THE LIFESPAN FOR 3-DIMENSIONAL QUASILINEAR WAVE EQUATIONS IN EXTERIOR DOMAINS

JOHN HELMS AND JASON METCALFE

Abstract. This article focuses on long-time existence for quasilinear wave equations with small initial data in exterior domains. The nonlinearity is permitted to fully depend on the solution at the quadratic level, rather than just the first and second derivatives of the solution. The corresponding lifespan bound in the boundaryless case is due to Lindblad, and Du and Zhou first proved such long-time existence exterior to star-shaped obstacles. Here we relax the hypothesis on the geometry and only require that there is a sufficiently rapid decay of local energy for the linear homogeneous wave equation, which permits some domains that contain trapped rays. The key step is to prove useful energy estimates involving the scaling vector field for which the approach of the second author and Sogge provides guidance.

1. Introduction. In this article, a lower bound of \( c/\epsilon^2 \), where \( \epsilon \) denotes the size of the Cauchy data, is established for the lifespan of solutions to quasilinear wave equations in exterior domains with Dirichlet boundary conditions. Here we examine nonlinearities which vanish to second order and may depend on the solution \( u \), rather than just its derivatives, at the quadratic level. The lifespan bound that is established is an analog of that which was obtained in [23] in the absence of a boundary. Similar lifespan bounds appeared in [6] (and could also be obtained via the methods of [24]) exterior to star-shaped obstacles. We permit much more general geometries and only require that there is a sufficiently rapid decay of local energy, with a possible loss of regularity. Due, e.g., to the seminal results [11, 12], this includes some domains which contain trapped rays.

Let us more specifically introduce the problem. Let \( \mathcal{K} \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary. Note that we shall not assume that \( \mathcal{K} \) is connected. We then examine the following quasilinear wave equation exterior to \( \mathcal{K} \)

\[
\begin{align*}
\Box u(t, x) &= Q(u, u', u''), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K}, \\
u(t, \cdot)|_{\partial \mathcal{K}} &= 0, \\
u(0, \cdot) &= f, \quad \partial_t u(0, \cdot) = g.
\end{align*}
\]

(1.1)

Here \( \Box = \partial_t^2 - \Delta \) is the d’Alembertian, and \( u' = \partial u = (\partial_t u, \nabla_x u) \) denotes the space-time gradient. The nonlinear term vanishes quadratically at the origin and is affine linear in \( u'' \), thus yielding a quasilinear equation. Without loss of generality, we may take \( 0 \in \mathcal{K} \subset \{|x| < 1\} \), and we shall do this throughout. While we shall state the lifespan bound for the scalar equation (1.1), the methods shall fully permit, even multiple speed, systems.

The authors were supported in part by NSF grant DMS-1054289. The second author was additionally supported by NSF grant DMS-0800678.

1
The nonlinearity $Q$ can be expanded as

$$Q(u, u', u'') = A(u, u') + B^{\alpha\beta}(u, u')\partial_\alpha\partial_\beta u$$

where $A(u, u')$ vanishes to second order at the origin and $B^{\alpha\beta}$ are functions which are symmetric in $\alpha, \beta$ and vanish to the first order at $(0, 0)$. Here we are using the summation convention where repeated indices are implicitly summed from 0 to 3, where $x_0 = t$ and $\partial_0 = \partial_t$.

For simplicity of exposition, we shall truncate at the quadratic level. As we are examining a small amplitude problem, it is clear that this shall not affect the long-time behavior. Upon doing so, we may write

$$Q(u, u', u'') = A(u, u') + b^{\alpha\beta}u\partial_\alpha u\partial_\beta u + b^{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha u\partial_\beta u$$

where $b^{\alpha\beta}$ and $b^{\alpha\beta\gamma}$ are constants which are symmetric in $\alpha, \beta$ and $A(u, u')$ is a quadratic form.

In order to solve (1.1), the Cauchy data are required to satisfy certain compatibility conditions. Letting $J_ku = \{\partial_x^\alpha u : 0 \leq |\alpha| \leq k\}$, for a formal $H^m$ solution $u$ of (1.1), we can write $\partial^\alpha_x u(0, \cdot) = \psi_k(J_k f, J_{k-1} g)$, $0 \leq k \leq m$. The functions $\psi_k$, which depend on $Q$, are called compatibility functions. The compatibility condition for $(f, g) \in H^m \times H^{m-1}$ simply requires that $\psi_k$ vanishes on $\partial K$ for all $0 \leq k \leq m - 1$. For smooth data, the compatibility conditions are said to hold to infinite order if this condition holds for all $m$. A more detailed exposition on compatibility can be found in, e.g., [17].

Our only assumption on the geometry of $K$ is that the associated local energy for solutions to the linear homogeneous wave equation decays sufficiently rapidly. For a clearer exposition, we shall assume that there is an exponential decay of local energy with a loss of a single degree of regularity, though it will be clear from the proof that a sufficiently rapid polynomial decay with a fixed finite loss of regularity will suffice.

More specifically, we shall assume that if $\Box u = 0$ and if supp $u(0, \cdot), \partial_t u(0, \cdot) \subset \{|x| < 10\}$, then

$$\|u'(t, \cdot)\|_{L^2(\mathbb{R}^3 \setminus K \cap \{|x| < 10\})} \lesssim e^{-at}\sum_{|\alpha| \leq 1} \|\partial_x^\alpha u(0, \cdot)\|_2$$

for some $a > 0$. The notation $A \lesssim B$ indicates that there is a positive unspecified constant $C$ (which may change from line to line) so that $A \leq CB$. Moreover, this $C$ will implicitly be independent of any parameters in our problem.

Local energy decay estimates such as (1.2) have a long history, which we shall only tersely describe. For nontrapping domains, one need not have the loss of regularity which appears in the right. See [22] for star-shaped obstacles and [34] for nontrapping domains. When there are trapped rays, it is known [36] that an estimate such as (1.2) cannot hold unless such a loss of regularity is permitted. Results in the positive direction in the presence of trapped rays begin with [11, 12] where such estimates are proved when $K$ consists of multiple convex obstacles subject to certain size/spacing conditions. More recent results include [1, 2, 4, 35, 41].

1To date, the authors are not aware of examples of three dimensional domains for which there is polynomial decay but not exponential decay, though the recent article [3] provides the most compelling evidence to date that such domains can be constructed.
We may now state our main theorem, which shows that for Cauchy data of size $\varepsilon$ solutions to (1.1) must exist up to a lifespan of $T_\varepsilon = c/\varepsilon^2$ for some small constant $c$.

**Theorem 1.1.** Let $\mathcal{K}$ be a smooth, bounded set for which the exponential decay of local energy (1.2) holds, and let $Q$ be as above. Suppose that the Cauchy data $f, g \in C^\infty(\mathbb{R}^3 \setminus \mathcal{K})$ are compactly supported and satisfy the compatibility conditions to infinite order. Then there exist constants $N$ and $c$ so that if $\varepsilon$ is sufficiently small and

$$
(1.3) \quad \sum_{|\mu| \leq N} \|\partial_\mu^2 f\|_2 + \sum_{|\mu| \leq N-1} \|\partial_\mu^2 g\|_2 \leq \varepsilon,
$$

then (1.1) has a unique solution $u \in C^\infty([0, T_\varepsilon] \times \mathbb{R}^3 \setminus \mathcal{K})$ where

$$
(1.4) \quad T_\varepsilon = c/\varepsilon^2.
$$

For simplicity of exposition, we are assuming here that the Cauchy data are compactly supported. It is likely that it would suffice to take the data to be small in certain weighted Sobolev norms.

On $\mathbb{R}_+ \times \mathbb{R}^3$, this lifespan was first proved in [23]. The dependence on the solution $u$ rather than just its first and second derivatives inhibits the energy methods which are typically employed to show such long time existence. Indeed, when the nonlinearity does not depend on $u$ at the quadratic level, solutions are known to exist almost globally, i.e. with a lifespan of $T_\varepsilon \approx \exp(c/\varepsilon)$. See [13].

The previous results [15, 16, 17, 18], [26, 27, 28], [24], [14] have focused on proving long-time existence for three dimensional wave equations in exterior domains when there is no dependence on $u$, and [25] examines the case that there is dependence at the cubic level and beyond. Of particular note is the paper [26] where long-time existence results were first established only assuming (1.2), and in particular, in domains which have trapped rays.

The current direction of research was initiated with the paper [6] where the same lower bound on the lifespan was shown exterior to star-shaped obstacles. A similar four dimensional problem was addressed in [5], and the techniques therein could also be applied to the exterior of three dimensional star-shaped obstacles. The current article extends these results to much more general geometries.

The method of proof shall utilize Klainerman’s method of invariant vector fields [21]. This has been adapted to the exterior domain setting with particularly notable contributions coming in [16, 18] and [20]. To this end, we set

$$
Z = \{\partial_\alpha, \Omega_{ij} = x_i \partial_j - x_j \partial_i : 0 \leq \alpha \leq 3, 1 \leq i < j \leq 3\}
$$

to be the generators of space-time translations and spatial rotations. We shall also denote $L = t\partial_t + r\partial_r$, which is the scaling vector field. A key fact is that

$$
[\Box, Z] = 0, \quad [\Box, L] = 2\Box.
$$

These vector fields thus preserve $\Box u = 0$, in the sense that if $u$ solves such an equation then $\Box Zu = 0$ and $\Box Lu = 0$. The Lorentz boosts $\Omega_{0k} = t\partial_k + x_k \partial_t$ have not yet been mentioned despite playing a key role when the method is applied on $\mathbb{R}_+ \times \mathbb{R}^3$. Though all of these vector fields have nice commutation properties with $\Box$, only $\partial_k$ is guaranteed to preserve the Dirichlet boundary conditions. While the other members of $Z$ do not preserve
the boundary conditions, their coefficients are bounded on the compact obstacle, and thus, approximately do. Indeed, these can be handled using elliptic regularity arguments and localized energy estimates as was initiated in [16]. On the other hand, the Lorentz boosts have unbounded normal component on \( \partial K \) and seem inadmissible for such boundary value problems. It is also worth noting that the Lorentz boosts have an associated speed, and they only commute with the d’Alembertian of the same speed, which renders them also difficult to use for multiple speed systems. While the scaling vector field has a bounded normal component on \( \partial K \), for long-time problems, its coefficients can be large in any neighborhood of the boundary. For this reason, we shall be required to use relatively few \( L \) compared to the vector fields \( Z \). This is an idea which originated in [18] and is further displayed in [26], [24, 25].

The star-shaped assumption on the geometry of \( K \) arises in [6] in order to prove energy estimates involving the scaling vector field \( L \). Indeed, using ideas akin to those from [18] which is in turn reminiscent of [32], it is shown that the worst boundary term (in terms of \( t \) dependence) which arises when proving an energy inequality for \( Lu \) has a beneficial sign. Developing an alternative method for handling these boundary terms was one of the major innovations of [26], and this article largely represents a combination of ideas from [6] and [26].

The star-shaped assumption arises in [5] in a related, though different, way. Long-time existence is shown there using only the vector fields \( Z \). This is accomplished by employing a class of localized energy estimates which are known to hold for small perturbations of the d’Alembertian exterior to star-shaped obstacles [27], [31] and iterating in a fashion which is akin to [16] and [27]. See also [28] for a further example of how a star-shaped assumption and the broader class of available localized energy estimates can simplify arguments.

