A symplectic approach to van den Ban’s convexity theorem

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Abstract. Let $G$ be a complex semisimple Lie group and $\tau$ a complex antilinear involution that commutes with a Cartan involution. If $H$ denotes the connected subgroup of $\tau$-fixed points in $G$, and $K$ is maximally compact, each $H$-orbit in $G/K$ can be equipped with a Poisson structure as described by Evens and Lu. We consider symplectic leaves of certain such $H$-orbits with a natural Hamiltonian torus action. A symplectic convexity theorem then leads to van den Ban’s convexity result for (complex) semisimple symmetric spaces.

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1. Introduction

In 1982, Atiyah [1] discovered a surprising connection between results in Lie theory and symplectic geometry. He proved a general symplectic convexity theorem of which Kostant’s linear convexity theorem (for complex semisimple Lie groups) is a corollary. In this context, the orbits relevant for Kostant’s theorem carry the natural symplectic structure of coadjoint orbits. The symplectic convexity theorem, which was found independently by Guillemin and Sternberg [4], states that the image under the moment map of a compact connected symplectic manifold with Hamiltonian torus action is a convex polytope. Subsequently, Duistermaat [2] extended the symplectic convexity theorem in a way that it could be used to prove Kostant’s linear theorem for real semisimple Lie groups as well.

Lu and Ratiu [10] found a way to put Kostant’s nonlinear theorem into a symplectic framework. For a complex semisimple Lie group $G$ with Iwasawa decomposition $G = NAK$, they regard the relevant $K$-orbit as symplectic leaves of the Poisson Lie group $AN$, carrying the Lu-Weinstein Poisson structure. Kostant’s nonlinear theorem for both complex and certain real groups then follows from the AGS-theorem or Duistermaat’s theorem.

In this paper, we want to give a symplectic interpretation of van den Ban’s convexity theorem for a complex semisimple symmetric space $(\mathfrak{g}, \tau)$, which is a generalization of Kostant’s nonlinear theorem for complex groups. For the
precise statement of van den Ban’s result we refer to Section 2. The main
difference in view of our symplectic approach is that van den Ban’s theo-
rem is concerned with orbits of a certain subgroup $H \subset G$ that are in general neither
symplectic nor compact. Since $G$ is complex we can use a method due to
Evens and Lu [3] to equip $H$-orbits in $G/K$ with a certain Poisson structure.
An $H$-orbit foliates into symplectic leaves, and on each leaf some torus acts in
a Hamiltonian way. The corresponding moment map $\Phi$ turns out to be proper,
and therefore the symplectic convexity theorem of Hilgert-Neeb-Plank [6] can
be applied, which describes the image under $\Phi$ in terms of local moment cones.
An analysis of those local moment cones shows that the image of $\Phi$ is the sum
of a compact convex polytope and a convex polyhedral cone, just as in van den
Ban’s theorem.
The case of van den Ban’s theorem for a real semisimple symmetric space is
dealt with in a separate paper [12]. It follows the symplectic approach of Lu
and Ratiu towards Kostant’s nonlinear convexity theorem. The main tool is a
generalized version of Duistermaat’s theorem for non-compact manifolds.

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2. Van den Ban’s theorem

The purpose of this section is to fix notation and to recall the state-
mement of van den Ban’s theorem.

Let $G$ be a real connected semisimple Lie group with finite center, equipped
with an involution $\tau$, i.e. $\tau$ is a smooth group homomorphism such that
$\tau^2 = id$.

Let $\mathfrak{g}$ be the Lie algebra of $G$. We write $H$ for an open subgroup of $G^\tau$, the
$\tau$-fixed points in $G$. Let $K$ be a $\tau$-stable maximal compact subgroup of $G$.
The corresponding Cartan involution $\theta$ on $\mathfrak{g}$ commutes with $\tau$ and induces
the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. If $\mathfrak{h}$ and $\mathfrak{q}$ denote the $\mathfrak{g}$
eigenspace of $\tau$ one obtains

$$\mathfrak{g} = (\mathfrak{k} \cap \mathfrak{h}) + (\mathfrak{p} \cap \mathfrak{h}) + (\mathfrak{k} \cap \mathfrak{q}) + (\mathfrak{p} \cap \mathfrak{q}).$$

We fix a maximal abelian subalgebra $\mathfrak{a}^{-\tau}$ of $\mathfrak{p} \cap \mathfrak{q}$. (In [14] this subalgebra is
denoted by $\mathfrak{a}_{pq}$.) In addition, we choose $\mathfrak{a}^\tau \subseteq \mathfrak{p} \cap \mathfrak{h}$ such that $\mathfrak{a} := \mathfrak{a}^\tau + \mathfrak{a}^{-\tau}$
is maximal abelian in $\mathfrak{p}$. Let $\Delta(\mathfrak{g}, \mathfrak{a}^{-\tau})$ and $\Delta(\mathfrak{g}, \mathfrak{a})$ denote the sets of roots
for the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}^{-\tau}$ and $\mathfrak{a}$, respectively.
Next, we choose a system of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{a})$ and define

$$\Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau}) = \{ \alpha|_{\mathfrak{a}^{-\tau}} : \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}), \alpha|_{\mathfrak{a}^{-\tau}} \neq 0 \}.$$ 

This leads to an Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k} = \mathfrak{n}^1 + \mathfrak{n}^2 + \mathfrak{a} + \mathfrak{k},$$
where

\[
\begin{align*}
    n &= \sum_{\alpha \in \Delta^+(g,a)} g^\alpha, \\
n^1 &= \sum_{\alpha \in \Delta^+(g,a), \alpha|_{a-\tau} \neq 0} g^\alpha = \sum_{\beta \in \Delta^+(g,a^{-\tau})} g^\beta, \\
n^2 &= \sum_{\alpha \in \Delta^+(g,a), \alpha|_{a-\tau} = 0} g^\alpha.
\end{align*}
\]

Here \( g^\alpha = \{ X \in g : [H,X] = \alpha(H)X \forall H \in a \} \) for \( \alpha \in \Delta(g,a) \), and similarly \( g^\beta \) is defined for \( \beta \in \Delta(g,a^{-\tau}) \).

Let \( N \) and \( A \) denote the analytic subgroups of \( G \) with Lie algebras \( n \) and \( a \), respectively. The Iwasawa decomposition \( G = NA \) on the group level has the middle projection \( \mu : G \rightarrow A \). We write \( pr_{a-\tau} : a \rightarrow a^{-\tau} \) for the projection along \( a^{-\tau} \).

For \( \beta \in \Delta^+(g,a^{-\tau}) \) define \( H_\beta \in a^{-\tau} \) such that \( H_\beta \perp \ker \beta, \beta(H_\beta) = 1 \), where \( \perp \) means orthogonality with respect to the Killing form \( \kappa \).

Note that the involution \( \theta \circ \tau \) leaves each root space \( g^\beta = (g^\beta)_+ \oplus (g^\beta)_- \) decomposes into \((+1)\)- and \((-1)\)-eigenspace with respect to \( \theta \circ \tau \).

For

\[
\Delta_- := \{ \beta \in \Delta(g,a^{-\tau}) : (g^\beta)_- \neq 0 \},
\]

let \( \Delta^+_\tau = \Delta_- \cap \Delta^+(g,a^{-\tau}) \). Define the closed cone

\[
\Gamma(\Delta^+_\tau) = \sum_{\beta \in \Delta^+_\tau} \mathbb{R}_+ H_\beta.
\]

Write \( W_{K\cap H} \) for the Weyl group

\[
W_{K\cap H} = N_{K\cap H}(a^{-\tau})/Z_{K\cap H}(a^{-\tau}).
\]

The convex hull of a Weyl group orbit through \( X \in a^{-\tau} \) will be denoted by \( \text{conv}(W_{K\cap H}.X) \).

