Exploring Born–Infeld electrodynamics using plasmas

D A Burton\textsuperscript{1,2}, R M G M Trines\textsuperscript{1,3}, T J Walton\textsuperscript{1,2} and H Wen\textsuperscript{1,2}

\textsuperscript{1} Department of Physics, Lancaster University, Lancaster, UK
\textsuperscript{2} Cockcroft Institute, Daresbury, UK
\textsuperscript{3} Rutherford Appleton Laboratory, Chilton, Didcot, UK

E-mail: d.burton@lancaster.ac.uk

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Abstract
The behaviour of large amplitude electrostatic waves in cold plasma is investigated in the context of Born–Infeld electrodynamics. Equations of motion for a relativistic electron fluid in a fixed ion background are established using an unconstrained action principle. A simple expression for the maximum electric field of ‘quasi-static’ electric waves in a cold Born–Infeld plasma is deduced and its properties are analysed. A lower bound on their wavelength is established and an approximation to their frequency is determined for ultrarelativistic phase velocities.

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1. Introduction

Recent years have seen renewed interest in nonlinear electrodynamics [1–6], in particular Born–Infeld theory [7], and its implications. Unlike their Maxwell counterparts, the Born–Infeld field equations are \textit{fundamentally} nonlinear; they are nonlinear even in the \textit{classical} vacuum. This is to be distinguished from the nonlinearities present in semi-classical theories induced from quantum electrodynamics, such as the Euler–Heisenberg one-loop effective action [8], which encode polarization due to the quantum vacuum.

Born and Infeld introduced their theory [7] in order to ameliorate the singular self-energy of a point charge, and a common feature of modern Born–Infeld-type theories is the existence of a maximum field strength. Furthermore, it was discovered that among the family of nonlinear generalizations of Maxwell theory, Born–Infeld electrodynamics possesses a number of highly attractive features; in particular, like the vacuum Maxwell equations, the vacuum Born–Infeld equations exhibit zero birefringence and their solutions have exceptional causal behaviour [9, 10]. Moreover, Born–Infeld theory shares a number of properties with the low energy dynamics of strings and branes [11].
Some of the most extreme conditions ever encountered in a terrestrial laboratory are
created when high-power laser pulses interact with matter. The laser pulse immediately
vaporizes the matter to form an intense laser-plasma providing novel avenues for generating
intense bursts of coherent electromagnetic radiation for a wide range of applications
in biological and material science [12]. In addition, laser-plasmas permit controllable
investigation of matter under extreme conditions that only occur naturally away from the
Earth. It is expected that the next generation of ultra-intense lasers will, for the first time, allow
controllable access to regimes where a host of different quantum electrodynamics phenomena
will be evident [13] and theories of nonlinear electrodynamics will be central to their study.
However, it is conceivable that Born–Infeld electrodynamics may play a role at field intensities
below where quantum electrodynamics is necessary [4].

Motivated by recent analyses of the propagation of Born–Infeld electromagnetic waves
in waveguides [2, 3] and in background uniform magnetic fields in unbounded space [4], this
paper addresses the aspects of the nonlinear electrodynamics of laser-plasmas.

A sufficiently short and intense laser pulse propagating through a plasma may create a
travelling longitudinal plasma wave whose velocity is approximately the same as the laser
pulse’s group velocity. However, it is not possible to sustain arbitrarily large electric fields;
substantial numbers of plasma electrons become trapped in the wave and are accelerated,
which dampens the wave (the wave ‘breaks’). Early theoretical investigation of nonlinear
plasma waves was undertaken by Akhiezer and Polovin [14], and later expounded by Dawson
[15] in the context of wave-breaking.

Wave-breaking is a fundamentally nonlinear phenomenon, and it is natural to explore the
properties of Born–Infeld plasmas from this perspective. For simplicity, in this paper we adopt
the cold plasma model for the electron fluid and generalize well-known results [14, 15], for the
cold Maxwell plasma, to the context of nonlinear electrodynamics. In particular, we obtain an
exact expression for the maximum electric field of electrostatic waves in a cold Born–Infeld
plasma and analyse their wavelength and frequency. There are numerous recent investigations
of large amplitude electrostatic waves in a warm Maxwell plasma; for example, see [16–19].

