GLOBAL WELL-POSEDNESS FOR THE 2D STABLE MUSKAT PROBLEM IN $H^{3/2}$

DIEGO CÓRDOBA AND OMAR LAZAR

Abstract. We prove a global existence result of a unique strong solution in $H^{5/2} \cap H^{3/2}$ with small $H^{3/2}$ norm for the 2D stable Muskat problem, hence allowing the interface to have arbitrary large finite slopes and finite energy (thanks to the $L^2$ maximum principle). The proof is based on the use of a new formulation of the Muskat equation that involves oscillatory terms. Then, a careful use of interpolation inequalities in homogeneous Besov spaces allows us to close the a priori estimates.

1. Introduction

In this paper, we are interested in the Muskat problem which was introduced in [28] by Morris Muskat in order to describe the dynamics of water and oil in sand. The Muskat problem models the motion of an interface separating two incompressible fluids in a porous medium. One can imagine the plane $\mathbb{R}^2$ splitted into two regions, say $\Gamma_1(t)$, $\Gamma_2(t)$ that evolve with time. We assume that the first region $\Gamma_1(t)$ is occupied by an incompressible fluid with density $\rho_1$ and the second region $\Gamma_2(t)$ is occupied by another one with density $\rho_2$. We further assume that both fluids are immiscible. The non-mixture condition allows one to consider the interface between these two fluids. This interface corresponds to their common boundary $\partial \Gamma_1(t)$ and $\partial \Gamma_2(t)$. The velocity in each region $\Gamma_i(t)$ ($i = 1$ or 2) is governed by the so-called Darcy’s law [18], which states that the velocity depends on the gradient pressure, the gravity and the density of the fluid (which is transported by the flow) via the following relation,

$$\frac{\mu}{\kappa} u(x, t) = -\nabla p - (0, g\rho), \quad (1.1)$$

where $\mu$ is the constant viscosity, $\kappa$ is the permeability of the porous media and $g$ is the gravity. For the sake of simplicity, we may, without loss of generality, assume that all those constants are equal to 1. The system is then driven by the following transport equation

$$\partial_t \rho + u \cdot \nabla \rho = 0 \quad (1.2)$$

since the fluids are incompressible we also have

$$\nabla \cdot u = 0 \quad (1.3)$$

Equations (1.1), (1.2) and (1.3) give rise to the so-called incompressible porous media system (IPM). Saffman and Taylor [32] pointed out that in 2D the Muskat problem is similar to the evolution of an interface in a vertical Hele-Shaw cell.

For the Muskat problem we can rewrite the IPM system in terms of the dynamics of the interface in between both fluids (see [1] and [15]). If we denote the interface by a planar curve $z(\alpha, t)$ and if as well we neglect surface tension, then the interface satisfies

$$\partial_t z(\alpha, t) = \frac{\rho_2 - \rho_1}{2\pi} \int \frac{z_1(\alpha, t) - \beta_1(t)}{|z(\alpha, t) - \beta(t)|^2} (\partial_\alpha z(\alpha, t) - \partial_\beta z(\beta, t)) d\beta,$$

2010 Mathematics Subject Classification. Primary 35A01, 35D30, 35D35, 35Q35, 35Q86.
Key words and phrases. Fluid interface, Muskat equation, Global strong solution, Regularity criteria.
where the curve $z$ is asymptotically flat at infinity i.e. $(z(\alpha, t) - (\alpha, 0)) \to 0$ as $|\alpha| \to \infty$. The point $(0, \infty)$ belongs to $\Gamma_1(t)$, whereas the point $(0, -\infty)$ belongs to $\Gamma_2(t)$. From elementary potential theory, we can derive explicit formulas for the velocity field $u$ and the pressure $p$ from the curve $z$.

A convenient way of studying the evolution of the interface is to consider this latter as a parametrized graph of a function. When the interface is a graph of a function $z(x, t) = (x, f(x, t))$, then this characterization is preserved by the system and $f$ satisfies the contour equation

$$f_t(x, t) = -\rho(\Lambda f + T(f)),$$

where $\rho$ is equal to $\rho = \frac{\rho_2 - \rho_1}{2}$, and the operator $\Lambda^\alpha$, $0 < \alpha < 2$, denotes the usual fractional Laplacian operator of order $\alpha$ and is defined as

$$\Lambda^\alpha f \equiv (-\Delta)^{\alpha/2} f = C_\alpha P.V. \int_\mathbb{R} \frac{f(x) - f(x - y)}{|y|^{1+\alpha}} dy$$

where $C_\alpha > 0$ is a positive constant. In particular, when $\alpha = 1$, the constant is equal to $\frac{1}{\pi}$.

The operator $\mathcal{H}$ denotes the Hilbert transform operator which is defined by

$$\mathcal{H}f = \frac{1}{\pi} P.V. \int_\mathbb{R} \frac{f(x - \alpha) - f(x)}{\alpha} d\alpha$$

In particular, one may easily check that $\partial_x \mathcal{H} = \Lambda$.

As for $T$, which is the nonlinear term, it is defined by

$$T(f) = \frac{1}{\pi} \int_\mathbb{R} \frac{\partial_x f(x) - \partial_x f(x - \alpha)}{\alpha} \left( \frac{f(x) - f(x - \alpha)}{\alpha} \right)^2 \frac{1}{1 + \left( \frac{f(x) - f(x - \alpha)}{\alpha} \right)^2} d\alpha.$$ (1.5)

Equivalently, the Muskat equation can be written as

$$(\mathcal{M}) : \left\{ \begin{array}{l} \partial_t f = \frac{\rho}{\pi} P.V. \partial_x \int \arctan \Delta_\alpha f \ d\alpha \\ f(0, x) = f_0(x), \end{array} \right.$$ 

where $\Delta_\alpha f \equiv \frac{f(x,t) - f(x,\alpha)}{\alpha} = \frac{\delta_\alpha f(x,t)}{\alpha}$.

Indeed, it is well known that linearizing $\mathcal{M}$ around the flat solution give rise to a (non-local) parabolic equation (see e.g. [1] and [15]). The equation is linearly stable if and only if the heavier fluid is below the interface (that is $\rho_2 > \rho_1$), otherwise we say that the curve is in the unstable regime. This is known as the Rayleigh-Taylor condition and is determined by the normal component of the pressure gradient jump at the interface having a distinguished sign (also called the Saffman-Taylor condition).

This equation has attracted the attention in the mathematical community since several years ago and we shall briefly sum up the results known regarding the Cauchy problem for $(\mathcal{M})$ in the stable regime ($\rho > 0$). First of all, let us recall that this equation has a maximum principle for $\|f(\cdot, t)\|_{L^\infty}$ and $\|f(\cdot, t)\|_{L^2}$. Indeed, it is shown in [16] that

$$\|f(t)\|_{L^\infty(\mathbb{R})} \leq \frac{\|f_0\|_{L^\infty(\mathbb{R})}}{1 + t},$$

Moreover, the authors showed in [16] that if $\|\partial_x f_0\|_{L^\infty} < 1$, then $\|\partial_x f(\cdot, t)\|_{L^\infty} < \|\partial_x f_0\|_{L^\infty}$ for all $t > 0$. On the other hand, there is also an $L^2$ maximum principle (see [11]), more precisely we have

$$\|f(T)\|_{L^2(\mathbb{R})}^2 + \int_0^T \int_\mathbb{R} \int_\mathbb{R} \log \left( 1 + \left( \frac{f(\alpha, s) - f(\beta, s)}{\alpha - \beta} \right)^2 \right) \ d\alpha d\beta ds = \|f_0\|_{L^2(\mathbb{R})}^2$$
which does not imply, for large initial data, a gain of regularity in the system. However, it was observed in [23] that a gain of regularity is possible even if we start with $L^2$ initial data, under the condition that the slope is initially less than 1 (see [23]). Recall that the Muskat equation has a scaling, if $f$ is a solution associated to the data $f_0$ so does the family $\lambda^{-1}f(\lambda x, \lambda t)$, $\lambda > 0$ associated to the data $\lambda^{-1}f_0(\lambda x)$. In particular, the Lipschitz norm $\dot{W}^{1,\infty}$ is critical as well as the homogeneous Sobolev norm $H^{3/2}$, and more generally, the whole family of homogeneous Besov spaces $\dot{B}^{1+p-1}_{p,q}$ with $(p,q) \in [1, +\infty]^2$ is critical with respect to the scaling of $\mathcal{M}$.

As far as the local well-posedness results are concerned, in [15], the authors proved local existence in $H^3$. The authors of [10] were able to lower the local theory to $H^2$. Recently, in [13], Constantin, Gancedo, Shvydkoy and Vicol have proved that the equation is locally well-posed in $\dot{W}^{p,2}$ with $p > 1$. There is another result by Matioc [27] where local existence is obtained in $H^s$, $s \in (3/2, 2]$. Instant analyticity is obtained in [9] from an initial data in $H^1$ (see also [27]).

If the heavier fluid is above the interface i.e. $\rho < 0$, then the equation ($\mathcal{M}$) is ill-posed in Sobolev spaces (see [15] and [3]). However, there exists weak solutions to the (IPM) system starting with an initial data with a jump of densities in the unstable regime. These solutions create a growing in time mixing zone around the initial interface where both densities $\rho_1$ and $\rho_2$ are mixed, for more details see [29], [34], [7] and [22].

As for the global well-posedness results, all the available theorems where done for data having small slopes $\|\partial_x f_0\|_{L^\infty} < 1$. These theorems are usually obtained by taking advantage of the parabolic nature of ($\mathcal{M}$) for very small initial data. The first global well-posedness result for small initial data is established in [15], which was inspired by the proof in [33] for a different setting (the fluids have same densities but different viscosities). In the critical framework, the authors of [11] and [12] obtained a global existence result of a unique strong solution in $H^3$, if initially the Wiener norm (i.e. those $f$ verifying that the Fourier transform of $\Lambda f$ is integrable) is smaller than 1/3 (which, in particular, implies a slope smaller than 1). As well, the authors of [11] were able to show the global propagation of the Lipschitz regularity provided the initial data has a Lipschitz norm less than 1. Recently, the authors of [13] proved that if initially the data is $\dot{W}^{2,p} \cap L^2$ with smallness in Lipschitz then there is global existence. In [27], the authors obtained the same kind of result in the space $H^{3/2+\epsilon}$ (for $\epsilon > 0$) with smallness in the same space. Furthermore, in [13], the authors obtained a regularity criteria, that as long as the slope has a uniform modulus of continuity there is existence. Recently, by taking advantage of this regularity criteria, it was proved in [6] a global existence of smooth solutions under a smallness assumption of some critical quantity (product of sup and inf of the slope) that have to be smaller than 1. In [30] they prove optimal decay estimates for higher derivatives in the Wiener norm. By adapting the proof of [11], Deng, Lei, and Lin [20] have been able to prove the existence of a global solution for arbitrarily large monotonic initial data. The key ingredient in the proof is that, under a monotonicity assumption, the slope of the interface still satisfies the $L^\infty$ maximum principle. Although, these solutions fail to be in $L^2$.

There is also several results of singularities for the Muskat equation ($\mathcal{M}$) in the stable regime. Indeed, it has been shown in [9] that there exists a family of initial data (with very large slope) where the interface reaches a regime in finite time in which is no longer a graph. Therefore there exists a time $T^*$ where the solution of ($\mathcal{M}$) blows-up: $\|f_x\|_{L^\infty}(T^*) = \infty$. In [8], they proved that there exists a class of analytic initial data in the stable regime for the Muskat problem such that the solution turns to the unstable regime and no longer belongs to $C^4$. Another recent physical scenario was studied in [14] where the authors prove that some solutions can pass from the stable to the unstable regime and return back to the stable regime before it breakdown. This shift of stability phenomena illustrates the high instability of the solutions to the Muskat equation even starting in the stable regime. Moreover, there is numerical evidence of initial data $\|\partial_x f_0\|_{L^\infty} = 50$
that develops an infinite slope in finite time (see [17]). Similar results in a confined porous media has been obtained; for example for global existence see [25] and for shift of stability see [24].

In this paper, we obtain the first global existence result of a unique solution having arbitrary large slope for the stable Muskat problem (\(\mathcal{M}\)) with finite \(L^2\) norm. All the previous global existence results known since then were assuming smallness of the slope, namely, \(\|\partial_x f_0\|_{L^\infty} < c\), where \(c \in (0,1]\).

The approach in this paper is completely new and is based on a reformulation of the usual Muskat equation (\(\mathcal{M}\)). This new formulation allows one to take advantage of the oscillations which are crucial in this problem. There is many ways of measuring smoothness while trying to do \textit{a priori} estimates in critical spaces and contrarily to (almost) all previous works in the Muskat equation we shall never split the study into high/low frequencies or small/big increment in the finite difference operator. At the contrary, we shall consider the interaction between both and the Besov spaces techniques would be the main tool to achieve this. It is worth saying that the new formulation of the problem turns out to be crucial to prove the main theorems of this paper since it gives new features that are very difficult to see in the original formulation (\(\mathcal{M}\)). If one tries to do \(\dot{H}^{3/2}\) estimates using Besov spaces techniques by using the classical formulation of the problem as stated in the introduction (\(\mathcal{M}\)), then one would quickly notice that there is a big issue to control the higher order terms, so a direct use of Besov space estimates would not give a satisfactory results.

This article completes the missing part in the understanding of the theory of global well-posedness of large solutions in the Lipschitz space for the stable Muskat problem. Indeed, the main result of this paper is the global well-posedness of strong solution in \(\dot{H}^{5/2} \cap \dot{H}^{3/2}\) under a smallness assumption on the \(\dot{H}^{3/2}\) norm of the initial data. It would not be possible to prove a global results for all data in the Lipschitz space since the authors of [14] have shown that there is solutions with initial data having a (relatively) high slopes that become singular in finite time showing the instability of the Cauchy problem associated to initial data in critical spaces.

The strategy of the proof is classical, we first establish the \textit{a priori} estimates and then we construct a solution \textit{via} classical compactness arguments that allow us to pass to the weak limit in a parabolic regularized Muskat equation.

The outline of this paper is as follows. In the next section, we state the main results. In the third section, we give the definitions of the spaces along with some harmonic analysis tools that we shall use throughout the article. The fourth section is devoted to the new formulation of the problem. The fifth section is the proof of the \textit{a priori} estimates in \(\dot{H}^{3/2}\). In the sixth section, we give the \textit{a priori} estimates in \(\dot{H}^{5/2}\). The last section is the proof of the main results.

2. Main results

The main result of this article is the following global existence theorem of a unique strong solution for small data in the critical Sobolev space.

\textbf{Theorem 2.1.} Assume \(f_0 \in \dot{H}^{5/2} \cap \dot{H}^{3/2}\) with \(\|f_0\|_{\dot{H}^{3/2}}\) small enough, then, there exists a unique strong solution \(f\) to equation (\(\mathcal{M}\)) which verifies \(f \in L^\infty([0,T], \dot{H}^{5/2}) \cap L^2([0,T], \dot{H}^{3})\), for all \(T > 0\).

\textbf{Remark 1.} By using the \(L^2\) maximum principle, one gets a unique solution to equation (\(\mathcal{M}\)) with finite energy.
Remark 2. It would be possible to lower the initial regularity by considering a data in \( H^s \) with \( 3/2 < s \leq 5/2 \). This would lead to tedious computations, the present article does only treat the case of regular enough data for the sake of simplicity.

In order to prove the main result, we shall prove a regularity criteria in terms of the control of the slopes that gives a condition for a weak solution to be strong.

**Theorem 2.2.** Let \( f_0 \in \dot{W}^{1,\infty} \cap \dot{H}^{3/2} \) and let \( T > 0 \), if one controls the \( L^\infty([0,T],\dot{W}^{1,\infty}) \) norm and if \( \| f_0 \|_{\dot{H}^{3/2}} \leq C(\| f_0 \|_{\dot{H}^{1,\infty}}) \) then there exists a unique weak solution \( f \) to equation (\( \mathcal{M} \)) which verifies \( f \in L^\infty([0,T],(\dot{H}^{3/2} \cap \dot{W}^{1,\infty}) \cap L^2([0,T],H^2) \) and in particular, \( \| f(T) \|_{\dot{H}^{3/2}} \leq \| f_0 \|_{\dot{H}^{3/2}} \).

