Abstract

Consider the following coloring process in a simple directed graph $G(V, E)$ with positive indegrees. Initially, a set $S$ of vertices are white, whereas all the others are black. Thereafter, a black vertex is colored white whenever more than half of its in-neighbors are white. The coloring process ends when no additional vertices can be colored white. If all vertices end up white, we call $S$ an irreversible dynamic monopoly (or dynamo for short) under the strict-majority scenario. An irreversible dynamo under the simple-majority scenario is defined similarly except that a black vertex is colored white when at least half of its in-neighbors are white. We derive upper bounds of $(2/3)|V|$ and $|V|/2$ on the minimum sizes of irreversible dynamos under the strict and the simple-majority scenarios, respectively. For the special case when $G$ is an undirected connected graph, we prove the existence of an irreversible dynamo with size at most $\lceil |V|/2 \rceil$ under the strict-majority scenario. Let $\epsilon > 0$ be any constant. We also show that, unless $\text{NP} \subseteq \text{TIME}(n^{O(\ln \ln n)})$, no polynomial-time, $((1/2 - \epsilon) \ln |V|)$-approximation algorithms exist for finding the minimum irreversible dynamo under either the strict or the simple-majority scenario. The inapproximability results hold even for bipartite graphs with diameter at most 8.
1 Introduction

Let $G(V, E)$ be a simple directed graph (or digraph for short) with positive indegrees. A simple undirected graph is interpreted as a directed one where each edge is accompanied by the edge in the opposite direction. In this paper, all graphs are simple and have positive indegrees. The following coloring process extends that of Flocchini et al. [4] by taking digraphs into consideration. Initially, all vertices in a set $S \subseteq V$ are white, whereas all the others are black. Thereafter, a black vertex is colored white when more than half of its in-neighbors are white. The coloring process proceeds asynchronously until no additional vertices can be colored white. If all vertices end up white, then $S$ is called an irreversible dynamo under the strict-majority scenario. An irreversible dynamo under the simple-majority scenario is defined similarly except that a black vertex is colored white when at least half of its in-neighbors are white. Tight or nearly tight bounds on the minimum size of irreversible dynamos are known when $G$ is a toroidal mesh [6, 14], torus cordalis, torus serpentinus [6], butterfly, wrapped butterfly, cube-connected cycle, hypercube, DeBruijn, shuffle-exchange, complete tree, ring [5, 10] and chordal ring [4].

Chang and Lyuu [1] show that $G(V, E)$ has an irreversible dynamo of size at most $(23/27)|V|$ under the strict-majority scenario. This paper improves their $(23/27)|V|$ bound to $(2/3)|V|$. Moreover, if $G$ is undirected and connected, our $(2/3)|V|$ upper bound can be further lowered to $\lceil |V|/2 \rceil$. Under the simple-majority scenario, we show that every digraph has an irreversible dynamo of size at most $|V|/2$. In the literature on fault-tolerant computing, an irreversible dynamo is interpreted as a set of processors whose faulty behavior leads all processors to erroneous results [4, 5, 6, 10, 13]. Under this interpretation, our upper bounds limit the number of adversarially placed faulty processors that any system can guarantee to tolerate without inducing erroneous results on all processors.

Under several randomized mechanisms for coloring the vertices, Kempe, Kleinberg and Tardos [8, 9] and Mossel and Roch [11] show $\left(1 - (1/e) - \epsilon\right)$-approximation algorithms for allocating a given number of seeds to color the most vertices white, where $\epsilon > 0$ is an arbitrary constant. Kempe, Kleinberg and Tardos [8] also show inapproximability results for allocating seeds in digraphs to color the most vertices white. This paper considers the related computational problem of finding a minimum irreversible dynamo given an undirected graph, which arises naturally because an extensive literature has been investigating the minimum size of irreversible dynamos [4, 5, 6, 10, 13]. We show that, unless $\text{NP} \subseteq \text{TIME}(n^{O(\log \log n)})$, no polynomial-time, $((1/2 - \epsilon)\ln |V|)-$approximation algorithms exist for the minimum irreversible dynamo, either under the strict or the simple-majority scenario. The inapproximability results hold even for bipartite graphs with diameter at most 8. In proving our inapproximability results, we make use of Feige’s [3] famous result on the inapproximability of finding a minimum dominating set in an undirected graph.

2
Variants on the coloring process appear in the literature. Given two alternative actions, Watts [16] argues that an individual in a social or economical system typically chooses an alternative based on the fraction of the neighboring individuals adopting it. Watts’ model assumes a sparse, undirected and random graph. There is also a random variable distributed in [0, 1], from which every vertex independently draws a ratio. Initially, a uniformly random set of vertices are white, leaving all the others black. Thereafter, a black vertex becomes white when the fraction of its white neighbors exceeds the above ratio. Finally, the coloring process ends when no additional vertices can be colored white. Watts gives theoretical and numerical results on the fraction of white vertices at the end. Gleeson and Cahalane [7] extend Watts’ work by deriving an analytical solution for the fraction of white vertices at the end in tree-like graphs. Samuelsson and Socolar [15] study a more general process called the unordered binary avalanche, which allows coloring mechanisms beyond the threshold-driven ones. Unlike the works mentioned above, we do not assume that the initially white vertices are uniformly and randomly distributed.

This paper is organized as follows. Section 2 gives the definitions. Sections 3–4 present upper bounds on the minimum size of irreversible dynamos for directed and undirected graphs, respectively. Section 5 presents inapproximability results on finding minimum irreversible dynamos.

2 Definitions

Let $G(V, E)$ be a simple directed graph (or digraph for short) [17] with positive indegrees. For $v \in V$, we denote by $N^\text{in}(v) \subseteq V \setminus \{v\}$ the set of vertices incident on an edge coming into $v$. Similarly, $N^\text{out}(v) \subseteq V \setminus \{v\}$ is the set of vertices incident on an edge going from $v$. Define $\deg^\text{in}(v) = |N^\text{in}(v)|$ and $\deg^\text{out}(v) = |N^\text{out}(v)|$ as the indegree and outdegree of $v$, respectively. For $X, Y \subseteq V$, we write $e(X, Y) = |(X \times Y) \cap E|$, i.e., the number of edges going from a vertex in $X$ to one in $Y$. An undirected graph is a directed one with every edge accompanied by an edge in the opposite direction. For a vertex $v$ of an undirected graph, we define $\deg(v) = \deg^\text{in}(v)$ and $N(v) = N^\text{in}(v)$ without loss of generality. Furthermore, define $N^*(v) = N(v) \cup \{v\}$. For any two vertices $x$ and $y$ of an undirected connected graph, let $d(x, y)$ be their distance, i.e., the number of edges on a shortest path between $x$ and $y$. For any $v \in V$ and nonempty $U \subseteq V$, denote $d(v, U) = \min_{u \in U} d(v, u)$ for convenience. For any $V' \subseteq V$, the subgraph of $G$ induced by $V'$ is denoted by $G[V'] = (V', E \cap (V' \times V'))$. That is, $G[V']$ has all the edges in $E$ with both endpoints in $V'$. For emphasis, we may sometimes write $N^\text{in}_G(v), N^\text{out}_G(v)$ and $N^*_G(v)$ for $N^\text{in}(v), N^\text{out}(v)$ and $N^*(v)$, respectively. Similarly, we may write $\deg^\text{in}_G(v), \deg^\text{out}_G(v)$ and $d_G(x, y)$ for $\deg^\text{in}(v), \deg^\text{out}(v)$ and $d(x, y)$, respectively. All graphs in this paper are simple and have positive indegrees.

A network $\mathcal{N}(G, \phi)$ consists of a digraph $G(V, E)$ with positive indegrees and
a function \( \phi : V \rightarrow \mathbb{N} \). The coloring process in \( \mathcal{N}(G, \phi) \) proceeds asynchronously. Initially, a set \( S \subseteq V \) of vertices, called the seeds, are white whereas all the others are black. Thereafter, a vertex \( v \) becomes white when at least \( \phi(v) \) of the vertices in \( N^\text{in}(v) \) are white. The coloring process ends when no additional vertices can be colored white. Let \( c(S, G, \phi) \subseteq V \) be the set of vertices that are white at the end given that \( S \) is the set of seeds. Define \( \text{min-seed}(G, \phi) = \min\{U \subseteq V, c(U, G, \phi) = V \mid U \} \), namely, the minimum number of seeds needed to color all vertices white at the end. Clearly, it does not matter in what sequences the vertices are colored white as they will end up with the same \( c(S, G, \phi) \).

