Set of Orthogonal Basis Functions over the Binocular Pupil

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Sets of orthogonal basis functions over circular areas—often representing pupils in optical applications—are known in the literature for the full circle (Zernike or Jacobi polynomials) and the annulus. Here, an orthogonal set is established if the area is two non-overlapping circles of equal size. The main free geometric parameter is the ratio of the pupil radii over the distance between both circles. Increasingly higher order aberrations—as defined for a virtual larger pupil in which both pupils are embedded—are fed into a Gram-Schmidt orthogonalization to distill one unique set of basis functions. The key effort is to work out the overlap integrals between a full set of primitive basis functions, which are chosen to be products of powers of the distance from the mid-point between both pupils by azimuthal functions of the Fourier type.

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I. AIM AND SCOPE

Manufacturing schemes of lenses and mirrors inevitably prefer circular cross-sections of beams, and the associated description of functions (aberrations) defined across these fields calls for basis functions on this circular support, the best-noted probably being the Zernike functions [2, 4, 17, 19, 21]. Masking a central circular portion of a circular beam leads to annular, ring-shaped regions, for which orthogonal basis sets are also established [8, 10, 14, 24, 26]. This work proceeds to the task of defining such a basis set for a two-beam interferometer, in which the input pupil is defined by two disconnected circular areas of equal radius [11].

We define the area of integration in a global spherical coordinate system centered in between the two apertures in Section II. Supposed anonymous functions defined over these apertures are expanded with a separation ansatz as products of powers of the distance to the origin of coordinates by the usual Fourier series in the azimuth, all integrals over products of these can be reduced to a generic integral, summarized in Section III. The value of this article lies in the the reduction formulas of two associated integrals in two appendices. Section IV proceeds with an application, the re-orthogonalization of the Zernike basis functions—defined over the larger area that encompasses both circles—with respect to the two circular regions that define the binocular pupil.

II. BINOCULAR GEOMETRY

Orthogonality of functions $f_k$ and $f_l$ over two-dimensional areas is defined through their product integrated over the area

$$(f_k, f_l) = \int \int f_k^* f_l d\Omega \sim \delta_{kl}. \quad (1)$$

In Cartesian coordinates $x$ and $y$, or circular coordinates with distance $\rho$ to the origin and azimuth $\theta$,

$$x = \rho \cos \theta; \quad y = \rho \sin \theta, \quad (2)$$

the differential is $d\Omega = dx dy$ or $d\Omega = \rho d\rho d\theta$. This manuscript deals with two-dimensional areas that are the sum of the interior of two circular pupils represented by

$$(x \pm R)^2 + y^2 \leq r^2, \quad (3)$$
where $R$ is half the distance between the two pupil centers, where $2R$ is the interferometric baseline, and $r$ is each pupil’s radius (Fig. 1). This transforms the integral operator into a sum over both circles,

$$
\int\int d\Omega = \int_{R-r}^{R+r} d\rho \rho \left[ \int_{2\rho R \cos \theta \leq r^2 - R^2 - \rho^2} d\theta + \int_{2\rho R \cos \theta \geq r^2 - \rho^2 + R^2} d\theta \right].
$$

(4)

Scaling distances in units of $R$, $z \equiv \rho/R$, leaves one essential shape parameter, $q \equiv r/R$. To avoid double-counting of areas, these must not overlap:

$$
0 \leq q \leq 1.
$$

(5)

The range of radial distances that lie inside the circle centered at $x = \pm R$ for some fixed direction $\theta$ is

$$
z = \cos \theta \pm \sqrt{q^2 - \sin^2 \theta},
$$

(6)

and for the other one centered at $x = -R$,

$$
z = -\cos \theta \pm \sqrt{q^2 - \sin^2 \theta}.
$$

(7)

In each of the two equations, the lower sign connects to the intersection with the circle rim that is closer to the origin, the upper sign to the intersection with the farther one.

III. GENERIC INTEGRAL

At the heart of this work is performing overlap integrals in analytical terms over the area described above; since this will be based on spanning the functional space with “primitive” basis functions of the type

$$
p_k \equiv p_{n_k, m_k}(z, \theta) \equiv z^{n_k} e^{im_k \theta},
$$

(8)

the generic integral reads

$$
A_{n, m}(q) \equiv \int_{|\sin \theta| \leq q} d\theta \int_{2z \cos \theta \leq q^2 - z^2} dz \int_{2z \cos \theta \geq z^2 - q^2 + 1} dz z^n e^{im\theta}, \quad n, m = 0, 1, 2, \ldots
$$

(9)
\( i \) is the imaginary unit, \( m \) the azimuthal frequency, and \( n \) the power to build a complete radial basis. With a sign tag \( P \) defined for both circles,

\[
P \equiv \begin{cases} 
  +1, & |\theta| < \pi/2, \\
  -1, & \pi/2 \leq |\theta| \leq \pi, 
\end{cases}
\]

(10)
The integration over \( z^{n+1} \) may be executed,

\[
(n + 2)A_{n,m}(q) = \int_{|\sin \theta| \leq q} d\theta \left[ \left( P \cos \theta + \sqrt{q^2 - \sin^2 \theta} \right)^{n+2} - \left( P \cos \theta - \sqrt{q^2 - \sin^2 \theta} \right)^{n+2} \right] e^{im\theta}.
\]

(11)
The contribution from the imaginary part proportional to \( \sin(m\theta) \) vanishes; \( e^{im\theta} \) can be replaced by \( \cos(m\theta) \) in this equation. Considering the coordinate transformation \( y \leftrightarrow -y \) or \( \theta \leftrightarrow -\theta \), the integral over all four quadrants can be reduced to an integral over the first and second quadrant and a factor of 2:

\[
(n + 2)A_{n,m}(q) = 2 \int_{\sin \theta \leq q} d\theta \left[ \left( P \cos \theta + \sqrt{q^2 - \sin^2 \theta} \right)^{n+2} - \left( P \cos \theta - \sqrt{q^2 - \sin^2 \theta} \right)^{n+2} \right] \cos(m\theta).
\]

(12)
Considering also the variable transformation \( \theta \leftrightarrow \pi - \theta \), the parities of the Chebyshev term \( T_m(\cos \theta) = \cos(m\theta) \), of \( \cos \theta \) and of \( P \), this vanishes for odd \( m \) and reduces to an integral over the first quadrant for even \( m \),

\[
\frac{(n + 2)A_{n,m}(q)}{4} = \int_{\sin \theta \leq q} d\theta \left[ \left( \cos \theta + \sqrt{q^2 - \sin^2 \theta} \right)^{n+2} - \left( \cos \theta - \sqrt{q^2 - \sin^2 \theta} \right)^{n+2} \right] \cos(m\theta).
\]

(13)
If we define the integrals

\[
B_{n+2,m}^\pm(q) \equiv \int_0^{\arcsin q} d\theta \left( \cos \theta \pm \sqrt{q^2 - \sin^2 \theta} \right)^{n+2} T_m(\cos \theta),
\]

(14)
this can be rephrased

\[
\frac{(n + 2)A_{n,m}(q)}{4} = \begin{cases} 
  0 & \text{for } m = 1, 3, 5, 7, \ldots \\
  B_{n+2,m}^+(q) - B_{n+2,m}^-(q) & \text{for } m = 0, 2, 4, 6, \ldots
\end{cases}
\]

