Explicit Construction of the Brownian Self-Transport Operator

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Abstract

Applying the technique of characteristic functions developed for one-dimensional regular surfaces (curves) with compact support, we obtain the distribution of hitting probabilities for a wide class of finite membranes on square lattice. Then we generalize it to multi-dimensional finite membranes on hypercubic lattice. Basing on these distributions, we explicitly construct the Brownian self-transport operator which governs the Laplacian transfer.

In order to verify the accuracy of the distribution of hitting probabilities, numerical analysis is carried out for some particular membranes.

Introduction

We are engaged in the physical problems that can be mathematically modeled by the Laplacian transfer problem [1]. These are: the diffusion through semi-permeable membranes and the electrode problem [2], the heterogeneous catalysis [3], the NMR in porous environment [4], etc.

Consider the diffusion of particles from the source (usually supposed planar) to the semi-permeable membrane with rather complex geometry, see Fig. 1a. In the steady-state regime, the concentration of particles $C$ obeys the Laplacian equation,

$$
\Delta C = 0.
$$

The flux of particles in the bulk obeys Fick’s law, $\Phi = - D \nabla C$, where $D$ is the diffusion coefficient. The flux across the surface is given by $\Phi_n = - W C$, where $W$ is the permeability of the membrane (the probability per unit time, surface, and concentration for a particle to cross the membrane). Equating these two fluxes, we obtain the mixed boundary condition,

$$
\frac{\partial C}{\partial n} = \frac{1}{\Lambda} C,
$$

often called also Fourier or Robin boundary condition. The physical parameter $\Lambda = D/W$ plays an important role in this treatment.

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Figure 1: The problem of the Laplacian transfer across a resistive irregular interface (a membrane or an electrode).

In the electrode problem (see Fig. 1b), there are a planar and an irregular electrodes (with an interface resistivity $r$), placed in an electrolyte with a resistivity $\rho$. The electric potential $V$ obeys the Laplace equation $\Delta V = 0$ in the bulk of the electrolyte. The boundary condition (2) is obtained by equating the current from the electrolyte, $-\nabla V/\rho$, to the current $-V/r$ crossing the electrode surface, with $\Lambda = r/\rho$.

For the heterogeneous catalysis, a catalyst of a complex geometry (with the reactivity $K$) is placed in a solution. When a molecule $A$ diffusing in the solution, reaches the catalytic surface, it transforms into $A^*$ with reaction rate $K$. As above, the concentration of molecules $A$ obeys the Laplace equation (1). The boundary condition (2) with $\Lambda = D/K$ is obtained due to the mass conservation.

The same arguments allow to describe the NMR in the porous environment by the Laplace equation under the mixed boundary condition. So, all these different phenomena are described by the same mathematical formalism (see [5], [6] for details).

The Laplacian transfer problem is significantly more complex than the corresponding Dirichlet or Neumann problem. In order to find the solution, Filoche and Sapoval [5] proposed the following program:

– to choose the appropriate discretization;
– to solve the discrete problem;
– to take the continuous limit.

For these purposes, they introduced the Brownian self-transport operator $Q$. This operator controls the properties of the Laplacian transfer and depends only on the geometry of membrane’s boundary. It can be easily defined for a given discrete membrane: $Q_{k,n}$ is the probability of first contact with the membrane on the $n$-th site if started from the $k$-th site (without touching other sites!). This is exactly the problem of the hitting probabilities on the lattice. So, if one can calculate $Q$ for any “reasonable” membrane, the Laplacian transfer problem would be solved (except the difficulties related with the continuous limit). In this paper, we construct the Brownian self-transport operator for the membranes with a rather general geometry.

Now we should specify what the term “reasonable membrane” means. For
the moment, we confine ourselves to the two-dimensional case (square lattice) in order to clarify the representation of our treatment. When two-dimensional results are obtained, we shall explain how they can be generalized for the multi-dimensional case. On a square lattice, the Laplacian equation (1) becomes

$$4C_{x,y} - C_{x+1,y} - C_{x-1,y} - C_{x,y+1} - C_{x,y-1} = 0,$$

i.e., diffusion can be modelized by simple random walks. We suppose that the membranes are quite regular. It means that one can choose such discretization that the number of singular points is negligible by comparison with the number of regular points (contrariwise to the case of fractal membranes). As we decided to work on the square lattice, we should represent the membrane as a sequence of horizontal and vertical segments. The last difficulty concerns with corners. The problem is that a corner point is connected with two lattice points whereas a regular point is connected with only one lattice point. We would like to be indemnified from these ambiguities. To avoid the corner points, we use the following rules:

- for the internal corner (see Fig. 2a, 2b), we carry the corner point from the boundary to the bulk of membrane.
- for the external corner (see Fig. 2c, 2d), we completely remove the corner point from the lattice. The links from the boundary to these points are also removed. Two other links are closed to each other.

This operation preserves the connectivity of the lattice and eliminates the singularities (corner points).

Section 1 introduces definitions and restrictions which allow us to reach final results. Section 2 is devoted to the technique developed in our previous paper [7] for one-dimensional regular surfaces with compact support. Note that the actual treatment is partially based on the previous results. The distribution of hitting probabilities for a two-dimensional finite membrane is obtained in Section 3. The generalization of these results for multi-dimensional case is described in Section 4. In Section 5, we explicitly construct the Brownian self-transport operator. Some important generalizations are given in Section 6.

2 The other reason to avoid the corner points is related with numerical simulations. Indeed, to verify our analytical results we should solve the problem of Laplacian transfer numerically. And here it is not clear how to treat the corner points.

3 These rules were described, in particular, in [6].
Section 7 describes some numerical results in order to verify the method. In the last section we make conclusions.

1 Definitions

Consider a square lattice $\mathbb{Z}^2$ on a plane. A path is a sequence (finite or infinite) of points $\{A_n\} \subset \mathbb{Z}^2$ such that $\forall n \ |A_n - A_{n-1}| = 1$, where $|A - B|$ denotes the distance between two points $A$ and $B$.

The sequence of points $S = \{(x_m, y_m) \in \mathbb{Z}^2 : m \in \mathbb{Z}\}$ is called membrane’s boundary if it obeys the following conditions:

1. **Bijection**: Index $m$ enumerates the points $(x_m, y_m)$, i.e., there is one-to-one correspondence (bijection) between the set of integer numbers $\mathbb{Z}$ and the set of boundary points $S$.

2. **Boundary**: $S$ separates the lattice $\mathbb{Z}^2$ in two disjoint sets $\mathcal{E}$ and $\mathcal{M}_0$—external and internal points.

3. **Accessibility**: Any point of $S$ is accessible from all external and internal points, i.e., for any $(x, y) \notin S$ and any $(x_m, y_m) \in S$ there exists a path $\{A_n\}$ such that $A_1 = (x, y)$ and $\{A_n\} \cap S = (x_m, y_m)$.

4. **Compactness**: There is only finite number $M$ of boundary points (in $S$) which do not lie on the horizontal axis, i.e., the non-plane part of the membrane has a finite size.

Set $\mathcal{M} = \mathcal{M}_0 \cup S$ is called finite membrane.

Once chosen, points $(x_m, y_m)$ are assumed fixed in all following calculations. We always use notations $x_m$ (or $x_n$) and $y_m$ (or $y_n$) for abscissae and ordinates of boundary points.

In order to simplify expressions and to avoid possible ambiguities, we introduce the following conventions:

- the bulk of membrane is placed in the lower half plane (except, possibly, a finite number of points);
- non-plane boundary points are enumerated by index $m = 1..M$, i.e.,

$$\forall m \in [1..M] \quad y_m \neq 0, \quad \forall m \notin [1..M] \quad y_m = 0.$$

Let us discuss the definition of the finite membrane. The boundary and accessibility conditions provide that boundary points take their places sufficiently close to each other, i.e., $\forall m \exists n \ (|x_m, y_m| - (x_n, y_n)) \leq \sqrt{2}$. The accessibility condition prohibits the existence of corner points (see Fig. 3a, 3b). Moreover, this condition forbids also the “diagonal” points (see Fig. 3c). This means that membrane’s boundary is composed of horizontal and vertical segments just as required. So, we can conclude that these two conditions provide all the necessary properties which were described in the Introduction.

The compactness condition is essential to achieve our goal. This condition means that the membrane is composed by three parts: two plane “tails” on the same height, and one intermediate (non-trivial) part that can be very complex but finite. The importance of the compactness condition was thoroughly discussed in the previous paper, and we do not repeat it here. Note that in

\[ I.e., \text{there is no path from one set to another which does not pass through a point in } S. \]
practice, this condition is not too restrictive because usually one considers the finite number of sites.

According to the bijection condition, we can define the function $J(x, y)$ which gives the index of the boundary points $(x, y)$, i.e., $J(x_m, y_m) = m$. Note that this function is defined only on the boundary points: if $(x, y) \notin S$, $J(x, y)$ has no value. We shall use the following convention:

if $(x, y) \notin S$, the object containing $J(x, y)$ is equal to 0.

For example, if we write $e^{iJ(x,y)\theta}$, we always mean $e^{iJ(x,y)\theta} \chi_S(x, y)$, where

$$\chi_S(x, y) = \begin{cases} 1, & \text{if } (x, y) \in S, \\ 0, & \text{if } (x, y) \notin S. \end{cases}$$

Note that it is just a useful convention to simplify the expressions.

We call all the points $\{(x, y) \in \mathbb{Z}^2 : y = n\}$ the $n^{th}$ level. We say that the membrane’s boundary lies between $(-N^*)^{th}$ and $N^{th}$ levels if

$$N = \max\{y_m\}, \quad N^* = -\min\{y_m\}.$$

The external point $A$ is called near-boundary point, if there exists $m$ such that $|A - (x_m, y_m)| = 1$, i.e., $A$ lies “near” the membrane’s boundary. The functions defined on these points, are called near-boundary functions (see below). We enumerate these points (and functions) by the same index $m$ as for boundary points. Note that usually we suppose $m \in [1, M]$, i.e., we take into account near-boundary points which lie near the non-plane part of the membrane.

