SCHRODINGER EQUATION FOR LAGRANGIAN PATH INTEGRAL WITH SCALING OF LOCAL TIME

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ABSTRACT

A method for deriving the Schrodinger equation for Lagrangian path integral with scaling of local time is given.

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1. Introduction

Scaling of time and exact path integration: In 1979 Duru and Kleinert’s important paper [1] on the exact solution for the H-atom problem opened the way to the exact path integral treatment of several potential problems. Prior to the work of Duru and Kleinert the exact solution to only a few problems could be given within the path integral scheme [2]. Though the original treatment of the H-atom by Duru and Kleinert using reparametrization of paths was a formal one, it did lead to an important new technique for exact path integration. Besides the use of the technique of adding new degrees of freedom with trivial dynamics and of point transformations, scaling of local time was utilized to complete exact path integration for several problems of quantum mechanics [3-14].

Let us consider a classical system described by a Hamiltonian $H(q,p)$ with the Hamilton’s equations of motion

$$\frac{dq^k}{dt} = \frac{\partial H}{\partial p^k} \quad (1)$$
$$\frac{dp^k}{dt} = -\frac{\partial H}{\partial q^k} \quad (2)$$

The solutions of these equations give $q$ and $p$ as functions of $t$ and initial values $q_0, p_0$ taken by $q$ and $p$ at specified time $t_0$. The two functions $q \equiv q(t)$, and $p \equiv p(t)$ describe the classical trajectory in phase space. Thus the time also plays the role of parameter specifying points on the classical paths. The classical path can also be specified in terms of a new parameter $\sigma = \sigma(t)$, such that $\sigma$ increases monotonically with time; in this paper $\sigma$ will called pseudo-time and could be a different function for each path. For the present we shall consider only those cases where $dt/d\sigma$ is a function of co-ordinates only, say, $\alpha(q)$. This reparametrization of paths can be described by means of an auxiliary Hamiltonian $\mathcal{H}$ given by

$$\mathcal{H} = f(q)(H - E) \quad (3)$$

Imposing the energy constraint $H(q,p) - E = 0$ on the solutions, the Hamil-
tons equations for $H$ become

$$\frac{\partial q^k}{\partial \sigma} = \alpha(q) \frac{\partial H}{\partial p} \quad (4)$$

$$\frac{\partial p^k}{\partial \sigma} = -\alpha(q) \frac{\partial H}{\partial q} \quad (5)$$

These equations along with $dt/d\sigma = \alpha(q)$ give the correct equations for the
classical motion of the particle.

Duru and Kleinert’s treatment of hydrogen atom makes use of formal
arguments involving reparametrization of paths in the path integral frame-
work of quantum mechanics and Kustarnheimo-Stiefel transformation [16].
They succeeded in relating the path integral for the H- atom to that for the
harmonic oscillator in four dimensions. Subsequent work by others authors
led to the important technique of scaling of local time. In this technique a
central role is played by an identity, which we call the scaling formula. For
two different problems, this formula relates path integrals for the propaga-
tors or for the energy dependent Green functions [17]. The two path integrals
appearing in the scaling formula correspond to the dynamical Hamiltonian
$H(q,p)$ and an auxiliary hamiltonian $\mathcal{H} = \alpha(q)(H(q,p) - E + \Delta V)$. Note
that in the quantum mechanical problem the auxiliary hamiltonian differs
from the corresponding classical expression given in (3) above; the difference
being by $O(\hbar^2)$ terms written as $\Delta V$. The exact form of $\Delta V$ depends on
details of the path integral scheme used.

Scaling and Hamiltonian Path Integral Quantization: In the exact treatment
of path integrals the scaling of local time is merely used as a tool to arrive at
the solution. The idea of scaling of local time in path integration has been
found useful in another way.

When one wants to set up a hamiltonian path integral quantization
scheme in arbitrary co-ordinates, the path integral with the classical hamil-
tonian does not lead to correct quantization rules. To recover the correct
Schrödinger equation one is, in general, forced to add, in an ad hoc manner,
terms of $O(\hbar^2)$ to the classical hamiltonian. The necessity of adding ad-hoc
$O(\hbar^2)$ terms is an unwelcome feature of hamiltonian path integral quanti-
zation schemes. Even here the idea of scaling of local time turns out to be
useful. It was first shown in ref. [18] that it is possible to avoid the need
of addition of these ad-hoc $O(\hbar^2)$ terms, and to formulate hamiltonian path integral quantization entirely in terms of classical hamiltonian provided one uses a suitable local scaling of time [18]. This is a distinct advantage over other hamiltonian path integral schemes and this once again underlines the importance of scaling of local time in the path integral framework.

