Existence of mild solutions for a system of partial differential equations with time-dependent generators

Amanda del Carmen Andrade-González
Universidad Autónoma de Aguascalientes
Departamento de Matemáticas y Física
Aguascalientes, Aguascalientes, Mexico.
acandra@correo.uaa.mx

José Villa-Morales
Universidad Autónoma de Aguascalientes
Departamento de Matemáticas y Física
Aguascalientes, Aguascalientes, Mexico.
jvilla@correo.uaa.mx

Abstract

We give sufficient conditions for global existence of positive mild solutions for the weak coupled system:

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \rho_1 t^{\rho_1 - 1} \Delta_{\alpha_1} u_1 + t^{\sigma_1} u_1^{\beta_1}, \quad u_1 (0) = \varphi_1, \\
\frac{\partial u_2}{\partial t} &= \rho_2 t^{\rho_2 - 1} \Delta_{\alpha_2} u_2 + t^{\sigma_2} u_2^{\beta_2}, \quad u_2 (0) = \varphi_2,
\end{align*}
\]

where \(\Delta_{\alpha_i}\) is a fractional Laplacian, \(0 < \alpha_i \leq 2\), \(\beta_i > 1\), \(\rho_i > 0\), \(\sigma_i > -1\) are constants and the initial data \(\varphi_i\) are positive, bounded and integrable functions.

Mathematics Subject Classification (2010). Primary 35K55, 35K45; Secondary 35B40, 35K20.

Keywords. weakly coupled system, existence of mild solutions, non autonomous initial value problem.
1 Introduction: statement of the results and overview

Let $i \in \{1, 2\}$ and $j = 3 - i$. In this paper we study the existence of positive mild solutions of

$$\frac{\partial u_i(t, x)}{\partial t} = \rho_i t^{\rho_i - 1} \Delta_{\alpha_i} u_i(t, x) + t^{\rho_i} u_j^\beta_j(t, x), \quad t > 0, \ x \in \mathbb{R}^d, \quad (1)$$

$$u_i(0, x) = \varphi_i(x), \quad x \in \mathbb{R}^d.$$  

where $\Delta_{\alpha_i} := -(-\Delta)^{\alpha_i/2}$, $0 < \alpha_i \leq 2$, is the $\alpha_i$-Laplacian, $\beta_i > 1$, $\rho_i > 0$, $\sigma_i > -1$ are constants and $\varphi_i$ are non negative, not identically zero, bounded integrable functions.

The associated integral system of (1) is

$$u_i(t, x) = \int_{\mathbb{R}^d} p_i(t, y - x) \varphi_i(y) dy + \int_0^t \int_{\mathbb{R}^d} p_i(t, s, y - x) s^{\rho_i} u_j^\beta_j(s, y) dy ds. \quad (2)$$

Here $p_i(t, x)$ denote the fundamental solution of $\frac{\partial}{\partial t} - \Delta_{\alpha_i}$ (in probability theory it is called the symmetric $\alpha_i$-stable density). We say that $(u_1, u_2)$ is a mild solution of (1) if $(u_1, u_2)$ is a solution of (2).

If there exist a solution $(u_1, u_2)$ of (1) defined in $[0, \infty) \times \mathbb{R}^d$, we say that $(u_1, u_2)$ is a (classical) global solution, on the other hand if there exists a number $t_e < \infty$ such that $(u_1, u_2)$ is unbounded in $[0, t] \times \mathbb{R}^d$, for each $t > t_e$, then we say that $(u_1, u_2)$ blows up in finite time. It is well known that a classical solution is a mild solution, but not vice versa. Therefore, if we give a sufficient condition for global existence of positive solutions to (2) then we do not necessary have a condition for global existence of classical solutions to (1). Here we are going to deal with (mild) global solutions.

Set $a \in \{1, 2\}$ for which

$$\alpha_a = \min\{\alpha_1, \alpha_2\} \quad \text{and} \quad b = 3 - a. \quad (3)$$

We also note that $i$ and $j$ are dummy variables, then if we define an expression for $i$ we obtain other similar expression for $j$, changing only the roles of the indices. For example, in the below inequality (4) is required $\tilde{x}_j$ and it is obtained from the definition of $\tilde{x}_i$ given in (5). We are going to follow this convention.

The main result is:

**Theorem 1** Addition to the above conditions on $\alpha_i$, $\beta_i$, $\rho_i$, $\sigma_i$ suppose that

$$\max\{\tilde{x}_i, \tilde{x}_j\} < \min\{1, \tilde{\rho}_i, \tilde{\rho}_j, \max\{\tilde{k}_i, \tilde{k}_j\}\}, \quad (4)$$

where

$$\tilde{x}_i = \frac{1 + \beta_i + \sigma_i(1 - \beta_i \beta_j)}{\beta_i(1 + \beta_j)}, \quad \tilde{\rho}_i = \rho_i - \sigma_i, \quad (5)$$

and
and
\[ \hat{k}_i = \frac{d\rho_i\rho_j(\beta_i\beta_j - 1) - (\alpha_j\rho_i\sigma_j + \alpha_i\beta_j\sigma_j)\beta_i}{\beta_i(\alpha_j\rho_i + \alpha_i\beta_j\rho_j)}. \] (6)

If
\[ \max\{\tilde{x}_i, \tilde{x}_j\} < \Delta < \min\left\{1, \tilde{\rho}_i, \tilde{\rho}_j, \max\{\tilde{k}_i, \tilde{k}_j\}\right\} \]
and \( \varphi_i \in L^\infty_T(\mathbb{R}^d) \cap L_T^2(\mathbb{R}^d) \), where
\[ r_i = \frac{d\rho_i\rho_j(\beta_i\beta_j - 1)}{\alpha_i\rho_j(1 + \beta_i) + \alpha_i\beta_j\rho_i + \beta_i\alpha_j\rho_i\sigma_j + \beta_i(\alpha_j\rho_i - \alpha_i\rho_j)\Delta}. \] (7)
then (2) has a unique global solution \((u_1, u_2)\). Moreover, there exists an \( \varepsilon > 0 \) such that if \( \|\varphi_i\|_{r_i} + \|\varphi_j\|_{r_j}^2 < \varepsilon \), then
\[ t^{\xi_i} \|u_i(t)\|_{s_i} \leq c\varepsilon, \quad \forall t > 0, \] (8)
where \( c \) is a positive constant
\[ s_i = \frac{d\rho_i\rho_j(\beta_i\beta_j - 1)}{\alpha_i\rho_j\sigma_i + \beta_i\alpha_j\rho_i\sigma_j + (\alpha_i\rho_j + \beta_i\alpha_j\rho_j)\Delta}. \] (9)

and
\[ \xi_i = \frac{\alpha_i\rho_j - \Delta\alpha_i\sigma_i + \alpha_i\beta_i\rho_j - \Delta\alpha_i\beta_i\rho_j}{\alpha_i\rho_j(\beta_i\beta_j - 1)}. \]

Since we are dealing with an integral equation we just need to have the solutions of (2) defined almost surely, this is what (8) tell us. But imposing more restrictions we have that the solutions of (2) are essentially bounded:

**Corollary 2** Assume the hypothesis of Theorem (7) and that
\[ \max\{\tilde{x}_i, \tilde{x}_j\} < \min\left\{1, \tilde{\rho}_i, \tilde{\rho}_j, \max\{\min\{\tilde{k}_i, \tilde{k}_j\}, \min\{\tilde{k}_i, \tilde{k}_j\}\}\right\}, \] (10)
where
\[ \tilde{k}_i = \frac{\alpha_i\rho_j(\beta_i\beta_j - 1) - (\alpha_j\rho_i\sigma_j + \alpha_i\beta_j\sigma_j)\beta_i}{\beta_i(\alpha_j\rho_i + \alpha_i\beta_j\rho_j)}. \]
then
\[ \|u_i(t)\|_\infty \leq c\|\varphi_i\|_\infty + ct^{\alpha_i - \beta_i}\xi_i - \rho_i\alpha_i/(\alpha_i\beta_i) + 1, \quad \forall t > 0. \]

**Remark 3** From the expression (10) of \( \tilde{k}_i \) we observe that \( \alpha_i \geq d \) implies \( \tilde{k}_i \leq \hat{k}_i \). Then, the bound (8) for the solutions of (2) imply they are essentially bounded. In others words, for small dimensions the solutions are essentially bounded and integrable.

