THE REARRANGEMENT-INVARIANT SPACE $\Gamma_{p,\phi}$

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Abstract. Fix $b \in (0, \infty)$ and $p \in (1, \infty)$. Let $\phi$ be a positive measurable function on $I_b := (0, b)$. Define the Lorentz Gamma norm, $\rho_{p,\phi}$, at the measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\rho_{p,\phi}(f) := \left[ \int_0^b f^{**}(t)^p \phi(t) dt \right]^{\frac{1}{p}},$$

in which $f^{**}(t) := t^{-1} \int_0^t f^*(s) ds$, where $f^*(t) := \mu_f^{-1}(t)$, with $\mu_f(s) := \{ x \in I_b : |f(x)| > s \}$.

Our aim in this paper is to study the rearrangement-invariant space determined by $\rho_{p,\phi}$. In particular, we determine its Kőthe dual and its Boyd indices. Using the latter a sufficient condition is given for a Caldéron-Zygmund operator to map such a space into itself.

1. Introduction

Let $(X, \mu)$ be a $\sigma$-finite measure space with $\mu(X) = b$ and denote by $\mathcal{M}(X)$ the set of $\mu$-measurable real-valued functions on $X$. This paper is concerned with the properties of certain rearrangement invariant spaces of functions in $\mathcal{M}(X)$. The norm of such a space is defined in terms of an index $p$, $1 < p < \infty$, and a positive locally integrable (weight) function $\phi$ on $I_b := (0, b)$ by

$$\rho_{p,\phi}(f) := \left[ \int_0^b f^{**}(t)^p \phi(t) dt \right]^{\frac{1}{p}}, \quad f \in \mathcal{M}(X).$$

Here,

$$f^{**}(t) := t^{-1} \int_0^t f^*(s) ds, \quad t \in I_b,$$

in which the decreasing rearrangement, $f^*$, is the inverse (in a generalized sense) of the distribution function, $\mu_f$, of $f$, where

$$\mu_f(\lambda) := \mu(\{ x \in X : |f(x)| > \lambda \}), \quad \lambda > 0.$$ 

We require

$$\int_1^\infty \phi(t)t^{-p}dt < \infty, \quad \text{if } b = \infty, \text{ and } \int_{I_b} \phi(t)t^{-p}dt = \infty, \quad \text{for all } b \in \mathbb{R}_+;$$

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otherwise, the space
\[ \Gamma_{p,\phi} = \Gamma_{p,\phi}(X) := \{ f \in \mathcal{M}(X) : \rho_{p,\phi}(f) < \infty \} \]
would, in the first case, consist only of the zero function and, in the second case, would be equal to the space \( L_1(X) \) of \( \mu \)-integrable functions on \( X \). Such weights \( \phi \) will be called non-trivial.

The spaces \( \Gamma_{p,\phi} \) are examples of rearrangement-invariant (r.i) Banach function spaces, which are defined by norms \( \rho \) whose characteristic property is that \( \rho(f) = \rho(g) \) whenever \( f, g \in \mathcal{M}(X) \) are equimeasurable in the sense that \( f^* = g^* \).

A key thing to know about a Banach function norm, \( \rho \), such as (1.1), is its associate norm, \( \rho' \), defined at \( g \in \mathcal{M}(X) \) by
\[
\rho'(g) = \sup_{f \in \mathcal{M}(X), \rho(f) \leq 1} \int_X |fg|d\mu.
\]
We will show that, when \( \Gamma_{p,\phi}(X) \not\supset L_\infty(X) \), or, equivalently, \( \int_{I_b} \phi(s)ds = \infty \), one has
\[
\rho'_{p,\phi}(g) \approx \rho_{p',\psi}(g), \quad g \in \mathcal{M}(X),
\]
where \( p' = \frac{p}{p-1} \) and \( \psi \) is a certain (dual) weight.

We motivate the choice of \( \psi \), in an appendix to the paper. For now, we just state our main result, namely,

**Theorem A** Let \( (X,\mu) \) be a \( \sigma \)-finite measure space with \( \mu(X) = b \). Fix \( p, 1 < p < \infty \), and suppose \( \phi \) is a non-trivial weight function on \( I_b \). Then,
\[
\rho'_{p,\phi}(g) \approx \rho_{p',\psi}(g) + \left[ \frac{\int_X |g|}{\int_{I_b} \phi} \right]^{p'}, \quad g \in \mathcal{M}(X),
\]
in which
\[
\psi(t) := \frac{t^{p+p'-1} \int_0^t \phi \int_t^b \phi(s)s^{-p}ds}{\left[ \int_0^t \phi + t^p \int_t^b \phi(s)s^{-p}ds \right]^{p'+1}}, \quad t \in I_b, \quad p' = \frac{p}{p-1}.
\]

A proof of this theorem has been given by the first author and L. Pick in [3] using so-called discretization methods. Our aim here is to give a new proof using more familiar techniques. Alternative descriptions of the function space dual to \( \Gamma_{p,\phi} \) can be found in [4] and [8].

The Boyd indices of an r.i. norm are essential to describing the action of such operators as those of Calderón-Zygmund on the space \( L_\rho(\mathbb{R}^n) \). These indices are defined in terms of the norm, \( h_\rho(s) \), of the dilation operator. Their calculation when \( \rho = \rho_{p,\phi} \) and \( \mu(X) = \infty \) is greatly simplified by the result in
Theorem B Fix an index $p, 1 < p < \infty$ and let $\phi$ be a non-trivial weight on $\mathbb{R}_+$. Take $\rho = \rho_{p, \phi}$ and at $s \in \mathbb{R}_+$ set
\[ h_\rho(s) := \sup \frac{\rho(f(t))}{\rho(f)} = \sup \frac{\rho(f^*(t))}{\rho(f^*)}, \quad 0 \neq f \in \mathcal{M}_+(\mathbb{R}_+). \]
Then,
\[ h_\rho(s) \approx \sup_{t \in \mathbb{R}_+} \left[ \int_0^s \phi(y)dy + s^{p_1p} \int_t^b \phi(y)y^{-p}dy \right]^{\frac{1}{p}}. \]

2. rearrangement-invariant spaces

Let $(X, \mu)$ be a $\sigma$-finite measure space with $\mu(X) = b$ and denote by $\mathcal{M}(X)$ the set of $\mu$-measurable real-valued functions on $X$ and by $\mathcal{M}_+(X)$ the nonnegative functions in $\mathcal{M}(X)$. A Banach function norm is a functional $\rho : \mathcal{M}_+(X) \to \mathbb{R}_+$ satisfying

(A1) $\rho(f) = 0$ if and only if $f = 0$ $\mu$-a.e.,
(A2) $\rho(cf) = c\rho(f), \ c \geq 0$,
(A3) $\rho(f + g) \leq \rho(f) + \rho(g)$,
(A4) $0 \leq f_n \uparrow f$ implies $\rho(f_n) \uparrow \rho(f)$,
(A5) $|E| < \infty$ implies $\rho(\chi_E) < \infty$,
(A6) $|E| < \infty$ implies $\int_E f d\mu \leq c_E(\rho) \rho(f)$, for some constant $c_E(\rho)$ depending on $E$ and $\rho$ but not on $f \in \mathcal{M}_+(X)$.