Path Integrals with local scaling of time: The hamiltonian path integral quantization of ref [6] has been further developed in later papers [19-20]. Motivated by the importance of local scaling of time in the path integral framework, we have introduced and studied properties of the hamiltonian path integrals with local scaling of time. These path integrals were called the path integrals of second kind as opposed to the standard path integral which we shall refer to the path integral of first kind. The path integral of second type was defined in terms of a path integral of first type for the auxiliary hamiltonian as introduced above.

One of the most important properties of the path integrals, with or without a scaling of time is the Schrodinger equation satisfied by it. The method for obtaining the Schrodinger equation for the standard path integral is very simple and well known [9]. For the path integral of second kind the method to derive Schrodinger equation has been given only the for hamiltonian form of path integrals in [19-20]. The derivation of the Schrodinger equation for $HPI2$ turns out to be rather complicated as compared to that for $HPI1$.

The aim of this paper is to introduce the lagrangian form of path integral with scaling and to establish the method of obtaining the Schrodinger equation for this second form of lagrangian path integrals.
2. Path Integrals with scaling with scaling

In this section we shall briefly recall the definition of the hamiltonian path integral of second type and to introduce lagrangian path integral with scaling.

Given a hamiltonian \( H(q,p) \) and a positive scaling function \( \alpha(q) \), we first define an auxiliary hamiltonian \( H(E;q,p) = \alpha(q)(H(q,p) - E) \). The hamiltonian path integral of second kind, denoted by \( \mathcal{K}[H,\alpha,\rho] \), depends on the hamiltonian \( H(q,p) \), local scaling function \( \alpha(q) \), and integration measure \( \rho \); \( \mathcal{K} \) is defined in terms of an ordinary hamiltonian path integral \( \mathcal{K}[\mathcal{H},\rho] \) for the auxiliary hamiltonian \( \mathcal{H} \), by means of the equation

\[
\mathcal{K}[H,\rho,\alpha](q_t,q_0) = \sqrt{\alpha(q_0)} \int \frac{dE}{2\pi\hbar} \exp(-iEt/\hbar) \int d\sigma \mathcal{K}[\mathcal{H}(E;q,p),\rho](q\sigma,q_0) \quad (6)
\]

This path integral \( \mathcal{K} \) was called HPI2, or the hamiltonian path integral of the second kind. It depends on the hamiltonian \( H(q,p) \), integration measure \( \rho \) and scaling function \( \alpha(q) \). In the right hand side, the definition of HPI2 involves another hamiltonian path integral HPI1 \( \mathcal{K}[\alpha(H - E),\rho] \) for the auxiliary hamiltonian

For a phase space function \( H(q,p) \) and integration measure \( \rho \), HPI1 has been defined in terms of what we have named short time propagator, to be called STP hereafter, and will be denoted by \( (q_2\epsilon||q_10) \). In terms of the STP the path integral \( K[H,\rho] \) is given by means of the following equations.

\[
K^{(N)}[H,\rho](q_t;q_0t_0) = \prod_{k=1}^{N-1} \rho(q_k) dq_k \prod_{j=0}^{N-1} (q_{j+1}\epsilon||q_j0) \quad (7)
\]

where

\[
K[H,\rho](q_t;q_0t_0) = \lim_{N \to \infty} K^{(N)}[H,\rho](q_t;q_0t_0) \quad (8)
\]

We make a few remarks about the definition of HPI1 and HPI2. The STP approximates the full path integral \( K[H,\rho] \) for short times. The path integral HPI1 is very similar to any other hamiltonian path integral existing in literature within the time slicing approach to the path integration. Finally, the above definition of HPI2 is such that for \( \alpha(q) = \text{constant} \), the HPI2 \( \mathcal{K}[H,\rho,\alpha] \) coincides with HPI1 \( K[H,\rho] \). Further details of the definition of
HPI1 and HPI2 and about the choice of STP can be found in our earlier papers[19- 20].

