Birational geometry in codimension 2 of symplectic resolutions

Baohua Fu

January 15, 2022

Abstract

We prove the conjecture that two projective symplectic resolutions for a symplectic variety $W$ are related by Mukai’s elementary transformations over $W$ in codimension 2 in the following cases: (i). nilpotent orbit closures in a classical simple complex Lie algebra; (ii). some quotient symplectic varieties.

1 Introduction

A symplectic variety is a complex algebraic variety $W$, smooth in codimension 1, such that there exists a regular symplectic form on its smooth part which can be extended to a global regular form on any resolution (see [Bea]). A resolution $\pi : X \to W$ is called symplectic if the lifted regular form on $X$ is non-degenerated everywhere. One can show that a resolution is symplectic if and only if it is crepant.

Let $W$ be a symplectic variety and $\pi : X \to W$ a symplectic resolution. Assume that $W$ contains a smooth subvariety $Y$ such that the restriction of $\pi$ to $P := \pi^{-1}(Y)$ makes $P$ a $\mathbb{P}^n$-bundle over $Y$. If $\text{codim}(P) = n$, then we can blow up $X$ along $P$ and then blow down along the other direction, which gives another (proper) symplectic resolution $\pi^+ : X^+ \to W$, provided that $X^+$ remains in our category of algebraic varieties. The diagram $X \to W \leftarrow X^+$ is called Mukai’s elementary transformation (MET for short) over $W$ with center $Y$. A MET in codimension 2 is a diagram which becomes a MET after removing subvarieties of codimension greater than 2.
**Conjecture 1.** Let $W$ be a symplectic variety which admits two projective symplectic resolutions $\pi : X \to W$ and $\pi^+ : X^+ \to W$. Then the birational map $\phi = (\pi^+)^{-1} \circ \pi : X \to X^+$ is related by a sequence of METs over $W$ in codimension 2.

Notice that since the two resolutions $\pi, \pi^+$ are both crepant, the birational map $\phi$ is isomorphic in codimension 1.

In [HY] (Conjecture 7.3), Hu and Yau made an analogue conjecture for birational maps between smooth projective holomorphic symplectic varieties. Regarding the many alike properties shared by hyperkähler manifolds and projective symplectic resolutions, we believe that their conjecture holds for projective symplectic resolutions.

This conjecture is true for four-dimensional symplectic varieties by the work of Wierzba and Wiśniewski ([WW]). The purpose of this note is to provide more evidence for this conjecture. Using results of Namikawa ([Nam]), we prove the following:

**Theorem 1.1.** Let $\mathcal{O}$ be a nilpotent orbit closure in a simple classical Lie algebra. Then any two (proper) symplectic resolutions for $\mathcal{O}$ are connected by a sequence of METs over $\mathcal{O}$ in codimension 2.

Then we consider the situation of quotient symplectic varieties. Let $V$ be a vector space and $G$ a finite subgroup of $\text{GL}(V)$, then $G$ acts naturally on $T^*V$ and the quotient $(T^*V)/G$ is a symplectic variety. In this case, we have the following:

**Theorem 1.2.** Let $G$ be a finite subgroup of $\text{GL}(V)$ such that for any codimension 2 subspace $H \subset V$, the set $\{ g \in G | V^g = H \}$ forms a single conjugacy class. Then for any two projective symplectic resolutions $\pi_i : Z_i \to (T^*V)/G$, $i = 1, 2$, the induced birational map $\phi : Z_1 \to Z_2$ is isomorphic in codimension 2.

Note that there are only finitely many codimension 2 subspaces $H$ such that the set $\{ g \in G | V^g = H \}$ is non-empty. This theorem has an interesting corollary.

**Corollary 1.3.** Let $G$ be a finite subgroup of $\text{GL}(2)$ such that the elements $g \in G$ whose eigenvalues are all different to 1 form a single conjugacy class. Then $(T^*\mathbb{C}^2)/G$ admits at most one projective symplectic resolution, up to isomorphisms.
In the proof, we obtain the following theorem (valid for a general $G$), which could be helpful in further studies. This also gives an illustration of philosophy of the McKay correspondence: how the geometry of a symplectic resolution of $(T^*V)/G$ is controlled by the group $G$.

**Theorem 1.4.** Let $V$ be a vector space and $G < GL(V)$ a finite sub-group. Suppose we have a projective symplectic resolution $\pi : Z \to (T^*V)/G$. Then:

(i) $V/G$ is smooth;

(ii) $Z$ contains a Zariski open set $U$ which is isomorphic to $T^*(V/G)$;

(iii) the morphism $\pi : T^*(V/G) \to (T^*V)/G$ is the natural one (c.f. section 3), which is independent of the resolution.

