Big Bangs and Bounces
on the
Brane

by

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ABSTRACT

In view of our accelerating universe, one of the outstanding theoretical issues is the absence of a quantum-gravitational description of de Sitter space. Although speculative, an intriguing circumvention may be found in the realm of brane-world scenarios; where the physical universe can be interpreted as a non-critical 3-brane moving in a higher-dimensional, static bulk. In this paper, we focus on the cosmological implications of a positively curved brane world evolving in the background of a “topological” anti-de Sitter black hole (i.e, Schwarzschild-like but with an arbitrary horizon topology). We show that the bulk black hole will typically induce either an asymptotically de Sitter “bounce” universe or a big bang/big crunch FRW universe, depending on a critical value of mass. Interestingly, the critical mass is only non-vanishing in the case of a spherical horizon geometry. We go on to provide a holographic interpretation of this curiosity.
1 Introduction

Empirical evidence, as deduced from astronomical data, has indicated that the physical universe is presently accelerating [1]. This observation, in turn, implies that the universe has a positive cosmological constant or, at least, some exotic form of matter that mimics this constant in the current epoch [1]. Naturally, one would want to incorporate this observational evidence into our current understanding (albeit limited) of quantum gravity and cosmology. Alas, any such prospect has proven to be a formidable challenge that is far from being resolved. Along with the roadblock of explaining the observed value of the cosmological constant, there is an apparent incompatibility between a positive-valued constant and quantum gravity as we comprehend it. Let us briefly elaborate on these points.

Explaining the Cosmological Constant: On a semi-classical level, the cosmological constant is expected to represent the vacuum energy density of spacetime. Unfortunately, such expectations fail to even remotely predict the observed value of the constant: $\sim 10^{-120}$ (in units of Planck mass to the fourth power). Indeed, this value is many orders of magnitude smaller than that of any naive theoretical prediction; the most optimistic being $\sim 10^{-60}$ (if supersymmetry is broken just above the current accelerator limits). This discrepancy implies some type of subtle mechanism (or combination thereof) that essentially cancels the vacuum density while, at the same time, leaving behind a small, very stable residue. In spite of many interesting attempts in the literature, this problem has yet to be resolved in a way that does not resort to fine-tuning methods or anthropic principles. For a discussion on these attempts and the topic in general, see [4, 5] and the references within.

On a more fundamental level, one might hope that string (or M) theory could, after a suitable process of compactification and supersymmetry breaking, give rise to a stable vacuum with a positive cosmological constant. Alas, this does not appear to be the case. Firstly, perturbative string theory, with broken supersymmetry, has inherent tachyonic instabilities. These instabilities, in turn, induce either a large, negative vacuum energy or the decoupling of gravity [1]. Secondly, for non-perturbative string theory (or matrix theory

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1 An example of the latter is a dilatonic scalar field that is slowly rolling in a stable potential; also known as "quintessence" [2, 3].

2 For a lay-physicist’s review on string/M theory and supersymmetry, see [6].
In the document, the vacuum states of interest are plagued by singular behavior, with this breakdown having been attributed to a relevant “no-go” theorem [4]. Moreover, serious attempts at circumventing this theorem are found to be pathological; for instance, wrong-sign kinetic terms and non-compact “compactification” manifolds. (See [10] for further discussion and references.)

**Incompatibility with Quantum Gravity**: Significant to this discussion is that the current acceleration of the universe implies an asymptotically (As) de Sitter (dS) future. Unfortunately, many of the quantum-gravitational aspects of dS spacetimes are poorly understood [7, 13, 14]. Much of the difficulty can be traced to dS space having a finite value for its entropy [15]. Moreover, this finite value serves as an upper bound on the observable entropy of any AsdS spacetime [16]. Consequently, it can be expected that, given a positive cosmological constant, gravity will be described by a finite-dimensional Hilbert space. This finite size is, however, directly incompatible with the infinite-dimensional Hilbert space that is inherent to string theory. Furthermore, the same incompatibility issue arises when dS space is considered from a holographic perspective. (Note that the holographic principle [17, 18] is expected to play a fundamental role in any quantum theory of gravity.) That is to say, it has been conjectured that, in analogy to anti-dS holography [19, 20, 21], dS space is dual to a conformal field theory (CFT). However, any CFT should be described (like string theory) by a Hilbert space of infinite dimensionality.

A further complication that arises in AsdS spacetimes (or any spacetime that gives rise to a future event horizon [11, 12]) is the unclear status of physical observables. First note that physical observables can normally be defined in terms of S-matrix elements between asymptotically free particle states [34, 35]. Significant to this picture is the existence of an infinitely large

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3Strictly speaking, an AsdS future need not be the case if the acceleration is explained by, for instance, a quintessence theory [4, 5]. Nevertheless, the same problematic features, as discussed in this section, have been shown to persist in this type of alternative scenario [11, 12].

4In particular, a Euclidean CFT that lives on the boundary at temporal infinity in dS space [22]. There has, since [22], been a plethora of investigations into this dS/CFT duality. Some very recent studies include [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. Many earlier papers have been referenced in [25, 26].

