Octonionic formulation of the fully symmetric Maxwell’s equations in 3 + 1 dimensions

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Abstract

Using octonions, more specifically, using a 4 × 4 matrix representation of octonions obtained with the help of algebraic properties of quaternions, we obtain the fully symmetric Maxwell’s equations (Maxwell’s equations with electric and magnetic charges and currents which is invariant under duality transformations).

1 Introduction

The algebra of octonions O forms the largest normed division algebra over the real numbers R, complex numbers C, and quaternions H (see e.g. [1]). The algebra of octonions O is an algebraic structure defined on the 8-dimensional real vector space R⁸ and is a non-associative extension of the algebra of quaternions H. These composition algebras are responsible for many important mathematical structures of interest for physicists [2, 3]. Octonions have been used in various ways to describe properties of quantum mechanics [4], field theory [5, 6], gauge theory [7, 8]. Electromagnetism has been expressed on octonionic algebras [9, 10, 11, 12], split octonion algebra [13] and octonion Dirac equations [14, 15].

In the presence of sources, the usual Maxwell’s equations are neither symmetric nor invariant with respect to the duality transformation between electric and magnetic fields. Dirac [16] proposed the existence of magnetic monopoles to symmetrise the Maxwell’s equations. He also argued that the existence of an isolated magnetic charge would imply the quantisation of electric charge. Dirac [17] also described the applications of quaternions to Lorentz transformations. Symmetric Maxwell’s equations are invariant under Lorentz and duality transformation. t’Hooft [18] and Polyakov [19] gave the idea that the magnetic monopoles may be found in Yang-Mill’s theory. In the present study, we describe the fully symmetric Maxwell’s equations by using the algebraic properties of
quaternions and octonions. This study will give an octonionic representation of fully symmetric Maxwell’s equations as a single equation.

2 Octonions

The octonionic algebra \( \mathbb{O} \) is an eight dimensional algebra over the field of the real numbers \( \mathbb{R} \). Its elements can be represented as

\[
O = e_0 y_0 + e_1 y_1 + e_2 y_2 + e_3 y_3 + e_4 y_4 + e_5 y_5 + e_6 y_6 + e_7 y_7
= e_0 y_0 + \sum_{a=1}^{7} e_a y_a ,
\]

where \( y_k (k = 0, 1, \ldots, 7) \) are real numbers, \( e_a (a = 1, 2, \ldots, 7) \) are imaginary octonion units and \( e_0 \) is the multiplicative unit element. The set of octets \( (e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7) \) is known as the octonion basis and its elements satisfy the following multiplication rules:

\[
e_0 = 1 , \quad i.e., \quad e_0 e_a = e_a e_0 = e_a
\]
\[
e_a e_b = - \delta_{ab} e_0 + f_{abc} e_c , \quad (a, b, c = 1, 2, \ldots, 7) ,
\]

where \( \delta_{ab} \) is the usual Krönecker symbol and the structure constants \( f_{abc} \) are completely antisymmetric, take the value 1 for the following combinations:

\[
f_{abc} = +1 \quad \forall (abc) = (123), (471), (257), (165), (624), (543), (736) ,
\]

and vanish otherwise. The octonion basis elements satisfy the following additional relations:

\[
[e_a, e_b] = 2 f_{abc} e_c ,
\]
\[
\{e_a, e_b\} = -2 \delta_{ab} e_0 ,
\]
\[
e_a (e_b e_c) \neq (e_a e_b) e_c ,
\]

where the brackets [ , ] and \( \{ , \} \) denote, as usual, the commutator and the anti-commutator, respectively.

Notice that the sub-algebra generated by \( (e_0, e_7) \) is isomorphic to the algebra \( \mathbb{C} \) of the complex numbers while the sub-algebra generated by \( (e_0, e_1, e_2, e_3) \) is isomorphic to the algebra \( \mathbb{H} \) of quaternions.

