Resummation of Threshold Logarithms in Effective Field Theory for DIS, Drell-Yan and Higgs Production

Ahmad Idilbi,1,∗ Xiangdong Ji,1,2,† and Feng Yuan3,‡

1Department of Physics, University of Maryland, College Park, Maryland 20742, USA
2Department of Physics, Peking University, Beijing, 100871, P. R. China
3RIKEN/BNL Research Center, Building 510A, Brookhaven National Laboratory, Upton, NY 11973

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Abstract

We apply the effective field theoretic (EFT) approach to resum the large perturbative logarithms arising when partonic hard scattering cross sections are taken to the threshold limit. We consider deep inelastic scattering, Drell-Yan lepton pair production and the standard model Higgs production through gluon-gluon fusion via a heavy-top quark loop. We demonstrate the equivalence of the EFT approach with the more conventional, factorization-based methods to all logarithmic accuracies and to all orders in perturbation theory. Specific EFT results are shown for the resummation up to next-to-next-to-next-to leading logarithmic accuracy for the above-mentioned processes. We emphasize the relative simplicity by which we derive most of the results and more importantly their clear physical origin. We find a new relation between the functions \( f(q,g) \) in the quark and gluon form factors and the matching coefficients in Drell-Yan and Higgs production, which may explain their universality believed to hold to all orders in perturbation theory.

∗Electronic address: idilbi@physics.umd.edu
†Electronic address: xji@physics.umd.edu
‡Electronic address: fyuan@quark.phy.bnl.gov
I. INTRODUCTION

Factorization theorems \cite{1} for inclusive hard scattering processes are our main tool by which we quantitatively analyze cross sections, with hadrons involved, when a generic hard scale $Q^2$ is taken to infinity. Taking the Drell-Yan (DY) lepton-pair production as an example, it is well known that the cross section can be expressed as a convolution of a hard scattering coefficient function (or “Wilson coefficient”) calculated perturbatively, and a non-perturbative, universal, parton distribution function (PDF) for each one of the incoming hadrons. Corrections to the factorized cross section scale as $1/(Q^2)^n$ where $n \geq 1$ up to some logarithmic ratios. However, it is also well known that fixed order, pQCD calculation of the Wilson coefficient yields singular distribution functions of the form

$$\alpha_s^k \left[ \frac{\ln^{m-1}(1-z)}{(1-z)} \right]_+, \, (m \leq 2k)$$

where $z = Q^2/\hat{s}$ and $\hat{s}$ is the total momentum squared of the incoming partons. The “plus” distributions are defined in the usual way. The appearance of such distributions is a result of an emission of soft and/or collinear gluons into the final state. When such distributions are Mellin transformed to the conjugate space, logarithms of the form $\alpha_s^k \ln^m N$ ($m = 2k, 2k-1, ..., 0$) show up where $N \equiv N \exp(\gamma_E)$ is the conjugate variable of $z$ and $\gamma_E$ is the Euler constant. In the limit $z \to 1$ or, equivalently, large $N$, fixed order perturbative calculation cannot be reliably trusted and an all order resummation of the large logarithms is needed. This is what is generically meant by “threshold resummation”. This notion has evolved during the last 20 years into one of the most studied and highly developed subjects within perurbative quantum chromodynamics (pQCD). Earlier studies \cite{2, 3} supplied a sound and rigorous (although complicated) treatment to perform such resummation. In both of these works, resummation is performed after establishing some sort of factorized cross section into well-defined quantities (at the operator level) that capture the physics at the hard, jet and soft scales. An integral transformation to the conjugate space is then applied in order to de-convolute the various terms in the cross section. Then, in the conjugate space, energy evolution equations are solved and the exponentials thus obtained contain the resummed large logarithms. Thus the perturbative expansion is put under control and the contributions obtained from the yet uncalculated higher orders in $\alpha_s$ reduce the theoretical uncertainty inherent in any fixed order calculation. Thereby better phenomenological studies can be carried out and better agreement with experimental data is usually obtained. More recent studies have further developed and refined this topic \cite{4, 5, 6, 7, 8, 9}.

In this paper we adopt the effective field theory approach (EFT) to resum the threshold large logarithms. This approach was first applied to deep inelastic (non-singlet) structure function in the limit $x \to 1$ where $x$ is the partonic Bjorken variable \cite{10}. Later on it was applied to the DY process in the limit $z \to 1$ \cite{11}. In both cases, resummation was performed up to next-to-leading logarithms (NLL). The implementation of the EFT methodology to resum threshold logarithms is made more concrete due to the recently developed “soft collinear effective theory” (SCET) \cite{12, 13}.

The SCET describes interactions between soft and collinear partons. It is the most appropriate framework to calculate contributions from the soft-collinear limit of the full QCD calculations (which is more commonly known as the “soft limit”). Therefore any perturbative calculations within SCET has to reproduce the same results as the full QCD in that limit. To $\mathcal{O}(\alpha_s)$, this has been verified explicitly for DIS and DY \cite{10, 11}. This is also
the case when one considers distributions at small transverse momentum \[14, 15\]. Moreover, it was also shown that the one loop diagrams (the form factor type of diagrams) calculated in SCET have the same infrared (IR) pole structure as the full QCD calculation \[10, 16\]. [The result in \[10\] for the collinear diagrams involve mixed poles of IR and ultraviolet (UV) divergences. This would be treated by applying the “zero-bin” subtraction \[17\].] These observations have to be valid to higher orders in the strong coupling. This allows us to extract the relevant quantities needed to perform resummation from the full QCD calculations as we shall see below.

The EFT resummation program described here is conceptually simple and has been explained in detail in \[18\]. The starting point (and again considering DY as an example) is the collinearly factorized inclusive cross section in moment space

\[
\sigma_N = \sigma_0 \cdot G_N(Q) \cdot q(Q, N) \cdot q(Q, N),
\]

(2)

where \(\sigma_0\) is the Born level cross section, \(q(Q, N)\) is the PDF of partons in hadrons and,

\[
G_N(Q) = |C(\alpha_s(Q^2))|^2 e^{I_1(Q/\mu_I, \alpha_s(Q^2))} \times M_N(\alpha_s(Q^2)) e^{I_2(Q/\mu_I, \alpha_s(\mu_I^2))} e^{I_3(Q/\mu_I, \alpha_s(Q))}.
\]

(3)

Explicit expressions for the various contributions in \(G_N\) will be given below; however, we want to comment on their physical origin. \(C(\alpha_s(Q^2))\) contains the non-logarithmic contribution of the purely virtual diagrams and the first exponent \(I_1\) contains all the logarithms originating from the same type of diagrams. Both quantities are obtained from the matching procedure at the scale \(Q\), and the running between \(Q\) and the intermediate scale \(\mu_I\). This, in turn, is controlled by the anomalous dimension of the EFT current to be denoted by \(\gamma_1\).

The intermediate scale \(\mu_I\) shows up when real gluons are emitted so one must consider the cross section with real gluon emissions. The result will have both soft and collinear divergences. When taking into account the IR poles from the virtual diagrams (in the EFT approach this is done by taking into account the contribution from the counterterms of the effective operators), the total contribution will contain only collinear divergences to be absorbed into a product of two PDFs. The conclusion is that the matching procedure at the intermediate scale is guaranteed to work to all orders in perturbation theory, following the factorization theorem, as long as the EFT used generates the full QCD results in the appropriate kinematical limit and one gets the matching coefficient \(M_N(\alpha_s(\mu_I^2))\) which by definition is finite in the non-regulated theory. This quantity has to be free of any logarithms. \(I_2\) collects all the logarithms that are due to the evolution of the PDF between \(\mu_I\) and the factorization scale \(\mu_F\). This is controlled by the anomalous dimension of the PDF to be denoted by \(\gamma_2\). \(I_3\) encodes all the contributions due to the running of the coupling constant between the matching scales \((Q, \mu_I)\) and the final factorization scale \(\mu_F\). All the large logarithms appear only in the exponents and the term \(|C(\alpha_s(Q^2))|^2 M_N(\alpha_s(Q^2))\) is free of any large logarithms. In Eq. (3) we have chosen \(\mu_F = Q\) for simplicity.

The above formalism must be contrasted with the more conventional, factorization based one. It will be shown that the EFT approach is equivalent to the other approaches and to all logarithmic accuracies. We will derive all the known ingredients needed to perform threshold resummation to next-to-next-to-next-to leading logarithmic accuracies (N^3LL) for DIS non-singlet structure function, DY process and the closely related Standard Model (SM) Higgs production through gluon-gluon fusion into a top quark loop. Moreover, the integrations in Eq. (3) are very easy to perform and we have carried out the integrations up to \(g^{(3)}\) that resums the NNLL. This calculation is to be compared with the ones explained in, e.g.,
Appendices A of Refs. [13, 20]. In this paper we use dimensional regularization in $d = 4 - 2\varepsilon$ to regulate both the UV and the IR divergences and we utilize the $\overline{\text{MS}}$ scheme throughout.

This paper is organized as follows. In Sec. II we derive the anomalous dimension of the quark and gluon effective currents up to $O(\alpha_s^3)$ and write down the matching coefficients at the scale $Q^2$ up to $O(\alpha_s^2)$. In Sec. III we obtain the matching coefficients at $\mu^2$ to $O(\alpha_s^2)$ and give our final expression for the resummed coefficient function $G_N$. There we also comment on the universality of the functions $f_{(q,g)}$ that enter the quark and gluon anomalous dimensions of the effective operators. In Sec. IV we compare the EFT approach with the conventional one and derive our main result that establishes the full equivalence of the two approaches. From that relation we obtain the recently calculated $D^{(3)}_{(q,g)}$ for DY and Higgs production and $B^{(3)}_{q}$ for DIS. We carry out the integration in the resummed coefficient function to illustrate the simplicity of the EFT results and obtain the well-known functions $g_i(\lambda_{(q,g)})$ for $i = 1, 2, 3$. Our conclusions are presented in Sec. V. In the Appendix we write down explicit expressions for soft and virtual limit in full QCD for all the processes we consider up to $O(\alpha_s^2)$ in $z$ space and in the conjugate space (for large moments).

II. ANOMALOUS DIMENSION AND MATCHING COEFFICIENTS FOR EFFECTIVE CURRENTS

The EFT approach for resummation starts from calculating the contributions at scale $Q^2$. Technically this is done by matching the full QCD theory currents to the EFT currents at the scale $Q^2$ by considering the purely virtual diagrams in the full theory. By doing this, we integrate out the hard modes of virtualities of order $Q^2$. The matching of the currents can be expressed as an operator expansion

$$J_{\text{QCD}} = C(Q^2/\mu^2, \alpha_s(\mu^2))J_{\text{eff}}(\mu) + \ldots,$$

where $C$ is the matching coefficient, $\mu$ is the factorization or renormalization scale of the effective current and ellipses denote higher-dimensional currents which will be ignored in this work. We will consider the quark vector current $J^\mu = \bar{\psi}\gamma^\mu\psi$ for DIS and DY cases and the gluon scalar current $J = G^{\mu\nu}G_{\mu\nu}$ for Higgs production in hadron colliders.

