1. Introduction

1.1. Let \( f \) and \( f' \) be germs of analytic functions on smooth complex analytic varieties \( X \) and \( X' \) and consider the function \( f \oplus f' \) on \( X \times X' \) given by \( f \oplus f'(x, x') = f(x) + f'(x') \). The Thom-Sebastiani Theorem classically states that the monodromy of \( f \oplus f' \) on the nearby cycles is isomorphic to the product of the monodromy of \( f \) and the monodromy of \( f' \) (in the original form of the Theorem \[18\] the functions were assumed to have isolated singularities). It is now a common idea that the Thom-Sebastiani Theorem is best understood by using Fourier transformation and exponential integrals because of the formula

\[
(1.1) \quad \int \exp(t(f \oplus f')) = \int \exp(tf) \cdot \int \exp(tf').
\]

Indeed, by using asymptotic expansions of such integrals for \( t \to \infty \), A. Varchenko proved a Thom-Sebastiani Theorem for the Hodge spectrum in the isolated singularity case \[19\] (see also \[13\]) and the general case has been announced by M. Saito \[17\], \[12\].

The aim of the present paper is to give a motivic meaning to equation (1.1) and to deduce a motivic Thom-Sebastiani Theorem. To explain our approach, we will begin by reviewing some known results on \( p \)-adic exponential integrals.

1.2. Let \( K \) be a finite extension of \( \mathbb{Q}_p \). Let us denote by \( R \) the valuation ring of \( K \), by \( P \) the maximal ideal of \( R \), and by \( k \) the residue field of \( K \). The cardinality of \( k \) will be denoted by \( q \), so \( k \simeq \mathbb{F}_q \). For \( z \) in \( K \), \( \text{ord} z \in \mathbb{Z} \cup \{+\infty\} \) denotes the valuation of \( z \), \( |z| = q^{-\text{ord} z} \), and \( \text{ac}(z) = z\pi^{-\text{ord} z} \), where \( \pi \) is a fixed uniformizing parameter for \( R \).

Let \( f \in R[x_1, \ldots, x_m] \) be a non constant polynomial. Let \( \Phi : R^m \to \mathbb{C} \) be a locally constant function with compact support. Let \( \alpha \) be a character of \( R^\times \), i.e. a morphism \( R^\times \to \mathbb{C}^\times \) with finite image. For \( i \) in \( \mathbb{N} \), set

\[
Z_{\Phi, f, i}(\alpha) = \int_{\{x \in R^m | \text{ord} f(x) = i\}} \Phi(x)\alpha(\text{ac}f(x))|dx|,
\]
where \( |dx| \) denotes the Haar measure on \( K^m \), normalized so that \( R^m \) is of measure 1.

We denote by \( \Psi \) the standard additive character on \( K \), defined by
\[
z \mapsto \Psi(z) = \exp(2i\pi \text{Tr}_{K/Q_p} z).
\]

For \( i \) in \( \mathbb{N} \), we consider the exponential integral
\[
E_{\Phi,f,i} = \int_{R^m} \Phi(x)\Psi(\pi^{-i+1} f(x))|dx|.
\]

Let \( \alpha \) be a character of \( R^\times \). The conductor of \( \alpha \), \( c(\alpha) \), is defined as the smallest \( c \geq 1 \) such that \( \alpha \) is trivial on \( 1 + P^c \), and one associates to \( \alpha \) the Gauss sum
\[
g(\alpha) = q^{1-c(\alpha)} \sum_{v \in (R/P^{c(\alpha)})^\times} \alpha(v)\Psi(v/\pi^{-c(\alpha)}).
\]

**Proposition 1.2.1.** (See §1 of [3].) For any \( i \) in \( \mathbb{N} \),
\[
E_{\Phi,f,i} = \int_{\{x \in R^m | \text{ord}_f(x) > i\}} \Phi(x)|dx| + (q - 1)^{-1} \sum_{\alpha} g(\alpha^{-1})Z_{\Phi,f,i,\alpha+1}(\alpha).
\]

Here \( i-c(\alpha)+1 \geq 0 \). If moreover the critical locus of \( f \) in \( \text{Supp} \Phi \) is contained in \( f^{-1}(0) \), then, for all except a finite number of characters \( \alpha \), the integrals \( Z_{\Phi,f,j}(\alpha) \) are zero for all \( j \).

**Corollary 1.2.2.** (Using Theorem 3.3 of [3].) Assume that \( \Phi \) is residual, i.e. that \( \text{Supp} \Phi \) is contained in \( R^m \) and that \( \Phi(x) \) depends only on \( x \) modulo \( P \), and that the critical locus of \( f \) in \( \text{Supp} \Phi \) is contained in \( f^{-1}(0) \). Assume furthermore that the divisor \( f = 0 \) has good reduction (in the sense that the conditions in Theorem 3.3 of [3] are satisfied). Then
\[
E_{\Phi,f,i} = \int_{\{x \in R^m | \text{ord}_f(x) > i\}} \Phi(x)|dx| + (q - 1)^{-1} \sum_{\alpha} g(\alpha^{-1})Z_{\Phi,f,i}(\alpha).
\]

So we see that \( p \)-adic exponential integrals may be expressed as linear combinations of \( p \)-adic integrals involving multiplicative characters with Gauss sums as coefficients.

When \( k \) is a field of characteristic 0, there is a \( k((t)) \)-analogue of \( p \)-adic integration, motivic integration, introduced by M. Kontsevich. In particular, it is possible by the results of [4] and [5] to define motivic analogues of the \( p \)-adic integrals \( Z_{\Phi,f,i}(\alpha) \) in [2.2] as elements of a Grothendieck group of Chow motives. Since in this analogy the \( k((t)) \)-case always has good reduction, it becomes quite natural to use equality [1.2.2] as a candidate for the definition of motivic exponential integrals and this is indeed what we do in this paper. To achieve this aim, we enlarge slightly our virtual motives by attaching virtual motives to Gauss sums.
in a way very similar to Anderson’s construction of ulterior motives [1]. But now equation (1.1) is no longer trivial, and the main result of the paper is that it still holds true for our motivic exponential integrals (Theorem 4.2.4). We deduce from this result a motivic analogue of the Thom-Sebastiani Theorem (Theorem 5.2.2). Passing to Hodge realization, this gives a proof of the Thom-Sebastiani Theorem for the Hodge spectrum (Corollary 6.2.4).

2. ADDING ULTEIOR MOTIVES TO THE GROTHENDIECK GROUP

2.1. We fix a base field $k$, which we assume throughout the paper to be of characteristic zero, and we denote by $\mathcal{V}_k$ the category of smooth and projective $k$-schemes. For an object $X$ in $\mathcal{V}_k$ and an integer $d$, we denote by $A^d(X)$ the Chow group of codimension $d$ cycles with rational coefficients modulo rational equivalence. Objects of the category $\mathcal{M}_k$ of (rational) $k$-motives are triples $(X, p, n)$ where $X$ is in $\mathcal{V}_k$, $p$ is an idempotent (i.e. $p^2 = p$) in the ring of correspondences $\text{Corr}^0(X, X) (= A^d(X \times X)$ when $X$ is of pure dimension $d)$, and $n$ is an integer. If $(X, p, n)$ and $(Y, q, m)$ are motives, then

$$\text{Hom}_{\mathcal{M}_k}((X, p, n), (Y, q, m)) = q \text{Corr}^{m-n}(X, Y) p.$$ 

Composition of morphisms is given by composition of correspondences. The category $\mathcal{M}_k$ is additive, $\mathbb{Q}$-linear, and pseudo-abelian, and there is a natural tensor product on $\mathcal{M}_k$. We denote by $h$ the functor $h : \mathcal{V}_k \to \mathcal{M}_k$ which sends an object $X$ to $h(X) = (X, \text{id}, 0)$ and a morphism $f : Y \to X$ to its graph in $\text{Corr}^0(X, Y)$. We denote by $\mathbf{L}$ the Lefschetz motive $\mathbf{L} = (\text{Spec} k, \text{id}, -1)$. There is a canonical isomorphism $h(\mathbf{P}^1_k) \simeq 1 \oplus \mathbf{L}$.

When $E$ is a field of characteristic zero, one defines similarly the category $\mathcal{M}_{k,E}$ of $k$-motives with coefficients in $E$, by replacing the Chow groups $A^j$ by $A^j \otimes_{\mathbb{Q}} E$. Let $K_0(\mathcal{M}_{k,E})$ be the Grothendieck group of the pseudo-abelian category $\mathcal{M}_{k,E}$. It is also the abelian group associated to the monoid of motives with coefficients in $E$. The tensor product on $\mathcal{M}_{k,E}$ induces a natural ring structure on $K_0(\mathcal{M}_{k,E})$. For $m$ in $\mathbb{Z}$, let $F^mK_0(\mathcal{M}_{k,E})$ denote the subgroup of $K_0(\mathcal{M}_{k,E})$ generated by $h(S, f, i)$, with $i - \dim S \geq m$. This gives a filtration of the ring $K_0(\mathcal{M}_{k,E})$ and we denote by $\tilde{K}_0(\mathcal{M}_{k,E})$ the completion of $K_0(\mathcal{M}_{k,E})$ with respect to this filtration.

Remark 2.1.1. We expect, but do not know how to prove, that the filtration $F^\ast$ on $K_0(\mathcal{M}_{k,E})$ is separated. This assertion is clearly implied by the conjectural existence (cf. [13] p.185) of additive functors $h^{\leq j} : \mathcal{M}_{k,E} \to \mathcal{M}_{k,E}$, $j \in \mathbb{Z}$, such that for any $X = h(S, f, i)$ in $\mathcal{M}_{k,E}$, the $h^{\leq j}(X)$ form a filtration of $X$ with $h^{\leq 2i-1}(X) = 0$, $h^{\leq 2i}(X) = X$, and $h^{\leq j}(X) = \mathbf{L} h^{\leq j-2}(X)$ for all $j$.

In particular, using the existence of weight filtrations, one obtains, without using any conjecture, that the kernel of the canonical morphism

$$K_0(\mathcal{M}_{k,E}) \to \tilde{K}_0(\mathcal{M}_{k,E})$$
is killed by étale and Hodge realizations.

2.2. We begin by recalling some material from [4]. Let \( G \) be a finite abelian group and let \( \hat{G} \) be its complex character group. We denote by \( \mathcal{V}_{k,G} \) the category of smooth and projective \( k \)-schemes with \( G \)-action. Let \( E \) be a subfield of \( \mathbb{C} \) containing all the roots of unity of order dividing \( |G| \). For \( X \) in \( \mathcal{V}_{k,G} \) and \( g \) in \( G \), we denote by \( [g] \) the correspondence given by the graph of multiplication by \( g \). For \( \alpha \) in \( \hat{G} \), we consider the idempotent \( f_\alpha := |G|^{-1} \sum_{g \in G} \alpha^{-1}(g)[g] \) in \( \text{Corr}^0(X, X) \otimes E \), and we denote by \( h(X, \alpha) \) the motive \( (X, f_\alpha, 0) \) in \( M_{k,E} \).

