VANISHING CAPILLARITY LIMIT OF THE
NON-CONSERVATIVE COMPRESSIBLE TWO-FLUID MODEL

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Abstract. In this paper, we consider the non-conservative compressible two-fluid model with constant viscosity coefficients and unequal pressure function in $\mathbb{R}^3$, we study the vanishing capillarity limit of the smooth solution to the initial value problem. We first establish the uniform estimates of global smooth solution with respect to the capillary coefficients $\sigma^+$ and $\sigma^-$, then by the Lion-Aubin lemma, we can obtain the unique smooth solution of the 3D non-conservative compressible two-fluid model with the capillary coefficients converges globally in time to the smooth solution of the 3D generic two-fluid model as $\sigma^+$ and $\sigma^-$ tend to zero. Also, we give the convergence rate estimates with respect to the capillary coefficients $\sigma^+$ and $\sigma^-$ for any given positive time.

1. Introduction. In this paper, we consider the non-conservative compressible two-fluid model in $\mathbb{R}^3$ as follows

\[
\begin{align*}
\alpha^+ + \alpha^- &= 1, \\
\partial_t (\alpha^+ \rho^+ u^+) + \text{div}(\alpha^+ \rho^+ u^+ u^+) + \alpha^+ \nabla P^+(\rho^+) &= 0, \\
\partial_t (\alpha^- \rho^- u^-) + \text{div}(\alpha^- \rho^- u^- u^-) + \alpha^- \nabla P^-(\rho^-) &= 0, \\
P^+(\rho^+) - P^-(\rho^-) &= f(\alpha^+ \rho^+) - f(\alpha^- \rho^-),
\end{align*}
\]

the variable $0 \leq \alpha^+(x,t) \leq 1$ is the volumetric rate of presence of fluid $+$, and $0 \leq \alpha^-(x,t) \leq 1$ is that of fluid $-$; $\rho^\pm(x,t) \geq 0$, $u^\pm(x,t)$ and $P^\pm(\rho^\pm) = A^\pm(\rho^\pm)^{\gamma^\pm}$ are the densities, the velocity and pressure of each phase. It is assumed that $\gamma^\pm \geq 1$, and $A^\pm > 0$ are constants; In the sequel, we set $A^+ = A^- = 1$ without loss of generality. $f(\cdot)$ is a smooth function with $f(\cdot) \in C^3([0, \infty])$, which is strictly decreasing

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near the equilibrium; $\tau^\pm$ are the viscous stress tensors

$$
\tau^\pm := \mu^\pm (\nabla_x u^\pm + \nabla_x^T u^\pm) + \lambda^\pm \text{div} u^\pm I_d,
$$

where $\sigma^\pm > 0$ are the capillary coefficients, the constant $\mu^\pm$ and $\lambda^\pm$ are shear and bulk viscosity coefficients, satisfying $\mu^\pm > 0$, and $2\mu^\pm + 3\lambda^\pm \geq 0$, which deduce $\mu^\pm + \lambda^\pm > 0$. This model is commonly used in industrial applications, such as nuclear, power, oil-and-gas, micro-technology and so on, here we can refer to [4]. Furthermore, internal viscous and capillary forces cannot be neglected for some applications as for instance wave breaking. Furthermore, this system is known as a two-fluid flow system with algebraic closure. Here, we include a capillary pressure term, i.e., we do not assume that the two phase pressures $P^+$ and $P^-$ are equal. The assumption about non-equal pressure functions $P^+ \neq P^-$ is quite natural. This amounts to including capillary pressure forces and is commonly included in modeling of two-phase flow in porous media. For more information about this model and related models, see for instance [2, 17, 24] and references cited therein.

Note that when $\alpha^+ \equiv 0$ (or $\alpha^- \equiv 0$) and $\sigma^\pm = 0$, the system (1.1) becomes the compressible Navier-Stokes system. There is a huge literature on the global existence and large time behaviors of smooth solutions to the compressible Navier-Stokes system, see, e.g. [8, 9, 16, 20, 21]. When $\alpha^+ \equiv 0$ (or $\alpha^- \equiv 0$), the system (1.1) is the compressible Navier-Stokes-Korteweg system. There are also many works on the well-posedness and large time behaviors of smooth solution to the compressible Navier-Stokes-Korteweg system. Hattori and Li [13, 14, 15] considered the local and global existence of smooth solutions of multi-dimensional model in Sobolev space. Danchin and Desjardins [7] proved the existence and uniqueness results of suitably smooth solution in the Besov space frame. Kotschote [19] proved the local existence of a strong solution. Next, Tan and Wang [29] obtained the optimal time decay of the system.

In recent years, some efforts were made on the existence and large time behavior of global solution to the non-conservative compressible two-fluid model (1.1). Bresch, Desjardins, Ghidaglia and Grenier [4] considered the existence of global weak solution in the periodic domain when $1 < \gamma^\pm < 6$, where two pressure functions are equal and the capillary effects are considered. Later, Bresch, Huang and Li [5] showed the existence of global weak solution in one-dimensional case without capillary effect (i.e, $\sigma^\pm = 0$), when $\gamma^\pm > 1$. We refer also to the recent work [12] which is in a gas-liquid context where a polytropic gas law is used for the gas phase whereas the liquid is assumed to be incompressible. Next, based on the detailed analysis of the Green’s function to the linearized system and elaborate energy estimates to the nonlinear system, Cui, Wang, Yao and Zhu [6] established the decay rates of classical solution to the model (1.1) with two equal pressure functions. Recently, Evje, Wang and Wen [11] obtained the global existence and decay rate of weak solution without capillary coefficient. Since the compressible fluid models of Korteweg type are the dispersion correction of the compressible Navier-Stokes (see [20]). Bian, Yao and Zhu [3] considered the vanishing capillarity limit of the compressible fluid models of korteweg type to the Navier–Stokes equations in whole space. It is worth mentioning the recent results of Pu and Guo [25]; they established the semiclassical limit of the the quantum hydrodynamic equations to the classical hydrodynamic equations. In this paper, we plan to study the vanishing capillarity limit of the non-conservative compressible two-fluid model with respect to the capillary coefficients $\sigma^+$ and $\sigma^-$. That is, as
the capillary coefficients $\sigma^+$ and $\sigma^-$ tend to zero, does the solution of the non-conservative compressible two-fluid model converge globally in time to the smooth solution of the generic two-fluid model without capillary coefficients in $\mathbb{R}^3$?

Firstly, the unequal pressures imply the differential identity

$$dP^+ - dP^- = df(\alpha^- \rho^-). \quad (1.2)$$

It is obvious that

$$dP^+ = s_+^2 d\rho^+, \quad dP^- = s_-^2 d\rho^-, \quad \text{where} \quad s_{\pm}^2 = \frac{dP_{\pm}}{d\rho_{\pm}}(\rho_{\pm}) = \gamma_{\pm} \frac{P_{\pm}(\rho_{\pm})}{\rho_{\pm}}, \quad (1.3)$$

here $s_{\pm}$ denote the sound speed of each phase separately. Let us introduce the variable

$$R_{\pm} = \alpha^\pm \rho^\pm, \quad (1.4)$$

which together with (1.1) allow one to write

$$d\rho^+ = \frac{1}{\alpha^+} (dR^+ - \rho^+ d\alpha^+), \quad d\rho^- = \frac{1}{\alpha^-} (dR^- + \rho^- d\alpha^+). \quad (1.5)$$

From (1.2) and (1.5), we can obtain

$$d\alpha^+ = \frac{\alpha^- s_+^2}{\alpha^- \rho^+ s_+^2 + \alpha^+ \rho^- s_-^2} dR^+ - \frac{\alpha^+ \alpha^-}{\alpha^- \rho^+ s_+^2 + \alpha^+ \rho^- s_-^2} \left( \frac{s_2^2}{\alpha^-} + f' \right) dR^- - \frac{s_2^2}{\alpha^+} dR^{- \prime}, \quad (1.6)$$

Putting the above equality into (1.5), we can get the following expressions

$$d\rho^+ = \frac{\rho^+ \rho^- s_-^2}{R^-(\rho^+)^2 s_+^2 + R^+(\rho^-)^2 s_-^2} \left( \rho^- dR^+ + \left( \rho^+ + \rho^+ \frac{\alpha^- f'}{s_+^2} \right) dR^- \right), \quad (1.7)$$

and

$$d\rho^- = \frac{\rho^+ \rho^- s_+^2}{R^-(\rho^+)^2 s_+^2 + R^+(\rho^-)^2 s_-^2} \left( \rho^- dR^+ + \left( \rho^+ - \rho^+ \frac{\alpha^- f'}{s_+^2} \right) dR^- \right). \quad (1.8)$$

In the end, from (1.3), (1.7), (1.8), we have

$$dP^+ = C^2 \left( \rho^- dR^+ + \left( \rho^+ + \rho^+ \frac{\alpha^- f'}{s_+^2} \right) dR^- \right),$$

and

$$dP^- = C^2 \left( \rho^- dR^+ + \left( \rho^+ - \rho^+ \frac{\alpha^- f'}{s_+^2} \right) dR^- \right),$$

where

$$C^2 = \frac{s_2^2}{\alpha^- \rho^+ s_+^2 + \alpha^+ \rho^- s_-^2}.$$

Next, from (1.1), we get the identity

$$\frac{R^+}{\rho^+} + \frac{R^-}{\rho^-} = 1, \quad (1.9)$$

thus

$$\rho^- = \frac{R^- \rho^+}{\rho^+ - R^+}.$$
From the pressure relation (1.14), we get
\[
\varphi(\rho^+) = P^+(\rho^+) - P^- \left( \frac{R^- \rho^+}{\rho^+ - R^+} \right) - f(R^-) = 0. \tag{1.10}
\]
We will use the implicit function theorem to define \(\rho^+\). Differentiating the above equation with respect to \(\rho^+\) for given \(R^+\) and \(R^-\), we have
\[
\varphi'(\rho^+) = s_+^2 + s_-^2 \frac{R^+ R^+}{(\rho^+ - R^+)^2}.
\]
By the definition of \(R^+\), it is natural to look for \(\rho^+\) which belongs to \((R^+, +\infty)\). Since \(\varphi' > 0\) in \((R^+, +\infty)\), and \(\varphi: (R^+, +\infty) \rightarrow (-\infty, +\infty)\), this determines that \(\rho^+ = \rho^+(R^+, R^-)\) is the unique solution of (1.10). From [1.19], [1.3], [1.11], \(\rho^\pm, \alpha^\pm\) and \(C\) are defined by
\[
\rho^\pm = \phi^\pm(R^+, R^-), \quad \alpha^\pm = \psi^\pm(R^+, R^-), \quad C = \zeta(R^+, R^-).
\]
Please refer to [4] (page 614) for more details.

Then the system (1.1) can be rewritten as follows:
\[
\begin{align*}
\partial_t R^\pm + \text{div}(R^\pm u^\pm) &= 0, \\
\partial_t (R^+ u^+) + \text{div}(R^+ u^+ \otimes u^+) + \psi^+ \varsigma^2 [\phi^- \nabla R^+ + \left( \phi^+ + \phi^+ f'' \right) R^+] \nabla R^- \\
&= \text{div} \left\{ \psi^+ [\mu^+(\nabla u^+ + \nabla^T u^+) + \lambda^+ \text{div} u^+ Id] \right\} + \sigma^+ R^+ \nabla \Delta R^+ + \\
\partial_t (R^- u^-) + \text{div}(R^- u^- \otimes u^-) + \psi^- \varsigma^2 \left[ \phi^- \nabla R^+ + \left( \phi^+ - \phi^+ f'' \right) R^+ \right] \\
&= \text{div} \left\{ \psi^- [\mu^-(\nabla u^- + \nabla^T u^-) + \lambda^- \text{div} u^- Id] \right\} + \sigma^- R^- \nabla \Delta R^-.
\end{align*}
\]
\tag{1.11}
We consider the Cauchy problem of (1.11):
\[
(R^+, u^+, R^-, u^-)(x,t)|_{t=0} = (R^+_0, u^+_0, R^-_0, u^-_0)(x,t), \quad x \in \mathbb{R}^3, \tag{1.12}
\]
and
\[
R^\pm(x,t) \to \hat{R}^\pm > 0, \quad u^\pm \to 0, \quad \text{as} \quad |x| \to \infty,
\]
where \(\hat{R}^\pm > 0\) denote the background doping profiles.

The following are the main results. Firstly, the global existence of smooth solution to the problem (1.11)-(1.12) is stated in the following theorem.

**Theorem 1.1.** Let \(\sigma^+, \sigma^- > 0\) be fixed. Assume \(\frac{s^2_{-}(1,1)}{\psi^{-}(1,1)} < f'(1) < \frac{\eta - s^2_{-}(1,1)}{\psi^{-}(1,1)}\) < 0, where \(\eta\) is a positive, small fixed constant, and there exists a constant \(\delta > 0\), such that if
\[
\| (R^+_0 - \hat{R}^+, R^-_0 - \hat{R}^-) \|_{H^+} + \| (u^+_0, u^-_0) \|_{H^0} \leq \delta,
\]
where \(\delta\) is small enough, then the problem (1.11)-(1.12) admits a solution globally in time and satisfies
\[
R^+(x,t) - \hat{R}^+, \quad R^-(x,t) - \hat{R}^- \in C^0(\mathbb{R}^+, H^4(\mathbb{R}^3)) \cap C^1(\mathbb{R}^+, H^2(\mathbb{R}^3)),
\]
\[
u^+(x,t), \quad u^-(x,t) \in C^0(\mathbb{R}^+, H^3(\mathbb{R}^3)) \cap C^1(\mathbb{R}^+, H^1(\mathbb{R}^3)),
\]
\[ \| (\bar{R}^+ - \bar{R}^-, \bar{R}^- - \bar{R}^-)(\cdot, t) \|_3 + \| (u^+, u^-)(\cdot, t) \|_3 \leq C(\| (R_0^+ - \bar{R}^+, R_0^- - \bar{R}^-) \|_4 + \| (u_0^+, u_0^-) \|_3), \]  
(1.13)

where \( C \) is a positive constant independent of \( \sigma^+ \) and \( \sigma^- \). 

**Remark 1.1.** Compare with [11], here we need to show the uniform estimates of the global smooth solution for the problem \([1.11]-[1.12]\) with respect to the capillary coefficients \( \sigma^+ \) and \( \sigma^- \).

