SPANNING SIMPLICIAL COMPLEXES OF UNI-CYCLIC GRAPHS

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ABSTRACT. In this paper, we introduce the concept of \( \Delta_s(G) \) associated to a simple finite connected graph \( G \). We give the characterization of all spanning trees of the uni-cyclic graph \( U_{n,m} \). In particular, we give the formula for computing the Hilbert series and \( h \)-vector of the Stanley Riesner ring \( k[\Delta_s(U_{n,m})] \). Finally, we prove that the spanning simplicial complex \( \Delta_s(U_{n,m}) \) is shifted hence \( \Delta_s(U_{n,m}) \) is shellable.

Key words : Primary Decomposition, Hilbert Series, \( f \)-vectors, \( h \)-vectors, spanning Trees.

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1. INTRODUCTION

Suppose \( G(V, E) \) is a finite simple connected graph with the vertex set \( V \) and edge-set \( E \), a spanning tree of a simple connected finite graph \( G \) is a subgraph of \( G \) that contains every vertex of \( G \) and is also a tree. We represent the edge-set of all spanning trees of a graph \( G \) by \( s(G) \). In this paper, for a finite simple connected graph \( G(V, E) \), we introduce the concept of spanning simplicial complexes by associating a simplicial complex \( \Delta_s(G) \) defined on the edge set \( E \) of the graph \( G \) as follows:

\[
\Delta_s(G) = \langle F_i \mid F_i \in s(G) \rangle
\]

It is always possible to associate \( \Delta_s(G) \) to any simple finite connected graph \( G(V, E) \) but the characterization of \( s(G) \) has been a problem in this regard.

For the uni-cyclic graphs \( U_{n,m} \), we prove some algebraic and combinatorial properties of spanning simplicial complex \( \Delta_s(U_{n,m}) \). Where, a uni-cyclic graph \( U_{n,m} \) is a connected graph over \( n \) vertices and containing exactly one cycle of length \( m \). In Proposition 3.1, we give the characterization of \( s(U_{n,m}) \). Moreover, we give characterizations of the \( f \)-vector and \( h \)-vector in Lemma 3.3 and Theorem 3.5 respectively, which enable us to device a formula to compute the Hilbert series of the Stanley Reisner ring \( k[\Delta_s(U_{n,m})] \) in Theorem 3.6. In the Theorem 3.8 we show that the spanning simplicial complex \( \Delta_s(U_{n,m}) \) is shifted. So, we have the corollary 3.9 that the spanning simplicial complex \( \Delta_s(U_{n,m}) \) is shellable.

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2. Basic Setup

In this section, we give some basic definitions and notations which we will follow in this paper.

**Definition 2.1.** A spanning tree of a simple connected finite graph $G(V,E)$ is a subtree of $G$ that contains every vertex of $G$.

We represent the collection of all edge-sets of the spanning trees of $G$ by $s(G)$, in other words;

$$s(G) = \{E(T_i) \subset E, \text{ where } T_i \text{ is a spanning tree of } G\}.$$

**Remark 2.2.** It is well known that for any simple finite connected graph spanning tree always exist. One can find a spanning tree systematically by *cutting-down method*, which says that spanning tree of a given simple finite connected graph is obtained by removing one edge from each cycle appearing in the graph.

For example by using *cutting-down method* for the graph given in figure 1 we obtain:

$$s(G) = \{\{e_2, e_3, e_4\}, \{e_1, e_3, e_4\}, \{e_1, e_2, e_4\}, \{e_1, e_2, e_3\}\}$$

![Fig. 1. $C_4$](image)

**Definition 2.3.** A Simplicial complex $\Delta$ over a finite set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$, with the property that $\{i\} \in \Delta$ for all $i \in [n]$, and if $F \in \Delta$ then $\Delta$ will contain all the subsets of $F$ (including the empty set). An element of $\Delta$ is called a face of $\Delta$, and the dimension of a face $F$ of $\Delta$ is defined as $|F| - 1$, where $|F|$ is the number of vertices of $F$. The maximal faces of $\Delta$ under inclusion are called facets of $\Delta$. The dimension of the simplicial complex $\Delta$ is:

$$\dim \Delta = \max \{\dim F | F \in \Delta\}.$$

We denote the simplicial complex $\Delta$ with facets $\{F_1, \ldots, F_q\}$ by

$$\Delta = \langle F_1, \ldots, F_q \rangle$$

**Definition 2.4.** For a simplicial complex $\Delta$ having dimension $d$, its *f-vector* is a $d + 1$-tuple, defined as:

$$f(\Delta) = (f_0, f_1, \ldots, f_d)$$

where $f_i$ denotes the number of $i$-dimensional faces of $\Delta$.

