NEW SYNCHRONIZATION INDEX OF NON-IDENTICAL NETWORKS

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Abstract. Recently, quantifying the level of the synchrony in non-identical networks has got considerable attention. In the first part of this paper, a new synchronization index for non-identical networks is proposed. Non-identical networks can be categorized into two main types. The first group consists of similar oscillators with miss-match in their parameters, and the second group is organized from completely different oscillators. The synchronizability of the second group of the non-identical networks is more challenging since the amplitude and frequencies of the different oscillators may not be matched. Thus, one way to investigate the limitation of the synchronizability of these networks is to explore the parameter space of their amplitude and frequency. In the second part of this research, the amplitude and frequency of each individual system of the non-identical network are considered as varying parameters and the effect of these parameters on the synchronizability of the network is measured with the proposed index. The results are compared with the conventional indexes, such as the root-mean-square error and phase synchrony with the help of Hilbert transform. The outcomes show that the new proposed synchronization index not only is simple and accurate, but also fast with short computational time. It is not affected by amplitude, phase, or polarity. It can detect the similarity in the fluctuations which is a sign of synchrony in the non-identical networks.

1. Introduction. Complex dynamical networks are studied in various scientific fields [17, 28] such as sociology [9], economy [16] and biology [13] especially neuroscience [2, 18, 39]. A dynamical network can be considered as some individual dynamical systems which have interaction with each other [7]. From the mathematical point of view, each node in a network is an individual system, and the interactions are considered as connection links [29]. Complex dynamical networks can be divided into two groups with respect to their dynamical systems (each node). The first group is known as identical networks in which all nodes are exactly the same dynamical systems [32]. The second group is non-identical networks. They are categorized into two different types themselves [1]. One type is a network with similar dynamical systems in each node but with different parameters. However, the more challenging type of non-identical networks is the one which consists of completely different dynamical systems in each node. Analyzing the dynamical properties of this kind of network such as collective behaviors, is more complicated than the others.

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An important collective behavior of the complex networks is synchronization [12]. Synchronization is an essential issue in dynamical networks which has been widely addressed in the literature [24, 33, 20, 34]. It is usually referred to the state of a network in which all the nodes share relatively the same dynamical motion properties [4, 19]. There are different analytical methods to investigate the synchronizability in identical networks [10, 22, 3]. Wu and Chua have proposed a conjecture for the relationship between the coupling strength and the eigenvalues of coupling matrices in linearly coupled arrays of oscillators [36]. Alternatively, master stability function (MSF), a method which investigates the linear stability of the synchronization manifold in identical networks, was proposed in 1998 by Pecora & Carrol [23]. However, analytical investigation of the synchronizability in the nonidentical networks is challenging since the classical synchronization manifold cannot be defined in such kind of networks [27, 5, 38]. In other words, the complete synchronization cannot occur. There are few works in the literature that have studied the synchronization in nonidentical networks. Hill et al. have proposed a global synchronization criterion for nonidentical networks in which certain boundedness is used to define the synchronization manifold [14]. Xiang et al. have constructed a Lyapunov function for all nodes of a nonidentical network [26]. This method is useful just for the networks in which all the nodes have the same equilibrium point. However, other kinds of synchronization as an intermediate regime can be detected in nonidentical networks such as phase synchrony (PS) [21]. In this method, there can be found a relation between the phase of the oscillators while their amplitudes are uncorrelated [25, 6]. Also, finding the phase of the oscillators, especially chaotic oscillators, is not simple [35]. The most general method to calculate the phase of the oscillators is the Hilbert transform. The other approach is the classical error function between different nodes which can be taken as a synchronization property. However, it cannot be accurate according to the fact that full synchronization cannot be achieved. Other researches used the statistical analysis to detect the similarity between time series of nonidentical networks. They have used methods like mutual information, coherence, transfer entropy, synchronization likelihood, Granger causality, and etc. [37]. These methods are sophisticated and need long computational time.

