Evolution of geodesic congruences in a gravitationally collapsing scalar field background

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Abstract

The evolution of timelike and null geodesic congruences in a non-static, inhomogeneous spacetime representing the gravitational collapse of a massless scalar field, is investigated in detail. We show explicitly how the initial values of the expansion, rotation and shear of a congruence, as well as the spacetime curvature along the congruence, influence the evolution and focusing of trajectories in different ways. The role of initial conditions on the focusing time is explored and highlighted. In certain specific cases, the expansion scalar is found to exhibit a finite jump (from negative to positive value) before focusing. The issue of singularity formation and the effect of the central inhomogeneity in the spacetime, on the evolution of the kinematic variables, is discussed. In summary, our analysis does seem to throw some light on how a family of trajectories evolve in a specific model of gravitational collapse.
I. INTRODUCTION

In General Relativity (GR), the behaviour of a family of test particles (described by a non-spacelike geodesic congruence), in a given spacetime background, is analysed by studying the evolution of three kinematic variables—expansion, shear and rotation (ESR). The evolution of the ESR along the congruence, is governed by the Raychaudhuri equations [1–3]. It is well known that the Raychaudhuri equations play a crucial role in the context of the Penrose-Hawking singularity theorems [4, 5].

The structure and geometric features of a given spacetime (encoded in the metric $g_{ij}$ and its derivatives) must necessarily be reflected in the evolution of the kinematic variables that characterise a geodesic congruence. Apart from initial conditions on the kinematic variables, geometric quantities (eg. the Ricci tensor, Ricci scalar, and the Weyl tensor) which appear in the Raychaudhuri equations, also influence the evolution of the congruence. Thus, knowing the kinematics of geodesic congruences surely helps in probing the spacetime geometry. In addition, we must accept that, observationally, the nature of any spacetime geometry (and the gravitational field it represents) can only be verified through a study of trajectories.

The evolution of the kinematic variables of timelike geodesic congruences have been extensively studied in various spacetime backgrounds in the recent past [6–8]. However, much of this earlier work has been in spacetimes which are static. Studies on the evolution of geodesic congruences in inhomogeneous, non-static spacetimes such as those representing gravitational collapse, have not been looked at yet. One might expect that issues specific to collapsing scenarios which include the formation of singularities, apparent horizons, conjectures and theorems related to singularities etc. may be understood through such studies.

An important and well-known result which follows from the Raychaudhuri equation for the expansion, is that of geodesic focusing. A congruence has a focal point if all geodesics in the family converge and meet there, at a finite value of the affine parameter. Geodesic focusing may be completely benign, i.e., a focal point may not be a curvature singularity but the geodesic family terminates at such a focal point, thereby defining the notion of geodesic incompleteness. On the other hand, curvature singularities must always be focal points of a geodesic congruence, and their existence also implies the geodesic incompleteness of a spacetime.

Even though much has been said and proved about geodesic focusing, questions do remain.
In particular, the questions we wish to analyse are largely related to detailed studies from which, some surprises may emerge. Some such questions are:

- Which quantities\textit{ clearly determine and control} the occurrence of focusing/defocusing of geodesic congruences in a given spacetime?

- What is the\textit{ precise} role of spacetime curvature and other geometric quantities in the evolution of congruences?

- To what\textit{ extent} do the \textit{initial values} of expansion, shear and rotation influence the occurrence of focusing/defocusing?

Adopting the methods developed in some of our earlier papers\cite{6-8}, in this article, we will try to answer some of these questions in the context of a class of non-static, inhomogeneous spacetimes representing gravitational collapse.

Of course, in order to proceed with our studies, we need a spacetime line element representing gravitational collapse. Among many available models, we choose an exact solution representing a scalar field collapse scenario in (3+1)-dimensions\cite{9}. Our choice is essentially governed by an available exact, non-static, inhomogeneous solution which is reasonably simple in its line element structure. However, note that in this solution, the scalar field is all-pervading (exists for all $r$ and $t$) and the collapse scenario is somewhat different from the usual pressure-less dust ball collapse (Oppenheimer-Snyder) or even the spherisymmetric collapse of a perfect fluid with pressure.

The above-mentioned spacetime, along with its geodesic structure is discussed in Sec. II. In Sec. III we briefly review the derivation of the Raychaudhuri equations for geodesic congruences. In Sec. IV-Ⅶ we solve the ESR evolution equations (along with the geodesic equations) numerically, and bring out certain aspects of the evolution kinematics of the congruence. Sec VIII summarises our results and suggests future avenues of work.

