Finite volume effects and dynamical chiral symmetry breaking in $\text{QED}_3$

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We investigate the impact of finite volume effects on the critical number of flavours, $N_f^c$, for chiral symmetry restoration in $\text{QED}_3$. To this end we solve a set of coupled Dyson-Schwinger equations on a torus. For order parameters such as the anomalous dimension of the fermion wave function or the chiral condensate we find substantial evidence for a large dependence on the volume. We observe a shift in $N_f^c$ from values in the range of $3.61 \leq N_f^c \leq 3.84$ in the infinite volume/continuum limit down to values below $N_f^c \leq 1.5$ at finite volumes in agreement with earlier results of Gusynin and Reenders in a simpler truncation scheme. These findings explain discrepancies in $N_f^c$ between continuum and lattice studies.

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I. INTRODUCTION

Dynamical chiral symmetry breaking in the theory of Quantum Electrodynamics in (2+1) dimensions, $\text{QED}_3$, \cite{1} has been studied quite intensively over the years. The problem is of considerable interest for two reasons. On the one hand, $\text{QED}_3$ has enough similarities to Quantum Chromodynamics (QCD) such that analogies to the more complicated non-Abelian case may be drawn. On the other hand, $\text{QED}_3$ itself is of considerable interest due to possible applications in condensed matter systems.

In particular, $\text{QED}_3$ with $N_f = 2$ flavours of massless fermions has been suggested as an effective low energy theory of high-$T_c$ cuprate superconductors \cite{2,3,4}. These possess an unconventional $d$-wave symmetry of the pairing condensate, with nodes at the electronic Fermi surface allowing for a description in terms of massless nodal quasi-particles. The quasi-particles do not couple to an external electromagnetic field and hence represent pure spin degrees of freedom; a spin-charge separation has taken place. The interaction of these spinons with the collective, topological excitations of the gap can be described by a $U(1)$ gauge theory. Since furthermore the motion of the quasi-particles is mainly confined to the two-dimensional copper-oxygen planes in these systems one ends up with quantum electrodynamics in (2+1) dimensions. Depending on whether the system is ordered or disordered it is either in an insulating quantum antiferromagnetic (AF) phase or a pseudogap (PG) phase with remnant properties of the underlying superconductor. In the gauge theory the AF-phase corresponds to a phase with broken chiral symmetry and long-range correlations due to massless photons. In the PG-phase the fermions are massless and the fermion wave function as well as the photon propagator develop power laws at small momenta with a fractional anomalous dimension \cite{2,3,5}.

The above considerations explain the interest in determining $N_f^c$, the critical number of fermion flavours for the chiral phase transition of $\text{QED}_3$. If $N_f^c > 2$ then the effective low energy theory is chirally broken at zero temperature. For the superconductor this means that the theory displays a phase transition between the superconducting and the antiferromagnetic phase when doping is varied \cite{2,3,6}. If on the other hand $N_f^c < 2$ the system goes from a superconducting to a pseudogap-phase when underdoped.

The value of $N_f^c$ has been investigated in a number of studies from Dyson-Schwinger equations (DSEs) in the continuum \cite{3,4,5,6,7,8,9,10,11,12,13,14} and lattice gauge theory \cite{15,16,17,18,19}. While the former allows for numerical as well as analytical studies in principle there remains the question of the importance of truncation artefacts. For $N_f^c$ a partial answer has been obtained in \cite{6}, where it was found that the details of the fermion-photon vertex only have a minor quantitative impact on $N_f^c$. In particular for all vertex dressings employed $N_f^c$ stayed well above $N_f^c = 2$.

In all studies of dynamical chiral symmetry breaking in continuum $\text{QED}_3$ a clear separation of scales has been found. The intrinsic scale of the theory is given by the dimensionful coupling constant $\alpha = N_f e^2 / 8$. Then a second and much lower scale is given by the dynamically generated fermion mass $M(0)$ in the chiral limit. Related to this one has small values for the other order parameter, the chiral condensate. On the lattice with its finite volume this separation of scales is hard to bridge. Consequently, recent studies for the number of flavours $N_f = 2$ \cite{17} and $N_f = 4$ \cite{17,18} determined bounds on the chiral condensate, but no definite value for $N_f^{\text{crit}}$ could be extracted. A definite signal for chiral symmetry breaking was obtained only for $N_f = 1$ \cite{18}. A very recent calculation claims that $N_f^c \approx 1.5$ \cite{19} with the caveat of volume and discretisation artefacts.

In general, the presence of an infrared cutoff due to the finite volume reduces the value of the critical number of flavours. This has been demonstrated by Gusynin and Reenders in a simple truncation scheme for the DSEs \cite{20}. They considered an approximation to the DSE for the fermion self-energy which neglects corrections to the fermion wave-function and the fermion-photon vertex. The photon is then given by its leading behaviour in the $1/N_f$ expansion. In this work we elaborate on
these findings by considering a more sophisticated truncation scheme which explicitly takes into account non-perturbative effects in all of these quantities. In addition we follow a different strategy to assess the volume effects by evaluating the system on a three-torus\(^1\).