**Remark 2.1.** Consider the Lie algebra \( g^{\theta \tau} \) of \( \theta \tau \)-fixed points in \( g \). It is reductive and its semisimple part \( g' = [g^{\theta \tau}, g^{\theta \tau}] \) admits a Cartan decomposition \( g' = t' + p' \) with \( t' \subset t, p' \subset p \). Due to our choice, \( a^{-\tau} \) is a maximal abelian subalgebra of \( p' \). The set of roots \( \Delta(g',a^{-\tau}) \) consists exactly of those reduced roots \( \beta \in \Delta(g,a^{-\tau}) \) for which \( (g^\beta)_+ \neq 0 \). Moreover, the Weyl group \( W' \) associated to \( g' \) coincides with \( W_{K\cap H} \).

We can now state the central theorem.
Let $G$ be a real connected semisimple Lie group with finite center, equipped with an involution $\tau$, and $H$ a connected open subgroup of $G^\tau$. For $X \in a^{-\tau}$, write $a = \exp X \in A^{-\tau}$. Then

$$(pr_{a^{-\tau}} \circ \log \circ \mu)(Ha) = \text{conv}(W_{K\cap H, X}) - \Gamma(\Delta^\pm).$$

**Remark 2.3.**

- The statement of the theorem above differs from the original in [14] by a minus sign in front of the conal part $\Gamma(\Delta^\pm)$. This is due to the fact that we consider the set $Ha$ and an Iwasawa decomposition $G = NAK$, whereas in [14] the set $aH \subset G = KAN$ is considered. Indeed, if we denote the two middle projections by $\mu : NAK \to A$ and $\mu' : KAN \to A$, then $\Gamma(\Delta^\pm) = \log \circ \mu'(H) = -\log \circ \mu(H)$.

- Van den Ban proved his theorem under the weaker condition that $H$ is an essentially connected open subgroup of $G^\tau$ (by reducing it to the connected case).

- If $\tau = \theta$ one obtains Kostant’s (nonlinear) convexity theorem. Note that in this case the group $H$ and the orbit $Ha$ are compact.

### 3. Poisson structure

Let $G$ be a connected and simply connected semisimple complex Lie group with Lie algebra $\mathfrak{g}$. The Cartan involutions on both group and Lie algebra level will be denoted by $\theta$. In addition, let $\tau$ be a complex antilinear involution (on $G$ and $\mathfrak{g}$) which commutes with $\theta$.

The Lie algebra $\mathfrak{g}$ decomposes into $(+1)$- and $(-1)$-eigenspaces with respect to both involutions $\theta$ and $\tau$.

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{h} + \mathfrak{q},$$

where $\mathfrak{k}$ and $\mathfrak{h}$ denote the $(+1)$-eigenspaces with respect to $\theta$ and $\tau$, respectively, and $\mathfrak{p}$ and $\mathfrak{q}$ denote the $(-1)$-eigenspaces.

The maximal compact subgroup $K$ of $G$ with Lie algebra $\mathfrak{k}$ is $\tau$-stable. Let $H$ denote the connected subgroup of $G$ consisting of $\tau$-fixed points. We will be interested in certain $H$-orbits in the symmetric space $G/K$. Each such orbit can be equipped with a Poisson structure as introduced by Evens and Lu. We briefly describe their method which can be found in [3, Section 2.2]. For details on Poisson Lie groups see e.g. [11].

Let $(U, \pi_U)$ be a connected Poisson Lie group with tangent Lie bialgebra $(\mathfrak{u}, \mathfrak{u}^*)$ and double Lie algebra $\mathfrak{d} = \mathfrak{u} \bowtie \mathfrak{u}^*$. The pairing

$$\langle v_1 + \lambda_1, v_2 + \lambda_2 \rangle := \lambda_1(v_2) + \lambda_2(v_1) \quad \forall v_1, v_2 \in \mathfrak{u}, \lambda_1, \lambda_2 \in \mathfrak{u}^*,$$

defines a non-degenerate symmetric bilinear form and turns $(\mathfrak{d}, \mathfrak{u}, \mathfrak{u}^*)$ into a Manin triple. We will identify $\mathfrak{d}^*$ with $\mathfrak{d}$ via $\langle . \rangle$.

Consider the following bivector $R \in \wedge^2 \mathfrak{d}$:

$$R(v_1 + \lambda_1, v_2 + \lambda_2) = \lambda_2(v_1) - \lambda_1(v_2) \quad \forall v_1, v_2 \in \mathfrak{u}, \lambda_1, \lambda_2 \in \mathfrak{u}^*.$$
In terms of a basis \( \{ v_1, \ldots, v_n \} \) for \( u \) and a dual basis \( \{ \lambda_1, \ldots, \lambda_n \} \) for \( u^* \) the bivector is represented by \( R = \sum_{i=1}^{n} \lambda_i \wedge v_i \).

Assume that \( D \) is a connected Lie group with Lie algebra \( \mathfrak{d} \), and assume that \( U \) is a connected subgroup of \( D \) with Lie algebra \( u \). Then \( D \) acts on the Grassmannian \( \text{Gr}(n, \mathfrak{d}) \) of \( n \)-dimensional subspaces of \( \mathfrak{d} \) via the adjoint action of \( D \) on \( \mathfrak{d} \) and therefore defines a Lie algebra antihomomorphism

\[
\eta: \mathfrak{d} \to \mathcal{X}(\text{Gr}(n, \mathfrak{d})),
\]

into the vector fields on \( \text{Gr}(n, \mathfrak{d}) \). Using the symbol \( \eta \) also for its multilinear extension we can define a bivector field \( \Pi \) on \( \text{Gr}(n, \mathfrak{d}) \) by

\[
\Pi = \frac{1}{2} \eta(R).
\]

Note that \( \Pi \) in general does not define a Poisson structure on the entire \( \text{Gr}(n, \mathfrak{d}) \). However, it does so on the subvariety \( \Sigma(\mathfrak{d}) \) of Lagrangian subspaces (with respect to \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{d} \), and on each \( D \)-orbit \( D.l \subset \Sigma(\mathfrak{d}) \).

The bivector \( R \) also gives rise to a Poisson structure \( \pi_- \) on \( D \) that makes \( (D, \pi_-) \) a Poisson Lie group:

\[
\pi_-(d) = \frac{1}{2} (r_d R - l_d R) \quad \forall d \in D.
\]

Here \( r_d \) and \( l_d \) denote the differentials of right and left translations by \( d \). Note that the restriction of \( \pi_- \) to the subgroup \( U \subset D \) coincides with the original Poisson structure \( \pi_U \) on \( U \), i.e. \( (U, \pi_U) \) is a Poisson subgroup of \( (D, \pi_-) \).

For \( l \in \Sigma(\mathfrak{d}) \) the \( D \)-orbit through \( l \) is not only a Poisson manifold with respect to \( \Pi \) but a homogeneous Poisson space under the action of \( (D, \pi_-) \). Moreover, the \( U \)-orbit \( U.l \) is a homogeneous \( (U, \pi_U) \)-space, since the Poisson tensor \( \Pi \) at \( l \) turns out to be tangent to \( U.l \). In fact, the tangent space at \( l \in D.l \) can be identified with \( \mathfrak{d}/n(l) \), where \( n(l) \) is the normalizer subalgebra of \( l \). In the case when \( n(l) = l \), we identify the cotangent space with \( l \) itself, and for \( X, Y \in l \) one obtains:

\[
\Pi(l)(X, Y) = \langle pr_u X, Y \rangle, \quad \text{i.e.} \quad \Pi(l)^2(X) = pr_u X,
\]

where \( pr_u : \mathfrak{d} \to u \) denotes the projection along \( u^* \).

Let \( U^* \) be the connected subgroup of \( D \) with Lie algebra \( u^* \). What has been said about the Poisson Lie group \( U \) is also true for its dual group \( U^* \), i.e. \( (U^*, \pi_{U^*}) \) is a Poisson Lie subgroup of \( (D, \pi_-) \) and the orbit \( U^*.l \) is a homogeneous \( (U^*, \pi_{U^*}) \)-space. It follows in particular that \( (U.l) \cap (U^*.l) \) contains the symplectic leaf through \( l \).