II. EXACT SOLUTION FOR SCALAR FIELD COLLAPSE

We begin by writing down the line element representing the gravitational collapse of a massless scalar field minimally coupled to (3+1)-dimensional gravity\cite{9}

$$ds^2 = (at + b) \left[ -f^2(r)dt^2 + f^{-2}(r)dr^2 \right] + R^2(r,t)(d\psi^2 + \sin^2 \psi d\phi^2) \quad (1)$$
where
\[ f^2(r) = \left(1 - \frac{2c}{r}\right)^\alpha \]
\[ R^2(r, t) = (at + b)r^2 \left(1 - \frac{2c}{r}\right)^{1-\alpha} \]
and the scalar field profile is given as,
\[ \Phi(r, t) = \pm \frac{1}{4\sqrt{\pi}} \ln \left[d \left(1 - \frac{2c}{r}\right)^{\frac{\alpha}{\sqrt{3}}} (at + b)^{\frac{\sqrt{3}}{2}}\right] \]
where \(a, b, c, d\) are constants and \(\alpha = \pm \frac{\sqrt{3}}{2}\). \(R(r, t)\) is the area radius.

The above line element and the scalar field constitutes a solution of the Einstein field equations given as
\[ R_{\alpha\beta} = 8\pi \partial_\alpha \phi \partial_\beta \phi \tag{2} \]
The plots for the scalar field profile are shown in Fig. 1. The curvature scalar for the metric is
\[ \mathcal{R} = \frac{12ca^2(r - c) - 3a^2r^2}{2r^2(at + b)^3} \left(1 - \frac{2c}{r}\right)^{-2-\alpha} + \frac{2c^2(1-\alpha^2)}{(at + b)r^4} \left(1 - \frac{2c}{r}\right)^{-2+\alpha} \]
It should be noted that curvature singularities are present at \(r = 2c\) and at \(t = -b/a\) for both values of \(\alpha\). Depending on the constants \(a, b\) and \(c\), the metric (1) represents different types of spacetimes. Details about these solutions are available in [9]. The constant \(c\) is related to the central inhomogeneity of the matter distribution. In the limit \(c \to 0\), the \(r\) dependence of the metric is removed and the spacetime becomes homogeneous [10, 11]. \(a < 0\) and \(a > 0\) represent black-hole-like and white-hole-like solutions, respectively. We will consider the case \(a < 0\). The apparent horizon is described by \(g^{\alpha\beta} \partial_\alpha R \partial_\beta R = 0\). The time of formation of the apparent horizon at a Lagrangian coordinate \(r\) is given by
\[ t_{ah}(r) = -\frac{b}{a} \pm \frac{r^2}{2} (1 - \frac{2c}{r})^{1-\alpha} [r - c(1+\alpha)]^{-1} \tag{3} \]
where plus (minus) sign is for \(a > 0\) (\(a < 0\)). For \(a < 0\), the two-sphere labelled by \(r\) gets trapped \((g^{\alpha\beta} \partial_\alpha R \partial_\beta R < 0)\) for \(t > t_{ah}(r)\).

The geodesics in the equatorial section \((\psi = \pi/2)\) of the collapsing spacetime are governed by
\[ \dot{\phi} = \frac{L}{R^2(z,t)} \tag{4} \]
\[ \dot{z} = f(z) \sqrt{f^2(z)\dot{t}^2 + \frac{sR^2(z,t) - L^2}{R^2(z,t)(at + b)}} \tag{5} \]
\[ \dot{t} + \frac{a}{(at + b)} \dot{t}^2 + 2 \frac{f'(z)}{f(z)} \dot{z} + \frac{as}{2f^2(z)(at + b)^2} = 0 \]  \hspace{1cm} (6)

where \( L \) is an integration constant representing angular momentum. In obtaining the Eqns. (4)-(6), we have used the transformation \( r - 2c = z \). Also, we have used the fact that the velocity vector \( u^\alpha = (\dot{t}, \dot{z}, \dot{\phi}) \) satisfies the constraint \( u^\alpha u_\alpha = s \), where \( s = -1 \) for timelike geodesics and \( s = 0 \) for null geodesics.

III. THE RAYCHAUDHURI EQUATIONS

A. Timelike geodesic congruence

Consider a congruence of timelike geodesics in a given spacetime background. The geodesic congruence may undergo isotropic expansion, shear and rotation. The kinematics of these quantities are investigated in the spacelike hypersurface orthogonal to the central geodesic. Therefore, one can define a transverse spatial metric \( h_{\alpha\beta} \) induced on the spacelike hypersurface as,

\[ h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta; \quad (\alpha, \beta = 0, 1, 2, 3) \]  \hspace{1cm} (7)

where the timelike vector field \( u^\alpha \) associated with the congruence is tangent to the geodesic at each point and satisfies the timelike constraint \( u^\alpha u_\alpha = -1 \). The transverse metric satisfies \( u^\alpha h_{\alpha\beta} = 0 \) implying that \( h_{\alpha\beta} \) is orthogonal to \( u^\alpha \). From the vector field \( u^\alpha \), one can define the velocity gradient tensor \( B_{\alpha\beta} = \nabla_\beta u_\alpha \). In four spacetime dimensions, the tensor \( B_{\alpha\beta} \) can be decomposed into its trace, symmetric traceless and anti-symmetric parts as

