Quadratic Differential Systems on \( \mathbb{R}^3 \) Having a non-Singular Semisimple Derivation

I. Burdujan

University of Agricultural Sciences and Veterinary Medicine, Iaşi, 700490, Romania

Abstract

The classification, up to a center-affinity, of the homogeneous quadratic differential systems defined on \( \mathbb{R}^3 \) that have at least a semisimple non-singular derivation, is achieved. It is proved that there exist four affine-equivalence classes of such systems, only.

2000 Mathematics Subject Classification: Primary 34G20, Secondary 34L30, 15A69

Keywords and phrases: homogeneous quadratic dynamical systems, semisimple derivation.

1 Introduction

Let us recall that the problem of classification up to an affine-equivalence of homogeneous quadratic differential systems is equivalent to the problem of classification up to an isomorphism of commutative binary algebras defined on the same ground space as the analyzed systems (e.g., see [12], [16], [1]).

Let \( k \) be a perfect field and \( A \) be a finite-dimensional vector space over \( k \). Recall that an endomorphism \( T \in \text{End} A \) is said to be semisimple if it is diagonalisable in an extension of \( k \), i.e. there exists a basis of \( A \) consisting of eigenvectors of \( T \) (see [11]). In fact, \( T \) is semisimple if and only if its minimal polynomial is separable.

In the following, we restrict our interest to the case when \( k \) is \( \mathbb{R} \) or \( \mathbb{C} \).

In general, a derivation of a \( k \)-algebra is uniquely represented, via its JORDAN-CHEVALLEY decomposition, as the sum of a semisimple derivation and a nilpotent derivation. Some classes of real 3-dimensional commutative algebras having a derivation were already classified, up to an isomorphism, in [2], [3] and [4]; more exactly, there were respectively obtained the isomorphism classes for algebras having either a derivation with complex eigenvalues ([2], [3]) or a nilpotent derivation ([4]). Accordingly, there were classified their corresponding homogeneous quadratic differential systems up to a center-affine equivalence.

---

1 Corresponding author,
Ilie Burdujan, E-mail address: ilieburdujan@uaiasi.ro, ilieburdujan@yahoo.com
address: 3, Mihail Sadoveanu Street, 700490, Iaşi, Romania
The aim of this paper is to classify, up to an isomorphism, the real 3-dimensional commutative algebras having at least a semisimple nonsingular derivation. The main result proved here is: \textit{there exist four isomorphism classes of real 3-dimensional commutative algebras having at least a semisimple nonsingular derivation}. For each of them are exhibited the main properties which allow to decide rapidly on the problem of their mutual isomorphism relationships. Really, the existence of a semisimple nonsingular derivation works like a strong constraint on the integral curves of the associated HQDS; more exactly, all nonsingular integral curves must be torsion free and, for a large part of them the curvature tensor have to be zero as well. On the other hand, from the algebraic point of view, the existence of a semisimple nonsingular derivation compels the solvability of analyzed algebras, yet the nilpotence of a large part of them.

In fact, we shall get the subalgebra lattices, the derivation algebras and the group of automorphisms for each class of analyzed algebras, because they are the most important invariants of any binary algebra. Especially, we are interested in finding the set $Ann\ A$ of annihilator elements, the set $N(A)$ of all nilpotent elements and the set $I(A)$ of all idempotent elements of each analyzed algebra $A$. Further, the corresponding homogeneous quadratic dynamical systems are classified up to a center-affine equivalence. Finally, let us remark that subalgebra lattice of $A(\cdot)$ allows to identify a natural partition $\mathcal{P}_A$ of the ground space $A$ of algebra whose sets will be named cells. In its turn, this partition has the property: if an integral curve of the associated homogeneous quadratic differential system meets a cell of partition $\mathcal{P}_A$ then it is entirely contained in that cell. It follows that the set of all integral curves decomposes into a partition subordinated to $\mathcal{P}_A$. Both these partitions are invariant under the natural action of $Aut\ A$. They could be useful tools in studying stability of singular solutions.

2 Preliminaries

Let $A$ be a finite dimensional vector space over a field $k$, $k[A]$ be the coordinate ring of $A$, $B = (e_1, e_2, \ldots, e_n)$ a basis in $A$ and $(x^1, x^2, \ldots, x^n)$ - the coordinate system assigned to $B$.