Remark 3. The definition of the weak solutions are easy to get. Indeed, we say that \( f \) is a weak solution to the Muskat problem if, for all \( \phi \in \mathcal{D}([0,T] \times \mathbb{R}) \), we have the following equality

\[
\int_0^T \int \phi_x(x,s)f(x,s)\,ds\,dx + \int \phi_x(x,0)f_0(x)\,dx = \int_0^T \int \phi_x(x,s)\left( \int_0^\infty \delta^{-1}e^{-\delta} \sin(\delta \Delta f) \,d\delta \right)\,dx\,ds.
\]

3. **Functional setting**

We shall use the homogeneous Sobolev space \( \dot{H}^s \), \(|s| < 1/2\), which is endowed with the (semi)-

\[
\| f \|_{\dot{H}^s} = \| A^s f \|_{L^2}
\]

We recall the definition of the homogeneous Besov spaces \( \dot{B}^s_{p,q}(\mathbb{R}) \) (see e.g. [5], [31]). Let \( (p,q,s) \in [1,\infty]^2 \times \mathbb{R} \), we say that a tempered distribution \( f \) (which is such that its Fourier transform is integrable near 0) belongs to the homogeneous Besov space \( \dot{B}^s_{p,q}(\mathbb{R}) \) if the following quantity, which is a (semi)-norm, is finite, that is

\[
\| f \|_{\dot{B}^s_{p,q}} = \left\| \left[ \mathbf{1}_{(0,1]}(s)\delta_y f + \mathbf{1}_{[1,2]}(s)(\delta_y \bar{f} + \delta_y f) \right]_{L^p} \right\|_{L^q(\mathbb{R},|y|^{-1}dy)} < \infty
\]

where \( \delta_y f(x) = f(x) - f(x-y) \) and \( \bar{f}(x) = f(x) - f(x+y) \).

Let us recall some classical embeddings (see e.g. [4], [2]), we have for all \((p_1,p_2,r_1,r_2) \in [1,\infty]^4\)

\[
\dot{B}^{s_1}_{p_1,r_1}(\mathbb{R}) \hookrightarrow \dot{B}^{s_2}_{p_2,r_2}(\mathbb{R}),
\]

where \( s_1 + \frac{1}{p_2} = s_2 + \frac{1}{p_1} \). We also have for all \((p_1,s_1) \in [2,\infty] \times \mathbb{R} \),

\[
\dot{B}^{s_1}_{p_1,r_1}(\mathbb{R}) \hookrightarrow \dot{B}^{s_1}_{p_1,r_2}(\mathbb{R}),
\]

for all \((r_1,r_2) \in [1,\infty] \) such that \( r_1 \leq r_2 \).

Let \((s_1,s_2) \in \mathbb{R}^2 \) so that \( s_1 < s_2 \), then for all \( \theta \in [0,1[ \) and \((p,r) \in [1,\infty]^2 \), we have the following real interpolation inequality

\[
\| f \|_{\dot{B}^{s_1+(1-\theta)s_2}_{p,1}} \leq \frac{C}{s_2 - s_1} \left( \frac{1}{\theta} + \frac{1}{1 - \theta} \right) \| f \|_{\dot{B}^{s_1}_{p,1}} \| f \|_{\dot{B}^{s_2}_{p,r}}. \tag{3.1}
\]

We shall use the following useful generalized Calderón commutator type estimate which was proved by Dawson, Macghan, and Ponce in [19]. Let \( \Phi \in \dot{W}^{k+l,\infty} \) and let us consider the commutator

\[
T_\Phi \delta^k_x f = [\mathcal{H}, \Phi] \, f,
\]

then, for all \( p \in ]1,\infty[ \) and \((k,l) \in \mathbb{N} \), the following estimate holds,

\[
\| T_\Phi \delta^k_x f \|_{\dot{W}^{k,l,p}} \leq C_{k,l} \| \Phi \|_{\dot{W}^{k+l,\infty}} \| f \|_{L^p}, \tag{3.2}
\]
for all \( f \in L^p \).

Throughout the article, we shall use the notation \( A \lesssim B \) if there exists a constant \( C > 0 \) depending only on controlled quantities such that \( A \leq CB \).

4. A new formulation of the stable Muskat equation

In this section, we give another formulation of the Muskat equation that will be useful when doing estimates (especially to control high regularity terms) in Besov spaces.

**Proposition 4.1.** Assume that \( f \) solves \((M)\) then \( f \) is a solution of \((\tilde{M})\) that is

\[
(M) : \begin{cases} 
 f_t(t, x) = \frac{\rho}{\pi} \text{P.V.} \int \partial_x \Delta f \int_0^\infty e^{-\delta} \cos(\delta \Delta f) \, d\delta \, d\alpha \\
 f(0, x) = f_0(x).
\end{cases}
\]

Reciprocally, if \( f \) solves \((\tilde{M})\) then \( f \) solves \((M)\). That is,

\[(\tilde{M}) \iff (M).\]

**Proof of Proposition 4.1**

Consider the following integrable function

\[
\mu(x) = \exp(-|x|)
\]

Its Fourier transform is well defined and we have,

\[
\hat{\mu}(\xi) = \frac{1}{1 + |\xi|^2}.
\]

Then, by considering the restriction of this Fourier transform onto \( \Delta f \) one readily arrives to \((\tilde{M})\). Conversely, if \( f \) is a solution of \((\tilde{M})\) then it is obviously also a solution of \((M)\).

We shall assume, without loss of generality, that \( \rho = \pi \). The purpose of the next section is to prove that one has nice *a priori* estimates in \( \dot{H}^{3/2} \).

5. A priori estimates in \( \dot{H}^{3/2} \)

In this section, we shall prove the following lemma

**Lemma 5.1.** Let \( T > 0 \), assume that \( f_0 \in \dot{H}^{3/2} \cap \dot{W}^{1,\infty} \) and that the \( L^\infty([0,T],\dot{W}^{1,\infty}) \) norm remains bounded, then if we denote by \( K \) the space-time Lipschitz norm of \( f \), we have

\[
\|f\|_{\dot{H}^{3/2}(T)}^2 + \frac{\pi}{1 + K^2} \int_0^T \|f\|_{\dot{H}^2}^2 \, ds \lesssim \|f_0\|_{\dot{H}^{3/2}} \left( \|f\|_{L^\infty([0,T],\dot{H}^{3/2})} + \|f\|_{L^\infty([0,T],\dot{H}^{1/2})} \right) \int_0^T \|f\|_{\dot{H}^2}^2 \, ds
\]

**Proof of Lemma 5.1** We multiply \( \Lambda^{3/2} f \) against \( \Lambda^{3/2} f_t \) and integrate with respect to the space variable, we obtain

\[
\frac{1}{2} \partial_t \|f\|_{\dot{H}^{3/2}}^2 = \int \mathcal{H} f_{xx} \int \partial_{xx} \Delta f \int_0^\infty e^{-\delta} \cos(\delta \Delta f) (x) \, d\delta \, d\alpha \, dx
\]

\[
- \int \mathcal{H} f_{xx} \int (\partial_x \Delta f)^2 \int_0^\infty e^{-\delta} \sin(\delta \Delta f) (x) \, d\delta \, d\alpha \, dx
\]

\[
= I_1 + I_2
\]

Obviously, the more singular term is \( I_1 \). The estimation of such a term requires a long splitting into several controlled terms. This is the aim of the next subsection.
5.1. Decomposition of the term $I_1$. The aim of this section is to make appear nice controlled terms via the use of a useful decomposition of $I_1$.

\[
I_1 = \int \mathcal{H} f_{xx} \left( \partial_{xx} \Delta_\alpha f - \partial_{xx} \bar{\Delta}_\alpha f \right) \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f(x)) \, d\delta \, d\alpha \, dx \\
- \int \mathcal{H} f_{xx} \int \partial_{xx} \Delta_\alpha f \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f(x)) \, d\delta \, d\alpha \, dx \\
= \int \mathcal{H} f_{xx} \left( \partial_{xx} \Delta_\alpha f - \partial_{xx} \bar{\Delta}_\alpha f \right) \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f(x)) \, d\delta \, d\alpha \, dx \\
+ \int \mathcal{H} f_{xx} \int \partial_{xx} \Delta_\alpha f \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f(x)) \, d\delta \, d\alpha \, dx \\
- \int \mathcal{H} f_{xx} \int \partial_{xx} \Delta_\alpha f \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f(x)) \, d\delta \, d\alpha \, dx
\]

hence, by symmetry, one obtains

\[
I_1 = \frac{1}{4} \int \mathcal{H} f_{xx} \left( \partial_{xx} \Delta_\alpha f - \partial_{xx} \bar{\Delta}_\alpha f \right) \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f(x)) + \cos(\delta \bar{\Delta}_\alpha f(x)) \, d\delta \, d\alpha \, dx \\
+ \frac{1}{2} \int \mathcal{H} f_{xx} \int \partial_{xx} \Delta_\alpha f \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f(x)) - \cos(\delta \bar{\Delta}_\alpha f(x)) \, d\delta \, d\alpha \, dx
\]

which may be rewritten as,

\[
I_1 = \frac{1}{2} \int \mathcal{H} f_{xx} \left( \partial_{xx} \Delta_\alpha f - \partial_{xx} \bar{\Delta}_\alpha f \right) \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2} (\Delta_\alpha f - \bar{\Delta}_\alpha f)) \cos(\frac{\delta}{2} (\Delta_\alpha f + \bar{\Delta}_\alpha f)) \, d\delta \, d\alpha \, dx \\
- \int \mathcal{H} f_{xx} \int \partial_{xx} \Delta_\alpha f \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2} (\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2} (\Delta_\alpha f + \bar{\Delta}_\alpha f)) \, d\delta \, d\alpha \, dx
\]

Finally, we get

\[
I_1 = - \int \mathcal{H} f_{xx} \left( \partial_{xx} \Delta_\alpha f - \partial_{xx} \bar{\Delta}_\alpha f \right) \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2} (\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin^2(\frac{\delta}{4} (\Delta_\alpha f + \bar{\Delta}_\alpha f)) \, d\delta \, d\alpha \, dx \\
+ \frac{1}{2} \int \mathcal{H} f_{xx} \left( \partial_{xx} \Delta_\alpha f - \partial_{xx} \bar{\Delta}_\alpha f \right) \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2} (\Delta_\alpha f - \bar{\Delta}_\alpha f)) \, d\delta \, d\alpha \, dx \\
- \int \mathcal{H} f_{xx} \int \partial_{xx} \Delta_\alpha f \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2} (\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2} (\Delta_\alpha f + \bar{\Delta}_\alpha f)) \, d\delta \, d\alpha \, dx
\]

\[
= I_{1,1} + I_{1,2} + I_{1,3}
\]

We need to further decompose the last term, namely, we write

\[
I_{1,3} = - \int \mathcal{H} f_{xx} \int \partial_{xx} \Delta_\alpha f \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2} (\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2} (\Delta_\alpha f + \bar{\Delta}_\alpha f)) \, d\delta \, d\alpha \, dx \\
= - \int \mathcal{H} f_{xx} \int \frac{f_{xx}(x) - f_{xx}(x - \alpha)}{\alpha} \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2} (\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2} (\Delta_\alpha f + \bar{\Delta}_\alpha f)) \, d\delta \, d\alpha \, dx \\
= \int \mathcal{H} f_{xx} \int \frac{f_{xx}(x) - f_{xx}(x)}{\alpha} \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2} (\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2} (\Delta_\alpha f + \bar{\Delta}_\alpha f)) \, d\delta \, d\alpha \, dx \\
- \int \mathcal{H} f_{xx} \int \frac{f_{xx}(x)}{\alpha} \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2} (\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2} (\Delta_\alpha f + \bar{\Delta}_\alpha f)) \, d\delta \, d\alpha \, dx
\]
5.1.1. Estimates of $I_{1,j}$ with $j = 1, 2, 3$. We shall first estimate $I_{1,1}$ and $I_{1,3}$ and then $I_{1,2}$.

5.1.1. Estimates of $I_{1,1}$

To control $I_{1,1}$, we use the continuity of the Hilbert transform on $L^2$ along with the embedding $H^{3/2} \hookrightarrow B_{1, \infty}^{1,2}$, then one gets

\[
I_{1,1} = -\int \mathcal{H} f_{xx} \int (\partial_{xx} \Delta \alpha f - \partial_{xx} \Delta \alpha f) \int_0^\infty e^{-\delta} \cos\left(\frac{\delta}{2}(\Delta f - \Delta f)\right) \sin^2\left(\frac{\delta}{4}(\Delta f + \Delta f)\right) d\delta d\alpha dx
\]

\[
\leq \frac{\Gamma(3)}{4} \|f\|_{H^2}^2 \int \left\|\Delta f + \Delta f\right\|_{L^\infty}^2 \frac{d\alpha}{|\alpha|^3}
\]

\[
\leq \frac{1}{2} \|f\|_{H^2}^2 \|f\|_{H^1_{1, \infty}}^2
\]

\[
\leq \frac{1}{2} \|f\|_{H^2}^2 \|f\|_{H^3/2}^2
\]

5.1.2. Estimates of $I_{1,3}$

Let us estimate $I_{1,3}$. Let us recall that

\[
I_{1,3} = \int \mathcal{H} f_{xx} \int f_x(x) - f_x(x - \alpha) \int_0^\infty e^{-\delta} \sin\left(\frac{\delta}{2}(\Delta f - \Delta f)\right) \sin\left(\frac{\delta}{2}(\Delta f + \Delta f)\right) d\delta d\alpha dx
\]

\[
- \frac{1}{2} \int \mathcal{H} f_{xx} \int f_x(x) - f_x(x - \alpha) \int_0^\infty \delta e^{-\delta} \partial_{x} D \cos\left(\frac{\delta}{2}(\Delta f - \Delta f)\right) \sin\left(\frac{\delta}{2}(\Delta f + \Delta f)\right) d\delta d\alpha dx
\]

\[
- \frac{1}{2} \int \mathcal{H} f_{xx} \int f_x(x) - f_x(x - \alpha) \int_0^\infty \delta e^{-\delta} \sin\left(\frac{\delta}{2}(\Delta f - \Delta f)\right) \partial_{x} S \cos\left(\frac{\delta}{2}(\Delta f + \Delta f)\right) d\delta d\alpha dx
\]

\[
- \int \mathcal{H} f_{xx} \int f_x(x) \int_0^\infty e^{-\delta} \sin\left(\frac{\delta}{2}(\Delta f - \Delta f)\right) \sin\left(\frac{\delta}{2}(\Delta f + \Delta f)\right) d\delta d\alpha dx
\]

\[
= \sum_{i=1}^4 I_{1,3,i}
\]

In the next subsection, we shall estimate the $I_{1,3,i}$ for $i = 1, \ldots, 4$. 

