TWO SCENARIOS ON A POTENTIAL SMOOTHNESS BREAKDOWN FOR THE THREE-DIMENSIONAL NAVIER–STOKES EQUATIONS

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Abstract. In this paper we construct two families of initial data being arbitrarily large under any scaling-invariant norm for which their corresponding weak solution to the three-dimensional Navier–Stokes equations become smooth on either [0, T1] or [T2, ∞), respectively, where T1 and T2 are two times prescribed previously. In particular, T1 can be arbitrarily large and T2 can be arbitrarily small. Therefore, a possible formation of singularities would occur after a very long or short evolution time, respectively.

We further prove that if a large part of the kinetic energy is consumed prior to the first (possible) blow-up time, then the global-in-time smoothness of the solutions follows for the two families of initial data.

1. Introduction. The Cauchy problem of the Navier–Stokes equations for the flow of a viscous, incompressible, Newtonian fluid can be written as

\[
\begin{aligned}
\partial_t v - \Delta v + \nabla p + v \cdot \nabla v &= 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\
\nabla \cdot v &= 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty).
\end{aligned}
\]

Here \(v\) represents the velocity of the fluid and \(p\) its pressure. It should be noted that the density and the viscosity have been normalized, as is always possible, by the rescaling argument on the time and space variable \(v(\frac{\nu t}{\rho}, \frac{x}{\rho})\) and \(\frac{1}{\rho}p(\frac{\nu t}{\rho}, \frac{x}{\rho})\).

To these equations we add an initial condition

\[
v(0) = v_0 \quad \text{in} \quad \mathbb{R}^3,
\]

where \(v_0\) is a smooth, divergence-free vector field.

Despite considerable effort invested by scientific community, the mechanisms governing the solutions to the three-dimensional Navier–Stokes equations remain unsolved. At the present time, we do not know yet whether smooth solutions to the three-dimensional Navier–Stokes equations on \(\mathbb{R}^3\) exist for all time. In other words, we do not know whether there are initially smooth solutions with finite kinetic

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energy of the Navier–Stokes equations that develop singularities in finite time, and this is a notoriously difficult question.

The mathematical existence theory developed so far supplies only partial answers to the smoothness of the Navier–Stokes equations. It is known that Navier–Stokes solutions are smooth on \([0, \infty)\) provided the initial velocity \(v_0\) satisfies a smallness condition for certain norm. Instead, if the initial data \(v_0\) are not assumed to be small, one can only guarantee existence on \([0, T)\), where \(T\) depends badly on some norm of \(v_0\).

1.1. Previous results. In 1934 Leray in his ground-breaking paper [15] established the first result of local- and global-in-time existence of smooth solutions to the three-dimensional Navier–Stokes equations on \(\mathbb{R}^3\). More precisely, Leray showed that there was a time interval \([0, T)\) for which \(L^\infty(\mathbb{R}^3)\)-solutions existed and hence they were smooth. He also proved that the Navier-Stokes equations had smooth solutions for all time under a smallness condition for certain norm. However, the smallness condition either on \(\|v_0\|\) or \(\|\nabla v_0\|\) is invariant under the scaling \(\lambda u(\lambda x, \lambda t)\) for all \(\lambda > 0\).

Fujita and Kato (1964) [7] established the local- and global-in-time existence of \(H^1(\mathbb{R}^3)\)-solutions. Twenty years later Kato [11] demonstrated that the three-dimensional Navier–Stokes equations are locally and globally well-posed in the \(L^3(\mathbb{R}^3)\) space. The smoothness of \(L^3(\mathbb{R}^3)\)-solutions being Leray-Hopf weak solutions is due to Escauriaza, Seregin and Sverak (2003) [6].

Afterwards came the work of Cannone (1995) [3] in the Besov spaces \(B_{q, \infty}^{-1+3/q}(\mathbb{R}^3)\) for \(q < \infty\). The next progress was the work of Koch and Tataru (2001) [12] in the \(BMO^{-1}(\mathbb{R}^3)\) space. Solving the Navier-Stokes problem in \(B_{q, \infty}^{-1+3/q}(\mathbb{R}^3)\) or \(BMO^{-1}(\mathbb{R}^3)\) allowed for constructing highly oscillating initial data \(v_0\) with \(\|v_0\|_{L^3(\mathbb{R}^3)}\) being large as long as \(\|v_0\|_{B_{q, \infty}^{-1+3/q}(\mathbb{R}^3)}\) or \(\|v_0\|_{BMO^{-1}(\mathbb{R}^3)}\) were small. Moreover, the smallness condition either on \(\|v_0\|_{B_{q, \infty}^{-1+3/q}(\mathbb{R}^3)}\) or \(\|v_0\|_{BMO^{-1}(\mathbb{R}^3)}\) led to global \(L^3(\mathbb{R}^3)\)-solutions which combined with being Leray–Hopf solutions implied smoothness globally in time. Finally, we mention the work of Lei and Lin (2014) [13] who proved the global-in-time well-posedness of solutions in the scaling invariant space

\[
\{ f \in \mathcal{D}'(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{-1} |\mathcal{F} f(\xi)| d\xi \},
\]

where \(\mathcal{F}\) stands for the Fourier transform.

A turning point appeared with the result of Bourgain and Pavlovic (2008) [2] dealing with the Navier–Stokes problem in \(B_{-1, \infty}^{-1}(\mathbb{R}^3)\). They showed that there were initial data in the Schwartz class \(S(\mathbb{R}^3)\) being arbitrarily small in \(B_{-1, \infty}^{-1}(\mathbb{R}^3)\) whose \(B_{-1, \infty}^{-1}(\mathbb{R}^3)\)-solutions become arbitrarily large after an arbitrarily short time. On the contrary, Chemin and Gallagher (2009) [4] showed that there existed global \(B_{-1, \infty}^{-1}(\mathbb{R}^3)\)-solutions if a certain nonlinear smallness condition was satisfied. These two last results broke the pattern followed for scaling invariant spaces in the above
indicated references – global-in-time well-posedness under a linear smallness condition for initial data. Even though Leray [15] already found nonlinear smallness conditions for proving the global-in-time existence of \( L^\infty(\mathbb{R}^3) \)-solutions. In this sense, Robinson and Sadowski (2014) [17] have recently been published a result of local well-posedness under a smallness condition for \( \| v_0 \|_{L^p(\mathbb{R}^3)} \int_0^T |\nabla u(s)|^2 u(s) ds \) where \( u(t) \) is the solution of the heat equation with the initial condition \( v_0 \).

A change in the philosophy of constructing large initial data \( v_0 \) was to look for special structures which allowed for proving global-in-time existence. In this sense, Mahalov and Nicolaenko (2003) [16] constructed large initial data \( v_0 \) by transforming the Navier–Stokes equations into a rotating fluid equation. In such a setting, it is known that Navier–Stokes solutions are globally well-posed. Chemin and Gallagher (2009) [4] proposed initial data which varied slowly in one direction. In these two examples the global-in-time well-posedness of two-dimensional Navier–Stokes equations is the crucial point in their proof.

Since our results rely on different ways of perturbing the Navier–Stokes equations for obtaining large solutions, we would like to mention some related works that study the concept of stability of solutions in certain spaces. Gallagher (2001) [8] proved that, for any sequence of initial data, their corresponding solution can be decomposed into a sum of orthogonal profiles bounded in \( \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \) plus a remainder which is small with respect to the \( L^3(\Omega) \)-norm. As a result, the stability of solutions in \( \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \) is proved for initial data in \( \dot{H}^{\frac{1}{2}}(\mathbb{R}^2) \cap L^3(\mathbb{R}^3) \) being bounded in \( \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \) and providing \( L^3(\mathbb{R}^3) \)-solutions. The space \( L^3(\mathbb{R}^3) \) could be changed to \( B_{q,\infty}^{-1+\frac{3}{q}}(\mathbb{R}^3) \) or \( BMO^{-1}(\mathbb{R}^3) \). This last result was extended, in [9], by Gallagher, Iritani and Plancho (2003) to the stability of solutions in \( B_p^{-1+\frac{3}{2}}(\mathbb{R}^3) \) and \( L^1(\mathbb{R}^3) \).

1.2. The contribution of this paper. Let us highlight the main contributions and how they differ from existing works concerning stability.

In this paper we will construct smooth initial data \( v_0 \) being arbitrarily large in any critical space that do not develop singularities up to a given time \( T_1 \) without appealing to the two-dimensional Navier–Stokes equations. To achieve such a result we make use of Kuto’s technique. More precisely, the method of proof is based on mild-solution theory for proving the global-in-time existence of \( L^3(\mathbb{R}^3) \)-solutions for small data \( v_0 \). The main difference is that we do not directly impose a smallness condition on the \( L^3(\mathbb{R}^3) \)-norm for \( v_0 \). In doing so, we decompose the original problem into a Stokes problem with an initial datum \( u_0 \) and a perturbed Navier–Stokes-like problem with an initial datum \( w_0 \). From these two subproblems, we will prove that the three-dimensional Navier–Stokes problem possesses \( L^3(\mathbb{R}^3) \)-solutions with initial data \( v_0 = u_0 + w_0 \), where \( u_0 \) has to be small in the \( L^3(\mathbb{R}^3) \)-norm and \( w_0 \) has to be small in the \( L^6(\mathbb{R}^3) \)-norm. As a consequence, \( v_0 \) is no longer small in any scaling-invariant space such as \( H^{\frac{1}{2}}(\mathbb{R}^2) \), \( L^3(\mathbb{R}^3) \), \( B^{-1+\frac{3}{2}}_{p,\infty}(\mathbb{R}^3) \) or \( BMO^{-1}(\mathbb{R}^3) \). This way we will rule out the smallness conditions for \( v_0 \). The result of Escauriaza, Seregin, and Sverák is the final ingredient to conclude with the construction of large initial data \( v_0 \) for the Navier–Stokes equations which provides smooth solutions on \([0, T_1]\), for \( T_1 \) being arbitrarily large. Consequently, the formation of potential singularities would have to be after \( T_1 \). This means that the system would preserve an enough amount of kinetic energy so that the solutions could blow up. On the other hand, if the \( L^3(\mathbb{R}^2) \)-value of the vorticity would keep large without blowing
up so that the kinetic energy would decay under a certain threshold on \([0, T_1]\), the solutions starting from our initial data remained smooth for all time.