We are interested in \( \phi \) being one of the following functions:

- **Strict majority**: \( \phi^\text{strict}(v) = \lceil (\text{deg}^\text{in}(v) + 1)/2 \rceil \); so a vertex \( v \) is colored white when more than half of the vertices in \( N^\text{in}(v) \) are white.
- **Simple majority**: \( \phi^\text{simple}(v) = \lfloor \text{deg}^\text{in}(v)/2 \rfloor \); so a vertex \( v \) is colored white when at least half of the vertices in \( N^\text{in}(v) \) are white.

A set \( S \subseteq V \) is called an irreversible dynamic monopoly (or irreversible dynamo for short) of \( \mathcal{N}(G, \phi^\text{strict}) \) if \( c(S, G, \phi^\text{strict}) = V \). Similarly, it is an irreversible dynamo of \( \mathcal{N}(G, \phi^\text{simple}) \) if \( c(S, G, \phi^\text{simple}) = V \). We may sometimes write \( \phi^\text{strict}_G \) and \( \phi^\text{simple}_G \) instead of \( \phi^\text{strict} \) and \( \phi^\text{simple} \) to emphasize the role of \( G \).

Given an undirected graph \( G(V, E) \), the problem **irreversible dynamo (strict majority)** asks for a minimum irreversible dynamo under the strict-majority scenario. Similarly, **irreversible dynamo (simple majority)** asks for one under the simple-majority scenario. An \( \ell \)-approximation algorithm for each of the above problems outputs an irreversible dynamo with size at most \( \ell \) times the minimum. A dominating set of an undirected graph \( G(V, E) \) is a set of vertices sharing at least one vertex with \( N^\ast_G(v) \) for each \( v \in V \). Given an undirected graph \( G(V, E) \), an \( \ell \)-approximation algorithm for the dominating set problem outputs a dominating set of \( G \) with size at most \( \ell \) times the minimum. Recall that an algorithm is said to run in polynomial time if its running time is polynomial in the length of its input.

The following fact is straightforward.

**Fact 1.** For any network \( \mathcal{N}(G(V, E), \phi) \) and any \( S, T \subseteq V \),

\[
c(S, G, \phi) \subseteq c(S \cup T, G, \phi).
\]

### 3 Irreversible dynamos of directed graphs

Let \( G(V, E) \) be a digraph with positive indegrees, \( k \) be a positive integer and \( \phi_{k/(k+1)}(v) \equiv \text{deg}^\text{in}(v) \cdot k/(k+1) \). This section derives upper bounds on \( \text{min-seed}(G, \phi_{k/(k+1)}) \). As corollaries, we obtain upper bounds on the minimum sizes of irreversible
dynamos under the strict and the simple-majority scenarios. For a partition $V = \bigcup_{i=1}^{k+1} V_i$ of $V$, define
\[
\eta(G, V_1, \ldots, V_{k+1}) \equiv \sum_{i=1}^{k+1} |c(V \setminus V_i, G, \phi_{k/(k+1)})|.
\]

An easy lemma follows.

**Lemma 2.** Let $G$ be a digraph with positive indegrees, $k$ be a positive integer and $V = \bigcup_{i=1}^{k+1} V_i$ be a partition. Then the following conditions are equivalent:

1. $\eta(G, V_1, \ldots, V_{k+1}) = (k + 1)|V|$.
2. $c(V \setminus V_i, G, \phi_{k/(k+1)}) = V$ for all $i \in [k+1]$.
3. $V_i \subseteq c(V \setminus V_i, G, \phi_{k/(k+1)})$ for all $i \in [k+1]$.

**Proof.** Items 1 and 2 are equivalent by noting that $|c(V \setminus V_i, G, \phi_{k/(k+1)})| \leq |V|$ for each $i \in [k+1]$. As $V \setminus V_i \subseteq c(V \setminus V_i, G, \phi_{k/(k+1)})$, $c(V \setminus V_i, G, \phi_{k/(k+1)}) = V$ if and only if $V_i \subseteq c(V \setminus V_i, G, \phi_{k/(k+1)})$.

The next lemma allows us to iteratively modify a partition of $V$ until one with $\eta(G, V_1, \ldots, V_{k+1}) = (k + 1)|V|$ is obtained.

**Lemma 3.** Let $k$ be a positive integer. Given a digraph $G$ with positive indegrees and a partition $V = \bigcup_{i=1}^{k+1} V_i$ with $\eta(G, V_1, \ldots, V_{k+1}) < (k + 1)|V|$, a partition $V = \bigcup_{i=1}^{k+1} V'_i$ satisfying
\[
\eta(G, V'_1, \ldots, V'_{k+1}) > \eta(G, V_1, \ldots, V_{k+1})
\]
can be found in polynomial time.

**Proof.** By the equivalence of Lemma 2(1) and (3), there exists an $i^* \in [k+1]$ with
\[
V_{i^*} \not\subseteq c(V \setminus V_{i^*}, G, \phi_{k/(k+1)}).
\]
Take any $v \in V_{i^*} \setminus c(V \setminus V_{i^*}, G, \phi_{k/(k+1)})$.

Clearly,
\[
|N^{in}(v) \cap (V \setminus V_{i^*})| < \frac{k}{k+1} |V|.
\]

This and the fact that $V \setminus V_{i^*} = \bigcup_{i \in [k+1] \setminus \{i^*\}} V_i$ is a partition of $V \setminus V_{i^*}$ into $k$ sets show the existence of a $j^* \in [k + 1] \setminus \{i^*\}$ with $|N^{in}(v) \cap V_{j^*}| < (1/(k+1))|V|$. Equivalently,
\[
|N^{in}(v) \setminus V_{j^*}| > \frac{k}{k+1}.
\]
Clearly, $i^*$ and $v$ can be found in polynomial time by calculating $c(V \setminus V_i, G, \phi_{k/(k+1)})$ for all $i \in [k+1]$. Then $j^*$ can be found in polynomial time by evaluating $|\mathcal{N}^\text{in}(v) \cap V_j|$ for all $j \in [k+1]$.

Now let $V'_{i^*} \equiv V_i \setminus \{v\}$, $V'_{j^*} \equiv V_j \cup \{v\}$ and $V'_h \equiv V_h$ for $h \in [k+1] \setminus \{i^*, j^*\}$. Clearly, $V = \bigcup_{i=1}^{k+1} V'_i$ is a partition of $V$. Trivially, for $h \in [k+1] \setminus \{i^*, j^*\}$,
\[ c(V \setminus V'_h, G, \phi_{k/(k+1)}) = c(V \setminus V_h, G, \phi_{k/(k+1)}). \]

Therefore,
\[ \eta(G, V'_1, \ldots, V'_{k+1}) - \eta(G, V_1, \ldots, V_{k+1}) = \left| c(V \setminus V'_{i^*}, G, \phi_{k/(k+1)}) \right| - \left| c(V \setminus V'_{j^*}, G, \phi_{k/(k+1)}) \right| + \left| c(V \setminus V'_{j^*}, G, \phi_{k/(k+1)}) \right| - \left| c(V \setminus V'_{i^*}, G, \phi_{k/(k+1)}) \right|. \] (2)

By the choice of $v$,
\[ v \notin c(V \setminus V'_{i^*}, G, \phi_{k/(k+1)}). \] (3)

As $v \notin V'_{i^*}$,
\[ v \in c(V \setminus V'_{i^*}, G, \phi_{k/(k+1)}). \] (4)

Relations (3)–(4) and the easily verifiable fact $V \setminus V'_{i^*} \subseteq V \setminus V'_{j^*}$ imply
\[ c(V \setminus V'_{i^*}, G, \phi_{k/(k+1)}) \subseteq c(V \setminus V'_{j^*}, G, \phi_{k/(k+1)}). \] (5)

