SOME PROPERTIES OF A CERTAIN SUBCLASS OF STRONGLY STARLIKE FUNCTIONS

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Abstract. By using the method of differential subordination, we study a certain subclass of strongly starlike functions which is denoted by $S^*_t(\alpha_1, \alpha_2)$, including of all normalized and analytic functions satisfying the following two-sided inequality:

$$-\frac{\pi \alpha_1}{2} < \arg \left\{ \frac{zf'(z)}{f(z)} \right\} < \frac{\pi \alpha_2}{2}, \quad |z| < 1,$$

where $0 < \alpha_1, \alpha_2 \leq 1$. The object of the present paper is to derive some certain inequalities for the desired class $S^*_t(\alpha_1, \alpha_2)$.

1. Introduction

Let $H$ be the class of all analytic functions $f$ in the open unit disk $\Delta = \{ z \in \mathbb{C} : |z| < 1 \}$. Also, let $A$ denote the class of functions $f$ of the form

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots,$$

which are analytic in $\Delta$. The subclass of $A$ consisting of all univalent functions $f(z)$ in $\Delta$ is denoted by $S$. We say that a function $f \in S$ is starlike of order $\alpha$, where $0 \leq \alpha < 1$ if, and only if,

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \Delta).$$

We denote by $S^*(\alpha)$ the class of starlike functions of order $\alpha$. The class $S^*(\alpha)$ was introduced by Robertson (see [16]). Also, we say that a function $f \in S$ is strongly starlike of order $\beta$, where $0 < \beta \leq 1$ if, and only if,

$$\arg \left\{ \frac{zf'(z)}{f(z)} \right\} < \frac{\pi \beta}{2} \quad (z \in \Delta).$$

The functions class of strongly starlike functions of order $\beta$ is denoted by $SS^*(\beta)$. The class $SS^*(\beta)$ was introduced independently by Stankiewicz (see [24], [25]) and by Brannan and Kirvan (see [1]). We remark that $SS^*(1) \equiv S^*(0) = S^*$, where $S^*$ denotes the class of starlike functions.

Let $f(z)$ and $g(z)$ be two analytic functions in $\Delta$. Then the function $f(z)$ is said to be subordinate to $g(z)$ in $\Delta$, written by $f(z) \prec g(z)$ or $f \prec g$, if there exists an analytic function $w(z)$ in $\Delta$ with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = g(w(z))$ for all $z \in \Delta$.

In the sequel, we consider the analytic function $G(z) := G(\alpha_1, \alpha_2, c)(z)$ as follows

$$(1.1) \quad G(\alpha_1, \alpha_2, c)(z) := \left( \frac{1 + cz}{1 - z} \right)^{(\alpha_1 + \alpha_2)/2} \quad (G(0) = 1, \ z \in \Delta),$$

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where \(0 < \alpha_1, \alpha_2 \leq 1\), \(c = e^{i\theta}\) and \(\theta = \frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1}\). Also, we consider the set \(\Omega_{\alpha_1, \alpha_2}\) as follows
\[
\Omega_{\alpha_1, \alpha_2} := \left\{ w \in \mathbb{C} : -\frac{\pi \alpha_1}{2} < \arg\{w\} < \frac{\pi \alpha_2}{2} \right\}.
\]
We note that the function \(G(z)\) is convex univalent in \(\Delta\) and maps \(\Delta\) onto \(\Omega_{\alpha_1, \alpha_2}\) (see [23]). Since
\[
\left(\frac{1 + cz}{1 - z}\right)^{(\alpha_1 + \alpha_2)/2} = \left(1 + \frac{(1 + cz)}{1 - z}\right)^{(\alpha_1 + \alpha_2)/2}
\]
\[
= 1 + \sum_{k=1}^{\infty} \binom{\alpha_1 + \alpha_2}{k} (1 + c)^k \left(\frac{z}{1 - z}\right)^k (z \in \Delta),
\]
using the binomial formula, we obtain
\[
G(z) = 1 + \sum_{n=1}^{\infty} \lambda_n z^n \quad (z \in \Delta),
\]
where
\[
\lambda_n := \lambda_n(\alpha_1, \alpha_2, c) = \sum_{k=1}^{n} \binom{n - 1}{k - 1} \binom{(\alpha_1 + \alpha_2)/2}{k} (1 + c)^k \quad (n \geq 1).
\]
We note that \(\lambda_n\) may be conveniently written in the form
\[
\lambda_n = \frac{(\alpha_1 + \alpha_2)(1 + c)}{2} F(1 - n, 1 - (\alpha_1 + \alpha_2)/2; 2; 1 + c) \quad (n \geq 1),
\]
where notation \(F\) stands for the Gauss hypergeometric function.

