Non-topological Domain Walls in a Model with Broken Supersymmetry

Leonardo Campanelli\textsuperscript{1,*} and Marco Ruggieri\textsuperscript{2,†}

\textsuperscript{1}Dipartimento di Fisica, Università di Bari, I-70126 Bari, Italy

\textsuperscript{2}Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

\textbf{Telephone number:} +81-75-753-7027

\textbf{Abstract}

We study non-topological, charged planar walls (Q-walls) in the context of a particle physics model with supersymmetry broken by low-energy gauge mediation. Analytical properties are derived within the flat-potential approximation for the flat-direction raising potential, while a numerical study is performed using the full two-loop supersymmetric potential. We analyze the energetics of finite-size Q-walls and compare them to Q-balls, non-topological solitons possessing spherical symmetry and arising in the same supersymmetric model. This allow us to draw a phase diagram in the charge-transverse length plane, which shows a region where Q-wall solutions are energetically favored over Q-balls. However, due to their finiteness, such finite-size Q-walls are dynamically unstable and decay into Q-balls in a time which is less than their typical scale-length.

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\textsuperscript{*}leonardo.campanelli@ba.infn.it

\textsuperscript{†}ruggieri@yukawa.kyoto-u.ac.jp

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I. INTRODUCTION

Solitons are “space-localized” states which appear in certain field theories (see [1] for a review). Depending on the nature of the boundary conditions, solitons can be categorized as either topological or non-topological.

In the former case, the non-perturbative solution of the equation of motions which corresponds to the soliton is characterized by a topological charge; its stability is then guaranteed by the conservation of the topological charge, which is usually zero for the vacuum and non-zero for the soliton.

In the case of a non-topological soliton [2–6], instead, the boundary conditions are the same as those of the vacuum state, while its stability, once certain conditions are fulfilled, is guaranteed by the conservation of a Noether charge associated to an additional global symmetry of the action.

An interesting class of non-topological solitons is constituted by spherical symmetric configurations known as Q-balls. Firstly introduced by Coleman [7], they were subsequently studied by Kusenko [8] who found general conditions under which Q-balls are allowed as the ground state of $U(1)$-charged scalar field theories.

Q-balls are solitons in more than one spatial dimension and the formal environment in which they are understood is similar to that of vacuum decay [9–12]. In order to avoid Derrick theorem [13], Q-ball configurations must be time-dependent. However, their properties can be made stationary once a suitable time dependence of the solution is imposed [7]:

$$\Phi(t, r) = \frac{1}{\sqrt{2}} e^{i\omega t} \phi(r), \quad (1)$$

where $\phi(r)$ is the real part of the the complex scalar field $\Phi(t, r)$ describing the Q-ball solution, and $\omega$ is a real number which represents frequency of rotation in internal $U(1)$ space. Spherical symmetry of the solution in the above equation is kept manifest by writing the dependence on the coordinates only by $r \equiv |r|$.

There exist several studies about Q-balls, which analyze both their general properties [14–17], and implications for astrophysics and cosmology [18, 19]. In particular, the role of Q-balls in cosmology is enlightened in Refs. [20–22], where it is shown that Q-balls arising in a supersymmetric particle physics model where supersymmetry is broken by low-energy gauge mediation are compelling candidate for baryonic dark-matter. We will refer to such a kind of Q-balls as “supersymmetric (SUSY) Q-balls”.

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Moreover, Q-balls within the Signum-Gordon model have been analyzed in Ref. [23], while gravitational wave emission from the fragmentation of a scalar condensate into Q-balls has been investigated in Ref. [24]. Finally, spinning Q-balls in several contexts have been also investigated [25].

In this article, we are mainly interested in studying solitonic solutions which are less symmetric than spherical symmetric Q-balls. They are characterized by planar symmetry in configuration space and were first studied by MacKenzie and Paranjape [26]: They are known as Q-walls.

The interest of this study is twofold: Firstly, the existence of solitonic configurations with higher symmetry, which are expected to be those with lower energy, do not forbid the existence of solitons with lower symmetry as excited states in dynamical processes in which the solitons are involved, such as scattering, fragmentation, evaporation, etc. Secondly, and to some extent also surprisingly, since we will find that there exists a region in the parameter space where finite-size, SUSY Q-walls solutions are energetically more favored than SUSY Q-balls, we argue that not always higher symmetry implies lower energy. Nevertheless, Q-walls cannot be considered as the true ground state of the theory. As a matter of fact, an analysis of finite-size effects reveals that Q-walls, even when they are energetically favored over Q-balls, decay in a finite amount of time, and the decay product are Q-balls with the same value of charge.

The plan of the paper is as follows. In section II, we review Q-ball solutions both within the context of a generic, complex scalar field theory and in a specific model of supersymmetry. In section III, we define and study Q-walls. We firstly derive some general results, independent on the analytic form of the potential. We then introduce Q-walls in the context of low-energy gauge mediation SUSY breaking and derive the equation of state of these solitons both in the flat-potential approximation and in the full potential cases. In Section IV, we address the stability of finite-size SUSY Q-walls, and analyze their energetics with respect to that of SUSY Q-balls. Finally, in Section V, we summarize our results and draw our conclusions.
II. Q-BALLS: AN OVERVIEW

In this section, we briefly review both the Q-ball solution introduced by Coleman in Ref. [7] and the supersymmetric Q-ball configuration firstly discussed by Dvali, Kusenko and Shaposhnikov [27].

