Continuation of root functionals of a system of polynomial equations and the reduction of polynomials modulo its ideal

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The notion of a root functional of a system of polynomials or an ideal of polynomials is a generalization of the notion of a root for a multiple root. The operation of continuation of root functionals and the operation of reduction of polynomials modulo the ideal are constructed on the basis of the operation of extension of bounded root functionals when the number of equations is equal to the number of unknowns and the number of roots is finite.

The notion of a root functional arose at the investigation of linear relations of polynomials with polynomial coefficients (syzygies) and is a generalization of the notion of a root for the case including also and multiple roots [1-6]. A bounded root functional characterize roots of a system polynomial coefficients (syzygies) and is a generalization of the notion of a root for the case of a multiple root. A linear functional this is an infinitely component object, therefore there arise the problem of its finite determination and operating by it in such a representation. An extension operation of bounded root functionals allows to continue a functional from its determination on the space of polynomials of the bounded degree, and also by this operations to reduce a polynomial modulo ideal, when the number of roots taking in account of multiplicity is finite. An extension operation is defined for a system of polynomials, in which the number of polynomials is equal to the number of variables.

Let \( R \) be a commutative ring with unity 1 and zero 0.

Let \( x = (x_1, \ldots, x_n) \) be variables, \( R[x] \) be the ring of polynomials in variables \( x \) with coefficients in \( R \).

In the paper we will use definition and assumption, given in [6].

Let \( L(x) \) be a functional in \( R[x] \), \( G(x) \) be a polynomial in \( R[x] \), denote by \( L(x).G(x) \) a functional with the following action: \( L(x).G(x).F(x) = L(x).G(x)F(x) \), where \( F(x) \in R[x] \).

**Definition 1.** Let \( x = (x_1, \ldots, x_n) \), \( y \equiv x \) be variables, \( f(x) = (f_1(x), \ldots, f_n(x)) \) be polynomials.

1. For a functional \( L(x) \) denote by \( [L(x)] \) the operator

\[
L(y).\det \begin{bmatrix}
\nabla f(x, y) & \nabla z(x, y)
\end{bmatrix}
\begin{bmatrix}
f(x)
1_x(x)
\end{bmatrix}
\equiv
L(y).\det \begin{bmatrix}
\nabla f(x, y) & \nabla z(x, y)
\end{bmatrix}
\begin{bmatrix}
f(y)
1_x(y)
\end{bmatrix}.
\]

Note that since \( \nabla x, y \) is linear over \( R \) operator, then operator \( [L(x)] \) is linear over \( R \).

2. For a functional \( L(x) \) and a polynomial \( F(x) \) denote by \( L(x).F(x) = [L(x)].F(x) \), then

\[
L(x).F(x) = L(y).\det \begin{bmatrix}
\nabla f(x, y) & \nabla F(x, y)
\end{bmatrix}
\begin{bmatrix}
f(x)
F(x)
\end{bmatrix} = L(y).\det \begin{bmatrix}
\nabla f(x, y) & \nabla F(x, y)
\end{bmatrix}
\begin{bmatrix}
f(y)
F(y)
\end{bmatrix}.
\]
3. For functionals $l(x_*)$ and $L(x_*)$ denote by $l(x_*) * L(x_*) = l(x_*) [L(x_*)]$, then

$$l(x_*) * L(x_*) = l(x_*) L(y_*). \det \begin{bmatrix} \nabla f(x, y) & \nabla_x (x, y) \\ f(x) & 1_x (x) \end{bmatrix} =$$

$$= l(x_*) L(y_*). \det \begin{bmatrix} \nabla f(x, y) & \nabla_x (x, y) \\ f(y) & 1_x (y) \end{bmatrix} .$$

This map $*$ we call an extension operation.

**Lemma 1.** Let $x = (x_1, \ldots, x_n)$ be variables, $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials, then for functionals $l(x_*)$, $L(x_*)$ and a polynomial $F(x) \in \mathbb{R}[x]$ there holds

$$l(x_*) * L(x_*) F(x) = l(x_*) L(x_*) * F(x) = l(x_*) [L(x_*)] F(x).$$

**Proof.**

$$l(x_*) * L(x_*) F(x) = (l(x_*) [L(x_*)]) F(x) = l(x_*) ([L(x_*)] F(x)) =$$

$$= l(x_*) L(x_*) * F(x).$$

**Lemma 2.** Let $x = (x_1, \ldots, x_n), y \simeq x$ be variables, $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials, then for any functional $l(x_*)$ there holds

