Covariant versions of Margolus-Levitin Theorem

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The Margolus-Levitin Theorem is a limitation on the minimal evolving time of a quantum system from its initial state to its orthogonal state. It also supplies a bound on the maximal operations or events can occur within a volume of spacetime. We compare Margolus-Levitin Theorem with other form of limitation on minimal evolving time, and present covariant versions of Margolus-Levitin Theorem in special and general relativity respectively. The relation among entropy bound, maximum information flow and computational limit are discussed, by applying the covariant versions of Margolus-Levitin Theorem.

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I. INTRODUCTION

In the view of quantum theory, the time-energy uncertainty relation is interpreted as the limitation of how fast can a quantum system evolve from its initial state to an distinguishable state, equivalently, to an orthogonal state. In this relation, $\Delta t$ can be understood as the minimal evolving time between two distinct states of the quantum system. The time-energy uncertainty relation $\Delta t \geq \pi \hbar / 2 \Delta E$ indicates the minimal evolving time is constrained by the energy spread of the system. Recently, Margolus and Levitin gave an other interpretation of Heisenberg principle. In their proposition, the average energy instead of the energy spread of the system is constraining the minimal evolving time $\Delta t$. The Margolus-Levitin Theorem has several advantages. For an isolated system the average energy is conserved while the energy spread can be arbitrary. More over, for complicated system the average energy is easy to determined while the energy spread require more information or assumption from the system. As a result, the Margolus-Levitin Theorem is widely used on a series of topics[1, 2, 3, 4, 5, 6, 7, 8, 9].

Some direct applications of Margolus-Levitin Theorem are taken to evaluate the speed of ultimate quantum computer[2], the total operations performed by the universe[3] and the accuracy of measuring the spacetime geometry[4, 8, 9]. Following Margolus-Levitin Theorem, once the average energy of the system is determined, the minimal evolving time between two distinct states is also determined. A quantum computer with every operation performed as quickly as the minimal evolving time has reached the maximum speed of the average energy. If we perceive the universe as such an quantum computer, as we already know the energy density and the age scale of the universe, we can work out the total operations performed by the universe from its beginning. This lead us to the third application, what is the accuracy of measuring the spacetime geometry? A spacetime region can be viewed as a container filled with operations or events. The accuracy of spacetime measurement depends on density and distribution of the operations or events. Accordingly, these applications of Margolus-Levitin Theorem are using abstract or concrete models, including quantum mechanical, cosmological, and quantum gravitational ones. Since the Margolus-Levitin Theorem is quantum mechanical, and is deduced in pure state system with a non-interacting hamiltonian. What is the validity of Margolus-Levitin Theorem on these applications?

In some previous work, Lloyd and his cooperators have discussed the extension of Margolus-Levitin Theorem in composite systems, mixed states, and entangled systems[6, 7]. They showed the Margolus-Levitin Theorem is still valid to estimate the minimal evolving time of these systems. Also, Lloyd have given out an attempting generalization of Margolus-Levitin Theorem in gravitational systems. What’s more, Zelinski and Zych have made modification on Margolus-Levitin Theorem to evaluate some specific systems more precisely[5]. In this letter we will discuss these generalizations and modifications. We will show that the generalized form of Margolus-Levitin Theorem proposed by Lloyd has failed to apply on some systems, i.e. the radiation dominating era of the universe. Lloyd’s form is covariant but not the correct covariant version for Margolus-Levitin Theorem. Thus we will give out our covariant

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versions of Margolus-Levitin Theorem, which correctly evaluate all relativistic systems and gravitational systems.

This paper will be organized as follow. In Sec. II, we revisit the Margolus-Levitin Theorem and its extension in frame of quantum mechanics. In Sec. III, the Margolus-Levitin Theorem on mixed state, composite system and entanglement were discussed. It shows that Margolus-Levitin Theorem as an fundamental limit on $\Delta t$ still works on these systems. In Sec. IV, we focus on constructing a covariant form of Margolus-Levitin Theorem when the special relativistic effect was taken into account. Since we still don’t have a complete quantum theory of gravity, we can’t obtain a similar theorem in general relativity from the fundamental level. We analyze this problem in a semiclassical way in Sec. V, and get a similar conclusion with Sec. IV. After that, the application of the formula of covariant quantum limits of computation is discussed in Sec. VI. We discuss the close relation among entropy bound, maximum information flow and ultimate computational limit, it is found that the order of magnitude of maximum logical operations $\#$ of a system during the amount of time of communicating from one side to the other is no less than the entropy of the system $S$, or the amount of information that the system can store. Remarkably, this naturally gives a physical interpretation of the inequality in the first set of hypotheses in [14]. Discussions and conclusions will be made in Sec. VII.

