Indirect Influences, Links Ranking, and Deconstruction of Networks

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Abstract

The PWP map was introduced by the second author as a tool for ranking nodes in networks. In this work we extend this technique so that it can be used to rank links as well. Applying the Girvan-Newman algorithm a ranking method on links induces a deconstruction method for networks, therefore we obtain new methods for finding clustering and core-periphery structures on networks.

1 Introduction

Three problems stand out for their centrality in the theory of complex networks, namely, hierarchization, clustering, and core-periphery. In hierarchization the aim is to find a ranking on the nodes of a network reflecting the importance of each node. Several methods have been proposed for such rankings, among them degree centrality, eigenvalue centrality, closeness centrality, betweenness centrality [23], Katz index [18], MICMAC of Godet [16], PageRank of Google [3, 19], Heat Kernel of Chung [7, 8], and Communicability of Estrada and Hatano [14]. We are going to use in this work the PWP method [10] which we review in Section 2 for comparison with other methods see [10, 12], and for applications and extensions see [4, 5, 11, 12]. Clustering consists in finding a suitable partition on the nodes of a network such that nodes within blocks are highly connected, and nodes in different blocks are weakly connected. The reviews [15, 21, 25] offer a fairly comprehensive picture of the many methods that have been proposed to attack this problem. Maximization of Newman’s modularity function and its extensions [2, 13, 17, 22] is a particularly interesting approach since it proposes a mathematical principle instead of an algorithm. Roughly speaking the core-periphery finding problem [24, 26, 27] consists in peeling a network as if it were an onion, discovering the rings out which it is built. The inner rings form the core of the network, the outer rings form its periphery.

In this work we argue that, within a certain framework, the three problems have a common root: a hierarchization method induces both a clustering finding method and a core-periphery finding method. Indeed, we provide three alternative methods for
Figure 1: Left: Weighted network \( W \). Right: Weighted network \( Y \).

reducing clustering and core-periphery finding to hierarchization: the first one via the dual network; the second one via the barycentric division network; the third one regards a link as a bridge, thus its importance is proportional to its functionality and to the importance of the lands it joins.

We work with double weighted directed networks, i.e. weights defined both on links and on nodes, formally introduced in Section 2 where we generalize the PWP method so that it can be applied to double weighted networks. In Section 3 we recall how a ranking on links induces, following the Girvan-Newman algorithm, a network deconstruction method. In Section 4 we introduce the dual construction for double weighted networks and use it to rank links, obtaining the corresponding clustering and core-periphery finding methods. In Section 5 we introduce the barycentric division construction for double weighted networks and consider the corresponding clustering and core-periphery finding methods. In Section 6 we introduce the bridge approach to link ranking and its corresponding clustering and core-periphery finding methods. In Section 7 we illustrate the notions introduced along the paper by applying them to a highly symmetric intuitively graspable network, and also to a more sophisticated network.

2 PWP on Double Weighted Networks

Let digraph be the category of directed networks, wdigraph the category of directed networks with weighted links, and wwdigraph the category of directed networks with weighted nodes and weighted links, i.e. objects in wwdigraph are tuples \((V, E, s, t, w, f)\) consisting of: 
- A directed network \((V, E, s, t)\) with set of nodes \(V\), set of links \(E\), and \((s, t) : E \rightarrow V \times V\) the source-target map. 
- A map \(f : V \rightarrow \mathbb{R}\) giving weight to nodes. 
- A map \(w : E \rightarrow \mathbb{R}\) giving weight to links. 

Figure 1 shows on the left the double weighted network \( W \), and on the right the double weighted network \( Y \) with the same underlying network and weights set to 1. A morphism \((\alpha, \beta) : (V_1, E_1, s_1, t_1, w_1, f_1) \rightarrow (V_2, E_2, s_2, t_2, w_2, f_2)\) in wwdigraph is given by a pair of maps \(\alpha : V_1 \rightarrow V_2\) and \(\beta : E_1 \rightarrow E_2\)
such that
\[(s_2, t_2) \circ \beta = (\alpha \times \alpha) \circ (s_1, t_1), \quad f_2(v_2) = \sum_{v_1 \in V_1, \alpha v_1 = v_2} f_1(v_1), \quad w_2(e_2) = \sum_{e_1 \in E_1, \beta e_1 = e_2} w_1(e_1).\]