In this paper, a new method of estimating the synchronization degree in nonidentical networks is proposed. This method concentrates on the pattern synchronization of the oscillations in the network and ignores the differences in the amplitude. In the next step, we are focused on another challenging issue in non-identical networks with different dynamical systems in each node. It is assigned to the investigation of the effect of form compatibility of the different dynamical systems in each node to the synchronization of the whole network. This issue can be summarized into the answer to a serious question: Which kinds of systems with what kinds of properties can be used in each node of non-identical networks? Indeed, besides the proper coupling strength, the overall dynamical properties of the different individual systems need to be (at least to some extent) consistent and matched with each other. In a nutshell, their dynamical form needs to be compatible with each other. For example, if the amplitude of the coupled variable in each node is very different from others (e.g., very low versus very high), it leads the networks to unbounded states. Therefore, the new proposed method of measuring synchronization degree is applied to a nonidentical network of the Lorenz system and the Rössler system. To find the optimum dynamical properties of the synchronization, the parameter space of the...
network is investigated. To this end, the amplitude and frequency of the Rössler system are considered to be varying parameters.

The rest of the paper is organized as follows. The new method of estimating the synchronization degree in a non-identical network is proposed in Section 2. Section 3 is dedicated to form compatibility which can be affected by amplitude and frequency of the Rössler oscillator in the network of Lorenz and Rössler oscillators. The results of different methods contain pattern synchrony with the new proposed method, mean square error, and phase synchrony with Hilbert transform are evaluated and compared with each other in section 4. Finally, section 5 is the conclusion.

2. New synchronization method. Consider a non-identical network with diffusive coupling which consists of $N$ node with different dynamical oscillators in each node as:

$$\dot{x}_i = f_i(x_i) + d \sum_{j=1}^{N} g_{ij}(x_j - x_i)$$

(1)

where $x_i$ is a vector which denotes the dynamical states of $i^{th}$ oscillator and $f_i$ are continuously differentiable functions. $d$ is the coupling strength, and coupling matrix is represented by a square laplacian matrix $G$. For simplicity, it is assumed that $N = 2$ (Lorenz oscillator in one node and Rössler oscillator is in the other node). Thus the Eq.1 changes to:

$$\dot{x}_L = s(y_L - x_L) + d(x_R - x_L)$$
$$\dot{y}_L = x_L(r - z_L) - y_L$$
$$\dot{z}_L = x_L y_L - \beta z_L$$
$$\dot{x}_R = (y_R - z_R) + d(x_L - x_R)$$
$$\dot{y}_R = x_R + ay_R$$
$$\dot{z}_R = b + (x_R - c)z_R$$

(2)

Both oscillators are in the chaotic mode where the parameters are set to $s = 10$, $r = 28$, $\beta = 8/3$, $a = c = 0.2$ and $b = 7$. Fig. 1 represent the phase space and time series of the network when $d = 0$. As the coupling strength of the network increases, the interaction of the network changes the inherent dynamics of the individual Lorenz and Rössler oscillators. It implies that the pattern of nodes becomes more correlated with each other, and the synchronization degree increases when the coupling strength increases. To shed light on it, Fig. 2 (b) represents the first states of each oscillator of Eq.2 with respect to each other. As it is illustrated in Fig. 2, increasing the coupling strength of the networks leads to the more synchronized networks and made the different systems to oscillate more correlated to each other.

2.1. Pattern synchrony. To quantify the degree of pattern synchronization in the network of Eq. 2, a similarity index between the fluctuations need to be defined. In such an index, the degree of synchronization should increase when the oscillators go up and down with respect to each other.

Fig. 3 illustrates the flowchart of the new methods which estimate the pattern synchronization degree in a non-identical network. The method begins with finding the peaks of the time series at each node in the network. Such peaks of a time series can be a representation of a whole oscillation. Therefore, the degree of the synchronization in the networks can be proportionate with the number of the nearby peaks in different time series of the network. The main purpose of this new approach
of finding the degree of synchronization in non-identical networks is counting the number of the close peaks in the time-series of each node with respect to others and put the sign of Not a Number (NaN) for those who are far from each other.