\[ B_{\alpha\beta} = \frac{1}{3} h_{\alpha\beta} \theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \]  \hspace{1cm} (8)
where $\theta = B^\alpha_\alpha$ is the expansion scalar (trace part), $\sigma_{\alpha \beta} = \frac{1}{2}(B_{\alpha \beta} + B_{\beta \alpha}) - \frac{1}{3}h_{\alpha \beta}\theta$ the shear tensor (symmetric traceless part) and $\omega_{\alpha \beta} = \frac{1}{2}(B_{\alpha \beta} - B_{\beta \alpha})$ the rotation tensor (antisymmetric part). By virtue of this construction, the shear and the rotation tensors satisfy $h^{\alpha \beta}\sigma_{\alpha \beta} = 0$ and $h^{\alpha \beta}\omega_{\alpha \beta} = 0$. We also have $g^{\alpha \beta}\sigma_{\alpha \beta} = 0$ and $g^{\alpha \beta}\omega_{\alpha \beta} = 0$. Since $u^\alpha\sigma_{\alpha \beta} = 0$ and $u^\alpha\omega_{\alpha \beta} = 0$, both $\sigma_{\alpha \beta}$ and $\omega_{\alpha \beta}$ are purely spatial (i.e., $\sigma^{\alpha \beta}\sigma_{\alpha \beta} > 0$ and $\omega^{\alpha \beta}\omega_{\alpha \beta} > 0$) and lie in the orthogonal hypersurface.

The evolution equation for the spatial tensor $B_{\alpha \beta}$ can be written as,

$$u^\gamma\nabla_\gamma B_{\alpha \beta} = -B_{\alpha \gamma}B^\gamma_\beta + R_{\gamma \beta \alpha \delta}u^\gamma u^\delta$$

where $R_{\gamma \beta \alpha \delta}$ is the Riemann tensor. The trace, symmetric traceless and anti-symmetric parts of the equation yield

$$\frac{d\theta}{d\lambda} + \frac{1}{3}\theta^2 + \sigma^2 - \omega^2 + R_{\alpha \beta}u^\alpha u^\beta = 0$$

$$u^\gamma\nabla_\gamma \sigma_{\alpha \beta} + \frac{2}{3}\theta\sigma_{\alpha \beta} + \sigma_{\alpha \gamma}\gamma^\gamma_{\beta} + \omega_{\alpha \gamma}\gamma^\gamma_\beta - \frac{1}{3}h_{\alpha \beta}(\sigma^2 - \omega^2) - C_{\gamma \beta \alpha \delta}u^\gamma u^\delta - \frac{1}{2}\tilde{R}_{\alpha \beta} = 0$$

$$u^\gamma\nabla_\gamma \omega_{\alpha \beta} + \frac{2}{3}\theta\omega_{\alpha \beta} + \sigma^\gamma_{\beta}\omega_{\alpha \gamma} - \gamma_{\alpha \omega_{\beta} \gamma} = 0$$

where $\sigma^2 = \sigma^{\alpha \beta}\sigma_{\alpha \beta}$, $\omega^2 = \omega^{\alpha \beta}\omega_{\alpha \beta}$, $\lambda$ is the affine parameter, $C_{\gamma \beta \alpha \delta}$ is the Weyl tensor and $\tilde{R}_{\alpha \beta} = (h_{\alpha \gamma}h_{\beta \delta} - \frac{1}{3}h_{\alpha \beta}h_{\gamma \delta})R^{\gamma \delta}$ is the transverse trace-free part of $R_{\alpha \beta}$. The equation for $\theta$ is a Riccati type equation, and is of considerable interest in the context of the singularity theorems. Redefining $\theta = 3\frac{E}{F}$, one can obtain the following Hill-type equation

$$\frac{d^2F}{d\lambda^2} + \frac{1}{3}(R_{\alpha \beta}u^\alpha u^\beta + \sigma^2 - \omega^2)F = 0.$$  

The analysis of focusing ($\theta \to -\infty$) or defocusing ($\theta \to \infty$) can be done by investigating the quantity $I = R_{\alpha \beta}u^\alpha u^\beta + \sigma^2 - \omega^2$. It is clear from (9) that the sufficient condition for geodesic focusing is $I > 0$. From the strong energy condition, we obtain $R_{\alpha \beta}u^\alpha u^\beta \geq 0$. Thus, from the focusing condition, it should be noted that rotation defies focusing while shear assists it.

In the context of this paper and from the Einstein-scalar equations we have, $R_{\alpha \beta} = 8\pi \partial_\alpha \phi \partial_\beta \phi$. Therefore,

$$I = \left[\sigma^2 - \omega^2 + 8\pi \left(\frac{d\phi}{d\lambda}\right)^2\right]$$

The third term on the right hand side above is always positive. Hence, the positivity of $I$ is crucially dependent on the sign of $\sigma^2 - \omega^2$. As we will see, $\sigma^2 - \omega^2$ may be positive or
negative over a certain domain of $\lambda$. It will also be observed that there can be a domain of $\lambda$ where $I < 0$. We will see in detail how this gives rise to certain characteristic new features in the evolution of the expansion scalar $\theta$.

