Nonexponential decay of Bose-Einstein condensates: a numerical study based on the complex scaling method

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Abstract We study the decay dynamics of an interacting Bose-Einstein condensate in the presence of a metastable trapping potential from which the condensate can escape via tunneling through finite barriers. The time-dependent decay process is reproduced by means of the instantaneous decay rates of the condensate at a given population of the quasi-bound state, which are calculated with the method of complex scaling. Both for the case of a double barrier potential as well as for the case of a tilted periodic potential, we find pronounced deviations from a monoexponential decay behaviour which would generally be expected in the absence of the atom-atom interaction.

1 Introduction

With the advent of optical lattices \(^1\)[1,2,3] and “atom chips” \(^4\)[4,5], it became possible to probe the transport properties of a Bose-Einstein condensate in the mesoscopic regime. The unprecedented degree of experimental control in these systems led to the observation of Bloch oscillations \(^6\)[6,7], the guided and free propagation of condensates through waveguide structures \(^8\)[8,9], the transport of condensates with “optical tweezers” \(^10\), as well as the realization of Josephson junctions \(^11\) and matter-wave interferometry \(^12\), to mention just a few examples. Those experiments typically involve rather small trapping potentials, with length scales that can be of the order of a few microns. In such geometries, decay mechanisms of the condensate become a relevant issue.

On the one hand, the condensed state is, at finite atom densities, subject to depletion, which is caused by the interaction with the thermal cloud and by three-body collisions. On the other hand, the condensate can escape from the trapping potential by tunneling through its barriers, if the chemical potential of the condensed atoms exceeds the background potential in the free space outside the trap. In that case, the self-consistent mean-field state of the condensate is no longer bound, but rather corresponds to a metastable “resonance” state, in a similar way as, e.g., doubly excited electronic states in the helium atom \(^13\).

From the theoretical point of view, various methods were used \(^14\)[14,15,16,17,18,19] to tackle the problem of how to treat “resonances”, i.e. stationary states that describe the escape of population from an open confinement potential, in the context of Bose-Einstein condensates. For linear systems, it is well known that this task is most conveniently accomplished by applying the method of “complex scaling” (or “complex rotation”) \(^20\)[20,21,22,23]. This technique essentially amounts to the complex dilations \(r \mapsto re^{i\theta}\) and \(-i\mathbf{\nabla} \mapsto -i\mathbf{\nabla}e^{-i\theta}\) of the position and momentum operators in the Hamiltonian that describes the quantum system under study. This transformation leads to a nonhermitean Hamiltonian with a complex eigenvalue spectrum the continuous part of which is rotated to the lower half of the complex energy plane. Resonances, i.e. decaying states with eigenvalues corresponding to poles of the resolvent below the real energy axis, are thereby uncovered and can be calculated using standard diagonalization techniques for complex matrices. This approach is essentially exact, in the sense that no a priori approximations are introduced in the complex dilation procedure. Positions and widths of resonances can therefore be calculated with high precision by means of the complex scaling procedure \(^22\)[22,23].

The generalization of this approach to Bose-Einstein condensates was recently accomplished in our previous studies \(^18\)[18,19], where we apply the complex scaling transformation to the nonlinear Gross-Pitaevskii equation that describes the mean-field dynamics of the condensate. In contrast to an alternative approach proposed by Moisyev and Cederbaum \(^17\), we explicitly take into account the complex nature of the wavefunction of the resonance state, which leads to a considerable complication of the problem due to the resulting nonanalyticity of the interaction term in the Gross-Pitaevskii equation. We showed in Ref. \(^15\)[15] how this complication can be tackled...
and how quasi-bound resonance states of the condensate can be calculated by means of a real-time propagation approach based on the complex-scaled Gross-Pitaevskii Hamiltonian.