As it is illustrated in Fig. 3, the new method starts with extracting the peaks of time series of the first state of each oscillator (since the coupling terms link the oscillators from the first state of each system). Then these peaks and their corresponding times are put to the vectors named $P_i$ and $T_i$, respectively. Due to the fact that different systems in the same run time can have different oscillations, the

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**Figures 1 and 2**

**Figure 1.** Phase space and time series of the Rössler and Lorenz oscillators, respectively.

**Figure 2.** (a) Time series of the first state of each oscillator of the network in different coupling strength. (b) The first state of each oscillator of network with respect to each other.
length of the $P_i$ vectors can be different. So the minimum length of the vectors has been chosen. The goal of the algorithm is to count the number of the adjacent peaks in time to quantify the synchronized oscillations in the network. Therefore, the algorithm searches the vector $T$ which is the absolute value of the differences between the $T_i$ vectors. After that, the algorithm enters the main loop which searches each element of the $T_i$ to find the values more than a pre-defined threshold. If the differences between vectors $T_i$ are more than a threshold, it means there is a mismatch between the oscillations of each system. Thus the algorithm searches to find out the system which is corresponding to these mismatches, let its corresponding element be the Nan (Not a number) and shift the other entries of the vector. Then the algorithm needs to go back and calculate the new vector $T$. This loop continues until the algorithm cannot find any mismatch between the time series. At last, the algorithm removes the Nan values and let the $D$ to be the number of the other elements of the vector $T$. Finally, $M$, which is the normalized version of the $D$ is calculated by dividing the $D$ to the whole length of vector $T$ before removing the Nan values.

To evaluate the new proposed method, the results of pattern synchrony is compared with two conventional indexes of synchronization in a non-identical network which are Root Mean Square Error (RMSE) and Phase Synchronization (PS) with respect to Hilbert transform.

Figure 3. The flowchart of the proposed method to estimate the synchronization degree on non-identical networks
2.2. **RMSE.** Root mean square error quantify the average squared difference between the amplitude of the time series during the run time which can be ruled for the network of Eq.2 as follows:

\[
E = \sqrt{\frac{\sum_{i=1}^{N} (x_R - x_L)^2}{N}}
\]

(3)

where \(x_R\) denote the first state of the Rössler oscillator, \(x_L\) is the first state of the Lorenz oscillator and \(N\) represent the duration of the time series or run time of the network.

2.3. **PS.** The idea of the phase synchronization (PS) claims that there would be a relation between the phases of the oscillators in the non-identical networks when their amplitudes are uncorrelated. It can be described by the average dynamics of the phase difference during the run time of the network:

\[
H = \frac{\sum_{i=1}^{N} |\phi_R - \phi_L|}{N}
\]

(4)

where \(\phi_R\) denote the phase of the Rössler oscillator, \(\phi_L\) is the phase of the Lorenz oscillator and \(N\) represent the duration of the time series or run time of the network. To calculate the phase of the oscillators, the Hilbert transform is used. Hilbert transform has been proposed by Gabor [11]. Gabor has proved that all information of the analytic signals can be compact in the amplitude of the signal. Moreover, the phase of the signals can be calculated just by knowing the amplitude. The analytic signal \(S(t)\) can be represented as:

\[
S(t) = r(t) + jI(t) = A(t)e^{j\phi(t)}
\]

\[
I(t) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{S(\tau)}{t-\tau} d\tau
\]

(5)

where \(r(t)\) and \(I(t)\) are the real and imaginary part of the signal. P.V. denote the Cauchy principal value of the integral. The results of these three methods are compared in Fig. 4. According to Fig. 2, the network is more synchronized, and the patterns of the two different oscillators are more correlated by increasing the coupling strength. Therefore, it is expected that the Error and PS show a decreasing pattern despite the new algorithm, which should be increasing. As it is represented in Fig. 4, the blue line shows the pattern synchrony with the new method and represents the degree of synchronization with respect to changing the coupling strength. It grows by increasing the coupling strength. However, the result of Error and PS methods which are shown by the red and green lines, are not appropriate and the trend of each line does not completely follow the expected decreasing pattern.