The remainder of the article is organized as follows. In Section 2, we gather our main energy and localized energy estimates. These largely represent a combination of the main estimates used in [6] as well as the techniques developed in [26] to permit the use of the scaling vector field when the obstacle is not star-shaped. In Section 3, we establish the main decay estimates which we shall utilize. The principal piece here is a \( L^1 - L^\infty \) estimate which is akin to those of Hörmander [10] as was adapted to remove the dependence on the Lorentz boosts by [18]. In Section 4, we prove the long time existence given by Theorem 1.1.

2. Energy estimates. In this section, we shall gather the main \( L^2 \) estimates which we shall require. These will primarily be energy estimates as well as weighted \( L^1 L^2 \) localized energy estimates for the solution and vector fields applied to the solution. That is, these will be variants of the energy estimate and the localized energy estimate

\[
\sup_{t \in [0,T]} \| u'(t, \cdot) \|_2 + \sup_{R \geq 1} R^{-1/2} \| u' \|_{L^2_t L^\infty_x([0,T] \times \{|x| < R\})} \lesssim \| u'(0, \cdot) \|_2
\]

\[
+ \inf_{u = f + g} \left( \int_0^T \| f(s, \cdot) \|_2 \, ds + \sum_{j \geq 0} \| \langle x \rangle^{1/2} g \|_{L^2_t L^\infty_x([0,T] \times \{|x| \approx 2^j\})} \right)
\]

which are known to hold on \( \mathbb{R}_+ \times \mathbb{R}^3 \). Estimates of this latter form originated in [33] and have subsequently appeared in, e.g., [7], [16], [18], [19], [37], [39], [40]. Their particular
utility for proving long-time existence in exterior domains was first recognized in [14], and they have played a primary role in nearly every such proof since. Also, see, e.g., [27, 28] and [30, 31].

2.1. Estimates for \( \|u\|_{L^2} \) on \( \mathbb{R}_+ \times \mathbb{R}^3 \). Here we shall gather the boundaryless \( L^2 \) estimates on \( u \), rather than on \( u' \), which we shall utilize in the sequel. We shall only require these in the boundaryless case as the Dirichlet boundary conditions permit the control

\[
\|u\|_{L^2_x((x\in \mathbb{R}^3 \setminus \mathcal{K} : |x|<2))} \lesssim \|u'\|_{L^2_x((x\in \mathbb{R}^3 \setminus \mathcal{K} : |x|<2))}
\]

and when a cutoff which vanishes on \( \{|x|<1\} \) is applied to \( u \), then it suffices to examine a boundaryless equation. The majority of these estimates were also utilized in [6]. In the sequel, we shall abbreviate \( L^2_x(\{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x|<2\}) \) as \( L^2_x(\{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x|<2\}) \).

We first state the variant of the localized energy estimates which we shall employ.

Lemma 2.1. Let \( u \) be a smooth function which vanishes for large \( |x| \) at each time \( t \). Then for \( T \geq 1 \)

\[
\|\langle x \rangle^{-3/4} u\|_{L^2_{t,x}(\{0,T\} \times \mathbb{R}^3)} + T^{-1/4} \|\langle x \rangle^{-1/4} u\|_{L^2_{t,x}(\{0,T\} \times \mathbb{R}^3)} 
\lesssim \|u'(0,\cdot)\|_2 + \inf_{h_0 \neq 0, h \neq 0} \left( \int_0^T \|f(s,\cdot)\|_{L^2_{t,x}} \, ds + \sum_{j\geq 0} \|\langle x \rangle^{1/2} g\|_{L^2_{t,x}(\{0,T\} \times \{\langle x \rangle \approx 2^j\})} \right).
\]

To obtain (2.3), we first note that over \( |x| > T \) it follows trivially from the energy inequality. To finish the proof, one need only dyadically decompose \( |x| < T \) and apply (2.1). The interested reader can find an alternate proof in [6].

We shall then use the following weighted Sobolev-type estimate from [4, 5].

Lemma 2.2. For \( n \geq 3 \) and \( h \in C_0^\infty(\mathbb{R}^n) \),

\[
\|h\|_{\dot{H}^{-1}(\mathbb{R}^n)} \lesssim \|h\|_{L^p_{t,x}(\{|x|<1\})} + \||x|^{-\frac{n-2}{2}} h\|_{L^2_{t,x}(\{|x|>1\})}.
\]

Here and throughout, the mixed norm represents

\[
\|f\|_{L^p_{t,x}(\mathbb{R}^n)} = \left( \int_0^\infty \left[ \int_{S^{n-1}} |f(r\omega)|^q \, d\omega \right]^{p/q} r^{n-1} \, dr \right)^{1/p}.
\]

By applying the energy inequality and (2.3) to the Riesz transforms of the solution and subsequently applying the preceding lemma, we obtain:

Proposition 2.3. Let \( u \) be a smooth function which vanishes for large \( |x| \) at each time \( t \). Then for \( T \geq 1 \)

\[
\|u\|_{L^\infty_{t,x}(\{0,T\} \times \mathbb{R}^3)} + T^{-1/4} \|\langle x \rangle^{-1/4} u\|_{L^2_{t,x}(\{0,T\} \times \mathbb{R}^3)} \lesssim \|u(0,\cdot)\|_2 + \|\partial_t u(0,\cdot)\|_{\dot{H}^{-1}}
+ \int_0^T \|\langle x \rangle^{-1/2} \partial u(s,\cdot)\|_{L^1_{t,x}(\{|x|>1\})} \, ds + \int_0^T \|\partial u(s,\cdot)\|_{L^6_{t,x}(\{|x|<1\})} \, ds.
\]

Further details of the proof can again be found in [6] and [5].

The following proposition will be used to control commutator terms when the solution in the exterior domain is cutoff away from the obstacle. It appears implicitly in [3].
Section 4] and utilizes arguments akin to those which appeared in [37, 16] which rely on Huygens’ principle. In higher dimensions, an alternate proof which does not rely on Huygens’ principle appeared in [5].

**Proposition 2.4.** Let \( u \) be a smooth solution to \( \Box u = G, \ u = 0 \) for \( t \leq 0 \). Suppose further that \( G(s,x) = 0 \) unless \( |x| < 3 \). Then,

\[
\|u\|_{L^\infty_t L^2_x([0,T] \times \mathbb{R}^3)} + T^{-1/4} \|\langle x \rangle^{-1/4} u\|_{L^2_{t,x}([0,T] \times \mathbb{R}^3)} \lesssim \|G\|_{L^2_t L^2_{t,x}([0,T] \times \mathbb{R}^3)}.
\]

In one case, we shall need an improvement on the estimate (2.4) when the forcing term is in divergence form. This follows easily from ideas of [8, 23]. See also [29] for an application in a context similar to the current study.

**Proposition 2.5.** Let \( v \) be a smooth solution to

\[
\Box v = \sum_{3} a_j \partial_j G, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3,
\]

\( v(0, \cdot) = \partial_t v(0, \cdot) = 0. \)

Then,

\[
\sup_{t \in [0,T]} \|v(t, \cdot)\|_2 + (T)^{-1/4} \|\langle x \rangle^{-1/4} v\|_{L^2_{t,x}([0,T] \times \mathbb{R}^3)} \lesssim \|G(0, \cdot)\|_{\dot{H}^{-1}} + \int_0^T \|G(s, \cdot)\|_2 \, ds.
\]

2.2. Energy estimates on \( \mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K} \). In this section, we examine the fixed time, \( L^2 \) energy estimates which will be used in the sequel. As we are proving long-time existence for quasilinear equations, we shall require such estimates for small perturbations of the d’Alembertian. Such an estimate is well-known for solutions satisfying Dirichlet boundary conditions. When vector fields are, however, applied to the solution and these boundary terms no longer vanish, some extra care is required, particularly when the time dependent vector field \( L \) occurs. In the remainder of this section, we are merely gathering results from [26, Section 2], and the interested reader is referred therein for detailed proofs.

In particular, we shall be studying smooth solutions \( u \) to

\[
\begin{align*}
\Box_\gamma u &= F, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K}, \\
\gamma u|_{\partial \mathcal{K}}(t, \cdot) &= 0, \\
u(0, \cdot) &= f, \quad \partial_t u(0, \cdot) = g.
\end{align*}
\]

where

\( \Box_\gamma u = (\partial^2_t - \Delta)u + \gamma^{\alpha\beta}(t,x) \partial_\alpha \partial_\beta u. \)

The perturbation is taken to satisfy \( \gamma^{\alpha\beta} = \gamma^{\beta\alpha} \) as well as

\[
\|\gamma^{\alpha\beta}(t, \cdot)\|_\infty \leq \frac{\delta}{1 + t}, \quad 0 < \delta \ll 1.
\]

We set \( e_0(u) \) to be the energy form

\[
e_0(u) = |u|^2 + 2\gamma^{0\alpha} \partial_\alpha u \partial_\beta u - \gamma^{\alpha\beta} \partial_\alpha u \partial_\beta u.
\]

Our first estimate concerns

\[
E_M(t) = E_M(u)(t) = \int_{\mathbb{R}^3 \setminus \mathcal{K}} \sum_{j=0}^M e_0(\partial_j^l u)(t, x) \, dx.
\]
As \( \partial_t^j \) preserves the Dirichlet boundary conditions, standard energy methods yield

**Lemma 2.6.** Fix \( M = 0, 1, 2, \ldots \) and assume that the \( \gamma^{\alpha \beta} \) are as above. Suppose that \( u \in C^\infty \) solves (2.7) and vanishes for large \( |x| \) for every \( t \). Then

\[
\partial_t E^{1/2}_M(t) \lesssim \sum_{j=0}^M \| \Box_j \partial_t^j u(t, \cdot) \|_2 + \| \gamma'(t, \cdot) \|_\infty E^{1/2}_M(t).
\]

(2.9)

Here, we have used the notation

\[
\| \gamma'(t, \cdot) \|_\infty = \sum_{\alpha, \beta, \mu = 0}^3 \| \partial_\mu \gamma^{\alpha \beta}(t, \cdot) \|_\infty.
\]

In the sequel, we shall frequently use the fact that

\[
\int_{\mathbb{R}^3 \setminus K} e_0(v)(t, x) \, dx \approx \| v'(t, \cdot) \|_2^2
\]

if (2.8) holds with \( \delta \) sufficiently small.

From these \( L^2 \) estimates for \( \partial_t \) applied to the solution \( u \), estimates for \( \partial_t^k u \) shall be obtained via elliptic regularity. The key lemma is

**Lemma 2.7.** For \( j, N = 0, 1, 2, \ldots \) fixed and for \( u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3 \setminus K) \) solving (2.7) and vanishing for large \( |x| \) for each \( t \), we have

\[
\sum_{|\alpha| \leq N} \| L^j \partial^\alpha u(t, \cdot) \|_2 \lesssim \sum_{k+j \leq N} \| L^j \partial^k u(t, \cdot) \|_2 + \sum_{|\alpha| + |\beta| \leq N + j} \| L^j \partial^\alpha \Box u(t, \cdot) \|_2.
\]

(2.10)

In order to prove estimates involving \( L \), we set \( \tilde{L} = \eta(x) r \partial_r + t \partial_t \) where \( \eta \in C^\infty(\mathbb{R}^3) \) with \( \eta(x) = 0 \) for \( x \in K \) and \( \eta(x) = 1 \) for \( |x| > 1 \). We note that \( \tilde{L} \) now preserves the Dirichlet boundary conditions. It, however, no longer commutes with \( \Box \). The commutators will be controlled using a combination of (1.2) and Huygens’ principle for the associated boundaryless d’Alembertian, which will be stated later. We set

\[
X_{k,j} = \int e_0(\tilde{L}^k \partial_t^j u)(t, x) \, dx.
\]

Associated to this energy, we have the following estimate.