In order to study the vanishing capillarity limit \( \sigma^\pm \to 0 \) of the global in time solution to \([1.11]-[1.12]\), we write solution to \([1.11]-[1.12]\) with the suffix \( \sigma^\pm \) as \((R_{\sigma^+}^+, u_{\sigma^+}^+, \bar{R}_{\sigma^+}, \bar{u}_{\sigma^+})\). Next, let \((\bar{R}^+, \bar{u}^+, \bar{R}^-, \bar{u}^-)\) be a solution to the following equations

\[
\begin{align*}
\partial_t \bar{R}^\pm + \text{div}(\bar{R}^\pm \bar{u}^\pm) &= 0, \\
\partial_t (\bar{R}^+ \bar{u}^+) + \text{div}(\bar{R}^+ (\bar{u}^+ \otimes \bar{u}^+) + \bar{\phi}^+ \xi^2 (\bar{\phi}^- \nabla \bar{R}^+ + \left( \bar{\phi}^+ + \bar{\phi}^+ f'(\bar{R}^-) \right) \nabla \bar{R}^-)) &= \text{div}\left\{ \bar{\phi}^+ [\mu^+ (\nabla \bar{u}^+ + \nabla^T \bar{u}^+) + \lambda^+ \text{div} \bar{u}^+ I] \right\}, \\
\partial_t (\bar{R}^- \bar{u}^-) + \text{div}(\bar{R}^- (\bar{u}^- \otimes \bar{u}^-) + \bar{\phi}^- \xi^2 (\bar{\phi}^- \nabla \bar{R}^+ + \left( \bar{\phi}^+ - \bar{\phi}^+ f'(\bar{R}^-) \right) \nabla \bar{R}^-)) &= \text{div}\left\{ \bar{\phi}^- [\mu^- (\nabla \bar{u}^- + \nabla^T \bar{u}^-) + \lambda^- \text{div} \bar{u}^- I] \right\},
\end{align*}
\]

with initial date

\[(\bar{R}^+, \bar{u}^+, \bar{R}^-, \bar{u}^-) \big|_{t=0} = (R_0^+, u_0^+, R_0^-, u_0^-)(x), \]  
(1.15)

which is obtained by letting \( \sigma^\pm \to 0 \) in \([1.11]-[1.12]\).

Then we have follow vanishing capillary limit \( \sigma^\pm \to 0 \).

**Theorem 1.2.** Let \((R_{\sigma^+}^+, u_{\sigma^+}^+, R_{\sigma^+}, u_{\sigma^+})(x,t)\) be the solution of the problem \([1.11]-[1.12]\) given by Theorem 1.1. Then there exists \((\bar{R}^+, \bar{u}^+, \bar{R}^-, \bar{u}^-)(x,t)\) such that, as the capillary coefficient \( \sigma^\pm \to 0 \), it holds that for any \( T > 0 \),

\[
\begin{align*}
R_{\sigma^+}^+ &\to \bar{R}^+ \text{ strongly in } C(0,T; C_b^1 \cap H^{3-s}_{\text{loc}}(\mathbb{R}^3)), \\
u_{\sigma^+}^+ &\to \bar{u}^+ \text{ strongly in } C(0,T; C_b^1 \cap H^{3-s}_{\text{loc}}(\mathbb{R}^3)), \\
R_{\sigma^+}^- &\to \bar{R}^- \text{ strongly in } C(0,T; C_b^1 \cap H^{3-s}_{\text{loc}}(\mathbb{R}^3)), \\
u_{\sigma^+}^- &\to \bar{u}^- \text{ strongly in } C(0,T; C_b^1 \cap H^{3-s}_{\text{loc}}(\mathbb{R}^3)), \quad s \in (0, \frac{1}{2}).
\end{align*}
\]

Here \((\bar{R}^+, \bar{u}^+, \bar{R}^-, \bar{u}^-)(x,t)\) is the global in time solution of the problem \([1.14]-[1.15]\) in \([0,T]\), satisfies

\[
\begin{align*}
\bar{R}^+ - \bar{R}^-, \bar{R}^+ - \bar{R}^- &\in C^0(R^+, H^3(\mathbb{R}^3)) \cap C^1(R^+, H^2(\mathbb{R}^3)), \\
\bar{u}^+, \bar{u}^- &\in C^0(R^+, H^3(\mathbb{R}^3)) \cap C^1(R^+, H^1(\mathbb{R}^3)),
\end{align*}
\]
\( \nabla (\bar{R}^+ - \bar{R}^+ + \bar{R}^+ - \bar{R}^-) \in L^2(\mathbb{R}^+, H^2(\mathbb{R}^3)) \), \( \nabla (\bar{u}^+ + \bar{u}^-) \in L^2(\mathbb{R}^+, H^3(\mathbb{R}^3)) \), which is a global solution for \( t > 0 \), and satisfies the following estimate for any \( t \in [0, T] \),

\[
\| (\bar{R}^+ - \bar{R}^+ + \bar{R}^+ - \bar{R}^- + \bar{u}^+, \bar{u}^-)(0, t) \|_3^2 \\
+ \int_0^t (\| \nabla (\bar{R}^+ - \bar{R}^+ + \bar{R}^+ - \bar{R}^- + \bar{u}^+)(s) \|_2^2 + \| \nabla (\bar{u}^+ + \bar{u}^-)(s) \|_3^2) ds \\
\leq C \| (R_0^+ - \bar{R}^+ + u_0^+, R_0^+ - \bar{R}^- + u_0^-)(0) \|_3^2,
\]

(1.16)

where \( C \) is a positive constant independent of \( \sigma^+ \) and \( \sigma^- \).

Finally, we give the convergence rate estimates from the solution of the problem (1.11)-(1.12) to the solution of the problem (1.14)-(1.15) for any given positive time.

**Theorem 1.3.** Let \( (R_{\sigma^+}^+, u_{\sigma^+}^+, R_{\sigma^-}^-, u_{\sigma^-}^-)(x, t) \) be the solution of the problem (1.11)-(1.12), and let \( (\bar{R}^+, \bar{u}^+, \bar{R}^-, \bar{u}^-)(x, t) \) be the global in time solution of the problem (1.14)-(1.15). Then for any fixed time \( t \in (0, \infty) \), we have

\[
\| (R_{\sigma^+}^+ - \bar{R}^+, u_{\sigma^+}^+ - \bar{u}^+, R_{\sigma^-}^- - \bar{R}^-, u_{\sigma^-}^- - \bar{u}^-) \|_2^2 \leq C \max(\sigma^+, \sigma^-) t e^{\beta t},
\]

(1.17)

where \( \beta \) and \( C \) are positive constants independent of \( t, \sigma^+ \) and \( \sigma^- \).

The rest of this paper is described as follows. In the section 2, we will give some useful lemmas which will be used later. In sections 3, we will reformulate the original problem. In the section 4, we will establish the uniform estimates of the some useful lemmas which will be used later. In sections 3, we will reformulate the original problem. In the section 5, we will establish the uniform estimates of the global-in-time solution. Finally, section 5 is devoted to the proof of the main results (Theorems 1.1-1.3).

**Notation.** Throughout this paper, \( C > 0 \) are generic positive constants independent of capillary coefficient \( \sigma^\pm \), which may vary in different estimates. \( \nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) \) and for a multi-index \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \), \( \partial^\alpha = (\partial^{\alpha_1}, \partial^{\alpha_2}, \partial^{\alpha_3}) \) and \( |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \). \( L^p(\mathbb{R}^3)(1 \leq p \leq \infty) \) denotes the space of measurable functions whose \( p \)-powers are integrable on \( \mathbb{R}^3 \), with the norm \( \| \cdot \|_{L^p} = \left( \int_{\mathbb{R}^3} |\cdot|^p \right)^{\frac{1}{p}} \), as \( p = 2 \), we denote the norm of \( L^2(\mathbb{R}^3) \) by \( \| \cdot \| \) for simplicity, and \( L^\infty(\mathbb{R}^3) \) is the space of bounded measurable functions on \( \mathbb{R}^3 \), with the norm \( \| \cdot \|_{L^\infty} = \text{ess sup}_x |\cdot| \). \( H^k(\mathbb{R}^3) \) is the sobolev space, \( k \in \mathbb{Z}_+ \), which stands for the space of \( L^2(\mathbb{R}^3) \)-functions whose derivatives (in the sense of distribution) up to \( k \)th order are also \( L^2(\mathbb{R}^3) \)-functions.

with norm \( \| \cdot \| = \left( \sum_{k=0}^k \| D^k \cdot \|_{L^2} \right)^{\frac{1}{2}} \). \( \nabla f \) denotes a set composed of all \( k \) order partial derivatives with respect to the variable \( x \) of the function \( f \). The notation “\( \langle \cdot, \cdot \rangle \)” stands for the inner-product in \( L^2(\mathbb{R}^3) \).

2. **Preliminaries.** In this section, we will give some useful lemmas which can be used in the next section.

**Lemma 2.1 (27).** Assume \( X \subset E \subset Y \) are Banach spaces and \( X \hookrightarrow E \hookrightarrow Y \). Then the following embeddings are compact:

(1) \( \{ \varphi : \varphi \in L^q(0,T;X), \frac{\partial \varphi}{\partial t} \in L^1(0,T;Y) \} \hookrightarrow L^q(0,T;E) \) if \( 1 \leq q \leq \infty \);
\[
(2) \left\{ \varphi : \varphi \in L^\infty(0,T;X), \frac{\partial \varphi}{\partial t} \in L^r(0,T;Y) \right\} \hookrightarrow C([0,T];E) \text{ if } 1 < r \leq \infty.
\]

**Lemma 2.2** (128). Let \( f \in H^2(\mathbb{R}^3) \). Then
\[
\begin{align*}
(1) & \| f \|_{L^\infty} \leq C \| \nabla f \|_{L^2} \| \nabla f \|_{H^1} ; \\
(2) & \| f \|_{L^6} \leq C \| \nabla f \| ; \\
(3) & \| f \|_{L^4} \leq C(\| f \| + \| \nabla f \|).
\end{align*}
\]

**Lemma 2.3** (11 [22]). Let \( m \geq 1 \) be an integer, then we have
\[
\| \nabla^m (fg) \|_{L^p} \leq C \| f \|_{L^{p_1}} \| \nabla^m g \|_{L^{p_2}} + C \| \nabla^m f \|_{L^{p_3}} \| g \|_{L^{p_4}},
\]
and
\[
\| \nabla^m (fg) - f \nabla^m g \|_{L^p} \leq C \| \nabla f \|_{L^{p_1}} \| \nabla^m g \|_{L^{p_2}} + C \| \nabla^m f \|_{L^{p_3}} \| g \|_{L^{p_4}}
\]
where \( p_1, p_2, p_3, p_4 \in [1, \infty] \) and
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
\]

3. Reformation of the original problem. In the section, we will reformulate the problem (1.11)-(1.12) in terms of the perturbed variables. Without loss of generality, we can take the constant equilibrium states \( \bar{R}^\pm = 1 \) and set
\[
n^\pm = R^\pm - 1,
\]
then the system (1.11)-(1.12) can rewritten as follows:
\[
\begin{align*}
\partial_t n^+ + \text{div} u^+ &= F_1, \\
\partial_t u^+ + \beta_1 \nabla u^+ + \beta_2 \nabla n^- - \nu_1^+ \Delta u^+ - \nu_2^+ \nabla \text{div} u^+ - \sigma^+ \nabla \Delta n^+ &= F_2, \\
\partial_t n^- + \text{div} u^- &= F_3, \\
\partial_t u^- + \beta_3 \nabla u^- + \beta_4 \nabla n^- - \nu_1^- \Delta u^- - \nu_2^- \nabla \text{div} u^- - \sigma^- \nabla \Delta n^- &= F_4,
\end{align*}
\]
with initial date
\[
(n^+, u^+, n^-, u^-) |_{t=0} = (n_0^+, u_0^+, n_0^-, u_0^-)(x) \\
= (R_0^+, u_0^+, R_0^-, u_0^-)(x) \to (0, 0, 0, 0),
\]
as \(|x| \to \infty\), where
\[
\begin{align*}
\beta_1 &= \frac{c^2(1,1)\phi^-(1,1)}{\phi^+(1,1)}, & \beta_2 &= \frac{c^2(1,1)\psi^-(1,1)}{s^2(1,1)} f'(1), & \beta_3 &= c^2(1,1), \\
\beta_4 &= \frac{c^2(1,1)\phi^+(1,1)}{\phi^-(1,1)} - \frac{c^2(1,1)\psi^+(1,1)}{s^2(1,1)} f'(1), & \nu_1^+ &= \frac{\mu^+}{\phi^+(1,1)}, & \nu_2^+ &= \frac{\mu^+ + \lambda^+}{\phi^+(1,1)} > 0,
\end{align*}
\]
and the source terms are

\[
F_1 = - \text{div}(n^+ u^+),
\]

\[
F_2 = -g_1^+(n^+, n^-) \partial_t n^+ - g_2^+(n^+, n^-) \partial_t n^- - (u^+ \cdot \nabla)u_i^+ + \mu^+ h_1^+(n^+, n^-) \partial_j n^+ \partial_j u_i^+ + \mu^+ h_2^+(n^+, n^-) \partial_j n^- \partial_j u_i^+ + \lambda^+ h_1^+(n^+, n^-) \partial_j n^+ \partial_j u_i^+ + \lambda^+ h_2^+(n^+, n^-) \partial_j n^- \partial_j u_i^+ + \mu^+ k^+(n^+, n^-) \partial_j^2 u_i^+ + (\mu^+ + \lambda^+) k^+(n^+, n^-) \partial_t \partial_j u_i^+,
\]

\[
F_3 = -\text{div}(n^- u^-),
\]

\[
F_4 = -g_1^-(n^+, n^-) \partial_t n^- - g_2^-(n^+, n^-) \partial_t n^+ - (u^- \cdot \nabla)u_i^- + \mu^- h_1^-(n^+, n^-) \partial_j n^+ \partial_j u_i^- + \mu^- h_2^-(n^+, n^-) \partial_j n^- \partial_j u_i^- + \lambda^- h_1^+(n^+, n^-) \partial_j n^+ \partial_j u_i^- + \lambda^- h_2^-(n^+, n^-) \partial_j n^- \partial_j u_i^- + \mu^- k^-(n^+, n^-) \partial_j^2 u_i^- + (\mu^- + \lambda^-) k^-(n^+, n^-) \partial_t \partial_j u_i^-,
\]