II. THE MARGOLUS-LEVITIN THEOREM

The Margolus-Levitin Theorem states a quantum system with energy expectation value $E$ above its ground state energy, could transform from one original state to its orthogonal state in the time $\Delta t$, where $\Delta t \geq \pi \hbar / 2E$ [1]. Firstly, let us consider an arbitrary quantum system could be decomposed as superposition of its energy eigenstates,

$$|\Psi(0) \rangle = \sum_i C_i |\Psi^i_{E_i} \rangle,$$

(1)

where $\Psi^i_{E_i}$ is the $i$th eigenstates of the system with energy $E_i$. We assumed the system have a discrete spectrum and the ground state energy is set zero. By quantum mechanics, after a time interval $t$, the system will transform to,

$$|\Psi(t) \rangle = \sum_i C_i e^{-i \hbar E_i t / \pi} |\Psi^i_{E_i} \rangle.$$

(2)

Then we set the inner product to be,

$$S(t) = \langle \Psi(0) | \Psi(t) \rangle = \sum_i |C_i|^2 e^{-i \hbar E_i t / \pi}.$$

(3)

In order to obtain the minimal value of evolving time, we use inequality $\cos(x) \geq 1 - 2/\pi [x + \sin(x)]$, which is valid for $x \geq 0$. Then,

$$Re(S) = \sum_i |C_i|^2 \cos \left( \frac{i \hbar E_i t}{\pi} \right)$$

$$\geq \sum_i |C_i|^2 \left[ 1 - \frac{2}{\pi} \left( \frac{E_i t}{\hbar} + \sin \left( \frac{E_i t}{\hbar} \right) \right) \right]$$

(4)

$$= 1 - \frac{2E}{\pi \hbar} t - \frac{2}{\pi} Im(S).$$

The system reaches its orthogonal state while we set $S(\Delta t) = 0$. Then we finally obtain $\Delta t \geq \pi \hbar / 2E$, where $E$ is the expectation value of the system energy. The ground state energy does not account for $t$, which only add a global phase factor in equations. Some examples which saturate the inequality have been found in [1], so $\pi \hbar / 2E$ gives a upper bound of how fast can a system transform to its orthogonal state. This is the main conclusion of Margolus-Levitin Theorem. If we use a different inequality such as:

$$\cos(x) \geq 1 - \frac{2}{\pi^\alpha} x^\alpha - \frac{2\alpha}{\pi} \sin(x),$$

(5)
FIG. 1: The value of $\pi \hbar / 2^{1/\alpha} \langle (E - E_0)^{\alpha} \rangle^{1/\alpha}$ for some specific configuration of quantum states are plotted as the vertical axis. In Figure (a) the initial state is $|\Psi(0)\rangle = 1/\sqrt{3}(|0\rangle + |1\rangle + |2\rangle)$. In Figure (b), $|\Psi(0)\rangle = \sqrt{0.99}/\sqrt{2}(|0\rangle + |1\rangle) + 0.1/\sqrt{2}(|10000\rangle + |10001\rangle)$. In all cases $\epsilon$ is set to be 1. It does not show there is constant $\alpha$ as an universal parameter to supply a tightest bound of minimal evolving time for all configurations of quantum states.

where $\alpha > 0$, then we obtain a more complicate form of the estimation:

$$\Delta t \geq \frac{\pi \hbar}{2^{1/\alpha} \langle (E - E_0)^{\alpha} \rangle^{1/\alpha}}, \quad \alpha > 0. \quad (6)$$

Here $E_0$ is the ground state energy, and $\langle \rangle$ denotes the expectation value on initial state. This new estimation ($\alpha \neq 1$) will get a higher bound than Margolus-Levitin Theorem ($\alpha = 1$) on $\Delta t$ in some special case, but we are not able to obtain a new bound with a constant $\alpha$, which limit $\Delta t$ more strictly than Margolus-Levitin Theorem universally, as in Fig. 1. On the other hand, the physical interpretation of the expression $\langle (E - E_0)^{\alpha} \rangle^{1/\alpha}$ with $\alpha \neq 1$ is obscure. The bound of Eq. (5) is more meaningful in mathematics than in physics. However, Margolus-Levitin Theorem relates the minimal evolving time to the average energy of the system, which offers a relatively clear physical picture. As a result, we still perceive Margolus-Levitin Theorem as a fine bound of $\Delta t$ in this letter, and all our generalizations are based on Margolus-Levitin Theorem.

III. MARGOLUS-LEVITIN THEOREM ON MIXED STATE, COMPOSITE SYSTEM AND ENTANGLEMENT

In quantum mechanics, mixed states, composite systems and entangled systems are always focused on as well as pure states, single and non-entangled ones. Some previous work shows that Margolus-Levitin Theorem as an fundamental limit on $\Delta t$ has its successful applications on these systems. Consider a general state $\rho$, and we assume $\rho$ can be decomposed with orthogonal energy eigenstates,

$$\rho(0) = \sum_{i,j} C_{ij} |\Psi_i\rangle \langle \Psi_j| \tag{7}$$

Here $\rho(0)$ can be viewed as a mixed state or a pure state, depending what value $C_{ij}$ takes. Time translation denoted by an unitary operator $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$ transform $\rho(0)$ to $\rho(t)$,

$$\rho(t) = \hat{U}(t) \rho(0) \hat{U}^\dagger(t) = \sum_{i,j} C_{ij} e^{-i\hat{H}t/\hbar} |\Psi_i\rangle \langle \Psi_j| e^{i\hat{H}t/\hbar}. \tag{8}$$

