THE TRACE METHOD FOR COTANGENT SUMS

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Abstract. This paper presents a combinatorial study of sums of integer powers of the cotangent which is a popular theme in classical calculus. Our main tool the realization of cotangent values as eigenvalues of a simple self-adjoint matrix with integer matrix. We use the trace method to draw conclusions about integer values of the sums and provide explicit evaluations; it is remarkable that throughout the calculations the combinatorics are governed by the higher tangent and arctangent numbers exclusively. Finally we indicate a new approximation of the values of the Riemann zeta function at even integer arguments.

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1. Introduction

It is a well-known fact that the trace of a matrix equals the sum of its eigenvalues

\[ \text{Tr } A = \sum \lambda_i, \]

counting algebraic multiplicities. This relation is respected by functional calculus and the identity

\[ \text{Tr } f(A) = \sum f(\lambda_i) \]

holds for arbitrary holomorphic functions, in particular, powers and polynomials. The trace method consists in the evaluation of this identity for particular matrices in order to obtain nontrivial combinatorial relations.

In the present paper we apply this method to cotangent sums of the form

\[ S(m, n, \alpha) = \sum_{k=0}^{n-1} \cot^m \frac{\alpha + k\pi}{n} \]

for \( \alpha \neq k\pi, \, n, m \in \mathbb{N}, \, n \geq 2 \), and the limit case

\[ S_0(m, n) = \sum_{k=1}^{n-1} \cot^m \frac{k\pi}{n}. \]

Finite cotangent sums are a recurrent theme in the mathematical literature. They arise in number theory in connection with Dedekind sums and topology [64, 40, 54], and more recently were used to evaluate the Riemann zeta function, see [63, Problem 141ff] for the apparently first occurrence of this connection, later rediscovered in [44, 61, 51, 8, 5, 27]; Berndt and Yeap [10] attribute the first occurrence of cotangent sums to [59, p. 155]. The recent literature on this topic is abundant, in particular concerning reciprocal relations [10, 19] and the question for which values of the parameters the sums (1.1) yield integer values is intriguing. For example, Byrne and Smith [14] proved that the sums are integer valued polynomials in \( n \) at the offset \( \alpha = \pi/4 \), found the leading terms and established recurrence relations. We were led to study such sums in connection with certain limit theorems arising in free probability, see [33, 32], where the matrices considered below arise in a natural way.

The most general expression for the sum (1.1) so far was obtained by Cvijović and Klinowski [26], who realized the cotangent values \( \cot^m \frac{\alpha + k\pi}{n} \) as roots of a polynomial and expressed the sums via Cramer’s rule applied to the Newton relations between elementary and power sum symmetric functions. In the present paper we go one step further and show that the polynomial found in [26] is in fact the characteristic polynomial of a simple matrix. Thus the trace method is applicable and we can draw certain conclusions about the sum (1.1). For example, if \( \cot \alpha \) is an integer, say, \( \alpha = \pi/4 \), it follows trivially that (1.1) evaluates to an integer, as was observed by different means in [14]. Moreover we obtain explicit expressions in terms of tangent numbers and derivative polynomials by extracting Taylor coefficients from suitable generating functions. The main results can be summarized as follows. First, the cotangent sum (1.1) is the trace of the matrix (2.1). Consequently it is a polynomial in \( \cot \alpha \) with integer coefficients and it follows immediately that the sum is integer valued whenever \( \cot \alpha \) is an integer. Moreover, the sums can be expressed in terms of tangent and arctangent numbers. In the simplest case (\( \cot \alpha = 0 \)) the odd power sums vanish and the even ones evaluate to

\[ S(2m, n, \pi/2) = (-1)^m n + \frac{1}{(2m-1)!} \sum_{k=1}^{m} n^{2k} A_{2m}^{(2k)} T_{2k-1}. \]

For general \( \alpha \), the coefficients of the polynomial

\[ S(m, n, \alpha) = \sum_{0 \leq k \leq [m/2]} p_{m,m-2k}(n) \cot^m \alpha. \]
can be expressed in terms of tangent and arctangent numbers as well

\[ p_{m,r}(n) = \begin{cases} S(m, n, \pi/2) & r = 0 \\ \frac{1}{r(m-1)!} \sum_{k=r}^{m} n^k A_m^{(k)} T^{(r)}_k & 1 \leq r \leq m \end{cases} \]

and as special cases we recover the sums

\[ S(2m + 1, n, \pi/4) = \sum_{k=1}^{n} (-1)^k \cot^{2m+1} \frac{2k-1}{4n} = \frac{1}{2(2m)!} \sum_{k=0}^{m} (2n)^{2k+1} A_{2m+1}^{(2k+1)} S_{2k}, \]

\[ S(2m, n, \pi/4) = \sum_{k=1}^{n} \cot^{2m} \frac{(2k-1)\pi}{4n} = (-1)^m n + \frac{1}{2(2m-1)!} \sum_{k=1}^{m} (2n)^{2k} A_{2m}^{(2k)} T_{2k-1}, \]

of Byrne and Smith [14] in terms of secant, tangent, and arctangent numbers, see Corollary 6.6. Finally we obtain an explicit formula for the sum (1.2)

\[ \sum_{k=1}^{n-1} \cot^{2m} \frac{k\pi}{n} = (-1)^m (n-1) - \frac{1}{(2m-1)!} \sum_{k=1}^{m} (-1)^k A_{2m}^{(2k)} \frac{4^k B_{2k}}{2k} (n^{2k} - 1). \]

which was previously evaluated by Berndt and Yeap in terms of Bernoulli numbers [10] (cf. also [62, 36, 27, 4, 29, 39]), see Corollary 6.8. Chu and Marini [20] wrote a systematic study of generating functions and we complement this in Section 4 by providing a generating function for arbitrary \( \alpha \).

Finite sums of trigonometric functions are a popular subject in terms of generating functions [20] and reciprocal relations [10, 19]. For explicit evaluations of sums of powers of cosines see [28, 47], for sines see [43] and for secants and cosecants see [37, 30]. For an evaluation of cosecant sums via the trace method see [58]; for the exponent \( m = 2 \) finite Fourier analysis is applicable [6].

It is perhaps interesting to note that the papers [15, 16] evaluate certain trigonometric sums using matrices with trigonometric entries and integer eigenvalues, while in the present paper we exploit integer matrices with trigonometric eigenvalues.

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2. Preliminaries on Linear Algebra and the Tangent Function

The main role in this paper is played by a certain matrix and its intricate relations to the tangent and cotangent functions.

2.1. A matrix. For scalars \( a, b, c \in \mathbb{C} \) we denote by \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C}) \) the matrix whose diagonal elements are equal to \( a \), whose upper-triangular entries are equal to \( b \) and whose lower-triangular elements are equal to \( c \), respectively. For simplicity of notation, we use the same letter \( J_n \) and \( B_n \) for the following matrices

\[ J_n := \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix} \quad \text{and} \quad B_n := i \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ -1 & 0 & 1 & \ldots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & -1 & -1 & \ldots & 0 \end{bmatrix}. \]

The first observation reveals that the entries of the sum (1.1) can be realized as eigenvalues of the following matrix and consequently the sum is the trace of the \( m \)-th power of this matrix.
Lemma 2.1. If $a = \cot \alpha$, then the characteristic polynomials $\chi_n(\alpha; \lambda)$ of the matrices

\begin{equation}
C_n = aJ_n + B_n = \begin{bmatrix}
a & a + i & \cdots & a + i \\
a - i & a & \cdots & a + i \\
\cdots & \cdots & \cdots & \cdots \\
a - i & a - i & \cdots & a
\end{bmatrix} \in M_n(\mathbb{C})
\end{equation}

satisfy the recurrence relation

\begin{equation}
\chi_n(\alpha; \lambda) = (2\lambda - 2a + w + \bar{w})\chi_{n-1}(\alpha; \lambda) - (\lambda - a + w)(\lambda - a + \bar{w})\chi_{n-2}(\alpha; \lambda)
\end{equation}

and have the following explicit expression

\begin{equation}
\chi_n(\alpha; \lambda) = \frac{(\cot \alpha + i)(\lambda - i)^n - (\cot \alpha - i)(\lambda + i)^n}{2i} = \text{Im}(\cot \alpha + i)(\lambda - i)^n
\end{equation}

(assuming $\lambda$ real). The eigenvalues are given by

$$\lambda_k = \cot \frac{\alpha + kn\pi}{n}, \text{ for } 0 \leq k \leq n - 1.$$

Proof. The spectrum of the matrix $C_n$ can be computed from its characteristic polynomial $\chi_n(\alpha; \lambda) = \det(\lambda I - C_n)$ using the following recurrence relation. Let $w = a + i$, then we have

$$\chi_n(\alpha; \lambda) = \begin{vmatrix}
\lambda - a & -w & -w & \cdots & -w \\
-\bar{w} & \lambda - a & -w & \cdots & -w \\
-\bar{w} & -\bar{w} & \lambda - a & -w & \cdots & -w \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\bar{w} & -\bar{w} & -\bar{w} & \lambda - a & \cdots & -w \\
-\bar{w} & -\bar{w} & -\bar{w} & -\bar{w} & \cdots & \lambda - a
\end{vmatrix}$$

we subtract the second row from the first row

$$= \begin{vmatrix}
\lambda - a + \bar{w} & -\lambda - w + a & 0 & 0 & \cdots & 0 \\
-\bar{w} & \lambda - a & -w & -w & \cdots & -w \\
-\bar{w} & -\bar{w} & \lambda - a & -w & \cdots & -w \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\bar{w} & -\bar{w} & -\bar{w} & -\bar{w} & \lambda - a & \cdots & -w \\
-\bar{w} & -\bar{w} & -\bar{w} & -\bar{w} & -\bar{w} & \cdots & \lambda - a
\end{vmatrix}$$

and the second column from the first column

$$= \begin{vmatrix}
2\lambda - 2a + w + \bar{w} & -\lambda - w + a & 0 & 0 & \cdots & 0 \\
-\lambda - w + a & \lambda - a & -w & -w & \cdots & -w \\
0 & -\bar{w} & \lambda - a & -w & \cdots & -w \\
0 & -\bar{w} & -\bar{w} & \lambda - a & -w & \cdots & -w \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & -\bar{w} & -\bar{w} & -\bar{w} & \lambda - a & \cdots & -w \\
0 & -\bar{w} & -\bar{w} & -\bar{w} & -\bar{w} & \cdots & \lambda - a
\end{vmatrix}$$

$$= (2\lambda - 2a + w + \bar{w})\chi_{n-1}(\alpha; \lambda) - (\lambda - a + w)(\lambda - a + \bar{w})\chi_{n-2}(\alpha; \lambda)$$

and the solution of this recurrence equation (with initial values $\chi_0(\alpha; \lambda) = 1$ and $\chi_1(\alpha; \lambda) = \lambda$) is

$$\chi_n(\alpha; \lambda) = \frac{w(\lambda - a + \bar{w})^n - \bar{w}(\lambda - a + w)^n}{w - \bar{w}} = \frac{(a + i)(\lambda - i)^n - (a - i)(\lambda + i)^n}{2i} = \text{Im}(a + i)(\lambda - i)^n.$$

