Stability in Graphs and Games

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Abstract

We study graphs and two-player games in which rewards are assigned to states, and the goal of the players is to satisfy or dissatisfy certain property of the generated outcome, given as a mean payoff property. Since the notion of mean-payoff does not reflect possible fluctuations from the mean-payoff along a run, we propose definitions and algorithms for capturing the stability of the system, and give algorithms for deciding if a given mean payoff and stability objective can be ensured in the system.

1 Introduction

Finite-state graphs and games are used in formal verification as foundational models that capture behaviours of systems with controllable decisions and possibly also the presence of adversarial environment. States correspond to possible configurations of a system, and edges describe how configurations can change. In a game, each state is owned by one of two players, and the player owning the state decides what edge will be taken. A graph is a game where only one of the players is present. When the choice of the edges is resolved, we obtain an outcome which is an infinite sequence of states and edges describing the execution of the system.

The long-run average performance of a run is measured by the associated mean-payoff, which is the limit average reward per visited state along the run. It is well known that memoryless deterministic strategies suffice to optimise the mean payoff, and the corresponding decision problem is in NP ∩ coNP for games and in P for graphs. If the rewards assigned to the states are multi-dimensional vectors of numbers, then the problem becomes coNP-hard for games [20].

Although the mean payoff provides an important metric for the average behaviour of the system, by definition it neglects all information about the fluctuations from the mean payoff along the run. For example, a “fully stable” run where the associated sequence of rewards is 1, 1, 1, 1, . . . has the same mean payoff (equal to 1) as a run producing n, 0, 0, . . . , n, 0, 0, . . . where a state with the reward n is visited once in n transitions. In many situations, the first run is much more desirable than the second one. Consider, e.g., a video streaming application which needs to achieve a sufficiently high bitrate (a long-run average number of bits delivered per second) but, in addition, a sufficient level of “stability” to prevent buffer underflows and overflows which would cause data loss and stuttering. Similar problems appear also in other contexts. For example, production lines should be not only efficient (i.e., produce the number of items per time unit as high as possible), but also “stable” so that the available stores are not overfilled and there is no “periodic shortage” of the produced items. A food production system should not only produce a sufficiently large amount of food per day on average, but also a certain amount of food daily. These and similar problems motivate the search for a suitable formal notion capturing the intuitive understanding of “stability”, and developing algorithms that can optimize the performance under given stability constraints.
That is, we are still seeking for a strategy optimizing the mean payoff, but the search space is restricted to the subset of all strategies that achieve a given stability constraint.

Since the mean-payoff \( mp(\lambda) \) of a given run \( \lambda \) can be seen as the average reward of a state visited along \( \lambda \), a natural idea is to define the stability of \( \lambda \) as sample variance of the reward assigned to a state along \( \lambda \). More precisely, let \( r_i \) be the reward if the \( i \)-th state visited by \( \lambda \), and let \( c_0, c_1, \ldots \) be an infinite sequence where \( c_i = (mp(\lambda) - r_i)^2 \). The long-run variance of the reward assigned to a state along \( \lambda \), denoted by \( va(\lambda) \), is the limit-average of \( c_0, c_1, \ldots \). The notion of long-run variance has been introduced and studied for Markov decision processes in [5]. If \( va(\lambda) \) is small, then large fluctuations from \( mp(\lambda) \) are rare. Hence, if we require that a strategy should optimize mean-payoff while keeping the long-run variance below a given threshold, we in fact impose a soft stability constraint which guarantees that “bad things do not happen too often”. This may or may not be sufficient.

In this paper, we are particularly interested in formalizing hard stability constraints which guarantee that “bad things never happen”. We introduce a new type of objectives called window-stability multi-objectives that can express a rich set of hard stability constraints, and we show that the set of all strategies that achieve a given window-stability multi-objective can be characterized by an effectively constructible finite-memory permissive strategy scheme. From this we obtain a meta-theorem saying that if an objective (such as mean-payoff optimization) is solvable for finite-state games (or graphs), then the same objective is solvable also under a given window-stability multi-objective constraint. We also provide the associated upper and lower complexity bounds demonstrating that the time complexity of our algorithms is "essentially optimal".

More specifically, a single window-stability objective (inspired by [7], see Related work below) is specified by a window length \( W \geq 1 \), a checkpoint distance \( D \geq 1 \), and two bounds \( \mu \) and \( \nu \). For technical reasons, we assume that \( D \) divides \( W \). Every run \( \lambda = s_0, s_1, s_2, \ldots \) then contains infinitely many checkpoints \( s_0, s_D, s_{2D}, s_{3D}, \ldots \). The objective requires that the average reward assigned to the states \( s_j, \ldots, s_{j+W-1} \), where \( s_j \) is a checkpoint, is between \( \mu \) and \( \nu \). In other words, the “local mean-payoff” computed for the states fitting into a window of length \( W \) starting at a checkpoint must be within the “acceptable” bounds \( \mu \) and \( \nu \). The role of \( W \) is clear, and the intuition behind \( D \) is the following. Since \( D \) divides \( W \), there are two extreme cases: \( D = 1 \) and \( D = W \). For \( D = 1 \), the objective closely resembles the standard “sliding window” model over data streams [12]: we require that the local mean-payoff stays within the acceptable bounds “continuously”, like the “local bitrate” in video-streaming. If \( D = W \), then the windows do not overlap at all. This is useful in situations when we wish to guarantee some time-bounded periodic progress. For example, if we wish to say that the number of items produced per day stays within given bounds, we set \( W \) so that it represents the (discrete) time of one day and put \( D = W \). However, there can be also scenarios when we wish to check the local mean-payoff more often then once during \( W \) transitions, but not “completely continuously”. In these cases, we set \( D \) to some other divisor of \( W \). A window-stability multi-objective is a finite conjunction of single window-stability objectives, each with dedicated rewards and parameters. Hence, window-stability multi-objectives allow for capturing more delicate stability requirements such as “a factory should produce between 1500 and 1800 gadgets every week, and in addition, within every one-hour period at least 50 computer chips are produced, and in addition, the total amount of waste produced in 12 consecutive hours does not exceed 500 kg.”

Our contribution can be summarized as follows:

A) We introduce the concept of window-stability multi-objectives.

B) We show that there is an algorithm which inputs a game \( G \) and a window-stability
multi-objective $\Delta$, and outputs a finite-state permissive strategy scheme for $\Delta$ and $G$. A finite-state permissive strategy scheme for $\Delta$ and $G$ is a finite-state automaton $\Gamma$ which reads the history of a play and constraints the moves of Player $\Box$ (who aims at satisfying $\Delta$) so that a strategy $\sigma$ achieves $\Delta$ in $G$ iff $\sigma$ is admitted by $\Gamma$. Hence, we can also compute a synchronized product $G \times \Gamma$ which is another game where the set of all strategies for Player $\Box$ precisely represents the set of all strategies for Player $\Box$ in $G$ which achieve the objective $\Delta$. Consequently, any objective of the form $\Delta \land \Psi$ can be solved for $G$ by solving the objective $\Psi$ for $G \times \Gamma$. In particular, this is applicable to mean-payoff objectives, and thus we solve the problem of optimizing the mean-payoff under a given window-stability multi-objective constraint. We also analyze the time complexity of these algorithms, which reveals that the crucial parameter which negatively influences the time complexity is the number of checkpoints in a window (i.e., $W/D$).

(C) We complement the upper complexity bounds of the previous item by lower complexity bounds that indicate that the time complexity of our algorithms is “essentially optimal”. Some of these results follow immediately from existing works [20, 7]. The main contribution is the result which says that solving a (single) window-stability objective is PSPACE-hard for games and NP-hard for graphs, even if all numerical parameters ($W$, $D$, $\mu$, $\nu$, and the rewards) are encoded in unary. The proof is based on novel techniques and reveals that the number of checkpoints in a window (i.e., $W/D$) is a crucial parameter which makes the problem computationally hard. The window stability objective constructed in the proof satisfies $D = 1$, and the tight window overlapping is used to enforce a certain consistency in Player $\Box$ strategies.

(D) For variance-stability, we argue that while it is natural in terms of using standard mathematical definitions, it does not prevent unstable behaviours. In particular, we show that the variance-stability objective may demand an infinite-memory strategy which switches between two completely different modes of behaviour with smaller and smaller frequency. We also show that the associated variance-stability problem with single-dimensional rewards is in NP for graphs. For this we use some of the results from [5] where the variance-stability is studied in the context of Markov decision processes. The main difficulty is a translation from randomized stochastic-update strategies used in [5] to deterministic strategies.

Related work. Multi-dimensional mean-payoff games were studied in [20], where it was shown that the lim-inf problem, relevant to our setting, is coNP-hard. Further, [10] studies memory requirements for the objectives, and [19] shows that for a “robust” extension (where Boolean combinations of bounds on the resulting vector of mean-payoffs are allowed) the problem becomes undecidable. Games with quantitative objectives in which both lower and upper bound on the target value of mean-payoff is given were studied in [14]. We differ from these approaches by requiring the “interval” bounds to be satisfied within finite windows, making our techniques and results very different.

As discussed above, we rely on the concept of windows, which was in the synthesis setting studied in [7] (see also [13]), as a conservative approximation of the standard mean-payoff objective. More concretely, the objectives in [7] are specified by a maximal window length $W$ and a threshold $t$. The task is to find a strategy that achieves the following property of runs: a run can be partitioned into contiguous windows of length at most $W$ such that in each window, the reward accumulated inside the window divided by the window length is at least $t$. The objective ensures a local progress in accumulating the reward, and it was not motivated by capturing stability constraints. There are also technical differences in solving these objectives and in the associated complexity bounds, mainly due to the absence of window overlapping (in particular, the PSPACE lower bound discussed in the point (C))
above does not carry over to the setting of \[2\]).

The notion of finite-state permissive strategy scheme is based on the concept of permissive strategies \[1\] and multi-strategies \[3\].

The notion of long-run variance has been introduced and studied for Markov decision processes in \[5\]. Since we consider deterministic strategies, none of our results is a special case of \[5\], and we have to overcome new difficulties as it is explained in Section 4.

More generally, our paper fits into an active field of multi-objective strategy synthesis, where some objectives capture the “hard” constraints and the other “soft”, often quantitative, objectives. Examples of recent results in this area include \[2\], where a 2-EXPTIME algorithm is given for the synthesis of combined LTL and mean-payoff objectives, \[8\], where a combination of parity and mean-payoff performance objectives is studied, or \[9\], where the controlling player must satisfy a given \(\omega\)-regular objective while allowing the adversary to satisfy another “environmental” objective.

