THE AVERAGE NUMBER OF BLOCK INTERCHANGES
NEEDED TO SORT A PERMUTATION AND A RECENT
RESULT OF STANLEY

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Abstract. We use an interesting result of probabilistic flavor concerning the
product of two permutations consisting of one cycle each to find an explicit
formula for the average number of block interchanges needed to sort a permuta-
tion of length \( n \).

1. Introduction

1.1. The main definition, and the outline of this paper. Let \( p = p_1p_2\cdots p_n \)
be a permutation. A block interchange is an operation that interchanges two blocks
of consecutive entries without changing the order of entries within each block. The
two blocks do not need to be adjacent. Interchanging the blocks \( p_i p_{i+1} \cdots p_{i+a} \) and
\( p_j p_{j+1} \cdots p_{j+b} \) with \( i + a < p_j \) results in the permutation

\[
p_1p_2\cdots p_{i-1}p_jp_{j+1}\cdots p_{j+b}p_{i+a+1}\cdots p_{j-1}p_ip_{i+1}\cdots p_{i+a}p_{j+b+1}\cdots p_n.
\]

For instance, if \( p = 3417562 \), then interchanging the block of the first two entries
with the block of the last three entries results in the permutation 5621734.

In this paper, we are going to compute the average number of block interchanges
to sort a permutation of length \( n \). The methods used in the proof are surprising
for several reasons. First, our enumeration problem will lead us to an interesting
question on the symmetric group that is very easy to ask and that is of probabilistic
flavor. Second, this question then turns out to be surprisingly difficult to answer–
the conjectured answer of one of the authors has only recently been proved by
Richard Stanley [6], whose proof was not elementary.

1.2. Earlier Results and Further Definitions. The first significant result on
the topic of sorting by block interchanges is by D. A. Christie [3], who provided a
direct way of determining the number of block interchanges necessary to sort any
given permutation \( p \). The following definition was crucial to his results.

Definition 1. The cycle graph \( G(p) \) of the permutation \( p = p_1p_2\cdots p_n \) is a directed
graph on vertex set \( \{0, 1, \ldots, n\} \) and \( 2n \) edges that are colored either black or gray
as follows. Set \( p_0 = 0 \).

(1) For \( 0 \leq i \leq n \), there is a black edge from \( p_i \) to \( p_{i-1} \), where the indices are
to be read modulo \( n + 1 \), and

(2) For \( 0 \leq i \leq n \), there is a gray edge from \( i \) to \( i + 1 \), where the indices are to be read modulo \( n + 1 \).

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See Figure 1.2 for two examples.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example.png}
\caption{The graphs $G(p)$ for $p = 1234$ and $p = 4213$.}
\end{figure}

It is straightforward to show that $G(p)$ has a unique decomposition into edge-disjoint directed cycles in which the colors of the edges alternate. Let $c(G(p))$ be the number of directed cycles in this decomposition of $G(p)$. The main enumerative result of [3] is the following formula. In the rest of this paper, permutations of length $n$ will be called $n$-permutations, for shortness.

**Theorem 1.** The number of block interchanges needed to sort the $n$-permutation $p$ is $\frac{n+1-c(G(p))}{2}$.

Note that in particular this implies that $n+1$ and $c(G(p))$ are always of the same parity. Christie has also provided an algorithm that sorts $p$ using $\frac{n+1-c(G(p))}{2}$ block interchanges. As the identity permutation is the only $n$-permutation that takes zero block interchanges to sort, it is the only $n$-permutation $p$ satisfying $c(G(p)) = n+1$.

Theorem 4 shows that in order to find the average number $a_n$ of block interchanges needed to sort an $n$-permutation, we will need the average value of $c(G(p))$ for such permutations. The following definition [4] will be useful.

**Definition 2.** The Hultman number $S_H(n, k)$ is the number of $n$-permutations $p$ satisfying $c(G(p)) = k$.

So the Hultman numbers are somewhat analogous to the signless Stirling numbers of the first kind that count $n$-permutations with $k$ cycles.