Thus we have to solve the equation

\begin{equation}
\text{Im}(a + i)(\lambda - i)^n = 0.
\end{equation}

To compute the zeros, write $a + i = r_0e^{i\alpha}$; i.e., $a = \cot \alpha$ and assume $\lambda - i = re^{-i\theta}$. Then equation (2.4) becomes $\text{Im} r_0e^{i\alpha}r^n e^{-i\theta} = 0$ and is equivalent to the equation $\sin(\alpha - n\theta) = 0$, that is,
\( \alpha - n\theta = -k\pi \) for some \( k \in \mathbb{Z} \). Thus the solutions of (2.4) can be written as \( \lambda_k = i + r_ke^{-i\theta_k} \) with \( \theta_k = \frac{\alpha + kn\pi}{n} \). Now our matrix is self-adjoint, all roots of the characteristic polynomial (2.4) are real and hence \( -1 = \text{Im}(\lambda_k - i) = -r_k \sin \theta_k; \) we conclude that \( r_k = \frac{1}{\sin \theta_k} \) and \( \lambda_k = \text{Re}(\lambda_k - i) = r_k \cos \theta_k = \cot \theta_k \). Consequently

\[
\chi_n(\alpha; \lambda) = \prod_{k=0}^{n-1} \left( \lambda - \cot \frac{\alpha + k\pi}{n} \right).
\]

□

Remark 2.2. An alternative formula for this polynomial can be found in [26, Formula (4)]. Indeed the coefficients of this polynomial are as follows

\[
\chi_n(\alpha; \lambda) = \text{Im}(a + i)(\lambda - i)^n
\]

\[
= \text{Im} \sum_{k=0}^{n} \binom{n}{k} (a + i)^k (-i)^{n-k}
\]

\[
= \text{Im} \sum_{k=0}^{n} \binom{n}{k}(a(-i)^{n-k} - (-i)^{n-k+1})\lambda^k
\]

\[
= \sum_{k=0}^{n} c_k \lambda^k
\]

where

\[
c_k = \begin{cases} \binom{n}{k} (-1)^{(n-k)/2} & n - k \text{ even} \\ a \binom{n}{k} (-1)^{(n-k+1)/2} & n - k \text{ odd} \end{cases}
\]

or equivalently,

\[
c_{n-k} = \binom{n}{k} \left( \cos \frac{k\pi}{2} + a \sin \frac{k\pi}{2} \right) = \begin{cases} \binom{n}{k} (-1)^{k/2} & k \text{ even} \\ a \binom{n}{k} (-1)^{(k+1)/2} & k \text{ odd} \end{cases}
\]

(cf. [26, Formula (4b)]).

In fact the discussion of [26] starts by showing that the characteristic polynomial \( \chi_n(\alpha; x) \) is related to the expression \( \sin \arccot x \). Indeed evaluation of the polynomial (2.3) at \( \lambda = \cot \theta \) and a few elementary manipulations yield the identity

\[
\chi_n(\alpha; \cot \theta) = \frac{\sin(n\theta - \alpha)}{\sin \alpha \sin^n \theta}.
\]

Remark 2.3. The recurrence relation (2.2) falls into the class of recurrence relations with constant coefficients which contains the Fibonacci numbers and the Chebyshev polynomials [13, 45], see [3] for a recent discussion.

2.2. Formulas for \( \tan(nx) \). A simple manipulation of the addition formulae for sine and cosine show that the tangent function obeys the addition rule

\[
(2.5) \quad \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}
\]

This rule is not practical for iteration and the following equivalent elegant formula proposed by Szmulowicz [60] is a convenient alternative

\[
\frac{1 + i \tan \sum x_k}{1 - i \tan \sum x_k} = \prod \frac{1 + i \tan x_k}{1 - i \tan x_k}
\]
It follows immediately from the identity
\[ e^{2ix} = \frac{1 + i \tan x}{1 - i \tan x} \]
and in particular, \( \tan(n \arctan z) \) is a rational function. Indeed
\[
\frac{1 + i \tan(nx)}{1 - i \tan(nx)} = \left( \frac{1 + i \tan x}{1 - i \tan x} \right)^n
\]
and thus
\[
\tan(nx) = i \left( \frac{1 + i \tan x}{1 - i \tan x} \right)^n
= i \left[ (1 - i \tan x)^n - (1 + i \tan x)^n \right]
\]
\[
(1 - i \tan x)^n + (1 + i \tan x)^n
\]
and
\[
\cot(nx) = \frac{(\cot x + i)^n + (\cot x - i)^n}{(\cot x + i)^n - (\cot x - i)^n}.
\]
Thus we obtain the well-known formula [7, item 16]
\[
\tan(n \arctan z) = i \left[ (1 - iz)^n - (1 + iz)^n \right]
\]
\[
(1 - iz)^n + (1 + iz)^n.
\]
comparing with the reciprocal polynomial of (2.3) at \( \alpha = \cot \alpha = 0 \) which is
\[
\tilde{p}_n(z) = z^n \chi_n(0; 1/z) = \frac{(1 - iz)^n + (1 + iz)^n}{2}
\]
we see that
\[
(2.7) \quad \tan(n \arctan z) = -\frac{1}{n + 1} \frac{\tilde{p}_n'(z)}{\tilde{p}_n(z)}.
\]

2.3. Formulas for \( \tan(nx - \alpha) \). In view of later applications we introduce a nonzero offset into equation (2.6) and obtain
\[
\frac{1 + i \tan(nx + \alpha)}{1 - i \tan(nx + \alpha)} = \left( \frac{1 + i \tan x}{1 - i \tan x} \right)^n \frac{1 + i \tan \alpha}{1 - i \tan \alpha}
\]
which after a few manipulations yields the identity
\[
\tan(nx + \alpha) = i \left( \frac{1 + i \tan x}{1 - i \tan x} \right)^n \frac{\cot \alpha + i}{\cot \alpha - i}
\]
\[
= i \left[ (\cot \alpha - i)(1 - i \tan x)^n - (\cot \alpha + i)(1 + i \tan x)^n \right]
\]
\[
(\cot \alpha - i)(1 - i \tan x)^n + (\cot \alpha + i)(1 + i \tan x)^n
\]
The reciprocal provides the following crucial identity for \( \cot \)
\[
(2.8) \quad \cot(n \arctan z - \alpha) = -i \left( \frac{(\cot \alpha + i)(1 - iz)^n + (\cot \alpha - i)(1 + iz)^n}{(\cot \alpha + i)(1 - iz)^n - (\cot \alpha - i)(1 + iz)^n} \right)
\]
which after comparison with the reciprocal polynomial
\[
\chi_n(\alpha; z) = z^n \chi_n(\alpha; 1/z) = \frac{(\cot \alpha + i)(1 - iz)^n - (\cot \alpha - i)(1 + iz)^n}{2i}
\]
identifies to
\[
\cot(n \arctan z - \alpha) = \frac{1}{n + 1} \frac{\chi_{n+1}(\alpha; z)}{\chi_n(\alpha; z)}
\]
2.4. **Derivatives of** \( \tan \) and \( \cot \). The higher derivatives of \( \tan z \) and \( \cot z \) are closely related, since \( \cot z = \tan \left( \frac{\pi}{2} - z \right) \). It is easy to see that there exist polynomials \( P_n(z) \) such that \( \frac{d^n}{dz^n} \tan z = P_n(\tan z) \); indeed these *derivative polynomials* satisfy the recursion

\[
P_{n+1}(x) = (1 + x^2)P_n'(x)
\]

and can be used to efficiently compute tangent and Bernoulli numbers [46]. Explicitly, these polynomials can be expressed via the geometric polynomials [12, (2.1)]

\[
\omega_n(x) = \sum_{k=0}^{n} \binom{n}{k} k! x^k
\]

as follows, see [12, (3.10–11)]:

\[
P_n(z) = (2i)^n(z + i) \omega_n \left( -\frac{iz + 1}{2} \right) = (-2i)^n(z - i) \sum_{k=0}^{n} \frac{k!}{2k} \binom{n}{k} (iz - 1)^k
\]

On the other hand (see [1, Lemma 2.1] or [12, (3.15)])

\[
\frac{d^n}{dz^n} \cot z = (-1)^n P_n(\cot z) = (2i)^n(\cot z - i) \sum_{k=1}^{n} \frac{k!}{2k} \binom{n}{k} (i \cot z - 1)^k
\]

and thus \((-1)^n P_n(x)\) serve as derivative polynomials for \( \cot \). Interest in these polynomials goes back at least to Ramanujan [9, Chapter 7, entry 11] and there is some literature, see for example [52, 17, 62, 41, 42, 35].

2.5. **Tangent and arctangent numbers.** The *tangent numbers* are the Taylor coefficients of the tangent function. They make up the odd part of the sequence of \( E_n \) of *Euler zigzag numbers*, which are given by the exponential generating function

\[
\tan(z) + \sec(z) = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n.
\]

The higher order tangent numbers [17] are defined as coefficients of the series

\[
\tan^k z = \sum_{n=k}^{\infty} \frac{T_n^{(k)}}{n!} z^n;
\]

Their bivariate generating function is

\[
T(x, z) = \sum_{k=1}^{\infty} x^k \tan^k z
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=1}^{n} \frac{T_n^{(k)}}{n!} x^k z^n
\]

\[
= x \tan z
\]

\[
= \frac{x}{1 - x \tan z}
\]

\[
= \sum_{n=0}^{\infty} \frac{T_n(x)}{n!} z^n
\]

where \( T_n(x) = \sum_{k=1}^{n} T_n^{(k)} x^k \). On the other hand, from the addition formula (2.5) we infer the exponential generating function of the derivative polynomials to be

\[
P(x, z) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} z^n = \frac{x + \tan z}{1 - x \tan z}
\]

cf. [38, (6.94)] and [41].
We can now deduce relations between these polynomial sequences. First, by direct comparison we find
\[ xP_n(x) = (1 + x^2)T_n(x). \]
On the other hand, differentiating with respect to \( x \) (resp. \( z \)) we find
\[ x \frac{\partial}{\partial x} P(x, z) = \frac{\partial}{\partial z} T(x, z). \]
Comparing coefficients we have
\[ xP'(x) = T_{n+1}(x); \]
evaluating (2.13) at \( x = 0 \) yields the initial value \( P_n(0) = T_n \) and we recover the explicit formula
\[ P_n(x) = T_n + \sum_{k=1}^{n+1} \frac{T^{(k)}_n}{k} x^k, \]
cf. [22, Theorem 1] and [18].

