A NOTE ON SHARP ONE-SIDED BOUNDS FOR THE HILBERT TRANSFORM

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Abstract. Let $\mathcal{H}$ denote the Hilbert transform on the circle. The paper contains the proofs of the sharp estimates

$$\frac{1}{2\pi} |\{\xi \in \mathbb{T} : |\mathcal{H}\xi| \geq 1\}| \leq \frac{4}{\pi} \arctan \left( \exp \left( \frac{\pi}{2} \|f\|_1 \right) \right) - 1, \quad f \in L^1(\mathbb{T}),$$

and

$$\frac{1}{2\pi} |\{\xi \in \mathbb{T} : |\mathcal{H}\xi| \geq 1\}| \leq \frac{\|f\|^2_2}{1 + \|f\|^2_2}, \quad f \in L^2(\mathbb{T}).$$

Related estimates for orthogonal martingales satisfying a subordination condition are also established.

1. Introduction

Let $\mathcal{H}$ denote the Hilbert transform on the unit circle $\mathbb{T}$ defined as the singular integral

$$\mathcal{H}f(e^{it}) = \frac{1}{2\pi} \text{p.v.} \int_{-\pi}^{\pi} f(e^{is}) \cot \frac{t - s}{2} ds, \quad f \in L^1(\mathbb{T}).$$

It is not a bounded operator on $L^1(\mathbb{T})$, but a classical result of Kolmogorov [4] states that

$$\frac{1}{2\pi} |\{\xi \in \mathbb{T} : |\mathcal{H}\xi| \geq 1\}| \leq c_1 \|f\|_1, \quad f \in L^1(\mathbb{T}),$$

for some universal constant $c_1$ (here and below the symbol $\|\cdot\|_p, p \in [1, 2]$, denotes the $p$-th norm with respect to the normalized Haar measure on $\mathbb{T}$).

Recently Osękowski [5] considered the following one-sided version of this estimate:

$$\frac{1}{2\pi} |\{\xi \in \mathbb{T} : \mathcal{H}f(\xi) \geq 1\}| \leq \|f\|_1, \quad f \in L^1(\mathbb{T}).$$

The above inequality is optimal: for every $c < 1$ there exists an integrable function $f$ such that $|\{\xi \in \mathbb{T} : \mathcal{H}f(\xi) \geq 1\}| > 2\pi c \|f\|_1$ (see [5]).

Our motivation comes from the following question. Let $m \in [0, 1]$ be a given number and suppose that a function $f \in L^1(\mathbb{T})$ satisfies $|\{\xi \in \mathbb{T} : \mathcal{H}f(\xi) \geq 1\}| = 2\pi m$. How small can $\|f\|_1$ be? (For similar problems, arising in the context of martingale transforms and the Haar system, consult the works of Burkholder [1] and Choi [2].) Clearly, inequality (1.1) gives some initial insight into this problem: we must have $\|f\|_1 \geq m$. However, this bound is not sharp (as will follow from Corollary 1.2 below).

To answer the above question, we establish a class of weak-type bounds.
**Theorem 1.1.** For every $c \in (0,1]$ and every function $f \in L^1(T)$ we have

$$
\frac{1}{2\pi} \left| \{ \xi \in T : \mathcal{H}^T f(\xi) \geq 1 \} \right| \leq c \| f \|_1 
+ 1 - \frac{2}{\pi} \arcsin(c) - \frac{2c}{\pi} \ln \left( \frac{1}{c} + \sqrt{1/c^2 - 1} \right).
$$

(1.2)

For every $c \in (0,1]$ the constant added on the right-hand side is optimal.

If we optimize the right-hand side of inequality (1.2) with respect to the parameter $c \in (0,1]$, we can rewrite the above statement in the following way.

**Corollary 1.2.** For every function $f \in L^1(T)$ we have

$$
\frac{1}{2\pi} \left| \{ \xi \in T : \mathcal{H}^T f(\xi) \geq 1 \} \right| \leq \frac{4}{\pi} \arctan \left( \exp \left( \frac{\pi}{2} \| f \|_1 \right) \right) - 1.
$$

(1.3)

Moreover, for every $m \in [0,1)$ there exists a function $f \in L^1(T)$ for which both sides of (1.3) are equal to $m$.

Equivalently: if a function $f \in L^1(T)$ satisfies $|\{ \xi \in T : \mathcal{H}^T f(\xi) \geq 1 \}| = 2\pi m$, then

$$
\| f \|_1 \geq \frac{2}{\pi} \ln \left( \tan \left( \frac{\pi}{4} (m + 1) \right) \right)
$$

and the right-hand side cannot be improved.

The second result contained in this note is the following one-sided version of the weak-type $(2,2)$ inequality for the Hilbert transform.

