CROSS NUMBER INVARIANTS OF FINITE ABELIAN GROUPS

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ABSTRACT. The cross number of a sequence over a finite abelian group $G$ is the sum of the inverse orders of the terms of that sequence. We study two group invariants, the maximal cross number of a zero-sum free sequence over $G$, called $k(G)$, introduced by Krause, and the maximal cross number of a unique factorization sequence over $G$, called $K_1(G)$, introduced by Gao and Wang. Conjectured formulae for $k(G)$ and $K_1(G)$ are known, but only some special cases are proved for either. We show structural results about maximal cross number sequences that allow us to prove an inductive theorem giving conditions under which the conjectured values of $k$ and $K_1$ must be correct for $G \oplus C_{p^a}$ if they are correct for a group $G$. As a corollary of this result we prove the conjectured values of $k(G)$ and $K_1(G)$ for cyclic groups $C_n$, given that the prime factors of $n$ are far apart. Our methods also prove the $K_1(G)$ conjecture for rank two groups of the form $C_n \oplus C_q$, where $q$ is the largest or second largest prime dividing $n$, and the prime factors of $n$ are far apart, and the $k(G)$ conjecture for groups of the form $C_n \oplus H_q$, where the prime factors of $n$ are far apart, $q$ is the largest prime factor of $n$, and $H_q$ is an arbitrary finite abelian $q$-group. Finally, we pose a conjecture about the structure of maximal-length unique factorization sequences over elementary $p$-groups, which is a major roadblock to extending the $K_1$ conjecture to groups of higher rank, and formulate a general question about the structure of maximal zero-sum free and unique factorization sequences with respect to arbitrary weighting functions.

1. Introduction

Let $(G, +)$ be a finite abelian group written additively, and let $G^*$ be the set of nonzero elements of $G$. For any subset $G_0 \subset G$, we define $G(G_0)$ to be the multiplicative free abelian group generated by $G_0$. Similarly, we define $\mathcal{F}(G_0) \subset G(G_0)$ to be the multiplicative free abelian monoid over $G_0$. A sequence over $G_0$ is an element of $\mathcal{F}(G_0)$. Elements of $G(G_0)$ are of the form

$$S = \prod_{g \in G_0} g^{v_g(S)},$$

where $v_g : G(G_0) \to \mathbb{Z}$ is the valuation function for $g$, satisfying $v_g(S) = 0$ for all but finitely many $g$ given any fixed $S$. If $S \in \mathcal{F}(G_0)$ then we have further that $v_g(S) \geq 0$ for all $g \in G_0$. The identity 1 of the monoid $\mathcal{F}(G_0)$ is the unique sequence satisfying $v_g(1) = 0$ for all $g \in G_0$. Given two sequences $S, T \in \mathcal{F}(G_0)$, we say that $T$ is a subsequence of $S$, or divides $S$, if $v_g(T) \leq v_g(S)$ for all $g \in G_0$. In such a case we may also write $T \mid S$.

By the greatest common divisor of two sequences $S$ and $T$ over $G_0$ we mean the sequence

$$\gcd(S, T) = \prod_{g \in G_0} g^{\min(v_g(S), v_g(T))}.$$

Let $\mathbb{N} = \mathbb{Z}_{\geq 0}$. An indexed sequence over a set $G_0 \subset G$ is a sequence over $G_0 \times \mathbb{N}$, i.e. an element of $\mathcal{F}(G_0 \times \mathbb{N})$. To each indexed sequence $S \in \mathcal{F}(G_0 \times \mathbb{N})$ we associate a unique sequence $\alpha(S)$, where the map $\alpha$ is given by extending $\alpha((g, n)) = g$ multiplicatively to
a monoid homomorphism \( \alpha : \mathcal{F}(G_0 \times \mathbb{N}) \to \mathcal{F}(G_0) \). This map \( \alpha \) is called the \textit{unlabelling homomorphism}. The valuation function on \( G \) is extended to indexed sequences naturally by composing it with the unlabelling homomorphism, i.e. \( \tau_g(S) = v_g(\alpha(S)) \). We consider elements of \( G_0 \times \mathbb{N} \) as indexed elements of \( G_0 \), and we define the \textit{order} of such an element \( (g, n) \) to be the order \( \text{ord}(g) \) of \( g \) in \( G \), i.e. the smallest positive integer \( m \) such that \( mg = 0 \) in \( G \).

A sequence (resp. indexed sequence) over a group \( G \) is defined as a sequence (resp. indexed sequence) over its set of nonzero elements \( G^\bullet \).

Define an indexed sequence over \( G_0 \) to be \textit{squarefree} if \( v_{(g, n)}(S) \leq 1 \) for all pairs \( (g, n) \in G_0 \times \mathbb{N} \). Unless otherwise specified all the indexed sequences we study are assumed to be squarefree.

If \( S \) is an indexed sequence over \( G \) and \( G_0 \) is a subset of \( G \), then define \( S_{G_0} \) to be

\[
S_{G_0} = \prod_{(g, n) \in G_0 \times \mathbb{N}} (g, n)^{v_{(g, n)}(S)}.
\]

If \( G_0 \) is a subgroup of \( G \) then \( S_{G_0} \) will simultaneously be considered an indexed sequence over \( G_0 \).

The \textit{sum} function \( \sigma : \mathcal{F}(G^\bullet) \to G \) is defined on a sequence \( S \) as

\[
\sigma(S) = \sum_{g \in G^\bullet} v_g(S) \cdot g.
\]

This function extends naturally to a sum function \( \overline{\sigma} : \mathcal{F}(G^\bullet \times \mathbb{N}) \to G \) on indexed sequences given by \( \overline{\sigma}(T) = \sigma(\alpha(T)) \).

Define the \textit{set of subsums}, or \textit{sumset}, of an indexed sequence \( S \) over \( G \) to be the set

\[
\Sigma(S) = \{ \overline{\sigma}(T) : T \mid S \}.
\]

We will say that \( S \) has \textit{full sumset} in \( G \) if \( \Sigma(S) = G \).

Overviews of progress on computing sumset sizes of several types of sequences are made in Chapter 5.2 of [13], Section 2 of [14], and Part 1 of [18]. Specialized versions of this problem, such as counting the number of subsequences of a given sum \([4]\), and counting sums of only short subsequences \([10]\), have been fruitful objects of study.