2 Nilpotent orbits

2.1 Stratified Mukai flops

Consider the nilpotent orbit $O = O_{[2k,1^{n-2k}]}$ in $\mathfrak{sl}_n$, where $2k \leq n$. The closure $\overline{O}$ admits exactly two symplectic resolutions given by

$$T^*G(k,n) \xrightarrow{\pi} \overline{O} \xleftarrow{\pi^+} T^*G(n-k,n),$$

where $G(k,n)$ (resp. $G(n-k,n)$) is the Grassmannian of $k$ (resp $n-k$) dimensional subspaces in $\mathbb{C}^n$. Let $\phi$ be the induced birational map $T^*G(k,n) \dashrightarrow T^*G(n-k,n)$.

It is shown by Namikawa (Nam Lemma 3.1) that when $2k < n$, $\pi$ and $\pi^+$ are both small and the diagram is a flop. This is the stratified Mukai flop of type $A_{k,n}$. When $2k = n$, the birational map $\phi$ is an isomorphism.

**Lemma 2.1.** If $n \neq 2k + 1$, then $\phi$ is an isomorphism in codimension 2. If $n = 2k + 1$, then $\phi$ is a MET over $\overline{O}$ in codimension 2.

**Proof.** The closure $\overline{O}$ consists of orbits $\{O_{[2i,1^{n-2i}]}\}_{0 \leq i \leq k}$. The fiber of $\pi$ (resp. $\pi^+$) over a point in $O_{[2i,1^{n-2i}]}$ is isomorphic to $G(k-i,n-2i)$ (resp. $G(n-k-i,n-2i)$). By a simple dimension count, one shows that the complement of $\pi^{-1}(O)$ (resp. $(\pi^+)^{-1}(O)$) is of codimension greater than 2 when $n \neq 2k + 1$, which proves that $\phi$ is isomorphic in codimension 2.

Now suppose that $n = 2k + 1$. Let $Y$ be the nilpotent orbit $O_{[2k-1,1^3]}$ and $P$ (resp. $P^+$) the preimage of $Y$ under $\pi$ (resp. $\pi^+$). Then $P$ is the
subvariety
\[
\{(\{F\}, x) \in G(k, 2k + 1) \times Y | \text{Img}(x) \subset F \subset \text{Ker}(x)\}\}
\]
\[\text{in } T^*G(k, 2k + 1) \subset G(k, 2k + 1) \times \overrightarrow{O}.\] The induced map \(P \to Y\) makes \(P\) a \(\mathbb{P}^2\)-bundle over \(Y\). Similarly \(P^+\) is the subvariety
\[
\{(\{F^+\}, x) \in G(k + 1, 2k + 1) \times Y | \text{Img}(x) \subset F^+ \subset \text{Ker}(x)\}\}
\]
\[\text{in } T^*G(k + 1, 2k + 1) \subset G(k + 1, 2k + 1) \times O.\] The induced map \(P^+ \to Y\) makes \(P^+\) a \(\mathbb{P}^2\)-bundle over \(Y\).

Let \(U = \mathcal{O} \cup Y\), which is open in \(\overrightarrow{O}\). The complement of \(\pi^{-1}(U)\) (resp. \((\pi^+)\)^{-1}(U)) is of codimension greater than 2. Notice that the \(\mathbb{P}^2\)-bundle \(P\) over \(Y\) is the dual of the \(\mathbb{P}^2\)-bundle \(P^+\) over \(Y\). One deduces that the diagram \(\pi^{-1}(U) \to U \leftarrow (\pi^+)\)^{-1}(U)\) is a MET over \(U\) with center \(P\), which concludes the proof. 

Notice that the precedent proof gives an explicit description of the center of the MET, which will be used later.

Now we introduce the stratified Mukai flops of type \(D\). Let \(\mathcal{O}\) be the orbit \(\mathcal{O}_{[2k-1, 12]}\) in \(\mathfrak{so}_{2k}\), where \(k \geq 3\) is an odd integer. Let \(G^+_{iso}(k), G^-_{iso}(k)\) be the two connected components of the orthogonal Grassmannian of \(k\)-dimensional isotropic subspace in \(\mathbb{C}^{2k}\) (endowed with a fixed non-degenerate symmetric form). Then we have two symplectic resolutions \(T^*G^+_{iso}(k) \to \overrightarrow{O} \leftarrow T^*G^-_{iso}(k)\). It is shown in [Nam] (Lemma 3.2) that this diagram is a flop and the two resolutions are both small.