This work is organised as follows. In section II we recall results for QED\(_3\) in the infinite volume/continuum limit. We present the Dyson-Schwinger equations for the fermion and photon propagators and discuss their asymptotic properties. In section III we recall general conditions for chiral symmetry breaking on a finite volume, formulate the DSEs in a box and discuss our numerical methods to solve these. In section IV we present our numerical results for the propagators at finite volume and the critical number of flavours \(N_f\) as a function of the box size. In section V we conclude.

II. QED\(_3\) IN THE INFINITE VOLUME/CONTINUUM LIMIT

A. The Dyson–Schwinger equations in QED\(_3\)

We consider QED\(_3\) with a four-component spinor representation for the Dirac algebra and \(N_f\) fermions. This allows a definition of chiral symmetry similar to the cases of QED\(_4\) and QCD\(_4\). With massless fermions, the Lagrangian has a \(U(2N_f)\) “chiral” symmetry, which is broken to \(SU(N_f) \times SU(N_f) \times U(1) \times U(1)\) if the fermions become massive. The order parameter for this symmetry breaking is the chiral condensate which can be determined e.g. via the fermion propagator.

The Dyson-Schwinger equations for the photon and fermion propagators in Euclidean space are given diagrammatically in Fig. 1. They read explicitly

\[
D^{-1}_{\mu\nu}(p) = D^{-1}_{\mu\nu}(p) \quad (1)
\]

\[
S^{-1}(p) = S^{-1}(p) + Z_1 e^2 \int \frac{d^3q}{(2\pi)^3} \text{Tr} [\gamma_\mu S(q)\Gamma_\nu(q, k) S(k)] ,
\]

\[
D^{-1}_{\mu\nu}(p) = D^{-1}_{\mu\nu}(p) \quad (2)
\]

with the momentum routing \(k_\mu = q_\mu - p_\mu\). Here \(N_f\) denotes the number of fermion flavours and \(Z_1\) is the renormalisation constant of the fermion-photon vertex \(\Gamma_\nu\).

In Landau gauge the general form of the dressed fermion propagator \(S(p)\) and the photon propagator \(D_{\mu\nu}(p)\) is given by

\[
S(p) = \frac{i\not{p} A(p^2) + B(p^2)}{p^2 A^2(p^2) + B^2(p^2)} , \quad (3)
\]

\[
D_{\mu\nu}(p) = \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{G(p^2)}{p^2} , \quad (4)
\]

with the photon dressing function \(G(p^2)\), the photon polarisation \(\Pi(p^2)\) and renormalisation function \(Z_3\) and the fermion dressing functions \(A(p^2)\) and \(B(p^2)\). These can be rearranged into the renormalisation group invariant fermion mass function \(M(p^2) = B(p^2)/A(p^2)\) and the fermion wave function \(Z_f(p^2) = 1/A(p^2)\). Another renormalisation group invariant is the ‘running coupling’ \(e^2 G(p^2)\) built from the renormalisation point dependent photon dressing function \(G(p^2)\) and the renormalised coupling \(e^2\). The bare renormalised fermion propagator is given by \(S^{-1}(p) = Z_2 i\not{p} + Z_m m\) where \(m\) is the renormalised current fermion mass, \(Z_m\) is the mass renormalisation function and \(Z_2\) the corresponding one for the fermion wave function. Note that in QED we have the Ward-Takahashi identity \(Z_1 = Z_2\).

The gauge dependence of the fermion mass and wave function and, correspondingly, the fermion-photon vertex has been a much debated issue in the past, see e.g. 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34 and references therein. There in particular the technical question of how to truncate the fermion-photon interaction to obtain a gauge covariant pattern of chiral symmetry breaking and restoration has been discussed. This issue is not quite settled yet, however there are clear indications 22, 32, 34 that Landau gauge is preferred in the sense that it allows for particularly simple approximation schemes. This is why we prefer Landau gauge in this work.

A range of ansätze for the fermion-photon vertex have been investigated in \(\mathcal{E}\). There it has been found

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\(^1\) A corresponding technique has been applied in QCD\(_4\) to determine finite volume effects in the quark and gluon propagators 21, 22, 23

\(^2\) In all other linear covariant gauges parametrised by the gauge parameter \(\xi\) a gauge dependent scale \(\frac{e^2}{\xi}\) appears which complicates matters considerably.
that the critical number of flavours $N_f$ obtained with the most elaborate construction, obeying the Ward-Takahashi identity, is almost similar to the one of the most simple ansatz, a bare renormalised vertex

$$\Gamma_{\nu} = Z_1 \gamma_{\nu}. \quad (5)$$

Therefore to keep matters as simple as possible we will only present results for the bare vertex approximation in this work. We did check, however, that more sophisticated ansätze do not alter the main conclusions presented below. In addition, a general analysis of the infrared behaviour of QED$_3$ in the framework of functional renormalisation group equations indicates that in Landau gauge the bare vertex may even be the best possible choice.

