THE COHERENT COHOMOLOGY RING OF AN ALGEBRAIC GROUP

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Abstract. Let $G$ be a group scheme of finite type over a field, and consider the cohomology ring $H^*(G)$ with coefficients in the structure sheaf. We show that $H^*(G)$ is a free module of finite rank over its component of degree 0, and is the exterior algebra of its component of degree 1. When $G$ is connected, we determine the Hopf algebra structure of $H^*(G)$.

1. Introduction

To each scheme $X$ over a field $k$, one associates the graded-commutative $k$-algebra $H^*(X) := \bigoplus_{i \geq 0} H^i(X, O_X)$ with multiplication given by the cup product. Any morphism of schemes $f : X \to X'$ induces a pull-back homomorphism of graded algebras $f^* : H^*(X') \to H^*(X)$, and there are Künneth isomorphisms $H^*(X) \otimes H^*(Y) \xrightarrow{\cong} H^*(X \times Y)$. When $X$ is affine, the “coherent cohomology ring” $H^*(X)$ is just the algebra $O(X)$ of global sections of $O_X$.

Now consider a $k$-group scheme $G$ with multiplication map $\mu : G \times G \to G$, neutral element $e_G \in G(k)$, and inverse map $\iota : G \to G$. Then $H^*(G)$ has the structure of a graded Hopf algebra with comultiplication $\mu^*$, counit $e_G^*$ and antipode $\iota^*$. If $G$ acts on a scheme $X$ and $F$ is a $G$-linearized quasi-coherent sheaf on $X$, then the cohomology $H^*(X, F)$ is equipped with the structure of a graded comodule over $H^*(G)$.

When $G$ is affine, the Hopf algebra $H^*(G) = O(G)$ uniquely determines the group scheme $G$. But this does not extend to an arbitrary group scheme $G$; for example, if $G$ is an abelian variety, then the structure of $H^*(G)$ only depends of $g := \dim(G)$. Indeed, by a result of Serre (see [Se59, Chap. 7, Thm. 10]), $H^*(G)$ is the exterior algebra $\Lambda^*(H^1(G))$; moreover, $H^1(G)$ has dimension $g$ and consists of the primitive elements of $H^*(G)$ (recall that $\gamma \in H^*(G)$ is primitive if $\mu^*(\gamma) = \gamma \otimes 1 + 1 \otimes \gamma$).

In the present article, we generalize this result as follows:

**Theorem 1.1.** Let $G$ be a group scheme of finite type over $k$. Then the graded algebra $H^*(G)$ is the exterior algebra of the $O(G)$-module $H^1(G)$, which is free of finite rank.

If $G$ is connected, then denoting by $P^*(G) \subset H^*(G)$ the graded subspace of primitive elements, we have an isomorphism of graded Hopf algebras

$$H^*(G) \cong O(G) \otimes \Lambda^*(P^1(G)).$$

Moreover, $P^i(G) = 0$ for all $i \geq 2$. 


As a consequence, the graded Lie algebra $P^*(G)$ equals $P^0(G) \oplus P^1(G)$, and hence is abelian; also, the vector space $P^1(G)$ is finite-dimensional. Note that $P^0(G)$ consists of the homomorphisms of group schemes $G \to \mathbb{G}_a$; this vector space is finite-dimensional in characteristic 0, but not in prime characteristics (already for $G = \mathbb{G}_a$).

When $G$ is an abelian variety and $k$ is perfect, the structure of $H^*(G)$ follows readily from that of connected graded-commutative Hopf algebras (see [Bo53, Thm. 6.1]) and from the isomorphism of $H^1(G)$ with the Lie algebra of the dual abelian variety (see [Mum74, §13, Cor. 3]). But for an arbitrary group scheme $G$, Theorem 1.1 is not a direct consequence of general structure results on Hopf algebras such as those of Cartier-Gabriel-Kostant (see [Sw69, Thm. 8.1.5]) and Milnor-Moore (see [MM65, §6]), since $H^*(G)$ is neither connected nor cocommutative. Also, returning to the setting of schemes, the $\mathcal{O}(X)$-module $H^*(X)$ is generally far from being free. For example, when $X$ is the punctured affine plane, $H^1(X)$ is a torsion module over $\mathcal{O}(X) = k[x, y]$ and is not finitely generated.

The proof of Theorem 1.1 is based on the affinization theorem (see [SGA3, Exp. VIB, Thm. 12.2]). It asserts that $G$ has a smallest normal subgroup scheme $H$ such that the quotient $G/H$ is affine; then $\mathcal{O}(G/H) \cong \mathcal{O}(G)$ via the quotient morphism $G \to G/H$, which is therefore identified with the canonical morphism $G \to \text{Spec} \mathcal{O}(G)$. In particular, the $k$-algebra $\mathcal{O}(G)$ is finitely generated. Moreover, $H$ is smooth, connected and contained in the center of the neutral component $G^o$; in particular, $H$ is commutative. Also, we have $\mathcal{O}(H) = k$, i.e., $H$ is “anti-affine”. In fact, $H$ is the largest anti-affine subgroup scheme of $G$; we denote it by $G_{\text{ant}}$.

By analyzing the quotient morphism $G \to G/G_{\text{ant}}$, we obtain an isomorphism of $\mathcal{O}(G)$-modules $\psi : H^*(G) \xrightarrow{\cong} \mathcal{O}(G) \otimes H^*(G_{\text{ant}})$ which identifies the pull-back $H^*(G) \to H^*(G_{\text{ant}})$ to $e^*_G \otimes \text{id}$ (Proposition 3.2). On the other hand, using the structure of anti-affine groups (see [Br09, SS09]) and additional arguments, we show that the Hopf algebra $H^*(G_{\text{ant}})$ is the exterior algebra of $H^1(G_{\text{ant}})$, a finite-dimensional vector space (Corollary 4.2 and Proposition 4.3). This implies the first assertion of Theorem 1.1.

When $G$ is connected, we show that the above map $\psi$ is an isomorphism of graded Hopf algebras (Proposition 3.3); moreover, $P^1(G) \cong H^1(G_{\text{ant}})$ via pull-back. This yields a description of the primitive elements which takes very different forms in characteristic 0 and in positive characteristics. We refer to Theorem 5.3 for the full statement, and mention a rather unexpected consequence: in positive characteristics, the group schemes $G$ such that $H^*(G) = k$ are trivial; in characteristic 0, they are exactly the fibered products $S \times_A E$, where $S$ is an anti-affine extension of an abelian variety $A$ by a torus, and $E$ is the universal vector extension of $A$ (Corollary 5.1).

A natural problem is to describe the coherent cohomology ring of group schemes over (say) discrete valuation rings. In this setting, a version of the affinization theorem is known (see [SGA3, Exp. VIB, Prop. 12.10]), but the structure of “anti-affine” group schemes is an open question.

This article is organized as follows. Section 2 collects preliminary results on linearized sheaves which should be well-known, but which we could not locate in the
literature under a form suited for our purposes. In Section 3 we show how to reduce
the structure of $H^*(G)$ to that of $H^*(G_{\text{ant}})$. The latter is determined in Section 4
and our main results (Theorems 1.1 and 5.3) are proved in Section 5 by putting
everything together.

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2. Linearized sheaves

Throughout this article, we consider schemes and their morphisms over a fixed field
$k$. Unless otherwise mentioned, schemes are assumed to be separated and of finite
type. We use [SGA3] as a general reference for group schemes, and fix such a group
scheme $G$.

We begin by recalling some notions on actions of group schemes (see [SGA3] Exp. I,
§6). A $G$-scheme is a scheme $X$ equipped with a $G$-action
$$\alpha : G \times X \to X, \quad (g, x) \mapsto g \cdot x,$$
i.e., with a morphism of schemes that satisfies the axioms of a group action. Given
two $G$-schemes $X, Y$, a morphism $u : X \to Y$ is $G$-equivariant if the diagram
$$\begin{array}{ccc}
G \times X & \xrightarrow{id \times u} & G \times Y \\
\downarrow \alpha & & \downarrow \beta \\
X & \xrightarrow{u} & Y
\end{array}$$
commutes, where $\beta$ denotes the $G$-action on $Y$; we also say that $u$ is a $G$-morphism.