As $V'_{j^*} = V_{j^*} \cup \{v\}$ and $v \notin \mathcal{N}^\text{in}(v)$, inequality (4) gives
\[ |\mathcal{N}^\text{in}(v) \setminus V'_{j^*}| = |\mathcal{N}^\text{in}(v) \setminus V_{j^*}| > \frac{k}{k+1} \cdot |V|. \]

Consequently, $v \in c(V \setminus V'_{j^*}, G, \phi_{k/(k+1)})$ and, therefore,
\[ c(V \setminus V'_{j^*}, G, \phi_{k/(k+1)}) = c(\{v\} \cup (V \setminus V'_{j^*}), G, \phi_{k/(k+1)}). \] (6)

Clearly,
\[ \{v\} \cup (V \setminus V'_{j^*}) = V \setminus V'_{j^*}. \]

This and Eq. (6) give
\[ c(V \setminus V'_{j^*}, G, \phi_{k/(k+1)}) = c(V \setminus V'_{j^*}, G, \phi_{k/(k+1)}). \] (7)

Inequalities (2), (5) and (7) complete the proof. \[\square\]

The main result of this section follows.
Theorem 4. Given a digraph $G(V, E)$ with positive indegrees and a positive integer $k$, a set $S \subseteq V$ with $c(S, G, \phi_{k/(k+1)}) = V$ and $|S| \leq (k/(k+1))|V|$ can be found in polynomial time.

Proof. By repeated applications of Lemma 3, a partition $V = \bigcup_{i=1}^{k+1} V_i$ with

$$\eta(G, V_1, \ldots, V_{k+1}) = (k+1)|V|$$

can be found in polynomial time. By the equivalence of Lemma 2(1) and (2), $c(V \setminus V_i, G, \phi_{k/(k+1)}) = V$ for all $i \in [k+1]$. Now take $S$ to be a smallest set among $V \setminus V_1, \ldots, V \setminus V_{k+1}$. Clearly, $|S| \leq (k/(k+1))|V|$.

Several theorems are immediate.

Theorem 5. For any digraph $G(V, E)$ with positive indegrees,

$$\min\text{-}\text{seed}(G, \phi^{\text{simple}}) \leq \left\lfloor \frac{|V|}{2} \right\rfloor.$$

Proof. Take $k = 1$ in Theorem 4.

Theorem 6. For any digraph $G(V, E)$ with positive indegrees,

$$\min\text{-}\text{seed}(G, \phi^{\text{strict}}) \leq \left\lfloor \frac{2 \cdot |V|}{3} \right\rfloor.$$

Proof. Take $k = 2$ in Theorem 4 and note that $\phi^{\text{strict}}(v) \leq (2/3)\deg^{\text{in}}(v)$ for all $v \in V$.

4 Irreversible dynamos of undirected graphs

We now turn to irreversible dynamos of undirected connected graphs. Let $G(V, E)$ be an undirected connected graph. A cut is an unordered pair $(S, V \setminus S)$ with $S \subseteq V$. We call a cut $(S, V \setminus S)$ proper if

\begin{align*}
|N(v) \cap S| &\leq |N(v) \setminus S|, \quad \forall v \in S, \quad (8) \\
|N(v) \setminus S| &\leq |N(v) \cap S|, \quad \forall v \in V \setminus S, \quad (9)
\end{align*}

and improper otherwise. So a proper cut is such that no vertex has more neighbors in the side $(S$ or $V \setminus S$) it belongs to than in the side it does not. The following fact is implicit in [12, pp. 303–304].

Fact 7. ([12, pp. 303–304]) Given an undirected graph $G(V, E)$ and an improper cut $(S, V \setminus S)$, a proper cut $(T, V \setminus T)$ with $e(T, V \setminus T) > e(S, V \setminus S)$ can be found in polynomial time.
A vertex \( v \in V \) is said to be bad with respect to (abbreviated w.r.t.) a cut \((S, V \setminus S)\) if \( |N(v) \cap S| = |N(v) \setminus S| \); it is good w.r.t. \((S, V \setminus S)\) otherwise. A connected component of \( G[S] \) or \( G[V \setminus S] \) is bad w.r.t. \((S, V \setminus S)\) if all its vertices are bad w.r.t. \((S, V \setminus S)\); it is good w.r.t. \((S, V \setminus S)\) otherwise. The set of connected components of \( G[S] \) that are bad w.r.t. \((S, V \setminus S)\) is denoted \( B(S) \). Similarly, \( B(V \setminus S) \) is the set of bad (w.r.t. \((S, V \setminus S)\)) connected components of \( G[V \setminus S] \). For \( v^* \in V \) and \( S \subseteq V \), define
\[
\psi(S, v^*) = e(S, V \setminus S) \cdot |V|^2 - \left[ \sum_{\hat{G}(V,E) \in B(S)} d\left(v^*, \hat{V}\right) + \sum_{\hat{G}(V,E) \in B(V \setminus S)} d\left(v^*, \hat{V}\right) \right].
\]

For a fixed \( v^* \in V \), we will keep refining cuts by increasing their \( \psi(\cdot, v^*) \)-values until a cut suitable for creating an irreversible dynamo results. One way to increase the \( \psi(\cdot, v^*) \)-values is to find larger cuts, as shown below.

**Lemma 8.** Let \( G(V, E) \) be an undirected connected graph and \( v^* \in V \). If two cuts \((A, V \setminus A)\) and \((B, V \setminus B)\) satisfy \( e(A, V \setminus A) > e(B, V \setminus B) \), then
\[
\psi(A, v^*) > \psi(B, v^*).
\]

**Proof.** In Eq. (10), the \( e(S, V \setminus S) \) term is multiplied by \( |V|^2 > |V|(|V| - 1) \). But the summations within the brackets of Eq. (10) evaluate to be at most \( |V|(|V| - 1) \) because \(|B(S)| + |B(V \setminus S)| \leq |V| \) and \( d(v^*, U) \leq |V| - 1 \) for any \( \emptyset \subseteq U \subseteq V \).

The following lemma is straightforward.

**Lemma 9.** For an undirected connected graph \( G(V, E) \) and \( v \in V \), every connected component of \( G[V \setminus \{v\}] \) shares a vertex with \( N_G(v) \).

**Proof.** As \( G \) is connected, any \( u \in V \setminus \{v\} \) can reach \( v \) by a path \( P \) in \( G \). Starting from \( u \) and going along with \( P \), a vertex in \( N_G(v) \) must be reached before arriving at \( v \). Hence the connected component of \( G[V \setminus \{v\}] \) containing \( u \) must have a vertex in \( N_G(v) \).

The next lemma shows that moving a bad vertex \( v \) across a cut does not change the cut size.

**Lemma 10.** Let \( G(V, E) \) be an undirected connected graph, \((S, V \setminus S)\) be a cut and \( v \in S \) be bad w.r.t. \((S, V \setminus S)\). Then
\[
e(S, V \setminus S) = e(S \setminus \{v\}, (V \setminus S) \cup \{v\}).
\]
Proof. As $v$ is bad w.r.t. $(S, V \setminus S)$, $|N(v) \cap S| = |N(v) \setminus S|$. Now Eq. (11) holds because (1) the edges incident on $v$ contribute $|N(v) \setminus S|$ to the lefthand side and $|N(v) \cap S|$ to the righthand side and (2) all other edges contribute the same amount to either side. \hfill \qed

The next lemma shows that moving a vertex $v$ across a cut does not change whether a connected component without vertices in $N_G^*(v)$ is bad.

**Lemma 11.** Let $G(V, E)$ be an undirected connected graph, $(S, V \setminus S)$ be a proper cut, $H(V_H, E_H)$ be a connected component of $G[S]$ or $G[V \setminus S]$, $v \in S$ and $N_G^*(v) \cap V_H = \emptyset$. Then

1. $H \in \mathcal{B}(S \setminus \{v\})$ if and only if $H \in \mathcal{B}(S)$.

2. $H \in \mathcal{B}((V \setminus S) \cup \{v\})$ if and only if $H \in \mathcal{B}(V \setminus S)$.

