THE BERGMAN ANALYTIC CONTENT OF PLANAR DOMAINS

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ABSTRACT. Given a planar domain $\Omega$, the Bergman analytic content measures the $L^2(\Omega)$-distance between $\bar{z}$ and the Bergman space $A^2(\Omega)$. We compute the Bergman analytic content of simply-connected quadrature domains with quadrature formula supported at one point, and we also determine the function $f \in A^2(\Omega)$ that best approximates $\bar{z}$. We show that, for simply-connected domains, the square of Bergman analytic content is equivalent to torsional rigidity from classical elasticity theory, while for multiply-connected domains these two domain constants are not equivalent in general.

1. Introduction

Recall that, for a bounded domain $\Omega$ in $\mathbb{C}$, the Bergman space $A^2(\Omega)$ is the Hilbert space of functions holomorphic in $\Omega$ that satisfy $\int_{\Omega} |f(z)|^2 dA(z) < \infty$. Extending previous studies on the approximation of $\bar{z}$ in analytic function spaces, in [6] the authors introduced the notion of Bergman analytic content, defined as the $L^2(\Omega)$-distance between $\bar{z}$ and the space $A^2(\Omega)$. They showed that the best approximation to $\bar{z}$ is 0 if and only if $\Omega$ is a disk, and that the best approximation is $\frac{1}{\pi}$ if and only if $\Omega$ is an annulus centered at the origin.

D. Khavinson and the first author in [5] reduced the problem of finding the best approximation to that of solving a Dirichlet problem in $\Omega$ with boundary data $|z|^2$. Using this fact, we will determine the best approximation and Bergman analytic content when $\Omega$ is a simply-connected, one-point quadrature domain.

Recall that a bounded domain $\Omega \subset \mathbb{C}$ is called a quadrature domain if it admits a formula expressing the area integral of any test function $g \in A^2(\Omega)$ as a finite sum of weighted point evaluations of $g$ and its derivatives:

$$
(1.1) \quad \int_D g(z) dA(z) = \sum_{m=1}^N \sum_{k=0}^{n_m} a_{m,k} g^{(k)}(z_m),
$$

where the points $z_m \in \Omega$ and constants $a_{m,k}$ are each independent of $g$. This class of domains is $C^\infty$-dense in the space of domains having a $C^\infty$-smooth boundary [2 Thm. 1.7], and the restricted class of quadrature domains for which $N = 1$ in (1.1) has the same density property. When $\Omega$ is a simply-connected quadrature domain with $N = 1$, the conformal mapping $\phi : \mathbb{D} \to \Omega$ is a polynomial, and by making a translation we may assume that the quadrature formula is supported at $\phi(0) = 0$. 
Theorem 1.1. Let $\Omega \subset \mathbb{C}$ be a simply-connected quadrature domain with quadrature formula supported at a single point (say the origin), and let $\phi : \mathbb{D} \to \Omega$ be the (polynomial) conformal map from the unit disk

$$\phi(z) = \sum_{k=1}^{n} a_k z^k.$$ 

Then the Bergman analytic content $\lambda_{A^2}(\Omega)$ of $\Omega$ is given by:

$$\lambda_{A^2}(\Omega) = \frac{\pi}{2^{1/2}} \left[ \sum_{m=1}^{2n-1} \frac{|c_m|^2}{m+1} - \sum_{k=1}^{n-k} k \left| \sum_{j=1}^{n-k} a_{k+j} \right| \right]^{1/2},$$

where

$$c_m := \sum_{k+j=m+1} k a_k a_j \quad 1 \leq k, j \leq n.$$ 

Moreover, the best approximation $f$ to $\bar{z}$ is the derivative $f = F'$ of $F = P \circ \phi^{-1}$, where

$$P(\zeta) = \frac{1}{2} \sum_{k=1}^{n} |a_k|^2 + \sum_{k=1}^{n-k} \sum_{j=1}^{n-k} a_{k+j} \zeta^k.$$ 

In [5], the authors characterized domains for which the best approximation to $\bar{z}$ is a polynomial, showing in particular that the only quadrature domain with this property is a disk. On the other hand, Theorem 1.1 reveals that for a large class of quadrature domains the best approximation $f$ has a primitive $F$ that becomes a polynomial in the right coordinate system.