**Definition 2.5. (Spanning Simplicial Complex )**

For a simple finite connected graph $G(V,E)$ with $s(G) = \{E_1, E_2, \ldots, E_s\}$ be the edge-set of all possible spanning trees of $G(V,E)$, we define a simplicial complex $\Delta_s(G)$ on $E$ such that the facets of $\Delta_s(G)$ are precisely the elements of $s(G)$, we call $\Delta_s(G)$ as the *spanning simplicial complex of $G(V,E)$*. In other words;

$$\Delta_s(G) = \langle E_1, E_2, \ldots, E_s \rangle.$$
For example; the spanning simplicial complex of the graph $G$ given in figure 1 is:

$$\Delta_s(G) = \langle \{e_2, e_3, e_4\}, \{e_1, e_3, e_4\}, \{e_1, e_2, e_4\}, \{e_1, e_2, e_3\} \rangle$$

We conclude this section with the definition of uni-cyclic graph $U_{n,m}$;

**Definition 2.6.** A uni-cyclic graph $U_{n,m}$ is a connected graph on $n$ vertices, and containing exactly one cycle of length $m$ (with $m \leq n$).

The number of vertices in $U_{n,m}$ equals the number of edges. In particular, if $m = n$ then $U_{n,m}$ is simply $n$-cyclic graph.

### 3. Spanning trees of $U_{n,m}$ and Stanley-Reisner ring $\Delta_s(U_{n,m})$

Throughout the paper, we fix the edge-labeling $\{e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ of $U_{n,m}$ such that $\{e_1, e_2, \ldots, e_m\}$ is the edge-set of the only cycle in $U_{n,m}$. In the following result, we give the characterization of $s(U_{n,m})$.

**Lemma 3.1.** Characterization of $s(U_{n,m})$

Let $U_{n,m}$ be the uni-cyclic graph with the edge set $E = \{e_1, e_2, \ldots, e_n\}$. A subset $E(T_i) \subset E$ will belong to $s(U_{n,m})$ if and only if $T_i = E \setminus \{e_i\}$ for some $i \in \{1, \ldots, m\}$. In particular;

$$s(U_{n,m}) = \{\tilde{E}_i \mid \tilde{E}_i = E \setminus \{e_i\} \text{ for all } 1 \leq i \leq m\}$$

**Proof.** As $U_{n,m}$ contains only one cycle of $m$ vertices, its spanning trees will be obtained by just removing one edge from the cycle of $U_{n,m}$ follows from [2,2]. Which implies that

$$s(U_{n,m}) = \{\tilde{E}_i \mid \tilde{E}_i = E \setminus \{e_i\} \text{ for all } 1 \leq i \leq m\}$$

□

We need the following elementary proposition in order to prove our next result.

**Proposition 3.2.** For a simplicial complex $\Delta$ over $[n]$ of dimension $d$, if $f_t = \binom{n}{t+1}$ for some $t \leq d$ then $f_i = \binom{n}{i+1}$ for all $0 \leq i < t$.

**Proof.** Suppose $\Delta$ be any simplicial complex over $[n]$ with dimension $d$ having $f_t = \binom{n}{t+1}$ for some $t \leq d$. It implies that $\Delta$ will contain all the subset of $[n]$ with the cardinality $t+1$ (which is $f_t = \binom{n}{t+1}$), then it is sufficient to prove that $\Delta$ will contain every subset of $[n]$ with the cardinality $|i|$ with $i \leq t$. Let us take any arbitrary subset $F$ of $[n]$ with $|F| < t+1$, then by adding more vertices to $F$ we can extend $F$ to $\tilde{F}$ with $|\tilde{F}| = t+1$, which is already in $\Delta$ therefore the assertion follows immediately from the definition of simplicial complex. Hence $\Delta$ will contain all the subsets of $[n]$ with the cardinality $\leq t$, that is

$$f_i = \binom{n}{i+1} \text{ for all } 0 \leq i < t.$$  

□

Our next result is the characterization of the $f$-vector of $\Delta_s(U_{n,m})$. 