The main purpose of this paper is to study the class \(S^*_\beta(\alpha_1, \alpha_2)\) which is provided below.

**Definition 1.1.** A function \(f \in \mathcal{A}\) belongs to the class \(S^*_\beta(\alpha_1, \alpha_2)\), if \(f\) satisfies the following two-sided inequality
\[
-\frac{\pi \alpha_1}{2} < \arg\left\{ \frac{zf'(z)}{f(z)} \right\} < \frac{\pi \alpha_2}{2} \quad (z \in \Delta),
\]
where \(0 < \alpha_1, \alpha_2 \leq 1\).

The class \(S^*_\beta(\alpha_1, \alpha_2)\) was introduced by Takahashi and Nunokawa (see [26]). We recall here the fact, that in [2] and in [3] a similar class was studied. It is clear that \(S^*_\beta(\alpha_1, \alpha_2) \subset S^*\) and that \(S^*_\beta(\alpha_1, \alpha_2)\) is a subclass of the class of strongly starlike functions of order \(\beta = \max\{\alpha_1, \alpha_2\}\), i.e. \(S^*_\beta(\alpha_1, \alpha_2) \subset S^*(\beta, \beta) \equiv SS^*(\beta)\).

In Geometric Function Theory there exist many certain subclasses of analytic functions which have been defined by the subordination relation, see for example [8], [9], [10], [11], [12], [13], [15], [22], [23]. It is clear that defining a class by using the subordination makes it easy to investigate its geometric properties. Below, we present a necessary and sufficient condition for functions to be the class \(S^*_\beta(\alpha_1, \alpha_2)\). Actually, we present the definition of the class \(S^*_\beta(\alpha_1, \alpha_2)\) by using the subordination.

**Lemma 1.1.** Let \(f(z) \in \mathcal{A}\). Then \(f \in S^*_\beta(\alpha_1, \alpha_2)\) if, and only if,
\[
z \frac{f'(z)}{f(z)} \prec G(z) \quad (z \in \Delta),
\]
where \(G(z)\) is defined in (1.1).

**Proof.** Let \(G(z)\) be given by (1.1). By (1.5), \(\{zf'(z)/f(z)\}\) lies in the domain \(\Omega_{\alpha_1, \alpha_2}\), where \(\Omega_{\alpha_1, \alpha_2}\) is defined in (1.2) and it is known that \(G(\Delta) = \Omega_{\alpha_1, \alpha_2}\). The function \(G(z)\) is univalent in \(\Delta\) and thus, by the subordination principle, we get (1.6). \(\square\)
For \( f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \) and \( g(z) = b_0 + b_1 z + b_2 z^2 + \cdots \), their Hadamard product (or convolution) is defined by \((f * g)(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \cdots\). The convolution has the algebraic properties of ordinary multiplication. Many of convolution problems were studied by St. Ruscheweyh in \cite{19} and have found many applications in various fields. The following lemma will be useful in this paper.

**Lemma 1.2.** (see \cite{21}) Let \( F(z), H(z) \in \mathcal{H} \) be any convex univalent functions in \( \Delta \). If \( f \prec F \) and \( g(z) \prec H(z) \), then

\[
(1.7) \quad f(z) * g(z) \prec F(z) * H(z) \quad (z \in \Delta).
\]

Following, one have an another useful lemma (see \cite{17}).