**B. Null geodesic congruence**

For a null vector field $u^\alpha$, $u^\alpha h_{\alpha\beta} \neq 0$, where $h_{\alpha\beta}$ is given by Eqn. (7). Therefore, the definition of the transverse induced metric given in Eqn. (7) will not work. For null geodesic congruence, the transverse metric is constructed as $h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha N_\beta + u_\beta N_\alpha$, where the auxiliary null vector $N^\alpha$ satisfies $N^\alpha N_\alpha = 0$, $N^\alpha u_\alpha = -1$. With this construction the transverse metric satisfies $u^\alpha h_{\alpha\beta} = 0$ and $N^\alpha h_{\alpha\beta} = 0$. The trace $h^{\alpha\beta}h_{\alpha\beta} = (n-2)$ indicates that, for a null geodesic congruence in $n$-spacetime dimensions, the transverse metric is $(n-2)$-dimensional. Therefore, for a null geodesic congruence in four dimensions, one can decompose $B_{\alpha\beta}$ as

$$B_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \quad (15)$$

With this decomposition, the equation for expansion, shear and rotation can be written down – these are available in [3]. In our work described below, we will show how the evolution of the kinematic variables (ESR) and focusing/defocusing effects depend on the initial conditions on ESR as well as the spacetime curvature.

**IV. EVOLUTION OF A TIMELIKE GEODESIC CONGRUENCE**

**A. Method of solution**

The equations (10)-(12) are nonlinear coupled ordinary differential equations. In the absence of analytical solutions, one has to solve this set of equations (along with the geodesic equations), numerically. However, instead of solving Eqns. (10)-(12), it is more convenient to solve Eqn (9), and subsequently, extract the expansion scalar $\theta$, shear tensor $\sigma_{\alpha\beta}$ and rotation tensor $\omega_{\alpha\beta}$ by taking the trace, symmetric traceless and anti-symmetric parts of $B_{\alpha\beta}$, respectively. The initial condition on $B_{\alpha\beta}$ can be easily constructed from the initial conditions on the ESR variables using (8). It may be pointed out that initial conditions on the velocity field $u^\alpha$, $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$ must satisfy the orthogonality conditions.
We first study the kinematic evolution of geodesic congruences for the black-hole-like solution \((a = -1, b = 1 \text{ and } c = 1)\). There is a timelike singularity at \(r = 2\), i.e., at \(z = 0\), and a spacelike singularity at \(t = 1\). For the latter case, the range of the time coordinate is \(-\infty < t \leq 1\). At time \(t = 1\), the whole spacetime collapses to the origin \(R = 0\). Therefore, for geodesics beginning at time \(t < 1\), the spacelike singularity at \(t = 1\) is a future directed singularity. We consider the geodesic congruence in the equatorial section \((\psi = \pi/2)\). Throughout the numerical evaluation, we have kept fixed the initial conditions on \(\{x^\alpha(\lambda), u^\alpha(\lambda)\}\).

**B. Evolution of kinematic variables for** \(\alpha = -\frac{\sqrt{3}}{2}\)

![Plots showing the dependence of the focusing affine parameter \(\lambda_f\) on the initial expansion, shear and rotation for \(\alpha = -\frac{\sqrt{3}}{2}\). Here the initial conditions are \(t(0) = -7.0\), \(z(0) = 1.0\), \(\phi(0) = 0\), \(\dot{t}(0) = 0.25\) and \(L = 1.0\).](image-url)

(a)\(\sigma_0^2 = 0\)  
(b)\(\omega_0^2 = 0\)  
(c)\(\theta_0 = 0\)
From the numerical evaluations, with different initial conditions, it is found that the congruence always exhibits focusing. We denote $\lambda_f$ as the value of the affine parameter at which focusing takes place. The dependence of $\lambda_f$ on the initial conditions of the ESR variables is presented in Fig. 2. The value $\lambda_f = 10.46$ corresponds to the time $t = 1$ when the geodesic congruence encounters a future spacelike singularity. Fig. 3 shows the variations of $g^{\alpha\beta} \partial_\alpha R \partial_\beta R$ and the area radius $R$ along the geodesic congruence with identically vanishing shear and rotation. It is clear that, as $\lambda \to 10.46$, the area radius $R$ tends to zero. Therefore, as $\lambda \to 10.46$, the congruence falls into the spacelike singularity. As observed in Fig. 3, the congruence encounters the apparent horizon at $\lambda_{ah} = 9.32$ after which it gets trapped ($g^{\alpha\beta} \partial_\alpha R \partial_\beta R < 0$) in the trapping region formed in the spacetime. The area radius $R$ initially increases till the congruence hits the apparent horizon and is subsequently subsumed in the trapped region before it falls into the singularity. Therefore, the congruence gets trapped before it falls into the singularity. This is because of the fact that, in the given collapsing spacetime, a two sphere labelled by $r$ becomes trapped before it becomes singular, i.e, before it collapses to the origin $R = 0$. The plateau-top region in Fig. 2 corresponds to the time (in terms of the affine parameter $\lambda$) of formation of the spacetime singularity. We now discuss the effect of initial expansion, shear and rotation as well as the curvature on the focusing time.