We obtain, by denoting $D = \Delta \alpha f - \Delta \alpha f$ and $S = \Delta \alpha f + \Delta \alpha f$, 

\[
I_{1,3} = \int \mathcal{H} f_{xx} \int f_x(x) - f_x(x - \alpha) \int_0^\infty e^{-\delta} \sin\left(\frac{\delta}{2}(\Delta \alpha f - \Delta \alpha f)\right) \sin\left(\frac{\delta}{2}(\Delta \alpha f + \Delta \alpha f)\right) d\delta d\alpha dx
\]

\[
- \frac{1}{2} \int \mathcal{H} f_{xx} \int f_x(x) - f_x(x - \alpha) \int_0^\infty \delta e^{-\delta} \partial_{x} D \cos\left(\frac{\delta}{2}(\Delta \alpha f - \Delta \alpha f)\right) \sin\left(\frac{\delta}{2}(\Delta \alpha f + \Delta \alpha f)\right) d\delta d\alpha dx
\]

\[
- \frac{1}{2} \int \mathcal{H} f_{xx} \int f_x(x) - f_x(x - \alpha) \int_0^\infty \delta e^{-\delta} \sin\left(\frac{\delta}{2}(\Delta \alpha f - \Delta \alpha f)\right) \partial_{x} S \cos\left(\frac{\delta}{2}(\Delta \alpha f + \Delta \alpha f)\right) d\delta d\alpha dx
\]

\[
- \int \mathcal{H} f_{xx} \int f_x(x) \int_0^\infty e^{-\delta} \sin\left(\frac{\delta}{2}(\Delta \alpha f - \Delta \alpha f)\right) \sin\left(\frac{\delta}{2}(\Delta \alpha f + \Delta \alpha f)\right) d\delta d\alpha dx
\]

\[
= \sum_{i=1}^4 I_{1,3,i}
\]
5.1.2.1. Estimates of $I_{1,3,1}$

This term is estimated as follows, by observing that $\dot{H}^2 \hookrightarrow \dot{B}^{3/2}_{\infty,2}$, we may for instance write

$$\begin{aligned}
|I_{1,3,1}| & \leq \|f\|_{\dot{H}^2} \int_0^\infty \delta e^{-\delta} \left\| f(x) - f(x - \alpha) \right\|_{L^2} \|f(x) - f(x + \alpha) - 2f(x)\|_{L^\infty} d\alpha \\
& \leq \Gamma(2)\|f\|_{\dot{H}^2} \left( \int \left\| f(x) - f(x - \alpha) \right\|_{L^2}^2 \frac{d\alpha}{|\alpha|^3} \right)^{1/2} \\
& \leq \|f\|_{\dot{H}^2} \|f\|_{\dot{B}^{1/2}_{2,2}} \|f\|_{\dot{B}^{3/2}_{\infty,2}} \\
& \leq \|f\|_{\dot{H}^2}^2 \|f\|_{\dot{H}^{3/2}}
\end{aligned}$$

5.1.2.2. Estimates of $I_{1,3,2}$

We first observe that, since we have

$$D = \Delta_\alpha f - \bar{\Delta}_\alpha f = \frac{f(x + \alpha) - f(x - \alpha)}{\alpha} = \frac{1}{\alpha} \int_0^\alpha f_x(x + s) + f_x(x - s) - 2f_x(x) \ ds + 2f_x(x),$$

therefore,

$$\partial_\alpha D = \frac{f_x(x + \alpha) + f_x(x - \alpha) - 2f_x(x)}{\alpha} = \frac{1}{\alpha} \int_0^\alpha f_x(x + s) + f_x(x - s) - 2f_x(x) \ ds$$

Also, since $S = \Delta_\alpha f + \bar{\Delta}_\alpha f = -\frac{(f(x+\alpha)+f(x-\alpha)-2f(x))}{\alpha}$, then we find that,

$$\partial_\alpha S = \bar{\Delta}_\alpha f_x - \Delta_\alpha f_x + \frac{f(x + \alpha) + f(x - \alpha) - 2f(x)}{\alpha^2},$$

and we infer that,

$$\begin{aligned}
I_{1,3,2} &= -\frac{1}{2} \int \mathcal{H} f_{xx} \int \frac{f_x(x) - f_x(x - \alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \frac{f_x(x + \alpha) + f_x(x - \alpha) - 2f_x(x)}{\alpha} \\
& \times \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)) \ d\alpha \ dx \\
& + \frac{1}{2} \int \mathcal{H} f_{xx} \int \frac{f_x(x) - f_x(x - \alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \frac{\int_0^\alpha (f_x(x + s) + f_x(x + s) - 2f_x(x)) \ ds}{\alpha^2} \\
& \times \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)) \ d\alpha \ dx \\
& = I_{1,3,2,1} + I_{1,3,2,2}.
\end{aligned}$$

We may estimate those two latter terms as follows. For the first one, it suffices to use the embedding $\dot{H}^1 \hookrightarrow \dot{B}^{1/2}_{\infty,2}$; indeed we have

$$\begin{aligned}
|I_{1,3,2,1}| & \leq \frac{1}{2} \|f\|_{\dot{H}^2} \int \left\| \frac{f_x(x) - f_x(x - \alpha)}{\alpha} \right\|_{L^2} \left\| \frac{f_x(x + \alpha) + f_x(x - \alpha) - 2f_x(x)}{\alpha} \right\|_{L^\infty} d\alpha \\
& \leq \frac{1}{2} \|f\|_{\dot{H}^2} \|f_x\|_{\dot{B}^{1/2}_{2,2}} \|f_x\|_{\dot{B}^{1/2}_{\infty,2}} \\
& \leq \frac{1}{2} \|f\|_{\dot{H}^2}^2 \|f\|_{\dot{H}^{3/2}}
\end{aligned}$$
For $I_{1,3,2,2}$, for some $q$, $r$ and $\tau$ (so that $1/r + 1/\tau = 1$) that will be chosen latter, we write

\[
|I_{1,3,2,2}| \leq \frac{1}{2} \|f\|_{H^2} \int \frac{\|f(x) - f(x - \alpha)\|_{L^\infty}}{|\alpha|^3} \int_0^\infty \delta e^{-\delta} \times |\alpha|^{q + \frac{1}{2}} \left( \int_0^\alpha \frac{\|f(x - s) + f(x + s) - 2f(x)\|_{L^2}^2}{s^{\frac{3}{2}q}} ds \right)^{1/r} \times \|\delta_\alpha f + \overline{\delta_\alpha f}\|_{L^\infty} d\delta d\alpha \\
\leq \frac{\Gamma(2)}{2} \|f\|_{H^2} \|f\|_{|B_{2,r}^{q+1/2}} \left( \int \frac{\|f(x) - f(x - \alpha)\|_{L^\infty}}{|\alpha|^3 - q - \frac{3}{2}} d\alpha \int \frac{\|\delta_\alpha f + \overline{\delta_\alpha f}\|_{L^\infty}^2}{|\alpha|^{4-2q-\frac{3}{2}}} d\alpha \int \frac{\|f\|_{L^\infty}}{\alpha^4} d\alpha \right)^{1/2} \\
\leq \frac{1}{2} \|f\|_{H^2} \|f\|_{|B_{2,r}^{q+1/2}} \|f\|_{|B_{3/2,2}^{1/2}} \\
\leq \frac{1}{2} \|f\|_{H^2} \|f\|_{|B_{3/2,2}^{1/2}} \|f\|_{|B_{3/2,2}^{1/2}}
\]

Then, by choosing $\tau = r = 2$ and $q = 1$, we obtain

\[
|I_{1,3,2,2}| \leq \frac{1}{2} \|f\|_{H^2}^2 \|f\|_{H^{3/2}}^2
\]

Finally, we have shown that

\[
|I_{1,3,2}| \leq \|f\|_{H^2}^2 \|f\|_{H^{3/2}}^2
\]

5.1.2.3. Estimates of $I_{1,3,3}$

We now estimate $I_{1,3,3}$, to do so, we use the following decomposition,

\[
I_{1,3,3} = -\frac{1}{2} \int \mathcal{H}f_{xx} \int \frac{f_x(x) - f_x(x - \alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \overline{\Delta_\alpha f})) \partial_\alpha S \cos(\frac{\delta}{2}(\Delta_\alpha f + \overline{\Delta_\alpha f})) d\delta d\alpha dx \\
= -\frac{1}{2} \int \mathcal{H}f_{xx} \int \frac{f_x(x) - f_x(x - \alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \overline{\Delta_\alpha f})) \Delta_\alpha f_x \cos(\frac{\delta}{2}(\Delta_\alpha f + \overline{\Delta_\alpha f})) d\delta d\alpha dx \\
+ \frac{1}{2} \int \mathcal{H}f_{xx} \int \frac{f_x(x) - f_x(x - \alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \overline{\Delta_\alpha f})) \Delta_\alpha f_x \cos(\frac{\delta}{2}(\Delta_\alpha f + \overline{\Delta_\alpha f})) d\delta d\alpha dx \\
- \frac{1}{2} \int \mathcal{H}f_{xx} \int \frac{f_x(x) - f_x(x - \alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \overline{\Delta_\alpha f})) \frac{f(x + \alpha) + f(x - \alpha) - 2f(x)}{\alpha^2} d\delta d\alpha dx \\
= \sum_{i=1}^3 I_{1,3,3,i}
\]

The control of $I_{1,3,3,1}$ is relatively easy, indeed, it suffices to write
\[ |I_{1,3,1}| \leq \frac{\Gamma(2)}{2} \| f \|_{H^2} \left( \int \frac{\| f_x(x) - f_x(x - \alpha) \|_{L^4}^2}{|\alpha|^2} d\alpha \right) \left( \int \frac{\| f_x(x) - f_x(x + \alpha) \|_{L^4}^2}{|\alpha|^2} d\alpha \right)^{1/2} \]
\[ \leq \frac{1}{2} \| f \|_{H^2} \| f_x \|_{\dot{B}^{1/2}_{4,2}}^2 \]
\[ \leq \frac{1}{2} \| f \|_{H^2} \| f \|_{\dot{B}^{7/4}_{4,2}}^2 \]
\[ \leq \frac{1}{2} \| f \|_{H^2} \| f \|_{\dot{H}^{3/2}} \]

As well, one may easily estimate \( I_{1,3,3,2} \) by writing
\[ |I_{1,3,3,2}| \leq \frac{\Gamma(2)}{2} \| f \|_{H^2} \left( \int \frac{\| f_x(x) - f_x(x - \alpha) \|_{L^4}^2}{|\alpha|^2} d\alpha \right) \left( \int \frac{\| f(x + \alpha) + f(x - \alpha) - 2f(x) \|_{L^4}^2}{|\alpha|^4} d\alpha \right)^{1/2} \]
\[ \leq \frac{1}{2} \| f \|_{H^2} \| f_x \|_{\dot{B}^{1/2}_{4,2}} \left( \int \frac{\| f_x(x) - f_x(x - \alpha) \|_{L^4}^2}{|\alpha|^2} d\alpha \right)^{1/2} \]
\[ \leq \frac{1}{2} \| f \|_{H^2} \| f \|_{\dot{B}^{3/2}_{4,2}} \left( \int \frac{\| f(x + \alpha) + f(x - \alpha) - 2f(x) \|_{L^4}^2}{|\alpha|^4} d\alpha \right)^{1/2} \]
\[ \leq \frac{1}{2} \| f \|_{H^2} \| f \|_{\dot{H}^{3/4}} \| f \|_{\dot{H}^{7/4}} \]
\[ \leq \frac{1}{2} \| f \|_{H^2} \| f \|_{\dot{H}^{3/2}} \]

Therefore,
\[ |I_{1,3,3} \leq \| f \|_{H^2} \| f \|_{\dot{H}^{3/2}} \] (5.1)

### 5.1.2.4. Estimates of \( I_{1,3,4} \)

It remains to estimate \( I_{1,3,4} \), to this end, we shall use the following useful decomposition in terms of controlled commutators, one may indeed rewrite \( I_{1,3,4} \), using the antisymmetry of \( \mathcal{H} \), as follows
\[
I_{1,3,4} = -\int \mathcal{H} f_{xx} \int \frac{f_{xx}(x)}{\alpha} \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2} (\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2} (\Delta_\alpha f + \bar{\Delta}_\alpha f)) d\delta d\alpha dx
\]
\[
= \frac{1}{2} \int f_{xx} \int_0^\infty \int_0^\infty e^{-\delta} \frac{1}{\alpha} \left[ \mathcal{H}, \sin(\frac{\delta}{2} (\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2} (\Delta_\alpha f + \bar{\Delta}_\alpha f)) \right] f_{xx} d\delta d\alpha dx.
\]
Hence, we may rewrite this term as a sum of two commutators, namely

\[
I_{1,3,4} = \frac{1}{2} \int \int \frac{f_{xx}(x) - f_{xx}(x - \alpha)}{\alpha} \int_0^\infty e^{-\delta} \left[ H, \sin\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) \sin\left(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)\right) \right] f_{xx} \times d\delta \, d\alpha \, dx
+ \frac{1}{2} \int \int \frac{f_{xx}(x - \alpha)}{\alpha} \int_0^\infty e^{-\delta} \left[ H, \sin\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) \sin\left(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)\right) \right] f_{xx} \times d\delta \, d\alpha \, dx.
\]

Finally, by integrating by parts one obtains,

\[
I_{1,3,4} = -\frac{\Gamma(2)}{2} \|f\|_{\dot{H}^2} \int \frac{\|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^2}}{\alpha} \int_0^\infty \frac{e^{-\delta}}{\alpha} \partial_{\alpha} \left[ H, \sin\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) \sin\left(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)\right) \right] f_{xx} \times d\delta \, d\alpha \, dx
- \frac{\Gamma(2)}{2} \|f\|_{\dot{H}^2} \int \frac{\|f_{xx}(x - \alpha) - f_{xx}(x)\|_{L^2}}{\alpha^2} \int_0^\infty e^{-\delta} \left[ H, \sin\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) \sin\left(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)\right) \right] f_{xx} \times d\delta \, d\alpha \, dx
+ \frac{1}{2} \int \int \frac{f_{xx}(x - \alpha) - f_{xx}(x)}{\alpha} \int_0^\infty e^{-\delta} \partial_{\alpha} \left[ H, \sin\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) \sin\left(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)\right) \right] f_{xx} \times d\delta \, d\alpha \, dx
= I_{1,3,4,1} + I_{1,3,4,2} + I_{1,3,4,3}
\]

Let us estimate \(I_{1,3,4,1}\), we use the commutator estimate (3.2) along with the embedding \(\dot{H}^2 \hookrightarrow \dot{B}_{\infty,4}^{3/2}\) to obtain

\[
|I_{1,3,4,1}| \leq \frac{\Gamma(2)}{2} \|f\|_{\dot{H}^2} \int \frac{\|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^2}}{\alpha} \frac{\|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^\infty}}{\alpha} \frac{\|f(x - \alpha) + f(x + \alpha) - 2f(x)\|_{L^\infty}}{\alpha} \, d\alpha
+ \frac{\Gamma(2)}{2} \|f\|_{\dot{H}^2} \int \frac{\|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^2}}{\alpha} \frac{\|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^\infty}}{\alpha} \frac{\|f(x - \alpha) + f(x + \alpha) - 2f(x)\|_{L^\infty}}{\alpha} \, d\alpha
\]

\[
\lesssim \|f\|_{\dot{H}^2} \left( \int \frac{\|f_{xx}(x) - f_{xx}(x - \alpha)\|^2_{L^2}}{\alpha^2} \, d\alpha \right)^{1/2} \left( \int \frac{\|f(x - \alpha) + f(x + \alpha) - 2f(x)\|^4_{L^\infty}}{\alpha^5} \, d\alpha \right)^{1/4}
\]

\[
\lesssim \|f\|_{\dot{H}^2} \|f\|_{\dot{H}^{3/2}} \|f\|_{\dot{B}_{1,4}^{1/2}} \|f_{xx}\|_{\dot{B}_{\infty,4}^{1/2}}
\]

