On Matrix Factorizations, 
Residue Pairings and 
Homological Mirror Symmetry

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Abstract

We argue how boundary $B$-type Landau-Ginzburg models based on matrix factorizations can be used to compute exact superpotentials for intersecting $D$-brane configurations on compact Calabi-Yau spaces. In this paper, we consider the dependence of open-string, boundary changing correlators on bulk moduli. This determines, via mirror symmetry, non-trivial disk instanton corrections in the $A$-model. As crucial ingredient we propose a differential equation that involves matrix analogs of Saito’s higher residue pairings. As example, we compute from this for the elliptic curve certain quantum products $m_2$ and $m_3$, which reproduce genuine boundary changing, open Gromov-Witten invariants.
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1. Introduction

1.1. Physical motivation

Topological open strings in connection with mirror symmetry (for overviews, see [1–3]) are useful for understanding certain non-perturbative phenomena related to $D$-branes on Calabi-Yau manifolds [4], and specifically, for computing exact, instanton-corrected effective superpotentials. So far, substantial progress (initiated in [5–7]) has been made for single or multiple parallel branes, for which the open strings are associated with “boundary preserving” vertex operators. Most of these works deal with non-compact geometries. There has been much less work on compact geometries (initiated in [8, 9]) and in particular very little work on intersecting branes, where “boundary changing” open string vertex operators come into play and the methods developed so far are not applicable.

Such brane configurations are particularly interesting for phenomenological applications, not the least because they naturally give rise to chiral fermions; the boundary changing operators correspond to matter fields in bi-fundamental gauge representations (for a review see eg. [10]). Such models can be represented by quiver diagrams, where the nodes correspond to branes and the maps between them to boundary changing open strings localized at the intersections. Essentially, the various terms of the effective potential on the world-volume are given by disk correlators of boundary changing vertex operators, summed over orderings corresponding to closed paths in the quiver diagram. However, most discussions stop here at the level of cohomology and charge selection rules.

But there is much more to the superpotential than just chasing arrows around a quiver: generically there are moduli from the parent Calabi-Yau space and possibly also open string (location and bundle) moduli from the branes, and the maps, and consequently the effective potential, depend on them. This dependence can be highly non-trivial due to infinite series of world-sheet instanton contributions. Obviously, for answering questions like what the vacuum structure of the full theory is, one needs to determine the dependence of the superpotential on these moduli, over the whole of the moduli space.

Mathematically the instanton problem corresponds to the counting of holomorphic maps from “polygon shaped” disks with $n$ boundary components into Calabi-Yau spaces, such that these boundaries lie across intersecting special Lagrangian cycles; see Fig. 1 for an illustration adapted to the elliptic curve. There we see that if we allow for generic, $(p, q)$ types of intersecting branes, the set of possible correlation functions becomes infinitely much richer as compared to non-intersecting brane configurations!

The mathematical framework that is designed to address precisely this kind of questions is homological mirror symmetry [3,11]. However, despite of that this has been an important ongoing topic in mathematics since more than 20 years, it has seen little use in physics. Indeed, for example, an explicit method for computing instanton corrections for intersecting
brane configurations on Calabi-Yau threefolds folds has been missing so far, even for the simplest amplitudes such as three-point functions.

Our purpose in the present paper is to make some modest steps into this direction, from an admittedly simple-minded physicist’s perspective; we hope that the physics intuition may help to further the development of the theory.

Let us be more specific. In topological strings, there are two somewhat antagonistic approaches to correlation functions. One takes a more algebraic, the other a more geometric viewpoint, involving period integrals, the variation of Hodge structures, etc. Obviously one would like to have a more unified understanding of both perspectives. A sensible first step would be to aim at the middle ground, namely by asking how the algebraic open string sector varies as fiber over the closed string moduli space.

This is what we will discuss in this paper, for open topological strings in the framework of twisted $N = (2, 2)$ superconformal field theory. More specifically, we will consider open string correlation functions on the disk $D$ of the form

$$B^{(A_0, \ldots, A_k)}(t) = \left< \Psi^{(A_0, A_1)}_{\alpha_0} \Psi^{(A_1, A_2)}_{\alpha_1} P \left( \int_{\partial D} \Psi^{(1)(A_2, A_3)}_{\alpha_2} \right) \cdots \left( \int_{\partial D} \Psi^{(1)(A_{k-1}, A_k)}_{\alpha_{k-1}} \right) \Psi^{(A_k, A_0)}_{\alpha_k} e^{-\sum t_i \int_D \phi_i^{(2)}} \right>, \tag{1}$$

which are perturbed by closed string deformations, $\phi$. Here $\Psi^{(A_i, A_j)}$ with $i \neq j$ are vertex operators that describe open strings that go from one boundary condition, or $D$ brane $\mathcal{L}_{A_i}$, to another one, $\mathcal{L}_{A_j}$. Physically they are localized at the intersection $\mathcal{L}_{A_i} \cap \mathcal{L}_{A_j}$. These are the boundary changing operators, in contrast to the boundary preserving ones, $\Psi^{(A_i, A_i)}$. 

Figure 1: In the $A$-model, the boundary changing correlators get contributions from world-sheet instantons that correspond to holomorphic maps from the disk world-sheet into the intersecting D-brane geometry (with appropriately matching boundary conditions). Notice that the instanton problem becomes infinitely rich when we allow for intersecting branes, even for the simplest example of the elliptic curve.
which are localized on one brane only. The superscripts denote the integrated 1- or 2-form descendants of the respective operators.

In terms of these correlators, the effective potential that is induced by the $D$-brane background is given by summing over all correlators pertaining to the given $D$-brane configuration:

$$W_{\text{eff}}(A_0,..,A_k)(s,t) = \sum_{k \geq 2} \frac{1}{k} s_{\alpha_0} s_{\alpha_1} \ldots s_{\alpha_k} B_{\alpha_0,..,\alpha_k}(t),$$

(2)

where $s_{\alpha}$ are the (not necessarily commuting) fields in the effective action that source the $\Psi_{\alpha}$. This amounts to summing over appropriate closed paths in the quiver diagram.

1.2. Mathematical setting

The general structure of open string correlators on the disk is well-known (see eg. [12, 13]): they can be written as

$$B_{\alpha_0,..,\alpha_k}(t) = \langle \langle \Psi_{\alpha_0} \otimes m_k(\Psi_{\alpha_1} \otimes \Psi_{\alpha_2} \otimes \ldots \otimes \Psi_{\alpha_k}) \rangle \rangle,$$

(3)

where the topological metric $\langle \langle \ast, \ast \rangle \rangle$ denotes a suitable, non-degenerate and cyclically symmetric inner product. Moreover, $m_k : \Psi \otimes k \rightarrow \Psi$ are certain higher multilinear, non-commutative products that take collections of operators as input and produce one operator as output. Correspondingly, the equations of motion arising from $W_{\text{eff}}$ take the form

$$\sum_{k \geq 0} m_k(\Psi \otimes k) = 0,$$

(4)

which are nothing but the Maurer-Cartan equations [14] which specify the locus of unobstructed deformations of the theory.

The products $m_k$, and thus the correlators built from them, satisfy a host of Ward identities, specifically the $A_{\infty}$ relations, and have certain cyclicity and integrability properties. These data together with the inner product and some other subsidiary conditions comprise what is called a Calabi-Yau $A_{\infty}$ algebra [15]. If there are several boundary components present, the $A_{\infty}$ algebra is promoted to an $A_{\infty}$ category. Moreover, since we have deformed the correlators by bulk moduli, we encounter deformed $m_k = m_k(t)$, which form what is called a curved $A_{\infty}$ structure. When combined with the bulk sector, this structure is promoted to an open-closed homotopy algebra [16]. All this has been discussed at length in the literature, and we do not want to spend more than a few remarks on this in the present context. Suffice it to refer the reader to refs. [12, 13, 17, 18] for more details, from a physics perspective.

In practice, the explicit evaluation of the correlators (1) is difficult, not the least because of contact terms that arise when the integrated insertions hit other operators, or $\int \phi^{(2)}$ hits the boundary. While the underlying algebraic structure organizes these contact terms, it is
Figure 2: Homological mirror symmetry between $A_\infty$ products $m_3 : \Psi^\otimes 3 \to \bar{\Psi}$. The value of the Fukaya product in the $A$-model is given by the exponentiated disk instanton action associated with the intersecting brane geometry. In the $B$-model the corresponding Massey product is computed by nested trees, involving propagators and lower trees.

not of direct help to actually compute the correlators. As we will see, one needs to augment the algebraic structure by certain differential equations.

An analogous problem appears already for closed strings, where special geometry in connection with mirror symmetry comes to the rescue [2]. Specifically, mirror symmetry is the statement that the topological $A$-model on a Calabi-Yau space $Y$ is equivalent to the topological $B$-model on the mirror manifold, $X$. This means that instanton corrected correlation functions in the $A$-model can be computed in terms of classical correlation functions in the simpler $B$-model, in the framework of topological Landau-Ginzburg models. For non-intersecting $D$-brane geometries, a corresponding geometrical framework of open mirror symmetry has been well developed [2,3] after the initial works [5–7,19].

As said before, our intention is to push the subject to more general, intersecting brane configurations, which is the arena of homological mirror symmetry. Given that this is mathematically highly sophisticated, it would be impossible to give any reasonable account here. Instead we will exhibit only some basic ideas to convey the motivation of what we want to do. In short, the basic statement is an isomorphism [11]:

$$Fuk(Y) \simeq D^b(Coh(X)),$$

(5)

where $Fuk(Y)$ and $D^b(Coh(X))$ denote the appropriately defined Fukaya category of special lagrangian cycles on the $A$-model side, and the bounded derived category of coherent sheaves (crudely: vector bundles on submanifolds) on the $B$-model side. It has been proved recently [20] for hypersurfaces in projective space. The relevance for the physics of $D$-branes has been recognized early on in refs. [21,22]. Equation (5) implies isomorphisms between the
$A_\infty$ products $m_k$ on the two sides, which are called Fukaya and Massey products in the $A$- and $B$-model, respectively. See Figure 2 for a visual representation. In physics language this amounts to an “equality” of the respective correlation functions. We put the word equality in quotation marks, since isomorphism means equality up to maps. The physicists are however interested in explicit expressions, not just in structural existence proofs of isomorphisms. In the closed string sector, this has been achieved [23] by applying methods of algebraic geometry, in particular the theory of Hodge variations, period integrals and associated flatness differential equations. This leads to an exact map, the mirror map

$$t(\alpha) : V^B(X) \simeq H^*(X, \wedge^* T_X) \to V^A(Y) \simeq QH^*(Y),$$

which maps the algebraic modulus $\alpha$ in the $B$-model into the flat coordinate $t$ of the $A$-model. Above, $V^B$ and $V^A$ denote the variation of Hodge structures [24, 25] on the both sides, respectively. They each comprise of data $V = (H^*, \nabla^{GM}, \langle\langle *, *\rangle\rangle)$, where $H^*$ is the relevant cohomology or quantum cohomology, $\nabla^{GM}$ the respective Gauss-Manin connection and $\langle\langle *, *\rangle\rangle$ a suitable inner product, or pairing.

A corresponding theory has been developed for homological mirror symmetry, which is the theory of non-commutative Hodge variations (see eg. [26–28]). The details not being important here, we will be schematic and refer the reader to the readable expositions in refs. [29,30]. The basic notion is an $A_\infty$ category $C$ with maps between objects $\mathcal{L}_i$:

$$\mathcal{C}_k(\mathcal{L}_0, \mathcal{L}_1, \ldots \mathcal{L}_k) = \text{Hom}^*(\mathcal{L}_0, \mathcal{L}_1) \otimes \text{Hom}^*(\mathcal{L}_1, \mathcal{L}_2) \cdots \otimes \text{Hom}^*(\mathcal{L}_{k-1}, \mathcal{L}_k).$$

With this one forms the Hochschild chain complex and its homology

$$CC_*(C) = \bigoplus_k \mathcal{C}_k(\mathcal{L}_0, \mathcal{L}_1, \ldots \mathcal{L}_k, \mathcal{L}_0),$$

$$HH_* (C) = H^*(CC_*(C), b),$$

where $b$ is the Hochschild differential. One may refine this to a semi-infinite variant and consider the so-called negative cyclic complex, $CC^-_*(C)$, and its cohomology, $HC^-_* (C)$. This involves the introduction of a spectral parameter, $u$. Together with the Gauss-Manin-Getzler connection $\nabla^{GMG}$ [31] and the pairing [32], this forms a structure $\mathcal{V}(C) = (HC^-_* (C), \nabla^{GMG}, \langle\langle *, *\rangle\rangle)$, which is a non-commutative analog of the semi-infinite variation of Hodge structures $V$ in the closed string theory. In the present context, there are two versions of this, one for the $A$-model side and one for the $B$-model side. Homological mirror symmetry then amounts to an isomorphism, in analogy to (6):

$$\mathcal{V}^A(Fuk(Y)) \simeq \mathcal{V}^B(D^b(Coh(X))).$$

It has been shown [29, 30] that (under suitable conditions) open-closed maps exist that provide isomorphisms

$$OC : \mathcal{V}^A(HC^-_*(Fuk(Y))) \to \mathcal{V}^A(QH^*(Y)),$$

$$\mathcal{V}^B(HC^-_*(D^b(Coh(X)))) \to \mathcal{V}^B(H^*(X)).$$
This means that the non-commutative Hodge structures of the open string theories map back to the Hodge structures of the bulk theories, and this can be used to show that homological mirror symmetry implies Hodge theoretical mirror symmetry [29,30].