## 2 Preliminaries

We use \(\mathbb{N}\), \(\mathbb{N}_0\), and \(\mathbb{Q}\) to denote the sets of positive integers, non-negative integers, and rationals, respectively. Given a set \(M\), we use \(M^*\) to denote the set of all finite sequences (words) over \(M\), including the empty sequence.

A game is a tuple \(G = (S, (S\square, S\diamond), E)\) where \(S\) is a non-empty set of states, \((S\square, S\diamond)\) is a partition of \(S\) into two subsets controlled by Player \(\square\) and Player \(\diamond\), respectively, and \(E \subseteq S \times S\) are the edges of the game such that for every \(s \in S\) there is at least one edge \((s, t) \in E\). A graph is a game such that \(S\diamond = \emptyset\). A run in \(G\) is an infinite path in the underlying directed graph of \(G\). An objective is a Borel property\(^1\) of runs. Note that the class of all objectives is closed under conjunction.

A strategy for player \(\odot\), where \(\odot \in \{\square, \diamond\}\) is a function \(\tau : S^*S\odot \to S\) satisfying that \((s, \sigma(hs)) \in E\) for all \(s \in S\odot\) and \(h \in S^*\). The sets of all strategies of Player \(\square\) and Player \(\diamond\) are denoted by \(\Sigma_G\) and \(\Pi_G\), respectively. When \(G\) is understood, we write just \(\Sigma\) and \(\Pi\).

A pair of strategies \((\sigma, \pi) \in \Sigma \times \Pi\) together with an initial state \(s\) induce a unique run \(\text{outcome}^\sigma_\pi\) in the standard way. We say that a strategy \(\sigma \in \Sigma\) achieves an objective \(\Phi\) in a state \(s\) if \(\text{outcome}^\sigma_\pi\) satisfies \(\Phi\) for every \(\pi \in \Pi\). The set of all \(\sigma \in \Sigma\) that achieve \(\Phi\) in \(s\) is denoted by \(\Sigma^\Phi(s)\). An objective \(\Phi\) is solvable for a given subclass \(G\) of finite-state games if there is an algorithm which inputs \(G \in G\) and its state \(s\), and decides whether \(\Sigma^\Phi(s) = \emptyset\). If \(\Sigma^\Phi(s) \neq \emptyset\), then the algorithm also outputs a (finite description of) \(\sigma \in \Sigma^\Phi(s)\).

We often consider strategies of Player \(\square\) tailored for a specific initial state. A finite sequence of states \(s_0, \ldots, s_n\) is consistent with a given \(\sigma \in \Sigma\) if \(s_0, \ldots, s_n\) is a finite path in the graph of \(G\), and \(\sigma(s_0, \ldots, s_i) = s_{i+1}\) for every \(0 \leq i < n\) where \(s_i \in V\). Given \(\sigma, \sigma' \in \Sigma\) and \(s \in S\), we say that \(\sigma\) and \(\sigma'\) are \(s\)-equivalent, written \(\sigma \equiv_s \sigma'\), if \(\sigma\) and \(\sigma'\) agree on all finite sequences of states initiated in \(s\) that are consistent with \(\sigma\). Note that if \(\sigma \equiv_s \sigma'\), then \(\text{outcome}^{\sigma, \pi}_s = \text{outcome}^{\sigma', \pi}_s\) for every \(\pi \in \Pi\).

A reward function \(g : S \to \mathbb{N}_0^k\), where \(k \in \mathbb{N}\), assigns non-negative integer vectors to the states of \(G\). We use \(\dim_g\) to denote the dimension \(k\) of \(g\), and \(\max_g\) to denote the maximal number employed by \(g\), i.e., \(\max_g = \max\{g(s)[i] \mid 1 \leq i \leq k, s \in S\}\). An objective is reward-based if its defining property depends just on the sequence of rewards assigned to the states visited by a run.

\(^1\) Recall that the set of all runs can be given the standard Cantor topology. A property is Borel if the set of all runs satisfying the property belongs to the \(\sigma\)-algebra generated by all open sets in this topology.
For every run $\lambda = s_0, s_1, \ldots$ of $G$, let $mp_\rho(\lambda) = \liminf_{n \to \infty} \frac{1}{n+1} \sum_{i=0}^{n} \rho(s_i)$ be the mean payoff of $\lambda$, where the $\liminf$ is taken component-wise. A mean-payoff objective is a pair $(\rho, \kappa)$, where $\rho : S \to \mathbb{N}_0^k$ is a reward function and $b \in \mathbb{Q}^k$. A run $\lambda$ satisfies a mean-payoff objective $(\rho, b)$ if $mp_\rho(\lambda) \geq b$.

Similarly, the long-run variance of the reward of a run $\lambda$ is defined by $va_\rho(\lambda) = \limsup_{n \to \infty} \frac{1}{n+1} \sum_{i=0}^{n} (\rho(s_i) - mp(\text{run}))^2$; intuitively, the long-run variance is a limit inferior of sample variances where the samples represent longer and longer run prefixes. A variance-stability objective is a triple $(\rho, b, c)$, where $\rho : S \to \mathbb{N}_0^k$ is a reward function and $b, c \in \mathbb{Q}^k$. A run $\lambda$ satisfies a variance-stability objective $(\rho, b, c)$ if $mp_\rho(\lambda) \geq b$ and $va_\rho(\lambda) \leq c$.

Let $W \in \mathbb{N}$ be a window size and $D \in \mathbb{N}$ a checkpoint distance such that $D$ divides $W$. For every $\ell \in \mathbb{N}_0$, the local mean payoff at the $\ell$th checkpoint in a run $\lambda$ is defined by $\text{lmpp}_{W,D,\rho,\ell}(\lambda) = \frac{1}{W} \sum_{i=0}^{W-1} \rho(s_{\ell, D+i})$ where $(\bar{v}/a)[i] = \bar{v}(i)/a$. Thus, every run $\lambda$ determines the associated infinite sequence $\text{lmpp}_{W,D,\rho,0}(\lambda)$, $\text{lmpp}_{W,D,\rho,1}(\lambda)$, $\text{lmpp}_{W,D,\rho,2}(\lambda), \ldots$ of local mean payoffs. A window-stability objective is a tuple $\Phi = (W, D, \rho, \mu, \nu)$, where $W, D \in \mathbb{N}$ such that $D$ divides $W$, $\rho : S \to \mathbb{N}_0^k$ is a reward function, and $\mu, \nu \in \mathbb{Q}^k$. A run $\lambda$ satisfies $\Phi$ if, for all $\ell \in \mathbb{N}$, we have that $\mu \leq \text{lmpp}_{W,D,\rho,\ell}(\lambda) \leq \nu$. A window-stability multi-objective is a finite conjunction of window-stability objectives.

In this paper, we study the solvability of variance-stability objectives, window-stability multi-objectives, and objectives of the form $\Delta \land \Psi$ where $\Delta$ is a window-stability multi-objective and $\Psi$ a mean-payoff objective.

### 3 The Window-Stability Multi-Objectives

This section is devoted to the window-stability multi-objectives and objectives of the form $\Delta \land \Psi$, where $\Delta$ is a window-stability multi-objective. In Section 3.1 we show how to solve these objectives for finite-state games, and we derive the corresponding upper complexity bounds. The crucial parameter which makes the problem computationally hard is the number of checkpoints in a window. In Section 3.2 we show that this blowup is unavoidable assuming the expected relationship among the basic complexity classes.

#### 3.1 Solving Games with Window-Stability Multi-Objectives

We start by recalling the concept of most permissive strategies which was introduced in [1]. Technically, we define permissive strategy schemes which suit better our needs, but the underlying idea is the same.

**Definition 1.** Let $G = (S, (S_\Box, S_\Diamond), E)$ be a game. A (finite-memory) strategy scheme for $G$ is a tuple $\Gamma = (\text{Mem}, \text{Up}, \text{Const}, \text{Init})$, where $\text{Mem} \neq \emptyset$ is a finite set of memory elements, $\text{Up} : S \times \text{Mem} \to \text{Mem}$ is a memory update function, $\text{Const} : S_\Box \times \text{Mem} \to 2^S$ is a constrainer such that $\text{Const}(s, m) \subseteq \{s' \in S \mid (s, s') \in E\}$, and $\text{Init} : S \to M$ is a partial function assigning initial memory elements to some states of $S$.

We require that $\text{Const}(s, m) \neq \emptyset$ for all $(s, m) \in \text{Reach} \text{Init}$ such that $s \in S_\Box$. Here, $\text{Reach} \text{Init}$ is the least fixed-point of $F : 2^{S \times \text{Mem}} \to 2^{S \times \text{Mem}}$ where $F(\Omega)$ consists of all $(s', m')$ such that $(s', m') \in \text{Init}$, or there is some $(s'', m'') \in \Omega$ such that $(s'', s') \in E$ and $\text{Up}(s', m'') = m'$; if $s'' \in S_\Box$, we further require $s' \in \text{Const}(s'', m'')$.

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2 Alternatively, we could stipulate $\text{Const}(s, m) \neq \emptyset$ for all $(s, m) \in S_\Box \times \text{Mem}$, but this would lead to technical complications in some proofs. The presented variant seems slightly more convenient.
We say that $\Gamma$ is memoryless if $\text{Mem}$ is a singleton. Every strategy scheme $\Gamma = (\text{Mem}, \text{Up}, \text{Const}, \text{Init})$ for a game $G = (S, (S_\square, S_\diamond), E)$ determines a game $G_\Gamma = (S \times \text{Mem}, (S_\square \times \text{Mem}, S_\diamond \times \text{Mem}), F)$, where

- for every $(s, m) \in S_\diamond \times \text{Mem}$, $((s, m), (s', m')) \in F$ if $\text{Up}(s, m) = m'$ and $(s, s') \in E$;
- for every $(s, m) \in S_\square \times \text{Mem}$ where $\text{Const}(s, m) \neq \emptyset$, we have that $((s, m), (s', m')) \in F$ if $\text{Up}(s, m) = m'$ and $(s, s') \in \text{Const}(s, m)$;
- for every $(s, m) \in S_\square \times \text{Mem}$ where $\text{Const}(s, m) = \emptyset$, we have that $((s, m), (s', m')) \in F$ if $s = s'$ and $m = m'$.

