Phenomenological theory of
two-dimensional quantum liquids

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Abstract

A phenomenological theory is presented for two-dimensional quantum liquids in terms of the Fermi surface geometry. It is shown that there is a one-to-one correspondence between the properties of an interacting electron system and its corresponding Fermi surface. By doing this, the concept of Fermi surface is generalized to include different topologies. It is shown that for a Fermi liquid the corresponding Fermi surface is rough. In the presence of a condensate, the Fermi surface is faceted, while for a ferromagnetic instability, the Fermi surface becomes a frozen solid. I also determine the surface tension, the step free energy, low lying excitations and other surface and transport properties of the Fermi surface. The different transitions between these phases are also determined. A non-Fermi liquid phase is shown to be a pre-roughening state of the Fermi surface, the properties of which are briefly analyzed.
I. INTRODUCTION

I present a completely new semi-classical approach to the quantum liquid theory in terms of true surface properties and geometry of the Fermi surface. Within such a description, it can be shown that there is a one-to-one correspondence between the properties of a two-dimensional interacting electron system and its one-dimensional Fermi surface. Contrary to popular belief and expectation, I will also prove that the Fermi liquid theory corresponds to a rough and not smooth Fermi surface. Some of the evidence can be applied to a three-dimensional interacting electron system.

The bulk of the paper is dedicated to the detailed analysis of the rough phase and the corresponding roughening transition. For completeness, I present in Sections II and III the general theory of equilibrium surfaces and the roughening transition. In addition, the theory of the more particular Pokrovsky-Talapov transition is presented in the Appendix. This standard surface physics theory is applied to Fermi surfaces in Section IV, while in Section V the possible transition between a rough and a flat faceted Fermi surface is described. Due to the present interest in non-Fermi liquid theories, I will also give a possible scenario in Section VI when and how such a phase could appear.

II. EQUILIBRIUM SURFACES

The theory of the equilibrium surfaces dates back to the celebrated work of Wulff (Wulff 1901) and by the time of Herring’s 1951 paper (Herring 1951) the theory was truly understood. The equilibrium surface shape is the shape that minimizes the total free energy of the surface at a fixed volume. This constrained minimization was originally solved by Wulff, which is an elegant construction and valued textbook material. To find the equilibrium surface shape, follow the rules (Wulff 1901): i) make a polar plot (the so-called Wulff plot) of the surface tension $\alpha(\hat{n})$ as a function of $\hat{n}$, ii) starting at the origin, draw the radius in direction $\hat{n}$ out to the Wulff plot, then iii) construct planes perpendicular to $\hat{n}$’s through the
intersection of the radius with the Wulff plot. The interior envelope of this family of planes is always a convex figure whose shape is that of the equilibrium surface.\(^1\)

In the following, I present the analytic derivation of the equilibrium surface shape based on the Landau and Andreev derivation (Landau and Lifshitz 1980), which is not the most general (it relies on \(α(ˆn)\) being piecewise differentiable) but it has the advantage of being clear and easy to follow.

As noted previously, the Wulff construction always produces a convex shape, which can be divided into a top and its corresponding mirror-image bottom surface, with respect to the \(+z\) and \(−z\) direction. For simplicity, hereafter I only deal with the top surfaces, as all the obtained results can be mapped to the bottom one. Throughout this paper I will use the notations: \(z(x, y)\) is the height of the surface with slopes \(∂z/∂x = h(x)\) and \(∂z/∂y = h(y)\), steps on the surface will have height \(H\) and width \(L\) with corresponding tilting angle \(θ\). With these notations the total free energy of the surface is:

\[
E_{\text{Tot}} = \int dxdy f(ˆn),
\]

where

\[
f(ˆn) = α(ˆn) \sqrt{1 + h^2(x) + h^2(y)} \tag{1}
\]

being the surface free energy per unit area. Minimizing the total surface free energy keeping the volume fixed is identical (Landau and Lifshitz 1980) to solving the variational equation

\[
\frac{∂f}{∂h(x)} \frac{∂δz}{∂x} + \frac{∂f}{∂h(y)} \frac{∂δz}{∂y} − 2λ δz = 0 .
\tag{2}
\]

where \(λ\) is a Lagrange multiplier. Eq. (2) transforms into

\[
\int dxdy \left[ \frac{∂f}{∂h(x)} \frac{∂δz}{∂x} + \frac{∂f}{∂h(y)} \frac{∂δz}{∂y} − 2λ δz \right] = 0 .
\tag{3}
\]

By assuming its variation to be equal to zero, for any height variation \(δz\), Eq. (3) can be partially integrated. The obtained result, again, must be valid for any \(δz\), accordingly a simple integration with a proper choice of the integration constant gives:

\[
\frac{∂}{∂x} \frac{∂f}{∂h(x)} + \frac{∂}{∂y} \frac{∂f}{∂h(y)} + 2λ = 0 .
\tag{4}
\]

\(^1\)The overall magnitude has no significance.
The solution of Eq. (4) is (Landau and Lifshitz 1980):

\[ f = \lambda [z - xh(x) - yh(y)] . \]  

Eq. (5) is non other than the Wulff plane perpendicular to \( \hat{n} \) and the interface profile will have the form:

\[
\begin{align*}
    x &= -\frac{1}{\lambda} \frac{\partial f}{\partial h(x)}, \\
    y &= -\frac{1}{\lambda} \frac{\partial f}{\partial h(y)} \quad \text{and} \quad z = \frac{1}{\lambda} \left[ f - h(x) \frac{\partial f}{\partial h(x)} - h(y) \frac{\partial f}{\partial h(y)} \right].
\end{align*}
\]  

The shape of the surface \( z(x, y) \) is obtained as a Legendre transform of the surface tension per unit area, and the surface free energy is the Legendre transform of the surface shape:

\[ f = z - x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} . \]

Note, that the above construction is so general that it applies to any equilibrium interface. As an example I present in Fig. 1 the equilibrium shape of a square with its Wulff plot. It is important to emphasize that in first approximation one would consider the surface tension of a square to be vanishing, as the surface radius is infinitely large. However, this is not the case. For a proper evaluation of the surface tension we must count all the broken bonds at the surface, or simply the number of bonds to which a periodic boundary condition is applied (number of periodic bonds). Consider, for simplicity, that for the square in Fig. 1 the interactions in \( z \) and \( x \) directions are equal to unity, then the calculated surface tension is: \( \alpha(\hat{n}) = |\sin \theta| + |\cos \theta| \).