**Lemma 1.3.** Let \( q(z) = \sum_{n=1}^{\infty} C_n z^n \) be analytic and univalent in \( \Delta \), and suppose that \( q(z) \) maps \( \Delta \) onto a convex domain. If \( p(z) = \sum_{n=1}^{\infty} A_n z^n \) is analytic in \( \Delta \) and satisfies the following subordination

\[
p(z) \prec q(z) \quad (z \in \Delta),
\]

then

\[
|A_n| \leq |C_1| \quad (n = 1, 2, 3, \ldots).
\]

The structure of this paper is the following. Early, we find a lower bound and an upper bound for \( \text{Re}\{zf'(z)/f(z)\} \), where \( f \in S^*_t(\alpha_1, \alpha_2) \). Moreover, as a corollary we show that if \( f \) is a strongly starlike function of order \( \beta \), then the upper bound for \( \text{Re}\{zf'(z)/f(z)\} \), where \( |z| \leq 1/3 \) is equal to \( 2^\beta \). Next, we present some subordination relations which will be useful in order to estimate the logarithmic coefficients. At the end, we estimate the coefficients of \( f \in S^*_t(\alpha_1, \alpha_2) \) and we will show how that the coefficient bounds are related to the well-known Bieberbach conjecture (see \cite{3}) proved by de Branges in 1985 (see \cite{5}).

**2. Some inequalities and subordination relations**

The first result of this paper is the following.

**Theorem 2.1.** Assume that \( f \in \mathcal{A} \). If \( f \in S^*_t(\alpha_1, \alpha_2) \), then

\[
(2.1) \quad \text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} \geq \left( \frac{1 - (2 \cos \frac{\theta}{2} + 1)r}{1 - r} \right)^{(\alpha_1 + \alpha_2)/2} \quad 0 \leq |z| = r \leq \frac{1}{2 \cos \frac{\theta}{2} + 1}
\]

and

\[
(2.2) \quad 0 < \text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} \leq \left( \frac{1 + (2 \cos \frac{\theta}{2} - 1)r}{1 - r} \right)^{(\alpha_1 + \alpha_2)/2} \quad 0 \leq |z| = r < 1,
\]

where \( 0 < \alpha_1, \alpha_2 \leq 1 \) and \( \theta = \frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1} \).

**Proof.** Let the function \( f \) be in the class \( S^*_t(\alpha_1, \alpha_2) \). Then by Lemma \cite{10} and by the definition of subordination, there exists a Schwarz function \( w(z) \), satisfying the following conditions

\[
w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \Delta)
\]

and such that

\[
\frac{zf'(z)}{f(z)} = \left( \frac{1 + cw(z)}{1 - w(z)} \right)^{(\alpha_1 + \alpha_2)/2} \quad (z \in \Delta).
\]

Define

\[
F(z) := \frac{1 + cw(z)}{1 - w(z)} \quad (z \in \Delta).
\]
It is clear that \( \text{Re}\{F(z)\} > 0 \) in the unit disk. We shall describe \( \text{Re}\{F(z)\} \) more precisely. From (2.3) we have

\[
|F(z) - 1| = \left| \frac{(1 + c)w(z)}{1 - w(z)} \right| \leq \frac{2|w(z)| \cos \frac{\theta}{2}}{1 - |w(z)|}.
\]

For \(|z| = r < 1\), using the known fact that (see [6])

\[
|w(z)| \leq |z|,
\]

this gives

\[
|F(z) - 1| \leq \frac{2r \cos \frac{\theta}{2}}{1 - r} \quad (|z| = r < 1).
\]

Thus, \( F(z) \) for \(|z| = r < 1\) lies in the disk which the center \( C = 1 \) and the radius \( R \) given by

\[
R := \frac{2r \cos \frac{\theta}{2}}{1 - r}.
\]

We note that the origin is outside of this disk for \(|z| < 1/(1 + 2 \cos(\theta/2))\), and so we obtain (2.1) and (2.2). This ends the proof of Theorem 2.1. □

Putting \( \alpha_1 = \alpha_2 = \beta \) in Theorem 2.1, we have:

**Corollary 2.1.** Let \( f \) be a strongly starlike function of order \( \beta \), where \( 0 < \beta \leq 1 \). Then

\[
\left( \frac{1 - 3r}{1 - r} \right)^\beta \leq \text{Re}\left\{ \frac{zf''(z)}{f'(z)} \right\} \leq \left( \frac{1 + r}{1 - r} \right)^\beta \quad (|z| = r \leq 1/3).
\]

In particular, if \( r = 1/3 \), then

\[
0 < \text{Re}\left\{ \frac{zf''(z)}{f'(z)} \right\} < 2^\beta.
\]

As a corollary, by [7] or [14, Theorem 3.2a], and by Theorem 2.1 we obtain a sufficient condition for functions belonging the class \( S^*_t(\alpha_1, \alpha_2) \).

