SOME INEQUALITIES ON \( h \)-CONVEX FUNCTIONS

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Abstract. In this paper, we state some characterizations of \( h \)-convex function is defined on a convex set in a linear space. By doing so, we extend the Jensen-Mercer inequality for \( h \)-convex function. We will also define \( h \)-convex function for operators on a Hilbert space and present the operator version of the Jensen-Mercer inequality. Lastly, we propound the complementary inequality of Jensen’s inequality for \( h \)-convex functions.

1. Introduction

Assume that \( I \) is an interval in \( \mathbb{R} \). Let us recall definitions of some special classes of functions.

We say that \([6]\) \( f : I \to \mathbb{R} \) is a Godunova-Levin function, or that \( f \) belongs to the class \( Q(I) \) if \( f \) is non-negative and for all \( x, y \in I \) and \( t \in (0, 1) \) we have

\[
f(tx + (1 - t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1 - t}.
\]

For \( s \in (0, 1] \), a function \( f : [0, \infty) \to [0, \infty) \) is said to be \( s \)-convex function, or that \( f \) belongs to the class \( K^2_s \), if

\[
f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y)
\]

for every \( x, y \in [0, \infty) \) and \( t \in [0, 1] \) (see [1]). Also, we say that \( f : I \to [0, \infty) \) is a \( P \)-function [4], or that \( f \) belongs to the class \( P(I) \), if for all \( x, y \in I \) and \( t \in [0, 1] \) we have

\[
f(tx + (1 - t)y) \leq f(x) + f(y).
\]

Throughout this paper, suppose that \( I \) and \( J \) are intervals in \( \mathbb{R} \), \( (0, 1) \subseteq J \) and functions \( h \) and \( f \) are real non-negative functions defined on \( J \) and \( I \), respectively.

In [10], Varošanec defined the \( h \)-convex function as follows:

Let \( h : J \subseteq \mathbb{R} \to \mathbb{R} \) be a non-negative function, \( h \neq 0 \). We say that \( f : I \to \mathbb{R} \)

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is a \( h \)-convex function, or that \( f \) belongs to the class \( SX(h, I) \), if \( f \) is non-negative and for all \( x, y \in I \), \( t \in (0, 1) \) we have
\[
 f(tx + (1 - t)y) \leq h(t)f(x) + h(1 - t)f(y) .
\]
If inequality (1.1) is reversed, then \( f \) is said to be \( h \)-concave, that is \( f \in SV(h, I) \).

If \( h(t) = t \), then all non-negative convex functions belong to \( SX(h, I) \) and all non-negative concave functions belong to \( SV(h, I) \). If \( h(t) = \frac{1}{t} \), then \( SX(h, I) = Q(I) \); if \( h(t) = 1 \), then \( SX(h, I) \supseteq P(I) \); and if \( h(t) = t^s \), where \( s \in (0, 1) \), then \( SX(h, I) \supseteq K_s^2 \).

A function \( h : J \to \mathbb{R} \) is said to be a super-additive function if
\[
 h(x + y) \geq h(x) + h(y) ,
\]
for all \( x, y \in J \). If inequality (1.2) is reversed, then \( h \) is said to be a sub-additive function. If the equality holds in (1.2), then \( h \) is said to be an additive function.

Function \( h \) is called a super-multiplicative function if
\[
 h(xy) \geq h(x)h(y) ,
\]
for all \( x, y \in J \) [10]. If inequality (1.3) is reversed, then \( h \) is called a sub-multiplicative function. If the equality holds in (1.3), then \( h \) is called a multiplicative function.

**Example 1.1.** [10] Consider the function \( h : [0, +\infty) \to \mathbb{R} \) by \( h(x) = (c + x)^{p-1} \). If \( c = 0 \), then the function \( h \) is multiplicative. If \( c \geq 1 \), then for \( p \in (0, 1) \) the function \( h \) is super-multiplicative and for \( p > 1 \) the function \( h \) is sub-multiplicative.

2. Preliminaries

In what follows we assume that \( \mathcal{H} \) and \( \mathcal{K} \) are Hilbert spaces, \( \mathbb{B}(\mathcal{H}) \) and \( \mathbb{B}(\mathcal{K}) \) are \( C^* \)-algebras of all bounded linear operators on the appropriate Hilbert space with identities \( I_\mathcal{H} \) and \( I_\mathcal{K} \), \( \mathbb{B}_h(\mathcal{H}) \) denotes the algebra of all self-adjoint operators in \( \mathbb{B}(\mathcal{H}) \). An operator \( A \in \mathbb{B}_h(\mathcal{H}) \) is called positive, if \( \langle Ax, x \rangle \geq 0 \) holds for every \( x \in \mathcal{H} \) and then we write \( A \geq 0 \). For \( A, B \in \mathbb{B}_h(\mathcal{H}) \), we say \( A \leq B \) if \( B - A \geq 0 \). We write \( A > 0 \) and say \( A \) is strictly positive operator, if \( A \) is a positive invertible operator. Let \( f \) be a continuous real valued function defined on an interval \( I \). The function \( f \) is called operator monotone if \( A \leq B \)
implies \( f(A) \leq f(B) \) for all \( A, B \) with spectra in \( I \). A function \( f \) is said to be operator convex on \( I \) if
\[
f(tA + (1 - t)B) \leq tf(A) + (1 - t)f(B),
\]
for all \( A, B \in \mathbb{B}(\mathcal{H}) \) with spectra in \( I \) and all \( t \in [0,1] \). A map \( \Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K}) \) is called positive if \( \Phi(A) \geq 0 \), whenever \( A \geq 0 \) and is said to be normalized if \( \Phi(I_{\mathcal{H}}) = I_{\mathcal{K}} \). We denote by \( \mathcal{P}[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{K})] \) the set of all positive linear maps \( \Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K}) \) and by \( \mathcal{P}_N[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{K})] \) the set of all normalized positive linear maps \( \Phi \in \mathcal{P}[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{K})] \). If \( \Phi \in \mathcal{P}_N[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{K})] \) and \( f \) is an operator convex function on an interval \( I \), then
\[
f(\Phi(A)) \leq \Phi(f(A)) \quad \text{(Davis-Choi-Jensen’s inequality) \quad (2.1)}
\]
for every self-adjoint operator \( A \) on \( \mathcal{H} \), whose spectrum is contained in \( I \), see [5].