Let \(\phi\) be the induced birational map from \(T^*G^+_{iso}(k)\) to \(T^*G^-_{iso}(k)\). Then a simple dimension count shows that:

**Lemma 2.2.** \(\phi\) is an isomorphism in codimension 2.

2.2 \(g = \mathfrak{sl}_n\)

Let \(\mathcal{O}\) be a nilpotent orbit in \(\mathfrak{sl}_n\) corresponding to the partition \(d = [d_1, \cdots, d_k]\) and \(x \in \mathcal{O}\). Let \((p_1, \cdots, p_s)\) be a sequence of integers such that \(d_i = \# \{j | p_j \geq i\}\). Fix a flag \(F := \{F_i\}\) of \(\mathbb{C}^n\) of type \((p_1, \cdots, p_s)\) such that \(xF_i \subset F_{i-1}\) for all \(i\). Such a flag is called a \textit{polarization} of \(x\). Every nilpotent element has only finitely many different polarizations.

Assume that \(p_{j-1} < p_j\) for some \(j\). Consider the map \(\alpha : F_j \to F_j/F_{j-2}\). The element \(x\) induces \(\bar{x} \in \text{End}(F_j/F_{j-2})\). We define a flag \(F'\) by \(F'_i = F_i\)
if \( i \neq j - 1 \) and \( F'_{i-1} = \alpha^{-1}(\operatorname{Ker}(\tilde{x})) \). By Lemma 4.1 [Nam], \( F' \) is again a polarization of \( x \) with type \((p_1, \ldots, p_j, p_{j-1}, \ldots, p_s)\).

Let \( P \) (resp. \( P' \)) be the stabilizer of \( F \) (resp. \( F' \)) in \( G = SL_n \). Then we obtain two symplectic resolutions \( T^*(G/P) \xrightarrow{\pi} \mathcal{O} \xleftarrow{p} T^*(G/P') \). Let \( \phi : T^*(G/P) \rightarrow T^*(G/P') \) be the induced birational map.

**Lemma 2.3.** (i) If \( p_j \neq p_{j-1} + 1 \), then \( \phi \) is isomorphic in codimension 2;
(ii) If \( p_j = p_{j-1} + 1 \), then \( \phi \) is a MET over \( \mathcal{O} \) in codimension 2.

**Proof.** To simplify the notations, set \( r = p_{j-1} \) and \( m = p_j + p_{j-1} \). Let \( \bar{F} \) be the flag obtained from \( F \) by deleting the subspace \( F_{j-1} \), which is also obtained from \( F' \) by the same manner. We denote by \( \bar{P} \subset G \) the stabilizer of \( \bar{F} \). Let \( X \) be the subvariety in \( G/\bar{P} \times \mathcal{O} \) consisting of the points \((\bar{E}, y)\) such that (i) \( y\bar{E}_i \subset \bar{E}_{i-1} \) for \( i \neq j - 1 \) and \( y\bar{E}_{j-1} \subset \bar{E}_{j-1} \); (ii) the induced map \( \bar{y} \in \operatorname{End}(\bar{E}_{j-1}/\bar{E}_{j-2}) \) satisfies \( \bar{y}^2 = 0 \) and \( \operatorname{rank}(\bar{y}) \leq r \).

The projection to the second factor of \( G/\bar{P} \times \mathcal{O} \) induces a morphism \( pr : X \rightarrow \mathcal{O} \). By the proof of Lemma 4.3 [Nam], the resolutions \( \pi, \pi' \) factorize through the map \( pr \), which gives a diagram \( T^*(G/P) \xrightarrow{\mu} X \xleftarrow{\bar{\mu}} T^*(G/P') \). By Lemma 4.3 [Nam], this diagram is locally a trivial family of stratified Mukai flops of type \( A_{r,m} \). By Lemma 2.1 if \( m \neq 2r + 1 \), then \( \phi \) is isomorphic in codimension 2, which proves claim (i).

Now assume that \( m = 2r + 1 \). Let \( Y \) be the subvariety in \( X \) consists of the points \((\bar{E}, y)\) such that the induced map \( \bar{y} \in \operatorname{End}(\bar{E}_{j-1}/\bar{E}_{j-2}) \) has rank \( r - 1 \). By the proof of Lemma 2.1 and Lemma 4.3 [Nam], the diagram \( T^*(G/P) \xrightarrow{\mu} X \xleftarrow{\bar{\mu}} T^*(G/P') \) is a MET over \( X \) in codimension 2 with center \( Y \).