2.4. **Extra example.** In the following, the result of the new pattern synchrony algorithm which is proposed in Sec. 2.1 is compared with RMSE and PS by using the nonidentical network consist of two different neuronal models. Synchronization in neural networks is one of the topics of current interest [30, 31]. To this end the Eq.1 which is a nonidentical network consists of two Hindmarsh-Rose (HR) [15] and
Figure 4. Compare the result of the pattern synchrony with new proposed method on Eq. 2 with the Error (RMSE) and PS with respect to Hilbert transform approaches concerning to changing the coupling strength (d). The blue, red and green lines are corresponding to the pattern synchrony, Error and PS methods, respectively.

FitzHug-Nagumo (FHN) [8] neuron models changes to:

\[
\begin{align*}
\dot{x}_{FHN} &= x_{FHN} + \frac{x_{FHN}^3}{3} - y_{FHN} + I_{FHN} + d(x_{HR} - x_{FHN}) \\
\dot{y}_{FHN} &= \frac{1}{\tau}(x_{FN} + a - by_{FN}) \\
\dot{x}_{HR} &= y_{HM} + 3x_{HM}^2 - x_{HM}^3 - z_{HM} + I_{HM} + d(x_{FHN} - x_{HR}) \\
\dot{y}_{HR} &= 1 - 5x_{HM}^2 - y_{HM} \\
\dot{z}_{HR} &= -rz_{HM} + rs(x_{HM} + 1.6)
\end{align*}
\]

(6)

where the parameters are set to \(I_{FZ} = 0.5, a = 0.7, b = 0.8, c = 12.5, I_{HM} = 3.2, r = 0.006\) and \(s = 4\). Fig. 5 represents the time series and state-space of the network when the coupling parameter is \(d = 0\). As it is shown in Fig. 5, the HR is chaotic, and FHN is periodic when the coupling strength is set to \(d = 0\). However, increasing the coupling strength leads the interaction flows in the connection links and changes the overall behavior of the network. Fig. 6 illustrates the first state of each oscillator by changing the coupling strength. As it is shown in Fig. 6, the dynamical pattern of each oscillator become more correlated during increasing the coupling strength. To quantify the degree of synchrony, the result of three mentioned methods is plotted in Fig. 7. The results of pattern synchrony with the help of the new proposed method, RMSE and phase synchrony with Hilbert transform are illustrated in Fig. 7 with respect to changing the coupling strength in the blue, red and green lines, respectively. There is a sudden maximum in pattern synchrony when the coupling strength is \(d = 0.7\) which does not exist in RMSE and phase synchrony. To check the accuracy of the pattern synchrony, the time series and state-space of the network is plotted in three different values of coupling strength which are before, during and after the \(d = 0.7\). As it is shown in Fig. 8, the pattern of the network in Fig. 8 (b)
Figure 5. Phase space and time series of the HR and FHN, respectively.

Figure 6. (a) Time series of the first state of each oscillator of the Eq. 6 in different coupling strength. (b) the first state of each oscillator of network with respect to each other.

is more synchronized than the Fig. 8 (a) and (c) which shows the accuracy of the new proposed method.

3. Form compatibility. Synchronization in a non-identical network with different oscillators in each node is affected by the various factors. One of the essential factors is the degree of the dynamical form compatibility of these different oscillators with each other. For example, the first time series of Fig. 2(a) represents the overall form of the uncoupled Lorenz (red) and Rössler (blue) oscillations in which the amplitude and frequency of the Rössler oscillations are lower than the Lorenz. In order to investigate the effect of form compatibility between the different oscillators
Figure 7. Compare the result of the pattern synchrony with the new proposed method on Eq. 6 with the Error (RMSE) and PS with respect to Hilbert transform approaches concerning to changing the coupling strength $d$. The blue, red and green lines are corresponding to the pattern synchrony, Error and PS methods, respectively.