On the other hand let us denote by \( A_n^{(k)} \) the arctangent numbers (see [21, p. 260] or [25]) defined by their exponential generating function
\[ \frac{(\arctan z)^k}{k!} = \sum_{n=k}^{\infty} \frac{A_n^{(k)}}{n!} z^n; \]
notice that \( A_n^{(k)} = 0 \) unless \( n-k \) is even and that up to sign these are the same as the coefficients of the hyperbolic arctangent function
\[ \frac{(\operatorname{atanh} z)^k}{k!} = \sum_{n=k}^{\infty} \frac{\tilde{A}_n^{(k)}}{n!} z^n. \]
The latter are nonnegative and
\[ A_n^{(k)} = (-i)^k \tilde{A}_n^{(k)}. \]

2.6. Derivatives of \( \arctan \). The derivatives of \( \arctan z \) are rational functions and it is easy to verify by induction that they are given by the following formulas
\[ \frac{d}{dz} \arctan z = \frac{1}{1 + z^2} = \frac{1}{2i} \left( \frac{1}{z-i} - \frac{1}{z+i} \right), \]
and thus
\[ \frac{d^m}{dz^m} \arctan z = \frac{i(-1)^m(m-1)!}{2} \left( (z-i)^{-m} - (z+i)^{-m} \right). \]

2.7. Faà di Bruno’s formula. In this section we briefly recall the combinatorics behind the composition of exponential generating functions. We prefer Rota’s approach via the incidence algebra of the set partition lattice for the conceptual proofs and concise formulas it provides; for details we refer the reader to [2] or the original paper [31].

We denote by \( \mathcal{P}(n) \) the lattice of partitions of the set \( \{1, 2, \ldots, n\} \) under refinement order. The number of classes (or blocks) a partition \( \nu \in \mathcal{P}(n) \) is called its size and denoted by \( |\nu| \).

Then Faà di Bruno’s formula can be interpreted as an isomorphism between the reduced incidence algebra of the partition lattices and exponential formal power series as follows. Let \( (a_n)_{n \geq 1} \) and \( (b_n)_{n \geq 1} \) be sequences and define a new sequence by the formula
\[ c_n = \sum_{\nu \in \mathcal{P}(n)} a_{|\nu|} \prod_{B \in \nu} b_{|B|}. \]
Then Faà di Bruno’s formula (see [57, Theorem 5.1.4] or [2, Proposition 5.9]) asserts that their exponential generating functions \( F_a(z) = \sum_{k=1}^{\infty} \frac{a_k}{k!} z^k \) and \( F_b(z) = \sum_{k=1}^{\infty} \frac{b_k}{k!} z^k \) satisfy the relation
\[ F_c(z) = F_b(F_a(z)). \]
Equivalently, given smooth functions $f$ and $g$, the $m$-th derivative of the composed function is
\begin{equation}
\frac{d^m}{dz^m}f(g(z)) = \sum_{\nu \in P(m)} f^{(|\nu|)}(g(z)) \prod_{B \in \nu} g^{(|B|)}(z).
\end{equation}

We single out two important functions, namely the $\zeta$-function
\[ \zeta(\nu, \rho) = 1, \]
which corresponds to the sequence $(1, 1, \ldots)$ and has generating functions $e^z - 1$, and the Möbius function, which is its inverse under convolution, and corresponds to the generating function $\log(1+z)$. In the forthcoming calculations only the values
\begin{equation}
\mu(\hat{n}, \nu) = \prod_{B \in \nu} (-1)^{|B|-1}(|B| - 1)!
\end{equation}
will be needed, see [56, Example 3.10.4]. We shall see that they appear when the derivatives (2.17) are inserted into Faà die Bruno’s formula (2.20).

### 3. Trace formula

In this section we apply the trace method to the matrix (2.1) in order to prove certain properties of the sum (1.1). The positivity of the coefficients in the expansion of the characteristic polynomial could be seen as a very special case the BMV conjecture [55]: if $A$ and $B$ are positive semi-definite matrices, then for all positive integers $m$, the polynomial in $t$, $\text{Tr}(A + tB)^m$, has only non-negative coefficients. The proof below shows that the assertion is also true whenever $A$ is an orthogonal projection of rank one and $B$ is a positive or antisymmetric self-adjoint matrix.

**Theorem 3.1.**

(i) The cotangent sum $(1.1)$ can be expressed as
\begin{equation}
S(m, n, \alpha) = \text{Tr} \left( (\cot \alpha J_n + B_n)^m \right)
\end{equation}

(ii) There are universal integer valued polynomials $p_{m,m-2k}(x)$ with rational coefficients such that the cotangent sum $(1.1)$ can be expressed as a polynomial of degree $m$ in $\cot \alpha$
\begin{equation}
S(m, n, \alpha) = \sum_{0 \leq k \leq [m/2]} p_{m,m-2k}(n) \cot^{m-2k} \alpha.
\end{equation}

Moreover, for any $n \in \mathbb{N}$, the coefficients $p_{m,m-2k}(n)$ are positive integers.

**Example 3.2.** For example, we have\footnote{We note in passing that there is a misprint in the formula for $S_5(g; \xi)$ in [26, p. 154].}
\[
\begin{align*}
S(1, n, \alpha) &= n \cot \alpha \\
S(2, n, \alpha) &= n^2 \cot^2 \alpha + n^2 - n \\
S(3, n, \alpha) &= n^3 \cot^3 \alpha + (n^3 - n) \cot \alpha \\
S(4, n, \alpha) &= n^4 \cot^4 \alpha + \frac{13}{5} (n^4 - n^2) \cot^2 \alpha + \frac{1}{3} (n^4 - 4n^2) + n \\
S(5, n, \alpha) &= n^5 \cot^5 \alpha + \frac{5}{3} (n^5 - n^3) \cot^3 \alpha + \frac{1}{3} (2n^5 - 5n^3 + n) \cot \alpha
\end{align*}
\]

It will be apparent from (6.1) later that indeed $S(m, n, \alpha)$ is a rational polynomial of degree $m$ in both $n$ and $\cot \alpha$; explicit expressions for the coefficients are computed in Corollary 6.7.

**Proof.** It is clear that the trace (3.1) is a polynomial of degree at most $n$ in $\cot \alpha$. Moreover since the entries of the matrices $J_n$ and $B_n$ are integers, the coefficients $p_{m,m-2k}(n)$ are integers as well. For positivity, we show that the mixed moments of $J_n$ and $B_n$ are positive. To see this, note that $P_n = \frac{1}{n} J_n$ is a self-adjoint projection of rank 1. It follows that for any matrix $C_n$, the compression $P_n C_n P_n$ lies in the 1-dimensional algebra generated by $P_n$, more precisely,
$P_n C_n P_n = \xi^T C_n \xi P_n$, where $\xi = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$ spans the image of $P_n$. For our matrix $B_n$ clearly $\xi^T B_n \xi = \sum b_{ij} = 0$ and by antisymmetry, also for odd powers $\xi^T B_n^k \xi = (-1)^k \xi^T B_n^k \xi = 0$. It follows that any mixed moment

$$\text{Tr}(J_n^{k_1} B_n^{l_1} J_n^{k_2} B_n^{l_2} \cdots J_n^{k_r} B_n^{l_r}) = n^{k_1 + \cdots + k_r} \text{Tr}(P_n^{k_1} B_n^{l_1} P_n^{k_2} B_n^{l_2} \cdots P_n^{k_r} B_n^{l_r})$$

$$= n^{k_1 + \cdots + k_r} \text{Tr}(P_n B_n^{l_1} P_n B_n^{l_2} P_n \cdots P_n B_n^{l_r} P_n)$$

$$= n^{k_1 + \cdots + k_r} \xi^T B_n^l \xi \xi^T B_n^l \xi \cdots \xi^T B_n^l \xi$$

$$= \begin{cases} 0 & \text{if some } l_j \text{ is odd} \\ > 0 & \text{if all } l_j \text{ are even}. \end{cases}$$

In particular, $S(m, n, \alpha)$ evaluates to an integer (natural number) whenever cot $\alpha$ is an integer (natural number). It was observed in [14] to the surprise of the authors that the sums in the next corollary are integer valued; explicit formulas are computed in Corollary 6.6 below. We will see later that even for noninteger values of cot $\alpha$ the sum may evaluate to an integer, e.g., for $n = 2$ and cot $\alpha = \frac{\pi}{2}$, Lucas numbers appear, see (5.1) below.

**Corollary 3.3.** The sums

$$S(2m - 1, n, \pi/4) = \sum_{k=1}^{n} (-1)^k \cot^{2m-1} \left(\frac{2k - 1}{4n}\pi\right)$$

$$S(2m, n, \pi/4) = \sum_{k=1}^{n} \cot^{2m} \left(\frac{2k - 1}{4n}\pi\right)$$

can be represented as integer-valued polynomials in $n$ of degrees $2m - 1$ and $2m$, respectively.