We will denote by \( \text{Sch}_{k,G} \) the category of separated schemes of finite type over \( k \) with \( G \)-action satisfying the following condition: the \( G \)-orbit of any closed point of \( X \) is contained in an affine open subscheme. This condition is clearly satisfied for \( X \) quasiprojective and insures the existence of \( X/G \) as a scheme. Objects of \( \text{Sch}_{k,G} \) will be called \( G \)-schemes and in this paper all schemes with \( G \)-action will be assumed to be \( G \)-schemes.

The following result, proved in [4] as a consequence of results in [7] and [20], generalizes to the \( G \)-action case a result which has been proved by Gillet and Soulé [3] and Guillén and Navarro Aznar [7].

**Theorem 2.2.1.** Let \( k \) be a field of characteristic 0. There exists a unique map
\[
\chi_c : \text{ObSch}_{k,G} \times \hat{G} \longrightarrow K_0(M_{k,E})
\]
such that

1. If \( X \) is smooth and projective with \( G \)-action, for any character \( \alpha \),
   \[
   \chi_c(X, \alpha) = [h(X, \alpha)].
   \]

2. If \( Y \) is a closed \( G \)-stable subscheme in a scheme \( X \) with \( G \)-action, for any character \( \alpha \),
   \[
   \chi_c(X \setminus Y, \alpha) = \chi_c(X, \alpha) - \chi_c(Y, \alpha).
   \]

3. If \( X \) is a scheme with \( G \)-action, \( U \) and \( V \) are \( G \)-invariant open subschemes of \( X \), for any character \( \alpha \),
   \[
   \chi_c(U \cup V, \alpha) = \chi_c(U, \alpha) + \chi_c(V, \alpha) - \chi_c(U \cap V, \alpha).
   \]

Furthermore, \( \chi_c \) is determined by conditions (1)-(2).

2.3. In this subsection we gather some elementary statements we shall need.

**Proposition 2.3.1.** Let \( k \) be a field of characteristic 0.

1. For any \( X \) in \( \text{ObSch}_{k,G} \),
   \[
   \chi_c(X) = \sum_{\alpha \in \hat{G}} \chi_c(X, \alpha).
   \]
(2) Let $X$ be in $\text{ObSch}_{k,G}$. Assume the $G$-action factors through a morphism of finite abelian groups $G \to H$. If $\alpha$ is not in the image of $\hat{H} \to \hat{G}$, then $\chi_c(X, \alpha) = 0$.

(3) Let $X$ and $Y$ be in $\text{ObSch}_{k,G}$ and let $G$ act diagonally on $X \times Y$. Then

$$\chi_c(X \times Y, \alpha) = \sum_{\beta \in \hat{G}} \chi_c(X, \beta) \cdot \chi_c(Y, \alpha \beta^{-1}).$$

(4) Let $X$ be in $\text{ObSch}_{k,H}$ and let $f : G \to H$ be a morphism of finite abelian groups. Then for any $\alpha$ in $\hat{G}$,

$$\chi_c(X, \alpha) = \sum_{f(\beta) = \alpha} \chi_c(X, \beta),$$

with $\hat{f} : \hat{H} \to \hat{G}$ the dual morphism between character groups.

Proof. Statements (1)-(3) are proven in Proposition 1.3.3 of [4]. They are all consequence of (4), whose proof is similar: the statement is obvious when $X$ is smooth projective, and the general case follows by additivity of $\chi_c$.

Lemma 2.3.2. Let $a$ be an integer and let $\mu_d(k)$ act on $\mathbb{G}_{m,k}$ by multiplication by $\xi^a$, $\xi \in \mu_d(k)$. For any non trivial character $\alpha$ of $\mu_d(k)$,

$$\chi_c(\mathbb{G}_{m,k}, \alpha) = 0.$$

Proof. This is Lemma 1.4.3 of [4].

We now discuss motivic Euler characteristics of quotients. The following lemma is well known.

Lemma 2.3.3. Let $X$ be a smooth projective scheme with $G$-action, $H$ a subgroup of $G$, and $\alpha$ a character of $G/H$. Assume the quotient $X/H$ is smooth. Then $h(X/H, \alpha) \simeq h(X, \alpha \circ \varrho)$, where $\varrho$ is the projection $G \to G/H$.

Proof. This is Lemma 1.5.1 of [4].

In general one has the following result, which was conjectured in [4] and proved in [2] by del Baño Rollin and Navarro Aznar.

Theorem 2.3.4. If $X$ is a scheme with $G$-action, $H$ a subgroup of $G$, and $\alpha$ a character of $G/H$, then $\chi_c(X/H, \alpha) = \chi_c(X, \alpha \circ \varrho)$, where $\varrho$ is the projection $G \to G/H$.

Remark 2.3.5. Theorem 2.3.4 is a direct consequence of Lemma 2.3.3 when $X$ is a smooth curve or when $X$ is smooth and may be embedded in a smooth projective scheme $Y$ with $G$-action such that the quotient $Y/H$ is smooth, and such that $Y \setminus X$ is the union of finitely many smooth closed $G$-stable subvarieties intersecting transversally and having smooth images in $Y/H$ which intersect transversally.
2.4. Jacobi motives. We fix an integer $d \geq 1$. We denote by $\mu_d(k)$ the group of $d$-roots of 1 in $k$ and by $\zeta_d$ a fixed primitive $d$-th root of unity in $\mathbb{C}$. We assume from now on that $k$ contains all $d$-roots of unity.

We set

$$A_d := K_0(M_{k, \mathbb{Q}(\zeta_d)}).$$

For $n \geq 1$, we consider the affine Fermat variety $F^d_n$ defined by the equation $x_1^d + \cdots + x_n^d = 1$ in $\mathbb{A}^n_k$, and its closure in $\mathbb{P}^n_k$, which we denote by $W^d_n$. Hence $W^d_n$ is defined in $\mathbb{P}^n_k$ by the equation $-X_0^d + \cdots + X_n^d = 0$, with $x_i = X_i/X_0$, $i \geq 1$.

The action of $\mu_d(k)$ on each coordinate induces a natural action of the group $\mu_d(k)^n$ on $F^d_n$. Hence, for $\alpha_1, \ldots, \alpha_n$, characters of $\mu_d(k)$, one defines the Jacobi motive $J(\alpha_1, \ldots, \alpha_n)$ as the element

$$J(\alpha_1, \ldots, \alpha_n) := \chi_c(F^d_n, (\alpha_1, \ldots, \alpha_n))$$

in $A_d$. It is clear that $J(\alpha_1, \ldots, \alpha_n)$ is symmetric in the $\alpha_i$'s. We also define $[\alpha(-1)] := \chi_c(x^d = -1, \alpha)$. Remark that $[\alpha(-1)] = 1$, if $k$ contains a $d$-th root of -1. We will need the following proposition, which is classical in other contexts.

**Proposition 2.4.1.** The following relations hold in $A_d$.

1. We have $J(1, 1) = L$.
2. We have $J(1, \alpha) = 0$ if $\alpha \neq 1$.
3. If $\alpha \neq 1$, $J(\alpha, \alpha^{-1}) = -[\alpha(-1)]$.
4. We have

$$J(\alpha_1, \alpha_2)[J(\alpha_1\alpha_2, \alpha_3) - \varepsilon] = J(\alpha_1, \alpha_2, \alpha_3) - \delta,$$

with $\varepsilon = \delta = 0$ if $\alpha_1\alpha_2 \neq 1$, $\varepsilon = 1$, $\delta = [\alpha_1(-1)](L - 1)$, if $\alpha_1\alpha_2 = 1$ and $\alpha_1 \neq 1$, and $\varepsilon = 1$, $\delta = L$, if $\alpha_1 = \alpha_2 = 1$.

**Proof.** Relation (1) follows directly from Remark 2.3.3 because the quotient of the curve $F^d_2$ by $\mu_d(k) \times \mu_d(k)$ is the affine line $\mathbb{A}_k^1$.

To prove (2), observe that the quotient of the curve $F^d_2$ by $\mu_d(k) \times \{1\}$ is the Kummer cover $x_2^d = 1 - x_1$ of the affine line. Hence, by Remark 2.3.3, $J(1, \alpha) = \chi_c(\mathbb{A}_k^1, \alpha)$, with the natural action of $\mu_d(k)$ on $\mathbb{A}_k^1$, and the assertion follows from Lemma 2.3.2 and Proposition 2.3.1.

Let us now prove (3). Observe that, the character $\alpha$ being non trivial, we have

$$\chi_c(F^d_2 \cap \{x_2 = 0\}, (\alpha, \alpha^{-1})) = 0.$$

Indeed, this follows for instance from Proposition 2.3.1. Hence

$$J(\alpha, \alpha^{-1}) = \chi_c(F^d_2 \setminus \{x_2 = 0\}, (\alpha, \alpha^{-1})).$$
Now we may identify $F_d^2 \setminus \{x_2 = 0\}$ with the affine curve $u^d - v^d = -1; v \neq 0,$ via the change of variable $u = x_1 x_2^{-1}, v = x_2^1.$ Taking in account the $\mu_d(k) \times \mu_d(k)$-action, we get

$$\chi_c(F_d^2 \setminus \{x_2 = 0\}, (\alpha, \alpha^{-1})) = \chi_c((u^d - v^d = -1; v \neq 0), (\alpha, 1)).$$

By Remark 2.3.5,

$$\chi_c((u^d - v^d = -1; v \neq 0), (\alpha, 1)) = \chi_c((u^d = v - 1; v \neq 0), \alpha)$$

$$= \chi_c(A^1_\alpha, \alpha) - \chi_c(u^d = -1, \alpha)$$

$$= -[\alpha(-1)],$$

because $\chi_c(A^1_\alpha, \alpha) = 0$ by Lemma 2.3.2 and Proposition 2.3.1.