**Proposition 2.8.** Let \( u \in C^\infty \) solve (2.7) with \( \gamma^{\alpha \beta} \) as above and vanish for large \( |x| \) for each \( t \). Then,

\[
\partial_t X^{1/2}_{k,j} \lesssim \| \gamma'(t, \cdot) \|_\infty X_{k,j}^{1/2} + \| \tilde{L}^k \partial_t^j \Box \gamma u(t, \cdot) \|_2 + \| \tilde{L}^k \partial_t^j \gamma^{\alpha \beta} \partial_\alpha \partial_\beta |u(t, \cdot)| \|_2 + \sum_{l \leq k - 1} \| L^l \partial_t^j \Box u(t, \cdot) \|_2 + \sum_{l + |\mu| \leq j + k} \| L^l \partial^\mu u'(t, \cdot) \|_{L^\infty(|x| < 1)}.
\]

(2.11)

In the sequel, we shall be choosing \( \gamma \) so that

\[
\| \gamma'(t, \cdot) \|_\infty \leq \frac{\delta}{1 + t}.
\]

(2.12)

By doing so, it will be easy to bootstrap the term involving \( \| \gamma'_\|_\infty \) upon integration over \([0, T]\) when \( T \) is appropriately bounded in terms of \( \delta \).
We finally state an energy estimate involving the full set of vector fields. Here, the associated boundary terms involve a loss of regularity, but they no longer involve the rotations. These boundary terms will be controlled using localized energy estimates which follow.

**Proposition 2.9.** For fixed $N_0$ and $m_0$, set

$$Y_{N_0,m_0}(t) = \sum_{|\mu|+k \leq N_0+m_0} \int_\mathbb{R} e_0(L^k Z^\mu \partial^\nu u)(t,x) \, dx.$$  

Suppose that (2.8) and (2.12) hold for $\delta$ sufficiently small. Then

$$\partial_t Y_{N_0,m_0} \lesssim Y_{N_0,m_0}^{1/2} \sum_{|\mu|+k \leq N_0+m_0} \|\partial_\gamma L^k Z^\mu \partial^\nu u(t,\cdot)\|_2 + \|\partial_\gamma' Y_{N_0,m_0} + \sum_{|\mu|+k \leq N_0+m_0+2} \|L^k \partial^\mu u'(t,\cdot)\|_{L^2(|x|<1)}^2.$$  

The above proposition contains a slight modification of what appeared previously in [26]. There one did not need to distinguish between the vector fields $Z$ and the one derivative $\partial$ in the definition of $Y_{N_0,m_0}$. Here we need this slight bit of additional precision. The proof follows the same argument. One needs to only apply standard energy methods to the principle terms and a trace theorem to the boundary terms which result upon doing such integrations by parts.

### 2.3. Localized energy estimates and boundary term estimates on $\mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K}$

In this section, we collect two additional results from [26]. The reader is referred there for proofs. Both estimates will concern solutions to the Dirichlet-wave equation

$$\begin{cases} \square w = G(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K}, \\
 w|_{\partial \mathcal{K}}(t,\cdot) = 0, \\
 w(t,x) = 0, & t \leq 0. \end{cases}$$  

When the estimates of Section 2.1 are applied away from the obstacle, the following, which strongly depends on (1.2), is used to handle the behavior near the boundary. This is also used to control the boundary term that appears in (2.13). For notational convenience, we set $S_T = [0,T] \times \mathbb{R}^3 \setminus \mathcal{K}$. 

Proposition 2.10. Suppose that $K \subset \{|x| < 1\}$ satisfies (1.2), and suppose that $w \in C^\infty$ solves (2.14). Then, for fixed $N_0$ and $m_0$, if $w$ vanishes for large $|x|$ for every fixed $t$,

\[
\sum_{|\mu|+j \leq N_0+m_0} \left\| L^j \partial^\mu w'(s, \cdot) \right\|_{L^2_t(L^\infty_x(S_T \cap \{|x| < 10\}))} \lesssim \int_0^T \sum_{|\mu|+j \leq N_0+m_0+1} \left\| \Box L^j \partial^\mu w(s, \cdot) \right\|_2 \, ds \\
+ \sum_{|\mu|+j \leq N_0+m_0-1} \left\| \Box L^j \partial^\mu w \right\|_{L^2_t(L^\infty_x(S_T))}.
\]

The second estimate shall be used to control the boundary term which arises in (2.11). The contributions from behavior near the obstacle are controlled using (1.2) and that away from the boundary follows from sharp Huygens’ principle after passing to a properly related boundaryless equation. See [26].

Proposition 2.11. Let $K \subset \{|x| < 1\}$ satisfy (1.2), and suppose that $w \in C^\infty$ solves (2.14). Then for fixed $N_0$ and $m_0$ and $t > 2$,

\[
\sum_{|\mu|+j \leq N_0+m_0} \int_0^t \left\| L^j \partial^\mu w(s, \cdot) \right\|_{L^2(|x| < 2)} \, ds \\
\lesssim \sum_{|\mu|+j \leq N_0+m_0+1} \left( \int_0^t \left( \int_0^s \left\| L^j \partial^\mu \Box w(\tau, \cdot) \right\|_{L^2(|x|-(s-\tau) | < 10)} \, d\tau \right) \, ds \right).
\]

3. **Pointwise decay estimates.** In this section, we gather the main decay estimates which will permit the necessary integrability to gain long-time existence. The first is a variant on standard weighted $\mathcal{L}^1$-$\mathcal{L}^\infty$ estimates (see, e.g., [9], [38]).

Proposition 3.1. Suppose that $w$ is a solution to the scalar inhomogeneous wave equation (2.14), and suppose that $K \subset \{|x| < 1\}$ is so that the decay of local energy (1.2) holds. Then

\[
(1 + t + |x|) |Z^\mu w(t, x)| \lesssim \int_0^t \int_{\mathbb{R}^3 \setminus K} \sum_{k \leq 1} \sum_{|\nu|+k \leq |\mu|+7} |L^k Z^\nu G(s, y)| \frac{dy \, ds}{|y|} \\
+ \int_0^t \sum_{k \leq 1} \left\| L^k \partial^\nu G(s, \cdot) \right\|_{L^2(|x| < 2)} \, ds.
\]

This is essentially [18, Theorem 4.1], though there solutions are studied exterior to star-shaped obstacles. For such domains, the associated version of (1.2) does not require a loss of regularity. In the current setting, as we allow for the possibility of trapped rays in our exterior domain and as such (1.2) has an associated loss of regularity, the right hand side of (3.1) reflects an additional vector field. See, also, [20, Theorem 3.1].

The second decay estimate is a weighted Sobolev lemma. See [20].
Lemma 3.2. Suppose that $h \in C^\infty(\mathbb{R}^3)$. Then for $R \geq 1$,

\begin{equation}
\|h\|_{L^\infty(R/2 < |x| < R)} \lesssim R^{-1} \sum_{|\alpha| \leq 2} \|Z^\alpha h\|_{L^2(R/4 < |x| < 2R)},
\end{equation}

and

\begin{equation}
\|h\|_{L^\infty(R < |x| < R+1)} \lesssim R^{-1} \sum_{|\alpha| \leq 2} \|Z^\alpha h\|_{L^2(R-1 < |x| < R+2)}.
\end{equation}

After localizing to the annulus, these estimates follow simply by applying Sobolev embedding on $\mathbb{R} \times S^2$ and then adjusting the volume element to match that of $\mathbb{R}^3$.

4. Proof of Theorem 1.1. Here we prove Theorem 1.1 via iteration. We shall first use local existence theory to reduce to the case of vanishing initial data.

Indeed by invoking, e.g., the local existence theory established in [17], if $\varepsilon$ in (1.3) is sufficiently small and $N$ is sufficiently large, then the existence of a smooth solution for $t \in [0, 2]$ satisfying

\begin{equation}
\sup_{0 \leq t \leq 2} \sum_{|\mu| \leq 102} \|\partial^\mu u(t, \cdot)\|_2 \leq C\varepsilon
\end{equation}

is guaranteed.

We shall use this local solution to reduce to the case of vanishing initial data. To this end, let $\eta \in C^\infty(\mathbb{R})$ with $\eta(t) \equiv 1$ for $t \leq 1/2$ and $\eta(t) \equiv 0$ for $t > 1$. Then $u_0 = \eta u$ solves

$$\Box u_0 = \eta Q(u, u', u'') + [\Box, \eta]u.$$ 

And solving (1.1) is then equivalent to showing that $w = u - u_0$ solves

\begin{equation}
\begin{cases}
\Box w = (1 - \eta)Q(u_0 + w, (u_0 + w)', (u_0 + w)'') - [\Box, \eta](u_0 + w), \\
|w|_{\partial K} = 0, \\
w(0, x) = \partial_t w(0, x) = 0.
\end{cases}
\end{equation}

We solve (4.2) via iteration. In particular, we let $w_0 \equiv 0$, and recursively define $w_m$ to solve

\begin{equation}
\begin{cases}
\Box w_m = (1 - \eta)Q(u_0 + w_{m-1}, (u_0 + w_{m-1})', (u_0 + w_{m-1})'') - [\Box, \eta]u, \\
w_m|_{\partial K} = 0, \\
w_m(0, x) = \partial_t w_m(0, x) = 0.
\end{cases}
\end{equation}
Our first goal is to show a form of boundedness. We set

\[(4.3)\] \[M_m(T) = \sup_{t \in [0, T]} \sum_{|\mu| \leq 100} \| \partial^\mu w_m(t, \cdot) \|_2 + \sum_{|\mu| \leq 95} \| \langle x \rangle^{-3/4} \partial^\mu w_m \|_{L^2_t(S_T)} + \sup_{t \in [0, T]} \sum_{|\mu| \leq 90} \| Z^\mu \partial^\mu w_m(t, \cdot) \|_2 + \sum_{|\mu| \leq 80} \langle T \rangle^{-1/4} \| \langle x \rangle^{-1/4} Z^\mu \partial^\mu w_m \|_{L^2_t(S_T)} + \sup_{0 \leq t \leq T} \sum_{|\mu| \leq 75} \| L Z^\mu \partial^\mu w_m(t, \cdot) \|_2 + \langle T \rangle^{-1/4} \sum_{|\mu| \leq 70} \| \langle x \rangle^{-1/4} L Z^\mu \partial^\mu w_m \|_{L^2_t(S_T)} + \sup_{0 \leq t \leq T} (1 + t) \sum_{|\mu| \leq 60} \| Z^\mu w_m(t, \cdot) \|_{\infty}.
\]

If \(M_0(T)\) denotes the above quantity with \(w_m\) replaced by \(u_0\), then (4.1) and finite propagation speed guarantees the existence of a constant \(C_0\) so that

\[(4.4)\] \[M_0(T) \leq C_0 \varepsilon.\]

We wish to inductively show that there is a uniform constant \(C_1\) so that

\[(4.5)\] \[M_m(T) \leq C_1 \varepsilon.\]

We label the terms of \(M_m(T)\) by \(I, II, \ldots, IX\).