Substituting this vertex into the fermion and photon DSEs, taking appropriate traces and contracting the photon DSE with the projector

$$P_{\mu \nu}(p) = \delta_{\mu \nu} - \zeta \frac{p_{\mu} p_{\nu}}{p^2} \quad (6)$$

with $\zeta = 1$ we arrive at

$$B(p^2) = Z_2 Z_m m + Z_3^2 e^2 \int \frac{d^3q}{(2\pi)^3} \frac{2B(q^2)}{q^2 A^2(q^2) + B^2(q^2)} \frac{G(k^2)}{k^2} \quad (7)$$

$$A(p^2) = Z_2 + Z_2^2 e^2 \int \frac{d^3q}{(2\pi)^3} \frac{2A(q^2)}{q^2 A^2(q^2) + B^2(q^2)} \frac{G(k^2)}{k^2} \left( \frac{k^2}{2p^2} + \frac{(p^2 - q^2)^2}{2k^2 p^2} \right) \quad (8)$$

$$\frac{1}{G(p^2)} = Z_3 + \Pi(p^2) = Z_3 - Z_3^2 e^2 N_f \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2 A^2(q^2) + B^2(q^2)} \frac{1}{k^2 A^2(k^2) + B^2(k^2)} \tilde{W}_1(p^2, q^2, k^2) \quad (9)$$

where we used $Z_1 = Z_2$ and the subtracted kernel

$$\tilde{W}_1(p^2, q^2, k^2) = W_1(p^2, q^2, k^2) - \frac{2k^2 (3 - \zeta)}{3p^2}, \quad (10)$$

with

$$W_1(p^2, q^2, k^2) = \frac{\zeta k^4}{p^2} + k^2 \left( \frac{1 - \zeta}{p^2} - \frac{2(q^2)}{p^4} \right) - 1 \quad (11)$$

in the photon equation. As explained in appendix A of ref. [5] the subtraction of the term proportional to $(3 - \zeta)$ in Eq. (10) is necessary to avoid spurious linear divergences in the photon-DSE generated by the regularisation procedure (a hard cutoff) used in the numerical treatment of the equations. The choice $\zeta = 1$, i.e. the transverse projection of the photon equation, is mandatory to avoid the back-reaction of spurious longitudinal terms into the right hand side of the photon equation. Compared to [5], where $\zeta = 3$ has been used, this treatment leads to a quantitatively improved value for the critical number of flavours, see below. Note, however, that the effects of varying $\zeta$ are quantitatively small in general; all qualitative conclusions derived in [5] and also here are independent of the choice of $\zeta$.

### B. Asymptotic behaviour of the propagators

An often used approximation to determine the asymptotic behaviour of the fermion and photon dressing functions has been the $1/N_f$-expansion. This expansion is equivalent to a perturbative expansion for small $e^2$ while keeping $\alpha = N_f e^2 / 8$ fixed. As QED$_3$ is an asymptotically free theory this expansion does provide correct answers in the ultraviolet. For the photon polarisation and the vector dressing function of the fermion one finds for $N_f$ massless fermion flavours [6]

$$\Pi(p^2 \gg \alpha) = \frac{N_f e^2}{8p} = \frac{\alpha}{p}, \quad (12)$$

$$A(p^2 \gg \alpha) = 1 \quad (13)$$

For the mass function of the fermion one can use the operator product expansion to obtain

$$M(p^2 \gg \alpha) = \frac{2 + \xi}{4} \frac{\langle \bar{\Psi} \Psi \rangle}{p^2}, \quad (14)$$

with the chiral condensate $\langle \bar{\Psi} \Psi \rangle$. Note that the condensate can also be determined from the trace of the fermion propagator:

$$\langle \bar{\Psi} \Psi \rangle = -4Z_2 \int \frac{d^3q}{(2\pi)^3} \frac{B(q^2)}{q^2 A^2(q^2) + B^2(q^2)} \quad (15)$$

In the infrared momentum region the $1/N_f$-expansion is clearly not sufficient. Here one has to resort to a self-consistent analysis of the DSEs in terms of asymptotic power-laws. This method is well developed in QCD$_4$ [32, 36, 37, 38] and has been adapted to QED$_3$ in Ref. [5]. Such an analysis is valid if no scales are present, i.e. in the deep infrared momentum region $p \ll \alpha$ and only in the absence of fermion masses. The appearance of infrared power laws with potentially fractional anomalous
dimensions is thus an indicative and characteristic property of the symmetric phase of QED$_3$.

For a bare fermion-photon vertex the infrared behaviour of the vector fermion dressing function and the photon polarisation in the chirally symmetric phase can be written as

$$A(p^2) \sim p^{2\kappa}, \quad \Pi(p^2) \sim p^{-1-4\kappa}. \quad (16)$$

In case of a dressed fermion-photon vertex according to the Ward-Takahashi identity the corresponding power-laws are

$$A(p^2) \sim p^{2\kappa}, \quad \Pi(p^2) \sim p^{-1-2\kappa}. \quad (17)$$

These expressions solve the DSEs in the chirally symmetric phase as described in detail in Ref. [5] (see also Ref. [32] for a re-derivation from a slightly different perspective).

