AUTONOMOUS FUNCTIONALS WITH ASYMPTOTIC 
\((p, q)\)-STRUCTURE.

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Abstract. We obtain local Lipschitz regularity for minima of autonomous integrals in the calculus of variations, assuming \(q\)-growth hypothesis and \(W^{1,p}\)-quasiconvexity only asymptotically, both in the sub-quadratic and the super-quadratic case.

1. Introduction

In this paper we study variational integrals of the type

\[ F(u) = \int_{\Omega} f(Du(x)) \, dx \quad \text{for } u : \Omega \to \mathbb{R}^N \]

where \(\Omega\) is an open bounded set in \(\mathbb{R}^n\), \(n \geq 2\), \(N \geq 1\). Here, the Lagrangian function \(f : \mathbb{R}^{Nn} \to \mathbb{R}\) is a continuous function satisfying the following properties:

1. **Regularity** - \(f \in C^2(\mathbb{R}^{nN}, \mathbb{R})\)
2. **\(q\)-Growth** - \(|f(z)| \leq \Gamma(1 + |z|^q)\)
3. **Asymptotical strict \(W^{1,p}\)-quasiconvexity** - There exists \(M >> 0\), \(\gamma > 0\) and a continuous function \(g \in W^{1,p}\) such that

\[ f(z) = g(z), \quad \forall z : |z| > M \]

and such that \(g\) is strictly \(W^{1,p}\)-quasiconvex, i.e. satisfies

\[ \int_{B_1} g(z + D\varphi) \geq g(z) + \gamma \int_{B_1} (1 + |D\varphi|^2)^{\frac{p}{2} - 1}|D\varphi|^2, \quad \forall z, \forall \varphi \in C_0^\infty(B_1, \mathbb{R}^N) \]

where in this text, \(p\) and \(q\) will always denote real numbers that satisfy the inequalities

\[ 1 < p \leq q < p + \frac{\min\{2, p\}}{2n} \]

We will study local \(W^{1,p}\)-minimizers of \(F\) i.e. functions \(u \in W^{1,p}(\Omega, \mathbb{R}^N)\) such that

\[ F(u + v) \geq F(u) \quad \forall v \in W^{1,p}_0(\Omega, \mathbb{R}^N) \]

and, in the following, we will simply refer to them as "minimizers".

In [9], T. Schmidt proved that if \(M = 0\), \(u\) is \(C^{1,\alpha}\) in an open dense subset of \(\Omega\). We will prove the following result:
**Theorem 1.** Let $f$ satisfy hypotheses (A.1),(A.2) and (A.3) and let $u$ be a local minimizer of the corresponding functional $\mathcal{F}$. Let $z_0 \in \mathbb{R}^{nN}$ such that $|z_0| > M + 1$ and assume there is a $x_0 \in \mathbb{R}^n$ with the property that

$$-\hat{B}_\rho(x_0) |Du - z_0|^p \to 0 \text{ as } \rho \to 0^+,$$

then $x_0 \in \text{Reg}(u)$, where $\text{Reg}(u) = \{x \in \Omega : u \text{ is Lipschitz in a neighbourhood of } x\}$. Moreover, $\text{Reg}(u)$ is a dense open subset of $\Omega$

2. Technical lemmas and definitions

**Lemma 2.** If $f$ is locally bounded from below, then the function $g$ in (A.3) can be chosen such that $g \leq f$.

Moreover, in this case, assuming (A.3) is equivalent to assuming the existence of a positive constant $M > 0$ big enough such that the following holds:

$$\text{(A.3')} \quad \int_{B_1} f(z + D\varphi) \geq f(z) + \gamma \int_{B_1} (1 + |D\varphi|^2)^{\frac{p}{2} - 1} |D\varphi|^2,$$

$$\forall z : |z| > M, \forall \varphi \in C^\infty_0(\Omega, \mathbb{R}^N).$$

**Proof.** The proof of the first part uses the theory of quasiconvex envelopes and it is identical to what is shown in ([4], Thm 2.5 (ii)), but we will repeat it for the convenience of the reader.

Let us start with the case $p > 2$.

Assume that $f(z) = g(z)$ for all $z : |z| > M$ and that $g$ satisfies strict quasiconvexity for a constant $\gamma > 0$, i.e.:

$$\text{(1)} \quad \int_{B_1} g(z + D\varphi(x)) \, dx \geq g(z) + \gamma \int_{B_1} (1 + |D\varphi(x)|^2)^{\frac{p}{2} - 1} |D\varphi|^2$$

Let $K = \sup_{|z| \leq M} (g - f)(z)$ and notice that $K < \infty$ since $f$ is locally bounded from below and $g$ is locally bounded.

Introduce an auxiliary function $h$ which is smooth and non-negative on $\mathbb{R}^{nN}$, with compact support and such that $|D^2h| \leq \gamma$ on $\mathbb{R}^{nN}$ and $h(z) \geq K$ for $|z| \leq M$.

We now claim that $\tilde{g} = g - h$ is uniformly strictly quasiconvex and satisfying $\tilde{g} \leq f$ for all $z \in \mathbb{R}^{nN}$ and $\tilde{g}(z) = f(z)$ for large enough $z$. Of course $\tilde{g}(z) = f(z)$ outside the support of $h(z)$ and $\tilde{g} \leq f$ is a trivial consequence of the fact that $h$ is always larger than the difference between $g$ and $f$.

To prove $\tilde{g}$ is strictly quasiconvex, let us consider, for any $\varphi \in C^\infty_0(\Omega, \mathbb{R}^N)$, the
quantity:
\[ h(z + D\varphi) - h(z) - Dh(z)D\varphi = \int_0^1 Dh(z + tD\varphi)D\varphi dt - Dh(z)D\varphi = \]
\[ = |D\varphi|^2 \int_0^1 \int_0^1 tD^2h(z + stD\varphi) ds dt \]
so
\[ h(z + D\varphi) \leq h(z) + Dh(z)D\varphi + \gamma |D\varphi|^2 \]
By passing to the integral average over \( B \) and changing all signs we obtain
\[ -\hat{\mathcal{B}}_1 f(z + D\varphi) \geq -h(z) - \gamma \hat{\mathcal{B}}_1 |D\varphi|^2. \]
By summing this inequality with (1) we obtain the quasiconvexity of \( \tilde{g} \).
To prove the second statement in the lemma, assume now \( g \leq f \) and choose \( z \) such that \(|z| > M\). We have:
\[ \int_{B_1} f(z + D\varphi) \geq \int_{B_1} g(z + D\varphi) \geq g(z) + \gamma \int_{B_1} (1 + |D\varphi|^2)^{\frac{p-1}{2}}|D\varphi|^2 = \]
\[ = f(z) + \gamma \int_{B_1} (1 + |D\varphi|^2)^{\frac{p-1}{2}}|D\varphi|^2. \]
Now, the only thing we need to change in the case \( p \leq 2 \) is to take a smooth function \( h \) such that \(|D^2h(z)| \leq \gamma(1 + |z|^2)^{\frac{p-1}{2}}\). This can easily be done by considering \( \tilde{h}(z) = h\left(\frac{z}{d}\right) \) (where \( h \) is the same function used in the proof of the case \( p > 2 \)) for large enough \( d \), i.e. for \( d > (1 + M^2)^{1-\frac{p}{2}} \).

Corollary 3. Let \( f \) satisfy (A.1), (A.2) and (A.3). Then it satisfies (A.3').