(15)
The table summarizes the results obtained in Appendix A as series expansions of \( A_{n,m}(q) \) for small \( n \) and \( m \).

| \( n \) | \( m \) | \( A_{n,m}(q) / \pi \) |
| --- | --- | --- |
| 0 | 0 | +2q^2 |
| 0 | 2 | +2q^2 - q^4 |
| 0 | 4 | +2q^2 - 4q^4 + 2q^6 |
| 0 | 6 | +2q^2 - 9q^4 + 12q^6 - 5q^8 |
| 0 | 8 | +2q^2 - 16q^4 + 40q^6 - 40q^8 + 14q^{10} |
| 0 | 10 | +2q^2 - 25q^4 + 100q^6 - 175q^8 + 140q^{10} - 42q^{12} |
| 1 | 0 | +2q^2 + 1/4q^4 + 1/96q^6 + 1/512q^8 + 5/8192q^{10} + 49/16384q^{12} + 63/524288q^{14} + 1089/16777216q^{16} + \ldots |
| 1 | 2 | +2q^2 - 3/4q^4 + 3/32q^6 + 5/512q^8 + 21/8192q^{10} + 63/65536q^{12} + 231/524288q^{14} + 3861/16777216q^{16} + \ldots |
| 1 | 4 | +2q^2 - 15/4q^4 + 75/32q^6 - 175/512q^8 - 315/8192q^{10} - 693/65536q^{12} - 2145/524288q^{14} - 32175/16777216q^{16} + \ldots |
| 1 | 6 | +2q^2 - 35/4q^4 + 1225/96q^6 - 3675/512q^8 + 8085/8192q^{10} + 7007/65536q^{12} + 15015/524288q^{14} + 182325/16777216q^{16} + \ldots |
| 2 | 0 | +2q^2 + q^4 |
| 2 | 2 | +2q^2 |
| 2 | 4 | +2q^2 - 3q^4 + 2q^6 - 1/2q^8 |
| 2 | 6 | +2q^2 - 8q^4 + 12q^6 - 8q^8 + 2q^{10} |
| 2 | 8 | +2q^2 - 15q^4 + 40q^6 - 50q^8 + 30q^{10} - 7q^{12} |
| 2 | 10 | +2q^2 - 24q^4 + 100q^6 - 200q^8 + 210q^{10} - 112q^{12} + 24q^{14} |
| 3 | 0 | +2q^2 + 9/4q^4 + 3/32q^6 + 1/512q^8 + 9/4096q^{10} + 3/65536q^{12} + 7/524288q^{14} + 81/16777216q^{16} + \ldots |
| 3 | 2 | +2q^2 + 5/4q^4 - 5/32q^6 + 5/512q^8 + 5/8192q^{10} + 7/65536q^{12} |
away from their common origin at $z = 0$, the relative strength of the variation introduced by the power $n$ and by the modulation $\propto \cos(m\theta)$ of the function in the integral kernel loses importance.

To lowest order in $q$, $A_{n,m}$ equals $2\pi q^2$, the total area of both circles. This is expected, because for circles far away from their common origin at $z = 0$, the relative strength of the variation introduced by the power $n$ and by the modulation $\propto \cos(m\theta)$ of the function in the integral kernel loses importance.

| $n$ | $m$ | Expansion |
|-----|-----|-----------|
| 0   | 1   | $2q^2$    |
| 1   | 0   | $2q^2$    |
| 1   | 1   | $2q^2$    |
| 2   | 0   | $2q^2$    |
| 2   | 1   | $2q^2$    |
| 3   | 0   | $2q^2$    |
| 3   | 1   | $2q^2$    |
| 3   | 2   | $2q^2$    |
| 4   | 0   | $2q^2$    |
| 4   | 1   | $2q^2$    |
| 4   | 2   | $2q^2$    |
| 4   | 3   | $2q^2$    |
| 5   | 0   | $2q^2$    |
| 5   | 1   | $2q^2$    |
| 5   | 2   | $2q^2$    |
| 5   | 3   | $2q^2$    |
| 5   | 4   | $2q^2$    |
| 6   | 0   | $2q^2$    |
| 6   | 1   | $2q^2$    |
| 6   | 2   | $2q^2$    |
| 6   | 3   | $2q^2$    |
| 6   | 4   | $2q^2$    |
| 6   | 5   | $2q^2$    |
| 7   | 0   | $2q^2$    |
| 7   | 1   | $2q^2$    |
| 7   | 2   | $2q^2$    |
| 7   | 3   | $2q^2$    |
| 7   | 4   | $2q^2$    |
| 7   | 5   | $2q^2$    |
| 7   | 6   | $2q^2$    |
| 8   | 0   | $2q^2$    |
| 8   | 1   | $2q^2$    |
| 8   | 2   | $2q^2$    |
| 8   | 3   | $2q^2$    |
| 8   | 4   | $2q^2$    |
| 8   | 5   | $2q^2$    |
| 8   | 6   | $2q^2$    |
| 8   | 7   | $2q^2$    |
| 9   | 0   | $2q^2$    |
| 9   | 1   | $2q^2$    |
| 9   | 2   | $2q^2$    |
| 9   | 3   | $2q^2$    |
| 9   | 4   | $2q^2$    |
| 9   | 5   | $2q^2$    |
| 9   | 6   | $2q^2$    |
| 9   | 7   | $2q^2$    |
| 9   | 8   | $2q^2$    |
| 9   | 9   | $2q^2$    |
| 10  | 0   | $2q^2$    |
| 10  | 1   | $2q^2$    |
| 10  | 2   | $2q^2$    |
| 10  | 3   | $2q^2$    |
| 10  | 4   | $2q^2$    |
| 10  | 5   | $2q^2$    |
| 10  | 6   | $2q^2$    |
| 10  | 7   | $2q^2$    |
| 10  | 8   | $2q^2$    |
| 10  | 9   | $2q^2$    |
| 10  | 10  | $2q^2$    |

**TABLE I:** Series expansion of $A_{n,m}(q)/\pi$. It is a polynomial in $q^2$ if $n + m$ is even, else shown in truncated form up to $O(q^{16})$. 
IV. GRAM-SCHMIDT ORTHOGONALIZATION

A. Procedure.

The integral evaluated in Section III allows to calculate the overlap integral (inner product) between any two functions expressed as linear combinations (“contractions”) of “primitive” basis functions of the form $g_n$ in the global circular coordinate system centered at the middle between the two sub-pupils, because the overlap between two of these is

\[(p_k, p_l) = \iint p_k^* p_l d\Omega = \iint z^{n_k} e^{-im_k\theta} z^{n_l} e^{im_l\theta} zd\theta d\theta = A_{n_k+n_l|m_k-m_l}(q). \quad (16)\]

