Nonlocality without nonlocality*

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Abstract

Bell’s theorem is purported to demonstrate the impossibility of a local “hidden variable” theory underpinning quantum mechanics. It relies on the well-known assumption of ‘locality’, and also on a little-examined assumption called ‘statistical independence’ (SI). Violations of this assumption have variously been thought to suggest “backward causation”, a “conspiracy” on the part of nature, or the denial of “free will”. It will be shown here that these are spurious worries, and that denial of SI simply implies nonlocal correlation between spacelike degrees of freedom. Lorentz-invariant theories in which SI does not hold are easily constructed: two are exhibited here. It is conjectured, on this basis, that quantum-mechanical phenomena may be modeled by a local theory after all.

1 Introduction

The violation of the Bell-CHSH inequality by quantum mechanics is commonly understood to undermine the possibility of “local” hidden-variable theories, i.e., theories which either supplement quantum mechanics with additional variables (e.g., actual particle positions in the Bohm theory (4)(5)), or replace the quantum-mechanical description with something else entirely. This “no-go” result is known as Bell’s theorem (1). The argument is essentially that the assumption of a certain kind of locality – known as ‘Bell locality’, ‘factorizability’ (12), ‘strong locality’ (15), or simply ‘locality’ (1) – is a sufficient condition to derive the inequality, and the predictions of quantum mechanics violate this inequality. (The locution “strong locality” suggests that there are weaker forms of locality, and indeed it was shown by Jarrett (15) (16) and by Shimony (21) that ‘strong locality’ is the conjunction of two weaker locality conditions, referred to by Shimony as “outcome independence” and “parameter independence”.)

However, there is an additional, nontrivial assumption that goes into the Bell argument, and this is the assumption of statistical independence (SI). Its role in the derivation was understood by Bell and others, but it has been little examined because it has for the most part been thought to be beyond question, since violations of it have seemed to most to entail either violations of “free will”, the possibility of “backward causation”, or some sort of cosmic conspiracy on the part of nature. I will show that, to the contrary, violations of SI entail none of these, and I will in fact offer in support of this contention two examples of classical, Lorentz-invariant theories that violate SI.

The paper proceeds as follows

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• Section 2 rehearses the way in which the constraints of factorizability and statistical independence come into the derivation of the Bell-CHSH inequality.

• Section 3 shows that SI entails that spacelike degrees of freedom are independent, and that violation of SI implies that spacelike degrees of freedom are not independent (rendering the term “degrees of freedom” something of a misnomer).

• In section 4, it is argued that violation of SI does not lead to problems with free-will, backward causation, etc.

• In section 5, two examples of SI-violating theories are offered.

• In section 6, it is shown that in fact violation of SI leads naturally to the “contextuality” (of the values of various degrees of freedom) demanded by the Kochen-Specker theorem.

• In section 7, we discuss future prospects for a theory with nonlocal constraints.

• Section 8 comprises some concluding remarks.

## 2 Bell’s theorem

The thought experiment at the core of the Einstein-Podolsky-Rosen (EPR) paper on the incompleteness of quantum mechanics (11) involves a pair of particles in an entangled state of position and momentum, a state which is an eigenstate of the quantum-mechanical operators representing the sum of the momenta and the difference of the positions of the particles. Quantum mechanics makes no definite predictions for the position and momentum of each particle, but does make unequivocal predictions for the position or momentum of one, given (respectively) the position or momentum of the other. EPR argued that this showed that quantum mechanics must be incomplete, since measurement of the position (or momentum) of one particle could not simultaneously give rise to a definite position (or momentum) of the other particle, on pain of violation of locality. They concluded that quantum mechanics, because it did not assign a position (or momentum) to the other particle beforehand, must be incomplete\footnote{The argument of the EPR paper is notoriously convoluted, but I follow (13) in regarding this as capturing Einstein’s understanding of the core argument.}

Bohm’s streamlined version of the EPR experiment (3) involves the spins of a pair of particles (either fermions or bosons) rather than their positions and momenta. Prepared in what has come to be known as a “Bell” state,

\[
\psi = \frac{1}{\sqrt{2}}(|+x\rangle_A |−x\rangle_B − |−x\rangle_A |+x\rangle_B),
\]

quantum mechanics predicts that a measurement of the component of spin of particle A in any direction (e.g., $\hat{z}$) is as likely to yield $+1$ as $−1$ (in units of $\hbar/2$), and so the expectation value $\bar{A}$ is 0. However, quantum mechanics also indicates that an outcome of $+1$ for a measurement of the spin of A in the $\hat{z}$ direction is guaranteed to yield an outcome of $−1$ for B for a measurement of the spin of B in the $\hat{z}$ direction. etc. This is directly analogous to the correlations between position and momentum measurements in the original EPR experiment.

