Edge effects in billiards: stickiness and long-lived states

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Abstract. The effect of rounded edges in the rectangular billiard is analyzed via the escape times statistic. Such realistic and sometimes intrinsic edges are shown to generate autosimilar structures and power law decays for the escape times statistic. As the rounding effects increase, stickiness and long lived trajectories are observed. Rounding edges of around 0.01% → 1% from the whole billiard size generate the sticky motion.

1. Introduction
The boundary of realistic physical devices always present irregularities like roughness, softness, defects, edges and border effects. Such sometimes intrinsic irregularities affect the dynamics of particles which collide with the boundary. From the theoretical point of view it is very difficult to describe, in general, the dynamics of colliding particles with irregular boundaries. Therefore in recent years more and more attention has been given to the description of particles confined inside boundaries (or billiards) which present some specific edges, softness etc. To mention some examples we have the edge roughness in quantum dots [1], unusual boundary conditions in two-dimensional billiards [2, 3], effects of soft walls [4] and edge collisions [5] of interacting particles in a 1D billiard, edge diffractions and the corresponding semiclassical quantization [6, 7], rounding edge [8] and edge corrections [9] in a resonator, deformation of dielectric cavities [10], among others. Although such irregularities are sometimes very small, they can change significantly the dynamics of quantum and classical systems.

From the nonlinear dynamics point of view, it has been shown that soft walls and soft edges do not destroy trajectories found in the hard-wall limit [11] and may induce the appearance of regular islands in phase space [4, 12, 13, 14]. Such regular islands inside the chaotic sea induce a “sticky” (or trapped) motion, which is a common phenomenon in conservative systems [15]. Such motion usually arises from broken Kolmogorov-Arnold-Moser (KAM) curves and generates a rich dynamics in quasi-integrable systems [15]. Stickiness may generate an anomalous behavior of the transport [15, 16, 17] and is therefore very important to be characterized in realistic physical systems. In the context of soft walls, the sticky motion has been observed theoretically and experimentally in the atom-optic billiard [18, 19] and has shown to affect the quantum conductance in the soft wall microwave billiard [20].

In this work we show that tiny rounding edges are able to change the escape times statistics, generating long lived trajectories and sticky motion. As a model we use the open rectangular billiard, but results should be valid in general for open billiards with edges.
2. Escape times statistics and the model

The escape times statistic is defined by [21]

\[ Q(\tau) = \lim_{N \to \infty} \frac{N_\tau}{N}, \]

(1)

where \( N \) is the total number of trajectories which escape the billiard and \( N_\tau \) is the number of trajectories which escape the billiard after the time \( \tau \). For systems with stickiness the escape times statistic decays as a power law [15] \( Q(\tau) \propto \tau^{-\gamma_{\text{esc}}} \), where \( \gamma_{\text{esc}} > 1 \) is the scaling exponent. For hyperbolic chaotic systems and long times the escape times statistic decays exponentially.

It is known that the diffusion exponent \( \mu \) from the MSD of the position \( \langle x^2 \rangle \sim t^\mu \), is related to \( \gamma_{\text{esc}} \) via \( \mu = 3 - \gamma_{\text{esc}} \) [16, 17].

The model used here is the open rectangular billiard with rounded edges (See Fig. 1). We consider a point-like collision of particles with the rounded edge. The edge is modelled by a quarter of a circle. The escape times is studied as a function of the initial incoming angles \( \theta_0 \) and the ratio \( R/L \). \( L \) is the billiard height, \( D \) its width and \( R \) is the radius of the semicircle which models the rounded edge. In all simulations \( L = 4 \), \( D = 10 \) and \( R \) changes. The escape point with constant aperture \( a = 1.0 \times 10^{-1} \) is located at \( D/2 \).

Without rounding edges \( (R/L = 0) \) the rectangular billiard is integrable and has zero Lyapunov exponents. It has marginally unstable periodic orbits, which are orbits bouncing perpendicularly between the parallel walls. They are obtained from the initial conditions \( \theta = 0, \pi/2 \). As we will see, for small values of the ratio \( R/L \) the rounded edges generate the sticky motion. Since marginally unstable periodic orbits in billiards are known [25, 26] to generate exponent \( \gamma_{\text{esc}} = 2 \) for the escape times, we also expect to see \( \gamma_{\text{esc}} \rightarrow 2 \) as \( R/L \) increases. For \( R/L = 1/2 \) our model is the Bunimovich stadium [22], which is known to be chaotic having an almost exponential escape time decay [23].

We start discussing the quantity \( Q(\tau) \) for different values of the ratio \( R/L \), where \( L \) is kept fixed. At times \( t = 0 \) particles always start at the escape point, towards the inner region of the billiard with an initial angle \( \theta_0 \). We use \( 10^3 \) initial conditions uniformly distributed in the interval \( 0.1 \leq \theta_0 \leq 1.5 \). For each initial condition the simulation is done until the particle escapes through the hole. The dynamics for \( -\pi/2 \leq \theta_0 \leq 0.0 \) is symmetric. Results for \( Q(\tau) \) are shown in the log-log plot of Fig. 2 for different values of \( R/L = 0.0, 1.0 \times 10^{-4}, 1.0 \times 10^{-3}, 1.0 \times 10^{-2}, 1.0 \times 10^{-1}, 0.5 \). For comparison the straight dashed line shows the power law decay with \( \gamma_{\text{esc}} = 2 \) (Straight lines...
in this plot mean a power law). First observation is that for $R/L = 0.0$ (no edge effects) $Q(\tau)$ did not has a power law. It should [24] obey a quasi-algebraic decay for short and intermediate times and a power law decay for large times. In fact, we did some simulations (not shown) using only initial conditions very close to the marginally unstable periodic orbit $\theta_0 \sim \pi/2$. We found in this case that $Q(\tau)$ has a power law tail for very long times with $\gamma_{esc} \sim 1.0$. This shows that marginally unstable periodic orbits without the focusing component from the edge generates the exponent $\gamma_{esc} \sim 1.0$. This agrees with results obtained in [25] for some specific initial conditions.

