Repeated Inverse Reinforcement Learning

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Abstract

How detailed should we make the goals we prescribe to AI agents acting on our behalf in complex environments? Detailed & low-level specification of goals can be tedious and expensive to create, and abstract & high-level goals could lead to negative surprises as the agent may find behaviors that we would not want it to do, i.e., lead to unsafe AI. One approach to addressing this dilemma is for the agent to infer human goals by observing human behavior. This is the Inverse Reinforcement Learning (IRL) problem. However, IRL is generally ill-posed for there are typically many reward functions for which the observed behavior is optimal. While the use of heuristics to select from among the set of feasible reward functions has led to successful applications of IRL to the problem of learning from demonstration (e.g., Abbeel et al., 2007), not identifying the reward function poses fundamental challenges to the question of how well and how safely the agent will perform when using the learned reward function in other tasks. This is particularly relevant because IRL is a possible approach to the concern about aligning the agent’s values/goals with those of humans for AI safety as society deploys more capable learning agents that impact more people in more ways (Russell et al., 2015; Amodei et al., 2016).

Adding AI safety concerns to IRL could take many forms: which human’s reward function matters?, which task should we watch the human perform?, how does the agent generalize what it learns from one task to other tasks?, etc. Here we focus solely on extending IRL to the generalization across tasks aspect of AI safety. We formalize multiple variations of a new repeated IRL problem in which the agent and (the same) human are placed in multiple tasks. We separate the reward function into two components, one which is invariant across tasks and can be viewed as intrinsic to the human, and a second that is task specific. As a motivating example, consider a human doing tasks throughout a work day, e.g., getting coffee, driving to work, interacting with co-workers, and so on. Each of these tasks has a task-specific goal but the human brings to each task intrinsic goals that correspond to maintaining health, financial well-being, not violating moral and legal principles, etc. In our repeated IRL setting, the agent presents a policy for each new task that it thinks the human would do. If the agent’s policy “surprises” the human by being sub-optimal, the human presents the agent with the optimal policy. The objective of the agent is to minimize the number of surprises to the human, i.e., to generalize the human’s behavior to new tasks.

Quite apart from the connection to AI safety, the repeated IRL problem we introduce and our results are of independent interest in resolving the question of unidentifiability

1. Introduction

One challenge in building AI agents that learn from experience is how to set their goals or rewards. In the Reinforcement Learning (RL) setting, one interesting answer to this question is inverse RL (or IRL) in which the agent infers the rewards of a human by observing the human’s policy in a task (Ng & Russell, 2000). Unfortunately, the IRL problem is ill-posed for there are typically many reward functions for which the observed behavior is optimal in a single task (Abbeel & Ng, 2004). While the use of heuristics to select from among the set of feasible reward functions has led to successful applications of IRL to the problem of learning from demonstration (e.g., Abbeel et al., 2007), not identifying the reward function poses fundamental challenges to the question of how well and how safely the agent will perform when using the learned reward function in other tasks. This is particularly relevant because IRL is a possible approach to the concern about aligning the agent’s values/goals with those of humans for AI safety as society deploys more capable learning agents that impact more people in more ways (Russell et al., 2015; Amodei et al., 2016).

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of rewards from observations in standard IRL. Our contributions include: (1) an efficient identification algorithm when the agent can choose the tasks in which it observes human behavior; (2) an upper bound on the number of total surprises when no assumptions are made on the tasks, along with a corresponding lower bound; (3) an extension to the setting where the human provides sample trajectories instead of complete behavior; and (4) identification guarantees when the agent can only choose the task rewards but is given a fixed task environment.

2. Markov Decision Processes (MDPs)

We are interested in environments that can be represented as MDPs. An MDP is specified by its state space $S$, action space $A$, initial state distribution $\mu \in \Delta(S)$, transition (or dynamics) function $P : S \times A \rightarrow \Delta(S)$, reward function $Y : S \rightarrow \mathbb{R}$, and discount factor $\gamma \in [0, 1)$. A policy $\pi : S \rightarrow A$ describes an agent’s behavior by specifying the action to take in each state. The (normalized) value function $\pi$ is defined as $V^\pi(s) = (1 - \gamma) \mathbb{E} \left[ \sum_{t=1}^{\infty} \gamma^{t-1} Y(s_t) | s_0 = s; \pi \right]$. Similarly, the Q-value function is $Q^\pi(s, a) = (1 - \gamma) \mathbb{E} \left[ \sum_{t=1}^{\infty} \gamma^{t-1} Y(s_t) | s_0 = s, a_0 = a; \pi \right]$. Where necessary we will use the notation $\eta^\pi_{s, a}$ to avoid ambiguity about the environment and the reward function used in computing $V^\pi$. Let $\pi^\ast : S \rightarrow A$ be an optimal policy, which maximizes $V^\pi$ and $Q^\pi$ in all states (and actions) simultaneously.

Given an initial distribution over states, $\mu$, a scalar value that measures the goodness of $\pi$ is defined as $\mathbb{E}_{s \sim \mu} [V^\pi(s)]$. We introduce some further notation to define $\mathbb{E}_{s \sim \mu} [V^\pi(s)]$ in vector-matrix form. Let $\eta^\pi_{\mu, P} \in \mathbb{R}^{\mid S \mid}$ be the normalized state occupancy under initial distribution $\mu$, dynamics $P$, and policy $\pi$, whose $s$-th entry is $(1 - \gamma) \mathbb{E} \left[ \sum_{t=1}^{\infty} \gamma^{t-1} I(s_t = s) | s_0 \sim \mu; \pi \right]$. This vector can be computed in closed-form as $\eta^\pi_{\mu, P} = (1 - \gamma) \left( \mu^T P^\pi - (I_{\mid S \mid} - \gamma P^\pi)^{-1} \right)^T$, where $P^\pi$ is an $|S| \times |S|$ matrix whose $(s, s')$-th element is $P(s'|s, \pi(s))$, and $I_{|S|}$ is the $|S| \times |S|$ identity matrix. For convenience we will also treat the reward function $Y$ as a vector in $\mathbb{R}^{\mid S \mid}$, and we have

$$\mathbb{E}_{s \sim \mu} [V^\pi(s)] = Y^T \eta^\pi_{\mu, P}. \quad (1)$$

3. Problem setup

Here we define the repeated IRL problem. The human’s reward function $\theta_\ast$ captures his/her safety concerns and intrinsic/general preferences. This $\theta_\ast$ is unknown to the agent and is the object of interest herein, i.e., if $\theta_\ast$ were known to the agent, the concerns addressed in this paper would be solved. We assume that the human cannot directly communicate $\theta_\ast$ to the agent but can evaluate the agent’s behavior in a task as well as demonstrate optimal behavior.

Formally, a task is defined by a pair $(E, R)$, where $E = (S, A, \mu, P, \gamma)$ is the task environment (i.e., an MDP without a reward function), and $R$ is the task-specific reward function (task reward). We assume that all tasks share the same $S, A, \gamma$, with $|A| \geq 2$, but may differ in the initial distribution $\mu$, dynamics $P$, and task reward $R$; all of the task-specifying quantities are known to the agent. In any task, the human’s optimal behavior is always with respect to the reward function $Y := \theta_\ast + R$. We emphasize again that $\theta_\ast$ is intrinsic to the human and remains the same across all tasks. Our use of task specific reward functions $R$ allows for greater generality than the usual IRL setting, but we note that our results apply equally to the case where the task reward is always zero.