A. General Properties

We consider a charged scalar field Φ whose lagrangian density is given by

\[ \mathcal{L} = \partial_{\mu} \Phi^* \partial^{\mu} \Phi - U(|\Phi|). \]  

(2)

Here \( U(|\Phi|) \) is a potential whose form will be specified later for the case of SUSY Q-balls. For the moment, we simply require it is invariant under a global \( U(1) \) transformation. We normalize the corresponding conserved Noether charge, \( q \), as

\[ q = \frac{1}{i} \int d^3 x (\Phi^* \dot{\Phi} - \dot{\Phi} \Phi^*) , \]  

(3)

where a dot indicates a derivative with respect to time. For a given field configuration \( \Phi \) the total energy is given by

\[ E = \int d^3 x \left[ |\dot{\Phi}|^2 + |\nabla \Phi|^2 + U(|\Phi|) \right] . \]  

(4)

In this paper, we are interested to solutions of the classical field equations that correspond to time-independent total energy \( E \) and to a fixed value, namely \( Q \), of the charge \( q \) in Eq. (3). Those requirements are satisfied by the choice in Eq. (1) with \( \omega \) considered as a Lagrange multiplier associated to \( Q \), and by the requirement that the physical solitonic configuration renders the functional

\[ \mathcal{E}_\omega \equiv E + \omega (Q - q) \]  

(5)

stationary with respect to independent variations of \( \Phi, \Phi^* \) and \( \omega \):

\[ \frac{\delta \mathcal{E}}{\delta \Phi} = 0, \quad \frac{\delta \mathcal{E}}{\delta \Phi^*} = 0, \quad \frac{\delta \mathcal{E}}{\delta \omega} = 0. \]  

(6)

The first two constraints lead to the equations of motion of the field \( \phi \), namely

\[ (\nabla^2 + \omega^2) \phi = \frac{\delta U(\phi)}{\delta \phi} , \]  

(7)
while the latter one is equivalent to the requirement that \( \phi \) carries a total charge \( q = Q \):

\[
Q = \omega \int d^3 x \, \phi^2 .
\]  

(8)

In three spatial dimensions, it is usually assumed that the profile function \( \phi \) is isotropic, that is \( \phi = \phi(r) \). In this case, the field equation reads

\[
\frac{d^2 \phi}{dr^2} + \frac{2}{r} \frac{d \phi}{dr} = \frac{\delta U(\phi)}{\delta \phi} - \omega^2 \phi .
\]  

(9)

Once we interpret the field \( \phi \) as the particle position \( x \) and \( r \) as the time \( t \), Eq. (9) is formally equal to the equation of motion in one dimension of a particle moving in the potential

\[
\tilde{U}(x) = -U(x) + \frac{1}{2} \omega^2 x^2
\]  

(10)

and subject to viscous damping. We will extensively use this Newtonian analogy throughout this paper. In passing, we note that the previous equations are formally similar to the equations for the vacuum decays [10, 11].

It has been shown [7] that, given the boundary conditions

\[
\frac{d \phi}{dr} \bigg|_{r=0} = 0 , \quad \phi \bigg|_{r\to\infty} = 0 ,
\]  

(11)
a solution to Eq. (9) exists, for \( \omega \) in the range \([\omega_0, m]\), where \( \omega_0 \equiv \sqrt{2U(\phi_0)}/\phi_0 \) and \( \phi_0 \) corresponds to the value of \( \phi \) which minimizes \( \sqrt{2U(\phi_0)}/\phi_0 \). Such a solution, which is an exact solution of the classical equation of motion and represents a non-topological soliton with charge \( Q \), is said to be a Q-ball. In the Newtonian interpretation, the Q-ball solution corresponds to a particle starting at \( t = 0 \) from a position \( x_0 \) with velocity \( v_0 = 0 \) and reaching the origin in an infinite amount of time.

In the field theory language, for each value of \( \omega \) the Q-ball solution is equivalent to the bounce for tunneling in three Euclidean dimensions in the potential \( \tilde{U} \) [6, 8–12]. Among the acceptable values of \( \omega \), the physical bounce solution is the one which minimizes the functional \( E_\omega \) in Eq. (5) for fixed value \( Q \) of the charge.

Once the Q-ball solution is found, one has to check its stability. To this end, several kinds of stabilities must be kept into account. First of all, as already pointed out by Coleman [7], the Q-ball could decay into a state consisting of plane waves of the quanta of the field \( \phi \) (that constitute the perturbative spectrum of the theory). Since the charge \( Q \) is conserved, the Q-ball can decay only into a state with the same value of the charge operator. The
perturbative spectrum consists of particles of mass \( m \). Therefore, if the energy of the Q-ball, given by Eq. (1), is lower than \( mQ \), then it can not decay into the quanta of the field. This is the absolute stability requirement:

\[
E < mQ .
\]  

(12)

Secondly, the Q-ball must be stable against quantum fluctuations. A general powerful result there exists, which states that stability against quantum fluctuations is achieved if the equation of state of the Q-ball satisfies the condition:

\[
\frac{\omega}{Q} \frac{dQ}{d\omega} \leq 0 .
\]  

(13)

(For a proof of the above criterion see, e.g., the article by F. Pacetti Correia and M. G. Schmidt [14], and Ref. [15].) Equation (13) is commonly known as classical stability criterion.

Finally, a Q-ball must be stable against fission into smaller Q-balls. As a matter of fact, this process is not forbidden by symmetry, as long as the total charge of the solitons in the final state is equal to the charge of the Q-ball in the initial state. It can be proved [1] that the requirement of stability against fission is equivalent to the following condition:

\[
\frac{d\omega}{dQ} < 0 .
\]  

(14)

A comparison between Eq. (13) and (14) shows that quantum stability requirement is a less stringent constraint with respect to stability against fission. However, in the whole regime in which our results are physically relevant, the relation \( \omega = \omega(Q) \) is always monotonous. As a consequence, in our context the two stability requirements are equivalent to each other.