We now want to apply this construction to our complex semisimple Lie algebra \( \mathfrak{g} \). In the above notation we will have \( \mathfrak{d} = \mathfrak{g} \), and the pairing \( \langle \cdot, \cdot \rangle \) will be given by the imaginary part, \( \Im \kappa \), of the Killing form \( \kappa \) on \( \mathfrak{g} \). Note that \( \mathfrak{k} \in \Sigma(\mathfrak{d}) \). Throughout the paper, we will identify the \( G \)-orbit through \( \mathfrak{k} \) with the symmetric space \( G/K \). In particular, orbits in \( G.\mathfrak{k} \) are identified with those in \( G/K \). Then we set \( u = \mathfrak{h} \), and it remains to define \( u^* \).
First we choose an appropriate Iwasawa decomposition of $g$. Recall the $\tau$-stable Cartan decomposition $g = \mathfrak{k} + \mathfrak{p}$. We fix a maximal abelian subalgebra $\mathfrak{a}^{\tau}$ in $\mathfrak{p} \cap \mathfrak{q}$. Then we can find an abelian subalgebra $\mathfrak{a}^\tau$ in $\mathfrak{p} \cap \mathfrak{h}$ such that $\mathfrak{a} = \mathfrak{a}^{\tau} + \mathfrak{a}^\tau$ is maximal abelian in $\mathfrak{p}$. We choose a positive root system, $\Delta^+(\mathfrak{g}, \mathfrak{a})$ by the lexicographic ordering with respect to an ordering of a basis of $\mathfrak{a}$, which was constructed from a basis of $\mathfrak{a}^{\tau}$ followed by a basis of $\mathfrak{a}^\tau$. This yields an Iwasawa decomposition $g = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ which is compatible with the involution $\tau$ in the following sense.

**Lemma 3.1.** For our choice of Iwasawa decomposition $g = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$, we have $\mathfrak{h} \cap \mathfrak{n} = \{0\}$. Besides, the centralizer of $\mathfrak{a}^{\tau}$ in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$.

**Proof.** Consider the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$,

$$\mathfrak{g} = (\mathfrak{a} + i\mathfrak{a}) + \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}^\alpha.$$

It is well-known [7, Proposition 6.70] that there are no real roots for a maximally compact Cartan subalgebra $(\mathfrak{a}^{-\tau} + \mathfrak{a}^\tau)$ of $\mathfrak{h}$, and therefore there are no $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$ such that $\alpha|_{\mathfrak{a}^{-\tau}} = 0$. By [8] Chapter VI, Lemma 3.3, this implies that $\tau(\mathfrak{g}^\alpha) \subset \bigoplus_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}^{-\alpha}$ for all $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$, and the claim $\mathfrak{h} \cap \mathfrak{n} = \{0\}$ follows immediately.

Since each $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$ does not vanish outside a hyperplane of $\mathfrak{a}^{-\tau}$, it follows that $\mathfrak{a}^{\tau}$ contains regular elements and its centralizer in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$.

$$\square$$

Consider the Cartan subalgebra $\mathfrak{c} = \mathfrak{z}(\mathfrak{a}^{-\tau})$ of $\mathfrak{g}$. Lemma 3.1 together with the properties of $\kappa$ implies that $\mathfrak{g} = \mathfrak{h} \oplus (\mathfrak{c}^{-\tau} \oplus \mathfrak{n})$ is a Lagrangian splitting with respect to the bilinear form $\Im$. In other words, $(\mathfrak{g}, \mathfrak{h}, (\mathfrak{c}^{-\tau} + \mathfrak{n}))$ is a Manin triple.

We can now define the desired Poisson manifolds using the method of Evens and Lu outlined above. We set

$$\mathfrak{d} = \mathfrak{g}, \ u = \mathfrak{h}, \ u^* = \mathfrak{c}^{-\tau} + \mathfrak{n}, \ (,) = \Im.$$

Let $C, C^{-\tau}, A$ and $N$ denote the analytic subgroups of $G$ with Lie algebras $\mathfrak{c}, \mathfrak{c}^{-\tau}, \mathfrak{a}$ and $\mathfrak{n}$, respectively. The group $H$ now has the structure of a Poisson Lie group. Its dual group is $H^* = C^{-\tau}N$. Fix $a \in A^{-\tau}$ and consider the base point $a.K \in G/K$. The $H$-orbit $P_a = Ha.K \in G/K$ is a Poisson homogeneous manifold with respect to the action by $(H, \pi_H)$. Also, the dual group orbit $H^*a.K$ is Poisson homogeneous with respect to $\pi_{H^*}$. For the symplectic leaf in $P_a$ through $a$, denoted by $M_a$, we have $M_a \subseteq Ha.K \cap H^*a.K$.

**Lemma 3.2.** The Poisson manifold $P_a$ is regular and equals the union of $A^*$-translates of $M_a$, i.e. each $p \in P_a$ can be written $p = a'm$ with unique $a' \in A^*, m \in M_a$. Moreover, $M_a = Ha.K \cap H^*a.K$. 

Proof. Consider the map \( M : A^r \times M_a \rightarrow P_a \).

First we will show that \( M \) is injective. The Poisson tensor \( \pi_H = \pi_- \) as defined in \([1]\) vanishes at each element \( c \in C^r \), since \( Ad(c) \) leaves both \( h \) and \( h^* = c^{-\tau} + n \) stable. Therefore \( a' \in A^r \) acts on \( P_a \) by Poisson diffeomorphisms and maps the symplectic leaf \( M_a \) onto the symplectic leaf \( M_{a' a} \). But \( M_{a_1 a} \neq M_{a_2 a} \) for \( a_1 \neq a_2 \in A^r \), following from the fact that \( M_{a_1 a} \) lies in \( H^*a_1 a.K = C^{-\tau}N_a a.K \) and the uniqueness of the Iwasawa decomposition.

At each point \( p \in P_a \) one can explicitly calculate the codimension of the symplectic leaf through \( p \) in \( P_a \), for instance by means of an infinitesimal version of Corollary 7.3 in \([9]\) and Theorem 2.21 in \([3]\). It follows that the codimension of the leaf through the point \( p = ha.K \) in the orbit \( P_a \) equals the dimension of the intersection of \( Ad(a) \mathfrak{k} \) and \( Ad(h^{-1}) \mathfrak{h}^* \), which is easily seen to be independent from the point \( p \in P_a \) and equal to the dimension of \( a^\tau \). Here we used the fact that the dimension of \( Ad(ha) \mathfrak{k} \cap \mathfrak{h}^* \) cannot exceed the dimension of \( a^\tau \), since the Killing form is negative definite on \( Ad(ha) \mathfrak{k} \) and a maximal negative definite subspace of \( \mathfrak{h}^* \) is \( ia^\tau \). This shows that \( P_a \) is a regular Poisson manifold, and that \( A^r M_a \) is a full dimensional subset of \( P_a \). Since \( A^r \) acts freely on \( P_a \) and \( P_a \) is a regular Poisson manifold, it can be represented as the union of such open subsets. The connectedness of \( P_a \) then implies that \( P_a = A^r M_a \).

Since \( A^r \) is connected and the union of \( A^r \)-translates of \( Ha.K \cap H^*a.K \) equals \( Ha.K \) and thus is also connected, it is easy to see that \( Ha.K \cap H^*a.K \) is connected as well. Besides, from the transversality we see that

\[
\dim(Ha.K \cap H^*a.K) = \dim(Ha.K) + \dim(H^*a.K) - \dim(G/K).
\]

Note that the first part of the proof implies that \( A^r a.K \cap H^*a.K = \{a.K\} \). Therefore, the codimension of \( Ha.K \cap H^*a.K \) in \( Ha.K \) is at least \( \dim(a^\tau) \).

But since \( M_a \) has codimension equal to \( \dim(a^\tau) \), and \( M_a \subseteq Ha.K \cap H^*a.K \), the last inclusion is actually an equality.

Consider the torus \( T = \exp(i a^{-\tau}) \subset H \). It acts on \( M_a \) in a symplectic manner, since \( \pi_H \) vanishes at each \( t \in T \). Moreover, the next lemma shows that this action is Hamiltonian with an associated moment map that is closely related to the middle projection \( \mu : G = NAK \rightarrow A \) of the Iwasawa decomposition.