While $\theta_\ast$ is private to the human, the agent has some prior knowledge on $\theta_\ast$, represented as a set of possible parameters $\Theta_0 \subset \mathbb{R}^{|\Theta|}$ that contains $\theta_\ast$. Throughout, we assume that the human’s reward has bounded and normalized magnitude, that is, $\|\theta_\ast\|_{\infty} \leq 1$.

A demonstration in $(E, R)$ means revealing $\pi^\ast$ to the agent, which optimizes for $Y := \theta_\ast + R$ under environment $E$. A common assumption in the IRL literature is that the full mapping is revealed, which can be unrealistic if some states are unreachable from the initial distribution. We address the issue by requiring only the state occupancy vector $\eta_{\mu, P}^\pi$.

In Section 7 we show that this also allows an easy extension to the setting where the human only demonstrates trajectories instead of providing a policy.

Under the above framework for repeated IRL, we consider two settings that differ in how the sequence of tasks are chosen. In both settings, we will want to minimize the number of demonstrations needed.

1. (Section 5) Agent chooses the tasks, observes the human’s behavior in each of them, and infers the reward function. In this setting where the agent is powerful enough to choose tasks arbitrarily, we will show that the agent will be able to identify the human’s reward function which of course implies the ability to generalize to new tasks.

2. (Section 6) Nature chooses the tasks, and the agent proposes a policy in each task. The human demonstrates a policy only if the agent’s policy is a mistake (a negative surprise), i.e., significantly suboptimal. In this setting we will derive upper and lower bounds on the number of mistakes our agent will make.

Here we differ (w.r.t. g.) from common IRL literature in assuming that reward occurs after transition.
4. The challenge of identifying rewards

Note that it is impossible to identify $\theta$, from watching human behavior in a single task. This is because any $\theta$ is fundamentally indistinguishable from an infinite set of reward functions that yield exactly the policy observed in the task. We introduce the idea of behavioral equivalence below to tease apart two separate issues wrapped up in the challenge of identifying rewards.

**Definition 1.** Two reward functions $\theta, \theta' \in \mathbb{R}^{|S|}$ are behaviorally equivalent in MDP tasks, if for any $(E, R)$, the set of optimal policies for $(R + \theta)$ and $(R + \theta')$ are the same.

We argue that the task of identifying the reward function should amount only to identifying the (behaviorally) equivalence class to which $\theta$ belongs. In particular, identifying the equivalence class is sufficient to get perfect generalization to new tasks. Any remaining unidentifiability is merely representational and of no real consequence. Next we present a constraint that captures the reward functions that belong to the same equivalence class.

**Proposition 1.** Two reward functions $\theta$ and $\theta'$ are behaviorally equivalent in MDP tasks if and only if $\theta - \theta' = c \cdot \mathbf{1}_{|S|}$ for some $c \in \mathbb{R}$, where $\mathbf{1}_{|S|}$ is an all-1 vector of length $|S|$.

**Proof.** To show that $\theta - \theta' = c \cdot \mathbf{1}_{|S|}$ implies behavioral equivalence, we note that for any policy $\pi$ the occupancy vector $\eta^{T}_{\mu, P}$ always satisfies $\mathbf{1}_{|S|}^T \eta^{T}_{\mu, P} = 1$, so $\forall \pi, |\theta^{T} \eta^{T}_{\mu, P} - \theta'^{T} \eta^{T}_{\mu, P}| = c$, and therefore the set of optimal policies is the same.

To show the other direction, we prove that if $\theta - \theta' \notin \text{span}(\{\mathbf{1}_{|S|}\})$, then there exists $(E, R)$ such that the sets of optimal policies differ. In particular, we choose $R = -\theta'$, so that all policies are optimal under $R + \theta' = 0$. Since $\theta - \theta' \notin \text{span}(\{\mathbf{1}_{|S|}\})$, there exists states $i$ and $j$ such that $\theta(i) + R(i) \neq \theta(j) + R(j)$. Suppose $i$ is the one with smaller sum of rewards, then we can make $j$ an absorbing state, and have two deterministic actions in $i$ that transition to $i$ and $j$ respectively. Under $R + \theta$, the self-loop in state $i$ is suboptimal, and this completes the proof.

5. Agent chooses the tasks

In this section, the protocol is that the agent chooses a sequence of tasks $\{(E_t, R_t)\}$. For each task $(E_t, R_t)$, the human reveals $\pi^*_t$, which is optimal for environment $E_t$ and reward function $\theta + R_t$. Our goal is to design an algorithm which chooses $\{(E_t, R_t)\}$ and identifies $\theta_*$ to a desired accuracy ($\epsilon$) using as few tasks as possible.

5.1. Omniscient identification algorithm

Theorem 1 shows that a simple algorithm can identify $\theta_*$ after only $O(\log(1/\epsilon))$ tasks, if any tasks may be chosen. Roughly speaking, the algorithm amounts to a binary search on each component of $\theta_*$ by manipulating the task reward $R_t$. See the proof for the algorithm specification.

**Theorem 1.** If $\theta_* \in \Theta_0 \subseteq \{\theta \in [-1, 1]^{|S|} : \theta(s_{\text{ref}}) = 0\}$, there exists an algorithm that outputs $\theta \in \mathbb{R}^{|S|}$ that satisfies $\|\theta - \theta_*\|_{\infty} \leq \epsilon$ after $O(\log(1/\epsilon))$ demonstrations.

**Proof.** The algorithm chooses the following fixed environment in all tasks: for each $s \in S \setminus \{s_{\text{ref}}\}$, let one action be a self-loop, and the other action transitions to $s_{\text{ref}}$. In $s_{\text{ref}}$, all actions cause self-loops. The initial distribution over states is uniformly at random over $S \setminus \{s_{\text{ref}}\}$.

Each task only differs in the task reward $R_t$ (where $R_t(s_{\text{ref}}) \equiv 0$ always). After observing the state occupancy of the optimal policy, for each $s$ we check if the occupancy is equal to 0. If so, it means that the demonstrated optimal policy chooses to go to $s_{\text{ref}}$ from $s$ in the first time step, and $\theta_* + R_t(s) \leq \theta_*(s_{\text{ref}}) + R_t(s_{\text{ref}}) = 0$; if not, we have $\theta_* + R_t(s) \geq 0$. Consequently, after each task we learn the relationship between $\theta_*(s)$ and $-R_t(s)$ on each $s \in S \setminus \{s_{\text{ref}}\}$, so conducting a binary search by manipulating $R_t(s)$ will identify $\theta_*$ to $\epsilon$-accuracy after $O(\log(1/\epsilon))$ tasks.

As noted before, once the agent has identified $\theta_*$ within an appropriate tolerance, it can compute a sufficiently-near-optimal policy for all tasks, thus completing the generalization objective through the far stronger identification objective in this setting.

