Ergodicity and exponential mixing of the real Ginzburg-Landau equation with a degenerate noise

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Abstract

In this paper, we establish the existence, uniqueness and attraction properties of an invariant measure for the real Ginzburg-Landau equation in the presence of a degenerate stochastic forcing acting only in four directions. The main challenge is to establish time asymptotic smoothing properties of the Markovian dynamics corresponding to this system. To achieve this, we propose a condition which only requires four noises.

Keywords: exponential mixing; Malliavin calculus; ergodic; real Ginzburg-Landau equation.

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1 Introduction and Main Results

1.1 Introduction

In this paper, we are concerned with the ergodicity of the stochastic real Ginzburg-Landau equation driven by Brownian motion on torus \( T = \mathbb{R}/2\pi\mathbb{Z} \) as follows

\[
\begin{aligned}
  &dU - \frac{\partial^2 U}{\partial z^2} \, dt - (U - U^3) \, dt = \sum_{k \in \mathbb{Z}_0} \beta_k e_k \, dW_k(t), \\
  &U|_{t=0} = U_0,
\end{aligned}
\]

(1.1)
where \( U : [0, \infty) \times T \to \mathbb{R} \), \( Z_0 \) is a subset of \( Z_* = \mathbb{Z} \setminus \{0\} \), \( \{\beta_k\}_{k \in \mathbb{Z}} \) are non-zero constants, \( \{W_k(t)\}_{k \in \mathbb{Z}} \) is one dimensional real-valued i.i.d Brownian motion sequence defined on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) and

\[
e_k(z) = \begin{cases} 
\sin(kz), & k \in \mathbb{Z} \cap [1, \infty), z \in T, \\
\cos(kz), & k \in \mathbb{Z} \cap (-\infty, -1], z \in T. 
\end{cases}
\]

Consider the following abstract equation on a Hilbert space \( H \),

\[
dU = F(U)dt + GdW_t, \quad U|_{t=0} = U_0.
\]

There is a wide literature devoted to proving uniqueness and associated mixing properties of invariant measures for nonlinear stochastic PDEs when \( GG^* \) is non-degenerate or mildly degenerate (see e.g. \([5, 15, 22, 27, 28]\) and references therein).

The purpose of this paper is to prove the exponential mixing for stochastic real Ginzburg-Landau equation \((1.1)\) when the random forcing is extremely degenerate to be several noises. There are several works related to this topic when the random forcing is extremely degenerate. We mention some of them which are relevant to our work.

- Hairer and Mattingly \([11, 12]\) considered stochastic 2D Navier-Stokes equations on a torus driven by degenerate additive noise. They established an exponential mixing property of the solution of the vorticity formulation for the 2D stochastic Navier-Stokes equations by using Malliavin calculus, although the noise is extremely degenerate (the noise only acts in four directions).

- Földes et al. \([9]\) was interested in the following stochastic Boussinesq equations

\[
\begin{align*}
\begin{cases} 
\mathrm{d}u + (u \cdot \nabla u)dt &= (-\nabla p + \nu_1 \Delta u + g \theta)dt, \quad \nabla \cdot u = 0 \\
\mathrm{d}\theta + (u \cdot \nabla \theta)dt &= \nu_2 \Delta \theta dt + \sigma \theta dW,
\end{cases}
\end{align*}
\]

where \( u = (u_1, u_2) \) denotes the velocity field, \( \theta \) is the temperature, \( g = (0, g)^T \) with \( g \neq 0 \) is a constant. The authors worked on the vorticity equations of \((1.2)\), which is given by

\[
\begin{align*}
\begin{cases} 
\mathrm{d}\omega + (u \cdot \nabla \omega - \nu_1 \Delta \omega) = g \partial_x \theta dt, \\
\mathrm{d}\theta + (u \cdot \nabla \theta - \nu_2 \Delta \theta) &= \sigma \theta dW.
\end{cases}
\end{align*}
\]

Although the forcing is extremely degenerate (only four directions in \( \theta \) have noise), the authors succeed to establish an exponential mixing property for the solution of equation \((1.3)\) by utilizing Malliavin calculus.

As stated above, all the authors in \([9, 11]\) established an exponential mixing property for the solution of vorticity equation instead of velocity equation. For our model, we can directly
deal with the velocity equation (1.1) due to its special structure. Let \( U_t \) be the solution to equations (1.2) or (1.3) and \( J_{0,t}\xi = DU_t(x)\xi \) be the effect on \( U_t \) of an infinitesimal perturbation of the initial condition in the direction \( \xi \). The authors of [9, 11] considered the vorticity formulation in order to obtain \( \mathbb{E}\|J_{0,t}\xi\|^p < \infty \). For equation (1.1), we can directly achieve it.

For the stochastic real Ginzburg-Landau equation, we mention the following results.

- For the stochastic real Ginzburg-Landau equation driven by Brownian motion, Hairer [10, Section 6] established an exponential mixing of the solution to (1.1) under the condition that the number of noises can be finite but should be sufficiently many. Our results in this article are stronger than that. Meanwhile, the random forcing of our model can be extremely degenerate to be only several noises.

- Xu [33] proved that the stochastic real Ginzburg-Landau equation driven by \( \alpha \)-stable process admits a unique invariant measure under some conditions. The noise in [33] is required to be non-degenerate.

- Mourrat and Weber [23] established a priori estimates for the dynamic \( \Phi^4_3 \) model on the torus which is independent of initial conditions. The \( \Phi^4_3 \) model is formally given by the stochastic partial differential equation

\[
\begin{aligned}
\partial_t X &= \Delta X - X^3 + mX + \xi, \quad \text{on } \mathbb{R}_+ \times [-1,1]^3, \\
X(0, \cdot) &= X_0
\end{aligned}
\]

where \( \xi \) denotes a white noise on \( \mathbb{R}_+ \times [-1,1]^3 \), and \( m \in \mathbb{R} \) is a parameter.

1.2 Main results

Let \( T = \mathbb{R}/2\pi \mathbb{Z} \) be equipped with the usual Riemannian metric, and let \( dz \) denote the Lebesgue measure on \( T \). Then

\[
H := \left\{ \xi \in L^2(T, \mathbb{R}); \int_T \xi(z)dz = 0 \right\}
\]

is a separable real Hilbert space with inner product

\[
\langle \xi, \eta \rangle = \int_T \xi(z)\eta(z)dz, \quad \forall \xi, \eta \in H
\]

and norm \( \|\xi\| = \langle \xi, \xi \rangle^{1/2} \).

It is well-known that

\[
\{e_k : k \in \mathbb{Z}_+\}.
\]

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is an orthonormal basis of $H$. For each $x \in H$, it can be represented by

$$x = \sum_{k \in \mathbb{Z}_0} x_k e_k.$$

Let $\Delta = \frac{\partial^2}{\partial z^2}$ be the Laplace operator on $H$, then

$$\Delta e_k = -\gamma_k e_k, \text{ with } k \in \mathbb{Z}_0, \gamma_k = |k|^2. \quad (1.4)$$

For $\sigma > 0$, we define

$$A := -\Delta,$$

$$H^{\sigma} = H^{\sigma, 2}(\mathbb{T}) := \left\{ x \in H : x = \sum_{k \in \mathbb{Z}_0} x_k e_k \text{ with } \|x\|_{H^{\sigma}}^2 := \sum_{k \in \mathbb{Z}_0} (1 + \gamma_k)^\sigma |x_k|^2 < \infty \right\},$$

$$D(A^{\sigma}) := \left\{ x \in H : x = \sum_{k \in \mathbb{Z}_0} x_k e_k \text{ with } \|x\|_{D(A^{\sigma})}^2 := \sum_{k \in \mathbb{Z}_0} |\gamma_k|^{2\sigma} |x_k|^2 < \infty \right\},$$

$$V^{\sigma} := \left\{ x \in D(A^{\sigma/2}), \text{ with } \|x\|_{\sigma} := \|x\|_{D(A^{\sigma/2})} < \infty \right\}.$$

For $\sigma > 0$, we denote by $H^{-\sigma}$ the dual space of $H^{\sigma}$. For the sake of convenience, we denote by $V = V^1$.

Set $N(U) = -U + U^3$ and

$$F(U) = -AU - N(U) = \Delta U + U - U^3.$$

Let $\{\theta_k\}_{k \in \mathbb{Z}_0}$ be the standard basis of $\mathbb{R}^{|\mathbb{Z}_0|}$, where $|\mathbb{Z}_0|$ denotes the number of the element belongs to the set $\mathbb{Z}_0$. We define a linear map $G : \mathbb{R}^{|\mathbb{Z}_0|} \rightarrow H$ such that

$$G\theta_k = \beta_k e_k, \quad (1.5)$$

where $\{\beta_k\}_{k \in \mathbb{Z}_0}$ is a sequence of non-zero numbers appeared in (1.1). We consider the stochastic forcing of the form

$$GdW_t = \sum_{k \in \mathbb{Z}_0} \beta_k e_k dW_k(t),$$

then (1.1) can be written as

$$dU = F(U)dt + GdW, \quad U|_{t=0} = U_0.$$
For any $n \geq 1$, we define $\mathcal{Z}_n$ recursively as follows:

$$\mathcal{Z}_n := \{ k + \ell + m : k \in \mathcal{Z}_{n-1}, \ell, m \in \mathcal{Z}_0 \}. \quad (1.6)$$

Our Hypothesis in this article is

**Hypothesis 1.1.**

(i) if $k \in \mathcal{Z}_0$, then $-k \in \mathcal{Z}_0$,

(ii) $\bigcup_{n=0}^{\infty} \mathcal{Z}_n = \mathcal{Z}_*$,

(iii) $|\mathcal{Z}_0| < \infty$.

To measure the convergence to equilibrium, we will use the following distance function on $H$

$$d(x, y) = 1 \wedge \delta^{-1}\|x - y\|. \quad (1.7)$$

where $\delta$ is a small parameter to be adjusted later on. The distance (1.7) extends in a natural way to a Wasserstein distance between probability measures by

$$d(\mu_1, \mu_2) = \sup_{\|\Phi\|_{d} \leq 1} \left| \int_H \Phi(x) \mu(dx) - \int_H \Phi(x) \nu(dx) \right|$$

where $\|\Phi\|_{d}$ denotes the Lipschitz constant of $\Phi$ in the metric $d$.

The transition function associated to (1.1) is given by

$$P_t(U_0, E) = \mathbb{P}(U(t, U_0) \in E) \text{ for any } U_0 \in H, E \in \mathcal{B}(H), t \geq 0, \quad (1.8)$$

where $\mathcal{B}(H)$ is the collection of Borel sets on $H$, $U(t, U_0)$ is the solution to equations (1.1) with initial value $U_0 \in H$. We also define the Markov semigroup $\{P_t\}_{t \geq 0}$ with $P_t : M_b(H) \to M_b(H)$ associated to (1.1) by

$$P_t \Phi(U_0) := \mathbb{E}_t \Phi(U(t, U_0)) = \int_H \Phi(U) P_t(U_0, dU) \text{ for any } \Phi \in M_b(H), t \geq 0, \quad (1.9)$$

where $M_b(H)$ is the space of bounded measurable functions on $H$ equipped with supremum norm. Denote by $C_b(H)$ the space of bounded continuous real-valued functions on $H$. Let $\mathcal{P}(H)$ be the collection of Borelian probability measures on $H$. The dual operator $P_t^*$ of $P_t$, which maps $\mathcal{P}(H)$ to itself, is given by

$$P_t^* \mu(A) := \int_H P_t(U_0, A) d\mu(U_0), \quad (1.10)$$

over $\mu \in \mathcal{P}(H)$. 

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Now we will give our main results in this paper.

**Theorem 1.1.** Assume Hypothesis (1.1) holds, then there exists a unique invariant measure \( \mu^* \) associated to (1.1) and for each \( t \geq 0 \) the map \( P_t \) is ergodic related to \( \mu^* \). Concretely, the following results hold.

(i) **(Exponential Mixing)** There are constants \( \delta > 0 \) and \( \gamma > 0 \) such that

\[
\sup_{\|\Phi\|_d \leq 1} \left| \mathbb{E}\Phi(U(t, U_0)) - \int_H \Phi(\bar{U})d\mu^*(\bar{U}) \right| \leq Ce^{-\gamma t},
\]

where \( C \) is a constant independent of \( U_0 \) and \( t \).

