Upper bound for the span of pencil graph

N Parvathi, AVimala Rani
Department of Mathematics,
SRM Institute of Science and Technology, Kattankulathur, Tamil Nadu, India.
parvathi.n@ktr.srmuniv.ac.in

Abstract An L(2,1)-Coloring or Radio Coloring or $\lambda$ coloring of a graph is a function $f$ from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x,y)=1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y)=2$, where $d(x,y)$ denotes the distance between $x$ and $y$ in $G$. The L(2,1)-coloring number or span number $\lambda(G)$ of $G$ is the smallest number $k$ such that $G$ has an L(2,1)-coloring with $\max \{f(v) : v \in V(G)\} = k$. The minimum number of colors used in L(2,1)-coloring is called the radio number $rn(G)$ of $G$ (Positive integer). Griggs and Yeh conjectured that $\lambda(G) \leq \Delta^2$ for any simple graph with maximum degree $\Delta \geq 2$. In this article, we consider some special graphs like, n-sunlet graph, pencil graph families and derive its upper bound of $\lambda(G)$ and $rn(G)$.

1. Introduction

In a private communication with J. Griggs, F. Roberts proposed the problem of efficiently assigning radio channels to transmitters at several locations, using nonnegative integers to several locations, using nonnegative integers to represent channels, so that close locations receive different channels, and channels for very close locations are at least two apart such that these channels would not interfere with each other.

The Channel assignment problem is that assigning a frequency to each radio transmitter so that interfering transmitters are assigned frequencies whose separation is not in a set of disallowed separations.[2]

The Channel assignment problem has been approached in many different ways, but the common was reducing the span of colors. Some researchers instead of reducing the span tried to minimize the order i.e. using minimum number of colors. There are considerable number of articles studying the (L(2,1))-Coloring. Most of the papers considered $\lambda$ as the parameter on particular classes of graphs and few papers considered $rn(G)$ as the parameter. Auxiliary graphs play an important role in connecting many useful networks. Griggs and Yeh’s Conjecture is true for the graphs we considered in this article.

In this article, we consider some special graphs like, n-sunlet graph, pencil graph families and derive its upper bound of $\lambda(G)$ and $rn(G)$.

2. Prepositions
1. [3] For every positive integer t, $\lambda(K_t) = t + 1$
2. [3] If T is a tree with $\Delta(T) = \Delta \geq 1$, then either $\lambda(T) = \Delta + 1$ or $\lambda(T) = \Delta + 2$ (Griggs and Yeh)
3. [3] If G is graph of order n, then $\lambda(G) \leq n + \chi(G) - 2$
4. [3] If G is a complete k-partite graph of order n, where $k \geq 2$, then $\lambda(G) = n + k - 2$
5. [3] If G is a graph with maximum degree $\Delta$, then $\lambda(G) \leq \Delta^2 + \Delta$ (Griggs and Yeh) and $\lambda(G) \leq \Delta^2 + \Delta - 1$. Kral and Skrekovski
6. [3] If G is a connected graph of diameter 2 with $\Delta(G) = \Delta$, then $\lambda(G) \leq \Delta^2$ (Griggs and Yeh)
7. [4] $\lambda_2(C_n) = 4$, C is the cycle (Griggs and Yeh)
8. [4] If G is outerplanar $\lambda_2(G) \leq \Delta + 8$. (Calamoneri and Petreschi improved bound to $\Delta + 2$ for $\Delta \geq 8$
9. [4] If G is triangular outerplanar then $\lambda_2(G) \leq \Delta + 6$.
10. Span of Wheel graph $C_nVK_1$ is $\lambda(W_n) = n + 1$
11. The radio chromatic number of path $P_n$ is $\Delta + 1$ for $n \geq 3$.
12. The radio chromatic number of comb graph G is $\Delta + 1$ for $n \geq 3$.
13. The radio chromatic number of star graph $K_{1,n}$ is $\Delta + 1$.
14. The radio chromatic number of cycle $C_n$

