Construction of $R$-matrices for symmetric tensor representations related to $U_q(\widehat{sl}_n)$

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Abstract
In this paper we construct a new factorized representation of the $R$-matrix related to the affine algebra $U_q(\widehat{sl}_n)$ for symmetric tensor representations with arbitrary weights. Using the 3D approach we obtain explicit formulas for the matrix elements of the $R$-matrix and give a simple proof that a ‘twisted’ $R$-matrix is stochastic. We also discuss symmetries of the $R$-matrix, its degenerations and compare our formulas with other results available in the literature.

Keywords: Yang–Baxter equation, quantum algebras, integrable systems

1. Introduction

Originally quantum groups were discovered in the context of quantum integrable systems and quantum inverse scattering method (see, for example, [1–3] and [4] for a review). They were formally introduced by Drinfeld and Jimbo [5, 6] as certain deformations of universal enveloping algebras of simple Lie algebras. The $R$-matrix as a solution of the Yang–Baxter equation [7, 8] plays the central role in the theory of quantum groups.

From the physical point of view different representations of quantum groups allow us to construct different spaces of physical states of quantum integrable models. The $R$-matrix becomes a linear operator acting in the tensor product of two arbitrary representations. In this paper we address the problem of finding explicit matrix elements for the $R$-matrix $R(\lambda)$ associated with symmetric tensor representations of the affine quantum algebra $U_q(\widehat{sl}_n)$. The parameter $\lambda$ here plays the role of a spectral parameter entering the evaluation homomorphism [6].

The problem of calculating quantum $R$-matrices related to the highest weight representations of the $U_q(\widehat{sl}_n)$ algebra has been considered by many authors. The two most known methods are a fusion procedure [2, 9, 10] and the method of spectral decomposition [6, 11].
Another method is based on the explicit evaluation of the universal $R$-matrix [12] in the tensor product of two highest weight representations but this method is technically challenging and has been successfully applied only for low rank algebras [13, 14]. We also mention the approach of [15] where the calculation of the higher-spin $sl(2)$ $R$-matrices is based on factorization properties of the $L$-operator.

In this paper we use the 3D approach developed in [16–18] and apply it to the case of symmetric representations of the $U_q(sl_n)$ algebra. Previously a closed formula for the higher-spin $R$-matrix of the 6-vertex model was obtained in [19] based on the positive solution of the tetrahedron equation [18]. It can be interpreted as the $R$-matrix of the higher-spin stochastic 6-vertex model [20–23]. Under a special choice of the spectral parameter [21] this model degenerates into the $q$-Hahn system which corresponds to the most general ‘chipping model’ introduced by Povolotsky [24]. Let us notice that the action of the $Q$-operator for the higher spin 6-vertex model (see section 6 in [25]) can be identified with the transition matrix of the Povolotsky’s chipping model in [24].

In the recent paper [26] the above higher spin stochastic 6-vertex model has been generalized to the case of symmetric representations of the higher rank $U_q(A_{n-1}^{(1)})$ (or $U_q(sl_{n+1})$). It was shown that even for a general $n$ the corresponding $R$-matrix satisfies the sum rule required for a stochastic interpretation. At a special point it gives a $n$ species generalization of the Povolotsky model. However, most of the results in [26] were obtained using the machinery of quantum groups. Our strategy is to derive explicit formulas for the $R$-matrix related to symmetric representations of $U_q(A_{n-1}^{(1)})$ extending the method of [19].

The structure of the paper is as follows. In section 2 we introduce the Boltzmann weights of the 3D model from [18] and a definition of a composite weight. In section 3 we consider the $n$-layer projection of the 3D model and obtain the formula for the $R$-matrix in the form of the $(n - 1)$-tuple sum. For $n = 2$ it corresponds to the formula from [19] up to a certain transformation. In section 4 we discuss symmetries of the $R$-matrix. In section 5 we consider degenerations and derive a factorization formula for the $R$-matrix. In section 6 we compare our formulas with other results available in the literature. In section 7 we introduce a stochastic $R$-matrix and give a simple proof of a sum rule. We also consider the corresponding $L$-operator and show that it is equivalent to the $L$-operator from [27]. In section 8 we discuss the results and appendix contains notations and some formulas used in the main text.

2. The 3D integrable model

In this section we recall a definition of the 3D integrable model with positive Boltzmann weights introduced in [18]. The Boltzmann weights of the model are constructed from matrix elements of an operator $R$ acting in the tensor product of three Fock spaces $\mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F}$ with the orthonormal basis $|n_1, n_2, n_3\rangle = |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle$, $n_i \in \mathbb{Z}_{\geq 0}$. The operator $R$ solves the tetrahedron equation

$$R_{123}R_{146}R_{246}R_{156} = R_{156}R_{246}R_{143}R_{123}. \quad (2.1)$$

With respect to the basis $|n_1, n_2, n_3\rangle$, the operator $R$ has the following matrix elements

$$R_{n_0'}^{n_0; n_1'; n_2', n_3'} = \langle n_1, n_2, n_3 | R | n_1', n_2', n_3' \rangle \quad (2.2)$$
with

\[
\mathbb{R}_{m_1,n_1,n_2}^{n_1',n_2'} = \delta_{n_1+n_2,n_1'+n_2'} \delta_{n_1+n_2,n_1'+n_2'} q^{-n_2(1+n_1+n_2)-n'_1}
\times \left[ m_1 + n_2 \atop n_1 \right] \varphi_{q^{-2n_2}, q^{-2n_1}} \left( q^{2}; q^{2} \right)_{n_1} \varphi_{q^{2}; q^{2} \varphi_{q^{-2n_2}, q^{-2n_1}}} \left( q^{2}; q^{2} \right)_{n_1} \varphi_{q^{2}; q^{2} \varphi_{q^{-2n_2}, q^{-2n_1}}} \left( q^{2}; q^{2} \right)_{n_1},
\]

(2.3)

where we used standard notations for \( q \)-series from appendix and \( \varnothing_1 \) is a basic hypergeometric series defined in (A.5).

We shall refer to the matrix representation of the operator \( \mathbf{R} \) as the 3D \( R \)-matrix. It is easy to write a matrix realization of the operator equation (2.1) in terms of matrix elements \( \mathbb{R}_{m_1,n_1,n_2}^{n_1',n_2'} \).

Note that in (2.3) we used a slightly different presentation of (2.3) compared to [18]. The two presentations are equivalent and related by a change \( r = n_2 - r \) in the summation variable. We also notice that due to conservation laws we always have \( n_2 \leq n_1 + n_2 \) and \( n'_1 \leq n_1 + n_2 \), so the hypergeometric function in (2.3) does not require a regularization. All nonzero elements in (2.3) are positive for \( 0 < q < 1 \) as explained in [18].

The \( \mathbf{R} \)-matrix (2.3) possesses a number of symmetries which are generated by two elementary ones

\[
\mathbb{R}_{m_1,n_1,n_2}^{n_1',n_2'} = \mathbb{R}_{n_1,n_2,m_1}^{n_1',n_2'}, \quad \mathbb{R}_{n_1,n_2,m_2}^{n_1,n_2'} = q^{n_2-n_2+n_2'-n_1'} (q^2; q^2)_{n_1} \mathbb{R}_{n_1,n_2,m_1}^{n_1,n_2'},
\]

(2.4)

They can be proved by using Heine’s transformations of \( \varnothing_1 \) series (A.6). We list here two other useful symmetries which follow from (2.4)

\[
\mathbb{R}_{m_1,n_1,n_2}^{n_1',n_2'} = q^{n_1+n_2+n_2'-n_1'} (q^2; q^2)_{n_1} \mathbb{R}_{n_1,n_2,m_1}^{n_1,n_2'},
\]

(2.5)

and

\[
\mathbb{R}_{n_1,n_2,m_2}^{n_1',n_2'} = q^{n_1+n_2+n_2'-n_1'} (q^2; q^2)_{n_1} \mathbb{R}_{n_1,n_2,m_1}^{n_1',n_2'}.
\]