(ii) **(Weak law of large numbers)** For the \( \delta > 0 \) in (i), any \( \Phi \) with \( \|\Phi\|_d \leq 1 \) and any \( U_0 \in H \), we have

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi(U(t, U_0))dt = \int_H \Phi(\bar{U})d\mu^*(\bar{U}) =: m_\Phi \quad \text{in probability.}
\]

(iii) **(Central limit theorem)** For the \( \delta > 0 \) in (i), any \( \Phi \) with \( \|\Phi\|_d \leq 1 \), every \( U_0 \in H \) and \( \xi \in \mathbb{R} \), we have

\[
\lim_{T \to \infty} \mathbb{P}\left( \frac{1}{\sqrt{T}} \int_0^T (\Phi(U(t, U_0)) - m_\Phi)dt < \xi \right) = \mathcal{X}(\xi),
\]

where \( \mathcal{X} \) is the distribution function of a normal random variable whose mean is equal to zero and variance is equal to

\[
\lim_{T \to \infty} \frac{1}{T} \mathbb{E}\left( \int_0^T (\Phi(U(t, U_0)) - m_\Phi)dt \right)^2.
\]

We emphasize that the constant \( C \) appeared in (1.11) is independent of the initial value \( U_0 \). This is one of the challenges in our paper.

Based on Theorem 1.1 the following result holds.

**Corollary 1.1.** For any \( n \geq 1 \), if \( Z_0 = \{-(n+1), -n, n, n+1\} \), the results of Theorem 1.1 hold.

### 1.3 The organization of this paper

This article is organized as follows: Section 2 is devoted to establishing some moment estimates. In Section 3, we present the proof of spectral properties for the Malliavin matrix \( \mathcal{M}_{0,t} \) of \( U_t \) in Theorem 3.1 and demonstrate a gradient estimate of \( P_t \) in Proposition 3.3. Finally, we give a proof of Theorem 1.1 in Section 4.
2 Some moment estimates

In this section, we establish some moment estimates which are useful in this paper. When $T > 0$ is a constant, we always denote by $C_T$ a constant depending on $T$ and it may change from line to line.

We say that $U_t = U(t, U_0)$ is a solution to (1.1) if it is $\mathcal{F}_t$-adapted,

$$ U \in C([0, \infty), H) \cap L^2_{loc}([0, \infty), V) \quad a.s., $$

and $U$ satisfies (1.1) in the mild sense, that is

$$ U_t = e^{-At}U_0 - \int_0^t e^{-A(t-s)}N(U_s)ds + \int_0^t e^{-A(t-s)}GdW_s. $$

The following proposition summarizes the basic well-posedness, regularity, and smoothness of equation (1.1).

**Proposition 2.1.** Given any $U_0 \in H$, there exists a unique solution $U : [0, \infty) \times \Omega \to H$ of (1.1) which is an $\mathcal{F}_t$-adapted process on $H$ satisfying (2.1).

For any $t \geq 0$ and any realization of the noise $W(\cdot, \omega)$, the map $U_0 \mapsto U(t, U_0)$ is Fréchet differential on $H$. For every fixed $U_0 \in H$ and $t \geq 0$, $W \mapsto U(t, W)$ is Frechet differential from $C((0,t), \mathbb{R}^{\mid Z_0\mid})$ to $H$. Moreover, $U$ is spatially smooth for all positive time, that is, for any $t_0 > 0$ and any $s > 0$,

$$ U \in C([t_0, \infty), H^s) \quad a.s. $$

Since we are considering the case of spatially smooth, additive noise, the proof of the well-posedness of (1.1) is standard and can be obtained following along the line of classical proof for the stochastic 2D Navier-Stokes equations (see e.g. [10]).

Let $U_t = U(t, U_0, W)$ be the solution of (1.1) with initial value $U_0$ and noise $W$. For any $\xi \in H$ and $s \geq 0$, $J_{s,t}\xi$ denotes the unique solution of

$$ \begin{cases} 
\partial_t J_{s,t}\xi + AJ_{s,t}\xi - J_{s,t}\xi + 3U^2_t J_{s,t}\xi = 0, \\
J_{s,s}\xi = \xi. 
\end{cases} $$

The Malliavin derivative $D : L^2(\Omega; H) \to L^2(\Omega, L^2(0,T,\mathbb{R}^{\mid Z_0\mid}) \times H)$ satisfies that for each $v \in L^2(0,T,\mathbb{R}^{\mid Z_0\mid})$

$$ \langle DU, v \rangle_{L^2(0,T,\mathbb{R}^{\mid Z_0\mid})} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( U(T, U_0, W + \epsilon \int_0^t vds) - U(T, U_0, W) \right). $$
we may infer that for $v \in L^2(\Omega, L^2(0, T, \mathbb{R}^{\lfloor Z_0 \rfloor}))$,

$$\langle DU, v \rangle_{L^2(0, T, \mathbb{R}^{\lfloor Z_0 \rfloor})} = \int_0^T J_{s,T} Gv(s) ds,$$

and hence, by the Riesz representation theorem,

$$D^j_s U_T = J_{s,T} G\theta_j,$$

for any $s \leq T, j = 1, \ldots, \lfloor Z_0 \rfloor$.

Here and below, we adopt the standard notation $D^j_s F := (DF)_j(s)$, that is, $D^j_s F$ is the $j$-th component of $DF$ evaluated at time $s$.

We define the random operator $A_{s,t} : L^2(s, t, \mathbb{R}^{\lfloor Z_0 \rfloor}) \rightarrow H$ by

$$A_{s,t}v := \int_s^t J_{r,t} Gv(r) dr.$$

Notice that, for any $0 \leq s < t$, the function $\varrho(t) := A_{s,t}v$ satisfies the following equation

$$\begin{cases}
\partial_t \varrho(t) + A\varrho(t) - \varrho(t) + 3U^2_t \varrho(t) = Gv(t), \\
\varrho(0) = 0.
\end{cases}$$

For any $s < t$, let $A^*_{s,t} : H \rightarrow L^2(s, t, \mathbb{R}^{\lfloor Z_0 \rfloor})$ be the adjoint of $A_{s,t}$, then

$$(A^*_{s,t}\xi)(r) = G^* K_{r,t} \xi,$$

for any $\xi \in H, r \in [s, t]$ where $G^* : H \rightarrow \mathbb{R}^{\lfloor Z_0 \rfloor}$ is the adjoint of $G$, and for $s < t, K_{s,t} \xi = J^*_{s,t} \xi$ is the solution of the following “backward” system

$$\partial_s \varrho^* = A\varrho^* + (\nabla N(U_s))^* \varrho^* = -(\nabla F(U_s))^* \varrho^*, \quad \varrho^*(t) = \xi. \quad (2.3)$$

We then define the Malliavin matrix

$$\mathcal{M}_{s,t} := A_{s,t} A^*_{s,t} : H \rightarrow H. \quad (2.4)$$

Observe that $\rho_t := J_{0,t} \xi - A_{0,t}v$ satisfies

$$\begin{cases}
\partial_t \rho_t + A_0^* - \rho_t + 3U^2_t \rho_t = -Gv(t), \\
\rho(0) = \xi.
\end{cases} \quad (2.5)$$

For any $t \geq s \geq 0$ let $J^{(2)}_{s,t} : H \rightarrow \mathcal{L}(H, \mathcal{L}(H))$ be the second derivative of $U$ with respect to an initial value $U_0$. In this paper, $\mathcal{L}(X) = \mathcal{L}(X, X)$ and $\mathcal{L}(X, Y)$ is the space of linear operators from $X$ to $Y$. Observe that for fixed $U_0 \in H$ and any $\xi, \xi' \in H$ the function
\( \varrho_t := J_{s,t}^{(2)}(\xi, \xi') \) is the solution of

\[
\partial_t \varrho_t + A \varrho_t - \varrho_t + 3U_t^2 \varrho_t + 6U_t \mathcal{J}_{s,t} \xi \mathcal{J}_{s,t} \xi' = 0, \quad g(s) = 0.
\]

For any \( \alpha \in (0, 1] \) and function \( g : [T/2, T] \to \mathbb{R} \), \( \|g\|_{C^\alpha[T/2,T]} \) is defined by

\[
\|g\|_{C^\alpha[T/2,T]} := \sup_{t_1 \neq t_2, t_1, t_2 \in [T/2, T]} \frac{|g(t_1) - g(t_2)|}{|t_1 - t_2|^\alpha}.
\]

For any \( \alpha \in (0, 1] \) and function \( f : [T/2, T] \to \mathbb{R} \), we define the semi-norms

\[
\|f\|_{C^\alpha([T/2,T], H)} := \sup_{t_1 \neq t_2, t_1, t_2 \in [T/2, T]} \frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|^\alpha}.
\]

**Lemma 2.1.** For any \( m > 0, T > 0 \), there exists a positive constant \( \gamma = \gamma_{m,T} \) such that

\[
\mathbb{E}\|U_t^m\|^2 \leq C_{T,m}(t^{-\gamma} + 1), \quad \forall t \in (0, T]. \tag{2.6}
\]

and

\[
\mathbb{E}\|U_t^m\|^2 \leq C_{T,m}(\|U_0^m\|^2 + 1), \quad \forall t \in (0, T], \tag{2.7}
\]

where \( C_{T,m} \) is a constant depending on \( T \) and \( m \).

**Proof.** Applying \( It\hat{o} \) formula to \( f(t) = \langle U_t, U_t^{2m-1} \rangle \), it gives

\[
d\|U_t^m\|^2 \leq \langle dU_t, U_t^{2m-1} \rangle + \langle U_t, (2m - 1)U_t^{2m-2}dU_t \rangle + C_m\|U_t^{m-1}\|^2 dt
\]

\[
\leq \langle dU_t, 2mU_t^{2m-1} \rangle + C_m(1 + \|U_t^m\|^2)dt
\]

\[
= \langle (\Delta U_t + U_t - U_t^3)dt, 2mU_t^{2m-1} \rangle + C_m(1 + \|U_t^m\|^2) dt + dM_t
\]

\[
= -(2m)(2m - 1)\|\partial \cdot U_t\|_t^{m-1} \leq 2m\|U_t^m\|^2 \leq 2m\|U_t^m\|^2 - 2m\|U_t^{m+1}\|^2 + C_m(1 + \|U_t^m\|^2) dt + dM_t,
\]

where \( M_t \) is a martingale, \( C_m \) is a constant depending on \( m \) and \( \beta_k \). For any \( s \leq t \leq T \), by Young’s inequality \( \|U_t^m\|^2 \leq C_{\varepsilon,m} + \varepsilon\|U_t^{m+1}\|^2, \forall \varepsilon > 0 \), one arrives at

\[
\mathbb{E}\|U_t^m\|^2 + m\mathbb{E}\int_s^t \|U_r^{m+1}\|^2 dr \leq C_{m,T}(1 + \mathbb{E}\|U_s^m\|^2). \tag{2.8}
\]

Note that

\[
(|a| + |b|)^p \leq 2^{p-1}(|a|^p + |b|^p), \quad \forall p > 1,
\]

we deduce that

\[
\int_s^t (\mathbb{E}\|U_r^m\|^2 + 1)^\lambda dr \leq C_{m,T}(\mathbb{E}\|U_t^m\|^2 + 1), \tag{2.9}
\]

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where \( \lambda = \lambda(m) = \frac{m+1}{m} > 1 \). By [23, Lemma 7.3], there exist an integer \( N \geq 1 \) and a sequence \( 0 = t_0 < t_1 < t_2 < \cdots < t_N = T \) such that for every \( k \in \{0, 1, \cdots, N-1\} \)

\[
E\|U_{t_{k+1}}^m\|^2 \leq C_{T,m}(t_{k+1}^{-\frac{1}{\alpha}} + 1).
\]

For any \( t \in [t_k, t_{k+1}) \), by (2.8), we obtain

\[
E\|U_{t_k}^m\|^2 \leq C_{T,m}(E\|U_{t_k}^m\|^2 + 1) \leq C_{T,m}(t_{k+1}^{-\frac{1}{\alpha}} + 1) \leq C_{T,m}(t_k^{-\frac{1}{\alpha}} + 1)
\]

which implies (2.6).

Let \( s = 0 \) in (2.8), we obtain the desired result (2.7). \( \square \)

Define \( \mathfrak{B}_n = \sum_{k \in \mathbb{Z}_0} \gamma_k^n \beta_k^2 \) (2.10)

for \( n \in \mathbb{N} \cup \{0\} \), where \( \gamma_k \) is defined by (1.8) and \( \beta_k \) is in (1.5).