In and of themselves, these phenomena offer no barrier to a hidden-variable theory, since it is straightforward to explain such correlations by appealing to a common cause – the source – and postulating that the particles emanate from this source in (anti)correlated pairs. However,
such an explanatory strategy must also account for the way that the anticorrelation drops off as the angle between the components of spin for the two particles increases (e.g., as \( \mathbf{A} \) rotates from \( \hat{x} \) toward \( \hat{z} \) while \( \mathbf{B} \) remains at \( \hat{x} \)). It was Bell’s great insight to note that the quantum theory implies that the anticorrelation is held onto more tightly than could be accounted for by any “local” theory. Bell showed that the predictions of a local theory must satisfy an inequality (a precursor to the Bell-CHSH inequality below), and that this inequality is violated by quantum theory for appropriate choices of the components of spin to be measured.

In order to understand the role of the locality assumption and the statistical independence assumption, let us briefly review the derivation of the Bell-CHSH inequality. The physical situation we are attempting to describe has the following form:

A source (represented by the ellipse) emits a pair of particles, or in some other way causes detectors \( \mathbf{A} \) and \( \mathbf{B} \) to simultaneously (in some frame) register one of two outcomes. The detectors can be set in one of two different ways, corresponding, in Bohm’s version of the EPR experiment, to a measurement of one of two different components of spin.

Let us now suppose that we have a theory that describes possible states of the particles by a discrete or continuous parameter \( \lambda \), describing either a discrete set of states \( \lambda_1, \lambda_2, \ldots \) or a continuous set. We will also suppose that the theory provides us with predictions for the average value \( \bar{A}(a, \lambda) \) and \( \bar{B}(b, \lambda) \) of measurements of properties \( a \) and \( b \) at detector \( \mathbf{A} \) and \( \mathbf{B} \) in any given state \( \lambda \). (The appeal to average values allows for stochastic theories, in which a given \( \lambda \) might give rise to any number of different outcomes, with various probabilities.) In general, one might suppose that \( \bar{A} \) also depended on either the detector setting \( b \) or the particular outcome \( B \) (i.e., \( \bar{A} = \bar{A}(a, \lambda, b, B) \)) and similarly for \( \bar{B} \). That it does not, that the expectation value \( \bar{A} \) in a given state \( \lambda \) does not depend on what one chooses to measure at \( \mathbf{B} \), or on the value of the distant outcome \( B \) (and vice-versa) is Bell’s locality assumption. Given this assumption, one can write the expression \( E(a, b, \lambda) \) for the expected product of the outcomes of measurements of properties \( a \) and \( b \) in a given state \( \lambda \) as

\[
E(a, b, \lambda) = \bar{A}(a, \lambda)\bar{B}(b, \lambda). \tag{2}
\]

This condition is also known as ‘factorizability’, deriving as it does from the fact that the joint probability of a pair of outcomes can be factorized into the product of the marginal probabilities of each outcome. We can thus represent the analysis of the experimental arrangement in this way:

\[
E(a, b, \lambda) = \bar{A}(a, \lambda)\bar{B}(b, \lambda)
\]

Now, a theory that accounts for our observations will presumably do so in part by giving a probability distribution \( P(\lambda) \) over the various possible states associated with a given “preparation” (a given set of circumstances in the region of the ellipse in the diagram above), and the expected outcome \( E(a, b) \) will then be given by the weighted sum (we restrict to discrete \( \lambda \) for
simplicity)

\[ E(a, b) = \sum_{\lambda} E(a, b, \lambda) P(\lambda|a, b) = \sum_{\lambda} \bar{A}(a, \lambda) \bar{B}(b, \lambda) P(\lambda|a, b) \]  

(3)

where \( P(\lambda|a, b) \) is the probability of \( \lambda \) given detector settings \( a \) and \( b \). Thus the expected value for the product of a measurement of spin components \( a_1 \) and \( b_2 \) is

\[ E(a_1, b_2) = \sum_{\lambda} \bar{A}(a_1, \lambda) \bar{B}(b_2, \lambda) P(\lambda|a_1, b_2) \]  

(4)

the sum of the products of the expected values of the outcome at A, the outcome at B in each state \( \lambda (\lambda_1, \lambda_2, \text{etc.}) \), and the probability of that state. If the probability of \( \lambda \) is independent of the detector settings \( a \) and \( b \), then one can replace \( P(\lambda|a, b) \) with \( P(\lambda) \). This is the condition known as Statistical Independence (SI), and as we shall now see, it is crucial to the derivation of Bell’s result.