The external points $(x, 0) \notin M$ are called ground points. The functions defined on these points, are called ground functions. Let us enumerate ground points $(\bar{x}_g, 0)$ by index $g \in [1, G]$ using bar notation for their abscissae. Note that there exist membranes without ground points (see Fig. 3a for example).

We introduce the outer normal $(\delta x_m, \delta y_m)$ for each site of the membrane’s boundary. Due to the accessibility condition, we have no corner points, therefore this vector is correctly defined. Note that $m$-th near-boundary point is just $(x_m + \delta x_m, y_m + \delta y_m)$, i.e., the outer normal is a vector directed from the boundary point to the corresponding near-boundary point.

Let $P_{x,y}(n)$ be the hitting probability, i.e., the probability of the first contact with the membrane on point $(x_n, y_n)$ if started from an external point $(x, y) \in \mathcal{E}$.
without touching other boundary points. Their characteristic functions \( \phi_{x,y}(\theta) \) are

\[
\phi_{x,y}(\theta) = \sum_{m=-\infty}^{\infty} P_{x,y}(m) e^{im\theta}.
\]

The inverse Fourier transform allows to obtain \( P_{x,y}(m) \),

\[
P_{x,y}(m) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-im\theta} \phi_{x,y}(\theta).
\]

2 Technique developed in [7]

Here we briefly present the technique which was developed in [7] to obtain the distribution of hitting probabilities for a regular surface with compact support. All the details can be found in this paper.

The discrete Laplacian equation (3) in terms of characteristic functions is simply

\[
\phi_{x,y}(\theta) = \frac{1}{4} \left[ \phi_{x+1,y}(\theta) + \phi_{x-1,y}(\theta) + \phi_{x,y-1}(\theta) + \phi_{x,y+1}(\theta) \right].
\]

(4)

Using the convolution properties of hitting probabilities and their normalization, one obtains the distribution of hitting probabilities for a planar surface,

\[
P_{x,y}^{\text{planar}}(m) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i(x-x_m)\theta} \varphi_{x,y}(\theta) = H_{x-x_m}^y,
\]

(5)

where function \( \varphi(\theta) \) is

\[
\varphi(\theta) = 2 - \cos \theta - \sqrt{(2 - \cos \theta)^2 - 1}.
\]

(6)

We wrote \( \varphi_{y}(\theta) \) given that for the lower half plane \((y < 0)\) one has exactly the same result due to reflection symmetry.

For the non-planar membrane, relations (3) remain valid for external points, but there are also membrane’s points where it is not true. Nevertheless, we preserve this form by adding the correction term which equals to 0 on external points.

In order to simplify manipulation with characteristic functions, we introduce vectors \( \Phi(y)(\theta) \) containing all \( \phi_{x,y}(\theta) \) on the \( y \)-th level with \(|x| \leq L \) (at the end of calculations \( L \to \infty \)). We can write (3) as

\[
A \Phi(y)(\theta) = \Phi(y-1)(\theta) + \Phi(y+1)(\theta) + \Delta \Phi(y)(\theta),
\]

where \( A \) is tridiagonal matrix with elements: \( A_{i,i} = 4, A_{i,i+1} = A_{i+1,i} = -1 \) (we also introduce an insignificant modification \( A_{-L,L} = A_{L,-L} = -1 \) to have the cyclic structure of \( A \)), \( \Delta \Phi(y) \) is the correction for membrane’s points. Matrix \( A \) has eigenvalues

\[
\lambda_h = 4 - 2 \cos \theta_h, \quad \text{with} \quad \theta_h = 2\pi h/(2L + 1), \quad h \in \{-L, \ldots, L\}
\]
and eigenvectors \( V_h \) whose \( l \)-th component is simply \( e^{il\theta} \). We decompose \( \Phi^{(y)}(\theta) \) and \( \Delta \Phi^{(y)}(\theta) \) on the base of \( V_h \),

\[
\Phi^{(y)}(\theta) = \frac{1}{2L+1} \sum_{h=-L}^{L} c_y(\theta, \theta_h)V_h, \quad \Delta \Phi^{(y)}(\theta) = \frac{1}{2L+1} \sum_{h=-L}^{L} \Delta c_y(\theta, \theta_h)V_h.
\]

Their coefficients \( c_y \) and \( \Delta c_y \) obey the recurrence relations

\[
\lambda_h c_y(\theta, \theta_h) = c_{y-1}(\theta, \theta_h) + c_{y+1}(\theta, \theta_h) + \Delta c_y(\theta, \theta_h).
\]  

(7)

So, the problem now is to find \( c_y(\theta, \theta_h) \). It can be solved in two steps. First, using (7) with certain conditions, we can express \( c_y \) in terms of \( \Delta c_y, \lambda_h \) and \( \varphi(\theta) \). Second, we have to find \( \Delta c_y \). The first step was made in [7]:

1. taking sufficiently large \( N_u \) and \( N_l \), we impose two conditions to close the recurrence relations (7),

\[
c_{N_u+1}(\theta, \theta_h) \approx \varphi(\theta)c_{N_u}(\theta, \theta_h), \quad c_{-N_l}(\theta, \theta_h) = 0.
\]  

(8)

Note that the first condition is an approximate relation:

1. we find the explicit solution of (7), i.e., we express \( c_y \) in terms of \( c_0 \) and \( \{\Delta c_y\} \) (see Conclusions for more details);

1. we take the limit \( N_u \to \infty \) and \( N_l \to \infty \) to obtain

\[
c_y(\theta, \theta_h) = \varphi^{(y)}(\theta_h)c_0(\theta, \theta_h) + \sum_{y'=N^*}^{N} \left[ \gamma^{(y')}(\theta_h) + \gamma^{(-y')}(\theta_h) \right] \Delta c_{y'}(\theta, \theta_h),
\]

(9)

where

\[
\gamma^{(y')}(\theta_h) = \sum_{j=1}^{\min\{y,y'\}} |\varphi(\theta_h)|^{2j-1+|y-y'|}
\]  

(10)

(here we use the convention that \( \sum_{j=a}^{b} \) is equal to 0 if \( b < a \), i.e., \( \gamma^{(y)} \) is equal to 0 if \( y \leq 0 \) or \( y' \leq 0 \)). Note that using only two simple identities,

\[
\varphi^2(\theta_h) - \lambda_h \varphi(\theta_h) + 1 = 0, \quad \gamma^{(y+1)}(\theta_h) - \lambda_h \gamma^{(y)}(\theta_h) + \gamma^{(y-1)}(\theta_h) = -\delta_{y,y'} (y > 0),
\]

(11)

one can verify directly, without intermediate steps of [7], that (7) is the solution of recurrence relations (7). Now we write \( \phi_{x,y}(\theta) \) as

\[
\phi_{x,y}(\theta) = \sum_{h=-L}^{L} e^{ix\theta_h} \left[ \varphi^{(y)}(\theta_h)c_0(\theta, \theta_h) + \sum_{y'=N^*}^{N} (\gamma^{(y')}(\theta_h) + \gamma^{(-y')}(\theta_h)) \Delta c_{y'}(\theta, \theta_h) \right].
\]

(12)

So, the first step of our program is completed. Note that all the above results are exactly the same as for a regular surface with compact support (by this reason we did not explain all details of calculation). Now we are ready to pass to the second step: to find \( \Delta c_{y'}(\theta, \theta_h) \) and \( c_0(\theta, \theta_h) \). These coefficients significantly depend on the membrane’s geometry. Thus we cannot use the expressions of [7], and we should recalculate \( \Delta c_{y'}(\theta, \theta_h) \) and \( c_0(\theta, \theta_h) \).
3 Two-dimensional finite membrane

3.1 Coefficients $\Delta c_{y'}$.

To calculate $\Delta c_{y'}$ we should define explicitly what $\Delta \Phi(y')$ is. We recall that these vectors were introduced in order to write correctly the expression (4) for the membrane’s points. In other words, relations (4) are satisfied automatically for any external point, but they should be imposed artificially for any point of the membrane.

For membrane’s point we have

$$\phi_{x,y}(\theta) = \begin{cases} e^{i\mathcal{J}(x,y)\theta}, & \text{if } (x,y) \in \mathcal{S}, \\ 0, & \text{if } (x,y) \in \mathcal{M}\setminus\mathcal{S}, \end{cases} \quad (13)$$

because particles cannot penetrate in the depth of the membrane. The first line of (13) represents the condition $P_{x_m,y_m}(n) = \delta_{m,n}$. Indeed, if a particle starts from the boundary point $(x_m, y_m)$, it should be immediately absorbed by this point. In other words, the probability to be absorbed by the $m$-th site is equal to 1 while the other sites have no chance. Expression (13) means that in $\Delta \Phi(y')$ there is only the contribution of boundary points and of membrane’s points near boundary. A simple verification shows that

$$\left(\Delta \Phi(y')\right)_{x'} = \chi \mathcal{M}(x', y') \left[ 4e^{i\mathcal{J}(x',y')\theta} - e^{i\mathcal{J}(x'+1,y')\theta} - e^{i\mathcal{J}(x'-1,y')\theta} - e^{i\mathcal{J}(x',y'-1)\theta} - \phi_{\mathcal{J}(x',y')}(|\theta|) \right], \quad (14)$$

where we enumerate near-boundary functions with the help of function $\mathcal{J}(x', y')$ (we use notation $\phi_{[m]}$ for the near-boundary function with index $m$). Again we insist to use our convention about $\mathcal{J}(x', y')$, i.e., in expression (14) there are only the terms where corresponding point lies on $\mathcal{S}$.

Using (14), we determine $\Delta c_{y'}$ according to its definition,

$$\Delta c_{y'}(\theta, \theta_h) = \sum_{x'=1}^{L} e^{-ix'\theta_h} (\Delta \Phi(y'))_{x'}.$$  

Usually there are a few nonzero components of $\Delta \Phi(y')$ for each $y'$. Indeed, according to the formula (14), $(\Delta \Phi(y'))_{x'}$ is defined by the boundary points and by the membrane’s points near the boundary. But on the $n$-th level there are a few such points, if $n \neq 0$. On the contrary, on the level zero there is an infinity of the boundary points due to the plane “tails”. Thus, the vector $(\Delta \Phi(-1))$ has exceptional structure. It contains the usual terms due to the non-trivial part of the membrane, and the contribution of plane “tails”.