The quickest, obvious way of obtaining some orthogonal basis set from any set of contracted primitive basis function is to diagonalize the overlap matrix containing all the overlap integrals between pairs of these basis functions [7]. To end up with some standardization of these orthogonal bases, we use the Gram-Schmidt procedure, which builds this set \[\{f_k\}\] incrementally. At each step of the procedure, an ansatz

\[f_{k+1}(z, \theta) = \beta_{k+1} \left[ g_{k+1}(z, \theta) + \sum_{l=1}^{k} \gamma_{k+1,l} f_l(z, \theta) \right] \quad (17)\]

is made for the next, \((k+1)\)st additional basis function \(f_{k+1}\), given a seed function \(g_{k+1}(z, \theta)\) plus the \(f_l\) generated by the earlier steps. Essentially, the projections of the seed along all earlier directions are subtracted, and the residual is normalized to unity. The request of orthogonality

\[\iint f_k^* f_l d\Omega = (f_k, f_l) = \delta_{kl}, \quad k, l = 1, \ldots , k + 1, \quad (18)\]

means the projection coefficients \(\gamma\) can be calculated from the \(k\) overlaps between the seed and the earlier basis functions,

\[\gamma_{k+1,l} = -(g_{k+1}, f_l), \quad l = 1, \ldots k. \quad (19)\]

The normalization \(\beta_{k+1}\) is finally computed from the self-overlap of the seed and the sum over the squared \(\gamma\),

\[1 = \beta^2_{k+1} \left[ (g_{k+1}, g_{k+1}) - \sum_{l=1}^{k} \gamma^2_{k+1,l} \right]. \quad (20)\]

The Gram-Schmidt methodology is well known in the literature for different geometric shapes of pupils [15, 23, 25]. Still, the procedure establishes different basis sets depending on the order in which the seeds \(g\) are fed into the procedure, and depending on which functional subspaces they span.

B. Zernike Seeds.

To define a unique set \[\{f_j\}\] of functions orthogonal over the binocular pupil, we may choose the real and imaginary parts of the primitives \[\{g_n\}\] in increasing order of complexity; ie, increasing order of aberration and increasing \(n\) and \(m\), as the seeds, as if one would subduce the Zernike polynomials \(Z_n(z, \theta)\) over the full super-pupil of radius \(R\) (normalized to \(z=1\)) in the Noll order of indexing [17] into the two sub-apertures. (We identify the variable \(z\) with the radial variable of the Zernike polynomials, although the corresponding Zernike radius would need to be \(1+q\), not 1, to cover both sub-apertures in full.) There is an “outer” loop over \(n = 0, 1, 2, \ldots\) and an “inner” loop over \(m = n\) (mod 2), \(\ldots\) \(n,\) considering only even \(n-m\):

\[g_1 = 1; \quad g_2 = z \cos \theta; \quad g_3 = z \sin \theta; \quad (21)\]

\[g_4 = 2z^2 - 1; \quad g_5 = z^2 \sin(2\theta); \quad g_6 = z^2 \cos(2\theta); \quad (22)\]

\[g_7 = (3z^3 - 2z) \sin \theta; \quad g_8 = (3z^3 - 2z) \cos \theta; \quad g_9 = z^3 \sin(3\theta); \quad g_{10} = z^3 \cos(3\theta); \quad \ldots \quad (23)\]

For this choice, the \(n+1\) basis functions from \(g_{(n+1)(n+1)/2}\) up to and including \(g_{(n+1)(n+2)/2}\) are associated with a polynomial of order \(n\) in \(z\). The arithmetic remains real-valued, because the “atoms” of the seeds are the separated...
real and imaginary part of (16) is split into
\[\int \int z^m z^n \cos(m\theta) \sin(n\theta) d\Omega = \frac{1}{2} A_{n_k + n_l, m_k + m_l} ; \]
\[\int \int z^m z^n \sin(m\theta) \cos(n\theta) d\Omega = \frac{1}{2} A_{n_k + n_l, m_l - m_k} ; \]
\[\int \int z^m z^n \cos(m\theta) \sin(n\theta) d\Omega = 0. \]

The first basis functions created with this recipe are discussed shortly in analytical form. We start with the “global common piston” \( g_1 \) which just needs to be normalized:
\[f_1 = \frac{1}{q} \sqrt{\frac{1}{2\pi}}. \] (27)

Next we feed what represents most of the differential piston, \( g_2 \), which turns out to be already orthogonal to \( f_1 \) and only needs to be normalized,
\[f_2 = \frac{1}{q} \sqrt{\frac{2}{\pi(4 + q^2)}} z \cos \theta. \] (28)

Next we feed \( g_3 \), some common sideways tilt perpendicular to the baseline between the two pupils,
\[f_3 = \frac{1}{q^2} \sqrt{\frac{2}{\pi}} z \sin \theta. \] (29)

The first case of nonzero overlap with an earlier basis function occurs when we use \( g_4 \) as a seed (some nodding tilt between the sub-pupils along the baseline), which has a nonzero component along \( f_1 \):
\[f_4 = \frac{1}{q^2} \sqrt{\frac{6}{\pi(12 + q^2)}} \left( z^2 - \frac{2 + q^2}{2} \right). \] (30)

Feeding \( g_5 \) to \( g_8 \) we get
\[f_5 = \frac{1}{q^2} \sqrt{\frac{3}{\pi(6 + q^2)}} z^2 \sin(2\theta). \] (31)
\[f_6 = \frac{1}{q} \sqrt{\frac{3}{\pi(18 + q^2)(12 + q^2)}} \left[ \frac{12 + q^2}{q^2} z^2 \cos(2\theta) - \frac{12}{q^2} z^2 + 5 \right]. \] (32)
\[f_7 = \frac{2}{q^2} \sqrt{\frac{1}{\pi(12 + q^2)}} \left[ 3z^3 \sin \theta - (3 + 2q^2)z \sin \theta \right]. \] (33)
\[f_8 = \frac{2}{q^2} \sqrt{\frac{1}{\pi Q_8(4 + q^2)}} \left[ 3(4 + q^2) z^3 \cos \theta - (12 + 21q^2 + 2q^4) z \cos \theta \right]. \] (34)

The abbreviation \( Q_8 = 288 + 48q^2 + 48q^4 + q^6 \) is used. Continuing with \( g_9 \),
\[f_9 = \frac{2}{q^2} \sqrt{\frac{1}{\pi(24 + q^2)(12 + q^2)}} \left[ \frac{12 + q^2}{q^2} z^3 \sin(3\theta) + 21z \sin(\theta) - \frac{36}{q^2} z^3 \sin \theta \right]. \] (35)
\[f_{10} = \frac{2}{q} \sqrt{\frac{1}{\pi Q_8(2304 + 704q^2 + 480q^4 + 60q^6 + q^8)}} \times \left[ \frac{Q_8}{q^2} z^3 \cos(3\theta) + 21(4 + 2q + q^2)(4 - 2q + q^2)z \cos(\theta) - \frac{12(24 + 8q^2 + 3q^4)}{q^2} z^3 \cos \theta \right]. \] (36)
\[ f_{11} = \frac{1}{q^2} \sqrt{\frac{10}{\pi Q_{11}(18 + q^2)}} \times \left[ 3(18 + q^2)z^4 - 3(24 + 26q^2 + q^4)z^2 + \frac{108 + 30q^2 + 36q^4 + q^6}{2} - 12(3 + q^2)z^2 \cos(2\theta) \right]. \] (37)