A strategy $\sigma \in \Sigma_G$ is admitted by $\Gamma$ in a given $s \in S$ if $\text{Init}(s) \neq \perp$ and for every finite path $s_0, \ldots, s_n$ in $G$ initiated in $s$ which is consistent with $\sigma$ there is a finite path $(s_0, m_0), \ldots, (s_n, m_n)$ in $G_\Gamma$ such that $m_0 = \text{Init}(s_0)$ and $s_{i+1} \in \text{Const}(s_i, m_i)$ for all $0 \leq i < n$ where $s_i \in S_\square$. Observe that if $\sigma$ is admitted by $\Gamma$ in $s$, then $\sigma$ naturally induces a strategy $\tau[\sigma, s] \in \Sigma_{G_\Gamma}$ which is unique up to $\equiv_{(s_0, m_0)}$. Conversely, every $\tau \in \Sigma_{G_\Gamma}$ and every $s \in S$ where $\text{Init}(s) \neq \perp$ induce a strategy $\sigma[\tau, s] \in \Sigma_G$ such that, for every finite path $(s_0, m_0), \ldots, (s_n, m_n)$ initiated in $(s, \text{Init}(s))$ which is consistent with $\tau$, we have that $\sigma[\tau, s][s_0, \ldots, s_n] = s_{n+1}$ if $\tau((s_0, m_0), \ldots, (s_n, m_n)) = (s_{n+1}, m_{n+1})$. Note that $\sigma[\tau, s]$ is determined uniquely up to $\equiv_s$.

**Definition 2.** Let $G$ be a game, $\Gamma$ a strategy scheme for $G$, $\Lambda_G \subseteq \Sigma_G$, $\Lambda_{G_\Gamma} \subseteq \Sigma_{G_\Gamma}$, and $s \in S$. We write $\Lambda_G \approx_s \Lambda_{G_\Gamma}$ if the following conditions are satisfied:

- Every $\sigma \in \Lambda_G$ is admitted by $\Gamma$ in $s$, and there is $\tau \in \Lambda_{G_\Gamma}$ such that $\tau[\sigma, s] \equiv_{(s, \text{Init}(s))} \tau$.
- For every $\tau \in \Lambda_{G_\Gamma}$ there is $\sigma \in \Lambda_G$ such that $\sigma[\tau, s] \equiv_s \sigma$.

Further, we say that $\Gamma$ is permissive for an objective $\Phi$ if $\Sigma_G^\Phi(s) \approx_s \Sigma_{G_\Gamma}(s)$ for all $s \in S$, where $\Sigma_G(s)$ is either $\emptyset$ or $\Sigma_G$, depending on whether $\text{Init}(s) = \perp$ or not, respectively.

The next proposition follows immediately.

**Proposition 3.** Let $G$ be a game, $\Phi, \Psi$ objectives, and $\Gamma$ a strategy scheme permissive for $\Phi$. Then, for every $s \in S$ we have that $\Sigma_G^\Phi \land \Psi(s) \approx_s \Sigma_{G_\Gamma}^\Psi(s)$.

Another simple but useful observation is that the class of objectives for which a permissive strategy scheme exists is closed under conjunction.

**Proposition 4.** Let $G = (S, (S_\square, S_\diamond), E)$ be a finite-state game, and $n \in \mathbb{N}$. Further, for every $1 \leq i \leq n$, let $\Gamma_i = (\text{Mem}_i, \text{Up}_i, \text{Const}_i, \text{Init}_i)$ be a strategy scheme for $G$ which is permissive for $\Phi_i$. Then there is a strategy scheme for $G$ with $\prod_{i=1}^n |\text{Mem}_i|$ memory elements computable in $O(|S|^2 \cdot |E| \cdot \prod_{i=1}^n |\text{Mem}_i|^2)$ time which is permissive for $\Phi_1 \land \cdots \land \Phi_n$.

As it was noted in [1], permissive strategy schemes do not exist for objectives which admit non-winning infinite runs that do not leave the winning region of player $\square$, such as reachability, Büchi, parity, mean-payoff, etc. On the other hand, permissive strategy schemes exists for “time bounded” variants of these objectives. Now we show how to compute a permissive strategy scheme for a given window-stability objective.

**Theorem 5.** Let $G = (S, (S_\square, S_\diamond), E)$ be a finite-state game and $\Phi = (W, D, g, \mu, \nu)$ a window-stability objective where $\text{dim}_W = k$. Then there is a strategy scheme $\Gamma$ with $W \cdot (\text{max}_g \cdot W)k \cdot (W/D)$ memory elements computable in $O(|S|^2 \cdot |E| \cdot W^2 \cdot (\text{max}_g \cdot W)^k \cdot (W/D))$ time which is permissive for $\Phi$.

**Proof.** Let $\ell = W/D$ and $V = \{0, \ldots, \text{max}_g \cdot (W-1)\}^k$. We put
Our aim is to construct $\Gamma$ so that for every run $s_0, s_1, \ldots$ in $G$, the memory elements in the corresponding run $(s_0, m_0), (s_1, m_1), \ldots$ in $G_r$, where $(s_0, m_0) \in \text{Init}$, satisfy the following. Let $n \in \mathbb{N}_0$, and let $m_n = (i, j, a_0, \ldots, a_\ell-1)$. Then

- $i = n \mod D$ is the number of steps since the last checkpoint, and $j = \min\{|n/D|, \ell-1\}$ is a bounded counter which stores the number of checkpoint visited, up to $\ell - 1$ (this information is important for the initial $W$ steps);
- for every $0 \leq r < \ell$, we put $c_r = n - r \cdot D - (n \mod D)$ if $n - r \cdot D - (n \mod D) \geq 0$, otherwise $c_r = n$. Intuitively, the state $s_{c_r}$ is the $r$-th previous checkpoint visited along $s_0, s_1, \ldots$ before visiting the state $s_n$ (see Figure 1). If the total number of checkpoints visited along the run up to $s_n$ (including $s_n$) is less than $r$, we put $c_r = n$. The vector $\alpha_r$ stored in $m_n$ is then equal to the total reward accumulated between $s_{c_r}$ and $s_n$ (not including $s_n$), i.e., $\alpha_r = \sum_{t=c_r}^{n-1} g(s_t)$ where the empty sum is equal to $\vec{0}$. In particular $m_0 = (0, 0, \vec{0}, \ldots, \vec{0})$.

Note that by Definition 1, we are obliged to define $Up(s, m)$ for all pairs $(s, m) \in S \times Mem$, including those that will not be reachable in the end. Let ‘$\oplus$’ be a bounded addition over $\mathbb{N}_0$ defined by $a \oplus b = \min\{a + b, \max a \cdot (W-1)\}$. We extend ‘$\oplus$’ to $V$ in the natural (component-wise) way. The function $Up$ is constructed as follows (consistent with the above intuition):

- For all $i, j \in \mathbb{N}_0$ such that $0 \leq i \leq D - 2$ and $0 \leq j \leq \ell - 1$, we put $Up(s, (i, j, a_0, \ldots, a_\ell-1)) = (i+1, j, a_0 \oplus g(s), \ldots, a_j \oplus g(s), a_{j+1}, \ldots, a_\ell-1)$.
- For all $j \in \mathbb{N}_0$ such that $0 \leq j \leq \ell - 2$, we put $Up(s, (D-1, j, a_0, \ldots, a_\ell-1)) = (0, j+1, 0, a_0 \oplus g(s), \ldots, a_j \oplus g(s), a_{j+1}, \ldots, a_\ell-2)$.
- $Up(s, (D-1, \ell-1, a_0, \ldots, a_\ell-1)) = (0, \ell-1, 0, a_0 \oplus g(s), \ldots, a_{\ell-2} \oplus g(s))$.

For every $(s, m) \in S \times Mem$, let $\text{succ}(s, m)$ be the set of all $(s', m') \in S \times Mem$ such that $(s, s') \in E$ and $Up(s, m) = m'$. Now we define a function $\mathcal{F} : 2^S \times Mem \rightarrow 2^S \times Mem$ such that, for a given $\Omega \subseteq S \times Mem$, the set $\mathcal{F}(\Omega)$ consists of all $(s, (i, j, a_0, \ldots, a_\ell-1))$ satisfying the following conditions:

- if $i = D-1$ and $j = \ell-1$, then $\nu \cdot W \leq \alpha_{\ell-1} + g(s) \leq \nu \cdot W$.
- if $s \in S_0$, then $\text{succ}(s, (i, j, a_0, \ldots, a_\ell-1)) \subseteq \Omega$.
- if $s \in S_\square$, then $\text{succ}(s, (i, j, a_0, \ldots, a_\ell-1)) \cap \Omega \neq \emptyset$.

Observe that $\mathcal{F}$ is monotone. Let $gf\text{ix}(\mathcal{F})$ be the greatest fixed-point of $\mathcal{F}$. For every $(s, m) \in S_\square \times Mem$, we put $\text{Const}(s, m) = \text{succ}(s, m) \cap gf\text{ix}(\mathcal{F})$. Further, the set $\text{Init}$

- $\text{Mem} = \{0, \ldots, D-1\} \times \{0, \ldots, \ell-1\} \times V$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{The information represented by the memory elements of $\Gamma$ (for $\ell = 3$).}
\end{figure}
consists of all \((s, (0, 0, \ldots, \bar{0}))\) \(\in \) \(gfix(\mathcal{F})\). It follows directly from the definition of \(\Gamma\) that \(\text{Const}(s, m) \neq \emptyset\) for all \((s, m) \in \text{Reach}(\text{Init})\) such that \(s \in S_\square\).

Since \(gfix(\mathcal{F})\) can be computed in \(O(|S|^2 \cdot |E| \cdot W^2 \cdot (\max_\varrho \cdot W)^{2k(W/D)})\) time by the standard iterative algorithm, the strategy scheme \(\Gamma = (\text{Mem}, \text{Up}, \text{Const}, \text{Init})\) can also be computed in this time. Further, observe the following:

\((A)\) Let \((s_0, m_0), (s_1, m_1), \ldots\) be a run in \(G\) such that \((s_0, m_0) \in \text{Init}\). Then \(s_0, s_1, \ldots\) is a run in \(G\) that satisfies the window-stability objective \(\Phi\).