A widely used function fidelity $F(\rho(0), \rho(t))$ is devoted to discriminate orthogonality on mixed states. When we set $F = 0$ to confirm the orthogonality, it is easy to obtain:

$$F(\rho(0), \rho(t)) = Tr \sqrt{\rho^{1/2}(0) \rho(t) \rho^{1/2}(0)} = \sum_i C_{ii} e^{-i\hat{E}_i t/\hbar} \tag{9}$$
\( E_i \) is energy value of \(|\Psi\rangle\) (Here the Hamiltonian is assumed to be diagonal). We can easily see that fidelity coincide with the inner product between initial and final states, while \( \rho \) is specialized to a pure state. Further deduction gives out \( \Delta t \geq \pi \hbar/2E \), and \( E = Tr(\hat{H}\rho) \) is the energy expectation value of the mixed state. Noting that we have not taken interacting Hamiltonian in our consideration yet. Directly, this generalization on mixed state lead us to investigate the self consistency of Margolus-Levitin Theorem on all kinds of composite quantum systems. For example, a mixed state \( \rho \) can be composed by several pure states \( (\rho = \sum_i p_i \rho_i) \), or, a pure state can be represented as directed product of several pure states, etc. A question is asked, what is relationship between the minimal time to orthogonal state \( \Delta t \) of the system and \( \Delta t_i \) of the \( i \)th subsystem? It is answered that in the latter case, \( \Delta t \) is no longer limited by \( \pi \hbar/2E \), but,

\[
\Delta t = max\left(\frac{\pi \hbar}{2E_i}\right).
\]

(10)

Here \( E_i \) is the energy expectation value of the \( i \)th subsystem, unless one of the subsystems has all the energy of the system[4]. It does mean a composite system may reach its orthogonal state with longer time than single ones with the same average energy. However, the total operations performed by the system should independent of its architecture[4]. In one second, a system with average energy \( E \) would perform as most \( 1/\Delta t = 2E/\pi \hbar \) operations, which is the exact summation of maximum operations performed by the subsystems\( (\sum_i 1/\Delta t_i = \sum_i 2E_i/\pi \hbar = 2E/\pi \hbar) \). This is another advantage of Margolus-Levitin Theorem, the operation rate of any system/subsystem limited by Margolus-Levitin Theorem is proportional to its energy expectation value, which is a fine extensive variable. In contrast, the operation rate limited by energy spread \( \Delta E \)[2, 6] or Eq. (9) is not additive generally, which could cause some severe problem.

So far we have discussed the minimal time to orthogonal state of a non-entangled system with a free Hamiltonian. Although the entangled case or interacting case of Margolus-Levitin Theorem is difficult to formulated, some testing calculation[6] implies a composite system with entanglement-generating Hamiltonian are capable to operate with a rate closer to the Margolus-Levitin Theorem bound than without it. The differences are dependent on the dominance of interacting, while Margolus-Levitin Theorem is still valid in these cases.

IV. A SPECIAL RELATIVISTIC FORM OF MARGOLUS-LEVITIN THEOREM

Margolus-Levitin Theorem enable us to evaluate how many operations or events can be discriminated at most of a 4-D space-time system, within a section of its history. The result should be invariant under any physical transformation of the observer. But apparently, the above form of the bound is not invariant in relativity. Assuming a system \( S \) with average energy \( E \) could be reported with \( \# \leq 2E\tau/\pi \hbar \) operations at most in time \( \tau \), in a rest frame \( O \). Now there is another observer in frame \( O' \) watching \( S \) is moving along the \( x \) axis with velocity \( v \), with an average energy \( E\gamma \) for a time interval \( \tau\gamma \) in the same history, where \( \gamma = 1/\sqrt{1 - v^2} \). The new limitation of operations in frame \( O' \) is \( \#' \leq 2E\gamma^2\tau/\pi \hbar \). This paradox does not even exist in special relativity. Accordingly, a moving clock ticks slower than a same static clock with a rate \( \gamma \) for time dilatation effect. And the quantities of the ticks or operations detected from the clock is independent of observers for causality. Margolus and Levitin has already give a testing form[1] of the bound of operation quantities detected in \( O' \), in a special relativity view, which is \( \# \leq 2px/\pi \hbar \). The inner product of 4-momentum \( p^\mu \) and 4-translation \( x^\mu \) guarantees Lorentz invariance of the bound. And this result is easily achieved by replacing quantum field theory of quantum mechanics in deduction. For simplicity we focus on a state \( \Psi \) composed by massive particles[19],

\[
\Psi = \sum_i C_i \Psi_{p_1, \sigma_1; p_2, \sigma_2; \ldots}^i.
\]

(11)

The indexes mean the \( i \)-th eigenstate of \( \Psi \) is occupied by an \( n_1 \)-th kind of particle with 4-momentum \( p_1 \) and spin \( \sigma_1 \) and so on. In quantum field theory, a time translation \((t, 0, 0, 0)\) is perceived as a trivial Lorentz transformation of the coordinates by \((-t, 0, 0, 0)\), as

\[
U(1, -t)\Psi = \sum_i e^{-i(p_1^0 + p_2^0 + \ldots)\tau} C_i \Psi_{p_1, \sigma_1; p_2, \sigma_2; \ldots}^i.
\]

(12)

With a Lorentz boost, the state \( \Psi \) transforms to,

\[
U(\Lambda, 0)\Psi = \sum_i C_i \times \sqrt{\frac{(\Lambda p_1)^0}{p_1^0}} \sum_{\sigma_1, \sigma_2, \ldots} D_{\sigma_1}^{(j_1)}(\Lambda, p_1)) \ldots \Psi_{\Lambda p_1, \sigma_1'; n_1; \Lambda p_2, \sigma_2', \ldots}.
\]

(13)

