On fourth Hankel determinant for functions associated with Bernoulli’s lemniscate

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Abstract

The aim of this paper is to find an upper bound of the fourth Hankel determinant $H_4(1)$ for a subclass of analytic functions associated with the right half of the Bernoulli’s lemniscate of the form $(x^2 + y^2)^2 - 2(x^2 - y^2) = 0$. The problem is also discussed for 2-fold and 3-fold symmetric functions. The key tools in the proof of our main results are the coefficient inequalities for class $P$ of functions with positive real part.

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1. Introduction

Let $A$ denote the family of all functions $f$ which are analytic in the open unit disc $U := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Therefore, each function $f \in A$ has a power series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \quad (1.1)$$

Also, let $S$ denote a subclass of $A$ which contains the univalent functions.

If $f$ and $g$ are analytic functions in $U$, then we say that $f$ is subordinate to $g$, denoted by $f \prec g$, if there exists an analytic function $w$ in $U$ with $w(0) = 0$ and $|w(z)| < |z|$ such that $f(z) = g(w(z))$. Moreover if the function $g$ is univalent in $U$, then we have $f(z) \prec g(z) \iff f(0) = g(0)$ and $f(U) \subset g(U)$.

Consider the subclass $\mathcal{S} \mathcal{L}$ of $A$ defined by

$$\mathcal{S} \mathcal{L} := \left\{ f \in A : \left| \left( \frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1, \quad z \in U \right\}.$$
The geometrical interpretation of the fact $f \in \mathcal{SL}$ is that, for any $z \in \mathbb{U}$, the ratio $zf'(z) / f(z)$ lies in the region bounded by the right half side of the Bernoulli’s lemniscate by the inequality $|u^2 - 1| < 1$. We can easily see that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}$, if and only if

$$\frac{zf'(z)}{f(z)} < \sqrt{1 + z},$$  \hspace{1cm} (1.2)

where the square root function is considered at principal branch, that is

$$\sqrt{1 + z} \bigg|_{z=0} = 1.$$  \hspace{1cm} (1.3)

Remark that the class $\mathcal{SL}$ was introduced by Sokól and Stankiewicz [21], and further studied by different authors in [2,11,17-20].

For a function $f \in \mathcal{A}$ of the form (1.1), the $q$-th Hankel determinant $H_q(n)$, with $q \geq 1$ and $n \geq 1$, was studied by Noonan and Thomas [14] and it is defined by

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \ldots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \ldots & a_{n+2q-2} \end{vmatrix}.$$  

Remarks 1.1. (i) It is well-known that the Fekete-Szegö functional $|a_3 - a_2^2|$ is $H_2(1)$, and Fekete and Szegö [9] generalized the estimate as $|a_3 - \lambda a_2^2|$ with $\lambda \in \mathbb{R}$ and $f \in \mathcal{S}$.

(ii) Moreover, we also know that the functional $|a_2 a_4 - a_3^2|$ is in fact $H_2(2)$.

(iii) The sharp upper bound of the second Hankel determinant for the familiar classes of starlike and convex functions was studied by Janteng, Halim, and Darus [12]. Thus, for $f \in \mathcal{S}^*$ and $f \in \mathcal{C}$ they obtained that $|a_2 a_4 - a_3^2| \leq 1$ and $8 |a_2 a_4 - a_3^2| \leq 1$, respectively. For second Hankel determinant see also [8].

(iv) In 2010, Babalola [5] considered the third Hankel determinant $H_3(1)$ and obtained the upper bound of the well-known classes of bounded-turning, starlike, and convex functions. Later, in 2013 Raza and Malik [16] investigated the upper bound of $H_3(1)$ for the class $\mathcal{SL}$, and they obtained that $|H_3(1)| \leq \frac{43}{576}$.

(v) Recently Arif et al. [3,4] have investigated $H_4(1)$ for some subclasses of univalent functions.