Proposition 3.3. Let $\Delta_s(U_{n,m})$ be the spanning simplicial complex of uni-cyclic graph $U_{n,m}$, then $\dim(\Delta_s(U_{n,m})) = n - 2$ and having the following f-vector $f(\Delta_s(U_{n,m})) = (f_0, f_1, \ldots, f_{n-2})$ with
\[
f_i = \begin{cases} 
\binom{n}{i+1}, & \text{for } i \leq m - 2; \\
\binom{n}{i+1} - \binom{n-m}{i-m+1}, & \text{for } m - 2 < i \leq n - 2.
\end{cases}
\]

Proof. Let $E = \{e_1, e_2, \ldots, e_n\}$ be the set of edges of $U_{n,m}$, then from (3.1);
\[
s(U_{n,m}) = \{\hat{E}_i \mid E_i = E \setminus \{e_i\} \text{ for all } 1 \leq i \leq m\}.
\]

Therefore, by definition 2.5 we have;
\[
\Delta_s(U_{n,m}) = \langle \hat{E}_1, \hat{E}_2, \ldots, \hat{E}_m \rangle.
\]

Since each facet $\hat{E}_i$ is of the same dimension $n - 2$ (as $|\hat{E}_i| = n - 1$), therefore $\Delta_s(U_{n,m})$ will be of dimension $n - 2$. Also, it is clear from the definition of $\Delta_s(U_{n,m})$ that $\Delta_s(U_{n,m})$ contains all those subsets of $E$ that do not contain $\{e_1, \ldots, e_m\}$.

Let us take any arbitrary subset $F \subset E$ consisting of $m - 1$ members. As $F$ consists of $m - 1$ elements, then it is clear that $\{e_1, \ldots, e_m\}$ can not appear in $F$; therefore, $F \in \Delta_s(U_{n,m})$. It follows that $\Delta_s(U_{n,m})$ contains all possible subsets of $E$ with the cardinality $m - 1$, therefore, $f_{m-2} = \binom{n}{m-1}$. Hence from (3.2) we have $f_i = \binom{n}{i+1}$ for all $i \leq m - 2$.

In order to prove the other case, we need to compute all the subsets of $F \subset E$ with $|F| = i (\geq m)$ containing the cycle $\{e_1, \ldots, e_m\}$. We have in total $n$ elements in $E$ and we are choosing $i$-elements out of it with the condition that $\{e_1, \ldots, e_m\}$ will be a part of it. By using the inclusion exclusion principle, we get that there are $\binom{n-m}{i+m+1}$ subsets of $E$ consisting of $i + 1 (\geq m)$ elements and containing the cycle $\{e_1, \ldots, e_m\}$. In total, we have $\binom{n}{i+1}$ subsets of $E$ with the cardinality $i+1$, therefore, we have the $f_i = \binom{n}{i+1} - \binom{n-m}{i-m+1}$ for $m - 2 < i \leq n - 2$. \hfill $\square$

For a simplicial complex $\Delta$ over $[n]$, one would associate to it the Stanley-Reisner ideal, that is, the monomial ideal $I_N(\Delta)$ in $S = k[x_1, x_2, \ldots, x_n]$ generated by monomials corresponding to non-faces of this complex (here we are assigning one variable of the polynomial ring to each vertex of the complex). It is well known that the Stanley-Reisner ring $k[\Delta] = S/I_N(\Delta)$ is a standard graded algebra. We refer the readers to [6] and [8] for more details about graded algebra $A$, the Hilbert function $H(A, t)$ and the Hilbert series $H_t(A)$ of a graded algebra.

Definition 3.4. Let $A$ be a standard graded algebra and
\[
h(t) = h_0 + h_1 t + \cdots + h_r t^r
\]
the (unique) polynomial with integral coefficients such that $h(1) \neq 0$ and satisfying
\[
H_t(A) = \frac{h(t)}{(1 - t)^d}
\]
where $d = \dim(A)$. The $h$-vector of $A$ is defined by $h(A) = (h_0, \ldots, h_r)$. 