The Bell-CHSH inequality (6), is:

\[ |E(a_1, b_1) - E(a_1, b_2)| + |E(a_2, b_2) + E(a_2, b_1)| \leq 2 . \]  

(5)

The beginning of the derivation goes as follows. First, write down the difference between expectation values for pairs of settings \( \langle a_1, b_1 \rangle \) and \( \langle a_1, b_2 \rangle \):

\[ E(a_1, b_1) - E(a_1, b_2) = \sum_{\lambda} \bar{A}(a_1, \lambda) \bar{B}(b_1, \lambda) P(\lambda|a_1, b_1) - \sum_{\lambda} \bar{A}(a_1, \lambda) \bar{B}(b_2, \lambda) P(\lambda|a_1, b_2) \]  

(6)

Assuming that SI holds, we can rewrite this as

\[ E(a_1, b_1) - E(a_1, b_2) = \sum_{\lambda} \bar{A}(a_1, \lambda) \bar{B}(b_1, \lambda) P(\lambda) - \sum_{\lambda} \bar{A}(a_1, \lambda) \bar{B}(b_2, \lambda) P(\lambda) \]  

(7)

The key step, which allows the introduction of \( E(a_2, b_2) \) and \( E(a_2, b_1) \), involves expanding this as

\[ E(a_1, b_1) - E(a_1, b_2) = \sum_{\lambda} \bar{A}(a_1, \lambda) \bar{B}(b_1, \lambda) P(\lambda)(1 \pm \bar{A}(a_2, \lambda) \bar{B}(b_2, \lambda)) - \sum_{\lambda} \bar{A}(a_1, \lambda) \bar{B}(b_2, \lambda) P(\lambda)(1 \pm \bar{A}(a_2, \lambda) \bar{B}(b_1, \lambda)) . \]  

(8)

This then leads to (5) via rearrangement of terms and manipulation using the relations \(|x| |y| = |xy| \) and \(|x + y| \leq |x| + |y| \). For our purposes, though, the crucial step is (7), in which essential use is made of SI. If we were not to assume SI, then (7) would revert to (6) and we would have to rewrite (8) as

\[ E(a_1, b_1) - E(a_1, b_2) = \sum_{\lambda} \bar{A}(a_1, \lambda) \bar{B}(b_1, \lambda) P(\lambda|a_1, b_1)(1 \pm \bar{A}(a_2, \lambda) \bar{B}(b_2, \lambda)) - \sum_{\lambda} \bar{A}(a_1, \lambda) \bar{B}(b_2, \lambda) P(\lambda|a_1, b_2)(1 \pm \bar{A}(a_2, \lambda) \bar{B}(b_1, \lambda)) \]  

(9)

which is simply invalid, since the new terms need not sum to zero anymore. Nor, for that matter, would they correspond to the desired \( E(a_2, b_2) \) and \( E(a_2, b_1) \). Thus without appeal to SI, there is no way to introduce the other two expectation values and derive the inequality.
3 Statistical independence revisited

The assumption of SI has been called into question only infrequently, but when it has, the critique has often been motivated by an appeal to the plausibility of Lorentz-invariant “backward causation”, whereby the change of detector settings gives rise to effects which propagate along or within the backward lightcone and thereby give rise to nontrivial initial correlations in the particle properties encoded in $\lambda$ (e.g., (8),(24), (19)). In this section, I will argue that this is an inappropriate way to motivate the rejection of SI, and that its rejection instead involves a relativistically nonproblematic commitment to nonlocal constraints on initial data.

Depicted in Figure 1 is a run in which the setting of A is changed from $a_1$ to $a_2$ while the particles (or whatever it is that emanates from the source) are in flight.

The different colors subsequent to the arrival of the particles at the two detectors correspond to distinct experimental outcomes.

Now a clear-thinking student of relativity should suspect that something is amiss with this argument, since all deterministic theories in use today already sanction a form of backward causation, in that they allow both prediction and retrodiction. All special and general-relativistic theories have a well-defined causal structure which makes no distinction, other than a conventional one, between future and past. Specifying the physical properties (the Cauchy data) at each point on either of the shaded surfaces suffices to determine the physical situation at $E$ (Figure 2). The future data determine the event $E$ just as much as the past data. And given an appropriate description of the future data—a description which adopts a “backward-directed” temporal orientation—one can regard these data as the cause of the event $E$.\footnote{For example, we seek the cause of an explosion in the past, but if the very same event were described from an inverted temporal perspective, described as an implosion, we would look in the opposite temporal direction.}

Let us put aside any qualms we might have regarding the notion of backward causation for the moment and examine the particular situation of the EPR-Bohm experiment more closely, in the hope that this will shed some light. Suppose we simply temporally invert the situation above, as in Figure 3. This looks like a pair of sources, A and B, emitting particles in the direction...
Figure 2: Past and future domains of dependence

Figure 3: EPR: Spacetime diagram with inverted time axis
of a common destination. Is it not reasonable to expect that the “final” state \( \lambda = (\lambda_A, \lambda_B) \) is correlated with the settings of the sources?