An indexed sequence \( S \) is \textit{zero-sum} if \( \overline{\sigma}(S) = 0 \), \textit{zero-sum free} if there does not exist \( 1 \neq T \mid S \) with \( \overline{\sigma}(T) = 0 \), and \textit{irreducible} or \textit{minimal zero-sum} if it differs from 1, is zero-sum, and has no nontrivial proper zero-sum subsequence.

Define the \textit{monoid of zero-sum indexed sequences} over \( G \) to be the monoid

\[
\mathcal{T}(G) = \{ S \in \mathcal{F}(G^\bullet \times \mathbb{N}) : \overline{\sigma}(S) = 0 \}.
\]

The set \( \mathcal{A}(G) \subset \mathcal{T}(G) \) is defined as

\[
\mathcal{A}(G) = \{ S \in \mathcal{T}(G) : S \text{ is irreducible} \}.
\]

We call \( Z(G) = \mathcal{F}(\mathcal{A}(G)) \) the \textit{factorization monoid} of \( G \). For any \( S \in \mathcal{A}(G) \) we denote by \([S]\) the corresponding generator in \( Z(G) \). Let \( \pi : Z(G) \to \mathcal{T}(G) \) denote the monoid homomorphism taking a formal product \( \prod_{i \leq m} [S_i] \) of irreducibles to the zero-sum indexed sequence \( \prod_{i \leq m} S_i \). An \textit{irreducible factorization} of a zero-sum indexed sequence \( S \in \mathcal{T}(G) \) is an element of \( \pi^{-1}(S) \). Equivalently, an irreducible factorization of \( S \) is a way of writing \( S = \prod_{i \leq m} S_i \), where the \( S_i \) are irreducible subsequences of \( S \). We say that \( S \) is a \textit{unique factorization indexed sequence} (UFIS) if \( |\pi^{-1}(S)| = 1 \).
Zero-sum indexed sequences and UFIS’s have interpretations in algebraic number theory when \(G\) is taken to be the ideal class group of the integer ring of an algebraic number field [1].

When \(S\) is an indexed sequence over \(G\), let \(|S| = \sum_{g \in G} v_g(S)\). This is referred to as the length of \(S\). Define the cross number of an indexed sequence \(S\) to be

\[ k(S) = \sum_{g \in G} \frac{v_g(S)}{\text{ord}(g)}. \]

In algebraic number theory it is often more natural to study the cross number of an indexed sequence than to study its length [1].

Three cross number invariants of a finite abelian group \(G\) are defined as follows. The little cross number of \(G\) is defined as

\[ k(G) = \max \{ k(S) : S \text{ is zero-sum free over } G \}, \]

the cross number of \(G\) is defined as

\[ K(G) = \max \{ k(S) : S \text{ is irreducible over } G \}, \]

and the \(K_1\) constant of \(G\) is defined as

\[ K_1(G) = \max \{ k(S) : S \text{ is a UFIS over } G \}. \]

The constants \(k(G)\) and \(K(G)\) have been studied intensely [1, 2, 5, 11, 15, 19]. The cross number is itself a number-theoretically natural alternative to the Davenport constant \(D(G)\) of a group, which is the maximal length of any irreducible indexed sequence over \(G\) [16]. Chapter 2 of the recent book [14] by Geroldinger and Ruzsa summarizes the main results on the Davenport constant.

Define \(P^-(n)\) and \(P^+(n)\) to be the smallest and largest primes, respectively, dividing \(n\). It is not difficult to show that

\[ k(G) + \frac{1}{\exp(G)} \leq K(G) \leq k(G) + \frac{1}{P^-(\exp(G))}, \]

so \(k(G)\) and \(K(G)\) are closely related. These inequalities follow by observing that removing any element of an irreducible indexed sequence leaves a zero-sum free indexed sequence. Write \(G\) as a direct sum of prime power order cyclic groups

\[ G = \bigoplus_{i=1}^{r} C_{p_i^{\alpha_i}}, \]

and define

\[ k^*(G) = \sum_{i=1}^{r} \left(1 - \frac{1}{p_i^{\alpha_i}}\right). \]

This is the conjectured value of \(k(G)\). We choose to study the little cross number because our methods apply more cleanly to it; maximal cross number irreducible indexed sequences generally have an extra term of order \(\exp(G)\), unlike maximal zero-sum free indexed sequences or UFIS’s, which generally only have terms of prime power order.

However, information about \(k(G)\) is weaker than information about \(K(G)\), since if

\[ K(G) = K^*(G) = k^*(G) + \frac{1}{\exp(G)}, \]
its conjectured value, then \( k(G) = k^*(G) \) for that group \( G \) as well. Krause and Zahlten conjectured the following \[19\], towards which the most recent progress has been the results of Geroldinger and Grynkwicz on the structure of maximal cross number irreducible indexed sequences \[12\].

**Conjecture 1.** The equality \( K(G) = K^*(G) \) holds for all finite abelian groups \( G \), and therefore \( k(G) = k^*(G) \) for all \( G \) as well.

Similarly, define

\[
K_1^*(G) = \sum_{i=1}^{r} \frac{p_i^{\alpha_i} - 1}{p_i^{\alpha_i} - p_i^{\alpha_i - 1}}.
\]

Gao and Wang made the analogous conjecture about the \( K_1 \) constant \[9\].

**Conjecture 2.** The equality \( K_1(G) = K_1^*(G) \) holds for all finite abelian groups \( G \).

Just as \( K(G) \) is the cross number variant of \( D(G) \), so too \( K_1(G) \) is the analog of the Narkiewicz constant \( N_1(G) \) of \( G \), which is the maximal length of a UFIS over \( G \). Narkiewicz defined this latter constant to quantify non-unique factorization in domains without unique factorization \[5, 6, 13, 21, 22\].

It is not difficult to see that indexed sequences of sufficient length or cross number over any nontrivial \( G \) will not be zero-sum free, irreducible, or unique factorization, so all of the above group invariants are finite.

In this paper we specifically study the constants \( k(G) \) and \( K_1(G) \). In both cases, the lower bound is known by construction \[9, 19\], so it suffices to prove

\[
k(G) \leq k^*(G),
\]

and

\[
K_1(G) \leq K_1^*(G),
\]

respectively. These have already been shown in the following special cases.

