Lifshitz Solutions of D=10 and D=11 supergravity

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Abstract

We construct infinite families of Lifshitz solutions of $D = 10$ and $D = 11$ supergravity with dynamical exponent $z = 2$. The new solutions are based on five- and seven-dimensional Einstein manifolds and are dual to field theories with Lifshitz scaling in 1+2 and 1+1 spacetime dimensions, respectively. When the Einstein spaces are Sasaki-Einstein, the solutions are supersymmetric.
1 Introduction

It is a tantalising possibility that holographic techniques in string/M-theory can be used to study strongly coupled condensed matter systems using weakly coupled theories of gravity. One focus has been on phase transitions that exhibit an anisotropic scaling of time and space:

\[ t \to \lambda z t, \quad x^i \to \lambda x^i \]  \hspace{1cm} (1.1)

where \( z \) is called the dynamical exponent. The case \( z = 1 \) arises when the critical point has full conformal invariance, and one might hope to model it using a solution of string/M-theory with an anti-de-Sitter (AdS) factor.

There have been two main approaches in trying to extend holographic technology to model field theories with \( z \neq 1 \). In the first approach, one generalises the AdS geometry to a Lifshitz(\( z \)) geometry of the form [1] (see also [2]):

\[ ds^2_{Lif} = -r^{2z} dt^2 + r^2 d\vec{x}^2 + \frac{dr^2}{r^2} \]  \hspace{1cm} (1.2)

The scaling invariance (1.1) is achieved by also scaling the holographic coordinate \( r \to \lambda^{-1} r \). In addition these geometries are also invariant under time translations and spatial translations and rotations. In the second approach, one instead considers a Schrödinger(\( z \)) geometry of the form [3][4]:

\[ ds^2_{Sch} = -r^{2z} (dx^+)^2 + 2r^2 dx^+ dx^- + r^2 d\vec{x}^2 + \frac{dr^2}{r^2} \]  \hspace{1cm} (1.3)

Here \( x^+ \) is identified with the dual field theory time coordinate \( t \) and the scaling (1.1) is achieved by also transforming the “extra” holographic coordinate \( x^- \) as \( x^- \to \lambda^{2-z} x^- \). These geometries have additional symmetries to those of Lifshitz(\( z \)). There are Killing vectors which generate non-relativistic boosts in the dual theory and the translation invariance of \( x^- \) is associated with conservation of particle number. In the special case that \( z = 2 \) the symmetry algebra is enlarged even further to include special conformal transformations and the full symmetry algebra is the Schrödinger algebra. It should be noted that compared to the AdS case the holographic dictionary for both Lifshitz and Schrödinger geometries is still very much in its infancy.

The initial constructions of these geometries have been in the context of “bottom up” models. In this approach one advocates a simple theory of gravity, typically in \( D = 4 \) or \( D = 5 \) dimensions, coupled to a small number of matter fields with some simple couplings. The main virtue of this approach is that one can easily start to investigate possible scenarios where string/M-theory might lead to new insights.
into condensed matter. One hopes that the solutions one finds exist somewhere in the landscape of string/M-theory (or perhaps provide a good approximation to such solutions) and hence correspond to bone-fide dual field theories. However, it is not clear if these hopes are realised, and the bottom up results could be misleading.

The alternative “top-down” approach aims at constructing holographic solutions directly in string/M-theory. In practice the focus is to construct solutions of $D=10/11$ supergravity\footnote{In certain limits, one can also consider probe branes in supergravity backgrounds.} possibly leaving issues such as perturbative and non-perturbative stability to future work. The main advantage of the top down approach is that the relevant solutions of string/M-theory will in fact correspond to dual field theories. Another advantage is that, somewhat surprisingly, it is often the case that one can explore infinite classes of solutions at the same time. Moreover, within these infinite classes one can find universal phenomena. The only disadvantage of the top down approach is that it is much harder to construct the solutions, and maybe harder still to construct solutions of direct relevance to condensed matter systems.

The original constructions of the Lifshitz and Schrödinger geometries were in the context of simple bottom up models. In the Schrödinger case top down solutions were subsequently constructed using duality transformations and/or consistent Kaluza-Klein reductions \cite{5,6,7}. These string/M-theory solutions have been significantly further generalised in \cite{8}-\cite{17}. By contrast, it has proved surprisingly difficult to construct top-down Lifshitz solutions. Indeed the difficulties led to the work of \cite{18} which proved a no-go theorem concerning their existence (see also \cite{19}). However, in \cite{20} three schematic constructions of top down solutions were discussed. More recently Balasubramanian and Narayan \cite{21} have found Lifshitz solutions of type IIB and $D=11$ supergravity with $z=2$. In this paper we will clarify and substantially generalise the solutions of \cite{21}.

Remarkably, our new Lifshitz solutions, all of which have $z=2$, can be (essentially) obtained from the same general class of type IIB and D=11 supergravity solutions that were used to obtain Schrödinger($z$) solutions (for various $z$) in \cite{13}. The solutions are constructed using five- and seven-dimensional Einstein spaces and for the special case when the Einstein space is Sasaki-Einstein, the solutions are supersymmetric, generically preserving two supersymmetries. As we shall see, our new solutions are closely related to the construction of Schrödinger($z$) solutions of \cite{13} with $z=0$.

In this paper we will use the notation that a Lif$_D(z)$ solution is one with $D$ non-
compact directions that is dual to a field theory in $D - 1$ spacetime dimensions. The solution must have the usual time and space translations, spatial rotations and Lifshitz scaling with dynamical exponent $z$. In this notation a $\text{Lif}_D(z = 1)$ solution with conformal invariance is the same as an $\text{AdS}_D$ solution. In general the solutions that we construct will have the property that the $D$-dimensional non-compact part of the geometry will depend on the coordinates of the internal compact dimensions. Because of this the solutions should be viewed directly in $D = 10/11$. However, our solutions include examples where the non-compact part of the geometry is independent of the internal coordinates and the solutions can also be obtained as solutions of a reduced $D$-dimensional theory of gravity. Developing the holographic dictionary for the latter class of solutions should be easier than for the former class.

In section 2, using arbitrary five-dimensional Einstein spaces, $E_5$, we construct $\text{Lif}_4(z = 2)$ solutions in type IIB supergravity. For a special sub-class we can T-dualise and uplift to obtain $\text{Lif}_4(z = 2)$ solutions of type IIA and D=11 supergravity, respectively. The most general class of solutions exist when $E_5$ has non-vanishing second Betti number and we illustrate with some explicit examples that include the Sasaki-Einstein spaces $T^{1,1}$ [22] and $Y^{p,q}$ [23].

Our constructions include solutions in $D = 11$ that consist of a direct product of a $\text{Lif}_4(z = 2)$ factor with a seven-dimensional compact manifold, with the latter a two-torus fibred over $E_5$. Moreover, the $\text{Lif}_4(z = 2)$ factor is independent of the coordinates of the compact manifold. The reason that these solutions evade the no-go theorem of [18] is simply that this theorem does not cover the most general type of four-form flux.

