ON PSEUDO $W_4$-SYMMETRIC MANIFOLDS

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Abstract. In this paper, we investigate several properties on a weakly symmetric structure on a Riemannian manifold.

1. Introduction

In [9], Pokhariyal defined some curvature tensors with the help of Weyl’s projective curvature tensor and studied their physical and geometrical properties. One of the curvature tensors introduced in [9] was the $W_4$-curvature tensor defined by

$$W_4(X, Y, Z, T) = R(X, Y, Z, T) + \frac{1}{n-1}(g(X, Z)r(Y, T) - g(X, Y)r(Z, T))$$

or in local coordinates,

$$W_{4ijkl} = R_{ijkl} + \frac{1}{n-1}(g_{ik}r_{jl} - g_{ij}r_{kl}),$$

where $R$ and $r$ are the Riemannian curvature tensor and the Ricci tensor, respectively. In [7], the author introduced a Riemannian manifold whose $W_4$-curvature tensor is second order recurrent, and studied the several properties of such a manifold on which some geometric conditions are imposed. In [2], Chalki introduced a type of Riemannian manifold $(M^n, g)$ whose curvature tensor $R_{ijkl}$ of type $(0, 4)$ satisfies the relation

$$R_{ijkl;m} = 2A_mR_{ijkl} + A_iR_{mjk} + A_jR_{imk} + A_kR_{ijm} + A_lR_{ijkm},$$

where $A$ is a nonzero 1-form and the semicolon denotes the covariant differentiation with respect to the metric tensor $g$. Such a manifold is called a pseudo symmetric manifold. This manifold has received a great deal of attention and is studied in considerable detail by many authors [2,3,4,5,6,8]. Motivated by the above studies, in the present
paper, we introduce a weakly symmetric structure on a Riemannian manifold called a pseudo $W_4$-symmetric manifold. More precisely, a Riemannian manifold $(M^n, g)$ is said to be pseudo $W_4$-symmetric if its $W_4$-curvature tensor $W_{4ijkl}$ of type $(0,4)$ fulfills the following condition:

$$W_{4ijkl;m} = 2A_m W_{4ijkl} + A_j W_{4imkl} + A_k W_{4ijml} + A_l W_{4ijkl},$$

where $A$ is an associated 1-form which is not zero. In particular, if the $W_4$-curvature tensor $W_{4ijkl}$ of $(M^n, g)$ satisfies

$$W_{4ijkl;m} = 0,$$

then we call the manifold a $W_4$-symmetric manifold. The purpose of this paper is to investigate the various properties of pseudo $W_4$-symmetric manifold on which some geometric conditions are imposed.

2. Some properties of $W_4$-curvature tensor

At first we show

**Theorem 2.1.** Let $(M^n, g)$ be a Riemannian manifold. If the second Bianchi identity for $W_4$-curvature tensor holds, then the Ricci tensor $r$ of $(M^n, g)$ is cyclic, i.e., $r_{kl;m} + r_{mk;l} + r_{lm;k} = 0$.

**Proof.** By virtue of the second Bianchi identity and (1), we have

$$W_{4ijkl;m} + W_{4imkl;j} + W_{4ijml;k}$$

(3) = $(g_{ik} r_{jl;m} - g_{ij} r_{kl;m}) + (g_{lm} r_{jk;i} - g_{ij} r_{mk;l}) + (g_{il} r_{jm;k} - g_{ij} r_{lm;k}).$

By the given condition, the second Bianchi identity for $W_4$-curvature, (3) reduces to

(4) 0 = $(g_{ik} r_{jl;m} - g_{ij} r_{kl;m}) + (g_{lm} r_{jk;i} - g_{ij} r_{mk;l}) + (g_{il} r_{jm;k} - g_{ij} r_{lm;k}).$

Multiplying (4) by $g^{ij}$, we get

$$1 - n) (r_{kl;m} + r_{mk;l} + r_{lm;k}) = 0,$$

showing that the Ricci tensor is cyclic. This completes the proof. □

Consequently we also obtain

**Theorem 2.2.** Let $(M^n, g)$ be a Riemannian manifold. If the second Bianchi identity for $W_4$-curvature tensor holds, then the scalar curvature $s$ of $(M^n, g)$ is constant.
Proof. According to Theorem 2.1, we have
\[ r_{kl;m} + r_{mk;l} + r_{lm;k} = 0. \]  
(5)  
Multiplying (5) by \( g^{kl} \), we obtain
\[ s_{;m} + r_{m;l}^l + r_{m;k}^k = 0, \]
which reduces to
\[ 2s_{;m} = 0 \]  
(6)
because the second Bianchi identity implies \( r_{m;l}^l = \frac{s_{;m}}{2} \). Therefore the relation (6) yields
\[ s = \text{constant}. \]
This completes the proof.  

A Riemannian manifold \((M^n, g)\) is said to be \(W_4\)-harmonic if its \(W_4\)-curvature tensor is harmonic, i.e.,
\[ W_{ijkl;m}^m = 0. \]
(7)
Concerning \(W_4\)-harmonic manifold, we have

**Theorem 2.3.** Let \((M^n, g)\) be a \(W_4\)-harmonic manifold. Then its scalar curvature \(s\) is constant.

Proof. From (1) and (7), it follows that
\[ 0 = R_{ijkl;m}^m + \frac{1}{n-1} (\delta_k^m r_{jl;m} - \delta_j^m r_{kl;m}) \]  
(8)
Since the second Bianchi identity implies
\[ R_{ijkl;m}^m = r_{jl;k} - r_{jk;l} \]
we have from (8)
\[ r_{jk;l} - r_{jl;k} = \frac{1}{n-1} (\delta_k^m r_{jl;m} - \delta_j^m r_{kl;m}) \]  
(9)
Multiplying (10) by \( g^{kl} \), we get
\[ 0 = \frac{1}{n-1} (r_{jl;m}^m - s_{;j}) \]  
(11)
Using the relation \( r_{jl;m}^m = \frac{s_{;j}}{2} \) obtained from the second Bianchi identity, we have from (11)
\[ s_{;j} = 0, \]
showing that the manifold has
\[ s = \text{constant}. \]
This completes the proof.

