GRAM DETERMINANT OF PLANAR CURVES

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Abstract. We investigate the Gram determinant of the bilinear form based on curves in a planar surface, with a focus on the disk with two holes. We prove that the determinant based on \( n - 1 \) curves divides the determinant based on \( n \) curves. Motivated by the work on Gram determinants based on curves in a disk and curves in an annulus (Temperley-Lieb algebra of type \( A \) and \( B \), respectively), we calculate several examples of the Gram determinant based on curves in a disk with two holes and advance conjectures on the complete factorization of Gram determinants.

1. Introduction

Let \( F^n_{0,0} \) be a unit disk with \( 2n \) points on its boundary. Let \( B_{n,0} \) be the set of all possible diagrams, up to deformation, in \( F^n_{0,0} \) with \( n \) non-crossing chords connecting these \( 2n \) points. It is well-known that \( |B_{n,0}| \) is equal to the \( n \)th Catalan number \( C_n = \frac{1}{n+1} \binom{2n}{n} \) \cite{10}. Accordingly, we will call \( B_{n,0} \) the set of Catalan states.

We will now generalize this setup. Let \( F_{0,k} \subset D^2 \) be a plane surface with \( k + 1 \) boundary components. \( F_{0,0} = D^2 \), and for \( k \geq 1 \), \( F_{0,k} \) is equal to \( D^2 \) with \( k \) holes. Let \( F^n_{0,k} \) be \( F_{0,k} \) with \( 2n \) points, \( a_0, \ldots, a_{2n-1} \), arranged counter-clockwise along the outer boundary, cf. Figure 1.

![Figure 1](image)

**Figure 1.** Throughout the paper, we number the points counter-clockwise beginning at the top of outer boundary. We label and differentiate between the holes.

Let \( B_{n,k} \) be the set of all possible diagrams, up to equivalence, in \( F^n_{0,k} \) with \( n \) non-crossing chords connecting these \( 2n \) points. We define equivalence as follows: for each
diagram $b \in \mathcal{B}_{n,k}$, there is a corresponding diagram $\gamma(b) \in \mathcal{B}_{n,0}$ obtained by filling the $k$ holes in $b$. We call $\gamma(b)$ the underlying Catalan state of $b$ (cf. Figure 2). In addition, a given diagram in $F_{0,k}^n$ partitions $F_{0,k}$ into $n+1$ regions. Two diagrams are equivalent if and only if they have the same underlying Catalan state and the labeled holes are distributed in the same manner across regions. Accordingly, $|\mathcal{B}_{n,k}| = (n+1)^{k-1}(\binom{2n}{n})$. We remark that in the cases $k = 0$ and $k = 1$, two diagrams are equivalent if they are homotopic, but for $k > 2$, this is not the case (for an example, see Figure 3).

In this paper, we define a pairing over $\mathcal{B}_{n,k}$ and investigate the Gram matrix of the pairing. This concept is a generalization of a problem posed by W. B. R. Lickorish for type $A$ (based on a disk, i.e. $k = 0$) Gram determinants, and Rodica Simion for type $B$ (based on an annulus, i.e. $k = 1$) Gram determinants, cf. [4] [5], [7] [8]. Significant research has been completed for the Gram determinants for type $A$ and $B$. In particular, P. Di Francesco and B. W. Westbury gave a closed formula for the type $A$ Gram determinant [3], [11]; a complete factorization of the type $B$ Gram determinant was conjectured by Gefry Barad and a closed formula was proven by Q. Chen and J. H. Przytycki [1] (see also [6]). The type $A$ Gram determinant was used by Lickorish to find an elementary construction of Reshetikhin-Turaev-Witten invariants of oriented closed 3-manifolds.

We specifically investigate the Gram determinant $G_n$ of the bilinear form defined over $\mathcal{B}_{n,2}$ and prove that $\det G_{n-1}$ divides $\det G_n$ for $n > 1$. Furthermore, we investigate the diagonal entries of $G_n$ and give a method for computing terms of maximal degree in $\det G_n$. We conclude the paper by briefly discussing generalizations of the Gram determinant and presenting some open questions.

2. Definitions for $\mathcal{B}_{n,2}$

Consider $F_{0,2}^n$, a unit disk with two holes, along with $2n$ points along the outer boundary. Denote the holes in $F_{0,2}^n$ by $\partial X_1$ and $\partial Y_1$, or more simply, just $X_1$ and $Y_1$. To differentiate between the two holes, we will always place $X_1$ to the left and $Y_1$ to the right if labels are not present.

Let $\mathcal{B}_n := \mathcal{B}_{n,2} := \{b_1^n, \ldots, b_{(n+1)}^{(2n)}\}$, the set of all possible diagrams, up to equivalence in $F_{0,2}^n$ with $n$ non-crossing chords connecting these $2n$ points. For simplicity, we will often use $b_i$ instead of $b_i^n$, when the number of points along the outer boundary can be inferred from context.
Let $X_2$ and $Y_2$ be the inversions\footnote{Inversion is an involution defined on the sphere $\mathbb{C} \cup \infty$ by $z \mapsto \frac{1}{\overline{z}}$.} of $X_1$ and $Y_1$, respectively, with respect to the unit disk, and let $\mathcal{S} = \{X_1, X_2, Y_1, Y_2\}$. Given $b_i \in \mathcal{B}_n$, let $b_i^*$ denote the inversion of $b_i$. Given $b_i, b_j \in \mathcal{B}_n$, we glue $b_i$ with $b_j^*$ along the outer boundary, respecting the labels of the marked points. $b_i$ and $b_j$ each contains $n$ non-crossing chords, so $b_i \circ b_j^*$ can have at most $n$ closed curves. The resulting diagram, denoted by $b_i \circ b_j^*$, is then a set of up to $n$ closed curves in the 2-dimensional sphere $(D^2 \cup (D^2)^*)$ with four holes: $X_1, X_2, Y_1, Y_2$ (we disregard the outer boundary, $\partial D^2$). Each closed curve partitions the set $\mathcal{S}$ into two sets. Two closed curves are of the same type if they partition $\mathcal{S}$ the same way. For each $b_i \circ b_j^*$, there are then up to eight types of disjoint closed curves, whose multiplicities we index by the following variables:

\begin{align*}
n_d &= \text{the number of curves with } \{X_1, X_2, Y_1, Y_2\} \text{ on the same side} \\
n_{x_1} &= \text{the number of curves that separate } \{X_1\} \text{ from } \{X_2, Y_1, Y_2\} \\
n_{x_2} &= \text{the number of curves that separate } \{X_2\} \text{ from } \{X_1, Y_1, Y_2\} \\
n_{y_1} &= \text{the number of curves that separate } \{Y_1\} \text{ from } \{X_1, X_2, Y_2\} \\
n_{y_2} &= \text{the number of curves that separate } \{Y_2\} \text{ from } \{X_1, X_2, Y_1\} \\
n_{z_1} &= \text{the number of curves that separate } \{X_1, X_2\} \text{ from } \{Y_1, Y_2\} \\
n_{z_2} &= \text{the number of curves that separate } \{X_1, Y_1\} \text{ from } \{X_2, Y_2\} \\
n_{z_3} &= \text{the number of curves that separate } \{X_1, Y_2\} \text{ from } \{X_2, Y_1\}
\end{align*}
Let $R := \mathbb{Z}[d, x_1, x_2, y_1, y_2, z_1, z_2, z_3]$, and $RB_n$ be the free module over the ring $R$ with basis $B_n$. We define a bilinear form $\langle \cdot, \cdot \rangle : RB_n \times RB_n \to R$ by:

$$\langle b_i, b_j \rangle = d^{n_d} x_1^{n_{x_1}} y_1^{n_{y_1}} y_2^{n_{y_2}} z_1^{n_{z_1}} z_2^{n_{z_2}} z_3^{n_{z_3}}$$

$\langle b_i, b_j \rangle$ is a monomial of degree at most $n$. Some examples of paired diagrams and their corresponding monomials, using examples from Figure 4, are given in Figure 5.

