GLOBAL EXISTENCE OF THE CRITICAL SEMILINEAR WAVE EQUATIONS WITH VARIABLE COEFFICIENTS OUTSIDE OBSTACLES

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Abstract

In this paper, we consider exterior problem of the critical semilinear wave equation in three space dimensions with variable coefficients and prove global existence of smooth solutions. Similar to the constant coefficients case, we show that the energy cannot concentrate at any point $(t, x) \in (0, \infty) \times \Omega$. For that purpose, following Ibrahim and Majdoub [6], we use a geometric multiplier close to the well-known Morawetz multiplier used in the constant coefficients case. Then we use comparison theorem from Riemannian Geometry to estimate the error terms. Finally, using Strichartz inequality as in Smith and Sogge [11], we get the global existence.

Keywords: exterior problem, variable coefficients wave equations, critical nonlinearity.

1 Introduction

In this paper we consider global existence of smooth solutions of the exterior problem

\[
\begin{aligned}
    &u_{tt} - \partial \left( a^{ij}(x)u_{x_j} \right) + u^5 = 0 \quad \text{on} \ (0, \infty) \times \Omega, \\
    &u(0, x) = f(x) \in C^\infty_0(\Omega), \quad u_t(0, x) = g(x) \in C^\infty_0(\Omega), \\
    &u(t, x) = 0 \quad x \in \partial \Omega,
\end{aligned}
\]

(1.1)

where $\Omega$ is the exterior of a smooth and compact obstacle $\varnothing \subset \mathbb{R}^3$, $A(x) = \left(a^{ij}(x)\right)$ are symmetric and positively definite matrices for all $x \in \Omega$, $a^{ij}(x)$ are smooth functions on $\Omega$. And assuming the data $(f, g)$ satisfies a necessary compatibility condition arising from the Dirichlet boundary condition. If $a^{ij} = \delta^{ij}$, which denotes the Kronecker delta function, we say problem (1.1) is of constant coefficients. In the case of critical nonlinear wave equation with constant coefficients, a wealth of results are available in the literature. For Cauchy

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problem, global existence of $C^2$-solutions in dimension $n = 3$ was first obtained by Rauch \cite{8}, assuming the initial energy to be small. In 1988, also for “large” data global $C^2$-solutions in dimension $n = 3$ were shown to exist by Struwe \cite{12} in the radially symmetric case. Grillakis \cite{4} in 1990 was able to remove the latter symmetry assumption and obtained the same result. Not much later, Kapitanskii \cite{7} established the existence of a unique, partially regular solution for all dimensions. Combining Strichartz inequality and Morawetz estimates, Grillakis \cite{5} in 1992 established global existence and regularity for dimensions $3 \leq n \leq 5$ and announced the corresponding results in the radial case for dimensions $n \leq 7$. Then Shatah and Struwe \cite{9} obtained global existence and regularity for dimensions $3 \leq n \leq 7$. They also proved the global well-posedness in the energy space in \cite{10} 1994. For the critical exterior problem in dimension 3, Smith and Sogge \cite{11} in 1995 proved global existence of smooth solutions. In 2008, Burq et al \cite{1} obtained the same result in 3-D bounded domain.

For the critical Cauchy problem with time-independent variable coefficients, Ibrahim and Majdoub \cite{6} in 2003 studied the existence of both global smooth for dimensions $3 \leq n < 6$ and Shatah-Struwe’s solutions for dimensions $n \geq 3$.

In this paper we consider the problem (1.1) with a general $A(x)$ and we refer to it as critical problem with variable coefficients. We define a metric $g = A^{-1}(x) = (a^{ij}(x))^{-1}$, $x \in \Omega$, then on the Riemannian manifold $(\Omega, g)$ we can introduce the distance function $\rho$. To derive the global existence, the key step is to show the $L^6$ part of the energy associated to (1.1) cannot concentrate at any point $(t_0, x_0)$, where $x_0 \in \overline{\Omega}$. Instead of the Morawetz multiplier $t\partial_t + r\partial_r + 1$, where $r = |x|$, we use a geometric multiplier following Ibrahim and Majdoub \cite{6}. That is: $t\partial_t + \rho \partial_\rho + 1$, where $\rho = \rho(x, x_0)$ is the distance function from some point $x$ to $x_0$ and $\partial_\rho = \nabla_g \rho = g^{ij}\rho_x \frac{\partial}{\partial x_i} = a^{ij}\rho_x \frac{\partial}{\partial x_i}$, and $\nabla_g$ here denotes the gradient on the Riemannian manifold. Then we use Hessian and Laplace comparison theorems from Riemannian Geometry to estimate the error terms. Finally we use Strichartz estimates to obtain the global existence, as in \cite{11}.

2 Main result

In this section we show the main result and proofs.

Following Ibrahim and Majdoub \cite{6}, we define:

$$g = A^{-1}(x) = (a^{ij}(x))^{-1} \quad x \in \Omega,$$

as a Riemannian metric on $\Omega$, and consider the couple $(\Omega, g)$ as a Riemannian manifold. For each $x \in \Omega$, the Riemannian metric $g$ induces the inner product and the norm on the tangent space $\Omega_x = \Omega$, by:

$$\langle X, Y \rangle_g = \langle A^{-1}(x)X, Y \rangle, \quad |X|^2_g = \langle X, X \rangle_g, \quad X, Y \in \Omega,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product of the Euclidean space. For $w \in H^1(\Omega)$, we have:

$$\nabla_g w = a^{ij}(x)w_{x_j} = A(x)\nabla w, \quad |\nabla_g w|^2_g = a^{ij}(x)w_{x_j}w_{x_j}, \quad x \in \Omega,$$

where $\nabla_g$ is the gradient of the Riemannian metric $g$, and $\nabla$ is the gradient on Euclidean space. Here and in the sequence, we use geometric convention of summing over upper and
In this paper we assume there are $c_1 > 0$ and $c_2 > 0$ such that
\[ c_1 |X|^2 \leq \langle A(x)X, X \rangle \leq c_2 |X|^2, \quad \text{for all } x \in \Omega, \, X \in \Omega, \]
then $|\nabla g w|_g \simeq |\nabla w|$.

We define the energy of the problem (1.1):
\[ E(t) = \frac{1}{2} \int_{\Omega} \left( u_t^2 + a^{ij}(x)u_{x_i}u_{x_j} + \frac{u^6}{3} \right) dx. \quad (2.4) \]

### 2.1 Global existence

The key to establish global existence for (1.1) is to show that: if the data $(f, g)$ has compact support, and if $u$ is a smooth solution to (1.1) in a half open strip $[0, t_0) \times \Omega$, then $u$ must be uniformly bounded by some constant in that strip. Then local existence and regularity theorems imply global existence and regularity. To establish the uniform bounds on $u$, by compactness it suffices to show that $u$ is bounded in a neighborhood of each given point $(t_0, x_0)$, where $x_0 \in \overline{\Omega}$.

**Theorem 2.1.** Suppose that $u \in C^\infty([0, t_0) \times \Omega)$ solves (1.1). Then if $x_0 \in \overline{\Omega}$, $u$ must be bounded in a neighborhood of $(t_0, x_0)$, and hence $u \in L^\infty([0, t_0) \times \Omega)$.

Let us now sketch the proof that $u$ cannot blow up at $(t_0, x_0)$. As in Grillakis [5] and Shatah and Struwe [9], the first key step is to show the $L^6$ part of the energy associated to (1.1) cannot concentrate at $(t_0, x_0)$:
\[ \lim_{t \nearrow t_0} \int_{\rho(x, x_0) \leq t - t_0} \frac{u^6}{6} dx = 0, \quad (2.5) \]
where $\rho$ is the distance function of the matrix $g$ from $x_0$ to $x \in \Omega$. If $A = (\delta^{ij})$, then $g$ is the standard metric of $\Omega$ and $\rho(x) = |x - x_0|$. For a general metric $g$, the structure of $\rho(x)$ is more complicated. For the properties of this function, see section 3.

The proof of (2.5) will be shown in section 2.2, following Struwe [12], exploiting an geometric multiplier mentioned above similar to the well-known Morawetz multiplier. However, extra error terms appear in the variable case. To overcome this difficulty, we apply Hessian and Laplace comparison theorem from differential geometry. If $x_0 \in \partial \Omega$, we apply the similar method as Burq et all used in [1] to control the boundary term.

To prove Theorem 2.1, The second key step is to use the Strichartz inequality to prove that $u$ is bounded near any point $(t_0, x_0)$, where $x_0 \in \Omega$. Our proof of this part is completely parallel to Smith and Sogge [11], for the convenience of the reader, we sketch the proofs as follows.