Furthermore, as mentioned in the introduction, a Banach function norm is said to be rearrangement invariant if $\rho(f) = \rho(g)$ whenever $f, g \in \mathcal{M}_+(X)$ are equimeasurable in the sense that $f^* = g^*$. The decreasing rearrangement, $f^*$, of $f \in \mathcal{M}(X)$ on $\mathbb{R}_+$ is defined as
\[ f^*(t) := \inf\{\lambda > 0 : \mu(\{x \in X : |f(x)| > \lambda\}) \leq t\}, \quad t \in I_b. \]
It satisfies the property that
\[ |\{t \in I_b : f^*(t) > \tau\}| = \mu(\{x \in X : |f(x)| > \tau\}), \quad f \in \mathcal{M}(X), \ \tau \in \mathbb{R}_+. \]

Now, although the mapping $f \mapsto f^*$ is not subadditive, the mapping $f \mapsto t^{-1} \int_0^t f^*(s)ds$ is, namely,
\[ t^{-1} \int_0^t (f + g)^*(s)ds \leq t^{-1} \int_0^t f^*(s)ds + t^{-1} \int_0^t g^*(s)ds, \quad \text{for all } f, g \in \mathcal{M}(X), \ t \in I_b. \]

The Kothe dual of a Banach function norm $\rho$ is another such norm, $\rho'$, with
\[ \rho'(g) := \sup_{\rho(f) \leq 1} \int_X fg\mu, \quad f, g \in \mathcal{M}_+(X). \]
It is obeys the Principle of Duality; that is,
\[ \rho'' := (\rho')' = \rho. \]

The space \( L_\rho(X) \) is the vector space
\[ \{ f \in \mathcal{M}(X) : \rho(|f|) < \infty \}, \]
together with the norm
\[ \|f\|_{L_\rho} := \rho(|f|). \]
This Banach space is said to be an r.i. space provided \( \rho \) is an r.i. function norm.

The norm, \( \rho_{p,\phi} \), defined in (1.1) in terms of an index \( p, 1 < p < \infty \), and a positive locally integrable (weight) function \( \phi \) on \( I_b \) is an r.i. norm;

If \( \rho \) is an r.i. function norm, then,
\[ (2.3) \quad \rho(\chi_{(0,t)}) = \frac{t}{\rho'(\chi_{(0,t)})}, \quad t \in I_b \]

The dilation operator, \( E_s \), \( s \in \mathbb{R}_+ \), given at \( f \in \mathcal{M}(\mathbb{R}_+) \), \( t \in \mathbb{R}_+ \), by
\[ (E_s f)(t) := f(st), \]
is bounded on any r.i. space \( L_\rho(\mathbb{R}_+) \) and the operator norm of \( E_{1/s} \) on \( L_\rho(\mathbb{R}_+) \) is denoted by \( h_\rho(s) \). The norm is determined on the non-negative decreasing functions in \( L_\rho(\mathbb{R}_+) \).

We define the lower and upper Boyd indices of \( L_\rho(\mathbb{R}_+) \) as
\[ i_\rho := \sup_{0 < s < 1} \frac{\log h_\rho(t)}{\log s} \quad \text{and} \quad I_\rho := \inf_{1 < t < \infty} \frac{\log h_\rho(t)}{\log s} \]

The operator norm of \( E_{1/s} \) on characteristic functions of the form \( \chi_{(0,a)} \), \( a \in \mathbb{R}_+ \), is denoted by \( M_\rho(s) \); thus,
\[ M_\rho(s) = \sup_{0 < a < \infty} \frac{\rho(\chi_{(0,as)})}{\rho(\chi_{(0,a)})}. \]
The so-called fundamental indices of \( \rho \) are defined in terms of \( M_\rho \) as
\[ i_\rho := \sup_{0 < s < 1} \frac{\log M_\rho(s)}{\log s} \quad \text{and} \quad I_\rho := \inf_{1 < s < \infty} \frac{\log M_\rho(s)}{\log s}. \]
Clearly,
\[ 0 \leq i_\rho \leq i_\rho \leq L_\rho \leq I_\rho \leq 1. \]

3. Weighted spaces

Fix \( b > 0 \) and let \( w \in \mathcal{M}_+(I_b) \), \( w > 0 \) a.e.. Given \( p, 1 < p < \infty \), the weighted Lebesgue space, \( L_p(w) \), is defined by the norm
\[ \left[ \int_0^b |f(t)|^p w(t) dt \right]^\frac{1}{p}, \quad f \in \mathcal{M}(I_b). \]
One readily shows that the Banach dual of $L^p(w)$ is the space $L^{p'}(w^{1-p'})$, $p' = \frac{p}{p-1}$, namely, the weighted Lebesgue space with norm
\[
\left[ \int_0^b |g(t)|^{p'} w(t)^{1-p'} dt \right]^\frac{1}{p'}, \quad g \in \mathcal{M}(I_b).
\]

In this section we consider the action of certain positive integral operators on such spaces. This action is expressed in terms of so-called weighted norm inequalities. The most basic ones involve the Hardy averaging operator and its dual, that is,
\[
(Pf)(t) := t^{-1} \int_0^t f(s)ds \quad \text{and} \quad (Qf)(t) := \int_t^b f(s) \frac{ds}{s}, \quad f \in \mathcal{M}_+(I_b), \quad t \in I_b.
\]

**Theorem 3.1** ([6]). Fix $b > 0$ and let $u$ and $v$ be weights on $I_b$. Then, for $1 < p \leq q < \infty$ one has the least constant $C > 0$ in the inequality
\[
\left( \int_0^b (u(t)(Pf)(t))^q dt \right)^{\frac{1}{q}} \leq C \left( \int_0^b (v(t)f(t))^p dt \right)^{\frac{1}{p}}, \quad f \in \mathcal{M}_+(I_b),
\]
equivalent to
\[
\sup_{0 < r < b} \left( \int_r^b \left( \frac{u(t)}{t} \right)^q dt \right)^{\frac{1}{q}} \left( \int_r^b v(t)^{-p'} dt \right)^{\frac{1}{p'}},
\]
and the least constant $C > 0$ in the inequality
\[
\left( \int_0^b (u(t)(Qf)(t))^q dt \right)^{\frac{1}{q}} \leq C \left( \int_0^b (v(t)f(t))^p dt \right)^{\frac{1}{p}}, \quad f \in \mathcal{M}_+(I_b),
\]
equivalent to
\[
\sup_{0 < r < b} \left( \int_r^b u(t)^q dt \right)^{\frac{1}{q}} \left( \int_r^b (tv(t))^{-p'} dt \right)^{\frac{1}{p'}}.
\]

An operator essentially built from $P$ and $Q$ when $b = \infty$ is the Stieltjes operator
\[
(Sf)(t) := \int_0^\infty \frac{f(s)}{s+ts} ds, \quad f \in \mathcal{M}_+(I_b).
\]
Clearly, for $f \in \mathcal{M}_+(\mathbb{R}_+)$, $t \in \mathbb{R}_+$,
\[
\frac{1}{2} [(Pf)(t) + (Qf)(t)] \leq (Sf)(t) \leq [(Pf)(t) + (Qf)(t)].
\]

The following results are given in Andersen [1] for $1 < p \leq q < \infty$ and in Sinnamon [7] for $1 < q < p < \infty$.

**Theorem 3.2.** Let $u$ and $v$ be weights on $\mathbb{R}_+$. Then, in the inequality
\[
\left( \int_0^\infty (Sf)(t)^q u(t) dt \right)^{\frac{1}{q}} \leq K \left( \int_0^\infty f(t)^p v(t) dt \right)^{\frac{1}{p}}, \quad f \in \mathcal{M}_+(\mathbb{R}_+),
\]
(3.5) \[
\sup_{t>0} \left( \int_0^\infty \left( \frac{t}{s+t} \right)^q u(s) \, ds \right)^\frac{1}{q} \left( \int_0^\infty \frac{v(t)^{1-p'}}{(s+t)^{p'}} \, ds \right)^\frac{1}{p'},
\]
when \( 1 < p \leq q < \infty \), and to
\[
\left[ \int_0^\infty \left[ \int_0^\infty \left( \frac{t}{s+t} \right)^q u(s) \, ds \right]^\frac{1}{p} \left[ \int_0^\infty \frac{v(t)^{1-p'}}{(s+t)^{p'}} \, ds \right]^\frac{1}{p'} \right]^\frac{pq}{p-q} u(t) \, dt \right]^\frac{1}{q-1} \frac{1}{p},
\]
when \( 1 < q < p < \infty \).