Now, if the time-dependent generators are the same and we take a specific initial data, then we get bounds for the solutions of (2). In particular, this means that for these choice of parameters there are global non trivial solutions for (2) with suitable initial conditions.
Theorem 4 If

\[ \alpha_i = \alpha_j = \alpha, \quad \rho_i = \rho_j = \rho \leq 1, \]

and

\[ \frac{1 + \max\{\sigma_i + \beta_i(1 + \sigma_j), \sigma_j + \beta_j(1 + \sigma_i)\}}{\beta_i \beta_j - 1} < \frac{d \rho}{\alpha}, \]

then there exists an \( \varepsilon > 0 \) such that if

\[ \varphi_i(x) = \varepsilon p(1, x), \quad x \in \mathbb{R}^d, \]

then

\[ u_i(t, x) \leq c \varepsilon (1 + t)^{-k} (1 + t^\rho)^d/\alpha p(1 + t^\rho, x), \quad \forall (t, x) \in [0, \infty) \times \mathbb{R}^d, \]

where \( c \) and \( k \) are positive constants.

In [4] Fujita shown (for the case \( \alpha_1 = \alpha_2 = 2, \rho_1 = \rho_2 = 1, \sigma_1 = \sigma_2 = 0 \)
and \( \varphi_1 = \varphi_2 \) in (1)) that \( d = \alpha_1/\beta_1 \) is the critical dimension for blow up of
(classical) solutions of (1): if \( d > \alpha_1/\beta_1 \), then (1) admits a global solution for
all sufficiently small initial conditions, whereas if \( d < \alpha_1/\beta_1 \), then for any non
vanishing initial condition the solution blow up.

Since Fujita’s pioneering work there are in the actuality a lot extensions. For
example, some works consider bounded domains, systems of equations, others
consider more general generators like elliptic operators, fractional operators, etc
(see [1], [3], [9], [8], [10], [12] and the references there in).

In this more general context some new phenomenon occurs. We mention
some of them:

- In general, the election of \( r_i \) in \( \| \varphi_i \|_r \) depend of the choice of \( \Delta \). But
  if, \( \alpha_j \rho_1 = \alpha_i \rho_j \) then \( r_i \) is independent of \( \Delta \). As a particular case, if
  \( \alpha_1 = \alpha_2 = 2 \) and \( \rho_1 = \rho_2 = 1 \) in (1) then our result coincides with the
  Uda result (see Theorem 4.2 in [13]).

- We observed that the estimations for global solutions depend on the generator
  of \( \Delta_{\alpha_a} \). On the other hand, the blow up estimations depends on
  \( \Delta_{\alpha_b} \) (see the results in [14]). From the interpretations of (1) given in the
  introduction of [9] we could say that the blow up depends of the slow
  (diffusion) motion of the particles and contrary the global existence of the
  fast motion of the particles.

- In the literature, the usual way of deal with the estimations required for
  the solutions of (2) is throw the properties of the heat equation, now we
  do not have such properties. To derive \( L^p \) bounds for the solutions we use
  a comparison result and the Banach fix point theorem (see Lemmas [3] and
  [10]). A similar method was used in the proof of Theorem 4.

In applied mathematics it is well known the importance of the study of
equations like (1). In fact, for example, they arise in fields like molecular biology,
hydrodynamics and statistical physics [11]. Also, notice that generators of the form \( g_i(t) \Delta_{\alpha_i} \) arise in models of anomalous growth of certain fractal interfaces [7].

The paper is organized as follows. In Section 2 we give some properties of the symmetric \( \alpha \)-stable densities and provide some preliminary results. In Section 3 we prove the main result, its corollary and Theorem 4.

2 Preliminary results

Let us start dealing with some properties of \( p_i \).

**Lemma 5** Let \( s, t > 0 \) and \( x, y \in \mathbb{R}^d \), then

1. \( p_i(ts, x) = t^{-d/\alpha_i} p_i(s, t^{1/\alpha_i} x) \).
2. \( p_i(t, x) \geq (s/t)^{d/\alpha_i} p_i(s, x) \), for \( t \geq s \).

**Proof.** See Section 2 in [12].

**Lemma 6** There exists a constant \( c \geq 1 \) such that

\[
p_i(t, x) \leq c p_a(t^{\alpha_a/\alpha_i}, x), \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}^d,
\]

where \( \alpha_a \) is defined in (3).

**Proof.** The inequality (11) follows from Lemma 2.4 in [6].

In what follows we will use \( c \) to denote a positive and finite constant whose value may vary from place to place.

For each bounded and measurable function \( f : \mathbb{R}^d \to \mathbb{R} \) we have the semi-group property (in probability it is called the Chapman-Kolmogorov equation):

\[
\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(z) p_i(t, y - z) dz \right) p_i(s, x - y) dy = \int_{\mathbb{R}^d} f(y) p_i(t + s, x - y) dy.
\]

**Lemma 7** Let \( \mu \geq 1 \), then

\[
\|p_a(t, \cdot)\|_\mu = ct^{-\frac{d}{\alpha_a}(1-\frac{1}{\mu})}.
\]

**Proof.** By (1) in Lemma 5 we get

\[
\|p_a(t, \cdot)\|_\mu = t^{-d\mu/\alpha_a} \int_{\mathbb{R}^d} p_a(1, t^{-1/\alpha_a} x)^\mu dx.
\]

The change of variable \( z = t^{-1/\alpha_a} x \) implies

\[
\|p_a(t, \cdot)\|_\mu = t^{\frac{d}{\alpha_a}(1-\mu)} \int_{\mathbb{R}^d} p_a(1, z)^\mu dz.
\]
From Theorem 2.1 in [2] we have
\[ \lim_{|x| \to \infty} p_a(1, x) = 0. \]
Hence there exists \( r > 0 \) such that
\[ p_a(1, x) \leq 1, \quad \forall |x| > r. \]
Using this information we have
\[
\int_{\mathbb{R}^d} p_a(1, x)^\mu \, dx \leq \int_{|x| \leq r} p_a(1, x)^\mu \, dx + \int_{|x| > r} p_a(1, x) \, dx \\
\leq \int_{|x| \leq r} p_a(1, x)^\mu \, dx + \int_{\mathbb{R}^d} p_a(1, x) \, dx \\
\leq \left\| p_a(1, \cdot)^\mu 1_{B(0, r)}(\cdot) \right\|_{\infty} + 1 < \infty.
\]
We used that \((p_a(1, \cdot))^\mu\) is a continuous function on the compact set \( B(0, r) \) (the closed ball with center at the origin and radius \( r > 0 \)) and \( p_a(1, x) \) is a density.

We shall later require several times the following auxiliary tool.

