BIFURCATION FOR A FREE BOUNDARY PROBLEM MODELING THE GROWTH OF NECROTIC MULTILAYERED TUMORS

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Abstract. In this paper we study bifurcation solutions of a free boundary problem modeling the growth of necrotic multilayered tumors. The tumor model consists of two elliptic differential equations for nutrient concentration and pressure, with discontinuous terms and two free boundaries. The novelty is that different types of boundary conditions are imposed on two free boundaries. By bifurcation analysis, we show that there exist infinitely many branches of non-flat stationary solutions bifurcating from the unique flat stationary solution.

1. Introduction. During the recent several decades, numerous mathematical tumor models have been proposed, for exploring the intrinsic mechanism of tumor growth. Many interesting mathematical problems and challenges arise and have attracted a lot of attention. In this paper, we study a mathematical model for dormant solid tumors in necrotic phase.

The tumor under considered is cultivated on an impermeable support membrane in laboratory with new cell cultivation technique, the tumor region is strip-like and consists of multilayered tumor cells (cf. [17], [18]). Let \( \Omega^+ \) and \( \Omega^- \) be proliferating region and necrotic region occupied by proliferating cells and necrotic cells, respectively, with the form of

\[
\Omega^+ = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : \eta(x) < y < \rho(x), x \in \mathbb{R}^{n-1}\},
\]

\[
\Omega^- = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : 0 < y < \eta(x), x \in \mathbb{R}^{n-1}\},
\]

where \( \eta(x) \) and \( \rho(x) \) are unknown functions satisfying \( 0 < \eta(x) < \rho(x) \) for \( x \in \mathbb{R}^{n-1} \) and \( n \in \mathbb{N} \). Denote free boundaries \( \Gamma^+ = \text{graph}\{\rho(x)\}, \Gamma^- = \text{graph}\{\eta(x)\} \) and fixed bottom boundary \( \Gamma_0 = \text{graph}\{0\} \). Let \( \sigma(x, y), \nu(x, y) \) and \( p(x, y) \) be the nutrient concentration, cell velocity and pressure within tumor at location \((x, y)\), respectively. Since necrotic cells do not consume nutrient, by diffusion we have

\[
\Delta \sigma = \sigma \chi_{\Omega^+} \quad \text{in} \quad \Omega^+ \cup \Omega^-,
\]

where \( \chi_{\Omega^+} \) is the characteristic function of \( \Omega^+ \), which equals to 1 on \( \Omega^+ \) and 0 outside \( \Omega^+ \). The solid tumor has porous medium structure, and Darcy’s law \( \nu = -\nabla p \) is
available within tumor. Since necrotic cells will not produce new cells, by combining
mass conservation law we have
\[-\Delta p = \mu(\sigma - \bar{\sigma})\chi_{\Omega^+} - \nu\chi_{\Omega^-} \quad \text{in} \quad \Omega^+ \cup \Omega^-,
\]
where \(\chi_{\Omega^-}\) is the characteristic function of \(\Omega^-,\) \(\mu, \nu\) and \(\bar{\sigma}\) are positive constants,
among of which \(\mu\) represents the proliferation rate of proliferating cells, \(\nu\) represents
the removal rate of necrotic cells, \(\bar{\sigma}\) is a threshold nutrient level for the balance of
cell apoptosis and mitosis. We assume tumor receives constant nutrient supply
through \(\Gamma^+\), and keeps compact by cell-to-cell adhesive, we have
\[\sigma = \bar{\sigma}, \quad p = \gamma \kappa \quad \text{on} \quad \Gamma^+,
\]
where \(\bar{\sigma} \geq 0\) is the nutrient concentration in the surroundings and \(\gamma > 0\) is
the rate of cell-to-cell adhesiveness, and \(\kappa\) is the mean curvature of \(\Gamma^+\). Necrosis may occur
at the nutrient level \(\hat{\sigma}\), and we have
\[\sigma = \hat{\sigma} \quad \text{on} \quad \Gamma^-.
\]
It is natural to assume \(0 < \hat{\sigma} < \bar{\sigma} < \bar{\sigma}\). We also impose
\[\sigma, \, p, \, \partial_n \sigma, \, \partial_n p \text{ are continuous across } \Gamma^-,
\]
where \(\partial_n\) denotes the outward normal derivative with respect to \(\Omega^+\). The nutrient
and necrotic cells do not pass across \(\Gamma_0\), which implies that
\[\partial_n \sigma = 0, \quad \partial_n p = 0 \quad \text{on} \quad \Gamma_0.
\]
Finally, by noting that \(\mathbf{v} \cdot \mathbf{n} = -\partial_n p\), where \(\mathbf{n}\) is the unit outward normal field on
\(\Gamma^+\), for dormant tumor we have
\[\partial_n p = 0 \quad \text{on} \quad \Gamma^+.
\]
Tumor model (1)–(7) is a free boundary problem, since \(\Gamma^+\) and \(\Gamma^-\) need to be
determined with unknown functions \(\sigma\) and \(p\) together. We refer to [3] for a similar
mathematical model modeling solid tumor spheroid with a necrotic core.

If equation (7) is replaced by
\[V_n = -\partial_n p \quad \text{on} \quad \Gamma^+,
\]
where \(V_n\) is the outward normal velocity of the upper free boundary \(\Gamma^+\) (time-
dependent), we get the corresponding evolutionary problem. In the non-necrotic
case, i.e., \(\Omega^- = \emptyset\), the model has only one free boundary \(\Gamma^+\), and results a Hele-
Shaw type problem. Cui and Escher [9] proved that the non-necrotic model is local
well-posed, the unique flat non-necrotic stationary solution (independent of \(x\)), is
asymptotically stable for \(\gamma > \gamma^*\) and unstable for \(0 < \gamma < \gamma^*\), where \(\gamma^*\) is some
positive threshold value of cell-to-cell adhesiveness. Zhou et al. [25] proved that
there exist infinitely many non-flat non-necrotic stationary solutions bifurcating
from the unique flat non-necrotic stationary solution. For similar illuminate results
of non-necrotic solid tumor spheroid models which have been extensively studied,
we refer to [2], [7], [8], [10], [12], [13], [14], [19], [20], [22], [23], [24] and references
cited therein.

