On warped product gradient \(\eta\)-Ricci solitons

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Abstract

If the potential vector field of an \(\eta\)-Ricci soliton is of gradient type, using Bochner formula, we derive from the soliton equation a Laplacian equation satisfied by the potential function \(f\). In a particular case of irrotational potential vector field we prove that the soliton is completely determined by \(f\). We give a way to construct a gradient \(\eta\)-Ricci soliton on a warped product manifold and show that if the base manifold is oriented, compact and of constant scalar curvature, the soliton on the product manifold gives a lower bound for its scalar curvature.

1 Introduction

Ricci flow, introduced by R. S. Hamilton \cite{Ham95}, deforms a Riemannian metric \(g\) by the evolution equation \(\frac{\partial}{\partial t} g = -2S\), called the ”heat equation” for Riemannian metrics, towards a canonical metric. Modeling the behavior of the Ricci flow near a singularity, Ricci solitons \cite{Ham03} have been studied in the contexts of complex, contact and paracontact geometries \cite{Cal04}.

The more general notion of \(\eta\)-Ricci soliton was introduced by J. T. Cho and M. Kimura \cite{ChoKim13} and was treated by C. Călin and M. Crasmareanu on Hopf hypersurfaces in complex space forms \cite{CalCra13}. We also discussed some aspects of \(\eta\)-Ricci solitons in paracontact \cite{Blaga16}, \cite{Blaga17} and Lorentzian para-Sasakian geometry \cite{Blaga18}.

A particular case of soliton arises when the potential vector field is the gradient of a smooth function. The gradient vector fields play a central rôle in the Morse-Smale theory \cite{Per03}. G. Y. Perelman showed that if the manifold is compact, then the Ricci soliton is
Warped product gradient $\eta$-Ricci solitons

In [13], R. S. Hamilton conjectured that a compact gradient Ricci soliton on a manifold $M$ with positive curvature operator implies that $M$ is Einstein manifold. In [11], S. Deshmukh proved that a Ricci soliton of positive Ricci curvature and whose potential vector field is of Jacobi-type, is compact and therefore, a gradient Ricci soliton. Different aspects of gradient Ricci solitons were studied in various papers. In [1], N. Basu and A. Bhattacharyya treated gradient Ricci solitons in Kenmotsu manifolds having Killing potential vector field. P. Petersen and W. Wylie discussed the rigidity of gradient Ricci solitons [19] and gave a classification imposing different curvature conditions [18].

The aim of our paper is to investigate some properties of gradient $\eta$-Ricci solitons. After deducing some results derived from the Bochner formula, we construct a gradient $\eta$-Ricci soliton on a warped product manifold and for the particular case of product manifolds, we show that if the base is oriented, compact and of constant scalar curvature, then we obtain a lower bound for the scalar curvature of the product manifold.

2 Bochner formula revisited

Let $(M,g)$ be an $m$-dimensional Riemannian manifold and consider $\xi$ a gradient vector field on $M$. If $\xi := \text{grad}(f)$, for $f$ a smooth function on $M$, then the $g$-dual 1-form $\eta$ of $\xi$ is closed, as $\eta(X) := g(X, \xi) = df(X)$. Then $0 = (d\eta)(X,Y) := X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X)$, hence:

$$g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X),$$

for any $X, Y \in \chi(M)$, where $\nabla$ is the Levi-Civita connection of $g$.

Also:

$$\text{div}(\xi) = \Delta(f)$$

and

$$\text{div}(\eta) := \text{trace}(Z \mapsto \sharp((\nabla \eta)(Z, \cdot))) = \sum_{i=1}^{m} (\nabla_{E_i} \eta)E_i = \sum_{i=1}^{m} g(E_i, \nabla_{E_i} \xi) := \text{div}(\xi),$$

for $\{E_i\}_{1 \leq i \leq m}$ a local orthonormal frame field with $\nabla_{E_i} E_j = 0$ in a point. From now on, whenever we make a local computation, we will consider this frame.
In this case, the Bochner formula becomes:

\[ \frac{1}{2} \Delta (|\xi|^2) = |\nabla \xi|^2 + S(\xi, \xi) + \xi(\text{div}(\xi)), \]

where \( S \) is the Ricci curvature of \( g \). Indeed:

\[ (\text{div}(\mathcal{L}_\xi g))(X) := \text{trace}(Z \mapsto Z((\nabla (\mathcal{L}_\xi g))(Z, \cdot, X))) = \sum_{i=1}^{m} (\nabla_{E_i}(\mathcal{L}_\xi g))(E_i, X) = \]

\[ = \sum_{i=1}^{m} \{ E_i((\mathcal{L}_\xi g)(E_i, X)) - (\mathcal{L}_\xi g)(E_i, \nabla_{E_i} X) \} = 2 \sum_{i=1}^{m} g(\nabla_{E_i} \nabla_{X} \xi - \nabla_{\nabla_{E_i} X} \xi, E_i) := \]

\[ := 2 \sum_{i=1}^{m} g(\nabla_{E_i}^2 X, E_i) = 2 \sum_{i=1}^{m} g(\nabla_{X,E_i}^2 \xi + R(E_i, X) \xi, E_i) := \]

\[ := 2 \sum_{i=1}^{m} g(\nabla_{X,E_i}^2, E_i) + 2 \text{trace}(Z \mapsto R(\nabla_{\xi} X)) := 2 \sum_{i=1}^{m} g(\nabla_{X,E_i} \xi - \nabla_{\nabla_{X} E_i} \xi, E_i) + 2S(X, \xi) = \]

\[ = 2 \sum_{i=1}^{m} g(\nabla_{E_i} \nabla_{E_i} \xi, E_i) + 2S(X, \xi) = 2 \sum_{i=1}^{m} X(g(\nabla_{E_i} \xi, E_i)) + 2S(X, \xi) = 2 X(\text{div}(\xi)) + 2S(X, \xi), \]

where \( R \) is the Riemann curvature and \( S \) is the Ricci curvature tensor fields of the metric \( g \) and the relation (5), for \( X := \xi \), becomes:

\[ (\text{div}(\mathcal{L}_\xi g))(\xi) = 2\xi(\text{div}(\xi)) + 2S(\xi, \xi). \]