Any autonomous differential system $(S)$ on $A$ of the form

$$\dot{x}^i = a^i_{jk} x^j x^k \quad i, j, k \in \{1, 2, \ldots, n\}$$

(2.1)

with $a^i_{jk} = a^i_{kj}$ in $k$ and $\dot{x}^i = \frac{dx^i}{dt}$ is called a \textit{homogeneous quadratic differential system} (shortly, HQDS) on $A$. To any HQDS (2.1) there is naturally associated a commutative binary algebra $A(\cdot)$ defined by extending bi-linearly to $A \times A$ the mapping from $B \times B \rightarrow A$ defined by:

$$(e_j, e_k) \rightarrow e_j \cdot e_k = a^i_{jk} e_i.$$  

Conversely, the structure constants of any finite dimensional commutative algebra could be used as coefficients defining a system of type (2.1). This result opens the way for an algebraic approach of HQDSs.
In fact, the structure properties of $A(\cdot)$ (i.e., properties which are invariant under isomorphism action) enforce the existence of some qualitative properties of $(\mathcal{S})$. Conversely, the qualitative properties of $(\mathcal{S})$ induce structure properties for $A(\cdot)$.

More exactly, there exists a correspondence (not necessarily a map)

$$\{\text{the properties affinely invariant of any HQDE } (\mathcal{S})\}$$

$$\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sin
Accordingly, there exists a natural partition of the set of all integral curves for the corresponding HQDS.

It was proved that there exists a 1-to-1 mapping between affine-equivalent classes of HQDSs on \(A\) and isomorphism classes on commutative binary algebras on \(A\). Consequently, the problem of classification up to an affine-equivalence of HQDSs is equivalent to the problem of finding the isomorphism classes of commutative binary algebras. In order to solve this last problem, it is suitable to divide the class of commutative algebras into two disjoint parts:

(i) the class of algebras having at least a derivation,
(ii) the class of algebras having no derivation.

For algebras belonging to former class there exist more facilities for their studies than for the algebras in the second class. Indeed, the existence of any derivation implies the existence of a one-parametric group of automorphisms what assure the existence of suitable bases where structure constants of algebra have a simpler/simplest form, i.e. most of them are 0, 1 or other small integers. In the specific case under analysis in this paper, the isomorphism classes have representatives which have either 0 or 1 as structure constants in suitable bases.

3 Algebras having a semisimple derivation

Let \(A(\cdot)\) be the real 3-dimensional (nontrivial) commutative algebra associated with a HQDS in \(A\). For convenience, suppose that the real vector space \(A\) was identified with \(\mathbb{R}^3\) by means of any chosen basis.

Suppose that \(\tilde{D}\) is a nonzero semisimple derivation of \(A(\cdot)\). Then, algebra \(A(\cdot)\) has a derivation \(D\) having the spectrum of the form \(Spec D = (1, \lambda, \mu)\) where \(1, \lambda, \mu\) are the eigenvalues, distinct or not, of \(D\). It follows that there exists a basis \(B = (e_1, e_2, e_3)\) of \(A\) such that

\[
D(e_1) = e_1, \quad D(e_2) = \lambda e_2, \quad D(e_3) = \mu e_3.
\]

Since in basis \(B' = (f_1 = e_1, f_2 = e_3, f_3 = e_2)\) we get \(D(f_1) = f_1, \ D(f_2) = \mu f_2, \ D(f_3) = \lambda f_3, \) it results it is enough to analyze only the case \(Spec D = (1, \lambda, \mu)\) with \(\lambda \leq \mu\). Since the derivation is nonsingular it results necessarily \(\lambda \cdot \mu \neq 0\). Of course, \(A\) decomposes into a direct vector sum of eigenspaces of derivation \(D\).

In order to express analytically the existence of a derivation \(D\) for algebra \(A(\cdot)\), we consider the structure constants of \(A\) in basis \(B\) usually defined by

\[
e_i \cdot e_j = a_{ij}^k e_k. \tag{3.1}
\]

For convenience, we shall denote

\[
\begin{align*}
a_{11}^1 &= a & a_{11}^2 &= b & a_{11}^3 &= c & a_{11}^4 &= k & a_{11}^5 &= m & a_{11}^6 &= n \\
a_{12}^1 &= d & a_{12}^2 &= e & a_{12}^3 &= f & a_{12}^4 &= p & a_{12}^5 &= q & a_{12}^6 &= r \\
a_{13}^1 &= g & a_{13}^2 &= h & a_{13}^3 &= j & a_{13}^4 &= s & a_{13}^5 &= t & a_{13}^6 &= v. \tag{3.2}
\end{align*}
\]

Then \(D\) is a derivation for \(A\) if and only if the next conditions are fulfilled:
These equations impose to take into account of the natural decomposition of the plane $\mathbb{R}^2$ of variables $(\lambda, \mu)$ defined by means of the lines:

$$\lambda = 2, \quad \mu = 2, \quad 1 - 2\lambda = 0, \quad 1 - 2\mu = 0, \quad \mu - 2\lambda = 0, \quad \lambda - 2\mu = 0, \quad \mu - 1 - \lambda = 0, \quad \lambda - 1 - \mu = 0, \quad \lambda + \mu - 1 = 0.$$  \hfill (3.4)

The next result holds true.