Definition (Excess). Let \( \beta > 0 \) and let us consider
\[ V^\beta(z) = (1 + |z|^2)^{\frac{p-1}{2}}z \quad \text{and} \quad W^\beta(z) = (1 + |z|)^{\beta-1}z \]
with \( V \) and \( W \) comparable in the sense that there exists a constant \( c > 0 \) depending only on \( \beta \) such that for all \( z \) we have
\[ c^{-1}|W^\beta(z)| \leq |V^\beta(z)| \leq c|W^\beta(z)|. \]
We will often consider the quantity \(|V^\frac{p}{2}(z)|^2\) in our computations. The advantage of sometimes dealing with the equivalent quantity \(|W^\frac{p}{2}(z)|^2\) is the fact that it can be easily proven that
\[ z \mapsto |W^\frac{p}{2}(z)|^2 \]
is convex for all \( p \geq 1 \).
For \( u \in W^{1,p}(B_\rho(x_0), \mathbb{R}^N) \) and \( z \in \mathbb{R}^{nN} \) define the quantity
\[ \Phi_p(u, x_0, \rho, z) := \int_{B_\rho(z_0)} |V^\frac{p}{2}(Du - z)|^2 \]
and in particular, we call
\[ \Phi_p(u, x_0, \rho) := \Phi_p \left( u, x_0, \rho, \int_{B_\rho(x_0)} Du \right) \]
the excess function of the minimizer \( u \) in \( x_0 \).

In [9], T. Schmidt proved that if \( u \) is a \( W^{1,p} \)-minimizer of \( F \) on \( B_\rho(x_0) \), for all \( L > 0 \) and \( \alpha \in (0, 1) \) there exist an \( \varepsilon_0 > 0 \) such that if
\[ \Phi_p(u, x_0, \rho) \leq \varepsilon_0 \text{ and } \left| \int_{B_\rho(x_0)} Du \right| \leq L \]
then \( u \in C^{1,\alpha}_{\text{loc}}(B_\rho(x_0); \mathbb{R}^N) \).

We will now replicate his reasoning in the weaker hypothesis of asymptotic quasiconvexity.

To do so it will be first necessary to observe the following.

**Lemma 4.** If there exists \( z_0, |z_0| > M + 1 \) and \( x_0 \) such that:
\[
\int_{B_\rho(x_0)} \left| V^\tilde{F}_p(Du - z_0) \right|^2 \to 0 \text{ as } \rho \to 0^+
\]
then there exists \( r_1 = r_1(x_0, z_0) \) such that for all \( r < r_1 \)
\[
\left| \int_{B_r(x_0)} Du \right| > M + 1.
\]

**Proof.** Let \( |z_0| = M + 1 + \varepsilon \). Then by definition of limit there must be a \( r_1 \) (of course this depends on the specific values of \( x_0 \) and \( z_0 \)) such that for all \( r < r_1 \) we have:
\[
\int_{B_r(x_0)} \left| W^\tilde{V}_p(Du - z_0) \right|^2 \leq \left| W^\tilde{V}_p \left( \frac{\varepsilon}{2} \right) \right|^2, \quad \forall r < r_1
\]
where we have also used inequality (2), comparing \( V \) and \( W \).

which, by Jensen inequality means:
\[
\left| \int_{B_r(x_0)} Du - z_0 \right| \leq \frac{\varepsilon}{2}, \quad \forall r < r_1
\]
which gives:
\[
\left| \int_{B_r(x_0)} Du \right| \geq |z_0| - \frac{\varepsilon}{2} = M + 1 + \varepsilon - \frac{\varepsilon}{2} > M + 1, \quad \forall r < r_1.
\]

We start with a lemma by P. Marcellini (Step 2 of Thm 2.1 in [5]).

**Lemma 5.** Let \( f \) satisfy assumptions (A.1), (A.2) and (A.3) and \( z \) be such that \( |z| > M \). Then \( |Df(z)| \leq \Gamma_2(1 + |z|^{q-1}) \).
Proof. Let $z$ be such that $|z| > M$. As shown in ([9], proposition 5.2), quasiconvexity in a given $z$ implies rank-one convexity in $z$. This implies that the modulus of each partial derivative is bounded by the modulus of the difference quotient, which can be bounded by the $(q-1)$-th power of the argument by the inequality in hypothesis. In symbols, if $\varphi_i(\cdot) = f(\zeta_1, \ldots, \zeta_i-1, \zeta_i+1, \ldots, \zeta_N)$:

$$|\varphi_i'(\zeta_i)| \leq \left| \frac{\varphi(\zeta + |\zeta| + 1) - \varphi(\zeta)}{|\zeta| + 1} \right| \leq \Gamma_2(1 + |\zeta|^{q-1})$$

\[ \square \]

We proceed adapting a result from Acerbi and Fusco ([11], Lemma II.3)

**Lemma 6.** Choose any $L > 0$. Let $f$ satisfy (A.1), (A.2) and be such that $|Df(z)| \leq \Gamma_2(1 + |z|^{q-1})$ holds true for all $z : |z| > M$, then

- $|f(z + \eta) - f(z) - Df(z)\eta| \leq c_1 |V^{q/2}(\eta)|^2$
- $|Df(z + \eta) - Df(z)||\eta| \leq c_2 |V^{q-1}(\eta)|$

for all $\eta \in \mathbb{R}^{nN}$ and for all $z$ such that $M < |z| \leq L$, with $c_1$ and $c_2$ depending only on $f,n,N,L,\Gamma,\Gamma_2$ and $M$.

**Proof.** Choose $L > 0$ and $z$ such that $|z| \leq L$ for some $L \in \mathbb{R}$. We start proving the first of the two inequalities. If $|\eta| \leq 1$ we have:

$$|f(z + \eta) - f(z) - Df(z)\eta| \leq |D^2f(z + \theta\eta)||\eta|^2 \leq \max_{|z| \leq L+1} |D^2f(z)||\eta|^2$$

If $\eta > 1$ we start from:

$$|f(z + \eta) - f(z) - Df(z)\eta| = |Df(z + \eta)\eta - Df(z)\eta|$$

and then, if $|z + \eta\theta| \leq M$, we end with

$$|Df(z + \eta\theta)\eta - Df(z)\eta| \leq 2 \max_{|z| \leq \max\{L,M\}} |Df(z)||\eta|$$

otherwise, if $|z + \eta\theta| > M$, we conclude with:

$$|Df(z + \eta\theta)\eta - Df(z)\eta| \leq \Gamma_2(1 + |z + \eta\theta|^{q-1})|\eta| + \max_{|z| \leq L} |Df(z)||\eta| \leq c_1 |\eta|^q.$$
while if $|\eta| > 1$ and $|z + \eta| > M$ then:

$$|Df(z + \eta) - Df(z)| \leq \Gamma_2(1 + |z + \eta|^{q-1}) + \max_{|z| \leq L} |Df(z)| \leq c_2|\eta|^{q-1}.$$ 

\[ \square \]

### 3. Caccioppoli estimate

Next step is to obtain a Caccioppoli estimate adapting a proof by T. Schmidt (see \cite{[9]}, Lemma 7.3). To do so we need a few lemmas. The proofs can be found in \cite{[5]} and in \cite{[9]}. 

