On multivariable matrix spectral factorization method

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Abstract. Spectral factorization is a prominent tool with several important applications in various areas of applied science. Wiener and Masani proved the existence of matrix spectral factorization. Their theorem has been extended to the multivariable case by Helson and Lowdenslager. Solving the problem numerically is challenging in both situations, and also important due to its practical applications. Therefore, several authors have developed algorithms for factorization. The Janashia-Lagvilava algorithm is a relatively new method for matrix spectral factorization which has proved to be useful in several applications. In this paper, we extend this method to the multivariable case. Consequently, a new numerical algorithm for multivariable matrix spectral factorization is constructed.

Key words: Matrix spectral factorization, multivariable systems, unitary matrix functions.

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1. INTRODUCTION

Spectral factorization was initiated in the works of Wiener \cite{38} and Kolmogorov \cite{27} as the scalar spectral factorization problem in relation to linear prediction theory of stationary stochastic processes, it has been extended to the matrix case by Wiener and Masani \cite{40}. Their matrix spectral factorization (MSF) theorem asserts that if $S$ is a positive definite integrable $d \times d$ matrix function defined on the unit circle $\mathbb{T}$ in the complex plane, $S \in L^1(\mathbb{T})^{d \times d}$, which satisfies the Paley-Wiener condition

\begin{equation}
\log \det S \in L^1(\mathbb{T}),
\end{equation}

then it admits the factorization

\begin{equation}
S(t) = S_+(t)S_+^*(t).
\end{equation}

Here $S_+ \in \mathbb{H}^2(\mathbb{T})^{d \times d}$, i.e., $S_+$ can be analytically extended inside $\mathbb{T}$ to a square integrable matrix function (for exact definitions see Sect. 2) and $A^*$ stands for the Hermitian conjugate of $A$. The spectral factor $S_+$ can be selected outer and it is the unique up to a constant right unitary factor.

Representation (1.2) plays a crucial role in the study of systems of singular integral equations \cite{19}, in linear estimation \cite{26}, quadratic and $H^\infty$ control \cite{1, 15}, communications \cite{14}, wavelets and filter design \cite{7, 37}, Granger causality estimation in neuroscience \cite{8}, etc. In many of these applications, it is important to actually compute $S_+$ approximately for a given matrix function $S$ which becomes a challenging problem. Therefore, starting with Wiener’s
original efforts [41] to create a sound computational method of MSF, dozens of different algorithms have appeared in the literature (see the survey papers [29], [36] and references therein, and also [6], [23] for more recent results).

A novel approach to the approximate factorization problem (1.2), without imposing any restriction on $S$ beyond the necessary and sufficient condition (1.1) for the existence of spectral factorization, was originally developed by Janashia and Lagvilava in [24] for $2 \times 2$ matrices. This approach was subsequently extended to matrices of arbitrary dimension in [25], efficiently algorithmized in [12], and successfully applied, e.g., in [31].

Helson and Lowdenslager [21] further generalized Wiener-Masani MSF theorem to the multivariable case. To this end, let $N \geq 1$ be a positive integer and let $H_N \subset \mathbb{Z}^N$ be the half-plane of lattice points defined recursively: $H_1 = \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ and $H_N = \{(k_1, k_2, \ldots, k_N) \in \mathbb{Z}^N : k_1 > 0 \text{ or } k_1 = 0 \text{ and } (k_2, \ldots, k_N) \in H_{N-1}\}$. We say that $f \in L^1(\mathbb{T}_N)$ is of analytic type (with respect to the half-plane $H_N$) if $C_k\{f\} = 0$ for each $k \in \mathbb{Z}^N \setminus H_N$, where $C_k\{f\}$ are the Fourier coefficients of $f$. The set of such functions will be denoted by $\mathcal{A}(\mathbb{T}_N)$. Finally, let $\mathbb{H}^2(\mathbb{Z}_N^d) = \mathcal{A}(\mathbb{T}_N) \cap L^2(\mathbb{T}_N)$.

The Helson-Lowdenslager MSF theorem [21] asserts that if

\begin{equation}
0 < S \in L^1(\mathbb{T}_N)^{d \times d}
\end{equation}

and satisfies the condition

\begin{equation}
\log \det S \in L^1(\mathbb{T}_N),
\end{equation}

then there exists a unique (up to a constant unitary matrix) factorization

\begin{equation}
S(t) = S_+(t)S_+^*(t), \quad t \in \mathbb{T}_N,
\end{equation}

where $S_+ \in \mathbb{H}^2(\mathbb{T}_N)^{d \times d}$ is a matrix function of outer analytic type (see Sect. 2 for definitions).

Wiener-Masani MSF theorem is used to process vector data depending on a single parameter, e.g., stationary time series collected by simultaneous observations at several different locations. However, due to the complex nature of the phenomena, data might be dependent on several parameters, e.g., color images on 2-D screen, or 3-D tomographic medical images. In such situations, the Helson-Lowdenslager MSF theorem enters the scene. Therefore a lot of effort was put in the development of computational methods for $N$-D MSF [5], [30], [32], [4] [17], [2], [18]. Clearly, improved methods of such factorization will further increase the applicability of the Helson-Lowdenslager theorem.

In this paper we extend the Janashia-Lagvilava method of MSF to the multivariable case, and hence introduce a novel computational algorithm for the Helson-Lowdenslager matrix spectral factorization. The paper is organized as follows. In Section 2, we introduce necessary notation and preliminary observations. In Section 3, we consider the uniqueness of $N$-D MSF. Section 4 deals with multivariable scalar spectral factorization. In Sections 5 and 6, we give an essential component of the proposed multivariable MSF algorithm and present its general description. We prove the convergence properties of the method in Section 7 and provide some results of numerical simulations in Section 8. Finally, in the Appendix, we demonstrate the application of spectral factorization in Granger causality.

2. Notation and preliminary observations

Throughout the paper, a positive integer $N \geq 1$ denotes the dimension of the torus $\mathbb{T}_N$. The latter is equipped with the normalized Lebesgue measure $\mu_N = dt/(2\pi)^N$. The half-plane of lattice points $H_N$ is defined in the Introduction. Note that $H_N$ has the following
properties. i) $0 \in H_N$; ii) $k \in H_N$ if and only if $-k \notin H_N$ unless $k = 0$; iii) $k_1, k_2 \in H_N$ imply $k_1 + k_2 \in H_N$.

The complex conjugate of $a \in \mathbb{C}$ is denoted by $\overline{a}$ and $A^*$ stands for the Hermitian conjugate of $A \in \mathbb{C}^{d \times d}$. For any set $S$, the notation $S^{d \times d}$ is used for the set of $d \times d$ matrices with entries from $S$. For $M \in S^{d \times d}$ and $m \leq d$, $[M]_{m \times m}$ denotes the $m \times m$ leading principle submatrix of $M$. A matrix function $S$ is called factorable if (1.3) and (1.4) hold. The notation $S > 0$ means that it is positive definite a.e.

Let $L^p(\mathbb{T}^N)$, $p > 0$, be the standard Lebesgue space of $p$-integrable functions with usual definition of the norm $\|f\|_{L^p(\mathbb{T}^N)}$ for $p \geq 1$.

The Fourier coefficients of $f \in L^1(\mathbb{T}^N)$ are defined by the formula

$$C_k \{f\} = \int_{\mathbb{T}^N} f(t) t^{-k} d\mu_N,$$

where $t^k = t_1^{k_1} t_2^{k_2} \ldots t_N^{k_N}$ for $t = (t_1, t_2, \ldots, t_N) \in \mathbb{T}^N$, $k = (k_1, k_2, \ldots, k_N) \in \mathbb{Z}^N$, and $f \in A(\mathbb{T}^N)$ means that $C_k \{f\} = 0$ for each multi-index $k$ outside $H_N$ (as in the Introduction),

$$A(\mathbb{T}^N) := \{f \in L^1(\mathbb{T}^N) : C_k \{f\} = 0 \text{ for each } k \notin H_N\}.$$ 

On several occasions, we need to expand a function $f \in L^2(\mathbb{T}^N)$ into “Fourier” series with respect to the first variable

$$(2.1) \quad f(t_1, t_2, \ldots, t_N) = \sum_{k \in \mathbb{Z}} t_1^k C_k \{f\}(t_2, \ldots, t_N)$$

where $C_k \{f\} \in L^2(\mathbb{T}^{N-1})$.

For each $k$, the function $C_k \{f\}$ is defined a.e. on $\mathbb{T}^{N-1}$ by

$$C_k \{f\}(t_2, \ldots, t_N) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t_1, t_2, \ldots, t_N) t_1^{-k} dt_1,$$ 

and equation (2.1) holds for a.e. $(t_1, t_2, \ldots, t_N) \in \mathbb{T}^N$.

If a function $f \in L^2(\mathbb{T}^N)$ has the form

$$f(t_1, t_2, \ldots, t_N) = \sum_{k=0}^n t_1^k \alpha_k(t_2, \ldots, t_N) \text{ where } \alpha_k \in L^2(\mathbb{T}^{N-1}),$$ 

then we say that

$$(2.2) \quad f \in \mathcal{P}^n_+(\mathbb{T}_1^N).$$

For a function $f$ defined by (2.1), we let

$$(2.3) \quad \tilde{f}(t_1, t_2, \ldots, t_N) = \sum_{k \in \mathbb{Z}} t_1^{-k} C_k \{f\}(t_2, \ldots, t_N).$$

Clearly

$$\tilde{f}(t_1, t_2, \ldots, t_N) = \overline{f(t_1, t_2, \ldots, t_N)} \text{ a.e. on } \mathbb{T}^N.$$ 

For a measurable function $f : \mathbb{T}^N \to \mathbb{C}$ we define $f_{t_2, t_3, \ldots, t_N} : \mathbb{T} \to \mathbb{C}$ for a.a. $(t_2, t_3, \ldots, t_N) \in \mathbb{T}^{N-1}$ by

$$f_{t_2, t_3, \ldots, t_N}(t) = f(t, t_2, t_3, \ldots, t_N).$$

Obviously, because of Fubini's theorem,

$$(2.4) \quad f \in L^p(\mathbb{T}^N) \implies f_{t_2, t_3, \ldots, t_N} \in L^p(\mathbb{T}) \text{ for a.e. } (t_2, t_3, \ldots, t_N) \in \mathbb{T}^{N-1}.$$ 

For $f \in L^1(\mathbb{T}^N)$, let $\hat{f} \in L^1(\mathbb{T}^{N-1})$ be defined by

$$(2.5) \quad \hat{f}(t_2, t_3, \ldots, t_N) = \int_{\mathbb{T}} f_{t_2, t_3, \ldots, t_N}(t) d\mu_1 = \int_{\mathbb{T}} f(t, t_2, t_3, \ldots, t_N) d\mu_1.$$
Let
\[(2.6) \quad \mathbb{H}^p(\mathbb{T}^N) := \mathcal{A}(\mathbb{T}^N) \cap L^p(\mathbb{T}^N), \quad \text{where} \quad p \geq 1,
\]be the class of analytic type functions (defined in the Introduction for \( p = 2 \)). The following recursive characterization of \( \mathbb{H}^p(\mathbb{T}^N) \) will be useful in the sequel.