In this paper, we apply this method in order to calculate specific time-dependent decay processes of the condensate. Instead of a direct numerical integration of the time-dependent Gross-Pitaevskii equation (which is rather consumptive in CPU time for the evolution time scales under consideration), we compute, with complex scaling, the decay rates of the quasi-bound state of the condensate at various values for the effective interaction strength (which would be proportional to the number of atoms that are populating this quasi-bound state at a given instance of time). Then we integrate, on the basis of this information, a simple rate equation that directly describes the decay of quasi-bound population. This approach is comparatively efficient, avoids the introduction of artificial complex potentials at the grid boundaries in order to absorb the outgoing population (see, e.g., Ref. [14]), and provides physical insight that could be used to control the decay process in a similar way as for macroscopic tunneling of condensates in double well potentials (e.g., [24,25]).

The paper is organized as follows: In Section 2 we establish the general relation between resonance states of the stationary Gross-Pitaevskii equation and the actual time-dependent decay process of the condensate. We furthermore discuss how such resonance states can be calculated by the method of complex scaling, as was described in more detail in Ref. [18]. Section 3 contains the numerical results that we obtain for two paradigmatic examples of metastable confinement configurations: a harmonic trapping potential with Gaussian envelopes, and a tilted periodic lattice. We calculate the time-dependent decay of the condensate, which is, in both cases, characterized by a pronounced nonexponential nature.

2 The nonlinear complex scaling approach

We consider a Bose-Einstein condensate that is confined within a cylindrical matter-wave guide with transverse frequency $\omega_\perp$ and evolves in the presence of a longitudinal potential $V(x)$. In the “1D mean-field regime” [20] (where the confinement is strong enough to inhibit transverse excitations within the waveguide, but not as strong as to enter the Tonks-Girardeau regime [27]), the dynamics of the condensate is described by the one-dimensional time-dependent Gross-Pitaevskii equation

$$i \frac{\partial}{\partial t} \psi(x, t) = \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + g_0 |\psi(x, t)|^2 \right) \psi(x, t)$$

(1)

where $x$ denotes the coordinate along the waveguide. The longitudinal potential $V(x)$ is assumed to provide a local harmonic confinement with trapping frequency $\omega_\parallel$, from which the condensate can escape via tunneling through finite barriers. Dimensionless variables, defined by setting $\hbar = m = \omega_\parallel = 1$, are used throughout this paper. This means that we express all length scales (including the coordinate $x$) in units of $a_\parallel = \sqrt{\hbar/(m\omega_\parallel)}$, all energy scales in units of $\hbar\omega_\parallel$, and all time scales in units of $\omega_\parallel^{-1}$.

The effective one-dimensional interaction strength is, in these units, given by $g_0 = 2a_\parallel \omega_\perp$, where $a_\parallel$ denotes the s-wave scattering length of the atoms [27].

For the description of our theoretical approach, we specifically focus in the following on the double barrier potential

$$V(x) = \frac{1}{2} x^2 \exp(-\alpha x^2)$$

(2)

with $\alpha = 0.1$, which could be experimentally realized, e.g., with red- and blue-detuned laser beams that are tightly focused onto the waveguide. Obviously, $V(x)$ does not exhibit any bound state, and the eigenspectrum of the linear (noninteracting) Hamilton operator is fully continuous. There exist, however, quasi-bound states which are localized in the well around $x = 0$ and which give rise to resonances in the energy spectrum (corresponding to complex poles of the scattering matrix). In the case of noninteracting atoms ($g_0 = 0$), such resonance states are described by time-dependent wavefunctions of the form $\psi(x, t) = \psi(x) \exp(-iEt)$, where $\psi(x)$ satisfies the stationary Schrödinger equation for the complex eigenvalue $E = \mu - i\Gamma/2$ and exhibits outgoing (Siegert) boundary conditions $\psi(x) \rightarrow \psi_0 \exp(ik|x|)$, with $\text{Re}(k) > 0$ for $x \rightarrow \pm \infty$. This latter property expresses the fact that the wavefunction of the decaying state is characterized by a finite current of atoms that propagate away from the well.