**B. Supersymmetric Q-balls**

In this subsection, we summarize the main properties of Q-balls arising in a model with broken supersymmetry. In the present model [28], known as “gauge-mediation SUSY breaking”, a coupling between vector-like messenger fields and ordinary gauge multiplets breaks explicitly supersymmetry, the coupling constant among the two kinds of field being of order \( g \sim 10^{-2} \). Because of this coupling, the potential for a generic flat direction \( \phi \) does not longer vanish; instead, it is lifted up by an amount which has been computed at the lowest
non-vanishing (two-loop) order in Ref. [29]:

$$U(\chi) = \Lambda \int_0^1 dx \frac{\chi^2 - x(1-x) + x(1-x) \ln[x(1-x)\chi^2]}{[\chi^2 - x(1-x)]^2}.$$  \hfill (15)

Here, $\chi \equiv \phi/M$ and $M \equiv M_S/(2g)$, with $M_S$ the messenger mass scale. The value of the mass parameter $\Lambda^{1/4}$ is constrained as (see, e.g., Ref. [30]): $10^3 \text{GeV} \lesssim \Lambda^{1/4} \lesssim (g^{1/2}/4\pi)^{\sqrt{m_{3/2}/M_{\text{Pl}}}}$, where $M_{\text{Pl}} \sim 2.4 \times 10^{18} \text{GeV}$ is the reduced Planck mass and the gravitino mass, $m_{3/2}$, is in the range $100 \text{ keV} \lesssim m_{3/2} \lesssim 1 \text{GeV}$ [29, 30].

The asymptotic expressions of $U(\chi)$, for small and large $\chi$ are [29]:

$$\frac{U(\chi)}{\Lambda} \sim \begin{cases} 
\chi^2, & \text{if } \chi \ll 1, \\
(\ln \chi^2)^2 - 2 \ln \chi^2 + \frac{\pi^2}{3}, & \text{if } \chi \gg 1.
\end{cases} \hfill (16)$$

In the context of gauge-mediation SUSY breaking, a widely used approximation in studying Q-balls consists in replacing the full potential $U(\chi)$ with its asymptotic expansions (16), in which a plateau plays the role of the logarithmic rise for large values of $\chi$:

$$U(\phi) = \begin{cases} 
\frac{1}{2} m^2 \phi^2, & \phi \leq M, \\
\Lambda, & \phi \geq M,
\end{cases} \hfill (17)$$

where $m = \sqrt{2\Lambda}/M$ is the soft breaking mass which is of order $1 \text{TeV}$ [30]. We will refer to such an approximation as the flat-potential approximation. It has been shown that the approximated potential $U(\phi)$ allows Q-balls solutions as the non perturbative ground state of the model [19, 27]. Such states are known as SUSY Q-balls.

In the limit of large charges, one can compute analytically the most important characteristics of SUSY Q-balls. In particular, the Q-ball energy is given by [19, 27]:

$$\frac{E}{mQ_{\text{cr}}} \approx \frac{4\sqrt{2\pi}}{3} \left( \frac{Q}{Q_{\text{cr}}} \right)^{3/4}, \hfill (18)$$

valid for $Q \gg Q_{\text{cr}}$, where

$$Q_{\text{cr}} \equiv \Lambda/m^4 \hfill (19)$$

is the so-called critical charge.

Taking into account the full potential (15), only numerical calculations are feasible. In this case, one finds absolutely stable Q-ball solutions, $E/mQ < 1$, only if the charge of the soliton is larger than $Q_{\text{min}}$, where [17]

$$Q_{\text{min}} \approx 504Q_{\text{cr}}. \hfill (20)$$
Moreover, the exact relationship between the energy of a SUSY Q-ball and its charge, can be written as

\[ \frac{E}{mQ_{cr}} = \xi_E(Q) \left( \frac{Q}{Q_{cr}} \right)^{3/4}, \]  

(21)

where \( \xi_E \) is a slowly increasing function of \( Q \), computed in [17]. This function can be fitted by a simple analytical form, namely

\[ \xi_E(Q) = a + b \log_{10}(Q/Q_{cr}), \]  

(22)

with \( a \simeq -17.438, b \simeq 15.559, \) and \( p \simeq 0.352 \). The maximum percentage error of the function \( \xi_E \) with respect to its numerical value is less than 2.8% for \( Q \) in the range \( Q \in [Q_{\text{min}}, 7.2 \times 10^{37} Q_{cr}] \).

III. Q-WALLS

A. General Properties

Next we turn to the main object of our study, namely Q-walls. We define a Q-wall as a solitonic solution of the equation of motion possessing planar symmetry, namely such that \( \phi(r) = \phi(z) \), where \( z \) is the coordinate perpendicular to the wall. The field equation (7) for the wall profile \( \phi \) reads

\[ \frac{d^2 \phi}{dz^2} = \delta U(\phi) \frac{\delta U}{\delta \phi} - \omega^2 \phi, \]  

(23)

and it must be solved with the following boundary conditions

\[ \left. \frac{d\phi}{dz} \right|_{z=0} = 0, \quad \phi|_{z \to \pm \infty} = 0, \]  