A $G$-linearization of a quasi-coherent sheaf $\mathcal{F}$ on the $G$-scheme $X$ is an isomorphism
$$\Phi : \alpha^*(\mathcal{F}) \xrightarrow{\cong} p_2^*(\mathcal{F})$$
(where $p_2 : G \times X \to X$ denotes the projection) such that the following cocycle
condition holds: for any (commutative) $k$-algebra $R$ and for any $g, h \in G(R)$, we have
$$\Phi_{gh} = \Phi_h \circ h^*(\Phi_g),$$
where we denote by
$$\Phi_g : g^*(\mathcal{F}_R) \xrightarrow{\cong} \mathcal{F}_R$$
the isomorphism of sheaves over $X_R := \text{Spec}(R) \times X$ obtained from $\Phi$ by base change
with $g \times \text{id} : X_R \to G \times X$. A sheaf equipped with a $G$-linearization will be called a
$G$-sheaf.

Given two $G$-sheaves $\mathcal{F}, \mathcal{G}$ on a $G$-scheme $X$, a morphism of sheaves of $\mathcal{O}_X$-modules
$\varphi : \mathcal{F} \to \mathcal{G}$ is $G$-equivariant, or a $G$-morphism, if the square
$$\begin{array}{ccc}
g^*(\mathcal{F}_R) & \xrightarrow{\Phi_g} & \mathcal{F}_R \\
g^*(\varphi_R) \downarrow & & \varphi_R \downarrow \\
g^*(\mathcal{G}_R) & \xrightarrow{\Psi_g} & \mathcal{G}_R
\end{array}$$
commutes for any \( k \)-algebra \( R \) and any \( g \in G(R) \), where \( \Phi \) (resp. \( \Psi \)) denotes the linearization of \( F \) (resp. \( G \)). The \( G \)-sheaves and \( G \)-morphisms form an abelian category that we denote by \( \text{QCoh}^G(X) \); the coherent \( G \)-sheaves are the objects of a full abelian subcategory, \( \text{Coh}^G(X) \). By [Th87, Lem. 1.4], any \( G \)-sheaf is the direct limit of its coherent \( G \)-subsheaves.

If \( X = \text{Spec}(k) \), then a quasi-coherent sheaf is just a \( k \)-vector space \( V \), and a \( G \)-linearization, a linear representation of \( G \) on \( V \). So \( \text{QCoh}^G(X) \) is equivalent to the category \( \text{Mod}(G) \) of \( G \)-modules, and \( \text{Coh}^G(X) \), to the full subcategory \( \text{mod}(G) \) of finite-dimensional \( G \)-modules.

We will need the following variant of a result in [HL10, p. 94]):

**Lemma 2.1.** Let \( u : X \to Y \) be an equivariant morphism of \( G \)-schemes, and \( F \) (resp. \( G \)) a \( G \)-sheaf on \( X \) (resp. \( Y \)). Then the higher direct images \( R^i u_*(F) \) (\( i \geq 0 \)), and the pull-back \( u^*(G) \) are equipped with natural structures of \( G \)-sheaves. In particular, \( H^i(X, F) \) is a \( G \)-module for any \( G \)-sheaf \( F \).

If in addition \( u \) sits in a cartesian square of equivariant morphisms of \( G \)-schemes,

\[
\begin{array}{ccc}
X' & \xrightarrow{u'} & Y' \\
\downarrow v' & & \downarrow v \\
X & \xrightarrow{u} & Y,
\end{array}
\]

where \( v \) is flat, then the base change isomorphism

\[
\theta_{u,v} : v^* R^i u_*(F) \xrightarrow{\cong} R^i u'_* v'^*(F)
\]

is \( G \)-equivariant.

**Proof.** The assertions on the pull-back \( u^*(G) \) and the direct image \( u_*(F) \) are special cases of [SGA3, Exp. I, Rem. 6.5.2, Lem. 6.6.1]. The assertion on higher direct images is checked similarly; we provide additional details on base change isomorphisms (treated as equalities in [loc. cit.]) for completeness.

Since \( \beta \) is flat, the cartesian square (2.1) yields a base change isomorphism

\[
\theta_{u,\beta} : \beta^* R^i u_*(F) \xrightarrow{\cong} R^i (\text{id} \times u)_* \alpha^*(F).
\]

We obtain similarly an isomorphism

\[
\theta_{u,p_2} : p_2^* R^i u_*(F) \xrightarrow{\cong} R^i (\text{id} \times u)_* p_2^*(F).
\]

Thus, there is a unique isomorphism

\[
\Psi : \beta^* R^i u_*(F) \xrightarrow{\cong} p_2^* R^i u_*(F)
\]

such that the square

\[
\begin{array}{ccc}
\beta^* R^i u_*(F) & \xrightarrow{\theta_{u,\beta}} & R^i (\text{id} \times u)_* \alpha^*(F) \\
\downarrow \Psi & & \downarrow R^i (\text{id} \times u)_* \alpha^*(\Phi) \\
p_2^* R^i u_*(F) & \xrightarrow{\theta_{u,p_2}} & R^i (\text{id} \times u)_* p_2^*(F)
\end{array}
\]

is commutative.
commutes; then for any \( g \in G(R) \), the induced morphism \( \Psi_g : g^* R^i u_* (\mathcal{F}) \to R^i u_* (\mathcal{F}) \)

satisfies \( \Psi_g = R^i u_* (\Phi_g) \). To show that \( \Psi \) is a \( G \)-linearization of \( R^i u_* (\mathcal{F}) \), it

remains to check that \( \Psi_{g_1 g_2} = \Psi_{g_1} \circ \Psi_{g_2} \) for all \( g_1, g_2 \in G(R) \). Using the analogous condition for \( \Phi \) and the equality

\[
\theta_{u,g h} = \theta_{u,h} \circ h^* (\theta_{u,g}),
\]

this reduces to checking that \( \theta_{u,h} \circ h^* R^i u_* (\Phi_g) = R^i u_* h^* (\Phi_g) \circ \theta_{u,h}, \) i.e., the square

\[
\begin{array}{ccc}
R^i u_* h^* (\Phi_{g_1 g_2}) & \xrightarrow{\theta_{u,h}} & R^i u_* h^* (\Phi_g) \\
\downarrow & & \downarrow \\
R^i u_* h^* g^* (\mathcal{F}_R) & \xrightarrow{h^* R^i u_* (\Phi_g)} & R^i u_* h^* (\mathcal{F}_R)
\end{array}
\]

commutes. But this follows from the compatibility of base change isomorphisms with isomorphisms of sheaves.

Finally, the assertion on \( \theta_{u,v} \) is equivalent to the commutativity of the square

\[
\begin{array}{ccc}
g^* v^* R^i u_* (\mathcal{F}_R) & \xrightarrow{v^* (\theta_{u,v})} & v^* R^i u_* (\mathcal{F}_R) \\
\downarrow & & \downarrow \\
g^* R^i u'_* v'^* (\mathcal{F}_R) & \xrightarrow{R^i u'_* v'^* (\Phi_g)} & R^i u'_* v'^* (\mathcal{F}_R)
\end{array}
\]

for any \( g \in G(R) \), where the horizontal maps are induced by the linearizations. Using the equivariance of \( u, v, u', v' \), this amounts to the commutativity of the square

\[
\begin{array}{ccc}
v^* R^i u_* (g^* (\mathcal{F}_R)) & \xrightarrow{v^* (\theta_{u,v})} & v^* R^i u_* (\mathcal{F}_R) \\
\downarrow & & \downarrow \\
R^i u'_* v'^* (g^* (\mathcal{F}_R)) & \xrightarrow{R^i u'_* v'^* (\Phi_g)} & R^i u'_* v'^* (\mathcal{F}_R)
\end{array}
\]

which follows again from the compatibility of base change isomorphisms with isomorphisms of sheaves.

We also record the following variant of \cite[Exp. I, Prop. 6.6.2]{SGA3}:

**Lemma 2.2.** Let \( \mathcal{F} \) be a \( G \)-sheaf on a \( G \)-scheme \( X \). Then \( H^*(X, \mathcal{F}) \) is a graded left Hopf comodule over the graded Hopf algebra \( \mathcal{H}(G) \). Moreover, the comodule map

\[
\Delta : H^*(X, \mathcal{F}) \to H^*(G) \otimes H^*(X, \mathcal{F})
\]

is compatible with the \( G \)-module structure, i.e., the square

\[
\begin{array}{ccc}
H^*(X, \mathcal{F}) & \xrightarrow{\Delta} & H^*(G) \otimes H^*(X, \mathcal{F}) \\
\downarrow & & \downarrow g^* \times \text{id} \\
H^*(X_R, \mathcal{F}_R) & \xrightarrow{g^*} & H^*(X_R, \mathcal{F}_R)
\end{array}
\]

(2.4)

(where the left vertical arrow is the pull-back map) commutes for any \( k \)-algebra \( R \) and any \( g \in G(R) \).
We omit the proof which follows similar lines as that of Lemma 2.1, the functorial properties of base change isomorphisms being replaced with those of Künneth isomorphisms.