Figure 8. Compare the result of the pattern synchrony with the new proposed method on Eq. 6 in three different values of coupling strength which is (a) $d = 0.6$ (b) $d = 0.7$ and (c) $d = 0.8$.

of the non-identical network, the amplitude, and frequency of the Rössler oscillator are considered as two parameters. So the Eq. 2 changes to:

\[ \begin{align*}
\dot{x}_L &= s(y_L - x_L) + d(x_R - x_L) \\
\dot{y}_L &= x_L(r - z_L) - y_L \\
\dot{z}_L &= x_L y_L - \beta z_L \\
\dot{x}_R &= \frac{f}{A} [(A y_R - A z_R) + d(x_L - A x_R)] \\
\dot{x}_R &= \frac{f}{A} (A x_R + A a y_R) \\
x_R &= \frac{f}{A} (b + A (A x_R - c) z_R)
\end{align*} \]
where $A$ and $f$ are assumed to be the network parameters which are responsible for the amplitude and frequency of the Rössler oscillator. For instance, Fig. 5 represents the uncoupled time series of the Eq. 7 in different parameters when the coupling strength is $d=0$.

4. **Result.** In order to check the effect of form compatibility on the network synchronization, the synchronization degree of the Eq. 7 is investigated by pattern synchrony with new proposed method, Error, and PS with Hilbert transform methods. Fig. 10 shows the space of the amplitude and frequency parameters concerning changing the coupling strength with the help of the pattern synchrony (Sec. 2.1). The x-axis is responsible for different frequencies, and the y-axis shows the different amplitudes. The maximum synchronization degree is shown in light yellow. As it is illustrated in Fig. 10, each column is in the same color, which means that the synchronization degree is not affected by changing the amplitude. Since the new proposed algorithm calculates the synchronized oscillations regardless of their amplitude. In low coupling strength, the maximum synchronization degree is in the vicinity of $F = 5$ where the two oscillators are at the same speed. However, increasing the coupling strength changes the synchronization to get maximum in lower frequencies. Increasing the coupling strength leads to an unstable network, which is shown in white color. According to the fact that the synchronization degree is not affected by the amplitude of the Rössler oscillator in the Eq. 6, the amplitude is considered to be fixed at $A = 1$, and the parameter space of the frequency is investigated by changing the coupling strength. Fig. 11 represents the synchronization degree of the Eq. 7 by changing the parameter $F$ and coupling strength of the network.
Figure 10. Synchronization degree in the parameter space of Eq. 7 with respect to changing the coupling strength with the help of the pattern synchrony (Sec. 2.1). The synchronization of the network is constant in each column, while the amplitude is changing. Since the important point in new proposed method is to consider the pattern similarity, not the similarity in the amplitudes. The light yellow and dark blue represent the most and the least synchronized pattern, respectively. The unstable state of the network is shown in the white color.

Figure 11. Synchronization degree of the Eq. 7 with respect to changing the parameter F and coupling strength by the help of the proposed algorithm. The unbounded states of the network are shown in white color.

Fig. 12 shows the synchronization degree of the Eq. 7 by the help of RMSE method. This method is based on the fact that the maximum synchronization degree has a minimum error where states of both oscillators are in the same dynamics. This method is affected by both amplitude and frequency parameters; accordingly
it may not be able to detect the accurate synchronization degree of the network since nonidentical networks may show more synchronized patterns with different amplitudes. The dark blue shows the relatively minimum error or best synchronization degree which is in the vicinity of $F = 3$.

Fig. 13 shows the PS degree of the Eq. 7 at the amplitude and frequencies parameter space. The minimum phase error or the best PS is shown with the dark blue color, which is affected by both amplitude and frequency parameter. The optimum of the frequency parameter is in the range of $F \in [3, 5]$, and the optimum amplitude almost is where the amplitude of the Rössler oscillator is higher than the Lorenz oscillator. To evaluate the methods precisely, the time series of the network is plotted with the best parameters of each method in Fig. 14 at same coupling strengths. Each row of Fig. 14 contains the time series with the optimum parameters of the different method in the same coupling strength while each column is corresponding to the same method with different coupling strength. According to Fig. 14, the new method shows the synchronization pattern better than the other methods.

