On quasi-Poisson homogeneous spaces of quasi-Poisson Lie groups

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1 Introduction

The notion of Poisson Lie group and its infinitesimal counterpart, Lie bialgebra, was introduced by Drinfeld [4]. Later it was explained that these objects are quasiclassical limits of Hopf QUE algebras. In [5] the more general objects, quasi-Hopf QUE algebras, were introduced along with their quasiclassical limits, Lie quasi-bialgebras. The corresponding geometric objects, quasi-Poisson Lie groups, were first studied by Kosmann-Schwarzbach [8].

It is well known that Lie bialgebra structures on $\mathfrak{g}$ are in a natural 1-1 correspondence with Lie algebra structures on $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ such that $\mathfrak{g}$ and $\mathfrak{g}^*$ are subalgebras in $D(\mathfrak{g})$ and the natural bilinear form on $D(\mathfrak{g})$ is invariant. Respectively, in order to get a Lie quasi-bialgebra structure on $\mathfrak{g}$, one should drop the condition that $\mathfrak{g}^*$ is a subalgebra in $D(\mathfrak{g})$.

Along with (quasi-)Poisson Lie groups it is natural to study their (quasi-)Poisson actions [1, 2] and, in particular, (quasi-)Poisson homogeneous spaces. Drinfeld in [6] presented an approach to the classification of Poisson homogeneous spaces. Namely, he showed that if $G$ is a Poisson Lie group, $\mathfrak{g}$ is the corresponding Lie bialgebra, then the isomorphism classes of Poisson homogeneous $G$-spaces are essentially in a 1-1 correspondence with the $G$-orbits of Lagrangian subalgebras in $D(\mathfrak{g})$.

The main goal of this paper is to generalize this result to the quasi-Poisson case (see Theorem [3]). We also study the behavior of quasi-Poisson homogeneous spaces under twisting. Some examples showing the technique of Lagrangian subalgebras are also provided.

It also turns out that quasi-Poisson homogeneous spaces, as well as Poisson ones, are related to solutions of the classical dynamical Yang-Baxter equation (see [7, 10] for the Poisson case). This topic will be discussed in a forthcoming paper.

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2 Preliminaries

2.1 Notation

We will use the following normalization of the wedge product of multivector fields on a smooth manifold. If \( v \) is a \( m \)-vector field, \( w \) is a \( n \)-vector field, then

\[
v \wedge w = \frac{1}{n!m!} \text{Alt}(v \otimes w),
\]

where

\[
\text{Alt}(x_1 \otimes x_2 \otimes \cdots \otimes x_k) = \sum_{\sigma \in S_k} \text{sign}(\sigma)x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(k)}.
\]

We will denote by \([ , ]\) the Schouten bracket of multivector fields (see, e.g., [2]).

Let \( G \) be a Lie group, \( \mathfrak{g} = \text{Lie} G \) its Lie algebra. For any \( v \in \bigwedge^k \mathfrak{g} \) denote by \( v^\lambda \) (resp. \( v^\rho \)) the left (resp. right) invariant multivector field that corresponds to \( v \), i.e., \( v^\lambda(g) = (l_g)_*^{} v \), \( v^\rho(g) = (r_g)_*^{} v \) for all \( g \in G \), where \( l_g \) (resp. \( r_g \)) is the left (resp. right) translation by \( g \).

Suppose that \( G \) acts smoothly on a smooth manifold \( X \). Then for any \( v \in \mathfrak{g} \) we denote by \( v_X \) the corresponding vector field on \( X \), i.e.,

\[
(v_X f)(x) = \frac{d}{dt}\Big|_{t=0} f(\exp tv \cdot x)
\]

for any \( x \in X \). Similarly, for \( v \in \bigwedge^k \mathfrak{g} \) one can define the multivector field \( v_X \). For any \( x \in X \) consider the map \( \rho_x : G \to X \), \( \rho_x(g) = g \cdot x \). Then \( (\rho_x)_*^{} v = v_X(x) \) for \( v \in \mathfrak{g} \).

For any point \( x \in X \) we denote by \( H_x = \{ g \in G \mid g \cdot x = x \} \) its stabilizer. Let \( \mathfrak{h}_x = \text{Lie} H_x \subset \mathfrak{g} \).

Suppose now that \( X \) is a homogeneous \( G \)-space. In this case we will identify \( T_xX \) with \( \mathfrak{g}/\mathfrak{h}_x \) for all \( x \in X \). Fix \( x \in X \) and for any \( f \in \mathcal{C}^\infty X \) define \( f^G \in \mathcal{C}^\infty G \) by the formula \( f^G(g) = f(\rho_x)(g) = f(g \cdot x) \). Note that the mapping \( f \mapsto f^G \) is an isomorphism between the spaces of smooth functions on \( X \) and right \( H_x \)-invariant smooth functions on \( G \).

2.2 Quasi-Poisson Lie groups and quasi-Poisson actions

Following [1], we define the notion of quasi-Poisson Lie group and the notion of quasi-Poisson action.