6. Nature chooses the tasks

While Theorem 1 yields a strong identification guarantee, it also relies on a strong assumption, that $\{(E_t, R_t)\}$ may be chosen by the agent in an arbitrary manner. In this section, we let nature, who is allowed to be adversarial for the

\footnote{2While we present a proof that manipulates $R_t$, an only slightly more complex proof applies to the setting where all the $R_t$ are exactly zero and the manipulation is limited to the environment; see full details in the previous version of the paper on arXiv (Amin & Singh, 2016).}
purpose of the analysis, choose \(\{(E_t, R_t)\}\).

Generally speaking, we cannot obtain identification guarantees in such an adversarial setup. As an example, if \(R_t \equiv 0\) and \(E_t\) remains the same over time, we are essentially back to the classical IRL setting and suffer from the degeneracy issue. However, generalization to future tasks, which is our ultimate goal, is easy in this special case: after the initial demonstration, the agent can mimic it to behave optimally in all subsequent tasks without requiring further demonstrations.

More generally, if nature repeats similar tasks, then the agent obtains little new information, but presumably it knows how to behave in most cases; if nature chooses a task unfamiliar to the agent, then the agent is likely to err, but it may learn about \(\theta_*\) from the mistake.

To formalize this intuition, we consider the following protocol: the nature chooses a sequence of tasks \(\{(E_t, R_t)\}\) in an arbitrary manner. For every task \((E_t, R_t)\), the agent proposes a policy \(\pi_t\). The human examines the policy’s value under \(\mu_t\), and if the initial demonstration, the agent can mimic it to behave optimally in all subsequent tasks without requiring further demonstrations.

When we reduce MDPs to linear bandits, each element of \(D\) corresponds to an MDP policy, and the feature vector is the state occupancy of that policy.

As before, \(R, \theta_* \in \mathbb{R}^d\) are the task reward and the human’s unknown reward, respectively. The initial uncertainty set for \(\theta_*\) is \(\Theta_0 \subseteq [-1, 1]^d\). The value of the \(i\)-th action is calculated as \((\theta_* + R)^\top x^{(i)}\), and \(a^\star\) is the action that maximizes this value. Every round the agent proposes an action \(a \in D\), whose loss is defined as

\[
l_t = (\theta_* + R)^\top (x^{a^\star} - x^a).
\]

We now show how to embed the previous MDP setting in linear bandits.

Example 1. Given an MDP problem with variables \(S, A, \gamma, \theta_*, \Theta_0, \{E_t, R_t\}\), we can convert it into a linear bandit problem as follows. All variables with prime belong to the linear bandit problem, and we use \(v^\top\) to denote the vector \(v\) with the \(i\)-th coordinate removed.

- \(D = \{\pi : S \to A\}, d = |S| - 1\).
- \(\theta'_* = \Theta_0\), \(\Theta'_0 = \{\theta \in \Theta_0\}\).
- \(x^\pi = (\eta^\pi_{\mu_t, P_t})^{\text{ref}}\), \(R'_t = R_t^{\text{ref}}\).

Then for any sequence of policies chosen in the MDP problem, the corresponding sequence of actions in the linear bandit problem suffer exactly the same sequence of losses.

Note that there is a more straightforward conversion by letting \(d = |S|, \theta'_* = \theta_*\), \(\Theta_0 = \Theta_0\), \(x^\pi = \eta^\pi_{\mu_t, P_t}, R'_t = R_t\), which also preserves losses. We perform a more succinct conversion in Example 1 by canonicalizing both \(\theta_*\) (already assumed) and \(R_t\) (explicitly done here) and dropping the coordinate for \(s\text{ref}\) in all relevant vectors.

MDPs with linear rewards In IRL literature, a generalization of the MDP setting is often considered, that reward is linear in state features \(\phi(s) \in \mathbb{R}^d\) (Ng & Russell, 2000; Abbeel & Ng, 2004). In this new setting, \(\theta_*\) and \(R\) are reward parameters, and the actual reward is the dot product between the reward parameter and \(\phi(s)\). This new setting can also be reduced to linear bandits similarly to Example 1, except that the state occupancy is replaced by the discounted sum of expected feature values. Our main result, Theorem 2, will still apply automatically, but now the guarantee will only depend on the dimension of the feature space and has no dependence on \(|S|\). We include the conversion below but do not further discuss this setting in the rest of the paper.

Example 2. Consider an MDP problem with state features, defined by \(S, A, \gamma, d \in \mathbb{Z}^+, \theta_* \in \mathbb{R}^d, \Theta_0 \subseteq [-1, 1]^d, \{E_t, \phi_t \in \mathbb{R}^d, R_t \in \mathbb{R}^d\}\), where task reward and background reward in state \(s\) are \(\theta_* \phi_t(s)\) and \(R^\top \phi_t(s)\), respectively, and \(\theta_* \in \Theta_0\). Suppose \(\|\phi_t(s)\|_\infty \leq 1\) always.
Algorithm 1 Ellipsoid Algorithm for Repeated Inverse Reinforcement Learning

1: **Input:** $\Theta_0$.
2: $\Theta_1 := \text{MVEE}(\Theta_0)$.
3: for $t = 1, 2, \ldots$ do
4:   Nature reveals $(X_t, R_t)$.
5:   Learner plays $a_t = \arg\max_{a \in D} c_t^T x_t^a$ where $c_t$ is the center of $\Theta_t$.
6:   if $l_t > \epsilon$ then
7:     Human reveals $a_t$.
8:   else
9:     $\Theta_{t+1} = \text{MVEE}\{\{\theta \in \Theta_t : (\theta - c_t)^T (x_t^a - x_t^{a_t}) \geq 0\}\}$.
10: end if
11: $\Theta_{t+1} = \Theta_t$.
12: end for

holds, then we can convert it into a linear bandit problem as follows:

- $D = \{\pi : S \rightarrow A\}$; $d$, $\theta^*$, and $R_t$ remain the same.
- $x_t^a = (1 - \gamma) \sum_{h=1}^{\infty} \gamma^{h-1} E[\phi(s_h) | \mu_t, p_t, \pi_t]/d$.

Note that the division of $d$ in $x_t^a$ is for normalization purpose, so that $\|x_t^a\|/d \leq \|\phi\|_\infty \leq 1$.

6.2. Ellipsoid Algorithm for Repeated Inverse Reinforcement Learning

We propose Algorithm 1, and provide the mistake bound in the following theorem. Note that the pseudo-code also contains the formal protocol of the process.

**Theorem 2.** For $\Theta_0 = [-1, 1]^d$, the number of mistakes made by Algorithm 1 is guaranteed to be $O(d^2 \log(d/\epsilon))$.

To prove Theorem 2, we quote a result from linear programming literature in Lemma 1, which is found in standard lecture notes (e.g., O’Donnell 2011, Theorem 8.8; see also Grötschel et al. 2012, Lemma 3.1.34).

**Lemma 1 (Volume reduction in ellipsoid algorithm).** Given any non-degenerate ellipsoid $B$ in $\mathbb{R}^d$ centered at $c \in \mathbb{R}^d$, and any non-zero vector $v \in \mathbb{R}^d$, let $B^+$ be the minimum-volume enclosing ellipsoid (MVEE) of

$$\{u \in B : (u - c)^T v \geq 0\}.$$  

We have $\frac{\text{vol}(B^+)}{\text{vol}(B)} \leq e^{-\frac{\epsilon^2}{2(d+1)}}$.