3.2 Boundary points’ contribution

Let us calculate the contribution of the $m$-th boundary point to $\phi_{x,y}(\theta)$ (see the second term in (12)). During these calculations we consider the case $y > 0$, i.e., we write just $\gamma_{y'}$ omitting $\gamma_{-y'}$. The opposite case $y < 0$ is obtained by reflection of all ordinates with respect to the horizontal axis, $y \to -y, y' \to -y'$. 

8
and we obtain $\delta x$ of the boundary point. Let us consider the case when (the corresponding near-boundary function
general position (Fig. 4). We suppose that the boundary point does not lie near the corners (i.e.,
The third column represents the weight factor of corresponding point (see for-
E points $I$ will be considered separately in Appendices 9.1. Note that these points give the
them here (see Appendices for more details).
The boundary point $B = (x_m, y_m)$ has four neighbours: one external point $E$, one internal point $I$, and two boundary points $B_1$ and $B_2$ with indices $m_1$ and $m_2$. Let us write accurately the corrections for all these points. Using (14),
we obtain

$$B : \Delta \phi_{x_m,y_m}(\theta) = 4e^{im\theta} - \phi[m](\theta) - e^{im\theta} - e^{im\theta},$$

$$B_1 : \Delta \phi_{x_m-1,y_m}(\theta) = 4e^{im\theta} - \phi[m+1](\theta) - e^{im\theta} - e^{im\theta},$$

$$B_2 : \Delta \phi_{x_m+1,y_m}(\theta) = 4e^{im\theta} - \phi[m-1](\theta) - e^{im\theta} - e^{im\theta},$$

$$I : \Delta \phi_{x_m,y_m-1}(\theta) = -e^{im\theta},$$

$$E : \Delta \phi_{x_m,y_m+1}(\theta) = 0,$$

The third column represents the weight factor of corresponding point (see formula (13)). Now we group together all terms containing factor $e^{im\theta}$, and call such a group the boundary point’s contribution $Z_m$ (we also add to this group the corresponding near-boundary function $\phi[m]$),

$$Z_m = e^{im\theta - ix_m\theta_h} \times \left\{ \begin{array}{ll} (4 - 2 \cos \theta_h)\gamma(y) & \text{if } (\delta x_m, \delta y_m) = (0,1), \\ (4 - 2 \cos \theta_h)\gamma(y) & \text{if } (\delta x_m, \delta y_m) = (0,0), \\ (4 - e^{-i\theta_h})\gamma(y) & \text{if } (\delta x_m, \delta y_m) = (1,0), \\ (4 - e^{i\theta_h})\gamma(y) & \text{if } (\delta x_m, \delta y_m) = (-1,0), \end{array} \right.$$
Substituting this expression into \( \phi \) points on the level zero were called ground points, and they are enumerated by functions. The level zero can contain external and membrane’s points. Characteristic \( Z \) where each of them independently.

Following the definition of \( e^{-ix_m \theta h} \gamma_{y_m}^y \gamma_{y_m}^y \phi_{[m]}(\theta) \)

Using the second identity of (11), we write

Using the second identity of (11), we write

With the help of the outer normal \( (\delta x_m, \delta y_m) \), we can simplify (15),

3.3 Distribution of hitting probabilities

Now we can come back to the main problem. Expression (12) contains two terms which can be denoted as \( \phi_{x,y}^{(1)}(\theta) \) and \( \phi_{x,y}^{(2)}(\theta) \). We are going to simplify each of them independently.

Let us consider the first term,

Following the definition of \( c_y(\theta, \theta_h) \), we have

The level zero can contain external and membrane’s points. Characteristic functions \( \phi_{x,0}(\theta) \) for the membrane’s points are given by (13). The external points on the level zero were called ground points, and they are enumerated by index \( g = 1..G \). Thus,

Substituting this expression into \( \phi_{x,y}^{(1)}(\theta) \) and taking the limit \( L \to \infty \), we obtain

\[
\phi_{x,y}^{(1)}(\theta) = \sum_{x'=-\infty}^{\infty} e^{i\mathcal{J}(x',0)\theta} H^y_{x'-x'} + \sum_{g=1}^{G} H^y_{x'-x_g} \phi_{x_g,0}(\theta). \tag{17}
\]
This is the contribution of the level zero.

The second term,
\[ \phi^{(2)}_{x,y}(\theta) = \sum_{h=-L}^{L} \frac{e^{i\theta}}{2L+1} \sum_{y=-N'}^{N'} \sum_{x=-L}^{L} \gamma^{(y)}_{y'}(\theta_h) e^{-i\theta \theta_h} (\Delta \Phi(y'))_{x'}, \]
represents other levels except level zero. Summation over \( x' \) and \( y' \) can be replaced by the sum of the boundary points' contributions,
\[ \phi^{(2)}_{x,y}(\theta) = \sum_{h=-L}^{L} \frac{e^{i\theta}}{2L+1} \sum_{m=1}^{M} Z_m. \]

Using (16), we write the limit \( L \to \infty \) for the case \( y \geq 0 \),
\[ \phi^{(2)}_{x,y}(\theta) = \int_{-\pi}^{\pi} \frac{d\theta'}{2\pi} \sum_{m=1}^{M} e^{i(x-x_m)\theta'} \left[ e^{im\theta} \delta_{y,y_m} + e^{im\theta} \gamma^{(y)}_{y_m}(\theta') e^{-i\delta x_m \theta'} - \gamma^{(y)}_{y_m}(\theta') \phi^{[m]}(\theta) \right]. \]

Changing the order of summation and integration, we obtain
\[ \phi^{(2)}_{x,y}(\theta) = \sum_{m=1}^{M} \left[ e^{i\theta} \delta_{x,x_m} \delta_{y,y_m} + e^{i\theta} D^{y,y_m}_{x-x_m} - D^{y,y_m}_{x-x_m} \phi^{[m]}(\theta) \right] \quad (18) \]
where we introduced coefficients
\[ D^{y,y'}_{x} = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta} (\gamma^{(y)}_{y'}(\theta) + \gamma^{(-y)}_{-y'}(\theta)). \]

Note that function \( \gamma^{(-y)}_{-y'}(\theta) \) is added into the definition of \( D^{y,y'}_{x} \) in order to generalize it for the case \( y < 0 \). It means that formula (18) is already valid for the both cases \( y \geq 0 \) and \( y \leq 0 \). Using definitions (8) and (10), we rewrite this expression in terms of \( H^y_{x} \),
\[ D^{y,y'}_{x} = \begin{cases} \frac{\sin(\theta)}{\theta} & \text{if } y \geq 0, \\ \frac{\sin(\theta)}{\theta} & \text{if } y \leq 0. \end{cases} \quad (19) \]

We bring expressions (17) and (18) together to obtain
\[ \phi_{x,y}(\theta) = \tilde{\phi}_{x,y}(\theta) - \sum_{m=1}^{M} D^{y,y_m}_{x-x_m} \phi^{[m]}(\theta) + \sum_{g=1}^{G} H^{y}_{x-x_g} \phi^{(g)}_{x,y,g}(\theta) \quad (20) \]
where
\[ \tilde{\phi}_{x,y}(\theta) = \sum_{m=-\infty}^{\infty} e^{i\theta} \left( \delta_{x,x_m} \delta_{y,y_m} + D^{y,y_m}_{x-x_m} \right) \quad (21) \]

Here we should explain how (21) is obtained. This sum over \( m \) can be separated in two parts: the sum over \( m \in [1, M] \) and the sum over \( m \not\in [1, M] \). The first part contained in (20) corresponds to the boundary points lying on all
levels except level zero. The second part gives contributions of boundary points on the level zero, including the plane “tails”. In the case $y > 0$, expression (17) contains the sum
\[ \sum_{x'=-\infty}^{\infty} e^{iJ(x',0)\theta} H_{x-x'}^y, \]
which can be represented as
\[ \sum_{m \notin [1,M]} e^{im\theta} H_{x-x'}^y. \]
For such points $y_m = 0$, $\delta x_m = 0$. Supposing $\delta y_m = 1$ and using relation $D_{x}^{y,1} = H_{x}^{y}$ (for $y > 0$), we immediately obtain
\[ \sum_{m \notin [1,M]} e^{im\theta} D_{x-x_m-\delta x_m} y_m \]
that proves representation (21) for $\tilde{\phi}_{x,y}(\theta)$. Now, if $y < 0$, we should not write the contribution of plane “tails” because it is compensated by $\Delta c_{-1}$ (see the end of Section 3.1). In other words, the plane “tails” have no direct influence on the points in the lower half plane. The same concerns the case with $y > 0$ and $\delta y_m < 0$. In the case $y = 0$ coefficient $H_{x}^{y}$ becomes $\delta$-symbol which appears in expression (21) explicitly. We conclude that expressions (20) and (21) are valid for any external point $(x, y)$, in particular, with $y < 0$ (if such points exist).

Applying the inverse Fourier transform to (20), we obtain the distribution of hitting probabilities,
\[ P_{x,y}(n) = \delta_{x,x_n} \delta_{y,y_n} + D_{x-x_n-\delta x_n}^{y,y_n+\delta y_n} \sum_{m=1}^{M} D_{x-x_m-\delta x_m}^{y,y_m} P_{[m]}(n) + \sum_{g=1}^{G} H_{x-x_g}^{y} P_{x,y,0}(n). \]
(22)

### 3.4 Equations for near-boundary and ground functions

To complete our calculations, we should find the near-boundary functions $P_{[m]}(n)$ and ground functions $P_{x,y,0}(n)$ entering in expression (22). After that, one can use this expression for any $x$, $y$ and $n$. We take
\[ x = x_k + \delta x_k, \quad y = y_k + \delta y_k \quad \text{for} \quad k \in [1,M] \]
to obtain $M$ linear equations for the near-boundary functions $P_{[k]}(n), k \in [1,M]$
\[ P_{[k]}(n) = D_{x_k+\delta x_k-x_n-\delta x_n}^{y_k+\delta y_k,y_n+\delta y_n} \sum_{m=1}^{M} D_{x_k+\delta x_k-x_m-\delta x_m}^{y_k+\delta y_k,y_m} P_{[m]}(n) + \sum_{g=1}^{G} H_{x_k+\delta x_k}^{y_k+\delta y_k} P_{x,y,0}(n). \]
(23)

Let us introduce matrices,
\[ (D_{NN})_{k,m} = D_{x_k+\delta x_k-x_m-\delta x_m}^{y_k+\delta y_k,y_m}, \quad k \in [1,M], \ m \in [1,M] \]
\[ (D_{NG})_{k,m} = H_{x_k+\delta x_k-x_m}^{y_k+\delta y_k}, \quad k \in [1,M], \ m \in [1,G]. \]

See Conclusions for more detailed discussion on this topic.
If there is no ground functions, i.e., the non-trivial part of the membrane completely lies in the upper half plane, we take \( D_{NG} = 0 \).