Then the time translation \((t, 0, 0, 0)\) becomes spacetime translation after the boost,
Here, $\Lambda$ is a Lorentz transformation to the coordinate, $W$ is the Lorentz transformation belonging to the little group, and $D$ furnish a representation of the little group. More details can be found in [19]. Then the form of inner product between the two states becomes relevant to the Lorentz boost, $S = \Psi^\dagger U(1, -t)\Psi = \sum |C_i|^2 e^{-i(p'_i + p'_i + \ldots)t} = (U(\Lambda, 0)\Psi)^\dagger U(\Lambda, -\Lambda t)\Psi = \sum |C_i|^2 e^{-i(p'_i + p'_i + \ldots)x'}$. We still use inequality $\cos(x) \geq 1 - 2/\pi[x + \sin(x)]$ to obtain the special relativistic form of Margolus-Levitin Theorem, 

$$Re(S) = \sum_i |C_i|^2 \cos\left(\frac{i}{\hbar}(p'_i + \ldots)x') \right) \geq \sum_i |C_i|^2 [1 - \frac{2}{\pi}(\frac{(p'_i + \ldots)x'}{\hbar} + \sin(\frac{(p'_i + \ldots)x'}{\hbar}))]$$

$$= 1 - \frac{2p'x'}{\pi\hbar} - \frac{2}{\pi}Im(S).$$

Where $p'_i = \Lambda p_i$, $x' = \Lambda t = (1, \vec{v})t\gamma$, $\gamma = 1/\sqrt{1 - v^2}$ and $p' = \sum_i |C_i|^2 (p'_i + p'_i + \ldots)$ is the expectation value of 4-momentum of the system. A quick analysis gives the bound of minimal evolving time of orthogonal state of the system, 

$$\Delta t' \geq \frac{\pi\hbar}{2p' \cdot (1, \vec{v})}.$$ 

The physical picture of this inequality is confusing. Because time is not a Lorentz invariant variable, it is not wise for us to seek a relativistic invariant form of minimal evolving time between orthogonal states of the system. Although a system evolving for a time interval $\tau$ in the rest frame $O$, could be described undergoing a dilated time $\tau\gamma$ in the moving frame $O'$, the bound of quantities of operations or events in the same history is still invariant, 

$$\#_{bound} = \frac{2E\tau}{\pi\hbar} = \#'_{bound} = \frac{\tau\gamma}{\Delta t'} = \frac{2px}{\pi\hbar}.$$ 

Where we denote $p'$ and $\gamma(1, \vec{v})$ by $p$ and $x$ for simplification, and we denote the upper bound of quantities of operations by $\#_{bound}$. This result is inevitable, since $px$ is an direct covariant representation of $E\tau$ in the rest frame $O$. If the 3-components of $p$ are all zero—it means the system is relative static to observer—the upper bound $\#_{bound}$ degenerates to the original form $E\tau$. It shows Margolus-Levitin Theorem still works on systems static to observer, and evaluating a covariant bound of $\#$ is more practical than a covariant bound of $\Delta t$. Hence we should seek a generalized form of Margolus-Levitin Theorem to give the bound of $\#$ of a system other than $\Delta t$.

As long as the system $S$ is simple, it is possible to get the 4-momentum $p$ and 4-translation $x$ of the system, and to make an inner product. However in practice, it is hard to define 4-momentum and 4-translation of some complex system such as fluids or radiations. In most theoretical analysis the energy-momentum tensor $T_{ab}$ is more general than 4-momentum, so a generalized form of Margolus-Levitin Theorem should take $T_{ab}$ as an component of estimation of the bound. Apparently, an observer with unit timelike 4-velocity $u^a$ in Minkowski spacetime will obtain an energy current density $J_a = -T_{ab}u^b$ [21], as measuring a fluid with energy momentum tensor $T_{ab}$, and the current density satisfies $\partial^aJ_a = 0$. By using Gauss’s law, we can get the conservation of energy. Thus we can straightforwardly replace $p_n$ in Eq. [15] by $\int dV T_{ab}u^b$ (here a constant $c^2$ is omitted) and then we get, 

$$\# \leq \frac{2}{\pi\hbar} \int T_{ab}u^a \frac{dx^b}{d\tau} dV d\tau = \frac{2}{\pi\hbar} \int T_{ab}u^a u^b dV d\tau.$$ 

The equivalence between Eq. [19] and Margolus-Levitin Theorem can be proved by setting $T_{ab} = p_a p_b / E\delta^3(\vec{x} - \vec{x}(\tau))$ [20], noting that a 3-volume contracts by a factor $\gamma$ under boost of Lorentz transformation. Since $x$ is 4-translation
of the system, \( u^a = dx^a/d\tau \) is not an arbitrary observer’s 4-velocity, but the 4-velocities of observers at rest with respect to the system. This ensures the bound of the system with same history is not depending on different observers. The covariance of the generalized form is obvious. \( T_{ab}u^au^b \) is a scalar and \( dVd\tau \) is an invariant 4-volume unit in a flat background. The operational capacity of a system is now determined by \( T_{ab}u^au^b \) and the most quantities of operations or events can be detected is proportional to the size of the 4-volume of the system. It must be emphasized that the non-negative \( \# \) requires the system obeying the weak energy condition,

\[
T_{ab}u^au^b \geq 0.
\]

(20)

Though in some theory, system with matter violating weak energy condition is not unstable and is able to evolve from its initial state to an orthogonal state to perform operations, the validity of Margolus-Levitin Theorem on these systems is still to be evaluated carefully. We assert that our estimations of upper bound of quantities of operations are following the weak energy condition.