3 Representation of octonions in terms of 4×4 matrix

The algebra of octonions is homeomorphic to a four-dimensional algebra over the complex numbers. In order to see this, consider that the algebra of the complex numbers is isomorphic to the sub-algebra of \( \mathbb{O} \) generated by the elements \( (e_0, e_7) \), since \( (e_7)^2 = -e_0 \). Now, any octonion \( O = e_0 y_0 + e_1 y_1 + e_2 y_2 + e_3 y_3 + e_4 y_4 + e_5 y_5 + e_6 y_6 + e_7 y_7 \in \mathbb{O} \) can be written as

\[
O = (y_0 + e_7 y_7) e_0 + (y_1 + e_7 y_4) e_1 + (y_2 + e_7 y_5) e_2 + (y_3 + e_7 y_6) e_3 .
\]
Denoting

\[ Y_0 = y_0 + e_7 y_7, \]
\[ Y_j = y_j + e_7 y_{j+3}, \quad \forall j = 1, 2, 3, \]

equation (6) becomes

\[ O = Y_0 e_0 + Y_1 e_1 + Y_2 e_2 + Y_3 e_3. \] (7)

Equation (7) represents octonions in terms of elements of the sub-algebra of quaternions (the sub-algebra generated by \((e_0, e_1, e_2, e_3)\)), with “coefficients” in the sub-algebra \((e_0, e_7)\), isomorphic to the algebra of complex numbers. It is well known that the algebra of quaternions is isomorphic to the algebra of Pauli matrices through the identification

\[ e_0 = 1 \quad \text{and} \quad e_j = -i \sigma_j, \quad \forall j = 1, 2, 3, \] (8)

where the unit matrix \(1\) and the Pauli matrices \(\sigma_j\) are given, as usual, by

\[
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\] (9)

In order to obtain a representation of \(\mathbb{H}\) in terms of real matrices one has to proceed in the following way. Define \(J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\). Since \(J^2 = -1\), we may replace the imaginary number \(i\) by \(J\) in (8), through the following identifications:

\[ e_0 = 1 \otimes 1, \quad e_1 = -J \otimes \sigma_1, \quad e_2 = 1 \otimes (-i \sigma_2) = -1 \otimes J, \quad \text{and} \quad e_3 = -J \otimes \sigma_3. \]

This provides following identification of the elements of the quaternionic basis with real \(4 \times 4\) matrices:

\[
e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & -J \\ -J & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix}.
\]

The algebra of real \(4 \times 4\) matrices so obtained is isomorphic to the quaternionic algebra \(\mathbb{H}\). With these identifications we can represent the algebra of octonions in terms of an algebra of \(4 \times 4\) matrices whose entries are elements of the sub-algebra generated by \((e_0, e_7)\). Denoting by \(\pi(O)\) the representative of an octonion \(O\) given as (7), we have:

\[
\pi(O) = \begin{bmatrix} Y_0 & -Y_3 & -Y_2 & -Y_1 \\ Y_3 & Y_0 & Y_1 & -Y_2 \\ Y_2 & -Y_1 & Y_0 & Y_3 \\ Y_1 & Y_2 & -Y_3 & Y_0 \end{bmatrix}.
\] (10)

Identifying the entries \(Y_k\), \(k = 0, \ldots, 3\), above, with complex numbers, eq. (10) provides an homomorphism (not an isomorphism) between \(O\) and an algebra of \(4 \times 4\) complex matrices. Equation (10) suggests, in addition, a convenient definition of derivative operations, as we will see below.
4 Fields and a derivative operator

Let us now consider $Y_k$, $k = 0, \ldots, 3$, as functions $Y_k(t, x_1, x_2, x_3)$ defined in $\mathbb{R}^4$ taking values on the sub-algebra of $\mathbb{O}$ generated by $(e_0, e_7)$, i.e., taking values on the $e_0$-$e_7$ plane. Denote by $\partial_t$ partial derivative with respect to $t$ and by $\partial_{x_k}$ the partial derivative with respect to $x_k$, $k = 1, 2, 3$, and define a derivative operator $\partial$ in terms of quaternion basis elements as equation (8),

$$\partial = \partial_t e_0 + \partial_1 e_1 + \partial_2 e_2 + \partial_3 e_3 .$$

By analogy with (10), the derivative $\partial$ can be written in terms of $4 \times 4$ matrix as

$$\partial = \begin{bmatrix} \partial_t & -\partial_3 & -\partial_2 & -\partial_1 \\ \partial_3 & \partial_t & \partial_1 & -\partial_2 \\ \partial_2 & -\partial_1 & \partial_t & \partial_3 \\ \partial_1 & \partial_2 & -\partial_3 & \partial_t \end{bmatrix} .$$