15. The radio chromatic number of cycle
   $rn(C_n) = \begin{cases} 
   \Delta + 1 & n = 0 \mod 3 \\
   \Delta + 2 & n = 1 \mod 3 \\
   \Delta + 3 & n = 2 \mod 3.
   \end{cases}$
16. The radio chromatic number of Sunlet graph $S_n$
   $rn(S_n) = \begin{cases} 
   \Delta + 2 & n = 0 \mod 3 \\
   \Delta + 1 & Otherwise
   \end{cases}$
17. For $n \geq 3$, the radio chromatic number of line graph of Sunlet graph $L(S_n)$ is
   $rn(L(S_n)) = \begin{cases} 
   \Delta + 1 & n = 0 \mod 6, n = 0 \mod 10 \\
   \Delta + 2 & n = 0 \mod 25 + 10i, i = 0, 1, 2, ..., \Delta + 2 \mod 3, n = 0 \mod 3, n = 0 \mod 10, n = 0 \mod 6, i = 1, 2, ..., \Delta + 2 \mod 6, n = 0 \mod 6, i = 1, 2, ...
   \end{cases}$
18. Let $n \geq 4$, then the radio chromatic number of Middle graph sunlet graph $S_n$ is $\Delta + 2$.
19. Let $n \geq 5$, then the radio chromatic number of Total graph sunlet graph $S_n$ is
   $rn(T(S_n)) = \begin{cases} 
   \Delta + 1 & n = 0 \mod 10 \\
   \Delta + 2 & n = 0 \mod 5 \\
   \Delta + 2 & n = 0 \mod (k+5i), i = 0, 1, 2, ..., \Delta + 2 \mod 5, k = 8, 9 \\
   \Delta + 3 & n = 0 \mod (k+5i), i = 0, 1, 2, ..., \Delta + 3 \mod 6, k = 6, 7
   \end{cases}$
20. For $n \geq 4$ the radio chromatic number of Central graph of Sunlet graph
\[ \text{rn}(C(S_n)) = \begin{cases} 
\Delta + 4 & \text{n is even} \\
\Delta + 5 & \text{n is odd} 
\end{cases} \]

20. \cite{10} Let G be a Bistar graph of order 2n+2, then \( \text{rn}(B_{n,n}) = n+2 \)
21. \cite{10} For all n, \( \text{rn}(B^2_{n,n}) = 2n+2 \)
22. \cite{10} For all n, \( \text{rn}(D_2(B_{n,n})) = 2n+4 \)
23. \cite{10} For all n, \( \text{rn}(S'(B_{n,n})) = 2n+3 \)
24. \cite{10} For all n>0, \( \text{rn}(DS(B_{n,n})) = 2n+3 \)

3. Preliminaries

Definition 3.1.
\cite{2} An L(2,1)-Coloring or Radio Coloring or \( \delta \) coloring of a graph is a function \( f \) from the vertex set \( V(G) \) to the set of all nonnegative integers such that \( |c(x) - c(y)| \geq 2 \) if \( d(x,y) = 1 \) and \( |c(x) - c(y)| \geq 1 \) if \( d(x,y) = 2 \),
where \( d(x,y) \) denotes the distance between \( x \) and \( y \) in \( G \).

The L(2,1)-coloring number or span number \( \delta(G) \) of G is the smallest number \( k \) such that G has an L(2,1)-coloring with \( \max\{c(v) : v \in V(G)\} = k \).

The minimum number of colors used in L(2,1)-coloring is called the radio number \( \text{rn}(G) \) of G (Positive integer).

Definition 3.2.
\cite{6} Let \( n \) be a positive integer with \( n \geq 2 \). A pencil graph with 2n + 2 vertices, denoted by \( P_{2n} \), is a graph with the vertex set and the edge set as follows.
\[ V(P_{2n}) = \{u_i, v_i | i \in [0,n]\} \]
\[ E(P_{2n}) = \{u_i u_{i+1}, v_i v_{i+1} | i \in [1,n-1]\} \cup \{u_0 v_0, v_{n-1} u_{n} v_{n} u_{n} v_{0}\}. \]
It is easy to check that the diameter of \( P_{2n} \) is \( \text{diam}(P_{2n}) = d = \lfloor n/2 \rfloor + 1 \), for \( n \geq 2 \).

Definition 3.3.
\cite{9} The Line graph of a graph \( G \), denoted by \( L(G) \) is a graph whose vertices are edges of \( G \), and if \( u,v \in E(G) \) then \( uv \in E(L(G)) \) if \( u \) and \( v \) share a vertex in \( G \).

Definition 3.4.
\cite{9} The n-sunlet graph is a graph on 2n vertices obtained by attaching a pendant edge to the n-vertices of cycle \( C_n \).

Definition 3.5.
\cite{1} The number of colors used in a radio coloring with the minimum score (number of used colors) is the radio chromatic number \( \text{rn}(G) \) of G.

Definition 3.6.
The largest colors used in a radio coloring with the minimum price (value of the largest used color) is the radio chromatic valuerv(G) of G.

**Definition 3.7.**

The sum of used colors in a radio coloring with the minimum weight (sum of used color) is the radio chromatic cost rc(G) of G.

