Information capacity of optical fiber channels with zero average dispersion

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We study the statistics of optical data transmission in a noisy nonlinear fiber channel with a weak dispersion management and zero average dispersion. Applying path integral methods we have found exactly the probability density functions of channel output both for a non-linear noisy channel and for a linear channel with additive and multiplicative noise. We have obtained analytically a lower bound estimate for the Shannon capacity of considered nonlinear fiber channel.

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\textit{Introduction} The classical theorem of information theory\textsuperscript{1} states that the capacity of a power-constrained transmission in an additive Gaussian noise channel grows logarithmically with increase of the signal to noise ratio (SNR). Thus, an improvement of the capacity (maximum average information per symbol that can be transmitted through the channel) in such systems can be achieved by increase of the signal power assuming that the noise level is not affected. The Gaussian statistics of noise is a fundamental assumption in derivation of this widely known Shannon’s result. Properties and applications of bandlimited linear channels with additive white Gaussian noise (AWGN) form a foundation of modern information theory. It should be emphasized that the AWGN linear channel model is not just a simple mathematical construction, but is applied directly to many practical problems such as, for instance, deep-space communication. However, in some applications, nonlinear response of a transmission medium must be taken into account. Evidently, properties of nonlinear information channel can be significantly different from that for AWGN models. Interaction of noise and signal in nonlinear transmission channel can result in non-Gaussian statistics of received signals. The theory of non-Gaussian information channels though being an evident challenge for many decades is not yet well established compared to the success of AWGN models. Studies in this fundamental research area are additionally motivated by practical technological achievements and growing demand for efficient high-speed, high quality communications. Recent progress in fiber optics attracts much fresh interest to the information theory of non-Gaussian nonlinear communication channels\textsuperscript{2,3}. Optical fiber waveguides made of silica present low loss, ultra-high capacity, cost-efficient transmission media with many attractive features. Using optical amplifiers to recover signal power simultaneously at many carrier frequencies (channels) within the fiber bandwidth it is possible to transmit optical information data over thousands of kilometres. It is already well recognized, however, that the nonlinear response of the transmission medium (Kerr effect nonlinearity) plays a crucial role in limiting the aggregate capacity of optical fiber systems. Accumulation of nonlinear interactions with propagation along the transmission line makes fiber information channels essentially nonlinear. Evidently, nonlinear impairments (or in other words, a level of signal corruption due to nonlinearity) depend on the signal power. Therefore, in nonlinear channels an increase of the signal power would not necessarily improve the system capacity. Recently, in their pioneering work Mitra and Stark suggested that from the information theory perspective under certain conditions one can treat essentially nonlinear noisy channels as linear ones with effective multiplicative noise\textsuperscript{2}. Applying this idea to multi-channel optical fiber transmission systems they derived a heuristic linear model with multiplicative noise that presumably approximates some features of the original nonlinear channel. Though a connection between statistical properties of such an effective ”nonlinear noise” and system/signal characteristics is still a subject of further research and justification, this intuitive approach outlines a possible way to treat nonlinear transmission channels. In order to compute the Shannon capacity it is necessary to make one more step beyond determination of a conditional probability. Namely, one has to find the optimal input signal statistics (that is even more complicated functional problem). The channel capacity $C$ defined by Shannon is a maximum of the following functional (called mutual information) with respect to the statistics of input signal $X$, given by distribution function $p(X)$:

$$C = \max_{p(X)} \int DXYDYP(X,Y) \log_2 \frac{P(X,Y)}{P_{\text{out}}(Y)p(X)}.$$ (1)

Here $P(X,Y) = P(Y|X)p(X)$ is the joint distribution function of input $X$ and output $Y$: $P_{\text{out}}(Y) = \int DXP(Y|X)p(X)$, and all specific properties of a communication channel are given by the conditional probability $P(Y|X)$. To the best of our knowledge the only case for which there exists an explicit analytical solution of the corresponding functional optimization problem is when the joint distribution of input and output signals are Gaussian. In this case the Shannon capacity can be
Schrödinger equation: the main order is described by the stochastic nonlinear dispersion management (see for details e.g. [9], [10]) in approaches to the estimation of system capacity: first, responding transmission systems. We compare here two conditional probabilities we analyze the capacity of cor-

of the channel output for both models. Using our derived late analytically the probability density function (PDF) to examine the similarity and difference general problems in the information theory of nonlinear channels that can be treated analytically. Such solvable models can provide guidance to analysis of much more complicated problems in the information theory of nonlinear fiber channels.