**Bound for I:** Here we shall apply (2.9) and (2.10) \((j = 0)\) with

\[(4.6)\] \[\gamma^{\alpha \beta} = -(1 - \eta) \left[ b^{\alpha \beta}(u_0 + w_{m-1}) + b^{\alpha \beta}_\sigma \partial_\sigma (u_0 + w_{m-1}) \right].\]

By the inductive hypothesis as well as (4.1), we have (2.8) and (2.12) with \(\delta = C_1 \varepsilon\).

Upon integrating (2.3), applying (1.4), bootstrapping, and utilizing (2.10), it suffices to bound

\[\sum_{j \leq 100} \int_0^T \| \Box^n \partial_t^j w_m(t, \cdot) \|_2 dt + \sup_{t \in [0, T]} \sum_{|\mu| \leq 99} \| \partial^\mu \Box w_m(t, \cdot) \|_2.\]

Here we note that

\[\sum_{j \leq 100} \| \Box^n \partial_t^j w_m \|_2 \leq \sum_{|\nu| \leq 50} \| \partial^n (u_0 + w_{m-1}) \| \left( \sum_{|\mu| \leq 100} \| \partial^\mu (u_0 + w_m) \| + \sum_{|\mu| \leq 102} \| \partial^\mu u_0 \| \right) + \sum_{|\nu| \leq 51} \| \partial^n (u_0 + w_{m-1}) \| \left( \sum_{|\mu| \leq 100} \| \partial^\mu (u_0 + w_m) \| + \sum_{|\mu| \leq 102} \| \partial^\mu u_0 \| \right) + \sum_{|\mu| \leq 100} \| \Box^n \partial_t^j w_m \|_2 + \sum_{|\mu| \leq 100} \| \partial^\mu [\Box, \eta] u_0 \|.\]
By using term \( I, \, III, \) and \( IX \) of (4.3) as well as (4.4), we have
\[
\sum_{j \leq 100} \int_0^T \| \Delta_j \partial^j w_m(t, \cdot) \|_2 \, dt \lesssim (M_0(T) + M_{m-1}(T))(M_0(T) + M_m(T)) \int_0^T (1+t)^{-1} \, dt \\
\quad \quad \quad \quad + (M_0(T) + M_{m-1}(T))^2 \int_0^T (1+t)^{-1} \, dt + \varepsilon.
\]

We can argue similarly to control
\[
\sup_{t \in [0,T]} \sum_{|\mu| \leq 99} \| \partial^\mu \Delta w_m(t, \cdot) \|_2.
\]

By doing so, it follows that
\[
I \leq C(M_0(T) + M_{m-1}(T))(M_0(T) + M_{m-1}(T) + M_m(T)) \log(2 + T) + C_2 \varepsilon
\]
provided that \( C_2 \) is a constant which is chosen sufficiently large relative to the constant in (4.1). At this point, we shall permit \( C_2 \) to change from line to line but note that \( C_2 \) is completely independent of important parameters such as \( m, \, C_1, \, \varepsilon, \) and \( T. \)

**Bound for II:** Here we fix a smooth cutoff \( \beta \) which is identically 1 on \(|x| < 2\) and vanishes for \(|x| > 3\). For the multi-index \( \mu \) fixed, we first examine \((1 - \beta) \partial^\mu w_m\), which solves the boundaryless wave equation
\[
\Box (1 - \beta) \partial^\mu w_m = (1 - \beta) \partial^\mu \Box w_m - [\Box, \beta] \partial^\mu w_m
\]
with vanishing initial data. To this, we apply (2.3). Thus, in order to control II, we see that it suffices to bound
\[
\sum_{|\mu| \leq 95} \int_0^T \| \partial^\mu \Box w_m(s, \cdot) \|_2 \, ds + \sum_{|\mu| \leq 95} \| \partial^\mu w'_m \|_{L^2_{t,x}(S_T \cap \{|x| < 3\})}.
\]

Here we have applied (2.2) to the lower order piece of the commutator. To control the latter term, we utilize (2.13), which reduces the bound for II to controlling
\[
(4.7) \quad \sum_{|\mu| \leq 96} \int_0^T \| \partial^\mu \Box w_m(s, \cdot) \|_2 \, ds + \sum_{|\mu| \leq 94} \| \partial^\mu \Box w_m \|_{L^2_{t,x}(S_T)}.
\]

As
\[
(4.8) \quad \sum_{|\mu| \leq 96} | \partial^\mu \Box w_m | \lesssim \sum_{|\mu| \leq 96} | \partial^\mu (u_0 + w_{m-1}) | \sum_{|\mu| \leq 97} | \partial^\mu (u_0 + w_m) | \\
+ \sum_{|\mu| \leq 96} | \partial^\mu (u_0 + w_m) | \sum_{|\mu| \leq 49} | \partial^\mu (u_0 + w_{m-1}) | + \sum_{|\mu| \leq 49} | \partial^\mu (u_0 + w_{m-1}) | \sum_{|\mu| \leq 96} | \partial^\mu (u_0 + w_{m-1}) | \\
+ | u_0 + w_{m-1} |^2 + \sum_{|\mu| \leq 96} | \partial^\mu [\Box, \eta] u |.
\]
it follows from (4.3) (I, III, and IX) and (4.4) that
\[
\sum_{|\mu| \leq 97} \int_0^T \| \partial^\mu \Box w_m(s, \cdot) \|_2 \, ds \lesssim (M_0(T) + M_{m-1}(T))(M_0(T) + M_m(T)) \int_0^T s^{-1} \, ds \\
+ (M_0(T) + M_{m-1}(T))^2 \int_0^T s^{-1} \, ds + \varepsilon.
\]
The second term in (4.7) is simpler and can be controlled similarly. It follows that
\[
II \leq C(M_0(T) + M_{m-1}(T))(M_0(T) + M_{m-1}(T) + M_m(T)) \log(2 + T) + C_2 \varepsilon.
\]

**Bounds for III and IV, |\nu| = 0:** The primary estimate which shall be utilized is (2.4). This meshes well with every nonlinear term with the exception of those involving second derivatives, which is more difficult as we cannot properly control the second derivatives in the weighted \(L^2_{x,t}\) spaces. To get around this, we write the worst terms in divergence form and utilize (2.6).

To do so, we fix \(\beta\) as above. Over \(|x| < 3\), due to the Dirichlet boundary conditions, we have that \(\|w_m(s, \cdot)\|_{L^2(|x|<3)} \lesssim \|w_m'(s, \cdot)\|_{L^2(|x|<3)}\). Moreover, on such a compact set, the coefficients of \(Z\) are bounded, and \(\|Zw_m(s, \cdot)\|_{L^2(|x|<3)} \lesssim \|w_m'(s, \cdot)\|_{L^2(|x|<3)}\). Such terms are subject to the bound established for \(II\) above. Thus, it will suffice to control \((1 - \beta)Z^\mu w_m\) in the appropriate norms.

We note that for \(\mu\) fixed, \((1 - \beta)Z^\mu w_m\) solves the boundaryless equation
\[
\Box(1 - \beta)Z^\mu w_m = (1 - \beta)Z^\mu \Box w_m - [\Box, \beta]Z^\mu w_m
\]
and that the latter term is supported in \(|x| < 3\}. We must further decompose the first term in the right. We write
\[
(1 - \beta)Z^\mu \Box w_m = \partial_\alpha \left[ (1-\eta)(1-\beta) \left( b^{\alpha\beta}(u_0 + w_{m-1})\partial_\beta Z^\mu(u_0 + w_m) \\
+ b^{\alpha\beta}_\gamma \partial_\gamma(u_0 + w_{m-1})\partial_\beta Z^\mu(u_0 + w_m) \right) \right] + G_m(t, x).
\]
The key here is that the former term falls into the context of Proposition 2.5 while the latter term does not contain the case where the full number of vector fields lands on the term containing second derivatives.

By applying (2.6), (2.4) and (2.6), as well as the comments in the preceding paragraph, we obtain
\[
(4.9) \sup_{t \in [0, T]} \sum_{|\mu| \leq 90} \| (1 - \beta)Z^\mu w_m(t, \cdot) \|_2 + \sum_{|\mu| \leq 90} \langle T \rangle^{-1/4} \langle |x| \rangle^{-1/4} (1 - \beta)Z^\mu w_m \|_{L^2_{t,x}(S_T)} \\
\leq \int_0^T \| \langle x \rangle^{-1/2} G_m(s, \cdot) \|_{L^1_{t,L^2_x}} \, ds + \sum_{|\mu| \leq 90} \sum_{|\theta| \leq 1} \int_0^T \| \partial^\theta (u_0 + w_{m-1}) \|_{L^2} \langle x \rangle \, ds \\
+ \sum_{|\mu| \leq 90} \| \partial^\mu w_m' \|_{L^2_{t,x}(0,T \times \{|x|<3\})}.
\]
The last term is controlled by \(II\), which was appropriately bounded in the previous section.
We first examine the first term in the right. Analogous to (4.8), we have

\[ |G_m| \lesssim \sum_{|\nu| \leq 48} |Z^{\nu}(u_0 + w_{m-1})| \sum_{|\mu| \leq 90} |Z^{\mu}(u_0 + w_m')| \]

\[ + \sum_{|\nu| \leq 48} |Z^{\nu}(u_0 + w_m')| \sum_{|\mu| \leq 90, |\theta| \leq 1} |Z^{\mu}\partial^{\theta}(u_0 + w_{m-1})| \]

\[ + \sum_{|\nu| \leq 48} |Z^{\nu}(u_0 + w_{m-1})| \sum_{|\mu| \leq 90, |\theta| \leq 1} |Z^{\mu}\partial^{\theta}(u_0 + w_{m-1})| + \sum_{|\mu| \leq 90} |Z^{\mu}[\Box, \eta]|u|. \]

For the first term in the right side of (4.9), we can apply Sobolev embedding on \( S^2 \) and the Schwarz inequality to obtain

\[ \sum_{|\mu| \leq 90} \int_0^T \|\langle x\rangle^{-1/2}G_m\|_{L^2_t L^\infty_x} \, ds \lesssim \varepsilon \]

\[ + \sum_{|\nu| \leq 48} \|\langle x\rangle^{-1/4}Z^{\nu}(u_0 + w_{m-1})\|_{L^2_t L^4_x(S_T)} \sum_{|\mu| \leq 90} \|\langle x\rangle^{-1/4}Z^{\mu}(u_0 + w_m')\|_{L^2_t L^4_x(S_T)} \]

\[ + \sum_{|\nu| \leq 48} \|\langle x\rangle^{-1/4}Z^{\nu}(u_0 + w_m')\|_{L^2_t L^4_x(S_T)} \sum_{|\mu| \leq 90, |\theta| \leq 1} \|\langle x\rangle^{-1/4}Z^{\mu}\partial^{\theta}(u_0 + w_{m-1})\|_{L^2_t L^4_x(S_T)} \]

\[ + \sum_{|\nu| \leq 48} \|\langle x\rangle^{-1/4}Z^{\nu}(u_0 + w_{m-1})\|_{L^2_t L^4_x(S_T)} \sum_{|\mu| \leq 90, |\theta| \leq 1} \|\langle x\rangle^{-1/4}Z^{\mu}\partial^{\theta}(u_0 + w_{m-1})\|_{L^2_t L^4_x(S_T)}. \]