**Lemma 3.1** The set $\mathbb{R}^2 \setminus \{\lambda = 0\} \cup \{\mu = 0\}$ decomposes into the following disjoint subsets:

i) on the line $\lambda = \frac{1}{2} : \{(\frac{1}{2}, -\frac{1}{2})\}, \{(\frac{1}{2}, \frac{1}{2})\}, \{(\frac{1}{2}, 1)\}, \{(\frac{1}{2}, -\frac{3}{2})\}, \{(\frac{1}{2}, 2)\}$

and $\{(\frac{1}{2}, \mu) | \mu \notin \{-\frac{1}{2}, 2\}; \frac{1}{2}, 1, \frac{3}{2}, 2\}\}$

ii) on the line $\mu = 1 : \{(-1, -1)\}, \{(1, -1)\}, \{(2, 1)\}, \{(2, 2)\}, \{(2, 3)\}, \{(2, 4)\}, \{(2, \frac{1}{2})\}$

and $\{(\mu, 1) | \lambda \notin \{-1, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, 1, \frac{3}{2}, 2\}\}$

The following results are proved.

**Proposition 3.1** Any real 3-dimensional commutative algebra $A(\cdot)$ having at least a semisimple derivation with kernel $\{0\}$ has a derivation $D$ such that its spectrum $\text{Spec} \ D = (1, \lambda, \mu)$ has as $(\lambda, \mu)$ one of the pairs listed in Lemma 3.1, i)-ix).
**Proposition 3.2** Let \( A(\cdot) \) be any real 3-dimensional commutative algebra having at least a semisimple derivation with kernel \( \{0\} \). Then \( \text{Der} \ A \) contains at least a derivation having the spectrum in one (and only one) of the next nine families

1) \( \{ (1, \frac{1}{3}, -\frac{1}{3}), (1, -\frac{1}{3}, \frac{2}{3}), (1, -1, 2), (1, 2, -1), (1, -2, -1), (1, -1, 2) \}, \)
2) \( \{ (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}), (1, 2, 4), (1, 4, 2), (1, \frac{1}{2}, 2), (1, 2, \frac{1}{2}) \}, \)
3) \( \{ (1, \frac{1}{2}, \frac{3}{2}), (1, 2, 1), (1, 1, 2) \}, \)
4) \( \{ (1, \frac{1}{2}, 1), (1, 1, \frac{1}{2}), (1, 2, 2) \}, \)
5) \( \{ (1, \frac{1}{2}, \frac{3}{2}), (1, \frac{1}{2}, \frac{1}{2}), (1, 2, 3), (1, 3, 2), (1, \frac{2}{3}, \frac{1}{3}), (1, \frac{1}{3}, \frac{2}{3}) \}, \)
6) \( \{ (1, \frac{1}{2}, \mu) \mid \mu \in \mathbb{R} \setminus \{ -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, 1, \frac{3}{2}, 2 \} \} \cup \{ (1, 2, \mu) \mid \mu \in \mathbb{R} \setminus \{ -1, 0, \frac{1}{2}, 1, 2, 3, 4 \} \} \cup \{ (1, \lambda, \frac{1}{2}) \mid \lambda \in \mathbb{R} \setminus \{ -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}, 2 \} \} \cup \{ (1, \lambda, 2) \mid \lambda \in \mathbb{R} \setminus \{ -1, 0, \frac{1}{2}, 1, 2, 3, 4 \} \} \),
7) \( \{ (1, 2, \mu, \mu) \mid \mu \in \mathbb{R} \setminus \{ -1, 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, 1, 2 \} \} \cup \{ (1, \lambda, 2\lambda) \mid \lambda \in \mathbb{R} \setminus \{ -2, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2 \} \}, \)
8) \( \{ (1, \lambda, \lambda - 1) \mid \lambda \in \mathbb{R} \setminus \{ -1, 0, \frac{1}{2}, \frac{1}{4}, 1, \frac{3}{2}, 2 \} \} \cup \{ (1, \lambda, \lambda + 1) \mid \lambda \in \mathbb{R} \setminus \{ -2, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2 \} \} \cup \{ (1, \lambda, 1 - \lambda) \mid \lambda \in \mathbb{R} \setminus \{ -1, 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, 1, 2 \} \}. \)

Let us recall that whenever \( D \in \text{Der} \ A \) has \( \text{Spec} \ D = (\alpha, \beta, \gamma) \) then \( \hat{D} = kD \) is a derivation with \( \text{Spec} \ \hat{D} = (k\alpha, k\beta, k\gamma) \). This remark and Proposition 3.2 allow to prove the next result.