The anomalous dimensions of the effective currents that control the running (with $\mu$) are defined as

$$\gamma_1(\mu) = -\mu \frac{d \ln J_{\text{eff}}}{d\mu}.$$  

If the matrix elements of the currents in full QCD are independent of the factorization scale, such as quark vector and axial vector currents, the same anomalous dimensions are obtained from the matching coefficients of the effective currents

$$\gamma_1(\mu) = \mu \frac{d \ln C}{d\mu}.$$  

The anomalous dimension is a function of both $Q^2/\mu^2$ and $\alpha_s(\mu^2)$. In fact, it can be shown that it is a linear function of $\ln Q^2/\mu^2$ to all orders in perturbation theory [10];

$$\gamma_1 = A(\alpha_s) \ln Q^2/\mu^2 + B_1(\alpha_s),$$

where $A$ and $B_1$ have expansions in $a_s \equiv \alpha_s(\mu^2)/4\pi$: $A = \sum_i a_s^i A^{(i)}$ and $B_1 = \sum_i a_s^i B_1^{(i)}$, and $\alpha_s$ is the renormalized coupling constant.
To obtain the anomalous dimensions and the matching coefficients, we consider the simplest matrix element of the full QCD currents between on-shell massless quark and gluon states. They are just the on-shell form factors $F$. Since they are “physical” observables, there are no UV divergences, but there are IR ones. To all orders in $\alpha_s$, we can write

$$F = C(Q^2/\mu^2, \alpha_s(\mu^2)) S(Q^2/\mu^2, \alpha_s(\mu^2), 1/\epsilon), \quad (8)$$

where $S$ contains only infrared poles in dimensional regularization (i.e., no finite terms). $S$ can be regarded as the matrix element of the effective current $J_{eff}$ after renormalization has already been performed. In the effective theory, Feynman diagrams for $S$ have vanishing contributions in dimensional regularization because there are no scales in the integrals. This can be regarded as the result of cancellation of IR and UV poles. As such, the IR poles in $S$ may be treated as UV poles for the purpose of calculating the anomalous dimension

$$\gamma_1(\mu) = -\mu \frac{d \ln S}{d \mu}. \quad (9)$$

Since $C$ does not contain any pole part, we can also write

$$\gamma_1(\mu) = -\mu \frac{d \ln F}{d \mu} \bigg|_{\text{pole part}}. \quad (10)$$

Therefore, the perturbative results for $F$ up to any loop order can be used to calculate the anomalous dimensions to the same order.

The best way to see the physical content of the form factor is to consider a resummed form \[21, 22, 23\]

$$\ln F(\alpha_s) = \frac{1}{2} \int_0^{Q^2/\mu^2} d\xi \frac{d \xi}{\xi} \left( K(\alpha_s(\mu), \epsilon) + G(1, \alpha_s(\xi \mu, \epsilon), \epsilon) + \int_\xi^1 \frac{d \lambda}{\lambda} A(\alpha_s(\lambda \mu, \epsilon)) \right), \quad (11)$$

where $A$ is the anomalous dimension of the $K$ and $G$ functions,

$$A(\alpha_s) = \mu^2 \frac{dG}{d\mu^2} = -\mu^2 \frac{dK}{d\mu^2}, \quad (12)$$

and is in fact the same $A$ as in Eq. (9). $K$ contains only the IR poles, and therefore, the whole $K$-function can be constructed from the perturbative expansion $A = \sum_i a_i(\alpha_s)^i$. The function $G$ contains only the hard contribution, and has a perturbative expansion

$$G(1, \alpha_s, \epsilon) = \sum_i a_i^G(\epsilon)(\epsilon). \quad (13)$$

Thus $\ln F$ can be expressed entirely in terms of $G^i(\epsilon)$ and $A^i(\epsilon)$.

The anomalous dimensions $A_q$ for the quark vector current (DIS and DY) and $A_g$ for the gluon scalar current (Higgs production) have been calculated up to $\mathcal{O}(\alpha_s^3)$ \[24\],

$$A_{(q,g)}^{(1)} = 4C_{(q,g)},$$

$$A_{(q,g)}^{(2)} = 8C_F C_{(q,g)} \left[ \frac{67}{18} - \zeta_2 \right] C_A - \frac{5}{9} N_F,$$

$$A_{(q,g)}^{(3)} = 16C_{(q,g)} \left[ C_A^{2} \left( \frac{245}{24} - \frac{67}{9} \zeta_2 + \frac{11}{6} \zeta_3 + \frac{11}{5} \zeta_2^2 \right) + C_F N_F \left( -\frac{55}{24} + 2 \zeta_3 \right) + C_A N_F \left( -\frac{209}{108} + \frac{10}{9} \zeta_2 - \frac{7}{3} \zeta_3 \right) + N_F^2 \left( -\frac{1}{27} \right) \right]. \quad (14)$$
where \(C_{(q,g)} = C_F\) for the quark and \(C_A\) for the gluon. In this sense \(A\) is universal.

The expansion coefficients for the \(G\) function have been obtained up to 3-loops from explicit calculations of the quark and gluon form factors [25]:

\[
G^{(1)}_{(q,g)} = 2(B^{(1)}_{2,(q,g)} - \delta_g \beta_0) + f^{(1)}_{(q,g)} + \epsilon \tilde{G}^{(1)}_{(q,g)},
\]
\[
G^{(2)}_{(q,g)} = 2(B^{(2)}_{2,(q,g)} - 2\delta_g \beta_1) + f^{(2)}_{(q,g)} + \beta_0 G^{(1)}_{(q,g)} + \epsilon \tilde{G}^{(2)}_{(q,g)},
\]
\[
G^{(3)}_{(q,g)} = 2(B^{(3)}_{2,(q,g)} - 3\delta_g \beta_2) + f^{(3)}_{(q,g)} + \beta_1 G^{(1)}_{(q,g)} + \beta_0 \left[ \tilde{G}^{(2)}_{(q,g)} - \beta_0 \tilde{G}^{(1)}_{(q,g)} \right],
\]

(15)

where \(\delta_g\) is zero for quark and 1 for gluon. The \(B_2\)'s are the coefficients in front of the delta function \(\delta(x - 1)\) in the Altarelli-Parisi splitting function and have been calculated to the third order [24]:

\[
B^{(1)}_{2,q} = 3C_F,
\]
\[
B^{(2)}_{2,q} = 4C_FC_A \left( \frac{17}{24} + \frac{11}{3} \zeta_2 - 3\zeta_3 \right) - 4C_FN_F \left( \frac{1}{12} + \frac{2}{3} \zeta_2 \right) + 4C_F^2 \left( \frac{3}{8} - 3\zeta_2 + 6\zeta_3 \right),
\]
\[
B^{(3)}_{2,q} = 16C_AC_FN_F \left( \frac{5}{4} - \frac{167}{54} \zeta_2 + \frac{1}{20} \zeta_2^2 + \frac{25}{18} \zeta_3 \right) + 16C_FC_F^2 \left( \frac{151}{64} + \zeta_2 \zeta_3 - \frac{205}{24} \zeta_2 - \frac{247}{60} \zeta_2^2 + \frac{211}{12} \zeta_3 + \frac{15}{2} \zeta_5 \right) - 16C_A^2 C_F \left( \frac{1657}{576} - \frac{281}{27} \zeta_2 + \frac{1}{8} \zeta_2^2 + \frac{97}{9} \zeta_3 - \frac{5}{2} \zeta_5 \right) - 16C_A^2 N_F^2 \left( \frac{17}{144} - \frac{5}{27} \zeta_2 + \frac{1}{9} \zeta_3 \right) - 16C_F^2 N_F^2 \left( \frac{23}{16} - \frac{5}{12} \zeta_2 - \frac{29}{30} \zeta_2^2 + \frac{17}{6} \zeta_3 \right) + 16C_F^2 \left( \frac{29}{32} - 2\zeta_2 \zeta_3 + \frac{9}{8} \zeta_2 + \frac{18}{5} \zeta_2^2 + \frac{17}{4} \zeta_3 - 15\zeta_5 \right),
\]

(16)

for quarks, and

\[
B^{(1)}_{2,g} = \frac{11}{3} C_A - \frac{2}{3} N_F,
\]
\[
B^{(2)}_{2,g} = 4C_AN_F \left( -\frac{2}{3} \right) + 4C_A^2 \left( \frac{8}{3} + 3\zeta_3 \right) + 4C_F N_F \left( -\frac{1}{2} \right),
\]
\[
B^{(3)}_{2,g} = 16C_AC_FN_F \left( -\frac{241}{288} \right) + 16C_A N_F^2 \frac{29}{288} - 16C_A^2 N_F \left( \frac{233}{288} + \frac{1}{6} \zeta_2 + \frac{1}{12} \zeta_2^2 + \frac{5}{3} \zeta_3 \right) + 16C_A^2 \left( \frac{79}{32} - \zeta_2 \zeta_3 + \frac{1}{6} \zeta_2 + \frac{11}{24} \zeta_2^2 + \frac{67}{6} \zeta_3 - 5\zeta_5 \right) + 16C_F^2 N_F^2 \frac{11}{144} + 16C_F^2 N_F \frac{1}{16},
\]

(17)
for gluons. The universal functions \( f_{(q,g)} \) are given by

\[
\begin{align*}
f_{(q,g)}^{(1)} &= 0, \\
f_{(q,g)}^{(2)} &= C_{(q,g)} C_A \left[ \frac{808}{27} - \frac{22}{3} \zeta_2 - \frac{28}{3} \zeta_3 \right] + C_{(q,g)} N_F \left[ -\frac{112}{27} + \frac{8}{3} \zeta_2 \right], \\
f_{(q,g)}^{(3)} &= C_{(q,g)} C_A \left[ \frac{136781}{729} - \frac{12650}{81} \zeta_2 - \frac{1361}{3} \zeta_3 + \frac{352}{5} \zeta_2^2 + \frac{176}{3} \zeta_2 \zeta_3 + 192 \zeta_5 \right] \\
&\quad + C_{(q,g)} C_A^2 N_F \left[ -\frac{11842}{729} + \frac{2828}{81} \zeta_2 + \frac{728}{27} \zeta_3 - \frac{96}{5} \zeta_2^2 \right] + C_{(q,g)} C_F N_F \left[ -\frac{1771}{27} \right] \\
&\quad + 4 \zeta_2 + \frac{304}{9} \zeta_3 + \frac{32}{5} \zeta_2^2 \right] + C_{(q,g)} N_F^2 \left[ -\frac{2080}{729} - \frac{40}{27} \zeta_2 + \frac{112}{27} \zeta_3 \right].
\end{align*}
\] (18)

The tilted functions in Eq. (15) are not given here since they do not contribute to the anomalous dimension as their contribution to the form factors are canceled (they can be found in [24]).

Finally, the anomalous dimension of the effective currents can be expressed in terms of the \( A \) and \( G \) functions. If one writes \( \gamma_1 = \sum_i a_s^{(i)} \gamma_1^{(i)} \), then

\[
\gamma_1^{(i)} = A_1^{(i)} \ln Q^2 / \mu^2 + B_1^{(i)} + 2 i \delta_g \beta_{i-1},
\] (19)

where

\[
B_1^{(i)} = -2 B_2^{(i)} - f_1^{(i)},
\] (20)

and the QCD \( \beta \)-function is given by

\[
\beta(a_s) = -\frac{d \ln \alpha_s}{d \ln \mu^2} = \beta_0 a_s + \beta_1 a_s^2 + \ldots.,
\] (21)

with \( \beta_0 = 11C_A/3 - 2N_F/3 \). The above expression for \( \gamma_1 \) might work to all orders in perturbation theory. In the gluon case, the last term is present when the anomalous dimension is defined in terms of the matching coefficient \( C_g \) and is absent when it is defined in term of the effective current.

