Interference Between a Large Number of Independent Bose-Einstein Condensates

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Abstract

We study theoretically the interference patterns produced by the overlap of an array of Bose-Einstein condensates that have no phase coherence among them. We show that density-density correlations at different quasimomenta, which play an important role in two-condensate interference, become negligible for large $N$, where $N$ is the number of overlapping condensates. In order to understand the physics of this phenomenon, it is sufficient to consider the periodicity of the lattice and the statistical probability distribution of a random-walk problem. The average visibility of such interference patterns decreases as $N^{-1/2}$ for large $N$. 
Since the first experimental realizations of Bose-Einstein condensates (BEC) in the alkali-atomic gases [1, 2, 3], one of the first goals was to demonstrate the coherence of the matter waves in a BEC. That goal was first achieved in the classic interference experiment by Andrews et. al. [4]. In the following years, interference effects proved to be a valuable experimental tool in the study of cold atomic gases [5, 6, 7, 8, 9]. For example, they were used to demonstrate phase coherence within a single condensate [6], study squeezed states of a BEC in a double-well potential [7], identify the phase transition between the superfluid and Mott-insulator phases of a Bose gas in an optical lattice [8], and observe the dynamics of matter wave fields [5, 9]. With the increased interest in atomic gases trapped in optical lattices, interference experiments will undoubtedly provide a useful tool to probe the coherence properties of these systems. It is therefore important to understand the physics of interference patterns produced following the release and expansion of such atomic gases.

The phenomenon of interference between BEC’s manifests itself as the observation of a periodic modulation of the density caused by the overlap of two or more condensates. When two condensates with an equal number of atoms are made to overlap, such a density modulation occurs in every run of the experiment with 100% visibility, i.e. the density is ideally the square of a sinusoidal function, up to an envelope function resulting from the spatial finiteness of the system [4]. Therefore, one can say that there exist strong correlations that dictate that the density must vanish at the midpoint between two successive maxima. If one starts with two independent condensates, i.e. if the relative phase between the two condensates is initially unknown, the location of interference maxima takes a random value, keeping the distance between successive maxima fixed by the length and time scales in the experimental setup [4, 10]. Note that in the context of interference experiments, the term “independent condensates” describes both a statistical mixture of states with well-defined but unknown relative phases and Fock states, where one starts with well-defined values of the atom numbers in each condensate. In the latter case, the location of the emerging interference maxima takes a random value, just like any quantum variable whose wave function collapses at the time of measurement.

If three condensates are made to overlap, one must consider two relative phases, which can be taken as the relative phase between condensates 1 and 2 and that between condensates 2 and 3. It is therefore intuitive to think that the overlap of three condensates will still produce interference fringes, but with reduced visibility due to the randomness of the two degrees of freedom. Similarly, one would expect the interference between a larger number of condensates to produce interference fringes with a value of visibility that decreases and eventually vanishes with increasing number of condensates. In a recent experiment, Hadzibabic et. al. studied the interference of about 30 condensates, and they observed density modulations with 34% average visibility [11]. At first sight, that result might seem to contradict the above-presented intuitive guess. We shall demonstrate that there is no contradiction between the experiment and our intuition. Our intuition might fail, however, in predicting how quickly the visibility decreases with increasing number of condensates.

We shall consider the experimental situation studied in Ref. [11]. In that experiment, a BEC is loaded into a one-dimensional optical lattice such that a large number of lattice sites are filled with atoms (Note that the lattice sites are not equally-populated, but each one contains a large number of atoms [12]). With the choice of parameters in their setup, apart from determining the occupation of the different lattice sites, interatomic interactions...
can be neglected throughout the experiment. Furthermore the height of the optical lattice potential is substantially larger than the kinetic energy of an atom in the ground state of a given lattice site. Therefore, we find that the (single-particle) wave function describing the condensate is well-approximated by a sum of Gaussians centred at the different potential minima. Note that all the Gaussians have the same width, namely that given by the harmonic oscillator length near the potential minima. The trapping potential is then turned off, and the cloud expands ballistically. After a long expansion time, a picture is taken of the density distribution. In Ref. [11] the experiment was analysed theoretically by taking the initial wave function and propagating it in time to obtain numerically the density distribution after the expansion. In this Paper we shall try to identify the origin of the different physical aspects of the experiment analytically, and we verify that identification with some numerical calculations. Unlike the analysis of Ref. [11], in order to avoid complications that do not have any qualitative effect on the behaviour of main interest to us, we shall not worry about the finite resolution of the imaging device. Such effects can be taken into account relatively easily.

