Resolving the inner structure of QSO discs through fold-caustic-crossing events

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ABSTRACT

Although the bulk of the observed optical flux from the discs of intermediate-redshift lensed quasars is formed well outside the region of strong relativistic boosting and light bending, relativistic effects have an important influence on microlensing curves. The reason lies in the divergent nature of amplification factors near fold caustics, which are increasingly sensitive to small spatial size details. Higher-order disc images produced by strong light bending around the black hole may affect the amplification curves, making a contribution of up to several per cent near maximum amplification. In accordance with theoretical predictions, some of the observed high-amplification events possess fine structure. Here we consider three putative caustic-crossing events, one by SBS J1520+530 and two events for individual images of Einstein’s cross (QSO J2237+0305). Using relativistic disc models allows us to improve the fits but the required inclinations are high, $i \gtrsim 70^\circ$. Such high inclinations apparently contradict the absence of any strong absorption that is likely to arise if a disc is observed edge-on through a dust torus. Still, high inclinations are required only for the central parts of the disc, which allows the disc itself initially to be tilted by 60–90$^\circ$ with respect to the black hole and aligned toward the black hole equatorial plane near the last stable orbit radius. For SBS J1520+530, an alternative explanation for the observed amplification curve is a superposition of two subsequent fold-caustic crossings. While relativistic disc models favour black hole masses $\sim 10^{10} M_{\odot}$ (several times higher than the virial estimates) or small Eddington ratios, this model is consistent with the observed distribution of galaxies over peculiar velocities only if the black hole mass is $\lesssim 3 \times 10^8 M_{\odot}$.

Key words: accretion, accretion discs – gravitational lensing: micro – quasars: individual: QSO J2237+0305– quasars: individual: SBS J1520+530.

1 INTRODUCTION

The history of gravitational lensing may be traced back to the beginning of the 20th century and beyond (see Schmidt & Wambsganss (2010) and the references therein for a historical review). The first double quasar images produced by strong lenses were reported in Walsh, Carswell & Weymann (1979). The first quadruply lensed quasar, QSO J2237+0305, was discovered by Huchra et al. (1985). This object was also the first where effects of microlensing by the stars of the lensing galaxy were found (Irwin et al. 1989).

The principal difference between strong lensing and microlensing lies in the angular distance scale set by the Einstein–Chwolson radius:

$$\theta_{\text{Ein}} = \sqrt{\frac{4GM}{c^2}} \frac{D_{\text{LS}}}{D_{\text{S}}D_{\text{L}}} \simeq 2.8 \frac{M}{M_{\odot}} \frac{D_{\text{LS}}}{D_{\text{S}}} \frac{1\text{ Gpc}}{D_{\text{L}}} \text{ mas.} \tag{1}$$

Here, $M$ is the mass of the lensing object and $D_{\text{L,S,LS}}$ are angular size distances toward the lens (‘L’) and source (‘S’) and between the source and the lens (‘LS’), $D_{\text{LS}} = D_{\text{S}} - D_{\text{L}} \times (1+z_{\text{L}})/(1+z_{\text{S}})$ for a flat Universe (see for example Hogg 1999). Dependence on the lens mass leads to drastically different angular scales associated with strong lensing by galaxies and galaxy groups (arcsec and less). The individual images formed in the latter case cannot be resolved by contemporary instrumentation, but their amplification variations may be a valuable tool to resolve the spatial structure of the source (Chang &Refsdal 1984). Accretion discs around supermassive black holes at cosmological distances should have comparable or

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somewhat smaller angular sizes at $\lambda \sim 2000$ Å. Below, we will use spatial sizes projected on to the picture plane at the distance of the source. They differ from the angular sizes by a dimensional factor of $D_h$:

$$r_{\text{Ein}} = \sqrt{\frac{4GM}{c^2} \frac{D_h D_S}{D_L}} \simeq 4.3 \times 10^{16} \sqrt{\frac{M}{M_\odot}} \frac{D_S}{D_h} \frac{D_L}{1 \text{ Gpc}} \text{ cm.}$$  

(2)

The main difference between the microlensing by stellar-mass objects of our Galaxy and the microlensing effects accompanying strong lensing of quasars is in the optical depth. Angular distances between individual lensing bodies in a galaxy scale with the distance $D$ as $\alpha D^{-1}$, while Einstein–Chwolson radii decrease only as $\alpha D^{-0.5}$. If we define the microlensing optical depth as the total solid angle of the Einstein circles of the stellar population of a unit solid angle of the lensing galaxy (cf. Nityananda & Ostriker 1984), it may be written as

$$\tau = \frac{4\pi G \Sigma}{c^2} \frac{D_{LS} D_h}{D_S} \simeq 0.06 \frac{\Sigma}{100 M_\odot \text{ pc}^{-2}} \frac{D_{LS}}{D_h} \frac{D_L}{1 \text{ Gpc}}.$$  

(3)

Here, $\Sigma$ is the total surface density of the clumpy matter in the lensing galaxy. The smoothly distributed component (primarily unclumped dark matter) contributes only to strong lensing, unless it creates a strong shear that destoyes point-lens degeneracy (Nityananda & Ostriker 1984). The optical depth becomes considerably large ($\tau \gtrsim 0.1$) for lensed quasars$^1$ at Gpc distances.

Differences in optical depth make microlensing effects in distant lensing galaxies qualitatively different from the rare single- and double-lens events in our Galaxy. As a background source moves with respect to a single point-like lens, it has zero probability of undergoing infinite amplification (Paczynski 1986b). In this regime, amplification becomes sensitive to the size and the structure of the object only if its angular distance from the lens centre becomes comparably small. On the other hand, strong ($\tau \gtrsim 0.1$–0.5, depending on the underlying shear) microlensing by a population of point masses creates a network of fold caustics (Paczynski 1986a) where amplification is divergent and behaves as $\alpha \tau^{-1/2}$, where $d$ is the angular distance toward the fold (Chang & Refsdal 1984). In the case of quasar microlensing, every particular image traverses some fold caustic with a probability of about unity on a time-scale of several years (for the case of QSO J2237$+$0305, the relative frequency of high-amplification events was estimated as $\sim 1$ yr$^{-1}$ by Witt, Kayser & Refsdal (1993); other objects have smaller relative proper motions). A comprehensive review of microlensing was carried out by Wambsganss (2006).

At present, there are at least three different approaches to quasar microlensing that may be used to probe the structure of supermassive black hole accretion discs. The first is to gather statistics on image amplification and to compare single-epoch anomalous fluxes with the predictions of accretion-disc models, as was done by Pooley et al. (2007), Bate et al. (2008), Floyd, Bate & Webster (2009), Blackburne et al. (2011) and Jimenez-Vicente et al. (2012). In the first paper, a huge – about a factor of 10–100 – inconsistency was found between the sizes of accretion discs estimated by microlensing methods and the predictions of standard accretion-disc theory. Partially, this inconsistency may be attributed to the crude mass estimates resulting from applying broad-band magnitudes with some assumptions regarding bolometric correction and accretion efficiency. On the other hand, Morgan et al. (2010) use more accurate mass estimates based on the widths of broad emission lines and achieve much better consistency. Indeed, among the four objects common for the samples used in these two studies, only for one (QSO J2237$+$0305) are the masses determined by the two methods consistent within the uncertainties. For two objects (PG 1115$+$080 and RXJ 1131$-$1231), photometry-based mass estimates are about four times lower, while for the least massive one, SDSS 0924$+$0219, masses differ by an order of magnitude. However, the accretion-disc sizes measured by Morgan et al. (2010) are still several times larger than expected. The authors suppose that virial mass estimates may still be systematically lower by a factor of $\sim 3$.

The analysis method used by Morgan et al. (2010) was introduced by Kochanek (2004) and may be characterized as extensive light-curve fitting. A number of artificial amplification maps is generated and the observational light curve is compared with numerous simulated light curves with random parameters. This technique was applied in a large number of studies such as those of Eigenbrod et al. (2008b) and Hainline et al. (2012). Since the technique requires multiple observational points, it is perfect for Einstein’s cross and other targets of monitoring programmes.

Monte Carlo microlensing analysis is rather resource-consuming. As an alternative to computationally extensive methods, it is reasonable to study the individual high-amplification events that are most sensitive to the spatial properties of the source. Primarily, high-amplification events are associated with fold-caustic crossings, when a pair of new microlensing images appears or disappears and the point-source amplification diverges. Caustic crossings by standard discs are good models for some of the observed amplification events (Gil-Merino et al. 2006; Koptelova et al. 2007). It may be shown that point-source amplification during a caustic crossing is divergent for a disc with a small inner radius $r_\text{in}$ as $\mu \propto r_\text{in}^{-1/4}$, hence brightness maxima in microlensing curves are best for resolving the innermost parts of the disc (Agol & Krolik 1999). Indeed, as was shown by Jaroszynski, Wambsganss & Paczynski (1992), lensing curves differ for accretion discs of different inclinations and Kerr parameters of the accretor. Amplification curves primarily differ near their maxima.