(24)

analogous to the Coleman’s boundary conditions for the Q-balls. If we interpret \( \phi \) as a coordinate \( x \) and \( z \) as the time \( t \), then Eq. (23) is formally equal to the equation of motion of a particle moving in the effective potential [17]. The force acting on the particle, \( F = -\partial U/\partial x \), is conservative. In the case of spherical symmetric profile studied in the previous section, a viscous term was present. The difference among the two cases is a trivial consequence of the analytical form of the Laplacian operator in one and three spatial dimensions, respectively. As a consequence of the conservative nature of the effective force acting on our fictitious particle, for any given initial condition \( x(t = 0) \equiv x_0, (dx/dt)_{t=0} \equiv v_0, \) the
solution of the equation of motion $x(t)$ can be found by means of the quadrature of the first integral of energy:

$$t = \pm \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dy}{\sqrt{E_{\text{tot}} - \widetilde{U}(y)}},$$

(25)

where $m$ is the mass of the particle and $E_{\text{tot}} = mv_0^2/2 + \widetilde{U}(x_0)$. Translating Eq. (25) to our field theoretical problem, we can write the solution $\phi(z)$ of Eq. (23) in implicit form as

$$z = \pm \sqrt{\frac{1}{2}} \int_{\phi_0}^{\phi} \frac{dy}{\sqrt{\widetilde{U}(\phi_0) - \widetilde{U}(y)}},$$

(26)

where $\phi_0 \equiv \phi(z = 0)$. If we normalize the potential in such a way that $U(0) = 0$, then the quantity $\phi_0$ satisfies the condition

$$\widetilde{U}(\phi_0) = 0.$$ 

(27)

Our initial conditions correspond, in the problem of the motion of a particle in the potential $\widetilde{U}$, to solutions that describe the motion of the particle starting at $t = 0$ from some position $x_0$ with zero velocity $v_0$ and arriving to the point $x = 0$ at $t = +\infty$ with velocity $v_\infty = 0$.

For a generic potential $U$, we can argue the form of the Q-wall profile by looking at the approximate solutions of the full equation of motion, in the limit of small or large $z$. For small $z$ one can write $\phi(z) \simeq \phi_0 + cz^2$ [the condition $\phi'(z = 0) = 0$ has to be satisfied, see Eq. (24)]. Substituting into Eq. (23) we find $c = -(1/2)(\partial \widetilde{U}/\partial \phi)|_{\phi_0}$. Moreover, for large $z$ one has $\phi \to 0$ and can keep only the quadratic terms in the effective potential $\widetilde{U}(\phi) \simeq (\omega^2 - m^2)\phi^2/2$. In this case, the equation of motion (23) linearizes and the solution is easily found to be $\phi(z) \simeq \phi_0 e^{-z\sqrt{m^2 - \omega^2}}$. Summarizing, the asymptotical behavior of the wall profile is given by

$$\frac{\phi(z)}{\phi_0} = \begin{cases} 1 - \frac{1}{2} \frac{\partial \widetilde{U}}{\partial \phi}|_{\phi_0} z^2, & z \to 0, \\ e^{-z\sqrt{m^2 - \omega^2}}, & z \to +\infty. \end{cases}$$

(28)

Equation (28) defines a decay length,

$$\Delta = \frac{1}{\sqrt{m^2 - \omega^2}},$$

(29)

that we identify with the thickness of the wall.

As in the case of Q-balls, the existence of Q-walls is subject to the absolute stability condition Eq. (12). Due to planar symmetry of the solitonic solution, it is convenient to
introduce the surface densities of charge and energy, \( \sigma \) and \( \rho \), respectively as

\[
Q \equiv A\sigma, \quad E \equiv A\rho,
\]

(30)

where \( A \equiv \int d^2x \) is the surface area of the wall, and

\[
\rho = \int dz \left[ \frac{1}{2} \omega^2 \phi^2 + \frac{1}{2} (\partial_z \phi)^2 + U(\phi) \right],
\]

(31)

\[
\sigma = \omega \int dz \phi^2.
\]

(32)

Using definition (30), the absolute stability condition reads

\[
\rho < m\sigma.
\]

(33)

In the next section, we will focus on stable Q-walls in a scalar field theory defined by the supersymmetric potential (15). Before discussing the specific case, it is interesting to note here that, for any form of the potential in Eq. (31), the surface energy and charge of a Q-wall are connected by the following parametric equation of state:

\[
\rho(\omega) = S(\omega) + \omega \sigma(\omega),
\]

(34)

\[
\sigma(\omega) = -\frac{dS}{d\omega},
\]

(35)

where

\[
S(\omega) \equiv 2 \sqrt{2} \int_{\phi_0}^{\phi_0} d\phi \sqrt{-\tilde{U}(\phi)},
\]

(36)

and \( \phi_0 \) is the solution of Eq. (27). Equations (34) and (35) are nothing but the statement that the total energy density of the Q-wall is the Legendre transform of the action integral \( S(\omega) \). For a given form of the potential \( U \), one can eliminates the \( \omega \) parameter between Eqs. (34) and (35), and can then express the surface energy \( \rho \) as a function of the surface charge \( \sigma \). The resulting equation of state can be finally used to check the stability condition.

B. Supersymmetric Q-walls: Analytical results

In this subsection, we derive analytical properties of supersymmetric Q-walls. This is possible when considering the simplified case of flat potential (17). In the context of Q-balls, it was shown [17] that the using of the full potential simply induces logarithmic corrections to the power-law relationships among the various physical quantities of Q-ball. We will show later that this statement holds for Q-wall solutions as well.
Within the flat-potential approximation, we easily obtain the expressions for the surface energy and the charge:

\[ \rho = m\sigma h(\omega/m), \]
\[ \sigma = \sigma_{cr} g(\omega/m), \]

where

\[ h(x) \equiv x \left( 1 + \frac{1}{1 + \frac{1}{x^2} \arccos x} \right), \]
\[ g(x) \equiv \frac{1}{x^2} \left( \frac{x}{\sqrt{1-x^2}} + \arccos x \right), \]

The introduction of the critical charge,

\[ \sigma_{cr} \equiv M^2, \]

will be useful in the following.

