Square-free Strong Triangular Decomposition of Zero-dimensional Polynomial Systems

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ABSTRACT
Triangular decomposition with different properties has been used for various types of problem solving, e.g. geometry theorem proving, real solution isolation of zero-dimensional polynomial systems, etc. In this paper, the concepts of strong chain and square-free strong triangular decomposition (SFSTD) of zero-dimensional polynomial systems are defined. Because of its good properties, SFSTD may be a key way to many problems related to zero-dimensional polynomial systems, such as real solution isolation and computing radicals of zero-dimensional ideals. Inspired by the work of Wang and of Dong and Mou, we propose an algorithm for computing SFSTD based on Gröbner bases computation. The novelty of the algorithm is that we make use of saturated ideals and separate to ensure that the zero sets of any two strong chains have no intersection and every strong chain is square-free, respectively. On one hand, we prove that the arithmetic complexity of the new algorithm can be single exponential in the square of the number of variables, which seems to be among the rare complexity analysis results for triangular-decomposition methods. On the other hand, we show experimentally that, on a large number of examples in the literature, the new algorithm is far more efficient than a popular triangular-decomposition method based on pseudo-division. Furthermore, it is also shown that, on those examples, the methods based on SFSTD for real solution isolation and for computing radicals of zero-dimensional ideals are very efficient.

1 INTRODUCTION
Decomposing a polynomial system to finitely many triangular sets with corresponding zero decomposition, called triangular decomposition of polynomial systems, is one of the fundamental tools in computational ideal theory. Wu first proposed a triangular-decomposition algorithm for computing characteristic sets in [28], which was applied to geometry theorem proving. Since then, triangular-decomposition methods have been applied successfully to not only geometry theorem proving but also lots of problems with diverse backgrounds, such as automated reasoning, real solution isolation, real solution classification and computing the radical of a polynomial ideal [3, 8–10, 17, 28–35], to name a few. Different applications may require different types of triangular sets and triangular decomposition with specific properties (see the book [26] for reference). In the past several decades, many specific triangular sets or triangular systems have been defined, e.g. regular chains, normal chains, and square-free triangular sets. And, lots of triangular-decomposition algorithms have been proposed, see for example [1, 6, 7, 12, 13, 19, 20, 22–25, 34]. Nevertheless, classical triangular-decomposition algorithms are mainly based on factorization and pseudo-division and are well known not so efficient on big examples. Another significant but somehow neglected aspect is that, there are few results about the complexity of classical triangular-decomposition algorithms.

The Gröbner basis method, first proposed by Buchberger in [5], has been extensively studied and applied to lots of research fields. It is well known that the complexity of computing Gröbner bases can be double-exponential time in general case and single-exponential time for zero-dimensional ideals [18, 21]. There are some famous algorithms for computing Gröbner bases (see for example [14–16]) and corresponding tools are available in some computer algebra systems, e.g. Maple, Mathematica and Magma.

Then, one may ask whether there exists a connection between triangular sets and Gröbner bases and whether one can obtain triangular decomposition by Gröbner bases computation. The problem was well solved in Wang’s work [27] in 2016. The key concept is the W-characteristic set, which is a minimal triangular set extracted from a reduced Gröbner basis with respect to (w.r.t.) the lex ordering (written as reduced LEX Gröbner basis). Wang proved that if the variable ordering condition is satisfied for a W-characteristic set, then the regularity and normality of the W-characteristic set are equivalent. Later, Dong and Mou proposed some algorithms in [12, 13] for characteristic decomposition which is also a type of triangular decomposition consisting of normal chains. Owe to efficient computation of Gröbner bases, their algorithms perform
better on some complicated polynomial systems than the classical algorithms.

In this paper, we only consider zero-dimensional systems. We define a new type of triangular sets, namely strong chains (see Def. 3.1). And then, a new type of triangular decomposition, called square-free strong triangular decomposition (SFSTD) (see Def. 4.1), is introduced for real solution isolation. We observe that SFSTD can also be applied to computing radicals of zero-dimensional ideals. So, the main goal of this paper is to efficiently compute SFSTD of zero-dimensional systems. More formally, we have the following problem statement:

**Input:** A nonempty finite set $F \subseteq \mathbb{Q}[x_1, \ldots, x_n] \setminus \{0\}$.

**Output:** An SFSTD $\{T_1, \ldots, T_k\}$ of $F$, where each $T_i$ is a square-free strong chain, such that
- the zero set of $F$ equals the union of all zero sets of $T_i$, and
- any two zero sets of $T_i$ and $T_j$ ($i \neq j$) have no intersection, if $F$ is zero-dimensional; FAIL otherwise.

We list our main contributions as follows:

1. Based on [13, Algorithm 1], we propose an algorithm (Algorithm 1) for computing a type of triangular decomposition consisting of strong chains.
2. We propose an algorithm (Algorithm 3) for computing SFSTD.
3. We prove that the arithmetic complexity of Algorithm 3 can be single exponential in the square of the number of variables.
4. We implemented Algorithm 3 with Maple2021. Our experiments show that Algorithm 3 is far more efficient than the classical method for zs-rd decomposition in [3] (see the group of columns TD in Table 2).
5. By SFSTD, one can compute isolating cubes of real solutions to the system, and the radical of the ideal generated by the system, efficiently (see the groups of columns RSI and RA in Table 2).

The rest of this paper is organized as follows. In Section 2, we recall some basic concepts and existing results about the theories of Gröbner bases and triangular sets. In Section 3, we propose Algorithm 1. The termination and correctness of Algorithm 1 is guaranteed by Theorem 3.8. In Section 4, we propose Algorithm 3 and prove the termination and correctness (see Theorem 4.4). In Section 5, we analyze the arithmetic complexity of Algorithm 3. In Section 6, we present two applications of SFSTD: real solution isolation and computing the radical of a zero-dimensional ideal. In Section 7, we explain the implementation details and show the experimental results. Section 8 concludes the paper.