Now we can rewrite (23) as

\[
(I + D_{NN}) \begin{pmatrix} P_{[1]}(n) \\ \vdots \\ P_{[M]}(n) \end{pmatrix} - D_{NG} \begin{pmatrix} P_{x_1,0}(n) \\ \vdots \\ P_{x_G,0}(n) \end{pmatrix} = \begin{pmatrix} D_{y_1}^{y_1 + \delta y_1, y_n + \delta y_n} \\ \vdots \\ D_{y_M}^{y_M + \delta y_M, y_n + \delta y_n} \end{pmatrix}.
\]

(24)

If ground points exist, we take the discrete Laplacian equations (3) for these points in order to obtain conditions for ground functions, \( 4P_{x_g,0}(n) - P_{x_g-1,0}(n) - P_{x_g+1,0}(n) - P_{x_g,1}(n) - P_{x_g,-1}(n) = 0, \ g \in [1..G] \).

Substituting \( P_{x_g,1}(n) \) and \( P_{x_g,-1}(n) \) from (22) into these conditions, we obtain

\[
4P_{x_g,0}(n) - P_{x_g+1,0}(n) - P_{x_g-1,0}(n) = D_{y_1}^{y_1 + \delta y_n} + D_{y_M}^{y_M + \delta y_n} - \sum_{m=1}^{M} (D_{x_g}^{y_m} + D_{x_g}^{1-y_m}) P_{[m]}(n) + \sum_{g'=1}^{G} 2H_{x_g-x_g'} P_{x_g',0}(n).
\]

In matrix form,

\[
D_{GN} \begin{pmatrix} P_{[1]}(n) \\ \vdots \\ P_{[M]}(n) \end{pmatrix} + D_{GG} \begin{pmatrix} P_{x_1,0}(n) \\ \vdots \\ P_{x_G,0}(n) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} H_{x_1-x_n-\delta x_n}^{y_n+\delta y_n} \\ \vdots \\ H_{x_G-x_n-\delta x_n}^{y_n+\delta y_n} \end{pmatrix},
\]

(25)

where

\[
(D_{GG})_{k,m} = -H_{x_k-x_m}^{1} + \begin{cases} 
2, & \text{if } k = m, \\
-0.5, & \text{if } |x_k - x_m| = 1, \\
0, & \text{otherwise}
\end{cases}, \quad k \in [1,G], \ m \in [1,G],
\]

\[
(D_{GN})_{k,m} = \frac{1}{2} H_{x_k-x_m}^{y_m}, \quad k \in [1,G], \ m \in [1,M].
\]

We have two matrix equations (24) and (23) which allow to find near-boundary and ground functions,

\[
\begin{pmatrix} P_{[1]}(n) \\ \vdots \\ P_{[M]}(n) \end{pmatrix} = (I + D)^{-1} \begin{pmatrix} P_{[1]}^*(n) \\ \vdots \\ P_{[M]}^*(n) \end{pmatrix}
\]

(26)

where

\[
D = D_{NN} + D_{NG} D_{GG}^{-1} D_{GN}
\]

and

\[
\begin{pmatrix} P_{[1]}^*(n) \\ \vdots \\ P_{[M]}^*(n) \end{pmatrix} = \begin{pmatrix} D_{y_1}^{y_1 + \delta y_1, y_n + \delta y_n} \\ \vdots \\ D_{y_M}^{y_M + \delta y_M, y_n + \delta y_n} \end{pmatrix} + \frac{1}{2} D_{NG} D_{GG}^{-1} \begin{pmatrix} H_{x_1-x_n-\delta x_n}^{y_n+\delta y_n} \\ \vdots \\ H_{x_G-x_n-\delta x_n}^{y_n+\delta y_n} \end{pmatrix}.
\]

(27)

Using (25) again, we obtain ground functions,

\[
\begin{pmatrix} P_{x_1,0}(n) \\ \vdots \\ P_{x_G,0}(n) \end{pmatrix} = \frac{1}{2} D_{GG}^{-1} \begin{pmatrix} H_{x_1-x_n-\delta x_n}^{y_n+\delta y_n} \\ \vdots \\ H_{x_G-x_n-\delta x_n}^{y_n+\delta y_n} \end{pmatrix} - D_{GG}^{-1} D_{GN} (I + D)^{-1} \begin{pmatrix} P_{[1]}^*(n) \\ \vdots \\ P_{[M]}^*(n) \end{pmatrix}.
\]

(28)
Expression (22) is our main result. What have we done? Using the characteristic functions technique, we express the hitting probability $P_{x,y}(n)$ (for any $x$, $y$ and $n$) in terms of the explicit coefficients $D_{x,y}^n$ and finite number of coefficients $P_{[m]}(n)$ and $P_{x,0}(n)$ which can be calculated with the help of (24), (27) and (28).

Note that $P_{x,y}(n)$ depends on $n$ only through coordinates $x_n$ and $y_n$ of the $n$-th boundary point and through outer normal $(\delta x_n, \delta y_n)$ at this point. It means that obtained distribution of hitting probabilities does not depend on a choice of parametrization of the membrane. In other words, we can use any parametrization of the membrane’s boundary. This remark will be used in the next section.

4 Multi-dimensional membranes

We confined ourselves to the two-dimensional case in order to clarify the treatment. Now we are going to generalize the previous results for multi-dimensional membranes.

Consider $d$-dimensional hypercubic lattice. As above, we can define a finite membrane with the help of similar conditions as in Section 1. In particular, we suppose that membranes have a compact support (the compactness condition), i.e., there exists a $(d - 1)$-dimensional hyperplane (level zero) such that the membrane’s boundary is just its “finite” perturbation. We choose such coordinates that this hyperplane is defined by equation $x^{(d)} = 0$, its points are enumerated by multi-index $x = (x^{(1)}, \ldots, x^{(d-1)})$. Coordinate $x^{(d)}$ of the orthogonal direction is denoted $y$. Let us denote $e_k$ the unit vectors in $k$-th direction, $k \in \{1, \ldots, d - 1\}$.

As usual, we enumerate the boundary points by index $n$, $(x_n, y_n)$. The outer normal on $n$-th point is $(\delta x_n, \delta y_n)$. We can introduce the $n$-th level as a set of points lying on the hyperplane with $y = n$, i.e., $\{(x, y) : y = n, \forall x\}$.

The discrete Laplacian equation in $d$-dimensional case is

$$P_{x,y}(n) = \frac{1}{2d} \sum_{i=1}^{d-1} \left( P_{x+e_i,y}(n) + P_{x-e_i,y}(n) \right) + \frac{1}{2d} \left( P_{x,y+1}(n) + P_{x,y-1}(n) \right),$$

whence one can obtain the distribution of hitting probabilities for a planar multi-dimensional case,

$$H_{x}^{y} = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{d\theta_1 \cdots d\theta_{d-1}}{(2\pi)^{d-1}} e^{i(x \cdot \theta) \varphi^{[n]}(\theta_{i})}$$

(29)

where $(x \cdot \theta)$ is a scalar product of two $(d - 1)$-dimensional vectors $x$ and $\theta$; function $\varphi(\theta_{i})$ depends on $\theta_1, \ldots, \theta_{d-1}$,

$$\varphi(\theta_1, \ldots, \theta_{d-1}) = \left( d - \sum_{i=1}^{d-1} \cos \theta_i \right) - \sqrt{\left( d - \sum_{i=1}^{d-1} \cos \theta_i \right)^2 - 1}.$$

The crucial idea of our generalization is that all manipulations along the orthogonal direction remain valid. In particular, the expressions (11) and (10)
are true, if one takes \( \varphi(\theta_1, \ldots, \theta_{d-1}) \) instead of \( \varphi(\theta) \). It means that we can impose the same conditions (8) basing again on the technique proposed in \([7]\). Now one can repeat the previous calculations of Section 3

- to calculate coefficients \( \Delta c_{g_i} \);
- to obtain contributions of boundary points;
- to impose conditions for near-boundary and ground functions.

Having made these technical steps, we understand that the only coefficients \( H_y \) (and \( D_y \) as consequence) are different for \( d > 2 \), but the structure of solution is exactly the same. It means that we leave expression (22) without changes for \( d \)-dimensional case,

\[
P_{x,y}(n) = \delta_{x,x_n} \delta_{y,y_n} + D_{x-x_n-y_n} + \sum_{m=1}^{M} D_{x-x_m} P_{[m]}(n) + \sum_{g=1}^{G} H_{x-y_g} P_{x,y_g}(n).
\]

(30)

Here coefficients \( D_{x-y_n} \) are still defined with the help of (19) but for coefficients \( H_{x-y} \), we should use (29) instead of (3).

As above, we can obtain a set of linear equations for near-boundary and ground functions. We take

\[
P_{[k]}(n) = D_{x_k+y_k-x_n+y_n} + \sum_{m=1}^{M} D_{x_k-x_m+y_n} P_{[m]}(n) + \sum_{g=1}^{G} H_{x_k+y_g} P_{x_k,y_g}(n)
\]

for near-boundary functions, and

\[
P_{x_g,0}(n) = \frac{1}{2d} \sum_{i=1}^{d-1} (P_{x_g+y_i,0}(n) + P_{x_g-y_i,0}(n)) + \frac{1}{2d} (P_{x_g,1}(n) + P_{x_g,-1}(n))
\]

for ground functions. Using the same representation as in Section 3.4, we obtain the near-boundary and ground functions according to expressions (26) and (28).