where we define the nonlinear functions of \((n^+, n^-)\) by

\[
\begin{align*}
\begin{cases}
g_1^+(n^+, n^-) = \frac{(s^2 \phi^-)(n^+ + 1, n^- + 1)}{\phi^+(n^+ + 1, n^- + 1)} - \frac{(s^2 \phi^-)(1, 1)}{\phi^+(1, 1)}, \\
g_1^-(n^+, n^-) = \frac{(s^2 \phi^+)(n^+ + 1, n^- + 1)}{\phi^-(n^+ + 1, n^- + 1)} - \frac{(s^2 \phi^+)(1, 1)}{\phi^-}(1, 1) - \frac{(s^2 \psi^+)(n^+ + 1, n^- + 1)f'(n^- + 1)}{s^2_+(n^+ + 1, n^- + 1)} + \frac{(s^2 \psi^+)(1, 1)f'(1)}{s^2_+(1, 1)}, \\
g_2^+(n^+, n^-) = s^2(n^+ + 1, n^- + 1) - s^2(1, 1) + \frac{(s^2 \psi^-)(n^+ + 1, n^- + 1)f'(n^- + 1)}{s^2_+(n^+ + 1, n^- + 1)} - \frac{(s^2 \psi^-)(1, 1)f'(1)}{s^2_+(1, 1)}, \\
g_2^-(n^+, n^-) = s^2(n^+ + 1, n^- + 1) - s^2(1, 1),
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
h_1^+(n^+, n^-) = \frac{(s^2 \psi^-)(n^+ + 1, n^- + 1)}{|n^+ + 1| s^2_+(n^+ + 1, n^- + 1)}, \\
h_1^-(n^+, n^-) = -\frac{s^2(n^+ + 1, n^- + 1)}{\phi^-(n^+ + 1, n^- + 1)}, \\
h_2^+(n^+, n^-) = -\frac{s^2(n^+ + 1, n^- + 1)}{(s^2 \phi^+)(n^+ + 1, n^- + 1)} + \frac{(s^2 \psi^-)(n^+ + 1, n^- + 1)f'(n^- + 1)}{(s^2 \phi^+)(n^+ + 1, n^- + 1)}, \\
h_2^-(n^+, n^-) = \frac{(s^2 \psi^+)(n^+ + 1, n^- + 1)f'(n^- + 1)}{|n^- + 1| s^2_-(n^+ + 1, n^- + 1)} + \frac{(s^2 \psi^-)(n^+ + 1, n^- + 1)f'(n^- + 1)}{(s^2 \phi^-)(n^+ + 1, n^- + 1)}.
\end{cases}
\end{align*}
\]

\[
k^\pm(n^+, n^-) = \frac{1}{\phi^\pm(n^+ + 1, n^- + 1)} - \frac{1}{\phi^\pm(1, 1)}.
\]
Note that
\[ \beta_1 \beta_4 - \beta_2 \beta_3 = -\frac{c^2(1,1)f'(1)}{\phi^+(1,1)} > 0, \]
this will be used in Lemma 4.1 which is crucial for the proof of Proposition 4.1 below.

4. The a priori estimates. The global existence of the solution \((R^+, u^+, R^-, u^-)\) to the steady state \((1, \hat{0}, 1, \hat{0})\) can be easily translated into the global existence of the perturbed solution \((n^+, u^+, n^-, u^-)\). The global smooth solution \((n^+, u^+, n^-, u^-)\) to the problem (3.1)-(3.2) is constructed by the combination of the local existence and the a priori estimate. The proof of the existence of local solution is standard, refer for instance to [18] and we mainly concern about the uniform a priori estimates of \((n^+, u^+, n^-, u^-)\).

In the following, we assume that there exists a sufficiently small \(\delta > 0\), such that the following a priori assumptions holds
\[
\sup_{0 \leq t \leq T} \left\{ \| (n^+, n^-) \|_3 + \| (u^+, u^-) \|_3 + (\sigma^\pm)^2 \sum_{k=4} \| \nabla^k (n^+, n^-) \| \right\} \leq \delta. \tag{4.1}
\]
By (4.1) and the Sobolev inequality, we have
\[
\frac{1}{2} \leq n^+ + 1, n^- + 1 \leq 2;
\]
For some positive constant \(C\), we have
\[
|\langle k^+, k^+ \rangle (n^+, n^-)| \leq C|\langle n^+, n^- \rangle|, \quad |\langle h^+_1, h^+_2 \rangle (n^+, n^-)| \leq C,
\]
and
\[
|\partial^k_{n^+} (g^+, g^+, h^+_1, h^+_2, k^+)(n^+, n^-)| \leq C, \quad |\partial^k_{n^-} (g^+, g^+, h^+_1, h^+_2, k^+)(n^+, n^-)| \leq C, \quad \text{for } k \geq 1.
\]
These will be often used in the rest of this paper.

**Proposition 4.1 (A priori estimates).** Suppose \((n^+, u^+, n^-, u^-)(x, t)\) is the solution of the problem (3.1)-(3.2) in \([0, T]\). If \(\delta\) is small enough, then there exists a positive constant \(C\) independent of \(\sigma^+\) and \(\sigma^-\), such that for \(t \in [0, T]\), it holds that
\[
\| (n^+, n^-) \|^2_3 + \| (u^+, u^-) \|^2_3 + \sigma^\pm \sum_{k=3} \| \nabla^k \nabla (n^+, n^-) \|^2
\]
\[
+ \int_0^t (\| \nabla (n^+, n^-)(\cdot, s) \|^2_3 + \| \nabla (u^+, u^-)(\cdot, s) \|^2_3
\]
\[
+ \sigma^\pm \sum_{k=2} \| \nabla^k \Delta (n^+, n^-)(\cdot, s) \|^2 ds
\]
\[
\leq C(\| (n^0+, n^-) \|^2_2 + \| (u^0+, u^-) \|^2_2). \tag{4.2}
\]
Next, we give the proof of Proposition 4.1. For the sake of clarity, we divide the proof into the subsequent three lemmas.
Lemma 4.1. Let $\tilde{\tau} = \frac{\beta_1 \beta_4 - \beta_2 \beta_3}{\beta_1 \beta_3} > 0$. It holds that

$$
\frac{1}{2} \frac{d}{dt} \left\{ \frac{\beta_1}{\beta_2} n^+ + (\frac{\beta_1}{\beta_2})^{-1} n^- \right\}^2 + \tilde{\tau} n^- \right\}^2 
\frac{1}{2} \frac{d}{dt} \left\{ \frac{\beta_1}{\beta_2} n^+ \right\}^2 + \frac{1}{2} \frac{d}{dt} \left\{ \frac{\beta_4}{\beta_3} n^- \right\}^2 
\leq C\delta \| \nabla (n^+, n^-) \|^2,
$$

for $\delta > 0$ small enough, and some positive constant $C$, which is independent of $\sigma^+$ and $\sigma^-$. 

Proof. Rewrite (3.1) and (3.1) as follows

$$
\{ \partial_t n^+ + \text{div}(n^+ + 1)u^+ \} = 0, \partial_t n^- + \text{div}(n^- + 1)u^- = 0.
$$

Without loss of generality, we can assume $\sigma^+ < 1$. Multiplying (3.1) by $\frac{\beta_1}{\beta_2} n^+, \frac{\beta_1}{\beta_2} n^+ + 1)u^+, \frac{\beta_4}{\beta_3} n^-, \frac{1}{\beta_3} (n^- + 1)u^-$ respectively, summing them up and then integrating the resulting equality over $\mathbb{R}^3$ by parts, we have

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left\{ \frac{\beta_1}{\beta_2} (n^+ + 1)u^+ \right\}^2 
\frac{1}{\beta_2} \int_{\mathbb{R}^3} (n^+ + 1)u^+ \nabla u^+ \cdot \nabla u^+ \right\} dx 
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left\{ \frac{\beta_4}{\beta_3} (n^- + 1)u^- \right\}^2 
\frac{1}{\beta_3} \int_{\mathbb{R}^3} (n^- + 1)u^- \nabla u^- \cdot \nabla u^- \right\} dx 
= \frac{1}{2} \frac{1}{\beta_2} \int_{\mathbb{R}^3} \partial_t n^+ |u^+|^2 dx + \langle F_2, \frac{1}{\beta_2} (n^+ + 1)u^+ \rangle - \langle \nabla n^+, (n^+ + 1)u^+ \rangle 
\frac{1}{\beta_2} \int_{\mathbb{R}^3} (\nu_1^+ \nabla u^+ \cdot \nabla (n^+ + 1) \cdot u^+ + \nu_2^+ \text{div} u^+ \nabla (n^+ + 1) \cdot u^+) dx 
+ \langle \sigma^+ \nabla \Delta n^+, \frac{1}{\beta_2} (n^+ + 1)u^+ \rangle 
\frac{1}{\beta_3} \int_{\mathbb{R}^3} \partial_t n^- |u^-|^2 dx + \langle F_4, \frac{1}{\beta_3} (n^- + 1)u^- \rangle - \langle \nabla n^+, (n^- + 1)u^- \rangle 
\frac{1}{\beta_3} \int_{\mathbb{R}^3} (\nu_1^- \nabla u^- \cdot \nabla (n^- + 1) \cdot u^- + \nu_2^- \text{div} u^- \nabla (n^- + 1) \cdot u^-) dx 
+ \langle \sigma^- \nabla \Delta n^-, \frac{1}{\beta_3} (n^- + 1)u^- \rangle 
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10}.
$$

(4.5)
First, by using (4.4) and (4.4), we have
\[ I_1 = \frac{1}{2} \int_{\mathbb{R}^3} |u^+|^2 \text{div}((n^++1)u^+) \, dx \]
\[ = \frac{1}{2} \int_{\mathbb{R}^3} |u^+|^2 (\nabla(n^++1) \cdot u^+ + (n^++1) \text{div} u^+) \, dx \]
\[ \leq C \| (n^++1) \|_{L^\infty} \| u^+ \|_{L^3} \| u^+ \|_{L^6} \| \nabla u^+ \| + C \| u^+ \|_{L^6}^2 \| \nabla u^+ \| \]
\[ \leq C \delta \| \nabla u^- \|^2. \]

Similarly, we have
\[ I_6 \leq C \delta \| \nabla u^- \|^2. \]

Then, by using integration by part, Hölder inequality, Lemma 2.2, the a priori assumption (4.1) and Young inequality, we have
\[ |I_2| \leq C |(g_1^+(n^+, n^-) \nabla n^+, (n^++1)u^+)| + C |(g_2^+(n^+, n^-) \nabla n^-, (n^++1)u^+)| \]
\[ + C |(u^+ \cdot \nabla) u^+, (n^++1)u^+)\]
\[ + C |(h_1^+(n^+, n^-)(\nabla n^+ \cdot \nabla) u^+, (n^++1)u^+)| \]
\[ + C |(h_2^+(n^+, n^-)(\nabla n^- \cdot \nabla) u^+, (n^++1)u^+)| \]
\[ + C |(h_1^+(n^+, n^-) \nabla n^+ \nabla u^+, (n^++1)u^+)| \]
\[ + C |(h_2^+(n^+, n^-) \nabla n^- \nabla u^+, (n^++1)u^+)| \]
\[ + C |(\nabla u^+, \nabla [k^+(n^+, n^-)u^+](n^++1))| \]
\[ + C |(\nabla u^+, [k^+(n^+, n^-)u^+] \nabla (n^++1))| \]
\[ + C |(\text{div} u^+, \text{div}[k^+(n^+, n^-)u^+](n^++1))| \]
\[ + C |(\text{div} u^+, [k^+(n^+, n^-)u^+] \nabla u^+(n^++1))| \]
\[ \leq C \| g_1^+(n^+, n^-) \|_{L^3} \| \nabla n^+ \| \| u^+ \|_{L^6} + \| g_2^+(n^+, n^-) \|_{L^3} \| \nabla n^- \| \| u^+ \|_{L^6} \]
\[ + C \| u^+ \|_{L^6} \| \nabla u^+ \| \| u^+ \|_{L^6} + C \| \nabla u^+ \| \| u^+ \|_{L^6} \]
\[ + C \| \nabla u^- \| \| u^+ \|_{L^6} \]
\[ + C \| \nabla u^+ \| (\| k^+(n^+, n^-) \| \| u^+ \|_{L^6} + \| k^+(n^+, n^-) \|_{L^6} \| \nabla u^+ \|_{L^3} \]
\[ + C \| \nabla u^+ \| \| [k^+(n^+, n^-)u^+] \|_{L^6} \| \nabla (n^++1) \| \]
\[ \leq C \delta (\| \nabla (n^+, n^-) \|^2 + \| \nabla u^- \|^2). \]

In the same way, we get
\[ |I_7| \leq C \delta (\| \nabla (n^+, n^-) \|^2 + \| \nabla u^- \|^2). \]

Now, we estimate \( I_3 \) and \( I_8 \) by using (4.4) and (4.4), we have
\[ I_3 = - \int_{\mathbb{R}^3} \nabla n^- (n^++1) u^+ \, dx = \int_{\mathbb{R}^3} n^- \text{div}((n^++1)u^+) \, dx = -\langle n^-, \partial_t n^+ \rangle, \]
and
\[ I_8 = - \int_{\mathbb{R}^3} \nabla n^+ (n^++1) u^- \, dx = \int_{\mathbb{R}^3} n^+ \text{div}((n^++1)u^-) \, dx = -\langle n^+, \partial_t n^- \rangle. \]
Then, one deduces that
\[ I_3 + I_8 = -\langle n^-, \partial_t n^+ \rangle - \langle n^+, \partial_t n^- \rangle = -\partial_t \langle n^-, n^+ \rangle. \]

By using the Young inequality and Hölder inequality, it holds that
\[ |I_4| \leq C \| \nabla (n^+ + 1) \| \| \nabla u^+ \| \| u^+ \|_{L^\infty} \leq C \delta \| \nabla n^+ \|^2 + C \delta \| \nabla u^+ \|^2, \]
and
\[ |I_9| \leq C \| \nabla (n^- + 1) \| \| \nabla u^- \| \| u^- \|_{L^\infty} \leq C \delta \| \nabla n^- \|^2 + C \delta \| \nabla u^- \|^2. \]

In the end, for \( I_5 \) and \( I_{10} \), by using integration by parts, (4.4) and (4.3), we get
\[ I_5 = \langle \sigma^+ \nabla \Delta n^+, \frac{1}{\beta_2} (n^+ + 1) u^+ \rangle = -\langle \sigma^+ \Delta n^+, \frac{1}{\beta_2} \text{div}((n^+ + 1) u^+) \rangle \]
\[ = \langle \sigma^+ \nabla n^+, \frac{1}{\beta_2} \partial_t n^+ \rangle = -\langle \sigma^+ \nabla n^+, \frac{1}{\beta_2} \partial_t \nabla n^+ \rangle = -\frac{1}{2} \frac{d}{dt} \sigma^+ \| \nabla n^+ \|^2, \]
and
\[ I_{10} = -\frac{1}{2} \frac{d}{dt} \sigma^- \| \nabla n^- \|^2. \]

Putting the estimates of \( I_1 \) to \( I_{10} \) into (4.5), taking \( \delta \) sufficiently small, and noticing that
\[ \frac{\beta_4}{\beta_3} - \frac{\beta_2}{\beta_1} = \tilde{\tau} = \frac{\beta_1 \beta_4 - \beta_2 \beta_3}{\beta_1 \beta_3} = -\frac{\sigma^2 (1, 1) f'(1)}{\beta_1 \beta_3 \phi^+(1, 1)} > 0, \]
we get (4.3). Thus, the proof of Lemma 4.1 is completed. \( \square \)

Next, we will give the higher-order derivative estimates.