In section 3 we present analogous constructions of $\text{Lif}_3(z = 2)$ solutions of D=11 supergravity using seven-dimensional Einstein spaces, $E_7$. These new solutions again extend those discussed in [21]. We briefly conclude in section 4.

The paper contains two appendices. In appendix A we carry out a dimensional reduction of type IIB on $S^1 \times E_5$ and of $D = 11$ on $S^1 \times E_7$ to obtain constrained theories of gravity in $D = 4$ and $D = 3$, respectively, from which we can make contact with the bottom up constructions of the Lifshitz($z$) solutions. For the special case when $E_5 = T^{1,1}$ we show that there is a more elaborate and consistent Kaluza-Klein reduction on $S^1 \times T^{1,1}$. In appendix B we we provide an alternative verification of the supersymmetry of the $\text{Lif}_4(z = 2)$ solutions of $D = 11$ supergravity.

\footnote{Note that this is not possible in the case of $\text{AdS}_D$ solutions. Also note that this means that one can have a $\text{Lif}_D(z = 1)$ solution of D=10/11 supergravity that is not an $\text{AdS}_D$ solution.}
2 \text{ Lif}_4(z = 2) \text{ Solutions}

Our starting point is the following ansatz (essentially as in [13]) for the bosonic fields of type IIB supergravity:

\begin{align}
\text{ds}_{10}^2 &= \Phi^{-1/2} \left[ 2 dx^+ dx^- + h(dx^+)^2 + dx_1^2 + dx_2^2 \right] + \Phi^{1/2} ds^2(M_6) \\
F_5 &= dx^+ \wedge dx^- \wedge dx_1 \wedge dx_2 \wedge d\Phi^{-1} + *_M d\Phi \\
G &= dx^+ \wedge W \\
P &= g dx^+ 
\end{align}

(2.1)

where \(G\) is the complex three-form and the complex one-form \(P\) incorporates the axion and dilaton.\(^3\) Here \(\Phi, h, g\) are functions and \(W\) is a complex two-form all defined on the six-dimensional Ricci-flat manifold, \(M_6\), and they can all have a functional dependence on the coordinate \(x^+\). One finds that all the equations of motion are satisfied provided that

\begin{align}
\nabla^2_M \Phi &= 0 \\
dx^+ \wedge dW &= d *_M W = 0 \\
\nabla^2_M h &= -4g^2 \Phi - |W|_M^2 
\end{align}

(2.2)

where \(|W|_M^2 \equiv (1/2!)W^i_{ij}W^*_ij\) with indices raised with respect to the metric on \(M_6\). Observe that when \(h = W = 0\) we have the standard D3-brane class of solutions with a Ricci-flat transverse space. As we will review a little later, when \(M_6\) is a Calabi-Yau 3-fold, \(M_6 = CY_3\), supersymmetry is preserved for certain choices of \(W\) [13]. The general structure of these solutions is that of D3-branes transverse to a Ricci-flat space with a wave propagating on the world-volume, and carrying additional RR and NS magnetic 3-form flux.

We now specialise to the case that \(M_6\) is a metric cone over a five-dimensional compact Einstein manifold \(E_5\), \(ds^2(M_6) = dr^2 + r^2 ds^2(E_5)\). The Einstein metric is normalised so that its Ricci tensor is equal to four times the metric, the same as for a round five-sphere. When \(M_6 = CY_3\) then the Einstein manifold is Sasaki-Einstein. In order to get solutions with \text{Lif}_4(z = 2)\ symmetry we now set

\begin{align}
\Phi &= r^{-4} \\
h &= r^{-2} f 
\end{align}

(2.3)

\(^3\)Our conventions for type IIB supergravity [24, 25] are as in [26]. In particular, packaging the dilaton, \(\phi\), and the axion, \(C_0\), as \(\tau = C_0 + i e^{-\phi}\), then \(P = (i/2)e^{\phi} d\tau\) and \(G = i e^{\phi/2}(\tau dB - dC_2)\) where \(B, C_2\) are the NS and RR 2-form potentials, respectively.
where \( f \) is a function of the coordinates on \( E_5 \) and \( x^+ \). In addition \( W \) and \( g \) are taken to be a three-form and a function defined on \( E_5 \) and they are both also functions of \( x^+ \). The equations (2.2) are now solved provided that

\[
\begin{align*}
dx^+ \wedge dW &= d\ast_E W = 0 \\
-\nabla^2_E f + 4f &= 4|g|^2 + |W|^2_E
\end{align*}
\]

(2.4)

Observe that when \( g = W = 0 \), necessarily we have \( f = 0 \) since eigenvalues of the Laplacian on compact \( E_5 \) are negative. For a similar reason, given \( g, W \) any solution to the second equation is necessarily unique.

After relabelling \( x^- = t, x^+ = \sigma \)

(2.5)

the full solution now reads

\[
\begin{align*}
ds^2 &= r^2 \left[ 2d\sigma dt + dx_1^2 + dx_2^2 \right] + \frac{d\sigma^2}{r^2} + f \, ds^2 + ds^2(E_5) \\
&= -\frac{r^4}{f} dt^2 + r^2 \left( dx_1^2 + dx_2^2 \right) + \frac{d\sigma^2}{r^2} + f \left( d\sigma + \frac{r^2}{f} dt \right)^2 + ds^2(E_5) \\
F_5 &= 4r^3 \, d\sigma \wedge dt \wedge dr \wedge dx_1 \wedge dx_2 + 4 \text{Vol}_{E_5} \\
G &= d\sigma \wedge W \\
P &= gd\sigma
\end{align*}
\]

(2.6)

with \( f, g, W \) satisfying (2.4). Observe that when \( f = g = W = 0 \) we have the standard \( AdS_5 \times E_5 \) solution. The solutions of [21] can be recovered in the special case when \( W = 0 \), \( f \) and \( g \) are functions only of \( \sigma \) (i.e. are independent of the coordinates of \( E_5 \)) and furthermore that \( g \) is real (i.e. the axion is zero). By restricting to solutions with \( f > 0 \), following [21], we can view these as Lif\(_d\)(\( z = 2 \)) solutions by taking \( \sigma \) to parametrise a compact \( S^1 \). In particular, the full solution is invariant under the following scalings of the four non-compact directions, parameterised by \( t, x^1, x^2, r \):

\[
t \rightarrow \lambda^2 t, \quad x^i \rightarrow \lambda x^i, \quad r \rightarrow \lambda^{-1} r
\]

(2.7)

These solutions correspond to dual field theories in \( d = 3 \) spacetime dimensions with dynamical exponent \( z = 2 \).