**Lemma 2.2.** For any \( m \geq 0, T \geq 0 \), there exist some \( C = C_{T,m} \) such that

\[
E \sup_{t \in [0,T]} \mathcal{E}(t)^m \leq C(\|U_0\|^{2m} + 1).
\]

**Proof.** Let us set

\[
M_t = 2 \sum_{k \in \mathbb{Z}_0} \beta_k \int_0^t \langle U_r, e_k \rangle dW_k(r)
\]

By Itô formula, we have

\[
\|U_t\|^2 = \|U_0\|^2 + \int_0^t 2\langle U_r, dU_r \rangle + \mathfrak{B}_0 t
\]

\[
= \|U_0\|^2 + \int_0^t 2\langle U_r, -AU_r + U_r - U_r^3 \rangle dr + \mathfrak{B}_0 t + M_t
\]

\[
= \|U_0\|^2 + \int_0^t (-2\|U_r\|^2 + 2\|U_r\|^2 - \|U_r\|^4_{L^4}) dr + \mathfrak{B}_0 t + M_t. \tag{2.11}
\]

Note that the quadratic variation of \( M_t \) is equal to

\[
\langle M \rangle_t = 4 \sum_{k \in \mathbb{Z}_0} \beta_k^2 \int_0^t \langle U_s, e_k \rangle^2 ds \leq \gamma \int_0^t \|U_s\|^2 ds,
\]

where \( \gamma = 4 \sum_{k \in \mathbb{Z}_0} \beta_k^2 \). We rewrite (2.11) as follows

\[
\mathcal{E}(t) - \mathfrak{B}_0 t = \|U_0\|^2 + M_t - \frac{1}{2} \gamma \langle M \rangle_t + K_t
\]
where
\[ K_t = \int_0^t \left( -\|U_r\|_1^2 + 2\|U_r\|^2 - \|U_r\|_{L^4}^4 \right) dr + \frac{1}{2} \gamma \cdot \langle M \rangle_t \leq C_\gamma t. \]
In the above, \( C_\gamma \) is some constant depending on \( \gamma \) and in the last inequality, we have used
\[ \|U_r\|^2 \leq C_\varepsilon + \varepsilon \|U_r\|_{L^4}^4, \quad \forall \varepsilon > 0. \]
Therefore,
\[ \mathcal{E}(t) - (\mathfrak{B}_0 + C_\gamma) t \leq \|U_0\|^2 + M_t - \frac{\gamma}{2} \langle M \rangle_t. \]
By the supermartingale inequality (cf. [16, (7.57)]), we have
\[ \mathbb{P} \left( \sup_{t \in [0,T]} \mathcal{E}(t) - (\mathfrak{B}_0 + C_\gamma) t \geq \mathbb{E} (\mathcal{E}(t) - (\mathfrak{B}_0 + C_\gamma) t) + \mathbb{P} \left( \gamma M_t - \frac{\gamma^2}{2} \langle M \rangle_t \geq e^{\gamma \rho} \right) \right) \leq e^{-\gamma \rho}. \]
Note that if \( \xi \) and \( \eta \) are non-negative random variables, then
\[ \mathbb{E}_\xi^m \leq 2^{m-1} \left( \mathbb{E}(\xi - \eta)^m \mathbb{1}_{\{\xi > \eta\}} + \mathbb{E}\eta^m \right) = 2^{m-1} \int_{0}^{\infty} \mathbb{P}(\xi - \eta > \lambda^{1/m}) d\lambda + 2^{m-1} \mathbb{E}\eta^m. \]
Apply this inequality to \( \xi = \sup_t \mathcal{E}(t) \) and \( \eta = \mathfrak{B}_0 + C_\gamma t + \|U_0\|^2 \), we derive
\[ \mathbb{E} \sup_{t \in [0,T]} \mathcal{E}(t)^m \leq 2^{m-1} \int_{0}^{\infty} \exp (-\gamma \lambda^{1/m}) d\lambda + 2^{m-1} \mathbb{E}(\mathfrak{B}_0 + C_\gamma t + \|U_0\|^2)^m \]
which yields the desired result.

For any integer \( n \geq 0 \), we set
\[ \mathcal{E}(n, t) = t^n \|U_t\|_n + \int_0^t s^n \|U_s\|_{n+1}^2 ds. \]

**Lemma 2.3.** For any \( n, m \geq 0 \), there exists a constant \( \kappa = \kappa_{n,m} \) such that
\[ \mathbb{E} \sup_{t \in [0,T]} \mathcal{E}(n, t)^m \leq C_{n,m,T}(\|U_0\|^2 + 1). \tag{2.12} \]

**Proof.** The proof is based on the method of induction in \( n \). Let us set \( f_n(t) = t^n \langle A^n U_t, U_t \rangle \). By the Itô formula in [16, Theorem 7.7.5] and following similar arguments in the proof of
Proposition 2.4.12, we have
\[ f_n(t) = \int_0^t ns^{n-1}||U_s||_n^2 + 2s^n \langle A^n U_s, -AU_s + U_s - U_s^3 \rangle + \mathcal{B}_n s^n ds + M_t, \] (2.13)
where
\[ M_t = \sum_{k \in \mathbb{Z}_0} 2\beta_k \int_0^t \langle A^n U_s, e_k \rangle dW_k(s) = \sum_{k \in \mathbb{Z}_0} 2\beta_k \gamma_n \int_0^t \langle U_s, e_k \rangle dW_k(s) \]

The quadratic variation of \( M_t \) is equal to
\[ \langle M \rangle_t = 4 \sum_{k \in \mathbb{Z}_0} \beta_k^2 \gamma_n^2 \int_0^t \langle U_s, e_k \rangle^2 ds \leq \gamma \int_0^t ||U_s||^2 ds, \] (2.14)
where \( \gamma = 4 \sum_{k \in \mathbb{Z}_0} \beta_k^2 \gamma_n^2 \).

Obviously, we have the following identities
\[ \langle A^n U_s, AU_s \rangle = ||U_s||_{n+1}^2, \quad \langle A^n U_s, U_s \rangle = ||U_s||_n^2. \] (2.15)

Firstly, we consider the case \( n = 1 \). In view of
\[ \langle AU_s, U_s - U_s^3 \rangle = \int_T - \frac{\partial^2 U_s(z)}{\partial z^2} (U_s(z) - U_s(z)^3)dz \]
\[ = \int_T \frac{\partial U_s(z)}{\partial z} \left( \frac{\partial U_s(z)}{\partial z} - 3U_s(z)^2 \frac{\partial U_s(z)}{\partial z} \right)dz \leq ||U_s||_1^2 \] (2.16)
and by (2.13) (2.15), we obtain
\[ t||U_t||_1^2 + \int_0^t s||U_s||_2^2 ds \leq C_T \int_0^t ||U_s||_1^2 ds + Ct^2 + M_t. \]

By (2.14), we rewrite the above equality in the form
\[ \mathcal{E}(1, t) \leq C_T \int_0^t ||U_s||_1^2 ds + \frac{\gamma}{2} \langle M \rangle_t + C_T t + M_t - \frac{\gamma}{2} \langle M \rangle_t \]
\[ \leq C_T \int_0^t ||U_s||_2^2 ds + \frac{\gamma}{2} \int_0^t ||U_s||_1^2 ds + C_T t + M_t - \frac{\gamma}{2} \langle M \rangle_t \]
\[ \leq C_T \mathcal{E}(t) + C_T t + M_t - \frac{\gamma}{2} \langle M \rangle_t. \]

Combining the above inequality with Lemma 2.2 and following a similar argument as in the proof of Lemma 2.2, we finish the proof of the inequality (2.12) with \( n = 1 \).

Now, assume that for \( k \leq n - 1 \), the inequality (2.12) holds. By Sobolev embedding
theorem, we have
\[
\langle A^n U_s, U_s^3 \rangle = \sum_{|\alpha|=n} C_\alpha \langle D^\alpha U^3, D^\alpha U \rangle 
\leq C(1 + \|U_s\|_\infty^2) \|U_s\|_n^2 \leq C(1 + \|U_s\|_1^2) \|U_s\|_n^2,
\]  
(2.17)

where \(\|U_s\|_\infty = \sup_{z \in \mathcal{T}} |U_s(z)|\).

By utilizing (2.17) and (2.14) (2.15), we rewrite (2.13) in the form
\[
E(n,t) = \int_0^t [ns^{n-1} \|U_s\|_n^2 - s^n \|U_s\|_{n+1}^2 + 2s^n \|U_s\|_n^2 + 2s^n \langle A^n U_s, -U_s^3 \rangle + \mathfrak{W}_n s^n] \, ds + M_t 
\leq \int_0^t C_{T,n} (1 + s \|U_s\|_n^2) s^{n-1} \|U_s\|_n^2 \, ds + C_n s^n + M_t - \frac{\gamma}{2} \langle M \rangle_t + \frac{\gamma}{2} \langle M \rangle_t 
\leq C_{T,n} (E(1,t) + 1) \int_0^t s^{n-1} \|U_s\|_n^2 \, ds + C_n s^n + M_t - \frac{\gamma}{2} \langle M \rangle_t + \gamma \cdot T \cdot E(t).
\]

Therefore,
\[
E(n,t) - C_{T,n} (1 + E(1,t)) E(n-1,t) - C_n s^n + M_t - \frac{\gamma}{2} \langle M \rangle_t.
\]

By the supermartingale inequality, we have
\[
\mathbb{P} \left( \sup_{t \in [0,T]} (E(n,t) - C_{T,n} E(1,t) E(n-1,t) - C_n s^n) \geq \rho \right) \leq e^{-\rho}.
\]

Since the inequality (2.12) holds for \(k \leq n - 1\), using similar arguments as that in Lemma 2.2, the inequality (2.12) holds for \(k = n\).

**Lemma 2.4.** For any \(n, m, T > 0\) and \(0 < s \leq T\), there exists a positive constant \(\lambda = \lambda_{n,m,T}\) such that
\[
E \sup_{t \in [s,T]} \|U_t\|_m^m \leq C_{n,m,T} (s^{-\lambda} + 1).
\]

**Proof.** By Lemma 2.3 and Lemma 2.1 one sees that for some \(\kappa, \gamma > 0\)
\[
E \sup_{t \in [s,T]} \|U_t\|_m \leq C_{n,m,T} s^{-nm} E(\|U_s\|_1^2 + 1) \leq C_{n,m,T} s^{-nm}(s^{-\gamma} + 1).
\]

By setting \(\lambda = nm + \gamma\), we complete the proof.

**Lemma 2.5.** For each \(\xi \in H\) and \(0 < s < t \leq T\), we have the following pathwise estimates
\[
\|J_{s,t} \xi\| \leq \|\xi\|, \quad \|K_{s,t} \xi\| \leq \|\xi\|.
\]
(2.18)
Moreover, for each \( \tau \leq T \) and \( p \geq 1 \), there exists \( C = C_{T,p} \) such that

\[
\mathbb{E} \sup_{s < t \in [\tau, T]} \| J_{s,t}^{(2)}(\xi, \xi') \|^p \leq C\|\xi\|^p\|\xi'\|^p \tag{2.19}
\]

**Proof.** By (2.22), for any \( \xi \in H \), we deduce that

\[
d\|J_{s,t}\xi\|^2 = -2\langle A J_{s,t}\xi, J_{s,t}\xi \rangle dt + 2\langle J_{s,t}\xi, J_{s,t}\xi \rangle dt - (6U_t^2 J_{s,t}\xi, J_{s,t}\xi) dt
\leq -2\|J_{s,t}\xi\|^2 dt + 2\|J_{s,t}\xi\|^2 dt
\]

which implies

\[
\frac{d}{dt}\|J_{s,t}\xi\|^2 \leq 0 \tag{2.20}
\]

and

\[
\|J_{s,t}\xi\|^2 + 2 \int_s^t \|J_{s,r}\xi\|^2 dr \leq \|\xi\|^2 e^{2(t-s)}. \tag{2.21}
\]

By (2.20), one arrives at the first part of (2.18). Moreover, the second part of (2.18) follows by duality. It remains to prove (2.19).