**Lemma 4.2.** For \( k = 1, 2, 3 \), it holds that
\begin{equation}
\frac{1}{2} \frac{d}{dt} \left\{ \frac{\beta_1}{\beta_2} \| \nabla^k n^+ \|^2 + \frac{1}{\beta_2} \| \nabla^k u^+ \|^2 + \frac{1}{\beta_3} \| \nabla^k \nabla n^+ \|^2 \right\} + \frac{1}{4} \frac{d}{dt} \left( \nu_1^+ \| \nabla^k \nabla u^+ \|^2 + \nu_2^+ \| \nabla^k \text{div} u^+ \|^2 \right) \\
\leq C \delta (\| \nabla^k \nabla u^+ \|^2 + \| \nabla u^+ \|^2) + C \delta (\| \nabla^k (n^+, n^-) \|^2 \\
+ C \sum_{0 \leq k \leq 3} \sigma^+ \delta \| \nabla^k \nabla n^+ \|^2 + \frac{\beta_2}{\nu_1^+} \| \nabla^k n^- \|^2, \tag{4.6}
\end{equation}
and
\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \frac{\beta_4}{\beta_3} \| \nabla^k n^- \|^2 + \frac{1}{\beta_3} \| \nabla^k u^- \|^2 + \frac{1}{\beta_3} \| \nabla^k \nabla n^- \|^2 \right) + \frac{1}{4} \frac{d}{dt} \left( \nu_1^- \| \nabla^k \nabla u^- \|^2 + \nu_2^- \| \nabla^k \text{div} u^- \|^2 \right) \\
\leq C \delta (\| \nabla^k \nabla u^- \|^2 + \| \nabla u^- \|^2) + C \delta (\| \nabla^k (n^+, n^-) \|^2 \\
+ \sum_{0 \leq k \leq 3} C \sigma^- \delta \| \nabla^k \nabla n^- \|^2 + \frac{\beta_3}{\nu_1} \| \nabla^k n^+ \|^2, \tag{4.7}
\end{equation}
for \( \delta > 0 \) small enough, and some positive constants \( C \), which are independent of \( \sigma^+ \) and \( \sigma^- \).
Proof. Applying $\nabla^k$ to (3.1) and (3.1), we have
\[
\partial_t \nabla^k n^+ + \text{div}((n^+ + 1)\nabla^k u^+) = - \sum_{l<k} \text{div}(C^l_k \nabla^{k-l}(n^+ + 1)\nabla^l u^+),
\] (4.8)
and
\[
\partial_t \nabla^k n^- + \text{div}((n^- + 1)\nabla^k u^-) = - \sum_{l<k} \text{div}(C^l_k \nabla^{k-l}(n^- + 1)\nabla^l u^-).
\] (4.9)
Multiplying (4.8), $\nabla^k (3.1)_2$, (4.9) and $\nabla^k (3.1)_4$ by $\frac{\beta_1}{\beta_2} \nabla^k n^+$, $\frac{1}{\beta_2}(n^+ + 1)\nabla^k u^+$, $\frac{\beta_4}{\beta_3} \nabla^k n^-$ and $\frac{1}{\beta_3}(n^- + 1)\nabla^k u^-$ respectively, and then integrating over $\mathbb{R}^3$, we have from the summation of the resulting equalities that
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left\{ \frac{\beta_1}{\beta_2} |\nabla^k n^+|^2 + \frac{1}{\beta_2} (n^+ + 1)|\nabla^k u^+|^2 \right\} dx
+ \frac{1}{\beta_2} (n^+ + 1) \left( \nu^+_1 \int_{\mathbb{R}^3} |\nabla^k \nabla u^+|^2 dx + \nu^+_2 \int_{\mathbb{R}^3} |\nabla^k \text{div} u^+|^2 dx \right)
= - \sum_{l<k} \text{div}(C^l_k \nabla^{k-l}(n^+ + 1)\nabla^l u^+), \frac{\beta_1}{\beta_2} \nabla^k n^+
+ \frac{1}{\beta_2} \int_{\mathbb{R}^3} \partial_t n^+ |\nabla^k u^+|^2 dx - \langle \nabla^k \nabla n^-, (n^+ + 1)\nabla^k u^+ \rangle
- \frac{1}{\beta_2} \nu^+_1 \int_{\mathbb{R}^3} \nabla(n^+ + 1) \cdot \nabla^k u^+ \cdot \nabla^k u^+ dx
- \frac{1}{\beta_2} \nu^+_2 \int_{\mathbb{R}^3} \nabla^k \text{div} u^+ \cdot \nabla(n^+ + 1) \cdot \nabla^k u^+ dx
+ (\sigma^+ \nabla^k \nabla n^+ + \frac{1}{\beta_2} (n^+ + 1)\nabla^k u^+) + (\nabla^k F_2, \frac{1}{\beta_2} (n^+ + 1)\nabla^k u^+)
= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7,
\] (4.10)
and
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left\{ \frac{\beta_4}{\beta_3} |\nabla^k n^-|^2 + \frac{1}{\beta_3} (n^- + 1)|\nabla^k u^-|^2 \right\} dx
+ \frac{1}{\beta_3} (n^- + 1) \left( \nu^-_1 \int_{\mathbb{R}^3} |\nabla^k \nabla u^-|^2 dx + \nu^-_2 \int_{\mathbb{R}^3} |\nabla^k \text{div} u^-|^2 dx \right)
= - \sum_{l<k} \text{div}(C^l_k \nabla^{k-l}(n^- + 1)\nabla^l u^-), \frac{\beta_4}{\beta_3} \nabla^k n^-
+ \frac{1}{\beta_3} \int_{\mathbb{R}^3} \partial_t n^- |\nabla^k u^-|^2 dx - \langle \nabla^k \nabla n^+, (n^- + 1)\nabla^k u^- \rangle
- \frac{1}{\beta_3} \nu^-_1 \int_{\mathbb{R}^3} \nabla(n^- + 1) \cdot \nabla^k u^- \cdot \nabla^k u^- dx
- \frac{1}{\beta_3} \nu^-_2 \int_{\mathbb{R}^3} \nabla^k \text{div} u^- \cdot \nabla(n^- + 1) \cdot \nabla^k u^- dx
+ (\sigma^- \nabla^k \nabla n^- + \frac{1}{\beta_3} (n^- + 1)\nabla^k u^-) + (\nabla^k F_4, \frac{1}{\beta_3} (n^- + 1)\nabla^k u^-)
= J_8 + J_9 + J_{10} + J_{11} + J_{12} + J_{13} + J_{14},
\] (4.11)
Similarly, we can obtain

Firstly, by using Lemma 2.2 and Young inequality, the a priori assumption (4.1), integration by parts, we have

\[ |J_1| \]

\[ = \left| \frac{\beta_1}{\beta_2} \nabla^k n^+ \cdot \nabla (n^+ + 1) \right| + \left| \frac{\beta_1}{\beta_2} \nabla^k \nabla (n^+ + 1) \cdot (n^+ + 1) \right| \]

\[ \leq C \left( \| \nabla^k n^+ \|^2 \| \text{div} u^+ \|_{L^\infty} + C \sum_{l \neq 0, l < k} \left( \left\| \nabla^k \nabla (n^+ + 1) \cdot (n^+ + 1) \right\| \right) \right) \]

\[ + 3 \sum_{l \neq 0, l < k} \left( \left\| \nabla^k \nabla (n^+ + 1) \cdot (n^+ + 1) \right\| \right) \]

\[ \leq C \sum_{l \neq 0, l < k} \left( \left\| \nabla^k \nabla (n^+ + 1) \right\|^2 \right) \]
and

\[ |J_{11}| + |J_{12}| \leq C \| \nabla (n^- + 1) \|_{L^\infty} \| \nabla^k \nabla u^- \|_{L^\infty} \| \nabla^k u^- \|_{L^\infty} \leq C \delta \| \nabla^k \nabla u^- \|^2 \]

Next, by using (4.8), (4.9) and integration by parts, we have

\[ J_6 = -\langle \sigma^+ \nabla^k \Delta n^+ + \frac{1}{\beta_2} \text{div}(n^+ + 1) \nabla u^+) \rangle \]
\[ = \langle \sigma^+ \nabla^k \Delta n^+ + \frac{1}{\beta_2} \left\{ \partial_t \nabla^k n^+ + \sum_{l<k} \text{div}(C_k^l \nabla^{k-l}(n^+ + 1) \nabla u^+) \right\} \rangle \]
\[ = \frac{1}{\beta_2} \int_{\mathbb{R}^3} \sigma^+ \nabla^k \Delta n^+ \partial_t \nabla^k n^+ dx \]
\[ + \frac{1}{\beta_2} \int_{\mathbb{R}^3} \sigma^+ \nabla^k \Delta n^+(\nabla^k(n^+ + 1) \text{div} u^+ + \nabla^k \nabla(n^+ + 1) \cdot u^+) dx \]
\[ + \frac{1}{\beta_2} \int_{\mathbb{R}^3} \left( \sum_{l \neq 0} C_k^l (\nabla^{k-l} \nabla(n^+ + 1) \cdot \nabla^l u^+ \right) \]
\[ + \nabla^{k-l}(n^+ + 1) \nabla^l(\text{div} u^+) \sigma^+ \nabla^k \Delta n^+ dx \]
\[ + \nabla^{k-l}(n^+ + 1) \nabla^l(\text{div} u^+) \sigma^+ \nabla^k \Delta n^+ dx \]
\[ = -\frac{1}{2} \frac{d}{dt} \frac{1}{\beta_2} \int_{\mathbb{R}^3} \sigma^+ |\nabla^k \nabla n^+|^2 dx \]
\[ - \frac{1}{\beta_2} \int_{\mathbb{R}^3} \sigma^+ \nabla^k \nabla n^+(\nabla^k \nabla(n^+ + 1) \text{div} u^+ + \nabla^k(n^+ + 1) \nabla \text{div} u^+ \]
\[ + \nabla^k \nabla(n^+ + 1) \cdot u^+ + \nabla^k \nabla(n^+ + 1) \cdot \nabla u^+) dx \]
\[ - \frac{1}{\beta_2} \int_{\mathbb{R}^3} \left( \sum_{l \neq 0} C_k^l (\nabla^{k-l} \nabla(n^+ + 1) \cdot \nabla^l u^+ \right) \]
\[ + \nabla^{k-l}(n^+ + 1) \nabla^l(\text{div} u^+) + \nabla^{k-l}(n^+ + 1) \nabla^l(\text{div} u^+) \sigma^+ \nabla^k \nabla n^+ dx \]
\[ - \frac{1}{\beta_2} \int_{\mathbb{R}^3} \left( \sum_{l \neq 0} C_k^l (\nabla^{k-l} \nabla(n^+ + 1) \cdot \nabla^l u^+ + \nabla^{k-l} \nabla(n^+ + 1) \cdot \nabla^l u^+ \right) \]
\[ + \nabla^{k-l}(n^+ + 1) \nabla^l(\text{div} u^+) + \nabla^{k-l}(n^+ + 1) \nabla^l(\text{div} u^+) \sigma^+ \nabla^k \nabla n^+ dx \]
\[ \leq -\frac{1}{2} \frac{d}{dt} \frac{1}{\beta_2} \int_{\mathbb{R}^3} \sigma^+ |\nabla^k \nabla n^+|^2 dx \]
\[ + C \| (\sigma^+) \|^2 \nabla^k \nabla n^+ \| \| (\sigma^+) \|^2 \nabla^k \nabla n^+ \| \| \text{div} u^+ \|_{L^\infty} \]
In the end, we estimate \( J \) and \( J_{14} \). Using integration by parts and Lemmas 2.2, 2.3 we get

\[
\left| J_{14} \right| 
\leq C \left| \langle \nabla_{k-1} \{ g_2^\pm (n^+, n^-) \nabla n^+ \}, \nabla (n^+ + 1) \nabla_k u^+ + (n^+ + 1) \nabla_{k+1} u^+ \rangle \right|
\]
which gives

\begin{align*}
|J_2| & \leq C(\| \nabla^{k+1} u^+ \| + \| \nabla(n+1) \|_{L^\infty} \| \nabla^{k+1} u^+ \|)(\| g^+_1(n^+, n^-) \|_{L^\infty} \| \nabla^{k+1} n^+ \| \\
& + \| \nabla^{k+1} g^+_1(n^+, n^-) \|_{L^3} \| \nabla^{k+1} n^+ \| \\
& + C(\| \nabla^{k+1} u^+ \| + \| \nabla(n+1) \|_{L^\infty} \| \nabla^{k+1} u^+ \|)(\| g^+_2(n^+, n^-) \|_{L^\infty} \| \nabla^{k+1} n^- \| \\
& + \| \nabla^{k+1} g^+_2(n^+, n^-) \|_{L^3} \| \nabla^{k+1} n^- \|) \\
& + C(\| \nabla^{k+1} u^+ \| + \| \nabla(n+1) \|_{L^\infty} \| \nabla^{k+1} u^+ \|)
\end{align*}

\begin{align*}
&\leq C\delta(\| \nabla^k(n^+, n^-) \|^2 + \| \nabla^k\nabla(n^+, n^-) \|^2 + \| \nabla u^+ \|^2).
\end{align*}

Similarly, we have

\begin{align*}
|J_{14}| & \leq C\delta(\| \nabla^k(n^+, n^-) \|^2 + \| \nabla^k\nabla u^- \|^2 + \| \nabla u^- \|^2).
\end{align*}

Then, taking \( \delta \) appropriately small, and putting the estimates of \( J_1-J_7 \) into (4.10), and the estimates of \( J_8-J_{14} \) into (4.11), we get (4.6) and (4.7). This completes the proof of Lemma 4.2.

