Real Hamiltonian Forms of Affine Toda Models Related to Exceptional Lie Algebras

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Abstract. The construction of a family of real Hamiltonian forms (RHF) for the special class of affine 1 + 1-dimensional Toda field theories (ATFT) is reported. Thus the method, proposed in [1] for systems with finite number of degrees of freedom is generalized to infinite-dimensional Hamiltonian systems. The construction method is illustrated on the explicit nontrivial example of RHF of ATFT related to the exceptional algebras $E_6$ and $E_7$. The involutions of the local integrals of motion are proved by means of the classical $R$-matrix approach.

Key words: solitons; affine Toda field theories; Hamiltonian systems

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1 Introduction

To each simple Lie algebra rank $g$ one can relate Toda field theory (TFT) in 1 + 1 dimensions. It allows Lax representation: $[L, M] = 0$, where $L$ and $M$ are first order ordinary differential operators, see e.g. [2, 3, 4, 5, 6, 7]:

$$L\psi \equiv \left(i \frac{d}{dx} - i q_x(x,t) - \lambda J_0\right) \psi(x,t,\lambda) = 0, \quad q_x(x,t) = \sum_{k=1}^{r} q_k H_k,$$

$$M\psi \equiv \left(i \frac{d}{dt} - \frac{1}{\lambda} I(x,t)\right) \psi(x,t,\lambda) = 0,$$

whose potentials take values in $g$. Here $q(x,t) \in \mathfrak{h}$ is the Cartan subalgebra of $g$, $q(x,t) = \sum_{k=1}^{r} q_k \tilde{e}_k$ is its dual $r$-component vector, $r = \text{rank } g$. $H_k$ are the Cartan generators dual to the orthonormal basis elements $\tilde{e}_k$ in the root space, and

$$J_0 = \sum_{\alpha \in \pi} E_{\alpha}, \quad I(x,t) = \sum_{\alpha \in \pi} e^{-\langle \alpha, q(x,t) \rangle} E_{-\alpha}.$$

By $\pi_{\mathfrak{g}}$ we denote the set of admissible roots of $\mathfrak{g}$, i.e. $\pi_{\mathfrak{g}} = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ where $\alpha_1, \ldots, \alpha_r$ are the simple roots of $g$ and $\alpha_0$ is the minimal root of $g$. The corresponding TFT is known as the affine TFT. The Dynkin graph that corresponds to the set of admissible roots of $g$ is called
extended Dynkin diagrams (EDD). The equations of motion are of the form:

\[
\frac{\partial^2 \vec{q}}{\partial x \partial t} = \sum_{j=0}^{r} \alpha_j e^{-(\alpha_j, \vec{q}(x,t))}.
\]

The present paper extends the ideas of [8] and [9] to the ATFT related to the exceptional simple Lie algebra \(E_6\); for finite Toda chains see [10, 11, 1, 12].

2 The reduction group

The operators \(L\) and \(M\) are invariant with respect to the reduction group \(G_\mathbb{R} \simeq \mathbb{D}_h\) where \(h\) is the Coxeter number of \(g\). It is generated by two elements satisfying \(g_1^h = g_2^h = (g_1g_2)^2 = 1\) which allow realizations both as elements in \(\text{Aut}_g\) and in \(\text{Conf}_\mathbb{C}\). The invariance condition has the form \(2\):

\[
C_1(U(x,t,\kappa_1(\lambda))) = U(x,t,\lambda), \quad C_1(V(x,t,\kappa_1(\lambda))) = V(x,t,\lambda),
\]

\[
C_k(U(x,t,\kappa_k(\lambda))) = U(x,t,\lambda), \quad C_k(V(x,t,\kappa_k(\lambda))) = V(x,t,\lambda), \quad k = 2, 3.
\]

(2)

where \(U(x,t,\lambda) = -iq_x(x,t) - \lambda J_0\) and \(V(x,t,\lambda) = -\frac{1}{\lambda}I(x,t)\). Here \(C_k\) are automorphisms of finite order of \(g\), i.e. \(C_1^h = C_2^h = (C_1C_2)^2 = 1\) while \(\kappa_k(\lambda)\) are conformal mappings of the complex \(\Lambda\)-plane:

\[
\kappa_1(\lambda) = \omega \lambda, \quad \kappa_2(\lambda) = \lambda^*, \quad \kappa_3(\lambda) = (\omega \lambda)^*.
\]

where \(\omega = \exp(2\pi i/h)\). The algebraic constraints \(2\) are automatically compatible with the evolution. A number of nontrivial reductions of nonlinear evolution equations can be found in [13, 14].