For any \( U_0, \xi, \xi' \in H \), the function \( \varrho_t := J_{s,t}^{(2)}(\xi, \xi') \in H \) is the solution of

\[
\partial_t \varrho_t + A \varrho_t - \varrho_t + 3U_t^2 \varrho_t + 6U_t J_{s,t}\xi J_{s,t}\xi' = 0, \quad \varrho(s) = 0.
\]

Then

\[
\partial_t \|\varrho_t\|_H^2 + 2\langle A \varrho_t, \varrho_t \rangle - 2\|\varrho_t\|^2 + \langle 3U_t^2 \varrho_t + 6U_t J_{s,t}\xi J_{s,t}\xi', \varrho_t \rangle = 0.
\]

By Young’s inequality and Sobolev embedding theorem, it yields

\[
\partial_t \|\varrho_t\|^2 \leq -3\langle U_t^2 \varrho_t, \varrho_t \rangle - \langle 6U_t J_{s,t}\xi J_{s,t}\xi', \varrho_t \rangle + 2\|\varrho_t\|^2
\leq 12 \int_T ((J_{s,t}\xi)(z))^2 (J_{s,t}\xi')(z) dz + 2\|\varrho_t\|^2
\leq 12\|J_{s,t}\xi\|_\infty^2 \|J_{s,t}\xi'\|^2 + 2\|\varrho_t\|^2
\leq 12\|J_{s,t}\xi\|_H^2 \|\xi'\|^2 + 2\|\varrho_t\|^2.
\]

Thus, by (2.21), we have

\[
\|J_{s,t}^{(2)}(\xi, \xi')\|^2 \leq C_T \int_s^t \|J_{s,r}\xi\|^2 dr \|\xi'\|^2 \leq C_T \|\xi\|^2 \|\xi'\|^2
\]

which completes the proof of (2.19).
Lemma 2.6. For any $p \geq 2, T \geq 0$, there exists $C = C_{p,T}$ such that

$$
E \sup_{t \in [T/2, T]} \| \partial_t K_{t,T} \xi \|_{H^{-2}}^p \leq C \| \xi \|^p
$$

Proof. Noting that $\rho_t^* = K_{t,T} \xi$ satisfies the following equation

$$
\partial_t \rho^* = A \rho^* + (\nabla N(U(t)))^* \rho^* = - (\nabla F(U(t)))^* \rho^*, \quad \rho^*(T) = \xi,
$$

and

$$
\| A \rho^* \|_{H^{-2}} \leq \| \rho^* \|,
$$

$$
\| (\nabla N(U(t)))^* \rho^* \|_{H^{-2}} \leq \sup_{\| \psi \|_{H^2} \leq 1} | \langle (\nabla N(U(t)))^* \rho^*, \psi \rangle | \leq \sup_{\| \psi \|_{H^2} \leq 1} | \langle \rho^*, (\nabla N(U(t)) \psi \rangle | \leq C \sup_{\| \psi \|_{H^2} \leq 1} \| \rho^* \| \cdot \| U^2(t) \| + 1 \| \psi \| \| ^1 \cdot \| \psi \| _{1},
$$

by Lemmas 2.1, 2.5 we finish the proof of this lemma.

For any $N \geq 1$, define

$$H_N := \text{span} \{ e_k : 0 < |k| \leq N \},$$

along with the associated projection operators

$$P_N : H \to H_N$$

the orthogonal projection onto $H_N$,

$$Q_N := I - P_N.$$ 

Lemma 2.7. For every $p \geq 1, T > 0, \delta > 0$, there exists $N_* = N_* (p, T, \delta)$ such that for any $N \geq N_*$ one has

$$
E \| Q_N \mathcal{J}_{0,T} \|_{L(H,H)}^p \leq \delta, \quad E \| \mathcal{J}_{0,T} Q_N \|_{L(H,H)}^p \leq \delta (\| U_0^p \|^2 + 1).
$$

(2.22)

Here $\| \cdot \|_{L(X,Y)}$ denotes the operator norm of linear map between the given Hilbert spaces $X$ and $Y$.

Proof. For any $m \geq 1$, by the Itô formula in [16, Theorem 7.7.5] and following similar arguments in the proof of [16, Proposition 2.4.12], it holds that

$$
t^m (A \mathcal{J}_{0,t} \xi, \mathcal{J}_{0,t} \xi) = \int_0^t \left[ m s^{m-1} \langle A \mathcal{J}_{0,s} \xi, \mathcal{J}_{0,s} \xi \rangle + 2 s^m (A J_{0,s} \partial_s \mathcal{J}_{0,s} \xi) \right] ds
$$

$$
= \int_0^t \left[ m s^{m-1} \langle A \mathcal{J}_{0,s} \xi, \mathcal{J}_{0,s} \xi \rangle + 2 s^m (A J_{0,s} - A \mathcal{J}_{0,s} \xi + \mathcal{J}_{0,s} \xi - 3 U^2_s \mathcal{J}_{0,s} \xi) \right] ds
$$
\[
\begin{align*}
&= \int_0^t \left[(2s^m + ms^{m-1})\|J_0,s\xi\|^2_1 - 2s^m\|J_0,s\xi\|^2_2 - 6s^m(A_J0,s\xi, U_s^2 J_0,s\xi) \right]ds \\
&\leq \int_0^t \left[(2s^m + ms^{m-1})\|J_0,s\xi\|^2_1 - 2s^m\|J_0,s\xi\|^2_2 + 6s^m\|J_0,s\xi\|^2_2\|U_s\|_\infty^2\|J_0,s\xi\| \right]ds \\
&\leq \int_0^t \left[(2s^m + ms^{m-1})\|J_0,s\xi\|^2_1 - 2s^m\|J_0,s\xi\|^2_2 + 6s^m\frac{1}{6}\|J_0,s\xi\|^2_2 + 6\|U_s\|_4^2\|J_0,s\xi\|^2 \right]ds \\
&\leq \int_0^t \left[(2s^m + ms^{m-1})\|J_0,s\xi\|^2_1 + 36s^m\|U_s\|_4^2\|J_0,s\xi\|^2 \right]ds \\
&\leq C_{T,m}\|\xi\|^2 + C_T \int_0^t s^m\|U_s\|_4^2\|\xi\|^2 ds,
\end{align*}
\]

where in the last inequality, we have used (2.21). By the above inequality and Lemma 2.4, there exists a \( m > 1 \) such that
\[
\mathbb{E}(t^m\|J_0,t\xi\|^2_1)^p \leq C_{T,m}\|\xi\|^2_{2p}, \quad \forall t \in [0,T].
\] (2.23)

Fix this \( m \). Noting (2.23) and \( \|Q_Nv\| \leq \frac{1}{N}\|v\|_1, \forall v \in V \), we get
\[
\mathbb{E}\|Q_NJ_0,T\xi\|^p \leq \frac{1}{N^p}\mathbb{E}\|J_0,T\xi\|^p_1 \leq \frac{1}{N^p}(\mathbb{E}\|J_0,T\xi\|^{2p}_1)^{1/2} \leq \frac{C_{T,m,p}\|\xi\|^p_1}{N^p \cdot T^{m/2}},
\]
which implies the first part of (2.22).

Now, we consider the second part of (2.22). For any \( \xi \in H \), let \( \tilde{\xi} = Q_N\xi, \xi_t = J_0,t\tilde{\xi} \). Then \( \xi_t \) satisfies the following equation.
\[
\begin{align*}
\begin{cases}
\partial_t \xi_t = -A\xi_t + \xi_t - 3U_t^2\xi_t \\
\xi_0 = \tilde{\xi}.
\end{cases}
\end{align*}
\] (2.24)

Denote \( \xi_t^h = Q_N\xi_t, \xi_t^l = P_N\xi_t \). One easily sees that
\[
\begin{align*}
\begin{cases}
\partial_t \xi_t^h = -A\xi_t^h + \xi_t^h - Q_N(3U_t^2\xi_t) \\
\xi_0^h = \tilde{\xi}.
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
\partial_t\|\xi_t^h\|^2 &= 2\langle \xi_t^h, \partial_t\xi_t \rangle = 2\langle \xi_t^h, -A\xi_t + \xi_t - 3U_t^2\xi_t \rangle \\
&\leq (2 - 2N^2)\|\xi_t^h\|^2 + \|\xi_t^h\|^2 + C\|J_0,t\tilde{\xi}\|^2\|U_t\|_\infty^2 \\
&\leq (3 - 2N^2)\|\xi_t^h\|^2 + C_T\|U_t\|_4^2\|\tilde{\xi}\|^2.
\end{align*}
\]

By Gronwall inequality, for any \( s \leq t \), we have
\[
\|\xi_t\|^2 \leq \|\xi_s^h\|^2e^{-(2N^2-3)(t-s)} + e^{-((2N^2-3)t} \int_s^t (C_Te^{(2N^2-3)r}\|U_r\|_4^2\|\tilde{\xi}\|^2)dr
\]

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\[ \leq \| \xi_s \|^2 e^{-(2N^2-3)(t-s)} + C_T \frac{1}{2N^2 - 3} \sup_{r \in [s,t]} \| U_r \|^2 \| \tilde{\xi} \|^2. \]

Thus, for any \( p \geq 2 \), it holds that

\[ \mathbb{E} \| \xi_t^h \|^p \leq C_{T,p} \mathbb{E} \| \xi_s \|^p e^{-(2N^2-3)(t-s)} + C_{T,p} \frac{1}{2N^2 - 3} \mathbb{E} \sup_{r \in [s,t]} \| U_r \|^p \| \tilde{\xi} \|^p. \]

In the above inequality, let \( s = \frac{t}{2} \). By Lemma 2.4, for some \( \gamma > 0 \), we have

\[ \mathbb{E} \| \xi_t^h \|^p \leq C_{T,p} \| \tilde{\xi} \|^p [e^{-\frac{N^2 t}{4}} + \frac{1}{2N^2 - 3} t^{-\gamma}] \]

which yields

\[ \mathbb{E} \| \xi_T^h \|^p \leq \frac{\delta}{2} \| \xi \|^p \]

for \( N \) big enough.

Now, we consider the estimate of \( \xi_T^l \). For any \( 0 \leq t \leq T \), one sees that \( \xi_T^l \) satisfies the following equation

\[
\begin{align*}
\partial_t \xi_T^l &= -A \xi_T^l + \xi_T^l - P_N(3U_T^2 \xi_T^l), \\
\xi_T^l|_{t=0} &= 0.
\end{align*}
\]

We claim that for any \( \delta > 0 \), there exists \( N_* = N_*(p,T,\delta) \) such that for any \( N \geq N_* \) one has

\[ \mathbb{E} \| \xi_T^l \|^p \leq \frac{\delta}{2} (\| U_T \|^2 + 1) \| \xi \|^p. \]

Once we have proved this, combining (2.27) with (2.26), we obtain the second part of (2.22).

