GLOBAL WELL-POSEDNESS FOR THE $L^2$-CRITICAL HARTREE EQUATION ON $\mathbb{R}^n$, $n \geq 3$.

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Abstract. We consider the initial value problem for the $L^2$-critical defocusing Hartree equation in $\mathbb{R}^n$, $n \geq 3$. We show that the problem is globally well posed in $H^s(\mathbb{R}^n)$ when $1 > s > \frac{2(n-2)}{n-4}$. We use the “I-method” following [9] combined with a local in time Morawetz estimate for the smoothed out solution $I\phi$ as in [7].

1. Introduction

In this paper we study the initial value problem of the $L^2$-critical defocusing Hartree equation,

$$\begin{cases}
i\partial_t \phi + \frac{1}{2} \Delta \phi = (|x|^{-2} * |\phi|^2)\phi, & x \in \mathbb{R}^n, \ t > 0, \\
\phi(x, 0) = \phi_0(x) \in H^s(\mathbb{R}^n).
\end{cases}$$

Here $H^s(\mathbb{R}^n)$ denotes the usual inhomogeneous Sobolev space. (1.1) is meaningful in dimension $n \geq 3$, where the Hartree potential is locally integrable. The Hartree type equations arise in atomic and nuclear physics and is related to the mean-field theory with respect to wave functions describing boson systems. ([14], [27])

The local well-posedness results for $s \geq 0$ is shown by the Strichartz estimates similarly as polynomial type NLS. For $s > 0$ (1.1) is locally well-posed in the subcritical sense. More precisely, for any $\phi_0 \in H^s(\mathbb{R}^n)$, the lifetime span of the solution depends on the norm of the initial data, $\|\phi_0\|_{H^s}$. Whereas, for $s = 0$ the lifetime span depends on the profile of the initial data as well. The classical solutions to (1.1) enjoy the mass conservation law,

$$\|\phi(\cdot, t)\|_{L^2(\mathbb{R}^n)} = \|\phi_0(\cdot)\|_{L^2(\mathbb{R}^n)},$$

and the energy conservation law,

$$E[t] := \int_{\mathbb{R}^n} |\nabla \phi|^2 + (|x|^{-2} * |\phi|^2)|\phi|^2 \, dx.$$  \hspace{1cm} (1.2)

When $s \geq 1$, the energy conservation law (1.2) together with the subcritical local theory immediately yields the global well-posedness. But when $0 \leq s < 1$, where the energy could be infinite, the mass conservation law cannot imply the global well-posedness, since in the local theory for $L^2$ initial data, the lifetime $T = T(\phi_0)$ could go to zero for a fixed $L^2$ norm.

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The purpose of this paper is to extend the global well-posedness result below the energy norm. Our main theorem is as follows:

**Theorem 1.1.** Let \( n \geq 3 \). The initial value problem of \((1.1)\) is globally well-posed for initial data \( \phi_0 \in H^s(\mathbb{R}^n) \) when \( \frac{2(n-2)}{3n-4} < s < 1 \).

We use the \( I \)-method and the interaction Morawetz inequality, which were used in several literatures of the same type of results, \([5, 7, 9, 12, 13, 25]\). The idea of \( I \)-method, which introduced by Colliander et al. \([8]\), is to use a smoothing operator \( I \) which regularizes a rough solution up to the regularity level of a conservation law by damping high frequency part. In our example, when \( \phi \in H^s \) for \( s < 1 \), \( E(\phi) \) may not be finite, but for a smoothed function \( I\phi \), \( E(I\phi) \) is finite. Here, one doesn’t expect that \( E(I\phi) \) is conserved, since \( I\phi \) is not a solution to \((1.1)\). But if \( I \) operator is close to the identity operator in some sense, \( I\phi \) is close to a solution and \( E(I\phi) \) is almost conserved. In fact, we control the growth of \( E(I\phi)(t) \) in time.

In addition to \( I \)-method, we use the interaction Morawetz inequality. Colliander et al. introduced in \([9]\) a new Morawetz interaction potential for the nonlinear Schrödinger equation in three dimension.

\[
M[\phi(t)] := \int_{\mathbb{R}^n} |\phi(x,t)|^2 \left( \int_{\mathbb{R}^n} \text{Im} \left[ \bar{\phi}(y,t) \nabla \phi(y,t) \right] \cdot \frac{y-x}{|y-x|} dy \right).
\]

This is a generalization of the classical Morawetz potential, which has been studied in many literatures especially regarding on the dispersive property of the Schrödinger equations \([1, 16, 22]\). The above functional \((1.3)\) generates a new space-time \( L^4_t L^2_x \) estimate for the nonlinear Schrödinger equation with the relatively general defocusing power nonlinearity. Incorporating this with the *almost conservation law*, they proved the scattering of the equation and relaxed the low regularity assumption given in the previous work \([8]\).

In \([5]\) the authors showed the almost conservation law and Morawetz interaction potential approach worked as well with the Hartree equation in dimension 3. More precisely, when the defocusing Hartree nonlinearity is mass supercritical and energy subcritical case, which is \((|x|^{-\gamma} + |\phi|^2)\phi, \quad 2 < \gamma < 3\), the equation is globally well posed in \( H^s(\mathbb{R}^3) \), \( 1 > s > \max\left( \frac{1}{2}, \frac{4(\gamma-2)}{3\gamma-4} \right) \) and has scattering as well. In the \( H^1(\mathbb{R}^3) \) case, the same result was shown in \([18]\) and later the scattering part was simplified in \([26]\).

The interaction Morawetz inequality is extended to other dimensions \([29, 12, 7]\). But in the mass critical case, where the admissible norm is critical, the space-time norm grows in time. We follow the similar way to \([7, 12]\). Due to local in time Morawetz inequality we are able to control

\[
\|\phi\|_{L^{2(n-1)}_t L^{n-2}_x (0,T \times \mathbb{R}^n)} \leq T^{\frac{n-2}{4(n-1)}} \|\phi_0\|_{L^2}^\frac{1}{2} \|\phi\|_{n-1}^\frac{n-2}{n-2} \|\phi\|_{L^{\infty}_t H^\frac{1}{2}_x (0,T \times \mathbb{R}^n)}^\frac{1}{2} \tag{1.4}
\]

for an admissible pair \((\frac{4(n-1)}{n}, \frac{2(n-1)}{n-2})\). The same machinery in \([12]\) with the above inequality \((1.4)\) would yield the result that the global well-posedness of \((1.1)\) holds when \( 1 > s > \).
max \( \left( \frac{1}{2}, \frac{2(n-2)}{3n-4} \right) \). Since we allow the admissible space-time norm grows in time, we do not know whether scattering holds true. Note that the number \( \frac{2(n-2)}{3n-4} \) is lower than \( \frac{1}{2} \) in dimension 3. The restriction \( s > \frac{1}{2} \) is inevitable if relying on the inequality (1.4). In order to remove this restriction, we use the the inequality (1.4) for the smoothed out solution \( I\phi \). This idea was first introduced in [7, 13]. They showed it still holds true with negligible error. In our case we have (For detail see Lemma 4.2)

\[
\|I\phi\|_{L_t^4 L_x^{4(n-1)/n}} \leq T^{\frac{n-2}{4(2n-1)}} \left( \|\phi_0\|_{L_t^2} \|I\phi\|_{L_t^\infty H_x^{1/2}([0,T] \times \mathbb{R}^n)}^{\frac{n-2}{n-1}} + \|I\phi\|_{L_t^\infty H_x^{1/2}([0,T] \times \mathbb{R}^n)} + T^{\frac{n-2}{4(2n-1)}} \text{Error} \right)
\]

Since \( I\phi \) is in \( H^1 \) (in particular in \( \dot{H}^{1/2} \)), \( s \) may go below \( \frac{1}{2} \). We show that on the time interval where the local well-posedness the error term is very small. At the time we prepare this paper we are informed that Miao et.al. [25] use the same idea to remove the restriction \( s > \frac{1}{2} \) in the result of \( \dot{H}^{1/2} \)-subcritical Hartree equation as an improvement of [5]. On the other hand, Miao et. al. [23, 24] studied the focusing or defocusing \( L^2 \) critical Hartree equations as well. They established the global well-posedness and scattering for \( L^2 \) radial initial data and the blow up criterion to the focusing \( L^2 \) critical Hartree equation in \( \mathbb{R}^3 \).