Assuming identity (2.5) is hold, then combining with the Strichartz estimates we show that $u \in L^6_t L^{12}_x(K)$, where $K$ is the domain of influence for $(t_0, x_0)$:
\[ K = \{(t, x) : \rho(x, x_0) \leq t_0 - t, \quad (t, x) \in [0, t_0) \times \Omega\}. \quad (2.6) \]

Then Strichartz estimates shows that $u \in L^6_t L^{12}_x(K)$ implies $\partial_t u \in L^\infty_t L^6_x(K)$. A similar argument can be applied to show that $\nabla_x u \in L^\infty_t L^6_x(K)$, which is equivalent to $|\nabla_g w|_g \simeq $
$L_t^\infty L_x^6(K)$. We then use Hölder’s inequality to see that the total energy cannot concentrate at $(t_0, x_0)$, that is:
\[
\lim_{t \nearrow t_0} \frac{1}{2} \int_{x \in \Omega} \left( u_t^2 + a^{ij}(x)u_{x_i}u_{x_j} + \frac{u_6^2}{3} \right) dx = 0. \tag{2.7}
\]

Now we shall give more specific details with several lemmas. The first is to show the $L^6$ part of the energy cannot concentrate at any point, the second is the spacetime estimates for the wave equation, the third is standard and says that the energy associated with our equation is conserved; furthermore the energy inside spatial cross-sections of a backward light cone is monotonic decreasing in time.

**Lemma 2.2.** If $u \in C^\infty([0, t_0) \times \Omega)$ solves (1.1), and $x_0 \in \Omega$, then:
\[
\lim_{t \nearrow t_0} \int_{x \in \Omega} \frac{u_6(t, x)}{6} dx = 0. \tag{2.8}
\]

We postpone the proof of lemma 2.2 for the moment.

**Lemma 2.3.** For the solution to the exterior problem in the half open strip $[0, t_0) \times \Omega$:
\[
\begin{align*}
  u_{tt} - \frac{\partial}{\partial x_i} \left( a^{ij}(x)u_{x_j} \right) &= F(t, x) \quad \text{on} \quad (0, \infty) \times \Omega, \\
  u(0, x) &= f(x), \quad u_t(0, x) = g(x),
\end{align*}
\]
satisfies the estimates as follows:
\[
\|u\|_{L_t^{\frac{2q}{q-6}} L_x^q([0, t_0) \times \Omega)} \leq C \left( \|f\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)} + \|F\|_{L_t^1 L_x^q([0, t_0) \times \Omega)} \right)
\]
\[
6 \leq q < \infty. \tag{2.10}
\]

For the proof see Smith and Sogge [11].

**Lemma 2.4.** Let $u$ as above. Then
\[
u \in L_t^{\frac{2q}{q-6}} L_x^q(K), \quad \text{if} \quad 6 \leq q < \infty.
\]

**Proof.** Hölder’s inequality implies that if $6 < q < q_1$, then $L_t^\infty L_x^6 \cap L_t^{\frac{2q_1}{q_1}} L_x^{q_1} \subset L_t^{\frac{2q}{q-6}} L_x^q$. Since $u \in L_t^\infty L_x^6$ by conservation of energy, it therefore suffices to check that
\[
u \in L_t^{\frac{2q}{q-6}} L_x^q(K), \quad \text{if} \quad 10 \leq q < \infty. \tag{2.11}
\]

If $0 \leq s_1 < s_2 < t_0$, set
\[
K_{s_1}^{s_2} = K \cap ([s_1, s_2] \times \Omega),
\]
where $K$ is as above. Then, since $u$ is smooth and has relatively compact support in $[0, t_0) \times \Omega$, it suffices to show that for some fixed $0 < s_1 < t_0$, one has
\[
\sup_{s_2 \leq s_1, t_0} \|u\|_{L_t^{\frac{2q}{q-6}} L_x^q(K_{s_1}^{s_2})} < \infty, \quad \text{if} \quad 10 \leq q < \infty.
\]
To establish this inequality, we shall want to apply (2.10), and if the norm in the left is only taken over \( K_{x_1}^{t_2} \), then the norm involving \( F \) need only be taken over the same set, by Huygen’s principle. Thus

\[
\|u\|_{L_t^\infty L_x^q(K_{x_1}^{t_2})} \leq C_q E_0 + C_q \|u^5\|_{L_t^4 L_x^2(K_{x_1}^{t_2})},
\]

where \( E_0 \) denotes the initial energy of \( u \). If \( q > 10 \), another application of Hölder’s inequality yields

\[
\|u^5\|_{L_t^4 L_x^2(K_{x_1}^{t_2})} \leq \|u\|_{L_t^\infty L_x^6(K_{x_1}^{t_2})} \|u\|_{L_t^6 L_x^6(K_{x_1}^{t_2})},
\]

and consequently

\[
\|u\|_{L_t^{2q} L_x^q(K_{x_1}^{t_2})} \leq C_q E_0 + C_q \|u\|_{L_t^\infty L_x^6(K_{x_1}^{t_2})} \|u\|_{L_t^6 L_x^6(K_{x_1}^{t_2})}.
\]

Given \( \varepsilon > 0 \), (2.8) implies that we can choose \( s_1 \) close enough to \( t_0 \) so that

\[
C_q \|u\|_{L_t^{2q} L_x^q(K_{x_1}^{t_2})} < \varepsilon.
\]

If we choose

\[
\varepsilon < 2^{-\frac{2q}{q-6}} (C_q E_0)^{1-\frac{2q}{q-6}},
\]

then the following standard lemma implies that

\[
\|u\|_{L_t^{2q} L_x^q(K_{x_1}^{t_2})} \leq 2C_q E_0,
\]

giving us (2.11) and finishing the proof.

**Lemma 2.5.** Let \( 0 < C_0 < \infty \) and suppose that \( 0 \leq y(s) \in C([a, b]), \) with \( y(a) = 0, \) satisfies

\[
y(s) \leq C_0 + \varepsilon y(s)^\gamma.
\]

Then if \( \varepsilon < 2^{-\gamma} C_0^{1-\gamma} \) it follows that

\[
y(s) < 2C_0, \quad s \in [a, b).
\]

**Proof.** Since \( C_0 + \varepsilon y^\gamma - x < 0 \) if \( \varepsilon < 2^{-\gamma} C_0^{1-\gamma} \) and \( x = 2C_0, \) it follows that

\[
0 \leq C_0 + \varepsilon x^\gamma - x \quad \forall x \in [0, x_0] \implies x_0 < 2C_0.
\]

Since \( y(s) \) must be \( \leq \) the supremum of such \( x_0, \) the lemma follows.

To complete the Theorem 2.1, we shall use the following special case of lemma 2.4:

\[
u \in L_t^4 L_x^{12}(K).
\]

Since \( (\partial_t u)_t - \frac{\partial}{\partial x_i} (a^{ij}(x)(\partial_x u)_{x_j}) = -5u^4 \partial_t u, \) and \( \partial_t u \) has compact support, estimates (2.10) with \( q = 6 \) implies that, if \( 0 \leq s \leq t \leq t_0, \) then

\[
\|\partial_t u\|_{L_t^\infty L_x^2(K_s^{t_2})} \leq C(s) + C \|u^4 \partial_t u\|_{L_t^4 L_x^{12}(K_s^{t_2})} \leq C(s) + C \|u\|_{L_t^4 L_x^{12}(K_s^{t_2})} \|\partial_t u\|_{L_t^\infty L_x^2(K_s^{t_2})}.
\]
Hence if $s$ is close enough to $t_0$ so that $C\|u\|_{L^4_t L^2_x}^4 \leq \frac{1}{2}$, we conclude that $\partial_t u \in L^\infty_t L^6_x(K^t_s)$ with norm bounded by $2C(s)$ for all $t \in (s, t_0)$, which yields

$$\partial_t u \in L^\infty_t L^6_x(K).$$

If $x_0$ is interior to $\Omega$, a similar argument can be applied to show that $\nabla_x u \in L^\infty_t L^6_x(K)$, which implies $\nabla_x u \in L^\infty_t L^6_x(K)$. And from this and Hölder’s inequality we conclude that the total energy of $u$ cannot concentrate at $t_0, x_0$:

$$\lim_{t \to t_0} \frac{1}{2} \int_{\mathbb{R}^n \times [t_0, t]} \left( u^2(t, x) + a^{ij}(x) u_{x_i} u_{x_j} + \frac{u^6(t, x)}{3} \right) dx = 0. \quad (2.12)$$

If $x_0 \in \partial \Omega$, however, this argument breaks down since $\nabla_x u$ does not vanish on $\partial \Omega$, we cannot apply (2.10) to estimate it. For the way to deal with this problem, one can get the details from [11].