Now we give the formula for the $h$-vector of $k[\Delta_s(U_{n,m})]$;

**Theorem 3.5.** If $\Delta_s(U_{n,m})$ is a spanning simplicial complex of the uni-cyclic graph $U_{n,m}$ and $(h_i)$ is the $h$-vector of $k[\Delta_s(U_{n,m})]$, then $h_k = 0$ for $k > n - 1$ and

$$h_k = \begin{cases} \sum_{i=0}^{k} (-1)^{k-i} \binom{n-1-i}{k-i} \binom{n}{i}, & \text{for } k \leq m - 1; \\ \sum_{i=0}^{m-1} (-1)^{k-i} \binom{n-1-i}{k-i} \left[\binom{n}{i} - \binom{n-m}{i-m}\right], & \text{for } m - 1 < k \leq n - 1. \end{cases}$$

**Proof.** We know from [8] that, if $\Delta$ be any simplicial complex of dimension $d$ and $(h_i)$ be the $h$-vector of $k[\Delta]$, then $h_k = 0$ for $k > d + 1$ and

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{n-1-i}{k-i} f_{i-1} \quad \text{for } 0 \leq k \leq d + 1$$

The result follows by substituting the values $f_i$'s (from 3.3) in the above formula. □

One of our main results of this section is as follows;

**Theorem 3.6.** Let $\Delta_s(U_{n,m})$ be the spanning simplicial complex of $U_{n,m}$, then the Hilbert series of the Stanley-Reisner ring $k[\Delta_s(U_{n,m})]$ is given by,

$$H(k[\Delta_s(U_{n,m})], t) = \sum_{i=0}^{m-2} \frac{\binom{n}{i+1} t^{i+1}}{(1-t)^{i+1}} + \sum_{i=m-1}^{n-2} \frac{\binom{n}{i+1} - \binom{n-m}{i-m+1} t^{i+1}}{(1-t)^{i+1}} + 1.$$ 

**Proof.** We know from [8] that if $\Delta$ be any simplicial complex of dimension $d$ with $f(\Delta) = (f_0, f_1, \ldots, f_d)$ be its $f$-vector, then the Hilbert series of Stanley-Reisner ring $k[\Delta]$ is given by

$$H(k[\Delta], t) = \sum_{i=0}^{d} \frac{f_i t^{i+1}}{(1-t)^{i+1}} + 1.$$ 

The result immediately follows by substituting the values of $f_i$'s in the above formula from 3.3. □

Algebraic shifting theory was introduced by G. Kalai in [7], and it describes strong tolls to investigate one of the most interesting and powerful property of simplicial complexes.

**Definition 3.7.** A simplicial complex $\Delta$ on $[n]$ is **shifted** if, for $F \in \Delta$, $i \in F$ and $j \in [n]$ with $j > i$, one has $(F \setminus \{i\}) \cup \{j\} \in \Delta$.

**Theorem 3.8.** The spanning simplicial complex $\Delta_s(U(m,n))$ of the uni-cyclic graph is shifted.

**Proof.** From 2.5 and 3.1 we know that

$$\Delta_s(U_{n,m}) = \langle \hat{E}_1, \hat{E}_2, \ldots, \hat{E}_m \rangle.$$ 

It is sufficient to prove the shifted condition on the facets of the simplicial complex. For some facet $\hat{E}_i \in \Delta_s(U_{n,m})$ with $j \in \hat{E}_i$, we claim that
\((\hat{E}_i \setminus \{j\}) \cup \{k\} \in \{\hat{E}_1, \hat{E}_2, \ldots, \hat{E}_m\} \) with \( k \not\in \hat{E}_i \) and \( j < k \). By the definition of \( \hat{E}_i \), we have only one possibility for \( k \) that is \( k = i \) and \( j < i \leq m \), therefore, it is easy to see that \((\hat{E}_i \setminus \{j\}) \cup \{i\} = \hat{E}_i \) for all \( j < i \in \hat{E}_i \). Hence \( \Delta_s(U_{n,m}) \) is a shifted simplicial complex. □

The above theorem immediately implies the following result;

**Corollary 3.9.** The spanning simplicial complex \( \Delta_s(U(m,n)) \) is shellable.

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