As a consequence, the atomic population inside the well decays exponentially according to $\propto \exp(-\Gamma t)$ if the system is initially prepared in the energetically lowest resonance state. It is quite obvious that this is no longer true in the nonlinear case of interacting atoms ($g_0 \neq 0$). There, the tunnel coupling through the barriers explicitly depends, via the nonlinear term in the Gross-Pitaevskii equation, on the local atomic density, which in turn induces a temporal variation of the decay rate $\Gamma$. We therefore naturally obtain, as was also pointed out in Ref. [29], a nonexponential decay of the atomic density, the reproduction of which is the central aim of this paper.

Despite this complication, a description of the decay process of the interacting condensate in terms of instantaneous quasi-bound states can nevertheless be justified if the rate $\Gamma$ characterizing the temporal variation of the density inside the well is rather small compared to the chemical potential (which should generally be the case if the condensate escapes via tunneling through finite barriers). In such a quasi-stationary situation, we can employ an adiabatic ansatz where the condensate is assumed to remain always in the energetically lowest (and most stable) resonance state associated with a given instantaneous density $|\psi(x, t)|^2$. This resonance state is
formally defined, together with its associated complex eigenvalue \( E_g = \mu_g - i\Gamma_g/2 \), by the self-consistent solution of the nonlinear stationary equation

\[
H(\psi_g)\psi_g(x) = E_g\psi_g(x)
\]  
(3)

with

\[
H(\psi) \equiv -\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x) + g|\psi(x)|^2.
\]  
(4)

\( \psi_g(x) \) is normalized according to the condition

\[
N[\psi_g] \equiv \int_{-\infty}^{\infty} |\psi_g(x)|^2 w(x)dx = 1.
\]  
(5)

Here \( w(x) \) represents a weight function that measures the population inside the well. For the double barrier potential \( (2) \), a natural choice for this weight function would be \( w(x) = \theta(a - x)\theta(x + a) \) where \( a \equiv 1/\sqrt{\alpha} \) corresponds to the maximum of the potential and \( \theta(x) \) denotes the Heavyside step function.

In addition, \( \psi_g(x) \) should also satisfy outgoing boundary conditions, which would imply an asymptotic behaviour of the form \( \psi_g(x) \propto \exp[\int x k(x')dx'] \) for large \( x \to \infty \) (and a similar one for \( x \to -\infty \)) where the spatial dependence of the effective wavenumber \( k(x) \) accounts for the smooth variation of the self-consistent potential in Eq. (4). It was pointed out in Ref. [18] that this condition can only be fulfilled in an approximate way up to a given maximum spatial distance \( x_c \), at which the interaction energy \( g|\psi_g(x)|^2 \) starts to exceed the chemical potential \( \mu \). For \( x > x_c \), the self-consistent quasi-bound state would formally encounter a singularity, which reflects the fact that explicit time-dependence is expected beyond that critical distance.

On the basis of these resonance states, we can now formulate the adiabatic ansatz for \( \psi(x,t) \) as

\[
\psi(x,t) = \sqrt{N(t)}\psi_g(t)(x) \exp\left(-i\int_0^t E_g(t')dt'\right).
\]  
(6)

Here, the effective time-dependent interaction strength is given by \( g(t) \equiv g_0N(t) \) where \( N(t) \) denotes the time-dependent population inside the well, defined by

\[
N(t) \equiv \int_{-\infty}^{\infty} |\psi(x,t)|^2 w(x)dx.
\]  
(7)

Using the normalization condition [5] of the resonance state and taking into account the fact that its eigenvalue \( E_g = \mu_g - i\Gamma_g/2 \) is complex, one can straightforwardly derive the implicit expression

\[
N(t) = N_0 \exp\left(-\int_0^t \Gamma_g(t')dt'\right).
\]  
(8)

for the quasi-bound population of the condensate, which can also be formulated in terms of the ordinary differential equation

\[
\frac{dN}{dt} = -\Gamma_g(t)N(t),
\]  
(9)

with the initial condition \( N(0) = N_0 \). The time-dependent decay process of the condensate can therefore be entirely reproduced with comparatively little numerical effort if the decay rates \( \Gamma_g(t) \) of the instantaneous quasi-bound states \( \psi_g(t)(x) \) are known.