Now we give a proof of (2.27). Obviously, we have

\[
\begin{align*}
\partial_t \| \xi_T^l \|^2 &= \langle \xi_T^l, \partial_t \xi_T^l \rangle \\
&\leq -\langle \xi_T^l, 3U_T^2 \xi_T^l \rangle = -\langle \xi_T^l, 3U_T^2 (\xi_T^l + \xi_T^h) \rangle \\
&\leq |\langle \xi_T^l, 3U_T^2 \xi_T^h \rangle| \leq 3\| U_T \|^2_\infty \| \xi_T^l \| \| \xi_T^h \|.
\end{align*}
\]

On the set \( \{ \| \xi_T^l \| \neq 0 \} \), we define

\[ \tau = \sup \left\{ t \in [0,T], \| \xi_T^l \| = 0 \right\}. \]

Hence, for \( t \in (\tau, T) \), by (2.28), it holds that

\[ \partial_t \| \xi_T^l \| = \partial_t \sqrt{\| \xi_T^l \|^2} = \frac{\partial_t \| \xi_T^l \|^2}{2 \sqrt{\| \xi_T^l \|^2}} \leq C \| U_T \|^2_\infty \| \xi_T^h \|. \]
\[ \| \xi_T \| \leq \| \xi_T \| + C \int_T^T \| U_s \|_\infty^2 \| \xi_h \| ds \leq C \int_T^T \| U_s \|_2^2 \| \xi_h \| ds. \]

which implies
\[
\mathbb{E}\| \xi_T \|^p \leq C_p \mathbb{E} \left( \int_0^T \| U_r \|_2^2 \| \xi_h \| dr \right)^p + C_p \mathbb{E} \left( \int_t^T \| U_r \|_2^2 \| \xi_h \| dr \right)^p
\leq C_p \mathbb{E} \left( \int_0^T \| U_r \|_1 \| \xi_h \| dr \right)^p + C_p \mathbb{E} \left( \int_t^T \| U_r \|_2^2 \| \xi_h \| dr \right)^p
:= I_1 + I_2,
\]

where \( t > 0 \) is a small parameter to be adjusted later. As for \( I_1 \), by (2.18), Lemmas 2.2, 2.5 and Hölder inequality, we have
\[
I_1 \leq C_{T,p} \mathbb{E} \left[ \sup_{s \in [0,t]} \| U_s \|^p \cdot \left( \int_0^t \| U_r \|_1 dr \right)^p \right] \cdot \| \xi \|^p
\leq C_{T,p} \left[ \mathbb{E} \sup_{s \in [0,t]} \| U_s \|^{2p} \right]^{1/2} \left[ \mathbb{E} \left( \int_0^t \| U_r \|_1 dr \right)^{2p} \right]^{1/2} \cdot \| \xi \|^p
\leq C_{T,p} \left[ \mathbb{E} \sup_{s \in [0,t]} \| U_s \|^{2p} \right]^{1/2} \left[ t^p \cdot \mathbb{E} \left( \int_0^t \| U_r \|_1^2 dr \right)^p \right]^{1/2} \cdot \| \xi \|^p
\leq C_{T,p} t^{p/2} \| U_0 \|^{2p} \| \xi \|^p.
\]

Setting \( t \) small enough, one arrives at that
\[
I_1 \leq \frac{\delta}{4} (1 + \| U_0 \|^{2p}) \| \xi \|^p. \tag{2.30}
\]

Fix this \( t \). Since
\[
I_2 \leq C_{T,p} \mathbb{E} \left( \sup_{r \in [t,T]} \| U_r \|_2^{4p} \cdot \left( \int_t^T \| \xi_h \| dr \right)^p \right)
\leq C_{T,p} \left( \mathbb{E} \sup_{r \in [t,T]} \| U_r \|_2^{4p} \right)^{1/2} \left( \mathbb{E} \left( \int_t^T \| \xi_h \|^2 dr \right)^p \right)^{1/2}
\leq C_{T,p} \left( \mathbb{E} \sup_{r \in [t,T]} \| U_r \|_2^{4p} \right)^{1/2} \left( \mathbb{E} \int_t^T \| \xi_h \|^{2p} dr \right)^{1/2},
\]

by (2.25) and Lemma 2.4, we can choose \( N \) big enough such that
\[
I_2 \leq \frac{\delta}{4} \| \xi \|^p. \tag{2.31}
\]

Combining (2.31), (2.30) and (2.29), we obtain the second part of (2.27).
Using the same method as [9, Lemmas A.6, A.7] and [11], by Lemma 2.5, the following two lemmas hold.

**Lemma 2.8.** For $0 < s < t$, we have

$$
\|A_{s,t}\|_{L(L^2([s,t], \mathbb{R}^m), H)} \leq C \left( \int_s^t \|\mathcal{J}_{r,t}\|_{L(H,H)}^2 dr \right)^{1/2}
$$

where $C$ is a constant independent of $s, t$. Moreover, for any $\beta > 0$, the following hold

$$
\|A^*_{s,t}(M_{s,t} + I)^{-1/2}\|_{L(L^2([s,t], \mathbb{R}^m), H)} \leq 1,
$$

$$
\|(M_{s,t} + I\beta)^{-1/2}A_{s,t}\|_{L(L^2([s,t], \mathbb{R}^m), H)} \leq 1,
$$

$$
\|(M_{s,t} + I\beta)^{-1/2}\|_{L(H,H)} \leq \beta^{-1/2},
$$

$$
\|(M_{s,t} + I\beta)^{-1}\|_{L(H,H)} \leq \beta^{-1}.
$$

Observe that for $\tau \leq t$

$$
\mathcal{D}_\tau^j \mathcal{J}_{s,t} \xi = \begin{cases} 
\mathcal{J}_{s,t}^{(2)}(G\theta_j, \mathcal{J}_{s,\tau} \xi) & \text{if } s \leq \tau, \\
\mathcal{J}_{s,t}^{(2)}(\mathcal{J}_{\tau,s} G\theta_j, \xi) & \text{if } s > \tau.
\end{cases}
$$

**Lemma 2.9.** For any $\xi \in H, 0 \leq s \leq t \leq T$ and $p \geq 1$ we have the bounds

$$
\mathbb{E}\|\mathcal{D}_\tau^j \mathcal{J}_{s,t} \xi\|^p \leq C\|\xi\|^p,
$$

$$
\mathbb{E}\|\mathcal{D}_\tau^j A_{s,t}\|_{L(L^2([s,t], \mathbb{R}^m), H)}^p \leq C,
$$

$$
\mathbb{E}\|\mathcal{D}_\tau^j A^*_{s,t}\|_{L(H, L^2([s,t], \mathbb{R}^m)))}^p \leq C,
$$

where $C = C_{T,p}$.

3 **Spectral properties of Malliavin matrix $\mathcal{M}$**

For any $\alpha > 0, N \in \mathbb{N}$, we define

$$
\mathcal{S}_{\alpha, N} := \{ \phi \in H : \|P_N \phi\|^2 \geq \alpha \|\phi\|^2 \}.
$$

The aim of this section is to prove the following result:

**Theorem 3.1.** For any $N \geq 1, \alpha \in (0, 1]$ and $T > 0$, there exists a positive constant $\varepsilon^* = \varepsilon^*(\alpha, N, T) > 0$, such that, for any $n \geq 0$, and $\varepsilon \in (0, \varepsilon^*)$, there exists a measurable set $\Omega_{\varepsilon} = \Omega_{\varepsilon}(\alpha, N, T) \subseteq \Omega$ satisfying

$$
\mathbb{P}(\Omega_{\varepsilon}) \leq r(\varepsilon),
$$

(3.1)
where \( r = r(\alpha, N, T) : (0, \varepsilon^*) \to (0, \infty) \) is a non-negative, decreasing function with \( \lim_{\varepsilon \to 0} r(\varepsilon) = 0 \), and on the set \( \Omega_\varepsilon \),

\[
\inf_{\phi \in S_{\alpha,N}} \frac{\langle M_{0,T} \phi, \phi \rangle}{\|\phi\|^2} \geq \varepsilon. \tag{3.2}
\]

In order to prove this theorem, we show the details of Lie bracket computations in subsection 3.1 demonstrate Proposition 3.1 in subsection 3.2 and Proposition 3.2 in subsection 3.3. Finally, give a proof the of Theorem 3.1 in subsection 3.4.

### 3.1 Details of Lie bracket computations

For any Fréchet differentiable \( E_1, E_2 : H \to H \),

\[
[E_1, E_2](u) := \nabla E_2(u) E_1(u) - \nabla E_1(u) E_2(u).
\]

This operator \([E_1, E_2]\) is referred as the Lie bracket of two “vector fields" \( E_1, E_2 \). For any \( k, \ell, j \in \mathbb{Z}, m, m', m'' \in \{0, 1\} \), by calculating, for any \( u = u(z) \in H \)

\[
I_k^m(u) := [F(u), \cos(kz + \frac{\pi}{2} m)] = A \cos(kz + \frac{\pi}{2} m) + 3u^2 \cos(kz + \frac{\pi}{2} m) - \cos(kz + \frac{\pi}{2} m),
\]

\[
J_{k,\ell}^{m,m'}(u) := -[[F(u), \cos(kz + \frac{\pi}{2} m)], \cos(\ell z + \frac{\pi}{2} m')] = 6u \cos(kz + \frac{\pi}{2} m) \cos(\ell z + \frac{\pi}{2} m').
\]

\[
K_{k,\ell,j}^{m,mm''}(u) := -[J_{k,\ell}^{m,m'}, \cos(jz + \frac{\pi}{2} m'')] = 6 \cos(kz + \frac{\pi}{2} m) \cos(\ell z + \frac{\pi}{2} m') \cos(jz + \frac{\pi}{2} m''). \tag{3.3}
\]

Therefore, for any \( k, \ell, j \in \mathbb{Z} \), we have

\[
\cos((k + \ell + j)z) = \sum_{m,m', m'' \in \{0,1\}} C_i^{m,m',mm''} K_i^{m,m',mm''}(u) \]

\[
\sin((k + \ell + j)z) = \sum_{m,m', m'' \in \{0,1\}} C_2^{m,m',mm''} K_2^{m,m',mm''}(u) \tag{3.4}
\]

where \( C_i^{m,m',mm''}, i = 1, 2 \) are some constants depending on \( k, \ell, j, m, m', m'' \).

### 3.2 Quadratic forms: lower bounds

Denote

\[
\langle Q_N \phi, \phi \rangle := \sum_{0 \leq |k| \leq N} |\langle \phi, e_k \rangle|^2
\]
One easily sees that the following Proposition holds.

**Proposition 3.1.** Fix any integer $N \in \mathbb{N}$, then for any $U \in H$ and $\alpha \in (0, 1]$, 

$$\langle Q_N \phi, \phi \rangle \geq \frac{\alpha}{2} \|\phi\|^2$$

holds for every $\phi \in S_{\alpha,N}$.

### 3.3 Quadratic forms: upper bounds

The aim of this subsection is to prove the following proposition.

**Proposition 3.2.** Fix $T > 0$, for any $N \geq 1, \alpha \in (0, 1]$, there are positive constant $q_1 = q_1(\alpha, N, T), q_2 = q_2(\alpha, N, T)$ such that the following holds. There exists a positive constant $\varepsilon^* = \varepsilon^*(\alpha, N, T) > 0$, such that, for any $\varepsilon \in (0, \varepsilon^*)$, there exist a measurable set $\Omega_\varepsilon = \Omega_\varepsilon(\alpha, N, T) \subseteq \Omega$ and positive constants $C_1 = C_1(\alpha, N, T), C_2 = C_2(\alpha, N, T)$ such that

$$P((\Omega_\varepsilon)^c) \leq C_1 \varepsilon^{q_1}$$

and on the set $\Omega_\varepsilon$ one has

$$\langle M_{0,T} \phi, \phi \rangle \leq \varepsilon \|\phi\|^2 \Rightarrow \langle Q_N \phi, \phi \rangle \leq C_2 \varepsilon^{q_2} \|\phi\|^2$$

which is valid for any $\phi \in S_{\alpha,N}$.

![Figure 1](image.png)

**Figure 1:** An illustration of the structure of the lemmas that leads to the proof of Proposition 3.2. In this figure, $m \in \{0, 1\}, \ell \in \mathbb{Z}_0$. The solid arrows indicate that if one term is “small” then the other one “small” on a set of large measure(displayed below or left of the arrow), where the meaning of “smallness” is made precisely in each lemma. The dashed arrow with color green shows that the process is iterative. The dotted arrow with color red signify that the new element is generated as a linear combination of elements from the previous actually.

In the Figure 1 we give an illustration of the structure of lemmas in this subsection that lead to a proof of Proposition 3.2.

**Lemma 3.1.** For any $0 < \varepsilon < \varepsilon_0(T, \mathcal{E}_0)$, there exist a set $\Omega_{\varepsilon,\mathcal{M}}$ and a constant $C = C_T$ with

$$P(\Omega_{\varepsilon,\mathcal{M}}) \leq C \varepsilon$$
such that on the set $\Omega_{\varepsilon,M}$

$$\langle M_0,T\phi,\phi \rangle \leq \varepsilon \|\phi\|^2$$

$$\Rightarrow \sup_{t \in [T/2,T]} |\langle K_t,T\phi,\varepsilon_t \rangle| \leq \varepsilon^{1/8} \|\phi\|$$

(3.5)

for each $\ell \in Z_0$ and $\phi \in H$.