(2.6)

Let us notice that up to the factor \( q^{-n_2(1+n_1+n_2)-n'_1} \), the expression (2.3) is a polynomial in \( q^{2n_1'} \) and can be formally continued to negative values \( n_3, n'_3 < 0 \). So let us assume that \( n_i, n'_i \geq 0, i = 1, 2 \) and \( n_3, n'_3 \in \mathbb{Z} \) provided that all indices are still constrained by delta-functions entering (2.3). Then it is easy to find a transformation of matrix elements of the 3D \( R \)-matrix under the replacement \( q \rightarrow q^{-1} \)

\[
\mathbb{R}_{m_1,n_1,n_2}^{n_1',n_2'} \big|_{q^{-1}} = q^{n_1-n'_1+2n_3(2n_2+1)(n_2-n_1)} \prod_{i=1}^{3} \left( q^2; q^2 \right)_{n_i} \mathbb{R}_{n_1,n_2,m_1}^{n_1',n_2'}.
\]

(2.7)

Following [18] we define a composite weight

\[
\mathcal{S}_{i,j}^{L} (w) = \sum_{k} w^{k} \prod_{l=1}^{n} \mathbb{R}_{j_{l},i_{l},k_{l}}^{l, l, k_{l+1}}.
\]

(2.8)

where \( i = \{i_{1}, i_{2}, \ldots, i_{n} \}, j = \{j_{1}, j_{2}, \ldots, j_{n} \}, \) etc and \( k_{n+1} = k_{1} \).

This composite weight has a number of important properties. Firstly, the presence of delta functions in (2.3) imply global conservation laws, namely

\[
I = I', \quad J = J',
\]

(2.9)
and therefore the matrix \( \mathbb{S} \) with entries (2.8) will have a block-diagonal form indexed by integers \( I, J = 0, \ldots, \infty \). Secondly, standard arguments relating the tetrahedron and Yang–Baxter equations imply that (2.8) satisfies the Yang–Baxter equation

\[
\sum_{ij'k'} \mathbb{S}_{ij'k'}(w) \mathbb{S}_{ijk'}(w') \mathbb{S}_{ij'k'}(w') = \sum_{ij'k'} \mathbb{S}_{ij'k'}(w') \mathbb{S}_{ijk'}(w') \mathbb{S}_{ij'k'}(w),
\]

and hence defines the \( R \)-matrix [16]. This \( R \)-matrix is composite in the sense that it is a direct sum of ‘smaller’ \( R \)-matrices. It is the fact which follows from considering the conservation laws (2.9) applied to each component in (2.11) and noticing that the equation reduces to a tensor sum of an infinite number of the Yang–Baxter equations on subspaces indexed by the parameters \( I, J, K = 0, \ldots, \infty \) defined in (2.10).

In particular, it was argued in [16, 18] that the subspace for each parameter \( I \) is in fact the underlying space of the rank \( I \) symmetric tensor representation of \( U_q(\mathfrak{sl}_n) \) and the action of \( \mathbb{S} \) on this space is the corresponding \( R \)-matrix

\[
\mathbb{S}^{ij'}_{ij}(w) = \bigoplus_{I,J=0}^{\infty} R^{ij}_{ij'}(w).
\]

The case \( n = 2 \) in (2.8) was considered in [19] which resulted in a new formula for the matrix elements of the \( R \)-matrix for \( U_q(\mathfrak{sl}_2) \) acting in the tensor product of representations of highest weight \( I \) and \( J \). Setting \( I = J = 1 \) the formula gives the \( R \)-matrix for the 6-vertex model.

3. The \( n \)-layer projection

In this section we will generalize the result of [19] by taking the \( n \)-layer projection and construct a new formula for the \( U_q(\mathfrak{sl}_n) \) \( R \)-matrix acting in the tensor product of representations with weights \( \omega_I \) and \( \omega_J \) respectively.

First let us introduce some vector notations. We denote by \( i := \{i_1, \ldots, i_r \} \) a set of positive integers \( i_k \in \mathbb{Z}_{>0} \) with \( r \) components and define

\[
|i| = \sum_{s=1}^{r} i_s, \quad (i, j) = \sum_{s=1}^{r} i_s j_s.
\]

Addition is done component-wise and we introduce two permutations \( \sigma \) and \( \tau \) acting on \( k \) as

\[
\sigma \{k_1, \ldots, k_r\} = \{k_2, \ldots, k_r, k_1\},
\]

\[
\tau \{k_1, \ldots, k_r\} = \{k_r, k_{r-1}, \ldots, k_1\}
\]

of the vector coordinates. The dimension \( r \) can take values \( n \) and \( n - 1 \) as explained below.

The Kronecker delta function of two vectors is zero unless all their components match, i.e.

\[
\delta_{ij} = \prod_{s=1}^{r} \delta_{i_s j_s}.
\]

We also note that in discussing \( \mathbb{S}(w) \) and \( R^{ij}_{ij'}(w) \) the vectors \( i, j, i', j' \) have different dimensions. When we use \( \mathbb{S}(w) \), the \( n \)-layer composite weight, it is implied that the dimension \( r = n \). When we derive the expression for the \( R \)-matrix \( R^{ij}_{ij'} \), it is implied that the dimension
\[ r = n - 1 \] because by fixing \( I, J \) the relation (2.10) implies that we can remove one of the indices. Typically we choose to remove last components \( i_n, j_n, i_n', j_n' \) and replace them with \( I - |i|, J - |j| \) etc except in certain cases where it is more convenient to keep them. Of course, in evaluating final expressions the replacement has to be made regardless.

Combining (2.3) and (2.8) the composite weight \( \mathcal{S}^f_{ij} (w) \) can be written as

\[
\mathcal{S}^f_{ij} (w) = \delta_{i+j,i'+j'} \sum_{k \in \mathbb{Z}^n} \delta_{i-k,i'+j-k} w^{k_q} q^{-(|i|-|i'|)-(k,i+j'-j')} \prod_{s=1}^{n} \left[ i_s + j_s \right]_{q^{-2}} \\
\times \prod_{m \in \mathbb{Z}^n} \prod_{s=1}^{n} \frac{(q^{-2i_s}; q^{-2j_s}; q^2)_m}{(q^2, q^{-2i_s}; q^2)_m}, \quad (3.5)
\]

The above formula contains \( 2n \) summations. The \( n \) summations in \( k \) are infinite ranging from 0 to \( \infty \). The \( n \) summations in \( m \) are restricted by \( 0 \leq m_s \leq \min(i_s, j_s'), s = 1, \ldots, n \) due to the presence of Pochhammers symbols in the numerator. Let us also notice that all sums in \( m_s' \) truncate before the Pochhammer symbols in the denominator become zero. Therefore, there is no need for a regularization.

This formula is quite easy to simplify. The presence of delta functions in (3.5) lead to the following global conservation laws for the spin indices \( i, j, i', j' \),

\[ i_1 + \cdots + i_n = i_1' + \cdots + i_n' = I, \quad j_1 + \cdots + j_n = j_1' + \cdots + j_n' = J \quad (3.6) \]

which allows us to remove one of the indices from \( i, j, i', j' \) once we fix integers \( I, J \). Furthermore, we can also express \( k_2, \ldots, k_n \) in terms of \( k_1 \) by the relations

\[ k_{s+1} = k_s + i_s - i_s', \quad k_n = k_1 + \sum_{s=1}^{n-1} (i_s - i_s'), \quad (3.7) \]

which allows us to rewrite the sum in \( k \) as a single sum in \( k_1 \). However, some care must be taken in computing this sum. Note that when \( i_s' > i_s \) for some \( s \), the summation range of \( k_s \) implies contributions to the sum for negative values of \( k_{s+1} \) not included in the expression (3.5). These contributions turn out to be trivial. To see that we first notice that

\[ w^{n_1'; n_1} (n_3, n_3', n_3) = 0, \quad n_3 < 0, \quad n_3' \geq 0. \quad (3.8) \]