We employ the Einstein summation convention throughout this paper. Latin indices \( a, b, c \)
run over 0, 1, 2, 3 and units are used in which the speed of light \( c = 1 \) and the permittivity of
the vacuum \( \varepsilon_0 = 1 \).

Let \( (x^a) \) be an inertial coordinate system on Minkowski spacetime \((\mathcal{M}, g)\) where \( x^0 \) is
the proper time of observers at fixed Cartesian coordinates \((x^1, x^2, x^3)\) in the laboratory. The
metric tensor \( g \) has the form

\[
g = \eta_{ab} \, dx^a \otimes dx^b
\]

with

\[
\eta_{ab} = \begin{cases} 
-1 & \text{if } a = b = 0, \\
1 & \text{if } a = b \neq 0, \\
0 & \text{if } a \neq b 
\end{cases}
\]

and the Hodge map \( \star \) is induced from the 4-form \( \star 1 \) on \( \mathcal{M} \) where

\[
\star 1 = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.
\]

The plasma electrons are represented as a cold relativistic fluid; their worldlines are
trajectories of the unit normalized future-pointing timelike 4-vector field \( V \) on \( \mathcal{M} \) and the
0-form \( n \) is their proper number density. We are interested in the evolution of a plasma over
timescales during which the motion of the ions is negligible in comparison with the motion
of the electrons, and we assume that the ions are at rest and distributed homogeneously in the
laboratory frame. Their worldlines are trajectories of the vector field \( N_{\text{ion}} = n_{\text{ion}} \partial / \partial x^0 \) on \( M \) where \( n_{\text{ion}} \) is the ion number density (a positive constant) in the laboratory frame.

Maxwell’s equations may be written covariantly as
\[
dF = 0, \quad d \ast G = -q n_{\text{ion}} \ast \widetilde{V} - q_{\text{ion}} \ast \widetilde{N}_{\text{ion}}
\] (4)
where \( q < 0 \) is the charge on the electron and \( q_{\text{ion}} = Z |q| \) is the charge on an ion with \( Z \) the multiplicity of the ionization. The 1-forms \( \widetilde{V} \) and \( \widetilde{N}_{\text{ion}} \) are the metric duals of the vector fields \( V \) and \( N_{\text{ion}} \), respectively, i.e. the 1-form \( \widetilde{V} \) satisfies \( \widetilde{V}(U) = g(V, U) \) for all vector fields \( U \) on \( M \). The Maxwell 2-form \( F \) encodes the electric field \( E \) and the magnetic induction \( B \); the excitation 2-form \( G \) encodes the electric displacement \( D \) and the magnetic field \( H \). In Maxwell electrodynamics \( G = F \) in the vacuum; however, in Born–Infeld electrodynamics (4) is retained but the constitutive relations for \((D, H)\) in terms of \((E, B)\) are nonlinear in the vacuum. In Born–Infeld electrodynamics the constitutive relation for \( G \) in terms of \( F \) is
\[
G = \frac{1}{\sqrt{1 - \kappa^2 X - \kappa^4 Y^2 / 4}} \left( F - \frac{\kappa^2 Y^2}{2} \ast F \right)
\] (5)
where the invariants \( X \) and \( Y \) are
\[
X = \ast(F \wedge \ast F), \quad Y = \ast(F \wedge F)
\] (6)
and \( \kappa \) is a new constant of nature.

To motivate the field equations used herein, we begin section 2 by developing them using an unconstrained action principle [20]. We will briefly review how (5) arises and show that \( \tilde{V} \) satisfies the field equations
\[
\nabla_{\tilde{V}} \tilde{V} = \frac{q}{m} \iota_{\tilde{V}} F, \quad g(V, V) = -1
\] (7)
for a relativistic cold electron fluid, where \( q \iota_{\tilde{V}} F \) is the Lorentz 4-force acting on the electron fluid, \( \iota_{\tilde{V}} \) is the interior product with respect to \( V \), \( m \) is the electron rest mass and \( \nabla \) is the Levi-Civita connection on \( M \). This will be followed in section 3 by an exploration of the properties of large amplitude relativistic electrostatic waves.

2. Action principle

A succinct action principle for establishing (4) and (7) is formulated using a three-dimensional manifold \( B \) (often called a ‘material’ [21], or ‘body’, manifold), where each point in \( B \) corresponds to an integral curve of \( V \).