## 2 PRELIMINARY

In the section, we recall some basic concepts and existing results about the theories of Gröbner bases and triangular sets. The reader is referred to [2, 26] for more details.

### 2.1 Zero-dimensional Systems

Let $x_1, \ldots, x_n$ be $n$ variables and let $x$ denote the vector $(x_1, \ldots, x_n)$. Throughout the paper, we fix the variable ordering $x_1 < \cdots < x_n$. $\mathbb{Q}$ denotes the rational numbers and $\mathbb{C}$ denotes the complex numbers. Let $F$ be any polynomial set in $\mathbb{Q}[x]$. We denote by $F$ the ideal generated by $F$ in $\mathbb{C}[x]$. For any ideal $I \subseteq \mathbb{C}[x]$, $\mathbb{V}(I)$ denotes the affine variety $\{(a_1, \ldots, a_n) \in \mathbb{C} \mid f(a_1, \ldots, a_n) = 0\}$ for all $f \in I$. In particular, $\mathbb{V}(F) := \mathbb{V}(\langle F \rangle)$. For any polynomial $f \in \mathbb{Q}[x]$ with an admissible monomial ordering, $\mathbb{L}(f)$ denotes the leading term of $f$ and $\mathbb{L}(f)$ denotes the leading monomial of $f$.

### Definition 2.1

For $F \subseteq \mathbb{Q}[x]$, $F$ is a zero-dimensional system or $(F)$ is a zero-dimensional ideal, if $\mathbb{V}(F)$ is a finite set.

### Definition 2.2

Fix a monomial ordering and let $F \subseteq \mathbb{Q}[x]$. A finite subset $G = \{g_1, \ldots, g_t\} \subseteq \mathbb{Q}[x]$ is called a Gröbner basis of $(F)$ if $\langle LT(g_1), \ldots, LT(g_t) \rangle = \langle LT(F) \rangle$, where $LT((F)) = \{LT(f) \mid f \in (F)\}$.

### Proposition 2.3

Let $F \subseteq \mathbb{Q}[x_1, \ldots, x_n]$. The following statements are equivalent:

(a) $F$ is a zero-dimensional system,

(b) for every Gröbner basis $G$ of $(F),$ $G$ contains $n$ polynomials $g_1, \ldots, g_n$ such that $\mathbb{L}(g_i) = x_i^{k_i}$ ($k_i \geq 1$),

(c) there exist a monomial ordering $<$ and a Gröbner basis of $(F)$ with respect to $<$ which contains $n$ polynomials $g_1, \ldots, g_n$ such that $\mathbb{L}(g_i) = x_i^{k_i}$ ($k_i \geq 1$).

### 2.2 Triangular Sets

Let $f \in \mathbb{Q}[x] \setminus \mathbb{Q}$. We denote by $\mathbb{L}(f)$ the main variable of $f$ and by $\mathbb{L}(f)$ the initial (or leading coefficient w.r.t. $\mathbb{L}(f)$ of $f$. For any $F \subseteq \mathbb{Q}[x]$, $\mathbb{L}(f) := \{\mathbb{L}(f) \mid f \in F\}$ and $\mathbb{I}(F) := \{\mathbb{I}(f) \mid f \in F\}$. Let $T = \{T_1, \ldots, T_k\}$ be a finite nonempty list of nonconstant polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$. $T$ is called a triangular set if $\mathbb{L}(T_1) < \cdots < \mathbb{L}(T_k)$.

Let $f$ and $g$ be two polynomials in $\mathbb{Q}[x]$, $F \subseteq \mathbb{Q}[x]$ and $T = \{T_1, \ldots, T_k\} \subseteq \mathbb{Q}[x]$, be a triangular set. We denote by $\deg(f, x_i)$ the degree of $f$ w.r.t. a particular variable $x_i$ and by $\mathbb{L}(f)$ the separant of $f$, i.e. $\partial f/\partial x_i(f)$. The saturated ideal of $F$ w.r.t. $T$ is defined as $(F) := \{f^\infty \mid \{g \in \mathbb{Q}[x] \mid |\text{there exists } i \geq 0 \text{ such that } f^i g \in (F)\}$. And, we denote by $\mathbb{S}(T)$ the saturated ideal of $T$, namely $(T) := (\mathbb{I}(T_1) \cdots \mathbb{I}(T_k))$. The resultant of $f$ and $g$ w.r.t. $\mathbb{L}(g)$ is denoted by $\mathbb{R}(f, g)$, and the resultant of $f$ and $T$ is defined as $\mathbb{R}(f, T) := \mathbb{R}(\cdots \mathbb{R}(\mathbb{R}(\mathbb{R}(f, T_1), T_{i-1}), \ldots, T_1)$. $T$ is called a regular chain (or is said to be regular) if $\mathbb{L}(T_1) \neq 0$ and for each $i \leq i < l$, $\mathbb{R}(\mathbb{I}(T_i), [T_1, \ldots, T_{i-1}]) \neq 0$. $T$ is called a normal chain (or is said to be normal) if $\mathbb{L}(T)$ does not involve the main variables of $T$. It is clear that any normal chain is a regular chain, $T$ is said to be square-free if the discriminant of $T_1$ w.r.t. $\mathbb{L}(T_1)$ is not equal to 0, and for each $i \leq i < l$, $\mathbb{R}(\mathbb{S}(T_i), [T_1, \ldots, T_i]) \neq 0$.