What does this tell us for our problem, namely computing the open string correlation functions (1)? These correspond to the components of the Hochschild complex (8) and the statement is that their deformation theory is isomorphic to the one of the closed string. This by itself does not fix any correlator. We would need an analog of the mirror map (6) to explicitly tie the A- and B-models together. For this, we should impose some extra structure in the form of flatness equations. The question is how to do this in practice.

Flatness equations based on the Gauss-Manin-Getzler\(^1\) connection [31], \(\nabla_{GMG}^t\), are a central theme in the mathematical literature. However, the latter has been mostly concerned to map via OC all kinds of quantities to the bulk, closed string sector. This allows to evaluate pairings and higher correlators in the simpler, commutative theory. Specifically, for two-point functions, one might be tempted to write
\[
\langle\langle \Psi^{(A,B)}, \Psi^{(B,A)} \rangle\rangle_D = \langle\langle \text{OC} [\Psi^{(A,B)}], \text{OC} [\Psi^{(B,A)}] \rangle\rangle_{S_2},
\]
and consider flatness equations of the form
\[
0 = \langle\langle \nabla_{GMG}^t \Psi^{(A,B)}, \Psi^{(B,A)} \rangle\rangle_D = \langle\langle \text{OC} [\nabla_{GMG}^t \Psi^{(A,B)}], \text{OC} [\Psi^{(B,A)}] \rangle\rangle_{S_2}
\]
(12)
where the last line exhibits the intertwining property [29] of \(\nabla_{GMG}^t\) and \(\nabla_{GM}^t\) over OC. However, such correlators may make sense for the annulus, but not for the sphere or disk. Moreover, the map OC invariably vanishes on boundary changing operators. Indeed, OC as well as \(\nabla_{GMG}^t\) act on complete cycles \(\Psi^{(A_0,A_1)} \otimes \cdots \otimes \Psi^{(A_k,A_0)}\), but not on the individual maps \(\Psi^{(A_i,A_j)}\) when \(i \neq j\). Thus we cannot capture in this way the boundary changing sector we are interested in.

Rather, the differential operators we seek should act on the individual operators in a well-defined manner, even if these are boundary changing. This is then finally, in essence, the problem that we want to address: find an explicit construction of flatness differential equations of the form
\[
\langle\langle \nabla_t \Psi^{(A,B)}, \Psi^{(B,A)} \rangle\rangle_D = 0,
\]
(13)
that make sense especially also for boundary changing sectors. This will be a crucial ingredient for computing general correlation functions for intersecting branes.

\(^1\)There are two kinds, one associated with moduli deformations, \(t\), plus one associated with the spectral parameter \(u\) if we are interested in the semi-infinite extension of Hodge variations; we will suppress the latter, as for our current purposes this extension is not immediately relevant.
1.3. Content of the paper

Our strategy is guided by a specific realization of the open topological $B$-model, namely in terms of a 2d, $N = (2, 2)$ superconformal Landau-Ginzburg model based on matrix factorizations. The key point is an isomorphism

$$D^b(Coh(X)) \simeq \text{Cat}(MF,W_X) ,$$

where $\text{Cat}(MF,W_X)$ is the category of matrix factorizations of a function $W_X$. Here this function is given by the LG superpotential whose vanishing describes the Calabi-Yau manifold $X$ under consideration: $W_X = 0$ (in some weighted projected space). This isomorphism has been proved under certain assumptions, by Orlov [33], based in earlier ideas of Kontsevich.

For physicists this isomorphism allows to describe topological $B$ type D-branes in terms of a simple field theoretical model, in which all the elements of the abstract category, namely objects and maps between them, have a concrete realization in terms of matrix valued field operators. First works on this by physicists include [34]-[36], and a selection of works relevant for our current purposes is given by [38]-[50].

In the next section we will first briefly review well-known aspects of the Landau-Ginzburg model in the bulk, and study its deformations. Here we will use a less geometrical and more field theoretical language in terms of renormalization and contact terms. For this we will introduce the language of Saito’s higher residue pairings, which so far did not receive a lot of attention in the physics literature. That is why we will use simple terms, and also provide a sample computation for the elliptic curve. It demonstrates how one can formulate differential equations directly in terms of residue pairings.

In Section 3 we will first review some basic facts of matrix factorizations and their deformations. We then analyze the interplay between bulk-boundary and boundary-boundary contact terms. This leads in Section 3.3 to a proposal for a flatness differential equation, which is based on a higher variant of the Kapustin-Li supertrace residue that plays an open string analog of Saito’s first higher residue pairing.

In Section 4 we apply these ideas to open strings on the elliptic curve and demonstrate how one can explicitly compute, by solving the differential equation, open string correlators such as in (1). We then match the results of the $B$-model to the instanton geometry of the $A$-model mirror, thereby reproducing known results. Indeed, homological mirror symmetry is very well understood for the elliptic curve (see eg. [51–54]), and the structure of bundles over it and their sections (given by theta functions), are more or less explicitly known. Basically this is because the curve is flat and so one can simply read off the areas of disks using elementary geometry.

The main purpose of the present paper is not to compute new correlation functions, rather than to develop a method how to compute boundary changing, open Gromov-Witten
invariants directly from the $B$-model from “first principles”, without relying on prior knowledge about the $A$-model side (we put quotation marks because we present a proposal, not a proof). This will be important for later applications, \textit{eg.} to Calabi-Yau threefolds, where the $A$-model correlators are not known beforehand.

2. Recapitulation of the closed string, bulk theory

2.1. Topological Landau-Ginzburg $B$-model

The issue is to determine the dependence of the $B$-model $n$-point correlation functions on the flat complex structure moduli $t_a$ of a Calabi-Yau manifold, $X$. As per (6), via mirror symmetry these coincide with the Kähler moduli of the $A$-model on the mirror manifold, $Y$. Consider for example

$$C_{abc}(t) = \langle \phi_a \phi_b \phi_c e^{\sum t_a \int \phi_d^{(2)}} \rangle.$$  \hspace{1cm} (15)

Here, $\phi_a$ are zero-form cohomology representatives and the deformations

$$\int \phi^{(2)} \equiv \int d^2z \{G^-_{-1}, [G^+_{-1}, \phi]\}$$  \hspace{1cm} (16)

are given by their two-form descendants. Expanding out we can write

$$\langle \phi_a \phi_b \phi_c \int \phi^{(2)}_{d_1} \ldots \int \phi^{(2)}_{d_n} \rangle =: \langle \langle \phi_a, \ell_{n+2}(\phi_b, \phi_c, \phi_{d_1}, \ldots, \phi_{d_n}) \rangle \rangle,$$  \hspace{1cm} (17)

where the maps $\ell_n$ are the closed string versions of the $m_k$, which obey certain recursive $L_\infty$ relations. This statement by itself is not useful to actually compute the correlation functions. However, it is here where geometry beats algebra: due to the special geometrical properties of (topologically twisted) $N = (2, 2)$ superconformal theories, these correlators satisfy a host of identities \cite{55–57}. One of those is that correlators with $n > 3$ are obtained by taking derivatives of the three-point functions, for example

$$C_{abcd}(t) = \frac{\partial}{\partial t_d} \langle \phi_a \phi_b \phi_c e^{\sum t_e \int \phi_e^{(2)}} \rangle.$$  \hspace{1cm} (18)

The property (18) expresses an underlying Frobenius structure \cite{58,59} and implies crossing-type relations of the form

$$C_{ab}^e(t) C_{ced}(t) = C_{ad}^e(t) C_{ebd}(t).$$  \hspace{1cm} (19)

These serve as integrability condition for the existence of a prepotential $F$ such that

$$C_{abc}(t) = \frac{\partial}{\partial t_a} \frac{\partial}{\partial t_b} \frac{\partial}{\partial t_c} F(t).$$  \hspace{1cm} (20)
Note that (18) holds for a special choice of “flat” coordinates $t_a$ of the moduli space. Otherwise the derivatives would need to be replaced by covariant derivatives, whose connection pieces encode the contact terms. In this way, the flatness of coordinates is tied to the cancellation of contact terms.

The important insight of refs. [57, 60–63] and others was that the exact $n$-point correlators, including all contact term contributions, can be computed in an effective $B$-type, topologically twisted Landau-Ginzburg formulation of the theory. It is characterized by a superpotential, $W_0(x)$, which is a holomorphic, non-degenerate and quasi-homogeneous polynomial that depends on a number of LG fields, $x_i$, $i = 1, ..., N$. These are to be viewed as coordinates of the target Calabi-Yau space $X$ defined by $W_0(x) = 0$ in some projective (or weighted projective) space.

We will consider (mini-versal) deformations of the complex structure described by

$$W(x, t) = W_0(x) - \sum g_a(t)y_a(x) ,$$

where $y_a(x)$ is some basis of the Jacobi ring, $y_a(x) \in \text{Jac}(W) \equiv \mathbb{C}^N[x]/dW$. In this language, the flat zero-form operators are represented by polynomials in the LG fields defined by

$$\varphi_a(x, t) = \frac{\partial}{\partial t_a} W(x, t) .$$

To distinguish the various fields, we denote by $\phi(x, t)$ the marginal operators that are sourced by the moduli $t$, and by $\varphi(x, t)$ generic elements of the chiral ring that can also include relevant besides marginal ones. General polynomials are denoted by $\xi$. In terms of these operators, the three-point functions then localize in the IR to the Grothendieck multi-residue [60], i.e., to

$$C_{abc}(t) = \text{res}_{x=0} \frac{\varphi_a(x, t)\varphi_b(x, t)\varphi_c(x, t)}{d_1W \ldots d_NW} = \oint \frac{dx}{(dW)^N} \varphi_a(x, t)\varphi_b(x, t)\varphi_c(x, t) ,$$

from which all other correlators can be generated.

The issue thus is to determine the dependence of $g_a(t)$ on the flat coordinates. The most familiar way to determine the flat coordinates is to solve the Gauss-Manin, or Picard-Fuchs system of differential equations that act on period integrals. This system itself can be systematically derived via the variation of Hodge structures. This procedure à la Griffith-Dwork [64] is standard since many years (for reviews, see eg. [1, 65]), so we don’t embark on it further. Suffice it to mention that the core structure consists of the vanishing of the Gauss-Manin connection, $\nabla^{GM}$. Its primary component takes the form

$$\nabla^{GM}_{t_a}(\varphi_b) \equiv \frac{\partial}{\partial t_a} \varphi_b(x, t) = U(\phi_0 \varphi_b) = 0 ,$$

where

$$U(\phi_0 \varphi_b) \equiv d_i \left( \frac{\phi_0 \varphi_b}{d_iW} \right) ,$$

We have temporarily suppressed a $t$-dependent prefactor that is needed for the proper normalization. Moreover, the integration contour is a real cycle supported on $|d_iW| = \epsilon$. 

$^2$
and this is what determines the coupling function of $\varphi_a(x,t)$. The subscript $+$ denotes a projection: it instructs to expand the polynomial operator product according to the Hodge decomposition

$$\phi_a(x)\varphi_b(x) = C^{\varphi_c(x)}_a + p_{ab}^i(x) d_i W(x) ,$$

and then to drop negative powers of $x$; that is, the result is determined by the exact piece of the OPE, $U(\phi_a\varphi_b) = d_i p_{ab}^i(x)$.

Mathematically, the $U$ term in (24) arises from integration by parts under the period integral in order to reduce the pole order [64]. Physically it is a contact term [66] that arises from integrating out the massive, exact states in the OPE (26). Via (24), their effect is subsumed in the $t$-dependent renormalization of the coupling functions $g_a(t)$. Schematically:

$$\varphi_a(x,t) = \varphi_0^a(x) + t U(\ell_2(\phi,\varphi_0^a))|_{t=0} + 1/2 t^2 U(\ell_3(\phi,\phi,\varphi_0^a))|_{t=0} + \ldots .$$

(27)

The iterative integrating out amounts to going from an off-shell topological string field theory to an on-shell “minimal model” with non-trivial higher $L_\infty$ products $\ell_n$; it is the special geometry of the $N = (2,2)$ superconformal theories that allows to sum all these terms up in one swoop.

2.2. Higher residue pairings

We recapitulate the determination of flat coordinates by using a method that we find especially suitable for the generalization to the boundary theory. This is because it avoids period integrals, which we wouldn’t know how to generalize to matrix valued operators when we will face the generalization to open strings. It also seems more natural physically, as it makes direct contact with the underlying field theory. This is the theory of Saito’s higher residue pairings [67–70]. So far it has been rarely discussed in physics (see however: [66,71–74]).

We will be brief and present only a rough outline. For more accurate technical details we refer the reader to the recent expositions [75–77]. The basic objects in Saito theory are the higher residue pairings

$$K(\varphi_a, \varphi_b)(u) \equiv \sum_{\ell=0}^{\infty} u^\ell K^{(\ell)}(\varphi_a, \varphi_b) ,$$

(28)

where $u$ is the auxiliary degree 2, spectral parameter that plays an important role in the variation of semi-infinite Hodge structures [24–26]. For us, it has nothing to do with gravitational descendants, rather its significance is to provide a grading with respect to the number of iterated contact terms; thus only a finite number of powers of $u$ will be relevant.