But the Bochner formula states that for any vector field \( X \):

\[ (\text{div}(\mathcal{L}_X g))(X) = \frac{1}{2} \Delta (|X|^2) - |\nabla X|^2 + S(X, X) + X(\text{div}(X)) \]

and from (4) and (7) we deduce that:

\[ \Delta (|\xi|^2) - 2|\nabla \xi|^2 = 2S(\xi, \xi) + 2\xi(\text{div}(\xi)). \]

Remark that (5) can be written in terms of (1, 1)-tensor fields:

\[ \text{div}(L_{\xi} g) = 2d(\text{div}(\xi)) + 2iQ_{\xi} g, \]

where \( Q \) is the Ricci operator defined by \( g(QX, Y) := S(X, Y) \).
3 Gradient \( \eta \)-Ricci solitons

Consider now the equation:

\[
\mathcal{L}_\xi g + 2S + 2\lambda g + 2\mu \eta \otimes \eta = 0,
\]

where \( g \) is a Riemannian metric, \( S \) its Ricci curvature, \( \eta \) a 1-form and \( \lambda \) and \( \mu \) are real constants. The data \( (g, \xi, \lambda, \mu) \) which satisfy the equation (10) is said to be an \( \eta \)-Ricci soliton on \( M \) \cite{10}; in particular, if \( \mu = 0 \), \( (g, \xi, \lambda) \) is a Ricci soliton \cite{14}. If the potential vector field \( \xi \) is of gradient type, \( \xi = \text{grad}(f) \), for \( f \) a smooth function on \( M \), then \( (g, \xi, \lambda, \mu) \) is called gradient \( \eta \)-Ricci soliton.

**Proposition 3.1.** Let \((M, g)\) be a Riemannian manifold. If (10) defines a gradient \( \eta \)-Ricci soliton on \( M \) with the potential vector field \( \xi := \text{grad}(f) \) and \( \eta \) is the \( g \)-dual 1-form of \( \xi \), then:

\[
(\nabla X Q)Y - (\nabla Y Q)X = -\nabla_{X,Y}^2 \xi + \nabla_{Y,X} \xi + \mu(\nabla \xi \otimes \nabla \xi - \nabla \xi \otimes \nabla \xi)(X, Y),
\]

for any \( X, Y \in \chi(M) \), where \( Q \) stands for the Ricci operator.

**Proof.** As \( g(QX, Y) := S(X, Y) \), follows:

\[
\nabla \xi + Q + \lambda I_{\chi(M)} + \mu df \otimes \xi = 0.
\]

Then:

\[
(\nabla X Q)Y = - (\nabla X \nabla Y \xi - \nabla \nabla_{X,Y} \xi) - \mu\{g(Y, \nabla X \xi)\xi + df(Y)\nabla X \xi\} :=
\]

\[
-\nabla_{X,Y}^2 \xi - \mu\{g(Y, \nabla X \xi)\xi + df(Y)\nabla X \xi\}
\]

and using (11) we get the required relation. \( \square \)

**Theorem 3.2.** If (10) defines a gradient \( \eta \)-Ricci soliton on the \( m \)-dimensional Riemannian manifold \((M, g)\) and \( \eta \) is the \( g \)-dual 1-form of the gradient vector field \( \xi := \text{grad}(f) \), then:

\[
\frac{1}{2}(\Delta - \nabla \xi)(|\xi|^2) = |\text{Hess}(f)|^2 + \lambda|\xi|^2 + \mu|\xi|^2\{|\xi|^2 - 2\Delta(f)\}.
\]
PROOF. First remark that if \( \xi = \sum_{i=1}^{m} \xi^i E_i \), for \( \{E_i\}_{1 \leq i \leq m} \) a local orthonormal frame field with \( \nabla_{E_i} E_j = 0 \) in a point, then:

\[
\text{trace}(\eta \otimes \eta) = \sum_{i=1}^{m} |df(E_i)|^2 = \sum_{1 \leq i, j, k \leq m} \xi^j \xi^k g(E_i, E_j) g(E_i, E_k) = \sum_{i=1}^{m} (\xi^i)^2 = \sum_{1 \leq i, j \leq m} \xi^i \xi^j g(E_i, E_j) = |\xi|^2.
\]

(15)

Taking the trace of the equation (10), we obtain:

\[
div(\xi) + \text{scal} + m\lambda + \mu |\xi|^2 = 0
\]

(16) and differentiating it:

\[
d(div(\xi)) + d(\text{scal}) + \mu d(|\xi|^2) = 0.
\]

(17)

Then taking the divergence of the same equation, we get:

\[
div(\mathcal{L}_\xi g) + 2div(S) + 2\mu \cdot div(df \otimes df) = 0.
\]

(18) Substracting the relations (18) and (17) computed in \( \xi \), considering (6), (8) and using the fact that the scalar and the Ricci curvatures satisfy [19],

\[
d(\text{scal}) = 2div(S),
\]

we obtain:

\[
\frac{1}{2} \Delta (|\xi|^2) - |\nabla \xi|^2 + S(\xi, \xi) + \mu \{2(div(df \otimes df))(\xi) - \xi(|\xi|^2)\} = 0.
\]