**Definition 3.8.**

The difference between the smallest and the largest used colors in a radio coloring with the minimum gap (difference between the smallest and the largest used colors) is the radio chromatic bandwidth rb(G) of G.

**4. Main Results**

**Theorem 4.1.**

Let n be a positive integer at least 7, then the span number of Pencil graph (Pc_n) is 2Δ, ie (Pc_n)=2Δ.

**Proof:**

\[ V(Pc_n) = \{u_i,v_i|i\in[0,n]\} \]

The Coloring of the vertices is as follows

\[
\begin{align*}
\text{c}(u_0) &= 0 & \text{c}(v_0) &= 3 \\
\text{c}(u_1) &= 4 & \text{c}(v_1) &= 2 \\
\text{c}(u_2) &= 1 & \text{c}(v_2) &= 5 \\
\end{align*}
\]

The vertices \( V(Pc_n) = \{u_i,v_i|i\in[3,n-1]\} \) are colored as

(i) wheni \equiv 1 \mod 2

\[
\begin{align*}
\text{c}(u_i) &= 6 \\
\text{c}(v_i) &= 0 \\
\end{align*}
\]

(ii) wheni \equiv 0 \mod 4

\[
\begin{align*}
\text{c}(u_i) &= 2 \\
\text{c}(v_i) &= 4 \\
\end{align*}
\]

(iii) wheni \equiv 1 \mod 4, i\neq 1 \mod 2

\[
\begin{align*}
\text{c}(u_i) &= 0 \\
\text{c}(v_i) &= 6 \\
\end{align*}
\]

(iv) wheni \equiv 0 \mod 2, i\neq 0 \mod 4, i\neq 2

\[
\begin{align*}
\text{c}(u_i) &= 4 \\
\text{c}(v_i) &= 2 \\
\end{align*}
\]

For c(u)_n& c(v)_n

(i) When n \equiv 1 \mod 2, 0 \mod 4

\[
\text{c}(u_n) = 1
\]
(ii) When $n \equiv 0 \mod 2, 1 \mod 4$, $n \not\equiv 0 \mod 4$

\[
c(v_n) = 5
\]

\[
c(u_n) = 5
\]

\[
c(v_0) = 1
\]

The coloring of pencil graph in the above pattern is an optimal coloring.

**Note:**
1. For $n=2,6$, $\lambda(Pc_n) = \Delta + 2$
2. For $n=3$, $\lambda(Pc_n) = 2\Delta + 1$
3. For $n=4,5$, $\lambda(Pc_n) = 2\Delta$

**Theorem 4.2.**
Let $n$ be a positive integer at least 6, then the span number of the line graph of pencil graph

\[
\lambda(L(PC_n)) = \begin{cases} 
2\Delta & \text{if } n \text{ is even} \\
2\Delta + 1 & \text{if } n \text{ is odd}
\end{cases}
\]

**Proof:**

\[
V(PC_n) = \{ u_i, v_i, w_i | i \in [0, n] \}
\]

\[
E(PC_n) = \{ u_i u_{i+1}, w_i v_i, w_i v_{i+1} | 1 \leq i \leq n-1, v_0 u_0, w_0 u_0 | i=0, n, v_0 w_n, u_0 v_1 \}
\]

The coloring of the vertices is as follows

\[
c(w_i) = \begin{cases} 
2 & i = 0 \\
0 & i \text{ is odd} \\
1 & i \text{ is even } i \geq 2
\end{cases}
\]

\[
c(u_0) = 6 \\
c(v_1) = 4 = c(u_3) \\
c(u_1) = 3 \\
c(v_2) = 7 \\
c(u_2) = 8 \\
c(v_3) = 5
\]

[A] When $n$ is odd

The vertices $V(L(PC_n)) = \{ u_i, v_i | i \in [4, n-2] \}$ are colored as

(i) when $i \equiv 0 \mod 4$

\[
c(u_i) = 6 \\
c(v_i) = 3
\]

(ii) when $i \equiv 1 \mod 4$

\[
c(u_i) = 3 \\
c(v_i) = 7
\]

(iii) when $i \equiv 2 \mod 4$

\[
c(u_i) = 7 \\
c(v_i) = 4
\]

(iv) when $i \equiv 3 \mod 4$
c(u_i)=4  
c(v_i)=6
For c(u_{n-1}), c(u_n) & c(v_0)