![Figure 5](image)

**Figure 5.** From left to right:

$\langle b_2, b_4 \rangle = x_1$  $\langle b_5, b_2 \rangle = x_1 x_2$  $\langle b_6, b_2 \rangle = d z_1$  $\langle b_1, b_3 \rangle = x_2$

Let

$$G_n = (g_{ij}) = (\langle b_i, b_j \rangle)_{1 \leq i, j \leq (n+1)/2}$$

be the Gram matrix of the pairing on $B_n$. For example,

$$G_1 = \begin{bmatrix}
    d & y_2 & x_2 & z_2 \\
    y_1 & z_1 & z_3 & x_1 \\
    x_1 & z_3 & z_1 & y_1 \\
    z_2 & x_2 & y_2 & d
\end{bmatrix} \quad \text{up to ordering of } B_1 \text{ and}$$

$$\det G_1 = \det ((d + z_2)(z_1 + z_3) - (x_1 + y_1)(x_2 + y_2))$$

We remark that for $b_i, b_j \in B_n$, $\langle b_j, b_i \rangle$ can be obtained by taking $b_i \circ b_j^*$ and interchanging the roles of $X_1$ and $Y_1$ with $X_2$ and $Y_2$, respectively. Let $h_t$ be an involution on the entries of $G_n$ which interchanges the variables $x_1$ with $x_2$ and $y_1$ with $y_2$. It follows that $\langle b_i, b_j \rangle = h_t(\langle b_j, b_i \rangle)$. The transpose matrix is then given by:

$$^{t}G_n = (h_t((b_i, b_j)))$$

We note that the variables $d, z_1, z_2, z_3$ are preserved by $h_t$ (cf. Theorem 3.2(4)).

We can define more generally: given $A = \{b_{i_1}, b_{i_2}, \ldots, b_{i_p} \} \subseteq B_n$ and $B = \{b_{m_1}, b_{m_2}, \ldots, b_{m_q} \} \subseteq B_n$, let $\langle A, B \rangle$ be an $p \times q$ submatrix of $G_n$ given by:

$$\langle A, B \rangle = (\langle b_{i_j}, b_{m_j} \rangle)_{1 \leq i \leq p, 1 \leq j \leq q}$$

For example, we can express the matrix $G_n$ as $\langle B_n, B_n \rangle$. The $i^{th}$ row of $G_n$ can be written as $\langle b_i, B_n \rangle$. 
This paper is mostly devoted to exploring possible factorizations of $\det G_n$, and is the first step toward computing $\det G_n$ in full generality, which we conjecture to have a nice decomposition.

Let $i_0 : B_n \to B_{n+1}$ be the embedding map defined as follows: for $b_i \in B_n$, $i_0(b_i) \in B_{n+1}$ is given by adjoining to $b_i$ a non-crossing chord close to the outer boundary that intersects the outer circle at two points between $a_0$ and $a_{2n-1}$, cf. upper part of Figure 8.

We will also use a generalization of $i_0$, for which we need first the following definition. For any real number $\alpha$, consider the homeomorphism $r_\alpha : \mathbb{C} \to \mathbb{C}$ on the annulus $R' \leq |z| \leq 1$, which we call the $\alpha$-Dehn Twist, defined by:

$$r_\alpha(z) = ze^{i\alpha(1-(1-|z|)/(1-R'))}$$

Note that $r_\alpha(z) = z$ as $|z| = R'$. Therefore, we can extend the domain of $r_\alpha$ to $D^2$ by defining $r_\alpha(z) = z$ for $0 \leq |z| \leq R'$. Fix $R'$ such that a circle of radius $R'$ encloses $X_1$ and $Y_1$. Then $r_\alpha$ acts on $b_i \in B_n$ as a clockwise rotation of a diagram close to the outer boundary.

![Figure 7. A $\pi/4$-Dehn Twist. Note that $r_{2\pi}(b_i) = b_i$ (cf. Figure 3).](image-url)

Figure 6. A pictorial representation of curves used to define $G_1$.
Consider the \( k \)-conjugated embedding \( i_k : B_n \rightarrow B_{n+1} \) defined by:

\[
i_k(b_i) = r_{\pi/n+1}^{-k}i_0r_{\pi/n}^{-k}(b_i)
\]

Intuitively, if for \( b_i \in B_{n+1} \) there exists \( b_j \in B_n \) such that \( i_k(b_j) = b_i \), then \( b_i \) is composed of \( b_j \) and a non-crossing chord close to the outer boundary connecting \( a_k \) and \( a_{k-1} \). Figure 8.

![Figure 8](image)

Figure 8. An embedding \( b_i \mapsto i_0(b_i) \), top; a 1-conjugated embedding \( b_i \mapsto i_1(b_i) \), bottom; \( b_i \in B_4 \).

For every \( b_i \in B_n \), let \( p_k(b_i) \) be the diagram obtained by gluing to \( b_i \) a non-crossing chord connecting \( a_k \) and \( a_{k-1} \) outside the circle, and pushing the chord inside the circle. The properties of \( p_k \) will be explored in greater detail in Section 4. We conclude this section with a basic identity linking \( i_0 \) and \( p_0 \):

**Proposition 2.1.** For any \( b_i \in B_n \), \( b_j \in B_{n-1} \), \( b_i \circ i_0(b_j)^* = p_0(b_i) \circ b_j^* \).

3. Basic Properties of Gram Determinant

In this section, we prove basic properties of \( \det G_n \). In particular, we show that the determinant of \( G_n \) is nonzero.

**Lemma 3.1.** \( \langle b_i, b_j \rangle \) is a monomial of maximal degree if and only if \( \gamma(b_i) = \gamma(b_j) \).