5. **Conclusion.** The new algorithm, which is a novel synchronization index of the non-identical networks, has been proposed in this paper. It is based on counting the adjacent peaks of coupled time series in the network. The main goal of this method is finding the synchronized patterns in each node regardless of their amplitude. It is simple and accurate with a short computational time. Also, it is not affected by amplitude, phase or polarity. To evaluate the proposed method, it is compared with two conventional synchronization indexes of non-identical networks which are RMSE and PS with the help of Hilbert transform. Moreover, to check the form compatibility which consists of the compatible amplitude and frequency (speed) of oscillators in each node of the non-identical network, the amplitude, and frequency of the Rössler oscillator is considered as varying parameters. Then the parameter
Figure 13. Synchronization degree of the Eq. 7 in the parameter space of A & F with respect to changing the coupling strength by the help of the PS. The dark blue is responsible for the minimum phase error and best PS. The unstable state of the network is shown at the white color.

Figure 14. Comparison the time series of the Eq. 7 with the optimum parameters of the new proposed method, E & PS with Hilbert transform synchronization index when the coupling strength is set to (a) $d = 0.5$ and (b) $d = 1$. 
space of the network has been explored with the proposed algorithm, and the results are compared with conventional methods. The results showed that the new proposed synchronization index not only is simple and accurate, but also fast with short computational time. It is not affected by amplitude, phase, or polarity. It can detect the similarity in the fluctuations which is a sign of synchrony in the non-identical networks.