**Theorem 1.3.** For every $c \in [0,1]$ and every function $f \in L^2(T)$ we have

$$
\frac{1}{2\pi} \left| \{ \xi \in T : \mathcal{H}^T f(\xi) \geq 1 \} \right| \leq c^2 \| f \|_2^2 + (1 - c)^2.
$$

(1.4)

For every $c \in [0,1]$ the constant added on the right-hand side is optimal.

**Corollary 1.4.** For every function $f \in L^2(T)$ we have

$$
\frac{1}{2\pi} \left| \{ \xi \in T : \mathcal{H}^T f(\xi) \geq 1 \} \right| \leq \frac{\| f \|_2^2}{1 + \| f \|_2^2}.
$$

(1.5)

Moreover, for every $m \in [0,1)$ there exists a function $f \in L^2(T)$ for which both sides of (1.5) are equal to $m$.

As in [5] we in fact establish more general statements for orthogonal martingales which satisfy a subordination condition. We postpone further details concerning the probabilistic setting to Section 2, where we introduce all necessary definitions and formulate the probabilistic counterparts of Theorems 1.1 and 1.3.

In Section 3 we recall the construction and properties of the special function which was used to prove inequality (1.1). Then in Section 4 we explain how to modify this function in order to prove inequality (1.2). The sharpness of this inequality and the proof of Corollary 1.2 are presented in Section 5. An analogous discussion for inequality (1.3) can be found in Section 6.

In Section 7 we apply the results to find one-sided versions of the weak-type $(p,q)$ bounds on the Hilbert transform for $0 < q \leq p \in \{1,2\}$.

Finally, in Section 8 we give some remarks about one-sided versions of the weak-type $(p,p)$ inequalities for $1 < p < 2$. We also explain why inequalities (1.2) and (1.4) do not transfer to analogous estimates for the Hilbert transform on the real line.
2. Probabilistic setting

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, filtered by \((\mathcal{F}_t)_{t \geq 0}\), a nondecreasing family of sub-\(\sigma\)-algebras of \(\mathcal{F}\), such that \(\mathcal{F}_0\) contains all the events of probability 0. Let \(X = (X_t)_{t \geq 0}, Y = (Y_t)_{t \geq 0}\) be two adapted real martingales with continuous paths and let \([X,Y]\) denote their quadratic covariance process (see e.g. \([5, \text{Section} \, 3]\) for details). We say that the processes \(X\) and \(Y\) are orthogonal if \([X,Y]\) is constant almost surely. We say that \(Y\) is differentially subordinate to \(X\) if the process \([X,Y]_t - [Y,Y]_t\) is nondecreasing and nonnegative as a function of \(t\).

We establish the following results.

**Theorem 2.1.** Assume that \(X, Y\) are orthogonal martingales such that \(Y\) is differentially subordinate to \(X\) and \(Y_0 = 0\). Then, for every \(c \in (0,1]\),

\[
\begin{equation}
\Pr(\sup_{t \geq 0} Y_t \geq 1) \leq c \|X\|_1 + 1 - \frac{2}{\pi} \arcsin(c) - \frac{2c}{\pi} \ln \left(1/c + \sqrt{1/c^2 - 1}\right).
\end{equation}
\]

For every \(c \in (0,1]\) the constant added on the right-hand side is optimal.

**Theorem 2.2.** Assume that \(X, Y\) are orthogonal martingales such that \(Y\) is differentially subordinate to \(X\) and \(Y_0 = 0\). Then, for every \(c \in [0,1]\),

\[
\begin{equation}
\Pr(\sup_{t \geq 0} Y_t \geq 1) \leq c^2 \|X\|_2^2 + (1 - c)^2
\end{equation}
\]

For every \(c \in [0,1]\) the constant added on the right-hand side is optimal.

The Reader will easily formulate the probabilistic counterparts of Corollaries \([1,2]\) and \([3]\).

3. Special function from \([5]\)

Recall that the conformal mapping \(K(z) = (\sqrt{z} - 1/\sqrt{z})/2\) maps the upper half-plane \(H = \mathbb{R} \times (0, \infty)\) onto the set \(H \setminus \{ai : a \geq 1\}\). The inverse of \(K\) is given by the formula \(L(z) = 2z^2 + 1 + 2z\sqrt{z^2 + 1}\) (here and below we set \(\sqrt{z} = \sqrt{r}e^{i\theta}/2\) if \(z = re^{i\theta}, r \geq 0, \theta \in (-\pi, \pi]\).

Introduce the function \(U : H \to \mathbb{R}\) given by the Poisson integral

\[
U(\alpha, \beta) = \frac{1}{\pi} \int_0^\infty \frac{\beta \left(1 - \frac{1}{2} \sqrt{1 - t^{-1}}\right)}{(\alpha - t)^2 + \beta^2} dt, \quad (\alpha, \beta) \in H.
\]

The special function from \([5]\) which was used to prove \([1,1]\) is defined as follows: \(U(x,y) = U(L(x,y))\) for \((x,y) \in H \setminus \{ai : a \geq 1\}; U(x,y) = 1 - |x|\) for \((x,y) \in \mathbb{R} \times (-\infty, 0]; U(0,y) = 0\) for \(y \geq 1\). The next lemma sums up its properties (for the proof see \([3, \text{Section} \, 3]\)).