One can gain a good amount of insight into the physics of the problem by considering the different relevant length scales and what they correspond to before and after the expansion of the atomic cloud (here we assume that the expansion lasts long enough so that any structure seen in the final density distribution is larger than the original size of the cloud before the expansion). The relation between each length scale $x$ before the expansion and the length scale it produces after the expansion $X$ is given by $X \sim \sqrt{\frac{\hbar}{m}}x$. The size of the wave function inside each well in the optical lattice $\xi$ gives the overall size of the cloud after the expansion $\Xi \sim \sqrt{\frac{\hbar}{m}}\xi$. That latter length scale gives the Gaussian envelope of the density distribution after the expansion. The distance between lattice sites $d$ gives the period of the density modulation inside the Gaussian envelope $D \sim \sqrt{\frac{\hbar}{md}}$. The size of the entire cloud before the expansion $l$ gives the length scale of structure in the periodic function $L \sim \sqrt{\frac{\hbar}{ml}}$, i.e. changes in the density after the expansion can only occur on length scales of the order of or larger than $L$. In the following paragraphs we shall revisit the above considerations more quantitatively and develop a simple framework for thinking about the problem at hand.

First, we look at the wave function before the expansion. As explained above, it can be approximated by:

$$\psi(x) = \sum_j \alpha_j e^{i\theta_j} e^{-(x-jd)^2/2\xi^2},$$

(1)

where $\alpha_j e^{i\theta_j}$ are the amplitudes of the wave function at the different lattice sites, labelled by $j$ ($j=1,2,...,N$, where $N$ is the number of overlapping condensates). The Fourier transform of that wave function can be calculated straightforwardly to give the wave function in momentum space:

$$\Phi(p) = \frac{1}{\sqrt{2\pi}h} \int_{-\infty}^{\infty} e^{-ipx/h} \psi(x) dx = \frac{\xi}{\sqrt{h}} \left( \sum_{j=1}^{N} \alpha_j e^{i\theta_j} e^{-ipd_j/h} \right) e^{-p^2\xi^2/2h^2}$$

(2)

$$|\Phi(p)|^2 = \frac{\xi^2}{h} \left( \sum_{jj'=1}^{N} \alpha_j \alpha_{j'} e^{i\theta_j - \theta_{j'}} e^{-ipd(j-j'/h)} \right) e^{-p^2\xi^2/h^2}$$
\[
\frac{\xi^2}{\hbar} \left( \sum_{n=0}^{N-1} A_n \cos\left( \frac{ndp}{\hbar} + \delta_n \right) \right) e^{-\frac{p^2 \xi^2}{\hbar^2}},
\]

(3)

where

\[
A_o = \sum_{m=1}^{N} |\alpha_m|^2, \quad \delta_o = 0
\]

\[
A_n = 2 \left| \sum_{m=1}^{N-n} \alpha_m \alpha_{m+n} e^{i(\theta_m - \theta_{m+n})} \right|, \quad n = 1, 2, ..., N - 1
\]

(4)

and \( \delta_n \) is the argument (i.e. phase angle) of the expression inside the absolute value brackets in the expression for \( A_n \). Since the density distribution in real space after a long time of ballistic expansion reflects the density distribution in momentum space before the expansion, the above expression must contain all the information about the produced interference patterns. As explained in the previous paragraph, we can immediately see that the last exponential factor in Eq. (2) provides the Gaussian envelope of the density distribution. The term inside the brackets in Eq. (2), to which we shall refer as \( \phi(p) \), describes the periodic function inside the Gaussian envelope [note that \( \phi(p+2\pi\hbar/d) = \phi(p) \)]. Since the largest value of \( m \) is \( N - 1 \), \( p \) must change by at least \( \sim \hbar/(Nd) \) in order to see any substantial change in the density distribution [note that \( Nd \) is the size of the entire cloud before the expansion]. It is perhaps easiest to think about the above analysis in term of Bloch states, which describe the wave function of a particle in a periodic potential [13]. One can then think of \( \phi(p) \) as describing the probability amplitude of a given atom to be in the quasimomentum state \( p \) in the lowest Bloch band. It now suffices to consider the discrete set of momenta \( p_j = 2\pi\hbar j/(Nd) \), where \( j \) runs over the integers from \(-N/2+1\) to \(N/2\). The scale for density variations now becomes identified as the separation between adjacent quasimomentum states. In other words, once we have calculated \( \phi(p) \) at the discrete set of quasimomenta, we can connect the points with a smooth curve to find \( \phi(p) \) for any value of \( p \).