The anomalous dimensions could also be calculated from the matching coefficient \( C(Q^2/\mu^2, \alpha_s(\mu^2)) \) extracted from known results of the form factors. First, we take the logarithm of Eq. (8),

\[
\ln F = \ln C(Q^2/\mu^2) + \ln S(Q^2/\mu^2, 1/\epsilon).
\] (22)

Then we separate out the poles from the form factor logarithms, which belong to the \( S(Q^2/\mu^2, 1/\epsilon) \). The finite part left over is just the logarithm of the matching coefficient \( \ln C \) to any desired order. So, eventually, we will get the following result for the anomalous dimension valid to arbitrary order in \( \alpha_s \),

\[
\gamma_1 = \frac{d}{d \ln \mu} \{ \ln F |_{\text{finite part}} \},
\] (23)

where the form factor in the above equation has been renormalized (including coupling constant renormalization). Using the above equation, we have calculated the anomalous dimension for the quark and gluon currents in the effective theory up to three-loop order, and
they are exactly the same as Eq. [19]. We should point out here that the anomalous dimension of the quark current is the same for the scattering case (DIS) and for the annihilation case (DY).

To calculate the matching coefficients, $C(Q^2/\mu^2, \alpha_s(\mu^2)) = \sum_i a_i(\mu^2)C_i(Q^2/\mu^2)$ for DIS, DY and Higgs production, we need the expressions for the quark form factor (space-like case and time-like case) [26], and for SM Higgs production [27, 28, 29] up to the same order. It should be noted that for DY and Higgs cases, $C_i^{(i)}$ contains imaginary parts that need be taken into account. For our purposes, it is enough to keep the imaginary part for $C_i^{(1)}$ only. Normalizing $C^{(0)}$ to 1 we find for DIS,

$$C^{(1)}_{\text{DIS}}(Q^2/\mu^2) = C_F \left[ -\ln^2 \left( \frac{Q^2}{\mu^2} \right) + 3 \ln \left( \frac{Q^2}{\mu^2} \right) - 8 + \zeta_2 \right],$$

$$C^{(2)}_{\text{DIS}}(Q^2/\mu^2) = C_F^2 \left[ \frac{1}{2} \left( \ln^2 \left( \frac{Q^2}{\mu^2} \right) - 3 \ln \left( \frac{Q^2}{\mu^2} \right) + 8 - \zeta_2 \right)^2 + \left( \frac{3}{2} - 12\zeta_2 + 24\zeta_3 \right) \ln \left( \frac{Q^2}{\mu^2} \right) - \frac{1}{8} + 29\zeta_2 - 30\zeta_3 - \frac{44}{5} \zeta_2^2 \right] + C_F N_F \left[ -\frac{2}{9} \ln^3 \left( \frac{Q^2}{\mu^2} \right) + \frac{19}{9} \ln^2 \left( \frac{Q^2}{\mu^2} \right) - \left( \frac{209}{27} + \frac{4}{3} \zeta_2 \right) \ln \left( \frac{Q^2}{\mu^2} \right) \right] + 4085 + \frac{23}{9} \zeta_2 + \frac{2}{9} \zeta_3 \right] + C_F C_A \left[ \frac{11}{9} \ln^3 \left( \frac{Q^2}{\mu^2} \right) + \left( 2\zeta_2 - \frac{233}{18} \right) \ln^2 \left( \frac{Q^2}{\mu^2} \right) \right] + \left( \frac{2545}{54} - \frac{22}{3} \zeta_2 - 26\zeta_3 \right) \ln \left( \frac{Q^2}{\mu^2} \right) - \frac{51157}{648} - \frac{337}{18} \zeta_2 + \frac{313}{9} \zeta_3 + \frac{44}{5} \zeta_2^2 \right].$$

The logarithms in the above result have been presented in [14]. For DY we can simply get the $C^{(i)}_q$ by replacing each $\ln \left( \frac{Q^2}{\mu^2} \right)$ in $C^{(i)}_{\text{DIS}}$ with $\ln \left( \frac{Q^2}{\mu^2} \right) - i\pi$. This is just a result of the fact that the time-like quark form factor can be obtained from the space-like one by analytic continuation. For the Higgs production we set $M_H^2 = Q^2$ and we get

$$C^{(1)}_g(Q^2/\mu^2) = C_A \left[ -\ln^2 \left( \frac{Q^2}{\mu^2} \right) + 7\zeta_2 + 2i\pi^2 \ln \left( \frac{Q^2}{\mu^2} \right) \right],$$

$$\text{Re}[C^{(2)}_g(Q^2/\mu^2)] = C_A^2 \left[ \frac{1}{2} \ln^4 \left( \frac{Q^2}{\mu^2} \right) + \frac{11}{9} \ln^3 \left( \frac{Q^2}{\mu^2} \right) - \left( \frac{67}{9} - 17\zeta_2 \right) \ln^2 \left( \frac{Q^2}{\mu^2} \right) + \left( \frac{80}{27} - \frac{88}{3} \zeta_2 - 2\zeta_3 \right) \ln \left( \frac{Q^2}{\mu^2} \right) + \frac{5105}{162} + \frac{335}{6} \zeta_2 - \frac{143}{9} \zeta_3 + \frac{125}{10} \zeta_2^2 \right] + C_A N_F \left[ -\frac{2}{9} \ln^3 \left( \frac{Q^2}{\mu^2} \right) + \frac{10}{9} \ln^2 \left( \frac{Q^2}{\mu^2} \right) + \left( \frac{52}{27} + \frac{16}{3} \zeta_2 \right) \ln \left( \frac{Q^2}{\mu^2} \right) \right] - \frac{916}{81} - \frac{25}{3} \zeta_2 - \frac{46}{9} \zeta_3 \right] + C_F N_F \left[ 2 \ln^2 \left( \frac{Q^2}{\mu^2} \right) - \frac{67}{6} + 8\zeta_3 \right].$$

The logarithms in $C^{(i)}$ will be needed later on to show that the matching coefficients at $\mu_f$
are free of any logarithms, and we have not included the imaginary part of $C^{(2)}_g$ since it does not contribute to the accuracy in which we are interested.

Using Eq. (5) we can write down the solution of the renormalization group equations for DY and Higgs,

$$C_{(q,g)}(Q^2/\mu_I^2, \alpha_s(\mu_I^2)) = C_{(q,g)}(1, \alpha_s(Q^2)) \exp \left[ \frac{I_1(Q, \mu_I)}{2} \right], \quad (26)$$

where

$$I_1 = - \int_\mu^Q \tilde{\gamma}_{1,(q,g)} \frac{d\mu}{\mu}, \quad \tilde{\gamma}_{1,(q,g)} = \gamma_{1,(q,g)} - 2i\delta_g\beta_{i-1}. \quad (27)$$

The $C(1, \alpha_s(Q^2)) \equiv C(\alpha_s(Q^2))$ is just the non-logarithmic part of $C(Q^2/\mu^2, \alpha_s(\mu^2))$. For the Higgs case the last relation is a result of the $\mu$ dependence of $C_\phi(\mu)$ which enters into the effective lagrangian that one obtains after integrating out the top quark (see, e.g., [27]). This $\mu$-dependence is governed by anomalous dimension which we denote by $\gamma_T$ following the notation of [14]. There it was shown that

$$\gamma_T = a_s[-2\beta_0] + a_s^2[-4\beta_1], \quad (28)$$

so the conclusion is that the only effect of this anomalous dimension, when combined with anomalous dimension of the matching coefficient at the scale $Q^2$ is to cancel the $\beta_1$ terms in $\gamma_1$ for the Higgs case. For DIS we replace $C_q$ with $C_{DIS}$ which runs with the same $\gamma_{1,q}$.

In Eq. (26) we encounter the first of three exponentials. The other two will be obtained below. Since $\mu_I^2$ will later be identified with $Q^2/N^p$ where $p = 1$ for DIS and $p = 2$ for DY and Higgs cases, it is clear that the exponential includes large logarithms of the form mentioned in the introduction. We again stress the fact that $C(\alpha_s(Q^2))$ (for all three processes) and $\gamma_{1,(q,g)}$ are completely determined to a given $O(\alpha_s^k)$ by the knowledge of the form-factor calculation up to the same order.

### III. MATCHING COEFFICIENTS AT $\mu_I$ AND THE RESUMMED COEFFICIENT FUNCTIONS

In this section we show how to extract the matching coefficients at the intermediate scale to $O(\alpha_s^2)$ for DIS, DY and Higgs production from the known calculations of full QCD, and to obtain resummed expressions for the coefficient functions. Since we are interested in the threshold region, we need to consider only the partonic channels that give rise to the singular contributions in the limit $z \to 1$, i.e, $\delta(1-z)$ and the “plus” distributions, $D_i(z)$, where

$$D_i(z) \equiv \left[ \frac{\ln^i(1-z)}{1-z} \right]_+ \quad (29)$$

For DIS, DY and the Higgs processes, these channels are: $q + \gamma^* \to q$, $q + \bar{q} \to \gamma^*$ and $g + g \to H$, respectively.

To the accuracy we are interested in, the $O(\alpha_s^2)$ cross section from soft contributions are needed. The full QCD calculations for cross sections can be found in Refs. [26] for DY, in
Refs. [28, 29] for the Higgs production and in Refs. [30, 31] for DIS. The result in the soft limit can be written as

$$G^{(s+v)}(z) \equiv \sum_i a_i^s(\mu^2)G^{(i),(s+v)}(z)$$  \tag{30}$$

which contains both soft and virtual contributions, and where $G(z)$ is the inverse Mellin transform of $G_N$ in Eq. (2). Explicit expressions for $G^{(i),(s+v)}(z)$ with $i = 1, 2$ can be found in Ref. [32] for DIS, in Ref. [33] for DY and in Ref. [34, 35] for the Higgs production. [$G^{(0)}(z) = \delta(1 - z)$]. Using the following well-known Mellin transforms of $D_i(\bar{N})$ in the large $N$ limit

$$D_0(\bar{N}) = -\ln \bar{N},$$

$$D_1(\bar{N}) = \frac{1}{2} \ln^2 \bar{N} + \frac{1}{2} \zeta_2,$$

$$D_2(\bar{N}) = -\frac{1}{3} \ln^3 \bar{N} - \zeta_2 \ln \bar{N} - \frac{2}{3} \zeta_3,$$

$$D_3(\bar{N}) = \frac{1}{4} \ln^4 \bar{N} + \frac{3}{2} \zeta_2 \ln^2 \bar{N} + 2 \zeta_3 \ln \bar{N} + \frac{27}{20} \zeta_2^2,$$  \tag{31}$$

we get the $G^{(i),(s+v)}(\bar{N})$, $i = 0, 1, 2$. Explicit expressions are given in the Appendix. As we have already mentioned, the SCET is supposed to reproduce the same results.

To get the matching coefficient at the intermediate scale $\mu_I$, $\mathcal{M}_N = \sum_i a_i^s \mathcal{M}_N^{(i)}$, we need to factorize the virtual contribution from the following relation,

$$G_N^{(s+v)} \left( \frac{Q^2}{\mu^2}, \bar{N}, \alpha_s(\mu^2) \right) = |C \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right)|^2 \times \mathcal{M}_N \left( \frac{Q^2}{\mu^2}, \bar{N}, \alpha_s(\mu^2) \right).$$  \tag{32}$$

The content of this formula is simple: The finite part of the partonic cross section $G_N^{(s+v)}$ comes from both the purely virtual, form-factor type of Feynman diagrams, which are included in $|C|^2$, and from diagrams with at least one real gluon emitted into the final state, which are included in $\mathcal{M}_N$. However, there is a different way to look at it. The right-hand side is just the result of a two-step matching of the product of two full QCD currents where at each step we collect the relevant contribution to the cross section. The first step accounts for $|C|^2$ and the second one gives rise to $\mathcal{M}_N$. It should be noted that multiple matching procedure, as the one performed here, results in a multiplicative matching coefficients. We also mention that the above equation could formally be proved, inductively in $\alpha_s$, by considering the cross section within the effective theory itself and relating it to the full QCD calculation in the soft limit.

