Real-Time Peer-to-Peer Streaming Over Multiple Random Hamiltonian Cycles

Joohwan Kim* and R. Srikant†
Dept. of Electrical and Computer Engineering and Coordinated Science Laboratory
University of Illinois at Urbana-Champaign
Email: {∗joohwan, †rsrikant}@illinois.edu

Abstract
We are motivated by the problem of designing a simple distributed algorithm for Peer-to-Peer streaming applications that can achieve high throughput and low delay, while allowing the neighbor set maintained by each peer to be small. While previous works have mostly used tree structures, our algorithm constructs multiple random directed Hamiltonian cycles and disseminates content over the superposed graph of the cycles. We show that it is possible to achieve the maximum streaming capacity even when each peer only transmits to and receives from $Ω(1)$ neighbors. Further, we show that the proposed algorithm achieves the streaming delay of $Ω(\log N)$ when the streaming rate is less than $(1 - 1/K)$ of the maximum capacity for any fixed constant $K \geq 2$, where $N$ denotes the number of peers in the network. The key theoretical contribution is to characterize the distance between peers in a graph formed by the superposition of directed random Hamiltonian cycles, in which edges from one of the cycles may be dropped at random. We use Doob martingales and graph expansion ideas to characterize this distance as a function of $N$, with high probability.

Index Terms
Peer-to-Peer Networks, Streaming Media, Delay Analysis, Random Graph Theory

I. INTRODUCTION
Dissemination of multimedia content over the Internet is often accomplished using a central server or a collection of servers which disseminate the data to all clients interested in the content. Youtube is an example of such a model, where multiple large-capacity servers are used to meet the download demands of millions of users. In contrast, in a peer-to-peer (P2P) network, a small (low-capacity) server uploads the content to a small number of clients, and these clients and all other clients in the network then exchange content among themselves. The P2P approach is scalable since the network utilizes the upload capacities of all the clients (commonly known as peers) in the network: as more peers join the network, the download requirement increases but the available capacity also increases proportionally. In this paper, we are interested in designing P2P networks where each peer needs to keep track of only a small number of other peers in the network. Restricting the neighborhood size of each peer reduces the administrative overhead for the peers. Therefore, one of the key challenges is to design an algorithm to decide which peers should belong to the same neighborhood. Such algorithms are called “pairing” algorithms since they pair peers to be neighbors. The pairing algorithm must be lightweight, i.e., when new peers enter the network or when existing peers exit the network, the algorithm should incur low overhead to readjust the pairing relationships in the neighborhood. In this paper, we propose a pairing algorithm based on directed Hamiltonian cycles which has low overhead for node insertion and deletion. The insertion and deletion parts of our algorithm are the same as the algorithm proposed

This work has been supported by the National Science Foundation grant CNS 09-64081 and Army MURI's W911NF-08-1-0233 and W911NF-07-1-0287.
in [1] for constructing small diameter graphs using undirected Hamiltonian cycles for distributed hash table (DHT) applications. However, there are certain key differences: our algorithm requires edges to be directed for real-time streaming purposes and while small diameter is sufficient for fast lookup times in DHTs, it is not sufficient to ensure high throughput and low delay for streaming applications.

The pairing algorithm mentioned in the previous paragraph determines the topology of the network. Given the topology, the network must then decide how to disseminate content in the network to achieve the maximum possible capacity and low delay. Multimedia content is often divided into chunks and thus, the content dissemination algorithm is also called the chunk dissemination algorithm in the literature. Chunk dissemination is accomplished by a peer in two steps in each timeslot: the peer has to select a neighbor to receive a chunk (called neighbor selection) and then it has to decide which chunk it will transmit to the selected neighbor (called chunk selection). Thus, the practical contributions of the paper can be summarized as follows: we present a low-complexity, high-throughput and low-delay algorithm for pairing, neighbor selection and chunk selection in real-time P2P streaming networks. We emphasize that the goal of this paper is to study real-time data dissemination in P2P networks. This is in contrast to stored multimedia content dissemination (which is the bulk of Youtube’s data, for example) or file-transfer applications (such as in BitTorrent).

Our approach for pairing results in a graph formed by the superposition of multiple random directed Hamiltonian cycles over a given collection of nodes (peers). We will see that the performance analysis of our algorithms requires us to understand the distance (the minimum number of hops) from a given peer to all other peers in the graph. The main theoretical contribution of the paper is to characterize these distances with high probability through a concentration result using Doob martingales. Using this result, we show that our algorithm achieves $\Theta(\log N)$ delay with high probability, when the streaming rate is less than or equal to $(1 - \frac{1}{K})$ of the optimal capacity for any constant $K \geq 2$, where $N$ denotes the number of peers in the network.

This paper is organized as follows. In Section II, we review prior work in the area of real-time P2P networks. In Section III, we provide two examples to help the reader understand the advantage of using random Hamiltonian cycles. In Section IV, we present our P2P algorithm that constructs random Hamiltonian cycles and disseminates content over the cycles in a fully distributed manner. In Section V, we consider the streaming rate that can be achieved under our algorithm. In Sections VI and VII, we analyze the delay to disseminate chunks to all peers under our algorithm. In Section VIII, we conclude the paper.

II. RELATED WORK

We briefly review prior work in the area of real-time P2P networks. Prior work in the area can be broadly categorized as designing one of two types of networks: a structured P2P network or an unstructured P2P network. The structured P2P streaming approach focuses on constructing multiple overlay spanning trees that are rooted at the source [2]–[10]. In this approach, the real-time content arriving at the source is divided into multiple sub-streams and each sub-stream is delivered over one of the trees. Since this approach uses the tree structure, connectivity from the source to all peers is guaranteed. By managing the tree depth to be $\Theta(\log N)$, this approach can guarantee $\Theta(\log N)$ delay to disseminate a chunk of each sub-stream to all peers. However, the fundamental limitation of the structured P2P streaming is vulnerability to peer churn. It is well known that the complexity of constructing and maintaining $\Theta(\log N)$-depth trees grows as $N$ increases [9], [10]. Therefore, in a highly dynamic P2P network where peers frequently join and leave the network, the structure approach is not scalable.

Unstructured P2P networks overcome this vulnerability to peer churn. In unstructured P2P networks, peers find their neighboring peers randomly and get paired with them locally. As a neighboring peer leaves, a peer chooses another peer randomly as its new neighboring peer. Due to the distributed fashion of this peer pairing, unstructured P2P networks are robust to peer
churn, unlike the structured P2P networks. However, the fundamental limitation of unstructured P2P networks is weak connectivity. Since peers are paired randomly without considering the entire network topology, there may be some peers that are not strongly connected from the source, which results in poor throughput and delay. To ensure full connectivity in this approach, it is required that every peer should be paired with $\Theta(\log N)$ neighboring peers [11], or should constantly change their neighbors to find neighbors providing a better streaming rate [12]. However, in these approaches, delay performance is hard to guarantee because chunks have to be disseminated over an “unknown” network topology.

Another interesting line of work has studied gossip-based algorithms that disseminate information to all peers in a fashion similar to the spread of epidemics. By studying the dissemination delay under these gossip-based algorithms, we can analyze the delay for peers to disseminate chunks to all peers in a P2P network. The seminal work in [13] shows that gossiping requires $\Theta(\log N)$ time with high probability to disseminate a single chunk from the source to all peers. When there is a sequence of chunks arriving at the source, the latest-blind algorithm proposed in [14] is proven to deliver $(1 - e^{-1})$ fraction of chunks to all peers with $\Theta(\log N)$ delay with high probability. Later work in [15] proposed the latest useful algorithm that can deliver almost all chunks with $\Theta(\log N)$ delay with high probability. However, the basic assumption for analysis in this line of work is that the network is a complete graph where every peer has $N - 1$ outgoing edges to all other peers, and only simulations are used in [15] to evaluate the performance on a random graph with bounded degree. In contrast, it is shown in [16] that gossip-based algorithms can achieve $\Theta(\log N)$ delay, when the matrix representing the connectivity between peers is doubly stochastic and symmetric. However, only a small fraction of the optimal throughput can be guaranteed with $\Theta(\log N)$ delay.

We address all the aforementioned limitations using multiple random Hamiltonian cycles. While the structure of Hamiltonian cycles provides us with full connectivity from the source to all peers, random pairing within each cycle enables peers to cope with peer churn. Furthermore, the proposed chunk dissemination algorithm guarantees $\Theta(\log N)$ delay required for each chunk to be disseminated to all the peers for a near optimal throughput. One may be concerned about using cycles because the diameter $N$ of a cycle could result in poor delay performance. However, we address this concern in the next section.

### III. INDEPENDENT RANDOM HAMILTONIAN CYCLES

In a delay-sensitive application, such as P2P streaming, cycles (or line topologies) have been considered to be undesirable since their diameter is $N - 1$, where the diameter of a directed graph is defined as the maximum distance between any pair of nodes. Delivering information from a node to all the other nodes over a cycle requires $N - 1$ successive transmissions, which results in $\Theta(N)$ delay. In this section, we consider two examples which show that one can use a superposed graph of multiple cycles as an alternative to the tree structure for information dissemination.

Consider $N$ nodes numbered $1, 2, \cdots, N$. By permuting these nodes, we can make a random Hamiltonian directed cycle as shown in Fig. 1(a) (Since all the graphs that we will consider are directed graphs, we will skip mentioning “directed” from now on.) Make another random Hamiltonian cycle by independently permuting the $N$ nodes as shown in Fig. 1(b). Clearly, the diameter of each cycle is $N - 1$. An interesting question is the following: “if we superpose both cycles, what is the diameter of the superposed graph?” Interestingly, the diameter significantly reduces from $\Theta(N)$ to $\Theta(\log N)$ with high probability in the superposed graph.

Next, we consider a further modification of the two random cycle model. From the second cycle (Fig. 1(b), we remove each edge with some probability $0 < q < 1$ independently. If we

---

1In random graph theory, it is shown that the superposition of two undirected random Hamiltonian cycles has a distribution similar to an undirected random regular graph [17]. This regular graph is known to have $\Theta(\log N)$ diameter with high probability [18]. Combining both results, we can infer that the superposition of two random undirected Hamiltonian cycles has $\Theta(\log N)$ depth with high probability. From this, it is not very difficult to obtain a similar result for directed cycles. However, in [19], we establish this result more directly.
superpose the first cycle and the remaining edges in the second cycle, what will be the diameter of the graph? Since we have removed around $qN$ edges from the second cycle, the diameter will certainly increase. However, we will show in a later section that the order of the diameter still remains $\Theta(\log N)$.

These two examples imply that a graph formed from superposed Hamiltonian cycles has a small diameter of $\Theta(\log N)$. This means that the superposed graph can be a good alternative to a spanning tree with a bounded outdegree that has been widely used to achieve a logarithmic dissemination delay in P2P streaming. However, in the case of peer churn, the complexity of constructing and updating spanning trees (as in prior literature) subject to the constraints on the degree bound and the logarithmic depth increases dramatically with the network size. In contrast, the superposed graph is robust to peer churn because independent cycles are much easier to maintain. In the rest of this paper, we show how these properties of random superposed cycles can be used to construct a P2P network that can achieve high throughput and low delay.

IV. System Model

We assume that time is slotted, and every peer (including the source peer) in the network contributes a unit upload bandwidth, i.e., each peer can upload one chunk per timeslot. In this case, it is well known that the maximum streaming rate (the maximum reception rate guaranteed to each peer) is approximately one for a large network because the total upload bandwidth $N$ contributed by all peers (including the source) has to be shared by $N - 1$ peers (excluding the source) [6]–[8]. Due to the limited communication and computation overheads, we assume that each peer can only communicate with a constant number of neighbors, which does not increase with the network size. We assume that there is peer churn, so that the topology is dynamic as new peers join or existing peers leave.