**Proof.** Applying Lemma 2.1 to the matrix $\left[\begin{smallmatrix} 1 & 1+i \\ 1-i & 1 \end{smallmatrix}\right]_n$, we obtain its eigenvalues as

$$\lambda_k = \cot\left(\frac{\pi}{4n} + \frac{k}{n}\pi\right), \text{ for } k \in \{1, \ldots, n\},$$

because $\alpha = \arccot(1) = \frac{\pi}{4}$. Let us show how these are related the sums considered by Byrne and Smith [14]. Indeed the corresponding power sums are

$$\sum_{k=1}^{n} \cot^r \left(\frac{\pi}{4n} + \frac{k}{n}\pi\right) = \sum_{k=1}^{\lfloor n/2 \rfloor} \cot^r \left(\frac{\pi}{4n} + \frac{k}{n}\pi\right) + \sum_{k=\lfloor n/2 \rfloor+1}^{n} \cot^r \left(\frac{\pi}{4n} + \frac{k}{n}\pi\right)$$

and substituting cot$(\frac{\pi}{4n} + \frac{k}{n}\pi)$ into the second sum, we get

$$= \sum_{k=1}^{\lfloor n/2 \rfloor} \cot^r \left(\frac{\pi}{4n} + \frac{k}{n}\pi\right) + \sum_{k=0}^{n-\lfloor n/2 \rfloor-1} (-1)^r \cot^r \left(-\frac{\pi}{4n} + \frac{k}{n}\pi\right)$$

$$= \begin{cases} -\sum_{k=1}^{n} \cot^r \left(\frac{2k - 1}{4n}\pi\right) & \text{if } r \text{ is odd,} \\ \sum_{k=1}^{n} \cot^r \left(\frac{2k - 1}{4n}\pi\right) & \text{if } r \text{ is even}. \end{cases}$$

4. Generating functions

In the present section we compute the generating function of the cotangent sums (1.1), for fixed $n$, i.e.,

$$F_n(z, \alpha) = \sum_{m=0}^{\infty} S(m, n, \alpha) z^m,$$
which is the moment generating function of the matrix $\cot \alpha J_n + B_n$ with respect to the non-normalized trace. Moreover we will compute the moment generating function of the matrix $B_n$ with respect to the nonnormalized trace and with respect to the state $\omega$ with density matrix $P_n = \frac{1}{n} J_n$, that is,

$$\omega(C) = \text{Tr}(P_n C) = \frac{1}{n} \sum_{i,j} c_{ij} = \xi^T C \xi$$

where as above by $\xi$ we denote the unit vector $\xi = \frac{1}{\sqrt{n}} (1, 1, \ldots, 1)^T$ and $C = [c_{ij}]_{i,j=1}^n \in M_n(\mathbb{C})$. The moment generating functions

$$M_{xJ_n + B_n}(z) = \text{Tr}((I - z(xJ_n + B_n))^{-1}),$$

$$M_{B_n}(z) = \text{Tr}((I - zB_n)^{-1}),$$

with respect to the trace are easy to compute directly through the characteristic polynomials. On the other hand, direct computation of

$$\tilde{M}_{B_n}(z) = \omega((I - zB_n)^{-1}) = \text{Tr}(P_n (I - zB_n)^{-1})$$

requires information about the eigenvectors which we could not obtain. It will therefore be computed indirectly. The tangent function and its inverse will play a major role in these computations and we collect some facts about these functions first.

### 4.1. Generating function for cotangent sums.

**Proposition 4.1.** For fixed $n$ the ordinary generating function of the cotangent sums (1.1) is

$$(4.1) \quad F_n(z, \alpha) = \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} \cot^m \frac{\alpha + k \pi}{n} z^m = \frac{n}{1 + z^2} (1 - z \cot(n \arctan z - \alpha))$$

More generally, the moment generating function of the matrix pencil $xJ_n + B_n$ is

$$(4.2) \quad M_{xJ_n + B_n}(z) = \frac{n}{1 + z^2} \left(1 + z \frac{x + \tan(n \arctan z)}{1 - x \tan(n \arctan z)} \right).$$

**Proof.** Let $\theta_k = \frac{\alpha + k \pi}{n}$ and recall that we have realized $\cot \theta_k$ as roots of the polynomial (2.3). Thus we can write the generating function of the sequence (1.1) as the logarithmic derivative of this polynomial. Indeed,

$$g_n(z) = \sum_{k=0}^{n-1} \frac{1}{z - \cot \theta_k}$$

$$= \frac{\chi'_n(\alpha; z)}{\chi_n(\alpha; z)}$$

$$= n \frac{\cot \alpha + i)(z - i)^{n-1} - (\cot \alpha - i)(z + i)^{n-1}}{(\cot \alpha + i)(z - i)^n - (\cot \alpha - i)(z + i)^n}$$
then the ordinary generating function is
\[
F_n(z, \alpha) = \frac{1}{z} g_n \left( \frac{1}{z} \right)
= n \frac{(\cot \alpha + i)(1 - iz)^{n-1} - (\cot \alpha - i)(1 + iz)^{n-1}}{(\cot \alpha + i)(1 - iz)^n - (\cot \alpha - i)(1 + iz)^n}
= \frac{n}{1 + z^2} \frac{(\cot \alpha + i)(1 - iz)^n - (\cot \alpha - i)(1 + iz)^n}{(\cot \alpha + i)(1 - iz)^n + (\cot \alpha - i)(1 + iz)^n}
= \frac{n}{1 + z^2} \left( 1 - z \cot(n \arctan z - \alpha) \right)
\]

where in the last step we used identity (2.8). The general formula (4.2) follows by substituting \( \alpha = \arccot x \) and the addition formula for tangent (2.5).

Remark 4.2. In the cases \( \alpha = 0 \) (\( \alpha = \pi/2 \), resp.) formula (4.1) reproduces [20, Formula (A7.2) (resp. (C6.2))]. At a first glance for \( \alpha = 0 \) the sum diverges: \( \sum_{k=0}^{n-1} \cot^m \frac{k\pi}{n} = \pm \infty \). However [20, Formula (A7.1)] the sum starts at \( k = 1 \), i.e., \( \sum_{k=1}^{n-1} \cot^m \frac{k\pi}{n} \). Inspection of the partial fraction expansion of the generating function (4.1) however reveals that the term \( \frac{1}{1 - z \cot \theta} \) vanishes as \( \alpha \) tends to zero and the generating function becomes
\[
F_n(z, 0) = \sum_{k=1}^{n-1} \frac{1}{1 - z \cot \theta_k}
\]
and this is indeed the generating function of the sums \( \sum_{k=1}^{n-1} \cot^m \frac{k\pi}{n} \). In the case \( \alpha = \pi/2 \) formula (4.1) reproduces [20, Formula (C6.2)]. Indeed, since \( \cot(\alpha - \pi/2) = -\tan \alpha \) we have \( M_{B_n}(z) = \Tr((I - zB_n)^{-1}) = \frac{n(1 + z \tan(n \arctan z))}{1 + z^2} \).

4.2. A functional relation. In this section we indicate an algorithm to calculate the coefficients \( p_{m,m-2k}(n) \), which is the main contribution of this paper. The following lemma is a special case of cyclic Boolean convolution [49]; we reproduce the calculation here for the reader’s convenience.

Lemma 4.3. The generating functions \( F_n(z, \alpha) \) and \( \tilde{M}_B(z) \) satisfy the relation
\[
M_{xJ_n + B_n}(z) = \frac{nxz \frac{d}{dz} \tilde{M}_B(z)}{1 - n xz M_B(z)} + M_B(z)
\]

Proof. The first terms of the power series are easy to calculate
\[
M_{xJ_n + B_n}(z) = n + nxz + \sum_{m \geq 2} \Tr((xJ_n + B_n)^m) z^m
\]
and for \( m \geq 2 \) we expand the powers and arrange the resulting words according to the last letter:

\[
\text{Tr}((xJ_n + B_n)^m) = \text{Tr}\left((xJ_n)^m + B_n^m\right) \\
+ \sum_{\substack{k \geq 1 \\ p_0 \geq 0 \\ p_1, p_2, \ldots, p_k \geq 1 \\ q_0 + q_1 + \cdots + q_k + q_k = m}} B_{p_0}^n (xJ_n)^{q_0} B_{p_1}^n (xJ_n)^{q_1} B_{p_2}^n \cdots (xJ_n)^{q_k} B_{p_k}^n \\
+ \sum_{\substack{k \geq 1 \\ q_0 \geq 0 \\ p_1, p_2, \ldots, p_k \geq 1 \\ q_0 + q_1 + \cdots + q_k + q_k = m}} (xJ_n)^{q_0} B_{p_1}^n (xJ_n)^{q_1} B_{p_2}^n (xJ_n)^{q_2} \cdots B_{p_k}^n (xJ_n)^{q_k} \\
= \text{Tr}(B_n^m) + \text{Tr}((xJ_n)^m) \\
+ \sum_{\substack{k \geq 1 \\ q_0 \geq 0 \\ p_1, p_2, \ldots, p_k \geq 1 \\ q_0 + q_1 + \cdots + q_k + q_k = m}} (xJ_n)^{q_0 + q_1 + \cdots + q_k} \text{Tr}(PB_n^{p_1}) \text{Tr}(PB_n^{p_2}) \cdots \text{Tr}(PB_n^{p_k-1}) \text{Tr}(PB_n^{p_k+q_0}) \\
+ \sum_{\substack{k \geq 1 \\ q_0 \geq 0 \\ p_1, p_2, \ldots, p_k \geq 1 \\ q_0 + q_1 + \cdots + q_k + q_k = m}} (xJ_n)^{q_0 + q_1 + \cdots + q_k} \text{Tr}(PB_n^{p_1}) \text{Tr}(PB_n^{p_2}) \cdots \text{Tr}(PB_n^{p_k})
\]

Inserting this expansion into (4.3) we obtain

\[
M_{xJ_n + B_n}(z) = n + nxz + \sum_{m \geq 2} \text{Tr}(B_n^m) z^m + \sum_{m \geq 2} (nxz)^m \\
+ \sum_{k \geq 1} \left( \frac{nxz}{1 - nxz} \right)^k (\tilde{M}_{B_n}(z) - 1)^k \tilde{M}_{B_n}(z) \\
+ \frac{1}{1 - nxz} \sum_{k \geq 1} \left( \frac{nxz}{1 - nxz} \right)^k (\tilde{M}_{B_n}(z) - 1)^k \\
= \text{Tr}((I - zB_n)^{-1}) + \frac{nxz}{1 - nxz} \\
+ \frac{nxz}{1 - nxz} \frac{1}{1 - \frac{nxz}{1 - nxz} (M_{B_n}(z) - 1)} \tilde{M}_{B_n}(z) + \frac{1}{1 - nxz} \left( \frac{1}{1 - \frac{nxz}{1 - nxz} (M_{B_n}(z) - 1)} - 1 \right) \\
= M_{B_n}(z) + \frac{nxz}{1 - nxz} + \frac{nxz \tilde{M}_{B_n}(z)}{1 - nxz \tilde{M}_{B_n}(z)} + \frac{1}{1 - nxz \tilde{M}_{B_n}(z)} - \frac{1}{1 - nxz} \\
= M_{B_n}(z) + \frac{1 + nxz \tilde{M}_{B_n}(z)}{1 - nxz \tilde{M}_{B_n}(z)} - 1 \\
= M_{B_n}(z) + \frac{nxz(M_{B_n}(z) + \tilde{M}_{B_n}(z))}{1 - nxz \tilde{M}_{B_n}(z)}
\]
where

\[
\hat{M}_{B_n}(z) = \sum_{p_0 \geq 0, p \geq 1} \text{Tr}(PB_n^{p_0+p})z^{p_0+p} \\
= \sum_{m=1}^{\infty} \sum_{p_0 \geq 0, p \geq 1} \text{Tr}(PB_n^m)z^m \\
= \sum_{m=1}^{\infty} m \text{Tr}(PB_n^m)z^m \\
= z \frac{d}{dz} \hat{M}_{B_n}(z)
\]

Lemma 4.4. For any \( x \), the differential equation

\[
\frac{n x z g'(z)}{1 - nxg(z)} + M_{B_n}(z) = M_{xJ_n + B_n}(z)
\]

with initial condition \( g(0) = 1 \) has the unique solution \( g(z) = z\hat{M}_{B_n}(z) = \frac{1}{n} \tan(n \arctan z) \).