\((B)\) Let \((s, m) \notin gfix(\mathcal{F})\), and let \(\Gamma^*\) be a strategy scheme which is the same as \(\Gamma\) except for its constrainer \(\text{Const}^*\) which is defined by \(\text{Const}^*(s, m) = \text{suc}(s, m)\) for all \((s, m) \in S_\square \times \text{Mem}\). Then there is a strategy \(\pi^* \in \Pi_{G^*}\), such that for every strategy \(\sigma^* \in \Sigma_{G^*}\), we have that \(\text{outcome}^{(s, m)}\) visits a configuration \((t, (D - 1, t - 1, \alpha_0, \ldots, \alpha_{\ell - 1}))\) where \(\alpha_{\ell - 1} + g(t) < \mu \cdot W\) or \(\alpha_{\ell - 1} + g(t) > \nu \cdot W\).

Both \((A)\) and \((B)\) follow directly from the definition of \(\mathcal{F}\). Now we can easily prove that \(\Gamma\) indeed encodes the window-stability objective \(\Phi\), i.e., \(\Sigma_G^\Phi(s) \approx \Sigma_{G^*_G}(s)\) for all \(s \in S\).

Let \(\tau \in \Sigma_{G^*}(s)\). We need to show that \(\sigma[\tau, s]\) achieves the objective \(\Phi\) in \(S\). So, let \(\pi \in \Pi_{G^*}\), and let \(s_0, s_1, \ldots\) be the run \(\text{outcome}^{\sigma[\tau, s]}\). Obviously, there is a corresponding run \((s_0, m_0), (s_1, m_1), \ldots\) in \(G\) initiated in \((s, \text{Init}(s))\), which means that \(s_0, s_1, \ldots\) satisfies \(\Phi\) by applying \((A)\).

Now let \(\sigma \in \Sigma_G^\Phi(s)\). We need to show that \(\sigma\) is admitted by \(\Gamma\) in \(s\). Suppose it is not the case. If \(\text{Init}(s) = \bot\), then \((s, (0, 0, \ldots, \bar{0})) \notin gfix(\mathcal{F})\), and hence \(\sigma \notin \Sigma_{G^*}(s)\) by applying \((B)\).

If \(\text{Init}(s) \neq \bot\), there is a finite path \(s_0, s_1, \ldots, s_{n+1} \in S_\square\) of minimal length such that \(s_0 = s\), \(s_n \in S_\square\), and the corresponding finite path \((s_0, m_0), (s_1, m_1), \ldots, (s_n, m_n)\) in \(G\), where \(m_0 = \text{Init}(s)\) and \(m_{i+1} = \text{Up}(s_i, m_i)\) for all \(0 \leq i \leq n\), satisfies that \(s_{n+1} \notin \text{Const}(s_n, m_n)\). Note that for all \(s_0, s_1, \ldots\) in \(S_\square\) where \(i < n\) we have that \(s_{i+1} \in \text{Const}(s_i, m_i)\), because otherwise we obtain a contradiction with the minimality of \(s_0, s_1, \ldots, s_n, s_{n+1}\). Since \((s_{n+1}, m_{n+1}) \notin gfix(\mathcal{F})\), by applying \((B)\) we obtain a strategy \(\pi^* \in \Pi_{G^*}\), such that for every \(\sigma^* \in \Sigma_{G^*}\), we have that \(\text{outcome}^{\sigma^*[\tau, s]}\) visits a configuration \((t, (D - 1, t - 1, \alpha_0, \ldots, \alpha_{\ell - 1}))\) where \(\alpha_{\ell - 1} + g(t) < \mu \cdot W\) or \(\alpha_{\ell - 1} + g(t) > \nu \cdot W\). Let \(\pi \in \Pi_G\) be a strategy satisfying the following conditions:

- \(\text{outcome}^{\sigma[\tau, s]}\) starts with \(s_0, \ldots, s_{n+1}\).
- For all finite paths of the form \(s_0, s_1, \ldots, s_t\) in \(G\) such that \(s_t \in S_\square\), let \((s_0, m_0), \ldots, (s_t, m_t)\) be the unique corresponding finite path in \(G\).

We put \(\sigma(s_0, \ldots, s_t) = s_{t+1}\), where \(\sigma^*([(s_{n+1}, m_{n+1}), \ldots, (s_t, m_t)]) = (s_{t+1}, m_{t+1})\).

Clearly, the run \(\text{outcome}^{\sigma[\tau, s]}\) does not satisfy the objective \(\Phi\), which contradicts the assumption \(\sigma \in \Sigma_G^\Phi(s)\).

For every window-stability multi-objective \(\Delta = \Phi_1 \land \cdots \land \Phi_n\) where \(\Phi_i = (W_i, D_i, \vartheta_i, \mu_i, \nu_i)\), we put \(M_\Delta = \prod_{i=1}^n W_i \cdot (\max_\varrho \cdot W_i)^k \cdot (W_i/D_i)\), where \(k_i = \dim_\varrho\). As a direct corollary to Theorem 5 and Proposition 4 we obtain the following:

\(\blacktriangleright\) \textbf{Corollary 6.} Let \(G = (S, (S_\square, S_\square), E)\) be a finite-state game and \(\Delta\) a window-stability multi-objective. Then there is a permissive strategy scheme for \(\Delta\) with \(M_\Delta\) memory elements constructible in time \(O(|S|^2 \cdot |E| \cdot M_\Delta^2)\).

Now we can formulate a (meta)theorem about the solvability of objectives of the form \(\Delta \land \psi\), where \(\Delta\) is a window-stability multi-objective and \(\psi\) is a reward-based objective such that the time complexity of solving \(\Psi\) for a game \(G = (S, (S_\square, S_\square), E)\) and a reward function \(\varrho\) can be asymptotically bounded by a function \(f\) in \(|S|\), \(|E|\), \(\max_\varrho\), and \(\dim_\varrho\).
Theorem 7. Let $\Psi$ be a reward-based objective solvable in $O(|S| \cdot |E| \cdot \max_\varrho \cdot \dim_\varrho)$ time for every finite-state game $G = (S, (S, S_0), E)$ and every reward function $\varrho$ for $\Psi$. Further, let $\Delta$ be a window-stability multi-objective. Then the objective $\Delta \land \Psi$ is solvable in time
\[ O(\max(f(|S| \cdot M_{\Delta}, |E| \cdot M_{\Delta}, \max_\varrho, \dim_\varrho), |S|^2 \cdot |E| \cdot M_{\Delta}^3)) \]
for every finite-state game $G = (S, (S, S_0), E)$ and every reward function $\varrho$ for $\Psi$.

Note that Theorem 7 is a simple consequence of Corollary 6 and Proposition 3.

Since mean-payoff objectives are solvable in $O(|S| \cdot |E| \cdot \max_\varrho)$ time when $\dim_\varrho = 1$ and in $O(|S|^2 \cdot |E| \cdot \max_\varrho \cdot k \cdot (k \cdot |S| \cdot \max_\varrho)^{k^2+2k+1})$ time when $\dim_\varrho = k \geq 2$, we finally obtain:

Theorem 8. Let $G = (S, (S, S_0), E)$ be a finite-state game, $\Delta$ a window-stability multi-objective, and $\Psi = (\varrho, b)$ a mean-payoff objective. If $\dim_\varrho = 1$, then the objective $\Delta \land \Psi$ is solvable in time $O(|S|^2 \cdot |E| \cdot M_{\Delta}^2 \cdot \max_\varrho)$. If $\dim_\varrho = k \geq 2$, then the objective $\Delta \land \Psi$ is solvable in time $O(|S|^2 \cdot |E| \cdot M_{\Delta}^3 \cdot \max_\varrho \cdot k \cdot (k \cdot |S| \cdot \max_\varrho)^{k^2+2k+1})$.

Let us note that for a given window-stability multi-objective $\Delta$ and a given one-dimensional reward function $\varrho$, there exists the maximal bound $b$ such that the objective $\Delta \land (\varrho, b)$ is achievable. Further, this bound $b$ is rational and computable in time $O(|S|^2 \cdot |E| \cdot M_{\Delta}^2 \cdot \max_\varrho)$.

3.2 Lower Bounds for Window-Stability Objectives

We now focus on proving lower bounds for solving the window-stability objectives. More precisely, we establish lower complexity bounds for the problem whose instances are triples of the form $(G, s, \Phi)$, where $G$ is a game (or a graph), $s$ is a state of $G$, $\Phi = (W, D, \varrho, \mu, \nu)$ is a window-stability objective, and the question is whether there exists a strategy $\sigma \in \Sigma$ which achieves $\Phi$ in $s$. The components of $\Phi$ can be encoded in unary or binary, which is explicitly stated when presenting a given lower bound.

The main result of this section is Theorem 12 which implies that solving a window-stability objective is PSPACE-hard for games and NP-hard for graphs even if $\dim_\varrho = 1$, $D = 1$, and $W$ as well as the values $\varrho(s)$ for all $s \in S$ are encoded in unary. Note that an upper time complexity bound for solving these objectives is $O(|S|^2 \cdot |E| \cdot W \cdot (\max_\varrho \cdot W)^{|W/D|})$ by Corollary 6. Hence, the parameter which makes the problem hard is $|W/D|$.

As a warm-up, we first show that lower bounds for solving the window-stability objectives where the reward function is of higher dimension, or $W$, $D$, and the rewards are encoded in binary, follow rather straightforwardly from the literature. Then, we develop some new insights and use them to prove the main result.

Theorem 9. Solving the window-stability objectives (where $\dim_\varrho$ is not restricted) is EXPTIME-hard. The hardness result holds even if

1. the problem is restricted to instances where each component of each reward vector is in $\{-1, 0, 1\}$, or
2. the problem is restricted to instances where the reward vectors have dimension one (but the rewards are arbitrary binary-encoded numbers).

Proof. The result can be proven by a straightforward adaptation of the proof of EXPTIME-hardness of multi-dimensional fixed-window mean-payoff problem [7, Lemma 23 and 24]. The reductions in [7] that we can mimic are from the acceptance problem for polynomial-space alternating Turing machines (item 1.) and countdown games [16] (item 2.). Although
the fixed-window mean-payoff problem differs from ours (see Section 1), an examination of the proofs in [7] reveals that almost the same constructions work even in our setting. In particular, while the problem to which countdown games are reduced in [7] assumes two-dimensional rewards, in our setting we can restrict to single dimension due to window-stability objective imposing both a lower and an upper bound on local mean payoff.

The reductions in the previous theorem require that the window size $W$ is encoded in binary, as the windows need to be exponentially long in the size of the constructed graph. For the case when $W$ is given in unary encoding, the following result can be adapted from [7].

▶ **Theorem 10.** Solving the window-stability objectives (where $\dim \varrho$ is not restricted) where the window size $W$ is encoded in unary is PSPACE-hard, even if it is restricted to instances where the components of reward functions are in $\{-1, 0, 1\}$.