We now present our P2P streaming algorithm which consists of a peer-pairing algorithm and a chunk-dissemination algorithm. For convenience, we use the term chunk dissemination algorithm to describe the joint neighbor selection and chunk selection algorithms mentioned in the previous section. Our pairing algorithm is similar to the one in [1], except for the fact that we use directed edges. The fact that the edges are directed does not matter for adding or deleting nodes to the network; this part of our algorithm is identical to [1]. However, the fact that the edges are directed and the fact that we are interested in achieving the maximum streaming capacity make our work quite different from [1], where the only goal is to construct an expander graph (with undirected edges) in a distributed fashion. But it is important to understand the pairing algorithm to proceed further. Therefore, we present it next.
A. Peer Pairing Algorithm

Under our peer-pairing algorithm, every peer has \( M \geq 2 \) incoming edges and \( M \) outgoing edges as shown in Fig. 2. We number the incoming edges of each peer as the first, second, ..., \( M \)-th incoming edges of the peer and number its outgoing edges as the first, second, ..., \( M \)-th outgoing edges. The peer where the \( m \)-th outgoing edge ends is called the \( m \)-th child, and the peer where the \( m \)-th incoming edge begins is called the \( m \)-th parent. We assume that the \( M \) outgoing edges of a peer may end at the same peer, so that the number of children of a peer could be less than \( M \). Similarly, the \( M \) incoming edges of a peer may begin at the same peer, so that the number of parents of a peer could be less than \( M \). Under our algorithm, every peer receives chunks from its parents over its incoming edges and transmits received chunks to its children over its outgoing edges.

![Fig. 2. Basic structure of the peer-pairing algorithm for \( M = 2 \): Every peer maintains \( M \) incoming edges and \( M \) outgoing edges. The peers where the outgoing edges end are called children, and the peers where the incoming edges begin are called parents. Every peer \( v \) receives chunks from parents and transmits chunks to children.](image)

We next describe how every peer establishes its \( M \) incoming and \( M \) outgoing edges.

**Initially**, the network consists of only two peers, the source peer (which we call peer 1 throughout this paper) and the first peer to arrive at the network (which we call peer 2), as shown in Fig. 3(a). Each peer establishes its first, second, ..., \( M \)-th outgoing edges to the other peer, so that these edges are the first, second, ..., \( M \)-th incoming edges of the other peer. Letting \( V \) be the set of current peers, we define \( E_m \) to be the set of all \( m \)-th edges, i.e., \( E_m \equiv \{ (i, j) \in V^2 \mid j \text{ is the } m \text{-th child of } i \} \), for \( m = 1, 2, \cdots , M \). Initially, \( E_m \) is given by \( \{(1, 2), (2, 1)\} \) for all \( m \) because there are only two peers. We define \( L_m \equiv (V, E_m) \) to be the digraph consisting of the peer set \( V \) and the \( m \)-th edges, and call it layer \( m \) for \( m = 1, 2, \cdots , M \). Layer \( m \) represents the pairing between every peer and its \( m \)-th child. By superposing the \( M \) layers, the current network topology can be expressed as a multi-digraph \( L^* = (V, E^*) \), where \( E^* \) is a multiset defined as \( E^* = \{ E_1 \cup E_2 \cup \cdots \cup E_M \} \).

**When a new peer \( v \) arrives**, this peer independently chooses an edge from each layer uniformly at random and breaks into the chosen edges as Fig. 3(b). Specifically, if the peer \( v \) arrives and randomly chooses \( (p_m, c_m) \in E_m \) from layer \( m \), the peer \( v \) becomes a new \( m \)-th child of peer \( p_m \) and becomes a new \( m \)-th parent of peer \( c_m \). Each layer \( m \) will then be updated as

\[
V := V \cup \{v\}
\]
\[
E_m := E_m \cup \{(p_m, v), (v, c_m)\} \setminus (p_m, c_m), \forall m.
\]

In practice, this edge-breaking can be easily implemented. If a new peer arrives, it contacts a server to register its IP address. The server then chooses \( M(1 + \alpha) \) IP addresses uniformly at random with repetition and returns them to the peer. Here, \( \alpha > 0 \) is used in practice in case some peers are not reachable for some reason. But for the purpose of analysis later, we assume that \( \alpha = 0 \) and all peers are reachable. Among these addresses, the peer contacts \( M \) reachable peers and breaks into their first, second, ..., \( M \)-th outgoing edge, respectively.

**When an existing peer \( v \) leaves**, its parents and children will lose one of their neighbors as shown in Fig. 3(c). Let \( p_m \) and \( c_m \) be the parent and the child, respectively, of peer \( v \) in layer \( m \), i.e., \( (p_m, v), (v, c_m) \in E_m \). (It is easy to see that every peer always has exactly one parent and one child in each layer.) The parent \( p_m \) in each layer \( m \) then directly contacts the child \( c_m \).
Fig. 3. Peer pairings for $M = 2$: (a) Initially, peer 1 takes peer 2 as its first and the second children, and vice versa. (b) When peer 4 arrives to the network with existing peers 1, 2, and 3, it independently chooses an edge (marked 'X') from each layer uniformly at random and breaks into the chosen edges. (c) When peer 2 leaves, its incoming edge and outgoing edge (marked 'X') in each layer are reconnected.

In the same layer and takes the child as its new $m$-th child as shown in Fig. 3(c). In this case, the topology will change as follows:

$$V \leftarrow V \setminus \{v\}$$

$$E_m \leftarrow E_m \cup \{(p_m, v), (v, c_m)\}, \quad \forall m,$$

In practice, there is a chance that two or more successive ancestors of a peer in a layer leave the network simultaneously, which makes this edge-repairing impossible. This issue can easily be addressed by letting each peer remember the IP addresses of the several successive ancestors along the cycle in each layer. For the details, please refer to [19].

At any given time, the network topology $L^*$ that has been constructed by the pairing algorithm satisfies the property stated in the following lemma. The lemma and its proof are straightforward, given the pairing algorithm, but we present them below to highlight their importance to the analysis in the rest of the paper.

**Lemma 1:** $L_m$, representing each layer $m$ is a directed Hamiltonian cycle, i.e., every peer has exactly one incoming and one outgoing edge in each layer, and all the edges in $E_m$ form a single directed cycle. Hence, the superposed graph $L^*$ is an $M$-regular multi-digraph, i.e., every peer has exactly $M$ incoming edges and $M$ outgoing edges.

**Proof:** Initially, two peers form a single cycle in each layer. When we add a new peer to each layer consisting of a single cycle, the new peer simply breaks into an existing edge in a layer, maintaining the existing cycle. When we remove a peer from each layer, its incoming edge and its outgoing edge are reconnected, which also maintains the cycle. Hence, when peers join
chunk generated at timeslot \( K \) timeslots, the maximum chunk-generating rate under our algorithm is \( \color{1}, \color{2}, \ldots, \color{0} \)

dissemination algorithm that can provide provable throughput and delay bounds.

**B. Chunk Dissemination Algorithm**

While the pairing algorithm determines the network topology, the chunk dissemination algorithm determines how chunks are disseminated over a given topology. We here present our chunk dissemination algorithm that can provide provable throughput and delay bounds.

Assume that the source generates at most one chunk during every timeslot, except timeslots \( 0, K, 2K, \cdots \) for some integer \( K > 2 \). Since at most \( K - 1 \) chunks are generated during every \( K \) timeslots, the maximum chunk-generating rate under our algorithm is \( (1 - 1/K) \). We call the chunk generated at timeslot \( t \) chunk \( t \). Suppose there are \( K - 1 \) predetermined colors, numbered color 1, color 2, ..., color \( K - 1 \) and we color each chunk \( t \) with color \( (t \text{ mod } K) \). In other words, the chunks are colored from 1 through \( K - 1 \), and then again starting from 1, with the process repeating forever. We call the chunk with color \( k \) simply a \( \text{color-}k \) chunk. If a color-\( k \) chunk is generated at time \( t \) at the source, then color \( k \) chunks are also generated at time \( t + K, t + 2K \) and so on. If chunks are not generated periodically in this manner, then a smoothing buffer has to be used at the source to ensure that only \( K - 1 \) chunks are periodically generated for every \( K \) timeslots, and any other additional chunks are stored for later transmission. Thus, there will be a queuing delay at the source for storing the additional chunks which we ignore since our goal here is to characterize the scaling behavior of the end-to-end transmission delay from the source to all peers as a function of \( N \).

Recall that every peer can upload at most one chunk to one other peer in a timeslot. At the beginning of each timeslot, every peer \( \text{schedules} \) one of its outgoing edges, i.e., the peer selects an outgoing edge and uploads a chunk over that edge. Specifically, every peer \( i \) shares the same scheduling vector \( \Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_K) \), where \( \lambda_k \in \{1, 2, \cdots, M - 1\} \) for all \( k < K \) and \( \lambda_K = M \). Peer \( i \) schedules its outgoing edges, cycling through the elements in the scheduling vector. For example, if \( M = 3 \) and \( \Lambda = (1, 2, 1, 3) \), every peer repeats scheduling its first, second, first, and third outgoing edges sequentially. We note that the scheduling round of a peer need not be synchronized with the other peers, i.e., at a given timeslot, peers may schedule different types of outgoing edges.

Suppose peer \( v \) schedules the \( k \)-th edge in the scheduling vector (i.e., the outgoing edge in layer \( \lambda_k \) or equivalently the \( \lambda_k \)-th outgoing edge) at the beginning of timeslot \( t \). Let \( Q_{v,k}(t) \) be the set of the color-\( k \) chunks that peer \( v \) has received before timeslot \( t \) for \( 0 < k < K \). If \( k < K \), peer \( v \) chooses the chunk from \( Q_{v,k}(t) \) that was generated most recently (called the latest chunk) and uploads this chunk over the scheduled edge, regardless of whether or not the other end possesses the chunk. If \( k = K \), the peer transmits the latest chunk in \( Q_{v,\mu(v)}(t) \), where \( \mu(v) \) is a random variable uniformly chosen from \( \{1, 2, \cdots, K - 1\} \) when the peer joined the network. We also assume that \( \mu(v) \) does not change once it is determined. We call \( \mu(v) \) the coloring decision of
peer \( v \). During \( K \) timeslots of a scheduling round, peer \( v \) will transmit the latest chunks with color 1, color 2, ..., color \( K - 1 \), and color \( \mu(v) \) over the \( \lambda_1 \)-st, \( \lambda_2 \)-nd, \( \cdots \), \( \lambda_K \)-th outgoing edges, respectively. Since the scheduling rounds of peers are asynchronous, peers may transmit chunks with different colors at a given timeslot. Note that when a peer receives a chunk, this chunk will be unavailable for uploading till the next timeslot. Furthermore, we have assumed implicitly that only the latest chunk with each color is available for uploading at a peer. Thus, if a color-\( k \) chunk that is generated later than the latest chunk in \( Q_{v,k}(t) \) of peer \( v \) arrives at peer \( v \) at timeslot \( t \), the peer will not upload all the chunks received before timeslot \( t \). We will show later that all chunks are delivered to all the peers despite the fact that older chunks are discarded. In other words, we will prove that the older chunks have already been disseminated by a peer by the time they are discarded and so are no longer necessary from the point of view of data dissemination (although they may be retained for playout at the peer).

We have presented our chunk-dissemination algorithm running on top of the pairing algorithm. Besides our algorithm, other chunk dissemination algorithms, such as the random useful algorithm [14], the latest-blind algorithm [15], and the latest-useful algorithm [15], can be potentially used over the network topology that is constructed by the pairing algorithm. Our performance analysis is, however, only for the chunk dissemination algorithm proposed here.

C. Bounds on Streaming Rate and Delay

Our P2P algorithm will be evaluated using two metrics: streaming rate and delay.

**Streaming Rate**: What is the streaming rate achieved by our P2P algorithm? The streaming rate is defined as the chunk reception rate guaranteed to all peers. When peers contribute unit bandwidth, the total upload bandwidth \( N \) contributed by all peers (including the source) has to be shared by \( N - 1 \) peers (excluding the source). Thus, the download bandwidth per peer cannot exceed \( \frac{N}{N-1} \), which is approximated to one for large \( N \). Hence, the optimal streaming rate is close to one for a large network. In Section VI, we will show that our algorithm disseminates all the chunks to all peers, and achieves a streaming rate of \( 1 - \frac{1}{K} \), which is arbitrarily close to the optimal streaming rate for sufficiently large \( K \).

**Dissemination Delay**: What is the delay that can be achieved by our P2P algorithm? When each peer is allowed to disseminate chunks only to a constant number of neighbors, as in a real P2P topology, the fundamental limit of the delay required to disseminate a chunk to all peers is known to be \( \Omega(\log N) \). This limit is a lower bound on the delay to disseminate multiple chunks because the contention between multiple chunks at a peer can only increase the dissemination delay. In Section VI, we show that our algorithm achieves this fundamental limit, i.e., every chunk arriving at the source at rate \( (1 - \frac{1}{K}) \) is disseminated to all peers within \( \Theta(\log N) \) timeslots with high probability under our algorithm.