The pairings can be derived and represented in various different ways. We like to outline a physically motivated, field theoretical derivation which reflects the underlying path integral. Following the notation of ref. [78], we define the BRST operator related to $B$-type
supersymmetry, as well as a deformation of it by $u$:

$$Q_W = \bar{\partial} + \iota_{dW},$$  \hspace{1cm} (29) $$Q_{W,u} = Q_W + u \partial,$$  \hspace{1cm} (30) 

where:

$$\bar{\partial} \equiv \eta \bar{i} d \bar{i}, \quad \partial \equiv \partial_{\bar{\eta}} d_i, \quad \iota_{dW} \equiv d_i W \partial_{\eta_i}.$$  \hspace{1cm} (31) 

Here, $\eta \bar{i} \sim dz \bar{i} \in T^* X$ and $\theta_i \sim \partial/\partial z_i \in T X$ denote complex fermions that represent differential forms on $X$ in the usual manner. Moreover, the Hamiltonian is

$$\mathcal{H}_\lambda \equiv \Delta_{\bar{\partial}} + \ell_W + \lambda (L_0 + u \partial \iota_{dW}),$$  \hspace{1cm} (32) 

where $\Delta_{\bar{\partial}} = [\bar{\partial}, \bar{\partial}^\dagger]$ is the laplacian, $\ell_W = d_i d_j W \partial_{\eta_i} \partial_{\theta_j}$, and

$$L_0 = d_i d_j \bar{W}(\phi) \bar{\eta}^i \theta^j + ||d_i W(\phi)||^2$$  \hspace{1cm} (33) 

is the zero mode lagrangian. Note that

$$\mathcal{H}_\lambda = [Q_{W,u}, G_\lambda],$$  \hspace{1cm} (34) $$G_\lambda = \partial^\dagger + \lambda \iota_{dW},$$  \hspace{1cm} (35) $$\bar{\partial}^\dagger \equiv d_i \partial_{\eta_i}, \quad \iota_{dW} \equiv d_i \theta_i.$$  \hspace{1cm} (36) 

After the reduction to the zero modes of $\Delta_{\bar{\partial}}$ in the path integral, the expression for the one-point function, $\int e^{-\mathcal{H}_\lambda} \xi$, of some arbitrary polynomial $\xi$ takes the form:

$$\langle \xi(x) \rangle = \int dx d\eta d\theta \xi(x) e^{-\lambda (L_0 + u \partial \iota_{dW})}.$$  \hspace{1cm} (37) 

Since the Hamiltonian is BRST exact, the correlators are topological and (on-shell) independent of $\lambda$. For $\lambda \to \infty$, the second term in $L_0$ localizes the integral on the critical points of $W$ and contributes $\lambda^{-N} |H|^{-2}$, while the first term yields a factor of the Hessian $\bar{H}$ of $\bar{W}$. This also comes with a factor $\lambda^N$ so that the $N$ zero modes of each of $\theta$ and $\eta$ of this term dominate the path integral, and this implies that the potential $\theta$ and $\eta$ dependence of any other, non-exponential term drops at $\lambda \to \infty$. The sum over the critical points over the remaining $H^{-1}$ can then be converted, via a standard argument, to the usual residue formula. Thus we get

$$\langle \xi(x) \rangle(u) = \oint dx (dW)^N \sum u^\ell (\partial h)^\ell \xi(x)$$  \hspace{1cm} (38) 

as perturbation expansion in terms of the propagator

$$\partial h \equiv \partial \iota_{dW} / ||dW||^2 = d_i \frac{d_i W}{||dW||^2}.$$  \hspace{1cm}
Considering Morse coordinates near every critical point $x_i^0$ by writing $W \to W(x_i^0) + 1/2(x_i)^2$, and then taking the limit $x_i^* \to 0$ à l’Hôpital, we can effectively cancel out the non-holomorphic pieces in $h$.

All-in-all we arrive at

$$\langle \xi(x) \rangle = \oint \frac{dx}{(dW)^N} L(u, \xi) , \text{ where}$$

$$L(u, \xi) = \sum u^\ell \left( \frac{1}{d_j d_j W} \right)^\ell \xi(x).$$

This then gives the following representation of the higher residue pairings

$$K(\xi_a, \xi_b)(u) = \oint \frac{dx}{(dW)^N} L(u, \xi_a) L(-u, \xi_b).$$

The first, constant term in the expansion in $u$:

$$K^{(0)}(\varphi_a, \varphi_b) = \oint dx \frac{\varphi_a(x) \varphi_b(x)}{(dW)^N} = \eta_{ab},$$

is just the topological two-point function, or metric in field space. The next term can be written in the form:

$$K^{(1)}(\xi_a, \xi_b) = \frac{1}{2} \oint \frac{dx}{(dW)^N} \sum_j \left( \frac{d_j \xi_a(x) \xi_b(x) - \xi_a(x) d_j (\xi_b(x))}{d_j W(x)} \right).$$

We call this the “integrated operator form” of the residue pairings. We also introduce a “contact term form” of the pairings by writing

$$K_C(\xi_a, \xi_b)(u) = K^{(0)}(L_C(u, \xi_a), L_C(-u, -\xi_a)), \text{ where}$$

$$L_C(u, \xi) = \sum u^\ell \overset{\ell}{\overbrace{U(U(...U(\xi(x)...))},}$$

which is supposedly\textsuperscript{4} equivalent to the “integrated operator form”, (41). The importance of this equality for $\ell = 1$,

$$K_C^{(1)}(\xi_a, \xi_b) = K^{(1)}(\xi_a, \xi_b),$$

has been emphasized by Losev [66].

As will become clear later, the reason for this naming is that integrated operator insertions have structurally the form $\int \xi^{(2)} \sim d\xi/dW$, while contact terms involve a projection to polynomials, $U(\xi) = d(\xi/dW)_{++}$, and so can be directly translated to the renormalization of operators.

\textsuperscript{3}One needs to assume isolated critical points for this argument, and as usual this can be achieved by temporarily resolving the singularity by a small perturbation and invoking continuity. A more rigorous derivation proceeds by introducing a compact support and cutoff functions as explained in detail in [75].

\textsuperscript{4}We are unaware of literature where this has been addressed explicitly, except for $\ell = 1$. 

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2.3. Flatness equations \( \textit{sans périodes} \)

We now turn to the determination of flat coordinates and field bases in terms of higher residue pairings, reviewing only aspects that are of immediate interest for our purposes.

To start, we preliminarily adopt an overall normalization defined by

\[
K^{(0)}(H, 1) = 1 ,
\]

where \( H = \det d_id_j W \) denotes the Hessian of the potential \( W \) (which is in general not a flat cohomology representative). As is well known, the physical correlators involving flat cohomology representatives will need to have an extra rescaling factor, which is closely related to Saito’s primitive form and is essentially given by the fundamental period \( \omega_0 \) of the Calabi-Yau \( X \).

Next we introduce the notion of a “good basis” \([68, 69]\) of field operators \( \{ \varphi_a \} \), which is defined by

\[
\begin{align*}
K^{(0)}(\varphi_a, \varphi_b) &\equiv \eta_{ab} = \text{const.} , \\
K^{(t>0)}(\varphi_a, \varphi_b) &= 0 .
\end{align*}
\]

These equations are not quite sufficient for defining flat bases and coordinates. Consider the primary one:

\[
0 = \partial_t K^{(0)}(\varphi_a, \varphi_b)
\]

\[
= K^{(0)}(\partial_t \varphi_a, \varphi_b) + K^{(0)}(\varphi_a, \partial_t \varphi_b) - \int dx \frac{\varphi_a(x) \varphi_b(x)}{(dW)^N} \sum d_iW
\]

\[
= (K^{(0)}(\partial_t \varphi_a, \varphi_b) - K^{(1)}(\phi \varphi_a, \varphi_b)) + (K^{(0)}(\varphi_a, \partial_t \varphi_b) + K^{(1)}(\varphi_a, \phi \varphi_b)) .
\]

The peculiar term on the RHS represents in LG language the insertion of the integrated 2-form operator (16):

\[
\int (\phi^{(2)}) \leftrightarrow \frac{d\phi}{dW} ,
\]

and for this reason we called (43) the integrated operator form of the residue pairing. By the identity (46) we can rewrite it in terms of the contact term form

\[
0 = K^{(0)}(\partial_t \varphi_a - U(\phi \varphi_a), \varphi_b) + K^{(0)}(\varphi_a, \partial_t \varphi_b - U(\phi \varphi_b)) ,
\]

which reproduces the geometrical equation (24) (we assume the gauge \( U(\varphi_a) = 0 \) here). Either way, the equation can be compactly rewritten as

\[
0 = K(\nabla_t^{GM} \varphi_a, \varphi_b) + K(\varphi_a, \nabla_t^{GM} \varphi_b) ,
\]

where

\[
\nabla_t^{GM} = \partial_t - \frac{\phi}{u} .
\]
For complete flatness, requiring the constancy of the topological metric, eq. (49) is not sufficient. Rather one needs to impose the stronger ”chiral square root” of this equation, and this for all higher pairings as well:

\[ K^{(\ell)}(\nabla_{GM}^{G} \varphi_{a}, \varphi_{*}) = 0. \]  

(54)

For any given \( \varphi_{a} \), this equation must hold for arbitrary cohomology elements \( \varphi_{*} \) in a “good” basis (48), and it can give non-trivial constraints if the degrees of the \( \varphi_{*} \) are matched appropriately to the one of \( \varphi_{a} \), and to \( \ell \).

Equation (54) is in disguise what the first order form of the familiar Picard-Fuchs system instructs us to do. The essence of the story is this: by scanning over \( \ell \) and all possible “spectators” \( \varphi_{*} \), we sample all components of the Gauss-Manin connection. In general, there are higher order contact terms beyond what we wrote in (24). In the more familiar language of period integrals, these arise from multiple, iterated partial integrations, which reflect the structure of the Hodge filtration. These are encoded by the nested propagators \( U_{(25)} \), where the order of the nesting is measured by \( \ell \). Physically, testing the differential equation (54) against all physical operators samples all possible contact terms. In conjunction with (48), it determines the renormalized coupling functions \( g_{a}(t) \).

2.4. Example: the elliptic curve

Let us illustrate the method of residue pairings by an easy example computation. We will reproduce results that are known since long and have been discussed in the physics literature in eg. [63, 79–81].

We consider the simplest possible case, namely the cubic elliptic curve. It is defined as the hypersurface \( W = 0 \) in \( CP^{2} \) where

\[ W(x, t) = \zeta(t) \left[ \frac{1}{3} (x_{1}^{3} + x_{2}^{3} + x_{3}^{3}) - \alpha(t)x_{1}x_{2}x_{3} \right]. \]

(55)

Here \( \alpha(t) \) is the complex structure modulus whose dependence on the flat coordinate \( t \) is to be determined. With hindsight we have also performed an overall rescaling by a function \( \zeta(t) \) which needs to be determined as well. The marginal operator thus is

\[ \phi(x, t) = \partial_{t} W(x, t) = -\zeta(t)\alpha'(t)x_{1}x_{2}x_{3} + \frac{\zeta'(t)}{3\zeta(t)} \sum x_{i}d_{i}W(x, t). \]

(56)

where we exhibited that it has an exact piece.\(^{5}\) We are also interested in a perturbation by

\(^{5}\)Such exact pieces correspond to linear combinations of periods in the more familiar geometrical language.
the following relevant operators:

\[ \varphi_1(x,t) = \zeta(t)g_1(t)x_1, \]
\[ \varphi_2(x,t) = \zeta(t)g_2(t)x_2x_3, \]

and the main task will be to compute \( t \)-dependent correlation functions of those. This requires first of all to determine the functions \( \alpha(t), \zeta(t), g_a(t) \).

We have already used as ansatz an initial good basis of operators. Indeed all the operators obey eq. (48), and in particular we have

\[ K^{(1)}(\phi, \phi) = 0, \]

(59)
despite that \( \phi \) has a non-vanishing exact piece. This extra freedom is allowed due to the antisymmetry of \( K^{(1)} \).

We now in turn impose the various flatness equations. First the most trivial one:

\[ K(\nabla^G M_1, \phi) \equiv K^{(0)}(\partial_1, \phi) - K^{(1)}(\phi, \phi) = 0. \]

(60)
This shows that (59) is indeed a relevant property. Next, imposing

\[ K^{(0)}(\nabla^G M \phi, 1) \equiv K^{(0)}(\partial_\phi, 1) - K^{(1)}(\phi \cdot \phi, 1) = 0, \]

(61)
yields the following differential equation:

\[ \frac{\zeta'(t)}{\zeta(t)} = \frac{\alpha''(t)}{2\alpha'(t)} - \frac{3\alpha(t)^2\alpha'(t)}{2\Delta}, \]

(62)
where \( \Delta \equiv \alpha^3 - 1 \) is the discriminant of the curve. Up to a constant, it is solved by

\[ \zeta(t) = -i\sqrt{\frac{\alpha'(t)}{3\Delta}}. \]

(63)
The rescaling of \( W \) indeed coincides with the fundamental period of the elliptic curve:

\[ \zeta(t) = \varpi_0(\alpha) \equiv \frac{1}{3\alpha} F_1(1/3, 2/3, 1; 1/\alpha^3), \]

(64)
as expected on general grounds. It also relates to Saito’s idea of a primitive form, which in this context boils down to a rescaling by \( \varpi_0 \) [68,69,75].