Definition 1. Let \( G \) be a Lie group, \( \mathfrak{g} \) its Lie algebra, \( P_G \) a bivector field on \( G \), and \( \varphi \in \bigwedge^3 \mathfrak{g} \). A triple \((G, P_G, \varphi)\) is called a quasi-Poisson Lie group.
\[ PG \text{ is multiplicative, i.e., } PG(gg') = (l_g)_*PG(g') + (r_{g'})_*PG(g), \]  
(1)  
\[ \frac{1}{2}[PG, PG] = \varphi^\rho - \varphi^\lambda, \]  
(2)  
\[ [PG, \varphi^\rho] = 0. \]  
(3)

The notion of Poisson Lie group is a special case of the notion of quasi-Poisson Lie group. Namely, for any Poisson Lie group \((G, PG)\) the triple \((G, PG, 0)\) is a quasi-Poisson Lie group.

Consider the mapping \(\eta : G \to \mathfrak{g} \wedge \mathfrak{g}\) defined by
\[
\eta(g) = (r_{g}^{-1})_*PG(g).
\]
It is a \(\mathfrak{g} \wedge \mathfrak{g}\)-valued 1-cocycle of \(G\) with respect to the adjoint action of \(G\) on \(\mathfrak{g} \wedge \mathfrak{g}\), i.e.,
\[
\eta(g_1 g_2) = \eta(g_1) + \text{Ad}_{g_1} \eta(g_2).
\]

Here \(\text{Ad}_{g}(x \otimes y) = (\text{Ad}_{g} x) \otimes (\text{Ad}_{g} y)\). The cocyclicity of \(\eta\) is equivalent to the multiplicativity condition \(1\).

Consider \(\delta = d\eta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}\). It is a 1-cocycle of \(\mathfrak{g}\) with respect to the adjoint action of \(\mathfrak{g}\) on \(\mathfrak{g} \wedge \mathfrak{g}\), i.e.,
\[
\delta([x, y]) = \text{ad}_x \delta(y) - \text{ad}_y \delta(x),
\]
where \(\text{ad}_x(y \otimes z) = [x \otimes 1 + 1 \otimes x, y \otimes z] = \text{ad}_x y \otimes z + y \otimes \text{ad}_x z\).

**Definition 2.** Suppose \((G, PG, \varphi)\) is a quasi-Poisson Lie group, \(G\) acts smoothly on a smooth manifold \(X\), \(PX\) is a bivector field on \(X\). The action of \(G\) on \(X\) is called quasi-Poisson if
\[
PX(gx) = (l_g)_*PX(x) + (\rho_x)_*PG(g),
\]  
(4)  
\[
\frac{1}{2}[PX, PX] = \varphi_X
\]  
(5)

(here \(l_g\) denotes the mapping \(x \mapsto g \cdot x\)).

Let us consider the case \(\varphi = 0\), i.e., \(G\) is a Poisson Lie group. Then the condition \(5\) means that \(X\) is a Poisson manifold, and from \(4\) it follows that the action of \(G\) on \(X\) is Poisson.

**Definition 3.** Suppose that \((G, PG, \varphi)\) is a quasi-Poisson group, \(G\) acts smoothly on a manifold \(X\) equipped with a bivector field \(PX\), and this action is quasi-Poisson. We call \(X\) a quasi-Poisson homogeneous \(G\)-space if the action of \(G\) on \(X\) is transitive.

**Lemma 1.** Suppose that \((G, PG, \varphi)\) is a quasi-Poisson group, \(X\) is a homogeneous \(G\)-space, \(PX\) is a bivector field on \(X\). Then the condition \(4\) is equivalent to
\[
P_X(gx) = \text{Ad}_g PX(x) + \eta(g),
\]  
(6)
where \( \text{Ad}_g : \bigwedge^2 (\mathfrak{g}/h_x) \to \bigwedge^2 (\mathfrak{g}/h_{gx}) \) is the isomorphism of the vector spaces induced by the automorphism \( \text{Ad}_g : \mathfrak{g} \to \mathfrak{g} \), and \( \eta(g) \) is the image of \( \eta(g) \) in \( \bigwedge^2 (\mathfrak{g}/h_{gx}) \).

### 2.3 Lie quasi-bialgebras

Recall that a Poisson Lie structure on a Lie group \( G \) induces the structure of a Lie bialgebra on the Lie algebra \( \mathfrak{g} = \text{Lie} G \). A quasi-Poisson structure on a Lie group \( G \) induces a similar structure on \( \mathfrak{g} \). We follow [5] in defining the notion of Lie quasi-bialgebra.

**Definition 4.** Let \( \mathfrak{g} \) be a Lie algebra, \( \delta \in \mathfrak{g} \wedge \mathfrak{g} \) a \( \mathfrak{g} \wedge \mathfrak{g} \)-valued 1-cocycle of \( \mathfrak{g} \), and \( \varphi \in \bigwedge^3 \mathfrak{g} \). A triple \((\mathfrak{g}, \delta, \varphi)\) is called a **Lie quasi-bialgebra** if

\[
\frac{1}{2} \text{Alt}(\delta \otimes \text{id}) \delta(x) = \text{ad}_x \varphi \quad \text{for any } x \in \mathfrak{g},
\]

\[
\text{Alt}(\delta \otimes \text{id} \otimes \text{id}) \varphi = 0,
\]

where \( \text{ad}_x(a \otimes b \otimes c) = [x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x, a \otimes b \otimes c] \).

The equation (7) is called the quasi co-Jacobi identity.

If we set \( \varphi = 0 \), then the notion of Lie quasi-bialgebra coincides with the notion of Lie bialgebra. In this case the equation (7) becomes the ordinary co-Jacoby identity, and the condition (8) is obviously satisfied.