Caustic-crossing events allow us to study the structure of the innermost parts of accretion discs, where general relativity effects are important. Two processes are expected to influence the observed light curves considerably: light bending and Doppler boosting (due to matter motion in the disc as well as frame-dragging). Both make brightness distributions asymmetric and strongly dependent on the inclination angle. In this paper, we aim to estimate the influence of relativistic effects on the amplification curves created by straight caustic-crossing events. We also apply the results of our calculations to three high-amplification events and show that some of their features are probably connected to relativistic effects.

The paper is organized as follows: first we describe the archival observational data we use in Section 2. A simulation technique applying Kerr geodesic calculation software is considered in Section 3. Fitting results for the three putative caustic crossings are given in Section 4 and discussed in Section 5.

2 OBSERVATIONAL DATA

The current number of lensed quasars where microlensing effects were found is about several tens (Pooley et al. 2007; Morgan et al. 2010; Jimenez-Vicente et al. 2012). However, long homogeneous observational series exist for few objects. The best studied among

\footnotetext[1]{Here we do not distinguish between radio-loud quasars and radio-quiet ‘quasi-stellar objects’ (QSOs), and refer to both object types as ‘quasars’ or QSOs.}
Table 1. Basic information about the objects and the observational amplification curves used for analysis. For the QSO J2237+0305 image-A event, the numbers of GLITP data points for two reduction techniques, ISIS and PSF, are given separated by a slash.

| Source redshift | 1.855 | 1.695 |
|----------------|-------|-------|
| Lens redshift | 0.72  | 0.039 |
| Time span (V), JD − 245 0000 | – | 1400–1650 | 1200–1650 |
| Number of points (V) | – | 53/52 | 83 |
| Time span (R), JD − 245 0000 | 1200–3000 | 1450–1510 | – |
| Number of points (R) | 253 | 51/49 | – |

these is the Einstein cross (QSO J2237+0305), which is the subject of extensive photometric monitoring programmes such as the Optical Gravitational Lensing Experiment (OGLE) II (Woźniak et al. 2000) and III (Udalski et al. 2006). We analyse two high-amplification events that took place for two different images in the years 1999 and 2000. We also found a considerable amount of archival data on another object, SBS J1520+530, and interpret variations of the relative image amplifications as a manifestation of microlensing amplification. A brief summary of the observational data we use in this work is given in Table 1.

2.1 SBS J1520+530

The object is doubly imaged, with an average flux ratio for the two images A and B of about 2 (Burud et al. 2002). The lens is a relatively distant (z ≃ 0.7) late-type elliptical with estimated velocity dispersion of σ ∼ 200 km s\(^{-1}\), steep mass profile, central convergence of κ ≃ 0.5 and possible signatures of interaction with its environment (Auger et al. 2008). B is about three times closer to the lens centre, hence we expect a moderate microlensing optical depth for B and a small one (τ ≲ 0.1) for A. Hence we are inclined to interpret the observed flux-ratio variations as microlensing of the B image.

We used the three following sources of reduced photometric data (R-band magnitudes for both images) covering a time span of about seven years (see Fig. 1):

(i) 58 data points obtained with the Nordic Optical Telescope (NOT) and published by Burud et al. (2002);
(ii) 60 observations with the 1.5-m Russian-Turkish Telescope (RTT: Khamitov et al. 2006), kindly provided by I. Bikmaev and I. Khamitov;
(iii) 123 observations with the 1.5-m AZT-22 telescope in Maidanak. These data were described, analysed and published by Gaynullina et al. (2005).

All the photometry was performed in the standard optical $R$ band, which for a source redshift of 1.855 (Barkhouse & Hall 2001) corresponds to $\lambda \simeq 2000$–2500 Å in the source reference frame.

The relatively large amount of data allows us to improve the estimate of the delay time. Cross-correlating the time series for the two images, we find a broad peak at $\Delta t = 127.6 \pm 2.0$ d, which we interpret as the delay between the two images. Within the uncertainties (which are hereafter calculated for 90 per cent significance level), this value is consistent with the 130 d delay found by Burud et al. (2002) and Gaynullina et al. (2005) and with the 128-d estimate by Khamitov et al. (2006). This value of $\Delta t$ is subsequently used to shift the series of A and B fluxes and estimate flux ratios.

The merged time series consists of 241 unevenly sampled $R$-band observational points for each image (see Fig. 1). To minimize information losses but avoid unjustified interpolation (on periods of time longer than tens of days), we construct flux ratios by linearly interpolating the magnitudes between the observational points. The resulting flux-ratio series consists of points of two kinds: (i) B-image fluxes shifted backwards by $\Delta t$ and divided by the interpolated values of A-image flux and (ii) interpolated B-image fluxes divided by the A-image fluxes shifted forward by $\Delta t$. We did not interpolate over time gaps longer than 45 d (about 16 d in the frame comoving with the object); therefore the resulting time series contains only 360 points, not entirely independent (see also Section 4.1).

The flux-ratio curve (see Fig. 2) demonstrates a maximum at the Julian date JD $\sim 245$ 1500 and another one at JD $\sim 245$ 1800, which together form an about two-year-long structure similar to standard disc amplification curves (see the next section). For the remaining two-thirds of the curve, the B/A ratio is stable and close to 0.47 with a standard deviation of around 4 per cent. From the resulting time series, we excluded the final portion of the curve in range JD = 245 3000–245 3600, where the data are sparse and the observed variability is difficult to relate to the high-amplification event in the earlier part of the data (see Fig. 10 later). The resulting amplification curve consists of 253 data points.

### 2.2 QSO J2237+0305

We use $R$- and $V$-band photometric data on images A and C of QSO J2237+0305. The data were taken from two sources.

(i) OGLE-II Huchra’s lens monitoring programme (Woźniak et al. 2000). The data are available at http://ogle.astrouw.edu.pl/cont/4_main/len/huchra/huchra_ogle2.html and contain $V$-band magnitudes for the four images.

(ii) Gravitational Lenses International Time Project (GLITP) archive (Alcalde et al. 2002): $V$- and $R$-band magnitudes for all images. Reduced data are available at http://wela.astro.ulg.ac.be/themes/extragal/gravlens/bibdat/engl/lc_2237.html.

The light curves are shown in Fig. 3. The data sample is similar to that used by Koptelova et al. (2007). We use GLITP data reduced by two different techniques: ISIS photometry (the magnitudes are available on the web page given above) and point-spread function (PSF) fitting (the magnitudes are meant to be available for download but the link is broken, hence we asked Elena Shimanovskaya who has kindly provided us with these magnitudes). The two reduction techniques are compared in Alcalde et al. (2002). Magnitudes reduced by different techniques are generally consistent but deviate considerably (by about 0.03 mag) near the maximum of the image A high-amplification event. Unfortunately, the maximum coincides with a period of bad visibility of the object (January/February 2000). OGLE points show unreasonably large scatter near JD = 245 1500. Also, consistency between OGLE and GLITP data during this period is poor and insufficient for our purposes (see Fig. 3, left panel).

![Figure 2](image_url)
Better but still significant deviations exist between the GLITP magnitudes reduced with different methods (we show only ISIS-reduced data to prevent the figure from overcrowding; the two GLITP light curves are shown together later, in Fig. 12). Therefore we exclude OGLE data from subsequent analysis for the image-A event.

GLITP data during the maximum of image A brightness demonstrate a double-peaked structure that we are inclined to interpret as a signature of the inner disc structure. The overall behaviour near JD 245 1520–245 1540 suggests that the dip is real and has an amplitude of about 0.02 mag.

For the C image event, most GLITP observational points are far away from the high-amplification event maximum, and we finally fit only the V-band OGLE data.

There is strong evidence that the individual time delays for the images of QSO J2237+0305 are very small, such as several days (see Koptelova, Oknyanskij & Shimanovskaya 2006 and references therein). We can make use of the relative proximity of the lens, which makes the microlensing variability of individual images much faster and stronger than the intrinsic variability of the source. The studied high-amplification events last less than a year, and variability of individual images does not show any detectable correlation on these time-scales. The Einstein cross is known for its lack of variability: the maximal gradients of intrinsic variability are considerably smaller than microlensing trends (see for example Eigenbrod et al. 2008a).