**Lemma 8** Let \( \varphi_i : \mathbb{R}^d \to [0, \infty) \), \( f : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty) \) and \( g : [0, \infty) \times [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty) \) be continuous functions. Suppose that for each \( t \geq 0 \) and \( x \in \mathbb{R}^d \) the real-valued, non-negative continuous functions \( u_i, v_i \) satisfies
\[
u_i(t, x) > \int_{\mathbb{R}^d} f(t, x, y)\varphi_i(y) \, dy + \int_0^t \int_{\mathbb{R}^d} g(t, s, x, y)u_i^{\beta_i}(s, y) \, dy \, ds,
\]
and
\[
u_i(t, x) \leq \int_{\mathbb{R}^d} f(t, x, y)\varphi_i(y) \, dy + \int_0^t \int_{\mathbb{R}^d} g(t, s, x, y)v_i^{\beta_i}(s, y) \, dy \, ds.
\]
Then \( u_i(t, x) \geq v_i(t, x) \), for each \( (t, x) \in [0, \infty) \times \mathbb{R}^d \).

**Proof.** Define
\[
N_i = \{ t \geq 0 : u_i(t, x) > v_i(t, x), \quad \forall x \in \mathbb{R}^d \}.
\]
It is clear that \( N_i \neq \emptyset (0 \in N_i) \). Let \( t_i = \sup N_i \in [0, \infty] \). We have the following cases.
\[ t_i < \infty \text{ and } t_j < \infty: \text{ First observe that } \]
\[
\int_{\mathbb{R}^d} g(t, s, x, y)\{ u_j^{\beta_j}(s, y) - v_j^{\beta_j}(s, y) \} \, dy \geq 0,
\]

6
we used that the function \( r \mapsto r^{\beta_i} \) is increasing. This implies \( t_i \geq t_j \). Analogously we deduce \( t_j \geq t_i \). Therefore \( t_i = t_j \). The continuity of \((u_i - v_i)(\cdot, x)\)

\[
0 = u_i(t_i, x) - v_i(t_i, x) > \int_0^{t_i} \int_{\mathbb{R}^d} g(t, s, x, y)\{u_j^{\beta_i}(s, y) - v_j^{\beta_i}(s, y)\} dy ds \geq 0.
\]

Which is a contradiction.

\( t_i = \infty \) and \( t_j < \infty \) (or \( t_i < \infty \) and \( t_j = \infty \)): Here we have

\[
0 = u_j(t_j, x) - v_j(t_j, x) > \int_0^{t_j} \int_{\mathbb{R}^d} g(t, s, x, y)\{u_i^{\beta_j}(s, y) - v_i^{\beta_j}(s, y)\} dy ds \geq 0.
\]

This also leads to a contradiction.

In this way, the only possibility is \( t_i = \infty \) and \( t_j = \infty \).

**Remark 9** Analogously, if for each \((t, x) \in [0, \infty) \times \mathbb{R}^d\) the continuous functions \(u_i, v_i\) satisfies

\[
u_i(t, x) \geq \int_{\mathbb{R}^d} f(t, x, y)\varphi_i(y) dy + \int_0^t \int_{\mathbb{R}^d} g(t, s, x, y)u_j^{\beta_i}(s, y) dy ds,\]

and

\[
v_i(t, x) \leq \int_{\mathbb{R}^d} f(t, x, y)\varphi_i(y) dy + \int_0^t \int_{\mathbb{R}^d} g(t, s, x, y)v_j^{\beta_i}(s, y) dy ds,
\]

then \(u_i(t, x) \geq v_i(t, x)\), for each \((t, x) \in [0, \infty) \times \mathbb{R}^d\).

Let \( s_j \geq 1 \) and define

\[E_\tau = \{ u : [0, \tau] \to L^{\infty}(\mathbb{R}^d) \cap L^{s_j}(\mathbb{R}^d), \|u\| < \infty \},\]

where

\[\|u\| = \sup_{0 \leq t \leq \tau} \left\{ \|u(t)\|_\infty + \|u(t)\|_s \right\}.
\]

Let \( R > 0 \) and set

\[P_\tau = \{ u \in E_\tau : u \geq 0 \}, \quad B_R = \{ u \in E_\tau : \|u\| \leq R \}.
\]

Since \( E_\tau \) is a Banach space and \( P_\tau, B_R \) are closed subspaces of \( E_\tau \), then they are also Banach spaces. Let us also define the functions \( f_j : \{(t, s) \in \mathbb{R}^2 : t \geq s \geq 0\} \to \mathbb{R} \) as

\[f_j(t, s) = (t^\rho_j - s^\rho_j)^{\alpha_j/\alpha_s},\]

and \( f_j(t) := f_j(t, 0) \), for \( j \in \{1, 2\} \).
Lemma 10 If \( \varphi_j \in L_+^\infty(\mathbb{R}^d) \cap L_+^{r_j}(\mathbb{R}^d), \) \( r_j \geq 1, \) for \( j \in \{1, 2\} \) and \( c > 0, \) then the integral equation

\[
v_j(t, x) = c \int_{\mathbb{R}^d} p_a(f_j(t), y - x)\varphi_j(y)dy + c \int_0^t \int_{\mathbb{R}^d} s^\gamma p_a(f_j(t, s) + f_i(s), z - x)\varphi_i^\beta(z)dzds + c \int_0^t \int_{\mathbb{R}^d} s^{\gamma_j + \beta_j - 1} p_a(f_j(t, s) + f_i(s, r), z - x) \\
\times r^{\sigma_i} v_j^\beta(r, z)drdzds
\]

has a unique solution \( v_j \in L_+^\infty([0, \tau], L_+^c(\mathbb{R}^d) \cap L_+^{r_j}(\mathbb{R}^d)), \) for some \( \tau > 0, \) when

\[
s_j \geq r_j \quad \text{and} \quad s_j^\beta_j \geq r_i.
\]

Proof. Define the mapping \( F : B_R \cap P_\tau \to L_+^\infty([0, \tau], L_+^c(\mathbb{R}^d) \cap L_+^{r_j}(\mathbb{R}^d)), \) as

\[
F(\varphi)(t, x) = c \int_{\mathbb{R}^d} p_a(f_j(t), y - x)\varphi_j(y)dy + c \int_0^t \int_{\mathbb{R}^d} s^\gamma p_a(f_j(t, s) + f_i(s), z - x)\varphi_i^\beta(z)dzds + c \int_0^t \int_{\mathbb{R}^d} s^{\gamma_j + \beta_j - 1} p_a(f_j(t, s) + f_i(s, r), z - x) \\
\times r^{\sigma_i} v_j^\beta(r, z)drdzds.
\]

First we are going to see that \( F \) is onto \( B_R \cap P_\tau. \) We take \((t, x) \in [0, \tau] \times \mathbb{R}^d\) and see

\[
|F(\varphi)(t, x)| \leq c \|\varphi_j\|_\infty + c \|\varphi_i\|_\infty \int_0^t s^\gamma ds + c R^{\beta_j \beta_i} \int_0^t \int_0^s s^{\gamma_j + \beta_j - 1} r^{\sigma_i} drds
\]

\[
\leq c \|\varphi_j\|_\infty + c \|\varphi_i\|_\infty \tau^{\sigma_i + 1} + c R^{\beta_j \beta_i} \tau^{\sigma_i + 1 + \beta_j + \beta_j}.
\]