*Proof.* As $N_G^*(v) \cap V_H = \emptyset$ and $H$ is a connected component of $G[S]$ or $G[V \setminus S]$, $H$ must remain a connected component of $G[S \setminus \{v\}]$ or $G[(V \setminus S) \cup \{v\}]$, respectively. As $N_G^*(v) \cap V_H = \emptyset$, every $u \in V_H$ satisfies $|N_G(u) \cap S| = |N_G(u) \cap (S \setminus \{v\})|$ and $|N_G(u) \setminus S| = |N_G(u) \setminus (S \setminus \{v\})|$; so $u$ is bad w.r.t. $(S, V \setminus S)$ if and only if it is bad w.r.t. $(S \setminus \{v\}, (V \setminus S) \cup \{v\})$. Therefore, $H$ is bad w.r.t. $(S, V \setminus S)$ if and only if it is bad w.r.t. $(S \setminus \{v\}, (V \setminus S) \cup \{v\})$. \hfill \qed

Given a proper cut $(S, V \setminus S)$, the next two lemmas analyze how bad components evolve when a vertex is moved away from $S$. See Fig. 1 for illustration.

**Lemma 12.** Let $G(V, E)$ be an undirected connected graph, $(S, V \setminus S)$ be a proper cut, $G'(V', E')$ be a connected component of $G[S]$ and $v \in V'$. Then $\mathcal{B}(S \setminus \{v\}) \subseteq \mathcal{B}(S) \setminus \{G'\}$.

*Proof.* Let $G_1(V_1, E_1) = G'(V', E'), \ldots, G_k(V_k, E_k)$ be the connected components of $G[S]$. For $u \in N_G(v) \cap S$,

$$|N_G(u) \cap S| \leq |N_G(u) \setminus S|$$

because $(S, V \setminus S)$ is proper. Consequently, for every $u \in N_G(v) \cap S$,

$$|N_G(u) \cap (S \setminus \{v\})| = |N_G(u) \cap S| - 1 \leq |N_G(u) \setminus S| - 1 < |N_G(u) \setminus S| + 1 = |N_G(u) \setminus (S \setminus \{v\})|,$$

where both equalities follow from $v \in S$ and $u \in N_G(v) \cap S$.

Let $G_{11}, \ldots, G_{1h}$ be the connected components of $G[V_1 \setminus \{v\}]$, where $h \geq 0$ ($h = 0$ if and only if $V_1 = \{v\}$). Clearly, the connected components of
(a) Below the gray line are the connected components of $G[S]$, namely $G', G_2, \ldots, G_k$. One of the components, $G'$, contains a vertex $v$ whose neighbors are shown as double circles. As in the proof of Lemma 12, the connected components obtained by removing $v$ from $G'$ are $G_{11}, \ldots, G_{1h}$. Above the gray line are the connected components of $G[V \setminus S]$, i.e., $G_1, \ldots, G_\ell$.

(b) Continuing from Fig. 1(a), the connected components of $G[S \setminus \{v\}]$ and $G[(V \setminus S) \cup \{v\}]$ are shown below and above the gray line, respectively. In the proof of Lemma 12, $G_{11}, \ldots, G_{1h}$ are shown to be good w.r.t. $(S \setminus \{v\}, (V \setminus S) \cup \{v\})$ by showing that $v$'s neighbors in $S$ are good w.r.t. $(S \setminus \{v\}, (V \setminus S) \cup \{v\})$.

Figure 1: Figs. 1(a)–1(b) show how the connected components of $G[S]$ and $G[V \setminus S]$ change as a vertex $v$ is moved from $S$ to $V \setminus S$. The notations $G'$, $\bar{G}$, $G_{11}, \ldots, G_{1h}$ and $\hat{G}_1, \ldots, \hat{G}_\ell$ are from the proofs of Lemmas 12 and 14.
Clearly, $\bar{G}$ and $\hat{G}$ for $t$ for Lemma 14. As shown below.

Corollary 13. Let $G(V, E)$ be an undirected connected graph, $(S, V \setminus S)$ be a proper cut, $G'(V', E') \in \mathcal{B}(S)$, $v \in V'$ and $v^* \in V$. Then

$$\sum_{G(\hat{V}, \hat{E}) \in \mathcal{B}(V \setminus \{v\})} d\left(v^*, \hat{V}\right) \leq \left(\sum_{G(\hat{V}, \hat{E}) \in \mathcal{B}(S)} d\left(v^*, \hat{V}\right)\right) - d\left(v^*, V'\right).$$

(13)

Proof. Immediate from Lemma 12. □

For a proper cut $(S, V \setminus S)$ and $v \in S$, $G[(V \setminus S) \cup \{v\}]$ has a unique connected component $G(\hat{V}, \hat{E})$ that contains $v$. Every other connected component of $G[(V \setminus S) \cup \{v\}]$ that is bad w.r.t. $(S \setminus \{v\}, (V \setminus S) \cup \{v\})$ must also be bad w.r.t. $(S, V \setminus S)$, as shown below.

Lemma 14. Let $G(V, E)$ be an undirected connected graph, $(S, V \setminus S)$ be a proper cut, $v \in S$ and $G(\hat{V}, \hat{E})$ be the connected component of $G[(V \setminus S) \cup \{v\}]$ that contains $v$. Then

$$\mathcal{B}\left((V \setminus S) \cup \{v\}\right) \subseteq \mathcal{B}\left(V \setminus S\right) \cup \{\hat{G}\}.$$

Proof. Let $\hat{G}_1(\hat{V}_1, \hat{E}_1), \ldots, \hat{G}_t(\hat{V}_t, \hat{E}_t)$ be the connected components of $G(V \setminus S)$ and $t = |\{1 \leq i \leq t \mid \hat{V}_i \cap N_G(v) \neq \emptyset\}|$. Without loss of generality, suppose that $\hat{V}_i \cap N_G(v) \neq \emptyset$ for $1 \leq i \leq t$ and $\hat{V}_i \cap N_G(v) = \emptyset$ for $t + 1 \leq i \leq \ell$. Clearly, $\hat{G} = G[\{v\} \cup \hat{V}_1 \cup \cdots \cup \hat{V}_t]$. Besides $\hat{G}$, the other connected components of $G[(V \setminus S) \cup \{v\}]$ are $\hat{G}_{t+1}, \ldots, \hat{G}_t$. Hence to complete the proof, we only need to show that for $t + 1 \leq i \leq \ell$, $\hat{G}_i \in \mathcal{B}(V \setminus S) \cup \{v\}$ only if $\hat{G}_i \in \mathcal{B}(V \setminus S)$. By Lemma 12, with $\hat{G}_i$ playing the role of $H$, we need only check that $N_{\hat{G}_i}(v) \cap \hat{V}_i = \emptyset$ for $t+1 \leq i \leq \ell$, which is true because $v \in S = V \setminus (\hat{V}_1 \cup \cdots \cup \hat{V}_k)$ and $\hat{V}_i \cap N_G(v) = \emptyset$ for $t + 1 \leq i \leq \ell$. □

Corollary 15. Let $G(V, E)$ be an undirected connected graph, $(S, V \setminus S)$ be a proper cut, $v \in S$, $v^* \in V$ and $G(\hat{V}, \hat{E})$ be the connected component of $G[(V \setminus S) \cup \{v\}]$ that contains $v$. Then

$$\sum_{G(\hat{V}, \hat{E}) \in \mathcal{B}(V \setminus S) \cup \{v\}} d\left(v^*, \hat{V}\right) \leq \left(\sum_{G(\hat{V}, \hat{E}) \in \mathcal{B}(V \setminus S)} d\left(v^*, \hat{V}\right)\right) + d\left(v^*, V\right).$$

(14)
Proof. Immediate from Lemma 14.

We now arrive at the following key lemma, which allows us to repeatedly increase the \( \psi(\cdot, v^*) \)-values of cuts by moving one vertex at a time.

**Lemma 16.** Let \( G(V, E) \) be an undirected connected graph, \( (S, V \setminus S) \) be a proper cut, \( G'(V', E') \in \mathcal{B}(S) \), \( v \in V' \) and \( v^* \in V \setminus V' \). If \( d(v^*, v) = d(v^*, V') \), then \( \psi(S \setminus \{v\}, v^*) > \psi(S, v^*) \).