**Proposition 2.1.** Let \( f \in L^p(\mathbb{T}^N) \), where \( p \geq 1 \) and \( N \geq 2 \). Then
\[(2.7) \quad f \in \mathbb{H}^p(\mathbb{T}^N)
\]if and only if
\[(2.8) \quad f_{t_2,t_3,\ldots,t_N} \in \mathbb{H}^p(\mathbb{T})
\]for a.e. \((t_2,t_3,\ldots,t_N) \in \mathbb{T}^{N-1}\) and
\[(2.9) \quad \hat{f} \in \mathbb{H}^p(\mathbb{T}^{N-1}).
\]

**Proof.** We have \( f_{t_2,t_3,\ldots,t_N} \in L^p(\mathbb{T}) \) for a.e. \((t_2,t_3,\ldots,t_N) \in \mathbb{T}^{N-1}\) due to \((2.4)\), and it follows from definition \((2.5)\), Jensen’s inequality \((\int_T |g| d\mu_1)^p \leq \int_T |g|^p d\mu_1\), and Fubini’s theorem that \( \hat{f} \in L^p(\mathbb{T}^{N-1}) \).

Suppose \((2.7)\) holds. For each integer \( k < 0 \), the function \( h_k \in L^1(\mathbb{T}^{N-1}) \) defined by
\[(2.10) \quad h_k(t_2,t_3,\ldots,t_N) = \int_{\mathbb{T}} f(t,t_2,t_3,\ldots,t_N) t^{-k} d\mu_1
\]has all Fourier coefficients equal to 0, because
\[C_k\{h_k\} = \int_{\mathbb{T}^N} f(t,t) t^{-k} d\mu_1 d\mu_{N-1} = 0
\]for any \( k = (k_2,k_3,\ldots,k_N) \), where \( t = (t_2,t_3,\ldots,t_N) \), since \( f \in \mathcal{A}(\mathbb{T}^N) \). Hence, for a.e. \((t_2,t_3,\ldots,t_N) \in \mathbb{T}^{N-1}\), the integral in \((2.10)\) is equal to 0 for all \( k < 0 \), and therefore \((2.8)\) holds. For \( k = 0 \) and \( k \notin H_{N-1} \), we still have
\[0 = \int_{\mathbb{T}^N} f(t,t) t^{-k} d\mu_1 d\mu_{N-1} = \int_{\mathbb{T}^{N-1}} \hat{f}(t) t^{-k} d\mu_{N-1},
\]and therefore \( \hat{f} \in \mathcal{A}(\mathbb{T}^{N-1}) \) and \((2.9)\) holds.

Suppose now that \((2.8)\) and \((2.9)\) hold. Then \( f \in \mathcal{A}(\mathbb{T}^N) \) can be proved by direct application of Fubini’s theorem reversing the above obtained implications. Therefore \((2.7)\) holds. \(\square\)

Next, for convenience of presentation of the obtained results, we introduce the Hardy spaces
\[(2.11) \quad \mathbb{H}^p(\mathbb{T}^N) \quad \text{for} \quad p > 0.
\]For \( N = 1 \), the Hardy space \( \mathbb{H}^p = \mathbb{H}^p(\mathbb{T}) \) is defined for all \( p > 0 \) by
\[\mathbb{H}^p := \left\{ f \in \mathcal{A}(\mathbb{D}) : \sup_{\rho < 1} \int_0^{2\pi} |f(\rho e^{i\theta})|^p d\theta < \infty \right\}.
\]The functions from \( \mathbb{H}^p \), where \( p > 0 \), and their boundary values can be identified (see, e.g. \((2.8)\)). Therefore, we can assume that \( \mathbb{H}^p = \mathbb{H}^p(\mathbb{T}) \subset L^p(\mathbb{T}) \) for \( 0 < p \leq \infty \), and this definition agrees with \((2.3)\) for \( p \geq 1 \) and \( N = 1 \). (However, we can speak about the values of a function \( f \in \mathbb{H}^p(\mathbb{T}) \) inside the unit disk if necessary). Nevertheless, the definition \((2.6)\) cannot be extended to arbitrary \( p > 0 \) because the question whether \( f \in \mathcal{A}(\mathbb{T}^N) \) arises only when \( f \)
is integrable. However, the equivalent characterization of $\mathbb{H}^p(\mathbb{T}^N)$ according to Proposition 2.1 enables us to extend this definitions to (2.11).

**Definition 2.1.** Assume $\mathbb{H}^p(\mathbb{T}^1) = \mathbb{H}^p(\mathbb{T}) = \mathbb{H}^p$. We say that $f \in \mathbb{H}^p(\mathbb{T}^N)$, where $p > 0$ and $N \geq 2$, if and only if $f_{t_2,t_3,\ldots,t_N} \in \mathbb{H}^p$ for a.a. $(t_2,t_3,\ldots,t_N) \in \mathbb{T}^{N-1}$ and $f_1 \in \mathbb{H}^p(\mathbb{T}^{N-1})$, where $f_1$ is defined by the equality

$$f_1(t_2,t_3,\ldots,t_N) := f_{t_2,t_3,\ldots,t_N}(0) = f_{t_2,t_3,\ldots,t_N}(z)|_{z=0}.$$  

(2.12)

Note that this definition of $\mathbb{H}^p(\mathbb{T}^N)$ differs from the standard Hardy space defined in the theory of several complex variables for $N > 1$ (see [35, p. 84]).

**Remark 2.1.** It follows from Definition 2.1 that if $f \in \mathbb{H}^p(\mathbb{T}^N)$, then $f(z,t_2,\ldots,t_N)$ is defined for a.a. $(t_2,\ldots,t_N) \in \mathbb{T}^{N-1}$ and each $z \in \mathbb{D}$. Furthermore, for $1 \leq k < N$, $f(0,\ldots,0,z,t_{k+1},\ldots,t_N)$ is defined for a.a. $(t_k,t_{k+1},\ldots,t_N) \in \mathbb{T}^{N-k}$ and each $z \in \mathbb{D}$. Therefore, similarly to (2.12), one can define the function $f_k : \mathbb{T}^{N-k} \to \mathbb{C}$ by

$$f_k(t_{k+1},\ldots,t_N) := f(0,\ldots,0,0,t_{k+1},\ldots,t_N).$$  

(2.13)

If $p \geq 1$, then $f_1 = \hat{f}$ a.e. on $\mathbb{T}^{N-1}$, where $\hat{f}$ is defined by (2.5), and

$$f_k(t_{k+1},\ldots,t_N) = \int_{\mathbb{T}^{N-k}} f(\cdot,t_{k+1},\ldots,t_N) \, d\mu_k.$$  

(2.14)

In particular,

$$f(0) := f(0,0,\cdots,0,0) = \int_{\mathbb{T}^N} f \, d\mu_N = C_0 \{ f \}$$  

(2.15)

(the definition in (2.15) makes sense, and thus will be used, for all $p > 0$).

The prominent property of Hardy space functions

$$\int_{\mathbb{T}} \log |h(t)| \, d\mu_1 > -\infty,$$

for any $0 \neq h \in \mathbb{H}^p$ (see [34, Th. 17.17]), is no longer valid for arbitrary function $0 \neq f \in \mathbb{H}^p(\mathbb{T}^N)$ for $N > 1$, because it may happen that

$$\int_{\mathbb{T}^N} \log |f(t)| \, d\mu_N = -\infty$$  

(see a counterexample at [21, p. 176]). However, it is possible to single out the situations where (2.16) may occur.

**Lemma 2.1.** Let $f \in \mathbb{H}^p(\mathbb{T}^N)$, for $p > 0$ and $N \geq 2$, and suppose (2.16) holds. Then

$$f_{N-1} \equiv 0.$$  

(2.17)

**Proof.** We use the well-known estimation

$$\log |h(0)| \leq \int_{\mathbb{T}} \log |h(t)| \, d\mu_1$$  

(2.18)

for any $h \in \mathbb{H}^p$ (see [34, Th. 17.17]), which together with (2.12) implies that

$$\int_{\mathbb{T}^N} \log |f(t)| \, d\mu_N \geq \int_{\mathbb{T}^{N-1}} \log |f_1(t_2,t_3,\ldots,t_N)| \, d\mu_{N-1}.$$  

Hence, it follows from (2.16) that the second integral in (2.19) is also $-\infty$. 