**Lemma 3.3.** The action of \( T = \exp(i a^{-\tau}) \) on \( M_a \) is Hamiltonian with a moment map \( \Phi = \text{pr}_{a^{-\tau}} \circ \log \circ \mu \). Here, \( \text{pr}_{a^{-\tau}} : \mathfrak{a} \rightarrow a^{-\tau} \) denotes the projection along \( a^\tau \), and \( \mathfrak{t}^* \) is identified with \( a^{-\tau} \) via \( \Theta \).

Moreover, the moment map \( \Phi \) is proper.

**Proof.**

(1) \( \Phi = \text{pr}_{a^{-\tau}} \circ \log \circ \mu \) is a moment map.

Let \( b : G = NAK \rightarrow B = NA \) be the \( B \)-projection in the Iwasawa decomposition. We write \( \text{pr}_{a} : \mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{t} \rightarrow \mathfrak{a} \) for the middle projection on the Lie algebra level. Let \( Z \in \mathfrak{t} = ia^{-\tau}, h \in H \) and \( X \in \mathfrak{n} \). We denote by \( \Phi_Z \) the function obtained by evaluating \( \Phi \) at \( Z \), by \( \tilde{X}_{ha} \) the tangent vector of the vector field generated by \( X \) at the point \( ha.K \in M_a \) (for brevity we will write \( h.K \) simply as \( h \) henceforth,

\[
\text{pr}_{a^{-\tau}}(b(a^{-\tau})X_{ha}) = \text{pr}_{a^{-\tau}}(a^{-\tau}X_{ha}) = a^{-\tau}X_{ha}.
\]
without fear of confusion) and by $D \Phi_{b(ha)}$ the derivative of $\Phi$ at the point $b(ha)$. We have:

\[
d\Phi_Z(\dot{X}_{ha}) = \left. \frac{d}{ds} \Phi_Z(\exp(sX)ha) \right|_{s=0} = \left. \frac{d}{ds} \Phi(\exp(sX)ha), Z \right|_{s=0} = \langle \frac{d}{ds} \Phi(b(ha) \exp(sAd(b(ha)^{-1})X)), Z \rangle = \langle D\Phi_{b(ha)}(b(ha)^{-1})X, Z \rangle
\]

The second last step follows from the fact that $t$ and $\mathfrak{k} + a$ are orthogonal with respect to $\langle \, , \rangle$.

Note that $Ad(b(ha))Z \in Z + n$. With (2) this implies

\[
\Pi(ha)^\flat(d\Phi_Z(ha)) = pr_h Ad(b(ha))Z = Z.
\]

(2) $\Phi$ is proper.

This follows from Lemma 3.3 in [14], which states the properness of the map

\[
F_a : (H \cap L_0)\backslash H \to \mathfrak{a}^{-\tau}, \quad F_a(x) = \Phi(xa).
\]

In our case $L_0 = \exp(ia\mathfrak{a})^\tau$ (since $\delta_{\mathfrak{a}}(a^{-\tau}) = c$ by the argument in the proof of Lemma 3.1).

Properness of the map $F_a : TA^\tau\backslash H \to \mathfrak{a}^{-\tau}$ implies properness of the induced maps $F_a : A^\tau\backslash H \to \mathfrak{a}^{-\tau}$ and $F_a : A^\tau\backslash (H \cap aKa^{-1}) \to \mathfrak{a}^{-\tau}$. Since $A^\tau\backslash (H \cap aKa^{-1}) \cong M_a$ by Lemma 3.2 and since $F_a$ becomes $\Phi$ under this identification, the claim follows.

\[\Box\]

**Remark 3.4.** In case $\tau = \theta$ the Lu-Evens Poisson structure on $P_a = Ka.K$ coincides with the Lu-Weinstein symplectic structure, and Lemma 3.3 becomes Theorem 4.13 in [10].

### 4. Symplectic convexity

Throughout this section we assume $G$ to be complex and the involution $\tau$ to be complex antilinear. In this case we will interpret van den Ban’s theorem in the symplectic framework developed in Section 3. More precisely, it can be viewed as a corollary of a symplectic convexity theorem for Hamiltonian torus actions.

Van den Ban’s theorem describes the image of the group orbit $Ha$ under the map $pr_{a^{-\tau}} \circ \log \circ \mu$. Recall from Section 3 the symplectic manifold $M_a \subseteq Ha.K \subseteq G/K$ on which the torus $T = \exp(i\mathfrak{a}^{-\tau})$ acts in a Hamiltonian fashion (Lemma 3.3). The associated moment map is $\Phi = pr_{a^{-\tau}} \circ \log \circ \mu$. From Lemma 3.2 and from the $A^\tau$-invariance of $pr_{a^{-\tau}} \circ \log \circ \mu$ it follows that

\[
(pr_{a^{-\tau}} \circ \log \circ \mu)(Ha) = \Phi(M_a).
\]
This means that van den Ban’s theorem can be viewed as a description of the image of a symplectic manifold under an appropriate moment map.

The description of the image of the moment map is the content of a series of symplectic convexity theorems. Probably best known are the original theorems of Atiyah and Guillemin-Sternberg [1, 4]. The result needed here is a generalization of the AGS-theorems to a non-compact setting. Several versions can be found in the literature, e.g. [8, 13]. We will state the theorem as given in [6]. Recall that a subset $C$ of a finite dimensional vector space $V$ is called locally polyhedral iff for each $x \in C$ there is a neighborhood $U_x \subseteq V$ such that $C \cap U_x = (x + \Gamma_x) \cap U_x$ for some cone $\Gamma_x$. A cone $\Gamma$ is called proper if it contains no lines, otherwise $\Gamma$ is called improper.

**Theorem 4.1.** [6, Theorem 4.1(i)] Consider a Hamiltonian torus action of $T$ on the connected symplectic manifold $M$. Suppose the associated moment map $\Phi : M \to \mathfrak{t}^*$ is proper, i.e. $\Phi$ is a closed mapping and $\Phi^{-1}(Z)$ is compact for every $Z \subseteq \mathfrak{t}^*$. Then $\Phi(M)$ is a closed, locally polyhedral, convex set.

**Remark 4.2.** Theorem 4.1 in [6] contains more detailed information, in particular a description of the cones that span $\Phi(M)$ locally (part (v)). More precisely, for each $m \in M$ there is a neighborhood $U_{\Phi(m)} \subseteq \mathfrak{t}^*$ of $\Phi(m)$ such that $\Phi(M) \cap U_{\Phi(m)} = (\Phi(m) + \Gamma_{\Phi(m)}) \cap U_{\Phi(m)}$, where $\Gamma_{\Phi(m)} = \mathfrak{t}_m^+ + C_m$. Here, $\mathfrak{t}_m$ denotes the Lie algebra of the stabilizer $T_m$ of $m$, and $C_m \subseteq \mathfrak{t}_m^+$ denotes the cone which is spanned by the weights of the linearized action of $T_m$. The (nontrivial) fact that the cone $\Gamma_{\Phi(m)} = \mathfrak{t}_m^+ + C_m$ is actually independent of the choice of a preimage point of $\Phi(m)$ is also shown in [6].

Coming back to the symplectic manifold $M_a$, Lemma 3.3 shows that the moment map $\Phi = pr_{A^{-\tau}} \circ \log \circ \mu$ on $M_a$ is proper. Theorem 4.1 can therefore be applied and yields

$$\Phi(M_a) \text{ is a closed, locally polyhedral, convex set.}$$

We will now give a more detailed description of $\Phi(M_a)$. It turns out that the $T$-action on $M_a$ has (finitely many) fixed points. At each fixed point we can calculate the cones that locally span $\Phi(M_a)$. From this description it will follow that the entire set $\Phi(M_a)$ lies in a proper cone and can therefore be described entirely by the local data at the fixed points.

We begin by determining the $T$-fixed points.

