The Reidemeister torsion of high-dimensional long knots from configuration space integrals

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Abstract

In a previous article, we gave a more flexible definition of an invariant \((Z_k)_{k \in \mathbb{N} \setminus \{0, 1\}}\) of Bott, Cattaneo, and Rossi, which is a combination of integrals over configuration spaces for long knots \(\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+2}\), for odd \(n \geq 3\). This extended the definition of the invariant \((Z_k)_{k \in \mathbb{N} \setminus \{0, 1\}}\) to all long knots in asymptotic homology \(\mathbb{R}^{n+2}\), for odd \(n \geq 3\). In this article, we obtain a formula for \(Z_k\) in terms of linking numbers of some cycles of a surface bounded by the knot and we express the Reidemeister torsion of the knot complement in terms of \((Z_k)_{k \in \mathbb{N} \setminus \{0, 1\}}\), when \(n \equiv 1 \text{ mod } 4\).

1 Introduction

In [Bot96], Bott introduced an isotopy invariant \(Z_2\) of knots \(S^n \hookrightarrow \mathbb{R}^{n+2}\) in odd dimensional Euclidean spaces. The invariant reads as a linear combination of configuration space integrals associated to graphs by integrating some forms associated to the edges, which represent directions in \(\mathbb{R}^n\) or in \(\mathbb{R}^{n+2}\). The involved graphs have four vertices of two kinds, and four edges of two kinds.

This invariant was generalized to a whole family \((Z_k)_{k \in \mathbb{N} \setminus \{0, 1\}}\) of isotopy invariants of long knots \(\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+2}\), for odd \(n \geq 3\), by Cattaneo and Rossi in [CR05] and by Rossi in his thesis [Ros02]. The invariant \(Z_k\) involves graphs with \(2k\) vertices of two kinds and \(2k\) edges of two kinds.

In [Wat07, Corollary 4.9], Watanabe proved that these so-called Bott-Cattaneo-Rossi (BCR for short) invariants are finite type invariants with respect to some operations on long ribbon knots. His study allowed him to prove that the invariants \(Z_k\) are not trivial for even \(k \geq 2\), and that they are related to the Alexander polynomial for long ribbon knots. He obtained an exact formula for \(Z_2\) in terms of the Alexander polynomial for any long ribbon knot.

In [Let19], we introduced more flexible definitions for the invariants \(Z_k\). Our definitions allowed us to generalize these invariants in the larger setting of long knots inside asymptotic homology \(\mathbb{R}^{n+2}\) when \(n\) is odd \(\geq 3\).
In this article, we obtain a formula for the generalized $Z_k$ invariant in terms of linking numbers of some cycles of a surface bounded by the knot, which holds at least when $n \equiv 1 \mod 4$. Theorem 2.24 gives this formula for the rectifiable knots of Definition 2.20 which are particular long knots. More generally, Corollary 2.25 extends this formula to virtually rectifiable knots, which are the long knots $\psi$ such that the connected sum $\psi \# \cdots \# \psi$ of $r$ copies of $\psi$ is rectifiable for some $r \geq 1$. Section 5 shows that the connected sum of any long knot with three copies of itself is rectifiable when $n \equiv 1 \mod 4$.

In Theorem 2.29, we use Corollary 2.25 to express the Reidemeister torsion $T_\psi(t)$, for virtually rectifiable knots, as the following combination of integrals over configuration spaces:

$$T_\psi(e^h) = \exp \left( - \sum_{k \geq 2} Z_k(\psi) h^k \right).$$

This formula also determines the invariant $Z_k$ as an explicit function of the Alexander polynomials of the knot. To our knowledge, our induced explicit determination of $(Z_k)_{k \geq 2}$ is the first complete computation of an invariant defined from configuration space integrals in degree higher than five. Our formula for $(Z_k)_{k \geq 2}$ extends and refines the forementioned result of Watanabe [Wat07, Corollary 4.9] for virtually rectifiable knots.

In Section 2, we first give a self-contained definition of the invariant $Z_k$ of [Let19] using intersection numbers of preimages of propagators, where propagators are special chains in the two-point configuration space of the ambient asymptotic homology $\mathbb{R}^{n+2}$, which are presented in Definition 2.9. We state all the forementioned theorems in this section. Section 3 describes how to obtain the formula for $Z_k$ in terms of linking numbers, for rectifiable knots, using some suitable propagators. The details of the construction of such propagators for any rectifiable knot are presented in Section 4. In Section 6, we derive the formula of Theorem 2.29 for the Reidemeister torsion from Corollary 2.25.

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2 Definition of $(Z_k)_{k \geq 2}$ and main statements

2.1 Parallelized asymptotic homology $\mathbb{R}^{n+2}$ and long knots

In this article, $n$ is an odd integer $\geq 3$, and $M$ is an $(n+2)$-dimensional closed smooth oriented manifold, such that $H_*(M; \mathbb{Z}) = H_*(S^{n+2}; \mathbb{Z})$. Such a manifold is called a homology $(n+2)$-sphere. In such a homology sphere, choose a point $\infty$ and a closed ball $B_\infty(M)$ around this point. Fix an identification of this ball $B_\infty(M)$
with the complement $B_\infty$ of the open unit ball of $\mathbb{R}_{n+2}$ in $S_{n+2} = \mathbb{R}_{n+2} \cup \{\infty\}$, such that this smooth identification extends from a neighborhood of $B_\infty(M)$ to a neighborhood of $B_\infty$ in $S_{n+2}$. Let $M^o$ denote the manifold $M \setminus \{\infty\}$ and let $B^o_\infty(M)$ denote the punctured ball $B_\infty(M) \setminus \{\infty\}$, which is identified with the complement $B^o_\infty$ of the open unit ball in $\mathbb{R}_{n+2}$. Let $B(M)$ denote the closure of $M^o \setminus B^o_\infty$, so that the manifold $M^o$ can be seen as $M^o = B(M) \cup B^o_\infty$, where $B^o_\infty \subset \mathbb{R}_{n+2}$. The manifold $M^o$ endowed with the decomposition $M^o = B(M) \cup B^o_\infty$, where $B^o_\infty \subset \mathbb{R}_{n+2}$.

Long knots of such a space $M^o$ are smooth embeddings $\psi : \mathbb{R}_n \hookrightarrow M^o$ such that $\psi(x) = (0, 0, x) \in B^o_\infty$ when $||x|| \geq 1$, and $\psi(x) \in B(M)$ when $||x|| \leq 1$.

**Definition 2.1.** A parallelization of an asymptotic homology $\mathbb{R}_{n+2}$ is a bundle isomorphism $\tau : M^o \times \mathbb{R}_{n+2} \to TM^o$ that coincides with the canonical trivialization $\tau_0 : \mathbb{R}_{n+2} \times \mathbb{R}_{n+2} \to T\mathbb{R}_{n+2}$ on $B^o_\infty \times \mathbb{R}_{n+2}$. An asymptotic homology $\mathbb{R}_{n+2}$ with such a parallelization is called a parallelized asymptotic homology $\mathbb{R}_{n+2}$. An asymptotic homology $\mathbb{R}_{n+2}$ that admits a parallelization is called parallelizable. Given a parallelization $\tau$ and a point $x \in M^o$, $\tau_x$ denotes the isomorphism $\tau(x, \cdot) : \mathbb{R}_{n+2} \to T_xM^o$.

### 2.2 BCR diagrams

In this section, we describe the diagrams involved in the definition of the invariant $Z_k$, which are the BCR diagrams of [Let19, Section 2.2].

**Definition 2.2.** A BCR diagram is an oriented connected graph $\Gamma$, defined by a set $V(\Gamma)$ of vertices, decomposed into $V(\Gamma) = V_i(\Gamma) \sqcup V_e(\Gamma)$, and a set $E(\Gamma)$ of ordered pairs of distinct vertices, decomposed into $E(\Gamma) = E_i(\Gamma) \sqcup E_e(\Gamma)$, whose elements are called edges, where the elements of $V_i(\Gamma)$ are called internal vertices, those of $V_e(\Gamma)$ external vertices, those of $E_i(\Gamma)$ internal edges, and those of $E_e(\Gamma)$ external edges, and such that, for any vertex $v$ of $\Gamma$, one of the five following properties holds:

1. $v$ is external, with two incoming external edges and one outgoing external edge, and one of the incoming edges comes from a univalent vertex.

2. $v$ is internal and trivalent, with one incoming internal edge, one outgoing internal edge, and one incoming external edge, which comes from a univalent vertex.

3. $v$ is internal and univalent, with one outgoing external edge.

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1 Note that this implies that our graphs have neither loops nor multiple edges with the same orientation.
4. \( v \) is internal and bivalent, with one incoming external edge and one outgoing internal edge.

5. \( v \) is internal and bivalent, with one incoming internal edge and one outgoing external edge.

In the following, internal edges are depicted by solid arrows, external edges by dashed arrows, internal vertices by black dots, and external vertices by white dots, as in Figure 1 where all the five behaviors of Definition 2.2 appear.

![Figure 1: An example of a BCR diagram of degree 6](image)

Definition 2.2 implies that any BCR diagram consists of one cycle with some legs attached to it, where legs are external edges that come from a (necessarily internal) univalent vertex, and where the graph is a cyclic sequence of pieces as in Figure 2 with as many pieces of the first type than of the second type. In particular, a BCR diagram has an even number of vertices.

![Figure 2](image)

For any positive integer \( k \), set \( \underline{k} = \{1, \ldots, k\} \).

**Definition 2.3.** Define the **degree** of a BCR diagram \( \Gamma \) as \( \deg(\Gamma) = \frac{1}{2} \text{Card}(V(\Gamma)) \), and let \( \mathcal{G}_k \) denote the set of all BCR diagrams of degree \( k \). Note that a degree \( k \) BCR diagram has \( 2k \) edges.

A **numbering** of a BCR diagram \( \Gamma \) of degree \( k \) is a bijection \( \sigma : E(\Gamma) \to 2\underline{k} \). A **numbered BCR diagram** is a pair \((\Gamma, \sigma)\) where \( \Gamma \) is a BCR diagram and \( \sigma \) is a numbering of \( \Gamma \). Let \( \tilde{\mathcal{G}}_k \) denote the set of numbered BCR diagrams up to numbered graph isomorphisms.
2.3 Two-point configuration spaces

Let $X$ be a $d$-dimensional closed smooth oriented manifold, let $\infty$ be a point of $X$, and set $X^\circ = X \setminus \{\infty\}$. We give a short overview of a compactification $C_2(X^\circ)$ of the two-point configuration space $C_0^0(X^\circ) = \{(x, y) \in (X^\circ)^2 \mid x \neq y\}$, as defined in [Les15, Section 2.2].

If $P$ is a submanifold of a manifold $Q$, such that $P$ is transverse to $\partial Q$ and $\partial P = P \cap \partial Q$, its normal bundle $\mathfrak{N}P$ is the bundle whose fibers are $\mathfrak{N}_x P = T_x Q / T_x P$. A fiber $U\mathfrak{N}_P$ of the unit normal bundle $U\mathfrak{N}P$ of $P$ is the quotient of $\mathfrak{N}_x P \setminus \{0\}$ by the dilations $2$.

Here, we use the blow-up in differential topology, which replaces a compact submanifold $P$ of a compact manifold $Q$ as above with its unit normal bundle $U\mathfrak{N}P$. The obtained manifold is a smooth compact manifold. It is diffeomorphic to the complement in $Q$ of an open tubular neighborhood of $P$. Its interior is $Q \setminus (\partial Q \cup P)$, and its boundary is $U\mathfrak{N}P \cup (\partial Q \setminus \partial P)$ as a set.

Define the space $C_1(X^\circ)$ as the blow-up of $X$ along $\{\infty\}$. It is a compact manifold with interior $X^\circ$ and with boundary the unit normal bundle $S_{\infty}^{d-1}X$ to $X$ at $\infty$.

Blow up the point $(\infty, \infty)$ in $X^2$. In the obtained manifold, blow up the closures of the sets $\{\infty\} \times X^\circ$, $X^\circ \times \{\infty\}$ and $\Delta_X^\circ = \{(x, x) \mid x \in X^\circ\}$.

The obtained manifold $C_2(X^\circ)$ is compact and it comes with a canonical map $p_b: C_2(X^\circ) \to X^2$. Its interior is canonically diffeomorphic to the open configuration space $C_0^0(X^\circ) = \{(x, y) \in (X^\circ)^2 \mid x \neq y\}$, and $C_2(X^\circ)$ has the same homotopy type as $C_0^0(X^\circ)$. The manifold $C_2(X^\circ)$ is called the two-point configuration space of $X^\circ$. Its boundary is the union of:

- the closed part $p_b^{-1}(\{(\infty, \infty)\})$,
- the unit normal bundles to $X^\circ \times \{\infty\}$ and $\{\infty\} \times X^\circ$, which are $X^\circ \times S_{\infty}^{d-1}X$ and $S_{\infty}^{d-1}X \times X^\circ$,
- the unit normal bundle to the diagonal $\Delta_{X^\circ}$, which is identified with the unit tangent bundle $UXX^\circ$ via the map $[(u, v)]_{(x, x)} \in U\mathfrak{N}_{(x, x)} \Delta_{X^\circ} \mapsto [v - u]_x \in U_x X^\circ$.

The following lemma can be proved as [Les15, Lemma 2.2].

Lemma 2.4. When $X^\circ = \mathbb{R}^d$, the Gauss map

$$
C_0^0(\mathbb{R}^d) \to S^{d-1},
(x, y) \mapsto \frac{y - x}{\|y - x\|}
$$

extends to a smooth map $G: C_2(\mathbb{R}^d) \to S^{d-1}$.

$^2$Dilations are homotheties with positive ratio.
We now define an analogue of $G$ on the boundary of $C_2(M^o)$ for any parallelized asymptotic homology $\mathbb{R}^{n+2}$.

**Definition 2.5.** Let $(M^o, \tau)$ be a parallelized asymptotic homology $\mathbb{R}^{n+2}$. Identify the sphere $S^{n+1}_\infty$ with $S^{n+1}$ in such a way that $u \in S^{n+1}$ is the limit when $t$ approaches $+\infty$ of the map $t \in [\frac{1}{||u||}, +\infty] \mapsto t.u \in B^\infty \subset \mathbb{R}^{n+2}$. The boundary of $C_2(M^o)$ is the union of:

- the closed part $\partial_{\infty,\infty}C_2(M^o) = p_b^{-1}(\{\infty \times \infty\})$, which identifies with the similar part $\partial_{\infty,\infty}C_2(\mathbb{R}^{n+2}) \subset C_2(\mathbb{R}^{n+2})$,
- an open face $\partial_{\infty,M^o}C_2(M^o) = p_b^{-1}(\{\infty\} \times M^o) = S^{n+1}_\infty \times M^o = S^{n+1} \times M^o$,
- an open face $\partial_{M^o,\infty}C_2(M^o) = p_b^{-1}(M^o \times \{\infty\}) = M^o \times S^{n+1}$,
- an open face $\partial_{\Delta}C_2(M^o) = p_b^{-1}(\Delta_{M^o}) = UM^o$.

Define the smooth map $G_\tau : \partial C_2(M^o) \to S^{n+1}$ by the following formula:

$$G_\tau(c) = \begin{cases} 
G(c) & \text{if } c \in \partial_{\infty,\infty}C_2(M^o) = \partial_{\infty,\infty}C_2(\mathbb{R}^{n+2}), \\
-u & \text{if } c = (u,y) \in \partial_{\infty,M^o}C_2(M^o) = S^{n+1}_\infty \times M^o,
\end{cases}$$

$$u \text{ if } c = (x,u) \in \partial_{M^o,\infty}C_2(M^o) = M^o \times S^{n+1},$$

$$\frac{\tau^{-1}(u)}{||\tau^{-1}(u)||} \text{ if } c = [u]_x \in U_xM^o \subset UM^o = \partial_{\Delta}C_2(M^o).$$

In order to simplify the notations, for any configuration in one of the three above open faces, we write $c = (x,y,u)$ where $(x,y) = p_b(c)$, and $u$ denotes the coordinate in the previous definition, which is either in $S^{n+1}$ or in $U_xM^o$.

### 2.4 Configuration spaces

Let $\Gamma$ be a BCR diagram, and let $M^o$ be an asymptotic homology $\mathbb{R}^{n+2}$. Fix a long knot $\psi : \mathbb{R}^n \hookrightarrow M^o$. Let $C^0_\Gamma(\psi)$ denote the open configuration space

$$C^0_\Gamma(\psi) = \{c : \Lambda(\Gamma) \hookrightarrow M^o \mid \text{There exists } c_i : \Lambda_i(\Gamma) \hookrightarrow \mathbb{R}^{n} \text{ such that } c \big|_{\Lambda_i(\Gamma)} = \psi \circ c_i\}.$$

An element $c$ of $C^0_\Gamma(\psi)$ is called a configuration. Note that $c_i$ is uniquely determined by $c$. By definition, the images of the vertices under a configuration are pairwise distinct, and the images of the internal vertices are on the knot.

This configuration space is a non-compact smooth manifold. It admits a compactification $C_\Gamma(\psi)$, which is defined in [Ros02, Section 2.4, pp. 51-61].

**Theorem 2.6** (Rossi). There exists a compact manifold with ridges and edges $C_\Gamma(\psi)$, such that:

3as a subset of $\partial C_2(M^o)$.  

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• its interior is canonically diffeomorphic to $C_0^0(\psi)$,

• for any two internal vertices $v$ and $w$, the map $\left(c \in C_0^0(\psi) \mapsto (c(v), c(w)) \in C_2(\mathbb{R}^n)\right)$ extends to a smooth map $p_{v,w}^i: C_\Gamma(\psi) \to C_2(\mathbb{R}^n)$,

• for any two vertices $v$ and $w$, the map $\left(c \in C_0^0(\psi) \mapsto (c(v), c(w)) \in C_2(M^o)\right)$ extends to a smooth map $p_{v,w}^e: C_\Gamma(\psi) \to C_2(M^o)$,

• for any vertex $v$, the map $\left(c \in C_0^0(\psi) \mapsto c(v) \in C_1(M^o)\right)$ extends to a smooth map $p_v: C_\Gamma(\psi) \to C_1(M^o)$.

**Definition 2.7.** For any edge $f$ of $\Gamma$, which goes from a vertex $v$ to a vertex $w$, $C_f$ denotes the configuration space $C_2(\mathbb{R}^n)$ if $f$ is internal, and $C_2(M^o)$ if $f$ is external, and $p_f: C_\Gamma(\psi) \to C_f$ denotes the map $p_{v,w}^i$ if $f$ is internal, and $p_{v,w}^e$ if $f$ is external.

Orient the space $C_\Gamma(\psi)$ as follows. For any $i \in \mathbb{N}$, let $dY_i^v$ denote the $i$-th coordinate form of the internal vertex $v$ (parametrized by $\mathbb{R}^n$) and for any $i \in n + 2$, let $dX_i^v$ denote the $i$-th coordinate form of the external vertex $v$ (in an oriented chart of $M^o$). Split any external edge $e$ in two halves: the tail $e_-$ and the head $e_+$. Define a form $\Omega_{e\pm}$ for any half-edge $e\pm$ of an external edge $e$, as follows:

• for the head $e_+$ of a leg going to an external vertex $v$, $\Omega_{e_+} = dX_1^v$,

• for the head $e_+$ of an edge that is not a leg, going to an external vertex $v$, $\Omega_{e_+} = dX_2^v$,

• for the tail $e_-$ of an edge coming from an external vertex $v$, $\Omega_{e_-} = dX_3^v \wedge \ldots \wedge dX_{n+2}^v$,

• for any external half-edge $e\pm$ adjacent to an internal vertex $v$, $\Omega_{e\pm} = dY_1^v \wedge \ldots \wedge dY_n^v$.

Let $N_{T,i}(\Gamma)$ denote the number of internal trivalent vertices, and define the sign of a BCR diagram as $\varepsilon(\Gamma) = (-1)^{N_{T,i}(\Gamma) + \text{Card}(E_\Gamma)}$. The orientation of $C_\Gamma(\psi)$ is $\Omega(\Gamma) = \varepsilon(\Gamma) \wedge \Omega_{e\pm}$, where $\Omega_{e\pm} = \Omega_{e_-} \wedge \Omega_{e_+}$ for any external edge $e$.

**Lemma 2.8.** Let $\Gamma_k$ be the degree $k$ BCR diagram of Figure 3.
Then, $-C_{\Gamma_k}(\psi)$ is oriented by the coordinates $(c_i(v_j), c_i(w_j))_{j \in \mathbb{K}} \in (\mathbb{R}^n \times M^0)^k$.

Proof. For any $j \in \mathbb{K}$ let $j^+$ denote the integer

$$j^+ = \begin{cases} j + 1 & \text{if } j < k, \\ 1 & \text{if } j = k. \end{cases}$$

First note that $\varepsilon(\Gamma_k) = 1$ since there are $2k$ external edges and no internal trivalent vertices. For any $j \in \mathbb{K}$, $\ell_j$ denotes the leg from $v_j$ to $w_j$, and $f_j$ denotes the external edge of the cycle from $w_j$ to $w_{j^+}$ as in Figure 3. Set $dY_{v_j} = \bigwedge_{i=1}^{n+2} dX^i_{v_j}$, $dX^j_{w_j} = \bigwedge_{i=1}^{n+2} dX^i_{w_j}$ and $dX^{j^+}_{w_j} = \bigwedge_{i=1}^{n+2} dX^i_{w_{j^+}}$. We have $\Omega_{\ell_j} = dY_{v_j} \wedge dX^1_{w_j}$ and $\Omega_{f_j} = dX^2_{w_j} \wedge dX^2_{w_{j^+}}$. Then,

$$\Omega(\Gamma_k) = \bigwedge_{j=1}^{k} (dY_{v_j} \wedge dX^1_{w_j} \wedge dX^2_{w_j} \wedge dX^2_{w_{j^+}})$$

$$= dY_{v_1} \wedge dX^1_{w_1} \wedge dX^2_{w_1} \wedge \bigwedge_{j=2}^{k} (dX^2_{w_j} \wedge dY_{v_j} \wedge dX^1_{w_j} \wedge dX^2_{w_{j+1}}) \wedge dX^2_{w_1}$$

$$= - \bigwedge_{j=1}^{k} (dX^2_{w_j} \wedge dY_{v_j} \wedge dX^1_{w_j} \wedge dX^2_{w_j}) = - \bigwedge_{j=1}^{k} (dY_{v_j} \wedge dX_{w_j}). \quad \Box$$

### 2.5 Conventions about orientations and rational chains

From now on, homology groups are taken with rational coefficients unless otherwise mentioned, all the manifolds are oriented, and their boundaries are oriented with the "outward normal first" convention. The ordered products of manifolds are naturally oriented, and this orientation does not depend on the order if the manifolds are even-dimensional. The fibers of the normal bundle of an oriented submanifold $P$ of a manifold $Q$ are oriented in such a way that the orientation of $\mathfrak{N}_x P$ followed by the orientation of $T_x P$ is the orientation of $T_x Q$. The orientation of $\mathfrak{N}_x P$ is called the coorientation of $P$. The preimages of submanifolds
are oriented in such a way that coorientations are preserved. The intersection
\( \bigcap_{i=1}^{r} A_r \) of submanifolds is oriented in such a way that \( \mathcal{R}A_\cap \) is oriented as
\( \bigoplus_{i=1}^{r} \mathcal{R}A_r \). If \( A \) is an oriented manifold, \( -A \) denotes the same manifold, with op-
posite orientation. In this article, an embedded rational \( d \)-chain \( A \) of a manifold \( X \) is a finite rational combination of compact oriented \( d \)-submanifolds with ridges and corners with pairwise disjoint interiors of \( X \). The support \( \text{Supp}(A) \) of \( A \) is the union of these submanifolds, and the interior \( \text{int}(A) \) of \( A \) is the union of their interiors.

### 2.6 Propagators and first definition of \( Z_k \)

In this section, we define \( Z_k \) for long knots in parallelized asymptotic homology \( \mathbb{R}^{n+2} \). Let \((M^\circ, \tau)\) be a parallelized asymptotic homology \( \mathbb{R}^{n+2} \).

**Definition 2.9.** An **internal propagator** \(^4\) is an embedded rational \((n+1)\)-chain \( A \) in \( C_2(\mathbb{R}^n) \) such that \( \partial A = \frac{1}{2} (G|_{C_2(\mathbb{R}^n)})^{-1}(\{ -x_A, +x_A \}) \) for some \( x_A \in S^{n-1} \).

An **external propagator** of \((M^\circ, \tau)\) is an embedded rational \((n+3)\)-chain \( B \) in \( C_2(M^\circ) \) such that \( \partial B = \frac{1}{2} (G_\tau)^{-1}(\{ -x_B, +x_B \}) \) for some \( x_B \in S^{n+1} \).

For any \( k \geq 2 \), a \( k \)-family \( F_* = (A_i, B_i)_{i \in \mathbb{Z}} \) of propagators of \((M^\circ, \tau)\) is the data of \( 2k \) internal propagators \((A_i)_{i \in \mathbb{Z}} \) and \( 2k \) external propagators \((B_i)_{i \in \mathbb{Z}} \) of \((M^\circ, \tau)\).

The existence of such propagators follows from \([\text{Let19}, \text{Section 4.2}]\). We recall the discrete definition of the invariant \( Z_k \) from our previous article \([\text{Let19}, \text{Sections 2.7-2.8}]\).