Note that the component of bi-degree $(0,i)$, $\Delta^{(0,i)} : H^i(X, F) \to \mathcal{O}(G) \otimes H^i(X, F)$, is the comodule map for the $G$-module structure of $H^i(X, F)$; the image of $\Delta^{(0,i)}$ is the subspace of $G$-invariants, where $G$ acts on $\mathcal{O}(G) \otimes H^i(X, F)$ via its action on $\mathcal{O}(G)$ by right multiplication, and its action on $H^i(X, F)$ defined in Lemma 2.1.

We now turn to the behavior of linearized sheaves under torsors, and obtain a slight generalization of [HL10, Thm. 4.2.14]:

**Lemma 2.3.** Let $H$ be a group scheme, $X$ a $G \times H$-scheme, $Y$ an $H$-scheme, and $u : X \to Y$ a $G$-torsor which is also $H$-equivariant. Then the pull-back $u^*$ and the invariant direct image $u^*_G$ yield equivalences of categories

$$\text{QCoh}^H(Y) \cong \text{QCoh}^{G \times H}(X), \quad \text{Coh}^H(Y) \cong \text{Coh}^{G \times H}(X).$$

**Proof.** Consider first the case where $H$ is trivial. Then a $G$-linearization of a quasi-coherent sheaf $F$ is just a descent data for the faithfully flat morphism $u$. Thus, $u^*$ is an equivalence from $\text{QCoh}^H(Y)$ to $\text{QCoh}^{G \times H}(X)$, by [SGA1, Exp. VIII, Cor. 1.3]; it restricts to an equivalence from $\text{Coh}^H(Y)$ to $\text{Coh}^{G \times H}(X)$ by [loc. cit., Rem. 1.12]. Moreover, the natural map $F \to u^*_G u^*(F)$ is an isomorphism, as follows from [loc. cit., Cor. 1.7]. This proves the assertion in this case.

In the general case, note that $u^*(G)$ is a $G \times H$-sheaf on $X$ for any $H$-sheaf $G$ on $Y$, by Lemma 2.1 applied to the $G \times H$-equivariant morphism $u$ (relative to the trivial action of $G$ on $Y$). Conversely, for any $G \times H$-sheaf $F$ on $X$, the data of its $H$-linearization descends to an $H$-linearization of $u^*_G(F)$, as follows from descent for morphisms of sheaves (see [loc. cit., Cor. 1.2]). □

Next, consider a subgroup scheme $H$ of $G$ and an $H$-scheme $Y$. Then $H$ acts freely on $G \times Y$ via $h \cdot (g, y) := (gh^{-1}, hy)$; we assume that the quotient $X := (G \times Y)/H$ is a scheme, and denote it by $G \times^H Y$. We have a $G$-action on $X$ via left multiplication on $G$, and a cartesian square of $G$-morphisms

\[
\begin{array}{ccc}
G \times Y & \xrightarrow{p_1} & G \\
\downarrow r & & \downarrow q \\
X & \xrightarrow{f} & G/H,
\end{array}
\]

where $p_1$ denotes the projection, and $q$, $r$ the quotients by $H$. The fiber of $f$ at the base point of $G/H$ is identified to $Y$; let

$$j : Y \to X$$

be the corresponding closed immersion, and

$$p_2 : G \times Y \to Y$$

the projection. We may now state the following variant of [Th87, Lem. 1.3]:

**Lemma 2.4.** With the above notation, the pull-back $j^*$ and the composition $r^H \circ p_2^*$ yield equivalences $\text{QCoh}^G(X) \cong \text{QCoh}^H(Y)$ and $\text{Coh}^G(X) \cong \text{Coh}^H(Y)$. 
Proof. Applying Lemma 2.3 to the $G$-equivariant $H$-torsor $r : G \times Y \to X$ and to the $H$-equivariant $G$-torsor $p_2 : G \times Y \to Y$, we obtain that $r_*^H \circ p_2^*$ is an equivalence from $\text{QCoh}^H(Y)$ to $\text{QCoh}^G(X)$, and likewise for coherent sheaves. We now show that $j^*$ yields the inverse equivalence. Note that $j = r \circ s$, where

$$s : Y \to G \times Y, \quad y \mapsto (e_G, y)$$

is a section of $p_2$. Let $\mathcal{F}$ be a $G$-sheaf on $X$; then $\mathcal{F} = r_*^H p_2^*(\mathcal{G})$ for a unique $H$-sheaf $\mathcal{G}$ on $Y$. Thus, $j^*(\mathcal{F}) = s^* r_*^H p_2^*(\mathcal{G}) = s^* p_2^*(\mathcal{G}) = \mathcal{G}$. \hfill \qed

In particular, if $Y = \text{Spec}(k)$ then $X = G/H$; also, recall that an $H$-sheaf on $Y$ is just an $H$-module. Thus, Lemma 2.4 yields familiar equivalences of categories

$$\text{QCoh}^G(G/H) \cong \text{Mod}(H), \quad \text{Coh}^G(G/H) \cong \text{mod}(H).$$

For any $H$-module $M$, we denote by

$$\mathcal{L}_{G/H}(M) := q_*^H(\mathcal{O}_G \otimes M)$$

the associated $G$-sheaf on $G/H$, where we recall that $q : G \to G/H$ stands for the quotient morphism.

Returning to an arbitrary $H$-scheme $Y$, we obtain a variant of [Ja03, 5.19]:

**Lemma 2.5.** With the above notation, let $\mathcal{F}$ be a $G$-sheaf on $X = G \times^H Y$, and $\mathcal{G} := j^*(\mathcal{F})$ the corresponding $H$-sheaf on $Y$. Then for any $i \geq 0$, there is an isomorphism of $G$-sheaves

$$R^i f_*(\mathcal{F}) \cong \mathcal{L}_{G/H}(H^i(Y, \mathcal{G})).$$

**Proof.** The cartesian square (2.5), where $q$ is flat, yields a pull-back isomorphism

$$q^* R^i f_*(\mathcal{F}) \cong R^i (p_1)_* r^*(\mathcal{F})$$

which is a morphism of $G \times H$-sheaves in view of Lemmas 2.1 and 2.3. But $\mathcal{F} \cong r_*^H p_2^*(\mathcal{G})$ by Lemma 2.4 and hence $r^*(\mathcal{F}) \cong p_2^*(\mathcal{G})$ by Lemma 2.3 again. Thus,

$$q^* R^i f_*(\mathcal{F}) \cong R^i (p_1)_* p_2^*(\mathcal{G}) \cong \mathcal{O}_G \otimes H^i(Y, \mathcal{G}).$$

This yields the required isomorphism $R^i f_*(\mathcal{F}) \cong q_*^H(\mathcal{O}_G \otimes H^i(Y, \mathcal{G}))$. \hfill \qed

3. **Reduction to an anti-affine group**

We still consider a group scheme $G$ of finite type over $k$. The action of $G \times G$ on $G$ via left and right multiplication: $(x, y) : z := x z y^{-1}$, equips $\mathcal{O}_G$ with the structure of a $G \times G$-sheaf. By Lemmas 2.1 and 2.2, this defines a structure of $G \times G$-module on $H^*(G)$ which is compatible with its structure of graded Hopf algebra; in particular, with its structure of $\mathcal{O}(G)$-module. We now describe the $(G \times G)-\mathcal{O}(G)$-module $H^*(G)$ in terms of the largest anti-affine subgroup $G_{\text{ant}}$, by carefully keeping track of all the actions:

**Proposition 3.1.** For each integer $i \geq 0$, there is an isomorphism

$$\varphi_i : H^i(G) \cong (\mathcal{O}(G \times G) \otimes H^i(G_{\text{ant}}))^G,$$
where the right-hand side denotes the subspace of $G$-invariants for the action of $G$ on $\mathcal{O}(G \times G) \otimes H^i(G_{\text{ant}})$ via its action on $G \times G$ by right multiplication ($t \cdot (z, w) = (zt^{-1}, wt^{-1})$), and its action on $G_{\text{ant}}$ by conjugation.

