Metastability of superfluidity, excitations, and fluctuations in the presence of a uniformly moving defect in a Bose-Einstein condensate

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We solve the Gross-Pitaevskii and Bogoliubov equations to investigate the metastability of superfluidity in a Bose-Einstein condensate in the presence of a uniformly moving defect potential in a two-dimensional torus system. Through calculating the total energy and momentum as functions of the velocity of the moving defect, we find evidence for a swallowtail loop, which indicates the existence of metastable states and hysteresis. We also find that the first excited energy (energy gap) in the finite-sized torus vanishes at the critical velocity (the edge of the swallowtail), that it obeys one-fourth power-law scaling, and that the dynamical fluctuation of the density (amplitude of the order parameter) is strongly enhanced near the critical velocity. We confirm the validity of our results near the critical velocity by calculating the quantum depletion.

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I. INTRODUCTION

The breakdown of superfluidity is a long-standing but still central issue regarding quantum fluids [1–6]. It has been observed in experiments in which cold atomic gases have been trapped in simply connected geometries [7–12]. Recently, Bose-Einstein condensates (BECs) confined to multiply connected regions have been realized experimentally [13–24]. In these experiments, various properties of superfluidity have been exhibited, including critical velocity, vortex nucleation, decay of persistent current, phase slip, and hysteresis. The superfluid realized in multiply connected systems is topologically protected by the quantization of the circulation [2, 25]. Multiply connected systems are thus suitable for the exploration of the fundamental properties of superfluids.

Theoretical investigations have been carried out into the breakdown of superflow stability in cold atoms [26–45] using the Gross-Pitaevskii (GP) equation [46, 47]. For example, Frisch et al. [26] showed that, in the presence of a defect potential, a vortex pair is nucleated when the velocity is above a critical value. In a closely related work, nucleation of solitons was studied in one-dimensional systems in which the critical velocity was exceeded [28], and the results aid in our understanding of the relationship between the nucleation of a topological defect and the breakdown of superfluidity. From the viewpoint of nonlinear physics, the breakdown of superfluidity can be interpreted as a saddle-node bifurcation, as suggested by Pomeau, Rica, and Josserand [27–30], Huepe and Brachet [31–33], and Pham and Brachet [34–35].

In earlier theoretical studies, however, little was known about the excitation spectrum and fluctuation properties in multiply connected superfluids near and below the critical velocity in the presence of a defect potential. Furthermore, the quantum depletion (the number fraction of particles not in condensates) was not calculated at the time at which vortex nucleation occurred; it is thus not clear if the earlier results from the GP equations are valid near the critical velocity. In this paper, we will address these issues by direct numerical calculations based on the GP and Bogoliubov [48, 49] equations. Furthermore, as a basis for this, we also consider the metastability of superfluidity by calculating the energy diagram, i.e., the velocity dependence of the total energy in the moving frame (see, for example, Refs. [43, 50, 51]) and by performing a linear stability analysis. Some of our results were reported in Ref. [52]. In this full paper, we present a more extensive analysis of the metastability in the present system and results on quantum depletion near the critical velocity; material not reported in Ref. [52] includes energy diagrams, the existence or absence of a ghost vortex pair, and a swallowtail structure [53–55].

This paper is organized as follows: In Sec. II, we introduce our model. In Sec. III, we show the results of numerical calculations of the GP and Bogoliubov equations. In Sec. IIIA, the velocity dependence of the total energy and the total momentum is presented, and we discuss the relation between our results and a swallowtail loop in an optical lattice system [50]. In Sec. IIIB, we show the density and the phase profile below and above the critical velocity and discuss the appearance of a ghost vortex pair. In Secs. IIIC and IIDD, we demonstrate the properties of the excitation and the fluctuation. In particular, we discuss the scaling law of the first excitation energy near the critical velocity, and we consider its physical meaning. In Sec. IIIE, we show the results for quantum depletion and discuss the validity of the GP and the Bogoliubov approximation in this system. Finally, we summarize our results in Sec. IV.
II. MODEL

