Integral representations for matter fields with electromagnetic interaction in quantum Einstein gravity

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Integral representations for the electromagnetic field and the electron one are presented in the manifestly covariant operator formalism of quantum gravity. These representations contain not only the gravitational interaction but also the electromagnetic one, and satisfy the matter field equations. Some properties of them are investigated.

Subject Index B05, B39, E05

1. Introduction

In 1992, Abe and Nakanishi [1] proposed a method for solving the manifestly covariant operator formalism of quantum electrodynamics. It is based on the method for quantum field theories in the Heisenberg picture that has been developed by them since 1991 [2,3]. In Ref. [1], the electromagnetic field $A_\mu(x)$ and the electron one $\psi(x)$ are expanded in powers of $e^2$. Here $e$ denotes the electromagnetic coupling constant, and the conventional expression $eA_\mu$ is rewritten by $A_\mu$ in the Lagrangian density of gauge theory.

In the $e^2$-expansion, the $N$th-order field equations, and the $N$th-order equal-time commutation and anti-commutation relations are obtained recurrently. They yield inhomogeneous differential equations of the 4D commutators and anti-commutators between the fundamental fields in the $N$th order. These equations are set up as q-number Cauchy problems; their initial conditions are the above equal-time (anti-)commutation relations.

Abe and Nakanishi [1] noted that the inhomogeneous differential equations can be solved in terms of integral representations. For $u(x, y)$ satisfying

$$\left\{(\partial^2_0)^2 - (\partial^2_1)^2 - (\partial^2_2)^2 - (\partial^2_3)^2\right\}u(x, y) = f(x, y), \tag{1.1}$$

one has

$$u(x, y) = -\int d^4z \left[\theta(x^0 - z^0) - \theta(y^0 - z^0)\right]D(x - z)f(z, y)$$

$$+ \int d^3z \left[D(x - z)\frac{\partial}{\partial z^0}\cdot u(z, y) - D(x - z)\partial^0 u(z, y)\right]|_{z^0 = y^0}, \tag{1.2}$$

with the use of the Pauli–Jordan D function $D(x - z)$ [1,4] and of

$$\theta(x) \equiv \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{otherwise}. \end{cases} \tag{1.3}$$
Equation (1.2) is the basis of integral representations for the 4D commutation relations containing the electromagnetic field in the Minkowski spacetime.

Similarly, for \( v(x, y) \) satisfying
\[
(i\gamma^\mu [\partial_\mu^x - i A^{(0)}_\mu(x)] - m)v(x, y) = g(x, y),
\]
one has
\[
v(x, y) = \int d^4z[\theta(x^0 - z^0) - \theta(y^0 - z^0)]S^{(0)}(x, z)g(z, y)
+ \int d^3z\gamma^0v(z, y)|_{\mu = y^0},
\]
with the zeroth order of the electromagnetic field \( A^{(0)}_\mu(x) \) and the electron mass \( m \). In (1.5), \( S^{(0)}(x, y) \) is the zeroth order of the S function [1] defined by the following Cauchy problem:
\[
(i\gamma^\mu [\partial_\mu^x - i A^{(0)}_\mu(x)] - m)S^{(0)}(x, y) = 0, \quad S^{(0)}(x, y)|_0 = -i\gamma^0\delta^3(x - y),
\]
where \( |_0 \) denotes to set \( x^0 = y^0 \).

Equations (1.2) and (1.5) imply that we can also solve the matter field equations in quantum gravi-electrodynamics in terms of integral representations. These operator solutions are a starting point to treat the matter fields both with the gravitational interaction and with the electromagnetic one.

The purpose of the present paper is to give integral representations for the electromagnetic field and the electron one in quantum gravi-electrodynamics, and to investigate their properties. For this purpose, we use the quantum-gravity Pauli–Jordan D function [5,6] and a tensorial q-number commutator function [7] for the electromagnetic field, and introduce a quantum-gravity version of the S function for the electron one.

This paper is organized as follows. In the next section, we briefly review the manifestly covariant operator formalism for the electromagnetic field and the electron one in quantum gravi-electrodynamics. In Sect. 3, we introduce the quantum-gravity version of the S function and investigate its transformation properties. In Sect. 4, we propose integral representations for the electromagnetic field and the electron one, using the quantum-gravity Pauli–Jordan D function, the tensorial q-number commutator function, and the quantum-gravity S function; we show the transformation properties of these representations. The last section is devoted to discussion.

2. Covariant operator formalism of quantum gravi-electrodynamics

We consider the quantum coupled Einstein–Maxwell–Dirac system, whose total Lagrangian density is constructed by combining the following two Lagrangian densities: one [8] contains the gravitational field \( g_{\mu\nu}(x) \) and the electromagnetic field \( A_\mu(x) \), and the other [9,10] the vierbein field \( h_\mu^a(x) (a = 0, 1, 2, 3) \) and the electron field \( \psi(x) \). We thus obtain the matter Lagrangian density as follows:
\[
\mathcal{L}_M = -\frac{h}{4}g^{\mu\nu}g^{\lambda\nu}F_{\kappa\lambda}F_{\mu\nu} - h g^{\lambda \mu}A_\mu \partial_\lambda B + \alpha \frac{h}{2}B^2 - \frac{\hbar}{2}g^{\mu\nu}\partial_\mu \bar{C} \cdot \partial_\nu C
+ i \left[ \hbar \bar{\psi} \gamma^\mu (\partial_\mu + \omega_\mu) \psi - \bar{\psi} (\bar{\gamma}_\mu - \omega_\mu) \gamma^\mu \psi \right] + e\hbar \bar{\psi} \gamma^\mu A_\mu \psi - mh \bar{\psi} \psi.
\]
Here, \( h \equiv \det h_\mu^a \), \( F_{\kappa\lambda} \equiv \partial_\lambda A_\kappa - \partial_\kappa A_\lambda \), \( B \) is the electromagnetic auxiliary scalar field, \( C \) and \( \bar{C} \) are the electromagnetic Faddeev–Popov ghost scalar fields, \( \alpha \) denotes a gauge parameter; we use Greek
small letters for $GL(4)$ indexes and italic small letters for internal Lorentz ones. In the present paper, we prefer the conventional notation, $eA_\mu$, since we do not treat the $e^2$-expansion. As in Ref. [4], \( \bar{\psi} \equiv \psi^\dagger \gamma_0 \) and \( \gamma^\mu \equiv h^{\mu a} \gamma_a \) with the flat-space gamma matrices \( \gamma_a \) \( (a = 0, 1, 2, 3) \),

\[
\left\{ \gamma_a, \gamma_b \right\} = 2\eta_{ab}, \quad \eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1, \quad \eta_{ab} = 0 \quad \text{for} \quad a \neq b. \quad (2.2)
\]

The symbol \( \omega^\mu \) in (2.1) is defined by

\[
\omega^\mu \equiv \frac{1}{2} \sigma_{ab} \omega^{ab}_\mu, \quad (2.3)
\]

with

\[
\sigma_{ab} \equiv \frac{1}{4} (\gamma_a \gamma_b - \gamma_b \gamma_a), \quad (2.4)
\]

\[
\omega^{ab}_\mu \equiv \frac{1}{2} [h^{\rho a}(\partial^\mu h^{\rho b} - \partial^\rho h^{\mu b}) - h^{\rho b}(\partial^\mu h^{\rho a} - \partial^\rho h^{\mu a}) + h^\mu c h^{\rho a} h^{\sigma b}(\partial^\rho h^{\mu c} - \partial^\rho h^{\sigma c})]. \quad (2.5)
\]

Here, \( \omega^{ab}_\mu \) denotes the spin connection.