Here we have also applied (4.1). The above is, in turn,

\[ (T)^{1/2}(M_0(T) + M_{m-1}(T))(M_0(T) + M_{m-1}(T) + M_m(T)) + C_2 \varepsilon. \]

The second term in the right side of (4.9) is better behaved. Indeed, using IX, it follows immediately that this is

\[ \lesssim (\log(2 + T))(M_0(T) + M_{m-1}(T))(M_0(T) + M_m(T)). \]

Combining the above, we have established

\[ III |_{\nu = 0} + IV |_{\nu = 0} \leq C(M_0(T) + M_{m-1}(T))(M_0(T) + M_{m-1}(T) + M_m(T))(T)^{1/2} + C_2 \varepsilon. \]

**Bounds for III and IV**, \( |\nu| = 1 \): We, again, fix \( \beta \) as above and seek to control the contribution away from the boundary as the coefficients of \( Z \) are \( O(1) \) on the support of \( \beta \) and the corresponding terms near the boundary are controlled by II. Indeed, we apply (4.1) and (4.3) to \((1 - \beta)Z^{\mu}w_m\), which solves the boundaryless equation shown in the previous subsection. This yields

\[ (4.11) \sup_{t \in [0,T]} \sum_{|\mu| \leq 90} \|Z^{\mu}w_m(t, \cdot)\|_2 + \sum_{|\nu| \leq 90} \langle T\rangle^{-1/4} \|\langle x\rangle^{-1/4}Z^{\mu}w_m'\|_{L^2_t L^4_x(S_T)} \]

\[ \lesssim \sum_{|\mu| \leq 90} \int_0^T \|Z^{\mu}\Box w_m(s, \cdot)\|_2 \, ds + \sum_{|\mu| \leq 91} \|\partial^{\mu}w_m\|_{L^2_t L^4_x([0,T] \times \{|x| < 3\})}. \]
The last term in the above equation is subject to the bounds previously established for II.

Mimicking the arguments above, we obtain to following bound for the first term in the right side of (4.11):

\[
\int_0^T \sum_{|\nu| \leq 46} \|Z^\nu(u_0 + w_{m-1})(s, \cdot)\|_\infty \sum_{|\mu| \leq 90} \|Z^\mu \partial^2(u_0 + w_m)(s, \cdot)\|_2 \, ds \\
+ \int_0^T \sum_{|\nu| \leq 46} \|Z^\nu \partial(u_0 + w_m)(s, \cdot)\|_\infty \sum_{|\mu| \leq 90, |\nu| \leq 1} \|Z^\mu \partial^\nu(u_0 + w_{m-1})(s, \cdot)\|_2 \, ds \\
+ \int_0^T \sum_{|\nu| \leq 46} \|Z^\nu(u_0 + w_{m-1})(s, \cdot)\|_\infty \sum_{|\mu| \leq 90, |\nu| \leq 1} \|Z^\mu \partial^\nu(u_0 + w_m)(s, \cdot)\|_2 \, ds \\
+ \int_0^T \sum_{|\mu| \leq 90} \|Z^\mu[\square, \eta]u(s, \cdot)\|_2 \, ds.
\]

The key thing to note here is that as we are not utilizing estimates for time-dependent perturbations of \(\square\), we must face a term of the form \(\sum_{|\nu| \leq 90} \|Z^\nu w_m\|_2\), which shall be bounded in the following section. Utilizing terms III and IX of (4.3) as well as (4.1), it follows that this piece satisfies

\[
III \bigg|_{|\nu|=1} + \text{IV} \bigg|_{|\nu|=1} \leq C(M_0(T) + M_{m-1}(T))(M_0(T) + M_{m-1}(T) + M_m(T)) \log(2 + T) + C_2\varepsilon.
\]

**Bounds for III,** \(|\nu|=2\): With \(\gamma\) chosen as when bounding I, we have (2.12) and (2.13) with \(\delta = C_1\varepsilon\). We may, thus, apply (2.12). Upon integrating (2.13) in \(t\), applying (2.12) and (1.4), and bootstrapping, we need to control

\[
(4.12) \int_0^T \sum_{|\mu| \leq 90, |\nu| \leq 1} \|\square, Z^\nu \partial^\nu w_m(t, \cdot)\|_2 \, dt + \sum_{|\mu| \leq 92} \|\partial^\mu w'_m\|_{L^2_x([0, T] \times \{x < 1\})}.
\]

As above, the bound established for II applies to the latter term, and we need only control the former term.

Using the product rule, it follows that

\[
\sum_{|\nu| \leq 90} \|\square, Z^\nu \partial^\nu w_m\| \lesssim \sum_{|\nu| \leq 47} \|Z^\nu(u_0 + w_{m-1})\| \sum_{|\mu| \leq 90, |\nu| \leq 1} \|Z^\mu \partial^\nu(u_0 + w_m)\| + \sum_{|\mu| \leq 93} \|Z^\mu u_0\| \\
+ \sum_{|\nu| \leq 47} \|Z^\nu(u_0 + w_m)\| \sum_{|\mu| \leq 90, |\nu| \leq 2} \|Z^\mu \partial^\nu(u_0 + w_{m-1})\| \\
+ \sum_{|\nu| \leq 47} \|Z^\nu(u_0 + w_{m-1})\| \sum_{|\mu| \leq 90, |\nu| \leq 2} \|Z^\mu \partial^\nu(u_0 + w_m)\| + \sum_{|\mu| \leq 91} \|Z^\mu[\square, \eta]u\|.
\]
And hence, using (4.1),

\[ \int_0^T \sum_{|\mu| \leq 90} \sum_{|\nu| \leq 1} |\mu| \leq 90 |\nu| = 1 \| \Box \gamma Z^\mu \partial^\nu w_m(t, \cdot) \|_2 dt \lesssim \varepsilon \]

\[ + \int_0^T \sum_{|\nu| \leq 47} \| Z^\nu (u_0 + w_{m-1})(t, \cdot) \|_\infty \left[ \sum_{|\mu| \leq 90} \| Z^\mu \partial^\nu (u_0 + w_m)'(t, \cdot) \|_2 + \sum_{|\mu| \leq 93} \| Z^\mu u_0(t, \cdot) \| \right] dt \]

\[ + \int_0^T \sum_{|\nu| \leq 47} \| Z^\nu (u_0 + w_m)'(t, \cdot) \|_\infty \sum_{|\mu| \leq 90} \| Z^\mu \partial^\nu (u_0 + w_{m-1})(t, \cdot) \|_2 dt \]

\[ + \int_0^T \sum_{|\nu| \leq 47} \| Z^\nu (u_0 + w_{m-1})(t, \cdot) \|_\infty \sum_{|\mu| \leq 90} \| Z^\mu \partial^\nu (u_0 + w_{m-1})(t, \cdot) \|_2 dt. \]

We now use terms III and IX of (4.3). This immediately yields

\[ III \left|_{|\nu|=2} \leq C(M_0(T) + M_{m-1}(T))(M_0(T) + M_{m-1}(T) + M_m(T)) \log (2 + T) + C_2 \varepsilon. \]

**Bound for V:** It is here where our approach most differs from that of [6]. When proving energy estimate involving \( L \), [6] utilized the star-shapedness, as in [18], to show that the worst contribution on the boundary had a favorable sign. Our approach instead follows that of [26], which relies instead on (2.16). We first note that by (2.10), it suffices to estimate

\[ \sum_{j+k \leq 81} \| L^j \partial^k w'_m(t, \cdot) \|_2 + \sum_{|\mu|+k \leq 80} \| L^k \partial^\mu \Box w_m(t, \cdot) \|_2. \]

We further note that

\[ \sum_{j+k \leq 81} \| L^j \partial^k w'_m(t, \cdot) \|_2 \leq \sum_{k \leq 80} \| \tilde{L} \partial^k w'_m(t, \cdot) \|_2 + \sum_{|\mu| \leq 81} \| \partial^\mu w'_m(t, \cdot) \|_2. \]

As the latter term is subject to the previously established bounds for \( I \), it suffices to control

\[ \sum_{j \leq 80} \| \tilde{L} \partial^j w'_m(t, \cdot) \|_2 + \sum_{|\mu|+k \leq 80} \| L^k \partial^\mu \Box w_m(t, \cdot) \|_2. \]
Upon doing so, we see that the and then sum over those dyadic intervals. See [16] for a more detailed computation. For the terms on the third and fourth lines, we shall apply (3.2) on dyadic intervals. We note that

\[ \int_0^T \sum_{j+k \leq 81} \left( \| \hat{\mathcal{L}}^k \partial_t^j \Box \gamma w_m(t, \cdot) \|_2 + \| [\hat{\mathcal{L}}^k \partial_t^j, \gamma^\alpha \partial_t^\alpha \partial_t^\beta] w_m(t, \cdot) \|_2 \right) dt \\
+ \sup_{t \in [0, T]} \sum_{k+1 \leq 81} \| L^k \partial^\mu \Box w_m(t, \cdot) \|_2 \]

We note that

\[ \sum_{j+k \leq 81} \left( \| \hat{\mathcal{L}}^k \partial_t^j \Box \gamma w_m \| + \| [\hat{\mathcal{L}}^k \partial_t^j, \Box - \Box \gamma] w_m \| \right) \lesssim \sum_{|\nu| \leq 41} |\partial^\nu (u_0 + w_{m-1})| \sum_{|\mu| \leq 80} |L \partial^\mu w_m| \\
+ \left( \sum_{|\nu| \leq 41} (|\partial^\nu (w_{m-1} + u_0)| + |\partial^\nu (w_{m} + u_0)'|) \right) \sum_{|\mu| \leq 80} |L \partial^\mu (w_{m-1} + u_0)'| \\
+ \sum_{j+|\mu| \leq 41} \| L^j \partial^\mu (w_m + u_0)' \| \sum_{|\nu| \leq 81} |\partial^\nu (w_{m-1} + u_0)'| \\
+ |w_{m-1} + u_0|^2 + \sum_{|\nu| \leq 81} |\partial^\nu [\Box, \eta] u|. \]

For the last term, we are using the assumption that the Cauchy data are compactly supported and finite propagation speed in order to guarantee that the coefficients of \( L \) are \( O(1) \) on the support of \([\Box, \eta] u\).