3. Characterizations

Assume that \( C \) is a convex subset of a linear space \( X \) and \( f \) is an arbitrary real-valued function on \( C \). The non-negative function \( f : C \to \mathbb{R} \) is called \( h \)-convex function on \( C \), if \( f(tx + (1 - t)y) \leq h(t)f(x) + h(1 - t)f(y) \) for every \( x, y \in C \) and \( t \in [0,1] \).

Let \( x \) and \( y \) be two fixed elements in \( C \). Define the map \( f_{x,y} \) as follows:
\[
f_{x,y} : [0,1] \to \mathbb{R}, \quad f_{x,y}(t) = f(tx + (1 - t)y).
\]

The following theorem is a characterization of \( h \)-convex functions.

**Theorem 3.1** (First characterization). *With the above assumptions, the following statements are equivalent:

(i) \( f \) is a \( h \)-convex function on \( C \).

(ii) The mapping \( f_{x,y} \) is a \( h \)-convex function on \( [0,1] \), for any \( x, y \in C \).

**Proof.** First, assume that (i) holds. Let \( \alpha, \beta \in [0,1] \) such that \( \alpha + \beta = 1 \) and \( t_1, t_2 \in [0,1] \). Hence
\[
f_{x,y}(\alpha t_1 + \beta t_2) = f((\alpha t_1 + \beta t_2)x + (1 - \alpha t_1 - \beta t_2)y)
= f\left(\alpha(t_1x + (1 - t_1)y) + \beta(t_2x + (1 - t_2)y)\right)
\leq h(\alpha)f(t_1x + (1 - t_1)y) + h(\beta)f(t_2x + (1 - t_2)y)
= h(\alpha)f_{x,y}(t_1) + h(\beta)f_{x,y}(t_2),
\]
Theorem 3.2 (Second characterization). The following statements of $h$-convex functions hold:

(i) If $f$ is a $h$-convex function on $C$, then $f_t : C^2 \rightarrow \mathbb{R}$ by $f_t(x, y) = f(tx + (1 - t)y)$.

(ii) If $C$ is a cone in $X$ and $f_t$ is a $h$-convex function on $C^2$ for every $t \in (0, 1)$, then $f$ is a $h$-convex function on $C$.

Proof. (i) For fixed $t \in [0, 1]$ and $(x, y), (u, v) \in C^2$. Then for every $\alpha \in [0, 1]$

\[
f_t(\alpha x, y) + (1 - \alpha)u, v) = f_t(\alpha x + (1 - \alpha)u, \alpha y + (1 - \alpha)v) = f(t(\alpha x + (1 - \alpha)u) + (1 - t)(\alpha y + (1 - \alpha)v)) = f(\alpha(tx + (1 - t)y) + (1 - \alpha)(tu + (1 - t)v)) \leq h(\alpha)f(tx + (1 - t)y) + h(1 - \alpha)f(tu + (1 - t)v) = h(\alpha)f_t(x, y) + h(1 - \alpha)f_t(u, v),
\]

that is, $f_t$ is a $h$-convex function on $C^2$.

(ii) Let $x, y \in C$ and $t \in (0, 1)$. Since $C$ is cone in $X$, $C + C \subseteq C$ and $\alpha C \subseteq C$ for every $\alpha \geq 0$, then $t^{-1}x, (1 - t)^{-1}y \in C$ and $(t^{-1}x, 0), (0, 1 - t)^{-1}y) \in C^2$.

On the other hand, by $h$-convexity of $f_t$ on $C^2$, we have

\[
f(tx + (1 - t)y) = f_t(x, y) = f_t(t(t^{-1}x, 0) + (1 - t)(0, (1 - t)^{-1}y)) \leq h(t)f_t(t^{-1}x, 0) + h(1 - t)f_t(0, (1 - t)^{-1}y) = h(t)f(x) + h(1 - t)f(y),
\]

this means that $f_{x,y}$ is a $h$-convex function on $[0, 1]$.

Conversely, suppose that (ii) holds. For $t \in [0, 1]$ and $x, y \in C$, we have

\[
f(tx + (1 - t)y) = f_{x,y}(t) = f_{x,y}((1 - t)0 + t1) \leq h(1 - t)f_{x,y}(0) + h(t)f_{x,y}(1) = h(1 - t)f(y) + h(t)f(x),
\]

that is, $f$ is a $h$-convex function on $C$. □

Now, for fixed $t \in [0, 1]$, we define the function $f_t : C^2 \rightarrow \mathbb{R}$ by $f_t(x, y) = f(tx + (1 - t)y)$.

In the next theorem, we state a new characterization of $h$-convex functions.
therefore, $f$ is a $h$-convex function on $C$.

\[ \square \]

**Theorem 3.3** (Third characterization). Let $h$ be a strictly positive multiplicative function, then the following statements are equivalent:

(i) $f$ is a $h$-convex function.

(ii) If $(1 + s)x - sy \in C$, for every $x, y \in C$ and $s \geq 0$, then

\[ f((1 + s)x - sy) \geq h(1 + s)f(x) - h(s)f(y). \quad (3.1) \]

Proof. First, note that multiplicity of $h$ implies that $h\left(\frac{1}{t}\right) = \frac{1}{h(t)}$, for every $t > 0$ and $h(1) = 1$.

Assume that (i) holds. By using $x = \frac{1}{s+1}[(1 + s)x - sy] + \frac{s}{s+1}y$, we have

\[
f(x) = f\left(\frac{1}{s+1}[(1 + s)x - sy] + \frac{s}{s+1}y\right)
\]

\[
\leq h\left(\frac{1}{s+1}\right) f[(1 + s)x - sy] + h\left(\frac{s}{s+1}\right) f(y)
\]

\[
= \frac{h(1)}{h(s+1)} f[(1 + s)x - sy] + \frac{h(s)}{h(s+1)} f(y) \quad \text{(by multiplicity of $h$)}
\]

and therefore,

\[ f[(1 + s)x - sy] \geq h(1 + s)f(x) - h(s)f(y). \]

Now, suppose that (ii) holds. If $\alpha, \beta \in [0, 1]$ and $\alpha + \beta = 1$, then there exists $s \geq 0$ such that $\alpha = \frac{1}{s+1}$ and $\beta = \frac{s}{s+1}$. Put $z = \alpha x + \beta y$, hence $x = (1+s)z - sy$ and so

\[ f(x) = f((1 + s)z - sy) \geq h(1 + s)f(z) - h(s)f(y). \]