We can carry out the same reasoning for the function \( \hat{f}_1 \) instead of \( f \), and continuing recursively in the same manner, we will obtain

\[
\int_{T^{N-k}} \log |\hat{f}_k| \, d\mu_{N-k} = -\infty \text{ for } k = 1, 2, \ldots, N-1.
\]

Hence, we get \( \hat{f}_{N-1} \in \mathcal{H}^p(\mathbb{T}) \) and \( \int_{\mathbb{T}} \log |\hat{f}_{N-1}| \, d\mu_1 = -\infty \), which implies (2.17). \( \square \)

Next we define the set of outer type functions from \( \mathcal{H}^p(\mathbb{T}^N) \), \( p > 0 \), which is denoted by \( \mathcal{H}_O^p(\mathbb{T}^N) \). For \( N = n = 1 \), the definition is classical: \( 0 \not\equiv f \in \mathcal{H}^p \), where \( p > 0 \), is called outer, \( f \in \mathcal{H}_O^p(\mathbb{T}) \), if

\[
f(z) = c \cdot \exp \left( \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} + z \log |f(e^{i\theta})| \, d\theta \right), \quad |c| = 1,
\]

which is equivalent to (see \[34], Th. 17.17)

\[
\int_{\mathbb{T}} \log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \, d\theta.
\]

For \( N > 1 \), the definition will be given recursively.

**Definition 2.2.** We say that \( f \in \mathcal{H}_O^p(\mathbb{T}^N) \), \( p > 0 \), if and only if

\[
f_{t_2, t_3, \ldots, t_N} \in \mathcal{H}_O^p \text{ for a.e. } (t_2, t_3, \ldots, t_N) \in \mathbb{T}^{N-1}
\]

and \( \hat{f}_1 \in \mathcal{H}_O^p(\mathbb{T}^{N-1}) \), where \( \hat{f}_1 \) is defined by (2.12).

Note that the definition of \( \mathcal{H}_O^p(\mathbb{T}^N) \) coincides with the corresponding concept introduced in \[21\], p. 181] for \( p \geq 1 \). Namely, the following lemma holds.

**Lemma 2.2.** Let \( f \in \mathcal{H}^p(\mathbb{T}^N) \), \( p > 0 \). Then

\[
f \in \mathcal{H}_O^p(\mathbb{T}^N)
\]

if and only if

\[
\int_{\mathbb{T}^N} \log |f(t)| \, d\mu_N = \log |f(0)| > -\infty
\]

where \( f(0) \) is defined by (2.15).

**Proof.** As mentioned above, the equivalence of these two conditions is a well-known fact for \( N = 1 \), therefore, we need to consider the case \( N \geq 2 \).

Note first that \( f \in \mathcal{H}_O^p(\mathbb{T}^N) \) \( \implies \) \( \int_{\mathbb{T}^N} \log |f(t)| \, d\mu_N \neq -\infty \) since otherwise \( \hat{f}_{N-1} \equiv 0 \) by Lemma 2.1, which contradicts the definition of \( \mathcal{H}_O^p(\mathbb{T}^N) \) space.

On the other hand, both (2.23) and (2.24) are equivalent to the sequence of equations

\[
\int_{\mathbb{T}^N} \log |f| \, d\mu_N = \int_{\mathbb{T}^{N-1}} \log |\hat{f}_1| \, d\mu_{N-1} = \ldots = \int_{\mathbb{T}} \log |\hat{f}_N| \, d\mu_1 = \log |f(0)|
\]

(because of successive application of (2.19), where we have to have “=” instead of “≥”; see also (2.15)). Therefore, (2.23) and (2.24) are equivalent. \( \square \)

**Definition 2.3.** We say that a matrix function \( F \in H^p(\mathbb{T}^N)^{d \times d} \) is of outer type and use the notation \( F \in \mathcal{H}_O^p(\mathbb{T}^N)^{d \times d} \) if \( \det F \in \mathcal{H}_O^p(\mathbb{T}^N) \) for some \( r > 0 \).

Next, we introduce the following imbedded spaces, \( \mathcal{H}^p(\mathbb{T}^N) \subset \mathcal{H}^p(\mathbb{T}_{N-1}^N) \subset \ldots \subset \mathcal{H}^p(\mathbb{T}_1^N) \subset L^p(\mathbb{T}^N) \), which is used later.
Definition 2.4. For \( N \geq 2 \) and \( 1 \leq l < N \), we say that \( f \in \mathbb{H}^p(T^N) \) (resp. \( f \in \mathbb{H}^p_O(T^N) \)) if and only if \( f \in L^p(T^N) \) and, for a.a. \((t_{l+1}, t_{l+2}, \ldots, t_N) \in T^{N-l}\),
\[
f(\cdot, t_{l+1}, \ldots, t_N) \in \mathbb{H}^p(T^l) \quad \text{(resp. } f(\cdot, t_{l+1}, \ldots, t_N) \in \mathbb{H}^p_O(T^l) \text{)}
\]
as a function of variables \((t_1, t_2, \ldots, t_l)\).

The classes of matrix functions \( \mathbb{H}^{p,N} \) and \( \mathbb{H}^{p,O} \) are defined similarly.

Remark 2.2. Note that if \( f \in \mathbb{H}^p(T^N) \) and \( k \leq l \), then Remark 2.1 remains valid and \( \hat{f}_k \) can be defined by (2.13). Furthermore, if \( f, g \in \mathbb{H}^p(T^N) \) and \( |f| = |g| \) a.e. on \( T^N \), then
\[
|\hat{f}_{k-1}(z, t_{k+1}, t_{k+2}, \ldots, t_N)| = |\hat{g}_{k-1}(z, t_{k+1}, t_{k+2}, \ldots, t_N)|
\]
for a.a. \((t_{k+1}, t_{k+2}, \ldots, t_N) \in T^{N-k}\) and each \( z \in \mathbb{D} \) (it is assumed that \( \hat{f}_0 = f \)), consequently,
\[
|\hat{f}_k| = |\hat{g}_k| \quad \text{a.e. on } T^{N-k}
\]
for each \( k = 1, 2, \ldots, l \).

Remark 2.3. It follows from Definitions 2.1, 2.2, and 2.4 that if \( f \in \mathbb{H}^p_O(T^l) \), where \( l < N \), and
\[
g(t_1, t_2, \ldots, t_N) = f(t_1, t_2, \ldots, t_N)h(t_{l+1}, t_{l+2}, \ldots, t_N)
\]
for some \( h \in L^\infty(T^{N-l}) \), then \( g \in \mathbb{H}^p_O(T^N) \) as well. This fact is often tacitly used in what follows.

A matrix function \( U \in L^\infty(T^N) \) is called unitary if \( U(t)U^*(t) = I_d \) for a.e. \( t \in T^N \), where \( I_d \) stands for the \( d \times d \) unit matrix.

Integration of matrix functions and convergence of matrix valued sequences are understood entry-wise.

In Section 7 we use the following stability result on matrix spectral factorization proved in [110] for \( N = 1 \).

Theorem 2.1. ([110], Th. 1) Let \( 0 < S^{(n)} \in L^1(T) \), \( n = 0, 1, 2, \ldots \), be a sequence of positive definite integrable matrix functions such that
\[
\|S^{(n)} - S^{(0)}\|_{L^1(T)} \to 0 \quad \text{and} \quad \int_T \log \det S^{(n)}(t) \, dt \to \int_T \log \det S^{(0)}(t) \, dt.
\]

Then
\[
\|S^{(n)}_+ - S^{(0)}_+\|_{L^2(T)} \to 0.
\]

The proof of the following proposition, which is valid for arbitrary finite measure space, follows easily from the necessary and sufficient condition for the convergence in the norm:
\[
\|f_n - f\|_{L^p} \to 0, \quad \text{where } p \geq 1, \text{ if and only if } f_n \rightharpoonup f \text{ and } \sup_{n \geq k, \mu(E) < \delta} \int_E |f_n|^p \, d\mu \to 0 \text{ as } k \to \infty, \delta \to 0, \text{ where } \rightharpoonup \text{ stands for the convergence in measure.}
\]

Proposition 2.2. If \( h_n \in L^1 \), \( f_n \in L^p \), \( p \geq 1 \), \( n = 0, 1, \ldots \), \( f_n \rightharpoonup f_0 \), \( |f_n(t)|^p \leq |h_n(t)| \) and \( \|h_n - h_0\|_{L^1} \to 0 \), then \( \|f_n - f_0\|_{L^p} \to 0 \).

Corollary 2.1. If \( \|f_n - f\|_{L^p} \to 0, \quad p \geq 1, \quad \text{and } |u_n(t)| \leq 1, \quad n = 0, 1, \ldots, u_n \rightharpoonup u \), then \( \|f_nu_n - fu\|_{L^p} \to 0 \).
### 3. Uniqueness of multivariable matrix spectral factorization

Definitions 2.1 and 2.2 allow us to formulate generalized Smirnov’s theorem (namely, $L^p(\mathbb{T}) \ni f = g/h, g \in \mathbb{H}^q, h \in \mathbb{H}_O^r \implies f \in \mathbb{H}^p$; see [28, p.109]) for the N-dimensional case. This result is used to provide a simple proof of the uniqueness of the Helson-Lowdenslager MSF theorem (see Proposition 3.2 below). Note that the uniqueness is not discussed in the original formulation of this theorem in [21, 22].

**Proposition 3.1.** Suppose $f \in L^p(\mathbb{T}^N)$ can be represented as a ratio

\[ f = \frac{g}{h} \]

with $g \in \mathbb{H}^q(\mathbb{T}^N)$ and $h \in \mathbb{H}_O^r(\mathbb{T}^N)$, where $p, q, r > 0$ are arbitrary. Then

\[ f \in \mathbb{H}^p(\mathbb{T}^N). \]

First we prove the following

**Lemma 3.1.** If $f \in L^p(\mathbb{T}^N)$, $p > 0$, and $f_{t_2,t_3,\ldots,t_N} \in \mathbb{H}^p$ for a.e. $(t_2, t_3, \ldots, t_N) \in \mathbb{T}^{N-1}$, then (see (2.12))

\[ \hat{f}_1 \in L^p(\mathbb{T}^{N-1}). \]

**Proof.** Indeed, if $h \in \mathbb{H}^p, p > 0$, then $|h|^p = \exp(p \log |h|)$ is a subharmonic function in $\mathbb{D}$, and therefore $\int_0^{2\pi} |h(\rho e^{i\theta})|^p d\theta$ is increasing on $(0, 1)$ as a function of $\rho$ (see [16, §1.6]). Thus, $|h(0)|^p = \lim_{\rho \to 0^+}(1/2\pi) \int_0^{2\pi} |h(\rho e^{i\theta})|^p d\theta \leq \lim_{\rho \to 1^-}(1/2\pi) \int_0^{2\pi} |h(\rho e^{i\theta})|^p d\theta = \int_T |h|^p d\mu_1$. Consequently

\[ \int_{\mathbb{T}^{N-1}} |\hat{f}_1|^p d\mu_{N-1} = \int_{\mathbb{T}^{N-1}} |f(0, \cdot)|^p d\mu_{N-1} \leq \int_{\mathbb{T}^{N-1}} \left( \int_T |f(t, \cdot)|^p d\mu_1 \right) d\mu_{N-1} = \int_{\mathbb{T}^{N-1}} |f|^p d\mu_N. \]

Thus, (3.3) holds. \qed

**Proof of Proposition 3.1.** The proof can be carried out by induction with respect to $N$ and the goal is achieved by using Definitions 2.1 and 2.2. Indeed, for $N = 1$, the statement amounts to the above-mentioned generalized Smirnov’s theorem. Thus, we can make the assumption that the proposition is correct if we take $N = 1$ instead of $N$.