The generalized form of Margolus-Levitin Theorem has a direct application on cosmology. Consider a flat universe with constant \( \omega \) above -1, which describes the massive dust dominant era and radiation dominant era of the universe. We have Friedmann equations with a flat metric:

\[
H^2 = \frac{8\pi G}{3} \rho,
\]

(21)

\[
\dot{\rho} + 3H\rho(1 + \omega) = 0,
\]

(22)

\[
ds^2 = -dt^2 + a^2(t)[dr^2 + r^2dΩ].
\]

(23)

Here \( T_{ab} = (\rho + p)u_au_b + pg_{ab} \) is the energy-momentum tensor of the dominating matter, \( p \) is pressure density, \( \rho \) is energy density and we have \( p = \omega \rho \). Resolving Friedmann equations, we obtain,

\[
H = \frac{2}{3(1 + \omega)(t - t_0)}.
\]

(24)

We will omit the constant \( t_0 \) for following discussion. Substituting Eq. (21), Eq. (24) and the representation of \( T_{ab} \) into Eq. (19), and take the apparent horizon \( l_H = c/H \) as the physical bound of our universe, we obtain,

\[
\# \leq \frac{9c^5t^2(1 + \omega)}{4\pi\hbar G}.
\]

(25)

The result has the same order of Lloyd’s result \( \# \approx c^5t^2/G\hbar \). According to our result, the bounds of the quantities of events detected in our universe vary from different expanding history. This is different with Lloyd’s result, since his deduction has omitted the expanding details of the universe. The result depending on \( \omega \) can be explained that different expanding history dilute the energy density of the universe with different rate, while the expanding horizon contains more operational units from outside with different rate. Then the form of upper bound of the total events in the observed universe is different among different expanding eras, while it always scales similar to the area of the horizon. It is consistent with and complementary to the Bekenstein bound. Any appliance of Eq. (19) on the \( \omega = -1 \) matter dominating era is not successful, e.g. the cosmological constant. Since Margolus-Levitin Theorem is discussing average energy above the ground state, it is not reasonable applying Margolus-Levitin Theorem on cosmological constant, which could be interpreted to vacuum energy.

V. A GENERAL RELATIVISTIC FORM OF MARGOLUS-LEVITIN THEOREM

In the special relativity, we find that the upper bound of operations is associate with a conserved quantity of energy. However, in general relativity, there is not a notion of the local stress-energy tensor of the gravitational field. Nevertheless, for asymptotically flat spacetimes, conserved quantity of energy associated with asymptotic symmetries has been well defined at spatial and null infinity. Since there are lots of approaches to define conserved energy, we assert that our result may not be unique in the most general contexts.

In general relativity, the energy properties of matter also represented by a energy momentum tensor \( T_{ab} \), and the energy momentum current is still:

\[
J_a = -T_{ab}u^b.
\]

(26)
The local energy density \( \rho \) of matter as measured by a given observer with unit timelike four-velocity \( u^a \) is well defined by\[21]\[
\rho = T_{ab}u^a u^b. \tag{27}
\]
So, locally the energy in a cell with spacial volume \( \Delta V \) is \( T_{ab}u^a u^b \Delta V \), the maximum logical operations \( \Delta \#_{\text{bound}} \) in the cell within proper time \( \Delta \tau \) of the observer is:
\[
\Delta \#_{\text{bound}} = \frac{2}{\pi \hbar} T_{ab}u^a u^b \Delta \tau \Delta V. \tag{28}
\]

The form is very similar with Eq. (19). One would expect Eq. (19) may be still valid in general relativisty. However, the condition \( \nabla^a T_{ab} = 0 \) does not lead to a global conservation law in general relativity (\( \nabla^a J_a \) is generally non-zero, not like in special relativity). There is no conserved total energy in general relativity, unless there is a Killing vector \( \xi^a \) in the spacetime, i.e., \( \nabla[a \xi_b] = 0 \). In this case \( \nabla^a J_a = \nabla^a (T_{ab} \xi^b) = 0 \), and the conserved total energy associated with the Killing field \( \xi^a \) on a spacelike Cauchy hypersurface \( \Sigma \) can be defined as,
\[
E = \int_\Sigma T_{ab}n^a \xi^b d\Sigma, \tag{29}
\]
where \( n^a \) denotes the unit normal vector to \( \Sigma \). Here the spacetime \( (M, g_{ab}) \) have been foliated by Cauchy hypersurface \( \Sigma_t \) which is parameterized by a global time function \( t \), then the metric can be write in ADM form,
\[
d s^2 = g_{ab}dx^a dx^b = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{30}
\]

\( N \) is called the \textit{lapse function} which measures the difference between the coordinate time \( t \) and proper time \( \tau \) on curves normal to the Cauchy hypersurfaces \( \Sigma_t \), \( N^i \) is called the \textit{shift vector} which measures the difference between a spatial point \( p \) and the point one would reach if instead of following \( p \) from one Cauchy hypersurface to the next one followed a curve tangent to the normal \( n^a \), and \( h_{ij} \) is the intrinsic metric induced on the hypersurface by \( g_{ab} \). The timelike normal \( n^a \) defines the local flow of time in \( M \).