**Proposition 3.3** In order to classify the real 3-dimensional commutative algebras having a semisimple nonsingular derivation it is enough to restrict the study on the algebras having a derivation with spectrum of one of the following types:

\[
(1, -1, 2), (1, 1, 2), (1, 2, 2), (1, 2, 3), (1, 2, 4) \\
(1, 2, \lambda) \text{ with } \lambda \in \mathbb{R} \setminus \{-1, 0, \frac{1}{2}, 1, 2, 3, 4\} \\
(1, 2, 2\lambda) \text{ with } \lambda \in \mathbb{R} \setminus \{-\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, 1, \frac{3}{2}, 2\} \\
(1, \lambda, 2\lambda) \text{ with } \lambda \in \mathbb{R} \setminus \{-1, \frac{1}{2}, \frac{3}{2}, 1, 2\} \\
(1, \lambda, \lambda + 1) \text{ with } \lambda \in \mathbb{R} \setminus \{-2, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2\} \\
((1, \lambda, 1 - \lambda) \text{ with } \lambda \in \mathbb{R} \setminus \{-1, 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, 1, 2\}.
\]

In the following we will identify all these classes of algebras.

1) **Case** \( \text{Spec} \ D = (1, -1, 2) \)

There exists a basis \( B = (e_1, e_2, e_3) \) such that the multiplication table of algebra \( A(\cdot) \) has the form:

**Table T**
\[
\begin{tabular}{c|ccc}
   & \( e_1^2 = pe_3 \) & \( e_2^2 = 0 \) & \( e_3^2 = 0 \) \\
\hline
\( e_1e_2 \) & 0 & 0 & 0 \\
\( e_1e_3 \) & 0 & 0 & 0 \\
\( e_2e_3 = qe_1 \) & 0 & 0 & 0
\end{tabular}
\]

with \( p, q \in \mathbb{R} \).

**Subcase** \( pq \neq 0 \)

In the case when \( pq \neq 0 \) any such algebra is isomorphic to algebra:

**Table T1**
\[
\begin{tabular}{c|ccc}
   & \( e_1^2 = e_3 \) & \( e_2^2 = 0 \) & \( e_3^2 = 0 \) \\
\hline
\( e_1e_2 \) & 0 & 0 & 0 \\
\( e_1e_3 \) & 0 & 0 & 0 \\
\( e_2e_3 = e_1 \) & 0 & 0 & 0
\end{tabular}
\]
(indeed, it is enough to use the basis \((\frac{1}{p}e_1, \frac{1}{q}e_2, \frac{1}{p} e_3))\).

Properties of algebra \(A_1\) (with table T1):

- \(\text{Ann } A = \{0\}\), \(\mathcal{N}(A) = \mathbb{R}e_2 \cup \mathbb{R}e_3\), \(\mathcal{I}(A) = \emptyset\),
- 1-dimensional subalgebras: \(\mathbb{R}e_2\), \(\mathbb{R}e_3\),
- 2-dimensional subalgebras: \(\text{Span}_{\mathbb{R}}\{e_1, e_3\}\),
- ideals: \(\text{Span}_{\mathbb{R}}\{e_1, e_3\}\),
- \(A_2 = \text{Span}_{\mathbb{R}}\{e_1, e_3\}\),
- \(A\) is solvable: \(A = \mathbb{R}e_2 \oplus \text{Span}_{\mathbb{R}}\{e_1, e_3\}\) - vector direct sum of a null subalgebra and a nilpotent (maximal) ideal, i.e. this is a Weddeburn-Artin decomposition for \(A\),
- \(\text{Der } A = \mathbb{R}D\),
- \(\text{Aut } A = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & x^2 \end{pmatrix} \mid x \in \mathbb{R}^* \right\}\),
- the partition \(P_A\) of \(\mathbb{R}^3\), defined by the subalgebra lattice of \(A\), consists of:
  - the singletons covering the axes \(Ox^2\) and \(Ox^3\),
  - the half-planes delimited by axis \(Ox^3\) on \(x^1Ox^3\),
  - the half-spaces \(x^2 > 0\) and \(x^2 < 0\) (delimited by plane \(x^1Ox^3\)) less the points of axis \(Ox^2\),
- the partition \(P_A\) of \(A\) induces a partition of the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the axes \(Ox^2\) and \(Ox^3\),
  - the integral curves contained in each half-plane delimited by axis \(Ox^3\) on \(x^1Ox^3\),
  - the integral curves contained in each half-space delimited by plane \(x^1Ox^3\) which do not meet the axis \(Ox^2\).