Let \( \psi \) be a long knot of \( M^\circ \). Consider a \( k \)-family \( F_* = (A_i, B_i)_{i \in \mathbb{Z}} \) of propagators of \((M^\circ, \tau)\). For any BCR diagram \( \Gamma \), let \( P_\Gamma \) be the product map
\[
P_\Gamma: \quad C_\Gamma(\psi) \to \prod_{e \in E_\Gamma} C_2(\mathbb{R}^n) \times \prod_{e \in \bar{E}_\Gamma} C_2(M^\circ) = \prod_{e \in E_\Gamma} C_e
\]
\[
c \to (p_e(c))_{e \in E_\Gamma}
\]

The \( k \)-family \( F_* \) is in general position for \( \psi \) if, for any numbered BCR diagram \((\Gamma, \sigma) \in \tilde{G}_k \), and for any \( c \in C_\Gamma(\psi) \) such that \( P_\Gamma(c) \in \prod_{e \in E_\Gamma} A_{\sigma(e)} \times \prod_{e \in \bar{E}_\Gamma} B_{\sigma(e)} \)

- for any internal edge \( e \), \( p_e(c) \in \text{int}(A_{\sigma(e)}) \),
- for any external edge \( e \), \( p_e(c) \in \text{int}(B_{\sigma(e)}) \),

\(^4\)In \([\text{Let19}]\), propagators were called propagating chains.
the following transversality property is satisfied.

\[ \varepsilon(c) T_p(c) \left( \prod_{e \in E(\Gamma)} C_e \right) = T_p(T_c C_{\Gamma}(\psi)) + \prod_{e \in E(\Gamma)} T_{p(c)} \text{int}(A_{\sigma(e)}) \times \prod_{e \in E(\Gamma)} T_{p(c)} \text{int}(B_{\sigma(e)}) , \]

where \( \varepsilon(c) = \pm 1 \) is called the sign of the intersection point \( c \), and where the above equality is an equality between oriented vector spaces.

In the following, \( D^F_{e,\sigma} \) denotes the chain \( p^{-1}e(A_{\sigma(e)}) \) if \( e \) is internal, and the chain \( p^{-1}e(B_{\sigma(e)}) \) if \( e \) is external. The chain \( D^F_{e,\sigma} \) lies in \( C_{\Gamma}(\psi) \).

[Let19] Theorem 4.3 guarantees the existence of \( k \)-families of propagators in general position for any \( \psi \) and any \( k \). In [Let19] Theorem 2.13, we proved that the extended BCR invariant \( Z_k \) of [Let19] Theorem 2.10 can be computed as follows.

**Theorem 2.10.** Let \((M^o, \tau)\) be a parallelized asymptotic homology \( \mathbb{R}^{n+2} \), and let \( \psi \) be a long knot of \( M^o \). Let \( F_* = (A_i, B_i)_{i \in \mathbb{Z}} \) be a \( k \)-family of propagators of \((M^o, \tau)\) in general position for \( \psi \).

For any numbered BCR diagram \((\Gamma, \sigma) \in \tilde{G}_k \), the algebraic intersection number \( I_{F_*}(\Gamma, \sigma, \psi) \) of the chains \((D^F_{e,\sigma})_{e \in E(\Gamma)} \) inside \( C_{\Gamma}(\psi) \) makes sense. Furthermore,

\[ Z_k(\psi) = \frac{1}{(2k)!} \sum_{(\Gamma, \sigma) \in \tilde{G}_k} I_{F_*}(\Gamma, \sigma, \psi). \]

In [Let19] Theorem 2.10, we proved that this quantity depends neither on the choice of the propagators nor on the parallelization, and is invariant under ambient diffeomorphisms. In particular, it is an isotopy invariant for long knots.

### 2.7 Connected sum and general definition of \( Z_k \)

In this section, we review the connected sum defined in [Let19] Section 2.9.

Let \( M_1^o \) and \( M_2^o \) be two asymptotic homology \( \mathbb{R}^{n+2} \), respectively decomposed as \( B(M_1) \cup B_\infty^o \) and \( B(M_2) \cup B_\infty^o \). Let \( B_{\infty,2}^o \) be the complement in \( \mathbb{R}^{n+2} \) of the two open balls \( \mathbb{B}_1 \) and \( \mathbb{B}_2 \) of radius \( \frac{1}{4} \) and with respective centers \( (0,0,\ldots,0,-\frac{1}{2}) \) and \( (0,0,\ldots,0,\frac{1}{2}) \). For \( i \in \{1,2\} \) and \( x \) in \( \partial B(M_i) \subset \mathbb{R}^{n+2} \), define the map \( \varphi_i(x) = \frac{1}{4}x + (-1)^i \frac{1}{2} \), which is a diffeomorphism from \( \partial B(M_i) \) to \( \partial \mathbb{B}_i \).

---

This intersection number counts the intersection points of these chains with the previously defined signs and the coefficients of the rational chains. For more details, see [Let19] Section 4.1.
Set $M_1^o \sharp M_2^o = B_{\infty,2}^o \cup B(M_1) \cup B(M_2)$, where $B(M_i)$ is glued to $\partial \mathbb{R}^n$ via the map $\varphi_i$. Set $B(M_1^o \sharp M_2^o) = B(M_1^o) \cup B(M_2^o) \cup (B_{\infty,2}^o \setminus B_{\infty,2}^o)$. This defines an asymptotic homology $\mathbb{R}^{n+2}$, with two canonical injections $\iota_i: B(M_i) \to B(M_1^o \sharp M_2^o)$ for $i \in \{1, 2\}$.

If $M_1^o$ and $M_2^o$ contain two long knots $\psi_1$ and $\psi_2$, define the long knot $\psi_1 \sharp \psi_2$ of $M_1^o \sharp M_2^o$ by the following formula, for any $x \in \mathbb{R}^n$:

$$(\psi_1 \sharp \psi_2)(x) = \begin{cases} \iota_2(\psi_2(4.x_1, \ldots, 4.x_{n-1}, 4.x_n - 2)) & \text{if } ||x - (0, \ldots, 0, \frac{1}{2})|| \leq \frac{1}{2}, \\ \iota_1(\psi_1(4.x_1, \ldots, 4.x_{n-1}, 4.x_n + 2)) & \text{if } ||x - (0, \ldots, 0, -\frac{1}{2})|| \leq \frac{1}{2}, \\ (0, 0, x) \in B_{\infty,2}^o & \text{otherwise.} \end{cases}$$

Similarly, any two parallelizations $\tau_1$ and $\tau_2$ of $M_1^o$ and $M_2^o$ induce a parallelization $\tau_1 \sharp \tau_2$ of $M_1^o \sharp M_2^o$, which is well-defined up to homotopy. In particular, if $M_1^o$ and $M_2^o$ are parallelizable, then $M_1^o \sharp M_2^o$ is also parallelizable in the sense of Definition 2.1. We recall the result of [Let19, Theorem 2.17].

**Theorem 2.11.** For any long knots $\psi_1: \mathbb{R}^n \to M_1^o$ and $\psi_2: \mathbb{R}^n \to M_2^o$,

$$Z_k(\psi_1 \sharp \psi_2) = Z_k(\psi_1) + Z_k(\psi_2).$$

Let us state [Let19, Proposition 2.18].

**Proposition 2.12.** For any positive odd integer $n$, the connected sum of any asymptotic homology $\mathbb{R}^{n+2}$ with itself is parallelizable in the sense of Definition 2.1.

This allows us to extend $Z_k$ as follows.

**Definition 2.13.** Let $\psi$ be a long knot in an asymptotic homology $\mathbb{R}^{n+2}$. Define $Z_k(\psi) = \frac{1}{2}Z_k(\psi \sharp \psi)$, where $Z_k$ is the invariant of Theorem 2.10.

[Let19, Prop 2.20] implies that Theorem 2.11 is still valid for this extended $Z_k$.

### 2.8 Linking number

We use the following definition and basic properties of the linking number.

**Definition 2.14.** Let $X^d$ and $Y^{n+1-d}$ be two disjoint cycles of our homology $(n + 2)$-sphere $M$, with $d \in \mathbb{Z}$. Let $W_X$ and $W_Y$ be two chains with respective boundaries $X$ and $Y$, such that $W_X$ and $W_Y$ are transverse to each other. The *linking number* of $X$ and $Y$ is defined as the intersection number $\langle X, W_Y \rangle_M$, so that

$$\text{lk}(X, Y) = \langle X, W_Y \rangle_M = (-1)^{d+1} \langle W_X, Y \rangle_M.$$ 

Furthermore, since $n$ is odd, $\text{lk}(X^d, Y^{n+1-d}) = (-1)^{d+1} \text{lk}(Y^{n+1-d}, X^d).$
These linking numbers will appear in the computation of our invariant $Z_2$ because of the following lemma, which relates external propagators to linking numbers.

**Lemma 2.15.** Let $X^d$ and $Y^{n+1-d}$ be two disjoint cycles of $M^o$. For any external propagator $B$,

$$\text{lk}(X,Y) = \langle X \times Y, B \rangle_{C_2(M^o)}.$$ 

*Proof.* The class of the cycle $X \times Y$ is an element of $H_{n+1}(C_2(M^o))$. Lemma 3.3 implies that $H_{n+3}(C_2(M^o)) = 0$. Therefore, the intersection number $\langle X \times Y, B \rangle_{C_2(M^o)}$ only depends on the homology class $\langle X \times Y \rangle$. Let $W_X$ and $W_Y$ be chains such that $\partial W_X = X$ and $\partial W_Y = Y$ as above. For the proof, assume that $W_X$ and $W_Y$ are manifolds (the general case follows easily). Let $W_X$ be obtained from $W_X$ by removing a little ball $D^{d+1}$ with boundary $S^d_x$ around each point $x \in W_X \cap Y$. Then, $\partial(W'_X \times Y) = X \times Y - \sum_{x \in W_X \cap Y} S^d_x \times Y$, and

$$[X \times Y] = \sum_{x \in W_X \cap Y} [S^d_x \times Y].$$

For any $x \in W_X \cap Y$, assume that $D^{d+1}$ meets $W_Y$ transversely along an interval $[x, x']$. For any $x \in W_X \cap Y$, remove a little ball $D^{n+2-d}_x$ with boundary $S^{n+1-d}_x$ around the point $x'$ (which is the intersection of $S^d_x$ and $W_Y$) from $W_Y$, in order to get $[X \times Y] = \sum_{x \in W_X \cap Y} [S^d_x \times S^{n+1-d}_x].$

It suffices to prove that $(-1)^{d+1}\langle S^d_x \times S^{n+1-d}_x, B \rangle_{C_2(M^o)}$ is the sign of the intersection point $x$ in $W_X \cap Y$. Since the balls $D^{d+1}$ and $D^{n+2-d}$ can be taken arbitrarily small, assume without loss of generality that $M^o = \mathbb{R}^{n+2}$,

$$S^d_x = \{(x_1, \ldots, x_d+1, 0, \ldots, 0) \mid (x_1)^2 + \ldots + (x_d+1)^2 = 1\},$$

$$S^{n+1-d}_{x'} = \{(0, \ldots, x_{d+1}, \ldots, x_{n+2}) \mid (x_{d+1} - 1)^2 + (x_{d+2})^2 + \ldots + (x_{n+2})^2 = 1\},$$

and $B = \frac{1}{2}(G^{-1}((-e_{d+1})) + G^{-1}((+e_{d+1})))$,

where $e_{d+1}$ is the $(d+1)$-th vector of the canonical basis of $\mathbb{R}^{n+2}$. Now $\langle S^d_x \times S^{n+1-d}_{x'}, B \rangle_{C_2(\mathbb{R}^{n+2})}$ is the degree of the Gauss map $S^d_x \times S^{n+1-d}_{x'} \to S^{n+1}$, which is $(-1)^d$. Since $\langle B^{d+1}, S^{n+1-d}_{x'} \rangle_{\mathbb{R}^{n+2}} = -1$, this concludes the proof. □

### 2.9 Seifert surfaces and matrices

**Definition 2.16.** A Seifert surface for a long knot $\psi: \mathbb{R}^n \to M^o$ is an oriented connected $(n+1)$-submanifold $\Sigma$ of $M^o$ such that $\partial\Sigma = \psi(\mathbb{R}^n)$, such that $\Sigma \cap B(M)$ is compact, and such that $\Sigma \cap B^\infty_\theta = \{(r \cos(\theta), r \sin(\theta), \xi) \mid \xi \in \mathbb{R}^n, r \geq 0\} \cap B^\infty_\theta$ for some $\theta \in \mathbb{R}$.

Let $\psi$ be a long knot, let $\Sigma$ be a Seifert surface for $\psi$, and, for any $d \in \mathbb{N}$, let $b_d$ denote the $d$-th Betti number of $\Sigma$. Let $\Sigma^+$ denote a parallel surface obtained
from $\Sigma$ by slightly pushing it in the positive normal direction. For any cycle $z$ in $\Sigma$, let $z^+$ denote the image of $z$ in the parallel surface $\Sigma^+$.

**Definition 2.17.** Let $\Sigma$ be a Seifert surface for a long knot $\psi$. For any $d \in \mathbb{Z}$, let $([a^d_i])_{i \in \mathbb{Z}_d}$ and $([z^d_i])_{i \in \mathbb{Z}_d}$ be two bases of $H_d(\Sigma)$. We say that the bases $B = ([a^d_i])_{i \in \mathbb{Z}_d}$ and $\tilde{B} = ([z^d_i])_{i \in \mathbb{Z}_d}$ of the reduced homology $\overline{H}_*(\Sigma)$ are dual to each other if, for any $d \in \mathbb{Z}$, and any $(i, j) \in (\mathbb{Z}_d)^2$, $\langle [a^d_i], [z^d_{j-d}] \rangle_{\Sigma} = \delta_{i,j}$, where $\langle \cdot, \cdot \rangle_{\Sigma}$ denotes the intersection pairing of $\Sigma$ and $\delta_{i,j}$ denotes the Kronecker delta. Given such a pair $(B, \tilde{B})$ of dual bases of $\overline{H}_*(\Sigma)$, for any $d \in \mathbb{Z}$, define the Seifert matrices

$$V_d^+(B, \tilde{B}) = (\text{lk}(z^d_i, (a^d_{j-d})^+))_{1 \leq i, j \leq |d|}$$

and set $L_{k, \nu}(B, \tilde{B}) = \frac{1}{k} \sum_{d \in \mathbb{Z}} (-1)^{(d+1)} \text{Tr}(V_d^+(B, \tilde{B})^\nu V_d^+(B, \tilde{B})^{k-\nu})$ for any $k \geq 2$ and any $\nu \in \{0, \ldots, k\}$.

Note that pairs of dual bases as above exist thanks to Poincaré duality. The following immediate lemma describes how the $L_{k, \nu}$ behave under connected sum.

**Lemma 2.18.** Let $\Sigma_1$ and $\Sigma_2$ be Seifert surfaces for two long knots $\psi_1$ and $\psi_2$. For $i \in \{1, 2\}$, let $(B_i, \tilde{B}_i)$ be a pair of dual bases of $\overline{H}_*(\Sigma_i)$ as in Definition 2.17.

There exists a natural Seifert surface $\Sigma_{1,2}$ for the connected sum $\psi_1 \# \psi_2$ and a pair $(B_{1,2}, \tilde{B}_{1,2})$ of dual bases of $\overline{H}_*(\Sigma_{1,2})$ such that, for any $d \in \mathbb{Z}$,

$$V_d^+(B_{1,2}, \tilde{B}_{1,2}) = \begin{pmatrix} V_d^+(B_1, \tilde{B}_1) & 0 \\ 0 & V_d^+(B_2, \tilde{B}_2) \end{pmatrix}.$$

In particular, for any $k \geq 2$ and any $\nu \in \{0, \ldots, k\}$,

$$L_{k, \nu}(B_{1,2}, \tilde{B}_{1,2}) = L_{k, \nu}(B_1, \tilde{B}_1) + L_{k, \nu}(B_2, \tilde{B}_2).$$

### 2.10 Rectifiability and virtual rectifiability

Let $I(\mathbb{R}^n, \mathbb{R}^{n+2})$ denote the space of injections $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+2}$, and let $\iota_0$ be the standard injection $x \in \mathbb{R}^n \mapsto (0, 0, x) \in \mathbb{R}^{n+2}$. Let $D^n$ be the unit ball of $\mathbb{R}^n$, and see $\pi_n(I(\mathbb{R}^n, \mathbb{R}^{n+2}), \iota_0)$ as the set $[(\mathbb{R}^n, \mathbb{R}^n \setminus D^n), (I(\mathbb{R}^n, \mathbb{R}^{n+2}), \iota_0)]$ of homotopy classes of maps $\mathbb{R}^n \to I(\mathbb{R}^n, \mathbb{R}^{n+2})$ that map $\mathbb{R}^n \to D^n$ to $\iota_0$ among such maps.

**Lemma 2.19.** Let $M^0$ be a parallelizable asymptotic homology $\mathbb{R}^{n+2}$ and let $\psi$ be a long knot of $M^0$. For any parallelization $\tau$ of $M^0$, the tangent map $T\psi$ induces a map $\iota(\tau, \psi) : x \in \mathbb{R}^n \mapsto (\tau(\psi(x)))^{-1} \circ T_x\psi \in I(\mathbb{R}^n, \mathbb{R}^{n+2})$.

The class $[\iota(\tau, \psi)] \in \pi_n(I(\mathbb{R}^n, \mathbb{R}^{n+2}), \iota_0)$ is independent of the parallelization $\tau$. 

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Proof. Let $\tau$ and $\tau'$ be two parallelizations of $\mathcal{M}^\circ$. [Let19, Theorem 6.2] implies the existence of a smooth family $(\tau_t)_{t \in [0,1]}$ of parallelizations of $\mathcal{M}^\circ$ such that $\tau_0 = \tau$ and that $\tau_1$ and $\tau'$ coincide on $\psi(\mathbb{R}^n)$. This yields an homotopy $(\iota(\tau_t, \psi))_{t \in [0,1]}$ from $\iota(\tau, \psi)$ to $\iota(\tau', \psi)$.

**Definition 2.20.** Under the assumptions of Lemma 2.19, the class $\iota(\psi) = [\iota(\tau, \psi)] \in \pi_n(I(\mathbb{R}^n, \mathbb{R}^{n+2}), \iota_0)$ is called the rectangleivity obstruction of $\psi$. The long knot $\psi$ is **rectifiable** if its rectangleivity obstruction is zero.

In Section 5.1 we prove the following lemma.

**Lemma 2.21.** For any rectifiable knot $\psi$ of a parallelizable asymptotic homology $\mathbb{R}^{n+2}$, there exists a parallelization $\tau$ such that $\iota(\tau, \psi)$ is the constant map with value $\iota_0$.

For long knots in a possibly non-parallelizable asymptotic homology $\mathbb{R}^{n+2}$, we have the following weaker definition.

**Definition 2.22.** Let $\mathcal{M}^\circ$ be an asymptotic homology $\mathbb{R}^{n+2}$. A long knot $\psi : \mathbb{R}^n \hookrightarrow \mathcal{M}^\circ$ is **virtually rectifiable** if there exists a positive integer $r$ such that the connected sum $\psi^{(r)} = \psi \# \cdots \# \psi$ of $r$ copies of $\psi$ is rectifiable.

In Section 5 we establish the following lemma.

**Lemma 2.23.** If $n \equiv 1 \mod 4$, then any long knot in an asymptotic homology $\mathbb{R}^{n+2}$ is virtually rectifiable.

### 2.11 A formula for $Z_k$ in terms of linking numbers

In Section 3 we prove the following theorem.

**Theorem 2.24.** For any rectifiable knot $\psi$ in an asymptotic homology $\mathbb{R}^{n+2}$, any Seifert surface $\Sigma$ for $\psi$, and any pair $(\mathcal{B}, \tilde{\mathcal{B}})$ of dual bases of $H_*(\Sigma)$,

$$Z_k(\psi) = \sum_{\nu=1}^{k-1} \lambda_{k,\nu} L_{k,\nu}(\mathcal{B}, \tilde{\mathcal{B}}),$$

where $\lambda_{k,\nu} = \frac{1}{(k-1)!} \text{Card}(\{\sigma \in \mathcal{S}_{k-1} \mid \text{Card}\{i \in [k-2] \mid \sigma(i) < \sigma(i+1)\} = \nu - 1\})$, and where $L_{k,\nu}(\mathcal{B}, \tilde{\mathcal{B}})$ is introduced in Definition 2.17.

Theorem 2.24 yields the following more general corollary.
Corollary 2.25. For any virtually rectifiable knot $\psi$ in an asymptotic homology $\mathbb{R}^{n+2}$, any Seifert surface $\Sigma$ for $\psi$, and any pair ($B, \tilde{B}$) of dual bases of $\overline{H}_*(\Sigma)$,

$$Z_k(\psi) = \sum_{\nu=1}^{k-1} \lambda_{k,\nu} L_{k,\nu}(B, \tilde{B}).$$

In particular, this formula holds for any long knot when $n \equiv 1 \mod 4$.

Proof. Let $\psi$ be a virtually rectifiable knot, and let $r$ be an integer such that $\psi^{(r)}$ is rectifiable. Let $\Sigma$, $B$ and $\tilde{B}$ be as in the theorem. From Theorem 2.11, we get that $Z_k(\psi^{(r)}) = rZ_k(\psi)$. For any $k \geq 2$ and $\nu \in k-1$, Lemma 2.18 yields $L_{k,\nu}(B_r, \tilde{B}_r) = rL_{k,\nu}(B, \tilde{B})$, where ($B_r, \tilde{B}_r$) is a pair of dual bases of the reduced homology of a Seifert surface for $\psi^{(r)}$. On the other hand, Theorem 2.24 implies that $Z_k(\psi^{(r)}) = \sum_{\nu=1}^{k-1} \lambda_{k,\nu} L_{k,\nu}(B_r, \tilde{B}_r)$. This concludes the proof of Corollary 2.25. $lacksquare$

2.12 Alexander polynomials and Reidemeister torsion

We use the following formula of [Lev66, p.542] as a definition of Alexander polynomials.

Theorem 2.26 (Levine). The Alexander polynomials of the Seifert surface $\Sigma$ are defined, for $d \in \mathbb{Z}$, by the formula

$$\Delta_{d, \Sigma}(t) = \det \left( t^{-\frac{1}{2}} V_d^+(B, \tilde{B}) - t^{\frac{1}{2}} V_d^-(B, \tilde{B}) \right)$$

and do not depend on the choice of the pair ($B, \tilde{B}$) of dual bases of $\overline{H}_*(\Sigma)$.

Furthermore, for any $d \in \mathbb{Z}$, $V_d^+(B, \tilde{B}) - V_d^-(B, \tilde{B}) = I_{bd}$, so that $\Delta_{d, \psi}(1) = 1$.

If $\Sigma$ and $\Sigma'$ are two Seifert surfaces for $\psi$, then there exists an integer $a \in \mathbb{Z}$ such that $\Delta_{d, \Sigma'}(t) = t^a \Delta_{d, \Sigma}(t)$.

Lemma 2.27. For any Seifert surface $\Sigma$, and any $d \in \mathbb{Z}$, $\Delta_{d, \Sigma}(t^{-1}) = \Delta_{d+1, \Sigma}(t)$.

Furthermore, for any pair ($B, \tilde{B}$) of dual bases of the reduced homology of $\Sigma$,

$$\sum_{d \in \mathbb{Z}} (-1)^{d+1} \left( \text{Tr}(V_d^+(B, \tilde{B})) + \text{Tr}(V_d^-(B, \tilde{B})) \right) = 0,$$

and

$$\sum_{d \in \mathbb{Z}} (-1)^{d+1} \text{Tr}(V_d^-(B, \tilde{B})) = \frac{\chi(\Sigma) - 1}{2}.$$  

With the notations of [Lev66] Theorem 1, our Alexander polynomial $\Delta_{d, \Sigma}$ is the product $\prod_{i \in b_d} \lambda_i^d$.

This follows from the fact that the $(\lambda_i^d)$ from [Lev66] are defined from the knot up to multiplication by a monomial.
Proof. There exists a pair \((B, \bar{B})\) of dual bases \(B = ([a_i^d])_{i,d}\) and \(\bar{B} = ([z_i^d])_{i,d}\) of \(H_*(\Sigma)\) such that \(a_i^d = z_i^d\) for \(d > \frac{n+1}{2}\), and such that \(z_i^d = (-1)^d a_i^d\) for \(d < \frac{n+1}{2}\). It follows that for \(d \neq \frac{n+1}{2}\), \(\tau V^+(d, \bar{B}) = -V^+_{n+1-d}(B, \bar{B})\). This implies that \(\Delta_{d,\Sigma}(t^{-1}) = \Delta_{n+1-d,\Sigma}(d)\). Let \(B'\) be the basis defined from \(B\) by replacing \(a_i^d\) with \((-1)\frac{n+1}{2} a_i^d\) for any \(i \in \{1, \ldots, b_{\frac{n+1}{2}}\}\), and let \(\bar{B}'\) be the basis defined from \(\bar{B}\) by replacing \(z_i^d\) with \(a_i^d\), so that \((B', \bar{B}')\) is a pair of dual bases of \(\mathcal{H}_*(\Sigma)\). We have \(\tau V^\pm_{\frac{n+1}{2}}(B, \bar{B}) = -V^\pm_{\frac{n+1}{2}}(B', \bar{B}')\), hence \(\Delta^\pm_{\frac{n+1}{2}}(t^{-1}) = \Delta^\pm_{\frac{n+1}{2}}(t)\). The first point of the lemma follows.

This implies that for any \(d \in \mathbb{N}\), \(\Delta_{d,\Sigma}(1) + \Delta'_{n+1-d,\Sigma}(1) = 0\), so

\[
\sum_{d \in \mathbb{N}} (-1)^{d+1} \left( \text{Tr}(V^+_d(B, \bar{B})) + \text{Tr}(V^-_d(B, \bar{B})) \right) = 0.
\]

On the other hand, since \(V^+_d(B, \bar{B}) - V^-_d(B, \bar{B}) = I_{b_d}\) for any \(d \in \mathbb{N}\), we have

\[
\sum_{d \in \mathbb{N}} (-1)^{d+1} \left( \text{Tr}(V^+_d(B, \bar{B})) - \text{Tr}(V^-_d(B, \bar{B})) \right) = 1 - \chi(\Sigma).
\]

The two equations above imply the second and third point of the lemma. Note that since the Alexander polynomials do not depend on the choice of the bases, neither do these equalities.

We use [Mil68, Theorem p.131] to compute the Reidemeister torsion of the knot (i.e. the Reidemeister torsion of the knot complement) as follows.

**Definition 2.28.** The *Reidemeister torsion* \(\mathcal{T}_\psi(t)\) of a long knot \(\psi\) is defined as

\[
\mathcal{T}_\psi(t) = \prod_{d \in \mathbb{N}} (\Delta_{d,\Sigma}(t))^{(-1)^{d+1}} \in \mathbb{Q}(t),
\]

where \(\Sigma\) is a Seifert surface for \(\psi\). We have \(\mathcal{T}_\psi(1) = 1\) and \(\mathcal{T}_\psi(t^{-1}) = \mathcal{T}_\psi(t)\), so that the torsion does not depend on the surface \(\Sigma\).