(20) As

\[
(div(df \otimes df))(\xi) := \sum_{i=1}^{m} \{E_i(df(E_i)df(\xi)) - df(E_i)df(\nabla_{E_i} \xi)\} =
\]

\[
= \sum_{i=1}^{m} \{g(E_i, \xi)g(\nabla_{E_i} \xi, \xi) + g(\xi, \xi)g(E_i, \nabla_{E_i} \xi)\} = g(\nabla_{\xi} \xi, \xi) + |\xi|^2 \sum_{i=1}^{m} g(\nabla_{E_i} \xi, E_i) :=
\]

(21) := \frac{1}{2} |\xi|^2 div(\xi),
the equation (20) becomes:

\[
\frac{1}{2} \Delta(|\xi|^2) - |\nabla \xi|^2 + S(\xi, \xi) + 2\mu|\xi|^2 \text{div}(\xi) = 0.
\]

From the \(\eta\)-soliton equation (10), we get:

\[
S(\xi, \xi) = -\frac{1}{2} \xi(|\xi|^2) - \lambda|\xi|^2 - \mu|\xi|^4,
\]
and the equation (22) becomes:

\[
\frac{1}{2} \Delta(|\xi|^2) = |\nabla \xi|^2 + \frac{1}{2} \xi(|\xi|^2) + \lambda|\xi|^2 + \mu|\xi|^4 - 2\mu|\xi|^2 \text{div}(\xi).
\]

As \(\xi := \text{grad}(f)\) follows \(\text{Hess}(f) = \nabla(df)\) and \(|\nabla \xi|^2 = |\text{Hess}(f)|^2\).

**Remark 3.3.** For \(\mu = 0\) in Theorem 3.2, we obtain the relation for the particular case of gradient Ricci soliton [19].

**Remark 3.4.** i) Assume that \(\mu \neq 0\). Denoting by \(\Delta_\xi := \Delta - \nabla_\xi\), the equation (14) can be written:

\[
\frac{1}{2} \Delta_\xi(|\xi|^2) = |\text{Hess}(f)|^2 + |\xi|^2\{\lambda + \mu[|\xi|^2 - 2\Delta(f)]\},
\]
where \(\xi := \text{grad}(f)\). If \(\lambda \geq \mu[2\Delta(f) - |\xi|^2]\), then \(\Delta_\xi(|\xi|^2) \geq 0\) and from the maximum principle follows that \(|\xi|^2\) is constant in a neighborhood of any local maximum. If \(|\xi|\) achieve its maximum, then \(M\) is quasi-Einstein. Indeed, since \(\text{Hess}(f) = 0\), from (10) we have \(S = -\lambda g - \mu df \otimes df\). Moreover, in this case, \(|\xi|^2\{\lambda + \mu[|\xi|^2 - 2\Delta(f)]\} = 0\), which implies either \(\xi = 0\), so \(M\) is Einstein, or \(|\xi|^2 = 2\Delta(f) - \frac{\lambda}{\mu} \geq 0\). Since \(\Delta(f) = -\text{scal} - m\lambda - \mu|\xi|^2\) we get \(\mu(2\mu + 1)|\xi|^2 = -(2\mu \cdot \text{scal} + 2m\lambda\mu + \lambda)\). If \(\mu = -\frac{1}{2}\), the scalar curvature equals to \(\lambda(1 - m)\) and if \(\mu \neq -\frac{1}{2}\), it is either locally upper (or lower) bounded by \(-\frac{\lambda(1+2m\mu)}{2\mu}\), for \(\mu < -\frac{1}{2}\) (\(\mu > -\frac{1}{2}\), respectively). On the other hand, if the potential vector field is of constant length, then \(2\mu\Delta(f) \geq \lambda + \mu|\xi|^2\) equivalent to \(\mu(2\mu + 1)|\xi|^2 + (2\mu \cdot \text{scal} + 2m\lambda\mu + \lambda) \leq 0\) with equality for \(\Delta(f) = \frac{\lambda}{2\mu} + \frac{|\xi|^2}{2} \geq \frac{\lambda}{2\mu}\) and \(\text{Hess}(f) = 0\) which yields the quasi-Einstein case.

ii) For \(\mu = 0\), we get the Ricci soliton case [19].

**Proposition 3.5.** Let \((M, g)\) be an \(m\)-dimensional Riemannian manifold and \(\eta\) be the \(g\)-dual 1-form of the gradient vector field \(\xi := \text{grad}(f)\). If \(\xi\) satisfies \(\nabla \xi = I_{\chi(M)} - \eta \otimes \xi\), where \(\nabla\) is the Levi-Civita connection associated to \(g\), then:
1. $Hess(f) = g - \eta \otimes \eta$;

2. $R(X,Y)\xi = \eta(X)Y - \eta(Y)X$, for any $X, Y \in \chi (M)$;

3. $S(\xi, \xi) = (1 - m)|\xi|^2$.

**Proof.** 1. Express the Lie derivative along $\xi$ as follows:

$$2(Hess(f))(X,Y) = (\mathcal{L}_\xi g)(X,Y) := \xi(g(X,Y)) - g(\{\xi,X\},Y) - g(X,\{\xi,Y\}) =$$

$$= \xi(g(X,Y)) - g(\nabla_\xi X, Y) + g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) =$$

$$= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 2[g(X,Y) - \eta(X)\eta(Y)].$$

2. Replacing now the expression of $\nabla_\xi$ in $R(X,Y)\xi := \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]} \xi$, from a direct computation we get $R(X,Y)\xi = \eta(X)Y - \eta(Y)X$.

3. $S(\xi, \xi) := \sum_{i=1}^m g(R(E_i, \xi)\xi, E_i) = \sum_{i=1}^m \{[\eta(E_i)]^2 - \eta(\xi)\} = |\xi|^2 - m|\xi|^2$. 