Consequently,

\[ f(\alpha x + \beta y) = f(z) \leq \frac{1}{h(1+s)} f(x) + \frac{h(s)}{h(1+s)} f(y) \]

\[ = h\left(\frac{1}{1+s}\right) f(x) + h\left(\frac{s}{1+s}\right) f(y) \]

\[ = h(\alpha) f(x) + h(\beta) f(y), \]

this show that $f$ is $h$-convex function. \[ \square \]
Theorem 3.4. (i) Assume that $X$ is a real vector space and $f : X \to \mathbb{R}$ is an even $h$-convex function. Then
\[
\frac{f((1-2t)x) + f((2t-1)y)}{h(t) + h(1-t)} \leq f((1-t)x + ty) + f(tx + (1-t)y)
\]
\[
\leq [h(t) + h(1-t)][f(x) + f(y)] . \tag{3.2}
\]

(ii) Let $X$ be a topological vector space, $h$ be an integrable strictly positive function and $f$ be a continuous even $h$-convex function, then
\[
\frac{1}{2} \int_0^1 [f(tx) + f(ty)] \, dt \leq \int_0^1 [h(t) + h(1-t)]f(tx + (1-t)y) \, dt . \tag{3.3}
\]

In addition, if $h$ is super-additive, then
\[
\frac{1}{2h(1)} \left( \int_0^1 h(t) \, dt \right) \int_0^1 [f(tx) + f(ty)] \, dt \leq f(x) + f(y) . \tag{3.4}
\]

Proof. (i) Since $f$ is a $h$-convex function, hence the right inequality of (3.2) is clear. Set $a = (1-t)x + ty$ and $b = -tx - (1-t)y$, we have $(1-t)a + tb = (1-2t)x$ and $ta + (1-t)b = (2t-1)y$. Therefore
\[
f((1-2t)x) + f((2t-1)y)
\]
\[
= f((1-t)a + tb) + f(ta + (1-t)b)
\]
\[
\leq [h(t) + h(1-t)][f(a) + f(b)] \quad \text{(by the right inequality of (3.2))}
\]
\[
= [h(t) + h(1-t)]\left[ f((1-t)x + ty) + f(-tx - (1-t)y) \right]
\]
\[
= [h(t) + h(1-t)]\left[ f((1-t)x + ty) + f(tx + (1-t)y) \right] . \tag{3.5}
\]

(f is a even function)

We therefore deduce the desired inequality in (3.2).

(ii) By using (3.2) inequality, we get
\[
f((1-2t)x) + f((2t-1)y) \leq [h(t) + h(1-t)] \left[ f((1-t)x + ty) + f(tx + (1-t)y) \right] . \tag{3.5}
\]

Integrating each side of (3.5), we have
\[
\int_0^1 f((1-2t)x) \, dt + \int_0^1 f((2t-1)y) \, dt
\]
\[
\leq \int_0^1 [h(t) + h(1-t)]f((1-t)x + ty) \, dt + \int_0^1 [h(t) + h(1-t)]f(tx + (1-t)y) \, dt
\]
\[
= 2 \int_0^1 [h(t) + h(1-t)]f(tx + (1-t)y) \, dt . \tag{3.6}
\]

Since $f$ is even and by changing of variables $u = 1-2t$, yield
\[
\int_0^1 f((1-2t)x) \, dt = \int_0^1 f(tx) \, dt ,
\]
and similarly
\[ \int_0^1 f((2t-1)y) \, dt = \int_0^1 f(ty) \, dt. \]
Consequently, by (3.6), the proof of (3.3) completes.

Now, assume that \( h \) is supper-additive. Hence by (3.3), we have
\[
\frac{1}{2} \int_0^1 [f(tx) + f(ty)] \, dt \\
\leq \int_0^1 [h(t) + h(1-t)] f(tx + (1-t)y) \, dt \\
\leq h(1) \int_0^1 [h(t)f(x) + h(1-t)f(y)] \, dt \quad (h \text{ is super-additive and } f \text{ is } h\text{-convex})
\]
\[
= h(1) \left[ f(x) \left( \int_0^1 h(t) \, dt \right) + f(y) \left( \int_0^1 h(1-t) \, dt \right) \right]
\]
\[
= h(1) \left( \int_0^1 h(t) \, dt \right) [f(x) + f(y)],
\]
this show that (3.4) holds.

\[ \square \]

**Corollary 3.5.** (i) Assume that \( X \) is a real vector space and \( f : X \to \mathbb{R} \) is an even convex function. Then
\[
f((1-2t)x) + f((2t-1)y) \leq f((1-t)x + ty) + f(tx + (1-t)y) \leq f(x) + f(y).
\] (3.7)

(ii) Let \( X \) be a topological vector space and \( f \) be a continuous even convex function, then
\[
\frac{1}{2} \int_0^1 [f(tx) + f(ty)] \, dt \leq \int_0^1 f(tx + (1-t)y) \, dt \leq f(x) + f(y).
\] (3.8)

**Proof.** Enough put in Theorem 3.4, \( h(t) = t \).

\[ \square \]

**Corollary 3.6.** [9, Lemma 3.2]

(i) Assume that \( X \) is a real vector space and \( f : X \to \mathbb{R} \) is an even function in \( P(I) \). Then
\[
\frac{f((1-2t)x) + f((2t-1)y)}{2} \leq f((1-t)x + ty) + f(tx + (1-t)y) \leq 2(f(x) + f(y)).
\] (3.9)
(ii) Let $X$ be a topological vector space and $f$ be a continuous even function in $P(I)$, then
\[
\frac{1}{4} \int_0^1 [f(tx) + f(ty)] \, dt \leq \int_0^1 f(tx + (1-t)y) \, dt \leq f(x) + f(y).
\] (3.10)

Proof. In Theorem 3.4, put $h(t) = 1$. $\square$

Example 3.7. [9, Theorem 3.3] Let $(X, \| \cdot \|)$ be a normed space, $x, y \in X$ and $0 < p < 1$. Since $f(x) = \|x\|^p$ is an even continuous $P$-convex function, we have the following Hermit-Hadamard inequality
\[
\frac{\|x\|^p + \|y\|^p}{4(p+1)} \leq \int_0^1 \|(1-t)x + ty\|^p dt \leq \|x\|^p + \|y\|^p.
\]

4. Jensen-Mercer Type Inequality

In [8], Mercer proved that
\[
f\left(x_1 + x_n - \sum_{j=1}^n t_j x_j\right) \leq f(x_1) + f(x_n) - \sum_{j=1}^n t_j f(x_j).
\] (4.1)

where $x_j$'s also satisfy in the condition $0 < x_1 \leq x_2 \leq \cdots \leq x_n$, $t_j \geq 0$ and $\sum_{j=1}^n t_j = 1$.