In fact, it is not. Suppose we know some, but not all, of the data in the past lightcone of an event \( E \). Suppose, for instance, that we know that the white region is empty of physically significant data, and suppose we know the data in the red and blue ellipses, but not in the turquoise ellipses indicated by question-marks:

In such a case, we know nothing more about what to expect at \( E \) than if we had no information at all. Given the ability to choose data in the turquoise ellipsoids, we can make \( E \) whatever we want.

But now suppose we fill in the unspecified ellipses:

Then we know data on a Cauchy slice of the past lightcone of \( E \), and \( E \) is fully determined. It is this situation that is the appropriate parallel of the time-reversed EPR experiment; the newly-specified data correspond to the outcomes of the two trials.

Any plausibility that the particles might be causally correlated with the detector settings derives from a situation in which the detectors themselves are the sources of the particles, rather than mere conduits. In the EPR case, there are meaningful, (anti)correlated detection events, and in the time-reversed picture these detection events serve as additional data, additional sources. (One can move the slice so that it is prior to any outcome, but one will still have to contend with the fact that only a complete specification of the physics inside the detector, including the state of the particles, is sufficient to determine \( E \).)

The upshot, then, is that postulating a correlation between detector settings and the initial state \( \lambda \) (corresponding to \( E \) in the figure above) – i.e., dropping \( SI \) – amounts to postulating a correlation between the detector settings and the particle properties at any given time. I.e., the correlations are not causal – they are not brought about dynamically – but are properties of the state at any instant. The role of dynamics in a proper theory describing quantum phenomena is to enforce, not generate, such correlations. The challenge, as yet unmet, is to articulate the constraint which encodes these correlations. In section 6, we will examine two theories with
appropriately nonlocal constraints in order to develop intuition. But first, let us examine two other problems which are purported to arise in the rejection of SI.

4 Superdeterminism and free will

The idea that the rejection of SI involves not some dynamical process like “backward causation” but rather some preexisting and persisting correlations between subsystems has been broached before, under the terms ‘conspiracy theory’, ‘hyperdeterminism’, and ‘superdeterminism’. Bell (2), Shimony (22), Lewis (18) and others have suggested that proposing a correlation between detector settings and particle properties involves some sort of conspiracy on the part of nature. This is frequently accompanied by the charge that the existence of such correlations is a threat to “free will”. Let us address these worries.

4.1 Conspiracy

The idea that postulating a correlation between detector settings and particle properties involves a “conspiracy” on the part of nature appears to derive from the idea that it amounts to postulating that the initial conditions of nature have been set up in anticipation of our measurements. It might be supposed, analogously, that every time I telephone my friend Jenny at 867-5309, something – perhaps a cosmic ray – causes my message to be misdirected to the non-working number 867-5308, so that I appear to live in a Kafkaesque world in which my efforts to contact Jenny are forever stymied. This would appear to be a world in which nature, in the form of particularly vicious initial conditions, conspires against me to the point where I am driven to postulate that it is a law of nature that I cannot successfully contact Jenny (except, perhaps, on her mobile phone). But according to the way the story is told, it is really just an accident that I cannot successfully make contact. Similarly, the conspiracy theorist views the appeal to a failure of SI in order to explain the strange correlations predicted by quantum theory as an appeal to a vast conspiracy on the part of nature to set initial conditions in such a way as to ensure that experiments come out in accord with the quantum-mechanical predictions, so that every time I do an EPR experiment it just happens to be the case that the detectors are set in a way appropriate to generate the observed correlations.

What the conspiracy theorist is in effect doing is supposing a non-lawlike suspension of SI. That is, she is supposing that the laws of nature are ordinary, local, relativistic laws, without any nonlocal constraints, but that the initial conditions are such that it happens to turn out that the states of measuring apparatuses are nontrivially, and persistently, correlated with the states of the particles they eventually interact with. The idea seems to be that, were the initial conditions to have been somewhat different, the entire quantum-mechanical edifice would fall apart. Certainly, this is a theoretical possibility, but not a very happy one, for two reasons. Were one to maintain that the laws of nature are the ordinary classical ones, with no general, nonlocal constraints, and that quantum mechanics is the result of a highly special set of initial conditions, one would be foregoing the possibility of explaining the myriad phenomena accounted for by quantum mechanics which have nothing to do with measurement, such as the stability of matter or the black-body emission spectrum. Although a theory that purports to account for the full spectrum of quantum phenomena in a way that does not violate Bell’s locality assumptions must specify nontrivial correlations between spacelike degrees of freedom, it cannot do just that. Rather, the constraints in an SI-violating theory must account for the full range of phenomena accounted for by quantum mechanics and quantum field theory. Thus a truly
useful and predictive theory underpinning quantum phenomena is highly unlikely to have the 
*ad hoc* character which concerns the conspiracy theorist.