Moreover, we consider the equation at one level up:

\[ K^{(1)}(\nabla^G_t \phi, \phi) \equiv K^{(1)}(\partial_t \phi, \phi) - K^{(2)}(\phi \cdot \phi, \phi) = 0 \]

(65)
(all higher ones being empty). This yields the following non-linear differential equation:

\[ \{ z; t \} = -\frac{9 - 8z + 8z^2}{18z^2\Delta^2} z'^2, \]

(66)
where \( z \equiv 1/\alpha^3 \) and \( \{ z; t \} \equiv \frac{z'''}{z'} - \frac{3}{2}(\frac{z''}{z'})^2 \) is the Schwarzian derivative. Via standard arguments the solution of this equation is given by:

\[
\frac{3\alpha(\alpha^3 + 8)}{\Delta} = j(t)^{1/3},
\]

where \( j(t) = q^{-1} + 744 + \ldots \) is the familiar modular invariant function in terms of \( q = e^{2\pi it} \). This relation identifies \( \alpha(t) \) with the Hauptmodul of the modular subgroup \( \Gamma(3) \). Its inverse coincides with the well-known mirror map of the elliptic curve, given by the ratio of its periods: \( t(\alpha) = \varpi_1(\alpha)/\varpi_0(\alpha) = i/\sqrt{3}_2 F_1(1/3, 2/3, 1; 1 - 1/\alpha^3)/2 F_1(1/3, 2/3, 1; 1/\alpha^3) \).

Finally, we turn to the relevant operators. The equations to consider are

\[
K^{(0)}(\nabla_t^{GM} \varphi_1, \varphi_2) \equiv K^{(0)}(\partial_t \varphi_1, \varphi_2) - K^{(1)}(\phi \cdot \varphi_1, \varphi_2) = 0 ,
\]

\[
K^{(0)}(\varphi_1, \nabla_t^{GM} \varphi_2) \equiv K^{(0)}(\varphi_1, \partial_t \varphi_2) + K^{(1)}(\varphi_1, \phi \cdot \varphi_2) = 0 ,
\]

which give rise to \( g_{a}'/g_a = a[\zeta'/3\zeta - \alpha^2\alpha'/\Delta] \). The solutions are:

\[
g_a(t) = (\Delta(t)\zeta(t))^{a/3} , \quad a = 1, 2. \quad (69)
\]

We can similarly determine contact terms between the relevant operators, by defining them directly in terms of residue pairings, \( ie. \)

\[
CT[\varphi_a, \varphi_b] := \sum_{s} K^{(1)}((\varphi_a \cdot \varphi_b, \varphi_s) \varphi_s) ,
\]

(70)

where \( \varphi_* \) is the dual of \( \varphi_* \) with regard to the inner product defined by \( K^{(0)} \). Explicitly:

\[
CT[\varphi_1, \varphi_1] = 0 ,
\]

\[
CT[\varphi_1, \varphi_2] = -\frac{\zeta'}{\zeta^2\Delta} g_1 g_2 \cdot 1 ,
\]

\[
CT[\varphi_2, \varphi_2] = -\frac{\alpha}{\zeta^3\Delta^3} g_2^3 \cdot \varphi_1 .
\]

(73)

Usually these terms are included in the LG potential as higher order corrections, which leads to a redefinition of the flat fields. We don’t need to present these formulas here, the interested reader may consult refs. [79–81] for details.

Now we are in the position to determine correlation functions. First note that because of the flattening rescaling of \( W \) by \( \zeta \), we need to put a normalization factor to the inner product such as to restore (47), \( ie. \) \( \langle H \rangle = 1 \):

\[
\langle \langle \xi_a, \xi_b \rangle \rangle := \frac{1}{\zeta^4} K^{(0)}(\xi_a, \xi_b) .
\]

(74)

This is the natural normalization in the current setup, but might seem peculiar since usually correlators are rescaled by \( 1/\varpi_0^2 \). However all is fine because the operators have been consistently rescaled as well.
We now simply plug the renormalized operators in and find:

\[ \langle \langle \phi, 1 \rangle \rangle (t) = \langle \langle \varphi_1, \varphi_2 \rangle \rangle (t) = 1 , \]  
\[ \langle \varphi_1 \varphi_1 \varphi_1 \rangle (t) = \langle \langle \varphi_1, \varphi_1 \varphi_1 \rangle \rangle (t) = \zeta (t) , \]  
\[ \langle \varphi_1 \varphi_1 \varphi_{1,2} \varphi_{1,3} \rangle (t) = \langle \langle \varphi_1, \varphi_{1,2} \varphi_{1,3} \rangle \rangle (t) = \alpha (t) \zeta (t) , \]

(where \( \varphi_{1,i} = \varphi_1 |_{x_i \to x_i} \)). As an example for a higher point function, consider

\[ \langle \varphi_1 \varphi_1 \varphi_2 \int \varphi_2^{(2)} \rangle (t) = \langle \langle \varphi_1, \ell_3 (\varphi_1, \varphi_2, \varphi_2) \rangle \rangle , \]  
where

\[ \ell_3 (\varphi_1, \varphi_2, \varphi_2) = 2 CT [\varphi_1, \varphi_2] \cdot \varphi_2 + CT [\varphi_2, \varphi_2] \cdot \varphi_1 \]
\[ = - \frac{1}{\zeta \Delta} \left[ 2 \frac{\zeta'}{\zeta} g_1 g_2 \varphi_2 + \frac{\alpha}{\Delta} g_2^3 \varphi_1 \cdot \varphi_1 \right] . \]

The end result is then

\[ \langle \varphi_1 \varphi_1 \varphi_2 \int \varphi_2^{(2)} \rangle (t) = \frac{2 \alpha^2 \alpha'}{\Delta} - \frac{\alpha''}{\alpha'} (t) . \]

All of these correlators reproduce results that are known in the literature [63, 79–82]. Our purpose was to re-derive them in terms of a more field theoretical and less geometrical language that can be easier generalized to open strings.

3. Open String B-Model

3.1. Matrix factorizations

We like to generalize the considerations of the previous sections to the open string sector, i.e., to the B-model on the disk. As mentioned in Section 1.3, for the relevant boundary LG model the various possible B-type topological D-branes are one-to-one to the various possible matrix factorizations \( M(Q) \) of the bulk superpotential:

\[ M(Q) : \ Q(x,t) \cdot Q(x,t) = W(x,t) \ 1_{n \times n} . \]

Here \( Q(x,t) \) is the boundary BRST operator that can be represented by an odd \( n \times n \) dimensional matrix (so \( n \) is even), whose precise structure encodes the brane geometry we want to describe. This includes also a specific dependence of closed (\( t \)) and possibly open (\( u \)) string moduli. The dimension \( n \) of the Chan-Paton space can take arbitrary even values, and this reflects that there is in general an infinite number of possible (potentially reducible) brane configurations on a given CY space, \( X \). When \( n = 2^k \) for some integer \( k \), one can write \( Q \) compactly in terms of boundary fermions \( \pi, \bar{\pi} \) that form a Clifford algebra. This is what we will do, while not being essential.
In the following we will focus on the most canonical of such matrix factorizations, for which we make use of the quasi-homogeneity of the superpotential, \( W(x, t) = 1/2 \sum q_i x_i d_i W(x, t) \). Here the BRST operator is given by

\[
Q(x, t) = 1/2 q_i \pi_i x_i + \bar{\pi}_i d_i W(x, t), \quad \{\pi_i, \bar{\pi}_j\} = \delta_{ij}, \quad i, j = 1...N,
\]

where \( q_i \) are the \( R \)-charges of the LG fields.\(^6\)

The physical operators at the boundary are then given by the non-trivial cohomology classes of \( Q \). A new feature as compared to the bulk LG theory is that they are matrix valued. Moreover in general there are boundary preserving and boundary changing operators. Boundary preserving operators, denoted by \( \Psi^A \equiv \Psi^{(A,A)} \), are each tied to a single matrix factorization \( \mathcal{M}(Q^{(A)}) \), and are represented by \( n_A \times n_A \) matrices. Boundary changing operators \( \Psi^{(A,B)} \in \text{Hom}^s(\mathcal{M}(Q^{(A)}), \mathcal{M}(Q^{(B)})) \) are associated with pairs of matrix factorizations, \( Q^{(A)} \) and \( Q^{(B)} \). They are represented by not necessarily quadratic, \( n_A \times n_B \) dimensional matrices.

For a given pair of boundary conditions \((*,*)\), the ("on-shell") space of physical operators is defined by

\[
\mathcal{H}^{(*,*)}_P = \{ \Psi^{(*,*)} : [Q, \Psi^{(*,*)}] = 0, \Psi^{(*,*)} \neq [Q,...] \}, \quad (84)
\]

where \(^7\)

\[
[Q, \Psi^{(A,B)}] := Q^{(A)}_{n_A \times n_A} \Psi^{(A,B)}_{n_A \times n_B} - (-1)^s \Psi^{(A,B)}_{n_A \times n_B} Q^{(B)}_{n_B \times n_B}, \quad (85)
\]

with \( s = 0 \) or \( s = 1 \) depending on the statistics of \( \Psi \).

Another difference as compared to the bulk theory, where variations of cohomology elements map back to cohomology elements, is that in the boundary theory variations of matrix valued cohomology elements are in general not BRST invariant: \( [Q(t), \partial_t \Psi(t)] \neq 0 \). In other words, variations of such cohomology elements will generically map into the full, off-shell Hilbert space at the boundary. Thus we need to pay attention to the structure of the full Hilbert space, which is given by general, \( \mathbb{Z}_2 \) graded matrices \( M \) with polynomial entries in \( x \). By a basic theorem of Hodge and Kodaira, the latter can always be decomposed as follows:

\[
\mathcal{H} = \mathcal{H}_U \oplus \mathcal{H}_P \oplus \mathcal{H}_E. \quad (86)
\]

Here \( \mathcal{H}_U \) comprises the set of unphysical operators that are not annihilated by \( Q \) and \( \mathcal{H}_E \) comprises the exact operators that are \( Q \)-variations. Note that \emph{a priori} \( \mathcal{H}_P \) is defined only up to addition of operators in \( \mathcal{H}_E \), and similarly \( \mathcal{H}_U \) is defined only up to addition of operators

---

\(^6\)For simplicity, we put them all equal to each other, \( q_i = 2/N \). They can be reinstated easily for covering weighted projected spaces as well.

\(^7\)We will always denote by \( [-,-] \) the graded commutator which acts by definition on even and odd elements with the proper sign, and when boundary changing operators are involved, it implicitly acts from left and right in the manner defined here.
in both $\mathcal{H}_P$ and $\mathcal{H}_E$. A given decomposition is referred to as a cohomological splitting, and can be viewed as a gauge choice for the off-shell physics (see eg. [12,13]).

This is closely related to the choice of cohomology representatives. We have seen above that for the bulk theory, the proper choice of operators including definite exact pieces is an important ingredient in the determination of flat coordinates. Thus the question arises as to what preferred basis of operators to choose initially.

The choice of cohomological splitting corresponds to how the inverse of the BRST operator $Q$ is precisely defined. This inverse, or “homotopy” will also be needed for the computation of higher point correlation functions, in the form of the open string propagator $U_D$ that enters in the higher $A_\infty$ products. Its choice corresponds to the choice of an off-shell completion of the theory, and mathematically speaking corresponds to adopting a specific choice of the minimal model of the underlying $A_\infty$ algebra.

A good strategy [20] to construct the inverse of $Q$ is to regard the boundary sector as fundamental and the bulk sector as a perturbation by the superpotential, $W$. That is, we split

$$Q = Q_S + Q_W, \quad Q_S = 1/N \sum x_i \pi_i, \quad Q_W = \bar{\pi}_i d_i W,$$

and first invert $Q_S$. We must require that the inversion works properly on the full Hilbert space, i.e., on arbitrary matrices $M(x) \in \mathcal{H}$ with polynomial entries. To this end, let us make an ansatz

$$U_S = \kappa^{-1}(M) \bar{\pi}_i d_i,$$

and introduce projection operators $\Pi_* : \mathcal{H} \rightarrow \mathcal{H}_*$ with:

$$\Pi_E = Q_S \cdot U_S$$
$$\Pi_U = U_S \cdot Q_S$$
$$\Pi_P = 1 - \Pi_E^S - \Pi_U^S,$$

where $Q_S$ acts as graded commutator as in (85). We need to determine the coefficients $\kappa(M)$ such that these operators are indeed good projectors that satisfy $(\Pi_*)^2 \cdot M = \Pi_* \cdot M$. They depend on the concrete matrices on which $U_S$ acts, and for determining them, we need to adopt some normal ordered form for $M$ to map back to. For example,

$$M = \text{const}(a, b, c) \bar{\pi}^a \pi^b x^c,$$

where $a, b, c$ are multi-indices labeling all of $\bar{\pi}_i, \pi_j, x_k$. The condition of good projections then gives $\kappa(M) = \sum(a_i + c_i)$.

The full $U_D$ that inverts the complete boundary BRST operator $Q$ can then be obtained by a simple application of homological perturbation theory, i.e.,

$$U_D := U_S \cdot \sum (Q_W \cdot U_S)^l.$$
The sum terminates at a finite number of steps and so yields an exact result. By construction, \(Q\) and \(U_D\) satisfy \((\Pi_\ast)^2 = \Pi_\ast\) when used in the projectors (89), and so \(U_D\) is indeed a good propagator in the full theory with non-vanishing superpotential \(W\).