Expanding the above equation to the third order, one gets

$$G_N^{(1),s+v} = 2 \text{Re}[C^{(1)}] + \mathcal{M}_N^{(1)},$$

$$G_N^{(2),s+v} = |C^{(1)}|^2 + 2 \text{Re}[C^{(2)}] + 2 \text{Re}[C^{(1)}] \mathcal{M}_N^{(1)} + \mathcal{M}_N^{(2)},$$

$$G_N^{(3),s+v} = 2 \text{Re}[C^{(1)}C^{(2)*}] + 2 \text{Re}[C^{(3)}] + |C^{(1)}|^2 \mathcal{M}_N^{(1)} + 2 \text{Re}[C^{(1)}] \mathcal{M}_N^{(2)} + 2 \text{Re}[C^{(2)}] \mathcal{M}_N^{(1)} + \mathcal{M}_N^{(3)}.$$  \tag{33}$$

The above factorization is consistent with that considered in [36]. We get the following
result for DIS,

\[
\mathcal{M}_{N,\text{DIS}}^{(1)} = C_F \left[ 2L^2 + 3L + 7 - 4\zeta_2 \right],
\]

\[
\mathcal{M}_{N,\text{DIS}}^{(2)} = C_F^2 \left[ 2L^4 + 6L^3 + \left( \frac{37}{2} - 8\zeta_2 \right) L^2 + \left( \frac{45}{2} - 24\zeta_2 + 24\zeta_3 \right) L \right]
\]

\[+ C_F C_A \left[ \frac{22}{9} L^3 + \left( \frac{367}{18} - 4\zeta_2 \right) L^2 - \left( -\frac{3155}{54} + \frac{22}{3}\zeta_2 + 40\zeta_3 \right) L \right]
\]

\[- C_F N_F \left[ \frac{4}{9} L^3 + \frac{29}{9} L^2 - \left( \frac{4}{3} \zeta_2 - \frac{247}{27} \right) L \right]
\]

\[+ C_F^2 \left[ \frac{205}{8} - \frac{97}{2} \zeta_2 - 6\zeta_3 + \frac{122}{5} \zeta_2^2 \right] + C_F C_A \left[ \frac{53129}{648} - \frac{155}{6} \zeta_2 - 18\zeta_3 - \frac{37}{5} \zeta_2^2 \right]
\]

\[+ C_F N_F \left[ -\frac{4057}{324} + \frac{13}{3} \zeta_2 \right],
\]

(34)

where \( L = \ln \frac{\mu^2}{\bar{Q}^2} \). The above result has also been obtained in [39] where an explicit two-loop calculation of a suitably defined jet function was performed [43]. For DY, we get

\[
\mathcal{M}_{N,q}^{(1)} = C_F \left[ 2L^2 + 2\zeta_2 \right],
\]

\[
\mathcal{M}_{N,q}^{(2)} = C_F^2 \left( \frac{1}{2} \right) \left[ 2L^2 + 2\zeta_2 \right]^2 + C_A C_F \left[ \frac{22}{9} L^3 + \left( \frac{134}{9} - 4\zeta_2 \right) L^2 + \left( \frac{808}{27} - 28\zeta_3 \right) L \right]
\]

\[- C_F N_F \left[ \frac{4}{9} L^3 + \frac{20}{9} L^2 + \frac{112}{27} L \right]
\]

\[+ C_F C_A \left[ \frac{2428}{81} + \frac{67}{9} \zeta_2 - \frac{22}{9} \zeta_3 - 12\zeta_2^2 \right]
\]

\[+ C_F N_F \left[ -\frac{328}{81} - \frac{10}{9} \zeta_2 + \frac{4}{9} \zeta_3 \right],
\]

(35)

where \( L = \ln \frac{\mu^2 \bar{Q}^2}{\bar{Q}^2} \). And finally, for the Higgs case, we have

\[
\mathcal{M}_{N,g}^{(1)} = C_A \left[ 2L^2 + 2\zeta_2 \right],
\]

\[
\mathcal{M}_{N,g}^{(2)} = C_A^2 \left( \frac{1}{2} \right) \left[ 2L^2 + 2\zeta_2 \right]^2 + C_A C_A \left[ \frac{22}{9} L^3 + \left( \frac{134}{9} - 4\zeta_2 \right) L^2 + \left( \frac{808}{27} - 28\zeta_3 \right) L \right]
\]

\[- C_A N_F \left[ \frac{4}{9} L^3 + \frac{20}{9} L^2 + \frac{112}{27} L \right]
\]

\[+ C_A C_A \left[ \frac{2428}{81} + \frac{67}{9} \zeta_2 - \frac{22}{9} \zeta_3 - 12\zeta_2^2 \right]
\]

\[+ C_A N_F \left[ -\frac{328}{81} - \frac{10}{9} \zeta_2 + \frac{4}{9} \zeta_3 \right].
\]

(36)

For all three processes, we have \( G_N^{(0)} = 1 \).

From the above results it is clear that the logarithms \( L \) and \( \bar{L} \) vanish when we set: \( \mu^2 = \mu_0^2 \equiv Q^2/\bar{Q}^2 \). Of course, this has to be the case as the matching coefficients should be
logarithmically free, and we can write

$$\mathcal{M}_N\left(\frac{Q^2}{\mu^2}, N, \alpha_s(\mu^2)\right) = \mathcal{M}_N\left(\ln\left(\frac{Q^2}{N^2\mu^2}\right), \alpha_s(\mu^2)\right),$$

(37)

and for $\mu^2 = \mu_I^2 = \frac{Q^2}{N^2}$ we have

$$\mathcal{M}_N\left(\ln\left(\frac{Q^2}{N^2\mu_I^2}\right), \alpha_s(\mu^2)\right) = \mathcal{M}_N(\alpha_s(\mu_I^2)).$$

(38)

These observations are valid to all orders in perturbation theory[10] and they lead to a strong constraint on the anomalous dimensions of the effective operators on both sides of the matching scale. Another interesting feature emerges from the results of the DY and Higgs cases, $\mathcal{M}_{N,\alpha,s}^{(i)}$ and $\mathcal{M}_{N,\alpha,s}^{(i)}$, $i = 1, 2$: One can simply get the latter from the former by replacing the overall factor $C_F$ with $C_A$ in the non-Abelian part. The Abelian part exponentiates and hence all occurrence of $C_F$ shall be replaced by $C_A$. In this sense, the matching coefficients seem to be universal. This could be argued based on that in the soft gluon limit, only the color charges of annihilating quarks and gluons are relevant.

Following the same steps as we did after the first stage matching at $Q^2$, we need now to consider the running of the effective operators that were used to perform the matching at $\mu_I$. However at and below the scale $\mu_I$ they are just the conventional PDFs taken to the limit $z \to 1$. As such, the running of the effective operators (the PDFs) is governed by the well-known DGLAP (Dokshitzer-Gribov-Lipatov-Altarelli-Parizi) evolution equation with anomalous dimension

$$\gamma_{2,(q,g)}^N = A_{(q,g)} \ln N^2 - 2B_{2,(q,g)},$$

(39)

where $A_{(q,g)}$ and $B_{2,(q,g)}$ are given in Eqs. [14] and [15]. We include the running effects in

$$I_2 = 2 \int_{\mu_I}^{\mu_F} \frac{d\mu}{\mu} \gamma_{2,(q,g)},$$

(40)

where $\mu_F$ is the factorization scale for parton distributions.

The resummed factorization coefficient functions for DY and Higgs are

$$G_{N,(q,g)}(Q) = |C_{(q,g)}(\alpha_s(Q))|^2 e^{I_1(Q,\mu_I)} \times \mathcal{M}_{N,(q,g)}(\alpha_s(\mu_I)) e^{I_2(\mu_I, \mu_F)},$$

(41)

where we have omitted $C_2^2$ for Higgs production. [The definition of $I_1$ and $I_2$ differs by a minus sign from Ref. [11].] Anticipating the discussion of the next section, we will set the factorization scale $\mu_F = Q$. The above equation can be brought into an equivalent form by exploiting the running of $\alpha_s$ from $\mu_I$ to $Q$ in $\mathcal{M}_{N,(q,g)}(\alpha_s(\mu_I))$;

$$\mathcal{M}_{N,(q,g)}(\alpha_s(\mu_I^2)) = \mathcal{M}_{N,(q,g)}(\alpha_s(Q^2)) \exp[I_3],$$

(42)

where

$$I_3 = -2 \int_{\mu_I}^{Q} \frac{d\mu}{\mu} \triangle B_{(q,g)},$$

(43)
where
\[ \triangle B_{(q,g)} \equiv -\beta(\alpha_s) \frac{d \ln \mathcal{M}_{N,(q,g)}}{d \ln \alpha_s}. \] (44)

The last two equations are also true for the DIS case. Thus we write
\[ G_N(Q) = \mathcal{F}(\alpha_s(Q)) e^{I(\lambda,\alpha_s(Q))}, \] (45)
where \( \mathcal{F} = |C_{(q,g)}(\alpha_s(Q))|^2 \mathcal{M}_{(q,g)}(\alpha_s(Q)) \) depends only on \( \alpha_s(Q) \). The subscript \( N \) of \( \mathcal{M} \) has been omitted since there is not any large logarithmic dependence in the matching coefficients. \( I = I_1 + I_2 + I_3 \) is a function of \( \lambda = \beta_0 \ln N \alpha_s(Q) \) and \( \alpha_s(Q) \) with all leading and sub-leading large logarithms resummed.

Since the cross section \( \sigma_N \) in Eq. (2) is independent of the intermediate scale \( \mu_I \), then from Eq. (41) and the definitions of \( \gamma_1 \) and \( \gamma_2 \) we get the following relation for DY and Higgs:
\[ \frac{d \ln \mathcal{M}_{N,(q,g)}(\alpha_s(\mu^2),L)}{d \ln \mu} = [2\gamma_2 - 2\gamma_1]_{(q,g)} = 2[AL + f]_{(q,g)}, \] (46)
from which we get
\[ \frac{d \ln \mathcal{M}_{N,(q,g)}(\alpha_s(\mu^2),L)}{d \ln \mu} \bigg|_{\mu=\mu_I} = 2f_{(q,g)}(\alpha_s(\mu^2_I)), \quad \mu_I = \frac{Q}{\sqrt{N}} \] (47)
where \( A_{(q,g)} \) are given in Eq. (14) and \( f_{(q,g)} \) are given in Eq. (18). The last equation sheds light on the physical meaning of the functions \( f_{(q,g)} \): It is the anomalous dimension of the matching coefficient \( \mathcal{M} \) evaluated at the intermediate scale \( \mu_I \). Here we see that the universality of these functions could be explained by the fact that \( \mathcal{M}_{(q,g)} \) are themselves universal.

The last equation also shows the same \( A_{(q,g)} \) appears in the logarithmic parts of \( \gamma_1,(q,g) \) and \( \gamma_2,(q,g) \), because otherwise the logarithms at \( \mu_I \) do not cancel in \( \mathcal{M}_N \).

For DIS a similar analysis is performed, however, we have to consider only one-half of \( I_2 \) in Eq. (41) since we match onto a single PDF. With this we get
\[ \frac{d \ln \mathcal{M}_{N,DIS}(\alpha_s(\mu^2),L)}{d \ln \mu} = [2\gamma_2 - 2\gamma_1]_q = 2[AL + B_2 + f]_q, \] (48)
from which we obtain at the intermediate scale,
\[ \frac{d \ln \mathcal{M}_{N,DIS}(\alpha_s(\mu),L)}{d \ln \mu} \bigg|_{\mu=\mu_I} = 2[B_2 + f]_q(\alpha_s(\mu_I^2)), \quad \mu_I = \frac{Q}{\sqrt{N}} \] (49)
Here there is an extra contribution from \( B_2 \).