**Lemma 3.1.** Function \(U : \mathbb{R}^2 \to \mathbb{R}\) is continuous, symmetric with respect to the first variable: \(U(x,y) = U(-x,y)\), and enjoys the following four properties.

(i) For any \(x,y \in \mathbb{R}\) we have \(U(x,y) \geq 1_{\{y \leq 0\}} - |x|\).

(ii) For any \(x \in \mathbb{R}\) we have \(U(x,1) \leq 0\).

(iii) For any \(y \in \mathbb{R}\) the function \(U(\cdot, y) : x \mapsto U(x,y)\) is concave on \(\mathbb{R}\).

(iv) \(U\) is superharmonic.

Moreover the function \((x,y) \mapsto U(x,y) + |x|\) is bounded.

We will also need one additional property of the special function \(U\).

**Lemma 3.2.** For \(c \in [0,1]\) we have

\[
U(0,c) = 1 - \frac{2}{\pi} \arcsin(c) - \frac{2c}{\pi} \ln \left(1/c + \sqrt{1/c^2 - 1}\right).
\]
Proof. For \( c = 1 \) we see from the definition that \( U(0,1) = 0 \). Fix therefore a \( c \in (0,1) \) and denote \( A = 1 - 2c^2 \), \( B = 2c\sqrt{1-c^2} \). Then \( L(0,c) = (A,B) \) and \( A^2 + B^2 = 1 \). From the definition of \( U \) in the upper half-plane

\[
U(0,c) = \frac{\pi + 2 \arctan \left( \frac{\chi}{B} \right)}{2\pi} - B \frac{2}{\pi} \int_0^\infty \frac{\sqrt{1 - \sqrt{1 - t^2}}}{(A - t)^2 + B^2} dt.
\]

Notice that if we change the interval of integration in the remaining integral to \((1,\infty)\), then we will get exactly one half of the integral’s value (since we can substitute \(1/t \) instead of \( t \) for \( t \in (0,1) \) and use \( A^2 + B^2 = 1 \)). We then substitute \( t = s^2 \), \( 1 < t < \infty \), and what is left is to integrate a rational function.

\[
\frac{B}{2\pi} \int_0^\infty \frac{\sqrt{1 - \sqrt{1 - t^2}}}{(A - t)^2 + B^2} dt = \frac{B}{\pi} \int_1^\infty \frac{\sqrt{1 - t^{-1}}}{(A - t)^2 + B^2} dt = \frac{2B}{\pi} \int_1^\infty \frac{s^2 - 1}{s^4 - 2As + 1} ds
\]

Now we can substitute the values of \( A \) and \( B \) and use simple algebraic and trigonometric identities (remembering that \( A^2 + B^2 = 1 \)) to get

\[
U(0,c) = \frac{\pi + 2 \arctan \left( \frac{1 - \sqrt{1 - c^2}}{2c\sqrt{1-c^2}} \right)}{2\pi} - \frac{c}{\pi} \ln \left( \frac{1 + \sqrt{1-c^2}}{1 - \sqrt{1-c^2}} \right)
\]

\[
= 1 - \frac{2}{\pi} \arcsin(c) - \frac{2c}{\pi} \ln \left( \frac{1}{c} + \sqrt{1/c^2 - 1} \right),
\]

which is the assertion of the Lemma.

Remark 3.3. Lemma \[3.2\] allows us to brute-force check that function \( U(0,\cdot) \) is convex on \([0,\infty)\) simply by calculating its second derivative (an observation from the Proof of Corollary \[1.2\] in the next section simplifies the computations). This provides a new proof of \[3.2\] needed to prove property (iii) from Lemma \[3.1\] above.

4. Proofs of (2.1) and (1.2)

Proof of (2.1). Fix any \( c \in (0,1) \) \( c = 1 \) corresponds to inequality \[1.1\] and the limit case \( c = 0 \) to the trivial estimate of probability by 1) and introduce the function \( \tilde{U}(x,y) = U(cx, cy)/c \). Due to Lemma \[3.1\] this function is continuous, symmetric with respect to the first variable, and enjoys the following properties.

(i) For any \( x, y \in \mathbb{R} \) we have \( \tilde{U}(x,y) \geq \frac{1}{c}1_{\{y\leq0\}} - |x| \).

(ii) For any \( x \in \mathbb{R} \) we have \( \tilde{U}(x,1) = \tilde{U}(0,1) = U(0,1)/c \).