As pointed by Friedman [11], the novelty of necrotic tumor model is that there
exist two free boundaries \(\Gamma^+\) and \(\Gamma^-\), imposed with different types of boundary
conditions. We see that \(\sigma \equiv \bar{\sigma}\) in \(\Omega^-\), and the lower free boundary \(\Gamma^-\) is associated
to an obstacle problem. It is well-known that the regularity of free boundaries
of obstacle problems in high dimension is very difficult (cf. Caffarelli [4]), which
implies that the analysis of necrotic tumor model in high dimension becomes very
challenging and interesting. For a similar tumor spheroid model with necrotic core, Hao et al. [16] first studied bifurcation solutions in case \( \nu = 0 \), by formally reducing the model into a bifurcation problem, without a rigorous verification, they proved the existence of bifurcation branches of non-radial stationary solutions. Recently, Cui [6] studied asymptotic behavior of transient solutions by using Nash-Moser implicit function theorem, and linearized stability theorem for parabolic differential equations in Banach manifolds. Motivate by this work, in this paper we shall make a rigorous analysis and extend the bifurcation result to the necrotic multilayered tumor model (1)–(7) in case \( \nu > 0 \). It is worthy to note that the method employed here is available for necrotic tumor spheroid model, and the rigorous verification of [16] can be mended by some necessary modifications.

To state our main result, we denote the stationary solution of problem (1)–(7) by \( (\sigma, p, \eta, \rho) \), with \( \Gamma^- = \text{graph}\{\eta(x)\} \) and \( \Gamma^+ = \text{graph}\{\rho(x)\} \). As shown in [21], problem (1)–(7) has a unique flat stationary solution \( (\sigma_s, p_s, \eta_s, \rho_s) \) for \( \bar{\sigma} > \sigma^* \), where \( \sigma^* > \tilde{\sigma} \) is a constant depending only on \( \hat{\sigma} \) and \( \tilde{\sigma} \). Later on, we always fix \( \bar{\sigma} > \sigma^* \).

For the sake of simplicity, we only consider the case \( n = 2 \), and assume that
\[
\sigma(x, y), \ p(x, y), \ \eta(x), \ \rho(x) \text{ are } 2\pi\text{-periodic in } x \in \mathbb{R}.
\] (9)
The higher dimensional case \( n \geq 3 \) can be treated similarly. Identify \( \mathbb{S} = \mathbb{R}/2\pi\mathbb{Z} \). For any \( k \in \mathbb{N} \) and \( \alpha \in (0, 1) \), we also identify the usual \( 2\pi \)-periodic Hölder continuous function space \( C^{k+\alpha}_{\text{per}}(\mathbb{R}) = C^{k+\alpha}(\mathbb{S}) \).

Our main result is as follows:

**Theorem 1.1.** There exist a positive integer \( K \) and a positive null sequence \( \{\gamma_k\}_{k \geq K} \), such that free boundary problem (1)–(7) has infinitely many bifurcation branches of non-flat stationary solutions with free boundaries
\[
\rho = \rho_s + \varepsilon \cos(kx) + O(\varepsilon^2), \quad \eta = \eta_s + \varepsilon A_k \cos(kx) + O(\varepsilon^2),
\] (10)
and
\[
\gamma = \gamma_k + O(\varepsilon) \quad \text{for} \quad k \geq K,
\] (11)
where \( \varepsilon \) is a small real parameter, and \( A_k \) is some constant given by (71).

We shall first solve problem (1)–(6) for given \( \rho \) in a small neighborhood of \( \rho_s \), by using Nash-Moser implicit function theorem and the classical implicit function theorem in Banach spaces, the solution obtained in this way is smoothly dependent on \( \rho \), next by substituting it into (7), and taking \( \gamma \) as a bifurcation parameter, we can get a reduced equation
\[
G(\gamma, \rho) = 0.
\] (12)
With a linearization analysis and delicate computation, we give a sufficient and necessary condition for the existence of nontrivial solutions of the linearized problem at the flat stationary solution, to figure out all possible bifurcation points. Finally by using Crandall-Rabinowitz theorem we can get bifurcation branches of non-flat stationary solutions. We refer readers to our recent work [21] for asymptotic stability of the flat stationary solution.

Our result implies that the cell-to-cell adhesiveness \( \gamma \) plays an important role on tumor’s invasion. By the definition (43) of \( \gamma_k \), we can see that the proliferation rate \( \mu \) and the removal rate \( \nu \) can also be taken as bifurcation parameters, and they are also important on the stability of necrotic tumors.

The structure of the rest of this paper is arranged as follows. In the next section, we study the linearized problem at the flat stationary solution. In Section
we reduce free boundary problem (1)–(7) into a bifurcation problem and give a proof of Theorem 1.1. In the last section we make some conclusions and biological implications of our main result.

2. Linearized problem. In this section, we study the linearization of problem (1)–(7) at the flat stationary solution, and study the existence of nontrivial solutions. Recall from [21], there exists a constant $\sigma_*$ depending only on $\bar{\sigma}$ and $\bar{\sigma}$ such that problem (1)–(7) has a unique flat stationary solution $(\sigma_*, p_*, \eta_*, \rho_*)$ with $0 < \eta_* < \rho_*$, if and only if $\bar{\sigma} > \sigma_*$. For any given $\bar{\sigma} > \sigma_*$, the flat stationary solution has following expressions:

\[
\begin{align*}
\sigma_s(x, y) &= \begin{cases}
\frac{\bar{\sigma} \sinh(y - \eta_*) + \hat{\sigma} \sinh(\rho_s - y)}{\sinh(\rho_s - \eta_*)} & \text{for } x \in \mathbb{S}, \eta_* \leq y \leq \rho_s, \\
\hat{\sigma} & \text{for } x \in \mathbb{S}, 0 < y < \eta_*,
\end{cases} \\
p_s(x, y) &= \begin{cases}
\frac{\mu}{2} \hat{\sigma}(y^2 - \rho_s^2) + (\nu - \mu \bar{\sigma})(y - \rho_s)\eta_* + \mu(\bar{\sigma} - \sigma_s(y)) & \text{for } x \in \mathbb{S}, \eta_* \leq y \leq \rho_s, \\
\frac{\nu}{2} (y^2 - \eta_*^2) + \rho_0 & \text{for } x \in \mathbb{S}, 0 < y < \eta_*,
\end{cases}
\end{align*}
\]

where $\rho_0 = \frac{\mu}{2} (\hat{\sigma}(\eta_*^2 - \rho_s^2) + (\nu - \mu \bar{\sigma})(\eta_* - \rho_s)\eta_* + \mu(\bar{\sigma} - \hat{\sigma}))$.