A. GP Equation in a Laboratory Frame

We consider a system in which \( N \) bosons of mass \( m \) are confined in a two-dimensional torus \([−L/2,+L/2] \times [−L/2,+L/2] \). In the mean-field approximation, the physical properties of the system can be described by a complex order parameter (condensate wave function) \( \Psi_L(r_L, t) \), where \( r_L \) denotes the coordinate in the laboratory frame. The subscript \( L \) denotes the variables in the laboratory frame. The condensate wave function obeys the GP equation [46, 47]:

\[
\frac{i\hbar}{\partial t} \Psi_L(r_L, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi_L(r_L, t) + U(r_L + vt) \Psi_L(r_L, t) + g|\Psi_L(r_L, t)|^2 \Psi_L(r_L, t),
\]

where \( U(r_L + vt) \) represents a defect potential with velocity \(-v\) and \( g(>0)\) is the strength of the interaction. We use the Gaussian potential:

\[
U(r_L) \equiv U_0 \exp \left[-\left(\frac{r_L}{d}\right)^2\right],
\]

where \( U_0(>0) \) and \( d \) are the strength and the width of the potential, respectively. Throughout this paper, the velocity of the potential is in the direction of positive \( x \) (\( v \equiv ve_x \), where \( e_x \) is a unit vector in the direction of positive \( x \)). Periodic boundary conditions are imposed on \( \Psi_L(r_L, t) \), because there is a requirement that the condensate wave function should be single valued:

\[
\Psi_L(r_L + Le_x, t) = \Psi_L(r_L, t),
\]

\[
\Psi_L(r_L + Le_y, t) = \Psi_L(r_L, t),
\]

where \( e_y \) is a unit vector in the direction of positive \( y \). From this boundary condition, we can define the winding number:

\[
W \equiv \frac{1}{2\pi} \int_{-L/2}^{+L/2} dx_L \frac{\partial}{\partial x_L} \varphi_L(r_L, t),
\]

where \( \varphi_L(r_L, t) \) is the phase of the condensate wave function.

B. GP Equation in a Moving Frame

The GP equation in the laboratory frame [11] has an explicit dependence on time. In order to remove this, we perform a coordinate transformation from the laboratory frame to a moving frame [3, 6], as follows:

\[
\mathbf{r} \equiv r_L + vt,
\]

\[
\Psi(r, t) \equiv \exp \left(\frac{i}{\hbar} \frac{mv^2}{2} t + \frac{i}{\hbar} m \mathbf{v} \cdot \mathbf{r} \right) \Psi_L(r_L, t).
\]

Using Eqs. (6) and (7), the GP equation in the moving frame is given by

\[
\frac{i\hbar}{\partial t} \Psi(r, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(r, t) + U(r) \Psi(r, t) + g|\Psi(r, t)|^2 \Psi(r, t).
\]

As a result of transformation (17), the periodic boundary condition becomes twisted [59]:

\[
\Psi(r + Le_x, t) = e^{-i\mu t/\hbar} \Psi(r, t),
\]

\[
\Psi(r + Le_y, t) = \Psi(r, t).
\]

The stationary solution of the GP equation (8) is given by \( \Psi(r, t) = e^{-i\mu t/\hbar} \Psi(r) \), where \( \mu \) is the chemical potential. Substituting this relation into Eq. (8), we obtain the time-independent GP equation:

\[
-\frac{\hbar^2}{2m} \nabla^2 \Psi(r) + U(r) \Psi(r) + g|\Psi(r)|^2 \Psi(r) = \mu \Psi(r).
\]

The chemical potential \( \mu \) is determined by the condition

\[
N = \int dr |\Psi(r)|^2.
\]

C. Bogoliubov Equation

The Bogoliubov equation [48, 49] can be derived by linearizing the GP equation around the stationary solution \( \Psi(r) \). Substituting

\[
\Psi(r, t) \equiv e^{-i\mu t/\hbar} \left[ \Psi(r) + u_i(r)e^{-i\epsilon_i t/\hbar} - v^*_i(r)e^{i\epsilon_i t/\hbar} \right]
\]

into the time-dependent GP equation (8), and neglecting the higher-order terms of \( u_i(r) \) and \( v_i(r) \), we obtain the Bogoliubov equation

\[
\left[ \frac{\mathcal{L}}{g|\Psi(r)|^2} - g|\Psi(r)|^2 \right] \begin{bmatrix} u_i(r) \\ v_i(r) \end{bmatrix} = \epsilon_i \begin{bmatrix} u_i(r) \\ v_i(r) \end{bmatrix},
\]

\[
\mathcal{L} \equiv -\frac{\hbar^2}{2m} \nabla^2 + U(r) - \mu + 2g|\Psi(r)|^2,
\]

where \( u_i(r) \) and \( v_i(r) \) are the wave functions of the \( i \)-th excited state with an excitation energy \( \epsilon_i \). The boundary conditions for \( u_i(r) \) and \( v_i(r) \) are determined by the condition in which Eq. (13) satisfies the twisted periodic boundary conditions (10) and (11):

\[
u_i(r + Le_x) = e^{+imvL/\hbar} u_i(r),
\]

\[
v_i(r + Le_x) = e^{-imvL/\hbar} v_i(r),
\]

\[
u_i(r + Le_y) = u_i(r),
\]

\[
v_i(r + Le_y) = v_i(r).
\]
The wave functions for the excited states satisfy the following orthonormal conditions:

\[ \int dr [u_i^*(r)v_j(r) - v_i^*(r)u_j(r)] = \delta_{ij}, \quad (20) \]

\[ \int dr [u_i(r)v_j(r) - v_i(r)u_j(r)] = 0. \quad (21) \]