5In this context, the “norm” refers to asymptotically flat or anti-dS spacetimes (which have a vanishing or negative cosmological constant).
boundary at spatial infinity. Conversely, AsdS spacetimes have no such time-like boundary and, therefore, provide no obvious means for defining physical observables. One hypothetical circumvention is to consider the spacelike boundary of an AsdS spacetime at future infinity. However, because of the inevitable event horizon, this boundary ultimately shrinks to a singular point from the perspective of any given observer [11]. Another possibility might be to define the S-matrix as a correlation between states at past and future infinity. However, such a construction would require a global view of the spacetime that is outside the realm of “mere mortals” [13, 14].

Although the above discussion paints a bleak picture (at least, it was supposed to), some resolution may be possible in the realm of brane-world scenarios [36]. That is to say, one can consider the speculative viewpoint that the physical universe is “trapped” on a 3-brane (or 3+1-dimensional hypersurface) that is immersed within a higher-dimensional bulk spacetime. To be specific, we will focus on Kraus’ generalization [37] of the Randall-Sundrum model [38]; whereby a non-critical brane is moving through a static, anti-dS black hole background. Let us take note of the following points of pertinence to this scenario. (i) The brane dynamics will determine the cosmological evolution of the universe; that is, an observer will interpret the brane motion as a cosmological contraction or expansion. (ii) The cosmological constant is now an effective one that can be expressed strictly in terms of bulk parameters; namely, the curvature radius of the anti-dS bulk and the brane tension [37].

The reason that this brane-world picture may represent an “improvement” (over more conventional descriptions) is that the physical universe can now, in some holographic sense [11, 42, 43], be regarded as an anti-dS one. Significantly, the quantum-gravitational aspects of anti-dS theories are much better understood than their dS counterparts. Furthermore, the cosmological constant (as measured on the brane) can now be viewed as an input parameter [7] that has its origins in the bulk theory. One could then

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6It is typically assumed that, of the standard model particles, only the graviton is allowed to propagate off the brane. It is also commonly assumed that all but one of the “extra” bulk dimensions have been compactified to string-scale length. We will, henceforth, adopt these conventions.

7It is interesting to note that, although the graviton is free to propagate into the bulk, gravity will typically remain localized on the brane for the model of interest [39, 41].
hope that such a brane-world picture will eventually be realized out of a string-compactification scenario; perhaps, along the lines of [44]. Meanwhile, the extreme (deranged?) optimist may hope that string theory can even fix the bulk parameters and, therefore, vicariously fix the brane cosmology!

In the current paper, we investigate, as discussed above, the cosmological implications of a non-critical brane moving in a static, anti-dS black hole bulk. In particular, we are interested in the influence of the bulk parameters on the cosmology of a brane universe with a positive cosmological constant. This treatment can be viewed as a generalization of a recent work by Petkou and Siopsis [45]. These authors, however, studied an anti-dS Schwarzschild bulk, whereas the current analysis extends considerations to so-called “topological” anti-dS black holes [16]. Such a black hole solution is still Schwarzschild-like, but the horizon topology is allowed to be, in addition to spherical, flat or hyperbolic. We also note that the two papers interpret the results from somewhat different perspectives.

The remainder of the paper is organized as follows. In Section 2, after introducing the (arbitrary-dimensional) action and bulk solutions of interest, we consider the equation of motion for the dynamical brane. Following prior literature (for instance, [37]), we express this equation in a form that mimics the standard Friedmann cosmological equation.

In Section 3, with guidance from [15], we present solutions for the induced brane metric as a function of proper time. (Here, the focus is on the physically relevant case of a 3-brane with a positive cosmological constant.) The form of solution is shown to vary significantly, depending on the values of the relevant bulk parameters. We then classify these various solutions according to identifiable cosmologies. Particularly of interest, a “bounce” cosmology (i.e., a universe that is AsdS at both past and future infinity) is only possible when the black hole mass stays below a critical value. Moreover, this critical value is only non-vanishing when the horizon geometry is spherical.

In Section 4, we interpret the intriguing observations of the prior section from a holographic perspective. Section 5 ends with a summary and some further discussion.
2 A Brane-World Scenario

To begin the formal analysis, let us consider a Randall-Sundrum type of brane-world scenario [38]. More specifically, we are interested in an $n+1$-dimensional anti-dS bulk spacetime that contains an $n$-dimensional brane of positive tension. Assuming, for sake of simplicity, that there are no additional sources of energy/matter on the brane, we can express the gravitational action as follows:

$$I = \frac{1}{16\pi G_{n+1}} \int_M d^{n+1}x \sqrt{-g} \left( R - 2\Lambda_{n+1} \right) + \frac{1}{8\pi G_{n+1}} \int_{\partial M} d^n x \sqrt{-h} K + \frac{\sigma}{8\pi G_{n+1}} \int_{\partial M} d^n x \sqrt{-h}. \quad (1)$$

Here, $G_{n+1}$ is the $n+1$-dimensional Newtonian constant, $M$ signifies the bulk manifold, $\partial M$ represents the brane manifold, $g_{\mu\nu}$ is the bulk metric, $h_{\mu\nu}$ is the induced metric on the brane, $\Lambda_{n+1} < 0$ is the bulk cosmological constant, $K$ is the trace of the brane’s extrinsic curvature, and $\sigma$ is the brane tension. Note that the second integral is analogous to the Gibbons-Hawking surface term [47] and is necessary for a well-defined variational principle on the spacetime boundary (in this case, the brane).