III. THE ROUGHENING TRANSITION

The disappearance of a cusp, sharp edges, canonical points, etc from the Wulff plot are all related to phase transitions. From these, the most studied is connected to the disappearance of cusps. Since the first theoretical attempts by Burton and Cabrera in 1949 (Burton and Cabrera 1949), this singularity was usually called the roughening transition or the
phenomena of surface melting. These two denominations refer to the same thermodynamic transition. Historically roughening transition is usually used in the theoretical works where the emphasis is on the critical phenomena aspect of the phases transition, while surface melting is an approach from the point of view of surface physics and experimentalists. This may generate some confusion, but everything becomes clear if we follow the argument of Burton et al. (Burton and Cabrera 1949; Burton, Cabrera and Frank 1951) outlined in the following: the static properties of a two-dimensional surface are well described by a Hamiltonian

$$H = 2J \sum_{\langle i,j \rangle} |H_i - H_j|,$$  \hspace{1cm} (8)

where the sum runs over all pairs of nearest-neighbour sites. With the simplest choice for the height variables, $H_i = \{0, 1\}$, Eq. (8) is identical to the two-dimensional Ising model ($S_i = 2H_i - 1$) which undergoes a second-order phase transition at $k_B T_c = 2.27J$, below which $\langle H_i \rangle < 1/2$ and above which $\langle H_i \rangle = 1/2$. For the surface this means that above $T_c$ the surface is rough, as the surface would consist of 50% each atoms and holes, but the nucleation barrier for two-dimensional nucleation would vanish, accordingly the surface is melted.

The presence of a cusp in the Wulff plot, is due to the appearance of an absolute value in the expansion of the surface free energy or surface tension on the smooth side of a surface, e.g., for a one-dimensional interface:

$$f[h(x)] = f(0) + |h(x)| f_s + \ldots,$$  \hspace{1cm} (9)

or

$$\alpha(\theta) = \alpha_0 + |\theta| \alpha_1 + \ldots,$$  \hspace{1cm} (10)

Some authors use the disordering transition denomination.

This is the well-known Solid-on-Solid model for a crystal-vapor interface, or the "Kossel"-crystal.
respectively. The coefficient $f_s$ of $|h(x)|$ called the *step free energy* is the energy cost of creating a unit length of step or ledge of unit height. When $|h(x)|$ is finite, adjacent steps are well separated and the energy per unit step length is well defined. The size of a horizontal facet is proportional to the jump $\partial f/\partial h(x)$, or equivalently, to the jump in $\partial \alpha/\partial \theta$ [as $\tan \theta \equiv h(x)$] at the cusp which follows immediately from Eqs. (3) and (6): when $h(x)$ crosses zero, $x$ jumps from $+f_s/\lambda$ to $-f_s/\lambda$, indicating that the facet size is $2f_s/\lambda$. Accordingly, the roughening transition is characterized macroscopically by the disappearance of a cusp from the Wulff plot or governed microscopically by the vanishing of the step free energy. The Wulff plot will be cusped for facets with smooth surface, while the cusp disappears from the Wulff plot of rough rounded surfaces.

Usually, the singularity at this transition is in the same universality class as the Kosterlitz-Thouless (1973) transition. The *smooth* interfacial phase maps to the high-temperature, exponential $XY$ phase, while the *rough* interfacial phase maps to the low-temperature algebraic (Gaussian) $XY$ phase. Or in sine-Gordon language, the rough phase maps into a non-interacting, or Tomonaga-Luttinger model, while the smooth surface into a Luther-Emery model. However, there are other types of continuous transitions, which can give rise to a rough surface, such as Ising, Potts, Pokrovsky-Talapov (1978) or chiral that are more frequent, depending upon symmetry, number of states, interaction parameters, fields, chirality, etc. An analysis of the critical behaviour of these transitions is not the purpose of the present work. Such a microscopic approach to the rough surfaces in general and to the Fermi surface in particular will be published elsewhere (Gulacsi 1996).

Most real crystals grow under conditions that are far from equilibrium. The application of the present theory for a real surface is limited by the requirement of equilibrium. From this point of view the Fermi surface is an ideal surface as by construction it is in equilibrium.

\footnote{The fixed volume condition is guaranteed by the Luttinger theorem. In most of the cases throughout this work, the Luttinger theorem is local \textit{à la} Haldane (Haldane 1994).}
IV. SURFACE PROPERTIES OF FERMIS LIQUIDS

The Fermi liquid picture is both simple and profound. With a few basic assumptions it describes accurately the low-energy, long-wave length behaviour of interacting systems in particular and the thermodynamic properties of the whole system in general. The theory assumes that there is a one-to-one correspondence between the low-energy excitations of a non-interacting system and the low-energy excitations of the interacting system while preserving the quantum numbers\(^5\). That is, if the non-interacting system is characterized by some distribution function \(n(k)\) of the bare particles, by switching on the interaction adiabatically the eigenstates of the interacting system, called quasiparticles, are also described by same function \(n(k)\). Accordingly, the excitation of the system is simply measured by the departure \(\delta n(k)\) from the ground-state distribution:

\[
\delta n(k) = n(k) - n^0(k) .
\]  

(11)

For the ideal system, there exists a simple linear relation between the energy of a given state and the corresponding distribution function. When particle interaction is taken into account, the relation between the state energy, \(E\), and the quasiparticle distribution function, \(n(k)\), becomes much more complicated. It may be expressed in a functional form, \(E[n(k)]\), which one cannot in general specify explicitly. If, however, \(n(k)\) is sufficiently close to the ground-state distribution \(n^0(k)\), we can carry out a Taylor expansion of this functional. On writing \(n(k)\) in the form Eq. (11) and taking \(\delta n(k)\) to be small, or to extend over a small region in momentum space, the Landau free energy (Landau 1957) becomes:

\[
F - F_0 = \sum_k [\epsilon(k) - \mu] \delta n(k) + \frac{1}{2} \sum_{k,k'} f(k,k') \delta n(k) \delta n(k') + \ldots ,
\]  

(12)

which is the heart of the phenomenological theory of Fermi liquids (Pines and Nozières 1988) with \(f(k,k')\) representing the effective interaction between quasiparticles.

\(^5\)For simplicity in the following and throughout the present work only the spinless case will be discussed.
It is not my purpose to further analyze Eq. (12) from the Fermi liquid point of view\textsuperscript{6}, but only to establish its surface properties as a semi-classical interface. For simplicity, I restrict ourselves to one-dimensional Fermi surfaces\textsuperscript{7} (two-dimensional systems). It should be noted that some formulas or demonstrations that appear in the text are also valid or easily generalized for a two-dimensional Fermi surface (three-dimensional systems). The reader will be prompted when these examples arise.

Based on the definitions presented in Sections \textbf{II} and \textbf{III} it is immediate that Eq. (12) describes a rough surface because the step free energy is zero\textsuperscript{8}. As noted already by Pines and Nozières (1988) Eq. (12) looks like a Taylor expansion, but it is not. With the notations from the previous sections and only if $\mu$ and $\delta n(k)$ are finite, $F - F_0$ defined in Eq. (12) is of the order $h^2(k_x)$ because $\epsilon(k) - \mu \propto h(k_x)$ and $\delta n(k) \propto h(k_x)$. Accordingly, the corresponding step free energy, $f_s$, is strictly zero and by definition the surface is rough. The roughening transition is defined by the vanishing of the step free energy, and any smooth surface must characterized by Eq. (9). The only case where such a simple proof would not work is the case $\mu = 0$, which would correspond to a half-filled band systems, a case which I will analyze later.