Then, we estimate \(I_{1,3,4,2}\), we see that this term may be written as

\[
I_{1,3,4,2} = \frac{1}{2} \int H_{f_{xx}} \frac{f_{xx}(x - \alpha) - f_{xx}(x)}{\alpha^2} \int_0^\infty \int e^{-\delta} \sin\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) \sin\left(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)\right) \, d\delta \, d\alpha \, dx
+ \frac{1}{2} \int f_{xx} H_{f_{xx}}(x - \alpha) - H_{f_{xx}}(x) \int_0^\infty \int e^{-\delta} \sin\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) \sin\left(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)\right) \, d\delta \, d\alpha \, dx
= I_{1,3,4,2,1} + I_{1,3,4,2,2}
\]
To estimate $I_{1,3,4,2,1}$, we write

$$
|I_{1,3,4,2,1}| \leq \frac{1}{2} \|f\|^2_{H^2} \int \frac{\|f_x(x - \alpha) - f_x(x)\|_{L^2}}{\alpha^2} \int_0^\infty e^{-\frac{\delta}{\alpha}} \|\delta f + \delta f\|_{L^\infty} d\delta d\alpha
$$

$$
\leq \frac{1}{2} \|f\|^2_{H^2} \left( \int \frac{\|f_x(x - \alpha) - f_x(x)\|^2_{L^2}}{\alpha^2} d\alpha \right)^{1/2} \left( \int \|\delta f + \delta f\|^2_{L^\infty} d\alpha \right)^{1/2}
$$

$$
\leq \frac{1}{2} \|f\|^2_{H^2} \|f\|_{H^{3/2}} \|\delta f\|_{H^{3/2}}
$$

Since $\mathcal{H}$ is continuous on $L^2$, one gets the same control for $I_{1,3,4,2,2}$ so that one finally obtains

$$
|I_{1,3,4,2}| \leq \frac{1}{2} \|f\|^2_{H^2} \|f\|_{H^{3/2}}
$$

It therefore remains to estimate $I_{1,3,4,3}$, to do so, we use the following decomposition

$$
I_{1,3,4,3} = \frac{1}{4} \int f_{xx} \int \frac{\mathcal{H}f_x(x - \alpha) - \mathcal{H}f_x(x)}{\alpha} \int_0^\infty \int e^{-\frac{\delta}{\alpha}} \partial_\alpha (\delta f + \delta f) \cos \left( \frac{\delta}{2} (\delta f + \delta f) \right) d\delta d\alpha dx
$$

$$
- \frac{1}{4} \int \mathcal{H}f_{xx} \int \frac{f_x(x - \alpha) - f_x(x)}{\alpha} \int_0^\infty \int e^{-\frac{\delta}{\alpha}} \partial_\alpha (\delta f + \delta f) \cos \left( \frac{\delta}{2} (\delta f + \delta f) \right) d\delta d\alpha dx
$$

$$
- \frac{1}{4} \int f_{xx} \int \frac{\mathcal{H}f_x(x - \alpha) - \mathcal{H}f_x(x)}{\alpha} \int_0^\infty \int e^{-\frac{\delta}{\alpha}} \partial_\alpha (\delta f + \delta f) \cos \left( \frac{\delta}{2} (\delta f + \delta f) \right) d\delta d\alpha dx
$$

$$
- \frac{1}{4} \int \mathcal{H}f_{xx} \int \frac{f_x(x - \alpha) - f_x(x)}{\alpha} \int_0^\infty \int e^{-\frac{\delta}{\alpha}} \partial_\alpha (\delta f + \delta f) \cos \left( \frac{\delta}{2} (\delta f + \delta f) \right) d\delta d\alpha dx
$$

$$
= \sum_{i=1}^4 I_{1,3,4,3,i}
$$

Since we shall do $L^p$ estimates, $p \in (1, \infty)$ on the terms involving $\mathcal{H}$ one observes that these terms have the same regularity as the terms $I_{1,3,j}$ for $j = 2, 3$, it is therefore not really difficult to see that

$$
|I_{1,3,4,3,1} + I_{1,3,4,3,2}| \leq 2 \|f\|^2_{H^2} \|f\|_{H^{3/2}}^2
$$

and

$$
|I_{1,3,4,3,3} + I_{1,3,4,3,4}| \leq 3 \|f\|^2_{H^2} \|f\|_{H^{3/2}}^2
$$

Therefore, we have obtained

$$
|I_{1,3}| \lesssim \|f\|^2_{H^2} \left( \|f\|^2_{H^{3/2}} + \|f\|_{H^{3/2}}^2 \right).
$$ (5.2)
5.1.3. Estimates of $I_{1,2}$

Recall that $I_{1,2}$ is given by

$$I_{1,2} = \frac{1}{2} \int \mathcal{H}_{xx} \left( \partial_{xx} \Delta f - \partial_{xx} \tilde{\Delta} f \right) \int_0^\infty e^{-\delta} \cos\left(\frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) d\delta \, d\alpha \, dx$$

$$= \frac{1}{2} \int \mathcal{H}_{xx} \int \frac{1}{\alpha} \partial_\alpha \{ \delta f + \tilde{\delta} f \} \int_0^\infty e^{-\delta} \cos\left(\frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) d\delta \, d\alpha \, dx,$$

therefore, an integration by parts gives

$$I_{1,2} = \frac{1}{2} \int \mathcal{H}_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha} \int_0^\infty e^{-\delta} \cos\left(\frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) d\delta \, d\alpha \, dx$$

$$+ \frac{1}{2} \int \mathcal{H}_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha} \int_0^\infty \delta e^{-\delta} \partial_\alpha \{ \Delta f - \tilde{\Delta} f \} \sin\left(\frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) \times d\delta \, d\alpha \, dx.$$ 

Hence,

$$I_{1,2} = \frac{1}{2} \int \mathcal{H}_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha} \int_0^\infty e^{-\delta} \cos\left(\frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) d\delta \, d\alpha \, dx$$

$$+ \frac{1}{2} \int \mathcal{H}_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha} \int_0^\infty \delta e^{-\delta} \frac{f_x(x)}{\alpha} \sin\left(\frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) d\delta \, d\alpha \, dx$$

$$- \frac{1}{2} \int \mathcal{H}_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha} \int_0^\infty \delta e^{-\delta} \frac{f(x + \alpha) - f(x - \alpha)}{\alpha^2}$$

$$\times \sin\left(\frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) d\delta \, d\alpha \, dx.$$

Then, importantly, one has to remark that the first term of $I_{1,2}$, that is

$$\frac{1}{2} \int \mathcal{H}_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta} \cos\left(\frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) d\delta \, d\alpha \, dx,$$

contains a dissipatif term. Indeed, one may write,

$$\frac{1}{2} \int \mathcal{H}_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta} \cos\left(\frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) d\delta \, d\alpha \, dx$$

$$= -\pi \int \mathcal{H}_{xx} \Delta f \, dx - \int \mathcal{H}_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta} \sin^2\left(\frac{\delta}{4} (\Delta f - \tilde{\Delta} f) \right) d\delta \, d\alpha \, dx,$$

$$= -\pi \| f \|^2_{H^2} - \int \mathcal{H}_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta} \sin^2\left(\frac{\delta}{4} (\Delta f - \tilde{\Delta} f) \right) d\delta \, d\alpha \, dx,$$
As we shall see, the nonlinear term, namely

\[- \int \mathcal{H} f_{xx} \left( \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \right) \int_0^\infty e^{-\delta} \sin^2 \left( \frac{\delta}{4} (\bar{\Delta} f - \Delta f) \right) \, d\delta \, d\alpha \, dx\]

is not that singular since it is not difficult to see that one may control it by using the space $\dot{B}_{\infty,2}^1$ for instance. Hence, we have

\[
I_{1,2} = - \int \mathcal{H} f_{xx} \left( \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \right) \int_0^\infty e^{-\delta} \sin^2 \left( \frac{\delta}{4} (\bar{\Delta} f - \Delta f) \right) \, d\delta \, d\alpha \, dx \\
+ \frac{1}{2} \int \mathcal{H} f_{xx} \left( \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha} \right) \int_0^\infty \delta e^{-\delta} f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x) \, d\delta \, d\alpha \, dx \\
\times \sin \left( \frac{\delta}{2} (\bar{\Delta} f - \bar{\Delta} f) \right) \, d\delta \, d\alpha \, dx \\
- \frac{1}{2} \int \mathcal{H} f_{xx} \left( \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^3} \right) \int_0^\infty \delta e^{-\delta} \int_0^\alpha f_x(x - s) + f_x(x + s) - 2f_x(x) \, ds \, d\delta \, d\alpha \, dx \\
\times \sin \left( \frac{\delta}{2} (\bar{\Delta} f - \bar{\Delta} f) \right) \, d\delta \, d\alpha \, dx \\
- \pi \| f \|_{H^2}^2 \tag{5.3}
= I_{1,2,1} + I_{1,2,2} + I_{1,2,3} + I_{1,2,4}.
\]

We shall estimate the $I_{1,2, j}$, $j = 1, 2, 3$, in the next subsection.

### 5.1.3.1 Estimate of the $I_{1,2,1}$

We need to further decompose $I_{1,2,1}$ as follows

\[
I_{1,2,1} = - \int \mathcal{H} f_{xx} \left( \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \right) \int_0^\infty e^{-\delta} \sin^2 \left( \frac{\delta}{4} (\bar{\Delta} f - \Delta f) \right) \, d\delta \, d\alpha \, dx \\
= - \int \mathcal{H} f_{xx} \left( \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \right) \int_0^\infty e^{-\delta} \\
\times \sin \left( \frac{\delta}{4} \frac{1}{\alpha} \int_0^\alpha f_x(x + s) + f_x(x - s) - 2f_x(x) \, ds - \frac{\delta}{2} f_x(x) \right) \sin \left( \frac{\delta}{4} (\bar{\Delta} f - \Delta f) \right) \, d\delta \, d\alpha \, dx \\
= - \int \mathcal{H} f_{xx} \left( \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \right) \int_0^\infty e^{-\delta} \\
\times \sin \left( \frac{\delta}{4} \frac{1}{\alpha} \int_0^\alpha f_x(x + s) + f_x(x - s) - \frac{\delta}{2} f_x(x) \, ds \right) \sin \left( \frac{\delta}{4} (\bar{\Delta} f - \Delta f) \right) \cos \left( \frac{\delta}{2} f_x(x) \right) \sin \left( \frac{\delta}{4} (\bar{\Delta} f - \Delta f) \right) \, d\delta \, d\alpha \, dx \\
- \int \mathcal{H} f_{xx} \left( \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \right) \int_0^\infty e^{-\delta} \\
\times \cos \left( \frac{\delta}{4} \frac{1}{\alpha} \int_0^\alpha f_x(x + s) + f_x(x - s) - 2f_x(x) \, ds \right) \sin \left( \frac{\delta}{2} f_x(x) \right) \sin \left( \frac{\delta}{4} (\bar{\Delta} f - \Delta f) \right) \, d\delta \, d\alpha \, dx
\]
therefore, we find hence,

\[ I_{1,2,1} = - \int \mathcal{H} f_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta} \times \sin(\frac{\delta 1}{4 \alpha} \int_0^\alpha f_x(x + s) + f_x(x - s) - 2f_x(x) ds) \cos(\frac{\delta}{2} f_x(x)) \sin(\frac{\delta}{4} (\Delta_\alpha f - \Delta_\alpha f)) d\delta \, dx d\alpha dx \\
- \int \mathcal{H} f_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta} \times \cos(\frac{\delta 1}{4 \alpha} \int_0^\alpha f_x(x + s) + f_x(x - s) - 2f_x(x) ds) \sin(\frac{\delta}{2} f_x(x)) \sin(\frac{\delta}{4} (\Delta_\alpha f - \Delta_\alpha f)) d\delta \, dx d\alpha dx \\
- \int \mathcal{H} f_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta} \times \sin^2(\frac{\delta}{4} (\Delta_\alpha f - \Delta_\alpha f)) d\delta \, dx d\alpha dx,
\]

therefore, we find

\[ I_{1,2,1} = - \int \mathcal{H} f_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta} \times \sin(\frac{\delta 1}{4 \alpha} \int_0^\alpha f_x(x + s) + f_x(x - s) - 2f_x(x) ds) \cos(\frac{\delta}{2} f_x(x)) \sin(\frac{\delta}{4} (\Delta_\alpha f - \Delta_\alpha f)) d\delta \, dx d\alpha dx \\
+ 2 \int \mathcal{H} f_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta} \times \sin^2(\frac{\delta 1}{4 \alpha} \int_0^\alpha f_x(x + s) + f_x(x - s) - 2f_x(x) ds) \sin(\frac{\delta}{2} f_x(x)) \sin(\frac{\delta}{4} (\Delta_\alpha f - \Delta_\alpha f)) d\delta \, dx d\alpha dx \\
- \int \mathcal{H} f_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta} \times \sin(\frac{\delta}{2} f_x(x)) \sin(\frac{\delta 1}{4 \alpha} \int_0^\alpha f_x(x + s) + f_x(x - s) - 2f_x(x) ds) \cos(\frac{\delta}{2} f_x(x)) d\delta \, dx d\alpha dx \\
- \int \mathcal{H} f_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta} \times \sin^2(\frac{\delta}{4} (\Delta_\alpha f - \Delta_\alpha f)) \frac{\delta}{2} f_x(x)) d\delta \, dx d\alpha dx.
\]
Finally, we find that
\[
I_{1,2,1} = - \int \mathcal{H} f_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta} \xrightarrow{\text{by Minkowski's inequality,}}
\]
\[
= - \int \mathcal{H} f_{xx} \int f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x) \quad \int_0^\infty e^{-\delta}
\]
\[
+ 2 \int \mathcal{H} f_{xx} \int f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x) \quad \int_0^\infty e^{-\delta}
\]
\[
+ 2 \int \mathcal{H} f_{xx} \int f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x) \quad \int_0^\infty e^{-\delta}
\]
\[
+ 2 \int \mathcal{H} f_{xx} \int f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x) \quad \int_0^\infty e^{-\delta}
\]
\[
= \sum_{j=1}^5 I_{1,2,1,j}
\]

To control $I_{1,2,1,1}$ it suffices to write the following
\[
|I_{1,2,1,1}| \leq \|f\|_{L^2} \int_0^\infty \delta e^{-\delta} \|f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)\|_{L^\infty} \quad \int_0^\alpha \|f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)\|_{L^2} \quad ds \quad d\alpha \quad d\delta
\]

Using Minkowski's inequality,
\[
|I_{1,2,1,1}| \leq \|f\|_{L^2} \int_0^\infty \delta e^{-\delta} \int \|f_x(x - \alpha) - f_x(x)\|_{L^\infty} \quad \int_0^\alpha \|f_x(x - \alpha) - f_x(x)\|_{L^2} \quad ds \quad d\alpha \quad d\delta
\]
\[
+ \|f\|_{L^2} \int_0^\infty \delta e^{-\delta} \int \|f_x(x - \alpha) - f_x(x)\|_{L^\infty} \quad \int_0^\alpha \|f_x(x + \alpha) - f_x(x)\|_{L^2} \quad ds \quad d\alpha \quad d\delta
\]
\[
+ \|f\|_{L^2} \int_0^\infty \delta e^{-\delta} \int \|f_x(x + \alpha) - f_x(x)\|_{L^\infty} \quad \int_0^\alpha \|f_x(x + \alpha) - f_x(x)\|_{L^2} \quad ds \quad d\alpha \quad d\delta
\]
\[
+ \|f\|_{L^2} \int_0^\infty \delta e^{-\delta} \int \|f_x(x + \alpha) - f_x(x)\|_{L^\infty} \quad \int_0^\alpha \|f_x(x + \alpha) - f_x(x)\|_{L^2} \quad ds \quad d\alpha \quad d\delta
\]
It suffices to estimate one of the terms in the right hand side of the last inequality. Therefore, we shall estimate the first term
\[ i_{1,2,1,1} = \| f \|_{\dot{H}^2} \int_0^\infty \delta e^{-\delta} \int \frac{\| f(x - \alpha) - f_x(x) \|_{L^\infty}}{\| \alpha \|^3} \int_0^\alpha \| f_x(x - s) - f_x(x) \|_{L^2} \ ds \ d\alpha \ d\delta \]

Analogously to the Bernstein’s inequality when dealing with the dyadic decomposition, the idea here is to transfer a little bit of regularity from the first to the last integral and to try to control some Besov norm by real interpolation. More precisely, we write

\[ |i_{1,2,1,1}| \leq \Gamma(2) \| f \|_{\dot{H}^2} \int \frac{\| f(x - \alpha) - f_x(x) \|_{L^\infty}}{\| \alpha \|^3} \int_0^\alpha \| f_x(x - s) - f_x(x) \|_{L^2} \ ds \ d\alpha \]

\[ \leq \| f \|_{\dot{H}^2} \int \frac{\| f(x - \alpha) - f_x(x) \|_{L^\infty}}{\| \alpha \|^3} \left( \int \frac{\| f_x(x - s) - f_x(x) \|_{L^2}^r}{|s|^q} \ ds \right)^{1/r} d\alpha \]

Then, by real interpolation and by choosing \( q = 9/8, r = \bar{r} = 2 \) one obtains

\[ |i_{1,2,1,1}| \leq \| f \|_{\dot{H}^2} \| f \|_{B_{3/2,\infty}^{5/8}} \| f \|_{B_{3/2,\infty}^{1/4}} \| f \|_{B_{3/2,\infty}^{3/4}} \| f \|_{\dot{H}^2} \]

then using the interpolation inequality

\[ \| f \|_{B_{3/2,\infty}^{13/8}} \leq \| f \|_{\dot{H}^2} \| f \|_{B_{3/2,\infty}^{3/4}} \| f \|_{B_{3/2,\infty}^{1/4}} \]

along with the fact that \( \dot{H}^{1/2+\eta} \hookrightarrow \dot{B}_{\infty,\infty}^\eta \) for \( \eta = 3/2 \) and \( \eta = 1 \), one finally gets

\[ |i_{1,2,1,1}| \leq \| f \|_{\dot{H}^2} \| f \|_{\dot{H}^{3/2}} \]

It is not difficult to see that, as well, one has

\[ \sum_{i=1}^4 |I_{1,2,1,1}| \leq \| f \|_{\dot{H}^2}^2 \| f \|_{\dot{H}^{3/2}} \]  \quad (5.4)\]

Then, it remains to estimate \( I_{1,2,1,5} \), we write

\[ I_{1,2,1,5} = - \int \mathcal{H} f_{xx} \int \frac{f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta} \sin^2(\frac{\delta}{2} f_x(x)) d\delta \ d\alpha \ dx \]

\[ \leq \pi \frac{K^2}{1 + K^2} \| f \|_{\dot{H}^2}^2 \]  \quad (5.5)\]

where \( K = \| f_x \|_{L^\infty L^\infty} \).