Figure 2 displays a morphism in wwdigraph, represented by thick arrows, with \(W\) as domain. To each double weighted network on \([n] = \{1, ..., n\}\) we associate a matrix-vector pair \((D, f) \in M_n(\mathbb{R}) \times \mathbb{R}^n\) consisting of the adjacency matrix \(D\) and the vector \(f\) of weights on nodes:
\[D_{ij} = \sum_{e \in E, t(e) = i, s(e) = j} w(e) \quad \text{and} \quad f_j = \text{weight of } j.\]

A weighted network without multiple links on \([n]\) and the pair \((D, f)\) encode, essentially, the same information. For simplicity we usually work with networks without multiples links. Morphisms between weighted networks without multiples links are defined for matrix-vector pairs, say from \((D, f) \in M_n(\mathbb{R}) \times \mathbb{R}^n\) to \((E, g) \in M_m(\mathbb{R}) \times \mathbb{R}^m\) by a map \(\alpha : [n] \to [m]\) such that \(E_{ij} = \sum_{\alpha (k) = i, \alpha (l) = j} D_{kl}\) and \(g_j = \sum_{\alpha (l) = j} f_l\). We have maps digraph \(\to\) wdigraph \(\to\) wwdigraph where the first map gives weight 1 to links, and the second map keeps the weight on links and gives weight 1 to nodes. The product \((V_1 \times V_2, E_1 \times E_2, (s_1, s_2), (t_1, t_2), w_1 \times w_2, f_1 \times f_2)\) of double weighted networks \((V_1, E_1, s_1, t_1, w_1, f_1)\) and \((V_2, E_2, s_2, t_2, w_2, f_2)\) is such that \(w_1 \times w_2(e_1, e_2) = w_1(e_1)w_2(e_2)\) and \(f_1 \times f_2(v_1, v_2) = f_1(v_1)f_2(v_2)\). Similarly, disjoint union of double weighted networks can be defined.

The PWP map depends on a parameter \(\lambda \in \mathbb{R}_{\geq 0}\) and is useful for measuring indirect influences on networks. Assume as given a weighted directed network with associated matrix \(D \in M_n(\mathbb{R})\) measuring the direct influence that each node exerts on the other nodes. The PWP map \(T : M_n(\mathbb{R}) \to M_n(\mathbb{R})\) sends \(D\) to the matrix of indirect influences

![Figure 2: Morphism between double weighted directed graphs.](image)
Figure 3: Left: Dual Network $Y^*$. Right: Barycentric Network $Y^\circ$.

$T = T(D, \lambda)$ given by:

$$T(D, \lambda) = \frac{e^{\lambda D} - I}{e^\lambda - 1} = \sum_{k=1}^{\infty} \frac{D^k \lambda^k}{k!} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} D^k =$$

$$\sum_{k=1}^{\infty} \left( \sum_{j=0}^{\infty} B_j \frac{\lambda^{j+k-1}}{j!} \right) \frac{D^k}{k!} = \sum_{k=0}^{\infty} \left( \sum_{l=0}^{k} (k)_l B_{k-l} \frac{D^{l+1}}{(l+1)!} \right) \frac{\lambda^k}{k!} =$$

$$D + \frac{D^2}{2} \lambda + \left( \frac{D}{6} + \frac{D^3}{3} \right) \frac{\lambda^2}{2!} + \left( \frac{D^2}{4} + \frac{D^4}{4} \right) \frac{\lambda^3}{3!} + \left( - \frac{D}{30} + \frac{D^3}{3} + \frac{D^5}{5} \right) \frac{\lambda^4}{4!} + \cdots$$

where $(k)_l = \frac{k!}{(k-l)!}$ and $B_j \in \mathbb{Q}$ are the Bernoulli numbers. From the above expression we see that $T(D, \lambda)$ is a one-parameter deformation of the adjacency matrix $D$ in the sense that $T(D, 0) = D$; replacing $D$ by $T(D, \lambda)$ one obtains a one-parameter deformation of all network concepts defined in terms of the matrix $D$ of direct influences. As an example we introduce an one-parameter deformation of the Girvan-Newman modularity function that takes indirect influences into account. The Girvan-Newman modularity function $Q : \text{Par}[n] \rightarrow \mathbb{R}$, defined on the set of partitions on the nodes of a (non-negative) weighted directed network $D$, is given by