To prove (4) we will consider the morphism

$$f : F_d^2 \times (F_d^2 \setminus \{y_1 = 0\}) \longrightarrow F_d^3 \setminus \{z_1^d + z_2^2 = 0\}$$

given by

$$(x_1, x_2, (y_1, y_2)) \longmapsto (z_1 = x_1 y_1, z_2 = x_2 y_1, z_3 = y_2).$$

This morphism identifies $F_d^3 \setminus \{z_1^d + z_2^2 = 0\}$ with the quotient of $F_d^2 \times (F_d^2 \setminus \{y_1 = 0\})$ by the kernel $\Gamma$ of the morphism $\mu_d(k)^4 \rightarrow \mu_d(k)^3$ given by $(\xi_1, \xi_2, \xi_3, \xi_4) \mapsto (\xi_1 \xi_3, \xi_2 \xi_3, \xi_4)$. It follows from Remark 2.3.5 and Proposition 2.3.1 (3) that

$$\chi_c(F_d^2, (\alpha_1, \alpha_2)) \cdot \chi_c(F_d^2 \setminus \{y_1 = 0\}, (\alpha_1 \alpha_2, \alpha_3))$$

$$= \chi_c(F_d^3 \setminus \{z_1^d + z_2^2 = 0\}, (\alpha_1, \alpha_2, \alpha_3)).$$

Indeed, $f$ extends to the rational map

$$W_d^2 \times W_d^2 \longrightarrow W_d^3$$

given by

$$([X_0, X_1, X_2], [Y_0, Y_1, Y_2]) \longmapsto [Z_0 = X_0 Y_0, Z_1 = X_1 Y_1, Z_2 = X_2 Y_1, Z_3 = X_0 Y_2].$$

Now let $Z$ and $Z'$ denote respectively the blow-up of $W_d^2 \times W_d^2$ along $\{X_0 = 0\} \times \{Y_1 = 0\}$ and the blow-up of $W_d^3$ along $\{Z_0 = Z_3 = 0\} \cup \{Z_1 = Z_2 = 0\}.$

It is classical (see [14]) that $f$ extends to a morphism $\bar{f} : Z \mapsto Z'$, that the actions of $\mu_d(k)^4$ and $\mu_d(k)^3$ extend to actions on $Z$ and $Z'$ respectively, and that $\bar{f}$ identifies $Z'$ with the quotient of $Z$ by $\Gamma$. (Of course one could also use here Theorem 2.3.4 directly instead of Remark 2.3.5.) To complete the proof of (4), we only have to prove that $\chi_c(F_d^2 \cap \{y_1 = 0\}, (\alpha_1 \alpha_2, \alpha_3)) = \varepsilon$ and that $\chi_c(F_d^3 \cap \{z_1^d + z_2^2 = 0\}, (\alpha_1, \alpha_2, \alpha_3)) = \delta$. The first equality is clear since, by Proposition 2.3.1.

$$\chi_c(F_d^2 \cap \{y_1 = 0\}, (\alpha_1 \alpha_2, \alpha_3)) = \chi_c(y_1 = 0, \alpha_1 \alpha_2) \cdot \chi_c(y_2^2 = 1, \alpha_3).$$
To prove the second one, we remark that
\[ \chi_c(F^3_d \cap \{ z_1^d + z_2^d = 0 \} \cap \{ z_1 \text{ or } z_2 \neq 0 \}, (\alpha_1, \alpha_2, \alpha_3)) = \chi_c(\mu_d^{-1}, \alpha_1) \cdot \chi_c(G_{m,k}, \alpha_1 \alpha_2) \cdot \chi_c(\mu_d, \alpha_3). \]

This follows from Proposition 2.3.1, by using the change of variable
\[ u = z_1 z_2^{-1}, \quad v = z_2 \text{ and } w = z_3. \]
The result is now a consequence of Lemma 2.3.2, since
\[ \chi_c(F^3_d \cap \{ z_1^d + z_2^d = 0 \} \cap \{ z_1 = z_2 = 0 \}, (\alpha_1, \alpha_2, \alpha_3)) \text{ is equal to } 1 \text{ or } 0, \]
according whether \( \alpha_1 \) and \( \alpha_2 \) are both trivial or not.

2.5. We assume from now on that \( k \) contains all the roots of unity. When \( d \) divides \( d' \) we have a canonical surjective morphism of groups \( \mu_d(k) \to \mu_d(k) \)
given by \( x \mapsto x^{d'/d} \) which dualizes to a injective morphism of character groups \( \hat{\mu}_d(k) \to \hat{\mu}_{d'}(k) \). We set \( \hat{\mu}(k) := \lim_{\to} \hat{\mu}_d(k) \). We shall identify \( \hat{\mu}_d(k) \) with the subgroup of elements of order dividing \( d \) in \( \hat{\mu}(k) \).

We denote by \( F \) the subfield of \( C \) generated by the roots of unity and we set
\[ A := K_0(\mathcal{M}_{k,F}) \]
and
\[ \hat{A} := \hat{K}_0(\mathcal{M}_{k,F}). \]

We have a natural ring morphism \( A_d \to A \). When no confusion occurs, we will still denote by the same symbol the image in \( A \) of an element in \( A_d \). In particular, if \( \alpha_1 \) and \( \alpha_2 \) are elements of \( \hat{\mu}(k) \) which are images of elements \( \widehat{\alpha}_1 \) and \( \widehat{\alpha}_2 \) of \( \hat{\mu}_d(k) \), we denote by \( J(\alpha_1, \alpha_2) \) the image of \( J(\widehat{\alpha}_1, \widehat{\alpha}_2) \) in \( A \), which is independent from the choice of \( d \), by Remark 2.3.3. For \( \alpha \) in \( \hat{\mu}(k) \) one defines similarly \( [\alpha(-1)] \) in \( A \).

We will now consider the ring \( U \) obtained from the ring \( A \) by adding all the Gauss sums motives associated to \( \hat{\mu}(k) \). This construction is strongly reminiscent of Anderson’s contruction of “ulterior motives” \( \square \).

We define \( U \) as the free \( A \)-module with basis \( G_\alpha \), \( \alpha \) in \( \hat{\mu}(k) \). We define an \( A \)-algebra structure on \( U \) by putting the following relations :

\[
\begin{align*}
G_1 &= -1 \\
G_\alpha G_{\alpha^{-1}} &= [\alpha(-1)]L \quad \text{for } \alpha \neq 1 \\
G_{\alpha_1} G_{\alpha_2} &= J(\alpha_1, \alpha_2) G_{\alpha_1 \alpha_2} \quad \text{for } \alpha_1, \alpha_2, \alpha_1 \alpha_2 \neq 1.
\end{align*}
\]

**Proposition 2.5.1.** The algebra \( U \) is associative and commutative.

**Proof.** The commutativity is clear and the associativity follows directly from Proposition 2.4.1. \( \square \)
3. Motivic integrals of multiplicative characters

3.1. Let $X$ be a $k$-variety, i.e. a separated and reduced $k$-scheme of finite type. We will denote by $\mathcal{L}(X)$ the scheme of germs of arcs on $X$. It is a scheme over $k$ and for any field extension $k \subset K$ there is a natural bijection

$$\mathcal{L}(X)(K) \simeq \text{Mor}_{k\text{-schemes}}(\text{Spec } K[[t]], X)$$

between the set of $K$-rational points of $\mathcal{L}(X)$ and the set of germs of arcs with coefficients in $K$ on $X$. We will call $K$-rational points of $\mathcal{L}(X)$, for $K$ a field extension of $k$, arcs on $X$, and $\varphi(0)$ will be called the origin of the arc $\varphi$. More precisely the scheme $\mathcal{L}(X)$ is defined as the projective limit

$$\mathcal{L}(X) := \lim_{\leftarrow} \mathcal{L}_n(X)$$

in the category of $k$-schemes of the schemes $\mathcal{L}_n(X)$ representing the functor

$$R \mapsto \text{Mor}_{k\text{-schemes}}(\text{Spec } R[t]/t^{n+1}R[t], X)$$

defined on the category of $k$-algebras. (The existence of $\mathcal{L}_n(X)$ is well known (cf. [5]) and the projective limit exists since the transition morphisms are affine.) We shall denote by $\pi_n$ the canonical morphism, corresponding to truncation of arcs,

$$\pi_n : \mathcal{L}(X) \rightarrow \mathcal{L}_n(X).$$

The schemes $\mathcal{L}(X)$ and $\mathcal{L}_n(X)$ will always be considered with their reduced structure.

3.2. In [5], we defined the boolean algebra $\mathcal{B}_X$ of semi-algebraic subsets of $\mathcal{L}(X)$. We will refer to [5] for the precise definition, but we recall that if $A$ is a semi-algebraic subset of $\mathcal{L}(X)$, the image $\pi_n(A)$ is constructible in $\mathcal{L}_n(X)$, and that for $f : X \rightarrow Y$ a morphism of $k$-varieties, the image by $f$ of any semi-algebraic subset of $\mathcal{L}(X)$ is a semi-algebraic subset of $\mathcal{L}(Y)$. Both statements are consequences from results by Pas [3].

Let $A$ be a semi-algebraic subset of $\mathcal{L}(X)$. We call $A$ weakly stable at level $n \in \mathbb{N}$ if $A$ is a union of fibers of $\pi_n : \mathcal{L}(X) \rightarrow \mathcal{L}_n(X)$. We call $A$ weakly stable if it stable at some level $n$. Note that weakly stable semi-algebraic subsets form a boolean algebra.

Let $X$, $Y$ and $F$ be algebraic varieties over $k$, and let $A$, resp. $B$, be a constructible subset of $X$, resp. $Y$. We say that a map $\pi : A \rightarrow B$ is a piecewise morphism, if there exists a finite partition of $B$ in subsets $S$ which are locally closed in $Y$ such that $\pi^{-1}(S)$ is locally closed in $X$ and such that the restriction of $\pi$ to $\pi^{-1}(S)$ is a morphism of $k$-varieties. We say that a map $\pi : A \rightarrow B$ is a piecewise trivial fibration with fiber $F$, if there exists a finite partition of $B$ in subsets $S$ which are locally closed in $Y$ such that $\pi^{-1}(S)$ is locally closed in $X$ and isomorphic, as a variety over $k$, to $S \times F$, with $\pi$ corresponding under the isomorphism to the projection $S \times F \rightarrow S$. We say that the map $\pi$ is a piecewise trivial fibration over some constructible subset $C$ of $B$, if the restriction of $\pi$ to
\(\pi^{-1}(C)\) is a piecewise trivial fibration onto \(C\). Let \(X\) be an algebraic variety over \(k\) of pure dimension \(m\) and let \(A\) be a semi-algebraic subset of \(L(X)\). We call \(A\) stable at level \(n \in \mathbb{N}\), if \(A\) is weakly stable at level \(n\) and \(\pi_{i+1}(L(X)) \to \pi_i(L(X))\) is a piecewise trivial fibration over \(\pi_i(A)\) with fiber \(A^m_k\) for all \(i \geq n\).

We call \(A\) stable if it stable at some level \(n\). Note that the family of stable semi-algebraic subsets of \(L(X)\) is closed under taking finite intersections and finite unions. If \(X\) is smooth, then \(A\) is stable at level \(n\) if it is weakly stable at level \(n\).