3 Amplification-Curve Simulation Technique

3.1 Basic assumptions

All the three high-amplification events are good candidates for caustic crossings. We model them by convolving a straight caustic-crossing curve with a standard disc profile integrated over one dimension. Simplified flat and non-relativistic (Section 3.2) and general relativistic (Section 3.3) disc models were used. In both cases we consider a multi-blackbody disc with monochromatic intensity obeying the Planck law with the temperature determined by the standard accretion-disc model:

$$I \propto \frac{1}{\exp \left( \frac{h \nu}{k T(r)} \right) - 1},$$

$$= \frac{1}{\exp \left( \frac{r/r_d}{3/4} \left( 1 - \sqrt{r/r_i} \right)^{1/4} \right) - 1},$$

where $r$ is the radial coordinate and $r_d$ and $r_i$ are, correspondingly, the radial scale of the disc and its innermost stable orbit radius, defined as

$$r_d = \left( \frac{k \lambda}{hc} \right)^{4/3} \left( \frac{3 M G^2}{2 \eta \sigma \kappa} \right)^{1/3},$$

$$r_i = x_{ISCO}(a) \frac{GM}{c^2}.$$

Here, $k$, $\lambda$, and $h$ are the Boltzmann, Stefan–Boltzmann and Planck constants, $c$ is the speed of light, $l$ and $\eta$ are the Eddington ratio and overall accretion efficiency, and $\kappa$ is the Thomson opacity, $\kappa \approx 0.35 \text{ cm}^2 \text{ g}^{-1}$ for solar composition. The dimensionless innermost stable circular orbit radius $x_{ISCO}(a)$, as well as efficiency $\eta(a)$, may be found analytically as functions of the dimensionless Kerr parameter $a$ (Bardeen, Press & Teukolsky 1972).

Expression (4) was derived for monochromatic intensity (no matter whether $I_\nu$ or $I_\lambda$) but is, to a high accuracy, valid even for broadband photometry if most of the radiation comes in the form of a smooth continuum. The mean effective wavelength varies by less than 2 per cent for a spectral slope varying from $p = -1$ to $p = 1$, where $F_\nu \propto \nu^p$. The above definition of $r_d$ is identical to the characteristic radius used by Morgan et al. (2010). Note, however, that it is generally sufficiently smaller (by a factor of $\sim 2.44$ for a standard disc with no inner edge) than the half-light radius. We will...
see below that for the shape of the amplification curve, the quantity of principal importance is the ratio of the two radial scales given by (5) and (6):

$$X = \frac{r_0}{r_{in}} = \left( \frac{k \lambda}{h(1+z)} \right)^{4/3} \left( \frac{3}{2} \eta(\alpha) GM \sigma_{xt} \right)^{1/3} \left( \frac{1}{0.25} \right)^{1/3} \left( \frac{0.1}{\eta(\alpha)} \right)^{-1/3} \times \left( \frac{M}{10^8 M_\odot} \right)^{-1/3} x_{ISCO}(\alpha).$$

For fixed $X$, the differences between amplification curves are important only for high inclinations and are connected to relativistic effects, namely Doppler boosting and light bending. Variations of the Kerr parameter may change $X$ by a factor of $\max(x_{ISCO})/\min(x_{ISCO}) = 9$. Inversely, one may reasonably estimate $X$ by fitting the amplification curve and arrive at a tightly correlated pair of uncertain $a$ and $M$. Having reasonable mass estimates, one may thus make an estimate for $a$ and vice versa.

Below, we fix the Eddington ratio at $l = 0.25$. The black hole masses are estimated as $M = 8.8 \times 10^9 M_\odot$ for SBS J1520+530 and $M = 9 \times 10^8 M_\odot$ for QSO J2237+0305 by Morgan et al. (2010). These mass estimates were obtained using emission-line profiles (see references given by Morgan et al. (2010) and the virial relations (Vestergaard & Peterson 2006)). The accuracy of these estimates is low, about 0.3 dex. Our analysis generally yields higher masses, poorly consistent with the virial estimates (see Section 4 below). In fitting the observational data, we first find $X$, inclination $i$ and positional angle $\psi$ and then optimize for the mass and Kerr parameter, fixing $X$.

A caustic is considered a straight line defined by condition $y = v_{eff} \times (t-t_0)$ at the given time $t$. Let $x$ and $y$ be the coordinates in the picture frame along and across the caustic, respectively. The origin of this coordinate system coincides with the black hole centre. In the general case, the disc is inclined and it is convenient to use the coordinate system $(\alpha, \beta)$ defined by the major and the minor axes of the disc projection upon the picture frame (as in Dexter & Agol 2009). The two coordinate systems in the picture frame are connected by rotation by an angle of $\psi$, which has the meaning of the relative positional angle of the normal to the disc with respect to the normal to the caustic (see the sketch in Fig. 4). Without any loss of generality we assume that the relative motion of the source and lens is normal to the caustic itself. Integration over one direction is convenient to consider as projection upon the orthogonal direction.

The effective velocity $v_{eff}$ is calculated as the relative proper motion multiplied by the angular size distance $D_s$ toward the object. Our analysis is sensitive only to its component perpendicular to the caustic direction. The velocity is connected to the peculiar spatial velocities of the QSO ($v_S$) and the lensing galaxy ($v_L$) in the following way:

$$v_{eff} = \frac{v_S \cdot n}{1 + z_S} - \frac{v_L \cdot n}{1 + z_L} = \frac{v_S \cdot n}{1 + z_S} D_S - \frac{v_L \cdot n}{1 + z_L} D_L.$$  

(8)

The third term describes the peculiar motion of the observer; $v_S \sim 370 \text{ km s}^{-1}$ is known from the dipole constituent of cosmological microwave background (CMB) variations (see for example Lineweaver et al. 1996). Unit vector $n$ is normal to the caustic and lies in the picture plane. A reasonable limit for peculiar motion velocities of individual galaxies is $v \lesssim v_{max} \sim 2000 \text{ km s}^{-1}$, which corresponds to about 2–3 root-mean-square (rms) values for the radial peculiar velocity distribution (Raychaudhury & Saslaw 1996). This limit may be converted to the condition for the possible effective transverse velocity $v_{eff} \lesssim v_{max} \sqrt{(1 + z_S)^2 + (1 + z_L)^2} \times (D_S/D_l)^2$. For SBS J1520+530, $v_S$ and $v_L$ contributions are of the same order because $D_S/D_l \approx 1.2$. In this case, $v_{eff} \lesssim 1600 \text{ km s}^{-1}$. For QSO J2237+0305, the lens is about ten times closer ($D_S/D_l \approx 11$) and $v_{eff} \lesssim 20000 \text{ km s}^{-1}$.

### 3.2 Simplified standard disc

Besides the more sophisticated model described below, we use a simple disc approximation ignoring all relativistic effects. In this approximation and for the straight caustic case, inclination does not affect the observed shape of the amplification curve. This is valid for a thin disc of arbitrary inclination as long as the disc is thin enough ($hR \ll \cos i$, where $i$ is inclination).

We are interested in intensities integrated over one dimension. Here, we provide the general form for the integral over one direction (we make a substitute $t = x/y$):

$$I_1(y) = \int I(x, y) \, dx \propto \int \frac{\sqrt{J} \, y}{K} \, \frac{\exp \left( \frac{-x_0}{n} J \times (1 + t^2) \right)^{3/8} f^{-1/4}}{f - 1} \, df,$$

where

$$K = K(i, \psi) = \cos^2 \psi + \frac{\sin^2 \psi}{\cos^2 i},$$

$$J = J(i, \psi) = \sin^2 \psi + \frac{\cos^2 \psi}{\cos^2 i} - \frac{\sin^2 \psi \cos^2 \psi}{K(i, \psi)} \tan^4 i,$$

$$f = 1 - \left( \frac{1}{J} \frac{r_{in}^2}{y^2 + 1 + t^2} \right)^{1/4},$$

$$t_{in} = \left\{ \begin{array}{ll}
0 & \text{if } y \geq r_{in}/\sqrt{J}, \\
\sqrt{1 - \frac{1}{J} \left( \frac{r_{in}}{y} \right)^2} & \text{if } y < r_{in}/\sqrt{J}.
\end{array} \right.$$  

If the influence of the inner radius is negligible, the onedimensional intensity profile is identical to the face-on disc intensity.

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profile with the radial scale of \( r_0 = r_d / \sqrt{J} \). Even if the correction factor \( f \) is taken into account, profile shapes do not depend on the angles \( \psi \) and \( i \), because the integral (9) contains the angles and coordinates only in combinations \( \gamma \sqrt{J} / r_d \) and \( \gamma \sqrt{J} / r_m \), which affect only the stretch factor of the curve for given \( X \).

The influence of the inner disc edge is included in the single parameter \( X = r_d / r_m \) (see previous subsection) determining the shape of the amplification curve. In Fig. 5, the dependence of the amplification curve on this parameter is shown for a fixed black hole mass. The Kerr parameter is varied from \(-0.99 \) to \(0.99 \).