In this section, we present the Jensen-Mercer inequality for $h$-convex functions.

Theorem 4.1. [10, Theorem 19] Let $t_1, \cdots, t_n$ be positive real numbers ($n \geq 2$). If $h$ is a non-negative super-multiplicative function, $f$ is a $h$-convex function on $I$ and $x_1, \cdots, x_n \in I$, then
\[
f\left(\frac{1}{T_n} \sum_{j=1}^n t_j x_j\right) \leq \sum_{j=1}^n h\left(\frac{t_j}{T_n}\right) f(x_j),
\] (4.2)

where $T_n = \sum_{j=1}^n t_j$.

Lemma 4.2. Let $0 < x \leq y$ and $f$ be a $h$-convex function, then for every $z \in [x, y]$, there exists $\lambda \in [0, 1]$ such that
\[
f(x + y - z) \leq [h(\lambda) + h(1 - \lambda)][f(x) + f(y)] - f(z).
\] (4.3)

Moreover, if $h$ is super-additive, then
\[
f(x + y - z) \leq h(1)[f(x) + f(y)] - f(z).
\]
Proof. Since \( z \in [x, y] \), there exists \( \lambda \in [0, 1] \) such that
\[
z = \lambda x + (1 - \lambda)y.
\]

By using \( h \)-convexity of \( f \), we have
\[
f(x + y - z) = f((1 - \lambda)x + \lambda y)
\leq h(1 - \lambda)f(x) + h(\lambda)f(y)
= [h(\lambda) + h(1 - \lambda)][f(x) + f(y)] - [h(\lambda)f(x) + h(1 - \lambda)f(y)]
\leq [h(\lambda) + h(1 - \lambda)][f(x) + f(y)] - f(\lambda x + (1 - \lambda)y)
= [h(\lambda) + h(1 - \lambda)][f(x) + f(y)] - f(z).
\]

If \( h \) is super-additive, then \( h(\lambda) + h(1 - \lambda) \leq h(1) \). So the end part of theorem holds.

**Theorem 4.3.** Let \( f \) be a \( h \)-convex function on an interval containing the \( x_j \) \((j = 1, \cdots, n)\) such that \( 0 < x_1 \leq \cdots \leq x_n \), then
\[
f \left( x_1 + x_n - \sum_{j=1}^{n} t_j x_j \right) \leq \left( \sum_{j=1}^{n} h(t_j)[h(\lambda_j) + h(1 - \lambda_j)] \right) (f(x_1) + f(x_n)) - \sum_{j=1}^{n} h(t_j)f(x_j),
\]
where for every \( j = 1, \cdots, n \), there exists \( \lambda_j \in [0, 1] \) such that \( x_j = \lambda_j x_1 + (1 - \lambda_j)x_n \).

Proof. With the above assumption, we have
\[
f \left( x_1 + x_n - \sum_{j=1}^{n} t_j x_j \right)
= f \left( \sum_{j=1}^{n} t_j(x_1 + x_n - x_j) \right)
\leq \sum_{j=1}^{n} h(t_j)f(x_1 + x_n - x_j)
\leq \sum_{j=1}^{n} h(t_j)([h(\lambda_j) + h(1 - \lambda_j)] f(x_1) + f(x_n)) - f(x_j)
\leq \sum_{j=1}^{n} h(t_j)[h(\lambda_j) + h(1 - \lambda_j)] f(x_1) + \sum_{j=1}^{n} h(t_j)f(x_j)
\]
this completes the proof.
Corollary 4.4. With the assumptions of previous theorem, if \( h \) is a super-additive function such that for every probability vector \((t_1, \ldots, t_n)\), \( \sum_{j=1}^{n} h(t_j) \leq 1 \), then
\[
f \left( x_1 + x_n - \sum_{j=1}^{n} t_j x_j \right) \leq h(1) \left( f(x_1) + f(x_n) \right) - \sum_{j=1}^{n} h(t_j) f(x_j).
\]
Moreover, if \( h \) is multiplicative, then
\[
f \left( x_1 + x_n - \sum_{j=1}^{n} t_j x_j \right) \leq f(x_1) + f(x_n) - \sum_{j=1}^{n} h(t_j) f(x_j).
\]

5. Operator \( h \)-convex functions

In this section, we present the definition of operator \( h \)-convex function for operators acting on a Hilbert space.

Definition 5.1. Let \( h : J \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a non-negative function, \( h \neq 0 \). We say that \( f : I \rightarrow \mathbb{R} \) is an operator \( h \)-convex function, if \( f \) is non-negative and for all \( A, B \in \mathbb{B}(\mathcal{H}) \) with \( \sigma(A), \sigma(B) \subseteq I \) and \( t \in (0,1) \),
\[
f(tA + (1-t)B) \leq h(t)f(A) + h(1-t)f(B),
\]
where \( \sigma(A) \) and \( \sigma(B) \) are spectrum of \( A \) and \( B \), respectively.

If inequality (5.1) is reversed, then \( f \) is said to be operator \( h \)-concave.

If \( t = \frac{1}{2} \) in (5.1), then \( f \) is called \( h \)-mid-convex function.

Example 5.2. Assume that \( h \) is a function on \([0, \infty)\) such that \( h(t) \geq t \) and \( f(t) = t^2 \) on an interval \( I \subseteq \mathbb{R} \). Then \( f \) is operator \( h \)-mid-convex function. Because,
\[
h\left(\frac{1}{2}\right) \left(A^2 + B^2\right) - \left(\frac{A + B}{2}\right)^2
\]
\[
= h\left(\frac{1}{2}\right) \left(A^2 + B^2\right) - \frac{A^2 + AB + BA + B^2}{4}
\]
\[
= \frac{(4h(1/2) - 1) A^2 - AB - BA + (4h(1/2) - 1) B^2}{4}
\]
\[
\geq \frac{1}{4} \left(A^2 - AB - BA - B^2\right)
\]
\[
= \frac{1}{4}(A - B)^2 \geq 0.
\]
Now, we can prove the following theorem as Theorem 1.9 in [5]. So, we omit the proof it.

