions of the energy-momentum and angular momentum densities.

The averaged tensors

\[ <g t^b_a(P; v^l)>, <m T^b_a(P; v^l)>, <g M^{abc}(P; v^l)>, <m M^{abc}(P; v^l)> \] (41)

depend on the four-velocity \( \vec{v} \) of a fiducial observer \( O \) which is at rest at the beginning \( P \) of the normal coordinates \( \text{NC}(P) \) used for averaging and on some fundamental length \( L \). After fixing of the length \( L \) one can determine univocally these tensors along the world line of an observer \( O \).

In general one can unambiguously determine these tensors (after fixing \( L \)) in the whole spacetime or in some domain \( \Omega \) if in the spacetime or in the domain \( \Omega \) a geometrically distinguished timelike unit vector field \( \vec{v} \) is given.

One can try to establish\(^5\) the fundamental length \( L \) by using loop quantum gravity. Namely, one can take as \( L \) e.g., the smallest length \( l \) over which the classical model of the spacetime is admissible.

Following loop quantum gravity [18-28] one can say about continuous classical differential geometry already just a few orders of magnitude above the Planck scale, e.g., for distances \( l \geq 100L_P = 100\sqrt{\frac{G \hbar}{c^3}} \approx 10^{-33} \) m. So, one can take as the fundamental length \( L \) the value \( L = 100L_P \approx 10^{-33} \) m.

After fixing the fundamental length \( L \) we have the averaged energy-momentum and angular momentum tensors established with the same precise as the canonical superenergy and angular supermomentum tensors.

The averaged tensors (with \( L \) fixed or no)

\[ <m T^b_a(P; v^l)>, <g t^b_a(P; v^l)>, <m M^{abc}(P; v^l)>, <g M^{abc}(P; v^l)> \] (42)

\(^5\)But this is not necessary. One can effectively use the averaged energy-momentum and angular momentum tensors without fixing \( L \).
give us as good tool to a local (and also global) analysis of the gravitational and matter fields as the canonical superenergy and angular supermomentum tensors

\[ mS^b_a(P;v^l), \quad gS^b_a(P;v^l), \quad mM^{abc}(P;v^l), \quad gM^{abc}(P;v^l) \] (43)
give. For example, one can apply the averaged energy-momentum and angular momentum tensors to the all problems which have been analyzed in the papers [1-8].

In this paper we only apply the averaged gravitational energy-momentum tensor

\[ <g t^b_a(P;v^l)> \]
in order to decide if free vacuum gravitational field has any kind of energy-momentum; especially, if gravitational waves carry any energy-momentum? The problem arose recently because some authors conjectured [29-33], by using coordinate dependent pseudotensors and complexes, that the energy and momentum in general relativity are confined to the regions of non-vanishing energy-momentum tensor of matter and that the gravitational waves carry no energy and momentum. The argumentation is the following. For some solutions to the Einstein equations and in some special coordinates, e.g., in Bonnor’s spacetime [34] in Bonnor’s or in Kerr-Schild coordinates, the Einstein canonical gravitational energy-momentum pseudotensor (and other pseudotensors also) globally vanishes outside of the domain in which \( T^{ik} \neq 0 \). The analogical global vanishing of the canonical pseudotensor \( E t^b_a \) we have for the plane and for the plane-fronted gravitational waves in, e.g., null coreper [3,35]. But one should emphasize that all these results are coordinate dependent [3,7,35]. Moreover, one should interpret physically the global vanishing of the canonical pseudotensor (and other pseudotensors also) in some coordinates in vacuum as a global cancellation of the energy-momentum of the real gravitational field which has \( R_{iklm} \neq 0 \) with energy-momentum of the inertial forces field which has \( R_{iklm} = 0 \); not as a proof of vanishing of the energy-momentum of the real gravitational field. It is because the all used pseudotensors were entirely constructed from the Levi-Civita’s connection \( \Gamma^i_{kl} = \Gamma^i_{lk} \) which describes a mixture of the real gravitational field (\( R_{iklm} \neq 0 \)) and an inertial forces field (\( R_{iklm} = 0 \)).

In order to get the coordinate independent results about energy-momentum of the the real gravitational field one must use tensorial expressions which depend on curvature tensor,
like the averaged gravitational energy-momentum tensor $<_g t^b_a(P; v^l)>$ or like the canonical gravitational superenergy tensor $g^{S b}_a(P; v^l)$. These two tensors vanish iff $R_{iklm} = 0$, i.e., iff the spacetime is flat and we have no real gravitational field.

When calculated, the averaged gravitational energy-momentum tensor $<_g t^b_a(P; v^l)>$ always gives the positive-definite averaged free relative gravitational energy density and, in the case of the gravitational waves, its non-zero flux. It is easily seen from the our papers [1-8,35].

Thus, the conjecture about localization of the gravitational energy only to the regions of the non-vanishing energy-momentum tensor of matter cannot be correct for the real gravitational field which has $R_{iklm} \neq 0$.

In a similar way one can use the averaged gravitational angular momentum tensor $<_g M^{abc}(P; v^l)>$ to coordinate-independent analysis of the angular momentum of the real gravitational field.
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