V. THROUGHPUT AND DELAY ANALYSIS

In this section, we show that our algorithm achieves the streaming rate of \( (1 - \frac{1}{K}) \), i.e., each chunk arriving at the source at rate \( (1 - \frac{1}{K}) \) can be disseminated to all peers by our algorithm. To this end, we first characterize the graph over which color-\( k \) chunks are disseminated. We then show that no color-\( k \) chunks are dropped before being disseminated to all peers.

As described in Section IV-B during every scheduling round of a peer \( v \), peer \( v \) transmits the latest color-1 chunk, color-2 chunk, ..., color-(\( K - 1 \)) chunk, and color-\( \mu(v) \) chunk over its \( \lambda_1 \)-st, \( \lambda_2 \)-nd, ..., \( \lambda_{(K-1)} \)-st, \( M \)-th outgoing edges, respectively. Thus, color-\( k \) chunks are delivered over the \( \lambda_k \)-th outgoing edges from all peers (i.e., the edges in layer \( \lambda_k \)) and the \( M \)-th outgoing edges from peers \( v \) with \( \mu(v) = k \). If we define flow graph \( G_k \) \( (k = 1, 2, \cdots, K - 1) \) to be
the graph consisting of the edges carrying color-$k$ chunks, the flow graph can be expressed as a multi-digraph $G_k = (V, E_k \cup E_{M,k})$, where

$$\mathcal{E}_{M,k} = \{(i,j) \in E_M | \mu(i) = k\}.$$  

Thus, color-$k$ chunks are disseminated over flow graph $k$, where the out-degree of every peer is at most two. (See the example of the flow graphs for $K = 3$ and $M = 2$ in Fig. 4(b)).

---

Fig. 4. A network with 6 peers named 1 (source), 2, 3,..., 6, where $M = 2$, $K = 3$, $\Lambda = (1,1,2)$ and $(\mu(s),\mu(2),\cdots,\mu(6)) = (1,2,2,1,1,2)$. Flow graph $G_1$ consists of the edges in the first layer and the edges $(i,j)$ with $\mu(i) = 1$ in the second layer. Similarly, flow graph $G_2$ consists of the edges in the first layer and the edges $(i,j)$ with $\mu(i) = 2$ in the second layer.

We next study how color-$k$ chunks are disseminated over flow graph $G_k$. Recall that if a color-$k$ chunk is generated at timeslot $t$, chunks $t + K, t + 2K, \cdots$ are all of color $k$. We call these chunks later chunks of chunk $t$. Since our chunk dissemination algorithm transmits only the latest chunk of each color, if a peer has received both chunk $t$ and a later chunk, the peer will not transmit chunk $t$ any longer. Thus, if all the peers that have received chunk $t$ have also received a later chunk, chunk $t$ cannot be disseminated to the remaining peers. However, the following proposition shows that this scenario does not occur.

**Proposition 1:** Under our algorithm, if a peer receives chunk $t$ during timeslot $l \geq t$, this peer has received chunk $t - K$ no later than timeslot $l - K$.

**Proof:** Without loss of generality, fix $t = 0$. We prove by induction that if a peer receives chunk 0 (with color $k$) during timeslot $l \geq 0$, it has received chunk $-K$ before or during timeslot $l - K$. 

---

(a) Layer 1 (left) and layer 2 (right): The numbers on peers $v$ in layer 2 are the coloring decisions $\mu(v)$.

(b) Flow graphs $G_1$ (left) and $G_2$ (right)
Initially \((l = 0)\), chunk 0 arrives at the source during timeslot 0. Since the coloring queue moves one color-\(k\) chunk every \(K\) timeslots, chunk \(-K\) must have arrived at the dissemination queue of the source during timeslot \(-K\). Since the source is the only peer that has chunk 0 during timeslot 0, the statement is true for \(l = 0\).

We now assume that the statement is true for \(l \leq t' - 1\). We next show that the statement is also true for \(l = t'\). Consider a particular peer \(j\) that receives chunk 0 for the first time through an incoming edge \((i, j)\) during timeslot \(t'\). This implies that peer \(i\) has received chunk 0 for the first time during timeslot \(t'' \in \{t' - K, t' - K + 1, \ldots, t' - 1\}\). (If \(t'' \geq t'\), peer \(j\) cannot receive chunk 0 during timeslot \(t'\). If \(t'' < t' - K\), peer \(j\) must have received chunk 0 during timeslot \(t'\).)

By the induction hypothesis, peer \(i\) must have received chunk \(-K\) before or during timeslot \(t'' - K\). Thus, chunk \(-K\) has been the latest color-\(k\) chunk to peer \(i\) from timeslot \(t'' - K + 1\) to timeslot \(t''\). Since the schedule is cyclic, edge \((i, j)\) was scheduled during timeslot \(t' - K\). Since \(t'' - K + 1 \leq t' - K \leq t''\), chunk \(-K\) must have been transmitted during that timeslot over \((i, j)\). Thus, the statement is true for \(l = t'\).

By induction, if a peer first receives chunk 0 during timeslot \(l\), it has received chunk \(-K\) before or during timeslot \(l - K\).

Proposition 1 implies that if a peer receives chunk \(t\), it has at least \(K\) timeslots (one scheduling round) to distribute the chunk to its children before a later chunk arrives. Since the peer schedules each outgoing edge in \(G_k\) exactly once during every \(K\) timeslots for transmitting color-\(k\) chunks, the peer will transmit chunk \(t\) to its children in \(G_k\) before a later chunk arrives. Thus, every color-\(k\) chunk arriving at the source can be disseminated to all the peers that are connected from the source in \(G_k\), i.e., there exists a path from the source to the peers in \(G_k\). Since every flow graph \(G_k\) contains layer \(\lambda_k\), which is a Hamiltonian cycle, every peer is connected from the source. Thus, all chunks arriving at rate \((1 - 1/K)\) can eventually be disseminated to all peers under our chunk-dissemination algorithm.

In streaming applications, this throughput analysis is meaningless without a delay guarantee.

We next consider how fast each color-\(k\) chunk is disseminated over flow graph \(G_k\).

**Lemma 2**: Let \(d_k(v)\) be the shortest distance from the source to peer \(v\) in flow graph \(G_k\). If a color-\(k\) chunk arrives at the source during timeslot \(t\), peer \(v\) receives the chunk by timeslot \(t + Kd_k(v)\).

**Proof**: If a peer receives a color-\(k\) chunk, it transmits the chunk to its children in \(G_k\) during next \(K\) timeslots by Proposition 1. Thus, the time until peer \(v\) receives the chunk through the shortest path in \(G_k\) from the source to peer \(v\) does not exceed \(Kd_k(v)\).

Lemma 2 shows that the delay required to disseminate a color-\(k\) chunk to peer \(v\) is upper bounded by \(K \cdot d_k(v)\) timeslots. If we call the maximum distance \(\max_{v \in V} d_k(v)\) the depth \(d_k^*\) of \(G_k\), the delay to disseminate a color-\(k\) chunk to all peers is upper bounded by \(Kd_k^*\) timeslots. Thus, if \(d_k^*\) is \(\Theta(\log N)\) for all \(k = 1, 2, \ldots, K - 1\), the dissemination delay of our algorithm is upper bounded by \(\Theta(\log N)\) timeslots. In the next section, we prove that this is true with high probability.

**VI. DEPTH OF FLOW GRAPHS**

In this section, we consider the depth \(d_k^*\) of each flow graph \(G_k\). Since all the layers are random graphs, which are affected by the history of past peer churn, the corresponding flow graph is also a random graph. Thus, the depth \(d_k^*\) of the flow graph must also be a random variable. The objective of this section is to show the following proposition:

**Proposition 2**: For any \(\psi \in (0, q/2)\), the maximum distance \(d_k^*\) from the source to all other peers in flow graph \(G_k\) is \(O(\log_1 + \psi, N)\) with probability \(1 - O(\log_1 + \psi, N^\psi)\) for some positive constant \(\sigma^*\) and \(q = 1/(K - 1)\).

To prove Proposition 2, we follow the following three steps. First, to characterize random variable \(d_k^*\), we need to characterize the random graph \(G_k\). In this step, we show that there is an alternative way to construct the random graph which is stochastically equivalent to the construction
described in Section IV. In the second step, using the alternative construction, we will show that the number of peers within \(l\) hops from the source in \(G_k\) increases exponentially in \(l\) until the number is no larger than \(N/2\). In the last step, we show that the number of remaining peers that are not within \(l\) hops from the source reduces exponentially.

### A. Distribution of Flow Graphs

Consider two random multi-digraphs \(G' = (V, E')\) and \(G'' = (V, E'')\) that have the same peer set \(V\) and random edges. For every possible multi-digraph \(G\) with peer set \(V\), we say that these two random graphs have the same distribution if \(P[G' = G] = P[G'' = G]\). In this subsection, we consider how flow graph \(G_k\) is distributed and how to construct a random graph that has the same distribution as flow graph \(G_k\).

Recall that flow graph \(G_k\) is the superposed graph of layer \(\lambda_k\) and \((V, E_{M,k})\), a subgraph of layer \(M\). Thus, we first consider the distribution of each layer. As in the first example of Section III, construct a random Hamiltonian cycle by permuting the peers in \(M\). Thus, \(\lambda_k\) can be replaced with \(k\) for analysis.

We next consider how \((V, E_{M,k})\) is distributed. Note that \((V, E_{M,k})\) is a subgraph of layer \(M\) that consists of only the edges \((i, j) \in E_M\) with \(\mu(i) = k\). Since each peer \(i\) makes its coloring decision to be \(\mu(i) = k\) with probability \(q \equiv 1/(K-1)\), \((V, E_{M,k})\) can be seen as the graph made
from layer \( M \) by independently removing each edge with probability \( 1 - q \). Since layer \( M \) has the same distribution as \( H \), \((V,E_{M,k})\) has the same distribution as the graph \( H' \) that we obtain from \( H \) by removing each edge with probability \( 1 - q \).

Note that conditioned on peer set \( V \), layer \( \lambda_k \) and layer \( M \) are mutually independent because peer pairing in a layer has been independent from that in another layer, i.e., for any \( G' \subseteq C(V) \),

\[
P[ L_{\lambda_k} \in G', L_M \in G''] = P[ L_{\lambda_k} \in G'] P[ L_M \in G'']
\]

Thus, the graph that has the same distribution as \( G_k \) can be constructed from two independent random Hamiltonian cycles as follows:

**Proposition 4:** Construct two random Hamiltonian cycles \( H_1 \) and \( H_2 \) by permuting the peers in \( V \) independently for each. Remove each edge from \( H_2 \) with probability \( 1 - q \), where \( q = 1/(K - 1) \), and call the resulting graph \( H'_2 \). If we superpose \( H_1 \) and \( H'_2 \), the superposed graph \( H^* \) has the same distribution as flow graph \( G_k \) for \( k = 1, 2, \ldots, K - 1 \).

Note that \( H^* \) in Proposition 4 is identical to the graph in the second example of Section III. Thus, the maximum distance from a given node to all other nodes in that graph is stochastically equivalent to the depth of each flow graph. By proving the depth of \( H^* \) is \( \Theta(\log N) \) with high probability, we show that the depth of flow graph is also \( \Theta(\log N) \) with high probability.

There exist several ways to construct \( H_1 \) and \( H'_2 \). The simplest way is to permute the peers and connect this permutation of peers with edges. However, when we construct \( H_1 \) and \( H'_2 \) using this method, analyzing the depth of \( H^* \) is not straightforward. Instead, we use another equivalent process which provides us with a tractable construction amenable to analysis:

**Flow Graph Construction (FGC) Process:** Given peer set \( V \),

1) \( v_1 = 1 \) (source), \( Z = \{v_1\}, E^{(1)} = E^{(2)} = \emptyset, t = 1 \) (Here, the variable \( t \) is used to indicate that there are \( t - 1 \) edges in \( E^{(1)} \)), and

\[
\vec{\tau} \triangleq (\tau_1, \tau_2, \ldots, \tau_N),
\]

where \( \tau_1, \tau_2, \ldots, \tau_N \) are independent Bernoulli random variables with the same mean \( q = 1/(K - 1) \).

2) Start iteration \( t \): we will draw outgoing edges from \( v_t \).

3) Choose \( c_t \) from \( C(v_t, E^{(1)}) \) uniformly at random, where \( C(v, E) \) is the set of peers satisfying

**Condition 1:** For every \( c \in C(v, E) \),

(a) There is no edge ending at peer \( c \) in \( E \).

(b) Adding edge \((v, c)\) to graph \((V, E)\) does not incur a loop or a cycle unless the cycle is Hamiltonian.

4) Add \((v_t, c_t)\) to \( E^{(1)} \).