When \( f \) has dependence on \( \sigma \) and the coordinates on the Einstein space, a \( D = 4 \) perspective of the solutions is artificial and they should be viewed directly in \( D = 10 \). However, there is an interesting sub-class of solutions where \( f \) is a constant, and without loss of generality we can set \( f = 1 \). These solutions require \( 4 = 4|g|^2 + |W|^2_E \).
with $W$ a harmonic form on $E_5$, and we will present some explicit examples with $W \neq 0$ below. For this class the $D = 4$ non-compact part of the metric is precisely that of the original Lif$_4(z = 2)$ geometry of $[1]$. In appendix A we will make contact with the bottom up construction of $[1]$ by performing a dimensional reduction of type IIB supergravity on $S^1 \times E_5$.

For this class of solutions, and more generally for solutions where $f$, $g$ and $W$ are independent of the $\sigma$ coordinate, which implies that $W$ is a harmonic two-form on $E_5$, the vector $\partial_\sigma$ is a Killing vector that also preserves the fluxes. This will generate a global symmetry in the dual $d = 3$ dimensional field theory (as will any isometries of the Einstein space $E_5$ that also preserve $g$ and $W$). The most general solutions will also have a dependence on $\sigma$ and $\partial_\sigma$ will no longer be Killing.

It is worth highlighting that for the general class of solutions given in (2.6), (2.4), the vectors $-x^+ \partial_+ + x^i \partial_i$, $i = 1, 2$ are also Killing, where for this paragraph we have temporarily reverted $t, \sigma$ back to $x^-, x^+$, respectively. When $x^+$ is non-compact these Killing vectors generate the finite transformation

$$x^i \to x^i - u^i x^+; \quad x^- \to x^- + u^i x^i - \frac{1}{2} u^2 x^+$$

for constant $u^i$. This symmetry is explicitly broken by taking $x^+$ to be compact\footnote{To see this, note that if $x^+ \equiv x^+ + 2\pi R$ then $(x^+, x^-, x^i)$ and $(x^+ + 2\pi R, x^-, x^i)$, which parametrise the same point, do not get mapped to the same point by the finite transformation.}. In fact when $x^+$ is non-compact the solutions can be viewed as $z = 0$ Schrödinger solutions of the type studied in $[13]$ with $x^+$ playing the role of the time coordinate and $x^-$ the auxiliary coordinate (leading to conservation of particle number in the dual field theory). The transformation (2.8) then corresponds to the usual non-relativistic boosts. By contrast, here we have obtained Lif$_4(z = 2)$ solutions by switching the roles of $x^+$ and $x^-$ and then compactifying $x^+$.

It is curious that for the new Lif$_4(z = 2)$ solutions given in (2.6) the Killing vector $\partial_t$ is always null; this won’t be the case for the type IIA solutions which we obtain after T-duality and are presented after the following discussion of supersymmetry.

\section{2.1 Supersymmetry}

When the Einstein space is taken to be Sasaki-Einstein, or equivalently the Ricci-flat cone $M_6$ is $CY_3$, the Lif$_4(z = 2)$ solutions that we have presented can preserve supersymmetry. In fact this follows from the analysis of $[13]$. Specifically, if the two-form $W$ is of type $(1, 1)$ and primitive or of type $(0, 2)$ on the $CY_3$ cone then the
more general solutions (2.1) generically preserve 2 supersymmetries. More specifically, for the $D = 10$ metric in (2.1) we introduce the frame $e^+ = \Phi^{-1/4}dx^+$, $e^- = \Phi^{-1/4}(dx^- + \frac{5}{2}dx^+)$, $e^2 = \Phi^{-1/4}dx^1$, etc. and choose positive orientation to be given by $e^{+ - 23} \wedge \text{Vol}_{CY}$, where Vol$_{CY}$ is the volume element on CY$_3$. Consider first the special case where $g = h = W = 0$. Then, as usual, a generic CY$_3$ breaks $1/4$ of the supersymmetry, while the harmonic function $\Phi$ leads to a further breaking of $1/2$, the Killing spinors satisfying the additional projection $\Gamma^{+ - 23} \epsilon = i \epsilon$ (with the Killing spinors gaining a factor $\Phi^{-1/8}$). Switching on $g, h, W$ we find that we need to also impose $\Gamma^+ \epsilon = 0$ and $\Gamma^{ij} W_{ij} \epsilon = 0$ (and the spinors gain a dependence on $x^+$). Note that in the supersymmetric Lif$_4(z = 2)$ solutions, $W$ is a two-form on the $SE_5$ space and hence $W$ should be $(1, 1)$ on the cone, or, equivalently, $(1, 1)$ with respect to the local four-dimensional Kähler-Einstein base space associated with the $SE_5$.

We can also determine the supersymmetry algebra by constructing the Killing vector that arises from bi-linears of these Killing spinors. Specifically, following [16] we find that if we define

$$K^M = \bar{\epsilon} \Gamma^M \epsilon$$

then $K = \partial_t$. In other words the supersymmetry is squaring to the Killing vector generating time translations in the dual Lifshitz field theory. As noted above, for these type IIB solutions this Killing vector is null.

### 2.2 Type IIA and $D = 11$ pictures

Starting with the type IIB solutions given in (2.6) we can obtain analogous solutions of type IIA by performing a T-duality on the $\sigma$ direction. To do this we require that $f$ and $W$ are independent of $\sigma$. We also require $g$ to be independent of $\sigma$ which means that the dilaton and axion are constant and for simplicity we take them to be trivial so that $G = -(dB^{(2)} + i dC^{(2)})$. The function $f$ satisfies

$$4f - \nabla_E^2 f = |W|^2_E$$

Writing

$$W = dA^{(1)} + i dA^{(2)}, \quad D\sigma \equiv d\sigma - A^{(1)}$$
and performing the T-duality we obtain the type IIA solutions

\[
ds^2 = -\frac{r^4}{f} dt^2 + r^2 \left( dx_1^2 + dx_2^2 \right) + \frac{dr^2}{r^2} + \frac{1}{f} D\sigma^2 + ds^2 (E_5)
\]

\[
e^{2\phi} = \frac{1}{f}
\]

\[
B = - D\sigma \wedge \frac{r^2}{f} dt
\]

\[
F_2 = - dA^{(2)}
\]

\[
F_4 = 4r^3 dt \wedge dx_1 \wedge dx_2 \wedge dr + D\sigma \wedge \frac{r^2}{f} dt \wedge dA^{(2)}
\]

(2.12)

where \( F_4 \) is the type IIA RR four-form field strength satisfying \( dF_4 = H_3 \wedge F_2 \). It is interesting to observe that the Killing vector \( \partial_t \), generating the time-translations of the dual Lifshitz field theory, is now time-like. This is to be contrasted with the IIB solutions where it was null. We also observe that the \( \sigma \) circle direction is now, in general, non-trivially fibred over the Einstein space. In particular, the first Chern class of this circle bundle is given by the cohomology class of \( dA^{(1)} \) and this will lead, in general, to the six-dimensional internal space no longer having the topology of \( S^1 \times E_5 \).