(iii) For any \( y \in \mathbb{R} \) the function \( \tilde{U}(\cdot,y) : x \mapsto \tilde{U}(x,y) \) is concave on \( \mathbb{R} \).

(iv) \( \tilde{U} \) is superharmonic.

We now mimic the proof of inequality \[1.1\].

First note that \( \tilde{U} \) is not smooth since \( \tilde{U}(x,y) = 1/c - |x| \) for \( y \leq 0 \). Let \( g : \mathbb{R}^2 \to [0,\infty) \) be a smooth radial function supported on a ball of center \((0,0)\) and radius 1, satisfying \( \int_{\mathbb{R}^2} g = 1 \). For \( \delta > 0 \) define

\[
\tilde{U}^\delta(x,y) = \int_{\mathbb{R}^2} \tilde{U}(x + \delta r, y + \delta s)g(r,s)drds.
\]
Function $\tilde{U}^δ$ inherits properties (i) – (iv) in a slightly changed form. For example, (i) implies that
\[
\tilde{U}^δ(x, y) \geq \int_{\mathbb{R}^2} 1_{\{y + \delta s \leq 0\}} g(r, s) dr ds - \int_{\mathbb{R}^2} |x + \delta r| g(r, s) dr ds \\
\geq 1_{\{y \leq -\delta\}} - (|x| + \delta).
\]
We also have $\tilde{U}^δ(x, 1) \leq \tilde{U}(x, 1) \leq U(0, c)/c$, since $\tilde{U}$ is superharmonic and $g$ is radial. Properties (iii) and (iv) transfer directly to function $\tilde{U}^δ$. Summarizing, function $\tilde{U}^δ$ is smooth, symmetric with respect to the first variable, and enjoys the following properties.

(I) For any $x, y \in \mathbb{R}$ we have $\tilde{U}^δ(x, y) \geq \frac{1}{c} 1_{\{y \leq -\delta\}} - (|x| + \delta)$.

(II) For any $x \in \mathbb{R}$ we have $\tilde{U}^δ(x, 1) \leq U(0, c)/c$.

(III) For any $y \in \mathbb{R}$ the function $\tilde{U}^δ(\cdot, y) : x \mapsto \tilde{U}(x, y)$ is concave on $\mathbb{R}$.

(IV) $\tilde{U}^δ$ is superharmonic.

Let $X, Y$ be martingales as in the statement, localized if necessary in order to guarantee the integrability of all random variables below. Let $\tau = \inf\{t \geq 0 : Y_t \geq 1 + \epsilon\}$. The Itô’s formula gives
\[
\tilde{U}^δ(X_{\tau \wedge t}, 1 - Y_{\tau \wedge t}) = \tilde{U}^δ(X_0, 1 - Y_0) + I_1 + I_2/2,
\]
where
\[
I_1 = \int_{0^+}^{\tau \wedge t} \tilde{U}^δ_x(X_s, 1 - Y_s) dX_s - \int_{0^+}^{\tau \wedge t} \tilde{U}^δ_y(X_s, 1 - Y_s) dY_s,
\]
\[
I_2 = \int_{0^+}^{\tau \wedge t} \tilde{U}^δ_{xx}(X_s, 1 - Y_s) d[X]_s \\
- 2 \int_{0^+}^{\tau \wedge t} \tilde{U}^δ_{xy}(X_s, 1 - Y_s) d[X, Y]_s + \int_{0^+}^{\tau \wedge t} \tilde{U}^δ_{yy}(X_s, 1 - Y_s) d[Y]_s.
\]
We will estimate the expected values of the above sums separately.

We have $Y_0 = 0$, so (II) implies that $\tilde{U}^δ(X_0, 1 - Y_0) \leq U(0, c)/c$. Moreover $\mathbb{E}I_1 = 0$, since the stochastic integrals in this sum are martingales. The middle term in $I_2$ vanishes, because $X$ and $Y$ are orthogonal. Using $\tilde{U}^δ_{xx} \leq 0$ (which follows from (III)), the subordination of $Y$ to $X$, and property (IV) we see that $I_2 \leq 0$.

After taking the expected value of both sides we arrive at $\mathbb{E}\tilde{U}^δ(X_{\tau \wedge t}, 1 - Y_{\tau \wedge t}) \leq U(0, c)/c$. Hence, by (I),
\[
\mathbb{P}(Y_{\tau \wedge t} \geq 1 + \delta) \leq c\mathbb{E}[|X_{\tau \wedge t}| + \delta] + U(0, c).
\]

Letting $\delta \to 0$ we see that $\mathbb{P}(\sup_{t \geq 0} Y_{\tau \wedge t} > 1) \leq c\mathbb{E}[|X_{\tau \wedge t}| + U(0, c) \leq c\|X\|_1 + U(0, c)$. Therefore
\[
\mathbb{P}(\sup_{t \geq 0} Y_t \geq 1 + 2\epsilon) \leq \lim_{t \to \infty} \mathbb{P}(\sup_{t \geq 0} Y_{\tau \wedge t} \geq 1) \leq c\|X\|_1 + U(0, c).
\]