Moreover, $\varphi_1$ is a morphism of $(G \times G)\mathcal{O}(G)$-modules, where $\mathcal{O}(G)$ acts on the right-hand side via the algebra homomorphism

$$(\text{id} \times t^*) \circ \mu^* : \mathcal{O}(G) \longrightarrow \mathcal{O}(G \times G), \quad f \longmapsto ((z, w) \mapsto f(zw^{-1})),$$

and $G \times G$ acts via its action on $G \times G$ by left multiplication: $(x, y) \cdot (z, w) = (xz, yw)$.

Finally, the square

$$(3.2) \quad \begin{array}{ccc} H^i(G) & \longrightarrow & (\mathcal{O}(G \times G) \otimes H^i(G_{\text{ant}}))^G \\ \downarrow j^* & \quad & \delta^* \otimes \text{id} \downarrow \\ H^i(G_{\text{ant}}) & \longrightarrow & (\mathcal{O}(G) \otimes H^i(G_{\text{ant}}))^G \end{array}$$

commutes, where $j : G_{\text{ant}} \rightarrow G$ denotes the inclusion, $\delta : G \rightarrow G \times G$ the diagonal (so that $\delta^* : \mathcal{O}(G \times G) = \mathcal{O}(G) \otimes \mathcal{O}(G) \rightarrow \mathcal{O}(G)$ is the multiplication), and $\Delta$ the comodule map for the $G$-module structure of $H^i(G_{\text{ant}})$.

**Proof.** We identify the $G \times G$-scheme $G$ to the quotient $(G \times G)/\delta(G)$; similarly, the $G \times G$-scheme $G/G_{\text{ant}}$ is identified to $(G \times G)/K$, where

$$K := \delta(G)(G_{\text{ant}} \times G_{\text{ant}}) = \delta(G)(e_G \times G_{\text{ant}}).$$

Thus, $K$ is isomorphic to the semi-direct product $G_{\text{ant}} \ltimes G$, where $G$ acts on $G_{\text{ant}}$ by conjugation. Moreover, the quotient morphism $G \rightarrow G/G_{\text{ant}}$ is identified to the natural morphism

$$f : (G \times G)/\delta(G) = (G \times G) \times^K K/\delta(G) \cong (G \times G) \times^K G_{\text{ant}} \longrightarrow (G \times G)/K,$$

where $K$ acts on $G_{\text{ant}}$ via the action of $G_{\text{ant}}$ on itself by multiplication, and the action of $G$ by conjugation. In view of Lemma 2.5, this yields an isomorphism of $G \times G$-linearized sheaves of $\mathcal{O}_{G/G_{\text{ant}}}$-modules:

$$R^i f_* (\mathcal{O}_G) \cong q_*^K (\mathcal{O}_{G \times G} \otimes H^i(G_{\text{ant}})),$$

where $q : G \times G \rightarrow (G \times G)/K$ denotes the quotient morphism, and $K$ acts on $H^i(G_{\text{ant}})$ via its above action on $G_{\text{ant}}$. Since $G/G_{\text{ant}}$ is affine, we obtain an isomorphism of $(G \times G)-\mathcal{O}(G/G_{\text{ant}})$-modules

$$H^i(G) \cong (\mathcal{O}(G \times G) \otimes H^i(G_{\text{ant}}))^K.$$

But $\mathcal{O}(G/G_{\text{ant}}) \cong \mathcal{O}(G)$; moreover, the anti-affine group $G_{\text{ant}}$ acts trivially on its module $\mathcal{O}(G \times G) \otimes H^i(G_{\text{ant}})$. This yields the isomorphism (3.1).

To show the final assertion, note that $j^* : H^i(G) \rightarrow H^i(G_{\text{ant}})$ factors as the natural map

$$H^i(G) \longrightarrow H^i(G) \otimes_{\mathcal{O}(G)} k = H^i(G) \otimes_{\mathcal{O}(G/G_{\text{ant}})} k$$

(associated to $e^*_G : \mathcal{O}(G) \rightarrow k$, or equivalently, to $e^*_{G/G_{\text{ant}}} : \mathcal{O}(G/G_{\text{ant}}) \rightarrow k$), followed by the map

$$(3.3) \quad H^i(G) \otimes_{\mathcal{O}(G/G_{\text{ant}})} k \longrightarrow H^i(G_{\text{ant}})$$
obtained by base change in the cartesian square
\[
\begin{array}{ccc}
G_{\text{ant}} & \longrightarrow & \text{Spec}(k) \\
\downarrow & & \downarrow e_{G/G_{\text{ant}}} \downarrow \\
G & \longrightarrow & G/G_{\text{ant}}.
\end{array}
\]
But this base change map yields an isomorphism \( e_{G/G_{\text{ant}}}^* \) by Lemma 2.5, and hence \((3.3)\) is an isomorphism as well. This identifies \( j^* \) to \( \otimes_{\mathcal{O}(G)} k \) for the structure of \( \mathcal{O}(G) \)-module on \( (\mathcal{O}(G \times G) \otimes H^i(G_{\text{ant}}))^G \), and implies the commutativity of \((3.2)\) by using the fact that \( \delta(G) \) is the fiber at \( e_G \) of the morphism \( \mu \circ (\text{id} \times \iota) : G \times G \to G \), \( (z, w) \mapsto zw^{-1} \).

The above result entails a description of the comultiplication \( \mu^* : H^*(G) \longrightarrow H^*(G) \otimes H^*(G) \), since the latter is the comodule map for the \( G \)-action on \( H^*(G) \) via left multiplication on \( G \). Likewise, the counit and antipode may be described via the isomorphisms \((3.1)\). But the resulting determination of the Hopf algebra structure of \( H^*(G) \) is very indirect, and will not be developed here.

We now obtain a simpler version of Proposition 3.1, where the left and right \( G \)-actions take different forms:

**Proposition 3.2.** For each integer \( i \geq 0 \), there is an isomorphism of \( (G \times G) \)-\( \mathcal{O}(G) \)-modules
\[
\psi_i : H^i(G) \longrightarrow \mathcal{O}(G) \otimes H^i(G_{\text{ant}}),
\]
where \( G \times e_G \) (resp. \( e_G \times G \)) acts on the right-hand side via its action on \( G \) by left multiplication (resp. its action on \( G \) by right multiplication, and on \( H^i(G_{\text{ant}}) \) by conjugation on \( G_{\text{ant}} \)), and \( \mathcal{O}(G) \) acts by multiplication on itself. Moreover, the triangle
\[
\begin{array}{ccc}
H^i(G) & \longrightarrow & \mathcal{O}(G) \otimes H^i(G_{\text{ant}}) \\
\downarrow j^* & & \downarrow e_{G/G_{\text{ant}}} \otimes \text{id} \\
H^i(G_{\text{ant}}) & \longrightarrow & \end{array}
\]
commutes.

In particular, there are isomorphisms of vector spaces
\[
H^i(G)^{G \times e_G} \cong H^i(G)^{e_G \times G} \cong H^i(G_{\text{ant}}), \quad H^i(G)^{G \times G} \cong H^i(G_{\text{ant}})^G.
\]

**Proof.** Consider the automorphism \( u \) of \( G \times G \) given by \( u(z, w) = (zw^{-1}, w) \); then \( u(x, y) \cdot (z, w) \cdot t = (xzw^{-1}y^{-1}, ywt^{-1}) \). The induced automorphism \( u^* \) of \( \mathcal{O}(G \times G) = \mathcal{O}(G) \otimes \mathcal{O}(G) \) yields an isomorphism for any \( G \)-module \( M \):
\[
u^* \otimes \text{id} : (\mathcal{O}(G) \otimes \mathcal{O}(G) \otimes M)^G \longrightarrow \mathcal{O}(G) \otimes (\mathcal{O}(G) \otimes M)^G,
\]
where \( G \) acts on \( \mathcal{O}(G) \otimes \mathcal{O}(G) \otimes M \) (resp. on \( \mathcal{O}(G) \otimes M \)) via its right action on \( G \times G \) (resp. on \( G \)) and its given action on \( M \).

Also, recall the isomorphism of \( G \)-modules
\[
e_{G}^* \otimes \text{id} : (\mathcal{O}(G) \otimes M)^G \longrightarrow M,
\]
where the invariants in the left-hand side are taken for the $G$-action on $O(G)$ via right multiplication, and the given $G$-action on $M$ (this isomorphism is the inverse of the comodule map $\Delta : M \stackrel{\cong}{\rightarrow} (O(G) \otimes M)^G$). Thus, we obtain an isomorphism

$$(O(G \times G) \otimes M)^G \cong O(G) \otimes M,$$

which is also $O(G)$-linear for the action of $O(G)$ on $O(G) \otimes O(G)$ via $(id \times \iota^*) \circ \mu^*$, and $G \times G$-equivariant for the action on the left-hand side via left multiplication on $G \times G$, and on the right-hand side as in the statement.