For any quasi-Poisson Lie group \((G, P_G, \varphi)\) there exists a Lie quasi-bialgebra structure on \( \mathfrak{g} \) given by the 1-cocycle \( \delta = d e \eta \) and \( \varphi \). Conversely, to any Lie quasi-bialgebra there corresponds a unique connected and simply connected quasi-Poisson Lie group (see [9]).

Given any linear map \( \delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g} \) we can define the skew-symmetric bilinear operation on \( \mathfrak{g}^* \): for all \( l, m \in \mathfrak{g}^* \) set \( [l, m]_\delta = \delta^*(l \otimes m) \).

Recall that for any Lie quasi-bialgebra \((\mathfrak{g}, \delta, \varphi)\) one can construct the so-called double Lie algebra \( \mathcal{D}(\mathfrak{g}) \) (see [3]):

1. \( \mathcal{D}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^* \) as a vector space;
2. define the bilinear operation \([ , ]_{\mathcal{D}(\mathfrak{g})}\) on \( \mathcal{D}(\mathfrak{g}) \) by the following conditions:
   1. \([a, b]_{\mathcal{D}(\mathfrak{g})} = [a, b] \quad \text{for } a, b \in \mathfrak{g};\)
   2. \([l, m]_{\mathcal{D}(\mathfrak{g})} = [l, m]_\delta - (l \otimes m \otimes \text{id}) \varphi \quad \text{for } l, m \in \mathfrak{g}^*;\)
   3. \([a, l]_{\mathcal{D}(\mathfrak{g})} = \text{coad}_a l - \text{coad}_l a \quad \text{for } a \in \mathfrak{g}, l \in \mathfrak{g}^*.\)

where \( \text{coad}_l : \mathfrak{g} \to \mathfrak{g} \) is defined by \( \langle \text{coad}_l a, m \rangle = -\langle [l, m]_\delta, a \rangle = -\langle l \otimes m, \delta(a) \rangle \), and \( \text{coad}_a : \mathfrak{g}^* \to \mathfrak{g}^* \) is defined by \( \langle \text{coad}_a l, b \rangle = -\langle l, [a, b] \rangle \). Here and below \( \langle , \rangle \) denotes the standard pairing between \( \mathfrak{g} \) and \( \mathfrak{g}^* \).

We denote by \( Q( , ) \) the following invariant symmetric bilinear form on \( \mathcal{D}(\mathfrak{g}) \):

\[
Q(a + l, b + m) = \langle l, b \rangle + \langle m, a \rangle.
\]
Suppose \( G \) is a quasi-Poisson Lie group, \( \mathfrak{g} \) is the corresponding Lie quasi-bialgebra, \( \mathcal{D}(\mathfrak{g}) \) is its double Lie algebra. Then the adjoint action of \( G \) on \( \mathfrak{g} \) can be extended to the action of \( G \) on \( \mathcal{D}(\mathfrak{g}) \) defined by

\[
g \cdot (a + l) = \text{Ad}_g a + (l' \otimes \text{id})\eta(g) + l',
\]

where \( l' = (\text{Ad}_g^{-1})^*l \). The differential of this action is the adjoint action of \( \mathfrak{g} \) on \( \mathcal{D}(\mathfrak{g}) \).

### 3 Main results

In [6] the characterization of all Poisson homogeneous structures on a given homogeneous \( G \)-space in terms of Lagrangian subalgebras in \( \mathcal{D}(\mathfrak{g}) \) is presented. We generalize this result to the quasi-Poisson case.

Suppose \( G \) is a quasi-Poisson Lie group, \( X \) is a quasi-Poisson homogeneous \( G \)-space. Recall that we identify \( T_xX \) and \( \mathfrak{g}/\mathfrak{h}_x \) for all \( x \in X \). For any \( x \in X \) define

\[
L_x = \left\{ a + l \mid a \in \mathfrak{g}, \ l \in (\mathfrak{g}/\mathfrak{h}_x)^* = \mathfrak{h}_x^* \subset \mathfrak{g}^*, \ (l \otimes \text{id})P_X(x) = \overline{a} \right\},
\]

where \( \overline{a} \) is the image of \( a \) in \( \mathfrak{g}/\mathfrak{h}_x \).

**Lemma 2.** \( L_x \) is Lagrangian (that is, maximal isotropic) subspace in \( \mathcal{D}(\mathfrak{g}) \), and \( L_x \cap \mathfrak{g} = \mathfrak{h}_x \). \( \square \)

Denote by \( \Lambda \) the set of all Lagrangian subalgebras in \( \mathcal{D}(\mathfrak{g}) \).

**Theorem 3.** Suppose \( (G, P_G, \varphi) \) is a quasi-Poisson Lie group, \( (X, P_X) \) is a quasi-Poisson homogeneous \( G \)-space. Then the following statements hold:

1. \( L_x \) is a subalgebra in \( \mathcal{D}(\mathfrak{g}) \) for all \( x \in X \);
2. \( L_{gx} = g \cdot L_x \);
3. Thus we get a bijection between the set of all \( G \)-quasi-Poisson structures on \( X \) and the set of \( G \)-equivariant maps \( x \mapsto L_x \) from \( X \) to \( \Lambda \) such that \( L_x \cap \mathfrak{g} = \mathfrak{h}_x \) for all \( x \in X \).