**Proof.** As \( v \in V' \) and \( v^* \not\in V' \), \( d(v^*, v) > 0 \). As \( G \) is connected, there exists a vertex \( w \in N_G(v) \) with \( d(v^*, w) = d(v^*, v) - 1 \). We must have \( w \not\in V' \) because \( d(v^*, v) = d(v^*, V') \) says that \( v \) is among the vertices in \( V' \) that are closest to \( v^* \). Suppose for contradiction that \( w \in S \). Then the facts that \( v \in V' \), \( w \in N_G(v) \) and \( G'(V', E') \) is a connected component of \( G[S] \) force \( w \in V' \), a contradiction. So \( w \not\in S \) and, therefore,

\[
w \in (V \setminus S) \cup \{v\}. \tag{15}
\]

Trivially,

\[
v \in (V \setminus S) \cup \{v\}. \tag{16}
\]

Eqs. (15)–(16) and the fact that \( w \in N_G(v) \) put \( w \) and \( v \) in the same connected component of \( G[(V \setminus S) \cup \{v\}] \), denoted \( G(V, E) \). Note that

\[
d(v^*, \bar{V}) \leq d(v^*, w) = d(v^*, v) - 1 = d(v^*, V') - 1. \tag{17}
\]

Summing inequalities (13)–(14), we have

\[
\sum_{G(\bar{V}, E) \in \mathcal{B}(S \setminus \{v\})} d(v^*, \bar{V}) + \sum_{G(\bar{V}, E) \in \mathcal{B}(V \setminus S) \cup \{v\}} d(v^*, \bar{V}) \\
\leq \left( \sum_{G(\bar{V}, E) \in \mathcal{B}(S)} d(v^*, \bar{V}) + \sum_{G(\bar{V}, E) \in \mathcal{B}(V \setminus S)} d(v^*, \bar{V}) \right) - d(v^*, V') + d(v^*, \bar{V}) \\
\leq \left( \sum_{G(\bar{V}, E) \in \mathcal{B}(S)} d(v^*, \bar{V}) + \sum_{G(\bar{V}, E) \in \mathcal{B}(V \setminus S)} d(v^*, \bar{V}) \right) - 1, \tag{18}
\]

where the last inequality follows from inequality (17). As \( G' \) is bad w.r.t. \( (S, V \setminus S) \) and \( v \in V' \), Lemma 10 gives

\[
e(S, V \setminus S) = e(S \setminus \{v\}, (V \setminus S) \cup \{v\}).
\]

This and inequality (18) show that \( \psi(S \setminus \{v\}, v^*) > \psi(S, v^*) \). \( \square \)
The above lemma allows us to increase the $\psi(\cdot, v^*)$-values of cuts whenever there is a bad connected component of $G[S]$ that does not contain $v^*$. As the $\psi(\cdot, v^*)$-values are bounded from above, they cannot be increased forever. So repeatedly applying the above lemma will finally yield a cut where all bad connected components of $G[S]$ must contain $v^*$, meaning $|B(S)| = 1$ as $v^*$ cannot appear in two connected components. Such a result is stated below, which considers $G[V \setminus S]$ as well.

**Lemma 17.** Given an undirected connected graph $G(V, E)$, a proper cut $(S, V \setminus S)$ with

$$|B(S) \cup B(V \setminus S)| \leq 1$$

(19)

can be found in polynomial time.

**Proof.** Fix $v^* \in V$ arbitrarily. By Fact 7 a proper cut $(S_0, V \setminus S_0)$ can be found in time polynomial in $|V|$. If

$$|B(S_0) \cup B(V \setminus S_0)| \leq 1,$$

taking $S = S_0$ satisfies inequality (19).

Inductively, let $(S_i, V \setminus S_i)$ be a proper cut with

$$|B(S_i) \cup B(V \setminus S_i)| > 1,$$

(20)

where $i \geq 0$. We show how to compute a proper cut $(S_{i+1}, V \setminus S_{i+1})$ with

$$\psi(S_{i+1}, v^*) > \psi(S_i, v^*)$$

(21)

in time polynomial in $|V|$. The connected components of $G[S_i]$ and $G[V \setminus S_i]$ can be found in time polynomial in $|V|$ using the breadth-first search [2]. By inequality (20), we can pick an arbitrary $G'(V', E') \in B(S_i) \cup B(V \setminus S_i)$ with $v^* \notin V'$. Assume without loss of generality that $G' \in B(S_i)$; otherwise, we switch $S_i$ and $V \setminus S_i$ from the beginning. By computing $d(v^*, u)$ for every $u \in V'$ using the breadth-first search, we find a $v \in V'$ with $d(v^*, v) = d(v^*, V')$ in time polynomial in $|V|$. As $v \in V'$ and $G' \in B(S_i)$ is bad w.r.t. $(S_i, V \setminus S_i)$, Lemma 10 implies that

$$e(S_i, V \setminus S_i) = e(S_i \setminus \{v\}, (V \setminus S_i) \cup \{v\}).$$

(22)

By Lemma 16, $\psi(S_i \setminus \{v\}, v^*) > \psi(S_i, v^*)$. Therefore, if $(S_i \setminus \{v\}, (V \setminus S_i) \cup \{v\})$ is proper, then inequality (21) holds for a proper cut $(S_{i+1}, V \setminus S_{i+1})$ by taking $S_{i+1} = S_i \setminus \{v\}$. Otherwise, Fact 7 implies that a proper cut $(T, V \setminus T)$ with $e(T, V \setminus T) > e(S_i \setminus \{v\}, (V \setminus S_i) \cup \{v\})$ can be found in time polynomial in $|V|$. Hence by Eq. (22), $e(T, V \setminus T) > e(S_i, V \setminus S_i)$, implying $\psi(T, v^*) > \psi(S_i, v^*)$ by
Lemma 8. Again, inequality (21) holds for a proper cut \((S_{i+1}, V \setminus S_{i+1})\) by taking \(S_{i+1} = T\).

Now continue computing a proper cut \((S_{i+1}, V \setminus S_{i+1})\) with \(\psi(S_{i+1}, v^*) > \psi(S_i, v^*)\) from \((S_i, V \setminus S_i)\) until inequality (20) fails for some \(i \geq 0\). As \(|\psi(\cdot, \cdot)|\) is at most polynomial in \(|V|\), there is a \(k \in \mathbb{N}\), which is at most polynomial in \(|V|\), with

\[|\mathcal{B}(S_k) \cup \mathcal{B}(V \setminus S_k)| \leq 1,\]

completing the proof.

The above lemma provides us with a cut \((S, V \setminus S)\) where \(G[S]\) and \(G[V \setminus S]\) together have at most one bad (w.r.t. \((S, V \setminus S)\)) connected component. Next, we show that \(S\) or \(V \setminus S\) plus one vertex from the only bad component (if it exists) is an irreversible dynamo under the strict-majority scenario.

Theorem 18. Given an undirected connected graph \(G(V, E)\), an irreversible dynamo of \(\mathcal{N}(G, \phi^{\text{strict}})\) with size at most \(|V|/2\) can be found in polynomial time.

Proof. Lemma 17 says a proper cut \((S, V \setminus S)\) with \(|\mathcal{B}(S) \cup \mathcal{B}(V \setminus S)| \leq 1\) can be found in polynomial time in \(|V|\). (1) If \(|\mathcal{B}(S) \cup \mathcal{B}(V \setminus S)| = 1\), let \(x \in V\) be an arbitrary vertex of the unique member of \(\mathcal{B}(S) \cup \mathcal{B}(V \setminus S)\). (2) Otherwise, take any \(x \in V\).

Pick any connected component \(H(V_H, E_H)\) of \(G[V \setminus S]\). We next show that

\[V_H \cap c(S \cup \{x\}, G, \phi^{\text{strict}}) \neq \emptyset.\]  \hspace{1cm} (23)

If \(H \in \mathcal{B}(S) \cup \mathcal{B}(V \setminus S)\), then \(x \in V_H\) by our choice of \(x\) in case (1) above, proving inequality (23). Otherwise, \(H\) must be a good (w.r.t. \((S, V \setminus S)\)) connected component of \(G[V \setminus S]\). So there exists a vertex \(u \in V_H\) with \(|N_G(u) \setminus S| \neq |N_G(u) \cap S|\), which together with the properness of \((S, V \setminus S)\) yields

\[|N_G(u) \setminus S| < |N_G(u) \cap S|.