The condition satisfied by the potential vector field $\xi$, namely, $\nabla_\xi = I_{\chi (M)} - \eta \otimes \xi$, naturally arises if $(M, \varphi, \xi, \eta, g)$ is for example, Kenmotsu manifold [16]. In this case, $M$ is a quasi-Einstein manifold.

**Example 3.6.** Let $M = \{(x,y,z) \in \mathbb{R}^3, z > 0\}$, where $(x,y,z)$ are the standard coordinates in $\mathbb{R}^3$. Set

$$\varphi := -\frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial x} \otimes dy, \quad \xi := -z \frac{\partial}{\partial z}, \quad \eta := -\frac{1}{z} dz,$$

$$g := \frac{1}{z^2} (dx \otimes dx + dy \otimes dy + dz \otimes dz).$$

Then $(\varphi, \xi, \eta, g)$ is a Kenmotsu structure on $M$.

Consider the linearly independent system of vector fields:

$$E_1 := z \frac{\partial}{\partial x}, \quad E_2 := z \frac{\partial}{\partial y}, \quad E_3 := -z \frac{\partial}{\partial z}.$$ 

Follows

$$\varphi E_1 = -E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = 0,$$
Warped product gradient $\eta$-Ricci solitons

$\eta(E_1) = 0, \ \eta(E_2) = 0, \ \eta(E_3) = 1,$
$[E_1, E_2] = 0, \ [E_2, E_3] = E_2, \ [E_3, E_1] = -E_1$

and the Levi-Civita connection $\nabla$ is deduced from Koszul’s formula

\[
2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) -
- g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]),
\]

precisely

\[
\nabla_{E_1} E_1 = -E_3, \ \nabla_{E_1} E_2 = 0, \ \nabla_{E_1} E_3 = E_1,
\]
\[
\nabla_{E_2} E_1 = 0, \ \nabla_{E_2} E_2 = -E_3, \ \nabla_{E_2} E_3 = E_2,
\]
\[
\nabla_{E_3} E_1 = 0, \ \nabla_{E_3} E_2 = 0, \ \nabla_{E_3} E_3 = 0.
\]

Then the Riemann and the Ricci curvature tensor fields are given by:

\[
R(E_1, E_2)E_2 = -E_1, \ \ R(E_1, E_3)E_3 = -E_1, \ \ R(E_2, E_1)E_1 = -E_2,
\]
\[
R(E_2, E_3)E_3 = -E_2, \ \ R(E_3, E_1)E_1 = -E_3, \ \ R(E_3, E_2)E_2 = -E_3,
\]
\[
S(E_1, E_1) = S(E_2, E_2) = S(E_3, E_3) = -2.
\]

From (10) computed in $(E_i, E_i)$:

\[
2[g(E_i, E_i) - \eta(E_i)\eta(E_i)] + 2S(E_i, E_i) + 2\lambda g(E_i, E_i) + 2\mu \eta(E_i)\eta(E_i) = 0,
\]

for all $i \in \{1, 2, 3\}$, we have:

\[
2(1 - \delta_{i3}) - 4 + 2\lambda + 2\mu \delta_{i3} = 0 \quad \iff \quad \lambda - 1 + (\mu - 1)\delta_{i3} = 0,
\]

for all $i \in \{1, 2, 3\}$. Therefore, $\lambda = \mu = 1$ define an $\eta$-Ricci soliton on $(M, \varphi, \xi, \eta, g)$. Moreover, it is a gradient $\eta$-Ricci soliton, as the potential vector field $\xi$ is of gradient type, $\xi = \text{grad}(f)$, where $f(x, y, z) := -\ln z$.

Assume now that (10) defines a gradient $\eta$-Ricci soliton on $(M, g)$ with $\mu \neq 0$. Under the hypotheses of the Proposition 3.5, the equation (24) simplifies a lot. Compute:

\[
(25) \quad |\nabla \xi|^2 := \sum_{i=1}^{m} g(\nabla_{E_i} \xi, \nabla_{E_i} \xi) = \sum_{i=1}^{m} \{1 + (|\xi|^2 - 2)[\eta(E_i)]^2\} = m + |\xi|^2(|\xi|^2 - 2),
\]
for \( \{E_i\}_{1 \leq i \leq m} \) a local orthonormal frame field with \( \nabla_{E_i} E_j = 0 \) in a point,

\[
\xi(|\xi|^2) = \xi(g(\xi, \xi)) = 2g(\nabla \xi \xi, \xi) = 2(|\xi|^2 - |\xi|^4),
\]

(26)

\[
\xi(|\xi|^4) = 2|\xi|^2 \xi(|\xi|^2) = 4(|\xi|^4 - |\xi|^6).
\]

(27)

From the equation (10) we obtain:

\[
S(\xi, \xi) = -(\lambda + 1)|\xi|^2 - (\mu - 1)|\xi|^4.
\]

(28)

Using Proposition 3.5 and the relation (28), we get:

\[
|\xi|^2 = (m - 1 - \lambda)|\xi|^2 - (\mu - 1)|\xi|^4,
\]

(29)

so \( |\xi|^2(\mu - 1) = m - 2 - \lambda \) i.e. \( \xi \) is of constant length. Using (26) we get \( |\xi| = 1 \). It follows \( \lambda + \mu = m - 1 \) and we deduce:

**Theorem 3.7.** Under the hypotheses of the Proposition 3.5, if (10) defines a gradient \( \eta \)-Ricci soliton on \((M, g)\) with \( \mu \neq 0 \), then the Laplacian equation (24) becomes:

\[
\Delta(f) = \frac{m - 1}{\mu}.
\]

(30)

Therefore, the existence of a gradient \( \eta \)-Ricci soliton defined by (10) with the potential vector field \( \xi := \text{grad}(f) \), yields the Laplacian equation (30), and the soliton is completely determined by \( f \).