**Lemma 2.1.** If \( f \) satisfies the following subordination

\[
1 + \frac{zf''(z)}{f'(z)} \prec G(z) \quad |z| \leq \frac{1}{1 + 2 \cos \frac{\theta}{2}},
\]

then \( f \) satisfies

\[
\frac{zf'(z)}{f(z)} \prec G(z) \quad |z| \leq \frac{1}{1 + 2 \cos \frac{\theta}{2}},
\]

where \( G(z) \) is defined in (1.1).

**Proof.** Denote

\[
p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \Delta),
\]

where \( p \) is analytic and \( p(0) = 1 \). A simple calculation implies that

\[
p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}
\]

and by (2.4) we get

\[
p(z) + \frac{zp'(z)}{p(z)} \prec G(z).
\]

Since \( \text{Re}(p(z)) > 0 \) in \( \Delta \) and \( G(z) \) is convex, the desired result follows. □

In order to estimate of \( f \) (Growth Theorem), where \( f \in S^*_t(\alpha_1, \alpha_2) \) and the logarithmic coefficients of members of \( S^*_t(\alpha_1, \alpha_2) \), we need the following theorem.
Theorem 2.2. If \( f \in S^*_t(\alpha_1, \alpha_2) \), then
\[
(2.5) \quad \log \left\{ \frac{f(z)}{z} \right\} < \int_0^z \frac{G(t) - 1}{t} \, dt,
\]
where the function \( G \) is convex univalent of the form (1.1). Moreover,
\[
(2.6) \quad \tilde{G}(z) = \int_0^z \frac{G(t) - 1}{t} \, dt = \sum_{n=1}^\infty \frac{\lambda_n}{n} z^n,
\]
is convex univalent too, where \( \lambda_n \) is defined in (1.4).

Proof. The subordination relation (1.6) gives us
\[
(2.7) \quad z \left( \log \left\{ \frac{f(z)}{z} \right\} \right)' < G(z) - 1,
\]
where \( G(z) - 1 \) is convex univalent. For \( x \geq 0 \) the function
\[
\tilde{h}(x; z) = \sum_{k=1}^\infty \frac{(1 + x)z^k}{k + x}
\]
is convex univalent in \( \Delta \) (see [15]). Since, for
\[
\tilde{h}(0; z) = \sum_{k=1}^\infty \frac{z^k}{k},
\]
we have by (2.7)
\[
(2.8) \quad [g(z) \prec F(z)] \Rightarrow \left[ g(z) \ast \tilde{h}(0; z) \prec F(z) \ast \tilde{h}(0; z) \right],
\]
whenever \( F(z) \) is a convex univalent function. Because
\[
(2.9) \quad g(z) \ast \tilde{h}(0; z) = \int_0^z \frac{g(t)}{t} \, dt,
\]
then from (2.4), (2.8) and from (2.9), we have
\[
\int_0^z \log \{ f(t) \}' \, dt \prec \int_0^z \frac{G(t) - 1}{t} \, dt,
\]
this gives (2.5). Moreover,
\[
\tilde{G}(z) = \int_0^z \frac{G(t) - 1}{t} \, dt = \{ G(z) - 1 \} \ast \tilde{h}(0; z),
\]
where \( G(z) - 1 \) and \( \tilde{h}(0; z) \) are convex univalent functions. Since the class of convex univalent functions is preserved under the convolution (see [20]), therefore, we conclude that the function \( \tilde{G}(z) \) is convex univalent. This is the end of proof. \( \Box \)

Because \( \tilde{G}(z) \) is univalent, we may rewrite Theorem 2.2 as the following corollary.

Corollary 2.2. If \( f(z) \in S^*_t(\alpha_1, \alpha_2) \), then
\[
(2.10) \quad \frac{f(z)}{z} < \exp \int_0^z \frac{G(t) - 1}{t} \, dt
\]
where \( G \) and \( \tilde{G} \) are of the form (1.1) and (2.6), respectively.