**Proof of Theorem 2.** Whenever a mistake is made and the optimal action $a^*_t$ is revealed, we can induce the constraint $(R_t + \theta^*)^T (x_t^{a^*_t} - x_t^{a_t}) > \epsilon$. Meanwhile, since $a_t$ is greedy w.r.t. $c_t$, we have $(R_t + c_t)^T (x_t^{a^*_t} - x_t^{a_t}) \leq 0$, where $c_t$ is the center of $\Theta_t$ as in Line 5. Taking the difference of the two inequalities, we obtain

$$\langle \theta^* - c_t \rangle^T (x_t^{a^*_t} - x_t^{a_t}) > \epsilon. \quad (4)$$

Therefore, the update rule on Line 8 preserves $\theta^*$ in $\Theta_{t+1}$. Since the update makes a central cut through the ellipsoid, Lemma 1 applies and the volume shrinks by a multiplicative constant $e^{-\frac{\epsilon^2}{2(d+1)}}$ every time a mistake is made.

To prove the theorem, it remains to upper bound the initial volume and lower bound the terminal volume of $\Theta_t$. First we show that an update never eliminates $B_\infty(\theta^*, \epsilon/2)$, the $\ell_\infty$ ball centered at $\theta^*$ with radius $\epsilon/2$. This is because, any eliminated $\theta$ satisfies $\langle \theta + c_t \rangle^T (x_t^{a^*_t} - x_t^{a_t}) < 0$. Combining this with Equation 4, we have

$$\epsilon < \langle \theta^* - \theta \rangle^T (x_t^{a^*_t} - x_t^{a_t}) \leq \|\theta^* - \theta\|_\infty \|x_t^{a^*_t} - x_t^{a_t}\| \leq 2\|\theta^* - \theta\|_\infty.$$  

The last step follows from $\|x_t\|_1 \leq 1$. So we conclude that any eliminated $\theta$ should be $\epsilon/2$ far away from $\theta^*$ in $\ell_\infty$ distance. Therefore, we can lower bound the volume of $\Theta_t$ for any $t$ by that of $\Theta_0 \cap B_\infty(\theta^*, \epsilon/2)$, which contains an infinite-norm ball with radius $\epsilon/4$ in the worst case (when $\theta^*$ is one of $\Theta_0$’s vertices). To simplify calculation, we will further relax this $\ell_\infty$ ball to its inscribed $\ell_2$ ball.

Finally we put everything together: let $M_T$ be the number of mistakes made from round 1 to $T$, and $C_d$ be the volume of the unit sphere in $\mathbb{R}^d$, we have

$$\frac{M_T}{2(d+1)} \leq \log(\text{vol}(\Theta_1)) - \log(\text{vol}(\Theta_{T+1})) \leq \log(C_d(\sqrt{d})^d) - \log(C_d(\epsilon/4)^d) = d \frac{4\sqrt{d}}{\epsilon}.$$  

So $M_T \leq 2d(d+1) \log \frac{4\sqrt{d}}{\epsilon} = O(d^2 \log \frac{d}{\epsilon})$.

6.3. Lower bound

In Section 5, we get an $O(\log(1/\epsilon))$ upper bound on the number of demonstrations, which has no dependence on $|S|$ (which corresponds to $d + 1$ in linear bandits). Comparing Theorem 2 to 1, one may wonder whether the polynomial dependence on $d$ is an artifact of the inefficiency of Algorithm 1. We clarify this issue by proving a lower bound, showing that $O(\log(1/\epsilon))$ mistakes are inevitable in the worst case when nature chooses the tasks. We provide a proof sketch below, and the complete proof is deferred to Appendix D.

**Theorem 3.** For any randomized algorithm\(^3\) in the linear bandit setting, there always exists $\theta^* \in [-1, 1]^d$ and

\(^3\)While our Algorithm 1 is deterministic, randomization is often crucial for online learning in general (Shalev-Shwartz, 2011).
\{(X_t, R_t)\} which are fixed before the execution of the algorithm,\(^4\) such that the expected number of mistakes made by the algorithm under \(\theta_*\) and \(\{(X_t, R_t)\}\) is \(\Omega(d \log(1/\epsilon))\).

\textbf{Proof Sketch.} We randomize \(\theta\) by sampling each element i.i.d. from \(\text{Unif}([-1, 1])\). We will prove that there exists a strategy of choosing \((X_t, R_t)\) such that any algorithm’s expected number of mistakes is \(\Omega(d \log(1/\epsilon))\), which proves the theorem as max is no less than average.

In our construction, \(X_t = [0_d, e_{j_t}]\), where \(j_t\) is some index to be specified. Hence, every round the agent is essentially asked to decide whether \(\theta(j_t) \geq -R_t(j_t)\). The adversary’s strategy goes in phases, and \(R_t\) remains the same during each phase. Every phase has \(d\) rounds where \(j_t\) is enumerated over \([1, \ldots, d]\).

The adversary will use \(R_t\) to shift the posterior on \(\theta(j_t) + R_t(j_t)\) so that it is (approximately) centered around the origin; in this way, the agent has about 1/2 probability to make an error (regardless of the algorithm), and the posterior interval will be halved. Overall, the agent makes \(d/2\) mistakes in each phase, and there will be about \(\log(1/\epsilon)\) phases in total, which gives the lower bound. \(\square\)

\textbf{Applying the lower bound to MDPs} The above lower bound is stated for linear bandits. In principle, we need to prove lower bound for MDPs separately, because linear bandits are more general than MDPs for our purpose, and the hard instances in linear bandits may not have corresponding MDP instances. In Lemma 2, we show that a certain type of linear bandit instances can always be emulated by MDPs with the same number of actions, and the hard instances constructed in Theorem 3 indeed satisfy the conditions for such a type; in particular, we require the feature vectors to be non-negative and have \(\ell_1\) norm bounded by 1. As a corollary, an \(\Omega(|S| \log(1/\epsilon))\) lower bound for the MDP setting (even with a small action space \(|A| = 2\) follows directly from Theorem 3.

\textbf{Lemma 2 (Linear bandit to MDP conversion).} Let \((X, R)\) be a linear bandit task, and \(K\) be the number of actions. If every \(x_t\) is non-negative and \(\|x_t\|_1 \leq 1\), then there exists an MDP task \((E, R')\) with \(d + 1\) states and \(K\) actions, such that under some choice of \(s_{ref}\), converting \((E, R')\) as in Example 1 recovers the original problem.

The proof of this lemma is deferred to Appendix A.

\textbf{6.4. On identification when Nature Chooses Tasks} While Theorem 2 successfully controls the number of total mistakes, it completely avoids the identification problem and does not guarantee to recover \(\theta_*\). In this section we explore further conditions under which we can obtain identification guarantees when Nature chooses the tasks.

The first condition, stated in Proposition 2, implies that if we have made all the possible mistakes, then we have indeed identified the \(\theta_*\), where the identification accuracy is determined by the tolerance parameter \(\epsilon\) that defines what is counted as a mistake.