$$l(x_*) * 1 = l(y_*) \det \| \nabla f(x,y) \|. $$

**Proof.** Set $F(x) = 1$, then $\nabla F(x,y) = 0$. We have

$$l(x_*) * 1 = l(x_*) F(x) =$$

$$= l(y_*) \det \begin{bmatrix} \nabla f(x,y) & \nabla F(x,y) \\ f(x) & F(x) \end{bmatrix} = l(y_*) \det \begin{bmatrix} \nabla f(x,y) & 0 \\ f(x) & F(x) \end{bmatrix} =$$

$$= l(y_*) \det [\nabla f(x,y)] \cdot F(x) = l(y_*) \det [\nabla f(x,y)] \cdot 1 = l(y_*) \det [\nabla f(x,y)].$$

**Theorem 1.** Let $x = (x_1, \ldots, x_n)$ be variables, $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials, $\delta_f = \sum_{i=1}^n (\deg (f_i) - 1)$. Let $\forall i = 1, 2: L_i (x_*) \ annuls \ f(x)_{x}^{\leq \delta_f + \delta_i}$, where $\delta_i \geq 0$, then

$$L_1(x_*) * L_2(x_*) = L_2(x_*) * L_1(x_*) \ in \ \mathbb{R}[x^{\leq \delta_f + \delta_1 + \delta_2 + 1}].$$

**Proof.** This is the reformulation of theorem 4 in [6].

**Theorem 2.** Let $x = (x_1, \ldots, x_n)$ be variables, $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials, $\delta_f = \sum_{i=1}^n (\deg (f_i) - 1)$.  

1. Let $\forall i = 1, 2, 3: L_i (x_*) \ annuls \ f(x)_{x}^{\leq \delta_f + \delta_i}$, where $\delta_i \geq 0$, then

$$(L_1(x_*) * L_2(x_*)) * L_3(x_*) = L_1(x_*) * (L_2(x_*) * L_3(x_*)) \ in \ \mathbb{R}[x^{\leq \delta_f + \delta_1 + \delta_2 + \delta_3 + 2}].$$
2. Let \( \forall i = 1, 2: L_i(x_*) \text{ annuls } (f(x))_{\mathbb{R}}^{\delta_f + \delta_i} \), where \( \delta_i \geq 0 \), let \( F(x) \in \mathbb{R}[x] \), then

\[
[L_1(x_*) \ast L_2(x_*),] \cdot \text{det} \frac{f(x)}{x} \equiv [L_1(x_*)], [L_2(x_*),] \cdot \text{det} \frac{f(x)}{x}
\]

and

\[
[L_2(x_*), [L_1(x_*),] \cdot \text{det} \frac{f(x)}{x} \equiv [L_1(x_*), [L_2(x_*),] \cdot \text{det} \frac{f(x)}{x}.
\]

**Proof.** This theorem is non-trivial and its proof is laborious, therefore it will be given in the subsequent papers.

**Theorem 3.** Let \( x = (x_1, \ldots, x_n) \), \( y \simeq x \) be variables, \( f(x) = (f_1(x), \ldots, f_n(x)) \) be polynomials, \( \delta_f = \sum_{i=1}^{n} (\deg(f_i) - 1) \), let \( l(x_*) \text{ annuls } (f(x))_{\mathbb{R}} \), let \( F(x) \in \mathbb{R}[x] \), then:

1) \( l(x_*) \cdot F(x) = l(y_*) \cdot F(y) \cdot \text{det} \left\| \nabla f(x, y) \right\| \)

and

\( l(x_*) \cdot F(x) = [l(x_*),] \cdot F(x) = (l(y_*) \cdot \text{det} \left\| \nabla f(x, y) \right\|) \cdot F(y); \)

2) \( l(x_*) \cdot F(x) \in \mathbb{R}[x^{\leq \delta_f}]; \)

3) \( l(x_*) \cdot F(x) \text{ is uniquely determined, up to addend in } (f(x))_{\mathbb{R}}^{\delta_f}; \text{ under non-uniqueness of } \nabla f(x, y), \text{ and not depend on } \nabla F(x, y); \)

4) if \( F(x) \in (f(x))_{\mathbb{R}} \), then \( l(x_*) \cdot F(x) = 0. \)

**Proof 1,2.**

\[
l(x_*) \cdot F(x) = l(y_*) \cdot \text{det} \left\| \begin{array}{cc}
\nabla f(x, y) & \nabla F(x, y) \\
\frac{f(y)}{0} & F(y)
\end{array} \right\| = l(y_*) \cdot F(y) \cdot \text{det} \left\| \nabla f(x, y) \right\|.
\]

The second equality holds, since \( l(y_*) \text{ annuls } (f(y))_y \). The obtained polynomial \( \in \mathbb{R}[x^{\leq \delta_f}] \), since \( \text{det} \left\| \nabla f(x, y) \right\| \) has a degree \( \leq \delta_f \) in \( x \).