We shall now define lagrangian path integral with scaling in a fashion similar to the definition of HPI2. We shall restrict our attention to simple potential problems in one dimension; the generalization of our method to many degrees of freedom is straightforward. For systems with one degree of freedom we take the dynamical hamiltonian to be

\[ H(q, p) = \frac{p^2}{2M} + V(q) \] (9)

and the auxiliary hamiltonian is

\[ \mathcal{H}(q, p) = \alpha(q) \left( \frac{p^2}{2M} + V(q) - E \right) \] (10)

The corresponding dynamical and auxiliary lagrangians are given by

\[ L = \frac{M}{2} \left( \frac{\partial q}{\partial t} \right)^2 - V(q) \] (11)

\[ \mathcal{L} = \frac{M}{2\alpha(q)} \left( \frac{\partial q}{\partial t} \right)^2 - \alpha(q)(V(q) - E) \] (12)

In an approach similar to the hamiltonian path integrals we shall at first define a discrete lagrangian form for the path integral with scaling. This path integral will be denoted by \( \mathcal{K}[L, \alpha, \rho] \) and is defined by

\[ \mathcal{K}[L, \rho, \alpha](q_t, q_0) \equiv \sqrt{\alpha(q_0)\alpha(q_t)} \int \frac{dE}{2\pi\hbar} \exp(-iEt/\hbar) \int d\sigma K[\mathcal{L}, \rho](q, q_0) \] (13)

where the quantity \( K[\mathcal{L}, \rho](q, q_0) \) appearing in the above equation is an ordinary lagrangian path integral for the auxiliary lagrangian \( \mathcal{L} \) defined below within the time slicing approach.

In order to define the propagator \( K[\mathcal{L}, \rho] \) we start with the definition of STP \((q_{k+1}, \epsilon|q_k)\) corresponding to \( \mathcal{L} \) with measure \( \int \rho(q) dq \), for short pseudo time \( \epsilon = \sigma/N \). Define
To simplify our notation, as in the above equations, we shall use the notation $f_k$ to denote the mid point value $f((q_k+1 + q_k)/2)$ of the function $f$; we shall use a similar notation for other functions of $q$. The path integral $K[\mathcal{L}, \rho](q\sigma, q_0)$ is defined in terms of the STP $(q_{k+1} \epsilon | q_k)$.}

$$K[\mathcal{L}, \rho](q\sigma, q_0) = \lim_{N \to \infty} \int \left( \frac{N-1}{\prod_{j=1}^{N-1} dq_j} \right) \prod_{k=0}^{N-1} (q_{k+1} \epsilon | q_k)$$

In the above and in the rest of this paper $q_N$ stands for $q$. The method for deriving the Schrodinger equation for path integral HPI1 is well known [21]. In fact evaluation of path integral of limit $N \to \infty$ is needed. This is because for short times HPI1 is approximated by the STP and this makes the derivation of the Schrodinger equation for HPI1 very simple. However, the derivation of Schrodinger equation for HPI2 is rather complex and was given in our previous paper. This is because in case of HPI2 the STP cannot be used in the right hand side to derive the Schrodinger equation. Expression of HPI2 requires use of full HPI1 even for short times due to the fact that HPI2 is defined in terms of HPI1 integrated over $\sigma$. Therefore one has to insert expressions (14) and (18) into (13) and take limit $N \to \infty$ at the end. For the hamiltonian path integral with scaling the method of deriving the Schrodinger equation is given in our previous paper.

### 3. Schrodinger equation

We shall now give the corresponding method for the lagrangian path integral with scaling. We use this result to obtain the scaling formula which
expresses the lagrangian path integral with scaling to lagrangian path integral without scaling. In the following the results are stated and proved only for $\rho = 1$.

**Proposition**: The lagrangian path integral $K[L,1,f]$ satisfies

$$\lim_{t \to 0} K[L,1,f](q_t,q_0) = \delta(q - q_0)$$

and the wave function $\psi(q) \in L^2(R,dq)$ propagated by the path integral

$$K[L,1,f](q_t,q_0) = \sqrt{f(q)} \int \frac{dE}{2\pi\hbar} \exp(-iEt/\hbar) \int_0^\infty d\sigma K[L,\rho](q_\sigma,q_0)$$

satisfy the Schrodinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M} \frac{\partial^2 \psi}{\partial q^2} + (V + v_{ps})\psi$$

where

$$v_{ps} = -\frac{\hbar^2}{8M} \left\{ (f''/f) - (f'/f)^2 \right\}$$

is the one dimensional Pak-Sokmen potential.