**Theorem 5.3** (Jensen’s type operator inequality). Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert space. Assume that $h$ is non-negative super-multiplicative function and $f$ is a real valued function on an interval $I \subseteq \mathbb{R}$. Suppose that $A, A_j \in \mathcal{B}_h(\mathcal{H})$ such that $\sigma(A), \sigma(A_j) \subseteq I$ ($j = 1, \ldots, n$). Then the following conditions are equivalent:

(i) $f$ is operator $h$-mid-convex on $I$;
(ii) $f(C^*AC) \leq 2h(\frac{1}{2})C^*f(A)C$ for every self-adjoint operator $A : \mathcal{H} \to \mathcal{H}$ and isometry $C : \mathcal{K} \to \mathcal{H}$, i.e.; $C^*C = 1_K$;
(iii) $f(C^*AC) \leq 2h(\frac{1}{2})C^*f(A)C$ for every self-adjoint operator $A : \mathcal{H} \to \mathcal{H}$ and isometry $C : \mathcal{H} \to \mathcal{H}$;
(iv) $f\left(\sum_{j=1}^n C_j^*A_jC_j\right) \leq 2h(\frac{1}{2})\sum_{j=1}^n C_j^*f(A_j)C_j$ for every self-adjoint operator $A_j : \mathcal{H} \to \mathcal{H}$ and bounded linear operators $C_j : \mathcal{K} \to \mathcal{H}$; with $\sum_{j=1}^n C_j^*C_j = 1_K$ ($j = 1, \ldots, n$);
(v) $f\left(\sum_{j=1}^n C_j^*A_jC_j\right) \leq 2h(\frac{1}{2})\sum_{j=1}^n C_j^*f(A_j)C_j$ for every self-adjoint operator $A_j : \mathcal{H} \to \mathcal{H}$ and bounded linear operators $C_j : \mathcal{H} \to \mathcal{H}$; with $\sum_{j=1}^n C_j^*C_j = 1_H$ ($j = 1, \ldots, n$);
(vi) $f\left(\sum_{j=1}^n P_jA_jP_j\right) \leq 2h(\frac{1}{2})\sum_{j=1}^n P_jf(A_j)P_j$ for every self-adjoint operator $A_j : \mathcal{H} \to \mathcal{H}$ and projection $P_j : \mathcal{H} \to \mathcal{H}$; with $\sum_{j=1}^n P_j = 1_H$ ($j = 1, \ldots, n$).

Using an idea of [5] we prove the following result.

**Theorem 5.4** (Davis-Choi-Jensen’s inequality). Let $\Phi$ be a normalized positive linear map and $f$ be an operator $h$-convex function on an interval $I$, then

$$f(\Phi(A)) \leq 2h\left(\frac{1}{2}\right)\Phi(f(A)),$$

for every self-adjoint operator $A$ with $\sigma(A) \subseteq I$.

**Proof.** We know that a self-adjoint operator $A$ can be approximated uniformly by a simple function $A' = \sum_j t_j E_j$ where $\{E_j\}$ is a decomposition of the unit $I_H$. By using normality of $\Phi$, we get $\sum_j \Phi(E_j) = I_K$. By applying (iv) of
Theorem 5.3 to $C_j = \sqrt{\Phi(E_j)}$, we have

$$f(\Phi(A')) = f\left(\sum_j t_j \Phi(E_j)\right) = f\left(\sum_j C_j t_j C_j\right)$$

$$\leq 2h\left(\frac{1}{2}\right) \sum_j C_j f(t_j) C_j = 2h\left(\frac{1}{2}\right) \sum_j f(t_j) \Phi(E_j)$$

$$= 2h\left(\frac{1}{2}\right) \Phi\left(\sum_j f(t_j) E_j\right) = 2h\left(\frac{1}{2}\right) \Phi(f(A')).$$

Since $\Phi$ is continuous, the proof is complete. \qed

**Theorem 5.5.** Let $t_1, t_2, \cdots, t_n$ be positive real numbers ($n \geq 2$) such that $\sum_{j=1}^n t_j = 1$. If $h$ is non-negative super-multiplicative function and if $f$ is $h$-convex function on an interval $I \subseteq \mathbb{R}$, $A_1, \cdots, A_n$ are self-adjoint operators in $\mathcal{B}(\mathcal{H})$ such that $\sigma(A_j) \subseteq I$, then

$$f\left(\sum_{j=1}^n t_j A_j\right) \leq \sum_{j=1}^n h(t_j) f(A_j). \quad (5.3)$$

If $h$ is sub-multiplicative and $f$ is operator $h$-concave on $I$, then inequality (5.3) is reversed.

**Proof.** We prove this theorem by induction on $n$. If $n = 2$, then inequality (5.3) is equivalent to inequality (5.1) with $t = t_1$ and $1-t = t_2$. Assume that inequality (5.3) holds for $n - 1$. Then for $n$, we have

$$f\left(\sum_{j=1}^n t_j A_j\right) = f\left(t_n A_n + \sum_{j=1}^{n-1} t_j A_j\right)$$

$$= f\left(t_n A_n + (t_1 + \cdots + t_{n-1}) \sum_{j=1}^{n-1} \frac{t_j}{t_1 + \cdots + t_{n-1}} A_j\right)$$

$$\leq h(t_n) f(A_n) + h(t_1 + \cdots + t_{n-1}) f\left(\sum_{j=1}^{n-1} \frac{t_j}{t_1 + \cdots + t_{n-1}} A_j\right)$$

$$\leq h(t_n) f(A_n) + h(t_1 + \cdots + t_{n-1}) \sum_{j=1}^{n-1} h\left(\frac{t_j}{t_1 + \cdots + t_{n-1}} A_j\right)$$

$$\leq h(t_n) f(A_n) + \sum_{j=1}^{n-1} h(t_j) f(A_j)$$

$$= \sum_{j=1}^n h(t_j) f(A_j).$$
This completes the proof. □

**Corollary 5.6.** By assumptions of Theorem 5.5, if $\Phi \in \mathcal{P}_N[\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{K})]$, then

$$f \left( \sum_{j=1}^{n} t_j \Phi(A_j) \right) \leq \sum_{j=1}^{n} 2h \left( \frac{t_j}{2} \right) \Phi(f(A_j)).$$

(5.4)

If $h$ is sub-multiplicative and $f$ is operator $h$-concave on $I$, then inequality (5.4) is reversed.