3 Spectral properties of the Lax operator

The reduction conditions \(2\) lead to rather special properties of the operator \(L\). Along with \(L\) we will use also the equivalent system:

\[
\tilde{L}m(x,t,\lambda) \equiv \frac{d}{dx} - iq_x m(x,t,\lambda) - \lambda[J_0, m(x,t,\lambda)] = 0,
\]

(3)

where \(m(x,t,\lambda) = \psi(x,t,\lambda)e^{iJ_0x\lambda}\). Here \(\vec{q}_e\) is the potential which we choose to be a Schwartz-type function taking values in the complexified Cartan subalgebra \(h_\mathbb{C} \subset g\). This means that the boundary conditions for \(\vec{q}\) are determined up to the constant vector \(\vec{\rho}_0\):

\[
\vec{\rho}_0 = \lim_{x \to \pm\infty} \vec{q}(x,t).
\]

The change of the variables \(\vec{q}' = \vec{q} - \vec{\rho}_0\) does not affect the potential and the spectral data of \(\tilde{L}\), but they change the right hand side of the ATFT equations into:

\[
\frac{\partial^2 \vec{q}'}{\partial x \partial t} = \sum_{j=0}^{r} s_j \alpha_j e^{-(\alpha_j, \vec{q}'(x,t))},
\]

where \(s_j = \exp[-(\alpha_j, \vec{\rho}_0)]\) obviously satisfy the consistency conditions:

\[
\prod_{j=0}^{r} s_j^{n_j} = \exp \left( - \sum_{j=0}^{r} (n_j \alpha_j, \vec{\rho}_0) \right) = 1.
\]
where \( n_j \) are the minimal positive integers for which \( \sum_{j=0}^{r} n_j \alpha_j = 0 \). In the literature there are two canonical ways of fixing up the vector \( \vec{\rho}_0 \). The first one is to put \( s_j = 1 \) for all \( j \), the second is to have \( s_j = n_j \) for all \( j \).

The spectral properties of the Lax operator are rather involved due to the fact that both \( J_0 \) and \( I(x,t) \) have complex-valued eigenvalues. In fact all these eigenvalues are constant and are proportional to \( \omega_k \), \( k = 0, 1, \ldots, h - 1 \), where \( \omega = e^{2\pi i/h} \) and \( h \) is the Coxeter number of \( g \). As a result one finds that the continuous spectrum of \( L(\lambda) \) fills up \( 2h \) rays passing through the origin:

\[
\lambda \in l_\nu : \arg \lambda = \left( \frac{\nu - 1}{h} \right) \pi.
\]

For such operators one can introduce Jost solutions only for potentials on a compact support [15, 16]:

\[
\lim_{x \to \infty} \Psi(x,t,\lambda)e^{i\lambda J_0 x + (i/\lambda) I_0 t} = 1, \quad \lim_{x \to -\infty} \Phi(x,t,\lambda)e^{i\lambda J_0 x + (i/\lambda) I_0 t} = 1,
\]

where

\[
I_0 = \lim_{x \to \infty} I(x,t) = \sum_{k=0}^{r} E_{-\alpha_k}, \quad \alpha_k \in \pi(g).
\]

Then the corresponding scattering matrix \( T(t,\lambda) \) is defined by

\[
T(t,\lambda) = (\Psi(x,t,\lambda))^{-1}\Phi(x,t,\lambda).
\]

The compactness of the potential ensures that the Jost solutions \( \Phi(x,t,\lambda) \), \( \Psi(x,t,\lambda) \) and the scattering matrix \( T(t,\lambda) \) are meromorphic functions of \( \lambda \).