For future reference, we include the following useful identities:

$$\Lambda_{n+1} = -n(n-1)/2L^2, \quad (2)$$

$$h_{\mu\nu} = g_{\mu\nu} - \eta_\mu \eta_\nu, \quad (3)$$

$$K_{\alpha\beta} = h^\mu_\alpha h^\nu_\beta \nabla_\mu \eta_\nu, \quad (4)$$

where $L$ is the curvature radius of the anti-dS bulk and $\eta_\mu$ is the unit normal vector to the brane.

It can be shown that the bulk equations of motion are solved by the following (static) black hole spacetimes [46]:

$$ds^2_{n+1} = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2_{k,n-1}, \quad (5)$$

where:

$$f(r) = \frac{r^2}{L^2} + k - \frac{\omega_n M}{r^{n-2}}, \quad (6)$$
\[ \omega_n = \frac{16\pi G_{n+1}}{(n-1)V_n} \]  
(7)

Here, \( d\Omega_{k,n-1}^2 \) represents the line element of an \((n-1)\)-dimensional constant-curvature (Euclidean) hypersurface, \( V_n \) is the dimensionless volume of this hypersurface, and \( k \) and \( M \) are constants of integration. Without loss of generality, \( k \) can be set equal to +1, 0 or -1; describing a horizon geometry that is respectively spherical (i.e., the anti-dS Schwarzschild case), flat or hyperbolic. Meanwhile, \( M \) measures the conserved mass of the black hole and will be regarded as a non-negative quantity.

As shown elsewhere (for instance, [37]), if one assumes an evolving brane and an otherwise static bulk, the brane dynamics can be formulated so as to describe a Friedmann-Robertson-Walker (FRW) universe [49]. For sake of completeness, let us now run through this procedure. As an appropriate starting point, we introduce Gaussian normal coordinates in the proximity of the brane:

\[ ds_{n+1}^2 = dz^2 + h_{\mu\nu}dx^\mu dx^\nu, \]  
(8)

where \( z = 0 \) defines the position of the brane. Next, the brane coordinates should be expressed as functions of the proper time, \( \tau \), as measured by a brane observer. In view of the generic symmetry of the bulk solutions, one can write:

\[ x^\mu = (t(\tau), r(\tau), \theta_1, ..., \theta_{n-1}), \]  
(9)

where \( \theta_i \) denotes the \( \tau \)-independent coordinates of \( d\Omega_{k,n-1}^2 \).

At this point, it is useful to consider the velocity vector associated with a comoving test particle. That is:

\[ u^\mu \equiv \frac{dx^\mu}{d\tau} = \left( \dot{t}, \dot{r}, 0, ..., 0 \right), \]  
(10)

where here (and throughout) a dot denotes differentiation with respect to \( \tau \). Considering that this vector must satisfy \( u^\mu u_\mu = -1 \), we are able to obtain the following relation:

\[ f(r)\dot{r}^2 - \frac{1}{f(r)}\dot{t}^2 = 1. \]  
(11)

\(^8\)Technically speaking, the \( k = -1 \) hyperbolic solution supports a negative value for the mass [46]. However, we will disregard this controversial scenario, on the grounds that such a black hole is known to induce a negative energy in its holographic boundary theory [48]. Note that such a negative-energy theory would likely be a non-unitary one.
Substituting the above result into Eq.(3), we find that:

$$d s^2_n = -d r^2 + r^2(\tau) d \Omega^2_{n-1},$$

which notably describes a metric having a FRW form.

We can proceed further by incorporating the unit vector, $\eta^\mu$, which is normal to the brane. The explicit form of this vector can be realized by way of a pair of relations: $\eta^\mu \eta_\nu = 1$ and $u^\mu \eta_\mu = 0$. These equations are readily solved to yield:

$$\eta_\mu = \left(-\dot{r}, \dot{t}, 0, ..., 0\right).$$

With the above formalism, we are now suitably positioned to investigate the equations of motion for the brane. First note that the Israel jump conditions [50] lead, quite generically, to the following tensor equation (for instance, [37]):

$$K_{\mu\nu} = T_{\mu\nu} - \frac{1}{n-1} T_{\rho}^{\rho} h_{\mu\nu},$$

where $T_{\mu\nu}$ is the stress-energy tensor of the brane. For this matter-free brane model, the stress tensor is simply $T_{\mu\nu} = -\sigma h_{\mu\nu}$. Hence, the above condition conveniently reduces to:

$$K_{\mu\nu} = \frac{\sigma}{n-1} h_{\mu\nu}.$$}

We can calculate the extrinsic curvature by way of the prior formal definition (4). For current purposes, it is sufficient to consider the component $K_{\theta \theta}$; where $\theta = \theta_i$ for any choice of $i$. In this case:

$$K_{\theta \theta} = \nabla_{\theta} \eta_{\theta} = -\Gamma_{\theta \rho}^{\rho} \eta_{\rho} = rf(r)\dot{t}.$$