FIG. 3. Plots of variation of (a) $g^{\alpha\beta} \partial_\alpha R \partial_\beta R$ and (b) area radius $R$ along the geodesic congruence for $\alpha = -\frac{\sqrt{3}}{2}$. 

(a) 

(b)
1. Effect of initial expansion and shear on focusing

Generally, the effect of increasing (decreasing) the initial expansion ($\theta_0$) is to delay (prepone) the focusing. But, for a given $\omega_0$, Fig. 2(a) shows a peculiar non-monotonic dependence of $\lambda_f$ on $\theta_0$. For $\omega_0^2$ lying between two values $\omega_{11}^2$ and $\omega_{12}^2$ ($\omega_{11}^2 < \omega_{12}^2$), with decreasing $\theta_0$, $\lambda_f$ decreases up to a certain initial expansion $\theta_c$; below $\theta_c$, $\lambda_f$ increases suddenly and then starts decreasing again. Notice that the value of $\theta_c$ depends on $\omega_0^2$.

To understand the above-mentioned peculiar behaviour for $\omega_{11}^2 < \omega_0^2 < \omega_{12}^2$, we consider the initial conditions corresponding to the section $\omega_0^2 = 0.04$ in Fig. 2(a) and plot the evolution of the expansion $\theta(\lambda)$ in Fig. 4. It is observed that, till $\theta_0 = -0.2647$, $\lambda_f$ decreases with decreasing $\theta_0$. However, between $\theta_0 = -0.2647$ and $\theta_0 = -0.2648$, a transition (sudden change) in focusing time is noted; focusing is delayed. Therefore, for $\omega_0^2 = 0.04$, $\theta_c$ lies between $-0.2648$ and $-0.2647$. In Fig. 5, we investigate this peculiar behaviour in the focusing time by plotting the corresponding evolution of the scalars $R_{\alpha\beta}u^\alpha u^\beta$, $\sigma^2$, $\omega^2$ and $I = R_{\alpha\beta}u^\alpha u^\beta + \sigma^2 - \omega^2$. It is observed that, for $\theta_0 = 0.25$, $I$ diverges mainly because of the curvature term $R_{\alpha\beta}u^\alpha u^\beta$ and the divergence takes place as $\lambda \to 10.46$, implying that the focusing takes place due to the singularity formation (Fig. 5(a)). For $\theta_0 = 0.1$ or $-0.2647$, $I$ diverges because of the term $\sigma^2$ and divergence takes place much before the singularity formation, implying that the focusing takes place due to divergence of shear (Fig. 5(b), Fig. 5(c)). For $\theta_0 = -0.2648$, initially $\sigma^2$ and $\omega^2$ almost cancel each other in the expression of $I$ (Fig. 5(d)). However, midway during the evolution, over a short period, $\omega^2$ dominates.
FIG. 5. Plots of $R_{\alpha\beta} u^\alpha u^\beta$, $\sigma^2$, $\omega^2$ and $I$ ($= R_{\alpha\beta} u^\alpha u^\beta + \sigma^2 - \omega^2$) for the initial values of the ESR variables corresponding to Fig. 4.

over $\sigma^2$ making $I < 0$ (congruence starts defocusing). This induces a sharp transition/jump (from negative to positive) in the evolution of $\theta$ in Fig. 4. As the evolution proceeds further, $I$ again becomes positive and diverges because of the curvature term $R_{\alpha\beta} u^\alpha u^\beta$. Therefore,
the focusing is delayed (see Fig. 4) because of the dominance of rotation over shear, midway during the evolution. However, complete defocusing does not take place because  $R_{\alpha\beta}u^\alpha u^\beta$ diverges as the evolution proceeds towards the singularity formation time  $t = 1.0$. The amplitude of the jump in the evolution of the expansion scalar gets smaller as one makes the initial expansion $\theta_0$ more negative (Fig. 4). It may again be noted that, for the case $\theta_0 = -0.70$, focusing takes place entirely due to the divergence of shear (Fig. 5(f)), and the curvature singularity has no role in the focusing.

The $\theta_0 = constant$ sections of Fig. 2(b) indicate that, with increasing $\sigma_0^2$, focusing time decreases monotonically, i.e., initial shear always helps in focusing.

### 2. Effect of initial rotation on focusing

It is well-known that rotation always defies focusing. The $\sigma_0^2 = constant$ sections of Fig. 2(c) show the dependence of $\lambda_f$ on initial rotation $\omega_0$. It is clear that $\lambda_f$ increases with $\omega_0^2$ up to a certain critical value $\omega_c^2$. At $\omega_0^2 = \omega_c^2$, there is a sudden change in focusing time. To understand this transition, i.e., sudden change in focusing time, we plot $\theta(\lambda)$ for $\theta_0 = 0$ and $\sigma_0^2 = 0.75$ and note the change in focusing time for different initial rotation (Fig. 6). The corresponding plots for the ESR variables, $R_{\alpha\beta}u^\alpha u^\beta$ and $I$ are shown in Fig. 7. Clearly, one can note a sudden change in focusing time between $\omega_0^2 = 0.1731$ and $\omega_0^2 = 0.1735$. The transition takes place because of the dominance of $\omega^2(\lambda)$, midway during the evolution (Fig. 7(c) and Fig. 7(d)).