Before we close the introduction, we would like to add some remark on the \( L^2 \)-critical focusing case,

\[
\begin{aligned}
\begin{cases}
  i\partial_t \phi + \frac{1}{2} \Delta \phi = -(|x|^{-2} \ast |\phi|^2)\phi, \quad x \in \mathbb{R}^n, \ t > 0, \\
  \phi(x, 0) = \phi_0(x) \in H^s(\mathbb{R}^n).
\end{cases}
\end{aligned}
\]

Note that the local well-posedness proof in Section 2 equally works for the focusing case. The equation is known to have a ground state solution \( Q \), which solves

\[
\Delta Q - Q = -(|x|^{-2} \ast |Q|^2)Q.
\]

The existence of \( Q \) is proven in [21] with the decisive property of being the sharp constant of the Gagliardo-Nirenberg inequality such as

\[
\int_{\mathbb{R}^n} (|x|^{-2} \ast |u|^2)|u|^2(x)dx \leq \frac{2}{\|Q\|_{L^2}} \|u\|_{L^2}^2 \|
abla u\|_{L^2}^2.
\]

The uniqueness is open except \( n = 4 \), which was settled in [20] adapting E. Lieb’s uniqueness proof in [21].

The paper is organized as follows. In Section 2 we review the local well-posedness theorem using the Strichartz estimate. In Section 3 we give the definition of \( I \) operator, show the modified local well-posedness of \( I\phi \), and obtain the upper bound of time increment of the modified energy. In Section 4 we recall the almost interaction Morawetz inequality for \( I\phi \) and show the error bound. In Section 5 we conclude the proof of global well-posedness in Theorem 1.1.
Notations. Given $A, B$, we write $A \lesssim B$ to mean that for some universal constant $K > 2$, $A \leq KB$. We write $A \sim B$ when both $A \lesssim B$ and $B \lesssim A$. The notation $A \ll B$ denotes $B > 3 \cdot A$. We write $\langle A \rangle \equiv (1 + A^2)\frac{1}{2}$, and $\langle \nabla \rangle$ for the operator with Fourier multiplier $(1 + |\xi|^2)\frac{1}{2}$. The symbol $\nabla$ denotes the spatial gradient. We will often use the notation $\frac{1}{2} + \epsilon$ for some universal $0 < \varepsilon \ll 1$. Similarly, we write $\frac{1}{2} - \epsilon \ll \frac{1}{2} - \varepsilon$. We use the function space $L^1_tL^2_x$ and $H^{s,p}$ given norms by

$$
\|F\|_{L^1_tL^2_x([\mathbb{R}^n+1])} \equiv \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} |F(x,t)|^2 \, dx \right)^{\frac{2}{3}} \, dt \right)^{\frac{3}{2}},
$$

$$
\|u\|_{H^{s,p}([\mathbb{R}^n])} \equiv \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{\frac{s}{2}}\mathcal{F}u]\|_{L^p([\mathbb{R}^n])},
$$

where $\mathcal{F}$ is a fourier transform, $1 \leq p, q, r \leq \infty$.

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2. The local well-posedness

We refer $(q,r)$ the admissible pair when $2 \leq q < \infty$, $2 \leq r \leq \frac{2n}{n-2}$ and

$$
\frac{2}{q} + \frac{n}{r} = \frac{n}{2}
$$

and state the Strichartz inequality in dimension $n$.

Proposition 2.1. Suppose that $(q,r)$, $(\lambda, \eta)$ are any two admissible pairs. Suppose that $u(x,t)$ is a solution of the problem

$$
i\partial_t u(x,t) + \Delta u(x,t) = F(x,t), \quad (x,t) \in \mathbb{R}^n \times [0,T],
$$

(2.6)

for a data $u(0) \in H^s$, $F \in L^1_tH^{s,r'}([0,T] \times \mathbb{R}^n) \cap C_tH^{s,r}([0,T] \times \mathbb{R}^n)$ where $\lambda'$ and $\eta'$ are the Hölder conjugates of $\lambda$ and $\eta$, respectively. Then $u$ belongs to $L^1_tH^{s,r}([0,T] \times \mathbb{R}^n) \cap C_tH^{s,r}([0,T] \times \mathbb{R}^3)$ and we have the estimate

$$
\|u\|_{L^1_tH^{s,r}([0,T] \times \mathbb{R}^n)} \lesssim \|u(0)\|_{H^s(\mathbb{R}^n)} + \|F\|_{L^1_tH^{s,r'}([0,T] \times \mathbb{R}^n)}. 
$$

For the pure power nonlinearity $\lambda|u|^{\alpha}u$, the local well-posedness of $i\partial_t u + \frac{1}{2}\Delta u = \lambda|u|^{\alpha}u$ with the rough data $u(0) \in H^s, 0 < s < 1$ was proven in [2] (See also [3, 28]).

We define the Strichartz norm of functions $\phi : [0,T] \times \mathbb{R}^n \to \mathbb{C}$ by

$$
\|\phi\|_{S^0_T} = \sup_{(q,r) \text{ admissible}} \|\phi\|_{L^1_tL^2_x([0,T] \times \mathbb{R}^n)}.
$$

In particular $S^0_T \subset C_tL^2_x([0,T] \times \mathbb{R}^n)$. Then the Strichartz estimates may be written as

$$
\|\phi\|_{S^0_T} \leq \|\phi\|_{L^2} + \|(i\partial_t + \Delta)\phi\|_{L^1_tL^2_x([0,T] \times \mathbb{R}^n)},
$$

where $(q', r')$ is the conjugate of an admissible pair $(q, r)$.

The local existence theorem of (1.1) is as follows.
Theorem 2.1. For a given \( \phi_0 \in H^s(\mathbb{R}^n) \), \( 0 < s \), there exists a positive time \( T = T(\|\phi_0\|_{H^s}) \) and the unique solution \( \phi \) of (1.1), in \( \phi \in C_tH^s_x([0,T] \times \mathbb{R}^n) \cap S^s_T \) for every admissible pair \( (q,r) \), where

\[
\|\phi\|_{S^s_T} = \sup_{(q,r) \text{ admissible}} \|\nabla\|^s\phi\|_{L^q_tL^r_x([0,T] \times \mathbb{R}^n)}.
\]

Proof. Let \( S^L(t) \) be the flow map \( e^{it\Delta} \) corresponding to the the linear Schrödinger equation. Then the Duhamel formulation of (1.1) is

\[
\phi(t) = S^L(t)\phi_0 - i \int_0^t S^L(t - \tau)|x|^{-2} * |\phi|^2\phi(\tau)d\tau.
\]

We will show that the map \( A : \phi \rightarrow S^L(t)\phi_0 - i \int_0^t S^L(t - \tau)|x|^{-2} * |\phi|^2\phi(\tau)d\tau \) is a contraction mapping on the ball \( \|\phi\|_{S^s_T} \leq 2M \) when \( T \) is chosen later and \( \|\phi_0\|_{H^s} < M \).

Let us show \( A \) is well defined on \( X \). Applying the linear and the dual Strichartz estimates, we have

\[
\|A\phi\|_{S^s_T} \lesssim \|\phi_0\|_{H^s} + \|\phi\|_{L^q_tH^s} + \|\phi\|_{L^r_tH^s} \lesssim 2M \quad (2.7)
\]

for any admissible \( (\lambda, \eta) \). We recall the Leibnitz rule for fractional Sobolev spaces \([6, 30]\): For \( s > 0, 1 < p < \infty \),

\[
\|fg\|_{H^s,p} \lesssim \|f\|_{L^{q_1}}\|g\|_{H^{s,q_2}} + \|f\|_{L^{r_1}}\|g\|_{H^{s,r_2}}
\]

provided \( \frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2} \), with \( q_2, r_2 \in (1, \infty) \) and \( q_1, r_1 \in (1, \infty) \).