Theorem 2.3. Let \( \tilde{G} \) be of the form (2.6). If \( f(z) \in S^*_t(\alpha_1, \alpha_2) \), then
\[
(2.11) \quad r \exp \tilde{G}(-r) < |f(z)| < r \exp \tilde{G}(r),
\]
for each \( r = |z| < 1 \).
Proof. From (2.10), we have
\[ f(z) \in \exp \tilde{G}(|z| \leq r), \]
for each \( 0 < r < 1 \) and \(|z| \leq r\), where \( \exp \tilde{G}(z) \) is convex univalent and for each \( 0 < r < 1 \) the set \( \exp \tilde{G}(|z| \leq r) \) is a set symmetric with respect to the real axis.
Furthermore,
\[ \exp \tilde{G}(-r) \leq |\exp \tilde{G}(z)| \leq \exp \tilde{G}(r) \]
for each \( 0 < r < 1 \) and \(|z| \leq r\). Therefore, from (2.12) and we obtain (2.11). □

3. On Logarithmic Coefficients and Coefficients

The logarithmic coefficients \( \gamma_n \) of \( f(z) \) are defined by
\[ \log \left\{ \frac{f(z)}{z} \right\} = \sum_{n=1}^{\infty} 2 \gamma_n z^n \quad (z \in \Delta), \]
which play an important role for various estimates in the theory of univalent functions. For example, if \( f \in \mathcal{S} \), then we have
\[ \gamma_1 = \frac{a_2}{2} \quad \text{and} \quad \gamma_2 = \frac{1}{2} \left( a_3 - \frac{a_2^2}{2} \right) \]
and the sharp estimates
\[ |\gamma_1| \leq 1 \quad \text{and} \quad |\gamma_2| \leq \frac{1}{2} (1 + 2 e^{-2}) \approx 0.635 \ldots , \]
hold. For \( n \geq 3 \), the estimate of \( \gamma_n \) is much harder and no significant upper bounds for \(|\gamma_n|\) when \( f \in \mathcal{S} \) and is still open for \( n \geq 3 \). The sharp upper bounds for modulus of logarithmic coefficients are known for functions in very few subclasses of \( \mathcal{S} \). For functions in the class \( \mathcal{S}^* \), it is easy to prove that \(|\gamma_n| \leq 1/n\) for \( n \geq 1 \) and equality holds for the Koebe function. In the sequel, we estimate the logarithmic coefficients of \( f \in \mathcal{S}^*_t(\alpha_1, \alpha_2) \).

Theorem 3.1. Let \( 0 < \alpha_1, \alpha_2 \leq 1 \), \( c = e^{\pi i \theta} \) and \( \theta = \frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1} \). Let \( f \in \mathcal{S}^*_t(\alpha_1, \alpha_2) \) and the coefficients of \( \log(f(z)/z) \) be given by (3.1). Then
\[ |\gamma_n| \leq \frac{(\alpha_1 + \alpha_2)}{2n} \cos \frac{1}{2} \theta. \]
The result is sharp.

Proof. Let us \( f \in \mathcal{S}^*_t(\alpha_1, \alpha_2) \). With replacing (1.3) and (3.1) in (2.7), we have
\[ \sum_{n=1}^{\infty} 2n \gamma_n z^n \prec \sum_{n=1}^{\infty} \lambda_n z^n, \]
Applying Lemma 1.3 gives
\[ 2n|\gamma_n| \leq |\lambda_1|, \]
where
\[ \lambda_1 = \frac{(1 + c)(\alpha_1 + \alpha_2)}{2}. \]
Thus the desired inequality (3.2) follows. The equality holds for the logarithmic coefficients of the function \( z \mapsto z \exp \tilde{G}(z) \), where \( \tilde{G} \) is defined in (2.6). This completes the proof. □

If we take \( \alpha_1 = \alpha_2 = \beta \), in the above Theorem 3.1 we get the following result which previously is obtained by Thomas, see [27, Theorem 1].
Corollary 3.1. Let \( f \) be a strongly starlike function of order \( \beta \), where \( 0 < \beta \leq 1 \). Then the logarithmic coefficients of \( f \) satisfy the sharp inequality
\[
|\gamma_n| \leq \frac{1}{n} \beta.
\]
In particular, taking \( \beta = 1 \) gives us the estimate of logarithmic coefficients of starlike functions.