**Proof 3.** From 1 of the theorem see, that \( l(x_*) \cdot F(x) \) not depend on \( \nabla F(x, y) \). The functional \( l(x_*) \text{ annuls } (f(x))_{\mathbb{R}}^{\delta_f + d} \) for any \( d \geq 0 \), then by virtue of 2 of theorem 2 in [6] the polynomial \( l(x_*) \cdot F(x) \) is uniquely determined up to addend belonging to \( (f(x))_{\mathbb{R}}^{\delta_f + d} \), independently of the choice of \( \nabla f(x, y) \). For sufficiently large \( d, \max(\delta_f, \deg(F) - d - 1) = \delta_f \). Hence, \( l(x_*) \cdot F(x) \) is uniquely determined up to addend in \( (f(x))_{\mathbb{R}}^{\delta_f} \), independently of the choice of \( \nabla f(x, y) \).

**Proof 4.** By virtue of 1 of the theorem there holds

\[
l(x_*) \cdot F(x) = l(y_*) \cdot F(y) \cdot \text{det} \left\| \nabla f(x, y) \right\| = 0. \]

The last equality holds, since \( F(y) \in (f(y))_y \), and \( l(y_*) \text{ annuls } (f(y))_y \).
Theorem 4. Let $x = (x_1, \ldots, x_n)$, $y \simeq x$ be variables, $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials, $\delta_f = \sum_{i=1}^{n}(\deg(f_i) - 1)$, let $l(x_*)$ annuls $(f(x))_x$, $L(x_*)$ annuls $(f(x))^{\leq \delta_f + \delta}$, where $\delta \geq 0$, then:

1) $L(x_*) \cdot l(x_*) = l(x_*) \cdot L(x_*)$;
2) $l(x_*) \cdot L(x_*) = l(x_*) \cdot (L(y_*) \cdot \det \|\nabla f(x, y)\|)$;
3) $l(x_*) \cdot L(x_*)$ is uniquely determined, independently of the choice of $\nabla f(x, y)$, and not depend on the operator $\nabla_x(x, y)$;
4) $l(x_*) \cdot L(x_*)$ annuls $(f(x))_x$;
5) $l(x_*) \cdot L(x_*)$ not depend on the action of $L(x_*)$ outside $R[x^{\leq \delta_f}]$.

Proof 1. Since $l(x_*)$ annuls $(f(x))^{\leq \delta_f + d}$ for any $d \geq 0$, $L(x_*)$ annuls $(f(x))^{\leq \delta_f + \delta}$, then by virtue of theorem 1 $l(x_*) \cdot L(x_*) = L(x_*) \cdot l(x_*)$ in $R[x^{\leq \delta_f + d + 1}]$, and mean, and in the whole $R[x]$ by the arbitrariness of $d \geq 0$.

Proof 2. Let $F(x) \in R[x]$, then

$$l(x_*) \cdot L(x_*) \cdot F(x) = l(x_*) \cdot L(y_*) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla F(x, y) \\ f(x) & F(x) \end{bmatrix} =$$

$$= l(x_*) \cdot L(y_*) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla F(x, y) \\ 0 & F(x) \end{bmatrix} = l(x_*) \cdot L(y_*) \cdot \det \|\nabla f(x, y)\| \cdot F(x) =$$

$$= l(x_*) \cdot (L(y_*) \cdot \det \|\nabla f(x, y)\|) \cdot F(x).$$

From the arbitrariness of $F(x)$ we have the equality of functionals

$$l(x_*) \cdot L(x_*) = l(x_*) \cdot (L(y_*) \cdot \det \|\nabla f(x, y)\|).$$

Proof 3. From 2 see, that $l(x_*) \cdot L(x_*)$ not depend on $\nabla_x(x, y)$.