Proof. Observe that the considered expression can be rewritten as a first order linear equation in standard form

\[
g'(z) + q(z)g(z) = p(z)
\]

which has a unique solution. and direct verification shows that it is given by \( g(z) = \frac{\tan(n \arctan z)}{n} \).

\[\square\]

5. Combinatorial interpretation

In this section we indicate explicit combinatorial interpretations of the coefficients of polynomials (3.2) which express the value of trace of matrices in terms of Dyck paths and rooted binary trees. We emphasize that these coefficients \( p_{m,k} \) are nonzero, whenever \( m \) and \( k \) have the same parity.

5.1. Dimension 2. First let us record that for \( n = 2 \) at offset \( \cot \alpha_0 = \frac{1}{2} \) we recover the well-known sequence Lucas numbers (A000032 in the On-Line Encyclopedia of Integer Sequences [53]). Indeed, the characteristic polynomial (2.3) is

\[
\chi_2(\lambda) = \text{Im}(\frac{1}{2} + i)(\lambda - i)^2 = \lambda^2 - \lambda - 1
\]

and the roots are the golden ratios \( \phi_{\pm} = \frac{1 \pm \sqrt{5}}{2} \) with moments

(5.1) \[ S(m, 2, \alpha_0) = L_m = \phi_+^m + \phi_-^m \]

satisfying the recurrence relation

\[
L_m = \begin{cases} 
2 & m = 0 \\
1 & m = 1 \\
L_{m-1} + L_{m-2} & m \geq 2
\end{cases}
\]
5.2. Interpretation of $\text{Tr}(J_n B^{2m}_n)$ in terms of Dyck paths. For the general case we establish some recurrence relations. An explicit formula will be established in Corollary 6.10 below.

**Proposition 5.1.** The moments

$$d_{n,m} = \text{Tr}(J_n B^{2m}_n)$$

satisfy the recurrence

$$
\begin{align*}
d_{n,0} &= 1 + d_{n-1,0} = n, \\
d_{1,m} &= \delta_{0,m}, \\
d_{n,m} &= d_{n-1,m} + \sum_{k=0}^{m-1} d_{n-1,k} d_{n,m-k-1}.
\end{align*}
$$

which is reminiscent of the recurrence relations for the Motzkin numbers.

**Proof.** The function $Q_n(z) = \frac{\tan(n \arctan z)}{z}$ is rational by (2.7). Indeed

$$Q_n(z) = \frac{1}{z} \tan(\arctan z + (n-1) \arctan z) = \frac{z + \tan((n-1) \arctan z)}{z - z^2 \tan((n-1) \arctan z)} = \frac{1 + Q_{n-1}(z)}{1 - z^2 Q_{n-1}(z)}$$

or equivalently

$$Q_n(z) = 1 + Q_{n-1}(z) + z^2 Q_{n-1}(z) Q_n(z).$$

From Lemma 4.4 we infer that $\sum_{m=0}^{\infty} d_{n,m} z^{2m} = Q_n(z)$ and we can readily calculate the required recurrence for the moments $d_{n,m}$. \qed

The continued fraction of the rational function $Q_n(z)$ is finite and was computed in [50]:

$$Q_n(z) = \frac{n}{1 - \frac{(n+1)(n-1)z^2}{1 - \frac{(n+2)(n-2)z^2}{1 - \frac{(n+3)(n-3)z^2}{1 - \ldots}}}}$$

We can thus infer from Flajolet’s theory of continued fractions [34] the following formula for the moments $d_{n,m}$.

**Theorem 5.2.**

$$d_{n,m} = n \sum_{\pi \in D_m} w(\pi)$$

where the sum runs over Dyck paths of length at most $2m$ with weights $a_{k-1} = \frac{n-k}{2k-1}$, $b_k = \frac{n+k}{2k+1}$, $k = 1, 2, \ldots, n$.

**Example 5.3.** For $n = 3$ the generating function is

$$Q_3(z) = \frac{z^2 - 3}{3z^2 - 1} = 3 + 8 x^2 + 24 x^4 + 72 x^6 + 216 x^8 + 648 x^{10} + O(x^{11})$$

and indeed for $n = 3$ with weights

$$a_0 = \frac{2}{1}, \quad a_1 = \frac{1}{3}, \quad b_1 = \frac{4}{3}, \quad b_2 = \frac{5}{5}$$

we have
\[ d_{3,1} = 3 \cdot \left( \frac{2 \cdot 4}{1 \cdot 3} \right) = 8 \]
\[ d_{3,2} = 3 \cdot \left( \frac{2 \cdot 4}{1 \cdot 3} \cdot \frac{2 \cdot 4}{1 \cdot 3} + \frac{2 \cdot 1 \cdot 5 \cdot 4}{3 \cdot 5} \cdot \frac{2 \cdot 4}{1 \cdot 3} \right) \]
\[ = 3 \cdot \left( \frac{64}{9} + \frac{8}{9} \right) = 24 \]
\[ d_{3,3} = 3 \cdot \left( \frac{2 \cdot 1 \cdot 5 \cdot 4}{3 \cdot 5} \cdot \frac{2 \cdot 4}{1 \cdot 3} + \frac{2 \cdot 4}{1 \cdot 3} \cdot \frac{2 \cdot 4}{1 \cdot 3} \right) \]
\[ = 3 \cdot \left( 0 + \frac{8}{27} + \frac{64}{27} + \frac{512}{27} \right) = 72 \]

etc.

5.3. Interpretation of \( \text{Tr}(J_n B_{2n}^m) \) in terms of binary trees. Set \( e_{n,k} = d_{n,k-1} \) and \( d_{n,1} = 1 \), then recursion (5.3) can be rewritten more compactly as

\[
\begin{align*}
    e_{n,0} &= 1, \\
    e_{n,1} &= n, \\
    e_{1,m} &= \delta_{0,m-1}, \\
    e_{n,m} &= \sum_{k=1}^{m} e_{n-1,k} e_{n,m-k}.
\end{align*}
\]

which is reminiscent of the Catalan recurrence relations.

Definition 5.4. A rooted binary tree is a rooted tree in which each node has at most two children, one of which we distinguish as firstborn. We use the convention that the root is not a child and therefore does not count as a firstborn; our trees are unordered but we take the convention that firstborns are always drawn on the right. We denote by \( T_{n,m} \) the set of rooted binary trees with \( m \) leaves, such that each leaf has a brother and every path emanating from the root contains at most \( n - 1 \) firstborns. We note that the set \( T_{n,m} \) is empty unless \( m \leq n \).

For a rooted binary tree \( \tau \in T_{n,m} \) we denote by \( \text{Paths}(\tau) \) the set of maximal rooted paths. For such a path \( p \in \text{Paths}(\tau) \) we denote by \( r(p) \) the number of firstborn nodes occurring in \( p \) and its weight \( \omega(p) = n - r(p) \) which is a number between 1 and \( n \).

Theorem 5.5. Let \( 1 \leq m \leq n \), then

\[
e_{n,m} = \sum_{\tau \in T_{n,m}} \prod_{p \in \text{Paths}(\tau)} \omega(p).
\]
Figure 5.1. $T_{3,2}$ with corresponding weight of paths.

Figure 5.2. $T_{3,3}$ with corresponding weight of paths.

**Proof.** Let us denote the right-hand side of (5.4) by $c_{n,m}$. If $m = 1$ the root is the only node and does not count as a firstborn, therefore $e_{n,1} = c_{n,1} = n$. Moreover $T_{2,2}$ only contains one tree of weight $e_{2,2} = c_{2,2} = 2$. More generally $T_{n,2}$ contains $(n - 1)$ trees and $e_{n,2} = c_{n,2} = n(n - 1) + (n - 1)(n - 2) + \cdots + 2 \times 1$. So, it is sufficient to verify that $e_{n,m} = c_{n,m}$ for $n, m \geq 3$. Notice that any rooted binary tree can be viewed as one or two (non-empty because $n, m \geq 3$) rooted binary trees grafted onto a common root; see Fig. 5.1 and 5.2. Thus in order to create all possible binary trees we start with a root vertex, and one child (Case 1) or two children (Case 2a and 2b) with all possible choices of the subtrees trees as shown in the diagram below.

Case 1. Assume that the root has only one child $v_0$. Then every path from the root to a leaf with at most $n - 1$ firstborns consists of the first step and a path from $v_0$ with at most $n - 2$ firstborns. So the weight remains the same and the number of leaves remains $m$.

Case 2a. Let $\tau$ be such a tree and $p$ a path passing through the firstborn child $v_0$. Then we can consider the latter as root vertex of new binary tree in $T_{n-1,k}$ with $k$ leaves for $k \in \{1, \ldots, m - 1\}$. Denote by $p'$ the restriction of the path $p$ to this subtree. Observe that $p'$ contains at most $n - 2$ firstborns because $v_0$ already counts as a firstborn and the weights of $p$ and $p'$ coincide. Indeed $r(p) = r(p') + 1$ and so $\omega(p) = n - r(p) = n - 1 - r(p') = \omega(p')$.

Case 2b. Let now $p$ be a path passing through the other child, that is, $p'$ is a path in a tree from $T_{n,m-k}$ and again the weight does not change.

Finally we have

$$\sum_{\tau \in T_{n,m}} \prod_{p \in \text{Paths}(\tau)} \omega(p) = \sum_{k=1}^{m-1} \sum_{\tau \in T_{n,m-k}} \prod_{p \in \text{Paths}(\tau)} \omega(p) \sum_{\tau \in T_{n-1,k}} \prod_{p \in \text{Paths}(\tau)} \omega(p) + \sum_{\tau \in T_{n-1,m}} \prod_{p \in \text{Paths}(\tau)} \omega(p),$$
and now we can write

\[ c_{n,m} = \sum_{k=1}^{m-1} c_{n-1,k} c_{n,m-k} + c_{n-1,m} \]

Thus we see that \( c_{n,m} = e_{n,m} \). \( \square \)

5.4. **Interpretation of \( p_{m,k}(n) \) for \( k \gg 2 \)**. The combinatorial objects that we consider now are called circular binary forests.

**Definition 5.6.** Assume that \( m,k \in \mathbb{N} \) have the same parity. For \( k \in \{2, \ldots, m\} \) a circular binary forest \( T^E_{n,m,k} \) of degree \( k \) is a set of \( k \) binary trees as above arranged on a circle with a total number of \( m \) leaves, see Figure 5.3 for an example.