IV. COMPARISON WITH THE TRADITIONAL APPROACH AND EXPLICIT RESULTS TO N^3LL ORDER

In this section we will illustrate the equivalence of the EFT approach and the traditional one which relies on the refactorization of hard processes as we mentioned in the introduction.
The renormalon problem in the later approach arises from doing resummation uniformly for all moments, which will necessarily encounter small scale $Q/N^p$ at fixed $Q$ when $N$ is sufficiently large. The EFT approach avoids that by short-cutting the steps when this scale becomes of order $\Lambda_{QCD}$. We will start by showing this first for DY and Higgs production, then will turn to the DIS case. In the last subsection, we give the explicit form of the relevant integrals obtained in the EFT approach.

A. Drell-Yan and Higgs

One of the well-known forms used to express the coefficient function for DY and Higgs in moment space is the following [37]:

$$G_N(Q^2) = g_0(\alpha_s(Q^2))e^{I_\Delta} \Delta C(\alpha_s(Q^2)),$$

(50)

where we have normalized the Born term to 1. The $g_0$ has a conventional expansion form:

$$g_0 = \sum_i a_i^q g_0^i.$$ [In this subsection, we omitted the subscript $q$ and $g$, intended for DY and Higgs production.] The term $\Delta C$ has the only role of cancelling the non-logarithmic contributions that appear in the exponent. These contributions arise from the various $\zeta$-terms in the Mellin transform of the “plus” distributions. The Sudakov exponential term $I_\Delta$ is given by

$$I_\Delta = \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \left[ 2 \int_{Q^2}^{(1-z)Q^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) + D(\alpha_s((1-z)^2 Q^2)) \right],$$

(51)

where, as already mentioned, we set $\mu_F^2 = Q^2$. As noted above, $I_\Delta$ contains both a logarithmic and non-logarithmic contribution. The quantities, $g_0$, $A$ and $D$ have the usual expansion in $\alpha_s$ and they are already known up to $O(\alpha_s^3)$ [20]. The $A$ is identical to the logarithmic coefficient in $\gamma_1$ and $\gamma_2$. It is our aim to relate these quantities with those that appear in $G_N$ of Eq. (41). For this we follow the procedure outlined in Appendices A, B and C of [19].

The integral in $I_\Delta$ can be rewritten in terms of the already defined $I_1, I_2$ and $I_3$,

$$I \equiv I_1 + I_2 + I_3 = I_\Delta + \ln \Delta C(\alpha_s(Q^2)),$$

(52)

where the coefficient function $\Delta C$ does not depend on $\mu_I \sim Q/N$.

To prove the above relation, we first use following expansion:

$$z^{N-1} - 1 = -\bar{\Gamma} \left( 1 - \frac{\partial}{\partial \ln N} \right) \theta \left( 1 - z - \frac{1}{N} \right) + O(1/N),$$

(53)

where the $\bar{\Gamma}$ function is related to the usual gamma function,

$$\bar{\Gamma} \left( 1 - \frac{\partial}{\partial \ln N} \right) = 1 - \Gamma_2 \left( \frac{\partial}{\partial \ln N} \right) \left( \frac{\partial}{\partial \ln N} \right)^2,$$

(54)

where the first parenthesis in the right-hand side is the argument of the $\Gamma_2$ function, and

$$\Gamma_2(\epsilon) = \frac{1}{\epsilon^2} [1 - e^{-\gamma_E} \Gamma(1 - \epsilon)] = -\frac{1}{2} \zeta_2 - \frac{1}{3} \zeta_3 \epsilon - \frac{9}{40} \zeta_2^2 \epsilon^2 + O(\epsilon^3).$$

(55)
In Eq. (53) we used \( \frac{\partial}{\partial \ln N} f(\ln N) = (\partial/\partial \ln N) f(\ln N) \) for an arbitrary function \( f \). After some algebra, \( I_\Delta \) can be expressed as

\[
I_\Delta = -\Gamma \left( 1 - \frac{\partial}{\partial \ln N} \right) \left\{ \int_{Q^2/N^2} d\mu^2 \left[ A(\alpha_s(\mu^2)) \ln \frac{Q^2}{\mu^2} + \frac{1}{2} D(\mu^2) \right] 
+ \int_{Q^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) \ln \frac{N^2}{\mu^2} \right\}.
\]

The double derivative from \( \tilde{\Gamma} \) acting on the curly bracket above gives a contribution

\[
\Gamma_2 \left( \frac{\partial}{\partial \ln N} \right) \left[ \frac{\partial}{\partial \ln N} D(\alpha_s(Q^2/N^2)) - 4A(\alpha_s(Q^2/N^2)) \right].
\]

(57)

To compare \( I_\Delta \) with the exponent \( I = I_1 + I_2 + I_3 \), we express the latter in the form

\[
I_1 + I_2 + I_3 = -\left\{ \int_{Q^2/N^2} d\mu^2 \left[ A(\alpha_s(\mu^2)) \ln \frac{Q^2}{\mu^2} + (B_1 + \Delta B + 2B_2) \right] 
+ \int_{Q^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) \ln \frac{N^2}{\mu^2} \right\}.
\]

(58)

Matching the two integrals, we get

\[
-\int_{Q^2/N^2} d\mu^2 (B_1 + \Delta B + 2B_2)(\alpha_s(\mu^2)) \\
= \Gamma_2 \left( \frac{\partial}{\partial \ln N} \right) \left[ \frac{\partial}{\partial \ln N} D(\alpha_s(Q^2/N^2)) - 4A(\alpha_s(Q^2/N^2)) \right] \\
- \frac{1}{2} \int_{Q^2/N^2} d\mu^2 \frac{d\mu^2}{\mu^2} D(\alpha_s(\mu^2)) + \ln \Delta C(\alpha_s(Q^2^2)).
\]

(59)

The above equation can be solved by perturbative expansion in \( \alpha_s \).

If the equality given in Eq. (59) holds to all values of \( N \), then for \( N = 1 \) we get

\[
\ln \Delta C(\alpha_s(Q^2^2)) = -\Gamma_2(\partial_{\alpha_s}) \left[ \partial_{\alpha_s} D(\alpha_s(Q^2/N^2)) - 4A(\alpha_s(Q^2/N^2)) \right] \bigg|_{N=1},
\]

(60)

where we follow [19] and replace the derivative \( \partial/\partial \ln N \) with \( \partial_{\alpha_s} \) where

\[
\partial_{\alpha_s} \equiv 2 \frac{d\alpha_s(\mu^2)}{d\ln \mu^2} \frac{\partial}{\partial \alpha_s} = -2\beta(\alpha_s)\alpha_s \frac{\partial}{\partial \alpha_s}.
\]

(61)

and, hence,

\[
\left( \frac{\partial}{\partial \ln N} \right) f(\alpha_s(Q^2/N^2)) = \partial_{\alpha_s} f(\alpha_s(Q^2/N^2)),
\]

where \( f \) is an arbitrary function.
Applying one more \( \partial / \partial \ln N = \partial_\alpha \) on both sides of Eq. \((59)\) we get our master relation
\[
2(B_1 + \Delta B + 2B_2)(\alpha_s(\mu^2)) = D(\alpha_s(\mu^2)) + \partial_\alpha \Gamma_2(\partial_\alpha) [4A - \partial_\alpha D] (\alpha_s(\mu^2)).
\] (63)
which can easily be solved for \( D^{(i)} \) order by order in \( \alpha_s \). As an example, let us expand both sides up to \( O(\alpha_s^4) \). First, we work out the expansion of the \( \Delta B \) term. From Eq. \((63)\), we get
\[
\begin{align*}
\Delta B_{(q,g)}^{(0)} &= \Delta B_{(q,g)}^{(1)} = 0, \\
\Delta B_{(q,g)}^{(2)} &= -\beta_0 M_{N,(q,g)}^{(1)}, \\
\Delta B_{(q,g)}^{(3)} &= -\beta_0 \left[ 2M_{N,(q,g)}^{(2)} - \left( M_{N,(q,g)}^{(1)} \right)^2 \right] - \beta_1 M_{N,(q,g)}^{(1)}, \\
\Delta B_{(q,g)}^{(4)} &= -\beta_0 \left[ 3M_{N,(q,g)}^{(3)} - 3M_{N,(q,g)}^{(1)} M_{N,(q,g)}^{(2)} + \left( M_{N,(q,g)}^{(1)} \right)^3 \right] \\
&- \beta_1 \left[ 2M_{N,(q,g)}^{(2)} - \left( M_{N,(q,g)}^{(1)} \right)^2 \right] \\
&- \beta_2 M_{N,(q,g)}^{(1)}.
\end{align*}
\] (64)
Noticing that \( B_{1,(q,g)}^{(i)} + 2B_{2,(q,g)}^{(i)} = -f_{(q,g)}^{(i)} \) and using the expansion of \( \Gamma_2 \), we get \( D^{(i)} \)
\[
\begin{align*}
D_{(q,g)}^{(0)} &= D_{(q,g)}^{(1)} = 0, \\
D_{(q,g)}^{(2)} &= -2f_{(q,g)}^{(2)} + 2\Delta B_{(q,g)}^{(2)} + 4\beta_0 \zeta_2 A_{(q,g)}^{(1)}, \\
D_{(q,g)}^{(3)} &= -2f_{(q,g)}^{(3)} + 2\Delta B_{(q,g)}^{(3)} + 4\zeta_2 \beta_1 A_{(q,g)}^{(1)} + 8\zeta_2 \beta_0 A_{(q,g)}^{(2)} + \frac{32}{3} \zeta_3 \beta_0^2 A_{(q,g)}^{(1)}, \\
D_{(q,g)}^{(4)} &= -2f_{(q,g)}^{(4)} + 2\Delta B_{(q,g)}^{(4)} + 12\zeta_2 \beta_0 A_{(q,g)}^{(3)} + 8\zeta_2 \beta_1 A_{(q,g)}^{(2)} + 32\zeta_2 \beta_0 A_{(q,g)}^{(2)} \\
&+ \frac{80}{3} \zeta_3 \beta_0 \beta_1 A_{(q,g)}^{(1)} + \frac{216}{5} \zeta_2 \beta_0 A_{(q,g)}^{(1)} - 12\zeta_2 \beta_0^2 D_{(q,g)}^{(2)}.
\end{align*}
\] (66)
Thus, apart from the coupling-constant running effects, \( D \) is essentially \(-2f = 2B_1 + 4B_2\).

From the last two equations we see that in order to get \( D^{(k)} \), the only same order information needed is \( f^{(k)} \). All the quantities needed to calculate \( D^{(2)} \) and \( D^{(3)} \) are known and we get
\[
\begin{align*}
D_{(q,g)}^{(2)} &= C_{(q,g)} \left\{ C_A \left( -\frac{101}{27} + \frac{11}{3} \zeta_2 + \frac{7}{2} \zeta_3 \right) + N_F \left( \frac{14}{27} - \frac{2}{3} \zeta_2 \right) \right\}, \\
D_{(q,g)}^{(3)} &= C_{(q,g)} C_A^2 \left[ -\frac{594058}{729} + \frac{98224}{81} \zeta_2 + \frac{40144}{27} \zeta_3 - \frac{2992}{15} \zeta_2^2 - \frac{352}{3} \zeta_2 \zeta_3 - 384 \zeta_5 \right] \\
&+ C_{(q,g)} C_A N_F \left[ \frac{125252}{729} - \frac{29392}{81} \zeta_2 - \frac{2480}{9} \zeta_3 + \frac{736}{15} \zeta_2^2 \right] \\
&+ C_{(q,g)} C_F N_F \left[ \frac{3422}{27} - 32 \zeta_2 - \frac{608}{9} \zeta_3 - \frac{64}{5} \zeta_2^2 \right] \\
&+ C_{(q,g)} N_F^2 \left[ -\frac{3712}{729} + \frac{640}{27} \zeta_2 + \frac{320}{27} \zeta_3 \right],
\end{align*}
\] (68)
where $C_{(q,g)} = C_F$ for the DY case and $C_A$ for the Higgs case. The above results agree with the recent calculation in [9, 38, 41]. The result for the Higgs production has already been reported on in [18].