Next, we need to estimate the terms such as \( \| \nabla^k\nabla(n^+, n^-) \|^2 \), with \( 0 \leq k \leq 2 \) and \( \sum_{0 \leq k \leq 3} \sigma^+_k \| \nabla^k\nabla(n^+, n^-) \|^2 \), which are as follows.

**Lemma 4.3.** For \( k = 0, 1, 2 \), it holds that

\begin{align}
\frac{d}{dt}\left\{ \nu^+_2 + \nu^+_2 \| \nabla^k\nabla u^+ \|^2 + (\nabla^k u^+, \frac{1}{\beta_2} \nabla^k\nabla u^+) \right\} \\
+ \beta_2 \| \nabla^k\nabla u^+ \|^2 + (\nabla^k\nabla u^-, \frac{1}{\beta_2} \nabla^k\nabla u^-) + \frac{1}{\beta_2} \sigma^+ \| \nabla^k \Delta u^+ \|^2
\end{align}

\begin{align}
\leq \frac{1}{\beta_2} \| \nabla^k\nabla u^+ \|^2 + C\delta \| \nabla^k\nabla(n^+, n^-) \|^2 + C\delta \| \nabla^k\nabla u^+ \|^2_1,
\end{align}

and

\begin{align}
\frac{d}{dt}\left\{ \nu^-_3 + \nu^-_3 \| \nabla^k\nabla u^- \|^2 + (\nabla^k u^-, \frac{1}{\beta_3} \nabla^k\nabla u^-) \right\} \\
+ \beta_3 \| \nabla^k\nabla u^- \|^2 + (\nabla^k\nabla u^+, \frac{1}{\beta_3} \nabla^k\nabla u^+) + \frac{1}{\beta_3} \sigma^- \| \nabla^k \Delta u^- \|^2
\end{align}

\begin{align}
\leq \frac{1}{\beta_3} \| \nabla^k\nabla u^- \|^2 + C\delta \| \nabla^k\nabla(n^+, n^-) \|^2 + C\delta \| \nabla^k\nabla u^- \|^2_1,
\end{align}
for $\delta > 0$ small enough, and some positive constants $C$, which are independent of $\sigma^+$ and $\sigma^−$.

**Proof.** Applying $\nabla^k$ to (3.1), then multiplying the resulting identities by $\frac{1}{\beta_2} \nabla^k u^+$ and $\frac{1}{\beta_3} \nabla^k u^−$ respectively, integrating over $\mathbb{R}^3$, we get

$$
\frac{d}{dt} \left\{ \langle \nabla^k u^+, \frac{1}{\beta_2} \nabla^k u^+ \rangle \right\} + \frac{\beta_1}{\beta_2} \| \nabla^k u^+ \|^2 + \langle \nabla^k \nabla u^+, \nabla^k u^+ \rangle + \frac{1}{\beta_2} \sigma^+ \| \nabla^k \Delta u^+ \|^2
$$

$$
= \langle \nabla^k u^+, \frac{1}{\beta_2} \partial_t \nabla^k u^+ \rangle + \frac{\nu_1^+}{\beta_2} \langle \nabla^k \Delta u^+, \nabla^k u^+ \rangle + \frac{\nu_2^+}{\beta_2} \langle \nabla^k \Delta u^+, \nabla^k u^+ \rangle + \langle \nabla^k u^-, \frac{1}{\beta_2} \nabla^k u^+ \rangle
$$

$$
= H_1 + H_2 + H_3 + H_4.
$$

and

$$
\frac{d}{dt} \left\{ \langle \nabla^k u^-, \frac{1}{\beta_3} \nabla^k u^− \rangle \right\} + \frac{\beta_4}{\beta_3} \| \nabla^k u^− \|^2 + \langle \nabla^k \nabla u^−, \nabla^k u^− \rangle + \frac{1}{\beta_3} \sigma^− \| \nabla^k \Delta u^− \|^2
$$

$$
= \langle \nabla^k u^-, \frac{1}{\beta_3} \partial_t \nabla^k u^− \rangle + \frac{\nu_1^−}{\beta_3} \langle \nabla^k \Delta u^−, \nabla^k u^− \rangle + \frac{\nu_2^−}{\beta_3} \langle \nabla^k \Delta u^−, \nabla^k u^− \rangle + \langle \nabla^k u^−, \frac{1}{\beta_3} \nabla^k u^− \rangle
$$

$$
= H_5 + H_6 + H_7 + H_8.
$$

Using (3.1), (3.3) and integration by parts, we have

$$
H_1 = \langle \nabla^k \text{div} u^+, \frac{1}{\beta_2} \nabla^k \text{div} u^+ \rangle - \langle \nabla^k \text{div} u^+, \frac{1}{\beta_2} \nabla^k F_1 \rangle
$$

$$
= \frac{1}{\beta_2} \| \nabla^k \text{div} u^+ \|^2 - \langle \nabla^k \text{div} u^+, \frac{1}{\beta_2} \nabla^k F_1 \rangle,
$$

and

$$
H_5 = \langle \nabla^k \text{div} u^−, \frac{1}{\beta_3} \nabla^k \text{div} u^− \rangle - \langle \nabla^k \text{div} u^−, \frac{1}{\beta_3} \nabla^k F_3 \rangle
$$

$$
= \frac{1}{\beta_3} \| \nabla^k \text{div} u^− \|^2 - \langle \nabla^k \text{div} u^−, \frac{1}{\beta_3} \nabla^k F_3 \rangle.
$$

Using the Hölder inequality, Lemma 2.2-2.3 and the a priori assumption 4.1, we have

$$
|\langle \nabla^k \text{div} u^+, \frac{1}{\beta_2} \nabla^k F_1 \rangle|
$$

$$
\leq C \| \nabla^k \nabla u^+ \| \| \nabla^k \nabla u^+ \| + \| \nabla^k \n^+ \|_{L^5} \| \nabla^k \n^+ \|_{L^5}
$$

$$
+ C \| \nabla^k \nabla u^+ \| \| \nabla^k \nabla u^+ \| + \| \nabla^k u^+ \|_{L^6} \| \nabla^k \n^+ \|_{L^6}
$$

$$
\leq C \delta (\| \nabla^k \nabla u^+ \|^2 + \| \nabla^k \nabla u^+ \|^2).
$$
Similarly, we have

$$\left| \langle \nabla^k \div -, \frac{1}{\beta_3} \nabla^k F_3 \rangle \right| \leq C\delta (\| \nabla^k \nabla n^- \|^2 + \| \nabla^k \nabla u^- \|^2).$$

And one deduces that

$$|H_1| \leq \frac{1}{\beta_2} \| \nabla^k \div u^+ \|^2 + C\delta (\| \nabla^k \nabla n^- \|^2 + \| \nabla^k \nabla u^- \|^2),$$

$$|H_5| \leq \frac{1}{\beta_3} \| \nabla^k \div u^- \|^2 + C\delta (\| \nabla^k \nabla n^- \|^2 + \| \nabla^k \nabla u^- \|^2).$$

Then, by using (3.1), and (3.1)_3, Lemma 2.2 and Young inequality, yields

$$H_2 + H_3 = \frac{\nu_1^+ + \nu_3^+}{\beta_2} \langle \nabla^k \div u^+, \nabla^k \nabla n^+ \rangle$$

$$= -\frac{\nu_1^+ + \nu_3^+}{2\beta_2} \frac{d}{dt} \langle \nabla^k \nabla n^+, \nabla^k \nabla n^+ \rangle + \frac{\nu_1^+ + \nu_3^+}{\beta_2} \langle \nabla^k \div F_1, \nabla^k \nabla n^+ \rangle$$

$$\leq -\frac{\nu_1^+ + \nu_3^+}{2\beta_2} \left\| \nabla^k \nabla n^+ \right\|^2 + C\delta \left\| \nabla^k \nabla n^+ \right\|^2 + C\delta \left\| \nabla^k \nabla u^+ \right\|^2.$$

In the same way, we get

$$H_6 + H_7 \leq -\frac{\nu_1^- + \nu_3^-}{2\beta_3} \left\| \nabla^k \nabla n^- \right\|^2 + C\delta \left\| \nabla^k \nabla n^- \right\|^2 + C\delta \left\| \nabla^k \nabla u^- \right\|^2.$$

By using Lemmas 2.2, 2.3 and the a priori assumption (4.1), we have

$$|H_4|$$

$$\leq C|\langle \nabla^k [g_1^+(n^+, n^-) \nabla n^+], \nabla^k \nabla n^+ \rangle| + C|\langle \nabla^k [g_2^+(n^+, n^-) \nabla n^+], \nabla^k \nabla n^+ \rangle|$$

$$+ C|\langle \nabla^k [(u^+ \cdot \nabla) u^+], \nabla^k \nabla n^+ \rangle| + C|\langle \nabla^k [h_1^+(n^+, n^-) \nabla n^+ \cdot \nabla] u^+, \nabla^k \nabla n^+ \rangle|$$

$$+ C|\langle \nabla^k [h_2^+(n^+, n^-) \nabla n^+ \cdot \nabla^T u^+], \nabla^k \nabla n^+ \rangle|$$

$$+ C|\langle \nabla^k [h_3^+(n^+, n^-) \nabla n^+ \cdot \nabla^T u^+], \nabla^k \nabla n^+ \rangle|$$

$$+ C|\langle \nabla^k [h_4^+(n^+, n^-) \div u^+], \nabla^k \nabla n^+ \rangle|$$

$$+ C|\langle \nabla^k [h_5^+(n^+, n^-) \div u^+], \nabla^k \nabla n^+ \rangle| + C|\langle \nabla^k [h_6^+(n^+, n^-) \Delta u^+], \nabla^k \nabla n^+ \rangle|$$

$$+ C|\langle \nabla^k [k^+(n^+, n^-) \div u^+], \nabla^k \nabla n^+ \rangle|.$$
which gives

\[|H_4| \leq C \left( \| \nabla^k \nabla n^+ \| \left( \| g_1^+(n^+, n^-) \|_{L^\infty} \| \nabla^k \nabla n^+ \| + \| \nabla^k g_1^+(n^+, n^-) \|_{L^6} \| \nabla n^+ \|_{L^3} \right) \\
+ C \left( \| \nabla^k \nabla n^+ \| \left( \| u^+ \|_{L^\infty} \| \nabla^k \nabla u^+ \| + \| \nabla^k u^+ \|_{L^6} \| \nabla u^+ \|_{L^3} \right) \\
+ C \left( \| \nabla^k \nabla n^+ \| \left( \| \nabla^k [h_1^+(n^+, n^-) \nabla u^+] \|_{L^5} \| \nabla n^+ \|_{L^3} \\
+ \| [h_1^+(n^+, n^-) \nabla u^+] \|_{L^\infty} \| \nabla^k \nabla n^+ \| \right) \\
+ C \left( \| \nabla^k \nabla n^+ \| \left( \| \nabla^k [h_2^+(n^+, n^-) \nabla u^+] \|_{L^5} \| \nabla n^+ \|_{L^3} \\
+ \| [h_2^+(n^+, n^-) \nabla u^+] \|_{L^\infty} \| \nabla^k \nabla n^+ \| \right) \\
+ C \| \nabla^k \nabla n^+ \| \left( \| \nabla^k k^+(n^+, n^-) \|_{L^6} \| \nabla^2 u^+ \|_{L^3} \\
+ \| [k^+(n^+, n^-) \|_{L^\infty} \| \nabla^k+2 u^+ \| \right) \\
\right) \right) \\
\leq C\delta(\| \nabla^k \nabla (n^+, n^-) \|^2 + \| \nabla^k \nabla u^+ \|^2).
\]

Similarly,

\[|H_8| \leq C\delta(\| \nabla^k \nabla (n^+, n^-) \|^2 + \| \nabla^k \nabla u^- \|^2).
\]

Putting the estimates of \(H_1-\)H_4 into (4.14) and estimates of \(H_5-H_8\) into (4.15), taking \(\delta\) sufficiently small, we obtain (4.12) and (4.13).

**Proof of Proposition 4.1.** Since

\[-\frac{s^2(1,1)}{\psi^-(1,1)} < f'(1) < \frac{\eta - s^2(1,1)}{\psi^-(1,1)} < 0,
\]

where \(\eta\) is a positive, small fixed constant, then \(\beta_2 = \frac{\epsilon^2(1,1) + \epsilon^2(1,1)(\psi^-(1,1)f'(1))}{s^2(1,1)} < \eta \frac{\epsilon^2(1,1)}{s^2(1,1)}\) and \(\beta_4\) is bounded. Choosing \(\eta\) small such that \(\beta_2\) is a small positive constant.