In order to take \( A_\mu, C, \bar{C}, \) and \( \psi \) as the canonical variables, we replace \( L_M \) by

\[
\tilde{L}_M = -\frac{1}{4} g^{\kappa \mu} g^{\lambda \nu} F_{\kappa \lambda} F_{\mu \nu} + \partial_\kappa (h g^{\lambda \mu} A_\mu) \cdot B + \alpha \frac{1}{2} B^2 - i h g^{\mu \nu} \partial_\mu \bar{\psi} \cdot \partial_\nu C + i h \bar{\psi} \gamma^\mu (\partial^\mu + \omega^\mu) \psi - m h \bar{\psi} \psi. \quad (2.6)
\]

The discarded term is a total divergence,

\[
L_M - \tilde{L}_M = \partial_\kappa \left( -h g^{\lambda \mu} A_\mu B - i \frac{1}{2} \bar{\psi} \gamma^\lambda \psi \right), \quad (2.7)
\]

with the use of

\[
\partial_\kappa (h \gamma^\lambda) = h (\gamma^\lambda \omega^\kappa - \omega^\kappa \gamma^\lambda). \quad (2.8)
\]

The following field equations are derived from \( \tilde{L}_M \):

\[
\partial_\kappa \left[ h (g^{\kappa \mu} g^{\lambda \nu} - g^{\kappa \nu} g^{\lambda \mu}) \partial_\mu A_\nu \right] - h g^{\lambda \nu} \partial_\nu B = -eh \bar{\psi} \gamma^\lambda \psi, \quad (2.9)
\]

\[
\partial_\kappa (h g^{\lambda \mu} A_\mu) = \alpha h B = 0, \quad (2.10)
\]

\[
\partial_\mu (h g^{\mu \nu} \partial_\nu C) = 0, \quad (2.11)
\]

\[
\partial_\mu (h g^{\mu \nu} \partial_\nu \bar{C}) = 0, \quad (2.12)
\]

\[
i h \gamma^\mu (\partial_\mu + \omega_\mu) \psi - m h \bar{\psi} \psi = -eh \gamma^\mu A_\mu \psi. \quad (2.13)
\]

By virtue of (2.13), the electric current, \( -eh \bar{\psi} \gamma^\lambda \psi \), is conserved:

\[
\partial_\kappa (-eh \bar{\psi} \gamma^\lambda \psi) = 0. \quad (2.14)
\]

Therefore, the total divergence of (2.9) yields the field equation for \( B \),

\[
\partial_\mu (h g^{\mu \nu} \partial_\nu B) = 0. \quad (2.15)
\]
The canonical conjugates of $A_\lambda, C, \bar{C}$, and $\psi$ are defined by

\[ \pi_A^\lambda = \frac{\partial \tilde{L}}{\partial \dot{A}_\lambda} = -h (g^{0\mu}g^{\lambda\nu} - g^{0\nu}g^{\lambda\mu}) \partial_\mu A_\nu + hg^{0i}B, \]  
(2.16)

\[ \pi_C = \frac{\partial \tilde{L}}{\partial C} = ihg^{\mu0} \partial_\mu \bar{C}, \]  
(2.17)

\[ \pi_{\bar{C}} = \frac{\partial \tilde{L}}{\partial \bar{C}} = -ihg^{0\nu} \partial_\nu C, \]  
(2.18)

\[ \pi_\psi = \frac{\partial \tilde{L}}{\partial \dot{\psi}} = -i\bar{\psi} \gamma^0, \]  
(2.19)

respectively. Here, the functional derivative with respect to $C, \bar{C},$ or $\psi$ is made from the left of each operand. The equal-time canonical commutation and anti-commutation relations are set as follows:

\[ [\pi_A^\lambda, A'_\mu] = -i \delta^\lambda_\mu \delta^3, \]  
(2.20)

\[ \{\pi_C, C'\} = -i \delta^3, \]  
(2.21)

\[ \{\pi_{\bar{C}}, \bar{C}'\} = -i \delta^3, \]  
(2.22)

\[ \{\pi_\psi, \psi'\} = -i \delta^3, \]  
(2.23)

where $\delta^3$ denotes the spatial delta function $\Pi^3_{k=1} \delta(x^k - y^k).$ In the above and subsequently, a prime attached to a spacetime function means that its argument is not $x^\lambda$ but $y^\lambda$ where it is understood that $x^0 = y^0.$

Using the field equations, the canonical conjugates, and the equal-time canonical (anti-) commutation relations, we obtain various (anti-)commutation relations, e.g.,

\[ [A_\mu, B'] = i\delta^0_{\mu} \frac{\delta^3}{hg^{00}}, \]  
(2.24)

\[ [\dot{A}_\mu, A'_\nu] = i \left[ g_{\mu\nu} - (1 - \alpha) \frac{\delta^0_\mu \delta^0_\nu}{g^{00}} \right] \delta^3 \frac{\delta^0}{hg^{00}}, \]  
(2.25)

\[ [\dot{B}, A'_\nu] = i \partial_\nu \left( \frac{1}{hg^{00}} \right) \cdot \delta^3 + i \left( 2 \frac{\delta^0_k}{g^{00}} \delta^0_\nu - \delta^0_\nu \right) \partial^1_k \left( \frac{\delta^3}{hg^{00}} \right), \]  
(2.26)

\[ [B, B'] = 0, \]  
(2.27)

\[ [\dot{A}_\mu, \psi'] = 0, \quad [\dot{\psi}] = 0, \]  
(2.28)

\[ [\dot{B}, \psi'] = -e \frac{\delta^3}{hg^{00}} \psi, \quad [\dot{\psi}] = e \frac{\delta^3}{hg^{00}} \bar{\psi}, \]  
(2.29)

\[ \{\dot{C}, \psi'\} = 0, \quad \{\dot{C}, \bar{\psi}'\} = 0, \quad \{\dot{\bar{C}}, \psi'\} = 0, \quad \{\dot{\bar{C}}, \bar{\psi}'\} = 0, \]  
(2.30)

\[ \{\psi, \bar{\psi}'\} = \gamma^0 \frac{\delta^3}{hg^{00}}. \]  
(2.31)

Except for (2.28–2.30), these (anti-)commutation relations are the same as those in Refs. [4,8,10]. Note that the de Donder condition [4],

\[ \partial_\mu (hg^{\mu\nu}) = 0, \]  
(2.32)

is used in (2.25) and (2.26).
The action integral of $\tilde{L}_M$ in (2.6) is invariant under the gravitational BRST transformation. The corresponding charge \([4]\) is defined by

$$Q_G \equiv \int d^3x h^0 \nu (b_\rho \partial_\nu c^\rho - \partial_\nu b_\rho \cdot c^\rho),$$

(2.33)

where $b_\rho$ is the gravitational B-field and $c^\rho$ is the gravitational Faddeev–Popov ghost field. The (anti-)commutation relations between $Q_G$ and the matter fields are

$$[i Q_G, A_\mu] = -\kappa (\partial_\mu c^\rho \cdot A_\rho + c^\rho \partial_\rho A_\mu),$$

(2.34)

$$[i Q_G, B] = -\kappa c^\rho \partial_\rho B,$$

(2.35)

$$[i Q_G, C] = -\kappa c^\rho \partial_\rho C,$$

(2.36)

$$[i Q_G, \tilde{C}] = -\kappa c^\rho \partial_\rho \tilde{C},$$

(2.37)

$$[i Q_G, \psi] = -\kappa c^\rho \partial_\rho \psi.$$

(2.38)

with Einstein’s gravitational constant $\kappa$.