### 2.3 W-Characteristic Sets

#### Definition 2.4

Let $F \subseteq \mathbb{Q}[x]$ and $G \subseteq \mathbb{Q}[x]$ be the reduced Gröbner basis of $(F)$ w.r.t. the lex ordering $<_{lex}$. Define $G_i := \{g \in \mathbb{G} \mid \mathbb{L}(g) = x_i\}$ for $i = 1, \ldots, n$. For every monomial $G_i$, let $g_i \in G_i$ be the polynomial such that $\mathbb{L}(g_i) <_{lex} \mathbb{L}(g)$ for any $g \in G_i \setminus \{g_i\}$. The ordered list of all $g_i$ is called the $W$-characteristic set of $F$.

#### Theorem 2.5

Let $C = [C_1, \ldots, C_k]$ be the $W$-characteristic set of $F \subseteq \mathbb{Q}[x_1, \ldots, x_n]$, where $1 \leq i \leq n$. We have:

(a) $\mathbb{C} \subseteq (F) \subseteq \mathbb{L}(\mathbb{C})$,

(b) if the variable ordering condition is satisfied for a $W$-characteristic set $\mathbb{C}$, if the variables in $\{x_1, \ldots, x_n\} \setminus \{\mathbb{C}\}$ are ordered before $\mathbb{L}(\mathbb{C})$.

### 3 STD

In order to compute SFSTD, we need to compute a type of triangular decomposition consisting of strong chains first. And, the algorithm
3.1 The Definition of STD

Definition 3.1. Let \( T = \{ T_1, \ldots, T_n \} \subseteq \mathbb{Q}[x_1, \ldots, x_n] \) be a triangular set. \( T \) is called a strong chain (or is said to be strong) if \( \text{ini}(T) \subseteq \mathbb{Q} \setminus \{ 0 \} \) and \( \text{lv}(T_i) = x_i \) for \( i = 1, \ldots, n \). A strong chain is said to be reduced, if for each \( i \) (\( 1 \leq i \leq n \)), \( \text{ini}(T_i) = 1 \) and \( \text{deg}(T_i, x_i) > \text{deg}(T_j, x_j) \) where \( j > i \).

Any strong chain is a normal chain. And, strong chains have very good properties.

Proposition 3.2. Fix the lex monomial ordering and let \( T = \{ T_1, \ldots, T_n \} \subseteq \mathbb{Q}[x] \) be a strong chain. Then,

(a) \( \text{sat}(T) = \langle \text{deg}(T) \rangle \),
(b) \( \text{LM}(T_i) = x_i^k \ (k \geq 1) \) for \( i = 1, \ldots, n \),
(c) \( T \) is a LEX Gröbner basis, and
(d) \( T \) is a zero-dimensional system.

Further, \( T \) is reduced if and only if \( T \) is a reduced LEX Gröbner basis.

Proof. (a) It is because \( \text{ini}(T) \subseteq \mathbb{Q} \setminus \{ 0 \} \). (b) It is clear. (c) By (b), for any \( i \neq j \), the greatest common divisor of \( \text{LM}(T_i) \) and \( \text{LM}(T_j) \) is \( 1 \). Then, by [2, Lemma 5.66], we complete the proof. (d) It is obvious by (b), (c) and Proposition 2.3. \( \square \)

Definition 3.3. Let \( F \subseteq \mathbb{Q}[x] \) be a zero-dimensional system. A strong triangular decomposition (STD) of \( F \) is a finite set of strong chains \( \{ T_1, \ldots, T_n \} \subseteq \mathbb{Q}[x] \) such that

\[
\text{V}(F) = \bigcup_{i=1}^{n} \text{V}(T_i) \text{ and } \text{V}(T_i) \cap \text{V}(T_j) = \emptyset \text{ for any } i \neq j.
\]

3.2 Computing STD

3.2.1 The Algorithm

Given a nonempty finite polynomial set \( F \subseteq \mathbb{Q}[x] \setminus \{ 0 \} \), if \( F \) is zero-dimensional, then Algorithm 1 computes an STD of \( F \). (Note that when \( \text{V}(F) = \emptyset \), the output is an empty set.) Otherwise, the output is FAIL. The process of Algorithm 1 is as follows.

Let \( \Phi \) be a set of polynomial sets for STD (initialized as \( \{ F \} \)), and \( \text{ans} \) be a set of computed strong chains (initialized as \( \emptyset \)). In the while loop, for the first time, we pick \( F \) and compute the reduced LEX Gröbner basis \( G \) of \( F \). If there exists \( x_i \) such that for any \( g \in G \), \( \text{LM}(g) \neq x_i^k \ (k \geq 1) \), then \( F \) is not zero-dimensional. Otherwise, \( F \) is zero-dimensional. We extract the W-characteristic set \( \mathcal{C} \) of \( G \). Suppose that \( \mathcal{C} = \{ C_1, \ldots, C_m \} \), where \( m \leq n \).

I. If \( \mathcal{C} \) is a strong chain, then we add \( \mathcal{C} \) to \( \text{ans} \).
II. If \( \mathcal{C} \) is not strong, then we compute \( \kappa \) which is the smallest integer that makes \( \{ C_1, \ldots, C_{\kappa-1} \} \) not strong. Let \( G_{\text{sat}} \) be the reduced Gröbner basis of \( \{ C_1, \ldots, C_{\kappa-1} \} \) : \( \text{ini}(C_k) \) \( \kappa \) w.r.t. any monomial ordering. We update \( \Phi \) with \( G \cup \{ \text{ini}(C_k) \} \) and \( G \cup G_{\text{sat}} \).