**Theorem 3.2.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_1^t(\alpha_1, \alpha_2) \). Then we have
\[
|a_n| \leq \begin{cases} 
(\alpha_1 + \alpha_2) \cos \frac{1}{2} \theta, & n = 2, \\
\frac{\alpha_1 + \alpha_2}{n - 1} \cos \frac{1}{2} \theta \prod_{k=2}^{n-1} \left( 1 + \frac{\alpha_1 + \alpha_2}{k - 1} \cos \frac{1}{2} \theta \right), & n = 3, 4, \ldots,
\end{cases}
\]
where \( c = e^{\pi \theta}, \theta = (\alpha_2 - \alpha_1)/(\alpha_2 + \alpha_1) \) and \( 0 < \alpha_1, \alpha_2 \leq 1 \).

*Proof.* Let \( q(z) \) be defined by
\[
zf'(z) = q(z)f(z) \quad z \in \Delta.
\]
Then according to the assertion of Lemma 1.1, we have
\[
q(z) \prec G(z) \quad z \in \Delta,
\]
where \( G(z) \) defined by (1.4). If we let
\[
q(z) = 1 + \sum_{n=1}^{\infty} A_n z^n,
\]
then by Lemma 1.3 we see that the subordination (3.5) implies that
\[
|A_n| \leq (\alpha_1 + \alpha_2) \cos \frac{1}{2} \theta \quad (n = 1, 2, 3, \ldots).
\]
Now by equating the coefficients of \( z^n \) in both sides of equality (3.4) we have
\[
n a_n = A_{n-1} a_1 + A_{n-2} a_2 + \cdots + A_1 a_{n-1} + A_0 a_n \quad (n = 2, 3, \ldots),
\]
where \( A_0 = a_1 = 1 \). A simple calculation together with the inequality (3.6) yields that
\[
|a_n| = \frac{1}{n - 1} \times |A_{n-1} a_1 + A_{n-2} a_2 + \cdots + A_1 a_{n-1}|
\leq \frac{\alpha_1 + \alpha_2}{n - 1} \cos \frac{1}{2} \theta (|a_1| + |a_2| + \cdots + |a_{n-1}|)
= \frac{\alpha_1 + \alpha_2}{n - 1} \cos \frac{1}{2} \theta \sum_{k=1}^{n-1} |a_k|.
\]
Hence, we have \( |a_2| \leq (\alpha_1 + \alpha_2) \cos \frac{1}{2} \theta \). To prove the remaining part of the theorem, we need to show that
\[
\frac{\alpha_1 + \alpha_2}{n - 1} \cos \frac{1}{2} \theta \sum_{k=1}^{n-1} |a_k| \leq \frac{\alpha_1 + \alpha_2}{n - 1} \cos \frac{1}{2} \theta \prod_{k=2}^{n-1} \left( 1 + \frac{\alpha_1 + \alpha_2}{k - 1} \cos \frac{1}{2} \theta \right) \quad (n = 3, 4, 5 \ldots).
\]
Using induction and simple calculation, we could to prove the inequality (3.7). Hence, the desired estimate for \( |a_n| \) \((n = 3, 4, 5, \ldots)\) follows, as asserted in (3.6). This completes the proof of Theorem. \( \Box \)

Selecting \( \alpha_1 = \alpha_2 = \beta \), in the above Theorem 3.2 we may obtain bounds on coefficients of strongly starlike function of order \( \beta \), although they are not sharp when \( n = 3, 4, \ldots \).
Corollary 3.2. If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is a strongly starlike function of order \( \beta \), then
\[
|a_n| \leq \begin{cases} 
2\beta & n = 2, \\
\frac{2\beta}{n-1} \prod_{k=2}^{n-1} \left( 1 + \frac{2\beta}{k} \right) & n = 3, 4, \ldots,
\end{cases}
\]
where \( 0 < \beta \leq 1 \). The equality occurs for the function \( f_\beta(z) = z/(1-z)^{2\beta} \). Taking \( \beta = 1 \), we get the sharp estimate for the coefficients of starlike functions.

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