### 3.3 Kerr black hole disc of arbitrary inclination

Photons move along null geodesics that we calculate using the \texttt{geokerr} code (Dexter & Agol 2009). The disc was ascribed a constant relative thickness of \( h r = 0.01 \), which does not affect the results much as long as \( \cos i > h r \). The maximal inclination value we use is \( i = 88^\circ \), which implies \( \cos i \approx 0.03 \). We linearly interpolate the radial coordinate to the point where \( \cos i = \pm h r \) to determine the intrinsic intensity of the locus on the disc corresponding to the current values of \( a \) and \( \beta \). For this point, using the conserved angular momentum \( k_\phi \) and the energy at infinity \( k_i \) of the photon, we connect the observed photon energy (equal to \( k_i \)) to its energy in the frame corotating with the disc. The Doppler factor is calculated as

\[
\delta = \frac{v_{\text{obs}}}{v_{\text{em}}} = \frac{k_i}{k_i u'} = \frac{1}{u' (1 + \Omega^2)}. \tag{14}
\]

Here, \( u' = 1 / \sqrt{ - (g_{rr} + 2g_{r\phi} \Omega + g_{\phi\phi} \Omega^2) } \) is the null component of the disc rotation four-velocity (where the metric signature is \(-+\) and \( I = k_\phi / k_i = \sin i \) is the net angular momentum of the photon. For corotating discs, \( \Omega = \frac{1}{r^{3/2} + a} \)

in \( c^2 / GM \) units, while the radius \( r \) is in \( GM/c^2 \). Counter-rotating discs may be also considered in this formalism by changing the sign of \( a \). Hereafter we will use negative \( a \) values to describe the case of counter-rotation.

The observed intensity is integrated over a fixed (observer-frame) wavelength range and is therefore approximately proportional to the monochromatic intensity \( I \) at the central observer-frame frequency \( v_{\text{obs}} \). For a local blackbody-like spectrum (we consider \( I \) for convenience),

\[
I_v \approx \Delta v_{\text{obs}} \delta I_0 (v_{\text{em}}) \propto \left( \frac{\exp \left[ \frac{1}{\delta} (r/r_{\Delta})^{1/4} \right] - 1 \right)^{-1}. \tag{16}
\]

The Doppler boost thus contributes only through direct frequency shift. At large distances (\( r \gg r_m \) and \( r \gg r_d \)), its effect on the local intensity is still important, primarily due to the rapid fall-off of the Planck law with frequency. For inclined discs, the intensity distribution along the major axis is asymmetric even at large distances from the black hole. Up to the two leading terms in \( 1/r \), \( 1/8 \approx 1 + \sin i / \sqrt{r} \) and the relative intensity difference between the approaching \( (I^+) \) and receding \( (I^-) \) sides of an inclined disc is

\[
\frac{I^+-I^-}{I^+}\approx (r/r_d)^{1/4} \sin i \propto r^{1/4}. \tag{17}
\]

The above estimate was made on the assumption that the asymmetry is small. The actual value of the asymmetry is evidently limited by the value of max \( [(I^- - I^-)(I^+ + I^-)] \approx 1 \) when one side of the disc is much brighter than the other. The approaching side of an accretion disc is generally about twice as bright as the receding side at \( r \approx 10-100 GM/c^2 \). The weighted disc centre at large distances is effectively shifted by \( \Delta r \approx \sin i \times r^{1/2} \).

### 3.4 Calculation of amplification curves

For calculating the shapes of null geodesics, we use the public code of Dexter & Agol (2009). Then we take into account Doppler shifts and Doppler boosts in the way described above. For a given Kerr parameter \( a \), inclination \( i \) and relative positional angle \( \psi \), intensity is calculated on a rectangular grid that is log-uniform in \( r \) and \( \theta \). The two-dimensional brightness distribution is integrated over one dimension:

\[
I_t(y) = \int_{-\infty}^{+\infty} I_t(x, y) \, dx. \nonumber
\]

Then, the microlensing amplification curve for the straight caustic case may be calculated in the following way (see for example Koptelova et al. (2007), who used a similar approach):

\[
\mu(t) = \mu_0 \sqrt{\xi_0} \times \frac{\int I_t(x) \times \Delta y^{-1/2} \times \Theta(\Delta y) \, dy}{\int I_t(y) \, dy} + \mu_0. \tag{18}
\]

Here, \( \Delta y = y - \nu(t - t_0) \) and \( \Theta(x) \) is the Heaviside function: \( \Theta(x > 0) \equiv 1 \) and \( \Theta(x \leq 0) \equiv 0 \). The spatial scale factor \( \xi_0 \) is chosen according to Witt et al. (1993):

\[
\xi_0 \approx \sqrt{\frac{4GM_{\odot}}{c^2} [1 - \kappa_{c}] \frac{D_{ls}}{D_{l}}}. \tag{19}
\]
Here, $\kappa_r$ is the continuous contribution to the total convergence. If we adopt $\kappa_r = 0.5$, for SBS J1520+530 and QSO J2237+0305 the spatial scale is $\zeta_0 \simeq 3 \times 10^{10}$ and $1.3 \times 10^{17}$ cm, respectively.

The effective transverse velocity $v$ and the fold-crossing epoch $t_0$ are considered as free parameters. For every model (every given one-dimensional brightness distribution), $v$ and $t_0$ are optimized for by minimizing $\chi^2$ for the observational amplification curve. For the A image of QSO J2237+0305, two amplification curves were used for the two filters $R$ and $V$ and we fit both simultaneously. Our fitting has two additional parameters, $\mu_0$ and $\mu_1$, that we calculate using linear regression for each iteration of the optimization process. In the case of QSO J2237+0305, two $\mu_{0,1}$ pairs were used for the two photometric bands. The ratio of the two coefficients, $\mu_1/\mu_0$, has the meaning of caustic strength and depends only on the mass distribution in the lens (Kayser & Witt 1989).

Evidently, for the shape of the caustic-crossing amplification curve, the effects of general relativity are of primary importance. In Fig. 6, we show four light curves emerging from caustic crossings by a disc inclined by 60° around a $a = 0.6$ black hole, traversing a straight fold caustic in four directions: along the minor axis ($\psi = 0$ and 180°) and along the major ($\psi = 90$ and 270°). The effective transverse velocity and black hole mass used for Figs 6 and 7 were $5000 \text{km} \text{s}^{-1}$ and $10^8 M_\odot$, respectively. Due to their asymmetry, inclined discs produce amplification curves strongly dependent on the relative positional angle. Fig. 6 demonstrates all the main complications arising from relativistic effects. If the brighter part of the disc is amplified while its dimmer part is still unaffected by the caustic, the amplification curve may become two-peaked with the relative intensity of the two peaks strongly dependent on inclination, positional angle and black hole rotation parameter (Fig. 7). An accretion disc surrounding a rapidly rotating Kerr black hole observed at high inclination has a bright compact ‘hot spot’, which influences the amplification curve strongly. Relativistic effects are capable, in particular, of qualitatively explaining the observed fine structure of the amplification events under consideration.

Test runs reveal a small but considerable contribution from the photons belonging to the higher-order images produced by extreme light bending near the black hole (Beckwith & Done 2005). Their contribution to the total flux may be as high as several per cent, depending on the inclination and the Kerr parameter. The figure of 10 per cent given by Beckwith & Done (2005) is an overestimate for an optically thick disc, because at high inclinations and high Kerr parameters a considerable part of the black hole is covered by the disc. Simulated black hole shadows with one higher-order image visible are shown in Fig. 8. Higher orders have significantly smaller fluxes due to the gradually decreasing solid angle. The flux of the $n$th image loop decreases proportionally to its radial size $\Delta r$ (intensity is approximately conserved). During caustic crossing, the additional amplification factor is proportional to $\Delta r^{1/2}$, where the observed projected width of the approximately annular image is $\propto \Delta r$. The resulting contribution of the $n$th order image scales as $\propto \Delta r^{1/2} \propto \exp(-\pi n/b)$, where $b \simeq 2.7$ (in the so-called strong deflection limit; see Bozza 2010). Higher-order images are thus exponentially damped.

An example of higher-order contribution is given in Fig. 9. For high inclinations, it is generally about a couple of per cent, but are lower than 1 per cent for $i \approx 0$ and $a > 0$. For counter-rotating black holes, the size of the inner hole is largest and the contribution of the secondary images reaches 2–3 per cent for characteristic QSO black hole masses $\sim 10^8 M_\odot$. The effect is also important for larger black holes and at shorter wavelengths.

4 RESULTS

Fitting results using the simplified face-on disc and fully relativistic disc models are given in Tables 2 and 3, respectively.