The only change that we should make is concerned with matrix \( D_{GG} \). Indeed, we generalize this matrix as

\[
(D_{GG})_{k,m} = -H_{x_k-x_m}^1 + \begin{cases} 
-0.5, & \text{if } |x_k - x_m| = 1, \\
0, & \text{otherwise}.
\end{cases}
\]

It is important to stress that we used the remark at the end of Section 3: the distribution of hitting probabilities should not depend on a choice of parametrization. It allows to avoid a complex multi-dimensional parametrization of the membrane. To be more rigorous, one should introduce such a parametrization, recalculate distribution \( P_{x,y} \) again, and re-enumerate the boundary points by the single index in order to obtain (30). We omit these technical details.

Note that expression (30) has the same structure for any dimension of the lattice, the only distinction is contained in coefficients \( H_y \) which are individual for each \( d \). We give a useful asymptotics for coefficients \( H_y \),

\[
H_y \approx \frac{\Gamma(d/2)}{\pi^{d/2}} \frac{|y|}{(x^2 + y^2)^{d/2}}
\]

(31)

that is the multi-dimensional generalization of the well-known Cauchy distribution of hitting probabilities of the Brownian motion.
5 Brownian self-transport operator

As it was mentioned before, we are interested in the problems of Laplacian transfer, and consequently, in the Brownian self-transport operator $Q$. We recall that $Q_{k,n}$ is the probability that a random walker contacts at the first hit the $n$-th site of the membrane if started from the $k$-th site, without touching the other sites. Obviously, any path starting from $k$-th site of membrane must pass through the corresponding near-boundary point. Provided that we removed all the singular (corner) points, any boundary point has exactly one near-boundary point. It means that

$$Q_{k,n} = P_{[k]}(n),$$

i.e., the matrix $Q$ is composed of the near-boundary functions $P_{[k]}(n)$. If one denotes

$$Q^*_{k,n} = P^*_{[k]}(n),$$

expression (26) becomes

$$Q = (I + D)^{-1}Q^*.$$  (32)

Recall that the system of linear equations (23) was obtained only for non-trivial part of the membrane, in particular, matrix $Q$ has $M \times M$ elements. However, we are usually interested in the whole Brownian self-transport operator, including the plane “tails”.

To construct the Brownian self-transport operator for all sites of our interest, we can slightly modify definitions of matrices $D_{NN}$, $D_{NG}$ and $D_{GN}$. Let $A$ be the set of indices of sites for which one wants define $Q$. For example, if one needs to know $Q$ for all sites of the membrane, it is sufficient to take $A = \mathbb{Z}$. For the numerical simulations one takes a finite number of sites. In any case, $A$ must contain all non-trivial sites, i.e., $[1..M] \subset A$. We remark that summation over $m$ in (22) can be enlarged to all integer numbers from $-\infty$ to $\infty$. Indeed, for $m \notin [1, M]$ we have $y_m = 0$, and $D_{y,y_m} = 0$ according to (19). In particular, this summation covers all possible sites which can be of our interest. Now we rewrite the definitions of matrices $D_{NN}$, $D_{NG}$ and $D_{GN}$,

$$(D_{NN})_{k,m} = D_{y,k}^{y_m} + \delta_{y,k,y_m} x_k + \delta_{x,k} - x_m, \quad k \in A, \ m \in A$$
$$(D_{NG})_{k,m} = H_{y,k}^{y_m} + \delta_{y,k} x_k + \delta_{x,k} - \bar{x}_m, \quad k \in A, \ m \in [1, G].$$
$$(D_{GN})_{k,m} = \frac{1}{2} H_{y,m}^{y_m} x_k - x_m, \quad k \in [1, G], \ m \in A.$$ Matrix $D_{GG}$ remains unchanged, and

$$D = D_{NN} + D_{NG}D_{GG}^{-1}D_{GN}.$$ Evidently, matrix $Q^*$ (i.e., vectors $P^*_{[m]}$) should be also recalculated for all sites of the interest.

These modifications were made to obtain the Brownian self-transport operator in the matrix form (32). We stress that after finding the near-boundary and ground functions, one can use (22) directly to “compose” the matrix $Q$,

$$Q_{k,n} = D_{y,k}^{y_m} + \delta_{y,k,y_m} + \delta_{y,n} = M \sum_{m=1}^{M} D_{x_k}^{y_m} + \delta_{y,k,y_m} x_m P_{[m]}(n) + \sum_{g=1}^{G} H_{x_k}^{y_m} + \delta_{y,g} x_g P_{x_g,0}(n).$$
We can conclude that the Brownian self-transport operator is explicitly constructed for any reasonable multi-dimensional finite membrane. Now it can be useful to study the general properties of $Q^*$ and the spectral properties of $D$, and their dependencies on the geometry of the membrane. This work is not yet finished.

6 Generalizations

As we explained in Section 4, expression (22) can be generalized for the multi-dimensional case. One can go further by regarding more complex problems. In this section we consider the influence of different barriers, and the time-dependent distributions of hitting probabilities.

It is important to stress that we have obtained the Brownian self-transport operator for a membrane with infinite “tails” placed in the infinite space. However, usual physical membranes have a finite size, and they are placed in a closed volume (like a box or a cell). In order to apply our results to real membranes, these features should be taken into account, for example, by introducing absorbing or reflecting barriers. First, we consider the horizontal barriers, and then the vertical barriers.

6.1 Horizontal barriers

Suppose that there exists a horizontal barrier at the level $N+1 > N$ which absorbs or reflects all particles. In this case, it is not difficult to recalculate the distribution of hitting probabilities (22). Let us return to the recurrence relations (7). In order to close them, one can use a condition like

$$c_{N+1}(\theta, \theta_h) = \eta(\theta, \theta_h)c_N(\theta, \theta_h)$$

with a certain coefficient $\eta$ which can depend on $\theta$ and $\theta_h$. Now one can express $c_y$ in terms of $c_0$ and $\{\Delta c_{y'}\}$ (and of some explicit functions like $\eta$ and $\varphi$). This expression was found in (1),

$$c_y = f_y(N)\left(c_0 + \sum_{y'=1}^{y} \alpha_{y'} \Delta c_{y'}\right) + \alpha_y \sum_{y'=y+1}^{N} f_{y'}(N) \Delta c_{y'},$$

where

$$f_y(N) = \frac{\left[1 - \eta \varphi(\theta_h)\right] - \varphi^{2(N-y)+1}(\theta_h)\left[\varphi(\theta_h) - \eta\right]}{\left[1 - \eta \varphi(\theta_h)\right] - \varphi^{2N+1}(\theta_h)\left[\varphi(\theta_h) - \eta\right]} \varphi^y(\theta_h)$$

and

$$\alpha_y(\theta_h) = \sum_{j=0}^{y-1} \left[\varphi(\theta_h)\right]^{2j+1-y}.$$

When level $N$ of the barrier goes to infinity, function $f_y(N)$ tends to $f_y(\infty) = \varphi^y$, independently of the value of $\eta$. It simply means that one can choose any condition at infinity.
The absorbing barrier corresponds to \( \eta = 0 \) since the particles cannot reach the membrane from any point \((x, N + 1)\); thus

\[
f_y(N, \text{abs}) = \frac{1 - \varphi^{N+2-2y}}{1 - \varphi^{N+2}} \varphi^y.
\]

The reflecting barrier corresponds to \( \eta = 1 \) since the particles should have the same probability to reach the membrane from points \((x, N + 1)\) and \((x, N)\); thus

\[
f_y(N, \text{ref}) = \frac{1 + \varphi^{N+1-2y}}{1 + \varphi^{N+1}} \varphi^y.
\]

All other calculations are left unchanged, but we should replace \( \varphi^y \) by \( f_y(N, \text{abs}) \) or \( f_y(N, \text{ref}) \). It means that one can use expression (22) where coefficients \( \tilde{H}_y^x \) are replaced by modified ones,

\[
\tilde{H}_y^x = \begin{cases} 
0 & \text{if } y > N, \\
\int_{-\pi}^{\pi} \frac{d\theta'}{2\pi} e^{ix\theta'} f_y(N)(\theta') & \text{if } 1 \leq y \leq N, \\
H_{x-y} & \text{if } y < 0.
\end{cases}
\]

The last line represents the fact that the barrier in the upper half plane has no influence on the lower half plane.

Using this definition of coefficients \( \tilde{H}_y^x \), one can calculate the distribution of hitting probabilities for a general membrane with an additional barrier absorbing or reflecting particles. Note that these modifications are valid for any dimension.

### 6.2 Vertical barriers

Now we can limit the particle’s movement along the horizontal axis putting an absorbing or reflecting vertical barrier. Let us put two barriers at \( x = -L - 1 \) and \( x = L + 1 \) so that the non-trivial part of the membrane lies between these vertical lines. In Section 3 we represented the Laplacian equation (4) in matrix form using matrix \( A \)

\[
A_{i,i} = 4, \quad A_{i,i+1} = A_{i+1,i} = -1, \quad A_{-L,L} = A_{L,-L} = -1,
\]

and all other elements are 0. The last equality was imposed artificially to obtain the cyclic structure of matrix \( A \) that was convenient to have explicit eigenvectors \( V_h \). After all calculations, we took the limit \( L \to \infty \), therefore this modification vanished. Now we are working with finite \( L \), consequently, we should be more accurate with these conditions. Note that relation \( A_{-L,L} = A_{L,-L} = -1 \) corresponds to periodic or cyclic boundary condition, i.e., we maintain \( \phi_{L+1,y} = \phi_{-L,y} \) and \( \phi_{-L-1,y} = \phi_{L,y} \). In other words, we suppose that each pair of points \((L + 1, y)\) and \((-L, y)\) is identified or “glued together” (the same for points \((-L - 1, y)\) and \((L, y)\)).