4 Warped product \( \eta \)-Ricci solitons

Consider \((B, g_B)\) and \((F, g_F)\) two Riemannian manifolds of dimensions \( n \) and \( m \), respectively. Denote by \( \pi \) and \( \sigma \) the projection maps from the product manifold \( B \times F \) to \( B \) and \( F \) and by \( \tilde{\varphi} := \varphi \circ \pi \) the lift to \( B \times F \) of a smooth function \( \varphi \) on \( B \). In this context, we shall call \( B \) the base and \( F \) the fiber of \( B \times F \), the unique element \( \tilde{X} \) of \( \chi(B \times F) \) that is \( \pi \)-related to \( X \in \chi(B) \) and to the zero vector field on \( F \), the horizontal lift of \( X \) and the unique element \( \tilde{V} \) of \( \chi(B \times F) \) that is \( \sigma \)-related to \( V \in \chi(F) \) and to the zero vector field on \( B \), the vertical lift of \( V \). Also denote by \( \mathcal{L}(B) \) the set of all horizontal lifts of vector fields on \( B \), by \( \mathcal{L}(F) \) the set of all vertical lifts of vector fields on \( F \), by \( \mathcal{H} \) the
orthogonal projection of \( T_{(p,q)}(B \times F) \) onto its horizontal subspace \( T_{(p,q)}(B \times \{q\}) \) and by \( V \) the orthogonal projection of \( T_{(p,q)}(B \times F) \) onto its vertical subspace \( T_{(p,q)}(\{p\} \times F) \).

Let \( \varphi > 0 \) be a smooth function on \( B \) and

\[
g := \pi^* g_B + (\varphi \circ \pi)^2 \sigma^* g_F
\]

be a Riemannian metric on \( B \times F \).

**Definition 4.1.** [3] The product manifold of \( B \) and \( F \) together with the Riemannian metric \( g \) defined by (31) is called the warped product of \( B \) and \( F \) by the warping function \( \varphi \) (and is denoted by \( (M := B \times \varphi F, g) \)).

Remark that if \( \varphi \) is constant equal to 1, the warped product becomes the usual product of the Riemannian manifolds.

For simplification, in the rest of the paper we shall simply denote by \( X \) the horizontal lift of \( X \in \chi(B) \) and by \( V \) the vertical lift of \( V \in \chi(F) \).

Due to a result of J. Case, Y.-J. Shu and G. Wei [7], we know that for a gradient \( \eta \)-Ricci soliton \( (g, \xi := \text{grad}(f), \lambda, \mu) \) with \( \mu \in (-\infty, 0) \) and \( \eta = df \) the \( g \)-dual of \( \xi \), on a connected \( n \)-dimensional Riemannian manifold \( (M, g) \), the function

\[
e^{2\mu f} [\Delta(f) - |\xi|^2 - \frac{\lambda}{\mu}]
\]

is constant.

Choosing properly an Einstein manifold, a smooth function and considering the warped product manifold, we can characterize the gradient \( \eta \)-Ricci soliton on the base manifold as follows [7]. Let \( (B, g_B) \) be an \( n \)-dimensional connected Riemannian manifold, \( \lambda \) and \( \mu \) real constants such that \( -\frac{1}{\mu} \) is a natural number, \( f \) a smooth function on \( B \), \( k := \mu e^{2\mu f} [\Delta(f) - |\xi|^2 - \frac{\lambda}{\mu}] \) and \( (F, g_F) \) an \( m \)-dimensional Riemannian manifold with \( m = -\frac{1}{\mu} \) and \( S_F = kg_F \). Then \( (g, \xi := \text{grad}(f), \lambda, \mu) \) is a gradient \( \eta \)-Ricci soliton on \( (B, g_B) \) with \( \eta = df \) the \( g \)-dual of \( \xi \), if and only if the warped product manifold \( (M := B \times \varphi F, g) \) with the warping function \( \varphi = e^{-\frac{f}{m}} \) is Einstein manifold with \( S = \lambda g \).

Let \( S, S_B, S_F \) the Ricci tensors on \( M, B \) and \( F \) and \( \widetilde{S_B}, \widetilde{S_F} \) the lift on \( M \) of \( S_B \) and \( S_F \), which satisfy the following properties:

**Lemma 4.2.** [3] If \( (M := B \times \varphi F, g) \) is the warped product of \( B \) and \( F \) by the warping function \( \varphi \) and \( m > 1 \), then for any \( X, Y \in \mathcal{L}(B) \) and any \( V, W \in \mathcal{L}(F) \), we have:
1. $S(X, Y) = \widetilde{S}_B(X, Y) - \frac{m}{\varphi} H^\varphi(X, Y)$, where $H^\varphi$ is the lift on $M$ of $\text{Hess}(\varphi)$;

2. $S(X, V) = 0$;

3. $S(V, W) = \widetilde{S}_F(V, W) - \pi^*[\Delta(\varphi) + (m - 1)\frac{\|\text{grad}(\varphi)\|_F^2}{\varphi^2}]g(V, W)$.

Notice that the lift on $M$ of the gradient and the Hessian satisfy:

$$\text{grad}^\varphi(f) = \text{grad}(f),$$

$$\text{Hess}^\varphi(f)(X, Y) = \text{Hess}(f)(X, Y), \text{ for any } X, Y \in \mathcal{L}(B).$$

for any smooth function $f$ on $B$.

We shall construct a gradient $\eta$-Ricci soliton on a warped product manifold following [12].