Throughout this paper, length, energy, and time are normalized by the healing length \( \xi \equiv \hbar/\sqrt{mg_n0} \), \( \epsilon_0 \equiv gn_0 \), and \( \tau \equiv \hbar/\epsilon_0 \), respectively, where \( n_0 \equiv N/S \) (\( S \equiv L^2 \) is the area of the system) is the mean particle density. The velocity is normalized by the sound velocity \( v_s \equiv \sqrt{gn_0/m} \) or \( v_0 \equiv 2\pi\hbar/(mL) \).

Numerically solving the GP and Bogoliubov equations yields the condensate wave function, excitation spectra, and wave functions for the excited states. The methods we used for the numerical calculations are summarized in Appendix A.

### III. RESULTS

#### A. Energy and Momentum

First, we show the energy diagram, which represents the velocity dependence of the total energy in the moving frame. Using the energy diagram, we deduce the superfluid fraction (the nonclassical rotational inertia) and the critical velocity for a defect moving with constant velocity \( v \), and we discuss the metastability of the superflow states when \( v \) changes adiabatically.

The total energy in the moving frame is defined by

\[ E = \int dr \left[ \frac{\hbar^2}{2m} \nabla \Psi(r)^2 + U(r)\Psi(r)^2 + \frac{g}{2} |\Psi(r)|^4 \right], \quad (22) \]

which is shown in Fig. 1. We can see the periodicity of the total energy with respect to the velocity \( v \). The periodicity of the energy branch comes from that of the boundary condition \( \Psi \).

The lowest energy state under a given \( v \) is the ground state, and the other states are metastable. The metastability of each state is confirmed by calculating the excitation spectra of the Bogoliubov equation around each stationary state of the GP equation. The energy branches shown in Fig. 1 are almost parabolic, except in the vicinity of the termination points, which correspond to the critical velocity \( v_c \). Each branch can be specified by the winding number \( W \). For example, the red branch denoted by open circles (○), which continuously connects the ground state at \( v = 0 \), has \( W = 0 \), and the green branch, denoted by open triangles (△), has \( W = -1 \).

The winding number can serve as an adiabatic invariant under an adiabatic change of \( v \). Suppose that the system is in the ground state (\( W = 0 \)), and \( v \) increases adiabatically from 0 to \( v_c \). Then we expect that the system will evolve along the red branch, and the winding number will remain unchanged. We note that in order to see the dynamics under a change of \( v \), ring trap experiments \([19, 20]\) used the ground state of the noncirculating state as the initial condition.

For the GP equation, the adiabatic condition is determined by the Bogoliubov spectrum \([55, 60]\); in the present case, the dynamics can be regarded as adiabatic when there is little change in \( v \) within the time interval \( h/\Delta \) (where \( \Delta \) denotes the lowest excitation energy of the Bogoliubov spectrum). As we will show in III C, \( \Delta \) vanishes at \( v = v_c \). When \( v \) approaches \( v_c \), therefore, the adiabaticity condition is violated at a certain \( v \), and a transition from \( W = 0 \) to a circulating state (\( W = -1 \)) occurs. Here, this transition corresponds to a phase slip \([61, 62]\).

The flow properties of each branch can be seen in the velocity dependence of the total momentum shown in Figs. 2(a) and (b), where the total momentum of the moving frame and laboratory frame are, respectively, given by

\[ P = -\frac{i\hbar}{2} \int dr \left[ \Psi^*(r) \frac{\partial}{\partial x} \Psi(r) - \Psi(r) \frac{\partial}{\partial x} \Psi^*(r) \right], \quad (23) \]

\[ P_{\text{Lab}} = P - Nmv. \quad (24) \]

By symmetry, we see that the \( y \)-component of the total momentum is zero, and we thus consider only the \( x \)-component. For the red branch, the total momentum of the moving frame has a linear dependence for small \( v \); see Fig. 2(a). We find that the superfluid fraction \([63]\)

\[ \frac{\rho_s}{\rho} \equiv \frac{1}{Nm} \left. \frac{\partial P(v)}{\partial v} \right|_{v=0} \quad (25) \]

is almost unity, and the total momentum in the laboratory frame is almost zero in the red branch, except in the
vicinity of the critical velocity; see Fig. 2(b). This implies that the BEC is not dragged by the moving defect potential.