In this Letter we present a theoretical analysis of a physical model which describes the transmission of light signals in a noisy nonlinear fiber channel with zero average dispersion. To examine the similarity and difference between the effects of nonlinearity and multiplicative noise, in parallel, we study a linear model of the channel with both additive and multiplicative noise. We calculate analytically the probability density function (PDF) of the channel output for both models. Using our derived conditional probabilities we analyze the capacity of corresponding transmission systems. We compare here two approaches to the estimation of system capacity: first, based on Pinsker’s formula for input-output correlation matrix and, second, directly applying Shannon’s definition of the capacity.

The average propagation of a complex light envelope $E(z, t)$ in a noisy optical fibre line with the so-called weak dispersion management (see for details e.g. [9], [10]) in the main order is described by the stochastic nonlinear Schrödinger equation:

$$\frac{\partial E}{\partial z} = i < d > \frac{\partial^2 E}{\partial^2 z} + i|E|^2 E + n. \quad (2)$$

Here $n(z, t)$ is an additive complex white noise with zero mean and correlation function (see for notations [9])

$$< n(z, t) n^*(z', t') > = < n_0 > \delta(z - z') \delta(t - t'). \quad (3)$$

In the present Letter we restrict consideration to the case of weakly dispersion-managed fiber systems with zero average dispersion $< d > = 0$. The propagation equation then is effectively reduced to the Langevin equation for the regularized field $u(z) \equiv E(z, 0)$ with the regularized noise $\eta(z) \equiv n(z, 0)$.

$$\frac{du}{dz} - i|u|^2 u = \eta, \quad u(z = 0) = u_0, \quad (4)$$

Here $\eta(z)$ is a white noise with zero mean and correlation function $< \eta(z) \eta^*(z') > = D \delta(z - z')$, where $D = 2W < n_0 >$ is the regularized noise intensity. To restore the capacity for a bandwidth limited signal one has simply to multiply all the corresponding results by the channel bandwidth $W$.

Calculation of a conditional probability Some statistical properties of system [11] including higher-order momenta have been studied by Mecozzi [11]. However, the method used in [11] did not permit explicit computation of the PDF which is required in the analysis of system capacity. Therefore, to calculate the conditional probability $P(u, z|u_0)$ we apply here the so-called Martin-Siggia-Rose formalism [12] that presents the conditional PDF of the output as the following functional integral:

$$P(u, z|u_0) = \sum_{q(0)=u_0}^{+\infty} P' (r, \phi', z|r_0, \phi_0) e^{i m \int_0^z d\varsigma q(\varsigma)^2} \prod_{m=\infty}^{-\infty} \left[ q(0) - r_0 e^{i \phi_0} \right. \left. \right]$$

where the effective Lagrangian is defined as $L[q] = (2D)^{-1}|q' - i|q|^2 q|^2$. Integral [4] can be calculated analytically. The substitution $q(z) = \tilde{q}(z) \exp \left[ i \int_{0}^{z} d\varsigma \left( \tilde{q}(\varsigma)^2 \right) \right]$ brings Lagrangian to its free form. The Jacobian of this transform is unity and in the new variables the integral becomes Gaussian. After simple straightforward algebra it can be reduced to

$$P(u, z|u_0) = \sum_{m=\infty}^{+\infty} c_{im} e^{i m \phi} \prod_{m=\infty}^{-\infty} \left[ q(0) - m \right. \left. \right]$$

where the auxiliary “partition function” is

$$P' (r, \phi', z|r_0, \phi_0) \equiv \int_0^z Dq e^{i \phi} \exp \left[ i \int_{0}^{z} d\varsigma \left( \tilde{q}(\varsigma)^2 \right) \right]$$