**Outline of proof**: The integral over $E$ is performed first. The coefficient of $(-iE/\hbar)$ is $(t - \sum f_k)$ which gives a delta function $(t - \sum f_k)$. The integral can be done readily, remembering $\epsilon = \sigma/N$. The net effect of this is to replace $\epsilon$ everywhere by

$$\epsilon \to (\sum f_m) = \eta N/(\sum f)$$

with $\eta = t/N$ and there is an overall Jacobian factor coming from integration. This leaves us with the path integral $K$ of (21) as $N \to \infty$ limit of

$$K_N = \int \left( \prod_{k=1}^{N-1} dq_k \right) G \prod_{k=1}^{N-1} \left[ \sqrt{\frac{M}{2\pi i\hbar \eta}} F_k \exp \left\{ \frac{i}{\hbar} \left( \frac{\Delta^2_k}{2\eta M} F^{(k)} - \eta V_k F^{(k)} \right) \right\} \right]$$

with

$$G = \sqrt{f(q)} f(q_0)/(\sum_{m=0}^{N-1} f_m)$$

$$F^{(k)} = \left( \sum_{m=0}^{N-1} f_m \right) / (N f_k)$$
The propagator $K$ cannot be calculated for all time $t$ for arbitrary potential $V$. However, our aim here is to obtain the Schrodinger equation. Though for this purpose it is sufficient to calculate $K$ only for small $t$, nevertheless the $\lim_{N \to \infty}$ limit has to be computed. What we can do is to anticipate which terms would be $O(t), O(t^2)$, etc., and keep terms only up to order $t$. This can be done by recalling that each power of $\Delta_k$ should be counted as being of order $\sqrt{\eta}$. Thus we start from

$$
\psi(q, t) = \lim_{N \to \infty} \int K_N(qt, q_0) \psi(q_0, 0) dq_0
$$

and our strategy will be to Taylor expand everything so as to obtain the integrand in (28) as a function of $q, \Delta_{N-1}, \ldots, \Delta_0$ and to replace

$$
\left( \prod_{k=1}^{N-1} dq_k \right) dq_0
$$

by

$$
\prod_{k=0}^{N-1} d\Delta_k
$$

Next, keeping only appropriate powers of $\Delta$, we perform the $\Delta$ integrations. Lastly, we take the $N \to \infty$ limit. Before we start on this long but straightforward program, there is one small point which can be disposed right here. Recalling $\Delta_k = O(\sqrt{\eta})$, we can ignore the factors $1/F^{(k)}$ in the potential term because for finite $N$,

$$
\frac{1}{N} \sum_m f_m \approx \frac{1}{N} \sum_k f_k + O(\Delta_k) = f_k + O(\Delta_k)
$$

and we need not keep anything higher than order zero because $\eta$ is already a factor in the potential term. We obtain the same finite $t$ propagator whether we keep the $O(\Delta_k)$ terms coming from (31) or not. Here after we shall ignore the potential term altogether because it will get added to the final Schrodinger equation in the standard manner. We now perform the Taylor expansion.
\begin{align*}
\psi(q_0) &= \psi \left( q - \sum_{k=0}^{N-1} \Delta_k \right) \\
&= \psi - \left( \sum_{k=0}^{N-1} \Delta_k \right) \psi' + \frac{1}{2} \left( \sum_{k=0}^{N-1} \Delta_k \right)^2 \psi'' + \ldots \\
\text{(32)}
\end{align*}

From now on we shall omit explicit dependence on \( q \). Define

\[ \delta_k = \frac{(q_k + 1 + q_k)}{2} - q_N, \quad k = 0, 1, \ldots, N - 1 \]

(34)

In terms of \( \delta \)'s we have the following expansions.

\begin{align*}
\psi(q_0) &= \psi - \left( \sum_{k=0}^{N-1} \Delta_k \right) \psi' + \frac{1}{2} \left( \sum_{k=0}^{N-1} \Delta_k \right)^2 \psi'' + \ldots \\
&= \psi - \left( \sum_{k=0}^{N-1} \Delta_k \right) \psi' + \frac{1}{2} \left( \sum_{k=0}^{N-1} \Delta_k \right)^2 \psi'' + \ldots \\
\text{(33)}
\end{align*}

\[ f_k = f(q_k + \delta_k) = f + \delta_k f' + \frac{1}{2} \delta_k^2 (f'/f)^2 + \ldots \]

(35)

\[ 1/f_k = f^{-1} \left[ 1 - \delta_k f'/f - \frac{1}{2} \delta_k^2 f''/f + \delta_k^2 (f'/f)^2 + \ldots \right] \]

(36)

\[ F^{(k)} = 1 + (f'/f) \left( \frac{1}{N} \sum_{m=0}^{N-1} \delta_m - \delta_k \right) \]