Figure 3 shows a blow-up of the region near the critical velocity for \( W = 0 \). We note that the differential effective mass \((\kappa_e^{-1} = \partial P(v)/\partial v < 0)\) becomes negative near the critical velocity; this implies that the mass flow of the condensate in the moving frame decreases, although the velocity of the moving defect increases. Negative effective-mass states have been found in the GP equation for a BEC in an optical lattice near the critical velocity (see Fig. 7 in Ref. [64]). However, negative effective-mass states in an optical lattice are subject to dynamical instability (DI) [54, 56, 57, 64], while they are metastable in our case. This arises from differences in the boundary conditions. In an optical lattice, DI occurs when the crystal momentum of the excitation is \( \pi/\lambda \) and the period of the lattice is \( \lambda \). Our system can be regarded as having a periodic potential (with period \( L \)) and the period of the lattice is \( \lambda \). We thus believe that the findings presented in this section are generic for a superfluid torus near the critical velocity; see Fig. 2(b). This implies that the BEC is not dragged by the moving defect potential.

We next consider the spatial profiles of the density and the phase of the condensate wave function for a strong potential \( U_0 = 10\epsilon_0 \) and a weak potential \( U_0 = \epsilon_0 \).

Figures 4(a) and (b) show the spatial profiles of the density and phase, respectively, of the condensate wave function for \( L = 48\xi, U_0 = 10\epsilon_0, d = 2.5\xi \), and \( v = 0.43034v_\text{c} \approx 0.9999907v_\text{c} \). We find a vortex pair in the low-density region; this is called a ghost vortex.
FIG. 4. (Color online) (a) Density profile, and (b) phase profile, for $L = 48\xi$, $U_0 = 10\epsilon_0$, $d = 2.5\xi$, and $v = 0.43034v_c \approx 0.999997v_c$. White and black circles represent the position of the GVP.

pair (GVP) [66–68]. This is a (meta)stable stationary solution, because there is no anomaly in the excitation around this solution (see Sec. III C). The GVP is regarded as being pinned to the defect potential, and thus it could be expected to become depinned above the critical velocity. In fact, we calculated the real-time dynamics above the critical velocity and found that vortex nucleation occurred as shown in Figs. 5(a) and (b). Here, the real-time dynamics were produced by the Crank-Nicholson scheme.

Ghost vortices were first reported in Refs. [66, 67], which were theoretical studies of vortex invasions of rotating condensates. A GVP was found in a numerical study of condensates in the presence of an oscillating defect [68]. Our result yields the first example of a GVP accompanying a defect moving with a constant subcritical velocity.

However, a GVP does not appear in the presence of a weak potential; this is shown in Figs. 6(a) and (b), where we present the density and phase profiles for $L = 48\xi$, $U_0 = \epsilon_0$, $d = 2.5\xi$, and $v = 0.46085v_0 \approx 0.9999984v_c$. Typically, investigations are for a velocity in the range of $10^{-6} \lesssim |(v_c - v)/v_c| \leq 1$. The dynamics above the critical velocity in the weak-potential case are shown in Figs. 7(a) and (b). These figures clearly show that the vortex nucleation occurs even in the absence of a GVP in the initial conditions.

C. Excitations

We next present the results for excitations. Figure 8 shows energy spectra of the Bogoliubov equation as a function of $v$. In systems of finite size, the Bogoliubov spectra are discretized. The excitation energy is always positive, and thus the solutions shown in the figure are stable.

We now focus on the first excited energy, which we call an energy gap (denoted by $\Delta$). Figures 9(a) and 10(a) show the energy gap as a function of the velocity.

We first notice a linear decrease in the region in which the velocity is small. This reflects the energy gap in uniform systems, and it is given by $\Delta_{\text{uni}}/\epsilon_0 = 2\pi/(L/\xi) \left[ -v/v_c + \sqrt{\pi^2/(L/\xi)^2 + 1} \right]$. In fact, the solid black line in Fig. 9(a), which represents $\Delta_{\text{uni}}$, almost overlaps the numerical data for $U_0 \neq 0$ when the velocity is small, except near $v = 0$. The deviation between $\Delta_{\text{uni}}$ and the numerical data near $v = 0$ is due to the splitting
of levels, since the first excited state in a uniform system is fourfold degenerate.