### 2.13 The Reidemeister torsion in terms of BCR invariants

The following theorem is proved in Section 6.2.

**Theorem 2.29.** For any virtually rectifiable knot \(\psi\) of an asymptotic homology \(\mathbb{R}^{n+2}\), we have the following equality in \(\mathbb{Q}[[h]]\):

\[
\text{Ln}(\mathcal{T}_\psi(e^h)) = \sum_{d=1}^{n} (-1)^{d+1} \text{Ln}(\Delta_{d,\Sigma}(e^h)) = - \sum_{k \geq 1} Z_k(\psi) h^k.
\]

If \(n \equiv 1 \mod 4\), this formula holds for any long knot.
3 Computing $Z_k$ from admissible propagators

3.1 Admissible propagators

Let $M^n$ be a fixed parallelizable asymptotic homology $\mathbb{R}^{n+2}$ and let $\psi: \mathbb{R}^n \hookrightarrow M^n$ be a fixed long knot. Let $\psi_0: x \in \mathbb{R}^n \mapsto (0, 0, x) \in \mathbb{R}^{n+2}$ denote the trivial knot.

Fix a real number $R \geq 3$.

For $1 \leq r \leq R$, let $N^0_r$ denote the following neighborhood of the trivial knot $N^0_r = \{ x \in \mathbb{R}^{n+2} \mid x_1^2 + x_2^2 \leq r^2 \text{ or } ||x|| \geq \frac{2R}{r} \}$. Choose a neighborhood $N_R$ of $\psi(\mathbb{R}^n)$ in $M^n$ such that $N_R \cap B^\infty_\infty = N^0_R \cap B^\infty_\infty$, and such that $N_R \cap B(M)$ is a tubular neighborhood of $\psi(\mathbb{R}^n) \cap B(M)$. Choose a diffeomorphism $\Theta: N^0_R \rightarrow N_R$ such that $\Theta$ reads as the identity function on $N^0_R \cap B^\infty_\infty$, and such that $\Theta$ is a bundle isomorphism on $N^0_R \cap B(M)$ such that $\Theta \circ \psi_0 = \psi$.

**Lemma 3.1.** The knot $\psi$ is rectifiable if and only if there exists a parallelization $\tau$ of $M^n$ such that, for any $x \in N^0_R$, $T_x \Theta \circ (\tau_0)_x = \tau_{\Theta(x)}$, where $\tau_0$ denotes the canonical parallelization of $\mathbb{R}^{n+2}$ introduced in Definition 2.1.

**Proof.** If there exists such a $\tau$, then $T_{\psi_0(x)} \Theta \circ T_x \psi_0 = \tau_{\Theta(\psi_0(x))} \circ \tau_0$, so $\tau_{\psi(x)} \circ \tau_0 = T_x \psi$. Therefore, $\iota(\tau, \psi) = \tau_0$ and $\psi$ is rectifiable.

Let us prove the converse. Suppose that $\psi$ is rectifiable, and let $\tau_1$ be a parallelization such that $\iota(\tau_1, \psi)$ is the constant map of value $\tau_0$. Recall that if $(X, A)$ and $(Y, B)$ are two topological pairs, $[(X, A), (Y, B)]$ denotes the set of homotopy classes of maps $X \to Y$ that map $A$ to $B$. Let $\mathbb{D}^n$ denote the unit ball of $\mathbb{R}^n$. The map $x \in N^0_R \mapsto (\tau_1)_x^{-1} \circ T_x \Theta \circ (\tau_0)_x \in GL^+_{n+2}(\mathbb{R})$ induces an element $\kappa(\tau_1)$ of the set $[(N^0_R, N^0_R \cap B^\infty_\infty), (GL^+_{n+2}(\mathbb{R}), I_{n+2})] \cong [(N^0_R \cap B(M), N^0_R \cap \partial B(M)), (GL^+_{n+2}(\mathbb{R}), I_{n+2})]$. Since $N^0_R \cap B(M)$ is a disk bundle over $\psi_0(\mathbb{R}^n) \cap B(S^{n+2}) = \psi_0(\mathbb{D}^n)$, the restriction map

$$[(N^0_R \cap B(M), N^0_R \cap \partial B(M)), (GL^+_{n+2}(\mathbb{R}), I_{n+2})]$$

$$\rightarrow \left[ (\psi_0(\mathbb{D}^n), \psi_0(\partial \mathbb{D}^n)), (GL^+_{n+2}(\mathbb{R}), I_{n+2}) \right] \cong [(\mathbb{D}^n, S^{n-1}), (GL^+_{n+2}(\mathbb{R}), I_{n+2})]$$

is an isomorphism, and $[(N^0_R, N^0_R \cap B^\infty_\infty), (GL^+_{n+2}(\mathbb{R}), I_{n+2})] \cong \pi_n(GL^+_{n+2}(\mathbb{R}), I_{n+2})$. The fibers of the fibration $g \in GL^+_{n+2}(\mathbb{R}) \mapsto (g(0))^2 \times \mathbb{R}^n \in \mathbb{I}(\mathbb{R}^n, \mathbb{R}^{n+2})$ have the homotopy type of $SO(2)$. Since $n \geq 3$, this fibration induces an isomorphism $\pi_n(GL^+_{n+2}(\mathbb{R}), I_{n+2}) \rightarrow \pi_n(\mathbb{I}(\mathbb{R}^n, \mathbb{R}^{n+2}), I_0)$, where the obtained isomorphism $[(N^0_R, N^0_R \cap B^\infty_\infty), (GL^+_{n+2}(\mathbb{R}), I_{n+2})] \cong \pi_n(\mathbb{I}(\mathbb{R}^n, \mathbb{R}^{n+2}), I_0)$ maps $\kappa(\tau_1)$ to $[\iota(\tau_1, \psi)]$. By definition of $\tau_1$, $\iota(\tau_1, \psi) = \tau_0$, and $\kappa(\tau_1)$ is trivial. This proves the existence of a parallelization $\tau: N_R \times \mathbb{R}^{n+2} \rightarrow TN_R$ homotopic to $\tau_1$ such that, for any $x \in N^0_R$, $T_x \Theta \circ (\tau_0)_x = I_{n+2}$. Since $\tau_1 | N_R \times \mathbb{R}^{n+2}$ extends to $M^n \times \mathbb{R}^{n+2}$, it is immediate to see that $\tau$ also does. \[\square\]
Note that the previous proof also yields the following lemma, since $GL_{n+2}^+(\mathbb{R})$ and $SO(n+2)$ have the same homotopy type.

**Lemma 3.2.** For $n \geq 3$, $\pi_n(\mathcal{I}(\mathbb{R}^n, \mathbb{R}^{n+2}), \iota_0)$ is isomorphic to $\pi_n(SO(n+2), I_{n+2})$.

Suppose now that $\psi$ is rectifiable, and fix the identification $\Theta$ and the parallelization $\tau$ as in Lemma 3.1. For any $r \in [1, R]$, let $E_r$ denote the closure of $M^0 \setminus N_r$. With the identification induced by $\Theta$, $M^0$ reads $N_r^0 \cup E_r$.

**Notation 3.3.** In $N_R = N_R^0$, use the coordinates $x = (x_1, x_2, \overline{x}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$. For $r \in [1, R]$, $\mathcal{D}_\mu(r)$ denotes the disk $\{(x_1, x_2, \overline{0}) \mid x_1^2 + x_2^2 \leq r^2\} \subset N_r$. For $\theta \in \mathbb{R}$, set $L_0^\theta(r) = \{\pm (\rho \cos(\theta), \rho \sin(\theta), \overline{0}) \mid \rho \geq \frac{2R^2}{r}\}$, $L_0^\theta(r) = L_0^+ (r) + L_0^- (r)$, and orient these lines by $d\rho$.

Let $\Theta_2$ denote the diffeomorphism $C_2(N_R^0) \rightarrow C_2(N_R)$ induced by $\Theta$. We now define the main objects of this article, which will be used to compute $Z_k$.

**Definition 3.4.** Fix $\theta \in \mathbb{R}$. Fix Seifert surfaces $\Sigma^+$ and $\Sigma^-$ with disjoint interiors such that $\Sigma^+ \cap N_R = (\sigma \Sigma^\pm)^0 \cap N_R$, where $(\sigma \Sigma^\pm)^0 = \{\pm (r \cos(\theta), r \sin(\theta), \overline{x}) \mid \overline{x} \in \mathbb{R}^n, r \geq 0\}$. For any $r \in [1, R]$, let $\Sigma(r)$ denote the submanifold $E_r \cap (\Sigma^+ \cup \Sigma^-)$. The restriction of the Gauss map of $C_2(\mathbb{R}^{n+2})$ to $C_2(N_R^0)$ and the identification $C_2(N_1) \cong C_2(N_0^0)$ given by $\Theta_2$ induce a map $G_0 : C_2(N_1) \rightarrow \mathbb{S}^n + 1$.

An external propagator $B$ is called $R$-admissible (with respect to $(\Sigma^+, \Sigma^-, \psi)$) if:

- $B \cap p_b^{-1}(N_1 \times N_1) = \frac{1}{2} G_0^{-1}(\{-(\cos(\theta), \sin(\theta), \overline{0}), + (\cos(\theta), \sin(\theta), \overline{0})\})$.
- If $c \in B \cap p_b^{-1}(\psi(\mathbb{R}^n) \times M^0)$, then $p_2(c)$ lies in the closure $\Sigma^+ \cup \Sigma^-$ of $\Sigma^- \cap \Sigma^+$ in $C_1(M^0)$, where $p_2$ is the smooth map $p_2 : C_2(M^0) \rightarrow C_1(M^0)$ that extends $(x, y) \in C_2^0(M^0) \mapsto y \in M^0 \subset C_1(M^0)$.
- For any $r \in ]1, R - 1]$, $B \cap p_b^{-1}(N_r \times E_{r+1}) = \frac{1}{2} \left( \mathcal{D}_\mu(r) \times \Sigma(r + 1) - p_b^{-1}(L_0(r) \times E_{r+1}) \right)$.
- If $T$ denotes the smooth map of $C_2(M^0)$ such that for any $(x, y) \in C_2^0(M^0)$, $T(x, y) = (y, x)$, then $T(B) = B$. In particular, for any $r \in ]1, R - 1]$, $B \cap p_b^{-1}(E_{r+1} \times N_r) = \frac{1}{2} \left( \Sigma(r + 1) \times \mathcal{D}_\mu(r) + p_b^{-1}(E_{r+1} \times L_0(r)) \right)$.

In Section 4 we prove the following technical lemma.
Lemma 3.5. For any integer \( k \geq 1 \), any long knot \( \psi \), any pairwise distinct real numbers \((\theta_i)_{i \in 2k}\) in \([0, \pi]\), and any Seifert surfaces \((\Sigma_i^+, \Sigma_i^-)_{i \in 2k}\) such that \( \Sigma_i^+ \cap N_R = (\theta_i \Sigma^0) \cap N_R \), there exist external propagators \((B^R(\Sigma_i^+, \Sigma_i^-, \psi))_{i \in 2k}\), such that for any \( i \in 2k \), the propagator \( B^R(\Sigma_i^+, \Sigma_i^-, \psi) \) is \( R \)-admissible with respect to \((\Sigma_i^+, \Sigma_i^-, \psi)\).

Furthermore, we can fix such propagators \((B^R((\theta_0, \Sigma^0_0), (\theta_0, \Sigma^0_0), \psi_0))_{i \in 2k}\) for the trivial knot such that \( \Theta_2 \) maps the chain \( B^R((\theta_0, \Sigma^0_0), (\theta_0, \Sigma^0_0), \psi_0) \cap p^{-1}_b(N_R^0 \times N_R^0) \) to \( B^R(\Sigma_i^+, \Sigma_i^-, \psi) \cap p^{-1}_b(N_R \times N_R) \) for any \( i \in 2k \).

3.2 Use of admissible propagators to compute \( Z_k \)

3.2.1 Three reduction lemmas

We work with the following notations, and we fix the following setting for Section 3.2.

Setting 3.6.  

- The integer \( k \geq 2 \) is fixed.
- The real number \( R \) of the previous subsection is fixed to some arbitrary value \( R \geq k + 1 \).
- The numbers \((\theta_i)_{i \in 2k}\) are such that \( 0 \leq \theta_1 < \theta_2 < \cdots < \theta_{2k} < \pi \).
- For any \( i \in 2k \), \((\Sigma_i^0) = (\theta_i \Sigma^0)\).
- \((\Sigma_i^0)_{i \in 2k}\) are pairwise isotopic and parallel\(^8\) Seifert surfaces for \( \psi \), such that \( \Sigma_i^+ \cap N_R = (\Sigma_i^0) \cap N_R \), and such that for any \( (i, \varepsilon) \in 2k \times \{\pm\} \setminus \{(1, +)\} \), \( \Sigma_i^+ \cap E_1 \) is obtained from \( \Sigma_i^+ \cap E_1 \) by pushing it in the positive normal direction, so that the order of these surfaces in the positive normal direction is \((\Sigma_1^+, \ldots, \Sigma_{2k}^+, \Sigma_1^-, \ldots, \Sigma_{2k}^-)\).
- With the notations of Lemma 3.5, \( F^0_i = (A_i, B^0_i)_{i \in 2k} \) is a \( k \)-family of propagators of \((\mathbb{R}^{n+2}, \tau_0)\) in general position for \( \psi_0 \), such that for any \( i \in 2k \), \( B^0_i \) is an arbitrarily small perturbation of \( B^R((\Sigma_i^0)^0, (\Sigma_i^-)^0, \psi_0) \).
- With the notations of Lemma 3.5, \( F_i = (A_i, B_i)_{i \in 2k} \) is a \( k \)-family of propagators of \((M^0, \tau)\) in general position for \( \psi \) such that for any \( i \in 2k \), \( B_i \) is an arbitrarily small perturbation of \( B^R(\Sigma_i^+, \Sigma_i^-, \psi) \) and \( B_i \cap p^{-1}_b(N_R \times N_R) \) is the image of \( B^0_i \cap p^{-1}_b(N_R^0 \times N_R^0) \) under the identification \( \Theta_2 \).

\(^8\)They only meet along the knot.
For any edge $e$ of a numbered degree $k$ BCR diagram $(\Gamma, \sigma)$ as in Definition 2.3, define the chains $D_{e,\sigma} \subset C_\Gamma(\psi)$ and $D_{e,\sigma}^0 \subset C_0^\Gamma(\psi)$ as

$$D_{e,\sigma} = \begin{cases} p_e^{-1}(A_{\sigma(e)}) & \text{if } e \text{ is internal,} \\ p_e^{-1}(B_{\sigma(e)}) & \text{if } e \text{ is external,} \end{cases} \quad \text{and} \quad D_{e,\sigma}^0 = \begin{cases} p_e^{-1}(A^0_{\sigma(e)}) & \text{if } e \text{ is internal,} \\ p_e^{-1}(B^0_{\sigma(e)}) & \text{if } e \text{ is external.} \end{cases}$$

Lemma 3.7. If $\Gamma$ has an external edge from an internal vertex to an internal vertex, then, for any numbering $\sigma$, $\bigcap_{f \in E(\Gamma)} D_{f,\sigma} = \emptyset$, and $\bigcap_{f \in E(\Gamma)} D^0_{f,\sigma} = \emptyset$.

Proof. We first ignore the perturbations of the external propagators. Let $e = (v, w)$ be an external edge between two internal vertices of a numbered BCR diagram $(\Gamma, \sigma)$. For a configuration $c$ in $D_{e,\sigma}$, set $c(v) = p_v(c)$ and $c(w) = p_w(c)$, where $p_v$ and $p_w$ are the maps defined in Theorem 2.6. Since $v$ and $w$ are internal, we have $G_0(p_v(c)) \in \{0\}^2 \times S^n_{0,1}$. On the other hand, by Definition 3.4 of admissible propagators, since $(c(v), c(w)) \in N_1 \times N_1$, we have $G_0(p_v(c)) = \pm (\cos(\theta_{\sigma(e)}), \sin(\theta_{\sigma(e)}), 0)$. Thus, $D_{e,\sigma} = \emptyset$ and $\bigcap_{f \in E(\Gamma)} D_{f,\sigma} = \emptyset$. Similarly, $\bigcap_{f \in E(\Gamma)} D^0_{f,\sigma} = \emptyset$.

Now, note that the properties $\bigcap_{f \in E(\Gamma)} D_{f,\sigma} = \emptyset$ and $\bigcap_{f \in E(\Gamma)} D^0_{f,\sigma} = \emptyset$ are stable under small perturbations since the $D_{f,\sigma}$ and the $D^0_{f,\sigma}$ are compact. \hfill \qed

Lemma 3.8. Let $\Gamma \in G_k \setminus \{\Gamma_k\}$, where $\Gamma_k$ is the degree $k$ BCR diagram of Figure 3. For any numbering $\sigma$ of $\Gamma$, $\bigcap_{e \in E(\Gamma)} D_{e,\sigma} = \emptyset$ and $\bigcap_{e \in E(\Gamma)} D^0_{e,\sigma} = \emptyset$.

Proof. Fix a numbering $\sigma$. If $\Gamma$ has only internal vertices, conclude with Lemma 3.7. If $\Gamma \neq \Gamma_k$ and $V_e(\Gamma) \neq \emptyset$, then $\Gamma$ contains a maximal sequence $(w_1, \ldots, w_p)$ of consecutive external vertices with $p \geq 1$ like in Figure 3. Let $a$ be the bivalent vertex such that there is an external edge from $a$ to $w_1$ and let $b$ be the bivalent vertex such that there is an external edge from $w_p$ to $b$, and note that $a \neq b$.

![Figure 4: Notations for Lemma 3.8](image)

As in the previous proof, we first ignore the perturbations. Let $c$ in $\bigcap_{e \in E(\Gamma)} D_{e,\sigma}$. For any $i \in p$, since $p_{\ell_i}(c) \in B_{\sigma(\ell_i)}$ and $v_i$ is internal, $c(w_i) = p_{w_i}(c)$ lies in the closure $\overline{\Sigma^+_{p_{\sigma(\ell_i)}}} \cup \Sigma^-_{q_{\sigma(\ell_i)}}$ of $\Sigma^+_{p_{\sigma(\ell_i)}} \cup \Sigma^-_{q_{\sigma(\ell_i)}}$ in $C_1(M^\circ)$. Similarly, since $a$ is internal, $c(w_1) \in \overline{\Sigma^+_{p_{\sigma(f_1)}}} \cup \Sigma^-_{q_{\sigma(f_1)}}$. Then, $c(w_1)$ lies in the closure $\overline{\psi(\mathbb{R}^n)}$ of $\psi(\mathbb{R}^n)$ in $C_1(M^\circ)$.
The same argument now proves that $c(w_2)$ is in $\psi(\mathbb{R}^n)$. By induction, $c(w_i)$ lies in $\psi(\mathbb{R}^n)$ for any $i \in p$.

By construction of $C_T(\psi)$, $c$ is the limit of configurations $(c_t)_{t \in [0, 1]}$ of the interior of $C_T(\psi)$ when $t$ approaches 0. Since $c(w_i)$ is in $\psi(\mathbb{R}^n)$ for any $i \in p$ when $t = 0$, we can assume that $c_t$ maps all the vertices $(w_i)_{i \in p}$ in $N_1 \subset \mathbb{R}^{n+T_1}$, for any $t \in [0, 1]$. For any $t \in [0, 1]$, the vector $c_t(b) - c_t(a)$ is the sum of the vectors $c_t(w_1) - c_t(a), c_t(w_2) - c_t(w_1), \ldots, c_t(w_p) - c_t(w_{p-1})$, and $c_t(b) - c_t(w_p)$.

Since the propagators are admissible, and since $c(a)$, $c(b)$ and the $(c(w_i))_{i \in p}$ are in $N_1$, this implies that $G_0(c_t(a), c_t(b))$ is a linear combination of the vectors $((\cos(\theta_{\sigma(f_1)}), \sin(\theta_{\sigma(f_1)}) \overline{f}))_{i \in p+1}$. Thus, $G_0(c_t(a), c_t(b))$ is in $\mathbb{S}^1 \times \{0\}^n$ for any $t \in [0, 1]$. But since $a$ and $b$ are internal, for any $t \in [0, 1]$, $G_0(c_t(a), c_t(b))$ reads $(0, 0, \overline{x}_t)$ for some $\overline{x}_t \in \mathbb{S}^n$. This is a contradiction, so $\bigcap_{e \in E(\Gamma)} D_{e, \sigma} = \emptyset$. Similarly,

\[
\bigcap_{e \in E(\Gamma)} D_{e, \sigma}^0 = \emptyset.
\]

Again, this property is stable since the $D_{e, \sigma}$ and the $D_{e, \sigma}^0$ are compact. □

The two above lemmas allow us to reduce our study to the graph $\Gamma_k$. The following lemma will help us in this study in the next subsection.

**Lemma 3.9.** Let $\Gamma_k$ be the BCR diagram of Figure 3. If $c$ is a configuration of $\bigcap_{e \in E(\Gamma_k)} D_{e, \sigma}$ (resp. $\bigcap_{e \in E(\Gamma_k)} D_{e, \sigma}^0$), and if there exists $j \in \overline{k}$ such that $c(w_j) \in E_{k+1}$ (resp. $c(w_j) \in E_{k+1}^0$), then $c(w_i) \notin N_2$ for any $i \in \overline{k}$.

**Proof.** It suffices to prove the statement on $\bigcap_{e \in E(\Gamma_k)} D_{e, \sigma}$, the proof for $\bigcap_{e \in E(\Gamma_k)} D_{e, \sigma}^0$ is the same. Let us first ignore the perturbations, and assume without loss of generality that $j = k$, so that $c(w_k) \in E_{k+1}$.

Since $v_k$ is internal, and since the propagators are admissible, $c(w_k) \in \Sigma_{\sigma(f_k)}(k+1)$. Let us prove that $c(w_{k-1}) \notin N_k$. Assume by contradiction that $c(w_{k-1}) \in N_k$. Since the external propagators are admissible, $p_{f_k-1}(c) = (c(w_{k-1}), c(w_k)) \in D_{\mu}(k) \times \Sigma_{\sigma(f_{k-1})}(k+1) \cup p_{k-1}^{-1}(L_{\sigma(f_{k-1})}(k) \times E_{k+1})$. Since the surfaces $\Sigma_{\sigma(f_{k-1})}(k+1)$ and $\Sigma_{\sigma(f_k)}(k+1)$ are disjoint, $c(w_k) \notin \Sigma_{\sigma(f_{k-1})}(k+1)$. Thus, $c(w_{k-1}) \in L_{\sigma(f_{k-1})}(k) \subset C_1(M^s)$. But since $v_{k-1}$ is internal, $c(w_{k-1}) \in \Sigma_{\sigma(\ell_{k-1})}(k) \subset C_1(M^o)$, which is impossible since $L_{\sigma(f_{k-1})}(k)$ and $\Sigma_{\sigma(\ell_{k-1})}(k)$ do not intersect in $C_1(M^o)$. Thus, $c(w_{k-1}) \notin N_k$. By induction, we prove that $c(w_i) \notin N_{i+1}$ for any $i \in \overline{k}$.

Since the set $p_{w_k}^{-1}(E_{k+1}) \cap \left( \bigcup_{j \in \overline{k}} p_{w_j}^{-1}(N_2) \right)$ is compact, the property of the lemma is stable under small perturbations and the lemma is therefore true for small enough perturbations. □

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3.2.2 A first formula for the contribution of $\Gamma_k$

For any $(\Gamma, \sigma) \in \mathcal{G}_k$, set $\Delta_{\Gamma, \sigma}Z_k = IF_*(\Gamma, \sigma, \psi) - IF_0^*(\Gamma, \sigma, \psi_0)$, so that

$$Z_k(\psi) - Z_k(\psi_0) = \frac{1}{(2k)!} \sum_{(\Gamma, \sigma) \in \mathcal{G}_k} \Delta_{\Gamma, \sigma}Z_k.$$

Lemma 3.8 implies that $\Delta_{\Gamma, \sigma}Z_k = 0$ if $\Gamma$ is not isomorphic to $\Gamma_k$.

Since $R \geq k + 1$, $B_i \cap p_b^{-1}(N_{k+1} \times N_{k+1}) = \Theta_2(B_i \cap p_b^{-1}(N_{k+1} \times N_{k+1}))$. This yields the following lemma.

**Lemma 3.10.** Let $\langle \cdot, \ldots, \cdot \rangle_X$ denote the algebraic intersection of several chains of a manifold $X$ such that their codimensions add up to $\dim(X)$. Let $X_1(\Gamma_k)$ (respectively $X_0^0(\Gamma_k)$) denote the subset of $C_{\Gamma_k}(\psi)$ (respectively $C_{\Gamma_k}(\psi_0)$), whose elements are the configurations that map at least one vertex to $E_{k+1}$. For any edge $e$ of $\Gamma_k$, set $D^{(1)}_{e,\sigma} = D_{e,\sigma} \cap X_1(\Gamma_k)$ and $D^{(1),0}_{e,\sigma} = D_{e,\sigma} \cap X_0^0(\Gamma_k)$.

The chains $(D^{(1)}_{e,\sigma})_{e \in E(\Gamma_k)}$ and $(D^{(1),0}_{e,\sigma})_{e \in E(\Gamma_k)}$ are transverse, and

$$\Delta_{\Gamma, \sigma}Z_k = \langle (D^{(1)}_{e,\sigma})_{e \in E(\Gamma_k)} \rangle_{X_1(\Gamma_k)} - \langle (D^{(1),0}_{e,\sigma})_{e \in E(\Gamma_k)} \rangle_{X_0^0(\Gamma_k)}.$$

In this subsection, we prove the following lemma.