This proof should work equally well for two-dimensional Fermi surfaces for the non-

\textsuperscript{6}The excitation of Eq. (12), i.e. the sound modes and the particle-hole continuum, will be briefly analyzed in Section \textbf{VII}.

\textsuperscript{7}The surface under study, being defined in the momentum space, the local curvilinear surface coordinate is $k_x$, $k_z(k_x)$ denotes the local height of the surface with slope $\partial k_z / \partial k_x = h(k_x)$. The tilting angle remains $\theta$. It is also important to emphasize that the steps are associated with the world lines of fermions (Pokrovsky-Talapov 1978).

\textsuperscript{8}By construction Eq. (12) describes density fluctuations of a liquid surface, rather than elastic waves propagating on a smooth facet.

\textsuperscript{9}See, Eq. (19) for an expansion of a rough surface.
interacting case. However, some problems will appear if we consider the two-dimensional Fermi surface, a quantum interface case, which is analyzed in detail by Gulacsi (1996). For a one-dimensional Fermi surface, the above argument is based on a nearest-neighbour type effective interaction. However, at large length scales the discreteness of the step heights is irrelevant, the surface always can be described by a Gaussian model, having the properties of a transverse vibrating elastic network. Due to this, the obtained transition (for details see, Section V) is not destroyed if the interaction between quasiparticles is of short-range type or even has the form $f(k, k') \sim g/|k_x - k'_x|^{\gamma}$ with $\gamma \leq 2$ and $g > 0$. From these interactions the $g/|k_x - k'_x|^2$ form is the most probable one (Ashcroft and Mermin 1976) based on a Hartree-Fock theory, or simply because the elastic or bipolar interactions between steps on an equilibrium surface have this form (see, Section V).

The Fermi surface is indeed rough for the non-interacting case, and it can be directly proven by the Wulff plot. Equations (1) through Eq. (7) enable one to reconstruct the Wulff plot from the equilibrium surface shape. For an isotropic and non-interacting system with $\epsilon_F = h^2|k_F|^2/2m$ we immediately obtain that $\alpha(\theta) = \epsilon_F$ becomes independent of $\theta$, as in the case for an ordinary fluid. Accordingly it has no cusps; that is, the Fermi surface is rough. In this simple case, following Nozières (1991), we can determine the Lagrange multiplier of the Fermi surface appearing in Eqs. (3): $\lambda = v_F/4m$.

On a lattice the half-filled and doped case must be treated separately. Namely, for

\footnote{Fisher and Weeks (1983) and Fradkin (1983) and others have contested the Andreev and Pashkin (1978) argument about the roughness of the quantum liquid surfaces at zero temperature. However, new arguments appear in the favor of the roughness, namely as presented in Footnote 14, the width of the steps on a quantum surface at $T \to 0$ are very large, so the quantum effects are becoming smaller, accordingly the quantum surface becomes classical before the complete delocalization of the interface.}

\footnote{For simplicity consider a square lattice.}
\[ H_0 = -t \sum_{\langle i,j \rangle} (c_i^\dagger c_j + h.c.) + \mu \sum_i c_i^\dagger c_i, \]
with \( \mu = -2t \delta \), where \( \delta \) denotes the hole doping, using Eq. (7), the Wulff plot corresponding to the Fermi surface \( \mu = 2t \sum \cos k_i \) will be smooth for \( \mu \neq 0 \) and cusped for \( \mu = 0 \). A similar argument used in the previous paragraphs is that \( \delta n(k) \propto h(k_x) \) and also \( \epsilon(k) - \mu \propto h(k_x) \) and for \( \mu \neq 0 \), i.e., \( F - F_0 \propto h^2(k_x) \), while for \( \mu = 0 \) only \( \delta n(k) \propto h(k_x) \), so \( F - F_0 \propto h(k_x) \), accordingly the step free energy is finite. In this case, the Fermi surface and its Wulff plot are identical to Fig. 1 (rotated by \( \pi/2 \)). That is, the Fermi surface is faceted and the facets are smooth. However, in this case, the appearance of the facets can be attributed to the presence of the van Hove singularity. The step free energy is finite but small close to the van Hove singularity, increasing as we approach the singularity. This behaviour is caused by the vanishing group velocity, \( |\nabla \epsilon(k)| \), which explicitly shows up in evaluating Eq. (7). The step free energy, however, cannot grow infinitely large, is bounded by the crystal structure, but produces a facet with the largest possible flat surface, and this is the Fermi surface of the half-filled band case. The above arguments for the half-filled band case are also valid for a two-dimensional Fermi surface.

In conclusion, in a non-interacting system the change of the topology of the Fermi surface, i.e. the appearance of flat facets can be attributed to the presence of the van Hove singularity. For an interacting case it is difficult to determine the Wulff plot because we do not know the exact form of the self energy, \( \Sigma_{\text{Re}}(k, \omega) \). However, it can be shown that the Fermi liquid picture in general, and Eq. (12) in particular, is not valid close to half filling. For example, for the two-dimensional Hubbard model, second-order perturbation theory gives for the \( f(k, k') \) function (Honner 1995):

\[ f(k, k') = \frac{4U^2 \Delta k}{N t \pi^2 \sqrt{16 \sin^2 \Delta k - \Delta^2 \epsilon}} \ln \left| \frac{4 \sin \Delta k + \sqrt{16 \sin^2 \Delta k - \Delta^2 \epsilon}}{4 \sin \Delta k - \sqrt{16 \sin^2 \Delta k - \Delta^2 \epsilon}} \right|, \quad (13) \]

where \( 2 \Delta k = k_x - k'_x = k_y - k'_y \) and \( \Delta \epsilon = \cos k'_x - \cos k_x + \cos k'_y - \cos k_y \). The \( f(k, k') \) from Eq. (13) is bounded only in the vicinity of \((\pi/2, \pi/2)\) with a maximum value of \( 2U^2 / N t \pi^2 \) and is divergent as we approach the van Hove singularity. That is, the adiabatic switching on the interaction picture of the Fermi liquid theory is no longer valid.