**Theorem 3.** If \( G \) is a group of one of the following forms, then \( K(G) = K^*(G) \), and hence \( k(G) = k^*(G) \).

1. \[11\] \( G \) is a finite abelian \( p \)-group.
2. \[15\] \( G = C_{p^m} \oplus C_{p^n} \oplus C_q^s \) with distinct primes \( p, q \) and \( m, n, s \in \mathbb{N} \).
3. \[15\] \( G = \oplus_{i=1}^{r} C_{p_i} \oplus C_q^s \) with distinct primes \( p_1, \ldots, p_r, q \), and integers \( n_1, \ldots, n_r, s \in \mathbb{N} \), such that either \( r \leq 3 \) and \( p_1p_2 \cdots p_r \neq 30 \) or \( p_k \geq k^3 \) for every \( 1 \leq k \leq r \).

**Theorem 4.** \[20\] If \( G \) is a group of one of the following forms, then \( K_1(G) = K_1^*(G) \).

1. \( C_{p^m} \oplus C_p \), \( p \) prime;
2. \( C_{p^m} \oplus C_{p^2} \), \( p, q \) distinct primes;
3. \( C_{p^m} \oplus C_r^s \), \( p \neq r \) prime, \( r = 2, 3 \).

To improve on these theorems, we develop structural results on the extremal indexed sequences of interest. The following definition, motivated by an argument of Girard \[17\], will be helpful to introduce.

**Definition 5.** A zero-sum free indexed sequence \( S \) over \( G \) is **dense** if \( k(S) = k(G) \) and \(|S| = \min\{|T| : k(T) = k(G), T \text{ is zero-sum free}\}\).

Similarly, UFIS \( S \) is **dense** if \( k(S) = K_1(G) \) and \(|S| = \min\{|T| : k(T) = K_1(G), T \text{ is a UFIS}\}\).
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In Section 2, we will show that subject to certain conditions dense indexed sequences of both types have few elements of each order, and mostly elements of prime power order. In the rest of this introduction we define the conditions under which these arguments hold and state our main results.

For convenience, we will call a term whose order has more than one prime factor a cross term, following Kriz [20]. The main goal of our arguments is to show the nonexistence of cross terms in dense indexed sequences over certain groups.

We introduce the following definition, which will prove essential in Sections 2 and 3. It turns out that when the primes dividing \( \exp(G) \) are far apart, it is easier to bound the cross terms in dense indexed sequences over \( G \).

**Definition 6.** A prime \( p \) is **wide with respect to** a positive integer \( n \) if \( p \nmid n \) and, given that the prime factorization of \( n \) is \( n = q_1^{\alpha_1} \cdots q_r^{\alpha_r} \), the inequality

\[
\frac{p}{p-1} \geq \prod_{j=1}^{r} \frac{q_j^{\alpha_j+1} - 1}{q_j^{\alpha_j+1} - q_j^{\alpha_j}}
\]

holds. In such a case we write \( p \prec n \). The empty product is taken to be 1. A positive integer \( n = q_1^{\alpha_1} \cdots q_r^{\alpha_r} \) is **wide** if, assuming that \( q_1 < q_2 < \cdots < q_r \), for each \( i \in [1, r-1] \),

\[
q_i \prec \prod_{j=i+1}^{r} q_j^{\alpha_j}.
\]

A prime \( p \) is **2-wide with respect to** \( n = q_1^{\alpha_1} \cdots q_r^{\alpha_r} \) if \( p \nmid n \) and

\[
\frac{p^2 + 2p - 2}{p^2} \geq \prod_{j=1}^{r} \frac{q_j^{\alpha_j+1} - 1}{q_j^{\alpha_j+1} - q_j^{\alpha_j}},
\]

in which case we write \( p \prec_2 n \). Also, \( n \) is **2-wide** if, given \( q_1 < q_2 < \cdots < q_r \), we have for each \( i \in [1, r-1] \),

\[
q_i \prec_2 \prod_{j=i+1}^{r} q_j^{\alpha_j}.
\]

The main drawback of considering cross number instead of length is that when \( \exp(G) \) has many prime factors cross number is difficult to handle. Our methods extend previous results about \( k(G) \) and \( K_1(G) \) when \( \exp(G) \) has a small number of prime factors to cases where \( \exp(G) \) is wide and 2-wide, respectively.

Our main result towards the Conjecture 1 is the following.

**Theorem 7.** If \( G \) is a finite abelian group and \( p \) is a prime satisfying \( p \prec \exp(G) \), then

\[
k(C_p^{\alpha} \oplus G) = k(G) + k(C_p^{\alpha})
\]

for all \( \alpha \in \mathbb{N} \). In particular, if \( k(G) = k^*(G) \) then \( k(C_p^{\alpha} \oplus G) = k^*(C_p^{\alpha} \oplus G) \) as well.

We can prove the following by applying this result to Theorem 3 part (1).

**Corollary 8.** If \( G \) is a finite abelian group, \( \exp(G) \) is wide, and the \( p \)-components of \( G \) are all cyclic except possibly the \( P^+(\exp(G)) \)-component, then \( k(G) = k^*(G) \).
Explicitly, these are the groups of the form \( G = C_{p_1^{\alpha_1}} \oplus C_{p_2^{\alpha_2}} \oplus \cdots \oplus C_{p_{r-1}^{\alpha_{r-1}}} \oplus H_{p_r}, \) where \( H_{p_r} \) is an arbitrary finite abelian \( p_r \)-group. Similarly, if we apply Theorem 7 to Theorem 3 part (2), we get the following. For \( a, b \in \mathbb{N} \), the interval notation \([a, b]\) will denote the sets of integers \( \{m : a \leq m \leq b\} \).

**Corollary 9.** If \( G \) is a finite abelian group of the form
\[
G = C_{p_1^{\alpha_1}} \oplus C_{p_2^{\alpha_2}} \oplus \cdots \oplus C_{p_{r-1}^{\alpha_{r-1}}} \oplus C_{p_r^{\alpha_r}} \oplus C_{p_{r+1}^{\alpha_{r+1}}},
\]
where the \( p_i \) are distinct primes satisfying \( p_i < p_{i+1} \cdots p_r^{-1} p_r \), for all \( i \in [1, r-2], \alpha_{r-1} \geq \alpha_{r-2}^* \), and \( s \) is a nonnegative integer, then \( k(G) = k^*(G) \).