As was described in detail in Ref. [18], the calculation of the decay rates can be achieved by the method of complex scaling. This technique essentially amounts to the application of the nonunitary mapping

\[
\psi(x) \mapsto \psi^{(\theta)}(x) \equiv R_\theta\psi(x) = e^{i\theta/2}\psi(xe^{i\theta})
\]  
(10)

to the wavefunction \( \psi(x) \), which corresponds to the complex dilation \( x \mapsto xe^{i\theta} \) of the position operator. Applying this transformation to the linear stationary Schrödinger equation \( H_0\psi = E\psi \) with \( H_0 \) being defined through Eq. (4) via \( H_0 \equiv H(\psi = 0) \) yields the complex stationary equation

\[
H_0^{(\theta)}\psi^{(\theta)}(x) = E\psi^{(\theta)}(x)
\]  
(11)

with the complex scaled Hamiltonian

\[
H_0^{(\theta)} \equiv R_\theta H_0 R_\theta^{-1} = -\frac{1}{2}\exp(-2i\theta)\frac{\partial^2}{\partial x^2} + V(xe^{i\theta}).
\]  
(12)

The spectral properties of this nonhermitean Hamiltonian are widely discussed in the literature on complex scaling [20,21,22,23]. While bound states of the original Hamiltonian \( H_0 \) (which are absent in our particular case) remain bound after the complex dilation (as long as \( \theta < \pi/4 \)), the continuum states are “rotated” in the complex energy plane, in such a way that their eigenvalues are located along the axis \( E = \epsilon e^{-2i\theta} \) with real positive \( \epsilon \). This rotation uncovers the spectral resonances of the system, which correspond to the poles of the analytical continuation of the Green function \( G = (E - H_0 + i\delta)^{-1} \) to the lower half part of the complex energy plane. Those resonances turn into discrete complex eigenvalues \( E_n = \mu_n - i\Gamma_n/2 \) under complex dilation, and are represented by normalizable eigenfunctions that can be straightforwardly calculated by diagonalizing \( H_0^{(\theta)} \) in any numerical basis.

The generalization of this approach to the nonlinear case would be comparatively straightforward if the replacement \(|\psi(x)|^2 \rightarrow |\psi(x)|^2 \) in the nonlinear Gross-Pitaevskii Hamiltonian \( \hat{H} \) could be justified. For this particular case, the implementation of the complex scaling technique was explained in detail in Ref. [17]. In reality, however, the wavefunction of the resonance state is intrinsically complex due to the outgoing boundary conditions (i.e., \( \psi(x) \propto \exp[ik|x|] \) for \( |x| \to \infty \)), and the resulting nonanalyticity in the Hamiltonian \( \hat{H} \) introduces a major complication of the problem. Formally, a second analytic wavefunction \( \overline{\psi} \) needs to be introduced, which coincides with the complex conjugate of \( \psi \) on the real axis, i.e.

\[
\overline{\psi}(x) \equiv \psi^*(x) \quad \text{for real } x,
\]  
(13)
and which is independently transformed under the nonunitary dilation operator $R_\theta$, i.e.

$$\overline{\psi}(x) \rightarrow \overline{\psi}^{(\theta)}(x) \equiv R_\theta \overline{\psi}(x) = e^{i\theta/2} \overline{\psi}(xe^{i\theta}).$$ (14)