Proof. Note that

$$\langle M_0,T\phi,\phi \rangle = \sum_{\ell \in Z_0} (\beta_\ell)^2 \int_0^T \langle e_\ell, K_r,T\phi \rangle^2 dr$$

Define the function $g_\phi(\cdot): [T/2, T] \to \mathbb{R}^+$,

$$g_\phi(t) := \sum_{\ell \in Z_0} (\beta_\ell)^2 \int_0^t \langle e_\ell, K_r,T\phi \rangle^2 dr$$

then

$$g'_\phi(t) = \sum_{\ell \in Z_0} (\beta_\ell)^2 \langle e_\ell, K_t,T\phi \rangle^2$$

$$g''_\phi(t) = 2 \sum_{\ell \in Z_0} \beta_\ell^2 \langle e_\ell, K_t,T\phi \rangle \langle e_\ell, \partial_t K_t,T\phi \rangle$$

Let

$$\Omega_{\varepsilon,M} = \bigcap_{\phi \in H, \|\phi\|=1} \left\{ \sup_{t \in [T/2,T]} |g_\phi(t)| \geq \varepsilon \text{ or } \sup_{t \in [T/2,T]} |g'_\phi(t)| \leq \varepsilon^{1/4} \right\}.$$

It is obvious that (3.5) holds on $\Omega_{\varepsilon,M}$. Setting $\alpha = 1$ in [9, Lemma 6.2], by Lemmas 2.5, 2.6 one arrives at that

$$\mathbb{P}(\Omega_{\varepsilon,M}) \leq \mathbb{P} \left( \bigcup_{\phi \in H, \|\phi\|=1} \left\{ \sup_{t \in [T/2,T]} |g_\phi(t)| \leq \varepsilon \text{ and } \sup_{t \in [T/2,T]} |g'_\phi(t)| \geq \varepsilon^{1/4} \right\} \right)$$

$$\leq C \varepsilon \sum_{\ell \in Z_0} (\beta_\ell)^4 E \left[ \sup_{t \in [T/2,T], \|\phi\|=1} |\langle e_\ell, K_t,T\phi \rangle \langle e_\ell, \partial_t K_t,T\phi \rangle|^2 \right]$$

$$\leq C \varepsilon,$$

which completes the proof of this lemma.

Lemma 3.2. Fix $k \in Z, m \in \{0,1\}$. For any $0 < \varepsilon < \varepsilon_0(T)$, there exist a set $\Omega_{\varepsilon,k}^{1,m}$ and $C = C_{k,m,T}$ with

$$\mathbb{P}(\Omega_{\varepsilon,k}^{1,m}) \leq C \varepsilon,$$
such that on the set $\Omega^{1,m}_{\varepsilon,k}$, it holds that for any $\phi \in H$

$$\sup_{t \in [T/2,T]} |\langle \mathcal{K}_t \phi, \cos(kx + \frac{\pi}{2}m) \rangle| \leq \varepsilon \|\phi\| \Rightarrow \sup_{t \in [T/2,T]} |\langle \mathcal{K}_t \phi, I_k^m(U_t) \rangle| \leq \varepsilon^{1/10} \|\phi\|. \quad (3.6)$$

**Proof.** Define $g_\phi(t) := \langle \mathcal{K}_t \phi, \cos(kx + \frac{\pi}{2}m) \rangle$, $\forall t \in [0,T]$. Observing (2.3), one has

$$
g_\phi'(t) = \langle \mathcal{K}_t \phi, [F(U_t), \cos(kx + \frac{\pi}{2}m)] \rangle = \langle \mathcal{K}_t \phi, I_k^m(U_t) \rangle. $$

Let $\alpha = \frac{1}{4}$ and define

$$\Omega^{1,m}_{\varepsilon,k} = \bigcap_{\phi \in H, \|\phi\|=1} \left\{ \sup_{t \in [T/2,T]} |g_\phi(t)| \geq \varepsilon \text{ or } \sup_{t \in [T/2,T]} |g_\phi'(t)| \leq \varepsilon^{\alpha/2(1+\alpha)} \right\}.$$  

Then on $\Omega^{1,m}_{\varepsilon,k}$, (3.6) holds. By [9, Lemma 6.2], it holds that

$$P\left((\Omega^{1,m}_{\varepsilon,k})^c\right) \leq \frac{\varepsilon}{10} \left( \sup_{\phi \in H, \|\phi\|=1} \left\{ \sup_{t \in [T/2,T]} |g_\phi(t)| \leq \varepsilon \text{ and } \sup_{t \in [T/2,T]} |g_\phi'(t)| \geq \varepsilon^{\alpha/2(1+\alpha)} \right\} \right) \leq C\varepsilon \mathbb{E}\left[ \sup_{\phi \in H, \|\phi\|=1} \|g_\phi^{(2)}\|_{C^0[T/2,T]}^{1/2} \right]. \quad (3.7)$$

Note that

$$g_\phi'(t) = \langle \mathcal{K}_t \phi, I_k^m(U_t) \rangle = \langle \mathcal{K}_t \phi, Af + 3U_t^2 f - f \rangle,$$

where $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function given by $f(z) = \cos(kz + \frac{\pi}{2}m)$. We have

$$
\|g_\phi\|_{C^0[T/2,T]} \
\leq C \sup_{t \in [T/2,T]} |\partial_t \langle \mathcal{K}_t \phi, A f \rangle| + C \|\langle \mathcal{K}_t \phi, U_t^2 f \rangle\|_{C^0[T/2,T]} + C \sup_{t \in [T/2,T]} |\partial_t \langle \mathcal{K}_t \phi, f \rangle| \\n\leq C \sup_{t \in [T/2,T]} |\partial_t \langle \mathcal{K}_t \phi, A f \rangle| + C \sup_{t \in [T/2,T]} \|\partial_t \mathcal{K}_t \phi\|_{H^{-2}} \sup_{t \in [T/2,T]} \|U_t^2 f\|_2 \\n+ C \sup_{t \in [T/2,T]} \|\mathcal{K}_t \phi\| \cdot \|U_t^2 f\|_{C^0([T/2,T],H)} + C \sup_{t \in [T/2,T]} |\partial_t \langle \mathcal{K}_t \phi, f \rangle|.
$$

By (1.1), Lemma 2.4 and

$$\frac{\|(U_{t_1}^2 - U_{t_2}^2) f\|}{|t_1 - t_2|^\alpha} \leq \left( \|U_{t_1} f\| + \|U_{t_2} f\| \right) \cdot \frac{\|U_{t_1} - U_{t_2}\|}{|t_1 - t_2|^\alpha} \leq \left( \|U_{t_1}\| + \|U_{t_2}\| \right) \cdot \frac{\|U_{t_1} - U_{t_2}\|}{|t_1 - t_2|^\alpha},$$

we obtain

$$\mathbb{E}\|U_t^2 f\|_{C^0([T/2,T],H)}^p \leq C_{T,p}, \quad \forall p \geq 1.$$
Therefore, by Lemmas 2.4, 2.5, 2.6 and the above equality, one arrives at that

\[ E \left[ \sup_{\phi: \|\phi\| = 1} \|g_\phi^\prime\|^{2/\alpha}_{C^{\alpha}[T/2, T]} \right] \leq C_T. \]

Combining the above inequality with (3.7), the proof is completed. \( \square \)

**Lemma 3.3.** Fix any \( k \in \mathbb{Z}, m \in \{0, 1\} \). For any \( 0 < \varepsilon < \varepsilon_0(T) \), there exist a set \( \Omega_{\varepsilon,k}^m \) and \( C = C_{T,k} \) with

\[ P((\Omega_{\varepsilon,k}^m)^c) \leq C \varepsilon^{1/27}, \]

(3.8)

such that on the set \( \Omega_{\varepsilon,k}^m \), it holds that for any \( \phi \in H \)

\[ \sup_{t \in [T/2, T]} |\langle K_{t,T} \phi, I_{k,k}^m(U_t) \rangle| \leq \varepsilon \|\phi\| \]

\[ \Rightarrow \sup_{\ell, j \in \mathbb{Z}_0} \sup_{t \in [T/2, T]} |\beta_\ell \beta_j| \cdot |\langle K_{t,T} \phi, e_\ell e_j \cos(kx + \frac{\pi}{2}m) \rangle| \leq \varepsilon^{1/9} \|\phi\|. \]

(3.9)

**Proof.** Denote \( L_t = U_0 + \int_t^{T} F(U_s) ds \), then we have

\[ I_{k,k}^m(U_t) = A \cos(kz + \frac{\pi}{2}m) - \cos(kz + \frac{\pi}{2}m) + 3U_t^2 \cos(kz + \frac{\pi}{2}m) \]

\[ = A \cos(kz + \frac{\pi}{2}m) - \cos(kz + \frac{\pi}{2}m) + 3 \left( L_t + \sum_{k \in \mathbb{Z}_0} \beta_k e_k W_k(t) \right)^2 \cos(kz + \frac{\pi}{2}m) \]

\[ = A \cos(kz + \frac{\pi}{2}m) - \cos(kz + \frac{\pi}{2}m) \]

\[ + 3 \left( L_t^2 + 2L_t \sum_{\ell \in \mathbb{Z}_0} \beta_\ell e_\ell W_\ell(t) + \sum_{\ell, j \in \mathbb{Z}_0} \beta_\ell \beta_j e_\ell e_j W_\ell(t) W_j(t) \right) \cos(kz + \frac{\pi}{2}m). \]

Therefore,

\[ \langle K_{t,T} \phi, I_{k,k}^m(U_t) \rangle \]

\[ = \langle K_{t,T} \phi, A \cos(kz + \frac{\pi}{2}m) - \cos(kz + \frac{\pi}{2}m) + 3L_t^2 \cos(kz + \frac{\pi}{2}m) \rangle \]

\[ + 6 \sum_{\ell \in \mathbb{Z}_0} \langle K_{t,T} \phi, L_t \beta_\ell e_\ell \cos(kz + \frac{\pi}{2}m) \rangle W_\ell(t) \]

\[ + 3 \sum_{\ell, j \in \mathbb{Z}_0} \langle K_{t,T} \phi, \beta_\ell \beta_j e_\ell e_j \cos(kz + \frac{\pi}{2}m) \rangle W_\ell(t) W_j(t) \]

\[ := A_0(t) + \sum_{\ell \in \mathbb{Z}_0} A_\ell W_\ell(t) + \sum_{\ell, j \in \mathbb{Z}_0} A_{\ell,j} W_\ell(t) W_j(t). \]
By Lemmas 2.4-2.6 for any $T, p > 0$, we have

$$
\mathbb{E}\left[ \sup_{s \neq t \in [T/2, T]} \left| \frac{A_0(t) - A_0(s)}{|t - s|} + \sum_{\ell \in \mathbb{Z}_0} \frac{|A_\ell(t) - A_\ell(s)|}{|t - s|} + \sum_{\ell, j \in \mathbb{Z}_0} \frac{|A_{\ell,j}(t) - A_{\ell,j}(s)|}{|t - s|} \right|^p \right] \leq C_{T, p}.
$$

(3.10)

Define

$$
\mathcal{N}_1(\phi) := \sup_{s \neq t \in [T/2, T]} \left| \frac{A_0(t) - A_0(s)}{|t - s|} + \sum_{\ell \in \mathbb{Z}_0} \frac{|A_\ell(t) - A_\ell(s)|}{|t - s|} + \sum_{\ell, j \in \mathbb{Z}_0} \frac{|A_{\ell,j}(t) - A_{\ell,j}(s)|}{|t - s|} \right|,
$$

$$
\mathcal{N}_0(\phi) := \sup_{s \neq t \in [T/2, T]} \left| A_0(t) + \sum_{\ell \in \mathbb{Z}_0} |A_\ell(t)| + \sum_{\ell, j \in \mathbb{Z}_0} |A_{\ell,j}(t)| \right|.
$$

By [9, Theorem 6.4], there exists a set $\Omega_\varepsilon^\#$ such that

$$
\mathbb{P}(\Omega_\varepsilon^\#^c) \leq C\varepsilon,
$$

(3.11)

and on $\Omega_\varepsilon^\#$, we have

$$
\sup_{t \in [T/2, T]} |\langle K_{t,T}\phi, I_k^m(U) \rangle| \leq \varepsilon \Rightarrow \begin{cases} 
\text{either } \mathcal{N}_0(\phi) \leq \varepsilon^{1/9}, \\
\text{or } \mathcal{N}_1(\phi) \geq \varepsilon^{-1/27}.
\end{cases}
$$

(3.12)

Therefore, we obtain

$$
\sup_{t \in [T/2, T]} |\langle K_{t,T}\phi, I_k^m(U) \rangle| \leq \varepsilon \Rightarrow \mathcal{N}_0(\phi) \leq \varepsilon^{1/9}
$$

(3.13)

on a set

$$
\Omega_{\varepsilon,k}^{2,m} := \Omega_\varepsilon^\# \cap \bigcap_{\phi \in H, \|\phi\|=1} \{ \mathcal{N}_1(\phi) < \varepsilon^{-1/27} \}.
$$

Combining (3.13) with the following fact

$$
|\langle K_{t,T}\phi, \beta_\ell e_\ell \beta_j e_j \cos(\kappa z + \frac{\pi}{2} m) \rangle| = |\beta_\ell \beta_j| \cdot |\langle K_{t,T}\phi, e_\ell e_j \cos(\kappa z + \frac{\pi}{2} m) \rangle|,
$$

one arrives at (3.9). The desired result (3.8) is implied by (3.10) and (3.11).