Our main theorem towards the \( K_1 \) conjecture, proved in Section 3, is the following.

**Theorem 10.** If \( G \) is a finite abelian group and \( p \) is a prime with \( p \prec_2 \exp(G) \), then
\[
K_1(C_{p^\alpha} \oplus G) = K_1(C_{p^\alpha}) + K_1(G)
\]
for all \( \alpha \in \mathbb{N} \). In particular, if \( K_1(G) = K_1^*(G) \) then \( K_1(C_{p^\alpha} \oplus G) = K_1^*(C_{p^\alpha} \oplus G) \) as well.

This theorem, combined with the currently known values of \( K_1 \) listed in Proposition 4, proves Conjecture 2 for the following cases.

**Corollary 11.** If \( n \) is a 2-wide positive integer, and \( q = P^+(n) \), then \( K_1(C_n) = K_1^*(C_n) \) and \( K_1(C_n \oplus C_q) = K_1^*(C_n \oplus C_q) \).

These are the first cases of Conjecture 2 known to be true when \( \exp(G) \) has arbitrarily many prime factors. In Section 4 we will strengthen this result to the case when \( q \) is the second largest prime factor of \( n \).

In Section 2, we prove the key structural lemmas that are necessary to the proper bounding of dense indexed sequences over the groups under consideration. Using this machinery, we will prove our main theorems in Section 3. Additionally, in Section 4 we will use a similar argument to derive an inductive result on \( p \)-groups, generalizing a theorem of Kriz [20]. This will also strengthen Corollary 11 to account for \( q \) being the second largest prime factor of \( n \). Finally, in Section 5 we pose two conjectures about dense UFIS’s over elementary \( p \)-groups \( C_{p}^{k} \), which are a major roadblock to the resolution of Conjecture 2, and discuss generalizing zero-sum problems by picking other weighting functions for indexed sequences instead of the length or cross number.

### 2. The Structure of Dense Indexed Sequences

Henceforth, we will decompose a finite abelian group \( G \) in the canonical form
\[
(2.1) \quad G = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{k_i} C_{p_i^{\alpha_{i,j}}},
\]
where \( p_1, p_2, \ldots, p_r \) are distinct primes and for each \( i \in [1, r], \) we assume \( \alpha_{i,1} \geq \alpha_{i,2} \geq \alpha_{i,3} \geq \cdots \geq \alpha_{i,k_i} \). If \( n > k_i \) then \( \alpha_{i,n} \) is taken to be zero.

We generalize a definition from [2].

**Definition 12.** By **amalgamating** a subsequence \( T \) of a indexed sequence \( S \) we mean the operation \( S \mapsto ST^{-1}(\sigma(T), n) \), i.e. that of replacing \( T \) with its sum. The index \( n \) is the smallest nonnegative integer for which \( ST^{-1}(\sigma(T), n) \) is squarefree.
Amalgamating any subsequence of a UFIS preserves its unique factorization, and amalgamating any subsequence of a zero-sum free indexed sequence keeps it zero-sum free. We will show that amalgamation can often be performed without decreasing $k(S)$.

Our key result is the following “Amalgamation Lemma,” which has two parts, one for each type of indexed sequence. Informally, we say that an indexed sequence $S$ contains $n$ elements of order $\ell$ if

$$\sum_{\text{ord}(g) = \ell} \bar{v}_g(S) = n.$$  

**Lemma 13.** (Amalgamation Lemma.) Let $G$ be a group of the form (2.1), and suppose that $\alpha_{i,1} > \alpha_{i,2}$ for some $i \in [1, r]$. Let $\ell$ be a positive integer divisible by $p_i^{\alpha_{i,2} + 1}$.

If $S$ is a dense zero-sum free indexed sequence over $G$, then $S$ contains at most $p_i - 1$ elements of order $\ell$.

If $S$ is a dense UFIS over $G$, then $S$ contains at most $p_i$ elements of order $\ell$.

For the proof of the UFIS case, we will require the following lemma, which generalizes Lemma 2.2 from [6].

**Lemma 14.** If $G$ is a finite abelian group and $S$ is an indexed sequence over $G$, then $S$ divides a UFIS if and only if for any two zero-sum subsequences $U$ and $V$ of $S$, $\gcd(U, V)$ is also zero-sum.

**Proof.** In one direction, suppose $S$ divides a UFIS $T$ and $S$ has two zero-sum subsequences $U$ and $V$ for which $\gcd(U, V) \neq 0$. In particular, there are some irreducible subsequences $U'\mid U$ and $V'\mid V$ which have nontrivial greatest common divisor. Then, factorizations of $T(U')^{-1}$ and $T(V')^{-1}$ give rise to distinct factorizations of $T$, so we get

$$|\pi^{-1}(T)| \geq |\pi^{-1}(T(U')^{-1})| + |\pi^{-1}(T(V')^{-1})| \geq 2,$$

contradicting the unique factorization of $T$.

In the other direction, suppose that $S$ has no two zero-sum subsequences with nonzero sum greatest common divisor. Then, we claim that either $S$ is zero-sum and thus already a UFIS, or else $S(-\sigma(S), n)$ is a UFIS, where we choose $n$ so that $S(-\sigma(S), n)$ squarefree.

In the first case, if $S$ is already zero-sum then it must have unique factorization. For, if it had two distinct irreducible factorizations $S = U_1 U_2 \cdots U_m = V_1 V_2 \cdots V_n$, then some $V_i$ intersects $U_1$ but $\gcd(V_i, U_1)$ cannot be zero-sum, contradicting our assumption.

In the second case, suppose that $T = S(-\sigma(S), n)$ does not have unique factorization. We can reduce to the first case by showing that no two zero-sum subsequences of $T$ have nonzero sum greatest common divisor. Suppose otherwise; let two zero-sum subsequences $U$ and $V$ of $T$ satisfy $\sigma(gcd(U, V)) \neq 0$. If $(-\sigma(S), n)$ divides $U$, we can replace $U$ by $TU^{-1}$, and assume $(-\sigma(S), n)$ does not divide $U$. Similarly, we assume $(-\sigma(S), n)$ does not divide $V$. Thus we have two zero-sum subsequences of $T(-\sigma(S), n)^{-1} = S$ whose greatest common divisor is not zero-sum. This is a contradiction, so we’re done.  