**Proof.** If $\Phi \in \mathcal{P}_N[\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{K})]$, then

$$f \left( \sum_{j=1}^{n} t_j \Phi(A_j) \right) \leq \sum_{j=1}^{n} h(t_j) f(\Phi(A_j))$$

$$\leq \sum_{j=1}^{n} 2h(t_j) h \left( \frac{1}{2} \right) \Phi(f(A_j)) \quad \text{(by (5.2))}$$

$$\leq \sum_{j=1}^{n} 2h \left( \frac{t_j}{2} \right) \Phi(f(A_j)) \quad \text{(by super-multiplicity of $h$)}.$$

□

For convenience, let $\varphi(t)$ be a real valued continuous function on the interval $[m, M]$. Define

$$\mu_\varphi = \frac{\varphi(M) - \varphi(m)}{M - m}, \quad \nu_\varphi = \frac{M\varphi(m) - m\varphi(M)}{M - m}.$$

We remark that a straight line $\ell(t) = \mu_\varphi t + \nu_\varphi$ is a line thought two points $(m, \varphi(m))$ and $(M, \varphi(M))$.

Notice that, if $\varphi(t) = t$, then $\mu_\varphi = 1$ and $\nu_\varphi = 0$, if $\varphi(t) = 1$, then $\mu_\varphi = 0$ and $\nu_\varphi = 1$, and if $\varphi(t) = \frac{1}{t}$, then $\mu_\varphi = -\frac{1}{mM}$ and $\nu_\varphi = \frac{mM}{mM}$.

**Theorem 5.7.** Let $A_1, A_2, \cdots, A_n \in \mathcal{B}(\mathcal{H})$ be self-adjoint operators with spectra in $[m, M]$ for some scalars $m, M$ and $\Phi_1, \Phi_2, \cdots, \Phi_n \in \mathcal{P}_N[\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{K})]$ and $t_1, \cdots, t_n$ non-negative real numbers with $\sum_{j=1}^{n} t_j = 1$. If $f$ on $[m, M]$ is operator $h$-convex function and $h$ on the interval $J$ is super-multiplicative
operator convex function, then
\[
f \left( mI_K + MI_K - \sum_{j=1}^{n} t_j \Phi_j(A_j) \right) \leq h \left( \frac{\sum_{j=1}^{n} t_j \Phi_j(A_j) - mI_K}{M - m} \right) f(m) + h \left( \frac{MI_K - \sum_{j=1}^{n} t_j \Phi_j(A_j)}{M - m} \right) f(M)
\]
\[
\leq (\mu h + 2\nu h) \left( f(m) + f(M) \right) I_K - \sum_{j=1}^{n} h(t_j) \Phi_j(f(A_j)).
\]

**Proof.** Define the function \( g : [m, M] \to \mathbb{R} \) by \( g(t) = f(m + M - t) \). Since \( f \) is continuous and \( h \)-convex on \([m, M]\), so the same is true for \( g \). Consequently, for every \( t \in [m, M] \)
\[
f(t) \leq h \left( \frac{t - m}{M - m} \right) f(M) + h \left( \frac{M - t}{M - m} \right) f(m),
\]
and
\[
g(t) \leq h \left( \frac{t - m}{M - m} \right) g(M) + h \left( \frac{M - t}{M - m} \right) g(m).
\]
Since \( \sum_{j=1}^{n} t_j = 1 \), \( \Phi_j(I_{H}) = I_K \) and \( mI_{H} \leq A_j \leq MI_{H} \) \((j = 1, \cdots, n)\), we conclude that \( mI_K \leq \sum_{j=1}^{n} t_j \Phi_j(A_j) \leq MI_K \). Now, by using functional calculus and (5.6) and super-multiplicity of \( h \), we get
\[
h(t_j) f(A_j) \leq h(t_j) h \left( \frac{A_j - mI_K}{M - m} \right) f(M) + h(t_j) h \left( \frac{MI_K - A_j}{M - m} \right) f(m)
\]
\[
\leq h \left( \frac{t_j A_j - mt_j I_K}{M - m} \right) f(M) + h \left( \frac{Mt_j I_K - t_j A_j}{M - m} \right) f(m)
\]
\[
\leq \left[ \mu h \left( \frac{t_j A_j - mt_j I_K}{M - m} \right) + \nu h I_K \right] f(M)
\]
\[
+ \left[ \mu h \left( \frac{Mt_j I_K - t_j A_j}{M - m} \right) + \nu h I_K \right] f(m).
\]

Hence, by linearity of \( \Phi_j \) for every \( j = 1, \cdots, n \) and the inequality (5.8), we have
\[
h(t_j) \Phi_j(f(A_j)) \leq \left[ \mu h \left( \frac{t_j \Phi_j(A_j) - mt_j I_K}{M - m} \right) + \nu h I_K \right] f(M)
\]
\[
+ \left[ \mu h \left( \frac{Mt_j I_K - t_j \Phi_j(A_j)}{M - m} \right) + \nu h I_K \right] f(m).
\]
By summing of all \( j = 1, \ldots, n \)

\[
\sum_{j=1}^{n} h(t_j) \Phi_j(f(A_j)) \leq \left[ \mu_h \left( \frac{\sum_{j=1}^{n} t_j \Phi_j(A_j) - m I_K}{M - m} \right) + \nu_h I_K \right] f(M) \quad (5.9)
\]

\[
+ \left[ \mu_h \left( \frac{\sum_{j=1}^{n} t_j \Phi_j(A_j)}{M - m} \right) + \nu_h I_K \right] f(m)
\]

\[
= \mu_h \left[ \sum_{j=1}^{n} t_j \Phi_j(A_j) + \nu_f I_K \right] + \nu_h (f(m) + f(M)) I_K
\]

Also, using similar way and the (5.7) inequality, we have

\[
g \left( \sum_{j=1}^{n} t_j \Phi_j(A_j) \right) \leq h \left( \sum_{j=1}^{n} t_j \Phi_j(A_j) - m I_K \right) f(m) + h \left( \frac{MI_K - \sum_{j=1}^{n} t_j \Phi_j(A_j)}{M - m} \right) f(M)
\]

or equivalently,

\[
f \left( m I_K + MI_K - \sum_{j=1}^{n} t_j \Phi_j(A_j) \right) = g \left( \sum_{j=1}^{n} t_j \Phi_j(A_j) \right)
\]

\[
\leq h \left( \sum_{j=1}^{n} t_j \Phi_j(A_j) - m I_K \right) f(m) + h \left( \frac{MI_K - \sum_{j=1}^{n} t_j \Phi_j(A_j)}{M - m} \right) f(M)
\]

\[
\leq \left[ \mu_h \left( \frac{\sum_{j=1}^{n} t_j \Phi_j(A_j) - m I_K}{M - m} \right) + \nu_h I_K \right] f(m)
\]