**Lemma 2.6.** If $u \in C^\infty([0, t_0] \times \Omega)$ is a solution to (1.1), then

$$\frac{1}{2} \int_{\Omega} \left( u^2 + a^{ij}(x) u_{x_i} u_{x_j} + \frac{u^6}{3} \right) dx \quad (2.13)$$

is equal to a fixed constant $E_0 < \infty$ for all $0 \leq t < t_0$. Additionally, if $0 \leq s < t < t_0, x_0 \in \Omega$, then

$$\frac{1}{2} \int_{\mathbb{R}^n \times [t_0, t]} \left( u^2(t, x) + a^{ij}(x) u_{x_i}(t, x) u_{x_j}(t, x) + \frac{u^6(t, x)}{3} \right) dx \leq \frac{1}{2} \int_{\mathbb{R}^n \times [t_0, t]} \left( u^2(s, x) + a^{ij}(x) u_{x_i}(s, x) u_{x_j}(s, x) + \frac{u^6(s, x)}{3} \right) dx. \quad (2.14)$$

**Proof.** To prove the conservation of energy one multiplies both sides of the equation $u_t - \frac{\partial}{\partial x_i}(a^{ij}(x) u_{x_j}) + u^5 = 0$ by $\partial_t u$ to obtain the identity

$$\frac{\partial}{\partial t} \left( \frac{u^2}{2} + \frac{a^{ij}(x) u_{x_i} u_{x_j}}{2} + \frac{u^6}{6} \right) - \frac{\partial}{\partial x_i} \left( u_t a^{ij}(x) u_{x_j} \right) = 0. \quad (2.15)$$

Thus,

$$0 = \frac{\partial}{\partial t} \int_{\Omega} \left( \frac{u^2}{2} + \frac{a^{ij}(x) u_{x_i} u_{x_j}}{2} + \frac{u^6}{6} \right) dx - \int_{\Omega} \frac{\partial}{\partial x_i} \left( u_t a^{ij}(x) u_{x_j} \right) dx.$$

And since the last term is always zero, by the divergence theorem, due to the fact that $\partial_t u = 0$ on $\partial \Omega$ and $u(t, x) = 0$ for $|x| > C + t$, we see that (2.15) implies that (2.13) must be constant, as desired.

To prove the other half of lemma 2.6 we need to define the energy flux across part of the domain of dependence of a point.

To do this, we first need to introduce some more notation. First of all, if $0 \leq s < t < t_0$, set

$$K^t_s = K \cap ([s, t] \times \Omega),$$

where $K$ is as above. And let $M^t_s$ denote the "mantle" associated with it:

$$M^t_s = \partial K^t_s \cap ([s, t] \times \Omega).$$
Also, let $d\sigma$ denote the induced Lebesgue measure on $M_s^t$ and $\nu = \nu(\rho, x) = \frac{\nabla \rho}{\sqrt{1 + |\nabla \rho|^2}}$ denotes the unit normal through $(\rho, x) \in M_s^t$. If we let $e(u)$ be the vector field arising from (2.15),

\[
e(u) = \left( \frac{u_t^2 + a^{ij}(x)u_{ix}u_{xj}}{2} + \frac{u^6}{6}, -u_t a^{ij}(x)u_{xj} \right),
\]

then we can define the "energy flux" across $M_s^t$:

\[
Flux(u, M_s^t) = \int_{M_s^t} \langle e(u), \nu \rangle d\sigma
\]

\[
= \int_{M_s^t} \frac{1}{2} \left( \frac{u_t^2 + a^{ij}(x)u_{ix}u_{xj}}{2} + \frac{u^6}{6} \right) - u_t a^{ij}(x)u_{xj}\rho_x \left( x, \rho_x \right) d\sigma
\]

\[
\ge \int_{M_s^t} \frac{1}{2} \left( \frac{u_t^2 + a^{ij}(x)u_{ix}u_{xj}}{2} + \frac{u^6}{6} \right) - \frac{1}{2} \left[ u_t^2 + (a^{ij}(x)u_{xj})^2 \right] d\sigma
\]

\[
\ge \int_{M_s^t} \frac{1}{2} \left( \frac{u_t^2 + a^{ij}(x)u_{ix}u_{xj}}{2} + \frac{u^6}{6} \right) - \frac{1}{2} \left[ u_t^2 + (a^{ij}(x)u_{xj})^2 \right] d\sigma
\]

\[
= \int_{M_s^t} \frac{u^6}{6 \sqrt{1 + |\nabla \rho|^2}} d\sigma \ge 0,
\]

since $|\nabla \rho|^2 = g^{ij}(x)\rho_x \rho_x = a^{ij}(x)\rho_x \rho_x = 1$. Also, Cauchy-Schwarz inequality is used to prove the above inequality. If we integrate (2.15) over $K_s^t$ we arrive at the "flux identity":

\[
\frac{1}{2} \int_{\rho(x, x_0) \le t \le t_0} \left( u_t^2(t, x) + a^{ij}(x)u_{ix}(t, x)u_{xj}(t, x) + \frac{u^6(t, x)}{3} \right) dx + Flux(u, M_s^t)
\]

\[
= \frac{1}{2} \int_{\rho(x, x_0) \le t \le t_0-s} \left( u_t^2(s, x) + a^{ij}(x)u_{ix}(s, x)u_{xj}(s, x) + \frac{u^6(s, x)}{3} \right) dx,
\]

that is

\[
E(u, D(t)) + Flux(u, M_s^t) = E(u, D(s)),
\]

(2.16)

where

\[
E(u, D(t)) = \frac{1}{2} \int_{\rho(x, x_0) \le t \le t_0-t} \left( u_t^2 + a^{ij}(x)u_{ix}u_{xj} + \frac{u^6}{3} \right) dx.
\]

Since $Flux(u, M_s^t) \ge 0$, we see (2.16) implies (2.11), which completes the proof.

And we conclude from (2.15) that $t \rightarrow E(u, D(t))$ is a non-increasing function on $[0, t_0]$. It is also bounded, since $E(u, D(t)) \le E(t) \le E_0 < \infty$, on account of our assumptions on the data. Hence, $E(u, D(t))$ and $E(u, D(s))$ in (2.16) must approach a common limit. This in turn gives the important fact that

\[
Flux(u, M_s^t) \rightarrow 0, \quad \text{as} \quad s \rightarrow t.
\]

(2.17)
Given $\varepsilon > 0$, from the identity (2.12), we can find a $0 < t_1 < t_0$ so that
\[
\frac{1}{2} \int_{\rho(x, x_0) \leq t_0-t_1} \left( u_t^2 + a^{ij}(x)u_{x_i}u_{x_j} + \frac{u^6}{3} \right) (t_1, x) \, dx < \frac{\varepsilon}{2}.
\]
By dominated convergence, there is a $\delta > 0$ so that
\[
\frac{1}{2} \int_{\rho(x, x_0) \leq \delta + t_0-t} \left( u_t^2 + a^{ij}(x)u_{x_i}u_{x_j} + \frac{u^6}{3} \right) (t_1, x) \, dx < \varepsilon.
\]
Then by the monotonicity of energy (2.14), yields
\[
\int_{\rho(x, x_0) \leq \delta + t_0-t} \frac{u^6(t, x)}{6} \, dx < \varepsilon, \quad t_1 \leq t \leq t_0.
\]
Let
\[
K^\delta = \{(t, x) : \rho(x, x_0) < \delta + t_0-t, \ (t, x) \in [0, t_0) \times \Omega \}.
\]
For $\varepsilon$ sufficiently small, we can repeat the proof of lemma 2.4 with $K$ replaced by $K^\delta$, to conclude that
\[
u \in L^4_t L^{12}_x(K^\delta).
\]
Combining with lemma 2.6 as above we can now argue as before to conclude that
\[
\partial_t u \in L^\infty_t L^6_x(K^\delta), \quad \nabla_x u \in L^\infty_t L^6_x(K^\delta),
\]
which implies $u \in L^\infty(\mathbb{K}^\delta)$ by Sobolev’s theorem. Since $u$ vanishes outside of a relatively compact subset of $[0, t_0) \times \Omega$, we can cover its support by finitely many of these sets $\mathbb{K}^\delta$. Hence, $u \in L^\infty([0, t_0) \times \Omega)$, which implies that $u$ can be extended to a global solution.