The non-logarithmic contribution $F_{(q,g)}(Q^2) = \sum_i a^i F_{(q,g)}^{(i)} = |C(Q^2)|^2 M_{N_i}(Q^2)$ can be calculated from the already-known results for $C_{(q,g)}^{(i)}(Q^2)$ and $M_{N_i,(q,g)}^{(i)}(\alpha_s(Q^2))$, or we can simply read them from the well-known results for $G_i, (s+v)(Q^2)$ through Eq. (32) and Eq. (33);

$$F_g^{(1)} = 16C_F(\zeta_2 - 1),$$
$$F_g^{(2)} = C_F^2 \left[ \frac{511}{4} - 198\zeta_2 - 60\zeta_3 + \frac{552}{5}\zeta_2^2 \right]
+ C_F C_A \left[ -\frac{1535}{12} + \frac{376}{3}\zeta_2 + \frac{604}{9}\zeta_3 - \frac{92}{5}\zeta_2^2 \right]
+ C_F N_F \left[ \frac{127}{6} - \frac{64}{3}\zeta_2 + \frac{8}{9}\zeta_3 \right],$$

for DY lepton-pair production. For the Higgs case, we have

$$F_g^{(1)} = 16\zeta_2 C_A,$$
$$F_g^{(2)} = C_A^2 \left[ 93 + \frac{1072}{9}\zeta_2 - \frac{308}{9}\zeta_3 + 92\zeta_2^2 \right]
+ C_A C_F \left[ -\frac{1535}{12} + \frac{376}{3}\zeta_2 + \frac{604}{9}\zeta_3 - \frac{92}{5}\zeta_2^2 \right]
+ C_A N_F \left[ -\frac{80}{3} - \frac{160}{9}\zeta_2 + \frac{88}{9}\zeta_3 \right] + C_F N_F \left[ -\frac{67}{3} + 16\zeta_3 \right].$$

The above results agree with the $g_{01}$ and $g_{02}$ in [38]. The $\gamma_E$ terms in the results of [38] are due to the use of $N$ instead of $\overline{N}$ as in our case. It is very simple to also reproduce these terms. We also notice that their results for the $g_{0i}$ do not include the contributions from the non-logarithmic terms in $I_\Delta$.

### B. DIS

For the DIS case there are essentially two major differences. The first is that the $D$ term in $I_\Delta$ is zero to all orders in $\alpha_s$ [6, 40]. The second one comes from the “jet function” which encodes the effects of collinear gluon emission from the outgoing parton. So for DIS, the traditional approach yields the following expression for the exponent in the coefficient function $G_N(Q^2)$,

$$I_{DIS} = \int_0^1 dz \frac{z^{N-1} - 1}{1-z} \left[ \int_{Q^2}^{(1-z)Q^2} \frac{d\mu^2}{\mu^2} A_q(\alpha_s(\mu^2)) + B_q(\alpha_s((1-z)Q^2)) \right],$$

where again we set $\mu_F^2 = Q^2$. We have used $B$ here so that it will not be confused with $B_i$’s introduced earlier.
We now follow the same procedure as for the DY case, rewriting

\[ I_{\text{DIS}} = -\tilde{\Gamma} \left(1 - \frac{\partial}{\partial \ln \overline{N}}\right) \left\{ \int_{Q^2}^{Q^2/\overline{N}} \frac{d\mu^2}{\mu^2} \left[ A_q(\alpha_s(\mu^2)) \ln \frac{Q^2}{\mu^2} + B_q(\mu^2) \right] \right. \]

\[ + \left. \int_{Q^2}^{Q^2/\overline{N}} \frac{d\mu^2}{\mu^2} A_q(\alpha_s(\mu^2)) \ln \overline{N} \right\}, \]  

(72)

On the other hand, our result for DIS reads

\[ I_1 + I_2 + I_3 = -\left\{ \int_{Q^2}^{Q^2/\overline{N}} \frac{d\mu^2}{\mu^2} \left[ A_q(\alpha_s(\mu^2)) \ln \frac{Q^2}{\mu^2} + (B_{1,q} + \Delta B_{\text{DIS}} + B_{2,q}) \right] \right. \]

\[ + \left. \int_{Q^2}^{Q^2/\overline{N}} \frac{d\mu^2}{\mu^2} A_q(\alpha_s(\mu^2)) \ln \overline{N} \right\}. \]  

(73)

Matching the two results above, and noting that

\[ \left( \frac{\partial}{\partial \ln \overline{N}} \right) f(\alpha_s(Q^2/\overline{N})) = \frac{1}{2} \partial_{\alpha_s} f(\alpha_s(Q^2/\overline{N})), \]  

(74)

we get the final relation between EFT and traditional approaches for the DIS case;

\[ (B_{1,q} + \Delta B_{\text{DIS}} + B_{2,q})(\alpha_s(\mu^2)) \]

\[ = B_q(\alpha_s(\mu^2)) + \frac{1}{2} \partial_{\alpha_s} \Gamma_2 \left( \frac{1}{2} \partial_{\alpha_s} \left[ A_q - \frac{1}{2} \partial_{\alpha_s} B_q \right] (\alpha_s(\mu^2)) \right), \]  

(75)

from which we can solve for \( B_q^{(i)} \). Up to third order we have

\[ B_q^{(1)} = -B_{2,q}^{(1)}, \]

\[ B_q^{(2)} = -B_{2,q}^{(2)} - f_q^{(2)} + \Delta B_{\text{DIS}}^{(2)} + \frac{1}{2} \zeta_2 \beta_0 A_q^{(1)}, \]

\[ B_q^{(3)} = -B_{2,q}^{(3)} - f_q^{(3)} + \Delta B_{\text{DIS}}^{(3)} + \beta_0 \zeta_2 A_q^{(2)} + \frac{1}{2} \zeta_2 \beta_1 A_q^{(1)} + \frac{2}{3} \zeta_3 \beta_0^2 A_q^{(1)}. \]  

(76)
Therefore, apart from running effects, $B_q$ is essentially $-B_{2,q} - f_q$. More explicitly, we get

$$B_q^{(1)} = -3C_F,$$

$$B_q^{(2)} = C_F^2 \left[ -\frac{3}{2} + 12\zeta_2 - 24\zeta_3 \right] + C_F C_A \left[ -\frac{3155}{54} + \frac{44}{3}\zeta_2 + 40\zeta_3 \right] \\
+ C_F N_F \left[ \frac{247}{27} - \frac{8}{3}\zeta_2 \right],$$

$$B_q^{(3)} = C_F^3 \left[ -\frac{29}{2} - 18\zeta_2 - 68\zeta_3 - \frac{288}{5}\zeta_2^2 + 32\zeta_2\zeta_3 + 240\zeta_5 \right] \\
+ C_A C_F^2 \left[ -46 + 287\zeta_2 - \frac{712}{3}\zeta_3 - \frac{272}{5}\zeta_2^2 - 16\zeta_2\zeta_3 - 120\zeta_5 \right] \\
+ C_A C_F \left[ \frac{599375}{729} + \frac{32126}{81}\zeta_2 + \frac{21032}{27}\zeta_3 - \frac{652}{15}\zeta_2^2 - \frac{176}{3}\zeta_2\zeta_3 - 232\zeta_5 \right] \\
+ C_F^2 N_F \left[ \frac{5501}{54} - 50\zeta_2 + \frac{32}{9}\zeta_3 \right] \quad \text{(77)} \\
+ C_A C_F N_F \left[ \frac{160906}{729} - \frac{9920}{81}\zeta_2 - \frac{776}{9}\zeta_3 + \frac{208}{15}\zeta_2^2 \right].$$

Those results agree with the ones in Ref. [20]. Similar to the case of DY and Higgs, we get after simple calculation

$$\mathcal{F}^{(1)}_{\text{DIS}} = 16C_F(-9 - 2\zeta_2),$$

$$\mathcal{F}^{(2)}_{\text{DIS}} = C_F^2 \left[ \frac{331}{8} + \frac{111}{2}\zeta_2 - 66\zeta_3 + \frac{4}{5}\zeta_2 \right] \\
+ C_F C_A \left[ \frac{5465}{72} - \frac{1139}{18}\zeta_2 - \frac{464}{9}\zeta_3 + \frac{51}{5}\zeta_2 \right] \\
+ C_F N_F \left[ \frac{457}{36} + \frac{85}{9}\zeta_2 + \frac{4}{9}\zeta_3 \right]. \quad \text{(78)}$$

Again these results agree with $g_{01}^{\text{DIS}}$ and $g_{02}^{\text{DIS}}$.

### C. Drell-Yan Coefficient Function Using DIS Parton Distributions

If one calculates the Drell-Yan coefficient function in terms of the DIS parton distributions, one has

$$\Delta_N = G_{N,q}/G_{N,\text{DIS}} \sim \int_0^1 dz z^{N-1} \int_{(1-z)Q^2}^{(1-z)^2Q^2} \frac{d\mu^2}{\mu^2} A_q(\alpha(\mu^2)) \\
+ D_q(\alpha_s((1-z)^2Q^2) - 2B_q((1-z)Q^2)), \quad \text{(79)}$$

We have seen from the last two subsections that if one ignores the running effects, $D_q \sim 2B_1 + 4B_2$ and $B_q \sim B_1 + B_2$. Hence the last two terms in the above equation is just $\sim 2B_2$ in EFT, negative of the coefficient in front of $\delta(1-x)$ in the DGLAP splitting function.
D. Performing the Integrals

Another way to compare the EFT results with the traditional ones is to carry out the integral \( I_1 + I_2 + I_3 \) directly, and compare the final form of the resummed result. We wish also to show that the way we arrive at the final result is much simpler than the existing one in the literature.