Let $(B, g_B)$ be a Riemannian manifold, $\varphi > 0$ a smooth function on $B$ and $f$ a smooth function on $B$ such that:

$$S_B + \text{Hess}(f) - \frac{m}{\varphi} \text{Hess}(\varphi) + \lambda g_B + \mu df \otimes df = 0,$$

where $\lambda, \mu$ and $m > 1$ are real constants.

Take $(F, g_F)$ an $m$-dimensional manifold with $S_F = kg_F$, for $k := \pi^*[\varphi^2 + \varphi \Delta(\varphi) + (m - 1)\|\text{grad}(\varphi)\|_F^2 - \varphi(\text{grad}(f))(\varphi)]_F$, where $\pi$ and $\sigma$ be the projection maps from the product manifold $B \times F$ to $B$ and $F$, respectively, and $g := \pi^*g_B + (\varphi \circ \pi)^2\sigma^*g_F$ a Riemannian metric on $B \times F$. Then, for $\xi := \text{grad}(f \circ \pi)$, if consider $\mu = 0$ in (35), $(g, \xi, \lambda)$ is a gradient Ricci soliton on $B \times_\varphi F$ called the warped product Ricci soliton [12].

With the above notations, we prove that:

**Theorem 4.3.** Let $(B, g_B)$ be a Riemannian manifold, $\varphi > 0$, $f$ two smooth functions on $B$, let $m > 1$, $\lambda, \mu$ be real constants satisfying (35) and $(F, g_F)$ an $m$-dimensional Riemannian manifold. Then $(g, \xi, \lambda, \mu)$ is a gradient $\eta$-Ricci soliton on the warped product manifold $(B \times_\varphi F, g)$, where $\xi = \text{grad}(\tilde{f})$ and the $1$-form $\eta$ is the $g$-dual of $\xi$, if and only if:

$$S_B = -\text{Hess}(f) + \frac{m}{\varphi} \text{Hess}(\varphi) - \lambda g_B - \mu df \otimes df$$

and

$$S_F = kg_F,$$

where $k := \pi^*[\varphi^2 + \varphi \Delta(\varphi) + (m - 1)\|\text{grad}(\varphi)\|_F^2 - \varphi(\text{grad}(f))(\varphi)]_F$. 
Proof. The gradient \( \eta \)-Ricci soliton \((g, \xi, \lambda, \mu)\) on \((B \times \varphi F, g)\) is given by:

\[
Hess(\tilde{f}) + S + \lambda g + \mu \eta \otimes \eta = 0.
\]

Then for any \( X, Y \in \mathcal{L}(B) \) and for any \( V, W \in \mathcal{L}(F) \), from Lemma 4.2 we get:

\[
H^I(X,Y) + \tilde{S}_B(X,Y) - \frac{m}{\varphi} H^F(X,Y) + \lambda g_B(X,Y) + \mu df(X)df(Y) &= 0 \\
H^I(V,W) + \tilde{S}_F(V,W) - \pi^*[\varphi \Delta(\varphi) + (m-1)|grad(\varphi)|^2 - \lambda \varphi^2]|_F g(V,W) &= 0
\]

and using the fact that

\[
H^I(V,W) = (Hess(\tilde{f}))(V,W) = g(\nabla_V(grad(\tilde{f})), W) = \pi^*\left[\frac{grad(f)(\varphi)}{\varphi}\right]_F \varphi^2 g_F(V,W),
\]

we obtain:

\[
\tilde{S}_F(V,W) = \pi^*[\varphi \Delta(\varphi) + (m-1)|grad(\varphi)|^2 - \varphi(grad(f))(\varphi) - \lambda \varphi^2]|_F g_F(V,W).
\]

Conversely, notice that the left-hand side term in \((38)\) computed in \((X,V)\), for \( X \in \mathcal{L}(B) \) and \( V \in \mathcal{L}(F) \) vanishes identically and using again Lemma 4.2 for each situation \((X,Y)\) and \((V,W)\), we can recover the equation \((38)\) from \((36)\) and \((37)\).

Remark 4.4. In the case of product manifold \( (\varphi = 1) \), notice that the equation \((35)\) defines a gradient \( \eta \)-Ricci soliton on \( B \) and the chosen manifold \((F, g_F)\) is Einstein \((S_F = -\lambda g_F)\), so a gradient \( \eta \)-Ricci soliton on the product manifold \( B \times F \) can be naturally obtained by ”lifting” a gradient \( \eta \)-Ricci soliton on \( B \).

Remark 4.5. If for the function \( \varphi \) and \( f \) on \( B \) there exists two constants \( a \) and \( b \) such that \( \nabla(grad(\varphi)) = \varphi[aI_{\chi(B)} + bdf \otimes grad(f)] \), then \( Hess(\varphi) = \varphi(ag_B + bdf \otimes df) \) and \((g_B, grad(f), \lambda - ma, \mu - mb)\) is a gradient \( \eta \)-Ricci soliton on \( B \).

Let us make some remark on the class of manifolds that satisfy the condition \((35)\):

\[
S_B + Hess(f) - \frac{m}{\varphi} Hess(\varphi) + \lambda g_B + \mu df \otimes df = 0,
\]

for \( \varphi > 0 \), \( f \) smooth functions on the oriented and compact Riemannian manifold \((B, g_B)\), \( \lambda, \mu \) and \( m > 1 \) real constants. Denote by \( \xi := grad(f) \).