This is a good place to point out that in this paper, we will sometimes discuss cycles of the permutation $p$ in the traditional sense, which are not to be confused with the directed cycles of $G(p)$, counted by $c(G(p))$. Following [4], the number of cycles of the permutation $p$ will be denoted by $c(\Gamma(s))$. Indeed, the cycles of a permutation $p$ are equivalent to the directed cycles of the graph $\Gamma(p)$ in which there is an edge from $i$ to $j$ if $p(i) = j$. For instance, if $p = 1234$, then $c(\Gamma(p)) = 4$, while $c(G(p)) = 5$.

The following recent theorem of Doignon and Labarre [4] brings the Hultman numbers closer to the topic of enumerating permutations according to their cycle structure (in the traditional sense). Let $S_n$ denote the symmetric group of degree $n$.

**Theorem 2.** The Hultman number $S_H(n, k)$ is equal to the number of ways to obtain the cycle $(12\cdots n(n+1)) \in S_{n+1}$ as a product $qr$ of permutations, where $q \in S_{n+1}$ is any cycle of length $n+1$, and the permutation $r \in S_{n+1}$ has exactly $k$ cycles, that is $c(\Gamma(r)) = k$. 
2. Our Main Result

The following immediate consequence of Theorem 2 is more suitable for our purposes.

**Corollary 1.** The Hultman number \( \mathcal{S}_H(n, k) \) is equal to the number of \((n + 1)\)-cycles \( q \) so that the product \((12 \cdots n(n+1))q\) is a permutation with exactly \( k \) cycles, that is, \( c(\Gamma((12 \cdots n(n+1))q)) = k \).

**Example 1.** For any fixed \( n \), we have \( \mathcal{S}_H(n, n+1) = 1 \) since \( c(G(p)) = n+1 \) if and only if \( p \) is the identity permutation. And indeed, there is exactly one \((n+1)\)-cycle (in fact, one permutation) \( q \in S_{n+1} \) so that \((12 \cdots n(n+1))q\) has \( n+1 \) cycles, namely \( q = (12 \cdots n(n+1))^{-1} = (1(n+1)n \cdots 2) \).

In other words, finding the average of the numbers \( c(G(p)) \) over all \( n \)-permutations \( p \) is equivalent to finding the average of the numbers \( c(\Gamma((12 \cdots n(n+1))q)) \), where \( q \) is an \((n+1)\)-cycle.

Let us consider the product \( s = (12 \cdots n)z \), where \( z \) is a cycle of length \( n \). Let us insert the entry \( n+1 \) into \( z \) to get the permutation \( z' \) so that \( n+1 \) is inserted between two specific entries \( a \) and \( b \) in the following sense.

\[
z'(i) = \begin{cases} 
  z(i) & \text{if } i \notin \{a, n+1\}, \\
  n+1 & \text{if } i = a, \text{ and } \\
  b & \text{if } i = n+1.
\end{cases}
\]

See Figure 2 for an illustration.

![Figure 2. How \( z' \) is obtained from \( z \).](image)

The following proposition is the first step towards describing how the Hultman numbers grow.

**Proposition 1.** Let \( a, b, \) and \( z' \) be defined as above, and let \( s' = (12 \cdots (n+1)z') \). Then we have

\[
c(\Gamma(s')) = \begin{cases} 
  c(\Gamma(s)) - 1 & \text{if } 2 \leq a, \text{ and } a - 1 \text{ and } z(1) \text{ are not in the same cycle of } s, \\
  c(\Gamma(s)) + 1 & \text{if } 2 \leq a, \text{ and } a - 1 \text{ and } z(1) \text{ are in the same cycle of } s, \text{ and} \\
  c(\Gamma(s)) + 1 & \text{if } a = 1.
\end{cases}
\]
Proof. Let us assume first that \( a \geq 2 \), and that \( a - 1 \) is in a cycle \( C_1 \) of \( s \), and \( z(1) \) is in a different cycle \( C_2 \) of \( s \). Let \( C_1 = ((a - 1)b \cdots n(z(1)) \cdots) \) and let \( C_2 = (z(1) \cdots n) \). After the insertion of \( n + 1 \) into \( z \), the newly obtained permutation \( s' = (12 \cdots (n + 1)z') \) sends \( a - 1 \) to \( n + 1 \), then \( n + 1 \) to \( z(1) \), then leaves the rest of \( C_1 \) unchanged till its last entry. Then it sends \( n \) back to \( z'(n + 1) = b \), from where it continues with the rest of \( C_1 \) with no change. So in \( s' \), the cycles \( C_1 \) and \( C_2 \) are united, the entry \( n + 1 \) joins their union, and there is no change to the other cycles of \( s \). See Figure 2 for an illustration.