Specializing for the DY and Higgs case, the integral is then,

\[
I_1 + I_2 + I_3 = \int_{Q^2/N^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left[ A_{(q,g)}(\alpha_s(\mu^2)) \ln \frac{\mu^2 N^2}{Q^2} - (\Delta B_{(q,g)} - f_{(q,g)}) \right].
\]  

We also need the solution of the renormalization group equation for \( \alpha_s(\mu^2) \). Adopting the notation of Ref. [19] we have

\[
\alpha_s(\mu^2) = \frac{\alpha_s(Q^2)}{l} \left\{ 1 - \frac{\alpha_s(Q^2)}{l} \frac{b_1}{b_0} \ln l + \left( \frac{\alpha_s(Q^2)}{l} \right)^2 \left[ \frac{b_1^2}{b_0^2} (\ln^2 l - \ln l + l - 1) - \frac{b_2}{b_0} (\ln l - 1) \right] + \mathcal{O}(\alpha_s(Q^2)) \right\},
\]

where \( l = 1 + b_0 \alpha_s(Q^2) \ln \mu^2/Q^2 \) and \( b_i = \frac{1}{(4\pi)^{i-1}} \beta_i \). Let us start with the contribution of the \( A_{(q,g)}^{(1)} \) term. Changing the integration variable from \( \mu^2 \) to \( l \), this contribution gives

\[
I_{A_1} = \frac{A_{(q,g)}^{(1)}}{4\pi b_0} \int_{1-2\lambda}^{1} \frac{dl}{l} \left\{ 1 - \alpha_s(Q^2) \frac{b_1}{b_0} \ln l + \left( \frac{\alpha_s(Q^2)}{l} \right)^2 \left[ \frac{b_1^2}{b_0^2} (\ln^2 l - \ln l + l - 1) - \frac{b_2}{b_0} (\ln l - 1) \right] \right\} \left( 2 \ln N + \frac{l - 1}{b_0 \alpha_s(Q^2)} \right),
\]

where \( \lambda \equiv b_0 \alpha_s(Q^2) \ln \frac{N}{l} \). The last equation includes a pattern that repeats itself when other contributions are included. Taking as a working rule that \( \ln \frac{N}{l} \sim (1/\alpha_s(Q^2)) \), the last two terms give rise to comparable contributions, however inside the curly brackets we have expansion in \( \alpha_s(Q^2) \). Thus the hierarchy is manifest. Carrying out the integrals in Eq. (82) is very simple and we get

\[
I_{A_1} = \ln \frac{N}{l} \left\{ \frac{A_{(q,g)}^{(1)}}{4\pi b_0} \left[ 2 \lambda + (1 - 2\lambda) \ln(1 - 2\lambda) \right] \right\}
\]

\[
+ \frac{A_{(q,g)}^{(1)} b_1}{4\pi b_0^4} \left[ 2 \lambda + \ln(1 - 2\lambda) + \frac{1}{2} \ln^2(1 - 2\lambda) \right]
\]

\[
+ \alpha_s(Q^2) \frac{A_{(q,g)}^{(1)} b_1^2}{4\pi b_0^4} \left[ 2 \lambda^2 + 2 \lambda \ln(1 - 2\lambda) + \frac{1}{2} \ln^2(1 - 2\lambda) \right] \frac{1}{1 - 2\lambda}.
\]

Expanding the \( \lambda \)-terms in the last equation, we get a sum of the form \( \alpha_s^n(Q^2) \ln^{n+1} N \) from the first term, \( \alpha_s^n(Q^2) \ln^n N \) from the second term, and \( \alpha_s^{n+1} \ln^n N \) from the last term. These are commonly called: leading logarithms (LL), next-to-leading logarithms (NLL) and next-to-next-to leading logarithms (NNLL), respectively. Higher logarithmic accuracies follow easily.
Consider now the contribution from $A_{(q,g)}^{(2)}$. Similar to the $A_{(q,g)}^{(1)}$ contribution we get

$$I_{A_2} = \frac{A_{(q,g)}^{(2)}}{(4\pi)^2 b_0} \int_{1-2\lambda}^1 \frac{dl}{l^2} \alpha_s(Q^2) \left[ 1 - 2\alpha_s(Q^2) \frac{b_1}{b_0} \ln l + O(\alpha_s^3(Q^2)) \right]$$

$$\times \left( 2 \ln N + \frac{l - 1}{b_0 \alpha_s(Q^2)} \right), \quad (84)$$

so we see that $A_{(q,g)}^{(2)}$ does not contribute to the LL but starts from NLL. This contribution is

$$I_{A_2} = -\frac{A_{(q,g)}^{(2)}}{(4\pi)^2 b_0} [2\lambda + \ln(1 - 2\lambda)] - \alpha_s(Q^2) \frac{A_{(q,g)}^{(2)} b_1}{(4\pi)^2 b_0} [2\lambda + 2\lambda^2 + \ln(1 - 2\lambda)] . \quad (85)$$

From the $A_{(q,g)}^{(3)}$ term we get

$$I_{A_3} = \frac{A_{(q,g)}^{(3)}}{(4\pi)^3 b_0} \int_{1-2\lambda}^1 \frac{dl}{l^3} \alpha_s^2(Q^2) \left[ 1 + O(\alpha_s(Q^2)) \right] \left( 2 \ln N + \frac{l - 1}{b_0 \alpha_s(Q^2)} \right), \quad (86)$$

which is a NNLL contribution;

$$I_{A_3} = \alpha_s(Q^2) \frac{A_{(q,g)}^{(3)}}{(4\pi)^3 b_0^2} \frac{2\lambda}{1 - 2\lambda} . \quad (87)$$

The contribution from the term $\Delta B^{(i)} - f^{(i)}$ starts at NNLL accuracy since this term vanishes for $i = 0, 1$. From Eq. (56) we have $\Delta B_{(q,g)}^{(2)} - f_{(q,g)}^{(2)} = (1/2)(D_{(q,g)}^{(2)} - 4\beta_0 \zeta_2 A_{(q,g)}^{(1)})$. The contribution of this term gives

$$I_{B_2} = -\frac{1}{(4\pi)^2 b_0} \frac{1}{\alpha_s(Q^2)} [\Delta B_{(q,g)}^{(2)} - f_{(q,g)}^{(2)}] \int_{1-2\lambda}^1 \frac{dl}{l^2} \alpha_s^2(Q^2), \quad (88)$$

which is a NNLL contribution;

$$I_{B_2} = \alpha_s(Q^2) \frac{1}{(4\pi)^2 b_0} \left[ 4\beta_0 \zeta_2 A_{(q,g)}^{(1)} - D_{(q,g)}^{(2)} \right] \frac{\lambda}{1 - \lambda} . \quad (89)$$

Writing the sum of all contributions already obtained in the form of

$$I_{A_1} + I_{A_2} + I_{A_3} + I_{B_2} = \ln N g_{(q,g)}^{(1)} + g_{(q,g)}^{(2)} + \alpha_s(Q^2) g_{(q,g)}^{(3)} , \quad (90)$$
we get

\[ g_{(q,g)}^{(1)}(\lambda) = \frac{A_{(q,g)}^{(1)}}{4\pi b_0} \left[ \frac{2\lambda + (1 - 2\lambda) \ln(1 - 2\lambda)}{\lambda} \right], \]

\[ g_{(q,g)}^{(2)}(\lambda) = -\frac{A_{(q,g)}^{(2)}}{(4\pi)^2b_0^2} \left[ 2\lambda + \ln(1 - 2\lambda) \right] + \frac{A_{(q,g)}^{(1)} b_1}{4\pi b_0^3} \left[ 2\lambda + \ln(1 - 2\lambda) + \frac{1}{2} \ln^2(1 - 2\lambda) \right], \]

\[ g_{(q,g)}^{(3)}(\lambda) = \left[ \frac{4\xi_2 A_{(q,g)}^{(1)}}{4\pi} - \frac{D_{(q,g)}^{(2)}}{(4\pi)^2b_0} \right] \frac{\lambda}{1 - 2\lambda} + \frac{A_{(q,g)}^{(1)} b_1^2}{4\pi b_0^3} \left[ 2\lambda + 2\lambda \ln(1 - 2\lambda) + \frac{1}{2} \ln^2(1 - 2\lambda) \right] \]

\[ + \frac{A_{(q,g)}^{(1)} b_2}{4\pi b_0^3} \left[ 2\lambda + \ln(1 - 2\lambda) + \frac{2\lambda^2}{1 - 2\lambda} \right] + \frac{2A_{(q,g)}^{(3)} b_3}{(4\pi)^3b_0^2} \frac{\lambda^2}{1 - 2\lambda} \]

\[ - \frac{A_{(q,g)}^{(2)} b_1}{(4\pi)^2b_0^3} \left[ 2\lambda + 2\lambda^2 + \ln(1 - 2\lambda) \right] \frac{1}{1 - 2\lambda}. \]  

(91)

The above functions sum the large logarithms to LL, NLL and N^2LL, respectively. It is straightforward to get also the \( \alpha_s^2 g^{(4)} \) which resums the \( \text{N}^3\text{LL} \). It will contain contributions from \( A_{(q,g)}^{(i)} \) up to \( i = 4 \) and from \( D_{(q,g)}^{(2)} \) and \( D_{(q,g)}^{(3)} \). The yet uncalculated quantity \( A_{(q,g)}^{(4)} \) is the only missing piece to complete the \( \text{N}^3\text{LL} \) resummation program. The above results for \( g^{(i)} \) agree with those in [19, 42]. We remind the reader that we have set the factorization scale and the renormalization scale equal to \( Q^2 \) and the \( \gamma_E \) dependence is hidden in \( \gamma \) used throughout. The analysis for the DIS case can be performed similarly and one also gets agreement with the known results.

V. CONCLUSION

Threshold resummation of logarithmic enhancements due to soft gluon radiation has been performed using the methodology of effective field theory. This method works to any desired (subleading) logarithmic accuracy and it is completely equivalent to the more conventional, factorization-based techniques. This has been illustrated to all three inclusive processes we considered: DIS, DY and the SM Higgs production.

Conceptually and technically, however, this approach is much less complicated and it is physically more transparent than other ones. Working perturbatively in moment space (and for large values of \( N \)) we found that one does not need to introduce any additional nonperturbative quantities (other than the conventional PDFs), as is usually the case in the traditional approaches. All the quantities needed to get the resummed coefficient functions are straightforwardly obtained from fixed-order calculations of the form factors (which supply the \( C^{(i)} \) and the \( \gamma_1^{(i)} \)), the Altarelli-Parisi splitting kernels (which supply the \( \gamma_2^{(i)} \)) and the cross section for real gluon emission in the soft limit (from which we get the \( \mathcal{M}^{(i)} \)). It should be mentioned that the given treatment of DIS is applicable only in the Bjorken limit where one takes \( Q^2 \) to infinity first. However, for finite (but large) values of \( Q^2 \) where the scale \( Q^2 (1 - x)^2 \) would emerge, a different treatment is needed.

The method discussed in this paper can be extended straightforwardly to other processes.
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Note: While this paper is in writing, a paper by T. Becher and M. Neubert appeared in archive [hep-ph/0605050] which also uses EFT to resum the large logarithms for DIS. In their paper, the jet function is similar to the matching coefficient $\mathcal{M}$ here.
APPENDIX