5) If \( c_t \notin Z \), add \( c_t \) to \( Z \) and then let \( v_{|Z|} = c_t \).

6) If \( \tau_t = 1 \),

a) Choose \( c_t' \) from \( C(v_t, E^{(2)}) \) uniformly at random.

b) Add \((v_t, c_t')\) to \( E^{(2)} \).

c) If \( c_t' \notin Z \), add \( c_t' \) to \( Z \) and then let \( v_{|Z|} = c_t' \).

If \( \tau = 0 \), no edge is added to \( E^{(2)} \), and no peer is added to \( Z \) in this step. In this case, we set \( c_t' = \infty \).

7) If \( t < N \), increase \( t \) by one and go to Step 2.

8) Return \( E^{(1)} \) and \( E^{(2)} \).

For given peer set \( V \), we can construct two random graphs \((V, E^{(1)})\) and \((V, E^{(2)})\). The next proposition shows that if we superpose these graphs, the resulting graph has the same distribution as \( H^* \).

**Proposition 5:** Random graphs \((V, E^{(1)})\) and \((V, E^{(2)})\) constructed by the FGC process are mutually independent and have the same distribution as \( H_1 \) and \( H'_2 \), respectively.
In the rest of this subsection, we provide the intuition of the proof. The detailed proof is provided in Appendix B. When we construct \((V, E^{(1)})\) using the FGC process, we iteratively pick a peer \(v_t\) that does not have an outgoing edge and draw an edge from it to a random peer \(c_t\) that does not incur a non-Hamiltonian cycle. Thus, after drawing \(N\) edges in this manner, the resulting graph \((V, E^{(1)})\) will be a Hamiltonian cycle in \(C(V)\). Since we have chosen \(c_t\) uniformly at random among the candidates not incurring a non-Hamiltonian cycle, the resulting Hamiltonian cycle \((V, E^{(1)})\) is uniformly distributed in \(C(V)\).

We now consider how \((V, E^{(2)})\) has the same distribution as \(H_2^c\). Recall that we have obtained \(H_2^c\) by independently removing each edge with probability \(1-q\) from a random Hamiltonian cycle. Hence, if we draw a random Hamiltonian cycle as we have drawn skipping some edges, should have the same distribution as \(H_2^c\). Say the resulting graph \(H^t\). Instead of removing edges after completing the random Hamiltonian cycle, we now draw a random edge from each peer \(v_t\) with \(\tau_t = 0\), as we did for \((V, E^{(1)})\), and stop drawing once we finish drawing edges from the peers. Say the resulting graph \(H''\). Then, \(H''\), which we have drawn skipping some edges, should have the same distribution as \(H^t\), which we have drawn deleting some edges from a random Hamiltonian cycle. Since the process of drawing \(H''\) is identical to the way how the FGC process constructs \((V, E^{(2)})\), both \((V, E^{(2)})\) and \(H_2^c\) have the same distribution.

We finally show that \((V, E^{(1)})\) and \((V, E^{(2)})\) constructed by the FGC process are mutually independent. At each iteration, we have chosen the children \(c_t\) and \(c'_t\) of peer \(v_t\) independently of each other. Further, \(\tau_t\) is chosen independently of \(c_t\) and \(c'_t\). Hence, after \(N\) iterations, \(E^{(1)}\) and \(E^{(2)}\) are mutually independent, and thus so are the resulting graphs \((V, E^{(1)})\) and \((V, E^{(2)})\).

Overall, \((V, E^{(1)}, E^{(2)})\) that we have drawn using the FGC process will have the same distribution as \(H^*\) by Proposition 4. Thus, the FGC process can be seen as another way to construct flow graph \(G_k\). Note that we do not propose this process to construct the network topology in practice. We use this process for analysis and use our peer-pairing algorithm in practice, which results in random graphs with the same distribution. In the rest of this section, we analyze the depth of \(H^*\). Our analysis to spread over the next two subsections:

1) In Section VI-B, we show that it is possible to reach the closest \(N/2\) nodes from the source node in \(O(\log N)\) hops.
2) In Section VI-C, we show that we can reach all the other nodes from the set of the closest \(N/2\) nodes in another \(O(\log N)\) hops.

B. Edge Expansion of Flow Graph \(G_k\)

Before we present our proof that the closest \(N/2\) nodes from the source can be reached in \(O(\log N)\) hops, we first present some intuition behind the result. Let \(E^{(1)}(t), E^{(2)}(t),\) and \(Z(t)\) be \(E^{(1)}, E^{(2)},\) and \(Z,\) respectively, at the end of iteration \(t\) in the FGC process. By definition, \(Z(t)\) is the set of peers \(\{v_1, \ldots, v_t\}\) and their children at the end of iteration \(t\). Given these definitions, the proof can be broken into three major steps:

**Step (i):** We show that \(E[|Z(t)|] \geq (1+\psi)t\) for \(t \leq N/2\) and some \(\psi > 0\), where \(z(t) = |Z(t)|\). This is shown in Proposition 6.

**Step (ii):** We show that \(z(t) \geq (1+\psi)t\) w.h.p. in Proposition 7.

**Step (iii):** Finally, we relate the above concentration result to the distance between the source node and its \(N/2\) closest peers in Proposition 8. The intuition behind this result is as follows: As we can see in Fig. 5, \(Z(t)\) is the set of peers that are within one hop from the set of the first \(t\) peers, i.e., \(\{v_1, \ldots, v_t\}\). Similarly, \(Z(|Z(t)|)\) is the set of peers that are within one hop from the set of the first \(|Z(t)|\) peers, i.e., \(\{v_1, \ldots, v_{|Z(t)|}\}\). In general, if we iteratively define \(Z^{(h)}(t) = Z(|Z^{(h-1)}(t)|)\) for \(h > 0\) where \(Z^{(0)}(t) = \{v_1, \ldots, v_t\}\), \(Z^{(h)}(t)\) is the set of peers
that are within one hop from $Z^{(h-1)}(t)$. Thus, the result of Step (ii) leads to
\begin{align*}
z^{(h)}(t) &= z(z^{(h-1)}(t)) \geq (1 + \psi)z^{(h-1)}(t) \\
& \geq \cdots \geq (1 + \psi)^h t, \tag{2}
\end{align*}
where $z^{(h)} = |Z^{(h)}|$. In other words, the graph expands at least at rate $1 + \psi$ as shown in Fig. 5, and thus, it needs $O(\log_{1+\psi} N)$ steps to cover the first $N/2$ peers.

Now, we are ready to make the above argument precise. To derive the mean of $Z(t)$ in Step (i), we focus on how $Z(t)$ increases. At iteration $t$, if peer $v_t$ chooses $c_t$ from $Z^C(t-1)$, adding this peer to $Z(t)$ will increase $z(t)$ by one from $z(t-1)$. In addition, if $\tau_t = 1$ and peer $c'_t$ is not in $Z(t) = Z(t-1) \cup \{c_t\}$, adding peer $c'_t$ will also increase $z(t)$ by one. Hence, the increment of $z(t)$ at iteration $t$ is given by
\begin{equation}
z(t) = z(t-1) + 1_{\{c_t \notin Z(t-1)\}} + \tau_t \cdot 1_{\{c'_t \notin Z(t-1) \cup \{c_t\}\}} \tag{3}
\end{equation}
At each iteration, the increment of $z(t)$ is either 0, 1, or 2, and thus $z(t) - z(0) \leq 2t$. Initially, the increment is 2 with high probability because $Z(t-1)$ contains a few peers compared with $V$.

Therefore, $z(t)$ will increase fast initially. As $t$ increases, i.e., as $Z(t-1)$ contains more peers, the probability that the increment is 0 or 1 increases, and thus $z(t)$ will increase at a slower rate. Define
\[ G(t) = \{ E^{(1)}(t), E^{(2)}(t) \} \]
as the graph drawn right after iteration $t$. Since all outgoing edges of $v_1, \cdots, v_t$ are determined at this moment, $G(t)$ determines $Z(t)$ and $z(t)$. The following proposition shows how the mean of $z(t)$ evolves conditioned of $G(l)$ (for any $l \leq t$).

**Proposition 6:** For any integer $l$ in $[0, t]$,
\[
E[N - z(t) | G(l), \tau]
= \frac{N - t - 1}{N - l - 1} \left( \frac{N - \sum_{j=1}^{t} \tau_j - 1}{N - \sum_{j=1}^{l} \tau_j - 1} \right) (N - z(l)). \tag{4}
\]

The proof is provided in Appendix C. Since $\tau_1, \tau_t, \cdots, \tau_N$ are independent Bernoulli random variables with mean $q$, we have $E[\sum_{j=1}^{t} \tau_j] = qt$. By taking $l = 0$, we can obtain $E[z(t)]$ from
Proposition 6
\[ E[N - z(t)] = (N - t - 1) \left( 1 - \frac{tq}{N - 1} \right) \]
\[ \Rightarrow E[z(t)] = 1 + \frac{1}{t} + q(1 - \frac{t}{N - 1}). \tag{5} \]

For \( t \leq N/2 \), we can see that the minimum of \( E[z(t)/t] \) is attained at \( t = N/2 \), and the minimum is greater than \( 1 + q/2 \). Thus, we have
\[ \frac{E[z(t)]}{t} > 1 + \frac{q}{2}, \quad \text{for } t \leq \frac{N}{2}. \]

In other words, the expected number of outgoing edges from the set of peers \( v_1, v_2, \ldots, v_t \) is at least \( \psi t \), which corresponds to Step (i).

We next show Step (ii) by showing that \( z(t)/t \) is concentrated around its mean with high probability, and thus is larger than \( 1 + q/2 \) with high probability. Using a Doob Martingale and the Azuma-Hoeffding bound, we have the following result:

**Proposition 7:** For \( \psi \in (0, \frac{q}{2}) \) and \( t \leq \frac{N}{2} \),
\[ P[z(t) > (1 + \psi)t] > 1 - \exp(\sigma t), \]
where \( q = 1/(K - 1) \) and \( \sigma = (q/2 - \psi)^2/8 \).

The proof is provided in Appendix D. This result corresponds to the result of Step (ii).

We finally show how the distance from the source to peer \( v \) in the random graph \( H^* \) constructed by the FGC process. Since \( Z(t) \) is the set of the first \( t \) peers and their children, \( \max_{t \leq i \leq z(t)} d(v_i) \) is concentrated around its mean with high probability. Using a Doob Martingale and the Azuma-Hoeffding bound, we have the following result:

**Proposition 8:** For \( \psi \in (0, \frac{q}{2}) \),
\[ P[d(v_{N/2}) < \theta] > 1 - \frac{e^\sigma \log_{1+\psi} \frac{N}{2}}{N\sigma}, \]
where \( \theta = \log_{1+\psi} \frac{N}{2} \) and \( \sigma = (q/2 - \psi)^2/8 \).