After applying this bound to $X/(1 + 2\epsilon)$, $Y/(1 + 2\epsilon)$ and then letting $\epsilon \to 0$ we finally conclude that
\[
\mathbb{P}(\sup_{t \geq 0} Y_t \geq 1) \leq c\|X\|_1 + U(0, c).
\]

By Lemma 4.2 this finishes the proof of (2.1). □

Proof of (1.2). Let $B$ be a planar Brownian motion starting from $0 \in \mathbb{C}$ and let $\tau = \inf\{t \geq 0 : |B_t| = 1\}$. Denote the harmonic extensions of $f$ and $H^f$ to the unit disk by $u$ and $v$ respectively. They satisfy the Cauchy-Riemann equations and $v(0) = 0$. It follows from the Itô’s formula that the martingales $X = u(B_{\tau \wedge t})$ and
and \( Y = (v(B_{\tau,M}))_{t \geq 0} \) are orthogonal, \( Y \) is differentially subordinate to \( X \) and \( Y_0 = 0 \). Moreover \( B_\tau \) is distributed uniformly on the unit circle. Therefore the martingale inequality (2.1) applied to \( X \) and \( Y \) gives us inequality (1.2). \( \Box \)

5. Sharpness of (1.2) and (2.1), Proof of Corollary 1.2

**Sharpness of (2.1).** Let \( X = (X_t)_{t \geq 0}, Y = (Y_t)_{t \geq 0} \) be such that \( (X, 1 - Y) \) is a planar Brownian motion starting from point \((0, 1)\) and killed upon hitting the boundary of the set \( H \setminus \{ai : a \geq 1/c\} \). Then \( X, Y \) are orthogonal martingales, \( Y \) is subordinate to \( X \) and \( Y_0 = 0 \).

Introduce the stopping time \( \tau = \inf\{t > 0 : 1 - Y_t = 0\} \). The process \( \langle \tilde{U}(X_t, 1 - Y_t) \rangle_{t \geq 0} \) is a martingale of mean \( \tilde{U}(0, 1) = U(0, c)/c > 0 \) and hence

\[
U(0, c) = \mathbb{E} \tilde{U}(X_1, 1 - Y_1) = \mathbb{E} [\langle 1 - c|X_t| \rangle_{\tau \leq t} + \langle U(cX_t, c(1 - Y_t)) \rangle_{\tau > t}]
\]

\[
= \mathbb{P}(\tau \leq t) - c\mathbb{E}|X_t| + \mathbb{E} \langle U(cX_t, c(1 - Y_t)) \rangle_{\tau > t}.
\]

We now let \( t \to \infty \). On the set \( \{\tau = \infty\} \) we almost surely have \( \lim_{t \to \infty} X_t = 0 \) and \( \lim_{t \to \infty} c(1 - Y_t) = 1 \), hence \( \lim_{t \to \infty} U(cX_t, c(1 - Y_t)) = c|X_t| > 0 \). From Lemma 3.1 we know that the function \( (x, y) \mapsto U(x, y) + |x| \) is bounded, so by the dominated convergence theorem the third term in the above sum vanishes. Moreover the expected values \( \mathbb{E}|X_t| \) converge monotonically to \( \|X\|_1 \) since \( X \) is a martingale. Hence

\[
U(0, c) = \mathbb{P}(\tau < \infty) - c\|X\|_1 = \mathbb{P}(\sup_{t \geq 0} Y_t \geq 1) - c\|X\|_1.
\]

This (combined with Lemma 3.2) ends the proof of sharpness of (2.1). \( \Box \)

**Remark 5.1.** In the above example we can calculate \( \mathbb{P}(\sup_{t \geq 0} Y_t \geq 1) \) and \( \|X\|_1 \) explicitly. We know that \( z \mapsto L(cz) \) maps the set \( H \setminus \{ai : a \geq 1/c\} \) to the upper half-plane. The expression \( \mathbb{P}(\sup_{t \geq 0} Y_t \geq 1) \) is equal to the probability that the Brownian motion starting from point \( L(0, c) = (1 - 2c^2, 2c\sqrt{1 - c^2}) \) and killed upon hitting the boundary of \( H \) will terminate on \( (0, \infty) \times \{0\} \) and is therefore equal to

\[
\frac{\pi + 2\arctan \left( \frac{1 - 2c^2}{\pi \sqrt{1 - c^2}} \right)}{2\pi} = 1 - \frac{2}{\pi} \arcsin(c).
\]