Therefore, we find that

\[ \sum_{i=1}^5 |I_{1,2,1,i}| \leq \| f \|_{\dot{H}^2}^2 \left( \| f \|_{\dot{H}^{3/2}} + \| f \|_{\dot{H}^{3/2}}^2 \right) + \pi \frac{K^2}{1 + K^2} \| f \|_{\dot{H}^2}^2 \]  \quad (5.6)\]
5.1.3.2. Estimate of $I_{1,2,2}$

To estimate $I_{1,2,2}$, we observe that

$$I_{1,2,2} = -\frac{1}{2} \int \mathcal{H} f_{xx} \int \frac{f_x(x) + f_x(x + \alpha) - 2f_x(x)}{\alpha} \int_0^\infty \delta e^{-\delta} f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x) \, dx \times \frac{\sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))}{\alpha} \, d\alpha \, dx$$

$$\leq \frac{\Gamma(2)}{2} \|f\|_{H^2} \|f_x\|_{\tilde{B}^{1/2}_{\infty,2}} \|f\|_{H^{3/2}}$$

$$\leq \|f\|_{H^2}^2 \|f\|_{H^{3/2}}$$

To estimate $I_{1,2,3}$ we write

$$I_{1,2,3} = -\frac{1}{2} \int \mathcal{H} f_{xx} \int \frac{f_x(x) + f_x(x + \alpha) - 2f_x(x)}{\alpha^3} \int_0^\infty \delta e^{-\delta} \int_0^\alpha f_x(x) + f_x(x + s) - 2f_x(x) \, dx \times \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \, d\alpha \, dx$$

$$\leq \frac{1}{4} \|f\|_{H^2} \int_0^\infty \delta^2 e^{-\delta} \left( \frac{\|f_x(x) + f_x(x + \alpha) - 2f_x(x)\|_{L^\infty}}{|\alpha|^{3-q-\frac{1}{p}}} \right) \times \left( \int \frac{\|f_x(x + s) - f_x(x)\|_{L^2}}{|s|^{q}} \, ds \right)^{1/r} \, d\alpha \, d\delta$$

$$\leq \frac{1}{4} \|f\|_{H^2} \int_0^\infty \delta^2 e^{-\delta} \left( \frac{\|f_x(x) + f_x(x + \alpha) - 2f_x(x)\|_{L^\infty}}{|\alpha|^{3-q-\frac{1}{p}}} \right) \times \left( \int \frac{\|f_x(x + s) - f_x(x)\|_{L^2}}{|s|^{q}} \, ds \right)^{1/r} \, d\alpha \, d\delta$$

Then, by choosing $q = 9/8$, $r = \bar{r} = 2$, and by real interpolation (to get a control of $\tilde{B}^{11/8}_{\infty,1}$) along with classical homogeneous Besov embeddings one gets

$$|I_{1,2,3}| \leq \frac{1}{2} \|f\|_{H^2} \|f\|_{\tilde{B}^{13/8}_{2,2}} \|f\|_{\tilde{B}^{11/8}_{\infty,1}}$$

$$\leq \frac{1}{2} \|f\|_{H^2} \|f\|_{\tilde{B}^{1/4}_{\infty,1}} \|f\|_{\tilde{B}^{3/4}_{\infty,1}} \|f\|_{\tilde{B}^{1/4}_{\infty,2}} \|f\|_{\tilde{B}^{3/4}_{\infty,2}}$$

$$\leq \frac{1}{2} \|f\|_{H^2} \|f\|_{H^{3/2}}$$
Recalling that (5.3) is a dissipative term and by (5.5), and by we have proved that

\[ |I_1| \lesssim \| f \|_{H^2}^2 (\| f \|_{H^{5/2}}^2 + \| f \|_{H^{3/2}}^2) - \pi \| f \|_{H^2}^2 + \pi \frac{K^2}{1 + K^2} \| f \|_{H^2}^2 \]

Finally,

\[ |I_1| \lesssim \| f \|_{H^2}^2 (\| f \|_{H^{5/2}}^2 + \| f \|_{H^{3/2}}^2) - \frac{\pi}{1 + K^2} \| f \|_{H^2}^2 \]

(5.7)

5.1.2. Estimate of \( I_2 \)

To estimate

\[ I_2 = - \int \mathcal{H} f_{xx} \int (\partial_x \Delta f)^2 \int_0^\infty \delta e^{-\delta} \sin(\delta \Delta f(x)) \, d\delta \, d\alpha \, dx, \]

it suffices to write that

\[ |I_2| \leq \Gamma(2) \| f \|_{H^2}^2 \left( \int \frac{\| f_x(x) - f_x(x - \alpha) \|_{L^2}^4}{\alpha^2} \, d\alpha \right)^{1/2} \left( \int \frac{\| f_x(x) - f_x(x - \alpha) \|_{L^\infty}^2}{\alpha^2} \, d\alpha \right)^{1/2} \]

and therefore, we obtain

\[ |I_2| \leq \| f \|_{H^2}^2 \| f_x \|_{H^{1/2}}^2 \| f \|_{H^{3/2}} \]

\[ \leq \| f \|_{H^2}^2 \| f \|_{H^{3/2}} \]

Therefore, we get that

\[ \frac{1}{2} \partial_t \| f \|_{H^{3/2}}^2 + \frac{\pi}{1 + K^2} \| f \|_{H^2}^2 \leq C \| f \|_{H^2}^2 \left( \| f \|_{H^{5/2}}^2 + \| f \|_{H^{3/2}} \right) \]

(5.9)

And then integrating in time \( s \in [0, T] \) one gets the desired energy inequality (6.1). Therefore, if \( \| f_0 \|_{H^{3/2}} \) is smaller than some \( C(K) \) that depends only on \( K \), then the solution is in \( L^\infty ([0, T], L^2) \cap L^2 ([0, T], H^2) \). This concludes the \( H^{5/2} \)-estimates and therefore Lemma 6.1 is proved.

\[ \square \]

6. A PRIORI ESTIMATES IN \( H^{5/2} \) ESTIMATES

In this section we shall prove the following lemma

**Lemma 6.1.** Let \( T > 0 \) and \( f_0 \in H^{5/2} \cap H^{3/2} \) so that \( \| f_0 \|_{H^{3/2}} < C(\| f_{0,x} \|_{L^\infty}) \), then we have

\[ \| f \|_{H^{5/2}}^2 + \frac{\pi}{1 + M^2} \int_0^T \| f \|_{H^3}^2 \, ds \lesssim \| f_0 \|_{H^{3/2}} \left( \| f \|_{L^\infty ([0, T], H^{3/2})} + \| f \|_{L^\infty ([0, T], H^{3/2})} \right) \int_0^T \| f \|_{H^3}^2 \, ds \]

where \( M \) is the space-time Lipschitz norm of \( f \).

**Proof of Lemma 6.1** We have

\[ \frac{1}{2} \partial_t \| f \|_{H^{5/2}}^2 = \int \mathcal{H} f_{xx} \int \partial_x^3 \Delta f \int_0^\infty e^{-\delta} \cos(\delta \Delta f) \, dx \, d\alpha \, d\delta \]

\[ - \int \mathcal{H} \partial_x^3 f \int \partial_x \Delta f \partial_{xx} \Delta f \int_0^\infty \delta e^{-\delta} \sin(\delta \Delta f) \, dx \, d\alpha \, d\delta \]

\[ - \int \mathcal{H} \partial_x^3 f \int (\partial_x \Delta f)^3 \int_0^\infty \delta^2 e^{-\delta} \cos(\delta \Delta f) \, dx \, d\alpha \, d\delta \]

\[ = T_1 + T_2 + T_3. \]
We may decompose the first term as follows

\[
T_1 = - \int \mathcal{H} \partial_x^3 f \int (\partial_x^2 \Delta_a f - \partial_x^2 \Delta_a f) \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_a f - \Delta_a f)) \sin^2(\frac{\delta}{4}(\Delta_a f + \Delta_a f)) \, d\delta \, d\alpha \, dx
+ \int \mathcal{H} \partial_x^3 f \int \frac{f_{xx}(x) - f_{xx}(x - \alpha)}{\alpha^2} \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_a f - \Delta_a f)) \sin(\frac{\delta}{2}(\Delta_a f + \Delta_a f)) \, d\delta \, d\alpha \, dx
- \frac{1}{2} \int \mathcal{H} \partial_x^3 f \int \frac{f_{xx}(x) - f_{xx}(x - \alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \partial_a D \cos(\frac{\delta}{2}(\Delta_a f - \Delta_a f)) \sin(\frac{\delta}{2}(\Delta_a f + \Delta_a f)) \, d\delta \, d\alpha \, dx
- \frac{1}{2} \int \mathcal{H} \partial_x^3 f \int \frac{f_{xx}(x) - f_{xx}(x - \alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \partial_a S \sin(\frac{\delta}{2}(\Delta_a f - \Delta_a f)) \cos(\frac{\delta}{2}(\Delta_a f + \Delta_a f)) \, d\delta \, d\alpha \, dx
- \int \mathcal{H} \partial_x^3 f \int \partial_x^3 f(x) \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_a f - \Delta_a f)) \sin(\frac{\delta}{2}(\Delta_a f + \Delta_a f)) \frac{1}{\alpha} \, d\delta \, d\alpha \, dx
+ \frac{1}{2} \int \mathcal{H} \partial_x^3 f \int (\partial_x^2 \Delta_a f - \partial_x^2 \Delta_a f) \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_a f - \Delta_a f)) \sin^2(\frac{\delta}{4}(\Delta_a f + \Delta_a f)) \, d\delta \, d\alpha \, dx
= \sum_{i=1}^6 T_{1,i}
\]

We then estimate the \(T_{1,i}, j = 1, \ldots, 6\).

### 6.1. Estimate of \(T_{1,1}\)

To control \(T_{1,1}\), we use the continuity of the Hilbert transform in \(L^2\) along with the embedding 
\(\dot{H}^{3/2} \hookrightarrow \dot{B}^{1}_{\infty,2}\); then one gets

\[
T_{1,1} = - \int \mathcal{H} \partial_x^3 f \int (\partial_x^2 \Delta_a f - \partial_x^2 \Delta_a f) \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_a f - \Delta_a f)) \sin^2(\frac{\delta}{4}(\Delta_a f + \Delta_a f)) \, d\delta \, d\alpha \, dx
\leq \frac{\Gamma(3)}{4} \|f\|_{\dot{H}^3}^2 \int \|\delta_a f + \bar{\delta}_a f\|_{L^\infty}^2 \, d\alpha
\leq \frac{1}{2} \|f\|_{\dot{H}^3}^2 \|f\|_{\dot{B}^1_{\infty,2}}^2
\leq \frac{1}{2} \|f\|_{\dot{H}^3}^2 \|f\|_{\dot{H}^{3/2}}^2
\]

It remains to estimate,

\[
\sum_{j=2}^5 T_{1,j} = \int \mathcal{H} \partial_x^3 f \int \partial_x^3 f(x) - \partial_x^3 f(x - \alpha) \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_a f - \Delta_a f)) \sin(\frac{\delta}{2}(\Delta_a f + \Delta_a f)) \times d\delta \, d\alpha \, dx
- \frac{1}{2} \int \mathcal{H} \partial_x^3 f \int \frac{\partial_x^3 f(x) - \partial_x^3 f(x - \alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \partial_a D \cos(\frac{\delta}{2}(\Delta_a f - \Delta_a f)) \sin(\frac{\delta}{2}(\Delta_a f + \Delta_a f)) \times d\delta \, d\alpha \, dx
- \frac{1}{2} \int \mathcal{H} \partial_x^3 f \int \frac{\partial_x^3 f(x) - \partial_x^3 f(x - \alpha)}{\alpha} \int_0^\infty \delta e^{-\delta} \partial_a S \sin(\frac{\delta}{2}(\Delta_a f - \Delta_a f)) \cos(\frac{\delta}{2}(\Delta_a f + \Delta_a f)) \times d\delta \, d\alpha \, dx
- \int \mathcal{H} \partial_x^3 f \int \partial_x^3 f(x) \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_a f - \Delta_a f)) \sin(\frac{\delta}{2}(\Delta_a f + \Delta_a f)) \frac{1}{\alpha} \, d\delta \, d\alpha \, dx
\]
6.2. Estimate of $T_{1,2}$

To estimate $T_{1,2}$, it suffices to write

$$|T_{1,2}| \leq \|f\|_{H^{3}} \int_{0}^{\infty} \delta e^{-\delta} \|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^{\infty}} \|f(x - \alpha) + f(x + \alpha) - 2f(x)\|_{L^{2}} \, d\alpha \, d\alpha$$

$$\leq \Gamma(2)\|f\|_{H^{3}} \left( \int \|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^{2}}^{2} \, d\alpha \int \|f(x - \alpha) + f(x + \alpha) - 2f(x)\|_{L^{2}}^{2} \, d\alpha \right)^{1/2}$$

$$\leq \|f\|_{H^{3}} \|f_{xx}\|_{B_{3/2}^{1/2}} \|f\|_{B_{3/2}^{5/2}}$$

$$\leq \|f\|_{H^{3}}^{2} \|f\|_{B_{3/2}^{5/2}}$$

6.3. Estimate of $T_{1,3}$

The control of $T_{1,3}$ is done thanks to the following decomposition,

$$T_{1,3} = -\frac{1}{2} \int \mathcal{H} \partial_{x}^{3} f \int \frac{f_{xx}(x) - f_{xx}(x - \alpha)}{\alpha} \int_{0}^{\infty} \delta e^{-\delta} \frac{f_{x}(x + \alpha) + f_{x}(x - \alpha) - 2f_{x}(x)}{\alpha}$$

$$\times \cos(\frac{\delta}{2}(\Delta_{a}f - \bar{\Delta}_{a}f)) \sin(\frac{\delta}{2}(\Delta_{a}f + \bar{\Delta}_{a}f)) \, d\alpha \, dx$$

$$+ \frac{1}{2} \int \mathcal{H} \partial_{x}^{3} f \int \frac{f_{xx}(x) - f_{xx}(x - \alpha)}{\alpha} \int_{0}^{\infty} \delta e^{-\delta} \int_{0}^{\alpha} f_{x}(x - s) + f_{x}(x + s) - 2f_{x}(x) \, ds$$

$$\times \cos(\frac{\delta}{2}(\Delta_{a}f - \bar{\Delta}_{a}f)) \sin(\frac{\delta}{2}(\Delta_{a}f + \bar{\Delta}_{a}f)) \, d\alpha \, dx$$

$$= T_{1,3,1} + T_{1,3,2}$$

We have

$$|T_{1,3,1}| \leq \frac{1}{2} \|f\|_{H^{3}} \int \|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^{\infty}} \|f_{x}(x + \alpha) + f_{x}(x - \alpha) - 2f_{x}(x)\|_{L^{4}} \, d\alpha$$

$$\leq \|f\|_{H^{3}} \|f\|_{B_{\infty,2}^{5/2}} \|f\|_{B_{3/2}^{5/2}}$$

$$\leq \|f\|_{H^{3}}^{2} \|f\|_{B_{3/2}^{5/2}}$$

For $T_{1,3,2}$, we write

$$|T_{1,3,2}| \leq \frac{1}{2} \|f\|_{H^{3}} \int \|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^{\infty}} \int_{0}^{\infty} \delta e^{-\delta}$$

$$\times |\alpha|^{3/2} \left( \int_{0}^{\alpha} \|f_{x}(x - s) + f_{x}(x + s) - 2f_{x}(x)\|_{L^{2}}^{2} \, ds \right)^{1/2}$$

$$\times \|\Delta_{a}f + \bar{\Delta}_{a}f\|_{L^{\infty}} \, d\alpha$$

$$\leq \frac{\Gamma(3)}{2} \frac{\|f\|_{H^{3}} \|f\|_{B_{3/2}^{5/2}}}{\|f\|_{B_{3/2}^{3/2}}} \int \|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^{\infty}} \|\Delta_{a}f + \bar{\Delta}_{a}f\|_{L^{\infty}} \, d\alpha$$

$$\leq \|f\|_{H^{3}} \|f\|_{B_{3/2}^{3/2}} \left( \int \|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^{\infty}}^{2} \, d\alpha \int \|\Delta_{a}f + \bar{\Delta}_{a}f\|_{L^{\infty}}^{2} \, d\alpha \right)^{1/2}$$

$$\leq \|f\|_{H^{3}} \|f\|_{B_{3/2}^{3/2}} \|f\|_{B_{3/2}^{3/2}}$$

$$\leq \|f\|_{H^{3}} \|f\|_{B_{3/2}^{3/2}}$$

$$\leq \|f\|_{H^{3}} \|f\|_{B_{3/2}^{3/2}}$$

$$\leq \|f\|_{H^{3}} \|f\|_{B_{3/2}^{3/2}}$$
where we have chosen $r = r = 2$ and then by choosing $q = 1$ we obtain

$$|T_{1,3,2}| \leq \|f\|_{H^2} \|f\|_{H^{3/2}} \|f\|_{H^{5/2}} \|f\|_{H^2}$$
$$\leq \|f\|_{H^3}^2 \|f\|_{H^{3/2}}^2$$