**Proof:** For simplicity, let \( \phi = 1 + \psi \) for some \( \psi \in (0, q/2) \). Define event \( A_\phi \triangleq \{ z(\phi^t \log N) > \phi^{t+1} \log N \} \). From (6):
\[ d(v_{z(\phi^t \log N)}) - d(v_{\phi^t \log N}) \leq 1. \]

If \( A_\phi \) is true, \( z(\phi^t \log N) > \phi^{t+1} \log N \geq [\phi^{t+1} \log N] \). Since \( d(v_t) \) is non-decreasing in \( t \), the above inequality can be rewritten as
\[ d(v_{\phi^{t+1} \log N}) - d(v_{\phi^t \log N}) \leq 1. \]
(Recall that we have abused notation $v_t$ such that $v_t = v_{[t]}$ for non-integer $t$.) If $A_t$ is true for $t = 0, 1, 2, \ldots, t_0 - 1$, we obtain
\[
d(v_{\phi \log N}) - d(v_{\log N}) \leq 1
\]
\[
d(v_{\phi^2 \log N}) - d(v_{\phi \log N}) \leq 1
\]
\[
\vdots
\]
\[
d(v_{\phi^{t_0} \log N}) - d(v_{\phi^{t_0 - 1} \log N}) \leq 1
\]
\[
\Rightarrow d(v_{\phi^{t_0} \log N}) - d(v_{\phi^N \log N}) \leq t_0
\]
\[
\Rightarrow d(v_{\phi^{t_0} \log N}) \leq t_0 + d(v_{\log N}) \leq t_0 + \log N.
\]
Take $t_0 = \lceil \log_\phi \frac{N}{2} - \log_\phi \log N \rceil$. For large $N$,
\[
d(v_{\frac{N}{2}}) \leq d(v_{\phi^{t_0} \log N})
\]
\[
< \log_\phi \frac{N}{2} - \log_\phi \log N + 1 + \log N < \theta.
\]
Using the union bound, the probability of $d(v_{N/2}) < \theta$ can be expressed as
\[
P \left[ d(v_{\frac{N}{2}}) < \theta \right] \geq P \left[ \bigcap_{t=0}^{t_0-1} A_t \right]
\]
\[
= 1 - P \left[ \bigcup_{t=0}^{t_0-1} A_t^C \right] \geq 1 - \sum_{t=0}^{t_0-1} P [A_t^C]
\]
(7)
We next find an upper bound on $P [A_t^C]$. By the definition of $A_t$,
\[
P [A_t^C] = P \left[ z(\phi^t \log N) \leq \phi^{t+1} \log N \right]
\]
\[
=P \left[ z(\phi^t \log N) \leq (1 + \psi_t) \phi^{t \log N} \right],
\]
(8)
where
\[
1 + \psi_t = \frac{\phi^{t+1} \log N}{\phi^{t \log N}}.
\]
Suppose $\phi^t \log N = a + b$, where $a$ is an integer and $b \in [0, 1)$. Since $\phi < 2$, we have $\phi^{t+1} \log N = \phi \cdot a + \phi \cdot b < \phi \cdot a + 2$. Thus, $1 + \psi_t < \phi + 2 / (\phi^t \log N) = \phi + o(1)$, which is in $(1, 1 + q/2)$ for large $N$. Applying this and Proposition 7 to (8) for large $N$ and $0 \leq t < t_0$,
\[
P [A_t^C] < e^{-\sigma_t \phi^t \log N}
\]
\[
< e^{-\sigma_{\min} \phi^t \log N - 1}
\]
\[
\leq e^{-\sigma_{\min} \log N - 1},
\]
where $\sigma_t = (q/2 - \psi_t)^2 / 8$ and $\sigma_{\min} = \min_{0 \leq t < t_0} \sigma_t$. Since $1 + \psi_t < 1 + \psi + o(1)$, we have $\sigma_{\min} \geq \sigma$ for sufficiently large $N$. Applying this to (7), we have
\[
P \left[ d(v_{\frac{N}{2}}) < \theta \right]
\]
\[
> 1 - \sum_{t=0}^{t_0-1} e^{-\sigma \log N} e^\sigma
\]
\[
\geq 1 - t_0 e^{-\sigma \log N} e^\sigma
\]
\[
> 1 - \frac{e^\sigma \log_\phi N}{N^\sigma},
\]
for large $N$.
Since $d(v_t) \leq d(v_{N/2})$ if $t < N/2$, this proposition shows that the closest $N/2$ peers are within $O(\log N)$ hops from the source, which corresponds to the result of Step (iii).  

In this subsection, we have found that the distance from the source to \( v_{N/2} \) is \( \Theta(\log N) \) by analyzing edge expansion in the early phase \( t \leq N/2 \). To show the distance from the source to the farthest peer \( v_N \), we will analyze the distance from \( v_{N/2} \) to \( v_N \).

C. Contraction of the Remaining Graph

To analyze the distance of the remaining nodes from the source, we need to show that the distance from closest \( N/2 \) nodes from the source to the remaining \( N/2 \) nodes is \( O(\log N) \). However, we cannot use the same approach as in the previous subsection because the edge-expansion analysis using Proposition 9 is not valid for \( t > N/2 \). Indeed, we can infer from (5) that \( E[z(t)/t] \) reduces to one for large \( N \), which indicates almost zero expansion.

Instead of edge expansion, we focus of the contraction on the number of remaining peers. Recall that \( Z(t)(N/2) \) is the set of peers that are within \( h \) hops from the closest \( N/2 \) peers (i.e., \( \{v_1, \cdots, v_{N/2}\} \)), and \( z(t)(N/2) \) is the number of such peers. Hence, \( N - z(t)(0) \) is the number of peers that are \( h + 1 \) or more hops away from the closest \( N/2 \) peers. Using this notation, we show that \( O(\log N) \) peers are not within \( O(\log N) \) hops from the closest \( N/2 \) peers, i.e., \( N - z(\Theta(\log N))(N/2) = O(\log N) \). We deal with the final \( O(\log N) \) peers separately at the end.

Before we prove this, we present the intuition behind the proof:

**Step (i):** To observe the contraction of the number of remaining peers at each iteration \( t \), we define the contraction ratio at iteration \( t \) as

\[
F(t) \triangleq \frac{N - z(t)}{N - t}.
\]

Since a small \( F(t) \) means a large number of peers are within one hop from \( \{v_1, \cdots, v_t\} \), we want \( F(t) \) to be small for a faster contraction. By proving that \( F(t) \) is a supermartingale, we first show that the mean of the contraction ratio at any iteration \( t' > t \) is no larger than that at iteration \( t \), i.e., \( E[F(t)] \geq E[F(t')] \), in Lemma 3. Further, we show that the contraction ratio at iteration \( t' \) is no larger than that at iteration \( t \) with high probability, i.e., \( F(t) \geq F(t') \) for \( t < t' \) in Proposition 9. This implies that the contraction at iteration \( t' \) is no smaller than that at iteration \( t \). From this, we can conclude that, if we achieve a small contraction ratio at \( t = N/2 \), then we will have a small contraction ratio afterwards.

**Step (ii):** In this step, we show that the result in Step (i) holds with high probability for all iterations \( t_0 < t_1 < t_2 < \cdots < t_D \) that satisfy \( N - t_h = F(t_h^0)(N - t_0) \) for \( 0 < h \leq D \), where \( D \) is an appropriately chosen number which is of the order of \( \log N \), i.e., the contraction ratio at each iteration \( t_h \) is upper bounded by \( F(t_0) \), i.e.,

\[
F(t_0) \geq F(t_h), \quad \forall 0 < h \leq D, \tag{9}
\]

with high probability. Next, it is proven that, if (9) is true, we have \( z(D)(t_0) \geq t_D \). By the definition of \( t_h \), we conclude that

\[
N - z(D)(t_0) \leq F(t_0^D)(N - t_0)
\]

with high probability. This means that, if \( t_0 = \lfloor N/2 \rfloor \), the number of peers that are not within \( D \) hops from the first \( N/2 \) peers contracts exponentially in \( D \) if the initial contraction ratio \( F(t_0) \) is upper bounded by some constant in \((0, 1)\). In other words, this means that almost all peers are within \( O(\log N) \) hops from the first \( N/2 \) peers with high probability. The detailed proof of Step (ii) is in Proposition 10.

**Step (iii):** Finally, we show that \( F([N/2]) \) is upper bounded by some constant in \((0, 1)\) with high probability in Lemma 4. With this bound on \( F([N/2]) \) and the exponential contraction in Step (ii), we show that all peers except \( \log N \) peers must be within \( \Theta(\log N) \) hops from the first \( N/2 \) peers with high probability. We then show that the maximum distance to the remaining peers is also \( O(\log N) \). Overall, the maximum distance to all peers from the first \( N/2 \) peers is \( \Theta(\log N) \). We show this in Proposition 11.
For Step (i), we need to compare two contraction ratios \( F(t_0) \) and \( F(t) \) for \( t > t_0 \). As an initial step, we show that the mean \( E[F(t)] \) of the contraction ratio does not increase. Recall that \( G(t) \) is the graph drawn up to the end of iteration \( t \). Since \( G(t) \) determines \( z(t) \), it also determines \( F(t) \). The next lemma shows that \( F(t) \) is a supermartingale conditioned on \( G(t) \).

**Lemma 3:** \( F(t) \) is a supermartingale, i.e., for \( t > 0 \),

\[
E[F(t) \mid G(t-1)] \leq F(t-1).
\]

**Proof:** Note that if \( l \leq t \), we have \( N - \sum_{j=1}^{t} \tau_j - 1 \leq N - \sum_{j=1}^{l} \tau_j - 1 \) since \( \sum_{j=1}^{t} \tau_j \) is the number of edges in \( E^{(2)} \) up to iteration \( t \). Thus, from Proposition 6 we have

\[
E[N - z(t) \mid G(t-1), \vec{\tau}]
\leq \frac{N - t - 1}{N - (t - 1) - 1} (N - z(t - 1))
< \frac{N - t}{N - (t - 1)} (N - z(t - 1)).
\]

If we divide both sides by \( N - t \), we have \( E[F(t) \mid G(t-1), \vec{\tau}] \leq F(t-1) \). Since the upper bound is independent of \( \vec{\tau} \), we have the result in the lemma.

The lemma implies that for any \( t > t_0 \), the expected contraction ratio \( E[F(t)] \leq E[F(t_0)] \).

Using the Azuma-Hoeffding inequality, we can prove that \( F(t_0) \geq F(t) \) with high probability.

**Proposition 9:** For every \( \epsilon > 0 \) and \( 0 < t_0 < t \), we have

\[
P[F(t) - F(t_0) > \epsilon \mid G(t_0)] < \exp \left( -\frac{\epsilon^2 (N - t)}{8} \right).
\]

**Proof:** We use the Azuma-Hoeffding inequality for supermartingale \( F(t) \). To this end, we first need to find an upper bound of \( |F(t) - F(t-1)| \). Note that

\[
F(t) - F(t-1) = \frac{N - z(t)}{N - t} - \frac{N - z(t-1)}{N - t + 1}.
\]

(10)

Since \( z(t) - z(t-1) \leq 2 \) by (3), (10) is lower bounded by

\[
\frac{N - z(t) - (N - z(t-1))}{N - t + 1} \geq -\frac{2}{N - t + 1}.
\]

Since \( z(t) \geq z(t-1) \geq t - 1 \), (10) is upper bounded by

\[
\frac{N - z(t-1)}{(N - t)(N - t + 1)} \leq \frac{2}{N - t + 1}.
\]

Hence, \( |F(t) - F(t-1)| \) must be upper bounded by \( 2/(N - t + 1) \).

We now apply this Lipschitz difference to the Azuma-Hoeffding bound:

\[
P[F(t) - F(t_0) \geq \epsilon \mid G(t_0)]
< \exp \left( -\frac{\epsilon^2}{2 \sum_{l=t_0+1}^{t} \frac{4}{(N - l + 1)^2}} \right)
= \exp \left( -\frac{\epsilon^2}{8 \sum_{l=N-t_0}^{N-1} \frac{1}{l^2}} \right).
\]

(11)

For \( 0 < a < b \), we have

\[
\sum_{l=a+1}^{b} \frac{1}{l^2} \leq \int_{a}^{b} \frac{1}{x^2} dx \leq \frac{1}{a}.
\]

Applying the above for \( a = N - t \) and \( b = N - t_0 \), the R.H.S. of (11) is upper bounded by \( \exp(-\epsilon^2 (N - t)/8) \).
This proposition implies that the contraction ratio at a given iteration $t_0$ will be an upper bound of that at a later iteration, which corresponds to Step (i).

In Step (ii), we show that the exponential contraction of the remaining peers holds with high probability over multiple iterations using the result in Step (i). After establishing this, we show in the next proposition that all but log $N$ peers are within $O(\log N)$ hops from the closest $N/2$ peers from the source.

**Proposition 10:** Fix $t_0 = \lfloor N/2 \rfloor$. Conditioned on $G(t_0)$, let $\phi = F(t_0) + \epsilon$ for an arbitrary $\epsilon \in (0, 1 - F(t_0))$. Then, for large $N$,

$$\begin{align*}
P \left[ N - z(D(\phi))(t_0) \leq \log N \middle| G(t_0) \right] \\
> 1 - D(\phi) \cdot N^{-\epsilon^2/8},
\end{align*}$$

(12)

where $D(\phi) = \left\lfloor \log_{1/\phi}((N - t_0)/\log N) \right\rfloor$.

**Proof:** Define $t_h$ such that $N - t_h = \lfloor \phi^h(N - t_0) \rfloor$ for $h > 0$. Define event $A_h = \{F(t_h) \leq \phi\}$ for $h \geq 0$. Since $F(t_0) = \phi - \epsilon$, $A_0$ is always true. For convenience, let $D = D(\phi)$. From Proposition 9, the probability that $A_0, A_1, \cdots, A_D$ are all true is lower bounded as

$$P \left[ \cap_{h=0}^{D-1} A_h \middle| G(t_0) \right]$$

$$\geq 1 - P \left[ A_0^c \cap G(t_0) \right] - \sum_{h=1}^{D-1} P \left[ A_h^c \cap G(t_0) \right]$$

$$\geq 1 - \sum_{h=1}^{D-1} \exp \left( -\frac{\epsilon^2(N - t_h)}{8} \right)$$

$$\geq 1 - (D - 1) \exp \left( -\frac{\epsilon^2(N - t_{D-1})}{8} \right).$$

(13)

For large $N$, we have

$$N - t_{D-1} = \lfloor \phi^{D-1}(N - t_0) \rfloor > \phi^D(N - t_0)$$

$$\geq \log N.$$

Hence, the R.H.S. of (13) is lower bounded by $1 - (D - 1) \exp(-\epsilon^2 \log N/8) > 1 - D \cdot N^{-\epsilon^2/8}$, which is equal to the lower bound in (12).