(here $u = re^{i \phi}$, $u_0 = r_0 e^{i \phi_0}$). The effective action decomposes into sum of the classical part and a fluctuating part that does not depend on the limits. The fluctuating field is calculated by expanding over the complete set of eigenfunctions of the operator $-\partial^2_z + k_m^2$ satisfying zero boundary conditions at $z' = 0$ and $z' = z$. Omitting de-

tails of these operations we present a final expression for the conditional probability of our nonlinear channel:

$$P(u, z|u_0) = \frac{1}{2\pi D} \sum_{m=\infty}^{+\infty} c_{im} e^{i m \phi} \prod_{m=\infty}^{-\infty} \left[ q(0) - m \right. \left. \right]$$

where $c_{im} = \frac{1}{2\pi D} \sum_{m=\infty}^{+\infty} e^{i m \phi} \prod_{m=\infty}^{-\infty} \left[ q(0) - m \right. \left. \right]$.
here \( q_m = k_m r_0 / (D \sinh(k_m z)) \), \( k_m = \sqrt{2imD} \) and \( I_{|m|} \) is the modified Bessel function.

Next we establish an analogy between the considered nonlinear channel (NLCH) and a linear channel with multiplicative noise (LMNCH):
\[
\begin{align*}
 u' - ivu &= \eta, \quad u(z = 0) = u_0, \\
 < \eta'(z)\eta(z) >= D\delta(z - z'), \\
 < v(z)v(z') >= D'\delta(z - z')
\end{align*}
\]
Applying a similar procedure to above we derive the conditional probability function of the form Eq.(8) with replacement \( P_m \rightarrow \tilde{P}_m \), where
\[
\tilde{P}_m(r, z|r_0) = \frac{1}{Dz} e^{-m' r^2/2} I_{|m|} \left( \frac{rr_0}{Dz} \right) e^{-r^2 + r'^2/2}. \tag{12}
\]
Note that if the information is transmitted using only signal power (the so-called intensity modulation - direct detection systems) \( r = |u| \) the conditional probability takes the form (after integration in polar coordinates \( r, \phi \) over phase \( \phi \))
\[
P_0(r, z|r_0) = \frac{1}{Dz} I_0 \left( \frac{r_0 r}{Dz} \right) e^{-r^2 + r'^2/2}. \tag{13}
\]
Note that in both cases (nonlinear and effective multiplicative noise channels) formulae \( \S \) and \( \U \) yield the same result.

Channel capacity First we revise the procedure commonly used in the recent literature for the channel capacity estimation. We demonstrate here that the consideration based on pair correlation functions \( \S \) can lead to results very different from the Shannon capacity and, therefore, should be used with caution. Some authors \( \S \) instead of using the original Shannon definition calculate capacity by exploiting a simpler Pinsker formula based on a complex self-conjugate input-output correlation matrix \( \C_{\alpha\beta} \):
\[
\C_G = \log_2 \left| \frac{\text{Det diag}(\C_{\alpha\beta})}{\text{Det} \ C_{\alpha\beta}} \right|, \quad C_{\alpha\beta} \equiv < u_\alpha u_\beta^* > \tag{14}
\]
Here indices \( \alpha, \beta = \text{input, output} \); and brackets stand for the average over noise \( \eta \) for non-linear problem and \( \eta \) and \( \nu \) for double noise model and over statistics of the input signal \( u_0 \) (which is assumed to be Gaussian). Defined in this way the Gaussian capacity \( \C_G \) coincides with the Shannon capacity for the case of Gaussian joint input-output distributions which corresponds to the linear channel with additive noise \( \S \). For nonlinear channels or channels with multiplicative noise the Gaussian capacity \( \U \) represents the lower estimate for the true Shannon capacity \( \C \) (see \( \U \)).