The fact that \( \Phi(x,t,\lambda) \) and \( \Psi(x,t,\lambda) \) are also Jost solutions of \( M(\lambda) \) means that \( T(t,\lambda) \) evolves according to:

\[
i \frac{dT}{dt} - \frac{1}{\lambda} [I_0, T(t,\lambda)] = 0.
\] (4)

The limiting procedure to non-compact potentials can be done only for the so-called fundamental analytical solutions (FAS) \( m_\nu(x,t,\lambda) \), \( \lambda \in \Omega_\nu \), i.e. \( (\nu - 1)\pi/h \leq \arg \lambda \leq \nu\pi/h \). Their construction is outlined in [15, 16] for generic complex-valued \( J_0 \). Our Lax pair is special, because it satisfies the reduction condition [2] with the Coxeter automorphism [17]:

\[
\bar{C}_1(X) \equiv C_1^{-1}X C_1, \quad C_1 = e^{2\pi i H_\rho/h}, \quad \rho = \frac{1}{2} \sum_{\alpha > 0} \alpha;
\]

obviously \( C_1^h = 1 \) and \( \bar{C}_1(J_0) = \omega^{-1} J_0 \). The FAS \( m_\nu(x,t,\lambda) \) of [2] are defined only in the sector \( \Omega_\nu \) and satisfy:

\[
\bar{C}_1(m_\nu(x,t,\omega \lambda)) = m_{\nu-2}(x,t,\lambda), \quad \lambda \in l_{\nu-2}.
\]

Thus the inverse scattering problem for \( L(\lambda) \) can be formulated as a Riemann–Hilbert problem for \( m_\nu(x,t,\lambda) \) on the continuous spectrum \( \Sigma \equiv \bigcup_{\nu=1}^{2h} l_\nu \).

Skipping the details, we remark that equation (4) allows one to show that ATFT possess generating functionals of the integrals of motion. This can be shown easily if the potential \( q(x,t) \) is such that \( T(t,\lambda) \) can be diagonalized:

\[
T(t,\lambda) = u_0^{-1}(t,\lambda) D(\lambda) u_0(t,\lambda).
\] (5)
All factors in the above equation take values in the Lie group $G$ with the Lie algebra $g$ and $D(\lambda)$ is a diagonal matrix. Then there exist a set of functions $\tau_k(\lambda)$, $k = 1, \ldots, r$ such that:

$$D(\lambda) = \exp \left( \sum_{k=1}^{r} \frac{2\tau_k(\lambda)}{\alpha_k, \alpha_k} H_{\alpha_k} \right),$$

where $H_{\alpha_k}$ are the Cartan generators of $g$ corresponding to the simple root $\alpha_k$. Inserting (5) and (6) into (4) one finds that

$$\frac{dD}{dt} = 0,$$

i.e. $$\frac{d\tau}{dt} = 0,$$

for all $k = 1, \ldots, r$. Obviously all eigenvalues of the scattering matrix $T(t, \lambda)$ will be expressed in terms of $\tau_k$. Each $\tau_k(\lambda)$ can be expanded over the negative powers of $\lambda$:

$$\tau_k(\lambda) = \sum_{s=0}^{\infty} I_s^{(k)} \lambda^{-s},$$

and all the coefficients $I_s^{(k)}$ will be integrals of motion.

### 4 Real Hamiltonian forms

The Lax representations of the ATFT models (see e.g. [2, 3, 4, 18, 6] and the references therein) are related mostly to the normal real form of the Lie algebra $g$, see [20].

Our aim here is to:

1) generalize the ATFT to complex-valued fields $\vec{q}^C = \vec{q}^0 + i\vec{q}^1$, and to

2) describe the family of RHF of these ATFT models.

We also provide a tool generalizing of the one in [1] for the construction of new inequivalent RHF’s of the ATFT. The ATFT for the algebra $sl(n)$ can be written down as an infinite-dimensional Hamiltonian system as follows:

$$\frac{dq_k}{dt} = \{q_k, H_{ATFT}\}, \quad \frac{dp_k}{dt} = \{p_k, H_{ATFT}\},$$

$$H_{ATFT} = \int_{-\infty}^{\infty} dx \left( \frac{1}{2}(\vec{p}(x,t), \vec{p}(x,t)) + \sum_{k=0}^{r} e^{-\langle \vec{q}(x,t), \alpha_k \rangle} \right),$$

where $\vec{q}(x,t)$ and $\vec{p} = \partial \vec{q} / \partial x$ are the canonical coordinates and momenta satisfying canonical Poisson brackets:

$$\{p_k(x), q_j(y)\} = \delta_{jk} \delta(x-y).$$

Next we define the involution $C$ acting on the phase space $\mathcal{M}$ as follows:

1) $C(F(p_k, q_k)) = F(C(p_k), C(q_k))$,

2) $C(\{F(p_k, q_k), G(p_k, q_k)\}) = \{C(F), C(G)\}$,

3) $C(H(p_k, q_k)) = H(p_k, q_k)$.