$$Q(\pi) = \sum_{i \sim j} \left( \frac{D_{ij}}{m} - \frac{d^\text{in}_i d^\text{out}_j}{m} \right)$$

where the sum is over pair of nodes in the same block of $\pi$, and

$$m = \sum_{i,j \in [n]} D_{ij} > 0, \quad d^\text{in}_i = \sum_{j \in [n]} D_{ij}, \quad d^\text{out}_j = \sum_{i \in [n]} D_{ij}.$$ 

Turning on indirect influences we obtain the one-parameter deformation of the modularity function $Q_\lambda : \text{Par}[n] \rightarrow \mathbb{R}$ ($\lambda \in \mathbb{R}_{\geq 0}$, $Q_0 = Q$) given by

$$Q_\lambda(\pi) = \sum_{i \sim j} \left( \frac{T_{ij}(\lambda)}{M(\lambda)} - \frac{E_i(\lambda) E_j(\lambda)}{M(\lambda) M(\lambda)} \right)$$

where $M(\lambda) = \sum_{i,j \in [n]} T_{ij}(\lambda) > 0$. 

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In several examples, including the networks study in the closing sections and quite a few randomly generated networks, $Q_\lambda$ is a monotonically decreasing function of $\lambda$, a result intuitively appealing since turning on indirect influences makes networks "more connected."

We use the matrix of indirect influences $T$ to impose rankings on nodes of networks. Let ranking($X$) be the set of rankings on $X$, i.e. a pre-order $\leq$ on $X$ for which there is a map $f : X \rightarrow \mathbb{N}$ such that $i \leq j$ if and only if $f(i) \leq f(j)$. Equivalently, a ranking is given by a partition on $X$ together with a linear ordering on the blocks of the partition, thus the exponential generating series for rankings is

$$\sum_{n=1}^{\infty} \frac{|\text{ranking}[n]|x^n}{n!} = \frac{x}{1-x} \circ (e^x - 1) = \frac{e^x - 1}{2 - e^x}.$$  

Rankings on $[n]$ are isomorphic if there is a bijection $\alpha : [n] \rightarrow [n]$ that preserves pre-orders. The number of isomorphism classes of rankings on $[n]$ is equal to the number of compositions on $n$ so, see [1], there are $2^{n-1}$ non-isomorphic rankings.

In our previous works [10, 11, 12] we used the maps $E, F, I : M_n(\mathbb{R}) \rightarrow \text{ranking}[n]$ where the rank of a node in the respective pre-orders is given, setting $T = T(D, \lambda)$, by

$$E_i = \sum_{j=1}^{n} T_{ij}, \quad F_i = \sum_{j=1}^{n} T_{ji}, \quad \text{and} \quad I_i = \sum_{j=1}^{n} (T_{ij} + T_{ji}).$$

We call these rankings the ranking by indirect dependence, indirect influence, and importance. The ranking by importance on networks $W$ and $Y$ from Figure 1 are:

| Nodes Ranking by Importance |
|-----------------------------|
| Network Y                   |
| 3 > 1 > 5 > 6 > 4 > 2     |
| Network W                   |
| 5 > 3 > 1 > 4 > 6 > 2     |

We extend the PWP map to double weighted networks by defining a map

$$T : M_n(\mathbb{R}) \times \mathbb{R}^n \rightarrow M_n(\mathbb{R})$$
sending a pair \((D, f)\) measuring direct influences and weight of nodes, to the matrix \(T = T(D, f, \lambda)\) measuring indirect influences among nodes. Let \(\bullet : M_n(\mathbb{R}) \otimes \mathbb{R}^n \to M_n(\mathbb{R})\) be the linear map given on \(D \otimes f\) by \((D \bullet f)_{ij} = D_{ij} f_j\).