Taking $M = H^i(G_{\text{ant}})$ and applying Proposition 3.1, we obtain the isomorphism $\psi_i$ and its compatibility properties. The assertions on invariants follow readily. □

The $G \times G$-invariants in $H^*(G)$ are related to the primitive elements as follows:

**Proposition 3.3.** For each integer $i \geq 1$, the space of homogeneous primitive elements of degree $i$ satisfies

$$P^i(G) \subset H^i(G)^{G \times G},$$

with equality for $i = 1$.

**Proof.** Recall that for any $G$-module $M$, we have $M^G = \{m \in M \mid \Delta(m) = 1 \otimes m\}$, where $\Delta : M \rightarrow O(G) \otimes M$ denotes the comodule map. Together with Lemma 2.2, it follows that

$$H^i(G)^{G \times G} = \{\gamma \in H^i(G) \mid \mu^{(0,i)}(\gamma) = 1 \otimes \gamma\},$$

where $\mu^{(0,i)} : H^i(G) \rightarrow O(G) \otimes H^i(G)$ denotes the component of bi-degree $(0,i)$ of the comultiplication

$$\mu^* : H^i(G) \rightarrow H^i(G \times G) = \bigoplus_{i_1,i_2;i_1+i_2=i} H^{i_1}(G) \otimes H^{i_2}(G).$$

Likewise,

$$H^i(G)^{G \times G} = \{\gamma \in H^i(G) \mid \mu^{(i,0)}(\gamma) = \gamma \otimes 1\}.$$

As a consequence,

$$H^i(G)^{G \times G} = \{\gamma \in H^i(G) \mid \mu^*(\gamma) - \gamma \otimes 1 - 1 \otimes \gamma \in \bigoplus_{i_1,i_2 > 0; i_1+i_2=i} H^{i_1}(G) \otimes H^{i_2}(G)\}.$$

This yields our statement. □

Next, we obtain a refinement of Proposition 3.2:

**Proposition 3.4.** If $G$ is connected, then we have equalities of subspaces of $H^*(G)$:

$$H^*(G)^{G \times G} = H^*(G)^{G \times G} = H^*(G)^G \times G$$

and this subspace is a graded Hopf subalgebra, isomorphic to $H^*(G_{\text{ant}})$ via $j^*$. Moreover, we have an isomorphism of graded Hopf algebras

$$H^*(G) \cong O(G) \otimes H^*(G)^{G \times G}.$$
Proof. By Proposition 3.2, we may identify the $G \times G$-module $H^*(G)$ to $\mathcal{O}(G) \otimes H^*(G_{\text{ant}})$, where $G \times G$ acts on $\mathcal{O}(G)$ via left and right multiplication, and $G \times G$ acts trivially on $H^*(G_{\text{ant}})$ (since $G_{\text{ant}}$ is central in $G$ in view of the connectedness assumption). Then

$$H^*(G)^{G \times G} = H^*(G)^{e_G \times G} = 1 \otimes H^*(G_{\text{ant}}),$$

since $\mathcal{O}(G)^{G \times G} = \mathcal{O}(G)^{e_G \times G} = k$. This proves the equalities.

The comultiplication

$$\mu^* : H^*(G) \longrightarrow H^*(G) \otimes H^*(G)$$

sends $H^*(G)^{G \times G}$ to $H^*(G)^{G \times e_G} \otimes H^*(G)^{e_G \times G}$, since the multiplication $\mu : G \times G \to G$ is equivariant for the action of $G \times G$ on itself via $(x, y) \cdot (z, w) = (xz, wy^{-1})$, and for the action of $G \times G$ on $G$ via left and right multiplication. Thus, $H^*(G)^{G \times G}$ is a subcoalgebra of $H^*(G)$. Likewise, the antipode $\iota^* : H^*(G) \to H^*(G)$ preserves $H^*(G)^{G \times G}$. Hence $H^*(G)^{G \times G}$ is a Hopf subalgebra; it is mapped isomorphically to $H^*(G_{\text{ant}})$ by the morphism of Hopf algebras $j^*$, in view of Proposition 3.2.

For the final assertion, note that the multiplication of $H^*(G)$ induces an isomorphism $\mathcal{O}(G) \otimes (1 \otimes H^*(G_{\text{ant}})) \xrightarrow{\cong} H^*(G)$, by Proposition 3.2 again. \qed

Remark 3.5. When $G$ is not connected, the three subspaces $H^*(G)^{G \times e_G}$, $H^*(G)^{e_G \times G}$ and $H^*(G)^{G \times G}$ of $H^*(G)$ are generally distinct.

For example, let $E$ be an elliptic curve, and $G$ the semi-direct product of $E$ with the group of order 2 acting on $E$ via multiplication by $\pm 1$. Then $G_{\text{ant}} = E$ and $H^*(G_{\text{ant}})^G = k$. Thus, $H^1(G)^{G \times G} = 0$, while $H^1(G)^{G \times e_G}$ and $H^1(G)^{e_G \times G}$ are two distinct copies of $k$ in $H^1(G) \cong k^2$. In particular, $P^1(G) = 0$.

4. **The cohomology algebra of an anti-affine group**

Throughout this section, we assume that $G$ is anti-affine, i.e., $\mathcal{O}(G) = k$. Recall from [Br09, §2] that $G$ sits in a unique extension of smooth connected commutative group schemes,

$$0 \to T \times U \to G \xrightarrow{\alpha} A \to 0,$$

where $A$ is an abelian variety, $T$ is a torus, and $U$ is unipotent; moreover, $U$ is trivial if $\text{char}(k) > 0$. Thus, we obtain two extensions

$$0 \to T \to G/U \xrightarrow{\alpha T} A \to 0,$$

$$0 \to U \to G/T \xrightarrow{\alpha U} A \to 0,$$

where $G/U$ and $G/T$ are anti-affine as well. Note that $G/U$ is a semi-abelian variety, and $G/T$ an extension of an abelian variety by a vector group; also, we have an isomorphism of group schemes

$$G \cong G/U \times_A G/T. \tag{4.1}$$

This yields a useful reduction to the case where $T$ is trivial:

**Proposition 4.1.** With the above notation, the pull-back under the quotient morphism $G \to G/T$ yields an isomorphism of graded Hopf algebras $H^*(G/T) \cong H^*(G)$. 
Proof. We may replace $k$ with any field extension, and hence assume that the torus $T$ is split.

Since the morphism $\alpha : G \to A$ (the quotient by $T \times U$) is affine, we have

\[(4.2) \quad H^*(G) = H^*(G, \mathcal{O}_G) = H^*(A, \alpha_*(\mathcal{O}_G)).\]

Moreover, \((4.1)\) yields an isomorphism

$$\alpha_*(\mathcal{O}_G) \cong (\alpha_T)_*(\mathcal{O}_{G/U}) \otimes_{\mathcal{O}_A} (\alpha_U)_*(\mathcal{O}_{G/T}).$$

So it suffices to show that the map

$$H^*(A, (\alpha_U)_*(\mathcal{O}_{G/T})) \to H^*(A, (\alpha_T)_*(\mathcal{O}_{G/U}) \otimes_{\mathcal{O}_A} (\alpha_U)_*(\mathcal{O}_{G/T}))$$

induced by the pull-back $\mathcal{O}_A \to (\alpha_T)_*(\mathcal{O}_{G/U})$, is an isomorphism.