Let us choose \( (\lambda', \eta') = (\frac{4}{3+s}, \frac{2n}{2n+s+1}) \). The fractional Leibnitz rule, Hardy-Sobolev inequality and Hölder’s inequality lead to

\[
\|(|x|^{-2} * |\phi|^2\phi)\|_{H^{s, \frac{2n}{2n+s+1}}} \lesssim \|(|x|^{-2} * |\phi|^2\phi)\|_{L^{\frac{12}{3+s}}H^{s, \frac{2n}{2n+s+1}}} + \|(|x|^{-2} * |\phi|^2\phi)\|_{L^{\frac{12}{3+s}}H^{s, \frac{2n}{2n+s+1}}} \lesssim \|\phi\|^2_{L^{\frac{12}{3+s}}(\mathbb{R}^n)} + \|\phi\|^2_{L^{\frac{12}{3+s}}(\mathbb{R}^n)} \lesssim 2\|\phi\|^2_{L^{\frac{12}{3+s}}(\mathbb{R}^n)} \lesssim \|\phi\|^2_{H^{s, \frac{2n}{2n+s+1}}}. \quad (2.8)
\]

By the Sobolev embedding we have

\[
\|(|x|^{-2} * |\phi|^2\phi)\|_{H^{s, \frac{2n}{2n+s+1}}} \lesssim \|\phi\|^3_{H^{s, \frac{2n}{2n+s+1}}}
\]

Combining this with (2.7) we find

\[
\|A\phi\|_{S^s_T} \lesssim \|\phi_0\|_{H^s} + \left( \int_0^T \|\phi\|_{H^{s, \frac{2n}{2n+s+1}}}^{\frac{3+s}{4}} dt \right)^{\frac{4}{3+s}} \lesssim \|\phi_0\|_{H^s} + T^s \|\phi\|_{L^{\frac{12}{3+s}}H^{s, \frac{2n}{2n+s+1}}(\mathbb{R}^n)} \lesssim \|\phi_0\|_{H^s} + T^s \|\phi\|_{S^s_T}.
\]
The local well-posedness time \( T \) is chosen as \( T \lesssim \| \phi_0 \|_{H^s}^{-2} \). Similarly, one can show that \( A \) is a contraction. And uniqueness assertion and continuous dependence on data follow in the same manner.

\[
T \lesssim \| \phi_0 \|_{H^s}^{-2}.
\]

### 3. Almost conservation law of the modified energy

In this section, we define the smoothing operator \( I_N \), which sends an \( H^s \) function to an \( H^1 \) function. We find a bound of the growth of \( E(I_N \phi)(t) \) in time.

The operator \( I_N \) is defined as in [9]. Let \( N \gg 1 \) be a parameter to be chosen later. Define

\[
\hat{I}_N f(\xi) \equiv m(\xi) \hat{f}(\xi),
\]

where the multiplier \( m(\xi) \) is smooth, radially symmetric, nonincreasing in \( |\xi| \) and satisfies

\[
m(\xi) = \begin{cases} 
1 & |\xi| \leq N \\
\left( \frac{N}{|\xi|} \right)^{1-s} & |\xi| \geq 2N.
\end{cases}
\]

We note that \( m(\xi) \) satisfies the Hörmander multiplier condition. As intended, the definition of \( m(\xi) \) gives the following relations between \( \| I_N \phi \|_{H^1} \) and \( \| \phi \|_{H^s} \) for \( 0 < s < 1 \);

\[
\| I_N \phi \|_{H^1(\mathbb{R}^n)} \lesssim \sum_{k \leq \log N} (1 + 2^k) \| P_k \phi \|_{L^2(\mathbb{R}^n)} + \sum_{k > \log N} N^{-s}(1 + 2^k)^s \| P_k \phi \|_{L^2(\mathbb{R}^n)}
\]

\[
\lesssim N^{-s} \| \phi \|_{H^s(\mathbb{R}^3)}
\]

\[
\| \phi \|_{H^s(\mathbb{R}^n)} \lesssim \sum_{k \leq \log N} (1 + 2^k)^s \| P_k I \phi \|_{L^2(\mathbb{R}^n)} + \sum_{k > \log N} (1 + 2^k) N^{s-1} \| P_k I \phi \|_{L^2(\mathbb{R}^n)}
\]

\[
\lesssim \| I_N \phi \|_{H^1(\mathbb{R}^n)},
\]

where \( P_k \phi \) is defined by \( \hat{P_k \phi}(\xi) = \varphi(\xi/2^k)\hat{\phi}(\xi) \) for a nonnegative smooth function \( \varphi \) with \( \text{supp } \varphi = \{ |\xi| \geq 2 \} \) and \( \sum_{k \in \mathbb{Z}} \varphi(2^{-k} \xi) = 1 \). What it follows we write \( I \) for \( I_N \) suppressing \( N \).

Let us define the iteration space \( Z_I(t) \) as

\[
Z_I(t) = \sup_{(q,r) \text{ admissible}} \| (\nabla) I \phi \|_{L_q^q L^r_x([0,t] \times \mathbb{R}^3)}.
\]

#### 3.1. Modified local theory

First of all, we prove a local well-posedness result for the modified solution \( I \phi \). This theorem is essentially similar to the local well-posedness proof at the critical regularity in [2]. But here we assume critical Strichartz norm of \( I \phi \) is small, instead of \( \phi \). Similar proofs are found in [7], [13].

**Lemma 3.1.** For given initial data \( \phi_0 \in H^s(\mathbb{R}^n) \) for \( 0 < s \), there are time \( T^* > 0 \) and a universal constant \( \delta > 0 \) satisfying the following:

1. The solution \( \phi(x,t) \) to (3.1) exists on \( [0,T^*] \times \mathbb{R}^n \),
If
\[ \| I \phi \|_{L_t^{4(n-1)/n} L_x^{2(n-1)/(n+2)}([0,T^\ast] \times \mathbb{R}^n)} \leq \delta, \]
then
\[ Z_I(T^\ast) \lesssim \| \langle \nabla \rangle I \phi_0 \|_{L^2(\mathbb{R}^n)}. \]

Proof.
The first part is from the local well-posedness theorem, Theorem 2.1. The second part is also done by the Strichartz estimate (2.1) in the Duhamel formula with \( \langle \nabla \rangle I \) operator:
\[ \langle \nabla \rangle I \phi(x,t) = S^L(t) \langle \nabla \rangle I \phi_0 - i \int_0^t S^L(t - \tau) \langle \nabla \rangle I(|x|^{-2} * |\phi|^2 \phi(\tau))d\tau. \]

For all \( 0 \leq t \leq T^\ast \), we have
\[ Z_I(t) \lesssim \| I \phi_0 \|_{H^1} + \| \langle \nabla \rangle I(|x|^{-2} * |\phi|^2 \phi) \|_{L_t^\gamma' L_x^{r'}} \]
\[ \lesssim \| I \phi_0 \|_{H^1} + \| (|x|^{-2} * \langle \nabla \rangle I|\phi|^2) \|_{L_t^\gamma' L_x^{r'}} + \| ((|x|^{-2} * |\phi|^2) \langle \nabla \rangle I \phi) \|_{L_t^\gamma' L_x^{r'}}, \]
where \( (\gamma, \rho) \) is admissible. In the previous step we have used Leibniz’s rule for \( \langle \nabla \rangle I \). Note that in the high frequency \( (|\xi| > N) \), \( I \) is a negative derivative, but \( \langle \nabla \rangle I \) is a positive fractional derivative. A simple modification of the proof of the fractional Leibniz rule works for it. Let us choose \( (\gamma, \rho) = (4, \frac{2n}{n+2}) \). In fact we can use any admissible pair satisfying \( \gamma \geq \frac{2(n-1)}{n-2} \). We first estimate \( \| (|x|^{-2} * |\phi|^2) \langle \nabla \rangle I \phi \|_{L_t^4 L_x^{2p}} \). By using Hölder’s, fractional Sobolev’s inequalities, we obtain
\[ \| (|x|^{-2} * |\phi|^2) \langle \nabla \rangle I \phi \|_{L_t^4 L_x^{2p}} \]
\[ \leq \| \phi \|_{L_t^{4q_1} L_x^{2q_2}}^2 \| \langle \nabla \rangle I \phi \|_{L_t^{q_2} L_x^{3}} \]
\[ \leq \| \phi \|_{L_t^{q_1} L_x^{2p}}^2 \| \langle \nabla \rangle I \phi \|_{L_t^{q_2} L_x^{3}} \]  \[ \leq \| \phi \|_{L_t^{q_1} L_x^{2p}}^2 Z_I, \]  \[ \text{(3.13)} \]
where
\[ \frac{3}{4} = \frac{1}{q_1} + \frac{1}{q_2}, \quad \frac{2n}{n+1} = \frac{1}{r_1} + \frac{1}{r_2}, \quad 1 + \frac{1}{r_1} = \frac{2}{n} + \frac{1}{p} \]
and \((q_2, r_2)\) is admissible. Due to scaling argument, \((2q_1, 2p)\) is expected to be admissible. Let \((2q_1, 2p) = (\frac{4(n-1)}{n}, \frac{2(n-1)}{n-2})\), then \((q_2, r_2) = (\frac{4(n-1)}{n-3}, \frac{2n(n-1)}{n^2-2n+3})\).