**Lemma 3.11.** Label $\Gamma_k$ as in Figure 3. Fix a pair $(B, \tilde{B})$ of dual bases of $\mathcal{P}_*(\Sigma^+_1)$ and set $B = ([a_{ij}])_{i \in \text{in}, j \in \text{out}}$ and $\tilde{B} = ([\tilde{a}_{ij}])_{i \in \text{in}, j \in \text{out}}$. For any $i \in \mathbb{Z}$, set

$$i^+ = \begin{cases} i + 1 & \text{if } i < k, \\ 1 & \text{if } i = k. \end{cases}$$

For any $i \in \mathbb{Z}$, any numbering $\sigma$ of $\Gamma_k$, and any $\hat{\varepsilon}: \mathbb{Z} \to \{\pm 1\}$, set $\sigma \varepsilon(\ell_i) = \sigma(\ell_i) + (1 - \hat{\varepsilon}(i))k$ and

$$\hat{\varepsilon}_{\sigma, \sigma}(i) = \begin{cases} +1 & \text{if } \sigma \varepsilon(\ell_i) < \sigma \varepsilon(\ell_i^+), \\ -1 & \text{otherwise}. \end{cases}$$

The invariant $Z_k$ reads

$$Z_k(\psi) = \frac{1}{k(2k)!2^k} \sum_{\sigma \in \text{Num}(\Gamma_k)} \sum_{\hat{\varepsilon}: \mathbb{Z} \to \{\pm 1\}} \sum_{d \in \mathbb{N}, p: k \to \mathbb{R}} \sum_{k \to b_d} (-1)^{d+1} \prod_{j \in \mathbb{Z}} \ln \left( z_{p(j)}^d, (a_{p(j)}^{n+1-d})^{\hat{\varepsilon}_{\sigma, \sigma}(i)} \right).$$

The rest of this section is devoted to the proof of Lemma 3.11.
Lemma 3.12. Let \( Y(\sigma) \) denote the manifold \( \prod_{i \in k} \Sigma_{\sigma(i)}(2) \). Similarly, set \( Y^0(\sigma) = \prod_{i \in k} \Sigma_{\sigma(i)}^0(2) \). For any \( i \in k \), set \( Y_i(\sigma) = \Sigma_{\sigma(i)}(2) \times \Sigma_{\sigma(i+1)}(2) \) and \( B_{i,\sigma} = B_{\sigma(f_i)} \cap Y_i(\sigma) \) and similarly define \( Y_i^0(\sigma) \) and \( B_{i,\sigma}^0 = B_{\sigma(f_i)}^0 \cap Y_i^0(\sigma) \). Let \( \pi_i \) denote the projection map \( Y(\sigma) \to Y_i(\sigma) \), and set \( P_{i,\sigma} = \pi_i^{-1}(B_{i,\sigma}) \). Similarly define \( P_{i,\sigma}^0 : Y^0(\sigma) \to Y_i^0(\sigma) \) and \( P_{i,\sigma}^0 \).

The chains \( (P_{i,\sigma})_{i \in k} \) are transverse, the chains \( (P_{i,\sigma}^0)_{i \in k} \) are transverse, and
\[
\Delta_{\Gamma_k,\sigma} Z_k = -\frac{1}{2k} \left( \langle P_{1,\sigma}, \ldots, P_{k,\sigma} \rangle_{Y(\sigma)} - \langle P_{1,\sigma}^0, \ldots, P_{k,\sigma}^0 \rangle_{Y^0(\sigma)} \right).
\]

Proof. Let \( X_2(\Gamma_k) \) denote the set of configurations such that all the external vertices are mapped to \( E_2 \), and set \( D_{e,\sigma}^{(2)} = 2D_{e,\sigma} \cap X_2(\Gamma_k) \). For simplicity, we assume that \( D_{e,\sigma}^{(2)} \) is a variety. Similarly define \( X_0(\Gamma_k) \) and \( D_{e,\sigma}^{(2),0} \). Lemma 3.9 ensures that the intersection of the supports of the \( D_{e,\sigma}^{(1)} \) is contained in \( \bigcap_{e \in E(\Gamma_k)} D_{e,\sigma}^{(2)} \). On the other hand, \( \left( \bigcap_{e \in E(\Gamma_k)} D_{e,\sigma}^{(2)} \right) \setminus \left( \bigcap_{e \in E(\Gamma_k)} \text{Supp}(D_{e,\sigma}^{(1)}) \right) \) is contained in \( p_0^{-1}((N_R)^k) \). Then,
\[
\langle D_{e,\sigma}^{(1)}, X_1(\Gamma_k) \rangle = \frac{1}{2k} \langle D_{e,\sigma}^{(2)}, X_2(\Gamma_k) \rangle + \rho_0,
\]
where \( \rho_0 \) does not depend on the knot. This implies that
\[
\Delta_{\Gamma_k,\sigma} Z_k = \frac{1}{2k} \left( \langle D_{e,\sigma}^{(2)}, X_2(\Gamma_k) \rangle - \langle D_{e,\sigma}^{(2),0}, X_2(\Gamma_k) \rangle \right).
\]

Let us check that \( \left( \varphi : c \in \bigcap_{i \in k} D_{e,\sigma}^{(2)} \mapsto (c(w_i))_{i \in k} \in Y(\sigma) \right) \) is well-defined and is an orientation-reversing diffeomorphism.

The definition of \( X_2(\Gamma_k) \) and of admissible propagators implies that if \( c \in \bigcap_{i \in k} D_{e,\sigma}^{(2)} \), then \( (c(v_i), c(w_i)) \in B_{\sigma(i)} \) for any \( i \in k \). Then, for any \( i \in k \), since \( v_i \) is internal and because of Definition 3.3 \( c(w_i) \in \Sigma_{\sigma(i)}(2) \). This implies that \( \varphi \) is actually valued in \( Y(\sigma) \). It is a diffeomorphism since the disk \( D_{\mu}(1) \) meets the knot in exactly one point.

Let us check that \( \varphi \) reverses the orientations. Let \( n_i(x) \) denote the positive normal direction to \( \Sigma_{\sigma(i)}(2) \) at \( x \). The normal bundle to \( \bigcap_{i \in k} D_{e,\sigma}^{(2)} \) at \( c \) is
\[
\mathcal{N}_c \left( \bigcap_{i \in k} D_{e,\sigma}^{(2)} \right) = \prod_{i \in k} \left( T_{c(v_i)} \psi(\mathbb{R}^n) \times \mathbb{R} \cdot n_i(c(w_i)) \right),
\]

\[\text{March 24, 2020}\]
and we proved in Lemma 2 that \( C_{\Gamma_k}(\psi) \) is oriented as
\[
T_c \Gamma_k(\psi) = -\prod_{i \in \mathcal{K}} \left( T_{c(w_i)} \psi(\mathbb{R}^n) \times T_{c(w_i)} M^o \right)
\]
\[
= -\prod_{i \in \mathcal{K}} \left( T_{c(w_i)} \psi(\mathbb{R}^n) \times \mathbb{R}.n_i(c(w_i)) \times T_{c(w_i)} \Sigma_{\sigma}(\ell_i) \right)
\]
\[
= -\left( \prod_{i \in \mathcal{K}} T_{c(w_i)} \psi(\mathbb{R}^n) \times \mathbb{R}.n_i(c(w_i)) \right) \times \left( \prod_{i \in \mathcal{K}} T_{c(w_i)} \Sigma_{\sigma}(\ell_i) \right),
\]
where the last equality comes from the fact that the Seifert surfaces are even-dimensional. This proves that \( D_{L,\sigma} = \bigcap_{i \in \mathcal{K}} D_{\ell_i,\sigma}^{(2)} \) is oriented as \(-\varphi^{-1}(Y(\sigma))\).

Let us state without proof the following easy lemma, which we will use in the rest of this proof.

**Lemma 3.13.** Let \( P \) and \( Q \) be two oriented submanifolds of an oriented manifold \( R \). Let \( \mathcal{N}^Q(P \cap Q) \) denote the normal bundle of \( P \cap Q \) as a submanifold of \( Q \). For any \( x \in P \cap Q \),
\[
\mathcal{N}^Q_x(P \cap Q) = (-1)^{(\dim(R) - \dim(Q)) (\dim(R) - \dim(P))} \mathcal{N}^P_x P.
\]

For any \( i \in \mathcal{K} \), the coorientation of the submanifolds \( D_{f_i,\sigma}^{(2)} \cap D_{L,\sigma} \) in \( D_{L,\sigma} \) and of the submanifolds \( D_{f_i,\sigma}^{(2)} \) in \( C_{\Gamma}(\psi) \) differ by a factor \((-1)^{k(n+1)(n+1)} = 1\), so
\[
\left\langle \left( C_{E_{\kappa}}^{(2)} \right)_{e \in E(\Gamma_k)} \right\rangle_{X_{1}(\Gamma_k)} = \left\langle \left( D_{f_{i,\sigma}}^{(2)} \right)_{e \in E(\Gamma_k)} \right\rangle_{X_{1}(\Gamma_k)} = \left\langle \left( D_{f_{i,\sigma}}^{(2)} \cap D_{L,\sigma} \right)_{e \in E(\Gamma_k)} \right\rangle_{X_{1}(\Gamma_k)} = \left\langle \varphi(D_{f_{i,\sigma}}^{(2)} \cap D_{L,\sigma}) \right\rangle_{Y(\sigma)},
\]
where the sign comes from the fact that \( \varphi \) reverses the orientation.

Now, for any \( i \in \mathcal{K} \), \( \varphi(D_{f_{i,\sigma}}^{(2)} \cap D_{L,\sigma}) \) is cooriented as \( D_{f_{i,\sigma}} = p_{f_i}^{-1}(B_{\sigma(f_i)}) \), i.e. as \( B_{\sigma(f_i)} \) in \( C_2(M^o) \). On the other hand, \( P_{i,\sigma} \) is cooriented as \( B_{\sigma(f_i)} \cap (\Sigma_{\sigma(\ell_i)}(2) \times \Sigma_{\sigma(\ell_i)}(2)) \) in \( \Sigma_{\sigma(\ell_i)}(2) \times \Sigma_{\sigma(\ell_i)}(2) \), i.e. as \( (-1)^{2(n+1)}B_{\sigma(f_i)} = B_{\sigma(f_i)} \) in \( C_2(M^o) \).

Because of the 2 factors in the definition of the \( D_{f_{i,\sigma}}^{(2)} \), this yields
\[
\left\langle \left( D_{f_{i,\sigma}}^{(2)} \right)_{e \in E(\Gamma_k)} \right\rangle_{X_{2}(\Gamma_k)} = -2^k \langle P_{i,\sigma}, \ldots, P_{k,\sigma} \rangle_{Y(\sigma)}.
\]

Similarly,
\[
\left\langle \left( D_{f_{i,\sigma}}^{(2)} \right)_{e \in E(\Gamma_k)} \right\rangle_{X_{0}(\Gamma_k)} = -2^k \langle P_{i,\sigma}^0, \ldots, P_{k,\sigma}^0 \rangle_{Y^0(\sigma)}.
\]

We are going to define a manifold \( \overline{Y(\sigma)} \) in which the chains \( P_{i,\sigma} \) and \( P_{i,\sigma}^0 \) embed, in order to compute intersection numbers of the previous chains with boundaries in terms of intersection numbers of cycles inside one common manifold.
Lemma 3.14. For $i \in 2\mathbb{Z}$, let $S^+_i$ denote the gluing of $\Sigma_i^+ \cap E_2$ and $-(\Sigma_i^0 \cap E_2)$ along their boundaries, set $S_i = S^+_i \cup S^-_i$, and let $S^\leq_3$ denote the set of points of $S_i$ that come from a point in $N_3$ or $N_3^0$ before the gluing. For any $i \in \mathbb{K}$, set $Y_i(\sigma) = S_{\alpha(i)} \times S_{\alpha(i)+1}$, and set $Y(\sigma) = \bigcup_{i \in \mathbb{K}} S_{\alpha(i)}$. There exist canonical projection maps $\pi_i: Y(\sigma) \to Y_i(\sigma)$ for any $i \in \mathbb{K}$. The chains $(P_{i,\sigma})_{i \in \mathbb{K}}$ and $(P^0_{i,\sigma})_{i \in \mathbb{K}}$ naturally embed into $Y(\sigma)$, and the chains $(B_{i,\sigma})_{i \in \mathbb{K}}$ and $(B^0_{i,\sigma})_{i \in \mathbb{K}}$ naturally embed into $Y_i(\sigma)$. With these notations,

- the boundaries $\partial B_{i,\sigma}$ and $\partial B^0_{i,\sigma}$ lie in $S^\leq_3 \times S^\leq_3$;

- for any $i \in \mathbb{K}$, there exists an $(n + 1)$-chain $\hat{B}_{i,\sigma}$ in $S^\leq_3 \times S^\leq_3$ such that $\partial \hat{B}_{i,\sigma} = \partial B_{i,\sigma} - \partial B_i$.

The manifold $S^\leq_3 \times S^\leq_3$ does not depend on the knot. The chains $(\hat{B}_{i,\sigma})_{i \in \mathbb{K}}$ can be chosen such that they do not depend on the knot, either.

Proof. Fix $i \in \mathbb{K}$. Since the Seifert surfaces are parallel, the chain $B_{i,\sigma}$ does not meet $Y_i(\sigma) \cap (N_2 \times E_3)$ or $Y_i(\sigma) \cap (E_3 \times N_3)$. Then, $\partial B_{i,\sigma}$ is necessarily contained in $\partial Y_i(\sigma) \cap (N_3 \times N_3)$. The same argument proves that $\partial B^0_{i,\sigma}$ is contained in $\partial Y^0_i(\sigma) \cap (N_3 \times N_3)$.

Therefore, the chain $Q_{i,\sigma} = \partial B^0_{i,\sigma} - \partial B_{i,\sigma}$ is a cycle of $S^\leq_3 \times S^\leq_3$. Since the propagators are standard inside $p_b^{-1}(N_3 \times N_3)$, the cycle $Q_{i,\sigma}$ does not depend on the knot.

For any $j \in \mathbb{K}$, let $\ell_j^\pm$ denote the boundary $\partial(\Sigma_j^\pm \cap E_2)$, which is involved in the gluing in the definition of $S_j^\pm$, and let $x_j^\pm \in \ell_j^\pm$. Since the product $S^\leq_3 \times S^\leq_3$ retracts onto $(\ell_\sigma(\ell_i) \cup \ell_\sigma(\ell_{i+1}) \times (\ell_\sigma(\ell_{i+1}) \cup \ell_\sigma(\ell_{i+1}))$, $H_n(S^\leq_3 \times S^\leq_3) = \mathbb{Q}^8$, with a basis given by the eight spheres $[\ell_\sigma(\ell_i) \times x_\ell^\prime(\ell_{i+1})]$ and $[x_\ell^\prime(\ell_{i+1}) \times \ell_\sigma(\ell_{i+1})]$ for $\varepsilon, \varepsilon' \in \{\pm\}$. Let $(s_\ell)^{1 \leq r \leq 8}$ denote these spheres.

The manifold $S^\leq_3 \times S^\leq_3$ contains $T^0_i = (S^\leq_3 \times S^\leq_3) \cap (S^0_{\alpha(i)}(2) \times S^0_{\alpha(i)+1}(2))$ and $T_i = (S^\leq_3 \times S^\leq_3) \cap (S^0_{\alpha(i)}(2) \times S^0_{\alpha(i)+1}(2))$. $T^0_i$ and $T_i$ are diffeomorphic to each other because the surfaces $\Sigma^0_i$ and $\Sigma_j$ are identical inside $N_3$ for any $j$. Denote by $\Theta_T: T^0_i \to T_i$ the induced diffeomorphism.

Note that the eight chains $(s_\ell)^{1 \leq r \leq 8}$ also define bases $([s_\ell])^{1 \leq r \leq 8}$ of $H_n(T^0_i)$ and $H_n(T_i)$. The definition of the spheres $(s_\ell)^{1 \leq r \leq 8}$ implies that $\Theta_T(s_\ell) = s_\ell$ for any $1 \leq r \leq 8$. Since the propagators do not depend on the knot inside $N_3 \times N_3$, $\Theta_T(\partial B^0_{i,\sigma}) = \partial B_{i,\sigma}$. The cycle $\partial B^0_{i,\sigma}$ defines a class in $H_n(T^0_i) = H_n(S^\leq_3 \times S^\leq_3)$.

This class reads $[\partial B^0_{i,\sigma}] = \sum_{r=1}^{8} \alpha_{i,r}[s_\ell]$ for some rational numbers $\alpha_{i,r}^{1 \leq r \leq 8}$. Applying $\Theta_T$ to this identity to get $[\partial B_{i,\sigma}] = \sum_{r=1}^{8} \alpha_{i,r}[s_\ell]$. This means that $[Q_i] = 0$ in $H_n(S^\leq_3 \times S^\leq_3)$ and proves the existence of $\hat{B}_{i,\sigma}$.
Since the cycle $Q_i \subset S_{\sigma(i)}^{\leq 3} \times S_{\sigma(i+1)}^{\leq 3}$ is independent of the knot, the chain $\hat{B}_{i,\sigma}$ can be chosen independently of the knot.

Lemma 3.15. Let $b_d$ denote the $d$-th Betti number of $S_1^+$. It is possible to choose two families of cycles $((a_{i,j}^d)^+)_{0 \leq d \leq n+1, j \in b_d}$ and $((z_{i,j}^d)^+)_{0 \leq d \leq n+1, j \in b_d}$ in $S_1^+$ such that:

- For any $d \in \mathbb{N}$ and any $j \in b_d$, $[[a_{i,j}^d]^+] = [a_{i,j}^d]$ and $[[z_{i,j}^d]^+] = [z_{i,j}^d]$, where the cycles $(a_{i,j}^d)^+$ and $(z_{i,j}^d)^+$ are defined in Lemma 3.14.

  For $d = n + 1$, $(a_{i,1,1}^{n+1})^+ = (z_{1,1}^{n+1})^+ = S_1^+$. For $d = 0$, $(a_{i,1,1}^0)^+ = (z_{1,1}^0)^+$ is a point.

  In particular, for any $d \in \{0, \ldots, n+1\}$, $[[a_{i,j}^d]^+]_{j \in b_d}$ and $[[z_{i,j}^d]^+]_{j \in b_d}$ are bases of $H_d(S_1^+)$, and for any $(j, j') \in b_d^2$, we have the duality relation $[[a_{i,j}^d]^+], [[z_{j,j'}^{n+1-d}]^+]_{S_1^+} = \delta_{j,j'}$.

- For any $d \in \mathbb{N}$, the cycles $((a_{i,j}^d)^+)_{j \in b_d}$ and $((z_{i,j}^d)^+)_{j \in b_d}$ are contained in $\Sigma_1^+ \cap E_{k+1} \subset S_1^+$.

- The point $(a_{i,1,1}^0)^+ = (z_{1,1}^0)^+$ is in $\partial(S_1^+ \cap E_2)$.

- For any $d > \frac{n+1}{2}$ and any $j \in b_d$, $(a_{i,j}^d)^+ = (z_{i,j}^d)^+$, and for any $d < \frac{n+1}{2}$ and any $j \in b_d$, $(z_{i,j}^d)^+ = (-1)^d(a_{i,j}^d)^+$.

Since all the Seifert surfaces $(\Sigma_1^+)_i \in \mathbb{L}$ are obtained from $\Sigma_1^+$ by pushing $\Sigma_1^+$ in the positive normal direction, these families yield similar families $((a_{i,j}^d)^+)_{0 \leq d \leq n+1, j \in b_d}$ and $((z_{i,j}^d)^+)_{0 \leq d \leq n+1, j \in b_d}$ in $H_* (S_1^+)$.

Proof. It is possible to choose two families such that the first three properties hold because the map $H_d(\Sigma_1^+ \cap E_3) \to H_d(S_1^+)$ induced by the inclusion is an isomorphism for $0 \leq d \leq n$, which can be deduced from Mayer-Vietoris formula.

Due to the symmetry (and antisymmetry) properties of the intersection number, we can also choose these chains such that the last property holds.

Lemma 3.16. For any $i \in \mathbb{L}$, define the cycle $\overline{B}_{i,\sigma} = B_{i,\sigma} - B_{i,\sigma}^0 + \hat{B}_{i,\sigma}$ of $Y_i(\sigma)$.

Its class in $H_{n+1}(Y_i(\sigma))$ reads

\[
\overline{[B_{i,\sigma}]} = \sum_{d \in \mathbb{L} \setminus \{p,q,e,e'\}} \sum_{(b_d)^2 \times \{\pm\}^2} \text{lk} \left( (z_{\sigma(\ell,i),p}^{n+1-d})^\epsilon, (a_{\sigma(\ell,i),q})^{\epsilon} \right) \left[ (a_{\sigma(\ell,i),p})^\epsilon \times (z_{\sigma(\ell,i),q})^{\epsilon} \right] + R^{\overline{B}}_{i,\sigma},
\]

where $R^{\overline{B}}_{i,\sigma}$ reads

\[
\sum_{d \in \{0,n+1\}} \sum_{(\epsilon,\epsilon') \in \{\pm\}^2} \alpha^{(i)}_{d,1,1,\epsilon,\epsilon'} \left[ (a_{\sigma(\ell,i),1})^{\epsilon} \times (z_{\sigma(\ell,i),1})^{\epsilon'} \right],
\]

with rational coefficients $\alpha^{(i)}_{d,1,1,\epsilon,\epsilon'}$ independent of the knot.
Proof. The families of chains \( ((a_{\sigma(l_i),p})^+)_{0 \leq d \leq n+1, p \in \mathbb{Z}} \) and \( ((z_{\sigma(l_i),p})^+)_{0 \leq d \leq n+1, p \in \mathbb{Z}} \) induce the two following bases of \( H_{n+1}(\Sigma_1^2) \):

\[
\begin{align*}
\left( (a_{\sigma(l_i),p})^+ \times (z_{\sigma(l_i),q})^+ \right)_{0 \leq d \leq n+1, 1 \leq p, q \leq b_{d,\epsilon,\epsilon'} \in \{\pm\}^2}, \\
\left( (z_{\sigma(l_i),p})^+ \times (a_{\sigma(l_i),q})^+ \right)_{0 \leq d \leq n+1, 1 \leq p, q \leq b_{d,\epsilon,\epsilon'} \in \{\pm\}^2}.
\end{align*}
\]

These bases are dual is the sense that for any \( p, p', q, q', d, d', \epsilon, \epsilon', \eta, \eta' \),

\[
\left\langle \left( a_{\sigma(l_i),p}^+ \times (z_{\sigma(l_i),q})^+ \right), \left( z_{\sigma(l_i),p'}^+ \times (a_{\sigma(l_i),q'})^+ \right) \right\rangle_{\Sigma_1^2} = \delta_{(d',p',\eta,\eta')}^d;
\]

where \( \delta_{x}^y \) is the Kronecker delta. There exist coefficients such that

\[
B_{i,\sigma} = \sum_{d=0}^{n+1} \sum_{(p, q, \epsilon, \epsilon') \in (b_d)^2 \times \{\pm\}^2} \alpha_{d,p,q,\epsilon,\epsilon'}^{(i)} \left( (a_{\sigma(l_i),p})^+ \times (z_{\sigma(l_i),q})^+ \right)
\]

For any \( d \in \mathbb{Z} \), and any \( (p, q, \epsilon, \epsilon') \in (b_d)^2 \times \{\pm\}^2 \),

\[
\begin{align*}
\alpha_{d,p,q,\epsilon,\epsilon'}^{(i)} & = \left\langle \left[ B_{i,\sigma} \right], \left( (z_{\sigma(l_i),p})^+ \times (a_{\sigma(l_i),q})^+ \right) \right\rangle_{\Sigma_1^2} \\
& = \left\langle B_{\sigma(f_i)}, \left( (z_{\sigma(l_i),p})^+ \times (a_{\sigma(l_i),q})^+ \right) \right\rangle_{C_2(M^2)} \\
& = \text{lk} \left( \left( (z_{\sigma(l_i),p})^+, (a_{\sigma(l_i),q})^+ \right) \right),
\end{align*}
\]

where the first equality comes from the duality of the bases above, the second one comes from the second point of Lemma 2.15 and the third one comes from Lemma 2.15.

Set \( R_{i,\sigma}^B = \sum_{d=0}^{n+1} \sum_{(p, q, \epsilon, \epsilon') \in (b_d)^2 \times \{\pm\}^2} \alpha_{d,1,1,\epsilon,\epsilon'}^{(i)} \left( (a_{\sigma(l_i),1})^+ \times (z_{\sigma(l_i),1})^+ \right) \). The duality allows us to compute the coefficients that appear in \( R_{i,\sigma}^B \), too. First, \( \alpha_{0,1,1,\epsilon,\epsilon'}^{(i)} = \left\langle \left[ B_{i,\sigma} \right], \left[ S_{\sigma(l_i)}^\epsilon \times (a_{\sigma(l_i),1})^\epsilon \right] \right\rangle_{\Sigma_1^2} \). The chain \( S_{\sigma(l_i)}^\epsilon \times (a_{\sigma(l_i),1})^\epsilon \) is contained in \( S_{\sigma(l_i)}^\epsilon \times \partial (S_{\sigma(l_i),1}) \cap E_2 \). Then, it only meets \( B_{i,\sigma} \) inside \( S_{\sigma(l_i)}(2) \times \partial (S_{\sigma(l_i),1}) \cap E_2 \).

