QUADRATIC CATEGORIES, KOSZUL RESOLUTIONS.

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Abstract.
The category of quadratic algebras has been endowed by Manin [6] with two tensor products. These products have been generalized to quadratic operads by Ginzburg and Kapranov [5], and to $n$-homogeneous algebras by Berger [2]. The purpose of this paper is to define an abstract notion of quadratic category such that the categories of quadratic algebras and quadratic operads are examples of this notion. We define Koszul complexes in this setting, representations of quadratic categories in the category of quadratic algebras, and Tannakian quadratic categories.

1. Introduction. A quadratic algebra is a quotient of a tensor algebra $T(V)$ of a finite dimensional vector space $V$, by an ideal $C$ generated by a subspace of $V \otimes V$. These algebras are $N$-graded algebras generated by their elements of degree 1, which satisfy quadratic relations. Quadratic algebras appear in different domains of mathematics, as in topology with the notion of Steenrood algebras, in differential geometry, the Clifford algebras are one of the main tools to study Spin-geometry, in group theory, symmetric and exterior algebras are very useful. In algebraic geometry, a projective scheme can be realized as a projective spectrum of a quadratic algebra, this is equivalent to saying that the category of quasi-coherent sheaves over a projective scheme $N$, is equivalent to the category of graded modules over a quadratic algebra.

Cohomology theories are defined in the general context of abelian categories with enough injective objects by applying the $\text{Hom}$ functor to resolutions. To compute cohomology groups, we need to define complexes which represent these resolutions, like the Chevalley complex in groups theory, the Koszul complex in Lie algebras theory, the Bar complex in associative algebras theory... Even at this stage, these canonical complexes are hardly tractable. This has motivated Priddy in [7] to define a generalized Koszul complex for quadratic algebras which allows to compute their cohomology when it is a resolution. The complex defined by Priddy is a generalization of the classical Koszul complex defined with the exterior and symmetric algebras. This is useful in practice since the Koszul complex is simpler than the Bar resolution.

The automorphisms group of quadratic algebras has been used in theoretical physics in the inverse scattering problem, and in low dimensional topology. It is in this context that Manin has endowed in [6] the category of quadratic algebras with two tensor products $\circ$ and $\bullet$, he has also defined the notion of quadratic dual which represents the Yoneda algebra of quadratic Koszul algebras. He has shown that there exists internal $\text{Hom}$ object in the category of quadratic algebras endowed with the tensor product $\bullet$.

An operad is an object which encodes operations. These objects have been defined in homotopy theories, and in [5] to study algebraic structures. The Manin tensor products have been adapted to the theory of quadratic operads by Ginzburg and Kapranov [5], in their paper they have defined the notion of Koszul resolution of a quadratic operad, which is an application of the general Koszul

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duality defined by Beilinson Ginzburg and Schechtman [1]. Recently, Berger [3] has defined the category of $n$-Koszul algebras. He has endowed this category with tensor products similar to the ones defined by Manin.

The purpose of this paper is to define a general notion of quadratic category endowed with two tensor products and a duality which satisfy some compatibility conditions. The categories of quadratic algebras, quadratic operads, and $n$-Koszul algebras are examples of quadratic categories. In this context, we show the following result as theorem 2.2:

Let $(C, o, \bullet, !)$ be a quadratic category, then for each objects $U$ and $V$ of $C$, $V o U^!$ is an internal Hom of the tensor category $(C, \bullet)$.

Tensor categories have been studied in [4] to determine properties of the cohomology ring of algebraic varieties. These authors have defined the notion of Tannakian category which is an Abelian rigid tensor category endowed with an exact faithful functor to a category of vector spaces, and have shown that a Tannakian category is equivalent to the category of representations of an affine group scheme. We adapt the study of these authors to quadratic categories by defining a quadratic Tannakian category: it is an Abelian quadratic category endowed with an exact faithful functor to the category of quadratic algebras, we show: (see theorem 4.5)

A quadratic Tannakian category is equivalent to the category of quadratic representations of an affine group scheme.

A projective scheme $U$, is the projective spectrum of a quadratic algebra $T_U$, we can associate to $U$ the quadratic Tannakian category $T'_U$ generated by $T_U$. The property of the group $H_U$ whose category of quadratic representations is equivalent to $T'_U$ seems to by an interesting object to study.

Many authors have tried to define non commutative algebraic geometry. The algebraic geometry of tensor categories is studied by Deligne, perhaps quadratic Tannakian categories represent the good framework for noncommutative algebraic geometry, and the category of quadratic algebras the motivic category in non commutative algebraic geometry.

2. Quadratic categories. The purpose of this part is to present the notion and properties of quadratic categories.

Definition 2.1.

A quadratic category $(C, o, \bullet)$ is a category $C$, endowed with two tensors products $o$ and $\bullet$, whose neutral elements are respectively $I_o$ and $I_\bullet$, and whose associativity constraints are respectively $c_o$ and $c_\bullet$. We suppose that the following properties are satisfied: For each objects $U, U_1, U_2, U_3$ of $C$, there exists an object $U^!$ of $C$, morphisms

$c_U : I_\bullet \to U o U^!, \quad d_U : U^! \to I_o, \quad f_{U, U_2, U_3} : (U_1 o U_2) \bullet U_3 \to U_1 o (U_2 \bullet U_3), \quad h_{U, U_2, U_3} : U_1 \bullet (U_2 o U_3) \to (U_1 \bullet U_2) o U_3$

such that the following diagrams are commutative:

\[
\begin{array}{c}
(U_1 \bullet (U_2 o U_3)) \bullet U_4 & \xrightarrow{f_{U_1, U_2 o U_3, U_4}} & U_1 \bullet ((U_2 o U_3) \bullet U_4) \\
\downarrow h_{U_1, U_2, U_3, U_4} & & \downarrow h_{U_1, U_2, U_3, U_4} \\
((U_1 \bullet U_2) o U_3) \bullet U_4 & \xrightarrow{f_{U_1, U_2, U_3, U_4}} & (U_1 \bullet U_2) o (U_3 \bullet U_4)
\end{array}
\]
Let $u_1 : U_1 \rightarrow U'_1$, $u_2 : U_2 \rightarrow U'_2$ and $u_3 : U_3 \rightarrow U'_3$ be three morphisms of $C$. The following diagrams are supposed to be commutative:

\[
\begin{array}{c}
(U_1 \circ U_2) \bullet (U_3 \circ U_4) & \overset{f_{U_1 \circ U_2, U_3 \circ U_4}}{\longrightarrow} & U_1 \circ (U_2 \bullet (U_3 \circ U_4)) \\
((U_1 \circ U_2) \bullet U_3) \circ U_4 & \overset{f_{U_1 \circ U_2, U_3 \circ U_4} \circ Id_{U_4}}{\longrightarrow} & (U_1 \circ (U_2 \bullet U_3)) \circ U_4 \\
\end{array}
\]

Let $h_{U_1, U_2, U_3, U_4}$ denote the isomorphism $(U_1 \circ U_2) \bullet (U_3 \circ U_4) \cong U_1 \circ (U_2 \bullet (U_3 \circ U_4))$.

\[
\begin{array}{c}
I_o \bullet U \overset{Id \circ Id}{\longrightarrow} (U \circ U') \bullet U \overset{h_{U, U'}}{\longrightarrow} U \circ (U' \bullet U) \overset{Id \circ Id}{\longrightarrow} U \circ I_o \\
\end{array}
\]

\[
\begin{array}{c}
U^1 \circ I_o \overset{Id \circ Id}{\longrightarrow} U^1 \circ (U \circ U') \overset{h_{U, U'} \circ Id}{\longrightarrow} (U^1 \circ U) \circ U^1 \overset{Id \circ Id}{\longrightarrow} I_o \circ U^1 \\
\end{array}
\]

The example who has motivated the construction of quadratic categories is the theory of quadratic algebras, recall the constructions of the tensor products defined by Manin in [6].

Let $U = T(U_1)/(C_1)$ and $V = T(V_1)/(C_2)$ be two quadratic algebras respectively isomorphic to the quotient of the tensor algebra of the finite dimensional $L$-vector spaces $U_1$ and $V_1$, by the ideals generated by $C_1 \subset U_1 \otimes U_1$ and $C_2 \subset V_1 \otimes V_1$. We endow the class of quadratic algebras with the structure of a category such that $Hom(U, V)$ is the set of morphisms of algebras $h : U \rightarrow V$ defined by a linear application $h_1 : U_1 \rightarrow V_1$ such that $(h_1 \otimes h_1)(C_1) \subset C_2$.