For an interacting electron system, the half-filled band case is mostly due to perfect
nesting, in the presence of which the Fermi surface becomes unstable to the formation of long-range order (e.g., charge-density-wave or spin-density-wave), or Mott insulator, etc., due to which a charge or spin gap (depending on the interactions) will open. This causes the divergence of the $f(k, k')$ function. In this case, the Fermi surface is not defined in the sense of Eq. (11) or (12). Within the present approach, based on the Fermi surface geometry, we can generalize its definition with the understanding that only its topology changes, which means that the Fermi surface becomes faceted. Previously, the Fermi surface was rounded, while now it has flat, faceted regions. The presence of a facet would immediately indicate that the Fermi surface cannot be excited by any small amount of energy. A minimum energy is required (the step free energy defined in Section II) to create steps on a flat and smooth facet and this minimum energy signals the presence of a gap in the excitation spectrum. In the next Section the width of the facet is determined, in first approximation, to be proportional to $\Delta/v_F$, where $\Delta$ is a mean-field gap and $v_F$ is the non-interacting Fermi velocity. Accordingly, the appearance of these facets on the Fermi surface is equivalent to an instability of the Fermi surface by which a condensate will emerge in the interacting electron system. The presence of facets in a two-dimensional Fermi surface will equally mean an instability of the Fermi surface towards a quasiparticle gap opening.

V. THE FACETING TRANSITION

The half-filled band analyzed in the previous Section is a particular case where all the Fermi surface is flat. Generally, facets appear on a surface gradually, meaning their width increases with variation of some parameters (doping, external magnetic or electric field) or temperature. In any case, the appearance of facets on the Fermi surface will always lead to nesting (due to the symmetries of the Brillouin zone) usually a partial one. As the surface of the facets increases, nesting becomes stronger and stronger until it reaches a completely nested Fermi surface similar to the half-filled band case. The transition from the rough surface into a faceted one, is a roughening transition by which the topology of the Fermi
surface changes completely, and the properties of which are determined in the following.

The surface free energy of a facet is given in Eq. (9). However, this term represents only non-interacting steps. To describe properly the faceting transition we need the knowledge of the interactions between the steps present on the surface. As the present discussion is a phenomenological one, we do not intend to subtract these interactions from perturbation theories but rather to analyze them from a surface physics point of view. There are several main sources of interaction on a surface. The first is the fact that we do not allow steps to cross, or create overhangs, as these states would require higher energy. Meaning, the steps will be restricted resulting in an entropic interaction. The contribution of such an entropic repulsion, we can understand by following Gruber and Mullins (1967): let us suppose for a moment that a single step is constrained to move between two walls situated at $k_x = \pm k_L$.

The presence of the walls produces an entropic repulsion between the wall and step. The probability density $P(n)$ for the step passing true $k_x = n$ is (Gruber and Mullins 1967):

$$P(n) = \frac{1}{k_L^2} \cos^2 \frac{\pi n}{2k_L}, \quad (14)$$

and the free energy of the step becomes $f_s = f_s(0) + \text{const} / k_L^2$. Now, replacing the fixed boundaries of the walls by neighbouring steps results in: $f_s = f_s(0) + \text{const} h^2(k_x)$, which introduced in Eq. (9) will give an $|h(k_x)| h^2(k_x)$ contribution.

Short-range interactions (Jayaprakash, Rottman and Saam 1984) cannot alter this behaviour. Namely, in a mean-field theory the contribution of a short-range interaction $V(k_x)$ will be $(1/L) \int dk_x V(k_x) h(k_x) G(k_x)$, where $G(k_x)$ is the non-interacting pair correlation function, $G(k_x) = \{1 - \sin^2[\pi k_x h(k_x)/L]/(\pi k_x h(k_x)/L)^2\}$. If $V$ is smaller than the distance between steps or the step width $14$, $G(k_x)$ may be expanded and the contribution of a

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12 The original proof was for a terrace-step-kink (TSK) type step.

13 If $P(n)$ would be a constant, than the step staying away from the walls would decrease its entropy and thereby its free energy with $1/L$.

14 The step width in the $T \to 0$ limit can be related (Balibar, Guthmann and Rolley 1993) to the
short-range interaction will be of order $h^4(k_x)$.

Other interactions (see e.g., Beijeren, van and Nolden 1987), e.g., elastic or dipole interactions, assume the $g/|k_x - k'_x|^2$ form on a surface with free energy contribution in the continuum limit with $P(k_x)$ given in Eq. (14), true the whole length of a facet at small $h(k_x)$ is:

$$\sim \int_{-(2|h(k_x)|)^{-1}}^{+(2|h(k_x)|)^{-1}} dk_x \frac{|h(k_x)|}{k_x + 1/2|h(k_x)|} \cos^2 \pi k_x h(k_x) \sim |h(k_x)| h^2(k_x) \quad .$$

(15)

Accordingly, on a facet the surface free energy from Eq. (14) will have the form:

$$f[h(k_x)] = f(0) + f_s |h(k_x)| + f_{\text{int}} |h(k_x)| h^2(k_x) + \ldots \quad ,$$

(16)

where in $f_{\text{int}}$, we can accumulate the effects of all the interactions present on the surface.

At this point, it is important to emphasize the following: i) All type of interactions on a surface are giving a contribution of $h^3_x$ in the expression of the surface free energy. Naively, one might expect the free energy due to interactions to be of the $h^2(k_x)$ order, however it is not. A contribution of the $h^2(k_x)$ type occurs only in a rough surface, see Eq. (13) or Eq. (12) and the discussions following it. ii) The presence of the interaction term in Eq. (16) is crucial for the faceting transition. The $f_{\text{int}} = 0$ corresponds to a mean-field type critical bahaviour (see e.g., Rottman and Wortis 1984) which may containe a tricritical point.

In the following I determine the interface profile. Continuing the arguments from Section (11), when $h(k_x)$ crosses zero, the horizontal width of the facet, $k_x$, jumps from $+f_s/\lambda$ to $-f_s/\lambda$, that is the facet width will be proportional to $f_s/\lambda$. In a mean-field approach the spectrum has the $\sqrt{\epsilon^2(k) + \Delta^2(k)}$ form. Using the continuum result from Section (14) for the Lagrange multiplier: $\lambda = v_F/4$, in first approximation it is obtained that the Fermi surface

so-called pinning or coupling strength of an effective sine-Gordon model, accordingly the steps are relatively wide.

15A tricritical point appears for an attractive interaction and bound states for a repulsive one.
facet width is proportional to $\Delta/v_F$. In Section VI it will be shown that the facet width is inverse proportional to the effective correlation length on the surface: $\xi^{-1} \propto f_s/\lambda \approx \Delta/v_F$.

The interface profile near the edges of a facet will have a universal shape. Using Eqs. (6) and (16) one obtains:

$$k_z = k_0^z \left[ 1 - \frac{2 f_{\text{int}}}{f(0)} |h(k_x)|^3 \right],$$
$$k_x = \pm k_0^x \left[ 1 + \frac{3 f_{\text{int}}}{f_s} h^2(k_x) \right],$$

(17)

where the $k_0^z = f(0)/\lambda$ and $k_0^x = f_s/\lambda$ notations were used. That is,

$$k_z - k_0^z = -\frac{2}{3 \sqrt{f_{\text{int}}}} |k_x - k_0^x|^{3/2}. $$

(18)

This equation establishes a universal (non-analytic) jump of the curvature, well-known in surface physics. The $3/2$ exponent is the Pokrovsky-Talapov (1978) exponent while the transition is called the Gruber-Mullins (1967) or the Pokrovsky-Talapov transition. The above description of the Pokrovsky-Talapov transition is mainly based on an effective nearest-neighbour model, however, as previously demonstrated, the Pokrovsky-Talapov exponent and correspondingly the transition is not destroyed by short-range interactions and not even by long-range interactions. The effects of the long-range interactions are analyzed in more detail the following Section, where it will be seen that the roughening transition remains unchanged, only a new phase may appear.