In order to introduce vertical barriers, one should change the boundary conditions. For two absorbing barriers one should simply remove the artificial condition, i.e., \( A_{-L,L} = A_{L,-L} = 0 \) given that \( \phi_{L+1,y} = \phi_{-L-1,y} = 0 \) (particles are absorbed by barriers). For two reflecting barriers one should also introduce the following modifications, \( A_{-L,-L} = A_{L,L} = 3 \) given that \( \phi_{L+1,y} = \phi_{L,y} \) and \( \phi_{-L-1,y} = \phi_{-L,y} \). Obviously, one can also consider the mixed case with one
absorbing barrier and one reflecting barrier. Eigenvalues and eigenvectors of these modified matrices have no explicit form. It is more convenient to use the initial matrix $A$ (cyclic boundary conditions) with additional corrections, i.e.,

$$A\Phi^{(y)} = \Phi^{(y-1)} + \Phi^{(y+1)} + \Delta\Phi^{(y)} + \Delta\tilde{\Phi}^{(y)},$$

where we introduced a new correction vector $\Delta\tilde{\Phi}^{(y)}$ whose components are equal to 0 except

$$(\Delta\tilde{\Phi}^{(y)})_{-L} = -\phi_{L,y}, \quad (\Delta\tilde{\Phi}^{(y)})_{L} = -\phi_{-L,y},$$

(absorption)

$$(\Delta\tilde{\Phi}^{(y)})_{-L} = -\phi_{L,y} + \phi_{-L,y}, \quad (\Delta\tilde{\Phi}^{(y)})_{L} = -\phi_{-L,y} + \phi_{L,y},$$

(reflection).

Now we can repeat the previous calculations to obtain a new distribution of hitting probabilities $P_{x,y}(n)$ instead of (22),

$$P_{x,y}(n) = \tilde{D}_{x-x_n,\delta x_n} + \sum_{m=1}^{M} \tilde{D}_{x-x_m} P_{m}(n) + \sum_{g=1}^{G} \tilde{H}^{y}_{x-x_{g}} P_{x-x_{g}}(n) - (33)$$

$$= \left\{ \begin{array}{ll}
\sum_{y'=1}^{\infty} [\tilde{D}^{y,y'}_{x} P_{y,y'}(n) + \tilde{D}^{y,y'}_{x+1} P_{y,y'}(n)], & \text{(absorption)}, \\
\sum_{y'=1}^{\infty} [\tilde{D}^{y,y'}_{x} - \tilde{D}^{y,y'}_{x+1}] [P_{y,y'}(n) - P_{y,y'}(n)], & \text{(reflection)},
\end{array} \right.$$  

where $\tilde{H}^{y}_{x}$ should be calculated as finite sum over $h$ from $-L$ to $L$,

$$\tilde{H}^{y}_{x} = \sum_{h=-L}^{L} \frac{e^{ih\theta_{y}}}{2L+1} | \theta_{y} | (\theta_{y}), \quad \theta_{y} = \frac{2\pi h}{2L+1},$$

and $\tilde{D}^{y,y'}_{x}$ is expressed according to (19) in terms of $\tilde{H}^{y}_{x}$.

We should make several remarks. First, for sufficiently large $L$ one can use the integral expression (3) for coefficients $H^{y}_{x}$ as an approximation. Unfortunately, such approximation cannot give an accurate result for $|x|$ near $L$. When we are not interested in points near barriers, it does not lead to the problems. But in some cases it can do. Therefore one should be careful working with finite membranes.

Second, for absorbing and reflecting barriers there appear new unknown functions $P_{x,L,y}(n)$ which can be called barrier functions. It means that one should write additional conditions to obtain a closed system of linear equations for near-boundary, ground and barrier functions. In order to obtain the condition corresponding to the barrier functions $\phi_{k,L,y}$, one can maintain simply

$$x = \pm L$$

in (33) taking different $y$. The problem is that there is an infinite number of barrier functions. A convenient solution is to consider two vertical barriers together with a horizontal barrier (that is the usual case for a real physical membrane). In this case, the number of barrier functions is just $N$. However, one should not forget to use coefficients $H^{y}_{x}$ and $D^{y,y'}_{x}$ corrected by introducing the horizontal barrier, see the previous subsection.

Third, one can try to use a rough approximation for a very large $L$. Indeed, if $|x| \ll L$, one has $D^{y,y'}_{x \pm L} \sim 1/L^2$, and the barrier functions can be neglected. It simply means that one uses the initial formula (22) to approximate the solution with two remote barriers.
Fourth, in the case of one horizontal and two vertical barriers there is a finite number of external points, i.e., one can solve the system of discrete Laplacian equations properly, without using probabilistic techniques. In other words, the essential advantage of the present formalism is the possibility to analyse the membranes with an infinite number of sites.

6.3 Time-dependent distribution of hitting probabilities

Previous results were related to the Laplacian equation, i.e., we considered distributions of hitting probabilities independent of time. Generalizing the problem, we can find the distribution of probabilities \( P_{x,y}^{(t)}(n) \) to hit the \( n \)-th site of the membrane at \( t \)-th step (here \( t \) is discrete time; for the Brownian motion it is continuous time) starting from an external point \( (x, y) \). From the technical point of view, it is convenient to consider the Laplacian transform of this distribution,

\[
P_{x,y}^{(λ)}(n) = \sum_{t=0}^{∞} λ^t P_{x,y}^{(t)}(n).
\]

Note that \( λ = 1 \) corresponds to the previous time-independent distribution. More generally, \( P_{x,y}^{(λ)}(n) \) gives the distribution of hitting probabilities if particles have killing rate \((1 − λ)\).

For the planar case, one can apply the same technique as usual to obtain the distribution

\[
P_{x,y}^{(λ)}(n) = \frac{π}{2} \int_{-π}^{π} dθ e^{i(x-n)θ} \varphi^y(θ; λ) = H^y_{x-n}(λ)
\]

with function

\[
\varphi(θ; λ) = \frac{2}{λ} - \cos θ - \sqrt{(\frac{2}{λ} - \cos θ)^2 - 1}.
\]

This expression can be easily rewritten for the multi-dimensional case. Now we repeat the previous calculations for general membranes to obtain exactly the same formula as (22),

\[
P_{x,y}^{(λ)}(n) = \frac{1}{d!} \left[ \frac{d^t}{dλ^t} \left( P_{x,y}^{(λ)}(n) \right) \right]_{λ=0}.
\]

Replacing function \( \varphi(θ; λ) \) by

\[
\varphi(θ_1, ..., θ_{d-1}; λ) = \left( \frac{d}{λ} - \sum_{i=1}^{d-1} \cos θ_i \right) - \sqrt{\left( \frac{d}{λ} - \sum_{i=1}^{d-1} \cos θ_i \right)^2 - 1},
\]

we easily generalize this time-dependent distribution of hitting probabilities to the multi-dimensional case.

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7 Numerical verifications

In this section we briefly present some numerical results to check the validity of the method.

For a particular non-trivial membrane, the validity of expressions (22) and (30) can be verified by comparing them with the results obtained by numerical simulations (simple random walks on the lattice). We have taken two- and three-dimensional membranes represented on Fig. 5 and Fig. 6.

In the first case (Fig. 5a), there are no ground functions, \( G = 0 \), and there are 155 near-boundary functions which can be calculated using (32). For the concave membrane (Fig. 5b), there is 31 ground functions and 155 near-boundary functions. On Fig. 7 we present the distribution of hitting probabilities \( P(20) \) (i.e., \( Q_{20,n} \)) obtained with the help of formula (22). The solid line corresponds to the convex membrane, the dashed line – to the concave membrane. These results are obtained for the membranes placed in the finite box. In other words, we take the horizontal absorbing barrier on the distance \( D = 100 \) from the ground level, and two vertical barriers under the cyclic boundary conditions (see Section 6). Moreover, we account the corner points’ corrections (see Appendices 9.1).

Fig. 8 shows the distribution of hitting probabilities \( P_{113}(n) \) (i.e., \( Q_{113,n} \)) for three-dimensional membranes (see Fig. 6): the solid line corresponds to the convex membrane, and the dashed line – to the concave membrane. These curves are obtained with the help of formula (22), where we used (29) for \( H^y_x \) instead of (20). We calculate these distributions for the membranes placed in the finite box which is formed by the horizontal absorbing barrier at \( D = 15 \), and by four vertical barriers with cyclic boundary conditions (see Section 6). Also we took into account the corner points’ corrections.

In order to confirm that the method works, we performed the Monte-Carlo simulations. Starting from the near-boundary point \((x_k + \delta x_k, y_k + \delta y_k)\), the particle walks at random on the lattice until it is absorbed either on the membrane’s boundary or on the horizontal barrier. Repeating this procedure, we obtain the numerical distribution of frequencies of visits for all boundary points. Dividing by the total number of walks \( N_t \), we calculate the approximate hit-
Figure 6: Three-dimensional membranes: (a) convex membrane \((N = 6, N^* = 0, M = 365)\); (b) concave membrane \((N = 0, N^* = 6, M = 365)\). There are 13 facets \(5 \times 5\) (including 8 facets on the ground level), and 8 segments \(5 \times 1\). Boundary points are enumerated by index \(n\) following the rules: 1st facet \((y = 0)\): \(n = (|z| - 1) \cdot 5 + x\), 2nd facet \((x = 6)\): \(n = (|z| - 1) \cdot 5 + y + 25\), 3rd facet \((y = 6)\): \(n = (|z| - 1) \cdot 5 + x + 50\), 4th facet \((x = 0)\): \(n = (|z| - 1) \cdot 5 + y + 75\), 5th facet \(|z| = 6\): \(n = (y - 1) \cdot 5 + x + 100\), 6th - 13th facets \((z = 0)\) - by the same way.

Figure 7: Distributions of hitting probabilities (in log-scale) \(Q_{k,n}\) with \(k = 20\): the probability of the first contact with points \((x_n, y_n)\) of membrane’s boundary if started from the point \((x_{20}, y_{20})\).
Figure 8: Distribution of hitting probabilities (in log-scale) $Q_{k,n}$ with $k = 113$ (the center of the fifth facet) for the three-dimensional convex and concave membranes.