Here $F(p_k, q_k)$, $G(p_k, q_k)$ and the Hamiltonian $H(p_k, q_k)$ are functionals on $\mathcal{M}$ depending analytically on the fields $q_k(x,t)$ and $p_k(x,t)$.
The set of admissible roots for this algebra is symmetries of the extended Dynkin diagrams derived in [6].

Calgebras with height 1. We show that the involutions in this Section we provide several examples of RHF of ATFT related to exceptional Kac–Moody constructed from the $Z$

The examples below illustrate the procedure outlined above and display new types of ATFT.

5.1  $E_6^{(1)}$ Toda field theories

The set of admissible roots for this algebra is

$$\begin{align*}
\alpha_1 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \\
\alpha_2 &= e_1 + e_2, \\
\alpha_3 &= e_2 - e_1, \\
\alpha_4 &= e_3 - e_2, \\
\alpha_5 &= e_4 - e_3, \\
\alpha_6 &= e_5 - e_4, \\
\alpha_0 &= -\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8),
\end{align*}$$

where $\alpha_1, \ldots, \alpha_6$ form the set of simple roots of $E_6$ and $\alpha_0$ is the minimal root of the algebra. This is the standard definition of the root system of $E_6$ embedded into the 8-dimensional Euclidean space $E^8$. The root space $E_6$ of the algebra $E_6$ is the 6-dimensional subspace of $E^8$ orthogonal to
the vectors $e_7 + e_8$ and $e_6 + e_7 + 2e_8$. Thus any vector $\vec{q}$ belonging to $\mathbb{E}_6$ has only 6 independent coordinates and can be written as:

$$\vec{q} = \sum_{k=1}^{5} q_k e_k + q_6 e_6', \quad e_6' = \frac{1}{\sqrt{3}}(e_6 + e_7 - e_8).$$

(8)

Let us fix up the action of the involution $\mathcal{C}$ on a generic vector $\vec{q}$ in $\mathbb{E}^8$ by:

$$\mathcal{C}(q_k) = -q_{5-k} + \frac{1}{2} \sum_{m=1}^{4} q_m, \quad \text{for} \quad k = 1, \ldots, 4,$n

$$= q_{13-k} - \frac{1}{2} \sum_{m=5}^{8} q_m, \quad \text{for} \quad k = 5, \ldots, 8.$$

(9)

This action is compatible with the $\mathbb{Z}_2$-symmetry $\mathcal{C}^\#$ of the extended Dynkin diagram (see Fig. 1) and reflects an involution of the Kac–Moody algebra $\mathbf{E}_6^{(1)}$, see [21]. It acts on the root space as follows:

$$\mathcal{C}^\# e_k = -e_{5-k} + \frac{1}{2} \sum_{m=1}^{4} e_m, \quad \text{for} \quad k = 1, \ldots, 4,$n

$$= e_{13-k} - \frac{1}{2} \sum_{m=5}^{8} e_m, \quad \text{for} \quad k = 5, \ldots, 8,$$

$$\mathcal{C}^\# \alpha_1 = \alpha_6, \quad \mathcal{C}^\# \alpha_3 = \alpha_5, \quad \mathcal{C}^\# \alpha_4 = \alpha_4, \quad \mathcal{C}^\# \alpha_k = \alpha_k, \quad k = 0, 2, 4.$$  

(10)

![Figure 1. $\mathbf{E}_6^{(1)} \rightarrow \mathbf{F}_4^{(1)}$.](image)