(i) When n ≡ 3 mod 4
   c(u_{n-1})=5c(v_n)=6  
c(u_n)=7c(v_0)=9

(ii) When n ≡ 1 mod 4
   c(u_{n-1})=6c(v_n)=5  
c(u_n)=9c(v_0)=7

[B] When n is even

The vertices V(L(P_{c_n})) = \{u_i, v_i | i \in [4, n-1]\} are colored as

(i) when i ≡ 0 mod 4
   c(u_i)=6  
c(v_i)=3

(ii) when i ≡ 1 mod 4
   c(u_i)=3  
c(v_i)=7

(iii) when i ≡ 2 mod 4
   c(u_i)=7  
c(v_i)=4

(iv) when i ≡ 3 mod 4
   c(u_i)=4  
c(v_i)=6

For c(u_n) & c(v_0)

(i) When n ≡ 2 mod 4
   c(u_n)=5  
c(v_0)=8

(ii) When n ≡ 0 mod 4
   c(u_n)=8  
c(v_0)=5

The coloring of line graph of pencil graph in the above pattern is an optimal coloring.

Note:

1. For n=2,3 λ(L(P_{c_n}))=2Δ+1
2. For n=4,5 λ(L(P_{c_n}))=2Δ
Theorem 4.3.
Let \( n \) be a positive integer, \( n \geq 4 \), then \( (S_n) = \Delta + 2 \), except when \( n = 10 \) and \( n \equiv 10 \mod 2 \).

Proof:

Case 4.3.1.

When \( n \equiv 0 \mod 3 \).

The coloring of the vertices are done in the following way

\[
c(u_i) = \begin{cases} 
1 & i \equiv 1 \mod 3, i = 1 \\
5 & i \equiv 0 \mod 3 \\
3 & \text{otherwise}
\end{cases}
\]

\[
c(v_i) = \begin{cases} 
4 & i \equiv 1 \mod 3, i = 1 \\
0 & \text{otherwise}
\end{cases}
\]

Case 4.3.2.

When \( n \equiv 2 \mod 3 \).

The vertices of \( u_i \) are colored in the following manner
\( c(u_n) = 4 \), \( c(u_{n-1}) = 0 \).

For \( V(S_n) = \{u_i | i \in [1, n-2]\} \)

\[
c(u_i) = \begin{cases} 
1 & i \equiv 1 \mod 3, i = 1 \\
3 & i \equiv 2 \mod 3, i = 2 \\
5 & i \equiv 0 \mod 3
\end{cases}
\]

The vertices of \( v_i \) are colored in the following manner
\( c(v_1) = 5 \), \( c(v_{n-1}) = c(v_n) = 2 \), \( c(v_{n-2}) = 1 \).

For \( V(S_n) = \{v_i | i \in [1, n-3]\} \)

\[
c(v_i) = \begin{cases} 
4 & i \equiv 1 \mod 3 \\
0 & \text{otherwise}
\end{cases}
\]

Case 4.3.3.

When \( n \equiv 1 \mod 3 \).

Subcase 4.3.3.1. When \( n = 4j, j \equiv 1 \mod 3 \).

The coloring of the vertices of \( S_n \) is as follows

\[
c(v_i) = \begin{cases} 
1 & i \equiv 3 \mod 4 \\
0 & \text{otherwise}
\end{cases}
\]
Subcase 4.3.3.1. When \( n = 6j + 1, j \equiv 1 \mod 3 \)

The coloring of the vertices of \( S_n \) is as follows

\[
c(u_i) = \begin{cases} 
2 & i \equiv 1 \mod 4, i = 1 \\
4 & i \equiv 2 \mod 4, i = 2 \\
1 & i \equiv 3 \mod 4, i = 3 \\
5 & i \equiv 0 \mod 4, i = 4 
\end{cases}
\]

\[
c(u_n) = \begin{cases} 
4 & i \equiv 1 \mod 4 \\
2 & i \equiv 2 \mod 4 \\
5 & i \equiv 3 \mod 4, i = 3 \\
1 & i \equiv 0 \mod 4 
\end{cases}
\]

\[
c(v_j) = \begin{cases} 
2 & i = 1 \\
1 & i = 2, 3 \\
3 & i \equiv 0 \mod 4 \\
0 & \text{otherwise}
\end{cases}
\]

The coloring of the sunlet graph in the above pattern is an optimal coloring.

**Note:**

For \( n = 3 \) \( (S_n) = \Delta + 2 \)

**Theorem 4.4.**

Let \( n \) be positive integer, \( n \geq 4 \), then \( (S_n) = \Delta + 3 \), when \( n = 10 \) and \( n \equiv 10 \mod 2 \).

**Proof:**

The coloring of the vertices are done in the following way

\[
c(u_i) = \begin{cases} 
1 & i \equiv 0 \mod 4 \\
2 & i \equiv 2 \mod 4 \\
4 & i \equiv 1 \mod 4 \\
5 & i \equiv 3 \mod 4, i = 3 
\end{cases}
\]

\[
c(u_n) = \begin{cases} 
0 & i = 1 \\
3 & i = 2 \\
6 & i = n 
\end{cases}
\]
The coloring of sunlet graph in the above pattern is an optimal coloring.