This gives \(u \in c(S, G, \phi^{\text{strict}})\) by definition, which implies \(u \in c(S \cup \{x\}, G, \phi^{\text{strict}})\) by Fact II. Again, inequality (23) holds.
Next, we prove that $V \setminus S \subseteq c(S \cup \{x\}, G, \phi^{\text{strict}})$. For this purpose, we need only show that every $w \in V_H$ belongs to $c(S \cup \{x\}, G, \phi^{\text{strict}})$ because $H$ is an arbitrary connected component of $G[V \setminus S]$. Let $u \in V_H \cap c(S \cup \{x\}, G, \phi^{\text{strict}})$, whose existence is guaranteed by inequality (23). As $w, u \in V_H$ and $H$ is a connected component of $G[V \setminus S]$, there is a path $x_0 = u, \ldots, x_t = w$ whose vertices are in $V_H$. We proceed to show that $w \in c(S \cup \{x\}, G, \phi^{\text{strict}})$ by induction. See Fig. 2 for illustration. The induction base is $x_0 \in c(S \cup \{x\}, G, \phi^{\text{strict}})$, which is true by construction. Inductively, assume $x_i \in c(S \cup \{x\}, G, \phi^{\text{strict}})$, $0 \leq i < t$. Clearly, $\{x_i\} \cup S \subseteq c(S \cup \{x\}, G, \phi^{\text{strict}})$; hence

$$\left| N_G(x_{i+1}) \cap c(S \cup \{x\}, G, \phi^{\text{strict}}) \right| \geq \left| N_G(x_{i+1}) \cap (\{x_i\} \cup S) \right| = \left| N_G(x_{i+1}) \cap \{x_i\} \right| + \left| N_G(x_{i+1}) \cap S \right| = 1 + \left| N_G(x_{i+1}) \cap S \right|.$$  \hspace{2cm} (24)

As $S$ is proper, $\left| N_G(x_{i+1}) \setminus S \right| \leq \left| N_G(x_{i+1}) \cap S \right|$, which together with inequality (24) gives

$$\left| N_G(x_{i+1}) \cap c(S \cup \{x\}, G, \phi^{\text{strict}}) \right| > \frac{N_G(x_{i+1})}{2},$$

thus $x_{i+1} \in c(S \cup \{x\}, G, \phi^{\text{strict}})$.

We have shown that $V \setminus S \subseteq c(S \cup \{x\}, G, \phi^{\text{strict}})$, which yields $V = c(S \cup \{x\}, G, \phi^{\text{strict}})$. By symmetry, $V = c((V \setminus S) \cup \{x\}, G, \phi^{\text{strict}})$. So both $S \cup \{x\}$ and $(V \setminus S) \cup \{x\}$ are irreversible dynamos of $N(G, \phi^{\text{strict}})$. To complete the proof, it remains to show that the smaller of $S \cup \{x\}$ and $(V \setminus S) \cup \{x\}$ has size at most $\lfloor |V|/2 \rfloor$. As $x$ lies in exactly one of $S$ and $V \setminus S$,

$$|V| = |S \cup \{x\}| + |(V \setminus S) \cup \{x\}| - 1,$$

forcing the smaller of $|S \cup \{x\}|$ and $|(V \setminus S) \cup \{x\}|$ to be at most $\lfloor (|V| + 1)/2 \rfloor = \lfloor |V|/2 \rfloor$. \hfill $\Box$

The bound of Theorem 18 cannot be lowered because $\text{min-seed}(G, \phi^{\text{strict}}) = \lfloor |V|/2 \rfloor$ when $G$ is the complete graph on $V$. That is, among all undirected connected graphs on $V$, the complete graph attains the maximum value for $\text{min-seed}(G, \phi^{\text{strict}})$. Under the interpretation of an irreversible dynamo as a set of processors whose faulty behavior leads all processors to erroneous results, therefore, fully interconnecting the processors maximizes the number of adversarially placed faulty processors needed to render all processors’ results erroneous.

## 5 Inapproximability

In this section, we establish inapproximability results on finding minimum irreversible dynamos. Given any undirected graph $G(V, E)$, we define an undirected
graph $G(V,E)$ as follows. First, define

$$X_v \equiv \{x_{v,i} \mid 1 \leq i \leq \deg_G(v)\}, v \in V,$$
$$Y_v \equiv \{y_{v,i} \mid 1 \leq i \leq \deg_G(v)\}, v \in V,$$
$$X \equiv \bigcup_{v \in V} X_v,$$
$$Y \equiv \bigcup_{v \in V} Y_v,$$
$$W \equiv \bigcup_{v \in V} \{w_v\}.$$

Then define $G(V,E)$ by

$$V \equiv V \cup W \cup X \cup Y \cup \{z_1, z_2\} \cup \{g_1, g_2\},$$
$$E \equiv \{(v,x) \mid v \in V, x \in X_v\}
\cup \{(w_v,u) \mid v \in V, u \in N^*_G(v)\}
\cup \{(w_v,y) \mid v \in V, y \in Y_v\}
\cup \{(y,z_1) \mid y \in Y\}
\cup \{(y,z_2) \mid y \in Y\}
\cup \{(z_1,g_1)\}
\cup \{(z_2,g_2)\}.$$

For convenience, define

$$B_v \equiv \{w_v\} \cup N^*_G(v) \cup \left( \bigcup_{u \in N^*_G(v)} X_u \right), \ v \in V.$$

As every edge in $E$ has an endpoint in $V \cup Y \cup \{g_1, g_2\}$ and the other in $X \cup W \cup \{z_1, z_2\}$, $G$ is bipartite [17]. See Fig. 3 for illustration.

Clearly, $G$ can be constructed in polynomial time from $G$. As $G$ clearly has no isolated vertices, the networks $N(G, \phi^\text{strict}_G)$ and $N(G, \phi^\text{simple}_G)$ as well as their coloring processes are all well-defined. Below are some easy facts about $G$.

**Lemma 19.** For any $v \in V$,

1. $N_G(w_v) = Y_v \cup N^*_G(v)$.
2. $|Y_v| = \deg_G(v)$.
3. $\deg_G(v) + 1 = \phi^\text{strict}_G(w_v)$.
4. $Y \cup \{g_1, g_2\} \subseteq c(\{z_1, z_2\}, G, \phi^\text{strict}_G)$.
5. $X_v \cup \{w_v\} \subseteq c(\{v, z_1, z_2\}, G, \phi^\text{strict}_G)$.
6. $N_G(v) = X_v \cup \left( \bigcup_{u \in N^*_G(v)} \{w_u\} \right)$.  


Figure 3: Suppose $N_G(v) = \{v_1, v_2, v_3\}$. From bottom to top, the vertices in $\bigcup_{u \in N^*_G(v)} \mathcal{X}_u$, $N^*_G(v)$, $\{w_u \mid u \in N^*_G(v)\}$, $\bigcup_{u \in N^*_G(v)} \mathcal{Y}_u$, $\{z_1, z_2\}$ and $\{g_1, g_2\}$ are shown. Lines represent the edges of $\mathcal{G}$. A vertex is labeled $\mathcal{X}$ or $\mathcal{Y}$ if it belongs to the respective sets. The vertices in $B_v$ are filled with light gray. As every edge has an endpoint in double circle and the other in single circle, $\mathcal{G}$ is bipartite.
Lemma 19. \( |\mathcal{X}_v| = \deg_G(v). \)

8. \( |\bigcup_{u \in N_G(v)} \{w_u\}| = \deg_G(v) + 1. \)

9. \( \deg_G(v) + 1 = \phi_G^{\text{strict}}(v). \)

**Proof.** Items 1–4 are immediate from the definitions. So we prove item 5 next. By item 4, \( \mathcal{Y}_v \subseteq c(\{v, z_1, z_2\}, G, \phi_G^{\text{strict}}); \) thus, trivially,

\[ \mathcal{Y}_v \cup \{v\} \subseteq c(\{v, z_1, z_2\}, G, \phi_G^{\text{strict}}). \]  

(25)

By items 1–3, \( \mathcal{Y}_v \cup \{v\} \) is a subset of \( N_G(w_v) \) and has size \( \phi_G^{\text{strict}}(w_v) \), which together with relation (25) implies

\[ w_v \in c(\{v, z_1, z_2\}, G, \phi_G^{\text{strict}}) \]

by the coloring process. This and the trivial fact \( \mathcal{X}_v \subseteq c(\{v\}, G, \phi_G^{\text{strict}}), \) which holds because \( N_G(x) = \{v\} \) for \( x \in \mathcal{X}_v \), prove item 5.