In the present investigation, we determine the upper bound of $H_4(1)$ for the subclass $\mathcal{SL}$ of analytic and normalized functions in $\mathbb{U}$. To prove our main results we need the following definition and lemmas.

We recall the class $\mathcal{P}$ of analytic functions $p$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \ z \in \mathbb{U},$$  \hspace{1cm} (1.4)

with $\text{Re} p(z) > 0$ in $\mathbb{U}$. The class $\mathcal{P}$ is known as the class of functions with positive real part.

It is well-known (see, for example, [6] or [10, p. 80]) that, if $p \in \mathcal{P}$ and has the form (1.4), then the following sharp coefficient estimates hold:

$$|c_n| \leq 2, \ n \in \mathbb{N}.$$  \hspace{1cm} (1.5)

Lemma 1.2. If $p \in \mathcal{P}$ and has the form (1.4), then

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1^2|}{2},$$

where the above inequality is proved in [1].
Lemma 1.3. [7] If \( p \in \mathcal{P} \) and has the form (1.4), then
\[
|c_{n+k} - \mu c_n c_k| < 2 \text{ for } 0 \leq \mu \leq 1.
\]
This result is due to Ravichandran and Verma [15].

Lemma 1.4. If \( p \in \mathcal{P} \) and has the form (1.4), then
\[
|Jc_1^3 - Kc_1 c_2 + Lc_3| \leq 2 (|J| + |K - 2J| + |J - K + L|).
\]

Proof. It is easy to see that
\[
|Jc_1^3 - Kc_1 c_2 + Lc_3| = |J(c_3 - 2c_1 c_2 + c_1^3) + (K - 2J)(c_3 - c_1 c_2) + (J - K + L)c_3|
\leq |J||c_3 - 2c_1 c_2 + c_1^3| + |K - 2J||c_3 - c_1 c_2| + |J - K + L||c_3|
\leq 2 (|J| + |K - 2J| + |J - K + L|),
\]
where we have used the Lemma 1.3 for \( \mu = 1, n = 1, k = 2 \), and a result due to Libra and Zlotkiewicz [13].

Lemma 1.5. [16] If \( f \in \mathcal{S} \mathcal{L} \) and has the form (1.1), then
\[
|a_3 - a_2^2| \leq \frac{1}{4}.
\]

Lemma 1.6. If \( f \in \mathcal{S} \mathcal{L} \) and has the form (1.1), then
\[
|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{4}, \quad |a_4| \leq \frac{1}{6}, \quad |a_5| \leq \frac{1}{8}.
\]
These estimates are sharp.

The first three bounds were obtained by Sokól [19] and the bound for \( |a_5| \) was proved in [15].

Lemma 1.7. If \( f \in \mathcal{S} \mathcal{L} \) and has the form (1.1), then
\[
|a_2 a_4 - a_3^2| \leq \frac{1}{16}.
\]
This result was found by Sokól [19].

2. Main results

Theorem 2.1. If \( f \in \mathcal{S} \mathcal{L} \) and of the form (1.1), then
\[
|a_3 a_5 - a_4^2| \leq 0.080574496.
\]

Proof. If \( f \in \mathcal{S} \mathcal{L} \), by using the subordination relation (1.2), it follows that
\[
\frac{zf'(z)}{f(z)} < \Phi(z), \quad (2.1)
\]
where \( \Phi(z) = \sqrt{1 + z} \) is considered at principal branch (1.3). From (2.1), there exists a function \( w \), analytic in the unit disk \( \mathbb{U} \), with \( |w(z)| \leq 1 \) in \( \mathbb{U} \), such that
\[
\frac{zf''(z)}{f'(z)} = \Phi(w(z)), \quad z \in \mathbb{U}. \quad (2.2)
\]