**Theorem 4.5.**

For $n \geq 3$, the radio number of subdivision graph of $S_n$, $rn(S(S_n)) = \begin{cases} \Delta + 2 & n = 1 \text{mod} 3 \\ \Delta + 1 & \text{otherwise} \end{cases}$

**Proof:**

The subdivision graph of $S_n$ be defined as graph on $4n$ vertices. V($S(S_n)$) = \{u_i, v_i, w_i, x_i | 1 \leq i \leq n\}

The coloring of vertices of $S(S_n)$ is as follows
\[
c(x_n) = \begin{cases} 1 & n = 1 \\ 5 & n = 5 \\ 7 & \text{otherwise} \end{cases}
\]
\[
c(v_n) = \begin{cases} 3 & n = 1 \text{mod} 3, n = 1 \\ 1 & \text{otherwise} \end{cases}
\]
\[
c(u_n) = \begin{cases} 1 & n = 1 \text{mod} 3, n = 1, n \neq n \\ 3 & n = 0 \text{mod} 3 \\ 7 & n = n \\ 5 & \text{otherwise} \end{cases}
\]
\[
c(w_n) = \begin{cases} 9 & n = n \\ 3 & n = 1 \text{mod} 3, n = 1 \\ 5 & n = 0 \text{mod} 3 \\ 1 & \text{otherwise} \end{cases}
\]

The coloring of $S(S_n)$ in this manner gives optimal coloring.

Also the values of some parameters are found from the above coloring namely

- Radio Chromatic value $rv(S(S_n)) = 9$
- Radio Chromatic cost $rc(S(S_n)) = 25$
- Radio Chromatic bandwidth $rb(S(S_n)) = 8$

**Theorem 4.6.**

For $n \geq 4$, the radio number of pencil graph $PC_n$, $rn(PC_n) = \begin{cases} \Delta + 2 & \text{if } n \text{ is even} \\ \Delta + 3 & \text{if } n \text{ is odd} \end{cases}$

**Proof:**

*Case 4.6.1.*
When \( n \) is even, the coloring of the vertices \( c:V(\text{PC}_n) \rightarrow \{1,3,5,7,9\} \) is as follows:

\[
c(\nu_i) = \begin{cases} 
1 & i \equiv 1 \mod 2, i \equiv 0 \mod 4 \\
3 & i \equiv 1 \mod 4, i \equiv 1 \mod 4 \\
5 & i \equiv 0 \mod 4 \\
7 & \text{otherwise}
\end{cases}
\]

\[
c(\nu_j) = \begin{cases} 
1 & i \equiv 1 \mod 4 \\
3 & i \equiv 1 \mod 4, i \not\equiv 1 \mod 4 \\
7 & i \equiv 0 \mod 4 \\
9 & i = 0, 2 \\
5 & \text{otherwise}
\end{cases}
\]

**Case 4.6.2.**

When \( n \) is odd, the coloring of the vertices \( c:V(\text{PC}_n) \rightarrow \{1,3,5,7,9\} \) is as follows:

\[
c(\nu_i) = \begin{cases} 
1 & i \equiv 1 \mod 2, i \equiv 0 \& i \equiv 1 \mod 4 \\
3 & i \equiv 1 \mod 4, i \equiv 1 \mod 4 \\
5 & i \equiv 0 \mod 4 \\
9 & n \& i \equiv 1 \mod 4 \\
11 & i \equiv 0 \\
7 & \text{otherwise}
\end{cases}
\]

\[
c(\nu_j) = \begin{cases} 
1 & i \equiv 1 \mod 4, i \equiv n \\
3 & i \equiv 1 \mod 2, i \not\equiv 1 \mod 4 \\
7 & i \equiv 0 \mod 4 \\
9 & i = 2, n \& i \equiv 1 \mod 4 \\
11 & i = 0 \\
5 & \text{otherwise}
\end{cases}
\]

The coloring of \( S(S_n) \) in this manner gives optimal coloring.

Also the values of some parameters are found from the above coloring namely

- Radio Chromatic value \( r\text{v}(\text{PC}_n) = 11 \)
- Radio Chromatic cost \( r\text{c}(\text{PC}_n) = 36 \)
- Radio Chromatic bandwidth \( r\text{b}(\text{PC}_n) = 10 \)

**Note**

\( r\text{n}(\text{PC}_n) = \Delta + 3: n = 2, 3 \)

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