Therefore,

$$|T_{1,3}| \lesssim \|f\|_{H^3}^2 (\|f\|_{H^{3/2}} + \|f\|_{H^{3/2}}^2)$$

6.4. Estimate of $T_{1,4}$

We now estimate $T_{1,4}$, we first write that

$$T_{1,4} = -\frac{1}{2} \int \mathcal{H} \partial_\alpha f \int \frac{f_{xx}(x) - f_{xx}(x - \alpha)}{\alpha} \int_0^\infty \delta e^{-\delta \sin \left(\frac{\delta}{2} (\Delta_\alpha f - \Delta_\alpha f)\right)} \partial_\alpha S \cos \left(\frac{\delta}{2} (\Delta_\alpha f + \Delta_\alpha f)\right) d\delta d\alpha dx$$

$$= \frac{1}{2} \int \mathcal{H} \partial_\alpha^3 f \int \frac{f_{xx}(x) - f_{xx}(x - \alpha)}{\alpha} \int_0^\infty \delta e^{-\delta \sin \left(\frac{\delta}{2} (\Delta_\alpha f - \Delta_\alpha f)\right)} \Delta_\alpha f x \cos \left(\frac{\delta}{2} (\Delta_\alpha f + \Delta_\alpha f)\right) d\delta d\alpha dx$$

$$+ \frac{1}{2} \int \mathcal{H} \partial_\alpha^3 f \int \frac{f_{xx}(x) - f_{xx}(x - \alpha)}{\alpha} \int_0^\infty \delta e^{-\delta \sin \left(\frac{\delta}{2} (\Delta_\alpha f - \Delta_\alpha f)\right)} \Delta_\alpha f x \cos \left(\frac{\delta}{2} (\Delta_\alpha f + \Delta_\alpha f)\right) d\delta d\alpha dx$$

$$- \frac{1}{2} \int \mathcal{H} \partial_\alpha^3 f \int \frac{f_{xx}(x) - f_{xx}(x - \alpha)}{\alpha} \int_0^\infty \delta e^{-\delta \sin \left(\frac{\delta}{2} (\Delta_\alpha f - \Delta_\alpha f)\right)} f(x + \alpha) + f(x - \alpha) - 2f(x)$$

$$\times \cos \left(\frac{\delta}{2} (\Delta_\alpha f + \Delta_\alpha f)\right) d\delta d\alpha dx$$

$$= \sum_{j=1}^3 T_{1,4,j}$$

The estimate of $T_{1,4,1}$ is relatively easy, indeed, it suffices to write

$$|T_{1,4,1}| \leq \frac{\Gamma(2)}{2} \int \|f\|_{H^3} \left( \int \frac{\|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^\infty}}{|\alpha|^2} d\alpha \int \frac{\|f_{x}(x) - f_{x}(x + \alpha)\|_{L^2}}{|\alpha|^2} d\alpha \right)^{1/2}$$

$$\leq \frac{1}{2} \|f\|_{H^3} \|f\|_{B^{5/2}_{\infty,2}} \|f\|_{B^{3/2}_{2,2}}$$

As well, one may easily estimate $T_{1,4,2}$ by writing

$$|T_{1,4,2}| \leq \frac{\Gamma(2)}{2} \int \|f\|_{H^3} \left( \int \frac{\|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^\infty}}{|\alpha|^2} d\alpha \int \frac{\|f_{x}(x) - f_{x}(x + \alpha)\|_{L^2}}{|\alpha|^2} d\alpha \right)^{1/2}$$

$$\leq \frac{1}{2} \|f\|_{H^3} \|f\|_{B^{5/2}_{\infty,2}} \|f\|_{B^{3/2}_{2,2}}$$

$$\leq \frac{1}{2} \|f\|_{H^3}^2 \|f\|_{H^{3/2}}$$
For $T_{1,4,3}$, it suffices to write

$$|T_{1,4,3}| \leq \frac{\Gamma(2)}{2} \|f\|_{\dot{H}^{3}} \left( \int \frac{\|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^\infty}}{|\alpha|^2} d\alpha \int \frac{\|f(x + \alpha) + f(x - \alpha) - 2f(x)\|_{L^2}^2}{|\alpha|^4} d\alpha \right)^{1/2}$$

$$\leq \frac{1}{2} \|f\|_{\dot{H}^{3}} \left( \int \frac{\|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^\infty}}{|\alpha|^2} d\alpha \int \frac{\|f(x + \alpha) + f(x - \alpha) - 2f(x)\|_{L^2}^2}{|\alpha|^4} d\alpha \right)^{1/2}$$

$$\leq \frac{1}{2} \|f\|_{\dot{H}^{3}} \|f_{xx}\|_{\dot{H}^{1/2}} \|f\|_{\dot{H}^{3/2}}$$

Therefore,

$$|T_{1,4}| \lesssim \|f\|_{\dot{H}^{3}} \|f\|_{\dot{H}^{3/2}} \quad (6.1)$$

It remains to estimate $T_{1,5}$, this is the purpose of the next subsection.

### 6.5. Estimate of $T_{1,5}$

We first rewrite $T_{1,5}$ in term of controlled commutators

$$T_{1,5} = - \int \mathcal{H} \partial_{f}^3 f_{xx} \int \frac{\partial_{f}^3 f(x)}{\alpha} \int_0^\infty e^{-\delta} \sin \left( \frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) \sin \left( \frac{\delta}{2} (\Delta f + \tilde{\Delta} f) \right) d\delta d\alpha dx$$

$$= \frac{1}{2} \int \frac{\partial_{f}^3 f}{\alpha} \int_0^\infty e^{-\delta} \int \frac{1}{\alpha} \left[ \mathcal{H}, \sin \left( \frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) \sin \left( \frac{\delta}{2} (\Delta f + \tilde{\Delta} f) \right) \right] \partial_{f}^3 f d\delta d\alpha dx$$

$$= \frac{1}{2} \int \frac{\partial_{f}^3 f}{\alpha} \int_0^\infty e^{-\delta} \left[ \mathcal{H}, \sin \left( \frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) \sin \left( \frac{\delta}{2} (\Delta f + \tilde{\Delta} f) \right) \right] \partial_{f}^3 f d\delta d\alpha dx$$

Finally, by integrating by parts one obtains

$$T_{1,5,1} = - \frac{1}{2} \int \int \frac{f_{xx}(x) - f_{xx}(x - \alpha)}{\alpha} \int_0^\infty e^{-\delta} \partial_{\alpha} \left[ \mathcal{H}, \sin \left( \frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) \sin \left( \frac{\delta}{2} (\Delta f + \tilde{\Delta} f) \right) \right] \partial_{f}^3 f d\delta d\alpha dx$$

$$- \frac{1}{2} \int \int \frac{f_{xx}(x - \alpha) - f_{xx}(x)}{\alpha} \int_0^\infty e^{-\delta} \left[ \mathcal{H}, \sin \left( \frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) \sin \left( \frac{\delta}{2} (\Delta f + \tilde{\Delta} f) \right) \right] \partial_{f}^3 f d\delta d\alpha dx$$

$$+ \frac{1}{2} \int \int \frac{f_{xx}(x - \alpha) - f_{xx}(x)}{\alpha} \int_0^\infty e^{-\delta} \partial_{\alpha} \left[ \mathcal{H}, \sin \left( \frac{\delta}{2} (\Delta f - \tilde{\Delta} f) \right) \sin \left( \frac{\delta}{2} (\Delta f + \tilde{\Delta} f) \right) \right] \partial_{f}^3 f d\delta d\alpha dx$$

$$= T_{1,5,1} + T_{1,5,2} + T_{1,5,3}$$

By using the generalized Calderón commutator estimate (3.2) along with some classical Besov embeddings, we may control $I_{1,5,1}$ as follows
\[ |T_{1,5,1}| \lesssim \|f\|_{\dot{H}^3}^2 \int \frac{\|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^2}}{\alpha} \frac{\|f_x(x) - f_x(x - \alpha)\|_{L^\infty}}{\alpha} \frac{\|f(x - \alpha) + f(x + \alpha) - 2f(x)\|_{L^\infty}}{\alpha} \, d\alpha \\
+ \|f\|_{\dot{H}^3}^2 \int \frac{\|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^2}}{\alpha} \frac{\|f_x(x) - f_x(x + \alpha) + f_x(x + \alpha) - 2f_x(x)\|_{L^\infty}}{\alpha} \, d\alpha \\
\lesssim \|f\|_{\dot{H}^3}^2 \left( \int \frac{\|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^2}^2}{\alpha^2} \, d\alpha \right)^{1/2} \left( \int \frac{\|f(x - \alpha) + f(x + \alpha) - 2f(x)\|_{L^\infty}^4}{\alpha^3} \, d\alpha \right)^{1/4} \\
\times \left( \int \frac{\|f_x(x) - f_x(x - \alpha)\|_{L^\infty}^4}{\alpha^3} \, d\alpha \right)^{1/4} \\
+ \|f\|_{\dot{H}^3}^2 \left( \int \frac{\|f_{xx}(x) - f_{xx}(x - \alpha)\|_{L^2}^2}{\alpha^2} \, d\alpha \right)^{1/2} \left( \int \frac{\|f_x(x - \alpha) + f_x(x + \alpha) - 2f_x(x)\|_{L^\infty}^2}{\alpha^2} \, d\alpha \right)^{1/2} \\
\lesssim \|f\|_{\dot{H}^3}^2 \|f\|_{\dot{H}^{5/2}} \left( \|f\|_{\dot{B}^{1/4}_{\infty,4}} \|f\|_{\dot{B}^{3/2}_{\infty,4}} + \|f\|_{\dot{B}^{3/2}_{\infty,4}} \right) \\
\leq \|f\|_{\dot{H}^3}^2 \|f\|_{\dot{H}^{5/2}} \left( \|f\|_{\dot{B}^{1/4}_{\infty,4}} \|f\|_{\dot{B}^{3/2}_{\infty,4}} + \|f\|_{\dot{B}^{3/2}_{\infty,4}} \right) \\
\lesssim \|f\|_{\dot{H}^3}^2 \|f\|_{\dot{H}^{5/2}}^2 \left( \|f\|_{\dot{H}^{5/2}} \|f\|_{\dot{B}^{3/2}_{\infty,4}} + \|f\|_{\dot{H}^2} \right) \\
\lesssim \|f\|_{\dot{H}^3}^2 \|f\|_{\dot{H}^{5/2}}^2 + \|f\|_{\dot{H}^3}^2 \|f\|_{\dot{H}^{3/2}} \\
\]

Then, we estimate \( T_{1,5,2} \), we see that this term may be rewritten as

\[
T_{1,5,2} = \frac{1}{2} \int \mathcal{H} \partial_x^3 f \int \frac{f_{xx}(x - \alpha) - f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \sin\left( \frac{\delta}{2} (\Delta_\alpha f - \bar{\Delta}_\alpha f) \right) \sin\left( \frac{\delta}{2} (\Delta_\alpha f + \bar{\Delta}_\alpha f) \right) \, d\delta \, d\alpha \, dx \\
+ \frac{1}{2} \int \partial_x^3 f \int \mathcal{H} f_{xx}(x - \alpha) - \mathcal{H} f_{xx}(x) \int_0^\infty e^{-\delta} \sin\left( \frac{\delta}{2} (\Delta_\alpha f - \bar{\Delta}_\alpha f) \right) \sin\left( \frac{\delta}{2} (\Delta_\alpha f + \bar{\Delta}_\alpha f) \right) \, d\delta \, d\alpha \, dx \\
= T_{1,5,2,1} + T_{1,5,2,2}
\]

To estimate \( T_{1,5,2,1} \), we use the embedding \( \dot{H}^2 \hookrightarrow \dot{B}^{3/2}_{\infty,2} \) along with classical interpolation inequalities to get

\[
|T_{1,5,2,1}| \lesssim \|f\|_{\dot{H}^3}^2 \int \frac{\|f_{xx}(x - \alpha) - f_{xx}(x)\|_{L^2}}{\alpha^2} \int_0^\infty e^{-\delta} \frac{\|\Delta_\alpha f + \bar{\Delta}_\alpha f\|_{L^\infty}}{\alpha} \, d\delta \, d\alpha \\
\lesssim \|f\|_{\dot{H}^3}^2 \left( \int \frac{\|f_{xx}(x - \alpha) - f_{xx}(x)\|_{L^2}^2}{\alpha^2} \, d\alpha \right)^{1/2} \left( \int \frac{\|\Delta_\alpha f + \bar{\Delta}_\alpha f\|_{L^\infty}^2}{\alpha^4} \, d\alpha \right)^{1/2} \\
\lesssim \|f\|_{\dot{H}^3}^2 \|f\|_{\dot{H}^{3/2}} \|f\|_{\dot{H}^2} \\
\lesssim \|f\|_{\dot{H}^3}^2 \|f\|_{\dot{H}^{3/2}}
\]

Since \( \mathcal{H} \) is continuous on \( L^2 \), one gets the same control as the term above, namely

\[
|T_{1,5,2}| \lesssim \|f\|_{\dot{H}^3}^2 \|f\|_{\dot{H}^{3/2}}
\]
It remains to estimate $T_{1,5,3}$, to do so, we use the following decomposition

$$T_{1,5,3} = -\frac{1}{4}\int \partial_x^3 f \int \frac{Hf_{xx}(x) - Hf_{xx}(x)}{\alpha} \int_0^\infty \delta e^{-\delta} \partial_\alpha (\Delta_\alpha f - \bar{\Delta}_\alpha f) \cos\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) \times \sin\left(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)\right) d\delta \, d\alpha \, dx$$

$$- \frac{1}{4}\int \partial_x^3 f \int \frac{Hf_{xx}(x) - Hf_{xx}(x)}{\alpha} \int_0^\infty \delta e^{-\delta} \partial_\alpha (\Delta_\alpha f - \bar{\Delta}_\alpha f) \cos\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) \times \sin\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) d\delta \, d\alpha \, dx$$

$$- \frac{1}{4}\int \partial_x^3 f \int \frac{f_{xx}(x) - f_{xx}(x)}{\alpha} \int_0^\infty \delta e^{-\delta} \partial_\alpha (\Delta_\alpha f + \bar{\Delta}_\alpha f) \cos\left(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)\right) \times \sin\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) d\delta \, d\alpha \, dx$$

Since we shall do $L^p$ estimates, $p \in (1, \infty)$ on the terms involving $H$ one observes that these terms has the same regularity as the terms $I_{1,3,2}$ and therefore one analogously infers that,

$$|T_{1,5,3}| \lesssim \|f\|_{\dot{H}^3}^2 \|f\|_{\dot{H}^{3/2}}^2 + \|f\|_{\dot{H}^3}^2 \|f\|_{\dot{H}^{3/2}}^2,$$

hence,

$$|T_{1,5}| \lesssim \|f\|_{\dot{H}^3}^2 (\|f\|_{\dot{H}^{3/2}}^2 + \|f\|_{\dot{H}^{3/2}}^2).$$