Moreover, if a different decomposition of (1) into a Navier–Stokes problem and a perturbed Navier–Stokes-like problem is used, we will be able to prove that there exist Leray–Hopf weak solutions becoming smooth on \([T_2, \infty)\) for any given time \(T_2\). Then we infer that potential singularities could only occur on \((0, T_2)\), for \(T_2\) being arbitrarily small. The most kinetic energy would be consumed on \((0, T)\) so that the solutions could not experience new singularities on \([T_2, \infty)\).

In this paper we do not use the perturbation theory as a way of studying stability of solutions but a way of constructing large solutions to the Navier–Stokes equations. Particularly, if we used the stability theory developed for some space \(X\), with \(X\) being \(H^1(\mathbb{R}^2)\), \(L^3(\mathbb{R}^3)\), \(B^{1+\frac{2}{3}}_{q, \infty}(\mathbb{R}^3)\) or \(BMO^{-1}(\mathbb{R}^3)\), we would obtain that there exists a number \(\varepsilon\) (small enough) such that if \(\|v_0 - u_0\|_X \leq \varepsilon\), we have

\[
\|u(t) - v(t)\|_X \leq E\|v_0 - u_0\|_X \quad \text{for all} \quad t \in [0, T],
\]

where \(\varepsilon > 0\) and \(E > 0\) depend on some energy norms of the solution \(u(t)\). This would provide that the perturbed solution \(v(t)\) would have an initial datum satisfying \(\|v_0\|_X \leq \varepsilon + \|u_0\|_X\). But in order for the solution \(u(t)\) to exist on \([0, T]\) one requires some smallness condition for \(u_0\). Then, the solution \(v(t)\) would inherit a smallness condition for \(v_0\) and therefore would not be large.

2. Statement of problem.

2.1. Notation. As usual, \(L^p(\mathbb{R}^3)\), \(1 \leq p \leq \infty\), denotes the space of \(p\)-integrable, Lebesgue-measurable, \(\mathbb{R}^3\)-valued functions defined on \(\mathbb{R}^3\), and \(W^{1,p}(\mathbb{R}^3)\) denotes the space of functions \(v \in L^p(\mathbb{R}^3)\) such that \(\nabla v \in L^p(\mathbb{R}^3)\), where \(\nabla\) is the gradient operator in the distributional sense. In particular, when \(p = 2\), we denote \(H^1(\mathbb{R}^3) = W^{1,2}(\mathbb{R}^3)\). Moreover, \(C^\infty(\mathbb{R}^3)\) and \(C^\infty(\mathbb{R}^3 \times [0, T])\) are the spaces of infinitely continuously differentiable functions in \(\mathbb{R}^3\) and \(\mathbb{R}^3 \times [0, T]\), respectively, with \(0 < T \leq \infty\), and \(C^\infty_0(\mathbb{R}^3)\) and \(C^\infty_0(\mathbb{R}^3 \times [0, T])\) consist of those functions of \(C^\infty(\mathbb{R}^3)\) and \(C^\infty(\mathbb{R}^3 \times [0, T])\) with compact supports in \(\mathbb{R}^3\) and \(\mathbb{R}^3 \times [0, T]\), respectively. The Schwartz space is denoted as \(\mathcal{S}(\mathbb{R}^3)\) representing the space of rapidly decreasing infinitely continuously differentiable functions on \(\mathbb{R}^3\).

For \(X\) a Banach space, \(L^p(0, T; X)\) denotes the space of \(p\)-integrable, Bochner-measurable, \(X\)-valued functions on \((0, T)\). Moreover, \(C^0([0, \infty); X)\) is the set of all continuous \(X\)-valued functions on \([0, \infty)\).

We let \(\mathcal{P}\) be the Helmholtz-Leray operator onto the space of divergence-free functions in \(L^p(\mathbb{R}^3)\) with \(1 < p < \infty\).

2.2. The Navier–Stokes equations. In this paper the concept of weak solutions for the Navier–Stokes problem (1)–(2) will be understood in the sense of Leray and Hopf (see \([15, 10]\)).

**Definition 2.1.** A function \(v(t)\) is said to be a Leray–Hopf weak solution of problem (1)–(2) if:

\[
v \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)) \quad \text{with} \quad \nabla \cdot v = 0,
\]

and
To do this, we use a regularization approximation procedure so that all the estimates that follow are rigorously set up.

Let \( v_0 \in L^2(\mathbb{R}^3) \) be a divergence-free vector field. Then there exists at least one Leray–Hopf weak solution to \((1)-(2)\) on \([0,T]\).

Next we introduce the concept of strong (or regular) solutions to \((1)-(2)\).

A weak solution \( v(t) \) to problem \((1)-(2)\) is said to be a strong solution on \([0,T]\) if there exists a number \( M_v > 0 \) such that

\[
\sup_{t \in [0,T]} ||\nabla v||_{L^2(\mathbb{R}^3)} \leq M_v.
\]

The key point for proving that solutions to the Navier–Stokes equations are smooth is to obtain that Leray–Hopf weak solutions are strong indeed, of course, for smooth initial data.

Here we announce our two main results.

**Theorem 2.4.** Let \( T > 1 \) be given. Then there exist smooth, divergence-free initial data \( v_0 \) arbitrarily large under any scaling-invariant norm such that their corresponding Leray–Hopf solution \( v(t) \) to \((1)-(2)\) is smooth on \([0,T]\).

**Theorem 2.5.** Let \( 0 < T < 1 \) be given. Then there exist initial data \( v_0 \) arbitrarily large under any scaling-invariant norm such that there exists at least a Leray–Hopf solution \( v(t) \) to \((1)-(2)\) which is smooth on \([T,\infty)\).

By a scaling-invariant norm, we refer to the norms associated to the spaces \( H^{\frac{4}{3}}(\mathbb{R}^2), L^3(\mathbb{R}^3), B_{2,\infty}^{-1+\frac{4}{3}}(\mathbb{R}^3) \) or \( BMO^{-1}(\mathbb{R}^3) \).

Throughout this paper, different positive constants will appear due to interpolations and embeddings among spaces. Thus, \( C \) will always be the maximum of all of these constants in the previous steps, and \( K \) and \( K' \) will stand for constants depending on the initial data.

3. **Proof of Theorem 2.4.** In proving Theorem 2.4 we need to introduce a suitable approximation procedure so that all the estimates that follow are rigorously set up. To do this, we use a regularization à la Leray. That is, we replace the nonlinearity \( \mathbf{v} \cdot \nabla \mathbf{v} \) by \((\rho_\varepsilon \ast \mathbf{v}) \cdot \nabla \mathbf{v}_\varepsilon\), where \( \rho \in C^\infty_0(\mathbb{R}^3) \) such that \( \rho \geq 0 \) and \( \int_{\mathbb{R}^3} \rho(x) \, dx = 1 \) and \( \rho_\varepsilon(x) = \frac{1}{\varepsilon} \rho\left(\frac{x}{\varepsilon}\right) \) for all \( \varepsilon > 0 \), to get

\[
\begin{aligned}
\partial_t \mathbf{v}_\varepsilon - \Delta \mathbf{v}_\varepsilon + \nabla p_\varepsilon + (\rho_\varepsilon \ast \mathbf{v}_\varepsilon) \cdot \nabla \mathbf{v}_\varepsilon &= 0 & \text{in} & & \mathbb{R}^3 \times (0,\infty), \\
\nabla \cdot \mathbf{v}_\varepsilon &= 0 & \text{in} & & \mathbb{R}^3 \times (0,\infty),
\end{aligned}
\]

associated with the regularized initial condition \( \mathbf{v}_\varepsilon(0) = \mathbf{v}_0 \in S(\mathbb{R}^3) \). In virtue of [14, Th. 13.1], this procedure gives rise to a solution pair \((\mathbf{v}_\varepsilon, p_\varepsilon) \in C^\infty(\mathbb{R}^3 \times [0,\infty)) \times C^\infty(\mathbb{R}^3 \times [0,\infty))\). On dealing with the above equations, it is preferably
better to avoid the pressure. For this, we apply the Helmholtz-Leray operator \( P \) to (6) to get
\[
\begin{align*}
\partial_t v + \Delta v + P((\rho_x \ast v_x) \cdot \nabla v_x) &= 0, \\
v_x(0) &= v_0,
\end{align*}
\]
(7)
where we have utilized the fact that \(-P \Delta v = -\Delta P v = -\Delta v\), since \( P \) commutes with derivatives of any order. This is true when solving equations (1) on all of \( \mathbb{R}^3 \).

Our first step is to modify equation (7) in order to easily produce a family of subproblems: a Stokes problem and a Navier–Stokes-like perturbation. In doing so, observe that if \( v = w + u \), then
\[
(\rho_x \ast v) \cdot \nabla v = (\rho_x \ast w) \cdot \nabla w + (\rho_x \ast u) \cdot \nabla w + (\rho_x \ast w) \cdot \nabla u + (\rho_x \ast u) \cdot \nabla u.
\]
Thus let \( u \in C^\infty(\mathbb{R}^3 \times [0, \infty)) \) be the solution to the Stokes problem
\[
\begin{align*}
\partial_t u - \Delta u &= 0, \\
u(0) &= u_0,
\end{align*}
\]
(8)
and let \( w \in C^\infty(\mathbb{R}^3 \times [0, \infty)) \) be the solution to the perturbation problem
\[
\begin{align*}
\partial_t w - \Delta w + P((\rho_x \ast w) \cdot \nabla w) + P((\rho_x \ast u) \cdot \nabla u) \\
+ P((\rho_x \ast w) \cdot \nabla u) + P((\rho_x \ast u) \cdot \nabla u) &= 0, \\
w(0) &= w_0.
\end{align*}
\]
(9)

It is clear that defining \( v = u + w \) and adding (8) and (9) yields (7) for \( v_0 = u_0 + w_0 \). From now on, for simplicity in exposition, we handle (9) without regularizing, although it must be taken into account in order to justify all the computations in this work.