The quadratic algebra $U \circ V$ is the quotient of the tensor algebra $T(U_1 \otimes V_1)$ by the ideal generated by $t_{23}(C_1 \otimes C_2)$. The isomorphism $t_{23} : U_1^{\otimes 2} \otimes V_1^{\otimes 2} \rightarrow (U_1 \otimes V_1)^{\otimes 2}$ is defined by $t_{23}(u_1 \otimes u_1' \otimes v_1 \otimes v_1') = u_1 \otimes u_1' \otimes v_1 \otimes v_1'$.

The tensor product $U \circ V$ is the quotient of the tensor algebra $T(U_1 \otimes V_1)$ by the ideal generated by $t_{23}(U_1^{\otimes 2} \otimes C_2 + C_1 \otimes V_1^{\otimes 2})$.

The quadratic dual $U^1$ of $U$ is the quadratic algebra $T(U_1^*)/(C_1^*)$, where $U_1^*$ is the dual vector space of $U_1$, and $C_1^*$ the annihilator of $C_1$ in $U_1^* \otimes U_1^*$.

The neutral element $I_o$ of $o$ is $T(L)$, and the neutral element of $\cdot$, $I_o$ is $L$. The map $c_U : I_o \rightarrow U \circ U^1$ is defined by the map $c_{U^1} : L \rightarrow U_1 \otimes U_1^*$, $c_{U^1}(1) \rightarrow \sum_{i=1}^n u_i \otimes u_i^*$, where $(u_1, ..., u_n)$ is a basis of $U_1$, and $(u^1, ..., u^n)$ its dual basis. The map $d_U : U^1 \bullet U \rightarrow I_o$ is defined by the duality $U_1 \otimes U_1^* \rightarrow L$. 

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Let $U^i = T(U_i)/(C_i), i = 1, 2, 3$. The ideal $C$ which defines $(U^1 o U^2) \bullet U^3$ is $t_{23}(t_{23}(U_1^2 \otimes C_2 + C_1 \otimes U_2^2) \otimes C_3)$, and the ideal $C'$ which defines $U^1 o (U^2 \bullet U^3)$ is $t_{23}(U_1^2 \otimes (t_{23}(C_2 \otimes C_3)) + C_1 \otimes (U_2 \otimes U_3)^2)$. We remark that $C$ is contained in $C'$, the associativity constraint $(U_1 \otimes U_2) \otimes U_3 \rightarrow U_1 \otimes (U_2 \otimes U_3)$ of vector spaces projects to define the quadratic constraint $f_{U_1, U_2, U_3}$.

The ideal $D$ which defines $U^1 \bullet (U^2 o U^3)$ is $t_{23}(C_1 \otimes t_{23}(U_2^2 \otimes C_3 + C_2 \otimes U_3^2))$. The ideal $D'$ which defines the algebra $(U^1 \bullet U^2) o U^3$ is $t_{23}((U_1 \otimes U_2)^2 \otimes C_3 + t_{23}(C_1 \otimes C_2) \otimes U_3^2)$. We remark that $D'$ contains $D$ this implies that the associativity constraint for vector spaces $U_1 \otimes (U_2 \otimes U_3) \rightarrow (U_1 \otimes U_2) \otimes U_3$ projects to define the quadratic associativity constraint $h_{U_1, U_2, U_3}$.

**Theorem 2.2.**

Let $C$ be a quadratic category, $L$, and $N$ two objects of $C$. The contravariant functor:

$$C \rightarrow C$$

$$U \rightarrow \text{Hom}(U \bullet L, N)$$

is representable by $\text{No} L^1$.

**Proof.**

We define a map between the functors $U \rightarrow \text{Hom}(U \bullet L, N)$ and $U \rightarrow \text{Hom}(U, \text{No} L^1)$ by assigning to each element $u$ in $\text{Hom}(U \bullet L, N)$, the element $u'$ in $\text{Hom}(U, \text{No} L^1)$ defined by:

$$U \rightarrow U \bullet I_\bullet \overset{\text{Id}_U \bullet c_{U,L}}{\rightarrow} U \bullet (\text{Lo} L_1) \overset{\text{h}_{U,V,L,L'}}{\rightarrow} (U \bullet L) o L \overset{\text{uo} \text{Id}_L}{\rightarrow} \text{No} L^1$$

We define a map between the functors $U \rightarrow \text{Hom}(U, \text{No} L^1)$ and $U \rightarrow \text{Hom}(U \bullet N, L)$ by assigning to each element $v$ in $\text{Hom}(U, \text{No} L^1)$ the element $v''$ in $\text{Hom}(U \bullet L, N)$ defined by:

$$U \bullet L \overset{\text{Id}_L}{\rightarrow} (\text{No} L^1) \bullet L \overset{\text{f}_{\text{N,L,L'}}}{\rightarrow} \text{No}(L \bullet L) \overset{\text{Id}_{\text{No} L^1}}{\rightarrow} N$$

The correspondence $u \rightarrow u'$ and $v \rightarrow v''$ are morphisms of functors since they are compositions of morphisms of functors. We have to show that the correspondence defined on $\text{Hom}(U \bullet L, N)$ by $u \rightarrow (u'')$ is the identity morphism of the functor $U \rightarrow \text{Hom}(U \bullet L, N)$. We have:

$$(u'')^\sim = (U \rightarrow U \bullet I_\bullet \overset{\text{Id}_U \bullet c_{U,L}}{\rightarrow} U \bullet (\text{Lo} L_1) \overset{\text{h}_{U,V,L,L'}}{\rightarrow} (U \bullet L) o L \overset{\text{uo} \text{Id}_L}{\rightarrow} \text{No} L^1) \bullet L \overset{\text{f}_{\text{N,L,L'}}}{\rightarrow} \text{No}(L \bullet L) \overset{\text{Id}_{\text{No} L^1}}{\rightarrow} N$$

Let $(C, \otimes)$ be a tensor category, and $u : U \rightarrow U', u' : U' \rightarrow U''$, $v : V \rightarrow V'$, $v'' : V' \rightarrow V''$ arrows of $C$, $(u' \otimes u') \circ (u \otimes v) = (u' \circ u) \otimes (u' \circ v)$. Applying this fact to the tensor product $\bullet$ at the first line of the previous equality, we obtain:

$$(u'')^\sim = U \bullet L \rightarrow (U \bullet I_\bullet) \bullet L \overset{\text{Id}_U \bullet c_{U,L}}{\rightarrow} (U \bullet (\text{Lo} L_1)) \bullet L \overset{\text{h}_{U,V,L,L'}}{\rightarrow} (U \bullet L) o L \overset{\text{uo} \text{Id}_L}{\rightarrow} (\text{No} L^1) \bullet L \overset{\text{Id}_{\text{No} L^1}}{\rightarrow} N$$
\[
\frac{f_{N,L^1,L}}{\rightarrow} No(L^1 \cdot L) Id_{Nod^L} \frac{f_{N,L^1,L}}{\rightarrow} N
\]

Applying property (2.6) we obtain
\[
((U \bullet L) o L^1) \bullet L^\left(\frac{uoId_{L}}{\rightarrow} f_{L}^L\right) (N o L^1) \bullet L^\left(\frac{f_{N,L^1,L}}{\rightarrow} No(L^1 \bullet L) Id_{Nod^L} \frac{f_{N,L^1,L}}{\rightarrow} N\right)
\]

= \((U \bullet L) o L^1) \bullet L^\left(\frac{f_{U \bullet L \bullet L \bullet L}}{\rightarrow} f_{L}^L\right) (U \bullet L) o (L^1 \bullet L)^{uoId_{L} \bullet L} No(L^1 \bullet L) \frac{f_{N,L^1,L}}{\rightarrow} N
\]

We deduce that
\[
(u'') = U \bullet L \rightarrow (U \bullet I_o) \bullet L^\left(\frac{Id_{U \cdot L \cdot L} \bullet f_{L}^L}{\rightarrow} f_{L}^L\right) ((U \bullet L) o L^1) \bullet L
\]

\[
\frac{f_{U \bullet L \bullet L \bullet L}}{\rightarrow} (U \bullet L) o (L^1 \bullet L)^{uoId_{L} \bullet L} No(L^1 \bullet L) \frac{f_{N,L^1,L}}{\rightarrow} N
\]