4.1 SBS J1520+530

Fitting with the plain face-on disc model allows us to recover the size of the inner hole approximately. The number of degrees of freedom is 253 -- 5 = 248 here (an ‘optimistic’ number, see below). We fixed the mass of the black hole at $3 \times 10^8 M_\odot$ and localized an absolute minimum near $a = -0.4$. The corresponding $X$ value is 1–2. The minimum is consistent with physical $-1 < a < 1$ only for black hole masses in the range $(1.6 - 3.5) \times 10^8 M_\odot$. The best-fitting amplification curve is also shown in Figures 10 and 11 by a dashed line. Here we limited the effective velocity by the value of $v_{\text{eff,max}} = 2000 \text{km} \text{s}^{-1}$. If we relax this limitation, the fit may be improved (up to $\chi^2/\text{DOF} \sim 190$, where ‘DOF’ denotes degrees of freedom) by increasing the mass of the black hole and the transverse velocity.

A total of 708 fully relativistic models were calculated covering the possible ranges of inclination $0 < i < 90^\circ$, relative position angle $0 < \psi < 360^\circ$ and Kerr parameter for corotating ($1 \times a > 0$) and counter-rotating ($-1 < a < 0$) cases. For every model, we make an optimization run finding the best-fitting $v_{\text{eff}}$ and $t_0$ and two amplification parameters $\mu_0$ and $\mu_1$. Thus the model one-dimensional
Figure 7. Amplification curves for an accretion disc around a black hole with $a = 0.6$ inclined by $i = 60^\circ$ (upper panel) and $i = 85^\circ$ (lower panel) for variable positional angle $\psi$. Every horizontal slice corresponds to one curve. The scale is linear; the maximal shade corresponds to maximal brightness.

The brightness distribution fixes only the shape of the amplification curve, which may then be stretched and shifted in time ($t_0$ and $v_{eff}$) and in amplification factors ($\mu_{0,1}$). For Kerr disc models, we first fix the mass of the black hole to $0.88 \times 10^9 M_\odot$ and then, after finding the global minimum at $a = -0.9^{+0.4}_{-0.05}$, $i = 80 \pm 5^\circ$ and $\psi = 340 \pm 4^\circ$, fix $X$, $i$ and $\psi$ and vary the mass and rotation parameter.

In Tables 2 and 3, we give the best-fitting parameters for the amplification curve of SBS J1520+530 fitted with the simplified disc model and with a Kerr disc with the best-fitting parameters. A fully relativistic disc provides a much better fit (see below). However, the observed details in the amplification curve are even sharper and more profound. This is the possible signature of a non-trivial structure of the inner parts of the disc resulting from either disc tilts and warps (see below, Section 5.4) or additional energy input from black hole rotation (Agol & Krolik 2000).

The apparently low values of $\chi^2$ should be addressed separately. The procedure that we used to reproduce the amplification curve (see Section 2) minimizes the loss of observational information but generates two sets of flux ratios (A image fluxes interpolated over the observational epochs for B and vice versa) that are not entirely independent. Therefore, the effective number of degrees of freedom (DOF) is not the number of flux-ratio data points ($N_F = 253$) minus the number of parameters ($N_P = 5$ for simplified disc and $N_P = 8$ for Kerr metric) but something lower. Therefore in the tables we give two estimates for the number of DOFs: one optimistic $DOF_1 = N_F - N_P$ and one pessimistic $DOF_2 = N_F/2 - N_P \simeq 120$. The real value is somewhere in between. For the simplified disc model, the number of parameters is less by 3 ($i$ and $\psi$ lacking, and $X$ instead of $a$ and $M$).

For both DOF estimates, the difference in $\chi^2$ is significant: the $F$-test gives the probability of $7 \times 10^{-10}$ (optimistic) and $4 \times 10^{-5}$ (pessimistic DOF estimate) of the more sophisticated model providing a better fit accidentally. The measured best-fit parameters, primarily the high inclination, are suspicious but not entirely unphysical. We propose that the real disc should be inclined with respect to the black hole plane and its inner parts should thus be...
black hole shadows for $i = 80^\circ$ and two Kerr parameters, $a = 0$ and 0.9. The secondary image is clearly seen for the Schwarzschild black hole and is visible as an inclined nearly straight line at the approaching (left) side in the other case. A bar below the black hole has length equal to the innermost stable orbit radius (6 and $\sim 2.3$ in $GM/c^2$ units, respectively).

![Figure 8](image_url)  
**Figure 8.** Black hole shadows for $i = 80^\circ$ and two Kerr parameters, $a = 0$ and 0.9. The secondary image is clearly seen for the Schwarzschild black hole and is visible as an inclined nearly straight line at the approaching (left) side in the other case. A bar below the black hole has length equal to the innermost stable orbit radius (6 and $\sim 2.3$ in $GM/c^2$ units, respectively).

![Figure 9](image_url)  
**Figure 9.** Relative difference between the microlensing curves calculated in the single-image assumption and taking into account all the resolved secondary loops. We set $M = 10^{10} M_\odot$, $a = 0$, $i = 80^\circ$ and $\psi = 0$ (solid line) and $90^\circ$ (dotted line). The effective transverse velocity is $v_{\text{eff}} = 1000 \text{ km s}^{-1}$; the vertical line marks the instance of caustic crossing by the centre of the black hole.

strongly warped and distorted. We discuss this issue in more detail in Section 5.4.

An alternative to a single fold caustic amplifying a relativistic strongly inclined disc is a more complex structure of several caustics or cusps. We find excellent agreement on fitting the amplification curve with a simplified disc traversing two fold caustics sequentially in one direction at one velocity (Fig. 11). The best-fitting parameters are, for fixed $a = 0.2$ and $M = \mathcal{L} \times 10^8 M_\odot$, $v_{\text{eff}} = (1.35 \pm 0.22) \times 10^4 \text{ km s}^{-1}$, $t_{\text{fold}} = 2451440 \pm 12 \text{ d}$ for the first fold and $t_{\text{fold}} = 2451810 \pm 30 \text{ d}$ for the second, and the ratio of caustic strengths $l = 0.84 \pm 0.06, \chi^2 = 160/244(118)$. The strengths of the two caustics are about 0.7 and 0.6 and the weaker one is delayed by $\Delta t = 367 \pm 8 \text{ d}$. The worst problem for this interpretation of the observational data is the high required transverse velocity. The spatial size of the disc is $\propto M^{2/3}$, and the mass should be sufficiently decreased to match the required transverse velocity to the observed peculiar velocity distribution for galaxies. Setting $M = 3 \times 10^8 M_\odot$ allows us to obtain a reasonable fit ($\chi^2 \simeq 162$) for a relatively small transverse velocity $v_{\text{eff}} \sim 2 \times 10^3 \text{ km s}^{-1}$.

### 4.2 QSO J2237+0305

Simplified disc fits appear even worse for the case of QSO J2237+0305. Qualitatively, the situation is understandable: a

| Table 2. Results of amplification-curve fitting with the simplified accretion-disc model. Effective velocity is normalized by the disc radial scale (R-band for SBS J1520+530, V-band for QSO J2237+0305) in $10^{15}$-cm units. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                 | $r_d/r_{\text{in}}$ | $v_{\text{eff}}/r_d,15$, km s$^{-1}$ | $t_0$, JD $-$245 0000 d | $\mu_1/\mu_0$ | $\chi^2/\text{DOF}$ |
| SBS J1520+530   | $1.5^{+0.6}_{-0.4}$ | $740^{+80}_{-60}$ | $1676.3^{+4.7}_{-0.1}$ | $1.05 \pm 0.17$ | $209/246(120)$ |
| QSO J2237+0305 (A): ISIS | $2.0 \pm 0.03(R)$ | $-2900 \pm 100$ | $1460 \pm 5$ | $2.4 \pm 0.3(R)$ | $3.9 \pm 0.4(V)$ | $560/99$ |
|                 | $1.6 \pm 0.02(V)$ | $2.6 \pm 0.13(R)$ | $-2920 \pm 120$ | $1455.0 \pm 2$ | $4.5^{+0.5}_{-0.2}(R)$ | $6.0^{+0.4}_{-0.2}(V)$ | $122/96$ |
| QSO J2237+0305 (C) | $> 4.4$ | $6100^{+400}_{-300}$ | $1389.0^{+0.2}_{-0.1}$ | $0.66 \pm 0.02$ | $183/77$ |
Table 3. Results of amplification-curve fitting with an inclined Kerr disc model.