**Proof.** \( b_i \circ b_j^* \) has \( n \) closed curves if and only if each closed curve is formed by exactly two arcs, one in \( b_i \) and one in \( b_j^* \). Hence, any two points connected by a chord in \( b_i \) must also be connected by a chord in \( b_j \), so \( \gamma(b_i) = \gamma(b_j) \). \( \square \)

**Theorem 3.1.** \( \det G_n \neq 0 \) for all integers \( n \geq 1 \).

\(^2\)Throughout this paper, we use \( a_k \) and \( a_{k-1} \) to denote two adjacent points along the outer boundary, where \( k \) is taken modulo 2n.
Proof. Assume \( \langle b_i, b_j \rangle \) is a monomial of maximal degree consisting only of the variables \( d \) and \( z_1 \). Because \( \gamma(b_i) = \gamma(b_j) \) by Lemma 3.1, it follows that any two points connected in \( b_i \) are also connected in \( b_j \). Each connection in \( b_i \) can be drawn in four different ways with respect to \( X \) and \( Y \), since there are two ways to position the chord relative to each hole. Because \( \langle b_i, b_j \rangle \) is assumed to consist only of the variables \( d \) and \( z_1 \), it follows that each pair of arcs that form a closed curve in \( b_i \circ b_j^* \) either separates \( \{X_1, X_2\} \) from \( \{Y_1, Y_2\} \) or has \( \{X_1, X_2, Y_1, Y_2\} \) on the same side of the curve. One can check each of the four cases to see that this condition implies that any two arcs that form a closed curve in \( b_i \circ b_j^* \) must be equal, so \( b_i = b_j \). Using Laplacian expansion, this implies that the product of the diagonal of \( G_n \) is the unique summand of degree \( n(n+1)(2^n) \) in \( \det G_n \) consisting only of the variables \( d \) and \( z_1 \).

We need the following notation for the next theorem: let \( f : \alpha_1 \leftrightarrow \alpha_2 \) denote a function \( f \) which acts on the entries of \( G_n \) by interchanging variables \( \alpha_1 \) with \( \alpha_2 \). We can extend the domain of \( f \) to \( G_n \). Let \( f(G_n) \) denote the matrix formed by applying \( f \) to all the individual entries of \( G_n \).

Let \( h_1, h_2, h_3 \) be involutions acting on the entries of \( G_n \) with the following definitions:

1. \( h_1 : x_1 \leftrightarrow y_1 \quad z_1 \leftrightarrow z_3 \)
2. \( h_2 : x_2 \leftrightarrow y_2 \quad z_1 \leftrightarrow z_3 \)
3. \( h_3 = h_1 h_2 \) \( : x_1 \leftrightarrow y_1 \quad x_2 \leftrightarrow y_2 \)
4. \( h_t : x_1 \leftrightarrow x_2 \quad y_1 \leftrightarrow y_2 \)

Theorem 3.2.

1. \( \det h_1(G_1) = -\det G_1 \), and for \( n > 1 \), \( \det h_1(G_n) = \det G_n \).
2. \( \det h_2(G_1) = -\det G_1 \), and for \( n > 1 \), \( \det h_2(G_n) = \det G_n \).
3. \( \det h_3(G_n) = \det G_n \).
4. \( \det h_t(G_n) = \det G_n \).

Proof. For (1), note that \( h_1(G_n) \) corresponds to exchanging the positions of the holes \( X_1 \) and \( Y_1 \) for all \( b_i \in B_n \). \( b_j^* \) is unchanged, so \( h_1 \) can be realized by a permutation of rows. For states where \( X_1 \) and \( Y_1 \) lie in the same region, their corresponding rows are unchanged by \( h_1 \). The number of such states is given by \( \frac{1}{n+1} |B_n| \). Thus, the total number of row transpositions is equal to

\[
\frac{1}{2} \left( |B_n| - \left( \frac{1}{n+1} \right) |B_n| \right) = \frac{n}{2} \binom{2n}{n} = \binom{n(n+1)}{2} C_n
\]

where \( C_n = \frac{1}{n+1} \binom{2n}{n} \). It is a known combinatorial fact that \( C_n \) is odd if and only if \( n = 2^m - 1 \) for some \( m \), \( \mathbb{Z} \). Hence, \( C_n \) is odd implies that

\[
\frac{n(n+1)}{2} = \frac{2^m(2^m-1)}{2} = 2^{m-1}(2^m - 1)
\]

which is even for all \( m > 1 \). Thus, \( h_1(G_n) \) can be obtained from \( G_n \) by an even permutation of rows for \( n > 1 \), so \( \det h_1(G_n) = \det G_n \). \( h_1(G_1) \) is given by an odd number of row
transpositions on \( G_1 \), so \( \det h_1(G_1) = - \det G_1 \).

(2) can be shown using the same method of proof as before. \( h_2(G_n) \) corresponds to exchanging the positions of the holes \( X_2 \) and \( Y_2 \) for all \( b_i \in B_n \). \( h_2 \) can thus be realized by a permutation of columns, and the rest of the proof follows in a similar fashion as the previous one. Since \( h_2(G_n) \) can be obtained from \( G_n \) by an even permutation of columns for \( n > 1 \), \( \det h_2(G_n) = \det G_n \). \( h_2(G_2) \) is given by an odd number of column transpositions on \( G_1 \), so \( \det h_2(G_1) = - \det G_1 \), which proves (2).

Since \( h_3 = h_1 h_2 \), it follows immediately that \( \det h_3(G_n) = \det G_n \) for \( n > 1 \). The sum of two odd permutations is even, so the equality also holds for \( n = 1 \), which proves (3). (4) follows because \( \det h_t(G_n) = (\det t G_n) = \det G_n \).

\[ \begin{align*}
(1) \quad g_1 &: x_1 \leftrightarrow -x_1, x_2 \leftrightarrow -x_2, z_2 \leftrightarrow -z_2, z_3 \leftrightarrow -z_3 \\
(2) \quad g_2 &: y_1 \leftrightarrow -y_1, y_2 \leftrightarrow -y_2, z_2 \leftrightarrow -z_2, z_3 \leftrightarrow -z_3 \\
(3) \quad g_3 &: x_1 \leftrightarrow -x_1, y_2 \leftrightarrow -y_2, z_1 \leftrightarrow -z_1, z_2 \leftrightarrow -z_2 \\
(4) \quad g_1 g_2 &: x_1 \leftrightarrow -x_1, x_2 \leftrightarrow -x_2, y_1 \leftrightarrow -y_1, y_2 \leftrightarrow -y_2 \\
(5) \quad g_1 g_3 &: x_2 \leftrightarrow -x_2, y_2 \leftrightarrow -y_2, z_1 \leftrightarrow -z_1, z_3 \leftrightarrow -z_3 \\
(6) \quad g_2 g_3 &: x_1 \leftrightarrow -x_1, y_1 \leftrightarrow -y_1, z_1 \leftrightarrow -z_1, z_3 \leftrightarrow -z_3 \\
(7) \quad g_1 g_2 g_3 &: x_2 \leftrightarrow -x_2, y_1 \leftrightarrow -y_1, z_1 \leftrightarrow -z_1, z_2 \leftrightarrow -z_2
\end{align*} \]

**Theorem 3.3.** \( \det G_n \) is preserved under the following involutions on variables:

1. \( g_1 : x_1 \leftrightarrow -x_1, x_2 \leftrightarrow -x_2, z_2 \leftrightarrow -z_2, z_3 \leftrightarrow -z_3 \)
2. \( g_2 : y_1 \leftrightarrow -y_1, y_2 \leftrightarrow -y_2, z_2 \leftrightarrow -z_2, z_3 \leftrightarrow -z_3 \)
3. \( g_3 : x_1 \leftrightarrow -x_1, y_2 \leftrightarrow -y_2, z_1 \leftrightarrow -z_1, z_2 \leftrightarrow -z_2 \)
4. \( g_1 g_2 : x_1 \leftrightarrow -x_1, x_2 \leftrightarrow -x_2, y_1 \leftrightarrow -y_1, y_2 \leftrightarrow -y_2 \)
5. \( g_1 g_3 : x_2 \leftrightarrow -x_2, y_2 \leftrightarrow -y_2, z_1 \leftrightarrow -z_1, z_3 \leftrightarrow -z_3 \)
6. \( g_2 g_3 : x_1 \leftrightarrow -x_1, y_1 \leftrightarrow -y_1, z_1 \leftrightarrow -z_1, z_3 \leftrightarrow -z_3 \)
7. \( g_1 g_2 g_3 : x_2 \leftrightarrow -x_2, y_1 \leftrightarrow -y_1, z_1 \leftrightarrow -z_1, z_2 \leftrightarrow -z_2 \)