6.6. Estimate of $T_{1,6}$

We first rewrite $T_{1,6}$ as follows, by integrating by parts, one finds

$$T_{1,6} = \frac{1}{2}\int \partial_x^3 f \int (\partial_x^3 \Delta_\alpha f - \partial_x^3 \bar{\Delta}_\alpha f) \int_0^\infty e^{-\delta} \cos\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) d\delta \, d\alpha \, dx$$

$$= \frac{1}{2}\int \partial_x^3 f \int \frac{1}{\alpha} \partial_\alpha \{\partial_\alpha f_{xx} + \bar{\partial}_\alpha f_{xx}\} \int_0^\infty e^{-\delta} \cos\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) d\delta \, d\alpha \, dx$$

$$= \frac{1}{2}\int \partial_x^3 f \int \frac{f_{xx}(x) - f_{xx}(x) + f_{xx}(x) + \alpha - 2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \cos\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) d\delta \, d\alpha \, dx$$

$$+ \frac{1}{2}\int \partial_x^3 f \int \frac{f_{xx}(x) - f_{xx}(x) + f_{xx}(x) + \alpha - 2f_{xx}(x)}{\alpha} \int_0^\infty \delta e^{-\delta} \times \partial_\alpha \{\Delta_\alpha f - \bar{\Delta}_\alpha f\} \sin\left(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\right) d\delta \, d\alpha \, dx$$
and we obtain that,

\[ T_{1,6} = -\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \sin^2\left(\frac{\delta}{4}(\Delta_\alpha f - \Delta f)\right) \, d\delta \, d\alpha \, dx \]

\[ + \frac{1}{2} \int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha} \int_0^\infty \delta e^{-\delta} f_{x}(x - \alpha) + f_{x}(x + \alpha) - 2f_{x}(x) \, d\alpha \, dx \]

\[ \times \sin\left(\frac{\delta}{2}(\Delta_\alpha f - \Delta f)\right) \, d\delta \, d\alpha \, dx \]

\[ - \frac{1}{2} \int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha^3} \int_0^\infty \delta e^{-\delta} \int_0^\alpha f_{x}(x - s) + f_{x}(x + s) - 2f_{x}(x) \, ds \]

\[ \times \sin\left(\frac{\delta}{2}(\Delta_\alpha f - \Delta f)\right) \, d\delta \, d\alpha \, dx \]

\[ - \pi \int \mathcal{H}\partial_x^3 f \Lambda f_{xx} \, dx \]

\[ = T_{1,6,1} + T_{1,6,2} + T_{1,6,3} + T_{1,6,4} \]

In order to estimate \( T_{1,6} \) we shall estimate the \( T_{1,6,j} \), for \( j = 1, 2, 3 \), \( T_{1,6,4} \) being a dissipative term. This is the purpose of the next subsection.

6.6.1 Estimate of the \( T_{1,6,j} \), \( j = 1, 2, 3, 4 \)

We start with \( T_{1,6,1} \), we observe that this term is quite singular, however, by using the oscillating term, one may actually still get a control of this term. More precisely, we write

\[ T_{1,6,1} = -\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \sin^2\left(\frac{\delta}{4}(\Delta_\alpha f - \Delta f)\right) \, d\delta \, d\alpha \, dx \]

Then, we write,

\[ T_{1,6,1} = -\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \]

\[ \times \sin\left(\frac{\delta}{4} \frac{1}{\alpha} \int_0^\alpha f_{x}(x + s) + f_{x}(x - s) - 2f_{x}(x) \, ds + \frac{\delta}{2} f_{x}(x)\right) \sin\left(\frac{\delta}{4}(\Delta_\alpha f - \Delta f)\right) \, d\delta \, d\alpha \, dx \]

therefore,

\[ T_{1,6,1} = -\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \]

\[ \times \sin\left(\frac{\delta}{4} \frac{1}{\alpha} \int_0^\alpha f_{x}(x + s) + f_{x}(x - s) - 2f_{x}(x) \, ds \cos\left(\frac{\delta}{2} f_{x}(x)\right) \sin\left(\frac{\delta}{4}(\Delta_\alpha f - \Delta f)\right) \right) \, d\delta \, d\alpha \, dx \]

\[ - \int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \]

\[ \times \cos\left(\frac{\delta}{4} \frac{1}{\alpha} \int_0^\alpha f_{x}(x + s) + f_{x}(x - s) - 2f_{x}(x) \, ds \right) \sin\left(\frac{\delta}{2} f_{x}(x)\right) \sin\left(\frac{\delta}{4}(\Delta_\alpha f - \Delta f)\right) \, d\delta \, d\alpha \, dx, \]
thus

\[ T_{1,6,1} = - \int H\partial_\alpha^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \]

\[ \times \sin\left(\frac{\delta}{2}\right) \int_0^\alpha f_x(x + s) + f_x(x - s) - 2f_x(x) \, ds \cos\left(\frac{\delta}{2}f_x(x)\right) \sin\left(\frac{\delta}{2}(\Delta f - \Delta f)\right) \, d\delta \, d\alpha \, dx \]

\[ + 2 \int H\partial_\alpha^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \]

\[ \times \sin^2\left(\frac{\delta}{2}\right) \int_0^\alpha f_x(x + s) + f_x(x - s) - 2f_x(x) \, ds \sin\left(\frac{\delta}{2}f_x(x)\right) \sin\left(\frac{\delta}{4}(\Delta f - \Delta f)\right) \, d\delta \, d\alpha \, dx \]

\[ - \int H\partial_\alpha^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \]

\[ \times \sin\left(\frac{\delta}{2}\right) f_x(x) \sin\left(\frac{\delta}{4}(\Delta f - \Delta f)\right) \, d\delta \, d\alpha \, dx, \]

then, by developing the last term, namely

\[ - \int H\partial_\alpha^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \]

\[ \times \sin\left(\frac{\delta}{2}\right) f_x(x) \sin\left(\frac{\delta}{4}(\Delta f - \Delta f)\right) \, d\delta \, d\alpha \, dx, \]

we find,

\[ T_{1,6,1} = - \int H\partial_\alpha^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \]

\[ \times \sin\left(\frac{\delta}{2}\right) f_x(x) \sin\left(\frac{\delta}{4}(\Delta f - \Delta f)\right) \, d\delta \, d\alpha \, dx \]
Finally, we find that

\[\begin{align*}
T_{1,6,1} &= -\int \mathcal{H} \partial_\alpha^3 f \int f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x) \frac{1}{\alpha^2} \int_0^\infty e^{-\delta} \\
&\times \sin(\frac{\delta}{4} \int_0^\alpha f_x(x + s) + f_x(x - s) - 2f_x(x) \, ds) \cos(\frac{\delta}{2} f_x(x)) \sin(\frac{\delta}{4} (\Delta \alpha f - \Delta \alpha f)) \, d\delta \, d\alpha \, dx \\
&+ 2 \int \mathcal{H} \partial_\alpha^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \\
&\times \sin^2(\frac{\delta}{8} \int_0^\alpha f_x(x + s) + f_x(x - s) - 2f_x(x) \, ds) \sin(\frac{\delta}{2} f_x(x)) \sin(\frac{\delta}{4} (\Delta \alpha f - \Delta \alpha f)) \, d\delta \, d\alpha \, dx \\
&- \int \mathcal{H} \partial_\alpha^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \\
&\times \sin^2(\frac{\delta}{2} f_x(x)) \sin(\frac{\delta}{8} \int_0^\alpha f_x(x + s) + f_x(x - s) - 2f_x(x) \, ds) \, d\delta \, d\alpha \, dx \\
&- \int \mathcal{H} \partial_\alpha^3 f \int \frac{f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \sin^2(\frac{\delta}{2} f_x(x)) \, d\delta \, d\alpha \, dx \\
&= \sum_{j=1}^5 T_{1,6,1,j}
\end{align*}\]

To control \(T_{1,6,1,1}\) we write

\[|T_{1,6,1,1}| \leq \|f\|_{\dot{H}^3} \int_0^\infty \delta e^{-\delta} \|f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x)\|_{L^\infty} |\alpha|^3 \]

\[\times \int_0^\alpha \|f_x(x - s) + f_x(x + s) - 2f_x(x)\|_{L^2} \, ds \, d\alpha \, d\delta\]

Then, Minkowski’s inequality gives

\[|T_{1,6,1,1}| \leq \|f\|_{\dot{H}^3} \int_0^\infty \delta e^{-\delta} \int \frac{\|f_{xx}(x - \alpha) - f_{xx}(x)\|_{L^\infty}}{|\alpha|^3} \]

\[\times \int_0^\alpha \|f_x(x - s) - f_x(x)\|_{L^2} \, ds \, d\alpha \, d\delta\]

\[+ \|f\|_{\dot{H}^3} \int_0^\infty \delta e^{-\delta} \int \frac{\|f_{xx}(x + \alpha) - f_{xx}(x)\|_{L^\infty}}{|\alpha|^3} \]

\[\times \int_0^\alpha \|f_x(x + s) - f_x(x)\|_{L^2} \, ds \, d\alpha \, d\delta\]

\[+ \|f\|_{\dot{H}^3} \int_0^\infty \delta e^{-\delta} \int \frac{\|f_{xx}(x + \alpha) - f_{xx}(x)\|_{L^\infty}}{|\alpha|^3} \]

\[\times \int_0^\alpha \|f_x(x + s) - f_x(x)\|_{L^2} \, ds \, d\alpha \, d\delta\]

\[+ \|f\|_{\dot{H}^3} \int_0^\infty \delta e^{-\delta} \int \frac{\|f_{xx}(x - \alpha) - f_{xx}(x)\|_{L^\infty}}{|\alpha|^3} \]

\[\times \int_0^\alpha \|f_x(x - s) - f_x(x)\|_{L^2} \, ds \, d\alpha \, d\delta\]
It not difficult to see that it suffices to estimate one of those terms, let us estimate the first term, namely

\[ A = \|f\|_{H^3}^{10/3} \int_0^\alpha \delta e^{-\delta} \int \frac{\|f_{xx}(x - \alpha) - f_{xx}(x)\|_{L^\infty}}{|\alpha|^3} \int_0^\alpha \|f_x(x - s) - f_x(x)\|_{L^2} \, ds \, d\alpha \, d\delta, \]

to this end, we write (where \( r \) and \( \bar{r} \) are conjugate exponent)

\[
|A| \leq \Gamma(2) \|f\|_{H^3}^{10/3} \int_0^\pi \frac{\|f_{xx}(x - \alpha) - f_{xx}(x)\|_{L^\infty}}{|\alpha|^3} \int_0^\alpha \|f_x(x - s) - f_x(x)\|_{L^2} \, ds \, d\alpha \\
\leq \|f\|_{H^3}^{10/3} \|f_{xx}\|_{B_{\infty,1}^{13/8}} \|f\|_{B_{\infty,\infty}^{11/12}} \|f\|_{B_{\infty,\infty}^{11/12}} \left( \int_0^\alpha \|f_x(x - s) - f_x(x)\|_{L^2}^{r} \, ds \right)^{1/r} \, d\alpha \\
\leq \|f\|_{H^3}^{10/3} \|f_{xx}\|_{B_{\infty,1}^{13/8}} \|f\|_{B_{\infty,\infty}^{11/12}} \|f\|_{B_{\infty,\infty}^{11/12}}
\]

where in the last step we have chosen \( q = 9/8 \), \( r = \bar{r} = 2 \), then by interpolation of \( B_{\infty,1}^{10/3} \), we find,

\[
|A| \leq \|f\|_{H^3}^{10/3} \|f\|_{B_{2,2}^{13/8}} \|f\|_{B_{\infty,\infty}^{11/12}} \|f\|_{B_{\infty,\infty}^{11/12}}.
\]

Since,

\[
\|f\|_{B_{2,2}^{13/8}} \leq \|f\|_{H^3}^{10/3} \|f\|_{H^{1/2}}^{11/12} \|f\|_{H^{1/2}}^{11/12}
\]

then, by using classical Besov embeddings, one finally gets

\[
|A| \leq \|f\|_{H^3}^2 \|f\|_{H^{3/2}}^2
\]

and one may easily see that

\[
\sum_{i=1}^{4} T_{1,6,1,i} \lesssim \|f\|_{H^3}^2 \|f\|_{H^{3/2}}^2 \quad (6.3)
\]

As for the term \( T_{1,6,1,5} \), it suffices to see that

\[
T_{1,6,1,5} = - \int \mathcal{H} \partial_x^2 f \int f_{xx}(x - \alpha) + f_{xx}(x + \alpha) - 2f_{xx}(x) \frac{e^{-\delta} \sin^2(\frac{\delta}{2} f_x(x)) \, d\delta \, d\alpha \, dx}{\alpha^2} \\
= 2\pi \int (\mathcal{H} \partial_x^2 f)^2 \int_0^\alpha e^{-\delta} \sin^2(\frac{\delta}{2} f_x(x)) \, d\delta \, dx \leq \pi \frac{M^2}{1 + M^2} \|f\|_{H^3}^2 \quad (6.4)
\]

where \( M = \|f_x(x, t)\|_{L^\infty \times L^\infty} \). Therefore, we find

\[
\left| \sum_{i=1}^{5} T_{1,6,1,i} \right| \leq 2 \|f\|_{H^3}^2 \|f\|_{H^{3/2}}^2 + \pi \frac{M^2}{1 + M^2} \|f\|_{H^3}^2 \quad (6.5)
\]

To estimate \( T_{1,6,2} \), it suffices to write that
\[
T_{1,6,2} = -\int \mathcal{H} \partial_3^2 f \int \frac{f_{xx}(x-\alpha) + f_{xx}(x+\alpha) - 2f_{xx}(x)}{\alpha} \, d\alpha \int_0^\infty \delta e^{-\delta} \frac{f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)}{\alpha} \, d\alpha
\times \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \, d\delta \, d\alpha \, dx
\leq \|f\|^2_{H^3} \left( \int \|f_{xx}(x-\alpha) + f_{xx}(x+\alpha) - 2f_{xx}(x)\|_{L^\infty} \frac{\|f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)\|_{L^2}}{\alpha^2} \, d\alpha \right)^{1/2}
\leq \|f\|_{H^3} \|f\|_{\dot{B}_{6/2}^{5/2}} \|f\|_{\dot{B}_{2,2}^{3/2}}
\leq \|f\|_{H^3} \|f\|_{H^{3/2}}
\]

We now estimate \( T_{1,6,3} \), let \( q \in [0, 2] \) and \( r, \bar{r} \) so that \( 1/r + 1/\bar{r} = 1 \), we write
\[
T_{1,6,3} = \int \mathcal{H} \partial_3^2 f \int \frac{f_{xx}(x-\alpha) + f_{xx}(x+\alpha) - 2f_{xx}(x)}{\alpha^3} \int_0^\infty \delta e^{-\delta} \int_0^\alpha f_x(x-s) + f_x(x+s) - 2f_x(x) \, ds \times \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \, d\delta \, d\alpha \, dx
\leq \|f\|_{H^3} \int_0^\infty e^{-\delta} \frac{\|f_{xx}(x-\alpha) + f_{xx}(x+\alpha) - 2f_{xx}(x)\|_{L^\infty}}{\alpha^3} \, d\alpha \times \left( \int_0^\alpha \|f_x(x-s) - f_x(x)\|_{L^2} + \|f_x(x+s) - f_x(x)\|_{L^2} \, ds \right) \, d\alpha \, d\delta
\leq \|f\|_{H^3} \int_0^\infty \delta e^{-\delta} \frac{\|f_{xx}(x-\alpha) + f_{xx}(x+\alpha) - 2f_{xx}(x)\|_{L^\infty}}{\alpha^{3-q}} \, d\alpha \times \left( \int \frac{\|f_x(x+s) + f_x(x+s) - 2f_x(x)\|_{L^2}}{\|s\|^{r-\|s\|}} \, ds \right)^{1/r} \, ds \, d\alpha \, d\delta
\leq \|f\|_{H^3} \|f_{xx}\|_{\dot{B}_{5/2}^{3/2}} \|f\|_{\dot{B}_{5/2}^{3/2}}
\]