For the terms on the third and fourth lines, we shall apply (3.2) on dyadic intervals and then sum over those dyadic intervals. See [16] for a more detailed computation. Upon doing so, we see that the \( L^2 \) norm of the terms on the third and fourth lines above is bounded by

\[ \sum_{j+|\mu| \leq 43} \| \langle x \rangle^{-1/4} L^j Z^\mu (w_{m-1} + u_0) \|_2 \\
\times \left( \sum_{|\nu| \leq 81} (\| \langle x \rangle^{-3/4} \partial^\nu (w_m + u_0)' \|_2 + \| \langle x \rangle^{-3/4} \partial^\nu (w_{m-1} + u_0)' \|_2 + \| \langle x \rangle^{-3/4} \partial^\nu u_0 \|_2) \right) \\
+ \sum_{j+|\mu| \leq 41} \| \langle x \rangle^{-1/4} L^j Z^\mu (w_m + u_0)' \|_2 \sum_{|\nu| \leq 81} \| \langle x \rangle^{-3/4} \partial^\nu (w_{m-1} + u_0)' \|_2. \]
When integrated in $t$, we apply the Schwarz inequality and utilize terms $II$ and $XIII$ of (4.3) to establish control.

For the terms in the right side of (4.14) which are on the first and second lines, we apply $IX$ and $V$ of (4.3). And control for the second to last term in (4.14) follows from terms $IX$ and $III$ of (4.3).

Arguing as such yields the bound

$$
\int_0^T \sum_{j+k \leq 81} \sum_{k \leq 1} \left( \| \tilde{L}^k \partial_t^j \Box_{x} w_m(t, \cdot) \|_2 + \| [\tilde{L}^k \partial_t^j, \gamma^\alpha \partial_x \gamma^\beta \partial_x] w_m(t, \cdot) \|_2 \right) dt
\leq C(M_0(T) + M_{m-1}(T))(M_0(T) + M_{m-1}(T))(T)^{1/4} + C_2 \varepsilon.
$$

The integrand of last term in (4.13) is also controlled by the right side of (4.14). As there is no time integral, it can be bounded much more easily just using Sobolev embedding and terms $I$ and $V$ of (4.3). No loss of $(T)^{1/4}$ is necessitated here. The third term in (4.13) was previously controlled while establishing the bound for, say, $II$. For this term, as we saw previously, logarithmic losses would suffice.

It only remains to establish control for

$$
\sum_{|\mu| \leq 81} \int_0^T \| \partial^\mu w_m'(t, \cdot) \|_{L^2(|x| < 1)} dt.
$$

We apply (2.16) and see that it suffices to bound

$$
\sum_{|\mu| \leq 81} \int_0^T \int_0^s \| \partial^\mu \Box w_m(\tau, \cdot) \|_{L^2(|x| - (s-\tau)| \leq 10)} d\tau ds.
$$

Similar to (4.8), we have

$$
\sum_{|\mu| \leq 81} |\partial^\mu \Box w_m| \lesssim \sum_{|\nu| \leq 41} |\partial^\nu (u_0 + w_{m-1})| \sum_{|\mu| \leq 82} |\partial^\mu (u_0 + w_m)'| + \sum_{|\nu| \leq 41} |\partial^\nu (u_0 + w_{m-1})' + \sum_{|\mu| \leq 81} |\partial^\mu (u_0 + w_{m-1})| \sum_{|\mu| \leq 81} |\partial^\mu (u_0 + w_{m-1})'| + |u_0 + w_{m-1}|^2 + \sum_{|\mu| \leq 81} |\partial^\mu [\Box, \eta] u|.
$$
With the exception of the last term above, for which we instead use (4.1), we apply (3.3) to see that
\[
\sum_{|\mu| \leq 81} \| \partial^\mu \Box w_m \|_{L^2(|x| - (s - \tau) < 10)} 
\lesssim \sum_{|\nu| \leq 43} \| \langle x \rangle^{-1/4} Z^\nu (u_0 + w_{m-1}) \|_{L^2(|x| - (s - \tau) < 20)} \sum_{|\mu| \leq 82} \| \langle x \rangle^{-3/4} \partial^\mu (u_0 + w_m) \|_{L^2(|x| - (s - \tau) < 20)} 
\]
\[
+ \sum_{|\nu| \leq 43} \| \langle x \rangle^{-1/4} Z^\nu (u_0 + w_m) \|_{L^2(|x| - (s - \tau) < 20)} \sum_{|\mu| \leq 81} \| \langle x \rangle^{-3/4} \partial^\mu (u_0 + w_{m-1}) \|_{L^2(|x| - (s - \tau) < 20)} 
\]
\[
+ \sum_{|\nu| \leq 81} \| \langle x \rangle^{-1/4} Z^\nu (u_0 + w_{m-1}) \|_{L^2(|x| - (s - \tau) < 20)} \sum_{|\mu| \leq 81} \| \partial^\mu [\Box, \eta] u \|_{L^2(|x| - (s - \tau) < 10)} + \sum_{|\mu| \leq 81} \| \partial^\mu [\Box, \eta] u \|_{L^2(|x| - (s - \tau) < 10)} 
\]
Since the sets \{ ||x| - (j - \tau) < 20 \} have finite overlap as \( j \) ranges over the nonnegative integers, it follows that upon integrating in \( \tau \) and \( s \) that
\[
\sum_{|\mu| \leq 81} \int_0^T \int_0^s \| \partial^\mu \Box w_m \|_{L^2(|x| - (s - \tau) < 10)} \, d\tau \, ds 
\lesssim \sum_{|\nu| \leq 43} \| \langle x \rangle^{-1/4} Z^\nu (u_0 + w_{m-1}) \|_{L^2_t (S_T)} \sum_{|\mu| \leq 82} \| \langle x \rangle^{-3/4} \partial^\mu (u_0 + w_m) \|_{L^2_t (S_T)} 
\]
\[
+ \sum_{|\nu| \leq 43} \| \langle x \rangle^{-1/4} Z^\nu (u_0 + w_m) \|_{L^2_t (S_T)} \sum_{|\mu| \leq 81} \| \langle x \rangle^{-3/4} \partial^\mu (u_0 + w_{m-1}) \|_{L^2_t (S_T)} 
\]
\[
+ \sum_{|\nu| \leq 81} \| \langle x \rangle^{-1/4} Z^\nu (u_0 + w_{m-1}) \|_{L^2_t (S_T)} \sum_{|\mu| \leq 81} \| \partial^\mu [\Box, \eta] u \|_{L^2_t (S_T)} 
\]
Here we have also employed (4.1). And thus, we see that this boundary term is
\[
\leq C(M_0(T) + M_{m-1}(T))(M_0(T) + M_{m-1}(T) + M_m(T))^1/2 + C_2 \varepsilon, 
\]
and hence, when combined with the above
\[
V \leq C(M_0(T) + M_{m-1}(T))(M_0(T) + M_{m-1}(T) + M_m(T))^1/2 + C_2 \varepsilon. 
\]

**Bound for VI:** The arguments to bound terms VI, VII, and VIII follow those of the corresponding terms with no \( L \) quite closely. Indeed, for VI, we, as in the bound for II, apply (2.3) to \( L \partial^\mu w_m \) cutoff away from the boundary and (4.13) to both the remaining portion as well as the compactly supported commutator which results from cutting off above. It remains then to control
\[
(4.15) \sum_{j+|\mu| \leq 77} \int_0^T \| L \partial^\mu \Box w_m (s, \cdot) \|_2 \, ds + \sum_{j+|\mu| \leq 75} \| L \partial^\mu \Box w_m \|_{L^2_t (S_T)}. 
\]
By Sobolev embeddings, it suffices to control the first term.
We must now take a little care with the location of the scaling vector field. Playing the role of \(4.8\), we have

\[
\sum_{j+|\mu| \leq 77} |L^j \partial^\mu \Box w_m| \lesssim \sum_{|\nu| \leq 40} |\partial^\nu (u_0 + w_{m-1})| \sum_{j+|\mu| \leq 78} |L^j \partial^\mu (u_0 + w_m)'| \\
+ \sum_{j+|\nu| \leq 40} |L^j \partial^\nu (u_0 + w_{m-1})| \sum_{|\mu| \leq 78} |\partial^\mu (u_0 + w_m)'| \\
+ \sum_{j+|\nu| \leq 77} |L^j \partial^\nu (u_0 + w_{m-1})| \sum_{|\mu| \leq 78} |\partial^\mu (u_0 + w_m)'| \\
+ \sum_{j+|\nu| \leq 77} |\partial^\nu (u_0 + w_{m-1})| \sum_{j+|\mu| \leq 78} |L^j \partial^\mu (u_0 + w_m)'| \\
+ \sum_{j+|\nu| \leq 77} |\partial^\nu (u_0 + w_{m-1})| \sum_{j+|\mu| \leq 78} |\partial^\mu (u_0 + w_m)'| \\
+ |u_0 + w_{m-1}| \sum_{j \leq 1} |L^j (u_0 + w_{m-1})| + \sum_{j+|\mu| \leq 78} |L^j \partial^\mu [\Box, \eta] u|.
\]

When the scaling vector field is on the higher order term, we shall generally utilize terms \(V\) and \(IX\) of \(4.3\). Alternatively, when the scaling vector field is on the lower order term, we utilize \(3.2\). More specifically, we decompose dyadically in \(x\), apply \(3.2\), and apply the Schwarz inequality in both the dyadic summation variable and in \(t\) as above. Upon doing so, we can bound utilizing \(II\), \(IV\) and \(VII\) instead. For the second to last term, \(VII\) and \(IX\) are employed. And finally, \(4.11\) provides the bound for the final term.

We illustrate arguing in this fashion by examining the \(L^1([0, T]; L^2(\mathbb{R}^3 \setminus K))\)-norm of the first two terms in the right. Indeed, the norm of these terms is bounded by

\[
\int_0^T \sum_{|\nu| \leq 40} \| \partial^\nu (u_0 + w_{m-1})(s, \cdot) \|_\infty \sum_{j+|\mu| \leq 78} \| L^j \partial^\mu (u_0 + w_m)'(s, \cdot) \|_2 ds \\
+ \sum_{j+|\mu| \leq 42} \| (x)^{-1/4} L^j Z^\mu (u_0 + w_{m-1}) \|_{L^2_t L^\infty_x(S_T)} \sum_{|\nu| \leq 78} \| (x)^{-3/4} \partial^\nu (u_0 + w_m)' \|_{L^2_t L^\infty_x(S_T)}.
\]

And using \(XI, V, VIII\), and \(II\), this is

\[
\lesssim (M_0(T) + M_{m-1}(T))(M_0(T) + M_m(T))^1/4.
\]