Now take any \( u \in N^*_G(v) \). Equivalently, \( v \in N_G(u) \), which implies \( v \in N_G(w_u) \) by item 1. Equivalently, \( w_u \in N_G(v) \). Now that \( \bigcup_{u \in N^*_G(v)} \{w_u\} \subseteq N_G(v) \), item 6 obtains. The remaining items follow immediately.

A set \( D \subseteq V \) is called a dominating set of \( G \) if it shares at least one vertex with \( N^*_G(v) \) for each \( v \in V \). The next lemma shows that adding \( z_1 \) and \( z_2 \) to a dominating set of \( G \) produces an irreversible dynamo of \( G \) under the strict-majority scenario.

**Lemma 20.** If \( D \subseteq V \) is a dominating set of \( G(V, E) \), then \( c(D \cup \{z_1, z_2\}, G, \phi_G^{\text{strict}}) = \mathcal{W} \).

**Proof.** Consider the coloring process in \( N(G, \phi_G^{\text{strict}}) \) with \( D \cup \{z_1, z_2\} \) as the set of seeds. Pick \( v \in V \) arbitrarily. All the vertices in \( \mathcal{Y} \cup \{g_1, g_2\} \) will be white by Lemma 19. In particular, all the vertices in \( \mathcal{Y}_v \) will be white. Since \( D \cap N^*_G(v) \neq \emptyset \) by the definition of dominating sets, at least one vertex in \( N^*_G(v) \) is a seed, i.e., a white vertex initially. In total, at least \( |\mathcal{Y}_v| + 1 \) vertices in \( \mathcal{Y}_v \cup N^*_G(v) \) will be white. In other words, at least \( \phi_G^{\text{strict}}(w_v) \) vertices in \( N_G(w_v) \) will be white because \( |\mathcal{Y}_v| + 1 = \phi_G^{\text{strict}}(w_v) \) by Lemma 19 and Lemma 20, and \( \mathcal{Y}_v \cup N^*_G(v) = N_G(w_v) \) by Lemma 19. So \( w_v \in c(D \cup \{z_1, z_2\}, G, \phi_G^{\text{strict}}) \), implying

\[ \mathcal{W} \subseteq c(D \cup \{z_1, z_2\}, G, \phi_G^{\text{strict}}). \]  

(26)

For each \( v \in V \), relation (26) and Lemma 19 imply at least \( |\bigcup_{u \in N^*_G(v)} \{w_u\}| \) vertices in \( N_G(v) \) will be white. Furthermore, \( |\bigcup_{u \in N^*_G(v)} \{w_u\}| = \phi_G^{\text{strict}}(v) \) by Lemma 19 and Lemma 20. In summary, at least \( \phi_G^{\text{strict}}(v) \) vertices in \( N_G(v) \) will be white, and as a result, every \( v \in V \) will be white. Finally, all the vertices in \( \mathcal{X} \) will be white once all those in \( V \) are white.
Below we show that every irreversible dynamo of $N(G, \phi_G^{\text{strict}})$ has a non-empty intersection with $B_v$ for every $v \in V$.

**Lemma 21.** For each $v \in V$, every irreversible dynamo $S$ of $N(G, \phi_G^{\text{strict}})$ satisfies $S \cap B_v \neq \emptyset$.

**Proof.** Recall that $B_v = \{w_v\} \cup N^*_G(v) \cup (\bigcup_{u \in N_G^*(v)} X_u)$. We proceed to show that every $\alpha \in B_v$ satisfies

$$|N_G(\alpha) \cap B_v| > \frac{\deg_G(\alpha)}{2} \quad (27)$$

in three cases below according to whether $\alpha$ is $w_v$, a member of $N^*_G(v)$ or a member of $\bigcup_{u \in N_G^*(v)} X_u$:

- **$\alpha = w_v$:** By Lemma 19(1), $N_G(\alpha) \cap B_v = N^*_G(v)$. So $|N_G(\alpha) \cap B_v| = |N^*_G(v)| = \deg_G(v) + 1$. By Lemma 19(3), $\deg_G(v) + 1 = \phi_G^{\text{strict}}(\alpha) > \deg_G(\alpha)/2$. Therefore, $|N_G(\alpha) \cap B_v| = \deg_G(v) + 1 > \deg_G(\alpha)/2$.

- **$\alpha \in N^*_G(v)$:** Clearly, $\alpha \in V$ and $v \in N^*_G(\alpha)$. By Lemma 19(6), $X_\alpha \cup \{w_v\} \subseteq N_G(\alpha)$. So

$$X_\alpha \cup \{w_v\} \subseteq N_G(\alpha) \cap B_v \quad (28)$$

by the definition of $B_v$. By Lemma 19(7) and (9),

$$|X_\alpha \cup \{w_v\}| = \deg_G(\alpha) + 1 = \phi_G^{\text{strict}}(\alpha) > \frac{\deg_G(\alpha)}{2}.$$ 

This and relation (28) give $|N_G(\alpha) \cap B_v| \geq |X_\alpha \cup \{w_v\}| > \deg_G(\alpha)/2$.

- **$\alpha \in X_u$ where $u \in N^*_G(v)$:** Clearly, $N_G(\alpha) = \{u\} \subseteq B_v$. So $|N_G(\alpha) \cap B_v| = 1 > 1/2 = \deg_G(\alpha)/2$.

Having verified inequality (27) for all $\alpha \in B_v$,

$$|N_G(\alpha) \cap (V \setminus B_v)| = |N_G(\alpha)| - |N_G(\alpha) \cap B_v| \leq \deg_G(\alpha) - |N_G(\alpha) \cap B_v| < \frac{\deg_G(\alpha)}{2} \quad (29)$$

Next, suppose for contradiction that at least one vertex in $B_v$ ends up white in the coloring process in $N(G, \phi_G^{\text{strict}})$ with $V \setminus B_v$ as the set of seeds. Let $\alpha^* \in B_v$ be colored white first among all vertices in $B_v$. Then $\alpha^*$ must have at least
\(\phi^\text{strict}(\alpha^*)\) vertices in \(\mathcal{V} \setminus B_v\) by the coloring process, contradicting inequality (29). Consequently,

\[
c(\mathcal{V} \setminus B_v, \mathcal{G}, \phi^\text{strict}_\mathcal{G}) = \mathcal{V} \setminus B_v \neq \mathcal{V}.
\] (30)

Now if \(S \cap B_v = \emptyset\), then \(S \subseteq \mathcal{V} \setminus B_v\) and

\[
c(S, \mathcal{G}, \phi^\text{strict}_\mathcal{G}) \subseteq c(\mathcal{V} \setminus B_v, \mathcal{G}, \phi^\text{strict}_\mathcal{G})
\]

by Fact [1]. This and inequality (30) contradict the premise that \(S\) is an irreversible dynamo of \(\mathcal{N}(\mathcal{G}, \phi^\text{strict}_\mathcal{G})\). \(\square\)

The following Lemma shows that \(\mathcal{G}\) is a bipartite graph with diameter at most 8 if \(G\) has no isolated vertices.

**Lemma 22.** Assume that \(G\) has no isolated vertices. Then \(\mathcal{G}\) is a bipartite graph with diameter at most 8.

**Proof.** Partition \(\mathcal{V}\) into \(\mathcal{V}_1 = \mathcal{V} \cup \mathcal{Y} \cup \{g_1, g_2\}\) and \(\mathcal{V}_2 = \mathcal{X} \cup \mathcal{W} \cup \{z_1, z_2\}\). It is immediate from the definition of \(\mathcal{G}\) that each edge in \(\mathcal{E}\) has an endpoint in \(\mathcal{V}_1\) and the other in \(\mathcal{V}_2\). So \(\mathcal{G}\) is bipartite.