The analytic continuation of the stationary Gross-Pitaevskii equation to the complex domain yields then

$$H^{(\theta)}(\psi)\psi^{(\theta)}(x) = E\psi^{(\theta)}(x)$$ (15)

where the complex scaled nonlinear Hamiltonian is given by

$$H^{(\theta)}(\psi) = H_0^{(\theta)} + ge^{-i\phi}\overline{\psi}(x)\psi^{(\theta)}(x).$$ (16)

The lowest resonance state of the condensate can be calculated by a real-time propagation approach [18], i.e. by numerically propagating $\psi^{(\theta)}$ under the time-dependent Gross-Pitaevskii equation

$$\frac{i}{\partial \tau} \psi^{(\theta)}(x) = H^{(\theta)}(\psi_x)\psi^{(\theta)}(x)$$ (17)

in the complex scaled system, where $\tau$ represents a fictitious numerical “time” parameter (which is unrelated to the physical time evolution in the actual decay process). In practice, $\psi^{(\theta)}_x(x)$ is expanded on a spatial grid, and an implicit finite-difference scheme is employed to carry out the mapping $\psi^{(\theta)}_x \rightarrow \psi^{(\theta)}_{x+\delta x}$ for small time steps $\delta \tau$. If $\psi^{(\theta)}_x$ is renormalized after each propagation step in order to satisfy the condition [5], the integration of Eq. (17) necessarily converges to the most stable resonance state of the complex scaled Hamiltonian, which corresponds to the quasi-bound state with the smallest decay rate [30].

The major numerical difficulty in this approach lies in the evaluation of the nonlinear term in the complex scaled Hamiltonian [10]. Indeed, $\overline{\psi}^{(\theta)}(x)$ is not identical to $[\psi^{(\theta)}(x)]^*$, the complex conjugate of $\psi^{(\theta)}(x)$, and needs to be evaluated according to the relation

$$\overline{\psi}^{(\theta)}(x) = R_\theta \left( R_{-\theta} \overline{\psi}^{(\theta)}(x) \right)$$ (18)

which requires explicit back- and forward-rotations of the complex scaled condensate wavefunction. In practice, these rotations (which are also used to evaluate the normalization condition [5]) are numerically performed by mapping the grid representation of the wavefunction into a nonorthogonal set of analytic Gaussian orbitals $\phi_\nu(x)$ that are centered around different positions along the grid, and by using a transformation matrix that contains the overlap integrals $\int \phi_\nu(x)e^{i\theta}\phi_\nu(x)dx$ as elements (see Ref. [18] for more details). Such an operation, however, is known to be potentially unstable [31] and requires great care in the numerical implementation. It is therefore not obvious to which extent unlimited precision in the decay rates of the quasi-bound states can be achieved within this nonlinear complex scaling approach.

3 Calculation of time-dependent decay processes

3.1 Double barrier potential

Despite this latter complication, we find that the chemical potentials and decay rates of the self-consistent quasi-bound states of the double barrier potential [2] can be calculated in this way with rather good accuracy, even in case of very strong nonlinearities where the resonance level lies close the barrier height of $V(x)$. This was explicitly verified in Ref. [18] by comparing the resulting values for $\mu_\gamma$ and $\Gamma_\gamma$ with the ones that are obtained from an alternative approach, which was based on the real-time propagation of the original (i.e., unscaled) Gross-Pitaevskii equation in the presence of absorbing boundaries. Good agreement was generally found between the two approaches [18].

The chemical potentials and decay rates of the lowest resonance state are plotted in Fig. 1 as a function of the effective interaction strength $g$. Quite intuitively $\mu_\gamma$ increases with increasing $g$ due to the presence of the mean-field interaction energy in the nonlinear Gross-Pitaevskii Hamiltonian. This increase of the chemical potential results in a dramatic enhancement of the decay rate $\Gamma_\gamma$, which can be explained by the fact that the effective imaginary action integral that semiclassically determines the tunneling rate through the barriers is appreciably reduced with increasing energy. It was pointed out in Refs. [14][16][18] that an attractive interaction between the atoms leads to a stabilization of the resonance state, i.e., to a reduction of the chemical potential to values below $\mu = 0$ where the associated decay rate would vanish. For the double barrier potential under consideration, this stabilization process would occur at $g \approx -1.1$.