The remaining terms above are handled in a directly analogous way, which yields

\[
VI \lesssim C(M_0(T) + M_{m-1}(T))(M_0(T) + M_{m-1}(T) + M_m(T))^1/4 + C_2 \varepsilon.
\]
Bound for VII and VIII with $|\nu| = 0$: We fix $\beta$ as above. It suffices to bound $(1 - \beta) L Z^\mu w_m$ as the Dirichlet boundary conditions and the boundedness of the coefficients of $Z$ on the support of $\beta$ allow us to control $\beta L Z^\mu w_m$ using term VI. For $\mu$ fixed, we decompose

$$(1 - \beta) L Z^\mu \Box w_m$$

$$= \partial_\alpha \left[ (1 - \eta)(1 - \beta)(b^\alpha \beta(u_0 + w_{m-1}) L Z^\mu \partial_\beta(u_0 + w_m) + b^\gamma \partial_\gamma(u_0 + w_{m-1}) L Z^\mu \partial_\beta(u_0 + w_m)) \right]$$

$$+ \tilde{G}_m(t, x).$$

We apply (2.6), (2.4), and (2.5). The latter is used for simply be cited to control this piece. The last term is controlled by $VII$ of (4.10) where every term on the right is permitted at most one occurrence of (4.16) and using Sobolev embedding on $L^2$ on the support of every $\mu$ fixed, we decompose

$$(1 - \beta) L Z^\mu \Box w_m$$

$$= \partial_\alpha \left[ \sum_{\lambda \leq 70} \partial_\lambda \eta(1 - \eta)(1 - \beta)(b^\alpha \beta(u_0 + w_{m-1}) L Z^\mu \partial_\lambda(u_0 + w_m) + b^\gamma \partial_\gamma(u_0 + w_{m-1}) L Z^\mu \partial_\lambda(u_0 + w_m)) \right]$$

$$+ \tilde{G}_m(t, x).$$

We apply (2.6), (2.4), and (2.5). The latter is used for $[\Box, \beta] L Z^\mu w_m$. This yields

$$\sup_{t \in [0, T]} \sum_{|\nu| \leq 70} \| (1 - \beta) L Z^\mu w_m(t, \cdot) \|_2 + \sum_{|\nu| \leq 70} \langle T \rangle^{-1/4} \| (x)^{-1/2} (1 - \beta) L Z^\mu w_m \|_{L^2_{t,x}(S_T)}$$

$$\lesssim \int_0^T \langle \xi \rangle^{-1/2} \tilde{G}_m(s, \cdot) \|_{L^1_{t,x}} \, ds + \sum_{|\nu| \leq 70} \sum_{|\theta| \leq 1} \int_0^T \| \partial_\theta(u_0 + w_{m-1}) \|_{L^2_{t,x}(S_T)} \, ds$$

$$+ \sum_{|\nu| \leq 70} \| L \partial_\nu w_m \|_{L^2_{t,x}(0, T) \times \{|x| < 3\}}.$$

The last term is controlled by VI, and the bound previously established for VI shall simply be cited to control this piece.

To control the first term in the right of (4.10), we need not be as precise with the location of the scaling vector fields, and indeed, we can crudely utilize the obvious analog of (4.10) where every term on the right is permitted at most one occurrence of $L$. Upon doing so and using Sobolev embedding on $S^2$ as well as (4.4), we obtain

$$\sum_{|\nu| \leq 70} \int_0^T \langle \xi \rangle^{-1/2} \tilde{G}_m \|_{L^1_{t,x}} \, ds \lesssim \varepsilon$$

$$+ \sum_{|\nu| \leq 38} \sum_{j \leq 1} \int_0^T \langle \xi \rangle^{-1/4} \langle L^j Z^\nu(u_0 + w_{m-1}) \|_{L^2_{t,x}(S_T)} \sum_{|\nu| \leq 70} \sum_{|\theta| \leq 1} \int_0^T \langle \xi \rangle^{-1/4} \langle L^k Z^\mu(u_0 + w_m) \|_{L^2_{t,x}(S_T)}$$

$$+ \sum_{|\nu| \leq 70} \sum_{|\theta| \leq 1} \int_0^T \langle \xi \rangle^{-1/4} \langle L^j Z^\mu(u_0 + w_{m-1}) \|_{L^2_{t,x}(S_T)} \sum_{|\nu| \leq 70} \sum_{|\theta| \leq 1} \int_0^T \langle \xi \rangle^{-1/4} \langle L^k Z^\mu(u_0 + w_{m-1}) \|_{L^2_{t,x}(S_T)}.$$

Each of these individual factors is controlled either by $\langle T \rangle^{1/4} IV$ or $\langle T \rangle^{1/4} VIII$. Thus, the first term in the right of (4.10) is

$$\leq C(T)^{1/2} (M_0(T) + M_{m-1}(T))(M_0(T) + M_{m-1}(T) + M_m(T)) + C_2 \varepsilon.$$
For the second term in the right side of (4.16), we may simply use IX and VII to immediately see that it is

$$\leq C(M_0(T) + M_{m-1}(T))(M_0(T) + M_m(T))\log(2 + T).$$

And thus, by combining the above bounds, we see that

$$VII\bigg|_{\nu=0} + VIII\bigg|_{\nu=0} \leq C\langle T \rangle^{1/2}(M_0(T) + M_{m-1}(T))(M_0(T) + M_m(T)) + C_2\varepsilon.$$

**Bound for VII and VIII with \(\nu = 1\):** Here, again, it suffices to control the given norm when \(w_m\) is replaced by \((1 - \beta)w_m\). As the coefficients of \(Z\) are \(O(1)\) on the support of \(\beta\), the corresponding \(\beta w_m\) terms are subject to the bounds previously established for \(V\) and \(VI\).

Applying (3.2) and (2.4) to \((1 - \beta)LZ^\mu w_m\), one obtains

$$\sup_{t \in [0,T]} \sum_{|\mu| \leq 70} \|LZ^\mu w'_m(t, \cdot)\|_2 + \sum_{|\nu| \leq 70} \langle T \rangle^{-1/4}\|\langle x \rangle^{-1/4}LZ^\mu w'_m\|_{L^2_{t,x}(S_T)}$$

$$\leq \sum_{|\mu| \leq 70} \int_0^T \|L^j Z^\mu w_m(s, \cdot)\|_2 ds + \sum_{|\mu| \leq 71} \|L\partial_j w_m\|_{L^2_{t,x}([0,T] \times \{|x| < 3\})}.$$ 

The bound for term \(VI\) applies to the latter term on the right side.

Here we need to pay attention to the location of \(L\) for terms involving \(\partial^2 w_m\), but for the other terms we may be more crude and simply permit up to one occurrence of \(L\) on each factor. For everything except the case that all of the vector fields land on \(\partial^2 w_m\), we dyadically decompose, apply (3.2), and use the Cauchy-Schwarz inequality in the dyadic variable as well as \(t\). Upon doing so, we obtain that first term in the right side of the preceding equation is controlled by

$$\int_0^T \sum_{|\nu| \leq 36} \|Z^\nu(u_0 + w_{m-1})(s, \cdot)\|_\infty \sum_{|\mu| \leq 70} \|LZ^\mu \partial^2(u_0 + w_m)(s, \cdot)\|_2 ds$$

$$+ \sum_{|\nu| \leq 38, j \leq 1} \|\langle x \rangle^{-1/2}L^j Z^\nu(u_0 + w_{m-1})\|_{L^2_{t,x}(S_T)} \sum_{|\mu| \leq 70} \|\langle x \rangle^{-1/2}Z^\mu \partial(u_0 + w_m)\|_{L^2_{t,x}(S_T)}$$

$$+ \sum_{|\nu| \leq 38, j \leq 1} \|\langle x \rangle^{-1/2}L^j Z^\nu(u_0 + w_m)\|_{L^2_{t,x}(S_T)} \sum_{|\mu| \leq 70, k \leq 1} \|\langle x \rangle^{-1/2}L^k Z^\mu \partial^\nu(u_0 + w_{m-1})\|_{L^2_{t,x}(S_T)}$$

$$+ \sum_{|\nu| \leq 38, j \leq 1} \|\langle x \rangle^{-1/2}L^j Z^\nu(u_0 + w_{m-1})\|_{L^2_{t,x}(S_T)} \sum_{|\mu| \leq 70, k \leq 1} \|\langle x \rangle^{-1/2}L^k Z^\mu \partial^\nu(u_0 + w_{m-1})\|_{L^2_{t,x}(S_T)}$$

$$+ \int_0^T \sum_{|\mu| \leq 70, j \leq 1} \|L^j Z^\mu \partial(\cdot, \eta|u(s, \cdot)|_2)\|_2 ds.$$ 

The first term is bounded using VII and IX, while for the next three, term VIII is primarily used. The only exception is the second factor in the second term where IV is
applied. The estimate (4.1) is used to control the last term. This shows that the above yields
\[ VII \mid_{|\nu|=1} + VIII \mid_{|\nu|=1} \leq C(M_0(T)+M_{m-1}(T))(M_0(T)+M_{m-1}(T)+M_m(T))(T)^{1/2} + C_2 \varepsilon. \]

Because the weights that appear in the $L^2_{t,x}$ norms are $\langle x \rangle^{-1/2}$ rather than $\langle x \rangle^{-1/4}$, it would be rather easy to replace $(T)^{1/2}$ in this bound by $\log(2+T)$. As this will not improve the final lifespan, we choose to not lengthen the argument in order to show this.

**Bound for VII with** $|\nu|=2$: With $\gamma$ chosen as in (4.6), which satisfies (2.8) and (2.12) with $\delta = C_1 \varepsilon$, we employ (2.13). After integrating, applying (2.12), using that $T \leq T_\varepsilon$ and bootstrapping, it suffices to bound
\[ \int_0^T \sum_{|\mu| \leq 70} \sum_{|\nu| \leq 1} \|\Box \gamma \mathcal{L}Z^\nu \partial^\nu w_m(t, \cdot)\|_2 dt + \sum_{|\mu| \leq 72} \|L^k \partial^\mu w_m\|_{L^2_{t,x}([0,T] \times \{|x|<1\})}. \]

The latter term is controlled by VI of (4.3), and the bound proved previously for that term suffices.

Here, we have
\[
\sum_{|\mu| \leq 70} \sum_{|\nu| \leq 37} |Z^\nu(u_0+w_{m-1})| \left[ \sum_{|\mu| \leq 70} |L^j Z^\nu(u_0+w_m)| + \sum_{|\mu| \leq 73} |\partial^\mu u_0| \right] \\
+ \sum_{|\nu| \leq 37} \sum_{j \leq 1} \left[ \sum_{|\mu| \leq 71} |Z^\nu(u_0+w_m)| + \sum_{|\mu| \leq 74} |\partial^\mu u_0| \right] \\
+ \sum_{|\nu| \leq 37} \sum_{j \leq 1} \left[ \sum_{|\mu| \leq 70} |L^j Z^\nu(u_0+w_{m-1})| \right] \\
+ \sum_{|\nu| \leq 37} \sum_{j \leq 1} \left[ \sum_{|\mu| \leq 72} |L^j Z^\nu(u_0+w_m)| \right] \\
+ \sum_{|\nu| \leq 37} \sum_{j \leq 1} \left[ \sum_{|\mu| \leq 72} |L^j Z^\nu(u_0+w_{m-1})| \right] \\
+ \sum_{|\nu| \leq 37} \sum_{j \leq 1} \left[ \sum_{|\mu| \leq 72} |\partial^\mu [\Box, \eta] u| \right].
\]

Here we have used the assumption that the Cauchy data are compactly supported and finite propagation speed guarantees that the coefficients of $L$ and $Z$ are $O(1)$ on the supports of $u_0$ and $[\Box, \eta] u$.