4. Proof of Theorem A.

The following lemma is a key element in the proof of the Theorem A. In it and in the rest of the section, it will simplify things if we write \( \psi \) in the form
\[
\psi = \frac{(P\phi)(Q_p\phi)}{[(PQ_p)(\phi)]^{p+1}},
\]
where
\[
(P\phi)(t) = t^{-1} \int_0^t \phi(s) \, ds \quad \text{and} \quad (Q_p\phi)(t) = pt^{-1} \int_t^b \phi(s)s^{-p} \, ds.
\]

Lemma 4.1. Fix \( p \) and \( b \) with, \( 1 < p < \infty \) and \( 0 < b \leq \infty \). Suppose \( \phi \) is a non-trivial weight on \( I_b \) and let \( \psi \) be given by (4.1). Then, there exists \( C > 0 \), independent of \( f, g \in \mathcal{M}_+(I_b) \), such that

(i) \[
\int_{I_b} fg \left[ \frac{P\phi}{(PQ_p)(\phi)} \right]^{\frac{1}{p}+1} \leq C \left( \int_{I_b} f^{p\psi} \phi \right)^{\frac{1}{p}} \left[ \int_{I_b} g^{p\psi} \psi \right]^{\frac{1}{p}} + \int_{I_b} g \phi \right]^{\frac{1}{p}},
\]
if \( f \downarrow \), and

(ii) \[
\int_{I_b} fg \left[ \frac{Q_p\phi}{(PQ_p)(\phi)} \right]^{\frac{1}{p}+1} \leq C \left( \int_{I_b} f(t)^p \phi(t)t^{-p} \, dt \right)^{\frac{1}{p}} \left( \int_{I_b} g(t)^p \psi(t) \, dt \right)^{\frac{1}{p}},
\]
if \( f \uparrow \).

Proof. (i) We have
\[
\frac{p'}{p'} + 1 \int_{I_b} fg \left[ \frac{P\phi}{(PQ_p)(\phi)} \right]^{\frac{1}{p}+1}
= \int_{I_b} g(t) \int_0^t f(s) \left( \int_0^s \phi \right)^{\frac{1}{p'}} \phi(s) \, ds [t (PQ_p)(\phi)(t)]^{-\frac{1}{p'}} \, dt, \quad \text{since} \quad f \downarrow,
\]
\[
= \int_{t_0}^t f(t) \left( \int_0^t \phi \right) \frac{1}{p'} \int_t^b g(s \left( PQ_p \right) (\phi)(s))^{-\frac{1}{p'}} ds \phi(t) dt, \quad \text{by Fubini’s theorem},
\]
\[
= \int_{I_b} f(t) \left( \int_0^t \phi \right) \frac{1}{p'} \left[ \int_0^s g \left( PQ_p \right) (\phi)(s))^{-\frac{1}{p'}} \right]_t^b + \left( \frac{1}{p'} + 1 \right) \int_t^b \int_0^s g \left( PQ_p \right) (\phi)(s))^{-\frac{1}{p'}} \left( Q_p \phi \right)(s) ds \phi(t) dt
\]
\[
= \int_{I_b} f(t) \left( \int_0^t \phi \right) \frac{1}{p'} \left[ \int_{I_b} g \left[ \int_{I_b} \phi \right] + \left( \frac{1}{p'} + 1 \right) \int_t^b \int_0^s g \left( PQ_p \right) (\phi)(s))^{-\frac{1}{p'}} \left( Q_p \phi \right)(s) ds \phi(t) dt
\]
\[
\leq (p + 1)^2 \left[ \int_{I_b} f^p \phi \right] \frac{1}{p'} \left[ \int_{I_b} g \left[ \int_{I_b} \phi \right] + \left( \frac{1}{p'} + 1 \right) \int_t^b \int_0^s g \left( PQ_p \right) (\phi)(s))^{-\frac{1}{p'}} \left( Q_p \phi \right)(s) ds \phi(t) dt
\]
\[
\leq (p + 1)^2 \left[ \int_{I_b} f^p \phi \right] \frac{1}{p'} \left[ \int_{I_b} g \left[ \int_{I_b} \phi \right] + \left( \frac{1}{p'} + 1 \right) \int_t^b \int_0^s g \left( PQ_p \right) (\phi)(s))^{-\frac{1}{p'}} \left( Q_p \phi \right)(s) ds \phi(t) dt \right]^\frac{1}{p'}
\]
\[
\leq (p + 1)^2 \left[ \int_{I_b} f^p \phi \right] \frac{1}{p'} \left[ \int_{I_b} g \left[ \int_{I_b} \phi \right] + \left( \frac{1}{p'} + 1 \right) \int_t^b \int_0^s g \left( PQ_p \right) (\phi)(s))^{-\frac{1}{p'}} \left( Q_p \phi \right)(s) ds \phi(t) dt \right]^\frac{1}{p'}
\]
\[
\leq (p + 1)^2 \left[ \int_{I_b} f^p \phi \right] \frac{1}{p'} \left[ \int_{I_b} g \left[ \int_{I_b} \phi \right] + \left( \frac{1}{p'} + 1 \right) \int_t^b \int_0^s g \left( PQ_p \right) (\phi)(s))^{-\frac{1}{p'}} \left( Q_p \phi \right)(s) ds \phi(t) dt \right]^\frac{1}{p'}
\]
\[
\leq (p + 1)^2 \left[ \int_{I_b} f^p \phi \right] \frac{1}{p'} \left[ \int_{I_b} g \left[ \int_{I_b} \phi \right] + \left( \frac{1}{p'} + 1 \right) \int_t^b \int_0^s g \left( PQ_p \right) (\phi)(s))^{-\frac{1}{p'}} \left( Q_p \phi \right)(s) ds \phi(t) dt \right]^\frac{1}{p'}
\]

in which the third inequality was obtained using Hölder’s inequality with respect to the measure \( \phi(t) dt \).
The proof of (i) will be complete if we can show, that

$$\int_{I_b} \left( \int_t^b \int_0^s g \left[ s \left( P Q_p \right) \left( \phi \right) \left( s \right) \right]^{-\frac{1}{p'}} \left( Q_p \phi \right) \left( s \right) ds \right)^{p'} \phi(t) \int_0^t \phi dt$$

is dominated by a constant multiple of $\int_{I_b} \left( P g \right)(t)^{p'} \psi(t) dt$. To this end, let

$$H(t) := \int_0^s g \left[ s \left( P Q_p \right) \left( \phi \right) \left( s \right) \right]^{-\frac{1}{p'}} \left( Q_p \phi \right) \left( s \right), \quad s \in I_b$$

so that the assertion reads

$$\int_{I_b} \left( \int_t^b H(s) ds \right)^{p'} \phi(t) \int_0^t \phi dt$$

$$\leq C \int_{I_b} H(t)^{p'} \left[ t \left( P Q_p \right) \left( \phi \right) \left( t \right) \right]^{p'} \left( \left( Q_p \phi \right) \left( t \right) \right)^{-p'-1} \int_0^t \phi(t) dt$$