Let $\Omega$ be a domain in the plain, bounded by finitely many Jordan curves $\Gamma_0, \ldots, \Gamma_n$, with $\Gamma_0$ the outer boundary curve. Then the torsional rigidity of $\Omega$ equals

$$\rho(\Omega) := \int_{\Omega} |\nabla \nu|^2 \, dA,$$

where $\nu$ solves the Dirichlet problem

$$\begin{cases}
\Delta \nu = -2 & \text{in } \Omega \\
\nu|_{\Gamma_0} = 0 \\
\nu|_{\Gamma_i} = c_i & i = 1, \ldots, n,
\end{cases}$$

where the constants $c_i$ are not known a priori but are determined from the conditions

$$\int_{\Gamma_i} \partial_n \nu ds = 2a_i, \quad i = 1, \ldots, n,$$

where $\partial_n$ denotes differentiation in the direction of the outward normal, $ds$ is the arclength element, and $a_i$ is the area enclosed by $\Gamma_i$ (cf. [1], pp. 63-66)). The function $\nu$ is referred to as the “stress function” in elasticity theory, and the torsional rigidity measures the resistance to twisting of a cylindrical beam with cross section $\Omega$. 
In [5] the inequality
\[ \sqrt{\rho(\Omega)} \leq \lambda_{A^2}(\Omega). \]
was shown to hold for simply-connected domains. We show that this is an equality for
simply-connected domains.

**Theorem 1.2.** Suppose \( \Omega \) is a bounded, simply-connected domain. Then
\[ \lambda_{A^2}(\Omega)^2 = \rho(\Omega). \]

Theorem 1.2 led us to notice that some of the methods and examples in the current
paper overlap with classical studies of torsional rigidity [7, Ch. 22], and while the explicit
formulas in Theorem 1.1 appear to be new, our proof based on conformal mapping is very
similar to the procedure described in [7, Sec. 134]. The square of Bergman analytic
content is not in general equivalent to torsional rigidity. This follows from explicit
computations for doubly-connected domains such as the annulus (see Section 4).

As an extension of Theorem 1.1 it would be interesting to determine the best approxi-
mation to \( \bar{z} \) and the Bergman analytic content for general quadrature domains. Since
the problem again reduces to solving a Dirichlet problem with data \( z \bar{z} \), the procedure
developed in [3] for solving the Dirichlet problem with rational (in \( z \) and \( \bar{z} \)) data seems
promising.

**Outline.** We prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3. We discuss
examples in Section 4. In Section 5 we revisit the class of domains considered in [5] for
which the best approximation to \( \bar{z} \) is a monomial.

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Theorem 1.2 as a conjecture during conversations at the conference “Completeness
problems, Carleson measures, and spaces of analytic functions” at Mittag-Leffler.

2. **Proof of Theorem 1.1**

By the definition of Bergman analytic content, we have
\[ \lambda_{A^2}(\Omega) = \| \bar{z} - f \|_2, \]
where \( f \) is the projection of \( \bar{z} \) onto \( A^2(\Omega) \). By the Pythagorean theorem we then have that
\[ \lambda_{A^2}(\Omega) = \left( \int_\Omega |\bar{z}|^2 dA(z) - \int_\Omega |f(z)|^2 dA(z) \right)^{1/2} = \left( \int_D |\phi\phi'|^2 dA - \int_D |f \circ \phi|^2 |\phi'|^2 dA \right)^{1/2}, \]
where we have changed variables \( z = \phi(\zeta) \), \( dA(z) = |\phi'(\zeta)|^2 dA(\zeta) \). The first term
\[ \int_D |\phi\phi'|^2 dA = \int_D |\phi\phi'|^2 dA \]
is simply the square of the Bergman norm of a polynomial \( \phi\phi' \):
\[ \int_D |\phi\phi'|^2 dA = \pi \sum_{m=1}^{2n-1} \frac{|c_m|^2}{m+1}, \]
where
\[ c_m := \sum_{k+j=m+1} ka_k a_j \quad 1 \leq k, j \leq n, \]
are the coefficients in the expansion of the product $\phi \cdot \phi'$.