Let \( d' \) be the partition of \( n \) given by (possibly we need to re-order these parts):

\[
d'_i = \begin{cases} 
   d_i, & \text{if } i \neq r, r + 2 \\
   d_r - 1, & \text{if } i = r \\
   d_{r+2} + 1, & \text{if } i = r + 2.
\end{cases}
\]

Then one can verify that the morphism \( pr : X \rightarrow \mathcal{O} \) maps \( Y \) isomorphically to the nilpotent orbit \( \mathcal{O}_{d'} \), which shows that the diagram \( T^*(G/P) \xrightarrow{\mu} \mathcal{O} \xleftarrow{p} T^*(G/P') \) is a MET in codimension 2 over \( \mathcal{O} \) with center \( \mathcal{O}_{d'} \).

Notice that the precedent proof gives an explicit way to find out the MET center in \( \mathcal{O} \). Here we give an example.
Example 2.4. (Example 4.6 [Nam]). Let $\mathcal{O} = \mathcal{O}_{[3,2,1]} \subset \mathfrak{sl}_6$ and $x \in \mathcal{O}$. Then $x$ has six polarizations $P_{\sigma(1),\sigma(2),\sigma(3)}$ of flag type $(\sigma(1),\sigma(2),\sigma(3))$, where $\sigma$ are permutations. Let $Y_{i,j,k} = T^*(SL_6/P_{i,j,k})$, which gives a symplectic resolution for $\overline{\mathcal{O}}$. Then $Y_{321} \dashrightarrow Y_{231}$ is a MET in codimension 2 with center $\mathcal{O}_{[3,1^3]}$; $Y_{231} \dashrightarrow Y_{213}$ is isomorphic in codimension 2; $Y_{213} \dashrightarrow Y_{123}$ is a MET in codimension 2 with center $\mathcal{O}_{[2^3]}$ and so on. If a center appears twice in a sequence, then it is not really a MET center. For example, the birational map $Y_{321} \dashrightarrow Y_{132}$ is a MET in codimension 2 with center $\mathcal{O}_{[2^3]}$, but over the orbit $\mathcal{O}_{[3,1^3]}$, it is an isomorphism.

2.3 $\mathfrak{g} = \mathfrak{so}(V)$ or $\mathfrak{sp}(V)$

Let $V$ be an $n$-dimensional vector space endowed with a non-degenerate bilinear symmetric (resp. anti-symmetric) form for $\mathfrak{g} = \mathfrak{so}(V)$ (resp. $\mathfrak{g} = \mathfrak{sp}(V)$). Let $\epsilon = 0$ if $\mathfrak{g} = \mathfrak{so}(V)$ and $\epsilon = 1$ if $\mathfrak{g} = \mathfrak{sp}(V)$.

Let $P_{\epsilon}(n)$ be the set of partitions $\mathbf{d}$ of $n$ such that $\#\{i|d_i = l\}$ is even for every integer $l$ with $l \equiv \epsilon$ (mod 2). These are exactly those partitions which appear as the Jordan types of nilpotent elements of $\mathfrak{so}(V)$ or of $\mathfrak{sp}(V)$. Let $q$ be a non-negative integer such that $q \neq 2$ if $\epsilon = 0$. Define $Pai(n,q)$ to be the set of partitions $\mathbf{e}$ of $n$ such that $e_i \equiv 1 \pmod 2$ if $i \leq q$ and $e_i \equiv 0 \pmod 2$ if $i > q$. For $\mathbf{e} \in Pai(n,q)$, let $I(\mathbf{e}) = \{j|j \equiv n+1 \pmod 2, e_j \equiv \epsilon \pmod 2, e_j \geq e_{j+1} + 2\}$.

The Spaltenstein map $S : Pai(n,q) \rightarrow P_{\epsilon}(n)$ is defined as

$$S(\mathbf{e})_j = \begin{cases} e_j - 1, & \text{if } j \in I(\mathbf{e}) \\ e_j + 1, & \text{if } j - 1 \in I(\mathbf{e}) \\ e_j, & \text{otherwise}. \end{cases}$$

It is proved in [Hes] that for a nilpotent element of type $\mathbf{d}$, its polarization types are determined by $S^{-1}(\mathbf{d})$. For a sequence of integers $(p_1,\cdots,p_k)$, we define $\mathbf{e} = ord(p_1,\cdots,p_k)$ to be the partition given by $e_i = \#\{j|p_j \geq i\}$.