We only need to show that the probability in (13) is upper bounded by that in (12). To prove this, we show by induction that, if $A_0, \cdots, A_h$ are true, $z^{(h+1)}(t_0) > t_{h+1}$ is also true. For $h = 0$, since $N - z^{(1)}(t_0)$ is an integer, we have

$$F(t_0) = \frac{N - z(t_0)}{N - t_0} \leq \phi$$

$$\Rightarrow N - z(t_0) \leq \phi(N - t_0)$$

$$\Rightarrow N - z(t_0) \leq \lfloor \phi(N - t_0) \rfloor = N - t_1.$$

Thus, we have $t_1 \leq z(t_0) = z^{(1)}(t_0)$.

We assume that, if $A_0, \cdots, A_{h-1}$ are true, $t_h \leq z^{(h)}(t_0)$ is also true. We then show that if $A_0, \cdots, A_h$ are true, $z^{(h+1)}(t_0) > t_{h+1}$ is also true. If $A_h$ is true, i.e., $F(t_h) \leq \phi$, we have

$$N - z(t_h) \leq \phi(N - t_h) = \phi \lfloor \phi^h(N - t_0) \rfloor$$

$$\Rightarrow N - z(t_h) \leq \lfloor \phi \phi^h(N - t_0) \rfloor$$

$$\leq \lfloor \phi^{h+1}(N - t_0) \rfloor = N - t_{h+1}.$$

(14)

By the induction hypothesis, we have $t_h \leq z^{(h)}(t_0)$. Since $z(t)$ is non-decreasing, we have $z(t_h) \leq z(z^{(h)}(t_0)) = z^{(h+1)}(t_0)$. Applying this to (14), we have $t_{h+1} \leq z^{(h+1)}(t_0)$. 

By induction, we conclude that, if \(A_0, \ldots, A_{D-1}\) are true, we have \(t_D \leq z(D)(t_0)\), which is equivalent to \(N - z(D)(t_0) < \log N\). By definition of \(t_D\), we have \(N - t_D < \phi^D(N - t_0) < \log N\). Thus, if \(A_0, \ldots, A_{D-1}\) are true, we have \(N - z(D)(t_0) \leq \log N\). Hence, the probability in (13) is upper bounded by that in (12), which leads to the result of this proposition. 

From Proposition 10, if \(\phi\) is upper bounded by a constant less than one, it follows that \(D(\phi) = \Theta(\log N)\). This means that all peers except at most \(\log N\) peers are within \(O(\log N)\) hops from the first \(N/2\) peers with high probability. To show that all peers are also within \(O(\log N)\) hops from the first \(N/2\) peers, we only need to show that there exists the upper bound on \(F(N/2)\) and the distance to the remaining \(\log N\) peers from the other peers is also \(O(\log N)\). We show these in Step (iii).

In Step (iii), we first show that \(F(N/2)\) is upper bounded:

**Lemma 4:** For any \(\epsilon \in (0, q/2)\), we have

\[
P \left[ F \left( \left\lfloor \frac{N}{2} \right\rfloor \right) \geq 1 - \frac{q}{2} + \epsilon \right] \leq \exp \left( -\frac{\epsilon^2}{32} \left\lfloor \frac{N}{2} \right\rfloor \right),
\]

for sufficiently large \(N\).

**Proof:** Let \(t = \lfloor N/2 \rfloor\), \(\epsilon' = \epsilon/2\), and \(\psi = q/2 - \epsilon'\), where we recall that \(q = 1/(K-1)\). We can rewrite Proposition 7 as follows:

\[
P \left[ z(t) \leq (1 + \frac{q}{2} - \epsilon')t \right] \leq \exp \left( -\frac{\epsilon'^2}{8} \left\lfloor \frac{N}{2} \right\rfloor \right),
\]

Note that

\[
z(t) \leq (1 + \frac{q}{2} - \epsilon')t
\]

\[
\Leftrightarrow \frac{N - z(t)}{N - t} \geq \frac{N}{t} - 1 - \frac{q}{2} + \epsilon'.
\]

Since \(N/t - 1 \geq 1\) and \(t \geq (N - 1)/2\), we have

\[
\frac{N}{t} - 1 - \frac{q}{2} + \epsilon' < 2 + \frac{2}{N - 1} - 1 - \frac{q}{2} + \epsilon' < 1 - \frac{q}{2} + 2\epsilon',
\]

if \(\epsilon' > 2/(N-1)\), which is true for sufficiently large \(N\). Hence, we have \(P \left[ F(t) \geq 1 - \frac{q}{2} + 2\epsilon' \right] \leq P \left[ z(t) \leq (1 + q/2 - \epsilon')t \right]\). Since \(\epsilon = 2\epsilon'\), we have the result of this lemma.

Finally, we prove that all peers are within \(\Theta(\log N)\) hops from the first \(N/2\) peers using all the results in this subsection.

**Proposition 11:** For any \(\psi \in (0, q/2)\),

\[
P \left[ d(v_N) - d(v_{N/2}) \leq \theta \right]
\]

\[
\geq 1 - \frac{\log 1 + \psi}{N^{\sigma'}} - \exp(-\frac{\sigma'}{4} N),
\]

where \(\sigma' = (q/2 - \psi)^2/32\).

**Proof:** Let \(D^* = D(1 - q/2 + \epsilon)\) and \(t_0 = \lfloor N/2 \rfloor\). To show this proposition, we first show the following inequality:

\[
P \left[ d(v_N) - d(v_{t_0}) \leq \log N + D^* \right]
\]

\[
\geq P \left[ d(v_N) - d(v_{N - \log N}) \leq \log N, \right.
\]

\[
d(v_{N - \log N}) - d(v_{t_0}) \leq D^* \]

\[
= P \left[ d(v_{N - \log N}) - d(v_{t_0}) \leq D^* \right]
\]

\[
\geq P \left[ z(D^*)(t_0) \geq N - \log N \right].
\]
We have obtained (15) from the fact that \( d(v_N) - d(v_{N - \log N}) \leq \log N \) is always true because \( d(v_{t+1}) - d(v_t) \leq d(v_t) - d(v_1) \leq 1 \) by (4). Recall that \( Z^{(D^*)}(t_0) \) is the set of peers that are within \( D^* \) hops from the first \( t_0 \) peers, i.e., \( \{v_1, v_2, \ldots, v_{t_0}\} \). Thus, if \( v_{N - \log N} \in Z^{(D^*)}(t_0) \) (i.e., \( z^{(D^*)}(t_0) \geq N - \log N \)), then peer \( v_{N - \log N} \) must be within \( D^* \) hops from the first \( t_0 \) peers, i.e., \( d(v_{N - \log N}) - d(v_{t_0}) \leq D^* \). Hence, (16) follows from (15).

Let \( G \) be the set of all possible \( G(t_0) \)'s that satisfy \( F(t_0) \leq 1 - q/2 + \varepsilon/2 \). (Recall that \( G(t_0) \) determines \( F(t_0) \).) Then, (16) is lower bounded by

\[
P \left[ z^{(D^*)}(t_0) \geq N - \log N, G(t_0) \in G \right] = P \left[ z^{(D^*)}(t_0) \geq N - \log N \right] |G(t_0) \in G|
\]

\[
\times P[G(t_0) \in G].
\]

(17)

Note that \( D(\phi) \) is non-decreasing. If \( G(t_0) \in G \), then \( F(t_0) \leq 1 - q/2 + \varepsilon/2 \), and thus \( D(F(t_0) + 0) \leq D(1 - q/2 + \varepsilon/2) = D^* \). Using Proposition 10 we can find that (17) is lowered bounded by

\[
P \left[ z^{(D^*)}(t_0) \geq N - \log N | G(t_0) \in G \right] \geq P \left[ z^{(D(F(t_0) + 0) + \varepsilon/2)}(t_0) \geq N - \log N | G(t_0) \in G \right]
\]

\[
\geq 1 - \max_{F(t_0): G(t_0) \in G} D(F(t_0) + 0) \cdot N^{-\varepsilon^2/32}
\]

\[
\geq 1 - D^* \cdot N^{-\varepsilon^2/32}.
\]

(19)

From Lemma 4, we have found that (18) is lower bounded by \( 1 - \exp(-\varepsilon^2 t_0/128) \). Applying (19) and this to (17) and (18), respectively, we have

\[
P \left[ d(v_N) - d(v_{t_0}) \leq \log N + D^* \right]
\]

\[
\geq 1 - D^* \cdot N^{-\varepsilon^2/32} - \exp(-\varepsilon^2 t_0/128).
\]

Since \( D^* \leq \left[ \frac{\log N}{1 + q/2 - \varepsilon/2} \right] \leq \frac{\log N}{1 + q/2 - \varepsilon/2} \), we have \( \log N + D^* \leq \theta \), where \( \theta \) was defined in Proposition 8. Using this and \( \psi = q/2 - \varepsilon \) to the above, we finally have the result of the proposition.

This proposition shows that the maximum distance from the first \( N/2 \) peers to all peers is \( O(\log_{1+\phi} N) \) with high probability for some \( \phi \in (0, q/2) \).

We can prove the main theorem of this paper, Proposition 2. In the previous subsection, we have shown that the maximum distance from \( v_1 \) to the first \( N/2 \) peers is \( O(\log_{1+\psi} N) \) with probability \( 1 - O(\log_{1+\psi} N/N^\theta) \). In this subsection, we have shown that the maximum distance from the first \( N/2 \) peers to all other peers is also \( O(\log_{1+\psi} N) \) with probability \( 1 - O(\log_{1+\psi} N/N^\theta') \). Combining both results using the union bound, we can conclude that the maximum distance from the source peer to all other peers is \( O(\log_{1+\psi} N) \) with probability \( 1 - O(\log_{1+\psi} N/N^\theta') \).

VII. DIAMETERS OF FLOW GRAPHS

We have shown that the maximum distance from the source to all peers in a flow graph is \( O(\log N) \) with high probability. Using this result, we show that the diameter of the flow graph is also \( O(\log N) \) with high probability, i.e., the distance between any pair of peers in a flow graph is \( O(\log N) \) with high probability.

To analyze the diameter, we consider a flow graph with reversed edges. Specifically, for a given multi-di-graph \( G \), we reverse the direction of each edge and denote the resulting graph by \( \pi(G) \). By definition, the distance from the source to peer \( v \) in \( H^* \) is the same as the distance from peer \( v \) to the source in \( \pi(H^*) \). Thus, the maximum distance from the source to all peers in \( H^* \) is the same as the maximum distance from all peers to the source in \( \pi(H^*) \).
Lemma 5: Let $d^*$ be the maximum distance from the source to all peers in $H^*$, and let $\tilde{d}^*$ be the maximum distance from all peers to the source in the same graph $H^*$. Then, $d^*$ is identically distributed as $\tilde{d}^*$, i.e., for all $d \geq 0$

$$P[d^* \leq d] = P[\tilde{d}^* \leq d].$$

Proof: We first need to show that $H^*$ and $\pi(H^*)$ have the same distribution. We will then show that this result lead to the same distribution of $d^*$ and $\tilde{d}^*$. Recall that $H^*$ is the superposed graph of $H_1$ and $H_2^*$, and thus $\pi(H^*)$ is the superposed graph of $\pi(H_1)$ and $\pi(H_2^*)$. It is easy to see that $H_1$, $H_2$, $\pi(H_1)$ and $\pi(H_2)$ have the same distribution, i.e., for any Hamiltonian cycle $G$ and $l \in \{1, 2\}$,

$$P[H_l = G] = P[\pi(H_l) = G] = \frac{1}{(N-1)!}.$$ 

For a Hamiltonian cycle $G = (V, E)$, fix a subgraph $G' = (V, E')$, where $E'$ is a subset of $E$. Recall that we have constructed $H_2^*$ by independently removing each edge with probability $1 - q$ from $H_2$. Thus, we have

$$P[H_2^* = G' \mid H_2 = G] = q^{|E'|}(1-q)^{|E'|-|E|}$$

$$= P[H_2^* = \pi(G) \mid H_2 = \pi(G)]$$

$$= P[\pi(H_2^*) = G' \mid \pi(H_2) = G].$$

Since $H_2$ and $\pi(H_2)$ have the same distribution, $H_2^*$ and $\pi(H_2^*)$ must have the same distribution. Therefore, $H^*$ and $\pi(H^*)$ have the same distribution.