The method of bounding the above terms in $L^1_t([0,T]; L^2_x)$ depends on the location of the scaling vector field. We will illustrate the method on the terms in the third and fourth lines above. The remaining pieces are controlled in a directly analogous manner. When the scaling vector field is on the higher order factor, we shall use IX and VII (and III...
in the case that no \(L\) appears). When \(L\) lands on the lower order piece, we shall instead apply (3.2) as above. Doing so gives the following upper bound on the \(L^1([0, T]: L^2)\) norm of the terms in the third and fourth lines above:

\[
\int_0^T \sum_{|\mu| \leq 37} \|Z^\mu(u_0 + w_m)'(t, \cdot)\|_\infty \sum_{|\mu| \leq 70 \atop |\nu| \leq 2} \|L^j Z^\mu \partial^\nu (u_0 + w_{m-1})(t, \cdot)\|_2 dt
\]

\[
+ \sum_{|\nu| \leq 39 \atop j \leq 1} \|\langle x \rangle^{-1/2} L^j Z^\nu (u_0 + w_m)'\|_{L^2_{t,x}(S_T)} \sum_{|\mu| \leq 72} \|\langle x \rangle^{-1/2} Z^\mu (u_0 + w_{m-1})\|_{L^2_{t,x}(S_T)}
\]

By now citing IX, VII, and VIII of (4.3) and arguing similarly for the remaining nonlinear terms, we see that

\[
\text{VII}\big|_{|\nu|=2} \leq C(M_0(T) + M_{m-1}(T))(M_0(T) + M_{m-1}(T) + M_m(T))(T)^{1/2} + C_2 \varepsilon.
\]

**Bound for IX:** Here, we apply (3.1), and we are left with bounding (4.17)

\[
\int_0^T \int_{|\mu| + j \leq 67} \sum_{j \leq 1} |L^j Z^\mu \square w_m(s, y)| \frac{dy}{|y|} + \int_0^T \sum_{|\mu| + j \leq 64} \|L^j \partial^\mu \square w_m(s, \cdot)\|_{L^2(|x| < 2)} ds.
\]

For the first term, we use an analog of (4.8) for the given vector fields and apply the Schwarz inequality. Upon doing so, we have

\[
\sum_{|\mu| + j \leq 69} \|\langle x \rangle^{-1/2} L^j Z^\mu (u_0 + w_m)\|_{L^2_{t,x}(S_T)} \sum_{|\mu| + j \leq 68} \|\langle x \rangle^{-1/2} L^j Z^\mu (u_0 + w_{m-1})\|_{L^2_{t,x}(S_T)}
\]

\[
+ \left( \sum_{|\mu| + j \leq 68} \|\langle x \rangle^{-1/2} L^j Z^\mu (u_0 + w_{m-1})\|_{L^2_{t,x}(S_T)} \right)^2 + C_2 \varepsilon.
\]

Control for this term follows from IV and VIII of (4.3). And the second term of (4.17) was previously controlled in the process of bounding term VI of (4.3). It follows that

\[
\text{IX} \leq C(M_0(T) + M_{m-1}(T))(M_0(T) + M_{m-1}(T) + M_m(T))(T)^{1/2} + C_2 \varepsilon.
\]

**Boundedness of \(M_m(T)\):** If we combine the estimates for terms I, ..., IX just established, it follows that

\[
M_m(T) \leq C(M_0(T) + M_{m-1}(T))(M_0(T) + M_{m-1}(T) + M_m(T))(T)^{1/2} + C_2 \varepsilon
\]

where \(C_2\) is now a fixed constant which is independent of \(m, \varepsilon, \) and \(T\). If \(C_1\) is chosen so that \(C_1 > 2C_2\) and if we apply (4.4) as well as the inductive hypothesis, it follows that

\[
M_m(T) \leq C\varepsilon(\varepsilon + M_m(T))(T)^{1/2} + \frac{C_1}{2} \varepsilon.
\]

If we use (4.4), then if \(c\) in (4.4) is sufficiently small we may bootstrap in such a way that we obtain (4.5) if \(\varepsilon\) is small enough, as desired.
Convergence of the sequence \( \{w_m\} \): We now complete the proof of Theorem 1.1 by showing that the sequence \( \{w_m\} \) is Cauchy. Standard results show that the limiting function solves (4.2), which is equivalent to the existence promised in Theorem 1.1.

Indeed, if we set

\[
A_m(T) = \sup_{t \in [0, T]} \sum_{|\mu| \leq 90} \| \partial^\mu (w_m - w_{m-1})(t, \cdot) \|_2,
\]

we may argue quite similarly to the above. Upon doing so and using (4.5), it can be shown that

\[
A_m(T) \leq \frac{1}{2} A_{m-1}(T)
\]

for \( T \leq T_\epsilon \), which completes the proof.

REFERENCES

[1] N. Burq and M. Zworski: Geometric control in the presence of a black box. J. Amer. Math. Soc. 17 (2004), 443–471.
[2] H. Christianson: Dispersive estimates for manifolds with one trapped orbit. Comm. Partial Differential Equations 33 (2008), 1147–1174.
[3] H. Christianson and J. Wunsch: Local smoothing for the Schrödinger equation with a prescribed loss, preprint. (ArXiv: 1103.3908)
[4] Y. Colin de Verdière and B. Parisse: Équilibre instable en régime semi-classique. I. Concentration microlocale. Comm. Partial Differential Equations 19 (1994), 1535–1563.
[5] Y. Du, J. Metcalfe, C. D. Sogge and Y. Zhou: Concerning the Strauss conjecture and almost global existence for nonlinear Dirichlet-wave equations in 4-dimensions. Comm. Partial Differential Equations 33 (2008), 1487–1506.
[6] Y. Du and Y. Zhou: The life span for nonlinear wave equation outside of star-shaped obstacle in three space dimensions. Comm. Partial Differential Equations 33 (2008), 1455–1486.
[7] K. Hidano and K. Yokoyama: A remark on the almost global existence theorems of Keel, Smith, and Sogge. Funkcial. Ekvac. 48 (2005), 1–34.
[8] L. Hörmander: On the fully nonlinear Cauchy problem with small data. II. Microlocal analysis and nonlinear waves (Minneapolis, MN, 1988–1989), 51–81, IMA Vol. Math. Appl., 30, Springer, New York, 1991.
[9] L. Hörmander. Lectures on nonlinear hyperbolic differential equations. Springer-Verlag, Berlin, 1997.
[10] L. Hörmander: \( L^1, L^\infty \) estimates for the wave operator. Analyse mathématique et applications, 211–234, Gauthier-Villars, Montrouge, 1988.
[11] M. Iwawa: Decay of solutions of the wave equation in the exterior of two convex bodies. Osaka J. Math. 19 (1982), 459–509.
[12] M. Iwawa: Decay of solutions of the wave equation in the exterior of several convex bodies. Ann. Inst. Fourier (Grenoble) 38 (1988), 113–146.
[13] F. John and S. Klainerman: Almost global existence to nonlinear wave equations in three space dimensions. Comm. Pure Appl. Math. 37 (1984), 443–455.
[14] S. Katayama and H. Kubo: An alternative proof of global existence for nonlinear wave equations in an exterior domain. J. Math. Soc. Japan 60 (2008), 1135–1170.
[15] M. Keel, H. F. Smith, and C. D. Sogge: On global existence for nonlinear wave equations outside of convex obstacles. Amer. J. Math. 122 (2000), 805–842.
[16] M. Keel, H. F. Smith, and C. D. Sogge: Almost global existence for some semilinear wave equations, J. Anal. Math. 87 (2002), 265-279.
[17] M. Keel, H. F. Smith, and C. D. Sogge: Global existence for a quasilinear wave equation outside of star-shaped domains. J. Funct. Anal. 189 (2002), 155–226.
[18] M. Keel, H. F. Smith, and C. D. Sogge: Almost global existence for quasilinear wave equations in three space dimensions. J. Amer. Math. Soc. 17 (2004), 109–153.
[19] C. E. Kenig, G. Ponce, and L. Vega: On the Zakharov and Zakharov-Schulman systems. J. Funct. Anal. 127 (1995), 204–234.
[20] S. Klainerman: Uniform decay estimates and the Lorentz invariance of the classical wave equation. Comm. Pure Appl. Math. 38 (1985), 321–332.
[21] S. Klainerman: The null condition and global existence to nonlinear wave equations. Lect. Appl. Math. 23 (1986), 293–326.
[22] P. D. Lax, C. S. Morawetz, and R. S. Phillips: Exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle. Comm. Pure Appl. Math. 16 (1963), 477–486.
[23] H. Lindblad: On the lifespan of solutions of nonlinear wave equations with small initial data. Comm. Pure Appl. Math. 43 (1990), 445–472.
[24] J. Metcalfe, M. Nakamura, and C. D. Sogge: Global existence of solutions to multiple speed systems of quasilinear wave equations in exterior domains. Forum Math. 17 (2005), 133–168.
[25] J. Metcalfe, M. Nakamura, and C. D. Sogge: Global existence of quasilinear, nonrelativistic wave equations satisfying the null condition. Japan. J. Math. 21 (2005), 391–472.
[26] J. Metcalfe and C. D. Sogge: Hyperbolic trapped rays and global existence of quasilinear wave equations. Invent. Math. 159 (2005), 75–117.
[27] J. Metcalfe and C. D. Sogge: Long time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods. SIAM J. Math. Anal. 38 (2006), 188–209.
[28] J. Metcalfe and C. D. Sogge: Global existence of null-form wave equations in exterior domains. Math. Z. 256 (2007), 521–549.
[29] J. Metcalfe and C. D. Sogge: Global existence for high dimensional quasilinear wave equations exterior to star-shaped obstacles. Discrete Cts. Dyn. Sys. 28 (2010), 1589–1601.
[30] J. Metcalfe and D. Tataru: Global parametrices and dispersive estimates for variable coefficient wave equations. Math. Ann., to appear. (arXiv: 0707.1191)
[31] J. Metcalfe and D. Tataru: Decay estimates for variable coefficient wave equations in exterior domains. Advances in Phase Space Analysis of Partial Differential Equations, In Honor of Ferruccio Colombini’s 60th Birthday. Progress in Nonlinear Differential Equations and Their Applications, Vol. 78, 2009, p. 201–217.
[32] S. Morawetz: The decay of solutions of the exterior initial-boundary value problem for the wave equation. Comm. Pure Appl. Math. 14 (1961), 561–568.
[33] S. Morawetz: Time decay for the nonlinear Klein-Gordon equations. Proc. Roy. Soc. Ser. A 306 (1968), 291–296.
[34] S. Morawetz, J. Ralston and W. Strauss: Decay of solutions of the wave equation outside nontrapping obstacles. Comm. Pure Appl. Math. 30 (1977), 87–133.
[35] S. Nonnenmacher and M. Zworski: Semiclassical resolvent estimates in chaotic scattering. Appl. Math. Res. Express. 1 (2009), 74–86.
[36] J. Y. Ralston: Solutions of the wave equation with localized energy. Comm. Pure Appl. Math. 22 (1969), 807–823.
[37] H. F. Smith and C. D. Sogge: Global Strichartz estimates for nontrapping perturbations of the Laplacian. Comm. Partial Differential Equations 25 (2000), 2171–2183.
[38] C. D. Sogge. Lectures on nonlinear wave equations, 2nd edition, International Press, Boston, MA, 2008.
[39] J. Sterbenz: Angular regularity and Strichartz estimates for the wave equation. With an appendix by Igor Rodnianski. Int. Math. Res. Not. 2005, 187–231.
[40] W. A. Strauss: Dispersal of waves vanishing on the boundary of an exterior domain. Comm. Pure Appl. Math. 28 (1975), 265–278.
[41] J. Wunsch and M. Zworski: Resolvent estimates for normally hyperbolic trapped sets, preprint. (ArXiv: 1003.4640)

Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250
E-mail address: johelms@email.unc.edu, metcalfe@email.unc.edu