|                  | $\frac{r_d}{r_{in}}$ | $a$ | $M$ | $i$ | $\psi$ | $v_{eff}$ | $t_0$ | $\mu_1/\mu_0$ | $\chi^2$/DOF |
|------------------|----------------------|-----|-----|-----|--------|-----------|-------|----------------|-------------|
| SBS J1520+530:   | 2.15 ± 0.05          | -0.6–0.2 | 1.7–3.4 | 80 ± 10 | 335 ± 5 | 1500–2400 | 1678 ± 5 | 0.61 ± 0.05 | 176/244(118) |
| QSO J2237+0305:  |                      |     |      |     |        |           |       |                |             |
| A image, ISIS    | 1.7 ± 0.1 (R)        | 0.2^{+0.1}_{-0.2} | 7^{+2}_{-1} | 71^{+15}_{-5} | 96 ± 5 | -(3 ± 1) 10^4 | 1509.9 ± 1.4 | 0.91 ± 0.06 (R) | 333/96   |
|                  | 1.4 ± 0.1 (V)        |     |      |     |        |           |       |                |             |
| A image, PSF     | 1.7^{+0.6}_{-0.2} (R) | -0.3–0.8 | 5^{+3}_{-2} | 70^{+10}_{-20} | 90^{+30}_{-60} | -(1.5–3.0) 10^4 | 1512^{+8}_{-5} | 0.96 ± 0.07 (R) | 98/93   |
|                  | 1.4^{+0.5}_{-0.02} (V) |     |      |     |        |           |       |                |             |
| C image          | 2.1^{+0.7}_{-0.2}    | 0.97±0.02 | $10^{+10}_{-3}$ | 85 ± 5 | 53 ± 10 | (2.0 ± 0.2) 10^4 | 1309 ± 4 | 2.09 ± 0.05 | 88/75     |

Figure 10. $R$-band B/A flux ratio for SBS J1520+530 fitted with a Schwarzschild face-on ‘plain disc’ without relativistic effects (dashed curve) and a Kerr multi-blackbody disc of arbitrary inclination and spin (solid). The parameters of the best-fitting Kerr disc model are $a = -0.5, M = 2 \times 10^9 M_\odot, i = 80^\circ, \psi = 330^\circ$. The lower panel shows residuals with respect to the Kerr disc model.

plain-disc symmetric brightness distribution is unable to reproduce the two-peak structure of the image-A event as well as the narrow-peak feature of the image C event. Both are relatively well reproduced if the inclination is high ($i \gtrsim 70^\circ$).

Running more than 2000 models, we find a reasonable fit for the A-image amplification curve (Fig. 12). For fitting, we use GLITP data reduced by the two standard techniques separately. PSF-fitting data have larger (but probably more reliable) statistical errors, hence the minimal $\chi^2$ is considerably smaller. Higher rotation parameters (up to $\sim 0.9$) and higher masses (up to $\sim 10^{10} M_\odot$) produce apparently worse fits but are still allowed if one increases the observational uncertainties by a factor of several. Inclination $i$ and positional angle $\psi$ are well constrained, because the notable two-peak structure requires a certain disc orientation (see Fig. 7). At the moment we are satisfied with a qualitative agreement because the data definitely have some additional error sources, possibly connected to the bad visibility of the object near the amplification curve maximum. The amplification curve shape is similar for both filters; even the two maxima seen in the $R$-band curve are also visible in the $V$ band.
Resolving QSO discs

It is impossible to explain the observed amplification curve of the C image without a relatively large $a \gtrsim 0.95$. The rapid rise of the curve in Fig. 13 cannot be reproduced unless the accretion disc has a bright spot naturally explained by high-rotation-parameter, high-mass models. The predicted mass is very high, $M \gtrsim 7 \times 10^9 M_\odot$. If the C-image amplification is a bona fide caustic-crossing event, then virial mass measurements underestimate the masses of quasars by a factor of several.

5 DISCUSSION

5.1 What do caustic crossings really probe for?

The fine structure of high-amplification events in the light curves of microlensed quasars is an effect naturally expected in the framework of thin accretion-disc models. Replacing the accretion-disc brightness distribution by some symmetrical structure with brightness uniformly decreasing with distance would produce a light curve with a single nearly symmetrical peak with duration time-scale $r_d/v_{\text{eff}}$ defined by the size of the emitting region (see for example the analytical solutions given in Schneider & Weiss (1987), Appendix B). Its shape near the maximum is roughly parabolic. In this case, the amplification-curve maximum is always convex unless the brightness distribution has a non-monotonic or asymmetric shape. Interpreting individual humps in the amplification curves as individual caustic-crossing events leads to unphysically large transverse velocities several times larger than the upper limits estimated in Section 3.1.

As was shown in Section 3.2, the only parameter affecting the amplification curve shape in the simplified disc case is $X = r_d/r_{\text{in}}$. The disc-size scale itself is degenerate with the effective transverse velocity. Amplification curves are primarily sensitive to the combinations $r_{\text{in}}/v_{\text{eff}}\sqrt{\mathcal{J}}$ and $r_d/v_{\text{eff}}\sqrt{\mathcal{J}}$ (see Section 3.2), which have the physical meaning of traversal times for the central hole in the disc and the disc radial scale itself, respectively. This also holds for the general relativistic case as long as the disc is nearly face-on.

Relativistic effects break these degeneracies at inclinations $i \gtrsim 60^\circ$. Amplification-curve properties become sensitive to the two angles $i$ and $\psi$, but the dependence on $a$ is weaker and similar light curves emerge for similar values of $X$ if the inclination and positional angle are the same.

Interpretation of the three events relies on their consideration as caustic-crossing events. For QSO J2237+0305, this is justified by the works of Gil-Merino et al. (2006); Koptelova et al. (2007), who performed detailed simulations to check whether artificial amplification maps produce caustic crossings similar (in some sense) to the observed peaks in the light curves of individual images.

For the case of SBS J1520+530, optical depths to microlensing are relatively small. The central convergence of $\kappa_\text{c} = 0.5$ sets an absolute upper limit for it. The optical depth to microlensing is smaller than $\kappa_\text{c}$ unless the stars in the optical path are strongly clumped, which is unlikely for an elliptical galaxy (the probability of accidentally hitting a globular cluster is $\sim 10^{-5}$). If dark matter makes a significant contribution to the gravitational field of the lensing galaxy, optical depths should be decreased proportionally toward the probable value of $\sim 0.1$ for the stronger lensed B image. Image A is...
about three times further from the centre of the lens, which makes it a much worse candidate for microlensing. The caustic-net structure in this case is determined by small shears distorting single-lens amplification patterns (Kofman et al. 1997). Therefore, there is a strong probability of passing a close caustic pair. However, in this case the caustics are traversed in opposite directions, which should produce a pattern completely different from the observed amplification curve.

In this work, we used Kerr-metric geodesics and relativistic Doppler effects, but the disc model was not fully relativistic. We used the temperature law of Shakura & Sunyaev (1973) instead of the more accurate relativistic thin-disc law provided by Novikov & Thorne (1973) and Page & Thorne (1974). The difference in the two laws lies in the correction factor, which decays more smoothly inwards and is generally smaller for the fully relativistic thin-disc case. We tested the difference between the amplification curves calculated using these two temperature laws and found that for relevant black hole parameters \( M \lesssim 10^{10} \, M_\odot \) and \( a > -0.9 \) the difference is smaller than 1 per cent of the peak amplification. It is more important to consider the effects of non-trivial inner boundary condition, shock formation and rapid disc-tilt change in the inner parts of the disc and then pass on to finer effects, like the general-relativity corrections to the temperature law.

5.2 Caustic strengths

For SBS J1520+530, the amplification-curve amplitude is moderate and may be characterized by a caustic strength of \( k = \mu_1/\mu_0 \sim 0.6 \). Using the estimates made by Witt et al. (1993), one may expect a mean caustic strength of

\[
\langle k \rangle \sim 0.56 (\langle m \rangle/\tau)^{1/4}.
\]

where \( \tau \simeq 0.1-0.5 \) is the optical depth to microlensing and \( \langle m \rangle \sim 0.1-0.5 \) is the mean microlens mass in solar units. Note that the dependence on both the optical depth and the mean mass is very weak, thus making the caustic strength a good consistency check for the model. In the case of SBS J1520+530, observational data argue for \( \langle m \rangle/\tau \sim 1 \), consistent with a moderately subsolar mean stellar mass. The low end of the mass function of red and brown dwarfs is poorly known and probably variable (Bastian, Covey & Meyer 2010). There are indications for a low mean stellar/substellar mass \( (\langle m \rangle \sim 0.1) \) in the old stellar population of elliptical galaxies (van Dokkum & Conroy 2010), therefore the optical depth to microlensing is most likely relatively small, \( \tau \sim 0.1 \). If the amplification curve for SBS J1520+530 is interpreted as a superposition of two caustic crossings, the two folds have similar strengths of 0.7 and 0.6, also consistent with a stellar population rich in low-mass objects.

For QSO J2237+0305 events, the amplitudes are higher. Qualitatively, this was expected, because the lens is a spiral galaxy and the number of massive stars among the microlenses should be higher, with a mean mass of \( \sim 1 \, M_\odot \). The mean caustic strength \( \langle k \rangle \simeq 2 \) (as for the image C event) only if \( \langle m \rangle/\tau \sim 100 \, M_\odot \) and hence \( \langle m \rangle \sim 40 \, M_\odot \) (numerical simulations suggest that the optical depth to microlensing is \( \tau \sim 0.4 \) for Einstein’s cross: Bate et al. 2011), which is considerably higher than one might expect even from a young stellar population. In most simulations with \( \tau \sim 0.5 \) (see for example Schechter & Wambsganss 2002; Schechter, Wambsganss & Lewis 2004), predicted amplification-factor distributions are broad...
and allow caustics with strengths exceeding the mean value of $k$ by a factor of $\sim 2$. The reason why our estimates do not contradict theory is the selection effect: we consider the two brightest and most evident high-amplification events, while the total number of fainter events present in the light curves of the four images in the roughly three-year interval of the OGLE-II monitoring programme is about 10 (if the mean distance between fold traversals is about a year).