The involution $\mathcal{C}^\#$ splits the root space $\mathbb{E}_6$ into a direct sum of its eigensubspaces: $\mathbb{E}_6 = \mathbb{E}_+ \oplus \mathbb{E}_-$ with dim $\mathbb{E}_+ = 4$, dim $\mathbb{E}_- = 2$. The vectors:

$$\vec{e}_1 = \frac{1}{2}(e_5 - \sqrt{3}e_6), \quad \vec{e}_2 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4), \quad \vec{e}_3 = \frac{1}{2}(-e_1 - e_2 + e_3 + e_4),$$

$$\vec{e}_4 = \frac{1}{2}(-e_1 + e_2 - e_3 + e_4), \quad \vec{e}_5 = \frac{1}{2}(-e_1 + e_2 + e_3 - e_4), \quad \vec{e}_6 = \frac{1}{2}(\sqrt{3}e_5 + e_6').$$

form an orthonormal basis in $\mathbb{E}_6$. The first four satisfy $\mathcal{C}^\# \vec{e}_k = \vec{e}_k, k = 1, \ldots, 4$, so they span $\mathbb{E}_+$; the last two span $\mathbb{E}_-$ because $\mathcal{C}^\# \vec{e}_j = -\vec{e}_j, j = 5, 6$. In terms of $\vec{e}_k$ the admissible root system of $\mathbf{F}_4^{(1)}$ takes the standard form:

$$\beta_0 = -\vec{e}_2 - \vec{e}_1, \quad \beta_1 = \frac{1}{2}(e_1 - \vec{e}_2 - \vec{e}_3 - \vec{e}_4),$$

$$\beta_2 = \vec{e}_2 - \vec{e}_3, \quad \beta_3 = \vec{e}_4, \quad \beta_4 = \vec{e}_3 - \vec{e}_4.$$

satisfying $\beta_0 + 2\beta_1 + 2\beta_2 + 4\beta_3 + 3\beta_4 = 0$. 
Let us take the complex vector \( \vec{q}(x,t) = q^0(x,t) + i q^1(x,t) \in \mathbb{E}_6 \) (i.e., of the form (8)) and let \( \vec{p}(x,t) = \partial \vec{q}/\partial x \). Let us denote their projections onto \( \mathbb{E}_\pm \) by \( \vec{q}_\pm \) and \( \vec{p}_\pm \) respectively. Then the densities \( H_1^R, \omega_1^R \) for the RHF of AFTF equal:

\[
H_1^R = \frac{1}{2} \left( (p_+^0(x,t), p_+^0(x,t)) - (p_-^0(x,t), p_-^0(x,t)) \right) + e^{-\langle \vec{q}_+^0(x,t) \rangle_{\beta_0}} + 2e^{-\langle \vec{q}_+^0(x,t), \beta_1 \rangle} \cos(\langle \vec{q}_+^1(x,t), \vec{e}_5 + \sqrt{3} \vec{e}_6 \rangle) + 2e^{-\langle \vec{q}_+^0(x,t), \beta_2 \rangle} + 4e^{-\langle \vec{q}_+^0(x,t), \beta_3 \rangle} \cos(\langle \vec{q}_+^1(x,t), \vec{e}_5 \rangle) + 3e^{-\langle \vec{q}_+^0(x,t), \beta_4 \rangle},
\]

\[
\omega_1^R = (\delta \vec{p}_+(x) \wedge \delta \vec{q}_+(x)) - (\delta \vec{p}_-(x) \wedge \delta \vec{q}_-(x)),
\]

If we put \( \vec{q}_-(x,t) = 0 \) then also \( \vec{p}_-(x,t) = 0 \) and we get the reduced ATFT related to the Kac–Moody algebra \( F_4^{(1)} \).

5.2 \( E_7^{(1)} \) Toda field theories

The set of admissible roots for this algebra is

\[ \alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \quad \alpha_2 = e_1 + e_2, \]
\[ \alpha_3 = e_2 - e_1, \quad \alpha_4 = e_3 - e_2, \quad \alpha_5 = e_4 - e_3, \quad \alpha_6 = e_5 - e_4, \]
\[ \alpha_7 = e_6 - e_5, \quad \alpha_0 = e_7 - e_8, \]

where \( \alpha_1, \ldots, \alpha_7 \) form the set of simple roots of \( \mathbb{E}_7 \) and \( \alpha_0 \) is the minimal root of the algebra. This is the standard definition of the root system of \( \mathbb{E}_7 \) embedded into the 8-dimensional Euclidean space \( \mathbb{E}^8 \). The root space \( \mathbb{E}_7 \) of the algebra \( \mathbb{E}_7 \) is the 7-dimensional subspace of \( \mathbb{E}^8 \) orthogonal to the vector \( e_7 + e_8 \). Thus any vector \( \vec{q} \) belonging to \( \mathbb{E}_7 \) has 7 independent coordinates and can be written as:

\[
\vec{q} = \sum_{k=1}^{6} q_k e_k + q e_7', \quad e_7' = \frac{1}{\sqrt{2}}(e_7 - e_8).
\] (11)