The presence of ideal \(\text{Span}_{\mathbb{R}}\{e_1, e_3\}\) assures the existence of a decoupled subsystem defined on the null algebra \(\mathbb{R}e_2 \cong A/\text{Span}_{\mathbb{R}}\{e_1, e_3\}\).

Since \(A\) is solvable, the associated HQDS has a linear prime integral and, consequently, each nonsingular integral curve is torsion-free.

Note that algebra \(A\) in neither associative nor power-associative.

Subcase \(pq = 0\)

Of course, the case when \(p = q = 0\) gives the null algebra which is not of any interest. It remains to analyze the cases

\[(i) \ p \neq 0, \ q = 0, \quad (ii) \ p = 0, \ q \neq 0.\]

(i) In this case any such algebra is isomorphic to algebra:

\[
\begin{array}{ccc}
  e_1^2 &=& e_3 \\
  e_2^2 &=& 0 \\
  e_1 e_2 &=& 0 \\
  e_1 e_3 &=& 0 \\
  e_2 e_3 &=& 0
\end{array}
\]

Table T’2
Changing the basis \((e_1, e_2, e_3)\) in basis \((e_2, e_3, e_1)\) the multiplication table of algebra \(A(\cdot)\) gets the form:

**Table T2**

\[
\begin{array}{ccc}
  e_1^2 & = 0 & e_2^2 = 0 & e_3^2 = e_2 \\
  e_1e_2 & = 0 & e_1e_3 & = 0 & e_2e_3 & = 0
\end{array}
\]

**Properties of algebra \(A_2\) (with table T2):**

- \(\text{Ann } A = \text{Span}_\mathbb{R}\{e_1, e_2\}, \mathcal{N}(A) = \text{Span}_\mathbb{R}\{e_1, e_2\}, \mathcal{I}(A) = \emptyset,\)
- 1-dimensional subalgebras: \(\mathbb{R}u\) for each \(u \in \mathcal{N}(A),\)
- 2-dimensional subalgebras: \(\text{Span}_\mathbb{R}\{e_2, pe_1 + qe_3\} (p^2 + q^2 \neq 0),\)
- ideals: \(\mathbb{R}u\) for \(u \in \text{Span}_\mathbb{R}\{e_1, e_2\}, \text{Span}_\mathbb{R}\{e_2, pe_1 + qe_3\} (p^2 + q^2 \neq 0),\)
- \(A^2 = \mathbb{R}e_2,\)
- \(A\) is a nilpotent associative algebra,
- \(\text{Der } A = \left\{ \begin{array}{ccc} x & 0 & u \\
  y & 2z & v \\
  0 & 0 & z \end{array} \right\} | x, y, z, u, v \in \mathbb{R},\)
- \(\text{Aut } A = \left\{ \begin{array}{ccc} x & 0 & u \\
  y & z^2 & v \\
  0 & 0 & z \end{array} \right\} | x, y, z, u, v \in \mathbb{R}, \; xz \neq 0,\)
- the partition \(\mathcal{P}_A\) of \(\mathbb{R}^3,\) defined by the lattice of subalgebras of \(A,\) consists of:
  - the singletons covering the plane \(x^1Ox^2,\)
  - the half-planes delimited by axis \(Ox^2\) on each plane containing \(Ox^2\) less \(x^1Ox^2,\)
- the partition \(\mathcal{P}_A\) of \(A\) induces a partition of the set of integral curves of the associated HQDS consisting of:
  - the singletons consisting of singular solutions covering the plane \(x^1Ox^2,\)
  - the integral curves contained in each half-plane delimited by axis \(Ox^2\) on any plane passing through \(Ox^2\) less \(x^1Ox^2,\)

As each nonsingular integral curve lies on a 2-dimensional subalgebra, it results its torsion is zero. The presence of a large family of ideals implies the existence of corresponding family of prime integrals and, consequently, each nonsingular integral curve has also a zero curvature tensor. More exactly, each nonsingular integral curve lies on a parallel to \(Ox^2.\)

(ii) In this case, by using the changes of bases \((e_1, \frac{1}{q}e_2, e_3)\) and \((e_3, e_2, e_1),\) it follows that any analyzed algebra is isomorphic to algebra:

**Table T3**

\[
\begin{array}{ccc}
  e_1^2 & = 0 & e_2^2 = 0 & e_3^2 = 0 \\
  e_1e_2 & = e_3 & e_1e_3 & = 0 & e_2e_3 & = 0
\end{array}
\]