Let us prove that \( B_{i,\sigma} \cap (S_{\sigma(l_i)}(2) \times \partial (S_{\sigma(l_i),1}) \cap E_2) \) lies in \( b_{d,\epsilon,\epsilon'} \) \((N_3 \times N_3) \). A configuration in this intersection was in \( E_3 \times \partial N_2 \), it would be in \( \Sigma_{\sigma(l_i)}(3) \times \partial (S_{\sigma(l_i),1}) \cap (E_3 \times \partial N_2) \). Since \( \Sigma_{\sigma(l_i)}(2) \cap \Sigma_{\sigma(l_i),1}(2) = \emptyset \) and \( \Sigma_{\sigma(l_i),1}(2) \cap \partial (S_{\sigma(l_i),1}) = \emptyset \), this is impossible. Then, \( \langle B_{i,\sigma}, S_{\sigma(l_i)}^\epsilon \times (a_{\sigma(l_i),1})^\epsilon \rangle_{S_{\sigma(l_i)} \times S_{\sigma(l_i),1}} \) only counts intersection points in \( b_{d,\epsilon,\epsilon'} \) \((N_3 \times N_3) \). By construction, this implies that this intersection number does not depend on the knot. Similarly, \( \left\langle B_{i,\sigma}, S_{\sigma(l_i)}^\epsilon \times (a_{\sigma(l_i),1})^\epsilon \right\rangle_{S_{\sigma(l_i)} \times S_{\sigma(l_i),1}} \) does not depend on the knot, and \( \langle B_{i,\sigma}, S_{\sigma(l_i)}^\epsilon \times (a_{\sigma(l_i),1})^\epsilon \rangle_{S_{\sigma(l_i)} \times S_{\sigma(l_i),1}} \) does not depend on the knot because of Lemma 3.13. This proves that \( \alpha_{0,1,1,\epsilon,\epsilon'}^{(i)} \) does not depend on the knot. The same argument proves that \( \alpha_{d,1,1,\epsilon,\epsilon'}^{(i)} \) does not depend on the knot, so that the coefficients of \( R_{i,\sigma}^B \) do not depend on the knot.
Lemma 3.17. Let $J$ denote the set of tuples $(d, p, q, \hat{\varepsilon})$ such that $d \in \mathbb{N}$, $(p, q) \in (b_d)^2$, and $\hat{\varepsilon}$ is a map $\hat{\varepsilon} : k \to \{\pm\}$. For any $i \in k$, set $\overline{P_i} = P_i - P_i^0 + \pi_i^{-1}(B_i^0)$. With these notations, \[
abla \{P_i, \sigma\} = R_{i, \sigma} + \sum_{(d, p, q, \hat{\varepsilon}) \in J} \text{lk}\left(z_{n+1-d}^{d, \hat{\varepsilon}(i)}(\sigma) \times (a_{\sigma(i), p}^d)^{\hat{\varepsilon}(i)} \times (z_{n+1-d}^{d, \hat{\varepsilon}(i+1)}(\sigma)) \times \prod_{j \notin \{i, i+1\}} S_{\sigma(j)}^\varepsilon(j)\right), \]
where $R_{i, \sigma}$ reads \[
R_{i, \sigma} = \sum_{d \in \{0, n+1\}} \sum_{\hat{\varepsilon} : k \to \{\pm\}} \alpha_{d, \hat{\varepsilon}, \hat{\varepsilon}}^{(i)}\left((a_{\sigma(i), p}^d)^{\hat{\varepsilon}(i)} \times (z_{\sigma(i), p}^{n+1-d} \hat{\varepsilon}(i+1)) \times \prod_{j \notin \{i, i+1\}} S_{\sigma(j)}^\varepsilon(j)\right), \]
with coefficients $(\alpha_{d, \hat{\varepsilon}, \hat{\varepsilon}}^{(i)})_{d, \hat{\varepsilon}}$ that do not depend on the knot.

Proof. We have $\overline{P_i} = \pi_i^{-1}(B_i^0)$, so Lemma 3.16 implies \[
\overline{P_i} = R_{i, \sigma} + \sum_{d \in \mathbb{N}} \sum_{\hat{\varepsilon} : \mathbb{N} \to \{\pm\}} \text{lk}\left(z_{\sigma(i), p}^{d, \hat{\varepsilon}(i)}(\sigma) \times (a_{\sigma(i), p}^d)^{\hat{\varepsilon}(i)} \times (z_{\sigma(i), p}^{n+1-d} \hat{\varepsilon}(i+1)) \times \prod_{j \notin \{i, i+1\}} S_{\sigma(j)}^\varepsilon(j)\right), \]
with $R_{i, \sigma}$ as in the lemma with $\alpha_{d, \hat{\varepsilon}}^{(i)} = \alpha_{d, 1, 1, \hat{\varepsilon}(i), \hat{\varepsilon}(i+1)}^{(i)}$. But for any $d \in \mathbb{N}$, any $p$ and $q$ in $\mathbb{N}^2$ and any $(\varepsilon, \varepsilon') \subseteq \{\pm\}$, \[
\pi_i^{-1} \left((a_{\sigma(i), p}^{d})^{\varepsilon} \times (z_{\sigma(i), p}^{n+1-d})^\varepsilon\right) = \sum_{\hat{\varepsilon} : k \to \{\pm\} \to \{\pm\}} \eta_{\hat{\varepsilon}} \left((a_{\sigma(i), p}^{d})^{\hat{\varepsilon}} \times (z_{\sigma(i), p}^{n+1-d})^{\hat{\varepsilon}} \times \prod_{j \notin \{i, i+1\}} S_{\sigma(j)}^{\hat{\varepsilon}(j)}\right) \]
for some signs $(\eta_{\hat{\varepsilon}})_{\hat{\varepsilon}}$. Since for any $\hat{\varepsilon}$ the chain $(a_{\sigma(i), p}^{d})^{\hat{\varepsilon}} \times (z_{\sigma(i), p}^{n+1-d})^{\hat{\varepsilon}} \times \prod_{j \notin \{i, i+1\}} S_{\sigma(j)}^{\hat{\varepsilon}(j)}$ is cooriented by $(z_{\sigma(i), p}^{n+1-d})^{\hat{\varepsilon}} \times (a_{\sigma(i), p}^{d})^{\hat{\varepsilon}}$, the signs $(\eta_{\hat{\varepsilon}})_{\hat{\varepsilon}}$ are all positive. The lemma follows.

Lemma 3.18. Let $J'$ denote the set of tuples $(d, p, q, \hat{\varepsilon})$ such that $d \in \{0, \ldots, n+1\}$, $(p, q) \in (b_d)^2$, and $\hat{\varepsilon} : k \to \{\pm\}$. We have $J \subseteq J'$. For $(d_i, p_i, q_i, \hat{\varepsilon}_i)_{i \in k} \in (J')^k$, \[
\left\langle \left((a_{\sigma(i), p_i}^{d_i})^{\hat{\varepsilon}_i(1)} \times (z_{\sigma(i), p_i}^{n+1-d_i})^{\hat{\varepsilon}_i(1+1)} \times \prod_{j \notin \{i, i+1\}} S_{\sigma(j)}^{\hat{\varepsilon}_i(j)}\right)_{i \in k} \right\rangle = \begin{cases} (-1)^d & \text{if } d_1 = \ldots = d_k, \hat{\varepsilon}_1 = \ldots = \hat{\varepsilon}_k, \text{ and for any } i \in k, q_i = p_i, \\ 0 & \text{otherwise.} \end{cases} \]
Proof. If we do not have \( \hat{\varepsilon}_1 = \ldots = \hat{\varepsilon}_k \) the intersection is empty. If we do not have \( d_1 = \ldots = d_k \), there exists an integer \( i \in \mathbb{N} \) such that \( d_i > d_{i+} \) and the chains \((z^n_{\sigma(\ell_i+),q_i}, \hat{\varepsilon}_i, a^d_{\sigma(\ell_i+),p_i})\) do not intersect, up to small perturbations, so the intersection number of the lemma is zero. Let us now assume \( \hat{\varepsilon}_1 = \ldots = \hat{\varepsilon}_k = \hat{\varepsilon} \) and \( d_1 = \ldots = d_k = d \). The chain \((a^d_{\sigma(\ell_i),p_i})\) is cooriented by \(-1)^d \mathcal{N}(a^d_{\sigma(\ell_i),p_i})\) \( \times \mathcal{N}(z^n_{\sigma(\ell_i+),q_i}) \). Then, the normal bundle of the intersection of the lemma is oriented as

\[
\prod_{i \in \mathbb{N}} \left( (-1)^d \mathcal{N}(a^d_{\sigma(\ell_i),p_i}) \hat{\varepsilon}_i \times \mathcal{N}(z^n_{\sigma(\ell_i+),q_i}) \hat{\varepsilon}_i \right)
\]

This implies that

\[
\left\langle \left( \left( a^d_{\sigma(\ell_i),p_i} \hat{\varepsilon}_i \times (z^n_{\sigma(\ell_i+),q_i}) \hat{\varepsilon}_i \times \prod_{j \in \{i,i+\}} S^{d(j)}_{\sigma(\ell_j)} \right) \right) \right\rangle_{i \in \mathbb{N}} = (-1)^d \prod_{i \in \mathbb{N}} \left( \left( a^d_{\sigma(\ell_i+),p_i} \hat{\varepsilon}_i \times (z^n_{\sigma(\ell_i+),q_i}) \hat{\varepsilon}_i \right) \right)_{\mathcal{N}(\sigma(\ell_i+))} = (-1)^d \prod_{i \in \mathbb{N}} \delta_{q_i,p_i+}
\]

and concludes the proof. \( \square \)

**Lemma 3.19.** For any numbering \( \sigma \) of \( \Gamma_k \),

\[
\Delta_{\Gamma_k,\sigma} Z_k = \frac{1}{2k} \sum_{d \in \mathbb{N}} \sum_{z \in \mathbb{N}} \sum_{p=0}^{b_d} \left(-1\right)^{(d+1)} \prod_{i \in \mathbb{N}} \text{lk} \left( (z^n_{\sigma(\ell_i),p(i)}), (a^d_{\sigma(\ell_i+),p(i)}) \hat{\varepsilon}_i \right).
\]

**Proof.** Lemmas 3.17 and 3.18 imply that

\[
\langle \mathcal{T}_{1,\sigma}, \ldots, \mathcal{T}_{k,\sigma} \rangle_{\mathcal{N}(\sigma)} = \sum_{d \in \mathbb{N}} \sum_{z \in \mathbb{N}} \sum_{p=0}^{b_d} \left(-1\right)^d \prod_{i \in \mathbb{N}} \text{lk} \left( (z^n_{\sigma(\ell_i),p(i)}), (a^d_{\sigma(\ell_i+),p(i)}) \hat{\varepsilon}_i \right) + \rho_1,
\]

where \( \rho_1 \) does not depend on the knot. For any \( i \in \mathbb{N} \) and \( j \in \{0, 1, 2\} \), set

\[
P_i,\sigma(j) = \begin{cases} 
P_i,\sigma & \text{if } j = 0, \\ \rho_0 & \text{if } j = 1, \\ P_{i,\sigma} = \frac{1}{P_{i,\sigma}}(\hat{B}_{i,\sigma}) & \text{if } j = 2, \end{cases}
\]
so that \( \overline{P_{i,\sigma}} = P_{i,\sigma}(0) - P_{i,\sigma}(1) + P_{i,\sigma}(2) \). By transversality, \( \bigcap_{i \in \mathcal{K}} \text{Supp}(\overline{P_{i,\sigma}}) \subset \bigcap_{i \in \mathcal{K}} (\text{int}(P_{i,\sigma}(0)) \cup \text{int}(P_{i,\sigma}(1)) \cup \text{int}(P_{i,\sigma}(2))) \), so that

\[
\langle \overline{P_{1,\sigma}}, \cdots, \overline{P_{k,\sigma}} \rangle_{Y(\sigma)} = \sum_{j: k \rightarrow \{0,1,2\}} (-1)^{j(1)+\cdots+j(k)} \langle (P_{i,\sigma}(j(i)))_{i \in \mathcal{K}} \rangle_{Y(\sigma)}.
\]

For \( r \in [3, R - 1] \), let \( S_r^\leq \) denote the set of points of \( S_i \) that come from a point of \( N_r \) or \( N_r^0 \) in the gluing that defines \( S_i \) in Lemma 3.14. Let us prove that for a configuration \( c \in \bigcap_{i \in \mathcal{K}} P_{i,\sigma}(j(i)), \) if \( c(w_p) \in S_r^\leq \) for some \( r \in [3, R - 1] \) and \( p \in \mathcal{K} \), then \( c(w_p^+) \in S_r^{\leq r+1} \).

- If \( j(p) = 0 \), then \( (c(w_p), c(w_p^+)) \in B_{\delta}(f_p) \). If \( c(w_p^+) \) was in \( E_{r+1} \), then we would have \( (c(w_p), c(w_p^+)) \in p_b^{-1}(N_r \times E_{r+1}) \), so \( c(w_p) \in L_{\theta_{\sigma}(f_p)}(r) \) or \( c(w_p^+) \in \Sigma_{\sigma}(f_p)(r + 1) \). Since the Seifert surfaces \( (\Sigma_{\sigma}^j)_{j \in \mathcal{K}, \varepsilon = \pm} \) are pairwise parallel, this is impossible and \( c(w_p^+) \in \Sigma_{\sigma}(f_p)(2) \cap N_{r+1} ^0 \subset S_r^{\leq r+1} \).
- If \( j(p) = 1 \), we similarly prove that \( c(w_p^+) \in \Sigma_{\sigma}(f_p)(2) \cap N_{r+1} ^0 \subset S_r^{\leq r+1} \).
- If \( j(p) = 2 \), since \( \hat{B}_{\rho, \sigma} = \pi_1(\overline{P_{\rho,\sigma}}) \subset S_r^{\leq 3} \times S_r^{\leq 3} \) and \( r \geq 2 \), then \( c(w_p^+) \in S_r^{\leq 3} \subset S_r^{\leq r+1} \).

A finite induction proves that if \( j \) takes the value 2, then the intersection number \( \langle (P_{i,\sigma}(j(i)))_{i \in \mathcal{K}} \rangle_{Y(\sigma)} \) only counts configurations in \( \prod_{i \in \mathcal{K}} S_r^{\leq R} \), where these chains are independent of the knot. This implies that

\[
\langle \overline{P_{1,\sigma}}, \cdots, \overline{P_{k,\sigma}} \rangle_{Y(\sigma)} = \langle P_{1,\sigma}, \cdots, P_{k,\sigma} \rangle_{Y(\sigma)} + (-1)^k \langle P_{1,\sigma}^0, \cdots, P_{k,\sigma}^0 \rangle_{Y(\sigma)} + \rho_2,
\]

where \( \rho_2 \) is independent of the knot. Note that the quantity \( \rho_3 = -\rho_2 + ((-1)^{k+1} - 1) \langle P_{1,\sigma}^0, \cdots, P_{k,\sigma}^0 \rangle_{Y(\sigma)} \) does not depend on the knot. Lemma 3.12 reads

\[
\Delta_{\Gamma_{k,\sigma}} Z_k = -\frac{1}{2k} \left( \langle P_{1,\sigma}, \cdots, P_{k,\sigma} \rangle_{Y(\sigma)} - \langle P_{1,\sigma}^0, \cdots, P_{k,\sigma}^0 \rangle_{Y(\sigma)} \right)
\]

\[
= -\frac{1}{2k} \langle \overline{P_{1,\sigma}}, \cdots, \overline{P_{k,\sigma}} \rangle_{Y(\sigma)} + \rho_3
\]

\[
= \frac{1}{2k} \sum_{\mathcal{A} \in \mathcal{Z} : 
\mathcal{A} \rightarrow \{ \pm \}} \sum_{p \in \mathcal{K}} \sum_{k \in \mathcal{H}_p} (-1)^{d+1} \prod_{i \in \mathcal{K}} \text{lk} \left( (z_{\sigma(\ell_i), p(i)}^{n+1-d})^{\varepsilon(i)}, (a_{\sigma(\ell_i+), p(i+)}^d)^{\varepsilon(i)} \right)
\]

\[
- \rho_2 + \rho_3.
\]

If \( \psi \) is the trivial knot, this formula yields \( \rho_1 + \rho_3 = 0 \). This concludes the proof of Lemma 3.19. \( \square \)
Note that

- for any $\sigma \in \text{Num}(\Gamma_k)$, there are exactly $k$ numberings $\sigma'$ of $\Gamma_k$ such that $(\Gamma, \sigma)$ and $(\Gamma, \sigma')$ are isomorphic as numbered graphs,

- for any $(\sigma, i) \in \text{Num}(\Gamma_k) \times \mathbb{K}$, if $\Sigma = \Sigma_1^+$ and if $\Sigma^+$ denote the surface obtained from $\Sigma$ by pushing $\Sigma_1^+$ in the positive normal direction, then $(\Sigma_{\sigma(\ell)}^{\epsilon(i)}, \Sigma_{\sigma(\ell_+)}^{\epsilon(i+1)})$ is isotopic to $(\Sigma, \Sigma^+)$ if $\epsilon_{\sigma(i)} = +1$ and to $(\Sigma^+, \Sigma)$ if $\epsilon_{\sigma(i)} = -1$.

Therefore, Lemma 3.19 and the definition of the Seifert surfaces in Setting 3.6 imply Lemma 3.11.

### 3.3 Proof of Theorem 2.24

Lemma 3.11 can be rephrased as follows in terms of Seifert matrices.

**Lemma 3.20.** Fix a pair $(B, \tilde{B})$ of dual bases of $\overline{H}_*(\Sigma)$, and set $B = ([a_{ij}])_{0 \leq d \leq n, i \in \mathbb{K}}$ and $\tilde{B} = ([z_{ij}])_{0 \leq d \leq n, i \in \mathbb{K}}$. For any $d \in \mathbb{N}$, define the matrices $V_\pm^d = V_\pm^d(B, \tilde{B})$ as in Definition 2.17. For any numbering $\sigma$ of $\Gamma_k$ and any map $\hat{\epsilon}: \mathbb{K} \to \{\pm\}$, let $\epsilon_{\hat{\epsilon}, \sigma}$ be defined as in Lemma 3.11, and let $N(\hat{\epsilon}, \sigma)$ be the number of integers $i \in \mathbb{K}$ such that $\epsilon_{\hat{\epsilon}, \sigma} = +1$. For any $\nu \in \{0, \ldots, k\}$, set

$$
\lambda_{k, \nu}^{(0)} = \frac{1}{2^k(2k)!} \text{Card}\{(\hat{\epsilon}, \sigma) \in \{\pm\}^k \times \text{Num}(\Gamma_k) \mid N(\hat{\epsilon}, \sigma) = \nu\}.
$$

With these notations,

$$
Z_k(\psi) = \frac{1}{k} \sum_{\nu=1}^{k-1} \sum_{d=1}^{n} (-1)^{d+1} \lambda_{k, \nu}^{(0)} \text{Tr}(V_+^d)^\nu(V_-^d)^{k-\nu}).
$$

**Proof.** Note that for any $k \geq 2$, $\lambda_{k,0}^{(0)} = \lambda_{k,k}^{(0)} = 0$. \hfill $\Box$

In order to prove Theorem 2.24 it remains to prove the following lemma.

**Lemma 3.21.** For any $k \geq 2$ and any $\nu \in \mathbb{K}$, $\lambda_{k, \nu}^{(0)} = \lambda_{k, \nu}$ with the notations of Theorem 2.24.

**Proof.** For any $(\hat{\epsilon}, \sigma) \in \{\pm\}^k \times \text{Num}(\Gamma_k)$, define

$$
F_0(\hat{\epsilon}, \sigma): i \in \mathbb{K} \mapsto \sigma(\ell_i) + (1 - \epsilon(i))k \in 4\mathbb{K},
$$

and let $F_1(\hat{\epsilon}, \sigma) \in \mathfrak{S}_4$ be the permutation such that for any $i \in \mathbb{K}$,

$$
F_1(\hat{\epsilon}, \sigma)(i) = 1 + \text{Card}\{j \in \mathbb{K} \mid F_0(\hat{\epsilon}, \sigma)(j) < F_0(\hat{\epsilon}, \sigma)(i)\}.
$$

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By definition, $N(\xi, \sigma)$ is the number of elements $i \in k$ such that $F_0(\xi, \sigma)(i) < F_0(\xi, \sigma)(i^+)$. Since for any $i \in k$,

$$\left(F_0(\xi, \sigma)(i) < F_0(\xi, \sigma)(i^+)\right) \iff \left(F_1(\xi, \sigma)(i) < F_1(\xi, \sigma)(i^+)\right),$$

$N(\xi, \sigma) = N_1(F_1(\xi, \sigma))$, where for any $\rho \in \mathcal{G}_k$, $N_1(\rho) = \text{Card}\{i \in k \mid \rho(i) < \rho(i^+)\}$.

Let $\rho \in \mathcal{G}_k$ act on $(\xi, \sigma) \in \{\pm\}^k \times \text{Num}(\Gamma_k)$ as

$$\rho \cdot (\xi, \sigma) = (\xi \circ \rho^{-1}, \sigma_{\rho}),$$

where for any $i \in k$,

$$\begin{cases}
\sigma_{\rho}(f_i) = \sigma(f_i), \\
\sigma_{\rho}(\ell_i) = \sigma(\ell_{\rho^{-1}(i)}).
\end{cases}$$

With these notations, $F_1(\rho \cdot (\xi, \sigma)) = F_1(\xi, \sigma) \circ \rho^{-1}$. This implies that all the fibers of $F_1$ have same cardinality $\frac{2^k (2k)!}{k!}$, so that

$$\lambda_{k,\upsilon}^{(0)} = \frac{1}{k!} \text{Card} \left\{ \{\sigma \in \mathcal{G}_k \mid N_1(\sigma) = \upsilon\} \right\}.$$

Let $\sigma_+ \in \mathcal{G}_k$ denote the permutation such that $\sigma_+(i) = i^+$ for any $i \in k$. The subgroup $G$ generated by $\sigma_+$ in $\mathcal{G}_k$ is cyclic of order $k$. Let $G$ act on $\mathcal{G}_k$ in such a way that $\sigma_+ \cdot \sigma = \sigma \circ (\sigma_+)^{-1}$ for any $\sigma \in \mathcal{G}_k$. We have $N_1(\sigma) = N_1(\sigma_+ \cdot \sigma)$ for any $\sigma \in \mathcal{G}_k$, and each orbit is of cardinality $k$. The subgroup $G' = \{\sigma \in \mathcal{G}_k \mid \sigma(k) = k\}$ contains exactly one element of each orbit. For any $\sigma \in G'$, $N_1(\sigma) = 1 + N_2(\sigma_{k-1})$, where for any $\sigma' \in \mathcal{G}_{k-1}$, $N_2(\sigma') = \text{Card}(\{i \in k-2 \mid \sigma'(i) < \sigma'(i+1)\})$.

Therefore,

$$\lambda_{k,\upsilon}^{(0)} = \frac{1}{(k-1)!} \text{Card} \left\{ \{\sigma' \in \mathcal{G}_{k-1} \mid N_2(\sigma') = \upsilon - 1\} \right\} = \lambda_{k,\upsilon}.$$

4 Construction of admissible propagators

4.1 Preliminary setting

In this section, we prove Lemma [3.3]. It suffices to prove the following result.

**Lemma 4.1.** Fix a rectifiable long knot $\psi : \mathbb{R}^n \hookrightarrow M^0$, a diffeomorphism $\Theta : N_R^0 \to N_R$ as before Lemma [3.4], and a parallelisation $\tau$ as in Lemma [3.4]. Fix two real numbers $\theta \in \mathbb{R}$, and $R \geq 3$. Fix Seifert surfaces $\Sigma^\pm$ such that $\Sigma^\pm \cap N_R = (\theta \Sigma^\pm)^0 \cap N_R$.

Under these assumptions, there exist $R$-admissible propagators for $(\Sigma^+, \Sigma^-, \psi)$ as in Definition [3.4]. Furthermore, it is possible to choose $R$-admissible propagators $B$ (for $(\Sigma^+, \Sigma^-, \psi)$) and $B_0$ (for $((\theta \Sigma^+)^0, (\theta \Sigma^-)^0, \psi_0)$) such that $\Theta_2(B_0 \cap p_R^{-1}(N_R^0) \times N_R^0)) = B \cap p_R^{-1}(N_R \times N_R)$, where $\Theta_2 : C_2(N_R^0) \to C_2(N_R)$ is the diffeomorphism induced by $\Theta : N_R^0 \to N_R$.
From now on, we assume without loss of generality that $\theta = 0$ and $R = 3$, and we prove Lemma 4.1 until the end of Section 4.

Fix Seifert surfaces $\Sigma^\pm$ as in Definition 3.4. Identify a neighborhood $N_3$ of the knot with the neighborhood $N_3^0$ of the trivial knot in $\mathbb{R}^{n+2}$ defined as the union of the cylinder $\{x \in \mathbb{R}^{n+2} \mid x_1^2 + x_2^2 \leq 9\}$ and the complement of the open ball of center 0 and radius $\frac{2\sqrt{2}}{3} = 6$. In this setting, $(G_t|_{(\partial C^2(M^o)) \cap p_0^{-1}(N_3 \times N_3)})$ extends to a smooth map $G_0 : p_0^{-1}(N_3 \times N_3) \to S^{n+1}$, which is the restriction of the Gauss map of $C^2(\mathbb{R}^{n+2})$ to $p_0^{-1}(N_3 \times N_3) = p_0^{-1}(N_3^0 \times N_3^0)$.

Define the following subsets:

$$X_0 = p_0^{-1}(N_1 \times N_1), \quad X_1 = p_0^{-1}\left(\bigcup_{r \in [1,2]} E_{r+1} \times N_r\right), \quad X_2 = p_0^{-1}\left(\bigcup_{r \in [1,2]} N_r \times E_{r+1}\right),$$

$$Y_1 = p_0^{-1}((N_2 \cap E_1) \times N_1), \quad Y_2 = p_0^{-1}(N_1 \times (N_2 \cap E_1)).$$

$$X = X_0 \cup X_1 \cup X_2, \quad Y = Y_1 \cup Y_2, \quad \text{and} \quad W = C^2(M^o) \setminus (X \cup Y).$$

Figure 5 shows this decomposition of $C^2(M^o)$, where $X$ is in black, $Y$ in gray, and $W$ in white.

Let $(e_i)_{1 \leq i \leq n+2}$ denote the canonical basis of $\mathbb{R}^{n+2}$. The disks $D_\mu(r)$ and the lines $L_0^+(r)$ are defined in Notation 3.3. Define the following chains in $X$:

- $B_{X_0} = G_0^{-1}(\{e_1\}) \cap p_0^{-1}(N_1 \times N_1)$
- $B_{X_1} = \bigcup_{r \in [1,2]} (\Sigma^- \cap E_{r+1}) \times D_\mu(r) + p_0^{-1}(E_{r+1} \times L_0^+(r))$
Lemma 4.2. That $B_{X_2}$ is naturally oriented. We orient $B_{X_1}$ such that the inclusions $(\Sigma^- \cap E_{r+1}) \times D_{\mu}(r) + p_b^{-1}(E_{r+1} \times L_0^b(r)) \to B_{X_1}$ preserve the orientation for any $r \in [0, 1, 2]$. The chain $B_{X_2}$ is similarly oriented. We are going to define a chain $B \subset C_2(M^3)$ such that $\partial B = G^{-1}_r(\{e_1\})$ and $B \cap \text{int}(X_i) = \text{int}(B_{X_i})$ for $i \in \{0, 1, 2\}$.