Applying (2.1) we obtain that
\[
(U \bullet (L o L^1)) \bullet L^\left(\frac{h_{U \cdot L \cdot L} \bullet f_{L}^L}{\rightarrow} f_{L}^L\right) ((U \bullet L) o L^1) \bullet L^\left(\frac{f_{U \bullet L \bullet L \bullet L}}{\rightarrow} f_{L}^L\right) (U \bullet L) o (L^1 \bullet L)
\]

= \((U \bullet (L o L^1)) \bullet L^\left(\frac{h_{U \cdot L \cdot L} \bullet f_{L}^L}{\rightarrow} f_{L}^L\right) (U \bullet (L o L^1)) \bullet L^\left(\frac{f_{U \bullet L \bullet L \bullet L}}{\rightarrow} f_{L}^L\right) (U \bullet (L o L^1)) \bullet L^\left(\frac{h_{U \cdot L \cdot L} \bullet f_{L}^L}{\rightarrow} f_{L}^L\right) (U \bullet L) o (L^1 \bullet L)
\]

We deduce that
\[
(u'') = U \bullet L \rightarrow (U \bullet I_o) \bullet L^\left(\frac{Id_{U \cdot L \cdot L} \bullet f_{L}^L}{\rightarrow} f_{L}^L\right) ((U \bullet L) o L^1) \bullet L^\left(\frac{h_{U \cdot L \cdot L} \bullet f_{L}^L}{\rightarrow} f_{L}^L\right) (U \bullet (L o L^1)) \bullet L^\left(\frac{h_{U \cdot L \cdot L} \bullet f_{L}^L}{\rightarrow} f_{L}^L\right) (U \bullet (L o L^1)) \bullet L^\left(\frac{h_{U \cdot L \cdot L} \bullet f_{L}^L}{\rightarrow} f_{L}^L\right) (U \bullet L) o (L^1 \bullet L)
\]

Using the fact that \(o\) is a tensor product we deduce that:
\[
(u') = U \bullet L \rightarrow (U \bullet I_o) \bullet L^\left(\frac{Id_{U \cdot L \cdot L} \bullet f_{L}^L}{\rightarrow} f_{L}^L\right) ((U \bullet L) o L^1) \bullet L^\left(\frac{h_{U \cdot L \cdot L} \bullet f_{L}^L}{\rightarrow} f_{L}^L\right) (U \bullet (L o L^1)) \bullet L^\left(\frac{h_{U \cdot L \cdot L} \bullet f_{L}^L}{\rightarrow} f_{L}^L\right) (U \bullet (L o L^1)) \bullet L^\left(\frac{h_{U \cdot L \cdot L} \bullet f_{L}^L}{\rightarrow} f_{L}^L\right) (U \bullet L) o (L^1 \bullet L)
\]

Using property (2.5), we obtain that:
\[
U \bullet (L o L^1) \frac{h_{U \cdot L \cdot L} \bullet f_{L}^L}{\rightarrow} (U \bullet L) o (L^1 \bullet L) \frac{Id_{U \cdot L \cdot L} \bullet f_{L}^L}{\rightarrow} (U \bullet L) o (L^1 \bullet L)
\]
\[
= U \bullet (L_o(L \bullet L))^{Id_U \bullet (Id_L \bullet L)} \xrightarrow{Id_U \bullet (Id_L \bullet L)} U \bullet (LoI_o)^{b_{U,L,I_o}} \xrightarrow{(U \bullet L) \circ I_o}
\]

We deduce that

\[
(u')'' = U \bullet L \to (U \bullet I_o) \bullet L^{(Id_U \bullet c_L) \bullet Id_L} \xrightarrow{Id_U \bullet (Id_L \bullet L)} U \bullet (LoL^!) \bullet L^{c_{(U \bullet LoL^!) \bullet L}} \xrightarrow{Id_U \bullet (Id_L \bullet L)} U \bullet ((LoL^!) \bullet L)
\]

Applying (2.3) we deduce that

\[
U \bullet L \to (I_o \bullet L)^{Id_U \bullet (Id_L \bullet L)} \xrightarrow{Id_U \bullet (Id_L \bullet L)} U \bullet ((LoL^!) \bullet L)
\]

This implies that:

\[
(u')'' = U \bullet L \xrightarrow{U \bullet L} N
\]

We have to show now that \((v'')' = v\). We have:
\[(v')^1 = U \rightarrow U \cdot I^* \xrightarrow{Id_{U,L}^*} U \cdot (LoL)^*_{U,L,L} (U \cdot L)_{oL} \rightarrow \]

\[((U \cdot L)^* \xrightarrow{Id_{U,L}^*} (NoL') \cdot L \xrightarrow{f_{N,L,F}} \text{No}(L' \cdot L) \xrightarrow{Id_{N,F}^*} N)_{oL} \rightarrow NoL'\]

Using the fact that \(o\) is a tensor product, we obtain that:

\[((U \cdot L)^* \xrightarrow{Id_{U,L}^*} (NoL') \cdot L \xrightarrow{f_{N,L,F}} \text{No}(L' \cdot L))_{oL} \rightarrow NoL'\]

This implies that

\[(v')^1 = U \rightarrow U \cdot I^* \xrightarrow{Id_{U,L}^*} U \cdot (LoL)^*_{U,L,L} \]

\[(U \cdot L)_{oL} \rightarrow ((NoL') \cdot L)_{oL} \xrightarrow{f_{N,L,F}^*} (No(L' \cdot L))_{oL} \rightarrow (NoL')_{oL} \rightarrow NoL' \rightarrow NoL'\]

Using property (2.5) we obtain that:

\[U \cdot (LoL)^*_{U,L,L} \rightarrow (U \cdot L)_{oL} \rightarrow ((NoL') \cdot L)_{oL} \rightarrow \]

\[U \cdot (LoL)^*_{U,L,L} \rightarrow (U \cdot L)_{oL} \rightarrow ((NoL') \cdot L)_{oL} \rightarrow \]

\[U \cdot (LoL)^*_{U,L,L} \rightarrow (U \cdot L)_{oL} \rightarrow ((NoL') \cdot L)_{oL} \rightarrow \]

Thus

\[U \rightarrow U \cdot I^* \xrightarrow{Id_{U,L}^*} U \cdot (LoL)^*_{U,L,L} \rightarrow (U \cdot L)_{oL} \rightarrow ((NoL') \cdot L)_{oL} \rightarrow \]

\[U \rightarrow U \cdot I^* \xrightarrow{Id_{U,L}^*} U \cdot (LoL)^*_{U,L,L} \rightarrow (U \cdot L)_{oL} \rightarrow ((NoL') \cdot L)_{oL} \rightarrow \]

Since \(o\) is a tensor product, \(Id_{L\cdot L} = Id_{L\cdot oL} \). Thus using the fact that \(\cdot\) is a tensor product, we have:

\[U \cdot I^* \xrightarrow{Id_{U,L}^*} U \cdot (LoL)^*_{U,L,L} \rightarrow (NoL') \cdot (LoL') \rightarrow \]
\[ U \cdot I \xrightarrow{v \cdot \text{Id}_L} (\text{NoL}) \cdot I \xrightarrow{\text{Id}_V \cdot \text{Id}_L} (\text{NoL}) \cdot (\text{LoL}) \]

This implies that

\[ U \rightarrow U \cdot I \xrightarrow{\text{Id}_L \cdot v} U \cdot (\text{LoL}) \xrightarrow{h_{U,L,L'}} (U \cdot L) \text{LoL} \xrightarrow{(\text{Id}_L \cdot \text{Id}_L)} ((\text{NoL}) \cdot L) \text{LoL} \]

\[ U \xrightarrow{v} (\text{NoL}) \cdot I \xrightarrow{\text{Id}_V \cdot \text{Id}_L} (\text{NoL}) \cdot (\text{LoL}) \xrightarrow{h_{\text{NoL},L,L'}} ((\text{NoL}) \cdot L) \text{LoL} \]

Using the fact that the associative constraint \( c \) is a functorial isomorphism, we deduce that

\[ (v')' = U \xrightarrow{v} (\text{NoL}) \cdot I \xrightarrow{\text{Id}_V \cdot \text{Id}_L} (\text{NoL}) \cdot (\text{LoL}) \xrightarrow{h_{\text{NoL},L,L'}} ((\text{NoL}) \cdot L) \text{LoL} \]

\[ f_{N,L,L'} \xrightarrow{\text{Id}_L \cdot \text{Id}_L} (\text{No}(L \cdot L)) \text{LoL} \xrightarrow{c_{N,L,L'}} (\text{No}(L \cdot L) \text{LoL}) \text{LoL} \xrightarrow{\text{Id}_L \cdot \text{Id}_L} \]