**Corollary 4.** There is a bijection between the set of all isomorphism classes of quasi-Poisson homogeneous \( G \)-spaces and the set of \( G \)-conjugacy classes of pairs \( (L, H) \), where \( L \subset \mathcal{D}(\mathfrak{g}) \) is a Lagrangian subalgebra, \( H \) is a closed subgroup in \( G_L = \{ g \in G \mid g \cdot L = L \} \), and \( L \cap \mathfrak{g} = \text{Lie} H \). \( \square \)

The rest of this section is devoted to the proof of Theorem 3. We start with a technical lemma.
Lemma 5. Let $P$ be a bivector field on a smooth manifold $X$. Define \( \{ f_1, f_2 \} = P(df_1, df_2) \) for all $f_1, f_2 \in C^\infty X$. Then
\[
\oint \{ \{ f_1, f_2 \}, f_3 \} = -\frac{1}{2} \left[ [P, P] (df_1, df_2, df_3) \right],
\]
where $\oint$ denotes the sum over all cyclic permutations of $f_1, f_2, f_3$.

Proof. Straightforward computation.

Lemma 6. $L_{gx} = g \cdot L_x$ iff (4) holds.

Proof. By definition,
\[
L_x = \{ a + l \mid a \in \mathfrak{g}, l \in (\mathfrak{g}/\mathfrak{h}_x)^*, (l \otimes \text{id})P_X(x) = \overline{a} \},
\]
\[
L_{gx} = \left\{ a' + l' \mid a' \in \mathfrak{g}, l' \in (\mathfrak{g}/\mathfrak{h}_{gx})^*, (l' \otimes \text{id})P_X(gx) = \overline{a'} \right\}.
\]
It is enough to check that
\[
g \cdot L_x = \left\{ a' + l' \mid a' \in \mathfrak{g}, l' \in (\mathfrak{g}/\mathfrak{h}_{gx})^*, (l' \otimes \text{id})(\text{Ad}_g P_X(x) + \eta(g)) = \overline{a'} \right\}.
\]
Consider $a' + l' = g \cdot (a + l), a \in \mathfrak{g}, l \in (\mathfrak{g}/\mathfrak{h}_x)^*$, that is,
\[
l' = (\text{Ad}_g^{-1})^* l, \quad a' = \text{Ad}_g a + (l' \otimes \text{id})\eta(g).
\]
We have
\[
(l' \otimes \text{id})(\text{Ad}_g P_X(x) + \eta(g)) = (l \otimes \text{id})(\text{Ad}_g^{-1} \otimes \text{id})(\text{Ad}_g \otimes \text{Ad}_g) P_X(x) + (l' \otimes \text{id})\eta(g) = \text{Ad}_g(l \otimes \text{id}) P_X(x) + (l' \otimes \text{id})\eta(g).
\]
So $(l' \otimes \text{id})(\text{Ad}_g P_X(x) + \eta(g)) = \overline{a'}$ if and only if $a + l \in L_x$. This proves the required equality.

Now we are heading for the first statement of the theorem.

Let $e_i$ form a basis in $\mathfrak{g}$, $\partial_i$ (resp. $\partial'_i$) be the right (resp. left) invariant vector field on $G$ that corresponds to $e_i$.

Suppose $\eta(g) = \eta^{ij}(g)e_i \wedge e_j$. Then $P_G = \eta^{ij} \partial_i \wedge \partial_j$. Choose any $r \in \bigwedge^2 \mathfrak{g}$ such that the image of $r$ in $\bigwedge^2 (\mathfrak{g}/\mathfrak{h}_x)$ equals $P_X(x)$. Define
\[
\text{CYB}(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}].
\]

Lemma 7. Assume that (4) holds. Then the image of
\[
\varphi - \text{CYB}(r) + \frac{1}{2} \text{Alt}(\delta \otimes \text{id})(r)
\]
in $\bigwedge^3 (\mathfrak{g}/\mathfrak{h}_x)$ vanishes iff (5) holds.
Proof. From (4) it follows that
\[ P_X(gx)(d_{gx}f_1, d_{gx}f_2) = ((l_g)_*P_X(x) + (\rho_x)_*P_G(g)) (d_{gx}f_1, d_{gx}f_2) = P_X(x)(d_x(f_1 \circ l_g), d_x(f_2 \circ l_g)) + P_G(g)(d_g(f_1 \circ \rho_x), d_g(f_2 \circ \rho_x)) =\]
\[ r(d_x(f_1 \circ l_g)^G, d_x(f_2 \circ l_g)^G) + P_G(g)(d_gf_1^G, d_gf_2^G) = (r^\lambda(g) + P_G(g))(d_gf_1^G, d_gf_2^G).\]

For any \( f_1, f_2 \in C^\infty G \) define the bracket
\[ \{f_1, f_2\}(g) = (r^\lambda(g) + P_G(g))(d_gf_1, d_gf_2).\]

Using Lemma 3 we see that
\[ \oint \{f_1, f_2\} x, f_3) x (g \cdot x) = \oint \{f_1^G, f_2^G\}, f_3^G\}(g) = -\frac{1}{2}[P_G + r^\lambda; P_G + r^\lambda](df_1^G, df_2^G, df_3^G)(g).\]

Lemma 8. \([P_G + r^\lambda, P_G + r^\lambda] = 2(\varphi^o - \varphi^\lambda + \text{CYB}(r)^\lambda - \frac{1}{7} \text{Alt}(\delta \otimes \text{id})(r)^\lambda)\]