Next, the Lagrangian density $\tilde{L}_M$ is invariant under the internal Lorentz BRST transformation. The corresponding charge \([4,9]\) is defined by

$$Q_L \equiv \int d^3x h^0 \nu [s_{ab}(D_\nu t)^{ab} - \partial_\nu s_{ab} \cdot t^{ab} + i \partial_\nu \tilde{t}_{ab} \cdot t^{bc} t^a_c],$$

(2.39)

with

$$(D_\mu t)^{ab} \equiv \partial_\mu t^{ab} + \omega^{ac}_\mu t^{bc} - \omega^{bc}_\mu t^{ca}.$$  

(2.40)

Here, $s_{ab}$ is the internal Lorentz B-field, and $t^{ab}$ and $\tilde{t}_{ab}$ are the internal Lorentz Faddeev–Popov ghost fields; these fields are anti-symmetric with respect to the indexes $a$ and $b$. The (anti-)commutation relations between $Q_L$ and the matter fields are

$$[i Q_L, A_\mu] = 0,$$

(2.41)

$$[i Q_L, B] = 0,$$

(2.42)

$$[i Q_L, C] = 0,$$

(2.43)

$$[i Q_L, \tilde{C}] = 0,$$

(2.44)

$$[i Q_L, \psi] = -\frac{t^{ab}}{2} \sigma_{ab} \psi.$$  

(2.45)

Moreover, the Lagrangian density $\tilde{L}_M$ is invariant under the electromagnetic BRST transformation except for the total divergence in (2.7). The corresponding charge \([8]\) is defined by

$$Q_B \equiv \int d^3x h^0 \nu (B \partial_\nu C - \partial_\nu B \cdot C),$$

(2.46)

The (anti-)commutation relations between $Q_B$ and the matter fields are

$$[i Q_B, A_\mu] = \partial_\mu C,$$

(2.47)

$$[i Q_B, B] = 0,$$

(2.48)

$$[i Q_B, C] = 0,$$

(2.49)

$$[i Q_B, \tilde{C}] = i B,$$

(2.50)

$$[i Q_B, \psi] = i e C \psi.$$  

(2.51)
The charge $Q_B$ commutes with $h_\mu^a$, $b_\rho$, and $s_{ab}$, and anti-commutes with $c^0$, $\bar{c}_\rho$, $t_{ab}^a$, and $\bar{t}_{ab}$. Of course, $\bar{c}_\rho$ is the gravitational Faddeev–Popov anti-ghost field [4].

The above BRST charges give us the subsidiary conditions,

$$Q_G|\text{phys}\rangle = 0, \quad Q_L|\text{phys}\rangle = 0, \quad Q_B|\text{phys}\rangle = 0,$$

(2.52)

to define the physical subspace of the indefinite-metric Hilbert space.

3. Quantum-gravity S function for spinor fields

In this section, we have a quantum-gravity version of the S function for Dirac fields.

3.1. Introducing $S(x, y; m)$

The 4D anti-commutation relation between a free Dirac field $\psi(x)$ and its Dirac conjugate $\bar{\psi}(y)$ is given by

$$\{\psi(x), \bar{\psi}(y)\} = i \hat{S} (x - y; m),$$

(3.1)
in the Minkowski spacetime [4]. Here, $\hat{S} (x - y; m)$ is the ordinary S function for Dirac fields and satisfies

$$\left[ i (\gamma_0 \partial_0^a - \gamma_1 \partial_1^a - \gamma_2 \partial_2^a - \gamma_3 \partial_3^a) - m \right] \hat{S} (x - y; m) = 0,$$

(3.2)

$$\hat{S} (x - y; m)|_0 = -i \gamma_0 \delta^3,$$

(3.3)

with the use of the flat-space gamma matrices in (2.2). The Hermitian conjugate of (3.1) yields

$$\left[ \hat{S} (x - y; m) \right]^\dagger = -\gamma_0 \hat{S} (y - x; m) \gamma_0.$$

(3.4)

Comparing Eqs. (1.6) and (1.7) with Eqs. (3.2) and (3.3), and taking account of (2.31) based on the vierbein formalism, we define a quantum-gravity S function by the following Cauchy problem:

$$i \gamma^\mu(x) \left[ \partial_\mu^a + \omega_\mu(x) \right] S(x, y; m) - m S(x, y; m) = 0,$$

(3.5)

$$S(x, y; m)|_0 = -i \frac{\gamma_0^0}{h g_{00}} \delta^3.$$

(3.6)

Here, the role of $\omega_\mu$ in (3.5) corresponds to that of $A^{(0)}_{\mu}$ in (1.6).

The quantum-gravity S function $S(x, y; m)$ is a bilocal operator and is not a function of $x - y$ alone, since it is not translationally invariant. In general, $S(x, y; m)$ does not commute with $h_\mu^a(x)$.

We can regard the quantum-gravity version of (3.4) as

$$[S(x, y; m)]^\dagger = -\gamma_0 S(y, x; m) \gamma_0.$$

(3.7)

Therefore, we obtain

$$S(x, y; m) \left[ \partial_\mu^a - \omega_\mu(y) \right] \gamma^{\mu}(y) i + S(x, y; m) m = 0$$

(3.8)

from (3.5).
If \( e = 0 \), then we can solve (2.13) in terms of an integral representation,

\[
\psi(x) = \int d^3y J^0(x, y; m),
\]

with

\[
J^\lambda(x, y; m) \equiv i S(x, y; m)h(y)y^\lambda(y)\psi(y).
\]

The bilocal “current” \( J^\lambda(x, y; m) \) is conserved with respect to \( y \) by virtue of (2.8) and (3.8). Therefore, the right-hand side of (3.9) is independent of \( y^0 \) and reduces to \( \psi(x) \) by setting \( y^0 = x^0 \) via (3.6).

Inserting (3.9) into the right-hand side of (3.10), we have

\[
\psi(x) = \int d^3z \int d^3y i S(x, y; m)h(y)y^0(y)i S(y, z; m)h(z)y^0(z)\psi(z).
\]

Comparing (3.9) with (3.11), we find an integral representation,

\[
S(x, z; m) = \int d^3y K^0(x, y, z; m),
\]

with

\[
K^\lambda(x, y, z; m) \equiv i S(x, y; m)h(y)y^\lambda(y)S(y, z; m).
\]

The nonlocal “current” \( K^\lambda(x, y, z; m) \) is conserved with respect to \( y \) by virtue of (2.8), (3.5), and (3.8). Thus, the right-hand side of (3.12) is independent of \( y^0 \) and reduces to \( S(x, z; m) \) by setting \( y^0 = x^0 \) via (3.6).

Using (2.31) and (3.9), we obtain the 4D anti-commutation relation between \( \psi(x) \) and \( \bar{\psi}(z) \) as follows:

\[
\{ \psi(x), \bar{\psi}(z) \} = i S(x, z; m) + R(x, z; \bar{\psi}),
\]

where \( R(x, z; \bar{\psi}) \) contains a commutator between \( S(x, y; m) \) and \( \bar{\psi}(z) \).