A proof of Theorem 10 is obtained by adapting a proof from [7, Lemma 25], where a reduction from generalized reachability games is given.

The results of [7] do not yield lower bounds for window-stability objectives with one-dimensional reward functions in which either the windows size or the rewards are encoded in unary. In our setting, for the case of binary rewards/ unary window size one can come up with NP-hardness for graphs and PSPACE-hardness for games via reductions from the Subset-Sum problem and its quantified variant [17], respectively. Similarly, for unary rewards/ binary window size a PSPACE-hardness for games via reduction from emptiness of 1-letter alternating finite automata [15] seems plausible. We do not follow these directions, since we are able to prove even stronger and somewhat surprising result: solving window-stability objectives with one-dimensional reward functions is PSPACE-hard for games and NP-hard for graphs even if all the numbers in the input instance are encoded in unary. The proof of this result requires a new proof technique sketched below.

We rely on reductions from special variants of the SAT and QBF problems. An instance of the Balanced-3-SAT problem is a propositional formula $\varphi$ in a 3-conjunctive normal form which contains an even number of variables. Such an instance is positive if and only if $\varphi$ admits a satisfying assignment which maps exactly half of $\varphi$’s variables to 1 (true). We can also define a quantified variant, a Balanced-QBF problem: viewing a quantified Boolean formula $\psi = \exists x_1 \forall x_2 \cdots \exists x_{n-1} \forall x_n \varphi$ (where $\varphi$ is quantifier-free), as a game between player controlling existentially quantified variables (who strives to satisfy $\varphi$) and player controlling universal variables (who aims for the opposite), we ask whether the existential player can enforce assignment mapping exactly half of the variables to 1 and satisfying $\varphi$ (a formal definition of Balanced-QBF is given in Appendix B). The following lemma is easy.

▶ **Lemma 11.** The Balanced-QBF problem is PSPACE-complete. The Balanced-3-SAT is NP-complete.

Let $G$ be a finite-state game and $\Phi = (W, D, \varrho, \mu, \nu)$ a window-stability objective. An instance $(G, s, \Phi)$ is small if $\dim \varrho = 1$, and $W$, $D$, $\max \varrho$, and the numerators and denominators of the fully reduced forms of $\mu$ and $\nu$, are bounded by the number of states of $G$.

▶ **Theorem 12.** Solving the window-stability objectives with one-dimensional reward functions is PSPACE-hard for games and NP-hard for graphs, even if it is restricted to small instances.

Proof (sketch). We proceed by reductions from Balanced-3-SAT for graphs and from Balanced-QBF for games. As the reductions are somewhat technical, we explain just their core idea. The complete reduction can be found in Appendix B.
Assume a formula $\varphi$ in 3-CNF with variables $\{x_1, \ldots, x_n\}$, $n$ being even. Consider the graph $G$ in Figure 2. Both the “upper” gadget (consisting of unprimed states) and the “lower” gadget (with primed states) represent a standard “assignment choice” gadget, in which Player $\square$ selects an assignment to variables in $\varphi$ (e.g. choosing an edge going to $t_1$ from $s_1$ corresponds to setting variable $x_1$ to true etc.). With no additional constraints, $\square$ can choose different assignments in the two gadgets, and she may change the assignment upon every new traversal of the lower gadget. Now assign reward $1$ to states that correspond to setting some variable to true and $0$ to all the other states, let window size $W = 2n$, checkpoint distance $D = 1$, $\mu = \frac{n}{2}$, and $\nu = \frac{n}{2} + \frac{1}{3n}$ (say). In order to satisfy the window-stability objective $(W, D, \rho, \mu, \nu)$ from $s_1$, $\square$ has to select a balanced assignment in the upper gadget and moreover, mimic this assignment in all future points in the lower gadgets. The necessity of the first requirement is easy. For the second, assume that there is some $\ell$ such that in the $\ell$-th step of the outcome $\lambda$ the player chooses to go from, say, $s_1$ to $t_1$ (or from $s'_1$ to $t'_1$), while in the $(\ell + 2n)$-th step she goes from $s'_1$ to $f'_1$. Then the rewards accumulated within windows starting in the $\ell$-th and $(\ell + 1)$-th step, respectively, differ by exactly one. Thus, $|\text{mp}_W(D, t_1(\lambda) - \text{mp}_W(D, t_{1+1}(\lambda))| = 1/2n > 1/3n$, which means that the local mean payoffs at the $\ell$-th and $(\ell + 1)$-th checkpoint cannot both fit into the interval $[\mu, \nu]$.

Note that we use the balanced variant of 3-SAT and QBF, as to set up $\mu$ and $\nu$ we need to know in advance the number of variables assigned to true.

Once we force the player to commit to some assignment using the above insight, we can add more copies of the “primed” gadget that are used to check that the assignment satisfies $\varphi$. Intuitively, we form a cycle consisting of several such gadgets, one gadget per clause of $\varphi$, the gadgets connected by paths of suitable length (not just by one edge as above). In each clause-gadget, satisfaction of the corresponding clause $C$ by the chosen assignment is checked by allowing the player to accrue a small additional reward whenever she visits a state representing satisfaction of some literal in $C$. This small amount is then subtracted and added again on a path that connects the current clause-gadget with the next one: subtracting forces the player to satisfy at least one literal in the previous clause-gadget (and thus accrue the amount needed to “survive” the subtraction) while adding ensures that this “test” does not propagate to the next clause-gadget. Rewards have to be chosen in a careful way to prevent the player from cheating. For PSPACE-hardness of the game version we simply let the adversary control states in the initial gadget (but not in clause-gadgets) corresponding to universally quantified variables.\[\square\]
4 The variance-stability problem

In this section, we prove the results about variance-stability objectives promised in Section 4.

Theorem 13. The existence of a strategy achieving a given one-dimensional variance-stability objective for a given state of a given graph is in NP. Further, the strategy may require infinite memory.

Proof. The proof is based on adaptation of techniques for the hybrid variance from [5] to the non-stochastic setting. In particular, we consider the graph as a Markov decision process. Subsequently, we apply results of [5] and reduce variance-stability problem to the problem of finding an appropriate solution of a negative semi-definite program, which belongs to NP.

As part of the proof we show how to construct strategies that move through edges with the frequencies determined by the program. The proof is considerably complicated by the fact that the frequencies may be irrational and thus further limit processes are needed. Details can be found in Appendix C.

We now show that variance-stability objectives may require strategies with infinite memory. Consider the graph in Figure 3 and the variance-stability objective which requires to achieve the mean payoff at least 3/2 and long-run variance at most 9/4. Observe that there is an infinite-memory strategy solving the variance-stability which works as follows: We start in the state A, the strategy proceeds in infinitely many phases. In the $n$-th phase it goes $n$ times from A to B and back. Afterwards it goes to D, makes 2n steps on the loop on D, and then returns back to A. One can easily show, using the same technique as in the proof of Theorem 13, that the mean payoff converges along this run. The limit is obviously $4/2 + 0/2 + 1/2 = \frac{3}{2}$ since the −10 reward is obtained with zero frequency. The long-run variance is $\frac{1}{9} \left( \frac{-\frac{3}{2}}{2} \right)^2 + \frac{1}{4} \left( \frac{4 - \frac{3}{2}}{2} \right)^2 + \frac{1}{2} \left( 1 - \frac{3}{2} \right)^2 = \frac{2}{3}$. Now we show that there is no finite-memory strategy achieving the mean payoff $\frac{3}{2}$ and the long-run variance $\frac{9}{4}$. Note that the maximal mean payoff achievable (without any constraints) in the graph is 2.

Assume that there is a finite memory strategy $\sigma$ yielding mean payoff $x$ with $3/2 \leq x \leq 2$, and variance at most $9/4$. We first argue that $\sigma$ visits C with zero frequency. Denote by $f_Y$ the frequency of state Y. Because $x = 0 \cdot f_A + 4 \cdot f_B + (-10) \cdot f_C + 1 \cdot f_D$ by the definition of mean payoff, and also $f_A = f_B$ and $f_D = 1 - f_A - f_B - f_C$ by the definition of our graph, we have $f_A = \frac{x + 11 \cdot f_C - 1}{2}$ and $f_D = 2 - x - 12 \cdot f_C$. Thus, the variance is

$$f_A \cdot (0 - x)^2 + f_B \cdot (4 - x)^2 + f_C \cdot (-10 - x)^2 + f_D (1 - x)^2$$

$$= \frac{x - 1}{2} \cdot \left( (0 - x)^2 + (4 - x)^2 \right) + (2 - x) \cdot (1 - x)^2$$

$$+ f_C \cdot \left( \frac{11}{2} \cdot (0 - x)^2 + (4 - x)^2 \right) + (-10 - x)^2 - 12 \cdot (1 - x)^2. \right)$$

Using calculus techniques one can easily show that the first term is at least $9/4$ for all $x \in [3/2, 2]$, while the parenthesized expression multiplied by $f_C$ is positive for all such $x$. Hence $f_C = 0$. But any finite-memory strategy that stays in C with frequency 0 either eventually loops on D, in which case the mean payoff is only 1, or it eventually loops on A and B, in which case the variance is 4.

Even finite-memory strategies that would approximate the desired variance-stability (up to some $\varepsilon > 0$ error) must behave in a rather peculiar way: Infinitely many times stay in $\{A, B\}$ for a large number of steps (depending on $\varepsilon$) and also stay in C for a large number of steps. Hence, if the strategy was applied to a real-life system, a user would observe two disjoint repeating phases, one with low mean payoff but high instability, and one with low stability and high mean payoff.
Figure 3 One player game in which there is an infinite-memory strategy $\sigma$ such that $mp(outcome^\sigma,\pi) \geq 3/2$ and $va(outcome^\sigma,\pi) \leq 9/4$ (here $\pi$ is the only “trivial” strategy of the environment). However, there is no finite-memory $\sigma$ with this property.

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Technical Appendix

A Proofs for Section 3.1

Proposition 4 Let $G = (S, (S_0, S_2), E)$ be a finite-state game, and $n \in \mathbb{N}$. Further, for every $1 \leq i \leq n$, let $\Gamma_i = (\text{Mem}_i, \text{Up}_i, \text{Const}_i, \text{Init}_i)$ be a strategy scheme for $G$ which is permissive for $\Phi_i$. Then there is a strategy scheme for $G$ with $\prod_{i=1}^n |\text{Mem}_i|$ memory elements computable in $O(|S|^2 \cdot |E| \cdot \prod_{i=1}^n |\text{Mem}_i|^2)$ time which is permissive for $\Phi_1 \land \cdots \land \Phi_n$.