Lemma 3.4. For any $n \in \mathbb{N}$, and $q_n, C_n > 0$, there exist $p_{n+1}, q_{n+1}, C_{n+1} > 0$, a constant $C = C(n, T)$, and a set $\Omega_{\varepsilon,n}$ with

$$
\mathbb{P}(\Omega_{\varepsilon,n}^c) \leq C\varepsilon^{p_{n+1}},
$$

(3.14)
such that on the set $\Omega_{\varepsilon,n}$, it holds

$$
\sum_{k \in \mathbb{Z}^n} \sup_{t \in [T/2,T]} |\langle K_{t,T}\phi, e_k \rangle| \leq C_n \varepsilon^{q_n} \|\phi\|
$$

$$
\Rightarrow \sum_{k \in \mathbb{Z}^{n+1}} \sup_{t \in [T/2,T]} |\langle K_{t,T}\phi, e_k \rangle| \leq C_{n+1} \varepsilon^{q_{n+1}} \|\phi\|.
$$

**Proof.** For any $n \geq 0$, by Hypothesis 1.1 and the definition of $\mathbb{Z}_n$, one sees that

$$
\forall k \in \mathbb{Z}_n \Rightarrow -k \in \mathbb{Z}_n.
$$

Thus, on the set $\{\sum_{k \in \mathbb{Z}_n} \sup_{t \in [T/2,T]} |\langle K_{t,T}\phi, e_k \rangle| \leq C_n \varepsilon^{q_n} \|\phi\|\}$, it holds that

$$
\sup_{t \in [T/2,T], k \in \mathbb{Z}_n, m \in \{0,1\}} |\langle K_{t,T}\phi, \cos(kz + \frac{\pi}{2} m) \rangle| \leq C_n \varepsilon^{q_n} \|\phi\|.
$$

By Lemma 3.2, for any $k \in \mathbb{Z}_n, m \in \{0,1\}$, there exist a set $\Omega_{\varepsilon,k}^{1,m}, C = C_{k,m,T}$ and $p_n' > 0$ with

$$
\mathbb{P}(\Omega_{\varepsilon,k}^{1,m}) \leq C \varepsilon^{p_n'},
$$

such that on the set $\Omega_{\varepsilon,k}^{1,m}$, it holds that

$$
\sup_{t \in [T/2,T]} |\langle K_{t,T}\phi, \cos(kz + \frac{\pi}{2} m) \rangle| \leq C_n \varepsilon^{q_n} \|\phi\|
$$

$$
\Rightarrow \sup_{t \in [T/2,T]} |\langle K_{t,T}\phi, I_m^m(U_t) \rangle| \leq C_{n+1} \varepsilon^{q_{n+1}} \|\phi\|
$$

(3.16)

for some $C'_{n+1}, q'_{n+1} > 0$.

By Lemma 3.3, for any $k \in \mathbb{Z}_n, m \in \{0,1\}$, there exist $p_n'', C_{n+1}, q_{n+1}$ and a set $\Omega_{\varepsilon,k}^{2,m}$ such that on $\Omega_{\varepsilon,k}^{2,m}$,

$$
\sup_{t \in [T/2,T]} |\langle K_{t,T}\phi, I_k^m(U_t) \rangle| \leq C_{n+1} \varepsilon^{q_{n+1}} \|\phi\|
$$

$$
\Rightarrow \sup_{t \in [T/2,T], \ell, j \in \mathbb{Z}_0} |\langle K_{t,T}\phi, e_{\ell} e_j \cos(kz + \frac{\pi}{2} m) \rangle| \leq C_{n+1} \varepsilon^{q_{n+1}} \|\phi\|,
$$

(3.17)

and

$$
\mathbb{P}(\Omega_{\varepsilon,k}^{2,m}) \leq C \varepsilon^{p_n''}.
$$

(3.18)

Let

$$
\Omega_{\varepsilon,n} = \cap_{k \in \mathbb{Z}_n, m \in \{0,1\}} \Omega_{\varepsilon,k}^{1,m} \cap \Omega_{\varepsilon,k}^{2,m}.
$$

26
By (3.16) and (3.17), on the set $\Omega_{\varepsilon,n}$, it holds that

$$\sum_{k \in \mathbb{Z}, t \in [T/2, T]} \sup_{\ell, j \in \mathbb{Z}_0} |\langle K_{t,T} \phi, e_k \rangle| \leq C_n \varepsilon^{q_n} \|\phi\|$$

$$\Rightarrow \sup_{t \in [T/2, T]} \sup_{k \in \mathbb{Z}_0} \left| \langle K_{t,T} \phi, e_k \rangle \right| \leq C_{n+1} \varepsilon^{q_{n+1}} \|\phi\|$$

for some $C_{n+1}, q_{n+1} > 0$. Since (3.14) holds for $n = 0$, on the set $\Omega_{\varepsilon,n}$, it also holds that

$$\sum_{k \in \mathbb{Z}_0, t \in [T/2, T]} \left| \langle K_{t,T} \phi, e_k \rangle \right| \leq C_n \varepsilon^{q_n} \|\phi\|$$

$$\Rightarrow \sup_{t \in [T/2, T]} \sup_{k \in \mathbb{Z}_0} \left| \langle K_{t,T} \phi, e_k \rangle \right| \leq C_{n+1} \varepsilon^{q_{n+1}} \|\phi\|.$$ 

Therefore, based on (3.3) (3.4) and (3.15) (3.18), we complete the proof.

---

### 3.4 Proof of Theorem 3.1

The aim of this subsection is to give the proof of Theorem 3.1.

**Proof.** First, we recall the definition of $\Omega_{\varepsilon,M}$ in Lemma 3.1 and let $C_0 = 1, q_0 = \frac{1}{8}$. For any $n \in \mathbb{N}$, after we have defined the constant $C_n, q_n$, we define $p_{n+1}, q_{n+1}, C_{n+1}, \Omega_{\varepsilon,n}$ by Lemma 3.4.

Let

$$\Omega_{\varepsilon} = \Omega_{\varepsilon,M} \cap \bigcap_{n=1}^{\infty} \Omega_{\varepsilon,n}.$$ 

Noting $K_{t,T} \phi = \phi$ for $t = T$, by Lemmas 3.1 3.4 for some positive constants $p_N, q_N$, $C = C(T,N)$, we have

$$\mathbb{P}(\Omega_{\varepsilon})^c \leq C \varepsilon^{p_N},$$

and

$$\langle M_0, T \phi, \phi \rangle \leq \varepsilon \|\phi\|^2 \Rightarrow \langle Q_N \phi, \phi \rangle \leq C \varepsilon^{q_N} \|\phi\|^2,$$

which is valid on the set $\Omega_{\varepsilon}$ for any $\phi \in \mathcal{S}_{\alpha,N}$. The proof of Proposition 3.2 is finished. 

---
\[ \varepsilon \in (0, \varepsilon^*) \]

\[ \frac{\alpha}{2} > C_2 \varepsilon^{q_2}, \tag{3.19} \]

where, \( C_2, q_2 \) are the constants appeared in Proposition 3.2.

By Proposition 3.2 for some \( C_1, q_1 > 0 \), we have

\[ \mathbb{P}((\Omega_\varepsilon)^c) \leq C_1 \varepsilon^{q_1}. \]

On the set \( \Omega_\varepsilon \), for any \( \phi \in \mathcal{S}_{\alpha,N} \), if

\[ \langle M_{0,T} \phi, \phi \rangle < \varepsilon \| \phi \|^2, \]

by Proposition 3.1 and Proposition 3.2, we have

\[ \frac{\alpha}{2} \| \phi \|^2 \leq \langle Q \phi, \phi \rangle \leq C_2 \varepsilon^{q_2} \| \phi \|^2, \]

which contradicts with (3.19). Therefore, (3.2) holds on the set \( \Omega_\varepsilon \) and we complete the proof of Theorem 3.1.

Using Theorem 3.1, we obtain the following gradient estimate. The method to prove this Proposition is classical in this paper. One can see [9][11][12][13] etc.

**Proposition 3.3.** For some \( \gamma_0 > 0 \) and every \( \eta > 0, U_0 \in H \), the Markov semigroup \( \{P_t\}_{t \geq 0} \) defined by (1.9) satisfies the following estimate

\[ \| \nabla P_t \Phi(U_0) \| \leq C \left( \sqrt{P_t(||\Phi||^2)(U_0)} + e^{-\gamma_0 t} \sqrt{P_t(||\nabla \Phi||^2)(U_0)} \right) \]

for every \( t > 0 \) and \( \Phi \in C_b(H) \), where \( C \) is a constant independent of \( t, U_0 \) and \( \Phi \).

**Proof.** Our proof is very similar to that in [9] Section 3] except some little changes.

We build the control \( v \) and derive the associated \( \rho_t = \mathcal{J}_{0,t} \xi - \mathcal{A}_{0,t} v \) in (2.5) using the same iterative construction as that in [9]. Denote by \( v_{s,t} \) the control \( v \) restricted to the time interval \([s, t] \). Obviously, \( \rho_0 = \xi \) and \( \rho_t \) depends on \( \xi, t, v_{0,t} \). For each even non-negative integer \( n \in 2\mathbb{N} \), having determined \( v_{0,n} \) and \( \rho_n \), we set

\[ v_{n,n+1}(r) = (A_{n,n+1}^*(\mathcal{M}_{n,n+1} + I\beta)^{-1} \mathcal{J}_{n,n+1} \rho_n)(r), \quad v_{n+1,n+2}(r) = 0, \]

for \( r \in [n, n+2] \), where \( \beta = \beta(n) > 0 \) is to be determined in (3.22) below.

We define

\[ \mathcal{R}_{n,n+1}^{\beta} := \beta(\mathcal{M}_{n,n+1} + I\beta)^{-1}. \]
As that in [9], we split \( \rho_{n+2} = \rho_{n+2}^H + \rho_{n+2}^L \), where
\[
\rho_{n+2}^H = J_{n+1,n+2} P_N \mathcal{R}_{n,n+1}^\beta J_{n,n+1} \rho_n, \quad \rho_{n+2}^L = J_{n+1,n+2} P_N \mathcal{R}_{n,n+1}^\beta J_{n,n+1} \rho_n.
\] (3.20)

By (2.6), for some absolute constant \( C_0 > 1 \), we have
\[
\mathbb{E}(1 + \|U_{n+1}^8\|_2^2) |\mathcal{F}_n| \leq C_0.
\]

Set \( \delta = \frac{1}{2C_0} \). By the above inequality, Lemma 2.8 and (2.18) (2.22), one sees that
\[
\mathbb{E}(\|\rho_{n+2}^H\|_8^8 |\mathcal{F}_n) \leq \|\rho_n\|_8^8 \mathbb{E}(\|J_{n+1,n+2} Q N\|_n^8 |\mathcal{F}_{n+1}^8) \cdot \|J_{n,n+1}\|_8^8 |\mathcal{F}_n)
\leq \|\rho_n\|_8^8 \mathbb{E}(\delta (1 + \|U_{n+1}^8\|_2^2)) \cdot \|J_{n,n+1}\|_8^8 |\mathcal{F}_n)
\leq C_0 \delta \|\rho_n\|_8^8
\] (3.21)

for appropriate \( N = N(\delta) \). Fix such an \( N \) in (3.20). Following the lines in the [9, Lemma 3.1], noting Lemmas 2.5-2.9 and Theorem 3.1, there exists \( \beta = \beta(n) > 0 \) such that
\[
\mathbb{E}(\|\rho_{n+2}^L\|_8^8 |\mathcal{F}_n) \leq \delta \|\rho_n\|_8^8.
\] (3.22)

By (3.21) and (3.22), we have
\[
\mathbb{E}(\|\rho_{n+2}\|_8^8 |\mathcal{F}_n) \leq 2^7 \mathbb{E}(\|\rho_{n+2}^L\|_8^8 + \|\rho_{n+2}^H\|_8^8 |\mathcal{F}_n) \leq 2^7 \cdot 2C_0 \delta \|\rho_n\|_8^8 = \frac{1}{2} \|\rho_n\|_8^8
\]

which implies that, for any even non-negative integer \( n \), we have
\[
\mathbb{E}(\|\rho_n\|_8^8 \leq 2^{-n/2} \|\xi\|_8^8.
\] (3.23)

Based on (3.23), the estimates in Section 2, following the lines in [9, Section 3], we deduce that
\[
\sup_{\|\xi\|_8^8 = 1, t \geq 0} \mathbb{E} \left| \int_0^t v \cdot dW \right| \leq C
\]
and for some \( \gamma_0 > 0 \),
\[
\sup_{\|\xi\|_8^8 = 1} \mathbb{E} \|\rho_t\|_2^2 \leq C e^{-\gamma_0 t}.
\]

By [9, Section 3.1], we complete our proof.

\[ \square \]
4 Proof of Theorem 1.1

Let $H$ be a Banach space. Recall that
\[
d(x, y) = 1 \land \delta^{-1} \|x - y\|, \quad \forall x, y \in H,
\]
where $\delta$ is a small parameter to be adjusted later on. On the set
\[
Pr_1(H) := \left\{ \mu \in Pr(H) : \int_H d(0, u) d\mu(u) < \infty \right\},
\]
the metric $d$ induces a Wasserstein-Kantorovich distance defined by
\[
d(\mu_1, \mu_2) = \sup_{\|\Phi\|_d \leq 1} \left| \int_H \Phi(x) \mu(dx) - \int_H \Phi(x) \nu(dx) \right|
\]
where $\|\Phi\|_d$ denotes the Lipschitz constant of $\Phi$ in the metric $d$.