![Figure 6](image_url)

**FIG. 6.** Figures showing the role of initial rotation $\omega_0^2$ in focusing of the congruence, for $\theta_0 = 0.0, \sigma_0^2 = 0.75$. 

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FIG. 7. Plots of $R_{\alpha\beta} u^\alpha u^\beta$, $\sigma^2$, $\omega^2$ and $I = R_{\alpha\beta} u^\alpha u^\beta + \sigma^2 - \omega^2$ for the initial values of the ESR variables corresponding to Fig. 6.

C. Evolution of kinematic variables for $\alpha = \sqrt{3}/2$

As in the previous case, here too, focusing always takes place. In Fig 8 we show the dependence of the focusing affine parameter $\lambda_f$ on the initial values of the ESR variables. In this case, the singularity formation time $t = 1.0$ corresponds to $\lambda_f = 4.82$.

1. Effect of initial expansion and shear on focusing

The $\omega_0^2 = constant$ sections of Figs. 8(a) and 8(b) and $\sigma_0^2 = constant$ sections of Fig. 8(c) indicate that, up to a certain initial expansion, focusing time increases with increasing $\theta_0$. 

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(a) $\sigma_0^2 = 0.0$

(b) $\sigma_0^2 = 0.35$

(c) $\omega_0^2 = 0.0$

(d) $\theta_0 = 0.0$

FIG. 8. Plots showing the dependence of the focusing affine parameter $\lambda_f$ on the initial expansion, shear and rotation for $\alpha = \sqrt{\frac{3}{2}}$. Here the initial conditions are $t(0) = -10.0$, $z(0) = 0.3$, $\phi(0) = 0$, $\dot{t}(0) = 3.0$ and $L = 4.0$.

Above this certain value, focusing time is independent of $\theta_0$ because, focusing always takes place at the singularity. The dependence of focusing time on the initial shear is the same as that in the case with $\alpha = -\sqrt{\frac{3}{2}}$.

2. Effect of initial rotation on focusing

The $\theta_0 = constant$ sections of Fig. 8(a) indicate that, with zero initial shear, the focusing time is independent of rotation. But, with a sufficient non-zero initial shear, focusing time depends on the initial rotation (see $\sigma_0^2 = constant$ sections of Fig. 8(d) and Fig. 9(b)). This is due to the fact that, in the absence of any spacetime singularity formation, the congruence
would have focused beyond the singularity formation time $t = 1.0$. A sufficient initial shear focuses the congruence before it falls to the singularity. This focusing now can be delayed by choosing some initial rotation. Thus, sufficient initial shear and rotation affects the focusing behaviour of the congruence (Fig. 8(d) and Fig. 9(b)).

Fig. 10 demonstrates the reason why $\lambda_f$ is independent of initial rotation for $\sigma^2_0 = 0$. From the figures, it is clear that, unlike the case for $\alpha = -\sqrt{3} \over 2$, the rotation drops to small values and hence does not have a significant effect on $I$, as the evolution proceeds. Therefore, the evolution of $I$ is completely controlled either by the curvature term (Fig. 10(a) and Fig. 10(b)) or by the shear (Fig. 10(c) and Fig. 10(d)). The transition in $\lambda_f$ in Fig. 9(b) can be explained in the same way as that for the case $\alpha = -\sqrt{3} \over 2$.

![Graphs of $\theta$ vs $\lambda$ for different initial conditions](image)

(a)$\theta_0 = 0.0, \sigma^2_0 = 0.0$

(b)$\theta_0 = 0.0, \sigma^2_0 = 1.0$

FIG. 9. The role of the initial rotation $\omega^2_0$ on the evolution of the expansion scalar $\theta$ and focusing of the congruence with $\alpha = \sqrt{3} \over 2$.

From the above analysis, we observe that focusing always takes place. To illustrate this further, we draw three schematic diagrams in Fig. 11. The solid circle, red dot and blue dot represent the position of the apparent horizon, the singularity at $z = 0$ and the initial position of the congruence, respectively. The region outside (inside) the apparent horizon is trapped (untrapped). The apparent horizon shrinks in size with time. As the evolution proceeds, the congruence hits the apparent horizon (second diagram) at some value of $\lambda$. Subsequently, the congruence gets trapped and falls into the singularity (third diagram). Therefore, there is no case where the congruence does not hit the horizon. Therefore, we must always have focusing though this may be benign (not happening at a curvature singularity) for certain initial conditions, as discussed above.
V. ANALYSIS OF JUMP IN THE EXPANSION SCALAR