The abbreviation \( Q_{11} \equiv 2880 + 240q^2 + 78q^4 + q^6 \) is used.

\[ f_{12} = \frac{1}{q^2} \sqrt{\frac{5}{\pi Q_{11}(64000 + 10240q^2 + 1840q^4 + 128q^6 + q^8)}} \times \left[ \frac{4Q_{11}}{q^2} z^4 \cos(2\theta) - \frac{3(3840 + 3200q^2 + 400q^4 + 84q^6 + q^8)}{q^2} z^2 \cos(2\theta) - 21(160 + 108q^2 + 4q^4 + q^6) \right.

\[ + \left. \frac{12(960 + 1040q^2 + 38q^4 + 13q^6)}{q^2} z^2 - \frac{60(192 + 8q^2 + 3q^4)}{q^2} z^4 \right]. \] (38)

\[ f_{13} = \frac{1}{q^2} \sqrt{\frac{5}{\pi(640 + 80q^2 + 56q^4 + q^6)(6 + q^2)}} \left[ 4(6 + q^2)z^4 \sin(2\theta) - 3(8 + 12q^2 + q^4)z^2 \sin(2\theta) \right]. \] (39)

Figs. 2–4 illustrate the first eighteen of these functions for \( q = 1/2 \).

V. INTERFEROMETRIC SIGNAL

Expansions in orthogonal bases lead to accelerated book-keeping: the integral over the square of a function becomes the sum of the squared expansion coefficients (Parseval’s equation). In Maxwellian electrodynamics, the function is one of the two polarizations of the electric field vector \( \mathcal{E} \), and the simplification addresses how much total energy passes through the cross-section. If the circles are the entrance to a two-beam interferometer, this addresses computation of the photometric signal.

For a field expanded in the primitive basis,

\[ \mathcal{E}(z, \theta) = \sum_k \eta_k z^{n_k} e^{im_k \theta}, \] (40)

the interferometric signal correlates values at conjugated points \( P = \pm 1, x = P + s \cos \phi, y = s \sin \phi \), sharing the same local radial coordinate \( s \) and azimuth \( \phi \) (Fig. 1). The coordinate transformation to global \((z, \theta)\) circular coordinates are

\[ z = \sqrt{x^2 + y^2} = \sqrt{1 + 2Ps \cos \phi + s^2}; \quad \theta = \arctan(y/x) = (\pi +) \arctan \frac{s \sin \phi}{P + \cos \phi}. \] (41)

where (+\( \pi \)) indicates that \( \pi \) is to be added for \( P = -1 \) if \( \arctan \) denotes the principal value. The interferometric signal (spatial autocorrelation) is calculated by multiplying \( \mathcal{E} \) at two conjugated points in the pupils, a distance \( 2R \) apart in the \( x \)-direction, and integrating over \( s \) and \( \phi \). After transformation of \( \mathcal{E} \) at \((z, \theta)\) to the individual \((s, \phi)\) coordinates, the interferometric signal breaks down into a sum over products of the expansion coefficients in terms of these shifted/scaled polynomials. As we have set up each \( f_k \) as a linear combination of Zernike Polynomials \( Z_{k \leq k}(r, \theta) \), the route to transformations to polar coordinates originating at the two circle’s centers is known from the literature.

We summarize this in our notation; this is off-topic in the sense that it is not related to the orthogonality introduced above.

The interferometric intensity is \( I_{P=-1}(s, \phi) \mathcal{E}_{P=+1}(s, \phi) dsd\phi = \sum_k \eta_k^* \eta_l I_{k,l}, \) the generic information contained in

\[ I_{k,l} = \int_0^q dsd \int_0^{2\pi} d\phi (z^{n_k} e^{-im_k \theta})|_{P=-1}(z^{n_l} e^{im_l \theta})|_{P=+1} \] (42)

\[ = \int_0^q dsd \int_0^{2\pi} d\phi (1 - 2s \cos \phi + s^2)^{n_k/2} (1 + 2s \cos \phi + s^2)^{n_l/2} e^{-im_k |\pi + \arctan \frac{s \sin \phi}{P + \cos \phi}|} e^{im_l \arctan \frac{s \sin \phi}{P + \cos \phi}}. \] (43)
FIG. 2: The basis functions $f_1$ to $f_6$ for $q = 1/2$. The level of $f_i = 0$ is indicated with small horse-shoe rims around both sub-pupils.

We expand the integrand in a power series of $s$, integrate term by term and show the results in form of the first terms of a power series in $q$. The mean and excess of the four parameters,

$$m^{(+)} \equiv (m_k + m_l)/2; \quad m^{(-)} \equiv (m_k - m_l)/2; \quad n^{(+) \equiv (n_k + n_l)/2; \quad n^{(-)} \equiv (n_k - n_l)/2 \quad (44)$$
FIG. 3: The basis functions $f_7$ to $f_{12}$ for $q = 1/2$. 
FIG. 4: The basis functions $f_{13}$ to $f_{18}$ for $q = 1/2$. 
are defined to compress the notation.

\[
(-1)^{m_k} I_{k,l} = \pi q^2 - \left[ 2m^+ - n^- \right] \frac{\pi q^4}{8}
- \left[ (2m^+ - n^-)^2 + 2(2m^- - n^+)^2 \right] \frac{\pi q^6}{192}
+ \left[ 2m^+ - n^- \right] \left[ 8 + (2m^+ - n^-)^2 + 6(2m^- - n^+) \right] \frac{\pi q^8}{9216}
+ \left[ (2m^+ - n^-)^4 + 4(2m^+ - n^-)^2 \right] \left\{ 8 + 3(2m^- - n^+) \right\}
+ 12(2m^- - n^+)(4 + 2m^+ - n^-) \frac{\pi q^{10}}{37280}
- \left[ 2m^+ - n^- \right] \left[ 384 + (2m^+ - n^-)^4 + 20(2m^- - n^-)^2(4 + 2m^- - n^+) \right]
+ 20(2m^- - n^+)(20 + 3(2m^- - n^+)) \frac{\pi q^{12}}{88473600}
- \left[ (2m^+ - n^-)^6 + 10(2m^- - n^-)^4 \right] \left\{ 16 + 3(2m^- - n^+) \right\}
+ 4(2m^+ - n^-)^2 \left\{ 736 + 420(2m^- - n^+) + 45(2m^- - n^+)^2 \right\}
+ 120(2m^- - n^+)(8 + 2m^- - n^-)(4 + 2m^- - n^-) \frac{\pi q^{14}}{14863564800} + \ldots
\] (45)

The symbol \(\dagger\) indicates that bracket to its left represents the product of two factors. The first factor is the bracketed term as written; the second factor is the term after the substitutions \(n^- \rightarrow -n^-\) and \(m^- \rightarrow -m^-\). So in the second factor of the product, some sign flips occur whenever the total power of the \(m^-\) and \(n^-\) is odd.

It is equivalent to two re-expansions of the field at shifted centers of the sub-pupils [6] followed by areal integration. Selection rules are implicit; if \(2m^- - n^+\) or \(2m^+ - n^-\) vanish, many coefficients in the power series become zero.