This follows from (2.5) since the factor \( 1/(q^2; q^2)_n \) becomes zero and all other factors are nonzero. Now let us look at the product in (2.8) and assume that there are contributions from negative values for some \( k_s, s = 1, \ldots, n \). All \( k_s \) can not be negative, since \( k_1 \geq 0 \). Since the product is cyclic, we will always find at least one factor \( w^{i_s'; i_s} (k_s, k_s) \) such that \( k_s < 0 \) and \( k_{s+1} \geq 0 \). This factor will be equal to zero because of (3.8). Therefore, all factors which contain some negative \( k_s \) automatically disappear and we can safely sum over \( k_1 \) from 0 to \( \infty \) in (3.5) with substitutions (3.7). As one can easily see the sum on \( k_1 \) becomes a geometric series which converges provided

\[ w q^{-(l-1)} < 1, \quad 0 < q < 1. \quad (3.9) \]

Once this condition is satisfied for \( w = \lambda^2 > 0 \), the sum in (2.8) has all positive terms, since all matrix elements of the 3D R-matrix (2.3) are positive. Restricting the result to fixed positive values \( I, J \) we get the expression for matrix elements of the operator \( R^{(n)}_{ij} (w) \) in (2.12). The result reads
\[ [R_{l,j}^{(n)}(\lambda)]_{l,j}^{\psi,f'} = \delta_{l+j,f'+j'} q^\psi \prod_{s=1}^{n} \left[ i_s + j_s \right] \sum_{m \in \mathbb{Z}_{2^n}} \frac{2^{l+2} \sum_{i=0}^{n-1} m_i (i-i')} {1 - \lambda^2 2^{2l} q^{-l-j}} \]

(3.10)

\[ \Psi = -2(i,j) + (i',j') - (I - |i|)(J - |j|) + I (|j'| - |i| - 1) + \sum_{1 \leq k < |l| \leq n-1} (i_k' j_k' - i_k j_k). \]

(3.11)

Here in the lhs of (3.10) and in the expression for the phase factor (3.11) we used \((n - 1)\)-component indices, see (3.1) with \(r = n - 1\). However, in the rhs of (3.10) for compactness we kept \(n\)-component external indices assuming that we need to substitute \(i_{n-1}, j_{n-1}, i'_{n-1}, j'_{n-1}\) from (3.6). The formula has \(n\) summation indices \(m_1, m_2, \ldots, m_n\) which truncate after finitely many terms. Finally we notice that the sum \(\sum_{k \geq 1}\) in (3.10) taken over \(n \geq k \geq l \geq 1\) can be restricted to the values \(n - 1 \geq k \geq l \geq 1\), since it is equal to zero for \(k = n\) due to (3.6).

The case \(n = 2\) of (3.10) was given in (75) of [18]. This formula generates elements of a \((l+1) - n - 1\) dimensional matrix determined by indices \(0 \leq |i|, |f'| \leq I, 0 \leq |j|, |j'| \leq J\).

As the next step we shall evaluate one sum in (3.10) and reduce the total number of summations to \(n - 1\). We use the same method as in [19].

We start with the Lagrange interpolating formula

\[ \sum_{x=0}^{l} \frac{x}{q^x} P_k(x) = P_k(q), \]

(3.12)

which is valid for any polynomial \(P_k(x)\) of degree at most \(k\). First we define a new variable

\[ l = m_1 + \cdots + m_n \]

(3.13)

which runs from 0 to \(l\) and use \(l\) instead of \(m_n\). Then one can rewrite (3.10) as

\[ [R_{l,j}^{(n)}(\lambda)]_{l,j}^{\psi,f'} = \delta_{l+j,f'+j'} (-1)^{|I|+|J|} \frac{q^{i-2(l+j)+2(j+f') - (l-|i|)(J-|j|)+I(|j'| - |i| - 1) + \sum_{s=0}^{n-1} (i_s' j_s') - i_s j_s}} {\prod_{s=1}^{n-1} \left[ i_s + j_s \right]} \]

\[ \times \sum_{s=0}^{l} \frac{q^{i_s} (q^{-2i_s}; q^2)_{|i|} (q^{-2j_s}; q^2)_{|j|}} {\prod_{s=1}^{l-1} \left[ q^{-2l_j} q^{-2l_i}; q^2 \right]_{|i|+|j|} P(q^{2l})}. \]

(3.14)

The summation in \(l\) matches (3.12) with \(k = l, x = \lambda^2 q^{l+j}\) and

\[ P_l(q^{2l}) = q^{2(l+|i|+|j|) + \sum_{m=0}^{l} 2^{l+m} \sum_{i,j} (i,j) \prod_{s=1}^{n-1} (q^{-2i_s}; q^{-2j_s}; q^2)_{m_i} \]

\[ \times (q^{-2i}; q^2)_{|i|} (q^{-2|j|}; q^2)_{|j|} \]

(3.15)

The polynomial \(P_l(x)\) in (3.15) has degree of at most \(l\) and therefore we can replace the sum in \(l\) in (3.14) with the right-hand side of (3.12) to find the expression
\[
[R_{J_iJ_j}^{(n)}(\lambda)]_{ij} = \delta_{i+j+i'j'} [A_{J_iJ_j}^{(n)}(\lambda)]_{ij} B_{J_iJ_j}(\lambda) q^{(i-j)(i'-j') - (i+1)(i'-1) + \sum_{s=1}^{3} (i_s j_s + j'_s i'_s)} \times \sum_{m \in \mathbb{Z}^{+}} \left( \lambda^2 q^{2-2m}; q^2 \right)_{m_1} \left( \lambda^2 q^{2-2m}; q^2 \right)_{m_2} \left( \lambda^2 q^{2-2m}; q^2 \right)_{m_3} \prod_{s=1}^{n-1} \left[ i_s + j_s \right] \left[ j'_{s} \right] \frac{\left( \lambda^2 q^{2-2m}; q^2 \right)_{m+1}}{\left( \lambda^2 q^{2-2m}; q^2 \right)_{m+1}}.
\]

(3.16)

All external and summation indices in (3.16) have \( n - 1 \) components and the coefficients \( A_{J_iJ_j}^{(n)}(\lambda)_{ij} \) and \( B_{J_iJ_j}(\lambda) \) are given by

\[
[A_{J_iJ_j}^{(n)}(\lambda)]_{ij} = \frac{\left( \lambda^2 q^{2-2m}; q^2 \right)_{m_1} \left( \lambda^2 q^{2-2m}; q^2 \right)_{m_2} \left( \lambda^2 q^{2-2m}; q^2 \right)_{m_3}}{\left( \lambda^2 q^{2-2m}; q^2 \right)_{m_1} \left( \lambda^2 q^{2-2m}; q^2 \right)_{m_2} \left( \lambda^2 q^{2-2m}; q^2 \right)_{m_3}} \prod_{s=1}^{n-1} \left[ i_s + j_s \right] \left[ j'_{s} \right].
\]

(3.17)

\[
B_{J_iJ_j}(\lambda) = q^{-(i-j)} \frac{\left( \lambda^2 q^{2-2m}; q^2 \right)_{m+1}}{\left( \lambda^2 q^{2-2m}; q^2 \right)_{m+1}}.
\]

(3.18)

The formula (3.16) gives the answer for the matrix elements of the \( \widehat{U}_{J} \) \( R \)-matrix acting on the space \( V_I \otimes V_J \)

\[
V_I \equiv \{ |i \rangle \}, \quad |i \rangle \leq I.
\]

(3.19)

It follows from the tetrahedron equation for the 3D \( R \)-matrix (2.3) that (3.16) satisfies the Yang–Baxter equation

\[
R_{J_iJ_j}^{(n)}(\lambda \mu) R_{J_jK_k}^{(n)}(\mu) = R_{J_iK_k}^{(n)}(\mu \lambda) R_{J_iJ_j}^{(n)}(\lambda)
\]

(3.20)

for any \( I, J, K \in \mathbb{Z}_+ \). However, one will notice that the coefficient \( B_{J_iJ_j}(\lambda) \) is just a constant not depending on indices. We find it convenient to set this factor to 1. In what follows, we will use (3.16) with \( B_{J_iJ_j}(\lambda) = 1 \) unless stated otherwise. In this normalization we have

\[
[R_{J_iJ_j}^{(n)}(\lambda)]_{00} = 1.
\]

(3.21)

The main reason we do this is because (3.16) is now well defined even when \( I, J \in \mathbb{C} \). Although the 3D model projection outlined in this paper satisfies the Yang–Baxter equation for integral weights by construction, the equation (3.20) remains valid even for complex weights \( I, J, K \in \mathbb{C} \). The proof closely follows the arguments given in [26].