Let $\mathcal{O}$ be a nilpotent orbit of type $\mathbf{d}$ in $\mathfrak{g}$ and $x \in \mathcal{O}$. Let $(p_1,\cdots,p_k,q,p_k,\cdots,p_1)$ be a sequence of integers such that $\mathbf{e} : = ord(p_1,\cdots,p_k,q,p_k,\cdots,p_1)$ is in $Pai(n,q)$ and $S(\mathbf{e}) = \mathbf{d}$. Let $F$ be an isotropic flag (i.e. $F_i^\perp = F_{2k+1-i}, \forall i$) in $V$ of type $(p_1,\cdots,p_k,q,p_k,\cdots,p_1)$ such that $xF_i \subset F_{i-1}$.

Assume that $p_{j-1} < p_j$ for some $j$. Consider the map $\alpha : F_j \rightarrow F_j/F_{j-2}$. The element $x$ induces $\bar{x} \in End(F_j/F_{j-2})$. We define another flag $F'$ by
$F_i' = F_i$ if $i \neq j-1, 2k+2-j$, $F_{j-1}' = \alpha^{-1}(\text{Ker}(\bar{x}))$ and $F_{2k+2-j}' = (F_{j-1}')^\perp$. By Lemma 4.2 \cite{Nam}, $F'$ is again a polarization of $x$. We denote by $P$ (resp. $P'$) the stabilizer of $F$ (resp. $F'$). Then we obtain two symplectic resolutions $T^*(G/P) \xrightarrow{\varphi} \mathcal{O} \xleftarrow{\varphi} T^*(G/P')$. Let $\phi$ be the induced birational map from $T^*(G/P)$ to $T^*(G/P')$.

**Lemma 2.5.**

(i) If $p_j \neq p_{j-1} + 1$, then $\phi$ is isomorphic in codimension 2.

(ii) If $p_j = p_{j-1} + 1$, then $\phi$ is a MET in codimension 2 over $\mathcal{O}$.

The proof goes along the same line as that in Lemma 2.3. The difference is the definition of the partition $d'$ in the proof of (ii). Here we have $r = p_{j-1}$ and $p_j = r + 1$. Let $e'$ be the partition (after re-ordering if necessary) defined by

$$e'_j = \begin{cases} 
eq j \neq r, r + 2, & \\
eq r - 2, & j = r, \\
eq r + 2, & j = r + 2. \\
\end{cases}$$

Then $e' \in \text{Pai}(n, q)$. Now we should define $d' = S(e')$. In this case, $\phi$ is a MET in codimension 2 over $\mathcal{O}$ with center $\mathcal{O}_{d'}$.

**Example 2.6.** (Example 4.7 \cite{Nam}). Let $\mathcal{O} = \mathcal{O}_{[4, 12]}$ be the nilpotent orbit in $\mathfrak{so}_{10}$. Take an element $x \in \mathcal{O}$, then $x$ has four polarizations $P_{3223}^+, P_{3223}^-, P_{2332}^+, P_{2332}^-$. Let $Y_{3223}^+ = T^*(G/P_{3223}^+)$ and so on. Then $Y_{3223}^+ \dashrightarrow Y_{2332}^+$ is a MET in codimension 2 over $\mathcal{O}$ with center $\mathcal{O}_{[3^2, 2^2]}$.

**2.4 Proof of Theorem 1.1**

Let $\mathcal{O}$ be a nilpotent orbit in a classical simple Lie algebra $\mathfrak{g}$. By \cite{Nam}, every (proper) symplectic resolution for $\mathcal{O}$ is of the form $T^*(G/P) \to \mathcal{O}$ for some polarization $P$ of $\mathcal{O}$. Assume that we have two symplectic resolutions $T^*(G/P_i) \to \mathcal{O}, i = 1, 2$, then by the proof of Theorem 4.4 \cite{Nam}, we can reach $T^*(G/P_2) \to \mathcal{O}$ from $T^*(G/P_1) \to \mathcal{O}$ by using the operations in section 2.2 and 2.3, possibly by using another operation which is a locally trivial family of stratified Mukai flops of type $D$ (thus isomorphic in codimension 2 by Lemma 2.2). Now Lemma 2.3 and 2.5 give the theorem.
Let $V$ be an $n$-dimensional vector space and $G$ a finite subgroup of $GL(V)$. Then $G$ acts naturally on $T^*V \simeq V \oplus V^*$, preserving the symplectic form on $T^*V$. By $[\text{Bea}]$, $W := (T^*V)/G$ is a symplectic variety. We will study projective symplectic resolutions of $W$.

Consider the $C^\ast$-action on $T^*V$ defined by $\lambda(v, v') = (v, \lambda v')$ for $\lambda \in C^\ast$ and $(v, v') \in V \oplus V^*$. This action commutes with the action of $G$, thus we obtain an action of $C^\ast$ on $W$. The fixed point set under this action is identified with $V/G$. If we denote by $\omega$ the symplectic form on the smooth part of $W$, then $\lambda^*\omega = \lambda \omega$ for all $\lambda \in C^\ast$.