We can uplift these solutions to \( D = 11 \) on a circle parametrised by \( \chi \) and we obtain

\[
ds^2 = f^{1/3} \left[ -\frac{r^4}{f} dt^2 + r^2 \left( dx_1^2 + dx_2^2 \right) + \frac{dr^2}{r^2} \right] + f^{-2/3} \left[ D\sigma^2 + D\chi^2 \right] + f^{1/3} ds^2 (E_5)
\]

\[
G_4 = dt \wedge d \left( r^4 dx_1 \wedge dx_2 + \frac{r^2}{f} D\chi \wedge D\sigma \right)
\]

(2.13)

where \( D\chi \equiv d\chi - A^{(2)} \). It is illuminating to rewrite these \( D = 11 \) solutions in the form arising from performing a T-duality and then uplifting the more general solutions (2.1):

\[
ds^2 = -H_1^{-2/3} H_2^{-2/3} dt^2 + H_1^{-2/3} H_2^{1/3} \left( dx_1^2 + dx_2^2 \right) + H_1^{1/3} H_2^{-2/3} \left[ D\sigma^2 + D\chi^2 \right]
\]

\[
+ H_1^{1/3} H_2^{1/3} \left[ dr^2 + r^2 ds^2 (E_5) \right]
\]

\[
G_4 = dt \wedge dx_1 \wedge dx_2 \wedge d(H_1^{-1}) + dt \wedge d(H_2^{-1} D\chi \wedge D\sigma)
\]

(2.14)

In the Lif_{4} (z = 2) solutions \( H_1 = r^{-4}, H_2 = f r^{-2} \). The general structure of the solution is that of two membranes intersecting in the time direction: the world-volume of one of the membranes is where the dual field theory resides and the other membrane is wrapped over a two-torus, parametrised by \( \sigma, \chi \), which is fibred over an
overall transverse six-dimensional Ricci-flat space. Note that the NS and RR 3-form flux in the type IIB picture now manifests itself through the fibration.

For the special case that the Einstein space is Sasaki-Einstein, these type IIA and $D = 11$ solutions will preserve supersymmetry provided that $W$ is of type $(1,1)$ and primitive on the $CY_3$ cone. This follows because the type IIB Killing spinors are independent of the $\sigma$ direction and hence are preserved under the T-duality transformation. The Killing vector that can be constructed from the generic $D = 11$ Killing spinors is again $\partial_t$ which, as we have noted above, is now timelike. We provide an independent derivation of this in appendix B.

For the special case when $f = 1$ (which can occur, for example, when $E_5 = T^{1,1}$ as we show below), this solution is a direct product of a Lif$_4(z = 2)$ geometry with a compact seven-dimensional internal space, the latter a two-torus fibration over $E_5$. The reason that the solutions evade the no-go theorem of [18] is simply because [18] did not consider the most general flux compatible with Lif$_4(z = 2)$ symmetry.

2.3 Explicit Examples

Type IIB solutions with $W = 0$ and $g \neq 0$ can be found for any choice of $E_5$. Indeed for real $g$ (i.e. the axion is zero) these solutions were constructed in [21]. More interesting solutions have $W \neq 0$. Since $W$ is harmonic on $E_5$, the latter must have non-trivial two- and three-cycles. Thus there are no such solutions on $S^5$. A simple non-supersymmetric solution can be constructed for $E_5 = S^2 \times S^3$. Let us discuss in a bit more detail some supersymmetric examples using Sasaki-Einstein spaces, first considering $E_5 = T^{1,1}$ and then $E_5 = Y^{p,q}$.

The metric for $T^{1,1}$ can be written
\[
ds^2(E_5) = \frac{1}{9}(d\psi - \cos \theta_1 d\phi_1 - \cos \theta_2 d\phi_2)^2 + \frac{1}{6}(d\theta_1^2 + \sin \theta_1 d\phi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin \theta_2 d\phi_2^2)
\] (2.15)

For $W$ we choose
\[
W = \frac{k}{\sqrt{18}} (\sin \theta_1 d\theta_1 \wedge d\phi_1 - \sin \theta_2 d\theta_2 \wedge d\phi_2)
\] (2.16)

where $k = k(\sigma)$ and it is simple to check that it is harmonic (and thus closed and co-closed) on $T^{1,1}$. The unique solution to (2.4) is given by
\[
f = k^2 + |g|^2
\] (2.17)

and is independent of the coordinates on $T^{1,1}$ In particular, solutions that are independent of $\sigma$, which covers the type IIA and $D = 11$ solutions, have constant $f$. It
is straightforward to check that $W$ is of type $(1,1)$ and primitive on the cone over $T^{1,1}$ (the conifold), and hence this whole family of solutions preserve supersymmetry. Note that this construction can be immediately adapted to the Einstein spaces $T^{p,q}$ \[27\] to obtain analogous non-supersymmetric solutions.

We next turn to examples where $E_5 = Y^{p,q}$. The Sasaki-Einstein metric can be written in the canonical form \[23\]
\[
ds^2(Y^{p,q}) = \frac{1}{9}(d\psi' + \sigma)^2 + ds_4^2
\]
where
\[
ds_4^2 = \frac{1 - y}{6} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{dy^2}{w(y)q(y)} + \frac{1}{36}w(y)q(y)(d\beta + \cos \theta d\phi)^2
\]
is a locally defined Kähler-Einstein metric, and
\[
\sigma = y(d\beta + \cos \theta d\phi) - \cos \theta d\phi
\]
and
\[
w(y) = \frac{2(a - y^2)}{1 - y}, \quad q(y) = \frac{a - 3y^2 + 2y^3}{a - y^2}
\]
For $W$ we choose
\[
W = \frac{1}{\sqrt{72}} d \left[ \frac{1}{1 - y} (d\beta + \cos \theta d\phi) \right]
\]
It can easily be checked that $W$ is co-closed on $Y^{p,q}$ and furthermore, that it is $(1,1)$ and primitive with respect to the local four-dimensional KE metric $ds_4^2$ and hence is also on the corresponding CY$_3$. To see that this two-form is globally defined on the whole Sasaki-Einstein it is helpful to use the coordinates defined by
\[
\alpha = -\frac{1}{6}(\beta + \psi'), \quad \psi = \psi'
\]
In these coordinates the metric can be written
\[
ds^2 = w(y)(D\alpha)^2 + \frac{dy^2}{w(y)q(y)} + \frac{q(y)}{9}D(\psi)^2 + \frac{1 - y}{6}(d\theta^2 + \sin^2 \theta d\phi^2)
\]
where
\[
D\alpha = d\alpha + \frac{a - 2y + y^2}{6(a - y^2)} D\psi, \quad D\psi = d\psi - \cos \theta d\phi
\]
As discussed in detail in \[23\], in these coordinates one can show that there is a circle fibration, parametrised by $\alpha$ over a globally defined four-dimensional base. This base
space, in turn, is a two-sphere fibration, parametrised by $\psi, y$ with $y_1 \leq y \leq y_2$ where $y_i$ are two suitable roots of $q(y)$, over the round two-sphere, parametrised by $(\theta, \phi)$. In this construction $a$ and the roots $y_i$ are fixed by two relatively prime integers $p > q > 0$:

\[
\begin{align*}
    a &= \frac{1}{2} - \frac{p^2 - 3q^2}{4p^3} \sqrt{4p^2 - 3q^2} \\
    y_1 &= \frac{1}{4p} \left(2p - 3q - \sqrt{4p^2 - 3q^2}\right) \\
    y_2 &= \frac{1}{4p} \left(2p + 3q - \sqrt{4p^2 - 3q^2}\right)
\end{align*}
\] (2.25)

In this construction the one-form $D\alpha$ is the globally defined one-form of the circle fibration. Furthermore the two-form $dy \wedge D\psi$ is also globally defined. Writing $W$ in the new coordinates we conclude that it is globally defined.