At first, (4.3) + \(\sum_{1 \leq k \leq 3} (4.6) + d_1 \sum_{0 \leq k \leq 2} (4.12)\) gives

\[\frac{d}{dt} \left\{ \frac{1}{2} \| \nabla n^+ \| + (\frac{\beta_1}{\beta_2})^{-1} \| n^- \|^2 + \frac{\tau}{2} \| n^- \| + \frac{1}{4\beta_2} \| u^+ \|^2 \\
+ \frac{1}{4\beta_3} \| u^- \|^2 + \frac{1}{2\beta_3} \sigma^- \| \nabla n^- \|^2 \right\}
\]

\[+ \frac{d}{dt} \left\{ \sum_{1 \leq k \leq 3} \frac{\beta_1}{2\beta_2} \| \nabla^k n^+ \|^2 + \sum_{0 \leq k \leq 3} \frac{1}{4\beta_2} \| \nabla^k u^+ \|^2 \right\}.
\]
\[
\begin{align*}
+ \frac{1}{2\beta_2} \sigma^+ & \sum_{0 \leq k \leq 3} \| \nabla^k \nabla n^+ \|^2 + \\
+ \frac{d}{dt} \left\{ \sum_{0 \leq k \leq 2} \frac{d_1 (\nu_1^+ + \nu_2^+)}{2\beta_2} \| \nabla^k \nabla n^+ \|^2 + \sum_{0 \leq k \leq 2} \frac{d_1}{\beta_2} (\nabla^k u^+, \nabla^k \nabla n^+) \right\} \\
+ \sum_{0 \leq k \leq 3} \frac{1}{4\beta_2} (\nu_1^+ \| \nabla^k \nabla u^+ \|^2 + \nu_2^+ \| \nabla^k \div u^+ \|^2) \\
+ \frac{1}{4\beta_3} (\nu_1^- \| \nabla u^- \|^2 + \nu_2^- \| \div u^- \|^2) \\
+ \sum_{0 \leq k \leq 2} \frac{d_1}{\beta_2} \| \nabla^k \nabla n^+ \|^2 + \sum_{0 \leq k \leq 2} d_1 (\nabla^k \nabla n^-, \nabla^k \nabla n^+) \\
+ \frac{d_1}{\beta_2} \sigma^+ \sum_{0 \leq k \leq 2} \| \nabla^k \Delta n^+ \|^2 \\
\leq \sum_{0 \leq k \leq 2} \frac{d_1}{\beta_2} \| \nabla^k \div u^+ \|^2 + C \sum_{0 \leq k \leq 2} \delta \| \nabla^k \nabla u^+ \|^2_1 \\
+ \sum_{1 \leq k \leq 3} C\delta \| \nabla^k (n^+, n^-) \|^2 + C \sum_{1 \leq k \leq 3} \sigma^+ \delta \| \nabla^k \nabla n^+ \|^2 + \sum_{1 \leq k \leq 3} \frac{\beta_2}{\nu_1^+} \| \nabla^k n^- \|,
\end{align*}
\]

where we take \( d_1 \leq \min \left\{ \frac{\nu_2^+}{8}, \sqrt{\frac{\beta_1 + 2 (\nu_1^+ + \nu_2^+)}{2}} \right\} \) and \( \delta \) small enough, then it holds that

\[
\begin{align*}
\frac{d}{dt} \left\{ F_1(t) + \frac{1}{2} \left\| \sqrt{\frac{\beta_1}{\beta_2}} n^+ + \left( \sqrt{\frac{\beta_1}{\beta_2}} \right)^{-1} n^- \right\|^2 + \frac{\tilde{\pi}}{2} \| n^- \|^2 \right. \\
+ \frac{1}{4\beta_3} \| u^- \|^2 + \frac{1}{2\beta_3} \sigma^- \| \nabla n^- \|^2 \right\} \\
+ \sum_{0 \leq k \leq 3} \frac{1}{12\beta_2} (\nu_1^+ \| \nabla^k \nabla u^+ \|^2 + \nu_2^+ \| \nabla^k \div u^+ \|^2) \\
+ \frac{1}{4\beta_3} (\nu_1^- \| \nabla u^- \|^2 + \nu_2^- \| \div u^- \|^2) \\
+ \sum_{0 \leq k \leq 2} \frac{d_1}{\beta_2} \| \nabla^k \nabla n^+ \|^2 + \sum_{0 \leq k \leq 2} d_1 (\nabla^k \nabla n^-, \nabla^k \nabla n^+) \\
+ \frac{d_1}{2\beta_2} \sigma^+ \sum_{0 \leq k \leq 2} \| \nabla^k \Delta n^+ \|^2 \\
\leq \sum_{1 \leq k \leq 3} C\delta \| \nabla^k (n^+, n^-) \|^2 + \sum_{1 \leq k \leq 3} \frac{\beta_2}{\nu_1^+} \| \nabla^k n^- \|,
\end{align*}
\]

(4.17)
where $F_1$ is defined by

$$F_1 = \sum_{1 \leq k \leq 3} \frac{\beta_1}{2\beta_2} \| \nabla^k u^+ \|^2 + \sum_{0 \leq k \leq 2} \frac{d_1}{2\beta_2} (\nu_1^+ + \nu_2^+) \| \nabla^k \nabla u^+ \|^2$$
$$+ \sum_{0 \leq k \leq 3} \frac{1}{4\beta_2} \| \nabla^k u^+ \|^2 + \frac{1}{2\beta_2} \sigma^+ \sum_{0 \leq k \leq 3} \| \nabla^k \nabla u^+ \|^2$$
$$+ \sum_{0 \leq k \leq 2} \frac{d_2}{\beta_2} (\nabla^k u^+, \nabla^k \nabla u^+).$$

Since $d_1 \leq \sqrt{\beta_1 + d_1 (\nu_1^+ + \nu_2^+)}$, there exists a constant $D_1$ independent of $\delta, \sigma^+, \sigma^-$ and $\beta_2$, such that

$$\frac{1}{\beta_2 D_1} \left( \sum_{1 \leq k \leq 3} \| \nabla^k u^+ \|^2 + \sum_{0 \leq k \leq 3} \| \nabla^k u^+ \|^2 \right)$$
$$+ \frac{1}{2\beta_2} \sigma^+ \sum_{0 \leq k \leq 3} \| \nabla^k \nabla u^+ \|^2 \leq F_1$$
$$\leq \beta_2 D_1 \left( \sum_{1 \leq k \leq 3} \| \nabla^k u^+ \|^2 + \sum_{0 \leq k \leq 3} \| \nabla^k u^+ \|^2 \right)$$
$$+ \frac{1}{2\beta_2} \sigma^+ \sum_{0 \leq k \leq 3} \| \nabla^k \nabla u^+ \|^2.$$

Then \((4.17) + \sum_{1 \leq k \leq 3} d_2 (4.7)\) gives

$$\frac{d}{dt} \left\{ F_1(t) + \frac{1}{2} \| \sqrt{\frac{\beta_1}{\beta_3}} \nabla^k u^+ + (\sqrt{\frac{\beta_1}{\beta_3}})^{-1} \nabla^k n^- \|^2 + \frac{\sigma^-}{\beta_3} \| n^- \|^2 \right\}$$
$$+ \frac{d}{dt} \left\{ \sum_{1 \leq k \leq 3} \frac{d_2 \beta_4}{2\beta_3} \| \nabla^k u^- \|^2 + \sum_{0 \leq k \leq 3} \frac{d_2}{4\beta_3} \| \nabla^k u^- \|^2 \right\}$$
$$+ \frac{d_2}{2\beta_3} \sigma^- \sum_{1 \leq k \leq 3} \| \nabla^k \nabla u^- \|^2$$
$$+ \sum_{0 \leq k \leq 3} \frac{1}{12\beta_2} (\nu_1^+ \| \nabla^k \nabla u^+ \|^2 + \nu_2^+ \| \nabla^k \nabla u^+ \|^2) + \sum_{0 \leq k \leq 2} \frac{d_1 \beta_1}{\beta_2} \| \nabla^k \nabla u^+ \|^2.$$
\begin{align}
&+ \sum_{0 \leq k \leq 2} d_1 (\nabla^k \nabla n^-, \nabla^k \nabla n^+) + \frac{d_1}{2 \beta_2} \sum_{0 \leq k \leq 2} \| \nabla^k \Delta n^+ \|^2 \\
&+ \sum_{0 \leq k \leq 3} \frac{d_2}{2 \beta_3} \left( \nu_1^- \| \nabla^k \nabla u^- \|^2 + \nu_2^- \| \nabla^k \text{div} u^- \|^2 \right) \\
&\leq C \sum_{0 \leq k \leq 3} \delta \| \nabla^k \nabla u^- \|^2 + C \sum_{0 \leq k \leq 3} \| \nabla^k (n^+, n^-) \|^2 + C \sum_{1 \leq k \leq 3} \sigma^- \delta \| \nabla^k \nabla n^- \|^2 \\
&+ \sum_{1 \leq k \leq 3} \frac{d_2 \beta_3}{\nu_1^+} \| \nabla^k n^+ \|^2 + \sum_{1 \leq k \leq 3} \frac{\beta_2}{\nu_1^+} \| \nabla^k n^- \|^2 ,
\end{align}

(4.18)

where \( d_2 \) is a fixed positive constant satisfying \( d_2 \leq \min \left\{ \frac{1}{2} \frac{d_1 \nu_1^+}{\beta_3} \right\} \), and by using the smallness of \( \delta \), we have

\begin{align}
&\frac{d}{dt} \left\{ \int_1 (t) + \frac{1}{2} \| \sqrt{\frac{\beta_1}{\beta_2}} n^+ + \left( \sqrt{\frac{\beta_1}{\beta_2}} \right)^{-1} n^- \|^2 + \frac{\gamma}{2} \| n^- \|^2 \\
&+ \frac{1}{4 \beta_3} \| u^- \|^2 + \frac{1}{2 \beta_3} \sigma^- \| \nabla n^- \|^2 \right\} \\
&+ \frac{d_2}{2 \beta_3} \| \nabla^k n^- \|^2 + \sum_{1 \leq k \leq 3} \frac{d_2}{4 \beta_3} \| \nabla^k u^- \|^2 \\
&+ \frac{d_2 \sigma^-}{2 \beta_3} \sum_{1 \leq k \leq 3} \| \nabla^k \nabla n^- \|^2 \\
&+ \sum_{0 \leq k \leq 3} \frac{d_1 \beta_1 (1/\beta_2 - 1)}{\nu_1^+} \| \nabla^k \nabla u^+ \|^2 \\
&+ \sum_{0 \leq k \leq 2} \frac{d_1}{2 \beta_2} (\nabla^k \nabla n^-, \nabla^k \nabla n^+) + \frac{d_1}{2 \beta_2} \sum_{0 \leq k \leq 2} \| \nabla^k \Delta n^+ \|^2 \\
&+ \sum_{0 \leq k \leq 3} \frac{d_2}{6 \beta_3} (\nu_1^- \| \nabla^k \nabla u^- \|^2 + \nu_2^- \| \nabla^k \text{div} u^- \|^2) \\
&\leq C \sum_{1 \leq k \leq 3} \delta \| \nabla^k (n^+, n^-) \|^2 + C \sum_{1 \leq k \leq 3} \sigma^- \delta \| \nabla^k \nabla n^- \|^2 + \sum_{1 \leq k \leq 3} \frac{\beta_2}{\nu_1^+} \| \nabla^k n^- \|^2 .
\end{align}

(4.19)
Furthermore, \((4.19) + d_3 \sum_{0 \leq k \leq 2} \) gives

\[
\frac{d}{dt} \left\{ F_1(t) + \frac{1}{2} \| \sqrt{\frac{\beta_1}{\beta_2}} n^+ + \left( \sqrt{\frac{\beta_1}{\beta_2}} \right)^{-1} n^- \|^2 + \frac{\tau}{2} n^- \| n^- \|^2 \right. \\
+ \frac{1}{4\beta_3} \| u^- \|^2 + \frac{1}{2\beta_3} \| \nabla n^- \|^2 \left\{ \\
+ \frac{d}{dt} \left\{ \sum_{1 \leq k \leq 3} \frac{d_2\beta_4}{2\beta_4} \| \nabla^k n^- \|^2 + \sum_{1 \leq k \leq 3} \frac{d_2}{4\beta_3} \| \nabla^k u^- \|^2 \\
+ \frac{d_2}{2\beta_3} \| \nabla^k \nabla n^- \|^2 \right\} \right. \\
+ \frac{d}{dt} d_3 \left\{ \sum_{0 \leq k \leq 2} \frac{d_3\beta_4}{\beta_3} \| \nabla^k \nabla n^- \|^2 + \sum_{0 \leq k \leq 2} d_3 \langle \nabla^k \nabla n^-, \nabla^k \nabla n^+ \rangle \\
+ \sum_{0 \leq k \leq 2} \frac{d_3\beta_4}{\beta_3} \| \nabla^k \nabla n^- \| \| \nabla^k \nabla n^- \|^2 + \sum_{0 \leq k \leq 2} d_3 \langle \nabla^k \nabla n^-, \nabla^k \nabla n^+ \rangle \\
+ \frac{d_1}{\beta_3} \| \nabla^k \Delta n^+ \| \| \nabla^k \nabla n^+ \| + \sum_{0 \leq k \leq 2} d_1 \beta_1 \left( \frac{1}{\beta_2} - 1 \right) \| \nabla^k \nabla n^+ \|^2 \\
+ \sum_{0 \leq k \leq 2} \frac{1}{\beta_2^2} \left( \nu_1^+ \| \nabla^k \nabla u^+ \|^2 + \nu_2^+ \| \nabla^k \nabla u^+ \|^2 \right) + \sum_{0 \leq k \leq 2} d_1 \langle \nabla^k \nabla n^-, \nabla^k \nabla n^+ \rangle \\
+ \frac{d_1}{\beta_3} \| \nabla^k \Delta n^+ \|^2 + \sum_{0 \leq k \leq 2} d_2 \| \nabla^k \nabla n^- \|^2 + \sum_{0 \leq k \leq 2} d_3 \| \nabla^k \nabla n^- \|^2 \\
\leq C \sum_{0 \leq k \leq 3} \sigma^- \delta \| \nabla^k \nabla n^- \|^2 + \sum_{1 \leq k \leq 3} \frac{\beta_2}{\nu_1^+} \| \nabla^k n^- \| + \sum_{0 \leq k \leq 2} \frac{d_3}{\beta_3} \| \nabla^k \nabla u^- \|^2 \\
+ C \sum_{0 \leq k \leq 2} \delta \| \nabla^k \nabla (n^+, n^-) \|^2 + C \sum_{0 \leq k \leq 2} \delta \| \nabla^k \nabla u^- \|^2, \tag{4.20}
\]

where \( d_3 \leq \min \left\{ \frac{d_2\nu_2^+}{24}, \sqrt{\frac{d_2^2\beta_4 + d_3d_4(\nu_1^+ + \nu_2^+)}{2}} \right\} \) and \( \beta_2 \leq \min \left\{ \frac{1}{2}, \frac{d_3\beta_4\nu_1^+}{\beta_3} \right\}, \)

then by using the smallness of \( \delta \), we have

\[
\frac{d}{dt} \left\{ F_1(t) + F_2(t) + \frac{1}{2} \| \sqrt{\frac{\beta_1}{\beta_2}} n^+ + \left( \sqrt{\frac{\beta_1}{\beta_2}} \right)^{-1} n^- \|^2 + \frac{\tau}{2} n^- \| n^- \|^2 \right. \\
+ \sum_{0 \leq k \leq 2} \frac{d_3\beta_4}{4\beta_3} \| \nabla^k \nabla n^- \|^2 \\
+ \sum_{0 \leq k \leq 2} d_3 \langle \nabla^k \nabla n^-, \nabla^k \nabla n^+ \rangle + \frac{d_3}{2\beta_3} \sigma^- \sum_{0 \leq k \leq 2} \| \nabla^k \Delta n^- \|^2
\]
\[
+ \sum_{0 \leq k \leq 2} \frac{d_1 \beta_1}{2} \left( \frac{1}{\beta_2} - 1 \right) \| \nabla^k \nabla n^+ \|^2 \\
+ \sum_{0 \leq k \leq 3} \frac{1}{2 \beta_2} (\nu_1^+ \| \nabla^k \nabla u^+ \|^2 + \nu_2^+ \| \nabla^k \text{div} u^+ \|^2) + \sum_{0 \leq k \leq 2} d_1 (\nabla^k \nabla n^-, \nabla^k \nabla n^+) \\
+ \frac{d_1}{2 \beta_2} \sigma^+ \sum_{0 \leq k \leq 2} \| \nabla^k \Delta n^+ \|^2 + \sum_{0 \leq k \leq 2} \frac{d_2}{2 \beta_3} (\nu_1^- \| \nabla^k \nabla u^- \|^2 + \nu_2^- \| \nabla^k \text{div} u^- \|^2) \\
\leq 0,
\]