For the \( i \)-th time (\( i \geq 2 \)), we pick \( F \) from \( \Phi \) and compute the W-characteristic set \( \mathcal{C} \) of \( F \). Then, repeat (I) and (II) in the above paragraph.

Algorithm 1: STD

Input: a nonempty finite set \( F \subseteq \mathbb{Q}[x] \setminus \{ 0 \} \) and the vector \( \Phi \)
Output: \( \text{ans} = \{ T_1, \ldots, T_n \} \), a finite set of strong chains such that

\[
\text{V}(F) = \bigcup_{i=1}^{n} \text{V}(T_i) \text{ and } \text{V}(T_i) \cap \text{V}(T_j) = \emptyset \text{ for any } i \neq j,
\]

\( T_i\text{_{Cans}} \) if \( F \) is zero-dimensional, FAIL otherwise.

1. \( \text{ans} \leftarrow \emptyset \), \( \Phi \leftarrow \{ F \} \), \( \text{num} \leftarrow 0 \)
2. while \( \Phi \neq \emptyset \) do.
3. choose \( P \) from \( \Phi \) and set \( \Phi \leftarrow \Phi \setminus \{ P \} \)
4. \( G \leftarrow \text{the reduced LEX Gröbner basis of } \{ P \} \)
5. if \( G \neq \{ 1 \} \) then.
6. if \( \text{num} = 1 \) then there exists \( x_i \) such that for any \( g \in G \), \( \text{LM}(g) \neq x_i^k \ (k \geq 1) \) then
7. return FAIL.
8. \( \mathcal{C} \leftarrow \{ C_1, \ldots, C_m \} \) (\( m \leq n \)) which is the W-characteristic set of \( G \)
9. \( \text{ans} \leftarrow \text{ans} \cup \{ \mathcal{C} \} \)
10. else \( \mathcal{C} \) is a strong chain then.
11. \( \text{ans} \leftarrow \text{ans} \cup \{ \mathcal{C} \} \)
12. else \( C_k \leftarrow \text{the first polynomial of } \mathcal{C} \) \( \mathcal{C} \) not strong
13. \( G_{\text{sat}} \leftarrow \text{the reduced Gröbner basis of } \{ C_1, \ldots, C_{\kappa-1} \} : \text{ini}(C_k) \) \( \kappa \) w.r.t. any monomial ordering.
14. \( \Phi \leftarrow \{ G \cup \{ \text{ini}(C_k) \} \} \cup \{ G \cup G_{\text{sat}} \} \)
15. return \( \text{ans} \).
**Proof.** Let \( V_1 := V(T) \) and \( V_2 := V(T) \setminus V(f) \). Since \( T \) is a zero-dimensional system by Proposition 3.2 (d), \( V_2 \) is an affine variety. By [32, Theorem 2.2], \( \text{res}(f, T) = 0 \) if and only if \( V(T) \cap V(f) \neq \emptyset \), i.e., \( V_1 \neq \emptyset \) and \( V_2 \) is a proper subset of \( V_1 \). Then, it is equivalent to that there exists \( g \in \mathcal{I}(V_2) \setminus \mathcal{I}(V_1) \). Note that \( fg \in \mathcal{I}(V_1) \). Then, there exists some integer \( k_2 \geq 1 \) such that \( (fg)^{k_2} \in \langle T \rangle \). Let \( k_2 \) be the smallest integer number such that \( f^{k_2}g^{k_1} \in \langle T \rangle \). Since \( g \notin \mathcal{I}(V_1) \), \( k_2 \geq 1 \). Thus, we take \( g = f^{k_2}g^{k_1} \) which is not in \( \langle T \rangle \). \( \square \)

**Theorem 3.8.** Algorithm 1 terminates correctly.

**Proof.** (Correctness) For the input polynomial set \( F \), let \( G_0 \) be the reduced LEX Gröbner basis of \( F \). By Proposition 2.3, if there exists \( x_i \) such that for any \( g \in G_0 \), \( \text{LM}(g) \neq x_i^k \) \((k \geq 1)\), then \( F \) is not zero-dimensional. Algorithm 1 returns FAIL in Line 8.

Otherwise, \( F \) is zero-dimensional. In every loop, we pick a polynomial set \( P \) from \( \Phi \). Let \( G = (G \neq \{1\}) \) be the reduced LEX Gröbner basis of \( (P) \) and \( C = \{C_1, \ldots, C_m\} \) \((m \leq n)\) be the \( W \)-characteristic set of \( G \). If \( C \) is strong, then we add \( C \) to the output set in Line 12. If \( C \) is not strong, \( \Phi \) is updated with two sets in Line 16. Therefore, we only need to prove that

1. \( \forall g \in G \), where \( C \) is the strong \( W \)-characteristic set.
2. \( (G \cup \{\text{ini}(C_k)\}) \cap \forall g \in G \setminus \text{sat} = \emptyset \) and \( \forall g \in G = (G \cup \{\text{ini}(C_k)\}) \cup \forall g \in G \cap \text{Sat} \), where \( C_k \) is the first polynomial that makes \( \{C_1, \ldots, C_k\} \) not strong and \( G \setminus \text{Sat} \) is a Gröbner basis of \( \{C_1, \ldots, C_k\} : \text{ini}(C_k)^\infty \).

By Lemma 3.5, (I) is clear. It remains to prove (II). Since \( \{C_1, \ldots, C_k\} \) is strong, by Proposition 3.2 (d), \( \{C_1, \ldots, C_k\} \) is zero-dimensional. So, by Proposition 3.6, \( (G \cup \text{Sat}) = (G \cap \{\text{ini}(C_k)\}) \cup (G \cup \text{Sat}) = (G \cap \{\text{ini}(C_k)\}) \cap (G \cup \{\text{ini}(C_k)\}) \). Then, \( (G \cup \{\text{ini}(C_k)\}) \) is strictly larger than \( (G) \). It remains to prove \( \forall g \in G \cap \{\text{ini}(C_k)\} \).