The independent variables of the action introduced below are the electromagnetic 1-form \( A \) on \( M \) and a submersion \( f \) from \( M \) to \( B \):
\[
f : M \longrightarrow B \quad (8)
\]
\[
x^a \mapsto \xi^A = f^A(x).
\] (9)
The map \( f \) identifies an integral curve of \( V \) with its corresponding point in \( B \), where upper case Latin indices are \( A, B = 1, 2, 3 \). The inverse image \( f^{-1}(P) \) of the point \( P \in B \) is the worldline of an idealized ‘material’ particle in the electron fluid and \( V \) may be determined from \( f \) as the unique timelike future-pointing vector field satisfying
\[
df^A(V) = 0
\] (10)
and
\[
g(V, V) = -1.
\] (11)
In the present approach, material properties of the electron fluid are encoded in so-called material tensors and forms on $B$, and fields on $\mathcal{M}$ are induced from material tensors on $B$ using $f$. In particular, the number of electrons $N[\Sigma]$ occupying the spacelike hypersurface $\Sigma$ may be induced from a 3-form $\Omega$ on $B$ as

$$N[\Sigma] = \int_\Sigma f^*\Omega,$$

where $f^*$ denotes the pull-back map, induced from $f$, on forms on $B$ to forms on $U$. The electron number current 3-form $j = f^*\Omega$ is identically closed,

$$dj = 0,$$

where $d$ is the exterior derivative, and (14) follows because $\Omega$ is a top degree form on $B$ and $df^* = f^*d$.

A variational principle yielding (7) for $V$ and (4) for the potential 1-form $A$, where $F = dA$, employs the action functional $S$ where

$$S[A, f] = \int_M [\mathcal{L}_{\text{EM}}(X, Y) \ast 1 - m\sqrt{j \cdot j} \ast 1 - qA \wedge j - q_{\text{ion}}A \wedge j_{\text{ion}}].$$

The 0-form $\mathcal{L}_{\text{EM}}(X, Y) \ast 1$ is a local function of the invariants $X$ and $Y$ only. For Maxwell electrodynamics $\mathcal{L}_{\text{EM}} = X/2$, whereas for Born–Infeld electrodynamics $\mathcal{L}_{\text{EM}} = \mathcal{L}_{\text{BI}}$ where

$$\mathcal{L}_{\text{BI}}(X, Y) = \frac{1}{\kappa^2}(1 - \sqrt{1 - \kappa^2 X - \kappa^4 Y^2/4}).$$

The 3-form $j_{\text{ion}} = n_{\text{ion}}dx^1 \wedge dx^2 \wedge dx^3$ is the background ion number current and the 0-form $j \cdot j$ is the square of the magnitude of $j$,

$$j \cdot j = \ast^{-1}(j \wedge \ast j),$$

where $\ast^{-1}$ is the inverse of the Hodge map $\ast$ on forms on $M$.

The first term $\mathcal{L}_{\text{EM}}(X, Y) \ast 1$ in the Lagrangian 4-form in (15) depends only on $A$ and $g$, the second term $m\sqrt{f \cdot f} \ast 1$ depends only on $f$ and $g$, the third term $qA \wedge j$ couples $f$ and $A$ and the fourth term $q_{\text{ion}}A \wedge j_{\text{ion}}$ depends only on $A$.

We will show in the following that the field equations (4) and (7) for $n$, $V$ and $A$, respectively, are recovered from those for $f$ and $A$ by introducing the electron proper number density $n$ and 4-velocity field $V$ as

$$n = \sqrt{j \cdot j},$$

$$\tilde{V} = \ast^{-1}j/n.$$

2.1. Nonlinear generalization of the Maxwell equations

Equations for the electromagnetic field $F$ arise upon seeking stationary variations of $S$ with respect to $A$:

$$\delta_A S = \int_M \left[\frac{\partial \mathcal{L}_{\text{EM}}}{\partial X} \delta_A X + \frac{\partial \mathcal{L}_{\text{EM}}}{\partial Y} \delta_A Y \right] \ast 1 - \delta A \wedge (qj + q_{\text{ion}}j_{\text{ion}}),$$

where $\delta_A \alpha$ denotes the variation of a form $\alpha$ with respect to $A$. Using (6) and $\ast \ast \alpha = -\alpha$ where $\alpha$ is a 4-form on $M$, it follows