In $N_3 = N_3^0 \subset \mathbb{R}^{n+2}$, use the coordinates $x = (x_1, x_2, \mathbf{r}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$. In these coordinates, and for any $r \in [0, 1, 2]$, $N_r = \{(x_1, x_2, \mathbf{r}) \mid x_1^2 + x_2^2 \leq r^2 \text{ or } ||x|| \geq \frac{1}{r}\}$.

We also define the coordinates $x = (x_1, x_2, h_x, \omega_x)$, such that $h_x \in \mathbb{R}_+$ and $\omega_x \in S^{n-1}$. This will help us in drawing the next figures in $\mathbb{R}^2 \times \mathbb{R}^3$ with $\omega_x$ fixed in $S^{n-1}$. For example, Figure 6 depicts $N_r$.

![Figure 6: The neighborhood $N_r$.](image)

Set $R_r = \frac{18}{\sqrt{2}}$ and $h_r = \sqrt{R_r^2 - r^2}$, so that $\partial N_r = \partial_c N_r \cup \partial_s N_r$, with $\partial_c N_r = \{(x_1, x_2, h_x, \omega_x) \mid x_1^2 + x_2^2 = r^2, h_x \leq h_r\}$ and $\partial_s N_r = \{(x_1, x_2, h_x, \omega_x) \mid x_1^2 + x_2^2 = R_r^2 - h_x^2, h_x \leq h_r\}$.

We are going to define a chain $B_{Y_1} \subset Y_1$ such that $\partial(B_{X_0} + B_{Y_1} + B_{X_1}) \subset \partial(X_0 \cup Y_1 \cup X_1)$. For any $y \in N_3$, define

$$D^0(y, -e_1) = \{x \in N_1 \mid \text{there exists } t \geq 0 \text{ such that } x = y - t.e_1\},$$

so that $B_{X_0} = \{(x, y) \mid y \in N_1, x \in D^0(y, -e_1)\}$. We orient $D^0(y, -e_1)$ with $dt$, so that $B_{X_0}$ is oriented by $dy \wedge dt$.

Lemma 4.2. The boundary $\partial B_{X_0}$ splits into three pieces $G^{-1}_r(\{e_1\}) \cap \partial C_2(M^3) \cap p_b^{-1}(N_1 \times N_1)$, $\partial_1 B_{X_0} = B_{X_0} \cap p_b^{-1}(\partial N_1 \times N_1)$ and $\partial_2 B_{X_0} = B_{X_0} \cap p_b^{-1}(N_1 \times \partial N_1)$. The piece $\partial_1 B_{X_0}$ is exactly $\{(x, y) \mid x \in D^0(y, -e_1) \cap \partial N_1, y \in N_1\}$. For any $y \in N_1$, define the following points:

- If $h_y \leq h_1$, $|y_2| \leq \sqrt{R_1^2 - h_y^2}$, and $y_1 \geq \sqrt{R_1^2 - y_2^2 - h_y^2}$, $x^-(y)$ and $x^+(y)$ are the two intersection points of $D^0(y, -e_1)$ with the sphere $\{x \mid ||x|| = R_1\}$.

They coincide when $|y_2| = \sqrt{R_1^2 - h_y^2}$.  

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If \( h_y \leq h_1, |y_2| \leq 1, \) and \( y_1 \geq \sqrt{R_1^2 - y_2^2 - h_y^2}, \) \( x_c^-(y) \) and \( x_c^+(y) \) are the two intersection points of \( D^0(y, -e_1) \) with the cylinder \( \{ x | x_1^2 + x_2^2 = 1 \} \).

If \( h_y \leq h_1, \) and \( y_1^2 + y_2^2 \leq 1, \) \( x_s^-(y) \) and \( x_s^+(y) \) are the two intersection points of \( D^0(y, -e_1) \) with the cylinder or the sphere as above.

More precisely, we have the formulas
\[
x_c^\pm(y) = \left( \pm \sqrt{1 - y_2^2}, y_2, y \right), \quad x_s^\pm(y) = \left( \pm \sqrt{R_1^2 - y_2^2 - h_y^2}, y_2, y \right),
\]
when they make sense.

For any \( y \in N_1 \):
\[
D^0(y, -e_1) \cap \partial N_1 = \begin{cases} 
\emptyset & \text{if } h_y > h_1 \text{ or } |y_2| > \sqrt{R_1^2 - h_y^2} \\
\{ x_s^-(y), x_s^+(y) \} & \text{if } h_y \leq h_1, 1 < |y_2| \leq \sqrt{R_1^2 - h_y^2} \\
\{ x_c^-(y), x_c^+(y), x_s^-(y), x_s^+(y) \} & \text{if } h_y \leq h_1, |y_2| \leq 1, \\
\{ x_s^-(y), x_s^+(y) \} & \text{if } h_y \leq h_1 \text{ and } y_1^2 + y_2^2 \leq 1.
\end{cases}
\]

Figure 7 depicts the four possible cases, with the conventions of Section 4.1.

\[10\] They coincide when \( |y_2| = 1. \)

\[11\] They coincide when \( h_y = h_1. \)
Then, $\partial_1 B_{X_0}$ splits into six faces:

- The faces $\partial^\pm_{s,o} B_{X_0} = \{(x^\pm_s(y), y) \mid h_y \leq h_1, |y_2| \leq \sqrt{R_1^2 - h_y^2}, y_1 \geq \sqrt{R_1^2 - y_2^2 - h_y^2}\}$, oriented by $\mp dy$.

- The faces $\partial^\pm_{c,o} B_{X_0} = \{(x^\pm_c(y), y) \mid h_y \leq h_1, |y_2| \leq 1, y_1 \geq \sqrt{R_1^2 - y_2^2 - h_y^2}\}$, oriented by $\pm dy$.

- The face $\partial_{e,i} B_{X_0} = \{(x^-_e(y), y) \mid h_y \leq h_1, y_1^2 + y_2^2 \leq 1\}$, oriented by $-dy$.

- The face $\partial_{s,i} B_{X_0} = \{(x^-_s(y), y) \mid h_y \leq h_1, y_1^2 + y_2^2 \leq 1\}$, oriented by $+dy$.

**Lemma 4.3.** The boundary of $B_{X_1}$ is the union of:

- The face $\partial_2 B_{X_1} = \partial(\Sigma^- \cap E_2) \times D_\mu(1)$.

- The face $\partial_{E,\mu} B_{X_1} = - \bigcup_{r \in [1,2]} \partial(\Sigma^- \cap E_{r+1}) \times \partial D_\mu(r)$.

- The face $\partial_3 B_{X_1} = (\Sigma^- \cap E_3) \times \partial D_\mu(2)$.

- The face $\partial_{E,L} B_{X_1} = \bigcup_{r \in [1,2]} (\partial E_{r+1}) \times \{(R_r, 0, \overline{1})\}$.

- The face $\partial_E B_{X_1} = \partial E_2 \times L^+_0(1)$.

- The face $\partial L B_{X_1} = -E_3 \times \partial L^+_0(2)$.

- The face $\partial_{\infty} B_{X_1} = G_{r^{-1}}(\{e_1\}) \cap p_0^{-1}(E_2 \times \{\infty\})$.

Among these faces, $\partial_1 B_{X_1}$ and $\partial_E B_{X_1}$ are contained in $\partial Y_1$. We are going to extend the half-line $D^0(y, -e_1)$ inside $E_1$ in order to cancel the faces of $\partial_1 B_{X_0}$ and these faces $\partial_1 B_{X_1}$ and $\partial_E B_{X_1}$. The goal of Section 4.2 is to obtain the following lemma.

**Lemma 4.4.** There exists a chain $B_{Y_1} \subset Y_1$ such that the codimension 1 faces of $B_{Y_1}$ are:

- the faces $-\partial_{e,i} B_{X_0}, -\partial_{s,i} B_{X_0}, -\partial^\pm_{s,o} B_{X_0}$, and $-\partial^\pm_{c,o} B_{X_0}$,

- the faces $-\partial_1 B_{X_1}$ and $-\partial_E B_{X_1}$,

- the face $\partial_{\infty} B_{Y_1} = \{(x, x = \infty, u = e_1) \mid x \in N_2 \cap E_1\}$, oriented by $-dx$,

- faces $(\partial B_{Y_1})_{1 \leq i \leq 3}$, which are contained in $p_0^{-1}((N_2 \cap E_1) \times N_1) \subset \partial W$ and described in Lemmas 4.2, 4.3, and 4.4.

---

12The union $- \bigcup_{r \in [1,2]} \partial(\Sigma^- \cap E_{r+1}) \times \partial D_\mu(r)$ is oriented as $- [1,2] \times \partial(\Sigma^- \cap E_{r+1}) \times \partial D_\mu(r)$.

13In $M^\circ$, $\partial L^+_0(2)$ reduces to the point $(R_2, 0, \overline{1})$ with a negative sign.
4.2 Construction of the chain $B_{Y_1}$

4.2.1 Cancellation of the faces $\partial_e B_{X_0}$ and $\partial_s B_{X_0}$

In this section, set $Y_c = \{y \in N_1 \mid 0 < h_y \leq h_1, y_1^2 + y_2^2 \leq 1\}$. Let $y$ be a point of $Y_c$.

If $h_y \geq h_2$, define $D^1(y, -e_1) = \{x - te_1 \in N_2 \cap E_1 \mid t > 0\}$, and orient it by $dt$. If $h_y \leq h_2$, define $D^1(y, -e_1)$ as the union of the following oriented arcs.

- The line segment $L_c^-(y) \subset N_2$ from $x_c^-(y)$ to $\partial N_2$ with direction $-e_1$. Let $x_c^-(y) = (-2 \cos(\eta_c), -2 \sin(\eta_c), \overline{y})$ be the intersection point of this line with $\partial N_2$ (with $\eta_c \in [-\frac{\pi}{2}, \frac{\pi}{2}]$).

- The circular arc from $x_c^-(y)$ to $x_c^S(y) = (-2, 0, \overline{y})$ given by the formula $t \in [0, 1] \mapsto (-2 \cos((1-t)\eta_c), -2 \sin((1-t)\eta_c), \overline{y})$.

- The arc of longitude $\{x \in \partial(\Sigma^- \cap E_2) \mid h_x \geq h_y, \omega_x = \omega_y\}$, from $x_c^S(y)$ to $x_c^+(y) = \left(-\sqrt{R_2^2 - h_y^2}, 0, \overline{y}\right)$.

- The circular arc from $x_c^S(y)$ to the point $x_c^L(y) = \left(-\sqrt{R_2^2 - h_y^2 - y^2}, y, \overline{y}\right)$, given by $t \in [0, 1] \mapsto \left(-\sqrt{R_2^2 - h_y^2 - y^2} \cos(\eta_s), -\sqrt{R_2^2 - h_y^2} \sin(\eta_s), \overline{y}\right)$, where $\eta_s \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ satisfies $x_c^L(y) = \left(-\sqrt{R_2^2 - h_y^2} \cos(\eta_s), -\sqrt{R_2^2 - h_y^2} \sin(\eta_s), \overline{y}\right)$.

- The line segment $L_s^-(y) \subset N_2$ from $x_s^L(y)$ to $x_s^-(y)$, which has direction $-e_1$.

Figure 8 depicts the curve $D(y, -e_1) = D^0(y, -e_1) \cup D^1(y, -e_1)$, where the dotted part on the right is not in the plane but in the longitude.

Figure 8: The curve $D(y, -e_1)$ in $N_3$ (left) or in $\Pi_y = \{x \mid \overline{x} = \overline{y}\}$ (right)

Lemma 4.5. Set $B_c = \{(x, y) \mid y \in Y_c, x \in D^1(y, -e_1)\}$ and orient it by $dy \wedge dt$, where $dt$ represents the orientation of $D^1(y, -e_1)$. The codimension 1 faces of $B_c$ are:

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• the face $-\partial_{e_1}B_{X_0}$,
• the face $-\partial_{s_1}B_{X_0}$,
• the face $\partial_B Y_1 = \{(x,y) \mid y \in \partial_c N_1, x \in D_1(y,-e_1)\}$, oriented by $dt \wedge \Omega(\partial_c N_1)$,
• the face $-\partial_t B_{X_1} = -\partial(\Sigma^- \cap E_2) \times \mathbb{D}_\mu(1)$.

If $(x,y) \in B_c$, and if $y \in \psi(\mathbb{R}^n)$, then $x \in \Sigma^-$.

Proof. The first two faces and their orientations directly follow from the construction of $D_1(y,-e_1)$. The next two faces correspond to $y_1^2 + y_2^2 = 1$, and $h_y = 0$, respectively. The face corresponding to $h_y = h_1$ is of codimension 2, since $D_1(y,-e_1)$ reduces to a point. Note the cancellation at $h_y = h_2$ since $x^c \in (y) = x^c(y)$ and $x^c \in (y) = x^c(y)$ for such a $y$.

\[ \square \]

4.2.2 Cancellation of $\partial_{e_1} B_{X_0}$, $\partial_{s_1} B_{X_0}$, $\partial_{s_0} B_{X_0}$, and $\partial_{t} B_{X_0}$

Set $\mathcal{Y}_s = \{y \mid h_y \leq h_1, 0 < |y_2| \leq \sqrt{R_1^2 - h_y^2}, y_1 \geq \sqrt{R_1^2 - h_y^2 - y_2^2}\}$.

In this section, for any $y \in \mathcal{Y}_s$, we are going to extend $D_0(y,-e_1)$ to a curve $D(y,-e_1)$ in $N_2$ such that $\partial D(y,-e_1) = -\{y\} \in M^o$. In order to do so, we will connect $x^- \in (y)$ and $x^+ \in (y)$, and, when they exist, we will connect $x^- \in (y)$ and $x^+ \in (y)$.

We split $\mathcal{Y}_s$ in three parts $\mathcal{Y}_s^1 = \{y \in \mathcal{Y}_s \mid h_2 \leq h_2, |y_2| \geq \sqrt{R_2^2 - h_y^2}\}$,

$\mathcal{Y}_s^2 = \{y \in \mathcal{Y}_s \mid h_y \leq h_2, 2 < |y_2| \leq \sqrt{R_2^2 - h_y^2}\}$, and $\mathcal{Y}_s^3 = \{y \in \mathcal{Y}_s \mid h_y \leq h_2, 0 < |y_2| \leq 2\}$.

First case: $y \in \mathcal{Y}_s^1$

In this case, the half-line starting at $y$ with direction $-e_1$ is contained in $N_2$, so we set $D(y,-e_1) = \{x \in N_2 \mid G(x,y) = e_1\}$.

Second case: $y \in \mathcal{Y}_s^2$

In this case, the half-line $\{x \in N_2 \mid G(x,y) = e_1\}$ meets $\partial_s N_2$ in two points $x^\pm_{s,2}(y)$ as in Figure 9. Let $\gamma_s(y)$ denote the circular arc contained in the half-circle $\partial_s N_2 \cap \{x \mid x_2y_2 > 0, \overline{\tau} = \overline{\gamma}\}$ from $x^\pm_{s,2}(y)$ to $x^\pm_{s,2}(y)$. Then, the line $D(y,-e_1)$ is the union of $\{x \in N_2 \mid G(x,y) = e_1\}$ and $\gamma_s(y)$.

Third case: $y \in \mathcal{Y}_s^3$

In this case, the half-line $\{x \in N_2 \mid G(x,y) = e_1\}$ meets $\partial_s N_2$ in two points $x^\pm_{s,2}(y)$ and meets $\partial_s N_2$ in two points $x^\pm_{s,2}(y)$ as in Figure 9. Let $\gamma_s(y)$ be defined as in the previous case, and let $\gamma_c(y)$ be the circular arc from $x^\pm_{s,2}(y)$ to $x^\pm_{c,2}(y)$ in the half-circle $\partial_c N_2 \cap \{x \mid x_2y_2 > 0, \overline{\tau} = \overline{\gamma}\}$. Then, the line $D(y,-e_1)$ is the union of $\{x \in N_2 \mid G(x,y) = e_1\}$, $\gamma_c(y)$, and $\gamma_s(y)$.

\[ \text{which coincide if } |y_2| = 2. \]
Figure 9 depicts the curves $D^1(y, -e_1) = D(y, -e_1) \cap (N_2 \cap E_1)$ in the plane $\Pi_y = \{ x \mid \overline{x} = \overline{y} \}$ for different values of $y$ in $\mathcal{Y}$, $s$. The two plain circles depict the boundary of $N_1$ and the two dotted circles depict the boundary of $N_2$. The orientations are given in the picture by the arrows.

**Figure 9:** The curves $D^1(y, -e_1)$ in $\Pi_y$.

**Lemma 4.6.** Set $B_s^{(1)} = \{(x, y) \mid y \in \mathcal{Y}, x \in D^1(y, -e_1)\} \subset Y_1$, and orient it with $dy \wedge dt$, where $dt$ is the orientation of the lines $D(y, -e_1)$. Set $\mathcal{Y}^0_y = \{ y \mid h_y \leq h_2, y_1 \geq \sqrt{R_1^2 - h_y^2}, y_2 = 0 \}$.

The codimension 1 faces of $B_s^{(1)}$ are:

- the faces $-\partial_{s,o}^+ B_{X_0}$, $-\partial_{s,o}^- B_{X_0}$, $-\partial_{c,o}^+ B_{X_0}$, and $-\partial_{c,o}^- B_{X_0}$,
- the face $\partial_{\infty} B_{Y_1}$,
- the face $\partial_2 B_{Y_1} = \{(x, y) \mid y \in \mathcal{Y}, y_1 = \sqrt{R_1^2 - h_y^2 - y_2^2}, x \in D^1(y, -e_1)\}$, which is contained in $p_{y}^{-1}((N_2 \cap E_1) \times \partial N_1)$, and which is oriented by $-dy_2 \wedge dh_y \wedge d\omega_y \wedge dt$,
- the face $\partial_3 B_s^{(1)} = \{(x, y) \mid y \in \mathcal{Y}^0, x \in C_s(y) \cup C_c(y)\}$ where $C_s(y)$ denotes $\Pi_y \cap \partial_s N_2$, oriented as a direct circle of the plane $\Pi_y$ (i.e. as the boundary of a disk), and $C_c(y)$ denotes the intersection $\Pi_y \cap \partial_c N_2$, with the opposite orientation. $\partial_3 B_s^{(1)}$ is oriented by $dt \wedge dy_1 \wedge dh_y \wedge d\omega_y$, where $dt$ is the orientation of the circle in which $x$ lies.

**Proof.** The first four faces follow from the fact that the line $D(y, -e_1)$ extends $D^0(y, -e_1)$. When $y \in \mathcal{Y}^0$ and $h_y = h_2$, $\gamma_s(y)$ and $\gamma_c(y)$ cancel each other, so that there is no face corresponding to $h_y = h_2$. Note that there is no discontinuity when $|y_2| = \sqrt{R_1^2 - h_y^2}$ since $\gamma_s(y)$ reduces to a point. There is no discontinuity when $|y_2| = 2$ either since $\gamma_c(y)$ reduces to a point.
When \( y_1 \) approaches infinity, we obtain the face \( \partial_\infty B_{Y_1} \), when \( y_1 \) goes to \( \sqrt{R_1^2 - h_y^2 - y_2^2} \), we obtain the face \( \partial_2 B_{Y_1} \), and when \( y_2 \) approaches 0, we obtain the face \( \partial_3 B_s^{(1)} \).

\[ \square \]

### 4.2.3 Cancellation of the face \( \partial_3 B_s^{(1)} \)

For any \( y \in Y^0_s \) such that \( h_y > 0 \), define \( A(y) \) as the annulus \( \{ x \in \partial N_2 \mid \omega_x = \omega_y, h_x \geq h_y \} \) and orient it in such a way that its boundary is \( C_s(y) \cup C_c(y) \).

**Lemma 4.7.** Set \( B_s^{(2)} = \{ (x, y) \mid y \in Y^0_s, h_y > 0, x \in A(y) \} \), and orient this chain by \( -\Omega(A(y)) \wedge dy_1 \wedge dh_y \wedge d\omega_y \), where \( \Omega(A(y)) \) denotes the orientation of the annulus \( A(y) \) in which \( x \) lies. The codimension 1 faces of \( B_s^{(2)} \) are:

- The face \( -\partial_3 B_s^{(1)} \).
- The face \( -\partial E_2 \times L_0^+ (1) = -\partial E B_{X_1} \).
- The face \( \partial_3 B_{Y_1} = \{ (x, y) \mid 0 < h_y \leq h_2, y_1 = \sqrt{R_1^2 - h_y^2}, y_2 = 0, x \in A(y) \} \), oriented by \( \Omega(A(y)) \wedge dh_y \wedge d\omega_y \), and contained in \( p_0^{-1}(N_2 \cup E_1) \times \partial N_1 \).

**Proof.** The face \( -\partial_3 B_s^{(1)} \) corresponds to the boundary of \( A(y) \). The face \( \partial E B_{X_1} \) appears when \( h_y \) approaches zero. The face corresponding to \( h_y = h_2 \) is of codimension 2, since \( A(y) \) degenerates to a circle. When \( y_1 = \sqrt{R_1^2 - h_y^2} \), we obtain the face \( \partial_3 B_{Y_1} \), and when \( y_1 \) approaches infinity, we obtain a face contained in \( \{ (x, y = \infty, u = e_1) \mid x \in \partial N_2 \} \), thus of codimension at least two.

\[ \square \]

### 4.2.4 Proof of Lemma 4.4 and definition of the chain in \( X \cup Y \)

Set \( B_{Y_1} = B_c + B_s^{(1)} + B^{(2)} \). The chain \( B_{Y_1} \) satisfies the conditions of Lemma 4.4.

Let \( S: C_2(N_3) \rightarrow C_2(N_3) \) and \( T: C_2(M^o) \rightarrow C_2(M^o) \) be the smooth maps defined on the interior of their respective domains by the formulas \( S(x, y) = (-x, -y) \) and \( T(x, y) = (y, x) \), and set \( B_{Y_2} = -ST(B_{Y_1}) \) and \( B_{X \cup Y} = B_{X_0} + B_{Y_1} + B_{X_1} + B_{Y_2} + B_{X_2} \). By construction, we have the following lemma.

**Lemma 4.8.** Let \( G_r \) be the map of Definition 2.5.

The chain \( \partial B_{X \cup Y} - G_r^{-1}(\{ e_1 \}) \) defines a cycle \( \delta_W \) of \( \partial W \subset W \).

**Proof.** For any \( 1 \leq i \leq 3 \), set \( \partial_i B_{Y_2} = -ST(\partial_i B_{Y_1}) \). Set \( \partial_i B_{X_2} = -\partial L_{5i} (2) \times E_3 \) and \( \partial_{\mu} B_{X_2} = (\partial \mathbb{D}_{\mu} (2)) \times (\Sigma^+ \cap E_3) \). Set \( \partial_{\mu} B_{X_2} = -ST(\partial_{\mu} B_{X_1}) \), and \( \partial_{E,L} B_{X_2} = -ST(\partial_{E,L} B_{X_1}) \), where the faces \( \partial_{\mu} B_{X_1} \) and \( \partial_{E,L} B_{X_1} \) are defined in Lemma 4.3.

The boundary of \( B_{X \cup Y} \) is the union of:

- The faces \( (\partial_i B_{Y_1})_{1 \leq i \leq 3} \) and \( (\partial_i B_{Y_2})_{1 \leq i \leq 3} \).
• The faces $\partial_{\ell,i}B_{X_i}$, $\partial_{\mu}B_{X_i}$, $\partial_{E,L}B_{X_i}$, $\partial_{L}B_{X_i}$ for $i \in \{1, 2\}$.

• The face $G_{\tau}^{-1}(\{e_1\}) \cap (X \cup Y) = (G_{\tau}(\partial C_2(M^\circ) \cup E_1))^{-1}(\{e_1\})$.

All the previous faces except the last one are in $\partial W$. Making the difference with $G_{\tau}^{-1}(\{e_1\})$ replaces the last part with $-G_{\tau}^{-1}(\{e_1\}) \cap W = -(G_{\tau}(\partial E_1))^{-1}(\{e_1\})$, which is contained in $\partial W$. □

4.3 Extension of the chain to $W$

4.3.1 Construction of $B_W$ up to Lemma 4.11

In this section, we prove that the cycle $\delta_W$ of Lemma 4.9 is null-homologous in $W$.

**Lemma 4.9.** There exists a chain $B_W \subset W$ such that $\partial B_W = -\delta_W$.

**Proof of Lemma 4.11 assuming Lemma 4.9.** Let $B_W$ be like in the lemma, so that $\partial (B_W + B_{X \cup Y}) = G_{\tau}^{-1}(\{e_1\})$. Set $B_T = \frac{1}{2}(B_W + B_{X \cup Y} + T(B_W + B_{X \cup Y}))$, so that $\partial B_T = \frac{1}{2}G_{\tau}^{-1}(\{-e_1, e_1\})$. Note also that if $c = (x, y) \in B_T$, and if $(x, y) \in p_0^{-1}(\psi(\mathbb{R}^n) \times M^\circ)$, the definition of $(B_{X_i})_{i \leq 3}$ and the construction of $B_{c}$ in Lemma 4.9 imply that $y$ lies in the closure $\Sigma^- \cup \Sigma^+$ of $\Sigma^\circ$ in $C_1(M^\circ)$.

This proves the first assertion of Lemma 4.11. It remains to prove that admissible propagators can be chosen so that $\partial B_W = \partial W$ as stated in the second part of Lemma 4.11.

The previous work with the trivial knot and the surfaces $((0\Sigma^+)^0, (0\Sigma^-)^0)$, yields an admissible propagator $B_0$ for $((0\Sigma^+)^0, (0\Sigma^-)^0, \psi_0)$.

Set $W_2 = p_b^{-1}(N_3 \times N_3) \cap W$ and $W_3 = W \setminus W_2$. Set $B_{W_2}^T = B_0 \cap W_2$.