(4.21)

here

\[
F_2(t) = \sum_{0 \leq k \leq 3} \frac{d_2 \beta_4}{2 \beta_3} \| \nabla^k n^- \|^2 + \frac{1}{2 \beta_3} \| u^- \|^2 \\
+ \sum_{1 \leq k \leq 3} \frac{d_2}{4 \beta_3} \| \nabla^k u^- \|^2 + \frac{1}{2 \beta_3} \sigma^- \| \nabla n^- \|^2 \\
+ \frac{d_2}{2 \beta_3} \sigma^- \sum_{0 \leq k \leq 3} \| \nabla^k \nabla n^- \|^2 + d_3 \sum_{0 \leq k \leq 2} \frac{\nu_1^- + \nu_2^-}{2 \beta_3} \| \nabla^k \nabla n^- \|^2 \\
+ \frac{d_3}{\beta_3} \sum_{0 \leq k \leq 2} \langle \nabla^k u^-, \nabla^k \nabla n^- \rangle.
\]

Since \( d_3 \leq \sqrt{\frac{d_2^2 \beta_4 + d_2 d_3 (\nu_1^- + \nu_2^-)}{2}} \), thus there exists a positive constant \( D_2 \), such that

\[
\frac{1}{D_2} \left( \sum_{0 \leq k \leq 3} \| \nabla^k n^- \|^2 + \sum_{0 \leq k \leq 3} \| \nabla^k u^- \|^2 + \sigma^- \sum_{0 \leq k \leq 3} \| \nabla^k \nabla n^- \|^2 \right) \leq F_2(t)
\]

\[
\leq D_2 \left( \sum_{1 \leq k \leq 3} \| \nabla^k n^- \|^2 + \sum_{0 \leq k \leq 3} \| \nabla^k u^- \|^2 + \sigma^- \sum_{0 \leq k \leq 3} \| \nabla^k \nabla n^- \|^2 \right).
\]

Next, we take \( \eta \) sufficiently small, such that

\[
\beta_2 \leq \min \left\{ \frac{1}{2} \frac{d_3 \beta_4 \nu_1^+}{\beta_3}, \frac{1}{1 + \frac{4 \beta_2 (d_1 + d_2)^2}{d_1 d_2 \beta_1 \beta_2}} \right\},
\]

then

\[
\frac{1}{D_3} \left( \sum_{0 \leq k \leq 2} \| \nabla^k \nabla n^+ \|^2 + \sum_{0 \leq k \leq 2} \| \nabla^k \nabla n^- \|^2 \right) \leq
\]

the sum of the second, third, fifth, and the seventh terms on LHS of (4.21)

\[
\leq D_3 \left( \sum_{0 \leq k \leq 2} \| \nabla^k \nabla n^+ \|^2 + \sum_{0 \leq k \leq 2} \| \nabla^k \nabla n^- \|^2 \right),
\]
for some positive constant $D_3$. Denote

$$H(t) = F_1(t) + F_2(t) + \frac{1}{2} \left\| \frac{\beta_1}{\beta_2} n^+ + \left( \sqrt{\frac{\beta_1}{\beta_2}} \right)^{-1} n^- \right\|^2 + \frac{\tilde{\tau}}{2} \left\| n^- \right\|^2.$$ 

From the above, there exists a constant $\tilde{C}$ independent of $\delta$, $\sigma^+$ and $\sigma^-$, such that

$$\frac{1}{\tilde{C}} \left( \| (n^+, n^-) \|^2_3 + \| (u^+, u^-) \|^2_3 + \sigma^+ \sum_{0 \leq k \leq 3} \| \nabla^k \nabla (n^+, n^-) \|^2 \right) \leq H(t)$$

$$\leq \tilde{C} \left( \| (n^+, n^-) \|^2_3 + \| (u^+, u^-) \|^2_3 + \sigma^+ \sum_{0 \leq k \leq 3} \| \nabla^k \nabla (n^+, n^-) \|^2 \right).$$

In the end, there exists a positive constant $\tilde{C}_1$ independent of $\delta$, $\sigma^+$ and $\sigma^-$, such that the following inequalities holds

$$\frac{d}{dt} \left( \| (n^+, n^-) \|^2_3 + \| (u^+, u^-) \|^2_3 + \sigma^+ \sum_{0 \leq k \leq 3} \| \nabla^k \nabla (n^+, n^-) \|^2 \right)$$

$$+ \tilde{C}_1 \left( \| \nabla (n^+, n^-) \|^2_2 + \| \nabla (u^+, u^-) \|^2_2 + \sigma^+ \sum_{k=2} \| \nabla^k \Delta (n^+, n^-) \|^2 \right) \leq 0,$n

provided that $\delta$ is small enough, and $\beta_2$ is sufficiently small. Then by the Gronwall inequality, we complete the proof of Proposition 4.1. \hfill \Box

5. **Proof of the main results.** In this section, we will give the proofs of Theorems 1.1-1.3.

**Proof of Theorem 1.1.** Theorem 1.1 is a direct conclusion of the combination of the local existence and the global a priori estimate in Proposition 4.1. We omit the proof here, see [29] for the details. \hfill \Box

**Proof of Theorem 1.2.** According to (4.2), using a continuity argument, one can prove the global solution of the problem (1.1)-(1.2) with any small $\sigma^+$ and $\sigma^-$, provided that $\| (R_0^+ - \tilde{R}^+, R_0^- - \tilde{R}^-) \|_4 + \| (u_0^+, u_0^-) \|_3 \geq 0$ is small enough.

Let $(R_{\sigma^ \pm}^+, u_{\sigma^ \pm}^+, R_{\sigma^ -}^-, u_{\sigma^ -}^-)$ be the solution to the problem (1.1)-(1.2). Then from (4.2), we have the uniform estimates with respect to $\sigma^+$ and $\sigma^-:

$$\| (R_{\sigma^ \pm}^+, u_{\sigma^ \pm}^+, R_{\sigma^ -}^-, u_{\sigma^ -}^-) \|^2_3 + \| (u_{\sigma^ \pm}^+, u_{\sigma^ -}^-) \|^2_2 + \sigma^+ \sum_{k=3} \| \nabla^k \nabla (R_{\sigma^ \pm}^+, R_{\sigma^ -}^- - \tilde{R}^-) \|^2 \leq C,$n

$$\int_0^t \left( \| \nabla (R_{\sigma^ \pm}^+, R_{\sigma^ -}^- - \tilde{R}^-) \|^2_2 + \| \nabla (u_{\sigma^ \pm}^+, u_{\sigma^ -}^-) \|^2_3 + \sigma^+ \sum_{k=2} \| \nabla^k \Delta (R_{\sigma^ \pm}^+, R_{\sigma^ -}^- - \tilde{R}^-) \|^2 \right) ds \leq C.$$
Furthermore, from (1.11), we have
\[ \| \partial_t (R_{\sigma^+}^+, R_{\sigma^-}^-) (\cdot, t) \|_2 \leq C, \quad \| \partial_t (u_{\sigma^+}^+, u_{\sigma^-}^-) (\cdot, t) \|_1 \leq C, \]
the constants \( C \) are independent of \( \sigma^+ \) and \( \sigma^- \), thus these uniform estimates and Lemma 2.1 imply the existence of a subsequence still denoted by \((R_{\sigma^+}^+, u_{\sigma^+}^+, R_{\sigma^-}^-, u_{\sigma^-}^-)\), such that
\[
R_{\sigma^+}^+ \to \tilde{R}^+ \quad \text{strongly in} \quad C(0, T; C_b^1 \cap H^{3-s}_{\text{loc}}(\mathbb{R}^3)), \\
u_{\sigma^+}^+ \to \tilde{u}^+ \quad \text{strongly in} \quad C(0, T; C_b^1 \cap H^{3-s}_{\text{loc}}(\mathbb{R}^3)), \\
R_{\sigma^-}^- \to \tilde{R}^- \quad \text{strongly in} \quad C(0, T; C_b^1 \cap H^{3-s}_{\text{loc}}(\mathbb{R}^3)), \\
u_{\sigma^-}^- \to \tilde{u}^- \quad \text{strongly in} \quad C(0, T; C_b^1 \cap H^{3-s}_{\text{loc}}(\mathbb{R}^3)), \quad s \in (0, \frac{1}{2}), \\
\sigma^+ \nabla \Delta R_{\sigma^+}^+ \to 0 \quad \text{strongly in} \quad L^2(0, T; H^{1}_{\text{loc}}(\mathbb{R}^3)),
\]
as \( \sigma^\pm \to 0 \), and the limit function \((\tilde{R}^+, \tilde{u}^+, \tilde{R}^-, \tilde{u}^-)\) satisfies (1.14)-1.15). This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Let \((R_{\sigma^+}^+, u_{\sigma^+}^+, R_{\sigma^-}^-, u_{\sigma^-}^-)(x, t)\) be the solution of the problem (1.11)-(1.12), and let \((\tilde{R}^+, \tilde{u}^+, \tilde{R}^-, \tilde{u}^-)(x, t)\) be the solution of the problem 1.14-1.15. Then from Theorem 1.1 and the similar result in [11], we have
\[
\| R_{\sigma^+}^+ - \tilde{R}^+ \|_3 + \| R_{\sigma^-}^- - \tilde{R}^- \|_3 + \| u_{\sigma^+}^+ - \tilde{u}^+ \|_3 \leq C(\| R_{\sigma^+}^+ - \tilde{R}^+ \|_4^2 + \| u_{\sigma^+}^+ - \tilde{u}^+ \|_3^3),
\]
and
\[
\| R_{\sigma^-}^- - \tilde{R}^- \|_3 \leq C(\| R_{\sigma^-}^- - \tilde{R}^- \|_4^2 + \| u_{\sigma^-}^- - \tilde{u}^- \|_3^3).
\]
Define
\[
N^+ = n_{\sigma^+}^+ - \tilde{n}^+, \quad N^- = n_{\sigma^-}^- - \tilde{n}^-, \quad U^+ = u_{\sigma^+}^+ - \tilde{u}^+, \quad U^- = u_{\sigma^-}^- - \tilde{u}^-.
\]
We rewrite (1.14) as follows
\[
\begin{align*}
\partial_t \tilde{n}^+ + \text{div} \tilde{u}^+ &= F_1(\tilde{n}^+, \tilde{u}^+, \tilde{n}^-, \tilde{u}^-), \\
\partial_t \tilde{u}^+ + \beta_1 \nabla \tilde{n}^+ + \beta_2 \nabla \tilde{n}^- - \nu_1^\pm \Delta \tilde{u}^+ - \nu_2^\pm \text{div} \tilde{u}^+ &= F_2(\tilde{n}^+, \tilde{u}^+, \tilde{n}^-, \tilde{u}^-), \\
\partial_t \tilde{u}^- + \text{div} \tilde{u}^- &= F_3(\tilde{n}^+, \tilde{u}^+, \tilde{n}^-, \tilde{u}^-), \\
\partial_t \tilde{u}^- + \beta_3 \nabla \tilde{n}^- + \beta_4 \nabla \tilde{n}^+ - \nu_1^\pm \Delta \tilde{u}^- - \nu_2^\pm \text{div} \tilde{u}^- &= F_4(\tilde{n}^+, \tilde{u}^+, \tilde{n}^-, \tilde{u}^-),
\end{align*}
\]
where \( \tilde{n}^+ = \tilde{R}^+ - \tilde{R}^- \) and \( \tilde{n}^- = \tilde{R}^- - \tilde{R}^- \). Then from (3.1) and (5.3), we have
\[
\begin{align*}
\partial_t N^+ + \text{div} U^+ &= -\text{div}(n_{\sigma^+}^+ U^+ + \tilde{n}^+ N^+), \\
\partial_t U^+ + \beta_1 \nabla N^+ + \beta_2 \nabla N^- - \nu_1^\pm \Delta U^+ - \nu_2^\pm \text{div} U^+ - \sigma^+ \nabla \Delta n_{\sigma^+}^+ &= H^+, \\
\partial_t N^- + \text{div} U^- &= -\text{div}(n_{\sigma^-}^- U^- + \tilde{n}^- N^-), \\
\partial_t U^- - \beta_3 \nabla N^+ + \beta_4 \nabla N^- - \nu_1^\pm \Delta U^- - \nu_2^\pm \text{div} U^- - \sigma^- \nabla \Delta n_{\sigma^-}^- &= H^-.
\end{align*}
\]
where