Thus, if we define the function \( p \) by
\[
p(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \ldots, \quad z \in \mathbb{U}, \quad (2.3)
\]
it follows that \( p \in \mathcal{P} \) and
\[
w(z) = \frac{p(z) - 1}{p(z) + 1}, \quad z \in \mathbb{U}.
\]
From (2.2) and the above relation we obtain
\[\frac{zf'(z)}{f(z)} = \sqrt{\frac{2p(z)}{p(z) + 1}}, \quad z \in \mathbb{U}. \tag{2.4}\]

Now, according to the power series expansions (1.1) and (1.4), a simple computation shows that
\[\sqrt{\frac{2p(z)}{p(z) + 1}} = 1 + \frac{1}{4}c_1z + \left(\frac{1}{4}c_2 - \frac{5}{32}c_1^2\right)z^2 + \left(\frac{1}{4}c_3 - \frac{5}{16}c_1c_2 + \frac{13}{128}c_1^3\right)z^3 + \left(-\frac{141}{2048}c_1c_2 - \frac{39}{128}c_1^2c_2 - \frac{5}{32}c_2^2 + \frac{1}{4}c_4 - \frac{5}{16}c_1c_3\right)z^4 + \ldots, \tag{2.5}\]
and
\[zf'(z) = 1 + az + \left(2a_3 - a_2^2\right)z^2 + \left(3a_4 - 3a_2a_3 + a_2^3\right)z^3 + \ldots, \quad z \in \mathbb{U}. \tag{2.6}\]

By comparing (2.5) and (2.6), we have
\[a_2 = \frac{1}{4}c_1, \tag{2.7}\]
\[a_3 = \frac{1}{8}\left(c_2 - \frac{3}{8}c_1^2\right), \tag{2.8}\]
\[a_4 = \frac{1}{12}\left(c_3 - \frac{7}{8}c_1c_2 + \frac{13}{64}c_1^3\right), \tag{2.9}\]
\[a_5 = \left(-\frac{49}{6414c_1} + \frac{17}{384}c_1^2c_2 - \frac{11}{192}c_1c_3 - \frac{1}{32}c_2^2 + \frac{1}{16}c_4\right), \tag{2.10}\]
\[a_6 = -\frac{223}{7680}c_1^3c_2 - \frac{3}{80}c_1^2c_3 + \frac{77}{1920}c_1c_4 - \frac{3}{64}c_1c_4 - \frac{5}{96}c_2c_3 + \frac{181}{40960}c_1^5 + \frac{1}{20}c_5, \tag{2.11}\]
\[a_7 = \frac{323}{4608}c_1^2c_2c_3 - \frac{17}{384}c_2c_4 - \frac{19}{480}c_1c_5 - \frac{13}{576}c_2^3 + \frac{19}{1536}c_3^2 + \frac{1}{24}c_6 - \frac{32203}{11796480}c_1^6 - \frac{4717}{184320}c_1^3c_3 + \frac{33}{1024}c_1^2c_4 - \frac{7457}{18432}c_1^2c_2 - \frac{30211}{1474560}c_1^4c_2\tag{2.12}.

From (2.8), (2.9), and (2.10), we obtain
\[|a_3a_5 - a_4^2| = \left|\frac{89}{147456}c_2c_1^4 + \frac{31}{18432}c_1^2c_2^2 + \frac{23}{4608}c_2c_1c_3 - \frac{1}{256}c_2^2 + \frac{1}{128}c_2c_4 + \frac{103}{1179648}c_1^6 \right|.

Now, re-arranging the above equation, we have
\[|a_3a_5 - a_4^2| = \left|\frac{89}{589824}c_1^4 \left(c_2 - \frac{c_1^2}{2}\right) + \frac{27}{16384}c_2 \left(\frac{253}{486}c_1^2 - c_2\right) \left(c_2 - \frac{c_1^2}{2}\right)
- \frac{3}{9216}c_1c_3 \left(\frac{5}{92}c_1^2 - c_2\right) + \frac{1}{144}c_3 \left(\frac{23}{64}c_1c_2 - c_3\right)
- \frac{1}{256}c_2 \left(\frac{37}{64}c_1^2 - c_4\right) - \frac{1}{256}c_4 \left(\frac{3}{4}c_1^2 - c_2\right)\right|.