Since $l(x_*)$ annuls $(f(x))^{\leq \delta_f + d}$ for any $d \geq 0$, $L(x_*)$ annuls $(f(x))^{\leq \delta_f + \delta}$, then by virtue of 1 of theorem 3 in [6] $l(x_*) \cdot L(x_*)$ is uniquely determined in $R[x^{\leq \delta_f + \delta + d + 1}]$ independently of the choice of $\nabla f(x, y)$, and mean, is uniquely determined in the whole $R[x]$ by the arbitrariness of $d \geq 0$.

Proof 4. From 2 see, that $l(x_*) \cdot L(x_*)$ annuls $(f(x))_x$, since $l(x_*)$ annuls $(f(x))_x$.

Proof 5. From 2 see, that $l(x_*) \cdot L(x_*)$ not depend on the action of $L(x_*)$ outside $R[x^{\leq \delta_f}]$, since $\det \|\nabla f(x, y)\|$ has a degree $\leq \delta_f$ in $y$.

Definition 2. Let $x = (x_1, \ldots, x_n)$, $y \simeq x$ be variables, let $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials. A functional $E_1(x_*)$ we call a unit root functional of polynomials $f(x)$, if it annuls $(f(x))_x$, and $E_1(x_*) \cdot 1 = E_1(y_*) \cdot \det \|\nabla f(x, y)\| = 1 + f(x) \cdot g(x)$. A functional $E'(x_*)$ we call a unit bounded root functional of polynomials $f(x)$, if it annuls $(f(x))^{\leq \delta_f + \varepsilon}$, where $\varepsilon \geq 0$, and $E'(x_*) \cdot 1 = E'(y_*) \cdot \det \|\nabla f(x, y)\| = 1 + f(x) \cdot g(x)$.

Lemma 3. Let $x = (x_1, \ldots, x_n)$, $y \simeq x$ be variables, $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials. Let a functional $E(x_*)$ annuls $(f(x))_x$, and $E(x_*) \cdot 1 = 1 \in (f(x))_x$, let $E'(x_*) = E(x_*)$ in $R[x^{\leq \delta_f + \varepsilon}]$, where $\varepsilon \geq 0$. Then $E'(x_*)$ annuls $(f(x))^{\leq \delta_f + \varepsilon}$ and $E'(x_*) \cdot 1 = 1 \in (f(x))_x$. 22
**Proof.** $E'(x_*) \cdot 1 = E'(y_*) \cdot \det \|\nabla f(x, y)\| = E(y_*) \cdot \det \|\nabla f(x, y)\| = E(x_*) \cdot 1$, since $E'(y_*) = E(y_*)$ in $R[y \leq \delta_f + \varepsilon]$ and $\det \|\nabla f(x, y)\|$ has a degree $\leq \delta_f$ in $y$. Hence, $E'(x_*) \cdot 1 - 1 = E(x_*) \cdot 1 - 1 \in (f(x))_x$. Since $E'(x_*) = E(x_*)$ in $R[x \leq \delta_f + \varepsilon]$ and $E(x_*)$ annuls $(f(x))_{x}^{\delta_f + \varepsilon}$, then and $E'(x_*)$ annuls $(f(x))_{x}^{\delta_f + \varepsilon}$.

**Theorem 5.** Let $x = (x_1, \ldots, x_n)$, $y \simeq x$ be variables, $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials, $\delta_f = \sum_{i=1}^{n} (\deg(f_i) - 1)$. Let a functional $E'(x_*)$ annuls $(f(x))_{x}^{\delta_f + \varepsilon}$, where $\varepsilon \geq 0$, and $E'(x_*) \cdot 1 - 1 \in (f(x))_x$, then:

1) if $F(x) \in R[x \leq d]$, then $E'(x_*) \cdot F(x) \in R[x \leq \max(\delta_f, d - 1)]$

and $F(x) - E'(x_*) \cdot F(x) \in (f(x))_x \cap R[x \leq \max(\delta_f, d)]$;

2) if $l(x_*)$ annuls $(f(x))_x$, then $l(x_*) = E'(x_*) \cdot l(x_*) = l(x_*) \cdot E'(x_*)$;

3) if $l(x_*)$ annuls $(f(x))_x$ and $L'(x_*) = l(x_*)$ in $R[x \leq \delta_f + \varepsilon]$, then $l(x_*) = E'(x_*) \cdot L'(x_*) = L'(x_*) \cdot E'(x_*)$ in $R[x \leq \delta_f + \varepsilon + 1]$;

4) if a functional $L(x_*)$ annuls $(f(x))_x \cap R[x \leq \delta_f + \varepsilon]$ where $\varepsilon \geq 0$, then $L(x_*) = E'(x_*) \cdot L(x_*) = L(x_*) \cdot E'(x_*)$ in $R[x \leq \delta_f + \varepsilon]$ and $E'(x_*) \cdot L(x_*) = L(x_*) \cdot E'(x_*)$ annuls $(f(x))_x \cap R[x \leq \delta_f + \varepsilon + 1]$.