$$X \ast 1 = -F \wedge \ast F, \quad Y \ast 1 = -F \wedge F.$$
and
\[ \delta A X \star 1 = \delta A (X \star 1) = -2 d \delta A \wedge \star F, \tag{22} \]
\[ \delta A Y \star 1 = \delta A (Y \star 1) = -2 d \delta A \wedge F. \tag{23} \]
Hence,
\[ \delta A S = \int_M \delta A \wedge \left[ -2d \left( \frac{\partial L_{\text{EM}}}{\partial X} \star F + \frac{\partial L_{\text{EM}}}{\partial Y} F \right) - (qj + q_{\text{ion}} j_{\text{ion}}) \right], \tag{24} \]
where the variation \( \delta A \) is chosen to have compact support on \( M \), and an integration by parts has been used in the final step. By demanding that the action is stationary under all such variations we obtain
\[ d \star G = -qj - q_{\text{ion}} j_{\text{ion}}, \tag{25} \]
where
\[ \star G = 2 \left( \frac{\partial L_{\text{EM}}}{\partial X} \star F + \frac{\partial L_{\text{EM}}}{\partial Y} F \right). \tag{26} \]
The field equations (4) are obtained by introducing \( n \) and \( V \) using (18) and (19), the ion number 4-current \( N_{\text{ion}} \):
\[ \tilde{N}_{\text{ion}} = \star^{-1} j_{\text{ion}}, \tag{27} \]
and noting \( dF = 0 \) since \( F \) is an exact 2-form. Equation (5) immediately follows from (16) and (26) with \( L_{\text{EM}} = L_{\text{BI}} \) and using \( \star \star \beta = -\beta \) where \( \beta \) is a 2-form on \( M \).

Semi-classical field theories induced from quantum electrodynamics may also be encoded in a Lagrangian 0-form \( L_{\text{EM}} (X, Y) \) and, in particular, the Euler–Heisenberg one-loop Lagrangian [8] is a quadratic polynomial of the invariants \( X \) and \( Y \). The nonlinearity in the Euler–Heisenberg Lagrangian is a perturbation to the Maxwell Lagrangian, and one can argue that quantum corrections to the Born–Infeld Lagrangian \( L_{\text{BI}} (X, Y) \) in the weak field regime, where \( \kappa^2 |X| \ll 1 \) and \( \kappa^2 |Y| \ll 1 \), should be of a similar form to the quadratic terms in the Euler–Heisenberg Lagrangian [6]. However, one of the purposes of the present study is to explore the behaviour of a Born–Infeld plasma when the electric field strength is commensurate with \( 1/\kappa \) and we assume in the following that \( 1/\kappa \) is sufficiently low for quantum processes not to markedly intrude.

2.2. Field equations for the electron fluid

The field equations (7) describing the electron fluid are obtained by seeking stationary variations of \( S \) with respect to \( f \). Using (18) it follows
\[ \delta f \sqrt{j} \cdot j = \frac{1}{\sqrt{j} \cdot j} [\star^{-1} (\delta j \wedge \star j)], \tag{28} \]
where \( \delta f \alpha \) denotes the variation of a form \( \alpha \) with respect to \( f \). To proceed further we express \( j = f^* \Omega \) explicitly in terms of the components \( \{f^A\} \) of \( f \), namely
\[ \delta j = \frac{1}{3!} \left( \frac{\partial \Omega_{ABC}}{\partial \xi^D} \circ f \right) \delta f^D df^A \wedge df^B \wedge df^C + \frac{1}{2!} (\Omega_{ABC} \circ f) \delta f^A \wedge df^B \wedge df^C \]
\[ = \iota W d j + \iota W j, \tag{29} \]
where
\[ \iota W = \frac{1}{3!} \Omega_{ABC} (\xi^A) d \xi^A \wedge d \xi^B \wedge d \xi^C \quad \text{and} \quad j = f^* \Omega = \frac{1}{3!} (\Omega_{ABC} \circ f) df^A \wedge df^B \wedge df^C. \tag{30} \]
have been used. The vector field $W$ is

$$W = \delta f^A W_A,$$

where $\{W_A\}$ is a basis for $V$ orthogonal vector fields on $\mathcal{M}$, and the frame $\{V, W_A\}$ and coframe $\{-\tilde{V}, df^A\}$ are naturally dual, i.e.