Let $Z \xrightarrow{\pi} W$ be a projective symplectic resolution. By $[\text{Ka1}]$, the $C^\ast$-action on $W$ lifts to a $C^\ast$-action on $Z$ in such a way that $\pi$ is $C^\ast$-equivariant. If we denote by $\Omega$ the symplectic form extending $\pi^*\omega$ to the whole of $Z$, then $\lambda^*\Omega = \lambda \Omega$ for any $\lambda \in C^\ast$.

Let $Z_{C^\ast}$ be the points of $Z$ fixed by the $C^\ast$-action.

**Lemma 3.1.** There exists a connected component $Y$ of $Z_{C^\ast}$ such that $\pi : Y \to V/G$ is an isomorphism. In particular, $V/G$ is smooth.

This is proved by Kaledin (the proof of Theorem 1.7 $[\text{Ka1}]$). The following is proved in $[\text{Fu2}]$, but since $[\text{Fu2}]$ will never be published, we include the proof here. Let $U = \{z \in Z | \lim_{\lambda \to 0} \lambda \cdot z \in Y\}$.

**Lemma 3.2.** $U$ is isomorphic to $T^*Y$, and the induced $C^\ast$-action on $T^*Y$ is the natural action: $\lambda(y, v') = (y, \lambda v')$, for any $y \in Y$, $v' \in T^*_yY$ and $\lambda \in C^\ast$.

**Proof.** The precedent lemma shows that $Y$ is isomorphic to $V/G$, in particular $\dim(Y) = n$. For any point $y \in Y$, the action of $C^\ast$ on $Z$ induces a weight decomposition

$$T_yZ = \bigoplus_{p \in \mathbb{Z}} T_y^pZ,$$

where $T_y^pZ = \{v \in T_yZ | \lambda^* v = \lambda^p v\}$, and $T_yY$ is identified with $T_y^0Z$. The relation $\lambda^*\Omega = \lambda \Omega$ gives a duality between $T_y^p(Z)$ and $T_y^{1-p}(Z)$. In particular, $\dim(T_y^pZ) = \dim(T_y^1Z) = \dim Y = n$, so $T_y^pZ = 0$ for all $p \neq 0, 1$, which gives a decomposition $T_yZ = T_yY \oplus T_y^1Z$. Furthermore $Y$ is Lagrangian with respect to $\Omega$.

By the work of Bialynicki-Birula ($[\text{BB}]$), the decomposition $T_yZ = T_yY \oplus T_y^1Z$ shows that $U$ is a vector bundle of rank $n$ over $Y$, so $U$ is identified with the total space of the normal bundle $N$ of $Y$ in $Z$. Now we establish an isomorphism between $N$ and $T^*Y$ as follows. Denote by $\Omega_{can}$ the canonical
symplectic structure on $T^*Y$. Take a point $y \in Y$, and a vector $v \in N_y$. Since $Y$ is Lagrangian in the both symplectic spaces, there exists a unique vector $w \in T^*_y Y$ such that $\Omega_y(v, u) = \Omega_{can,y}(w, u)$ for all $u \in T^*_y Y$. We define the map $i : N \to T^*Y$ to be $i(v) = w$. It is clear that $i$ is a $\mathbb{C}^*$-equivariant isomorphism.

Now we will study in more detail the morphism: $\pi : T^*Y \simeq T^*(V/G) \to (T^*V)/G$. We denote by $p : V \to V/G$ the natural projection and $p_* : TV \to T(V/G)$ the induced tangent morphism.

We define a morphism $\tilde{p} : T^*(V/G) \to (T^*V)/G$ as follows: take a point $[x] \in V/G$ and a co-vector $\alpha \in T^*_x(V/G)$. We define a co-vector $\beta \in T^*_{[x]}V$ by $< \beta, v >= < \alpha, p_*(v) >$ for all $v \in T_x V$. Then we put $\tilde{p}([x], \alpha) = [x, \beta]$.