![Figure 5.3. A circular forest; firstborns are marked with an extra circle](image)

The weight of a forest \( F = (\tau_1, \tau_2, \ldots, \tau_k) \) is the product

\[ \omega(F) = \prod_{\tau \in F} \omega(\tau). \]

**Proposition 5.7.** Assuming that \( m \) and \( k \) \((k \neq 0)\) have the same parity, then

\[ p_{m,k}(n) = \sum_{F \in T^E_{n,m,k}} \omega(F). \]

**Proof.** From the proof of Theorem (3.1), we see that

\[ p_{m,k}(n) = \sum_{l_0, l_1, \ldots, l_k \geq 0 \atop l_1 \ldots l_{k-1}, l_0 + l_k \text{ even} \atop \sum l_i = m-k} \text{Tr}(B_{n}^{l_0} B_{n}^{l_1} J_{n} B_{n}^{l_2} \cdots J_{n} B_{n}^{l_k}) \]

\[ \sum_{l_0, l_1, \ldots, l_k \geq 0 \atop l_1 \ldots l_{k-1}, l_0 + l_k \text{ even} \atop \sum l_i = m-k} \text{Tr}(J_{n} B_{n}^{l_0 + l_k}) \prod_{1 \leq i \leq k-1} \text{Tr}(J_{n} B_{n}^{l_i}) \]

This can be visualized in terms of forests, see Figure 5.3. \( \square \)

5.5. **Interpretation of the constant term \( \text{Tr}(B_{n}^{2m}) \)**. The moment generating function of \( B_{n} \) is

\[ M_{B_{n}}(z) = \frac{n(1 + z \tan(n \arctan z))}{1 + z^2} = \frac{n + n z^2 Q_n(z)}{1 + z^2}. \]
If we expand the generating function in powers of $z$, then we obtain
\[ \text{Tr}(B_n^{2m}) = nd_{n,m-1} - \text{Tr}(B_n^{2m-2}) = nd_{n,m-1} - nd_{n,m-2} + nd_{n,m-3} + \cdots + nd_{n,0}(-1)^{m-1} + n(-1)^m \]
\[ = n \sum_{i=0}^{m} d_{n,m-1-i}(-1)^i = n \sum_{i=0}^{m} e_{n,m-i}(-1)^i \]

6. Explicit analytic evaluation of cotangent sums

In this section we study the Taylor series expansions of the generating function (4.1) and obtain closed formulas in terms of derivative polynomials.

6.1. A Möbius inversion. As a first step we extract the coefficients of the generating function using Faà di Bruno’s formula.

**Theorem 6.1.** The cotangent sum (1.1) can be expressed as

\[ S(m, n, \alpha) = (-1)^{m/2} n \mathbb{1}_m \text{ even} + \frac{(-i)^m}{(m-1)!} \sum_{\nu \in \mathcal{P}^{\text{odd}}(m)} P_{\nu - 1}(\cot \alpha)(in)^{|\nu|} \mu(\hat{\nu}, \nu) \]

where $(-1)^nP_n(x)$ are the derivative polynomials for cot (2.10), $\mathcal{P}^{\text{odd}}(n)$ is the set of partitions with odd blocks only and $\mu(\hat{\nu}, \nu)$ is the Möbius function of the partition lattice (2.21).

**Proof.** We start by expressing the generating function (4.1) in terms of the functions $f(z) = \ln(|\sin(z - \alpha)|)$ and $g(z) = n \arctan z$. Indeed observe that
\[ \frac{n}{1 + z^2} (1 - z \cot(n \arctan z - \alpha)) = \frac{n}{1 + z^2} - \frac{d}{dz} f(g(z)). \]
and moreover the Leibniz rule of order $m$ implies
\[ \frac{d^m}{dz^m} \left( \frac{d}{dz} f(g(z)) \right) = m \frac{d^m}{dz^m} f(g(z)) + z \frac{d^{m+1}}{dz^{m+1}} f(g(z)) \]
thus
\[ \frac{d^m}{dz^m} \left( \frac{d}{dz} f(g(z)) \right) \bigg|_{z=0} = m \frac{d^m}{dz^m} f(g(z)) \bigg|_{z=0}. \]

Now we can apply Faà di Bruno’s formula (2.20) for the $m$-th derivative of a composed function and obtain
\[ \frac{d^m}{dz^m} f(g(z)) = \sum_{\nu \in \mathcal{P}(m)} f^{(\nu)}(g(z)) \prod_{B \in \nu} g^{(|B|)}(z) \]
\[ = \sum_{\nu \in \mathcal{P}(m)} \cot(|\nu| - 1)(g(z) - \alpha) \prod_{B \in \nu} \frac{ni(-1)^{|B|}(|B| - 1)!}{2} ((z - i)^{-|B|} - (z + i)^{-|B|}) \]
\[ = \sum_{\nu \in \mathcal{P}(m)} (-1)^{|\nu| - 1} P_{\nu - 1}(\cot(g(z) - \alpha)) \left( -\frac{ni}{2} \right)^{|\nu|} \mu(\hat{\nu}, \nu) \prod_{B \in \nu} ((z - i)^{-|B|} - (z + i)^{-|B|}) \]
where we recognize the Möbius function of the partition lattice (2.21). Now at $z = 0$ we have $g(0) = 0$ and
\[ (-i)^{-k} - i^{-k} = i^k - (-i)^k = \begin{cases} 0 & k \text{ even} \\ 2i^k & k \text{ odd} \end{cases} \]
moreover if $\nu$ is odd then $|\nu| \equiv m \mod 2$ and we obtain
\[ \frac{d^m}{dz^m} f(g(z)) \bigg|_{z=0} = (-i)^m \sum_{\nu \in \mathcal{P}^{\text{odd}}(m)} P_{\nu - 1}(\cot \alpha)(ni)^{|\nu|} \mu(\hat{\nu}, \nu) \]
finally the Taylor coefficients of \( \frac{n}{1 + z^2} \) contribute \( ni^m \) for even \( m \) and the claim follows. \( \square \)

**Remark 6.2.** From the Newton identities between power sum and elementary symmetric polynomials we conclude

\[
\sum_{l_1 < l_2 < \cdots < l_k} \cot \frac{\alpha + l_j \pi}{n} = (-1)^k c_{n-k} = \begin{cases} \binom{n}{k} (-1)^{k/2} & \text{k even} \\ \cot \alpha \binom{n}{k} (-1)^{(1-k)/2} & \text{k odd} \end{cases}
\]

In the case \(|B| = 1\) this reduces to Theorem 6.1 with \( m = 1 \) and for \(|B| = n\) this is a consequence of the well-known identities

\[
\sum_{k=0}^{n-1} \sin \left( \frac{k\pi}{n} + z \right) = 2^{1-n} \sin(nz) \quad \text{and} \quad \prod_{k=0}^{n-1} \cos \left( \frac{k\pi}{n} + z \right) = 2^{1-n} \sin \left( nz + \frac{\pi}{2} n \right).
\]

For further literature about trigonometric multiple cotangent sums similar to those in (6.2), we refer the reader to [10, Section 6] and [64].

### 6.2. An evaluation in terms of derivative polynomials.

We can apply Rota’s calculus to further simplify the expression (6.1). 

**Corollary 6.3.** The cotangent sums \((1.1)\) evaluate to

\[
S(m, n, \alpha) = (-1)^{m/2} n \mathbb{1}_m \text{ even} + \frac{1}{(m-1)!} \sum_{k=1}^{m} n^k A_m^{(k)} P_{k-1}(\cot \alpha)
\]

where \( A_m^{(k)} \) are the arctangent numbers \((2.15)\); note that these are alternating \((2.16)\).

**Proof.** We extract the essential part of the formula (6.1) and arrive at the expression

\[
\sum_{\nu \in \mathcal{P}^{odd}(m, \nu)} P_{|\nu|-1}(\cot \alpha)(ni)^{|\nu|} \mu(\hat{0}_m, \nu) = \sum_{k=1}^{m} c_{m,k}(ni)^k P_{k-1}(\cot \alpha)
\]

where

\[
c_{m,k} = \sum_{\nu \in \mathcal{P}^{odd}(m, \nu)} \mu(\hat{0}_m, \nu)
\]

This sum can be evaluated using the combinatorial convolution \((2.18)\) by setting

\[
f_k = \begin{cases} (k-1)! & \text{for odd } k \\ 0 & \text{else} \end{cases}
\]

and \( g_k = t^k \) and the generating functions are

\[
F_f(z) = \sum_{k \text{ odd}} \frac{(k-1)!}{k!} z^k = \frac{1}{2} (\log(1 + z) - \log(1 - z)) = \frac{1}{2} \log \frac{1 + z}{1 - z} = \text{atanh } z
\]

and

\[
F_g(z) = \sum_{k=1}^{\infty} \frac{t^k}{k!} z^k = e^{tz} - 1;
\]

hence by \((2.19)\)

\[
F_g(F_f(z)) = e^{t \text{atanh } z} - 1;
\]

and the coefficient of \( t^k \) yields the desired coefficient \( c_{m,k} = \tilde{A}_m^{(k)} \) and from \((2.16)\) we gather the correct sign. \( \square \)
Remark 6.4. Comtet [21, p. 260] asserts that the arctangent numbers are inverse to the derivative polynomials. This means that the standard monomials can be expanded as a linear combination of tangent polynomials as follows [25, Formula (2.14)]:

\[(6.4) \quad x^m = \frac{1}{(m-1)!} \sum_{k=1}^{m} A_m^{(k)} P_{k-1}(x) + (-1)^{m/2} \mathbb{1}_m \text{ even}\]

Let us explain now that the similarity of this formula with (6.3) is not a coincidence. Indeed, using the property that the derivative polynomials linearize the cotangent power and the simple formula \(S(1, n, \alpha) = \text{Tr} C_n = n \cot \alpha\) allow for the following alternative straightforward proof:

\[
\sum_{s=1}^{n} \cot^m s \pi/n = n(-1)^{m/2} \mathbb{1}_m \text{ even} + \frac{1}{(m-1)!} \sum_{s=1}^{n} \sum_{k=1}^{m} A_m^{(k)} P_{k-1}(\cot \frac{\alpha + s \pi}{n})
\]

\[
= n(-1)^{m/2} \mathbb{1}_m \text{ even} + \frac{1}{(m-1)!} \sum_{s=1}^{m} A_m^{(k)} n^{k-1} (-1)^{k-1} \frac{d^{k-1}}{d\alpha^{k-1}} \cot \frac{\alpha + s \pi}{n}
\]

\[
= n(-1)^{m/2} \mathbb{1}_m \text{ even} + \frac{1}{(m-1)!} \sum_{k=1}^{m} A_m^{(k)} n^{k-1} (-1)^{k-1} \frac{d^{k-1}}{d\alpha^{k-1}} \sum_{s=1}^{n} \cot \frac{\alpha + s \pi}{n}
\]

\[
= n(-1)^{m/2} \mathbb{1}_m \text{ even} + \frac{1}{(m-1)!} \sum_{k=1}^{m} A_m^{(k)} n^{k-1} P_{k-1}(\cot \alpha)
\]

Remark 6.5. We are grateful to an anonymous referee who brought the paper [47] to our attention, where an inverse relation for Chebyshev Polynomials analogous to (6.4) is used to evaluate cosine power sums with proof similar to Remark 6.4.