Consider now a field of octonions $\mathbb{R}^4 \ni (t, x_1, x_2, x_3) \mapsto O(t, x_1, x_2, x_3) \in \mathbb{O}$ in the representation (10). The matrix representation (12) of the derivative $\partial$ suggests to define the derivative $\partial \pi(O)$ of an octonionic field $O$ by taking the matrix product of (12) with (10):

$$\partial \pi(O) = \begin{bmatrix} \partial_0 Y_0 - \partial_1 Y_1 - \partial_2 Y_2 - \partial_3 Y_3 \\ \partial_0 Y_0 - \partial_1 Y_1 + \partial_2 Y_2 - \partial_3 Y_3 \\ \partial_0 Y_0 + \partial_1 Y_1 + \partial_2 Y_2 - \partial_3 Y_3 \\ \partial_0 Y_0 + \partial_1 Y_1 - \partial_2 Y_2 - \partial_3 Y_3 \end{bmatrix} ,$$

The matrix elements in equation (13) are analogous to the corresponding matrix elements in equation (10), that has a compact form as equation (7). This suggests to define a derivative operator on an octonionic field $O(t, x_1, x_2, x_3)$ by imposing $\pi(\partial O) = \partial \pi(O)$. One gets,

$$\partial O = [\partial_1 Y_0 - \partial_1 Y_1 - \partial_2 Y_2 - \partial_3 Y_3] e_0 + [\partial_1 Y_0 + \partial_1 Y_1 + \partial_2 Y_2 - \partial_3 Y_3] e_1 + [\partial_1 Y_0 + \partial_1 Y_2 + \partial_3 Y_1 - \partial_2 Y_3] e_2 + [\partial_2 Y_0 + \partial_3 Y_1 + \partial_1 Y_2 - \partial_2 Y_1] e_3 \in \mathbb{O} ,$$

or, in a more compact form,

$$\partial O = [\partial_1 Y_0 - \partial_1 Y_3] e_0 + \left[ \partial_1 Y_0 + \partial_1 Y_3 + \left( \nabla \times \vec{Y} \right)_j \right] e_j ,$$

where $\vec{Y}$ is the 3-vector with components $(Y_1, Y_2, Y_3)$. Above we used Einstein’s summation convention (the summation over $j$ being from 1 to 3). This expression will be used to describe the fully symmetric Maxwell’s equations in the next section.

5 Generalised Maxwell’s equations generated by octonions

Until now we considered the functions $Y_k$, $k = 0, \ldots, 3$, as functions of $t$ and of space coordinates $x_j$, $j = 1, 2, 3$. The $e_0$-$e_7$ plane in the octonions algebra $\mathbb{O}$ is a sub-algebra isomorph to the algebra of the complex numbers $\mathbb{C}$. Our next step is to consider $t$ as an octonionic variable $t \mapsto e_0 t$ and to perform an analytic continuation of the functions $Y_k$, to this complex plane, by performing a sort of Wick rotation $e_0 t \mapsto e_7 x_0$ (with $x_0 \in \mathbb{R}$). Formally, by this procedure of analytic continuation of
the functions $Y_k(t, \vec{x})$ remain functions taking values on the $e_0$–$e_7$ plane, i.e., we may, as before, write

$$Y_k(e_\tau x_0, \vec{x}) = e_0 Y_k^{(0)}(x_0, \vec{x}) + e_\tau Y_k^{(1)}(x_0, \vec{x}),$$

with $Y_k^{(0)}$ and $Y_k^{(1)}$ being real function analogous to the real and imaginary parts of $Y_k$.

By this procedure, the time derivatives $\partial_t$ are replaced in (14) by $-e_\tau \partial_{x_0} \equiv -e_\tau \partial_0$. Hence, (14) becomes

$$\partial O = \left[-e_\tau \partial_0 \left(Y_0^{(0)} + e_\tau Y_0^{(1)}\right) - \partial_j \left(Y_j^{(0)} + e_\tau Y_j^{(1)}\right) \right] e_0$$

$$+ \left[\partial_j \left(Y_0^{(0)} + e_\tau Y_0^{(1)}\right) - e_\tau \partial_0 \left(Y_j^{(0)} + e_\tau Y_j^{(1)}\right) + \left(\nabla \times \left(Y_0^{(0)} + e_\tau Y_0^{(1)}\right)\right)_{\vec{j}}\right] e_j$$