(see e.g. \( \circ \) and compare with the calculations in the proof of Lemma 3.2). Hence

\[
\|X\|_1 = \frac{U(0, c) - (1 - 2\arcsin(c)/\pi)}{c} = \frac{2}{\pi} \ln \left( \frac{1}{c} + \sqrt{1/c^2 - 1} \right).
\]

**Sharpness of (1.2).** Consider a conformal mapping \( N : H \setminus \{ai : a \geq 1/c\} \to D(0, 1) \) which maps the set \( H \setminus \{ai : a \geq 1/c\} \) onto the open unit disk and satisfies \( N(0, 1) = (0, 0) \) (\( N \) is the composition of the mapping \( z \mapsto L(cz) \), which maps the set \( H \setminus \{ai : a \geq 1/c\} \) onto the upper half-plane, and the homography

\[
z \mapsto -2 \left( \frac{z - \text{Re} L(ci)}{\text{Im} L(ci)} + i \right)^{-1} - i,
\]

which maps the upper half-plane onto the unit disk). We define two conjugate harmonic functions: for \( r \in (0, 1), t \in (-\pi, \pi] \) we set \( u(re^{it}) = \text{Re} N^{-1}(re^{it}), v(re^{it}) = 1 - \text{Im} N^{-1}(re^{it}) \). Functions \( u, v \) have radial limits almost surely and the Hilbert transform of the function

\[
f(e^{it}) = \lim_{t \to 1^-} u(re^{it})
\]

is the function

\[
g(e^{it}) = \lim_{t \to 1^-} v(re^{it}).
\]
Moreover, \(|\{\xi \in T : g(\xi) \geq 1\}/(2\pi) = P(\sup_{t \geq 0} Y_t \geq 1)\) and \(\|f\|_1 = \|X\|_1\), where \(X, Y\) are like in the above proof of sharpness of (2.1). This finishes the proof of sharpness of (1.2). \(\square\)

**Proof of Corollary 1.2.** For \(c \in (0, 1)\) denote \(P(c) = 1 - 2 \arcsin(c)/\pi\), \(E(c) = 2 \ln(1/c + \sqrt{1/c^2 - 1})/\pi\), so that \(U(0, c) = P(c) - cE(c)\). A bit lengthy computation reveals that we identically have \(P'(c) - cE'(c) = 0\). Hence, for a given function \(f\) in \(L^1(T)\) the derivative of the right-hand side of (1.2) with respect to \(c \in [0, 1]\) is equal to zero whenever \(\|f\|_1 - E(c) = 0\), that is if \(c = 2 \exp\left(\frac{\pi}{2}\|f\|_1\right)/(1 + \exp(\pi\|f\|_1))\). Substituting such \(c\) into (1.2) reduces it to

\[
\frac{1}{2\pi} \left| \{\xi \in T: \mathcal{H}^T f(\xi) \geq 1\} \right| \leq c\|f\|_1 + P(c) - cE(c) = P(c),
\]

which can be further simplified to

\[
\frac{1}{2\pi} \left| \{\xi \in T: \mathcal{H}^T f(\xi) \geq 1\} \right| \leq \frac{4}{\pi} \arctan\left(\exp\left(\frac{\pi}{2}\|f\|_1\right)\right) - 1.
\]

As for sharpness, it is enough to take \(f\) which is optimal in (1.2) for the value of \(c\) we have chosen. \(\square\)

### 6. Sketch of proofs of (2.2), (1.4) and (1.5)

The proofs of inequalities (2.2), (1.4) and (1.5) are analogous to those of inequalities (2.1), (1.2) and (1.3), but we have to exploit properties of a different special function. Consider the function \(U_2 : \mathbb{R}^2 \to \mathbb{R}\) defined by the formula

\[
U_2(x, y) = \begin{cases} 
1 - x^2 & \text{if } y \geq 0, \\
(1 - y)^2 - x^2 & \text{if } 0 < y \leq 1, \\
-x^2 & \text{if } y > 1.
\end{cases}
\]

It is continuous and clearly enjoys the following properties.

1. For any \(x, y \in \mathbb{R}\) we have \(U_2(x, y) \geq 1_{\{y \leq 0\}} - x^2\).
2. For any \(x \in \mathbb{R}\) we have \(U_2(x, 1) \leq 0\).
3. For any \(y \in \mathbb{R}\) the function \(U_2(\cdot, y) : x \mapsto U(x, y)\) is concave on \(\mathbb{R}\).
4. \(U_2\) is superharmonic.

To prove inequality (2.2) we have to use the function \(\widetilde{U}_2 : \mathbb{R}^2 \to \mathbb{R}\) defined as \(\widetilde{U}_2(x, y) = U_2(x, cy)/c^2\) and follow the reasoning from Section 4.