In order to prove our main result, we need to write (8) and (9), by using the Fourier transform, as
\[
u(t) = K_t \ast u_0
\]
(10)
and
\[
w(t) = K_t \ast w_0 + \int_0^t K_{t-s} \ast (P(w \cdot \nabla w) + P(u \cdot \nabla w) + P(w \cdot \nabla u) + P(u \cdot \nabla u)) ds,
\]
(11)
where \( K_t = \frac{1}{(4\pi t)^2} e^{-\frac{|x|^2}{4t}} \), for all \( t > 0 \), is the heat kernel.

At this point we emphasize that, from (10) and (11), we obtain the Duhamel integral form of (7):
\[
u(t) = K_t \ast v_0 + \int_0^t K_{t-s} \ast (P(v \cdot \nabla v)) ds,
\]
(12)
with \( v_0 = u_0 + w_0 \). The equivalence between equations (7) and (12) and equations (9) and (11) are ensured due to the regularity of \( v \) or, more precisely, \( v_x \). Particularly, we require \( v \in C^0([0, \infty); L^q(\mathbb{R}^3) \cap W^{1,3}(\mathbb{R}^3))^3 \), which implies that \( w \in C^0([0, \infty); L^4(\mathbb{R}^3) \cap W^{1,3}(\mathbb{R}^3))^3 \), since \( u \in C^0([0, \infty); L^q(\mathbb{R}^3) \cap W^{1,3}(\mathbb{R}^3))^3 \), where \( q > 3 \).

The following proposition is concerned with some properties of \( K_t \). The proof is straightforward by using the properties of the convolution operator and the particular structure of \( K_t \).

\[ \text{The interested reader is referred to [14, Th. 15.3] for a proof of local-in-time existence in the case of the unregularized Navier-Stokes that can be straightforwardly adapted to the regularized one and combined with an a priori estimate to show that the solution exists globally in time.Obviously, such an a priori estimate will have a bad dependence on the regularization parameter.} \]
Proposition 1. It follows that, for all $1 < p \leq q < \infty$,
\[
\|K_t * f\|_{L^q(\mathbb{R}^3)} \leq C t^{-\left(\frac{3}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^3)},
\]
(13)
\[
\|\nabla K_t * f\|_{L^q(\mathbb{R}^3)} \leq C t^{-\left(1 + \frac{3}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^3)},
\]
(14)
where $C > 0$ is a constant that does not depend on $f$.

Proof. We will use the following property for the convolution operator:
\[
\|K_t * f\|_{L^q(\mathbb{R}^3)} \leq \|K_t\|_{L^{r}(\mathbb{R}^3)} \|f\|_{L^p(\mathbb{R}^3)}
\]
for $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$.

Then we have
\[
\|K_t * f\|_{L^q(\mathbb{R}^3)} \leq \|K_t\|_{L^{r}(\mathbb{R}^3)} \|f\|_{L^p(\mathbb{R}^3)}
\]
for \(\frac{1}{p} = \frac{1}{q} + \frac{1}{s}\).

From now on, we will assume $3 < q$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$ which implies that $3 > p > \frac{3}{2}$.

We will denote
\[
\beta(a,b) = \int_0^1 \gamma^{a-1}(1-\gamma)^{b-1} d\gamma
\]
(16)
for all $a, b > 0$.

Next we provide some estimates uniform with respect to the regularization parameter $\varepsilon$ for the solution to problem (11) under a certain smallness condition for $u_0$ and $w_0$, respectively.

Lemma 3.1. Let $T > 1$ be given, and let $u_0 \in \mathcal{S}(\mathbb{R}^3)$ and $w_0 \in \mathcal{S}(\mathbb{R}^3)$ be two divergence-free vector fields. Then there exists $K > 0$ such that if
\[
T^{\frac{2}{3}} \max\{\|u_0\|_{L^q(\mathbb{R}^3)}, \|\nabla u_0\|_{L^3(\mathbb{R}^3)}\} < \frac{1}{2^{\frac{1}{2}} C}
\]
(17)
and
\[
\|w_0\|_{L^3(\mathbb{R}^3)} < \frac{1}{32 C^2},
\]
(18)
where $C > 0$ is a universal constant coming essentially from (13), (14), (15), and (16), we have
\[
t^{\frac{1}{2}(1-\frac{3}{p})} \|w(t)\|_{L^q(\mathbb{R}^3)} < K
\]
(19)
and
\[
t^{\frac{1}{2}} \|\nabla w(t)\|_{L^3(\mathbb{R}^3)} < K
\]
(20)
for all $t \in [0, T]$.

Proof. Let us choose $K$ to be such that
\[
K = C \|w_0\|_{L^3(\mathbb{R}^3)} + CK^2 + 2CT^{\frac{1}{2}} \max\{\|u_0\|_{L^q(\mathbb{R}^3)}, \|\nabla u_0\|_{L^3(\mathbb{R}^3)}\} K
\]
\[+ CT \max\{\|u_0\|_{L^q(\mathbb{R}^3)}, \|\nabla u_0\|_{L^3(\mathbb{R}^3)}\}^2
\]
(21)
or, equivalently,
\[
0 = C \|u_0\|_{L^3(\mathbb{R}^3)} + CK^2 + (2CT^{\frac{1}{2}} \max\{\|u_0\|_{L^q(\mathbb{R}^3)}, \|\nabla u_0\|_{L^3(\mathbb{R}^3)}\} - 1) K
\]
\[+ CT \max\{\|u_0\|_{L^q(\mathbb{R}^3)}, \|\nabla u_0\|_{L^3(\mathbb{R}^3)}\}^2.
\]
On noting that
\[ 2CT^2 \max\{\|u_0\|_{L^6(\mathbb{R}^3)}, \|
abla u_0\|_{L^6(\mathbb{R}^3)}\} - 1 < -\frac{1}{2} \]
and
\[ \frac{1}{4} - 4C^2(\|w_0\|_{L^6(\mathbb{R}^3)} + T \max\{\|u_0\|_{L^6(\mathbb{R}^3)}, \|
abla u_0\|_{L^6(\mathbb{R}^3)}\})^2 > 0 \]
hold from (17) and (18), we obtain that the polynomial (21) has a positive root satisfying
\[ 0 < K \leq \frac{\frac{1}{2} - \sqrt{\frac{1}{4} - 4C^2(\|w_0\|_{L^6(\mathbb{R}^3)} + T \max\{\|u_0\|_{L^6(\mathbb{R}^3)}, \|
abla u_0\|_{L^6(\mathbb{R}^3)}\})^2}}{2C}. \]

At this point we recall that \( w \in C^0([0, \infty); L^6(\mathbb{R}^3) \cap W^{1,3}(\mathbb{R}^3)) \); therefore,
\[ \lim_{t \to 0^+} t^{\frac{1}{2}(1 - \frac{q}{2})} \|w(t)\|_{L^6(\mathbb{R}^3)} = 0 \]
and
\[ \lim_{t \to 0^+} t^\frac{1}{2} \|\nabla w(t)\|_{L^6(\mathbb{R}^3)} = 0. \]

We claim that (19) and (20) are satisfied for our choice of \( K \). If not, that is, if (19) and/or (20) fail/s for some \( t^* \in (0, T] \), let \( t_* \) be the first time for which
\[ t_*^{\frac{1}{2}(1 - \frac{2}{q})} \|w(t_*)\|_{L^6(\mathbb{R}^3)} = K \]  \hspace{1cm} (22)
and/or
\[ t_*^\frac{1}{2} \|\nabla w(t_*)\|_{L^6(\mathbb{R}^3)} = K. \]  \hspace{1cm} (23)

Let us select \( t = t_* \) in (11) and bound the right-hand side as follows. First of all, observe, from (13) and (14), that
\[ \|K_{t_*} * w_0\|_{L^6(\mathbb{R}^3)} \leq C t_*^{-(1 - \frac{2}{q})} \|w_0\|_{L^6(\mathbb{R}^3)} \]
and
\[ \|\nabla K_{t_*} * w_0\|_{L^3(\mathbb{R}^3)} \leq C t_*^{-\frac{1}{2}} \|\nabla w_0\|_{L^3(\mathbb{R}^3)}. \]

We have, by (13) and (15), that
\[
\| \int_0^{t_*} K_{t_* - s} * \mathcal{P}(w \cdot \nabla w) ds \|_{L^6(\mathbb{R}^3)} \leq \int_0^{t_*} \|K_{t_* - s} * \mathcal{P}(w \cdot \nabla w)\|_{L^6(\mathbb{R}^3)} ds \\
\leq C \int_0^{t_*} (t_* - s)^{-\frac{1}{2}} \|\mathcal{P}(w \cdot \nabla w)\|_{L^6(\mathbb{R}^3)} ds \\
\leq C \int_0^{t_*} (t_* - s)^{-\frac{1}{2}} \|w\|_{L^6(\mathbb{R}^3)} \|\nabla w\|_{L^3(\mathbb{R}^3)} ds \\
< CK^2 \int_0^{t_*} (t_* - s)^{-\frac{1}{2}} s^{-\frac{1}{2}} (1 - \frac{2}{q}) \frac{1}{2} ds \\
\leq CK^2 t_*^{-\frac{1}{2}(1 - \frac{2}{q})} \beta(\frac{3}{2q}, \frac{1}{2}) \leq CK^2 t_*^{-\frac{1}{2}(1 - \frac{2}{q})}.
\]