In order to compute $\int_{\Omega} |f(z)|^2 dA(z)$, we first find $f$ explicitly. By [5, Thm. 1], $f = F'$, where $u = \text{Re}(F)$ solves the Dirichlet problem
\[
\begin{cases}
\Delta u = 0 \\
u|_{\partial \Omega} = |z|^2.
\end{cases}
\]
Changing coordinates using the conformal map $\phi$, we obtain a harmonic function $\tilde{u} = u \circ \phi$ that solves the following Dirichlet problem in the unit disk:
\[
\begin{cases}
\Delta \tilde{u} = 0 \\
\tilde{u}|_{\mathbb{T}} = \frac{\phi \bar{\phi}}{2}.
\end{cases}
\]
Now, on $\mathbb{T}$ we have that $\phi \bar{\phi} = P(\zeta) + \overline{P(\zeta)}$, where
\[ P(\zeta) = \frac{1}{2} \sum_{k=1}^{n} |a_k|^2 + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} a_{k+j} \overline{a_j} \zeta^k. \]
Since this is a harmonic polynomial, we have that $\tilde{u}(\zeta) = \text{Re}(P(\zeta))$. Thus, $F \circ \phi = P$, and so by the chain rule $(f \circ \phi)(\phi') = p$, where
\[ p(\zeta) = P'(\zeta) = \sum_{k=1}^{n-1} k \sum_{j=1}^{n-k} a_{k+j} \overline{a_j} \zeta^{k-1}. \]
Calculating the Bergman norm of this polynomial, we find that
\[
\int_{\Omega} |f(z)|^2 dA(z) = \int_{\mathbb{T}} |f \circ \phi|^2 |\phi'|^2 dA = \sum_{k=1}^{n-1} k \sum_{j=1}^{n-k} |a_{k+j} a_j|^2.
\]
Combining (2.1) and (2.2), the result follows.

3. Proof of Theorem 1.2

Recall that if $\Omega$ is a simply connected domain, the torsional rigidity $\rho = \rho(\Omega)$ is given by equation (1.3)
\[
\rho = \int_{\Omega} |\nabla \nu|^2 dA,
\]
where $\nu$ is the unique solution to the Dirichlet problem
\[
\begin{cases}
\Delta \nu = -2 \\
\nu|_{\partial \Omega} = 0.
\end{cases}
\]
Consider the function $u(z) := \nu(z) + \frac{|z|^2}{2}$. Then $u$ solves the Dirichlet problem stated in the proof of Theorem 1.1:

$$
\begin{cases}
\Delta u = 0 \\
u|_{\partial \Omega} = \frac{|z|^2}{2}.
\end{cases}
$$

Thus, $u = \text{Re}(F)$, where $f = F'$ is the best approximation to $\bar{z}$.

Letting $\nu$ denote the torsion function, we have:

\[
\rho(\Omega) = \int_\Omega |\nabla \nu|^2 dA \\
= \int_\Omega |2\partial_\nu|^2 dA \\
= \int_\Omega |2\partial_\nu u - 2\partial_\nu \frac{|z|^2}{2}|^2 dA \\
= \int_\Omega |F' - \bar{z}|^2 dA \\
= \int_\Omega |\bar{z} - f|^2 dA \\
= \int_\Omega |z|^2 - |f|^2 dA \\
= \lambda_{A^2}(\Omega)^2,
\]

and this completes the proof.

4. Examples

4.1. Epicycloids. Let us consider the one-parameter family of domains $\Omega$ with conformal map $\phi : \mathbb{D} \to \Omega$, given by $\phi(z) = z + az^n$, with $0 \leq a \leq 1/n$.

Applying Theorem 1.1, we immediately obtain:

$$
\lambda_{A^2}(\Omega) = \sqrt{\frac{\pi (1 + 4a^2 + na^4)}{2}}.
$$

When $a = 1/n$ the domain develops cusps (the case $n = 4$ is plotted in Figure 4.1). The case $n = 2$ and $a = 1/2$ is a cardioid (cf. [8, Sec. 58]).
4.2. The annulus. The following example shows that Theorem 1.2 does not hold in general for multiply-connected domains. Let \( \Omega = \{z : r < |z| < R\} \) be the annulus. The best approximation to \( \bar{z} \) in \( A^2(\Omega) \) is \( f(z) = \frac{C}{z} \), where

\[
C = \frac{R^2 - r^2}{2(\log R - \log r)}
\]

(cf. [5] and [6]). Following the proof in Section 2 we have that

\[
\lambda_{A^2}(\Omega)^2 = \int_{\Omega} |z|^2 - |f|^2 \, dA.
\]