Firstly, we prove that \( C = \{C_1, \ldots, C_n\} \) with \( 1 \forall g \in G \) is zero-dimensional, by the proof of the correctness, every polynomial set in \( \Phi \) is also zero-dimensional. Then, by Proposition 2.3 and Definition 2.4, we have \( 1 \forall g \in G \).

Secondly, we prove that \( \text{res}(\text{ini}(C_k)), \{C_1, \ldots, C_k\} = 0 \). Note that \( C_k \) is the first polynomial that makes \( \{C_1, \ldots, C_k\} \) not strong. Then, \( C \) and \( \{C_1, \ldots, C_k\} \) are not normal, but \( \{C_1, \ldots, C_k\} \) is normal. Since \( 1 \forall g \in G \), \( \{x_1, \ldots, x_n\} \), the variable ordering condition is satisfied for \( C \). So, by Theorem 2.5 (b), \( \{C_1, \ldots, C_k\} \) is not regular. Then, \( \text{res}(\text{ini}(C_k)), \{C_1, \ldots, C_k\} = 0 \).

Finally, we prove \( (G) \subseteq (G \cup \text{Sat}) \), which is equivalent to proving that there exists \( g \in (G \setminus \text{sat}) \). Note that \( \{C_1, \ldots, C_k\} \) is a strong chain in \( \{x_1, \ldots, x_n\} \). So, by the conclusion in the above paragraph and by Proposition 3.7, there exists \( g \in \mathcal{C}[x_1, \ldots, x_n] \setminus \{\langle C_1, \ldots, C_k \rangle \} \) such that \( g \in \text{ini}(C_k) \). Recall that \( G = \{C_1, \ldots, C_k\} : \text{ini}(C_k)^\infty \). So, we have \( g \in (G \cap \{\text{ini}(C_k)\}) \).

**Corollary 3.9.** If \( \{T_1, \ldots, T_s\} \) is an STD of a zero-dimensional system \( F \subseteq \mathbb{Q}[x] \) computed by Algorithm 1, then for each \( i, T_i \) is reduced and \( (F) \subseteq \langle T_i \rangle \).

**Proof.** It is clear by the proof of Theorem 3.8, by Proposition 3.2 and by Lemma 3.5.

**Remark 3.10.** Except for the following two aspects, Algorithm 1 is similar to [13, Algorithm 1]. In order to guarantee the zero sets to be pairwise disjoint, Algorithm 1 computes \( \{C_1, \ldots, C_k\} : \text{ini}(C_k)^\infty \) in Line 15 instead of the ideal quotient of \( \{C_1, \ldots, C_k\} \) by \( \text{ini}(C_k) \) (see [13, Algorithm 1-Line 19]). And Algorithm 1 can detect whether the input system is zero-dimensional.

## 4 SFSTD

A popular method for computing square-free/regular chains is the method of relatively simplicial decomposition (see [32, Chapter 2] for more details), which is based on subresultant computation and pseudo-division. In this section, we discuss how to compute SFSTD by means of Gröbner bases. We propose Algorithm 2 for SFSTD of strong chains in Section 4.1 and Algorithm 3 for SFSTD of general zero-dimensional systems in Section 4.2.

**Definition 4.1.** Let \( F \subseteq \mathbb{Q}[x] \) be a zero-dimensional system. A square-free strong triangular decomposition (SFSTD) of \( F \) is an STD \( \{T_1, \ldots, T_s\} \) of \( F \), where \( T_i \) is square-free for \( i = 1, \ldots, s \).

### 4.1 Computing SFSTD of Strong Chains

Algorithm 2 computes an SFSTD of any strong chain \( T \subseteq \mathbb{Q}[x_1, \ldots, x_n] \). Let \( \Phi \) be a set of strong chains for SFSTD (initialized as \( \{T\} \)), and \( \Psi \) be a set of computed square-free strong chains (initialized as \( \emptyset \)). Every loop step, we pick a strong chain \( \Psi = \{P_1, \ldots, P_n\} \) from \( \Phi \) and remove it from \( \Phi \), until \( \Phi \) is empty.

1. If \( \Psi \) is square-free, then we add \( \Psi \) to \( \Phi \).
2. If \( \Psi \) is not square-free, then we compute \( k \) \((k \geq 1)\) which is the smallest integer such that \( \{P_1, \ldots, P_k\} \) is not square-free. If \( k = 1 \), then \( \Phi \) is updated with the strong chains \( \{\xi_1, P_2, \ldots, P_k\} \), \( \ldots, \{\xi_m, P_2, \ldots, P_n\} \), where \( \xi_i \) is an irreducible factor of \( P_1 \). If \( k > 1 \), we compute the reduced Gröbner basis \( G_{\text{sat}} \) of \( \{P_1, \ldots, P_k\} : \text{sep}(P_k)^\infty \) w.r.t. any monomial ordering. And, by Algorithm 1, we compute an STD of \( \Psi \cup \{\text{sep}(P_k)\} \) and an STD of \( \Psi \cup G_{\text{sat}} \). \( \Phi \) is updated with the strong chains in the two STD.

Algorithm 2 is illustrated on the following example.