### 3.2. Transformation properties

In the vierbein formalism of the quantum gravi-electrodynamics, there are the affine, the gravitational BRST, the internal Lorentz BRST, and the electromagnetic BRST symmetries. We investigate the transformation properties of the quantum-gravity S function \( S(x, y; m) \) with respect to these four symmetries.

Let \( \hat{P}_\lambda \) and \( \hat{M}_\lambda^\kappa \) be the translation generator and the \( GL(4) \) one [4], respectively. In what follows, we show that the affine transformation property of \( S(x, y; m) \) is given by

\[
[i \hat{P}_\lambda, S(x, y; m)] = (\partial_\lambda^x + \partial_\lambda^y)S(x, y; m),
\]

\[
[i \hat{M}_\lambda^\kappa, S(x, y; m)] = (x^\kappa \partial_\lambda^x + y^\kappa \partial_\lambda^y)S(x, y; m).
\]

Here, it is sufficient to prove only (3.16), since \( \hat{P}_\lambda \) can formally be regarded as \( \hat{M}_\lambda^5 \) with \( x^5 = y^5 = 1 \).
We define the difference between both sides of (3.16) as follows:

\[ A^\kappa \lambda (x, y; m) \equiv [i \hat{M}^\kappa \lambda, S(x, y; m)] - (x^\kappa \partial^\lambda_\kappa + y^\kappa \partial^\lambda_\kappa)S(x, y; m). \]  

(3.17)

We then set up a Cauchy problem for \( A^\kappa \lambda (x, y; m) \): we apply \( \{ i \gamma^\mu(x) [\partial^\mu_\kappa + \omega_\mu(x)] - m \} \) and the condition \( x^0 = y^0 \) to (3.17). Using (2.8), (3.5), (3.6), and (3.8), we obtain

\[
\begin{align*}
\{ i \gamma^\mu(x) [\partial^\mu_\kappa + \omega_\mu(x)] - m \} A^\kappa \lambda (x, y; m) \\
= -i [i \hat{M}^\kappa \lambda, \gamma^\mu(x)] [\partial^\mu_\kappa + \omega_\mu(x)]S(x, y; m) - i \gamma^\mu(x) [i \hat{M}^\kappa \lambda, \omega_\mu(x)]S(x, y; m) \\
+ i x^\kappa [\partial^\mu_\kappa \gamma^\mu(x) \cdot [\partial^\mu_\kappa + \omega_\mu(x)]S(x, y; m) + \gamma^\mu(x) \partial^\mu_\kappa \omega_\mu(x) \cdot S(x, y; m)] \\
- i \gamma^\kappa(x) \partial^\kappa_\kappa S(x, y; m),
\end{align*}
\]

(3.18)

(3.18)

The right-hand sides of (3.18) and (3.19) vanish by virtue of the following commutators:

\[
\begin{align*}
[i \hat{M}^\kappa \lambda, h] &= x^\kappa \partial_\kappa h + \delta^\kappa_\lambda h, \\
[i \hat{M}^\kappa \lambda, g^{\rho\sigma}] &= x^\kappa \partial_\kappa g^{\rho\sigma} - \delta^\kappa_\lambda g^{\rho\sigma} - \delta^\rho_\kappa g^{\kappa\sigma}, \\
[i \hat{M}^\kappa \lambda, \gamma^\mu] &= x^\kappa \partial_\kappa \gamma^\mu - \delta^\kappa_\lambda \gamma^\mu, \\
[i \hat{M}^\kappa \lambda, \omega_\mu] &= x^\kappa \partial_\kappa \omega_\mu + \delta^\kappa_\lambda \omega_\mu;
\end{align*}
\]

(3.20)

(3.21)

(3.22)

(3.23)

these are derived from the affine transformation of \( h_\mu^\alpha \) [4],

\[ [i \hat{M}^\kappa \lambda, h_\mu^\alpha] = x^\kappa \partial_\kappa h_\mu^\alpha + \delta^\kappa_\lambda h_\mu^\alpha. \]  

(3.24)

Therefore, we find

\[ A^\kappa \lambda (x, y; m) = 0. \]  

(3.25)

Hence the set of (3.15) and (3.16) is proved. We then obtain the Hermitian conjugates of (3.15) and (3.16),

\[ [i \hat{M}^\kappa \lambda, S(x, y; m)]^\dagger = - (x^\kappa \partial^\lambda_\kappa + y^\kappa \partial^\lambda_\kappa) \left[ \tilde{\gamma}^0_0 S(y, x; m) \tilde{\gamma}^0_0 \right], \]  

(3.26)

via (3.7).

We next show that the gravitational BRST transformation of \( S(x, y; m) \) is given by

\[ [i Q_G, S(x, y; m)] = -\kappa \left[ c^0(x) \partial^0_0 S(x, y; m) + S(x, y; m) \partial^0_0 \cdot c^0(y) \right]. \]  

(3.27)

We define the difference between both sides of (3.27) as follows:

\[ \mathcal{G}(x, y; m) \equiv [i Q_G, S(x, y; m)] + \kappa \left[ c^0(x) \partial^0_0 S(x, y; m) + S(x, y; m) \partial^0_0 \cdot c^0(y) \right]. \]  

(3.28)

Applying \( \{ i \gamma^\mu(x) [\partial^\mu_\kappa + \omega_\mu(x)] - m \} \) and the condition \( x^0 = y^0 \) to (3.28), and using (2.8), (3.5), (3.6), and (3.8), we obtain

\[
\begin{align*}
\{ i \gamma^\mu(x) [\partial^\mu_\kappa + \omega_\mu(x)] - m \} \mathcal{G}(x, y; m) \\
&= -i [i Q_G, \gamma^\mu(x)] [\partial^\mu_\kappa + \omega_\mu(x)]S(x, y; m) - i \gamma^\mu(x) [i Q_G, \omega_\mu(x)]S(x, y; m) \\
&- i \kappa [c^0(x) \partial^0_0 \gamma^\mu(x) \cdot [\partial^\mu_\kappa + \omega_\mu(x)]S(x, y; m) + c^0(x) \gamma^\mu(x) \partial^\mu_\kappa \omega_\mu(x) \cdot S(x, y; m) \\
&- \partial^\mu_\kappa c^0(x) \cdot y^\rho(x) \partial^\rho_\mu S(x, y; m)],
\end{align*}
\]

(3.29)
\( \mathcal{G}(x, y; m)_0 = \left[ i Q_G, -i \frac{\gamma_0}{h g^{00}} \delta^3 \right] - \frac{i \kappa}{h g^{00}} (c^\rho \partial_\rho \gamma_0 - \partial_\rho c^0 \cdot \gamma^\rho) \delta^3 \\
+ \frac{i \kappa \gamma_0}{(h g^{00})^2} (\partial_\rho (c^\rho h g^{00}) - 2 \partial_\rho c^0 \cdot h g^{00}) \delta^3. \) (3.30)

The right-hand sides of (3.29) and (3.30) vanish by virtue of the following commutators:

\[
[ i Q_G, h ] = -\kappa \partial_\mu (hc^\mu), \\
[ i Q_G, g^{\lambda\mu} ] = \kappa \left( \partial_\nu c^\mu \cdot g^{\lambda\nu} + \partial_\nu c^\lambda \cdot g^{\mu\nu} - c^\nu \partial_\nu g^{\lambda\mu} \right), \\
[ i Q_G, \gamma^\mu ] = \kappa \left( \partial_\nu c^\mu \cdot \gamma^\nu - c^\nu \partial_\nu \gamma^\mu \right), \\
[ i Q_G, \omega_\mu ] = -\kappa \left( \partial_\mu c_\nu \cdot \omega_\nu + c^\nu \partial_\nu \omega_\mu \right);
\]

these are derived from the gravitational BRST transformation of \( h_{\mu}^a \) [4,10],

\[
[ i Q_G, h_{\mu}^a ] = -\kappa \left( \partial_\mu c^\rho \cdot h_{\rho}^a + c^\rho \partial_\rho h_{\mu}^a \right).
\] (3.35)

Therefore, we find

\[
\mathcal{G}(x, y; m) = 0.
\] (3.36)

Hence (3.27) is proved. We then obtain the Hermitian conjugate of (3.27),

\[
[ i Q_G, \mathcal{S}(x, y; m)]^\dagger = \kappa \gamma_0 \left[ c^\nu (y) \partial_\nu \mathcal{S}(y, x; m) + \mathcal{S}(y, x; m) \delta^\nu_\rho \cdot c^\rho (x) \right] \gamma_0,
\] (3.37)

via (3.7).