**Definition 1.** The PWP map \(T : M_n(\mathbb{R}) \times \mathbb{R}^n \to M_n(\mathbb{R})\) is given for \(\lambda \in \mathbb{R}_{\geq 0}\) by

\[
T(D, f, \lambda) = \frac{e^{\lambda D \bullet f} - 1}{e^\lambda - 1} = \sum_{k=1}^{\infty} \frac{(D \bullet f)^k \lambda^k}{k!} = \sum_{k=0}^{\infty} \left( \sum_{l=0}^{k} \frac{(k)! B_{k-l} (D \bullet f)^{l+1}}{(l+1)!} \right) \lambda^k.
\]

Next we give an explicit formula for the entries of the PWP matrix \(T\) of indirect influences for double weighted networks, which implies the probabilistic interpretation for \(T\) given below.

**Proposition 2.** Let \((V, E, s, t, w, f)\) be a double weighted network, the indirect influence of node \(v\) on node \(u\) according to the PWP map is given by

\[
T_{uv}(\lambda) = \frac{1}{e^\lambda - 1} \sum_{k=1}^{\infty} \sum_{\{e_k \neq \emptyset\}} w(e_k) f(s e_k) \cdots w(e_1) f(s e_1) \frac{\lambda^k}{k!}.
\]

Equivalently, the matrix of indirect influences \(T(D, f, \lambda)\) is given by

\[
T_{ij} = \frac{1}{e^\lambda - 1} \sum_{k=1}^{\infty} \left( \sum_{i=i_k, \ldots, i_0=j} D_{i_k i_{k-1}} f_{i_{k-1}} \cdots D_{i_2 i_1} f_{i_1} \right) \frac{\lambda^k}{k!}.
\]

We regard \(\frac{\lambda^k}{(e^\lambda - 1)k!}\) as a probability measure on \(\mathbb{N}_{>0}\). Note that \(\frac{\lambda^k}{k!} \leq \frac{\lambda^{k+1}}{(k+1)!}\) if and only if \(k \leq \lambda - 1\), therefore \(\frac{\lambda^k}{(e^\lambda - 1)k!}\) achieves its maximum at \(\lfloor \lambda \rfloor\). The mean and variance of \(\frac{\lambda^k}{(e^\lambda - 1)k!}\) are respectively \(\frac{\lambda e^\lambda}{e^\lambda - 1}\) and \(\frac{\lambda e^{2\lambda} - \lambda e^\lambda - \lambda^2 e^\lambda}{(e^\lambda - 1)^2}\). By Chebyshev’s theorem we have for \(a > 0\) and \(l \in \mathbb{N}_{>0}\) that

\[
\text{prob}\left( \left| l - \frac{\lambda e^\lambda}{e^\lambda - 1} \right| \geq a \frac{\lambda e^{2\lambda} - \lambda e^\lambda - \lambda^2 e^\lambda}{(e^\lambda - 1)^2} \right) \leq \frac{1}{a^2}.
\]
For example setting $\lambda = 1, a = 10$ we get that $\text{prob}(l \geq 9) \leq 10^{-2}$.

Proposition 2 implies a probabilistic interpretation for $T_{ij}$ under the assumption that $D_{ij}, f_i \in [0,1]$ give the probabilities that the link $j \rightarrow i$ and the node $i$ be active, respectively. Under these assumptions it is natural to let the probability that a path $(i_0, i_1, ..., i_k)$ be active be $D_{i_k i_{k-1}} f_{i_{k-1}} \cdots D_{i_1 i_0} f_{i_0} \frac{\lambda^k}{(e^\lambda - 1)k!}$, i.e. being active is an independent property among the components of a path, links and nodes; a length dependent correcting factor $\frac{\lambda^k}{(e^\lambda - 1)k!}$ is included making long path less likely to be active.

**Theorem 3.** Consider a double weighted network with $D_{ij}, f_i \in [0,1]$ giving the probabilities that the link $j \rightarrow i$ and the node $i$ be active, respectively. The indirect influence $T_{ij}$ of node $j$ on node $i$ is the expected number of active paths from $j$ to $i$.