In a similar way, the other term is also estimated as follows:
\[ \| (|x|^{-2} * \langle \nabla \rangle I|\phi|^2) \|_{L_t^\gamma' L_x^{r'}} \leq \| \phi \|_{L_t^{4(n-1)/n} L_x^{2(n-1)/(n+2)}}^2 Z_I \]
\[ \text{(3.14)} \]

Now we estimate \( \| \phi \|_{L_t^{4(n-1)/n} L_x^{2(n-1)/(n+2)}}^2 \). We decompose \( \phi \) into its frequency localized pieces, \( \phi = P \leq N \phi + \sum_{j=1}^{\infty} P_{N_j} \phi \), where \( N_j = 2^{k_j} \) and and \( k_j \)'s are consecutive integers starting
from \([\log N]\) indexed by \(j = 1, 2, 3 \cdots\). By triangle inequality we get

\[
\|\phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}} \leq \|P_{\leq N} \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}} + \sum_{j=1}^{\infty} \|P_{N_j} \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}} + \sum_{j=1}^{\infty} \|P_{N_j} \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}} \|P_{N_j} \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}}.
\]

From the definition of \(I\) operator we have the followings:

\[
\|P_{\leq N} \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}} = \|I \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}},
\]

\[
\|P_{N_j} \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}} \lesssim N_j^{1-s} N^{s-1} \|IP_{N_j} \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}}.
\]

Putting these together into (3.15), we obtain

\[
\|\phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}} \lesssim \|P_{\leq N} \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}} + \sum_{j=1}^{\infty} N_j^{-s+\epsilon} N^{s-1} \|IP_{N_j} \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}} + \|\langle \nabla \rangle IP_{N_j} \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}} \|\langle \nabla \rangle I \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}}.
\]

Ignoring \(N_j^{s-1} \leq 1\) and using the fact that \(\|P_{N_j} f\|_{L^p} \lesssim \|f\|_{L^p}\), one can sum up over \(j\), if \(s > \epsilon\). Thus, we have

\[
\|\phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}} \lesssim \|P_{\leq N} \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}} + \|I \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}} \|\langle \nabla \rangle I \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}}.
\]

Hence, from (3.12) we conclude

\[
Z_I \lesssim \|I \phi_0\|_{H^1} + Z_I \|I \phi\|^2_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}} + Z_I^{3-2\epsilon} \|I \phi\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}}.
\]

Choosing sufficiently small \(\delta\) and \(T^*\), we conclude the proof. \(\square\)

3.2. Almost conservation law. We show the almost conservation law of the modified energy.
The usual energy \(1.2\) is shown to be conserved by differentiating in time

\[
\frac{d}{dt} E(\phi)(t) = \int_{\mathbb{R}^n} 2\text{Re} \partial_t \phi (2(|x|^{-2} * |\phi|^2)\phi - \Delta \phi - 2\partial_t \phi) + (|x|^{-2} * \partial_t |\phi|^2)|\phi|^2 - (|x|^{-2} * |\phi|^2)\partial_t |\phi|^2 \, dx
\]

\[
= \int_{\mathbb{R}^n} (|x|^{-2} * \partial_t |\phi|^2)|\phi|^2 - (|x|^{-2} * |\phi|^2)\partial_t |\phi|^2 \, dx = 0,
\]

using the equation \(1.1\). Since \(I\phi\) is not a solution to the equation \(1.1\), \(E(I\phi)(t)\) is not conserved. But still we have a control of the time increment of the modified energy \(E(I\phi)(t)\). Differentiating \(E(I\phi)(t)\) in time, we obtain

\[
\frac{d}{dt} E(I\phi)(t) = \int_{\mathbb{R}^n} 2\text{Re} \partial_t I\phi [2(I(|x|^{-2} * |\phi|^2)\phi) - \Delta I\phi - 2i\partial_t I\phi] \, dx.
\]

Then we have

\[
E(I\phi(T)) - E(I\phi(0)) = 4\text{Re} \int_0^T \int_{\mathbb{R}^n} \partial_t I\phi [(|x|^{-2} * |I\phi|^2)I\phi - I((|x|^{-2} * |\phi|^2)\phi)] \, dx \, dt := E_T(t)
\]

(3.16)

The following proposition shows that \(E(I\phi)\) is an almost conserved quantity.

**Proposition 3.1.** Assume we have \(s > 0\), \(N \gg 1\), \(\phi_0 \in C_0^\infty(\mathbb{R}^n)\), and a solution of \(1.1\) on a time interval \([0, T]\) for which

\[
\|I\phi\|_{L_t^4 L_x^{4(n-1)/n} \cap L^2_{x} (0,T) \times \mathbb{R}^n} \lesssim \delta.
\]

Assume in addition that \(\|\nabla I\phi_0\| \lesssim 1\). Then we conclude that for all \(t \in [0, T]\),

\[
E(I\phi)(t) = E(I\phi_0) + O(N^{-1+}).
\]

**Proof of Proposition 3.1.**

We compute in the frequency space. Applying the Parseval formula to \(E_T\) in (3.16), we obtain

\[
E_T = \text{Re} \int_0^T \int_{\mathbb{R}^n} \left( 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) \prod_{j=1}^4 |\xi_2 + \xi_3|^{-2} \partial^4 \phi(\xi_1) \overline{\partial^4 \phi(\xi_2) \partial^4 \phi(\xi_3) \partial^4 \phi(\xi_4)} \, d\xi_1 d\xi_2 d\xi_3 d\xi_4 \, dt.
\]

(3.17)
Now if we use equation (1.1) to substitute for $\partial_t I\phi$ in (3.17), then it is split into two terms as follows:

\[
E_a = \left| \int_0^T \sum_{j=1}^4 e^{i \sigma_0 (\xi_1, \xi_2, \xi_3, \xi_4)} \int_0^T \sum_{j=1}^4 e^{i \sigma_0 (\xi_1, \xi_2, \xi_3, \xi_4)} \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \left| \xi_2 + \xi_3 \right|^{-(n-2)} \right. \\
\left. \times |\Delta T_\phi(\xi_1) T_\phi(\xi_2) T_\phi(\xi_3) T_\phi(\xi_4)| d\xi_1 d\xi_2 d\xi_3 d\xi_4 dt \right|
\]

\[
E_b = \left| \int_0^T \sum_{j=1}^4 e^{i \sigma_0 (\xi_1, \xi_2, \xi_3, \xi_4)} \int_0^T \sum_{j=1}^4 e^{i \sigma_0 (\xi_1, \xi_2, \xi_3, \xi_4)} \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \left| \xi_2 + \xi_3 \right|^{-(n-2)} \right. \\
\left. \times |\tilde{T}(|x|^{-2} * \phi^2 \phi)|^{j+1} \tilde{T}(\xi_1) \tilde{T}(\xi_2) \tilde{T}(\xi_3) \tilde{T}(\xi_4)| d\xi_1 d\xi_2 d\xi_3 d\xi_4 dt \right|
\]

In both cases, we break down $\phi$ into Littlewood-Paley pieces $\phi_j$, each localized in $2^k$, in frequency, $(\xi_j) \sim 2^{kj} = N_j$, $k_j = 0, 1, 2, \cdots$, and then use a version of Coifman-Meyer estimate for a class of multiplier operators.

**Proposition 3.2** (Proposition 6.1 in [3]). Let $\sigma(\xi)$ be infinitely differentiable so that for all $\alpha \in N^n$ and all $\xi = (\xi_1, \cdots, \xi_k) \in \mathbb{R}^n$. Then there is a constant $c(\alpha)$ with

\[
|\partial^\alpha \sigma(\xi)| \leq c(\alpha) (1 + |\xi|)^{-|\alpha|}.
\]

Let the multi-linear operator $\Lambda$ be given

\[
[\Lambda(f_1, \cdots, f_k)](x) = \int_{\mathbb{R}^n} e^{i x (\xi_1 + \cdots + \xi_k)} \sigma(\xi_1, \cdots, \xi_k) \left| \xi_2 + \xi_3 \right|^{-(n-2)} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \cdots \hat{f}_k(\xi_k) \ d\xi_1 \cdots d\xi_k
\]

for $k \geq 2$. Then we have

\[
\|\Lambda(f_1, \cdots, f_k)\|_{L^p} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \cdots \|f_k\|_{L^{p_k}}
\]

where $(p, p_i)$ is related by $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k}$.