Now let us deal with the \( L_+^{r_j}(\mathbb{R}^d) \) norm. By Jensen inequality (see Theorem 14.16 in [13]) and using \( r_j \leq s_j \)

\[
\left\| \int_{\mathbb{R}^d} p_a(f_j(t), y - x)\varphi_j(y)dy \right\|_{s_j} \leq \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_a(f_j(t), y - x)\varphi_j^s(y)dydx \right)^{1/s_j}
\]

\[
= \left( \int_{\mathbb{R}^d} \varphi_j^s(y)dy \right)^{1/s_j}
\]

\[
\leq \left( \int_{\mathbb{R}^d} \varphi_j(y)^{r_j} \|\varphi_j\|_\infty \|\varphi_j\|_\infty dy \right)^{1/s_j}
\]

\[
= \|\varphi_j\|_\infty^{1-r_j/s_j} \|\varphi_j\|_\infty^{r_j/s_j}.
\]
Analogously, since $s_j \beta_j \geq r_i$ we have by Minkowski integral inequality (see Theorem 23.69 in [15]) and Jensen inequality

\[
\left\| \int_0^t \int_{\mathbb{R}^d} s^{\sigma_i} p_a(f_j(t, s) + f_i(s), z - \cdot) \varphi^i(z) dz ds \right\|_{s_j} \leq \int_0^t s^{\sigma_i} \int_{\mathbb{R}^d} p_a(f_j(t, s) + f_i(s), z - x) \varphi^i(z) dz ds
\]

\[
\leq \int_0^t s^{\sigma_i} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_a(f_j(t, s) + f_i(s), z - x) \varphi^i(z) dz dx \right)^{1/s_j} ds
\]

\[
= c t^{\sigma_j+1} \left( \int_{\mathbb{R}^d} \beta^i_s(z) dz \right)^{1/s_j} \leq c t^{\sigma_j+1} \| \varphi_i \|_{r_i/s_j} \| \varphi_i \|_{\infty}^{\beta_j-r_i/s_j}.
\]

Using that $\beta_i \beta_j \geq 1$ we get in the third summand in [14]

\[
\left\| \int_0^t \int_{\mathbb{R}^d} s^{\sigma_j} p_a(f_j(t, s) + f_i(s), r, z - \cdot) \varphi^i_j(z) (r, z) ddrdz \right\|_{s_j} \leq \int_0^t \int_{\mathbb{R}^d} s^{\sigma_j} p_a(f_j(t, s) + f_i(s), r, z - \cdot) \varphi^i_j(z) (r, z) ddrdz
\]

\[
\leq \int_0^t \int_{\mathbb{R}^d} s^{\sigma_j} \left( \int_{\mathbb{R}^d} \varphi^i_j(z) (r, z) dz \right)^{1/s_j} ddrdz
\]

\[
\leq \int_0^t \int_{\mathbb{R}^d} s^{\sigma_j} \left( \int_{\mathbb{R}^d} \varphi^i_j(z) (r, z) dz \right)^{1/s_j} ddrdz
\]

\[
\leq c \left( \sup_{r \leq t} \| \varphi(r, \cdot) \|_{\infty} \right)^{\beta_j \beta_i - 1} \left( \sup_{r \leq t} \| \varphi(r, \cdot) \|_{s_j} \right) t^{\sigma_j+\beta_j+\beta_j \sigma_1 + 1}
\]

\[
\leq c R^{\beta_j \beta_i \sigma_j + \beta_j + \beta_j \sigma_1 + 1}.
\]

If we take $R$ large enough such that

\[
\frac{R}{2} \geq \| \varphi_j \|_{\infty}^{1-r_j/s_j} \| \varphi_j \|_{r_j/s_j} + c \| \varphi_j \|_{\infty},
\]

and $\tau$ small enough such that

\[
c \left\| \varphi_j \right\|_{\infty}^{\beta_j \sigma_j + 1} + c R^{\beta_j \beta_i \sigma_j + \beta_j + \beta_j \sigma_1 + 1}
\]

\[
+ c t^{\sigma_j+1} \left( \sup_{r \leq t} \| \varphi_j \|_{r_j/s_j} \right) \left( \sup_{r \leq t} \| \varphi_j \|_{s_j} \right) \| \varphi_j \|_{\infty}^{\beta_j-r_i/s_j} + c R^{\beta_j \beta_i \sigma_j + \beta_j + \beta_j \sigma_1 + 1} \leq \frac{R}{2},
\]

then for each $\varphi \in B_R \cap P_\tau$,

\[
\| | F(\varphi) \| | = \sup_{r \leq \tau} \| F(\varphi)(t, \cdot) \|_{\infty} + \sup_{r \leq \tau} \| F(\varphi)(t, \cdot) \|_{s_j} \leq R.
\]

Now let us see that $F$ is a contraction. Let $\varphi, \psi \in B_R \cap P_\tau$,

\[
| F(\varphi)(t, x) - F(\psi)(t, x) | \leq c \int_0^t \int_{\mathbb{R}^d} s^{\sigma_j+\beta_j-1} \int_{\mathbb{R}^d} p_a(f_j(t, s) + f_i(s, r), z - x)
\]

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_a(f_j(t, s) + f_i(s, r), z - x) \varphi_j(z) \psi^j(z) dz dx ds.
\]
\begin{align*}
\times r^\beta_j \sigma_i & \left| \varphi(r, z)^{\beta_j \beta_i} - \psi(r, z)^{\beta_j \beta_i} \right| dr dz ds.
\end{align*}

Using the elementary inequality

\[ |s^p - r^p| \leq p(\max\{s, r\})^{p-1}|s - r|, \quad s, r > 0, \quad p \geq 1, \]

we have

\begin{align*}
|F(\varphi)(t, x) - F(\psi)(t, x)| & \leq c \int_0^t s^{\sigma_j + \beta_j - 1} \int_0^s r^\beta_j \sigma_i \beta_j \beta_i R^{\beta_j \beta_i - 1} dr ds \times \|\varphi - \psi\| \\
& \leq c \tau^{\sigma_j + \beta_j + \beta_i - 1} \|\varphi - \psi\|.
\end{align*}

Also we choose \( \tau > 0 \) small enough such that

\[ c \tau^{\sigma_j + \beta_j + \beta_i - 1} < 1. \]

From this we have that \( F \) is a contraction, then the result follows from Banach fix point theorem.

**Lemma 11** Suppose that \( \varphi_i \in L^\infty_+(\mathbb{R}^d) \cap L^r_i(\mathbb{R}^d) \), \( r_i \geq 1 \), for \( i \in \{1, 2\} \). Then there exits a local solution \((u_1, u_2)\) of (4). Moreover, there exits \( \bar{T} > 0 \) such that \( u_i \in L^\infty([0, \bar{T}], L^\infty_i(\mathbb{R}^d)) \cap L^s_i(\mathbb{R}^d)) \), for any \( s_i \) satisfying (13) and

\begin{align*}
& s_i \geq r_i, \quad s_j \geq \beta_i, \quad s_i \beta_i \geq s_j, \quad (15) \\
& \frac{\beta_i}{s_j} - \frac{1}{s_i} < \frac{\alpha_i}{d} \quad (16)
\end{align*}

**Proof.** Proceeding as in Lemma 10 we can find a real number \( T > 0 \) such that \((u_1, u_2)\) is a solution of (2) in \([0, T] \times \mathbb{R}^d\) (see, for example, Theorem 3 in [14]). From (2) and (13)

\begin{align*}
u_j(t, x) & \leq c \int_{\mathbb{R}^d} p_a(f_j(t), y - x) \varphi_j(y) dy \\
& + c \int_0^t \int_{\mathbb{R}^d} p_a(f_j(t, s), y - x)s^{\sigma_j}u_i^{\beta_j}(s, y) dy ds. \quad (17)
\end{align*}