The chain $\delta_{W_3}^T = \frac{1}{2}(\delta_W + T(\delta_W)) + \partial B_{W_2}^T$ is a cycle of $W_3$, which is null-homologous in $W$ because of Lemma 4.9. Since $W_3$ is a deformation retract of $W$, this implies that $\delta_{W_3}^T$ is a null-homologous cycle in $W_3$. Therefore, there exists $B_{W_3}$ such that $\partial B_{W_3}^T = -\delta_{W_3}^T$. Since $T(\delta_{W_3}^T) = \delta_{W_3}^T$, choose $B_{W_3}^T$ such that $T(B_{W_3}^T) = B_{W_3}^T$.

Set $B = \frac{1}{2}(B_{X \cup Y} + T(B_{X \cup Y})) + B_{W_2}^T + B_{W_3}^T$. Since the boundary of $\frac{1}{2}(B_{X \cup Y} + T(B_{X \cup Y})) + B_{W_2}^T$ is $\frac{1}{2}G_{\tau}^{-1}(\{-e_1, e_1\}) + \frac{1}{2}(\delta_W + T(\delta_W)) + \partial B_{W_2}^T$, the chain $B$ is as requested by Lemma 4.11. □

The rest of this section is devoted to the proof of Lemma 4.9.

Set $W_1 = p_b^{-1}(E_1 \times E_1)$. Note that $W \leftrightarrow W_1$ is a homotopy equivalence. In order to prove Lemma 4.9, it suffices to prove that the class $[\delta_W] \in H_{n+2}(W_1)$ is null. Lemma 4.9 directly follows from the following two lemmas.

**Lemma 4.10.** Let $M_W$ denote the cycle $p_b^{-1}((x, x) \mid x \in \partial \mathbb{D}(2)) = U M^\circ \partial \mathbb{D}(2)$.

With these notations, $H_{n+2}(W_1) = Q[M_W]$.  

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Proof. $W_1$ is nothing but $C_2(E_1)$, and $E_1$ is homotopic to the complement of $\psi(\mathbb{R}^n) \cup \{ \infty \} \subset S^{n+2}$. Let $\Delta_{E_1}$ denote the diagonal of $E_1 \times E_1$. The construction of the configuration space $C_2(E_1)$ implies that $C_2(E_1)$ has the homotopy type of its interior $E_1 \times E_1 \setminus \Delta_{E_1}$, so that $H_{n+2}(W_1) \cong H_{n+2}(E_1 \times E_1 \setminus \Delta_{E_1})$.

The Alexander duality implies that $H_* (E_1)$ is non trivial only in degree 0 and 1, and that $H_1 (E_1)$ is generated by $[\partial \mathbb{D}_\mu(2)]$. Then, $H_* (E_1) = 0$ for $* > 2$, and the long exact sequence associated to $E_1 \times E_1 \setminus \Delta_{E_1} \hookrightarrow E_1$ yields an isomorphism from $H_{n+3}(E_1^2, E_1^2 \setminus \Delta_{E_1})$ to $H_{n+2}(E_1^2 \setminus \Delta_{E_1})$.

The excision theorem yields an isomorphism between $H_{n+3}(E_1^2, E_1^2 \setminus \Delta_{E_1})$ and $H_{n+3}(\mathcal{N}(\Delta_{E_1}), \mathcal{N}(\Delta_{E_1}) \setminus \Delta_{E_1})$, where $\mathcal{N}(\Delta_{E_1})$ denotes a tubular neighborhood of $\Delta_{E_1}$. Since $M^0$ is parallelizable, $\mathcal{N}(\Delta_{E_1})$ is diffeomorphic to the trivial disk bundle $\Delta_{E_1} \times \mathbb{D}^{n+2}$, and

$$H_{n+3}(\mathcal{N}(\Delta_{E_1}), \mathcal{N}(\Delta_{E_1}) \setminus \Delta_{E_1}) \cong H_{n+3}(\Delta_{E_1} \times \mathbb{D}^{n+2}, \Delta_{E_1} \times (\mathbb{D}^{n+2} \setminus \{0\}))$$

$$\cong H_1(\Delta_{E_1}) \otimes H_{n+2}(\mathbb{D}^{n+2}, \partial \mathbb{D}^{n+2})$$

$$\cong H_1(\Delta_{E_1}) \otimes H_{n+1}(\partial \mathbb{D}^{n+2})$$

$$= \mathbb{Q}[\partial \mathbb{D}_\mu(2)] \otimes [\partial \mathbb{D}^{n+2}].$$

Therefore, $H_{n+2}(W_1) \cong \mathbb{Q}[\partial \mathbb{D}_\mu(2)] \otimes [\partial \mathbb{D}^{n+2}]$. This identification maps $[M_W]$ to $\pm[\partial \mathbb{D}_\mu(2)] \otimes [\partial \mathbb{D}^{n+2}]$. \qed

**Lemma 4.11.** There exists an $(n+2)$-chain $D_W$, with $\partial D_W \subset \partial W_1$, such that:

- $D_W$ is dual to $M_W$: $\langle D_W, M_W \rangle_{W_1} = \pm 1$.

- The intersection number $\langle D_W, \delta W \rangle_{W_1}$ is zero.

Since this lemma implies that $[\delta W] = 0 \in H_{n+2}(W_1)$, it implies Lemma 4.9. We are left with the proof of Lemma 4.11.

We will construct the chain $D_W = D_1 + D_2 + D_3$ as the sum of a chain $D_1$ defined in Lemma 4.12, a chain $D_2$ defined in Lemma 3.16, and a chain $D_3$ defined in Lemma 4.17.

### 4.3.2 Construction of the chain $D_1$

Fix a Seifert surface $\Sigma^0$ parallel to those used in the construction of the chain $B_{\Sigma \cup \gamma}$, such that $\Sigma^0 \cap N_3 = \{(r \cos(\frac{\varphi}{6}), r \sin(\frac{\varphi}{6}), \varphi) \mid \varphi \in \mathbb{R}, r \geq 0\} \cap N_3$, and let $\Sigma'$ denote $\Sigma^0 \cap E_1$. Fix an embedding $\varphi: [-1, 1] \times \Sigma' \to E_1$, such that $\varphi(0, x) = x$ for any $x \in \Sigma'$. This allows us to define a normal vector $n_x = \frac{\partial \varphi(0, x)}{||\partial \varphi(0, x)||}$ for any $x \in \Sigma'$. Let $\Sigma^+ \subset \Sigma'$ denote the parallel surface $\varphi(\{1\} \times \Sigma')$, and, for any $x \in \Sigma'$, let $x^+$ denote the associated point $\varphi(1, x)$ in $\Sigma^+$. Assume without loss of generality that $\Sigma^+ \cap N_3 = \{(r \cos(\frac{\varphi}{6}), r \sin(\frac{\varphi}{6}), \varphi) \mid \varphi \in \mathbb{R}, r \geq 0\} \cap (E_1 \cap N_3)$.

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**Lemma 4.12.** Set 

$$D_1 = \overline{p_{\delta}}^{-1}\{(\varphi(0,x),\varphi(t,x)) \mid (t,x) \in [0,1] \times \Sigma')\}.$$  

The closure adds the configurations \((x,x,[n_d]) \in \partial \Delta C_2(M^o)\) where \(x \in \Sigma'\). The intersection 

$$D_1 \cap M_W$$  

consists of the configuration \(c = (x_0,x_0,[n_{\alpha_0}])\) where \(x_0\) is the unique intersection point of \(\partial \mathbb{D}_\mu(2)\) and \(\Sigma'\). Orient \(D_1\) as \([0,1] \times \Sigma'\).

The boundary of \(D_1\) is the union of three codimension 1 faces:

- The face \(\Delta(\Sigma',\Sigma'^+)^+ = \{(x,x^+) \mid x \in \Sigma'\}\), oriented as \(\Sigma'\).
- The face \(\partial_1 D_1 = \{(x,x,[n_d]) \mid x \in \Sigma'\}\), oriented as \(\Sigma'\).
- The face \(\partial_2 D_1 = \{(x,\varphi(t,x)) \mid 0 < t \leq 1, x \in \partial \Sigma'\}\), oriented as \(-[0,1] \times \partial \Sigma'\).

Furthermore, the last two faces are contained in \(\partial W_1\).

**Proof.** This is a direct check. \(\square\)

Our chain \(D_W\) will be defined from \(D_1\) by gluing other pieces in order to cancel the face \(\Delta(\Sigma',\Sigma'^+)^+\), which is not contained in \(\partial W_1\).

### 4.3.3 Construction of the chain \(D_2\)

Let \(S'\) denote the closed surface obtained by gluing a disk \(\mathbb{D}^{n+1}\) and \(\Sigma'\) along their boundaries. The surface \(S'\) is oriented as \(\Sigma' \cup -\mathbb{D}^{n+1}\). Let \(S' \times S'^+\) denote the product of two copies of \(S'\), where the coordinates read \((x,y^+)\), so that \(\Sigma' \times \Sigma'^+ \subset W_1\) naturally embeds into \(S' \times S'^+\). Set \(\Delta(S',S'^+) = \{(x,x^+) \mid x \in S'\}\), and orient it as \(S'\).

**Notation 4.13.** Choose two families \((a^d_i)_{0 \leq d \leq n+1,1 \leq i \leq b_d}\) and \((z^d_i)_{0 \leq d \leq n+1,1 \leq i \leq b_d}\) of chains of \(S'\) such that:

- For any \(d \in \{0,\ldots,n+1\}\), the families \((a^d_i)_{1 \leq i \leq b_d}\) and \((z^d_i)_{1 \leq i \leq b_d}\) are two bases of \(H_d(S')\).
- For any \(d \in \{0,\ldots,n+1\}\), and any \((i,j) \in (b_d)^2\), \((\{a^d_i\},\{z^{n+1-d}_j\})_{S'} = \delta_{i,j}\).
- For any \(d \in \mathbb{Z}\) and any \(i \in b_d\), the chains \(a^d_i\) and \(z^d_i\) are contained in \(\Sigma' \cap E_3\).
- The chains \(a^0_1\) and \(z^0_1\) are two distinct points of \(\partial \Sigma'\), and \(a^{n+1}_1 = z^{n+1}_1 = S'\).
- For any \(d > \frac{n+1}{2}\), and any \(j \in b_d\), \(a^d_j = z^d_j\), and for any \(d < \frac{n+1}{2}\) and any \(j \in b_d\), \(z^d_j = (-1)^d a^d_j\).

Such a choice is possible as in Lemma 3.15 and the previous chains induce similar families \((a^d_i)^+_{0 \leq d \leq n+1,1 \leq i \leq b_d}\) and \((z^d_i)^+_{0 \leq d \leq n+1,1 \leq i \leq b_d}\) in \(S'^+\).
Lemma 4.14. We have the following equality in $H_{n+1}(S' \times S^+)$:

$$[\Delta(S', S^+)] = \sum_{d=0}^{n+1} \sum_{i \in \mathbb{D}} [a_i^d \times (z_i^{n+1-d})^+] .$$

Proof. The Künneth formula implies that $H_{n+1}(S' \times S^+)$ admits the two families $([a_i^d \times (z_i^{n+1-d})^+])_{0 \leq d \leq n+1, (i,j) \in \mathbb{D}}$ and $([z_i^d \times (a_j^{n+1-d})^+])_{0 \leq d \leq n+1, (i,j) \in \mathbb{D}}$ as bases. For any $(d, d') \in \{0, \ldots, n+1\}^2$, any $(i, j) \in \mathbb{D}$, and any $(i', j') \in \mathbb{D}^2$, we have the following duality property:

$$\langle [a_i^d \times (z_i^{n+1-d})^+], [z_j^{n+1-d'} \times (a_j^{d'})^+] \rangle_{S' \times S^+} = \delta_{i,i'} \delta_{j,j'} \delta_{d,d'} .$$

There exist coefficients such that $[\Delta(S', S^+)] = \sum_{d=0}^{n+1} \sum_{i \in \mathbb{D}} a_{i,j}^d [a_i^d \times (z_i^{n+1-d})^+]$, and the duality property above and the definition of $\Delta(S', S^+)$ yield

$$a_{i,j}^d = \langle [\Delta(S', S^+)], [z_i^{n+1-d} \times (a_j^{d})^+] \rangle_{S' \times S^+} = \delta_{i,j} .$$

Let $\mathbb{D}$ denote the $(n+1)$-disk $\mathbb{D}^{n+1}$, which we glued to $\Sigma'$ above. To express $\Delta(\Sigma', \Sigma^+)$ from this lemma, we study $\Delta(\mathbb{D}, \mathbb{D}^+) = \Delta(S', S^+) - \Delta(\Sigma', \Sigma^+)$. 

Lemma 4.15. There exists a chain $D_\delta$ in $\partial \Sigma' \times \partial \Sigma^+$ such that the chain $c^1_\delta = D_\delta - a_1^0 \times \mathbb{D}^+ - \mathbb{D} \times (z_1^0)^+ + \Delta(\mathbb{D}, \mathbb{D}^+)$ is a null-homologous cycle of $\mathbb{D} \times \mathbb{D}^+$.

Proof. Let $S$ (respectively $S^+$) denote the sphere that bounds the disk $\mathbb{D}$ (respectively $\mathbb{D}^+$), which we glued to $\Sigma'$ (respectively $\Sigma^+$). Note that $S = -\partial \Sigma'$. Assume without loss of generality that $a_1^0$ is the North Pole $P_N$ of the sphere $S$, and that $z_1^0$ is the South Pole $P_S$. Similarly define $P_S^+$ and $P_N^+$.

For any $x \in S \setminus \{P_N\}$, there exists a unique shortest geodesic parametrized with constant speed $(y^+_{x}(t))_{0 \leq t \leq 1}$ on the sphere $S^+$ going from $x^+$ to $P_N^+$. Set $D_\delta = \{(x, y^+_{x}(t)) | 0 \leq t \leq 1, x \in S \setminus \{P_N\}\}$, and orient it as $[0, 1] \times (S \setminus \{P_N\})$.

The boundary of $D_\delta$ is the union of three codimension 1 faces:

- The face $\{P_N\} \times S^+$.
- The face $+S \times \{P_N^+\}$.
- The face $-\partial \Delta(\mathbb{D}, \mathbb{D}^+)$.

The first face appears when $x$ approaches $P_N$, the second one when $t = 1$, and the third one when $t = 0$. This implies that $c^1_\delta$ is a cycle. Since $H_{n+1}(\mathbb{D} \times \mathbb{D}^+) = 0$, $c^1_\delta$ is null-homologous. 

\[ \square \]
Lemma 4.16. There exists a chain $D_2 \subset \Sigma' \times \Sigma'^+ \subset W_1$, such that $D_2 \cap \partial W_1 \subset \partial D_2$, and such that

$$\partial D_2 = \left( \sum_{d \in \mathbb{N}} \sum_{i \in \mathbb{N}_d} a_i^d \times (z_i^{n+1-d})^+ \right) + a_1^0 \times \Sigma' + \Sigma' \times (z_1^0)^+ + D_\delta - \Delta(\Sigma', \Sigma'^+).$$

Proof. Let $c_3^2 = c_3^0 - \Delta(S', S'^+) + \sum_{d=0}^{n+1} \sum_{i \in \mathbb{N}_d} a_i^d \times (z_i^{n+1-d})^+$.

Lemma 4.13 and Lemma 4.15 imply that $c_3^2$ is null-homologous in $S' \times S'^+$. Now $c_3^2$ reads

$$c_3^2 = \left( \sum_{d \in \mathbb{N}} \sum_{i \in \mathbb{N}_d} a_i^d \times (z_i^{n+1-d})^+ \right) + a_1^0 \times \Sigma' + \Sigma' \times (z_1^0)^+ + D_\delta - \Delta(\Sigma', \Sigma'^+).$$

Therefore, $c_3^2$ is a cycle of $\Sigma' \times \Sigma'^+$ and the class $[c_3^2]$ is null in $H_{n+1}(S' \times S'^+)$. The Künneth formula proves that $([a_i^d \times (z_i^{n+1-d})^+])_{1 \leq d \leq n, 1 \leq i \leq b_d}$ is a basis of $H_{n+1}(\Sigma' \times \Sigma'^+)$. Since it is a subfamily of the basis $([a_i^d \times (z_i^{n+1-d})^+])_{0 \leq d \leq n+1, 1 \leq i \leq b_d}$ of $H_{n+1}(S' \times S'^+)$, the inclusion map $H_{n+1}(\Sigma' \times \Sigma'^+) \to H_{n+1}(S' \times S'^+)$ is injective, and $[c_3^2] = 0$ in $H_{n+1}(\Sigma' \times \Sigma'^+)$. \(\square\)

At this point, $\partial(D_1 + D_2)$ is the sum of a chain contained in $\partial W_1$ and the chain $\left( \sum_{d \in \mathbb{N}} \sum_{i \in \mathbb{N}_d} a_i^d \times (z_i^{n+1-d})^+ \right)$, which is not contained in $\partial W_1$. It remains to define the chain $D_3$ in order to cancel $\left( \sum_{d \in \mathbb{N}} \sum_{i \in \mathbb{N}_d} a_i^d \times (z_i^{n+1-d})^+ \right)$.

### 4.3.4 Construction of the chain $D_3$

Recall that the unit normal bundle to the diagonal of $M^o \times M^o$ has been identified with the unit tangent bundle $UM^o$ of $M^o$, and that it is a piece of $\partial C_2(M^o)$.

Lemma 4.17. There exists a chain $D_3 \subset p_b^{-1}(E_3 \times E_3)$, which meets $\partial p_b^{-1}(E_3 \times E_3)$ only along $\partial D_3$, and such that $\partial D_3$ is the union of:

- The faces $-a_i^d \times (z_i^{n+1-d})^+$, for $d \in \mathbb{N}$ and $i \in b_d$.
- A finite collection of fibers $\varepsilon(x_i)U_{x_i}M \subset \partial W_1 \cap \partial C_2(M^o)$, for $1 \leq i \leq m$.

Furthermore, $\sum_{i=1}^m \varepsilon(x_i) = \frac{x(\Sigma')^{-1}}{2}$.
Proof. Recall that $H_*(E_3) = H_*(S^1)$. Then, for any $d \in \{2, \ldots, n\}$ and any $i \in b_d$, there exists $A_i^{d+1} \subset E_3$ such that $\partial A_i^{d+1} = a_i^d$. For any $i \in b_1$, there exists $(A_i^2)^0 \subset M^0$ such that $\partial(A_i^2)^0 = a_i^1$. Since

$$\langle (A_i^2)^0, \psi(R^n) \cup \{\infty\} \rangle_M = \langle (A_i^2)^0, \partial(\Sigma^+ \cup \{\infty\}) \rangle_M = [\partial((A_i^2)^0 \cap (\Sigma^+ \cup \{\infty\})) + \langle a_i^1, \Sigma^+ \cup \{\infty\} \rangle_M = 0,$$

the chain $(A_i^2)^0$ meets the knot in an even number of points $(x_1, \ldots, x_{2r})$ such that $x_{2i}$ and $x_{2i+1}$ have opposite signs. Cut $(A_i^2)^0$ along a disk $\delta_i$ around each of these points, and glue an annulus $[0, 1] \times S^1$ between $\partial\delta_i$ and $\partial\delta_{2i+1}$ for each $i$, so that the obtained chain $A_i^2$ does not meet the knot and the boundary of $A_i^2$ is $a_i^1$. It can be assumed that $A_i^2$ is contained in $E_3$.

Assume without loss of generality that the chains $(A_i^{d+1})_{i,d}$ have been chosen such that $A_i^{d+1}$ and $(z_i^{n+1-d})^+$ are transverse for any $i$ and $d$. Set $K_{i,d} = A_i^{d+1} \cap (z_i^{n+1-d})^+$ and $K = \bigcup_{d \in \mathbb{N}} \bigcup_{i \in b_d} K_{i,d}$. Define

$$D_3 = -\sum_{d \in \mathbb{N}} \sum_{i \in b_d} p_b^{-1} \left( (A_i^{d+1} \times (z_i^{n+1-d})^+) \setminus \Delta \right),$$

so that

$$\partial D_3 = -\sum_{d \in \mathbb{N}} \sum_{i \in b_d} a_i^d \times (z_i^{n+1-d})^+ + \sum_{x \in K} \varepsilon(x). U_x M,$$

where $(-1)^d \varepsilon(x)$ is the sign of the intersection point. For $d \in \mathbb{N}$ and $i \in b_d$,

$$\sum_{x \in K_{i,d}} \varepsilon(x) = (-1)^{d+1} \left\langle A_i^{d+1}, (z_i^{n+1-d})^+ \right\rangle_{M^0} = \text{lk} \left( a_i^d, (z_i^{n+1-d})^+ \right) = (-1)^{d+1} [V_{n+1-d}]_{i,i},$$

where $V_{n+1-d} = V_{n+1-d}(B, \tilde{B})$ as in Definition 2.14, with $B = (a_i^d)_{i,d}$ and $\tilde{B} = (z_i^d)_{i,d}$, so that

$$\sum_{x \in K} \varepsilon(x) = \sum_{d \in \mathbb{N}} \sum_{i \in b_d} \sum_{x \in K_{i,d}} \varepsilon(x) = \sum_{d \in \mathbb{N}} (-1)^{d+1} \text{Tr}(V_{n+1-d}).$$

Conclude with Lemma 2.27.

\hfill \Box

4.3.5 End of the proof of Lemma 4.11

Set $D_W = D_1 + D_2 + D_3$. By construction, $\partial D_W$ is the union of:

- The faces $\partial_1 D_1$ and $\partial_2 D_1$ of Lemma 4.12
- The faces $D_5, a_1^0 \times \Sigma^+$, and $\Sigma' \times (z_1^0)^+$ of Lemma 4.16
since the points $\partial L_2$, they are contained in $\partial N$ that $\partial W$ 

All these faces are contained in $\partial W_1$. Since $D_2 \subset \Sigma' \times \Sigma^+$, the chain $D_2$ does not meet $M_W$, which is contained in the diagonal. Since $D_3 \subset p_b^{-1}(E_3 \times E_3)$, $\langle D_3, M_W \rangle_{W_1} = 0$, and $\langle D_{W}, M_W \rangle_{W_3} = \pm 1$.

It remains to check that $\langle D_{W}, \delta_{W} \rangle_{W_3} = 0$ for the cycle $\delta_{W}$ of Lemma 4.8. Note that $\langle \delta_{W} \rangle = [\delta_{W} - \partial(B_{X \cup Y} \cap W_1)]$. But $\delta'_{W} = \delta_{W} - \partial(B_{X \cup Y} \cap W_1)$ is the union of:

- The faces $\partial_{\mu, 1} B = (\Sigma^- \cap E_2) \times (\partial D_{\mu}(1))$ and $\partial_{\mu, 2} B = (\partial D_{\mu}(1)) \times (\Sigma^+ \cap E_2)$.
- The faces $-\partial L_0^-(1) \times E_2$ and $-E_2 \times \partial L_0^+(1)$.
- The faces $\langle \partial_{1} B_{Y_1} \rangle_{1 \leq i \leq 3}$ and $\langle \partial_{2} B_{Y_2} \rangle_{1 \leq i \leq 3}$.
- The face $-G_{\tau}^{-1}(\{e_1\}) \cap W_1$.

The faces $\partial L_0^-(1) \times E_2, E_2 \times \partial L_0^+(1), \partial_{\mu, 1} B$ and $\partial_{\mu, 2} B$ cannot meet $D_{W}$: indeed, they are contained in $\partial N_1 \times E_2$ or $E_2 \times \partial N_1$, so they do not meet $D_1$ or $D_3$, and since the points $\partial L_0^-(1)$ are on the Seifert surfaces $\Sigma^+$ and $\Sigma^-$, and since $\Sigma'$ and $\Sigma^+$ do not meet the surfaces $\Sigma^\pm$, these faces do not meet $D_2$, either.

For $1 \leq i \leq 3$, let us study the intersection of the faces $\partial_{1} B_{Y_1} \subset p_b^{-1}((N_2 \cap E_1) \times \partial N_1)$ and $\partial_{2} B_{Y_2} \subset p_b^{-1}(\partial N_1 \times (N_2 \cap E_1))$ with $D_{W}$:

- They cannot meet $D_3$, which is contained in $p_b^{-1}(E_3 \times E_3)$.
- They could meet $D_1$ along $D_1 \cap \partial(X \cup Y) = \partial_{2} D_1$, which is composed of configurations where the two points are in $\partial N_1$. The choice of the longitudes $\partial \Sigma'$ and $\partial \Sigma^+$, and the description of these faces in Lemmas 4.3, 4.6, 4.7 imply that any configuration $c = (x, y) \in \partial N_1 \times \partial N_1$ in one of these faces is such that $\frac{y-x}{||y-x||} = e_1$. Figure 10 shows that this never happens when $(x, y) \in \partial_{2} D_1$. 

Figure 10: Dotted line: The surfaces $\Sigma'$ and $\Sigma^+$ inside $N_3 \cap \Pi_x$ for any $x$ such that $h_x < h_1$. Dashed line: The points $x \in N_2 \cap E_1$ such that $(x, (z_1^0)^+)$ or $(a_1^0, x)$ lies in $\partial_{1} B_{Y_1}$ or $\partial_{2} B_{Y_2}$.
Eventually, they could meet $D_2$ along $a_0^1 \times \Sigma^+$ and $\Sigma' \times (z_0^1)^+$, which would necessarily happen inside $a_1^0 \times (\Sigma'^+ \cap (N_2 \cap E_1))$ or $(\Sigma' \cap (N_2 \cap E_1)) \times (z_0^1)^+$. Assume without loss that $a_1^0 = (\cos(\pi/6), \sin(\pi/6), 0)$ and that $(z_0^1)^+ = (18 \cos(\pi/3), 18 \sin(\pi/3), 0)$. In this case, we get no intersection points, as it can be seen on Figure 10.

Therefore, these faces do not meet $D_W$.

We are left with the proof that $\langle D_W, -G_\tau^{-1}(\{e_1\}) \cap W_1 \rangle = 0$. We will use the following lemma since this intersection is contained in the faces of $D_W$.

**Lemma 4.18.** Let $P$ be an oriented manifold with boundary, let $Q$ be a submanifold of $P$, and let $R$ be a submanifold of $\partial P$. Assume that

- the submanifold $Q$ meets $\partial P$ along its boundary: $Q \cap \partial P \subset \partial Q$, and this intersection is transverse,
- the submanifolds $Q \cap \partial P$ and $R$ are transverse in $\partial P$.