The property (2.2) implies that:

\[ (\text{NoL}) \cdot (\text{LoL}) \xrightarrow{h_{\text{NoL},L,L'}} ((\text{NoL}) \cdot L) \text{LoL} \xrightarrow{f_{N,L,L'}} (\text{No}(L \cdot L)) \text{LoL} \xrightarrow{c_{N,L,L'}} (\text{No}(L \cdot L) \text{LoL}) \]

This implies that:

\[ (v')' = U \xrightarrow{v} (\text{NoL}) \cdot I \xrightarrow{\text{Id}_V \cdot \text{Id}_L} (\text{NoL}) \cdot (\text{LoL}) \]

\[ f_{N,L,L'} \xrightarrow{\text{Id}_L \cdot \text{Id}_L} (\text{No}(L \cdot L)) \text{LoL} \xrightarrow{c_{N,L,L'}} (\text{No}(L \cdot L) \text{LoL}) \xrightarrow{\text{Id}_L \cdot \text{Id}_L} \]

The property (2.5) implies that:

\[ (\text{NoL}) \cdot I \xrightarrow{\text{Id}_V \cdot \text{Id}_L} (\text{NoL}) \cdot (\text{LoL}) \]

\[ f_{N,L,L'} \xrightarrow{\text{Id}_L \cdot \text{Id}_L} (\text{No}(L \cdot L)) \text{LoL} \xrightarrow{c_{N,L,L'}} (\text{No}(L \cdot L) \text{LoL}) \]

This implies that:
\[(v^n)' = U \xrightarrow{v} (NoL') \bullet I_\bullet \xrightarrow{f_{N,L,I}} No\langle L' \bullet I_\bullet \rangle\]

\[\xrightarrow{Id_N o (Id_N \bullet c_L)} No\langle L' \bullet (LoL') \rangle \xrightarrow{Id_N o h_{L,L}} No\langle (L' \bullet L) o L' \rangle \rightarrow NoL' \]

The property (2.4) implies that:

\[L' \rightarrow L' \bullet I_\bullet \xrightarrow{Id_L} L' \bullet (LoL') \xrightarrow{h_{L,L}} (L' \bullet L) o L' \rightarrow L' \]

is the identity of \(L'\). This implies that \((v^n)' = v\) since \(o\) is a tensor product. □

The previous theorem implies that the tensor category \((C, \bullet, o)\) is endowed with an internal \(\text{Hom}(U,V) = VoL'\). The general properties of internal \(\text{Hom}\) (see [4] Definition 1.6 page 109) implies the existence of an isomorphism:

\[d_{U_1,U_2} : \text{Hom}(U_1,U_2) \bullet U_1 \rightarrow U_2\]

The map \(d_{U_1,U_2}\) is the composition:

\[(U_2 o U_1') \bullet U_1 \xrightarrow{f_{U_2,V_1,U_1}} U_2 o (U_1' \bullet U_1) \xrightarrow{Id_{U_2} o d_{U_1}} U_2 o I_o \rightarrow U_2.\]

There exists a map:

\[l_{U_1,U_2,U_3} : \text{Hom}(U_2,U_3) \bullet \text{Hom}(U_1,U_2) \rightarrow \text{Hom}(U_1,U_3)\]

The map \(l_{U_1,U_2,U_3}\) is the composition:

\[(U_3 o U_2') \bullet (U_2 o U_1') \xrightarrow{f_{U_2,U_1,U_1'}} U_3 o (U_2' \bullet (U_2 o U_1')) \xrightarrow{Id_{U_3} o h_{U_2,U_1'}} U_3 o ((U_2' \bullet U_2) o U_1') \xrightarrow{Id_{U_3} o (d_{U_2,U_1'})} U_3 o U_1'.\]

**Proposition 2.3.**

Let \((C, \bullet, o)\) be a quadratic category. There exists a canonical isomorphism \((U \bullet V)' \rightarrow V' o U'\).

**Proof.**

The general properties of tensor categories imply the existence of an isomorphism (see [4] 1.6.3 page 110):

\[\text{Hom}(U_1 \bullet U_2, U_3) \rightarrow \text{Hom}(U_1, \text{Hom}(U_2, U_3))\]

we obtain an isomorphism:

\[U_3 o (U_1 \bullet U_2)' \rightarrow (U_3 o U_2') o U_1'\]

Suppose that \(U_3 = I_o\), the previous isomorphism induces an isomorphism:
\[(U_1 \cdot U_2)^! \rightarrow U_2^! \circ U_1^!\]

Let \( h : U \rightarrow U' \) be a map. We define the morphism \( h^! : U'^! \rightarrow U'^! \) as follows:

\[
U'^! \rightarrow U'^! \cdot I_\bullet\rightarrow U'^! \cdot (UoU')^! \rightarrow U'^! \cdot U \rightarrow U'^! \circ (Id_{U'}\circ h^!\circ Id_{U'}) \rightarrow U'^! \circ U \rightarrow U'^! \circ U' \rightarrow U'^!\]

**Proposition 2.4.**

Let \((C, o, \bullet)\) be a quadratic category, for every object \(U\) of \(C\), \(U^!\) is unique up to an isomorphism.

**Proof.**

The object \(U^!\) represents the contravariant functor \(V \rightarrow Hom_C(V \bullet U, I_\circ).\) This implies that this object is unique up to an isomorphism. \(\square\)

We denote by \(\text{hom}(U, V)\) the object \(\text{Hom}(U, V)^1\), the paragraph before proposition 3, implies the existence of a map \(l_{U_1, U_2, U_3}.\) The dual of \(l_{U_1, U_2, U_3}\) defines a map:

\[
l_{U_1, U_2, U_3} : \text{hom}(U_1, U_3) \rightarrow \text{hom}(U_1, U_2) \circ \text{hom}(U_2, U_3)
\]

The morphism \(l_{U, U, U}\) endows \(\text{Hom}(U, U)\) with a product \(l_{U, U, U} = l_U : \text{Hom}(U, U) \circ \text{Hom}(U, U) \rightarrow \text{Hom}(U, U),\) and the morphism \(l_{U_1, U_2, U_3} = l_U : \text{hom}(U, U) \rightarrow \text{hom}(U, U) \circ \text{hom}(U, U)\) endows \(\text{hom}(U, U)\) with a coproduct. There exists a canonical product on \(\text{hom}(U, U)\) defined by

\[
(U^! \circ U) \circ (U^! \circ U) \rightarrow U^! \circ U^! \circ U^! \circ U
\]

**Proposition 2.5.**

The object \((\text{hom}(U, U), l_U)\) is an algebra, and the object \((\text{hom}(U, U), l_U)\), is a coalgebra. This means that the following diagrams commute:

\[
\begin{aligned}
\text{Hom}(U, U) \cdot (\text{Hom}(U, U) \circ \text{Hom}(U, U)) & \xrightarrow{Id_{\text{Hom}(U, U)} \circ l_U} \text{Hom}(U, U) \circ \text{Hom}(U, U) \\
\downarrow c_\bullet (\text{Hom}(U, U), \text{Hom}(U, U), \text{Hom}(U, U)) & \downarrow l_U \\
(\text{Hom}(U, U) \circ \text{Hom}(U, U)) \circ \text{Hom}(U, U) & \xrightarrow{l_U \circ Id_{\text{Hom}(U, U)} \circ l_U} \text{Hom}(U, U) \circ \text{Hom}(U, U) \circ \text{Hom}(U, U) \\
\downarrow & \downarrow l_U \\
\text{Hom}(U, U) \circ \text{Hom}(U, U) & \xrightarrow{Id_{\text{Hom}(U, U)} \circ l_U} \text{Hom}(U, U)
\end{aligned}
\]

These two diagrams endow \(\text{Hom}(U, U)\) with the structure of an algebra. The next two diagrams endow \(\text{hom}(U, U)\) with the structure of a coalgebra.
\[\begin{array}{ccc}
\text{hom}(U,U) & \xrightarrow{\iota_U} & \text{hom}(U,U) \\
\downarrow \iota_U & & \downarrow \iota_U \circ \text{Id}_{\text{hom}(U,U)} \\
\text{hom}(U,U) \circ \text{hom}(U,U) & \xrightarrow{\text{Id}_{\text{hom}(U,U)} \circ \iota_U} & \text{hom}(U,U) \circ \text{hom}(U,U) \\
\downarrow c_o & & \downarrow \text{Id}_{\text{hom}(U,U)} \circ d_U \\
\text{hom}(U,U) & \xrightarrow{\iota_U} & \text{hom}(U,U) \\
\downarrow & & \downarrow \text{Id}_{\text{hom}(U,U)} \circ d_U \\
\text{hom}(U,U) & \xrightarrow{\circ} & \text{hom}(U,U)
\end{array}\]

Proof.