3.3. We fix an integer \(d \geq 1\) and assume \(k\) contains all \(d\)-roots of unity. Let \(f : X \to G_{m,k}\) be a morphism of \(k\)-varieties. For any character \(\alpha\) of order \(d\) of \(\mu_d(k)\), one may define an element \([X, f^*L_\alpha]\) of \(A_d\) as follows.

The morphism \([d] : G_{m,k} \to G_{m,k}\) given by \(x \mapsto x^d\) is a Galois covering with Galois group \(\mu_d(k)\). We consider the fiber product

\[
\tilde{X}_{f,d} \to X
\]

\[
\downarrow \quad \downarrow f
\]

\[
G_{m,k} \to G_{m,k}.
\]

The scheme \(\tilde{X}_{f,d}\) is endowed with an action of \(\mu_d(k)\), so we can define

\([X, f^*L_\alpha] := \chi_c(\tilde{X}_{f,d}, \alpha)\).

More generally, if \(X\) is constructible in some \(k\)-variety and if \(f : X \to G_{m,k}\) is a piecewise morphism, one may define \([X, f^*L_\alpha] = \sum_{S \in S}[S, f^*_S L_\alpha]\), by taking an appropriate partition \(S\) of \(X\) into locally closed subvarieties, using the additivity of \(\chi_c\).

The following statement follows directly from the additivity of \(\chi_c\).

**Lemma 3.3.1.** Let \(X\) and \(Y\) be constructible in some \(k\)-varieties, and let \(f : X \to G_{m,k}\) and \(g : Y \to X\) be piecewise morphisms. Assume that \(g\) is piecewise trivial fibration with fiber \(F\). For any character \(\alpha\) of order \(d\) of \(\mu_d(k)\), the following holds

\([Y, (f \circ g)^*L_\alpha] = \chi_c(F)[X, f^*L_\alpha]\). □

3.4. Let \(X\) be an algebraic variety over \(k\) of pure dimension \(m\) and let \(f : X \to A^1_k\) be a morphism of \(k\)-varieties. By the very definition of semi-algebraic subsets, the set

\[
\{\text{ord}_t f = i\} := \{\varphi \in L(X) \mid \text{ord}_t f \circ \varphi = i\}
\]

is a semi-algebraic subset of \(L(X)\), for any integer \(i \geq 0\). One defines similarly the semi-algebraic subset \(\{\text{ord}_t f > i\}\). We will denote by \(\tilde{f}_i\) the mapping \(\{\text{ord}_t f = i\} \to G_{m,k}\) which to a point \(\varphi\) in \(\{\text{ord}_t f = i\}\) associates the constant term of the series \(t^{-i}(f \circ \varphi)\). (Sometimes we shall use the same notation \(f_i\) to denote the natural extension \(\{\text{ord}_t f \geq i\} \to A^1_k\).) Now let \(W\) be a stable semi-algebraic
subsets of $\mathcal{L}(X)$ which is contained in $\{\text{ord}_t f = i\}$ for some $i$. Choose an integer $n \geq i$ such that $W$ is stable at level $n$. The mapping $\tilde{f}_i$ factors to a piecewise morphism

$$\tilde{f}_{i|\pi_n(W)} : \pi_n(W) \rightarrow G_{m,k}.$$

Let $\alpha$ be a character of order $d$ of $\mu_d(k)$, which we also view as an element of $\hat{\mu}(k)$. By Lemma 3.3.1, the virtual motive $[\pi_n(W), \tilde{f}_{i|\pi_n(W)}^* L_{\alpha}] L^{-(n+1)m}$ is independent of $n$. So we may define

$$\int_{W} \alpha(\text{ac}f) d\mu$$

as the image of

$$[\pi_n(W), \tilde{f}_{i|\pi_n(W)}^* L_{\alpha}] L^{-(n+1)m}$$

in $A$.

**Proposition 3.4.1.** Let $X$ be an algebraic variety over $k$ of pure dimension $m$ and let $f : X \rightarrow \mathbb{A}^1_k$ be a morphism of $k$-varieties. For any $\alpha$ in $\hat{\mu}(k)$, there exists a unique map

$$W \mapsto \int_{W} \alpha(\text{ac}f) d\mu$$

from $B_X$ to $\hat{A}$ satisfying the following three properties.

3.4.2. If $W \in B_X$ is stable and contained in $\{\text{ord}_t f = i\}$ for some $i$, then $\int_{W} \alpha(\text{ac}f) d\mu$ coincides with the image of $\int_{W'} \alpha(\text{ac}f) d\mu$ in $\hat{A}$. 

3.4.3. If $W \in B$ is contained in $\mathcal{L}(S)$ with $S$ a closed subvariety of $X$ with $\dim S < \dim X$, then $\int_{W} \alpha(\text{ac}f) d\mu = 0$.

3.4.4. Let $W_i$ be in $B_X$ for each $i \in \mathbb{N}$. Assume that the $W_i$'s are mutually disjoint and that $W := \bigcup_{i \in \mathbb{N}} W_i$ is semi-algebraic. Then the series

$$\sum_{i \in \mathbb{N}} \int_{W_i} \alpha(\text{ac}f) d\mu$$

is convergent in $\hat{A}$ and converges to $\int_{W} \alpha(\text{ac}f) d\mu$.

Moreover we have

3.4.5. If $W$ and $W'$ are in $B_X$, $W \subset W'$, and if $\int_{W'} \alpha(\text{ac}f) d\mu$ belongs to the closure $F^m\hat{A}$ of $F^mA$ in $\hat{A}$, then $\int_{W} \alpha(\text{ac}f) d\mu$ belongs to $F^m\hat{A}$.

**Proof.** Completely similar to the proof of Proposition 3.2 of [5].

In the paper [5], we also defined simple functions $W \rightarrow \mathbb{Z}$, for $W$ in $B_X$. A typical example of a simple function is the following. Consider a coherent sheaf of ideals $\mathcal{I}$ on $X$ and denote by $\text{ord}_t \mathcal{I}$ the function $\text{ord}_t \mathcal{I} : \mathcal{L}(X) \rightarrow \mathbb{N} \cup \{+\infty\}$ given by $\varphi \mapsto \min_g \text{ord}_t g(\varphi)$, where the minimum is taken over all $g$ in the stalk $\mathcal{L}_{\pi_0(\varphi)}$ of $\mathcal{I}$ at $\pi_0(\varphi)$. The function $\text{ord}_t \mathcal{I}$ is a simple function.
Hence, for $W$ in $B_X$ and $\lambda : W \to \mathbb{Z} \cup \{+\infty\}$ a simple function, we can define
\[
\int_W \alpha(ac(f)) L^{-\lambda} d\mu := \sum_{n \in \mathbb{Z}} \int_{W \cap \lambda^{-1}(n)} \alpha(ac(f)) d\mu L^{-n}
\]
in $\hat{A}$, whenever the right hand side converges, in which case we say that $\alpha(ac(f)) L^{-\lambda}$ is integrable on $W$. If the function $\lambda$ is bounded from below, then $\alpha(ac(f)) L^{-\lambda}$ is integrable on $W$, because of (3.4.5).

For $Y$ a $k$-variety, we denote by $\Omega^1_Y$ the sheaf of differentials on $Y$ and by $\Omega^d_Y$ the $d$-th exterior power of $\Omega^1_Y$. If $Y$ is smooth and $F$ is a coherent sheaf on $Y$ together with a natural morphism $\iota : F \to \Omega^d_Y$, we denote by $I(F)$ the sheaf of ideals on $Y$ which is locally generated by functions $\iota(\omega)/d\gamma$ with $\omega$ a local section of $F$ and $d\gamma$ a local volume form on $Y$, and by $\text{ord}_x F$ the simple function $\text{ord}_x I(F)$.

**Proposition 3.4.6.** Let $h : Y \to X$ be a proper birational morphism, with $Y$ smooth, let $W$ be a semi-algebraic subset of $\mathcal{L}(X)$ and let $\lambda : \mathcal{L}(X) \to \mathbb{N}$ be a simple function. Then
\[
\int_W \alpha(ac(f)) L^{-\lambda} d\mu = \int_{h^{-1}(W)} \alpha(ac(f \circ h)) L^{-\lambda \circ h - \text{ord}_x h^*(\Omega^d_X)} d\mu.
\]

**Proof.** Follows directly from Lemma 3.4 of [3]. \qed

4. **Motivic exponential integrals and the main result**

4.1. We set $\hat{U} := U \otimes_A \hat{A}$. We will also consider the subring $A_{\text{loc}}$ of $\hat{A}$ generated by the image of $A$ in $\hat{A}$ and the series $(1 - L^{-n})^{-1}$, $n \in \mathbb{N} \setminus \{0\}$. We denote by $U_{\text{loc}}$ the tensor product $U \otimes_A A_{\text{loc}}$, which is naturally a subring of $\hat{U}$.

4.2. Let $X$ be an irreducible algebraic variety over $k$ and let $f : X \to \mathbb{A}^1_k$ be a morphism of $k$-varieties. Let $D$ be the divisor defined by $f = 0$ in $X$. By a very good resolution of $(X, D)$, we mean a couple $(Y, h)$ with $Y$ is a smooth and connected $k$-scheme of finite type, $h : Y \to X$ a proper morphism, such that the restriction $h : Y \setminus h^{-1}(D \cup \text{Sing}X) \to X \setminus (D \cup \text{Sing}X)$ is an isomorphism, and such that the ideal sheaf $\mathcal{I}(h^*(\Omega^d_X))$ is invertible (this last condition is of course irrelevant when $X$ is smooth), and such that the union of $(h^{-1}(D))_{\text{red}}$ and the support of the divisor associated to $\mathcal{I}(h^*(\Omega^d_X))$ has only normal crossings as a subscheme of $Y$. Such resolutions always exist by Hironaka’s Theorem.

Let $E_i$, $i \in J$, be the irreducible (smooth) components of $(h^{-1}(D \cup \text{Sing}X))_{\text{red}}$. Let $W$ be a reduced subscheme of $f^{-1}(0)$. We call a very good resolution $(Y, h)$ of $(X, D)$ a very good resolution of $(X, D, W)$ if $(h^{-1}(W))_{\text{red}}$ is the reunion of some $E_i$’s.
Let $d$ be a positive integer $\geq 1$. We will say $d$ is big with respect to $(f,g,W)$ if
\[
\int_{\pi_0^{-1}(W)} \alpha(acf) L^{-ord_d} d\mu = 0
\]
whenever the order of $\alpha$ does not divide $d$.

The following Proposition will be proven together with Theorem 4.2.4.