Note that this construction is not unique: we may equally well define an action of \(U_D\) from the right and consider a different normal ordering: \(M = \text{const}(a, b, c) \pi^a \bar{\pi}^b x^c\); this gives \(\kappa(M) = \sum(b_i + c_i)\). In general we may consider linear combinations which also invert \(Q\):

\[
U_D(\epsilon) := \frac{1}{2}(1 + \epsilon)U_D L + \frac{1}{2}(1 - \epsilon)U_D R, \tag{92}
\]

where \(\epsilon\) parametrizes the normal ordering ambiguity which leads to an ambiguity of the cohomological splitting of \(\mathcal{H}\) induced by the projectors \(\Pi_\ast(\epsilon)\). Fortunately, the ambiguity cancels when acting on BRST closed operators in \(\mathcal{H}_P \oplus \mathcal{H}_E\), and won’t play a role in the following. However, we note as a side remark that

\[
U_D(\epsilon) \cdot Q = -N(R + \frac{1}{2} \epsilon 1), \tag{93}
\]

where \(R = \sum \bar{\pi}_i \pi_i\) denotes the (diagonal) matrix of \(U(1)_R\) charges in the Chan-Paton space. Thus we can formally associate the normal ordering ambiguity with the grade of the \(D\)-brane, which too amounts to an arbitrary overall shift of the \(R\)-charge.

Having defined a cohomological splitting on the space of matrices, we can now write a preferred basis of the physical operators in closed form:

\[
\Psi_{i_1...i_k} := \Pi_P \cdot \pi_{i_1}...\pi_{i_k} \in \mathcal{H}_P, \quad k \leq N. \tag{94}
\]

By construction they are BRST closed but not exact: \(Q \cdot \Psi_{i_1...i_k} = 0 = U_D \cdot \Psi_{i_1...i_k}\) (ie., satisfy a Siegel-type gauge). This basis is analogous to a polynomial basis \(\{y_a(x)\}\) of \(\text{Jac}(W)\) with \(U(y) = 0\), in the bulk theory.

To conclude this section, we hasten to clarify a potential confusion: for the operators (94), where are the labels for the boundaries? The point is that LG models describe orbifolds of geometrical theories, with self-intersecting branes. As we will review later, the boundary labels of \(\Psi_{i_1...i_k}^{(A,B)}\) appear only after un-orbifolding, which leaves the form of the operators invariant.

### 3.2. Joint Bulk-Boundary Deformation Problem

In the following, we shall be interested in bulk deformations of the open string TFT defined by the matrix factorization (82); see [39, 41, 43, 44, 48–50, 83–86] for some previous works on deformations of matrix factorizations. We will denote the closed string moduli by \(t_a\) as before and possible open string moduli of a brane by \(u_\alpha\) (not to be confused with the spectral parameter \(u\)). With “moduli” we refer to operators with \(R\)-charge \(q = 2\) in the bulk or \(q = 1\) on the boundary, so that \(t_a\) and \(u_\alpha\) are dimensionless and can appear in correlation
functions in a non-polynomial way. On the other hand, we will denote relevant, “tachyonic”
deformations coupling to boundary changing operators $\Psi_\alpha$ by $s_\alpha$; being dimensionful, they
can appear in the effective potential only in a polynomial way.

Strictly speaking, because in general an effective potential $W_{\text{eff}}(t,u,s)$ will be generated,
some or all of these deformations won’t be true moduli but will be obstructed (possibly at
higher order). The true moduli space consists of the sub-locus of the joint $(t,u,s)$ deformation
space that preserves the factorization, \( Q(t,u,s)^2 = W(t) \). It can be shown [84,86] that
this supersymmetry preserving locus coincides with the critical locus, $\partial_{u,s} W_{\text{eff}}(t,u,s) = 0$,
of the effective superpotential.

In the following, we will consider only unobstructed, supersymmetry preserving factoriza-
tion loci, and concentrate on bulk-deformed boundary operators, $\Psi = \Psi(t)$; we will consider
possible boundary moduli $u_\alpha$ as frozen. More specifically, we will work to all orders in the
bulk perturbation but only infinitesimally in the boundary changing open string sector:\(^8\)
\[ Q(t,s) = Q(t) + s_\alpha \Psi_\alpha(t) . \]  (95)

Note that factorization is preserved to lowest order as long as \([Q(t),\Psi_\alpha(t)] = 0\). An ob-
struction, and thus a superpotential, can arise only if several boundary changing operators
are switched on simultaneously. Thus, for the purpose of determining “flat” representatives
$\Psi = \Psi(t)$, we can restrict our considerations to linear order in $s$ and consider the theory as
un-obstructed at this order (and thus on-shell, making computations well-defined in CFT).

We have seen in the previous section how to systematically construct canonical representa-
tives of the boundary cohomology, and even obtain explicit expressions (94) for them for
the class of factorizations we consider. However, in order to obtain actual correlation func-
tions, there is more left to do, namely the $t$-dependent renormalization $g(t)$ of the operators
$\Psi(t)$ still needs to be determined. This is in fact the main part of the work, which it is
usually neglected in the discussion of matrix factorizations.

Before we will discuss this in the next section, note that the operators that couple to
unobstructed, supersymmetry-preserving deformations parametrized by $t_\alpha$ are not just given
by the flat bulk fields defined by $\phi_\alpha(x,t) = \partial_\alpha W(x,t)$. Indeed, the reason [87] for introducing
boundary degrees of freedom in the first place had been to cancel the $B$-type supersymmetry
variation of the bulk superpotential $W$ that arises on world-sheets with boundaries. This im-
plies that the factorization-preserving deformations of $W$ that we consider, must necessarily
be accompanied by simultaneous deformations of $Q$ by certain boundary operators, $\gamma_\alpha \in H_U$.
Differently said, we deal here with a joint open/closed deformation problem where the de-
formations are locked to a common un-obstructed deformation locus via the factorization
condition (82). By definition, these induced boundary counter terms are not cohomology

\(^8\)For this to make sense if $\Psi$ is a boundary changing operator (which we assume), one should view $Q$ as
a block matrix containing $Q_A$ and $Q_B$ in the diagonal, perturbed by the off-diagonal element $s_\alpha \Psi^{(A,B)}_\alpha$. 22
elements but rather obey:
\[ [Q(t), \gamma_a(t)] = \phi_a(t)|_{\partial D} 1. \] (96)

That the bulk modulus \( \phi \) must be BRST exact at the boundary for the deformation to be un-obstructed, also follows from abstract reasoning (in that degree-two deformations in \( \text{Ext}^2(*) \) determine obstructions [14] and so must be trivial by assumption). From (96) and factorization it follows that
\[ \gamma_a(t) = \partial_a Q(t), \] (97)
up to BRST closed pieces. The point is that only the combined perturbation:
\[ \sum t_a \int \tilde{\gamma}_a \equiv \sum t_a \left( \int_{\partial D} \phi_a^{(2)} 1 - \int_{\partial D} \gamma_a^{(1)} \right), \] (98)
is preserved by the total, \( B \)-type BRST operator at the boundary,
\[ Q_{\text{tot}} = Q_{W}|_{\partial D} 1 + [Q, *], \] (99)
while the individual terms are not. This follows from (96) and the descent equation, \([Q, \phi^{(2)}] = \partial_{\perp} \phi^{(1)}\), where \( \phi^{(1)} \) is a one-form along the boundary of the disk. Physically, (98) implements the subtraction of the “Warner” contact term \( \int_{\partial D} \phi^{(1)} \) of \( \phi \) with the boundary, and mathematically represents the natural invariant pairing on the relative (co-)homology of the disk.

The coupled bulk-boundary perturbation (98) is thus a quite peculiar observable: despite neither of its individual building blocks belongs to the physical cohomology of the boundary theory, it deforms correlation functions. Whenever discussing perturbed correlation functions such as the one written in (1), one should implicitly use such combined, “relative” perturbations to cancel the bulk-boundary contact terms.

A key role is again played by contact terms, in particular the contact terms that arise when the insertion \( \int_{\partial D} \gamma^{(1)} \) hits other operators at the boundary:
\[ [Q, \int_{\partial D} \gamma^{(1)}] \Psi = \left( \int_{\partial D} \phi^{(1)} \right) \Psi + [\gamma, \Psi]. \] (100)
This follows from the descent equation \([Q, \gamma^{(1)}] = \phi^{(1)} + \partial_{\|} \gamma\). The second term on the RHS must then be cancelled by the \( t \)-variation of \( \Psi(t) \). Indeed, inverting the BRST operator in \( \partial_t [Q, \Psi] = 0 \) we can write \( \partial_t \Psi = -U_D([\gamma, \Psi]) \), whose \( Q \)-variation reproduces this term. Note that \([\gamma, \Psi] \in \mathcal{H}_E \) and thus the inversion is well-defined.

More succinctly, defining \( \Psi_g(t) \equiv g(t)\Psi(t) \) one might be tempted to write the following differential equation:
\[ \nabla_t^U \Psi_g(t) = \frac{g'(t)}{g(t)} \Psi_g(t), \text{ where} \]

\[ \nabla_t^U \equiv \frac{\partial}{\partial t} + U_D([\gamma, *]), \text{ \( \nabla_t^U : \mathcal{H}_P \to \mathcal{H}_P \)}, \] (101)
and wonder whether this is already the differential equation we are after. Actually, it is not, which is no surprise given its tautological derivation. First, the inversion of $Q$ is determined only up to BRST closed pieces in $\mathcal{H}_P \oplus \mathcal{H}_E$, and thus $g'(t)$ is not really determined. Moreover, by recalling how $\Psi(t)$ is defined in (94) in terms of $U_D$, and by explicitly following the action of $\partial_t$ through the latter’s definition in (91), it turns out that (101) is in fact an identity.

This means that the contact term that arises if $\int \gamma$ hits $\Psi$ is already taken care of by the action of $\partial_t$, and so no constraint on $g(t)$ can be obtained just from (101) alone. Rather, we expect that $g(t)$ is determined by the interplay between the integrated insertions (98) and this contact term. See Fig. 3 for a pictorial summary of the contact terms.

### 3.3. Flatness and boundary pairings

We now turn to a concrete realization of the desired flatness equations in LG language. For this we need to consider correlation functions. The most important quantity is a generalization of the Grothendieck multi-residue pairing to matrix factorizations, which was found by Kapustin and Li [34,36,88–90]. It defines an inner product at the boundary as follows:

$$\langle \Psi_\alpha, \Psi_\beta \rangle_{\partial D} \equiv \eta_{\alpha\beta} = K^{(0)}(1, \text{str}[dQ^\wedge n \cdot \Psi_\alpha \cdot \Psi_\beta]) ,$$

(102)
which is known to be non-degenerate. This pairing is well-defined for cohomology elements, and maps $\mathcal{H}_P \otimes \mathcal{H}_P \rightarrow \mathbb{C}$. This means that it satisfies trace property and cyclicity of $n$-point functions only on-shell. Off-shell where $\Psi_* \in \mathcal{H}_U$, cyclicity is violated so this pairing does not define a Calabi-Yau structure proper [15], without modifications. This problem has been addressed in refs. [49, 91], where correction terms were determined that turn (102) into a good off-shell pairing.

In the following, we will not be concerned about off-shell properties of the pairing. For now, focusing only on on-shell properties, we like to rewrite the Kapustin-Li supertrace-residue pairing in a symmetric form as follows:

$$K_{KL}^{(0)}(\Psi_\alpha, \Psi_\beta) \equiv K_{KL}^{(0)}(1; \Psi_\alpha, \Psi_\beta), \quad \text{with}$$

$$K_{KL}^{(0)}(\xi; \Psi_\alpha, \Psi_\beta) = K_{KL}^{(0)}(\xi, \sum_{k} \text{str}[dQ^\wedge k \cdot \Psi_\alpha \cdot dQ^\wedge (n-k) \cdot \Psi_\beta])$$

with (103)

$$\equiv \frac{(-1)^n}{(n+1)!} \sum_{k=0}^{n} (-1)^k |\Psi_\alpha| \sum_{\epsilon_1 \cdots \epsilon_n} \xi \int \Psi_\alpha dQ^\wedge k \cdot \Psi_\beta,$$

where $|\Psi|$ denotes the $\mathbb{Z}_2$-grade of $\Psi$. This form obtains naturally when one performs a derivation from the path integral [78], analogous to what we outlined in Section (2.2) for the bulk theory.

In view of our previous discussion, an obvious question is how to extend this to higher pairings, for example, by introducing the spectral parameter $u$. Such an extension has been constructed by Shklyarov [90].

Note however that the spectral parameter $u$ has degree (or charge) 2, which is matched to the superpotential $W$ and so matches insertions of $d/dW$ in the higher residue pairings. On the other hand, at the boundary the relevant cohomology is determined by $Q$, which has degree 1, so inversions of $Q$ (resp. contact terms) should formally be counted in terms of a degree 1, and not a degree 2 variable. This is closely tied to the fact that bulk deformations involve two fermionic integrations in $\int D\phi^{(2)}$, while boundary deformations involve only one in $\int_{\partial D} \Psi^{(1)}$. That is, the spectral parameter $u$ seems to be a genuine bulk quantity that does not capture all aspects of the boundary theory. It appears that for our purposes we need a different extension of (103), not in the direction $u$, but rather in a different, morally speaking anti-commuting direction.

