Conformal quantum mechanics and Fick-Jacobs equation

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Abstract

It is found a relation between conformal quantum mechanics and Fick-Jacobs equation, which describes diffusion in channels. This relation is given between a family of channels and a family of conformal Hamiltonians. In addition, it is shown that a conformal Hamiltonian is associated with two channels with different geometry. Furthermore, exact solutions for Fick-Jacobs equation are given for this family of channels.

1 Introduction

Recently, mathematical techniques developed in an area has been employed to study systems from other different areas. In this subject, an amazing result is given by $AdS_{d+1}/CFT_d$ duality, which allows a relation between $(d + 1)$-dimensional gravitational theory and certain classes of $d$-dimensional Yang-Mills theories [1]. The conformal group is very important in this duality, in fact this group is the the largest symmetry group of special relativity.

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Now, the Schrödinger group is a non-relativistic conformal group \[3, 4\]. This last group is the symmetry group for the free Schrödinger equation and has been important to study non-relativistic \(AdS_{d+1}/CFT_d\) duality \[5, 6\]. To study \(AdS_2/CFT_1\) correspondence, the so-called conformal quantum mechanics has been proposed as \(CFT_1\) dual to \(AdS_2\), see \[7, 8\]. The conformal quantum mechanics is invariant under Schrödinger group and has been employed to study problems from black-holes to atomic physics \[9, 10, 11\]. Furthermore, the simplest model of diffusion is described by the Fick equation and Sophus Lie showed that this equation is invariant under Schrödinger group \[12\]. Other studies about diffusion phenomena and Schrödinger group can be seen in \[13\]. Then, the conformal symmetry, relativistic or non-relativistic, is very important to understand diverse aspects of different systems.

Now, when the diffusion is in a channel, which has the shape of surface of revolution with cross sectional area \(A(x)\), the Fick equation has to be changed to Fick-Jacobs equation \[17\]

\[
\frac{\partial C(x,t)}{\partial t} = \frac{\partial}{\partial x} \left[ D_0 A(x) \frac{\partial}{\partial x} \left( \frac{C(x,t)}{A(x)} \right) \right],
\]

(1)

where \(C(x,t)\) is the particle concentration and \(D_0\) is the diffusion coefficient. This last equation is important to study diffusion in biological channels or zeolites \[18, 19, 20, 21, 22, 23, 24\]. The Fick-Jacobs equation does not look like the free Schrödinger equation, but it can be mapped to Schrödinger equation with an effective potential \[25\].

In this paper we will show that the Fick-Jacobs equation is equivalent to conformal quantum mechanics for a family of channels. Then for this family of channels the Fick-Jacobs equation is invariant under Schrödinger group. Also, it is found that the equivalence is given between a family of channels and a family of conformal Hamiltonians. In addition, it is shown that a conformal Hamiltonian is associated with two channels with different geometry. For these channels an exact solution for the Fick-Jacobs equation is given.

This paper is organized in the following way: in section 2 a brief review about Schrödinger group and conformal quantum mechanics is given; in section 3 it is shown that the Fick-Jacobs equation is equivalent to conformal quantum mechanics for a set of particular channels and an exact solution for this equation is given. Finally, in section 4 a summary is given.
2 Schrödinger group

The free Schrödinger equation
\[
i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{x}, t), \tag{2}\]
is invariant under the following transformation: Galileo transformation \(x'_i = x_i + v_i t\), rotations \(x'_i = R_{ij} x_j\), space-time translation \(t' = t + a, x'_i = x_i + x_0 t\), anisotropic scaling \(t' = b^2 t, x'_i = bx_i\) and special conformal transformation \([3, 4]\)
\[
t' = \frac{t}{1 + at}, \quad x'_i = \frac{x_i}{1 + at}. \tag{3}\]

Some work about Schrödinger group and conformal symmetry can be seen in [26, 27, 28, 29, 30, 31, 32, 33].