As \(G\) has no isolated vertices, \(\text{deg}_G(v) > 0\) for all \(v \in \mathcal{V}\). Hence \(\mathcal{Y}_v \neq \emptyset\) and \(\mathcal{X}_v \neq \emptyset\) for all \(v \in \mathcal{V}\) by Lemma [19][2] and (7), respectively. To show that \(\mathcal{G}\) has diameter at most 8, it suffices to establish \(d_{\mathcal{G}}(u, z_1) \leq 4\) for all \(u \in \mathcal{V}\), which is true because for each \(v \in \mathcal{V}\), \(x \in \mathcal{X}_v\) and \(y \in \mathcal{Y}_v\),

\[
\begin{align*}
P_1(v, x, y) &\equiv (x, v, w_v, y, z_1), \\
P_2(v, x, y) &\equiv (g_1, z_1), \\
P_3(v, x, y) &\equiv (g_2, z_2, y, z_1)
\end{align*}
\]

are all paths of \(\mathcal{G}\) by definition. \(\square\)

The following fact is due to Feige [3].

**Fact 23.** ([3]) Let \(\epsilon > 0\) be any constant. If dominating set has a polynomial-time, \(((1 - \epsilon) \ln N)\)-approximation algorithm for \(N\)-vertex graphs without isolated vertices, then \(NP \subseteq \text{TIME}(n^{O((\ln \ln n))})\).

We now relate the inapproximability of irreversible dynamo (strict majority) with that of dominating set.

**Theorem 24.** Let \(\epsilon > 0\) be any constant. If irreversible dynamo (strict majority) has a polynomial-time, \(((1/2 - \epsilon) \ln N)\)-approximation algorithm for \(N\)-vertex graphs, then \(NP \subseteq \text{TIME}(n^{O((\ln \ln n))})\).
Proof. We will prove the stronger statement that, if \textsc{irreversible dynamo (strict majority)} has a polynomial-time, \((1/2 - \epsilon) \ln N\)-approximation algorithm \textsc{ALG} for bipartite graphs with \(N\) vertices and diameter at most 8, then \(\text{NP} \subseteq \text{TIME}(n^{O(\ln \ln n)})\). Given an undirected graph \(G(V, E)\) without isolated vertices, \(G\) is a bipartite graph with diameter at most 8 by Lemma 22. The construction of \(G\) followed by the calculation of \(S = \text{ALG}(G)\) can be done in time polynomial in \(|V|\). Our assumption on \textsc{ALG} implies that \(S\) is an irreversible dynamo of \(N(G, \phi_G^{\text{strict}})\) with

\[
|S| \leq \left(\frac{1}{2} - \epsilon\right) \cdot \ln |V| \cdot \text{min-seed} (G, \phi_G^{\text{strict}}) \\
\leq [(1 - 2\epsilon) \cdot \ln |V| + O(1)] \cdot \text{min-seed} (G, \phi_G^{\text{strict}}). \tag{31}
\]

Above, the second inequality follows from \(|V| = O(|V|^2)\), which is easily verified given items 2 and 7 of Lemma 19. Denote by \(\gamma(G)\) the size of any minimum dominating set of \(G\). By Lemma 20

\[
\text{min-seed} (G, \phi_G^{\text{strict}}) \leq \gamma(G) + 2. \tag{32}
\]

With \(G\) and \(S\) in hand,

\[
\tilde{D} \equiv \{ u \in V \mid S \cap \{w_u\} \cup \{u\} \cup X_u \neq \emptyset \} \tag{33}
\]

can clearly be constructed in time polynomial in \(|V|\). As \(B_v = \{w_v\} \cup N^*_G(v) \cup (\bigcup_{u \in N^*_G(v)} X_u), \)

\[
B_v \subseteq \bigcup_{u \in N^*_G(v)} (\{w_u\} \cup \{u\} \cup X_u). \tag{34}
\]

For each \(v \in V\), Lemma 21 says \(S \cap B_v \neq \emptyset\). Hence relation (34) implies the existence of a \(u^* \in N^*_G(v)\) with \(S \cap \{w_{u^*}\} \cup \{u^*\} \cup X_{u^*} \neq \emptyset\), equivalently, \(u^* \in \tilde{D}\). Consequently, \(\tilde{D} \cap N^*_G(v) \neq \emptyset\) for all \(v \in V\), i.e., \(\tilde{D}\) is a dominating set of \(G\).

Now,

\[
|\tilde{D}| = \sum_{v \in \tilde{D}} 1 \leq \sum_{u \in \tilde{D}} |S \cap \{w_u\} \cup \{u\} \cup X_u| \leq |S|, \tag{35}
\]

where the first inequality follows from Eq. (33). Inequalities (31)–(32) and (35) yield

\[
|\tilde{D}| \leq [(1 - 2\epsilon) \cdot \ln |V| + O(1)] \cdot (\gamma(G) + 2), \tag{36}
\]
implying the existence of a constant $C$ with $|\tilde{D}| \leq (1 - \epsilon) \cdot \ln |V| \cdot \gamma(G)$ for $\min\{|V|, \gamma(G)\} > C$.

We have shown that (1) $\tilde{D}$ can be found in time polynomial in $|V|$, (2) $\tilde{D}$ is a dominating set of $G$ and (3) $|\tilde{D}| \leq (1 - \epsilon) \cdot \ln |V| \cdot \gamma(G)$ for $\min\{|V|, \gamma(G)\} > C$. When $\min\{|V|, \gamma(G)\} \leq C$, a minimum dominating set of $G$ can be found by brute force in time polynomial in $|V|$. Hence NP $\subseteq$ TIME($n^{O(\ln \ln n)}$) by Fact 23.

Analogous to the strict-majority case, the following result can be proved for IRREVERSIBLE DYNAMO (SIMPLE MAJORITY).

**Theorem 25.** Let $\epsilon > 0$ be any constant. If IRREVERSIBLE DYNAMO (SIMPLE MAJORITY) has a polynomial-time, $((1/2 - \epsilon) \ln N)$-approximation algorithm for $N$-vertex graphs, then NP $\subseteq$ TIME($n^{O(\ln \ln n)}$).

**Proof.** We will show NP $\subseteq$ TIME($n^{O(\ln \ln n)}$) if IRREVERSIBLE DYNAMO (SIMPLE MAJORITY) has a polynomial-time, $((1/2 - \epsilon) \ln N)$-approximation algorithm for bipartite graphs with $N$ vertices and diameter at most 8. By Theorem 24, we need only show that every vertex of $G$ has an odd degree, so that the strict and the simple-majority scenarios coincide. By Lemma 19(6)–(8), $\deg_G(v) = 2 \cdot \deg_G(v) + 1$ is odd for each $v \in V$. By Lemma 19(1)–(2), $\deg_G(w) = \deg_G(v) + |N_G^*(v)| = 2 \cdot \deg_G(v) + 1$, also odd for each $v \in V$. The vertices in $\{g_1, g_2\}$, $X$ and $Y$ have odd degrees of 1, 1 and 3 in $G$, respectively. By definition,

$$\deg_G(z_1) = |\{g_1\} \cup Y| = 1 + \sum_{v \in V} |Y_v| = 1 + \sum_{v \in V} \deg_G(v) = 1 + 2 \cdot |E|$$

is odd, where the last equality holds because each edge in $E$ is counted twice in $\sum_{v \in V} \deg_G(v)$. Finally, $\deg_G(z_2)$ is odd by symmetry. \qed

## 6 Conclusions

We improve Chang and Lyuu’s $|V|^{(23/27)}$ upper bound to $(2/3) |V|$ on the minimum size of irreversible dynamos under the strict-majority scenario. Our technique also gives a $|V|/2$ upper bound on the minimum size of irreversible dynamos under the simple-majority scenario. The upper bound under the strict-majority scenario can be lowered to $|V|/2$ for undirected connected graphs.

We have proved inapproximability results on IRREVERSIBLE DYNAMO (STRICT MAJORITY) and IRREVERSIBLE DYNAMO (SIMPLE MAJORITY). An interesting direction of research is to design approximation algorithms for special types of graphs.

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