Before we will discuss this, let us mention an instance where an extension in the $u$-direction appears to be useful for our purposes: namely we can consider an intermediate "bulk-boundary" pairing of the form

$$K_{bb}^{(0)}(\varphi_a, \Psi_\beta) := \Psi_\beta \cdot \text{str}[dQ^\wedge n \cdot \Psi_\beta] : \mathcal{H}_{\text{closed}} \otimes \mathcal{H}_{\text{open}} \rightarrow \mathbb{C}.$$

That is, we treat the image of the open-closed map,

$$OC[\ast] \equiv \text{str}[dQ^\wedge n \ast] : \mathcal{H}_{\text{open}} \rightarrow \mathcal{H}_{\text{closed}}, \quad \mathcal{H}_P \rightarrow \text{Jac}(W),$$

25
like any other bulk operator. In analogy to (94) we may propose as further condition for a “good” basis of operators

$$\begin{align*}
K_{bb}^{(0)}(\varphi_a, \Psi_\beta) &= \eta_{a\beta} = \text{const.} \quad (107) \\
K_{bb}^{(\ell>0)}(\varphi_a, \Psi_\beta) &= 0.
\end{align*}$$

While we do not have a proof for this requirement, we will see later that these conditions make good sense at least in the context of an explicit example.

However note, importantly, that the bulk-boundary pairing (105) affects only boundary preserving operators: $\Psi = \Psi^{(A, A)}$. As mentioned in the introduction, this is because the open-closed map vanishes identically on boundary changing operators, $\Psi^{(A, B)}$ with $A \neq B$. Thus $K_{bb}^{(\ell)}$ is insensitive to the intrinsically new features of boundaries, namely the ones that cannot be mapped to the bulk theory.

Let us return to our task and try to find a pairing that is useful for our purpose, namely ultimately formulating flatness equations. We will be guided by considering deformations of correlators in LG language, in analogy to what we have reviewed for the bulk theory. That is, requiring constancy of the inner product (102) yields

$$0 = \frac{\partial}{\partial t} K_{KL}^{(0)}(\Psi_\alpha, \Psi_\beta)$$

$$= K_{KL}^{(0)}(\frac{\partial}{\partial t} \Psi_\alpha, \Psi_\beta) + K_{KL}^{(0)}(\Psi_\alpha, \frac{\partial}{\partial t} \Psi_\beta) - K_{KL}^{(0)}(\sum_i \frac{d_i \phi}{d W}; \Psi_\alpha, \Psi_\beta)$$

$$+ K^{(0)}(1, \sum_k \text{str}[(\frac{\partial}{\partial t} dQ^k) \cdot \Psi_\alpha \cdot dQ^{(n-k)} \cdot \Psi_\beta]) + K^{(0)}(1, \sum_k \text{str}[dQ^k \cdot \Psi_\alpha \cdot (\frac{\partial}{\partial t} dQ^{(n-k)}) \cdot \Psi_\beta]).$$

The new ingredient is the action of $\partial_t$ on the $dQ$’s. This is in line of what we discussed in the previous section, namely that we deal here with a coupled bulk-boundary deformation problem. Eq. (108) explicitly manifests in LG language the heuristic correspondence:

$$\left(\int_D \phi^{(2)} - \int_{\partial D} \gamma^{(1)}\right) \leftrightarrow \left(\frac{d\phi}{dW} - \frac{d\gamma}{dQ}\right).$$

Here “$d\gamma/dQ$” has the meaning to drop a $dQ$ and replace it with its $t$-derivative, in all the proper locations.\(^9\)

We presented our problem in a way that suggests a generalization of the flatness equations as follows. We see from the structure of (108) that the kind of higher boundary pairing we are after, should involve $d/dQ$ (of degree $-1$) instead of $d/dW$ (degree $-2$). Thus we are

\(^9\)Recall that the $dQ$’s and thus $d\gamma$’s can be different for the two boundary segments, namely if these are associated with different matrix factorizations, $\mathcal{M}(A)$ and $\mathcal{M}(B)$. Relatedly, implicit in the notation $\int_{\partial D} \gamma^{(1)}$ is that there can be different boundary segments of the disk $D$, and the insertions must be done according to the respective boundary conditions.
lead to propose as first higher supertrace-residue pairing:

\[
K_{KL}^{(1)}(\Psi_\alpha, \Psi_\beta) := \frac{(-1)^{n+1}}{(n+1)!} \sum_{k=1}^{n} \frac{1}{\sum_{i=1}^{n}} \sum_{\epsilon_{i_1 \ldots i_n}} \times
\]

\[
2 \oint \mathrm{str} \left[ \left( \frac{d_{i_1} Q}{d_{i_1} W} \ldots \frac{d_{i_k} Q}{d_{i_k} W} \frac{d_{i_{k+1}} Q}{d_{i_{k+1}} W} \ldots \frac{d_{i_n} Q}{d_{i_n} W} \Psi_\beta \right) - \left( \frac{d_{i_1} Q}{d_{i_1} W} \ldots \frac{d_{i_k} Q}{d_{i_k} W} \Psi_\alpha \frac{d_{i_{k+1}} Q}{d_{i_{k+1}} W} \ldots \frac{d_{i_n} Q}{d_{i_n} W} \Psi_\beta \right) \right],
\]

which satisfies \( K_{KL}^{(1)}(\Psi_\alpha, \Psi_\beta) = (-1)^{||\Psi_\alpha||+1} K_{KL}^{(1)}(\Psi_\beta, \Psi_\alpha). \) In terms of this we can rewrite equation (108) as

\[
0 = \frac{\partial}{\partial t} K_{KL}^{(0)}(\Psi_\alpha, \Psi_\beta)
\]

\[
= -K_{KL}^{(0)} \left( \sum_i \frac{d_i \phi}{d_i W}; \Psi_\alpha, \Psi_\beta \right)
\]

\[
+ K_{KL}^{(0)} \left( \frac{\partial}{\partial t} \Psi_\alpha, \Psi_\beta \right) + K_{KL}^{(0)}(\Psi_\alpha, \frac{\partial}{\partial t} \Psi_\beta)
\]

\[
+ K_{KL}^{(1)}(\gamma \cdot \Psi_\alpha, \Psi_\beta) + K_{KL}^{(1)}(\Psi_\alpha, \gamma \cdot \Psi_\beta).
\]

This is supposed to be the boundary analog of eq. (49) in the bulk theory.

In fact, this equation represents an identity: the cohomology elements as defined in (94) are already normalized such that all inner products are constant, if the overall normalization is chosen such the one-point function of the top element is constant: \( K_{KL}^{(0)}(\Psi_{12 \ldots N}, 1) = K_{KL}^{(0)}(H, 1) = 1. \) Thus, eq. (111) by itself does not have too a great significance.

Indeed, requiring that the topological boundary two-point function (102)

\[
\langle \Psi_\alpha^{(A,B)} \Psi_\beta^{(B,A)} \rangle_{\partial D} = \eta_{\alpha \beta},
\]

be constant, is of little help for fixing the operators, because it is invariant under the relative rescaling \( \Psi^{(A,B)} \rightarrow g(t) \Psi^{(A,B)}, \ \Psi^{(B,A)} \rightarrow g(t)^{-1} \Psi^{(B,A)}. \) Thus, we cannot determine the independent renormalization factors from (111). However we need to know them because, for example, the three-point function \( \langle \Psi^{(A,B)} \Psi^{(B,C)} \Psi^{(C,A)} \rangle \) will be proportional to \( g(t)^3. \) This is why we need to find differential equations that determine the relative moduli-dependent renormalization factors for all fields individually, and not just of their products. It also relates back to what we said in the Introduction: as long as we consider only closed cycles of operators, \( \Psi^{(A,B)} \Psi^{(B,C)} \ldots \Psi^{(s,A)} \in HH_s(Cat(MF, W)) \), we cancel out important information, namely the one that intrinsically goes beyond the bulk theory.

Thus all boils down to one basic and crucial problem, namely to as to how to split equation (111) into two separate, stronger ones. Without any deeper insights, this is an ambiguous
problem, namely what forbids us to add and subtract terms to the individual pieces such that they cancel out in the sum? Alarmingly, in the end, practically all correlations functions that we want to compute will depend on this split!

The only pragmatic way we see, is to be guided by analogy of the bulk (cf., (54)) and by the structure of (111), and define “relative” connections by

$$K^{(0)}_{KL}(\nabla_t \Psi_\alpha, \Psi_\beta) := K^{(0)}_{KL}(\partial_t \Psi_\alpha, \Psi_\beta) + K^{(1)}_{KL}(\gamma \cdot \Psi_\beta) - \frac{1}{2} K^{(0)}_{KL}(\sum_i \frac{d_i \phi}{d_i W}; \Psi_\alpha, \Psi_\beta)$$  \hspace{1cm} (113)

$$K^{(0)}_{KL}(\Psi_\alpha, \nabla_t \Psi_\beta) := K^{(0)}_{KL}(\Psi_\alpha, \partial_t \Psi_\beta) + K^{(1)}_{KL}(\gamma \cdot \Psi_\alpha, \Psi_\beta) - \frac{1}{2} K^{(0)}_{KL}(\sum_i \frac{d_i \phi}{d_i W}; \Psi_\alpha, \Psi_\beta).$$

It is convenient to rewrite the differential equations into a common mode and a relative mode part as follows:

$$K^{(0)}_{KL}(\nabla_t \Psi_\alpha, \Psi_\beta) + K^{(0)}_{KL}(\Psi_\alpha, \nabla_t \Psi_\beta) = 0 \hspace{1cm} (114)$$

$$K^{(0)}_{KL}(\nabla_t \Psi_\alpha, \Psi_\beta) - K^{(0)}_{KL}(\Psi_\alpha, \nabla_t \Psi_\beta) = 0. \hspace{1cm} (115)$$

This is what we propose, without proof, for an explicit realization of the flatness equations (13) that we advertized in the Introduction. As said, the first equation can be satisfied by a judicious common mode normalization, which we assume (ratios of correlators will be invariant under changes of the overall normalization, anyway). The more non-trivial, new information is in the second equation (115) which samples the relative normalization of the operators.

A few remarks are in order.

First, note that the covariant derivatives (113) are manifestly written in terms of integrated insertions only. As mentioned in the previous section, the contact terms between $\gamma$ and $\Psi$ are already taken care of by the $t$ derivatives acting on the $\Psi$’s. One may wonder whether there could be additional, contact terms directly between $\phi$ and the $\Psi$’s as well. In fact, it is known that the possible stable degenerations of the punctured disk do not include such factorizations, rather degenerations involving bubbling off disks appear only when boundary operators hit each other; see again Figure 3. So $\phi$ can enter only as integrated operator, which means it should enter symmetrically with respect to the $\Psi$’s, precisely as it does in (113).

To see the structure of the differential equations more clearly, it is helpful to disentangle the scalar-valued renormalization of the operators, $g(t)$, from the boundary contact term. Using (101) we can rewrite

$$K^{(0)}_{KL}(\nabla_t \Psi_\alpha, \Psi_\beta)$$

$$= K^{(0)}_{KL}(g'(t) \Psi_\alpha, \Psi_\beta) - K^{(0)}_{KL}(U_D([\gamma, \Psi_\alpha], \Psi_\beta) + K^{(1)}_{KL}(\Psi_\alpha, \gamma \cdot \Psi_\beta) - \frac{1}{2} K^{(0)}_{KL}(\sum_i \frac{d_i \phi}{d_i W}; \Psi_\alpha, \Psi_\beta), \hspace{1cm} (116)$$

28
where we have implicitly rescaled $\Psi_\alpha$ by $g(t)$ and $\Psi_\beta$ by $1/g(t)$. This exhibits the interplay between the contact term of $\gamma$ with $\Psi_\alpha$, and the integrated insertions. In a sense, the renormalization factor $g(t)$ is determined by a mismatch between these terms.

Second, it may appear counter-intuitive that the middle terms on the RHS in (113) look switched. Actually being not contact terms but integrated insertions, there is no reason why $\gamma$ should stick locally to the operators on which we take derivatives. From their origin, they sample the $dQ's$ rather then the $\Psi's$. One can check that precisely the combinations given in (113) have good covariance properties under transformations $Q \rightarrow V(t)^{-1}QV(t)$. This will become evident in the example that we will discuss below. Also, note that the ordering of $\gamma$ with respect to the $\Psi's$ does not matter, up to signs.

Finally, what about higher pairings $K^{(\ell)}_{KL}$ for $\ell > 1$? Higher versions with more derivatives acting on the $\Psi's$ can be constructed in analogy to (110). Due to the anti-commuting nature of this kind of pairings and the limited number of $dQ's$, it is clear that there can be just a finite number of them. Such higher versions would play a role in a more thorough treatment. However, at this point we are not sufficiently certain about the mathematical logic of such higher pairings, so we prefer to leave this issue to later work. For now, we content ourselves to test the proposed equation (115) for the simplest possible case, namely for the cubic elliptic curve.

### 3.4. Example: the elliptic curve revisited

#### 3.4.1. B-Model computations

We now re-visit open string mirror symmetry for the cubic elliptic curve. In the physics literature this has been discussed in refs. [40, 41, 43–45]. We consider the canonical matrix factorization (83)

$$
Q(x,t) = \frac{1}{3\pi_i x_i + \bar{\pi}_i d_i W(x,t)} \qquad Q(x,t) \cdot Q(x,t) = W(x,t),
$$

(117)

$$
W(x,t) = \zeta(t) \left[ \frac{1}{3} \sum x_i^3 - \alpha(t)x_1 x_2 x_3 \right],
$$

(118)

with $\{\pi_i, \bar{\pi}_j\} = \delta_{ij}$, $i, j = 1, 2, 3$. It corresponds to a special, irreducible point of a continuous family of otherwise reducible matrix factorizations; for details see ref. [43]. This means that the open string moduli $u_\alpha$ (locations of D0 branes) are frozen, and can be put to zero in a suitable coordinate system. We will give some more geometrical information later.