### VI. FOURIER REPRESENTATION

All basis functions \(f_j\) are linear superpositions of the primitive type [8] after the \(\cos(m \theta)\) are replaced by \([\exp(i m \theta) + \exp(-i m \theta)]/2\) and the \(\sin(m \theta)\) by \([\exp(i m \theta) - \exp(-i m \theta)]/(2i)\). The two-dimensional Fourier Transform of these is

\[
A_{n,m}(q, \sigma) = \int \exp(2\pi i \sigma \cdot z) z^n \exp(i m \theta) d\Omega
\] (46)

for some wave number \(\sigma\). These integrals are calculated by individually translating each of the two circular areas to the origin of coordinates as described in [11].

\[
A_{n,m}(q, \sigma) = \sum_{P=\pm 1} e^{2\pi i P \sigma_x} \int_{s \leq q} e^{2\pi i \sigma_s \cdot z} z^n e^{im \theta} d^2 s,
\] (47)

where \(\sigma_x\) is the component of \(\sigma\) along the baseline axis. The coordinate transformation may interpret \(ze^{i \theta}\) as a complex variable. Since only the cases of even \(n - m\) need to be considered,

\[
z^n e^{im \theta} = z^n (ze^{i \theta})^m = (x^2 + y^2)^{(n-m)/2}(x + iy)^m
\] (48)

is expanded in multinomials of \(x\) and \(y\), \(x\) is replaced by \(x + P\). Finally the substitutions

\[
x = s \cos \varphi = \frac{s}{2} (e^{i \varphi} + e^{-i \varphi}); \quad y = s \sin \varphi = \frac{s}{2i} (e^{i \varphi} - e^{-i \varphi})
\] (49)

express \(z^n e^{im \theta}\) in the circular coordinates centered at \(x = P\) with radial coordinate \(s\) and azimuth \(\varphi\). Even powers of \(P\) are dropped because \(P^2 = 1\). Table II demonstrates the cases for small \(n\) and small non-negative \(m\).

This reduces each \(A_{n,m}(q, \sigma)\) to a finite sum of integrals over centered circles of radius \(q\),

\[
\int_{s \leq q} e^{2\pi i \sigma_s \cdot z} z^n e^{im \varphi} d^2 s = e^{im \varphi} \int_0^q s^{n+1} ds \int_0^{2\pi} e^{2\pi i \sigma_s \cos \varphi} e^{im \varphi} d\varphi
\] (50)

\[
= 2\pi e^{im \varphi} \frac{1}{|m|} \int_0^q s^{n+1} J_{|m|}(2\pi s \sigma) ds = 2\pi e^{im \varphi} \frac{1}{|m|} \frac{1}{(2\pi \sigma)^{n+2} g_{n+1,|m|}(2\pi \sigma q)},
\] (51)
\[ \frac{\pi}{\delta} \equiv \frac{\pi}{m}, \quad n > m. \]

The integrals \( A_{n,m}(q) \) in (49) are just the special case of zero momentum, \( \sigma = 0 \). In this limit, (50) simplifies to
\[ \int_{-\infty}^{\infty} s^n \exp(i m \theta) d \theta = 2 \pi n_{0} n^{n+2} / (n + 2), \]
which proposes an alternative to compute Table II.

We visualize the Fourier representations of the first 12 basis functions in Figures 5 and 6. The \( f_j(x, y) \) have a well defined parity with respect to reflection across the origin, i.e., the terms in (24) – (33) are superpositions of \( z^n \sin(n \theta) \) and \( z^n \cos(n \theta) \) with a fixed, common parity \((-1)^m\) in each line. As a consequence, the \( f_j(\sigma_x, \sigma_y) \) are purely real or purely imaginary, and only the non-vanishing of the two components is shown.

\( f_1 \) in Figure 5 is the familiar diffraction limited point spread function of the two-beam interferometer [11]. \( f_1 \) in Figure 2 is the convolution of the double pinhole mask with a single circular telescope pupil of area \( \pi q^2 \). Its Fourier representation is the product of the circular Airy disk centered at \( \sigma = 0 \) by the hyperbolic pattern of fringes with their narrower width along \( \sigma_x \) as determined by the baseline length.

**VII. SUMMARY**

Orthogonality of functions over the area of two non-overlapping circles is defined according to the algebraic standards. The fundamental areal integral for functions that are of simple analytical format in the spherical coordinate system attached to the center of symmetry has been recursively reduced to polynomials or Gaussian hypergeometric functions of the normalized circle radius. This defines a set of orthogonal basis functions over the common area, if the Zernike basis functions, for example, provide the input to the Gram-Schmidt procedure. The first few of these have been written down in analytical form. For the higher-order aberrations, the complete information for numerical instantanization has been presented.
FIG. 5: The Fourier transforms of basis functions $f_1$ to $f_6$ for $q = 1/2$.

On a side note, computation of the interferometric signal of the pupil-beam recombination given an expansion in the radial-azimuthal coordinates has been reduced to a double sum of over a generic overlap integral.

APPENDIX A: AUXILIARY INTEGRAL $B$

1. Two Recursion Strategies.

This section deals with the evaluation of the integrals defined in [14]. Splitting off and expanding a square of the $(n + 2)$nd power, we obtain

$$B_{n+2,m}^\pm(q) = (q^2 - 1)B_{n,m}^\pm(q) + 2X_{n+1,m}^\pm(q),$$  \hspace{1cm} (A1)

where we have introduced

$$X_{n+1,m}^\pm(q) \equiv \int_0^{\arcsin q} \cos \theta \left[ \cos \theta \pm \sqrt{q^2 - \sin^2 \theta} \right]^{n+1} \cos(m\theta) \cos \theta d\theta.$$  \hspace{1cm} (A2)

This defines a first strategy to evaluate $B_{n+2,m}(q)$: recursive reduction of the first lower index in steps of two at the expense of implementing the $X_{n+1,m}(q)$ integrals—those to be treated in Appendix B. Starting the recursion from
FIG. 6: The Fourier transforms of basis functions $f_7$ to $f_{12}$ for $q = 1/2$.

For even $n$, one ends up at

$$B_{0,m}^\pm(q) = \frac{1}{m} \sin(m \arcsin q) \simeq 0. \quad (A3)$$

This is also correct in the limit $m \to 0$, and equivalent to zero as the difference $B_{0,m}^+ - B_{0,m}^-$ is formed. The symbol $\simeq$ is reserved in this script to indicate that terms on the right hand side have been removed which cancel if differences $B^+(q) - B^-(q)$ (common pair of subscripts with both $B$) or differences $X^+(q) - X^-(q)$ (common pair of subscripts with both $X$) are calculated. For odd $n$, the recursion [(A1)] terminates at

$$B_{1,m}^\pm(q) = \int_{0}^{\arcsin q} d\theta \left( \cos \theta \pm \sqrt{q^2 - \sin^2 \theta} \right) \cos(m\theta) \quad (A4)$$

$$\simeq \pm \int_{0}^{\arcsin q} d\theta \sqrt{q^2 - \sin^2 \theta} T_m(\cos \theta). \quad (A5)$$

