Teaching stochastics to bachelors of mathematics: computer simulation for conceptual understanding

Elena Kuznetsova and Natalia Zhbanova*

Department of Automation and Informatics, Lipetsk State Technical University, 30, Moskovskaya St., Lipetsk, 398600, Russia

*E-mail: zbanoid@gmail.com

Abstract. The professional training of mathematics students is notable for the large volume and complexity of the content of educational material. In this regard, the problem of conceptual understanding is relevant. Since many concepts in mathematics have a high level of abstraction, studying them, it is essential to create an image of an object through various forms of knowledge representation. Computer simulation in the study of probability theory and its applications (stochastics) help students understand the essence of random phenomena and forms the ability to analyze statistical data. Students whose curriculum includes a computer simulations workshop in the study of stochastics, not only solve problems better but also higher evaluate the value of probabilistic ideas and methods and have a lower level of anxiety. Due to the use of simulations in teaching mathematics students on stochastics, we use a computer as a means of automation of calculations, means of learning, and a tool of knowledge, which corresponds to the principle of the integrated use of information computer technology in the educational process.

1. Introduction

Today, mathematics is not only science that studies abstract objects, but also a tool for solving the real world's significant problems. Nevertheless, as scientific publications show, insufficient attention is paid to students' preparation in mathematical and IT fields. The training of students in these areas has some features that must be considered in the teaching process. First of all, students of mathematical and IT fields have mathematical abilities, a high level of general intelligence, and motivation [1]. Secondly, future mathematicians' professional training is notable for the large volume, the complexity of the content and depth of the studied sections of mathematics. For most professions, mathematics is a tool for solving a specific, reasonably narrow range of problems. At the same time, mathematicians need conceptual understanding, theoretical thinking, knowledge of mathematical methods, and programming skills at a higher professional level. Also, mathematics, as a cultural phenomenon, has several features. First, mathematics builds and studies the abstract world, its objects and laws sometimes have no analogs in everyday life experience. Secondly, the essential characteristics of mathematics are a highly formalized language, and the logical accuracy of formulations, proofs, and conclusions. Besides, the development of mathematics, as a science, also takes place on an abstract plane, submits to its internal laws, which do not have a direct connection with the phenomena of the real world. These mathematical knowledge features sometimes lead to a misunderstanding of the essence and sources of formalism in the study of mathematics, when students master only the form, not filling it with specific content. So the Russian mathematician and teacher KHinchin called a severe misunderstanding a situation in which the
formalism of mathematical knowledge is mixed with the requirement of formal logical rigor of substantiation of mathematical truths [2]. As studies have shown [3], mathematics students understand and accept the intrinsic complexity of subjects. Indeed, many of the disciplines that make up the Applied Mathematics curriculum have a high level of abstraction of concepts, the understanding of which is a challenge. The use of computer simulations in the educational process allows us to achieve the necessary balance between theory, experiment, and computing, contributing to the conceptual understanding of the studied disciplines and the formation of professional competencies of future mathematicians [4]. The purpose of the article is to investigate the possibilities of integrating computer simulations into the formation of probability theory concepts when teaching mathematics students on stochastics.

2. Literature review
Many studies in the field of teaching mathematics to one degree or another related to the problem of understanding: not a single teacher of mathematics set the goal of learning to achieve knowledge at the level of memory-reproduction. Following tradition, in mathematics, conceptual and procedural knowledge is distinguished: understanding concepts and problem-solving skills [5–6]. Studies have shown that these two types of knowledge are interacting closely [7–9]. However, in recent years more attention has been paid to conceptual understanding [10]. The articles [2; 11–13] are devoted to the problems of overcoming formalism and achieving an "understanding" assimilation of educational material in the process of teaching mathematics in Russia.

The analysis of works on the methodology of teaching mathematics allows us to highlight the following aspects of understanding in mathematics.