For $x_0 \in \partial\Omega$, an additional argument is needed since $\nabla_x u$ does not vanish on $\partial\Omega$. Here we skip this step as the method is just totally the same as Smith and Sogge used in [11].

### 2.2 Nonconcentration of $L^6$ part of energy

Now we prove lemma 2.2. For that purpose we need several lemmas about differential geometry. And we work on $\Omega$ with metric $g = \langle \cdot, \cdot \rangle_g$ given by (2.1).

**Lemma 2.7.** Let $f$ be function and $X \in \Omega_x$ be vector field. Then, we have
\[
\langle \nabla_g f, \nabla_g (X(f)) \rangle_g = \langle \nabla_g f X, \nabla_g f \rangle_g + X(\frac{1}{2} \nabla_g f^2)_g, \quad x \in \Omega.
\]
We shall prove this identity in section 3.

Since for any vector field $Y, Z \in \Omega_x$, we have
\[
\langle \nabla_Y \nabla_g (\rho^2), Z \rangle_g = Y \langle \nabla_g (\rho^2), Z \rangle_g - \langle \nabla_g (\rho^2), \nabla_Y Z \rangle_g
\]
\[
= YZ(\rho^2) - \langle \nabla_Y Z \rangle_g(\rho^2)
\]
\[
= D^2 \rho^2(Y, Z),
\]

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where $D^2 \rho^2$ is the Hessian of the function $\rho^2$, if we replace $X$, $f$ in the equality (2.18) with $\nabla g(\frac{1}{2} \rho^2)\ n$, $u$ respectively, we get

$$
\langle \nabla g u, \nabla g (X(u)) \rangle_g = \langle \nabla g u, \nabla g \left( \nabla g \left( \frac{1}{2} \rho^2 \right) (u) \right) \rangle_g = \langle \nabla g u, \nabla g (\rho a^{ij} \rho_{x_j} u_{x_i}) \rangle_g
$$

$$
= \langle \nabla g u, \nabla g (\frac{1}{2} \rho^2) \rangle_g + \langle \nabla g (\frac{1}{2} \rho^2) \nabla g u, \nabla g u \rangle_g
$$

$$
= \frac{1}{2} D^2 \rho^2 (\nabla g u, \nabla g u) + \langle \nabla g (\frac{1}{2} \rho^2) \nabla g u, \nabla g u \rangle_g
$$

(2.18)

**Lemma 2.8.** If the sectional curvature $\kappa$ of the Riemannian manifold $(\Omega, g)$ satisfies

$$
-a^2 \leq \kappa \leq a^2
$$

then for the distance function $\rho$ on $(\Omega, g)$, $\forall X, Y \in \Omega_x$, we have

$$
\lim_{\rho \to 0} \frac{\partial}{\partial x_i}(\rho a^{ij} \rho_{x_j}) = \lim_{\rho \to 0} \left( \triangle_g \left( \frac{1}{2} \rho^2 \right) - \frac{1}{2} g^{lm} \frac{\partial g_{lm}}{\partial x_i} \rho \nabla g \rho \right) = 3,
$$

(2.19)

$$
\lim_{\rho \to 0} \frac{1}{2} D^2 \rho^2 (X, Y) = \langle X, Y \rangle_g
$$

(2.20)

where $\triangle_g$ is the Laplace operator on $(\Omega, g)$.

For the proof see section 3.

**Lemma 2.9.** Assume that $u$ is a weak solution to (1.1), then we have

$$
\left\| \frac{\partial u}{\partial \nu} \right\|_{L^2((0, t_0) \times \partial \Omega)} \leq CE(u)^\frac{1}{2},
$$

(2.21)

where $\frac{\partial u}{\partial \nu}$ is the trace to the boundary of the exterior normal derivative of $u$.

**Proof.** Similar to the constant case in Burq et al [1], take $Z \in C^\infty(\Omega, T\Omega)$ a vector field whose restriction to $\partial \Omega$ is equal to $\frac{\partial}{\partial \nu}$ and compute for $0 < T < t_0$

$$
\int_0^T \int_\Omega \left[ \left( \frac{\partial^2}{\partial t} - \frac{\partial}{\partial x_i} \left( a^{ij} \frac{\partial}{\partial x_j} \right) \right), Z \right] u(t, x) \cdot u(t, x) dx dt
$$

$$
= \int_0^T \int_\Omega \left[ \left( \frac{\partial^2}{\partial t} - a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial a^{ij}}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{\partial a^{ij}}{\partial x_j} \frac{\partial}{\partial x_i} \right) \right] u \cdot Z u - Z \left( \frac{\partial^2}{\partial t} - a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial a^{ij}}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{\partial a^{ij}}{\partial x_j} \frac{\partial}{\partial x_i} \right) u \cdot Z u dx dt.
$$

Integrate by parts, we obtain

$$
\int_0^T \int_\Omega \left[ \left( \frac{\partial^2}{\partial t} - \frac{\partial}{\partial x_i} \left( a^{ij} \frac{\partial}{\partial x_j} \right) \right), Z \right] u(t, x) \cdot u(t, x) dx dt
$$

$$
= \int_0^T \int_\Omega \frac{\partial}{\partial x_i} \left[ (Z u) a^{ij} \frac{\partial u}{\partial x_j} \right] dx dt + \int_0^T \int_\Omega -(Z u) u^5 + Z (u^5) u dx dt
$$

(2.22)

$$
+ \int_\Omega \partial_t (Z u) \cdot u dx \bigg|_0^T - \int_\Omega (Z u) \cdot \partial_t u dx \bigg|_0^T.
$$

From the assumption of the coefficients $a^{ij}$, and noting that on $[0, T] \times \partial \Omega$, $\nabla_x u = (\partial_x u) \nu$, we have

$$
\int_0^T \int_\Omega \frac{\partial}{\partial x_i} \left[ (Z u) a^{ij} \frac{\partial u}{\partial x_j} \right] dx dt = \int_0^T \int_{\partial \Omega} \frac{\partial u}{\partial \nu} a^{ij} \frac{\partial u}{\partial x_j} \nu_i \sigma dt
$$

$$
= \int_0^T \int_{\partial \Omega} \frac{\partial u}{\partial \nu} a^{ij} (\partial_x u) \nu_i \nu_j \sigma dt \geq C \int_0^T \int_{\partial \Omega} |\frac{\partial u}{\partial \nu}|^2 \sigma dt.
$$

(2.23)
Remark now that if $Z = \sum_j b_j \frac{\partial}{\partial x_j}$, then integration by parts yields (using the Dirichlet boundary condition)

$$\left| \int_\Omega -(Zu)^5 + Z(u^5) u \, dx \, dt \right| = \frac{4}{6} \left| \int_0^T \int_\Omega Z(u^6)(t, x) \, dx \, dt \right|$$

$$= \frac{4}{6} \int_0^T \int_\Omega \sum_j \frac{\partial b_j}{\partial x_j} u^6 \, dx \, dt \leq CE(u).$$

while

$$\left| \left[ \int_\Omega \partial_t (Zu) \cdot u \, dx \right]_0^T - \left[ \int_\Omega (Zu) \cdot \partial_t u \, dx \right]_0^T \right| \leq CE(u).$$

and $\left[ (\partial_t^2 - \frac{\partial}{\partial x_i}(a^{ij}(x) \frac{\partial}{\partial x_j})), Z \right] = -\left[ \frac{\partial}{\partial x_i}(a^{ij}(x) \frac{\partial}{\partial x_j}), Z \right]$ as a second order differential operator in the $x$ variable is continuous from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$ and consequently

$$\left| \int_0^T \int_\Omega \left[ (\partial_t^2 - \frac{\partial}{\partial x_i}(a^{ij}(x) \frac{\partial}{\partial x_j})), Z \right] u(t, x) \cdot u(t, x) \, dx \, dt \right| \leq CE(u).$$

As the constants are uniform with respect to $0 < T < t_0$, collecting (2.22), (2.23), (2.24), (2.25) and (2.26) yields (2.21).

**Proof of lemma 2.2.** Following Ibrahim and Majdoub [6], we use a geometric multiplier. For the sake of notation it is convenient to shift $(t_0, x_0) \in \mathbb{R} \times \Omega$ to the origin.