\textbf{Proposition 2. Consider the linear bandit setting. If there exists \(T_0\) such that for any round \(t \geq T_0\), no more mistakes can be ever made by the algorithm for any choice of \((E_t, R_t)\) and any tie-breaking mechanism, then we have \(\theta_* \in B_\epsilon(c_{T_0}, \epsilon)\).}

\textbf{Proof.} Assume towards contradiction that \(\|c_{T_0} - \theta_*\|_\infty > \epsilon\). We will choose \((R_t, x^{(1)}_t, x^{(2)}_t)\) to make the algorithm err. In particular, let \(R_t = -c_{T_0}\), so that the algorithm acts greedily with respect to \(0_d\). Since \(0_d x^{(2)}_t \equiv 0\), any action would be a valid choice for the algorithm.

On the other hand, \(\|c_{T_0} - \theta_*\|_\infty > \epsilon\) implies that there exists a coordinate \(j\) such that \(e_j^\top (\theta_* - c_{T_0}) > \epsilon\), where \(e_j\) is a basis vector. Let \(x^{(1)}_t = 0_d\) and \(x^{(2)}_t = e_j\). So the value of action 1 is always 0 under any reward function (including \(\theta_* + R_t\)), and the value of action 2 is \((\theta_* + R_t)^\top x^{(2)}_t = (\theta_* - c_{T_0})^\top e_j\), whose absolute value is greater than \(\epsilon\). At least one of the 2 actions is more than \(\epsilon\) suboptimal, and the algorithm may take any of them, so the algorithm can err again. \(\square\)

While Proposition 2 shows that identification is guaranteed if the agent exhausts the mistakes, the agent has no ability to actively fulfill this condition when Nature chooses tasks. For a stronger identification guarantee, we may need to grant the agent some freedom in choosing the tasks.

\textbf{Identification with fixed environment} Here we consider a setting that fits in between Section 5 (completely active) and Section 6.1 (completely passive), where the environment \(E\) (hence the induced feature vectors \(\{x^{(1)}, x^{(2)}, \ldots, x^{(K)}\}\)) is given and fixed, and the agent can arbitrarily choose the task reward \(R_t\). The goal is to obtain an identification guarantee in this new intermediate setting.

Unfortunately, a degenerate case can be easily constructed that prevents the revelation of any information about \(\theta_*\). In particular, if \(x^{(1)} = x^{(2)} = \ldots = x^{(K)}\), i.e., the environment is completely uncontrolled, then all actions are equally optimal and nothing can be learned.

More generally, if for some non-zero vector \(v\) we have \(v^\top x^{(1)} = v^\top x^{(2)} = \ldots = v^\top x^{(K)}\), then we may never recover \(\theta_*\) along the direction of \(v\). In fact, Proposition 1

\footnote{This means that the lower bound can be realized by an oblivious adversary, who cannot adapt the tasks to the realization of the random variables drawn by the algorithm.}
can be viewed as an instance of this result where \( v = 1_{|S|} \) (recall that the entries of the state occupancy vector always sum up to 1), and that is why we have to remove such redundancy in Example 1 in order to discuss identification in MDPs. Therefore, to guarantee identification in a fixed environment, the feature vectors must be substantially different in all directions, and we capture this intuition by defining a diversity score spread(\( X \)) (Definition 2) and showing that the identification accuracy depends inversely on the score (Theorem 4).

**Definition 2.** Given the feature matrix \( X = \begin{bmatrix} x^{(1)} & x^{(2)} & \cdots & x^{(K)} \end{bmatrix} \) whose size is \( d \times K \), define spread(\( X \)) as the \( d \)-th largest singular value of \( \bar{X} := X(I_K - \frac{1}{K}1_K1_K^\top) \).

**Theorem 4.** For a fixed feature matrix \( X \), if spread(\( X \)) > 0, then there exists a sequence \( R_1, R_2, \ldots, R_T \) with \( T = O(d^2 \log(d/\epsilon)) \) and a sequence of tie-break choices of the algorithm, such that after round \( T \) we have \( \| c_T - \theta_* \|_\infty \leq \frac{\epsilon \sqrt{(K-1)/2}}{\text{spread}(X)} \).

**Proof.** It suffices to show that in any round \( t \), if \( \| c_t - \theta_* \|_\infty > \frac{\epsilon \sqrt{(K-1)/2}}{\text{spread}(X)} \), then \( l_t > \epsilon \). The bound on \( T \) follows directly from Theorem 2. Similar to the proof of Proposition 2, our choice of the task reward is \( R_t = -c_t \), so that any \( a \in A \) would be a valid choice of \( a_t \), and we will choose the worst action. Note that for all \( a, a' \in D \),

\[
    l_t = (\theta_* + R_t)^\top (x^{a_t} - x^{a_t'}) \geq (\theta_* - c_t)^\top (x^a - x^a').
\]

So it suffices to show that there exists \( a, a' \in D \), such that \( (\theta_* - c_t)^\top (x^a - x^a') > \epsilon \). Let \( y_t = \theta_* - c_t \), and the precondition implies that \( \| y_t \|_2 \geq \| y_t \|_\infty \geq \frac{\epsilon \sqrt{(K-1)/2}}{\text{spread}(X)} \).

Define a matrix of size \( K \times (K - 1) \)

\[
    D = \begin{bmatrix}
    1 & 1 & \cdots & 0 \\
    -1 & 0 & \cdots & 0 \\
    0 & -1 & \cdots & 0 \\
    \vdots & & \ddots & \\
    0 & 0 & \cdots & -1 \\
    0 & 0 & \cdots & 1
    \end{bmatrix},
\]

Every column of this matrix contains exactly one 1 and one -1, and the columns enumerate all possible positions of them. With the help of this matrix, we can rewrite the desired result \((3 a, a' \in A, \text{s.t. } (\theta_* - c_t)^\top (x^a - x^a') > \epsilon)\) as \( \| y_t^\top XD \|_\infty \geq \epsilon \). We relax the LHS as \( \| y_t^\top XD \|_\infty \geq \| y_t^\top XD \|_2 / \sqrt{K(K-1)} \), and will provide a lower bound on \( \| y_t^\top XD \|_2 \). Note that

\[
    y_t^\top XD = y_t^\top (\bar{X} + (X - \bar{X}))D = y_t^\top \bar{X}D,
\]

because every row of \((X - \bar{X})\) is some multiple of \( 1_K^\top \) (recall Definition 2), and every column of \( D \) is orthogonal to \( 1_K \). Let \( \bar{c} \) be the vector normalized to unit length,

\[
    \| y_t^\top \bar{X}D \|_2 = \| y_t \|_2 \| y_t^\top \bar{X}D \|_2 = \| y_t \|_2 \| y_t^\top \bar{X} \|_2 \| y_t^\top XD \|_2.
\]

We lower bound each of the 3 terms. For the first term, we have the precondition \( \| y_t \|_2 > \frac{\epsilon \sqrt{(K-1)/2}}{\text{spread}(X)} \). The second term is \( \bar{X} \) left multiplied by a unit vector, so its \( \ell_2 \) norm can be lower bounded by the smallest non-zero singular value of \( \bar{X} \) (recall that \( \bar{X} \) is full-rank), which is spread(\( X \)).