\[
\eta_* = \frac{\mu}{\nu} \left(\sqrt{\bar{\sigma}^2 - \hat{\sigma}^2} - \hat{\sigma} \ln(\hat{\sigma} + \sqrt{\bar{\sigma}^2 - \hat{\sigma}^2}) + \bar{\sigma} \ln \bar{\sigma}\right),
\]

and

\[
\rho_* = \eta_* + \ln(\bar{\sigma} + \sqrt{\bar{\sigma}^2 - \hat{\sigma}^2}) - \ln \hat{\sigma}.
\]

By a direct computation, for $x \in \mathbb{S}$, there hold

\[
\begin{align*}
\partial_y^2 \sigma_s(x, \eta_*) &= \hat{\sigma}, & \partial_y^2 \sigma_s(x, \eta_*^-) &= 0, & \partial_y \sigma_s(x, \rho_*) &= \sqrt{\bar{\sigma}^2 - \hat{\sigma}^2}, \\
\partial_y^2 p_s(x, \eta_*) &= -\mu(\bar{\sigma} - \hat{\sigma}), & \partial_y^2 p_s(x, \eta_*^-) &= \nu, & \partial_y p_s(x, \rho_*) &= 0, \\
\partial_y \sigma_s(x, \eta_*) &= 0, & \partial_y \sigma_s(x, 0) &= 0, & \partial_y p_s(x, 0) &= 0.
\end{align*}
\]

Next, we study the linearization of free boundary problem (1)–(7) at the flat stationary solution $(\sigma_*, p_*, \eta_*, \rho_*)$. Let

\[
\sigma(x, y) = \sigma_s(x, y) + \varepsilon \phi(x, y), \quad p(x, y) = p_s(x, y) + \varepsilon \psi(x, y), \\
\eta(x) = \eta_* + \varepsilon \xi(x), \quad \rho(x) = \rho_* + \varepsilon \zeta(x),
\]

where $\phi$, $\psi$, $\xi$ and $\zeta$ are unknown functions of $2\pi$-periodic in $x$.

By substituting (20) into equations (1), (2), and collecting all first order $\varepsilon$-terms, we easily get

\[
\Delta \phi = \begin{cases}
\phi & \text{for } x \in \mathbb{S}, \eta_* < y < \rho_*, \\
0 & \text{for } x \in \mathbb{S}, 0 < y < \eta_*,
\end{cases}
\]

\[
- \Delta \psi = \begin{cases}
\mu \phi & \text{for } x \in \mathbb{S}, \eta_* < y < \rho_*, \\
0 & \text{for } x \in \mathbb{S}, 0 < y < \eta_*.
\end{cases}
\]

By (3) and (17) we have

\[
\sigma(x, \rho_* + \varepsilon \zeta(x)) - \sigma_s(x, \rho_*) \\
= \sigma_s(x, \rho_* + \varepsilon \zeta(x)) + \varepsilon \phi(x, \rho_* + \varepsilon \zeta(x)) - \sigma_s(x, \rho_*) \\
= \varepsilon \phi(x, \rho_*) + \varepsilon \partial_y \sigma_s(x, \rho_*) \zeta(x) + O(\varepsilon^2) \\
= \varepsilon \phi(x, \rho_*) + \varepsilon \sqrt{\bar{\sigma}^2 - \hat{\sigma}^2} \zeta(x) + O(\varepsilon^2) = 0.
\]
Since the mean curvature on the curve \( y = \rho + \varepsilon \zeta(x) \) is given by
\[
\kappa \bigg|_{y=\rho_x + \varepsilon \zeta} = -\varepsilon (1 + \varepsilon \zeta^2)^{-\frac{3}{2}} \zeta_{xx} = -\varepsilon \zeta_{xx} + O(\varepsilon^2),
\]
from (3) and (18) we have
\[
p(x, \rho + \varepsilon \zeta(x)) - p(x, \rho)
= p_s(x, \rho + \varepsilon \zeta(x)) + \varepsilon \psi(x, \rho + \varepsilon \zeta(x)) - p_s(x, \rho)
= \varepsilon \psi(x, \rho) + \varepsilon \partial_y p_s(x, \rho_\varepsilon) \zeta(x) + O(\varepsilon^2)
= \varepsilon \psi(x, \rho) + O(\varepsilon^2) = -\varepsilon \gamma \zeta_{xx} + O(\varepsilon^2).
\]
Thus the linearizations of equation (3) are given by
\[
\phi(x, \rho_\varepsilon) = -\sqrt{\sigma^2 - \bar{\sigma}^2} \zeta(x) \quad \text{for} \ x \in \mathbb{S}. \tag{23}
\]
\[
\psi(x, \rho_\varepsilon) = -\gamma \zeta_{xx}(x) \quad \text{for} \ x \in \mathbb{S}. \tag{24}
\]
Similarly, by using (17)–(19), we have that the linearizations of equations (4)–(6) are
\[
\phi(x, \eta_\varepsilon) = 0, \quad \text{for} \ x \in \mathbb{S}, \tag{25}
\]
\[
\partial_y \phi(x, \eta^+_\varepsilon) - \partial_y \phi(x, \eta^-_\varepsilon) = -\bar{\sigma} \xi(x) \quad \text{for} \ x \in \mathbb{S}, \tag{26}
\]
\[
\psi(x, \eta^+_\varepsilon) - \psi(x, \eta^-_\varepsilon) = 0, \quad \text{for} \ x \in \mathbb{S}, \tag{27}
\]
\[
\partial_y \psi(x, \eta^+_\varepsilon) - \partial_y \psi(x, \eta^-_\varepsilon) = (\mu(\bar{\sigma} - \bar{\sigma}) + \nu) \xi(x) \quad \text{for} \ x \in \mathbb{S}, \tag{28}
\]
\[
\partial_y \phi(x, 0) = 0 \quad \text{for} \ x \in \mathbb{S}, \tag{29}
\]
\[
\partial_y \psi(x, 0) = 0 \quad \text{for} \ x \in \mathbb{S}. \tag{30}
\]
On the boundary \( y = \rho + \varepsilon \zeta(x) \), the unit outward normal direction \( \mathbf{n}_\varepsilon = \frac{(-\varepsilon \zeta_{x}, 1)}{\sqrt{1 + \varepsilon^2 \zeta_{xx}^2}} \).