The value of $\kappa$ as a function of $N_f$ can then be determined from the DSEs using a procedure given in detail in Ref. [5]. The solutions for a bare vertex with and without the subtracted term in Eq. (10) is plotted in Fig. 2. For $N_f > 1.5$ the exact solutions can be represented by the fits

$$\kappa_{\text{sub}} = \frac{0.142}{N_f} + \frac{0.002}{N_f^2} + O(1/N_f^3), \quad (18)$$

$$\kappa_{\text{bare}} = \frac{0.137}{N_f} + \frac{0.058}{N_f^2} + O(1/N_f^3). \quad (19)$$

A corresponding fit for the bare vertex and a Ward-Identity improved vertex for $\zeta = 3$ is given in Ref. [5]. The difference between (18) and (19) represents the systematic uncertainty in our numerical calculation due to the cutoff regularisation procedure used. For dimensional regularisation the subtraction of the term in (10) is not necessary and therefore the unsubtracted result Eq. (19) should be viewed as the exact one for the bare vertex truncation scheme. In principle one can reproduce (19) also in a numerical treatment of the DSEs with a hard cutoff when vertex corrections around the cutoff scale are included, see Ref. [33] for details. However, this procedure is much more involved than the simple subtraction scheme of Eq. (10). Since the difference between (18) and (19) is rather small in the interesting region above $N_f = 2$ and therefore unimportant for all of the conclusions of the present work we resort to the simple subtraction scheme (10) and consequently reproduce (18) in our numerics.

The critical number of flavours $N_f^c$ where chiral symmetry is restored can be determined analytically from the DSE for the scalar fermion dressing function $B(p^2)$. Again we refer the reader for details to Ref. [5] and merely state the result

$$(N_f^c)_{\text{bare}} \approx 3.84, \quad (N_f^c)_{\text{sub}} \approx 3.61. \quad (20)$$

The corresponding result from the functional renormalisation group is $(N_f^c)_{\text{bare}} \approx 3.64$ in agreement with our result for the unsubtracted equation. Numerical results for Ward-identity improved vertices lead to results in the range 3.5–4 [5]. These numbers can be contrasted with $(N_f^c)_{1/N_f = 32/\pi^2} \approx 3.24$ from the $1/N_f$-expansion [7]. Note that all these results are far above $N_f = 2$ relevant for the description of high-$T_c$ cuprate superconductors as discussed in the introduction.

Finally we wish to emphasise that these power law solutions can only be obtained when the full structure of the propagator DSEs is taken into account. The authors of Refs. [8, 10] did not find a solution corresponding to a symmetric phase in their truncation scheme because the feedback from the function $A$ onto the vacuum polarisation is not taken into account. This then prohibits the appearance of power laws and therefore does not allow for the appearance of the chirally symmetric phase and should be discarded [5, 32].

III. QED$_3$ AT FINITE VOLUMES

A. Chiral symmetry breaking in a box

Before we embark on our investigation, let us recall the general finite volume behaviour of the chiral condensate [35]. The fermion propagator in its spectral representation is given by

$$S_A(x, y) = \sum_n \frac{u_n(x) u_n^\dagger(y)}{m - i\lambda_n}, \quad (22)$$

where $u_n(x)$ and $\lambda_n$ are eigenfunctions and eigenvalues of the Euclidean Dirac operator, $D u_n(x) = \lambda_n u_n(x)$. The gauge field $A$ is treated as an external field. These eigenfunctions occur either as zero modes or in pairs of opposite eigenvalues. Setting $x = y$, integrating over $x$
and neglecting the zero mode contributions, one obtains
\[
\frac{1}{V} \int_{V} S_{A}(x, x) = -\frac{2m}{V} \sum_{\lambda_{n} > 0} \frac{1}{m^{2} + \lambda^{2}_{n}}. \quad (23)
\]

The chiral condensate can be deduced by averaging the left hand side of this equation over all gauge field configurations and then taking the infinite volume limit to give
\[
\langle \bar{q}q \rangle = -2m \int_{0}^{\infty} d\lambda \frac{\rho(\lambda)}{m^{2} + \lambda^{2}}, \quad (24)
\]

where \(\rho(\lambda)\) is the mean level density of the spectrum, which becomes dense in the infinite volume limit. In the chiral limit, \(m \to 0\), only the infrared part of the spectrum contributes and one finally arrives at the Banks-Casher relation \([40]\)
\[
\langle \bar{q}q \rangle = -\pi \rho(0). \quad (25)
\]

If the two limits are interchanged, \(i.e.\) if one takes the chiral limit before the infinite volume limit, one has a discrete sum in Eq. (23) and the infrared part of the spectrum cannot trigger a non-vanishing chiral condensate: chiral symmetry is restored. If, however, at a given volume the explicit fermion mass \(m\) is not too small, one can still observe the spontaneous formation of a quark condensate. If the factor \((m^{2} + \lambda^{2}_{n})^{-1}\) varies only slightly with \(n\), the sum in Eq. (23) can still be replaced by an integral and Eq. (25) remains valid. For this to be a legitimate approximation one needs \(m \gg \Delta \lambda \sim 1/V \rho(\lambda) = \pi/(V \langle \bar{q}q \rangle)\), at the lower end of the spectrum. Thus one obtains the condition \([39]\)
\[
V m \langle \bar{q}q \rangle \gg \pi. \quad (26)
\]

This relation reveals the crux of the matter. In principle, if the volume is large enough, the necessary quark masses are academically small and may even be neglected in the numerical treatment. What counts as \(large\) in this context, however, depends sensitively on the size of the chiral condensate. In QED, the condensate becomes extremely small already well below the critical number of flavours \(N_{f}^{c}\) of the chiral phase transition \([2]\). Thus although a formulation on a finite volume may do well for \(N_{f} = 1\), all signals of dynamical chiral symmetry breaking will be lost already well below the \(N_{f}^{c}\) of the theory in the infinite volume/continuum limit. This behaviour will be quantified below.