Consider a particular matrix element of the Yang–Baxter equation \( (i, j, k) \) \( (i', j', k') \) with fixed external indices \( I = (i_0, \ldots, i_{n-1}) \), etc. Due to the conservation law in (3.16) we have \( |i + j + k| = |i' + j' + k'| \equiv m \) and all summation indices in (3.20) will also be limited by \( m \). Choose an integer \( N > m \) and assume that integer weights \( I, J, K > N \). It is easy to see that all denominators in the \( R \)-matrices entering the Yang–Baxter equation are nonzero and (3.20) becomes the equality of two rational functions in variables \( x = q^{-l}, y = q^{-j} \) and \( z = q^{-k} \). After eliminating denominators we can rewrite (3.20) as equality of two polynomials in three variables \( x, y, z \). The degree of these polynomials grows as a fixed polynomial in \( N \). Now we know that the Yang–Baxter equation (3.20) is true for infinitely many integer variables \( I, J, K > N \). It can only happen if (3.20) reduces to a polynomial identity in \( x, y, z \in \mathbb{C} \). Therefore, the Yang–Baxter equation with the \( R \)-matrix (3.16) and normalization (3.21) is satisfied for \( I, J, K \in \mathbb{C} \). In this case it defines the infinite dimensional \( R \)-matrix corresponding to Verma module representations of \( \hat{U}_{J} \).

To illustrate how formula (3.16) works, let us consider a special case \( n = 2 \). In this case matrix elements are indexed by indices \( i, j, i', j' \), (3.16) becomes a single sum which is given
by
\[
[R_{I,J}^{(2)}(\lambda)]_{i,j}^{f,g} = \delta_{i+j,i'+j'} q^{i'+i-j-j'} [i+j]_q \left( \frac{\lambda^{2i} q^{i+j}; q^2}{\lambda^{2i'} q^{i'+j'}; q^2} \right)_{i+j} \frac{\lambda^{2i} q^{i+j}; q^2}{\lambda^{2i'} q^{i'+j'}; q^2}.
\]

This is a truncated and balanced basic hypergeometric series \(\phi_3\) for the elements of the \(U_q(\mathfrak{sl}_2)\) R-matrix. This case was already studied in [19] and the formula given there is of the same type as (3.22) but with different arguments. Most notably, the hypergeometric sum in [19] is a polynomial in the spectral parameter \(\lambda\) while (3.22) is a rational function.

Using the Sears transform (A.7) we can transform the sum in (3.22) to (5.8) in [19] by identifying \(n = i, a = q^{-2i}, b = \lambda q^{i-j}, c = \lambda^2 q^{2i+j-2i-2j}, d = \lambda^2 q^{2i+j-2i}, e = q^{-2i-2j}\) and \(f = \lambda^2 q^{2i+j-2i-2j}\). Let us note that (5.8) in [19] requires a regularization but the expression (3.22) is free from any divergences.

One of the problems with (3.16) is that the hypergeometric sum is a rational function in \(\lambda\). For integer \(I\) the whole expression (3.16) is a polynomial in \(\lambda\) up to an overall factor \((\lambda^2 q^{i-j}; q^2)_{i+j+1}\). However, when both weights \(I, J\) are complex, this factor is no longer a polynomial in \(\lambda\) and no polynomial normalization exists. In this case we can adapt the normalization (3.21) when matrix elements of the R-matrix are rational functions of \(\lambda\).

As mentioned before, using the Sears’ transformation the hypergeometric sum in (3.16) for \(n = 2\) can be transformed into a hypergeometric polynomial in \(\lambda\) up to simple \(q\)-binomial factors. The authors are aware of the \(\Lambda_n\) multivariable generalizations of the Sears’ transformation in the literature, but they do not appear to be applicable to our expression for \(n \geq 3\).

Focusing on each summation index in (3.16) one can easily see that it is a \(\phi_3\) basic hypergeometric series but it is not balanced and only one of the hypergeometric series has a \(q^2\) argument so the \(A_1\) Sears’ transformations does not apply.

We expect that a formula with a hypergeometric sum being a polynomial in \(\lambda\) still exists but it probably requires a new yet to be discovered identity for multivariable hypergeometric series.

4. Symmetries and special cases

In this section we discuss symmetries of the R-matrix \(R_{I,J}^{(n)}(\lambda)\) given by (3.16). They can be derived from the corresponding symmetries of the 3D R-matrix generated by (2.4). It is actually more convenient to use (2.5) and (2.6) since we need to to keep a position of the 3rd ‘hidden’ direction where we take the trace. Applying these transformations to the factors in (2.8) we find two symmetries

\[
[R_{I,J}^{(n)}(\lambda)]_{i,j}^{f,g} = \lambda^{2([f]-[i])} [R_{I,J}^{(n)}(\lambda)]_{i,f}^{g,j},
\]

\[
[R_{I,J}^{(n)}(\lambda)]_{i,j}^{f,g} = q^{2([f]-2([i]))} \lambda^{2([f]-[i])} \prod_{j=1}^{n} \frac{(q^2; q^2)_j (q^2; q^2)_j}{(q^2; q^2)_j (q^2; q^2)_j} [R_{I,J}^{(n)}(\lambda)]_{i,f}^{g,j}.
\]

Let us explain some notations here. In the previous section we mentioned that for the R-matrix \(R_{I,J}^{(n)}(\lambda)\) we are using \(n - 1\)-component indices, i.e. \(i = \{i_1, \ldots, i_{n-1}\}\) with the last \(n\)th component \(i_n = I - |f|\) removed and similar for \(j\)’s. However, in (4.2) the product in the rhs is
taken over $s = 1, \ldots, n$ where for $s = n$ we substitute the last component as above, i.e. $i_n = I - |i|$, $j_n = J - |j|$, etc. The transformation $\tau$ is defined in (3.2).

In addition, in (4.2) we used a notation $[i, j]$ for a convolution of $n$-component indices, i.e.

$$[i, j] = (i, j) + (I - |i|)(J - |j|).$$

(4.3)

There is also a symmetry of the $R$-matrix which corresponds to the cyclic permutation of the $n$ 3D $R$-matrices in the ‘hidden’ direction. Let us introduce the notation

$$\tilde{i} = [I - |i|, i_1, \ldots, i_{n-2}].$$

(4.4)

which is equivalent to the permutation $\sigma^{-i}$ for the $n$-tuple $i$ but with the last component removed. Here we assume that $I, J \in \mathbb{Z}_+$. Performing a cyclic shift in (2.8) we easily obtain

$$[R^{(n)}_{i,j}(\lambda)]_{I,J}^{I',J'} = \lambda^{2(|I'|-|I|)}[R^{(n)}_{i,j}(\lambda)]_{I,J}^{I',J'}. $$

(4.5)

For example, when $n = 2$ this corresponds to $i \rightarrow I - i$ and similarly for other indices.

The last symmetry follows from the transformation of the 3D $R$-matrix (2.7). After simple calculations one can obtain the following result

$$[R^{(n)}_{i,j}(\lambda, q)]_{I,J}^{I',J'} = q^{(|I'|-|I|)}[R^{(n)}_{i,j}(\lambda^{-1}, q^{-1})]_{I,J}^{I',J'}. $$

(4.6)

Finally, when $I = J$ and $\lambda = 1$ the $R$-matrix reduces to permutation operator

$$R^{(n)}_{i,j}(1) = \mathcal{P}_{1,2} $$

(4.7)

which can be seen from (5.2) in the next section.