But, this holds by Theorem 3.1 inasmuch as

$$\left( \int_0^t \phi \left( s \right) \int_0^s \phi ds \right)^{\frac{1}{p'}} \left( \int_t^b \left[ s \left( P Q_p \right) \left( \phi \right) \left( s \right) \right]^{-p} \left( Q_p \phi \right) \left( s \right) \left( \int_0^s \phi \right)^{1-p} ds \right)^{\frac{1}{p'}}$$

$$= 2^{-\frac{1}{p'}} \left( \int_0^t \phi \right)^{\frac{2}{p'}} \left( \int_t^b \left[ s \left( P Q_p \right) \left( \phi \right) \left( s \right) \right]^{-p} \left( Q_p \phi \right) \left( s \right) \left( \int_0^s \phi \right)^{1-p} ds \right)^{\frac{1}{p'}}$$

$$\leq 2^{-\frac{1}{p'}} \left( \int_0^t \phi \right)^{\frac{2}{p'}} \left( \int_t^b \left[ s \left( P Q_p \right) \left( \phi \right) \left( s \right) \right]^{-p} d \left[ s \left( P Q_p \right) \left( \phi \right) \left( s \right) \right] \right)^{\frac{1}{p'}}$$

$$= -\frac{2^{-\frac{1}{p'}}}{p-1} \left( \int_0^t \phi \right)^{\frac{1}{p'}} \left( \left[ s \left( P Q_p \right) \left( \phi \right) \left( s \right) \right]^{-p+1} \left| b \right| \phi \right)^{\frac{1}{p'}}$$

$$\leq \frac{2^{-\frac{1}{p'}}}{p-1} \left( \frac{\int_0^t \phi}{t \left( P Q_p \right) \left( \phi \right) \left( t \right)} \right)^{\frac{1}{p'}}$$

$$= \frac{2^{-\frac{1}{p'}}}{p-1} \left( \frac{\int_0^t \phi}{\int_0^t \phi + \left( Q_p \phi \right) \left( t \right)} \right)^{\frac{1}{p'}}$$

$$\leq \frac{2^{-\frac{1}{p'}}}{p-1}.$$

(ii) To begin, suppose $b = \infty$. Making the change of variable $t \to t^{-1}$ three times in a row and setting $\tilde{f}(y) = f(y^{-1})$, $\tilde{g}(y) = g(y^{-1})$, $\tilde{\phi}(y) = \phi(y^{-1})y^{p-2}$, we obtain

$$\int_0^\infty \tilde{f} \tilde{g} \left[ \frac{Q_p \phi}{\left( P Q_p \right) \left( \phi \right)} \right]^{\frac{1}{p'}+1} dt = \int_0^\infty \tilde{f} \tilde{g} \left[ \frac{\int_0^\infty \phi \left( s \right) s^{-p} ds}{t^p \int_0^{t^{-1}} Q_p \phi} \right]^{\frac{1}{p'}+1} dt.$$
\[
\int_{0}^{\infty} \tilde{f} \left[ \frac{\int_{0}^{t} \phi}{e^{t} \int_{0}^{s} \phi ds} \right]^{\frac{1}{p}+1} dt
= \int_{0}^{\infty} \tilde{f} \left[ \frac{\int_{0}^{t} \phi}{t (Q_{p} P)(\phi)(t)} \right]^{\frac{1}{p}+1} dt
= \int_{0}^{\infty} \tilde{f} \left[ \frac{P_{\phi}}{(PQ_{p})(\phi)} \right]^{\frac{1}{p}+1}.
\]

Thus, from (i), there follows, since \( \tilde{f} \downarrow \),
\[
\int_{0}^{\infty} f g \left[ \frac{Q_{p} \phi}{(PQ_{p})(\phi)} \right]^{\frac{1}{p}+1} \leq C \left[ \int_{0}^{\infty} \tilde{f} \phi \right]^{\frac{1}{p}} \left[ \left( \int_{0}^{\infty} (P g)^{p} \psi \right) \right]^{\frac{1}{p}} + \frac{C}{\left[ \int_{0}^{\infty} \tilde{f} \phi \right]^{\frac{1}{p}}},
\]
with
\[
\tilde{\psi} = \frac{(P \tilde{\phi})(Q_{p} \tilde{\phi})}{(PQ_{p})(\tilde{\phi})}.
\]

Now, the change of variable \( t \to t^{-1} \) yields
\[
\int_{0}^{\infty} \tilde{f}(t) \tilde{\phi}(t) dt = \int_{0}^{\infty} \tilde{f}(t^{-1}) \tilde{\phi}(t^{-1}) t^{-2} dt
= \int_{0}^{\infty} f(t) \phi(t) t^{-p} dt,
\]
and
\[
\int_{0}^{\infty} \tilde{g}(t) dt = \int_{0}^{\infty} \tilde{g}(t^{-1}) t^{-2} dt = \int_{0}^{\infty} g(t) dt.
\]

And
\[
\int_{0}^{\infty} \tilde{\phi}(t) dt = \int_{0}^{\infty} \tilde{\phi}(t^{-1}) t^{-2} dt = \int_{0}^{\infty} \phi(t) t^{-p} dt = \infty.
\]

Again,
\[
\tilde{\psi}(t) = \frac{t^{-1} \int_{0}^{t} \phi(s) ds t^{-p} \int_{t}^{\infty} \phi(s) s^{-p} ds}{(PQ_{p})(\tilde{\phi})(t)}^{\frac{1}{p+1}}
= \frac{t^{-1} \int_{0}^{t} \phi(s^{-1}) s^{-2} ds t^{-p} \int_{t}^{\infty} \phi(s^{-1}) s^{-2} ds}{\left[ \int_{t}^{\infty} \frac{t^{-1} \phi(s^{-1}) s^{-2} ds + \frac{p-1}{p} \int_{0}^{t} \phi(s^{-1}) s^{-2} ds} \right]^{\frac{1}{p+1}}}
= \frac{t^{-1} \int_{0}^{t} \phi(s) ds t^{-p} \int_{t}^{\infty} \phi(s) s^{-p} ds}{\left[ \int_{0}^{\infty} \frac{t^{-1} \phi(s^{-1}) s^{-2} ds + \frac{p-1}{p} \int_{0}^{t} \phi(s) ds} \right]^{\frac{1}{p+1}}}
\]
\[ \frac{t^{p'-2}(P\phi)(t^{-1})(Q\nu\phi)(t^{-1})}{[(PQ\rho)(\phi)(t^{-1})]^{p'+1}} = \psi(t^{-1})t^{p'-2}. \]

So,

\[ \int_0^\infty (P\tilde{g})(t)p\tilde{\psi}(t)dt = \int_0^\infty \left(t^{-1} \int_0^t g(s^{-1})s^{-2}ds\right)^{p'} \psi(t^{-1})t^{p'-2}dt \]
\[ = \int_0^\infty \left(t^{-1} \int_0^t g(s^{-1})s^{-2}ds\right)^{p'} \psi(t^{-1})t^{-2}dt \]
\[ = \int_0^\infty \left(t^{-1} \int_0^\infty g(s)ds\right)^{p'} \psi(t)dt. \]

This completes the proof of (ii) when \( b = \infty \). In the case \( b < \infty \), a similar argument works if we replace the transformation \( t \to t^{-1} \) by \( t \to (b-t)^{-1} \).