The general solution to (2.4) will be given by $f = \bar{f} + |g|^2$ where $\bar{f}$ satisfies

\[
-4\bar{f} + \frac{2}{1-y} \left[(a - 3y^2 + 2y^3) \bar{f}'\right]' + \frac{1}{(1-y)^4} = 0
\] (2.26)

If $g = 0$ (as in the type IIA and $D = 11$ solutions) it is important that $\bar{f}$ (and hence $f$) is strictly positive in the interval $y_1 \leq y \leq y_2$. We have not found an analytic expression for the function $\bar{f}$ but we numerically solved it for a few values of $a$, for specific values of $p$ and $q$, as shown in figure 1. Note that $\bar{f}$ monotonically increases with $a$ and also that it diverges as $a \to 1$, which is expected since $a = 1$ is the case of $S^5$, where there are no solutions with $W \neq 0$.

It would be interesting to generalise these solutions by replacing $Y^{p,q}$ with the more general $L^{a,b,c}$ Sasaki-Einstein metrics of [28]. In the above examples, where the topology of $E_5$ is $S^2 \times S^3$, we have $H^2(E_5, \mathbb{Z}) = \mathbb{Z}$ and hence the circle bundle over $E_5$ appearing in the type IIA metric in (2.12) is specified by an integer $n$, the Chern number. Taking $n = \pm 1$ gives a total space of topology $S^3 \times S^3$ (taking it to be $n \neq \pm 1$ would instead lead to a non-simply connected total space, which can always be lifted to the simply connected cover with $n = \pm 1$). This lifts to $D = 11$ solutions (2.14) with internal space of topology $S^3 \times S^3 \times S^1$. 

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Figure 1: Numerical solutions for the function $\bar{f}(y)$ for $(p,q) = (4,1)$ (i.e. $a \sim 0.10$) - dark blue; $(3,1)$ (i.e. $a \sim 0.18$) - green; $(2,1)$ (i.e. $a \sim 0.39$) - red; $(3,2)$ (i.e. $a \sim 0.64$) - cyan. For each case we have plotted $\bar{f}$ just for the values $y \in [y_1, y_2]$.

# 3 \textit{Lif}_3(z = 2) \textit{Solutions}

We consider the ansatz (essentially as in \cite{13}) for the bosonic fields of $D = 11$ supergravity given by

$$
\begin{align*}
ds^2 &= \Phi^{-2/3} \left[ 2 \, dx^+ dx^- + h \, (dx^+)^2 + dx^2 \right] + \Phi^{1/3} ds^2 (M_8) \\
G &= dx^+ \wedge dx^- \wedge dx \wedge d\Phi^{-1} + dx^+ \wedge V
\end{align*}
$$

(3.1)

where $\Phi$, $h$ are functions and $V$ is a three-form on the eight-dimensional Ricci-flat space $M_8$, and they can all have a dependence on the coordinate $x^+$. Our conventions for $D = 11$ supergravity \cite{29} are as in \cite{30}. One finds that all the equations of motion are satisfied provided that

$$
\begin{align*}
\nabla_M^2 \Phi &= 0 \\
dx^+ \wedge dV = d \ast_M V &= 0 \\
\nabla_M^2 h &= -|V|^2_M
\end{align*}
$$

(3.2)

where $|V|^2_M \equiv (1/3!) V^{ijk} V_{ijk}$ with indices raised with respect to the metric on $M_8$. When $h = V = 0$ we have the standard M2-brane class of solutions with a transverse Ricci-flat space $M_8$. When $M_8$ is a Calabi-Yau four-fold, $M_8 = CY_4$, supersymmetry is preserved for certain choices of $V$ \cite{13}, as we discuss below. The general structure
of these solutions is that of M2-branes transverse to the Ricci-flat space with a wave propagating on the world-volume, that in addition carry magnetic four-form flux.

We now specialise to the case that $M_8$ is a metric cone over a seven-dimensional Einstein manifold $E_7$, $ds^2(M_8) = dr^2 + r^2ds^2(E_7)$. The Einstein metric is normalised so that its Ricci tensor is equal to six times the metric, the same as for a round seven-sphere. When $M_8 = CY_4$ the Einstein manifold is Sasaki-Einstein $SE_7$. In order to get solutions with $\text{Lif}_3(z = 2)$ symmetry we now set

$$\Phi = r^{-6}$$  
$$h = fr^{-4}$$  

where $f$ is a function of the coordinates on the Einstein space and $x^+$. In addition $V$ is taken to be a three-form defined on $E_7$ and a function of $x^+$. The equations (3.2) are solved provided that

$$dV = d*EV = 0$$  
$$8f - \nabla_E^2 f = |V|_E^2$$  

(3.4)

For a given $V$, any solution of the second equation is unique. In particular, if $V = 0$ then $f = 0$. We note that this ansatz is similar to that considered in appendix A.1 of [21], however their ansatz implicitly assumes that $|V|^2 = \text{constant}$ and $f$ did not have any dependence on the Einstein space.

Introducing the new coordinates

$$\rho = r^2, \quad x \to x/2, \quad x^+ = \sigma/2, \quad x^- = t/2$$  

(3.5)

the full solution reads

$$ds^2 = \frac{1}{4} \left[ \rho^2 (2d\sigma dt + dx^2) + \frac{d\rho^2}{\rho^2} + f d\sigma^2 \right] + ds^2(E_7)$$

$$= \frac{1}{4} \left[ -\frac{\rho^4}{f} dt^2 + \rho^2 dx^2 + \frac{d\rho^2}{\rho^2} \right] + \frac{f}{4} \left[ d\sigma + \frac{\rho^2}{f} dt \right]^2 + ds^2(E_7)$$

$$G = \frac{3}{8} \rho^2 d\sigma \wedge dt \wedge dx \wedge d\rho + \frac{1}{2} d\sigma \wedge V$$  

(3.6)

with $f, V$ satisfying (3.4). Observe that when $f = V = 0$ we have the standard $AdS_4 \times E_7$ solution.