**B. The DSEs on a torus**

On a compact manifold, the photon and fermion fields have to obey appropriate boundary conditions in the time direction. These have to be periodic for the photon fields and antiperiodic for the fermions. For computational reasons it is highly advantageous, though not necessary, to choose the same conditions in the spatial directions. We choose the box to be of equal length in all directions, \(L_{1} = L_{2} = L_{3} \equiv L\), and denote the corresponding volume \(V = L^{3}\). Together with the boundary conditions this leads to discretised momenta in momentum space. Thus all momentum integrals appearing in the Dyson-Schwinger equations are replaced by sums over Matsubara modes.

On a torus with antiperiodic boundary conditions for the fermion fields, the momentum integral changes into a sum of Matsubara modes,
\[
\int \frac{d^{3}q}{(2\pi)^{3}} \langle \cdot \cdot \cdot \rangle \longrightarrow \frac{1}{L^{3}} \sum_{n_{1}, n_{2}, n_{3}} \langle \cdot \cdot \cdot \rangle, \quad (27)
\]

counting momenta \(\mathbf{q}_{\text{m}} = \sum_{i=1,3} (2\pi/L)(n_{i} + 1/2)\hat{e}_{i}\), where \(\hat{e}_{i}\) are Cartesian unit vectors in Euclidean momentum space. For the photon with periodic boundary conditions the momentum counting goes like \(\mathbf{q}_{\text{m}} = \sum_{i=1,3} (2\pi/L)(n_{i})\hat{e}_{i}\). For the numerical treatment of the equations it is convenient to rearrange these summations such that they represent a spherical coordinate system \([21]\), see Fig. 3 for an illustration. We then write
\[
\frac{1}{L^{3}} \sum_{n_{1}, n_{2}, n_{3}} \langle \cdot \cdot \cdot \rangle = \frac{1}{L^{3}} \sum_{j, m} \langle \cdot \cdot \cdot \rangle, \quad (28)
\]

where \(j\) counts spheres with \(\mathbf{q}_{\text{m}} = \text{const. and } m\) numbers the grid points on a given sphere. The corresponding momentum vectors are denoted \(\mathbf{q}_{m,j}\) and their absolute values are given by \(q_{m,j} = |\mathbf{q}_{m,j}|\). It is then a simple matter to introduce the torus equivalent of an \(O(3)-\text{invariant cut-off by restricting } j\) to an interval \([1, N]\). This proce-
The resulting DSEs are then given by

\[ B(p^2_{i,l}) = Z_3 Z_{m,m} + Z_2 \frac{e^2}{L^3} \sum_{j,m}^N q_{j,m}^2 A^2(q_{j,m}^2) + B^2(q_{j,m}^2) \frac{G(k_{i,l,j,m}^2)}{k_{i,l,j,m}^2} \]

\[ A(p^2_{i,l}) = Z_2 + Z_2 \frac{e^2}{L^3} \sum_{j,m}^N q_{j,m}^2 A^2(q_{j,m}^2) + B^2(q_{j,m}^2) \frac{G(k_{i,l,j,m}^2)}{k_{i,l,j,m}^2} \left( -\frac{k_{i,l,m}^2}{2p_{i,l}^2} + \frac{(p_{i,l}^2 - q_{j,m}^2)^2}{2k_{i,l,m}^2 p_{i,l}^2} \right) \]

\[ \frac{1}{G(p^2_{i,l})} = Z_3 - Z_2 \frac{e^2 N_f}{L^3} \sum_{j,m}^N q_{j,m}^2 A^2(q_{j,m}^2) + B^2(q_{j,m}^2) \frac{1}{k_{i,l,j,m}^2} \]

\[ \frac{1}{G(p^2_{i,l})} = Z_3 - Z_2 \frac{e^2 N_f}{L^3} \sum_{j,m}^N q_{j,m}^2 A^2(q_{j,m}^2) + B^2(q_{j,m}^2) \frac{1}{k_{i,l,j,m}^2} \]

\[ \frac{1}{G(p^2_{i,l})} = Z_3 - Z_2 \frac{e^2 N_f}{L^3} \sum_{j,m}^N q_{j,m}^2 A^2(q_{j,m}^2) + B^2(q_{j,m}^2) \frac{1}{k_{i,l,j,m}^2} \]

Note that the momentum argument \( k \) of the photon self-energy in the fermion DSE is the difference \( k_{i,l,j,q} = p_{i,l} - q_{j,q} \) of two antiperiodic Matsubara momenta and thus lives on a momentum grid corresponding to periodic boundary conditions, as it should.

The DSEs can be solved numerically employing well established methods. Our numerical method on the torus is outlined in Ref. [22], the corresponding continuum method as well as details on the renormalisation procedure of the DSEs are given in Ref. [8].