**Proposition 4.3.** The $T$-fixed points in $M_a$ are exactly those elements of the form $w(a).K \in G/K$ with $w \in W_{K \cap H} = N_{K \cap H}(a^{-\tau})/Z_{K \cap H}(a^{-\tau})$.

**Proof.** Recall that for $a \in A^{-\tau}$ we view the symplectic manifold $M_a$ as a submanifold of the $H$-orbit in $G/K$ through the base point $a.K \in G/K$. Clearly, each element $w(a).K \in G/K$ with $w \in W_{K \cap H}$ is $T$-fixed. To see that $w(a).K$ lies in $M_a$, note that $w(a).K \in H^*a.K$ since $w(a) \in A^{-\tau}$. On the other hand, there exists $k \in K \cap H$ such that $w(a) = k a k^{-1}$, which implies $w(a).K \in Ha.K$. Therefore, $w(a).K \in Ha.K \cap H^*a.K = M_a$ by Lemma 4.2.
Conversely, assume that $cpa.K \in M_n$ with $c \in K^\tau, p \in \exp(p^\tau)$ is $T$-fixed. Since $M_n$ lies in the orbit of the dual group $H^* = NC^{-\tau}$ there are elements $n \in N, b \in A^{-\tau}, k \in K$ such that $cpa = nbk$. Since nb.K \in G/K is a $T$-fixed point,

$$tnt^{-1}b \in nbK \quad \forall \ t \in T.$$ 

The Lie subalgebra $n$ is $T$-invariant, so by the uniqueness of the Iwasawa decomposition $tnt^{-1} = n$ for all $t \in T$. But since $\alpha_{a^{-\tau}} \neq 0$ for all $\alpha \in \Delta(g, a)$ this can happen only for $n = e$. This implies $cpa = bk$. Symmetrizing the last equation yields

$$cpa \theta(cp) = cpa^2pc^{-1} = b^2.$$ 

Applying $\theta \circ \tau$ to (3) gives

$$cp^{-1}a^2p^{-1}c^{-1} = b^2.$$ 

We multiply (3) by (4) from the right and from the left and obtain

$$cpa^4p^{-1}c^{-1} = b^4 = cp^{-1}a^4pc^{-1}.$$ 

But then $pa^4p^{-1} = p^{-1}a^4p$, i.e. $p^2$ and $a^4$ commute (and are self-adjoint). Therefore, $p$ and $a^2$ also commute, and we can combine equations (3) and (4) to

$$cp^2a^2c^{-1} = b^2 = cp^{-2}a^2c^{-1}.$$ 

This shows $p^2 = p^{-2}$ or $p = e$. But then (4) implies $cac^{-1} = b$. Since both $a$ and $b$ lie in $A^{-\tau}$ and since $c \in K^\tau = K \cap H$, there is some element $w \in W_{K\cap H}$ such that $w(a) = b$ (Recall from Remark 2.1 that $W_{K\cap H}$ is the Weyl group of the reductive Lie algebra $g^{\theta_\tau} = (t \cap h) + (p \cap q)$ of $\theta$-fixed points of $g$). The $T$-fixed point $cpa.K \in M_n$ can therefore be written as $cpa.K = b.K = w(a).K$ for some $w \in W_{K\cap H}$.

Recall our choice of base point $a = \exp(X)$ and the identification $t^\tau \cong a^{-\tau}$. We now describe the image of the moment map $\Phi(M_n) \in a^{-\tau}$ in the neighborhood of a fixed point image $\Phi(w(a).K) = w(X)$. From Theorem 4.4 (and Remark 4.4) we know that locally $\Phi(M_n)$ looks like $w(X) + \Gamma_{w(X)}$ for some cone $\Gamma_{w(X)} \in a^{-\tau}$. The next Lemma describes $\Gamma_{w(X)}$ in terms of the vectors $H_\beta$ for (reduced) roots $\beta \in \Delta(g, a^{-\tau})$ defined in Section 2.

**Lemma 4.4.** Let $a = \exp X$ with $X \in a^{-\tau}$ and $w \in W_{K\cap H}$. The local cone $\Gamma_{\Phi(w(a), K)} = \Gamma_{w(X)} \subseteq a^{-\tau}$ is the cone spanned by the union of the following two sets.

$$\{-\beta(w(X))H_\beta : \beta \in \Delta^+(g, a^{-\tau}), (g^\beta)_+ \neq 0\}$$

and

$$\{-H_\beta : \beta \in \Delta^+(g, a^{-\tau}), (g^\beta)_- \neq 0\}$$

**Proof.** We are adapting the argument from [6 page 155] to our setting. To determine the local cone $\Gamma_{w(X)}$ it is enough to consider the linearized action of $T$ on the tangent space $V_{w(a), K} := T_{w(a), K}M_n$. Darboux’s theorem guarantees the existence of a $T$-equivariant symplectomorphism of a neighborhood of $w(a).K \in$...
expressed in terms of the Poisson tensor. The formula for the Poisson tensor at $w^\#$

\[\Pi_{w,\mathcal{K}}(Z.Y, Y) \quad \forall \ Y \in V_{w,\mathcal{K}}, Z \in \mathfrak{t}.\]

Here, $\Omega_{w,\mathcal{K}}$ denotes the symplectic form on the symplectic vector space $V_{w,\mathcal{K}}$. Since $T$ acts symplectically on $V_{w,\mathcal{K}}$ the notation $Z.Y$ makes sense as the linear action of an element $Z \in \mathfrak{sp}(V_{w,\mathcal{K}})$ on a vector $Y \in V_{w,\mathcal{K}}$.

In appropriate symplectic coordinates $q_1, p_1, \ldots, q_n, p_n$ we have $\Omega_{w,\mathcal{K}} = \sum_i dq_i \wedge dp_i$ and the matrix representation for the linear map defined by $Z \in \mathfrak{t}$ is

\[
\begin{pmatrix}
0 & \alpha_1(Z) & \cdots & \alpha_n(Z) \\
-\alpha(Z) & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & -\alpha_n(Z) & 0 \\
\end{pmatrix}
\begin{pmatrix}
q_1 \\
p_1 \\
\vdots \\
q_n \\
p_n \\
\end{pmatrix}
\]

The moment map takes the form

\[
\Phi(q_1, p_1, \ldots, q_n, p_n) = \Phi(w,\mathcal{K}) + \sum_{i=1}^n \alpha_i \left( \frac{1}{2} q_i^2 + p_i^2 \right).
\]

In terms of the symplectic coordinates on $V_{w,\mathcal{K}}$ chosen above, the matrix representations for the symplectic form $\Omega_{w,\mathcal{K}}$ and the corresponding Poisson tensor $\Pi_{w,\mathcal{K}}$ just differ by a factor of $(-1)$. The moment map can then be expressed in terms of the Poisson tensor.

\[
\Phi(Z, \varphi) = -\Pi_{w,\mathcal{K}}(Z, \varphi, \varphi) \quad \forall \ \varphi \in V_{w,\mathcal{K}}^*, Z \in \mathfrak{t}.
\]

(Recall the bijection $\Pi^\sharp : V_{w,\mathcal{K}}^* \to V_{w,\mathcal{K}}$. Then

\[
Z \varphi = (\Pi^\sharp)^{-1}(Z \langle \varphi, \varphi \rangle),
\]

where the dot on the right hand side has been explained above.)