We also notice a sharp decrease in the energy gap near the critical velocity. To characterize this behavior, we used the function \( \Delta = \Delta_0 \left( \frac{v_c - v}{v_c} \right) \), where \( \Delta_0, v_c, \) and \( c \) are parameters adjusted to fit the data. We used four data points near the critical velocity. The detailed data are shown in Table 1 of Ref. [52]. Figures 9(b) and 10(b) show the results. We determined that the scaling for the energy gap \( \Delta = \Delta_0 \left( \frac{v_c - v}{v_c} \right)^{1/4} \) near the critical velocity.

From this scaling law, we can see that this is adiabatic, since there should be little change in \( v \) over the time scale \( \left( \frac{\hbar}{\Delta_0} \right) \left( \frac{v_c - v}{v_c} \right)^{-1/4} \). The scaling law for \( \Delta \) is reminiscent of the Hamiltonian saddle node (HSN) bifurcation [33, 35], where a stable stationary solution (elliptic fixed point) and an unstable stationary solution (hyperbolic fixed point) merge at the critical velocity (bifurcation point). The normal form of the HSN bifurcation is given by

\[
\frac{d^2}{dt^2} u(t) = \mu - \beta u(t)^2,
\]

where \( u(t) \) is the amplitude of the critical mode, \( \mu \propto 1 - v/v_c \) denotes a bifurcation parameter, and \( \beta \) is a constant related to the system parameters. If we assume that the system exhibits HSN bifurcation, we can explain the one-fourth power law because the eigenvalue of the linearized version of Eq. (26) around the stable fixed point is proportional to \( \mu^{1/4} \). Physically, the normal form (26) shows that the breakdown of the metastable state is caused by the disappearance of the energy barrier, as described in Fig. 11. We have consistently observed that the metastability and adiabaticity of the dynamics disappear simultaneously at the critical velocity.

We note that a similar mechanism for the destabilization of the metastable state has been found in attractive BECs in harmonic traps [69]. The attractive BEC collapses when the number of atoms in the harmonic trap exceeds a critical value. Near the collapse point, the monopole mode, which is a low-lying excitation in the attractive BEC, obeys a one-fourth power law, as found by variational calculations [69] and numerical calculations [70]. In Ref. [70], it was reported that the HSN bifurcation also occurs in this system. We anticipate that the origin of the scaling law is the same as that for our system.

However, there is some evidence for a saddle-node bifurcation near the critical velocity of a superfluid in the presence of an obstacle [33, 35, 71, 72]. The saddle-node bifurcation yields a time scale that is proportional to the
we obtain square root of the distance from the bifurcation points.

In future work, we will seek to determine under what conditions the bifurcation near the critical velocity of a superfluid is a saddle node or a Hamiltonian saddle node. Another area of future work is to derive the normal form conditions the bifurcation near the critical velocity of a superfluid is a saddle node or a Hamiltonian saddle node.

In the previous subsection, we discussed only the excitation spectra. Here, we will show the results for excited states on the wave functions.

Using the wave functions of the excited states, we can obtain the properties of the fluctuations. Substituting Eq. (13) into \( n(r, t) = |\Psi(r, t)|^2 \) and \( \Psi(r, t)/|\Psi(r, t)| \), and neglecting the higher-order terms of \( u_i(r) \) and \( v_i(r) \), we obtain

\[
n(r, t) = |\Psi(r)|^2 + 2\text{Re} \left[ \delta n_i(r) e^{-i\epsilon_i t/\hbar} \right],
\]

and

\[
\frac{\Psi(r, t)}{|\Psi(r, t)|} = e^{-i\epsilon t/\hbar} e^{i\phi(r)}
\]

\[
\times \left\{ 1 + \frac{i}{|\Psi(r)|^2} \text{Im} \left[ \delta P_i(r) e^{-i\epsilon t/\hbar} \right] \right\},
\]

where the local density fluctuation \( \delta n_i(r) \) and the local phase fluctuation \( \delta P_i(r) \) for mode \( i \) are defined by

\[
\delta n_i(r) = \Psi^*(r) u_i(r) - \Psi(r) v_i(r),
\]

\[
\delta P_i(r) = \Psi^*(r) u_i(r) + \Psi(r) v_i(r).
\]

In Fig. 12, we show the spatial profiles of the density and phase fluctuations for the first excited state. We can see that both the density and phase fluctuations are enhanced when the velocity of the moving defect decreases. The enhancement for the density fluctuation is a few orders of magnitude greater than that for the phase fluctuation. We plot the energy dependence of the density fluctuations in Fig. 13. We note that the spectral intensity shifts to lower energy and is enhanced when the velocity approaches the critical value. Similar behavior has been seen in one-dimensional systems, and was related to soliton nucleation.