3. Pseudo $W_4$-symmetric manifolds

Let $(M^n, g)$ be a Riemannian manifold. A vector field $A^\sharp$ is said to be an associated vector field of 1-form $A$ if it satisfies the relation

$$g(X, A^\sharp) = A(X)$$

for each vector field $X$ on $M^n$. Concerning pseudo $W_4$-symmetric manifold, we obtain

**Theorem 3.1.** Let $(M^n, g)$ be a pseudo $W_4$-symmetric manifold. If its scalar curvature $s$ is constant, then

$$r(X, A^\sharp) = -\frac{s}{2} g(X, A^\sharp),$$

where $A^\sharp$ is the associated vector field of 1-form $A$ in (2).

**Proof.** Multiplying (2) by $g^{il}$ and then multiplying the relation obtained thus by $g_{jk}$, we have

$$s_m = 2A_m s + g^{il} A_i r_{ml} + g^{jk} A_j r_{mk} + g^{jk} A_k r_{jm} + g^{il} A_l r_{im}$$

or equivalently

$$\nabla_X s = 2A(X)s + 4r(X, A^\sharp),$$

where $\nabla$ denotes the covariant derivative with respect to the metric tensor $g$. By virtue of $s=$constant, the last relation reduces to

$$r(X, A^\sharp) = -\frac{1}{2} A(X)s = -\frac{1}{2} s g(X, A^\sharp).$$

This completes the proof.

A Riemannian manifold $(M^n, g)$ is said to be Einstein if its Ricci tensor is proportional to the metric tensor $g$, that is,

$$r = \frac{s}{n} g.$$

Note that the scalar curvature $s$ of an Einstein manifold is constant [1].

As a consequence, we obtain

**Theorem 3.2.** Let $(M^n, g)$ be a pseudo $W_4$-symmetric manifold. If the manifold is Einstein, then its Ricci tensor vanishes identically.
Proof. Since the Einstein condition implies \( s = \text{constant} \), we have from Theorem 3.1

\[
(12) \quad r(X, A^z) = -\frac{1}{2} s g(X, A^z).
\]

On the other hand, according to the Einstein condition \( r = \frac{s}{n} g \), we obtain

\[
(13) \quad r(X, A^z) = \frac{s}{n} g(X, A^z).
\]

From (12) and (13), it follows that

\[
(14) \quad -\frac{s}{2} g(X, A^z) = \frac{s}{n} g(X, A^z).
\]

Substitute \( X = A^z \) into (14), then we get from \( ||A|| \neq 0 \)

\[
-\frac{s}{2} = \frac{s}{n} = 0,
\]

showing that the Einstein condition \( r = \frac{s}{n} g \) implies

\[
r = 0.
\]

This completes the proof. \( \square \)

Let \((M^n, g)\) be a Riemannian product manifold \((M^p \times M^{n-p}, \hat{g} + \tilde{g})\). In local coordinates, we adopt the Latin indices (resp. the Greek indices) for tensor components which are constructed on \((M^p, \hat{g})\) (resp. \((M^{n-p}, \tilde{g})\)). Therefore, the Latin indices take the values from 1, \ldots, \( p \) whereas the Greek indices run over the range \( p + 1, \ldots, n \). Now we can state the followings.

**Theorem 3.3.** Let a Riemannian manifold

\[
(M^n, g) = (M^p \times M^{n-p}, \hat{g} + \tilde{g})
\]

be a pseudo \( W_4 \)-symmetric manifold. Then either one decomposition manifold \((M^p, \hat{g})\) is flat or the other decomposition manifold \((M^{n-p}, \tilde{g})\) is \( W_4 \)-symmetric.

Proof. Since any tensor components of \( W_4 \) and its covariant derivatives with both Latin and Greek indices together should be zero, we have from \( W_{4ijkl,\alpha} = 0 \) and (2)

\[
0 = 2A_\alpha W_{4ijkl},
\]
which leads to either \( A = 0 \) on \((M^{n-p}, \tilde{g})\) or \( W_4 = 0 \) on \((M^p, \hat{g})\). In case of \( A = 0 \) on \((M^{n-p}, \tilde{g})\), we obtain from (2)

\[
W_{4\alpha\beta\gamma\delta;\mu} = 2A_\mu W_{4\alpha\beta\gamma\delta} + A_\alpha W_{4\mu\beta\gamma\delta} + A_\beta W_{4\alpha\mu\gamma\delta} + A_\gamma W_{4\alpha\beta\mu\delta} + A_\delta W_{4\alpha\beta\gamma\mu} = 0,
\]

showing that the decomposition manifold \((M^{n-p}, \tilde{g})\) is \( W_4 \)-symmetric. The other case \( W_4 = 0 \) on \((M^p, \hat{g})\) tells us from (1) that

\[
0 = R_{ijkl} + \frac{1}{n-1}(g_{ik}r_{jl} - g_{ij}r_{kl}).
\]

Multiplying (15) by \( g^{il} \), we get

\[
0 = r_{jk}.
\]

From (15) and (16), it follows that

\[
R_{ijkl} = 0,
\]

showing that the decomposition manifold \((M^p, \hat{g})\) is flat. This completes the proof.

\[\square\]

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