The energy defined in Eq. (29) is just the total energy on \( \Sigma \) observed by an observer with 4-velocity \( \xi^a \), the maximum logical operations of \( \Sigma \) over proper time \( \int dt \) is bounded by
\[
\# \leq \frac{2}{\pi \hbar} \int T_{ab}n^a \xi^b d\Sigma dt. \tag{31}
\]
Alternatively, from the definition of energy density \( \rho \) in Eq. (27), we suspect the maximal logical operations the fluid over time \( \int dt \) would have the form,
\[
\# \leq \frac{2}{\pi \hbar} \int N T_{ab}u^a u^b d\Sigma dt. \tag{32}
\]
This definition of energy coincides with the definition in (29) when the Killing vector \( \xi^a \) satisfies,
\[
\xi^a = Nu^a \text{ and } u^a = n^a. \tag{33}
\]
This means Eq. (32) is valid when the spacetime manifold have symmetries which can generate time killing vectors normal to \( \Sigma \). We can write \( d\mathcal{V} = Nd\Sigma dt \) as the proper volume element of spacetime 4-volume \( \mathcal{V} \), then
\[
\# \leq \frac{2}{\pi \hbar} \int N T_{ab}u^a u^b d\Sigma dt = \frac{2}{\pi \hbar} \int_\mathcal{V} T_{ab}u^a u^b d\mathcal{V}, \tag{34}
\]
Obviously, this form of Margolus-Levitin Theorem is covariant and consistent with Eq. (19) in special relativity. So far we know little about quantum gravity, we are not able to give a deduction of the generalized form of Margolus-Levitin Theorem from the fundamental level. However, the picture of the generalized form is proved to be compatible with quantum mechanics and general relativity.

Consider a fluid with energy momentum tensor \( T_{ab} = (\rho + p)u_a u_b + pg_{ab} \) spreads in a spacetime region with gravitational field \( g_{ab} \). We assume the distribution of the fluid is static and spherical symmetric, and the size of the
system is \( l \), the bound of the integral is omitted for simplification. Then the upper bound of quantities of operations is represented by,

\[
\# \leq \frac{2}{\pi \hbar} \int \rho(r)r^2 \, dr \, d\Omega \, dt.
\]  

(35)

Here \( \Omega \) is the area of a unit 2-sphere, and \( d\Omega = dr \, d\Omega \). It seems there is no difference between Eq. (35) and the special relativistic form. However, the gravitational effect has already been included in Eq. (35). The Margolus-Levitin Theorem relates the minimal evolving time to the average energy of the system, while the average energy here should be the sum of energy not gained from gravity and energy gained from gravity. Here the energy not gained from gravity is \( E_{\text{ng}} = \int 1/\sqrt{-g_{00}} \rho(r) r^2 \, d\Omega \, dt \), and the energy gained from gravity is \( E_g = -\int \rho(r) [1/\sqrt{-g_{00}} - 1]^2 \, dr \, d\Omega \). \( E_g \) is below zero, and is lowering down the total average energy. This implicates that the system performs operations slower in gravitational field. While since a system in gravitation field has the same average energy with an other system without considering gravity, as measured by an observer from infinity, they have the same bound of quantities of operations, even one of them performs slower. This can be explained from the other aspect as following. An operation unit of the system takes \( \Delta T \) at infinity to reach its orthogonal state. Because of the time dilation effect of general relativity, if we put the unit into gravitational field, the minimal evolving time will expands to \( \Delta t = 1/\sqrt{-g_{00}} \Delta T \) as measured by observers at infinity. However, the physical size of system in gravitational field is \( \sqrt{-g_{00}} \) times larger than the size as measured by observers at infinity. If the outside of the static spherical system is vacuum, we can obtain \(-g_{00}(r)g_{11}(r) = 1\). As a result, the system in gravitational field could maximally perform as many operations as the system with the same average energy out of gravitational field. Negative binding energy keeps the bound fixed although the operation units are slowed and the amount of operation units become large. This is the similar mechanism with the analysis entropy bound of gravitational system in monster-like state\([18]\). Also, the generalized form of Margolus-Levitin Theorem gives a complimentary explanation to the accuracy of spacial measurement with the analysis entropy bound of gravitational system in monster-like state\([18]\). The generalized form of Margolus-Levitin Theorem relates the minimal evolving time to the average energy of the system, while the average energy here should be the sum of energy not gained from gravity and energy gained from gravity. Here the energy not gained from gravity is \( E_{\text{ng}} = \int 1/\sqrt{-g_{00}} \rho(r) r^2 \, d\Omega \, dt \), and the energy gained from gravity is \( E_g = -\int \rho(r) [1/\sqrt{-g_{00}} - 1]^2 \, dr \, d\Omega \). \( E_g \) is below zero, and is lowering down the total average energy. This implicates that the system performs operations slower in gravitational field. While since a system in gravitation field has the same average energy with an other system without considering gravity, as measured by an observer from infinity, they have the same bound of quantities of operations, even one of them performs slower. This can be explained from the other aspect as following. An operation unit of the system takes \( \Delta T \) at infinity to reach its orthogonal state. 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Also, the generalized form of Margolus-Levitin Theorem gives a complimentary explanation to the accuracy of spacial measurement \( \delta l \sim l^{1/3} \). Since the spacetime is distorted by gravity, the operations and events are not spreading homogeneously between spacial section and temporal section of the 4-volume. In the characteristic time of communication \( l/c \), each operation unit of the system can perform only of order one operation. Then the relax of temporal resolution from \( \delta t \sim l^{1/2} \) (non-gravitational case) to \( \delta t \sim l^{1/3} \) corresponds to the enhancement of spacial resolution from \( \delta l \sim l^{1/2} \) (non-gravitational case) to \( \delta l \sim l^{2/3} \). While in both cases the maximal quantities of events contained in a 4-volume \( l^4 \) are both scaling as \( l^2 \).