\[
\begin{align*}
\text{Figure 3.} \quad \text{If } a - 1 \text{ and } z(1) \text{ are in different cycles of } s, \text{ those cycles will turn into one.}
\end{align*}
\]

Let us now assume that \( a \geq 2 \), and that \( a - 1 \) and \( z(1) \) are both in the same cycle \( C \) of \( s \). Then \( C = ((a - 1)b \cdots n(z(1)) \cdots) \). After the insertion of \( n + 1 \) into \( z \), the newly obtained permutation \( s' = (12 \cdots (n + 1)z') \) sends \( a - 1 \) to \( n + 1 \), then \( n + 1 \) to \( z(1) \), cutting off the part of \( C \) that was between \( a - 1 \) and \( n \). So \( C \) is split into two cycles, the cycle \( C' = ((a - 1)(n + 1)z(1)) \cdots \) and the cycle \( C'' = (b \cdots n) \). Note that \( s'(n) = b \) since \( z'(n + 1) = b \). See Figure 2 for an illustration.

Finally, if \( a = 1 \), then \( s'(n + 1) = (n + 1) \), and the rest of the cycles of \( s \) do not change. \( \square \)

Let \( TC_n \) denote the set of \( n \)-permutations that can be obtained as a product of two cycles of length \( n \). Let \( a_n \) be the average number of cycles of the elements of \( TC_n \).

Lemma 1. For all positive integers \( m \), we have \( a_{2m+2} = a_{2m+1} + \frac{1}{2m+1} \).

Proof. We apply Proposition \( \[ \] \) with \( n \) replaced by \( n + 1 = 2m + 1 \). This means \( n + 1 \) is an odd number. So \( z' \) is a cycle of length \( 2m + 2 \) obtained from a cycle \( z \) of length \( 2m + 1 \) through the insertion of the maximal element \( 2m + 2 \) into one of \( 2m + 1 \) possible positions. By Proposition \( \[ \] \) some of these insertions increase \( c(G(s)) \) by one, and others decrease \( c(G(s)) \) by one, depending on whether \( a - 1 \) and \( z(1) \) are in the same cycle of \( s \) or not. The question is, of course, how many times they will be in the same cycle of \( s \).
The average number of block interchanges needed to sort a permutation and a recent result of St

\[ z(1) \]

\[ a-1 \]

\[ b \]

\[ n+1 \]

\[ n \]

**Figure 4.** If \( a-1 \) and \( z(1) \) are in the same cycle of \( s \), that cycle will split into two cycles.

This question is easily seen to be equivalent to the following question.

**Question 1.** Let \( i \) and \( j \) be two fixed elements of the set \( \{1, 2, \ldots, h\} \). Select an element \( p \) of \( TC_h \) at random. What is the probability that \( p \) contains \( i \) and \( j \) in the same cycle?

The first author of this article has conjectured that the answer to this question was \( 1/2 \) for odd \( h \). This conjecture was recently proved by Richard Stanley [6], who also settled the question for even values of \( h \).