Proof. Intuitively, a strategy scheme $\Gamma = (\text{Mem}, \text{Up}, \text{Const}, \text{Init})$ which is permissive for $\Phi_1 \land \cdots \land \Phi_n$ is obtained as a “synchronized product” of all $\Gamma_i$. We put

\[
\text{Mem} = \text{Mem}_1 \times \cdots \times \text{Mem}_n
\]

and define the memory update function by

\[
\text{Up}(s, (m_1, \ldots, m_n)) = (\text{Up}_1(s, m_1), \ldots, \text{Up}_n(s, m_n)).
\]

The constrainer requires more care. One might be tempted to define $\text{Const}(s, (m_1, \ldots, m_n))$ as the intersection of all $\text{Const}_i(s, m_i)$. However, this intersection may be empty, and hence must identify all elements of $S \times \text{Mem}$ which can reach such “problematic” $(s, (m_1, \ldots, m_n))$. Therefore, we define a function $F : 2^{S \times \text{Mem}} \rightarrow 2^{S \times \text{Mem}}$ as follows: For a given $\Omega \subseteq S \times \text{Mem}$, the set $F(\Omega)$ consists of all $(s, m) \in S \times \text{Mem}$ satisfying one of the following conditions:

\[
\begin{align*}
\text{Let } gfix(F) \text{ be the greatest fixed-point of } F. \text{ The constrainer } \text{Const} \text{ is defined as follows:} \\
\text{Const}(s, (m_1, \ldots, m_n)) &= \text{consists of all } s' \in \bigcap_{i=1}^n \text{Const}_i(s, m_i) \text{ such that } (s', \text{Up}(s, m_i)) \in \Omega, \text{ where } m = (m_1, \ldots, m_n); \\
\text{Init} &= \{(s, (m_1, \ldots, m_n)) \mid (s, m_i) \in \text{Init}_i(s) \text{ for all } 1 \leq i \leq n\} \cap gfix(F).
\end{align*}
\]

Finally, we put

\[
\text{Init} = \{(s, (m_1, \ldots, m_n)) \mid (s, m_i) \in \text{Init}_i(s) \text{ for all } 1 \leq i \leq n\} \cap gfix(F).
\]

Observe that $F$ is monotone, and $gfix(F)$ is computable in $O(|S|^2 \cdot |E| \cdot \prod_{i=1}^n |\text{Mem}_i|^2)$ time by the standard iteration algorithm. Hence, $\Gamma$ is also computable in $O(|S|^2 \cdot |E| \cdot \prod_{i=1}^n |\text{Mem}_i|^2)$ time. It is easy to check that $\Gamma$ is permissive for $\Phi_1 \land \cdots \land \Phi_n$. □

B Proofs for Section 3.2

First we formally define the Balanced-QBF problem. Let $\psi = Q_1x_1 \cdots Q_nx_n \varphi$ be a quantified boolean formula in a 3-CNF prenex normal form (3-CNF-PNF), i.e. for each $1 \leq i \leq n$ we have $Q_i \in \{\forall, \exists\}$ and $\varphi$ is a quantifier-free formula in 3-CNF containing only variables from $\{x_1, \ldots, x_n\}$. A model of $\psi$ is a rooted directed tree $T$ satisfying the following properties:

\[
\begin{align*}
\text{nodes of } T \text{ are labelled by elements of } \{1, \ldots, n+1\}, \text{ the root being labelled by 1, all leaves by } n+1, \text{ and children of each node labelled } i \leq n \text{ being labelled by } i+1; \\
\text{edges in } T \text{ are labelled by truth values 0, 1};
\end{align*}
\]
We have states of the form if \( Q_i = \forall \), then all nodes labelled by \( i \) have exactly two children, one connected via edges labelled by 0 and the other by edges labelled by 1; if \( Q_i = \exists \), all nodes labelled by \( i \) have exactly one child;

for each leaf \( v \), the truth assignment induced by the unique path \( p \) from the root to \( v \) (i.e. \( x_i \) is assigned the truth value labelling the unique edge on \( p \) outgoing from an \( i \)-labelled node) satisfies \( \varphi \).

A model \( T \) is balanced if \( n \) is even and all the assignments induced by root-leaf paths in \( T \) assign 1 to exactly half of the \( n \) variables. In a Balanced-QBF problem we are given a formula \( \psi \) in a 3-CNF-PNF and we ask whether this formula admits a balanced model.

**Lemma 11** The Balanced-QBF problem is PSPACE-complete. The Balanced-3-SAT is NP-complete.

**Proof.** We prove the PSPACE-completeness of the Balanced-QBF problem, the result for the satisfiability variant can be obtained via identical reasoning. The membership in PSPACE can be easily tested by a non-deterministic polynomial-space Turing machine in the same manner as for the standard QBF problem. To prove the hardness result we employ a polynomial reduction from standard QBF. Let \( \psi = Q_1 x_1 \cdots Q_n x_n \varphi \) be a formula in a 3-CNF-PNF. Clearly, \( \psi \) is true iff it has a model. We construct a new formula \( \psi' \) (again in 3-CNF-PNF) such that \( \psi \) has a model iff \( \psi' \) has a balanced model. We introduce new existentially quantified variables \( x_{n+1}, \ldots, x_{2n} \) and extend \( \varphi \) with \( n \) clauses of the form \( x_i \lor \neg x_i, n + 1 \leq i \leq 2n \).

Then every model \( T \) of \( \psi \) can be easily extended into balanced model of \( \psi' \) as follows: for each leaf \( v \) of \( T \) we count the number \( o \) of 1's on the path from the root to \( v \). We then append to \( v \) a path \( p \) of length \( n \) such that first \( n - o \) edges on \( p \) are labelled by 1 and the remaining ones by 0 (the nodes on \( p \) are labelled according to the definition of a model).

Conversely, every balanced model \( T' \) of \( \psi' \) can be pruned into a model of \( T \) by simply removing all nodes with label \( i > n + 1 \).

**Theorem 12** Solving the window-stability objectives with one-dimensional reward functions is PSPACE-hard for games and NP-hard for graphs, even if it is restricted to small instances.

**Proof.** We prove the PSPACE-hardness for games, the result for graphs will then follow easily. We proceed by reduction from the Balanced-QBF problem. Let \( \psi = Q_1 x_1 \cdots Q_n x_n \varphi \) be a quantified boolean formula in 3-CNF-PNF with \( n \) even. We show how to construct, in polynomial time, a game \( G \), a state \( s \) of \( G \), and a window-stability objective \( \Phi = (W,D,\varrho,\mu,\nu) \) where \( \dim_{\varrho} = 1 \) such that \( (G,s,\Phi) \) is a small instance and \( \psi \) admits a balanced model iff there is a strategy of player \( \Box \) in game \( G \) achieving the stability objective \( (W,D,\varrho,\mu,\nu) \) in the state \( s \).

Let \( \varphi = C_1 \land \cdots \land C_m \), where each \( C_j \) is a clause. We put \( W = 2(n+m), D = 1, \mu = \frac{\nu}{2^{W}}, \) and \( \nu = \frac{2(1+1)}{2^{W}+1} \). We then construct the game \( G \) and the reward function \( \varrho \) out of several gadgets: for each \( 0 \leq j \leq m \) we have gadgets \( G^\text{sat}_j \) and \( G^\text{force}_j \).

The gadget \( G^\text{sat}_0 \) consists of states \( s_0^{i,1}, t_0^{i,1}, t_0^{i,0}, 1 \leq i \leq n \). A state \( s_0^{i} \) belongs to Player \( \Box \) iff \( Q_i = \forall \), otherwise it belongs to \( \Diamond \). All the other states belong to Player \( \Box \). For each \( 1 \leq i \leq n \) there are edges \( (s_0^{i}, t_0^{i+1}), (s_0^{i}, t_0^{i-1}) \) and for \( i \) and \( n \) we have edges \( (t_0^{1}, s_0^{i+1}), (t_0^{0}, s_0^{i-1}) \). States of the form \( s_0^{i}, 1 \leq i \leq n \), receive reward 1, all other states have reward 0.

For each \( 1 \leq j \leq m \), the gadget \( G^\text{sat}_j \) is defined similarly, but with certain differences: We have states \( s_j^{i}, t_j^{i+1}, t_j^{i,0}, 1 \leq i \leq n \), all of them belonging to \( \Box \). Moreover, for each...
variable \( x_k \) in \( C_j \) we have Player \( \square \)'s state \( r^{k,z(j,k)}_j \), where \( z(j,k) = 0 \) or \( 1 \) depending on whether \( x_k \) appears negated in \( C_j \) or not (if it appears both negated and unnegated, any choice can be taken, for concreteness we put \( z(j,k) = 1 \)). For each \( 1 \leq i < n \) we have edges \( (s^i_j, s^{i,1}_j), (s^i_j, t^{0,0}_j), (t^{i,1}_j, s^{i+1}_j), (t^{i,0}_j, s^{i+1}_j) \). Additionally we have edges \( (s^n_j, t^{n,1}_j), (s^n_j, t^{n,0}_j) \), and for each variable \( x_k \) in \( C_j \) an edge \( (s^k_j, r^{k,z(j,k)}_j) \) and, if \( k < n \), an edge \( (r^{k,z(j,k)}_j, s^k_j) \).

Rewards are assigned as follows: states of the form \( t^{i,1}_j \) for some \( i \) have reward \( \frac{11}{10} \) or \( \frac{1}{10} \) depending on whether \( z(j,k) = 1 \) or not, and all other states in \( G^\text{sat}_j \) have reward 0.

We connect the gadgets \( G^\text{sat}_j \) with gadgets \( G^\text{force}_j \). For each \( 0 \leq j \leq m \) the gadget \( G^\text{force}_j \) consists of states \( u^j_{0,1} \), \( \ell \leq 2m \) and edges \( (u^j_{0,1}, u^j_{\ell+1}) \), \( 1 \leq \ell < 2m \). We have

\[
\rho(u^j_{\ell}) = \begin{cases} 
-\frac{1}{10} & \text{if } j > 0 \text{ and } \ell = 2j - 1 \\
\frac{1}{10} & \text{if } j > 0 \text{ and } \ell = 2j \\
0 & \text{otherwise.}
\end{cases}
\]

Finally, the connection of aforementioned gadgets is realized as follows: for each \( 0 \leq j \leq m \) we add edges \( (i^n_0, u^0_j), (i^n_0, u^0_j) \) and \( (u^{2m}_j, s^0_j \mod m) \). Moreover, for each \( 1 \leq j \leq m \), if the above construction yields a state of the form \( r^{n,z(j,n)}_j \), then we also add an edge \( (r^j_{i,n,z(j,i)}, u^0_j) \).