Proof. Using the graded anticommutativity of Schouten bracket, we get
\[ [P_G + r^\lambda, P_G + r^\lambda] = [P_G, P_G] + 2[P_G, r^\lambda] + [r^\lambda, r^\lambda].\]

From (2) it follows that
\[ [P_G, P_G] = 2(\varphi^o - \varphi^\lambda).\]

We will calculate the rest of the terms on the right hand side using coordinates. Let \( r = r^{ij}e_i \wedge e_j \). Then \( r^\lambda = r^{ij}\partial_i^l \wedge \partial_j^l \), and
\[ [r^\lambda, r^\lambda] = -4r^{\mu\nu}r^{ij}[[\partial^l_{\mu}, \partial^l_{\nu}] \wedge \partial_i^l \wedge \partial_j^l =\]
\[ -4r^{\mu\nu}r^{ij} \text{Alt} (\partial_i^l, \partial_j^l) = -\frac{1}{2} \text{Alt}(\delta \otimes \text{id})(r)^\lambda = 2 \text{CYB}(r)^\lambda.\]

Now we prove that \([P_G, r^\lambda] = -\frac{1}{2} \text{Alt}(\delta \otimes \text{id})(r)^\lambda\). We have
\[ [P_G, r^\lambda] = [\eta^{\mu\nu}\partial_{\mu} \wedge \partial_{\nu}, r^{ij}\partial_i^l \wedge \partial_j^l] =\]
\[ r^{ij}(\partial_i^l, \eta^{\mu\nu}\partial_{\mu} \wedge \partial_{\nu} - [\partial_i^l, \eta^{\mu\nu}\partial_{\mu}] \wedge \partial_j^l \wedge \partial_{\nu}) = 2r^{ij}(\partial_i^l, \eta^{\mu\nu}\partial_{\mu} \wedge \partial_{\nu} \wedge \partial_j^l).\]

Using the cyclicity of \( \eta \), we get
\[ \partial_i^l \eta^{\mu\nu}(g)e_\mu \wedge e_\nu = \left. \frac{d}{dt}\right|_{t=0} \eta^{\mu\nu}(g \exp te_i)e_\mu \wedge e_\nu =\]
\[ \left. \frac{d}{dt}\right|_{t=0} (\eta^{\mu\nu}(g)e_\mu \wedge e_\nu + \text{Ad}_g(\eta^{\mu\nu}(\exp te_i)e_\mu \wedge e_\nu)) =\]
\[ \left. \frac{d}{dt}\right|_{t=0} \eta^{kl}(\exp te_i)(\text{Ad}_g)^{l}_{k}(\text{Ad}_g)^{l}_{k}e_\mu \wedge e_\nu =\]
\[ \partial_i^l \eta^{kl}(\text{Ad}_g)^{l}_{k}(\text{Ad}_g)^{l}_{k}e_\mu \wedge e_\nu,\]
where \( \text{Ad}_g e_k = (\text{Ad}_g)^\mu_k e_\mu \). So, \( \partial'_{\mu_1} \eta^{\mu_2}(g) = \partial'_{\mu_1} \eta^{kl}(e)(\text{Ad}_g)^\mu_k (\text{Ad}_g)^\nu_l \).

Continuing our calculations, we have

\[
\begin{align*}
[P_G, r^\lambda](g) &= -2r^{ij}(\partial'_{\mu_1} \eta^{\mu_2})(g) \partial_{\nu_i}(g) \wedge \partial_{\nu_j}(g) = \\
-2r^{ij} \partial'_{\mu_1} \eta^{kl}(e)(\text{Ad}_g)^\mu_k (\text{Ad}_g)^\nu_l \partial_{\nu_i}(g) \wedge \partial_{\nu_j}(g) = \\
-2r^{ij} \partial'_{\mu_1} \eta^{\mu_2}(e) \partial'_{\nu_i}(g) \wedge \partial'_{\nu_j}(g) = \\
-2r^{ij} \partial'_{\mu_1} \eta^{\mu_2}(e) \operatorname{Alt}(\partial'_{\nu_i}(g) \otimes \partial'_{\nu_j}(g)) = \\
-\frac{r^{ij}}{2} \operatorname{Alt}(\delta \otimes \text{id})r^\lambda(g).
\end{align*}
\]

\( \square \)

Now we finish the proof of Lemma 9. From the definition of a quasi-Poisson action it follows that

\[
\oint \{ f_1, f_2 \} x \cdot f_3 x (g \cdot x) = -\varphi x (df_1, df_2, df_3)(g \cdot x) = \\
-\varphi^G (df_1^G, df_2^G, df_3^G)(g).
\]

It means that for all \( f_1, f_2, f_3 \in C^\infty X \) we have

\[
\left( \varphi - \text{CYB}(r) + \frac{1}{2} \operatorname{Alt}(\delta \otimes \text{id})r \right)^\lambda (df_1^G, df_2^G, df_3^G) = 0.
\]

Consequently, for all \( l, m, n \in \mathfrak{h}_x^\perp \) we get

\[
\langle \varphi - \text{CYB}(r) + \frac{1}{2} \operatorname{Alt}(\delta \otimes \text{id})r, l \otimes m \otimes n \rangle = 0,
\]

which proves the statement of the lemma. \( \square \)

**Lemma 9.** Assume that \( 11 \) holds. Then \( L_x \) is a subalgebra in \( \mathcal{D}(g) \) if and only if the image of the tensor \( \varphi + \frac{1}{2} \operatorname{Alt}(\delta \otimes \text{id})r - \text{CYB}(r) \) in \( \wedge^3 (\mathfrak{g}/\mathfrak{h}_x) \) vanishes.