- The integrity and systematic nature of mathematical knowledge. Therefore, it is so dangerous when any element of the system is not understood.
- The rationale for each step of the logical conclusion. If any step is skipped, the whole conclusion collapses, so it is vital to choose the severity level from the beginning.
- Mastering the skills of working with the symbolic language (translation of mathematical facts into the symbolic language and, conversely, see facts behind the symbols). We need to pay attention to the process of introducing new symbols.
- Interpretation of the theory, the ability to give an example or counterexample.
- The ability to apply systems of concepts in solving problems in a new situation.
- In teaching the probabilistic sections of mathematics (stochastics), the problem of conceptual understanding has several features because probability theory occupies a particular position among many other mathematical disciplines. On the one hand, it is characterized by the unity of the formal-structural properties of the mathematical apparatus; on the other hand, the concepts of causality, chance, and probability are philosophical categories. The definition of the basic concepts of probability theory is impossible without revealing the relationship of philosophical ideas about the need, chance, possibility, probability, the formation of a unique style of thinking, gaining experience in stochastic modeling [14]. A computer simulation workshop provides an opportunity to accumulate experience with probabilistic distributions. This capability allows further mental modeling of random events and processes and effectively conducts probabilistic forecasting when solving creative research problems [15-18].

3. Methods
The use of computer simulations in the educational process is based on the didactic capabilities of ICT: the possibility of an interactive dialogue; computer visualization of educational information; computer modeling; storage of large volumes of information and providing easy access to it; automation of calculation processes and information retrieval activities; automation of processes of information and methodological support, organizational management of educational activities and monitoring of learning outcomes [19].
At the same time, in the process of studying at a university for the productive use of ICTs and the fullest possible realization of their didactic opportunities, it is necessary to take into account such aspects as training features and professionally essential qualities in the field of ICT for students in this field; psychological features of higher education; features of the scientific content of the discipline; the level of training of students; class form (lecture, practical lesson, laboratory work, independent work); the specifics of the studied section and local pedagogical goals (training, control, testing). ICTs have the following functions in the process of teaching the students majoring in applied mathematics: performing calculations, searching, and storing information, a teaching tool, a cognition tool.

The development of a theoretical basis and the integration of computer modeling in the practice of teaching stochastics are based on the analysis of scientific literature, analysis and generalization of educational experience, the results of an educational experiment, and the study of students' assessments.  

4. Discussion

II In [12], two methods for the formation of mathematical concepts are distinguished: classification-operational and updated (ontological). In the first method, the concept is introduced through the genus and species differences, while the main actions are the identification of significant characters and classification. The second method involves creating an image of a concept through various forms of representing knowledge. The classification-operational way of forming concepts is effective for sciences that study the real objects of the material world, such as chemistry, physics, and biology. The actualized method of forming concepts is useful for those who have a high level of abstraction or are fundamental in any new theory. 

An example of such concepts is the basic concepts of probability sections of mathematics (stochastics). Based on the study of scientific research on the philosophy and history of probability theory, systems of key concepts were formulated [14]. For probability theory, this is randomness, probability, a random event, a random variable, probability distributions, numerical characteristics of random variables, joint distributions of random variables, independence, and correlation. For mathematical statistics, this is sampling, population, distribution parameter, estimation of distribution parameter, bias, consistency of estimates, confidence interval, hypothesis, and statistical conclusion. The formation of systems of key concepts needs increased attention.

As experience has shown, computer simulation is a useful tool for the formation of probability theory concepts, making it possible to practice the principle of visual modeling. We have developed task systems in the disciplines Probability Theory and Mathematical Statistics, Theory of Stochastic Processes, Econometrics, Mathematical Modeling, which enable students to create an image of the concept under study, to identify its characteristics and properties.

The objectives of a computer workshop on probability theory and mathematical statistics are:

- assistance in understanding the probabilistic nature of the studied objects, deeper penetration into the essence of random phenomena;
- active, meaningful assimilation of theoretical principles, probabilistic concepts, and laws;
- the formation of the basic skills necessary for the analysis and processing of data using a computer, the development of the skills of statistical inference;
- acquisition of stochastic modeling skills.

It is assumed that the most effective is the combination of such organizational forms as working with a teacher, independent work (with teacher's advice), homework, and laboratory work. Laboratory work involves both the use of popular software products (Excel, Statistica, Etc.) and the creation of students' programs using stochastic functions.

A computer workshop in the course of stochastics allows to implement didactic principles:

- visualization (for example, taking advantage of graphical analysis);
• accessibility, feasibility, and individualization of training (the ability to build classes and the formation of tasks, taking into account the level of training of students and educational tasks set by the teacher);
• consciousness and activity (tasks for independent work, links to information sources, research tasks).

The actualized method for the formation of probability theory concepts involves the experience of the stochastic activity, many alternative approaches, various representations of the same object: symbolic, graphic, computer model. To achieve understanding in the new material, we focus on those aspects that include new knowledge in the knowledge mastered earlier. The task of the workshop is formulated so that there is a problem that needs to be addressed. In the formulation of the problem, both positive and negative examples are essential.

