A Fast Algorithm for the Discrete Core/Periphery Bipartitioning Problem

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Abstract

Various methods have been proposed in the literature to determine an optimal partitioning of the actors in a network into core and periphery subsets. However, these methods either work only for relatively small input sizes, or do not guarantee an optimal answer. In this paper, we propose a new algorithm to solve this problem. This algorithm is efficient and exact, allowing the optimal partitioning for networks of several thousand actors to be computed in under a second. We also show that the optimal core can be characterized as a set containing the actors with the highest degrees in the original network.

1 Introduction

A concept that is prevalent in the field of social network analysis is the core/periphery model. Such models arise in many fields of research, ranging from corporate structure (Barsky, 1999) and world economics (Smith and White, 1992) to scientific citation networks (Mullins et al, 1977; Doreian, 1985) and Japanese monkeys (Corradino, 1990).

As discussed in Borgatti and Everett (1999), a discrete core/periphery model can be formulated as follows: consider a set of n actors, labelled 1, 2, . . . , n, and suppose that certain pairs of these actors interact. The idea behind the model is that the actors can be partitioned into a cohesive subgraph (a ‘core’) and a loosely-connected ‘periphery’. A simple example is a star graph, where the only ties that exist are those connecting a distinguished node (1, say) to each of the other nodes. Then node 1 forms the core, and the others form the periphery.

Several algorithms have been suggested for finding an optimal or near-optimal decomposition of such a set into its core and peripheral parts. The simplest approach is to try all possible subsets as the ‘core’, and pick the one

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that works best. However (as noted by Boyd et. al., 2006), there are exponentially many such subsets, so this becomes infeasible quite rapidly as $n$ increases. It therefore appears to be necessary to resort to heuristics, or prune the search space in some way. Algorithms based on the former approach include the genetic algorithm of Borgatti et. al. (2002) in the UCINET software package, as well as algorithms based on simulated annealing and the Kernighan-Lin algorithm, considered by Boyd et al (2006). An example of an algorithm which prunes the search space can be found in the recent paper of Brusco (2011), which develops an exact algorithm based on the branch-and-bound technique that is feasible for networks with up to about 60 actors.

In this light, the main result of this paper might seem surprising: namely, that it is possible to solve this problem exactly and efficiently, without resorting to heuristics or pruning! This is true for both symmetric and asymmetric networks. The solutions that will be described in this paper are very fast, and therefore easily scalable to large networks. The basis of the algorithm is a greedy procedure that systematically picks agents with maximal degree to form part of the ‘core’, and we will also prove that this algorithm gives an optimal solution.

2 Statement of the Problem

We adopt a similar formulation to that used in Brusco (2011), and first consider the case of symmetric networks (in which $A_{ij} = A_{ji}$ for all $i$ and $j$). The symmetric core/periphery bipartitioning problem is defined as follows:

- There are $n$ actors, labelled $1, 2, \ldots, n$, and an $n \times n$ binary adjacency matrix $A$ such that $A_{ij} = 1$ if actor $i$ interacts with actor $j$, and $A_{ij} = 0$ otherwise. (We do not consider self-interactions, and assume that, for each $i$, we have $A_{ii} = 0$.)

- Define $S = \{1, \ldots, n\}$. We wish to find a proper, non-empty ‘core’ subset $S_1 \subset S$ such that the following quantity is minimized:

$$Z(S_1) = \sum_{(i < j) \in S_1} \mathbb{I}(A_{ij}=0) + \sum_{(i < j) \notin S_1} \mathbb{I}(A_{ij}=1)$$

(1)

(Here, we have employed the indicator function $\mathbb{I}(P)$, which is equal to 1 if the predicate $P$ is true, and 0 if $P$ is false.)