\[
+ \left[ \mu_h \left( \frac{MI_K - \sum_{j=1}^{n} t_j \Phi_j(A_j)}{M - m} \right) + \nu_h I_K \right] f(M)
\]

\[
= \mu_h \left[ f(m) I_K + f(M) I_K - \left( \frac{MI_K - \sum_{j=1}^{n} t_j \Phi_j(A_j)}{M - m} \right) f(m) \right.
\]

\[
+ \left. \frac{\sum_{j=1}^{n} t_j \Phi_j(A_j) - m I_K}{M - m} f(M) \right) \right] + \nu_h (f(m) + f(M)) I_K
\]

\[
= \mu_h \left[ (f(m) + f(M)) I_K - \left( \mu_f \sum_{j=1}^{n} t_j \Phi_j(A_j) + \nu_f I_K \right) \right] + \nu_h (f(m) + f(M)) I_K
\]

\[
\leq (\mu_h + 2\nu_h) (f(m) + f(M)) I_K - \sum_{j=1}^{n} h(t_j) \Phi_j(f(A_j)).
\]
In the final inequality, we use the inequality (5.9), and we obtain desired inequalities (5.5). □

Corollary 5.8. (i) Let \( h(t) = t \) in Theorem 5.7, then we have

\[
\begin{align*}
& f \left( mI_K + MI_K - \sum_{j=1}^{n} t_j \Phi_j(A_j) \right) \\
& \leq \left( \frac{\sum_{j=1}^{n} t_j \Phi_j(A_j) - mI_K}{M - m} \right) f(m) + \left( \frac{MI_K - \sum_{j=1}^{n} t_j \Phi_j(A_j)}{M - m} \right) f(M) \\
& \leq f(m)I_K + f(M)I_K - \sum_{j=1}^{n} t_j \Phi_j(f(A_j)).
\end{align*}
\]

(ii) Let \( h(t) = 1 \) in Theorem 5.7, then we have

\[
\begin{align*}
& f \left( mI_K + MI_K - \sum_{j=1}^{n} t_j \Phi_j(A_j) \right) \\
& \leq 2 \left( f(m) + f(M) \right) I_K - \sum_{j=1}^{n} t_j \Phi_j(f(A_j)).
\end{align*}
\]

With similar proof of Theorem 5.7, we have the following proposition.

Proposition 5.9. Let \( A_1, A_2, \cdots, A_n \in \mathcal{B}(\mathcal{H}) \) be self-adjoint operators with spectra in \([m, M]\) for some scalars and \( \Phi_1, \Phi_2, \cdots, \Phi_n \in \mathcal{P}[\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{K})] \) positive linear maps with \( \sum_{j=1}^{n} \Phi_j(I_H) = I_K \). If \( f \) on \([m, M]\) is operator \( h \)-convex function and \( h \) on the interval \( J \) is operator convex function, then

\[
\begin{align*}
& f \left( mI_K + MI_K - \sum_{j=1}^{n} \Phi_j(A_j) \right) \\
& \leq h \left( \frac{\sum_{j=1}^{n} \Phi_j(A_j) - mI_K}{M - m} \right) f(m) + h \left( \frac{MI_K - \sum_{j=1}^{n} \Phi_j(A_j)}{M - m} \right) f(M) \\
& \leq (\mu_h + 2\nu_h) \left( f(m) + f(M) \right) I_K - \sum_{j=1}^{n} \Phi_j(f(A_j)).
\end{align*}
\]

Corollary 5.10. [7, Theorem 1] Let \( h(t) = t \) in Proposition 5.9, then we have

\[
\begin{align*}
& f \left( mI_K + MI_K - \sum_{j=1}^{n} \Phi_j(A_j) \right) \\
& \leq \left( \frac{\sum_{j=1}^{n} \Phi_j(A_j) - mI_K}{M - m} \right) f(m) + \left( \frac{MI_K - \sum_{j=1}^{n} \Phi_j(A_j)}{M - m} \right) f(M) \\
& \leq f(m)I_K + f(M)I_K - \sum_{j=1}^{n} \Phi_j(f(A_j)).
\end{align*}
\]
Theorem 5.11. Suppose that $A_j \in \mathcal{B}_h(\mathcal{H})$ with $\sigma(A_j) \in [m, M]$ ($m < M$), $\Phi_j \in \mathcal{P}_N[\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{K})]$ ($j = 1, \ldots, n$) and $t_1, \ldots, t_n \geq 0$ such that $\sum_{j=1}^{n} t_j = 1$. If $f, g \in C([m, M])$ and $F(u, v)$ is a real valued continuous function defined on $U \times V$, where $f[m, M] \subseteq U, g[m, M] \subseteq V$ and $F$ is an operator monotone function relative to the first component $u$ and $f$ is a non-negative operator monotone $h$-convex and $h$ is super-multiplicative operator convex function on $J$, then

$$F \left[ \sum_{j=1}^{n} h(t_j)\Phi_j(f(A_j)), g \left( \sum_{j=1}^{n} t_j\Phi_j(A_j) \right) \right] \leq \max_{m \leq t \leq M} F \left[ \mu_h \mu_f t + \mu_h \nu_f + \nu_h (f(m) + f(M)), g(t) \right] I_K. \quad (5.10)$$

Proof. With the above assumptions and similar proof of previous theorem, we have

$$\sum_{j=1}^{n} h(t_j)\Phi_j(f(A_j)) \leq \mu_h \left[ \mu_f \sum_{j=1}^{n} t_j\Phi_j(A_j) + \nu_f I_K \right] + \nu_h (f(m) + f(M)) I_K.$$ 
Consequently,

$$F \left[ \sum_{j=1}^{n} h(t_j)\Phi_j(f(A_j)), g \left( \sum_{j=1}^{n} t_j\Phi_j(A_j) \right) \right] \leq F \left[ \mu_h \left[ \mu_f \sum_{j=1}^{n} t_j\Phi_j(A_j) + \nu_f I_K \right] + \nu_h (f(m) + f(M)) I_K, g \left( \sum_{j=1}^{n} t_j\Phi_j(A_j) \right) \right] \leq \max_{m \leq t \leq M} F \left[ \mu_h \mu_f t + \mu_h \nu_f + \nu_h (f(m) + f(M)), g(t) \right] I_K,$$
and we have the desired inequality (5.10). \hfill \Box