On the other hand, the hypothesis of the proposition implies that

\[ f_{t_2,t_3,\ldots,t_N} = g_{t_2,t_3,\ldots,t_N} / h_{t_2,t_3,\ldots,t_N} \in \mathbb{H}^p \]

due to the one-dimensional theorem, as long as (2.4) holds and Definitions 2.1 and 2.2 imply that $g_{t_2,t_3,\ldots,t_N} \in \mathbb{H}^q$ and $h_{t_2,t_3,\ldots,t_N} \in \mathbb{H}_O^r$. This in turn implies (3.3) by Lemma 3.1. We also have $\hat{f}_1 = f(0, \cdot) = g(0, \cdot)/h(0, \cdot) = \hat{g}_1/\hat{h}_1$ with $\hat{g}_1 \in \mathbb{H}_O(\mathbb{T}^{N-1})$ and $\hat{h}_1 \in \mathbb{H}_O^p(\mathbb{T}^{N-1})$ (by virtue of Definitions 2.1 and 2.2). Hence, $\hat{f}_1 \in \mathbb{H}^p(\mathbb{T}^{N-1})$ by the assumption of the induction and (3.2) holds by Definition 2.1. \qed

**Remark 3.1.** Using Hölder’s inequality and induction similar to the proof above, one can prove that if $f \in \mathbb{H}^p(\mathbb{T}^N)$ and $g \in \mathbb{H}^q(\mathbb{T}^N)$, then $f g \in \mathbb{H}_{\overline{O}}^{p+q}(\mathbb{T}^N)$. (Therefore, the exponent $r$ in Definition 2.3 can be taken equal to $p/d$.) Furthermore, if $f \in \mathbb{H}_O^p(\mathbb{T}^N)$ and $g \in \mathbb{H}_O^q(\mathbb{T}^N)$ then $f g \in \mathbb{H}_O^{p+q}(\mathbb{T}^N)$.

We are now ready to prove the uniqueness of factorization in the Helson-Lowdenslager MSF theorem, which is similar to the one presented in [9] for the Wiener-Masani MSF theorem.
Proposition 3.2. Let $0 < S \in L^1(\mathbb{T}^N)^{d \times d}$ and suppose \[(1.4)\] holds. If
\[(3.4)\] are two spectral factorizations of $S$ with spectral factors of outer type, $S_+, \Xi_+ \in \mathbb{H}_O^2(\mathbb{T}^N)^{d \times d}$, then there exists a constant unitary matrix $U \in \mathbb{C}^{d \times d}$ such that
\[(3.5)\]

Proof. The equations in \[(3.4)\] imply that
\[(\Xi_+^{-1}S_+)(S_+^*(\Xi_+^*)^{-1}) = I_d,\]
so that $U = \Xi_+^{-1}S_+$ is a unitary matrix function, i.e., $U^{-1} = U^*$ a.e. on $\mathbb{T}^N$. The entries of a unitary matrix are bounded. Hence
\[U \in L^\infty(\mathbb{T}^N)^{d \times d}.\]
Since $\det \Xi_+$ is of outer analytic type and $\Xi_+^{-1} = (\det \Xi_+)^{-1} \text{adj}(\Xi_+)$, we have
\[U = \frac{V}{w}, \quad \text{where } V \in \mathbb{H}^q(\mathbb{T}^N)^{d \times d} \text{ and } w \in \mathbb{H}^r(\mathbb{T}^N) \text{ for some } q, r > 0\]
(see Remark 3.1). Hence we can apply Proposition 3.1 for the entries of $U$ and conclude that
\[U \in \mathbb{H}^\infty(\mathbb{T}^N)^{d \times d}.\]
By changing the roles of $\Xi_+$ and $S_+$ in this discussion, we get
\[U^{-1} \in \mathbb{H}^\infty(\mathbb{T}^N)^{d \times d}.\]
Hence,
\[U, U^* \in \mathbb{H}^\infty(\mathbb{T}^N)^{d \times d}.\]
which means that entries of $U$ have all Fourier coefficients except $C_0$ equal to 0. Consequently, they are constant. \[\square\]

4. Multivariable scalar spectral factorization

Helson-Lowdenslager spectral factorization theorem in the scalar case asserts that: if $0 < f \in L^1(\mathbb{T}^N)$ and $\int_{\mathbb{T}^N} \log f \, d\mu_N > -\infty$, then there exists a unique (up to a constant factor of modulus 1) function $f_+ \in \mathbb{H}_O^2(\mathbb{T}^N)$ such that
\[(4.1)\]
\[f = |f_+|^2 \text{ a.e. on } \mathbb{T}^N.\]
If $N = 1$, then the function $f_+$ can be written explicitly: $f_+(t) = \lim_{r \to 1-} f_+(rt)$, where
\[f_+(z) = c \cdot \exp \left( \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log f(e^{i\theta}) \, d\theta \right), \quad |c| = 1, \quad |z| < 1\]
(cf. 2.20). Note that $f_+$ can be also written as
\[f_+(t) = c \cdot \sqrt{f(t)} \exp \left( \frac{1}{2}iS(\log f)(t) \right),\]
where $S(f)$ stands for the conjugate of $f \in L^1(\mathbb{T})$:
\[S(f)(e^{i\tau}) = \frac{1}{2\pi} f(P) \int_0^{2\pi} f(e^{i\theta}) \cot \frac{\tau - \theta}{2} \, d\theta.\]
In the multivariable case, it is sufficient for our purposes to construct a factorization
\[f = f_{+,1\,\overline{f_{+,1}}} ,\]
where $f_{+,1} \in H^2_O(T_N^1)$. Such factorization can be written in the explicit form

\begin{equation}
(4.2)
\quad f_{+,1}(t) = \sqrt{f(t)} \exp \left( iS_1(\log \sqrt{f(t)}) \right)
\end{equation}

if we introduce the singular operator $S_1 : L^1(T^N) \to L^p(T^N)$, $p < 1$, with respect to the first variable by the formula

\begin{equation}
S_1(h)(e^{i\tau}, t_2, \ldots, t_N) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}, t_2, \ldots, t_N) \cot \frac{\tau - \theta}{2} d\theta.
\end{equation}

Note that we can write $f_+$ in (4.1) explicitly as

\begin{equation}
(4.3)
\quad f_+(t_1, t_2, \ldots, t_N) = f_{+,1}(t_1, t_2, \ldots, t_N) \prod_{k=1}^{N-1} \exp \left( iS_1(\hat{f}_k(t_{k+1}, \ldots, t_N)) \right)
\end{equation}

where $\hat{f}_k : T^{N-k} \to \mathbb{R}$, $k = 1, 2, \ldots, N - 1$, are the functions defined by (cf. (2.13)):

\begin{equation}
(4.4)
\quad \hat{f}_k(t_{k+1}, \ldots, t_N) = \int_{T_k} \log \sqrt{f(\cdot, t_{k+1}, \ldots, t_N)} d\mu_k
\end{equation}

In particular, for $2 \leq l \leq N$, we have $f = f_{+,l} f_{+,l}^*$, where

\begin{equation}
(5.1)
\quad S(t) = S_{+,1}(t) S_{+,1}^*(t)
\end{equation}

where

\begin{equation}
(5.2)
\quad S_{+,1} \in H^2(T_N^1)^{d \times d},
\end{equation}

of a matrix (1.3) which satisfies (1.4). The existence of such factorization follows from the corresponding 1-D Wiener-Masani theorem since Fubini’s theorem guarantees that, for a.e. $(t_2, \ldots, t_N) \in T^{N-1}$,

\begin{equation}
S(\cdot, t_2, \ldots, t_N) \in L^1(T^1) \quad \text{and} \quad \log \det S(\cdot, t_2, \ldots, t_N) \in L^1(T^1).
\end{equation}

Below we provide constructive procedures for factorization (5.1). These procedures stem from the corresponding 1-D MSF algorithm proposed in [25]. The main idea which demonstrates the possibility of such generalization is presented in our recent paper [13].

**Procedure 1.** We perform lower-upper factorization

\begin{equation}
(5.3)
\quad S(t) = M_1(t) M_1^*(t)
\end{equation}
where
\[
M_1(t) = \begin{pmatrix}
    f_1(t) & 0 & \cdots & 0 & 0 \\
    \xi_{21}(t) & f_2(t) & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    \xi_{d-1,1}(t) & \xi_{d-1,2}(t) & \cdots & f_{d-1}(t) & 0 \\
    \xi_{d1}(t) & \xi_{d2}(t) & \cdots & \xi_{d,d-1}(t) & f_d(t)
\end{pmatrix}
\]

with \( f_i \in \mathbb{H}_O^2(T_1^N) \), \( 1 \leq i \leq d \), and \( \xi_{ij} \in L^2(T_1^N) \), \( 2 \leq i \leq d, 1 \leq j < i \). As in the \( N = 1 \) case, (5.3) can be achieved by pointwise Cholesky factorization of \( S(t) \) and then applying formula (4.2) for the diagonal entries. Note that \( \det M_1 \in \mathbb{H}_O^2(T_1^N) \).