The approximate values $\tilde{Q}_{k,n}$ for each pair of sites $k$ and $n$. When $N_t$ goes to infinity, the approximate values $\tilde{Q}_{k,n}$ tend to the exact values $Q_{k,n}$. These simulations represent the well-known Bernoulli trials. In order to estimate the accuracy of values $\tilde{Q}_{k,n}$ for the finite $N_t$, we use the central limit theorem. It gives the natural measure $\Delta Q_{k,n}$ of deviations $Q_{k,n} - \tilde{Q}_{k,n}$,

$$\Delta Q_{k,n} \sim \sqrt{\frac{Q_{k,n}}{N_t}},$$

where $\sigma \approx \sqrt{Q_{k,n}}$ is the dispersion of the Bernoulli trials.

Taking different values of $N_t$ and performing the Monte-Carlo simulations for all possible sites $k$ and $n$, we compute the maximal deviation of the approximate value $\tilde{Q}_{k,n}$ from the exact value $Q_{k,n}$,

$$\alpha = \max_{k,n} \left\{ \frac{|Q_{k,n} - \tilde{Q}_{k,n}|}{\sqrt{Q_{k,n}}} \right\} \sqrt{N_t}.$$

Values of $\alpha$ for different membranes are given in Table 1.

| Membranes   | $10^5$ | $10^6$ | $10^7$ |
|-------------|--------|--------|--------|
| 2D convex   | 4.99   | 4.02   | 4.43   |
| 2D concave  | 4.96   | 4.91   | 4.46   |
| 3D convex   | 5.33   | 4.44   | 4.46   |
| 3D concave  | 6.33   | 4.59   | 4.46   |

Table 1. Monte-Carlo verifications of the method’s work. For $N_t \in \{10^5, 10^6, 10^7\}$, the maximal deviation $\alpha$ is calculated.

These results confirm that formulae (22) and (30) work correctly.
8 Discussion and conclusions

In this paper, we defined a wide class of finite membranes which play an important role for the discrete Laplacian transfer problem. We based on the technique of characteristic functions developed in our previous paper. In particular, expressions (8) are used to close the recurrence relations (7). We remind that expression $c_{N_+1} \approx \varphi c_{N_u}$ was the central approximation of (7). In order to obtain this approximate relation, we supposed that from a remote line $y = N_u$ (for large $N_u$) the membrane can be viewed as almost translationally invariant object (along the horizontal axis). Then the explicit solution of recurrence relations (7) had been found. Finally, we took the limit $N_u \to \infty$ (and $N_l \to \infty$ for the lower half plane) to obtain expression (9) for coefficients $c_y(\theta, \theta_h)$. It means that influence of approximate relation (8) vanishes. In other words, the approximate relation (8) is taken at infinity ($N_u \to \infty$), i.e., this condition has no influence on the solution. Moreover, one can verify directly, without any approximation, that (9) is the exact solution of recurrence relations (7).

We stress that the essential result of our calculations is the form of solution (22) which expresses the distribution of hitting probabilities for a general membrane, $P_{x,y}(n)$, in terms of corresponding planar distribution, $H_x^n$. In other words, we explicitate how the membrane’s geometry should change the planar distribution. It means that having solved the planar problem, one can easily generalize its solution for the case with a rather complex geometry. For example, when the problem with horizontal barrier for the planar case was solved, we just replaced old coefficients $H_x^n$ by the new ones. The same ideas were used for the time-dependent distribution and for the multi-dimensional generalization. Note that these motivations are frequently used in the theory of conformal transforms. Indeed, taking an appropriate conformal transform, one can map the initial complex set into a simple set (like a half plane or a disk), solve the simplified problem, and then reconstruct the solution by the inverse conformal transform. Note also that the conformal theory works only in two-dimensional case while the present approach is valid for any dimension of the lattice.

8.1 Ground functions

Let us briefly discuss the role of ground functions. From the beginning, the upper and the lower half planes were considered separately (for example, see two terms $\gamma_{y'}^y$ and $\gamma_{-y'}^{-y}$ in equation (5)). The technique proposed in (7) consists to *step down* from the $N_u$-th and $(-N_l)$-th levels to the level zero. It allows to express coefficients $c_y$ in terms of $\Delta c_y$. But we cannot step down directly from $N_u$-th to $(-N_l)$-th level, because there appears an infinity of near-boundary functions due to $\varphi_0$, and we do not achieve our goal\footnote{For the same reason the compactness condition was imposed at the beginning.}. In other words, we cannot “pass” through the level zero in our treatment. On the other hand, a random walk started at the upper half plane may hit the boundary point in the lower half plane (if such point exists). This walk passes the level zero through an external point, i.e., through a ground point. Therefore we can say that ground functions “connect” the solutions for the upper and lower half planes.

Consider as example the concave membrane (Fig. 5b). The probability to hit the $n$-th site in the lower half plane if started from a point $(x, y)$ in the upper...
half plane becomes

\[ P_{x,y}(n) = \sum_{g=1}^{G} H_{x-\bar{x}}^{y} P_{\bar{x},0}(n). \]

We simply took (22) under condition that \( y > 0, y_m < 0 \) for \( m \in [1,M] \). This formula has a clear probabilistic sense: to hit the \( n \)-th site, the walk should reach one of the ground points (enumerated by \( g \)) with probability \( H_{y,x-\bar{x}}^y \), and after that hit the \( n \)-th site with probability \( P_{\bar{x},0}(n) \).

### 8.2 Probabilistic sense of coefficients \( D_{x}^{y,y'} \)

Now we clarify the nature of coefficients \( D_{x}^{y,y'} \) which seem to be artificial at the first sight. First of all, they have no purely probabilistic sense (for example, they can have values greater than 1). To understand the sense of these coefficients, consider the simple problem: to find the probability to hit a point \( (x_0, y_0) \) if started from \( (x, y) \) without touching the horizontal axis. The solution of this problem is given in Appendices 9.2,

\[ P_{[(x,y)\rightarrow(x_0,y_0)]} = \frac{D_{y}^{y_0}}{D_{y,y_0}}. \]  

(34)

Using this solution, we can rewrite (22) for \( (x, y) \notin S \) as

\[ P_{x,y}(n) = P_{[(x,y)\rightarrow(x_n,\delta x, y_n, y_n+\delta y_n)]} D_{0}^{y_n+\delta y_n, y_n} + \sum_{m=1}^{M} P_{[(x,y)\rightarrow(x_m,y_m)]} D_{0}^{y_m, y_m} P_{[m]}(n) + \sum_{g=1}^{G} H_{x-\bar{x}}^{y} P_{\bar{x},0}(n). \]  

(35)

Now this expression has more clear sense: we take all possible ways to reach near-boundary point \( (x_n, 1) \) without touching the horizontal axis, and then subtract all ways which go through membrane’s points. Coefficients \( D_{0}^{y_m, y_m} \) and \( D_{0}^{y_n+\delta y_n, y_n+\delta y_n} \) have combinatoric origin: they allow to account different possibilities to go through the membrane. According to definition \( (19) \),

\[ D_{0}^{y,y} = \sum_{j=1}^{y} H_{0}^{2j-1}, \]

and for two-dimensional case the approximation

\[ D_{0}^{y,y} \approx \frac{\ln y}{2\pi} + \text{const} \quad (\text{const} \approx 0.3675) \]

gives \( D_{0}^{y,y} \) with a sufficiently high accuracy.

Writing (33) for the planar membrane, one has

\[ P_{x,y}(n) = P_{[(x,y)\rightarrow(x_n,1)]} D_{0}^{1,1}, \]

i.e., a particle reaches near-boundary point \( (x_n, 1) \) with probability \( P_{[(x,y)\rightarrow(x_n,1)]} \), and then it hits corresponding boundary point \( (x_n, 0) \) with probability \( D_{0}^{1,1} = H_{0}^{1} \) just as required.

\[ \text{We write } H_{x-\bar{x}}^{y} \text{ because in the upper half plane there is no “perturbation”, the concave membrane completely lies in the lower half plane.} \]
8.3 Coefficients $D_{x-x'}^{y,y'}$ like Green’s functions

Coefficients $D_{x-x'}^{y,y'}$ can be viewed from another standpoint. A direct verification shows that

$$\mathcal{L}D_{x-x'}^{y,y'} = \delta_{x,x'}\delta_{y,y'}, \quad D_{x-x'}^{y,0} = 0,$$

where the discrete Laplacian $\mathcal{L}$ operates on coordinates $(x, y)$ or $(x', y')$ (we remind that $D_{x-x'}^{y,y'} = D_{y,y'}^{x,x'}$). For example, in two-dimensional case we have

$$\mathcal{L}u_{x,y} = 4u_{x,y} - u_{x+1,y} - u_{x-1,y} - u_{x,y+1} - u_{x,y-1}.$$

It means that coefficients $D_{x-x'}^{y,y'}$ can be treated as Green’s functions of the discrete Dirichlet problem in the upper half plane. Consequently, the solution of the general Dirichlet problem in the same (planar) geometry,

$$\mathcal{L}u_{x,y} = f_{x,y}, \quad u_{x,0} = 0,$$

is given as

$$u_{x,y} = \sum_{x',y'} D_{x-x'}^{y,y'} f_{x',y'}.$$

Unfortunately, this result has no direct connection with our actual problem of general membranes. Nevertheless, we can use the following trick. What violates the using of Green’s function $D_{x-x'}^{y,y'}$? The answer is that there exist the boundary and internal points of the membrane with $y > 0$. Obviously, the Laplacian equation becomes invalid on these points. However, we can correct this situation by introducing a certain function $f_{x,y}(n)$ (here $n$ is a parameter). In other words, we impose the Laplacian equation artificially for the boundary and internal points. The correction function is equal to 0 for any external point. Moreover, $f_{x,y}$ equals to 0 for almost all internal point except the internal layer whose points are nearest to the membrane’s boundary. For the boundary points we have

$$f_{x_m,y_m}(n) = 4P_{x_m,y_m}(n) - P_{x_m-1,y_m}(n) - P_{x_m+1,y_m}(n) - P_{x_m,y_m+1}(n) - P_{x_m,y_m-1}(n),$$

whereas for the nearest internal layer

$$f_{x_m-\delta x_m,y_m-\delta y_m}(n) = -P_{x_m,y_m}(n).$$

Now we can write the general solution in terms of Green’s functions,

$$P_{x,y}(n) = \sum_{m=1}^{M} \left[ D_{x-x_m}^{y,y_m} f_{x_m,y_m}(n) + D_{x-x_m+\delta x_m}^{y,y_m-\delta y_m} f_{x_m-\delta x_m,y_m-\delta y_m}(n) \right].$$