Compute
\[
\langle \nabla p(x, \mathbf{n}_\varepsilon) \big|_{y=\rho_\varepsilon + \varepsilon \zeta}, y = \rho_\varepsilon + \varepsilon \zeta \rangle = \partial_y (p_\varepsilon + \varepsilon \psi) \big|_{y=\rho_\varepsilon + \varepsilon \zeta} + O(\varepsilon^2)
= \varepsilon (\partial^2_{\bar{\sigma}} p_\varepsilon \zeta + \partial_y \psi) \big|_{y=\rho_\varepsilon} + O(\varepsilon^2)
= \varepsilon \left[ \partial_y \psi \big|_{y=\rho_\varepsilon} - \mu(\bar{\sigma} - \bar{\sigma}) \zeta(x) \right] + O(\varepsilon^2).
\]
Then we get that the linearization of equation (7) is
\[
\partial_y \psi(x, \rho_\varepsilon) - \mu(\bar{\sigma} - \bar{\sigma}) \zeta(x) = 0 \quad \text{for} \ x \in \mathbb{S}. \tag{31}
\]

Next, we study nontrivial solutions of problem (21)–(31). Let \( \phi, \psi, \xi \) and \( \zeta \) be smooth functions with the following form of
\[
\begin{align*}
\phi(x, y) &= \sum_{k \in \mathbb{Z}} \phi_k(y) e^{ikx}, \\
\psi(x, y) &= \sum_{k \in \mathbb{Z}} \psi_k(y) e^{ikx}, \\
\xi(x) &= \sum_{k \in \mathbb{Z}} \xi_k e^{ikx}, \\
\zeta(x) &= \sum_{k \in \mathbb{Z}} \zeta_k e^{ikx},
\end{align*}
\]
where \( \phi_k(y) \) and \( \psi_k(y) \) are unknown functions, \( \xi_k \) and \( \zeta_k \) are unknown coefficients for each \( k \in \mathbb{Z} \).
Substituting (32) into (21)–(30), we obtain that for each \( k \in \mathbb{Z} \),
\[
\begin{align*}
\phi_k'' - k^2 \phi_k &= \phi_k & \text{for } & \eta_k < y < \rho_s, \\
\phi_k(y) &= 0 & \text{for } & 0 < y \leq \eta_s, \\
\phi_k' (\eta^+_k) &= -\xi_k, \\
\phi_k(\rho_s) &= -\sqrt{\sigma^2 - \sigma^2 \zeta_k},
\end{align*}
\]
and
\[
\begin{align*}
\psi_k'' - k^2 \psi_k &= -\mu \phi_k & \text{for } & \eta_s < y < \rho_s, \\
\psi_k'' - k^2 \psi_k &= 0 & \text{for } & 0 < y < \eta_s, \\
\psi_k' (\eta^+_s) &= \psi_k'(\eta^-_s) + (\mu(\bar{\sigma} - \tilde{\sigma}) + \nu) \xi_k, \\
\psi_k(\eta^+_s) &= \psi_k(\eta^-_s), \\
\psi_k(\rho_s) &= \gamma k^2 \zeta_k, \\
\psi_k(0) &= 0.
\end{align*}
\]
It is easy to verify that for any given \( \xi_k \), the solution of problem (33) is
\[
\phi_k(y) = \begin{cases} -\frac{\sinh \sqrt{k^2 + 1}(y - \eta_s) \sqrt{\sigma^2 - \sigma^2 \zeta_k}}{\sinh \sqrt{k^2 + 1}(\rho_s - \eta_s)} & \text{for } \eta_s < y < \rho_s, \\ 0 & \text{for } 0 < y < \eta_s, \end{cases}
\]
and
\[
\xi_k = \frac{\sqrt{k^2 + 1} \sqrt{\sigma^2 - \sigma^2 \zeta_k}}{\sigma \sinh \sqrt{k^2 + 1}(\rho_s - \eta_s)} \quad \text{for } k \in \mathbb{Z}.
\]
Similarly, by substituting the above solution into problem (34) for \( k \neq 0 \), we obtain
\[
\psi_k(y) = \begin{cases} -\mu \phi_k(y) + (\gamma k^2 - \mu \sqrt{\sigma^2 - \sigma^2 \zeta_k}) \cosh ky \cosh k\rho_s & \text{for } \eta_s < y < \rho_s, \\
& + \frac{c_k \sinh k(\rho_s - y)}{\sinh k(\rho_s - \eta_s)} & \text{for } 0 < y < \eta_s, \\
& (\gamma k^2 - \mu \sqrt{\sigma^2 - \sigma^2 \zeta_k}) \cosh ky \cosh k\rho_s + \frac{c_k \cosh ky \cos \eta_s}{\cos \eta_s} & \text{for } 0 < y < \eta_s, \end{cases}
\]
where
\[
c_k = \frac{(\mu \sigma - \nu) \xi_k}{k[\coth k(\rho_s - \eta_s) + \tanh k\eta_s]} \quad \text{for } k \neq 0, k \in \mathbb{Z}.
\]
Note that \( \xi_0 = \zeta_0 \) follows from (16) and (36). By solving problem (34) for \( k = 0 \), we obtain
\[
\psi_0(y) = \begin{cases} \left[ \mu \sqrt{\sigma^2 - \sigma^2} \left( \frac{\sinh(y - \eta_s)}{\sinh(\rho_s - \eta_s)} - 1 \right) \right] \zeta_0 & \text{for } \eta_s < y < \rho_s, \\
& + (\mu \bar{\sigma} - \nu)(\rho_s - y) \zeta_0 & \text{for } 0 < y < \eta_s. \end{cases}
\]
By substituting (37) into the left side of equation (31), we compute
\[
\psi_k'(\rho_s) - \mu(\bar{\sigma} - \tilde{\sigma}) \zeta_k
\]
\[
= -\mu \phi_k'(\rho_s) + (\gamma k^2 - \mu \sqrt{\sigma^2 - \sigma^2}) \zeta_k k \tanh k\rho_s
\]
By using this formula and (39), we compute

\[
\lambda_k := \gamma k^3 \tanh k \rho_s + \mu \sqrt{\bar{\sigma}^2 - \bar{\sigma}^2} \left[ \sqrt{k^2 + 1} \coth \sqrt{k^2 + 1} (\rho_s - \eta_s) - k \tanh k \rho_s \right]
\]

\[
+ \frac{(-\mu \bar{\sigma} + \nu) \sqrt{k^2 + 1} \sqrt{\bar{\sigma}^2 - \bar{\sigma}^2}}{\bar{\sigma} \sinh (\rho_s - \eta_s) \sinh \sqrt{k^2 + 1} (\rho_s - \eta_s) \left[ \coth k (\rho_s - \eta_s) + \tanh k \eta_s \right] - \mu (\bar{\sigma} - \bar{\sigma})}.
\]

From (16), we can infer that

\[
\coth (\rho_s - \eta_s) = \frac{\bar{\sigma}}{\sqrt{\bar{\sigma}^2 - \bar{\sigma}^2}}.
\]