On the rough side of the surface, however, a simple Taylor expansion cannot be performed. As presented in the Appendix; on the rough side the free energy will always have the form:

$$f[h(k_x)] - f(0) \propto h^2(k_x) + \ldots .$$

(19)

similar to Eq. (12).

Other characteristics of the roughening transition can be re-written for the Fermi surface, e.g., the universal jump in the curvature of the Fermi surface before and after the transition, or it can be shown that near the edge of the facets on the rough curved surfaces the steps are spontaneously generated with a square root density.
As presented in the Appendix, a Pokrovsky-Talapov transition appears by varying the a doping, or external magnetic or electric fields. As a function of temperature, however, such a transition is a Kosterlitz-Thouless transition if the width of the facet vanishes at the transition, otherwise is a standard second-order phase transition.

VI. FERMI AND NON-FERMI LIQUIDS

The conclusions from the previous Sections were that the rough phase of the Fermi surface corresponds to the metallic (Fermi liquid) state of a system, while a faceting (Pokrovsky-Talapov type) transition will lead to an instability of the Fermi surface corresponding to the appearance of some kind of condensate. In both cases, the surface may be characterized by the density (occupation probability) distribution \( \rho(k) \propto \delta n(k) \), where \( \delta n(k) \) was defined in Eq. (11). As mentioned in Section I, in the rough phase the density-density correlation function has an algebraic decay but an exponential decay for the smooth faceted phase. It is important to emphasize that these correlation functions appear in both translational and orientational order of the surface, which are decoupled (Haldane 1994), with the understanding that the orientational fluctuations cannot change the shape of the surface.

The disordered (rough) phase is a non-interacting or Tomonaga-Luttinger liquid, that is

\[
\langle \rho(k_x) \rho(0) \rangle = \frac{K}{\pi^2 k_x^2} - K' \cos\left[ \frac{\pi k_x h(k_x)}{k_x^2} \right], \tag{20}
\]

where \( K \) is the well-known roughness constant. Equation (20) can be viewed as the local slope operator so, the height-height correlation function

\[
\langle [h(k_x) - h(0)]^2 \rangle = 2 \int_0^{k_x} dk'_x \ (k_x - k_y) \langle \rho(k_x) \rho(0) \rangle, \tag{21}
\]

will have the well-known form:

\[
\langle [h(k_x) - h(0)]^2 \rangle \approx \frac{2K}{\pi^2} \ln k_x + \frac{K'}{2h_x^2} \cos\left[ \frac{\pi k_x h(k_x)}{k_x^2} \right], \tag{22}
\]

with the universal constant \( K \) taking the \( K = 1 \) value, corresponding to the notations used in Eqs. (20) and (21). It can be seen from Eq. (20) that the non-interacting part adequately
describes the properties of surface. The second term of Eq. (20) which is the characteristic
collection of the Tomonaga-Luttinger models, is not affecting the height-height correlation
function. Accordingly, the low-energy and long-wave length properties of the metallic phase
are non-interacting in nature. By this we confirmed the Fermi liquid picture.

The faceted phase on the other hand, maps into a Luther-Emery phase:

\[ \langle \rho(k_x)\rho(0) \rangle \propto \exp \left( -\frac{k_x}{\xi} \right), \] (23)

where \( \xi \propto \lambda/f_s \approx v_F/\Delta \) is the effective correlation length on the surface and \( \Delta \) the quasi-
particle gap. The \( f_s \) is step free energy defined in Section IV. In Section V, the above result
was previously derived from a different point of view, namely using surface minimization
arguments and determining the Lagrange multiplier of the surface.

In the orientational order, i.e. in the \( k_z \) direction, the situation is not so simple and
detailed microscopic calculations are needed (Gulacsi 1996). However, we know that in the
metallic phase (rough surface) there must be a step singularity\(^\text{16}\) in \( n(k) \propto \rho(k) \) of the
\( n(k) = \theta(k - k_F) \) form with \( \theta(x) \) being the step function. While, in the smooth phase of the
Fermi surface there are no singularities, e.g. in a mean-field description \( n(k) \approx 1/[\Delta^2 + v_F^2 k^2] \).

Concerning a non-Fermi liquid type behaviour, it seems (based on the above presented
analysis) that there are no other phases left. The exact solutions of nearest-neighbour in-
teracting lattice models, or surface models, exhibits three phases (see, Appendix). First,
one which is ferroelectric, corresponding to a completely locked (or frozen) solid surface,
corresponding to a ferromagnetic type transition, i.e., a Pomeranchuk (1958) type insta-

\[ ^{\text{16}} \text{The presence of singularities as we cross a surface is not a surprise. In the case of a standard solid-
liquid (or solid-gas) interface the mass density has a singularity at the surface} \delta \rho = \rho_{\text{solid}} - \rho_{\text{liquid}}, \]

which determines the Lagrange multiplier defined in Section IV (Nozières 1991).

\[ ^{\text{17}} \text{The frozen ferroelectric phases of the vertex models are the only ones that exhibit true long-range
order corresponding to a ferromagnetic state.} \]
corresponds to the rough and flat (faceted) surfaces, respectively. In this context, one of the most appealing possibilities to generate a liquid state, that is a metallic phase, but not a Fermi liquid, would be a phase intermediate between the rough and flat surfaces, which shares the properties of both. That is, a new phase between these two such that the new surface would be characterized by an exponential decay of the translational order but an algebraic decay of the correlation function of the orientational order. The existence of such a surface (sometimes called a hexatic fluid) is still controversial in the surface community (Bauer 1987), however, it would correspond to the disordered flat surface introduced by den Nijs (1989) which is in a pre-roughening state, corresponding to the presence of a ”Haldane gap” type gap. Being so, it has what we could call hidden topological order.

The appearance of such a phase is due to short-range interactions (Nijs, den 1989). In Section V, I have shown that next-nearest-neighbour or short-range interactions cannot disturb or change the Pokrovsky-Talapov melting transition. However, what they can do, is to generate the above introduced disordered flat surface (Nijs, den 1989).

An algebraic decay of the orientational order, means that \( n(k) \) will have some singularity as we cross \( k_F \), and the surface would correspond to a metallic phase. However, the metallic phase is not a Fermi liquid because the translational order is gapped. The step free energy is zero in such a non-Fermi liquid phase (Nijs, den 1989), so based on the definitions introduced in Section II the surface is ”rough”, but the height-height correlations are not divergent and thus the reason why such a phase is refered to as pre-roughening phase.