**Example 4.2.** Consider the strong chain \( T = \{T_1, T_2\} = \{x^2 - 1, y^2 - 2xy + 1\} \subseteq \mathbb{Q}[x, y] \) with \( x < y \). \( T \) is not square-free and \( T_2 \) is the first polynomial that makes it not square-free. Since \( \text{sep}(P_2)^\infty = \{1\} \) where \( \text{sep}(P_2) = 2y - 2x \), we only compute an STD of \( \Psi \cup \{\text{sep}(P_k)\} \) by Algorithm 1, which is \( \{0, y \} \). \( \Phi \) is updated with the strong chain \( \{x^2 - 1, y - x\} \). Because the strong chain is square-free, Algorithm 3 terminates with \( \Phi = \emptyset \) and \( \{x^2 - 1, y - x\} \) is an SFSTD of \( T \).

**Theorem 4.3.** Algorithm 2 terminates correctly.
4.2 Computing SFSTD

For any zero-dimensional system \( F \subseteq \mathbb{Q}[\bar{x}] \), Algorithm 3 computes an SFSTD of \( F \). The process of the computation is as follows. We first compute an STD of \( F \) by Algorithm 1 and then compute an SFSTD of every strong chain in the STD by Algorithm 2. The union of all SFSTDs is an SFSTD of \( F \). If the input system is not zero-dimensional, Algorithm 3 returns FAIL in Line 3.

**Theorem 4.4.** Algorithm 3 terminates correctly.

**Proof.** It is obvious by Theorem 3.8 and Theorem 4.3. \( \square \)

## 5 ARITHMETIC COMPLEXITY ANALYSIS

In the section, we analyze the complexity of our algorithms. Here, we only consider arithmetic complexity which counts the number of field operations (not bit operations). We first introduce the concept of multiplicity and some results we will use later.

For any point \( p = (a_1, \ldots, a_n) \in \mathbb{C}^n \), we denote by \( C[x]_p \) the set \( \left\{ f \mid f, g \in C[x], g(a_1, \ldots, a_n) \neq 0 \right\} \). For any ideal \( I \subseteq C[x] \), we denote \( \dim(I) \) the dimension of the \( C \)-vector space \( C[x]/I \). For any \( F \subseteq C[x] \), define \( \deg(F) := \dim(F \cap F) \) and denote by \( \deg(P) \) the maximum degree of elements of \( P \).

**Definition 5.1 ([11, Chap. 4, Def. 2.1]).** Let \( I \subseteq C[x] \) be a zero-dimensional ideal and \( p \in \mathfrak{V}(I) \). The multiplicity \( m_p \), denoted \( M_p(I) \), is the dimension of the \( C \)-vector space \( C[x]/pC[x]_p \).

**Theorem 5.2 ([11, Chap. 4, Cor. 2.5]).** Let \( I \subseteq C[x] \) be a zero-dimensional ideal. We have \( \dim(I) = \sum_{\alpha \in \mathfrak{V}(I)} M_p(\alpha) \).

**Theorem 5.3.** (Prop. 8.1) \([21, Thm. 3]\). Let \( P \subseteq C[x_1, \ldots, x_n] \) be a zero-dimensional system, and \( G \) be the reduced Gröbner basis of \( P \) w.r.t. any monomial ordering. Then,

(a) \( \deg(G) \leq \deg(P) \)

(b) the arithmetic complexity of computing \( G \), denoted \( \text{CGB}(n, \deg(P)) \), can be polynomial in \( \deg(P)^n \).

Remark that in Theorem 5.3, \( \text{CGB}(n, \deg(P)) \) depends on algorithms used for computing Gröbner bases.

### 5.1 Complexity of Algorithm 1

Let \( F \subseteq \{x_1, \ldots, x_n\} \) be a zero-dimensional system, \( d := \deg(F) \) and \( D := \max(d, \dim(F)) \) in Section 5.1 and Section 5.2. For convenience, we assume \( \mathfrak{V}(F) \neq \emptyset \).

**Remark 5.4.** By Bézout’s theorem, \( D \leq d^n \).

**Theorem 5.5.** Algorithm 1 computes an STD of \( F \) with the arithmetic complexity

\[
(2 \dim(F) - 1) \cdot (\text{CGB}(n, D) + \text{CGB}(n, \deg(F))),
\]

which can be polynomial in \( D^n \).

In order to prove Theorem 5.5, we prepare some lemmas. Given the input \( F \), suppose that Algorithm 1 terminates with \( N (N \geq 1) \) times of loops. Let \( P_i \) be the picked polynomial set in the \( i \)-th loop. Consider a binary tree \( T \) with \( P_1, \ldots, P_N \) as nodes. If \( \{P_i\} \neq \{1\} \) and the W-characteristic set of \( P_i \) is not strong, the node
Lemma 5.7. The number of leaves of tree $T$ is at most $\dim(f)$. The number of nodes of tree $T$ is at most $2\dim(f) - 1$, i.e., $N \leq 2\dim(f) - 1$.

Proof. Note that $\mathcal{V}(P_i) \geq 1$ for $i = 1, \ldots, N$. Then, by (4) of Lemma 5.7, the number of leaves is at most $\dim(f)$. Note that every node is either a leaf or has two child nodes. So, the number of nodes is at most $2\dim(f) - 1$.

Proof of Theorem 5.5. By Lemma 5.9, the number of times of loops $N \leq 2\dim(f) - 1$. In each loop, the most complicated computation is Gröbner bases computation in Line 5 and Line 15. By Lemma 5.8 (b), the computation in Line 5 has the complexity $\mathcal{O}(n^d)$. Note that the saturated ideal in Line 15 is equal to $\langle C_1, \ldots, C_{k-1}, 1 - r \text{ini}(C_i) \rangle \cap \mathcal{C}[x_1, \ldots, x_{k-1}]$, where $r$ is a new variable. So, by Lemma 5.8 (a), the complexity of the computation in Line 15 is $\mathcal{O}(\mathcal{G}(k, \dim(f)))$. Note that $k \leq n$. Thus, (1) is proved. By Theorem 5.3 (b), (1) can be polynomial in $d^m$.