**Properties of algebra \(A_3\) (with table T3):**

- \(\text{Ann } A = \mathbb{R}e_3, \; \mathcal{N}(A) = \text{Span}_\mathbb{R}\{e_1, e_3\} \cup \text{Span}_\mathbb{R}\{e_2, e_3\}, \; \mathcal{I}(A) = \emptyset,\)
- 1-dimensional subalgebras: \(\mathbb{R}u\) for each \(u \in \mathcal{N}(A),\)
- 2-dimensional subalgebras: \(\text{Span}_\mathbb{R}\{e_3, pe_1 + qe_2\} (p^2 + q^2 \neq 0),\)
- ideals: \(\mathbb{R}e_3, \; \text{Span}_\mathbb{R}\{e_3, pe_1 + qe_2\} (p^2 + q^2 \neq 0),\)
- \(A^2 = \mathbb{R}e_2,\)
• A is a nilpotent associative algebra,
• $\text{Der } A = \begin{bmatrix} x & 0 & 0 \\ 0 & z & 0 \\ y & v & x + z \end{bmatrix}$ with $x, y, z, v \in \mathbb{R}$,
• $\text{Aut } A = \tilde{H} \cup h \cdot H$ where
  \[
  H = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & z & 0 \\ y & v & xz \end{bmatrix} \mid x, y, z, v \in \mathbb{R}, xz \neq 0 \right\}
  \text{ and } h = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
  (in fact, $H$ is a normal divisor for $\text{Aut } A$, $h^2 = \text{Id}$ and $\text{Aut } A/H \cong \mathbb{Z}_2$).
• the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  • the singletons covering the planes $x^1Ox^3$ and $x^2Ox^3$,
  • the half-planes delimited by axis $Ox^3$ on each plane containing $Ox^3$ less the planes $x^1Ox^3$ and $x^2Ox^3$,
• the partition $\mathcal{P}_A$ of $A$ induces a partition of the set of integral curves of the associated HQDS consisting of:
  • the singletons consisting of singular solutions covering the planes $x^1Ox^3$ and $x^2Ox^3$,
  • the integral curves lying on the half-planes delimited by axis $Ox^3$ of each plane containing $Ox^3$ less the planes $x^1Ox^3$ and $x^2Ox^3$.

Each nonsingular integral curve lies on a subalgebra so that it is torsion-free.

2) Case $\text{Spec } D = (1, 1, 2)$

There exists a basis $\mathcal{B} = (e_1, e_2, e_3)$ such that the multiplication table of algebra $A(\cdot)$ has the form:

\begin{align*}
\text{Table } T & \quad e_1^2 = pe_3 \\
& \quad e_2^2 = qe_3 \\
& \quad e_3^2 = 0 \\
& \quad e_1e_2 = re_3 \\
& \quad e_1e_3 = 0 \\
& \quad e_2e_3 = 0
\end{align*}

with $p, q, r \in \mathbb{R}$.

**Subcase $pqr \neq 0$**

Each such algebra is isomorphic to algebra:

\begin{align*}
\text{Table } T' & \quad e_1^2 = e_3 \\
& \quad e_2^2 = \lambda e_3 \\
& \quad e_3^2 = 0 \\
& \quad e_1e_2 = e_3 \\
& \quad e_1e_3 = 0 \\
& \quad e_2e_3 = 0
\end{align*}

with $\lambda \neq 0$ (indeed, it is enough to use the basis $(e_1, \frac{p}{r}e_2, pe_3)$). We have to consider the subcases:

- $(i) \lambda = 1$
- $(ii) \lambda < 1 (\lambda \neq 0)$
- $(iii) \lambda > 1$

$(i)$ **Subcase $\lambda = 1$**
There exists a basis $B = (e_1, e_2, e_3)$ such that the multiplication table of algebra $A(\cdot)$ gets the form $T_2$, so that this algebra is isomorphic to algebra $A_2$.

(iii) Subcase $\lambda < 1$, $\lambda \neq 0$

Let $s_1 \neq s_2$ be the distinct solutions of equation $\lambda s^2 + 2s + 1 = 0$. Then, in basis $(e_1 + s_1 e_2, e_1 + s_2 e_2, 2\frac{\lambda-1}{\lambda} e_3)$ where $\lambda s_i^2 + 2s_i + 1 = 0$ (for $i = 1, 2$) the multiplication table of algebra $A(\cdot)$ gets the form $T_3$, so that this algebra is isomorphic to algebra $A_3$.