We plot the function \( h(\omega/m) \) in the upper panel of Fig. 1. Since \( h = \rho/m\sigma \), the absolute stability condition requires \( h < 1 \). Numerically, we find that the above condition is fulfilled for \( \omega \lesssim 0.68 \, m \). Using Eq. (38) we then find that there exists a minimal charge \( \sigma_{\text{min}} \approx 3.00 \sigma_{cr} \) such that, for \( \sigma < \sigma_{\text{min}} \), SUSY Q-wall are classically unstable and decay into particles of mass \( m \). For completeness, in the lower panel of Fig. 1 we plot the function \( g(\omega/m) \). It is interesting to note that there exists a critical value of \( \omega \), namely \( \omega \approx 0.85 \, m \), above which Q-walls are unstable against quantum fluctuations and fission (see Eq. (13)). However, this is not relevant for our discussion, since in the region of quantum and fission instability, Q-walls are already unstable against decay into free quanta.

In Fig. 2, we plot \( \omega \) and \( \rho \) versus the charge \( \sigma \). The values of \( \omega \) (which we indicate as points in the plots) are found numerically by solving Eq. (38) as a function of \( \omega \). The corresponding solutions are then inserted in Eq. (37) to find the equation of state of a Q-wall, that is \( \rho \) as a function of \( \sigma \). Dotted lines represent the analytical expressions of \( \omega \) and \( \rho \) for large values of the charge (\( \sigma \gg \sigma_{cr} \)):

\[ \frac{\omega}{m} \approx \left( \frac{\pi}{2} \right)^{1/2} \left( \frac{\sigma}{\sigma_{cr}} \right)^{-1/2}, \]
\[ \frac{\rho}{m\sigma_{cr}} \approx (2\pi)^{1/2} \left( \frac{\sigma}{\sigma_{cr}} \right)^{1/2}, \]

which are in agreement with the results of Ref. [26].
FIG. 1. Upper panel. The function $h = \rho/m\sigma$ as a function of $\omega$. When $h > 1$, SUSY Q-walls are absolutely unstable and decay into particles of mass $m$. Lower panel. The function $g = \sigma/\sigma_c$ against $\omega$. The vertical dashed line corresponds to $\omega \approx 0.85m$. For charges larger than this critical value of $\omega$, Q-walls are unstable against fission and against quantum fluctuations.

Next we compute the Q-wall profile. From Eq. (26) we find:

$$\frac{\phi(z)}{\phi_0} = \begin{cases} 
\cos(\omega z), & z \leq Z, \\
\cos(\omega Z) e^{-(z-Z)/\Delta}, & z \geq Z,
\end{cases}$$

where the wall spread, $Z$, is defined by the condition $\phi(Z) = M$, with $\Delta$ defined in Eq. (29). Solving Eq. (27) we find $\phi_0 = mM/\omega$ and in turns $Z = (1/\omega) \arccos(\omega/m)$.

The analytic form of the Q-wall profile in the above equation suggests the introduction
FIG. 3. The total width, $\delta$ (points), and the spread, $Z$ (empty squares), of a SUSY Q-wall as a function of the charge, together with their analytical expressions for large charges (dotted line). The difference between $\delta$ and $Z$ is the thickness of the wall $\Delta$, which goes to zero for large charge.

FIG. 4. *Upper panel.* $\phi_0$ as a function of the charge (points) with its analytical expression for large charges (dotted line). *Lower panel.* Susy Q-wall profile for different values of the charge: $\sigma = 10^5 \sigma_{cr}$ (continuous line), $\sigma = 50 \sigma_{cr}$ (long dashed line) $\sigma = 10 \sigma_{cr}$ (dashed line), $\sigma = \sigma_{\text{min}} \approx 3.00 \sigma_{cr}$ (dotted line).

of the wall *width*, that we denote by $\delta$, as

$$\delta \equiv Z + \Delta . \quad (45)$$

In Fig. 3, we show $\delta$ (points in the plot) and $Z$ (empty squares in the plot) as a function of the charge, together with their analytical expressions for large charges (dotted line):

$$\frac{\delta}{m^{-1}} \approx \frac{Z}{m^{-1}} \approx \left( \frac{\pi}{2} \right)^{1/2} \left( \frac{\sigma}{\sigma_{cr}} \right)^{1/2} . \quad (46)$$

It is worth noting that the limit of large charges is equivalent to neglect the thickness of the wall with respect to its spread.
Finally, we plot in Fig. 4 the quantity $\phi_0$ as a function of the charge (points in the plot), and compare it with its analytical expression for large charges (dotted line):

$$\frac{\phi_0}{M} \approx \left( \frac{2}{\pi} \right)^{1/2} \left( \frac{\sigma}{\sigma_{cr}} \right)^{1/2}.$$  

(47)

In the same figure, we plot the Q-wall profile for different values of the charge.

C. Supersymmetric Q-walls: Full Potential

We now present our results for the full potential (15). In analogy with the approximate case discussed in the previous section, we define the quantity

$$h(\omega) \equiv \rho/m\sigma,$$  

(48)

(compare with Eq. (37)). We find numerically that absolute stability condition, $h < 1$, is fulfilled for values of $\omega$ less than a maximal value

$$\omega_{max} \approx 0.93m$$  

(49)

(see Fig. 5). This in turn means that there exists a minimum charge, above which SUSY Q-walls are unstable and than decay into the quanta of the field. Numerically we find

$$\sigma_{min} \approx 1.23\sigma_{cr}.$$  

(50)

(It is now clear why we introduced the critical charge: It turns to be, approximatively, the value of the charge defining the limit between stable and unstable Q-walls.)