Using the elementary inequality

\[ (s + r)^q \leq 2^{q-1}(s^q + r^q), \quad q \geq 1, \quad s, r \geq 0, \]

in the previous estimation we have

\begin{align*}
u_j(t, x) & \leq c \int_{\mathbb{R}^d} p_a(f_j(t), y - x) \varphi_j(y) dy \\
& + c \int_0^t \int_{\mathbb{R}^d} p_a(f_j(t, s), y - x)s^{\sigma_j} u_i^{\beta_j}(s, y) dy ds.
\end{align*}
\begin{align*}
&\times \left( \int_{\mathbb{R}^d} c_p a(f_i(s), z - y) \varphi_i(z) \, dz \right)^{\beta_j} \, dy \, ds \\
&+ c \int_0^t \int_{\mathbb{R}^d} c_p a(f_j(t, s), y - x) s^{\sigma_j} \\
&\times \left( \int_0^z \int_{\mathbb{R}^d} c_p a(f_j(s, r), z - y) r^{\sigma_j} u_j^{\beta_j}(r, z) \, dz \, dr \right)^{\beta_j} \, dy \, ds.
\end{align*}

By Jensen inequality we obtain
\begin{align*}
\left( \int_{\mathbb{R}^d} c_p a(f_i(s), z - y) \varphi_i(z) \, dz \right)^{\beta_j} &\leq c^{\beta_j} \int_{\mathbb{R}^d} \varphi_i^{\beta_j}(z) p_a(f_i(s), z - y) \, dz,
\end{align*}
and using again Jensen inequality
\begin{align*}
\left( \int_0^s \int_{\mathbb{R}^d} c_p a(f_i(s, r), z - y) r^{\sigma_j} u_j^{\beta_i}(r, z) \, dz \, dr \right)^{\beta_j} \\
&\leq c^{\beta_j} \left( \int_0^s \int_{\mathbb{R}^d} p_a(f_i(s, r), z - y) \, dz \, dr \right)^{\beta_j - 1} \\
&\times \int_0^s \int_{\mathbb{R}^d} p_a(f_i(s, r), z - y) r^{\beta_j} s^{\beta_i} u_j^{\beta_i}(r, z) \, dz \, dr \, dy \, ds.
\end{align*}

Taken into account this inequalities we deduce from (18)
\begin{align*}
u_j(t, x) &< c \int_{\mathbb{R}^d} p_a(f_j(t), y - x) \varphi_j(y) \, dy \\
&+ c \int_0^t \int_{\mathbb{R}^d} p_a(f_j(t, s), y - x) s^{\sigma_j} \int_{\mathbb{R}^d} \varphi_i^{\beta_j}(z) p_a(f_i(s), z - y) \, dz \, dy \, ds \\
&+ c \int_0^t \int_{\mathbb{R}^d} p_a(f_j(t, s), y - x) s^{\sigma_j + \beta_j - 1} \\
&\times \int_0^s \int_{\mathbb{R}^d} p_a(f_i(s, r), z - y) r^{\beta_j} s^{\beta_i} u_j^{\beta_i}(r, z) \, dz \, dr \, dy \, ds.
\end{align*}

It is easy to see that the semigroup property \([12]\) yields
\begin{align*}
u_j(t, x) &< c \int_{\mathbb{R}^d} p_a(f_j(t), y - x) \varphi_j(y) \, dy \\
&+ c \int_0^t \int_{\mathbb{R}^d} s^{\sigma_j} p_a(f_j(t, s) + f_i(s), z - x) \varphi_i^{\beta_j}(z) \, dz \, ds \\
&+ c \int_0^t \int_{\mathbb{R}^d} s^{\sigma_j + \beta_j - 1} \int_0^s p_a(f_j(t, s) + f_i(s, r), z - x) \\
&\times r^{\beta_j} s^{\beta_i} u_j^{\beta_i}(r, z) \, dr \, dz \, ds.
\end{align*}

By comparison Lemma \([5]\) (see Remark \([4]\)) we have
\begin{align*}
u_j(t, x) \leq v_j(t, x), \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}^d,
\end{align*}
and Lemma 10 implies $u_j \in L^\infty([0, \tau], L^\infty(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d)$. From 2 and Minkowski inequality (see Theorem 16.17 in 15) one has

$$
\|u_i(t)\|_{s_i} \leq c \left\| \int_{\mathbb{R}^d} p_a(f_i(t), y - \cdot) \varphi_i(y)dy \right\|_{s_i} + c \left\| \int_0^t \int_{\mathbb{R}^d} p_a(f_i(t, s), y - \cdot) s^\sigma u_j^{\beta_i}(s, y)dyds \right\|_{s_i} := cJ_1 + cJ_2.
$$

(19)

Let us estimate $J_1$ first. Since $s_i \geq r_i$, the Young inequality and Lemma 7 implies

$$
J_1 \leq \|\varphi_i\|_{r_i} \|p_a(f_i(t), \cdot)\left(1 + \frac{r_i}{s_i} - \frac{1}{s_i}\right)^{-1} = c\|\varphi_i\|_{r_i} t^{-\frac{d\varphi_i}{r_i} \left(\frac{1}{r_i} - \frac{1}{s_i}\right)}.
$$

Now using that $s_j \geq \beta_i$, $s_i \beta_i \geq s_j$, the Minkowski integral inequality, Young inequality and Lemma 7 we get

$$
J_2 \leq \int_0^t \left\| \int_{\mathbb{R}^d} p_a(f_i(t, s), y - \cdot) u_j^{\beta_i}(s, y)dy \right\|_{s_j} s_i^{\sigma_i} ds
\leq c \int_0^t \left( t^{\rho_i} - s^{\rho_i} \right)^{-\frac{d\varphi_i}{r_i} \left(\frac{1}{r_i} - \frac{1}{s_i}\right)} \|u_j(s)\|_{s_j}^{\beta_i} s_i^{\sigma_i} ds.
$$

A change of variable in the above integral allow us to write

$$
J_2 \leq \frac{c}{\rho_i} t^{-\frac{d\varphi_i}{r_i} \left(\frac{1}{r_i} - \frac{1}{s_i}\right) + \sigma_i + 1} \left( \sup_{s \leq t} \|u_j(s)\|_{s_j} \right)^{\beta_i} \times \int_0^1 \left(1 - s\right)^{-\frac{d\varphi_i}{r_i} \left(\frac{1}{r_i} - \frac{1}{s_i}\right)} s_i^{\frac{\sigma_i}{\alpha_i} + \frac{1}{\alpha_i} - 1} ds.
$$

Putting this together we have that condition 16 implies that $\|u_i(t)\|_{s_i}$ is bounded for $t < \min\{\tau, T\}$.