By \([Br09, \S 2.1]\), there is an isomorphism of sheaves of $\mathcal{O}_A$-modules

\[(4.3) \quad (\alpha_T)_*(\mathcal{O}_{G/U}) \cong \bigoplus_{\lambda \in \hat{T}} \mathcal{L}_\lambda,\]

where $\hat{T}$ denotes the character group of $T$, and each $\mathcal{L}_\lambda$ is an algebraically trivial invertible sheaf on $A$. Moreover, $\mathcal{L}_0 \cong \mathcal{O}_A$ but $\mathcal{L}_\lambda$ is non-trivial for any $\lambda \neq 0$. In view of \([Mum74, \S 8, p. 76]\), it follows that

\[(4.4) \quad H^*(A, \mathcal{L}_\lambda) = 0 \quad (\lambda \neq 0).\]

On the other hand, since $\alpha_U$ is a torsor under the vector group $U$, the sheaf $(\alpha_U)_*(\mathcal{O}_{G/T})$ has an increasing filtration by coherent subsheaves indexed by the non-negative integers, with subquotients being the structure sheaf $\mathcal{O}_A$. (Indeed, $(\alpha_U)_*(\mathcal{O}_{G/T}) = \mathcal{L}_{(G/T)/U}(\mathcal{O}(U))$, and the $U$-module $\mathcal{O}(U)$ has an increasing filtration with subquotients being the trivial module $k$). By \((4.4)\), this yields

$$H^*(A, \mathcal{L}_\lambda \otimes_{\mathcal{O}_A} (\alpha_U)_*(\mathcal{O}_{G/T})) = 0 \quad (\lambda \neq 0).$$

Together with \((4.3)\), this completes the proof. \hfill \Box

**Corollary 4.2.** If $\text{char}(k) > 0$, then the pull-back $\alpha^* : H^*(A) \to H^*(G)$ is an isomorphism of graded Hopf algebras.

In view of these results, we may assume that $\text{char}(k) = 0$, and $G$ is an extension of the abelian variety $A$ by the vector group $U$. Recall that there is a universal such extension, $0 \to V \to E \to A \to 0$, where $V := H^1(A)^\vee$ (the dual vector space of $H^1(A)$, viewed as an additive group). Moreover, by \([Br09, \S 2.2]\), $E$ is anti-affine and we have a commuting diagram of extensions

\[(4.5) \quad \begin{array}{ccccccccc}
0 & \to & V & \to & E & \to & A & \to & 0 \\
\gamma & & \downarrow & & \beta & & \downarrow & & \text{id} \\
0 & \to & U & \to & G & \to & A & \to & 0,
\end{array}\]

where the classifying map $\gamma$ is surjective.
Proposition 4.3. With the above notation and assumptions, the homomorphism \( \alpha^* : H^*(A) \to H^*(G) \) is surjective, and its kernel is the ideal of \( H^*(A) = \Lambda^*(H^1(A)) = \Lambda^*(V^\vee) \) generated by the image of \( \gamma^* : U^\vee \to V^\vee \) (the transpose of \( \gamma : V \to U \)). In particular, we have an isomorphism of graded Hopf algebras \( H^*(G) \cong \Lambda^*(W^\vee) \), where \( W := \ker(\gamma) \).

Proof. We argue by induction on \( \dim(U) \). If \( U = 0 \), then \( G = A \) and the statement is obvious. So we assume that \( U \neq 0 \), and choose a non-zero \( u \in U \). This yields an exact sequence of vector groups

\[
0 \longrightarrow \mathbb{G}_a \longrightarrow U \longrightarrow U' \longrightarrow 0,
\]

where \( 1 \in \mathbb{G}_a \) is sent to \( u \). We also obtain a derivation \( D \) of the \( k \)-algebra \( \mathcal{O}(U) \), given by \( D(f) := \frac{\partial}{\partial t} f(x + tu) \vert_{t=0} \). Equivalently, \( D \) is the vector field associated with \( u \) viewed as a point of the Lie algebra of \( U \). Then \( D \) is surjective and its kernel is \( \mathcal{O}(U)^{\mathbb{G}_a} \cong \mathcal{O}(U') \); in other words, we have an exact sequence of \( U \)-modules

\[
0 \longrightarrow \mathcal{O}(U') \longrightarrow \mathcal{O}(U) \longrightarrow \mathcal{O}(U) \longrightarrow 0.
\]

Next, let \( G' := G/\mathbb{G}_a \) so that \( G' \) sits in two extensions

\[
0 \longrightarrow \mathbb{G}_a \longrightarrow G \longrightarrow G' \longrightarrow 0,
\]

where \( 0 \in \mathbb{G}_a \) is sent to \( u \). We also obtain a derivation \( D \) of the \( k \)-algebra \( \mathcal{O}(U) \), given by \( D(f) := \frac{\partial}{\partial t} f(x + tu) \vert_{t=0} \). Equivalently, \( D \) is the vector field associated with \( u \) viewed as a point of the Lie algebra of \( U \). Then \( D \) is surjective and its kernel is \( \mathcal{O}(U)^{\mathbb{G}_a} \cong \mathcal{O}(U') \); in other words, we have an exact sequence of \( U \)-modules

\[
0 \longrightarrow \mathcal{O}(U') \longrightarrow \mathcal{O}(U) \longrightarrow \mathcal{O}(U) \longrightarrow 0.
\]

By \([1,2]\), we have \( H^*(G') = H^*(A, \alpha_*(\mathcal{O}_G)) \); likewise, \( H^*(G') = H^*(A, \alpha'_*(\mathcal{O}_{G'})) \). Moreover, \( \alpha_*(\mathcal{O}_G) \) (resp. \( \alpha'_*(\mathcal{O}_{G'}) \)) is the \( G \)-sheaf on \( A = G/U \) associated to the \( U \)-module \( \mathcal{O}(U) \) (resp. \( \mathcal{O}(U') \)). But the exact sequence \((4.6)\) yields an exact sequence of \( G \)-sheaves

\[
0 \longrightarrow \alpha_*(\mathcal{O}_G) \longrightarrow \alpha'_*(\mathcal{O}_{G'}) \longrightarrow \alpha_*(\mathcal{O}_G) \longrightarrow 0,
\]

and hence a long exact sequence of cohomology groups

\[
\cdots \longrightarrow H^{i-1}(G) \longrightarrow H^{i-1}(G') \longrightarrow H^i(G') \longrightarrow H^i(G) \longrightarrow H^i(G') \longrightarrow \cdots
\]

Now \( D \) acts on \( H^*(G) \) via the action of the Lie algebra of \( G \) arising from the \( G \)-action by (left or right) multiplication. But the anti-affine group \( G \) acts trivially on its module \( H^*(G) \), and hence \( D \) acts by 0. This yields short exact sequences

\[
(4.7) \quad 0 \longrightarrow H^{i-1}(G) \longrightarrow H^i(G') \longrightarrow H^i(G) \longrightarrow 0.
\]

In particular, \( \varphi^* : H^*(G') \to H^*(G) \) is surjective. Using the induction assumption, it follows that the natural homomorphism \( \psi : \Lambda^*(H^1(G)) \to H^*(G) \) is surjective. On the other hand, by \([1,7]\), the Poincaré polynomial \( P_{H^*(G)}(t) := \sum_{i \geq 0} \dim(H^i(G)) t^i \) satisfies \( P_{H^*(G')}(t) = (1 + t) P_{H^*(G)}(t) \). By the induction assumption again, this yields \( P_{H^*(G)}(t) = (1 + t)^n \), where \( n = \dim(H^1(G') - 1 = \dim(H^1(G)) \). Thus, \( H^*(G) \) and \( \Lambda^*(H^1(G)) \) have the same Poincaré polynomial; hence \( \psi \) is an isomorphism.

To complete the proof, it remains to construct an isomorphism \( W^\vee \cong \Lambda^*(H^1(G)) \), compatible with pull-backs of anti-affine extensions of \( A \) by vector groups. We do this in two steps.
First, we construct an isomorphism $W^\vee \xrightarrow{\cong} \text{Ext}^1(G, \mathbb{G}_a)$ compatible with such pull-backs. For this, consider the exact sequence of commutative group schemes $0 \to U \to G \to A \to 0$ and the associated long exact sequence

$$0 \to \text{Hom}(A, \mathbb{G}_a) \to \text{Hom}(G, \mathbb{G}_a) \to \text{Hom}(U, \mathbb{G}_a) \to \text{Ext}^1(A, \mathbb{G}_a) \to \text{Ext}^1(G, \mathbb{G}_a) \to \text{Ext}^1(U, \mathbb{G}_a).$$

We have $\text{Hom}(A, \mathbb{G}_a) = 0 = \text{Hom}(G, \mathbb{G}_a)$, since $G$ is anti-affine; also, $\text{Hom}(U, \mathbb{G}_a) = U^\vee$, $\text{Ext}^1(A, \mathbb{G}_a) = V^\vee$ and $\text{Ext}^1(U, \mathbb{G}_a) = 0$. Moreover, the pushout map $\partial : \text{Hom}(U, \mathbb{G}_a) \to \text{Ext}^1(A, \mathbb{G}_a)$ is identified to the transpose of the classifying map $\gamma : V \to U$; thus, $\text{Coker}(\partial) \cong \text{Coker}(\gamma^\vee) = W^\vee$. This yields the required isomorphism.