With this information, we can now quantitatively reproduce the time-dependent decay process of the condensate by means of the integration of the rate equation [3]. For this purpose, we use the values of the decay rates that are calculated with the complex scaling method at the equidistant interaction strengths $g = 0, 0.1, 0.2 \ldots$, and employ a cubic interpolation to obtain intermediate values of $\Gamma$. As initial value of the effective interaction strength, we consider $g(t = 0) = 4$, where the chemical potential of the quasi-bound state lies already rather close to the barrier height of the potential. In the specific case of a condensate of $^{87}$Rb atoms that encounters the longitudinal and transverse confinement frequencies $\omega_\parallel = \omega_\perp = 2\pi \times 10^3$ Hz, this would imply that about $N(0) = 4/g_0 \approx 100$ atoms are initially localized in the single well [32].

The result of the integration is displayed in the two lower panels of Fig. 1. We see a clearly nonexponential decay of the bound population, which reflects the fact that the decay rate decreases with decreasing interaction strength $g$. As a consequence, a rather large number of atoms leave the trap during the first 1000 units
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3.2 Tilted periodic potential

Nonexponential features can also be observed in the presence of relatively small interaction strengths, namely if the confinement potential permits the possibility of resonant tunneling. This is, for instance, the case for the tilted periodic potential

\[ V(x) = \sin^2(x/2) + Fx \]

that is experimentally realized with optical lattices [12] or, within the atom chip context, by means of periodic sequences of microfabricated wires [33]. In this nonlinear Wannier-Stark system, the possibility of resonant tunneling arises if the local ground state of one of the wells is nearly degenerate with the first excited state of the adjacent well. As was pointed out in Ref. [29], this near-degeneracy would give rise to a significant enhancement of the condensate’s decay rate.

As in our previous study [19], we assume that the condensate is initially confined within one single well of the lattice. The method of complex scaling can again be used to calculate the chemical potential and decay rate of the self-consistent quasi-bound state in the presence of the interaction, even though the tilted potential [19] leads to an asymptotic spatial behaviour of the continuum states that is substantially different from the previous double-barrier problem. As was described in detail in Ref. [19], additional complications arise in this potential (such as the existence of many different self-consistent resonance states with identical decay rates) and technical modifications need to be implemented in order to achieve good convergence of the real-time propagation method. In analogy with the double barrier potential, the weight function that characterizes the bound population inside the well according to Eq. (5) is given by \( w(x) = \theta(x) - \theta(x - \pi) \).

The resonance-enhanced decay process of the condensate in this tilted lattice is displayed in Fig. 2, for the tilt strength \( F = 0.1412 \) at which the level of the noninteracting local ground state in each well lies slightly above the level of the first excited state in the adjacent well on the left-hand side. As initial value for the nonlinearity, we consider \( g(t = 0) = 0.25 \), which would correspond to an attractive interaction between the atoms (which could be realized, e.g., by using condensates with \(^7\)Li atoms or by applying Feshbach tuning techniques [34]). This attractive nonlinearity lowers the chemical potential of the self-consistent quasi-bound state, in such a way that it becomes shifted below the level of the first excited (noninteracting) state in the adjacent well. During the time evolution, the loss of population leads to an increase of the chemical potential, which, at about \( t \approx 2500 \) time units, approaches the resonance. At that point, the decay rate becomes drastically increased resulting in a significant escape of atoms from the well. As a consequence, the chemical potential of the quasi-bound state quickly moves out of resonance, and the decay rate becomes again reduced.