Multiply the equation $u_{tt} - \frac{\partial}{\partial x_i}(a^{ij}(x) u_{x_j}) + u^5 = 0$ by $tu_t + \rho a^{lm} \rho_{x_m} u_{x_l} + u$. By (2.15) it is easy to see the contribution from the first term is

$$\frac{\partial}{\partial t} \left[ t \left( \frac{1}{2} u_t^2 + a^{ij}(x) u_{x_i} u_{x_j} + \frac{u^6}{3} \right) \right] - \frac{1}{2} (u_t^2 + a^{ij}(x) u_{x_i} u_{x_j} + \frac{u^6}{3}) - \frac{\partial}{\partial x_i} (tu_t a^{ij}(x) u_{x_j}) = 0.$$

Similarly, we compute

$$0 = \left( u_{tt} - \frac{\partial}{\partial x_i} (a^{ij}(x) u_{x_j}) + u^5 \right) \left( \rho a^{lm} \rho_{x_m} u_{x_l} \right).$$

For

$$\rho a^{lm} \rho_{x_m} u_{x_l} u_{tt} = (\rho a^{lm} \rho_{x_m} u_{x_l} u_t)_t - \rho a^{lm} \rho_{x_m} u_{x_l} u_t$$

$$= (\rho a^{lm} \rho_{x_m} u_{x_l} u_t)_t - \frac{1}{2} \rho a^{lm} \rho_{x_m} \frac{\partial u_t^2}{\partial x_l}$$

$$= (\rho a^{lm} \rho_{x_m} u_{x_l} u_t)_t - \frac{1}{2} \rho a^{lm} \rho_{x_m} u_{x_l} \frac{\partial (\rho a^{lm} \rho_{x_m} u_t^2)}{\partial x_l} - \frac{\partial}{\partial x_l} (\rho a^{lm} \rho_{x_m} u_t^2).$$
Using (2.18) in lemma 2.7, we have

\[
\rho a^{lm} \rho x_m u_{xi} \frac{\partial}{\partial x_i} (a^{ij} (x) u_{x_j})
= \frac{\partial}{\partial x_i} \left( \rho a^{lm} \rho x_m u_{xi} a^{ij} (x) u_{x_j} \right) - \frac{\partial}{\partial x_i} \left( \rho a^{lm} \rho x_m u_{xi} a^{ij} (x) u_{x_j} \right) \\
= \frac{\partial}{\partial x_i} \left( \rho a^{lm} \rho x_m u_{xi} a^{ij} (x) u_{x_j} \right) - \langle \nabla_g u, \nabla_g (\rho a^{lm} \rho x_m u_{xi}) \rangle_g \\
= \frac{\partial}{\partial x_i} \left( \rho a^{lm} \rho x_m u_{xi} a^{ij} (x) u_{x_j} \right) - \frac{1}{2} D^2 \rho^2 (\nabla_g u, \nabla_g u) - \nabla_g \left( \frac{1}{2} \rho^2 \right) \left( \frac{1}{2} \nabla_g u^2 \right) \\
= \frac{\partial}{\partial x_i} \left( \rho a^{lm} \rho x_m u_{xi} a^{ij} (x) u_{x_j} \right) - \frac{1}{2} D^2 \rho^2 (\nabla_g u, \nabla_g u) - \rho a^{lm} \rho x_m \frac{\partial}{\partial x_i} \left( \frac{1}{2} \nabla_g u^2 \right) \\
= \frac{\partial}{\partial x_i} \left( \rho a^{lm} \rho x_m u_{xi} a^{ij} (x) u_{x_j} \right) - \frac{1}{2} D^2 \rho^2 (\nabla_g u, \nabla_g u) \\
- \frac{1}{2} \frac{\partial}{\partial x_i} (a^{ij} u_{xi} u_{x_j} \rho a^{lm} \rho x_m) + \frac{1}{2} \frac{\partial}{\partial x_i} (\rho a^{lm} \rho x_m) |\nabla_g u|^2.
\]

and

\[
\rho a^{lm} \rho x_m u_{xi} u^5 = \frac{1}{6} \left( \rho a^{lm} \rho x_m \frac{\partial u^6}{\partial x_i} \right) = \frac{1}{6} \left[ \frac{\partial}{\partial x_i} (\rho a^{lm} \rho x_m u^6) - u^6 \frac{\partial}{\partial x_i} (\rho a^{lm} \rho x_m) \right].
\]

Finally,

\[
0 = u \left( u_t - \frac{\partial}{\partial x_i} (a^{ij} (x) u_{x_j}) + u^5 \right) = (uu_t)_t - u^2 - \frac{\partial}{\partial x_i} (uu^{ij} u_{x_j}) + u_x a^{ij} (x) u_{x_j} + u^6 \\
= \frac{\partial}{\partial t} (uu_t) - \frac{\partial}{\partial x_i} (uu^{ij} u_{x_j}) + |\nabla_g u|^2 + u^6 - u_t^2.
\]

Adding, we obtain that

\[
(tu_t + \rho a^{lm} \rho x_m u_{xi} + u)(uu_t - \frac{\partial}{\partial x_i} (a^{ij} (x) u_{x_j}) + u^5) \\
= \partial_t (tQ + uu_t) - \frac{\partial}{\partial x_i} (tP) + R = 0,
\]

where

\[
Q = \frac{1}{2} \left( u_t^2 + a^{ij} (x) u_{xi} u_{x_j} + u^6 \right) + \frac{u_t \rho a^{ij} \rho x_i u_{x_i}}{t},
\]

\[
P = \frac{\rho a^{ij} \rho x_i}{t} \left[ \frac{1}{2} (u_t^2 - a^{lm} \rho x_m u_{xi} - \frac{u^6}{3}) + a^{ij} u_{x_j} \left( u_t + \frac{\rho a^{lm} \rho x_m u_{xi}}{t} + \frac{u}{t} \right) \right],
\]

\[
R = \left( \frac{1}{2} \frac{\partial}{\partial x_i} (\rho a^{ij} \rho x_j) - \frac{3}{2} \right) u_t^2 + \frac{1}{2} D^2 \rho^2 (\nabla_g u, \nabla_g u) \\
+ \left( \frac{1}{2} - \frac{1}{2} \frac{\partial}{\partial x_i} (\rho a^{ij} \rho x_j) \right) |\nabla_g u|^2 + \left( \frac{5}{6} - \frac{1}{6} \frac{\partial}{\partial x_i} (\rho a^{ij} \rho x_j) \right) u^6.
\]

Note that the boundary of the truncated cones \( K^T_S \) is

\[
\partial K^T_S = (\{ [S, T] \times \partial \Omega \} \cap K^T_S) \cup M^T_S \cup D(T) \cup D(S),
\]

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for \( K^T_S, M^S_T, D(T) \) are as above. Thus if \( \nu_{\partial \Omega} \) denotes the outward unit normal for \( \Omega \), integrating the identity \((2.28)\) over the truncated cones \( K^T_S \), we get

\[
0 = \int_{D(T)} (TQ + u_t u) dx - \int_{D(S)} (SQ + u_t u) dx + \int_{M^S_T} \frac{tQ + u_t u - tP \cdot \nabla \rho}{\sqrt{1 + |\nabla \rho|^2}} d\sigma
\]

\[
+ \int_{([S,T] \times \partial \Omega) \cap K^T_S} \nu_{\theta \Omega} \cdot (-tP) d\sigma + \int_{K^T_S} Rdtdx.
\]

(2.29)

First we compute the second to the last term. Note that on \([S,T] \times \partial \Omega\),

\[
\nabla_x u = (\partial_x u) \nu, \quad u = u_t = 0.
\]

Thus,

\[
P = \frac{\rho a^{lm} \rho_{xm}}{t} \left( -\frac{1}{2} a^{ij} u_x u_x \right) + a^{lm} u_x \frac{\rho a^{ij} \rho_{xj} u_{x_i}}{t},
\]

\[
\nabla_g u = a^{ij} u_{x_j} = (\partial_x u) a^{ij} \nu_j,
\]

so

\[
P = \frac{\rho a^{lm} \rho_{xm}}{t} \left( -\frac{1}{2} (\partial_x u)^2 a^{ij} \nu_j (\partial_x u) \nu_i \right) + (\partial_x u) a^{lm} \nu_m (\partial_x u) a^{ij} \rho a^{ij} \nu_j \nu_i
\]

\[
= \frac{\rho a^{lm} \rho_{xm}}{t} \left( -\frac{1}{2} (\partial_x u)^2 a^{ij} \nu_j \nu_i \right) + a^{lm} \nu_m (\partial_x u)^2 \rho \nabla_g \rho \cdot \nu.
\]

Finally we get

\[
-t\nu \cdot P = \frac{1}{2} \rho a^{lm} \rho_{xm} \nu_l (\partial_x u)^2 a^{ij} \nu_j \nu_i - a^{lm} \nu_m (\partial_x u)^2 \rho a^{ij} \rho_{xj} \nu_i
\]

\[
= -\frac{1}{2} \rho a^{lm} \nu_l \nu_m (\partial_x u)^2 \rho a^{ij} \rho_{xj} \nu_i = -\frac{1}{2} a^{lm} \nu_l \nu_m (\partial_x u)^2 \rho \nabla_g \rho \cdot \nu.
\]