Let us fix up the action of the involution \( \mathcal{C} \) on a generic vector \( \vec{q} \) in \( \mathbb{E}^8 \) by equation (9). This action is compatible with the \( \mathbb{Z}_2 \)-symmetry \( \mathcal{C}^\# \) of the extended Dynkin diagram (see Fig. 2) and reflects an involution of the Kac–Moody algebra \( \mathbb{E}_7^{(1)} \), see [21]. It acts on the root space as in equation (10) above.

![Figure 2. The \( \mathbb{E}_7^{(1)} \) \( \rightarrow \mathbb{E}_6^{(2)} \) RHF of affine TFT.](image)

However its action on the root space \( \mathbb{E}_7 \) is different. It splits \( \mathbb{E}_7 \) into a direct sum of its eigensubspaces: \( \mathbb{E}_7 = \mathbb{E}_+ \oplus \mathbb{E}_- \) with \( \dim \mathbb{E}_+ = 4 \), \( \dim \mathbb{E}_- = 3 \). The vectors:

\[ \vec{e}_1 = \frac{1}{2\sqrt{2}}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - \sqrt{2} e_7'), \quad \vec{e}_3 = \frac{1}{\sqrt{2}}(-e_1 + e_4), \]
\[ \vec{e}_2 = \frac{1}{2\sqrt{2}}(-e_1 - e_2 - e_3 - e_4 + e_5 - e_6 - \sqrt{2} e_7'), \quad \vec{e}_4 = \frac{1}{\sqrt{2}}(-e_2 + e_3), \]
form an orthonormal basis in $\mathbb{E}_+$. Indeed, they satisfy $(\bar{e}_j, \bar{e}_k) = \delta_{jk}$ and $C^\# \bar{e}_k = \bar{e}_k$, $k = 1, \ldots, 4$. In terms of $\bar{e}_k$ the admissible root system of $\mathbb{E}^{(2)}_6$ takes the standard form:

\[
\begin{align*}
\beta_0 &= -2\bar{e}_2 - \bar{e}_1, \\
\beta_1 &= \bar{e}_2 - \bar{e}_3, \\
\beta_2 &= \bar{e}_3 - \bar{e}_4, \\
\beta_3 &= 2\bar{e}_4, \\
\beta_4 &= \bar{e}_1 - \bar{e}_2 - \bar{e}_3 - \bar{e}_4,
\end{align*}
\]

satisfying $\beta_0 + 2\beta_1 + \beta_2 + 3\beta_3 + 2\beta_4 = 0$.

The vectors \(\bar{e}_5 = \frac{1}{2}(-e_1 + e_2 + e_3 - e_4)\), \(\bar{e}_6 = \frac{1}{\sqrt{2}}(e_5 + e_6)\), \(\bar{e}_7 = \frac{1}{2}(-e_5 + e_6 - \sqrt{2}e_7)\), span $\mathbb{E}_-$ because $C^\# \bar{e}_j = -\bar{e}_j$, $j = 5, 6, 7$.

Let us take the complex vector $\bar{q}(x, t) = q^0(x, t) + iq^1(x, t) \in \mathbb{E}_7$ (i.e., of the form (11)) and let $\bar{p}(x, t) = \partial \bar{q}/\partial x$. Let us denote their projections onto $\mathbb{E}_\pm$ by $\bar{q}_\pm$ and $\bar{p}_\pm$ respectively. Then the densities $\mathcal{H}^R_1$, $\omega^R_1$ for the RHF of ATFT equal:

\[
\mathcal{H}^R_2 = \frac{1}{2} \left((\bar{p}^0_+(x, t), \bar{p}^0_+(x, t)) - (\bar{p}^0_-(x, t), \bar{p}^0_-(x, t))\right) + 2e^{-(\bar{q}^0_-(x, t), \beta_0)} \cos((\bar{q}^1_-(x, t), \bar{e}_7))
\]