6.3. Special cases. Let us evaluate formula (6.3) at certain offsets. We start with the elementary evaluations at \(\alpha = \pi/2\) and \(\alpha = \pi/4\). The first sum yields the constant coefficient \(p_{m0}(n)\) from Theorem 3.1 it vanishes for odd \(m\) and equals the free cumulants of the generalized tetilla law, see [33, Proposition 4.10]. The remaining coefficients are computed in Corollary 6.7 below. The second sum provides an explicit formula for the sums considered by Byrne and Smith [14].

Corollary 6.6.

\[(6.5) \quad S(2m, n, \pi/2) = (-1)^m n + \frac{1}{(2m-1)!} \sum_{k=1}^{m} n^{2k} A_{2m}^{(2k)} T_{2k-1}\]

\[
S(m, n, \pi/4) = (-1)^{m/2} n \mathbb{1}_m \text{ even} + \frac{1}{2(m-1)!} \sum_{k=1}^{m} (2n)^{k} A_m^{(k)} E_{k-1}
\]

Proof. The evaluation of the generating function (2.13) yields

\[P(0, z) = \tan z \quad P(1, z) = \frac{1 + \tan z}{1 - \tan z} = \tan(2z) + \sec(2z)\]

and we conclude that \(P_n(0) = T_n\) and the expansion (2.14) at \(x = 1\) yields the identity [46, (10)]

\[2^n E_n = P_n(1) = T_n + \sum_{k=1}^{n+1} \frac{T_n^{(k)}}{k} =: E_n^B\]

where \(E_n^B\) are also known as Euler numbers of type B [48]. See [32, Remark 4.6] for another, similar identity for \(E_n^B\).
6.4. Explicit evaluation of the coefficients \( p_{m,r} \). We observed in Theorem 3.1 that the cotangent sum (1.1) can be expressed as a bivariate polynomial in \( n \) and \( \cot \alpha \) with rational coefficients. We can identify these coefficients explicitly by applying the expansion (2.14) to the evaluation (6.3).

**Corollary 6.7.** The integer valued polynomials \( p_{m,r}(n) \) appearing in (3.2) have the following explicit expressions

\[
p_{2m,0}(n) = S(2m, n, \pi/2) \\
p_{m,r}(n) = \frac{1}{r(m-1)!} \sum_{k=r}^{m} n^k A_m^{(k)} T_k^{(r)}.
\]

6.5. Evaluation of the sum of Berndt and Yeap. Finally let us give an alternative and somewhat simpler expression for the summation formula of Berndt and Yeap [10, Corollary 2.2]

\[
\sum_{k=1}^{n-1} \cot^{2m} \frac{k\pi}{n} = (-1)^m n - (-1)^m 2^{2m} \sum_{j_0,j_1,j_2,\ldots,j_{2m}\geq 0} n^{2j_0} \prod_{p=0}^{2m} \frac{B_{2j_p}}{(2j_p)!}.
\]

**Corollary 6.8.** The sum \( S_0(2m, n) \) can be evaluated as follows

\[
\sum_{k=1}^{n-1} \cot^{2m} \frac{k\pi}{n} = (-1)^m (n-1) - \frac{1}{(2m-1)!} \sum_{k=1}^{m} (-1)^k A_{2m}^{(2k)} \frac{4^k B_{2k}}{2k} (n^{2k} - 1).
\]

**Proof.** The sum \( S_0(2m, n) \) can be obtained from the general formula \( S(2m, n, \alpha) \) after removing the singular term at \( k = 0 \) and then taking the limit \( \alpha \to 0 \). Let

\[
S_0(2m, n, \alpha) = \sum_{k=1}^{n-1} \cot^{2m} \frac{\alpha + k\pi}{n} = S(2m, n, \alpha) - \cot^{2m} \frac{\alpha}{n},
\]

then \( S_0(2m, n, \alpha) = \lim_{\alpha \to 0} S_0(2m, n, \alpha) \). First we linearize the singular term according to formula (6.4) and combine it with the summation formula (6.3) to obtain

\[
S_0(2m, n, \alpha) = (-1)^m (n - 1) + \frac{1}{(2m-1)!} \sum_{k=1}^{2m} A_{2m}^{(k)} (n^k \cot \alpha - \cot^{(k)} (\alpha/n)).
\]

Next we replace the polynomial evaluation by the derivative according to (2.10) and we see that

\[
n^{k+1} P_k (\cot \alpha) - P_k (\cot (\alpha/n)) = (-1)^k (n^{k+1} \cot^{(k)} (\alpha) - \cot^{(k)} (\alpha/n))
\]

At this point it is convenient to recall the series expansion of cotangent

\[
\cot z = \frac{1}{z} + \sum_{p=1}^{\infty} (-1)^p \frac{2^{2p} B_{2p} z^{2p-1}}{(2p)!}
\]

to observe that the derivatives of the singular term \( 1/z \) cancel and we can express the difference in terms of the analytic part

\[
\gamma(z) = \cot z - \frac{1}{z} = \sum_{p=1}^{\infty} (-1)^p \frac{2^{2p} B_{2p} z^{2p-1}}{(2p)!}
\]

and find

\[
\lim_{\alpha \to 0} n^{k+1} P_k (\cot \alpha) - P_k (\cot (\alpha/n)) = (-1)^k \lim_{\alpha \to 0} n^{k+1} \gamma^{(k)} (\alpha) - \gamma^{(k)} (\alpha/n))
\]

\[
= (-1)^k (n^{k+1} - 1) \gamma^{(k)} (0)
\]

\[
= \begin{cases} 
0 & k \text{ even} \\
-(-1)^{(k+1)/2} (n^{k+1} - 1) \frac{2^{k+1} B_{k+1}}{k+1} & k \text{ odd}
\end{cases}
\]
and finally
\[ \lim_{\alpha \to 0} S_0(2m, n, \alpha) = (-1)^m (n - 1) - \frac{1}{(2m - 1)!} \sum_{k \text{ even}} (-1)^{k/2} A_m^{(k)} (n^k - 1) \frac{2^k B_k}{k}. \]

**Remark 6.9.** The generating function of the Euler zigzag numbers (2.11) is related to the generating function (2.13)
\[ \tan(z) + \sec(z) = \frac{1 + \tan(z/2)}{1 - \tan(z/2)} = \sum_{n=0}^{\infty} P_n(1) \frac{z^n}{2^n n!} \]
and comparing with the explicit formula for the derivative polynomials (2.9) we conclude the following identity:
\[ E_n = -(-i)^n \sum_{k=0}^{n} \frac{k!}{2k} \binom{n}{k} (i - 1)^{k+1}. \]
See [24] for other evaluations of the derivative polynomials at rational angles.

**Corollary 6.10.** Extracting the linear coefficient of (6.3) we can obtain an explicit expression for the moments (5.2)
\[ \text{Tr}(J_n B_n^{2m}) = \frac{1}{(2m)!} \binom{A_m^{(1)}}{m+1} n + \sum_{k=1}^{2m+1} T_{k-1}^{(k)} A_{m+1}^{(k)} \]
where \( T_n^{(k)} \) are the higher tangent numbers (2.12).

### 7. Concluding Remarks

In this section we connect the algebraic and analytic approach and give some final remarks.

#### 7.1. Another explicit formula for \( \alpha = \frac{\pi}{2} \)
From [38, Problem 76 on P. 317, Answer on P. 559], we infer the identity (cf. [11, (3.29)])
\[ \omega_n (-1/2) = \sum_{k=1}^{m} (-1)^k \frac{k!}{2k} \binom{m}{k} = \begin{cases} \frac{2}{m+1} (1 - 2^{m+1}) B_{m+1} & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even}. \end{cases} \]
If we plug in \( \alpha = \frac{\pi}{2} \) into Equation (6.1) we will take into account the equation (2.9) then for \( m \) even (for \( m \) odd the sum is zero) our sums can be written in terms of Bernoulli numbers (which frequently appear in trigonometric sums, see [10, 23, 4, 29, 39])
\[ S(m, n, \pi/2) = (-1)^{m/2} n + \sum_{\nu \in \mathbb{P}_{\text{odd}}(m), |\nu| \text{ is even}} \frac{(-1)^{m/2} \nu! (2n)^{|\nu|}}{(m-1)! |\nu|} (1 - 2^{|\nu|}) B_{|\nu|}. \]

#### 7.2. Asymptotic analysis and derivative.
In order to investigate asymptotic properties formula from Theorem 6.1 it is sufficient to consider the contribution of the singleton partition and we obtain
\[ \lim_{n \to \alpha} \frac{1}{n^m} \sum_{k=0}^{n-1} \cot^{m} \frac{\alpha + k \pi}{n} = \begin{cases} \frac{1}{(m-1)!} P_{m-1}(\cot \alpha) & \text{if } m > 1 \\ \cot \alpha & \text{if } m = 1. \end{cases} \]
In particular from equation (3.1) we infer the asymptotic expression
\[ \frac{1}{(m-1)!} P_{m-1}(z) = \lim_{n \to \alpha} \text{Tr} \left[ \left( z \begin{bmatrix} 1/n & 1/n \\ 1/n & 1/n \end{bmatrix} + \begin{bmatrix} 0 & i/n \\ -i/n & 0 \end{bmatrix} \right)^m \right] \text{ for } m > 1. \]
Similarly we prove that the derivatives of tangent and cotangent can be approximated by simple matrices.
Finally we examine the limit formula for $\alpha = \frac{\pi}{2}$. From Section 7.1 we conclude
\[
\lim_{n \to \infty} \frac{1}{n^m} \sum_{k=0}^{n-1} \cot^m \frac{\pi}{2n} + \frac{k\pi}{n} = \begin{cases} 
\frac{(-1)^{m/2+1}2^m (2m-1)B_m}{m!} & \text{if } m \text{ is even} \\
0 & \text{if } m \text{ is odd}.
\end{cases}
\]
Indeed inspecting formula (6.5) immediately yields the asymptotics
\[
\sum_{k=0}^{n-1} \cot^{2m} \left( \frac{\pi}{2n} + \frac{k\pi}{n} \right) = (-1)^{m+1} A_{2m} (2^{2m} - 1) n^{-2m} \frac{2^{2m} B_{2m}}{(2m)!} + O(n^{-m-2})
\]
and since $A_{2m} = 1$ this yields the desired limit.