$$d f^A(W_B) = \delta^A_B, \quad \tilde{V}(W_A) = 0, \quad d f^A(V) = 0, \quad \tilde{V}(V) = -1,$$

where $\delta^A_B$ is the Kronecker delta. Since $j$ satisfies (14) it follows that

$$\delta f j = df_w j$$

using (29) and so

$$\delta f (\sqrt{j} \cdot j \star 1) = \frac{1}{\sqrt{j} \cdot j} \delta f j \wedge \star j = df_w j \wedge \frac{1}{\sqrt{j} \cdot j} \star j.$$

Varying (15) with respect to $f$ yields

$$\delta f S = -\int_{\mathcal{M}} \left[ \frac{m}{\sqrt{j} \cdot j} \delta f j \wedge \star j + q A \wedge \delta f j \right],$$

where the variation $\delta f^A$ is chosen to have compact support on $\mathcal{M}$. An integration by parts yields

$$\delta f S = \int_{\mathcal{M}} \iota_W j \wedge d \left( \frac{m}{\sqrt{j} \cdot j} \star j - q A \right)$$

and by demanding that $S$ is stationary under all suitable variations $\delta f^A$ we obtain

$$\iota_W j \wedge d \left( \frac{m}{\sqrt{j} \cdot j} \star j - q A \right) = 0.$$

Eliminating $j$ in favour of $n$ and $V$ using (18) and (19) yields

$$\iota_W \iota_V (m d\tilde{V} - q F) = 0$$

and since $\{V, W_A\}$ is a frame on $\mathcal{M}$ it follows

$$\iota_V d\tilde{V} = \frac{q}{m} \iota_V F.$$

Since $g(V,V)$ is constant it may be shown $\iota_V d\tilde{V} = \nabla_V \tilde{V}$ (see, for example, [22]) and (7) immediately follows from (39).

3. Nonlinear electrostatic waves

The remainder of this paper focuses on properties of large-amplitude longitudinal electrostatic waves propagating parallel to the $x^3$-axis with phase velocity $v$ (with $0 < v < 1$) in the laboratory frame. We introduce the pair $\{e^1, e^2\}$

$$e^1 = v \, dx^3 - dx^0, \quad e^2 = dx^3 - v \, dx^0$$

and note that the orthonormal coframe $\{\gamma e^1, \gamma e^2, dx^1, dx^2\}$ is adapted to observers moving at velocity $v$ along $x^3$ (i.e. observers in the ‘wave frame’), where the Lorentz factor $\gamma = 1/\sqrt{1 - v^2}$. We seek a 4-velocity field $V$ of the form

$$\tilde{V} = \mu(\xi)e^1 + \psi(\xi)e^2.$$
where $\zeta = x^3 - v_x x^0$ is the wave’s phase and $e^2 = d\zeta$. We have adopted the so-called quasi-static approximation; the pointwise dependence of $\mu$ and $\psi$ is on $\zeta$ only. Using $g(V, V) = -1$ the component $\psi$ is found as

$$\psi = -\sqrt{\mu^2 - \gamma^2},$$

(42)

where the sign of $\psi$ is chosen to ensure that the velocity $\gamma e^2(V)$ of the electron fluid in the wave frame is non-positive. Thus, the speed of the electrons is slower than the phase speed of the wave, except at wave-breaking where the electrons catch the wave.

In the present analysis we assume that the electromagnetic field is due entirely to the electron fluid and ion background. The magnetic field vanishes and the only non-zero component of the electric field is in the $x^3$ direction; it follows that the Maxwell 2-form $F$ is

$$F = E \, dx^0 \wedge dx^3,$$

(43)

where $E$ is the $x^3$ component of the electric field.