For $m = 0$ this value will be given in [(A16)]. For larger $m$, a mixture of partial integrations and the product rule for Chebyshev polynomials [1, 22.7.24] generates a recursion for the second index, again in steps of two [18]:

$$B_{1,m}^\pm(q) \simeq \frac{2(m-2)(1-2q^2)}{m+1} B_{1,m-2}^\pm(q) + \frac{5-m}{m+1} B_{1,m-4}^\pm(q). \quad (A6)$$
Values to start this recursion at small even \( m \) are discussed in Section \([A2] \)—this implies that \( n + m \) is odd and does not happen for the Gram-Schmidt seeds proposed in Section \([I] \). At small odd \( m \) the recursion starts from

\[
B^\pm_{1,1}(q) = \pm \frac{\pi}{4} q^2; \quad B^\pm_{1,3}(q) = \pm \frac{\pi}{4} q^2(1 - q^2).
\]  

(A7)

An alternative second strategy to evaluate \( B_{n+2,m}(q) \) looks as follows: Binomial expansion of the \((n + 2)\)nd power in the integrand of \((14) \) yields

\[
B^\pm_{n+2,m}(q) \approx \pm \sum_{s=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-s+1}{s} (q^2 - 1)^s \sum_{u=0}^{n-1-2s} \binom{n+1-2s}{2u+1-u} \left[ B^\pm_{1,u+m}(q) + B^\pm_{1,u-m}(q) \right].
\]  

(A8)

Here, the prime at the sum symbol means the term for \( u = 0 \), if it occurs, is to be halved. The differences to the first strategy are

- No evaluation of the \( X^\pm_{n+1,m}(q) \) is needed. The contents of Appendix \([B] \) can be ignored.
- The decrement of the first index of \( B^\pm_{n+2,m} \) via \((A8) \) down to 1 comes at the cost of an increment of some second indices, eventually a recourse to \((A6) \).
- Use of \((A8) \) in conjunction with \((A9) \) eventually uses \((A7) \), whereas \((A7) \) and the case of odd \( m \) are made irrelevant for the first strategy through \((19) \).

Explicit examples of \((A8) \) for small \( n \) are:

\[
B^\pm_{2,m} \approx B_{1,m+1} + B_{1,m-1}, \quad m \geq 2;
\]  

\((A9) \)

\[
B^\pm_{3,m} \approx B_{1,m+2} + B_{1,m-2} + (1 + q^2) B_{1,m}, \quad m \geq 2;
\]  

\((A10) \)

\[
B^\pm_{4,2} \approx 2(1 + q^2) B_{1,1} + (1 + 2q^2) B_{1,3} + B_{1,5};
\]  

\((A11) \)

\[
B^\pm_{4,m} \approx B_{1,m-3} + (1 + 2q^2) B_{1,m-1} + (1 + 2q^2) B_{1,m+1} + B_{1,m+3}, \quad m \geq 3;
\]  

\((A12) \)

\[
B^\pm_{5,2} \approx (1 + 3q^2) B_{1,0} + (2 + 4q^2 + q^4) B_{1,2} + (1 + 3q^2) B_{1,4} + B_{1,6};
\]  

\((A13) \)

\[
B^\pm_{5,3} \approx (2 + 3q^2) B_{1,1} + (1 + 4q^2 + q^4) B_{1,3} + (1 + 3q^2) B_{1,5} + B_{1,7};
\]  

\((A14) \)

\[
B^\pm_{5,m} \approx B_{1,m-4} + (1 + 3q^2) B_{1,m-2} + (1 + 4q^2 + q^4) B_{1,m} + (1 + 3q^2) B_{1,m+2} + B_{1,m+4}, \quad m \geq 4.
\]  

\((A15) \)

The argument \( q \) at all the \( B_{\ldots}(q) \) has been omitted for brevity.

2. The case of odd \( n + m \).

With a substitution \( \sin \theta = \xi \), \( B_{1,0} \) can be written as a superposition of complete Elliptic Integrals of the first and second kind:

\[
B^\pm_{1,0}(q) = \int_0^{\arccos \sqrt{1 - q^2}} (\cos \theta \pm \sqrt{q^2 - \sin^2 \theta}) d\theta
\]  

\[
= q \pm \int_0^q \sqrt{q^2 - \xi^2} d\xi = q \pm [E(q^2) - (1 - q^2)K(q^2)].
\]  

(A16)
A merger of the series expansions of the Elliptic Integrals yields

\[ B_{1,0}^\pm(q) \simeq \pm [E(q^2) - (1 - q^2)K(q^2)]. \]  

(A17)

Due to a logarithmic singularity at \( q^2 = 1 \), the power series converges poorly if the argument \( q^2 \) of the Elliptic Integrals approaches unity \([3, 12]\). \( B_{1,0}^\pm(q) \) is the first anchor value for \((A6)\). The second is

\[ B_{1,2}^\pm(q) \simeq \pm \int_0^{\arcsin q} \sqrt{q^2 - \sin^2 q \cos(2\theta)} d\theta \]

\[ \simeq \pm \frac{1}{3}(1 - q^2)K(q^2) \pm \frac{1}{3}(2q^2 - 1)E(q^2). \]  

(A18)

(A19)

With the aid of \((A6)\), expressions for \( B_{1,m}^\pm(q) \) are bootstrapped from \((A17)\) and \((A19)\). This sequence starts:

\[ B_{1,4}^\pm(q) \simeq \pm \frac{1}{15}(q^2 - 1)(8q^2 - 1)K(q^2) \pm \frac{1}{15}(-16q^4 + 16q^2 - 1)E(q^2); \]  

(A20)

\[ B_{1,6}^\pm(q) \simeq \pm \frac{1}{105}(q^2 - 1)(128q^4 - 80q^2 + 3)K(q^2) \pm \frac{1}{105}(2q^2 - 1)(128q^4 - 128q^2 + 3)E(q^2); \]  

(A21)

\[ B_{1,8}^\pm(q) \simeq \pm \frac{1}{315}(q^2 - 1)(1024q^6 - 1152q^4 + 288q^2 - 5)K(q^2) \pm \frac{1}{315}(-2048q^8 + 4096q^6 - 2496q^4 + 448q^2 - 5)E(q^2). \]  

(A22)

**APPENDIX B: AUXILIARY INTEGRAL X**

The auxiliary integrals \((A2)\) are put into an algebraic format by the substitution \( \sin \theta = qz \),

\[ X_{n+1,m}^\pm(q) = q \int_0^{1} \left( \sqrt{1 - q^2 z^2} \pm q \sqrt{1 - z^2} \right)^{n+1} T_m(\sqrt{1 - q^2 z^2})dz, \]  

(B1)

and then broken down through binomial expansion of the \((n + 1)\)st power and the explicit polynomial expression for the Chebyshev function \([1, 22.3.6]\) via \([3, 3.197.3]\)

\[ X_{n+1,m}^\pm(q) \simeq \pm 2^{m-3}\pi(n+1)!q^2 \sum_{j=0}^{\left\lfloor n/2 \right\rfloor} \frac{1}{(n + 2j)!j!} \left( \frac{q}{2} \right)^{2j} \]

\[ \times \begin{cases} \sum_{\sigma=0}^{\left\lfloor n/2 \right\rfloor} (-1)^\sigma \frac{m}{2(m-\sigma)} (m-\sigma)! F \left( j + \sigma - \frac{n+m}{2}, \frac{1}{2} | q^2 \right); & m > 0; \\ F \left( j - \frac{n}{2} + 1, \frac{1}{2} | q^2 \right); & m = 0. \end{cases} \]