**Proof.** To prove (1), we show that \( g_1 \) can be realized by conjugating the matrix \( G_n \) by a diagonal matrix \( P_n \) of all diagonal entries equal to \( \pm 1 \). The diagonal entries of \( P_n \) are defined as

\[ p_{ii} = (-1)^{q(b_i, F_x)} \]

where \( q(b_i, F_x) \) is the number of times \( b_i \) intersects \( F_x \) modulo 2, cf. Figure 9. The theorem follows because curves corresponding to the variables \( x_1, x_2, z_2 \) and \( z_3 \) intersect \( F_x \cup F_x^* \) in an odd number of points, whereas curves corresponding to the variables \( d, z_2, y_1 \) and \( y_2 \) cut it an even number of times.

![Figure 9](image-url)

More precisely, for

\[ g_{ij} = \langle b_i, b_j \rangle = d^{n_d x_1^{n_{x_1}} x_2^{n_{x_2}} y_1^{n_{y_1}} y_2^{n_{y_2}} z_1^{n_{z_1}} z_2^{n_{z_2}} z_3^{n_{z_3}}}, \]
the entry $g'_{ij}$ of $P_n G_n P_n^{-1}$ satisfies:
\[
g'_{ij} = p_{ij}p_{ij}p_{ij} = p_{ij}p_{ij}g_{ij}
\]
\[
= (-1)^{q(b_i, F_x) + q(b_j, F_x)} g_{ij}
\]
\[
= (-1)^{n_x_1 + n_x_2 + n_z_2 + n_z_3} g_{ij}
\]
\[
= d^{n_d} (-x_1)^{n_x_1} (-x_2)^{n_x_2} g_1^{n_y_1} g_2^{n_y_2} z_1^{n_z_1} (-z_2)^{n_z_2} (-z_3)^{n_z_3}
\]

For (2) and (3), we use the same method of proof as for (1). In (2), we use $F_y$ and $F_y \cup F_y^*$. In (3), we use $F_x$ and $F_x \cup F_y^*$. (4) through (7) follow directly from (1), (2) and (3). \qed

4. Terms of Maximal Degree in det $G_n$

Theorem 3.1 proves that the product of the diagonal entries of $G_n$ is the unique term of maximal degree, $n(n+1)\binom{2n}{n}$, in det $G_n$ consisting only of the variables $d$ and $z_1$. More precisely, the product of the diagonal of $G_n$ is given by
\[
\delta(n) = \prod_{b_i \in B_n} (b_i, b_i) = d^{1\alpha(n)} z_1^{\beta(n)}
\]
with $\alpha(n) + \beta(n) = n(n+1)\binom{2n}{n}$. $\delta(n)$ for the first few $n$ are given here:
\[
\delta(1) = d^2 z_1^2 \quad \delta(2) = d^{20} z_1^{16} \quad \delta(3) = d^{144} z_1^{96} \quad \delta(4) = d^{888} z_1^{512}
\]

Computing the general formula for $\delta(n)$ can be reduced to a purely combinatorial problem. We conjectured that $\beta(n) = (2n)^{4n-1}$ and this was in fact proven by Louis Shapiro using an involved generating function argument [9]. The result is stated formally below.

**Theorem 4.1.**
\[
\delta(n) = d^{n(n+1)\binom{2n}{n} - (2n)^{4n-1} z_1^{(2n)^{4n-1}}}
\]

Let $h(\det G_n)$ denote the truncation of $\det G_n$ to terms of maximal degree, that is, of degree $n(n+1)\binom{2n}{n}$. Each term is a product of $(n+1)\binom{2n}{n}$ entries in $G_n$, each of which is a monomial of degree $n$. By Lemma 3.1, $(b_i, b_j)$ has degree $n$ if and only if $b_i$ and $b_j$ have the same underlying Catalan state. There are $C_n = \frac{1}{n+1} \binom{2n}{n}$ elements in $B_n$. Divide $B_n$ into subsets corresponding to underlying Catalan states, that is, into subsets $A_1, \ldots, A_{C_n}$, such that for all $b_i, b_j \in A_k$, $\gamma(b_i) = \gamma(b_j)$. Then from Lemma 3.1 we have

**Proposition 4.1.**
\[
h(\det G_n) = \prod_{k=1}^{C_n} \det(A_k, A_k)
\]

Note that $\langle A_k, A_k \rangle$ are simply blocks in $G_n$ whose determinants can be multiplied together to give the highest terms in det $G_n$. Finding the terms of maximal degree in det $G_n$ can give insight into decomposition of det $G_n$ for large $n$. 
Example 1. $B_1$ corresponds to the single Catalan state in $B_{1,0}$. Thus, $\det G_1 = h(\det G_1)$, a homogeneous polynomial of degree 4.

Example 2. $B_2$ can be divided into two subsets, corresponding to the two Catalan states in $B_{2,0}$. We can thus find $h(\det G_2)$ by computing two $9 \times 9$ block determinants. The two Catalan states in $B_{2,0}$ are equivalent up to rotation, so the two block determinants are equal. Specifically, we have:

\[ h(\det G_2) = d^6(x_1 x_2 + x_2 y_1 + x_1 y_2 + y_1 y_2 - dz_1 - z_1 z_2 - d z_3 - z_2 z_3)^4 \\
- x_1 x_2 + x_2 y_1 + y_1 y_2 - dz_1 - z_1 z_2 - d z_3 - z_2 z_3)^4 \\
- x_1 x_2 z_1 - y_1 y_2 z_1 + dz_1^2 + x_2 y_1 z_3 + x_1 y_2 z_3 - dz_3^2)^2 \\
- 2 x_1 x_2 y_1 y_2 + dx_1 x_2 z_1 + dy_1 y_2 z_1 - d^2 z_1^2 + dx_2 y_1 z_3 + dx_1 y_2 z_3 - d^2 z_3^2)^2 \\
= d^6 \det G_1^4 \left( -x_1 x_2 z_1 - y_1 y_2 z_1 + dz_1^2 + x_2 y_1 z_3 + x_1 y_2 z_3 - dz_3^2 \right)^2 \\
\left( -2 x_1 x_2 y_1 y_2 + dx_1 x_2 z_1 + dy_1 y_2 z_1 - d^2 z_1^2 + dx_2 y_1 z_3 + dx_1 y_2 z_3 - d^2 z_3^2 \right)^2 \\

Example 3. $B_3$ can be divided into five subsets, corresponding to the five Catalan states in $B_{3,0}$. We can thus find $h(\det G_3)$ by computing the determinants of five blocks in $B_3$. The determinant of each block gives a homogeneous polynomial of degree 240/5 = 48. $B_{3,0}$ forms two equivalence classes up to rotation, so there are only two unique block determinants. For precise terms, we refer the reader to the Appendix.