Since $\tilde{V}$ and $F$ are elements of the subspace of forms on $\mathcal{M}$ generated by $\{dx^0, dx^3\}$ it follows

$$\tilde{V} \wedge \left(d\tilde{V} - \frac{q}{m} F\right) = 0.$$

(44)

The action of $\iota_V$ on (44) leads to

$$d\tilde{V} = \frac{q}{m} F$$

(45)

using (39) with $g(V, V) = -1$, and (41), (43) and (45) yield

$$E = \frac{1}{\gamma^2} \frac{m}{q} \frac{d\mu}{d\zeta}.$$

(46)

Since $d\xi \wedge F = 0$ and $dF = 0$ it follows

$$d\left(\frac{\partial L_{EM}}{\partial Y} F\right) = 0$$

(47)

and inserting (46), (41) into (4) yields

$$\frac{d}{d\xi} \left(2 \frac{\partial L_{EM}}{\partial X} \frac{d\mu}{d\zeta} \bigg|_{Y = 0}\right) = \frac{q^2}{m} \gamma^2 (n \mu - Z_n \gamma^2), \quad n = \frac{Z_n \gamma^2 v}{\sqrt{\mu^2 - \gamma^2}},$$

(48)

where

$$Y = 0$$

(49)

follows from (6) and (47) has been used. Hence,

$$\frac{d}{d\xi} \left(2 \frac{\partial L_{EM}}{\partial X} \frac{d\mu}{d\zeta} \bigg|_{Y = 0}\right) = \frac{q^2}{m} Z_n \gamma^4 \left(\frac{v \mu}{\sqrt{\mu^2 - \gamma^2}} - 1\right).$$

(50)

The product of (50) and $d\mu/d\zeta$ may be written as

$$\frac{d}{d\xi} \left(2 \frac{\partial L_{EM}}{\partial X} - L_{EM}\right) \bigg|_{Y = 0} = m Z_n (v \sqrt{\mu^2 - \gamma^2} - \mu) = 0$$

(51)

using

$$X = E^2 = \frac{1}{\gamma^2} \frac{m^2}{q^2} \left(\frac{d\mu}{d\zeta}\right)^2$$

(52)

which follows from (6) and (46).
Figure 1. The dashed curve shows $\mu$ versus $\zeta$ and the solid curve shows $E$ versus $\zeta$ (not to scale). Points of intersection of $E$ with the $\zeta$-axis are labelled I and III, and II is a turning point of $E$. Using (46) and $q < 0$ it follows $d\mu/d\zeta$ and $E$ are of the opposite sign.

3.1. Maximum amplitude oscillation

The square root in the right-hand side of (50) places a lower bound on the component $\mu$ of $\tilde{V}$. Suppose that over an oscillation $\mu$ almost attains the lowest possible value $\mu_I = \gamma$ at $\zeta = \zeta_I$ (see figure 1). Since $\mu_I$ is a turning point of $\mu$ it follows $E(\zeta_I) = 0$. At $\zeta = \zeta_{II}$ the electric field $E(\zeta_{II}) = -E_{\text{max}}$ where $E_{\text{max}}$ is the amplitude of the largest possible oscillation (see figure 1) and $dE/d\zeta|_{\zeta=\zeta_{II}} = 0$ with (50) leads to

$$\frac{\nu \mu_{II}}{\mu_{II}^2 - \gamma^2} - 1 = 0$$

where $\mu_{II} = \mu(\zeta_{II})$, and hence $\mu_{II} = \gamma^2$. Integration of (51) over $[\zeta_I, \zeta_{II}]$ yields

$$\left(2 \frac{\partial \mathcal{L}_{\text{EM}}}{\partial X} X - \mathcal{L}_{\text{EM}}\right)_{X=E_{\text{max}}^2, Y=0} + \mathcal{L}_{\text{EM}}_{X=0, Y=0} = mZn_{\text{ion}}(\gamma - 1).$$

For a Maxwell plasma $\mathcal{L}_{\text{EM}} = X/2$, which inserted into (54) yields the well-known wave-breaking limit due to Akhiezer and Polovin [14] and Dawson [15],

$$E_{\text{AP max}}^\gamma = \sqrt{2mZn_{\text{ion}}(\gamma - 1)} = m\omega_pc \sqrt{2(\gamma - 1)},$$

for a multiply-ionized relativistic cold Maxwell plasma, where

$$\omega_p = \sqrt{\frac{q^2Zn_{\text{ion}}}{m\varepsilon_0}}$$

is the plasma frequency and the speed of light $c$ and permittivity of the vacuum $\varepsilon_0$ have been restored.