If \( n + m \) is an even number—which is the case for the Gram-Schmidt procedure described in Section \([15, 11]\)—the hypergeometric series terminate and become polynomials of \( q^2 \) of order \((n + m)/2 - j - \sigma\) \([1, 15.4.1]\). The result can be tabulated in terms of power series coefficients \( a_j(n, m) \),

\[ X_{n+1,m}^\pm(q) \equiv \pi q \sum_{j=0}^{\left\lfloor n/2 \right\rfloor} a_j(n, m)(\pm q)^j. \]  

(B2)

Only the values with odd \( j \) are of interest, because our application eventually looks only at the differences \( X_{n+1,m}^+(q) - X_{n+1,m}^-(q) \) in which terms of even \( j \) cancel. The basic values for \( m = 0 \) are in Table \([11]\).
Table [IV] summarizes [B2] for small values of $n$ and $m$.

| $n$ | $m$ | $\sum_{j=1,3,5,7,\ldots} \alpha_j(n,m)q^j$ |
|-----|-----|---------------------------------|
| 0   | 0   | $+1/4q$                         |
| 0   | 2   | $+1/4q - 1/8q^3$                |
| 0   | 4   | $+1/4q - 1/2q^5 + 1/4q^7$      |
| 0   | 6   | $+1/4q - 9/8q^3 + 3/2q^7 - 5/8q^9$ |
| 0   | 8   | $+1/4q - 2q^5 + 5q^7 - 5q^9 + 7/4q^9$ |
| 1   | 0   | $+1/4q - 1/16q^3 - 1/128q^5 - 5/248q^7 - 35/32768q^9 - 147/262144q^{11}$ |
| 1   | 2   | $+1/4q - 5/16q^3 + 7/128q^5 + 15/248q^7 + 77/32768q^9 + 273/262144q^{11}$ |
| 1   | 4   | $+1/4q - 17/16q^3 + 95/128q^5 - 245/248q^7 - 483/32768q^9$ |
| 1   | 6   | $+1/4q - 37/16q^3 + 455/128q^5 - 4305/248q^7 + 9933/32768q^9$ |
| 1   | 8   | $+1/4q - 65/16q^3 + 1407/128q^5 - 26565/248q^7 + 213213/32768q^9$ |
| 2   | 0   | $+3/4q$                         |
| 2   | 2   | $+3/4q - 3/8q^3 + 1/8q^5$      |
| 2   | 4   | $+3/4q - 3/2q^5 + 5/4q^7 - 3/8q^9$ |
| 2   | 6   | $+3/4q - 27/8q^5 + 45/8q^9 - 33/8q^{11} + 9/8q^{13}$ |
| 2   | 8   | $+3/4q - 6q^5 - 17q^7 - 45/2q^9 + 57/4q^{11} - 7/2q^{13}$ |
| 3   | 0   | $+q + 3/8q^3 - 1/64q^5 - 1/1024q^7 - 3/16384q^9 - 7/131072q^{11}$ |
| 3   | 2   | $+q - 1/8q^3 + 7/64q^5 - 13/1024q^7 - 19/16384q^9 - 35/131072q^{11}$ |
| 3   | 4   | $+q - 13/8q^3 + 95/64q^5 - 625/1024q^7 + 1085/16384q^9$ |
| 3   | 6   | $+q - 33/8q^5 + 455/64q^7 - 6125/1024q^9 + 37485/16384q^{11}$ |
| 3   | 8   | $+q - 61/8q^7 + 1407/64q^9 - 32193/1024q^{11} + 383229/16384q^{13}$ |
| 4   | 0   | $+5/4q + 5/4q^3$                |
| 4   | 2   | $+5/4q + 5/8q^3$                |
| 4   | 4   | $+5/4q - 5/4q^3 + 5/8q^5 - 5/8q^7 + 1/8q^9$ |
| 4   | 6   | $+5/4q - 35/8q^5 + 15/2q^7 - 55/8q^9 + 13/4q^{11} - 5/8q^{13}$ |
| 4   | 8   | $+5/4q - 35/4q^7 + 25q^9 - 75/2q^{11} + 125/4q^{13} - 55/4q^{15} + 5/2q^{17}$ |
| 5   | 0   | $+3/2q + 45/16q^3 + 45/128q^5 - 15/2048q^7 - 9/32768q^9 - 9/262144q^{11}$ |
| 5   | 2   | $+3/2q + 33/16q^3 + 5/128q^5 + 45/2048q^7 - 57/32768q^9 - 29/262144q^{11}$ |

TABLE III: Expansion coefficients of [B2] at $m = 0$. 
\[ \begin{align*}
5 & \quad +3/2q - 3/16q^3 + 77/128q^5 - 735/2048q^7 + 3255/32768q^9 - 2009/262144q^{11} - 1071/2097152q^{13} - 6039/67108864q^{15} \\
&\quad - 50765/2147483648q^{17} + \ldots \\
5 & \quad +3/2q - 63/16q^3 + 837/128q^5 - 12915/2048q^7 + 112455/32768q^9 - 242109/262144q^{11} + 147609/2097152q^{13} + 312741/67108864q^{15} \\
&\quad + 1756755/2147483648q^{17} + \ldots \\
5 & \quad +3/2q - 147/16q^3 + 3245/128q^5 - 79695/2048q^7 + 1149687/32768q^9 - 4749129/262144q^{11} + 9810801/2097152q^{13} - 23146695/67108864q^{15} \\
&\quad - 47732685/2147483648q^{17} + \ldots \\
6 & \quad +7/4q + 21/4q^3 + 7/4q^5 \\
6 & \quad +7/4q + 35/8q^3 + 7/8q^5 \\
6 & \quad +7/4q + 7/4q^3 \\
6 & \quad +7/4q - 21/8q^3 + 35/8q^5 - 35/8q^7 + 21/8q^9 - 7/8q^{11} + 1/8q^{13} \\
6 & \quad +7/4q - 35/4q^3 + 91/4q^5 - 35q^7 + 133/4q^9 - 77/4q^{11} + 25/4q^{13} \\
&\quad - 7/8q^{15} \\
7 & \quad +2q + 35/4q^3 + 175/32q^5 + 175/512q^7 - 35/8192q^9 - 7/65536q^{11} \\
&\quad - 5/524288q^{13} - 25/16777216q^{15} - 175/536870912q^{17} + \ldots \\
7 & \quad +2q + 31/4q^3 + 119/32q^5 + 35/512q^7 + 77/8192q^9 - 35/65536q^{11} \\
&\quad - 13/524288q^{13} - 53/16777216q^{15} - 335/536870912q^{17} + \ldots \\
7 & \quad +2q + 19/4q^3 + 15/32q^5 + 63/512q^7 - 483/8192q^9 + 777/65536q^{11} \\
&\quad - 357/524288q^{13} - 585/16777216q^{15} - 2607/536870912q^{17} + \ldots \\
7 & \quad +2q - 1/4q^3 + 55/32q^5 - 957/512q^7 + 9933/8192q^9 - 30723/65536q^{11} \\
&\quad + 49203/524288q^{13} - 92565/16777216q^{15} - 155275/536870912q^{17} + \ldots \\
7 & \quad +2q - 29/4q^3 + 559/32q^5 - 13585/512q^7 + 213213/8192q^9 \\
&\quad - 1072071/65536q^{11} + 3270267/524288q^{13} - 20796633/16777216q^{15} \\
&\quad + 38053345/536870912q^{17} + \ldots \\
8 & \quad +9/4q + 27/2q^3 + 27/2q^5 + 9/4q^7 \\
8 & \quad +9/4q + 99/8q^3 + 21/2q^5 + 9/8q^7 \\
8 & \quad +9/4q + 9q^3 + 15/4q^5 \\
8 & \quad +9/4q + 27/8q^3 \\
8 & \quad +9/4q - 9/2q^3 + 21/2q^5 - 63/4q^7 + 63/4q^9 - 21/2q^{11} + 9/2q^{13} \\
&\quad - 9/8q^{15} + 1/8q^{17}
\end{align*}\]