We remark that Lemma 14 is still true if we stipulate that $U$ and $V$ are irreducible, with the same proof. This is convenient for the following argument.

**Proof.** (of Lemma 13) We assume $\ell | \exp(G)$, or the lemma is trivial.

Here is some notation and motivation that we will use in both cases. Let

$$S_\ell = \prod_{\text{ord}(g) = \ell} g_{v_g(S)},$$

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where the product is over all \( g \in G^* \times \mathbb{N} \) of order \( \ell \), and define \( H_k \) to be the subgroup of \( G \) consisting of all elements with order dividing \( k \), for any positive integer \( k \mid \text{exp}(G) \). Now, \( S_\ell \) can be thought of as an indexed sequence over \( H_\ell \). We show that if the conditions of the lemma are false then there is some subsequence of \( S_\ell \) of length at most \( p_1 \) that can be amalgamated into one nonzero element of \( G \) of order dividing \( \ell/p_1 \). Clearly, this amalgamation operation does not decrease \( k(S) \), but decreases \( |S| \), so \( S \) must not be dense, which is a contradiction.

Consider the quotient map \( Q : H_\ell \to H_\ell / H_{\ell/p_1} \), the codomain being isomorphic to \( C_{p_1} \) because \( p_1^{\alpha_{1,2}+1}/\ell \), and extend it via the unlabelling homomorphism to a monoid homomorphism \( \overline{Q} : \mathcal{F}(H_\ell \times \mathbb{N}) \to \mathcal{F}(C_{p_1} \times \mathbb{N}) \). Let \( T = \overline{Q}(S_\ell) \).

In the case that \( S \) is a dense zero-sum free indexed sequence, suppose \( |T| \geq p_1 \). Then, it is easy to show that \( T \) has a nontrivial zero-sum subsequence \( T_0 \) with length at most \( p_1 \). Let \( S_0 \in \overline{Q}^{-1}(T_0) \) be some corresponding subsequence of \( S \), with \( \overline{\sigma}(S_0) \neq 0 \) since \( S \) is zero-sum free.

But \( Q(\overline{\sigma}(S_0)) = 0 \), so it follows that \( \overline{\sigma}(S_0) \in \ker Q = H_{\ell/p_1} \), and so \( \text{ord}(\overline{\sigma}(S_0)) \mid \frac{\ell}{p_1} \). Thus \( S_0 \) can be amalgamated in the fashion described, and we have a contradiction. This proves that any dense zero-sum free indexed sequence \( S \) must have at most \( p_1 - 1 \) elements of order \( \ell \).

If \( S \) is a dense UFIS, suppose \( |T| \geq p_1 + 1 \). We claim that \( T \) must have two intersecting irreducible subsequences \( T_0 \) and \( T_0' \). Otherwise, it is a consequence of Lemma 14 that \( T \) divides a UFIS. However, \( |T| > p_1 = N_1(C_{p_1}) \), so this is impossible.

Let \( S_0 \in \overline{Q}^{-1}(T_0) \) and \( S_0' \in \overline{Q}^{-1}(T_0') \). Because \( K(C_{p_1}) = p_1 \), both \( |S_0| = |T_0| \leq p_1 \) and \( |S_0'| = |T_0'| \leq p_1 \) must hold. Because

\[
Q(\overline{\sigma}(\gcd(S_0, S_0'))) = \overline{\sigma}(\gcd(T_0, T_0')) \neq 0,
\]

we find that \( \gcd(S_0, S_0') \) is not zero-sum. Therefore, if \( \overline{\sigma}(S_0) = \overline{\sigma}(S_0') = 0 \), then \( S \) cannot be a UFIS, by Lemma 14. We may assume without loss of generality that \( \overline{\sigma}(S_0) \neq 0 \). Then \( S_0 \) is the subsequence of \( S_\ell \) we are seeking to amalgamate, and it satisfies all the desired conditions. It follows by contradiction that \( |S_\ell| \leq p_1 \), as desired.

As a consequence of Lemma 13 we have a bound on the number of cross terms in dense indexed sequences. To eliminate them altogether, we need a stronger hypothesis, namely that of wide (or 2-wide) exponent.

**Lemma 15.** Let \( G \) be of the form (2.1) with \( \alpha_{1,1} > \alpha_{1,2} \), and let \( a \in [\alpha_{1,2} + 1, \alpha_{1,1}] \).

If \( S \) is a dense zero-sum free indexed sequence over \( G \) and

\[
p_1 < p_2^{\alpha_{2,1}} p_3^{\alpha_{3,1}} \cdots p_r^{\alpha_{r,1}},
\]

then \( S \) contains at least \( p_1 - 1 \) elements of order \( p_1^a \).

If \( S \) is a dense UFIS over \( G \) and

\[
p_1 < p_2^{\alpha_{2,1}} p_3^{\alpha_{3,1}} \cdots p_r^{\alpha_{r,1}},
\]

then \( S \) contains at least \( p_1 - 1 \) elements of order \( p_1^a \).

**Proof.** We start with the zero-sum free case. Suppose for the sake of contradiction that \( S \) contains at most \( p_1 - 2 \) elements of order \( p_1^a \). Remove the subsequence \( S' \) of \( S \) consisting of all its elements with order divisible by \( p_1^a \), and replace them by the indexed sequence...
is the key result that makes the inequalities in the definitions of wideness and 2-wideness necessary. Henceforth, we always assume that \( p_1 \) is the smallest prime dividing \( \exp(G) \).