Using $\mathcal{L}_{\text{EM}} = \mathcal{L}_{\text{BI}}$ and (16), (54) and (55) it follows

$$E_{\text{BI max}}^\gamma = \frac{1}{\kappa} \sqrt{1 - \left[\kappa^2(E_{\text{max}}^\gamma)^2/2 + 1\right]^{-2}}$$

for a cold Born–Infeld plasma and $\lim_{\kappa \to 0} E_{\text{BI max}}^\gamma = E_{\text{max}}^\gamma$ as expected.
The choice \( \kappa = e_0 r_0^2 / |q| \) made by Born and Infeld, where \( r_0 \) is the classical radius of the electron, leads to \( \kappa \sim 10^{-22} \text{ m}^{-1} \). However, quantum electrodynamic vacuum effects are expected to become important at electric field strengths \( \sim 10^{18} \text{ V m}^{-1} \), and if Born–Infeld theory plays a role at the classical level it follows \( E_{\text{max}}^{\text{BI}} < 10^{18} \text{ V m}^{-1} \). The next generation of laser systems (such as Extreme Light Infrastructure (ELI) [23] and High Power Laser Energy Research system (HiPER) [24]), are expected to offer intensities \( \sim 10^{25} \text{ W cm}^{-2} \) corresponding to electric field strengths \( \sim 10^{16} \text{ V m}^{-1} \).

Although in practice \( \kappa E_{\text{max}}^{\text{BI}} \ll 1 \) it is worth noting that although the limit of \( E_{\text{max}}^{\text{NP}} \) as \( v \to c \) does not exist, using (57) it follows
\[
\lim_{v \to c} E_{\text{max}}^{\text{BI}} = \frac{1}{\kappa}
\]
which is the value of the electric field of a static point electron evaluated at the location of the point electron in Born and Infeld’s original theory [7].

### 3.2. Period and frequency of maximum amplitude waves

The period \( \lambda \) of the maximum amplitude oscillation of a cold Born–Infeld plasma is obtained from the solution to (50) with \( L_{\text{EM}} = L_{\text{BI}} \) and the initial conditions \( \mu(\zeta_I) = \gamma \) and \( d\mu/d\xi|_{\zeta_I} = 0 \). The particular first integral of (50) satisfying the initial conditions on \( \mu \) may be written as
\[
\left( \frac{d\mu}{d\xi} \right)^2 = \frac{q^2 \gamma^4}{m^2 \kappa} (1 - [\kappa^2 m Z_{\text{ion}} (v \sqrt{\mu^2 - \gamma^2} - \mu + \gamma) + 1]^{-2})
\]
and, using (59), consideration of the stationary points of \( \mu \) yields \( \gamma \leq \mu \leq \gamma (1 + v^2) \).

Furthermore, \( d\mu/d\xi > 0 \) for \( \zeta_I < \zeta < \zeta_{\text{III}} \) (see figure 1) where \( \mu(\zeta_I) = \gamma \) and \( \mu(\zeta_{\text{III}}) = \gamma (1 + v^2) \). Thus, using (59) the period \( \lambda \) of the maximum amplitude oscillation is
\[
\lambda = 2 (\zeta_{\text{III}} - \zeta_I)
\]
\[
= \frac{2}{\omega_p \gamma^2} \int_{\gamma}^{\gamma (1 + v^2)} \kappa \frac{1}{\sqrt{1 - [\kappa^2 (v \sqrt{\mu^2 - \gamma^2} - \mu + \gamma) + 1]^{-2}}} d\mu,
\]
where \( \kappa = \kappa m \omega_p / |q| \).

A lower bound on \( \lambda \) may be determined by noting \( 0 \leq (v \sqrt{\mu^2 - \gamma^2} - \mu + \gamma) \leq \gamma - 1 \) for \( \gamma \leq \mu \leq \gamma (1 + v^2) \). It follows
\[
\int_{\gamma}^{\gamma (1 + v^2)} \frac{1}{\sqrt{1 - [\kappa^2 (\gamma - 1) + 1]^{-2}}}
\]
and thus
\[
\lambda > \frac{2}{\omega_p \gamma^2} \int_{\gamma}^{\gamma (1 + v^2)} \kappa \frac{1}{\sqrt{1 - [\kappa^2 (\gamma - 1) + 1]^{-2}}} d\mu
\]
\[
= \frac{4 m \gamma v^2}{|q| E_{\text{max}}^{\text{BI}}}. \tag{62}
\]