Applying the triangle inequality, Lemma 1.2, and Lemma 1.3, we have
\[|a_3a_5 - a_4^2| \leq \frac{103}{589824} |c_1|^4 \left(2 - \frac{|c_1|^2}{2}\right) + \frac{27}{4096} \left(2 - \frac{|c_1|^2}{2}\right) + \frac{23}{2304} |c_1| + \frac{17}{288}. \tag{2.13}\]
Taking $|c_1| = y \in [0,2]$ in (2.13), it gives
\[
|a_{3a5} - a_4^2| \leq \frac{103}{589824} y^4 \left( 2 - \frac{y^2}{2} \right) + \frac{27}{4096} \left( 2 - \frac{y^2}{2} \right) + \frac{23}{2304} y + \frac{17}{288}.
\] (2.14)

The above function gets its maximum value at $y = 1.573483035$, in (2.14), we have
\[
|a_{3a5} - a_4^2| \leq 0.080574496.
\]

Theorem 2.2. If $f \in SL$ and of the form (1.1), then
\[
|a_{3a4} - a_{2a5}| \leq \frac{173}{532224} \sqrt{39963} + \frac{1}{24} \approx 0.1066468.
\]

Proof. From (2.7), (2.8), (2.9), and (2.10), we have
\[
|a_{3a4} - a_{2a5}| = \left| -\frac{59}{49152} c_1^5 + \frac{17}{3072} c_1^3 c_2 - \frac{1}{96} c_1^2 c_2 + \frac{1}{64} c_1 c_3 - \frac{1}{96} c_2 c_3 \right|.
\]

Now applying the triangle inequality, Lemma 1.2, and Lemma 1.3, we have
\[
|a_{3a4} - a_{2a5}| \leq \frac{77}{61444} |c_1| \left( 2 - \frac{|c_1|^2}{2} \right) + \frac{1}{24} + \frac{1}{32} |c_1|.
\] (2.15)

Let $|c_1| = y \in [0,2]$, then (2.15), becomes
\[
|a_{3a4} - a_{2a5}| \leq \frac{77}{12288} y \left( 2 - \frac{y^2}{2} \right) + \frac{1}{24} + \frac{1}{32} y.
\]

The above function has its maximum value at $y = \frac{2}{231} \sqrt{39963}$. This implies that
\[
|a_{3a4} - a_{2a5}| \leq \frac{173}{532224} \sqrt{39963} + \frac{1}{24} \approx 0.1066468.
\]

Theorem 2.3. If $f \in SL$ and of the form (1.1), then
\[
|a_5 - a_{2a4}| \leq \frac{7}{16}.
\]

Proof. From (2.7), (2.9), and (2.10), we obtain
\[
|a_5 - a_{2a4}| = \left| -c_1 \left( \frac{25}{2048} c_1^3 - \frac{1}{16} c_1 c_2 + \frac{5}{64} c_3 \right) - \frac{1}{16} \left( \frac{c_2^2}{2} - c_4 \right) \right|.
\]

Now by using the triangle inequality, Lemma 1.3, and Lemma 1.4, we have
\[
|a_5 - a_{2a4}| \leq \frac{7}{16}.
\]

Theorem 2.4. If $f \in SL$ and of the form (1.1), then
\[
|a_4 - a_{2a3}| \leq \frac{1}{6}.
\]

This result is sharp for the function $f(z) = z \exp \left( \int_0^{\frac{1+t^2}{t}} \right) = z + \frac{1}{6} z^4 - \frac{1}{144} z^7 + \cdots$.
\textbf{Proof.} From (2.7), (2.8), and (2.9), we have
\[|a_4 - a_2a_3| = \left| \frac{11}{384}c_3^3 - \frac{5}{48}c_1c_2 + \frac{1}{12}c_3 \right|.\]
Using Lemma 1.4, we obtain
\[|a_4 - a_2a_3| \leq \frac{1}{6}. \quad \square\]