As for sharpness, let \(X = (X_t)_{t \geq 0}, Y = (Y_t)_{t \geq 0}\) be such that \((X, 1 - Y)\) is a planar Brownian motion starting from point \((0, 1)\) and killed upon hitting the boundary of the strip \(\{(x, y) : 0 \leq y \leq 1/c\}\). Then \(X, Y\) are orthogonal martingales, \(Y\) is subordinate to \(X\) and \(Y_0 = 0\). Similarly as in Section 5 it is enough to apply the Doob’s optional sampling theorem to the martingale \((U_2(X_t, 1 - Y_t))_{t \geq 0}\) which has mean \(\widetilde{U}_2(0, 1) = U_2(0, c)/c^2 = (1 - c)^2/c^2 > 0\).

The calculations which occur during the proof of Corollary 1.4 are also straightforward. We leave the details to the Reader.

### 7. An application

For \(0 < q \leq 1\) and \(0 < q' \leq 2\) define

\[
c(1, q) = \sup_{x > 0} \frac{1}{x} \left[\frac{1}{x} \arctan\left(\exp\left(\frac{\pi}{2}x\right)\right) - 1\right]^{1/q},
\]

\[
c(2, q') = \sup_{x > 0} \frac{1}{x} \left(\frac{x^2}{1 + x^2}\right)^{1/q'} = \left(\frac{2}{q'}\right)^{1/q} \left(\frac{2}{q'} - 1\right)^{1/q' - 1/2}.
\]
Corollary 7.1. For $0 < q \leq 1$ and for every function $f \in L^1(T)$ we have
\[
\left(\frac{1}{2\pi} \left| \{x \in T : H^T f(x) \geq 1 \} \right| \right)^{1/q} \leq c(1, q) \|f\|_1.
\]

Corollary 7.2. For $0 < q \leq 2$ and for every function $f \in L^2(T)$ we have
\[
\left(\frac{1}{2\pi} \left| \{x \in T : H^T f(x) \geq 1 \} \right| \right)^{1/q} \leq c(2, q) \|f\|_2.
\]

Remark 7.3. Both above inequalities are sharp. Moreover, for $q > 1$ inequality (7.1) does not hold with any constant $c(1, q) < \infty$, since we know from Section 5 that there exist functions for which both sides of the inequality (1.1) tend to zero, but their quotient tends to one. Similarly, it makes no sense to consider (7.2) for $q > 2$.

Proof of Corollary 7.1. Using Corollary 1.2 and the definition of $c(1, q)$ we get
\[
\left(\frac{1}{2\pi} \left| \{x \in T : H^T f(x) \geq 1 \} \right| \right)^{1/q} \leq \left(\frac{4}{\pi} \arctan \left(\exp \left(\frac{\pi}{2\pi} \|f\|_1\right)\right) - 1\right)^{1/q} 
\leq c(1, q) \|f\|_1.
\]

Moreover the function
\[x \mapsto \left(\frac{4}{\pi} \arctan \left(\exp \left(\frac{\pi}{2\pi} x\right)\right) - 1\right)^{1/q}, \quad x > 0,
\]
attains its maximum at some positive $x = x_1$. By Corollary 1.2 there exists a function $f \in L^1(T)$ for which $\|f\|_1 = x_1$ and
\[
\frac{1}{2\pi} \left| \{x \in T : H^T f(x) \geq 1 \} \right| = \frac{4}{\pi} \arctan \left(\exp \left(\frac{\pi}{2\pi} x_1\right)\right) - 1.
\]
This finishes the proof of optimality of the obtained inequality. \hfill \Box

The proof Corollary 7.2 follows exactly the same lines and therefore we skip it.

8. Final remarks

We start this section by formulating an open problem: for a given $p \in (1, 2)$ identify the best constant $c(p)$ in an $L^p$-version of (1.1)
\[
\frac{1}{2\pi} \left| \{x \in T : H^T f(x) \geq 1 \} \right| \leq c(p) \|f\|_p.
\]
It would be even more interesting to find the sharp counterpart of estimate (1.3), that is to determine for each $p \in (1, 2)$ and any positive $c$ (smaller than $c(p)$) the best constant $d(p, c)$, such that
\[
\frac{1}{2\pi} \left| \{x \in T : H^T f(x) \geq 1 \} \right| \leq c \|f\|_p + d(p, c).
\]
The author believes that for $p \in (1, 2)$ the value of $c(p)$ is 1, but he was unable to solve the above problems. Nevertheless, we will try to shed some light on them.

For $p \in [1, 2]$ and $x \geq 0$ define
\[
R_p(x) = \sup \left\{ \frac{1}{2\pi} \left| \{x \in T : H^T f(x) \geq 1 \} \right| : \|f\|_p \leq x \right\}.
\]
In the previous sections we have shown that $R_1(x) = \frac{4}{\pi} \arctan \exp \left(\frac{\pi}{2\pi} x\right) - 1$ and $R_2(x) = x/(1 + x)$. One can moreover check that $R_1 \geq R_2$ (for example by comparison of derivatives of both sides).