Concerning the translational order, which is gapped, the surface exhibits flat surface facets similar to the previously analyzed faceted phase. The main difference is that, in the later phase the facets have a well-defined macroscopical width (see, Section III), while in a pre-rough phase only the average surface is flat. The surface structure is characterized by perfect antiferromagnetic order of the heights \( h(k_x) \) at random positions \( k_x \). Clearly, the surface structure is complex; that is why it is so difficult to describe analytically a non-Fermi liquid in higher dimensions. Because of the random positions of \( k_x \), no nesting can occur on the Fermi surface, accordingly no long-range order, or condensate either, and indeed,
the system will be metallic. The step free energy being zero, Eq. (12) is still valid, with a specific \( f(k, k') \). Within the present phenomenological approach the exact form of \( f(k, k') \) cannot be determined. However, some insights can be gained following a simple analysis: for a Fermi liquid, as mentioned previously, both orientational and transversal correlations are algebraic. Being so, there is a singularity in \( n(k) \) and \( Z = \{1 - \partial \Sigma_{\text{Re}}(k, \omega) [\omega = \epsilon(k)] / \partial \omega \}^{-1} \) is finite. Let us consider the pre-rough, non-Fermi liquid, phase. As previously discussed, the orientational order is still algebraic; some sort of singularity will appear in \( n(k) \) as we cross \( k_F \). The translational order being gapped \( Z \) is zero. So, the emerging non-Fermi liquid will have a singularity similar to a Tomonaga-Luttinger liquid in one-dimension. However, I must emphasize that the above analysis is approximate within a semi-classical approach to the Fermi surface, and a microscopic treatment is needed (Gulacsi 1996) to prove exactly the above statement.

**VII. CONCLUSIONS**

Based mainly on a semi-classical approach, I have demonstrated that there is a one-to-one correspondence between the properties of a two-dimensional interacting electron system and its one-dimensional Fermi surface. Though, some of the evidence can be generalized to a three-dimensional interacting electron system.

In a generic scenario, the Fermi surface can have three distinctive phases, namely the locked solid, which would correspond to a ferromagnetic type instability, a rough phase, which I have proven to be the metallic or Fermi liquid phase and a flat faceted phase, which is an instability of the Fermi surface against the appearance of a quasiparticle gap. By varying the interaction strength, external magnetic or electric, fields, or doping, we can cross these phases in the previously presented order, that is, frozen solid \( \iff \) rough, melted surface \( \iff \) flat, faceted surface. It is interesting to note, that there is no transition between frozen solid and flat faceted surface. The transitions between these phases are: \( i \) first or second-order from the frozen solid to rough phase. \( ii \) Kosterlitz-Thouless or Pokrovsky-
Talapov melting between the flat faceted and rough surfaces. The properties of the rough and faceted surfaces and the transition between these two phases were described in great detail in Sections \[\text{II} - \text{IX}\]. In some cases, a third phase can appear between the rough and flat surfaces, namely a pre-rough surface with hidden topological order\[\text{[18]}\]. So, the possible phases in the order of their appearance are:

\[
\text{Locked} \iff \text{Rough} \iff \text{Pre - rough} \iff \text{FlatFaceted},
\]

and these will correspond to

\[
\text{Ferromagnetic} \iff \text{Fermi Liquid} \iff \text{Non - Fermi Liquid} \iff \text{Condensate}.
\]

The corresponding transitions are: 

\(i\) first or second-order from the frozen solid to rough phase. 
\(ii\) Kosterlitz-Thouless or Pokrovsky-Talapov melting between the rough and pre-rough surfaces. 
\(iii\) first or second-order between the flat faceted and pre-rough phases. 

Such a scenario predicts for two-dimensional models (e.g., the Hubbard model) a non-Fermi liquid behavior close to half filling, that is between the low-density Fermi liquid behavior and the half-filled band case.

In regards to the rough phase, a true analysis of its properties requires a discussion of its dynamics. In the Fermi liquid theory there are the sound modes and also solutions with imaginary frequency that represents the particle-hole continuum. Immediately, it can be seen that, the zero sound corresponds to the bosonic excitations of the one-dimensional system. In a one-dimensional system, we would not expect a continuum to appear because of reduced phase space. However, treating the Fermi surface as an effective equilibrium surface we can attach separate orientational and translational degrees of freedom, as demonstrated in the previous Section. Such that the orientational bosonic fluctuations are the sound

\[\text{[18]}\] In connection with the following discussion: the pre-rough phase will have sound modes (the orientational order remains unchanged compared to the rough phase), however, the particle-hole continuum will disappear because the translational order is different.
modes, while the translational fluctuations can generate a particle-hole continuum. Being a rough phase, these fluctuations are Gaussian, in a discrete system the Lagrangian is: 
\[(g/4\pi) \int |\nabla \phi|^2, \] with \(\phi = \pi \ell/2\) and \(\ell = \pm 1\). We can identify this with the six-vertex model (see, Appendix) in the following way: in the eight-vertex model (Baxter 1990) the magnetic charge attached to the vertices seven and eight is \(m = \pm 1\), due to which the scaling dimension of a vortex operator is \(gm^2/2 = g/2\) and, accordingly, the scaling dimension of its corresponding fugacity is \(2 - gm^2/2 = 2 - g/2\). If the Boltzmann weights of vertices seven and eight are vanishingly small, the free energy is singular with an exponent inverse proportional to the fugacity which identified with the singular behaviour obtained by Baxter (Baxter 1990) gives: \(\cos(g\pi/4) = -\cos \mu\), where \(\cos \mu\) is the anisotropy parameter of the model (see, Appendix). Thus, the translation fluctuations of the Fermi surface will have a ground-state corresponding to that of the planar six-vertex model (Baxter 1990):

\[
\omega_0 = \frac{\cos \mu}{4} - \frac{\sin \mu}{2} \int_{-\infty}^{+\infty} dx \frac{\sinh(\pi - \mu)x}{\sinh \pi x \cosh \mu x}, \tag{26}
\]

with low lying states forming a particle-hole continuum bounded by

\[
\frac{\pi}{2} \frac{\sin \mu}{\mu} |\sin k| \leq \omega \leq \pi \frac{\sin \mu}{\mu} \left|\sin \frac{k}{2}\right|. \tag{27}
\]

Thus, not only bosons corresponding to zero sound will appear on the Fermi surface but also a particle-hole continuum.

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\[19\text{As already presented in Section [V], at large length scales the discreteness of the step heights is irrelevant, due to which the surface always can be described by a Gaussian model, having the properties of a transverse vibrating elastic network.}\]
In the past twenty years, incommensurate periodicities have been observed in many physical systems and have become the subject of considerable theoretical interest. Two incommensurate structures in contact can overcome their rigidity and form a common periodicity. This transition is known as the commensurate-incommensurate, or lock-in transition.