**Proof.** Consider the mapping \( R : \mathfrak{g}^* \to \mathfrak{g} \) that corresponds to \( r \in \wedge^2 \mathfrak{g} \):

\[
R(l) = (l \otimes \text{id})r = \sum_i l(r_i^l)r_i^\mu,
\]

where \( r = \sum_i r_i^l \otimes r_i^\mu \).

Then

\[
L_x = \{ a + l \mid a \in \mathfrak{g}, l \in (\mathfrak{g}/\mathfrak{h}_x)^*, (l \otimes \text{id})\overline{\tau} = \overline{\pi} \} = \\
\{ a + l \mid a \in \mathfrak{g}, l \in \mathfrak{h}_x^\perp, R(l) = \overline{\tau} \} = \{ l + R(l) \mid l \in \mathfrak{h}_x^\perp \} + \mathfrak{h}_x.
\]

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From Lemma 6 it follows that $h \cdot L_x = L_{hx} = L_x$ for any $h \in H_x$. Consequently, for all $a \in h_x$ we have $\text{ad}_a(L_x) \subset L_x$. So $L_x$ is a Lie subalgebra in $\mathcal{D}(g)$ if and only if $[l_1 + R(l_1), l_2 + R(l_2)] \in L_x$ for any $l_1, l_2 \in h_x^\perp$.

Choose any $l_1, l_2, l_3 \in h_x^\perp$. We are going to check that

$$Q([l_1 + R(l_1), l_2 + R(l_2)], l_3 + R(l_3)) =$$

$$\langle l_1 \otimes l_2 \otimes l_3, -\varphi + \text{CYB}(r) - \frac{1}{2} \text{Alt}(\delta \otimes \text{id})r \rangle.$$

Indeed,

$$\langle l_1 \otimes l_2 \otimes l_3, [r^{12}, r^{13}] \rangle = \langle l_1 \otimes l_2 \otimes l_3, \sum_{i,j} [r_i', r_j'] \otimes r''_i \otimes r''_j \rangle =$$

$$\langle l_1, \sum_{i,j} ([l_2, r''_i] r_i', [l_3, r''_j] r_j') \rangle = Q(l_1, [R(l_2), R(l_3)]) =$$

$$Q([l_1, R(l_2)], R(l_3)).$$

Similarly,

$$\langle l_1 \otimes l_2 \otimes l_3, [r^{12}, r^{23}] \rangle = Q([R(l_1), l_2], R(l_3)),$$

$$\langle l_1 \otimes l_2 \otimes l_3, [r^{13}, r^{23}] \rangle = Q([R(l_1), R(l_2)], l_3).$$

It is easy to see that $\frac{1}{2} \text{Alt}(\delta \otimes \text{id})r = (\delta \otimes \text{id})r + \tau(\delta \otimes \text{id})r + \tau^2(\delta \otimes \text{id})r$,

where $\tau(x \otimes y \otimes z) = z \otimes x \otimes y$. We have

$$\langle l_1 \otimes l_2 \otimes l_3, (\delta \otimes \text{id})r \rangle = \sum_i \langle l_1 \otimes l_2, \delta(r_i') \rangle \langle l_3, r''_i \rangle =$$

$$\sum_i \langle [l_1, l_2], \delta, l_3, r''_i \rangle r_i' = -Q([l_1, l_2], R(l_3)),$$

$$\langle l_1 \otimes l_2 \otimes l_3, \tau(\delta \otimes \text{id})r \rangle = -Q([R(l_1), l_2], l_3),$$

$$\langle l_1 \otimes l_2 \otimes l_3, \tau^2(\delta \otimes \text{id})r \rangle = -Q([l_1, R(l_2)], l_3),$$

$$\langle l_1 \otimes l_2 \otimes l_3, \varphi \rangle = -Q([l_1, l_2], l_3).$$

Adding up all the terms on the right hand side and using the fact that $Q([R(l_1), R(l_2)], R(l_3)) = 0$ we see that

$$Q([l_1 + R(l_1), l_2 + R(l_2)], l_3 + R(l_3)) =$$

$$\langle l_1 \otimes l_2 \otimes l_3, -\varphi + \text{CYB}(r) - \frac{1}{2} \text{Alt}(\delta \otimes \text{id})r \rangle.$$

The r.h.s. of this equality vanishes for any $l_1, l_2, l_3 \in (g/h_x)^*$ iff the image of $\varphi - \text{CYB}(r) + \frac{1}{2} \text{Alt}(\delta \otimes \text{id})r$ in $\wedge^3 (g/h_x)$ vanishes.

The l.h.s. vanishes for any $l_1, l_2, l_3 \in (g/h_x)^*$ iff $Q([l_1 + R(l_1), l_2 + R(l_2)], L_x)$ vanishes, i.e., since $L_x$ is maximal isotropic, iff $[l_1 + R(l_1), l_2 + R(l_2)] \in L_x$.