To view these as $\text{Lif}_3(z = 2)$ solutions we consider solutions with $f > 0$ and take the $\sigma$ coordinate to parametrise a compact internal $S^1$. The full solution is then invariant under Lifshitz scalings of the three non-compact directions, parametrised by
(t, x, r), with dynamical exponent \( z = 2 \). In appendix A.3 we show how this special sub-class of solutions can be obtained from a simple theory of gravity in \( D = 3 \) after dimensional reduction on \( S^1 \times E_7 \).

When \( f \) has dependence on \( \sigma \) and/or the \( E_7 \) space, the solutions should be viewed directly in \( D = 11 \). An interesting sub-class has constant \( f \) and, without loss of generality, we can take \( f = 1 \) and \( |V|^2 = 8 \). These solutions, and more generally solutions that are independent of \( \sigma \), will have \( \partial_\sigma \) as a Killing vector. These solutions also have \( -\sigma \partial_x + x \partial_t \) as a Killing vector and the corresponding finite transformation (see (2.8)) is broken by \( \sigma \) being compact. As in the type IIB case discussed in section 3, in a certain sense, these solutions are closely related to the Schrödinger(\( z \)) solutions of [13] with \( z = 0 \). We also observe that the Killing vector \( \partial_t \) is null in the full \( D = 11 \) solution. However, for the solutions independent of \( \sigma \) we can dimensionally reduce on \( \sigma \) to obtain a type IIA solution where \( \partial_t \) is time-like.

Some simple explicit examples can be obtained by taking \( E_7 = S^3 \times E_4 \) where \( S^3 \) is the round three-sphere and \( E_4 \) is an arbitrary four-dimensional Einstein space and \( V \) is taken to be the volume form on the \( S^3 \). We will postpone further constructions of explicit solutions to future work.

### 3.1 Supersymmetry

When \( E_7 \) is taken to be Sasaki-Einstein, or equivalently \( M_8 \) is taken to be a Calabi-Yau four-fold, these Lif\( _3(\, z = 2) \) solutions can preserve supersymmetry [13]. In particular, if we choose the three-form \( V \) to only have \((2, 1)\) plus \((1, 2)\) pieces and be primitive on the \( CY_4 \) then the more general solutions given in (3.1) generically preserve 2 supersymmetries [13]. More specifically, we introduce the frame \( e^+ = \Phi^{-1/6} dx^+, e^- = \Phi^{-1/6} (dx^- + \frac{1}{2} dx^+), e^2 = \Phi^{-1/6} dx^2, \) etc. and choose positive orientation to be given by \( e^{+2} \wedge \text{Vol}_{CY} \), where \( \text{Vol}_{CY} \) is the volume element on \( CY_4 \). Consider first the special case that \( h = V = 0 \). Then, as usual, a non-flat \( CY_4 \) breaks 1/8 of the supersymmetry, and the harmonic function \( \Phi \) can be added “for free” (the projection on the Killing spinors arising from the \( CY_4 \) automatically imply the projection \( \Gamma^{+2} \epsilon = -\epsilon \)). Switching on \( h, V \) we find that we need to also impose \( \Gamma^+ \epsilon = 0 \) and \( \Gamma^{ijk} V_{ijk} \epsilon = 0 \). As usual the skew-whiffed solutions, obtained by changing the sign of the four-form flux, generically don’t preserve any supersymmetry (apart from the special case when \( SE_7 = S^7 \)).

The supersymmetry algebra can be obtained by constructing the Killing vector arising as bi-linears in the Killing spinors. Following [13] we find that the anticom-
mutator of the supersymmetries gives the null Killing vector $\partial_t$.

4 Final Comments

We have constructed rich classes of Lif($z$) solutions of $D = 10$ and $D = 11$ supergravity with $z = 2$. This work opens up several avenues for further exploration and we conclude by briefly mentioning some of them.

It will be interesting to see if our solutions can be simply generalised to Lifshitz solutions with other values of $z$. Another direction will be to construct black hole solutions that asymptotically approach the new Lifshitz solutions. It will also be interesting to see if it is possible to construct solutions that interpolate between Schrödinger, Lifshitz and AdS geometries corresponding to RG flows in the dual field theories. It has been proposed [31] that Lifshitz geometries can naturally arise as the ground states of holographic superconductors and it will be interesting to see if our solutions appear in this way too.

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A Reduced Theories of Gravity

A.1 Reduction of type IIB on $S^1 \times E_5$ to $D = 4$ when $f = 1$

Lifshitz solutions were first constructed in a bottom up context in a $D = 4$ theory of gravity with cosmological constant coupled to a vector field and a two-form [1], or equivalently, to a single massive vector field [32]. We show how the special class of type IIB solutions, for arbitrary Einstein space, with $g = 0$, $W$ independent of $\sigma$ and $f = 1$, that we constructed in section 2 are related to these constructions. Specifically we dimensionally reduce on $S^1 \times E_5$, where $S^1$ is parametrised by $\sigma$, to obtain a truncated $D = 4$ theory of gravity coupled to a vector and a scalar field.
We consider the type IIB ansatz
\[ ds^2 = ds_4^2 + e^{2T} (d\sigma + A)^2 + ds^2 (SE_5) \]
\[ F_5 = 4e^T (d\sigma + A) \wedge \text{Vol}_4 + 4 \text{Vol} (SE_5) \]
\[ H = d\sigma \wedge W \]
with \( W \) a harmonic (i.e. closed and co-closed) form on \( SE_5 \) satisfying \( |W|^2 = 4 \), and trivial axion and dilaton. Here the vector field \( A \) and the scalar \( T \) are defined on the four-dimensional space corresponding to the line element \( ds_4^2 \). The equations of motion of type IIB supergravity are satisfied provided that we satisfy the \( D = 4 \) equations of motion
\[ R_{\mu\nu} = -4g_{\mu\nu} + 2A_\mu A_\nu + \nabla_\mu \nabla_\nu T + \partial_\mu T \partial_\nu T + \frac{1}{2} e^{2T} F_{\mu\lambda} F^\lambda_\nu \]
\[-\nabla^2 T - \partial_\mu T \partial^\mu T = -4 + 2e^{-2T} - \frac{1}{4} e^{2T} F_{\mu\nu} F^{\mu\nu} \]
\[ \nabla_\nu (e^{3T} F^\nu_\mu) = 4e^T A_\mu \]
\[ A^2 = -e^{-2T} \]
These equations can be obtained from the \( D = 4 \) action \( S = \int d^4x \sqrt{-g} L \) with Lagrangian given by
\[ L = e^T \left[ R + 12 - 2e^{-2T} - \frac{1}{4} e^{2T} F_{\mu\nu} F^{\mu\nu} - 2A^2 \right] \]
provided that we impose the constraint \( A^2 = -e^{2T} \) by hand in the equations of motion. Thus, the reduction does not comprise a “consistent KK reduction” in the technical sense. In the next section we will see how such a reduction can be achieved for the case \( E_5 = T^{1,1} \).