\[
H^+ = - \left[ g_1^+(n_{\sigma^+}, n_{\sigma^-}) \nabla N^+ + \left( g_1^+(n_{\sigma^+}, n_{\sigma^-}) - g_1^+(\bar{n}^+, \bar{n}^-) \right) \nabla \bar{n}^+ \right]
\]
\[
- \left[ g_2^+(n_{\sigma^+}, n_{\sigma^-}) \nabla N^+ + \left( g_2^+(n_{\sigma^+}, n_{\sigma^-}) - g_2^+(\bar{n}^+, \bar{n}^-) \right) \nabla \bar{n}^+ \right]
\]
\[
- (u_{\sigma^+} \cdot \nabla) U^+ + (U^+ \cdot \nabla) \bar{u}^+
\]
\[
+ \mu^+ [h_1^+(n_{\sigma^+}, n_{\sigma^-})(\nabla n_{\sigma^+} \nabla U^+ + \nabla N^+ \nabla \bar{u}^+)
\]
\[
+ (h_1^+(n_{\sigma^+}, n_{\sigma^-}) - h_1^+(\bar{n}^+, \bar{n}^-)) \nabla \bar{n}^+ \nabla \bar{u}^+ \]
\[
+ \mu^+ [h_2^+(n_{\sigma^+}, n_{\sigma^-})(\nabla n_{\sigma^+} \nabla T U^+ + \nabla N^+ \nabla \bar{u}^+)
\]
\[
+ (h_2^+(n_{\sigma^+}, n_{\sigma^-}) - h_2^+(\bar{n}^+, \bar{n}^-)) \nabla \bar{n}^+ \nabla \bar{u}^+ \]
\[
+ \mu^+ [h_3^+(n_{\sigma^+}, n_{\sigma^-})(\nabla n_{\sigma^+} \nabla U^+ + \nabla N^+ \nabla \bar{u}^+)
\]
\[
+ (h_3^+(n_{\sigma^+}, n_{\sigma^-}) - h_3^+(\bar{n}^+, \bar{n}^-)) \nabla \bar{n}^+ \nabla \bar{u}^+ \]
\[
+ \lambda^+ [h_1^+(n_{\sigma^+}, n_{\sigma^-})(\nabla n_{\sigma^+} \text{div} U^+ + \nabla N^+ \text{div} \bar{u}^+]
\]
\[
+ (h_1^+(n_{\sigma^+}, n_{\sigma^-}) - h_1^+(\bar{n}^+, \bar{n}^-)) \nabla \bar{n}^+ \nabla \bar{u}^+ \]
\[
+ \lambda^+ [h_2^+(n_{\sigma^+}, n_{\sigma^-})(\nabla n_{\sigma^+} \text{div} U^+ + \nabla N^+ \text{div} \bar{u}^+]
\]
\[
+ (h_2^+(n_{\sigma^+}, n_{\sigma^-}) - h_2^+(\bar{n}^+, \bar{n}^-)) \nabla \bar{n}^+ \nabla \bar{u}^+ \]
\[
+ \mu^+ [k^+(n_{\sigma^+}, n_{\sigma^-}) \Delta U^+ + (k^+(n_{\sigma^+}, n_{\sigma^-}) - k^+(\bar{n}^+, \bar{n}^-)) \Delta \bar{u}^+]
\]
\[
+ (\mu^+ + \lambda^+)[k^+(n_{\sigma^+}, n_{\sigma^-}) \text{div} U^+
\]
\[
+ (k^+(n_{\sigma^+}, n_{\sigma^-}) - k^+(\bar{n}^+, \bar{n}^-)) \nabla \text{div} \bar{u}^+],
\]

and

\[
H^- = - \left[ g_1^-(n_{\sigma^+}, n_{\sigma^-}) \nabla N^- + \left( g_1^-(n_{\sigma^+}, n_{\sigma^-}) - g_1^-(\bar{n}^+, \bar{n}^-) \right) \nabla \bar{n}^- \right]
\]
\[
- \left[ g_2^-(n_{\sigma^+}, n_{\sigma^-}) \nabla N^- + \left( g_2^-(n_{\sigma^+}, n_{\sigma^-}) - g_2^-(\bar{n}^+, \bar{n}^-) \right) \nabla \bar{n}^- \right]
\]
\[
- (u_{\sigma^-} \cdot \nabla) U^- + (U^- \cdot \nabla) \bar{u}^-
\]
\[
+ \mu^- [h_1^-(n_{\sigma^+}, n_{\sigma^-})(\nabla n_{\sigma^+} \nabla U^- + \nabla N^- \nabla \bar{u}^-)
\]
\[
+ (h_1^-(n_{\sigma^+}, n_{\sigma^-}) - h_1^-(\bar{n}^+, \bar{n}^-)) \nabla \bar{n}^+ \nabla \bar{u}^+ \]
\[
+ \mu^- [h_2^-(n_{\sigma^+}, n_{\sigma^-})(\nabla n_{\sigma^+} \nabla U^- + \nabla N^- \nabla \bar{u}^-)
\]
\[
+ (h_2^-(n_{\sigma^+}, n_{\sigma^-}) - h_2^-(\bar{n}^+, \bar{n}^-)) \nabla \bar{n}^+ \nabla \bar{u}^+ \]
\[
+ \mu^- [h_3^-(n_{\sigma^+}, n_{\sigma^-})(\nabla n_{\sigma^+} \nabla U^- + \nabla N^- \nabla \bar{u}^-)
\]
\[
+ (h_3^-(n_{\sigma^+}, n_{\sigma^-}) - h_3^-(\bar{n}^+, \bar{n}^-)) \nabla \bar{n}^+ \nabla \bar{u}^+ \]
\[
+ \mu^- [h_4^-(n_{\sigma^+}, n_{\sigma^-})(\nabla n_{\sigma^+} \nabla U^- + \nabla N^- \nabla \bar{u}^-)
\]
\[
+ (h_4^-(n_{\sigma^+}, n_{\sigma^-}) - h_4^-(\bar{n}^+, \bar{n}^-)) \nabla \bar{n}^+ \nabla \bar{u}^+ \],
\]
By using (5.1) and (5.2), we have
\[
1 \leq - \int_{\mathbb{R}^3} R^C \left[ N_+ - N_- + \nabla \cdot \mathbf{u} \right] - \int_{\mathbb{R}^3} \left( h_2^{-1}(n_{\sigma^+}, n_{\sigma^-}) - h_2(n_+^-, n_-^-) \right) \nabla \mathbf{n}^- \cdot \nabla T \mathbf{n}^-
\]
and summing them up, yields
\[
\frac{1}{2} \frac{d}{dt} \left( \| N_+ \|^2 + \| U_+ \|^2 \right) + \nu_1^+ \| \nabla U_+ \|^2 + \nu_2^+ \| \text{div} U_+ \|^2
\]
\[
= - \int_{\mathbb{R}^3} N_+ \text{div} U_+ dx - \int_{\mathbb{R}^3} \nabla \cdot (n_{\sigma^+} U_+ + \bar{u}^+ N_+) N_+ dx
\]
\[
- \int_{\mathbb{R}^3} \beta_1 \nabla N^+ \cdot U_+ dx - \int_{\mathbb{R}^3} \beta_2 \nabla N^- \cdot U_+ dx
\]
\[
+ \int_{\mathbb{R}^3} H^+ U_+ dx + \int_{\mathbb{R}^3} \sigma^+ \nabla \Delta n_{\sigma^+} \cdot U_+ dx,
\]
and
\[
\frac{1}{2} \frac{d}{dt} \left( \| N_- \|^2 + \| U_- \|^2 \right) + \nu_1^- \| \nabla U_- \|^2 + \nu_2^- \| \text{div} U_- \|^2
\]
\[
= - \int_{\mathbb{R}^3} N_- \text{div} U_- dx - \int_{\mathbb{R}^3} \nabla \cdot (n_{\sigma^-} U_- + \bar{u}^- N_-) N_- dx
\]
\[
- \int_{\mathbb{R}^3} \beta_3 \nabla N^+ \cdot U_- dx - \int_{\mathbb{R}^3} \beta_4 \nabla N^- \cdot U_- dx
\]
\[
+ \int_{\mathbb{R}^3} H^- U_- dx + \int_{\mathbb{R}^3} \sigma^- \nabla \Delta n_{\sigma^-} \cdot U_- dx.
\]
By using (5.1) and (5.2), we have
\[
- \int_{\mathbb{R}^3} N_+ \text{div} U_+ dx - \int_{\mathbb{R}^3} N_+ \text{div} (n_{\sigma^+} U_+ + \bar{u}^+ N_+) dx - \int_{\mathbb{R}^3} \beta_1 \nabla N^+ \cdot U_+ dx
\]
\[
- \int_{\mathbb{R}^3} \beta_2 \nabla N^- \cdot U_+ dx + \int_{\mathbb{R}^3} \sigma^+ \nabla \Delta n_{\sigma^+} \cdot U_+ dx
\]
\[
\leq C(\| N_+ \|^2 + \| N_- \|^2 + \| U_+ \|^2) + \epsilon(\| \nabla U_+ \|^2 + \| \text{div} U_+ \|^2) + C\sigma^+.
\]
Similarly, we can obtain

\[
- \int_{\mathbb{R}^3} N^- \text{div} U^- \, dx - \int_{\mathbb{R}^3} N^- \text{div} (n_{\sigma^z}^- U^- + \bar{u}^- N^-) \, dx - \int_{\mathbb{R}^3} \beta_3 \nabla N^+ \cdot U^- \, dx \\
- \int_{\mathbb{R}^3} \beta_4 \nabla N^- \cdot U^- \, dx + \int_{\mathbb{R}^3} \sigma^- \nabla \Delta n_{\sigma^z}^- \cdot U^- \, dx \\
\leq C(\|N^+\|^2 + \|N^-\|^2 + \|U^-\|^2) + c(\|\nabla U^-\|^2 + \|\text{div} U^-\|^2) + C\sigma^-.
\]

Furthermore, by using (5.1) and (5.2), we have

\[
\int_{\mathbb{R}^3} U^+ H^+ \, dx \\
\leq C(\|N^+\| \|\nabla q^+_1(n_{\sigma^z}^+, n_{\sigma^z}^-)\|_{L^\infty} \|U^+\| \\
+ C(\|N^+\| \|1(n_{\sigma^z}^+, n_{\sigma^z}^-)\|_{L^\infty} \|\text{div} U^+\| \\
+ C(\|N^+\| + \|N^-\|) \|\nabla \bar{u}^-\|_{L^\infty} \|U^+\| \\
+ C(\|N^+\| + \|N^-\|) \|\nabla \bar{u}^+\|_{L^\infty} \|U^+\| \\
+ C \|U^+\| \|\nabla U^+\| \|h^+_1(n_{\sigma^z}^+, n_{\sigma^z}^-)\|_{L^\infty} \\
+ C \|N^+\| \|\nabla \bar{u}^+_1(n_{\sigma^z}^+, n_{\sigma^z}^-)\|_{L^\infty} \|U^+\| \\
+ C(\|N^+\| + \|N^-\|) \|\nabla \bar{u}^+_2(n_{\sigma^z}^+, n_{\sigma^z}^-)\|_{L^\infty} \|U^+\| \\
+ C(\|N^+\| + \|N^-\|) \|\nabla \bar{u}^- + \bar{u}^+_2(n_{\sigma^z}^+, n_{\sigma^z}^-)\|_{L^\infty} \|U^+\| \\
- \mu^+ \int_{\mathbb{R}^3} k^+(n_{\sigma^z}^+, n_{\sigma^-})|\nabla U^+|^2 \, dx + C(\|\nabla U^+\| \|\nabla k^+(n_{\sigma^z}^+, n_{\sigma^-})\|_{L^\infty} \|U^+\| \\
+ C(\|N^+\| + \|N^-\|) \|\nabla U^+\|_{L^3} \|\Delta \bar{u}^+\|_{L^3} \\
+ C(\|N^+\| + \|N^-\|) \|\nabla U^+\|_{L^3} \|\nabla \text{div} \bar{u}^-\|_{L^3} \\
- (\mu^+ + \lambda^+) \int_{\mathbb{R}^3} k^+(n_{\sigma^z}^+, n_{\sigma^-})|\text{div} U^+|^2 \, dx \\
+ C(\|\text{div} U^+\| \|\nabla k^+(n_{\sigma^z}^+, n_{\sigma^-})\|_{L^\infty} \|U^+\| \\
\leq -\mu^+ \int_{\mathbb{R}^3} k^+(n_{\sigma^z}^+, n_{\sigma^-})|\nabla U^+|^2 \, dx - (\mu^+ + \lambda^+) \int_{\mathbb{R}^3} k^+(n_{\sigma^z}^+, n_{\sigma^-})|\text{div} U^+|^2 \, dx \\
+ C(\|N^+\|^2 + \|N^-\|^2 + \|U^+\|^2) + c(\|\nabla U^+\|^2 + \|\text{div} U^+\|^2).
\]
Similarly, we can obtain
\[
\int_{\mathbb{R}^3} U^{-} H^{-} \, dx \leq -\mu^{-} \int_{\mathbb{R}^3} k^{-}(n_{\sigma^\pm}, n_{\sigma^\pm})|\nabla U^{-}|^2 \, dx
\]
\[
- (\mu^{-} + \lambda^{-}) \int_{\mathbb{R}^3} k^{-}(n_{\sigma^\pm}, n_{\sigma^\pm})|\text{div} U^{-}|^2 \, dx
\]
\[
+ C(\| N^+ \|_2^2 + \| N^- \|_2^2 + \| U^- \|_2^2) + \varepsilon(\| \nabla U^- \|_2^2 + \| \text{div} U^- \|_2^2).
\]
Taking \(\varepsilon\) small enough, there exist two positive constants \(C_1\) and \(C_2\) independent of \(\sigma^+\) and \(\sigma^-\), such that
\[
\frac{d}{dt} (\| N^-, U^- \|_2^2 + \| (U^+, U^-) \|_2^2) + C_1 \| \nabla (U^+, U^-) \|_2^2
\]
\[
\leq C(\| N^+, N^- \|_2^2 + \| (U^+, U^-) \|_2^2) + C\sigma^+ + C\sigma^-.
\]
In the same way, we can obtain
\[
\frac{d}{dt} (\| \nabla (N^+, N^-) \|_2^2 + \| \nabla (U^+, U^-) \|_2^2) + C_2 \| \nabla^2 (U^+, U^-) \|_2^2
\]
\[
\leq C(\| \nabla (N^-, N^-) \|_2^2 + \| \nabla (U^+, U^-) \|_2^2) + C\sigma^+ + C\sigma^-,
\]
where \(C_2\) and \(C\) are two positive constants independent of \(\sigma^+\) and \(\sigma^-\). In the end, we add (5.5) to (5.6) and applying Gronwall inequality to the resulting inequality, it holds that
\[
\| (N^+, U^+, N^-, U^-) \|_1^2 \leq C\max\{\sigma^+, \sigma^-\} te^{\beta t}, \quad \text{for } t \in [0, \infty),
\]
for some positive constant \(\beta\) independent of \(\sigma^+\) and \(\sigma^-\). This completes the proof of Theorem 1.3.

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