**Lemma 3.3.** $\tilde{p}$ is well-defined.

**Proof.** Let $y = gx$ with $g \in G$. We consider $\beta' \in T^*_y V$ defined by $< \beta', w >= < \alpha, p_*(w) >$ for all $w \in T^*_y V = g_* T_x V$. Then $w = g_*v$ for some $v \in T_x V$. Now we have

$$< \beta', g_*v >= < \alpha, p_*g_*(v) > = < \alpha, p_*(v) > = < \beta, v >$$

for all $v \in T_x V$, which gives $\beta' = g^*\beta$. Then $[y, \beta'] = [gx, g^*\beta] = [x, \beta]$ in $(T^*V)/G$. \hfill $\square$

Notice that $p : V \to V/G$ is étale at a point $x \in V$ if and only if the stabilizer $G_x$ of $x$ in $G$ is trivial, thus $\tilde{p}|_{[x]} : T^*_x(V/G) \to (T^*_x V)/G$ is an isomorphism if and only if $G_x$ is trivial. Furthermore $\tilde{p}$ induces an identity on the zero section $V/G$.

**Lemma 3.4.** The morphism $\pi : T^*Y \simeq T^*(V/G) \to (T^*V)/G$ coincides with the morphism $\tilde{p}$.

**Proof.** Let $V_0 = V - \bigcup_{g \neq 1} V^g$, on which $G$ acts freely. $\tilde{p}$ induces an isomorphism $\tilde{p}_0 : T^*(V_0/G) \to (T^*V_0)/G$. We will show that $f := \tilde{p}_0^{-1} \circ \pi|_{T^*(V_0/G)} : T^*(V_0/G) \to T^*(V_0/G)$ is an identity. Notice that $f$ is an identity over the zero section $V_0/G$. Furthermore $f$ is $\mathbb{C}^*$-equivariant and it preserves the natural symplectic form on $T^*(V_0/G)$.

Take a point $[x] \in V_0/G$ and $\alpha \in T^*_x(V_0/G)$. We consider $\alpha$ as a vector in $T_{[x]}(T^*(V_0/G))$, then $f_*(\alpha) = \frac{d}{d\lambda}|_{\lambda=0} f(\lambda \alpha) = \frac{d}{d\lambda}|_{\lambda=0} \lambda f(\alpha) = f(\alpha)$. Now since $f$ preserves the symplectic form, we have $< \alpha, v >= f_*(\alpha), f_*(v) >=$

9
Let \( G \) be a finite subgroup of \( GL(V) \) such that for any codimension 2 subspace \( H \subset V \), the set \( \{ g \in G | V^g = H \} \) forms a single conjugacy class. Then for any two projective symplectic resolutions \( \pi_i : Z_i \to (T^*V)/G, i = 1, 2 \), the induced birational map \( \phi : Z_1 \dashrightarrow Z_2 \) is isomorphic in codimension 2.

Proof. Let \( g \) be an element such that \( H := V^g \) is of codimension 2. Let \( W_g = (T^*H)/C(g) \), where \( C(g) = Stab(H)/Cent(H) \) is the quotient of the subgroup \( Stab(H) \) of elements \( h \in G \) which preserve \( H \) by the subgroup \( Cent(H) \) of elements \( h \in Stab(H) \) which acts as identity on \( H \). \( W_g \) is of codimension 4 in \( W \), whose preimage by \( \pi_i \) is of codimension 2, by the semismallness of projective symplectic resolutions (see [Ka1]).

By the McKay correspondence (see [Ka2]), the number of codimension 2 components in \( \pi_i^{-1}(W_g) \) equals to the number of conjugacy classes of \( \{ h \in G | V^h = H \} \), which is 1 by the hypothesis.

Let \( U_i \) be the Zariski open subset in \( Z_i \) given by Lemma 3.2 and \( q_i : U_i \simeq T^*Y_i \to Y_i \) the natural projection. Let \( F_i \subset Y_i \simeq V/G \) be the subvariety \( H/C(g) \). By Lemma 3.4 and the explicite description of \( \tilde{p} \), we have \( q_i^{-1}(F_i) \subseteq \pi_i^{-1}(W_g) \). Notice that \( q_i^{-1}(F_i) \) is of codimension 2, so its closure is the unique codimension 2 component of \( \pi_i^{-1}(W_g) \), then the complement of \( U_i \cap \pi_i^{-1}(W_g) \) is of codimension at least 3 in \( \pi_i^{-1}(W_g) \). By Lemma 3.5 the birational map \( \phi \) induces an isomorphism between \( U_1 \) and \( U_2 \). The above arguments show that \( \phi \) gives an isomorphism between the generic points of the unique codimension 2 components in \( \pi_i^{-1}(W_g) \), for all \( g \in G \) such that \( V^g \) is of codimension 2 in \( V \), thus \( \phi \) is isomorphic in codimension 2.