Then, by choosing \( q = 2, \bar{r} = 2 \), we may get a control of \( \dot{B}_{\infty,1}^{3/2} \) by real interpolation and of \( \dot{B}_{\infty,1}^{5/2} \) as well, we find
\[
|T_{1,6,3}| \leq \|f\|_{H^3} \|f\|_{\dot{B}_{2,2}^{3/2}} \|f\|_{\dot{B}_{2,2}^{3/2}} \|f\|_{\dot{B}_{\infty,1}^{3/2}} \|f\|_{\dot{B}_{\infty,1}^{3/2}}
\leq \|f\|_{H^3} \|f\|_{H^{3/2}}
\]

Therefore, since \( T_{1,6,4} \) is a dissipative term and by (6.4), we get that
\[
|T_1| \lesssim \|f\|^2_{H^3}(\|f\|^2_{H^{3/2}} + \|f\|_{H^{3/2}}^2) - \pi \|f\|^2_{H^3} + \frac{\pi M^2}{1 + M^2} \|f\|^2_{H^3}
\]
which finally gives,
\[
|T_1| \lesssim \|f\|^2_{H^3}(\|f\|^2_{H^{3/2}} + \|f\|_{H^{3/2}}^2) - \frac{\pi}{1 + M^2} \|f\|^2_{H^3} \quad (6.6)
\]

6.7. Estimate of \( T_2 \)

To estimate those two terms namely
\[
T_2 = -3 \int \mathcal{H} \partial_3^2 f \int \partial_x \Delta_\alpha f \partial_x \Delta_\alpha f \int_0^\infty \delta e^{-\delta} \sin(\delta \Delta_\alpha f(x)) \, d\delta \, d\alpha \, dx,
\]
it suffices to write,

\[ |T_2| \lesssim \|f\|_{\dot{H}^3} \left( \int \frac{\|f_{xx} - f_{xx}(x - \alpha)\|_{L^\infty}^2}{\alpha^2} \, d\alpha \right)^{1/2} \left( \int \frac{\|f_x(x) - f_x(x - \alpha)\|_{L^\infty}^2}{\alpha^2} \, d\alpha \right)^{1/2} \]

and therefore,

\[ |T_2| \lesssim \|f\|_{\dot{H}^3} \|f_{xx}\|_{\dot{B}_{\infty}^{1/2}} \|f\|_{\dot{H}^2} \]

hence, using a classical Besov embedding along with Sobolev interpolations one gets

\[ |T_2| \lesssim \|f\|^2_{\dot{H}^3} \|f\|_{\dot{H}^{3/2}} \quad (6.7) \]

**6.8. Estimate of \(T_3\)**

It suffices to observe for instance that \(\dot{B}_{6,3}^{5/3} \hookrightarrow \dot{H}^2 = \left[ \dot{H}^{3/2}, \dot{H}^3 \right]_{\frac{5}{3}, \frac{1}{3}}\), so that

\[
T_3 = -2 \int \mathcal{H}_{\delta}^3 \int (\partial_x \Delta_\alpha f)^3 \int_{0}^{\infty} \delta^2 e^{-\delta} \cos(\delta \Delta_\alpha f) \, dx \, d\alpha \, d\delta
\]

\[
\lesssim \|f\|_{\dot{H}^3} \|f\|_{\dot{B}_{6,3}^{5/3}}^3
\]

\[
\lesssim \|f\|^2_{\dot{H}^3} \|f\|_{\dot{H}^{3/2}} \quad (6.8)
\]

Finally, by (6.6), (6.7) and (6.8) and by integrating in time \(s \in [0, T]\) we find that for all \(T > 0\)

\[ \|f\|_{\dot{H}^{5/2}}^2(T) + \frac{\pi}{1 + M^2} \int_{0}^{T} \|f\|_{\dot{H}^3}^2 \, ds \lesssim \|f_0\|_{\dot{H}^{5/2}} + P(f) \int_{0}^{T} \|f\|_{\dot{H}^3}^2 \, ds \]

where \(P(f) = \|f\|_{L^\infty([0, T], \dot{H}^{3/2})} + \|f\|_{L^\infty([0, T], \dot{H}^{3/2})}^2\). So that Lemma 6.1 is proved.

\[ \square \]

**7. PROOFS OF THEOREMS 2.1 AND 2.2**

We consider the following approximated system,

\[
(\check{\mathcal{M}}_\epsilon) \quad \begin{cases}
\partial_t f_\epsilon(t, x) - \frac{\rho}{\pi} \int \partial_x \Delta_\alpha f_\epsilon \int_{0}^{\infty} e^{-\delta} \cos(\delta \Delta_\alpha f_\epsilon) \, d\delta \, d\alpha - \epsilon \Delta f_\epsilon = 0 \\
\rho_\epsilon(x) \delta_0(x) = \rho_0(x) \ast \phi_\epsilon(x)
\end{cases}
\]

where \(\phi_\epsilon\) is a classical mollifier that is \(\phi_\epsilon(x) = e^{-1}\phi(x^{-1})\), \(\phi \in \mathcal{D}(\mathbb{R})\) so that \(\phi\) is nonnegative and \(\int \phi(x) \, dx = 1\). If the Lipschitz norm remains bounded on a time interval \([0, T]\) and if the \(\|f_0, \epsilon\|_{\dot{H}^{3/2}}\) is smaller than a constant that depends only on the \(L^\infty([0, T], \dot{W}^{1, \infty})\) norm then we may locally solve the equation and using the same \textit{a priori} estimate as proved in section 5 for the regularized equation \((\check{\mathcal{M}}_\epsilon)\) we may show that actually \(T_\epsilon = +\infty\). One has (uniformly in \(\epsilon\)) that

\[ \|f_0, \epsilon\|_{\dot{H}^{3/2}} \lesssim \|f_0\|_{\dot{H}^{3/2}} \]

as well as,

\[ \|f_0, \epsilon\|_{\dot{W}^{1, \infty}} \lesssim \|f_0\|_{\dot{W}^{1, \infty}} \]

and therefore the regularized initial data converges strongly in \(\dot{H}^{3/2} \cap \dot{W}^{1, \infty}\). Let \(\phi \in \mathcal{D}([0, T] \times \mathbb{R})\) be nonnegative, from the \textit{a priori} estimates we know that \(\phi f_\epsilon\) is bounded in \(L^\infty([0, T]; \dot{H}^{3/2} \cap \dot{W}^{1, \infty}) \cap L^2([0, T]; \dot{H}^2)\) and since those spaces a separable Banach spaces, thanks to the Banach-Alaoglu theorem, we may extract from these solutions \(f_\epsilon\) a subsequence \(\{f_{k, \epsilon}\}_{k \geq 0}\) that converges weakly to a solution \(f \in L^2([0, T]; \dot{H}^2)\) and \(*\)-weakly in \(L^\infty([0, T]; \dot{H}^{3/2})\). In order to obtain the
We know that up to 6 use the Rellich compactness theorem (see e.g. [10]) to get a nice bound on $\partial_t f_\epsilon$ locally in space and time, namely on

$$-\epsilon \lambda^2 f_\epsilon - \Lambda f_\epsilon - 2 \int \partial_x \Delta_\alpha f_\epsilon \int_0^\infty \delta e^{-\delta} \sin^2(\frac{\delta}{2} \Delta_\alpha f_\epsilon) \, d\delta \, d\alpha$$

Since $f_\epsilon$ is bounded in $L^2([0,T]; \dot{H}^2)$ then $\partial_t f_\epsilon$ is bounded in $L^2([0,T]; \dot{H}^1)$; it is not difficult to see that the contribution coming from the nonlinear part of $\partial_t f_\epsilon$ is a locally bounded sequence in $L^2 \dot{H}^{-1/4}$. Indeed, by using the dual form of the Sobolev embedding it suffices to have a bound on the $L^2 L^4/3$ norm of the nonlinearity. By controlling the product, we find that if $f_\epsilon \in \dot{B}^{3/2}_2 \cap \dot{B}^{3/4}_8$ then the nonlinear part of $\partial_t f$ is bounded in $L^2 \dot{H}^{-1/4}$. But, $f_\epsilon$ is locally bounded in $L^2 \dot{H}^{3/2} \cap L^2 \dot{H}^{9/8}$ and this latter space embeds into $\dot{B}^{3/2}_2 \cap \dot{B}^{3/4}_8$, therefore we get that the nonlinear term of $\partial_t f_\epsilon$ is a bounded sequence in $L^2 \dot{H}^{-1/4}$. Since the linear part is locally bounded in $L^2 \dot{H}^1$ hence we may use the Rellich compactness theorem (see e.g. [26]) to get the strong convergence of a subsequence in $(L^2 L^2)_{loc}$. Consequently, the nonlinear term converges in $\mathcal{D}'$, it is then classical to prove that the limit is a solution of the equation. For higher regularity data (that is $f_0 \in \dot{H}^{5/2} \cap \dot{H}^{5/2}$, with $f_0$ small enough in $\dot{H}^{3/2}$), using the a priori estimates proved in section 6 we know that up to an extraction, $f_\epsilon$ is a bounded sequence in $L^\infty([0,T], \dot{H}^{5/2})$ (in particular the space-time Lipschitz norm will remain bounded) and we can pass to the weak limit as well since Rellich gives also the strong compactness in $(L^2 L^2)_{loc}$ and we conclude the result.

By doing the same a priori estimates for the difference of two solutions in $\dot{H}^{3/2}$ (resp $\dot{H}^{5/2}$), one observes that the uniqueness follows easily from the regularizing effect together with the fact that the $L^\infty_t \dot{H}^{3/2}$ norm decays (resp $L^\infty_t \dot{H}^{5/2}$), so that Grönwall’s inequality gives the uniqueness in the usual way and we therefore omit the details.

□

Acknowledgments

Both D. C and O. L were supported by the National Grant MTM2014-59488-P from the Spanish government and ICMAT Severo Ochoa project SEV-2015-55. O. L was supported by the Marie-Curie Grant, acronym: TRANSIC, from the FP7-IEF program, and the ERC through the Starting Grant project H2020-EU.1.1.-63922.

References

[1] D.M. Ambrose. Well-posedness of two-phase Hele-Shaw flow without surface tension. European J. Appl. Math. 15, no. 5, 597-607, 2004.
[2] H. Bahouri, J.-Y Chemin and R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, 343p, Springer Verlag, 2011.
[3] T. Beck, P. Sosoe and P. Wong. Duchon-Robert solutions for the Rayleigh-Taylor and Muskat problems. J. Differ. Equations 256, no. 1, 206-222, 2014.
[4] J. Bergh and Lofstrom, Interpolation Spaces. An Introduction Grundlehren der mathematischen Wissenschaften 223, Berlin-Heidelberg-New York, Springer-Verlag 1976.
[5] Besov, O. V.: Investigation of a class of function spaces in connection with imbedding and extension theorems. (Russian) Trudy. Mat. Inst. Steklov 60 (1961), 42-81.
[6] S. Cameron, Global well-posedness for the 2D Muskat problem with slope less than 1, arXiv:1704.08401v3.
[7] A. Castro, D. Córdoba and D. Faraco. Mixing solutions for the Muskat problem, arxiv:1605.04822, 2016.
[8] A. Castro, D. Córdoba, C. L. Fefferman and F. Gancedo Breakdown of smoothness for the Muskat problem, Arch. Ration. Mech. Anal., 208, no. 3, 805-909 (2013).
[9] A. Castro, D. Córdoba, C. L. Fefferman, F. Gancedo and María López-Fernández. Rayleigh Taylor breakdown for the Muskat problem with applications to water waves. Annals of Math 175, no. 2, 909-948, 2012.
[10] A. Cheng, R. Granero-Belinchon and S. Shkoller, Well-posedness of the Muskat problem with $H^2$ initial data, Adv. Math., 286, 32-104, 2016.
[11] P. Constantin, D. Córdoba, F. Gancedo and R. M. Strain. *On the global existence for the Muskat problem*, J. Eur. Math. Soc. **15**, 201-227 (2013).

[12] P. Constantin, D. Córdoba, F. Gancedo, L. Rodríguez-Piazza and R. M. Strain. *On the Muskat problem: global in time results in 2D and 3D*. Amer. J. Math **138**, no. 6, 1455-1494, 2016.

[13] P. Constantin, F. Gancedo, R. Shvydkoy and V. Vicol. *Global regularity for 2D Muskat equations with finite slope* Ann. Inst. H. Poincaré Anal. Non Linéaire, **34**, no. 4, 1041-1074, 2017.

[14] D. Córdoba, J. Gómez-Serrano and A. Zlatoš. *A note on stability shifting for the Muskat problem II: Stable to Unstable and back to Stable* Anal. PDE **10** (2017), no 2, 367-378.

[15] D. Córdoba and F. Gancedo. *Contour dynamics of incompressible 3-D fluids in a porous medium with different densities* Comm. Math. Phys. **273** (2007), 2, 445-471.

[16] D. Córdoba and F. Gancedo. A maximum principle for the Muskat problem for fluids with different densities. Comm. Math. Phys., **286** (2009), no. 2, 681-696.

[17] D. Córdoba, J. Gómez-Serrano and A. Zlatoš. *A note in stability shifting for the Muskat problem* Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **373** (2015) no.2050, 2014-2024.

[18] H. Darcy. *Les Fontaines Publiques de la Ville de Dijon*. Dalmont, Paris, 1856

[19] L. Dawson, H. McGahagan and G. Ponce. *On the decay properties to a class of Schrodinger equations*, Proceedings of the American Mathematical Society, **136** (2008), no. 6, 2081-2090

[20] F. Deng, Z. Lei, and F. Lin *On the two-dimensional Muskat problem with monotone large initial data*, Communications on Pure and Applied Math, Volume **70**, Issue 6 (2017) 1115-1145.

[21] J. Escher and B.-V. Matioc. *On the parabolicity of the Muskat problem: Well-posedness, fingering, and stability results*. Z. Anal. Awend. **30**, 193-218, (2011).

[22] C. Führer, L. Székelyhidi Jr. Piecewise constant subsolutions for the Muskat problem, arxiv:1709.05155.

[23] F. Gancedo. *A survey for the Muskat problem and a new estimate*. SeMA J. **74**, no. 1, 21-35, 2017

[24] J. Gómez-Serrano, R. Granero-Belinchón. On turning waves for the inhomogeneous Muskat problem: a computer-assisted proof. Nonlinearity, **27**(6):1471-1498, 2014.

[25] R. Granero-Belinchón. Global existence for the confined Muskat problem. SIAM J. Math. Anal., **46**(2):1651-1680, 2014.

[26] P.-G. Lemarié-Rieusset. *The Navier-Stokes problem in the 21st century*. CRC Press, Boca Raton, FL, 2016.

[27] B.V. Matioc. *The Muskat problem in 2D: equivalence of formulations, well-posedness, and regularity results*, arxiv: 1610.05546, 2016.

[28] M. Muskat. *Two fluid systems in porous media. The encroachment of water into an oil sand*. Physics, **5**, (1934), 250-264.

[29] F. Otto. *Evolution of microstructure in unstable porous media flow: a relaxational approach*. Comm. Pure Appl. Math., **52**(7):873-915, 1999

[30] N. Patel and R.M. Strain. *Large Time Decay Estimates for the Muskat Equation*. Comm. Partial Differential Equations **42**, no. 6, 977-999, 2017.

[31] T. Runst and W. Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*. De Gruyter, Berlin, 1996.

[32] P.G. Saffman and Taylor. The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid. Proc. R. Soc. London, Ser. A **245**, 312-329, 1958.

[33] M. Siegel, R. Caflisch and S. Howison. *Global Existence, Singular Solutions, and Ill-Posedness for the Muskat Problem*. Comm. Pure and Appl. Math., **57**, (2004), 1374-1411

[34] L. Jr. Szekelyhidi. *Relaxation of the incompressible porous media equation*. Annales de l’ENS (4) **45**, no. 3, 491-509, 2012.

**Instituto de Ciencias Matemáticas (ICMAT), Consejo Superior de Investigaciones Científicas, Madrid, Spain**

*E-mail address: dcg@icmat.es*

**Instituto de Ciencias Matemáticas (ICMAT), Consejo Superior de Investigaciones Científicas, Madrid, Spain, and Departamento de Análisis Matemático & IMUS, Universidad de Sevilla, C/ Tarifa s/n, Campus Reina Mercedes, 41012 Sevilla, Spain**

*E-mail address: omar.lazar@icmat.es*