5. $\det G_{n-1}$ DIVIDES $\det G_n$

In this section, we prove that the Gram determinant for $n - 1$ chords divides the Gram determinant for $n$ chords. We need several lemmas:

Lemma 5.1. For any $b_i \in B_n$, $p_0(b_i) \in B_{n-1}$ if and only if $b_i$ contains no chord connecting $a_0$ and $a_{2n-1}$.

Proof. Suppose $a_0$ and $a_{2n-1}$ are not connected by a chord in $b_i$, say, $a_0$ is connected to $a_j$ and $a_{2n-1}$ is connected to $a_k$. Then $p_0(b_i)$ connects $a_0$ and $a_{2n-1}$ by a chord outside the outer boundary, and this chord does not form a closed curve. Because $a_j$ is connected to $a_0$ and $a_k$ is connected to $a_{2n-1}$, $p_0(b_i)$ contains single path from $a_k$ to $a_j$, which we can deform through isotopy so that it fits inside the outer circle. Thus, $p_0(b_i) \in B_{n-1}$, cf. Figure 10.

If $b_i$ contains an arc connecting $a_0$ and $a_{2n-1}$, then $p_0(b_i)$ contains a closed curve enclosing some subset of $\{X_1, Y_1\}$, and cannot be in $B_n$. \hfill \Box

Lemma 5.2. For any $b_i \in B_n$, if $p_0(b_i) \notin B_{n-1}$, there exists $b_{\alpha(i)} \in B_{n-1}$ such that, for all $b_j \in B_{n-1}$, one of the following is true:

1. $\langle p_0(b_i), b_j \rangle = d(b_{\alpha(i)}, b_j)$
2. $\langle p_0(b_i), b_j \rangle = x_1 \langle b_{\alpha(i)}, b_j \rangle$
3. $\langle p_0(b_i), b_j \rangle = y_1 \langle b_{\alpha(i)}, b_j \rangle$
4. $\langle p_0(b_i), b_j \rangle = z_2 \langle b_{\alpha(i)}, b_j \rangle$.
Figure 10. From $b_i$, we obtain $p_0(b_i)$ by adjoining a chord outside the outer boundary between $a_0$ and $a_{2n-1}$, and pushing the chord inside the boundary. If $b_i$ does not contain a chord connecting $a_0$ and $a_{2n-1}$, then $p_0(b_i) \in B_{n-1}$.

Proof. By Lemma 5.1, $b_i$ contains a chord connecting points $a_0$ and $a_{2n-1}$, so $p_0(b_i)$ must consist of some diagram in $B_{n-1}$ and a closed curve enclosing some subset of $\{X_1, Y_1\}$. The former is given by $\langle b_{\alpha(i)}(i) \rangle$ for some $b_{\alpha(i)} \in B_{n-1}$, and the latter curve is given by one of the following variables: $d, x_1, y_1, z_2$.

The previous two lemmas, combined with Proposition 2.1, leads to the following corollary.

Corollary 5.1. Let $A = \{1, d, x_1, y_1, z_2\}$. For any $b_i \in B_n$, there exists $b_{\alpha(i)} \in B_{n-1}$ and $c \in A$ such that $\langle b_{\alpha(i)}, i_0(B_{n-1}) \rangle = c \langle b_{\alpha(i)}(i), i_0(B_{n-1}) \rangle$.

That is, the rows of $\langle B_n, i_0(B_{n-1}) \rangle$ are each either equal to some row of $G_{n-1}$, or to some row of $G_{n-1}$ multiplied by one of the following variables: $d, x_1, y_1, z_2$. We now have all the lemmas needed for our main result of this section.

Theorem 5.1. For $n > 1$, $\det G_{n-1} | \det G_n$.

Proof. We begin by proving that for every row of the matrix $G_{n-1}$, there exists an equivalent row in the submatrix $\langle B_n, i_0(B_{n-1}) \rangle$ of $G_n$. Fix $b_i \in B_{n-1}$ and take the row of $G_{n-1}$ given by $\langle b_i, B_{n-1} \rangle$. We claim that the row in $\langle B_n, i_0(B_{n-1}) \rangle$ given by $\langle i_1(b_i), i_0(B_{n-1}) \rangle$ is equal to $\langle b_i, B_{n-1} \rangle$. In other words, $\langle i_1(b_i), i_0(B_{n-1}) \rangle$ is equal to the $i$th row of $G_{n-1}$, a fact which we leave to the reader for the moment, but will demonstrate explicitly in the next section, cf. Theorem 6.1.

Reorder the elements of $B_n$ so that $\langle i_0(B_{n-1}), i_0(B_{n-1}) \rangle$ forms an upper-leftmost block of $G_n$ and $\langle i_1(B_{n-1}), i_0(B_{n-1}) \rangle$ forms a block directly underneath $\langle i_0(B_{n-1}), i_0(B_{n-1}) \rangle$. 
This is illustrated below:

\[
G_n = \begin{pmatrix}
\langle \mathbf{B}_n, \mathbf{0} \rangle, \mathbf{B}_n^{-1} & \ast & \ast & \ast & \ast \\
\langle \mathbf{B}_n, \mathbf{B}_n^{-1} \rangle & \ast & \ast & \ast & \ast \\
0 & \ast & \ast & \ast & \ast \\
\end{pmatrix}
\]

Corollary 5.1 implies that every row of \( \langle \mathbf{B}_n, \mathbf{0} \rangle \) is a multiple of some row in \( G_{n-1} \). Let \( j_1, \ldots, j_k \) denote the indices of all rows of \( \langle \mathbf{B}_n, \mathbf{0} \rangle \) other than those in \( \langle \mathbf{B}_n, \mathbf{B}_n^{-1} \rangle \). Let \( G_n' \) be the matrix obtained by properly subtracting multiples of rows in \( \langle \mathbf{B}_n, \mathbf{0} \rangle \) from rows \( j_1, \ldots, j_k \) of \( G_n \) so that the submatrix obtained by restricting \( G_n' \) to rows \( j_1, \ldots, j_k \) and columns corresponding to states in \( \mathbf{0} \) is equal to 0:

\[
G_n' = \begin{pmatrix}
0 & \ast & \ast & \ast & \ast \\
G_{n-1} & \ast & \ast & \ast & \ast \\
0 & \ast & \ast & \ast & \ast \\
0 & \ast & \ast & \ast & \ast \\
0 & \ast & \ast & \ast & \ast \\
\end{pmatrix}
\]

Thus, \( G_n' \) restricted to the columns corresponding to states in \( \mathbf{0} \) contains precisely \( n^{2n-2} \) nonzero rows, each equal to some unique row of \( G_{n-1} \). The determinant of this submatrix is equal to \( \det G_{n-1} \). Since \( \det G_{n-1} \det G_n' \) and \( \det G_n' = \det G_n \), this completes the proof.

6. Further Relation Between \( \det G_{n-1} \) and \( \det G_n \)

As was first noted in the proof of Theorem 5.1, there exists a submatrix of \( G_n \) equal to \( G_{n-1} \). This section will be focused on identifying multiple nonoverlapping submatrices in \( G_{n-1} \) equal to multiples of \( G_{n-1} \). This will prove useful for simplifying the computation of \( \det G_n \). We start with the main lemma for this section and for Theorem 5.1.