Note that the propagators determined from the continuum version of the DSEs, Eqs. (7–9) are independent of the regularisation procedure. In our numerical calculations in the infinite volume/continuum limit we use a subtracted version of these equations and an O(3)-invariant UV-cutoff which can be sent to infinity at the end of each calculation. These DSEs therefore represent not only the infinite volume limit but also the continuum limit (in coordinate space) of the representation given by Eqs. (7–9) of the DSEs on a torus. We use the phrase infinite volume/continuum limit to indicate this simultaneous removal of both an ultraviolet and an infrared cutoff.

### IV. NUMERICAL RESULTS

#### A. Finite size effects: fixing the UV cutoff

Before we investigate the finite volume effects of various quantities of interest we have to clarify whether there are sizeable effects due to the finite size of the system corresponding to a fixed ultraviolet momentum cutoff. Part of these effects are removed by our choice of ‘cutting the edges’ of the torus, described in the previous section. The remaining finite size effects can be evaluated by varying the size of the cutoff. Here, a natural minimal cutoff is given by the intrinsic scale \( \alpha = N_f e^2 / 8 \) of QED3. Thus, working in the range \( 0 < N_f \leq 8 \), we anticipate that finite size effects are of minor importance for cutoffs larger than the intrinsic scale, i.e. \( \Lambda^2 \geq e^4 \). Indeed, this is the case as can be seen from the two plots in Fig. 4. Shown are the fermion mass function \( M(p^2) \) and wave function \( Z_f(p^2) \) as a function of scaled momentum \( p^2 / e^4 \) for various cutoffs between \( \Lambda_{UV}^2 = 0.15 e^4 \) and \( \Lambda_{UV}^2 = 39.5 e^4 \). The number of fermion flavours is chosen to be \( N_f = 1 \), similar results are obtained for other choices. The box volume for the fermions is related to the lowest momentum point \( p_{min}^2 \) by \( V = L^3 \) with \( L = \sqrt{3} \pi / p_{min} \). Here we use \( L = 390 / e^2 \). Note that for the photon the box volume is related to the lowest momentum point by \( V = L^3 \) with \( L = 2 \pi / p_{min} \). The same box length then results in a slightly different value for the lowest momentum point in the photon dressing function. We refrain from showing the photon explicitly here, since the finite size effects are similar to the ones for the fermions.

In Fig. 4 the finite size effects are visible only for the smallest cutoff. There are effects in both the infrared and ultraviolet momentum regions, where a number of momentum points deviate from the results with larger cutoffs. This is true for the renormalisation point dependent fermion wave function, normalised such that they match the continuum results, but also for the renormalisation point independent fermion mass function. The perhaps surprising observation that the variation of an ultraviolet cutoff also affects the infrared behaviour of the dressing functions implies a certain entanglement between the infrared and ultraviolet modes of a gauge theory. Similar effects have been found in four dimensional Yang-Mills theory, see e.g. [33] and references therein.

We also need to comment on the fact that we observe dynamical chiral symmetry breaking in our system at all, despite in practice working with a vanishing bare quark mass. In the continuum formulation this entails working in the chiral limit. However, due to the formal reasons discussed in section III.A this cannot be true on a finite volume, since taking the chiral limit before the infinite volume limit inevitably leads to the loss of dynamical chiral symmetry breaking. However, this is not what
we observe here. Indeed, the volumes we use are large enough to allow for extremely small bare fermion masses according to the relation of Eq. (26). In practice, this allows one to neglect the fermion mass entirely in the numerical treatment of the DSEs. However it is important to keep in mind that this means we are working close to the chiral limit, but not in the chiral limit.

We conclude this subsection with the observation that from about $\Lambda_{UV}^2 = e^4$ onwards almost no finite size effects are present. We therefore consider this value a lower bound for admissible cutoffs on a torus, in agreement with our general considerations concerning the natural scale in QED$_3$. This finding also agrees with corresponding results on finite size effects in lattice simulations, see [19] and Refs. therein. In order to be absolutely sure that cutoff effects do not play any role in what follows we use the somewhat larger cutoff $\Lambda_{UV}^2 = 2.35 e^4$ from now on.
B. Finite volume effects: towards the infinite volume limit

We are now in a position to study the finite volume effects occurring for the fermion and photon propagators on a torus. To this end we keep the ultraviolet cutoff of our system fixed and vary the infrared cutoff in the range of $L e^2 = 200 - 3000$. We show the resulting behaviour of the fermion mass and wave function as well as the photon together with the corresponding continuum results in Fig. 5. Again, we choose $N_f = 1$. The variation of the volume clearly results in the loss of a substantial amount of generated fermion mass when the volume gets smaller and smaller. For even smaller volumes than shown in the figure chiral symmetry is restored in agreement with the condition $V m|\langle \bar{q}q \rangle| \gg \pi$ discussed in the subsections III A and IV A. On the other hand, we observe that extremely large volumes are needed to account for the full effect of dynamical chiral symmetry breaking observed in the infinite volume/continuum limit. This is in marked contrast with the behaviour of the quark sector of QCD$_4$ and can be explained by a closer look at the scales involved. Whereas in QCD$_4$ the generated quark masses ($M(0) \approx 300 - 400$ MeV) are of the same order as the intrinsic scale of the system ($\Lambda_{QCD}^{MS} \approx 250$ MeV), the situation is clearly different in QED$_3$. Here our characteristic scale is of the order $\alpha = e^2/8$, whereas the generated fermion masses are of order $10^{-2} e^2$, as can be seen from the plot.