Besides we introduce, in analogy with Eq. (38), the function

$$g(\omega) = \frac{\sigma}{\sigma_{cr}}.$$  

(51)

We plot $g(\omega)$ in the lower panel of Fig. 5. It is interesting to compare Fig. 5 with Fig. 4, the latter being the result of the calculation within the flat-potential approximation: The derivative of the total charge with respect to $\omega$ is negative in the whole range of the allowed values of $\omega$ for the full potential case. The quantum stability condition (13) implies that Q-wall solution is classically stable in the whole range of $\omega$ (and stable against fission as well). On the other hand, in the case of the flat-potential approximation, we have already
FIG. 5. Upper panel. The ratio $h = \rho/m\sigma$ as a function of $\omega$. When $h > 1$, SUSY Q-walls are absolutely unstable and decay into particles of mass $m$. Lower panel. The function $g = \sigma/\sigma_{cr}$ as a function of $\omega$.

noted that there exists a critical value of $\omega$ above which the derivative changes sign, signaling a regime of classical instability.

In Fig. 6, we present $\omega$ and $\rho$ as a function of the charge (points in the plots). Inspired by Eqs. (42)-(43), we write $\omega$ and $\rho$ in the following way:

$$\omega/m = \xi_\omega(\sigma) \left( \frac{\sigma}{\sigma_{cr}} \right)^{-1/2}, \quad (52)$$

$$\rho/m\sigma_{cr} = \xi_\rho(\sigma) \left( \frac{\sigma}{\sigma_{cr}} \right)^{1/2}. \quad (53)$$

We checked that the functions $\xi$ are slowly varying functions of the charge $\sigma$, and indeed are well approximated by power functions of the logarithm of the charge:

$$\xi(\sigma) = [a + b\log_{10}(\sigma/\sigma_{cr})]^q. \quad (54)$$

They parameterize the deviation from the simple power-laws obtained in the flat-potential approximation. In Table I, we report the values of the coefficients $a$, $b$, $q$ found by least-squaring the numerical data for large charges. We also show the maximum percentage error of the functions $\xi$ with respect to their numerical values. The dotted lines in Fig. 6 are indeed the approximate expressions (52)-(53).

For the full potential case, we are no longer able to derive analytical Q-wall profiles. Therefore, we must use some phenomenological definition for the Q-wall width, $\delta$, in analogy with our previous definition (45). It is convenient to adopt the following definition:

$$\phi(\delta) \equiv 0.1 \phi_0. \quad (55)$$
TABLE I. Nonlinear fit of the functions $\xi$ defined in Eqs. (52)-(53), (54), (56) and (57). The quantity $\text{Err}\%$ represents the maximum percentage error of the functions $\xi$ with respect to their numerical values in the range $\sigma \in [10^4 \sigma_{cr}, 9.5 \times 10^8 \sigma_{cr}]$.

|   | $a$   | $b$   | $q$ | $\text{Err}\%$ |
|---|-------|-------|-----|-----------------|
| $\omega$ | -0.024 | 2.033 | 1   | 0.06%          |
| $\rho$   | -5.109 | 5.288 | 1.031 | 0.08%         |
| $\delta$ | -1.542 | 1.502 | -0.978 | 0.03%       |
| $\phi_0$ | -13.961 | 4.486 | 0.027 | 0.06%       |

FIG. 6. The parameter $\omega$ (upper panel) and the surface energy density $\rho$ (lower panel) as a function of the surface charge $\sigma$. Dotted lines represent the approximate expressions of $\omega$ and $\rho$ for large values of the charge.

In Fig. 7, we show $\delta$ as a function of the charge (points in the plot) with its approximate expression for large charges (dotted line):

$$\frac{\delta}{m^{-1}} = \xi_\delta(\sigma) \left( \frac{\sigma}{\sigma_{cr}} \right)^{1/2}.$$  

(56)

In Fig. 8, we finally show $\phi_0$ as a function of the charge (points in the plot) with its approximate expression for large charges (dotted line):

$$\frac{\phi_0}{M} = \xi_{\phi}(\sigma) \left( \frac{\sigma}{\sigma_{cr}} \right)^{1/2}.$$  

(57)

Also shown in the figure is the Q-wall profile for different values of the charge.

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FIG. 7. The total width of a SUSY Q-wall $\delta$ (points), as a function of the charge, together with its approximate expression for large charges (dotted line).

 FIG. 8. Upper panel. $\phi_0$ as a function of the charge (points) with its approximate expression for large charges (dotted line). Lower panel. SUSY Q-wall profile for different values of the charge: $\sigma = 10^3 \sigma_{cr}$ (continuous line), $\sigma = 50 \sigma_{cr}$ (dashed line), $\sigma = \sigma_{min} \simeq 1.23 \sigma_{cr}$ (dotted line).

IV. COMPARISON OF SUPERSYMMETRIC Q-WALLS AND Q-BALLS

Up to now, we have considered two dimensional Q-wall configurations whose surface area is then indefinitely large. In real situations, however, the surface area cannot be infinite. To retain all the above results about Q-walls, we must impose that the typical longitudinal dimension of a wall,

$$L \equiv \sqrt{A},$$

is much greater than the typical transverse dimension $\delta$ defined in Eq. (55).
A. Energetics of Finite-size Q-walls

In the case of finite-size Q-walls, we can associate to a Q-wall a finite charge $Q$ and a finite energy $E_w$ in the following way:

$$Q = \sigma L^2, \quad E_w = \rho L^2,$$

whenever the feasibility condition is satisfied:

$$\alpha \equiv \frac{L}{\delta} \gg 1.$$  \hspace{1cm} (60)

(In the following we assume, for definiteness, that $\alpha \geq 10$). The ideal (but unphysical) case of infinite wall surface is recovered for $L \to \infty$ or, equivalently, for $\alpha \to \infty$.