Equating this result with $\sigma r^2/(n-1)$, as prescribed by Eq.(15), we have:

$$\dot{t} = \frac{\sigma r}{(n-1)f(r)}.\quad (17)$$

Finally, let us substitute the above outcome into Eq.(11). Also defining the Hubble “constant” in the usual way (i.e., $H \equiv \dot{r}/r$), we obtain the following equation of motion for the brane:

$$H^2 = -\frac{\dot{f}(r)}{r^2} + \frac{\sigma^2}{(n-1)^2}$$

$$= \frac{k}{r^2} + \frac{\omega_n M}{r^n} - \frac{1}{L^2} + \frac{\sigma^2}{(n-1)^2}.$$
where Eq. (3) has been incorporated into the lower line.

Significantly, the above expression - which describes the cosmological evolution of the brane universe - takes on a Friedmann-like form. We can make this connection even more explicit by introducing an effective cosmological constant for the brane universe. The appropriate definition turns out to be:

$$
\Lambda_n \equiv \frac{(n-1)(n-2)}{2} \left[ \frac{\sigma^2}{(n-1)^2} - \frac{1}{L^2} \right] = \frac{(n-2)}{n} \left[ \frac{n}{2(n-1)} \sigma^2 - |\Lambda_{n+1}| \right],
$$

(19)
as this leads to:

$$
H^2 = \frac{2\Lambda_n}{(n-1)(n-2)} - \frac{k}{r^2} + \frac{\omega_n M}{r^n}.
$$

(20)

This above form can, in fact, be identified with the $n$-dimensional Friedmann equation for radiative matter (since $\rho_{\text{rad}} \sim r^{-n}$ is standard for an FRW universe). Notably, an anti-dS black hole in the bulk is, indeed, expected to induce radiative matter in its holographic dual [12, 13]. Also of interest, the horizon geometry of the black hole determines the topology of the spatial slicing in the brane universe.

In the “traditional” Randall-Sundrum model [38], one is supposed to fine tune the brane tension ($\sigma$) so that $\Lambda_n$ precisely vanishes (i.e., a critical brane). However, there is no reason, a priori, for such a fine tuning to occur. In fact, current observational evidence is in favor of a non-vanishing, positive cosmological constant [1]. On this basis, we will assume that $\Lambda_n > 0$ is always satisfied in the analysis to follow. Note that this stipulation translates into a critical value for the brane tension. More specifically, $\sigma^2 > \sigma_c^2$, where $\sigma_c = (n-1)/L$. This condition, in turn, leads to an interesting but very open question: can string theory (or any other microscopic framework) be used to constrain the value of the brane tension?

### 3 FRW Cosmology on the Brane

In this section, we will endeavor to solve the cosmological brane equation (20) for all acceptable values of the bulk integration constants; that is, $k$ and $\omega_n$. For other studies on non-critical, dynamical brane scenarios, see [51, 52, 53, 54].
To be succinct, let us now focus on the physically most interesting and analytically solvable case of \( n = 4 \) (also, \( \Lambda_n > 0 \)). We do, however, expect that the same qualitative features will persevere for larger values of \( n \) \(^{[5]}\); although analytical solutions may no longer be explicitly obtainable.

Let us begin here by appropriately re-expressing Eq.\((20)\):

\[
\dot{r}^2 = \frac{\Lambda_4}{3} r^2 - k + \frac{\omega_4 M}{r^2}. \tag{21}
\]

It proves to be convenient if we further incorporate the following definitions: \( \mathcal{H}^2 \equiv \Lambda_4/3 \) and \( x \equiv r^2 \). The cosmological equation then takes the form:

\[
\frac{\dot{x}^2}{4} = \mathcal{H}^2 x^2 - kx + \omega_4 M. \tag{22}
\]

One finds that the solutions of Eq.\((22)\) naturally separate into distinct classes as \( k \) changes discretely and as the quantity \( 4\omega_4 M\mathcal{H}^2 \) varies in relation to unity (or zero). Regarding the mass dependence, we can demonstrate this behavior by rewriting Eq.\((22)\) as follows:

\[
\frac{\dot{x}^2}{4} = 4\mathcal{H}^2 (x - x_+) (x - x_-), \tag{23}
\]

where:

\[
x_{\pm} = \frac{1}{2\mathcal{H}^2} \left[ k \pm \sqrt{k^2 - 4\omega_4 M\mathcal{H}^2} \right]. \tag{24}
\]