### 4.2 Free will

Another worry about giving up *SI* and postulating generic nonlocal, spacelike correlations has
to do with a purported threat to our “free will”. This particular concern has been the subject
of renewed debate in the last couple of years, prompted in part by an argument of Conway &
Kochen (7). The core of the worry is that if detector settings are correlated with particle
properties, this must mean that we cannot “freely choose” the detector settings. This worry,
however, appears to be based on a conception of free will which is incompatible with ordinary
determinism, as pointed out by ’t Hooft (25). Why is it any more of a threat to free will to have our “actions” correlated with other degrees of freedom than it is to have our actions be *determined* by the events in our past? (Conway and Kochen bite the bullet and argue that
even ordinary determinism is incompatible with free will.)

One might conceivably make the case that ordinary deterministic theories are fine in a way
that superdeterministic theories, with their nonlocal constraints, are not by arguing that, in
allowing that our actions are determined by the past, we are simply granting that our actions
arise from our own thoughts and inclinations. This (limited) sort of determination is actually
*essential* for free choice. This perspective is a version of what is called ‘compatibilism’ in
philosophy, the view that freedom of the will is compatible with determinism (14).

A problem for free will would then arise if it were the case that the nonlocal correlations
associated with an underlying superdeterministic theory somehow *prevented* an agent from acting
on its thoughts and inclinations. This is to say that the physical object identified as the “agent”
would exhibit behavior not explicable in terms of the influences on or in its past lightcone.
But this is not what is being contemplated here, for this would involve non-Lorentz-invariant
dynamics. What is at issue are Lorentz-invariant theories with *nonlocal constraints*.

### 5 Theories with nonlocal constraints: two examples

Theories which have constraints on initial data can be divided into two kinds, local and nonlocal.
The gauge field theories of the standard model are local, in that the constraint may be expressed
as a local condition, for example $\nabla \cdot E = 0$, the Gauss law (in *vacuo*). The locality of the
constraint means that specifying the field at every point outside of an open set surrounding a
point $x$ does not constrain the value of the field at $x$. Rather, the field in the neighborhood of
a point is constrained only by the field at the point.

Theories with nonlocal constraints are less familiar. These are theories in which specifying
the value of a field outside the neighborhood of a point $x$ constrains the field at $x$. We will now
consider two examples of such theories.

#### 5.1 Timelike Cauchy surfaces

Consider the theory of the massless scalar field, given by the wave equation $\square \phi = 0$. In two
space dimensions, this reads

\[
\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \phi = 0 \tag{10}
\]
where $\phi(x, t)$ is a twice-differentiable, real-valued field on spacetime. It is well-known that the Cauchy problem is well-posed, meaning that specifying the field $\phi(x)$ and its normal derivative $\partial \phi(x)/\partial t$ on a spacelike hyperplane $t = 0$ uniquely fixes the field at all other times. More important, it is also the case that a solution exists for any such data, meaning that the field and its time rate-of-change at each point are independently specifiable. Thus the ordinary initial value formulation of the wave equation has no constraints, either local or nonlocal.

On the other hand one can also specify initial data on a mixed (spacelike and timelike) hyperplane. Given data $\phi(x_1, t)$ and $\partial \phi(x_1, t)/\partial x_2$ on the hyperplane $x_2 = 0$, the data uniquely determine the solution, if a solution exists at all for that data. The fact that solutions do not exist for arbitrary data (except in the case of one space dimension) means that, as formulated, the initial value problem is not “well-posed.” However, it has recently been shown (10) that, just as in a gauge theory, one may write down a constraint on the initial data such that any initial data satisfying the constraint lead to a unique, stable solution of the equation. The resulting problem is well-posed.