However, for \( x \in \partial \Omega \), given that \( x_0 = 0 \in \partial \Omega \), we have

\[
\nabla_g \rho(x) = \vec{T} + \mathcal{O}(x), \quad \nu(x) = \nu(0) + \mathcal{O}(x),
\]

where \( \vec{T} \) is a unit vector tangent to \( \partial \Omega \) at \( x_0 = 0 \). Consequently, as \( \nu(0) \cdot \vec{T} = 0 \),

\[
\nabla_g \rho(x) \cdot \nu(x) = \mathcal{O}(|x|^2) = \mathcal{O}(\rho^2), \quad \text{for } x \in \partial \Omega.
\]

So the second to the last term in \((2.29)\) is bounded (using lemma 2.9) by

\[
\sup_{x \in K^T_S} \rho^2 \times \int_{(-1,0) \times \partial \Omega} \left( \frac{\partial u}{\partial v} \right)^2 d\sigma(x) dt \leq C|S|^2 E(u).
\]

For the first term

\[
\int_{D(T)} u_t u dx \leq \left( \int_{D(T)} u^0 dx \right)^2 \left( \int_{D(T)} u^2 dx \right)^\frac{1}{2} \left( \int_{D(T)} 1 dx \right)^\frac{1}{2}
\]

\[
\leq C|T| (E(u,D(T)))^\frac{1}{2} (E(u,D(T)))^\frac{1}{2},
\]
and

$$|\int_{D(T)} TQdx| \leq \int_{D(T)} \left| \frac{T}{2} (u_l^2 + a^{ij}(x)u_{x_i}u_{x_j} + \frac{u^6}{3}) + \frac{T u_t \rho a^{ij} \rho_{x_j} u_{x_i}}{t} \right| dx$$

$$\leq C|T|(E(u, D(T))) + |T| \int_{D(T)} |u_t a^{ij} \rho_{x_j} u_{x_i}| dx$$

$$\leq C|T|(E(u, D(T))) + |T| \int_{D(T)} \frac{u^2}{2} + \frac{a^{lm} \rho_{x_i} \rho_{x_m} a^{ij} u_{x_i} u_{x_j}}{2} dx$$

$$\leq C_1|T|(E(u, D(T))) \to 0,$$

as $T \to 0$.

So if $T \to 0$,

$$\int_{D(T)} (TQ + u_t u)dx \to 0.$$

Let $T \to 0$ in the identity (2.29), we conclude that

$$- \int_{D(S)} (SQ + u_t u)dx + \int_{M_S^0} \frac{tQ + uu_t - tP \cdot \nabla \rho}{\sqrt{1 + |\nabla \rho|^2}} d\sigma$$

$$+ \int_{(S,0 \times \partial \Omega) \cap K_s^0} \nu_{\partial \Omega} \cdot (-tP) d\sigma = - \int_{K_s^0} R d\tau dx.$$ (2.30)

Let

$$I = - \int_{D(S)} (SQ + u_t u)dx,$$

$$II = \int_{M_S^0} \frac{tQ + uu_t - tP \cdot \nabla \rho}{\sqrt{1 + |\nabla \rho|^2}} d\sigma.$$

On the surface $\rho = -t$, we have

$$tQ + uu_t - tP \cdot \nabla \rho$$

$$= \frac{t}{2} (u_l^2 + a^{ij}(x)u_{x_i}u_{x_j} + \frac{u^6}{3}) + u_t \rho a^{ij} \rho_{x_j} u_{x_i} + uu_t$$

$$- \rho a^{lm} \rho_{x_i} \rho_{x_m} \left[ \frac{1}{2} (u_l^2 - a^{ij}(x)u_{x_i}u_{x_j} - \frac{u^6}{3}) \right] - a^{ij} u_{x_i} u_{x_i} (u + uu_t + \rho a^{lm} \rho_{x_m} u_{x_l})$$

$$= -\rho u_t^2 + 2u_t \rho a^{ij} \rho_{x_j} u_{x_i} - \rho a^{ij} \rho_{x_j} u_{x_i} \rho_{x_i} u_{x_i} - uu_t$$

$$= -\rho \left( \frac{\rho a^{ij} \rho_{x_j} u_{x_i}}{\rho} - u_t \right)^2 - u \left( \frac{\rho a^{ij} \rho_{x_j} u_{x_i}}{\rho} - u_t \right).$$

If we parameterize $M_S^0$ by

$$\Omega \ni y \to (-\rho(y), y), \quad \rho \leq |S|,$$

and let $v(y) = u(-\rho(y), y)$, then $d\sigma = \sqrt{1 + |\nabla \rho|^2} dy$, and

$$\nabla v = u_t (-\nabla \rho) + \nabla u,$$

furthermore

$$a^{ij} \rho_{y_j} v_{y_i} = -a^{ij} \rho_{y_j} \rho_{y_i} u_t + a^{ij} \rho_{x_j} u_{x_i}$$