5.2 Complexity of Algorithms 2 & 3

Theorem 5.10. An SFSTD of a reduced strong chain $T \subseteq \mathbb{Q}[x]$ can be computed by Algorithm 2 within a complexity of polynomial in $\dim(T)^n$. An SFSTD of $F \subseteq \mathbb{Q}[x]$ can be computed by Algorithm 3 within a complexity of polynomial in $d^{m^2}$.

To prove Theorem 5.10, we prepare some lemmas first.

Given a reduced strong chain $T$, suppose that Algorithm 2 terminates with $M (M \geq 1)$ times of loops. Let $P_i$ be the picked reduced strong chain in the $i$-th loop (see Corollary 3.9). Consider a tree $T$ with some nodes $P_1, \ldots, P_M$ and some other nodes (1). If $P_i$ is not square-free, then the node $P_i$ has one or more reduced strong chains in Line 11 or Line 14 as child nodes. Otherwise, the node $P_i$ has no child node. If $P_i$ has only one child node, let the set (1) be its second child node. The root node of tree $T$ is $P_1 = T$.

We also define a value of every node $P_i$ as in (2).

Lemma 5.11. For every node $P$ of tree $\bar{T}$, $\deg(P) \leq \dim(P)$.

Proof. If $P = \{1\}$, the conclusion is clear. Otherwise, $P$ is a reduced strong chain. Then, by Proposition 3.2 and Theorem 5.3 (a), we complete the proof.

Lemma 5.12. For the tree $\bar{T}$, we have $\sum_{P_i \in \text{child}(P_i)} \mathcal{V}(P_i) \leq \dim(P_i)$ and $\sum_{P_i \text{ is a leaf}} \mathcal{V}(P_i) \leq \dim(\bar{T})$.

Proof. Note that in Line 11, $(P_1, \ldots, P_n) = (1, 2, \ldots, n) \cup \mathcal{V}(P_1, \ldots, P_n)$ for $i = 1, 2, \ldots, n$. Then, it is similar to the proof of Lemma 5.7.

Lemma 5.13. The number of nodes of tree $\bar{T}$ is at most $2\dim(T) - 1$ which implies $M \leq 2\dim(T) - 1$.
We implemented Algorithm 3, the methods NRSI and IRA with \(\tau\) we compute the isolating cubes of every square-free strong chain for computing Gröbner bases in Algorithm 1–Line 5&Line 15 and Algorithm 2–Line 13.

**Proof.** Note that if a node is not a leaf, then it has at least two child nodes. Then, it is similar to the proof of Lemma 5.9. \(\square\)

**Proof of Theorem 5.10.** Firstly, we analyze the complexity of Algorithm 2. By Lemma 5.13, the number of loop steps \(M \leq 2 \dim(T) - 1\). In each loop, the most complicated computation is in Line 13 and Line 14. Note that in Line 13, \((G_{sat}) = (P_1, \ldots, P_k - r \sep(P_k) \cap \bigcap_{i=1}^k x_i)\), where \(k \leq n\). \(r\) is a new variable and \(G_{sat}\) is a reduced Gröbner basis. So, by Lemma 5.11, the complexity of computing \(G_{sat}\) is \(\text{CB}(n+1, \dim(P))\), where \(P\) is the reduced strong chain chose in Line 3. By Theorem 5.3 (b), it can be polynomial in \(\dim(P)^n\). It remains to analyze the complexity of computation in Line 14. Similar to the proof of Lemma 5.8, \(\dim(P) \leq \dim(T)\) and \(\deg(G_{sat}) \leq \dim(P)\). And by Lemma 5.11, \(\deg(P) \leq \dim(P)\). Thus, it is clear that \(\deg(P \cup \{\sep(P_k)\})\), \(\dim(P \cup \{\sep(P_k)\})\), \(\deg(P \cup G_{sat})\) and \(\dim(P \cup G_{sat})\) are all less than or equal to \(\dim(T)\). Then, by Theorem 5.5, the complexity can be polynomial in \(\dim(T)^n\). Therefore, after multiplying M, the complexity of Algorithm 3 can still be polynomial in \(\dim(T)^n\).

Secondly, we analyze the complexity of Algorithm 3. By Theorem 5.5, the complexity of the calculation in Line 1 can be polynomial in \(D^n\). Suppose the reduced strong chains (see Corollary 3.9) computed in Line 1 are \(T_1, \ldots, T_k\). By Lemma 5.9, \(t \leq \dim(F)\). By (4) of Lemma 5.7, \(\dim(T_i) \leq \text{Value}(T_i) \leq \dim(F)\). Thus, by the conclusion in the above paragraph, the complexity of the calculation in Line 5 can be polynomial in \(\dim(F)^n\). Then, the complexity of Algorithm 3 can be polynomial in \(\max(D^n, \dim(F)^n) = D^n\). Since \(D < d^n\) (see Remark 5.4), the complexity can be polynomial in \(d^{n^2}\). \(\square\)

Recall that Theorem 5.5 and Theorem 5.10 talk about arithmetic complexity without analyzing the growth of the size of coefficients. In fact, the size of the coefficients in the algorithms may increase very fast.

### 6 TWO APPLICATIONS OF SFSTD

In the section, we present two applications of SFSTD: real solution isolation and computing radicals. Given a zero-dimensional system \(F \subseteq \mathbb{Q}[x]\), we first compute an SFSTD \(\{T_1, \ldots, T_k\}\) of \(F\) by Algorithm 3.

In order to compute the isolating cubes of real solutions of \(F\), we compute the isolating cubes of every square-free strong chain \(T_i\) by [33, Algorithm NREALZERO]. The method is called NRSI in Section 7.