**Proof of Theorem A.** We first show

\[ \rho_{p,\phi}'(g) \geq c \left(\rho_{p,\psi}'(g) + \frac{\int_{I_b} |g|}{\left[\int_{I_b} \phi\right]^\frac{1}{p}}\right), \tag{4.2} \]

for some \( c > 0 \) independent of \( g \in \mathcal{M}_+(I_b) \). To this end, it suffices, in view of (2.2), to find constants \( C, c > 0 \), independent of \( g \in \rho_{p,\phi}' \), to which there corresponds an \( f \in \mathcal{M}_+(I_b) \), with \( f \downarrow, \rho_{p,\phi}(f) \leq C \) and

\[ \int_{I_b} fg^* \geq c \left[\rho_{p,\psi}'(g^*) + \frac{\int_{I_b} g^*}{\left[\int_{I_b} \phi\right]^\frac{1}{p}}\right]. \tag{4.3} \]

Fixing \( g \), we seek

\[ f = Qh \]

for some \( h \) in \( \mathcal{M}_+(I_b) \).

We need a condition on \( h \) to guarantee \( \rho_{p,\phi}(Qh) < \infty \). But,

\[ \rho_{p,\phi}(Qh) = \left[\int_{I_b} ((PQ)h)^p \phi\right]^\frac{1}{p} \]
\[ = \left[\int_{I_b} (Ph + Qh)^p \phi\right]^\frac{1}{p} \]
\[ \leq 2 \left[\int_{I_b} (Sh)^p \phi\right]^\frac{1}{p} \]
\[ \leq B \left[\int_{I_b} h^p \psi^{1-p}\right]^\frac{1}{p}, \]
the last inequality being proved in the Appendix. The desired condition on \( h \) is thus

\[
\int_{I_b} h^p \psi^{1-p} < \infty.
\]

As pointed out in Section 3, the weighted Lebesgue norms

\[
\left[ \int_{I_b} g^{p'} \psi \right]^\frac{1}{p'} \quad \text{and} \quad \left[ \int_{I_b} h^p \psi^{1-p} \right]^\frac{1}{p}, \quad g, h \in \mathcal{M}_+(I_b),
\]

are dual to one another. Therefore, for our given \( g \in L^{p', \phi} \), there exists \( h_0 \in \mathcal{M}_+(I_b) \), such that

\[
\int_{I_b} h_0^p \psi^{1-p} \leq 1
\]

and

\[
\int_{I_b} g^* Qh_0 = \int_{I_b} h_0 P g^* \geq \frac{1}{2} \left[ \int_{I_b} (g^{**})^{p'} \psi \right]^\frac{1}{p'} = \frac{1}{2} \rho_{p', \psi}(g).
\]

If \( \int_I \phi < \infty \), the constant function with value

\[
\frac{1}{\left[ \int_{I_b} \phi \right]^\frac{1}{p}}
\]

will belong to \( \Gamma_{p, \phi} \) with norm 1 and

(4.4)

\[
f := Qh_0 + \frac{1}{\left[ \int_{I_b} \phi \right]^\frac{1}{p}}
\]

will satisfy

\[
\int_{I_b} f g^* \geq \frac{\int_{I_b} g^*}{\left[ \int_{I_b} \phi \right]^\frac{1}{p}}.
\]

Altogether, then, the function \( f \) defined in (4.4) has \( \rho_{p, \phi}(f) \leq C = B + 1 \) and satisfies (4.3) with \( c = \frac{1}{2} \).

We now prove the inequality opposite to (4.2), this being equivalent to

(4.5)

\[
\int_{I_b} f^* g^* \leq C \rho_{p, \phi}(f^*) \left[ \rho_{p', \psi}(g^*) + \frac{\int_{I_b} g^*}{\left[ \int_{I_b} \phi \right]^\frac{1}{p}} \right],
\]

in which \( C > 0 \) is independent of \( f, g \in \mathcal{M}(X) \).

It suffices to consider \( g^* \) of the form

\[
g^* = k + Qh, \quad k \geq 0 \quad \text{and} \quad h \in \mathcal{M}_+(I_b).
\]
For the term
\[
\frac{\int_{I_b} g^*}{\left[ \int_{I_b} \phi \right]^{\frac{1}{p}}}
\]
to be finite we require \( b = \mu(X) < \infty \) or \( \int_{I_b} \phi = \infty \). In either case, the term is dominated by an absolute constant times \( \rho_{p',\psi}(g^*) \) and is irrelevant.

We have only to consider those \( g^* \) of the form \( g^* = Qh, \ h \in \mathcal{M}_+(I_b) \). For such \( g^* \),

\[
\int_{I_b} f^* g^* = \int_{I_b} f^* Qh = \int_{I_b} h P(f^*) = \int_{I_b} f^{**} h \\
= \int_{I_b} f^{**} h \left[ (PQ_p) (\phi) \right]^{\frac{1}{p'} + 1} \left[ (PQ_p) (\phi) \right]^{-\frac{1}{p'}} \\
= p^{-\frac{1}{p'}} \int_{I_b} f^{**} h \left[ \frac{P\phi + Q\phi}{(PQ_p)(\phi)} \right]^{\frac{1}{p'} + 1} \\
\leq \left( \frac{2}{p} \right)^{\frac{1}{p'} + 1} \left[ \int_{I_b} f^{**} h \left[ \frac{P\phi}{(PQ_p)(\phi)} \right]^{\frac{1}{p'} + 1} + \int_{I_b} f^{**} h \left[ \frac{Q\phi}{(PQ_p)(\phi)} \right]^{\frac{1}{p'} + 1} \right] \\
= \left( \frac{2}{p} \right)^{\frac{1}{p'} + 1} [I_1 + I_2].
\]

(4.6)

Since \( f^{**} \downarrow \), Lemma 4.1 (i), gives

\[
I_1 \leq C \rho_{p,\phi}(f) \left[ \left( \int_{I_b} (Ph)^{p'} \psi \right)^{\frac{1}{p'}} + \frac{\int_{I_b} h}{\left[ \int_{I_b} \phi \right]^{\frac{1}{p}}} \right].
\]

(4.7)

But,

\[
g^{**} = Pg^* = (PQ)h = Ph + Qh \geq Ph
\]

and

\[
\int_{I_b} g^* = \int_{I_b} Qh = \int_{I_b} h,
\]

whence (4.7) implies

\[
I_1 \leq C \rho_{p,\phi}(f) \left[ \rho_{p',\psi}(g^*) + \frac{\int_{I_b} g^*}{\left[ \int_{I_b} \phi \right]^{\frac{1}{p}}} \right].
\]

(4.8)

Observing that \( \int_0^t f^* \uparrow \), Lemma 4.1 (ii), ensures

\[
I_2 \leq C \left[ \int_{I_b} \left( \int_0^t f^* \right)^p \phi(t) t^{-p} dt \right]^{\frac{1}{p'}} \left( \int_{I_b} (Qh)^{p'} \psi \right)^{\frac{1}{p'}} = C \rho_{p,\phi}(f^*) \rho_{p',\psi}(g^*)
\]

(4.9)

Combining (4.6), (4.8) and (4.9) yields (4.5) and thereby completes the proof. \( \square \)
Corollary 4.2. Let $\phi$ be a non-trivial weight function on $\mathbb{R}_+$, and $\psi$ its dual weight. Then,

\begin{equation}
(4.10) \quad t^{-\rho'} \left[ \int_0^t \psi(s) ds + t^{\rho'} \int_t^\infty \psi(s)s^{-\rho'} ds \right] \approx \left[ \int_0^t \phi(s) ds + t^\rho \int_t^\infty \phi(s)s^{\rho} ds \right]^{1-\rho'}.
\end{equation}

Proof. It is easy to see that

\[ \rho_{p,\phi}(\chi_{(0,t)}) = \left( \int_0^t \phi(s) ds + t^{\rho} \int_t^\infty \phi(s)s^{\rho} ds \right)^{1/\rho} \]

and

\[ \rho_{\rho',\psi}(\chi_{(0,t)}) = \left( \int_0^t \psi(s) ds + t^{\rho'} \int_t^\infty \psi(s)s^{-\rho'} ds \right)^{1/\rho'}. \]