In this appendix, we collect the coefficient functions for deep-inelastic scattering, Drell-Yan and the Higgs production (within the large top-quark mass effective theory) to $O(\alpha_s^2)$ in the soft limit of full QCD. They are used to extract the matching coefficients $M$ in Eqs. (34-36). As we have remarked in the main paper, these results must be reproduced by calculations of an EFT in which only the soft and collinear degrees of freedom are taken into account. For DIS (see Refs. \[30, 31\]), Drell-Yan (see Refs. \[26\]) and Higgs production (see Refs. \[28, 29\]), we have

\[
G_{\text{DIS}}^{(2), s+v}(x) = C_F^2 \left\{ \left[ 16C_1(x) + 12C_0(x) + \delta(1 - x) \left( \frac{9}{2} - 8\zeta_2 \right) \right] \ln^2 \left( \frac{Q^2}{\mu^2} \right) \\
+ \left[ 24C_2(x) - 12C_1(x) - (45 + 32\zeta_2)C_0(x) \right] \delta(1 - x) \left( -\frac{51}{2} - 12\zeta_2 + 40\zeta_3 \right) \ln \left( \frac{Q^2}{\mu^2} \right) \\
+ 8C_3(x) - 18C_2(x) - (27 + 32\zeta_2)C_1(x) + 2 \times 48 \times \left( -\frac{3}{4} \right) \zeta_3C_0(x) \\
+ \left( \frac{51}{2} + 36\zeta_2 + 64\zeta_3 \right) C_0(x) + \delta(1 - x) \left( \frac{331}{8} + 69\zeta_2 - 78\zeta_3 + 6\zeta_2^2 \right) \right\} \\
+ C_F N_F \left\{ \left[ \frac{4}{3} C_0(x) + \delta(1 - x) \right] \ln^2 \left( \frac{Q^2}{\mu^2} \right) + \left[ \frac{8}{3} C_1(x) - \frac{58}{9} C_0(x) \right] - \delta(1 - x) \left( \frac{19}{3} + \frac{16}{3} \zeta_2 \right) \ln \left( \frac{Q^2}{\mu^2} \right) + \frac{4}{3} C_2(x) - \frac{58}{9} C_1(x) + \left( \frac{247}{27} - \frac{8}{3} \zeta_2 \right) C_0(x) \\
+ \delta(1 - x) \left( \frac{457}{36} + \frac{38}{3} \zeta_2 + \frac{4}{3} \zeta_3 \right) \right\} \\
+ C_A C_F \left\{ \left[ -\frac{22}{3} C_0(x) - \frac{11}{2} \delta(1 - x) \right] \ln^2 \left( \frac{Q^2}{\mu^2} \right) \\
+ \left[ -\frac{44}{3} C_1(x) + \left( \frac{367}{9} - 8\zeta_2 \right) C_0(x) + \left( \frac{215}{6} + \frac{88}{3} \zeta_2 - 12\zeta_3 \right) \delta(1 - x) \right] \ln \left( \frac{Q^2}{\mu^2} \right) \\
- \frac{22}{3} C_2(x) + \left( \frac{367}{9} - 8\zeta_2 \right) C_1(x) + 36\zeta_3 C_0(x) \\
+ \left( -\frac{3155}{54} + \frac{44}{3} \zeta_2 + 4\zeta_3 \right) C_0(x) \\
+ \delta(1 - x) \left( -\frac{5465}{72} - \frac{251}{3} \zeta_2 + \frac{140}{3} \zeta_3 + \frac{71}{5} \zeta_2^2 \right) \right\}. \tag{92}
\]
\[ G_{q}^{(2),s+v}(z) = C_{F}^{2} \left\{ [64D_{1}(z) + 48D_{0}(z) + \delta(1 - z)(18 - 32\zeta)] \ln^{2} \left( \frac{Q^{2}}{\mu^{2}} \right) \\
+ [192D_{2}(z) + 96D_{1}(z) - (128 + 64\zeta)D_{0}(z) \\
+ \delta(1 - z)(-93 + 24\zeta + 176\zeta_{3})] \ln \left( \frac{Q^{2}}{\mu^{2}} \right) \\
+ 128D_{3}(z) - (256 + 128\zeta)D_{1}(z) + 256\zeta_{3}D_{0}(z) \\
+ \delta(1 - z) \left( \frac{511}{4} - 70\zeta_{2} - 60\zeta_{3} + \frac{8}{5}\zeta_{2} \right) \right\} \\
+ C_{F}N_{F} \left\{ \left[ \frac{8}{3}D_{0}(z) + 2\delta(1 - z) \right] \ln^{2} \left( \frac{Q^{2}}{\mu^{2}} \right) \\
+ \left[ \frac{32}{3}D_{1}(z) - \frac{80}{9}D_{0}(z) - \frac{34}{3}\delta(1 - z) \right] \ln \left( \frac{Q^{2}}{\mu^{2}} \right) \\
+ \frac{32}{3}D_{2}(z) - \frac{160}{9}D_{1}(z) + \left( \frac{224}{27} - \frac{32}{3}\zeta_{2} \right)D_{0}(z) + \delta(1 - z) \left( \frac{127}{6} - \frac{112}{9}\zeta_{2} + 8\zeta_{3} \right) \right\} \\
+ C_{A}C_{F} \left\{ \left( -\frac{44}{3}D_{0}(z) - 11\delta(1 - z) \right) \ln^{2} \left( \frac{Q^{2}}{\mu^{2}} \right) \\
+ \left[ -\frac{176}{3}D_{1}(z) + \left( \frac{536}{9} - 16\zeta_{2} \right)D_{0}(z) + \left( \frac{193}{3} - 24\zeta_{3} \right)\delta(1 - z) \right] \ln \left( \frac{Q^{2}}{\mu^{2}} \right) \\
- \frac{176}{3}D_{2}(z) + \left( \frac{1072}{9} - 32\zeta_{2} \right)D_{1}(z) + \left( -1616 \frac{27}{27} + \frac{176}{3}\zeta_{2} + 56\zeta_{3} \right)D_{0}(z) \\
+ \delta(1 - z) \left( -\frac{1535}{12} + \frac{592}{9}\zeta_{2} + 28\zeta_{3} - \frac{12}{5}\zeta_{2}^{2} \right) \right\}. \tag{93} \]
\[ G_{N, DIS}^{(2), s+v} = C_F^2 \left\{ \left[ 8 \ln^2 \overline{N} - 12 \ln \overline{N} + \frac{9}{2} \right] \ln^2 \left( \frac{Q^2}{\mu^2} \right) \\
+ \left[ -8 \ln^3 \overline{N} - 6 \ln^2 \overline{N} + (45 + 8\zeta_2) \ln \overline{N} - \frac{51}{2} - 18\zeta_2 + 24\zeta_3 \right] \ln \left( \frac{Q^2}{\mu^2} \right) \\
+ 2 \ln^4 \overline{N} + 6 \ln^3 \overline{N} - \left( \frac{27}{2} + 4\zeta_2 \right) \ln^2 \overline{N} \right. \\
\left. + \left( -\frac{51}{2} - 18\zeta_2 + 24\zeta_3 \right) \ln \overline{N} + \frac{315}{8} + \frac{111}{2} \zeta_2 - 66\zeta_3 + \frac{4}{5} \zeta_2^2 \right) \\
+ C_F N_F \left\{ \left[ -\frac{4}{3} \ln \overline{N} + 1 \right] \ln^2 \left( \frac{Q^2}{\mu^2} \right) + \left( \frac{4}{3} \ln^2 \overline{N} + \frac{58}{9} \ln \overline{N} - \frac{19}{3} - 4\zeta_2 \right) \ln \left( \frac{Q^2}{\mu^2} \right) \\
- \frac{4}{9} \ln^3 \overline{N} - \frac{29}{9} \ln^2 \overline{N} + \left( -\frac{247}{27} + \frac{4}{3} \zeta_2 \right) \ln \overline{N} \\
+ \frac{457}{36} + \frac{85}{9} \zeta_2 + \frac{4}{9} \zeta_3 \right) + C_A C_F \left\{ \left[ \frac{22}{3} \ln \overline{N} - \frac{11}{2} \right] \ln^2 \left( \frac{Q^2}{\mu^2} \right) \\
+ \left( -\frac{22}{3} \ln^2 \overline{N} - \left( \frac{367}{9} - 8\zeta_2 \right) \ln \overline{N} + \frac{215}{6} + 22\zeta_2 - 12\zeta_3 \right) \ln \left( \frac{Q^2}{\mu^2} \right) \\
+ \frac{22}{9} \ln^3 \overline{N} + \left( \frac{367}{18} - 4\zeta_2 \right) \ln^2 \overline{N} + \left( \frac{3155}{54} - \frac{22}{3} \zeta_2 - 40\zeta_3 \right) \ln \overline{N} \\
- \frac{5465}{72} - \frac{1139}{18} \zeta_2 + \frac{464}{9} \zeta_3 + \frac{51}{5} \zeta_2^2 \right) \right\}. \] \]

\[ G_{N, q}^{(2), s+v} = C_F^2 \left\{ \left[ 32 \ln^2 \overline{N} - 48 \ln \overline{N} + 18 \right] \ln^2 \left( \frac{Q^2}{\mu^2} \right) \\
+ \left[ -64 \ln^3 \overline{N} + 48 \ln^2 \overline{N} + (128 - 128\zeta_2) \ln \overline{N} - 93 + 72\zeta_2 + 48\zeta_3 \right] \ln \left( \frac{Q^2}{\mu^2} \right) \\
+ 32 \ln^4 \overline{N} - (128 - 128\zeta_2) \ln^2 \overline{N} + \frac{511}{4} - 198\zeta_2 - 60\zeta_3 + \frac{552}{5} \zeta_2^2 \right\} \\
+ C_F N_F \left\{ \left[ -\frac{8}{3} \ln \overline{N} + 2 \right] \ln^2 \left( \frac{Q^2}{\mu^2} \right) + \left[ \frac{16}{3} \ln^2 \overline{N} + \frac{80}{9} \ln \overline{N} - \frac{34}{3} + 16 \zeta_2 \right] \ln \left( \frac{Q^2}{\mu^2} \right) \\
- \frac{32}{9} \ln^3 \overline{N} - \frac{80}{9} \ln^2 \overline{N} - \frac{224}{27} \ln \overline{N} + \frac{127}{6} - \frac{192}{9} \zeta_2 + \frac{8}{9} \zeta_3 \right) \\
+ C_A C_F \left\{ \left[ \frac{44}{3} \ln \overline{N} - 11 \right] \ln^2 \left( \frac{Q^2}{\mu^2} \right) \right\}. \]
\[ + \left[ 88 \ln^2 N - \left( \frac{536}{9} - 16 \zeta_2 \right) \ln N + \frac{193}{3} - \frac{88}{3} \zeta_2 - 24 \zeta_3 \right] \ln \left( \frac{Q^2}{\mu^2} \right) \\
+ \frac{176}{9} \ln^3 N + \left( \frac{536}{9} - 16 \zeta_2 \right) \ln^2 N + \left( \frac{1616}{27} - 56 \zeta_3 \right) \ln N \\
- \frac{1535}{12} + \frac{1128}{9} \zeta_2 + \frac{604}{9} \zeta_3 - \frac{92}{5} \zeta_2^2 \right] \]  

(97)

\[ G_{N, g}^{(2), s+v} = C_A^2 \left\{ \left[ 32 \ln^2 N + \frac{44}{3} \ln N \right] \ln^2 \left( \frac{Q^2}{\mu^2} \right) \\
+ \left[ -64 \ln^3 N - \frac{176}{6} \ln^2 N - \left( \frac{536}{9} + 112 \zeta_2 \right) \ln N - 24 - \frac{176}{3} \zeta_2 + 24 \zeta_3 \right] \ln \left( \frac{Q^2}{\mu^2} \right) \\
+ 32 \ln^4 N + \frac{176}{9} \ln^3 N + \left( \frac{536}{9} + 112 \zeta_2 \right) \ln^2 N \\
+ \left( \frac{1616}{27} - 56 \zeta_3 \right) \ln N + 93 + \frac{1072}{9} \zeta_2 - \frac{308}{9} \zeta_3 + 92 \zeta_2^2 \\
+ C_A N_F \left\{ \left[ -\frac{8}{3} \ln N \right] \ln^2 \left( \frac{Q^2}{\mu^2} \right) + \left[ \frac{16}{3} \ln^2 N + \frac{80}{9} \ln N + 8 + \frac{32}{3} \zeta_2 \right] \ln \left( \frac{Q^2}{\mu^2} \right) \\
- \frac{32}{9} \ln^3 N - \frac{80}{9} \ln^2 N - \frac{224}{27} \ln N - \frac{80}{3} - \frac{160}{9} \zeta_2 - \frac{88}{9} \zeta_3 \right] \right\} \\
+ C_F N_F \left\{ 4 \ln \left( \frac{Q^2}{\mu^2} \right) - \frac{67}{3} + 16 \zeta_3 \right\} \right\} . \]  

(98)

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