Here we have utilized the fact that \( t_*^{\frac{1}{2}(1 - \frac{2}{q})} \|w(t)\|_{L^6(\mathbb{R}^3)} < K \) or \( t_*^\frac{1}{2} \|\nabla w(t)\|_{L^3(\mathbb{R}^3)} < K \) for all \( t \in (0, t_*) \), and the change of variable \( s = t_* \gamma \) to obtain \( \beta(\frac{3}{2q}, \frac{1}{2}) \).
Analogously, we obtain, from \( \|u(t)\|_{L^{\gamma}(\mathbb{R}^3)} \leq \|u_0\|_{L^{\gamma}(\mathbb{R}^3)} \) and \( \|\nabla u(t)\|_{L^{3}(\mathbb{R}^3)} \leq \|\nabla u_0\|_{L^{3}(\mathbb{R}^3)} \), that

\[
\int_0^{t^*} \|K_{t^*-s} \ast \mathcal{P}(u \cdot \nabla w)\|_{L^{\gamma}(\mathbb{R}^3)} ds \leq \int_0^{t^*} \|K_{t^*-s} \ast \mathcal{P}(u \cdot \nabla w)\|_{L^{\gamma}(\mathbb{R}^3)} ds
\]

\[
\leq C \int_0^{t^*} (t^* - s)^{-\frac{1}{2}} \|\mathcal{P}(u \cdot \nabla w)\|_{L^{p}(\mathbb{R}^3)} ds
\]

\[
\leq C \int_0^{t^*} (t^* - s)^{-\frac{1}{2}} \|u\|_{L^{\gamma}(\mathbb{R}^3)} \|\nabla w\|_{L^{3}(\mathbb{R}^3)} ds
\]

\[
\leq CK \|u_0\|_{L^{\gamma}(\mathbb{R}^3)} \int_0^{t^*} (t^* - s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds
\]

\[
\leq CK \|u_0\|_{L^{\gamma}(\mathbb{R}^3)} \beta\left(\frac{1}{2}, \frac{1}{2}\right)
\]

\[
\leq CK \|u_0\|_{L^{\gamma}(\mathbb{R}^3)} T^{\frac{1}{2}} t^*^{-\frac{1}{2}} (1 - \frac{1}{2})
\]

and

\[
\int_0^{t^*} \|K_{t^*-s} \ast \mathcal{P}(w \cdot \nabla u)\|_{L^{\gamma}(\mathbb{R}^3)} ds \leq C \|u_0\|_{L^{\gamma}(\mathbb{R}^3)} \|\nabla u_0\|_{L^{3}(\mathbb{R}^3)} t^* \beta(\frac{1}{2}, \frac{1}{2})
\]

\[
\leq C \|u_0\|_{L^{\gamma}(\mathbb{R}^3)} \|\nabla u_0\|_{L^{3}(\mathbb{R}^3)} T^{1-\frac{2}{q}} t^*^{-\frac{1}{2}} (1 - \frac{1}{2}).
\]

Applying the above estimates to (11), we obtain

\[
t^* \beta(1 - \frac{1}{2}) \|w(t^*)\|_{L^{\gamma}(\mathbb{R}^3)} < C \|w_0\|_{L^{3}(\mathbb{R}^3)} + CK^2 + CT^\beta(1 - \frac{1}{2}) \|u_0\|_{L^{\gamma}(\mathbb{R}^3)} K
\]

\[
+ CT^\frac{1}{2} \|\nabla u_0\|_{L^{3}(\mathbb{R}^3)} K
\]

\[
+ CT^\frac{1}{2} \|\nabla u_0\|_{L^{3}(\mathbb{R}^3)} K
\]

Since \( T > 1 \) and \( q > 3 \), we also have, from (21) that

\[
t^* \beta(1 - \frac{1}{2}) \|w(t^*)\|_{L^{\gamma}(\mathbb{R}^3)} < C \|w_0\|_{L^{3}(\mathbb{R}^3)} + CK^2 + CT^\beta \|u_0\|_{L^{\gamma}(\mathbb{R}^3)} K
\]

\[
+ CT^\frac{1}{2} \|\nabla u_0\|_{L^{3}(\mathbb{R}^3)} + CK^2
\]

\[
+ 2CT^\frac{1}{2} \max\{\|u_0\|_{L^{\gamma}(\mathbb{R}^3)}, \|\nabla u_0\|_{L^{3}(\mathbb{R}^3)}\} K
\]

\[
+ CT \max\{\|u_0\|_{L^{\gamma}(\mathbb{R}^3)}, \|\nabla u_0\|_{L^{3}(\mathbb{R}^3)}\}^2
\]

\[
= K.
\]

(24)
Moreover, we have, by (14) and (15), that
\[
\int_0^{t_*} \|\nabla K_{t_*-s} * \mathcal{P}(w \cdot \nabla w)\|_{L^p(\mathbb{R}^3)} ds \leq \int_0^{t_*} \|\nabla K_{t_*-s} * \mathcal{P}(w \cdot \nabla w)\|_{L^3(\mathbb{R}^3)} ds
\]
\[
\leq C \int_0^{t_*} (t_* - s)^{-\frac{2}{p'}} \|\mathcal{P}(w \cdot \nabla w)\|_{L^p(\mathbb{R}^3)} ds
\]
\[
\leq C \int_0^{t_*} (t_* - s)^{-\frac{2}{p'}} \|w\|_{L^p(\mathbb{R}^3)} \|\nabla w\|_{L^3(\mathbb{R}^3)} ds
\]
\[
< CK^2 \int_0^{t_*} (t_* - s)^{-\frac{2}{p'}} s^{-\frac{1}{2}} (1 - \frac{3}{4}) s^{-\frac{1}{2}} ds
\]
\[
\leq CK^2 t_*^{-\frac{1}{2}} \beta \left(\frac{1}{2} \left(\frac{1}{2} - \frac{3}{q} \right), \frac{3}{2q} \right) \leq CK^2 t_*^{-\frac{1}{2}}.
\]

Analogously,
\[
\int_0^{t_*} \|\nabla K_{t_*-s} * \mathcal{P}(u \cdot \nabla w)\|_{L^3(\mathbb{R}^3)} ds \leq \int_0^{t_*} \|\nabla K_{t_*-s} * \mathcal{P}(u \cdot \nabla w)\|_{L^3(\mathbb{R}^3)} ds
\]
\[
\leq C \int_0^{t_*} (t_* - s)^{-\frac{2}{p'}} \|\mathcal{P}(u \cdot \nabla w)\|_{L^p(\mathbb{R}^3)} ds
\]
\[
\leq C \int_0^{t_*} (t_* - s)^{-\frac{2}{p'}} \|u\|_{L^p(\mathbb{R}^3)} \|\nabla w\|_{L^3(\mathbb{R}^3)} ds
\]
\[
\leq CK \|u_0\|_{L^p(\mathbb{R}^3)} \int_0^{t_*} (t_* - s)^{-\frac{2}{p'}} s^{-\frac{1}{2}} ds
\]
\[
\leq CK \|u_0\|_{L^p(\mathbb{R}^3)} t_*^{-\frac{1}{2}} \beta \left(\frac{1}{2}, 1 - \frac{3}{2p} \right)
\]
\[
\leq CK \|u_0\|_{L^p(\mathbb{R}^3)} T^{\frac{1}{2}} t_*^{-\frac{1}{2}}.
\]

\[
\int_0^{t_*} \|\nabla K_{t_*-s} * \mathcal{P}(w \cdot \nabla u)\|_{L^3(\mathbb{R}^3)} ds \leq \int_0^{t_*} \|\nabla K_{t_*-s} * \mathcal{P}(w \cdot \nabla u)\|_{L^3(\mathbb{R}^3)} ds
\]
\[
\leq C \int_0^{t_*} (t_* - s)^{-\frac{2}{p'}} \|\mathcal{P}(w \cdot \nabla u)\|_{L^p(\mathbb{R}^3)} ds
\]
\[
\leq C \int_0^{t_*} (t_* - s)^{-\frac{2}{p'}} \|w\|_{L^p(\mathbb{R}^3)} \|\nabla u\|_{L^3(\mathbb{R}^3)} ds
\]
\[
\leq CK \|\nabla u_0\|_{L^3(\mathbb{R}^3)} \int_0^{t_*} (t_* - s)^{-\frac{2}{p'}} s^{-\frac{1}{2}} (1 - \frac{3}{4}) ds
\]
\[
\leq CK \|\nabla u_0\|_{L^3(\mathbb{R}^3)} \beta \left(\frac{1}{2} + \frac{3}{2p}, 1 - \frac{3}{2p} \right)
\]
\[
\leq CK \|\nabla u_0\|_{L^3(\mathbb{R}^3)} T^{\frac{1}{2}} t_*^{-\frac{1}{2}}
\]

and
\[
\int_0^{t_*} \|\nabla K_{t_*-s} * \mathcal{P}(u \cdot \nabla u)\|_{L^3(\mathbb{R}^3)} ds \leq C \|u_0\|_{L^3(\mathbb{R}^3)} \|\nabla u_0\|_{L^3(\mathbb{R}^3)} t_*^{-\frac{1}{2}} T^{\frac{1}{2}} (1 - \frac{1}{2}).
\]

Applying the above estimates to (11), we obtain
\[
t_*^{\frac{1}{2}} \|\nabla w(t_*)\|_{L^3(\mathbb{R}^3)} < C\|w_0\|_{L^3(\mathbb{R}^3)} + CK^2 + CT^{\frac{1}{2}} \|u_0\|_{L^3(\mathbb{R}^3)} K
\]
\[
+ CT^{\frac{1}{2}} \|\nabla u_0\|_{L^3(\mathbb{R}^3)} K + CT^{\frac{1}{2}} (1 - \frac{1}{2}) \|u_0\|_{L^3(\mathbb{R}^3)} \|\nabla u_0\|_{L^3(\mathbb{R}^3)}.
\]
Lemma 3.2. Let \( u_0 \in S(\mathbb{R}^3) \) and \( w_0 \in S(\mathbb{R}^3) \) be two divergence-free vector fields satisfying (17) and (18). Then there exists a number \( M_w > 0 \) such that the solution \( w(t) \) to (9) satisfies

\[
\sup_{t \in [0,T]} \| w(t) \|_{L^3(\mathbb{R}^3)} \leq M_w.
\]