Imposing the boundary condition $P_{x_m,y_m}(n) = \delta_{m,n}$ and considering step by step four possible directions of the outer normal, one can demonstrate that this formula can be written as

$$P_{x,y}(n) = \delta_{x,x_m} \delta_{y,y_m} + D_{x-x_n-\delta x_n}^{y,y_m+\delta y_m} - \sum_{m=1}^{M} D_{x-x_m}^{y,y_m} P_{m}(n).$$

Note that here we do not discuss neither contribution of plane “tails”, nor ground functions. One can easily complete these motivations in order to obtain formula (21). This approach is also valid for multi-dimensional case.
8.4 Comments on the Brownian self-transport operator

The important application of the distribution (22) of hitting probabilities is the explicit construction of the Brownian self-transport operator $Q$ which governs the Laplacian transfer. We obtained the explicit formula (32) for $Q$, and it opens the possibility of analytical researches in this field. Note that exact analytical expression of $Q$ even for a planar membrane allowed to obtain the important characteristics of the Laplacian transfer. [8]

On the other hand, expression (32) simplifies also the numerical treatment of the problem. Usually one uses computer simulations of random walks to calculate the matrix elements of $Q$. In order to obtain the whole matrix $Q$ with a high accuracy, one should make enormous number of random walks. Moreover, one walk can be very long, especially on lattices with $d > 2$. On the contrary, working with our approach, one just needs to manipulate in the framework of linear algebra. Once calculated, coefficients $H_y^x$ can be easily used for any membrane. Therefore, one just needs to “compose” the matrix $D$ for a given membrane, and to inverse $(I + D)$. The time required to make these operations depends only on the number of sites for which one calculates $Q$.

In order to adjust the formalism to a more realistic membrane, one can introduce different barriers. As we described above, the horizontal barrier can be introduced by simple modification of coefficients $H^x_y$ while the problem with vertical barriers is more complicated.

We conclude that the present treatment opens a wide field for further investigations.

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9 Appendices

9.1 Corner points’ corrections

In Section 3.2, we calculated the contributions of the boundary points supposing that these points are in general position. In other words, we did not consider the neighbourhood of the corners. Moreover, the procedure of corner points’ removing distorts the lattice near corners (see Fig. 3d), and we did not account the influence of such distortion. For sufficiently regular membranes, these corrections should not change the distribution of hitting probabilities globally. However, the lattice distortions can perturb significantly the solution locally, in the vicinity of the corners. It can be important for certain problems. Here we briefly explain how solution (22) can be improved in order to account such local distortions. Note that these corrections carry a particular character, they depend on the way how the lattice has been distorted. For example, if one would not like to remove the corners, there is no correction.
Why problems appear near the corners? Consider as example the corner point \((x_c, y_c)\) shown on Fig. 9.

We wrote the Laplacian equation (4) for any external point, in particular,

\[
4 \phi_{x_c, y_c+1} - \phi_{x_c+1, y_c+1} - \phi_{x_c-1, y_c+1} - \phi_{x_c, y_c+2} - \phi_{x_c, y_c} = 0.
\]

After that, we removed the corner point \((x_c, y_c)\) (thus one can take \(\phi_{x_c, y_c} = 0\)), and connected points \((x_c, y_c+1)\) and \((x_c-1, y_c)\) to each other by a liaison. It means that the true Laplacian equation is

\[
4 \phi_{x_c, y_c+1} - \phi_{x_c+1, y_c+1} - \phi_{x_c-1, y_c+1} - \phi_{x_c, y_c+2} - \phi_{x_c-1, y_c} = 0.
\]

We prefer to work with the previous general form, and for these purposes we introduce here the correction term \(\Delta \phi_{x_c, y_c+1}\) which is equal to \(\phi_{x_c-1, y_c}\). The same concerns the point \((x_c-1, y_c)\). So, each external corner gives two correction terms (in two-dimensional case). Note that these corrections have not been accounted above, they appear due to distortion of the lattice. We should also consider the Laplacian equation at point \((x_c, y_c)\) because this removed point has been accounted in the general treatment,

\[
4 \phi_{x_c, y_c} - \phi_{x_c+1, y_c} - \phi_{x_c-1, y_c} - \phi_{x_c, y_c+1} - \phi_{x_c, y_c-1} = \Delta \phi_{x_c, y_c}.
\]

Above we supposed that the left hand side is equal to 0 because the point \((x_c, y_c)\) was referred as external point. Now we calculate the corrections due to the lattice distortions, and for these purposes we write the correction term \(\Delta \phi_{x_c, y_c}\). Taking all these terms, we obtain the whole contribution of the corner point \((x_c, y_c)\),

\[
Z_c = -\left[ e^{-ix_c \theta \lambda_{y_c}} - e^{-ix_c \theta \lambda_{y_c+1}} \right] \phi_{x_c-1, y_c} - \left[ e^{-i(x_c-1) \theta \lambda_{y_c}} - e^{-i(x_c-1) \theta \lambda_{y_c+1}} \right] \phi_{x_c, y_c+1}
- e^{i \mathcal{J}(x_c, y_c-1) \theta \lambda_{y_c}} - e^{i \mathcal{J}(x_c+1, y_c) \theta \lambda_{y_c}}.
\]

Two last terms should be included into the contribution of boundary points with indices \(\mathcal{J}(x_c, y_c-1)\) and \(\mathcal{J}(x_c+1, y_c)\) respectively (exactly these two terms are missed if one considers the boundary points near corners, see Section 3.2). Two first terms involve the unknown characteristic functions \(\phi_{x_c-1, y_c}\) and \(\phi_{x_c, y_c+1}\).
that can be called corner functions. It means that in general case formula (22) becomes

\[ P_{x,y}(n) = P_{x,y}^0(n) - \sum_{\text{corners}} \left( D_{x-x_c} y_{c} - D_{x-x_c} \delta y_{c}^{(1)}(n) \right) \left( P_{x_c + \delta x_c^{(1)}, y_c + \delta y_c^{(1)}}(n) + \right) \]

\[ \left( D_{x-x_c} y_{c} - D_{x-x_c} \delta y_{c}^{(1)}(n) \right) \left( P_{x_c + \delta x_c^{(2)}, y_c + \delta y_c^{(2)}}(n) \right), \]

where \( P_{x,y}^0(n) \) denotes the right hand side of (22), and \((\delta x_c^{(1,2)}, \delta y_c^{(1,2)}) \) are two outer normals on the corner point \((x_c, y_c)\).

The corner functions \( P_{x_c + \delta x_c^{(1,2)}, y_c + \delta y_c^{(1,2)}}(n) \) can be calculated by the same way that was used for near-boundary functions, i.e., we close the system of linear equations and solve it. The last formula gives a right distribution for all sites of the membrane, including the points near corners.

### 9.2 One point problem

Consider the simple problem of finding the distribution of hitting probabilities \( P_{x,y}(n) \) on the horizontal axis if there exists an absorbing point \((0, y_0)\) with \( y_0 > 0 \). As consequence, we shall find the probability to hit the point \((0, y_0)\) if it started from \((x, y)\) without touching the horizontal axis.

Here we cannot apply the technique of the planar case given that
- there is no translational invariance;
- the Laplacian equation (3) is invalid on the point \((0, y_0)\);

On the contrary, we can use formula (8). There exists the only correction \( \Delta \Phi^{(y_0)} \) for the unique point \((0, y_0)\), i.e.,

\[ \Delta \Phi^{(y_0)} = -\phi_{1,0} - \phi_{-1,0} - \phi_{0,1} - \phi_{0,-1} \]

(we omitted \(4\phi_{0,0}\) because it equals to 0). Using formula (12) and taking the limit \( L \to \infty \), we obtain

\[ \phi_{x,y} = \int \frac{d\theta' d\phi'}{2\pi} e^{ix\phi'} \left( \phi^y(\theta') c_0(\theta, \theta') + \gamma(\theta') \Delta \Phi^{(y_0)}(\theta) \right), \]

where

\[ c_0(\theta, \theta') = \sum_{x=-\infty}^{\infty} e^{ix(\theta-\theta')} = 2\pi \delta(\theta - \theta'). \]

Applying the inverse Fourier transform, we obtain

\[ P_{x,y}(n) = H_{x-n}^y - D_{x}^y \left( P_{1,0}(n) + P_{-1,0}(n) + P_{0,1}(n) + P_{0,-1}(n) \right). \] (36)

Looking on this relation, we say that there are four near-boundary functions on four points around \((0, y_0)\). Taking corresponding \( x \) and \( y \) and summarizing four equations for near-boundary functions, we write the equation for their sum,

\[ \sum_{m=1}^{4} P_{m}(n) = \left( H_{1-n}^{y_0} + H_{-1-n}^{y_0} + H_{-n}^{y_0+1} + H_{-n}^{y_0-1} \right) \]
\[-(D_{1}^{0,0} + D_{-1}^{0,0} + D_{0}^{0,1,0} + D_{0}^{0,0,1,0}) \sum_{m=1}^{4} P_{m}(n).\]

Using the properties of coefficients $H^{y}_{x}$ and $D^{y,y'}_{x}$, we can simplify this relation and express the sum of near-boundary functions,

\[\sum_{m=1}^{4} P_{m}(n) = \frac{H_{0}^{y} D_{0}^{y,y}}{D_{0}^{y,y}}.\]

Substituting this sum into (36), we have

\[P_{x,y}(n) = H_{x-n}^{y} - H_{-n}^{y} \frac{D_{x}^{y,y}}{D_{0}^{y,y}}.\]

So, the problem is solved. The probability to hit the point $(0, y_{0})$ without touching the horizontal axis is

\[P_{x,y}^{(one)} = 1 - \sum_{n=-\infty}^{\infty} P_{x,y}(n) = \frac{D_{x}^{y,y}}{D_{0}^{y,y}}.\]

We hope that the consideration of this simple problem clarifies the sense of coefficients $D^{y,y'}_{x}$ and simplifies the understanding of general results.

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