Lloyd has given another covariant version of Margolus-Levitin Theorem \([4]\),

\[
\# \leq -\frac{2}{\pi \hbar} \int T_{ab} g^{ab} \, d\mathcal{V},
\]  

(36)

where the inequality is set in a rest frame with an inertial clock. For the fluid we have discussed above, the covariant quantity \(-T_{ab} g^{ab}\) can be simplified to \( \rho - 3p \). Then Eq. (36) is equivalent to Eq. (35) when the fluid is composed by massive dust\((\rho(r) = 0)\). For fluids\((\rho(r) \neq 0)\), Eq. (36) gets different results with Eq. (35). If we let the fluid to be radiations which is the dominating component of the universe in some era. The characteristic feature of radiations is \( p = 1/3 \rho \), which makes the maximal operations of the radiations obtained by Eq. (36) to be 0. It means the quantum units in radiations will never evolve to their orthogonal states. This result can be avoided if we take Eq. (35) as the covariant version of Margolus-Levitin Theorem, since the pressure density of the fluid is not contained apparently in the expression.

VI. ENTROPY BOUND, MAXIMAL INFORMATION FLOW AND COMPUTATIONAL LIMIT

The entropy bound measures the capability of how much the information can be coded in a system. The maximum entropy flow rate associated with maximum speed of information transfer, is said the limit of communication. And computational limit reflects the ability that the system can deal with information or produce new information. We will find those three bounds are compatible with each other. First, we will discuss Bekenstein system. With our covariant form of Margolus-Levitin Theorem, some result can be generalized to Bousso system. Bekenstein system and Bousso system will be defined below. For convenience, Planck units will be used in this section, i.e., \( c = G = \hbar = k = 1 \), where \( c \) is the speed of light, \( G \) is Newton constant, \( \hbar \) is Planck constant, and \( k \) is Boltzmann constant.

A. In Bekenstein System

For any complete, weakly self-gravitating, isolated matter system in ordinary asymptotically flat space, Bekenstein\([10, 23]\) has argued that the Generalized Second Law of thermodynamics\((\text{GSL})\) implies a bound to the
entropy-to-energy ratio

\[ \frac{S}{E} \leq 2\pi R, \]  

(37)

Here, \( E \) is the total energy of the matter system, \( R \) is the radius of the smallest sphere that fits around the matter system. The bound has been explicitly shown to hold in wide classes of equilibrium systems\(^2\text{22}\), we will call this kind of system that satisfies the conditions for the application of Benkenstein’s bound as **Bekenstein system**\(^1\text{11}\).

Since energy is well defined in **Bekenstein system**, it can be viewed as a computational system. Then, according to the Margolus-Levitin Theorem, the maximal logical operations that the computer can perform during the time \( t_{\text{com}} = 2R \) is

\[ \#_{\text{bound}} = 2Et_{\text{com}}/\pi = 4ER/\pi, \]  

(38)

\( t_{\text{com}} \) is just the time for sending a message from one side of the system to the other. Using the Bekenstein bound, i.e., Eq. (37), we have,

\[ \frac{\#_{\text{bound}}}{S} = \frac{4ER}{\pi S} \geq \frac{2}{\pi^2}, \]  

(39)

which is a lower bound of \( \#_{\text{bound}} \)-to-\( S \) ratio. We find \( \#_{\text{bound}} \) and \( S \) have the same order, which means the order of magnitude of maximum logical operations of a Bekenstein system \( \#_{\text{bound}} \) is no less than the entropy of the system \( S \) during an amount of time for communicating from one side to the other. In fact, Lloyd\(^2\text{2}\) states that the measurement of the degree of parallelization in an ultimate computer is \( t_{\text{com}}/t_{\text{flip}} \), where \( t_{\text{flip}} \) is the characteristic time for flipping a bit in the system. For a Bekenstein system \( t_{\text{com}}/t_{\text{flip}} \geq 2/\pi^2 \), which means the operation of a Bekenstein system can not be totally serial. Here, we just reinterpret in another way.

Why the operation of a computational system is always somewhat parallel? The answer is we can’t transfer information at an arbitrary speed. It has been broadly discussed\(^2\text{24}\) that the information flow rate is bounded by the configuration of the system. Bekenstein presents a bound on information flow \( \dot{I} \)

\[ \dot{I} \leq \frac{\pi E}{\ln 2}, \]  

(40)

It is a reinterpretation of the proposition in \( \text{22} \) as he said. This bound is consistent with previous work by Bremermann and Pendry etc\(^2\text{26}\). How much information can be transferred during the time \( t_{\text{com}} = 2R \) ? By using (40), it is easy to see that the information \( I \leq 2\pi ER/\ln 2 \), and the entropy taken by the information is \( S \leq 2\pi ER \). This means all the information stored in a system can be transferred in a time it takes communicate from one side to the other if we can make the communication channel reach it’s capacity limitation. It is also worth to note that equation (39) tell us the ultimate computer can produce new information as much as the ultimate communication channel can transfer, so that any computer will always be somewhat parallel.