**Proposition 4.2.1.** Let $X$ be an irreducible algebraic variety over $k$, let $f : X \to A^1_k$ and $g : X \to A^1_k$ be morphisms of $k$-varieties. Let $W$ be a reduced subscheme of $\pi^{-1}(0)$. Let $d$ be a positive integer $\geq 1$. Assume there exists a very good resolution $(Y,h)$ of $(X,f \circ g = 0,W)$ such that $d$ is a multiple of the multiplicities of the $E_i$'s in the divisor of $f \circ h$ on $Y$. Then $d$ is big with respect to $(f,g,W)$.

**Definition 4.2.2.** Let $X$ be an irreducible algebraic variety over $k$, let $f : X \to A^1_k$ and $g : X \to A^1_k$ be morphisms of $k$-varieties. Let $W$ be a reduced subscheme of $\pi^{-1}(0)$. For integers $i \geq 0$, we set
\[
\left(4.2.2\right) \int_{\pi_0^{-1}(W)} \exp(t^{-(i+1)}f) L^{-ord_d} d\mu := \int_{\pi_0^{-1}(W)\cap\{ord_d > i\}} L^{-ord_d} d\mu
\]
\[
+ \sum_{\alpha \in \hat{\mu}(k)} \frac{1}{L-1} G_{\alpha} \int_{\pi_0^{-1}(W)\cap\{ord_d = i\}} \alpha(acf) L^{-ord_d} d\mu
\]
in $\hat{U}$. The sum is finite since, by Proposition 4.2.1, there exists an integer $d$ which is big with respect to $(f,g,W)$.

**Remark 4.2.3.** The integral (4.2.2) belongs to $U_{loc}$ because of Proposition 5.1 in [3] and the definitions in 3.4.

If $f : X \to A^1_k$ and $f' : X' \to A^1_k$ are morphisms of varieties, we denote by $f \oplus f' : X \times X' \to A^1_k$ the morphism given by composition of the morphism $(f,f')$ with the addition morphism $\oplus : A^1_k \times A^1_k \to A^1_k$.

We can now state the main result of the paper.

**Main Theorem 4.2.4.** Let $X$ and $X'$ be irreducible algebraic varieties over $k$, let $f : X \to A^1_k$, $g : X \to A^1_k$, $f' : X' \to A^1_k$ and $g' : X' \to A^1_k$ be morphisms of $k$-varieties. Let $W$ (resp. $W'$) be a reduced subscheme of $f^{-1}(0)$ (resp. $f'^{-1}(0)$). For every $i \geq 0$,
\[
\left(4.2.4\right) \int_{\pi_0^{-1}(W \times W')} \exp(t^{-(i+1)}(f \oplus f')) L^{-ord_d} d\mu = \left(\int_{\pi_0^{-1}(W)} \exp(t^{-(i+1)}f) L^{-ord_d} d\mu\right) \cdot \left(\int_{\pi_0^{-1}(W')} \exp(t^{-(i+1)}f') L^{-ord_d} d\mu\right).
\]
Remark 4.2.5. Of course, by relations (2.5.1)-(2.5.3), (1.2.4) is equivalent to the following relations in \( A_{\text{loc}} \), where, for notational convenience, we will write \([\alpha; f; g]\) for the integrand \( \alpha(acf) L^{-\ord g} d\mu \),

(4.2.6) \[
\int_{\pi_0^{-1}(W \times W^\prime) \cap \{\ord f \neq f'\}} [\alpha; f \oplus f'; gg'] = \frac{1}{L-1} \sum_{\alpha_1 \neq 1 \alpha_2 \neq 1} \frac{1}{\alpha_1 \alpha_2} J(\alpha_1^{-1}, \alpha_2^{-1}) \cdot \\
\left( \int_{\pi_0^{-1}(W) \cap \{\ord f = i\}} [\alpha_1; f; g] \right) \cdot \left( \int_{\pi_0^{-1}(W') \cap \{\ord f' = i\}} [\alpha_2; f'; g'] \right) \\
- \frac{1}{L-1} \left( \int_{\pi_0^{-1}(W) \cap \{\ord f = i\}} [1; f; g] \right) \cdot \left( \int_{\pi_0^{-1}(W') \cap \{\ord f' = i\}} [\alpha; f'; g'] \right) + \cdots \\
+ \left( \int_{\pi_0^{-1}(W) \cap \{\ord f > i\}} [1; f; g] \right) \cdot \left( \int_{\pi_0^{-1}(W') \cap \{\ord f' = i\}} [\alpha; f'; g'] \right) + \cdots,
\]

for \( \alpha \neq 1 \) (here \( \cdots \) means “same term as before with \( (W, f, g) \) and \( (W', f', g') \) interchanged”) and

(4.2.7) \[
\int_{\pi_0^{-1}(W \times W^\prime) \cap \{\ord f \neq f'\}} [1; f \oplus f'; gg'] = \\
\frac{1}{L-1} \int_{\pi_0^{-1}(W \times W^\prime) \cap \{\ord f \neq f'\}} [1; f \oplus f'; gg'] = \\
\sum_{\alpha \neq 1} [\alpha(-1)] L \left( \int_{\pi_0^{-1}(W) \cap \{\ord f = i\}} [\alpha; f; g] \right) \cdot \left( \int_{\pi_0^{-1}(W') \cap \{\ord f' = i\}} [\alpha^{-1}; f'; g'] \right) \\
+ \left( \int_{\pi_0^{-1}(W) \cap \{\ord f > i\}} [1; f; g] \right) - \frac{1}{L-1} \int_{\pi_0^{-1}(W) \cap \{\ord f = i\}} [1; f; g] \cdot \left( \cdots \right).
\]

Again, the sums are finite by Proposition 4.2.1.

4.3. Proof of Proposition [4.2.1] and Theorem [4.2.4]. Applying Proposition 3.4.6 to very good resolutions of \( (X, f g = 0, W) \) and \( (X', f' g' = 0, W') \) and using additivity of \( \chi \), one reduces to the case when \( X \) is smooth of dimension \( m \), \( x_1, \ldots, x_m \) are regular functions on \( X \) inducing an étale map \( X \to A^m_k \), \( f = u \prod_{i=1}^r x_i^{n_i} \), with \( n_i > 0 \), \( g = v \prod_{i=1}^m x_i^{m_i} \), with \( m_i \geq 0 \), \( u \) and \( v \) are units, \( W \) is the union of the hypersurfaces \( x_i = 0 \), for \( i \in I \subset \{1, \ldots, r\} \), and similarly with \( f' \) for \( X', f', g' \) and \( W' \). We call this situation the DNC case.

We denote by \( F^n_d \) the open subvariety of \( F^n_d \) defined by \( x_i \neq 0, i = 1, \ldots n \). It is stable under the \( \mu_d(k)^n \)-action.

Proposition 4.3.1. Assume \( d \) is big with respect to \( (f, g, W) \) and \( (f', g', W') \).

For any \( \alpha \) in \( \widehat{\mu(k)} \) of order dividing \( d \) and any integer \( i \geq 0 \), the following relation
holds in $A$,

$$
\int_{\pi_0^{-1}(W \times W') \cap \{\text{ord}_i f = i\}} \frac{1}{L - 1} \sum_{\alpha_1 \alpha_2 = \alpha} \chi_c(\bar{F}_d^2, (\alpha_1^{-1}, \alpha_2^{-1})) 
\cdot \left( \int_{\pi_0^{-1}(W) \cap \{\text{ord}_i f = i\}} [\alpha_1; f; g] \right) 
\cdot \left( \int_{\pi_0^{-1}(W') \cap \{\text{ord}_i f' = i\}} [\alpha_2; f'; g'] \right).
$$

Here we keep the same notation for $\chi_c(\bar{F}_d^2, (\alpha_1^{-1}, \alpha_2^{-1}))$ and its image in $A$, and $\alpha_1$ and $\alpha_2$ are assumed to be of order dividing $d$.

**Proof.** By Proposition 3.4.6, one reduces to the DNC case. For $i = 0$ the result is clear, the domain of integration being empty, so we may assume $i \geq 1$. Now remark that $\pi_0^{-1}(W) \cap \{\text{ord}_i f = i\}$ is the disjoint union of semi-algebraic sets

$$
W_\gamma := \{\varphi \mid \text{ord}_i x_j(\varphi) = \gamma_j\},
$$

for $\gamma = (\gamma_1, \ldots, \gamma_r)$, with $\sum_{j=1}^r \gamma_j n_j = i$ and $\gamma_j > 0$ when $j \in I$. On $W_\gamma$ we may consider the function $\bar{x}_j : W_\gamma \to G_{m,k}$ which to a point $\varphi$ associates the constant term of the series $t^{-\gamma_j}x_j(\varphi)$, for $1 \leq j \leq r$. Set

$$
K = \{j \in \{1, \ldots, r\} \mid \gamma_j > 0\}.
$$

The assumptions made imply that $K$ is not empty. Now remark that on $W_\gamma$,

$$
\bar{f}_i = u \prod_{j \in K} \bar{x}_j^{n_j},
$$

with $u$ a unit which is a function of the $\bar{x}_j$'s for $j \notin K$ only. Hence there exists $\ell > 0$ such that

$$
W_\gamma = \pi_\ell^{-1}(Z \times G_{m,k}^K),
$$

with $Z$ a smooth variety, and such that the restriction of $\bar{f}_i$ to $W_\gamma$ is equal to

$$
(u \prod_{j \in K} t_j^{n_j}) \circ \pi_\ell,
$$

with $u$ a unit on $Z$ and $t_j$ the canonical coordinate on the corresponding $G_{m,k}^K$ factor. Let $a$ be the $\text{gcd}$ of the $n_i$’s. By using an appropriate torus isomorphism of $G_{m,k}^K$ and changing $Z$, one deduces that

$$
W_\gamma = \pi_\ell^{-1}(Z \times G_{m,k}),
$$

with $Z$ a smooth variety, and that the restriction of $\bar{f}_i$ to $W_\gamma$ is equal to $(ut^a) \circ \pi_\ell$, with $u$ a unit on $Z$ and $t$ the canonical coordinate on the $G_{m,k}$ factor. Now the result is a direct consequence of the next Lemma and the following Proposition. 

$\square$
Lemma 4.3.2. With the above notations and assumptions, if the order of \( \alpha_1 \) does not divide \( a \), then
\[
\int_{W_\gamma} [\alpha_1; f; g] = 0.
\]

Proof. It is a direct consequence of Lemma 1.4.4 of [4] (compare with the proof of Proposition 4.3.3).

Remark that Proposition 4.2.1 follows now directly from Lemma 4.3.2.