The dynamics near the critical velocity are often discussed in terms of the energy landscape, which is schematically depicted in Fig. 11. Near the critical velocity, the metastable and unstable states approach each other in the configuration space, and thus the landscape around the local minimum becomes flat along a particular direction in the coordinate space. This results in the vanishing of the energy gap, as shown in Sec. III C. The definition of the abscissa in the energy landscape is not clear in the general case. In a few cases, however, it was found to be associated with collective coordinates. In the present case, we can identify the abscissa...
the density and phase fluctuations are slightly different
scribed in Sec. III B. In this case, there is no GVP near the critical velocity, as de-
of a GVP on the fluctuations. For the weak potential
ation, we can see that the motion of \( \Psi \) from the metastable
state to the unstable state is accompanied by a density
or the amplitude of the Bogoliubov eigenstate that has
the lowest excitation energy. From the results in this sec-
tion, we can see that the motion of \( \Psi \) from the metastable
state to the unstable state is accompanied by a density
fluctuation near the defect potential.

We now consider the effect of the existence or absence
of a GVP on the fluctuations. For the weak potential
case, there is no GVP near the critical velocity; as de-
scribed in Sec. III B. In this case, the spatial patterns of
the density and phase fluctuations are slightly different
(data not depicted here). However, enhancement of the
density fluctuation also occurs in the absence of a GVP.

E. Quantum Depletion

Finally, we will present the results for quantum deple-
tion, which represents the number of atoms not in the
condensate. The expression for the quantum depletion is

\[
\frac{N_{\text{dep}}}{N} = \frac{1}{N} \sum_i \int dr |v_i(r)|^2.
\]

The condition for the GP and Bogoliubov approxima-
tions to be valid is \( N_{\text{dep}}/N \ll 1 \). Therefore, we can
check the self-consistency of these approximations by ex-
amining the quantum depletion.

In previous works, quantum depletion in nonuniform
systems was calculated in the limiting case; these stud-
ies used perturbative approaches and many-body calcu-
lations for the ground state \([78, 79]\). Our results are
the first for quantum depletion near the critical velocity
above which vortices nucleate in the vicinity of the defect
potential and metastable states in nonuniform systems.

We show the velocity dependence of the quantum de-
pletion in Fig. 14 [80]. If the system is uniform (\( U = 0 \))
and does not exhibit spontaneous translational symme-
try breaking, the quantum depletion does not depend on
the velocity because the velocity dependence of \( v_i(r) \)
is only present in the plane wave component \( e^{-i m \cdot r \cdot \hbar} \).
Our results show that the quantum depletion depends
on the velocity, due to the presence of the defect poten-
tial. When the velocity is small, the quantum depletion
is almost the same as that for uniform systems. This
behavior is consistent with that for the energy gap when
the velocity is small. Near the critical velocity, we find
that the quantum depletion increases steeply. The origin
of this increase is enhancement of the low-energy density
of states and density fluctuations. Within the range of \( v \)
used in our calculations, we do not find power-law scaling
of the quantum depletion, in contrast to that found for
the energy gap; see Fig. 14 (c), and note that the curves
are not straight lines in the log-log plot near the critical
velocity. There may be a narrow scaling region for the
quantum depletion in this system.

Although the quantum depletion increases near the
critical velocity, the value of \( N_{\text{dep}}/N \) is still much smaller
than unity for \( \delta \equiv 1/\sqrt{\eta \alpha \xi^2} = 0.1 \) and \( (\delta c - v)/v_c \gtrsim 10^{-6} \); see Fig. 14 (c). Here, \( \delta \) is the ratio between
the healing length and the mean particle distance. A small \( \delta \)
corresponds to the weakly interacting case. This shows
that the GP and Bogoliubov approximations are valid
even near the critical velocity, for sufficiently small \( \delta \).
We can easily obtain the quantum depletion for other
values of \( \delta \), because the \( \delta \) dependence of \( v_i(r) \) is given
by \( v_i(r; \delta) = \delta \times v_i(r; \delta = 1) \). Figure 14(a) shows the \( \delta \) de-
pendence of the quantum depletion. These results show
that the quantum depletion is much smaller than unity
near the critical velocity for \( \delta \lesssim 0.5 \), and for \( \delta > 1 \), the

![Image](image.jpg)
FIG. 12. (Color online) (a), (b), and (c): Spatial profiles of the density fluctuation for $L = 48\xi, U_0 = 5\epsilon_0$, and $d = 2.5\xi$ by the first excited state for $v = 0.1v_s, 0.42v_s, \text{and } v = 0.42655v_s$, respectively. (d), (e), and (f): Spatial profiles of the phase fluctuation for $L = 48\xi, U_0 = 5\epsilon_0$, and $d = 2.5\xi$, for the first excited state for $v = 0.1v_s, v = 0.42v_s, \text{and } v = 0.42655v_s$, respectively. The white circles represent the position of the GVP. Here, we set $1/\sqrt{n_0\xi^2} = 0.1$.