To prove the correctness of the reduction we employ additional handy notions. A path \( s_1 s_2 \ldots s_k \in S^* \) (here \( S \) is the state set of \( G \)) is an assignment path if it does not contain cycles and moreover, for each \( 1 \leq i \leq n \) it contains exactly one state from the set \( \{t^{i,1}_j, t^{0,0}_j, r^{i,z(j,i)}_j \mid 0 \leq j \leq m \} \). An assignment path \( h \) determines an assignment \( \eta_h \) such that \( \eta_h(x_i) = 1 \) if \( h \) contains a state of the form \( t^{i,1}_j \) or \( r^{i,1}_j \) for some \( j \), and \( \eta_h(x_i) = 0 \) otherwise.

We also extend the function \( \rho \) to finite paths in \( G \): for \( h = s_1, s_2, \ldots, s_k \) we put \( \rho(h) = \sum_{i=1}^k \rho(s_i) \). If \( h \) is an acyclic path and \( u_1, \ldots, u_k \) are all the states of the form \( u^j_{\ell} \) contained in \( h \), then we call the number \( F(h) = \sum_{i=1}^{k'} \rho(u_i) \) a force-reward of \( h \). We also denote by \( A(\lambda_k), B(\lambda_k) \) and \( C(\lambda_k) \) the numbers of occurrences of states of the form \( t^{i,1}_j \), \( r^{i,1}_j \) and \( r^{i,0}_j \) in \( \lambda_k \), respectively. Note that

\[
\rho(h) = A(h) + B(h) + (B(h) + C(h))/10 + F(h).
\]

Now let \( \lambda = s_0, s_1, \ldots \) be an arbitrary run in \( G \). For \( k \in \mathbb{N}_0 \) define a path \( \lambda_k = s_k, s_{k+1}, \ldots, s_{k+2(n+1)} \). The following claim easily follows from the construction of \( G \).

**Claim 17.** Let \( \lambda \) be an arbitrary run in \( G \). Then for every \( k \in \mathbb{N}_0 \) the path \( \lambda_k \) is an assignment path satisfying exactly one of the following conditions

1. \( \lambda_k \) contains states from exactly two distinct gadgets of the form \( G^\text{sat}_j \) and \( F(\lambda_k) = 0 \); or
2. \( \lambda_k \) contains states from exactly gadget of the form \( G^\text{sat}_j \) and \( F(\lambda_k) \in \left[ -\frac{1}{10}, \frac{1}{10} \right] \).

Moreover, in this case \( \lambda_k \) contains, for each \( 1 \leq i \leq n \) exactly one state from \( \{t^{i,1}_j, t^{0,0}_j, r^{i,z(j,i)}_j \} \) and the state \( s^j_i \).

A run \( \lambda \) in \( G \) is self-consistent if for all \( k, k' \in \mathbb{N}_0 \) the assignments induced by the assignment paths \( \lambda_k \) and \( \lambda_{k'} \) are identical A strategy \( \sigma \in \Sigma_G \) is self-consistent if for all \( \pi \in \Pi_G \) the run outcome \( \sigma^{\pi,\tau}_G \) is self-consistent.

Now assume that the formula \( \psi \) admits a balanced model \( T \). We define a strategy \( \sigma \) in \( G \) as follows: for each finite path \( h \in S^* \Sigma_G \) initiated in \( s^0_1 \) we have that
\[ h = s_0^1, t_0^1, l_1, s_0^2, t_0^2, \ldots, t_0^i, l_i, s_0^\ell, t_0^\ell \text{ for some } 1 \leq i \leq n \text{ and there is a node } v \in T \text{ such that } l_1, \ldots, l_i \text{ is the sequence of labels of edges on the unique path from root of } T \text{ to } v, \text{ then we put } \sigma(h) = t_{i, \ell}^{l, \epsilon}, \text{ where } \epsilon = \text{ the label of the unique edge in } T \text{ outgoing from } v. \]

- if \( h \) ends with a state of the form \( s_i^j \), where \( j > 0 \), then the length of \( h \) is at least \( 2(n + m) \).

The prefix of \( h \) of length \( 2n \) is an assignment path inducing an assignment \( \eta \). If \( i \) is the smallest number in \( \{1, \ldots, n\} \) such that either \( \eta(x_i) = 1 \) and \( C_j \) contains literal \( x_i \) or \( \eta(x_i) = 0 \) and \( C_j \) contains literal \( \neg x_i \) (but not literal \( x_i \)), then we put \( \sigma(h) = r_{j, \eta(x_i)}^{l, \epsilon} \), otherwise we put \( \sigma(h) = t_{i, \eta(x_i)}^{l, \epsilon} \).

In all other cases there is only one outgoing edge from the last state of \( h \), and \( \sigma \) selects this edge.

For \( h \) not initiated in \( s_0^1 \) we can define \( \sigma(h) \) arbitrarily. The second item in the definition of \( \sigma \) ensures that \( \sigma \) is self-consistent. Now let \( \pi \in \Pi_G \) be arbitrary and denote \( \lambda = outcome_{s_0^1}^\sigma \).

We need to prove that the local mean payoff in each checkpoint of \( \lambda \) lies between \( \mu \) and \( \nu \), or equivalently, that for all \( k \in \mathbb{N}_0 \) it holds \( \rho(\lambda_k) \in \left[\frac{\mu}{2}, \frac{\nu}{2} + \frac{1}{2}\right] \). From the first item in the definition of \( \sigma \) it follows that the assignment \( \eta \) induced by \( \lambda_0 \) (and thus also by all \( \lambda_k \), \( k \geq 0 \), by self-consistency) is balanced and satisfies \( \varphi \), since it can be obtained by a root-leaf traversal of \( T \). By \( [\dag] \) it holds \( \rho(\lambda_k) = A(\lambda_k) + B(\lambda_k) + (B(\lambda_k) + C(\lambda_k))/10 + F(\lambda_k) \). From balancedness of \( \eta \) we get \( A(\lambda_k) + B(\lambda_k) = n/2 \). Now we distinguish two cases. Either \( \lambda_k \) satisfies the first item in Claim \( [\dag] \). Then \( F(\lambda_k) = 0 \) and \( B(\lambda_k) + C(\lambda_k) \leq 2 \), since \( \sigma \) visits at most one state of the form \( r_{j, (i,j)}^{l, \epsilon} \) in each of the gadgets \( G_{j, \text{out}}^{\lambda, \sigma} \). In particular, \( \frac{\mu}{2} \leq \rho(\lambda_k) \leq \frac{\nu}{2} + \frac{1}{2}. \) Or \( \lambda_k \) satisfies the first item in Claim \( [\dag] \). In this case case \( B(\lambda_k) + C(\lambda_k) = 1: \lambda_k \) intersects just one \( G_{j, \text{out}}^{\lambda, \sigma} \) hence at most one state of the form \( r_{j, (i,j)}^{l, \epsilon} \), and at the same time it visits all states of the form \( s_i^j \) and one successor of each such state. Since \( \eta \) satisfies \( C_j \), from the definition of \( \sigma \) it follows that \( \lambda_k \) contains at least one \( r_{j, (i,j)}^{l, \epsilon} \). Moreover, from Claim \( [\dag] \) we get \( -\frac{1}{10} \leq F(\lambda_k) \leq 1/10 \). This again implies \( \frac{\mu}{2} \leq \rho(\lambda_k) \leq \frac{\nu}{2} + \frac{1}{2}. \)

For the other direction, let \( \sigma \in \Sigma_G \) be a strategy \( \sigma \) such that for all \( \pi \in \Pi_G \) the run \( outcome_{s_0^1}^\sigma \) satisfies the window-stability objective \( (W, D, \rho, \mu, \nu) \). We use \( \sigma \) to inductively construct a tree \( T \). We start with a single node labelled by \( 1 \). In each step, we do the following for all leaves \( v \) of the current tree: let \( x_i \) be the label of \( v \). If \( Q_i = \emptyset \), we append to \( v \) two children labelled by \( i + 1 \), labelling one of the new edges by 0 and the other by 1. If \( Q_i = \emptyset \), then let \( h = s_0^1, t_0^1, l_1, s_0^2, t_0^2, \ldots, t_0^i, l_i, s_0^\ell, t_0^\ell \) be the unique path in \( G \) from \( s_0^0 \) to \( s_0^\ell \) such that \( \ell_1, \ldots, \ell_{i-1} \) is the sequence of edge-labels on the unique path from root to \( v \) in the current tree. We append to \( v \) a child labelled by \( i + 1 \) connected via an edge labelled by \( x \in \{0, 1\} \) s.t. \( \sigma(h) = t_{1, x}^{l, \epsilon} \).

To show that \( T \) is a balanced model of \( \psi \) we need to prove that for all \( \pi \in \Pi_G \) the assignment \( \eta_\pi \) induced by \( (outcome_{s_0^1}^\sigma)_0 \) is balanced and satisfies \( \varphi \). So fix any \( \pi \) and denote \( \lambda = outcome_{s_0^1}^\sigma \). We start with proving that each \( \lambda_k \), \( k \in \mathbb{N}_0 \), induces a balanced assignment. Assume that there is \( k \) such that this is not true. Recall \( [\dag] \) and note that \( B(\lambda_k) + C(\lambda_k) \leq 6 \) (due to Claim \( [\dag] \) and since \( \varphi \) is in 3-CNF), \( F(\lambda_k) \in [-1/10, 1/10] \) (Claim \( [\dag] \)) and \( A(\lambda_k) + B(\lambda_k) \in \{0, 1, \ldots, n\} \setminus \{n/2\} \) (non-balancedness). From \( [\dag] \) it follows that either \( \rho(\lambda_k) \leq \frac{n}{2} - \frac{3}{10} \) or \( \rho(\lambda_k) \geq \frac{n}{2} + \frac{9}{10} \), a contradiction with \( \sigma \) satisfying the window-stability objective.