Proposition 4.3.3. Let \( u_1 : X_1 \to G_{m,k} \) and \( u_2 : X_2 \to G_{m,k} \) be morphisms of algebraic varieties over \( k \). Let \( a, b \) and \( d \) be in \( \mathbb{N} \setminus \{0\} \). Assume \( a \) and \( b \) divide \( d \). Denote by \((X_1 \times G_{m,k} \times X_2 \times G_{m,k})^0\) the complement in \( X_1 \times G_{m,k} \times X_2 \times G_{m,k} \) of the divisor of \( u_1 t_1^a + u_2 t_2^b \). For any character \( \alpha \) of \( \mu_d(k) \),
\[
\int_{X_1 \times G_{m,k} \times X_2 \times G_{m,k}} (u_1 t_1^a + u_2 t_2^b)^* L_\alpha = (L - 1) \sum_{\substack{\alpha_1, \alpha_2 = \alpha \\alpha_i^d = 1}} \chi_c(F_d^2, (\alpha_1^{-1}, \alpha_2^{-1}))[X_1, u_1^* L_{\alpha_1}][X_2, u_2^* L_{\alpha_2}].
\]

Proof. By definition the left hand side LHS of (4.3.3) is equal to
\[
\chi_c(u_1(x_1)t_1^a + u_2(x_2)t_2^b = w^d \bigg| (x_1, x_2, t_1, t_2, w) \in X_1 \times X_2 \times G_{m,k}^3, (1, 1, \alpha)\bigg).
\]

Here the action of \( \mu_d(k) \) is the standard one on the last \( G_{m,k} \) and trivial on the other factors. Hence, by Theorem 2.3.4,
\[
\text{LHS} = \chi_c\left(u_1(x_1) = v_1^a, u_2(x_2) = v_2^b, v_1 t_1^a + v_2 t_2^b = w^d\right| (x_1, x_2, t_1, t_2, v_1, v_2, w) \in X_1 \times X_2 \times G_{m,k}^5, (1, 1, \alpha)\right).
\]

Here the group action is the action of \( \mu_a(k) \times \mu_b(k) \times \mu_d(k) \) which is componentwise on the last three \( G_{m,k} \) factors and trivial on the others. Making the toric change of variable
\[
(t_1, t_2, v_1, v_2, w) \longrightarrow (T_1 = v_1 t_1 w^{-d/a}, T_2 = v_2 t_2 w^{-d/b}, v_1, v_2, w),
\]
this may be rewritten as
\[
\text{LHS} = \chi_c\left(u_1(x_1) = v_1^a, u_2(x_2) = v_2^b, T_1^a + T_2^b = 1\right| (x_1, x_2, T_1, T_2, v_1, v_2, w) \in X_1 \times X_2 \times G_{m,k}^5, (1, 1, \alpha)\right).
\]

By Proposition 2.3.1 (3) and Lemma 2.3.2 one deduces
\[
\text{LHS} = (L - 1) \sum_{\substack{\alpha_1 \in \mu_a(k) \\alpha_2 \in \mu_b(k)}} [X_1, u_1^* L_{\alpha_1}][X_2, u_2^* L_{\alpha_2}] \chi_c(F_d^2, (\alpha_1^{-1}, \alpha_2^{-1}, \alpha)).
\]
Here the \( \mu_a(k) \times \mu_b(k) \times \mu_d(k) \)-action on \( F_d^2 \) is given by
\[
(\xi_1, \xi_2, \xi_3) : (T_1, T_2) \mapsto (\xi_3^{-d/a} T_1, \xi_2 \xi_3^{-d/b} T_2).
\]
The result follows because, by Proposition 2.3.1 (4), \( \chi_c(\tilde{F}_d^2, (\alpha_1^{-1}, \alpha_2^{-1}, \alpha)) \) is equal to \( \chi_c(\tilde{F}_d^2, (\alpha_1^{-1}, \alpha_2^{-1})) \) (with the standard \( \mu_a(k) \times \mu_b(k) \)-action) if \( \alpha_1 \alpha_2 = \alpha \) and to 0 otherwise.

Proposition 4.3.4. Let \( X \) and \( X' \) be irreducible algebraic varieties over \( k \), let \( f : X \to A^1_k \), \( g : X \to A^1_k \), \( f' : X' \to A^1_k \) and \( g' : X' \to A^1_k \) be morphisms of \( k \)-varieties. Let \( W \) (resp. \( W' \)) be a reduced subscheme of \( f^{-1}(0) \) (resp. \( f'^{-1}(0) \)). Let \( i \geq 0 \) be an integer.

1. For any \( \alpha \neq 1 \) in \( \tilde{\mu}(k) \) and any integer \( j > i \),
\[
\int_{\pi_0^{-1}(W \times W') \cap \{\text{ord}_t f \oplus f' = j\}} [\alpha; f \oplus f'; gg'] = 0.
\]

2. For any integer \( j > i \),
\[
\int_{\pi_0^{-1}(W \times W') \cap \{\text{ord}_t f \oplus f' = j+1\}} [1; f \oplus f'; gg'] = L^{-1} \int_{\pi_0^{-1}(W \times W') \cap \{\text{ord}_t f = i\}} [1; f \oplus f'; gg'].
\]

Proof. As before one reduces to the DNC case and one uses the fact that
\[
\pi_0^{-1}(W) \cap \{\text{ord}_t f = i\}
\]
is the disjoint union of semi-algebraic sets
\[
W_\gamma := \{ \varphi | \text{ord}_x x_j(\varphi) = \gamma_j \},
\]
for \( \gamma = (\gamma_1, \ldots, \gamma_r) \), with \( \sum_{j=1}^r \gamma_j n_j = i \). We consider again the functions \( \bar{x}_j : W_\gamma \to G_{m_k} \). On \( W_\gamma \) we may write \( \bar{f}_i = v \prod_{1 \leq j \leq r} \bar{x}_j^{n_j} \) with \( v \) a unit, and similarly for \( f' \). Let \( \varphi \) be a point in \( W_\gamma \). We write
\[
x_j(\varphi(t)) = t^{\gamma_j} \bar{x}_j(1 + \sum_{k \geq 1} a_{k,j} t^k),
\]
for \( 1 \leq j \leq r \), and
\[
x_j(\varphi(t)) = \sum_{k \geq 0} a_{k,j} t^k,
\]
for $r < j \leq m$. Similar notation is used for $\varphi'$ in $W'_\gamma$. We may assume that $\gamma_1 \geq 1$. For $\ell > i$, the coefficient of $t_\ell$ in $f(\varphi(t)) + f'(\varphi'(t))$ is equal to

$$\sum_{j=1}^{r} n_j a_{\ell-i,j} \bar{f}_j + P + \sum_{j' = 1}^{r'} n_{j'} a_{\ell-i,j'} \bar{f}_{j'} + P', \tag{3}$$

where $P$ (resp. $P'$) is a polynomial in the variables $a_{k,j}$ and $\bar{x}_j$ (resp. $a'_{k,j'}$ and $\bar{x}'_{j'}$), with $k \leq \ell - i$, having as coefficients regular functions in $\pi_0(\varphi)$ (resp. $\pi_0(\varphi')$). Moreover, since $\gamma_1 \geq 1$, the polynomial $P$ does not involve the variable $a_{\ell-1,1}$ (but might contain the variable $a_{\ell-1,2}$ when $\gamma_2 = 0$).

Let $\Gamma_\ell$ be the locus of $\text{ord}_t(f \oplus f') \geq \ell$ in $\pi_\ell(W_\gamma \times W'_{\gamma'}) \subset \mathcal{L}_\ell(X \times X')$, and let $\Gamma_+^\ell$ be the locus of $\text{ord}_t(f \oplus f') > \ell$ in $\Gamma_\ell$. From (3), for $\ell$ replaced by $i + 1, \ldots, \ell$, it follows that $a_{\ell-i,1}$ does not appear in the equations defining the variety $\Gamma_\ell$, and that $\Gamma_+^\ell$ is the hypersurface of $\Gamma_\ell$ defined by equating (3) to zero.

Taking the function (3) as a new coordinate on $\Gamma_\ell$, instead of $a_{\ell-1,1}$, we see that $\Gamma_\ell \simeq \Gamma_+^\ell \times A_{1,k}$, with the mapping $(f \oplus f')_\ell : \Gamma_\ell \setminus \Gamma_+^\ell \to G_{m,k}$ (which is given by (3)) corresponding to the projection of $\Gamma_+^\ell \times G_{m,k}$ onto the last factor. Assertion (2) follows directly, and assertion (1) is now a consequence from Proposition 2.3.1 and Lemma 2.3.2.

We are now able to conclude the proof of Theorem 1.2.7. Remark first that

$$\chi_c(\tilde{F}^2_{d}, (\alpha_1, \alpha_2)) = \begin{cases} J(\alpha_1, \alpha_2) & \text{if } \alpha_1 \neq 1 \text{ and } \alpha_2 \neq 1 \\ -1 & \text{if } \alpha_1 \neq 1 \text{ and } \alpha_2 = 1 \text{ or } \alpha_1 = 1 \text{ and } \alpha_2 \neq 1 \\ L - 2 \text{ or } L & \text{if } \alpha_1 = 1 \text{ and } \alpha_2 = 1. \end{cases}$$

If $\alpha \neq 1$, relation (1.2.6) follows directly from Proposition 4.3.1 and Proposition 4.3.4 (1). Assume now $\alpha = 1$. We set

$$a_i := \int_{\pi_0^0(W) \cap \{\text{ord}_t f = i\}} [1; f; g] \quad \text{and} \quad A_i := \int_{\pi_0^1(W) \cap \{\text{ord}_t f > i\}} [1; f; g]$$

and define similarly $a'_i$ and $A'_i$. For $k \geq i$, we also set

$$a_{i,k} := \int_{\pi_0^0(W \times W') \cap \{\text{ord}_t (f \oplus f') = k\}} [1; f \oplus f'; g g']_{\{\text{ord}_t f = i\} \cap \{\text{ord}_t f' = i\}}.$$

Let us denote by RHS the right hand side of (1.2.7).