Since $\rho_{p,\phi}$ and $\rho_{\rho',\psi}$ are associate r.i. function norms, (4.10) now follows from (2.3).

\[ \square \]

Corollary 4.3. Fix $p \in (1, \infty)$ and suppose $\phi$ is a non-trivial weight function on $\mathbb{R}_+$, with

\[ \int_0^\infty \phi(t) dt = \infty. \]

Then,

\begin{equation}
(4.11) \quad \sup_{f \in \Omega_{0,1}(\mathbb{R}_+)} \frac{\int_0^\infty fg}{(\int_0^\infty f^{p} \phi)_{\frac{1}{p}}} \approx \left( \int_0^\infty (Sg)^{p'} \psi \right)^{\frac{1}{p'}}, \quad g \in \mathcal{M}_+(\mathbb{R}_+),
\end{equation}

in which $\psi$ is the weight dual to $\phi$ and

\[ \Omega_{0,1}(\mathbb{R}_+) := \{ f \in \mathcal{M}_+(\mathbb{R}_+) : tf(t) \uparrow \text{ and } f \downarrow \} \]

Proof. As pointed out in [2, p. 117], $f \in \Omega_{0,1}(\mathbb{R}_+)$ if and only if

\[ \frac{1}{2} t^{-1} \int_0^t h^*(s) ds \leq f(t) \leq 2t^{-1} \int_0^t h^*(s) ds, \]

for some $h \in \mathcal{M}_+(\mathbb{R}_+)$. Hence, the left side of (4.11), is equivalent to

\[ \sup_{h \in \mathcal{M}_+(\mathbb{R}_+)} \frac{\int_0^\infty t^{-1} \int_0^t h^*(s) ds g(t) dt}{\rho_{p,\phi}(h)} = \sup_{h \in \mathcal{M}_+(\mathbb{R}_+)} \frac{\int_0^\infty h^*(t) \int_t^\infty g(s) ds dt}{\rho_{p,\phi}(h)} \]

\[ = \rho'_{p,\phi} \left( \int_t^\infty g(s) \frac{ds}{s} \right) \]

\[ \approx \rho'_{\rho',\psi} \left( \int_t^\infty g(s) \frac{ds}{s} \right)^{1/\rho'}, \]

which yields (4.11), in view of (3.3), since

\[ \rho'_{\rho',\psi} \left( \int_t^\infty g(s) \frac{ds}{s} \right) = \left( \int_0^\infty \left( t^{-1} \int_0^t \int_s^\infty g(y) \frac{dy}{y} \right)^{p'} \psi(t) dt \right)^{\frac{1}{p'}}. \]
\[ \left( \int_0^\infty (Tf)^q \phi_2 \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty f^p \phi_1 \right)^{\frac{1}{p}}, \quad f \in \Omega_{0,1}(\mathbb{R}_+), \]

If and only if

\[ \left( \int_0^\infty (ST')(h)^q \psi_1 \right)^{\frac{1}{q}} \leq K \left( \int_0^\infty h^p \phi_2^{1-q'} \right)^{\frac{1}{p}}, \quad h \in \mathcal{M}_+(\mathbb{R}_+), \]

or

\[ \left( \int_0^\infty (TS)(h)^q \phi_2 \right)^{\frac{1}{q}} \leq K \left( \int_0^\infty h^p \psi_1^{1-q} \right)^{\frac{1}{p}}, \quad h \in \mathcal{M}_+(\mathbb{R}_+). \]

Here, \( K \approx C. \)

**Proof.** The reverse Hölder inequality ensures that (4.12) is equivalent to

\[ \frac{\int_0^\infty (Tf)h}{\left( \int_0^\infty h^q \phi_2^{1-q'} \right)^{\frac{1}{q}}} \lesssim \left( \int_0^\infty f^p \phi_1 \right)^{\frac{1}{p}}, \quad f \in \Omega_{0,1}(\mathbb{R}_+), h \in \mathcal{M}_+(\mathbb{R}_+), \]

or

\[ \frac{\int_0^\infty f(T'h)}{\left( \int_0^\infty f^p \phi_1 \right)^{\frac{1}{p}}} \lesssim \left( \int_0^\infty h^q \phi_2^{1-q'} \right)^{\frac{1}{q}}, \quad f \in \Omega_{0,1}(\mathbb{R}_+), h \in \mathcal{M}_+(\mathbb{R}_+). \]

In view of Corollary 4.3, (4.15) amounts to

\[ \rho^p \phi_1 ((ST')(h)) \lesssim \left( \int_0^\infty h^q \phi_2^{1-q'} \right)^{\frac{1}{q}}, \quad h \in \mathcal{M}_+(\mathbb{R}_+), \]

that is, (4.13). As we have

\[ \int_0^\infty (ST')h(t)g(t)dt = \int_0^\infty h(t)(TS)g(t)dt, \quad h, g \in \mathcal{M}_+(\mathbb{R}_+), \]

(4.13) is equivalent to (4.14), by the duality theorem for weighted Lebesgue spaces.
5. IMBEDDINGS AND BOYD INDICES

**Theorem 5.1.** Fix \( p, q \in (1, \infty) \). Suppose \( \phi_1 \) and \( \phi_2 \) are weights on \( \mathbb{R}_+ \), with \( \phi_1 \) and its dual weight \( \psi_1 \) as in Corollary 4.3. Then, the (possibly infinite) norm of the imbedding

\[
\Gamma_{p, \phi_1}(\mathbb{R}_+) \hookrightarrow \Gamma_{q, \phi_2}(\mathbb{R}_+)
\]

is equivalent to

\[
\sup_{t > 0} \left[ \frac{\int_0^t \phi_2(s) \, ds + t^q \int_t^\infty \phi_2(s) s^{-q} \, ds}{\int_0^t \phi_1(s) \, ds + t^p \int_t^\infty \phi_1(s) s^{-p} \, ds} \right]^{\frac{1}{q}}.
\]

if \( 1 < p \leq q < \infty \), and to

\[
\left[ \int_0^\infty \left[ \frac{\int_0^t \phi_2(s) \, ds + t^q \int_t^\infty \phi_2(s) s^{-q} \, ds}{\int_0^t \phi_1(s) \, ds + t^p \int_t^\infty \phi_1(s) s^{-p} \, ds} \right]^{\frac{q}{p-q}} \phi_2(t) \, dt \right]^{\frac{1}{q}}
\]

if \( 1 < q < p < \infty \).

**Proof.** The imbedding (5.1) is equivalent to an inequality of the form

\[
\rho_{q, \phi_2}(I f) \leq C \rho_{p, \phi_1}(f), \quad f \in \Omega_{0,1}(\mathbb{R}_+),
\]

or

\[
\left( \int_0^\infty (If)^q \phi_2 \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty f^p \phi_1 \right)^{\frac{1}{p}}, \quad f \in \Omega_{0,1}(\mathbb{R}_+),
\]

in which \( I \) is the identity operator. According to Theorem 4.4, (5.5) reduces to

\[
\left( \int_0^\infty (Sh)^q \phi_2 \right)^{\frac{1}{q}} \leq K \left( \int_0^\infty h^p \psi_1^{1-\frac{1}{p}} \right)^{\frac{1}{q}}, \quad h \in \mathcal{M}_+(\mathbb{R}_+);
\]

here, \( K \approx C \) and

\[
\psi_1(t) = \frac{(P\phi_1)(t)(Q_p\phi_1)(t)}{[(PQ_p)(\phi_1)]^{p+1}}.
\]