\textbf{Theorem 2.5.} If \(f \in \mathcal{S}_2\) and of the form (1.1), then
\[|a_3a_7 - a_4a_6| \leq \frac{125999}{589824}.\]

\textbf{Proof.} From (2.8), (2.9), (2.11), and (2.12), we have
\[\begin{align*}
|a_3a_7 - a_4a_6| &= \left| \frac{19}{12288}c_4^2 + \frac{4493}{83886080}c_8 - \frac{1}{240}c_3c_5 - \frac{17}{3072}c_2c_4 + \frac{7}{4608}c_2c_3 \\
&\quad + \frac{1}{192}c_2c_6 - \frac{1}{721}c_4c_2 - \frac{25}{9216}c_2c_3 + \frac{5}{5898240}c_1c_2 \\
&\quad + \frac{1}{31}c_3c_5 - \frac{127}{61440}c_2c_3 - \frac{1}{512}c_1c_6 + \frac{9799}{773}c_5c_3 - \frac{47}{3932160}c_4c_3 \\
&\quad + \frac{1}{184320}c_1c_2c_3 - \frac{1}{125}c_1c_2c_3 - \frac{331}{240}c_1c_2c_3 + \frac{11}{4096}c_2c_4 \\
&\quad + \frac{1}{256}c_1c_3c_4. \end{align*}\]
By re-arranging the above equation, we obtain
\[\begin{align*}
|a_3a_7 - a_4a_6| &= \left| \frac{241}{92160}c_2c_3 \left( \frac{299}{964}c_2c_1 - c_3 \right) + \frac{149}{24576}c_2^2 \left( \frac{299}{2235}c_3c_1 - c_4 \right) \\
&\quad - \frac{149}{49152}c_2^2 \left( \frac{4537}{47680}c_2^2 - c_4 \right) - \frac{1}{768}c_2(c_1c_5 - c_6) \\
&\quad + \frac{331}{1474560}c_1c_3 \left( \frac{2319}{2648}c_1c_2 - c_2 \right) - \frac{31}{30720}c_1c_3 \left( \frac{331}{1488}c_2c_3 - c_5 \right) \\
&\quad - \frac{141439}{188743680}c_1c_3 \left( \frac{40437}{288278}c_1c_2 - c_2 \right) \left( c_2 - \frac{c_3}{2} \right) + \frac{1}{240}c_3 \left( \frac{15}{16}c_1c_4 - c_5 \right) \\
&\quad - \frac{9619}{5242880}c_2^2 \left( \frac{169429}{173142}c_1c_2 - c_2 \right) \left( c_2 - \frac{c_3}{2} \right) + \frac{127}{30720}c_3 \left( \frac{15}{16}c_1c_4 - c_5 \right) \\
&\quad + \frac{41}{16384}c_1c_3 \left( \frac{47}{82}c_1c_2 - c_2 \right) \left( c_2 - \frac{c_3}{2} \right) + \frac{1}{256}c_6 \left( c_2 - \frac{c_3}{2} \right) \right|. \end{align*}\]
Applying the triangle inequality, Lemma 1.2, and Lemma 1.3, the above equation becomes
\[\begin{align*}
|a_3a_7 - a_4a_6| &\leq \frac{144139}{94371840}c_1^4 \left( 2 - \frac{|c_1|^2}{2} \right) + \frac{96409}{1966080} \left( 2 - \frac{|c_1|^2}{2} \right) \\
&\quad + \frac{215}{73728}c_1^3 + \frac{10649}{92160}. \quad (2.16) \end{align*}\]
Let \(|c_1| = y \in [0, 2]\), then (2.16) becomes
\[|a_3a_7 - a_4a_6| \leq \frac{144139}{94371840}y^4 \left( 2 - \frac{y^2}{2} \right) + \frac{96409}{1966080} \left( 2 - \frac{y^2}{2} \right) + \frac{215}{73728}y^3 + \frac{10649}{92160}.\]
Clearly, the above function is decreasing so by putting \(y = 2\), we have
\[|a_3a_7 - a_4a_6| \leq \frac{125999}{589824}.\]
Theorem 2.6. If \( f \in SL \) and of the form (1.1), then
\[
|a_4a_7 - a_5a_6| \leq 0.2210481986.
\]