(37)

\begin{align*}
&+ \frac{1}{2} \left( f''/f \right) \left( \frac{1}{N} \sum_{m=0}^{N-1} \delta_m^2 - \delta_k^2 \right) \\
&+ (f'/f)^2 \left( \delta_k^2 - \frac{1}{N} \sum_{m=0}^{N-1} \delta_m \right) \\
\text{(38)}
\end{align*}

\[ \Pi_k \sqrt{F^{(k)}} = 1 + \frac{1}{4} (f'/f)^2 \left[ \sum \delta_k^2 - \frac{1}{N} \left( \sum \delta_k \right)^2 \right] + \ldots \\
\text{(40)}
\]

\begin{align*}
G &= 1 - \frac{f'}{f} \left( \frac{1}{2} \sum_{k=0}^{N-1} \Delta_k + \frac{1}{N} \sum_{m=0}^{N-1} \delta_m \right) \\
&+ \frac{1}{2N} (f'/f)^2 \sum_{k=0}^{N-1} \Delta_k \sum_{m=0}^{N-1} \delta_m \\
&+ \frac{1}{N^2} (f'/f)^2 \left( \sum \delta_m \right)^2 - \frac{1}{8} (f'/f)^2 \left( \sum \Delta_k \right)^2 \\
&+ \frac{f''}{f} \left[ \frac{1}{4} \left( \sum \Delta_k \right)^2 - \frac{1}{2N} \sum \delta_m^2 \right] + \ldots \\
\text{(41)}
\end{align*}
Substituting these in \((25)\), we get the gaussian integral

\[
\psi(q,t) = \lim_{N \to \infty} \int_{-1}^{N} \prod_{k=0}^{N-1} \frac{D\Delta_k}{\sqrt{2\pi iN\hbar}} \exp \left[ \frac{i}{\hbar} \sum_{k=1}^{N-1} \frac{\Delta_k^2}{2\eta} \right] Q(\Delta_k)
\]

where

\[
Q(\Delta_k) = \left\{ \psi(q) - \psi'(q) \sum \Delta_k + \frac{1}{2} \psi''(q) (\sum \Delta_k)^2 \right\} X_1 X_2 X_3
\]

where \(X_1, X_2, \) and \(X_3\) are given by

\[
X_1 = 1 - (f'/f) \left( \frac{1}{2} \sum \Delta_k + \frac{1}{N} \sum \delta_m \right) + (f''/f) \left\{ \frac{1}{4} (\sum \Delta_k)^2 - \frac{1}{2N} \sum \delta_m^2 \right\}
\]

\[
+ (f''/f)^2 \left\{ \frac{1}{2N} (\sum \delta_k) (\sum \delta_m) + \frac{1}{N^2} (\sum \delta_m)^2 - \frac{1}{8} (\sum \Delta_k)^2 \right\}
\]

\[
X_2 = 1 + \frac{iM}{\hbar} (f'/f) \sum_k \frac{\Delta_k^2}{2\eta} (\sum (\delta_m/N) - \delta_k) + \frac{iM}{2\hbar} (f''/f) \sum_k \frac{\Delta_k^2}{2\eta} (\sum \delta_m^2/N - \delta_k^2)
\]

\[
+ \frac{iM}{\hbar} (f'/f)^2 \sum_k \frac{\Delta_k^2}{2\eta} (\delta_k^2 - (\delta_k/N) \sum \delta_m/N)
\]

\[
+ \frac{1}{2} (f'/f)^2 \left\{ \frac{iM}{\hbar} \sum_k \frac{\Delta_k^2}{2\eta} \left( \frac{1}{N} \sum \delta_m - \delta_k \right) \right\}^2
\]

\[
X_3 = 1 + \frac{1}{4} (f'/f)^2 \left\{ \sum \delta_m^2 - \frac{1}{N} (\sum \delta_m)^2 \right\}
\]

The above integrals are fairly straightforward even though laborious. The points to remember are as follows.