Proof. From (11), we have

\[
\| w(t) \|_{L^3(\mathbb{R}^3)} \leq \| K_t \ast w_0 \|_{L^3(\mathbb{R}^3)} + \int_0^t \| K_{t-s} \ast (P(w \cdot \nabla w) + ) \|_{L^3(\mathbb{R}^3)} ds
\]

\[
+ \int_0^t \| K_{t-s} \ast (P(u \cdot \nabla w) + P(w \cdot \nabla w) + P(u \cdot \nabla u) \|_{L^3(\mathbb{R}^3)} ds.
\]

Let us now bound each term on the right-hand side. We have, by (13), (17), and (18), that

\[
\| K_t \ast w_0 \|_{L^3(\mathbb{R}^3)} \leq C \| w_0 \|_{L^3(\mathbb{R}^3)}
\]

\[
\int_0^t \| K_{t-s} \ast P(w \cdot \nabla w) \|_{L^3(\mathbb{R}^3)} ds \leq C \int_0^t (t-s)^{-\frac{3}{p}(\frac{3}{p} - 1)} \| w \cdot \nabla w \|_{L^p(\mathbb{R}^3)} ds
\]

\[
\leq C \int_0^t (t-s)^{-\frac{3}{q}} \| w \|_{L^q(\mathbb{R}^3)} \| \nabla w \|_{L^3(\mathbb{R}^3)} ds
\]

\[
\leq C K^2 \int_0^t (t-s)^{-\frac{3}{q} - (1 - \frac{3}{2})} s^{\frac{3}{2}} ds
\]

\[
\leq C K^2 \beta \left( \frac{1}{2}, 1 - \frac{2}{3}, \frac{3}{q} \right) \leq C K^2
\]

Finally we conclude that the assertions in (22) and (23) are a contradiction in view of (24) and (25). This completes the proof. \( \Box \)
\[
\begin{align*}
\int_0^t \| K_{t-s} \cdot \mathcal{P}(u \cdot \nabla w) \|_{L^3(\mathbb{R}^3)} ds & \leq C \int_0^t (t-s)^{-\left(\frac{3}{q} - 1\right)\frac{1}{2}} \| u \cdot \nabla w \|_{L^5(\mathbb{R}^3)} ds \\
& \leq C \int_0^t (t-s)^{-\frac{1}{q}} \| u \|_{L^5(\mathbb{R}^3)} \| \nabla w \|_{L^5(\mathbb{R}^3)} ds \\
& \leq CK \int_0^t (t-s)^{-\frac{1}{q} s^{-\frac{1}{2}} (1 - \frac{3}{q})} \| u \|_{L^5(\mathbb{R}^3)} s^{-\frac{1}{q}} ds \\
& \leq CK \| u_0 \|_{L^5(\mathbb{R}^3)} \beta \left(\frac{1}{2} (2 - \frac{3}{q}), \frac{3}{q}\right) \\
& \leq CK \| u_0 \|_{L^5(\mathbb{R}^3)}.
\end{align*}
\]

Therefore, we obtain
\[
\| w(t) \|_{L^3(\mathbb{R}^3)} \leq C \| u_0 \|_{L^3(\mathbb{R}^3)} + CK^2 + 2C \| u_0 \|_{L^3(\mathbb{R}^3)} K + C \| u_0 \|_{L^3(\mathbb{R}^3)}^2 := M_w. \tag{26}
\]

In view of Lemmas 3.1 and 3.2, we have proved the existence of an \( L^3(\mathbb{R}^3) \)-solution to (7) on \([0, T]\) under certain smallness conditions for \( u_0 \) and \( w_0 \).

**Lemma 3.3.** Let \( u_0 \in \mathcal{S}(\mathbb{R}^3) \) and \( w_0 \in \mathcal{S}(\mathbb{R}^3) \) be two divergence-free vector fields satisfying (17) and (18), respectively. Then there exists \( M_v > 0 \) such that the solution \( v(t) \) to (7) with \( v_0 = u_0 + w_0 \) satisfies
\[
\sup_{t \in [0, T]} \| v(t) \|_{L^3(\mathbb{R}^3)} \leq M_v.
\]

**Proof.** First notice that, from (10), we have \( \| u(t) \|_{L^3(\mathbb{R}^3)} \leq \| u_0 \|_{L^3(\mathbb{R}^3)} := M_u \) for \( t \in [0, T] \). By Lemma 3.2, we have \( \| w(t) \|_{L^3(\mathbb{R}^3)} \leq M_w \) for all \( t \in [0, T] \). Therefore, if we define \( v(t) = u(t) + w(t) \), we obtain \( \| v(t) \|_{L^3(\mathbb{R}^3)} \leq M_u + M_w := M_v \) for all \( t \in [0, T] \), where \( v \) satisfies (7) on \([0, T]\) with \( v_0 = u_0 + w_0 \). \( \square \)
This implies that \( \| \cdot \|_{L^3(\mathbb{R}^3)} \)-norm. That is, we will use different scalings for \( u_0 \) and \( w_0 \) so that supercritical and subcritical norms increase and decrease oppositely with the \( L^3(\mathbb{R}^3) \)-norm being invariant. This way we avoid that the size of any norm of \( v_0 \) is no longer small, rather large.

Let \( \tilde{v}_0 \in \mathcal{S}(\mathbb{R}^3) \) be a divergence-free vector field. We are allowed to take \( \varepsilon > 0 \) such that \( \tilde{v}_0 = (1 - \varepsilon)\tilde{v}_0 + \varepsilon\tilde{v}_0 := u_{0,\varepsilon} + w_{0,\varepsilon} \) so that \( w_{0,\varepsilon} \) satisfies condition (18).

Next we define \( u_{0,\varepsilon}^\lambda = \lambda u_{0,\varepsilon}(\lambda x) \) and \( w_{0,\varepsilon}^\lambda = \lambda w_{0,\varepsilon}(\lambda x) \) for \( \lambda, \lambda > 0 \). Letting \( \lambda \to 0 \), we find that there exists \( \lambda_0 \) such that, for all \( \lambda \leq \lambda_0 \), it follows that condition (17) holds for \( u_{0,\varepsilon}^\lambda \). Moreover, for any \( \lambda \), we find that \( w_{0,\varepsilon}^\lambda \) fulfills condition (18) since the \( L^3(\mathbb{R}^3) \)-norm is scaling invariant. This last rescaling is not really necessary, but it allows us to construct initial data arbitrarily large under any supercritical norm. Thus we define \( v_0 = u_{0,\varepsilon}^\lambda + w_{0,\varepsilon}^\lambda \) whose \( L^3(\mathbb{R}^3) \)-norm remains almost invariant due to our special choice, i.e. \( \|v_0\|_{L^3(\mathbb{R}^3)} \leq \|u_0\|_{L^3(\mathbb{R}^3)} + \|w_0\|_{L^3(\mathbb{R}^3)} \leq (1 - \varepsilon)\|\tilde{v}_0\|_{L^3(\mathbb{R}^3)} + \varepsilon\|\tilde{v}_0\|_{L^3(\mathbb{R}^3)} \leq \|v_0\|_{L^3(\mathbb{R}^3)} \). Instead, supercritical norms can be arbitrarily large by doing \( \lambda \) to tend to \( \infty \) and subcritical norms by doing \( \tilde{\lambda} \) to tend to \( 0 \).

Another possibility to construct smooth initial data \( v_0 \) is as follows. Consider \( \tilde{u}_0 \in \mathcal{S}(\mathbb{R}^3) \) and \( \tilde{w}_0 \in \mathcal{S}(\mathbb{R}^3) \) to be two divergence-free vector fields and define \( v_0 = \tilde{u}_0 + \varepsilon \tilde{w}_0 := u_{0,\varepsilon} + w_{0,\varepsilon} \). Pick \( \lambda \) to be such that \( u_{0,\varepsilon}^\lambda \) satisfies condition (17) and \( \varepsilon \) to be such that \( w_{0,\varepsilon}^\lambda \) satisfies condition (18).

The following theorem was proved in [6] by Escauriaza, Seregin, and Šverák.

**Theorem 3.4.** Let \( v_0 \in \mathcal{S}(\mathbb{R}^3) \) be a divergence-free vector field. Assume that \( v(t) \) is one weak Leray–Hopf solution to (1)–(2) and satisfies the additional condition

\[
\sup\limits_{t \in [0, T]} \|v(t)\|_{L^3(\mathbb{R}^3)} < \infty.
\]

Then \( v(t) \) is a strong solution to (1)–(2) on \([0, T]\).

Therefore, Lemma 3.3 and Theorem 3.4 combined with Theorem 2.2 give that the solutions \( v(t) \) whose initial data \( v_0 \) can be decomposed as, for instance, \( v_0 = u_{0,\varepsilon}^\lambda + w_{0,\varepsilon}^\lambda \) with \( u_{0,\varepsilon} \in \mathcal{S}(\mathbb{R}^3) \) and \( w_{0,\varepsilon} \in \mathcal{S}(\mathbb{R}^3) \) being divergence-free vector fields fulfilling (17) (for certain \( \lambda \)) and (18) (for certain \( \varepsilon \)) are strong, and hence they are smooth on \([0, T]\). It proves Theorem 2.4 by appealing to the strong-weak uniqueness.