In order to demonstrate the unique features of the two-condensate interference problem, we discuss it in some more detail (we take \( N = 2 \) and \( \alpha_1 = \alpha_2 \equiv \alpha \)). The language of Bloch states is not suited to describe such a small value of \( N \). However, if one takes any two points in momentum space that are separated by \( \pi\hbar/d \), which corresponds to the distance between quasimomentum states for \( N = 2 \), one finds a clear density-density correlation, which can be expressed as:

\[
\frac{\langle |\phi(p)|^2 - \bar{\rho} \rangle (|\phi(p + \pi\hbar/d)|^2 - \bar{\rho})}{\bar{\rho}^2} = \frac{1}{2},
\]

(5)

where \( \bar{\rho} \) is the average value of \( |\phi(p)|^2 \) and represents the average density of the interference pattern, neglecting the Gaussian envelope. The effect of that correlation is displayed in its most spectacular form when one starts with a Fock state [10]. In that case, the detection of the first few atoms, combined with density-density correlations, is the mechanism responsible for the determination of the relative phase between the two condensates. Another manifestation of that density-density correlation is the vanishing density at the interference minima. The correlations can also be seen using the treatment of the previous paragraph. From Eq. (1) we find that \( A_o = A_1 = 2|\alpha|^2 \), which says that the visibility must be 100% in every run.
of the experiment. If one identifies the points of maximum and minimum density as the preferred quasimomentum states, the densities are automatically set at $2\rho$ and zero, respectively. This type of density-density correlations at the different quasimomenta decreases if we increase $N$. However, more intricate correlations appear as $N$ is increased, corresponding to the increasing number of $A_n$’s. Therefore, one might think that it is important to take into account the effects of all those correlations to describe the interference patterns correctly. We shall show below that neglecting all such correlations still yields a good approximation of the produced interference fringes.

We now proceed with the numerical calculations and comment on the results as we move along. Fig. 1 shows the average visibility $V$ as a function of the number of overlapping condensates $N$. Each point in Fig. 1 represents the average value of $10^3$ runs of the simulation. First, we perform what we refer to as the exact calculation. In each run, we generate random values for the phases of the condensates in the different lattice sites. From those phases we can calculate the probability amplitude $f(p)$ for an atom to be in any of the $N$ quasimomentum states $p = 2\pi \hbar j/(Nd)$, $j = -N/2 + 1, ..., N/2$. We then find a least-squares fit to the squares of those amplitudes (i.e. $|f(p)|^2$) of the form:

$$h(p) = h_o \left\{ 1 + V \cos \left( \frac{pd\hbar}{h} + \delta \right) \right\}.$$

We perform the exact calculation for both a Thomas-Fermi (TF) density distribution, where the initial occupation of the lattice sites has the form of an inverted parabola, and for the homogeneous case, where all the lattice sites are equally-populated. The reason why the average visibility is higher in the TF case than the uniform case for the same number of condensates is that in the TF case the edge condensates, which have a small number of atoms, have a small effect on the produced interference patterns, and that results in a smaller effective number of condensates. In the remainder of this Paper, we focus on the homogeneous case. When calculating the values $f(p)$ in the above simulations, we use the same values of the condensate phases for all the points $p$. We note, however, that the number of points that we generate in the function $f(p)$ is equal to the number of randomly-generated phases. Moreover, as the number of condensates $N$ increases, correlations in $f(p)$ at different points $p$ become more subtle. For example, two-point correlations of the form $\langle |f(p)|^2 |f(p')|^2 \rangle / \langle |f(p)|^2 \rangle \langle |f(p')|^2 \rangle - 1$ decrease and approach zero as $N \to \infty$. It is then natural to ask whether it is necessary to keep track of the condensate phases when we calculate $f(p)$ at the different points $p$. We answer that question by generating another set of simulations (for the homogeneous case), but now for each value of $p$ we calculate $f(p)$ by summing $N$ terms of the form $e^{i\theta}$ with randomly chosen values of the variable $\theta$. We can see from Fig. 1 that for $N \approx 20$ the two sets become almost indistinguishable, apart from statistical fluctuations. Although this calculation does not correspond to any simple physical statistical ensemble in the present context, it has the advantage of simplifying our thinking about the problem. When visualising the produced interference patterns, we no longer have to keep track of how the phase of the matter-wave field from each condensate changes as we move across the (imaginary) detection screen. We can simply think of a function where to each of the $N$ different values of quasimomentum on the $x$-axis we assign a randomly-generated number as the value of the function (we shall use that idea to give a simple derivation of the large $N$ behaviour of the average visibility $V$ below). The problem
can be simplified a little bit further as follows. For large $N$ the problem of adding $N$ terms of the form $e^{i\theta}$ with random values of $\theta$ is a two-dimensional random walk problem. As is well-known, the large-$N$ random walk problem in two dimensions has the probability density \[ g(r) = \frac{\pi}{2R^2} e^{-\pi r^2/4R^2}, \] where the variable $r$ is the distance from the origin, and $R$ is the mean value of $r$ [Note that $r$ corresponds to $|f(p)|$ in the present problem]. We generate a fourth set of points where we now generate each point $|f(p)|$ from the probability distribution in Eq. \[ g(r) \]. From Fig. 1, we can see that this new set of points agrees with the exact calculation just as well as the previous calculation, even for $N$ as low as 11, where approximating the probability distribution of the random-walk problem by Eq. \[ g(r) \] is not expected to be very accurate. For further comparison, we plot in Fig. 2 the probability density $P(V)$ to find a certain value of the visibility $V$ for $N = 11$ and $N = 51$. Fig. 2 shows the results for both the exact calculation and the calculation where correlations are neglected. We do that by running the simulation $10^5$ times, and then distributing the obtained values of $V$ into a histogram. Again, we find very good agreement between the two calculations. We note that if we look at higher-frequency components in the density distribution, i.e. terms corresponding to higher values of $n$ in Eq. \[ A_n \], we expect to see less agreement between our approximation and the exact calculation. The reason is that $\langle A_n \rangle$ decreases substantially as $n$ becomes of order $N$ in the exact calculation, whereas our approximation gives a value of $\langle A_n \rangle$ that is independent of $n$. The effect of those high frequency components, however, is rather difficult to see from a simple view of the interference patterns.