Before proceeding further, let us observe that for a given $L$ there exists a minimum charge defined by

$$Q_w^{\text{min}} = \sigma_{\text{min}} L^2.$$  \hspace{1cm} (61)

This immediately follows from Eq. (51). It is clear that for fixed values of $\alpha$, the various quantities defining a Q-wall, e.g. energy, total width, etc., will depend now on both $\alpha$ and $Q$:

$$E_w = E_w(Q, \alpha), \quad \delta = \delta(Q, \alpha), \quad \text{etc.},$$  \hspace{1cm} (62)

while the minimum charge only on $\alpha$:

$$Q_w^{\text{min}} = Q_w^{\text{min}}(\alpha).$$  \hspace{1cm} (63)

(In the case of infinite Q-walls the above quantities depend only on the surface charge $\sigma$, while $\sigma_{\text{min}}$ is fixed.)

As an example, let us write the expression of the energy and charge of a finite-size SUSY Q-wall in the flat-potential approximation:

$$\frac{E_w}{mQ_{cr}} \simeq \pi^{3/4} \left( \frac{Q}{Q_{cr}} \right)^{3/4} \alpha^{1/2},$$  \hspace{1cm} (64)

$$\frac{Q}{Q_{cr}} \simeq \pi \left( \frac{\sigma}{\sigma_{cr}} \right)^2 \alpha^2,$$  \hspace{1cm} (65)

where we remind that $Q_{cr} \equiv \Lambda/m^4$. The above equations are valid in the limit $\sigma \gg \sigma_{cr}$. In particular, for Eq. (64) this means $Q \gg (\sigma_{cr}/\sigma_{\text{min}}) Q_{w}^{\text{min}} \simeq Q_{w}^{\text{min}}/3$, where the minimum charge is

$$\frac{Q_w^{\text{min}}}{Q_{cr}} \simeq \pi \left( \frac{\sigma_{\text{min}}}{\sigma_{cr}} \right)^2 \alpha^2 \simeq 9\pi \alpha^2.$$  \hspace{1cm} (66)

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FIG. 9. The ratio between the energy of a SUSY Q-wall $E_w = \sigma L^2$ and the energy of a SUSy Q-ball $E_b$, as a function of the charge $Q$ and for different values of $\alpha = L/\delta$. From down to top: $\alpha = 10, 15, 30, 50, 100$. The horizontal dotted line is $E_w = E_b$. In the case of walls, the charge is $Q = \sigma L^2$, where $\sigma$ is the surface energy density and $L$ the longitudinal dimension of the wall. In a Q-wall configuration, $L$ is always much greater than the transverse dimension $\delta$.

Comparing Eq. (64) with Eq. (18), namely the energy of a SUSY Q-wall with the energy of a SUSY Q-ball possessing the same charge $Q$, we find that

$$\frac{E_w}{E_b} \approx \frac{3}{4\sqrt{2\pi}1/4} \alpha^{1/2} \approx 0.40 \alpha^{1/2},$$

valid for $Q \gg \max\{Q_b^{\text{min}}, Q_w^{\text{min}}\}$, where $Q_b^{\text{min}}$ is the minimum charge required for stable Q-balls. Since we are assuming $\alpha \geq 10$, we find that $E_w > E_b$ for $\alpha \in [10, \infty]$.

We want now to analyze the more realistic case of full supersymmetric potential giving rise to Q-walls and Q-balls. To this end, in Fig. 9, we show the ratio between the energy of a SUSY Q-wall $E_w$ and the energy of a SUSY Q-ball $E_b$, as a function of the charge $Q$ and for different values of $\alpha$. We find that $Q_w^{\text{min}} > Q_b^{\text{min}} \approx 504Q_{cr}$ for all $\alpha \geq 10$, and that large planar Q-walls ($\alpha \gtrsim 30$) are more energetic than Q-balls with the same charge.

However, it is very interesting to observe that there exist Q-wall configurations with moderate values of $\alpha$ ($10 \leq \alpha \lesssim 30$) that are less energetic of the corresponding Q-balls if the charge is in the range $Q_w^{\text{min}} \leq Q \leq Q^*$, where $Q_w^{\text{min}}$ and $Q^*$ as a function of $\alpha$ are shown in Fig. 10, $Q^*$ being defined by the equality

$$E_w(Q^*) \equiv E_b(Q^*).$$

Stated in other words, we find that for charges in the range

$$2.16 \times 10^4 Q_{cr} \leq Q \leq 2.06 \times 10^{11} Q_{cr}$$

valid for $Q \gg \max\{Q_b^{\text{min}}, Q_w^{\text{min}}\}$. 

\[\text{(67)}\]
(the lower and upper limits correspond to $Q = Q_w^{\text{min}}$ and $Q = Q^*$ evaluated at $\alpha = 10$) there always exists a Q-wall with $10 \leq \alpha \lesssim 30$ such that

$$E_w < E_b.$$  \hfill (70)

### B. Estimate of the Lifetime of a Finite-size Q-wall

Equation (70) establishes that, for particular values of the charge, finite-size Q-walls are energetically favored over Q-balls with the same value of the charge. The straightforward interpretation of this result would be that finite-size, non-topological Q-walls can possibly be the ground state of the scalar theory defined by Lagrangian (2). However, due to their finiteness, such Q-walls are dynamically unstable. It is clear, indeed, that under the action of their own tension, finite-size Q-walls with charge $Q$ will eventually shrink. The typical size of the Q-wall, when this happens, is found by equating its energy to that a Q-ball with same charge $Q$:

$$E_w(L_{\text{decay}}, Q) = E_b(Q).$$ \hfill (71)

Using the flat-potential approximation, it is straightforward to obtain

$$L_{\text{decay}} = \frac{2\sqrt{2\pi}}{3} \left( \frac{Q}{\Lambda} \right)^{1/4},$$ \hfill (72)

where we used Eqs. (43) and (59) and Eq. (18).