We first estimate a pointwise bound on the symbol

\[
1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \leq B(N_2, N_3, N_4)
\]

Factoring $B(N_1, N_2, N_3)$ out of the integral in $E_T$, it leaves a symbol $\sigma_1$, which satisfies the condition of Proposition 3.2, as the following:

\[
\sum_{N_1, N_2, N_3, N_4} B(N_2, N_3, N_4) \int_0^T \int_{\mathbb{R}^n} [\Lambda(\Delta T_\phi, \partial_\phi T_\phi, \partial_\phi T_\phi)](\xi_4) \tilde{T}(\phi_4)(\xi_4) d\xi_4 dt
\]

\[
+ \sum_{N_1, N_2, N_3, N_4} B(N_2, N_3, N_4) \int_0^T \int_{\mathbb{R}^n} [\Lambda(\partial_\phi T_{\phi_1}, \partial_\phi T_{\phi_2}, \partial_\phi T_{\phi_3})](\xi_4) \tilde{T}(\phi_4)(\xi_4) d\xi_4 dt
\]

where

\[
[\Lambda(f, g, h)](x) = \int_{\mathbb{R}^3} e^{i x (\xi_1 + \xi_2 + \xi_3)} \sigma_1(\xi_1, \xi_2, \xi_3) \left| \xi_2 + \xi_3 \right|^{-(n-2)} \hat{f}(\xi_1) \hat{g}(\xi_2) \hat{h}(\xi_3) \ d\xi_1 d\xi_2 d\xi_3
\]
and

$$\sigma_1(\xi_1, \xi_2, \xi_3) = 1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_1 + \xi_2 + \xi_3)} \left| 1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_1 + \xi_2 + \xi_3)} \right|.$$  

We shall show that

$$E_a + E_b \lesssim N^{-1+}(Z)(T)^p$$

for some $P > 0$.

For this aim, we claim that

$$\sum_{N_1,N_2,N_3,N_4} \int_0^T \int_{\mathbb{R}^n} B(N_2, N_3, N_4) \left[ \Delta I\phi_1, I\phi_2, I\phi_3 \right] (x, t) I\phi_4(x, t) \, dx \, dt$$

(3.19)

$$+ \sum_{N_1,N_2,N_3,N_4} \int_0^T \int_{\mathbb{R}^n} B(N_2, N_3, N_4) \left[ I\phi_1, I\phi_2, I\phi_3 \right] (x, t) I\phi_4(x, t) \, dx \, dt$$

(3.20)

$$\lesssim N^{-1+} (Z)(T)^4 + Z(T)^6.$$  

From Proposition 3.2 we have

$$\|\Delta(f, g, h)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \|h\|_{L^{p_3}}$$  

(3.21)

where $\frac{1}{p} = \frac{2}{n} - 1 + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$.

For the first term $E_a$, we use (3.21) and Hölder inequality to get

$$\left| \int_0^T \int_{\mathbb{R}^n} B(N_2, N_3, N_4) \left[ \Delta I\phi_1, I\phi_2, I\phi_3 \right] (x, t) I\phi_4(x, t) \, dx \, dt \right|$$

(3.22)

$$\lesssim \|\Delta I\phi_1\|_{L^{p_1}_{L^p}} \|I\phi_2\|_{L^{p_2}_{L^p}} \|I\phi_3\|_{L^{p_3}_{L^p}} \|I\phi_4\|_{L^{q_4}_{L^q}} \lesssim B(N_2, N_3, N_4) \frac{N_1}{N_2 N_3 N_4} (Z(T))^4.$$  

We reduce to show

$$\sum_{N_1,N_2,N_3,N_4} B(N_2, N_3, N_4) \frac{N_1}{N_2 N_3 N_4} \lesssim N^{-1+\epsilon}$$

(3.23)

By symmetry we may assume $N_2 \geq N_3 \geq N_4$. Then it suffices to consider the following three cases.

**Case 1: $N \gg N_2$.** We have $m(\xi_1) = 1$ since $\sum_i \xi_i = 0$. So, the symbol

$$1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)} = 0.$$
Case 2: \( N_2 \geq N \gg N_3 \geq N_4 \). Since \( \sum_i \xi_i = 0 \), we have \( N_1 \sim N_2 \). By the mean value theorem,

\[
|1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)}| \leq \frac{|m(\xi_2) - m(\xi_2 + \xi_3 + \xi_4)|}{m(\xi_2)} \lesssim \frac{\nabla m(\xi_2) \cdot (\xi_3 + \xi_4)}{m(\xi_2)} \lesssim \frac{N_3}{N_2}.
\]

Thus,

\[
(3.19) \lesssim \frac{1}{N_2N_4}(Z_1(T))^4 \lesssim N^{-1+\epsilon}N_2^{-\epsilon}(Z_1(T))^4
\]

Summing up with \( N_4, N_3, N_2 \), we have (3.23).

Case 3: \( N_2 \geq N_3 \gg N \). In this case we need to consider two subcases \( N_1 \sim N_2 \) and \( N_2 \gg N_1 \) since by \( \sum_i \xi_i = 0 \) the case \( N_1 \gg N_2 \) cannot happen.

For the first case, \( N_1 \sim N_2 \), we estimate

\[
|1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)}| \frac{N_1}{N_2N_3N_4} \lesssim \frac{1}{N_3m(\xi_3)N_4m(\xi_4)} \lesssim \frac{1}{N_3^2} \frac{N^s}{N_4m(\xi_4)} \sim N^{-1+\epsilon}N_3^{-\epsilon}
\]

since \( xm(x) \geq 1 \) for \( x \geq 1 \). We can sum up \( N_4, N_3 \) directly. But when summing up \( N_2 \), we use the Cauchy-Schwartz inequality with \( \phi_i = P_{N_i}I\phi \) as follows:

\[
\sum_N P_{N_1} \nabla I\phi \cdot P_{N_2} \nabla I\phi \leq \left( \sum_N (P_{N_1} \nabla I\phi)^2 \right).
\]

In the second case, \( N_2 \gg N_1 \), again by \( \sum_i \xi_i = 0 \), we have \( N_2 \sim N_3 \).

\[
|1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}| \frac{N_1}{N_2N_3N_4} \lesssim \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)} \frac{N_1}{N_2N_3N_4} \sim N_1m(\xi_1)\frac{1}{N_2} \frac{N^{2s}}{N_2} \cdot \frac{1}{N_4m(\xi_4)}
\]

For our purpose, we want to show

\[
\left( \frac{N}{N_2} \right)^{2s} N_1m(\xi_1) \lesssim N.
\]

If \( N_1 \leq N \), then \( m(\xi_1) = 1 \) and this is true. If \( N_1 \gg N \), then

\[
\left( \frac{N}{N_2} \right)^{2s} N_1m(\xi_1) = \frac{N^{1+s}}{N_2^s} \cdot \left( \frac{N_1}{N_2} \right)^s = N \left( \frac{NN_1}{N_2^s} \right)^s \lesssim N.
\]

This conclude the proof of (3.19). Now we turn to the estimate of \( E_b \). The above analysis is applied to \( E_b \), once we show the following lemma.
Lemma 3.2.

\[ \| P_M I ([|x|^{-2} \ast (\phi_1 \phi_2)] \phi_3) \|_{L^4_t L^2_x} \lesssim M(\mathcal{Z}_I(T))^3 \]  
\( (3.24) \)

**Proof.** We divide \( \phi \) into \( \phi = \phi_{lo} + \phi_{hi} \) where

\[
\text{supp} \hat{\phi}_{lo}(\xi, t) \subseteq \{ |\xi| < 2 \}
\]

\[
\text{supp} \hat{\phi}_{hi}(\xi, t) \subseteq \{ |\xi| > 1 \}
\]

In the case that all \( \phi \)'s are \( \phi_{lo} \) we simply estimate

\[
\begin{align*}
\| P_M (I [(|x|^{-2} \ast \phi_{lo} \bar{\phi}_{lo}) \phi_{lo}] \|_{L^4_t L^2_x} & = \| (|x|^{-2} \ast \phi_{lo} \bar{\phi}_{lo}) \phi_{lo} \|_{L^4_t L^2_x} \\
& \lesssim \| \phi_{lo} \|_{3 L^{12}_t L^{\frac{6n}{n-4}}_x} \\
& \lesssim \| \langle \nabla \rangle I \phi_{lo} \|_{3 L^{12}_t L^{\frac{6n}{n-4}}_x} \\
& \lesssim M(\mathcal{Z}_I(T))^3.
\end{align*}
\]