**Lemma 7.** Let $0 < r < s$ and $B_s \subset \Omega$. We define a bounded linear smoothing operator

$$T_{r,s} : W^{1,1}(\Omega; \mathbb{R}^N) \to W^{1,1}(\Omega; \mathbb{R}^N)$$

for $u \in W^{1,1}(\Omega; \mathbb{R}^N)$ and $x \in \Omega$ by

$$T_{r,s}u(x) := \int_{B_1} u(x + \theta(x)y) \, dy \text{ where } \theta(x) := \frac{1}{2} \max\{\min\{|x| - r, s - |x|\}, 0\}.$$ 

With this definition, for all $1 \leq p \leq q < \frac{n}{n-1}p$ and all $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ the following assertions are true:

1. $T_{r,s}u \in W^{1,p}(\Omega; \mathbb{R}^N),$
2. $u = T_{r,s}u$ almost everywhere on $(\Omega \setminus B_s) \cup B_r,$
3. $T_{r,s}u \in u + W^{1,p}_0(B_s \setminus B_r; \mathbb{R}^N),$
4. $|D T_{r,s}u| \leq c(n)T_{r,s}|Du|$ almost everywhere in $\Omega,$
5. $\|T_{r,s}u\|_{L^p(B_s \setminus B_r)} \leq c(n,p)\|u\|_{L^p(B_s \setminus B_r)}$, 
6. $\|DT_{r,s}u\|_{L^p(B_s \setminus B_r)} \leq c(n,p)\|Du\|_{L^p(B_s \setminus B_r)},$
7. $\|T_{r,s}u\|_{L^q(B_s \setminus B_r)} \leq c(n,p,q)(s-r)^{n-p-1} \left[ \sup_{t \in (r,s)} \frac{\tilde{\Xi}(t)-\tilde{\Xi}(r)}{t-r} + \sup_{t \in (r,s)} \frac{\tilde{\Xi}(s)-\tilde{\Xi}(t)}{s-t} \right]^{\frac{1}{p}},$
8. $\|DT_{r,s}u\|_{L^q(B_s \setminus B_r)} \leq c(n,p,q)(s-r)^{n-p-1} \left[ \sup_{t \in (r,s)} \frac{\Xi(t)-\Xi(r)}{t-r} + \sup_{t \in (r,s)} \frac{\Xi(s)-\Xi(t)}{s-t} \right]^{\frac{1}{p}},$
9. $\|V^\Xi(D T_{r,s}u)\|^{\frac{2}{p}} \leq c T_{r,s} \left[ \|V^\Xi(Du)\|^{\frac{2}{p}} \right]^{\frac{1}{2}} \forall p : 1 \leq p \leq 2 \text{ a.e. in } \Omega, c = c(n,p),$ 

where we used the abbreviations:

$$\tilde{\Xi}(t) := \|u\|_{L^p(B_t)}^p$$

and

$$\Xi(t) := \|Du\|_{L^p(B_t)}^p.$$
Another lemma that will be useful in obtaining the Caccioppoli estimate is the following, the proof of which can also be found in \cite{5}.

**Lemma 8.** Let $-\infty < r < s < +\infty$ and a continuous nondecreasing function $\Xi : [r, s] \to \mathbb{R}$ be given. Then there are $\tilde{r} \in [r, \frac{2r + s}{3}]$ and $\tilde{s} \in [\frac{r + 2s}{3}, s]$, for which hold:

$$\frac{\Xi(t) - \Xi(\tilde{r})}{t - \tilde{r}} \leq 3\frac{\Xi(s) - \Xi(r)}{s - r}$$

and

$$\frac{\Xi(\tilde{s}) - \Xi(t)}{\tilde{s} - t} \leq 3\frac{\Xi(s) - \Xi(r)}{s - r}$$

for every $t \in (\tilde{r}, \tilde{s})$.

In particular, we have $\frac{s - r}{3} \leq \tilde{s} - \tilde{r} \leq s - r$.

Now we can prove the Caccioppoli estimate.

**Lemma 9 (Caccioppoli Inequality).** Let $f$ satisfy $(A.1),(A.2)$ and $(A.3)$ for a given $M$. Choose any positive constant $L > M > 0$ and a consider $W^{1,p}$-minimizer $u \in W^{1,p}(B_{\rho}(x_0); \mathbb{R}^N)$ of $\mathcal{F}$ on $B_{\rho}(x_0)$. Then, for all $\zeta \in \mathbb{R}^N$ and $z \in \mathbb{R}^{nN}$ with $M < |z| < L + 1$, we have:

$$\Phi_p\left(u, x_0, \frac{\rho}{2}, z\right) \leq c \left[ h\left(\int_{B_{\rho}(x_0)} \left| Dv \right|^p + \left| \frac{v}{s - r} \right|^p \, dx\right) + \left(\Phi_p(u, x_0, \rho, z)\right)^\frac{q}{p}\right]$$

where we have set $h(t) := t + t^\frac{q}{p}$ and $v(x) = u(x) - \zeta - z(x - x_0)$ and where $c$ denotes a positive constant depending only on $n, N, p, q, \Gamma, L, M, \gamma$ and $\Lambda_L := \sup_{|z|\leq L+2} |Df^2(z)|$.

**Proof.** Assume for simplicity $x_0 = 0$ and choose 

$$\frac{\rho}{2} \leq r < s \leq \rho.$$

Define

$$\Xi(t) := \int_{B_t} \left[ \left| Dv \right|^p + \left| \frac{v}{s - r} \right|^p \right] \, dx.$$

We choose in addition $r \leq \tilde{r} < \tilde{s} \leq s$ as in Lemma \cite{8}. Let $\eta$ denote a smooth cut-off functions with support in $B_{\tilde{s}}$ satisfying $\eta \equiv 1$ in $B_{\tilde{r}}$ and $0 \leq \eta \leq 1$, $|\nabla \eta| \leq \frac{2}{s - r}$ on $B_{\rho}$. Using the operator from Lemma \cite{7} we set

$$\psi := T_{\tilde{r}, \tilde{s}}[(1 - \eta)v] \text{ and } \varphi := v - \psi.$$

Using properties (2) and (3) from lemma \cite{7} we have $\varphi \in W^{1,p}_0(B_{\tilde{s}}; \mathbb{R}^N)$ and $\varphi = v$ on $B_{\tilde{r}}$. Furthermore, we see

$$Du - z = Dv = D\varphi + D\psi \text{ on } B_{\rho}.$$
Using (A.3') (see Corollary 3), from lemma 2 we obtain that, for every \( z \) such that \( L + 1 > |z| > M \)

\[
\gamma \int_{B_r} |V^\frac{p}{q}(Dv)|^2 \, dx = \gamma \int_{B_z} |V^\frac{p}{q}(D\varphi)|^2 \, dx \leq \gamma \int_{B_z} |V^\frac{p}{q}(D\varphi)|^2 \, dx = \\
= \gamma \int_{B_z} (1 + |D\varphi|^2)^{\frac{p}{2} - 1} |D\varphi|^2 \, dx \leq \int_{B_z} [f(z + D\varphi) - f(z)] \, dx = \\
= \int_{B_z} [f(Du - D\psi) - f(Du)] \, dx + \int_{B_z} [f(Du) - f(Du - D\psi)] \, dx + \\
+ \int_{B_z} [f(z + D\psi) - f(z)] \, dx.
\]

Applying the minimality of \( u \) and lemma 6 and adding and subtracting \( Df(z)D\psi(x) \) we conclude that \( \forall z : L + 1 > |z| > M \),

\[
(5) \quad \gamma \int_{B_r} |V^\frac{p}{q}(Dv)|^2 \, dx \leq \\
\leq \int_{B_z} \left[ \int_0^1 (Df(z) - Df(Du - \tau D\psi)) \, d\tau D\psi + f(z + D\psi) - f(z) - Df(z)D\psi \right] \, dx \leq \\
\leq c \int_{B_z} \left[ \int_0^1 |V^{q-1}(Dv - \tau D\psi)| \, d\tau |D\psi| + |V^\frac{p}{q}(D\psi)|^2 \right] \, dx.
\]

Starting from now, we divide the proof in two cases, beginning from the case \( p > 2 \).