**Proof.** From (2.9), (2.10), (2.11), and (2.12), it follows that
\[
|a_4a_7 - a_5a_6| = \left| -\frac{1}{2304} c_1 c_3 c_5 + \frac{83}{18432} c_1 c_2 c_3^2 - \frac{1}{2304} c_2 c_3 c_4 - \frac{7}{2304} c_1 c_2 c_6 \\
+ \frac{583}{73728} c_1^2 c_2 c_4 + \frac{669}{655360} c_3 c_4^2 + \frac{1}{112} c_5^2 - \frac{3}{18432} c_3 c_2^2 \\
- \frac{184320}{3} c_1 c_2 c_3^2 - \frac{184320}{c_2^2 c_4} - \frac{3}{1280} c_1 c_2 c_4 + \frac{737280}{46080} c_1 c_2^3 c_5 \\
+ \frac{3}{1024} c_1^3 c_4 - \frac{20131}{181939328} c_1^9 - \frac{1}{320} c_4 c_5 - \frac{1310720}{137} c_1^3 c_4 + \frac{259}{c_1 c_2^2} c_1^2 \\
- \frac{6912}{13} c_3^3 + \frac{18432}{5} c_1^4 c_5 + \frac{18432}{c_2^2 c_6} - \frac{1105920}{c_1^3 c_3^2} - 23592960 c_1^2 c_2 \\
+ \frac{439633}{4529848320} c_1^7 c_2 - \frac{3}{8192} c_1 c_2 c_3^2 - \frac{515}{4718592} c_1 c_3 + \frac{1}{288} c_3 c_6 \right|.
\]

This implies that
\[
|a_4a_7 - a_5a_6| = \left| \frac{18934}{11796480} c_1^2 c_3 \left( \frac{2575}{18934} c_1^2 - c_2 \right) \left( c_2 - \frac{c_1^2}{2} \right) \\
+ \frac{1}{3840} c_2 \left( \frac{2167}{256} c_3 c_2 - c_5 \right) \left( c_2 - \frac{c_1^2}{2} \right) + \frac{3}{1024} c_1 c_4 \\
+ \frac{3}{4096} c_1^2 c_3 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{13}{9216} c_1 \left( \frac{5}{13} c_1 c_5 - c_6 \right) \left( c_2 - \frac{c_1^2}{2} \right) \\
+ \frac{332519}{566231040} c_1^3 c_2 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{10739}{56623104} c_1^3 \left( c_2 - \frac{c_1^2}{2} \right) \\
+ \frac{3431}{2949120} c_1 c_4 \left( \frac{1233}{6862} c_1^2 - c_2 \right) \left( c_2 - \frac{c_1^2}{2} \right) \\
+ \frac{169489}{1132462080} c_1^5 \left( \frac{100655}{677956} c_1^2 - c_2 \right) \left( c_2 - \frac{c_1^2}{2} \right) + \frac{3}{640} c_3 c_4 \left( c_2 - \frac{c_1^2}{2} \right) \\
- \frac{1}{288} c_5 \left( \frac{13}{24} c_3^2 - c_6 \right) + \frac{793056}{283115520} c_1 c_2 - c_3 \\
+ \frac{59}{11520} c_2 c_3 \left( \frac{965}{944} c_1 c_4 - c_4 \right) + \frac{1}{320} c_5 \left( \frac{7}{12} c_2^2 - c_4 \right) \\
+ \frac{5}{3072} c_1 c_2 \left( \frac{413}{1600} c_2 c_4 - c_6 \right) - \frac{1}{2304} c_3 c_1 c_5 \right|.
\]