### 5. Reductions and factorization

There are two special points in the spectral parameter $\lambda = q^{\frac{1}{2}(I-J)}$ where the multiple sum in (3.16) reduces to one nonzero summand. These specializations produce the $R$-matrix without difference property with weights $I, J$ playing the role of spectral parameters. With the normalization (3.21) we can choose $I, J \in \mathbb{C}$ and obtain the $R$-matrix acting in the tensor product of two Verma modules.

For the case of the $U_q(\mathfrak{s}\mathfrak{l}_2)$ algebra the importance of such reductions was first noticed in [21]. Under the choice $\lambda = q^{\frac{1}{2}(I-J)}$ the $U_q(\mathfrak{s}\mathfrak{l}_2)$ $R$-matrix of [19] reduces to the $R$-matrix of Povolotsky model [24] which satisfies stochasticity condition and defines a family of zero-range hopping models. A generalization of the Povolotsky model to arbitrary rank $n - 1$ was obtained in the recent paper [26].

So let us start with the case $\lambda = q^{\frac{1}{2}(I-J)}$, $I - J \in \mathbb{Z}_+$. The expression for the $R$-matrix (3.16) contains the factor $(\lambda^2q^{I-J}; q^2)^{|I|}$ outside the sum which has the argument 1 after the above substitution. This factor is always zero for $|J'| > 0$ unless it is canceled off by the factor $(\lambda^2q^{2I-J-|J'|} q^2)^{|J|}$ inside the sum. It can only happen when $|m| = |J'|$ or $m = J'$, since $m_s \leq \frac{1}{2}, s = 1, \ldots, n - 1$. Let us note that the argument fails when $J - I$ is a positive integer because the other factor in the denominator can cancel off the zero of $(\lambda^2q^{I-J}; q^2)^{|I|}$ and multiple summands survive.
After simple algebra one can derive from (3.16) the following result

\[
\left[ \tilde{R}_{l,j}^{(i)} \left( q \frac{J+I}{2} \right) \right]_{i,j}^{l,j} = \delta_{I,J} q^{i+j} \left( q^{2j} - q^{2I-2j} + q^{2j-I} - q^{2I+2j} + 2q^{j+2I} + 2q^{I+2j} + 2q^{2I+j} + 2q^{2j+I} - 2q^{2j} \right) 
\]

\[
\times \left( q^{2j-I} \right) \left( q^{2I-2j} \right) \left( q^{2j-I} \right) \left( q^{2I+2j} \right) \left( q^{2j+I} \right) \left( q^{2I+j} \right) \prod_{i=1}^{n-1} \left[ i' \right] \left[ j \right]. 
\]  

(5.1)

Similarly, we can make substitution \( \lambda = q^{\frac{J-I}{2}} \) for \( J-I \in \mathbb{Z} \). In this case the argument is the same except with the factors \( (\lambda q^{2j-I} q^{2j-I}) \) and \( (\lambda q^{2I-2j} q^{2I-2j}) \) and so the only summand that contributes is \( m = i \). Then we obtain

\[
\left[ \tilde{R}_{l,j}^{(i)} \left( q \frac{J-I}{2} \right) \right]_{i,j}^{l,j} = \delta_{I,J} q^{i+j} \left( q^{2j-I} - q^{2I-2j} + q^{2j-I} - q^{2I+2j} + 2q^{j+2I} + 2q^{I+2j} + 2q^{2I+j} + 2q^{2j+I} - 2q^{2j} \right) 
\]

\[
\times \left( q^{2j-I} \right) \left( q^{2I-2j} \right) \left( q^{2j-I} \right) \left( q^{2I+2j} \right) \left( q^{2j+I} \right) \left( q^{2I+j} \right) \prod_{i=1}^{n-1} \left[ i' \right] \left[ j \right]. 
\]  

(5.2)

Obviously these reductions are substantially simpler than the original \( R \)-matrix. As mentioned above \( I, J \) play role of the spectral parameters for these \( R \)-matrices and can now take arbitrary complex values.

In fact, one can construct the full \( R \)-matrix as a matrix product of (5.1) and (5.2). To explain this it is convenient to apply a simple similarity transformation in the first space and introduce

\[
\tilde{R}_{l,j}^{(i)}(\lambda) = U \otimes 1 \tilde{R}_{l,j}^{(i)}(\lambda) U^{-1} \otimes 1 
\]  

(5.3)

with

\[
U_{l,i'} = \delta_{l,i'} (\lambda q^{j-I-j})^{2|l|}. 
\]  

(5.4)

Now let us define two operators \( M \) and \( N \) acting in the tensor product of two Verma modules by

\[
M(q^l, q^j) = \tilde{R}_{l,j}^{(i)} \left( q \frac{I+J}{2} \right), \quad N(q^l, q^j) = \tilde{R}_{l,j}^{(i)} \left( q \frac{J-I}{2} \right), 
\]

(5.5)

where as usual \( \tilde{R}_{l,j}^{(i)}(\lambda) = \mathbb{P}_{l,j} R_{l,j}^{(i)}(\lambda) \), etc with \( \mathbb{P}_{l,j} \) being the permutation operator. Both operators \( M(q^l, q^j) \) and \( N(q^l, q^j) \) of complex arguments \( q^l, q^j \) are defined by its matrix elements via (5.1), (5.2) and (5.5).

With these notations one can easily derive from (3.16) the following factorization

\[
\tilde{R}_{l,j}^{(i)}(\lambda) = M \left( \frac{L+j}{2}, q^l \right) N \left( \frac{I+j}{2}, q^j \right). 
\]  

(5.6)

A similar factorization of the \( R \)-matrix appeared in [28] for the \( n = 2 \) case of the XXX chain.

We can also rewrite a factorization formula (5.6) for the matrix elements of the original \( R \)-matrix \( R_{l,j}^{(i)}(\lambda) \) as follows

\[
\left[ R_{l,j}^{(i)}(\lambda) \right]_{i,j}^{l,j} = \sum_{k+l=i+j} \tilde{M}_{l,i} \tilde{N}_{j,i}^{l,j} 
\]  

(5.7)
with

\[ \tilde{M}_{i,j}^{l,l'} = \delta_{i+j,i'+j'} q^{-2j} \prod_{s=1}^{n-1} \left[ j' \right] \left( \lambda q^{2-2j} ; q^{2} \right)^{j-j'-l} \left( \lambda q^{-2j+l} ; q^{-2} \right)^{j-l'} , \]

(5.8)

\[ \tilde{N}_{i,j}^{l,l'} = \delta_{i+j,i'+j'} q^{-2j} \prod_{s=1}^{n-1} \left[ j' \right] \left( \lambda q^{2-2j} ; q^{2} \right)^{j-j'-l} \left( \lambda q^{-2j+l} ; q^{-2} \right)^{j-l'} , \]

(5.9)

where we removed some gauge factors which cancel in the matrix product (5.7).

6. Comparison with other results

In this and next sections we will compare (3.16) with some other presentations of the \( U_q(\tilde{sl}_n) \) related \( R \)-matrix given in the literature. We will establish a connection with the standard the \( U_q(\tilde{sl}_n) \) -operator presented in [26] and also compare our results with some higher-spin examples of the \( U_q(\tilde{sl}_n) \) -matrix.

We start with some remarks regarding the coefficient \( l_{A_{IJ}}^{IJ} \) in (3.17). For specific elements of the \( R \)-matrix the \( q \)-Pochhammer symbols are finite as their arguments are integers.