Using this result, we show that $d^*$ and $\tilde{d}^*$ also have the same distribution. Let $G$ be the set of all possible flow graphs that satisfy $d^* \leq d$. Then, it is easy to see that $\pi G$ is the set of all possible flow graphs that satisfy $d^* \leq d$, where $\pi G = \{\pi(G) \mid G \in G\}$. Since $H^*$ and $\pi(H^*)$ have the same distribution, we have

$$P[H^* \in G] = P[\pi(H^*) \in G]$$

$$\iff P[H^* \in G] = P[H^* \in \pi G]$$

$$\iff P[d^* \leq d] = P[\tilde{d}^* \leq d].$$

Hence, $d^*$ and $\tilde{d}^*$ have the same distribution. 

Previously, we have shown that the maximum distance from the source to other peers in a flow graph is $\Theta(\log N)$ with high probability. Although this result is enough to show $\Theta(\log N)$ streaming delay, we can prove the following stronger result which was mentioned in the second example of Section III.

Proposition 12: For any $\psi \in (0, q/2)$, the diameter of flow graph $G_k$ is $O(\log_{1+\psi} N)$ with probability $1 - O(\log_{1+\psi} N / N^\sigma')$ for some positive constant $\sigma'$.

Proof: Let $d_{i,j}$ be the minimum distance from peer $i$ to peer $j$ in $H^*$. We show that $\max_{i,j} d_{i,j}$ is $O(\log_{1+\psi} N)$ with probability $1 - O(\log_{1+\psi} N / N^\sigma')$. Since $d_{i,j}$ is the minimum distance from peer $i$ to peer $j$, the length of the shortest path from $i$ to $j$ via the source (peer 1) is upper bounded by $d_{i,j}$. Thus, for any $d > 0$,

$$P \left[ \max_{(i,j) \in V^2} d_{i,j} \leq 2d \right]$$

$$\geq P \left[ \max_{(i,j) \in V^2} d_{i,1} + d_{1,j} \leq 2d \right]$$

$$\geq P \left[ \max_{i \in V} d_{i,1} + \max_{j \in V} d_{1,j} \leq 2d \right]$$

$$\geq 1 - P[d^* > d] - P[\tilde{d}^* > d]$$

$$= 1 - 2P[d^* > d].$$
In the last equation, we have used Lemma 5. In the previous section, we have shown that if 

\[ d = \Theta(\log_{1+\psi} N) , \]

\[ P[ d^* > d ] < O(\log_{1+\psi} N/N^\sigma) . \]

Thus, we have proven the proposition.

VIII. CONCLUSIONS

Instead of conventional approaches using multiple overlay trees, we have proposed a simple P2P streaming algorithm that consists of a simple pairing algorithm similar to the one proposed earlier for constructing distributed hash tables [1], but used here for streaming data. Our proposed chunk dissemination algorithm can deliver all chunks to all peers with \( \Theta(\log N) \) delay and achieves \((1 - 1/K)\) fraction of the optimal streaming capacity for any constant \( K \geq 2 \).

There are several issues that need to be addressed to implement our algorithm in practice. The first issue is one of modifying our chunk dissemination algorithm to accommodate peer churn. Even though our chunk dissemination algorithm shows that the network topology at any given moment can achieve both a near optimal throughput and \( \Theta(\log N) \) delay, peer arrivals and departures disrupt the topology continuously and hence, one needs practical solutions to account for this churn in the chunk dissemination algorithm as well as the delay analysis. The second issue is one of dealing with packet losses. Even in the wired Internet, packet losses are not uncommon, and therefore, a practical protocol must have provisions to recover from such losses. Finally, we have to deal with asynchronous transmissions, i.e., chunk transmissions will not occur in a time-slotted manner in the Internet for many reasons. The analysis gets much more involved in this case. Dealing with these practical issues is an important avenue for future work.

REFERENCES

[1] C. Law and K.-Y. Siu, “Distributed construction of random expander networks,” in Proceedings of IEEE INFOCOM, vol. 3, April 2003, pp. 2133 – 2143 vol.3.
[2] Y.-J. Chu and T.-H. Liu, “On the shortest arborescence of a directed graph,” Science Sinica, vol. 14, pp. 1396–1400, 1965.
[3] J. Edmonds, “Edge-disjoint branchings,” Combinatorial Algorithms, pp. 91–96, 1972.
[4] R. E. Tarjan, “Finding optimum branchings,” Networks, vol. 7, no. 1, pp. 25–35, 1977. [Online]. Available: http://dx.doi.org/10.1002/net.3230070103
[5] M. Castro, P. Druschel, A.-M. Kermarrec, A. Nandi, A. Rowstron, and A. Singh, “Splitstream: high-bandwidth multicast in cooperative environments,” SIGOPS Oper. Syst. Rev., vol. 37, no. 5, pp. 298–313, Oct. 2003. [Online]. Available: http://doi.acm.org/10.1145/1165380.1165474
[6] J. Li, P. A. Chou, and C. Zhang, “Mutualcast: an efficient mechanism for one-to-many content distribution,” in Proceedings of ACM SIGCOMM, Beijing, China, April 2005.
[7] J. Mundinger, R. Weber, and G. Weiss, “Optimal scheduling of peer-to-peer file dissemination,” J. of Scheduling, vol. 11, pp. 105–120, April 2008. [Online]. Available: http://dl.acm.org/citation.cfm?id=1349638.1349650
[8] R. Kumar, Y. Liu, and K. Ross, “Stochastic Fluid Theory for P2P Streaming Systems,” in Proceedings of IEEE INFOCOM, Anchorage, AK, April 2007.
[9] S. Liu, R. Zhang-Shen, W. Jiang, J. Rexford, and M. Chiang, “Performance bounds for peer-assisted live streaming,” in Proceedings of ACM SIGMETRICS, Annapolis, MD, 2008, pp. 313–324.
[10] S. Liu, M. Chen, S. Sengupta, M. Chiang, J. Li, and P. A. Chou, “P2p streaming capacity under node degree bound,” in Proceedings of IEEE ICDCS, 2010, pp. 587–598.
[11] C. Zhao, X. Lin, and C. Wu, “The streaming capacity of sparsely-connected p2p systems with distributed control,” in Proceedings of IEEE INFOCOM, 2011, pp. 1449–1457.
[12] X. Zhang, J. Liu, B. Li, and Y.-S. Yum, “Coolstreaming/donet: a data-driven overlay network for peer-to-peer live media streaming,” in Proceedings of IEEE INFOCOM, vol. 3, 2005, pp. 2102 – 2111.
[13] A. M. Frieze and G. R. Grimmett, “The shortest-path problem for graphs with random arc-lengths,” Discrete Applied Mathematics, vol. 10, no. 1, pp. 57 – 77, 1985.
[14] S. Sanghavi, B. Hajek, and L. Massoulié, “Gossiping with multiple messages,” in In IEEE Transactions on Information Theory, 2007, pp. 4640–4654.
[15] T. Bonald, L. Massoulié, F. Mathieu, D. Perino, and A. Twigg, “Epidemic Live Streaming: Optimal Performance Trade-Offs,” in Proceedings of ACM SIGMETRICS, Annapolis, MD, June 2008.
[16] D. Shah, Gossip Algorithms. Foundations and Trends in Networking, 2009, vol. 3, no. 1, pp. 1–125.
[17] J. H. Kim and N. C. Wormald, “Random matchings which induce hamilton cycles and hamiltonian decompositions of random regular graphs,” Journal of Combinatorial Theory, Series B, vol. 81, no. 1, pp. 20 – 44, 2001. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0095895600919919
A. Lemma 6

Let \( V \) be the set of peers that can be made of peer set \( V \). We define \( L \) as the set of all possible subgraphs of a graph in \( C(V) \) with \( J \) edges, i.e.,

\[
\tilde{C}(V, J) = \{(V, E) ||E| = J, \exists (V, E') \in C(V) \text{ s.t. } E \subset E'\}.
\]

It is not difficult to show that

\[
|\tilde{C}(V, J)| = \binom{N}{J} \frac{(N-1)!}{(N-J-1)!}.
\]

We have first chosen \( J \) peers among \( N \) peers, and then have chosen the number of ways in which we can draw outgoing edges from them without violating the Hamiltonian cycle constraint. Recall that \( H_2^* \) is obtained by randomly removing each edge with probability \( 1 - q \) from \( H_2 \), which is also uniformly distributed in \( C(V) \). Hence, conditioned on the fact that the number of remaining edges in \( H_2^* \) is \( J \), i.e., \( |E(H_2^*)| = J \), \( H_2^* \) is uniformly distributed in \( \tilde{C}(V, J) \). Thus, conditioned on \( |E(H_2^*)| = J \), the graph \( H^* \) that we obtain by superposing two independent graphs \( H_1 \) and \( H_2^* \) is uniformly distributed in

\[
C(V) \times \tilde{C}(V, J) \triangleq \{(V, E', E'')|(V, E') \in C(V), (V, E'') \in \tilde{C}(V, J)\}.
\]

B. Proof of Proposition 5

We show how \( H_1 \) and \( H_2^* \) are distributed. Recall that \( C(V) \) is the set of all possible Hamiltonian cycles that can be made of peer set \( V \). From Proposition 5, \( H_1 \) is uniformly distributed in \( C(V) \). Define \( \tilde{C}(V, J) \) to be the set of all possible subgraphs of a graph in \( C(V) \) with \( J \) edges, i.e.,

\[
\tilde{C}(V, J) = \{(V, E)||E| = J, \exists (V, E') \in C(V) \text{ s.t. } E \subset E'\}.
\]

It is not difficult to show that

\[
|\tilde{C}(V, J)| = \binom{N}{J} \frac{(N-1)!}{(N-J-1)!}.
\]

We have first chosen \( J \) peers among \( N \) peers, and then have chosen the number of ways in which we can draw outgoing edges from them without violating the Hamiltonian cycle constraint. Recall that \( H_2^* \) is obtained by randomly removing each edge with probability \( 1 - q \) from \( H_2 \), which is also uniformly distributed in \( C(V) \). Hence, conditioned on the fact that the number of remaining edges in \( H_2^* \) is \( J \), i.e., \( |E(H_2^*)| = J \), \( H_2^* \) is uniformly distributed in \( \tilde{C}(V, J) \). Thus, conditioned on \( |E(H_2^*)| = J \), the graph \( H^* \) that we obtain by superposing two independent graphs \( H_1 \) and \( H_2^* \) is uniformly distributed in

\[
C(V) \times \tilde{C}(V, J) \triangleq \{(V, E', E'')|(V, E') \in C(V), (V, E'') \in \tilde{C}(V, J)\}.
\]
From now on, we show how to relate this result to the distribution of \((V, E^{(1)}, E^{(2)})\) in the FGC process. For each graph \(G\) that can be \(H^*\), we need to show that
\[
P[H^* = G] = P \left[ (V, E^{(1)}, E^{(2)}) = G \right].
\] (20)

This is equivalent to showing the following:
\[
\sum_{j=0}^{N} P \left[ H^* = G \mid |E(H'_2)| = j \right] P \left[ |E(H'_2)| = j \right] = \sum_{j=0}^{N} P \left[ (V, E^{(1)}, E^{(2)}) = G \mid \sum_{j=1}^{N} \tau_j = j \right] P \left[ \sum_{j=1}^{N} \tau_j = j \right]
\] (21)

Note that \(|E(H'_2)|\) is a binomial random variable with parameter \((N, q)\) because we have removed each edge from \(H_2\) with probability \((1 - q)\). Since \(\tau_1, \ldots, \tau_N\) are independent Bernoulli random variables with mean \(q\), \(\sum_{j=1}^{N} \tau_j\) is also a binomial random variable with the same parameter. Hence, both \(\sum_{j=1}^{N} \tau_j\) and \(|E(H'_2)|\) have the same distribution, i.e., \(P \left[ |E(H'_2)| = j \right] = P \left[ \sum_{j=1}^{N} \tau_j = j \right]\) for all \(j\). Hence, if the graph \((V, E^{(1)}, E^{(2)})\) resulting from the FGC process is uniformly distributed in \(\mathcal{C}(V) \times \hat{\mathcal{C}}(V, J)\) conditioned on \(\sum_{j=1}^{N} \tau_j = J\), the equality in (21) is satisfied, and thus (21) is also satisfied. Hence, we now focus on showing that \((V, E^{(1)}, E^{(2)})\) is uniformly distributed in \(\mathcal{C}(V) \times \hat{\mathcal{C}}(V, J)\) conditioned on \(\sum_{j=1}^{N} \tau_j = J\).