**Lemma 6.1.** For any \( b_i, b_j \in \mathbf{B}_{n-1} \), \( \langle i_0(b_i), i_0(b_j) \rangle = \langle i_1(b_i), i_0(b_j) \rangle = \langle b_i, b_j \rangle \).

**Proof.** We begin with the equality \( \langle i_1(b_i), i_0(b_j) \rangle = \langle b_i, b_j \rangle \). By Proposition 2.1, \( i_1(b_i) \circ i_0(b_j) = p_0i_1(b_i) \circ b_j \ast \), so it suffices to prove that \( p_0i_1(b_i) = p_0r_{\pi n}i_0r_{\pi n-1}^{-1}(b_i) = b_i \). This is demonstrated pictorially:
Thus, \( \langle i_1(b_i), i_0(b_j) \rangle = \langle b_i, b_j \rangle \). Recall that \( \langle b_i, b_j \rangle = h_t(\langle b_j, b_i \rangle) \). From this and the previous equality, it follows that
\[
\langle i_0(b_i), i_0(b_j) \rangle = h_t(\langle i_1(b_j), i_0(b_i) \rangle) = h_t(\langle b_j, b_i \rangle) = h_t^2(\langle b_i, b_j \rangle) = \langle b_i, b_j \rangle.
\]
□

**Corollary 6.1.** \( \langle i_0(B_{n-1}), i_1(B_{n-1}) \rangle = \langle i_1(B_{n-1}), i_0(B_{n-1}) \rangle = G_{n-1} \).

**Lemma 6.2.** For any \( b_i, b_j \in B_{n-1} \), \( \langle i_0(b_i), i_0(b_j) \rangle = \langle i_1(b_i), i_1(b_j) \rangle = d \langle b_i, b_j \rangle \).

**Proof.** \( i_0(b_i) \circ i_0(b_j)^* \) is composed of \( b_i \circ b_j^* \) in addition to a chord close to the boundary glued with its inverse. The latter pairing gives a trivial circle. Thus, \( \langle i_0(b_i), i_0(b_j) \rangle = d \langle b_i, b_j \rangle \) for all \( b_i, b_j \in B_{n-1} \).

By symmetry, \( \langle i_1(B_{n-1}), i_1(B_{n-1}) \rangle = d G_{n-1} \). □

**Corollary 6.2.** \( \langle i_0(B_{n-1}), i_0(B_{n-1}) \rangle = \langle i_1(B_{n-1}), i_1(B_{n-1}) \rangle = d G_{n-1} \).

Using these two facts, we can construct from \( G_n \) a \((|B_n| - 2|B_{n-1}|) \times (|B_n| - 2|B_{n-1}|)\) matrix whose determinant is equal to \( \det G_n/(1 - d^2)^{n-1} \det G_{n-1}^2 \). This allows us to compute \( \det G_n \) with greater ease, assuming we know \( \det G_{n-1} \). This process is shown in the next theorem.

**Theorem 6.1.** There exists an integer \( k \geq 3 \) such that, for all integers \( n > 1 \),
\[
\det G_{n-1}^2 \mid \det G_n(1 - d^2)^k.
\]

3Clearly \( k \) is bounded above by \( (n + 1)(2^n) \), or even better, by \( |B_n| - 2|B_{n-1}| \). There are obviously better approximations possible, but we do not address them in this paper.
Proof. Order the elements of $B_n$, (or equivalently, the rows and columns of $G_n$) as shown in Theorem 5.1. We apply the procedure from Theorem 5.1 to construct $G_n'$, whose form is given roughly below:

$$
G_n' = \begin{pmatrix}
0 & (1 - d^2)G_{n-1} & * & * & * \\
G_{n-1} & dG_{n-1} & * & * & * \\
0 & * & * & * & * \\
0 & * & * & * & * \\
0 & * & * & * & * \\
0 & * & * & * & * \\
\end{pmatrix}
$$

Consider the block in $G_n'$ whose columns correspond to states in $i_1(B_{n-1})$ and whose rows correspond to states in neither $i_0(B_{n-1})$ nor $i_1(B_{n-1})$ (boxed above). Every row in this submatrix is a linear combination of two rows from $G_{n-1}$. More precisely, each row is of the form $a_1l_1 - a_2dl_2$, where $l_1$ and $l_2$ are two rows, not necessarily distinct, in $G_{n-1}$, and $a_1, a_2 \in A = \{1, d, x_1, y_1, z_2\}$. If we assume $(1 - d^2)$ is invertible in our ring, then each row is a linear combination of two rows from $(1 - d^2)G_{n-1}$. We then simplify $G_n'$ as follows:

Let $G_n''$ be the matrix obtained by properly subtracting linear combinations of the first $n\binom{2n-2}{n-1}$ rows of $G_n'$ from the rows which correspond to states in neither $i_0(B_{n-1})$ nor $i_1(B_{n-1})$ so that the submatrix obtained by restricting $G_n''$ to columns corresponding to states in $i_1(B_{n-1})$ and rows corresponding to states in neither $i_0(B_{n-1})$ nor $i_1(B_{n-1})$ is equal to 0:

$$
G_n'' = \begin{pmatrix}
0 & (1 - d^2)G_{n-1} & * & * & * \\
G_{n-1} & dG_{n-1} & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
\end{pmatrix}
$$

The block decomposition at this point proves that det $G_n''$ is equal to $(1 - d^2)^{n\binom{2n-2}{n-1}}(\det G_{n-1})^2$ times the determinant of the boxed block, which we denote by $\overline{G}_n$. The latter contains a power of $(1 - d^2)^{-1}$, whose degree is unspecified. Thus, $\det G_{n-1}^2|\det G_n''(1 - d^2)^k$ for some integer $k \geq 0$. We remind the reader that $G_n''$ is obtained from $G_n'$ via determinant preserving operations, and hence $\det G_n' = \det G_n$.

Note that if $\det G_n$ has fewer than $n\binom{2n-2}{n-1}$ powers of $(1 - d^2)^{-1}$, then $\det G_{n-1}^2|\det G_n$. It remains an open problem as to whether this is true. For an example of this decomposition, we refer the reader to the Appendix.