For completeness, we make the following two observations about the results of our simulations: (1) Using an unrestricted least-squares fit, we obtained values of the visibility greater than 1. That happened about 5% of the runs for $N = 11$ (homogeneous case), and did not occur for $N > 40$. In those cases we have used the value 1. The difference in the average visibility between the corrected and uncorrected sets was always under 2%. (2) In all the data sets, the standard deviation in the visibility converged to around 0.52 of the mean for large $N$. That value can be obtained analytically for the exact calculation by making the observation that the statistics of the measured values of the visibility, as given by Eq. \[ f(p) \], is described by a two-dimensional random-walk problem. Using Eq. \[ g(r) \], one can see that the ratio of the standard deviation to the mean of the distance in that problem is given by $\sqrt{4/\pi - 1} = 0.523$.

We now derive the large $N$ behaviour of $V$. One can do that by considering Eq. \[ A_n \]. It is straightforward to see that:

\[ \frac{\langle A_n^2 \rangle}{\langle A_n \rangle^2} = 4 \frac{N - n}{N^2}, \quad n = 1, 2, ..., N - 1. \] (8)

Moreover, since the calculation of $\langle A_n \rangle$ reduces to a two-dimensional random walk problem in the large $N$ limit [with $(N - n) \gg 1$], we find that $\langle A_n \rangle^2/\langle A_n^2 \rangle = \pi/4$, which gives the average visibility as $V = \langle A_1 \rangle/\langle A_n \rangle \to \sqrt{\pi/N}$. We plot that function on Fig. 1 for comparison with our simulations. We now derive the $N^{-1/2}$ behaviour using the approach where we neglect multiple-point correlations in $f(p)$. As explained above, $f(p)$ is periodic in $p$ with period $2\pi h/d$. We take that interval and divide it into $K$ regions of equal length,
where $K \gg 1$. We now take $N$ to be a large multiple of $K$, i.e. $N = MK$ with $M \gg 1$, so that each region contains $M$ points where the function $f(p)$ is to be evaluated. For large $K$ and $M$, we can replace all the points in each region by one point that represents the average value $\bar{F}$ of the function in that region. That coarse graining procedure adds a negligibly small correction to the value of the visibility that we obtain in the density fit. At each point, $f(p)$ is generated from a probability distribution with mean $\bar{f}$ and standard deviation $\Delta f$. Note that the ratio $\Delta f / \bar{f}$, which can be obtained from Eq. (7), is independent of $N$. With a simple statistics argument, we find that $\Delta F / \bar{F} = (1 / \sqrt{M}) \Delta f / \bar{f}$, where $\bar{F}$ and $\Delta F$ are the mean and standard deviation of $F$, respectively. The quantity $\Delta F / \bar{F}$ measures the relative deviations of the density from its mean. Since it is those deviations that produce the finite density modulation, the average visibility must also be proportional to $1 / \sqrt{M}$ for large $M$. It therefore follows that $V$ falls off as $N^{-1/2}$ for large values of $N$.