When all \( \phi \)'s are \( \phi_{hi} \), we use Bernstein inequality, Sobolev embedding and the Leibniz rule as following:

\[
\begin{align*}
\| 1_M P_M (I [(|x|^{-2} \ast \phi_{hi} \bar{\phi}_{hi}) \phi_{hi}] \|_{L^4_t L^2_x} & = \| (|x|^{-2} \ast \phi_{hi} \bar{\phi}_{hi}) \phi_{hi} \|_{L^4_t L^2_x} \\
& \lesssim \| \nabla^{-1} P_M (I [(|x|^{-2} \ast \phi_{hi} \bar{\phi}_{hi}) \phi_{hi}] \|_{L^4_t L^2_x} \\
& \lesssim \| \langle \nabla \rangle \frac{n}{3n-4} I [(|x|^{-2} \ast \phi_{hi} \bar{\phi}_{hi}) \phi_{hi}] \|_{3 \frac{2n^2 + n - 4}{(3n - 4)2n}} \\
& \lesssim \| \langle \nabla \rangle \frac{n}{3n-4} I \phi_{hi} \|_{3 L^{12}_t L^p_x} \\
& \text{where } \frac{1}{p} = -\frac{2}{n} + 1 + \frac{3n^2 + 1 - 4}{(3n - 4)2n} \\
& \lesssim \| \langle \nabla \rangle^1 \phi_{hi} \|_{3 L^{12}_t L^{\frac{6n}{n-4}}_x} \\
& \lesssim (\mathcal{Z}_I(T))^3.
\end{align*}
\]

The remaining \( lo - hi \) cases are controlled in a similar manner to the \( hi - hi \) case. We omit the detail here. \( \square \)

Hence, we have shown (3.19), (3.20) and so conclude the proof. \( \square \)

**4. Almost interaction Morawetz estimate in** \( \mathbb{R}^n, n \geq 3 \)**

In this section, we show the almost interaction Morawetz inequality. Let us start by recalling the higher dimensional interaction Morawetz inequality for a general nonlinearity. The interaction Morawetz inequality was developed in [9] in \( \mathbb{R}^3 \) and this higher dimensional
extension was derived in \cite{29}. We first recall higher dimensional interaction Morawetz
inequality for a general nonlinearity.

**Lemma 4.1** (\cite{29}, Proposition 5.5). Let $\phi$ solve

$$i\partial_t \phi + \frac{1}{2} \Delta \phi = \mathcal{N}$$

on $I \times \mathbb{R}^n$. Assume that $\text{Im}(\mathcal{N}\phi) = 0$.

Then, we have

$$- \int_I \int_{\mathbb{R}^n \times \mathbb{R}^n} \Delta \left( \frac{1}{|y-x|} \right) |\phi(x,t)|^2 |\phi(y,t)|^2 dx dy dt$$

$$+ 2 \int_I \int_{\mathbb{R}^n \times \mathbb{R}^n} |\phi(x,t)|^2 \frac{y-x}{|y-x|} \cdot \{\mathcal{N}, \phi\}(y,t) dx dy dt$$

$$\lesssim \|\phi_0\|_{L_x^2}^2 \|\phi\|_{L^{\infty}_x H^{\frac{1}{2}}([0,T] \times \mathbb{R}^n)}^2$$

where $\{f,g\} = \text{Re}(f \nabla \overline{g} - g \nabla \overline{f})$.

First, we apply \eqref{4.25} to the solution to \eqref{1.1}, where $\mathcal{N} = (|x|^{-2} * |\phi|^2) \phi$. A computation shows that the second term is positive.

the second term of \eqref{4.25} = $-2 \int |\phi(x,t)|^2 \frac{y-x}{|y-x|} \cdot \nabla_y \frac{1}{|y-z|^2} |\phi(z,t)|^2 |\phi(y,t)|^2 dx dy dz$

$$= 4 \int |\phi(x,t)|^2 |\phi(y,t)|^2 |\phi(z,t)|^2 \frac{y-z}{|y-z|^4} dx dy dz$$

$$= 2 \int |\phi(x,t)|^2 |\phi(y,t)|^2 |\phi(z,t)|^2 \left( \frac{y-x}{|y-x|} - \frac{z-x}{|z-x|} \right) \cdot \frac{y-z}{|y-z|^3} dx dy dz$$

$$\geq 0$$

By the same analysis as in \cite{29}, we obtain several estimates of space-time $L^q_x L^p_t$-norms.

**Proposition 4.1.** Let $\phi(t,x)$ be a classical solution to \eqref{1.1}. Then we have
when $n = 3$,

$$\|\phi\|_{L_t^4 L_x^6(\mathbb{R} \times \mathbb{R}^3)} \lesssim \|\phi_0\|_{L_x^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\phi(t)\|_{L^\infty_x H^\frac{1}{2}_x(\mathbb{R} \times \mathbb{R}^3)}^{\frac{1}{2}}$$

\(4.26\)

when $n \geq 4$,

$$\||\nabla| \frac{n-3}{n-2} \phi\|_{L_t^4 L_x^{\frac{2(n-1)}{n-2}}([0,T] \times \mathbb{R}^n)} \lesssim \|\phi_0\|_{L_x^2}^{\frac{1}{2}} \|\phi(t)\|_{L^\infty_x H^\frac{1}{2}_x([0,T] \times \mathbb{R}^n)}^{\frac{1}{2}}$$

\(4.27\)

$$\|\phi\|_{L_t^{2(n-1)} L_x^{2(n-1)}([0,T] \times \mathbb{R}^n)} \leq \|\phi_0\|_{L_x^2}^{\frac{1}{2}} \|\phi\|_{L^\infty_x H^\frac{1}{2}_x([0,T] \times \mathbb{R}^n)}^{n-1}$$

\(4.28\)

$$\|\phi\|_{L_t^{4(n-1)} L_x^{4(n-1)}([0,T] \times \mathbb{R}^n)} \leq \|\phi_0\|_{L_x^2}^{\frac{1}{2}} \|\phi\|_{L^\infty_x H^\frac{1}{2}_x([0,T] \times \mathbb{R}^n)}^{n-1}.$$ 

\(4.29\)
\[ L^2 \text{-critical Hartree equations} \]

**Proof.** A detailed proof is found in [29], Section 5. Here we give a sketch. In dimension \( n = 3 \), we have formally \(-\Delta \frac{1}{|x|} = 4\pi \delta \), and then (4.26) follows.

In higher dimension, \( n \geq 4 \), we obtain \(-\Delta \frac{1}{|x|} = \frac{n-3}{|x|^3} \). A convolution with \( \frac{1}{|x|^3} \) is essentially to take the fractional derivative \( |\nabla|^{-(n-3)} \). Hence we obtain from (4.25)

\[
|\nabla| -\frac{n-3}{2} |\phi|^2}_{L^1_{t,x}([0,T] \times \mathbb{R}^n)} \lesssim \|\phi_0\|_{L^2_{x}(\mathbb{R}^n)} \|\phi\|_{L^\infty_{t}H^{\frac{1}{2}}((0,T) \times \mathbb{R}^n)}.
\]

From Lemma 5.6 in [29]

\[
|\nabla| -\frac{n-3}{2} \phi \|_{L^1_{t,x}([0,T] \times \mathbb{R}^n)} \lesssim \|\phi_0\|_{L^2_{x}(\mathbb{R}^n)} \|\phi\|_{L^\infty_{t}H^{\frac{1}{2}}((0,T) \times \mathbb{R}^n)}.
\]

(4.30)

Interpolation between (4.30) and the trivial estimate

\[
\|\nabla^{\frac{1}{2}} \phi\|_{L^\infty_t L^2_x} \lesssim \|\phi\|_{L^\infty_t H^{\frac{1}{2}}_{x}}
\]

and using the Hölder’s inequality in time we have

\[
\|\phi\|_{L^{4(n-1)}_{t}L^\infty_{x}H^{\frac{1}{2}}_{n-2}((0,T) \times \mathbb{R}^n)} \leq T^\frac{n-2}{(n-1)} \|\phi_0\|_{L^2_{x}} \|\phi\|_{L^\infty_{t}H^{\frac{1}{2}}_{x}}.
\]

(4.31)

For the initial data below \( \dot{H}^{\frac{1}{2}} \), the above estimate is not useful since \( \dot{H}^{\frac{1}{2}} \)-norm of the solution may not be finite. To overcome this difficulty, we use the interaction Morawetz inequality into the smoothed solution \( I\phi \). Write the \( I \)-Hartree equation as the following:

\[
iI\phi_t + \frac{1}{2} \Delta I\phi = (|x|^{-2} * I\phi \overline{I\phi})I\phi + |I((|x|^{-2} * \overline{I\phi})\phi) - (|x|^{-2} * I\phi \overline{I\phi})I\phi|
\]

\[=: \mathcal{N}_{\text{good}} + \mathcal{N}_{\text{bad}}.\]