**Proof 1.** Since $E'(x_*)$ annuls $(f(x))_{x}^{\delta_f + \varepsilon}$ and $F(x) \in R[x \leq d]$, then by virtue of 1 of theorem 2 in [6] $E'(x_*) \cdot F(x) \in R[x \leq \max(\delta_f, d - 1)]$. Then

$$F(x) - E'(x_*) \cdot F(x) \in R[x \leq d] + R[x \leq \max(\delta_f, d - 1)] = R[x \leq \max(d, \max(\delta_f, d - 1))] = R[x \leq \max(\delta_f, d)]$$.

Moreover,

$$E'(x_*) \cdot F(x) =$$

$$= E'(y_*) \cdot \det \begin{vmatrix} \nabla f(x, y) & \nabla F(x, y) \\ f(x) & F(x) \end{vmatrix} \equiv E'(y_*) \cdot \det \begin{vmatrix} \nabla f(x, y) & \nabla F(x, y) \\ 0 & F(x) \end{vmatrix}$$

$$= (E'(y_*) \cdot \det \|\nabla f(x, y)\|) \cdot F(x) = (1 + f(x) \cdot g(x)) \cdot F(x) \frac{(f(x))_x}{\|\nabla f(x, y)\|} F(x)$$,

then $F(x) - E'(x_*) \cdot F(x) \in (f(x))_x$. Finally, $F(x) - E'(x_*) \cdot F(x) \in (f(x))_x \cap R[x \leq \max(\delta_f, d)]$.

**Proof 2.** Since $l(x_*)$ annuls $(f(x))_x$, $E'(x_*)$ annuls $(f(x))_{x}^{\delta_f + \varepsilon}$, then by virtue of 1 of theorem 4 there holds $E'(x_*) \cdot l(x_*) = l(x_*) \cdot E'(x_*)$, and by virtue of 2 of theorem 4

$$l(x_*) = l(x_*) \cdot (E'(y_*) \cdot \det \|\nabla f(x, y)\|) = l(x_*) \cdot (1 + f(x) \cdot g(x)) = l(x_*)$$.
The last equality holds, since \( l(x_n) \) annuls \((f(x))_x\), and since \( l(x_n) \cdot 1 = l(x_n) \).

**Proof 3.** Since \( L'(x_n) \) annuls \((f(x))_{x}^{\leq \delta_f+\delta}\), and \( L'(x_n) \) annuls \((f(x))_{x}^{\leq \delta_f+\varepsilon}\), then by virtue of theorem 1 \( L'(x_n) \ast L'(x_n) = L'(x_n) \ast L'(x_n) \) in \( R[x^{\leq \delta_f+\varepsilon+1}] \). Since and \( L'(x_n) = l(x_n) \) in \( R[x^{\leq \delta_f+\delta}] \), then by virtue of 3 of theorem 3 in [6] \( l(x_n) \ast E'(x_n) = L'(x_n) \ast L'(x_n) \) in \( R[x^{\leq \delta_f+\varepsilon+1}] \). By virtue of 2 of the theorem \( l(x_n) = l(x_n) \ast E'(x_n) \). Hence, there holds \( l(x_n) = L'(x_n) \ast E'(x_n) = E'(x_n) \) in \( R[x^{\leq \delta_f+\varepsilon+1}] \).