If we want to derive the formula for the \( L \) -operator as an \( n \times n \) -matrix with operator entries acting in the Verma modules spanned by \( \langle a | = \langle a | a, a, \ldots, a | \rangle \rangle \) we need to rewrite (3.17) in the form suitable for abstract values of \( j \) indices. This is achieved by a slight change of normalization of the \( R \)-matrix

\[ \tilde{R}_{i,j}^{(n)} (\lambda) = \sigma_{i,j} (\lambda) R_{i,j}^{(n)} (\lambda) , \]

(6.1)

with

\[ \sigma_{i,j} (\lambda) = - \lambda^{i-j} q^{i+j} \left( \lambda^{2} q^{2-2j} ; q^{2} \right)^{j-l_1} \left( \lambda^{2} q^{-2j} ; q^{2} \right)^{j-l_1}. \]

(6.2)

We also restore a coefficient \( B_{i,j} (\lambda) \) in (3.18) and define

\[ \tilde{A}_{i,j}^{(n)} (\lambda) = \sigma_{i,j} (\lambda) A_{i,j}^{(n)} (\lambda) B_{i,j} (\lambda) , \]

(6.3)

After simple calculations we obtain

\[ \tilde{A}_{i,j}^{(n)} (\lambda) = \frac{(-1)^{i+j} q^{i+j} \lambda^{i+j} \left( \lambda^{2} q^{2-2j} ; q^{2} \right)^{j-l_1} \prod_{s=1}^{n-1} \left( q^{2j} ; q^{2} \right)^{j-s} \left( q^{2j} ; q^{2} \right)^{j-l_1}}{\left( q^{2} q^{2j} ; q^{2} \right)^{j-l_1} \left( q^{2} q^{2j} ; q^{2} \right)^{j-l_1}} \]

(6.4)

and this expression is a finite product for integer \( i, i' \) and abstract values of \( j \)’s. Shortly speaking a change of normalization is equivalent to replacing the product \( A_{i,j}^{(n)} (\lambda) B_{i,j} (\lambda) \) in (3.16) with (6.4). The sum in (3.16) is still finite because it truncates by integer values of \( i \)’s.

It is easier to write down explicit formulas in original \( n \)-component notations. Introduce \( n \)-component vectors \( e_a = (0, \ldots, 0, 1, 0, \ldots, 0) \) with 1’s at the \( a \)th position from the left, \( j, k \in \mathbb{Z}_n^+ \) with \( |j| = |k| = J \). Then using (6.4) in (3.16) we obtain for the renormalized \( R \)-matrix (6.1)
It is easy to check that the extra twist factor $q^{[6]}(\lambda)$ in (6.10) drops out from the Yang–Baxter equation.

However, the matrix elements of $R^{(\gamma)}_{l,m}(z)$ in [26] are defined by the transposed action

$$[R^{(\gamma)}_{l,m}(z)]^\gamma_{\alpha,\beta} = \langle \gamma, \delta | R^{(\gamma)}_{l,m}(z) | \alpha, \beta \rangle.$$ (6.12)
We have checked that the relation (6.10) holds for the $U_q(A_n^{(1)})$ and $U_q(A_{n+1}^{(1)})$ $R$-matrices for all cases given in appendix A of [26].

It is also interesting to compare our reductions (5.1) and (5.2) with that obtained in [26]. In particular, we expect that the theorem 2 in [26]

\[ [R_{l,m}^K(q^{l-m})]_{a_0,b} = \delta_{a_0+b-\gamma,\delta} q^{\gamma \delta} m \prod_{s=1}^{l} \begin{bmatrix} \beta_s \\ \gamma_s \end{bmatrix}, \] (6.13)

\[ \psi = \sum_{1 \leq s \leq n+1} \alpha_s (\beta_s - \gamma_s) + \sum_{1 \leq s \leq n+1} (\beta_s - \gamma_s) \gamma_s, \] (6.14)

should correspond to the substitution $\lambda = q^{-1}$ given by (5.2). A direct calculation shows the relation (6.10) also holds in this case.

Now let us turn to the $U_q(sl_n)$ $L$-operator. When $J = 1$ the expression (6.5) further reduces to the trigonometric $n$-state $R$-matrix [29, 30]. We shall also use a twisted version of this $R$-matrix [31] which we give using notations of [32]

\[ R_{\alpha,\gamma}^{\beta,\delta}(\lambda) = \delta_{\alpha, \beta} \delta_{\gamma, \delta} (q - 1) (\lambda + \lambda^{-1} q^{-1}) + \delta_{\alpha, \beta} \delta_{\gamma, \delta} \rho_{\alpha, \gamma} (\lambda - \lambda^{-1}) + \delta_{\alpha, \beta} \delta_{\gamma, \delta} \sigma_{\alpha, \beta}, \] (6.15)

where

\[ \sigma_{\alpha, \beta} = \begin{cases} 0 & \text{if } \alpha = \beta, \\ (q - q^{-1}) \lambda & \text{if } \alpha < \beta, \\ (q - q^{-1}) \lambda^{-1} & \text{if } \alpha > \beta \end{cases} \] (6.16)

and $\rho_{\alpha, \beta}$ are nonzero complex parameters such that

\[ \rho_{\alpha, \alpha} = \rho_{\alpha, \beta} \rho_{\beta, \alpha} = 1, \quad \alpha, \beta = 1, \ldots, n. \] (6.17)

Setting all $\rho_{\alpha, \beta} = 1$ and taking convention that all indices $\alpha, \beta, \gamma, \delta = 1, \ldots, n$ in (6.15) denote positions of 1’s counted from the right, i.e. $\alpha \equiv \epsilon_{n-\alpha+1}$ we obtain that (6.15) is equivalent to (6.5) with $J = 1$.

Setting $I = J = 1$ in the Yang–Baxter equation (3.20) we obtain the $L$-operator algebra

\[ R_{1,2}(\lambda/\mu) L_1(\lambda) L_2(\mu) = L_2(\mu) L_1(\lambda) R_{1,2}(\lambda/\mu), \] (6.18)

where the $R_{1,2}(\lambda)$-matrix corresponds to the standard $U_q(A_{n-1}^{(1)})$ trigonometric $R$-matrix (6.5) with $J = 1$. The $L$-operators $L(\lambda)$ are identified with $R_{1,2}(\lambda)$ (6.5) acting in the ‘quantum’ space with the weight $K$.

To rewrite the $L$-operator in algebraic notations let us introduce Weil operators $X_k, Z_k$, $i = 1, \ldots, n$ acting in the space of $n$-component vectors $|j\rangle$, $j_i \in \mathbb{Z}$, $s = 1, \ldots, n$ and their conjugates such that

\[ Z_k |j\rangle = q^{k} |j\rangle, \quad X_k |j_1, \ldots, j_n\rangle = |j_1, \ldots, j_k + 1, \ldots, j_n\rangle, \] (6.19)

\[ \langle j| Z_k = q^{-k} \langle j, \ldots, j_k| \quad \langle j_1, \ldots, j_n| X_k = \langle j_1, \ldots, j_k - 1, \ldots, j_n| \] (6.20)

They satisfy the Weil algebra relations

\[ Z_k X_l = q^{k-l} X_l Z_k, \quad k, l = 1, \ldots, n. \] (6.21)

We can now define the $L$-operator $L(\lambda)$ as an $n \times n$ matrix with operator entries such that

\[ \langle j| L_{\alpha, \beta}(\lambda) |k\rangle = [R_{1,2}(\lambda)]_{\alpha, \beta}^{\gamma, \delta} |\psi\rangle \] (6.22)
Using (6.19)–(6.21) we obtain

\[
L_{\alpha, \beta}(\mu) = \begin{cases} 
[\mu Z_\alpha] & \text{if } \alpha = \beta, \\
\mu X_\alpha^{-1}X_\beta[Z_\alpha] \prod_{s=\beta}^{\alpha-1} Z_s & \text{if } \alpha > \beta, \\
\mu^{-1}qX_\alpha^{-1}X_\beta[Z_\alpha] \prod_{s=\alpha}^{\beta-1} Z_s^{-1} & \text{if } \alpha < \beta,
\end{cases}
\] (6.23)

where we defined a rescaled spectral parameter \( \mu = \lambda q^{1/2} \) and for any vector \( \ket{j} \), \( \ket{j} = J \)

\[
Z(j) = q^{\mu \ket{j}}, \quad Z = \prod_{s=1}^{n} Z_s.
\] (6.24)

In fact, we can consider (6.23) as an operator solution of the algebra (6.18) since a rescaling of the spectral parameter does not affect (6.18). The operator \( Z \) commutes with (6.23) and all representations are characterized by its complex eigenvalue \( q^{\mu} \).