We will also investigate the general case. If \( e \) is a generator of the component \( C_{\alpha_{1,1}} \) in \( G \), then we define this indexed sequence \( T \) as follows. First, define the sequence

\[
T_0 = e^{p_1 - 1}[p_1 e]^{p_1 - 1} \cdots [p_1^{\alpha_{1,1} - a} e]^{p_1 - 1},
\]

Then, choose \( T \in \alpha^{-1}(T_0) \) with \( ST \) squarefree. Since \( T \) has no subsequence sums with order not divisible by \( p_1 \), \( S_1 \) is still zero-sum free. It remains to show that \( k(S_1) \geq k(S) \), or equivalently that \( k(T) \geq k(S') \). Note that by the Amalgamation Lemma, we get an upper bound on \( k(S') \) simply by bounding the number of terms of each order, from which we can show

\[
k(T) - k(S') \geq \sum_{i=a}^{\alpha_{1,1}} \frac{p_1 - 1}{p_i} - \left( \frac{p_1 - 2}{p_1^a} \sum_{1 < d | n_1} \frac{p_1 - 1}{dp_i^a} \right),
\]

where \( n_1 = n/p_1^a \). Thus it suffices to show that if \( n' = n/p_1^{\alpha_{1,1}} \), then

\[
1 \geq (p_1 - 1) \left( \sum_{d | n'} \frac{1}{d} - 1 \right).
\]

Now, it is easy to calculate given the factorization \( n' = p_2^{\alpha_{2,1}} \cdots p_r^{\alpha_{r,1}} \) that

\[
\prod_{j=2}^{r} \frac{p_j^{\alpha_{j,1} + 1} - 1}{p_j^{\alpha_{j,1}} - p_j^{\alpha_{j,1}}} = \sum_{d | n'} \frac{1}{d},
\]

so it suffices to have

\[
\frac{p_1}{p_1 - 1} \geq \prod_{j=2}^{r} \frac{p_j^{\alpha_{j,1} + 1} - 1}{p_j^{\alpha_{j,1}} - p_j^{\alpha_{j,1}}}.
\]

This is exactly the wideness condition we assumed to be true. Finally, it is clear that \( |T| < |S'| \) if \( k(T) = k(S') \), so since we have \( k(T) \geq k(S) \), this contradicts the denseness of \( S \). Thus \( S \) has exactly \( p_1 - 1 \) elements of order \( p_1^a \).

For the second case, that \( S \) is a dense UFIS, suppose for the sake of contradiction that \( S \) contains at most \( p_1 - 2 \) elements of order \( p_1^a \). As before we replace the subsequence \( S' \) of all elements of \( S \) with order divisible by \( p_1^a \) by the indexed sequence \( T' \) consisting of \( p_1 \) elements of each order \( p_1^a, p_1^{a+1}, \ldots, p_1^{\alpha_{1,1}} \). Explicitly, this indexed sequence is chosen so that \( S(S')^{-1}T'' \) is squarefree and

\[
\alpha(T'') = e^{p_1 - 1}[(1 - p_1)e]^{p_1 - 1}[(1 - p_1)e]^{p_1 - 1} \cdots [p_1^{\alpha_{1,1} - a} e]^{p_1}.
\]

After replacement the indexed sequence can still be extended to a UFIS using Lemma 14, if it is not one already.

Again, we need to show \( k(S(S')^{-1}T') \geq k(S) \), i.e. \( k(T') \geq k(S') \). Transforming as in the zero-sum free case, except that \( T' \) now has at least \( 2 \) more elements of order \( p_1^a \) than does \( S' \), the requirement is exactly the 2-wideness condition we assumed. This is a contradiction once more, so \( S \) has at least \( p_1 - 1 \) elements of order \( p_1^a \). \( \square \)

Lemma 15 is the key result that makes the inequalities in the definitions of wideness and 2-wideness necessary. Henceforth, we always assume \( p_1 \) is the smallest prime dividing \( \exp(G) \).

We will mostly consider the case \( k_1 = 1 \), which allows us to pick any \( a \in [1, \alpha_{1,1}] \) for Lemma 15, but in Section 4 we will also investigate the general case.
3. Proof of Main Theorem

Combining the structural results of Section 2, we have enough to prove our main theorems. Let \( p \) and \( G \) satisfy the conditions of Theorems 7 and 10, so that \( p < \exp(G) \) or \( p < 2 \exp(G) \), respectively. Write \( p_1 = p, \alpha_{1,1} = \alpha \), and let the prime divisors of \( \exp(G) \) be \( p_2, p_3, \ldots, p_r \). Define \( G' = C_{p_1^{\alpha_{1,1}}} \oplus G \).

**Proof.** (of Theorem 7) Let \( S \) be a dense zero-sum free indexed sequence over \( G' \). By the Amalgamation Lemma and Lemma 15, we find that \( S \) has at exactly \( p_1 - 1 \) terms of each order \( p_1^a \), where \( a \in [1, \alpha_{1,1}] \). Let \( H = C_{p_1^{\alpha_{1,1}}} \) be the \( p_1 \)-component of \( G' \), and let \( S' = S_H \). Then \( k(S') = k(H) = k^*(H) \). It is not difficult to show from here that \( S' \) has full sumset in \( H \).

But then, we find that \( S(S')^{-1} \) cannot have any subsums lying in \( H \setminus \{0\} \), or else together with some subsequence of \( S' \) one could form a zero-sum subsequence of \( S \). Therefore, we find that even after projecting \( G' \rightarrow G \), the image of \( S(S')^{-1} \) is still zero-sum free. As a result, \( k(S(S')^{-1}) \leq k(G) \), since projection cannot decrease cross number. We have

\[
k(S) \leq k(H) + k(G) = k(C_{p_1^{\alpha_{1,1}}}) + k(G),
\]

as desired. It follows that \( k(G') = k^*(G') \) if \( k(G) = k^*(G) \), since \( k^* \) is additive over direct sums. \( \square \)

The proof of Theorem 10 is slightly more subtle, because Lemma 15 is too weak by itself.

**Proof.** (of Theorem 10) Let \( S \) be a dense UFIS over \( G' \). It follows from Lemma 15 that \( S \) contains at least \( p_1 - 1 \) terms of each order \( p_1^a \), for \( a \in [1, \alpha_{1,1}] \). Let \( H = C_{p_1^{\alpha_{1,1}}} \) be the \( p_1 \)-component of \( G \), and let \( S' = S_H \). We have already that \( S' \) is a subsequence of a UFIS over \( H \) and contains at least \( p_1 - 1 \) terms of each order dividing \( p_1^{\alpha_{1,1}} \). It is not difficult to show that \( S' \) must have full sumset over \( H \) under these conditions.

Now, we show that in fact \( S' \) has exactly \( p_1 \) elements of each of these orders. Suppose this is not the case, and choose \( a \) to be minimal for which \( S' \) contains only \( p_1 - 1 \) terms of order \( p_1^a \), forming a subsequence \( S''|S' \). We may assume \( \sigma(S'') \neq 0 \), since if \( \sigma(S'') = 0 \) then we could replace \( S'' \) in \( S' \) with a sequence of \( p_1 \) terms of the same order, increasing the cross number of \( S' \).