**Corollary 3.6.** Let \( G \) be a finite subgroup in \( GL(2) \) such that the elements \( g \in G \) whose eigenvalues are all different to 1 form a single conjugacy class. Then \( (T^*\mathbb{C}^2)/G \) admits at most one projective symplectic resolution, up to isomorphisms.
Proof. By the hypothesis, the set \( \{ g \in G | V^g = 0 \} \) forms a single conjugacy class, thus the two symplectic resolutions are isomorphic in codimension 2. By the work of [WW], any two projective symplectic resolutions for \((T^*\mathbb{C}^2)/G\) are connected by Mukai flops, so there is no flop at all, thus the two resolutions are isomorphic. \(\square\)

Remark 3.7. This theorem and the corollary can be regarded as an extension of Corollary 2.4 in [FN] (see also theorem 1.9 [Ka1]), where it is proved that if the complex reflections in \(G\) form a single conjugacy class, then the quotient admits at most one projective symplectic resolution (up to isomorphisms).

Example 3.8. Here we give an example to show that our assumption on \(G\) cannot be removed. Let \(x, y\) be the coordinates of \(\mathbb{C}^2\) and \(G\) the subgroup of \(GL(2)\) generated by the two elements: \(\sigma(x, y) = (x, -y)\) and \(\tau(x, y) = (y, x)\). Then \(G\) is the dihedral group of order 8, which acts on \(T^*(\mathbb{C}^2)\) (with coordinates \(x, z, y, w\)) as follows:

\[
\sigma(x, z, y, w) = (x, z, -y, -w), \quad \tau(x, z, y, w) = (y, w, x, z).
\]

The quotient \(T^*(\mathbb{C}^2)/G\) is isomorphic to \(Sym^2(S)\), where \(S = \mathbb{C}^2/\pm1\), which possesses exactly two non-isomorphic symplectic resolutions (for details, see Example 2.7 [FN]), one is connected to the other by a Mukai’s elementary transformation.

Example 3.9. Here we give an example to show that our corollary covers some situations where Corollary 2.4 in [FN] does not apply. Let \(G\) be the subgroup of \(GL(2)\) generated by the following two elements:

\[
\sigma : (x, y) \mapsto (-x, y); \quad \tau : (x, y) \mapsto (x, -y).
\]

There is only one element \(\sigma \circ \tau\) whose eigenvalues are different to 1. By the precedent Corollary, two projective symplectic resolutions for \((T^*\mathbb{C}^2)/G\) are isomorphic.

In fact, \((T^*\mathbb{C}^2)/G\) is isomorphic to the product of two \(A_1\)-singularities, thus it admits a symplectic resolution \(T^*\mathbb{P}^1 \times T^*\mathbb{P}^1 \to (T^*\mathbb{C}^2)/G\). Notice that the unique 2-dimensional fibre is isomorphic to \(\mathbb{P}^1 \times \mathbb{P}^1\), thus no MET over \((T^*\mathbb{C}^2)/G\) can be performed. By [WW], this is the unique (up to isomorphisms) projective symplectic resolution for \((T^*\mathbb{C}^2)/G\).
References

[Bea] Beauville, A., *Symplectic singularities*, Invent. Math. **139** (2000), 541–549

[BB] Bialynicki-Birula, A., *Some theorems on actions of algebraic groups*, Ann. of Math., II. Ser. **98** (1973), 480-497

[Fu1] Fu, B., *Symplectic resolutions for nilpotent orbits*, Invent. Math. **151** (2003), 167–186

[Fu2] Fu, B., *Symplectic resolutions for quotient singularities*, math.AG/0206288

[FN] Fu, B. and Namikawa, Y., *Uniqueness of crepant resolutions and symplectic singularities*, Ann. Inst. Fourier, **54** (2004), no. 1, 1–19

[Hes] Hesselink, W., *Polarization in the classical groups*, Math. Z., **160** (1978), 217–234

[HY] Hu, Y.; Yau, S.-T., *HyperKähler manifolds and birational transformations*, Adv. Theor. Math. Phys. **6** (2002), no. 3, 557–574

[Ka1] Kaledin, D., *On crepant resolutions of symplectic quotient singularities*, Selecta Math. (N.S.) **9** (2003), no. 4, 529–555

[Ka2] Kaledin, D., *McKay correspondence for symplectic quotient singularities*, Invent. Math. **148** (2002), no. 1, 151–175

[Nam] Namikawa, Y., *Birational Geometry of symplectic resolutions of nilpotent orbits*, math.AG/0404072

[WW] Wierzba, J.; Wiśniewski, J., *Small contractions of symplectic 4-folds*, Duke Math. J. **120** (2003), no. 1, 65–95

Labortoire J. Leray, Faculté des sciences
2, Rue de la Houssinière, BP 92208
F-44322 Nantes Cedex 03 - France

baohua.fu@polytechnique.org