Taking the trace of \((39)\), we obtain:

\[
scal_B + \Delta(f) - \frac{m}{\varphi} \Delta(\varphi) + n\lambda + \mu |\xi|^2 = 0.
\]
Warped product gradient $\eta$-Ricci solitons

Remark that:

\begin{equation}
|Hess(f) - \frac{\Delta(f)}{n} g_B|^2 := \sum_{1 \leq i,j \leq n} [Hess(f)(E_i, E_j) - \frac{\Delta(f)}{n} g_B(E_i, E_j)]^2 =
\end{equation}

\begin{equation}
= |Hess(f)|^2 - 2 \frac{\Delta(f)}{n} \sum_{i=1}^n g_B(\nabla_E \xi, E_i) + \frac{(\Delta(f))^2}{n} = |Hess(f)|^2 - \frac{(\Delta(f))^2}{n}.
\end{equation}

Also:

\begin{equation}
(div(Hess(f)))(\xi) := \sum_{i=1}^n (\nabla_{E_i} (Hess(f)))(E_i, \xi) = \sum_{i=1}^n [E_i(Hess(f)(E_i, \xi)) - Hess(f)(E_i, \nabla_{E_i} \xi)] =
\end{equation}

\begin{equation}
= \sum_{i=1}^n E_i(g_B(\nabla_{E_i} \xi, \xi)) - \sum_{i=1}^n g_B(\nabla_{E_i} \xi, \nabla_{E_i} \xi) = \sum_{i=1}^n g_B(\nabla_{E_i} \nabla_{E_i} \xi, E_i) - |\nabla \xi|^2 :=
\end{equation}

\begin{equation}
:= div(\nabla \xi) - |Hess(f)|^2
\end{equation}

and

\begin{equation}
div(\nabla \xi) := \sum_{i=1}^n g_B(\nabla_{E_i} \nabla_{E_i} \xi, E_i) = \sum_{i=1}^n E_i(g_B(\nabla_{E_i} \xi, E_i)) = \sum_{i=1}^n E_i(Hess(f)(\xi, E_i)) =
\end{equation}

\begin{equation}
= \sum_{i=1}^n (\nabla_{E_i} (Hess(f)(\xi)))(E_i) =: div(Hess(f)) (\xi),
\end{equation}

therefore:

\begin{equation}
(div(Hess(f)))(\xi) = div(Hess(f)(\xi)) - |Hess(f)|^2.
\end{equation}

Applying the divergence to (39), computing it in $\xi$ and considering (3), we get:

\begin{equation}
(div(Hess(f)))(\xi) = -(div(S_B))(\xi) + m(div(\frac{Hess(\varphi)}{\varphi}))(\xi) - \mu(\frac{1}{2}d(|\xi|^2) + \Delta(f)d||)^2(\xi) =
\end{equation}

\begin{equation}
= - \frac{d(\text{scal}_B)}{2} \langle \xi, \xi \rangle + m \varphi = div(Hess(\varphi)))(\xi) - m \varphi Hess(\varphi)(\text{grad}(\varphi), \xi) - \mu(\frac{1}{2}d(|\xi|^2)(\xi) + \Delta(f)||^2(\xi) =
\end{equation}

\begin{equation}
= - \frac{d(\text{scal}_B)}{2} + m \cdot div(Hess(\varphi))(\xi) - m \varphi Hess(\varphi, Hess(\varphi)) - \mu(\frac{1}{2}d(|\xi|^2)(\xi) + \Delta(f)||^2(\xi).
\end{equation}

From (40), (41), (42) and (43), we obtain:

\begin{equation}
(div(Hess(f))(\xi)) = |Hess(f) - \frac{\Delta(f)}{n} g_B|^2 - \frac{\text{scal}_B}{n} \Delta(f) + \frac{m}{n} \varphi \Delta(f) - div(\lambda \xi) -
\end{equation}
Warped product gradient $\eta$-Ricci solitons

$$- \frac{d}{2} (\text{scal}_B)(\xi) + m \cdot \text{div}(\text{Hess}(\varphi)(\xi)) - \frac{m}{\varphi} \langle \text{Hess}(f), \text{Hess}(\varphi) \rangle - \frac{\mu}{2} d(|\xi|^2)(\xi) - \frac{n+1}{n} \mu |\xi|^2 \Delta(f).$$

Integrating with respect to the canonical measure on $B$, we have:

$$\int_B d(\text{scal}_B)(\xi) = \int_B \langle \text{grad}(\text{scal}_B), \xi \rangle = - \int_B \langle \text{grad}_B, \text{div}(\xi) \rangle = - \int_B \text{scal}_B \cdot \Delta(f)$$

and similarly:

$$\int_B d(|\xi|^2)(\xi) = \int_B \langle \text{grad}(|\xi|^2), \xi \rangle = - \int_B \langle |\xi|^2, \text{div}(\xi) \rangle = - \int_B |\xi|^2 \cdot \Delta(f).$$

Using:

$$|\xi|^2 \cdot \text{div}(\xi) = \text{div}(|\xi|^2 \xi) - |\xi|^2$$

and integrating (44) on $B$, from the above relations and the divergence theorem, we obtain:

$$\int_B d(|\xi|^2)(\xi) = \int_B \langle \text{grad}(|\xi|^2), \xi \rangle = - \int_B \langle |\xi|^2, \text{div}(\xi) \rangle = - \int_B |\xi|^2 \cdot \Delta(f).$$

Using:

$$|\xi|^2 \cdot \text{div}(\xi) = \text{div}(|\xi|^2 \xi) - |\xi|^2$$

and integrating (44) on $B$, from the above relations and the divergence theorem, we obtain:

(45)

$$\frac{n-2}{2n} \int_B \langle \text{grad}(\text{scal}_B), \xi \rangle = \int_B |\text{Hess}(f) - \frac{\Delta(f)}{n} g_B|^2 - m \int_B \frac{1}{\varphi} \langle \text{Hess}(f), \text{Hess}(\varphi) \rangle +$$

$$+ \frac{m}{n} \int_B \frac{\Delta(\varphi)}{\varphi} \Delta(f) + \frac{n+2}{2n} \mu \int_B |\xi|^2.$$ 