Recall that a canonical basis of the non-trivial cohomology elements is given by (94), which by construction are in a generalized Siegel gauge, i.e., obey $U_D \cdot \Psi_{i_1...i_k} = 0$. Let us put
the relative renormalization factors that we need to determine, as follows:

\[
\begin{align*}
\Psi(t) &= g_0(t)^{-1} \mathbf{1} \\
\Psi_i(t) &= g_1(t) \Pi_P \cdot \pi_i \\
\Psi_{ij}(t) &= g_1(t)^{-1} \Pi_P \cdot \pi_i \pi_j, \\
\Psi_{ijk}(t) &= g_0(t) \Pi_P \cdot \pi_i \pi_j \pi_k.
\end{align*}
\]

(119)

The top element has charge 1 and represents a marginal operator that couples to the \( D0 \) brane modulus \( u \) (which we suppress). We will need only a few of these operators explicitly:

\[
\begin{align*}
\Psi_1(t) &= \pi_1 - 3 \zeta x_1 \bar{\pi}_1 + \frac{3}{2} \zeta \alpha \left[ (x_3 \bar{\pi}_2 + x_2 \bar{\pi}_3) \right], \\
g_1(t)\Psi_2(t) &= \pi_2 \bar{\pi}_3 + \frac{9}{4} \zeta^2 \left[ (2x_2^2 + x_1 \zeta \alpha) \bar{\pi}_1 \bar{\pi}_2 - (2x_3^2 + x_1 \alpha) \bar{\pi}_1 \bar{\pi}_3 \right] \\
&\quad + 9 \zeta^2 \left( x_2x_3 \frac{1}{4} (x_1^2 \alpha^2) \bar{\pi}_2 \bar{\pi}_3 - 3x_2 \zeta \bar{\pi}_2 \bar{\pi}_3 + 3x_3 \zeta^2 \bar{\pi}_3 \bar{\pi}_2 \\
&\quad + \frac{3}{2} \zeta \alpha \left[ x_3 \bar{\pi}_1 \pi_3 - x_2 \bar{\pi}_1 \pi_2 + x_1 \bar{\pi}_3 \pi_3 - x_1 \bar{\pi}_2 \pi_2 \right], \\
\end{align*}
\]

(120)

and

\[
\begin{align*}
g_0(t)\Psi_{123}(t) &= \pi_1 \pi_2 \pi_3 - \frac{27}{8} H \bar{\pi}_1 \bar{\pi}_2 \bar{\pi}_3 \\
&\quad + \frac{9}{4} \zeta^2 \left[ (x_1 \zeta x_3 + 2x_2^2) \bar{\pi}_1 \bar{\pi}_2 \pi_1 + (x_2 \zeta x_3 + 2x_1^2) \bar{\pi}_1 \bar{\pi}_2 \pi_2 - (x_1 \zeta x_2 + 2x_3^2) \bar{\pi}_1 \bar{\pi}_3 \pi_1 \right. \\
&\quad \left. - (x_2 \zeta x_3 + 2x_1^2) \bar{\pi}_1 \pi_3 \pi_3 + (x_1 \zeta x_2 + 2x_3^2) \bar{\pi}_2 \pi_3 \pi_2 + (x_1 \zeta x_3 + 2x_2^2) \bar{\pi}_3 \pi_3 \pi_3 \right] \\
&\quad + \frac{9}{4} \zeta^2 \left[ (4x_1 x_2 - x_3 \zeta x_2 \alpha^2) \bar{\pi}_1 \bar{\pi}_2 \pi_3 + (4x_2 x_3 - x_1 \zeta x_2 \alpha^2) \bar{\pi}_2 \pi_3 \pi_1 - (4x_1 x_3 - x_2 \zeta x_2 \alpha^2) \bar{\pi}_3 \pi_3 \pi_2 \right] \\
&\quad + \frac{3}{2} \zeta \alpha \left[ x_2 \bar{\pi}_1 \pi_1 \pi_2 - x_3 \bar{\pi}_1 \pi_1 \pi_3 + x_1 \bar{\pi}_2 \pi_1 \pi_2 + x_3 \bar{\pi}_2 \pi_2 \pi_3 - x_1 \bar{\pi}_3 \pi_1 \pi_3 + x_2 \bar{\pi}_3 \pi_2 \pi_3 \right] \\
&\quad - 3 \zeta \left[ x_1 \bar{\pi}_1 \pi_2 \pi_3 + x_2 \bar{\pi}_2 \pi_1 \pi_3 - x_3 \bar{\pi}_3 \pi_1 \pi_2 \right].
\end{align*}
\]

(121)

We also do some modifications: analogous to the rescaling by \( \zeta(t) \) of the superpotential \( W(x,t) \), we have an corresponding degree of freedom at the boundary, namely a rescaling of the boundary fermions as follows:

\[
\pi_i \rightarrow \rho(t) \pi_i, \quad \bar{\pi}_i \rightarrow \rho(t)^{-1} \bar{\pi}_i.
\]

(123)

This in particular affects the boundary counter term as follows:

\[
\gamma(t) \equiv \frac{\partial}{\partial t} Q(t) = U_D(\phi(t)) \mathbf{1} - 3 \frac{\rho'(t)}{\rho(t)} [Q, R], 
\]

(124)

where \( R \) is the matrix of \( R \)-charges and \( \phi \) is as given in (56). Moreover, in analogy to the bulk theory, where \( \phi \) had to be shifted by an exact piece (consistent with the condition of good
basis (94), we allow for a shift of $\Psi_{123}$ by an exact piece. It turns out that given the various possible $Q$-exact parameters, their image in the various residues is only one dimensional. So we write this extra exact piece conveniently in terms of just one parameter, $\lambda(t)$:

$$
\Psi_{123}(t) \to \Psi_{123}(t) - \lambda(t) \left[ Q, \Lambda \right],
$$

(125)

$$
\Lambda = 27^2 g_0(t) \rho(t)^{-2} x_2 \bar{\pi}_1 \bar{\pi}_2 \bar{\pi}_3 \bar{\pi}_2.
$$

(126)

We now assemble correlation functions from the Kapustin-Li and related pairings. First we rescale them like in the bulk, i.e., $\tilde{\mathcal{K}}(\ast) \equiv \zeta^{-3} \mathcal{K}(\ast)(\ldots, \ldots)$, up to a constant factor.

In particular,

$$
\left< \Psi_{\alpha}, \bar{\Psi}_{\beta} \right>_D \equiv \left< \left< \Psi_{\alpha}, \bar{\Psi}_{\beta} \right> \right> = \mathcal{K}^{(0)}_{KL}(\Psi_{\alpha}, \bar{\Psi}_{\beta}).
$$

(127)

It is easy to check the constancy of the boundary inner products:

$$
\left< \Psi(t) \bar{\Psi}_{ijk}(t) \right>_D = \left< \Psi_{i}(t) \bar{\Psi}_{jk}(t) \right>_D = \epsilon_{ijk}.
$$

(128)

Moreover we have for all operators $\tilde{\mathcal{K}}^{(1)}_{KL}(\Psi_{\ast}, \bar{\Psi}_{\ast}) = 0$.

We now compute the various pairings explicitly, and for this it suffices to consider the operators in eqs. (120-122). Let us start with $\Psi_{123}$ and consider the integrated bulk insertion first:

$$
\mathcal{K}^{(0)}_{KL} \left( \sum_i \frac{d_i \phi}{d_i \mathcal{W}}; \Psi_{123}, \Psi \right) = 3 \frac{\zeta'(t)}{\zeta(t)} + \frac{9 \alpha^2 \alpha'(t)}{4 \Delta} - 3 \lambda(t) = 6 \frac{\eta'(t)}{\eta(t)} - 3 \lambda(t).
$$

(129)

Something nice happens here, namely $\alpha(t)$ and $\zeta(t)$ conspire such as to produce the Dedekind function ($q = e^{2\pi it}$):

$$
\eta(t) \equiv q^{1/24} \prod_{n>0} (1 - q^n)(t) = \frac{(-\sqrt{3} \alpha'(t))^{1/4}}{\Delta^{1/8}}, \text{ therefore :}
$$

$$
\frac{\zeta'(t)}{\zeta(t)} = 2 \frac{\eta'(t)}{\eta(t)} - \frac{3 \alpha(t)^2 \alpha'(t)}{4 \Delta}.
$$

(130)

The nicety continues for all the other pairings:

$$
\tilde{\mathcal{K}}^{(0)}_{KL}(\partial_t \Psi_{123}, \bar{\Psi}) = \frac{g_0'(t)}{g_0(t)} \rho'(t) - \rho(t)^{-2} x_2 \bar{\pi}_1 \bar{\pi}_2 \bar{\pi}_3 \bar{\pi}_2
$$

(131)

$$
\tilde{\mathcal{K}}^{(0)}_{KL}(\Psi_{123}, \partial_t \bar{\Psi}) = -\frac{g_0'(t)}{g_0(t)}
$$

$$
\tilde{\mathcal{K}}^{(1)}_{KL}(\gamma \Psi_{123}, \bar{\Psi}) = 2 \frac{\eta'(t)}{\eta(t)} - \rho'(t) - 3 \lambda(t),
$$

$$
\tilde{\mathcal{K}}^{(1)}_{KL}(\Psi_{123}, \gamma \bar{\Psi}) = 2 \frac{\eta'(t)}{\eta(t)} + \rho'(t) + \lambda(t)
$$
Thus in the covariant derivatives (113) the $\rho$-dependence cancels out:

$$
\tilde{K}^{(0)}_{KL}(\nabla_t \Psi_{123}, \Psi) = \frac{g_0(t)}{g_0(t)} + \frac{\eta'(t)}{\eta(t)} + \frac{3}{2} \lambda(t) = -\tilde{K}^{(0)}_{KL}(\Psi_{123}, \nabla_t \Psi). \tag{132}
$$

Their sum cancels; thus the first differential equation (114) is satisfied identically, as expected. More interesting is the difference which depends on $\lambda(t)$:

$$
\tilde{K}^{(0)}_{KL}(\nabla_t \Psi_{123}, \Psi) - \tilde{K}^{(0)}_{KL}(\Psi_{123}, \nabla_t \Psi) = 2\frac{g_0(t)}{g_0(t)} + 2\frac{\eta'(t)}{\eta(t)} + 3\lambda(t) \tag{133}
$$

This is analogous to what happened in the bulk theory, where $\phi$ had to be shifted by an exact piece in order to obtain a trivial relative normalization factor between 1 and $\phi$. To fix $\lambda$, we invoke the proposed bulk-boundary pairing conditions (107):

$$
K^{(0)}_{bb}(1, \Psi_{123}) = g_0(t), \tag{134}
$$

$$
\tilde{K}^{(1)}_{bb}(\phi, \Psi_{123}) = g_0(t)\left(2\frac{\eta'(t)}{\eta(t)} + 3\lambda(t)\right).
$$

The second equation determines $\lambda(t)$ and so the differential equation finally turns into:

$$
g_0'(t) = 0, \tag{135}
$$

precisely as required. This then also fixes the first condition in (134). We thus see some degree of consistency of the procedure.

Now on to the more interesting sector, where we find:

$$
\tilde{K}^{(0)}_{KL}(\partial_t \Psi_1, \Psi_{23}) = \frac{g_1(t)}{g_1(t)} + \frac{\rho'(t)}{3\rho(t)} + \frac{2\eta'(t)}{3\eta(t)}, \tag{136}
$$

$$
\tilde{K}^{(0)}_{KL}(\Psi_1, \partial_t \Psi_{23}) = -\frac{g_1(t)}{g_1(t)} + \frac{2\rho'(t)}{3\rho(t)} + \frac{4\eta'(t)}{3\eta(t)},
$$

$$
\tilde{K}^{(1)}_{KL}(\gamma_1, \Psi_{123}) = \frac{2\eta'(t)}{\eta(t)} - \frac{2\rho'(t)}{3\rho(t)},
$$

$$
\tilde{K}^{(1)}_{KL}(\gamma_1, \Psi_{123}) = \frac{2\eta'(t)}{\eta(t)} - \frac{\rho'(t)}{3\rho(t)},
$$

and so

$$
\tilde{K}^{(0)}_{KL}(\nabla_t \Psi_1, \Psi_{23}) = \frac{g_1(t)}{g_1(t)} - \frac{\eta'(t)}{3\eta(t)} = -\tilde{K}^{(0)}_{KL}(\Psi_1, \nabla_t \Psi_{23}). \tag{137}
$$

Again, $\rho(t)$ cancels and the sum of both equations cancels. This then finally determines, up to a multiplicative constant:

$$
g_1(t) = \eta(t)^{1/3}. \tag{138}
$$
3.4.2. **B-Model correlators**

With the flattening renormalization factors at hand, we are ready to compute correlation functions. The simplest one is

\[ \langle \Psi_i \Psi_j \Psi_k \rangle_D \equiv \langle \langle \Psi_i, m_2(\Psi_j \otimes \Psi_k) \rangle \rangle = \eta(t) \epsilon_{ijk}, \]  

where the lowest \(A_\infty\) product is just matrix multiplication in the chiral ring:

\[ m_2(\Psi_j \otimes \Psi_k) \equiv \Psi_j \cdot \Psi_k = \eta(t) \Psi_{jk} + [Q, \ast]. \]  

Before we will discuss its significance below, let us first consider higher point correlators.