The difference between this constraint and the gauge-theoretic constraint is that the former is nonlocal while the latter is local. Specifically, the Cauchy data is given by functions $f$ and $g$ such that

$$f(x_1, t) := \phi(x_1, 0, t) = \int_{k_1^2 \geq \omega^2} \tilde{f}(k_1, \omega) e^{i(k_1 x_1 - \omega t)} dk_1 d\omega$$

$$g(x_1, t) := \frac{\partial \phi(x_1, 0, t)}{\partial x_2} = \int_{k_1^2 \geq \omega^2} \tilde{g}(k_1, \omega) e^{i(k_1 x_1 - \omega t)} dk_1 d\omega$$

where $\tilde{f}$ and $\tilde{g}$ are smooth functions of $k_1$ and $\omega$, related to $f$ and $g$ by the Fourier transform.

The upshot of this example is that the ordinary theory of the massless scalar field, formulated in terms of states specified on mixed spacelike and timelike hypersurfaces, is one in which a natural generalization of SI to the case of fields (rather than particle states and detector settings) is violated. I.e., the natural analogue of $P(\lambda|a,b) = P(\lambda)$ does not hold. For example, consider disjoint compact regions $A$, $B$ and $\Lambda$ on the initial data surface. Let $\lambda = (f(\Lambda), g(\Lambda))$ represent the state the field in $\Lambda$, and let $a = (f(A), g(A))$ and $b = (f(B), g(B))$ represent the detector settings $a$ and $b$. Then it is the case that, given a generic probability distribution on the space of initial data $f$ and $g$, the probability of $\lambda$ (the restriction of $f$ and $g$ to region $\Lambda$) will not be independent of $a$ and $b$. For example, if the functions $f$ and $g$ vanish in the regions $A$ and $B$, then it must be the case that they vanish in $\Lambda$ (otherwise $\Lambda$ would be a region in which $f$ and $g$ have compact support, which we know not to be the case).

Note that, despite the failure of SI in this context, we have perfectly a well-posed initial value problem, and we even have compact domains of dependence (see Figure 4). Here, data in $R$ determines data in the region out to $E$. The nonlocal constraint simply means that data in $R$ may not be specified freely.

### 5.2 Timelike compactification

Consider once more the wave equation, this time in one space dimension (for simplicity). And consider again the initial value problem, but on an ordinary spacelike hyperplane. However,

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3These mixed hypersurfaces are sometimes called “timelike” (17) or “non-spacelike” (9).
suppose that the spacetime on which the field takes values is compactified in the time direction, so that the entirety forms a cylinder (see Figure 5). This, too, is an example of a theory whose initial value formulation possesses a nonlocal constraint.

The reason for the constraint is of course that the solutions must be periodic. Whereas in the ordinary initial value problem, initial data may be any smooth functions $f(x) = \phi(x, 0)$ and $g(x) = \dot{\phi}(x, 0)$, we now require that $\phi(x, 0) = \phi(x, T)$ and $\dot{\phi}(x, 0) = \dot{\phi}(x, T)$, where $T$ is the circumference of the cylinder. Solutions to the wave equation can be written as sums of plane waves, with Fourier space representation

$$\hat{\phi}(k, t) = \hat{F}(k)e^{-ikt} + \hat{G}(k)e^{ikt},$$

Since these plane waves must have period $T$ (in the preferred frame dictated by the cylinder),
we have a constraint \( k = \frac{2\pi n}{T} \) (where \( n \) is a positive or negative integer), so that initial data are no longer arbitrary smooth functions of \( k \)

\[
\dot{\phi}(k, 0) = \hat{F}(k) + \hat{G}(k) \\
\dot{\phi}_t(k, 0) = -ik(\hat{F}(k) - \hat{G}(k))
\]

but are rather constrained by the requirement \( k = \frac{2\pi n}{T} \). Thus the initial data are the functions

\[
\phi(x, 0) = \frac{1}{\sqrt{T}} \sum_{n=-\infty}^{\infty} \hat{\phi}(\frac{2\pi n}{T}, 0)e^{i\frac{2\pi n}{T}x}dk \\
\phi_t(x, 0) = \frac{1}{\sqrt{T}} \sum_{n=-\infty}^{\infty} \hat{\phi}_t(\frac{2\pi n}{T}, 0)e^{i\frac{2\pi n}{T}x}dk
\]

i.e., they consist of arbitrary sums of plane waves with wave number \( k = \frac{2\pi n}{T} \).

The restriction to a discrete (though infinite) set of plane waves means that initial data do not have compact support; they are periodic in both space and time. Thus, as in the case of the mixed initial value problem, the data cannot be specified freely. However, for sufficiently large \( T \) or sufficiently small \( \Delta x \), the local physics is indistinguishable from the local physics in ordinary Minkowski spacetime. Only at distance scales on the order of \( T \) does the compact nature of the direction become evident in the repetition of the spatial structure.