TABLE IV: Table of \( \sum_j \alpha_j(n,m)q^j \), summed over odd \( j \) only. For odd \( n + m \), the series is shown up to \( O(q^{17}) \), indicated by the triple dots.
APPENDIX C: NOTATIONS

[ ] floor function; largest integer not greater than the argument

(.,.) scalar (inner) product between the two arguments [16]

\[ \sum_{u=\ldots} \] summation with the term of index \( u = 0 \) halved

\( \simeq \) equivalent upon subtraction of \( B^\pm \) or \( X^\pm \) of superscripts of opposite sign

\( A_{n,m}(q) \) the integral [9]

\( \alpha_{n,m}(q) \) the expansion of the \( X \) integral [12]

\( B_{n,m}^\pm(q) \) the integral [13]

\[ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \] the Beta Integral [9, 8.38]

\( \delta_{jk} \) Kronecker delta; equal to 1 if \( j = k \), equal to 0 if \( j \neq k \)

\( E(.) \) complete Elliptic Integral of the first kind [1, §17.3]

\( E \) electric field amplitude

\( f_k(q) \) \( k \)th orthogonal basis function of the binocular area

\( F(\frac{a}{c}, \ldots) \) Gaussian hypergeometric function [1, §15]

\( \varphi \) azimuth angle in spherical coordinates centered at circular sub-pupil

\( \gamma_{\ldots} \) abscissa sections of the Gram-Schmidt procedure

\( g_k \) \( k \)th input (“seed”) function to the Gram-Schmidt synthesis

\( g_{\ldots} \) Bessel Function integral [1, 11.3.1]

\( I \) interferometric intensity

\( i \) imaginary unit

\( J(.) \) Bessel function of the First Kind

\( j!! \) double factorial, \( = 1 \cdot 3 \cdot 5 \cdot 7 \cdots j \) if \( j \) is odd, \( = 2 \cdot 4 \cdot 6 \cdots j \) if \( j \) even. \( (-1)!! = 1 \)

\( K(.) \) complete Elliptic Integral of the second kind [1, §17.3]

\( q \) circle diameter in units of circle center distance

\( s \) radial distance to a circle center, \( 0 \leq s \leq q \)

\( \sigma, \sigma_x, \sigma_y, \sigma \) wave number, Cartesian coordinates, modulus

\( T_m(.) \) Chebyshev Polynomial the first kind of order \( m \) [1, §22]

\( \theta \) azimuth angle in the circular coordinates centered at midpoint between sub-pupils

\( X_{n,m}^\pm(q) \) the integral [A2]

\( Z_k(.) \) Zernike circle functions [17]

\( z \) radial distance to a center of symmetry \( 0 \leq z \leq 1 + q \)

[1] Abramowitz, M., and I. A. Stegun (eds.), 1972, Handbook of Mathematical Functions (Dover Publications, New York), 9th edition, ISBN 0-486-61272-1.

[2] Bhatia, A. B., and E. Wolf, 1952, Proc. Phys. Soc. B 65(11), 909.

[3] Campbell, C. E., 2003, J. Opt. Soc. Am. A 20(2), 209.

[4] Chong, C.-W., P. Raveendran, and R. Mukundan, 2003, Pattern Recogn. 36(3), 731.

[5] Cody, W. J., 1965, Math. Comp. 19(90), 249.

[6] Comastri, S. A., L. I. Perez, G. D. Perez, G. Martin, and K. Bastida, 2007, J. Opt. A: Pure Appl. Opt. 9(3), 209.

[7] Dai, G.-m., and V. N. Mahajan, 2007, Opt. Lett. 32(1), 74.

[8] Dai, G.-m., and V. N. Mahajan, 2007, J. Opt. Soc. Am. A 24(1), 139.

[9] Gradstein, I., and I. Ryzhik, 1981, Summen-, Produkt- und Integraltafeln (Harri Deutsch, Thun), 1st edition, ISBN 3-87144-350-6.

[10] Hou, X., F. Wu, L. Yang, and Q. Chen, 2006, Appl. Opt. 45(35), 8893.

[11] Hu, P. H., J. Stone, and T. Stanley, 1989, J. Opt. Soc. Am. A 6(10), 1595.

[12] Lee, D. K., 1990, Comp. Phys. Commun. 60(3), 319.

[13] Lundström, L., and P. Unsbo, 2007, J. Opt. Soc. Am. A 24(3), 569.

[14] Mahajan, V. N., 1981, J. Opt. Soc. Am. 71(1), 75.

[15] Mahajan, V. N., and G.-m. Dai, 2007, J. Opt. Soc. Am. A 24(9), 2994.

[16] McCarthy, D. W., E. M. Sabatke, R. J. Sarlot, P. M. Hinz, and J. H. Burge, 2000, in Interferometry in Optical Astronomy, edited by P. J. Lena and A. Quirrenbach (Int. Soc. Optical Engineering), volume 4006 of Proc. SPIE, pp. 659–672.
[17] Noll, R. J., 1976, J. Opt. Soc. Am. 66(3), 207.
[18] Novario, P. G., 2005, Electr. Trans. Num. Anal. 20, 198.
[19] Prata, A., Jr., and W. V. T. Rusch, 1989, Appl. Opt. 28(4), 749.
[20] Schwiegerling, J., 2002, J. Opt. Soc. Am. A 19(10), 1937.
[21] Sheppard, C. J. R., S. Campbell, and M. D. Hirschhorn, 2004, Appl. Opt. 43(20), 3963.
[22] Shu, H., L. Luo, G. Han, and J.-L. Coatrieux, 2006, J. Opt. Soc. Am. A 23(8), 1960.
[23] Swantner, W., and W. W. Chow, 1994, Appl. Opt. 33(10), 1832.
[24] Swantner, W. H., and W. H. Lowrey, 1980, Appl. Opt. 19(1), 161.
[25] Upton, R., and B. Ellerbroek, 2004, Opt. Lett. 29(24), 2840.
[26] Wang, J. Y., and D. E. Silva, 1980, Appl. Opt. 19(9), 1510.