To lower bound the last term, note that \( DD^\top = 2KI_K - 21_K1_K^\top \), and rows of \( \bar{X} \) are orthogonal to \( 1_K^\top \) and so is \( y_t^\top \bar{X} \), so

\[
    \| y_t^\top XD \|_2^2 \geq \inf_{\| z \|_2 = 1, z \in \pm 1_K} z^\top DD^\top z = \inf_{\| z \|_2 = 1, z \in \pm 1_K} z^\top (2KI_K - 21_K1_K^\top)z = \inf_{\| z \|_2 = 1, z \in \pm 1_K} 2Kz^\top z = 2K.
\]

Putting all the pieces together, we have

\[
    \| y_t^\top \bar{X}D \|_\infty \geq \| y_t \|_2 \| y_t^\top \bar{X} \|_2 \| y_t^\top XD \|_2 / \sqrt{d} > \frac{\epsilon \sqrt{(K-1)/2}}{\text{spread}(X)} \cdot \sqrt{2K} / \sqrt{K(K-1)} = \epsilon. \quad \Box
\]

The \( \sqrt{K} \) dependence in Theorem 4 may be of concern as \( K \) can be exponentially large. However, Theorem 4 also holds if we replace \( X \) by any matrix that consists of \( X \)’s columns, so we may choose a small yet most diverse set of columns as to optimize the bound. We also show in Appendix B that Theorem 4 is tight in the worst case.

**7. Working with trajectories**

In previous sections, we have assumed that the human evaluates the agent’s performance based on the state occupancy of the agent’s policy, and demonstrates the optimal policy in terms of state occupancy as well. In practice, we would like to instead assume that for each task, the agent rolls out a trajectory, and the human shows an optimal trajectory if he/she finds the agent’s trajectory unsatisfying. We are still concerned about upper bounding the number of total mistakes, and aim to provide a parallel version of Theorem 2.

Unlike in traditional IRL, in our setting the agent is also acting, which gives rise to many subtleties. First, the total reward on the agent’s single trajectory is a random variable, and may deviate from the expected value of its policy. Therefore, it is generally impossible to decide if the agent’s
policy is near-optimal, and instead we assume that the human can check if each action that the agent takes in the trajectory is near-optimal: when the agent takes action $a$ at state $s$, an error is counted if and only if $Q^\pi(s, a) < V^\pi(s) - \epsilon$.

While this resolves the issue on the agent’s side, how should the human provide his/her optimal trajectory? The most straightforward protocol is that the human rolls out a trajectory from the specified $\mu_t$. We argue that this is not a reasonable protocol for two reasons: (1) in expectation, the reward collected by the human may be less than that by the agent, which is due to us conditioning on the event that an error is spotted; (2) the human may not encounter the problematic state in his/her own trajectory, hence the information provided in the trajectory may be irrelevant.

To resolve this issue, we consider a different protocol where the human rolls out a trajectory using optimal policy from the very state where the agent errs.

Now we discuss how we can prove a parallel of Theorem 2 under this new protocol. First, let’s assume that the demonstration were still given in state occupancy induced by the optimal policy from the problematic state. In this case, we will not update $n$ and the terms of a single trajectory, we will not update $\theta$ with parameters $\Theta_t := \text{MVEE}(\Theta_t)$, $i \leftarrow 0$, $Z \leftarrow 0$, $Z^* \leftarrow 0$.

1. **Input:** $\Theta_0$, $H$, $n$.
   
   // variables ‘$'$ are converted as in Example 1.

2. $\Theta_t := \text{MVEE}(\Theta_t)$, $i \leftarrow 0$, $Z \leftarrow 0$, $Z^* \leftarrow 0$.

3. for $t = 1, 2, \ldots$ do
   
   4. Nature reveals $(E_t, R_t)$.
   
   5. Agent rolls-out a trajectory using $\pi_t$ greedily w.r.t. $c_t + R_t$, where $c_t$ is the center of $\Theta_t$.

6. if agent takes action $a$ in $s$ with $Q^\pi(s, a) < V^\pi(s) - \epsilon$ then
   
   7. Human produces an $H$-step trajectory from $s$, whose empirical state occupancy vector (excluding the $s_{ref}$ coordinate) is denoted as $\tilde{z}_{i:H}$.
   
   8. $i \leftarrow i + 1$, $Z^* \leftarrow Z^* + \tilde{z}_{i:H}$.
   
   9. Let $z_t$ be the state occupancy of $\pi_t$ from initial state $s$, and $Z \leftarrow Z + z_t$.

10. if $i = n$ then

11. \hspace{1em} $\Theta_{t+1} := \text{MVEE}\{(\theta \in \Theta_t : (\theta - c_t)^\top (Z^* - Z) \geq 0)\}$.

12. \hspace{1em} $i \leftarrow 0$, $Z \leftarrow 0$, $Z^* \leftarrow 0$.

13. else

14. \hspace{1em} $\Theta_{t+1} = \Theta_t$.

15. end if

16. else

17. \hspace{1em} $\Theta_{t+1} = \Theta_t$.

18. end if

19. end for

\[\tilde{O}(\frac{b^2 \log(\frac{d}{\epsilon})}{\epsilon}).\]

The proof of Theorem 5 is deferred to Appendix E.

### 8. Related work & Conclusions

Most existing work in IRL focused on inferring the reward function using data acquired from a fixed environment (Ng & Russell, 2000; Abbeel & Ng, 2004; Coates et al., 2008; Ziebart et al., 2008; Ramachandran & Amir, 2007; Syed & Schapire, 2007; Regan & Boutilier, 2010). There is prior work on using data collected from multiple — but exogenously fixed — environments to predict agent behavior (Ratliff et al., 2006). There are also applications where methods for single-environment MDPs have been adapted to multiple environments (Ziebart et al., 2008). Nevertheless, all these works consider the objective of mimicking an optimal behavior in the presented environment(s), and do not aim at generalization to new tasks.

In the economics literature, the problem of inferring an agent’s utility from behavior has long been studied under the heading of utility or preference elicitation (Chajewska et al., 2000; Von Neumann & Morgenstern, 2007; Regan & Boutilier, 2010).

$^3$A log $\log(1/\epsilon)$ term is suppressed in $\tilde{O}()$. 
Proceedings of the 25th international conference on Machine learning, pp. 144–151. ACM, 2008.

Grötschel, Martin, Lovász, László, and Schrijver, Alexander. Geometric algorithms and combinatorial optimization, volume 2. Springer Science & Business Media, 2012.

Hadfield-Menell, Dylan, Russell, Stuart J, Abbeel, Pieter, and Dragan, Anca. Cooperative inverse reinforcement learning. In Advances in Neural Information Processing Systems, pp. 3909–3917, 2016.

Ng, Andrew Y and Russell, Stuart J. Algorithms for inverse reinforcement learning. In Proceedings of the 17th International Conference on Machine Learning, pp. 663–670, 2000.

O’Donnell, Ryan. 15-859(E) – linear and semidefinite programming: lecture notes. Carnegie Mellon University, 2011. https://www.cs.cmu.edu/afs/cs.cmu.edu/academic/cl...s-2159, 2011.

Ramachandran, Deepak and Amir, Eyal. Bayesian inverse reinforcement learning. Urbana, 51:61801, 2007.

Ratliff, Nathan D, Bagnell, J Andrew, and Zinkevich, Martin A. Maximum margin planning. In Proceedings of the Twenty-Fifth Conference on Uncertainty in Artificial Intelligence, pp. 444–451. AUAI Press, 2009.

Regan, Kevin and Boutilier, Craig. Robust policy computation in reward-uncertain mdps using nondominated policies. In AAAI, 2010.