For satisfaction of \( \varphi \) we first show that \( \lambda \) is self-consistent. Assume the contrary. Then there is \( k \) such that the balanced assignments induced by \( \lambda_k \) and \( \lambda_{k+1} \) differ. Let \( s \) be the first state of \( \lambda_k \) and \( s' \) the last state of \( \lambda_{k+1} \). Note that \( |\rho(\lambda_k) - \rho(\lambda_{k+1})| = |\rho(s) - \rho(t)| \). Since \( \lambda_k \) and \( \lambda_{k+1} \) induce different assignments, it holds \( |\rho(s) - \rho(t)| \geq 9/10 \). But then at
least one of the values $\rho(\lambda_k), \rho(\lambda_{k+1})$ falls outside of the interval $[\frac{2}{5}, \frac{2}{5} + \frac{1}{4}]$, a contradiction with $\sigma$ satisfying the window-stability objective.

Now we prove that the assignment $\eta_\pi$ induced by $\lambda_0$ (and thus by all $\lambda_k$) satisfies $\varphi$. Assume that there is a clause $C_j$ not satisfied by $\eta_\pi$, and let $k$ be the smallest index such that $\lambda_k$ begins with a state $u_{j-1}^{2j-1}$ (at least one such $k$ must exist due to construction of $G$). Note that $\lambda_k$ ends with a state $u_j^{2j-1}$ and $F(\lambda_k) = -\frac{1}{10}$, and $G_j^{sat}$ is the only gadget of the form $G_j^{sat}$ intersected by $\lambda_k$. Since we assume that the assignment induced by $\lambda_k$ does not satisfy $C_j$, $\lambda_k$ does not contain any state of the form $r_j^{i_1,a_i}$. Hence, $B(\lambda_k) + C(\lambda_k) = 0$ and $A(\lambda_k) = n/2$ (as $\eta_\pi$ is balanced). From (1) it follows that $\rho(\lambda_k) = \frac{9}{2} - 1/10$, a contradiction with $\sigma$ satisfying the window-stability objective. This finishes the proof.

Note that if all the quantifiers in $\psi$ are existential (i.e. $\psi$ is an instance of Balanced-3-SAT), then $G$ does not contain any states of Player $\Diamond$. This shows the NP-hardness for graphs.

C Proofs for Section 4

**Theorem 13** The existence of a strategy achieving a given one-dimensional variance-stability objective for a given state of a given graph is in NP. Further, a finite description of a strategy achieving the objective is computable in exponential time, and the strategy may require infinite memory.

Let us consider a game $G = (S, (S_C, S_H), E)$ and an instance of the variance-stability problem determined by a reward function $\varphi$ together with a mean-payoff bound $b \in \mathbb{Q}$ and a variance bound $c \in \mathbb{Q}$. We assume that all outcomes are initiated in a fixed initial state $s$.

A frequency vector is a tuple $(f_e)_{e \in E} \in [0,1]^{|E|}$ with $\sum_{e \in E} f_e = 1$ and

$$\sum_{s' : (s', s) \in E} f_{(s', s)} = \sum_{s' : (s, s') \in E} f_{(s, s')}$$

for all $s \in S$. Now consider the following constraints:

$$mp := \sum_{s \in S} f_s \cdot \varphi(s) \geq a \quad (2)$$

$$va := \sum_{s \in S} f_s \cdot (\varphi(s) - mp)^2 \leq b \quad (3)$$

Here $f_s = \sum_{(s', s) \in E} f_{(s', s)}$ for every $s \in S$.

As every single player game is a special case of a Markov decision process, we may invoke Proposition 5. of [5] and obtain the following lemma.

**Proposition 19 ([5])**. Assume that there is a solution to the given variance-stability problem. Then there is a frequency vector $(f_e)_{e \in E}$ satisfying the inequalities (2) and (3). All $e \in E$ satisfying $f_e > 0$ belong to the same strongly connected component of $G$ reachable from $s$.

The above inequalities (2) and (3) can be turned into a negative semi-definite program, using techniques of [5], and hence decided in non-deterministic polynomial time [13]. To finish our algorithm, we need to show that a solution to the above inequalities can also be turned into a strategy which visits each $e \in E$ with the frequency $f_e$. 


Let $\lambda = s_0 s_1 \ldots$ be a run. Given $e \in E$ and $i \in \mathbb{N}$ we define

$$a_i^e(\lambda) = \begin{cases} 
1 & \text{if } (s_i, s_{i+1}) = e \\
0 & \text{otherwise}
\end{cases}$$

Lemma 20. Suppose $(f_e)_{e \in E}$ is a frequency vector such that all $e \in E$ satisfying $f_e > 0$ belong to the same strongly connected component reachable from the initial state $\bar{s}$. Then there is a strategy $\sigma_f$ with $\lim_{n \to \infty} \sum_{j=0}^n a_j^e(\lambda) = f_e$ for all $e \in E$, where $\lambda$ is the outcome under $\sigma_f$ (initiated in $\bar{s}$).

Proof. Let us assume, w.l.o.g., that $G$ itself is strongly connected. If, $(f_e)_{e \in E}$ is rational and all edges $e$ satisfying $f_e > 0$ induce a strongly connected graph, we may easily construct the strategy $\sigma_f$ as follows. We multiply all numbers $f_e$ with the least-common-multiple of their denominators and obtain a vector of natural numbers $f'_e$ that still satisfy the above flow equations. Now we may imagine the game as a multi-diagraph, where each edge $e$ has the multiplicity $f'_e$. It is easy to show that the flow equations are exactly equivalent to existence of a directed Euler cycle. From this Euler cycle in the digraph we immediately get a cycle in our game which visits each edge exactly $f'_e$ times. By repeating the path indefinitely we obtain a run with the desired frequencies $f'_e$ of edges.

Now consider a general frequency vector $(f_e)_{e \in E}$, i.e. the frequencies $f_e$ may be irrational and the graph induced by edges $e$ with $f_e > 0$ does not have to be strongly connected. Then we may still approximate $(f_e)_{e \in E}$ it by a sequence of rational vectors $(f'_e)_{e \in E}$, here $f'_e \to f_e$ as $i \to \infty$, satisfying the flow equations (this follows from the fact that all frequency vectors satisfying the flow equations form a closed polyhedron). For each of the vectors we have a finite cycle $c_i$ which, when repeated indefinitely, gives the frequencies $(f'_e)_{e \in E}$. We use $C^s_i$ for such infinite sequence, initiated in $s$, and, to make it easier to concatenate cycles, w.l.o.g. we suppose that for all $i$ we have $f'_e > 0$.

Let $\varepsilon_0 \varepsilon_1 \ldots$ be a strictly decreasing sequence of numbers converging to 0, with $\varepsilon_0 \leq 0.1$. For all $i$, let $L_i$ be a number such that for all $L' \geq L_i$ and all $s$ we have

$$\sum_{j=0}^{L'} a_j^e(C^s_i) \geq f'_e - \varepsilon_i$$

We define runs $\alpha_i$ and numbers $K_i$ as follows. We put $\alpha_1 = C^s_0$ and $K_1 = 1$. Further, we define a $\alpha_i$ by taking $\alpha_{i-1}$ for $K_{i-1}$ steps, and then concatenating $C^s_i$, where $s$ is the $(K_{i-1} + 1)$-th element of $\alpha_{i-1}$. We let $K_i > K_{i-1}$ be a number such that

$$\sum_{j=0}^{K_i} a_j^e(\alpha_{i-1}) \geq f'_e - \varepsilon_i \quad \text{and} \quad \varepsilon_i^2 \cdot K_i \geq L_{i+1}.$$ 

Let $\alpha$ be the limit of the sequences $\alpha_i$ (note that $\alpha$ agrees with any $\alpha_i$ on the first $K_i$ steps).

We claim that, for all $e$,

$$\lim_{n \to \varepsilon} \sum_{j=0}^n a_j^e(\alpha) \frac{n+1}{n+1} = f_e. \quad (4)$$

Let $n$ be a number such that $K_i \leq n \leq K_{i+1}$. We will show that, for all $e$,

$$\sum_{j=0}^{n} a_j^e(\alpha) \frac{n+1}{n+1} \geq \min\{f'_e, f'_{e+1}\} - 2 \cdot \varepsilon_i$$

which by the convergence of $f_e$ and $\varepsilon_i$ will show (4).
If \( n \leq K_i + L_{i+1} \), then

\[
\sum_{j=0}^{n} a_j^i(\alpha) + \sum_{j=K_i}^{n} a_j^i(\alpha_i) \geq K_i \cdot (f_e^i - \varepsilon_i) \geq \frac{K_i \cdot (f_e^i - \varepsilon_i)}{K_i + L_{i+1}} \geq \sum_{j=K_i+1}^{n} a_j^i(\alpha_i) \geq \sum_{j=K_i}^{n} a_j^i(\alpha_i) \geq \sum_{j=0}^{n} a_j^i(\alpha_i) \geq \sum_{j=K_i+1}^{n} a_j^i(\alpha_i) \geq K_i \cdot (f_e^i - \varepsilon_i) \geq \frac{K_i \cdot (f_e^i - \varepsilon_i)}{K_i + L_{i+1}} \geq f_e^i - 2 \cdot \varepsilon_i
\]

where the last-but-one inequality follows because

\[
K_i \cdot (f_e^i - \varepsilon_i) \geq (K_i + L_{i+1}) \cdot (f_e^i - 2 \cdot \varepsilon_i)
\]

\[
K_i \cdot (f_e^i - \varepsilon_i) \geq (K_i + \varepsilon_i^2) \cdot (f_e^i - 2 \cdot \varepsilon_i)
\]

\[
0 \geq -K_i \cdot \varepsilon_i + K_i \cdot \varepsilon_i^2 \cdot f_e^i - K_i \cdot 2 \cdot \varepsilon_i^3
\]

\[
0 \geq K_i \cdot (-\varepsilon_i + \varepsilon_i^2 \cdot f_e^i - 2 \cdot \varepsilon_i^3)
\]

and \((-\varepsilon_i + \varepsilon_i^2 \cdot f_e^i - 2 \cdot \varepsilon_i^3)\) is negative as \( \varepsilon_i \leq 0.1 \)

On the other hand, if \( K_i + L_{i+1} \leq n \leq K_{i+1} \), then

\[
\sum_{j=0}^{n} a_j^i(\alpha) = \sum_{j=0}^{K_i} a_j^i(\alpha) + \sum_{j=K_i+1}^{n} a_j^i(\alpha) \geq (K_i + 1) \cdot (f_e^i - \varepsilon_i) - (n + 1 - (K_i + 1)) \cdot (f_e^{i+1} - \varepsilon_{i+1}) \geq \min\{f_e^i, f_e^{i+1}\} - \varepsilon_i
\]