3 Proof of results

Proof of Theorem 1. The first steep will be to see that it is possible to choice $r_i, s_i, r_j$ and $s_j$ such that the conditions in Lemma 11 are satisfied. From 17 we obtain

$$
\|u_i(t)\|_{s_i} \leq c \|\varphi_i\|_{r_i} t^{-\xi_i} + c \int_0^t s_i^{\sigma_i} \left( t^{\rho_i} - s^{\rho_i} \right)^{-\delta_i} \|u_j(s)\|_{s_j}^{\beta_i} ds,
$$

(20)

where

$$
\xi_i = \frac{d\rho_i}{\alpha_i} \left(\frac{1}{r_i} - \frac{1}{s_i}\right) \text{ and } \delta_i = \frac{d}{\alpha_i} \left(\frac{\beta_i}{s_j} - \frac{1}{s_i}\right).
$$

(21)
Iterating the inequality (20)

\[ \|u_i(t)\|_{s_i} \leq c \|\varphi_i\|_{r_i} t^{-\xi_i} + c \|\varphi_j\|_{r_j} \int_0^t s^{\sigma_i - \beta_j \xi_i} (t^{\rho_i} - s^{\rho_i})^{-\delta_i} ds \]

\[ + c \int_0^t s^{\sigma_i} (t^{\rho_i} - s^{\rho_i})^{-\delta_i} \left( \int_0^s r^{\sigma_j} (s^{\rho_j} - r^{\rho_j})^{-\delta_j} \|u_i(r)\|_{s_i}^{\beta_j} dr \right)^{\beta_i} ds. \]

Let \( w_i(t) = t^{\xi_i} \|u_i(t)\|_{s_i} \), then

\[ w_i(t) \leq c \|\varphi_i\|_{r_i} + c \|\varphi_j\|_{r_j} t^{\xi_i} \int_0^t s^{\sigma_i - \beta_j \xi_i} (t^{\rho_i} - s^{\rho_i})^{-\delta_i} ds \]

\[ + c t^{\xi_i} \int_0^t s^{\sigma_i} (t^{\rho_i} - s^{\rho_i})^{-\delta_i} \left( \int_0^s r^{\sigma_j - \beta_j \xi_i} (s^{\rho_j} - r^{\rho_j})^{-\delta_j} dr \right)^{\beta_i} ds \]

\[ \times \left( \sup_{r \leq t} w_i(r) \right)^{\beta_j \beta_i}. \] (22)

Making some change of variables we obtain

\[ \int_0^t s^{\sigma_i - \beta_j \xi_i} (t^{\rho_i} - s^{\rho_i})^{-\delta_i} ds = \frac{1}{\rho_i} t^{\sigma_i - \beta_j \xi_i - \delta_i \rho_i + 1} \]

\[ \times \int_0^1 s^{\sigma_i - \beta_j \xi_i - \delta_i \rho_i + 1} (1 - s)^{-\delta_i} ds. \] (23)

This integral is convergent if

\[ \delta_i < 1, \quad \sigma_i - \beta_i \xi_j + 1 > 0. \] (24)

In the same way, the second integral in third term in the right hand side of (22) is finite if

\[ \delta_j < 1, \quad \sigma_j - \beta_j \xi_i + 1 > 0. \] (25)

Whence

\[ w_i(t) \leq c \|\varphi_i\|_{r_i} + c \|\varphi_j\|_{r_j} t^{\xi_i + \sigma_i - \beta_j \xi_i - \delta_i \rho_i + 1} \]

\[ + c t^{\xi_i} \int_0^t s^{\sigma_i + (\sigma_j - \beta_j \xi_i - \delta_j \rho_j + 1) \beta_i} (t^{\rho_i} - s^{\rho_i})^{-\delta_i} ds \left( \sup_{r \leq t} w_i(r) \right)^{\beta_j \beta_i}, \]

is well defined if

\[ \sigma_i + (\sigma_j - \beta_j \xi_i - \delta_j \rho_j + 1) \beta_i + 1 > 0. \] (26)

This can be written as

\[ w_i(t) \leq c \|\varphi_i\|_{r_i} + c \|\varphi_j\|_{r_j} t^{\eta_i} + c t^{\xi_i} \left( \sup_{r \leq t} w_i(r) \right)^{\beta_j \beta_i}, \]

where

\[ \eta_i = \xi_i + \sigma_i - \beta_i \xi_j - \delta_i \rho_i + 1. \] (27)
\[ \theta_i = \sigma_i + [\sigma_j - \beta_j \xi_i - \delta_j \rho_j + 1] \beta_i - \delta_i \rho_i + \xi_i + 1. \]  

(28)

Let us take
\[ \xi_i = (1 - \Delta) x_i \text{ and } \xi_j = (1 - \Delta) x_j, \]
for a convenient choice of \( \Delta > 0 \).

Assume that
\[ \rho_i \delta_i - \sigma_i = \Delta = \rho_j \delta_j - \sigma_j. \]

(29)

If
\[ \theta_i = 0, \]
then (28) implies
\[ 1 + \beta_i (1 - \beta_j) x_i + x_i = 0, \]
whence
\[ x_i = \frac{1 + \beta_i}{\beta_i \beta_j - 1}. \]

Also we want
\[ \eta_i = 0, \]
then (27) implies
\[ x_j = \frac{1 + \beta_j}{\beta_i \beta_j - 1}. \]

The assumption (29) and definitions in (21) yields
\[
\frac{\beta_i}{s_j} - \frac{1}{s_i} = \frac{\alpha_i}{d \rho_i} (\Delta + \sigma_i),
\]
\[
\frac{\beta_j}{s_i} - \frac{1}{s_j} = \frac{\alpha_j}{d \rho_j} (\Delta + \sigma_j).
\]

Solving such linear system of equations we find \( s_i \) and \( s_j \) given in (9). On the other hand, using (21) we get \( r_i \) and \( r_j \) given in (7). The conditions (24), (25) and (26) are satisfies if
\[ \max \{\tilde{x}_i, \tilde{x}_j\} < \Delta < \min \left\{1, \tilde{\rho}_i, \tilde{\rho}_j, \tilde{k}_i \right\}, \]

where \( \tilde{x}_i, \tilde{\rho}_i \) and \( \tilde{k}_i \) are defined in (5) and (6). For this election of \( r_i, s_i, r_j \) and \( s_j \) the conditions (13), (15) and (16) are also satisfied, then by Lemma 11 we have a local solution \((u_1, u_2)\) on \([0, \tilde{T}] \times \mathbb{R}^d\). If we change the rôles of \( i \) and \( j \) in the above procedure we have
\[ \max \{\tilde{x}_i, \tilde{x}_j\} < \Delta < \min \left\{1, \tilde{\rho}_i, \tilde{\rho}_j, \tilde{k}_j \right\}. \]

This implies that the correct condition on \( \Delta \) is
\[ \max \{\tilde{x}_i, \tilde{x}_j\} < \Delta < \min \left\{1, \tilde{\rho}_i, \tilde{\rho}_j, \max(\tilde{k}_i, \tilde{k}_j) \right\}. \]
As a second steep we are going to deal with the $L^s(R^d)$ boundedness of $u_i$. Conditions (30) and (31) implies

$$z_i(t) \leq c \left( \| \varphi_i \|_{r_i} + \| \varphi_j \|_{r_j}^{\beta_i} \right) + cz_i^{\beta_i, \beta_j}(t),$$

where

$$z_i(t) = \sup_{r \leq t} w_i(r).$$

If we take $\varphi_i, \varphi_j$ small enough such that

$$\| \varphi_i \|_{r_i} + \| \varphi_j \|_{r_j}^{\beta_i} < (2c)^{\frac{\beta_i, \beta_j}{1 - \beta_i, \beta_j}},$$

then

$$z_i(t) \leq 2c(\| \varphi_i \|_{r_i} + \| \varphi_j \|_{r_j}^{\beta_i}), \quad \forall t \geq 0.$$  

(34)

In fact, if (34) were false, since $z_i$ is continuous, the intermediate value theorem would imply that there exists $t_0 > 0$ such that

$$z_i(t_0) = 2c(\| \varphi_i \|_{r_i} + \| \varphi_j \|_{r_j}^{\beta_i}),$$

then of (32) we could conclude

$$\| \varphi_i \|_{r_i} + \| \varphi_j \|_{r_j}^{\beta_i} \geq (2c)^{\frac{\beta_i, \beta_j}{1 - \beta_i, \beta_j}},$$

this is a contradiction with (33). The second part of the statement follows from the definitions of $z_i$ and $w_i$ together with (34).