**Theorem 3.** [6] Let \( i \) and \( j \) be two fixed, distinct elements of the set \( [h] = \{1, 2, \ldots, h\} \), where \( h > 1 \). Let \( x \) and \( y \) be two randomly selected \( h \)-cycles over \( [h] \). Let \( P(h) \) be the probability that \( i \) and \( j \) are in the same cycle of \( xy \). Then

\[
P(h) = \begin{cases} 
\frac{1}{2} & \text{if } h \text{ is odd, and} \\
\frac{1}{2} - \frac{2}{(h-1)(h+2)} & \text{if } h \text{ is even.}
\end{cases}
\]

The proof of Lemma 1 is now straightforward. If \( a \neq 1 \), then \( a-1 \) and \( z(1) \) are equally likely to be in the same cycle or not in the same cycle of \( s \). Therefore, an increase of one or a decrease of one in \( c(G(s)) \) is equally likely. If, on the other hand, \( a = 1 \), which occurs in \( 1/(2m+1) \) of all cases, then \( c(G(s)) \) increases by one. So

\[
a_{2m+2} = \frac{2m}{2m+1}a_{2m+1} + \frac{1}{2m+1}(a_{2m+1}+1) = a_{2m+1} + \frac{1}{2m+1}.
\]

Note that the statement of Lemma 1 holds even when \( m = 0 \), since \( a_2 = 2 = a_1 + 1 \).

As Theorem 3 provided a formula for \( P(h) \) for even values of \( h \) as well as odd values, we can state and prove the analogous version of Lemma 1 for the integers not covered there.
Lemma 2. For all positive integers \( m \), we have

\[
a_{2m+1} = a_{2m} + \frac{m}{2(m+1)(m+2)} = a_{2m} + \frac{1}{2m} - \frac{1}{m(m+1)}.
\]

Proof. This is very similar to the proof of Lemma 1. If \( a \neq 1 \), which happens in \((2m-1)/(2m)\) of all cases, then the probability of \( a-1 \) and \( z(1) \) falling into the same cycle of \( s \) is \( \frac{1}{2} - \frac{2}{(2m+1)(2m+4)} \) by Theorem 1. By Proposition 1, in these cases \( c(G(s)) \) grows by one. If \( a = 1 \), which occurs in \( 1/(2m+2) \) of all cases, \( c(G(s)) \) always grows by one. So

\[
a_{2m+1} = \frac{2m-1}{2m} \cdot \left( \frac{1}{2} - \frac{2}{(2m-1)(2m + 2)} \right) (a_{2m} + 1) + \frac{2m-1}{2m} \cdot \left( \frac{1}{2} + \frac{2}{(2m-1)(2m+2)} \right) (a_{2m} - 1) + \frac{1}{2m} (a_{2m} + 1),
\]

which is equivalent to the statement of the lemma as can be seen after routine rearrangements. □

We are now in position to state and prove our formula for the average number \( a_n \) of cycles in elements of \( TC_n \).

Theorem 4. We have \( a_1 = 1 \), and

\[
a_n = \frac{1}{\lfloor (n-1)/2 \rfloor + 1} + \sum_{i=1}^{n-1} \frac{1}{i}.
\]

Note that this formula produces even the correct value even for \( n = 1 \), that is, it produces the equality \( a_1 = 1 \).

Proof. (of Theorem 4) The statement is now a direct consequence of Lemmas 1 and 2 if we note the telescoping sum \( \sum_{i=1}^{t} \frac{1}{i+1} = 1 - \frac{1}{t+1} \) obtained when summing the values computed in Lemma 2. □

Note that it is well-known that on average, an \( n \)-permutation has \( \sum_{i=1}^{n} \frac{1}{i} \) cycles. This is the average value of \( c(\Gamma(p)) \) for a randomly selected \( n \)-permutation. Theorem 4 shows that the average value of \( c(G(p)) \) differs from this by about \( 1/n \).

Finally, our main goal is easy to achieve.