Now, the Schrödinger equation for the 1-dimensional conformal quantum mechanics is given by
\[
i\hbar \frac{\partial \psi(x, t)}{\partial t} = H \psi(x, t), \quad H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + g \frac{x^2}{2}, \tag{4}\]
which is invariant under Schrödinger transformation. The classical system with the potential \(V(r) = gr^{-2}\) was first studied by Jacobi [34] and the quantum system was proposed by Jackiw [12]. Using the Schrödinger group generators, the spectrum of Hamiltonian [4] was found by de Alfaro, Fubini and Furlan [16]. The Hamiltonian [1] appears in different contexts, from black-holes to atomic physics [9, 10, 11]. In the next section we will show that this systems also appears in diffusion phenomena.

3 Conformal quantum mechanics and Fick-Jacobs equation

Using \(C(x, t) = \sqrt{A(x)} \psi(x, t)\), the Fick-Jacobs equation becomes
\[
\frac{\partial \psi(x, t)}{\partial t} = \left[ D_0 \frac{\partial^2}{\partial x^2} - \frac{D_0}{2\sqrt{A(x)}} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{A(x)}} \frac{\partial A(x)}{\partial x} \right) \right] \psi(x, t). \tag{5}\]
Then, if we propose \( \psi(x,t) = e^{-Et} \phi(x) \), we get the following Schrödinger equation
\[
E \phi(x) = H \phi(x),
\] (6)
where
\[
H = -D_0 \frac{\partial^2}{\partial x^2} + \frac{D_0}{2 \sqrt{A(x)}} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{A(x)}} \frac{\partial A(x)}{\partial x} \right). \quad (7)
\]
Now, the family of channels with cross sectional area \( A(x) = ax^{2\nu} \) is associated with the following family of Hamiltonians
\[
H = -D_0 \frac{\partial^2}{\partial x^2} + \frac{g}{x^2}, \quad g = D_0 \nu (\nu - 1). \quad (8)
\]
For each \( \nu \) we have a conformal quantum mechanics Hamiltonian (4). However, for each Hamiltonian (8) we have two channels, namely each Hamiltonian is associated with two \( \nu \) values. For example, \( \nu = 0 \) and \( \nu = 1 \), represent different sectional areas, but both cases give the same Hamiltonian
\[
H = -D_0 \frac{\partial^2}{\partial x^2}. \quad (9)
\]

The solution for the Schrödinger equation (6) with the Hamiltonian (8) is given by
\[
\phi_{\nu}(x) = |x|^{\frac{1}{2}} J_{\pm (2\nu - 1)} \left( \pm \sqrt{ \frac{E}{D_0} x } \right), \quad (10)
\]
where \( J_{\nu}(w) \) is the Bessel function of order \( \nu \). Then, if the channel has cross sectional area \( A(x) = ax^{2\nu} \), the solution for the Fick-Jacobs is given by
\[
C_{\nu}(x,t) = B e^{-Et} |x|^{\frac{2\nu + 1}{2}} J_{\pm (2\nu + 1)} \left( \pm \sqrt{ \frac{E}{D_0} x } \right), \quad (11)
\]
here \( B \) is a constant.

Notice that whether \( \nu = 0 \) the solution
\[
C_{\nu=0}(x,t) = e^{-Et} \left( B_1 \sin \left( \sqrt{ \frac{E}{D_0} x } \right) + B_2 \cos \left( \sqrt{ \frac{E}{D_0} x } \right) \right), \quad (12)
\]
is obtained. While if $\nu = 1$, the solution

$$C_{\nu=1}(x, t) = e^{-Et|x|} \left( B_1 \sin \left( \sqrt{\frac{E}{D_0}}x \right) + B_2 \cos \left( \sqrt{\frac{E}{D_0}}x \right) \right)$$

(13)

is gotten. We can see that $\nu = 0$ and $\nu = 1$ are associated with the same Hamiltonian, but the particle concentration is not the same.

4 Summary

In this paper we shown a relation between conformal quantum mechanics and Fick-Jacobs equation. This relation is given between a family of channels and a family of conformal Hamiltonians. It was found that a conformal Hamiltonian is associated with two channels with different geometry. In addition, exact solutions for Fick-Jacobs equation are given for this family of channels. This result is interesting, because the conformal quantum mechanics has been proposed as a realization of $AdS_2/CFT_1$ duality and Fick-Jacobs equation is employed to describe diffusion in biological channel. Then, it is possible that mathematical techniques from string theory can be employed to study some biological problems.

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