We recall the following abstract results. Then, we give a proof of Theorem 1.1.

**Theorem 4.1.** (See [12, Theorem 2.5].) Let $(P_t)_{t \geq 0}$ be a Markov semigroup over a Banach space $H$ satisfying

1. there exist constants $\alpha \in (0, 1)$, $C > 0$ and $T_1 > 0$ such that
\[
\|DP_t \Phi\|_\infty \leq C \|\Phi\|_\infty + \alpha_1 \|D\Phi\|_\infty, \quad (4.1)
\]
   for every $t \geq T_1$ and every Fréchet differentiable function $\Phi : H \to \mathbb{R}$;

2. for every $\delta > 0$, there exists a $T_2 = T_2(\delta)$ so that for any $t > T_2$ there exists an $a > 0$ so that
\[
\sup_{\Gamma \in C(P^*_t \delta_{U_0}, P^*_t \delta_{\tilde{U}_0})} \Gamma\{(U', U'') \in H \times H : \|U' - U''\| < \delta\} \geq a, \quad (4.2)
\]
   for every $U_0, \tilde{U}_0 \in H$. Here $\delta_U$ is the dirac measure concentrated at $U$, the operator $P^*_t$ is defined by (1.10) and $C(\mu_1, \mu_2)$ denotes the set of all measures $\pi$ on $H \times H$ such that $\pi(A \times H) = \mu_1(A)$ and $\pi(H \times A) = \mu_2(A)$ for every Borel set $A \subset H$.

Then, there exist constants $\delta > 0, \alpha < 1$ and $T > 0$ such that
\[
d(P^*_t \mu_1, P^*_t \mu_2) \leq \alpha d(\mu_1, \mu_2) \quad (4.3)
\]
for every pair of probability measures $\mu_1, \mu_2$ on $H$. In particular, $(P_t)_{t \geq 0}$ has a unique invariant measure $\mu_*$ and its transition probabilities converge exponentially fast to $\mu_*$.

**Theorem 4.2.** (See [14, Theorem 2.1].) Let $(P_t)_{t \geq 0}$ be a Feller Markov semigroup on a metric space $(H, d)$ with the continuity property: $\lim_{t \to 0} P_t \Phi(U_0) = \Phi(U_0)$ for all $\Phi \in C_b(H), U_0 \in$
Let $P_t(U_0, A)$ be the associated transition functions. Suppose that $(P_t)_{t \geq 0}$ satisfy

1) for some $C, \gamma > 0$ and every $\mu_1, \mu_2 \in Pr_1(H)$,

$$d(P^*_t \mu_1, P^*_t \mu_2) \leq Ce^{-\gamma t}d(\mu_1, \mu_2),$$

(4.4)

2) for every $R > 0$

$$\sup_{t \geq 0} \sup_{U_0 \in B_R} \int_H |d(0, U)|^3 P_t(U_0, dU) < \infty,$$

(4.5)

where $B_R := \{U_0 \in H, d(0, U_0) < R\}$.

Then, there exists a unique invariant probability measure $\mu_* \in Pr_1(H)$ such that for any $\Phi \in C^1(H)$ and any $U_0 \in H$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi(U(t, U_0)) dt = \int_H \Phi(\bar{U}) d\mu_*(\bar{U}) =: m_\Phi \text{ in probability.}$$

Moreover, the limit $\sigma^2 = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left( \int_0^T (\Phi(U(t, U_0)) - m_\Phi)dt \right)^2$ exists and

$$\lim_{T \to \infty} \mathbb{P} \left( \frac{1}{\sqrt{T}} \int_0^T (\Phi(U(t, U_0)) - m_\Phi) dt < \xi \right) = \chi_\sigma(\xi),$$

where $\chi_\sigma$ is the distribution function of a normal random variable with zero mean and variance $\sigma^2$.

We are now in a position to give a proof of Theorem 1.1

Proof. Recall that $U_t = U(t, U_0)$ is the solution of (1.1). For any $t > 0, U_0 \in H$ and $E \in B(H)$, $P_t(U_0, E), P_t$ and $P^*_t$ are defined by (1.8)-(1.10) respectively.

We divide our proof into two parts (a) and (b). In the first part (a), we use Theorem 4.1 to prove (1.11). In the second part (b), we use Theorem 4.2 to give a proof of (1.12) and (1.13).

(a) First, by Proposition 3.3 (4.1) holds. For any $r > 0$, we use $B_r$ to denote $\{U' \in H, \|U'\| \leq r\}$. Following the same way as that in [9 Page 2489], one arrives at that for any $\varepsilon, \delta > 0$ there exists $T_\varepsilon = T_\varepsilon(\varepsilon, \delta) \geq 0$ such that

$$\inf_{\|U\| \leq 2} P_T(U, B_\delta) > 0,$$

(4.6)

for any $T > T_\varepsilon$.

By Lemma 2.1, there exist positive constants $C_1$ and $\gamma$ such that for any $\varepsilon, \delta > 0$ and
$T > t = 1$, we have

\[ P_{T_1} (U_0, B_{\frac{1}{2}}) = \int_H P_t (U_0, dU) P_{T-t} (U, B_{\frac{1}{2}}) \]

\[ \geq \int_{B_{\frac{1}{2}}} P_t (U_0, dU) \inf_{U \in B_{\frac{1}{2}}} P_{T-t} (U, B_{\frac{1}{2}}) \geq (1 - \frac{E ||U||}{3}) \inf_{U \in B_{\frac{1}{2}}} P_{T-t} (U, B_{\frac{1}{2}}) \]

\[ \geq (1 - \frac{C_1 (1 + t^{-\gamma})}{3}) \inf_{U \in B_{\frac{1}{2}}} P_{T-t} (U, B_{\frac{1}{2}}) \geq (1 - \frac{2C_1}{3}) \inf_{U \in B_{\frac{1}{2}}} P_{T-t} (U, B_{\frac{1}{2}}), \quad (4.7) \]

where $U_t$ is the solution to equation (1.1) with initial value $U_0$. In the above inequality, we set $\frac{1}{2} = 4C_1$. By (4.6), there exists $T^* = T^* (\frac{1}{2}, \delta)$ such that for any $T > T^*$,

\[ \inf_{||U|| < 2} P_{T-t} (U, B_{\frac{1}{2}}) > 0. \]

Combining the above inequality with (4.7), noting $\frac{1}{2} = 4C_1$, one arrives at that

\[ \inf_{U_0 \in H} P_T (U_0, B_{\frac{1}{2}}) > 0 \quad (4.8) \]

for $T \geq T^*$.

For any $U_0, \tilde{U}_0 \in H$ and $T > 0$, we define $\tilde{\Gamma}_{U_0, \tilde{U}_0} \in Pr (H \times H)$ by

\[ \tilde{\Gamma}_{U_0, \tilde{U}_0} (A_1 \times A_2) := P_T (U_0, A_1) P_T (\tilde{U}_0, A_2) \quad \text{for any } A_1, A_2 \in B(H). \]

Then, by (4.8), we have

\[ \sup_{\Gamma \in C(P_T^\ast \delta_{U_0}, P_T^\ast \delta_{\tilde{U}_0})} \Gamma \{(U', U'') \in H \times H : ||U' - U''|| < \delta\} \]

\[ \geq \tilde{\Gamma}_{U_0, \tilde{U}_0} \{(U', U'') \in H \times H : ||U' - U''|| < \delta\} \]

\[ \geq P_T (U_0, B_{\frac{1}{2}}) \cdot P_T (\tilde{U}_0, B_{\frac{1}{2}}) \geq \left( \inf_{U_0 \in H} P_T (U_0, B_{\frac{1}{2}}) \right)^2 \]

which yields (4.2).

By Theorem 4.1, for some $\alpha < 1, T > 0$ and every pair of probability measures $\mu_1, \mu_2$ on $H$, we have

\[ d(P_T^\ast \mu_1, P_T^\ast \mu_2) \leq \alpha d(\mu_1, \mu_2). \]

Therefore, for some $C, \gamma > 0$ and every $\mu_1, \mu_2 \in Pr_1 (H)$, we have

\[ d(P_T^\ast \mu_1, P_T^\ast \mu_2) \leq C e^{-\gamma t} d(\mu_1, \mu_2). \quad (4.9) \]

Also by Theorem 4.1, $(P_t)_{t>0}$ has a unique invariant measure $\mu_\ast$. 

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In (4.9), letting \( \mu_1 = P_t^* \delta U_0 \) and \( \mu_2 = \mu^* \), one sees that
\[
d(P_t^* \delta U_0, P_t^* \mu^*) \leq Ce^{-\gamma t} d(\delta U_0, \mu^*),
\]
which implies
\[
\sup_{\|\Phi\|_d \leq 1} \left| P_t \Phi(U_0) - \int_H \Phi(z) \mu^*(dz) \right| \leq Ce^{-\gamma t}.
\]
We complete the proof of (1.11).

(b) By Itô formula and (1.1), for any \( \eta > 0 \), it gives that
\[
\eta \|U_t\|^2 - \eta \|U_0\|^2 + 2\eta \int_0^t \|U_s\|^2 ds
\]
\[
= \eta \mathcal{E}_0 t + 2\eta \int_0^t \langle U_s, GdW_s \rangle + 2\eta \int_0^t \langle U_s, U_s \rangle ds - 2\eta \int_0^t \langle U_s, U_s^3 \rangle ds
\]
\[
\leq \eta (\mathcal{E}_0 + 4\pi) t + 2\eta \int_0^t \|U_s, GdW_s \| - 2\eta \int_0^t \|U_s(z)\|^2 ds.
\]
Let \( \bar{U}(t) = \eta \|U_t\|^2, \bar{Z}(t) = \eta \|U_t\|^2 + \eta \|U_t\|^2 \), then we have
\[
\eta (\mathcal{E}_0 + 4\pi) - 2\eta \|U_s\|^2 - 2\eta \|U_s\|^2 \leq \eta (\mathcal{E}_0 + 4\pi) - 2\bar{Z}(t),
\]
\[
4\eta^2 \|U_s, G\|^2 \leq 4\eta \mathcal{E}_0 \bar{Z}(t).
\]
By [12, lemma 5.1], there exists \( \eta^* > 0 \), such that for any \( \eta \in (0, \eta^*) \)
\[
\mathbb{E} \left[ \exp \left\{ \eta \|U_t\|^2_H + \frac{1}{2} e^{-t/2} \int_0^t \eta \|U_s\|^2 ds \right\} \right] \leq C(\eta, \mathcal{E}_0) \exp \{\eta \|U(0)\|^2 e^{-t} \},
\]
which yields (4.5).

The Feller property and stochastic continuity of \( P_t \) follow immediately from the well-posedness properties of (1.1) as recalled in Proposition 2.1.

Therefore, by (4.9) and the arguments above, the conditions of Theorem 4.2 hold for \( P_t \) and we finish the proof of (1.12) and (1.13).

\[\Box\]

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