Integrating in polar coordinates we get that

\[
\int_{\Omega} |z|^2 \, dA = \frac{\pi}{2} (R^4 - R^2),
\]

and

\[
\int_{\Omega} \left| \frac{C}{z} \right|^2 \, dA = 2\pi C^2 \int_r^R \frac{1}{\rho} \, d\rho
\]

\[
= \frac{\pi}{2} \frac{(R^2 - r^2)^2}{\log R - \log r}.
\]

Thus, we have that

\[
\lambda_{A^2}(\Omega)^2 = \frac{\pi}{2} \left( (R^4 - r^4) - \frac{(R^2 - r^2)^2}{\log R - \log r} \right),
\]

which is smaller than the torsional rigidity [11, p. 64] of \( \Omega \):

\[
\rho(\Omega) = \frac{\pi}{2} (R^4 - r^4).
\]
So we find that neither Theorem 1.2 nor the inequality (1.4) hold for multiply-connected domains.

4.3. The annular region bounded by a pair of confocal ellipses. We consider the region $G$ between two confocal ellipses that is the image of an annulus $\Omega := \{z \in \mathbb{C} : r < |z| < R\}$ under a Joukowski map $\phi(z) = z + \frac{1}{z}$.

![Figure 4.2. The annular region $G$ when $r = 1.2$, $R = 2.5$.](image)

Following the proof in Section 2, the projection of $\bar{z}$ to the Bergman space is given by $f = F''$, where $u = \text{Re}(F)$ solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 \\ u|_{\partial G} = \frac{|z|^2}{2} \end{cases}.$$ 

The function $\tilde{u} = u \circ \phi$ is harmonic and solves the following Dirichlet problem in the annulus $\Omega := \{\zeta \in \mathbb{C} : r < |\zeta| < R\}$:

$$\begin{cases} \Delta \tilde{u} = 0 \\ \tilde{u}|_{\partial \Omega} = \frac{\phi^2}{2} \end{cases}.$$ 

We make the ansatz

$$2\tilde{u}(\zeta) = A + B \log |\zeta| + C(\zeta^2 + \overline{\zeta}^2) + D \left( \frac{1}{\zeta^2} + \frac{1}{\overline{\zeta}^2} \right).$$

The boundary condition gives:

$$2\tilde{u}(\zeta) = |\zeta|^2 + \frac{1}{|\zeta|^2} + \frac{\zeta}{\zeta} + \frac{\overline{\zeta}}{\overline{\zeta}}, \quad \text{on } \partial \Omega.$$
Using polar coordinates to parameterize the two circular boundary components \( z = r e^{i\theta} \) and \( z = R e^{i\theta} \), we obtain two equations:

\[
A + B \log r + 2 \left( Cr^2 + \frac{D}{r^2} \right) \cos(2\theta) = r^2 + \frac{1}{r^2} + 2 \cos(2\theta),
\]

\[
A + B \log R + 2 \left( CR^2 + \frac{D}{R^2} \right) \cos(2\theta) = R^2 + \frac{1}{R^2} + 2 \cos(2\theta),
\]

which implies the system of equations for \( A, B, C, D \)

\[
Cr^2 + \frac{D}{r^2} = 1,
\]

\[
CR^2 + \frac{D}{R^2} = 1,
\]

\[
A + B \log R = R^2 + \frac{1}{R^2},
\]

\[
A + B \log r = r^2 + \frac{1}{r^2}.
\]

Solving this (linear in \( A, B, C, D \)) system, we obtain:

\[
A = \frac{-\log r}{\log R - \log r} \left( R^2 + \frac{1}{R^2} \right) + \frac{\log R}{\log R - \log r} \left( r^2 + \frac{1}{r^2} \right),
\]

\[
B = \frac{1}{\log R - \log r} \left( R^2 + \frac{1}{R^2} - r^2 - \frac{1}{r^2} \right),
\]

\[
C = \frac{1}{R^2 + r^2},
\]

\[
D = \frac{r^2 R^2}{R^2 + r^2}.
\]