To keep volume effects small, this scale has to be well accommodated by the system on a box, which translates to a lowest momentum to be much smaller than $p^2 = 10^{-5} e^4$. Indeed, this is what we observe. Choosing the volume large enough that the lowest momentum is well below this scale we approach the infinite volume/continuum limit. This is true for all three dressing functions, the fermion mass function $M(p^2)$, the fermion wave function $Z_f(p^2)$ and the photon self energy $\Pi(p^2)$.

C. Finite volume effects: $Z_f(N_f)$ and $\kappa(N_f)$

Let us now examine the influence of finite volume effects on the critical value $N_f^c$, where the system undergoes a phase transition from the chirally broken into the chirally symmetric phase. This phase transition is marked by the change of the infrared asymptotics of the fermion wave function $Z_f(p^2)$: for $N_f < N_f^c$ this function is a constant in the infrared, whereas for $N_f > N_f^c$ it develops a power law with $N_f$-dependent exponent $\kappa$. In a sense, $\kappa$ can be viewed as an order parameter of this phase transition. We exhibit this behaviour of $\kappa$ by fitting a power law in the infrared to the fermion wave function

$$Z_f(p^2) = C (p^2)^{−\kappa}.$$  \hspace{1cm} (32)

which is related to the fermion vector dressing function by $Z_f = 1/A$. Both, the coefficient $C$ and the power $\kappa$ may depend on $N_f$.

To visualize this procedure the fermion wave function $Z$ is shown in the upper panel of Fig. 6 for a box with $L e^2 \approx 800$ for different numbers of flavours $N_f$. One clearly sees the aforementioned behaviour: for $N_f \leq 2.0$ the function develops a constant in the infrared region (i.e. $\kappa = 0$), whose value is proportional to $N_f$. The phase transition occurs for this specific volume for $2.0 < N_f < 2.2$. For $N_f \geq 2.2$ one observes a power law of the wave function at small momenta. The coefficient of this power law is still proportional to $N_f$, however the exponent $\kappa$ decreases with growing $N_f$ according to Eq. (18). Consequently we observe a decrease

\footnote{Note, however, that the phase transition is not a first or second order transition but has similar properties to the conformal transition of QED in four dimensions [5, 12].}
of the function $Z_f(p^2)$ with growing $N_f$ in the symmetric phase. As a result one could determine $N_f^c$ as the number of flavours for which $Z_f$ is maximal at a given infrared momentum. This is also the case in the infinite volume/continuum limit [5].

As an (equivalent) alternative we determine $N_f^c$ by fitting the power law (32) to our numerical results for $Z_f(p^2)$ in the infrared momentum region. There is a caveat here: similar to four-dimensional Yang-Mills theory one observes that the power law (32) can only be seen for momenta $1/L \ll p \ll e^2$. This behaviour is generic on a torus [23]. To obtain significant results for our infrared coefficients we therefore have to perform the fit for momentum points significantly larger than the lowest one. In practice we chose momenta from the fourth point onwards.

The resulting dependence of the exponent $\kappa$ on $N_f$ is plotted in the lower panel of Fig. 4 for several volumes of the box together with the corresponding function in the infinite volume/continuum limit. At infinite volume the function $\kappa(N_f)$ is zero for $N_f < N_f^c \approx 3.84$ and equal to the analytic result of Eq. (20) in the symmetric phase above $N_f^c$. For a given finite volume the phase transition is still indicated by a maximum in $\kappa(N_f)$. However, there is an additional region at $N_f < N_f^c(L)$, where the function rises slowly towards its maximal value. This region is generated by the finite infrared cutoff of the system in a box, which prevents the fermion wave function $Z_f(p^2)$ from bending towards a finite asymptotic value at $p^2 = 0$. This effect mimics a power law at values of $N_f$ where the system is still in the chirally broken phase. With increasing volume this region gets smaller and smaller until it reaches the sharp rise of the function in the infinite volume/continuum limit as shown in Fig. 6.

As a result we find a critical number of flavours which depends upon the volume of the torus. The explicit values are shown in Fig. 7 together with corresponding results from Ref. [20]. In the following we concentrate on the solutions close to the chiral limit and postpone the discussion of the (orange) curve with large bare fermion mass to the next subsection.