1. We need to keep only terms of order \(t\), therefore, a fortiori, \(O(t)\) terms.
2. The \(\delta'\)s are linear combinations of \(\Delta'\)s given by \((34)\)
3. There are \(1/\eta\) terms in \(X_2\) which means that \(O(\Delta^4)\) terms have to be kept for all such terms.
With these points in mind, (25) can be evaluated. The main labor is in computing the summations. For example, in the term \((f''/f)(-\sum \delta^2_m/N)\),

\[
\sum_{m=0}^{N-1} \delta^2_m = \frac{1}{4} \Delta^2_{N-1} + \sum_{m=0}^{N-2} \left( \sum_{j=m+1}^{N-1} \Delta_j + \frac{1}{2} \Delta_m \right)^2
\]

\[
= \frac{1}{4} \Delta^2_{N-1} + \sum_{m=0}^{N-2} \left( \sum_{j=m+1}^{N-1} \Delta_j + \frac{1}{2} \Delta_m \right) \left( \sum_{n=m+1}^{N-1} \Delta_n + \frac{1}{2} \Delta_m \right)
\]

\[
= \frac{1}{4} \Delta^2_{N-1} + \sum_{m=0}^{N-2} \left( \frac{1}{4} D^2_m + \Delta_m \sum_{j=m+1}^{N-1} \Delta_j + \sum_{j=m+1}^{N-1} \sum_{n=m+1}^{N-1} \Delta_j \Delta_n \right)
\]

which upon integration over \(\Delta\)'s will become

\[
\left( \sum_k \Delta^2_k/(2\eta) \right) \left( -\sum_m \delta^2_m/N \right) \frac{f''}{f} \psi(q)
\]

\[
= -\frac{i\hbar f''}{Nf} \left[ \frac{1}{4} + \sum_{m=0}^{N-2} \left( \frac{1}{4} + \sum_{j=m+1}^{N-1} \delta_{jm} + \sum_{n=m+1}^{N-1} \sum_{j=m+1}^{N-1} \delta_{jn} \right) \right] \psi(q)
\]

\[
= -\frac{i\hbar f''}{Nf} \left( \frac{1}{4} + \frac{1}{4} (N - 1) + 0 + \sum_{m=0}^{N-2} (N - m - 1) \right) \psi(q)
\]

\[
= -\frac{i\hbar f''}{fN^2} \left( \frac{1}{4} + \frac{1}{4} (N - 1) - \frac{1}{2} N(N - 1) + \ldots \right) \psi(q)
\]

\[
= -\frac{i\hbar f''}{2f} \psi
\]

In the last step we have taken the limit \(N \to \infty\). We omit the remaining details and quote the result, after reinstating the \(q\) dependence,

\[
\psi(q, t) \approx \psi(q) + \frac{i\hbar}{2m} \psi''(q) + \frac{i\hbar}{8m} \left[ (f'/f)^2 - (f''/f) \right] \psi(q)
\]

which gives us the Schrodinger equation of the proposition if we recall that the potential \(V\), as we mentioned above, could have been kept through all this analysis without any change.
Scaling Formula: As a byproduct of the above result we get the scaling formula

\[ \mathcal{L}[L, \alpha, 1] = K[L - v_{PS}, 1] \quad \text{(64)} \]

This result follows from the fact that the two sides satisfy the same equation and have the same value at \( t = 0 \).

Concluding remarks:

1. In this paper we have defined Lagrangian path integral with scaling and have established a method for obtaining the Schrödinger equation for this path integral of second kind only for a specific choice of the lagrangian STP, obviously the method is general and can be used for any other choice of STP. Also the method is applicable to systems with several degrees of freedom.

2. We have checked that the results obtained here are are in agreement with those obtained for the hamiltonian path integral with scaling obtained earlier. This check was performed by establishing a correspondence between the lagrangian form of STP used here and the hamiltonian forms of STP used in the earlier papers. This correspondence is obtained by first carrying out the momentum integrations in the hamiltonian STP of ref [19-20] and making repeated use of the McLaughlin Schulman trick [22].

3. To conclude we indicate possible new application of the method established in this paper. It is well known that semi-classical expression for the propagator, well known as the van Vleck Pauli-DeWitt formula [23-24], gives exact answer for propagator in several cases[26-28]. Recently there has been a renewed interest in finding the class of problems for which the semi-classical result or its suitable generalizations give exact answers [29].

In this connection it is intersting to ask if the class of problems for which WKB result is exact can be enlarged by use of scaling of time. Specifically, let as assume that the semi-classical propagator for the auxiliary Lagrangian \( \mathcal{L}(E; \dot{q}^K, q^K) \), or a suitable generalization, is inserted for \( K \) in the right hand side of the definition of \( K \). One can then pose the question what is the class of problems for which a solution for the scaling function \( \alpha(q) \) can be found such that (13) gives an
exact answer? For investigations of this type, the method of deriving the Schrodinger equation established here will certainly be very useful.

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