By using this formula and (39), we compute

\[
\psi'_0(\rho_s) - \mu (\bar{\sigma} - \bar{\sigma}) \zeta_0 = \mu \zeta_0 \sqrt{\bar{\sigma}^2 - \bar{\sigma}^2} \coth (\rho_s - \eta_s) - (\mu \bar{\sigma} - \nu) \zeta_0 - \mu (\bar{\sigma} - \bar{\sigma}) \zeta_0
\]

\[
= \mu \sigma \zeta_0 - (\mu \bar{\sigma} - \nu) \zeta_0 - \mu (\bar{\sigma} - \bar{\sigma}) \zeta_0
\]

\[
= \nu \zeta_0.
\]

From the definition (41), we see that \( \lambda_k \) can be regarded as a function of \( \gamma \). For any integer \( k \geq 1 \), define

\[
\gamma_k := \frac{1}{k^3 \tanh k \rho_s} \left\{ \mu (\bar{\sigma} - \bar{\sigma}) + \mu \sqrt{\bar{\sigma}^2 - \bar{\sigma}^2} \left[ k \tanh k \rho_s - \sqrt{k^2 + 1} \coth \sqrt{k^2 + 1} (\rho_s - \eta_s) \right] \right. \]

\[
+ \frac{(\mu \bar{\sigma} - \nu) \sqrt{k^2 + 1} \sqrt{\bar{\sigma}^2 - \bar{\sigma}^2}}{\bar{\sigma} \sinh (\rho_s - \eta_s) \sinh \sqrt{k^2 + 1} (\rho_s - \eta_s) \left[ \coth k (\rho_s - \eta_s) + \tanh k \eta_s \right]} \left\} \right.
\]

\[
\right.
\]

\[
\right. \).
\]

Obviously, we have

\[
\lambda_k = \lambda_k (\gamma) := k^3 (\gamma - \gamma_k) \tanh k \rho_s \quad \text{for } k \neq 0, k \in \mathbb{Z}.
\]

**Lemma 2.1.** There exists an integer \( K > 0 \), such that for \( k > K \), \( \gamma_k \) is positive, distinct, and monotone decreasing. Moreover, \( \lim_{k \to +\infty} \gamma_k = 0 \).

**Proof.** By a direct computation, we have

\[
\lim_{k \to +\infty} \tanh k \rho_s = \lim_{k \to +\infty} \tanh k \eta_s = \lim_{k \to +\infty} \coth k (\rho_s - \eta_s) = 1,
\]

\[
\lim_{k \to +\infty} \left( \sqrt{k^2 + 1} \coth \sqrt{k^2 + 1} (\rho_s - \eta_s) - k \tanh k \rho_s \right) = 0.
\]

With above two formulae, we easily obtain

\[
\lim_{k \to +\infty} k^3 \tanh k \rho_s \gamma_k = \mu (\bar{\sigma} - \bar{\sigma}) > 0,
\]

and

\[
\gamma_{k+1} - \gamma_k = -3 \mu (\bar{\sigma} - \bar{\sigma}) \cdot \frac{1}{k^2} + o \left( \frac{1}{k^2} \right), \quad \text{as } k \to +\infty.
\]

The desired result follows. \( \Box \)

From all above deductions, we have
Lemma 2.2. The linearized problem (21)–(31) has nontrivial solutions if and only if \( \gamma = \gamma_k \) for some \( k \geq 1 \).

Proof. Since \( \nu > 0 \), \( \psi_k'(\rho_s) - \mu(\bar{\sigma} - \bar{\sigma})\zeta_0 = \nu \zeta_0 = 0 \) if and only of \( \zeta_0 = 0 \). On the other hand, by (40), we see that \( \psi_k'(\rho_s) - \mu(\bar{\sigma} - \bar{\sigma})\zeta_k = 0 \) is equivalent to \( \lambda_k \zeta_k = 0 \) for \( k \neq 0 \). If \( \lambda_k = 0 \) for some \( k \neq 0 \), then for any given \( \zeta_k \neq 0 \), problem (21)–(31) has a nontrivial solution \( (\phi, \psi, \xi, \zeta) = (\phi_k(y), \psi_k(y), \xi_k, \zeta_k) e^{ikx} \) with \( \phi_k(y), \psi_k(y), \xi_k \), \( \zeta_k \) given by (35)–(37). The assertion follows from (44) immediately. \( \square \)

Remark 1. \( \lambda_k \) can be also regarded as a function of \( \nu \). For \( k \geq 1 \), \( k \in \mathbb{N} \), we define

\[
\nu_k := \mu \bar{\sigma} + \bar{\sigma} \sinh k(\rho_s - \eta_s) \sinh \sqrt{k^2 + 1(\rho_s - \eta_s)} \left\{ \mu(\bar{\sigma} - \bar{\sigma}) 
+ \mu \sqrt{\bar{\sigma}^2 - \bar{\sigma}^2} \left[ k \tanh kp_s - \sqrt{k^2 + 1} \coth \sqrt{k^2 + 1(\rho_s - \eta_s)} \right] - \gamma k^3 \tanh kp_s \right\} \coth k(\rho_s - \eta_s) + \tanh k\eta_s 
\sqrt{k^2 + 1} \coth \sqrt{k^2 - \bar{\sigma}^2}.
\]

Clearly, for any \( \gamma > 0 \), \( \lim_{k \to +\infty} \nu_k = -\infty \). We easily verify that for sufficiently small \( \gamma > 0 \), there exist finitely many points \( \nu_k > 0 \). Note that \( \lambda_k = 0 \) is equivalent to \( \nu = \nu_k \) for \( k \geq 1 \). By the proof of Lemma 2.2, we see that if \( \nu = \nu_k > 0 \), then the linearized problem (21)–(31) has nontrivial solutions. Similar result holds for regarding \( \lambda_k \) as a function of \( \mu \).

3. Bifurcation analysis. In this section, we first reduce free boundary problem (1)–(7) into a bifurcation problem, and then we show there exist infinitely many branches of non-flat stationary solutions bifurcating from the flat stationary solution \( (\sigma, \rho, \eta, \rho_s, \eta_s) \).