Regan, Kevin and Boutilier, Craig. Eliciting additive reward functions for markov decision processes. In IJCAI Proceedings-International Joint Conference on Artificial Intelligence, volume 22, pp. 2159, 2011.

Rothkopf, Constantin A and Dimitrakakis, Christos. Preference elicitation and inverse reinforcement learning. In Machine Learning and Knowledge Discovery in Databases, pp. 34–48. Springer, 2011.

Russell, Stuart, Dewey, Daniel, and Tegmark, Max. Research priorities for robust and beneficial artificial intelligence. AI Magazine, 36(4):105–114, 2015.

Shalev-Shwartz, Shai. Online learning and online convex optimization. Foundations and Trends in Machine Learning, 4(2):107–194, 2011.
Syed, Umar and Schapire, Robert E. A game-theoretic approach to apprenticeship learning. In *Advances in neural information processing systems*, pp. 1449–1456, 2007.

Von Neumann, John and Morgenstern, Oskar. *Theory of games and economic behavior (60th Anniversary Commemorative Edition)*. Princeton university press, 2007.

Ziebart, Brian D, Maas, Andrew L, Bagnell, J Andrew, and Dey, Anind K. Maximum entropy inverse reinforcement learning. In *AAAI*, pp. 1433–1438, 2008.
Appendix

A. Proof of Lemma 2

The construction is as follows. Choose \( s_{\text{ref}} \) as the initial state, and make all other states absorbing. Let \( R'(s_{\text{ref}}) = 0 \) and \( R' \) restricted on \( S \setminus \{s_{\text{ref}}\} \) coincide with \( R \). The remaining work is to design the transition distribution of each action in \( s_{\text{ref}} \) so that the induced state occupancy matches exactly one column of \( X \).

Fixing any action \( a \), and let \( x \) be the feature that we want to associate \( a \) with. The next-state distribution of \( (s_{\text{ref}}, a) \) is as follows: with probability \( p = \frac{1 - \|x\|_1}{1 - \gamma \|x\|_1} \), the next-state is \( s_{\text{ref}} \) itself, and the probability of transitioning to the \( j \)-th state in \( S \setminus \{s_{\text{ref}}\} \) is \( \frac{1 - \gamma}{1 - \gamma \|x\|_1} x(j) \). Given \( \|x\|_1 \leq 1 \) and \( x \geq 0 \), it is easy to verify that this is a valid distribution.

Now we calculate the occupancy of policy \( \pi(s_{\text{ref}}) = a \). The normalized occupancy on \( s_{\text{ref}} \) is

\[
(1 - \gamma)(p + \gamma p^2 + \gamma^2 p^3 + \cdots) = \frac{p(1 - \gamma)}{1 - \gamma p} = 1 - \|x\|_1.
\]

The remaining occupancy, with a total \( \ell_1 \) mass of \( \|x\|_1 \), is split among \( S \setminus \{s_{\text{ref}}\} \) proportional to \( x \). Therefore, when we convert the MDP problem as in Example 1, the corresponding feature vector is exactly \( x \), so we recover the original linear bandit problem. \( \square \)

B. Theorem 4 is tight in the worst case

We show that the theorem is tight up to a constant factor in the worst case. Let \( X = \begin{bmatrix} U & -U \end{bmatrix} \) where \( U_{d \times d} \) is any orthonormal matrix, so \( K = 2d \). This is a valid choice of \( X \) because its column vector \( x \) satisfies \( \|x\|_1 \leq \|x\|_\infty = 1 \). All \( d \) singular values of \( X \) are \( \sqrt{2} \), and \( X' = X \), so \( \text{spread}(X) = \sqrt{2} \), and the bound is \( \epsilon \sqrt{2d - 1}/2 = O(\sqrt{d}) \).

Since \( U \) is arbitrary, we choose its first row to be \( 1 \sqrt{d}/\sqrt{d} \). Then we choose an ellipsoid center \( c \) and \( \theta_* \) that are \( \epsilon \sqrt{d}/2 \) different from each other in \( \ell_\infty \) distance, and show that a mistake is impossible. In particular, let \( c \) be equal to \( \theta_* \) except on the first coordinate where they differ by \( \epsilon \sqrt{d}/2 \). Let \( a \) be the action taken by the algorithm and \( a^* \) be an optimal action, and \( R \) be any task reward, we have

\[
\text{loss} = (\theta_* + R)(a^* - x^a) \\
\leq (\theta_* + R)(a^* - x^a) - (c + R)(a^* - x^a) \\
= (\theta_* - c)(a^* - x^a) \\
= |\theta_* - c| \cdot \|a^*(1) - x^a(1)\|_\infty \\
\leq \epsilon \sqrt{d}/2 \cdot (2/\sqrt{d}) = \epsilon.
\]

In addition, note that the same construction also works if we rescale \( X \) with any multiplicative constant \( C \in (0, 1) \), hence the bound is tight in the worst case only not for \( \text{spread}(X) = \sqrt{d} \), but for a range \( \text{spread}(X) \in (0, \sqrt{2}] \).

C. Bounding the \( \ell_\infty \) distance between \( \theta_* \) and the ellipsoid center

To prove Theorem 5, we need an upper bound on \( ||\theta_* - c||_\infty \) for quantifying the error due to \( H \)-step truncation and sampling effects, where \( c \) is the ellipsoid center. As far as we know there is no standard result on this issue. However, a simple workaround, described below, allows us to assume \( ||\theta_* - c||_\infty \leq 2 \) without loss of generality.

Whenever \( ||c||_\infty > 1 \), there exists coordinate \( j \) such that \( |c_j| > 1 \). We can make a central cut \( e_j^T (\theta - c) < 0 \) or \( > 0 \) depending on the sign of \( c_j \), and replace the original ellipsoid with the MVEE of the remaining shape. This operation never excludes any point in \( \Theta_0 \), hence it allows the proofs of Theorem 2 and 5 to work. We keep making such cuts and update the ellipsoid accordingly, until the new center satisfies \( ||c||_\infty \leq 1 \). Since central cuts reduce volume substantially (Lemma 1) and there is a lower bound on the volume, the process must stop after finite number of operations. After the process stops, we have \( ||\theta_* - c||_\infty \leq ||\theta_*||_\infty + ||c||_\infty \leq 2 \).

D. Proof of Theorem 3

As a standard trick, we randomize \( \theta_* \) by sampling each element i.i.d. from \( \text{Unif}([-1, 1]) \). We will prove that there exists a strategy of choosing \( (X_t, R_t) \) such that any algorithm’s expected number of mistakes is \( \omega(d \log(1/\epsilon)) \), where the expectation is with respect to the randomness of \( \theta_* \) and the internal randomness of the algorithm. This immediately implies a worst-case result as max is no less than average (regarding the sampling of \( \theta_* \)).

In our construction, \( X_t = [0_d, e_j] \), where \( j_t \) is some index to be specified. Hence, every round the agent is essentially asked to decide whether \( \theta(j_t) \geq -R_t(j_t) \). The adversary’s strategy goes in phases, and \( R_t \) remains the same during each phase. Each phase has \( d \) rounds where \( j_t \) is enumerated over \( \{1, \ldots, d\} \). To fully specify the nature’s strategy, it remains to specify \( R_t \) for each phase.