The intuitive idea behind this formulation is that we wish to maximize the number of ties between actors in the core, and minimize the number of ties between actors in the periphery. In an ideal scenario, there would be ties between every pair of actors in the core, and no ties between any pair of actors in the periphery. Notice that ties between core actors and periphery actors do not appear in the expression for $Z(S_1)$; this is consistent with the goal of Boyd et al. (2006) of finding a bipartition that simultaneously maximizes connectivity in the core block and minimizes connectivity in the periphery block.
3 The Algorithm

We now present a simple algorithm that solves the above problem in \(O(n^2)\) time. Before doing this, however, we pause to make two definitions:

- The **degree** of a node \(i\) is the number of ties incident to \(i\). We represent this quantity by \(\text{deg}(i)\). It can be seen that \(\text{deg}(i) = \sum_{j \in S} a_{ij}\).

- Given a node \(i\), and a subset \(T \subseteq S\), we define \(\delta_T(i)\) to be the number of ties joining \(i\) with a node in \(T\). In other words, \(\delta_T(i) = \sum_{j \in T} a_{ij}\).

We now consider a restricted version of the problem, under the assumption that the number of actors in the core, \(S_1\), is fixed at the outset and is equal to \(k\) (where \(1 \leq k < n\)). There are therefore \(\frac{k(k-1)}{2}\) pairs of distinct actors in \(S_1\), and each pair either has a tie between them, or it does not. So we can write:

\[
\sum_{(i<j) \in S_1} \mathbb{I}_{\{A_{ij}=0\}} + \sum_{(i<j) \in S_1} \mathbb{I}_{\{A_{ij}=1\}} = \frac{k(k-1)}{2}. \tag{2}
\]

Furthermore, the number of ties contributed by each node \(i \not\in S_1\) to the periphery set is simply the degree of \(i\), less the number of ties joining \(i\) to a node in \(S_1\). We can therefore write:

\[
\sum_{(i<j) \not\in S_1} \mathbb{I}_{\{A_{ij}=1\}} = \frac{1}{2} \sum_{i \not\in S_1} \mathbb{I}_{\{A_{ij}=1\}} = \frac{1}{2} \sum_{i \not\in S_1} (\text{deg}(i) - \delta_{S_1}(i)). \tag{3}
\]

Using these two results, we can express \(Z(S_1)\) as follows:

\[
Z(S_1) = \sum_{(i<j) \in S_1} \mathbb{I}_{\{A_{ij}=0\}} + \sum_{(i<j) \not\in S_1} \mathbb{I}_{\{A_{ij}=1\}}
\]

\[
= \frac{k(k-1)}{2} - \sum_{(i<j) \in S_1} \mathbb{I}_{\{A_{ij}=1\}} + \frac{1}{2} \sum_{i \not\in S_1} (\text{deg}(i) - \delta_{S_1}(i))
\]

\[
= \frac{k(k-1)}{2} - \frac{1}{2} \sum_{i \in S_1} \delta_{S_1}(i) + \frac{1}{2} \sum_{i \not\in S_1} \text{deg}(i) - \frac{1}{2} \sum_{i \not\in S_1} \delta_{S_1}(i)
\]

\[
= \left( \frac{1}{2} \sum_{i \in S} \text{deg}(i) + \frac{k(k-1)}{2} \right) - \frac{1}{2} \left( \sum_{i \in S_1} \text{deg}(i) + \sum_{i \not\in S_1} \delta_{S_1}(i) \right)
\]

\[
= \left( \frac{1}{2} \sum_{i \in S} \text{deg}(i) + \frac{k(k-1)}{2} \right) - \sum_{i \in S_1} \text{deg}(i) \tag{4}
\]

where the final equality arises because

\[
\sum_{i \in S} \delta_{S_1}(i) = \sum_{i \in S} \sum_{j \in S_1} \mathbb{I}_{\{A_{ij}=1\}} = \sum_{j \in S_1} \sum_{i \in S} \mathbb{I}_{\{A_{ij}=1\}} = \sum_{j \in S_1} \text{deg}(j). \tag{5}
\]
If \( k \) is fixed, the terms in the first bracket are independent of the choice of \( S_1 \), so our problem reduces to finding an \( S_1 \) of size \( k \) such that the final term is maximized. Clearly, we should therefore take \( S_1 \) to consist of the \( k \) nodes with largest degree in \( S \). This can be done in \( O(n \log n) \) time, since it takes \( O(n \log n) \) time to sort the nodes in descending order of degree using a standard algorithm such as merge sort (Knuth, 1998), and a further \( O(k) \) time to construct \( S_1 \).