Then using (4.25) we obtain

\[
-\int_{I} \int_{\mathbb{R}^n \times \mathbb{R}^n} \Delta \left( \frac{1}{|y-x|} \right) |I\phi(x,t)|^2 |I\phi(y,t)|^2 dxdydt + 2 \int_{I} \int_{\mathbb{R}^n \times \mathbb{R}^n} |I\phi(x,t)|^2 \frac{y-x}{|y-x|} \cdot \{\mathcal{N}_{\text{good}}, I\phi\}(y,t) dxdydt
\]

\[+ 2 \int_{I} \int_{\mathbb{R}^n \times \mathbb{R}^n} |I\phi(x,t)|^2 \frac{y-x}{|y-x|} \cdot \{\mathcal{N}_{\text{bad}}, I\phi\}(y,t) dxdydt \leq \|I\phi\|_{L^\infty_t L^2_x} \|I\phi\|_{L^\infty_t \dot{H}^{\frac{1}{2}}((0,T) \times \mathbb{R}^n)}^2 \]

\[+ \int_{0}^{T} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\text{Im}(\mathcal{N}_{\text{bad}}(I\phi(t,y)) \nabla (I\phi(t,x)))| dxdydt \]

(4.32)
By the same computation as above, one can see the second term of (4.32) is positive. We wish the third term involving $N_{bad}$ to be small. Similarly to (4.31) we have
\[
\|I \phi\|_{L_t^4 L_x^{2(n-1)/n}([0,T] \times \mathbb{R}^n)} \lesssim T^{n/2(n-1)} \left( \|I \phi\|_{L_t^4 L_x^{2(n-1)/n}}^{1/2} + \|I \phi\|_{L_t^4 L_x^{2(n-1)/n}}^{1/2} + \text{Error} \right),
\]
where Error is defined in the lemma below.

**Lemma 4.2.** On a time interval $J$ where the local well-posedness in Theorem 2.1 holds true, we have that
\[
\text{Error} = \int \int_{\mathbb{R}^n \times \mathbb{R}^n} |I \phi(x,t)|^2 \frac{y - x}{|y - x|} \cdot \{N_{bad}, I \phi\}(y,t) dx dy dt
+ \int \int_{\mathbb{R}^n \times \mathbb{R}^n} \text{Im}(N_{bad} \nabla \phi(t,y)) \nabla (I \phi(t,x)) I \phi(t,x) dx dy dt \\
\lesssim \frac{1}{N_{1-}^2} Z_I(J)^6.
\]
In particular, if we assume $\|\langle \nabla \rangle I \phi_0\|_{L^2} \lesssim 1$ and $\|I \phi\|_{L_t^4 L_x^{2(n-1)/n}([0,T] \times \mathbb{R}^n)} \lesssim \delta$, then
\[
\text{Error} \lesssim \frac{1}{N_{1-}^2}.
\]

**Proof.** We rewrite the error term via $N_{bad} = I((|x|^{-2} \ast \phi \bar{\phi}) \phi) - (|x|^{-2} \ast I \phi \bar{\phi}) I \phi$:
\[
\begin{align*}
\text{Error} &= \int \int_{\mathbb{R}^n \times \mathbb{R}^n} |I \phi(x,t)|^2 \frac{y - x}{|y - x|} \cdot (N_{bad} \nabla \phi - I \phi \nabla N_{bad})(y,t) dx dy dt \\
&\quad + \int \int_{\mathbb{R}^n \times \mathbb{R}^n} \text{Im}(N_{bad} \nabla \phi(t,y)) \nabla (I \phi(t,x)) I \phi(t,x) dx dy dt \\
&\leq \int \int_{\mathbb{R}^n} |N_{bad}| \cdot |\nabla \phi| dy dt \|I \phi\|_{L_x^2}^2 + \int \int_{\mathbb{R}^n} |\nabla N_{bad}| \cdot |I \phi| dy dt \|I \phi\|_{L_x^2}^2 \\
&\quad + \|N_{bad}\|_{L^2_x L^2_t} \|I \phi\|_{L^6_x L_{6/5}^2_t}^2 \|\nabla \phi\|_{L^6_x L_{6/5}^2_t}^2 + \|\nabla N_{bad}\|_{L^2_x L^2_t} \|I \phi\|_{L^6_x L_{6/5}^2_t}^2 \|\nabla \phi\|_{L^6_x L_{6/5}^2_t}^2 \\
&\lesssim \|\langle \nabla \rangle N_{bad}\|_{L^2_x L^2_t} \|I \phi\|_{L^6_x L_{6/5}^2_t} \|\phi\|_{L^6_x L_{6/5}^2_t} \lesssim \|\langle \nabla \rangle [I((|x|^{-2} \ast \phi \bar{\phi}) \phi) - (|x|^{-2} \ast I \phi \bar{\phi}) I \phi]\|_{L^1_x L^6_t} \langle Z_I(J) \rangle^3.
\end{align*}
\]
We reduce to show
\[
\|\langle \nabla \rangle [I((|x|^{-2} \ast \phi \bar{\phi}) \phi) - (|x|^{-2} \ast I \phi \bar{\phi}) I \phi]\|_{L^1_x L^6_t} \lesssim \frac{1}{N_{1-}} \langle Z_I(J) \rangle^3.
\]
By Plancherel theorem in space, we have
\[
\mathcal{F} \nabla [I((|x|^{-2} \ast \phi \bar{\phi}) \phi) - (|x|^{-2} \ast I \phi \bar{\phi}) I \phi] (-\xi_1)
= \int \sum_{i=1}^4 i \xi_1 \left[ \frac{m(\xi_1) - m(\xi_2) m(\xi_3) m(\xi_4)}{m(\xi_2) m(\xi_3) m(\xi_4)} \right] \widehat{\phi}(\xi_2) \widehat{\phi}(\xi_3) \widehat{\phi}(\xi_4) \xi_2 + \xi_3 |^{-n-2} d\xi_2 d\xi_3 d\xi_4,
\]
where we ignored complex conjugates since they don’t make any differences. As we did in Section 3, we decompose $\phi$ into a sum of dyadic pieces. It is reduced to show

$$\sum_{N_2, N_3, N_4} \| \int_{\xi_i \sim N, i = 2, 3, 4} \sigma(\xi_2, \xi_3, \xi_4) \nabla \phi(\xi_2) \nabla \phi(\xi_3) \nabla \phi(\xi_4) |\xi_2 + \xi_3|^{-n/2} d\xi_2 d\xi_3 d\xi_4 \|_{L^1_{\xi_2}} \lesssim \frac{1}{N^3} (Z_1(T))^3$$

where

$$\sigma(\xi_2, \xi_3, \xi_4) = |\xi_1| \frac{m(\xi_1) - m(\xi_2)m(\xi_3)m(\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}$$

We use Proposition 3.2. Note that the exponent numerology $3 \cdot \frac{3n-4}{6n} = \frac{1}{2} + 1 - \frac{2}{n}$ and that $(3, \frac{6n}{3n-4})$ is admissible. Thus, once we show that

$$\frac{N}{|\xi_2\xi_3\xi_4|} \sigma(\xi_2, \xi_3, \xi_4) \lesssim 1,$$  \hspace{1cm} (4.36)

then we have

$$\| \int_{\xi_i \sim N, i = 2, 3, 4} \frac{1}{\xi_2\xi_3\xi_4} \sigma(\xi_2, \xi_3, \xi_4) \nabla \phi(\xi_2) \nabla \phi(\xi_3) \nabla \phi(\xi_4) |\xi_2 + \xi_3|^{-n/2} d\xi_2 d\xi_3 d\xi_4 \|_{L^1_{\xi_2}} \lesssim \frac{1}{N} (Z_1(T))^3.$$  

The proof of (4.36) is very similar to the proof of Proposition 3.1. So, a sketch is enough. We assume $N_2 \geq N_3 \geq N_4$ by symmetry and consider the following cases.

**Case 1:** $N \gg N_2$. The symbol is identically zero.

**Case 2:** $N_2 \geq N \gg N_3 \geq N_4$. Since $\sum_i \xi_i = 0$, we have $N_1 \sim N_2$. By the mean value theorem, we estimate

$$\left| 1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right| = \left| \frac{m(\xi_2) - m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)} \right| \lesssim \frac{\| \nabla m(\xi_2) \cdot (\xi_3 + \xi_4) \|}{m(\xi_2)} \lesssim \frac{N_3}{N_2}.$$  

Thus,

$$\frac{N}{|\xi_2\xi_3\xi_4|} \sigma(\xi_2, \xi_3, \xi_4) \lesssim \frac{N}{N_2N_3N_4} \frac{N_3}{N_2} \lesssim 1.$$

**Case 3:** $N_2 \geq N_3 \gtrsim N$. In this case we need to consider two subcases $N_1 \sim N_2$ and $N_2 \gg N_1$ due to $\sum_i \xi_i = 0$. 