Because \( S \) has \( p_1 \) elements of each order lower than \( a \), if \( H' = C_{p_1^a} \) then \( S_{H'} \) has full sumset over \( H' \). Because the elements of lower order all sum to zero, there can be at most one zero-sum free subsequence of \( S(S_{H'})^{-1} \) which has sum in \( H' \) and this sum must be of order exactly \( p_1^{a-1} \); otherwise, \( S \) would have two distinct intersecting irreducible subsequences.

Now, by removing one element \((b, m)\) from this zero-sum free subsequence of \( S(S_{H'})^{-1} \), if it exists, we are then allowed to insert \((-\sigma(S''), m)\) into \( S \), to construct an indexed sequence \( S(b, m)^{-1}(-\sigma(S''), m) \) which is a subsequence of a UFIS. This \( b \) can be picked to have order divisible by \( p_1^a \), since the sum of the indexed sequence containing it must have order \( p_1^a \). But then

\[
k(S(b, m)^{-1}(-\sigma(S''), m)) > k(S),
\]

so it follows that our assumptions were false and \( S \) contains exactly \( p_1 \) elements of order \( p_1^a \).

We can now follow the same argument as for the proof of 7, showing that \( k(S) \leq k_1(C_{p_1^{\alpha_{1,1}}}) + k_1(G) \) and proving the theorem from there. \( \square \)
As a direct consequence, Theorem 10 proves Conjecture 2 for all groups of the form $C_{p^a} \oplus C_{q^a}$ or $C_{p^a} \oplus C_{q^a} \oplus C_q$, with $p < q$, strengthening Proposition 4 part (2). This follows from the fact that

$$\frac{p^2 + 2p - 2}{p^2} \geq \frac{q}{q - 1} > \frac{q^\beta + 1 - 1}{q^\beta + 1 - q^\beta}$$

holds whenever $p < q$, for any $\beta \geq 1$.

4. An Inductive Result Lifting Exponents

Theorem 10 is an inductive result of the form: given a group $G$ of a certain structure, if $K_1(G) = K_1^*(G)$, then this is also true for some group containing $G$. In this section we derive more results of this form: if $G = \bigoplus C_{p^a}$, and $K_1(G) = K_1^*(G)$, we give conditions under which we can increase the largest exponent on some prime dividing $\exp(G)$. Specifically, we will be able to show this when $G$ is a $p$-group, and when $G$ is of the form $C_p \oplus C_{p^\alpha} \oplus C_{q^\beta}$. For the $p$-group case, we prove the following.

Proposition 16. If $p$ is prime, and $\alpha_1 \geq \max\{\alpha_2, \alpha_3, \ldots, \alpha_r\}$, then

$$K_1(C_{p^{\alpha_1+1}} \oplus C_{p^{\alpha_2}} \oplus \cdots \oplus C_{p^{\alpha_r}}) = K_1(C_{p^{\alpha_1}} \oplus C_{p^{\alpha_2}} \oplus \cdots \oplus C_{p^{\alpha_r}}) + \frac{1}{p^{\alpha_1}}.$$  

Proof. That the left hand side is at least the right hand side is immediate. In the other direction, the proof follows by applying Lemma 13 with $p_1 = p$ and $a = \alpha_1 + 1$. Let $G$ be the group on the left hand side of (4.1), and $H$ be the group on the right, identified naturally as a subgroup of $G$. We find that a dense UFIS over $G$ can have at most $p$ elements of order $p^{\alpha_1+1}$, by the Amalgamation Lemma, and the remaining terms form a subsequence of a UFIS over $H$. The result follows.

Proposition 16 generalizes Theorem 6 part (1) of Kriz, which states that $K_1(C_{p^a} \oplus C_{q^a}) \leq K_1(C_{p^a}) + K_1(C_{q^a}) - 1$ [20]. We now extend this technique to prove the following generalization of Theorem (4) part (2).

Proposition 17. If $G = C_p \oplus C_{p^\alpha} \oplus C_{q^\beta}$ for $p, q$ distinct primes and $\alpha, \beta$ positive integers, then $K_1(G) = K_1^*(G)$.

Proof. Theorem 10 is enough when $q$ is 2-wide with respect to $\{p\}$, i.e. when

$$\frac{q^2 + 2q - 2}{q^2} \geq \frac{p}{p - 1},$$

so we may assume the opposite.

Now, let $S$ be a dense UFIS over $G$, and let $S$ have $t_a$ elements of order $p^a$, for each $a \in [1, \alpha]$. Since $p$ is 2-wide with respect to $\{q\}$, by directly applying Lemma 15, we find $t_a \geq p - 1$ for each $a \geq 2$. For $a = 1$ we make the following special argument.

If $t_1 \leq 2p - 2$, then we can replace the subsequence of $S$ consisting of all elements with order divisible by $p$ with the canonical example consisting of $2p$ elements of order $p$ and $p$ of each order $p^a$ with $a > 1$. If the change in $k(S)$ is $\delta$, this $\delta$ is bounded below by an increase of $2/p$ plus a decrease of at most the total cross number of the cross terms of $S$. Of these, there are at most $q$ terms of each order $pq^b$ and at most $p$ of each order $p^{1+a}q^b$, where $a, b > 0$. 

Thus

\[(4.3) \quad \delta > \frac{2}{p} - q \sum_{b=1}^{\beta} \frac{1}{pq^b} - p \sum_{a,b>0} \frac{1}{p^{1+a}q^b} = \frac{2}{p} - \frac{q}{p(q-1)} - \frac{1}{(p-1)(q-1)}.\]

It remains to prove \(\delta > 0\). The right side of (4.3) is always nonnegative unless \(q = 2\), which is impossible because we assumed \(q\) is not 2-wide with respect to \(\{p\}\). Thus, we have shown that there are at least \(2p - 1\) elements of \(S\) of order \(p\), and the last argument of the proof of Theorem 10 proves that \(S\) must then have exactly \(2p\) elements of this order. Similarly, we find that \(S\) must have exactly \(p\) elements of each order \(p^a\) with \(a > 1\), so we can finish in the same way as in the proof of Theorem 10.

As a corollary, we can strengthen Corollary 11 by combining this result with Theorem 10.