For example, one of the essential concepts in the course of stochastics is a probability distribution, since probabilistic thinking is the ability to think in the language of probability distributions [14, 18].

When studying the course of probability theory, students consider practical tasks related to the experimental study of the properties of various distribution laws based on computer modeling. The importance of the step by step development of key concepts for the formation of a holistic view of the subject is essential. This idea applies in the study of the properties and probabilistic characteristics of the Cauchy distribution. On the one hand, the Cauchy distribution has similar characteristics with a normal distribution $N(0,1)$. In this case, its study will contribute to filling knowledge gaps on both topics and more effective memorization. On the other hand, an analysis of the differences between the two types of similar distributions will lead to a deep conceptual understanding of their characteristics and features.

The task’s essence is to generate random numbers in a mathematical package, distributed according to the normal law and the Cauchy law, followed by their presentation in graphical form and experimental verification of their properties. As well known, the density function for the Cauchy distribution is similar to the density function of the normal distribution. However, it has a significant feature: “thick tails” or “heavy tails.”

Students are invited to formulate differences in the results. A comparison of the histograms (figure 1 and figure 2) shows that for the normal distribution, all implementations were in the range ($-3; 3$). Simultaneously, the Cauchy distribution demonstrates several outliers that deviate significantly both in the positive and negative direction.

![Figure 1. Histogram of random variables generated by Normal distribution $N(0;1)$.](image-url)
Figure 2. Histogram of random variables generated by Cauchy distribution C(0;1).

The example shows how the presence of “heavy tails” manifests itself in practice. Computer simulation of random variables, the construction of their histograms, and visual comparative analysis allow students to form a clear idea of distributions and helps to master the basic concepts of the course of stochastics.

In the second stage of the study, students will have to check the Chebyshev theorem experimentally. The theorem says that the arithmetic mean of a sufficiently large number of random variables (with limited variances) loses its random character: if independent random variables have the same mathematical expectation equal to a, and their variances are bounded by the same constant C, then

\[ P \left( \left| \frac{X_1 + \cdots + X_n}{n} - a \right| \leq \epsilon \right) \geq 1 - \frac{C}{n\epsilon^2} \]

With an increase in the number of terms \( n \), the arithmetic mean of random variables \( \bar{X} = \frac{X_1 + \cdots + X_n}{n} \) converges in probability to mathematical expectation \( a \).

Students verify the theorem by calculating the numerical characteristics of the generated distributions, and experimentally verify the absence of a mathematical expectation (sample mean) for random variables distributed according to Cauchy.

As well known, the mathematical expectation of a normal distribution can be calculated in a standard way:

\[ M(x) = \int_{-\infty}^{+\infty} x \cdot f(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x \cdot e^{-\frac{x^2}{2}} dx = 0 \]

The mathematical expectation of the Cauchy distribution does not exist since the integral diverges:

\[ M(x) = \int_{-\infty}^{+\infty} x \cdot f(x) dx = \int_{-\infty}^{+\infty} \frac{x}{x^2 + 1} dx \]

Students test these statements by calculating a sample mean. Sample averages calculated over columns of generated random numbers with a normal distribution have stability: when the experiment
is repeated, the obtained values \( \bar{X} = \frac{X_1 + \ldots + X_n}{n} \) are in the vicinity of mathematical expectation \( M(X)=0 \). For sample means calculated from random variables with the Cauchy distribution, significant discrepancies are observed. The conclusion is drawn about the fulfillment of the Chebyshev theorem for normal distribution and not for the Cauchy distribution, and the reasons are analyzed.

It is assumed that as a result of the assignment, students will form images of such fundamental concepts as “distribution,” “numerical characteristics of random variables,” “distribution parameter,” “estimation of a distributed parameter.” Also, understanding (a new interpretation of previously studied topics), and application (the ability to use the material studied), are indicators of the qualitative assimilation of knowledge.

5. Results
Conceptual understanding allows students to use alternative approaches to solve the tasks better, apply the knowledge in a new context, and correctly assess the possibilities and limits of statistical methods in the study of real data. Besides, the attitude of students towards the study of stochastics is changing for the better. Students, whose curriculum includes a computer workshop, rate higher the significance of probabilistic methods and ideas, have a lower level of anxiety [20]. Thus, the targeted formation of conceptual knowledge through computer simulation positively affects not only the cognitive but also the value and emotional spheres.