Using the triangle inequality, Lemma 1.2, and 1.3, we have
\[
|a_4a_7 - a_5a_6| \leq \frac{9467}{1474560} |c_1|^2 \left( 2 - \frac{|c_1|^2}{2} \right) + \frac{15109}{368640} \left( 2 - \frac{|c_1|^2}{2} \right) \\
+ \frac{322063}{35389440} |c_1| \left( 2 - \frac{|c_1|^2}{2} \right) + \frac{323519}{141557760} |c_1|^3 \left( 2 - \frac{|c_1|^2}{2} \right) \\
+ \frac{169489}{566231040} |c_1|^5 \left( 2 - \frac{|c_1|^2}{2} \right) + \frac{23}{1152} |c_1| + \frac{20677}{184320}. \tag{2.17}
\]
Let \(|c_1| = y \in [0, 2]\), then (2.17) becomes

\[
|a_4a_7 - a_5a_6| \leq \frac{9467}{1474560}y^2 \left(2 - \frac{y^2}{2}\right) + \frac{15109}{368640} \left(2 - \frac{y^2}{2}\right) + \frac{322063}{3539440}y \left(2 - \frac{y^2}{2}\right)
+ \frac{323519}{14557760}y^3 \left(2 - \frac{y^2}{2}\right) + \frac{169489}{56621040}y^5 \left(2 - \frac{y^2}{2}\right) + \frac{23}{1152}y + \frac{20677}{184320}.
\]

As the above function attains its maximum value at \(y = 1.082047787\), so the above equation becomes

\[
|a_4a_7 - a_5a_6| \leq 0.2210481986.
\]

\( \square \)

**Theorem 2.7.** If \(f \in S_L\) and of the form (1.1), then

\[
|H_3(1)| \leq \frac{43}{576}.
\]

**Proof.** Since

\[
|H_3(1)| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = |a_3| |a_2a_4 - a_3^2| + |a_4| |a_4 - a_2a_3| + |a_5| |a_3 - a_2^2|.
\]

Using Lemma 1.6, Lemma 1.5, and Lemma 1.7, we get

\[
|H_3(1)| \leq \frac{43}{576}.
\]

\( \square \)

**Theorem 2.8.** If \(f \in S_L\) and of the form (1.1), then

\[
|H_4(1)| \leq 0.06786551485.
\]

**Proof.** Since

\[
|H_4(1)| \leq |a_2a_4 - a_3^2| |a_3a_7 - a_4a_6| + |a_2a_3 - a_4| |a_4a_7 - a_5a_6|
+ |a_5| \left\{ |a_3| |a_3a_5 - a_2^2| + |a_5| |a_5 - a_2a_4| + |a_6| |a_4 - a_2a_3| \right\}
+ |a_4| \left\{ |a_4| |a_3a_5 - a_2^2| + |a_5| |a_2a_5 - a_3a_4| + |a_6| |a_2a_4 - a_3^2| \right\}.
\]

Using Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4, Theorem 2.7, Theorem 2.5, Theorem 2.6, and Lemma 1.6, we have

\[
|H_4(1)| \leq 0.06786551485.
\]

\( \square \)

3. **Bounds of \(|H_{4,1}(f)|\) for the sets \(SL^{(2)}\) and \(SL^{(3)}\)**

Let \(m \in \mathbb{N} = \{1, 2, \ldots\}\). A domain \(\Lambda\) is said to be \(m\)-fold symmetric if a rotation of \(\Lambda\) about the origin through an angle \(2\pi/m\) carries \(\Lambda\) on itself. It is easy to see that, an analytic function \(f\) is \(m\)-fold symmetric in \(\mathbb{U}\), if

\[
f \left(e^{2\pi i/m}z\right) = e^{2\pi i/m}f \left(z\right), \quad (z \in \mathbb{U}).
\]