$$= e_0 \left[\partial_0 Y_0^{(1)} - \partial_j Y_j^{(0)} + \left(\partial_j Y_0^{(0)} - \partial_0 Y_j^{(1)} + \left(\nabla \times Y^{(0)}\right)_{\vec{j}}\right) e_j\right]$$

$$+ e_\tau \left[-\partial_0 Y_0^{(0)} - \partial_j Y_j^{(1)} + \left(\partial_j Y_0^{(1)} - \partial_0 Y_j^{(0)} + \left(\nabla \times Y^{(1)}\right)_{\vec{j}}\right) e_j\right]. \quad (16)$$

Now, we change our notations by writing

$$\vec{Y} = \vec{Y}^{(0)} + e_\tau \vec{Y}^{(1)} \equiv \vec{E} + e_\tau \vec{H},$$

$$\partial_0 Y_0 = \partial_0 Y_0^{(0)} + e_\tau \partial_0 Y_0^{(1)} \equiv -\rho_m + e_\tau \rho_e, \quad (18)$$

$$\partial_j Y_0 = \partial_j Y_0^{(0)} + e_\tau \partial_j Y_0^{(1)} \equiv (j_m)_j - e_\tau (j_e)_j, \quad \forall j = 1, 2, 3. \quad (19)$$

With this new notation, eq. (16) becomes

$$\partial O = e_0 \left[\rho_e - \partial_j E_j + \left((j_m)_j + (\partial_0 H_j) + \left(\nabla \times \vec{E}\right)_{\vec{j}}\right) e_j\right]$$

$$+ e_\tau \left[\rho_m - \partial_j H_j + \left(- (j_e)_j - (\partial_0 E_j) + \left(\nabla \times \vec{H}\right)_{\vec{j}}\right) e_j\right]. \quad (20)$$

By imposing $\partial O = 0$, the different components of equation (20) provide eight equations corresponding to the coefficients of $e_0$, $e_\tau$, $e_j$ and $e_\tau e_j$, $j = 1, 2, 3$. These equations can be more properly written in vector form:

$$\nabla \cdot \vec{E} = \rho_e, \quad (21)$$

$$\nabla \cdot \vec{H} = \rho_m, \quad (22)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{H}}{\partial t} - j_m, \quad (23)$$

$$\nabla \times \vec{H} = \frac{\partial \vec{E}}{\partial t} + j_e. \quad (24)$$

As we now recognise, equations (21)–(24) are the fully symmetric Maxwell’s equations and allow for the possibility of magnetic charges and currents, analogous to electric charges and currents. It is a well known fact that equations (21)–(24) are unchanged under the so-called duality transformations,
defined, for $\theta \in [0, 2\pi)$, by

$$\begin{pmatrix} \rho_e \\ \rho_m \end{pmatrix} \mapsto R(\theta) \begin{pmatrix} \rho_e \\ \rho_m \end{pmatrix}, \quad \begin{pmatrix} \vec{J}_e \\ \vec{J}_m \end{pmatrix} \mapsto R(\theta) \begin{pmatrix} \vec{J}_e \\ \vec{J}_m \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} \mapsto R(\theta) \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}$$

where $R(\theta) := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$. We should also recall that, using arguments from quantum mechanics, Dirac [16] argued that the existence of an isolated magnetic charge would imply the quantisation of electric charge.

From (21)–(24) we easily obtain the so-called continuity equations for the electric and magnetic charge densities and currents:

$$\partial_0 \rho_e + \nabla \cdot \vec{J}_e = 0 \quad \text{and} \quad \partial_0 \rho_m + \nabla \cdot \vec{J}_m = 0.$$ 

By using (18) and (19), these continuity equations imply, as one easily sees, $\Box Y_0^{(0)} = 0$ and $\Box Y_0^{(1)} = 0$, or simply $\Box Y_0 = 0$, where $\Box$ is the D’Alembertian operator.

6 Conclusion

We have considered a representation of octonions as a four dimensional algebra over complex numbers. After a suitable definition of a derivative operation over fields of octonions in $3+1$ dimensions we derived the fully symmetric Maxwell’s equations (i.e., with electric and magnetic charges and currents) as a single equation.

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