**B. In Bousso System**

The form of Bekenstein’s entropy bound is unapparently covariant. Bousso has already given out an apparently covariant entropy bound for gravitational systems. The system satisfying Bousso’s entropy bound is called **Bousso system**. We find that the correspond relation of Eq. (39) in **Bousso system** is still valid. We will discuss the validity of the \( \#_{\text{bound}} \)-to-\( S \) ratio lower bound in gravitational collapse system and cosmological models.

1. **Testing the bound in gravitational collapse**

In a gravitational collapsing system, the internal mass distribution is not homogeneous. Some regions are over-dense, and the light-rays which we used to send signals from one side to the other side are deflected. The signals might take intricate, long-winded path across the gravitational collapsing system\(^1\text{11}\). This means it takes more time to send the messages across the system, and more logical operations should be taken into account in Eq. (39). It is quite plausible that this mechanism will protect the \( \#_{\text{bound}} \)-to-\( S \) ratio lower bound from being violated.
2. Testing the bound in lightsheet

In [11], Bousso shows the covariant entropy bound resolved the limitation in using Fischler-Susskind bound[13] to closed FRW universe, and take it as a strong evidence for the validity of his conjecture. All the quantitative calculation were taken under the assumption that the entropy in the universe can be described by local entropy density $s$ and the matter content can be described by a perfect fluid, with stress tensor $T^{ab} = diag(-\rho, p, p, p)$. It is quite plausible that for this kind of matter the hypothesis in [14],

$$\pi(\lambda_{\infty} - \lambda) T_{ab} k^a k^b \geq |s_a k^a|,$$  \hspace{1cm} (41)

will be satisfied. This inequality is sufficient to prove the generalized Bousso Entropy bound and all the result in sec. IV of [11] is naturally.

If we also make the assumption that there is an entropy flux 4-vector $s^a$ associated with each lightsheet $L$ whose integral over $L$ is the entropy flux through $L$ as in [14]. Using the same notation in [14], the entropy flux through $L$ is given by

$$ S_L = \int_{B'} d^2 x \sqrt{\text{det} h(x)} \int_0^{\lambda_{\infty}(x)} d\lambda \ s(\lambda) A(\lambda). $$ \hspace{1cm} (42)

Here $x \equiv (x^1, x^2) = x^I$ is a coordinate system on the 2-surface $B$, $h_{\Gamma\Lambda}(x)$ is the induced 2-metric on $B$, $\lambda$ is an affine parameter, $k^a = (d/d\lambda)^a$ is tangent to each geodesic, and $\lambda_{\infty}(x)$ is the value of affine parameter at the endpoint of the generator which starts at $x$, $s \equiv -s_a k^a$ is non-negative for future directed timelike or null $s^a$, $A(\lambda) \equiv \exp[\int_0^\lambda d\bar{\lambda} \theta(\bar{\lambda})]$ is an area-decreasing factor in the given geodesic with $\theta$ interpreted as the expansion.

In the same way, the energy flux through $L$ is

$$ E_L = \int_{B'} d^2 x \sqrt{\text{det} h_{\Gamma\Lambda}(x)} \int_0^{\lambda_{\infty}(x)} d\lambda \ \rho(\lambda) A(\lambda), $$ \hspace{1cm} (43)

where $\rho = -J_a k^a = T_{ab} k^a k^b \geq 0$ if we assume the null convergence condition.

Noting that the amount of time for sending a signal from $B$ to $B'$ is just $\lambda_{\infty} - \lambda$ in Planck unit. Then, by using equation (41), (42) and (43), the $\#\text{bound-to-}S$ ratio lower bound hold on any lightsheet followed immediately, i.e.

$$ \frac{\#\text{bound}}{S} = \frac{2E_L(\lambda_{\infty} - \lambda)}{\pi S_L} \geq \frac{2}{\pi^2}. $$ \hspace{1cm} (44)

Now the physical meaning of the hypothesis is specific, and the computational limits and the entropy bound is related to each other very closely.

VII. CONCLUSIONS AND DISCUSSION

In this paper, we revised Margolus-Levitin Theorem and compared it with other form of limitation of minimal evolving time. We stated that Margolus-Levitin Theorem has a clear physical picture and has successful applications on composite systems, mixed state and entanglement systems. Following the spirit of Margolus-Levitin Theorem, we have presented covariant versions of Margolus-Levitin Theorem respectively in special relativity and general relativity. The covariant version in special relativity is deduced analytically, while the version in general relativity is still tentative but reasonable. After that we give some applications of the versions of Margolus-Levitin Theorem on cosmology and quantum gravitational systems. In the end, we discussed the relation between entropy bound, maximal information flow and computational limit. All the applications were consistent with previous works, and presented more details and physical pictures on those topics.

Although the covariant version of Margolus-Levitin Theorem in general relativity has its successful applications, we still need estimations to evaluate its validity more completely. So far the deduction shows the objects of relativistic version of Margolus-Levitin Theorem are configured on space-like surfaces, excluding null-surfaces. Because anything observed by an relativistic observer should be configured on the null-surfaces called light-cones, the deduction of Margolus-Levitin Theorem needs to be extended to null-surfaces.
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