This finishes the proof of the lemma. □
Suppose \( v \in \bigwedge^2(\mathfrak{g}/\mathfrak{h}_x) \). Consider the mapping \( v \mapsto L_v \), where

\[
L_v = \{ a + l \mid a \in \mathfrak{g}, l \in \mathfrak{g}/\mathfrak{h}_x, (l \otimes \text{id})v = \overline{a} \}.
\]

This is a bijection between \( \bigwedge^2(\mathfrak{g}/\mathfrak{h}_x) \) and the set of all Lagrangian subspaces \( L \subset \mathcal{D}(\mathfrak{g}) \) such that \( L \cap \mathfrak{g} = \mathfrak{h}_x \).

Further, there is a bijection between bivector fields \( P_X \) on \( X \) and smooth maps \( x \mapsto L_x \) from \( X \) to the set of all Lagrangian subspaces in \( \mathcal{D}(\mathfrak{g}) \) such that \( L_x \cap \mathfrak{g} = \mathfrak{h}_x \) for all \( x \in X \).

From Lemmas 6, 7 and 9 it follows that \((X, P_X)\) is a quasi-Poisson homogeneous \( G \)-space iff the corresponding map \( x \mapsto L_x \) is \( G \)-equivariant, subalgebra-valued, and \( L_x \cap \mathfrak{g} = \mathfrak{h}_x \) for all \( x \in X \).

This finishes the proof of Theorem 3.

### 4 Twisting

Let \( G \) be a Lie group. Suppose \((P_G, \varphi)\) and \((P'_G, \varphi')\) are quasi-Poisson structures on \( G \).

**Definition 5 (see [9]).** \((G, P'_G, \varphi')\) is obtained by twisting (by \( r \in \bigwedge^2 \mathfrak{g} \)) from \((G, P_G, \varphi)\) if

\[
P'_G = P_G + r^\lambda - r^\rho,
\]

\[
\varphi' = \varphi + \frac{1}{2} \text{Alt}(\delta \otimes \text{id})r - \text{CYB}(r).
\]

There is a similar relation on Lie quasi-bialgebras. Let \( \mathfrak{g} \) be a Lie algebra, \((\delta, \varphi)\) and \((\delta', \varphi')\) are Lie quasi-bialgebra structures on \( \mathfrak{g} \).

**Definition 6 (see [5, 9]).** \((\mathfrak{g}, \delta', \varphi')\) is obtained by twisting (by \( r \in \bigwedge^2 \mathfrak{g} \)) from \((\mathfrak{g}, \delta, \varphi)\) if

\[
\delta'(x) = \delta(x) + \text{ad}_x r \quad \text{for all } x \in \mathfrak{g},
\]

\[
\varphi' = \varphi + \frac{1}{2} \text{Alt}(\delta \otimes \text{id})r - \text{CYB}(r).
\]

Twisting is an equivalence relation.

If \((G, P'_G, \varphi')\) is obtained by twisting from \((G, P_G, \varphi)\) then the corresponding Lie quasi-bialgebra \((\mathfrak{g}, \delta', \varphi')\) is obtained by twisting from \((\mathfrak{g}, \delta, \varphi)\).

The converse holds if \( G \) is connected.

Denote by \( \mathcal{D}(\mathfrak{g}, \delta, \varphi) \) and \( \mathcal{D}(\mathfrak{g}, \delta', \varphi') \) the double Lie algebras of Lie quasi-bialgebras \((\mathfrak{g}, \delta, \varphi)\) and \((\mathfrak{g}, \delta', \varphi')\) respectively. The following result is obtained in [5].

**Theorem 10.** \((\mathfrak{g}, \delta', \varphi')\) is obtained by twisting from \((\mathfrak{g}, \delta, \varphi)\) if and only if there exists a Lie algebra isomorphism \( f_r : \mathcal{D}(\mathfrak{g}, \delta, \varphi) \rightarrow \mathcal{D}(\mathfrak{g}, \delta', \varphi') \) fixing all the elements of \( \mathfrak{g} \) and preserving the canonical bilinear forms on the doubles.
Suppose that \((G, P'_G, \varphi')\) is obtained by twisting from \((G, P_G, \varphi)\). Let \(r \in \Lambda \frac{1}{2} g\) be the corresponding bivector. Then \(f_r : D(g, \delta, \varphi) \to D(g, \delta', \varphi')\), 
\[f_r(a + l) = a + l + (l \otimes \text{id})r\] is the corresponding Lie algebra isomorphism.

Using \(f_r\) we can identify \(D(g, \delta, \varphi)\) and \(D(g, \delta', \varphi')\). Since \(f_r\) preserves the canonical bilinear forms, the sets of Lagrangian subalgebras under this identification are the same.

**Theorem 11.** Let \((X, P_X)\) be a homogeneous quasi-Poisson \((G, P_G, \varphi)\)-space. Then \((X, P_X - r_X)\) is a homogeneous quasi-Poisson \((G, P'_G, \varphi')\)-space, and the map \(P_X \mapsto P_X - r_X\) is a bijection between the set of all \((G, P_G, \varphi)\)- and \((G, P'_G, \varphi')\)-quasi-Poisson structures on \(X\).

**Proof.** Denote by \(\Lambda\) (resp. \(\Lambda'\)) the set of all Lagrangian Lie subalgebras in \(D(g, \delta, \varphi)\) (resp. \(D(g, \delta', \varphi')\)).