We start from the calculation of the correlation matrix. To perform noise averages we use either PDF \( \S \) or \( \U \). It is easy to find that \( C_{\text{in,in}} = \langle |u_0|^2 \rangle \equiv S \), \( C_{\text{out, out}} = \langle |u(z)|^2 \rangle = (S + N), \quad N \equiv 2Dz \) regardless the model. However, the cross-correlations \( C_{\text{in, out}} = \langle u_0 u^* \rangle \) are different
\[
C_{\text{in, out}} = \begin{cases} 
\frac{S \sech^2 k_1 z}{1+(S/N)k_1 z \tanh k_1 z}, & \text{NLCH} \\
Se^{-D'z/2}, & \text{LMNCH}
\end{cases} \tag{15}
\]
where \( k_1 = \sqrt{2iD} \). Note that SNR = \( S/N = s \) changes only due to variation of \( S \), while \( N = 2Dz \) is fixed as we consider here a fixed transmission distance. Substitution of the correlation matrix into the definition Eq.(14) yields the final result
\[
\C_G = \log_2 \left[ 1 + \frac{s}{(1+s)(a(z)|1 + s b(z))|1 - s} \right], \tag{16}
\]
where
\[
a(z) = \begin{cases} 
|\cosh k_1 z|^4, & \text{NLCH} \\
\frac{e^{D'z}}{Dz}, & \text{LMNCH}
\end{cases}
\]
\[b(z) = \begin{cases} 
k_1 z \tanh k_1 z, & \text{NLCH} \\
0, & \text{LMNCH}
\end{cases}\tag{17}\tag{18}
\]
It is seen from Eq.(15) that with increase of SNR \( \C_G \) decays to zero for the case of the nonlinear channel (similar to conclusions made in \( \S \) and \( \U \)) and tends to a constant for the case of the multiplicative noise channel. However, below we will show that in both cases the true Shannon capacity \( \C \) is unbounded and grows logarithmically with increase of \( S/N \) similar to the linear channel.

Direct estimate of the Shannon capacity Following Shannon \( \S \) we consider now the channel capacity \( \C \) defined as a maximum of the mutual information with respect to the statistics of input, \( u_0 \), given by distribution function \( p(u_0) \) under the fixed average input power \( S \):
\[
\C = \max_{p(u_0)} \int d^2 u d^2 u_0 P(u, u_0) \log_2 \frac{P(u, u_0)}{P_{\text{out}}(u)p(u_0)}. \tag{19}
\]
The conditional probability \( P(u|u_0) \) connecting output and input probabilities: \( P_{\text{out}}(u) = \int d^2 u_0 P(u|u_0)p(u_0) \) is given either by \( \S \) or \( \U \). Note that the Shannon definition allows one to obtain directly an estimate of capacity. Any arbitrary trial distribution \( p(u_0) \) provides for a certain low boundary estimate of the capacity \( \C \). The closer a trial function is to the optimal distribution of \( p(u_0) \) the better is our approximation of the true capacity. Applying the so-called Klein inequality for two arbitrary probability distribution functions \( P \) and \( \mathcal{P} \)
\[
\int d^2 u d^2 u_0 P(u, u_0) \log_2 \frac{P(u, u_0)}{P_{\text{out}}(u)p(u_0)} \geq 0 \tag{20}
\]
we obtain the following chain of inequalities:
\[
\C \geq \int d^2 u d^2 u_0 P(u, u_0) \log_2 \frac{P(u, u_0)}{P_{\text{out}}(u)p(u_0)} \geq \int d^2 u d^2 u_0 P(u, u_0) \log_2 \frac{P(u, u_0)}{P_{\text{out}}(u)p(u_0)} \tag{21}
\]
where \( P \) is an arbitrary PDF (by this we mean that it is non-negative and normalized) and \( p(u_0) \) is an arbitrary (not optimal) initial signal distribution. Next we exploit an arbitrariness of \( P \) and \( p(u_0) \) in \(^{21}\) by choosing \( p(u_0) = (2\pi)^{-1}p(r_0) \), \( P(u, u_0) = (2\pi)^{-2}P_0(r|r_0)p(r_0) \). Here we assumed that both an input distribution \( p(u_0) \) and \( P \) are phase independent and \( P_0(r|r_0) \) is the radial conditional probability given by Eq. \((13)\). Substitution of these trial functions into inequality Eq. \((21)\) brings it to the form

\[
C \geq \int dr dr' r r_P(r|r_0) p(r_0) \log_2 \frac{P_0(r|r_0)}{\int dr' r' P_0(r|r') p(r')}.
\]