Since its discovery in the context with dislocation theory (Frank and Merwe, van der 1949), the commensurate-incommensurate transition had a major role (for a review, see Bak 1982; or Nijs, den 1988) in understanding the physics of charge-density-waves, incommensurate lattice problems, the general problem of incommensurate periodicities, two-dimensional (XY) models, theory of phase transitions in magnetic systems and 1D quantum systems, and the correlation driven metal insulator (Mott) transition (Gulacsi and Bedell 1994b).

In most of the systems and models mentioned above, the problem of characterizing the commensurate-incommensurate transition has been reduced to the solution of the 1D sine-Gordon model. The level of incommensurability is measured by a soliton density, \( n_{sol}(\mu) \), where \( \mu \) is the chemical potential. The commensurate-incommensurate transition happens at \( \mu = E_{sol} \), where \( E_{sol} \) is the single soliton energy. For \( \mu < E_{sol} \) and \( n_{sol} = 0 \) the system is locked in a commensurate phase, while for \( \mu > E_{sol} \), i.e., \( n_{sol} \neq 0 \) the structure is incommensurate.

In the classical limit the commensurate-incommensurate transition occurs with the emergence of a soliton lattice in the soliton gap (Frank and Merwe, van der 1949; and Horowitz, 1982). At low soliton density, the solitons repel each other, with an exponential type interaction (Horowitz 1982), and the transition takes place with a logarithmic behavior \( n_{sol} \propto \ln^{-1}(\mu - E_{sol}) \), i.e., corresponding to a critical exponent \( \tilde{\beta} = 0 \). In the full quantum problem, it is known (Haldane 1982), that the soliton lattice will melt, giving rise to a Luttinger liquid which emerges at the top or bottom of the soliton gap. In this case the solitons will repel each other much stronger than in the classical limit, with an interaction proportional to \( n_{sol}^2 \), in the low soliton density limit. The incommensurability close to the
transition behaves as $n_{\text{sol}} \propto (\mu - \Delta)^{1/2}$, where $\Delta$ is the soliton gap, corresponding to a critical exponent $\bar{\beta} = 1/2$. This is the Pokrovsky-Talapov exponent (Pokrovsky and Talapov 1978). In some applications this transition is characterized with the variation of the thermodynamic potential, $\Omega = E(n_{\text{sol}}) - \mu n_{\text{sol}}$, for which we obtain $\Omega \propto n_{\text{sol}}^3 = (\mu - \Delta)^{3/2}$. From the experimental point of view, the relevant quantity to observe this transition is the charge susceptibility, $\chi_{\text{sol}} = (\partial \mu / \partial n_{\text{sol}})^{-1}$, which is divergent at the transition $\chi_{\text{sol}} \propto 1/n_{\text{sol}}$. This divergent behavior implies a divergence of the effective mass of the charge carriers, and accordingly the divergence of the specific heat coefficient. Thus, the basic characteristics of the Pokrovsky-Talapov transition are:

$$n_{\text{sol}}(\mu) \propto (\mu - \Delta)^{1/2}, \quad \Omega(\mu) \propto (\mu - \Delta)^{3/2}, \quad \chi_{\text{sol}}(\mu) \propto (\mu - \Delta)^{-1/2} \quad (28)$$

Or, as a function of the soliton density (incommensurability) are:

$$\mu(n_{\text{sol}}) \propto \Delta + n_{\text{sol}}^2, \quad E(n_{\text{sol}}) \propto \Delta n_{\text{sol}} + n_{\text{sol}}^3/2, \quad \chi(n_{\text{sol}}) \propto 1/n_{\text{sol}} \quad (29)$$

The Pokrovsky-Talapov transition, however, is typical for all integrable models. Its importance in connection to the 2D XY, Gaussian, Coulomb-Gas models but, most important to the equivalent (Chui and Weeks 1976; Beijeren, van, 1977; and Jayaprakash and Saam 1984) solid-on-solid surface models led a to thorough analysis of its properties. Recognizing (Nijs, den 1981; and B. Horowitz, Bohr, Kosterlitz and Schulz 1983) the equivalence between the Gaussian models and the direct field six-vertex model (defined below) led to the remarkable result that the six-vertex model in a direct field is an ideal model to describe the commensurate-incommensurate transition. The direct field in the six-vertex model correspond to the chemical potential in Eqs. (28) and (29), while the polarization of the six-vertex model is equivalent to the soliton density, $n_{\text{sol}}$.

The five- and six-vertex models are the simplest, exactly solvable model describing surface transitions. In general, the vertex models define a statistical mechanics on a (for simplicity) square lattice as follows. On each link there is a degree of freedom taking two values which is represented (traditionally) by an arrow, i.e. $\rightarrow$ and $\leftarrow$. To each vector therefore there
corresponds 16 possible configurations. To each of these we associate a Boltzmann weight \( \omega_i = \exp(-\varepsilon_i/k_B T) \). This general model is referred to as the sixteen-vertex model. The exact solution of this general ferroelectric model is not known. If we choose the Boltzmann weights, so that six vertices with two entering and two exiting arrows (or eight vertices including also the cases where four or zero arrows exit) have finite weight, then we define the six- or eight-vertex models. The six-vertex model is exactly solvable via the algebraic Bethe Ansatz, for details see Lieb and Wu (Lieb and Wu 1972) and Baxter (Baxter 1990). The exact solution has at most four independent variables (this is because the zero energy can be chosen arbitrary and the ice rule implies \( \omega_5 = \omega_6 \)). In this situation the model is called the asymmetric six-vertex model, in contrast to the symmetric six-vertex model, where the number of independent variables are two. The symmetric model in the presence of \( h \) and \( v \) external fields is equivalent to the asymmetric one. The energies of the six possible arrow configurations of the symmetric model are \( \varepsilon_1 = \varepsilon_2, \varepsilon_3 = \varepsilon_4 \) and \( \varepsilon_5 = \varepsilon_6 \). Similarly, the field-free five-vertex model is defined by three independent Boltzmann weights, e.g., \( \varepsilon_1, \varepsilon_2 = \infty, \varepsilon_3 = \varepsilon_4 \) and \( \varepsilon_5 = \varepsilon_6 \) and its exact solution can be determined via the algebraic Bethe Ansatz (Gulacsi, Beijeren, van and Levi 1993).

1.) For the six-vertex model, the exact free energy obtained with the algebraic Bethe Ansatz (Lieb and Wu 1972; and Baxter 1990) exhibits three different analytic forms for the three different phases, i.e., two frozen-in ferroelectric phases (with the polarization, \( p = \pm 1 \)), an antiferroelectric phase (\( p = 0 \)) and a disordered phase (paraelectric, \( p = 0 \)). The transitions to the ferroelectric phases is of first-order, while the transition from the paraelectric phase to the antiferroelectric one is of Kosterlitz-Thouless type.