The time evolution of the condensate was again calculated by integrating the rate equation (9), using decay rates that were calculated with the complex scaling method at the equidistant values \( g = 0, -0.0025, -0.005, \ldots \) of the interaction strength. The accuracy with
which these decay rates could be calculated was not as good as in the case of the double barrier potential \[19\] which is clearly reflected by the appearance of wiggles in the time evolution of which arise from numerical inaccuracies, do not leave significant traces in the time evolution of \(P(t)\).

This intermediate “burst” of atoms should be readily observable within existing experimental setups based on optical lattices or atom chips. By imposing a rather weak transverse confinement, the one-dimensional interaction strength (given by \(g(0) = 2a_{\hbar} \sqrt{\Gamma} \) \[27\]) can be appreciably reduced, what should allow one to suppress effects beyond the mean-field description of the condensate, and additional longitudinal potentials can, as in the experiment on tunneling well potential \[35\], be employed to prepare the condensate in one single well of the lattice. Obviously, the attractive interaction between the atoms is not a necessary condition for this nonexponential decay phenomenon: Indeed, the same effect could be induced with a repulsively interacting species at slightly weaker tilt strengths \(F\), where the noninteracting ground state of the well lies slightly below the first excited level of the adjacent well. In both cases of attractive and repulsive interaction, the intermediate enhancement of the decay rate should clearly manifest in the time-of-flight image of the condensate after the decay process, which would display a pronounced peak due to the effect of resonant tunneling.

4 Conclusion

In summary, we studied the time-dependent decay of Bose-Einstein condensates in mesoscopic trapping potentials that permit escape by tunneling through finite barriers. The decay process of the condensate was reproduced by integrating a simple rate equation for the quasi-bound population, using instantaneous decay rates that were computed by means of the method of complex scaling. This approach is rather efficient as compared to a direct numerical integration of the time-dependent Gross-Pitaevskii equation, and provides additional insight into the mechanisms that underly the decay of the condensate. Though only applied for one-dimensional configurations, the complex scaling approach can be straightforwardly generalized to three-dimensional decay problems, and might furthermore represent a convenient conceptual framework for treating resonances of the nonlinear Gross-Pitaevskii equation from the mathematical point of view.

With this approach, we calculated the decay of a Bose-Einstein condensate in a double barrier potential \[14,16,18\] as well as in a tilted periodic potential \[29,19\]. For this latter case, we find a strong intermediate enhancement of the tunneling rate, which arises due to a near-degeneracy with a quasi-bound state in another well of the periodic potential. This enhancement leads to a pronounced deviation from an exponential behaviour of the condensate’s escape dynamics, which could be controlled by suitable time-dependent variations of the tilt field, in a similar way as for macroscopic tunneling in double well potentials \[24,25\] and for nonlinear resonant transport through atomic quantum dots \[36\]. Such nonexponential effects should be readily observable in present-day state-of-the-art experiments on interacting matter waves in mesoscopic trapping potentials \[3,11,12,33,35,37\].

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Fig. 2 Decay of a Bose-Einstein condensate in the tilted periodic potential \[19\] with \(F = 0.1412\). The atomic cloud is assumed to be entirely localized within a single well of the lattice, and decays via tunneling through the barrier on the left-hand side of the well. This decay process can be substantially accelerated in the case of resonant tunneling, i.e., if the chemical potential of the condensate matches the energy of the first excited (noninteracting) state of the adjacent well (dashed horizontal lines; the thin solid lines mark the noninteracting ground levels of the wells). In contrast to the case of the double barrier potential, we consider here a relatively weak and attractive initial interaction strength, \(g(0) = -0.25\), which leads to an initial chemical potential that lies slightly below the level of this excited state (thick horizontal line in the upper panel). The time evolution of the quasi-bound population \(P(t) = N(t)/N(0)\), shown in the lower panels, clearly displays the characteristic signature of an intermediate resonant tunneling process. Note that the wiggles in the decay rates at large \(|g|\) (middle right panel), which arise from numerical inaccuracies, do not leave significant traces in the time evolution of \(P(t)\).
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