Since we have constructed \((V, E^{(1)})\) following Condition 1 in the FGC process, it is a Hamiltonian cycle, which belongs to \(\mathcal{C}(V)\). Due to the same reason, \((V, E^{(2)})\) is a subgraph of a Hamiltonian cycle, and thus it should belong to \(\hat{\mathcal{C}}(V, J)\) if \(\sum_{j=1}^{N} \tau_j = J\). Hence, conditioned on \(\sum_{j=1}^{N} \tau_j = J\), the resulting graph \((V, E^{(1)}, E^{(2)})\) belongs to \(\mathcal{C}(V) \times \hat{\mathcal{C}}(V, J)\).

We next show how \((V, E^{(1)}, E^{(2)})\) is distributed in \(\mathcal{C}(V) \times \hat{\mathcal{C}}(V, J)\). Let \(\vec{\tau} = (c_1, \ldots, c_N)\). For a given \(\vec{c}\), define \(\vec{\tau} = (c'_1; \tau_l = 1)\). Note that the FGC process randomly chooses \((\vec{c}, \vec{\tau}, \vec{c}')\). We prove by contradiction that a unique choice of \((\vec{c}, \vec{\tau}, \vec{c}')\) in the FGC process results in a unique \((V, E^{(1)}, E^{(2)})\). Assume to the contrary that two different choices \(\Delta = (\vec{c}, \vec{\tau}, \vec{c}')\) and \(\Delta = (\vec{c}, \vec{\tau}, \vec{c}')\) result in the same graph. Let \(t^*\) be the first iteration that both decisions are not the same, i.e., \(c_t = c_t, \tau_l = \tau_l\), and \(c'_t = c'_t\) for all \(t < t^*\) and \(l < t^*\) satisfying \(\tau_l = 1\), and \(c_t \neq c'_t\) or \(\tau_t \neq \tau_t\) or \(c'_t \neq c'_t\). Since the FGC process works identically up to iteration \(t^* - 1\) under both choices, \(v_t\) at iteration \(t^*\) must be also identical. Since both choices \(\Delta = (\vec{c}, \vec{\tau}, \vec{c}')\) and \(\Delta = (\vec{c}, \vec{\tau}, \vec{c}')\) differ at iteration \(t^*\), the outgoing edges of \(v_t\) will be different under both choices. Hence, the resulting graphs under both choices cannot be the same, which is a contradiction. Hence, a unique \((\vec{c}, \vec{\tau}, \vec{c}')\) results in a unique resulting graph \((V, E^{(1)}, E^{(2)})\), which we have proven to be in \(\mathcal{C}(V) \times \hat{\mathcal{C}}(V, J)\) in the previous paragraph.

We finally show \((V, E^{(1)}, E^{(2)})\) is uniformly distributed in \(\mathcal{C}(V) \times \hat{\mathcal{C}}(V, J)\) conditioned on \(\sum_{j=1}^{N} \tau_j = J\). From Lemma 6, \(c_t\) \((t < N)\) is chosen uniformly at random among \(N - t\) candidates. Hence, \(\vec{c}\) is chosen uniformly at random among \((N - 1)!\) possible combinations. Similarly, from Lemma 6, \(c_t\) \((t < N)\) is chosen uniformly at random among \(N - \sum_{j=1}^{t-1} \tau_j - 1\) candidates, and thus \(\vec{c}'\) is chosen uniformly at random among \((N - 1)(N - 2) \cdots (N - J)\) combinations if \(\sum_{j=1}^{N} \tau_j = J\). Conditioned on \(\sum_{j=1}^{N} \tau_j = J\), there are \(\binom{N}{J}\) candidates for \(\vec{\tau}\). It is easy to see that \(\vec{\tau}\) is uniformly distributed among these candidates. Thus, conditioned on \(\sum_{j=1}^{N} \tau_j = J\), the FGC process chooses \((\vec{c}, \vec{\tau}, \vec{c}')\) uniformly at random among \((\binom{N}{J})^2\) candidates. Since a unique \((\vec{c}, \vec{\tau}, \vec{c}')\) results in unique \((V, E^{(1)}, E^{(2)}) \in \mathcal{C}(V) \times \hat{\mathcal{C}}(V, J)\), \((V, E^{(1)}, E^{(2)})\) is uniformly distributed among \((\binom{N}{J})^2\) graphs in \(\mathcal{C}(V) \times \hat{\mathcal{C}}(V, J)\). Since the cardinality of \(\mathcal{C}(V) \times \hat{\mathcal{C}}(V, J)\) is \((\binom{N}{J})^2\), we can say that \((V, E^{(1)}, E^{(2)})\) is uniformly distributed over the entire set \(\mathcal{C}(V) \times \hat{\mathcal{C}}(V, J)\) conditioned on \(\sum_{j=1}^{N} \tau_j = J\).

As we mentioned right after (21), both \((V, E^{(1)}, E^{(2)})\) and \(H^*\) have the same distribution.
C. Proof of Proposition 6

We prove this result by induction. We first show that (4) is true for \( l = t \). Since \( G(t) \) has been defined as the graph drawn up to iteration \( t \), \( Z(t) \) and \( z(t) \) are deterministic conditioned on \( G(t) \). Thus, we can remove the expectation from \( N - z(t) \), which corresponds to (4) for \( l = t \).

We assume that (4) is true for \( l = l' + 1 \leq t \), i.e.,

\[
E[N - z(t) + G'(t+1), \bar{\tau}] \\
\frac{(N - t - 1)(N - \sum_{j=1}^{t} \tau_j - 1)}{(N - (l' + 1) - 1)(N - \sum_{j=1}^{l' + 1} \tau_j - 1)}.
\]

If we rewrite \( z(l' + 1) \) using (3) and take \( E[\cdot | G(l'), \bar{\tau}] \), we have

\[
E[N - z(t) + G'(l'), \bar{\tau}] \\
\frac{(N - t - 1)(N - \sum_{j=1}^{t} \tau_j - 1)}{(N - (l' + 1) - 1)(N - \sum_{j=1}^{l' + 1} \tau_j - 1)}.
\]

where \( X_1 = 1_{\{c_{l+1}' \notin Z(l')\}} \) and \( X_2 = \tau_{l+1} \), \( \sum_{j=1}^{t} \tau_j - 1 \) by Lemma 6 we have

\[
E[X_1 | G(l'), \bar{\tau}] \\
\frac{|V \setminus Z(l')|}{|C(v_{l+1}, E'((l'))|} = \frac{N - z(l')}{N - l' - 1}.
\]

Similarly, \( c_{l+1}' \) is chosen from \( C(v_{l+1}, E^{(2)}(l')) \) with uniform distribution. Since \( |C(v_{l+1}, E^{(1)}(l'))| = N - l' - 1 \) by Lemma 6 we have

\[
E[X_2 | G(l'), \bar{\tau}, X_1] \\
\frac{\tau_{l+1}|V \setminus (Z(l') \cup \{c_{l+1}'\})|}{|C(v_{l+1}, E^{(2)}(l'))|} \\
\frac{\tau_{l+1}(N - z(l') - X_1)}{(N - \sum_{j=1}^{l' + 1} \tau_j - 1)}.
\]

Taking \( E[\cdot | G(l'), \bar{\tau}] \) to both sides and applying (23), we have

\[
E[X_2 | G(l'), \bar{\tau}] \\
\frac{\tau_{l+1}(N - z(l'))}{N - \sum_{j=1}^{l' + 1} \tau_j - 1} N - l' - 2
\]

Applying (23) and (24) to the numerator of (22), we have

\[
N - z(l') - E[X_1 + X_2 | G(l'), \bar{\tau}] \\
= (N - z(l')) \left( 1 - \frac{1}{N - l' - 1} - \frac{\tau_{l+1}}{N - \sum_{j=1}^{l' + 1} \tau_j - 1} (N - l' - 2) \right)
\]

\[
= (N - z(l')) \frac{N - l' - 2 N - \sum_{j=1}^{l' + 1} \tau_j - 1}{N - l' - 1}.
\]

Adding the above to (22), we have (4) for \( l = l' \). By induction, (4) is true for \( 1 \leq l \leq t \).
D. Proof of Proposition 7

We first define the Doob martingale \( \{B_l\}_{0 \leq l \leq t} \) as \( B_l \equiv E[N - z(t) \mid G(l)] \). From (3), we find an upper bound on \( |B_l - B_{l-1}| \) for each \( l \) and then find the probability that the martingale concentrates around its mean using the Azuma-Hoeffding bound.

The upper bound \( \psi_l \) can be found from the following lemma.

**Lemma 7:** Let \( \tilde{B}_l = E[N - z(t)\mid G(l), \tilde{\tau}] \). For \( 1 \leq l \leq t \),

\[
|\tilde{B}_l - \tilde{B}_{l-1}| \leq 2.
\]

The proof is provided in Appendix E. Since the upper-bound is independent of \( \tilde{\tau} \), the same bound also holds for \( B_l \), i.e., \( |B_l - B_{l-1}| \leq 2 \). Using the Azuma-Hoeffding bound, we have

\[
P[B_t - B_0 > \alpha] < \exp\left(-\frac{\alpha^2}{2\sum_{j=1}^t 2^2}\right). \tag{25}
\]

From Proposition 6, \( B_t = N - z(t) \) and \( B_0 = (N - t - 1)(1 - \frac{t}{N-1}) \). Taking \( \alpha = 1 + t(q(1 - \frac{t}{N-1}) - \psi) \), (25) can be expressed as follows: for \( t \leq N/2 \),

\[
P[z(t) < (1 + \psi)t] < \exp\left(-\frac{\alpha^2}{8t}\right)
\]

\[
< \exp\left(-\frac{(\frac{q}{2} - \psi)^2t}{8t}\right)
\]

\[
= \exp\left(-\frac{(\frac{q}{2} - \psi)^2}{8}\right) = \exp(-\sigma t). \tag{26}
\]

In (26), we have used \( \alpha < (q/2 - \psi)t \) for \( t \leq N/2 \). Thus, we have proven Proposition 7.

E. Proof of Lemma 7

From Proposition 6, we have

\[
\tilde{B}_l - \tilde{B}_{l-1} = f(l)(N - z(l)) - f(l - 1)(N - z(l - 1)),
\]

where

\[
f(l) = \frac{N - t - 1}{N - l - 1} \left( \frac{N - \sum_{j=1}^l \tau_j - 1}{N - l - 1} \right).
\]

Since \( l \leq t \), we have \( f(l - 1) \leq f(l) \leq 1 \). Thus,

\[
\tilde{B}_l - \tilde{B}_{l-1} \geq f(l)(N - z(l) - (N - z(l - 1)))
\]

\[
\geq f(l)(-2) > -2.
\]

Since \( z(l - 1) \geq z(l) \), we have

\[
\tilde{B}_l - \tilde{B}_{l-1} \leq (N - z(l - 1))(f(l) - f(l - 1)).
\]
Taking $x_l \triangleq (N - l - 1)$ and $y_l \triangleq (N - \sum_{j=1}^{l} \tau_j - 1)$, we can simplify $(f(l) - f(l - 1))$ in the above as

\[
(f(l) - f(l - 1)) \\
\leq x_ly_l \left[ \frac{1}{x_ly_l} - \frac{1}{(x_l + 1)(y_l + \tau_l)} \right] \\
\leq x_ly_l \left[ \frac{1}{x_ly_l} - \frac{1}{(x_l + 1)(y_l + 1)} \right] \\
\leq x_ly_l \left[ \frac{(x_l + 1)(y_l + 1) - x_ly_l}{x_ly_l(x_l + 1)(y_l + 1)} \right] \\
\leq \frac{1}{y_l + 1} + \frac{1}{x_l + 1} \leq \frac{2}{x_l + 1}.
\]

For the last two inequalities, we have used $x_t \leq x_l$, $y_t \leq y_l$, and $x_l \leq y_l$. Since $N - z(l - 1) \leq N - l = x_l + 1$, we have $\tilde{B}_l - \tilde{B}_{l-1} \leq 2$. Overall, we can find the Lipschitz difference $|\tilde{B}_l - \tilde{B}_{l-1}| \leq 2$. 