**Remark 2.** It is easy to see that the solutions given in Theorem 2.4 satisfy the estimate:

\[
\|v(t)\|_{L^3(\mathbb{R}^3)} \leq (K + C\|u_0\|_{L^3(\mathbb{R}^3)})\|v(t)\|_{L^3(\mathbb{R}^3)}^\frac{1}{2} \quad \text{for all } t \in [0, T].
\]

This implies that \( \|v(T)\|_{L^3(\mathbb{R}^3)} \) can be as small as required provided that \( T \) is large. As a result, we can extend our solution to \([0, T^*]\) for \( T^* \) being possible large. See [14, Thm 15.3].
4. **Proof of Theorem 2.5.** We first decompose (7) as follows. Let $w_\varepsilon$ be the solution to the Navier–Stokes problem

$$
\begin{aligned}
\partial_t w_\varepsilon - \Delta w_\varepsilon + P((\rho_\varepsilon * w_\varepsilon) \cdot \nabla w_\varepsilon) &= 0, \\
w_\varepsilon(0) &= w_0,
\end{aligned}
$$

and let $u_\varepsilon$ be the solution to the perturbation problem

$$
\begin{aligned}
\partial_t u_\varepsilon - \Delta u_\varepsilon + P((\rho_\varepsilon * w_\varepsilon) \cdot \nabla u_\varepsilon) + P((\rho_\varepsilon * u_\varepsilon) \cdot \nabla w_\varepsilon) &= 0, \\
u_\varepsilon(0) &= u_0.
\end{aligned}
$$

As before, we drop the subscript $\varepsilon$ and the convolution operator from (27) and (28).

The following result is a consequence of Lemmas 3.1 and 3.2. In particular, we assume that we have $C > 1$ in Lemma 3.1.

**Lemma 4.1.** Let $w_0 \in S(\mathbb{R}^3)$ be a divergence-free vector field such that

$$
\|w_0\|_{L^3(\mathbb{R}^3)} \leq \frac{1}{4C}.
$$

Then there exists a smooth solution $w(t)$ to (27) on $[0, \infty)$ such that

$$
\sup_{t \in [0, \infty)} \|w(t)\|_{L^3(\mathbb{R}^3)} < K := \frac{1 - \sqrt{1 - 4C^2\|w_0\|_{L^3(\mathbb{R}^3)^2}}}{2C}.
$$

**Proof.** From (8) for $u_0 = 0$ and (9), we recover (27). By following the proof of Lemma 3.1, we obtain that (19) and (20) hold for

$$
0 = C\|w_0\|_{L^3(\mathbb{R}^3)} + CK^2 - K.
$$

Therefore,

$$
K = \frac{1 - \sqrt{1 - 4C^2\|w_0\|_{L^3(\mathbb{R}^3)^2}}}{2C}.
$$

In virtue of (26), we obtain (30).

**Lemma 4.2.** Let $0 < T < 1$ be given. Let $u_0 \in S(\mathbb{R}^3)$ and $w_0 \in S(\mathbb{R}^3)$ be two divergence-free vector fields such that

$$
\max_{t \in [0, T]} \|w(t)\|_{L^3(\mathbb{R}^3)} \leq \frac{1}{8C},
$$

and

$$
T^{-\frac{1}{2}}\|u_0\|_{L^2(\mathbb{R}^3)} < \frac{1}{8C}.
$$

Then there exists $t^* \in (0, T]$ such that

$$
\|u(t^*)\|_{L^3(\mathbb{R}^3)} < \frac{1}{8C}.
$$

**Proof.** Multiplying (28) by $u$ and integrating over $\mathbb{R}^3$ gives, after integration by parts,

$$
\frac{1}{2} \frac{d}{dt} \|u\|^2_{L^2(\mathbb{R}^3)} + \|\nabla u\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} u \cdot \nabla w \cdot u \, dx

\leq \|u\|_{L^6(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)} \|w\|_{L^3(\mathbb{R}^3)}

\leq C \|w\|_{L^3(\mathbb{R}^3)} \|\nabla u\|^2_{L^2(\mathbb{R}^3)}.
$$
From (31), we arrive at
\[ \frac{d}{dt} \| u \|^2_{L^2(\mathbb{R}^3)} + \| \nabla u \|^2_{L^2(\mathbb{R}^3)} \leq 0. \]

Integrating with respect to time, we get
\[ \sup_{t \in [0,T]} \| u(t) \|^2_{L^2(\mathbb{R}^3)} + \int_0^T \| \nabla u(s) \|^2_{L^2(\mathbb{R}^3)} ds \leq \| u_0 \|^2_{L^2(\mathbb{R}^3)}. \]

By interpolation, we write
\[ \| u(t) \|_{L^3(\mathbb{R}^3)} \leq C \| u(t) \|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \| \nabla u(t) \|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}. \]

Therefore,
\[ \int_0^T \| u(t) \|^4_{L^3(\mathbb{R}^3)} dt \leq C \| u_0 \|^4_{L^2(\mathbb{R}^3)} \]
and hence
\[ T \inf_{s \in [0,T]} \| u \|^4_{L^3(\mathbb{R}^3)} \leq C \| u_0 \|^4_{L^2(\mathbb{R}^3)} \]

and
\[ \inf_{s \in [0,T]} \| u \|^2_{L^2(\mathbb{R}^3)} \leq CT^{-\frac{1}{4}} \| u_0 \|^2_{L^2(\mathbb{R}^3)}. \]

If conditions (31) and (32) hold, there exists \( t^* \in (0,T] \) such that condition (33) is satisfied.

In order for condition (31) to hold, we need
\[ 1 - \sqrt{1 - 4C^2 \| w_0 \|^2_{L^3(\mathbb{R}^3)}} < \frac{1}{4}, \quad (34) \]
which holds from (30). Let us choose \( \lambda \) and \( \varepsilon \) such that \( v_0 = u_{0,\varepsilon} + w_{0,\varepsilon} \) with \( u_0 \) and \( w_0 \) satisfying (32) and (34), respectively. Thus, we arrive at
\[ \| v(t^*) \|^4_{L^3(\mathbb{R}^3)} \leq \| w(t^*) \|^4_{L^3(\mathbb{R}^3)} + \| u(t^*) \|^4_{L^3(\mathbb{R}^3)} < \frac{1}{4C}. \]

Then, by Lemma 4.1, we obtain
\[ \sup_{t \in [T,\infty)} \| v(t) \|^4_{L^3(\mathbb{R}^3)} \leq \frac{1}{2C}, \]

since \( v(t) \) is a solution of the regularized Navier–Stokes equations as \( w(t) \).

As a result of Theorem 3.4, we have accomplished to prove that the unregularized solutions \( v(t) \) whose initial data \( v_0 \) can be decomposed as \( v_0 = u_{0,\varepsilon} + w_{0,\varepsilon} \) with \( u_{0,\varepsilon} \in \mathcal{S}(\mathbb{R}^3) \) and \( w_{0,\varepsilon} \in \mathcal{S}(\mathbb{R}^3) \) being divergence-free vector fields fulfilling (34) and (32), respectively, are strong, and hence they are smooth on \( [T,\infty) \). We have used the same decomposition for \( v_0 \) as in the proof of Theorem 2.4. This way our initial conditions are arbitrarily large under any scaling-invariant norm. It proves Theorem 2.5.
5. **Additional results.** To complete the proof of Theorem 2.4 we show there exist initial data \( \mathbf{v}_0 \) which cannot be a priori decomposed as above. To do this, we just need to use, for instance, an \( L^3(\mathbb{R}^3) \)-stability result. The proof combines ideas from [6] for establishing local-in-time existence of \( L^3(\mathbb{R}^3) \)-solutions and from [9] for proving stability in Besov spaces.

**Theorem 5.1.** Let \( \mathbf{v}(t) \) be a smooth solution to (1)–(2) with an initial datum \( \mathbf{v}_0 = \mathbf{u}_0 + \mathbf{w}_0 \), where \( \mathbf{u}_0 \) and \( \mathbf{w}_0 \) are two smooth, divergence-free vector fields fulfilling (17) and (18), respectively. Then there exists \( \varepsilon = \varepsilon(\mathbf{v}) \) such that, for all initial data \( \mathbf{v}_0 \) with \( \|\mathbf{v}_0 - \mathbf{v}_0\|_{L^3(\mathbb{R}^3)} < \varepsilon \), the corresponding solution \( \mathbf{v}(t) \) with \( \mathbf{v}(0) = \mathbf{v}_0 \) satisfies

\[
\|\mathbf{v}(t) - \mathbf{v}(t)\|_{L^3(\mathbb{R}^3)} \leq C(\mathbf{v})\|\mathbf{v}_0 - \mathbf{v}_0\|_{L^3(\mathbb{R}^3)} \text{ for all } t \in [0, T].
\]

**Proof.** The proof is divided into two parts:

**Part I:** A priori estimates

To start with, define \( \mathbf{w}(t) := \mathbf{v}(t) - \mathbf{v}(t) \) to be the solution to

\[
\begin{aligned}
\partial_t \mathbf{w} - \Delta \mathbf{w} + \nabla q + \mathbf{v} \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v} + \mathbf{w} \cdot \nabla \mathbf{w} &= 0, \\
\nabla \cdot \mathbf{w} &= 0, \\
\mathbf{w}(0) &= \mathbf{w}_0 := \mathbf{v}_0 - \mathbf{v}_0.
\end{aligned}
\tag{36}
\]

We approximate \( \mathbf{w}(t) \) by the following iterative algorithm: find \( \mathbf{w}^{n+1}(t) \) as the solution of

\[
\begin{aligned}
\partial_t \mathbf{w}^{n+1} - \Delta \mathbf{w}^{n+1} + \nabla q^{n+1} + \mathbf{v} \cdot \nabla \mathbf{w}^{n+1} + \mathbf{w}^{n+1} \cdot \nabla \mathbf{v} + \mathbf{w}^{n+1} \cdot \nabla \mathbf{w}^{n+1} &= 0, \\
\nabla \cdot \mathbf{w}^{n+1} &= 0, \\
\mathbf{w}(0) &= \mathbf{w}_0,
\end{aligned}
\]

where \( \mathbf{w}^0 \) is defined by

\[
\begin{aligned}
\partial_t \mathbf{w}^0 - \Delta \mathbf{w}^0 + \nabla q^0 &= 0, \\
\nabla \cdot \mathbf{w}^0 &= 0, \\
\mathbf{w}^0(0) &= \mathbf{w}_0.
\end{aligned}
\]