Theorem 5.12. With assumptions of previous theorem, if $f$ is operator $h$-convex on $[m, M]$ and $h$ is operator convex function on $J$, then for every $\alpha \in \mathbb{R}$

$$\sum_{j=1}^{n} h(t_j)\Phi_j(f(A_j)) \leq \alpha g \left( \sum_{j=1}^{n} t_j\Phi_j(A_j) \right) + \beta I_K, \quad (5.11)$$
where

$$\beta = \max_{m \leq t \leq M} \left\{ \mu_h \mu_f t + \mu_h \nu_f + \nu_h (f(m) + f(M)) - \alpha g(t) \right\}. $$

In addition,

(i) If $\alpha g$ is concave, then

$$\beta \geq \max_{s \in \{m, M\}} \left\{ \mu_h f(s) + \nu_h (f(m) + f(M)) - \alpha g(s) \right\}. $$
(ii) If $\alpha g$ is strictly convex differentiable, then

$$\beta \leq \mu_h f(s) - \alpha g(s) + |\mu_h \mu_f - \alpha g'(s)|(M - m) + \nu_h (f(m) + f(M)),$$

where $s \in \{m, M\}$.

Proof. Let $\alpha \in \mathbb{R}$. Define $F(u, v) = u - \alpha v$. Since $F$ is operator monotone on a first variable $u$, hence by Theorem 5.11, we have

$$\sum_{j=1}^{n} h(t_j) \Phi_j(f(A_j)) - \alpha g \left( \sum_{j=1}^{n} t_j \Phi_j(A_j) \right)$$

$$\leq \max_{m \leq t \leq M} \{ \mu_h \mu_f t + \mu_h \nu_f + \nu_h (f(m) + f(M)) - \alpha g(t) \} I_K,$$

and we have the desired inequality (5.11).

Put $\Psi(t) = \mu_h (\mu_f t + \nu_f) + \nu_h (f(m) + f(M)) - \alpha g(t)$. In the case (i), if $\alpha g(t)$ is concave on $[m, M]$, then $\Psi$ is convex on $[m, M]$ and therefore

$$\beta = \max_{m \leq t \leq M} \Psi(t) = \max \{ \Psi(m), \Psi(M) \}$$

$$= \max_{s \in \{m, M\}} \{ \mu_h f(s) + \nu_h (f(m) + f(M)) - \alpha g(s) \}.$$

In the case (ii), if $\alpha g$ is strictly convex differentiable, then there exists $t_0 \in [m, M]$ such that $\beta = \max_{m \leq t \leq M} \Psi(t) = \Psi(t_0)$. So, if $s \in \{m, M\}$, then

$$\beta = \mu_h (\mu_f t_0 + \nu_f) + \nu_h (f(m) + f(M)) - \alpha g(t_0)$$

$$= \mu_h f(s) + \mu_h \mu_f (t_0 - s) - \alpha g(t_0) + \nu_h (f(m) + f(M))$$

$$= \mu_h f(s) - \alpha g(s) + \mu_h \mu_f (t_0 - s) - \alpha g(t_0) + \alpha g(s) + \nu_h (f(m) + f(M))$$

$$\leq \mu_h f(s) - \alpha g(s) + \mu_h \mu_f (t_0 - s) - \alpha g'(s)(t_0 - s) + \nu_h (f(m) + f(M))$$

($\alpha g$ is strictly convex differentiable)

$$\leq \mu_h f(s) - \alpha g(s) + |\mu_h \mu_f - \alpha g'(s)|(M - m) + \nu_h (f(m) + f(M)).$$

\[ \square \]

Corollary 5.13 (Complementary inequality of Jensen’s inequality). Let $A_j, \Phi_j$ and $t_j$ ($j = 1, \cdots, n$) be as in Theorem 5.12. If $f \in \mathcal{C}([m, M])$ is a function which is nonnegative, strictly $h$-convex and twice differentiable, then for every $\alpha \in \mathbb{R}^+$

$$\sum_{j=1}^{n} h(t_j) \Phi_j(f(A_j)) \leq \alpha f \left( \sum_{j=1}^{n} t_j \Phi_j(A_j) \right) + \beta I_K,$$

(5.12)
where $\beta = \mu_h \mu f t_0 + \mu_h \nu_f + \nu_h (f(m) + f(M)) - \alpha f(t_0)$ and

$$t_0 = \begin{cases} f^{-1} \left( \frac{1}{\alpha} \mu_h \mu_f \right) & \text{if } \alpha f'(m) < \mu_h \mu_f < \alpha f'(M), \\ m & \text{if } \alpha f'(m) \geq \mu_h \mu_f, \\ M & \text{if } \alpha f'(M) \leq \mu_h \mu_f. \end{cases}$$

REFERENCES

[1] W.W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen, Publ. Inst. Math. 23 (1978) 13–20.

[2] V. Darvish, S.S. Dragomir, H.M. Nazari and A. Taghavi, Some inequalities associated with the Hermite-Hadamard inequalities for operators $h$-convex functions, Acta et Commentationes Universitatis Trnavensis de Mathematica, 21 (2017), 287–297.

[3] T.H. Dinh and K.T.B. Vo, Some inequalities for operator $(p, h)$-convex functions, Linear Multilinear A. 66 (2018), 580–592.

[4] S.S. Dragomir, J. Pečarić and E.E. Persson, Some inequalities of Hadamard type, Soochow J. Math. 21 (1995) 335–341.

[5] T. Furuta, J. Mićić Hot, J.E. Pečarić and Y. Seo, Mond-Pečarić method in operator inequalities, Element, Zagreb, 2005.

[6] E.K. Godunova and V.I. Levin, Neravenstva dlja funkci širokogo klassa, soderžaščego vypuklye, monotonnye i nekotorye drugie vidy funkci, in: Vyčislitel. Mat. i. Mat. Fiz. Mežvuzov. Sb. Nauč. Trudov, MGPI, Moskva, (1985) 138–142.

[7] A. Matković, J. Pečarić and I. Perić, A variant of Jensen’s inequality of Mercer’s type for operators with applications, Linear Algebra Appl. 418 (2006), 551–564.

[8] A. McD. Mercer, A variant of Jensen’s inequality, J. Ineq. Pure and Appl. Math., 4 (2003), Article 73. [ONLINE: http://www.emis.de/journals/JIPAM/article314.html].

[9] J. Rooin, S. Habibzadeh and M.S. Moslehian, Jensen inequalities for $P$-class functions, Period Math. Hung. 77 (2018), 261–273.

[10] S. Varošanec, On $h$-convexity, J. Math. Anal. Appl., 326 (2007), 303–311.

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