We pause for a moment to comment on one of the calculations by Hadzibabic et. al. [11]. They calculated numerically the minimum and maximum densities of every shot and averaged the results over many shots. They found that the average minimum density $\rho_{\text{min}} \sim N^{-1}$ and the average maximum density $\rho_{\text{max}} \sim \ln N$, so that $\rho_{\text{min}} / \rho_{\text{max}} \rightarrow 0$ as $N \rightarrow \infty$. In our approach we can obtain the large-$N$ behaviour of those quantities straightforwardly by requiring that:

$$
N \int_0^{\sqrt{\rho_{\text{min}}}} g(r) dr \sim 1 \quad \text{(9)}
$$

$$
N \int_{\sqrt{\rho_{\text{max}}}}^{\infty} g(r) dr \sim 1 \quad \text{(10)}
$$

Solving those equations in the limit $N \rightarrow \infty$ gives the above asymptotic behaviour. In Ref. [11] that result is described as being an increasing contrast with increasing $N$. However, one must be careful when using that definition of contrast, since the visibility of interference fringes decreases with increasing $N$, as we have shown above and as mentioned in Ref. [11]. Note also that the width of those features, i.e. the density maxima and minima, decreases with increasing number of condensates.

We finally comment briefly on the three-dimensional case. As above, using arguments of Bloch states we can see that the density distribution after the expansion will have the same symmetries as the reciprocal lattice. For example, if the real lattice is cubic, the reciprocal lattice will also be cubic, and so will be the density distribution after the expansion. In the sense that an interference pattern is a periodic modulation of the density, we would then expect to see interference patterns in any number of dimensions. One complication arises when one tries to image such interference patterns. With present-day imaging techniques, one can only obtain two-dimensional pictures. Therefore, if one simply shines a laser beam at the condensate and looks at the shadow of that beam, only the total density along the imaging direction is measured. That has the effect of reducing the visibility of the interference pattern from its value for a two-dimensional lattice. That can be seen by considering first the two-dimensional case, which is produced by generating a set of random numbers with mean value $\bar{f}$ and standard deviation $\Delta f$ at the quasimomentum states. In the three-dimensional case, we instead have to add $M$ such randomly-generated numbers at each quasimomentum state (in the reduced two-dimensional space; here we consider a cubic lattice for simplicity), where $M$ is the number of lattice sites in the third dimension. The mean of the sum is
given by $M\bar{f}$, whereas the standard deviation of the sum is given by $\sqrt{M}\Delta f$. Therefore the deviations from the mean in the measured density will be reduced by a factor $M^{-1/2}$, leading to a reduced visibility. It is possible, however, to image slices of the expanded cloud as explained in Ref. [4]. Using that technique, the three dimensional density distribution should be accessible.

In conclusion, we have considered the interference patterns produced by the overlap of an array of independent Bose-Einstein condensates. We have shown that density-density correlations at different quasimomenta do not play any significant role in producing the interference fringes, in contrast to what might be intuitively expected. In order to understand the main features of the produced interference patterns, it is sufficient to consider the periodicity of the lattice, and to identify the probability distribution for the occupation of a given quasimomentum state. We have demonstrated the above statement by reproducing a number of features of the interference patterns with remarkable accuracy using an approximation where we neglect those density-density correlations. In particular, the average visibility decreases as $N^{-1/2}$ with increasing number of condensates $N$. Our results help explain the behaviour observed in recent experiments on the interference of atomic clouds released from optical lattices.

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**Figures**

Fig. 1: Average visibility $V$ as a function of number of condensates $N$ on a logarithmic scale. Circles and stars represent the exact calculation for the Thomas-Fermi and the homogeneous case, respectively. X’s are generated by summing $N$ random numbers of unit modulus at each value of quasimomentum. +’s are generated using the asymptotic two-dimensional random walk probability distribution at each value of quasimomentum.

Fig. 2: Probability density $P(V)$ as a function of visibility $V$ for $N = 11$ (a) and $N = 51$ (b). X’s and +’s are obtained from the exact calculation and the calculation where density-density correlations are neglected, respectively. The solid line is the asymptotic two-dimensional random walk probability density corresponding to a mean distance $\sqrt{\pi/11} = 0.53$ (a) and $\sqrt{\pi/51} = 0.25$ (b).
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