In view of the above consideration, let us define the following (dimensionless) mass parameter: \( \tilde{M} \equiv 4\omega_4 M\mathcal{H}^2 \). It then follows that the solutions should be categorized according to \( \tilde{M} = 0 \), \( 0 < \tilde{M} < 1 \), \( \tilde{M} = 1 \) and \( \tilde{M} > 1 \), as well as \( k = +1, k = 0 \) and \( k = -1 \). As it turns out, Eq.\((22)\) can readily be solved for each of these scenarios \( (4 \times 3 = 12 \text{ in all})\), and we summarize the results - case by case - immediately below.\(^{[10]}\)

**CASE (1a) \( \tilde{M} = 0 \) and \( k = +1 \):**

\[
r^2 = \frac{1}{\mathcal{H}^2} \cosh^2(\mathcal{H}\tau). \tag{25}
\]

**CASE (1b) \( \tilde{M} = 0 \) and \( k = 0 \):**

\[
r^2 = \frac{1}{\mathcal{H}^2} \exp(\pm 2\mathcal{H}\tau). \tag{26}
\]

\(^{10}\)Note that we have omitted any solution in which \( r^2 < 0 \) for all values of \( \tau \).
CASE (1c) $\tilde{M} = 0$ and $k = -1$:

$$r^2 = \frac{1}{\mathcal{H}^2} \sinh^2(\mathcal{H} \tau). \quad (27)$$

CASE (2a) $0 < \tilde{M} < 1$ and $k = +1$;

Scenario (I):

$$r^2 = \frac{1}{2\mathcal{H}^2} \left[ 1 + \sqrt{1 - \tilde{M} \cosh(2\mathcal{H} \tau)} \right], \quad (28)$$

Scenario (II):

$$r^2 = \frac{1}{2\mathcal{H}^2} \left[ 1 - \sqrt{1 - \tilde{M} \cosh(2\mathcal{H} \tau)} \right]. \quad (29)$$

CASE (2b) $0 < \tilde{M} < 1$ and $k = 0$:

$$r^2 = \frac{\sqrt{\tilde{M}}}{2\mathcal{H}^2} \sinh(\pm 2\mathcal{H} \tau). \quad (30)$$

CASE (2c) $0 < \tilde{M} < 1$ and $k = -1$:

$$r^2 = \frac{1}{2\mathcal{H}^2} \left[ \sqrt{1 - \tilde{M} \cosh(2\mathcal{H} \tau)} - 1 \right]. \quad (31)$$

CASE (3a) $\tilde{M} = 1$ and $k = +1$;

Scenario (I):

$$r^2 = \frac{1}{2\mathcal{H}^2} \left[ 1 + \exp(\pm 2\mathcal{H} \tau) \right], \quad (32)$$

Scenario (II):

$$r^2 = \frac{1}{2\mathcal{H}^2} \left[ 1 - \exp(\pm 2\mathcal{H} \tau) \right]. \quad (33)$$

CASE (3b) $\tilde{M} = 1$ and $k = 0$:

$$r^2 = \frac{1}{2\mathcal{H}^2} \sinh(\pm 2\mathcal{H} \tau). \quad (34)$$

CASE (3c) $\tilde{M} = 1$ and $k = -1$:

$$r^2 = \frac{1}{2\mathcal{H}^2} \left[ \exp(\pm 2\mathcal{H} \tau) - 1 \right]. \quad (35)$$
CASE (4a) $\tilde{M} > 1$ and $k = +1$:

$$r^2 = \frac{1}{2\mathcal{H}^2} \left[ 1 + \sqrt{\tilde{M} - 1} \sinh(\pm 2\mathcal{H}\tau) \right]. \quad (36)$$

CASE (4b) $\tilde{M} > 1$ and $k = 0$:

$$r^2 = \frac{\sqrt{\tilde{M}}}{2\mathcal{H}^2} \sinh(\pm 2\mathcal{H}\tau). \quad (37)$$

CASE (4c) $\tilde{M} > 1$ and $k = -1$:

$$r^2 = \frac{1}{2\mathcal{H}^2} \left[ \sqrt{\tilde{M} - 1} \sinh(\pm 2\mathcal{H}\tau) - 1 \right]. \quad (38)$$

In spite of the wide array of solutions, a careful inspection reveals that all of the above can be classified according to one of six possible cosmologies. Let us discuss each of these in turn.

(i) The brane universe is a “pure” dS spacetime, which occurs for cases 1a, 1b and 1c. This particular outcome could have easily been anticipated, inasmuch as $\tilde{M} = M = 0$ translates into an FRW theory with no matter and a positive cosmological constant; cf. Eq. (21). The alert reader may have noticed that the three solutions only agree asymptotically. However, all three still describe dS space, just with different choices of spatial slicing. (For a discussion on the various ways of foliating dS space and the associated global properties, see [57].) The topology of the slicing is, as one might have expected, perfectly correlated with the horizon topology of the bulk black hole.