We have only to show that the first two diagrams are commutative since the two last are their dual. The fact that first two diagrams commute follows from general properties of tensor categories.

**Proposition 2.6.**

Let \((\mathcal{C}, \bullet, o, !)\) be a quadratic category. The quadratic dual \(I^!\) of \(I\) is isomorphic to \(I_o\).

**Proof.**

The theorem \(2\) implies that the functor defined on \(\mathcal{C}\) by \(U \rightarrow \text{Hom}_\mathcal{C}(U, I_o) = \text{Hom}_\mathcal{C}(U \bullet I, I_o)\) is representable by \(I_o \circ I^!\) and \(I_o\). Since \(I_o \circ I^!\) is isomorphic to \(I^!\), we deduce that \(I^!\) is isomorphic to \(I_o\).

**Definition 2.7.**

Let \((\mathcal{C}, \bullet, o, !)\) be a quadratic category, the commutative constraints \(c'_o\) of \((\mathcal{C}, \bullet)\) and \(c'_o\) of \((\mathcal{C}, o)\) is a quadratic braiding if for every objects \(U_1, U_2\) and \(U_3\) of \(\mathcal{C}\), the following diagram is commutative:

\[
\begin{array}{cccccc}
U_1 \bullet (U_2 \circ U_3) & \xrightarrow{c'_o(U_1 \bullet U_2 \circ U_3)} & (U_2 \circ U_3) \bullet U_1 & \xrightarrow{c'_o(U_2 \circ U_3) \circ \text{Id}_{U_1}} & (U_2 \circ U_3) \bullet U_1 & \\
\downarrow h_{U_1, U_2, U_3} & & \downarrow & & \downarrow f_{U_3, U_2, U_1} \\
(U_1 \bullet U_2) \circ U_3 & \xrightarrow{c'_o(U_1 \bullet U_2 \circ U_3)} & U_3 \circ (U_1 \bullet U_2) & \xrightarrow{\text{Id}_{U_3 \circ U_2, U_1} \circ c'_o(U_1, U_2)} & U_3 \circ (U_2 \bullet U_1)
\end{array}
\]

**Proposition 2.8.**

Let \((\mathcal{C}, o, \bullet, !)\) be a quadratic category endowed with a quadratic braiding, and \(U\) an object of \(\mathcal{C}\), then \(U^{!\!}\) is isomorphic to \(U\). In particular \(I^!\) isomorphic to \(I^!\).

**Proof.**

Let \(U\) be an object of \(\mathcal{C}\), consider the morphisms

\[c'_o: I_o \xrightarrow{c'_o} U \circ U^{!} \xrightarrow{c'_o(U, U^{!})} U^{!} \circ U\]
Since the square of $c_U$ and $d_U$ and dual of $U^!$ are verified by $c'_U$ and $d'_U$ and $U$. We have:

$$d'_U : U \otimes c'_{(U,U')} U^! \to U \otimes I_o$$

We are going to show that the universal properties verified by $c_U$ and $d_U$ and dual of $U^!$ are verified by $c'_U$ and $d'_U$ and $U$.

$$U^! \to I_o \otimes c'_{(U,U')} U^! \otimes f_{U^!,U,U'} f_{U^!,U,U'} \otimes c'_{(U,U')} U^! \to U^! \otimes I_o \otimes U \otimes I_o$$

$$= U^! \to I_o \otimes c'_{(U,U')} U^! \otimes f_{U^!,U,U'} f_{U^!,U,U'} \otimes c'_{(U,U')} U^! \otimes U$$

The compatibility property of the quadratic braiding implies:

$$(U o U^!) \otimes U^! \otimes c'_{(U,U')} U^! \otimes f_{U^!,U,U'} f_{U^!,U,U'} \otimes c'_{(U,U')} U^! \otimes U$$

Since the square of $c'_o$ is the identity, this implies that:

$$U^! \to I_o \otimes c'_{(U,U')} U^! \otimes f_{U^!,U,U'} f_{U^!,U,U'} \otimes c'_{(U,U')} U^! \otimes I_o$$

$$= U^! \to I_o \otimes c'_{(U,U')} U^! \otimes f_{U^!,U,U'} f_{U^!,U,U'} \otimes c'_{(U,U')} U^! \otimes U$$

Since the commutative constraint of $\bullet$ is a functorial isomorphism, we deduce that:

$$U^! \to I_o \otimes c'_{(U,U')} U^! \otimes f_{U^!,U,U'} f_{U^!,U,U'} \otimes c'_{(U,U')} U^! \otimes (U o U^!)$$

$$= U^! \to I_o \otimes c'_{(U,U')} U^! \otimes f_{U^!,U,U'} f_{U^!,U,U'} \otimes c'_{(U,U')} U^! \otimes (U o U^!) = U^! \to U^! \otimes I_o$$

This implies that
Using the compatibility property of the braiding, we have:

\[ U^1 \rightarrow I_s \bullet U^1 f_{U,U^1}^{c_s,Id_{U^1}} (UoU) \bullet U^1 f_{U^1,U^1}^{Id_{U^1},oU} (U^1 o (U \bullet U^1)) \rightarrow U^1 o I_o \]

\[ = U^1 \rightarrow U^1 f_{U^1,U^1}^{Id_{U^1},oU} (U \bullet U^1) \rightarrow U^1 o (U \bullet U^1) \rightarrow U^1 o I_o \]

Using the fact that \( c_o \) is a quadratic braiding, we obtain that \((U^1 \bullet U) o U^1 f_{U^1,U^1}^{c_o,Id_{U^1}} (UoU) \rightarrow U^1 o (U^1 \bullet U) o U^1 f_{U^1,U^1}^{Id_{U^1},oU} (U \bullet U^1)\) is a functorial isomorphism and the square of \( c' \) is the identity of \( U^1 \).

We also have:

\[ U \rightarrow U \bullet I_s \rightarrow U \bullet (U^1 o U) \rightarrow (U \bullet U^1) o U \rightarrow (U \bullet U^1) o U \rightarrow U \]

\[ = U \rightarrow U \bullet I_s \rightarrow U \bullet (UoU) \rightarrow (U \bullet U^1) o Id_{U^1} \rightarrow I_o U \rightarrow U \]

Using the compatibility property of the braiding, we have:

\[ h_{U,U^1}^{Id_{U^1},oU} (U \bullet U^1) o U^1 f_{U^1,U^1}^{c_o,(U^1,U^1)} (UoU) \rightarrow (U \bullet U^1) o U^1 f_{U^1,U^1}^{Id_{U^1},oU} (U \bullet U^1) \rightarrow U^1 o I_o U \rightarrow U \]