**Proof.** Let $\Omega$ be a probability space provided with independent random variables $\hat{D}_{ij}, \hat{f}_i : \Omega \rightarrow \{0,1\}$ such that $E(\hat{D}_{ij}) = D_{ij}$ and $E(\hat{f}_i) = f_i$. Let $(\Omega \times \mathbb{N}_{\geq 1}, p)$ be the probability space with $p(w, k) = p(w)\frac{\lambda^k}{(e^\lambda - 1)k!}$, and consider the random variables $\hat{T}_{ij} : \Omega \times \mathbb{N}_{\geq 1} \rightarrow [0,1]$ given by

$$\hat{T}_{ij}(w, k) = \sum_{i=i_k, ..., i_0=j} D_{i_k i_{k-1}}(\omega) f_{i_{k-1}}(\omega) \cdots D_{i_1 i_0}(\omega) f_{i_0}(\omega).$$

The expected number of active paths $E(\hat{T}_{ij})$ from $j$ to $i$ is given by

$$\sum_{k=1}^{\infty} \sum_{i=i_k, ..., i_0=j} E(\hat{D}_{i_k i_{k-1}}) E(\hat{f}_{i_{k-1}}) \cdots E(\hat{D}_{i_1 i_0}) E(\hat{f}_{i_0}) \frac{\lambda^k}{(e^\lambda - 1)k!} = \sum_{k=1}^{\infty} D_{i_k i_{k-1}} f_{i_{k-1}} \cdots D_{i_1 i_0} f_{i_0} \frac{\lambda^k}{(e^\lambda - 1)k!} = T_{ij}.$$
3 Network Deconstruction from Links Ranking

Applying the Girvan-Newman [17] clustering algorithm we describe how to deconstruct a network assuming as given a method for ranking links on networks. Let the connected components of a directed network be the equivalence classes of nodes under the equivalence relation generated by adjacency. The Girvan-Newman algorithm iterates the following procedures: -Compute the ranking of links. -Remove the links of highest rank. The algorithm stops when there are no further links, and outputs a forest of rooted trees, dendrogram, determined by the following properties: -The roots are the connected components of the original network. -The leaves are the nodes of the original network. -The internal nodes are the connected components of the various networks that arise as the procedures above are iterated. -There is an arrow from node $a$ to node $b$ if and only if $a \subset b$ and there is no node $c$ such that $a \subset c \subset b$.

Application of this algorithm provides a network deconstruction as it gradually eliminates links until reaching the network with no links. Reading the resulting forest from leaves to roots we obtain a reconstruction of our network, a genealogy of the various interrelated components of the network. The properties of the network revealed by the deconstruction procedure very much depend, as we shall see, on the choice of ranking among links.

We address the finding of rankings on links by reducing it to finding rankings on nodes: assuming a ranking method on nodes we propose three ranking methods on links, two of them obtained by applying geometric procedures (dual and barycentric constructions) to build a new network from the given one, so that nodes of the new network encode
information on the links of the original network. The third one formalizes the intuition that a link is sort of a bridge, and thus its importance is proportional to its functionality and to the importance of the nodes that it joins. The first two methods use the ranking of nodes by importance, the third method uses the rankings by indirect dependence and by indirect influence.

The original application of the Girvan-Newman [17] deconstruction algorithm to clustering uses the ranking on links given by the betweenness degree, i.e. the number of geodesics (length minimizing directed paths) passing trough a given link. By eliminating links of high betweenness the deconstruction process uncovers clusters. Running the deconstruction algorithm with the three links ranking methods proposed below we obtain new clustering algorithms. As we are free to choose the ranking on links in the deconstruction algorithm, we may as well apply the rankings opposite to the rankings mentioned in the previous paragraph and eliminate links of the lowest importance, thus uncovering core-periphery structures, with the periphery being the nodes that become isolated early on, and the core being the nodes in resilient connected components.

4 Dual Double Weighted Networks

We introduce the dual of a double weighted directed network via the map

\( (\cdot)^\ast : \text{wwdigraph} \to \text{wwdigraph} \)

sending a network \( G \) to its dual network \( G^\ast \). Figure 3 displays the dual network \( Y^\ast \) to the double weighted network \( Y \) from Figure 1. Given network \((V,E,s,t,w,f)\) its dual
network \((V^*, E^*, s^*, t^*, w^*, f^*)\) is such that:

- \(V^* = E\) and \(E^* = \{(e, v, h) \in E \times V \times E \mid te = v = sh\}\). For \(e \in V^*\) set \(f^*(e) = w(e)\).