Let \( 0 < r_1, r_2 < (\rho_s - \eta_s)/4 \) and \( 0 < \alpha < 1 \). Denote

\[
B(\rho_s, r_1) := \{ \rho \in C^{4+\alpha}(\mathbb{S}) : \|\rho - \rho_s\|_{C^{4+\alpha}(\mathbb{S})} < r_1 \}, \quad (46)
\]

\[
B(\eta_s, r_2) := \{ \eta \in C^{4+\alpha}(\mathbb{S}) : \|\eta - \eta_s\|_{C^{4+\alpha}(\mathbb{S})} < r_2 \}. \quad (47)
\]

For \( (\rho, \eta) \in B(\rho_s, r_1) \times B(\eta_s, r_2) \), we denote

\[
\Omega_\rho = \{(x, y) \in \mathbb{S} \times \mathbb{R} : 0 < y < \rho(x) \}, \quad \Gamma_\rho = \{(x, y) \in \mathbb{S} \times \mathbb{R} : y = \rho(x) \},
\]

\[
\Omega_\eta = \{(x, y) \in \mathbb{S} \times \mathbb{R} : 0 < y < \eta(x) \}, \quad \Gamma_\eta = \{(x, y) \in \mathbb{S} \times \mathbb{R} : y = \eta(x) \},
\]

and

\[
\Sigma_{\rho, \eta} = \{(x, y) \in \mathbb{S} \times \mathbb{R} : \eta(x) < y < \rho(x) \}.
\]

By the maximum principle, we see that for given \( (\rho, \eta) \in B(\rho_s, r_1) \times B(\eta_s, r_2) \), the solution of problem (1) and (4) satisfies

\[
\sigma(x, y) \equiv \bar{\sigma} \quad \text{in} \quad \Omega_\eta.
\]

Hence for the solution \( \sigma \) in \( \Omega_\rho \), we only need to solve the following problem

\[
\begin{cases}
\Delta \sigma = \sigma & \text{in} \quad \Sigma_{\rho, \eta}, \\
\sigma = \bar{\sigma} & \text{on} \quad \Gamma_\rho, \\
\partial_n \sigma = 0 & \text{on} \quad \Gamma_\eta, \\
\sigma = \bar{\sigma} & \text{on} \quad \Gamma_\eta.
\end{cases}
\]

Lemma 3.1. There exists a constant \( r \in (0, r_1) \), such that for any \( \rho \in B(\rho_s, r) \), problem (48) has a unique solution \( (\sigma, \eta) \) satisfying \( \sigma \in C^{4+\alpha}(\Sigma_{\rho, \eta}) \) and \( \eta \in B(\eta_s, r_2) \cap C^\infty(\mathbb{S}) \). Moreover, the mapping \( \rho \mapsto (\sigma, \eta) \) from \( B(\rho_s, r) \) to \( C^{4+\alpha}(\Sigma_{\rho, \eta}) \times (B(\eta_s, r_2) \cap C^\infty(\mathbb{S})) \) is smooth.
Next, we solve the solution operator. Then by using Nash-Moser implicit function theorem (see Theorem 3.3.1 in Part III of Hamilton [15]), we obtain that there exist a sufficiently small constant \( r \in (0, r_1) \), and a unique smooth mapping \( S \) from \( B(\rho_s, r) \) to \( B(\rho_s, r_2) \) such that
\[
S(\rho_s) = \eta_s \quad \text{and} \quad A(\rho, S(\rho)) = 0.
\]
By letting \( \sigma = \sigma(x, y; \rho, S(\rho)) \) and \( \eta = S(\rho) \), we see that \((\sigma, \eta)\) is the solution of problem (48), and the mapping \( \rho \mapsto (\sigma, \eta) \) is smooth. For more details, we refer to the proof of Lemma 3.2 in [21]. We omit it here. \( \square \)

From Lemma 3.1, for given \( \rho \in B(\rho_s, r) \), we have
\[
\sigma = \begin{cases} 
\sigma(x, y; \rho, \eta) & \text{in } \Sigma_{\rho, \eta}, \\
\hat{\sigma} & \text{in } \Omega_\eta,
\end{cases}
\quad \text{and} \quad \eta = S(\rho). \tag{49}
\]
Next, we solve the solution \( p \) of problem (1)–(6) in \( \Omega_\rho \). Denote
\[
p^+ = p|_{\Sigma_{\rho, \eta}} \quad \text{and} \quad p^- = p|_{\Omega_\eta}.
\]
It is easy to see that \( p^+ \) and \( p^- \) satisfy the following problem:
\[
\begin{cases}
\Delta p^+ = -\mu(\sigma - \hat{\sigma}) & \text{in } \Sigma_{\rho, \eta}, \\
\Delta p^- = \nu & \text{in } \Omega_\eta, \\
p^+ = \gamma \kappa(\rho) & \text{on } \Gamma_\rho, \\
p^+ = p^- & \text{on } \Gamma_\eta, \\
\partial_n p^+ = \partial_n p^- & \text{on } \Gamma_\eta, \\
\partial_y p^- = 0 & \text{on } \Gamma_0,
\end{cases} \tag{50}
\]
where \( \kappa(\rho) = -\rho_{xx}/(1 + \rho_y^2)^{3/2} \) is the mean curvature of \( \Gamma_\rho \).

**Lemma 3.2.** There exists a sufficiently small constant \( \delta \in (0, r) \) such that for any \( \rho \in B(\rho_s, \delta) \), with \( \sigma \) and \( \eta \) given by (49), problem (50) has a unique solution \((p^+, p^-) \in C^{2+\alpha}(\Sigma_{\rho, \eta}) \times C^{2+\alpha}(\Omega_\eta)\), and the mapping \( \rho \mapsto (p^+, p^-) \) is smooth in \( B(\rho_s, \delta) \).

**Proof.** For any given \( \rho \in B(\rho_s, r) \) and \( g \in C^{2+\alpha}(\Sigma) \), we consider
\[
\begin{cases}
\Delta p^+ = -\mu(\sigma - \hat{\sigma}) & \text{in } \Sigma_{\rho, \eta}, \\
p^+ = g & \text{on } \Gamma_\eta, \\
p^+ = \gamma \kappa(\rho) & \text{on } \Gamma_\rho, \\
\partial_y p^- = 0 & \text{on } \Gamma_0.
\end{cases} \tag{51}
\]
By Lemma 3.1 and the classical regularity theory of elliptic differential equations, the above problem has a unique solution \((p^+, p^-)\) denoted by
\[
p^+ := p^+(x, y; \rho, g) \in C^{2+\alpha}(\Sigma_{\rho, \eta}) \quad \text{and} \quad p^- := p^-(x, y; \rho, g) \in C^{2+\alpha}(\Omega_\eta), \tag{52}
\]
respectively. Moreover, the mapping \((\rho, g) \mapsto (p^+, p^-)\) is smooth in \(B(\rho_s, r) \times C^{2+\alpha}(S)\).