(ii) Depending on the choice of sign, the brane universe either begins with a “big bang” or ends with a “big crunch”; that is, begins or ends at a singular ($r = 0$) surface. This occurs for cases 2b, 2c, 3b, 3c, 4a, 4b and 4c. To be more specific about the form of these solutions, let us first consider the

\textsuperscript{11}For the cases in question, it is readily confirmed that the curvature (of the induced brane metric) does indeed diverge when $r = 0$. Hence, these are “true” singularities, beyond which the spacetime should not be analytically continued.
big bang cosmological picture. In this case, the brane universe comes into existence at a time that can be regarded as \( \tau = 0 \) without loss of generality.\(^{12}\) After the bang, the universe expands monotonically (with time) towards an \( \text{AsdS} \) future; that is, \( r \) grows exponentially as \( \tau \to +\infty \). Meanwhile, the big crunch picture is simply the time-reversed version of the preceding description.

(iii) The brane universe is \( \text{AsdS} \) in the infinite past and approaches a static universe (i.e., \( H = \dot{r}/r = 0 \)) in the infinite future. (The solutions can also describe the time reverse of this description, depending on the choice of sign.) This quite unorthodox cosmology (first recognized in [45]) only occurs for scenario I of case 3a. Note that such an occurrence is highly model dependent; requiring a spherical horizon geometry with \( \tilde{M} \) set precisely equal to unity.

(iv) The brane universe interpolates between a singularity (either a big bang or big crunch) and an asymptotically static universe. This somewhat pathological cosmology only occurs for scenario II of case 3a and (as discussed above) is highly model dependent.

(v) The brane universe is \( \text{AsdS} \) in both the infinite past and the infinite future (but not pure dS). This type of cosmology strictly occurs for scenario I of case 2a. Note that such a universe can be said to “bounce” when it reverses from a contracting phase to an expanding one at \( \tau = 0 \). Further note that, although the size of the universe is minimal at the bounce surface, it still does not become vanishingly small. Such a cosmological model is very interesting, considering that bounce universes can not be trivially constructed. That is to say, adding a finite energy density to an otherwise purely dS spacetime typically induces a big bang or crunch.\(^{23}\)

(vi) The brane universe begins with a big bang and ends, after a finite time period, with a big crunch. This cosmology only occurs for scenario II of case 2a.

\(^{12}\)The singular surfaces in question do not always coincide with \( \tau = 0 \). However, \( r = 0 \) does always coincide with a finite value of \( \tau \). Hence, we are free to rescale the proper time coordinate to agree with the usual conventions.
It may be of relevance that, besides pure dS space (i), only the bounce cosmology (v) will eternally avoid the interior of the black hole horizon\textsuperscript{13}. The big bang cosmologies, on the other hand, actually start out at the black hole singularity. Although this latter scenario appears problematic, it can still be resolved with an analytical continuation to Euclidean time \textsuperscript{43}. As discussed in the cited paper, the Euclidean version of the big bang (brane) universe manages to remain outside of the bulk horizon during its entire evolution.

Generally speaking, it is interesting to note that the brane cosmology knows very little about the black hole mass, except for its relation to a pair of critical values; namely, \( M = 0 \) and \( M = 1 \). In support of this notion, we point out that the hyperbolic and exponential functions depend only on the proper time, the brane tension and the bulk cosmological constant. (The factors in front of these functions are, of course, essentially meaningless.) Conversely, the brane universe will always be able to make a clear distinction between different choices of the bulk topological parameter, \( k \).

\section{Holographic Interpretation}

To help make sense of the prior analysis, let us summarize, for each type of horizon topology, how the brane cosmology qualitatively varies as a function of increasing black hole mass.

Firstly, we consider the case of a spherical horizon geometry (i.e., \( k = +1 \)), which translates into an anti-dS Schwarzschild black hole. Starting at \( M = 0 \) and then “turning on” the mass, we initially observe a bounce cosmology; that is, an AsdS spacetime in both the infinite past and infinite future\textsuperscript{14}. As \( M \) continues to increase, the bounce cosmology persists until a critical value of mass, \( M = M_c \equiv (4\omega_4\mathcal{H}^2)^{-1} \), is reached\textsuperscript{15}. Exactly at this critical point,

\footnotesize\textsuperscript{13}Setting \( f(r) = 0 \) and \( n = 4 \) in Eq.(3), we find the horizon location \( (r = r_H) \) to be given by \( r_H^2 = \frac{L^2}{2} \left[ \sqrt{k^2 + 4\omega_4 M/L^2} - k \right] \).

\footnotescript{14}Note that a pure dS spacetime, which occurs only for \( M = 0 \), is really just a special type of bounce cosmology and will be regarded as such in this section. It is also worth noting that the slightest perturbation to the brane world would cause an \( M = 0 \) geometry to lose whatever “esteemed” status it did have.

\footnotescript{15}This critical mass value was first identified in \textsuperscript{43}. 

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\end{footnotesize}
one finds a somewhat pathological cosmology: static in either the infinite past or future. On the other hand, for any $M > M_c$, we observe a typical FRW cosmological picture; either a spacetime that begins with a big bang and evolves towards an AsdS future, or else the time-reversed (big crunch) scenario.