7. Stochastic \( R \)-matrix

Let us define another \( R \)-matrix \( S_{ij}(\lambda) \) by

\[
[S_{ij}(\lambda)]_{i'j'}^{ij} = \rho_{ij}^{i'j'} [R_{ij}^{(n)}(\lambda)]_{i'j'}^{ij}, \quad i, j, i', j' \in \mathbb{Z}_+^{n-1},
\] (7.1)

with

\[
\rho_{ij}^{i'j'} = q^{(i-j)(i'-j')+\sum_{1 \leq k < c \leq i} (k-i') + (i'-j')(j'+j)} = q^{(i-j)(i'-j') - j(i+k) + \sum_{1 \leq k < c \leq i} (k-i')}.
\] (7.2)

In [26] \( S_{ij}(z) \) was given in terms of \( R_{ij}^{(n)}(z) \) with \( z = \lambda^2 \). Here we defined \( S_{ij}(\lambda) \) in terms of \( R_{ij}^{(n)}(\lambda) \) using the relation (6.10). Using quantum group arguments it was shown in [26] that (7.1) solves the Yang–Baxter equation and satisfies the stochasticity condition

\[
\sum_{ij} [S_{ij}(\lambda)]_{i'j'}^{ij} = 1.
\] (7.3)

We can now give the direct proof of (7.3) using the explicit formula (3.16) for the \( R \)-matrix.

To do that we find it convenient to follow notations of [26]. Introduce the function

\[
\Phi_\mu(\gamma|\beta; \mu, \lambda) = q^{\mu \frac{1}{\lambda} \frac{(\lambda; q)_\beta (\mu; q)_\gamma}{(\mu; q)_\beta (\lambda; q)_\gamma}} \prod_{s=1}^{n-1} \left[ \frac{\beta_s}{\gamma_s} \right],
\] (7.4)

\[
\xi = \sum_{1 \leq i < k \leq n} (\beta_i - \gamma_i) \gamma_k.
\] (7.5)

where \( \alpha, \beta, \gamma, \delta \in \mathbb{Z}_+^{n-1}, \lambda, \mu \in \mathbb{C} \). This function satisfies the following sum rule

\[
\sum_{i} \Phi_\mu(ij; \lambda, \mu) = 1.
\] (7.6)

Note that the sum in (7.6) is always finite since the summand is equal to zero unless \( 0 \leq i \leq j \), i.e. \( 0 \leq i_s \leq j_s \) for all \( s = 1, \ldots, n - 1 \). The relation (7.6) can be easily proved by induction in \( n \), see [26] for details.
Using these definitions and the expansion (5.7) \( R_{i,j}^{(n)}(\lambda) \) can be expressed as
\[
[R_{i,j}^{(n)}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} q^{(i'j')-(ij)+J(i+j)+J(i+j)+\sum_{a,b}((a,b)^2-2ab)} \times \sum_{m+n-i+j} \Phi_q(j|m; q^{-2j}, \lambda^2 q^{-1-j}) \times q^{2(m+n-i+j)^2} \times \Phi_q(n|m; q^{-2n-j}, \lambda^2 q^{-1-j})
\]
(7.7)
where we imply that the sum is taken over \( m, n \in \mathbb{Z}^+ \) with the sum \( m + n = i + j \) fixed.

Using this presentation of \( R_{i,j}^{(n)}(\lambda) \) in terms of \( \Phi \) we can rewrite the expression for matrix elements of \( S_{i,j}(\lambda) \) as
\[
[S_{i,j}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \sum_{m+n=i+j} \Phi_q(\lambda^2 q^{(i'j')-(ij)+J(i+j)} | m-j|m; q^{-2j}, \lambda^2 q^{-1-j}) \times q^{2(m+n-i+j)^2} \times \Phi_q(n|m; q^{-2n-j}, \lambda^2 q^{-1-j})
\]
(7.8)
This expression can be simplified using symmetries of the function \( \Phi \). Substituting the explicit form of \( \Phi(7.4) \) one can easily check that
\[
\Phi_q(m-j|m; \mu/j, \mu) = \Phi_q(j|m; \lambda, \mu) q \sum_{c,d}((m-c,d)^2-2(c,d)) \mu^{-|j|}|m|.
\]
(7.9)
Then we can rewrite (7.8) in a factorized form
\[
[S_{i,j}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \sum_{m+n=i+j} \Phi_q\left(m-j|m; q^{-1-j}, \lambda^2 q^{-1-j}\right) \Phi_q\left(n|m; \lambda^2 q^{-1-j}, q^{-2j}\right).
\]
(7.10)
Now the relation (7.3) becomes trivial. Indeed, for fixed \( i', j' \) we have
\[
\sum_{i,j} [S_{i,j}(\lambda)]_{i,j}^{i',j'} = \sum_{i+j=i'+j'} \Phi_q\left(m-j|m; q^{-1-j}, \lambda^2 q^{-1-j}\right) \Phi_q\left(n|m; \lambda^2 q^{-1-j}, q^{-2j}\right)
\]
\[
= \sum_{m+n=i'+j'} \Phi_q\left(n|m; \lambda^2 q^{-1-j}, q^{-2j}\right) \sum_{i+j=m+n} \Phi_q\left(m-j|m; q^{-1-j}, \lambda^2 q^{-1-j}\right)
\]
\[
= \sum_{m+n=i'+j'} \Phi_q\left(n|m; \lambda^2 q^{-1-j}, q^{-2j}\right) = 1,
\]
(7.11)
where we used twice the relation (7.6).

Setting \( \lambda = q^{(j-j')/2} \) in (7.8) and using relations
\[
\Phi_q(|i'| j; 1, \mu) = \delta_{i,0}, \quad \Phi_q(|i| j; \mu, \mu) = \delta_{i,j},
\]
(7.12)
we obtain two nontrivial degenerations of the \( R \)-matrix \( S_{i,j}(\lambda) \)
\[
[S^{(1)}(\mu, \nu)]_{i,j}^{i',j'} = \frac{[S_{i,j}(q^{(j-j')/2})]}_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \delta_{i,j} \Phi_q(|i| j; \mu, \nu)
\]
(7.13)
and
\[
[S^{(2)}(\mu, \nu)]_{i,j}^{i',j'} = \frac{[S_{i,j}(q^{(j-j')/2})]}_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \delta_{i,j} \Phi_q(|i| j; \nu, \mu) q^{-2j} \sum_{c,d}((c,d)^2-2(c,d)) \mu^{-|j|}|j|.
\]
(7.14)
where \( \mu = q^{-2j}, \nu = q^{-2j} \) play the role of (complex) spectral parameters. Similar formulas for the \( R \)-matrix \( R_{i,j}^{(n)}(\lambda) \) have been already obtained in (5.1) and (5.2).
We can now derive the formula for the $L$-operator corresponding to the stochastic $R$-matrix (7.1). First, we choose $I = 1$, $J \in \mathbb{Z}_+$ and $i = e_\alpha, i' = e_\beta$. Let us notice that the exponent of the $q$-factor in (7.1) can be compactly written in $n$-component notations as follows

$$\sum_{1 \leq k < l \leq n} (k_j - k'_j) = \sum_{k=1}^{\alpha} - \sum_{k=1}^{\beta} =: \delta_{\alpha, \beta}.$$  (7.15)

In particular, for $J = 1$ it simplifies to

$$\rho_{e_\alpha, e_\gamma} = q^{\delta_{\alpha, \gamma}}$$  (7.16)

for $e_\alpha + e_\gamma = e_\beta + e_\delta$ with

$$\epsilon_{\alpha, \gamma} = \begin{cases} 1, & \alpha > \gamma, \\ 0, & \alpha = \gamma, \\ -1, & \alpha < \gamma. \end{cases}$$  (7.17)

Let us comment that (7.16) corresponds to the case

$$\rho_{h, \gamma} = q^{\epsilon_{\alpha, \gamma}}$$  (7.18)

in (6.15). It was shown in [32] that (7.18) leads to a factorization of the $L$-operators at roots of unity. It would be interesting to understand further a relation between stochasticity and factorization of $L$-operators.