5.3 Black hole masses and disc sizes

The relative size of the inner disc hole and the observed strength of relativistic effects favour black hole masses significantly higher than virial estimates. For SBS J1520+530 and QSO J2237+0305, the virial masses estimated using the width of the C iv $\lambda$1549 resonance line are equal to $8.8 \times 10^8$ (Peng et al. 2006) and $9 \times 10^8 M_\odot$ (Yee & De Robertis 1991), correspondingly, with a proposed uncertainty of about a factor of 2. Single caustic-crossing event fitting argues for considerably higher masses, $(2\sim3) \times 10^9 M_\odot$ and $(7\sim9) \times 10^9 M_\odot$.

Altering black hole masses by a factor of $\sim 3$ was proposed by Morgan et al. (2010) to explain the inconsistency between the accretion-disc sizes estimated using microlensing statistics and the accretion-disc size predictions following from the standard disc theory. Note that in our study the degeneracies are different from those in Morgan et al. (2010). While the Monte Carlo light-curve analysis is sensitive to the disc size itself ($r_d \propto (ln) \sqrt{M^2}$), the shapes of caustic-traversal events primarily reflect the ratio of the disc radial size scale to its inner boundary size ($X \propto (ln) \sqrt{M}^{-1/3}$). To reproduce our results without altering black hole masses, we should decrease the Eddington ratio by a factor of several rather than the accretion efficiency, as was proposed by Morgan et al. (2010).

To distinguish between higher black hole masses and lower $ln$ ratios, one may use either photometric data or different-technique accretion-disc size estimates. The central black hole mass may be estimated using amplification-corrected magnitudes. The observed flux from a thin disc is calculated by integrating the observed intensity (indices ‘obs’ and ‘em’ refer here to the radiation intensity in the observer frame and in the frame comoving with the QSO):

$$F_\nu = \int I_{\nu,\text{obs}} \, d\Omega$$
$$= \frac{2\pi}{D^2 \times (1+z)^3 \cos i} \times \int I_{\nu,\text{em}} [\nu/(1+z)] \frac{R}{dR},$$

where $D = D(z)$ is angular size distance. This distance scale is known for its non-monotonic dependence on redshift. A maximal distance of 1.7 Gpc is reached at the intermediate redshift of $z \approx 1.6$ for standard $\Lambda$CDM. Most of the well-studied lensed quasars have redshifts close to this value.

For $I_\nu$, we substitute the Planck law with the temperature law determined by standard accretion-disc theory as

$$T = \left( \frac{3 \sqrt{G^2 M^2 l}}{2 \sigma_{\text{kc}} \eta} \right)^{1/4} R^{-3/4}.$$

Here we neglect the correction term, which has only a small influence on the integral flux, since the area of the disc affected by the term is about $(r_{in}/2.44r_d)^2 \approx 0.1$ times smaller. For integration limits set to 0 and $+\infty$, the integral is reduced to the following
The observed flux is finally expressed as

\[
F_o = \frac{4\pi h c_\text{em} \cos i}{c^2} D^2 \times (1 + z)^3 \int \frac{R \, dR}{\exp \left( \frac{\nu}{kT(R)} \right) - 1}
\]

\[
= 8\pi \left( \frac{2}{3} \right)^{1/3} \Gamma(8/3) \zeta(8/3) \frac{k^{1/3} \theta_{\text{obs}}^{1/3}}{h_\nu^{1/2} \nu^{1/2} \sigma^{1/2}} \times (GM^2/\eta) \cos i \times \frac{1}{D^2 \times (1 + z)^3}
\]

\[
\simeq 6.9 \cos i \left( \frac{1}{\eta} \right)^{2/3} \left( \frac{\theta_{\text{obs}}}{1\mu} \right)^{-1/3} \left( \frac{M}{10^8 M_\odot} \right)^{4/3}
\]

\[
\times \left( \frac{D}{1 \text{Gpc}} \right)^{-2} (1 + z)^{-5/3} \mu\text{Jy}.
\]

This flux may be used to estimate the black hole mass independently. In particular, for the HST F814W filter (we used the calibration given in Holtzman et al. (1995); this allows us to use the magnitudes given by Morgan et al. (2010) in Table 1), the mass may be estimated photometrically as follows:

\[
M \simeq 2.7 \times 10^7 \left( \frac{D}{1 \text{Gpc}} \right)^{3/2} (1 + z)^2
\]

\[
\times \sqrt{\frac{2}{\pi}} \cos^{-3/4} i \times 10^{-0.3(1-19)} M_\odot.
\]

Following the original work, where a similar formula was used to estimate the photometric radius, we denote the magnitude by \( I \) and use the amplification-corrected values of \( I(\text{QSO} \, J2237+0305) = 17.9 \pm 0.44 \) mag and \( I(\text{SBSJ} \, J1520 + 530) = 18.92 \pm 0.13 \) mag. Photometric masses are closer to the virial mass estimates. These estimates are degenerate with the \( \ell \eta \) ratio in a way different from the results of our light-curve analysis, which allows us to reach consistency if the Eddington ratio is low, \( \ell \eta \sim 0.1 \). In Fig. 14, we show the estimates for \( \ell \eta \) and black hole mass made with different methods. Monte Carlo amplification-curve fitting is sensitive to \( r_d \propto M^{2/3} \times (\ell \eta)^{1/3} \), while the observed amplification-corrected flux scales as \( F_o \propto M^{2/3} \times (\ell \eta)^{2/3} \), which results in parallel bands in the graph. For QSO J2237+0305, the microlensing and photometric radii are consistent within the uncertainties. Our data may be made consistent with photometry and virial masses for narrow ranges of parameter values. In particular, for QSO J2237+0305 there is a parameter where all three bands intersect, \( M \simeq (1.6-1.8) \times 10^8 M_\odot \) and \( \ell \eta \simeq 0.5-0.6 \). In the case of SBS J1520+530, photometry, virial and light-curve fitting intersect for \( M \simeq (0.8-1.1) \times 10^8 M_\odot \) and \( \ell \eta \simeq 0.7-1.3 \). However, in this case the microlensing radius found by Morgan et al. (2010) is considerably higher.

Consistency between different methods of radius and mass estimates allows us to suggest that the microlensing radii found by Morgan et al. (2010) are subject to some bias that in some cases increases the effective radius by a factor of several. This may be contamination from a source of much larger angular size (such as unresolved starlight or the broad-line region) or some optically thin scatterer. These effects were considered in the original work and seem to explain the observed discrepancy qualitatively.

Since in general there are inconsistencies between the accretion-disc radii obtained by different methods, we refrain from any final conclusions regarding the masses of the central black holes in QSO discs. Still it seems plausible that the apparently large disc sizes may be connected to larger black hole masses, smaller Eddington ratios and high inclinations (the issue of these will be considered in detail in the next subsection). We also do not exclude the possibility that some non-standard accretion regimes may influence the observational properties of quasar accretion discs. In particular, low accretion rates may lead to an optically thin advection-dominated region in the inner parts of the disc. The innermost stable orbit radius will be overestimated in this case, resulting in black hole masses and accretion-disc sizes biased toward higher values. We will consider the effects of non-standard accretion regimes on microlensing variability in subsequent papers.

Larger black hole masses also mean larger disc sizes. For black hole masses \( M \sim 10^{10} M_\odot \), disc size in the considered wavelength range becomes \( \sim 10^{16} \text{cm} \), which is comparable to the Einstein radius \( r_{\text{Ein}} \) in the case of SBS J1520+530. The straight caustic approximation is violated in this case because \( \theta_{\text{Ein}} \) also sets the mean curvature radius of fold caustics as well as the mean distance between individual folds for intermediate \( \tau \) (Gaudi & Petters 2002). For the case of QSO J2237+0305, the Einstein–Chwolson radius is larger because of the smaller distance \( D_c \). Most of the results of this work refer to the inner parts of the disc, which makes the effects of caustic curvature even smaller. For future more accurate studies, we propose using the more precise parabolic model (Fluke & Webster 1999).