The local cone $\Gamma_{w(X)}$ is just $\Phi(V_{w,\mathcal{K}}^*)$, i.e. it consists exactly of the weights

\[
\{ \ Z \mapsto -\Pi_{w,\mathcal{K}}(Z, \varphi, \varphi) : \ \varphi \in V_{w,\mathcal{K}}^* \}
\]

Recall that we identify the cotangent space $T_{w,\mathcal{K}}^*(G/K)$ with $\text{Ad}(w,\mathcal{K})$. The formula for the Poisson tensor at $w,\mathcal{K}$ says that for $Y_1, Y_2 \in \mathfrak{t},$

\[
\Pi_{w,\mathcal{K}}(\text{Ad}(w,\mathcal{K})) Y_1, \text{Ad}(w,\mathcal{K}) Y_2 = (\text{pr}_\mathfrak{g} \text{Ad}(w,\mathcal{K})) Y_1, \text{Ad}(w,\mathcal{K}) Y_2).
\]

Note that $T_{w,\mathcal{K}}^*(G/K) = T_{w,\mathcal{K}}^* M_a \oplus (T_{w,\mathcal{K}}^* M_a)_{\mathfrak{t}}$. Both $T_{w,\mathcal{K}}^* M_a$ and $(T_{w,\mathcal{K}}^* M_a)_{\mathfrak{t}}$ are stable under the action of $T$. Moreover, $T_{w,\mathcal{K}}^* M_a = \Pi^\sharp (T_{w,\mathcal{K}}^* (G/K))$ by the definition of the symplectic leaf $M_a$. Hence,
for \( \varphi \in T_{w(a)}^* M_a, \psi \in (T_{wM_a})^\perp \) and \( Z \in \mathfrak{k} \), one obtains

\[
\Pi_{w(a), K}(Z, (\varphi + \psi), (\varphi + \psi)) = (\varphi + \psi).\Pi_{w(a), K}^\perp(Z, (\varphi + \psi)) = \varphi.\Pi_{w(a), K}^\perp(Z \varphi + Z \psi, \varphi) = - (Z \varphi + Z \psi).\Pi_{w(a), K}^\perp(\varphi) = \Pi_{w(a), K}(Z \varphi, \varphi)
\]

In view of \((5)\) and \((2)\) (from Section 3) it follows that the local cone is given by

\[
\Gamma_{w(X)} = \{ Z \mapsto -\langle pr_b[Z, Ad(w(a))Y], Ad(w(a))Y \rangle : Y \in \mathfrak{t} \}. \tag{6}
\]

In order to determine the weights in \((6)\) we will construct a basis \( \{v_1, \ldots, v_r\} \) for \( \mathfrak{t} \) with two main features.

1. For each \( v_i \) we determine explicitly an element \( H_i \in \mathfrak{a}^r \) such that

\[
\langle pr_b[Z, Ad(w(a))v_i], Ad(w(a))v_i \rangle = \Im(\kappa(H_i, Z)) \quad \forall Z \in \mathfrak{k}.
\]

2. \( \langle pr_b[Z, Ad(w(a))v_i], Ad(w(a))v_j \rangle = 0 \) for all \( Z \in \mathfrak{k} \) whenever \( i \neq j \).

Once such a basis is found each \( Y \in \mathfrak{t} \) can be written as a linear combination \( Y = \sum_{i=1}^N c_i v_i \). Then, for \( Z \in \mathfrak{k} \),

\[
\langle pr_b[Z, Ad(w(a))Y], Ad(w(a))Y \rangle = \langle pr_b[Z, Ad(w(a))\sum_{i=1}^N c_i v_i], Ad(w(a))\sum_{i=1}^N c_i v_i \rangle = \sum_{i=1}^N c_i^2 \langle pr_b[Z, Ad(w(a))v_i], Ad(w(a))v_i \rangle = \sum_{i=1}^N c_i^2 \Im(\kappa(H_i, Z))
\]

In view of \((5)\) it then follows that \( \Gamma_{w(X)} \) is the cone spanned by the vectors \( H_i \).

Recall the weight space decomposition of \( \mathfrak{g} \) with respect to \( \mathfrak{a}^r \).

\[
\mathfrak{g} = \mathfrak{a}^r + i\mathfrak{a}^r + i\mathfrak{a}^r + i\mathfrak{a}^r + \sum_{\beta \in \Delta(\mathfrak{g}, \mathfrak{a}^r)} \mathfrak{g}^\beta
\]

Each \( \mathfrak{g}^\beta \) is stable under the involution \( \theta^r \), hence decomposes into \((+1)\)- and \((-1)\)-eigenspaces \( \mathfrak{g}^\beta = (\mathfrak{g}^\beta)_+ \oplus (\mathfrak{g}^\beta)_- \). We first consider certain bases for \( \mathfrak{g}^\beta = (\mathfrak{g}^\beta)_+ \) and \( \mathfrak{g}^\beta = (\mathfrak{g}^\beta)_- \). Each \( \mathfrak{g}^\beta \) is stable under the adjoint action of \( \mathfrak{a}^r \). For the corresponding weight space decomposition we write

\[
\mathfrak{g}^\beta = \sum_{\eta \in \Delta(\mathfrak{g}^\beta, \mathfrak{a}^r)} \mathfrak{g}^{\beta, \eta}
\]

Note that \( \mathfrak{g}^{\beta, \eta} \) is equal to the eigenspace \( \mathfrak{g}^\alpha \subset \mathfrak{n} \) for \( \alpha \in \Delta(\mathfrak{g}, \mathfrak{a}) \) if and only if \( \alpha_{|\mathfrak{a}^r} = \beta \) and \( \alpha_{|\mathfrak{a}^r} = \eta \). Also, if \( \mathfrak{g}^{\beta, \eta} = \mathfrak{g}^\alpha \), then \( \beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^r) \) if and only if...
α ∈ Δ⁺(g, a). The involutions τ and θ transform the eigenspaces as follows
\[ \tau(g^{β,η}) = g^{−β,η}, \quad θ(g^{β,η}) = g^{−β,−η}, \quad θτ(g^{β,η}) = g^{β,−η} \]

For each eigenspace \( g^{β,η} \) fix a vector \( X_{β,η} \) that spans \( g^{β,η} \) as a complex vector space. If \( η \neq 0 \) we define
\[ A_{β,η} = X_{β,η} + θτX_{β,η}, \quad B_{β,η} = X_{β,η} − θτX_{β,η}. \]
We obtain the following (complex) basis for the reduced root space \( g^β \)
\[ \{X_{β,0} \} \cup \{A_{β,η} : η ≠ 0 \} \cup \{B_{β,η} : η ≠ 0 \} \]
The important feature of this basis is that it consists of eigenvectors of the complex linear involution \( θτ \). Indeed, \( θτA_{β,η} = A_{β,η}, θτB_{β,η} = −B_{β,η} \) and \( X_{β,0} \) might be a \((+1)-\) or a \((-1)-\)eigenvector of \( θτ \). Therefore, a basis for \( (g^β)^+ \) is given by the \( A_{β,η} \)'s and possibly \( X_{β,0} \). A basis for \( (g^β)^− \) is given by the \( B_{β,η} \)'s and possibly \( X_{β,0} \) (if it is not contained in \( g^β = (g^β)^+. \)

The desired (real) basis for \( t \) now consists of a basis for \( Zt(a) = Zt(a−τ) = ia−τ + ia+τ \) and the following set.
\[ \bigcup_{β ∈ Δ⁺(g, a−τ)} \{X_{β,0} + θX_{β,0}\} \cup \{iX_{β,0} + θiX_{β,0}\} \]
(7)\[ \bigcup \{A_{β,η} + θA_{β,η} : η ≠ 0\} \cup \{iA_{β,η} + θiA_{β,η} : η ≠ 0\} \]
\[ \bigcup \{B_{β,η} + θB_{β,η} : η ≠ 0\} \cup \{iB_{β,η} + θiB_{β,η} : η ≠ 0\} \}
We can now calculate the weights appearing in \( Zt(a) \) for each basis element. We fix \( Z = iH ∈ t = ia−τ \). Recall that \( a = expX \), therefore \( w(a) = exp(w(X)) \). First we make two short auxiliary calculations. For a vector \( C_β ∈ g^β \) which is also a \( θτ \)-fixed point,
\[ [Z, Ad(w(a)), (C_β + θC_β)] = iβ(H)w(a)βC_β − iβ(H)w(a)−βθC_β \]
\[ = β(H)w(a)−β(iC_β + θiC_β) + β(H)(w(a)β − w(a)−β)iC_β \]
In the second line, the first summand lies in \( h \) the second in \( c−τ + n \). For \( D_β ∈ g^β \) such that \( θτD_β = −D_β, the h ⊕ (c−τ + n) \) decomposition is different:
\[ [Z, Ad(w(a)), (D_β + θD_β)] = iβ(H)w(a)βD_β − iβ(H)w(a)−βθD_β \]
\[ = β(H)w(a)−β(−iD_β + θiD_β) + β(H)(w(a)β + w(a)−β)iD_β \]
Now, for \( A_{β,η} \), which lies in \( g^β \) and satisfies \( θτA_{β,η} = A_{β,η} \), we compute
(8) \[ \langle pr_h[Z, Ad(w(a)), (A_{β,η} + θA_{β,η})], Ad(w(a)), (A_{β,η} + θA_{β,η}) \rangle \]
\[ = \langle β(H)w(a)−β(iA_{β,η} + θiA_{β,η}), w(a)βA_{β,η} + w(a)−βθA_{β,η} \rangle \]
\[ = β(H)w(a)−ββ\langle iA_{β,η}, θA_{β,η} \rangle + β(H)\langle θiA_{β,η}, A_{β,η} \rangle \]
\[ = (w(a)−2β − 1) Re(α_{β,η}, θA_{β,η}) β(H) \]
\[ = (w(a)−2β − 1) Re(α_{β,η}, θA_{β,η}) κ(H_{β, H}) \]
\[ = (w(a)−2β − 1) Re(α_{β,η}, θA_{β,η}) κ(H_{β, Z}) \]
We can replace $A_{\beta, \eta}$ with $iA_{\beta, \eta}$ in the above calculation and obtain