Theorem 5. The average number of block interchanges needed to sort an \( n \)-permutation is

\[
b_n = n - \frac{1}{\lfloor n/2 \rfloor + 1} - \sum_{i=2}^{n} \frac{1}{i}.
\]

Proof. By Theorem 1 and Theorem 4 the average number value of \( c(G(p)) \) over all permutations \( p \) of length \( n \) is \( a_{n+1} = \frac{1}{\lfloor n/2 \rfloor + 1} + \sum_{i=1}^{n} \frac{1}{i} \). Our claim now immediately follows from Theorem 4. □

So the average number of block interchanges needed to sort an \( n \)-permutation is close to \( (n - \log n)/2 \).
3. Remarks and Further Directions

Richard Stanley’s proof of Theorem 3 is not elementary. It uses symmetric functions, exponential generating functions, integrals, and a formula of Boccaara [1]. A more combinatorial proof of the stunningly simple answer for the case of odd \( k \) would still be interesting.

As pointed out by Richard Stanley [7], there is an alternative way to obtain the result of Theorem 3 without using Theorem 3, but that proof in turn uses symmetric functions and related machinery. It is shown in Exercises 69(a) and 69(c) of [8] that

\[(1) \quad P_n(q) = \sum_{p \in TC_n} q^{c(\Gamma(p))} = \frac{1}{(n+1)} \sum_{i=0}^{[(n-1)/2]} c(n+1, n-2i)q^{n-2i},\]

where, as usual, \( c(n, k) \) is a signless Stirling number of the first kind, that is, the number of permutations of length \( n \) with \( k \) cycles. Now \( a_n \) can be computed by considering \( P_n(1) \), which in turn can be computed by considering the well-known identity

\[F_{n+1}(x) = \sum_{k=1}^{n+1} c(n+1, k)x^k = x(x+1) \cdots (x+n),\]

and then evaluating \( F'_{n+1}(1) + F'_{n+1}(-1) \).

The present paper provides further evidence that the cycles of the graph \( G(p) \) have various enumerative properties that are similar to the enumerative properties of the graph \( \Gamma(p) \), that is, the cycles of the permutation \( p \). This raises the question as to which well-known properties of the Stirling numbers, such as unimodality, log-concavity, real zeros property, hold for the Hultman numbers as well. (See for instance Chapter 8 of [2] for definitions and basic information on these properties.) A simple modification is necessary since \( S_H(n, k) = 0 \) if \( n \) and \( k \) are of the same parity. So let

\[Q_n(q) = \begin{cases} \sum_{p \in TC_n} q^{c(\Gamma(p))/2} & \text{if } n \text{ is even} \\ \sum_{p \in TC_n} q^{c(\Gamma(p)+(1)/2} & \text{if } n \text{ is odd.} \end{cases}\]

While the coefficients of \( P_n(q) \) are all the Hultman numbers \( S_H(n-1, 1), S_H(n-1, 2), \ldots, S_H(n-1, n-1) \), the coefficients of \( Q_n(q) \) are the nonzero Hultman numbers \( S_H(n-1, k) \).

Clearly, \( Q_n(q) = P_n(q^2) \) if \( n \) is even, and \( Q_n(q) = qP_n(q^2) \) if \( n \) is odd. However, Exercise 69(b) of [8] shows that all roots of \( P_n(q) \) have real part 0. Hence the roots of \( Q_n(q) \) are all real and non-positive, from which the log-concavity and unimodality of the coefficients of \( Q_n(q) \) follows. This raises the question of whether there is a combinatorial proof for the latter properties, possibly along the lines of the work of Bruce Sagan [5] for the Stirling numbers of both kinds. Perhaps it is useful to note that (1) and Theorem 2 imply that

\[S_H(n, k) = \begin{cases} c(n+2, k)/(n+2) & \text{if } n-k \text{ is odd,} \\ 0 & \text{if } n-k \text{ is even.} \end{cases}\]

Finally, to generalize in another direction, we point out that it is very well-known (see, for example, Chapter 4 of [2]), that if we select a \( n \)-permutation \( p \) at random, and \( i \) and \( j \) are two fixed, distinct positive integers at most as large as \( n \), then the probability that \( p \) contains \( i \) and \( j \) in the same cycle is 1/2. Theorem 3 shows that if \( n \) is odd, then the set \( TC_n \) behaves just like the set \( S_n \) of all permutations in this
aspect. This raises the question whether there are other naturally defined subsets of \( n \)-permutations in which this phenomenon occurs.

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