**Procedure 2.** The factor \( S_{+,1} \) is represented as
\[
S_{+,1}(t) = M_1(t)U_2(t)U_3(t)\ldots U_r(t).
\]
Each \( U_m, m = 2, 3, \ldots, d \), has the form
\[
U_m(t) = \begin{pmatrix}
    U_m(t) & 0 \\
    0 & I_{r-m}
\end{pmatrix},
\]
where \( U_m \) is a unitary matrix function with a special structure
\[
U_m(t) = \begin{pmatrix}
    u_{11}(t) & u_{12}(t) & \cdots & u_{1,m-1}(t) & u_{1m}(t) \\
    u_{21}(t) & u_{22}(t) & \cdots & u_{2,m-1}(t) & u_{2m}(t) \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    u_{m-1,1}(t) & u_{m-1,2}(t) & \cdots & u_{m-1,m-1}(t) & u_{m-1,m}(t) \\
    \hat{u}_{m1}(t) & \hat{u}_{m2}(t) & \cdots & \hat{u}_{m,m-1}(t) & \hat{u}_{mm}(t)
\end{pmatrix},
\]
(see (2.3)) with \( u_{ij} \in \mathbb{H}_O^\infty(T_1^N) \), such that
\[
\det U_m(t) = 1 \text{ for a.e. } t \in T_1^N
\]
and
\[
[Q_m]_{m \times m} := [M_1U_2 \ldots U_m]_{m \times m} \in (\mathbb{H}_O^2(T_1^N))_{m \times m}.
\]

**Procedure 3.** The unitary matrix functions (5.6) are constructed recursively. We assume that \( U_2, U_3, \ldots, U_{m-1} \) have already been constructed and obtain \( U_m \) by the following steps:

**Step 1.** Consider the matrix function \( F \) of the form
\[
F(t) = \begin{pmatrix}
    1 & 0 & 0 & \cdots & 0 & 0 \\
    0 & 1 & 0 & \cdots & 0 & 0 \\
    0 & 0 & 1 & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & 1 & 0 \\
    \zeta_1(t) & \zeta_2(t) & \zeta_3(t) & \cdots & \zeta_{m-1}(t) & f_m(t)
\end{pmatrix},
\]
where the last row of \( F \) is the same as the last row of \( [Q_{m-1}]_{m \times m} \). We have \( \zeta_i \in L^2(T_1^N) \), \( 1 \leq i \leq m - 1 \), and \( f_m \in \mathbb{H}_O^2(T_1^N) \).

Suppose
\[
f_m(t_1, t_2, \ldots, t_N) = \sum_{k=0}^\infty t_k^N \gamma_k(t_2, \ldots, t_N), \text{ where } \gamma_k \in L^2(T_1^{N-1}),
\]
\[
\zeta_i(t_1, t_2, \ldots, t_N) = \sum_{k=-\infty}^\infty t_k^N \alpha_{i,k}(t_2, \ldots, t_N), \text{ where } \alpha_{i,k} \in L^2(T_1^{N-1}),
\]
and

\[ \zeta_{+,i}(t_1, \cdot) = \sum_{k=0}^{\infty} t^k_1 \alpha_{i,k}(\cdot) \quad \text{and} \quad \zeta_{-,i}(t_1, \cdot) = \sum_{k=-\infty}^{-1} t^k_1 \alpha_{i,k}(\cdot). \]

**Step 2.** Decompose \( F \) as

\[ F(t) = F_+(t)F_-(t), \]

where \( F_+ \) and \( F_- \) have the same structure as \( F \) while their last rows are replaced by

\[ [\zeta_{+,1}, \zeta_{+,2}, \ldots, \zeta_{+, (m-1)}, 1] \quad \text{and} \quad [\zeta_{-,1}, \zeta_{-,2}, \ldots, \zeta_{-, (m-1)}, f_m], \]

respectively.

**Step 3.** For a sufficiently large \( n \), approximate the matrix function \( F_- \) by \( F_{-}^{(n)} \) of the same structure as (5.9) but with the last row replaced by

\[ [\zeta_{-1}^{(n)}, \zeta_{-2}^{(n)}, \ldots, \zeta_{-(m-1)}^{(n)}, f_m^{(n)}], \]

where

\[ \zeta_{-,i}^{(n)}(t_1, \cdot) = \sum_{k=-n}^{-1} t^k_1 \alpha_{i,k}(\cdot) \quad \text{and} \quad f_m^{(n)}(t_1, \cdot) = \sum_{k=0}^{n} t^k_1 \gamma_k(\cdot). \]

**Step 4.** For the matrix function \( F_{-}^{(n)} \), construct the corresponding unitary matrix function \( U_{m}^{(n)} \) of the form (5.7), where \( u_{ij} \in P_+^n(T_1^N) \) (see (2.2)), which satisfies (5.8) such that

\[ F_{-}^{(n)} U_{m}^{(n)} \in (P_+^n(T_1^N))^{m \times m}. \]

This construction can be realized pointwise for a.e. \((t_2, t_3, \ldots, t_N) \in \mathbb{T}^{N-1}\) by the corresponding 1-D theorem proved in [25] (see Theorem 1 and its proof therein).

The matrix function \( U_m \) is obtained as a limit of \( U_{m}^{(n)} \), as \( n \to \infty \). The convergent properties of the algorithm is analyzed in Sections 7.

As it is done in [25], for a simplicity of the presentation, we can assume that each \( \xi_{ij} \) in (5.24) is approximated by

\[ \xi_{ij}(t) \approx \xi_{ij}^{(i-j)n}(t) = \sum_{k=-(i-j)n}^{\infty} t^k_1 C_{1k} \{ \xi_{ij} \}(t_2, \ldots, t_N), \]

and then we get the approximation of (5.5)

\[ S_{+,1}^{(n)}(t) = M_{1}^{(n)}(t) U_{2}^{(n)}(t) U_{3}^{(2n)}(t) \cdots U_{r}^{((r-1)n)}(t) =: M_{1}^{(n)}(t) U^{(n)}(t), \]

where \( M_{1}^{(n)} \) is obtained from (5.4) by making the approximations (5.10) and each \( U_{m}^{((m-1)n)} \) is constructed according to the above described procedures taking \((m-1)n\) instead of \(n\) in Step 3. We recall that \( \det U^{(n)} = 1 \) a.e. and, therefore,

\[ S_{+,1}^{(n)} \in \mathbb{H}^2(T_1^N)^{d \times d} \quad \text{for all} \quad n = 1, 2, \ldots \]

The convergence

\[ \| S_{+,1}^{(n)} - S_{+,1}^{(n)} \|_{L^2(T_N)} \to 0 \quad \text{as} \quad n \to \infty \]

is proved in Section 7.
6. Description of the multivariable MSF method

In this section, we outline a general scheme of the proposed N-D MSF method which can be utilized as a computational algorithm.

Recursively with respect to $l$, we factorize the matrix \( S \) as

\[
S(t) = S_{+,l}(t)S^*_{+,l}(t),
\]

\( l = 1, 2, \ldots, N \), where

\[
S_{+,l} \in \mathbb{H}^2(T^N_l)_{d \times d}.
\]

By virtue of Definitions 2.4, factorization (1.5) is achieved as soon as we reach \( l = N \), i.e.

\[
S_+ = S_{+,N}.
\]

The basic procedure is the factorization

\[
S(t) = S_{+,1}(t)S^*_{+,1}(t)
\]

described in the previous section (for uniqueness purposes, we assume that \( S_{+,1}(0, t_2, \ldots, t_N) \) is positive definite in (6.3), however, it does not play any role).

For a factorable \( S \in L^1(T^N)d \times d \), where \( N \geq 2 \), we denote the resulting factor \( S_{+,1} \in \mathbb{H}^2(T^N_1)_{d \times d} \) by

\[
S_{+,1} =: \mathcal{S}F_N[S].
\]

In addition, we define the matrix function

\[
\hat{S}_{+,1} \in L^2(T^{N-1})_{d \times d}
\]

for a.e. \( (t_2, \ldots, t_N) \in T^{N-1} \) by the equation (cf. Remark 2.1)

\[
\hat{S}_{+,1}(t_2, \ldots, t_N) = S_{+,1}(0, t_2, \ldots, t_N) = \frac{1}{2\pi} \int_T S_{+,1}(t, t_2, \ldots, t_N) \, dt.
\]

Relation (6.3) holds because of (5.2) and the Fubini theorem. We also have

\[
\int_{T^N} \log |\det S_{+,1}| \, dm_1 = \frac{1}{(2\pi)^N} \int_{T^{N-1}} \left( \int_T \log |\det S_{+,1}(t_1, t_2, \ldots, t_N)| \, dt_1 \right) \, dt_2 \ldots \, dt_N
\]

\[
= \frac{1}{(2\pi)^N} \int_{T^{N-1}} \log |\det S_{+,1}(0, t_2, \ldots, t_N)| \, dt_2 \ldots \, dt_N = \int_{T^{N-1}} \log |\det \hat{S}_{+,1}| \, dm_{N-1}
\]

The second equality holds due to (2.21) which, together with (1.4) and (6.3), implies that

\[
\log |\det \hat{S}_{+,1}| \in L^1(T^{N-1}).
\]

Applying the operator \( \mathcal{S}F_N \) defined by (6.4), we proceed with factorization (6.1) as follows: The relations (6.5) and (6.8) imply that

\[
S_1 := \hat{S}_{+,1}\hat{S}^*_{+,1} \in L^1(T^{N-1})_{d \times d}
\]

is a factorable matrix function and the operator \( \mathcal{S}F_{(N-1)} \) can be applied to it. Consequently, \( U_2 := \hat{S}^{-1}_{+,1}\mathcal{S}F_{(N-1)}[S_1] \) is a unitary matrix function and if we define

\[
S_{+,2}(t_1, t_2, \ldots, t_N) := S_{+,1}(t_1, t_2, \ldots, t_N)U_2(t_2, \ldots, t_N),
\]

we get

\[
S(t) = S_{+,2}(t_1, t_2, \ldots, t_N)U_2(t_2, \ldots, t_N), \quad S_{+,2} \in \mathbb{H}^2(T^N_2)_{d \times d}.
\]

Similarly, if the factorization

\[
S(t) = S_{+,l-1}(t)S^*_{+,l-1}(t),
\]
has already been constructed, where \( S_{+,l-1} \in \mathbb{H}^2(I_{l-1}O)^{d \times d} \), then we define

\[
\hat{S}_{+,l-1}(t_1, \ldots, t_N) = S_{+,l-1}(0, \ldots, 0, t_1, \ldots, t_N) = \int_{I_{l-1}} S_{+,l-1}(\cdot, t_1, \ldots, t_N) \, d\mu_{l-1}
\]

and

\[
S_{l-1} = \hat{S}_{+,l-1} \hat{S}_{+,l-1}^*,
\]

apply the operator \( S_F(N-l+1) \) to \( S_{l-1} \) to get a unitary matrix function

\[
U_l(t_1, \ldots, t_N) := \hat{S}_{+,l-1}^{-1}(t_1, \ldots, t_N) S_F(N-l+1)[S_{l-1}](t_1, \ldots, t_N),
\]

and obtain

\[
S_{+,l}(t_1, \ldots, t_N) = S_{+,l-1}(t_1, \ldots, t_N) U_l(t_1, \ldots, t_N)
\]

which satisfies (6.1) and (6.2).