$$
\langle \rho_b[Z, \text{Ad}(w(a))(iA_{\beta, \eta} + \theta iA_{\eta, \beta})], \text{Ad}(w(a))(iA_{\beta, \eta} + \theta iA_{\eta, \beta}) \rangle
$$

$$
= (w(a)^{-2\beta} - 1) \Re \kappa(iA_{\beta, \eta}, \theta iA_{\eta, \beta}) \beta(H)
$$

$$
= (w(a)^{-2\beta} - 1) \Re \kappa(A_{\beta, \eta}, \theta A_{\eta, \beta}) \Im \kappa(H_{\beta}, Z)
$$

Carrying out the calculation for $B_{\beta, \eta}$ (which are $(-1)$-eigenvectors of $\theta \tau$) we obtain a result of a different nature

$$
\langle \rho_b[Z, \text{Ad}(w(a))(B_{\beta, \eta} + \theta B_{\eta, \beta})], \text{Ad}(w(a))(B_{\beta, \eta} + \theta B_{\eta, \beta}) \rangle
$$

$$
= -(w(a)^{-2\beta} + 1) \Re \kappa(B_{\beta, \eta}, \theta B_{\eta, \beta}) \Im \kappa(H_{\beta}, Z),
$$

and

$$
\langle \rho_b[Z, \text{Ad}(w(a))(iB_{\beta, \eta} + \theta iB_{\eta, \beta})], \text{Ad}(w(a))(iB_{\beta, \eta} + \theta iB_{\eta, \beta}) \rangle
$$

$$
= -(w(a)^{-2\beta} + 1) \Re \kappa(B_{\beta, \eta}, \theta B_{\eta, \beta}) \Im \kappa(H_{\beta}, Z).
$$

If $X_{\beta,0}$ is fixed by $\theta \tau$, then

$$
\langle \rho_b[Z, \text{Ad}(w(a))(X_{\beta,0} + \theta X_{\beta,0})], \text{Ad}(w(a))(X_{\beta,0} + \theta X_{\beta,0}) \rangle
$$

$$
= (w(a)^{-2\beta} - 1) \Re \kappa(X_{\beta,0}, \theta X_{\beta,0}) \Im \kappa(H_{\beta}, Z),
$$

and

$$
\langle \rho_b[Z, \text{Ad}(w(a))(iX_{\beta,0} + \theta iX_{\beta,0})], \text{Ad}(w(a))(iX_{\beta,0} + \theta iX_{\beta,0}) \rangle
$$

$$
= (w(a)^{-2\beta} - 1) \Re \kappa(X_{\beta,0}, \theta X_{\beta,0}) \Im \kappa(H_{\beta}, Z).
$$

The case that $\theta \tau X_{\beta,0} = -X_{\beta,0}$ leads to

$$
\langle \rho_b[Z, \text{Ad}(w(a))(X_{\beta,0} + \theta X_{\beta,0})], \text{Ad}(w(a))(X_{\beta,0} + \theta X_{\beta,0}) \rangle
$$

$$
= -(w(a)^{-2\beta} + 1) \Re \kappa(X_{\beta,0}, \theta X_{\beta,0}) \Im \kappa(H_{\beta}, Z),
$$

and

$$
\langle \rho_b[Z, \text{Ad}(w(a))(iX_{\beta,0} + \theta iX_{\beta,0})], \text{Ad}(w(a))(iX_{\beta,0} + \theta iX_{\beta,0}) \rangle
$$

$$
= -(w(a)^{-2\beta} + 1) \Re \kappa(X_{\beta,0}, \theta X_{\beta,0}) \Im \kappa(H_{\beta}, Z).
$$

Moreover, for $Y \in \mathfrak{g}(\alpha)$ one easily checks that

$$
\langle \rho_b[Z, \text{Ad}(w(a))Y], \text{Ad}(w(a))Y \rangle = 0.
$$

Note that the coefficient of $\Im \kappa(H_{\beta, \gamma})$ in (9) and (11) is always positive. Therefore, basis vectors of $\mathfrak{t}$ which are $(-1)$-eigenvectors of $\theta \tau$ contribute the set $\{-H_{\gamma} : \beta \in \Delta^+(\mathfrak{g}, a^{-\gamma}), (g^\beta)_{-} \neq 0 \}$ to $\Gamma_w(X)$.

On the other hand, the coefficient of $\Im \kappa(H_{\beta, \gamma})$ in (8) and (10) depends on the value of $\beta(w(X))$. If $\beta(w(X)) = 0$ this coefficient is zero. If $\beta(w(X)) > 0$ the coefficient is positive, and if $\beta(w(X)) < 0$ it is negative. Therefore, basis vectors of $\mathfrak{t}$ which are $(+1)$-eigenvectors of $\theta \tau$ contribute the set $\{-\beta(w(X))H_{\beta} : \beta \in \Delta^+(\mathfrak{g}, a^{-\gamma}), (g^\beta)^+ \neq 0 \}$ to $\Gamma_w(X)$.
The fact that $\langle pr_{i}[Z, Ad(w(a))v_i], Ad(w(a))v_j\rangle = 0$ holds for all $Z \in \mathfrak{t}$ whenever $i \neq j$ follows from general properties of the Killing form.

The conclusion is that the cone $\Gamma_{w(X)} = \Phi(V^*_{w(a), K})$ is generated by the weights

$$\{-\beta(w(X))H_\beta : \beta \in \Delta^+(g, a^{\perp}), (g^{\beta})_+ \neq 0\} \cup \{-H_\beta : \beta \in \Delta^+(g, a^{\perp}), (g^{\beta})_- \neq 0\},$$

as asserted.

\[\square\]

**Corollary 4.5.** The image of the moment map $\Phi(M_a)$ is contained in the set $w'(X) + \Gamma_+$, where $w' \in W_{K \cap H}$ is such that $\beta(w'(X)) \geq 0$ for all $\beta \in \Delta^+(g, a^{\perp})$ and $\Gamma_+$ is the proper cone $\Gamma_+ = \text{cone}(-H_\beta : \beta \in \Delta^+(g, a^{\perp}))$.