Moreover, we show that the internal Lorentz BRST transformation of \( \mathcal{S}(x, y; m) \) is given by

\[
[ i \mathcal{Q}_L, \mathcal{S}(x, y; m)] = -\frac{1}{2} \left[ t^{ab}(x) \delta_{ab} \mathcal{S}(x, y; m) - \mathcal{S}(y, x; m) \delta_{ab} t^{ab}(y) \right].
\] (3.38)

We define the difference between both sides of (3.38) as follows:

\[
\mathcal{V}(x, y; m) \equiv [ i \mathcal{Q}_L, \mathcal{S}(x, y; m) ] + \frac{1}{2} \left[ t^{ab}(x) \delta_{ab} \mathcal{S}(x, y; m) - \mathcal{S}(y, x; m) \delta_{ab} t^{ab}(y) \right].
\] (3.39)

Applying \( \{ i \gamma^\mu(x)[\partial_\mu^a + \omega_\mu(x)] - m \} \) and the condition \( x^0 = y^0 \) to (3.39), and using (2.8), (2.40), (3.5), and (3.6), we obtain

\[
\{ i \gamma^\mu(x)[\partial_\mu^a + \omega_\mu(x)] - m \} \mathcal{V}(x, y; m) \\
= -i [ i \mathcal{Q}_L, \gamma^\mu(x)] [\partial_\mu^a + \omega_\mu(x)] \mathcal{S}(x, y; m) - i \gamma^\mu(x) [i \mathcal{Q}_L, \omega_\mu(x)] \mathcal{S}(x, y; m) \\
+ i t^{ab} h_{\mu}^a \gamma_b \left[ \partial_\mu^a + \omega_\mu(x) \right] \mathcal{S}(x, y; m) + \frac{i}{2} \gamma^\mu(D_{\mu} t)^{ab} \delta_{ab} \mathcal{S}(x, y; m),
\] (3.40)

\[
\mathcal{V}(x, y; m)_0 = \left[ i \mathcal{Q}_L, -i \frac{\gamma_0}{h g^{00}} \delta^3 \right] + i \frac{t^{ab}}{h g^{00}} h_{\mu}^a \gamma_b \delta^3.
\] (3.41)

The right-hand sides of (3.40) and (3.41) vanish by virtue of the following commutators:

\[
[ i \mathcal{Q}_L, h ] = 0,
\] (3.42)

\[
[ i \mathcal{Q}_L, \gamma^\mu ] = t^{ab} h_{\mu}^a \gamma_b,
\] (3.43)

\[
[ i \mathcal{Q}_L, \omega_{\mu}^{ab} ] = (D_{\mu} t)^{ab},
\] (3.44)

these are derived from the internal Lorentz BRST transformation of \( h_{\mu}^a \) [4],

\[
[ i \mathcal{Q}_L, h_{\mu}^a ] = -t^{a}_{b} h_{\mu}^b.
\] (3.45)
Therefore, we find
\[ \mathcal{V}(x, y; m) = 0. \] (3.46)

Hence (3.38) is proved. We then obtain the Hermitian conjugate of (3.38),
\[ [i Q_L, \mathcal{S}(x, y; m)] = \frac{1}{2} \gamma_0 \left[ \sigma^{ab}(y) \mathcal{S}(x, y; m) - \mathcal{S}(y, x; m) \sigma^{ab}(x) \right] \gamma_0, \] (3.47)
via (3.7).

In addition, let us denote the electromagnetic BRST transformation of \( \mathcal{S}(x, y; m) \) by \( \mathcal{B}(x, y; m) \):
\[ [i Q_B, \mathcal{S}(x, y; m)] = \mathcal{B}(x, y; m). \] (3.48)

The bilocal operator \( \mathcal{B}(x, y; m) \) should satisfy the electromagnetic BRST transformations of (3.5) and (3.6):
\[ \{ i \gamma^\mu(x) \left[ \partial_\mu + \omega_\mu(x) \right] - m \} \mathcal{B}(x, y; m) = 0, \] (3.49)
\[ \mathcal{B}(x, y; m)|_0 = 0. \] (3.50)

We thus find
\[ \mathcal{B}(x, y; m) = 0. \] (3.51)

4. Integral representations

If \( e = 0 \) in (2.9), then we have the integral representation for \( A_\mu \) [7],
\[ A_\mu(x) = \int d^3 y \mathcal{J}_\mu^0(x, y), \] (4.1)
with
\[ \mathcal{J}_\mu^0(x, y) \equiv \mathcal{D}_{\mu\nu}(x, y) \hat{\partial}_\nu \cdot h(y) \left[ g^{\lambda\rho}(y) g^{\nu\sigma}(y) - g^{\lambda\nu}(y) g^{\rho\sigma}(y) \right] A_\sigma(y) \]
\[ - \mathcal{D}_{\mu\nu}(x, y) h(y) \left[ g^{\lambda\rho}(y) g^{\nu\sigma}(y) - g^{\lambda\nu}(y) g^{\rho\sigma}(y) \right] \hat{\partial}_\lambda A_\sigma(y) \]
\[ + \mathcal{D}_{\mu\nu}(x, y) h(y) g^{\lambda\nu} B(y) - \partial_\lambda \mathcal{D}(x, y) \cdot h(y) g^{\lambda\sigma}(y) A_\sigma(y). \] (4.2)