By \(S^{(m)}\), we mean the set of \(m\)-fold univalent functions having the following Taylor series form

\[
f \left(z\right) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad (z \in \mathbb{U}). \quad (3.1)
\]
The sub-family $SL^{(m)}$ of $S^{(m)}$ is the set of $m$-fold symmetric starlike functions associated with lemniscate of Bernoulli. More intuitively, an analytic function $f$ of the form (3.1) belongs to the family $SL^{(m)}$, if and only if
\[
\frac{zf'(z)}{f(z)} = \sqrt{\frac{2p(z)}{p(z) + 1}} \text{ with } p \in \mathcal{P}^{(m)},
\]
where the set $\mathcal{P}^{(m)}$ is defined by
\[
\mathcal{P}^{(m)} = \left\{ p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} c_{mk} z^{mk}, \ (z \in \mathbb{U}) \right\}.
\]

(3.2)

Now we can prove the following theorem.

**Theorem 3.1.** Let $f \in SL^{(2)}$ be of the form (3.1). Then
\[
|H_{4,1}(f)| \leq \frac{13}{3072}.
\]

**Proof.** Since $f \in SL^{(2)}$, therefore there exists a function $p \in \mathcal{P}^{(2)}$ such that
\[
\frac{zf'(z)}{f(z)} = \sqrt{\frac{2p(z)}{p(z) + 1}}.
\]
For $f \in SL^{(2)}$, using the series form (3.1) and (3.2) when $m = 2$, we can write
\[
a_3 = \frac{1}{8} c_2, \ a_5 = -\frac{1}{32} c_2^2 + \frac{1}{16} c_4, \ a_7 = \frac{19}{1536} c_2^3 - \frac{17}{384} c_4 c_2 + \frac{1}{24} c_6.
\]
It is clear that for $f \in SL^{(2)}$,
\[
H_{4,1}(f) := a_3 a_5 a_7 - a_3^3 a_7 + a_5^2 a_5^2 - a_5^3.
\]
Therefore
\[
H_{4,1}(f) = -\frac{4}{786432} \left( \frac{1}{4} c_2^2 - c_4 \right) \left( 20 \left( \frac{7}{20} c_2^2 - c_4 \right) c_2^2 + \left( 16 c_2 c_6 + 48 \left( c_2 c_6 - c_4^2 \right) \right) \right).
\]
Using Lemma 1.3 and the triangle inequality, we get
\[
|H_{4,1}(f)| \leq \frac{8}{786432} \left( 160 + 64 + 192 \right) = \frac{13}{3072}.
\]
Hence the proof is complete.

**Theorem 3.2.** If $f \in SL^{(3)}$ be of the form (3.1), then
\[
|H_{4,1}(f)| \leq \frac{8}{3456}.
\]

**Proof.** Since $f \in SL^{(3)}$, therefore there exists a function $p \in \mathcal{P}^{(3)}$ such that
\[
\frac{zf'(z)}{f(z)} = \sqrt{\frac{2p(z)}{p(z) + 1}}.
\]
For $f \in SL^{(3)}$, using the series form (3.1) and (3.2) when $m = 3$, we can write
\[
a_4 = \frac{1}{12} c_3, \ a_7 = -\frac{13}{576} c_2^3 + \frac{1}{24} c_6.
\]
It is clear that for $f \in SL^{(3)}$,
\[
H_{4,1}(f) := -a_4^2 a_7 + a_4^4.
\]
Therefore
\[ H_{4,1}(f) = \frac{17}{82944} c_3^4 - \frac{1}{3456} c_3^2 c_6 = -\frac{c_3^2}{3456} \left( c_6 - \frac{58752}{82944} c_3^2 \right). \]

Using Lemma 1.3 and triangle inequality, we get
\[ |H_{4,1}(f)| \leq \frac{8}{3456}. \]

Hence the proof is complete. \( \square \)

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