Hence, using (58) and (62) it follows that \( \lambda \) diverges at least as fast as \( \gamma \) in the limit \( v \to c \). Thus, although the Lorentz force on an electron trapped in a maximum amplitude wave is finite, the estimate \( |q| E_{\text{max}}^{\text{BI}} \lambda / 2 = 2 m \gamma v^2 \) of the work done on the electron over half of a period of the wave diverges in the limit \( v \to c \) and, in principle, it is possible to accelerate electrons to arbitrarily high energies in the present classical theory. Furthermore, consideration of the
electromagnetic stress–energy–momentum tensor $T_{EM}$ arising from metric variations of the first term in (15) [4] yields an electromagnetic mass–energy density $\varrho_{EM}$ (in the lab frame) that diverges in the limit $v \to c$ for a wave on the verge of breaking. The tensor $T_{EM}$ satisfies

$$T_{EM}(T, U) = \frac{\partial L_{EM}}{\partial X} \iota_{T} \star (\iota_{U} F \wedge \star F - F \wedge \iota_{U} \star F) + \left( L_{EM} - X \frac{\partial L_{EM}}{\partial X} - Y \frac{\partial L_{EM}}{\partial Y} \right) g(T, U),$$

(63)

where $T, U$ are the arbitrary vectors, and the 0-form $\varrho_{EM}$ is

$$\varrho_{EM} = T_{EM}(\partial/\partial x^0, \partial/\partial x^0).$$

(64)

Using (43), (63) and (64) it follows

$$\varrho_{EM} = \left( 2X \frac{\partial L_{EM}}{\partial X} - L_{EM} \right) \bigg|_{Y=0}$$

(65)

for the class of solutions to (4) and (7) investigated in this paper. The Born–Infeld energy density $\varrho_{BI}$ obtained using (65) and (16),

$$\varrho_{BI} = \frac{1}{\kappa^2} \frac{1}{\sqrt{1 - \kappa^2 X}} (1 - \sqrt{1 - \kappa^2 X}),$$

(66)

diverges in the limit $E \to 1/\kappa$.

The behaviour of (60) for $\hat{k}^2 \gamma \ll 1$ and $\gamma \gg 1$ may be extracted by changing the integration variable to $\chi = \mu/\gamma^3$. Using (60) it follows

$$\lambda = \frac{2\gamma}{\omega_p} \int_{\gamma^{-2}}^{1+\gamma^2} \frac{1}{\sqrt{2I}} \left[ 1 + \frac{3}{4} \kappa^2 I + O(\kappa^4 I^2) \right] d\chi,$$

(67)

where

$$I = \gamma^3 (v \sqrt{\chi^2 - \gamma^{-4} - \chi + \gamma^{-2}})$$

$$= \gamma \left( 1 - \frac{1}{2} \chi \right) + O(\gamma^{-1}).$$

(68)

Since $\zeta = x^3 - vx^0$, the (angular) frequency $\omega_{BI}$ of the electrostatic waves measured in the lab frame is

$$\omega_{BI} = \frac{2\pi v}{\lambda} = \frac{2\pi}{\lambda} + O(\gamma^{-2})$$

(69)

and (67), (68) and (69) yield

$$\omega_{BI} \approx \omega_{AP} \left[ 1 - \left( \frac{\kappa m \omega_{p} c}{2q} \right)^2 \gamma \right],$$

(70)

where the speed of light $c$ has been restored and $\omega_{AP}$ is the (angular) frequency of electrostatic waves of a cold Maxwell plasma for $\gamma \gg 1$ [14]:

$$\omega_{AP} = \frac{\pi}{2\sqrt{2\gamma} \omega_{p}}.$$
4. Conclusion

In practice, a typical laser-plasma experiment currently has $\gamma \sim 10^{-100}$ and $\omega_p \sim 10^{14} \text{rad s}^{-1}$; using $\kappa \sim 10^{-18} \text{m V}^{-1}$ it follows $(\omega_B/\omega_A - 1) \sim 10^{-13} - 10^{-12}$. Although this value is currently out of reach of terrestrial experiments, we suggest that it may be possible to investigate the ramifications of (57), (62) and (70) for large amplitude electric waves in the environment of astrophysical objects such as quasars and magnetars. In any case, a number of significant items have been omitted from the above calculation, such as the finite temperature of the plasma and ion motion. Future work will include more thorough evaluation of the consequences of non-zero $\kappa$ in the context of more comprehensive plasma models.

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