FIG. 13. (Color online) Energy and $y$-dependence of the density fluctuations for $L = 48\xi, U_0 = 5\epsilon_0$, and $d = 2.5\xi$, (a) $v = 0.1v_s$, (b) $v = 0.42v_s$, and (c) $v = 0.42655v_s$, at $x = 0$. Here, we set $1/\sqrt{n_0\xi^2} = 0.1$.

Bogoliubov approximation breaks down near the critical velocity. In conclusion, our results are valid close to the critical velocity if $\delta$ is not very large.

**IV. SUMMARY**

In summary, we investigated the metastability, excitations, and fluctuations of the BEC with a uniformly moving defect in a two-dimensional torus system. We first calculated the velocity dependence of the total energy and the total momentum. Our results suggest that the system has a swallowtail loop and a negative effective-mass region near the edge of the swallowtail, as is the case for optical lattice systems. In contrast to optical lattice systems, the negative effective-mass states are metastable. This difference arises from the boundary conditions. Our boundary condition prohibits the instability when the crystal momentum is $\pi/L$, and DI occurs in optical lattice systems. We also found the existence
of GVPs in stationary states in the presence of a strong defect.

Using the results of the GP equation, we solved the Bogoliubov equation and obtained the excitation spectra. We determined that near the critical velocity, the scaling of the energy gap followed a one-fourth power law. This implies an algebraic divergence of the characteristic time scale toward the critical velocity and a violation of the adiabaticity condition at the critical velocity. From the viewpoint of nonlinear physics, this scaling law can be regarded as an HSN bifurcation, which describes the disappearance of a metastable state protected by an energy barrier. We also point out that the same scaling law appears in the attractive BEC in a harmonic trap near the collapse point.

From wave functions of the excited states \( u_i(r) \) and \( v_i(r) \), we obtained the fluctuation properties and showed that the density (amplitude of the order parameter) fluctuations are enhanced near the critical velocity.

We also calculated the quantum depletion and found that it increased near the critical velocity. The validity of the GP and the Bogoliubov approximations were discussed on the basis of these results.

In future work, we will attempt to determine why the Hamiltonian saddle-node bifurcation appears at the critical velocity and to derive the normal form from the GP and Bogoliubov equations.

A possible further extension of the present work is to study the effects of quantum fluctuations on the metastability of superfluidity. In particular, these effects are crucial for cases that are not weakly interacting and that are near the critical velocity. In fact, a nonzero drag force acting on a defect below the critical velocity in a one-dimensional system has been reported in Ref. [81]. This phenomenon is due to quantum fluctuations. It is important to understand the effects of quantum fluctuations on vortex nucleation.

Another extension of the present work is to study multicomponent systems, such as spinor BECs [82, 83], and to reveal the effects of the internal degrees of freedom on the metastability of the superfluidity.

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Appendix A: Methods of Numerical Calculations

In this appendix, we explain the methods used for the numerical calculations. Similar methods were used in our previous work \[84].

In order to obtain the solutions of the time-independent GP equation \[11\], we used imaginary time propagation. The imaginary-time GP equation is given by

\[
-\hbar \frac{\partial}{\partial t} \Psi(r, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(r, t) + U(r) \Psi(r) - \mu(t) \Psi(r, t) + g|\Psi(r, t)|^2 \Psi(r, t),
\]

where \( \mu(t) \) is the time-dependent chemical potential, whose time dependence is determined by the total number of particles \[12\]. The time-independent solution of Eq. \[11\] coincides with the original GP equation \[11\].