Setting \( R := B_\delta \setminus B_\tau \), recalling \( \psi \equiv 0 \) on \( B_\tau \) and some elementary properties of \( V \), i.e.

\[
(6) \quad |V^\beta(A + B)| \leq c |V^\beta(A)| + |V^\beta(B)|
\]

and

\[
(7) \quad \min\{t^2, t^p\} |V^\frac{p}{q}(A)|^2 \leq |V^\frac{p}{q}(tA)|^2 \leq \max\{t^2, t^p\} |V^\frac{p}{q}(A)|^2
\]

(see [9], Definition 6.1), we infer:

\[
(8) \quad \int_{B_r} |V^\frac{p}{q}(Dv)|^2 \, dx \leq c \left[ \int_R |V^\frac{p}{q}(D\psi)|^2 \, dx + \int_R |V^{q-1}(Dv)||D\psi| \, dx \right] =: c[I_1 + I_2]
\]

Let us introduce the abbreviation

\[
\Delta := \int_{B_\delta \setminus B_\tau} \left[ |V^\frac{p}{q}(Dv)|^2 + \left| V^\frac{p}{q} \left( \frac{v}{s - r} \right) \right|^2 \right] \, dx.
\]
Using properties (6) and (8) of lemma \( q < \frac{mp}{n+1} \) and lemma \( \mathfrak{s} \) we get:

\[
(9) \quad I_1 \leq c \left[ \int_R |D\psi|^2 \, dx + \int_R |D\psi|^q \, dx \right] \leq \\
\leq c \left[ \int_R |D[(1-\eta)v]|^2 \, dx + \int_R |DT_{t,\delta} [(1-\eta)v]|^q \, dx \right] \\
\leq c \left[ \Delta + (s-r)^n \left( \sup_{t \in (\tilde{r}, \tilde{s})} \frac{\Xi(t) - \Xi(\tilde{r})}{t - \tilde{r}} \right)^{\frac{2}{p}} + \\
+ (s-r)^n \left( \sup_{t \in (\tilde{r}, \tilde{s})} \frac{\Xi(\tilde{s}) - \Xi(t)}{\tilde{s} - t} \right)^{\frac{2}{p}} \right] \leq \\
\leq c \left[ \Delta + (s-r)^n \left( \frac{\Delta}{(s-r)^n} \right)^{\frac{2}{p}} \right].
\]

Using \( q < p + \frac{1}{n} < p + 1 \) and Hölder’s inequality we can treat \( I_2 \) in a similar fashion:

\[
(10) \quad I_2 \leq c \int_R (|Dv| |D\psi| + |Dv|^{q-1} |D\psi|) \, dx \leq \\
\leq c \left[ \left( \int_R |Dv|^2 \, dx \right)^{\frac{1}{q}} \left( \int_R |D\psi|^2 \, dx \right) + \\
+ \left( \int_R |Dv|^p \, dx \right)^{\frac{q-1}{p}} \left( \int_R |D\psi|^\frac{p}{p+1-q} \, dx \right)^{\frac{p+1-q}{p}} \right] \leq \\
\leq c \left[ \Delta + (s-r)^n \left( \frac{\Delta}{(s-r)^n} \right)^{\frac{2}{p}} \right].
\]

For the last inequality, notice that \( \int_R |Dv|^2 \, dx \) and \( \int_R |Dv|^p \, dx \) are obviously less than \( \int_{B_r \setminus B_{\rho}} |V^\frac{q}{p}(D\psi)|^2 \), which is less than \( \Delta \), and that \( \int_R |D\psi|^2 \, dx < \Delta \).

Combining \( \mathfrak{s} \), (9) and (10) we arrive at

\[
\int_{B_{\rho}} |V^\frac{q}{p}(Dv)|^2 \, dx \leq C_1 \left[ \Delta + (s-r)^n \left( \frac{\Delta}{(s-r)^n} \right)^{\frac{2}{p}} \right],
\]

where \( C_1 \) denotes a positive fixed constant depending on \( n, N, p, q, \Gamma, \gamma, L, \Lambda_L \).

Adding \( C_1 \int_{B_{\rho}} |V^\frac{q}{p}(Dv)|^2 \, dx \) on both sides and dividing by \( 1 + C_1 \) we see:

\[
(11) \quad \int_{B_{\rho}} |V^\frac{q}{p}(Dv)|^2 \, dx \leq \frac{C_1}{1 + C_1} \int_{B_{\rho}} |V^\frac{q}{p}(Dv)|^2 \, dx + \int_{B_{\rho'}} \left| V^\frac{q}{p} \left( \frac{v}{s-r} \right) \right|^2 \, dx + \\
+ (s-r)^n \left( \frac{1}{(s-r)^n} \int_{B_{\rho'}} \left| V^\frac{q}{p}(Dv) \right|^2 \, dx \right)^{\frac{2}{p}}.
\]
Using Lemma 6.6 from [9], we have:

\[
\int_{B_{\rho/2}} |V^\frac{q}{p}(Dv)|^2 \, dx \leq c \left[ \int_{B_\rho} \left| V^\frac{q}{p} \left( \frac{v}{\rho} \right) \right|^2 \, dx + \left( \int_{B_\rho} \left| V^\frac{q}{p}(Dv) \right|^2 + \left| V^\frac{q}{p} \left( \frac{v}{\rho} \right) \right|^2 \, dx \right)^{\frac{q}{p}} \right],
\]

which proves the claim in the case \( p > 2 \).

We now approach the proof in the case \( p \leq 2 \) restarting from (5) and using a different argument.

We use the notations of the previous case except for the following modification:

\[
\Xi(t) := \int_{B_t} \left[ \left| V^\frac{q}{p}(Dv) \right|^2 + \left| V^\frac{q}{p} \left( \frac{v}{\rho - r} \right) \right|^2 \right] \, dx.
\]

Exactly as before, we reach

\[
\int_{B_r} |V^\frac{q}{p}(Dv)|^2 \, dx \leq c \int_{B_2} \left[ \int_0^1 |V^{q-1}(Dv - \tau D\psi)| \, d\tau |D\psi| + |V^\frac{q}{p}(D\psi)|^2 \right] \, dx.
\]

By Acerbi and Fusco in [2], Lemma 2.1, it was proven that for any \( z_1, z_2 \in \mathbb{R}^{nN} \) one has:

\[
\int_0^1 (1 + |z_1 + tz_2|^2)^{\frac{p-2}{2}} \, dt \leq c(1 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}.
\]

In our case, we get:

\[
\int_{B_r} |V^\frac{q}{p}(Dv)|^2 \, dx \leq c \left[ \int_{B_r} \left| V^\frac{q}{p}(D\psi) \right|^2 \, dx + \int_{B_r} (1 + |Dv|^2 + |D\psi|^2)^{\frac{p-2}{2}} \left( |Dv| + |D\psi| \right) \, dx \right] =: c[(I) + (II)].
\]