Since, by Proposition 2.4.1, $\chi_c(\tilde{F}^2_{d}, (\alpha, \alpha^{-1}))$ is equal to $- [\alpha(-1)]$ if $\alpha \neq 1$ and to $L - 2$ if $\alpha = 1$, we deduce from Proposition 4.3.1 the relation

$$\text{RHS} = - \frac{L}{L - 1} a_{i,i} + a_i a'_i + A_i A'_i - \frac{1}{L - 1} a_i A'_i - \frac{1}{L - 1} a'_i A_i =$$

$$- \frac{L}{L - 1} \left( a_i a'_i - \sum_{i < \ell} a_{i,\ell} \right) + a_i a'_i + A_i A'_i - \frac{1}{L - 1} a_i A'_i - \frac{1}{L - 1} a'_i A_i.$$
The left hand side LHS of (4.2.7) is equal to
\[ A_i A'_i + \sum_{k \leq i < \ell} a_{k,\ell} - \frac{1}{L - 1} \left( a_i A'_i + a'_i A_i + \sum_{k < i} a_{k,i} - \sum_{i < \ell} a_{i,i} \right). \]

Hence we obtain
\[ \text{LHS} - \text{RHS} = \sum_{k < i < \ell} a_{k,\ell} - \frac{1}{L - 1} \sum_{k < i} a_{k,i} = \sum_{k < i < \ell} a_{k,\ell} - \frac{L - 1}{1 - L - 1} \sum_{k < i} a_{k,i}. \]

The result now follows, since one deduces from Proposition 4.3.4 (2) that, for fixed \( k \) and \( i \), with \( k < i \),
\[ \sum_{i < \ell} a_{k,\ell} = \frac{L - 1}{1 - L - 1} a_{k,i}. \]

5. Motivic Thom-Sebastiani Theorem

5.1. Let \( B \) be any of the rings \( A_{\text{loc}}, \hat{A}, U_{\text{loc}}, \hat{U} \). We consider the ring of Laurent polynomials \( B[T, T^{-1}] \) and its localisation \( B[T, T^{-1}]_{\text{rat}} \) obtained by inverting the multiplicative family generated by the polynomials \( 1 - L^a T^b \), \( a, b \in \mathbb{Z} \), \( b \neq 0 \). Remark in this definition we could restrict to \( b > 0 \) or to \( b < 0 \). Hence, by expanding denominators into formal series, there are canonical embeddings of rings
\[ \exp_T : B[T, T^{-1}]_{\text{rat}} \hookrightarrow B[T^{-1}, T] \]
and
\[ \exp_{T^{-1}} : B[T, T^{-1}]_{\text{rat}} \hookrightarrow B[[T^{-1}, T]]. \]
Here \( B[T^{-1}, T] \) (resp. \( B[[T^{-1}, T]] \)) denotes the ring of series \( \sum_{i \in \mathbb{Z}} a_i T^i \) with \( a_i = 0 \) for \( i \ll 0 \) (resp. \( i \gg 0 \)). By taking the difference \( \exp_T - \exp_{T^{-1}} \) of the two expansions one obtains an embedding
\[ \tau : B[T, T^{-1}]_{\text{rat}} / B[T, T^{-1}] \hookrightarrow B[[T^{-1}, T]], \]
where \( B[[T^{-1}, T]] \) is the group of formal Laurent series with coefficients in \( B \).

Let \( \varphi = \sum_{i \in \mathbb{Z}} a_i T^i \) and \( \psi = \sum_{i \in \mathbb{Z}} b_i T^i \) be series in \( B[[T^{-1}, T]] \). We define their Hadamard product as the series
\[ \varphi \ast \psi := \sum_{i \in \mathbb{Z}} a_i b_i T^i. \]

**Proposition 5.1.1.** Let \( \varphi \) and \( \psi \) be series in \( B[[T^{-1}, T]] \). If they belong to the image of \( \tau \), then their Hadamard product \( \varphi \ast \psi \) is also in the image of \( \tau \).

**Proof.** Let \( P_1 \) and \( P_2 \) be in \( B[T, T^{-1}]_{\text{rat}} \). From the formula
\[ \frac{1}{(1 - L^a T^d)(1 - L^b T^d)} = (L^a - L^b)^{-1} \left( -\frac{L^a}{1 - L^a T^d} - \frac{L^b}{1 - L^b T^d} \right) \]
it follows that there exists \( d \in \mathbb{N} \setminus \{0\} \) such that, modulo \( B[T, T^{-1}] \), both \( P_1 \) and \( P_2 \) are \( B \)-linear combinations of elements \( T^r(1 - L^a T^d)^{-k} \), with \( r, a \in \mathbb{Z}, k \in \mathbb{N} \setminus \{0\} \). Thus, modulo \( B[T, T^{-1}] \), both \( \exp_T(P_1) \) and \( \exp_T(P_2) \) are \( B \)-linear combinations of elements of the form

\[
\varrho := \sum_{n \in \mathbb{N}} f(n) L^{na} T^{nd+r}
\]

in \( B[T, T^{-1}]_{\text{rat}} \), with \( a \in \mathbb{Z}, r \in \mathbb{N}, r < d \), and \( f \) a polynomial with coefficients in \( \mathbb{Q} \) such that \( f(\mathbb{Z}) \subset \mathbb{Z} \). We claim that

\[
\tau(\varrho) = \sum_{n \in \mathbb{Z}} f(n) L^{na} T^{nd+r}.
\]

Indeed, if \( f(n) = \binom{k+n-1}{k-1} \), then \( \varrho = T^r(1 - L^a T^d)^{-k} \), and an explicit calculation, using the relation

\[
\binom{k-m-1}{k-1} = (-1)^{k-1} \binom{m-1}{k-1},
\]

proves the claim in this special case. Hence the claim holds for any \( f \), considering \( f \) as a linear combination of such special \( f \)'s. The Hadamard product of elements of the form \( \varrho \) has again the same form. Thus the claim implies that the Hadamard product commutes with \( \tau \) for elements of the form \( \varrho \), which implies the result by the previous considerations. \( \square \)

Let us denote by \( B[[T]]_{\text{rat}} \) the intersection of \( B[[T]] \) with the image of \( \exp_T \). It follows from the above proposition that \( B[[T]]_{\text{rat}} \) is stable by Hadamard product.

Let \( \varphi = \exp_T(P) \) be in \( B[[T]]_{\text{rat}} \). We denote by \( \lambda(\varphi) \) the constant term in the expansion of \( \exp_T^{-1}(P) \).

**Proposition 5.1.2.** Let \( \varphi \) and \( \psi \) be series in \( TB[[T]]_{\text{rat}} \). Then

\[
\lambda(\varphi \ast \psi) = -\lambda(\varphi) \cdot \lambda(\psi).
\]

**Proof.** Let us remark that \( \lambda(\varphi) \) only depends upon the class of \( \varphi \) modulo additive translation by \( TB[T] \). Hence we may assume there exists \( P \) and \( Q \) in \( B[T, T^{-1}]_{\text{rat}} \) such that \( \exp_T(P) = \varphi, \exp_T(Q) = \psi, \exp_T^{-1}(P) \) and \( \exp_T^{-1}(Q) \) belong to \( B[[T^{-1}]] \). By Proposition 5.1.1 there exists \( R \) in \( B[T, T^{-1}]_{\text{rat}} \) such that \( \exp_T(R) = \varphi \ast \psi \) and \( \exp_T^{-1}(R) = -\exp_T^{-1}(P)\exp_T^{-1}(Q) \). The result follows. \( \square \)

5.2. Let \( X \) be an irreducible algebraic variety over \( k \) of pure dimension \( m \) and let \( f : X \to \mathbb{A}^1_k \) be a morphism. Let \( W \) be a reduced subscheme of \( f^{-1}(0) \). We set

\[
E_{W,f}(T) = \sum_{i>0} \left[ \int_{\pi_0^{-1}(W)} \exp(t^{-(i+1)} f) d\mu \right] T^i
\]
in $U_{\text{loc}}[[T]]$. For any $\alpha$ in $\hat{\mu}(k)$, we set

$$Z_{W,f,\alpha}(T) = \sum_{i>0} \left[ \int_{\pi_0^{-1}(W) \cap \text{ord}_tf=i} \alpha(acf) \, d\mu \right] T^i,$$

in $A_{\text{loc}}[[T]]$. When $X$ is smooth, $Z_{W,f,\alpha}(T)$ is equal to the natural image in $A_{\text{loc}}[[T]]$ of $\int_f (f^*, \alpha)$, with the notation of [1], setting $T = L^{-s}$. Hence it follows from Theorem 2.2.1 of [1] that $Z_{W,f,\alpha}(T)$ belongs to $A_{\text{loc}}[[T]]_{\text{rat}}$. This still holds when $X$ is no more smooth by resolution of singularities and Proposition 3.4.6 (adapting the proof of Theorem 2.2.1 of [1] in a straightforward way).

We set

$$S^\psi_{\alpha,W,f} := \frac{L}{1-L} \lambda(Z_{W,f,\alpha}(T))$$

in $\hat{A}$. When $X$ is smooth and $\alpha$ is of order $d$, we defined in Definition 4.1.2 of [1] an element $S_{\alpha,x}$ of $A_d$, well defined modulo $(L-1)$-torsion, for $x$ a closed point in $f^{-1}(0)$, which corresponds to the $\alpha$-equivariant part of the motivic Euler characteristic of nearby cycles. Its image in $\hat{A}$ is just what we call now $S^\psi_{\alpha,x,f}$.

Remark there is no $(L-1)$-torsion in $\hat{A}$ (see also Remark 2.1.1). By Theorem 2.2.1 and Lemma 4.1.1 of [1], $S^\psi_{\alpha,W,f}$ belongs to the image of $\hat{A}$, even when $X$ is no more smooth, by resolution of singularities and Proposition 3.4.6.

To deal with vanishing cycles we set $S^\phi_{\alpha,W,f} = S^\psi_{\alpha,W,f}$ for $\alpha \neq 1$, and $S^\phi_{\alpha,W,f} = S^\psi_{\alpha,W,f} - \chi_c(W)$ for $\alpha = 1$.

**Proposition 5.2.1.** The series $E_{W,f}(T)$ belongs to $U_{\text{loc}}[[T]]_{\text{rat}}$ and

$$\lambda(E_{W,f}(T)) = -L^{-m} \sum_{\alpha \in \hat{\mu}(k)} \sum_{G_{\alpha^{-1}} S^\phi_{\alpha,W,f}}.$$

**Proof.** By the very definitions,

$$E_{W,f}(T) = \frac{1}{L-1} \sum_{\alpha \in \hat{\mu}(k)} G_{\alpha^{-1}} Z_{W,f,\alpha}(T) + P(T)$$

with

$$P(T) = \sum_{i \geq 0} \int_{\pi_0^{-1}(W) \cap \{\text{ord}_tf>i\}} \pi_0^{-1}(W) \cap \{\text{ord}_tf>0\}} \, d\mu \, T^i - \int_{\pi_0^{-1}(W) \cap \{\text{ord}_tf>0\}} \, d\mu.$$

One may write $P(T)$ as

$$P(T) = (T - 1)^{-1} \left( Z_{W,f,1}(T) - \int_{\pi_0^{-1}(W) \cap \{\text{ord}_tf>0\}} \, d\mu \right) - \int_{\pi_0^{-1}(W)} \, d\mu.$$

Since it follows from Theorem 2.2.1 of [1] that $Z_{W,f,1}(T)$ belongs to $A_{\text{loc}}[[T]]_{\text{rat}}$ and that $\exp_{T^{-1}}(Z_{W,f,1}(T))$ belongs to $A_{\text{loc}}[[T^{-1}]]$, we deduce that $\lambda(P(T)) = -L^{-m} \chi_c(W)$. 