Summarizing, we get

\[
S_+ = S_{+,1} U_2 U_3 \ldots U_N.
\]

In order to satisfy the uniqueness condition, we can take the spectral factor which is positive definite at the origin

\[
S_+(z)(S_+(0))^{-1} \sqrt{S_+(0)(S_+(0))^*}.
\]

Note also that

\[
\hat{S}_{+,l}(t_{l+1}, \ldots, t_N) = \hat{S}_{+,l-1}(0, t_{l+1}, \ldots, t_N).
\]

In actual computations, the equations presented in this section are changed with approximations. For sufficiently large positive integers \( n_1, n_2, \ldots, n_N \), we proceed as follows: First we approximate \( S_{+,1} \) as it is described in Procedure 3 of the previous section

\[
S_{+,1} \approx S^{(n_1)}_{+,1} =: S_{N}^{(n_1)}[S].
\]

Then we compute \( \hat{S}^{(n_1)}_{+,1}(t_2, \ldots, t_N) = S^{(n_1)}_{+,1}(0, t_2, \ldots, t_N) \), approximate

\[
S_1 \approx \hat{S}^{(n_1)}_{+,1} = S^{(n_1)}_{+,1}(\hat{S}_{+,1}^{(n_1)})^*,
\]

and obtain its approximate factor \( S_F^{(n_2)}(S_1^{(n_1)}) \) which gives an approximation of \( S_{+,2} \) as (cf. (6.10))

\[
S_{+,2} \approx S^{(n_1 n_2)}_{+,2} = S^{(n_1)}_{+,1}(S_1^{(n_1)})^{-1} S_F^{(n_2)}(S_1^{(n_1)}) =: S^{(n_1)}_{+,1} U_2^{(n_1 n_2)}.
\]

Continuing in this manner, we obtain

\[
S_+ = S_{+,N} \approx S^{(n_{1 \ldots n_N})}_{+,N} = S^{(n_1)}_{+,1} U_2^{(n_1 n_2)} \ldots U_N^{(n_{1 \ldots n_{N-1}} n_N)},
\]

where the following functions are defined recursively

\[
U_l^{(n_1 n_2 \ldots n_l)} = (\hat{S}^{(n_{1 \ldots n_{l-1}})}_{+,l-1})^{-1} S_F^{(n_l)}(S_{l-1}^{(n_{1 \ldots n_{l-1}})}),
\]

\[
\hat{S}^{(n_{1 \ldots n_{l-1}})}_{+,l-1}(t_1, \ldots, t_N) = S^{(n_{1 \ldots n_{l-1}})}_{+,l-1}(0, \ldots, 0, t_1, \ldots, t_N) = \int_{I_{l-1}} S^{(n_{1 \ldots n_{l-1}})}_{+,l-1}(\cdot, t_1, \ldots, t_N) \, d\mu_{l-1}.
\]

\[
S_{l-1}^{(n_{1 \ldots n_{l-1}})} = \hat{S}_{+,l-1}^{(n_{1 \ldots n_{l-1}})}(\hat{S}_{+,l-1}^{(n_{1 \ldots n_{l-1}})})^*,
\]

\[
S_{+,l}^{(n_1 n_2 \ldots n_l)} = S_{+,l-1}^{(n_1 n_2 \ldots n_l)} U_l^{(n_1 n_2 \ldots n_l)}.
\]
Hence the convergence “restricted to hyperplanes” (7.1) follows.

**Lemma 7.1.**

Then

$$f_n \rightarrow f \quad \text{in } L^p(\mathbb{T}^N),$$

if for each $l = 2, 3, \ldots, N$ and for a.e. $(t_l, t_{l+1}, \ldots, t_N) \in \mathbb{T}^{N-l+1}$,

$$f_n(\cdot, t_l, \ldots, t_N) \rightarrow f(\cdot, t_l, \ldots, t_N) \quad \text{in } L^p(\mathbb{T}^{l-1})$$
as functions of variables $(t_1, t_2, \ldots, t_{l-1})$. It is assumed that $f_n \rightarrow f$ in $L^p(\mathbb{T}^N)$ as well.

**Remark 7.1.** Simple examples show that, in general, $f_n \rightarrow f \not\Rightarrow f_n \rightarrow f$ (in the same $L^p(\mathbb{T}^N)$). This happens because convergence in norm does not imply convergence almost everywhere.

For convenience of references, we prove some lemmas which easily follow from the Fubini theorem.

**Lemma 7.1.** Let $f \in L^2(\mathbb{T}^N)$, and let

$$f_n(t_1, t_2, \ldots, t_N) = \sum_{k=-n}^{n} t_1^k C_{1k}(f)(t_2, \ldots, t_N).$$

Then

$$f_n \rightarrow f \quad \text{in } L^2(\mathbb{T}^N).$$

**Proof.** For a.a. $(t_1, \ldots, t_N) \in \mathbb{T}^{N-l+1}$, we have

$$f_{t_l \ldots t_N} := f(\cdot, t_l, \ldots, t_N) \in L^2(\mathbb{T}^{l-1})$$
and

$$C_{1k}(f)(t_2, \ldots, t_N) = C_{1k}(f_{t_l \ldots t_N})(t_2, \ldots, t_{l-1}) \quad \text{for a.a. } (t_2, \ldots, t_{l-1}) \in \mathbb{T}^{l-2}.$$Therefore, for a.a. $(t_l, \ldots, t_N) \in \mathbb{T}^{N-l+1}$,

$$\int_{\mathbb{T}^{l-1}} \left| f_{t_l \ldots t_N}(t_l, \ldots, t_{l-1}) - \sum_{k=-n}^{n} t_1^k C_{1k}(f)(t_2, \ldots, t_N) \right|^2 dt_l \ldots dt_{l-1} =$$

$$\int_{\mathbb{T}^{l-2}} \left( \int_{\mathbb{T}} \left| f_{t_l \ldots t_N}(t_l, \ldots, t_{l-1}) - \sum_{k=-n}^{n} t_1^k C_{1k}(f_{t_l \ldots t_N})(t_2, \ldots, t_{l-1}) \right|^2 dt_1 \right) dt_l \ldots dt_{l-1} \rightarrow 0$$
since the second integral in the last expression converges to 0 for a.a. $(t_2, \ldots, t_{l-1}) \in \mathbb{T}^{l-2}$ (due to pointwise application of the Parseval’s identity) and it is majorized by

$$\int_{\mathbb{T}} \left| f_{t_l \ldots t_N}(t_l, \ldots, t_{l-1}) \right|^2 dt_1 \in L^2(\mathbb{T}^{l-2}).$$

Hence the convergence “restricted to hyperplanes” (7.1) follows. □
For a function \( f \in L^1(\mathbb{T}^N) \) and \( 1 \leq j \leq N - 1 \), slightly abusing the notation (cf. (2.5)), let \( \hat{f} \in L^1(\mathbb{T}^{N-j}) \) be the function defined by
\[
\hat{f}(t_{j+1}, \ldots, t_N) = \int_{T_j} f(\cdot, t_{j+1}, \ldots, t_N) \, d\mu_j.
\]

**Lemma 7.2.** Let
\[
f_n \rightharpoonup f \quad \text{in} \quad L^p(\mathbb{T}^N),
\]
where \( p \geq 1 \). Then, for each \( j = 1, 2, \ldots, N - 1 \),
\[
(7.2)
\]
\[
\hat{f}_n \rightharpoonup \hat{f} \quad \text{in} \quad L^p(\mathbb{T}^{N-j}).
\]

**Proof.** Let \( j + 1 < l \leq N \). For a.a. \( (t_l, t_{l+1}, \ldots, t_N) \in \mathbb{T}^{N-l+1} \), we have
\[
\left\| \hat{f}_n(\cdot, t_l, \ldots, t_N) - \hat{f}(\cdot, t_l, \ldots, t_N) \right\|_{L^p(\mathbb{T}^{l-j-1})}^p
\leq \frac{1}{2\pi} \int_{\mathbb{T}^{l-j-1}} \left| \int_{T_l} f_n(\cdot, t_{j+1}, \ldots, t_N) \, d\mu_j - \int_{T_l} f(\cdot, t_{j+1}, \ldots, t_N) \, d\mu_j \right| \, dt_{j+1} \cdots dt_l
\leq \frac{1}{2\pi} \int_{\mathbb{T}^{l-j-1}} \left( \int_{T_l} \left| f_n(\cdot, t_{j+1}, \ldots, t_N) - f(\cdot, t_{j+1}, \ldots, t_N) \right| \, d\mu_j \right) \, dt_{j+1} \cdots dt_l
= \left\| f_n(\cdot, t_l, \ldots, t_N) - f(\cdot, t_l, \ldots, t_N) \right\|_{L^1(\mathbb{T}^{l-1})}^p \to 0,
\]
which implies (7.2). \( \square \)