### 7. Future Directions

In this section, we discuss briefly generalizations of the Gram determinant and present a number of open questions and conjectures.
7.1. The case of a disk with \( k \) holes. We can generalize our setup by considering \( F_{0,k}^n \), a unit disk with \( k \) holes, in addition to \( 2n \) points, \( a_0, \ldots, a_{2n-1} \), arranged in a similar way to points in \( F_{0,2}^n \). For \( b_i, b_j \in B_{n,k} \), let \( b_i \circ b_j^* \) be defined in the same way as before. Each paired diagram \( b_i \circ b_j^* \) consists of up to \( n \) closed curves on the 2-sphere \((D^2 \cup (D^2)^* )\) with \( 2k \) holes. Let \( S \) denote the set of all \( 2k \) holes. We differentiate between the closed curves based on how they partition \( S \). We define a bilinear form by counting the multiplicities of each type of closed curve in the paired diagram. In the case \( k = 2 \), we assigned to each paired diagram a corresponding element in a polynomial ring of eight variables, each variable representing a type of closed curve. In the general case, the number of types of closed curves is equal to

\[
\frac{2|S|}{2} = 2^k = 2^{2k-1}
\]

so we can define the Gram matrix of the bilinear form for a disk with \( k \) holes and \( 2n \) points with \((n + 1)^{k-1}(2n) \times (n + 1)^{k-1}(2n)\) entries, each belonging to a polynomial ring of \( 2^{2k-1} \) variables. We denote this Gram matrix by \( G_{F_{0,k}}^n \). For \( n = 1 \) and \( k = 3 \), we can easily write this \( 8 \times 8 \) Gram matrix. For purposes of notation, let us denote the holes in \( F_{0,3}^n \) by \( \partial_1, \partial_2 \) and \( \partial_3 \), and their inversions by \( \partial_{-1}, \partial_{-2} \) and \( \partial_{-3} \), respectively. Hence, each closed curve in the surface encloses some subset of \( S = \{ \partial_1, \partial_{-1}, \partial_2, \partial_{-2}, \partial_3, \partial_{-3} \} \). Let \( x_{a_1,a_2,a_3} \) denote a curve separating the set of holes \( \{ \partial_{a_1}, \partial_{a_2}, \partial_{a_3} \} \) from \( S - \{ \partial_{a_1}, \partial_{a_2}, \partial_{a_3} \} \). We can similarly define \( x_{a_1,a_2} \) and \( x_{a_1} \). The Gram matrix is then:

\[
G_{F_{0,3}}^1 = \begin{pmatrix}
  d & x_{-3} & x_{-2} & x_{-2,-3} & x_{-1} & x_{-1,-3} & x_{-1,-2} & x_{1,2,3} \\
  x_{3} & x_{3,-3} & x_{2,-3} & x_{1,-1,2} & x_{1,-1,3} & x_{1,1,2} & x_{1,1,3} & x_{1,1,2,3} \\
  x_{2} & x_{2,-3} & x_{2,-2} & x_{1,-1,2} & x_{1,-1,3} & x_{1,1,2} & x_{1,1,3} & x_{1,1,2,3} \\
  x_{2,3} & x_{1,-1,-2} & x_{1,-1,-3} & x_{1,-1} & x_{1,-1,-3} & x_{1,-1,-2} & x_{1,-1} & x_{1,-1,-3} \\
  x_{1} & x_{1,-1,3} & x_{1,-1,2} & x_{1,-2,3} & x_{1,-2,1} & x_{1,-2,3} & x_{1,-2,1} & x_{1,-2,3} \\
  x_{1,3} & x_{1,3,-3} & x_{1,-2,3} & x_{1,-1,3} & x_{1,-1,2} & x_{1,-1,3} & x_{1,-1,2} & x_{1,-1,3} \\
  x_{1,2} & x_{1,2,-3} & x_{1,-2,3} & x_{1,-1,3} & x_{1,-1,2} & x_{1,-1,3} & x_{1,-1,2} & x_{1,-1,3} \\
  x_{1,2,3} & x_{1,-1,-2} & x_{1,-1,-3} & x_{1,-1} & x_{1,-2,3} & x_{1,-2,1} & x_{1,-2,3} & x_{1,-2,1} \\
\end{pmatrix}
\]

It would be tempting to conjecture that the determinant of the above matrix has a straightforward decomposition of the form \((u + v)(u - v)\). We found that it is the case for the substitution \( x_{a_1} = x_{a_1,a_2} = 0 \) with \( a_1, a_2 \in \{-3, -2, -1, 1, 2, 3\} \) (see Appendix). However in general, the preliminary calculation suggests that \( \det G_{F_{0,3}}^1 \) may be an irreducible polynomial.

Finally, we observe that many results we have proven for \( \det G_{n}^{F_{0,2}} \) holds for general \( \det G_{n}^{F_{0,k}} \). For example, \( \det G_{n}^{F_{0,k}} \neq 0 \) and \( \det G_{n}^{F_{0,k-1}} | \det G_{n}^{F_{0,k}} \). In the specific case of \( \det G_{n}^{F_{0,3}} \) we conjecture that the diagonal term is of the form \( \delta(n) = d^{a(n)}(x_{-1,-1}x_{2,-2}x_{3,-3})^{\beta(n)} \), where \( a(n) + 3\beta(n) = n(n + 1)^2(2n) \) and \( \beta(n) = n(n + 1)^4(n + 1) \).

7.2. Speculation on factorization of \( \det G_{n} \). Section 5 establishes that \( \det G_{n-1} | \det G_{n} \), but we conjecture that there are many more powers of \( \det G_{n-1} \) in \( \det G_{n} \). Indeed, even
in the base case, \( \det G_1^k \mid \det G_2 \) for \( k \) up to 4. Finding the maximal power of \( \det G_{n-1} \) in \( \det G_n \) in the general case is an open problem and can be helpful toward computing the full decomposition of \( \det G_n \).

Examining the terms of highest degree in \( \det G_n \), that is, \( h(\det G_n) \) may also yield helpful hints toward the full decomposition. In particular, we note that:

\[
\det G_1^4 | h(\det G_2) \quad \text{and} \quad \left( \frac{h(\det G_2)^6}{\det G_1^9} \right) | h(\det G_3)
\]

We can conjecture that

\[
\left( \frac{\det G_2^6}{\det G_1^9} \right) | \det G_3
\]

so it follows that \( \det G_1^{15} \mid \det G_3 \). We therefore offer the following conjecture:

**Conjecture 1.** \( \det G_1^{(\frac{2^n}{n-1})} \mid \det G_n \) for \( n \geq 1 \).

In addition, we also offer the following conjecture, motivated by observations of \( \det G_1 \) and \( \det G_2 \):

**Conjecture 2.** Let \( H_n \) denote the factors of \( \det G_n \) not in \( \det G_{n-1} \), that is, \( H_n \mid \det G_n \) and \( \gcd(H_n, \det G_{n-1}) = 0 \). Then \( (H_n)^{2n} \mid \det G_n \).

**Conjecture 3.** Let, as before, \( R = \mathbb{Z}[d, x_1, x_2, y_1, y_2, z_1, z_2, z_3] \) and \( R_1 \) be a subgroup of \( R \) of elements invariant under \( h_1, h_2, h_t \), and \( g_1, g_2, g_3 \). Similarly, let \( R_2 \) be a subgroup of \( R \) composed of elements \( w \in R \) such that \( h_1(w) = h_2(w) = -w \) and \( h_t(w) = g_1(w) = g_2(w) = g_3(w) \). Then

1. \( \det G_n = u^2 - v^2 \), where \( u \in R_1 \) and \( v \in R_2 \).
2. \( \det G_n = \prod \alpha(u^2_\alpha - v^2_\alpha) \), where \( u_\alpha \in R_1 \) and \( v_\alpha \in R_2 \), and \( u_\alpha - v_\alpha \) and \( u_\alpha + v_\alpha \) are irreducible polynomials.
3. \( \det G_n = \prod_{i=1}^{n}(u_i^2 - v_i^2)^{(\alpha_{i-1})} \), where \( u_i \in R_1 \) and \( v_i \in R_2 \).