The submanifolds $Q$ and $R$ are transverse in $P$ and $\langle Q, R \rangle_P = \langle \partial Q \cap \partial P, R \rangle_{\partial P}$.

**Proof.** The lemma follows from a direct computation. □

The only configurations of $D_W$ where the two points collide with $u = \tau_x(e_1)$ are:

- Those coming from the faces $\varepsilon(x_i)U_x, M \subset \partial D_3$. Their contribution is

\[
\langle D_3, -G_\tau^{-1}(\{e_1\}) \cap W_1 \rangle_{W_1} = \left\langle \sum_{i=1}^m \varepsilon(x_i)U_x, M, -G_\tau^{-1}(\{e_1\}) \cap W_1 \right\rangle_{\partial W_1} = -\sum_{i=1}^m \varepsilon(x_i) = 1 - \chi(\Sigma')/2
\]

- Those coming from $\partial_1 D_1$. Assume without loss of generality that $e_1$ is a regular value of the map $\varphi_n : x \in \Sigma' \mapsto \tau_x^{-1}(n_x) \in S^{n+1}$. Their contribution is

\[
\langle D_1, -G_\tau^{-1}(\{e_1\}) \cap W_1 \rangle_{W_1} = \langle \partial_1 D_1, -G_\tau^{-1}(\{e_1\}) \cap W_1 \rangle_{\partial W_1} = + \deg_{e_1}(\varphi_n),
\]

where $\deg_y(\varphi_n)$ is the differential degree of $\varphi_n$ at $y$, and where the plus sign comes from the fact that the face $\partial_1 D_1$ of Lemma 4.12 is oriented as $-\Sigma'$. 48
The proof of Lemma 4.11 is now completed by the following lemma.

**Lemma 4.19.** Let \( \varphi_n \) be the map \( x \in \Sigma' \mapsto \tau_x^{-1}(n_x) \in \mathbb{S}^{n+1} \).

The differential degree of \( \varphi_n \) may be extended to the constant map on \( \mathbb{S}^{n+1} \) with value \( \frac{\chi(\Sigma') - 1}{2} \).

**Proof.** Note that for any \( x \in \Sigma' \cap N_3, \varphi_n(x) = (\cos(\frac{2\pi}{3}), \sin(\frac{2\pi}{3}), 0) \). All the boundary of \( \Sigma' \) is mapped by \( \varphi_n \) to one point in \( \mathbb{S}^{n+1} \). This implies that the differential degree of \( \varphi_n \) does not depend on the chosen regular value in \( \mathbb{S}^{n+1} \).

Assume without loss that \( \varphi_n \) admits \( -e_1 \) and \( e_1 \) as regular values.

For any \( x \in \Sigma' \), define the projection \( X(x) \) of \( \tau_x(e_1) \) on \( T_x \Sigma' \) along the direction \( n_x \) (which is the only vector of \( T_x \Sigma' \) that can be expressed as \( \tau_x(e_1) - \lambda n_x \) for some \( \lambda \in \mathbb{R} \)). This defines a tangent vector field \( X \) on \( \Sigma' \), whose zeros are the points such that \( \varphi_n(x) = \pm e_1 \). Around such a zero \( z \), \( \varphi_n \) is a local diffeomorphism from a disk around \( z \) to a disk inside \( \mathbb{S}^{n+1} \). In this setting, the index \( i(X, z) \) of the zero is \( +1 \) if and only if this local diffeomorphism preserves the orientation. This implies that \( \sum_{z \text{ zero of } X} i(X, z) = \deg_{e_1}(\varphi_n) + \deg_{-e_1}(\varphi_n) \). Since \( \deg(\varphi_n) \) does not depend on the regular value, \( \deg(\varphi_n) = \frac{1}{2} \sum_{z \text{ zero of } X} i(X, z) \).

Let \( \mathbb{D} \subset \Sigma' \) be the set \( \{(r \cos(\frac{\pi}{6}), r \sin(\frac{\pi}{6}), \pi) \mid 1 \leq r \leq 2, \pi \in \mathbb{R}^n \}\) \( \cap E_1 \), as depicted in Figure 11. This is an \( (n + 1) \)-disk on which \( X \) takes a constant value \( X_0 \neq 0 \). Change the vector field \( X \) on \( \mathbb{D} \) so that it keeps the same value on \( \partial \mathbb{D} \setminus \partial \Sigma' \) but is going outwards on all \( \mathbb{D} \cap \partial \Sigma' \). The obtained vector field \( X' \) is going outwards on \( \partial \Sigma' \) and \( X'|_{\mathbb{D}} \) is going outwards on \( \partial \mathbb{D} \) as in Figure 11. The zeros of \( X' \) are the union of those of \( X \) with same indices (which are in \( \Sigma' \setminus \mathbb{D} \)) and those of \( X'|_{\mathbb{D}} \).

In this setting, Poincaré-Hopf theorem (see for example [Mil65, Section 6, p 35]) yields \( \sum_{z \text{ zero of } X'} i(X', z) = \chi(\Sigma' \cup \mathbb{D}) = \chi(\Sigma') \), and \( \sum_{z \text{ zero of } X'|_{\mathbb{D}}} i(X', z) = 1 \).

The difference of these two formulas gives \( \sum_{z \text{ zero of } X} i(X, z) = \chi(\Sigma') - 1 \), and implies the lemma.

Figure 11: Left: The surface \( \Sigma' \) with the darker disk \( \mathbb{D} \), and the vector field \( X \). The hashed area depicts \( \Sigma' \cap E_2 \), which is not necessarily a disk as in the picture.

Right: The modified field \( X' \) on \( \mathbb{D} \), which points outwards on the boundary.
5 On virtual rectifiability

5.1 Proof of Lemma 2.21

Lemma 5.1. Let \((I_t)_{0 \leq t \leq 1}\) be a homotopy of maps \((\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{D}^n) \to (\mathbb{I}(\mathbb{R}^n, \mathbb{R}^{n+2}), i_0)\) with \(I_0(\mathbb{R}^n) = \{i_0\}\). Let \(\mathcal{G}\) denote the space of smooth maps from \(\mathbb{R}^n\) to \(GL_{n+2}(\mathbb{R})\) that map \(\mathbb{R}^n \setminus \mathbb{D}^n\) to \(I_{n+2}\).

There exists a continuous map \(t \in [0, 1] \mapsto g_t \in \mathcal{G}\), such that for any \((t, x) \in [0, 1] \times \mathbb{R}^n\), \(I_t(x) = g_t(x) \circ I_0(x)\), and such that, for any \(x \in \mathbb{R}^n\), \(g_0(x) = I_{n+2}\).

Proof. Set \(g_0(x) = I_{n+2}\) for any \(x\). Endow \(\mathbb{R}^{n+2}\) with its canonical Euclidean structure and let \(P_{t,x}\) denote the orthogonal complement of \(I_t(x)(\mathbb{R}^n)\) in \(\mathbb{R}^{n+2}\). Let \(\pi_{t,x}\) denote the orthogonal projection on \(P_{t,x}\). Set \(f(t, t_0, x) = \min_{z \in \pi_{0, x}(\mathbb{R}^n)} ||\pi_{t, x}(z)||\).

Since \(f\) maps the complement of the compact \([0, 1]^2 \times \mathbb{R}^n\) to \(1\), it is uniformly continuous. Fix \(\delta > 0\) so that for any \((t, t_0, x)\) and \((t', t_0', x')\) with \(|t-t'|+|t_0-t_0'|+|x-x'|| < \delta\), \(|f(t', t_0', x') - f(t, t_0, x)| < \frac{1}{2}\), and for any \(j \in \mathbb{N}\), set \(t_j = \min(j, 2, 1)\).

We are going to define \(g_t\) on each \([t_j, t_{j+1}]\).

Note that \(P_{0,x} \cap P_{t,x}^\perp = \{0\}\) if and only if \(f(t, 0, x) > 0\). Since \(f(0, 0, x) = 1\) for any \(x\), we have \(P_{0,x} \cap P_{t,x}^\perp = \{0\}\) for \(0 \leq t \leq t_1\). For \(0 \leq t \leq t_1\), define \(g_t(x)\) by the following formula:

\[
\forall z = (z_1, z_2, \overline{z}) \in \mathbb{R}^{n+2} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, g_t(x)(z_1, z_2, \overline{z}) = \pi_{t,x}(\pi_{0,x}(z)) + I_t(x)(\overline{z})
\]

Since \(P_{0,x} \cap P_{t,x}^\perp = \{0\}\), \(\pi_{t,x}\) defines an isomorphism from \(P_{0,x}\) to \(P_{t,x}\). Thus, \(g_t(x)\) is an isomorphism. For \(t_k \leq t \leq t_{k+1}\) and \(x \in \mathbb{R}^n\) define \(g_t(x)\) so that

\[
\forall z = (z_1, z_2, \overline{z}) \in \mathbb{R}^{n+2}, g_t(x)(z_1, z_2, \overline{z}) = \pi_{t,x}(\pi_{t,x}(\cdots \pi_{t,0,x}(z) \cdots)) + I_t(x)(\overline{z}).
\]

Since \(f(t, t_k, x) \geq f(t_k, t_k, x) - \frac{1}{2} = \frac{1}{2}\), the above method proves that \(g_t(x)\) is an isomorphism. This defines a family \((g_t)_{0 \leq t \leq 1}\) as required by the lemma.

Proof of Lemma 2.24. Let \(\tau\) be a parallelization such that the class \([\iota(\tau, \psi)]\) of Lemma 2.19 is zero, so that there exists \((I_t)_{0 \leq t \leq 1}\) as in Lemma 5.1 with \(I_1 = \iota(\tau, \psi)\). Let \((\tilde{g}_t)_{0 \leq t \leq 1}\) be a smooth approximation of the map \((g_t)_{0 \leq t \leq 1}\) of Lemma 5.1 such that for any \(x \in \mathbb{R}^n\), \(\tilde{g}_0(x) = I_{n+2}\) and \(I_1(x) = \iota(\tau, \psi)(x) = \tilde{g}_1(x) \circ I_0(x)\). Assume without loss of generality that \((t \in [0, 1] \mapsto \tilde{g}_t \in \mathcal{G})\) is constant on a neighborhood of \(\{0, 1\}\). Take a tubular neighborhood \(N\) of \(\psi(\mathbb{R}^n)\) and identify \(N\) with \(\psi(\mathbb{R}^n) \times \mathbb{D}^2\) with coordinates \((\psi(x), r, \theta)\). For any \(y = (\psi(x), r, \theta) \in N\), set \(\tau_y = (\tau_e)_y \circ \tilde{g}_1 \circ \tau_e\). This defines a map \(\tau : N \times \mathbb{R}^{n+2} \to TN\), which extends to a map \(\tau' : M^o \times \mathbb{R}^{n+2} \to TM^o\), by setting \((\tau')_y = (\tau_e)_y\) when \(y \not\in N\). This construction ensures that \(\iota(\tau', \psi) = I_0\), and \(\tau'\) is a parallelization of \(M^o\).
5.2 Case \( n \equiv 5 \mod 8 \)

We use the following Bott periodicity theorem, which is proved in [Bot57].

**Theorem 5.2.** [Bott] For any \( k \geq 0 \), and any \( N \geq 1 \),

\[
\pi_N(\text{SO}(N + 2 + k), I_{N+2+k}) = \begin{cases} 
0 & \text{if } N \equiv 2, 4, 5 \text{ or } 6 \mod 8, \\
\mathbb{Z}/2\mathbb{Z} & \text{if } N \equiv 0 \text{ or } 1 \mod 8, \\
\mathbb{Z} & \text{if } N \equiv 3 \text{ or } 7 \mod 8.
\end{cases}
\]

This yields the following corollary, which is the first assertion of Lemma 2.23.

**Corollary 5.3.** Suppose \( n \equiv 5 \mod 8 \), and let \( M^\circ \) be an asymptotic homology \( \mathbb{R}^{n+2} \). If \( M^\circ \) is parallelizable, then all long knots \( \psi: \mathbb{R}^n \hookrightarrow M^\circ \) are rectifiable.

**Proof.** As stated in Lemma 5.2, \( \pi_n(\mathcal{I}(\mathbb{R}^n, \mathbb{R}^{n+2}), \iota_0) = \pi_n(\text{SO}(n + 2), I_{n+2}) \). Since \( n \equiv 5 \mod 8 \), \( \pi_n(\mathcal{I}(\mathbb{R}^n, \mathbb{R}^{n+2}), \iota_0) = 0 \). Then \( \pi_n(\mathcal{I}(\mathbb{R}^n, \mathbb{R}^{n+2}), \iota_0) = 0 \), and, if \( M^\circ \) is parallelizable, the hypothesis of Lemma 2.21 is satisfied for any knot.

In the non-parallelizable case, \( M^\circ \# M^\circ \) is parallelizable because of Proposition 2.12, and the previous argument applies to \( \psi \# \psi \).

\( \square \)

5.3 Case \( n \equiv 1 \mod 8 \) and connected sum of long knots

The following lemma concludes the proof of Lemma 2.23.

**Lemma 5.4.** When \( n \equiv 1 \mod 8 \), for any long knot \( \psi \) in a parallelizable asymptotic homology \( \mathbb{R}^{n+2} \), the connected sum \( \psi \# \psi \) is rectifiable. Therefore, for any long knot \( \psi \) in a (possibly non-parallelizable) asymptotic homology \( \mathbb{R}^{n+2} \), the connected sum \( \psi \# \psi \# \psi \# \psi \) is rectifiable.

**Proof.** Let \( (M^\circ, \tau) \) be a parallelized asymptotic homology \( \mathbb{R}^{n+2} \), let \( (M^\circ \# M^\circ, \tau \# \tau) \) be the induced connected sum, and fix a long knot \( \psi: \mathbb{R}^n \hookrightarrow M^\circ \). Since \( \psi \# \psi \) is defined by stacking two copies of the knot, \( \iota(\tau \# \tau, \psi \# \psi) \) is the map defined by stacking two copies of \( \iota(\tau, \psi) \). In terms of homotopy classes in \([([\mathbb{R}^n, B^\infty_{\infty,n}], (\mathcal{I}(\mathbb{R}^n, \mathbb{R}^{n+2}), \iota_0)] = \pi_n(\mathcal{I}(\mathbb{R}^n, \mathbb{R}^{n+2}), \iota_0) \), this implies \([\iota(\tau \# \tau, \psi \# \psi)] = 2[\iota(\tau, \psi)]. \) Lemma 5.2 and Theorem 5.2 yield \( \pi_n(\mathcal{I}(\mathbb{R}^n, \mathbb{R}^{n+2}), \iota_0) = \mathbb{Z}/2\mathbb{Z} \). This implies \([\iota(\tau \# \tau, \psi \# \psi)] = 0 \). Lemma 2.21 implies that \( \psi \# \psi \) is rectifiable.

In the non-parallelizable case, \( M^\circ \# M^\circ \) is parallelizable because of Proposition 2.12 and the previous argument applies to \( \psi \# \psi \).

\( \square \)

Note that since \( \pi_n(\text{SO}(n + 2), I_{n+2}) = \mathbb{Z} \) for \( n \equiv 3 \mod 4 \), the same method implies that \( \psi \# \psi \) is virtually rectifiable if and only if \( \psi \# \psi \) is rectifiable (otherwise the class \( \langle \psi \# \psi \rangle \) of Definition 2.22 has infinite order). This argument together with Corollary 5.3 and Lemma 5.4 yields the following remark.
Remark 5.5. Let $M^o$ be an asymptotic homology $\mathbb{R}^{n+2}$ and let $\psi$ be a long knot of $M^o$. Then, $\psi$ is virtually rectifiable if and only if $\psi \bar{\psi} \psi \bar{\psi}$ is rectifiable.

6 Proof of Theorem 2.29

6.1 A generating series for the $(\lambda_{k,\nu})_{k \geq 2, \nu \in k-1}$

In this section, we prove the following result for the coefficients $(\lambda_{k,\nu})_{k \geq 2, \nu \in k-1}$ of Theorem 2.24.

Lemma 6.1. For any $k \geq 2$, set $L_k(X) = \sum_{\nu=1}^{k-1} \lambda_{k,\nu} X^\nu$, set $L_1(X) = \frac{X+1}{2}$, and define the formal power series $L(X,Y) = \sum_{k \geq 1} L_k(X) Y^{k-1} \in \mathbb{Q}[[X,Y]]$. Then,

$$L(X,Y) = \frac{1 - X}{2} \frac{1 + X \exp((1 - X)Y)}{1 - X \exp((1 - X)Y)}.$$ 

In order to prove Lemma 6.1, we first obtain an induction formula for the coefficients $(\lambda_{k,\nu})_{k \geq 2, \nu \in k-1}$ in Lemma 6.2. We next derive an induction formula for the polynomials $(L_k)_{k \geq 1}$ in Lemma 6.3, and a differential equation on $L(X,Y)$ in Lemma 6.4.

Lemma 6.2. Extend the definition of the coefficients $(\lambda_{k,\nu})_{k \geq 2, \nu \in k-1}$ to $(k, \nu) \in \mathbb{Z} \times \mathbb{Z}$ by setting $\lambda_{k,\nu} = 0$ when $\nu \notin \{1, \ldots, k-1\}$ or $k \leq 1$. For any $k \geq 3$,

$$(k-1)\lambda_{k,\nu} = \lambda_{k-1,\nu} + \lambda_{k-1,\nu-1} + \sum_{r=2}^{k-2} \sum_{p=0}^{r-2} \lambda_{r,p} \lambda_{k-r,\nu-p}.$$ 

Proof. By definition, for any $k \geq 3$ and any $\nu \in k-1$, $\lambda_{k,\nu} = \frac{1}{(k-1)!} \text{Card}(\{\sigma \in \mathcal{S}_{k-1} | N(\sigma) = \nu - 1\})$, where $N(\sigma) = \text{Card}(\{i \in k-2 | \sigma(i) < \sigma(i+1)\})$.

Let $\sigma \in \mathcal{S}_{k-1}$, and set $r_\sigma = \sigma^{-1}(k-1)$, $I_\sigma = \{1, \ldots, r_\sigma - 1\}$, and $J_\sigma = \{r_\sigma + 1, \ldots, k-1\}$. Let $i_\sigma: \sigma(I_\sigma) \rightarrow I_\sigma$ and $j_\sigma: \sigma(J_\sigma) \rightarrow J_\sigma$ denote the two only such maps that are strictly increasing bijections. The permutation $\sigma$ induces two permutations $\sigma_1 = i_\sigma \circ \sigma|_{I_\sigma} \in \mathcal{S}_{I_\sigma}$ and $\sigma_2 = j_\sigma \circ \sigma|_{J_\sigma} \in \mathcal{S}_{J_\sigma}$.

- If $r_\sigma = k-1$, $N(\sigma_1) = N(\sigma) - 1$ and $\sigma_2 \in \mathcal{S}_0$.
- If $r_\sigma = 1$, $N(\sigma_2) = N(\sigma)$, and $\sigma_1 \in \mathcal{S}_0$.
- If $2 \leq r_\sigma \leq k-2$, then $N(\sigma) = 1 + N(\sigma_1) + N(\sigma_2)$, since the elements $i \in k-2$ such that $\sigma(i) < \sigma(i+1)$ are taken into account in $N(\sigma_1)$ if $i < r_\sigma - 1$, in $N(\sigma_2)$ if $i \geq r_\sigma + 1$, and since $\sigma(r_\sigma - 1) < \sigma(r_\sigma)$ and $\sigma(r_\sigma) > \sigma(r_\sigma + 1)$.

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Now, note that \( \sigma \) is equivalent to the data of \((r_\sigma, \sigma(I_\sigma), \sigma_1, \sigma_2)\), and that, for a given \(r_\sigma\), there are \(\binom{k-2}{r_\sigma-1}\) possible choices of \(\sigma(I_\sigma)\). This yields the induction formula of the lemma.

**Lemma 6.3.** The polynomial \( L_2(X) \) is \(X\), and for any \(k \geq 3\),

\[
L_k(X) = \frac{1}{k-1} \sum_{r=1}^{k-1} L_r(X)L_{k-r}(X).
\]

**Proof.** The first point of the lemma is immediate since \(\lambda_{2,1} = 1\). For \(k \geq 3\),

\[
(k-1)L_k(X) = \sum_{\nu \in k-1} (k-1)\lambda_{k,\nu}X^\nu
= \sum_{\nu \in k-1} \left(\lambda_{k-1,\nu} + \lambda_{k-1,\nu-1} + \sum_{r=2}^{k-2} \sum_{p \geq 0} \lambda_{r,p} \lambda_{k-r-p,\nu-p}\right)X^\nu
= L_{k-1}(X) + XL_{k-1}(X) + \sum_{r=2}^{k-2} \sum_{\nu \in k-1} \sum_{p \geq 0} \lambda_{r,p} X^p \lambda_{k-r-p,\nu-p}X^{\nu-p}
= (X+1)L_{k-1}(X) + \sum_{r=2}^{k-2} L_r(X)L_{k-r}(X)
= \sum_{r=1}^{k-1} L_r(X)L_{k-r}(X),
\]

since \(L_1(X)L_{k-1}(X) + L_{k-1}(X)L_1(X) = (X+1)L_{k-1}(X).\)

**Lemma 6.4.** \(L(X,Y)\) satisfies the differential equation

\[
\frac{\partial L}{\partial Y}(X,Y) = (L(X,Y))^2 - \left(\frac{1-X}{2}\right)^2.
\]

**Proof.** Indeed,

\[
\frac{\partial L}{\partial Y}(X,Y) = \sum_{k \geq 2} L_k(X)(k-1)Y^{k-2}
= L_2(X) + \sum_{k \geq 3} \sum_{r=1}^{k-1} L_r(X)Y^{r-1}L_{k-r}(X)Y^{k-r-1}
= L_2(X) + \sum_{k \geq 1} t_k(X)Y^k,
\]

where \((L(X,Y))^2 = \sum_{k \geq 0} t_k(X)Y^k\). Since \(L(X,Y) = \frac{(X+1)}{2} + \sum_{k \geq 2} L_k(X)Y^{k-1}\), we have \(t_0(X) = \frac{(X+1)^2}{4}\), and

\[
\frac{\partial L}{\partial Y}(X,Y) = X + L(X,Y)^2 - \frac{(X+1)^2}{4} = L(X,Y)^2 - \left(\frac{1-X}{2}\right)^2.
\]
Proof of Lemma 6.1. Since $|L_k(x)| \leq 1$ for any $x \in [-1, 1]$, $L(X, Y)$ defines a power series that converges at least on $]-1, 1[$. Fix $x \in ]0, \frac{1}{2}[$, and set $u_x(t) = L(x, t)$ for any $t \in ]-1, 1[$. The function $u_x$ satisfies the equation $u'_x = (u_x)^2 - \left(\frac{1-x^2}{2}\right)^2$. Set $a = \frac{1-x^2}{2}$, and note that

$$\int_0^t \frac{u'_x(t)}{(u_x(t))^2 - a^2} dt = \frac{1}{2a} \left( \ln \left( \frac{u_x(t) - a}{u_x(t) + a} \right) - \ln \left( \frac{u_x(0) - a}{u_x(0) + a} \right) \right),$$

so that, for any $t$,

$$\ln \left( \frac{(u_x(t) - a)(u_x(0) + a)}{(u_x(t) + a)(u_x(0) - a)} \right) = 2at.$$

Since $u_x(0) = \frac{x+1}{2}$ and $a = \frac{1-x^2}{2}$, this yields the formula of Lemma 6.1. Both sides of the formula of Lemma 6.1 are power series with a convergence domain containing a disk around $(0, 0)$, so that the formula also holds for the formal power series. □

6.2 The formula with the Reidemeister torsion

Lemma 6.5. For any virtually rectifiable long knot $\psi$,

$$\sum_{k \geq 2} Z_k(\psi) h^k = -\ln(T_\psi(e^h)).$$

Proof. Let $\Sigma$ be a Seifert surface for $\psi$, let $(\mathcal{B}, \mathcal{\tilde{B}})$ be a pair of dual bases of $H_*(\Sigma)$, and set $V^+_d = V^+_d(\mathcal{B}, \mathcal{\tilde{B}})$. Corollary 2.25 yields

$$\sum_{k \geq 2} Z_k(\psi) h^k = \sum_{d \in \mathbb{Z}} \sum_{k \geq 2} \sum_{\nu \in k-1} (-1)^{d+1} \lambda_{k, \nu} \frac{h^k}{k} \text{Tr} \left((V^+_d)^\nu (V^-_d)^{k-\nu}\right)$$

$$= \sum_{d \in \mathbb{Z}} (-1)^{d+1} \text{Tr} \left( M(hV^+_d, hV^-_d) \right),$$

where $M(X, Y) = \sum_{k \geq 2} \sum_{\nu \in k-1} \frac{1}{k} \lambda_{k, \nu} X^\nu Y^{k-\nu}$. Note that

$$M(X, Y) = \int_0^Y \left( L(XY^{-1}, T) - L_1(XY^{-1}) \right) dT.$$
Lemma 6.1 and basic integral calculus yield

\[ M(X,Y) = \int_0^Y \left( \frac{1 - XY^{-1}}{2} \left( 1 + \frac{2X \exp((1 - XY^{-1})T)}{Y - X \exp((1 - XY^{-1})T)} \right) - \frac{XY^{-1} + 1}{2} \right) dT \]

Therefore,

\[ M(hV^+_d, hV^-_d) = \sum_{d \in \mathbb{N}} (-1)^{d+1} \text{Tr} \left( \frac{h}{2} (V^+_d + V^-_d) - \ln(e^{-\frac{h}{2}V^+_d} - e^{\frac{h}{2}V^-_d}) \right) \]

where the second equality uses Lemma 2.27 and the fact that \( \text{Tr}(\ln(I + H)) = \ln(\det(I + H)) \) for \( H \in \mathcal{M}_n(\mathbb{C}) \) sufficiently small.

\[ \sum_{k \geq 2} Z_k(\psi) h^k = \sum_{d \in \mathbb{N}} (-1)^d \text{Tr} \left( \frac{h}{2} (V^+_d + V^-_d) - \ln(e^{-\frac{h}{2}V^+_d} - e^{\frac{h}{2}V^-_d}) \right) \]

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