\[
+ 4e^{-(\bar{q}^0_-(x, t), \beta_1)} \cos((\bar{q}^1_-(x, t), \frac{1}{2}(\bar{e}_5 + \sqrt{2}\bar{e}_6 - \bar{e}_7)))
\]

\[
+ 6e^{-(\bar{q}^0_+(x, t), \beta_3)} \cos((\bar{q}^1_+(x, t), \bar{e}_5)) + 4e^{-(\bar{q}^0_+(x, t), \beta_4)},
\]

\[
\omega^R_2 = (\delta \bar{p}_+(x) \wedge \delta \bar{q}_+(x)) - (\delta \bar{p}_-(x) \wedge \delta \bar{q}_-(x)).
\]

Again, if we put $\bar{q}_-(x, t) = 0$ then also $\bar{p}_-(x, t) = 0$ and we get the reduced ATFT related to the Kac–Moody algebra $\mathbb{E}_6^{(2)}$ [6].

6 Classical R-matrix method and ATFT

There are several methods to approach the Hamiltonian properties of the ATFT, see e.g. [13, 18, 11, 19]. One of the effective methods is based on the well known classical $R$-matrix [22] which is introduced by:

\[
\{U(x, \lambda) \otimes U(y, \mu)\} = [R(\lambda, \mu), U(x, \lambda) \otimes 1 + 1 \otimes U(y, \mu)] \delta(x - y),
\]

where $\{U(x, \lambda) \otimes U(y, \mu)\}_{ij,kl} = \{U_{ij}(x, \lambda), U_{kl}(y, \mu)\}$. In order to apply this definition effectively we make use of another Lax operator for the ATFT:

\[
\tilde{L} \psi(x, \lambda) \equiv i \frac{d\psi}{dx} + U(x, \lambda) \tilde{\psi}(x, \lambda) = 0,
\]

\[
U(x, \lambda) = -\frac{i}{2} \sum_{j=1}^r q_{j,x} H_j - \lambda \sum_{j=0}^r e^{-(\alpha_j, \phi)/2} E_{\alpha_j},
\]

which is gauge equivalent to $L$ [11]. Since the Poisson brackets are introduced by (17) the left hand side of (12) takes the form:

\[
\{U_{ij}(x, \lambda) \otimes U_{kl}(y, \mu)\} = \frac{i}{4} \sum_{k=0}^r e^{-(\alpha_k, \phi)/2} (\mu H_{\alpha_k} \otimes E_{\alpha_k} - \lambda E_{\alpha_k} \otimes H_{\alpha_k}) \delta(x - y).
\]
Thus equation (12) becomes an over-determined set of equations for $R(\lambda, \mu)$ which is solved by [22]:

$$R(\lambda, \mu) = \frac{1}{4i} \left( \lambda^h + \mu^h \right) + \frac{1}{2i} \sum_{k=1}^{r} H_k \otimes H_k + \frac{1}{2i} \sum_{\alpha \in \Delta} \left( \frac{\lambda^p(\alpha)}{\mu} \right) \frac{\mu^h E_{\alpha} \otimes E_{-\alpha}}{\lambda^h - \mu^h}$$

$$= \frac{1}{4i \sinh(h \eta/2)} \left( \cosh(h \eta/2) \sum_{k=1}^{r} H_k \otimes H_k + \sum_{\alpha \in \Delta} e^{(p(\alpha)-h/2)\eta} E_{\alpha} \otimes E_{-\alpha} \right),$$

where $h$ is the Coxeter number of $\mathfrak{g}$, $\eta = \ln(\lambda/\mu)$ and $p(\alpha)$ is the height of the root $\alpha$ modulo $h$. If $\alpha = \sum_{j=1}^{r} n_{\alpha,j} \alpha_j$ where $n_{\alpha,j}$ are integers, then $p(\alpha) = \sum_{j=1}^{r} n_{\alpha,j}$.