Notice that if \( w_1 = u_1^2 - v_1^2 \) and \( w_2 = u_2^2 - v_2^2 \), then \( w_1 w_2 = (u_1 u_2 + v_1 v_2)^2 - (u_1 v_2 + u_2 v_1)^2 \).

We have little confidence in Conjecture 3(3). It is closely, maybe too closely, influenced by the Gram determinant of type B (\( \det G^B_n = \det G_n^{F_{0,1}} \)). That is

**Theorem 7.1.** (\[14\])

\[
\det G_n^B = \prod_{i=1}^{n} (T_i(d)^2 - a^2)^{(2^{n-1})}
\]

where \( T_i(d) \) is the Chebyshev polynomial of the first kind:

\[
T_0 = 2, \quad T_1 = d, \quad T_i = d T_{i-1} - T_{i-2};
\]

\( d \) and \( a \) in the formula, correspond to the trivial and the nontrivial curves in the annulus \( F_{0,1} \), respectively.
8.3. $\hat{G}_2$ (defined in Theorem 6.1), after simplification

\[
\left\{ \begin{array}{l}
x_{2y_2} - dz_2 \\
0 \\
-dx_1 + x_2 z_1 - y_1 z_2 + y_2 z_3 \\
-x_1^2 + y_1^2 + z_1^2 + z_2^2 \\
-dy_1 + y_2 z_1 - x_1 z_2 + x_2 z_3 \\
0 \\
-d^2 + x_2^2 + y_2^2 - z_2^2 \\
y_2 - dx_2 z_2 \\
-x_2 z_2 + x_1 z_3 \\
dy_2 - x_2 z_2 \\
-dx_1 + x_2 z_1 - x_2 y_2 + y_2 z_3 \\
-x_1^2 + y_1^2 + z_1^2 + z_2^2 \\
-y_2 y_1 + dy_2 \\
2y_2 - d^2 y_2 - dx_2 z_2 \\
2y_1 - dz_2 + x_1 z_1 \\
dx_1 + x_2 z_1 - x_2 y_2 + y_2 z_3 \\
-x_1^2 + y_1^2 + z_1^2 + z_2^2 \\
\end{array} \right.
\]

\[
\det \hat{G}_2 = \frac{\det G_2}{(1 - d^2)^4 \det G_1^2}
\]

8.4. $\det G_2$

\[
\det G_2 = -d^6 (-x_1 x_2 + x_2 y_1 + x_1 y_2 - y_1 y_2 + d_1 - z_1 z_2 - d_3 + x_2 z_3) (x_1 x_2 - x_2 y_1 - x_1 y_2 - y_1 y_2 + d_1 + z_1 z_2 + d_3 + x_2 z_3)
\]

\[
(8d^2 - 24 - 8y_1 + 2d_1^2 y_1^2 - 8x_2^2 + 2d_2^2 y_2^2 + 2x_1^2 x_2^2 + 2d_3^2 y_3^2 + 2d_1 d_2 y_1 y_2 - 8y_1^2 + 2d_1^2 y_2^2 + 2y_2^2 y_3^2 + 8x_1 y_1 y_2 - 2d_1^2 y_1^2 y_2^2 - 2y_1^2 y_2^2)
\]

\[
(8d^2 - 24 - 8y_1 + 2d_1^2 y_1^2 - 8x_2^2 + 2d_2^2 y_2^2 + 2x_1^2 x_2^2 + 2d_3^2 y_3^2 + 2d_1 d_2 y_1 y_2 - 8y_1^2 + 2d_1^2 y_2^2 + 2y_2^2 y_3^2 + 8x_1 y_1 y_2 - 2d_1^2 y_1^2 y_2^2 - 2y_1^2 y_2^2)
\]
8.5. Terms of maximal degree in \( \det G_3 \)

\[ h(\det G_3) = h(\det G_2) \cdot \det G_1^{30} - 2 \cdot \det G_1^{30} \cdot \omega^3 \]

\[ = d^{66} (-x_1 x_2 + x_2 y_1 + x_1 y_2 - y_1 y_2 + d z_1 - z_1 z_2 + z_2 z_3)^{15} \]

\[ (-x_1 x_2 - x_2 y_1 - x_1 y_2 + y_1 y_2 + z_1 z_2 + z_2 z_3)^{15} \]

\[ (-x_1 x_2 z_1 - y_1 y_2 z_1 + d z_1^2 + x_2 y_1 z_3 + x_1 y_2 z_3 - d z_3^2)^{12} \]

\[ (2 x_1 x_2 y_1 y_2 - d x_1 x_2 z_1 - d y_1 y_2 z_1 + d^2 z_1^2 - d x_1 y_2 z_3 - d x_1 y_2 z_3 + d^2 z_3^2)^{12} \]

\[ (x_1 x_2 y_1 y_2 z_1 - d x_1 x_2 z_1^2 - d y_1 y_2 z_1^2 + d^2 z_1^3 - x_1 x_2 y_1 y_2 z_3 + d x_2 y_1 z_3^2 + d x_1 y_2 z_3^2 - d^2 z_3^3)^{3} \]

\[ (x_1 x_2 y_1 y_2 z_1 - d x_1 x_2 z_1^2 - d y_1 y_2 z_1^2 + d^2 z_1^3 + x_1 x_2 y_1 y_2 z_3 - d x_2 y_1 z_3^2 - d x_1 y_2 z_3^2 + d^2 z_3^3)^{3} \]

8.6. \( \det G_3 \) with substitution \( x_1 = x_2 = y_1 = y_2 = z_2 = 0 \)

\[ \det G_3 |_{x_1 = x_2 = y_1 = y_2 = z_2 = 0} = (-2 + d)^{16}(-1 + d)^{30} (1 + d)^{4}(2 + d)^{16}(-3 + d)^{30}(z_1 - z_3)^{30}(z_1 + z_3)^{30} \]

\[ (z_1^2 - z_1 z_3 + z_3^2)(z_1^2 + z_1 z_3 + z_3^2)(-2d^2 - 2z_1^2 + d^2 z_1^2 - 2z_3^2 + d^2 z_3^2)^{12} \]

\[ (-3d^2 - z_1^2 + d^2 z_1^2 + z_1 z_3 - d^2 z_1 z_3 - z_3^2 + d^2 z_3^2)^2(-3d^2 - z_1^2 + d^2 z_1^2 - z_1 z_3 + d^2 z_1 z_3 - z_3^2 + d^2 z_3^2)^2 \]

8.7. \( \det G_1^{F_{0,3}} \) with substitution \( x_{a_1} = x_{a_1 a_2} = 0 \) for all variables of the form \( x_{a_1} \) and \( x_{a_1 a_2} \)

\[ \det G_1^{F_{0,3}} |_{x_{a_1} = x_{a_1 a_2} = 0} = -(d - x_1, 2, 3)(d + x_1, 2, 3) \]

\[ (x_1, 2, -2, 3, -1, 3, -1, 2, -x_1, 3, -3, x_1, -1, -1, -3 - x_1, 2, -3, x_1, -1, -1, -3 - x_1, 1, -2, x_1, -1, -2, -3 - x_1, 2, -2, x_1, 3, -3, x_1, -2, -3 + x_1, 2, -3, x_1, -2, 3, x_1, -2, -3)^2 \]
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