We can now return to the original problem and treat the case in which \( k \) is unknown. Assume that the nodes are sorted in descending order of degree, and that the resulting list of nodes is \( \{v_1, v_2, \ldots, v_n\} \). We can then determine the optimal \( S_1 \) by iterating through the possible values of \( k \) and calculating the optimal \( Z(S_1) \) for each, and finally taking the best one.

Note that we need not repeat the calculation from scratch in each iteration, because of the following observation: the addition of \( v_k \) to the optimal set increases the value of \( Z \) by

\[
\left( \frac{k(k-1)}{2} - \sum_{i=1}^{k} \deg(v_i) \right) - \left( \frac{(k-1)(k-2)}{2} - \sum_{i=1}^{k-1} \deg(v_i) \right) = k - 1 - \deg(v_k).
\]

Initially, the core set is empty, and so the starting value of \( Z \) is

\[
Z(\emptyset) = \sum_{(i<j) \in S} \Pi(A_{ij}) = \frac{1}{2} \sum_{i \in S} \deg(i). \tag{7}
\]

The full algorithm can therefore be specified as follows:

1. Calculate and store the degrees of each node. Then sort the nodes in descending order of degree, to get a list of nodes \( \{v_1, v_2, \ldots, v_n\} \).

2. Set \( Z_{\text{best}} := \infty \) and \( k_{\text{best}} := 0 \). (Note: instead of \( \infty \), a suitably large upper bound, such as \( n^2 \), can be used.)

3. Set \( Z := \frac{1}{2} \sum_i \deg(i) \).

4. For each \( k \) from 1 to \( n-1 \), inclusive: set \( Z := Z + k - 1 - \deg(v_k) \). Then, if \( Z < Z_{\text{best}} \), set \( Z_{\text{best}} := Z \) and \( k_{\text{best}} := k \).

5. Set \( S_1 := \{v_1, \ldots, v_{k_{\text{best}}}\} \).

6. Return \( S_1 \).

Reading the input (i.e., the adjacency matrix describing the network) takes \( O(n^2) \) time, and so does calculating the degrees of each node. Sorting the nodes takes \( O(n \log n) \) time, and all other operations take \( O(n) \) time, so the algorithm runs in \( O(n^2) \) time. If the input data is presented in the form of an adjacency list (i.e., as a set of \( n \) lists such that the \( i \)-th list contains the neighbours of the \( i \)-th actor), or simply as a list of existing ties, the algorithm would run in \( O(n \log n + m) \) time, where \( m \) is the number of ties in the network.
This algorithm is therefore a significant improvement on both the branch-and-bound and the heuristic approaches. The branch-and-bound method provides an optimal answer, but is slow; the heuristic approaches do not guarantee an optimal answer. The algorithm just described provides an optimal answer, and does so quickly.

As an aside, it is possible to improve the main part of this algorithm further. Let $Z_k$ be the value of $Z$ after the $k$-th iteration of the algorithm. Notice that the sequence \( \{\deg(v_k) : 1 \leq k < n\} \) is non-increasing. Therefore, there exists a $k^*$ such that $Z_1 \geq Z_2 \geq \ldots \geq Z_{k^*} \leq Z_{k^*+1} \leq \ldots \leq Z_{n-1}$. This observation allows us to determine the optimum value of $k$ in $O(\log n)$ time, by binary searching on $k$ to find the largest $k^*$ such that $k^* - 1 - \deg(v_{k^*}) \leq 0$. Once we have found this optimal value, we pick our core to be $S_1 = \{v_1, \ldots, v_{k^*}\}$, as before. However, this does not lead to an order-of-magnitude improvement in the time complexity, because, e.g., it still takes $O(n \log n)$ time to sort the nodes at the beginning of the algorithm.