- For \((e, v, h) \in E^*\) set \((s^*, t^*)(e, v, h) = (e, h)\) and \(w^*(e, v, h) = \frac{f(v)}{\text{out}(v)\text{in}(v)}\), where \(\text{out}(v)\) and \(\text{in}(v)\) are the out-degree and in-degree of node \(v\).

The dual construction applied to networks without multiple links may be identified with the map \((\ )^* : M_n(\mathbb{R}) \times \mathbb{R}^n \rightarrow M_{n^2}(\mathbb{R}) \times \mathbb{R}^{n^2}\), where \(M_{n^2}(\mathbb{R})\) is the space of maps \([n]^2 \times [n]^2 \rightarrow \mathbb{R}\). The map \((\ )^*\) sends \((D, f)\) to the pair \((D^*, f^*)\) given by \(f^*_{(i,j)} = D_{ij}\); \(D^*_{(i,j)(l,k)} = 0\) if either \(j \neq l\), or \(D_{ij} = 0\), or \(D_{lk} = 0\); and \(D^*_{(i,j)(j,k)} = \frac{f_j}{\text{out}(j)\text{in}(j)}\) if \(D_{ij} \neq 0\) and \(D_{lk} \neq 0\).

A node-ranking map \(R_n : M_n(\mathbb{R}) \times \mathbb{R}^n \rightarrow \text{ranking}[n]\) on weighted networks gives rise to link-ranking map \(R^*_n : M_n(\mathbb{R}) \times \mathbb{R}^n \rightarrow \text{ranking}[n]^2\) given by \(R^*_n = R_n^2 \circ (\ )^*\). Looking at the node-ranking maps \(E, F, I : M_n(\mathbb{R}) \times \mathbb{R}^n \rightarrow \text{ranking}[n]\) introduced in Section 2, we obtain the corresponding link-ranking maps by indirect dependence, influence, and importance \(E^*, F^*, I^* : M_n(\mathbb{R}) \times \mathbb{R}^n \rightarrow \text{ranking}[n]^2\).

**Theorem 4.** The PWP matrix of indirect influences on the dual double weighted network \((V^*, E^*, s^*, t^*, w^*, f^*)\) is given for \(e, f \in E\) by

\[
T^*_{ef} = \frac{1}{e^\lambda - 1} \sum_{k=1}^{\infty} \left( \sum_{e=e_{k+1}, e_{k+1}, \ldots, e_0 = f} f(te_{k-1})w(e_{k-1}) \cdots f(te_0)w(e_0) \right) \frac{\lambda^k}{k!},
\]
or in matrix notation

\[ T^*_{(m,l)(i,j)} = \frac{1}{e^{\lambda} - 1} \sum_{k=1}^{\infty} \left( \sum_{l=i_k, \ldots, i_1=i} f_{i_k}D_{i_{k-1}i_{k-2}} \cdots f_{i_1}D_{i_{i_1}j} \right) \lambda^k k! \]

Using the dual network \( Y^* \) one gets the following ranking on the links of network \( Y \):

| Ranking the links of network | Y dual method |
|----------------------------|---------------|
| Influence                  | Influence     |
| 56 > 24 > 64 > 31 > 13 > 43 > 41 > 63 > 15 > 35 |
| Dependence                 | Dependence    |
| 31, 35 > 56 > 13, 15 > 64, 63 > 43, 41 > 24 |
| Importance                 | Importance    |
| 56 > 31 > 13 > 35 > 15 > 64 > 63 > 43 > 41 > 24 |

We are ready to apply the deconstruction method from Section 3, regarded as a clustering method, ranking links of \( Y \) by importance using the dual method. Figure 4 displays on the left the various stages as we deconstruct network \( Y \) until the bare network is reached. Note that nodes are separated into various components only at the very last step where all remaining links have the same importance, as no concatenation of links is even possible. Thus there is only one cluster encompassing all nodes. Figure 5 shows the core-periphery finding process for network \( Y \) considering the rank of links by importance and using the dual construction, revealing a core consisting of four rings: the core \( \{1, 3\} \), second ring \( \{5, 6\} \), third ring \( \{4\} \) and the periphery \( \{2\} \).