Using the fact that \( c_o \) is a functorial isomorphism and the square of \( c' \) is the identity, this implies that:

\[ U \rightarrow U \bullet I_s \rightarrow U \bullet (U^1 o U) \rightarrow (U \bullet U^1) o U \rightarrow (U \bullet U^1) o U \rightarrow U \]

\[ = U \rightarrow U \bullet I_s \rightarrow U \bullet (UoU) \rightarrow (U^1 o U) \rightarrow U \]

\[ = U \rightarrow U \bullet I_s \rightarrow U \bullet (UoU) \rightarrow (U^1 o U) \rightarrow U \]

\[ = U \rightarrow U \bullet I_s \rightarrow U \bullet (UoU) \rightarrow (U^1 o U) \rightarrow U \]

\[ = U \rightarrow U \bullet I_s \rightarrow U \bullet (UoU) \rightarrow (U^1 o U) \rightarrow U \]

\[ = U \rightarrow U \bullet I_s \rightarrow U \bullet (UoU) \rightarrow (U^1 o U) \rightarrow U \]

\[ = U \rightarrow U \bullet I_s \rightarrow U \bullet (UoU) \rightarrow (U^1 o U) \rightarrow U \]

\[ = U \rightarrow U \bullet I_s \rightarrow U \bullet (UoU) \rightarrow (U^1 o U) \rightarrow U \]

\[ = U \rightarrow U \bullet I_s \rightarrow U \bullet (UoU) \rightarrow (U^1 o U) \rightarrow U \]

\[ = U \rightarrow U \bullet I_s \rightarrow U \bullet (UoU) \rightarrow (U^1 o U) \rightarrow U \]

\[ = U \rightarrow U \bullet I_s \rightarrow U \bullet (UoU) \rightarrow (U^1 o U) \rightarrow U \]
Using the fact that \( \cdot \) is a braided tensor product and the fact that the square of \( e'_o \) is the identity, we deduce that:

\[
U \longrightarrow U \cdot U \xrightarrow{I_d \cdot e'_o} U \cdot (U_o U) \xrightarrow{f_{U_o U}} U \cdot (U \cdot U) \xrightarrow{e'_o} (U \cdot U) \longrightarrow (U_o U) \cdot U
\]

This implies that:

\[
U \longrightarrow I_o \cdot U \xrightarrow{I_d \cdot e'_o} U \cdot (U_o U) \xrightarrow{h_{U_o U}} (U \cdot U) \longrightarrow U I_o
\]

Using property 2.3, and the fact that \( o \) is a braided tensor product we deduce that

\[
U \longrightarrow U \cdot I_o \xrightarrow{I_d \cdot e'_o} U \cdot (U_o U) \xrightarrow{h_{U_o U}} (U \cdot U) \longrightarrow U o I_o \rightarrow U
\]

is the identity.

Since the quadratic dual is unique up to an isomorphism, we deduce that \( U \) is a quadratic dual of \( U^! \).

\[ \square \]

**Proposition 2.9.**

Let \( (C, o, \cdot, \!) \) be a quadratic category endowed with a quadratic braiding. The contravariant functor \( P : U \rightarrow U^! \) is an equivalence of category.

**Proof.**

Let \( U_1^! \) and \( U_2^! \) be objects of \( C \). Using the previous result, we can suppose that \( U_1^! = U_1 \) and \( U_2^! = U_2 \). The theorem 2.2, implies the existence of a bijection between \( \text{Hom}_C(U_1 \cdot U_2, I_o) \) and \( \text{Hom}_C(U_1, U_2) \), and the existence of an isomorphism between \( \text{Hom}_C(U_2 \cdot U_1, I_o) \) and \( \text{Hom}_C(U_2, U_1^!) \). The commutative constraint \( e'_o \) defines an isomorphism between \( \text{Hom}_C(U_1 \cdot U_2^!, I_o) \) and \( \text{Hom}_C(U_2^! \cdot U_1, I_o) \). The object \( U_1^! \) represents the functor \( n_{U_1} : V \rightarrow \text{Hom}_C(V \cdot U_1, I_o) \). This implies that the correspondence defined on \( C \) by \( U \rightarrow U^! \) is functorial, and using the Yoneda lemma, we deduce that the morphisms between \( n_{U_2} \) and \( n_{U_1} \) are given by \( \text{Hom}_C(U_2^! \cdot U_1, I_o) \). This implies that \( P \) is fully faithful. \[ \square \]

**Definition 2.10.**

Let \( (C, o, \cdot, \!) \) be a quadratic category endowed with a quadratic braiding. We denote by \( C' \) the subcategory of \( C \), such that for each object \( U \) of \( C' \), there exists an object \( V \) of \( C \) such that \( U \) is isomorphic to \( V o I_o \). \( (C', o) \) is a subtensor category of \( (C, o) \). We say that the quadratic
category \((C, \circ, \bullet, !)\) is quadratic rigid if for every objects, \(U_1\) and \(U_2\) of \(C'\), \(U_1U_2 = U_1 \bullet U_2\) and the restriction of the braided associativity constraints \(f_{U_1, U_2, U_3}\) and \(h_{U_1, U_2, U_3}\) to \(C'\) coincide with the associativity constraint of \(\circ\) and \(\bullet\).

Let \((C, \circ, \bullet, !)\) be a rigid quadratic category, and \(U\) an object of \(C'\), we define \(U^*\) to be \(U^!oI_\bullet\). We have \(I_\circ = I_oI_\bullet\). This implies that \(I_\circ\) is an object of \(C'\). Thus \(I_\circ \bullet I_\circ = I_oI_\bullet = I_.\) We deduce that for each object \(U\) of \(C'\), \(UoU^!\) is isomorphic to \((UoI_\bullet)oU^!\) which is isomorphic to \(Uo(U^!oI_\bullet)\).

Since \(U\) is an object of \(C'\), \(UoU^!\) is isomorphic to \(U \bullet (U^!oI_\bullet) = U \bullet U^*\).

**Proposition 2.11.**

Let \(U\) be an object of \(C'\) the map \(c_U\), and the map

\[
d^*_{U^!} : Uo(U^!oI_\bullet) = U \bullet (U^!oI_\bullet) \xrightarrow{h_{U^!, U^!}} (U \bullet U^!)oI_\bullet \xrightarrow{d^*_{U^!}oId} I_oI_\bullet = I_\circ
\]

defines on \((C', \bullet)\) the structure of a rigid tensor category.

**Proof.**
We have to show that for each element \(U\) in \(C'\):

\[
U \to U \bullet I_\circ \xrightarrow{Id^*oI^*} U \bullet (U^!oU) = U \bullet (U^* \bullet U) \xrightarrow{h_{U^!, U^!}} (U \bullet U^*)oU \xrightarrow{d^*_{U^!}oId} U
\]

is the identity of \(U\), and

\[
U^* \to I_\circ \bullet U^* \xrightarrow{e^*_{U^!}oId^*} (U^!oU) \bullet U^* = (U^* \bullet U) \bullet U^* \xrightarrow{f_{U^!, U}oId^*} U^*oU (U^* \bullet U^*) \xrightarrow{Id^*oId} U^*
\]

is the identity. These assertions follows from the fact that \(U\) is the dual of \(U^!\). In the second assertion, we multiply (2.4) applied to \(U^!\) by \(I_\circ\). This implies that \(U^*\) is a dual of \(U\) since the category \(C\) is braided we deduce from Deligne Milne (see [4]) that the category \(C'\) is rigid.

**Definition 2.12.**

Let \((C, \circ, \bullet, !)\) be a quadratic rigid category, and \(h : U \to U\) a morphism. We can define \(hoId_{I_\circ} : UoI_\circ \to UoI_\circ\). We define \(Trace(h)\) to be the trace of \(hoId_{I_\circ}\). This is the endomorphism of \(I_\circ\) defined by:

\[
I_\circ \xrightarrow{e_{U^!}oId} U^!oU \xrightarrow{Id^*oI^*} U^!oI_\circ \xrightarrow{Id} I_\circ
\]

We denote by \(\text{rank}(U)\) the trace of \(Id_U\). Let \(h, h' : U \to U\), \(Trace(hh') = Trace(h)Trace(h')\).

The ring \(\text{Hom}(I_\circ, I_\circ)\) is commutative since \(I_\circ\) is the neutral element of \((C, \bullet)\).

**Proposition 2.13.**

Let \((C, \circ, \bullet, !)\) be a quadratic rigid category, and \(C''\) be the subcategory of \(C\) such that for every object \(U\) of \(C''\), there exists an object \(V\) of \(C\) such that \(U = V \bullet I_o\). The category \((C'', \circ, I_o)\) is a rigid tensor category.

**Proof.**

The restriction of the contravariant functor \(P : C \to C\) defined on \(C\) by \(U \to U^!\) to \(C'\) defines an isomorphism between the tensor categories \(C'\) and \(C''\).

**Definition 2.14.**
Let $(C,o,\bullet,!)$ a quadratic category, and $h : U \to V$ a morphism. The morphism $h' : U^! \to V^!$ is a contragredient of $h$ if and only if $hoh' \circ c_U = c_V$, and $d_V \circ (h' \bullet h) = d_U$.

**Proposition 2.15.**

Let $(C,o,\bullet,!)$ be a braided quadratic category. Then a map $h$ which has a contragredient is invertible.

**Proof.**

Let $h' : U^! \to V^!$ be a contragredient of the map $h : U \to V$. We are going to show that $h' \circ h^! = Id_{U^!}$ and $h^! \circ h' = Id_{V^!}$, since the category is braided, this implies that $h$ is invertible.