Here, \( \mathcal{D}_{\mu\nu}(x, y) \) is a tensorial q-number commutator function, and \( \mathcal{D}(x, y) \) is the quantum-gravity Pauli–Jordan D function [5–7]. These functions are defined by the following Cauchy problems:
\[ \hat{\partial}_k^x \left[ h \left( g^{x_k} g^{\sigma\mu} - g^{x_k} g^{\sigma\lambda} \right) \hat{\partial}_\nu \mathcal{D}_{\mu\nu}(x, y) \right] - h g^{x_\tau} \cdot \hat{\partial}_\nu \mathcal{D}(x, y) \hat{\partial}_\nu = 0, \] (4.3)
\[ \hat{\partial}_k^x \left[ h g^{x_\mu} \mathcal{D}_{\mu\nu}(x, y) \right] + \alpha h \cdot \mathcal{D}(x, y) \hat{\partial}_\nu = 0, \] (4.4)
\[ \mathcal{D}_{\mu\nu}(x, y)|_0 = 0, \] (4.5)
\[ \hat{\partial}_0^x \mathcal{D}_{\mu\nu}(x, y)|_0 = - \left[ g_{\mu\nu} - (1 - \alpha) \frac{\delta_0^\mu \delta_0^\nu}{g^{00}} \right] \frac{\delta^3}{h g^{00}}, \] (4.6)

and
\[ \hat{\partial}_k^x \left[ h g^{x_\mu} \hat{\partial}_k^x \mathcal{D}(x, y) \right] = 0, \] (4.7)
\[ \mathcal{D}(x, y)|_0 = 0, \] (4.8)
\[ \hat{\partial}_0^x \mathcal{D}(x, y)|_0 = - \frac{\delta^3}{h g^{00}}. \] (4.9)
Since the above functions satisfy
\[ [\mathcal{D}_{\mu\nu}(x, y)]^\dagger = -\mathcal{D}_{\nu\mu}(y, x), \quad [\mathcal{D}(x, y)]^\dagger = -\mathcal{D}(y, x), \] (4.10)
we obtain the Hermitian conjugates of (4.3), (4.4), (4.6), (4.7), and (4.9) [5–7] as follows:
\[ \left[ \mathcal{D}_{\mu\nu}(x, y) \bar{\mathbf{\delta}}_{\mu}^{\nu} \cdot h (g^{\rho\sigma} g^{\nu k} - g^{\nu\sigma} g^{\rho k}) \right] \bar{\mathbf{\delta}}_{\sigma}^{\nu} - \mathbf{\delta}_{\mu}^{\nu} \mathcal{D}(x, y) \bar{\mathbf{\delta}}_{\tau}^{\nu} \cdot h g_{\tau k} = 0, \]
(4.11)
\[ \left[ \mathcal{D}_{\mu\nu}(x, y) h g^{\nu\rho} \right] \bar{\mathbf{\delta}}_{\rho}^{\nu} + \alpha \mathbf{\delta}_{\mu}^{\nu} \mathcal{D}(x, y) \cdot h = 0, \]
(4.12)
\[ \mathcal{D}_{\mu\nu}(x, y) \bar{\mathbf{\delta}}_{\nu}^{\mu} = \left[ g_{\mu\nu} - (1 - \alpha) \frac{\delta_0^0 \delta_0^0}{g_{00}} \right] \frac{\delta^3}{h g_{00}}, \]
(4.13)
\[ \mathcal{D}(x, y) \bar{\mathbf{\delta}}_{\nu}^{\mu} = \frac{\delta^3}{h g_{00}}, \]
(4.14)
\[ \mathcal{D}(x, y) \bar{\mathbf{\delta}}_{\nu}^{\mu} = 0, \]
(4.15)

On the basis of (4.11), (4.12), and (4.14), the bilocal “current” \( \mathcal{J}_{\mu}^{\lambda}(x, y) \) in (4.2) is conserved with respect to \( y \); therefore, the right-hand side of (4.1) is independent of \( y^0 \), and reduces to \( A_\mu(x) \) at \( y^0 = x^0 \).

If \( \epsilon \neq 0 \) in (2.9), then we extend (4.1) as follows:
\[ A_\mu(x) = - \int d^4 z [\theta(x^0 - z^0) - \theta(y^0 - z^0)] \mathcal{D}_{\mu\nu}(x, z) [-e h(z) \bar{\psi}(z) \gamma^\nu(z) \psi(z)] \]
\[ + \int d^3 z \mathcal{J}_{\mu}^0(x, z)|_{z^0 = y^0}. \]
(4.16)

The relation between (2.9) and (4.16) is analogous to that between (1.1) and (1.2). Here note that \( \mathcal{J}_{\mu}^{\lambda}(x, z) \) in (4.2) satisfies
\[ \mathcal{J}_{\mu}^{\lambda}(x, z) \bar{\mathbf{\delta}}_{\lambda}^{\mu} = -\mathcal{D}_{\mu\nu}(x, z) [-e h(z) \bar{\psi}(z) \gamma^\nu(z) \psi(z)], \]
(4.17)
because of (2.9), (2.10), (4.11), and (4.12).

Differentiating the right-hand side of (4.16) with respect to \( y^0 \), we find
\[ \int d^4 z [\theta(y^0 - z^0) - \theta(y^0 - z^0)] \mathcal{D}_{\mu\nu}(x, z) [-e h(z) \bar{\psi}(z) \gamma^\nu(z) \psi(z)] + \int d^3 z \mathcal{J}_{\mu}^0(x, z)|_{z^0 = y^0} = 0, \]
(4.18)
via (4.17). Namely, the expression in (4.16) is independent of \( y^0 \). We thus set \( y^0 = 0 \) in (4.16) for the sake of convenience:
\[ A_\mu(x) = - \int d^4 z \epsilon(x^0, z^0) \mathcal{D}_{\mu\nu}(x, z) [-e h(z) \bar{\psi}(z) \gamma^\nu(z) \psi(z)] + \int d^3 z \mathcal{J}_{\mu}^0(x, z)|_{z^0 = 0}, \]
(4.19)
with
\[ \epsilon(x^0, z^0) \equiv [\theta(x^0 - z^0) - \theta(y^0 - z^0)]|_{y^0 = 0}. \]
(4.20)

Using (4.2), (4.5), (4.8), (4.9), (4.13), and (4.17), we can reduce the right-hand side of (4.19) to \( A_\mu(x) \). In addition, we can solve (2.15) in terms of the integral representation for \( B [6,7] \),
\[ B(x) = \int d^3 z \left[ \mathcal{D}(x, z) \bar{\mathbf{\delta}}_{\nu}^{\mu} \cdot h(z) g_{0\nu}^0(z) B(z) - \mathcal{D}(x, z) h(z) g_{0\nu}^0(z) \bar{\mathbf{\delta}}_{\nu}^{\mu} B(z) \right]. \]
(4.21)

Of course, this representation at \( z^0 = 0 \) and (4.19) are consistent with the field equation (2.9) and the gauge-fixing condition (2.10).
On the other hand, if \( e \neq 0 \) in (2.13), then we extend (3.9) as follows:

\[
\psi(x) = \int d^4 z [\theta(x^0 - z^0) - \theta(y^0 - z^0)] S(x, z; m) [-e h(z) \gamma^\mu(z) A_\mu(z) \psi(z)] \\
+ \int d^3 z \mathcal{J}^0(x, z; m) |_{z^0 = y^0}.
\]  

(4.22)

The relation between (2.13) and (4.22) is analogous to that between (1.4) and (1.5). Here note that \( \mathcal{J}^\lambda(x, z; m) \) in (3.10) satisfies

\[
\mathcal{J}^\lambda(x, z; m) \overset{\rightarrow}{\delta^\lambda_x} = S(x, z; m) [-e h(z) \gamma^\mu(z) A_\mu(z) \psi(z)],
\]  

(4.23)

because of (2.8), (2.13), and (3.8).