Motivic Thom-Sebastiani Theorem 5.2.2. Let \( X \) and \( X' \) be irreducible algebraic varieties over \( k \) of pure dimension \( m \) and \( m' \). Let \( f : X \to A_k^1 \) and \( f' : X' \to A_k^1 \) be morphisms of \( k \)-varieties. Let \( W \) (resp. \( W' \)) be a reduced subscheme of \( f^{-1}(0) \) (resp. \( f'^{-1}(0) \)). Then
\[
\sum_{\alpha} G_{\alpha^{-1}}S^\phi_{\alpha,W \times W',f \circ f'} = \left( \sum_{\alpha} G_{\alpha^{-1}}S^\phi_{\alpha,W,f} \right) \cdot \left( \sum_{\alpha} G_{\alpha^{-1}}S^\phi_{\alpha,W',f'} \right).
\]

Proof. By Theorem 4.2.4, \( \text{gr}_{\leq 0}E_{W \times W',f \circ f'} = E_{W,f} \ast E_{W',f'} \). The series \( E_{W,f} \) and \( E_{W',f'} \) having no constant term, the result follows from Proposition 5.1.2 and Proposition 5.2.1.

6. Hodge realization and the Hodge spectrum

6.1. In this section we will assume \( k = \mathbb{C} \). For \( d \geq 1 \), there is an embedding of \( \hat{\mu}_d(\mathbb{C}) \) in \( \mathbb{Q}/\mathbb{Z} \) given by \( \alpha \mapsto a \) with \( \alpha(e^{2\pi i/d}) = e^{2\pi ia} \). This gives an isomorphism \( \hat{\mu}(\mathbb{C}) \simeq \mathbb{Q}/\mathbb{Z} \). We denote by \( \gamma \) the section \( \mathbb{Q}/\mathbb{Z} \to [0,1) \).

A \( \mathbb{C} \)-Hodge structure of weight \( n \) is just a finite dimensional bigraded vector space \( V = \bigoplus_{p+q=n} V^{p,q} \), or, equivalently, a finite dimensional vector space \( V \) with decreasing filtrations \( F^\cdot \) and \( F^\cdot \) such that \( V = F^p \oplus F^q \) when \( p + q = n + 1 \). One defines similarly a rational \( \mathbb{C} \)-Hodge structure of weight \( n \), by allowing \( p \) and \( q \) to belong to \( \mathbb{Q} \) but still requiring \( p + q \in \mathbb{Z} \).

We denote by \( K_0(\text{MHS}_\mathbb{C}) \) the Grothendieck group of the abelian category of \( \mathbb{C} \)-Hodge structures (it is also the Grothendieck group of the abelian category of complex mixed Hodge structures) and by \( K_0(\text{RMHS}_\mathbb{C}) \) the Grothendieck group of the abelian category of rational \( \mathbb{C} \)-Hodge structures.

The Hodge realization functor induces a morphism \( H : A \to K_0(\text{MHS}_\mathbb{C}) \) which is zero on the kernel of the morphism \( A \to \hat{A} \) by Remark 2.1.1. Hence we shall also write \( H(A) \) for \( A \) in the image of \( A \) by this morphism.

This morphism may be extended to a morphism \( H : U \to K_0(\text{RMHS}_\mathbb{C}) \) as follows. For \( p \) and \( q \) in \( \mathbb{Q} \) with \( p + q \in \mathbb{Z} \), we denote by \( H^{p,q} \) the class of the rank 1 vector space with bigrading \( (p,q) \). We set \( H(G_1) = -1 \) and \( H(G_\alpha) = -H^1 - \gamma(\alpha) - \gamma(\alpha) \) for \( \alpha \neq 0 \). This is compatible with the relations 2.5.1 and 2.5.3 since, by a standard calculation (see [14]),
\[
H(J_{\alpha_1,\alpha_2}) = -H^1 - \gamma(\alpha_1) + \gamma(\alpha_2) - \gamma(\alpha_1 + \alpha_2), \gamma(\alpha_1) + \gamma(\alpha_2) - \gamma(\alpha_1 + \alpha_2),
\]
when \( \alpha_1 \neq 0, \alpha_2 \neq 0 \) and \( \alpha_1 \alpha_2 \neq 0 \). Similarly as before, since \( H \) vanishes on the kernel of the morphism \( U \to \hat{U} \), we extend it to the image of this morphism.

6.2. For \( X \) a complex algebraic variety, we denote by \( \text{MHM}(X) \) the abelian category of mixed modules on \( X \) constructed by M. Saito in [8, 10]. In the definition of mixed Hodge modules it is required that the underlying perverse sheaf is defined over \( \mathbb{Q} \). To allow some more flexibility we will also use the
category MHM′(X) of bifiltred D-modules on X which are direct factors of objects of MHM(X) as bifiltred D-modules. We denote by $D^b(MHM(X))$ and $D^b(MHM′(X))$ the corresponding derived categories. Let $f : X \to \mathbb{A}^1_C$ be a morphism. We denote by $\psi^H_f$ and $\phi^H_f$ the nearby and vanishing cycle functors for mixed Hodge modules as defined in [10] and $T_s$ the semi-simple part of the monodromy operator. One should note that $\psi^H_f$ and $\phi^H_f$ on mixed Hodge modules correspond to $\psi_f[-1]$ and $\phi_f[-1]$ on the underlying perverse sheaves. If $M$ is a mixed Hodge module on $X$ we denote by $\psi^H_fM,\alpha$ the object of MHM′($X$) which corresponds to the eigenspace of $T_s$ for the eigenvalue $\exp(2\pi i\gamma(\alpha))$. These definitions extend to the Grothendieck group of the abelian category MHM′($X$).

For any object $K$ of $D^b(MHM(X))$, we denote by $\chi_c(X,K)$ the class of $\pi_p!(K)$ in $K_0(MHS_C)$, where $p$ is the projection onto Spec $C$. Clearly this definition may be extended to $D^b(MHM′(X))$.

**Theorem 6.2.1.** Let $X$ be a smooth and connected complex algebraic variety of dimension $m$, $f : X \to \mathbb{A}^1_C$ be a morphism and let $i_W : W \hookrightarrow f^{-1}(0)$ be a reduced subscheme of $f^{-1}(0)$. The following equalities hold

$$H(S^\psi_{\alpha,w,f}) = (-1)^{m-1}\chi_c(W, i^*_W \psi^H_{f,\alpha} C^H_X[m])$$

and

$$H(S^\phi_{\alpha,w,f}) = (-1)^{m-1}\chi_c(W, i^*_W \phi^H_{f,\alpha} C^H_X[m]).$$

**Proof.** When $W$ is a point the first equality is Theorem 4.2.1 of [4]. The proof of the general case is completely similar. The second equality follows directly from the first one. \qed

**Definition 6.2.2.** Let $X$ be a smooth and connected complex algebraic variety of dimension $m$, $f : X \to \mathbb{A}^1_C$ be a morphism and let $i_W : W \hookrightarrow f^{-1}(0)$ be a reduced subscheme of $f^{-1}(0)$. We set

$$\tilde{\chi}_c(W, i^*_W \psi^H_f C^H_X[m]) = \sum_{\alpha} H(G_{\alpha^{-1}})\chi_c(W, i^*_W \psi^H_{f,\alpha} C^H_X[m])$$

and

$$\tilde{\chi}_c(W, i^*_W \phi^H_f C^H_X[m]) = \sum_{\alpha} H(G_{\alpha^{-1}})\chi_c(W, i^*_W \phi^H_{f,\alpha} C^H_X[m]).$$

We deduce from Theorem 5.2.2 and Theorem 5.2.1 the following Corollary.

**Corollary 6.2.3.** Let $X$ and $X'$ be smooth and connected complex algebraic varieties of pure dimension $m$ and $m'$. Let $f : X \to \mathbb{A}^1_C$ and $f' : X' \to \mathbb{A}^1_C$ be morphisms of algebraic varieties. Let $W$ (resp. $W'$) be a reduced subscheme of...
Let $X$ be a smooth complex algebraic variety of pure dimension $m$, let $f : X \to \mathbb{A}^1_C$ be a morphism of algebraic varieties and let $x$ be a closed point of $f^{-1}(0)$. Let us recall the definition of the spectrum $\text{Sp}(f,x)$ given in \cite{16} and \cite{11} (which differs from that of \cite{17} by multiplication by $t$). Let $H$ be a complex mixed Hodge structure with an automorphism $T$ of order dividing $d$. One defines the Hodge spectrum of $(H,T)$ as $H\text{Sp}(H,T) = \sum_{\alpha \in \mathbb{Z}} n_{\alpha} t^{\alpha} \in \mathbb{Z}[t^{-\frac{1}{d}},t^{\frac{1}{d}}]$, with $n_{\alpha} = \dim \text{Gr}^p_H \lambda$, for $\lambda = \exp(2\pi i \alpha)$ and $p = [\alpha]$, where $H_{\lambda}$ is the eigenspace of $T$ with eigenvalue $\lambda$, and $F$ is the Hodge filtration. This definition extends to the Grothendieck group of the abelian category of complex mixed Hodge structures with an automorphism $T$ of order dividing $d$. Note that $H\text{Sp}(H(k), T) = t^{-k} H\text{Sp}(H,T)$, where $(k)$ is the Tate twist. We denote by $\iota$ the $\mathbb{Z}$-algebra automorphism of $\mathbb{Z}[t^{-\frac{1}{d}},t^{\frac{1}{d}}]$ defined by $\iota(t^{\frac{1}{d}}) = t^{-\frac{1}{d}}$. Now one defines $\text{Sp}(f,x)$ as

$$\text{Sp}(f,x) := t^{m} \iota \left( \sum_{j \in \mathbb{Z}} (-1)^j H\text{Sp}(H^j \phi_f^H C^H_X[m], T_s) \right).$$

**Corollary 6.2.4.** Let $X$ and $X'$ be smooth and connected complex algebraic varieties of pure dimension $m$ and $m'$. Let $f : X \to \mathbb{A}^1_C$ and $f' : X' \to \mathbb{A}^1_C$ be morphisms of algebraic varieties. Let $x$ and $x'$ be closed points in $f^{-1}(0)$ and $f'^{-1}(0)$. Then

$$\text{Sp}(f \oplus f', (x,x')) = \text{Sp}(f, x) \cdot \text{Sp}(f', x').$$

Corollary 6.2.4 was first proved by A. Varchenko in \cite{19} when $f$ and $f'$ have isolated singularities (see also \cite{13}). The general case is due to M. Saito (unpublished, but see \cite{17}, \cite{12}).

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