Using property 2.5, we deduce that $h^! \circ h' = $$U^! \xrightarrow{\varepsilon'_U} U^! \bullet (UoU^!) \xrightarrow{h^! \bullet U}(V^! \bullet U)oU^! \xrightarrow{(Id_{U^!} \circ h) \circ Id_{U^!}} (V^! \bullet V)oU^! \xrightarrow{d_V} U^!$$$

Using the fact that $d_V \circ (h' \bullet h) = d_U$, and property 2.4, we deduce that $h^! \circ h' = Id_{U^!}$. The proof that $h^! \circ h' = Id_{U^!}$ is similar.

**Definition 2.16.**

A quadratic functor $H : (C,o,\bullet,!) \to (C',o',\bullet',!)$ is a functor of tensor categories $H : (C,o) \to (C',o')$ which commutes with the quadratic dual, that is $H(U^!) = H(U)^!$.

**Proposition 2.17.**

Let $(C,o,\bullet,!)$ and $(C',o',\bullet',!)$ be quadratic braided categories, and $F,F' : (C,o,\bullet,!) \to (C',o',\bullet',!)$ be two quadratic functors every morphism $u : F \to F'$ is an isomorphism.

**Proof.**

The morphism $u : F \to F'$ is defined by a family of morphisms $u_U : F(U) \to F'(U)$ where $U$ is an object of $C$. The morphism $u_{U^!}$ is a contragredient of $u_U$. This implies that $u$ is an isomorphism. The inverse of $u$ is the family of maps $u_{U^!}^{-1}$.

We can generalize the morphisms defined by Manin [6] for quadratic algebras as follows:

We have the morphism $c_{I_o} : I_o \to I_o \circ I_o^! = I_o$.

Let $U$ and $V$ be objects of $C$, we have a morphism $U \bullet V \rightarrow UoV$ defined by

$$U \bullet (V\circ I_o) \xrightarrow{c_{U \circ I_o}} (V\circ I_o) \xrightarrow{f_{V \circ I_o}} V \circ I_o \circ U \xrightarrow{Id_{V \circ I_o} \circ c_{I_o}} V \circ (I_o \bullet U) \xrightarrow{c_{U \circ I_o}} UoV$$

**3. Koszul complexes.** In this part, we suppose that the category $C$ is additive, and is contained in an abelian category $C'$. Let $h$ be an element of $\text{Hom}(U,U)$, we denote by $d'_h$ the image of $h$ by the isomorphism $\text{Hom}(U,U) \rightarrow \text{Hom}(I_o,UoU^!)$. We can construct the map:

$$d_h : UoU^! \rightarrow I_o \bullet (UoU^!) \xrightarrow{d'_h} (UoU^!) \bullet (UoU^!) \xrightarrow{\text{Id}_{UoU^!}} UoU^!$$

**Definition 3.1.**
The category $C$ is an $n$-Koszul category, if for every object $U$, and each map $h : U \to U$, $(dh)^{n+1} = 0$. The category $C$ is a Koszul category if it is 1-Koszul. This is equivalent to saying that $(dh)^2 = 0$.

Let $C$ be a Koszul category. We denote by $D_U$ the endomorphism $d_{d_U}$, and define the first Koszul complex to be $L(U) = (U \otimes U^!, D_U)$. We say that $U$ is Koszul if $L(U)$ is exact. Let

$$U \xrightarrow{d_0} U_1 \to U_2 \to \cdots$$

be a resolution of $U$ in $C'$. We say that this resolution is a Koszul resolution, if there exists an object $V$ of $C$ endowed with a differential $\alpha_V$, such that there exists embedding $\alpha_U : U \otimes U^! \to V$, $e_p : U_p \to V$, such that the following squares are commutative:

$$U_p \xrightarrow{d_p} U_{p+1}$$

$$\downarrow e_p \downarrow e_{p+1}$$

$$V \xrightarrow{\alpha_V} V$$

$$U \otimes U^! \xrightarrow{\alpha_U} V$$

$$\downarrow D_U \downarrow \alpha_V$$

$$U \otimes U^! \xrightarrow{\alpha_U} V$$

### 3.1. Koszul complexes of algebras

We suppose that $C$ is a Koszul category, the objects of $C$ are graded algebras defined over a field $F$. We denote by $U_i$ the $i$-component of $U$, and we suppose that $U_0 = F$, the map $D_U$ is a left multiplication by an element $\alpha_U$ of $U \otimes U^!$, we suppose that $C$ is stable by the usual tensor product of $F$-vector spaces, and there exists an embedding $U \otimes V \to U \otimes V$. We denote by $\alpha_U$ the image of $\alpha_U$ by this embedding. We suppose that $\alpha_U \in U_1 \otimes U^!_1$. The family $(U \otimes U^!, \alpha_U)_{i \in \mathbb{N}}$ is a Koszul complex called the first algebra Koszul complex. If this complex is exact, the algebra is called a Koszul algebra.

**Theorem 3.2.**

Let $U$ be an object of $C$, if the complex $(U \otimes U^!, \alpha_U)$ is acyclic then $\text{Ext}_U(F, F) = U^!$.

**Proof.**

Suppose that $(U \otimes U^!, \alpha_U)$ is acyclic. This implies that:

$$0 \to U \to U \otimes U^! \to U \otimes U^!_1 \to U \otimes U^!_{i+1} \to \cdots$$

is a $U$-resolution of $U$. The tensor product $(U \otimes_\mathbb{F} U^!_1) \otimes_U F$ is isomorphic to $(U \otimes_U F) \otimes_\mathbb{F} U^!_1$. The tensor product $U \otimes_U F$ is $F$, since the left-module of $F$ verifies $I(U)F = 0$ where $I(U)$ is the augmentation ideal. This implies that $(U \otimes_U F) \otimes_\mathbb{F} U^!_1 = U^!_1$ and the multiplication by $\alpha_U$ induces the zero map between $U^!_i$ and $U^!_{i+1}$. We deduce that that $\text{Ext}_U(F, F) = U^!$.
3.2. Second Koszul complex. Let $U^{\dagger}$ be the $F$-dual of $U$, $U \otimes U^{\dagger}$ is embedded in $\text{Hom}_U(U \otimes U^{\dagger}, U)$ as follows: let $u_1, u_2$ be elements of $U$, $v_1$ an element of $U^{\dagger}$, and $v_2$ an element of $U$. We define $(u_1 \otimes v_1)(u_2 \otimes v_2) = v_1(v_2)u_1u_2$ if $l = p$, $(u_1 \otimes v_1)(u_2 \otimes v_2) = 0$ if $l \neq p$. We can define the differential $D'_U$ on $U \otimes U^{\dagger}$ by setting $D'_U(h)(u) = h(\alpha_U(u))$. The complexes $(U \otimes U, \alpha_U)$ and $(U \otimes U^{\dagger}, D'_U)$ are dual each other.

3.3. Quadratic algebras. One of the main objectives of specialists in Koszul structures is to construct a Koszul resolution of an object $U$. The reason of this, is the fact that using this resolution we can easily compute $U$-homology and cohomology. The usual Bar complex allows to compute the homology of an algebra. Recall its definition. Let $U$ be an algebra, $\epsilon : U \to F$ the augmentation of $U$, and $I(U)$ the kernel of $\epsilon$. The Bar complex of $U$ is the tensor product $B(U) = U \otimes T(I(U)) \otimes U$. Its elements are denoted by $u[u_1 : \ldots : u_p]u'$. The Bar complex is bigraded the degree of $v = u[u_1 : \ldots : u_p]u'$ is $(m, n)$ where $m = \sum_{i=1}^{l=p} \text{degree}(u_i) + \text{degree}(u) + \text{degree}(u')$, and $n = p$ is the homological degree. The converse of theorem 3.2 for quadratic algebras is shown by Priddy [7] using a spectral sequence. Here is an elementary proof:

**Proposition 3.3.**

Let $U$ be a quadratic algebra, if $\text{Ext}_U(F, F) = U^1$, then $U$ is a Koszul algebra.