Differentiating the right-hand side of (4.22) with respect to \( y^0 \), we find

\[
- \int d^4 z \delta(y^0 - z^0) S(x, z; m) [-e h(z) \gamma^\mu(z) A_\mu(z) \psi(z)] + \int d^3 z \delta_0^y [\mathcal{J}^0(x, z; m) |_{z^0 = y^0}] = 0,
\]  

(4.24)

via (4.23). Namely, the expression in (4.22) is independent of \( y^0 \). We thus set \( y^0 = 0 \) in (4.22) for the sake of convenience:

\[
\psi(x) = \int d^4 z \epsilon(x^0, z^0) S(x, z; m) [-e h(z) \gamma^\mu(z) A_\mu(z) \psi(z)] + \int d^3 z \mathcal{J}^0(x, z; m) |_{y^0 = 0}.
\]  

(4.25)

Using (3.6) and (4.23), we can reduce the right-hand side of (4.25) to \( \psi(x) \). Here, the Dirac conjugate of (4.25) is given by

\[
\bar{\psi}(x) = - \int d^4 z \epsilon(x^0, z^0) [-e h(z) \bar{\psi}(z) \gamma^\mu(z) A_\mu(z)] S(z, x; m) + \int d^3 z [\mathcal{J}^0(x, z; m)]^\dagger |\bar{y}^0 = 0,\]

(4.26)

via (3.7).

Let us investigate the properties of the integral representations (4.19) and (4.25) under the affine, the gravitational BRST, the internal Lorentz BRST, and the electromagnetic BRST transformations.

In order to obtain the affine transformation of (4.19), we use (3.20–3.22), (4.17), the affine transformation of \( D_{\mu\nu}(x, y) \) [7],

\[
[i \hat{P}_\lambda, D_{\mu\nu}(x, y)] = (\delta^\alpha_\lambda + \delta^\gamma_\lambda) D_{\mu\nu}(x, y),
\]  

(4.27)

\[
[i \hat{M}^{\alpha}_\lambda, D_{\mu\nu}(x, y)] = (x^\kappa \delta^\alpha_\kappa + y^\kappa \delta^\gamma_\kappa) D_{\mu\nu}(x, y) + \delta^\alpha_\mu D_{\lambda\nu}(x, y) + \delta^\gamma_\nu D_{\mu\lambda}(x, y),
\]  

(4.28)

and that of \( D(x, y) \) [5,6],

\[
[i \hat{P}_\lambda, D(x, y)] = (\delta^\alpha_\lambda + \delta^\gamma_\lambda) D(x, y),
\]  

(4.29)

\[
[i \hat{M}^{\alpha}_\lambda, D(x, y)] = (x^\kappa \delta^\alpha_\kappa + y^\kappa \delta^\gamma_\kappa) D(x, y).
\]  

(4.30)

We thus find

\[
[i \hat{P}_\lambda, A_\mu(x)] = \delta^\alpha_\lambda A_\mu(x) - \int d^4 z \delta^\alpha_\kappa \epsilon(x^0, z^0) D_{\mu\nu}(x, z)[-e h(z) \bar{\psi}(z) \gamma^\nu(z) \psi(z)],
\]  

(4.31)

\[
[i \hat{M}^{\alpha}_\lambda, A_\mu(x)] = \delta^\alpha_\mu A_\mu(x) + \delta^\gamma_\alpha A_\lambda(x) \]

\[
- \int d^4 z \delta^\gamma_\kappa \epsilon(x^0, z^0) D_{\mu\nu}(x, z)[-e h(z) \bar{\psi}(z) \gamma^\nu(z) \psi(z)].
\]  

(4.32)

In the right-hand sides of these equations, the 4D integral terms reduce to 3D surface integral ones. Hence they vanish.
Furthermore, using (3.15), (3.16), (3.20), (3.22), and (4.23), we obtain the affine transformation of (4.25) as follows:

\[
[i \hat{P}_\lambda, \psi(x)] = \partial^\lambda \psi(x) + \int d^4z \partial^\lambda \{\epsilon(x^0, z^0)S(x, z; m)[-eh(z)\gamma^\mu(z)A_\mu(z)\psi(z)]
+ \int d^3z [\partial^\lambda J^0(x, z; m) - \delta^\lambda_\mu \partial^\mu J^\mu(x, z; m)]|_{\varphi=0},
\]

(4.33)

\[
[i \hat{M}^\lambda_\psi, \psi(x)] = \epsilon^\lambda x^\psi \partial^\psi \psi(x) + \int d^4z \partial^\lambda \{\epsilon(x^0, z^0)S(x, z; m)[-eh(z)\gamma^\mu(z)A_\mu(z)\psi(z)]
+ \int d^3z [\partial^\lambda \{z^\psi J^0(x, z; m) - \delta^\lambda_\mu \partial^\mu \rho_\mu J^\mu(x, z; m)]|_{\varphi=0}.
\]

(4.34)

In the right-hand sides of these equations, the 4D integral terms and the 3D integral ones reduce to 3D surface integral terms and surface integral ones, respectively. Hence they vanish.

Next, the properties of \(D_{\mu\nu}(x, y)\) and \(D(x, y)\) \([7]\) with respect to the gravitational BRST transformation are given by

\[
[i Q_G, D_{\mu\nu}(x, y)] = -\kappa \left[\partial^\nu c^\rho(x) \cdot D_{\rho\nu}(x, y) + c^\rho(x) \partial^\nu D_{\mu\nu}(x, y)
+ D_{\mu\sigma}(x, y) \cdot \partial^\nu c^\sigma(y) + D_{\mu\nu}(x, y) \partial^\nu \cdot c^\sigma(y)\right],
\]

(4.35)

\[
[i Q_G, D(x, y)] = -\kappa \left[c^\rho(x) \partial^\nu D(x, y) + D(x, y) \partial^\nu \cdot c^\rho(y)\right].
\]

(4.36)

Using (2.34), (2.35), (2.38), (3.31–3.33), (4.17), (4.35), and (4.36), we obtain the gravitational BRST transformation of (4.19) as follows:

\[
[i Q_G, A_\mu(x)] = -\kappa \partial^\mu c^\rho(x) \cdot A_\rho(x) - \kappa c^\rho(x) \partial^\mu A_\mu(x)
+ \kappa \int d^4z \partial^\mu \{\epsilon(x^0, z^0)D_{\mu\nu}(x, z)c^\nu(z)[-eh(z)\bar{\psi}(z)\gamma^\nu(z)\psi(z)]\}.
\]

(4.37)

This 4D integral term vanishes, since it reduces to the 3D surface integral one.

In parallel, using (2.34), (2.38), (3.27), (3.31), (3.33), and (4.23), we obtain the gravitational BRST transformation of (4.25) as follows:

\[
[i Q_G, \psi(x)] = -\kappa c^\rho(x) \partial^\rho \psi(x)
- \kappa \int d^4z \partial^\rho \{\epsilon(x^0, z^0)S(x, z; m)c^\rho(z)[-eh(z)\gamma^\nu(z)A_\nu(z)\psi(z)]
+ \kappa \int d^3z \partial^\rho \{J^0(x, z; m)c^\rho(z) - J^\rho(x, z; m)c^\rho(z)\}|_{\varphi=0}.
\]

(4.38)

In the right-hand side, the integral terms vanish because of the similar reasons in (4.33) and (4.34). Moreover, the internal Lorentz BRST transformations of \(D_{\mu\nu}(x, y)\) and \(D(x, y)\) are defined as follows:

\[
[i Q_L, D_{\mu\nu}(x, y)] \equiv I_{\mu\nu}(x, y),
\]

(4.39)

\[
[i Q_L, D(x, y)] \equiv I(x, y).
\]