**Proof 4.** Let \( F(x) \in R[x^{\leq \delta_f+\delta}] \), then \( \max(\delta_f, \delta_f+\delta) = \delta_f+\delta \), since \( \delta \geq 0 \). By virtue of the second statement of 1 of the theorem

\[
F(x) - E'(x_n) \ast F(x) \in (f(x))_x \cap R[x^{\leq \max(\delta_f, \delta_f+\delta)}] = (f(x))_x \cap R[x^{\leq \delta_f+\delta}_x],
\]

and therefore is annulled by \( L(x_n) \). We have, by using of the second equality of lemma 1,

\[
0 = L(x_n).F(x) - E'(x_n) \ast F(x) = L(x_n).F(x) - L(x_n) \ast E'(x_n) \ast F(x),
\]

hence, from the arbitrariness of \( F(x) \in R[x^{\leq \delta_f+\delta}] \), \( L(x_n) = L(x_n) \ast E'(x_n) \) in \( R[x^{\leq \delta_f+\delta}] \). Since \( E'(x_n) \) annuls \((f(x))_{x}^{\leq \delta_f+\varepsilon}\), \( L(x_n) \) annuls \((f(x))_{x}^{\leq \delta_f+\delta} \cap R[x^{\leq \delta_f+\delta}] \), and mean, and \((f(x))_{x}^{\leq \delta_f+\delta} \), then by virtue of theorem 1 \( E'(x_n) \ast L(x_n) = L(x_n) \ast E'(x_n) \) in \( R[x^{\leq \delta_f+\delta+\varepsilon+1}] \), and hence, and in \( R[x^{\leq \delta_f+\delta}] \).

Let \( F(x) \in (f(x))_{x}^{\leq \varepsilon} \cap R[x^{\leq \delta_f+\delta+\varepsilon+1}] \). Since \( E'(x_n) \) annuls \((f(x))_{x}^{\leq \delta_f+\varepsilon}\), then by virtue of the first statement of 1 \( E'(x_n) \ast F(x) \in (f(x))_{x}^{\leq \delta_f+\delta} \cap R[x^{\leq \delta_f+\delta}] \), and by virtue of 3 of theorem 2 in [6] \( E'(x_n) \ast F(x) \in (f(x))_{x}^{\leq \varepsilon+1} \), hence, the polynomial \( E'(x_n) \ast F(x) \in (f(x))_{x} \cap R[x^{\leq \delta_f+\delta}] \). From the last it follows that there holds the equality \( L(x_n) \ast E'(x_n).F(x) = L(x_n).E'(x_n) \ast F(x) = 0 \), since \( L(x_n) \) annuls \((f(x))_{x} \cap R[x^{\leq \delta_f+\delta}] \).

Hence, from the arbitrariness of a polynomial \( F(x) \in (f(x))_{x} \cap R[x^{\leq \delta_f+\delta+\varepsilon+1}] \), the functional \( L(x_n) \ast E'(x_n) \) annuls \((f(x))_{x} \cap R[x^{\leq \delta_f+\delta+\varepsilon+1}] \).

**Theorem 6.** Let \( x = (x_1, \ldots, x_n) \), \( y \simeq x \) be variables, \( f(x) = (f_1(x), \ldots, f_n(x)) \) be polynomials, \( \delta_f = \sum_{i=1}^{n} (\deg(f_i) - 1) \). Let \( E(x_n) \) annuls \((f(x))_{x} \) and \( E(x_n) \ast 1 - 1 \in (f(x))_{x} \), then:

1) if \( F(x) \in R[x^{\leq \delta}] \), then
\[
E(x_n) \ast F(x) \in R[x^{\leq \delta}] \]

and
\[
F(x) - E(x_n) \ast F(x) \in (f(x))_{x} \cap R[x^{\leq \max(\delta_f, \delta_f+\varepsilon)}];
\]

2) if \( l(x_n) \) annuls \((f(x))_{x} \), then
\[
l(x_n) = E(x_n) \ast l(x_n) = l(x_n) \ast E(x_n) = E(x_n) \ast l(y_n) \cdot \det \|\nabla f(x, y)\|; \]

3) if \( l(x_n) \) annuls \((f(x))_{x} \), and \( L'(x_n) = l(x_n) \) in \( R[x^{\leq \delta_f+\delta}] \), where \( \delta \geq 0 \), then
\[
l(x_n) = E(x_n) \ast L'(x_n) = L'(x_n) \ast E(x_n); \]

4) if a functional \( L(x_n) \) annuls \((f(x))_{x} \cap R[x^{\leq \delta_f+\delta}] \), where \( \delta \geq 0 \), then

\[
L(x_n) = E(x_n) \ast L(x_n) = L(x_n) \ast E(x_n) \in R[x^{\leq \delta_f+\delta}];
\]

note that \( E(x_n) \ast L(x_n) = L(x_n) \ast E(x_n) \) annuls \((f(x))_{x} \).
5) if \( l(x_*) \) annuls \( (f(x))_x \), then it is uniquely determined its the action on \( \mathbb{R}[x^{≤ δ_f + δ}] \).