The basic step in the proof of the convergence is the following

**Theorem 7.1.** Let \( S^{(n)} \), \( n = 1, 2, \ldots \), and \( S \) satisfy (1.3),
\[
S^{(n)} \rightharpoonup S \quad \text{in} \quad L^1(\mathbb{T}^N)
\]
and, for a.a. \( (t_2, t_3, \ldots, t_N) \in \mathbb{T}^{N-1} \),
\[
\int_{\mathbb{T}} \log \det S^{(n)}(t_1, t_2, \ldots, t_N) \, dt_1 \to \int_{\mathbb{T}} \log \det S(t_1, t_2, \ldots, t_N) \, dt_1.
\]

Suppose
\[
(7.4)
S^{(n)}(t) = S^{(n)}_{+1}(t) \left( S^{(n)}_{+1}(t) \right)^* \quad \text{is the factorization of } S^{(n)} \text{ defined according to Section 5. Then}
\]
\[
S^{(n)}_{+1} \rightharpoonup S_{+1} \quad \text{in} \quad L^2(\mathbb{T}^N).
\]

**Proof.** By virtue of 1-dimensional Theorem 2.5 for a.a. \( (t_2, t_3, \ldots, t_N) \in \mathbb{T}^{N-1} \), we have
\[
S^{(n)}_{+1}(\cdot, t_2, \ldots, t_N) \to S_{+1}(\cdot, t_2, \ldots, t_N) \quad \text{in} \quad L^2(\mathbb{T}).
\]
This implies the convergence in measure for each \( l = 2, \ldots, N \) and a.a. \( (t_l, \ldots, t_N) \in \mathbb{T}^{N-l+1} \):
\[
S^{(n)}_{+1}(\cdot, t_l, \ldots, t_N) \Rightarrow S_{+1}(\cdot, t_l, \ldots, t_N) \quad \text{on} \quad \mathbb{T}^{l-1}.
\]
Equation (7.4) guarantees that the squares of absolute values of the entries of matrix functions \( S^{(n)}_{+1}(\cdot, t_1, \ldots, t_N) \) are bounded by diagonal entries of \( S^{(n)}(\cdot, t_1, \ldots, t_N) \) which are convergent in \( L^1(\mathbb{T}^{l-1}) \) by virtue of the definition of the convergence ("restricted to hyperplanes") in (7.3). Therefore, Proposition 2.2 implies that
\[
S^{(n)}_{+1}(\cdot, t_l, \ldots, t_N) \to S_{+1}(\cdot, t_l, \ldots, t_N) \quad \text{in} \quad L^1(\mathbb{T}^{l-1}).
\]
Hence (7.5) holds. \( \square \)
Theorem 7.1 implies the convergence

\begin{equation}
S_{+1}^{(n_1)} \to S_{+1} \text{ in } L^2(\mathbb{T}^N)
\end{equation}

for \(S_{+1}^{(n_1)}\) defined by \((5.11)\), which in particular contains \((5.12)\). Indeed, \(S_{+1}^{(n_1)}\) is a spectral factor of \(M_{1}^{(n_1)}(M_{1}^{(n_1)})^*\) which can be taken in the role of \(S^{(n)}\) in Theorem 7.1. By virtue of Lemma 7.1 and H"older’s inequality, we have

\[M_{1}^{(n_1)}(M_{1}^{(n_1)})^* = M_1 M_1^* = S \text{ in } L^1(\mathbb{T}),\]

and determinants are also equal for each \(n_1\)

\[\det \left( S_{+1}^{(n_1)}(S_{+1}^{(n_1)})^* \right) = \det (M_1 M_1^*) = \det S\]

because of the structure of matrices \(\hat{S}_{+1}^{(n_1)}\) and \(M_{1}^{(n_1)}\), see \((5.11)\). Hence, the hypothesis of Theorem 7.1 are satisfied and \((7.6)\) holds.

The important observation is that as the factorization proceeds the determinants of the obtained matrices remain unchanged, namely

\begin{equation}
\det S_{l}^{(n_1 \ldots n_l)} = \det S_{l} \text{ for each } l = 2, \ldots, N.
\end{equation}

This can be justified recursively by using the definitions \((6.4)\), \((6.17)\), \((6.11)\), \((6.12)\), \((6.19)\), \((6.20)\), and the property \((6.16)\):

\[\det S_{l-1}^{(n_1 \ldots n_{l-1})} = \det S_{l-1} \Rightarrow \left| \det \left( SF_{(N-l+1)}[S_{l-1}^{(n_1 \ldots n_{l-1})}] \right) \right| = \left| \det \left( SF_{(N-l)}[S_{l-1}] \right) \right|\]

(see Remark 2.2 and also \((6.7)\)) \(\Rightarrow \left| \det \hat{S}_{+l}^{(n_1 \ldots n_l)} \right| = \left| \det \hat{S}_{+l} \right| \Rightarrow \det S_{l}^{(n_1 \ldots n_l)} = \det S_{l}.

We are now ready to show

\begin{equation}
S_{+,N}^{(n_1 \ldots n_N)} \to S_{+,N} = S_+ \text{ in } L^2(\mathbb{T}^N),
\end{equation}

and consequently \((6.22)\), by using again the recursive steps. The convergence \((7.6)\) can be used as a starting point. Now assume that

\[S_{+,l-1}^{(n_1 \ldots n_{l-1})} \to S_{+,l-1} \text{ in } L^2(\mathbb{T}^N),\]

This implies that (see \((6.11)\) and \((6.19)\)

\[\hat{S}_{+,l-1}^{(n_1 \ldots n_{l-1})} \to \hat{S}_{+,l-1} \text{ in } L^2(\mathbb{T}^{N-l+1}),\]

by virtue of Lemma 7.2, which in turn implies that (see \((6.12)\) and \((6.20)\)

\[S_{l-1}^{(n_1 \ldots n_{l-1})} \to S_{l-1} \text{ in } L^1(\mathbb{T}^{N-l+1})\]

because of H"older’s inequality. Combining this with \((7.7)\), we obtain

\[SF_{(N-l+1)}[S_{l-1}^{(n_1 \ldots n_{l-1})}] \to SF_{(N-l)}[S_{l-1}] \text{ in } L^2(\mathbb{T}^{N-l+1}).\]

by virtue of Theorem 7.1. Consequently (see \((6.13)\) and \((6.18)\)), \(U_{1}^{(n_1 \ldots n_l \ldots n_1)} \Rightarrow U_{1}\) and applying Corollary 2.2 we get (see \((6.14)\) and \((6.21)\)

\[S_{+l}^{(n_1 \ldots n_l)} = S_{+,l-1}^{(n_1 \ldots n_{l-1})} U_{1}^{(n_1 \ldots n_l)} \to S_{+l} \text{ in } L^2(\mathbb{T}^N).\]
8. Numerical simulations

The computer code for numerical testing of the proposed algorithm was written in MATLAB. A simple outer type polynomial matrix

\[
A = \begin{pmatrix}
4 + y - xy^{-1} + x + 2xy & 1 + 2y + x + xy \\
1 + y + 2xy^{-1} + 2x + 2y & 5 + y + xy^{-1} - x + xy
\end{pmatrix}
\]

(we relabel \( x = t_1 \) and \( y = t_2 \); the integer coefficients are also chosen for notational simplicity) was designed, which is positive definite at the origin, and \( S \) was constructed as

\[
S = AA^*.
\]

Then (8.1) is an exact spectral factorization of

\[
S = \begin{pmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{pmatrix},
\]

where

\[
S_{11} = 9x^{-1}y^{-1} + 9x^{-1} - x^{-1}y - x^{-1}y^2 - 2y^{-2} + 8y^{-1} + 30 + 8y - 2y^2 - xy^{-2} - xy^{-1} + 9x + 9xy;
\]

\[
S_{12} = 9x^{-1}y^{-1} + 11x^{-1} + 9x^{-1}y + 4x^{-1}y^2 - 2y^{-2} + 6y^{-1} + 16 + 17y + 5y^2 - xy^{-2} + xy^{-1} + 9x + 7xy;
\]

\[
S_{21} = 7x^{-1}y^{-1} + 9x^{-1} + x^{-1}y - x^{-1}y^2 + 5y^{-2} + 17y^{-1} + 16 + 6y - 2y^2 + 4xy^{-2} + 9xy^{-1} + 11x + 9xy;
\]

\[
S_{22} = 7x^{-1}y^{-1} + 8x^{-1}y + 3x^{-1}y^2 + 5y^{-2} + 12y^{-1} + 43 + 12y + 5y^2 + 3xy^{-2} + 8xy^{-1} + 7xy;
\]

In less than 2 seconds, a complete 16 digit accuracy of MatLab double precision has been achieved on a computer with the following characteristics: Intel(R) Core(TM) i7 8650U CPU, 1.90 GHz, RAM 16.00 Gb.

As was mentioned in the Introduction, there exist several multivariable MSF methods available in the literature. However, none of them provide numerical examples of factorized multivariable matrices. Therefore, we were unable to carry out a comparative analysis of the proposed method based on numerical simulations.

9. Appendix: Application to Granger causality

Granger causality \[20\] has emerged in recent years as one of the leading statistical techniques in neuroscience for inferring directions of neural interactions and information flow in the brain from collected multidimensional data. In this Appendix, for illustrative purposes, we demonstrate the application of spectral factorization in Granger causality. The basic idea can be traced back to Wiener \[39\].