Theorem \(\square\) gives us the \(G\)-equivariant map \(x \mapsto L_x\) from \(X\) to \(\Lambda\) such that \(L_x \cap g = h_x\) defined by

\[L_x = \{a + l | a \in g, \ l \in h_x, \ (l \otimes \text{id})P_X(x) = \overline{a}\}\]

On the other hand, consider the map \(x \mapsto L'_x\) from \(X\) to the set of subspaces in \(D(g, \delta', \varphi')\) corresponding to \(P_X - r_X\):

\[L'_x = \{a + l | a \in g, \ l \in h_x, \ (l \otimes \text{id})(P_X(x) - r_X) = \overline{a}\}\]

It is easy to see that \(f_r(L_x) = L'_x\). Since \(f_r\) is a Lie algebra isomorphism, preserves the canonical bilinear forms on the doubles and commutes with the action of \(G\) on the doubles, the map \(x \mapsto L'_x\) is a \(G\)-equivariant map from \(X\) to \(\Lambda'\). Since \(f_r\) fixes all the points of \(g\), we have \(L'_x \cap g = h_x\). From Theorem \(\square\) it follows that \(P_X - r_X\) defines a \((G, P'_G, \varphi')\)-quasi-Poisson structure on \(X\).

Obviously, the map \(P_X \mapsto P_X - r_X\) from the set of all \((G, P_G, \varphi)\)-quasi-Poisson structures on \(X\) to the set of all \((G, P'_G, \varphi')\)-quasi-Poisson structures on \(X\) is injective. Similarly, the map \(P'_X \mapsto P'_X + r_X\) transforms a \((G, P'_G, \varphi')\)-structure to a \((G, P_G, \varphi)\)-structure. Thus, we have a bijection. \(\square\)

**5 Examples**

Recall that if \((G, P_G)\) is a Poisson Lie group, then the homogeneous \(G\)-spaces \(X = \{x\}\) and \(Y = G\) admit the structure of Poisson homogeneous \((G, P_G)\)-spaces. Here we consider the quasi-Poisson case.

**Example 1.** Let \((G, P_G, \varphi)\) be a quasi-Poisson Lie group, \(X = \{x\}\) is a homogeneous \(G\)-space, \(P_X = 0\) is the only bivector field on \(X\). Then the (trivial) action of \(G\) on \(X\) is quasi-Poisson. The corresponding Lagrangian subalgebra is \(g\).
Example 2. Consider the action of a connected quasi-Poisson Lie group $(G, P_G, \varphi)$ on $Y = G$ by left translations. By Theorem 3 there is a bijection between the set of $G$-quasi-Poisson structures on $Y$ and the set of $G$-conjugacy classes of Lagrangian subalgebras $L \subset D(g)$ such that $L \cap g = 0$.

The map $r \mapsto L_r = \{a + l \in D(g) \mid (l \otimes \text{id})r = a\}$ from $\bigwedge^2 g$ to the set of Lagrangian subspaces in $D(g)$ transversal to $g$ is a bijection. On the other hand, $L_r$ is a Lie subalgebra iff $\varphi + \frac{1}{2} \text{Alt}(\delta \otimes \text{id})r - \text{CYB}(r) = 0$.

Thus $Y$ can be a quasi-Poisson homogeneous $G$-space if and only if $G$ is obtained by twisting from a Poisson Lie group. In this case there is a 1-1 correspondence between the solutions of the equation

$$\text{CYB}(r) - \frac{1}{2} \text{Alt}(\delta \otimes \text{id})r = \varphi$$

and $(G, P_G, \varphi)$-quasi-Poisson structures on $Y$ given by $P_Y = P_G + r^\lambda$.

Let us also introduce the following purely quasi-Poisson example.

Example 3. Suppose $g$ is a finite-dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form $(\cdot | \cdot)$. Let $G$ be a connected Lie group such that $\text{Lie } G = g$. Consider the “Manin quasi-triple” (see [1]) $(a, a_1, a_2)$, where $a = g \times g$,

$$a_1 = \{(x, x) \mid x \in g\} \simeq g, \quad a_2 = \{(x, -x) \mid x \in g\},$$

and $a$ is equipped with a non-degenerate invariant symmetric bilinear form $((a, b), (c, d)) \mapsto \frac{1}{2} ((a|c) - (b|d))$. It is easy to calculate that the corresponding Lie quasi-bialgebra structure on $g$ is given by $\delta = 0$, $\varphi = [\Omega^{12}, \Omega^{23}] = -\text{CYB}(\Omega)$, where $\Omega \in (S^2 g)^g$ corresponds to $(\cdot | \cdot)$. This Lie quasi-bialgebra gives rise to the quasi-Poisson Lie group $(G, 0, \varphi)$.

Pick any $g \in G$, and consider the Lagrangian subalgebra

$$L_g = \{(x, y) \mid y = \text{Ad}_g x\} \subset a.$$

It can be shown that it corresponds to the quasi-Poisson homogeneous space $(C_g, P)$, where $C_g \subset G$ is the conjugacy class of $g$, and

$$P(g) = (r_g \otimes l_g - l_g \otimes r_g)(\Omega).$$

Moreover, one can show that $(G, P)$ is a quasi-Poisson $G$-manifold with respect to the action by conjugation, and $(C_g, P)$ are “quasi-Poisson $G$-submanifolds” of $(G, P)$ (see [2], where this example was introduced and studied for a compact Lie group $G$).
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