Proof of Corollary 2

Let us consider (17), then

$$\| u_i(t) \|_\infty \leq c \left\| \int_{\mathbb{R}^d} p_a(f_i(t), y - \cdot) \varphi_i(y) dy \right\|_\infty + c \left\| \int_0^t \int_{\mathbb{R}^d} p_a(f_i(t, s), y - \cdot) s^{\sigma} u_j^{\beta_i}(s,y) dy ds \right\|_\infty.$$  

(35)

Since $p_a(f_i(t), \cdot)$ is a density

$$\left\| \int_{\mathbb{R}^d} p_a(f_i(t), y - \cdot) \varphi_i(y) dy \right\|_\infty \leq \| \varphi_i \|_\infty \left\| \int_{\mathbb{R}^d} p_a(f_i(t), y - \cdot) dy \right\|_\infty = \| \varphi_i \|_\infty.$$  

To estimate the second term in (35) we use Hölder inequality (see Theorem 16.14 in [15]) and Lemma 7

$$\int_{\mathbb{R}^d} p_a(f_i(t, s), y - \cdot) s^{\sigma} u_j^{\beta_i}(s,y) dy ds = \| p_a(f_i(t, s), \cdot - x) u_j^{\beta_i}(s, \cdot) \|_1$$

$$\leq \| p_a(f_i(t, s), \cdot - x) \|_{(1 - \frac{\beta_i}{\sigma})^{-1}}.$$  

15
\[ \times |u_j^{\beta_i}(s, \cdot)|_{\alpha_i}^{\beta_i} \]
\[ = c(t^{\rho_i} - s^{\rho_i})^{-\frac{d \beta_i}{\alpha_i \sigma_j}} |u_j(s, \cdot)|_{\alpha_i}^{\beta_i}. \]

From (35) one has
\[ ||u_i(t)||_{\infty} \leq c||\varphi_i||_{\infty} + c \int_0^t (t^{\rho_i} - s^{\rho_i})^{-\frac{d \beta_i}{\alpha_i \sigma_j}} |u_j(s, \cdot)|_{\alpha_i}^{\beta_i} s^{\sigma_i} ds, \]
and (35) implies
\[ ||u_i(t)||_{\infty} \leq c ||\varphi_i||_{\infty} + c \int_0^t (t^{\rho_i} - s^{\rho_i})^{-\frac{d \beta_i}{\alpha_i \sigma_j}} |u_j(s, \cdot)|_{\alpha_i}^{\beta_i} s^{\sigma_i} ds \]
\[ = c ||\varphi_i||_{\infty} + c t^{\sigma_i - \beta_i \xi_i - \frac{d \beta_i}{d \rho_i}} \int_0^t (1 - s) - \frac{d \beta_i}{\alpha_i \sigma_j} \frac{1}{\sigma_i} (s^{\sigma_i} - \beta_i \xi_i) + \frac{1}{\sigma_i} ds. \]

Proceeding as in (23) we deduce that the above integral is fine if (24) holds and
\[ \frac{d \beta_i}{\alpha_i} < \beta_j. \]

The definition (36) of \( s_j \) impose the condition
\[ \Delta < \left( \frac{\alpha_i}{d} \right)[d \rho_i \rho_j(\beta_i \beta_j - 1)] - (\alpha_i \rho_i \sigma_j + \alpha_i \beta_j \rho_j \sigma_i) \beta_i \beta \left( \alpha_i \rho_i + \alpha_i \beta_j \rho_j \right) := \hat{k}_i. \]

Whence
\[ \Delta < \min \left\{ 1, \tilde{\rho}_j, \tilde{\rho}_i, \hat{k}_i, \hat{k}_i \right\}. \]

Changing the rôles of \( i \) and \( j \) we have the condition (10).

\[ \blacksquare \]

**Proof of Theorem 4.** Here we consider the equation (17) with
\[ \alpha_i = \alpha_j = \alpha \quad \text{and} \quad \rho_i = \rho_j = \rho, \]
and we write \( p(t, x) \) instead of \( p_a(t, x) \). Under this considerations we study the solution of
\[ u_i(t, x) \leq \int_{\mathbb{R}^d} cp(t^\rho, y - x) \varphi_i(y) dy + \int_0^t \int_{\mathbb{R}^d} cp(t^\rho - s^\rho, y - x) s^{\sigma_i} u_j^{\beta_i}(s, y) dy ds, \]
with the initial condition
\[ \varphi_i(x) = \varepsilon p(1, x), \quad x \in \mathbb{R}^d. \]

Define the functions \( g, h : (0, \infty) \to (0, \infty) \) as
\[ g(t) = (e^t - 1)^{\rho}, \quad h(t) = [g(t) + 1]^{1/\alpha}. \]
Through a change of variable the inequality can be transformed into

\[ u_i(g^{1/\rho}(t), h(t)x) \leq c\varepsilon \int_{\mathbb{R}^d} p(g(t), y - h(t)x) p(1, y) dy \]

\[ + c \int_0^{t-1} \int_{\mathbb{R}^d} p(g(t) - s^\rho, y - h(t)x) s^{\sigma_i} u_j^{\beta_i}(s, y) dy ds \]

\[ = c\varepsilon p(g(t) + 1, h(t)x) \]

\[ + c \int_0^{t} \int_{\mathbb{R}^d} p(g(t) - g(s), y - h(t)x) g^{\sigma_i/\rho}(s) \]

\[ \times u_j^{\beta_i}(g^{1/\rho}(s), y) dy ds \]

\[ = c\varepsilon (h(t))^{-d} p(1, x) \]

\[ + c \int_0^{t} \int_{\mathbb{R}^d} p(g(t) - g(s), h(s)y - h(t)x) g^{\sigma_i/\rho}(s) \]

\[ \times u_j^{\beta_i}(g^{1/\rho}(s), h(s)y) dy ds. \]

Setting \( \bar{u}_i(t, x) = u_i(g^{1/\rho}(t), h(t)x) \), \( t \geq 0, \ x \in \mathbb{R}^d \), we have

\[ \bar{u}_i(t, x) \leq c\varepsilon (h(t))^{-d} p(1, x) \]

\[ + c \int_0^{t} \int_{\mathbb{R}^d} g^{\sigma_i/\rho}(s) e^s h^d(s) \]

\[ \times \int_{\mathbb{R}^d} p(g(t) - g(s), h(s)y - h(t)x) u_j^{\beta_i}(s, y) dy ds. \]