**Proof.** For any \( ε ≥ 0 \) the functional \( E(x_*) \) annuls \( (f(x))^{≤ δ_f + ε} \).

**Proof 1.** From 1 of theorem 5 it follows that

\[
F(x) - E(x_*) * F(x) ∈ (f(x))_x \cap \mathbb{R}[x^{≤ \max(δ_f, δ)]}
\]

and \( ∀ ε ≥ 0 : E(x_*) * F(x) ∈ \mathbb{R}[x^{≤ \max(δ_f, δ - ε - 1)}] \), hence, \( E(x_*) * F(x) ∈ \mathbb{R}[x^{≤ δ_f}] \).

**Proof 2.** From 2 of theorem 5 it follows that

\[
l(x_*) = E(x_*) * l(x_*) = l(x_*) * E(x_*) ,
\]

and since \( E(x_*) \) and \( l(x_*) \) annul \( (f(x))_x \), then from 2 of theorem 4 it follows that

\[
l(x_*) * E(x_*) = E(x_*) * (l(y_0) * \det ∇f(x, y)) \).
\]

**Proof 3 and 5.** From 3 of theorem 5 it follows that

\[
l(x_*) = E(x_*) * L'(x_*) = L'(x_*) * E(x_*) \in \mathbb{R}[x^{≤ δ_f + δ + ε + 1}] .
\]

From the arbitrariness of \( ε ≥ 0 \) we obtain that

\[
l(x_*) = E(x_*) * L'(x_*) = L'(x_*) * E(x_*) \in \mathbb{R}[x] .
\]

Since the equality holds for any \( L'(x) \) such that \( L'(x_*) = l(x_*) \in \mathbb{R}[x^{≤ δ_f + δ}] \), then \( l(x_*) \) is uniquely determined its the action in \( \mathbb{R}[x^{≤ δ_f}] \).

**Proof 4.** The first statement it follows from the first statement of 4 of theorem 5. The second statement it follows from 1 and 4 of theorem 4, since \( L(x_*) \) annuls \( (f(x))^{≤ δ_f + δ} \) and \( E(x_*) \) annuls \( (f(x))_x \).

**Theorem 7.** Let \( x = (x_1, \ldots, x_n) \), \( y \simeq x \) be variables, \( f(x) = (f_1(x), \ldots, f_n(x)) \) be polynomials, \( δ_f = \sum_{i=1}^n (\deg(f_i) - 1) \). Let \( E'(x_*) \) annuls \( (f(x))_x \), or annuls \( (f(x))^{≤ δ_f + ε} \), where \( ε ≥ 0 \), and \( E'(x_*) * 1 - 1 \in (f(x))_x \). Then:

1) \( \mathbb{R}[x]/(f(x))_x \) coincide with the set of all elements of the form \( E'(x_*) * F(x)/(f(x))_x \), where \( F(x) ∈ \mathbb{R}[x^{≤ δ_f}] \). Moreover, \( E'(x_*) * F(x) ∈ \mathbb{R}[x^{≤ δ_f}] \).

2) \( (f(x))_x \cap \mathbb{R}[x^{≤ δ_d}] \), where \( δ_d ≥ δ_f \), coincide with the set of all elements of the form \( F(x) - E'(x_*) * F(x) \), where \( F(x) ∈ \mathbb{R}[x^{≤ δ_d}] \).

**Proof.** If the functional \( E'(x_*) \) annuls \( (f(x))_x \), then for any \( ε ≥ 0 \) it annuls \( (f(x))^{≤ δ_f + ε} \).

Therefore the statement it suffices to prove for the case, when the functional \( E'(x_*) \) annuls \( (f(x))^{≤ δ_f + ε} \) for \( ε ≥ 0 \).

Consider the sequence of polynomials \( G_0(x) \), \( ∀ p ≥ 0 : G_{p+1}(x) = E'(x_*) * G_p(x) \).