(iii) Subcase $\lambda > 1$

In basis $(\sqrt[2]{\lambda-1} e_1, e_1 - e_2, (\lambda-1)e_3)$ the multiplication table of algebra $A(\cdot)$ gets the form:

| Table T4 | $e_1^2 = e_3$ | $e_2^2 = e_3$ | $e_3^2 = 0$ |
|----------|---------------|---------------|--------------|
| $e_1e_2 = 0$ | $e_1e_3 = 0$ | $e_2e_3 = 0$ |

Properties of algebra $A_4$ (with table T4):

- $Ann A = \mathbb{R}e_3$, $N(A) = \mathbb{R}e_3$, $I(A) = \emptyset$,
- 1-dimensional subalgebras: $\mathbb{R}e_3$,
- 2-dimensional subalgebras: $Span_{\mathbb{R}}\{e_3, pe_1 + qe_2\}$ ($p^2 + q^2 \neq 0$),
- ideals: $\mathbb{R}e_3$, $Span_{\mathbb{R}}\{e_3, pe_1 + qe_2\}$ ($p^2 + q^2 \neq 0$),
- $A^2 = \mathbb{R}e_3$,
- $A$ is a nilpotent associative algebra,
- $Der A = \begin{bmatrix} x & -y & 0 \\ y & x & 0 \\ z & v & 2x \end{bmatrix}$ with $x, y, z, v \in \mathbb{R}$,
- $Aut A = \begin{bmatrix} \rho \cos \theta & -\rho \sin \theta & 0 \\ \rho \sin \theta & \rho \cos \theta & 0 \\ z & v & \rho^2 \end{bmatrix} | \rho, z, v \in \mathbb{R}, \rho > 0, \theta \in [0, 2\pi]$,
- the partition of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  - the singletons covering axis $Ox^3$,
  - the half-planes delimited by axis $Ox^3$ on each plane containing $Ox^3$,
- the partition $P_A$ of $A$ induces a partition of the set of integral curves of the associated HQDS consisting of:
  - the singletons consisting of singular solutions covering the axis $Ox^3$,
  - the integral curves lying on the half-planes delimited by axis $Ox^3$ of each plane containing $Ox^3$.

Each nonsingular integral curve lies on a subalgebra such that it is torsion-free. There exists a large family of linear prime integrals in accordance with the family of ideals and, consequently, each nonsingular integral curve has a zero curvature tensor. More exactly, each nonsingular integral curve lies on a parallel to $Ox^3$.

Subcase $pqr = 0$

In this case we have to analyze the following six possibilities:
(1) \( p = 0, q \neq 0, r = 0 \).  
(2) \( p = 0, q = 0, r \neq 0 \).  
(3) \( p = 0, q \neq 0, r \neq 0 \).  
(4) \( p \neq 0, q = 0, r = 0 \).  
(5) \( p \neq 0, q = 0, r \neq 0 \).  
(6) \( p \neq 0, q \neq 0, r = 0 \).

(1) This algebra is isomorphic to algebra \( A_2 \).  
(2) This algebra is isomorphic to algebra \( A_3 \).  
(3) This algebra is isomorphic to algebra \( A_3 \).  
(4) This algebra is isomorphic to algebra \( A_2 \).  
(5) By using the changes of bases \((p, q, e_1, e_2, e_3)\) followed by \((2e_1 - e_2, 2e_2, 2e_3)\) it results that this algebra is isomorphic to algebra \( A_3 \).  
(6) If \( pq > 0 \) this algebra is isomorphic to algebra \( A_4 \). In case \( pq < 0 \) there exists a basis such that the multiplication table of algebra \( A(\cdot) \) gets the form:

\[
\begin{array}{ccc}
&e_1^2 = e_3 & e_2^2 = -e_3 & e_3^2 = 0 \\
e_1e_2 & = 0 & e_1e_3 & = 0 & e_2e_3 & = 0
\end{array}
\]

This algebra has \( \mathcal{N}(A) = \{ x(e_1 \pm e_2) + ze_3 \mid x, z \in \mathbb{R} \} \). Then the basis \((\frac{1}{2}(e_1 + e_2), \frac{1}{2}(e_1 - e_2), \frac{1}{2}e_3)\) assures us that \( A(\cdot) \) is isomorphic with an algebra of type \( A_3 \).

**Case Spec \( D = (1, 2, 2) \)**

Looking at the set of derivations for algebra \( T_2 \) we see that it contains a derivation \( D \) with \( \text{Spec } D = (1, 2, 2) \), namely, the derivation corresponding to 
\[
\beta = 2, \quad \rho = 1, \quad \gamma = \lambda = \mu = 0;
\]
accordingly, the class of algebras having a derivation with spectrum \( (1, 2, 2) \) is contained into the class of algebras of type \( T_2 \).