For two jointly stationary processes \( \ldots, X_{-1}, X_0, X_1, X_2 \ldots \) and \( \ldots, Y_{-1}, Y_0, Y_1, Y_2 \ldots \), let

\[
X_{n+1} = \sum_{k=0}^{\infty} a_k X_{n-k} + \varepsilon_n
\]

be the autoregressive representation of the process \( X \), and let

\[
X_{n+1} = \sum_{k=0}^{\infty} b_k X_{n-k} + \sum_{k=0}^{\infty} c_k Y_{n-k} + \eta_n
\]

be its joint representation, where \( \varepsilon \) and \( \eta \) are corresponding noise terms. It is assumed that \( X_k, Y_k \) belong to a Hilbert space \( \mathcal{H} \) and

\[
\langle Z_n^i, Z_{n+k}^j \rangle = \int_{\mathbb{T}} t^{-k} d\nu_{ij}, \tag{9.1}
\]
where \( i, j = 1, 2; \) \( Z^1 = X, Z^2 = Y; \) and \( \nu = (\nu_{ij}) \) is a matrix spectral measure defined on \( T. \) Therefore, the value \( \sigma_1 = \|\varepsilon_n\| \) measures the accuracy of the autoregressive prediction of \( X_n \) based on its previous values, whereas the value \( \Sigma_1 = \|\eta_n\| \) represents the accuracy of predicting the same value \( X_n \) based on the previous values of both \( X \) and \( Y. \) According to Wiener [39] and Granger [20], if \( \Sigma_1 \) is less than \( \sigma_1 \) in some suitable statistical sense, then \( Y \) is said to have a causal influence on \( X. \) This causal influence is quantified by (see [8])

\[
F_{Y \rightarrow X} = \ln \frac{\sigma_1}{\Sigma_1}.
\]

Recently the concept of multi-step Granger causality has also been introduced [8] with

\[
F_{Y \rightarrow X}^L = \ln \frac{\sigma_L}{\Sigma_L},
\]

where

\[
\sigma_L = \inf_{a_k} \|X_{n+L} - \sum_{k=0}^\infty a_k X_{n-k}\|
\]

and

\[
\Sigma_L = \inf_{b_k, c_k} \|X_{n+L} - \sum_{k=0}^\infty b_k X_{n-k} - \sum_{k=0}^\infty c_k Y_{n-k}\|.
\]

As usually is the case in applications, let us assume below that the stationary processes are regular and non-deterministic. Therefore, the spectral measure is absolutely continuous

\[
\nu(t) = S(t) \, dt
\]

and the matrix function \( S \) satisfies the factorability condition (1.1) (see [33]).

By virtue of (9.8), the process \( \ldots, X_{-1}, X_0, X_1, X_2 \ldots \) is unitary equivalent to \( \{t^n\}_{n \in \mathbb{Z}} \) in the Hilbert space \( L^2(d\nu_{11}). \) Therefore, if \( \nu_{11}(t) = f(t) \, dt \) and \( f(t) = f_+(t) f_+(t) \) is the spectral factorization of \( f, \) then (9.4) can be expressed by

\[
\sigma_L = \sqrt{\sum_{k=0}^{L-1} |C_k\{f_+\}|^2} = \|P_{L-1}[f_+]\|,
\]

where \( P_L \) stands for the projection operator acting as \( P_L : \sum_{k=0}^\infty c_k t^k \rightarrow \sum_{k=0}^L c_k t^k. \) Indeed,

\[
\sigma_L^2 = \inf_{a_k} \|x_{n+L} - \sum_{k=0}^\infty a_k x_{n-k}\|^2 = \inf_{a_k} \frac{1}{2\pi} \int_T \left| t^L - \sum_{k=0}^\infty a_k t^k \right|^2 f(t) \, dt
\]

\[
= \inf_{a_k} \frac{1}{2\pi} \int_T \left( t^L - \sum_{k=0}^\infty a_k t^k \right) f_+(t) \left( t^L - \sum_{k=0}^\infty a_k t^k \right) f_+(t) \, dt
\]

\[
= \inf_{a_k} \left\| t^L P_{L-1}[f_+] + \sum_{k=0}^\infty C_{k+L} \{f_+\} t^k - \sum_{k=0}^\infty a_k t^k f_+(t) \right\|^2
\]

\[
= \left\| t^L P_{L-1}[f_+] \right\|^2 + \inf_{a_k} \left\| \sum_{k=0}^\infty C_{k+L} \{f_+\} t^k - \sum_{k=0}^\infty a_k t^k f_+(t) \right\|^2 = \left\| P_{L-1}[f_+] \right\|^2.
\]

The infimum in the previous line is equal to zero because of the Beurling theorem (see, e.g. [28]).

This reasoning can be extended to the matrix case, and it can be proved that

\[
\Sigma_L = \sqrt{\sum_{k=0}^{L-1} (|C_k\{S_{11}^+\}|^2 + |C_k\{S_{12}^+\}|^2)},
\]

where \( S_{11}^+ \) and \( S_{12}^+ \) are the corresponding entries in the spectral factor of \( S_+. \) Indeed, using the notation \( \|(a, b)\|_2^2 = \|a\|_2^2 + \|b\|_2^2 \) for \( a \) and \( b \) from \( L^2(T) \) and similarly to the scalar case,
we get in the matrix case (cf. [40, 7.9]):

$$
\Sigma_L^2 = \inf_{N, \alpha, \beta} \| x_{n+L} - \sum_{k=0}^{N} \alpha_k x_{n-k} - \sum_{k=0}^{N} \beta_k y_{n-k} \|^2 .
$$

$$
= \inf_{N, \alpha, \beta} \frac{1}{2\pi} \int_T \left( (t-L, 0) - \sum_{k=0}^{N} (\alpha_k, \beta_k) t^k \right) S(t) \left( (t-L, 0)^T - \sum_{k=0}^{N} (\alpha_k, \beta_k)^* t^{-k} \right) dt =
$$

$$
\inf_{N, \alpha, \beta} \frac{1}{2\pi} \int_T \left( (t-L, 0) - \sum_{k=0}^{N} (\alpha_k, \beta_k) t^k \right) S^+(t) \left( S^+(t)^* \right) \left( (t-L, 0)^T - \sum_{k=0}^{N} (\alpha_k, \beta_k)^* t^{-k} \right) dt =
$$

$$
\inf_{N, \alpha, \beta} \left\| t-L (P_{L-1}[S_{11}], P_{L-1}[S_{12}]) + \sum_{k=0}^{\infty} \left( C_{k+L} \{ S_{11}^+ \}, C_{k+L} \{ S_{12}^+ \} \right) t^k - \sum_{k=0}^{N} (\alpha_k, \beta_k) t^k S^+(t) \right\|_2^2
$$

$$
= \left\| P_{L-1}[S_{11}^+] \right\|_2^2 + \inf_{N, \alpha, \beta} \left\| \sum_{k=0}^{\infty} \left( C_{k+L} \{ S_{11}^+ \}, C_{k+L} \{ S_{12}^+ \} \right) t^k - \sum_{k=0}^{N} (\alpha_k, \beta_k) t^k S^+(t) \right\|_2^2
$$

$$
= \| P_{L-1}[S_{11}^+] \|^2_2 + \| P_{L-1}[S_{12}^+] \|^2_2 .
$$

Since $S^+$ is an outer analytic matrix function, the last infimum is $0$ by the vector generalization of Beurling theorem (see, e.g. [11]). Thus (9.6) holds.

Due to (9.3), (9.5), and (9.6), we have

$$
F^L_{Y \rightarrow X} = \ln \frac{\sum_{k=0}^{L-1} | C_k \{ f_{+} \} |^2}{\sum_{k=0}^{L-1} \left( | C_k \{ S_{11}^+ \} |^2 + | C_k \{ S_{12}^+ \} |^2 \right)}. 
$$

Consequently, the spectral factorization plays a crucial role in estimating the Granger causality.

The multivariable matrix spectral factorization algorithm proposed in this paper provides perspective to develop the multivariate Granger causality theory for the situation where the collected data depends on more than one parameter (e.g. when we have a spatio-temporal dependence of random variables on indices). For simplicity of notation, we assume that the number of these parameters is two and we have jointly stationary processes $X_{nm}$ and $Y_{nm}$. Then there exists a spectral measure $\nu = (\nu_{ij})$ defined on $T^2$ such that

$$
\langle Z_{i,n}^j, Z_{m+k,n+l}^j \rangle = C_{kl} = \int_T t_1^{-k} t_2^{-l} d\nu_{ij}(t_1, t_2),
$$

where $Z^1 = X$ and $Z^2 = Y$. Again we assume that $\nu$ is absolutely continuous: $\nu(t) = S(t) dt$, and $S$ satisfies the factorability condition (11.4). The half-plane $H_2 \subset Z^2$ imposes the “temporal order” in $Z^2$ then, namely

$$(n, m) \geq (k, l) \iff (n-k, m-l) \in H_2$$

and the corresponding Granger causality can be defined by

$$
F^{LM}_{Y \rightarrow X} = \ln \frac{\sigma_{LM}}{\Sigma_{LM}},
$$

where

$$
\sigma_{LM} = \inf_{a_{kl}} \left\| X_{n+L,m+M} - \sum_{(k,l) \in H_2} a_{kl} X_{n-k,m-l} \right\|
$$

and

$$
\Sigma_{LM} = \inf_{b_{kl}, c_{kl}} \left\| X_{n+L,m+M} - \sum_{(k,l) \in H_2} b_{kl} X_{n-k,m-l} - \sum_{(k,l) \in H_2} c_{kl} Y_{n-k,m-l} \right\|.
$$
Denoting the (multivariable) scalar spectral factor of $S_{11}$ by $f_+$ and the (multivariable) matrix spectral factor of $S$ by $S^+$, one can show that
\[
\sigma_{LM}^2 = \sum_{(0,0) \leq (k,l) < (L,M)} |C_{kl}\{f_+\}|^2
\]
and
\[
\Sigma_{LM}^2 = \sum_{(0,0) \leq (k,l) < (L,M)} (|C_{kl}\{S_{11}^+\}|^2 + |C_{kl}\{S_{12}^+\}|^2),
\]
similarly to (9.5) and (9.6). Therefore, similarly to (9.7), we have
\[
F_{Y \rightarrow X}^{LM} = \ln \frac{\sum_{(0,0) \leq (k,l) < (L,M)} |C_{kl}\{f_+\}|^2}{\sum_{(0,0) \leq (k,l) < (L,M)} (|C_{kl}\{S_{11}^+\}|^2 + |C_{kl}\{S_{12}^+\}|^2)}.
\]

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