Also define

\[ w_i(t, x) = c\varepsilon e^{-(\theta_i + \eta_i)t} p(1, x), \ t \geq 0, \ x \in \mathbb{R}^d, \]

where

\[ \theta_i = 1 + \sigma_i + \frac{\beta_i (1 + \sigma_j)}{\beta_i \beta_j - 1}, \]

and \( \eta_i \) is a positive number to be fixed. Observe that

\[ A_i(t, x) = \int_0^t \int_{\mathbb{R}^d} g^{\sigma_i/\rho}(s) e^s h^d(t) \]

\[ \times \int_{\mathbb{R}^d} p(g(t) - g(s), h(s)y - h(t)x) u_j^{\beta_i}(s, y) dy ds \]

\[ \leq (c\varepsilon)^{\beta_i} \int_0^t \int_{\mathbb{R}^d} g^{\sigma_i/\rho}(s) e^s h^d(t) \]

\[ \times \int_{\mathbb{R}^d} p(g(t) - g(s), h(s)y - h(t)x) \]

\[ \times e^{- (\theta_j + \eta_j) s^{\beta_i}} p(1, y) dy ds \left( \sup_{z \in \mathbb{R}^d} p(1, z) \right)^{\beta_i - 1}. \]
Using property (1) in Lemma 5 and the unimodality of $p(1,\cdot)$ ($p(1,x) \leq p(1,0)$, for each $x \in \mathbb{R}^d$) we get

$$A_i(t,x) < (c\varepsilon)^{\beta_i} \int_0^t e^{s\sigma_i + s - (\theta_j + \eta_j)\beta_i} \times \left( h(t) - g(t) - g(s) \right) + 1 \frac{h(t)}{h(s)} x \hspace{1mm} ds \sigma_i \beta_i,$$

$$= (c\varepsilon)^{\beta_i} (p(1,0))^{\beta_i - 1} \int_0^t e^{s\sigma_i + s - (\theta_j + \eta_j)\beta_i} ds \sigma_i \beta_i.$$

Since $\rho \leq 1$, then

$$(e^s - 1)^\rho + 1 \geq e^\rho s, \quad s \geq 0.$$ To see this consider the function $\tilde{f} : [1, \infty) \rightarrow \mathbb{R}$,

$$\tilde{f}(s) = (s - 1)^\rho + 1 - s^\rho,$$

and observe that $\tilde{f}'(s) \geq 0$. Taking into account this,

$$c\varepsilon h(t)^{-d} p(1,x) + cA_i(t,x)$$

$$\leq c\varepsilon e^{-d\rho t/\alpha} p(1,x) + (c\varepsilon)^{\beta_i} p(1,0)^{\beta_i - 1} p(1,x) \frac{e^{(\sigma_i + 1 - (\theta_j + \eta_j)\beta_i)t} - 1}{\sigma_i + 1 - (\theta_j + \eta_j)\beta_i}$$

$$:= J_3 + J_4.$$ (38)

Now let us estimate $J_3$ and $J_4$. For $J_3$ we have

$$J_3 = e^{(\theta_i + \eta_i - d\rho t/\alpha)} w_i(t,x) < \frac{1}{2} w_i(t,x),$$

because

$$\theta_i < \frac{d\rho}{\alpha},$$

and taking $\eta_i$ small enough. And for $J_4$,

$$J_4 = (c\varepsilon)^{\beta_i - 1} p(1,0)^{\beta_i - 1} e^{(\theta_i + \eta_i + \sigma_i + 1 - (\theta_j + \eta_j)\beta_i)t - e^{(\theta_i + \eta_i)t}}$$

$$\sigma_i + 1 - (\theta_j + \eta_j)\beta_i$$

$$\leq (c\varepsilon)^{\beta_i - 1} p(1,0)^{\beta_i - 1} e^{(\theta_i + \eta_i + \sigma_i + 1 - (\theta_j + \eta_j)\beta_i)t}$$

$$\sigma_i + 1 - (\theta_j + \eta_j)\beta_i$$

$$w_i(t,x).$$

If we take

$$\eta_j > \frac{\eta_i}{\beta_i},$$

then

$$\theta_i + \eta_i + \sigma_i + 1 - (\theta_j + \eta_j)\beta_i < 0,$$

here we used that $\theta_i + \sigma_i + 1 = \theta_j \beta_i$. This yields

$$J_4 \leq (c\varepsilon)^{\beta_i - 1} p(1,0)^{\beta_i - 1} w_i(t,x).$$

18
Therefore, if \( \varepsilon > 0 \) is small enough then
\[
J_4 \leq \frac{1}{2} w_i(t, x). \tag{40}
\]
From (38), (39) and (40) we get
\[
c\varepsilon h(t)^{-d} p(1, x) + cA_i(t, x) < w_i(t, x).
\]
This means that
\[
w_i(t, x) > c\varepsilon h(t)^{-d} p(1, x) + c \int_0^t g^{\sigma_i/\rho}(s) e^s h_i(s) \, ds
\times \int_{\mathbb{R}^d} p(g(t) - g(s), h(s)y - h(t)x) w_{ij}(s, y) dy ds.
\]
By the comparison Lemma 8 we have
\[
u_i \left( g^{1/\rho}(t), h(t)x \right) \leq c\varepsilon e^{-\theta_i + \eta_i} p(1, x), \quad \forall (t, x) \in [0, \infty) \times \mathbb{R}^d.
\]
The results follows from (1) in Lemma 9.

Acknowledgment
This work was partially supported by the grant No. 118294 of CONACyT. Moreover, Villa-Morales was also supported by the grant PIM13-3N of UAA.

References

[1] X. Bai, \textit{Finite time blow-up for a reaction-diffusion system in bounded domain}. Z. Angew. Math. Phys. DOI 10.1007/s00033-013-0330-4 (2013).

[2] R.M. Blumental, R.K. Getoor, \textit{Some theorems on stable processes}. Trans. Amer. Math. Soc. \textbf{95} (1960), 263-276.

[3] M. Escobedo, M.A. Herrero, \textit{Boundedness and blow up for a semilinear reaction-diffusion system}. J. Differ. Equ. \textbf{89} (1991), 176-202.

[4] H. Fujita, \textit{On the blowing up of solutions of the Cauchy problem for} \( u_t = \Delta u + u^{1+\alpha} \). J. Fac. Sci. Univ. Tokyo Sect. I \textbf{13} (1966), 109-124.

[5] M. Guedda, M. Kirane, \textit{A note on nonexistence of global solutions to a nonlinear integral equation}. Bull. Belg. Math. Soc. Simon Stevin \textbf{6} (1999), 491-497.

[6] J.A. López-Mimbela, J. Villa, \textit{Local time and Tanaka formula for a multitype Dawson-Watanabe superprocess}. Math. Nachr. \textbf{279} (2006), 1695-1708.
[7] J.A. Mann Jr., W.A. Woyczyński, *Growing Fractal Interfaces in the Presence of Self-similar Hopping Surface Diffusion*. Phys. A 291 (2001), 159-183.

[8] K. Mochizuki, Q. Huang, *Existence and behavior of solutions for a weakly coupled system of reaction-diffusion equations*. Methods Appl. Anal. 5 (1998), 109-124.

[9] A. Perez, J. Villa, *Blow-up for a system with time-dependent generators*. ALEA 7 (2010), 207-215.

[10] Y.W. Qi, H.A. Levine, *The critical exponent of degenerate parabolic systems*. Z. Angew. Math. Phys. 44 (1993), 549-265.

[11] M.F. Shlesinger, G.M. Zaslavsky, U. Frisch (Eds), *Lévy Flights and Related Topics in Physics*. Lecture Notes in Physics 450. Springer-Verlag, Berlin, 1995.

[12] S. Sugitani, *On nonexistence of global solutions for some nonlinear integral equations*. Osaka J. Math. 12 (1975), 45-51.

[13] Y. Uda, *The critical exponent for a weakly coupled system of the generalized Fujita type reaction-diffusion equations*. Z. Angew. Math. Phys. 46 (1995), 366-383.

[14] J. Villa-Morales, *Blow up of mild solutions of a system of partial differential equations with distinct fractional diffusions*. arXiv:1208.4001v3 (2013).

[15] J. Yeh, *Real Analysis, theory of measure and integration*. 2nd Edition, World Scientific, 2006.