To estimate (I), we use the obvious property that

\[
(1 + |z_1|^2)^{\frac{q}{p}} \leq 1 + (1 + |z_1|^2)^{\frac{p-2}{2}} |z_1|^2
\]

and Hölder and Young inequalities, obtaining:

\[
(I) \leq \int_{B_r} \left( 1 + \left| V^\frac{q}{p}(D\psi) \right|^2 \right)^{\frac{2q}{p+2}} \left| V^\frac{q}{p}(D\psi) \right|^2 \, dx \leq c \left[ \int_{B_r} \left| V^\frac{q}{p}(D\psi) \right|^2 \, dx + \int_{B_r} \left| V^\frac{q}{p}(D\psi) \right|^\frac{2q}{p} \, dx \right] =: c \left[ \int_{B_r} \left| V^\frac{q}{p}(D\psi) \right|^2 \, dx + (III) \right] \leq c\Delta + (III).
\]
Now, using properties 7 and 9 from Lemma 7 and Lemma 8 we have

\[(III) \leq c \int_R \left( T_{\tilde{r},\tilde{s}} \left[ V^{\tilde{z}}(D[(1-\eta)v]) \right] \right)^q dx \leq \]
\[\leq c(\tilde{s} - \tilde{r})^n \left( \sup_{t \in (\tilde{r},\tilde{s})} \frac{(\tilde{s} - \tilde{r})1-n}{\tilde{s} - \tilde{r}} \int_{B_{\tilde{s}} \setminus B_{\tilde{r}}} |V^{\tilde{z}}(D[(1-\eta)v])|^2 dx \right) \]
\[+ \sup_{t \in (\tilde{r},\tilde{s})} \frac{(\tilde{s} - \tilde{r})1-n}{\tilde{s} - \tilde{r}} \int_{B_{\tilde{s}} \setminus B_{\tilde{r}}} \left| V^{\tilde{z}}(D[(1-\eta)v]) \right|^{\tilde{z}} dx \]
\[\leq c(s - r)^n \left( (s - r)^{1-n} \left( \sup_{t \in (\tilde{r},\tilde{s})} \frac{\Xi(t) - \Xi(\tilde{r})}{t - \tilde{r}} + \sup_{t \in (\tilde{r},\tilde{s})} \frac{\Xi(\tilde{s}) - \Xi(t)}{\tilde{s} - t} \right) \right)^{\frac{2}{p}} \leq c(s - r)^n \left( \frac{\Delta}{(s - r)^n} \right)^{\frac{2}{p}} \]

So we have proved that:

\[(I) \leq c \left[ \Delta + (s - r)^n \left( \frac{\Delta}{(s - r)^n} \right)^{\frac{2}{p}} \right] \]

To estimate (II), we make repeated use of Young inequality and of the fact that

\[1 + |z_1|^2 + |z_2|^2 \leq 1 + (1 + |z_1|^2 + |z_2|^2) \frac{2q}{p} (|z_1|^2 + |z_2|^2) \]

In particular

\[(II) \leq c \left[ \int_R (1 + |Dv|^2 + |D\psi|^2) \frac{2q}{p} (|Dv| + |D\psi|) dx \right. \]
\[+ \left. \int_R (1 + |Dv|^2 + |D\psi|^2)^{(p-2)} \frac{2q}{p} (|Dv|^2 + |D\psi|^2)^{\frac{p}{2}} \right] \left[ |Dv| + |D\psi| \right] dx \leq \]
\[\leq \left[ \int_R \left| V^{\tilde{z}}(Dv) \right|^2 dx + \int_R \left| V^{\tilde{z}}(D\psi) \right|^2 dx \right. \]
\[+ \left. \int_R \left| V^{\tilde{z}}(Dv) \right|^{\frac{2q}{p}} dx + \int_R \left| V^{\tilde{z}}(D\psi) \right|^{\frac{2q}{p} - 1} \left| V^{\tilde{z}}(D\psi) \right| dx \right] \]

The first three summands on the right hand side of this inequality are easily controlled by \(c\Delta\) or have already been encountered throughout the proof, so that we only need to estimate

\[(IV) := \int_R \left| V^{\tilde{z}}(Dv) \right|^{\frac{2q}{p} - 1} \left| V^{\tilde{z}}(D\psi) \right| dx. \]

Using \(q < \frac{3}{2p}\), Hölder inequality yields:

\[(IV) \leq \left( \int_R \left| V^{\tilde{z}}(Dv) \right|^2 dx \right)^{\frac{2q}{2p} - \frac{q}{p}} \left( \int_R \left| V^{\tilde{z}}(D\psi) \right|^{\frac{2q}{2p}} dx \right)^{\frac{4q - 2q}{2p}} \]
The first factor can be easily estimated by $\Delta^{\frac{2q-p}{2p}}$, while using the fact that $q < p + \frac{p}{2n}$, and hence $q < \frac{3}{2}p$, we can estimate the other factor as we did for $(III)$, so that

$$(IV) \leq c(s-r)^n \left(\frac{\Delta}{(s-r)^n}\right)^{\frac{q}{p}}$$

In conclusion, we have

$$\int_{B_r} \left| V^p(Dv) \right|^2 dx \leq c \left[ \Delta + (s-r)^n \left(\frac{\Delta}{(s-r)^n}\right)^{\frac{q}{p}} \right]$$

and we finish the proof as we did for $p > 2$. □

**Remark 1 (Schmidt’s Remark 7.4 in [9])**. Let us mention that in the case $q = p$, the inequality (4) holds without the second term on its right-hand side. This can be inferred directly from the proofs. However, in the case $q > p$ we will see that this second term is arbitrarily small. This is the reason why we call (4) a "Caccioppoli inequality".

## 4. Almost $\mathcal{A}$-harmonicity

Consider a bilinear form $\mathcal{A}$ on $\mathbb{R}^{nN}$. We assume that the upper bound

$$(13) \quad |\mathcal{A}| \leq \Lambda$$

with $\Lambda > 0$ holds and that the Legendre-Hadamard condition

$$(14) \quad \mathcal{A}(\zeta x^T, \zeta x^T) \geq \lambda |x|^2 |\zeta|^2 \quad \text{for all } x \in \mathbb{R}^n, \zeta \in \mathbb{R}^N$$

with ellipticity constant $\lambda > 0$ is satisfied.

We say that $h \in W^{1,1}_{loc}(\Omega, \mathbb{R}^N)$ is $\mathcal{A}$-harmonic on $\Omega$ iff

$$\int_{\Omega} \mathcal{A}(Dh, D\varphi) \, dx = 0$$

holds for all smooth $\varphi : \Omega \to \mathbb{R}^N$ with compact support in $\Omega$.

The following two lemmas, whose proof can be found in ([9], Lemma 7.8, 7.7 and 6.8) will enable us to approximate $W^{1,p}$-minimizers with functions that are $\mathcal{A}$-harmonic.

**Lemma 10.** Let $f$ satisfy $(A.1)$ and $(A.3')$ for a given $M > 0$. Choose any $M > 0$. Then, for any given $z$ such that $|z| > M$, we have that $\mathcal{A} = D^2 f(z)$ satisfies the Legendre-Hadamard condition

$$\mathcal{A}(\zeta x^T, \zeta x^T) \geq \lambda |x|^2 |\zeta|^2 \quad \text{for all } x \in \mathbb{R}^n \text{ and } \zeta \in \mathbb{R}^N$$

with ellipticity constant $\lambda = 2\gamma$. 

Proof. Let $u$ be the affine function $u(x) = zx$ with $z$ such that $|z| > M$. Quasiconvexity in $z$ ensures that $u$ is a $W^{1,p}$-minimizer of the functional $F$ induced by $f$ and that the function:

$$G_\varphi(t) = \mathcal{F}_{|B_1}(u + t\varphi) - \gamma \int_{B_1} (1 + |tD\varphi|^2)^{\frac{p}{2} - 1}|tD\varphi|^2 \, dx$$

has a minimum in $t = 0$ for any $\varphi \in W^{1,p}_0(B_1, \mathbb{R}^N)$ and, in the same way as it is done in [6], Prop. 5.2, from $G'(0) = 0$ and $G''(0) \geq 0$ the Legendre-Hadamard condition will follow.