Next, we show that the natural map

$$u : \text{Ext}^1(G, \mathbb{G}_a) \to \text{H}^1(G, \mathcal{O}_G) = \text{H}^1(G)$$

that associates to each extension the class of the corresponding $\mathbb{G}_a$-torsor, is an isomorphism. For this, we argue again by induction on $\dim(U)$. If $U = 0$, then $G = A$ and the assertion is exactly [Sc59, Chap. 7, Thm. 7]. For an arbitrary $U$, let $W', G'$ be as above; then the exact sequence $0 \to \mathbb{G}_a \to G \to G' \to 0$ yields a commutative diagram of exact sequences

$$
\begin{array}{cccccc}
0 & \longrightarrow & k & \longrightarrow & \text{Ext}^1(G', \mathbb{G}_a) & \xrightarrow{\varphi'} \text{Ext}^1(G, \mathbb{G}_a) & \longrightarrow & 0 \\
& & \downarrow{\text{id}} & & \downarrow{u'} & & \downarrow{u} \\
0 & \longrightarrow & k & \longrightarrow & \text{H}^1(G') & \xrightarrow{\varphi''} \text{H}^1(G) & \longrightarrow & 0,
\end{array}
$$

since $\text{Hom}(G', \mathbb{G}_a) = \text{Hom}(G, \mathbb{G}_a) = \text{Hom}(\mathbb{G}_a, \mathbb{G}_a) = k$ and $\text{Ext}^1(\mathbb{G}_a, \mathbb{G}_a) = 0$; the bottom exact sequence is (1.7) for $i = 1$. By the induction assumption, $u'$ is an isomorphism; it follows that so is $u$.

**Remark 4.4.** We present an alternative proof of Proposition 4.3 which is more conceptual but less self-contained. We first claim that

\begin{equation}
(4.8) \quad \text{H}^i(E) = 0 \quad (i > 0).
\end{equation}

This has been proved by G. Laumon in an unpublished preprint (see [La96, Thm. 2.4.1]); we provide another argument as follows.

Since the morphism $\beta : E \to A = E/V$ is affine, we have $\text{H}^i(E) = \text{H}^i(A, \beta_*(\mathcal{O}_E))$. Moreover, $\beta_*(\mathcal{O}_E)$ is the $E$-sheaf on $A$ corresponding to the $V$-module $\mathcal{O}(V)$. The latter may be characterized as the injective hull of the trivial module $k$ (the unique simple module).

Now recall that the category $\text{mod}(V)$ of finite-dimensional $V$-modules is equivalent to $\text{Coh}^E(A)$, via $M \mapsto \mathcal{L}_{E/V}(M)$. Moreover, each coherent $E$-sheaf $\mathcal{F}$ on $A$ has a finite increasing filtration with subquotients being the structure sheaf $\mathcal{O}_A$, i.e., $\mathcal{F}$ is the sheaf of local sections of a unipotent vector bundle. In fact, this yields an equivalence from $\text{Coh}^E(A)$ to the category $\text{Uni}(A)$ of unipotent vector bundles on $A$ (see [Br12, Rem. 3.13(ii)])]. Also, recall that $\text{Uni}(A)$ is equivalent to the category $\text{Coh}_0(A)$ of coherent sheaves on the dual abelian variety $\hat{A}$ supported at the origin,
via the Fourier-Mukai transform that assigns to a coherent sheaf on \( \hat{A} \) supported at 0, the sheaf \( (p_1)_*([P \otimes \hat{\mathcal{P}} p_2(F)]) \) on \( A \); here \( p_1, p_2 \) denote the projections from \( A \times \hat{A} \), and \( \mathcal{P} \) stands for the Poincaré bundle on \( A \times \hat{A} \) (see [Muk78, Thm. 4.12]).

Thus, we obtain an equivalence of abelian categories from \( \text{mod}(V) \) to \( \text{Coh}_0(\hat{A}) \). By taking direct limits, this extends to an equivalence of abelian categories \( \text{Mod}(\hat{V}) \rightarrow \text{QCoh}(\hat{A}) \). Thus, \( F \) sends \( \mathcal{O}(V) \) to the injective hull \( I \) of the residue field \( k(0) \). By [Muk78, Thm. 4.12] again, we have

\[
H^i(A, \mathcal{L}_{E/V}(M)) \cong \text{Ext}^i_{\mathcal{O}_{\hat{A},0}}(k(0), F(M))
\]

for any finite-dimensional \( V \)-module \( M \) and any \( i \geq 0 \). Since cohomology commutes with direct limits, it follows that

\[
H^i(A, \beta_*(\mathcal{O}_E)) = H^i(A, \mathcal{L}_{E/V}(\mathcal{O}(V))) \cong \text{Ext}^i_{\mathcal{O}_{\hat{A},0}}(k(0), F(\mathcal{O}(V))) = \text{Ext}^i_{\mathcal{O}_{\hat{A},0}}(k(0), I).
\]

But the latter vanishes for any \( i > 0 \); this yields (4.8).

Next, recall that \( H^*(G) = H^*(\Lambda, \alpha_*(\mathcal{O}_G)) \), where \( \alpha_*(\mathcal{O}_G) \) is the \( E \)-sheaf on \( A = E/V \) corresponding to the \( V \)-module \( \mathcal{O}(V/W) \). The Koszul complex yields a resolution of this \( V \)-module,

\[
0 \rightarrow \mathcal{O}(V) \otimes \Lambda^n(W^\vee) \rightarrow \mathcal{O}(V) \otimes \Lambda^{n-1}(W^\vee) \rightarrow \cdots \rightarrow \mathcal{O}(V) \rightarrow \mathcal{O}(V/W) \rightarrow 0,
\]

where \( n := \dim(W) \), and hence an exact sequence

\[
0 \rightarrow \beta_*(\mathcal{O}_E) \otimes \Lambda^n(W^\vee) \rightarrow \beta_*(\mathcal{O}_E) \otimes \Lambda^{n-1}(W^\vee) \rightarrow \cdots \rightarrow \beta_*(\mathcal{O}_E) \rightarrow \alpha_*(\mathcal{O}_G) \rightarrow 0.
\]

Moreover, \( H^i(A, \beta_*(\mathcal{O}_E)) \) vanishes for all \( i > 0 \), by (4.8). So we obtain an isomorphism

\[
H^*(G) = H^*(\Lambda, \alpha_*(\mathcal{O}_A)) \cong \Lambda^*(W^\vee).
\]

Also, note that \( W^\vee = V^\vee / \text{Im}(\gamma) \cong H^1(A) / \text{Im}(\gamma) \). So, when \( U \) is trivial, we recover the isomorphism \( H^*(A) \cong \Lambda^*(H^1(A)) \); for an arbitrary \( U \), we obtain the required isomorphism.

5. The main results

5.1. Proof of Theorem 4.1. Recall that \( G \) denotes a group scheme of finite type over \( k \), and \( j : G_{\text{ant}} \rightarrow \hat{G} \) the inclusion of the largest anti-affine subgroup.

By Proposition 4.2 there is an isomorphism of graded \( \mathcal{O}(G) \)-modules \( \psi : H^*(G) \rightarrow \mathcal{O}(G) \otimes H^*(G_{\text{ant}}) \) which identifies \( j^* : H^*(G) \rightarrow H^*(G_{\text{ant}}) \) with \( \epsilon_G \otimes \text{id} \). Moreover, by Corollary 4.2 (when \( \text{char}(k) > 0 \)) and Proposition 4.3 (when \( \text{char}(k) = 0 \)), the natural map \( \Lambda^*(H^1(G_{\text{ant}})) \rightarrow H^*(G) \) is an isomorphism, and the \( k \)-vector space \( H^1(G_{\text{ant}}) \) is finite-dimensional. Thus, \( H^*(G) \) is free of finite rank as a graded module over \( \mathcal{O}(G) \); recall that the latter algebra is finitely generated. Moreover, the natural homomorphism of graded \( \mathcal{O}(G) \)-modules

\[
\varphi : \Lambda^*_\mathcal{O}(G)(H^1(G)) \rightarrow H^*(G)
\]

is an isomorphism at the \( k \)-point \( \epsilon_G \). Since \( \varphi \) is \( G \times G \)-equivariant, it is an isomorphism everywhere.