Define a mapping \(F : B(\rho_s, r) \times C^{2+\alpha}(S) \to C^{1+\alpha}(S)\) as follows:

\[
F(\rho, g) = \partial_\nu p^+(x, \eta(x); \rho, g) - \partial_\nu p^-(x, \eta(x); \rho, g),
\]
for \(\rho \in B(\rho_s, r)\) and \(g \in C^{2+\alpha}(S)\). Then problem (50) is equivalent to equation \(F(\rho, g) = 0\). Clearly, \(F \in C^\infty(\mathcal{B}(\rho_s, r) \times C^{2+\alpha}(S), C^{1+\alpha}(S))\), and \(F(\rho_s, p_0) = 0\), where \(p_0 = p_s(\eta_s)\).

Next, we compute the Fréchet derivative \(D_y F(\rho_s, p_0)\) and show it is an isomorphism from \(C^{2+\alpha}(S)\) onto \(C^{1+\alpha}(S)\). Note that \(\eta_s = \mathcal{S}(\rho_s), \sigma_s = \sigma(x, y; \rho_s, \eta_s), \kappa(\rho_s) = 0\) and

\[
p_s|_{\Sigma_{\rho_s, \eta_s}} = p^+(x, y; \rho_s, \eta_s), \quad p_s|_{\Omega_{\eta_s}} = p^-(x, y; \rho_s, \eta_s).
\]

By a similar deduction as in Section 2, we have

\[
D_y F(\rho_s, p_0)h = -\partial_y q^+(x, \eta_s) + \partial_y q^-(x, \eta_s) \quad \text{for} \quad h \in C^{2+\alpha}(S),
\]
where \(q^+\) and \(q^-\) satisfy the following problems

\[
\Delta q^+ = 0 \quad \text{in} \quad \Sigma_{\rho_s, \eta_s}, \quad q^+ = h \quad \text{on} \quad \Gamma_{\eta_s}, \quad q^+ = 0 \quad \text{on} \quad \Gamma_{\rho_s},
\]
\[
\Delta q^- = 0 \quad \text{in} \quad \Omega_{\eta_s}, \quad q^- = h \quad \text{on} \quad \Gamma_{\eta_s}, \quad \partial_y q^- = 0 \quad \text{on} \quad \Gamma_0.
\]

For any \(h \in C^\infty(S)\) with the expression \(h(x) = \sum_{k \in \mathbb{Z}} h_k e^{ikx}\), a direct computation shows that

\[
D_y F(\rho_s, p_0)h = \sum_{k \in \mathbb{Z}} f_k h_k e^{ikx},
\]
where \(f_0 = (\rho_s - \eta_s)^{-1}\) and \(f_k = k(\coth k(\rho_s - \eta_s) + \tanh k\eta_s)\) for \(k \neq 0, k \in \mathbb{Z}\). It is easy to verify that

\[
\sup_{k \in \mathbb{Z}} |k| \left| \frac{1}{f_k} \right| < +\infty, \quad \sup_{k \in \mathbb{Z}} |k|^2 \left| \frac{1}{f_{k+1}} - \frac{1}{f_k} \right| < +\infty, \quad \sup_{k \in \mathbb{Z}} |k|^3 \left| \frac{1}{f_{k+2}} - \frac{2}{f_{k+1}} + \frac{1}{f_k} \right| < +\infty.
\]

Then by Theorem 4.5 of [1], we have

\[
[D_y F(\rho_s, p_0)]^{-1} \in L(C^{1+\alpha}(S), C^{2+\alpha}(S)).
\]

Hence by the classical implicit function theorem in Banach spaces, we obtain the desired result. The proof is complete. \(\square\)

By Lemma 3.1 and Lemma 3.2, for any give \(\rho \in B(\rho_s, \delta)\), problem (1)–(6) has a unique solution \((\sigma, p, \eta)\) with \(\sigma, \eta\) given by (49). Since the solution \(p\) is also depending on \(\gamma\), we denote \(p := \mathcal{R}(\gamma, p)\). Define a mapping \(G : \mathbb{R}^+ \times B(\rho_s, \delta) \to C^{1+\alpha}(S)\) by

\[
G(\gamma, \rho) = \partial_\nu \mathcal{R}(\gamma, \rho)|_{\Gamma_{\rho}} \quad \text{for} \quad \gamma \in \mathbb{R}^+, \rho \in B(\rho_s, \delta),
\]
then by substituting the solution \(p = \mathcal{R}(\gamma, \rho)\) into (7) we obtain

\[
G(\gamma, \rho) = 0.
\]
Clearly, \(\rho = \rho_s\) is a solution of equation (58), i.e.,

\[
G(\gamma, \rho_s) = 0 \quad \text{for all} \quad \gamma > 0.
\]
Moreover, by Lemma 3.1 and Lemma 3.2 we have

\[
G \in C^\infty(\mathbb{R}^+ \times B(\rho_s, \delta), C^{1+\alpha}(S)).
\]
Denote by $D_\rho G(\gamma, \rho)$ the Fréchet derivative of $G$ with respect to $\rho$ at $(\gamma, \rho)$. We have the following result:

**Lemma 3.3.** For any $\zeta \in C^\infty(\mathbb{S})$ given by $\zeta(x) = \sum_{k \in \mathbb{Z}} \zeta_k e^{ikx}$, there holds

$$D_\rho G(\gamma, \rho_s) \zeta = \sum_{k \in \mathbb{Z}} \lambda_k(\gamma) \zeta_k e^{ikx},$$

(61)

where $\lambda_1(\gamma)$ is given by (44) for $k \neq 0$, and $\lambda_0(\gamma) = \nu$.

**Proof.** By the above reduction, problem (1)–(7) is equivalent to equation (58), so their linearizations at the flat stationary solution are also equivalent. It implies that

$$D_\rho G(\gamma, \rho_s) \zeta = \partial_\rho \psi|_{\gamma=\rho_s} - \mu(\sigma - \partial) \zeta$$

for $\zeta \in C^{4+\alpha}(\mathbb{S})$, (62)

where $\psi$ satisfies problem (21)–(30). Hence, by (40)–(44) we immediately get (61). The proof is complete. \hfill \Box

From Lemma 2.2 and Lemma 3.3, we see that if $\gamma \notin \{\gamma_k\}_{k \geq 1}$, then $(\gamma, \rho_s)$ is not a bifurcation point. It means that non-flat stationary solutions may only bifurcate at $(\gamma_k, \rho_s)$ for $k \geq 1$.