As pointed out in the introduction (3), higher point correlators involve higher \(A_\infty\) products. These can be recursively assembled in terms of the boundary propagator \(U_D\) defined in (91) and lower products, forming nested trees such as in Figure 2. The functional complexity, on the other hand, is largely governed by the proper “flat” renormalization factors, which are usually neglected in this context.

The first non-trivial \(A_\infty\) product is defined by

\[ m_3(\Psi_\alpha \otimes \Psi_\beta \otimes \Psi_\gamma) = U_D(\Psi_\alpha \cdot \Psi_\beta) \cdot \Psi_\gamma - (-1)^{|\Psi_\alpha|} \Psi_\alpha \cdot U_D(\Psi_\beta \cdot \Psi_\gamma), \]  

whose \(Q\)-variation measures the non-associativity of projected OPE’s:

\[ [Q, m_3(\Psi_\alpha \otimes \Psi_\beta \otimes \Psi_\gamma)] = \Pi_P(m_2(\Psi_\alpha \otimes \Psi_\beta)) \cdot \Psi_\gamma - \Psi_\alpha \cdot \Pi_P(m_2(\Psi_\beta \otimes \Psi_\gamma)). \]  

With this a particularly nice correlator can be computed explicitly, by inserting a “weak bounding chain” [14] \(\Psi_s \equiv -1/3 \sum s_i \Psi_i\) into (141). This yields

\[ m_3(\Psi_s \otimes \Psi_s \otimes \Psi_s) = \eta(t) W(s, t) \mathbf{1}, \]  

where \(W(x, t)\) is the LG superpotential defined in (118). The interpretation of this is that these are the two non-vanishing terms of the Maurer-Cartan equation (4), where the zeroth product: \(m_0 = -\eta(t) W(s, t) \mathbf{1}\) represents the curvature term of the deformed \(A_\infty\) algebra. Such a solution of the Maurer-Cartan equation with non-zero \(m_0\) is called “weakly obstructed”.

Going back to physics, note that product \(m_3\) in (143) leads to the following correlator

\[ \langle \langle \Psi_{123}, m_3(\Psi_s \otimes \Psi_s \otimes \Psi_s) \rangle \rangle = \langle \int \Psi_{123}^{(1)} \Psi_s \Psi_s \Psi_s \rangle_D = \frac{\partial}{\partial u} W_{eff}(s, t, u)|_{u=0} \]  

\[ = \eta(t) W(s, t), \]  

where \(u\) is the open string modulus that couples to the marginal boundary operator \(\Psi_{123}\). This describes an obstruction that appears if all three \(s_i\) are switched on simultaneously.
3.4.3. Instantons in the $A$-Model

So far, we have considered the topological $B$-model, where the specific brane geometry in question is encoded in the canonical matrix factorization (117) of $W$. All quantities depend on $t$ which is the flat coordinate associated with the algebraic complex structure deformation of $W$. As it is usual for LG models, the underlying geometry is the one of an orbifold, here $T_2/\mathbb{Z}_3$.

In the present situation we have a self-intersecting brane configuration, so the operators $\Psi_i$ and $\Psi_{ij}$ are localized on intersections despite not literally being boundary-changing. After undoing the orbifold, we obtain an equivariant matrix factorization with 3 different branes, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, with $RR$ charges $(r, c_1)$ given by $(2, 1), (-1, 1)$ and $(-1, 2)$, resp. (up to monodromy). The operators localized at the intersections then gain corresponding equivariant labels that govern the selection rules for correlators, $\Psi_i \rightarrow \Psi_i^{(A,B)}$, $\Psi_{ij} \rightarrow \Psi_{ij}^{(B,A)}$ and $\Psi_{ijk} \rightarrow \Psi_{ijk}^{(A,A)}$, etc. This can be visualized with help of the quiver diagram in Figure 4. For background on such equivariant matrix factorizations, see e.g., [38, 43].

We now consider the mirror geometry, where $t$ has the interpretation as a Kähler modulus. The mirror geometry is given by an orbisphere with three punctures, $P^1_{333} \simeq T_2/\mathbb{Z}_3$, where again there is just one brane: namely the Seidel special lagrangian which intersects itself three times [92]. The fundamental domain is just one-third as compared to the one of the curve, which is why $t$ is implicitly rescaled by a factor of 3. Undoing the orbifold, the Seidel lagrangian unfolds into three different, pairwise intersecting special lagrangians $\mathcal{L}_{1,2,3}$ on $T_2$. These are shown, on the covering plane, in Figure 1.

Figure 4: Quiver diagram depicting the spectrum of boundary preserving and boundary changing operators.
A-model correlation functions for these brane configurations are well-known \cite{41,93,94} essentially because for the flat torus the instanton contributions can be just read-off by measuring the areas of polygons. These results are particular examples of the general story laid out in refs. \cite{51–53}. The correlators have the structure of generalized theta functions, and in our case the three-point functions are very simple:

\[
\langle \Psi_1^{(A_1,A_2)}(u_1,u_2)\Psi_1^{(A_2,A_3)}(u_2,u_3)\Psi_1^{(A_3,A_1)}(u_3,u_1) \rangle_D = \alpha_1(\sum u_i, t) \tag{145}
\]
\[
\langle \Psi_1^{(A_1,A_2)}(u_1,u_2)\Psi_2^{(A_2,A_3)}(u_2,u_3)\Psi_3^{(A_3,A_1)}(u_3,u_1) \rangle_D = \alpha_2(\sum u_i, t) \tag{146}
\]
\[
\langle \Psi_1^{(A_1,A_2)}(u_1,u_2)\Psi_3^{(A_2,A_3)}(u_2,u_3)\Psi_2^{(A_3,A_1)}(u_3,u_1) \rangle_D = \alpha_3(\sum u_i, t),
\]

where

\[
\alpha_\ell(u,t) = e^{2\pi i(\ell/3-1/12)} \Theta \left[ \frac{(1-\ell)/3-1/2}{-1/2} \right] \frac{\nu}{3u,3t}, \tag{147}
\]

If the brane moduli are switched off: \(u_\alpha = 0\), these indeed reproduce our result (139) due to \(\alpha_1(0,t) = 0\) and \(\alpha_2(0,t) = -\alpha_3(0,t) = \eta(t)\).

This gives then the following A-model interpretation of the \(\eta\)-function that we find from the \(B\)-model: its first term, \(\eta(t) \sim q^{1/24} + \ldots\), measures the area of the smallest triangle as shown in Figure 1, which is 1/24th of the area of the fundamental domain. The higher powers take the higher wrappings into account.

This result on open-closed Gromov-Witten invariants is by no means new and is contained in the previously cited works. However, there the \(t\) dependent normalization was put by hand in order to fit the known areas of polygons, and was not computed. In a sense, the explicit mirror map between the two sides of the isomorphism (9) was missing. Our point was to make predictions for the A-model starting from the \(B\)-Model, without the need to fix functions by hand. This will be important for later applications, eg. to Calabi-Yau threefolds, where the A-model correlators are not known beforehand.

For the four-point amplitude (144), there is a reassuring relation to the works \cite{92}. The authors used matrix factorizations as well, however on the A-model side, which is surprising because in physics, matrix factorizations appear for \(B\)-type supersymmetry. What they have done, in a beautiful way, is to interpret the individual matrix entries of \(Q\) as maps which count polygons, and thereby implicitly reproduce the \(B\)-model matrices directly in terms of A-model variables. Thus \(Q\) serves as an \(A_\infty\) functor that maps from \(Fuk(T_2, \mathcal{L}_\alpha)\) into \(\text{Cat}(\mathcal{M}\mathcal{F}, W)\). At some point they had to impose a “quantum” product by hand, and this is precisely the manifestation of the structure constant \(\eta(t)\) in the product \(m_2\) (140), which arises from our \(B\)-model flatness equations. This is the most basic manifestation of the open string mirror symmetry between quantum products in the A-model, and classical products in the \(B\)-model; see Figure 5.
Figure 5: Quantum and classical products $m_2$ in the topological $A$- and $B$-models, resp. The former features genuine boundary changing open Gromov-Witten invariants. The latter is entirely determined by renormalization factors computed by the differential equation (115).

It was also shown in [92] that by counting triangles one obtains as curvature term of the $A_\infty$ algebra:

$$m_0(s, t) = \left[ \varphi(t) \sum s_i^3 - \psi(t)s_1s_2s_3 \right] 1 ,$$

where $\varphi(t)$ and $\psi(t)$ are certain modular functions. By comparison with (143), we reproduce them as follows: $\varphi \simeq \eta \xi$ and $\psi \simeq \eta \xi \alpha$, up to constants. In view of (144), we can explain the results of [92] by simply taking $u$-derivatives of (145): $\partial_u \alpha_1(u)|_{u=0} \simeq \eta \xi$, and $\partial_u \alpha_2(u)|_{u=0} = -\partial_u \alpha_3(u)|_{u=0} \simeq \eta \xi \alpha$. In this way, their results can be directly understood from $B$-model open string correlation functions.

4. Summary and outlook

In this paper we made a proposal as to how to compute correlation functions in B-type topological strings that involve boundary changing operators. This is important because in physics this corresponds to computing superpotentials for the largest class of string backgrounds with $D$-branes, namely ones involving intersecting branes. This class is infinitely richer than backgrounds without intersecting branes, and extra significance lies also in the fact that, to our knowledge, such moduli-dependent, boundary changing correlators have never been computed (though determined by hand for the elliptic curve).

Summarizing, our main result is a differential equation that is formulated in terms of certain residue pairings. It generalizes the flatness equations of the bulk theory on the sphere, whose primary component looks

$$K^{(0)}(\nabla^G M \varphi, *) \equiv K^{(0)}(\partial_\varphi, *) - K^{(1)}(\phi \varphi, *) = 0 .$$

Here $K^{(0)}(\cdot, \cdot)$ are Saito’s higher residue pairings, where $K^{(0)}(\cdot, \cdot)$ is nothing but the topological metric, and $K^{(1)}(\cdot, \cdot)$ has extra insertions of $\frac{d}{dW}$ in it. One can package all the
pairings into one quantity, by summing \( K(u)(*,*) = \sum u^\ell K^{(\ell)}(\cdot,\cdot) \), where \( u \) is the degree 2 spectral parameter. Then one can concisely write

\[
K(u)(\nabla^\text{GM}_t \varphi, *) = 0, \quad \text{with} \quad \nabla^\text{GM}_t = \partial_t - \frac{\partial_t W}{u}.
\]

Our generalization to the boundary theory looks formally similar, except that the pairings are formulated in terms of matrix factorizations which underlie \( B \)-type topological LG models on the disk. The basic pairing, namely the topological metric, is given by the Kapustin-Li supertrace residue formula, \( K^{(0)}_{KL} \) given in (102). The next higher pairing that we consider, \( K^{(1)}_{KL} \) in (110), has extra insertions of \( \frac{d}{dQ} \) in it, where \( Q \) is the BRST operator that is defined by the factorization \( Q^2 = W \). In terms of these, the flatness equation then looks

\[
\langle \langle \nabla_t \Psi^{(A,B)}, \Psi^{(B,A)} \rangle \rangle_D = 0,
\]

where the “relative” boundary-bulk connection is given in (113).

The key role at the boundary is played not by \( \phi = \partial_t W \), but by the boundary counterterm \( \gamma = \partial_t Q \). This has to do with how stable degenerations of the disk work: there is no direct contact term between \( \phi \) and boundary fields. This also reflects that at the boundary, the relevant cohomology is given by the one of \( Q \), and not of \( Q_W = \bar{\partial} + \iota_{dW} \). Morally speaking, this suggests to define a boundary connection by \( \nabla_t = 1\partial_t - \frac{\partial_t Q}{v} \), where \( v \) is a formal anti-commuting parameter of degree 1 which plays the role of the spectral parameter \( u \) in the bulk.

Whether there is more flesh to this than a naive analogy to the bulk, depends on whether one can meaningfully define higher pairings \( K^{(\ell)}_{KL} \) which would involve more insertions of \( \frac{d}{dQ} \). These should reflect the filtrated Hodge structure associated with the boundary BRST operator \( Q \). Related questions are how the general definition of a “good basis” in terms of higher residues would look like, in relation to the operator basis that we defined in (94). That should also include a boundary-bulk pairing \( K^{(\ell)}_{bb}(\cdot,\cdot) \) as defined in (105), as for \( \ell = 0 \) it is an important ingredient of the axiomatic definition of open topological strings [15,95,96].

All-in-all, conjecturally we would have for a “good basis”:

\[
\begin{align*}
K^{(0)}(\varphi_a,\varphi_b) &= \eta_{ab}, & K^{(\ell>0)}(\varphi_a,\varphi_b) &= 0 \\
K^{(0)}_{bb}(\varphi_a,\Psi_\beta) &= \eta_{a\beta}, & K^{(\ell>0)}_{bb}(\varphi_a,\Psi_\beta) &= 0 \\
K^{(0)}_{KL}(\Psi_\alpha,\Psi_\beta) &= \eta_{a\beta}, & K^{(\ell>0)}_{KL}(\Psi_\alpha,\Psi_\beta) &= 0
\end{align*}
\]

The resolution of these questions would require a deeper understanding of the underlying mathematics, which is beyond the scope of the paper. Sorting them out would likely be important for the application to higher dimensional Calabi-Yau spaces. This was our main motivation for the present work and we intend to report on it in the future.

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