By Theorem 3.2, the least possible \( K \) in (5.6) is equivalent to

\[
\sup_{t > 0} \left[ \int_0^\infty \psi_1(s) \left( \frac{t}{s+t} \right)^q \phi_2(s) \, ds \right]^{\frac{1}{q}} \left[ \int_0^\infty \left( \frac{t}{s+t} \right)^q \phi_2(s) \, ds \right]^{\frac{q}{p-q}} \phi_2(t) \, dt \right]^{\frac{1}{q}}
\]

when \( 1 < p \leq q < \infty \), and to

\[
\left[ \int_0^\infty \left[ \int_0^\infty \psi_1(s) \left( \frac{t}{s+t} \right)^q \phi_2(s) \, ds \right]^{\frac{(p-1)q}{p-q}} \int_0^\infty \left( \frac{t}{s+t} \right)^q \phi_2(s) \, ds \right]^{\frac{q}{p-q}} \phi_2(t) \, dt \right]^{\frac{1}{q}}.
\]
when \(1 < q < p < \infty\). But,
\[
\int_0^\infty \left( \frac{t}{s+t} \right)^q \phi_2(s) ds \approx \int_0^t \phi_2(s) ds + t^q \int_t^\infty \phi_2(s) s^{-q} ds
\]
and
\[
\int_0^\infty \psi_1(s) \psi_2(s) ds \approx t^{-p'} \left[ \int_0^t \psi_1(s) ds + t^p \int_t^\infty \psi_1(s) s^{-p} ds \right]^{1-p'},
\]
by Corollary 4.2, so (5.7) becomes (5.2) and (5.8) becomes (5.3).

\[\square\]

**Theorem 5.2.** Fix an index \(p, 1 < p < \infty\) and suppose \(\phi\) is a non-trivial weight on \(\mathbb{R}_+\). Take \(\rho = \rho_{p,\phi}\) on \(\mathcal{M}_+(\mathbb{R}_+)\). Then,
\[
(5.9) \quad h_\rho(t) \approx M_\rho(t) \approx \sup_{s \in \mathbb{R}_+} \left[ \frac{\int_s^t \phi(y)dy + s^p \int_s^\infty \phi(y)y^{-p}dy}{\int_0^s \phi(y)dy + s^p \int_s^\infty \phi(y)y^{-p}dy} \right]^{\frac{1}{p}}, \quad t \in \mathbb{R}_+,
\]
and
\[
(5.10) \quad i_\rho = i_{\rho}, \quad I_\rho = I_{\rho}.
\]

**Proof.** For \(f \in \mathcal{M}_+(\mathbb{R}_+), f\) decreasing, we have
\[
\left( E_{\frac{t}{s}}f \right)^{**} (s) = f^{**} \left( \frac{s}{t} \right), \quad s \in \mathbb{R}_+,
\]
so
\[
\rho_{p,\phi} \left( E_{\frac{t}{s}}f \right) = \rho_{p,\phi}(f),
\]
where
\[
\phi(s) = t\phi(st).
\]
Thus, for \(t \in \mathbb{R}_+,
\[
h_\rho(t) = \sup_{f \in \mathcal{M}_+(\mathbb{R}_+)} \frac{\rho_{p,\phi} \left( E_{\frac{t}{s}}f \right)}{\rho_{p,\phi}(f)}
\]
\[
= \sup_{f \in \mathcal{M}_+(\mathbb{R}_+)} \frac{\rho_{p,\phi}(f)}{\rho_{p,\phi}(f)}
\]
\[
\approx \sup_{s \in \mathbb{R}_+} \frac{\rho_{p,\phi} \left( \chi_{(0,s)} \right)}{\rho_{p,\phi}(\chi_{(0,s)})}, \quad \text{by Theorem 5.1}
\]
\[
\approx \sup_{s \in \mathbb{R}_+} \left[ \frac{\int_0^s t\phi(y)dy + s^p \int_s^\infty t\phi(y)y^{-p}dy}{\int_0^s \phi(y)dy + s^p \int_s^\infty \phi(y)y^{-p}dy} \right]^{\frac{1}{p}}
\]
\[ \approx \sup_{s \in \mathbb{R}^+} \left[ \int_{st}^{st} \phi(y)dy + sp \int_{st}^{\infty} \phi(y)y^{-p}dy \right]^{\frac{1}{p}} \]
\[ \approx \sup_{s \in \mathbb{R}^+} \frac{\rho_{p,\phi}(X_{(0,st)})}{\rho_{p,\phi}(X_{(0,s)})} \]
\[ = M_{\rho}(t). \]

\[ \lim_{\varepsilon \to 0^+} \int_{\varepsilon < |x| < N} K(x)dx. \]

\[ \int_{|x| < R} \| x \| K(x)dx \leq C_2 R. \]

\[ \int_{|x| > 2|y|} |K(x - y) - K(x)|dx \leq C_3. \]

\[ |K(x_1 - y) - K(x_2 - y)| \leq C_4 \frac{|x_1 - x_2|}{|x_3 - y|^{n+1}}. \]

The Calderón-Zygmund operator, \( T_K \), with kernel \( K \), is the singular integral operator
\[ (T_K f)(x) := \lim_{\varepsilon \to 0^+} \int_{|x - y| > \varepsilon} K(x - y)f(y)dy, \quad x \in \mathbb{R}^n, \]
which is defined a.e. for all \( f \in \mathcal{M}(\mathbb{R}^n) \) with
\[ \int_{\mathbb{R}^n} \frac{|f(y)|}{1 + |y|^{n+1}}dy < \infty. \]
Theorem 6.1. Fix $p, 1 < p < \infty$, and suppose the weight $\phi$ on $\mathbb{R}_+$ satisfies
\[ \int_0^\infty \phi(s) \min\left[1, s^{-p}\right] ds < \infty \quad \text{and} \quad \int_0^\infty \phi(s) \max\left[1, s^{-p}\right] ds = \infty. \]

Denote by $\psi$ the function defined in (4.1).

Let $T_K$ be a CZ operator. Then, one has
\begin{equation}
T_K : \Gamma_{p,\phi}(\mathbb{R}^n) \to \Gamma_{p,\phi}(\mathbb{R}^n)
\end{equation}
if there exists $c, 0 < c < 1$, such that for all $t \in \mathbb{R}_+$,
\begin{equation}
\int_0^{ct} \phi(s) ds + c^p t^p \int_t^\infty \phi(s) s^{-p} ds \leq \frac{1}{2} \left[ \int_0^t \phi(s) ds + t^p \int_t^\infty \phi(s) s^{-p} ds \right]
\end{equation}
\begin{equation}
\int_0^{ct} \psi(s) ds + c^p t^p \int_t^\infty \psi(s) s^{-p} ds \leq \frac{1}{2} \left[ \int_0^t \psi(s) ds + t^p \int_t^\infty \psi(s) s^{-p} ds \right].
\end{equation}

Proof. Let $\rho$ be an r.i. norm on $\mathfrak{M}_+(\mathbb{R}_+)$ defined in terms of r.i. norm $\rho$ on $\mathfrak{M}_+(\mathbb{R}_+)$ by $\rho(f) = \overline{\rho}(f^*)$. It is shown in [9] that
\[ T_K : L_\mu(\mathbb{R}^n) \to L_\mu(\mathbb{R}^n) \]
provided
\[ \lim_{s \to 0^+} sh(s) = 0 = \lim_{s \to \infty} h(s), \]
where $h(s) = h_\rho\left(\frac{1}{s}\right)$. In terms of $h_\rho(s)$ and $h_\overline{\rho}(s) = sh(s)$, these conditions read
\begin{equation}
\lim_{s \to 0^+} h_\rho(s) = 0 = \lim_{s \to 0^+} h_\overline{\rho}(s).
\end{equation}
The inequalities
\[ h_\rho(s_1 s_2) \leq h_\rho(s_1) h_\rho(s_2) \quad \text{and} \quad h_\overline{\rho}(s_1 s_2) \leq h_\overline{\rho}(s_1) h_\overline{\rho}(s_2), \quad s_1, s_2 \in \mathbb{R}_+, \]
imply that, given $\varepsilon > 0$, (6.3) is equivalent to the existence of $c, 0 < c < 1$, for which $h_\rho(c) < \varepsilon$ and $h_\overline{\rho}(c) < \varepsilon$. By Theorem B, then, (6.3) is equivalent to (6.2), when $\overline{\rho} = \rho_{p,\phi}$. \qed

Remark 6.2. The condition (6.2) is also necessary for (6.1) when, for example, $T_K$ is the Hilbert transform or one of the Riesz transforms.