**Corollary 18.** If \(n\) is a 2-wide positive integer, and \(q\) is its largest or second largest prime factor, then \(K_1(C_n) = K_1^*(C_n)\) and \(K_1(C_n \oplus C_q) = K_1^*(C_n \oplus C_q)\).

5. Concluding Remarks

The study of unique factorization over elementary \(p\)-groups \(C^n_p\) is particularly important to Conjecture 2 because every nonzero element of \(C^n_p\) has order \(p\), so we have \(k(S) = \frac{1}{p} |S|\) for all \(p\). This problem is also interesting because the study of zero-sum sequences over \(p\)-groups lends itself to algebraic techniques from representation theory, see for example Chapter 5.5 of [1] or [23, 24, 25], giving strong results that are not available in the general case.

Study of the Narkiewicz constant \(N_1(G)\) is not limited by the same obstacles as study of \(K_1(G)\) in the case that \(\exp(G)\) has many prime factors; the \(N_1\) conjecture has been shown for all groups of rank at most 2 [6, 7, 8].

However, in the rank direction \(N_1(G)\) and \(K_1(G)\) have proved equally difficult to compute. Moreover, it suffices to prove the following statement to resolve the elementary \(p\)-group case in both the \(N_1\) and \(K_1\) conjectures.

**Conjecture 19.** If \(G = C^n_p\), then \(N_1(G) = np\).

Conjecture 19 has been shown for the following cases.

**Proposition 20.** [3] If \(G\) is one of the following groups then \(N_1(G) = np\).

1. \(G = C^n_2, C^n_3\).
2. \(G = C^n_p\).
3. \(G = C^n_5, C^n_4, C^n_7\).

Note that although case 3 was not specifically mentioned by Gao [3], the same methods in that paper apply. Now we transform Conjecture 19 into a problem about finding certain sub-sequences of zero-sum free indexed sequences over \(G\). Again, we begin with some structural results about maximal-length UFIS’s over \(C^n_p\).

Henceforth, let \(S\) be a UFIS over \(G = C^n_p\) that factors into irreducibles as \(S = U_1U_2 \cdots U_t\).

By Proposition 1 of [3], at least \(|S| - n(p - 1)\) of the \(U_i\) have odd length, and furthermore

\[(5.1) \quad \prod_{i=1}^t |U_i| \leq |G| = p^n.\]
We say a $U_i$ is \textit{optimal} in $S$ if it cannot be replaced by any longer irreducible indexed sequence while preserving the unique factorization of $S$. Notice that the only property necessary to satisfy is the following.

**Proposition 21.** [3] If $U_1$ and $S'$ are an irreducible indexed sequence and a UFIS, respectively, then $U_1S'$ is a UFIS if and only if $\Sigma(U_1) \cap \Sigma(S') = \{0\}$.

Thus $U_i$ is optimal in $S$ if and only if it is the longest irreducible indexed sequence over $G$ which has sumset equal to $\{0\} \cup (G\setminus \Sigma(S'))$, where $S' = SU_i^{-1}$.

Note that if $|U_i| \leq p$ for all $i$, then $|S| \leq np$ by the concavity of the logarithm and (5.1). Henceforth a zero-sum indexed sequence is called \textit{short} if it has length at most $p$ and \textit{long} otherwise. A zero-sum free indexed sequence is called \textit{short} if it has length less than $p$ and \textit{long} otherwise. The following conjecture would resolve the case of longer $U_i$.

**Conjecture 22.** If $T$ is a zero-sum free indexed sequence over $G = C_p^k$, then there exists a subsequence $T_0$ of $T$ with length less than $p$ such that no other subsequence of $T$ has the same sum as $T_0$.

Note that the statement is trivial for $|T| < p$, since we can just take $T_0 = T$, and the sum is unique because $T$ is zero-sum free. Also, it suffices to show that $T$ has a proper subsequence $T_0$ (not necessarily with length less than $p$) satisfying this property, since we can apply the result inductively. To see why this conjecture implies Conjecture 19, we claim that any maximal UFIS over $G$ must have only short irreducible subsequences. Otherwise, remove one element from a long $U_i$ and apply Conjecture 22 to the remaining zero-sum free subsequence to find that $U_i$ can be replaced by a pair of short irreducible indexed sequences of longer total length.

Next, we offer the following strengthening of Conjecture 22.

**Conjecture 23.** If $T$ is a zero-sum free indexed sequence over $G = C_p^k$, then there exists a cyclic subgroup $H$ of $G$ such that $T_H \neq 1$ and $H \cap \Sigma(TH^{-1}) = \emptyset$.

To see that Conjecture 23 implies Conjecture 22, take $T_0 = T_H$.

For the study of the group invariants $k$, $K_1$, and $K_1$ when $\exp(G)$ has many prime factors, our methods rely on conditions such as wideness and 2-wideness. We are hopeful that a fully general solution removing these conditions, or improvements on the inequalities in the wide and 2-wide conditions, can be made.

If we weight indexed sequences more severely in the following manner we can drop the wideness and 2-wideness conditions. Define weighting function $f$ to be the totally multiplicative function on the positive integers with $f(p_i) = 1/2^i$, where $p_i$ is the $i$-th prime. If $S$ is an indexed sequence over a finite abelian group $G$ define

$$k(S, f) = \sum_{g \in G} \overline{p}_g(S) \cdot f(\text{ord}(g)).$$

Let $k(G, f), k^*(G, f), K_1(G, f)$ and $K_1^*(G, f)$ be defined in the natural ways. It is not difficult to show the following using our methods, in place of Theorems 7 and 10. Recall that $P^-(n)$ is the smallest prime dividing $n$.

**Proposition 24.** If $p < P^- (\exp(G))$ and $f$ is defined as above, then $k(C_{p^\alpha} \oplus G, f) = k(G, f) + k(C_{p^\alpha}, f)$ and $K_1(C_{p^\alpha} \oplus G, f) = K_1(C_{p^\alpha}, f) + K_1(G, f)$. 

Note that $D(G)$ and $N_1(G)$ are, respectively, the cases of $K(G, f)$ and $K_1(G, f)$ when $f = 1$, and no additive result like Proposition 24 is true for either of them. A study of other choices of weighting functions $f$, and the structures of the dense indexed sequences that result, might shed light on all of the conjectures we are interested in.

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