**Proof 1.** Let \( G_0(x) ∈ \mathbb{R}[x^{≤ δ_d}] \), since \( E'(x_*) \) annuls \( (f(x))^{≤ δ_f + ε} \), by virtue of the second statement of 1 of theorem 5 there hold

\[
G_0(x) - G_1(x) = G_0(x) - E'(x_*) * G_0(x) ∈ (f(x))_x ,
\]

..............................

\[
G_{p-1}(x) - G_p(x) = G_{p-1}(x) - E'(x_*) * G_{p-1}(x) ∈ (f(x))_x .
\]
Hence, \( S(x) = G_0(x) - E'(x_*) \star G_{p-1}(x) = G_0(x) - G_p(x) \in (f(x))_x \), then there holds \( G_0(x) = E'(x_*) \star G_{p-1}(x) + S(x) \). And by virtue of the first statement of 1 of theorem 5 there hold
\[
G_1(x) = E'(x_*) \star G_0(x) \in \mathbb{R}[x^{\leq \max(\delta, d-(\varepsilon+1))}],
\]
\[
G_p(x) = E'(x_*) \star G_{p-1}(x) \in \mathbb{R}[x^{\leq \max(\delta, d-p(\varepsilon+1))}].
\]

For sufficiently large \( p \) there holds \( \max(\delta_f, d - (p - 1) \cdot (\varepsilon + 1)) = \delta_f \). Hence, \( F(x) = G_{p-1}(x) \in \mathbb{R}[x^{\leq \delta_f}] \). We finally obtain, that any polynomial \( G_0(x) \) is of the form \( G_0(x) = E'(x_*) \star F(x) + S(x) \), where \( S(x) \in (f(x))_x \), \( F(x) \in \mathbb{R}[x^{\leq \delta_f}] \). Hence, any polynomial in \( \mathbb{R}[x]/(f(x))_x \) is of the form \( E'(x_*) \star F(x)/(f(x))_x \), where \( F(x) \in \mathbb{R}[x^{\leq \delta_f}] \).

Otherwise, if \( F(x) \in \mathbb{R}[x^{\leq \delta_f}] \), then \( E'(x_*) \star F(x)/(f(x))_x \in \mathbb{R}[x]/(f(x))_x \). Moreover, since \( E'(x_*) \) annuls \( (f(x))_x^{\leq \delta_f + \varepsilon} \), then by virtue of the first statement of 1 of theorem 5 there holds \( E'(x_*) \star F(x) \in \mathbb{R}[x^{\leq \max(\delta, \delta_f - \varepsilon - 1)}] = \mathbb{R}[x^{\leq \delta_f}] \).

**Proof 2.** Let \( G_0(x) \in (f(x))_x \cap \mathbb{R}[x^{\leq d'}] \), then \( G_0(x) \in (f(x))_x^{\leq d} \) for some \( d \). Since \( E'(x_*) \) annuls \( (f(x))_x^{\leq \delta_f + \varepsilon} \), by virtue of 3 of theorem 2 in [6] there hold
\[
G_1(x) = E'(x_*) \star G_0(x) \in (f(x))_x^{\leq d-(\varepsilon+1)},
\]
\[
G_p(x) = E'(x_*) \star G_{p-1}(x) \in (f(x))_x^{\leq d-p(\varepsilon+1)}.
\]

For sufficiently large \( p : d - p \cdot (\varepsilon + 1) < 0 \), hence, \( G_p(x) = 0 \). Set \( P = p \), then
\[
G_0(x) = \sum_{p=0}^{P-1} (G_p(x) - E'(x_*) \star G_p(x)) = \left( \sum_{p=0}^{P-1} G_p(x) \right) - E'(x_*) \star \left( \sum_{p=0}^{P-1} G_p(x) \right).
\]

By virtue of the first statement of 1 of theorem 5 \( \forall p \geq 0 : G_p(x) \in \mathbb{R}[x^{\leq \max(\delta, d'-p(\varepsilon+1))}] \subseteq \mathbb{R}[x^{\leq d'}] \), since \( d' \geq \delta_f \). Hence, any \( G_0(x) \in (f(x))_x \cap \mathbb{R}[x^{\leq d'}] \) is of the form \( F(x) - E'(x_*) \star F(x) \), where \( F(x) = \sum_{p=0}^{P-1} G_p(x) \in \mathbb{R}[x^{\leq d'}] \).

Otherwise, let \( F(x) \in \mathbb{R}[x^{\leq d'}] \); then by virtue of the second statement of 1 of theorem 5 the polynomial \( F(x) - E'(x_*) \star F(x) \) belongs to \( (f(x))_x \cap \mathbb{R}[x^{\leq \max(\delta, d')}] = (f(x))_x \cap \mathbb{R}[x^{\leq d'}] \), since \( d' \geq \delta_f \).

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