In the last section, we noticed that, midway during the evolution of the congruence, the dominance of rotation over shear (which makes \( I < 0 \)), leads to a sharp transition (from negative to positive value) in the evolution of expansion of the congruence. As the evolution proceeds further, because of the curvature term, \( I \) diverges to positive infinity, thereby causing the focusing of the congruence. Let us now ask and analyse the question – what would happen if the curvature term does not diverge as the evolution proceeds? For example, for static spacetimes, as the family of outgoing timelike geodesics evolves, the curvature term \( R_{\alpha\beta}u^\alpha u^\beta \) becomes less and less significant. This is also true for this non-static case except at the end of the evolution process. Therefore, largely, the value and sign of \( I \) is determined by the shear and the rotation terms. Thus, we may have the following sub-cases:

(1) During the evolution, if the shear dominates over the rotation, ultimately making \( I > 0 \),
FIG. 11. Schematic diagrams explaining the focusing of congruence. The solid circle indicates the (shrinking) apparent horizon and the blue curve traces the central geodesic of the congruence.

we eventually have focusing following a jump.

(2) If \( I < 0 \) always and goes to zero as \( \lambda \to \infty \), \( \theta \) goes to zero after a sharp transition from negative to positive value. This has been observed in the kinematic study of a family of projectile trajectories [12] and also in the kinematic study of deformable media without stiffness [13].

(3) If \( I \) oscillates between positive and negative values, we have periodic oscillations in the expansion scalar; focusing does not take place. This has been observed in the kinematic study of a family of trajectories in the two dimensional isotropic harmonic oscillator, a charged particle in an electromagnetic field [12] and also in the kinematic study of deformable media with stiffness [13]. In [12], \( I = \sigma^2 - \omega^2 \) and the curvature term is a constant, given by \( \alpha \); the total term \( (I + \text{curvature}) \) oscillates between positive and negative values.

(4) If \( I < 0 \) always and diverges to negative infinity at some \( \lambda \), we have complete defocusing of the congruence.

One of the above cases could have occurred for our non-static spacetime, had the singularity not formed. But, as the evolution proceeds towards the singularity formation time, the curvature term, and hence \( I \), diverges to positive infinity. Therefore, we have focusing following a sharp transition in the expansion scalar. If the rotation had not dominated over shear, midway during the evolution, we could have observed direct focusing without any intermediate jump in the expansion scalar.
VI. EFFECT OF \( a \) AND \( c \)

In this section, we study the effect of \( a \) and \( c \) on the kinematic evolution. The central inhomogeneity \( c \) plays an important role in the evolution of the timelike congruences. For the initial conditions in Fig. 2, the timelike geodesic equations can not be solved in the limit \( c \to 0 \) or \( a \to 0 \). Therefore, we take different initial conditions on \( \{x^\alpha(\lambda), u^\alpha(\lambda)\} \) which can be used to solve the timelike geodesic equations for all values of \( c \) and \( a \). The dependence of the time to singularity, quantified by \( \lambda_f \), on the initial expansion and rotation for different values of \( a \) and \( c \) are shown in Fig. 12. The sharp rise of \( \lambda_f \) in Fig. 12(a) and Fig. 12(b) indicates that \( \lambda_f \to \infty \), i.e., focusing does not occur. Therefore, in the static case \( (a = 0) \), irrespective of the presence/absence of the central inhomogeneity \( c \), there exists a region in the space of initial conditions for which focusing does not takes place; the expansion scalar goes to zero (Fig. 13(a) and Fig. 13(b)) as \( \lambda \to \infty \). On the other hand, in the collapsing case \( (a \neq 0) \), we always have focusing in finite time. Interestingly, the presence of the central inhomogeneity introduces a peculiar behaviour (i.e., the observed jump) in the evolution of the expansion scalar (Fig. 13). From Figs. 12(c)-12(f), it is clear that the jump behaviour is absent for \( c = 0 \) and also for large value of \( c \). From the geodesic equations, we notice that, for a given initial condition, \( \dot{\phi} \sim (z + 2c)^{-(1+\alpha)} \). Therefore, for large \( c \), \( \dot{\phi} \approx 0 \) which means that the congruence is almost radial. Thus, the burst in congruence rotation, as observed for moderate values of \( c \), is damped out. This in turn smooths out the jump in the evolution of the expansion scalar. This may be the reason behind the gradual disappearance of the peculiar behaviour in the variation of the time to singularity with initial conditions, as observed in Figs. 12(d)-12(f).

VII. EVOLUTION OF A NULL GEODESIC CONGRUENCE

To study the null geodesics congruence, one needs to find out the auxiliary null vector \( N^\alpha \) from the conditions \( N^\alpha N_\alpha = 0 \) and \( N^\alpha u_\alpha = -1 \). Notice that the choice of \( N^\alpha \) is not unique. Here, we take \( N^\psi = 0 \), \( N^\phi = 0 \). The other components are obtained by solving the conditions satisfied by \( N^\alpha \). These are given by