For volumes that are not too large, our solutions and the ones reported in Ref. [20] can be well fitted by a form

$$N_f^c = a - b/(Le^2)^{1/3}.$$  \hspace{1cm} (33)

For our results we obtain $a = 3.23$ and $b = 10.64$, shown as dashed line in the plot. Interestingly, to high precision the exponent of the box length in this fit is given by $-1/3$, although we do not have a good explanation for exactly this behaviour. In terms of $(1/L)^{1/3}$ the fit then suggests a linear extrapolation to the infinite volume limit. However, it turns out that this linear behaviour breaks down for extremely large volumes. This is evident for the results from Gusynin and Reenders [20]. The deviation from the linear behaviour is quantitatively important: for the truncation of Gusynin and Reenders a linear extrapolation to the infinite volume results in $N_f^c \approx 2.52$ as compared to the infinite volume result $N_f^c = 3.2$. (Note that the curve gets extremely steep for the very largest volumes, which are not plotted.) For our results we find a value of $N_f^c \approx 3.23$ for the linear extrapolation, whereas the analytical infinite volume result is $N_f^c \approx 3.84$. Though this difference is not huge, it highlights the need for non-linear extrapolation procedures to the infinite volume limit. Unfortunately our largest volumes are not yet large enough to penetrate this region of nonlinearity.

D. Non-vanishing bare fermion masses

In this subsection we investigate the consequences of explicit bare fermion masses $m_0$ in the Lagrangian of QED$_3$ on the chiral phase transition. To this end we first determined the properties of the system in the infinite volume/continuum limit by solving the corresponding DSEs for a range of different bare fermion masses. For any given mass and varying $N_f$ we find that the fermion wave function never develops a pure power law in the infrared. Instead there is a region $0 < p < \Lambda_{IR}$, where the function is constant and a region $\Lambda_{IR} < p < \alpha$ where a power law is present. The scale $\Lambda_{IR}$ depends on the explicit fermion mass as well as on the number of flavours. We conclude from this that away from the chiral limit the infrared exponent $\kappa$ is no longer an order parameter. As concerns the global behaviour of $Z_f(p^2)$ we still observe the behaviour discussed above Eq. (33): at a small enough momentum $p^2$ the value of the function $Z_f(p^2)$ increases with $N_f$ up to a certain point at a criti-
direction and periodic ones in the two spatial directions. This results in $L_{\text{dett}} = \pi/p_{\text{min}}$ on the lattice. The couplings are related by $e^2 \approx 1/(\beta a)$, where $\beta$ is the dimensionless inverse coupling on the lattice and $a$ the lattice spacing. With $e^2 \approx 1$ and $\beta \sim \mathcal{O}(1)$ one then obtains $L_{\text{DSE}} e^2 \approx \sqrt{3} L_{\text{dett}} \beta$. This means that we should compare the results of contemporary lattice calculations on $8^3$-lattices with our values for $L \approx 140$. From the plot of Fig. 7 we then find chiral symmetry breaking for $N_f = 1$, whereas at $N_f = 1.5$ the system is in the chiral symmetric phase. This is in agreement with recent lattice results \[19\] and shows that these results are compatible with our value of $N_f^c \approx 3.8$ in the infinite volume/continuum limit.

Finally we investigate the behaviour of the chiral condensate on a torus as a function of the explicit fermion mass. To this end we extract the condensate from our fermion mass function with the help of Eq. (15) at a given fixed ultraviolet momentum cutoff. Our results for $N_f = 1.5$ and different box lengths $L$ are shown in Fig. 8. Clearly, for small volumes the condensate decreases linearly with decreasing bare fermion mass $m_0$ and extrapolates to zero in the chiral limit. For large enough boxes this behaviour changes and we find a finite value in this limit in agreement with results from our continuum DSEs. It is interesting to note that the curves for small volumes are almost degenerate. A volume analysis in this region would therefore indicate weak finite volume effects where in fact there are large effects when the volume is increase further. Again, this result should serve as a caveat for the interpretation of lattice results.

V. SUMMARY

In this work we investigated volume effects on the chiral phase transition of QED$^3$ as a function of the number of flavours, $N_f$. To this end we solved a coupled system of Dyson-Schwinger equations for the fermion and photon dressing functions in the infinite volume/continuum limit and on a three-torus. We worked in a truncation scheme that in the infinite volume/continuum limit reproduces a critical number of flavours $N_f^c \approx 3.61 - 3.84$, a number close to the one obtained with more involved approximation schemes \[2\].

Examining the same system on a torus we found considerable volume effects. These can be explained due to the presence of scales of vastly different magnitude in QED$^3$. On the one hand one has the natural scale $\alpha/e^2 = N_f/8$, which is of order one. On the other hand, the dynamically generated fermion masses are orders of magnitude smaller. For example, one has $M(0) = 0.05 e^2$, for $N_f = 1$ as can be seen from Fig. 5. While the first scale remains of the same order, the second one rapidly decreases when the number of flavours becomes larger. When the generated fermion mass drops below the infrared cutoff of the system, given by the inverse of the box length, chiral symmetry breaking disappears and the
system falls into the chirally symmetric phase. We quantified these effects and presented results for the critical number of flavours as a function of the box length. They agree qualitatively with corresponding results of [20], determined in a simpler truncation scheme and a different set-up (continuum limit with finite infrared cutoff). Quantitative differences are small.

Our results confirm the notion that lattice calculations at $N_f = 1.5$ or $N_f = 2$ need very large physical volumes to see dynamical chiral symmetry breaking, let alone quantify their finite volume effects. Contemporary lattice results cannot yet accommodate for these and consequently find a system in the chirally symmetric phase in quantitative agreement with our findings. We have shown that these results are nicely compatible with a critical number of flavours $N_f \approx 3.61 - 3.84$ in the infinite volume/continuum limit.

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