**Proof.**

Suppose that $U = T(V)/C$. We compute $\text{Ext}_U(F, F)$ using the bar complex. We have $F \otimes B(U) \otimes F = T(I(U))$. This implies that $\text{Ext}_U(F, F) = H^*(T(I(U))^*)$. We denote by $D'$ the differential of this complex. The algebra $T(V^*)$ is contained in $T(I(U)^*)$ since $I(U)$ contains $V$. We use the homological degree to define the graduation of $T(I(U)^*)$. We obtain that $T(I(U)^*)_0 = T(V^*)$, and $T(I(U))_1 = \sum(V \otimes^a (T(V)/C) \otimes V \otimes^a)^*$. The kernel of the restriction of $D'$ to $T(V^*)$ is $U^1$. The complex $(I(U) \otimes U^{\dagger}, D'_U)$ is contained in the Bar complex $T((I(U), D)$. The result follows from the fact that $H^{p\cdot l}(T(I(U))) = H_{p\cdot l}(T(I(U))) = 0$ if $l \neq p$. 

4. Representations of quadratic categories. In this section we study representations of quadratic categories to the category of quadratic algebras, and adapt the results of [4].

**Definition 4.1.**

Let $U = L[V]/C$ be a quadratic algebra. We denote by $\text{Aut}(U)$. The automorphisms group of $U$. An element $h$ of $\text{Aut}(U)$ is defined by an automorphism $h_1$ of $V$ such that $(h_1 \otimes h_1)(C) = C$. Consider an algebraic group $H$, a quadratic representation of $H$, in $U$ is a morphism $\rho : H \to \text{Aut}(U)$.

**Proposition 4.2.**

Let $H$ an algebraic group, the set of quadratic representations of $H$ is a quadratic category.

**Proof.**

Let $\rho_1 : H \to \text{Aut}(U_1)$, and $\rho_2 : H \to \text{Aut}(U_2)$ be two quadratic representations, $\rho_1 \circ \rho_2$ is the representation defined as follows: for each $h \in H$ $(\rho_1 \circ \rho_2)(h) : U_1 \otimes U_2 \to U_1 \otimes U_2$ is the morphism $\rho_1(h) \circ \rho_2(h)$.

The map $(\rho_1 \circ \rho_2)(h)$ is the automorphism of $\rho_1(h) \circ \rho_2(h)$ of $U_1 \bullet U_2$.

The map $\rho_1(h)$ is the automorphism $(\rho_1(h))^1$ of $U_1^1$.

The category $C_H$ of quadratic representations endowed with the tensor products $\circ$ and $\bullet$ that we have just defined is a quadratic category.

The respective neutral elements $I_0^H$ and $I_\bullet^H$ for $\circ$ and $\bullet$ are the respective trivial representations $H \to \text{Aut}(I_0)$ and $H \to \text{Aut}(I_\bullet)$. 

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Let $V$ be a vector space, $V$ is a quadratic algebra endowed with the zero product. In this case $C = V^\otimes 2$. Thus the category of representations of $H$ is embedded in the category of quadratic representations of $H$.

**Definition 4.3.**

A morphism of the category $C_H$ of quadratic representations is defined by a family of automorphisms $e_U$ of $U$, where $\rho : H \to \text{Aut}(U)$ is an object of $C_H$, $U = T(U_1)/C$ such that $e_U$ commutes with the action of $H$ on $U_1$. We suppose that:

\[ e_{U \otimes V} = e_U \circ e_V \]
\[ e_{U \otimes V} = e_U \circ e_V \]
\[ e_{U'} = e_{U'} \]

We denote $\text{Aut}(C_H)$ the group of automorphisms of $C_H$.

**Proposition 4.4.** The natural embedding $H \to \text{Aut}(C_H)$ is an isomorphism.

**Proof.**

The proof follows from the corresponding result for tensor categories since the category of quadratic representations contains the category of representations (see [4]).

**Theorem 4.5.**

Let $(C, o, \bullet, !)$ be a quadratic braided abelian category such that $\text{End}(I_\bullet)$ is the ground field $L$, and $P$ the category of quadratic algebras defined over the field $L$. Suppose that there exists an exact faithful functor $F : C \to P$, then $(C, o, \bullet, !)$ is equivalent to the category of quadratic representations of an affine group scheme.

**Proof.**

Let $P'$ be the category whose objects are couples $(N, T)$ where $N$ is a finite dimensional vector space, and $T$ a subspace of $N \otimes N$. A morphism $h : (N, T) \to (N', T')$ is a linear map $h : N \to N'$ such that $(h \otimes h)(T) \subset T'$. The category $P'$ is endowed with the tensor product defined by $(N, T) \otimes (N', T') = (N \otimes N', t_{23}(T \otimes T'))$. The functor $D : P \to P'$, $U = T(U_1)/C \to (U_1, C)$ is fully faithful, and it is a tensor functor when $P$ is endowed with the tensor product $\bullet_P$ defined by $T(U_1)/C \bullet_P T(U'_1)/C' = T(U_1 \otimes U'_1)/t_{23}(C \otimes C')$.

Let $(C, o, \bullet, !)$ be a quadratic braided abelian category endowed with an exact faithful quadratic functor $F : C \to P$, then $D \circ F : C \to P'$ is exact and faithful. In Deligne Milne [4], it is shown that $C$ is equivalent to the category of comodules over a coalgebra $\mathcal{H}$. Since the object of $P'$ are finite dimensional spaces as in Deligne Milne [4], we can endow $\mathcal{H}$ with a structure of a bialgebra since every morphism between two quadratic functors is an isomorphism, and deduce that $(C, o, \bullet, !)$ is equivalent to the category of quadratic representations of the affine group scheme $H$ defined by $\mathcal{H}$.

In the proof of theorem 5, we can define on $P'$ the following tensor structure: Let $(U, T)$, and $(U', T')$ be two elements of $P'$, $(U, T) \otimes (U', T') = (U \otimes U', t_{23}(U \otimes T' + T \otimes U'))$. This defines on the coalgebra $\mathcal{H}$ another algebra product.

**Definition 4.6.**

A neutral braided Tannakian quadratic category is a quadratic category $(C, o, \bullet, !)$ such that there exists an exact faithful $L$-linear functor $F : C \to P$, where $P$ is the category of quadratic algebras. The functor $F$ is called a fibre functor. If $L'$ is a $L$-algebra, a $L'$-fiber functor is a fiber functor $F$ such that $DoF$ (see proof of theorem 5 for the definition of $D$) takes its values in the
category of $L'$-modules.

**Theorem 4.7.**

Let $(C, o, \bullet, !)$ be a Tannakian quadratic category, $L$ a field and $\text{Aff}_L$ the category of affine schemes defined over $L$. We denote by $\text{Fib}(C)$ the category of fiber functors of $C$. The functor $\text{Fib}(C) \to \text{Aff}_L, F \to H$ is a gerbe over $\text{Aff}_L$ The fiber of $\text{Spec}(L')$ where $L'$ is a $L$-algebra is the $L'$-valued fiber functor.

**Proof.**

Let $F' : C(o, \bullet, !) \to P$ be another fiber $L'$-functor, we have to show that $\text{Hom}(F', F')$ is representable by a $H$-torsor over $\text{Spec}(L')$ where $H$ represents $F$. The composition of a fiber functor and the functor $D$ defined in the proof of theorem 5 defines fiber a functor $C$ to the category of $L'$-module. We apply the corresponding result for Tannakian category to $(F_0(C), \bullet)$ [4].

5. Quadratic motives. Let $L$ be a field, and $V_L$ the category of projective schemes defined on $L$. A cohomological functor $F : V \to (C, \otimes)$, where $(C, \otimes)$ is a tensor abelian category, which verifies standards properties that verified cohomologies theories like Kunneth formula $F(U \times U') = F(U) \otimes F(U')$. The category of motives $F_0 : V_L \to N_L$ is an initial object in the category of cohomological functors, that is for every cohomological functor $F : V \to C$, there exists a functor of abelian categories $F_C : N_L \to C$ such that $F = F_C \circ F_0$. The construction of a category of motives is given in Deligne-Milne [4].

Let $U$ be a quadratic algebra. We denote by $\text{Proj}(U)$ the non commutative projective scheme defined by $U$. This is the category of $U$-graded modules up to the modules of finite length. The category of coherent sheaves over every projective scheme is equivalent to the non commutative projective scheme defined by a quadratic algebra. This realization is not unique, that is the category of coherent sheaves defined on a projective scheme can be equivalent to the non commutative projective schemes defined by two non isomorphic quadratic algebras.

We define the category of quadratic motives to be the category of $L$-quadratic algebras this is an abelian category.

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