If \( G \) is connected, then we have isomorphisms of graded Hopf algebras

\[
H^*(G) \cong \mathcal{O}(G) \otimes H^*(G)^{G \times G} \cong \mathcal{O}(G) \otimes H^*(G_{\text{ant}})
\]
by Proposition 3.4. Since $P^i(G) \subset H^i(G)^{G \times G}$ for $i \geq 1$ (Proposition 3.3), we obtain that $P^i(G) \subset P^i(G_{\text{ant}})$ via pull-back. But in view of the structure of $H^i(G_{\text{ant}})$, we have $P^1(G_{\text{ant}}) = H^1(G_{\text{ant}})$ and $P^i(G_{\text{ant}}) = 0$ for $i \geq 2$. Thus, $P^i(G) = 0$ for $i \geq 2$ as well. Moreover, by Proposition 3.3 again, $P^1(G) = H^1(G)^{G \times G}$ and hence we obtain an isomorphism

\[(5.1) \quad j^* : P^1(G) \xrightarrow{\cong} H^1(G_{\text{ant}}).\]

This completes the proof of Theorem 1.1.

Combining Proposition 3.4, Corollary 4.2 and Proposition 4.3, we also obtain the characterization of ‘acyclic’ group schemes mentioned in the introduction:

**Corollary 5.1.** Let $G$ be a group scheme of finite type over $k$. Then $H^i(G) = k$ if and only if $G$ is trivial when $\text{char}(k) > 0$, resp. $G \cong S \times A$, $E$ when $\text{char}(k) = 0$, where $S$ is an anti-affine extension of an abelian variety $A$ by a torus, and $E$ is the universal vector extension of $A$.

### 5.2. The primitive elements.

From now on, we assume that $G$ is connected; we will describe the space $P^1(G)$ of homogeneous primitive elements of degree 1, in terms of the structure of $G$.

By [Ra70, Lem. IX 2.7], $G$ has a normal connected linear subgroup scheme $L$ such that the quotient $G/L$ is an abelian variety; we denote by $
abla : G \twoheadrightarrow A := G/L$

the quotient homomorphism. In characteristic 0, one easily sees that $L$ is the largest connected linear (or, equivalently, affine) subgroup scheme of $G$. In particular, $L$ is unique; we then set $L := G_{\text{aff}}$ and $A := A(G)$. This does not extend to positive characteristics, since we may replace $A$ with its quotient by any infinitesimal subgroup scheme. Yet when $G$ is smooth, there exists a (unique) smallest subgroup scheme $L$ as above (see [BLR90, Thm. 9.2.1]); we denote again $L$ by $G_{\text{aff}}$, and $G/L$ by $A(G)$. For example, if $G$ is anti-affine, then $G_{\text{aff}} = T \times U$ with the notation of Section 4.

Returning to an arbitrary connected group scheme $G$, the largest abelian quotient $A$ is related to $A(G_{\text{ant}})$ as follows:

**Proposition 5.2.** With the above notation and assumptions, we have a commutative square

\[(5.2) \quad \begin{array}{ccc}
G_{\text{ant}} & \xrightarrow{j} & G \\
\downarrow \alpha_{\text{ant}} & & \downarrow \alpha \\
A(G_{\text{ant}}) & \xrightarrow{\varphi} & A,
\end{array}\]

where $\varphi$ is an isogeny.

**Proof.** The homomorphism $\alpha \circ j : G_{\text{ant}} \twoheadrightarrow A$ sends the smooth connected affine group scheme $T \times U = \text{Ker}(\alpha_{\text{ant}})$ to the origin of the abelian variety $A$, and hence factors through $\alpha_{\text{ant}}$. This shows the existence of $\varphi$. The quotient of $A$ by the image of $\alpha \circ j$ is connected and proper, but also affine as a quotient group scheme of $G/G_{\text{ant}}$. Thus, $\alpha \circ j$ is surjective, and hence so is $\varphi$. Finally, since $\text{Ker}(\alpha \circ j) = L \cap G_{\text{ant}}$, we obtain
an isomorphism \( \text{Ker}(\varphi) \cong (L \cap G_{\text{ant}}) / \text{Ker}(\alpha_{\text{ant}}) = (L \cap G_{\text{ant}}) / (G_{\text{ant}})_{\text{aff}} \). The latter quotient is a linear subgroup scheme of the abelian variety \( G_{\text{ant}} / (G_{\text{ant}})_{\text{aff}} = A(G_{\text{ant}}) \), and hence is finite. Thus, \( \varphi \) is an isogeny.

\[ \square \]

**Theorem 5.3.** Keep the above notation. If \( \text{char}(k) > 0 \), then \( \dim(P^1(G)) = \dim(A) \). If \( \text{char}(k) = 0 \), then \( P^1(G) \) is the image of \( \alpha^* : H^1(A(G)) \to H^1(G) \).

**Proof.** If \( \text{char}(k) > 0 \), then \( \dim(P^1(G)) = \dim(H^1(G_{\text{ant}})) = \dim(H^1(A(G_{\text{ant}}))) \) by (5.1) and Corollary 4.2. Thus, \( \dim(P^1(G)) = \dim(A(G_{\text{ant}})) \). But \( \dim(A(G_{\text{ant}})) = \dim(A) \) by Proposition 5.2. This proves the first assertion.

For the second assertion, note that \( \text{Im}(\alpha^*) \) is contained in \( P^1(G) \). Also, the commutative square (5.2) yields a commutative square of pull-backs

\[
\begin{array}{ccc}
H^1(A(G)) & \xrightarrow{\varphi^*} & H^1(A(G_{\text{ant}})) \\
\alpha^* \downarrow & & \downarrow \alpha^*_{\text{ant}} \\
P^1(G) & \xrightarrow{j^*} & H^1(G_{\text{ant}}),
\end{array}
\]

where \( j^* \) is the isomorphism (5.1); moreover, \( \alpha^*_{\text{ant}} \) is surjective in view of Proposition 4.3. Thus, it suffices to show that \( \varphi^* \) is an isomorphism. But since \( \varphi \) is an isogeny, there exists an isogeny \( \psi : A(G) \to A(G_{\text{ant}}) \) such that \( \varphi \circ \psi \) is the multiplication \( n_{A(G)} \) for some positive integer \( n \), and \( \psi \circ \varphi = n_{A(G_{\text{ant}})} \). Moreover, \( n_{A(G)}^* : H^1(A(G)) \to H^1(A(G)) \) is just multiplication by \( n \), and similarly for \( n_{A(G_{\text{ant}})}^* \) (as follows e.g. from the isomorphism \( H^1(A, \mathcal{O}_A) \cong \text{Ext}^1(A, \mathbb{G}_a) \) for any abelian variety \( A \), and from the bilinearity of \( \text{Ext}^1 \)). It follows that \( \varphi \) is indeed an isomorphism. \( \square \)

When \( \text{char}(k) > 0 \), it may happen that \( \alpha^* : H^1(A(G)) \to H^1(G) \) is zero while \( P^1(G) \neq 0 \), as shown by the following:

**Example 5.4.** Let \( p := \text{char}(k) \) and let \( E \) be an elliptic curve such that the \( p \)-torsion subgroup scheme \( E_p \) (the kernel of the multiplication \( p_E \)) is isomorphic to \( \mathbb{Z}/p\mathbb{Z} \times \mu_p \), where \( \mu_p \) denotes the multiplicative group of \( p \)-th roots of unity; this holds if \( E \) is ordinary and the base field \( k \) is sufficiently large. We may view \( E_p \) as a subgroup scheme of \( \mathbb{G}_a \times \mathbb{G}_m \); then

\[
G := E \times^{E_p} (\mathbb{G}_a \times \mathbb{G}_m)
\]

is a smooth connected commutative group scheme. We have

\[
G_{\text{aff}} = E_p \times^{E_p} (\mathbb{G}_a \times \mathbb{G}_m) \cong \mathbb{G}_a \times \mathbb{G}_m
\]

and \( A(G) = E / E_p \cong E \); moreover, \( G_{\text{ant}} = E \times^{E_p} E_p \cong E \). Hence \( G_{\text{ant}} = A(G_{\text{ant}}) \), and the isogeny \( \varphi : A(G_{\text{ant}}) \to A(G) \) is identified to the quotient morphism \( E \to E / E_p \), that is, to \( p_E : E \to E \). Thus, \( \varphi^* : H^1(E) \to H^1(E) \) is zero. In view of the commutative diagram (5.3), where \( j^* \) is an isomorphism and \( \alpha^*_{\text{ant}} = \text{id} \), it follows that \( \alpha^* = 0 \). But \( P^1(G) \cong H^1(G_{\text{ant}}) \cong k \).
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