Secondly, let us consider the flat ($k = 0$) and hyperbolic ($k = -1$) horizon geometries; that is, the topological analogues $\text{[16]}$ of the anti-dS Schwarzschild black hole. In essence, we observe the same type of mass dependence as described above, except that the critical mass value is now a vanishing one ($M_c = 0$) and there is no longer a pathology at $M = M_c$. Keep in mind that we are interpreting a pure dS spacetime as a bounce geometry.$^\text{[19]}

The generic existence of such a bulk mass threshold - effectively separating bounce from big bang (or big crunch) cosmologies on the brane - can be viewed as a direct consequence of the holographic principle. To substantiate this claim, let us next consider the following discussion.

We begin here by recalling (from the literature) a pair of relevant holographic bounds. Firstly, Bousso has demonstrated $\text{[16]}$ that the finite entropy of dS space serves as an upper bound on the observable entropy of a spacetime with a positive cosmological constant. $^\text{[17]}$ Secondly, on the basis of the Bousso bound, Balasubramanian, de Boer and Minic have conjectured $\text{[58]}$ a mass bound that also applies to spacetimes with a positive cosmological constant. More to the point, if such a spacetime has a total energy $^\text{[18]}$ in excess of pure dS space, then the Balasubramanian et al conjecture predicts the existence of a cosmological singularity; that is, a big bang or big crunch. Thus, a singularity-free AsdS spacetime (i.e., a bounce cosmology) can only be possible when this mass bound has not been exceeded.

Let us now apply the above concepts to our non-critical brane universe. It is clear from Eq.$\text{(21)}$ that a massive black hole in the bulk induces an

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16 The reader may have noticed that $r = 0$ at $\tau = 0$ for a dS spacetime with hyperbolic spatial slicing; cf. Eq.$\text{(27)}$. However, this surface is only an apparent singularity (i.e., the curvature remains finite), beyond which the spacetime can indeed be analytically continued $\text{[57]}$. Also of interest, the solution for dS space with flat spatial slices (cf. Eq.$\text{(26)}$) only covers one half of the total manifold. One obtains the other half by reversing the sign in the exponential.

17 This bound was rigorously verified for, in particular, spherically symmetric spacetimes having “physically realistic” matter $\text{[16]}$.

18 For the purposes of this discussion, any relevant mass or energy should be defined in analogy to that of Brown and York’s quasi-localized energy $\text{[59, 60]}$. 

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energy density, in the form of holographic radiative matter, on the brane. Moreover, this holographic brane matter is directly proportional to the mass of the black hole. Since $M = 0$ translates into a pure dS brane universe (cf. Eqs. (25-27)), one might naively expect that any $M > 0$ would exceed the conjectured mass bound \cite{58}. That is to say, any non-vanishing value of $M$ could be expected to induce a big bang or big crunch in the brane cosmology. This observation fits in very nicely with the $k = 0$ and $k = -1$ scenarios, where the critical mass value is exactly at zero. However, the situation is not so pleasantly simple when we consider a spherical ($k = +1$) horizon geometry. In this case, the bounce cosmology persists for $M > 0$, in spite of an apparent violation of the holographic mass bound.

The resolution of the above paradox lies in the precise nature of Boussou’s entropic bound \cite{16}; again noting that this is the critical antecedent for the mass bound in question \cite{58}. Technically speaking, the holographic entropy bound limits only the “accessible” entropy in a relevant spacetime (i.e., one with a positive-valued cosmological constant). This qualification on the entropy makes an important distinction: accessible degrees of freedom are those which can both influence and be influenced by a given experiment and inaccessible degrees of freedom are those which cannot. To put it another way, any given observer is causally limited to a specific domain of a relevant spacetime; that is, the observer’s so-called “causal diamond” \cite{16}. Consequently, when an observer attempts to measure the “entropy of the universe”, the actual measurement will only reflect the contents of her causal diamond and not, necessarily, the entire spacetime manifold. (Keep in mind that accessible entropy is an observer-dependent property, so that, in spite of appearances, the entropic upper bound is not really a global concept.)

In view of the above discussion, it follows that the upper bound on mass should not, strictly speaking, apply to the global energy of the spacetime. Rather, the litmus test for the bound should be the “accessible energy” as determined by a given observer. Hence, a brane cosmology with $M > 0$ (i.e., with a total energy in excess of pure dS space) will not necessarily induce a singularity. That is to say, if no observer can capably measure a mass in excess of the conjectured bound, then the bounce cosmology should persist.

\footnote{The philosophy that underlies this distinction is the principle of black hole complementarity \cite{11}. That is, physical consistency in a black hole setting renders causally inaccessible regions as being operationally meaningless. In this regard, it is significant that the causal structure of black hole and dS spacetimes are quite similar \cite{13}.}
It is interesting to note that the existence/non-existence of a singularity is a global property, whereas the holographic mass bound is most accurately viewed as a local concept. Nonetheless, this is not a contradiction, insofar as any given observer may or may not be in causal contact with the singularity (if there is one).