### 6 Contextuality

The Kochen-Specker theorem points toward a kind of contextuality in quantum mechanics, and indeed in any theory in which the properties of a system are understood to be independent of the properties of other systems. The theorem shows that, for systems described by quantum mechanics, the properties of these systems cannot consistently be assigned values if the values respect a certain seemingly natural criterion called ‘functional composition’ \(^\text{(20)}\). Functional composition is the assumption that the value \( v(\hat{A}\hat{B}) \) of an observable \( \hat{A}\hat{B} \) which is the product of commuting observables \( \hat{A} \) and \( \hat{B} \) is equivalent to the product \( v(\hat{A})v(\hat{B}) \) of the values of each observable, as long as the observables commute. Given this assumption, one can show that the following set of operators, representing spin observables for a system composed of two spin-1/2 particles, cannot simultaneously be assigned values in a way that is consistent with the requirement that the values belong to the eigenvalue spectrum of the operators

\[
I \otimes \hat{\sigma}_z \quad \hat{\sigma}_z \otimes I \\
\hat{\sigma}_z \otimes I \quad I \otimes \hat{\sigma}_z \\
\hat{\sigma}_x \otimes \hat{\sigma}_z \quad \hat{\sigma}_z \otimes \hat{\sigma}_x
\]

\[(13)\]

Rather, the value assigned to a given observable must depend on whether it is being measured along with the other (commuting) observables in its row, or the other observables in its column.

Recent work on generalizing the Kochen-Specker result to any theory admitting an operational characterization shows that there is a sense in which any theory that reproduces the predictions of quantum mechanics must be contextual \(^\text{(23)}\). Without going into unnecessary detail, the general idea is that any theory reproducing the predictions of quantum theory must be such that the probabilities for various outcomes must in general depend on which other properties are (simultaneously) measured. Such a result, however, is utterly unsurprising in a theory with nonlocal constraints, so long as one recognizes that the detector orientations themselves
are part of the system, since the nonlocal constraint means that the degrees of freedom of the
detectors are not independent of those describing the particles. Indeed, from a non-operational,
closed-system point of view, one may view the contextuality implicit in the Kochen-Specker
theorem as implying the existence of a nonlocal constraint. This sheds light on the relationship
between Bell’s theorem and the Kochen-Specker theorem, in that K-S essentially shows that
any local hidden-variable theory must violate statistical independence, while Bell shows that
any statistically independent theory must violate locality.

7 Future directions

Neither of the two examples above, examples of theories with nonlocal constraints, appear to
have any direct connection to quantum mechanics, though the mixed initial value problem might
be so related. It is certainly worth investigating what sort of theory emerges if one takes, for
example, the wave equation on three space and two time dimensions (called an ‘ultrahyperbolic’
equation) and considers data on an initial Cauchy surface of 3 + 1 dimensions. Such a theory will
also have nonlocal constraints, and might give rise to interesting behavior when the extra time
dimension is averaged over in such a way as to generate an effective theory in 3 + 1 spacetime
dimensions. The obvious difficulty is that, if the extra time dimension is not compact, there may
be no obvious choice of measure over which to average.

Perhaps more intriguingly, one might ponder the way in which the ordinarily superfluous
gauge degrees of freedom of modern gauge theories might serve as nonlocal hidden variables.
The vector potential in electrodynamics, for example, ordinarily plays no direct physical role: only
derivatives of the vector potential, which give rise to the electric and magnetic fields, correspond
to physical “degrees of freedom” in classical and quantum electrodynamics. The
Aharonov-Bohm effect shows that the vector potential does play an essential role in the quantum
theory, but the effect is still gauge-invariant. One might nevertheless conjecture that there is
an underlying theory in which the potential does play a physical role, one in which the physics is
not invariant under gauge transformations. The indeterminacy we associate with quantum
theory may then arise via epistemic limitations. More specifically, it may be impossible for us
to directly observe the vector potential, and the uncertainties associated with quantum theory
may arise from our ignorance as to its actual (and nonlocally constrained) value. From this
perspective, quantum theory would be an effective theory which arises from modding out over
the gauge transformations.

Finally, recent work on decoherence and the emergence of classicality suggests that the
emergence of classicality requires very special quantum states. For worlds with a large number
of subsystems, hence a high Hilbert space dimension, only a measure zero subset of the total set
of quantum states gives rise to distinctively classical behavior. Thus it seems quite reasonable
to suppose a similarly strong constraint on the states of a hidden-variable theory.

4 One might also ponder the connection with the closely related way in which energy enters into classical,
nongravitational physics. In the absence of gravity, only differences in energy are held to be observable, but
when gravity enters the picture, absolute values of energy are understood to be relevant. Or course, this in turn
leads to the cosmological constant problem when one attempts to couple a quantum theory of matter to classical
gravitation.