Testing by \( |\mathbf{w}^{n+1}||\mathbf{w}^{n+1}| \), we obtain

\[
\frac{1}{3} \frac{d}{dt} \|\mathbf{w}^{n+1}\|_{L^3(\mathbb{R}^3)}^2 + \|\mathbf{w}^{n+1}\|_{L^6(\mathbb{R}^3)}^2 + \frac{4}{9} \|\nabla |\mathbf{w}^{n+1}|\|_{L^2(\mathbb{R}^3)}^2 \\
= \int_{\mathbb{R}^3} \nabla q^{n+1} \cdot |\mathbf{w}^{n+1}| \mathbf{w}^{n+1} \, dx \\
- \int_{\mathbb{R}^3} \nabla \cdot (\mathbf{v} \mathbf{w}^{n+1} + \mathbf{w} \mathbf{w}^{n+1}) \cdot |\mathbf{w}^{n+1}| \mathbf{w}^{n+1} \, dx.
\]

Integrating by parts, we estimate each term on the right hand side as follows. For the pressure term, applying the divergence operator to (36), we first observe that

\[
- \Delta q^{n+1} = \nabla \cdot (\mathbf{v} \mathbf{w}^{n+1} + \mathbf{w} \mathbf{w}^{n+1}) \text{ in } \mathbb{R}^3.
\tag{37}
\]

The Calderon-Zygmund inequality applied to (37) (see [17, Lm 5.1] for a proof) implies that

\[
\|q^{n+1}\|_{L^{5/2}(\mathbb{R}^3)} \leq C\|\mathbf{w}^{n+1}\|_{L^5(\mathbb{R}^3)}(\|\mathbf{w}^n\|_{L^5(\mathbb{R}^3)} + \|\mathbf{v}\|_{L^5(\mathbb{R}^3)}).
\]
Thus,
\[ \int_{R^3} \nabla q^{n+1} \cdot |w^{n+1}|w^{n+1} \, dx = - \int_{R^3} q^{n+1}\nabla \cdot (w^{n+1}|w^{n+1}) \, dx \]
\[ = \int_{R^3} q^{n+1}\nabla \cdot w^{n+1}|w^{n+1} \, dx \]
\[ + \int_{R^3} q^{n+1}w^{n+1}\nabla w^{n+1} \cdot \frac{w^{n+1}}{|w^{n+1}|} \, dx \]
\[ \leq \|q^{n+1}\|_{L^\frac{5}{3}(R^3)}^\frac{3}{2}\|w^{n+1}\|_{L^5(R^3)}^\frac{3}{2}\|w^{n+1}\|_{L^2(R^3)}^\frac{3}{2}\|\nabla w^{n+1}\|_{L^2(R^3)}. \]

Next the interpolation inequality \[\|\cdot\|_{L^\frac{5}{3}(R^3)} \leq C\|\cdot\|_{L^2(R^3)}^{\frac{3}{2}}\|\nabla \cdot \|_{L^2(R^3)}^{\frac{3}{2}} \] leads to
\[ \|w^{n+1}\|_{L^5(R^3)} = \|w^{n+1}\|_{L^\frac{5}{3}(R^3)}^{\frac{3}{2}} \|w^{n+1}\|_{L^2(R^3)}^{\frac{3}{2}} \leq C\|w^{n+1}\|_{L^3(R^3)}^{\frac{3}{2}}\|\nabla w^{n+1}\|_{L^2(R^3)}^{\frac{3}{2}} \] (38)

From (38) and Young’s inequality, we arrive at
\[ \int_{R^3} \nabla q^{n+1} \cdot |w^{n+1}|w^{n+1} \, dx \leq C\|w^{n+1}\|_{L^3(R^3)}^{\frac{3}{2}}\|w^{n+1}\|_{L^5(R^3)}^{\frac{3}{2}}\|w^{n+1}\|_{L^2(R^3)}^{\frac{3}{2}} \]
\[ + C\|w^{n+1}\|_{L^\frac{5}{3}(R^3)}^{\frac{3}{2}}\|w^{n+1}\|_{L^5(R^3)}^{\frac{3}{2}}\|v\|_{L^5(R^3)} \]
\[ + \gamma\|w^{n+1}\|_{L^5(R^3)}^{\frac{3}{2}}\|\nabla w^{n+1}\|_{L^2(R^3)}^{2} + \delta\|\nabla w^{n+1}\|_{L^2(R^3)}^{2}. \]

The other term for the pressure term is also bounded as:
\[ \int_{R^3} q^{n+1}w^{n+1}\nabla w^{n+1} \cdot w^{n+1} \, dx \leq C\|w^{n+1}\|_{L^\frac{5}{3}(R^3)}^\frac{3}{2}\|w^{n+1}\|_{L^5(R^3)}^\frac{3}{2}\|w^{n+1}\|_{L^\frac{5}{3}(R^3)}^{\frac{3}{2}} \]
\[ + C\|w^{n+1}\|_{L^\frac{5}{3}(R^3)}^\frac{3}{2}\|w^{n+1}\|_{L^5(R^3)}^\frac{3}{2}\|w\|_{L^5(R^3)}^{\frac{3}{2}} \]
\[ + \gamma\|w^{n+1}\|_{L^\frac{5}{3}(R^3)}^\frac{3}{2}\|\nabla w^{n+1}\|_{L^2(R^3)}^{2} + \delta\|\nabla w^{n+1}\|_{L^2(R^3)}^{2}. \]

In the same way, we bound the remainder terms:
\[ \int_{R^3} \nabla \cdot (vw^{n+1}) \cdot |w^{n+1}|w^{n+1} \, dx = - \int_{R^3} vw^{n+1}\nabla w^{n+1} |w^{n+1} \, dx \]
\[ + \int_{R^3} vw^{n+1}\nabla w^{n+1} \cdot w^{n+1} \, dx \]
\[ \leq C\|w^{n+1}\|_{L^\frac{5}{3}(R^3)}^{\frac{3}{2}}\|w^{n+1}\|_{L^5(R^3)}^{\frac{3}{2}}\|v\|_{L^5(R^3)}^{\frac{3}{2}} \]
\[ + \gamma\|w^{n+1}\|_{L^\frac{5}{3}(R^3)}^{\frac{3}{2}}\|\nabla w^{n+1}\|_{L^2(R^3)}^{2} + \delta\|\nabla w^{n+1}\|_{L^2(R^3)}^{2} \]
\[ \int_{R^3} \nabla \cdot (w^{n+1}v) \cdot |w^{n+1}|w^{n+1} \, dx \leq C\|w^{n+1}\|_{L^\frac{5}{3}(R^3)}^{\frac{3}{2}}\|w^{n+1}\|_{L^5(R^3)}^{\frac{3}{2}}\|v\|_{L^5(R^3)}^{\frac{3}{2}} \]
\[ + \gamma\|w^{n+1}\|_{L^\frac{5}{3}(R^3)}^{\frac{3}{2}}\|\nabla w^{n+1}\|_{L^2(R^3)}^{2} + \delta\|\nabla w^{n+1}\|_{L^2(R^3)}^{2}. \]
and
\[ \int_{\mathbb{R}^3} \nabla \cdot (w^n w^{n+1}) \cdot |w^{n+1}| w^{n+1} \, dx \leq C \|w^{n+1}\|_{L^1(\mathbb{R}^3)}^{\frac{1}{2}} \|w^n\|_{L^1(\mathbb{R}^3)}^{\frac{1}{2}} \|w^{n+1}\|_{L^1(\mathbb{R}^3)}^{\frac{3}{2}} \\
+ \gamma \|w^{n+1}\|_{L^1(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla w^{n+1}\|_{L^2(\mathbb{R}^3)}^2 \\
+ \delta \|\nabla w^{n+1}\|_{L^2(\mathbb{R}^3)}^2. \]