As a matter of fact, from $G''(0) \geq 0$, we obtain:

$$\int_{B_1} \frac{\partial^2 F}{\partial z_k \partial z_j}(z_0) D_k \varphi^\alpha D_j \varphi^\beta \, dx \geq 2\gamma \int_{B_1} |D\varphi^2| \, dx$$

for every $\varphi \in C^1_c(B_1, \mathbb{R}^N)$. Let us $\varphi = \lambda + i\mu$ and write (15) for $\lambda$ and for $\mu$, i.e.:

$$\int_{B_1} \frac{\partial^2 F}{\partial z_k \partial z_j}(z_0) D_k \lambda^\alpha D_j \lambda^\beta \, dx \geq 2\gamma \int_{B_1} |D\lambda^2| \, dx$$

and

$$\int_{B_1} \frac{\partial^2 F}{\partial z_k \partial z_j}(z_0) D_k \mu^\alpha D_j \mu^\beta \, dx \geq 2\gamma \int_{B_1} |D\mu^2| \, dx$$

we obtain:

$$\int_{B_1} \frac{\partial^2 F}{\partial z_k \partial z_j}(z_0) [D_k \lambda^\alpha D_j \lambda^\beta + D_k \mu^\alpha D_j \mu^\beta] \, dx \geq 2\gamma \int_{B_1} |D\lambda^2| + |D\mu^2| \, dx$$

and hence:

$$Re \int_{B_1} \frac{\partial^2 F}{\partial z_k \partial z_j}(z_0) D_k \varphi^\alpha D_j \varphi^\beta \, dx \geq 2\gamma \int_{B_1} |D\varphi|^2 \, dx$$

Now, consider any $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^N$, $\tau \in \mathbb{R}$ and $\Psi(x) \in C_c^\infty(B_1, \mathbb{R})$ and take $\varphi$ to be $\varphi(x) = \eta e^{i\tau \xi \cdot x} \psi(x)$. Since $\varphi^n(x) = \eta^n \psi(x) e^{i\tau \xi \cdot x}$, we have

$$\int_{B_1} \frac{\partial^2 F}{\partial z_k \partial z_j}(z_0)\eta^\alpha \eta^\beta [\tau^2 \xi_k \xi_j \psi^2 + D_k \psi D_j \psi] \, dx \geq 2\gamma |\eta|^2 \int_{B_1} (|D\psi|^2 + \tau^2 |\xi|^2 |\Psi(x)|^2) \, dx.$$

Dividing by $\tau^2$ and letting $\tau \to \infty$ we get:

$$\int_{B_1} \frac{\partial^2 F}{\partial z_k \partial z_j}(z_0)\xi_k \xi_j \eta^\alpha \eta^\beta \psi^2(x) \, dx \geq 2\gamma |\eta|^2 |\xi|^2 \int_{B_1} \psi^2(x) \, dx$$

and since this holds for all $\Psi \in C_c^\infty(B_1, \mathbb{R})$ the proposition is proved. \qed

Remark 2. Assume $f \in C^2_{\text{loc}}(\mathbb{R}^{nN})$. Then for each $L > 0$, there is a modulus of continuity $\omega_L : [0, +\infty[ \to [0, +\infty]$ satisfying $\lim_{z \to 0} \omega_L(z) = 0$ such that for all
We may assume, without loss of generality, that
\[ |z_1| \leq L, \ |z_2| \leq L + 1 \Rightarrow |D^2f(z_1) - D^2f(z_2)| \leq \omega_L(|z_1 - z_2|^2). \]
Moreover, \( \omega_L \) can be chosen such that the following properties hold:

1. \( \omega_L \) is non-decreasing,
2. \( \omega_L^2 \) is concave,
3. \( \omega_L^2(z) \geq z \) for all \( z \geq 0 \).

**Lemma 11.** Let \( f \) satisfy (A.1),(A.2),(A.3) for a given \( M > 0 \). Choose any \( L > M > 0 \) and take \( u \in W^{1,p} \) to be a \( W^{1,p} \)-minimizer of \( F \) on some ball \( B_\rho(x_0) \), where \( q \leq p + 1 \). Then for all \( z : M < |z| \leq L \) and \( \varphi \in C^\infty_c(B_\rho(x_0)) \) we have

\[
\left( \int_{B_\rho(x_0)} D^2f(z)(Du - z, D\varphi) \, dx \right) \leq c\sqrt{\Phi_p\omega_L(\Phi_p)} \sup_{B_\rho(x_0)} |D\varphi|.
\]

where \( \Phi_p := \Phi_p(u, x_0, \rho, z) \), the constant \( c \) depends only on \( n, N, p, q, \Gamma, L \) and \( \omega_L \) is the abovementioned modulus of continuity (see also [9]).

**Proof.** The proof of a similar result, in [9], will be adapted and explicitly repeated for the convenience of the reader.

We may assume, without loss of generality, that \( x_0 = 0 \) and \( \sup_{B_\rho} |D\varphi| = 1 \). Setting \( v(x) := u(x) - zx \), the Euler equation of \( F \) gives

\[
\left( \int_{B_\rho} D^2f(z)(Dv, D\varphi) \, dx \right) \leq \int_{B_\rho} |D^2f(z)(Dv, D\varphi) + Df(z)D\varphi - Df(Du)D\varphi| \, dx
\]

Now we estimate the integrand on the right-hand side.

On the set \( \{ x \in B_\rho : |Dv| \leq 1 \} \) we have \( |Dv|^2 \leq 2 |V^\sharp(Dv)|^2 \). Using this, Remark 2 and the concavity of \( \omega_L \) we have:

\[
|D^2f(z)(Dv, D\varphi) + Df(z)D\varphi - Df(Du)D\varphi| \leq \int_0^1 |D^2f(z) - D^2f(z + tDv)| \, dt |Dv| \leq \omega_L(|Dv|^2) |Dv| \leq c\omega_L \left( |V^\sharp(Dv)|^2 \right) |V^\sharp(Dv)|.
\]

On the set \( \{ x \in B_\rho : |Dv| \geq 1 \} \), Lemma 6 implies

\[
|D^2f(z)(Dv, D\varphi) + Df(z)D\varphi - Df(Du)D\varphi| \leq \sup_{|z| \leq L + 1} |D^2f(z)||Dv| + c |V^{q-1}(Dv)| \leq c |Dv|^\max\{q-1,1\} \leq c |V^\sharp(Dv)|^2.
\]
Combining (21), (22) and (23) and noticing that property 3 of $\omega_L$ stated in Remark 2 implies that

$$\max\left\{\omega_L\left(\left|V^2(Du)\right|^2\right), \frac{1}{2}\right\} = \omega_L\left(\left|V^2(Du)\right|^2\right)$$

we have

$$\int_{B_\rho} |D^2 f(z)(Du, D\varphi)| \leq c \int_{B_\rho} \omega_L\left(\left|V^2(Du)\right|^2\right) \left|V^2(Du)\right| \, dx.$$ 

Now we apply Hölder to obtain

$$\int_{B_\rho} |D^2 f(z)(Du, D\varphi)| \leq c \left[\int_{B_\rho} \omega_L^2\left(\left|V^2(Du)\right|^2\right) \right]^\frac{1}{2} \left[\int_{B_\rho} \left|V^2(Du)\right|^2 \right]^\frac{1}{2}$$

Now, by Jensen, using the concavity of $\omega_L^2$, to obtain

$$\int_{B_\rho} |D^2 f(z)(Du, D\varphi)| \leq c \sqrt{\Phi_p} \omega_L(\Phi_p).$$

This completes the proof.