$$= -u_t + a^{ij} \rho_{x_j} u_{x_i}.$$
so
\[
II = \int_{M_S^{t}} \frac{tQ + uu_t - tP : \nabla \rho}{\sqrt{1 + |\nabla \rho|^2}} \rho \sigma d\sigma
\]
\[
= - \int_{M_S^{t}} \frac{1}{\sqrt{1 + |\nabla \rho|^2}} \left[ \rho \left( \frac{\rho a^{ij} \rho_j \rho_{x_i} u_{x_i} - u_t}{\rho} \right)^2 + u \left( \frac{\rho a^{ij} \rho_j \rho_{x_i} u_{x_i}}{\rho} - u_t \right) \right] d\sigma
\]
\[
= - \int_{\{y \in \Omega : \rho \leq |S|\}} \rho (a^{ij} \rho_j \rho_{y_i} v_{y_i})^2 + v (a^{ij} \rho_j \rho_{y_i} v_{y_i}) dy
\]
\[
= - \int_{\{y \in \Omega : \rho \leq |S|\}} \frac{1}{\rho} \left| \rho a^{ij} \rho_j \rho_{y_i} v_{y_i} + v \right|^2 dy + \int_{\{y \in \Omega : \rho \leq |S|\}} \frac{v^2}{\rho} + \frac{\rho v a^{ij} \rho_j \rho_{y_i} v_{y_i}}{\rho} dy.
\]
For
\[
\int_{\{y \in \Omega : \rho \leq |S|\}} \frac{\rho v a^{ij} \rho_j \rho_{y_i} v_{y_i}}{\rho} dy
\]
\[
= \int_{\{y \in \Omega : \rho \leq |S|\}} a^{ij} \rho_j \rho_{y_i} \left[ \frac{1}{2} v^2 \right] dy
\]
\[
= \int_{\{y \in \Omega : \rho \leq |S|\}} \frac{\partial}{\partial y_i} (a^{ij} \rho_j \rho_{y_i} \frac{1}{2} v^2) - \frac{1}{2} v^2 \frac{\partial}{\partial y_i} (a^{ij} \rho_j \rho_{y_i}) dy
\]
\[
= \int_{\rho = |S|} \frac{a^{ij} \rho_j \rho_{y_i} 1}{|\nabla \rho|} \frac{1}{2} v^2 ds - \int_{\{y \in \Omega : \rho \leq |S|\}} \frac{1}{2} v^2 \frac{\partial}{\partial y_i} (\frac{\rho a^{ij} \rho_j \rho_{y_i}}{\rho}) dy
\]
\[
= \int_{\rho = |S|} \frac{u^2}{2 |\nabla \rho|} ds - \int_{\{y \in \Omega : \rho \leq |S|\}} \frac{v^2}{2 \rho} \frac{\partial}{\partial y_i} (\rho a^{ij} \rho_j \rho_{y_i}) - 1] dy,
\]
where $\sigma$ is the induced Lebesgue measure on the surface $\rho = -t$. And so we have
\[
II = - \int_{\{y \in \Omega : \rho \leq |S|\}} \frac{1}{\rho} \left| \rho a^{ij} \rho_j \rho_{y_i} v_{y_i} + v \right|^2 dy + \int_{\{y \in \Omega : \rho \leq |S|\}} \frac{v^2}{\rho} + \frac{\rho v a^{ij} \rho_j \rho_{y_i} v_{y_i}}{\rho} dy
\]
\[
= - \int_{\{y \in \Omega : \rho \leq |S|\}} \frac{1}{\rho} \left| \rho a^{ij} \rho_j \rho_{y_i} v_{y_i} + v \right|^2 dy + \int_{\rho = |S|} \frac{u^2}{2 |\nabla \rho|} ds
\]
\[
+ \int_{\{y \in \Omega : \rho \leq |S|\}} \frac{v^2}{\rho} \frac{3}{2} - \frac{1}{2 \partial y_i} (\rho a^{ij} \rho_j \rho_{y_i}) \right] dy \tag{2.31}
\]
\[
= \int_{M_S^{t}} \frac{t (a^{ij} \rho_j \rho_{x_i} u_{x_i} - u_t + \frac{u}{2})^2}{\sqrt{1 + |\nabla \rho|^2}} d\sigma + \int_{\rho = |S|} \frac{u^2}{2 |\nabla \rho|} ds
\]
\[
+ \int_{\{y \in \Omega : \rho \leq |S|\}} \frac{v^2}{\rho} \frac{3}{2} - \frac{1}{2 \partial y_i} (\rho a^{ij} \rho_j \rho_{y_i}) \right] dy.
\]
In $D(S) = \{ x \in \Omega : \rho(x) < -S \}$, $t = S$, and we have
\[
SQ + uu_t = \frac{S}{2} (u_t^2 + a^{ij} u_{x_i} u_{x_j} + \frac{u^6}{3}) + u_t (u + \rho a^{ij} \rho_j \rho_{x_i} u_{x_i}).
\]
For the second in the right side, using Cauchy-Schwarz inequality in lemma 3.2 we have
\[
u_t (u + \rho a^{ij} \rho_j \rho_{x_i} u_{x_i}) \leq |S| \left[ u_t^2 + u (\rho a^{ij} \rho_j \rho_{x_i} u_{x_i})^2 \right]
\]
\[
\leq |S| \left[ u_t^2 + u (\rho a^{ij} \rho_j \rho_{x_i} u_{x_i})^2 \right] \leq |S| \left[ u_t^2 + \frac{u (\rho a^{ij} \rho_j \rho_{x_i} u_{x_i})^2}{\rho^2} \right] \leq |S| \left[ u_t^2 + \frac{u (\rho a^{ij} \rho_j \rho_{x_i} u_{x_i})^2}{\rho^2} + \frac{2u (\rho a^{ij} \rho_j \rho_{x_i} u_{x_i})}{\rho^2} \right].
\]
As $S < 0$, we get
\[
SQ + uu_t \leq \frac{Su_6}{6} - \frac{Su^2}{2\rho^2} - \frac{Supa^ij\rho x_i u_{x_i}}{\rho^2},
\]
so
\[
I = - \int_{D(S)} (SQ + u_t)dx \\
\geq -S \int_{D(S)} \frac{u_6}{6} dx + S \left( \frac{1}{2} \int_{D(S)} \frac{u^2}{\rho^2} dx + \int_{D(S)} \frac{upa^ij\rho x_i u_{x_i}}{\rho^2} dx \right) \quad (2.32)
\]
\[
= |S| \int_{D(S)} \frac{u_6}{6} dx + S \left( \frac{1}{2} \int_{D(S)} \frac{u^2}{\rho^2} dx + \int_{D(S)} \frac{upa^ij\rho x_i u_{x_i}}{\rho^2} dx \right).
\]
Similarly we compute
\[
\int_{D(S)} \frac{upa^ij\rho x_i u_{x_i}}{\rho^2} dx \\
= \int_{D(S)} \frac{a^{ij} \rho x_j \rho x_i}{\rho} dx \\
= \int_{D(S)} \frac{\rho^2}{2} \partial_{x_i} \left( \frac{a^{ij} \rho x_j \rho x_i}{\rho} \right) dx \\
= \int_{rho=|S|} \frac{u^2 a^{ij} \rho x_j \rho x_i}{2|\nabla \rho|} ds - \int_{rho=|S|} \frac{u^2}{2} \partial_{x_i} \left( \frac{\rho a^{ij} \rho x_j}{\rho^2} \right) dx \\
= \int_{rho=|S|} \frac{u^2}{2|\nabla \rho|} ds - \int_{rho=|S|} \frac{u^2}{2} \left[ \frac{1}{\rho^2} \partial_{x_i} \left( \rho a^{ij} \rho x_j \right) - \rho a^{ij} \rho x_j + 2\rho x_i \right] dx \\
= \int_{rho=|S|} \frac{u^2}{2|\nabla \rho|} ds - \int_{rho=|S|} \frac{u^2}{2} \left[ \partial_{x_i} \left( \rho a^{ij} \rho x_j \right) - 2 \right] dx. \quad (2.33)
\]
Combining (2.32) and (2.33), we quickly get
\[
I \geq |S| \int_{D(S)} \frac{u_6}{6} dx - \int_{rho=|S|} \frac{u^2}{2|\nabla \rho|} ds + S \int_{D(S)} \frac{u^2}{2\rho^2} \left[ 3 - \rho a^{ij} \rho x_j \right] dx. \quad (2.34)
\]
By continuity, the sectional curvature is uniformly bounded near $x_0$. Then following from (2.30), (2.31), (2.34), and lemma 2.8, using Cauchy-Schwarz inequality in lemma 3.2,
we have

\[ |S| \int_{D(S)} \frac{u^6}{6} \, dx \leq I + \int_{\rho = |S|} \frac{u^2}{2|\nabla \rho|} \, ds - S \int_{D(S)} \frac{u^2}{2\rho^2} \left[ 3 - \frac{\partial}{\partial x_i} (\rho a^{ij} \rho_{x_j}) \right] \, dx \]

\[ = -II - \int_{K^0_S} R dtdx + \int_{(S, 0) \cap K^0_S} v_{0\Omega} \cdot (tP) \, d\sigma \]

\[ + \int_{\rho = |S|} \frac{u^2}{2|\nabla \rho|} \, ds - S \int_{D(S)} \frac{u^2}{2\rho^2} \left[ 3 - \frac{\partial}{\partial x_i} (\rho a^{ij} \rho_{x_j}) \right] \, dx \]

\[ = \int_{M^0_S} \frac{|t| (a^{ij} \rho_{x_i} u_{x_j} - u_t + \frac{u}{\rho})^2}{\sqrt{1 + |\nabla \rho|^2}} \, d\sigma - \int_{\{y \in \Omega; \rho \leq |S|\}} \frac{v^2}{\rho} \left[ \frac{3}{2} - \frac{1}{2} \frac{\partial}{\partial y_i} (\rho a^{ij} \rho_{y_j}) \right] \, dy \]

\[ - S \int_{D(S)} \frac{u^2}{2\rho^2} \left[ 3 - \frac{\partial}{\partial x_i} (\rho a^{ij} \rho_{x_j}) \right] \, dx + \int_{(S, 0) \cap K^0_S} v_{0\Omega} \cdot (tP) \, d\sigma - \int_{K^0_S} R dtdx \]

\[ \leq C |S| \int_{M^0_S} (u_t - \rho a^{ij} \rho_{x_i} u_{x_j})^2 \, d\sigma + C \int_{M^0_S} \frac{|t|^2}{\rho} \, d\sigma + C \int_{\{y \in \Omega; \rho \leq |S|\}} \frac{v^2}{\rho} \, dy \]

\[ + C |S| \int_{D(S)} \frac{u^2}{2\rho^2} \rho dtdx + C \int_{K^0_S} \rho (u_t^2 + a^{ij} u_{x_i} u_{x_j} + u^6) dtdx + C |S|^2 E(u) - \frac{1}{3} \int_{K^0_S} u^6 dtdx \]

\[ \leq C_1 |S| \int_{M^0_S} (u_t^2 + a^{ij} u_{x_i} u_{x_j}) \, d\sigma + C \left( \int_{M^0_S} |t|^{-\frac{3}{2}} \, d\sigma \right) \left( \int_{M^0_S} u^6 d\sigma \right)^{\frac{1}{2}} \]

\[ + C_2 \left( \int_{M^0_S} 1 \, d\sigma \right)^{\frac{1}{2}} \left( \int_{M^0_S} u^6 d\sigma \right)^{\frac{1}{2}} + C_3 |S|^2 \left( \int_{D(S)} u^6 d\sigma \right)^{\frac{1}{2}} \]

\[ + C \int_{K^0_S} \rho (u_t^2 + a^{ij} u_{x_i} u_{x_j} + u^6) dtdx + C |S|^2 E(u) \]

\[ \leq C_4 |S| Flu(x(u, M^0_S)) + C_5 (|S| + |S|^2) \left( Flu(x(u, M^0_S)) \right)^{\frac{1}{4}} + C_6 |S|^2 (E(u, D(S)))^{\frac{1}{4}} \]

\[ + C \int_{K^0_S} \rho (u_t^2 + a^{ij} u_{x_i} u_{x_j} + u^6) dtdx + C |S|^2 E(u) \]

\[ \leq C_4 |S| Flu(x(u, M^0_S)) + C_5 (|S| + |S|^2) \left( Flu(x(u, M^0_S)) \right)^{\frac{1}{4}} + C_6 |S|^2 E_0^{\frac{1}{2}} \]

\[ + C \int_{K^0_S} \rho (u_t^2 + a^{ij} u_{x_i} u_{x_j} + u^6) dtdx + C |S|^2 E(u). \]