5 Barycentric Division of Double Weighted Networks

The barycentric division of double weighted networks is a construction that places nodes and links of networks on the same footing. Figure 3 shows the barycentric division \( Y^o \)
of network $Y$. This construction allows to compare the importance of nodes with the importance of links, thus providing a precise formulation of the question of whether a network is dominated by actors or by relations. The barycentric division map

$$(\ )^\circ : \text{wwdigraph} \to \text{wwdigraph}$$

turns nodes and links of $(V, E, s, t, w, f)$ into nodes of the network $(V^\circ, E^\circ, s^\circ, t^\circ, w^\circ, f^\circ)$ defined as follows:

- $V^\circ = V \sqcup E$ and $E^\circ = \{ (v, e) \in V \times E \mid v = se \} \sqcup \{ (e, v) \in E \times V \mid te = v \}$.
- For $(v, e) \in E^\circ$ set $(s^\circ, t^\circ)(v, e) = (v, e) \in V^\circ \times V^\circ$.
- For $(e, v) \in E^\circ$ set $(s^\circ, t^\circ)(e, v) = (e, v) \in V^\circ \times V^\circ$.
- For $(e, v) \in E^\circ$ set $w^\circ(v, e) = w^\circ(e, v) = 1$.
- For $v \in V \subseteq V^\circ$ set $f^\circ(v) = f(v)$. For $e \in E \subseteq V^\circ$ set $f^\circ(e) = w(e)$.

The barycentric construction applied to networks without multiple links may be identified with the map $(\ )^\circ : M_n(\mathbb{R}) \times \mathbb{R}^n \to M_{n^2+n}(\mathbb{R}) \times \mathbb{R}^{n^2+n}$, Given a node-ranking map $R_n : M_n(\mathbb{R}) \times \mathbb{R}^n \to \text{ranking}[n]$ on double weighted networks, we construct the link-ranking map $R_n^\circ : M_n(\mathbb{R}) \times \mathbb{R}^n \to \text{ranking}[n]^2$ given by $R_n^\circ = r \circ R_{n^2+n} \circ (\ )^\circ$, where $r : \text{ranking}[n^2+n] = \text{ranking}([n]^2 \sqcup [n]) \to \text{ranking}[n]^2$ is the restriction map.
Figure 12: Core of Network $T$ using Barycentric Division

**Theorem 5.** The PWP matrix of indirect influences on the barycentric division double weighted network $(V^o, E^o, s^o, t^o, w^o, f^o)$ is given for $e, f \in E$ by

$$T_{ef}^o = \frac{1}{e^\lambda - 1} \sum_{k=1}^{\infty} \left( \sum_{e=e_k, \ldots, e_0=f} f(te_{k-1})w(e_{k-1}) \cdots f(te_0)w(e_0) \right) \frac{\lambda^{2k}}{(2k)!}. $$

or in matrix notation

$$T^o_{(m,l)(i,j)} = \frac{1}{e^\lambda - 1} \sum_{k=1}^{\infty} \left( \sum_{l=i_k, \ldots, i_1=i} f_{i_k}D_{i_ki_{k-1}} \cdots f_{i_1}D_{i_1j} \right) \frac{\lambda^{2k}}{(2k)!}. $$

Considering the barycentric division network $Y^o$ and the ranking on nodes by indirect dependence, indirect influence, and importance of its nodes we obtain the ranking on the links of network $Y$ given by:

| Ranking on Nodes of Network $Y^o$ |
|-----------------------------------|
| Influence                         |
| $6 > 4 > 3, 1 > 56 > 24, 64 > 63, 13, 43, 31, 41 > 5 > 2 > 35, 15$ |
| Dependence                        |
| $3 > 5 > 1 > 35, 31 > 4 > 56 > 15, 13 > 43, 41 > 6 > 64, 63 > 24 > 2$ |
| Importance                        |
| $3 > 1 > 4 > 5 > 6 > 31 > 56 > 13 > 35 > 43, 41 > 15 > 64 > 63 > 24 > 2$ |

Figure 4 displays on the right the clustering process for network $Y$ based on the barycentric construction using the ranking by importance on links, again we obtain just one cluster component $\{1, 2, 3, 4, 5, 6\}$. Figure 5 displays the core-periphery finding process for network $Y$ based on the barycentric construction, yielding the same result as with the dual construction: core $\{1, 3\}$ and subsequent peripheral outer rings $\{5, 6\}, \{4\}, \{2\}$. 