**Proof.** From Theorem 4.1 and Remark 4.2 we know that there is a neighborhood $U_{w'(X)} \subseteq a^{\perp}$ of $w'(X)$ such that $\Phi(M_a) \cap U_{w'(X)} = (w'(X) + \Gamma_{w'(X)}) \cap U_{w'(X)}$. Lemma 4.4 implies that $\Gamma_{w'(X)} \subseteq \Gamma_+$. Suppose there exists some $Z \in \Phi(M_a)$ such that $Z \not\in w'(X) + \Gamma_+$. Since $\Phi(M_a)$ is convex the line segment $w'(X)Z$ lies entirely in $\Phi(M_a)$. Fix some $Y \in w'(X)Z \cap U_{w'(X)}$ with $Y \neq w'(X)$. Then $Y \in \Phi(M_a) \cap U_{w'(X)} \subseteq w'(X) + \Gamma_{w'(X)} \subseteq w'(X) + \Gamma_+$. But this implies $Z \in w'(X) + \Gamma_+$. Therefore, $\Phi(M_a) \subseteq w'(X) + \Gamma_+$. The cone $\Gamma_+$ is proper since it is spanned by vectors $-H_\beta$ associated to positive roots $\beta$.

The special property of $\Phi(M_a)$ stated in the corollary allows us to describe $\Phi(M_a)$ entirely in terms of the local cones $\Gamma_{w(X)}$ associated to the fixed points, as the following proposition shows.

**Proposition 4.6.** Let $C$ be a closed, convex, locally polyhedral set (in some finite dimensional vector space $V$). Denote by $\Gamma_c$ the local cone at $c \in C$ (i.e. there is a neighborhood $U_c \subset V$ of $c$ such that $C \cap U_c = (c + \Gamma_c) \cap U_c$). Suppose $C \subset x + \Gamma$ for some $x \in V$ and some proper cone $\Gamma \subset V$. Then

$$C = \bigcap_{\Gamma_c \text{ proper}} (c + \Gamma_c),$$

i.e. $C$ is completely determined by the local cones that are proper.

**Proof.** For any $c \in C$ we write $d_c$ for the dimension of the maximal subspace contained in $\Gamma_c$. (In particular, $d_c = 0$ means that $\Gamma_c$ is proper.) First we will show that if $d_c > 0$, then $c \in c' + \Gamma_{c'}$ for some $c'$ with $d_{c'} < d_c$.

If $d_c > 0$, then $\Gamma_c$ contains a line, say $L$. Since $C$ lies in a proper cone, $(c+L) \cap C$ is semi-bounded. We pick an endpoint $c'$ of $(c+L) \cap C$. Since $C$ is closed, $c' \in C$, and clearly $c \in c' + \Gamma_{c'}$. Convexity of $C$ implies that if a line $L'$ is contained in $\Gamma_{c'}$ then $L' \subset \Gamma_{c'}$ for each inner point $\tilde{c}$ of $(c+L) \cap C$. In particular, $d_{c'} \leq d_c$.

On the other hand, $\Gamma_{c'}$ does not contain the line $L \subset \Gamma_{c'}$. Therefore, $d_{c'} < d_c$.

Now, the assumptions on $C$ imply

$$C = \bigcap_{c \in C} (c + \Gamma_c)$$
If we set \( n = \dim(V) \) the above arguments lead to
\[
C = \bigcap_{d_c \leq n} (c + \Gamma_c) = \bigcap_{d_c \leq n-1} (c + \Gamma_c) = \cdots = \bigcap_{d_c = 0} (c + \Gamma_c)
\]

We are now ready to give the desired description of \( \Phi(M_n) \) which is the content of van den Ban’s theorem.

**Theorem 4.7.** The set \( \Phi(M_n) = (pr_a \circ \log \circ \mu)(H a) \) is the sum of a compact convex set and a closed (proper) cone \( \Gamma \). More precisely, for \( a = \exp X \),
\[
\Phi(M_n) = \text{conv}(\mathcal{W}_{K \cap H} \cdot X) + \Gamma,
\]
with
\[
\Gamma = \text{cone}\{-H_\beta : \beta \in \Delta^+(g, a^{-\tau}), (g^3)_- \neq 0\}
\]

**Proof.** The image \( \Phi(M_n) \) is closed, convex and locally polyhedral. Moreover, by Corollary 4.4 it is contained in \( w'(X) + \Gamma_+ \) for some proper cone \( \Gamma_+ \). Proposition 4.6 implies that \( \Phi(M_n) \) is determined by the local cones that are proper. According to Remark 4.2, a local cone \( \Gamma_{\Phi(M)} \) can be proper only if \( t_m = t \), i.e. if \( m \) is a \( T \)-fixed point. The \( T \)-fixed points have been characterized in Proposition 4.3, so Proposition 4.6 yields
\[
\Phi(M_n) = \bigcap_{w \in \mathcal{W}_{K \cap H}} (w(X) + \Gamma_w(X)),
\]
with \( \Gamma_w(X) \) as in Lemma 4.3.

The sum \( \text{conv}(\mathcal{W}_{K \cap H} \cdot X) + \Gamma \) is closed, convex and locally polyhedral as well. As a sum of a compact set and the proper cone \( \Gamma \) it is contained in \( x + \Gamma \) for some \( x \in a^{-\tau} \), hence Proposition 4.6 is applicable. First we want to see at which points in \( \text{conv}(\mathcal{W}_{K \cap H} \cdot X) + \Gamma \) the local cone is proper. Let \( c \in \text{conv}(\mathcal{W}_{K \cap H} \cdot X) \) and \( \gamma \in \Gamma \). Clearly, the local cone at \( c + \gamma \) is improper unless \( \gamma = 0 \). But then \( c + \gamma = c \) is contained in a convex set with extremal points \( \{w(X) : w \in \mathcal{W}_{K \cap H}\} \).

The local cone can be proper only if \( c + \gamma \) is one of those extremal points. Proposition 4.6 now gives
\[
\text{conv}(\mathcal{W}_{K \cap H} \cdot X) + \Gamma = \bigcap_{w \in \mathcal{W}_{K \cap H}} (w(X) + \Gamma_w'(X)).
\]

Here, \( \Gamma_w'(X) \) denotes the local cone of \( \text{conv}(\mathcal{W}_{K \cap H} \cdot X) + \Gamma \) at \( w(X) \). To finish the proof it is sufficient to show that \( \Gamma_w'(X) = \Gamma_w(X) \).

Clearly, \( \Gamma_w'(X) = \Gamma_w''(X) + \Gamma \), where \( \Gamma = \text{cone}\{-H_\beta : \beta \in \Delta^+(g, a^{-\tau}), (g^3)_- \neq 0\} \) as before and \( \Gamma_w''(X) = \text{cone}\{w'(X) - w(X) : w' \in \mathcal{W}_{K \cap H}\} \). From Lemma 4.4 we know that \( \Gamma_w(X) \) contains the cone \( \Gamma \). Moreover, the set \( \Phi(M_n) \) is convex and contains all points \( w(X) \), and therefore contains \( \text{conv}(\mathcal{W}_{K \cap H} \cdot X) \).

This implies that its local cone at \( w(X) \), i.e. \( \Gamma_w(X) \), contains \( \Gamma_w''(X) \) as well. Therefore, \( \Gamma_w(X) \supseteq \Gamma_w''(X) + \Gamma = \Gamma_w'(X) \).
Each root $\beta \in \Delta(g, a^{-\tau})$ defines the isomorphism

$$s_\beta : a^{-\tau} \to a^{-\tau}, Z \mapsto Z - 2\frac{\beta Z}{\langle \beta, \beta \rangle} H_\beta.$$

In view of Remark 2.1, the Weyl group $W' = W_{K \cap H}$ consists exactly of those $s_\beta$ for which $(g^\beta)_+ \neq 0$. In particular, $s_\beta(w(X)) \in W_{K \cap H}$ for all $\beta \in \Delta^+(g, a^{-\tau})$ for which $(g^\beta)_+ \neq 0$. The identity $s_\beta(w(X)) - w(X) = -2\frac{\beta(w(X))}{\langle \beta, \beta \rangle} H_\beta$ implies

$$\text{cone}\{-\beta(w(X))H_\beta : \beta \in \Delta^+(g, a^{-\tau}), (g^\beta)_+ \neq 0\} \subseteq \Gamma''_w.$$ 

With Lemma 4.4, we obtain $\Gamma_{w(X)} \subseteq \Gamma''_{w(X)} + \Gamma = \Gamma''_{w(X)}$. □

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