(2.35)

We put some specific computations in the last part of section 3, such as the term

\[ \int_{M^0_S} |t|^{-\frac{3}{2}} \, d\sigma. \]
Then combing with (2.17) and (2.35) we have
\[
\int_{D(S)} \frac{u^6}{6} \, dx \leq \text{CFlux}(u, M_0^6) + C(1 + |S|)(\text{Flux}(u, M_0^6))^\frac{1}{2}
\]
\[+ C_6|S|E_0^\frac{1}{3} + \frac{C_6}{|S|} \int_{K_0^2} \rho(u_0^2 + a^{ij}u_{x_i}u_{x_j} + u^6) \, dt \, dx + C|S|E(u)\]
\[\to 0 \quad \text{as } S \to 0.
\]
which completes the proof of lemma 2.2.

3 Appendix

In this section we give some definition and proofs about Riemannian Geometry.

Definition 3.1. Distance function
Suppose \((M, g)\) is a Riemannian manifold. For \(x, y \in M\), we define a function \(d : M \times M \to [0, \infty)\):
\[
d(x, y) = \inf \{ L(\gamma) \mid \gamma \text{ is a piecewise smooth curve joining } x \text{ and } y \}\). (3.1)
If \(M\) is connected, the distance \(d(x, y)\) is well defined, since there are piecewise smooth curves joining \(x\) and \(y\). In this case, we can see the function \(d\) satisfies the three properties of distance.

Lemma 3.2. Cauchy-Schwarz inequality
If \(A\) is a symmetric, nonnegative \(n \times n\) matrix, then for \(x, y \in \mathbb{R}^n\) we have
\[
|\sum_{i,j=1}^n a^{ij}x_iy_j| \leq \left( \sum_{i,j=1}^n a^{ij}x_i^2 \right)^{\frac{1}{2}} \left( \sum_{i,j=1}^n a^{ij}y_i^2 \right)^{\frac{1}{2}}. \quad (3.2)
\]

Lemma 3.3. Suppose \(M\) is a Riemannian manifold, and \(O \in M\), let
\[
\rho : M \to [0, \infty) \quad \rho(x) = d(x, O),
\]
then \(\rho^2 \in C^\infty(M)\) in a neighborhood of \(O\), denoted by \(U_O\). And in \(U_O\) we have
\[
|\nabla g\rho|^2_g = g^{ij}\rho_x \rho_{x_j} = 1, \quad D^2\rho^2 > 0,
\]
where \((g^{ij}) = (g_{ij})^{-1}\).

Proof of lemma 2.7. First we compute
\[
X \left( \frac{1}{2} |\nabla g f|_g^2 \right) = \frac{1}{2} X < \nabla g f, \nabla g f >_g
\]
\[= \nabla_X \nabla g f, \nabla g f >_g
\]
\[= \nabla \nabla g f, \nabla g f >_g + < [X, \nabla g f], \nabla g f >_g
\]
\[= \nabla \nabla g f, \nabla g f >_g + [X, \nabla g f] f
\]
\[= \nabla \nabla g f, \nabla g f >_g + X \nabla g f(f) - \nabla g f X(f)
\]
\[= \nabla \nabla g f, \nabla g f >_g + X < \nabla g f, \nabla g f >_g \nabla g f < X, \nabla g f >_g
\]
\[= \nabla \nabla g f, \nabla g f >_g + X < \nabla g f, \nabla g f >_g
\]
\[- < \nabla \nabla g f, \nabla g f >_g - < X, \nabla \nabla g f \nabla g f >_g.
\]
So we get

\[ < X, \nabla g f \nabla g f >_g = X \left( \frac{1}{2} |\nabla g f|_g^2 \right) \]

Hence one can finish the proof as follows

\[ < \nabla g f, \nabla g (X(f)) >_g = \nabla g f < X, \nabla g f >_g \]
\[ = < \nabla g f X, \nabla g f >_g + < X, \nabla g f \nabla g f >_g \]
\[ = < \nabla g f X, \nabla g f >_g + X \left( \frac{1}{2} |\nabla g f|_g^2 \right) \]  

**Proof of lemma 2.8.** To do this we need some computation and an additional lemma, that is lemma 3.4 as below. Let \( G \) denote \( \text{det}(g_{ij}) \), first we compute

\[
\frac{\partial}{\partial x_i} (\sqrt{G} g^{ij} \frac{\partial}{\partial x_j} \left( \frac{1}{2} \rho^2 \right)) = \frac{\partial}{\partial x_i} (\sqrt{G} \rho g^{ij} \rho x_j) = \frac{\partial}{\partial x_i} (\sqrt{G}) \rho \nabla g \rho + \sqrt{G} \frac{\partial}{\partial x_i} (\rho g^{ij} \rho x_j).
\]

So

\[
\frac{\partial}{\partial x_i} (\rho g^{ij} \rho x_j) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x_i} (\sqrt{G} g^{ij} \frac{\partial}{\partial x_j} \left( \frac{1}{2} \rho^2 \right)) - \frac{1}{2G} \frac{\partial}{\partial x_i} g^{lm} \frac{\partial}{\partial x_i} \rho \nabla g \rho
\]
\[
= \Delta_g \left( \frac{1}{2} \rho^2 \right) - \frac{1}{2G} g^{lm} \frac{\partial}{\partial x_i} \rho \nabla g \rho \]
\[ (3.3) \]

where \( \Delta_g \) is the Laplace operator on the Riemannian manifold \((M, g)\).

**Lemma 3.4.** Suppose \( M \) is a connected Riemannian manifold, \( x \in M \), \( \rho \) is the distance function from some point to \( x \). If the sectional curvature \( \kappa \) of \( M \) satisfies

\[-a^2 \leq \kappa \leq a^2,\]

where \( a \) is a positive real number. Then in \( M \setminus \{x\} \), we have

\[
1 + 2a \rho \cot \rho \leq \Delta_g \left( \frac{1}{2} \rho^2 \right) \leq 1 + 2a \rho \coth \rho,
\]
\[ (3.4) \]

\[
a \rho \cot \rho \leq D^2 \left( \frac{1}{2} \rho^2 \right) \leq a \rho \coth \rho.
\]
\[ (3.5) \]

It is a classical comparison theorem about the Hessian and Laplace of the distance function, and one can find the proof in many books about Riemannian Geometry such as Cheeger and Ebin [2], Greene and Wu [3].

Combining (3.3) and (3.4), we quickly get the identity (2.19) in lemma 2.8. And the identity (2.20) can be easily obtained from (3.5).

Now we introduce the geodesic polar coordinates. In this coordinate system, the metric can described as follows

\[
ds^2 = d\rho^2 + \rho^2 g_{11} d\theta^2 + 2\rho^2 g_{12} d\theta d\varphi + \rho^2 g_{22} d\varphi^2,
\]

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and then the estimates of the integrate in the geodesic ball or on the mantle can be easily get. For example, we estimate $\int_{D(S)} \frac{1}{\rho^2} dx$ and $\int_{M_0} |t|^{-\frac{3}{2}} d\sigma$ in identity (2.35):

$$\int_{D(S)} \frac{1}{\rho^2} dx = \int_0^{|S|} \int_0^{2\pi} \int_{-\pi}^{\pi} \frac{\rho^2 \sqrt{G}}{\rho^\frac{3}{2}} d\varphi d\theta d\rho$$

$$\leq C \int_0^{S} \rho^\frac{1}{2} d\rho$$

$$\leq C |S|^\frac{3}{2},$$

$$\int_{M_0} |t|^{-\frac{3}{2}} d\sigma = \int_0^{|S|} |t|^{-\frac{3}{2}} dt \int_0^{2\pi} \int_{-\pi}^{\pi} |t|^2 \sqrt{G} d\varphi d\theta$$

$$\leq C \int_0^{|S|} \sqrt{t} dt$$

$$\leq C |S|^\frac{3}{2}.$$

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