Similar results hold for the rest of cases listed in Proposition 3.3. More exactly, the following assertions are valid:

- \( \text{Spec } D = (1, 2, 3) \) corresponds to derivations of type \( T_2 \) for 
  \[
  \beta = 3, \quad \rho = 1, \quad \gamma = \lambda = \mu = 0,
  \]
- \( \text{Spec } D = (1, 2, 4) \) corresponds to derivations of type \( T_2 \) for  
  \[
  \beta = 4, \quad \rho = 1, \quad \gamma = \lambda = \mu = 0,
  \]
- \( \text{Spec } D = (1, 2, a) \) corresponds to derivations of type \( T_2 \) for  
  \[
  \beta = a, \quad \rho = 1, \quad \gamma = \lambda = \mu = 0,
  \]
- \( \text{Spec } D = (1, a, 2a) \) corresponds to derivations of type \( T_2 \) for  
  \[
  \beta = 1, \quad \rho = a, \quad \gamma = \lambda = \mu = 0,
  \]
The existence of a semisimple nonsingular derivation for a real 3-dimensional commutative algebra implies the existence of a large family of symmetries reflected by the existence of suitable bases where the most part of structure constants become zero (see Theorem 3.1). In these bases, the corresponding homogeneous quadratic differential systems get the simplest form (see Theorem 3.2). This implies the existence of geometric symmetries for the set of all its integral curves. From algebraic point of view it follows that the existence of a semisimple nonsingular derivation involves the solvability of these algebras even the nilpotence property for most part of them. Moreover, algebras $A_2 - A_4$ are necessarily associative, while algebra $A_1$ is neither associative nor power-associative. Further, each ideal of algebra $A$ allows
to identify a decoupled subsystem of the associated HQDS. Consequently, from geometrical point of view, let us remark that all its nonsingular integral curves have zero torsion, because each such algebra is solvable and there exists at least a linear prime integral of the associated system. Moreover, Theorem 3.2 assures that integral curves of systems 2)-4) have both curvature and torsion tensors zero, so that these curves are lying on lines parallel to one of the coordinate axes.

References

[1] Burdujan I., Quadratic differential systems, (Romanian), Pim, Iaşi, 2008.
[2] Burdujan I., Homogeneous quadratic dynamical systems on $\mathbb{R}^3$ having derivations with complex eigenvalues, Libertas Mathematica XXVIII (2008) 69–92.
[3] Burdujan I., A classification of a class of homogeneous quadratic dynamical systems on $\mathbb{R}^3$ with derivations, Bull. I.P. Iaşi, Sect. Matematica, Mecanica teoretica, Fizica, LIV (2008) 37–47.
[4] Burdujan I., Classification of Quadratic Differential Systems on $\mathbb{R}^3$ having a nilpotent of order 3 Derivation, Libertas Mathematica XXIX (2009) 47-64.
[5] T. Date, Classifications and Analysis of Two-dimensional Real Homogeneous Quadratic Differential Equation Systems, J. Diff. Eqs. 32 (1979) 311–334.
[6] Kaplan J. L., Yorke J. A., Nonassociative real algebras and quadratic differential equations, Nonlinear Analysis, Theory, Methods and Applications, v. 3,(1), 1979 , 49–51.
[7] I. Kaplansky, Algebras with many derivations, ”Aspects of Mathematics and its Applications” (ed. J.A. Barroso), Elsevier Science Publishers B.V., 1986, 431.
[8] K.M. Kinyon , A.A. Sagle, Quadratic Dynamical Systems and Algebras, J. of Diff. Eqs. 117 (1995) 67–127 .
[9] Kinyon K. M., Sagle A. A., Automorphisms and Derivations of Differential Equations and Algebras, Rocky Mountain J. of Math., v. 24, no. 1, 1994, 135–153.
[10] Krasnov Y., Kononovich A., Osharovitch G., On a structure of the fixed point set of homogeneous maps, Discrete and Continuous Dynamical Systems Ser.S, v.6, no.4, 2013, 1017-1027.
[11] Lang S. Algebra (3rd edition), Springer-Verlag. 2002.
[12] L. Markus, Quadratic Differential Equations and Non-associative Algebras, in ”Contributions to the Theory of Nonlinear Oscillations” Annals of Mathematics Studies, no. 45, Princeton University Press, Princeton, N. Y., 1960.
[13] A.A. Sagle, R. Walde, Introduction to Lie groups and Lie algebras, Academic Press, New York, 1973.
[14] R.D. Schafer, An Introduction to Nonassociative Algebras, Academic Press, New York, London, 1966.
[15] N.I. VULPE, K.S. SIBIRSKII, Geometrical Classification of Quadratic differential systems (Russian), Differentialnye Uravnenje 13 no. 5 (1977) 803–814.

[16] S. WALKER, Algebras and differential equations, Hadronic Press, Palm Harbor, 1991.