We denote by \(\sqrt{\mathcal{I}}\) the radical of an ideal \(I \subseteq \mathbb{C}[\bar{x}]\). We claim that \(\sqrt{\mathcal{I}} = \bigcap_{i=1}^k \mathcal{T}_i\). The proof of the claim is as follows. Since \(\{T_1, \ldots, T_k\}\) is an SFSTD of \(F\), we have \(\forall(T) = \bigcup_{i=1}^k \forall(T_i)\). Then, \(\forall(T) = \forall(\bigcap_{i=1}^k \mathcal{T}_i)\). Note that every \(\mathcal{T}_i\) is a square-free strong chain. Then, by [4, Corollary 3.3] and by Proposition 3.2 (a), \(\mathcal{T}_i\) is radical. Thus, we complete the proof. The method for computing \(\sqrt{\mathcal{I}}\) by the intersection of ideals is called IRA in Section 7.

### 7 EXPERIMENTS

We implemented Algorithm 3, the methods NRSI and IRA with Maple2021, where we use the Maple command Groebner[Basis] for computing Gröbner bases in Algorithm 1–Line 5&Line 15 and Algorithm 2–Line 13.

In the section, we explain implementation details and show the experimental results of partial testing examples. All testing examples, code and experimental results are available online via: https://github.com/lihaokun/StrongSfTriDec. All tests were conducted on 16-Core Intel Core i7-12900KF@3.20GHz with 128GB of memory and Windows 11.

#### 7.1 Description of the Experimentation

Testing examples are collected from the literatures [3, 23, 33] and the website http://homepages.math.uic.edu/~jan/demo.html. We just get rid of the ones that are repeated or not zero-dimensional. Owing to space constraints, we only present 44 “difficult” examples (total 151 examples) in Table 1. Timings are in seconds. ‘OT’ means out of the timing 3600 seconds, and “LOSS” means kernel connection lost during calculation of Maple. The column “sys” denotes the name of the polynomial system. The column “n/d” stands for the number of variables/maximum degree of elements in the system.

We record the time to compute triangular decomposition by Algorithm 3 (see the column Algorithm 3) and two Maple commands (see the column mp-rc) in the group of columns TD. The two commands used are RegularChains[Triangularize] with the options output=lazard and radical=yes, and RegularChains [ChainTools][SeparateSolutions]. The first one, which is also used in [3], decomposes the system to a finite number of square-free regular chains. The second one ensures the zero sets of any two regular chains have no intersection. In fact, since a strong chain is a regular chain, SFSTD is stronger than such decomposition.

In the group of columns RSI, we record the time of real solution isolation computed by the method NRSI (see the column NRSI), the Mathematica12 command Solve (see the column mt-solve) and the Maple command RootFinding[Isolate] (see the column mp-rt).

In the group of columns RA, we record the time to compute radicals by the method IRA (see the column IRA) and the Maple command PolynomialIdeals[Radical1] (see the column mp-radical).

#### 7.2 Statistical Experimental Results

We show the statistical experimental results of all 151 examples in Table 2. The number of examples that can be solved within 3600 seconds is recorded in the row Solved. The number of LOSS (OT) examples is recorded in the row LOSS (OT). We record the sum of the computing time of all solved examples (written as solved time) in the row Time (Solved). For every LOSS or OT example, we record their computing time as 3600 seconds. And, the sum of the computing time of all examples is recorded in the row Time.

For triangular decomposition, Algorithm 3 performs significantly better than mp-rc. There are 31 examples which can only be solved by Algorithm 3. And the solved time of Algorithm 3 is a half of that of mp-rc. One main reason why Algorithm 3 performs better is that the outputs of Algorithm 3 usually have less components. Denote by \(m_1\) and \(m_2\) the numbers of components computed by Algorithm 3 and mp-rc on the same example, respectively. We observe that \(m_1 < m_2\) for 61 examples. Especially, for 54 of those 61 examples, we have \(m_1 \leq \frac{1}{2} m_2\). On the contrary, there are no examples where \(m_1 > m_2\). And \(m_1 = m_2\) for 54 examples where \(m_1 = m_2 = 1\) for 39 examples.

For real solution isolation, mt-solve performs better than NRSI on small examples which can be solved in 2 seconds, but the solved
time of NRSI is approximately 450 seconds less than that of mtsolve. And, there are 5 difficult examples solved successfully by NRSI which cannot be solved by mtsolve. NRSI solves 3 examples that mp-r does not, while mp-r solves 2 examples that NRSI does not. However, the solved time of NRSI is approximately 10000 seconds less than that of mp-r. It is worth noting that for the system katsura8, the computing time of NRSI is approximately 3 times that of mp-radical. Since it is difficult to compute intersections of ideals (see the systems redcyc7 and kss3), IRA does not perform as well as we expect.

8 CONCLUSION
In the paper, we propose an algorithm for computing SFSTD and prove that the arithmetic complexity can be single exponential time (note that there are few results about the complexity of triangular-decomposition algorithms). Our algorithm is partly inspired by [13, Algorithm 1] and thus is based on Gröbner bases. The novelty of our algorithm is that we make use of separant and saturated ideals to ensure that every strong chain is square-free and the zero sets of any two strong chains have no intersection, respectively. It is worth noting that although SFSTD is stronger than zs-rc decomposition in [3], our algorithm is much more efficient than the classical method in experiments. The only disadvantage of our algorithm is that a computed SFSTD of a big system always has huge coefficients. So, it sometimes takes a large amount of time to compute isolating cubes of every square-free strong chain or compute intersections of ideals. We will consider giving a bit complexity analysis of our algorithm in the future.

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