For the first case, $N_1 \sim N_2$, we estimate
\[ \frac{NN_1}{N_2N_3N_4} \left| 1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right| \lesssim \frac{N}{N_3m(\xi_3)N_4m(\xi_4)} \]
\[ = \frac{N}{N_3(\frac{N}{N_3})^{1-s}N_4m(\xi_4)} \sim \frac{N^s}{N_3^s} \cdot \frac{1}{N_4m(\xi_4)} \]
\[ \lesssim 1, \]
where used $xm(x) \geq 1$ for $x \geq 1$.

In the second case, $N_2 \gg N_1$, again by $\sum_i \xi_i = 0$, we have $N_2 \sim N_3$.

\[ \left| 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right| \frac{NN_1}{N_2N_3N_4} \lesssim \frac{m(\xi_1)}{m(\xi_2)m(\xi_4)} \frac{NN_1}{N_2N_3N_4} \]
\[ \sim N_1m(\xi_1) \frac{1}{N^2N_2^s} \cdot \frac{N}{N_4m(\xi_4)} \]

For our purpose we want to show
\[ \left( \frac{N}{N_2} \right)^{2s} N_1m(\xi_1) \frac{1}{N} \lesssim 1. \]

If $N_1 \leq N$, then $m(\xi_1) = 1$ and
\[ \left( \frac{N}{N_2} \right)^{2s} \frac{N_1}{N} \lesssim 1. \]

If $N_1 \gtrsim N$, then
\[ \left( \frac{N}{N_2} \right)^{2s} N_1m(\xi_1) \frac{1}{N} = \frac{N^s}{N_2^s} \cdot \left( \frac{N_1}{N_2} \right)^s \lesssim 1. \]

\[ \square \]

5. Proof of Main Theorem

We combine the interaction Morawetz estimate and Proposition 4.1 with a scaling argument to prove the following statement giving a uniform bound in terms of the $H^s$-norm of the initial data.

**Proposition 5.1.** Suppose $\phi(x,t)$ is a global in time solution to (1.1) from data $\phi_0 \in C_0^\infty(\mathbb{R}^n)$. Then for a given large $T$ we have
\[ \|\phi(T)\|_{H^s} \lesssim \|\phi_0\|_{H^s} T^{\alpha(s,n)} \]
(5.37)
as long as $\frac{2(n-2)}{3(n-4)} < s < 1$. The positive number $\alpha(s,n)$ depends on $s$ and $n$.

**Remark 5.1.** Since $T$ is arbitrarily large, the a priori bound on the $H^s$ norm gives the global well-posedness in the range of $\frac{2(n-2)}{3(n-4)} < s < 1$. 
Proof. The equation (1.1) is invariant over scaling of

$$\phi^\lambda(x, t) \equiv \lambda^{-\frac{n}{2}} \phi\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right).$$

According to

$$\|\nabla I\phi_0^\lambda\|_{L^2(\mathbb{R}^n)}^2 \lesssim \left( N^{1-s} \lambda^{-s} \|\phi_0\|_{H^s(\mathbb{R}^n)} \right)^2,$$

we choose \(\lambda \approx N^{\frac{1}{n-2}}\). (5.38)

in order to normalize \(\|\nabla I\phi_0^\lambda\|_{L^2(\mathbb{R}^n)} \lesssim O(1)\). The second term of the modified energy \(E(I\phi_0^\lambda)\) is treated as follows,

$$\|x|^{-2} * |I\phi_0^\lambda|^2|I\phi_0^\lambda|^2\|_{L^1(\mathbb{R}^n)} \lesssim \|x|^{-2} * |I\phi_0^\lambda|^2\|_{L^n(\mathbb{R}^n)} \|I\phi_0^\lambda\|_{L^\infty(\mathbb{R}^n)}^2 \lesssim \|I\phi_0^\lambda\|_{L^2(\mathbb{R}^n)}^4 \|I\phi_0^\lambda\|_{L^\infty(\mathbb{R}^n)}^2 \lesssim \|I\phi_0^\lambda\|_{H^1(\mathbb{R}^n)}^2$$

by Sobolev embedding.

Hence we have \(E(I\phi_0^\lambda) \lesssim 1\). The remaining proof is similar to the proof of Theorem 5.1 in [7] with necessary modification on exponents. As we have already seen, Hartree type nonlinearity behaves smoother than the polynomial type \(|\phi|^{\frac{n}{n-2}}\).

Let us pick a time \(T_0\) arbitrarily large, and let us define

$$S := \{0 < t \leq \lambda^2 T_0 : \|I\phi_0^\lambda\|_{L_t^\frac{4(n-1)}{n} L_x^\frac{2(n-1)}{n-2} ([0,T] \times \mathbb{R}^n)} \leq K T_0^{\frac{n-2}{4(n-1)}} \}$$

with \(K, N\) a constant to be chosen later. We claim that \(S\) is the whole interval \([0, \lambda^2 T_0]\).

Assuming not, there exists a time \(T \in (0, \lambda^2 T_0)\) so that

$$KT^{\frac{n-2}{4(n-1)}} < \|I\phi_0^\lambda\|_{L_t^\frac{4(n-1)}{n} L_x^\frac{2(n-1)}{n-2} ([0,T] \times \mathbb{R}^n)} < 2KT^{\frac{n-2}{4(n-1)}}$$

(5.39)

by continuity.

We now split the interval \([0,T]\) into consecutive subintervals \(J_k, k = 1, \cdots, L\) so that

$$\|I\phi_0^\lambda\|_{L_t^\frac{4(n-1)}{n} L_x^\frac{2(n-1)}{n-2} (J_k \times \mathbb{R}^n)} \leq \delta$$

where \(\delta\) defined as in Lemma 3.1. Note that

$$L \sim \frac{(2K)^{\frac{4(n-1)}{n}} T^{\frac{n-2}{n}}}{\delta}$$

due to (5.39). From Proposition 3.1 we know that for any \(0 < s < 1\)

$$\sup_{[0,T]} E(I\phi^\lambda(t)) \lesssim E(I\phi_0^\lambda) + L N^{-1+}.$$
By substituting $\lambda = N^{\frac{1-s}{n}}$, the above is equal to
\[
N^{1 - \frac{1-s}{n}} \frac{2(n-2)}{n} \sim \frac{(2K)^{\frac{4(n-1)}{n}}}{\delta} T_0^{\frac{n-2}{n}} .
\] (5.40)

Thus we choose $N$ as above for arbitrary $T_0$ as long as $\frac{2(n-2)}{3n-4} < s < 1$.

On the other hand we have that in Lemma 4.2,
\[
\|I\phi^\lambda\|_{L_t^\infty L_x^2(R^n)} \lesssim T_0^{-\frac{n-2}{n}} \left( \|I\phi^\lambda\|_{L_t^\infty L_x^2(R^n)}^{\frac{4(n-1)}{n}} + \|I\phi^\lambda\|_{L_t^\infty H_x^\frac{1}{2}(R^n)} \right)
\]
\[
+ T_0^{\frac{n-2}{n}} \int_0^T \text{Error} \, dt.
\]

We know that $\int_{J_k} \text{Error} \, dt \lesssim N^{-1+}$ on each $J_k$. Hence summing up all the $J_k$'s, we find
\[
\int_0^T \text{Error} \, dt \lesssim LN^{-1+} \lesssim 1
\]
by the choice of $\lambda$, $N$ as (5.38), (5.40). Thus with the trivial bound $\|I\phi\|_{H_x^\frac{1}{2}} \lesssim \|I\phi\|_{H^1}$, we have
\[
\|I\phi^\lambda\|_{L_t^\infty L_x^2(R^n)} \lesssim T_0^{\frac{n-2}{n}} .
\]
This estimate contradicts (5.39) for a proper choice of $K$.

Therefore, we conclude $S = [0, \lambda^2 T_0]$ and $T_0$ can be arbitrary large. In addition we also have that for $s > \frac{2(n-2)}{3n-4}$
\[
\|I_N\phi^\lambda(\lambda^2 T_0)\|_{H_x^1} = O(1),
\]
from which we estimate
\[
\|\phi(T_0)\|_{H^s} \lesssim \|\phi\|_{L^2} + \lambda^s \|I\phi^\lambda(\lambda^2 T_0)\|_{H^s}
\]
\[
\lesssim \lambda^s \|I\phi^\lambda(\lambda^2 T_0)\|_{H_x^\frac{1}{2}} \lesssim \lambda^s \lesssim N^{1-s}
\]
\[
\lesssim T_0^{\alpha(s,n)}
\]
where $\alpha(s, n) = \frac{(n-2)s(1-s)}{s(3n-4)-2(n-2)}$. \qed

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