If we knew that for any given $x \geq 0$ the function $p \mapsto R_p(x)$ is nonincreasing with respect to the parameter $p \in [1, 2]$, then we could conclude that $c(p) = 1$ for
\( p \in [1, 2] \). Indeed, from \( x \geq R_1(x) \geq R_p(x) \) we would get \( c(p) \leq 1 \). Moreover, from \( R_p \geq R_2 \) and the fact \( R_2 \) that is strictly convex and tangent to the linear function with slope 1 it would follow that we cannot have \( c(p) < 1 \).

**Remark 8.1.** For \( x \geq 1 \) the function \( p \mapsto R_p(x) \) is monotonous. Indeed, for \( p \in [1, 2] \) and \( x \geq 1 \) we have

\[
R_1(x) \geq R_1(x^{1/p}) \geq R_p(x) \geq R_2(x^{2/p}) \geq R_2(x),
\]

where we used monotonicity of \( R_1, R_2 \) in the first and last inequality, and Jensen’s inequality (if \( \|f\|_2^2 \geq x^{2/p} \), then \( \|f\|_p^p \leq x \); if \( \|f\|_p^p \leq x \), then \( \|f\|_1 \leq x \)) in the two middle passages. Unfortunately, in order to find \( c(p) \) it is crucial to control \( R_p(x) \) for small \( x \).

**Remark 8.2.** We would be able to prove inequality \( \mathbf{8.1} \) if we found a superharmonic function \( U_p \) on the plane, satisfying

\[
U_p(x, y) \geq 1_{\{y \geq 0\}} - c(p)\|x\|^p
\]

and \( U_p(0, 1) \leq 0 \). Recall that for \( p = 1 \) the function was equal to the function \( (x, y) \mapsto 1_{\{y \geq 0\}} - |x| \) on the set \( \{(x, y) : y \geq 0\} \cup \{(0, y) : y \geq 1\} \) and was harmonic on the complement of this set. For \( p = 2 \) the function \( U_2 \) was equal to the function \( (x, y) \mapsto 1_{\{y \geq 0\}} - |x|^2 \) on the set \( \{(x, y) : y \geq 0\} \cup \{(x, y) : y \geq 1\} \) and was harmonic on the set \( \{(x, y) : 0 < y < 1\} \). This suggests that for each \( p \) there should be a domain \( \Omega_p \) contained in \( \{(x, y) : y > 0\} \) such that \( U_p \) is harmonic on \( \Omega_p \) and equal to the function \( (x, y) \mapsto 1_{\{y \geq 0\}} - c(p)\|x\|^p \) outside \( \Omega_p \). For a similar phenomenon arising in the context of martingale transforms, where the corresponding problems were completely solved, see the works of Osękowski [6] and Suh [8].

The last remark addresses the problem of extending the results of the preceding sections to the Hilbert transform on the real line.

**Remark 8.3.** The problem investigated in this note does not make sense in the nonperiodic setting. By a “blowing-up the circle” argument inequality \( \mathbf{8.1} \) implies the following sharp estimate for the Hilbert transform on the real line

\[
|\{x \in \mathbb{R} : \mathcal{H}^R f(x) \geq 1\}| \leq \|f\|_1, \quad f \in L^1(\mathbb{R})
\]

(see [3] for details). Suppose that for some positive \( c \) there exists a finite constant \( D(c) \) for which

\[
|\{x \in \mathbb{R} : \mathcal{H}^R f(x) \geq 1\}| \leq c\|f\|_1 + D(c)
\]

holds for every integrable function \( f : \mathbb{R} \to \mathbb{R} \). We will show that then \( c \) has to be greater or equal than 1 (in which case we can simply take \( D(c) = 0 \)). For \( f \in L^1(\mathbb{R}) \) and \( \lambda > 0 \) define \( f_\lambda(x) = f(x/\lambda) \). Then \( \|f_\lambda\|_1 = \lambda\|f\|_1, \mathcal{H}^R f_\lambda(x) = \mathcal{H}^R f(x/\lambda) \) and

\[
|\{x \in \mathbb{R} : \mathcal{H}^R f_\lambda(x) \geq 1\}| = |\{x \in \mathbb{R} : \mathcal{H}^R f(x/\lambda) \geq 1\}| = \lambda |\{x \in \mathbb{R} : \mathcal{H}^R f(x) \geq 1\}|.
\]

After applying \( \mathbf{8.3} \) to \( f_\lambda \), dividing both sides by \( \lambda \) and letting \( \lambda \to \infty \), we arrive at

\[
|\{x \in \mathbb{R} : \mathcal{H}^R f(x) \geq 1\}| \leq c\|f\|_1.
\]

Our claim follows from the arbitrariness of \( f \in L^1(\mathbb{R}) \) and the sharpness of \( \mathbf{8.2} \).

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