One question (at least) still remains to be answered; namely, why is the critical mass value finite when $k = +1$ but otherwise zero? This apparent discrepancy can be qualitatively accounted for as follows. First of all, it is useful to recall that the black hole topology effectively determines the foliation of the spacetime on the brane. That is, a black hole with a spherical/flat/hyperbolic horizon geometry induces a brane universe that is foliated by spherical/flat/hyperbolic spatial slices; cf. Eq. (20). Now focusing on the flat and hyperbolic cases, let us point out that the constant-time slices in these spacetimes are typically infinite in extent (since $d\Omega^2_{k \leq 0, n-1}$ contains an unbounded coordinate). Hence, even the slightest increase in energy density on the brane (i.e., the slightest increase in $M$) can lead to significant perturbations in the underlying dS geometry. Hence, on an intuitive basis, even an infinitesimal value for $M$ can be expected to violate the mass bound (for at least one hypothetical observer) and thus collapse the spacetime.

Next, let us concentrate on the case of a spherical topology. The constant time slices on the brane are now strictly compact (since $d\Omega^2_{k=1, n-1}$ contains only angular coordinates). Hence, a relatively small increase in energy density on the brane should give rise to a proportionally small increase in an observer’s accessible energy. It thus follows, again in an intuitive sense, that the underlying dS geometry (for $k = 1$) should be somewhat resilient to an increasing black hole mass, as long as $M$ does not get too large. This notion is, of course, in agreement with the analysis of the prior section. To reiterate, the bounce universe can persist until reaching some critical value for the energy density; above which, the holographic mass bound finally fails and a singular collapse ensues. It is also worth pointing out that, for general FRW cosmologies (i.e., independent of the brane-world scenario), bounce universes are significantly more favorable when the spacetime is foliated by spherical slices.

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20 Or to put it another way, if a tree falls in the forest, does it always make a sound?
5 Conclusion

In summary, we have considered the cosmological implications of a dynamical brane-world scenario. In particular, we have studied a non-critical brane moving in the static background of an anti-de Sitter black hole. The novelty of our treatment, as compared to an earlier work [45], was in allowing for all possible horizon topologies of the Schwarzschild-like black hole in the bulk.

After introducing the relevant action and bulk solutions (of arbitrary dimensionality), we went on to formulate the equation of motion for the brane. Moreover, we were able to recast this expression into a form that mimics the Friedmann equation for radiative matter. Significantly, the energy density of this holographically induced matter is directly proportional to the mass of the bulk black hole. Meanwhile, the horizon geometry of the black hole fixes the topology of the spatial slices in the brane universe.

In the next phase of the analysis, we focused on the physically relevant scenario of a 4-dimensional brane universe with a positive cosmological constant [1]. Under these conditions, we were able to re-express the brane cosmological equation in a readily solvable form. Various solutions were presented for all relevant values of the parameters, \( M \) and \( k \), describing the bulk geometry. The most interesting feature was the existence of a critical value for the black hole mass, \( M = M_c \), which is finite for a spherical \((k = +1)\) topology but is otherwise \((k = 0, k = -1)\) vanishing. Significantly, when \( M \leq M_c \), one obtains singularity-free, asymptotically de Sitter bounce cosmologies (including pure de Sitter space when \( M = 0 \)). Conversely, if \( M > M_c \), then only FRW cosmologies having a big bang (or big crunch) are possible.

In the penultimate section of the paper, we went on to interpret this critical mass from a holographic perspective. In particular, we utilized holographic bounds that limit the accessible entropy [16] and matter [58] of any asymptotically de Sitter spacetime. On this basis, we were able to provide ample justification for the generic existence of such a critical mass value. Moreover, holographic arguments enabled us to explain why \( M_c \) takes on a finite value exclusively in the \( k = +1 \) case.

It should be clarified that our holographic interpretation was conspicuously of a qualitative nature. It would be interesting if one could further

\[ \text{One notable exception being the finely tuned case of } M = M_c \text{ and } k = +1. \text{ This choice of bulk parameters yields a somewhat pathological cosmology that is asymptotically static in either the infinite past or the future.} \]
apply holographic arguments on a quantitative level and directly predict the critical mass values. In this regard, we note the existence of two distinct holographic dualities that could well come into play. These being: (i) a dual relation between the anti-de Sitter bulk and a conformal field theory (CFT) living on the brane [19, 20, 21] and (ii) a duality between the de Sitter brane and a Euclidean CFT that lives at temporal infinity [22, 58]. However, progress along these lines may be impeded by a limited understanding of the dual boundary theories. That is to say, in spite of the success of these holographic dualities, the CFTs in question can best be classified as abstractions. Moreover, the de Sitter-based correspondence currently has an unclear status [63]. Nevertheless, we hope to address this intriguing matter in a future study.

On a less speculative note, one might extend the prior treatment by considering more exotic black holes in the bulk or by introducing other forms of matter onto the brane. With regard to the latter scenario, possibilities include conventional dust matter, stiff matter (which also appears as a natural consequence of electrostatic charge in the bulk [64]) and a $M^2$ term. This last contribution appears when the induced brane energy is directly calculated via a Hamiltonian method [56]. Once again, we defer such prospects to a future time.

6 Acknowledgments

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