From the twisted periodic boundary conditions \[9\] and \[10\], we can expand the condensate wave function as a series of plane waves, as follows:

\[
\Psi(r, t) = \sum_G \langle C_q + G|t \rangle e^{i(q + G) \cdot r},
\]

where \( C_q + G|t \rangle \) is an expansion coefficient, \( q \equiv mv/h \), and \( n_1 \) and \( n_2 \) \( \in \mathbb{Z} \). Substituting Eq. \[12\] into Eq. \[11\] and using the orthonormal condition \( \int dre^{i(q + G') \cdot r} = S\delta_{G,G'} \), we obtain the imaginary time GP equation for \( C_q + G|t \rangle \):

\[
-\frac{\hbar^2}{2m} (q + G)^2 - \mu \langle C_q + G|t \rangle
+ \frac{1}{S} \sum_{G'} \tilde{U}(G - G') \langle C_q + G'|t \rangle C_q + G|t \rangle

+ gn_0 \sum_{G, \Delta G} \langle C_{q+G'+\Delta G}|t \rangle \langle C_{q+G}|t \rangle C_{q+G'+\Delta G}|t \rangle,
\]

where \( \tilde{U}(k) \) is the Fourier transformation of the external potential:

\[
\tilde{U}(k) \equiv \int dr e^{-ik \cdot r} U(r).
\]

The total particle number condition for \( C_q + G|t \rangle \) is given by

\[
1 = \sum_G |\langle C_q + G|t \rangle|^2.
\]

From the boundary conditions \[10\], \[17\], \[18\], and \[19\], the wave functions \( u_i(r) \) and \( v_i(r) \) can be expanded as a series of plane waves, as follows:

\[
u_i(r) = \frac{1}{\sqrt{S}} \sum_G B_{q+G,i} e^{-i(q + G) \cdot r},
\]

\[
u_i(r) = \frac{1}{\sqrt{S}} \sum_G A_{q+G,i} e^{i(q + G) \cdot r},
\]

where \( A_{q+G,i} \) and \( B_{q+G,i} \) are the expansion coefficients of mode \( i \). The normalization condition for the wave functions of the excited states becomes

\[
\sum_G (|A_{q+G,i}|^2 - |B_{q+G,i}|^2) = 1.
\]

Substituting Eqs. \[12\], \[17\], and \[18\] into Eq. \[14\], we obtain the Bogoliubov equation for the expansion coefficients:

\[
D_G A_{q+G,i} + \frac{1}{S} \sum_{G'} \tilde{U}(G - G') A_{q+G',i}
+ 2g n_0 \sum_{G'} S_{G,G'} A_{q+G',i} B_{q+G',i}

- g n_0 \sum_{G'} W_{G,G'} B_{q+G',i} = \epsilon_i A_{q+G,i},
\]

\[
- D_G B_{q+G,i} - \frac{1}{S} \sum_{G'} \tilde{U}^*(G - G') B_{q+G',i}

- 2g n_0 \sum_{G'} S_{G,G'} B_{q+G',i} + g n_0 \sum_{G'} W_{G,G'} A_{q+G,i} = \epsilon_i B_{q+G,i},
\]

where we have introduced the following variables in order to simplify the notation:

\[
D_G \equiv \frac{\hbar^2}{2m} (q + G)^2 - \mu,
\]

\[
S_{G,G'} = \sum_{G''} \langle C_{q+G'' + G'} - G C_{q+G''}|t \rangle,
\]

\[
W_{G,G'} = \sum_{G''} \langle C_{q+G'' + G'} - G C_{q+G''}|t \rangle.
\]

Numerically diagonalizing Eqs. \[10\] and \[11\], we obtain the excited states and the wave functions of the excited states.

We introduced the cutoff \( G_c \) to calculate the summation of \( G \). We used the cutoff wave number (the number of bases) \( G_c \xi = 7.82(4973), 10.1(8227), \) and \( 11.4(10557) \) for \( L = 32\xi \) and \( 6.71(8227), 7.59(10557), \) and \( 8.38(12893) \) for \( L = 48\xi \). We checked that the cutoff dependence of the present results is negligibly small, other than for the calculation of the quantum depletion (see Ref. \[80\]).
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The upper portion of the swallowtail corresponds to the saddle point of the energy landscape. According to Ref. [62], the solution of the saddle point contains the vortices. In fact, the unstable solutions containing the vortices were reported in similar systems [33].

We use extrapolation to evaluate the quantum depletion. The method is as follows: In uniform two-dimensional systems, the cutoff dependence of the quantum depletion is given by $O(G_c^{-2})$, where $G_c$ is the cutoff wavenumber. Therefore, we assume the same cutoff dependence of the quantum depletion even in the presence of the defect potential. By adjusting $a$ and $b$, we can fit the function $N_{dep}/N = a - b/(G_c \xi)^2$ to three data points, and in this way, we can calculate the quantum depletion. The error bars for the fitting are not shown in Figs. 14 and 15 because these are sufficiently small in the range of our graph.