We have

\[
(f \circ \phi)\phi' = \frac{B}{2\zeta} + C\zeta - \frac{D}{\zeta^3},
\]

and thus the square of the Bergman norm of \( f \) is

\[
\int_G |f(z)|^2 \, dA(z) = \int_{\Omega} |f \circ \phi|^2 |\phi'|^2 \, dA \]

\[
= \int_{\Omega} \left| \frac{B}{2\zeta} + C\zeta - \frac{D}{\zeta^3} \right|^2 \, dA \]

\[
= \frac{\pi}{2} \left( B^2 (\log R - \log r) + C^2 (R^4 - r^4) + D^2 \left( \frac{1}{r^4} - \frac{1}{R^4} \right) \right).
\]
The square of the Bergman norm of $\bar{z}$ is
\[
\int_G |z|^2 \, dA(z) = \int_{\Omega} |\phi \phi'|^2 \, dA \\
= \int_{\Omega} \left| \left( \zeta + \frac{1}{\zeta} \right) \left( 1 - \frac{1}{\zeta^2} \right) \right|^2 \, dA(\zeta) \\
= \int_{\Omega} \left| \zeta - \frac{1}{\zeta^3} \right|^2 \, dA(\zeta) \\
= \frac{\pi}{2} \left( R^4 - r^4 + \frac{1}{r^4} - \frac{1}{R^4} \right).
\]

Thus, $\lambda_{A^2}(G)^2 = \int_G |z|^2 \, dA(z) - \int_G |f(z)|^2 \, dA(z)$ is given by:
\[
\frac{\pi}{2} \left( R^4 - r^4 + \frac{1}{r^4} - \frac{1}{R^4} \right) - \frac{1}{\log R - \log r} \left( R^2 + \frac{1}{R^2} - r^2 - \frac{1}{r^2} \right)^2 - \frac{2}{R^2 + r^2} \left( R^4 - r^4 + \frac{1}{r^4} - \frac{1}{R^4} \right).
\]

5. Domains for which the best approximation is a monomial

The domains defined by $C\text{Re}(z^n) - |z|^2 + 1 > 0$ represent an interesting class of examples (cf. [5]). These are the domains for which the best approximation to $\bar{z}$ is a monomial, namely, $\frac{Cn}{2} z^{n-1}$. However, as indicated in Figure 5.1, there are values of $C$ for which the set \( \{ z : C\text{Re}(z^n) - |z|^2 + 1 > 0 \} \) does not have a bounded component, and $\bar{z}$ is no longer in $L^2(\Omega)$. Here we address the question of what range of $C$ leads to a bounded component.

![Figure 5.1](image_url)

Figure 5.1. The region \( \{ z : \frac{1}{2}\text{Re}(z^3) - |z|^2 + 1 > 0 \} \) does not have a bounded component.
Proposition 5.1. The set \( \{ z : C \text{Re}(z^n) - |z|^2 + 1 > 0 \} \) has a bounded component whenever

\[
C \leq \frac{2(n-2)^{\frac{n-2}{n^2}}}{n^2}.
\]

Proof. Take \( z = re^{i\theta} \) and let \( f(r,\theta) := C \cos(n\theta)r^n - r^2 + 1 \) be the defining function of the domain in polar coordinates. We will show that when

\[
C \leq \frac{2(n-2)^{\frac{n-2}{n^2}}}{n^2}
\]

we have \( f(R,\theta) \leq 0 \) for all \( \theta \), where \( R = \left( \frac{2}{nC} \right)^{1/(n-2)} \). Since the region \( \{ z : C \text{Re}(z^n) - |z|^2 + 1 > 0 \} \) clearly contains the origin, this ensures that it has a component entirely contained in the disk \( |z| < R \).

It is enough to show that \( f(R,0) \leq 0 \) since we have \( f(R,\theta) \leq f(R,0) \).

The function \( F(r) := f(r,0) = Cr^n - r^2 + 1 \), has derivative \( F'(r) = Cnr^{n-1} - 2r \), with a critical point at \( R = \left( \frac{2}{nC} \right)^{1/(n-2)} \), which by the first derivative test is a local minimum. Plugging this critical point into \( F \), we find that

\[
C \left( \frac{2}{nC} \right)^{n/(n-2)} - \left( \frac{2}{nC} \right)^{2/(n-2)} + 1 \leq 0
\]

precisely when

\[
C \leq \frac{2(n-2)^{\frac{n-2}{n^2}}}{n^2}.
\]

\[\square\]

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