Lemma 12. Fix $1 < p < \infty$, $0 < \lambda \leq \Lambda < \infty$ and $\varepsilon > 0$. Then there is a $\delta(n, N, p, \Lambda, \lambda, \varepsilon) > 0$ such that the following assertion holds:

For all $s \in (0, 1)$, for all $A$ satisfying (13) and (14) and for each $u \in W^{1,p}(B_\rho(x_0); \mathbb{R}^N)$ with:

$$\int_{B_\rho(x_0)} |V^2(Du)|^2 \, dx \leq s^2$$

and

$$\int_{B_\rho(x_0)} A(Du, D\varphi) \, dx \leq s\delta \sup_{B_\rho(x_0)} |D\varphi|$$

for all smooth $\varphi : B_\rho(x_0) \to \mathbb{R}^N$ with compact support in $B_\rho(x_0)$ there is an $A$-harmonic function $h \in C^\infty_{\text{loc}}(B_\rho(x_0), \mathbb{R}^N)$ with

$$\sup_{B_{\rho/2}(x_0)} |D h| + \rho \sup_{B_{\rho/2}(x_0)} |D^2 h| \leq c$$

and

$$\int_{B_{\rho/2}(x_0)} \left|V^2\left(\frac{u - sh}{\rho}\right)\right|^2 \, dx \leq s^2 \varepsilon.$$ 

Here $c$ denotes a constant depending only on $n, N, p, \Lambda, \lambda$. 
5. Excess decay estimate

Proposition 13. Let \( z_0 \) be s.t. \( |z_0| > M + 1 \) and \( x_0 \) be s.t.
\[
\lim_{\rho \to 0} \int_{B_{\rho}(x_0)} \left| V^*_p(Du(x) - z_0) \right|^2 = 0
\]
then
\[
\Phi_p(u, x_0, \rho) \to 0 \quad \text{as} \quad \rho \to 0.
\]

Proof. Let \((Du)_\rho := \int_{B_{\rho}(x_0)} |Du|\). We have, using (3), (2) and convexity of \(|W^*_p(z)|^2\):
\[
\Phi_p(u, x_0, \rho) = \int_{B_{\rho}(z_0)} |V^*_p[Du - (Du)_\rho]|^2 \, dx \leq \\
\leq c \left[ \int_{B_{\rho}(z_0)} |V^*_p[Du - z_0]|^2 \, dx + |V^*_p[z_0 - (Du)_\rho]|^2 \right] \leq \\
\leq c \int_{B_{\rho}(z_0)} |V^*_p[Du - z_0]|^2 \, dx \to 0
\]

Finally, we can prove

Lemma 14. Assume \( q \) and \( p \) are real numbers such that \( q < p + \frac{\min\{2, p\}}{2n} \).
Let \( f \) satisfy assumptions (A.1), (A.2) and (A.3) for a given \( M > 0 \).
Choose any \( L > M + 1 > 0 \), \( \alpha \in (0, 1) \), \( z_0 \in \mathbb{R}^{nN} \) such that \( |z_0| > M + 1 \).
Then there are constants \( \varepsilon_0 > 0 \), \( \theta \in (0, 1) \) and a radius \( \rho^* > 0 \) depending on \( n, N, L, p, q, \Gamma, \alpha, \gamma, x_0, z_0 \) and \( \Lambda_L := \max_{B_{\rho^*+2}} |D^2 f| \) and with \( \varepsilon_0 \) depending additionally on \( \omega_L \) such that the following holds.
Consider \( u \) a \( W^{1,p} \)-minimizer of \( \mathcal{F} \) on \( B_{\rho^*}(x_0) \), with \( \rho < \rho^* \) and \( x_0 \in \mathbb{R}^n \) satisfying
\[
\lim_{\rho \to 0} \int_{B_{\rho}(x_0)} \left| V^*_p(Du(x) - z_0) \right|^2 = 0.
\]
If the following conditions hold
\[
(23) \quad \Phi_p(u, x_0, \rho) \leq \varepsilon_0
\]
and
\[
(24) \quad |(Du)_{x_0, \rho}| \leq L
\]
then
\[
\Phi_p(u, x_0, \theta \rho) \leq \theta^{2\alpha} \Phi_p(u, x_0, \rho).
\]
Proof. Let $z_0$ be such that $|z_0| > M+1$ and $x_0$ any point such that $\lim_{\rho \to 0} \int_{B_{\rho}(x_0)} |Du(x) - z_0|^p = 0$. In what follows, for simplicity of notation, we assume that $x_0 = 0$ and we abbreviate
\[ z = (Du)_\rho := \int_{B_{\rho}} Du \, dx \]
and
\[ \Phi_p(\cdot) := \Phi_p(u, 0, \cdot). \]
where $\rho > 0$ is any positive value small enough (smaller than a $\rho^*$ that will be determined throughout the proof).

Since the claim is obvious in the case $\Phi_p(\rho) = 0$ we can assume $\Phi_p(\rho) \neq 0$.

Setting
\[ w(x) := u(x) - zx \quad \text{and} \quad s := \sqrt{\Phi_p(\rho)}, \]
we have by definition of $\Phi_p(\rho)$,
\[ \int_{B_{\rho}} |V^p(Dw)|^2 \, dx = s^2 = \Phi_p(\rho). \]

Next we will approximate by $A$-harmonic functions, where $A := D^2 f(z)$.

If we choose $\rho < \rho^* := r_1(z_0)$ as in Lemma 4 we have $|z| > M+1$, hence, from $|A| \leq \max_{B_{L+2}} |D^2 f| =: \Lambda_L$ and Lemma 10 we deduce that $A$ satisfies (13) with a bound $\Lambda_L$ and (14) with ellipticity constant $2\gamma$. Lemma 11 yields the estimate:
\[ \left| \int_{B_{\rho}} A(D\omega, D\varphi) \, dx \right| \leq C_2 \omega_L(\Phi_p(\rho)) \sup_{B_{\rho/2}} |D\varphi| \]
for all $\rho < \rho^*$ and for all smooth functions $\varphi : B_{\rho} \to \mathbb{R}^N$ with compact support in $B_{\rho}$, where $C_2$ is a positive constant depending on $n, N, p, q, \Gamma, L, \Lambda_L$.

For $\varepsilon > 0$ to be specified later, we fix the corresponding constant $\delta(n, N, p, \Lambda_L, \gamma, \varepsilon) > 0$ from Lemma 12.

Now, let $\varepsilon_0 = \varepsilon_0(n, N, p, \Lambda_L, \gamma, \varepsilon)$ be small enough so that (23) implies:
\[ (25) \quad C_2 \omega_L(\Phi_p(\rho)) \leq \delta \]
\[ (26) \quad s = \sqrt{\Phi_p(\rho)} \leq 1. \]

We apply Lemma 12. The lemma ensures the existence of an $A$-harmonic function $h \in C^\infty_0(B_{\rho}; \mathbb{R}^N)$ such that
\[ \sup_{B_{\rho/2}} |Dh| + \rho \sup_{B_{\rho/2}} |D^2 h| \leq c \]
where $c = c(n, N, p, \Lambda_L, \gamma)$ and
\[ (27) \quad \int_{B_{\rho/2}} \left| V^p \left( \frac{w - sh}{\rho} \right) \right|^2 \, dx \leq s^2 \varepsilon. \]
Now fix $\theta \in (0, 1/4]$. Taylor expansion implies the estimate:

$$ \sup_{x \in B_{2\theta \rho}} |h(x) - h(0) - Dh(0)x| \leq \frac{1}{2} (2\theta \rho)^2 \sup_{x \in B_{\rho/2}} |D^2 h| \leq c \theta^2 \rho. $$