(22)

Evaluation of the r.h.s. of this inequality for any trial function leads to an estimate of a lower bound for the Shannon capacity. Substituting \( P(r, z|r_0) \) from \(^{15}\), and considering a Gaussian trial function for input statistics \( p(r_0) = (2/S) \exp(-r_0^2/S) \), after simple algebra we obtain:

\[
C \geq C_0(s) = \ln(1 + s) - 2s + F_1(s) \tag{23}
\]

\[
F_1(s) = s^{-1} \int_0^\infty dx x K_0(x\sqrt{1 + s^{-1}}) I_0(x) \ln I_0(x)
\]

where \( I_0 \) and \( K_0 \) are modified Bessel functions and \( s = S/N \) is the SNR. Then the main contribution from the integral \( F_1(s) \) to the asymptotic behavior of \( C_0(s) \) for large \( s \) comes from the region \( x \gg 1 \). Using the asymptotic expansion of modified Bessel functions we get

\[
C \geq \frac{1}{2} \ln s + O(1).
\]

This proves that \( C_0 \) and hence the Shannon capacity \( C \) are both unbounded as \( S/N \rightarrow \infty \).

Our result in particular shows that a naive straightforward application of the Pinsker formula for evaluation of the capacity of a nonlinear channel as, for instance, in \(^{19}\) can lead to wrong conclusions regarding the asymptotic behavior of the capacity with \( S/N \rightarrow \infty \). Note that for the specific problem considered here it is possible to modify the definition of \( C_{\alpha \beta} \) to obtain correct asymptotics for capacity using input-output correlation matrix. Indeed, calculation of \( C_G \) constructed with correlators \( \langle r^2 \rangle_S = S, \quad \langle r \rangle_S = S + N, \quad \langle r_0 \rangle_S = S F_2(s), \)

\[
F_2(s) = (2s^2)^{-1} \int_0^\infty dx x^2 I_0(x) K_0(x\sqrt{1 + s^{-1}}),
\]

leads to \( C_G = \ln(1 + s)/(1 + s [1 - F_2(s)]) \). Taking into account that \( F_2(s \rightarrow \infty) \rightarrow 1 \) one can see that it gives the correct asymptotic behavior for capacity. Unfortunately, there is no general recipe for choosing the correct correlators in the Pinsker formula.

**Discussion and conclusions** We have examined the statistics of optical data transmission in a noisy nonlinear fiber channel with a weak dispersion management and zero average dispersion. We have also studied similarity and difference between effects of nonlinearity and multiplicative noise, considering in parallel a linear channel with multiplicative (and additive) noise. Using analytically calculated conditional PDF we analyzed the Shannon transmission capacity for both models. We did manage to find analytically a lower bound estimate for the Shannon capacity of the nonlinear fiber channel considered here. We revise the Pinsker formula which has been used without justification in some recent works and show that the Gaussian capacity defined through the pair correlation functions should be used with caution in the case of nonlinear transmission channels. To incorporate the optimization procedure inherent for the Shannon definition one needs to elaborate all possible correlators and find those which are essential, i. e. are much greater than others. Those correlators may then be used in the Pinsker formula to provide a simple and tractable expression for the channel capacity. That would not be necessary if the Shannon definition could be worked out. Unfortunately, it is hardly the case for any more or less practical problem of interest. Another important result of our analysis is that nonlinearity and multiplicative noise do not necessarily degrade input-output correlations in the same way. Therefore, relating the nonlinear problem to a linear one with multiplicative noise has to be carefully justified for each specific transmission system model.

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