(4.40)
These satisfy
\begin{align}
\partial^x_k \left[ \hbar \left( g^{k \lambda} g^{\sigma \mu} - g^{k \mu} g^{\sigma \lambda} \right) \partial^x_{\lambda} I_{\mu \nu}(x, y) \right] - h g^{\sigma \tau} \partial^x_{\tau} I(x, y) \partial^x_{\nu} &= 0, \\
\partial^x_k \left[ h g^{k \mu} I_{\mu \nu}(x, y) \right] + \alpha h I(x, y) \partial^x_{\nu} &= 0,
\end{align}
(4.41)
and
\begin{align}
\partial^x_{\mu} \left[ h g^{\mu \nu} \partial^x_{\nu} I(x, y) \right] &= 0, \\
I(x, y)_{|0} &= 0, \\
\partial^x_{0} I(x, y)_{|0} &= 0,
\end{align}
(4.42)
on the basis of (3.42), (3.45), (4.3–4.9). We thus obtain
\begin{align}
I_{\mu \nu}(x, y) &= 0, \\
I(x, y) &= 0.
\end{align}
(4.43)
(4.44)
(4.45)
(4.46)
(4.47)

Using (2.41), (2.42), (2.45), (3.38), (3.42), and (3.45), we find that the integral representation (4.19) for \(A_\mu\) is invariant under the internal Lorentz BRST transformation. On the other hand, applying (2.41), (2.45), (3.38), (3.42), and (3.45) to the anti-commutation relation between \(Q_\La\) and the integral representation (4.25) for \(\psi\), we obtain the form of the right-hand side of (2.45).

The functions \(D_{\mu \nu}(x, y)\) and \(D(x, y)\) are invariant under the electromagnetic BRST transformation [7]. Therefore, the transformation of (4.19) is given by
\begin{align}
\{i Q_\La, A_\mu(x)\} &= \int d^3 z \left\{ \partial^z_{\lambda} \tilde{\partial}^\tau_{\rho} \cdot (z) \left[ g^{0 \rho} (z) g^{\nu \sigma} (z) - g^{0 \nu} (z) g^{\rho \sigma} (z) \right] \partial^z_{\nu} C(z) \\
&\quad - \partial^z_{\mu} D(x, z) h(z) g^{0 \sigma} (z) \partial^z_{\sigma} C(z) \right\} |_{\epsilon = 0},
\end{align}
(4.50)
via (2.47), (2.48), and (2.51). Since the first term of this integrand does not involve \(\partial_0 C\), we integrate it by parts. Using (4.11), we consequently reduce the right-hand side of (4.50) to \(\partial_\mu C\) based on the integral representation for \(C\) [6],
\begin{align}
C(x) &= \int d^3 z \left[ \tilde{\partial}^\tau_{\rho} \cdot (z) g^{0 \rho} (z) \partial^z_{\nu} C(z) - D(x, z) h(z) g^{0 \sigma} (z) \partial^z_{\sigma} C(z) \right].
\end{align}
(4.51)

In parallel, the transformation of (4.25) is given by
\begin{align}
\{i Q_\La, \psi(x)\} &= i e C(x) \psi(x) + \int d^4 z \left[ \epsilon(x^0, z^0) \tilde{S}(x, z; m) \{ - e h(z) \gamma^k (z) C(z) \psi(z) \} \right],
\end{align}
(4.52)
by virtue of (2.8), (2.47), (2.51), (3.6), (3.8), and (3.51). This 4D integral term reduces to the 3D surface integral one. Hence it vanishes.

5. Discussion

In the present paper, we have briefly reviewed the manifestly covariant operator formalism for the electromagnetic field and the electron one in quantum Einstein gravity; we have treated these fields as matter ones with the electromagnetic interaction. The field equations (2.9) and (2.13) contain inhomogeneous terms with the electromagnetic coupling constant \(e\). We have introduced a
quantum-gravity S function \( S(x, y; m) \) and investigated it; this function enables us to treat integral representations for Dirac fields. Using \( D_{\mu\nu}(x, y) \), \( D(x, y) \), and \( S(x, y; m) \), we have solved (2.9), (2.10), and (2.13) in terms of integral representations for \( A_\mu \) and \( \psi \). These representations include not only the gravitational interaction but also the electromagnetic one. We have verified the transformation properties of these integral representations with respect to the affine, the gravitational BRST, and the electromagnetic BRST symmetries.

Because of the inhomogeneous terms in (2.9) and (2.13), the 4D integral terms involving the factor \( \epsilon(x^0, z^0) \) appear in (4.19) and (4.25). These terms are “retarded” superpositions on \( x^0 > z^0 > 0 \) for \( x^0 > 0 \), and “advanced” superpositions on \( x^0 < z^0 < 0 \) for \( x^0 < 0 \). In addition, these terms are contributions from the electric current via \( D_{\mu\nu}(x, z) \) in (4.19) and from the electromagnetic interaction via \( S(x, z; m) \) in (4.25), respectively. The factor \( \epsilon(x^0, z^0) \) in (4.19) or (4.25) yields a “time-order” between \( x^0 \) and \( z^0 \), while \( D_{\mu\nu}(x, z) \), \( D(x, z) \), and \( S(x, z; m) \) are not functions of \( x - z \) alone. Thus, we can regard the inhomogeneous terms in (2.9) and (2.13) as origins of “time-evolution” in the operator level.

On the other hand, if a physical vacuum \( |0\rangle \) is translationally invariant,

\[
\hat{P}_x |0\rangle = 0, \tag{5.1}
\]

then the vacuum expectation values of \( A_\mu(x) \) and \( \psi(x) \) are independent of \( x^\lambda \) via (4.31) and (4.33). Therefore, taking the vacuum expectation values of the integral representations (4.19) and (4.25),

\[
\langle 0| A_\mu(x) |0\rangle &= -\int d^4z \, \epsilon(x^0, z^0) \langle 0| D_{\mu\nu}(x, z) \left[ -\epsilon h(z) \bar{\psi}(z) \gamma^\nu(z) \psi(z) \right] | 0\rangle \\
&\quad + \int d^3z \langle 0| J^0_{\mu}(x, z) |0\rangle |_{z^0=0} \tag{5.2},
\]

\[
\langle 0| \psi(x) |0\rangle &= \int d^4z \, \epsilon(x^0, z^0) \langle 0| S(x, z; m) \left[ -\epsilon h(z) \gamma^\mu(z) A_\mu(z) \psi(z) \right] | 0\rangle \\
&\quad + \int d^3z \langle 0| \chi^0(x, z; m) |0\rangle |_{z^0=0} \tag{5.3},
\]

we find that the \( x^\lambda \)-dependence of the first term in each right-hand side cancels that of the second term.

References

[1] M. Abe and N. Nakanishi, Prog. Theor. Phys. 88, 975 (1992).
[2] M. Abe and N. Nakanishi, Prog. Theor. Phys. 85, 391 (1991).
[3] N. Nakanishi, Prog. Theor. Phys. 111, 301 (2004).
[4] N. Nakanishi and I. Ojima, Covariant Operator Formalism of Gauge Theories and Quantum Gravity (World Scientific, Singapore, 1990).
[5] N. Nakanishi, Prog. Theor. Phys. 68, 947 (1982).
[6] H. Kanno and N. Nakanishi, Prog. Theor. Phys. 73, 496 (1985).
[7] R. Yoshida, Prog. Theor. Exp. Phys. 2013, 123B03 (2013).
[8] N. Nakanishi, Prog. Theor. Phys. 63, 656 (1980).
[9] N. Nakanishi, Prog. Theor. Phys. 62, 779 (1979).
[10] N. Nakanishi, Prog. Theor. Phys. 62, 1101 (1979).