PROPOSITION 4.6. Let $(B, g_B)$ be an oriented and compact Riemannian manifold, $f$ a smooth function on $B$, let $m > 1$, $\lambda, \mu$ be real constants satisfying (35) (for $\varphi = 1$) and $(F, g_F)$ be an $m$-dimensional Einstein manifold with $S_F = -\lambda g_F$. If $(g, \xi, \lambda, \mu)$ is a gradient $\eta$-Ricci soliton on the product manifold $(B \times F, g)$, where $\xi = \text{grad}(\tilde{f})$ and the 1-form $\eta$ is the $g$-dual of $\xi$, then:

(46)

$$\frac{n-2}{2n} \int_B \langle \text{grad}(\text{scal}_B), \xi \rangle = \int_B |\text{Hess}(f) - \frac{\Delta(f)}{n} g_B|^2 + \frac{n+2}{2n} \mu \int_B |\xi|^2.$$ 

Let now consider the product manifold $B \times F$, in which case (40) (for $\varphi = 1$) becomes:

(47)

$$\text{scal}_B + \Delta(f) + n\lambda + \mu |\xi|^2 = 0$$

and integrating it on $B$, we get:

(48)

$$\mu \int_B |\xi|^2 = - \int_B \text{scal}_B - n\lambda \cdot \text{vol}(B).$$

Replacing it in (46), we obtain:

(49)

$$\frac{n-2}{2n} \int_B \langle \text{grad}(\text{scal}_B), \xi \rangle + \frac{n+2}{2n} \int_B \text{scal}_B = \int_B |\text{Hess}(f) - \frac{\Delta(f)}{n} g_B|^2 - \frac{n+2}{2} \lambda \cdot \text{vol}(B).$$
Proposition 4.7. Let \((B, g_B)\) be an oriented, compact and complete \(n\)-dimensional \((n > 1)\) Riemannian manifold of constant scalar curvature, \(\varphi > 0\), \(f\) two smooth functions on \(B\), let \(m > 1\), \(\lambda\), \(\mu\) be real constants satisfying (39). If one of the following two conditions hold:

1. \(\varphi = 1\) and \(\lambda = -\frac{\text{scal}_B}{n}\);

2. there exists a positive function \(h\) on \(B\) such that \(\text{Hess}(f) = -h \cdot \text{Hess}(\varphi)\) and \(\mu \geq 0\),

then \(B\) is conformal to a sphere in the \((n + 1)\)-dimensional Euclidean space.

Proof. 1. From (49) we obtain:
\[
\int_B |\text{Hess}(f) - \frac{\Delta(f)}{n} g_B|^2 = \frac{n+2}{2}(\frac{\text{scal}_B}{n} + \lambda)\mu \cdot \text{vol}(B),
\]
so \(\text{Hess}(f) = \frac{\Delta(f)}{n} g_B\) which implies by [22] that \(B\) is conformal to a sphere in the \((n + 1)\)-dimensional Euclidean space.

2. From the condition \(\text{Hess}(f) = -h \cdot \text{Hess}(\varphi)\) we obtain \(\Delta(f) = -h \Delta(\varphi)\) and replacing them in (45), we get:
\[
\int_B |\text{Hess}(f) - \frac{\Delta(f)}{n} g_B|^2 + \frac{n+2}{2n} \mu \int_B |\xi|^2 = 0.
\]

From \(\mu \geq 0\) we deduce that \(\text{Hess}(f) = \frac{\Delta(f)}{n} g_B\) and according to [22], we get the conclusion.

Finally, we state a result on the scalar curvature of a product manifold admitting an \(\eta\)-Ricci soliton:

Proposition 4.8. Let \((B, g_B)\) be an oriented and compact Riemannian manifold of constant scalar curvature, \(f\) a smooth function on \(B\), let \(m > 1\), \(\lambda\), \(\mu\) be real constants satisfying (35) (for \(\varphi = 1\)) and \((F, g_F)\) be an \(m\)-dimensional Einstein manifold with \(S_F = -\lambda g_F\). If \((g, \xi, \lambda, \mu)\) is a gradient \(\eta\)-Ricci soliton on the product manifold \((B \times F, g)\), where \(\xi = \text{grad}(\tilde{f})\) and the 1-form \(\eta\) is the \(g\)-dual of \(\xi\), then the scalar curvature of \(B \times F\) is \geq -(n + m)\lambda.

Proof. From (49) we deduce that \(\frac{n+2}{2}(\frac{\text{scal}_B}{n} + \lambda) \cdot \text{vol}(B) = \int_B |\text{Hess}(f) - \frac{\Delta(f)}{n} g_B|^2 \geq 0\) and since \(\text{scal}_F = -m\lambda\), we get the conclusion.
We end these considerations by giving an example of gradient $\eta$-Ricci soliton on a product manifold.

Example 4.9. Let $(g_M, \xi_M, 1, 1)$ be the gradient $\eta$-Ricci soliton on the Riemannian manifold $M = \{(x, y, z) \in \mathbb{R}^3, z > 0\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$, with the metric $g_M := \frac{1}{z^2}(dx \otimes dx + dy \otimes dy + dz \otimes dz)$ (given by Example 3.6) and let $S^3$ be the 3-sphere with the round metric $g_S$ (which is Einstein with the Ricci tensor equals to $2g_S$). By Remark 4.4 we obtain the gradient $\eta$-Ricci soliton $(g, \xi, 1, 1)$ on the "generalized cylinder" $M \times S^3$, where $g = g_M + g_S$ and $\xi$ is the lift on $M \times S^3$ of the gradient vector field $\xi_M = \text{grad}(f)$, where $f(x, y, z) := -\ln z$.

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Warped product gradient $\eta$-Ricci solitons

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