Increasing the rounding effects to $R/L = 1.0 \times 10^{-4}$, the qualitative behaviour of $Q(\tau)$ starts to change (when compared to the case $R/L = 0.0$) for times $\tau \gtrsim 1.0 \times 10^3$, i.e. for those trajectories which stay longer inside the billiard. This means that the very few trajectories which collide with the rounded edges tend to stay longer inside the billiard and also change the qualitative behavior of $Q(\tau)$. Since the probability to collide with the edges increases in time, just very long-lived trajectories essentially affect $Q(\tau)$. For rounded edges of size $R/L = 1.0 \times 10^{-3}$ and $R/L = 1.0 \times 10^{-2}$ we observe a power law decay for times $\tau \gtrsim 2 \times 10^3$. In all cases we found that the long-lived trajectories can be fitted by an escape exponent of $\gamma_{esc} \sim 2.0$, as can be seen by comparing these decays with the straight dashed line for $\gamma_{esc} = 2.0$ (see Fig. 2). For $R/L = 1.0 \times 10^{-1}$ the power law behavior is not relevant anymore and the decay is better fitted by an exponential. For $R/L = 0.5$ no power law decay is obtained (for this time and hole aperture) since this is the stadium billiard with totally chaotic behavior. This agrees with the almost exponential escape times decay observed [23] for the Bunimovich stadium when the aperture is small enough.

The escape times from the long-lived trajectories present relevant characteristic of sticky motion for $R/L = 10^{-4}, 10^{-3}, 10^{-2}$. In other words, sticky motion and long living trajectories start to occur for very small rounding edges. Therefore edges of around $0.01\% \rightarrow 1\%$ of the whole billiard size are sufficient for this purpose. Visually such rounded edges are almost negligible.

3. Self-similar escape times

Figure 3 shows the behavior of the escape times as a function of the initial angles $\theta_0$ and for values (a) $R/L = 0.0$ and (b)-(c) $R/L = 1.0 \times 10^{-3}$. For $R/L = 0.0$ we observe in Fig. 3(a)
that the escape times alternate between minima (with values around $\tau \sim 10$) and maxima (with values around $\tau \sim 10^3$). For angles $\theta_0 \rightarrow 1.5$ (almost horizontal trajectories) the escape times increase since trajectories need more time to reach the escape point which lies in the horizontal wall. The minima in the escape times are related to the shortest trajectories. Some of these

![Figure 3. Semi-log plot of escape times as a function of initial angle $\theta_0$ for (a) $R/L = 0.0$, (b) $R/L = 1.0 \times 10^{-3}$ (c) magnification of (b).](image-url)

trajectories are shown schematically in Fig. 1, where $A_n$ refers to the trajectory which collides once with the rounded edge and $n$ times with the horizontal boundary. Besides trajectory $P$ ($\theta_0 \sim 0.0$), which collides directly with the wall in front of the escape point and leaves the billiard, the shortest trajectory is labelled by letter $E$ (Fig. 1) and has $\theta_0 = \arctan (L/D) \sim 0.89$. This agrees with the smaller minimum shown in Fig. 3(a). The width of the minima is related to the aperture $a \sim 0.4$. Close to the border of these widths, larger escape times are observed and are related to trajectories which jump over the escape point, staying longer inside the billiard. As the rounded edge increases to $R/L = 10^{-3}$, the escape times increase very much, as shown in Fig. 3(b). Trajectories with escape times close to $2 \times 10^4$ are observed. It means that edge effects strongly increase the time that trajectories will stay inside the billiard. More interesting is that for small edges the escape times show a self-similar structure as a function of $\theta_0$, as can be seen in the magnification shown in Fig. 3(c), where the escape times are plotted for initial angles in the small interval $[0.8956, 0.8966]$. This means that no matter how small the interval of initial angles, small rounded edges always induce long-lived trajectories and stickiness.

4. Summary

Rounded edges effects in a rectangular billiard are shown to strongly influence the dynamics. We show that rounded edges of about 0.01% → 1% are sufficient to generate the sticky motion and long lived trajectories inside the billiard. The sticky motion is characterized here by the escape times statistics decay. Self-similar escape times are also observed, showing that no matter how small is the interval of initial angles, tiny rounded edges always induce long-lived trajectories and stickiness.
Very recently [27] it was shown that edge effects are essential for the directional emission in optical mushroom-shaped billiards, where marginally unstable periodic orbits are generated close to the edge. From results of the present work we suggest that, due to stickiness and long-lived trajectories, mushroom-shaped billiards with rounded edges could be more efficient to generate the directional emission. In addition, in the periodic orbit quantization of the stadium billiard it was shown [7] that important contributions are due to edge orbits. Such contributions should also increase with rounded edges. Stickiness are also expected to strongly influence the transport of particles along billiard systems or periodic potentials in classical [28, 29, 30] and quantum mechanics [31, 32, 33].

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