### 5.4 Inclinations of QSO discs

High inclinations \( (i \gtrsim 70^\circ) \) are required if we interpret the three high-amplification events considered as caustic crossings. All three events show fine structure near their maxima. It is possible to explain this by highly inclined, relativistic discs around black holes with masses considerably higher than the virial estimates. One amplification curve (QSO J2237+0305, image C) allows us to restrict both the Kerr parameter and the mass to a high accuracy. Still it should be noted that these estimates are sensitive to the inner structure of the disc and therefore model-dependent. High inclinations are needed because the observational data require asymmetric brightness distributions.
Still, there are strong reasons why QSO discs should not be inclined by $\gtrsim 70^\circ$.

(i) The outer-disc rim has a non-negligible thickness and is able to obscure the central UV-radiating parts if the inclination is $i \gtrsim \pi/2 - h/R$, where $hR \sim 0.02$ is the relative disc thickness (Shakura & Sunyaev 1973); this is important if the disc has a very high inclination $i \gtrsim 85^\circ$.

(ii) A stricter limit is set by dust tori detected spectrally for most quasars. Hot dust emission is observed from QSO J2237+0305 (Agol, Jones & Blaes 2000), consistent with a large-scale ($R \gtrsim 0.03$ pc) structure of dust intercepting a considerable part of the quasar luminosity and reradiating the absorbed luminosity in the infrared range.

(iii) A disc observed edge-on should appear underluminous. For a thin disc, the observed luminosity is $\cos i$ times lower than for the face-on case; for a concave disc the decrease is stronger. Thus the real luminosity should be a factor of $\cos^{-1} i \sim 10$ higher. In the case of QSO J2237+0305, strong observational evidence for small inclination ($i > 60^\circ$ is ruled out at a 95 per cent confidence level) is given in Poindexter & Kochanek (2010).

The strength of the last argument is diminished by the high inferred masses of the central black holes. A black hole mass an order of magnitude higher leads to an Eddington luminosity an order of magnitude higher. This makes the observed flux consistent with the microlensing disc-size estimates. Also, Poindexter & Kochanek (2010) did not consider relativistic effects, which have the potential to distort the dependence of the observed flux on inclination significantly. For a moderate black hole mass of $10^8 M_\odot$ and $a = 0.6$, the luminosity dependence on inclination deviates by more than 20 per cent at $i > 40^\circ$ and by more than 50 per cent at $i > 70^\circ$ (see Fig. 15).

An interesting clue is that SBS J1520+530 is a broad absorption-line (BAL) quasar (Chavushyan et al. 1997). Although BAL QSO are generally considered as being viewed nearly edge-on (Elvis 2000), some of these objects have strongly variable bright radio emission, suggesting a collimated relativistic outflow observed at low inclination (Zhou et al. 2006). It is still an open question as to how one can collimate a jet in a direction different from the normal to the disc plane.

A possible solution is that the angular momenta of the accreting black holes in SBS J1520+530 and QSO J2237+0305 are misaligned with the angular momentum of the infalling matter. Since there are no reasons for a supermassive black hole to be aligned with the accreting matter at large distances, this is probably the case. The inner parts of the flow should be tilted and warped. Most investigations predict alignment of the inner parts of the flow, known as the Bardeen–Petterson effect (see Bardeen & Petterson 1975), normally inside the $R \sim 10 r_{in}$ region where the observed UV radiation is emitted. A similar misalignment may be proposed as a possible reason for the apparent inconsistency between the inclinations inferred from radio and optical properties of radio-loud BAL quasars.

In Fig. 16 we show the predicted amplification curve for a thin accretion disc with zero inclination at infinite distance but tilted toward the plane of the black hole inclined by $\theta_{in}$ to the picture plane (and to the plane of the disc at large distances). For inclination, we use the law $\theta(r) = \theta_{in} \exp (-r/r_{in})$. The rotation law for an inclined disc was taken from Shakura (1987), while the temperature law was assumed to be standard. Every annulus was considered flat, which is of course an over-simplification but still serves the main goal of showing that fold-caustic traversal curves are primarily sensitive to the innermost parts of the accretion disc.

### 5.5 Influence of disc tilts upon the unification scheme

In the case of random orientation of the black hole spin and the orbital angular momentum of the matter entering the disc, an average disc will have a tilt of $\theta \sim 90^\circ$ at large distances. Half of the discs should be counter-rotating. In case of random misalignment, half of the disc planes should have inclinations larger than $60^\circ$ with respect to the black hole rotation frame. A tilted disc around a Kerr black hole is usually proposed to gradually change its inclination with decreasing radius, approaching alignment at the innermost stable orbit. Linear analysis performed by Zhuravlev & Ivanov (2011) for a thin disc around a nearly Schwarzschild black hole confirms this conclusion only for the case of considerably large viscosity (see

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**Figure 15.** Flux dependence on inclination for an accretion disc around an $M = 10^8 M_\odot$, $a = 0.6$ black hole. The dotted line shows the $\cos i$ law for comparison.

**Figure 16.** Simulated amplification curves of inclined (solid line) and tilted (dotted) discs. For both curves, $a = 0.9$, $M = 10^8 M_\odot$, maximal inclination is $80^\circ$, $\psi \simeq 0$, $v_{eff} = -10000 \text{ km s}^{-1}$. See Section 5.4 for details.
the original work for details). The authors also show the stability of counter-rotating discs around slowly rotating black holes. On the other hand, numerical simulations made by Fragile & Anninos (2005) and Dexter & Fragile (2011) suggest that the disc evolution toward alignment should take place in its innermost parts, at several Schwarzschild radii.

Tilted discs may become a serious obstacle to the formation and observability of relativistic jets. Once formed, relativistic outflows will be stopped by the higher-pressure material of the disc or dust torus. If jet formation and propagation into the rarefied galactic medium becomes impossible in the case of strongly inclined discs, one may try to explain the observed radio brightness dichotomy of QSOs by different regimes of alignment, as was proposed by Lawrence & Elvis (2010) for Seyferts and radio galaxies. The presence of a thick torus with a tilt of $\theta_i \sim 30^\circ$ leads to about 75 per cent probability that the jet will be stopped either by the torus or by the tilted disc itself. The real fraction of radio-weak sources is higher, about 90 per cent (see for example Hewitt, Foltz & Chaffee 2001).

The importance of disc tilts for jet formation may also have a connection to the formation of ‘localized’ X-ray radiation found in some QSOs. In particular, in the two recent works by Zimmer, Schmidt & Wambsganss (2011) and Chen et al. (2011) the properties of the X-ray-emitting region in QSO J2237+0305 are recovered using microlensing curves. In Chen et al. (2011), it is concluded that this region is smaller and more variable than any existing extended jet or corona model can explain. Among other possibilities, the ‘failed jet’ model by Ghisellini, Haardt & Matt (2004) is put forward. If the jet possesses enough energy to escape the gravitational well but is stopped by the disc matter, the source of X-ray emission may be connected to the hot spot in the disc where its energy is dissipated. The relativistic termination shock remains invisible, but the bow shock where the ram pressure is balanced by the pressure of the disc should make an important contribution to the observed X-ray emission. Also, the spatial size of the X-ray-bright spot may be small if the jet is sufficiently collimated. A stopped jet model may be checked by considering correlations between different properties of X-ray and radio emission from quasars.

6 CONCLUSIONS

We show that general relativity effects have considerable influence on the amplification curves of microlensed quasars. This effect is more profound if the disc is strongly inclined. Having better data on hand (smaller observational errors, larger homogeneity and better temporal coverage) may allow us to resolve the structure of high-amplification events with better precision and to use them for accurately measuring disc tilts and black hole rotation parameters of lensed quasars.

Some implications may be made even now. In particular, the discs in objects SBS J1520+530 and QSO J2237+0305 are seen at high inclinations. The apparent contradiction between the high observed fluxes and lack of strong absorption signatures in the spectra may be resolved if one considers a disc that has low inclination ($i \lesssim 60^\circ$) at tens of $GM/c^2$ but is seen nearly edge-on in its inner parts. A similar picture is expected if the initial angular momentum of the accreted matter is strongly misaligned with that of the black hole.

The high probability of effectively measuring a high inclination may be qualitatively understood as a very strong tilt and warp near the last stable orbit. In this case there should be, to a high probability, some radius at which the disc lies perpendicular to the picture frame. The contribution of this annulus should be enhanced by Doppler boost. In other words, if a disc has a range of inclinations due to tilts and warps, higher inclinations will make a larger contribution to the microlensing amplification curves.

In the analysed data we do not find any challenges for the standard accretion-disc model. However, they bring our attention toward the role of disc tilts, which has never been considered in the framework of QSO microlensing. Also, some accretion-disc properties are better understood if we alter the accretion-disc parameters such as Eddington ratio and accretion efficiency. The apparently high black hole masses and disc sizes required by microlensing effects may be connected to the low accretion efficiency and contamination by larger angular scales.

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We became aware of the papers by Bogdanov & Cherepashchuk (2002, 2004) where amplification curves considered in this work were analysed. Authors restore one-dimensional brightness distributions and propose to connect their shapes to relativistic effects.

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