REFERENCES

[1] G. R. ´Avila, J. Kurths, J.-L. Guisset and J.-L. Deneubourg, How do small differences in nonidentical pulse-coupled oscillators induce great changes in their synchronous behavior?, The European Physical Journal Special Topics, 223 (2014), 2759–2773.
[2] D. S. Bassett and O. Sporns, Network neuroscience, Nature Neuroscience, 20 (2017), 353–364.
[3] V. N. Belykh, I. V. Belykh and M. Hasler, Connection graph stability method for synchronized coupled chaotic systems, Physica D: Nonlinear Phenomena, 195 (2004), 159–187.
[4] S. Boccaletti, J. Almendral, S. Guan, I. Leyva, Z. Liu, I. Sendiña-Nadal, Z. Wang and Y. Zou, Explosive transitions in complex networks’ structure and dynamics: Percolation and synchronization, Physics Reports, 660 (2016), 1–94.
[5] S. Boccaletti, J. Brogad, F. Arecchi and H. Mancini, Synchronization in nonidentical extended systems, Physical Review Letters, 83 (1999), 536.
[6] S. Boccaletti, J. Kurths, G. Osipov, D. Valladares and C. Zhou, The synchronization of chaotic systems, Physics Reports, 366 (2002), 1–101.
[7] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez and D.-U. Hwang, Complex networks: Structure and dynamics, Physics Reports, 424 (2006), 175–308.
[8] R. FitzHugh, Mathematical models of threshold phenomena in the nerve membrane, The Bulletin of Mathematical Biophysics, 17 (1955), 257–278.
[9] L. C. Freeman, Research Methods in Social Network Analysis, Routledge, 2017.
[10] H. Fujisaka and T. Yamada, Stability theory of synchronized motion in coupled-oscillator systems, Progress of Theoretical Physics, 69 (1983), 32–47.
[11] D. Gabor, Theory of communication. part I: The analysis of information, Journal of the Institution of Electrical Engineers-Part III: Radio and Communication Engineering, 93 (1946), 429–441.
[12] J. Gao, B. Barzel and A.-L. Barabási, Universal resilience patterns in complex networks, Nature, 530 (2016), 307–312.
[13] M. Gosak, R. Marković, J. Dolenšek, M. S. Rupnik, M. Marhl, A. Stožer and M. Perc, Network science of biological systems at different scales: A review, Physics of Life Reviews, 24 (2018), 118–135.
[14] D. J. Hill and J. Zhao, Global synchronization of complex dynamical networks with non-identical nodes, in 2008 47th IEEE Conference on Decision and Control, IEEE, 2008, 817–822.
[15] J. L. Hindmarsh and R. Rose, A model of neuronal bursting using three coupled first order differential equations, Proceedings of the Royal society of London. Series B. Biological sciences, 221 (1984), 87–102.
[16] D. Y. Kenett and S. Havlin, Network science: A useful tool in economics and finance, Mind & Society, 14 (2015), 155–167.
[17] D. Y. Kenett, M. Perc and S. Boccaletti, Networks of networks—an introduction, Chaos, Solitons & Fractals, 80 (2015), 1–6.
[18] J. Ma and J. Tang, A review for dynamics of collective behaviors of network of neurons, Science China Technological Sciences, 58 (2015), 2038–2045.
[19] S. Majhi, B. K. Bera, D. Ghosh and M. Perc, Chimera states in neuronal networks: A review, Physics of Life Reviews, 28 (2019), 100–121.
[20] S. Majhi, D. Ghosh and J. Kurths, Emergence of synchronization in multiplex networks of mobile rössler oscillators, Physical Review E, 99 (2019), 012308, 13pp.
[21] A. Y. Mutlu and S. Aiyviente, Multivariate empirical mode decomposition for quantifying multivariate phase synchronization, EURASIP Journal on Advances in Signal Processing, 2011 (2011), 615717.
[22] V. Patidar and K. Sud, Identical synchronization in chaotic jerk dynamical systems, Electronic Journal of Theoretical Physics, 3 (2006), 33–70.
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[23] L. M. Pecora and T. L. Carroll, Master stability functions for synchronized coupled systems, Physical Review Letters, 80 (1998), 2109.
[24] A. Pikovsky, M. Rosenblum and J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences, Cambridge University Press, Cambridge, 2001.
[25] F. A. Rodrigues, T. K. D. Peron, P. Ji and J. Kurths, The kuramoto model in complex networks, Physics Reports, 610 (2016), 1–98.
[26] M. Rosenblum, A. Pikovsky, J. Kurths, C. Schäfer and P. A. Tass, Phase synchronization: From theory to data analysis, in Handbook of Biological Physics, 4 (2001), 279–321.
[27] M. G. Rosenblum, A. S. Pikovsky and J. Kurths, Phase synchronization of chaotic oscillators, Physical Review Letters, 76 (1996), 1804.
[28] I. Stamova, T. Stamov and X. Li, Global exponential stability of a class of impulsive cellular neural networks with supremums, International Journal of Adaptive Control and Signal Processing, 28 (2014), 1227–1239.
[29] S. H. Strogatz, Exploring complex networks, Nature, 410 (2001), 268–276.
[30] X. Sun, J. Lei, M. Perc, J. Kurths and G. Chen, Burst synchronization transitions in a neuronal network of subnetworks, Chaos: An Interdisciplinary Journal of Nonlinear Science, 21 (2011), 016110.
[31] X. Sun, M. Perc, J. Kurths and Q. Lu, Fast regular firings induced by intra-and inter-time delays in two clustered neuronal networks, Chaos: An Interdisciplinary Journal of Nonlinear Science, 28 (2018), 106310, 10pp.
[32] U. K. Verma, A. Sharma, N. K. Kamal, J. Kurths and M. D. Shrimali, Explosive death induced by mean-field diffusion in identical oscillators, Scientific Reports, 7 (2017), 7936.
[33] C. Wang and J. Ma, A review and guidance for pattern selection in spatiotemporal system, International Journal of Modern Physics B, 32 (2018), 1830003, 15pp.
[34] Q. Wang, G. Chen and M. Perc, Synchronous bursts on scale-free neuronal networks with attractive and repulsive coupling, PLoS One, 6 (2011), e15851.
[35] D. Wen, Y. Zhou and X. Li, A critical review: Coupling and synchronization analysis methods of eeg signal with mild cognitive impairment, Frontiers in Aging Neuroscience, 7 (2015), 54.
[36] C. W. Wu and L. O. Chua, On a conjecture regarding the synchronization in an array of linearly coupled dynamical systems, IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, 43 (1996), 161–165.
[37] J. Xiang and G. Chen, On the v-stability of complex dynamical networks, Automatica, 43 (2007), 1049–1057.
[38] X. Yang, X. Li, Q. Xi and P. Duan, Review of stability and stabilization for impulsive delayed systems, Mathematical Biosciences & Engineering, 15 (2018), 1495–1515.
[39] X. Zhang, X. Li, J. Cao and F. Miaadi, Design of memory controllers for finite-time stabilization of delayed neural networks with uncertainty, Journal of the Franklin Institute, 355 (2018), 5394–5413.

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