6. Conclusion
It is necessary to recall two essential principles of using computers in the educational process when teaching stochastics. The first one implies using a computer as a means of calculation (for calculating probability values solving problems or calculating the numerical characteristics of a sample in data analysis). The second one implies using a computer as a cognitive tool (in stochastic computer modeling). In the first case, routine calculations are avoided. In the second, a perspective opens up, both cognitively and for understanding the connection between informatics and mathematics, the natural sciences, and the humanities, which contributes to the development of intuition and research skills in situations of uncertainty and choice, and activates cognitive activity. Computer modeling in teaching stochastics allows the implementation of the principle of the integrated use of ICT in the educational process. The computer is used both as a learning tool and as a cognition tool.

References
[1] Kuznetsova E and Matytcina M 2018 A multidimensional approach to training mathematics students at a university: improving the efficiency through the unity of social psychological and pedagogical aspects Int. J. Math. Educ. Scienc. Techn. 49 401-16
[2] Khinchin AYa 1983 On formalism in the school teaching of mathematics Pedagogicheskie stat’i. (Moscow: Izdatel’stvo Akademii pedagogicheskix nauk RSFSR) p 204
[3] Kuznetsova E 2019 Evaluation and interpretation of student satisfaction with the quality of the university educational program in applied mathematics Teach. Math. Applic.: Int. J. IMA 38 107-19
[4] Teodoro V D and Neves R G 2011 Mathematical modelling in science and mathematics education Comp. Phys. Communic. 182 8-10
[5] Hiebert J and Lefevre P 1986 Conceptual and procedural knowledge: The case of mathematics ed J Hibert (New York: Routledge) p 113-33
[6] Baroody A J, Feil Y and Johnson A R 2007 An alternative reconceptualization of procedural and conceptual knowledge J. Res. Math. Educ. 38 115-31
[7] Byrnes JP and Wasik B A 1991 Role of conceptual knowledge in mathematical procedural learning Develop. Psych. 27 777-86
[8] Byrnes J P 1992 The conceptual basis of procedural learning Cogn. Develop 7 235-57
[9] Rittle-Johnson B and Schneider M 2014 Developing conceptual and procedural knowledge of
mathematics Eds R Cohen Kadosh and A Dowker (Oxford: Oxford University Press) pp 1102-18

[10] Crooks N M and Alibali M W 2014 Defining and measuring conceptual knowledge in mathematics Develop. Rev. 34 344-77

[11] Brejtigam E K and Karakozov S D 2010 The integrity of the system of basic concepts in the study of mathematics at school and university Mir nauki, kultur’ obrazovaniya 3 190-4

[12] Vladimirtseva S A 2008 The main directions of development of the theory of the formation of mathematical concepts in school Mir nauki, kultur’, obrazovaniya 4 103-7

[13] Kuznetsova E V 2013 To the question of the relationship of knowledge and understanding in the process of teaching mathematics Prepodavatel’ XXI vek 3-1 52-7

[14] Kuznetsova E 2019 Probabilistic ideas and methods in undergraduate mathematics: axiological aspects IEJME: Math. Education 14 363-73

[15] Fielding-Wells J 2018 Dot plots and hat plots: supporting young students emerging understandings of distribution, center and variability through modeling ZDM Math. Educ 50 1125-38

[16] Konold C, Harradine A and Kazak S 2007 Understanding distributions by modeling them International J. Comp. Math. Learn. 12 217-30

[17] Pfannkuch M, Budgett S, Fewster R, Fitch M, Pattenwise S, Wild C and Ziedins I 2016 Probability Modeling and Thinking: What Can We Learn from Practice? Stat. Educ. Res. J. 15 11-37

[18] Steel E A, Liermann M and Guttorp P 2019 Beyond Calculations: A Course in Statistical Thinking, Amer. Statistician 73 392-401

[19] Robert I V 2008 Theory and methods of informatization of education (psychological, pedagogical and technological aspects) (Moscow: Institut informatizatsii obrazovaniya) p 274

[20] Kuznetsova E V and Vlasova K A 2019 Identification of differences in the attitude of students to the discipline of probability theory Vestnik Nizhegorodskogo universiteta im. N.I. Lobachevskogo. Seriya: Sotsial’nyye nauki 2 166-72