If we set \( T = 0 \) the equations of motion can be derived from the Lagrangian
\[ L = \left[ R + 10 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 2A^2 \right] \]
describing a massive vector, with \( \text{mass}^2 = 4 \), coupled to gravity plus cosmological constant as in [1][32], provided that we impose both \( A^2 = -1 \) and \( F_{\mu\nu} F^{\mu\nu} = -8 \) by hand.

As a consistency check, one can directly verify that the Lif_4 (\( z = 2 \)) solution
\[ ds^2_3 = -r^4 dt^2 + r^2 (dx_1^2 + dx_2^2) + \frac{dr^2}{r^2} \]
\[ A = r^2 dt \]
with \( T = 0 \) solves the above equations of motion.
A.2 Consistent KK reduction of type IIB on $S^1 \times T^{1,1}$ to $D = 4$

We now consider the special case when the Einstein space is $T^{1,1}$. For this case we can dimensionally reduce on $T^{1,1} \times S^1$ to obtain an unconstrained $D = 4$ theory of gravity coupled to three scalar fields and a vector field. It is highly likely that one can substantially extend this construction, probably consistent with supersymmetry, (see [33]-[42]). We expect these results to be important in future developments on these solutions.

We first introduce the notation for parametrising $T^{1,1}$

$$ds_i^2 = \frac{1}{6} \left( d\theta_i^2 + \sin^2 \theta_i \, d\phi_i^2 \right)$$

$$J_i = \frac{1}{6} \sin \theta_i \, d\theta_i \wedge d\phi_i$$

$$D\psi = d\psi - \cos \theta_1 \, d\phi_1 - \cos \theta_2 \, d\phi_2 \quad (A.6)$$

We then consider the type IIB reduction ansatz

$$ds_{10}^2 = ds_4^2 + e^{2T} \left( d\sigma + A \right)^2 + e^{2V} \frac{1}{9} D\psi^2 + e^{2U} \left( ds_1^2 + ds_2^2 \right)$$

$$F_5 = 4 e^{T-V-4U} Vol_4 \wedge (d\sigma + A) + \frac{4}{3} D\psi \wedge J_1 \wedge J_2$$

$$H = \sqrt{2} \left( d\sigma + dk \right) \wedge (J_1 - J_2) \quad (A.7)$$

with non-trivial ten dimensional dilaton $\phi$ and vanishing axion. Here $T$, $U$, $V$, $\phi$ and $k$ are scalars and $A$ is a vector field defined on the four-dimensional space corresponding to the line element $ds_4^2$. We note that, if desired, one can remove the $dk$ term from $H$ by redefining $\sigma \rightarrow \sigma - k$, and this transformation indicates that $k$ is a St"uckelberg scalar for the vector field $A$, as we shall see below. We also note that after $T$-duality on the $\sigma$ direction this then uplifts to the following ansatz for $D = 11$ supergravity

$$ds_{11}^2 = e^{-2(\phi-T)/3} \left[ ds_4^2 + e^{-2T} \left( d\sigma + A \right)^2 + \frac{1}{9} e^{2V} D\psi^2 + e^{2U} \left( ds_1^2 + ds_2^2 \right) \right] + e^{4(\phi-T)/3} d\chi^2$$

$$F_4 = 4 e^{T-V-4U} Vol_4 + d \left[ (A - dk) \wedge (d\sigma + A) \right] \wedge d\chi$$

$$dA = \sqrt{2} \left( J_1 - J_2 \right) \quad (A.8)$$

After substituting the type IIB ansatz [A.7] into the type IIB equations of motion, we obtain $D = 4$ equations of motion (for the $D = 11$ ansatz we obtain an equivalent set of equations). Specifically, from the type IIB dilaton and three-form equation of motion we obtain

$$e^{-T-V-4U} \nabla_\mu \left( e^{T+V+4U} \nabla^\mu \phi \right) = -2 e^{-\phi-4U} \left[ e^{-2T} + (dk - A)^2 \right]$$

$$d \left( e^{-\phi+T+V} *_4 (A - dk) \right) = 0 \quad (A.9)$$
From the type IIB Einstein equations we obtain

\[
6e^{-2U} - 2e^{2V-4U} - \nabla^2 U - \partial_\mu T \partial^\mu U - 4\partial_\mu U \partial^\mu U - \partial_\mu U \partial^\mu V = 4e^{-2V-8U} + \frac{1}{2}e^{-\phi-4U} [e^{-2T} + (dk - A)^2]
\]

\[
4e^{2V-4U} - \nabla^2 V - \partial_\mu T \partial^\mu V - 4\partial_\mu U \partial^\mu V - \partial_\mu V \partial^\mu V = 4e^{-2V-8U} - \frac{1}{2}e^{-\phi-4U} [e^{-2T} + (dk - A)^2]
\]

\[
- \frac{1}{2}e^{-2T-V-4U} \nabla_\mu (e^{3T+V+4U} F^\mu_\nu) = 2e^{-\phi-4U-T} (\partial_\nu k - A_\nu)
\]

\[
- \nabla^2 T - \partial_\mu T \partial^\mu T - 4\partial_\mu T \partial^\mu U - \partial_\mu T \partial^\mu V + \frac{1}{4}e^{2T} F_{\mu\nu} F^{\mu\nu} = -4e^{-2V-8U} + \frac{1}{2}e^{-\phi-4U} [3e^{-2T} - (dk - A)^2]
\]

\[
R_{\mu\nu} - \nabla_\mu \nabla_\nu T - \partial_\mu T \partial_\nu T - \frac{1}{2}e^{2T} F_{\mu\lambda} F^{\nu\lambda} - 4(\nabla_\mu \nabla_\nu U + \partial_\mu U \partial_\nu U)
\]

\[
- (\nabla_\mu \nabla_\nu V + \partial_\mu V \partial_\nu V) = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - 4e^{-2V-8U} g_{\mu\nu}
\]

\[
+ \frac{1}{4}e^{-\phi} \left[ 8e^{-4U} (\partial_\mu k - A_\mu) (\partial_\nu k - A_\nu) - 2g_{\mu\nu} e^{-4U} [e^{-2T} + (dk - A)^2] \right] \quad (A.10)
\]

These equations can all be obtained from the $D = 4$ action $S = \int d^4x \sqrt{-g} \mathcal{L}$ with Lagrangian given by

\[
\mathcal{L} = e^{T+4U+V} \left[ R + 12 \partial U^2 + 8 \partial U \partial V + 8 \partial T \partial U + 2 \partial T \partial V - \frac{1}{4}e^{2T} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial \phi^2 \right.
\]

\[
+ 24e^{-2U} - 4e^{2V-4U} - 8e^{-2V-8U} - 2e^{-\phi-4U-2T} - 2e^{-\phi-4U} (dk - A)^2 \right]
\]

\[
(A.11)
\]