4 Generalization to Asymmetric Networks

In the version of the problem described by Brusco (2011), the underlying networks were allowed to be symmetric or asymmetric. We now consider the asymmetric case. The definition of the matrix $A$ then changes slightly: we now have $A_{ij} = 1$ if there is a tie from actor $i$ to actor $j$, and $A_{ij} = 0$ otherwise. The objective is now to find a proper subset $S_1 \subset S$ such that

$$Z(S_1) = \frac{1}{2} \sum_{(i<j) \in S_1} (\mathbb{I}_{A_{ij}=0} + \mathbb{I}_{A_{ji}=0}) + \frac{1}{2} \sum_{(i<j) \notin S_1} (\mathbb{I}_{A_{ij}=1} + \mathbb{I}_{A_{ji}=1})$$

is minimized.

To solve this version of the problem, we introduce a symmetric weight function $w(i,j) = \frac{1}{2}(\mathbb{I}_{A_{ij}=1} + \mathbb{I}_{A_{ji}=1})$, for any two nodes $i \neq j$. Then we can write

$$Z(S_1) = \sum_{(i<j) \in S_1} (1 - w(i,j)) + \sum_{(i<j) \notin S_1} w(i,j). \tag{9}$$

Finally, we redefine $\deg(i) = \sum_{j \in S} w(i,j)$, and $\delta_T(i) = \sum_{j \in T} w(i,j)$. It is now straightforward to check that the analysis in Section 3 carries over to this case, after we replace $\mathbb{I}_{A_{ij}=0}$ with $(1-w(i,j))$, and $\mathbb{I}_{A_{ij}=1}$ with $w(i,j)$. Therefore, the algorithm in Section 3 still holds (albeit with a modified definition of degree), and its time complexity remains unchanged.

5 Tests of the Algorithm

For input graphs with $n$ up to about 1000, the algorithm runs in under a second. This could be sped up significantly if the graph is sparse ($m \ll n^2$) and the data
is presented in the form of an adjacency list (or a list of ties), since the algorithm then takes $O(n \log n)$ time and can therefore handle networks with $n$ up to about 50000 in under a second. (These estimates are conservative.)

As a check, the algorithm described in Section 3 was tested, together with the brute force algorithm (which tries every possible subset of $S$ as the core and is therefore guaranteed to produce the optimal answer), on 100 random input cases with $5 \leq n \leq 25$. Both algorithms produced the same answer each time, and our algorithm is noticeably faster.

6 Conclusions

We have presented an exact, efficient algorithm to solve a discrete core/periphery bipartitioning problem. This algorithm outperforms both the heuristic and exhaustive search methods that have so far been used, and vastly increases the sizes of the problems that can be tackled.

We also offer the qualitative insight that the actors which make up the core are simply the ones with the most connections in the original network. As the actors with highest degree are inserted into the core, the size of the core increases until it hits a well-defined threshold. Beyond this threshold, it becomes less attractive to add new actors to the core because the degrees of the entering actors are not large enough to compensate for the core’s increasing size.

Note that this particular formulation of the core/periphery bipartitioning problem is solved by choosing the most central nodes to lie in the core, where ‘centrality’ in this case is defined as degree centrality. However, other measures of centrality are often used (Wasserman and Faust, 1994), and it may be possible to formulate alternative definitions of a core/periphery bipartitioning in which the optimal solution takes into account the betweenness or closeness centralities of the actors. Furthermore, it would be interesting to try and extend the algorithm presented in this paper to other variants of the core/periphery bipartitioning problem, some of which have continuous (as opposed to discrete) formulations.

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