5 't Hooft has also gestured in this direction - see (25), p. 7.
8 Conclusion

The ideas sketched in the previous section are preliminary, of course, and they are only two of many possible ways to construct theories which feature nonlocal constraints. What the reader should take away from this paper, if nothing else, is the idea that it is not all that difficult to construct nonlocal theories which nevertheless local in the sense of being Lorentz-invariant and not allowing superluminal signaling, and that such theories are quite promising as deterministic or stochastic models of many of the curious phenomena described by quantum mechanics.

References

[1] J. Bell. On the Einstein–Podolsky–Rosen paradox. Physics, 1:195–200, 1964.
[2] J. Bell. La nouvelle cuisine. In M. Bell, K. Gottfried, and M. Veltman, editors, John S. Bell on the Foundations of Quantum Mechanics, pages 216–234. World Scientific, Singapore, 2001.
[3] D. Bohm. Quantum Theory. Prentice-Hall, Englewood Cliffs, 1951.
[4] D. Bohm. A suggested interpretation of the quantum theory in terms of hidden variables. 1. Phys. Rev., 85:166–179, 1952.
[5] D. Bohm. A suggested interpretation of the quantum theory in terms of hidden variables. 2. Phys. Rev., 85:180–193, 1952.
[6] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt. Proposed experiment to test local hidden variable theories. Physical Review Letters, 23:880–884, 1969.
[7] J. Conway and S. Kochen. The free will theorem. Foundations of Physics, 36:1441, 2006.
[8] O. Costa De Beauregard. S Matrix, Feynman Zigzag and Einstein Correlation. Phys. Lett., A67:171–174, 1978.
[9] R. Courant. Methods of Mathematical Physics, Vol. II: Partial Differential Equations. Interscience, New York, 1962.
[10] W. Craig and S. Weinstein. On determinism and well-posedness in multiple time dimensions. 2008. arXiv.org:0812.0210.
[11] A. Einstein, B. Podolsky, and N. Rosen. Can quantum-mechanical description of reality be considered complete? Physical Review, 47:777–780, 1935.
[12] A. Fine. Correlations and physical locality. In P. Asquith and R. Giere, editors, PSA 1980, Volume 2, pages 535–556. Philosophy of Science Association, E. Lansing, MI, 1981.
[13] A. Fine. The Shaky Game: Einstein, Realism and the Quantum Theory. University of Chicago Press, Chicago, 1986.
[14] D. Hume. A Treatise of Human Nature. Oxford Univ. Press, Oxford, 2000. orig. pub. 1739–1740.
[15] J. Jarrett. On the physical significance of the locality conditions in the Bell arguments. Noûs, 18:569–589, 1984.
[16] J. Jarrett. Bell’s theorem: a guide to the implications. In J. Cushing and E. McMullin, editors, *Philosophical Consequences of Quantum Theory*, pages 60–79. Univ. of Notre Dame Press, Notre Dame, 1989.

[17] F. John. *Partial Differential Equations*. Springer-Verlag, fourth edition, 1991.

[18] P. Lewis. Conspiracy theories of quantum mechanics. *British Journal for the Philosophy of Science*, 57:359–381, 2006.

[19] H. Price. *Time’s Arrow and Archimedes’ Point: New Directions for the Physics of Time*. Oxford University Press, Oxford, 1996.

[20] M. Redhead. *Incompleteness, Nonlocality, and Realism*. Clarendon Press, Oxford, 1989.

[21] A. Shimony. Controllable and uncontrollable non-locality. In S. Kamefuchi, H. Ezawa, and M. Namiki, editors, *Foundations of Quantum Mechanics in Light of New Technology*. The Physical Society of Japan, Tokyo, 1984. Reprinted in A. Shimony, *Search for a Naturalistic World View, Vol II*, Cambridge University Press, Cambridge, 1993, 130–139.

[22] A. Shimony. An exchange on local beables. In *Search for a Naturalistic World View, Vol II*, pages 163–170. Cambridge University Press, Cambridge, 1993.

[23] R. W. Spekkens. Contextuality for preparations, transformations and unsharp measurements. *Physical Review A*, 71:052108, 2005.

[24] R. I. Sutherland. Bell’s theorem and backwards in time causality. *Int. J. Theor. Phys.*, 22:377–384, 1983.

[25] G. ’t Hooft. On the free will postulate in quantum mechanics. 2007. arXiv:quant-ph/0701097.

[26] S. Weinstein. Decoherence and the nonemergence of classicality. 2008. arXiv.org:0807.3376v2.