Introducing a vertical field, \( v \) in the symmetric model is still easily solvable with the algebraic Bethe Ansatz. All the above mentioned transitions are \textit{second-order} now, as a function of temperature and Pokrovsky-Talapov as function of the external field. The largest

\[ \]
eigenvalue of the transfer matrix is determined from the Bethe Ansatz, \( z(p) \) in notation of (Lieb and Wu). The exact free energy is obtained as \(-F/k_B T = \max z(p)_{-1 \leq p \leq 1}\). All the three transitions are similar in nature, accordingly we describe only the antiferroelectric one, \( p \to 0 \). The free energy in the ordered phase (smooth faceted surface) is determined by (Eq. (304) from Lieb and Wu 1972):

\[
z(p) \propto z'(0) p + \frac{1}{6} z'''(0) p^3 + \ldots,
\]

(30)

with \( z'' = 0 \), i.e., there is no second-order contribution. Eq. (30) is identical to Eq. (29).

Close to the transition \( p \propto (v - k_B T|z'(0)|)^{1/2} \) (Eq. (340) from Lieb and Wu 1972), \( \Omega \propto (v - k_B T|z'(0)|)^{3/2} \) (Eq. (342) from Lieb and Wu 1972) and the magnetic susceptibility \( \chi \propto (v - k_B T|z'(0)|)^{-1/2} \) which is equivalent to Eq. (28). The same result is obtained for the full asymmetric six-vertex model, i.e., with both \( h \) and \( v \) being present, however, due to the complexity of the resulting equations numerical analysis is required.

In the disordered phase (rough surface), however, the free energy has the form (Eq. (316) from Lieb and Wu 1972):

\[
z(p) \propto \mu - \pi 4 \cos(\frac{\pi \Psi_0}{2\mu}) p^2 + \ldots,
\]

(31)

where \( \cos \mu \) is the anisotropy parameter of the model and \( \Psi_0 = (1 + \omega_1 e^{i\mu})/(e^{i\mu} + \omega_1) \).

In connection with Section IV it is important to emphasize the fact that no linear term appears in Eq. (31). In this part of the phase diagram (i.e., on the rough surface) Taylor expansions cannot be performed, as compared to Eq. (30), result of the Eq. (31) or (33) type are obtained directly from an integral equation.

2.) In the five-vertex model, in contrast to the symmetric six-vertex model, the arrow-reversal symmetry is broken. Again there are three different analytic forms of the exact free energy (Gulacsi, Beijeren, van and Levi 1993) corresponding to two frozen-in ferroelectric phases \( (p = \pm 1) \), one antiferroelectric phase \( (p = 0) \) (similar to the symmetric six-vertex model) and a ferrielectric \( (-1 \leq p \leq 1) \) phase. All the transitions are of second-order. From all these phases, the ferrielectric and ferroelectric phases are connected to surfaces (see later)
due to which I briefly present their critical behaviour. In this case, e.g., $p \to 1$ and the free energy in the ordered phase (smooth facets) is (Eq. (A7) and (A8) from Gulacsi, Levi and Tosatti 1994):

$$f(1 - p) \propto f(0) + k_B T \left[ \frac{\omega_1}{\omega_1 - \omega_3} \ln \left( \frac{\omega_1}{\omega_3} \right) + 2 \ln \left( \frac{\omega_3}{\omega_5} \right) \right] \frac{1 - p}{2}$$

$$- k_B T \frac{\pi^2}{6} \frac{\omega_1 \omega_3 (\omega_1 + \omega_3)}{(\omega_1 - \omega_3)^3} \ln \left( \frac{\omega_1}{\omega_3} \right) \left( \frac{1 - p}{2} \right)^3 + \ldots . \quad (32)$$

As in the six-vertex model, see Eq. (30) there is no second-order term. Close to the transition $1 - p$, $\Omega$ and the susceptibility behaves similar to the previously presented six-vertex case or Eq. (28). The free energy on the ferrielectric side (rough surface) has a particular simple form in the non-interacting, zero external field limit:

$$f(1 - p) \propto f(0) + \pi^2 k_B T \left( \frac{1 - p}{2} \right)^2 + \ldots . \quad (33)$$

while for the full formula see Eq. (21) from Gulacsi et al. (Gulacsi, Levi and Tosatti 1994).

Because of the simplicity of Eq. (33) the five-vertex model is most adequate to model the Pokrovsky-Talapov transition.

As a third example, I present briefly a variant of the Pokrovsky-Talapov transition appearing on surfaces, similar to Section V, known also as Gruber-Mullins (1967) transition. Surfaces of simple cubic crystals are described by the five-vertex model (Gulacsi, Levi and Tosatti 1994), while surfaces of body centered crystals with the six-vertex model (Beijeren, van, 1977; and Jayaprakash and Saam 1984) As presented in Section II, knowing the free energy of the surface we can determine the Cartesian coordinates, $z(x, y)$, of the corresponding surface from the Wulff construction. The free energy of the surface we obtain directly from the exact solution of the five- or six-vertex model, depending on the crystal structure. For the simple cubic surfaces, $p \to 1$ and $1 - p \propto (x - x_0)^{1/2}$, while for surface of body centered crystals $p \to 0$ and $p \propto (x - x_0)^{1/2}$. In both cases, $z - z_0 \propto (x - x_0)^{3/2}$ and the average spacing between steps $\propto (x - x_0)^{-1/2}$, equations identical to Eq. (28).

As mentioned above, a general characteristic of the (Luttinger) liquid phase preceding the Pokrovsky-Talapov transition is that its quasiparticles strongly repel each other. The
basic equation obtained for the ground-state energy in Eq. (29) can be interpreted as follows: the linear term in $n_{\text{sol.}}$ sums up the energy contributions of the non-interacting quasiparticles (solitons, holons, surface steps, etc., depending on the model under consideration). The missing quadratic term in $n_{\text{sol.}}$ shows that there is no effective attraction between the quasiparticles (see below) and the $n_{\text{sol.}}^3$ term suggests of the presence of strong repulsion. This repulsion is discussed in detail in Section [V].

As all the previous proof were given for an effective nearest-neighbour interacting systems, it is very important to realize that the Pokrovsky-Talapov transition is very stable with respect to different types of interactions. Short range interactions, with range much less than the average distance between the quasiparticles contribute to the ground-state energy by an order of $n_{\text{sol.}}^4$, thus, leaving the Pokrovsky-Talapov transition unaltered as it was proven in Section [V], where also the cases of long-range interactions was analyzed.

By the above examples we want to emphasize that the integrable models have common ground-state properties\textsuperscript{21} thus, a Pokrovsky-Talapov type transition will exist in all related models.

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\textsuperscript{21}It should be emphasized that the above mentioned mappings between different models establishes an equivalence between the eigenvalues of the different models. That is, the ground-state energies and thus, the thermodynamic properties of the different models can be made equivalent. However, non of these mapping schemes preserves the correlation functions.
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XII. FIGURE CAPTION

Fig. 1. The surface of a square (solid line) and its corresponding Wulff plot (dashes line).