\[
N^t = \frac{1}{\sqrt{|g_{tt}|}} \frac{1}{\sqrt{\left(g_{rr}u^r + \sqrt{|g_{tt}|}u^t\right)}}, \quad N^r = -\frac{1}{\sqrt{g_{rr}}} \frac{1}{\sqrt{\left(g_{rr}u^r + \sqrt{|g_{tt}|}u^t\right)}} \tag{16}
\]
Here, the initial conditions on $x^\alpha(\lambda)$ and $u^\alpha(\lambda)$ are the same as those for the timelike case in Fig. 2. Fig. 14 shows the dependence of focusing affine parameter on initial expansion and shear. Here, $\lambda_f = 8.98$ corresponds to the time $t = 1.0$. Fig. 15 shows the effect of shear on the evolution of expansion. The corresponding plots for $R_{\alpha\beta}u^\alpha u^\beta$, $\sigma^2$ and $R_{\alpha\beta}u^\alpha u^\beta + \sigma^2$ are shown in Fig. 16. In this case, we take initial rotation to be zero. For $\sigma_0^2 = 0.2$, focusing takes place mainly because of the divergence of curvature term (Fig. 16(a)) and the congruence falls into the singularity. For $\sigma_0^2 = 0.6$, $\sigma^2$ competes with the curvature term (16(b)). For $\sigma_0^2 = 0.7$ and $\sigma_0^2 = 0.8$, the curvature terms remains small and focusing takes place because of the divergence of shear (Fig. 16(c) and Fig. 16(d)) indicating that the focusing takes place before the congruence falls into the singularity.

VIII. SUMMARY AND CONCLUSION

In this work, we have demonstrated how the distinct roles of shear, rotation and spacetime curvature affect the detailed behaviour of trajectories in a gravitational scalar field collapse scenario. The dependence of the time to singularity $\lambda_f$ on the initial conditions have been spelt out in detail. The difference in the nature of the spacetime geometry for the two different values of $\alpha$ is shown to be reflected in the quantitative behaviour of the congruence of geodesics. We summarize the key findings as follows.

- For the cases in which a spacelike curvature singularity is formed, a (timelike) congruence is eventually trapped and focuses, either by hitting the curvature singularity, or due to intersection of geodesics in finite time. In the latter case, a peculiar influence of the initial expansion and rotation on the time to singularity is observed.

- For initial conditions on the kinematic variables in a certain range, the occurrence of congruence singularity due to intersection of geodesics in finite time is observed to be completely avoided. It is found that the expansion scalar exhibits a jump (from negative (contracting) to positive (expanding)) triggered by a burst in the rotation of the congruence. The congruence, however, focuses due to the curvature singularity.

- For a large negative value of the initial expansion (initially rapidly collapsing congruence), the congruence singularity is driven by build-up of shear. The singularity
occurs much earlier than the time taken by the central geodesic to hit the curvature singularity.

• The central inhomogeneity parameter $c$ is observed to influence the geodesic behaviour, and for moderate values, introduces a peculiar jump (burst) behaviour in the evolution of the expansion (rotation) of the congruence. However, for high values of $c$, the geodesics tend to be radial which damps out the observed burst in congruence rotation. This in turn smooths out the peculiar jump in the evolution of the expansion scalar.

• Null geodesic congruences have also been studied briefly. Generic features have been found to be similar to the timelike case for vanishing initial rotation. However, further detailed studies are required for a more complete understanding of the null case.

We have chosen a very specific exact solution for our studies. The solution, by no means, represents a realistic collapse scenario. However, the advantage of using this solution is related to its exact nature, which is, indeed rare, especially for space and time dependent (i.e. non-static) line elements. We hope to study more realistic collapse scenarios in our future investigations.

ACKNOWLEDGEMENT

Rajibul Shaikh acknowledges the Council of Scientific and Industrial Research (CSIR), India for providing support through a fellowship.

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FIG. 12. Plots showing the dependence of the focusing affine parameter $\lambda_f$ on the initial expansion and rotation for $\alpha = -\frac{\sqrt{3}}{2}$, $t(0) = 0.0$, $z(0) = 2.0$, $\phi(0) = 0$, $\dot{t}(0) = 2.0$ and $L = 2.2$. Here $\sigma_0^2 = 0.0$. 

(a) $a = 0.0$, $c = 0.0$

(b) $a = 0.0$, $c = 1.0$

(c) $a = -0.01$, $c = 0.0$

(d) $a = -0.01$, $c = 1.0$

(e) $a = -0.01$, $c = 8.0$

(f) $a = -0.01$, $c = 20.0$
FIG. 13. Plots showing the evolution of expansion for the initial conditions in Fig. 12. Here $\sigma_0^2 = 0.0, \omega_0^2 = 0.016$

FIG. 14. Plots showing the dependence of the focusing affine parameter $\lambda_f$ on the initial expansion and shear for $\alpha = -\frac{\sqrt{3}}{2}$. 

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FIG. 15. Figures showing the role of initial shear $\sigma_0^2$ in focusing of the congruence for $\theta_0 = 1.0, \omega_0^2 = 0.0$.

FIG. 16. Plots of $R_{\alpha\beta}u^\alpha u^\beta$, $\sigma^2$ and $I = R_{\alpha\beta}u^\alpha u^\beta + \sigma^2$ for the initial values of the ESR variables corresponding to Fig. Here $\omega^2 = 0$ identically.