Euler’s identity $\zeta(2k) = \frac{(-1)^{k+1} (2k)! B_{2k}}{2^{2k} (2k)!}$ and the preceding discussions give rise to a new approximation of the values of the Riemann zeta function at even integer arguments, namely
\[
\zeta(2k) = \lim_{n \to \infty} \frac{\pi^{2k} \text{Tr} \left( \frac{0}{2n} \right)^{2k}}{2n^{2k} (2^{2k} - 1)} \quad \text{for } k \in \mathbb{N}.
\]

Approximation of the Riemann zeta function for even values by powers of cotangent is well studied, see [63, 61, 5, 27].

REFERENCES

1. V. S. Adamchik, *On the Hurwitz function for rational arguments*, Appl. Math. Comput. 187 (2007), no. 1, 3–12.
2. Martin Aigner, *Combinatorial theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 234, Springer-Verlag, Berlin-New York, 1979.
3. Milica Andelić, Zhibin Du, and Emrah Ózdemir, *A matrix approach to some second-order difference equations with sign-alternating coefficients*, J. Difference Equ. Appl. 26 (2020), no. 2, 149–162.
4. M. H. Annaby and R. M. Asharabi, *Exact evaluations of finite trigonometric sums by sampling theorems*, Acta Math. Sci. Ser. B (Engl. Ed.) 31 (2011), no. 2, 408–418.
5. Tom M. Apostol, *Another elementary proof of Euler’s formula for $\zeta(2n)$*, Amer. Math. Monthly 80 (1973), 425–431.
6. Matthias Beck and Mary Halloran, *Finite trigonometric character sums via discrete Fourier analysis*, Int. J. Number Theory 6 (2010), no. 1, 51–67.
7. M. Beeler, R.W. Gosper, and R. Schroeppel, *Hakmem*, Tech. Report AIM-239, MIT, 1972, https://dspace.mit.edu/handle/1721.1/6086.
8. M. Benze and A.A. Jagers, *Aufgabe 828*, Elem. Math. 35 35 (1980), no. 5, 123.
9. Bruce C. Berndt, *Ramanujan’s notebooks. Part I*, Springer-Verlag, New York, 1985.
10. Bruce C. Berndt and Boon Pin Yeap, *Explicit evaluations and reciprocity theorems for finite trigonometric sums*, Adv. in Appl. Math. 29 (2002), no. 3, 358–385.
11. Khristo N. Boyadzhiev, *A series transformation formula and related polynomials*, Int. J. Math. Math. Sci. (2005), no. 23, 3849–3866.
12. , *Derivative polynomials for tanh, tan, sech and sec in explicit form*, Fibonacci Quart. 45 (2007), no. 4, 291–303 (2008).
13. R.G. Buschman, *Fibonacci numbers, Chebyshev polynomials, generalizations and difference equations*, Fibonacci Quart. 1 (1963), 1–7, 19.
14. Graeme J. Byrne and Simon J. Smith, *Some integer-valued trigonometric sums*, Proc. Edinburgh Math. Soc. (2) 40 (1997), no. 2, 393–401.
15. F. Calogero and A. M. Perelomov, *Properties of certain matrices related to the equilibrium configuration of the one-dimensional many-body problems with the pair potentials $V_1(x) = -\log \sin x$ and $V_2(x) = 1/\sin^2 x$*, Comm. Math. Phys. 59 (1978), no. 2, 109–116.
16. , *Some Diophantine relations involving circular functions of rational angles*, Linear Algebra Appl. 25 (1979), 91–94.
17. L. Carlitz and Richard Scoville, *Tangent numbers and operators*, Duke Math. J. 39 (1972), 413–429.
18. Ching-Hua Chang and Chung-Wei Ha, *Central factorial numbers and values of Bernoulli and Euler polynomials at rationals*, Numer. Funct. Anal. Optim. 30 (2009), no. 3-4, 214–226.
19. Wenchang Chu, *Reciprocal relations for trigonometric sums*, Rocky Mountain J. Math. 48 (2018), no. 1, 121–140.
20. Wenchang Chu and Alberto Marini, Partial fractions and trigonometric identities, Adv. in Appl. Math. 23 (1999), no. 2, 115–175.
21. Louis Comtet, Advanced combinatorics, enlarged ed., D. Reidel Publishing Co., Dordrecht, 1974, The art of finite and infinite expansions.
22. Djurdje Cvijović, Derivative polynomials and closed-form higher derivative formulae, Appl. Math. Comput. 215 (2009), no. 3, 3002–3006.
23. , Summation formulae for finite cotangent sums, Appl. Math. Comput. 215 (2009), no. 3, 1135–1140.
24. , Values of the derivatives of the cotangent at rational multiples of $\pi$, Appl. Math. Lett. 22 (2009), no. 2, 217–220.
25. , Higher-order tangent and secant numbers, Comput. Math. Appl. 62 (2011), no. 4, 1879–1886.
26. Djurdje Cvijović and Jacek Klinowski, Finite cotangent sums and the Riemann zeta function, Math. Slovaca 50 (2000), no. 2, 149–157.
27. Djurdje Cvijović, Jacek Klinowski, and H. M. Srivastava, Some polynomials associated with Williams’ limit formula for $\zeta(2n)$, Math. Proc. Cambridge Philos. Soc. 135 (2003), no. 2, 199–209.
28. C. M. da Fonseca and Victor Kowalenko, On a finite sum with powers of cosines, Appl. Anal. Discrete Math. 7 (2013), no. 2, 354–377.
29. Carlos M. da Fonseca, M. Lawrence Glasser, and Victor Kowalenko, Basic trigonometric power sums with applications, Ramanujan J. 42 (2017), no. 2, 401–428.
30. , Generalized cosecant numbers and trigonometric inverse power sums, Appl. Anal. Discrete Math. 12 (2018), no. 1, 70–109.
31. Peter Doubilet, Gian-Carlo Rota, and Richard Stanley, On the foundations of combinatorial theory. VI. The idea of generating function, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory, 1972, pp. 267–318.
32. Wiktor Ejsmont and Franz Lehner, The free tangent law, 2020, preprint, arXiv:2004.02679.
33. , Sums of commutators in free probability, 2020, submitted, arXiv:2002.06051.
34. P. Flajolet, Combinatorial aspects of continued fractions, Discrete Math. 32 (1980), no. 2, 125–161.
35. Ghislain R. Franssens, Functions with derivatives given by polynomials in the function itself or a related function, Anal. Math. 33 (2007), no. 1, 17–36.
36. Ira M. Gessel, Generating functions and generalized Dedekind sums, Electron. J. Combin. 4 (1997), no. 2, Research Paper 11, approx. 17, The Wilf Festschrift (Philadelphia, PA, 1996).
37. Peter J. Grabner and Helmut Prodinger, Secant and cosecant sums and Bernoulli-Nörlund polynomials, Quaest. Math. 30 (2007), no. 2, 159–165.
38. Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, Concrete mathematics, second ed., Addison-Wesley Publishing Company, Reading, MA, 1994.
39. Yuan He, Explicit expressions for finite cotangent sums, J. Math. Anal. Appl. 484 (2020), no. 1, 123702, 24.
40. F. Hirzebruch and D. Zagier, The Atiyah-Singer theorem and elementary number theory, Publish or Perish, Inc., Boston, Mass., 1974, Mathematics Lecture Series, No. 3.
41. Michael E. Hoffman, Derivative polynomials for tangent and secant, Amer. Math. Monthly 102 (1995), no. 1, 23–30.
42. , Derivative polynomials, Euler polynomials, and associated integer sequences, Electron. J. Combin. 6 (1999), Research Paper 21, 13.
43. S. R. Holcombe, Falling coupled oscillators and trigonometric sums, Z. Angew. Math. Phys. 69 (2018), no. 1, Paper No. 19, 21.
44. Finn Holme, En enkel beregning av $\sum_{k=1}^{\infty} \frac{1}{k^p}$, Nordisk Mat. Tidsskr. 18 (1970), 91–92.
45. A. F. Horadam, Basic properties of a certain generalized sequence of numbers, Fibonacci Quart. 3 (1965), 161–176.
46. Donald E. Knuth and Thomas J. Buckholtz, Computation of tangent, Euler, and Bernoulli numbers, Math. Comp. 21 (1967), 663–688.
47. Xingxing Lv and Shimeng Shen, On Chebyshev polynomials and their applications, Adv. Difference Equ. (2017), Paper No. 3443, 6.
48. Shi-Mei Ma, On $\gamma$-vectors and the derivatives of the tangent and secant functions, Bull. Aust. Math. Soc. 90 (2014), no. 2, 177–185.
49. F. Lehner O. Arizmendi, T. Hasebe, Cyclic boolean and monotone independence, in preparation, 2019.
50. Kamilla Oliver and Helmut Prodinger, The continued fraction expansion of Gauss’ hypergeometric function and a new application to the tangent function, Transactions of the Royal Society of South Africa 67 (2012), no. 3, 151–154.
51. Ioannis Papadimitriou, A simple proof of the formula $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$, Amer. Math. Monthly 80 (1973), 424–425.
52. I. J. Schwatt, An introduction to the operations with series, Second edition, Chelsea Publishing Co., New York, 1962.
53. N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, published electronically at https://oeis.org, 2019.

54. Anthony Sofo, *General order Euler sums with multiple argument*, J. Number Theory **189** (2018), 255–271.

55. Herbert R. Stahl, *Proof of the BMV conjecture*, Acta Math. **211** (2013), no. 2, 255–290.

56. Richard P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997.

57. **______**, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.

58. John R. Stembridge and John Todd, *On a trigonometrical sum*, Linear Algebra Appl. **35** (1981), 287–295.

59. M. Stern, *Über einige Eigenschaften der Funktion Ex*, J. Reine Angew. Math. **59** (1861), 146–162.

60. Frank Szmulowicz, *New analytic and computational formalism for the band structure of n-layer photonic crystals*, Phys. Lett. A **345** (2005), 469–477.

61. Kenneth S. Williams, *On \(\sum_{n=1}^{\infty} (1/n^{2k})\)*, Math. Mag. **44** (1971), 273–276.

62. Kenneth S. Williams and Nan Yue Zhang, *Evaluation of two trigonometric sums*, Math. Slovaca **44** (1994), no. 5, 575–583, Number theory (Račkova dolina, 1993).

63. A. M. Yaglom and I. M. Yaglom, *Challenging mathematical problems with elementary solutions. Vol. II*, Dover Publications, Inc., New York, 1987, Reprint of the 1967 edition.

64. Don Zagier, *Higher dimensional Dedekind sums*, Math. Ann. **202** (1973), 149–172.

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