We can now derive the formula for the $L$-operators corresponding to the stochastic $R$-matrix $S_{IJ}(\lambda)$. Using (7.15) for general $J$ and (6.5) one can write it in terms of Weil generators (6.2) similar to (6.23) in a compact form

$$L_{\alpha, \beta}(\mu) = \mu^{e_{\alpha, \beta}} X_\alpha^{-1} X_\beta [\mu^{e_{\alpha, \beta}} Z_\gamma] \prod_{\gamma=1}^{n} Z_\gamma^{e_{\alpha, \gamma}}.$$  (7.19)

It satisfies the algebra

$$S_{12}(\mu/\nu) L_{\alpha, \beta}(\mu) \otimes L_{\gamma, \delta}(\nu) = L_{\alpha, \beta}(\nu) \otimes L_{\gamma, \delta}(\mu) S_{12}(\mu/\nu),$$  (7.20)

where $S_{12}(\lambda)$ is given by (7.1) with $I = J = 1$. This $L$-operator was first obtained in [32] in a slightly different form. The root of unity condition $q^N = 1$ used there does not affect the local structure of the algebra (7.20).

Choosing the eigenvalue of the operator $Z$ in (6.24) as $C$ one can rewrite (7.19) as

$$L_{\alpha, \beta}(\mu) = \mu^{e_{\alpha, \beta}} C X_\alpha^{-1} X_\beta (1 - \mu^{-2^{e_{\alpha, \beta}} Z_\gamma^{-2}}) \prod_{s=\alpha+1}^{n} Z_s^{-2}.$$  (7.21)

This $L$-operator contains two complex parameters $\mu$ and $C = q^J$, where $J$ can be identified with the weight of representation. As well known one can multiply the $L$-operator (7.21) by arbitrary complex parameters $a_i$ (‘horizontal’ fields) from the left without affecting the Yang-Baxter relation. It immediately follows from the property

$$[A_1 \otimes A_2, S_{12}(\mu)] = 0.,$$  (7.22)

where $A = [a_1, \ldots, a_n]$.

We can also remove one pair of Weyl operators $Z_i, X_i$ by setting

$$Z_i = C \prod_{i=2}^{n} Z_i^{-1}, \quad X_i \equiv 1.$$  (7.23)
Let us introduce another set of operators
\[ k_i = q^{-2}Z_{i+1}^{-2}, \quad \phi^+_i = X_{i+1}^{-1}(1 - Z_{i+1}^{-2}), \quad \phi_i = X_{i+1}, \quad i = 1, \ldots, n - 1 \] (7.24)
instead of \( Z_i, X_i, i = 2, \ldots, n \). Each set \( k_i, \phi_i, \phi^+_i \) forms a \( q \)-oscillator algebra
\[ \phi k = q^2 k \phi, \quad \phi^+ k = q^{-2} k \phi^+, \quad \phi \phi^+ - q^2 \phi^+ \phi = 1 - q^2. \] (7.25)

If we now choose
\[ a_1 = -\mu C, \quad a_i = \frac{\mu v}{C} q^{2(i-1-n)}, \quad i = 2, \ldots, n \] (7.26)
and make a change of variables
\[ C = \frac{\sqrt{\mu v}}{q^n}, \quad \mu = q \sqrt{\frac{x}{v}}, \] (7.27)
then we get exactly the \( L \)-operator from the recent paper by Garbali et al [27]
\[ L_{i,j}^{GGW}(x) = a_i L_{i,j}(\mu), \] (7.28)
with \( L_{i,j}(\mu) \) given by (7.21). Therefore, the \( L \)-operator \( L_{i,j}^{GGW}(x) \) corresponds to the standard \( U_q(\mathfrak{sl}(n)) \) \( L \)-operator for symmetric representations in the presence of twist and ‘horizontal’ fields.

8. Conclusion

In this paper we have constructed a new formula of the \( R \)-matrix \( R(\lambda) \) acting in the tensor product of two symmetric representations of the quantum group \( U_q(\mathfrak{sl}(n)) \). The method is based on calculating the \( n \)-layer projection of the 3D integrable model introduced in [16–18]. The final result (7.7) can be represented in the factorized matrix form with both factors given by a simple product formula (7.4).

The structure of this factorized representation is quite interesting. The weights of representations enter the result algebraically with no poles at integer values, so the formula equally applies to finite-dimensional and infinite-dimensional representations. For integer weights we only need to restrict matrix elements to basis vectors from finite-dimensional submodules. However, the internal sum in (7.7) can include vectors beyond finite-dimensional blocks.

Following [26] we also introduced a stochastic \( R \)-matrix (7.1). A factorized representation (7.10) makes the proof of stochasticity almost trivial. All matrix elements of the \( R \)-matrix are positive provided that the condition (3.9) is satisfied. Therefore, it defines a discrete time Markov process with positive probabilities.

One of the possible directions of future research is to construct stochastic models for other Lie algebras. The quantum group approach of [26] suggests that this may be possible and a similar factorization of the \( R \)-matrix can exist for other cases.

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Appendix

Here we list standard definitions in $q$-series which we need in the main text

\[(a; q)_{\infty} := \prod_{i=0}^{\infty} (1 - a q^i), \quad (a; q)_{n} = \frac{(a; q)_{\infty}}{(a q^n; q)_{\infty}} \quad (A.1)\]

\[(a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}} \quad (A.2)\]

\[\prod_{i=1}^{m} (a_i; q)_{n_i} = \prod_{i=1}^{m} (a_i; q)_{n_i} \quad (A.3)\]

\[\prod_{i=1}^{m} (q; q)_{n_i} = \prod_{i=1}^{m} (q; q)_{n_i} \quad (A.4)\]

We also define a basic hypergeometric series

\[r+1+ r_1 \phi_1 \left( \begin{array}{c} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{array} \right | q, x \right) = \sum_{i \geq 0} \frac{(a_1, \ldots, a_{r+1}; q)_i}{(q, b_1, \ldots, b_r; q)_i} x^i. \quad (A.5)\]

In the main text we use Heine’s transformations of $\phi_1$ series (III.1)–(III.3) in [33]

\[2 \phi_1 \left( \begin{array}{c} a, b \\ c \end{array} \right | q, z \right) = \frac{(az, b; q)_{\infty}}{(c, z; q)_{\infty}} 2 \phi_1 \left( \begin{array}{c} c/b, z \\ az \end{array} \right | q, b \right) \]

\[= \frac{(c/b, az; q)_{\infty}}{(c, z; q)_{\infty}} 2 \phi_1 \left( \begin{array}{c} abz/c, b \\ bz \end{array} \right | q, c/b \right) \]

\[= \frac{(abz/c; q)_{\infty}}{(z; q)_{\infty}} 2 \phi_1 \left( \begin{array}{c} c/a, c/b \\ z \end{array} \right | q, abz/c \right) \quad (A.6)\]

and Sears’s transformation of terminating $\phi_3$ series

\[4 \phi_3 \left( \begin{array}{c} q^{-n}, a, b, c \\ d, e, f \end{array} \right | q, q \right) = \frac{(a, ef/ab, ef/ac; q)_n}{(e, f, ef/ac; q)_n} 4 \phi_3 \left( \begin{array}{c} q^{-n}, e, f, ef/abc \\ ef/ab, ef/ac, q^{-n} \end{array} \right | q, q \right) \quad (A.7)\]

provided that def $= abc q^{-1-n}$, see (III.16) in [33].

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