Using (6) and (7) together with what we have obtained we get:

$$ \int_{B_{2\rho}} \left| V^\# \left( \frac{w(x) - sh(0) - sDh(0)x}{2\theta \rho} \right) \right|^2 \, dx \leq $$

$$ \leq c \left[ \theta^{-n - \max\{2, p\}} \int_{B_{\rho/2}} \left| V^\# \left( \frac{w - sh}{\rho} \right) \right|^2 \, dx + \right. $$

$$ + \left. \int_{B_{2\rho}} \left| V^\# \left( \frac{h(x) - h(0) - Dh(0)x}{2\theta \rho} \right) \right|^2 \, dx \right] \leq $$

$$ \leq c \left[ \theta^{-n - \max\{2, p\}} s^2 \varepsilon + \left| V^\# (\theta s) \right|^2 \right] \leq $$

$$ \leq c \left[ \theta^{-n - \max\{2, p\}} s^2 \varepsilon + \theta^2 s^2 \right] $$

Setting $\varepsilon := \varepsilon(\theta) = \theta^{n+2 + \max\{2, p\}}$ (so, remember that $\varepsilon$ and hence $\delta$ and $\varepsilon_0$ depend on whatever $\theta$ we wish to choose) and recalling the definitions of $w$ and $s$ we have:

$$ \int_{B_{2\rho}} \left| V^\# \left( \frac{u(x) - z x - s(h(0) + Dh(0)x)}{2\theta \rho} \right) \right|^2 \, dx \leq c \theta^2 \Phi_p(\rho). $$

On the other hand, we remark that, using the definition of $s$ and properties of $h$:

$$ |sDh(0)|^2 \leq c^2 \Phi_p(\rho) $$

We can take $\varepsilon_0$ small enough such that (23) implies also:

$$ s \leq \frac{1}{c} $$

and that would imply

$$ |sDh(0)|^2 \leq 1. $$

Using this fact together with (29) and (6) we get

$$ \Phi_p(2\theta \rho, z + sDh(0)) \leq $$

$$ \leq c \left[ (2\theta)^{-n} \int_{B_{\rho}} \left| V^\# (Du - z) \right|^2 \, dx + \left| V^\# (sDh(0)) \right|^2 \right] \leq $$

$$ \leq c \left[ \theta^{-n} \left( \Phi_p(\rho) + |sDh(0)|^2 \right) \right] \leq c \theta^{-n} \Phi_p(\rho). $$

Now we need to use (21) with $\zeta = sh(0)$ and $z + sDh(0)$ instead of $z$, and we can be sure that $|z + sDh(0)| > M$ because $|sDh(0)| \leq 1$.

Now, we can combine (28) and (31) and Caccioppoli inequality (21) with $\zeta = sh(0)$.
and \( z + sDh(0) \) instead of \( z \), and we get
\[
\Phi_p(\theta \rho, z + sDh(0)) \leq c \left[ \theta^2 \Phi_p(\rho) + \theta^{\frac{2q}{p}} \Phi_p(\rho) + \theta^{-n} \Phi_p(\rho)^{\frac{2}{p}} \right].
\]
Thereby the condition \( |z + sDh(0)| \leq L + 1 \) of Lemma 9 can be deduced from (30).

Now, if \( \varepsilon_0 \) is chosen small enough, depending on \( \theta \), (23) implies the following:
\[
\Phi_p(\theta \rho, z + sDh(0)) \leq c \theta^2 \Phi_p(\rho).
\]

For \( q = p \), however, the last inequality holds without further assumptions since the last term on the right hand side of (52) does not occur (see Remark 1).

Using Lemma 6.2 in [9] (written in the same notation as ours except for \( A \) instead of \( z \)) we deduce from the previous inequality:
\[
\Phi_p(\theta \rho) \leq C_3 \theta^2 \Phi_p(\rho),
\]
where \( C_3 > 0 \) depends on \( n, N, p, q, \Gamma, \gamma, \Lambda_L, L \). Finally, we choose \( \theta \in (0, \frac{1}{4}] \) (depending on \( \alpha \) and whatever \( C_3 \) depends on) small enough such that
\[
C_3 \theta^2 \leq \theta^2
\]
holds, and \( \varepsilon_0 \) small enough such that (25), (26), (30), (33) follow from (23).

Taking into account (31) and (35) the proof of the proposition is complete. □

The following adaptation of ([9], Lemma 7.10) is then a trivial consequence of this last lemma.

**Lemma 15.** Assume \( q \) and \( p \) are real numbers such that \( q < p + \frac{\min\{2,p\}}{2n} \).

Let \( f \) satisfy assumptions (A.1), (A.2) and (A.3) for a given \( M > 0 \).

Choose any \( L > 2M + 2 > 0 \), \( \alpha \in (0,1) \), \( z_0 \in \mathbb{R}^{nN} \) such that \( |z_0| > M + 1 \).

Then there is a constant \( \tilde{\varepsilon}_0 > 0 \) and a radius \( \rho^* > 0 \) depending on \( n, N, L, p, q, \Gamma, \alpha, \gamma, x_0, z_0 \) and \( \Lambda_L := \max |D^2 f| \) and with \( \tilde{\varepsilon}_0 \) depending additionally on \( \omega_L \) such that the following holds.

Consider \( u \) a \( W^{1,p} \)-minimizer of \( \mathcal{F} \) on \( B_{\rho}(x_0) \), with \( \rho < \rho^* \) and \( x_0 \in \mathbb{R}^n \) satisfying
\[
\lim_{\rho \to 0} \int_{B_\rho(x_0)} |Du(x) - z_0|^p = 0.
\]

If the following conditions hold
\[
\Phi_p(u, x_0, \rho) \leq \tilde{\varepsilon}_0
\]
and
\begin{equation}
(Du)_{x_0, \rho} \leq \frac{L}{2}
\end{equation}
then there is a constant $c$ depending on $n, N, L, p, q, \Gamma, \alpha, \gamma, x_0, z_0$ such that
$$\Phi_p(u, x_0, r) \leq c \left( \frac{r}{\rho} \right)^{2\alpha} \Phi_p(u, x_0, \rho)$$
for any $r < \rho$.

**Regularity**

Now we are able to prove our main result

**Proof of Theorem** Let $x_0$ be such that $\exists z_0 : |z_0| > M + 1$ with the property that
$$\int_{B_\rho(x_0)} |Du - z_0|^p \to 0 \text{ as } \rho \to 0.$$
and choose any $\alpha \in (0, 1)$ and $L = 4|z_0| > M$.
Then there is $r_2 > 0$ small enough such that $|f_{B_\rho(x_0)} Du| < L/2$ for all $\rho < r_2$ and, since $\Phi_p(u, x_0, \rho) \to 0$ as $\rho \to 0^+$, there is $r_3$ such that, for all $\rho < r_3$, $\Phi_p(u, x_0, \rho) < \varepsilon_0$.
Applying lemma we have that $Du$ belongs to the Morrey-Campanato space $L^{\lambda, 2}$ with $\lambda = 2\alpha + n > n$ so that, because of the continuous immersion $L^{\lambda, 2} \hookrightarrow C^{0,0, \alpha}$ we obtain that $Du \in C^{0,0, \alpha}$ and so $u \in C^{1,\alpha}(B_\rho(x_0))$ choosing $\rho < \min\{\rho^*, r_2, r_3\}$. So $x_0 \in \text{Reg}(u)$.
Of course $\text{Reg}(u)$ is an open set by definition.

We will now argue by contradiction to prove that it is dense. Assume there is a point $x \in \Omega$ and a radius $r > 0$ such that $B_r(x)$ is entirely outside $\text{Reg}(u)$. Since $Du \in L^p \subseteq L^1$, by Lebesgue-Besicovitch theorem, for almost all points $y$ in $B_r(x)$ this would mean that $\lim_{r \to 0} \int_{B_r(y)} |Du| < M$ and so $|Du|$ is essentially bounded by $M$ in $B_r(x)$, which contradicts the hypothesis that $B_r(x)$ is outside $\text{Reg}(u)$.

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