In the 1st phase, \( R_t \equiv 0 \). For each coordinate \( j \), the information revealed to the agent is one of the following: \( \theta(j) > \epsilon, \theta(j) \geq -\epsilon, \theta(j) < -\epsilon, \theta(j) \leq \epsilon \). For clarity we first make an simplification, that the revealed information is either \( \theta_j > 0 \) or \( \theta_j \leq 0 \); we will deal with the subtleties related to \( \epsilon \) at the end of the proof.

In the 2nd phase, we fix \( R_t \) as

\[
R_t(j) = \begin{cases} 
-1/2 & \text{if } \theta(j) \geq 0, \\
1/2 & \text{if } \theta(j) < 0.
\end{cases}
\]
Since $\theta_*$ is randomized i.i.d. for each coordinate, the posterior of $\theta_* + R_t$ conditioned on the revealed information is $\text{Unif}[-1/2, 1/2]$, for any algorithm and any interaction history. Therefore the 2nd phase is almost identical to the 1st phase except that the intervals have shrunk by a factor of 2. Similarly in the 3rd phase we use $R_t$ to offset the posterior of $\theta_* + R_t$ to $\text{Unif}[-1/4, 1/4]$, and so on.

In phase $m$, the half-length of the interval is $2^{-m+1}$, and the probability that a mistake occurs is at least $1/2 - \epsilon/2^{-m+2}$ for any algorithm. The whole process continues as long as this probability is greater than 0. By linearity of expectation, we can lower bound the total mistakes by the sum of expected mistakes in each phase, which gives

$$
\sum_{2^{-m+1} \geq \epsilon} d(1/2 - \epsilon/2^{-m+2}) \geq \sum_{2^{-m+1} \geq \epsilon} d \cdot 1/4 \geq \lfloor \log_2(1/\epsilon) \rfloor d / 4.
$$

The above analysis made a simplification that the posterior of $\theta_* + R_t$ in phase $m$ is $[-2^{-m+1}, 2^{-m+1}]$. We now remove the simplification. Note, however, that the actual posterior cannot be too different from this simplified version, and their end points can differ by at most $\epsilon$. So the error probability is at least $1/2 - 2\epsilon/(2^{-m+2} - 2\epsilon)$. The rest of the analysis is similar: we count the number of mistakes until the error probability drops below $1/4$, and in each of these phases we get at least $d/4$ mistakes in expectation. The number of such phases is given by

$$
1/2 - 2\epsilon/(2^{-m+2} - 2\epsilon) \geq 1/4,
$$

which is satisfied if $2^{-m+2} \geq 6\epsilon$, so $m \geq \lfloor \log_2 \frac{2}{6\epsilon} \rfloor$. This completes the proof.

E. Proof of Theorem 5

Since the update rule is still in the format of a central cut through the ellipsoid, Lemma 1 applies. It remains to show that the update rule preserves $\theta_*$ and a certain volume around it, and then we can follow the same argument as for Theorem 2.

Fixing a mini-batch, let $t_0$ be the round on which the last update occurs, and $\Theta = \Theta_{t_0}, c = c_{t_0}$. Note that $\Theta_t = \Theta$ during the collection of the current mini-batch and does not change, and $c_t = c$ similarly.

For each $i = 1, 2, \ldots, n$, define $z_{i}^{*,H}$ as the expected value of $z_i^{*,H}$, where expectation is with respect to the randomness of the trajectory produced by the human, and let $z_i^*$ be the infinite-step expected state occupancy. Note that $z_i^{*,H}, z_i^*, z_{i}^{*} \in \mathbb{R}^{|S|-1}$ because the occupancy on $s_{\text{ref}}$ is not included.

As before, we have $\theta_*^T (z_i^* - z_i) > \epsilon$ and $c^T (z_i^* - z_i) \leq 0$, so $(\theta_* - c)^T (z_i^* - z_i) > \epsilon$. Taking average over $i$, we get

$$(\theta_* - c)^T (\frac{1}{n} \sum_{i=1}^{n} z_i^* - \frac{1}{n} \sum_{i=1}^{n} z_i) > \epsilon.$$  

What we will show next is that $(\theta_* - c)^T (\frac{\hat{Z}^* - \hat{Z}}{n}) > \epsilon/3$ for $\hat{Z}^*$ and $\hat{Z}$ on Line 11, which implies that the update rule is valid and has enough slackness for lower bounding the volume of $\Theta_i$ as before. Note that

$$(\theta_* - c)^T (\frac{\hat{Z}^* - \hat{Z}}{n}) = (\theta_* - c)^T (\frac{1}{n} \sum_{i=1}^{n} z_i^* - \frac{1}{n} \sum_{i=1}^{n} z_i) - (\theta_* - c)^T (\frac{1}{n} \sum_{i=1}^{n} z_i^{*,H} - \frac{1}{n} \sum_{i=1}^{n} z_i^{*,H}) - (\theta_* - c)^T (\frac{1}{n} \sum_{i=1}^{n} z_i^* - \frac{1}{n} \sum_{i=1}^{n} z_i^*)$$

Here we decompose the expression of interest into 3 terms. The 1st term is lower bounded by $\epsilon$ as shown above, and we will upper bound each of the remaining 2 terms by $\epsilon/3$.

For the 2nd term, since $\|z_i^* - z_i^*\|_1 \leq \gamma^H$, the $\ell_1$ norm of the average follows the same inequality due to convexity, and we can bound the term using Hölder's inequality given $\|\theta_* - c\|_\infty \leq 2$ (see details of this result in Appendix C).

To verify that the choice of $H$ in the theorem statement is appropriate, we can upper bound the 2nd term as

$$2\gamma^H = 2((1 - (1 - \gamma))^{1/2}) log(6/e) \leq 2e^{-log(6/e)} = \frac{2}{\epsilon^2}.$$

For the 3rd term, fixing $\theta_*$ and $c$, the partial sum $\sum_{i=1}^n (\theta_* - c)^T (\hat{z}_i^* - z_i^*)$ is a martingale. Since $\|z_i^* - z_i^*\|_1 \leq 1$, $\|z_i^{*,H} - z_i^{*,H}\|_1 \leq 1$, and $\|\theta_* - c\|_\infty \leq 2$, we can initiate Lemma 3 by letting $b = 4$, and setting $n$ to sufficiently large to guarantee that the 3rd term is upper bounded by $\epsilon/3$ with high probability.

Given $(\theta_* - c)^T (\frac{\hat{Z}^* - \hat{Z}}{n}) > \epsilon/3$, we can follow exactly the same analysis as for Theorem 2 to show that $B_{\infty}(\theta, \epsilon/6)$ is never eliminated, and the number of updates can be bounded by $2d(d + 1) log \frac{12\sqrt{\epsilon}}{d \epsilon}$. The number of total mistakes is the number of updates multiplied by $n$, the size of the mini-batches. Via Lemma 3, we can verify that the choice of $n$ in the theorem statement satisfies $\|\sum_{i=1}^n (\theta_* - c)^T (\hat{z}_i^* - z_i^*)\| \leq n\epsilon/3$ with probability at least $1 - \delta/ \left( 2d(d + 1) log \frac{12\sqrt{\epsilon}}{d \epsilon} \right)$. Union bounding over all updates and the total failure probability can be bounded by $\delta$. 

}\]