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6 Bridge Approach to Link Ranking

Our third method for ranking links on double weighted network is based on the idea that the importance of a link is proportional to the importance of the nodes that it connects and to its functionality. Assume that we have already computed the dependence and influence of nodes as in Section 2. The dependence $E(e)$, influence $F(e)$, and importance $I(e)$ of link $e$ in a double weighted network are given, according to the bridge approach, by $E(e) = E(se)f(se)$, $F(e) = w(e)F(te)$, and $I(e) = E(se)f(se) + w(e)F(te)$. The following result is consequence of Proposition 2.

**Theorem 6.** The importance of link $e$ in a double weighted directed network is given, according to the bridge approach, by

$$I(e) = \frac{f(se)}{e^\lambda - 1} \sum_{k=1}^{\infty} \sum_{e_k, \ldots, e_1 \in \Gamma} w(e_k)f(se_k) \cdots w(e_1)f(se_1) \frac{\lambda^k}{k!} +$$

$$\frac{w(e)}{e^\lambda - 1} \sum_{k=1}^{\infty} \sum_{e_k, \ldots, e_1 \in \Gamma} w(e_k)f(se_k) \cdots w(e_1)f(se_1) \frac{\lambda^k}{k!}.$$

Equivalently, in matrix notation

$$I_{ij} = \frac{f_i}{e^\lambda - 1} \sum_{k=1}^{\infty} \sum_{i_k, \ldots, i_1} D_{i_k} f_{i_k} \cdots D_{i_2} f_{i_1} \frac{\lambda^k}{k!} + \frac{D_{ij}}{e^\lambda - 1} \sum_{k=1}^{\infty} \sum_{i_k, \ldots, i_1} D_{i_k} f_{i_k} \cdots D_{i_1} f_{i_1} \frac{\lambda^k}{k!}.$$
7 Symmetric and Generic Network Examples

As a rule one expects the dual, barycentric, and bridge methods to yield different results, as their explicit formulae given above indicate, nevertheless in some cases they do agree. In this section we consider a highly symmetric network $S$, shown on the left of Figure 6, coming with intuitively clear clustering and core-periphery structures. Network $S$, with 12 nodes and 15 links, consists of three directed 4-cycles connected through 3 nodes forming an additional directed cycle. The barycentric network $S^\circ$, with 25 nodes and 30 links, and the dual network $S^\ast$, with 15 nodes and 21 links, are shown in the center and right hand side of Figure 6. Although $S^\circ$ and $S^\ast$ are different networks, our three methods yield the same clustering and core-component structure, and indeed the outputs are what one may naively expect: the clusters are the directed cycles \{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}, the core are the nodes \{4, 6, 9\} connecting these cycles, with a second layer formed by the nodes adjacent to the core \{1, 3, 5, 7, 10, 12\}, and the periphery being the nodes \{2, 8, 11\} attached to the second layer.

Finally, we test our methods on a more sophisticated network $T$ with 48 nodes and 242 links shown in Figure 7, which we borrowed from [25]. Applying our three clustering methods to $T$ until the obtained clusters are trees or cycles, we obtain the networks displayed in Figures 8, 9, 10, which although not identical are actually pretty similar. Note however that the barycentric method yields a pretty large cluster with 19 nodes. The number of steps required to reach such clusterings with our three methods are also quite similar. The core of network $T$ according to our three methods are shown in Figures 11, 12, 13, respectively. Again the outputs are pretty consistent, and were obtained in roughly the same number of steps.

8 Conclusion

We have shown that the problems of hierarchization, clustering, and core-periphery finding are intimately related. Indeed any hierarchization method, together with a suitable choice of network constructions, leads to clustering and core-periphery finding methods. We considered three construction, namely, the dual, barycentric subdivision, and bridge constructions on double weighted networks. Applying this philosophy together with the PWP method for ranking nodes we obtain new clustering and core-periphery methods, which we computed in three toy models. We also applied the PWP map to obtain a one-parameter deformation of the modularity function. Our methods can be readily be modified to use other definitions for the matrix of indirect influences, instead of the PWP map.
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