7. Appendix

It is our purpose here to give an heuristic argument to motivate the choice of $\psi$ in (4.1) when $\phi$ is a non-trivial weight on $I_b$ satisfying $\int_{I_b} \phi = \infty$.

Now,
\[ \rho_{p,\phi}'(g) = \sup_{f \in \mathfrak{M}_+(I_b)} \frac{\int_0^b f^*(t) g^*(t) dt}{\left[ \int_0^b f^{**}(t) \phi(t) dt \right]^{\frac{1}{p}}} =: I(g), \quad g \in \mathfrak{M}_+(I_b). \]
It suffices to consider \( f(t) = \int_t^b h(s) \frac{ds}{s} \) for some \( h \in \mathcal{M}_+(I_b) \), \( h \neq 0 \) a.e. Since, in that case,

\[
\int_0^b f^*(t)g^*(t) = \int_0^b \int_t^b h(s) \frac{ds}{s} g^*(t) dt = \int_0^b h(t)g^{**}(t) dt
\]

and

\[
f^{**}(t) = t^{-1} \int_0^t \int_s^b h(y) \frac{dy}{y} ds = t^{-1} \int_0^t h(s) ds + \int_t^b h(s) \frac{ds}{s} \approx (Sh)(t), \quad t \in I_b,
\]

we have

\[
I(g) = \sup_{h \in \mathcal{M}_+(I_b)} \frac{\int_0^b h(t)g^{**}(t) dt}{\left[ \int_0^b (Sh)(t)^p \phi(t) dt \right]^\frac{1}{p}}.
\]

If \( \bar{\phi} \) is such that

\[
(7.1) \quad \int_0^b (Sh)^p \phi \leq C \int_0^b h^p \bar{\phi}, \quad h \in \mathcal{M}(I_b),
\]

then,

\[
I(g) \geq C^{-1} \sup_{h \in \mathcal{M}_+(I_b)} \frac{\int_0^b h(t)g^{**}(t) dt}{\left[ \int_0^b (Sh)(t)^p \phi(t) dt \right]^\frac{1}{p}}.
\]

This suggests we take \( \psi(t) = (\bar{\phi}(t))^{1-p'} \) where \( \bar{\phi} \) is, in some sense the smallest weight such that (7.1) holds. Andersen’s condition (3.5) for (7.1) leads us to solve for \( \phi(t)^{1-p'} \) in the equation

\[
(7.2) \quad \left[ \int_0^b \frac{\phi(s)}{(s+t)^p} ds \right]^\frac{1}{p} \left[ \int_0^b \left( \frac{t}{s+t} \right)^{p'} \bar{\phi}(s)^{1-p'} ds \right]^\frac{1}{p'} = 1,
\]

or, what is equivalent,

\[
\int_0^t \phi(s)^{1-p'} ds + t^{p'} \int_t^b \phi(s)^{1-p'} s^{-p'} ds = \left[ t^{-p} \int_0^t \phi(s) ds + \int_t^b \phi(s) s^{-p} ds \right]^{1-p'}.
\]

Differentiation with respect to \( t \) yields

\[
t^{p'-1} \int_t^b \phi(s)^{1-p'} s^{-p'} ds = \int_0^t \phi(s) ds \left[ t^{-p} \int_0^t \phi(s) ds + \int_t^b \phi(s) s^{-p} ds \right]^{-p'}.
\]

Differentiating again with respect to \( t \) we get

\[
\bar{\phi}(t)^{1-p'} = \frac{p p'^{p'-1} \int_0^t \phi(s) ds \int_t^b \phi(s) s^{-p} ds}{\left[ \int_0^t \phi(s) ds + t^p \int_t^b \phi(s) s^{-p} ds \right]^{p'+1}} - \frac{t^{p'} \phi(t)}{\left[ \int_0^t \phi(s) ds + t^p \int_t^b \phi(s) s^{-p} ds \right]^{p'}}.
\]
It seems we essentially have

\begin{equation}
\overline{\phi}(t)^{1-p'} = \frac{(P\phi)(t)(Q_p\phi)(t)}{[(P\phi)(t) + (Q_p\phi)(t)]^{p'+1}}.
\end{equation}

The weight \( \hat{\phi}(t) \) given by

\[
\hat{\phi}^{1-p'}(t) = \left[ \int_0^t \phi(s)ds + tp \int_t^b \phi(s)s^{-p}ds \right]^{p'} = \frac{\phi(t)}{[(P\phi)(t) + (Q_p\phi)(t)]^{p'}}
\]

is readily shown to satisfy Andersen’s condition \((7.2)\) and, hence, so will

\[
\frac{(P\phi)(t)(Q_p\phi)(t)}{[(P\phi)(t) + (Q_p\phi)(t)]^{p'+1}} = \overline{\phi}(t)^{1-p'} + \hat{\phi}(t)^{1-p'}.
\]

Now, \( \overline{\phi}(t) \) will be better then \( \hat{\phi}(t) \) in \((7.1)\) if

\[
\int_0^b \hat{g}^{**}(t)^{p'} \hat{\phi}(t)^{1-p'} dt \leq C \int_0^b g^{**}(t)^{p'} \overline{\phi}(t)^{1-p'} dt.
\]

One readily infers from Theorem 5.1 that this will be so if and only if

\[
\int_0^t s^{p'-1} \int_s^b \hat{\phi}(y)^{1-p'} y^{-p'} dy ds \leq C \int_0^t s^{p'-1} \int_s^b \overline{\phi}(y)^{1-p'} y^{-p'} dy ds.
\]

But,

\[
\int_s^b \overline{\phi}(y)^{1-p'} y^{-p'} dy \approx \int_s^b y^{-p'} \frac{(P\phi)(y)(Q_p\phi)(y)}{[(P\phi)(y) + (Q_p\phi)(y)]^{p'+1}} dy = \int_s^b y^{p-1} \int_0^y \phi(z) dz \int_0^b \phi(z) z^{-p} dz dy
\]

\[
= -\frac{1}{p'} \int_s^b \int_0^y \phi(z) dz \frac{d}{dz} \left[ \int_0^y \phi(z) dz + y^p \int_y^b \phi(z) z^{-p} dz \right]^{p'} dy
\]

\[
= -\frac{1}{p'} \int_0^y \phi(z) dz \left[ \int_0^y \phi(z) dz + y^p \int_y^b \phi(z) z^{-p} dz \right]^{p'} \bigg|_s^b
\]

\[
+ \frac{1}{p'} \int_s^b \phi(y) \left[ \int_0^y \phi(z) dz + y^p \int_y^b \phi(z) z^{-p} dz \right]^{p'} dy
\]

\[
\geq \frac{1}{p'} \int_s^b \hat{\phi}(y)^{1-p'} y^{-p'} dy,
\]

if \( \int_0^b \phi(z) dz = \infty. \)
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