The relation (12) allows one to derive the Poisson brackets between the matrix elements of the fundamental solutions $T_{x_0}(x, \lambda)$ and $T_{x_0}^{-1}(x, \lambda)$ (or the scattering matrix $T_{x_0}(\lambda)$) of $L$ which are defined by:

$$LT_{x_0}^+(x, \lambda) = 0, \quad \lim_{x \to \pm x_0} T_{x_0}^\pm(x, \lambda)e^{-i\lambda x J_0} = 1,$$

$$T_{x_0}(\lambda) = (T_{x_0}^-(x, \lambda))^{-1}T_{x_0}^+(-x, \lambda).$$

Then

$$\{ T_{x_0}^\pm(x, \lambda) \otimes T_{x_0}^\pm(y, \mu) \} = [R(\lambda, \mu), T_{x_0}^\pm(x, \lambda) \otimes T_{x_0}^\pm(y, \mu)],$$

and

$$\{ T_{x_0}(\lambda) \otimes T_{x_0}(\mu) \} = [R(\lambda, \mu), T_{x_0}(\lambda) \otimes T_{x_0}(\mu)].$$

(13)

These results hold true for potentials on compact support provided we choose $x_0$ large enough so that $\tilde{q}_x = 0$ for $|x| > x_0$.

With $T_{x_0}(t, \lambda)$ we can associate a set of generating functionals $\tau_k(x_0, \lambda)$ of the integrals of motion

$$T_{x_0}(t, \lambda) = u_0^{-1}(x_0, t, \lambda) D_{x_0}(\lambda) u_0(x_0, t, \lambda), \quad D_{x_0}(\lambda) = \exp \left( \sum_{k=1}^{r} 2\tau_k(x_0, \lambda) H_{\alpha_k} \right).$$

Again we can write the expansion

$$\tau_k(x_0, \lambda) = \sum_{s=0}^{\infty} I_{x_0,k}^{(s)} \lambda^{-s}.\]$$

An important consequence of (13) is that the functions $\tau_k(x_0, \lambda)$ are in involution. Indeed from [13] there follows that:

$$\{ \text{tr} T_{x_0}^k(\lambda), \text{tr} T_{x_0}^p(\mu) \} = 0,$$

(14)

for any pair of integers $k, p$. Obviously $\text{tr} T^k(\lambda)$ can be expressed in terms of the invariants $\tau_k(\lambda)$ [6] of the scattering matrix $T(\lambda) \in \mathcal{G}$. Therefore from (14) we find that

$$\{ \tau_k(x_0, \lambda), \tau_m(x_0, \mu) \} = 0, \quad 1 \leq k, m \leq r,$$

i.e. the integrals of motion $I_k^{(s)}$ are all in involution:

$$\{ I_k^{(s)}, I_k^{(n)} \} = 0, \quad 0 \leq k, m \leq r, \quad s, n \geq 0.$$
In order to derive the corresponding results for the RHF of the ATFT we have to use the fact that the automorphism $\mathcal{C}$ induces an automorphism $\mathcal{C}^\vee$ on the Lie algebra $\mathfrak{g}$ and on the Lax operator as follows [23]:

$$C^\vee(H_j) = H_{C^\#(e_j)}, \quad C^\vee E_\alpha = E_{C^\#(\alpha)}, \quad L(C(q^*), x, \lambda^*) = C^\vee(L(q^*, x, \lambda))^*.$$ 

These relations allow one to prove that the fundamental solution $T_{x_0}(x, \lambda)$ and the scattering matrix $T_{x_0}(\lambda)$ for the corresponding RHF model has the properties:

$$(T_{x_0}(x, \lambda^*))^* = C^\vee(T_{x_0}(x, \lambda)), \quad (T_{x_0}(\lambda^*))^* = C^\vee(T_{x_0}(\lambda)),$$

and as a consequence: $(\tau_k(\lambda^*))^* = \tau_{\bar{k}}(\lambda))$, where $\bar{k}$ is defined through $C^\#(\alpha_k) = \alpha_{\bar{k}}$.

7 Conclusions

The RHF of the ATFT models related to the exceptional Kac–Moody algebras $E_6^{(1)}$ and $E_7^{(1)}$ are constructed. These models generalize the ones in [6] since they contain two types of fields $\vec{q}_+(x, t)$ and $\vec{q}_-(x, t)$ with different properties with respect to the involution $\mathcal{C}$. The models in [6] contain only fields invariant with respect to $\mathcal{C}$.

We outlined the derivation of the Hamiltonian properties through the classical $R$-matrix approach.

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