In order to estimate the lifetime of a finite-size Q-wall, we follow the approach of Ref. [26]. We introduce an effective Lagrangian defining the dynamics of a Q-wall of size $L$:

$$\mathcal{L}_w = \frac{1}{2} m_w(L) \dot{L}^2 - V_w(L),$$ \hfill (73)

where $m_w(L) = E_w(L)$ and $V_w(L) = E_w(L)$ are the effective mass and tension of the wall, respectively. The equation of motion for $L(t)$ is then

$$\frac{1}{2} \ddot{L} \dot{L}^2 + L = L_i,$$ \hfill (74)

where we have indicate with $L_i$ the wall scale-length when the wall begins to shrink at the time $t_i$. The lifetime of the wall, i.e. the time interval between $t_i$ and the time when the wall decays into a Q-ball, $t_{\text{decay}} \equiv t(L_{\text{decay}})$, is given by integrating Eq. (74):

$$\tau_w = t_{\text{decay}} - t_i = \frac{\pi}{2\sqrt{2}} L_i f(L_{\text{decay}}/L_i),$$ \hfill (75)
FIG. 10. Phase diagram of non-topological solitons in a model with broken supersymmetry via low-energy gauge mediation. The regions denoted by B (W) and N correspond, respectively, to domains where Q-balls (finite-size Q-walls) and unbound quanta are stable (meta-stable). The dotted line represents the minimum charge possessed by a finite-size, SUSY Q-wall, $Q_{w}^{\text{min}}$, at the varying of the ratio $\alpha$ between longitudinal and transverse dimension of the wall. The continuous line is the charge $Q^{*}$ defined by $E_{w}(Q^{*}) = E_{b}(Q^{*})$, as a function of $\alpha$. The dashed line is the the minimum charge required for a stable SUSY Q-ball, $Q_{b}^{\text{min}} \approx 504Q_{\text{cr}}$. W is a meta-stability region for Q-walls: Although finite-size Q-walls are less energetic than Q-balls in this region, finite-size effects cause Q-walls to decay into Q-balls with the same value of the charge.

where

$$f(x) \equiv \frac{2}{\pi} \left[ \sqrt{x(1-x)} + \arccos \sqrt{x} \right].$$  \hspace{1cm} (76)

We note that $f(x)$ is a decreasing function of $x$ such that $f(0) = 1$ and $f(1) = 0$. Consequently, we have

$$\tau_{w} \leq \frac{\pi}{2\sqrt{2}}L_{i} \approx 1.11L_{i}.$$  \hspace{1cm} (77)

C. Phase diagram

The previous discussions can be summarized in Fig. [10] where the regions denoted by B, W, and N correspond, respectively, to domains in which Q-balls, Q-walls, and unbound quanta are stable (N corresponds to normal phase, as it is customary to define the unpaired phase for superfluids and superconductors). In particular, the region denoted by W is
bounded by two transition lines from a ground state made of Q-walls to the ground state made by Q-balls. In this region, Eq. (70) is satisfied. However, since finite-size Q-walls are dynamically unstable, it is more appropriate to call the region W as the meta-stability region of Q-walls.

This region has to be understood as follows: If a point \((\alpha, Q)\) lies in the region W, then the energy of the corresponding planar-symmetric, finite-size soliton is lower than the energy of a spherical-symmetric soliton possessing the same charge. However, that planar-symmetric soliton has a finite lifetime (due to its dynamical instability) and its charge-conserving decay is a spherical-symmetric soliton.

V. CONCLUSIONS

We have investigated a particular class of non-topological solitons possessing planar symmetry, known as Q-walls, in the context of a supersymmetric particle physics model with supersymmetry broken by low-energy gauge mediation.

On general grounds, solutions with spherical symmetry (Q-balls) are expected to be favored over the ones with planar symmetry. Q-walls, then, might be interpreted as excited states of Q-balls. The study of such excited states is interesting for they may form as intermediate states when dynamics is involved – e.g. in processes of Q-balls collision, as well as in soliton production via fragmentation of the Affleck-Dine condensate.

Firstly, we have discussed general analytic properties of Q-wall configurations arising in field theories invariant under a global \(U(1)\) transformation and without relying on any specific form of the potential giving rise to them. This analysis is therefore relevant for any model whose potential allows for Q-walls.

Secondly, we have analyzed the peculiar properties of Q-walls within a specific supersymmetric model, namely that in which supersymmetry is broken at the quantum level by virtue of low-energy gauge mediation. The use of an approximate form of the two-loop potential that breaks supersymmetry has led us to simple analytic results, in agreement with previous works on this subject.

Also, we have studied Q-wall solutions considering the exact form of the supersymmetric potential. Only numerical results have been obtained which are, however, in fairly close agreement with those derived in the approximate case.
Finally, we have compared the energy of a finite-size Q-wall with that of a Q-ball, at fixed charge $Q$. The result is summarized in the “phase diagram” of Fig. 10 in which a region exists –region W in Fig. 10– where Q-walls are less energetic than Q-balls.

Such a region is, however, a meta-stability region since an effective analysis of stability has revealed that finite-size Q-walls are unstable with respect to the decay into Q-balls.

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