Adjusting \( \gamma \) and \( \delta \) adequately, we find that
\[ \frac{1}{3} \frac{d}{dt} \|w^{n+1}\|_{L^3(\mathbb{R}^3)}^3 + \frac{1}{3} \|w^{n+1}\|_{L^3(\mathbb{R}^3)}^{\frac{2}{3}} \|\nabla w^{n+1}\|_{L^2(\mathbb{R}^3)}^2 + \frac{2}{9} \|\nabla|w^{n+1}|\|_{L^2(\mathbb{R}^3)}^2 \leq C \|w^{n+1}\|_{L^3(\mathbb{R}^3)}^{\frac{2}{3}} \|w^n\|_{L^3(\mathbb{R}^3)}^{\frac{2}{3}} \|w^{n+1}\|_{L^3(\mathbb{R}^3)}^{\frac{2}{3}} \|v\|_{L^3(\mathbb{R}^3)}^{\frac{2}{3}} \]
Integrating over \((T_i, T_{i+1})\), where \( \{T_i\}_{i=1}^M \) are to be determined later on, yields
\[ \|w^{n+1}(t)\|_{L^3(\mathbb{R}^3)}^3 \leq \|w^{n+1}(T_i)\|_{L^3(\mathbb{R}^3)}^3 \\
+ C \int_{T_i}^{T_{i+1}} \|w^{n+1}\|_{L^3(\mathbb{R}^3)}^{\frac{2}{3}} \|w^n\|_{L^3(\mathbb{R}^3)}^{\frac{1}{3}} \|w^{n+1}\|_{L^3(\mathbb{R}^3)}^{\frac{2}{3}} ds \\
+ C \int_{T_i}^{T_{i+1}} \|w^{n+1}\|_{L^3(\mathbb{R}^3)}^{\frac{2}{3}} \|w^n\|_{L^3(\mathbb{R}^3)}^{\frac{2}{3}} \|v\|_{L^3(\mathbb{R}^3)}^{\frac{2}{3}} ds \\
\leq \|w^{n+1}(T_i)\|_{L^3(\mathbb{R}^3)}^3 + \frac{1}{2} \|w^{n+1}\|_{L^3(\mathbb{R}^3)}^3 + C \|w^n\|_{L^3(\mathbb{R}^3)}^3 + C \|v\|_{L^3(\mathbb{R}^3)}^3 \]
In particular, this shows
\[ \|w^{n+1}\|_{L^3(T_i, T_{i+1}; L^3(\mathbb{R}^3))} \leq C \|w^{n+1}(T_i)\|_{L^3(\mathbb{R}^3)} \\
+ C \|w^n\|_{L^3(T_i, T_{i+1}; L^3(\mathbb{R}^3))} \|w^{n+1}\|_{L^3(T_i, T_{i+1}; L^3(\mathbb{R}^3))} \]
which implies that
\[ \int_{T_i}^{T_{i+1}} \left( \frac{1}{2} \|w^{n+1}\|_{L^2(\mathbb{R}^3)}^2 + \frac{2}{9} \|\nabla w^{n+1}\|_{L^2(\mathbb{R}^3)}^2 \right) ds \]
\[ \leq C \|w^{n+1}(T_i)\|_{L^3(\mathbb{R}^3)} \\
+ C \|w^n\|_{L^3(T_i, T_{i+1}; L^3(\mathbb{R}^3))} \|w^{n+1}\|_{L^3(T_i, T_{i+1}; L^3(\mathbb{R}^3))} \]
\[ + C \|v\|_{L^3(T_i, T_{i+1}; L^3(\mathbb{R}^3))} \|w^{n+1}\|_{L^3(T_i, T_{i+1}; L^3(\mathbb{R}^3))}. \]
We now use (38) together with (40) to get
\[
\|w^{n+1}\|_{L^5(T,T_{i+1};L^5(\mathbb{R}^3))} \leq C\left\| w^{n+1} \right\|_{L^\infty(T,T_{i+1};L^3(\mathbb{R}^3))} \left\| \nabla w^{n+1} \right\|^2_{L^2(T,T_{i+1};L^2(\mathbb{R}^3))}
\leq C\left\| w^{n+1} (T_i) \right\|_{L^3(\mathbb{R}^3)}
+ C\left\| w^{n} \right\|_{L^5(T,T_{i+1};L^5(\mathbb{R}^3))} \left\| w^{n+1} \right\|_{L^5(T,T_{i+1};L^5(\mathbb{R}^3))}
+ C\left\| w \right\|_{L^5(T,T_{i+1};L^5(\mathbb{R}^3))} \left\| w^{n+1} \right\|_{L^5(T,T_{i+1};L^5(\mathbb{R}^3))},
\]
(41)

\[C > 0\]
due to (43).

Then, on applying (42) and (46) to (39) and (41), respectively, we find
\[
\|w^{n+1}\|_{L^5(T,T_{i+1};L^5(\mathbb{R}^3))} \leq (2C)^i+1\|w_0\|_{L^3(\mathbb{R}^3)}
\]
(44)

and
\[
\|w^{n+1}\|_{L^\infty(T,T_{i+1};L^5(\mathbb{R}^3))} \leq (2C)^i+1\|w_0\|_{L^3(\mathbb{R}^3)},
\]
(45)

for all \(i \in \{0, \ldots, M-1\}\).

For \(i = 0\), let us suppose that
\[
\|w^n\|_{L^5(0,T_i;L^5(\mathbb{R}^3))} \leq \frac{1}{4C}.
\]
(46)

Then, on applying (42) and (46) to (39) and (41), respectively, we find
\[
\|w^{n+1}\|_{L^5(0,T_i;L^5(\mathbb{R}^3))} \leq 2C\|w_0\|_{L^3(\mathbb{R}^3)},
\]
and
\[
\|w^{n+1}\|_{L^\infty(0,T_i;L^5(\mathbb{R}^3))} \leq 2C\|w_0\|_{L^3(\mathbb{R}^3)},
\]
whereupon we deduce that estimates (44) and (45) are satisfied for \(i = 0\). In particular, this implies that
\[
\|w^{n+1}\|_{L^5(0,T_i;L^5(\mathbb{R}^3))} \leq \frac{1}{4C}
\]
due to (43).

In general, for \(i \geq 1\), assume that estimates (44) and (45) hold for \(i - 1\). Then if we argue as before, but now assuming that
\[
\|w^n\|_{L^5(T,T_{i+1};L^5(\mathbb{R}^3))} \leq \frac{1}{4C},
\]
(47)

we see that
\[
\|w^{n+1}\|_{L^5(T,T_{i+1};L^5(\mathbb{R}^3))} \leq 2C\|w^{n+1}(T_i)\|_{L^3(\mathbb{R}^3)}
\]
and
\[
\|w^{n+1}\|_{L^\infty(T,T_{i+1};L^5(\mathbb{R}^3))} \leq 2C\|w^{n+1}(T_i)\|_{L^3(\mathbb{R}^3)},
\]
By the induction hypothesis
\[
\|w^{n+1}(T_i)\|_{L^3(\mathbb{R}^3)} \leq (2C)^i\|w_0\|_{L^3(\mathbb{R}^3)},
\]
we arrive at estimates (44) and (45). Thus, again, from (43), we obtain
\[
\|w^{n+1}\|_{L^6(T_i,T_{i+1};L^5(\mathbb{R}^3))} \leq \frac{1}{4C}.
\]

Therefore, from (45), we have
\[
\|w^{n+1}(t)\|_{L^6(\mathbb{R}^3))} \leq (2C)^M\|w_0\|_{L^6(\mathbb{R}^3)} \quad \text{for all } t \in [0,T].
\]

We now need to apply an induction argument to prove that (48) is satisfied for all \(n \in \mathbb{N}\). But this is easy to check since (46) and (47) are true for \(n = 0\) from (43).

A standard passage to the limit completes the proof of (35).

**Remark 3.** Similar techniques to those in the proof of Theorem 5.1 were developed in [1, 17] for obtaining estimates in the \(L^3(\mathbb{R}^3)\)-norm for solutions to the Navier–Stokes equations.

The question that remains open is whether our particular solutions provided by Theorem 2.4 can develop singularities on \((T, \infty)\) for all \(T \in [1, 17]\) for obtaining estimates in the Stokes equations. Similar techniques to those in the proof of Theorem 5.1 were developed in [1, 17] for obtaining estimates in the \(L^3(\mathbb{R}^3)\)-norm for solutions to the Navier–Stokes equations.

In virtue of assumption (A), we infer that
\[
\|\nabla \times \hat{v}(t)\|_{L^2(\mathbb{R}^3)} = \|\nabla \times \tilde{v}(t)\|_{L^2(\mathbb{R}^3)}.
\]

We now need to apply an induction argument to prove that (48) is satisfied for all \(n \in \mathbb{N}\). But this is easy to check since (46) and (47) are true for \(n = 0\) from (43).

A standard passage to the limit completes the proof of (35).

**Theorem 5.2.** Let \(T > 1\). Assume that assumption (A) holds. Then the solution \(v(t)\) to (1)–(2) provided by Theorem 2.4 with \(v_0 = u_{0,\epsilon} + w_{0,\epsilon}^\lambda\) are smooth on \([0, \infty)\).

**Proof.** From (5), we find
\[
\frac{1}{2}\|v(T)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^T \|\nabla v(s)\|_{L^2(\mathbb{R}^3)}^2 \, ds = \frac{1}{2}\|v_0\|_{L^2(\mathbb{R}^3)}^2.
\]

In virtue of assumption (A), we infer that \(\|v(T)\|_{L^2(\mathbb{R}^3)} < 2\varepsilon\). Moreover, we have
\[
\frac{1}{2}\|v(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_t^T \|\nabla v(s)\|_{L^2(\mathbb{R}^3)}^2 \, ds = \frac{1}{2}\|v(T)\|_{L^2(\mathbb{R}^3)}^2
\]

for all \(t \in [\frac{T}{2}, T]\).

As in the proof of Lemma 4.2, we write
\[
\|v(t)\|_{L^2(\mathbb{R}^3)} \leq C\|v(t)\|_{L^2(\mathbb{R}^3)} \|\nabla v(t)\|_{L^2(\mathbb{R}^3)}.
\]

Taking the fourth power of both sides and integrating over \([\frac{T}{2}, T]\) yields
\[
\int_{\frac{T}{2}}^T \|v(t)\|_{L^2(\mathbb{R}^3)}^4 \, dt \leq C\|v(T)\|_{L^2(\mathbb{R}^3)}^4.
\]
Therefore,
\[ \frac{T}{2} \inf_{s \in \left[ \frac{T}{2}, T \right]} \| \mathbf{v}(s) \|_{L^3(R^3)} \leq \frac{C}{4} \| \mathbf{v}(\frac{T}{2}) \|_{L^2(R^3)} \]
and hence
\[ \inf_{s \in \left[ \frac{T}{2}, T \right]} \| \mathbf{v}(s) \|_{L^3(R^3)} \leq C 2^{-\frac{1}{4}} T^{-\frac{1}{4}} \| \mathbf{v}(\frac{T}{2}) \|_{L^2(R^3)} < C \| \mathbf{v}(\frac{T}{2}) \|_{L^2(R^3)} \].

Let us choose \( \varepsilon < \frac{1}{2C^2} \). Then we find that there exists \( t^* \in (0, T] \) such that it follows that
\[ \| \mathbf{v}(t^*) \|_{L^3(R^3)} < \frac{1}{4C}. \]

We are now allowed to apply Lemma 4.1 to obtain that \( \| \mathbf{v}(t) \|_{L^3(R^3)} \leq \frac{1}{4C} \) for all \( t \in [T, \infty) \). Moreover, we know that \( \| \mathbf{v}(t) \|_{L^3(R^3)} \leq M_t \) for all \( t \in [0, T] \) by Lemma 3.3. As a result of Theorem 3.4, the solution \( \mathbf{v}(t) \) is smooth globally in time. \( \square \)

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