We can also rewrite this in the Einstein-frame by defining $g = e^{-T-4U-V} g_E$ and we obtain $S = \int d^4x \sqrt{-g_E} \mathcal{L}_E$ with

\[
\mathcal{L}_E = R_E + e^{-T-4U-V} (24e^{-2U} - 4e^{2V-4U} - 8e^{-2V-8U} - 2e^{-\phi-4U-2T})
\]

\[
- 2e^{-\phi-4U} (dk - A)^2 - 12 \partial U^2 - \frac{3}{2} \partial V^2 - 4 \partial U \partial V - \frac{3}{2} \partial T^2 - 4 \partial T \partial U
\]

\[
- \partial T \partial V - \frac{1}{4}e^{3T+4U+V} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial \phi^2 \quad (A.12)
\]

One can easily check that the equations of motion reduce to those consider in the last subsection after setting $\phi = U = V = k = 0$. Note that a key reason why this more general consistent KK reduction exists for $T^{1,1}$ is that the form of $W$ has the property that the non-zero components of $W^2_{ij}$ are proportional to $\delta_{ij}$ on $T^{1,1}$. This will not be the case\[ for general $E_5$ and $W$.

\[5\] Although we note that it is also true for the non-supersymmetric case of $S^2 \times S^3$ with $W$ proportional to the volume form on $S^2$ and similarly for $T^{p,q}$.
A.3 Reduction of $D = 11$ on $S^1 \times E_7$ to $D = 3$

We show how the special class of $D = 11$ solutions of section [3] for arbitrary Einstein space, with $V$ independent of $\sigma$ and $f = 1$ can be obtained from a $D = 3$ of gravity after dimensional reduction on $S^1 \times E_7$.

Consider the following ansatz for $D = 11$ supergravity

$$ds^2 = ds_3^2 + \frac{e^{2T}}{4}(d\sigma + A)^2 + ds^2(E_7)$$

$$G_4 = 3e^T d\sigma \wedge \text{Vol}_3 + \frac{1}{2} d\sigma \wedge V$$

(A.13)

Here $V$ is a harmonic (closed and co-closed) form on $E_7$ satisfying $|V|^2_{E_7} = 8$ and $ds_3^2$, $A$ and $T$ are a three-dimensional metric, vector potential and scalar field, respectively. We find that the $D = 11$ equations of motion are satisfied provided that the following $D = 3$ equations are satisfied

$$R_{\mu\nu} = \nabla_\mu \nabla_\nu T + \nabla_\mu T \nabla_\nu T - 12g_{\mu\nu} + \frac{1}{8} e^{2T} F_{\mu\rho} F^\rho_\nu + A_\mu A_\nu$$

$$-\nabla^2 T - \partial_\nu T \partial^\nu T = -12 + 4e^{-2T} - \frac{1}{16} e^{2T} F_{\mu\nu} F^{\mu\nu}$$

$$\nabla_\nu (e^{3T} F^\nu_\mu) = 8e^T A_\mu$$

$$A_\mu A^\mu = -4e^{-2T}$$

(A.14)

These equations can all be obtained from the $D = 3$ action $S = \int d^3x \sqrt{-g} \mathcal{L}$ with Lagrangian given by

$$\mathcal{L} = e^T \left[R + 24 - 4e^{-2T} - \frac{1}{16} e^{2T} F_{\mu\nu} F^{\mu\nu} - A^2 \right]$$

(A.15)

provided that we impose the constraint $A_\mu A^\mu = -4e^{-2T}$ by hand in the equations of motion. If we set $T = 0$ we have a theory of gravity with cosmological constant and a massive vector with mass $^2 = 8$, but we have to now impose $A^2 = -4$ and $F_{\mu\nu} F^{\mu\nu} = -128$ by hand. One can directly check that the Lif$_3(z = 2)$ solution in $D = 3$

$$ds_3^2 = \frac{1}{4} \left[-\rho^4 dt^2 + \rho^2 dx^2 + \frac{d\rho^2}{\rho^2} \right]$$

$$A = \rho^2 dt$$

(A.16)

with $T = 0$ solves these equations. We expect that more elaborate and consistent KK reductions can be made for special choices of $E_7$ such as $E_7 = S^3 \times E_4$. 

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B Checking supersymmetry for Lif$_4(z = 2)$ solutions in $D = 11$

We consider the $D = 11$ ansatz

$$ds^2_{11} = -\Delta^2 dt^2 + \Delta^{-1} \left[ H_1^{-1} (dx_1^2 + dx_2^2) + H_2^{-1} (D\chi_1^2 + D\chi_2^2) + ds^2(CY_3) \right]$$

$$G_4 = dt \wedge d[J_{SU(5)}]$$

(B.1)

where

$$\Delta \equiv H_1^{-1/3} H_2^{-1/3}, \quad dD\chi_i = -W_i$$

(B.2)

and the functions $H_1$, $H_2$ and the two-forms $W_i$ are defined on the Calabi-Yau threefold $CY_3$. The two-form $J_{SU(5)}$ is defined to be

$$J_{SU(5)} = H_1^{-1} dx_1 \wedge dx_2 + H_2^{-1} D\chi_1 \wedge D\chi_2 + J_{CY}$$

(B.3)

where $J_{CY}$ is the Kähler-form on $CY_3$.

We demand that this is a supersymmetric solution of $D = 11$ supergravity with a time-like Killing spinor, by demanding that it has an $SU(5)$ structure satisfying the conditions given in [30]. The $SU(5)$ structure is given by $J_{SU(5)}$ and the $(5,0)$ form $\Omega_{SU(5)}$ defined by

$$\Omega_{SU(5)} = H_1^{-1/2} H_2^{-1/2} (dx_1 + i dx_2) \wedge (D\chi_1 + i D\chi_2) \wedge \Omega_{CY}$$

(B.4)

where $\Omega_{CY}$ is the holomorphic $(3,0)$ form on $CY_3$. The relevant conditions that need to be imposed are [30]

$$d \left( \Delta^{-3} [J_{SU(5)}]^4 \right) = 0$$

$$d \left( \Delta^{-3/2} \Omega_{SU(5)} \right) = 0$$

(B.5)

The first equation implies that $W = W_1 + iW_2$ is primitive and the second one implies that its type $(0,2)$ component is missing. In order that we solve the $D = 11$ equations of motion we need to also impose the equation of motion for the four-form which gives

$$\nabla_{CY}^2 H_1 = 0$$

$$\nabla_{CY}^2 H_2 = - |W|^2$$

(B.6)

Note that in [30] $J_{SU(5)}$, $\Omega_{SU(5)}$ were denoted by $\Omega$, $\chi$, respectively.

In the language of [30] note that these imply, in particular, that $W_4 = 3d \log \Delta$ and $W_5 = -12d \log \Delta$. 

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After the above remarks we observe see that our solution (2.13) does indeed lie within this class with

\[ ds^2(CY_3) = dr^2 + r^2 ds^2(SE_5) \]

\[ H_1 = r^{-4}, \quad H_2 = fr^{-2} \]  

(B.7)  

with \( W \) defined on \( SE_5 \) and \((1, 1)\) and primitive on \( CY_3 \), and \( f \) satisfying equation (2.10).

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