Notice that the dimension of kernel space of $D_\rho G(\gamma_k, \rho_s)$ is even, so the well-known Crandall-Rabinowitz bifurcation theorem (see Theorem 5.1 of [5]) is not applicable for equation (58) directly. To overcome this difficulty, for any integer $k, m \geq 1$ and number $\alpha \in (0, 1)$, we introduce Banach space

$$X_k^{m+\alpha} := \text{the closure of the span}\{\cos(jkx), j = 0, 1, 2, \cdots\} \text{ in } C^{m+\alpha}(\mathbb{S}).$$

It is easy to verify that for the mean curvature of the curve $\Gamma_\rho$, we have

$$\kappa(\cdot) \in C^\infty(X_k^{m+2+\alpha}, X_k^{m+\alpha}).$$

Then by using (60), and a similar argument of Lemma 4.3 in [26], we can show

$$G \in C^\infty(\mathbb{R}^+ \times (B(\rho_s, \delta) \cap X_k^{4+\alpha}), X_k^{1+\alpha}).$$

(63)

With all above preparations, we give a proof of our main result.

**The proof of Theorem 1.1.** Denote

$$\mathcal{G}(\cdot, \cdot) = \text{the restriction of } G(\cdot, \cdot) \text{ on } \mathbb{R}^+ \times X_k^{4+\alpha}.$$  

Obviously, we have

$$\mathcal{G}(\gamma, \rho_s) = 0 \quad \text{for all } \gamma > 0.$$  

(64)

By (63) and Lemma 3.3, for any $\zeta(x) = \sum_{j=0}^{\infty} \zeta_j \cos(jkx) \in C^\infty(\mathbb{S})$,

$$D_\rho \mathcal{G}(\gamma, \rho_s) \zeta = \nu \zeta_0 + \sum_{j=1}^{\infty} j^3 k^3 (\tanh(jk \rho_s)) (\gamma - \gamma_j) \zeta_j \cos(jkx).$$

(65)

Let $k \geq K$. Then from Lemma 2.1 we easily get

$$\text{Ker}D_\rho \mathcal{G}(\gamma_k, \rho_s) = \text{span}\{\cos(kx)\},$$

(66)

$$\text{Im}D_\rho \mathcal{G}(\gamma_k, \rho_s) \text{ has codimension } 1,$$

(67)

and

$$D_\gamma D_\rho \mathcal{G}(\gamma_k, \rho_s) \cos(kx) = k^3 \tanh(k \rho_s) \cos(kx) \notin \text{Im} D_\rho \mathcal{G}(\gamma_k, \rho_s).$$

(68)

Hence all conditions of Crandall-Rabinowitz bifurcation theorem are satisfied, and it follows that $(\gamma_k, \rho_s)$ is a bifurcation point of equation $\mathcal{G}(\gamma, \rho) = 0$. It means that
there exist a number $\tau_k > 0$ and a smooth mapping $\varepsilon \mapsto (\gamma_\varepsilon, \rho_\varepsilon)$ from $(-\tau_k, \tau_k)$ to $\mathbb{R}^+ \times (B(\rho_s, \delta) \cap X^{4+\alpha})$ with

$$
\gamma_\varepsilon = \gamma_k + O(\varepsilon) \quad \text{and} \quad \rho_\varepsilon = \rho_s + \varepsilon \cos(kx) + O(\varepsilon^2),
$$

such that $G(\gamma_\varepsilon, \rho_\varepsilon) = G(\gamma_k, \rho_s) = 0$. Since problem (1)–(7) is equivalent to equation $G(\gamma, \rho) = 0$, by combining (35)–(38), problem (1)–(7) has a family of non-flat stationary solutions bifurcating from $(\gamma_k, \rho_s)$ for $k \geq K$, with the upper free boundary $\Gamma^+ = \text{graph}\{\rho_\varepsilon\}$, and the lower free boundary $\Gamma^- = \text{graph}\{\eta_\varepsilon\}$, where

$$
\eta_\varepsilon = \eta_s + \varepsilon A_k \cos(kx) + O(\varepsilon^2),
$$

and

$$
A_k = \frac{\sqrt{k^2 + 1} \sqrt{\sigma^2 - \bar{\sigma}^2}}{\bar{\sigma} \sinh \sqrt{k^2 + 1}(\rho_s - \eta_s)} \quad \text{for} \quad k \geq K.
$$

The proof is complete. \(\square\)

4. Conclusions and biological implications. In this paper, we study a free boundary problem modeling dormant necrotic tumors, which has two disjoint free boundaries with boundary conditions of different types. As pointed out by Friedman (see page 235 of [11]), necrotic tumor model is novel in free boundary problems and many challenges arise in rigorous mathematical analysis.

By taking $\gamma$ as a bifurcation parameter, and using bifurcation analysis we prove that there exist infinitely many non-flat stationary solutions with the upper free boundary $y = \rho_s + \varepsilon \cos(kx) + O(\varepsilon^2)$, the lower free boundary $y = \eta_s + \varepsilon A_k \cos(kx) + O(\varepsilon^2)$, and $\gamma = \gamma_k + O(\varepsilon)$ for sufficiently large $k \in \mathbb{N}$. As shown in [12], [13], [20], [22], [23], [25] for non-necrotic tumor models, these bifurcation solutions shaped like ‘fingers’, are associated to the invasion of tumors into the surroundings. Hence the cell-to-cell adhesiveness $\gamma$ plays an important role on tumor invasion.

Our results also show that bifurcation point $\gamma_k$ is dependent on the proliferation rate $\mu$ and the removal rate $\nu$, so $\mu$ and $\nu$ can also be taken as bifurcation parameters. In fact, from (43) and (44), we easily show there exist infinitely many bifurcation points $\mu_k$, by taking $\mu$ as a bifurcation parameter, which is similar as shown in necrotic tumor spheroid model (see [16]) and non-necrotic tumor models (cf. [13], [23]). But for $\nu$, if we take $\nu = 0$, then by (65) we have $\text{Ker} D_G(\gamma_k, \rho_s) = \text{span}\{1, \cos(kx)\}$ of dimension 2, and the above bifurcation analysis based on Crandall-Rabinowitz theorem is not applicable any more. It is different from the result of [16], where bifurcations were obtained in case $\nu = 0$. This difference implies that the tumor’s shape and structure may have important effects on tumor growth. On the other hand, by Remark 1 we can also show that there exist finitely many bifurcation points $\nu_k$ for sufficiently small $\gamma$, it implies that $\nu$ plays an important role on tumor invasion, similarly as $\gamma$ and $\mu$, and these three parameters should be measured and examined accurately in laboratory and clinical experiments.

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