Gap generation in the BCS model with finite range temporal interaction

Vieri Mastropietro

Dipartimento di Matematica, Università di Roma “Tor Vergata”
Via della Ricerca Scientifica, I-00133, Roma

Abstract. In the [BCS] paper the theory of superconductivity was developed for the BCS model, in which the (instantaneous) interaction is only between fermions of opposite momentum and spin. Such model was analyzed by variational methods, finding that a superconducting behavior is energetically favorable. Subsequently it was claimed that in the thermodynamic limit the BCS model is equivalent to the (exactly solvable) quadratic mean field BCS model; a rigorous proof of this claim is however still lacking. In this paper we consider the BCS model with a finite range temporal interaction, and we prove rigorously its equivalence with the mean field BCS model in the thermodynamic limit if the range is long enough, by a (uniformly convergent) perturbation expansion about mean field theory.

1. Introduction and main results

Bardeen, Cooper and Schreiffer [BCS] developed their theory describing superconductors by the BCS model, in which the interaction has infinite range and only fermions of opposite momentum and spin (Cooper pairs) interact; the Hamiltonian of this model is

\[ H_{BCS} = \sum_\sigma \int_V d\vec{x} a_{\vec{x},\sigma}^+ \left( -\frac{\partial^2}{2m} \right) a_{\vec{x},\sigma} - \frac{\lambda}{V} \left[ \sum_\sigma \int_V d\vec{x} a_{\vec{x},\sigma}^+ a_{\vec{x},-\sigma} \right] \sum_\sigma \int_V d\vec{y} a_{\vec{y},\sigma}^+ a_{\vec{y},-\sigma} \] (1.1)

where \( a_{\vec{x},\sigma}^+ \) are creation or annihilation fermionic field operators with spin \( \sigma \) in a \( d \)-dimensional box with side \( L \) and \( V = L^d \), \( m \) is the mass and \( \lambda > 0 \) is the (attractive) coupling. By a variational procedure it was found that it was energetically favorable to form a superconducting phase. Later on it was realized that the properties of such superconducting phase are identical to the ones of the mean field BCS model, an exactly solvable model in which the interaction is quadratic and the Hamiltonian has the form

\[ H_{MF} = |\Delta|^2 + \sum_\sigma \int_V d\vec{x} a_{\vec{x},\sigma}^+ \left( -\frac{\partial^2}{2m} \right) a_{\vec{x},\sigma} - \sqrt{\lambda} |\Delta| \left[ \sum_\sigma \int_V d\vec{y} a_{\vec{y},\sigma}^+ a_{\vec{y},-\sigma} \right] - \sqrt{\lambda} |\Delta| \left[ \sum_\sigma \int_V d\vec{y} a_{\vec{y},\sigma}^+ a_{\vec{y},-\sigma} \right] \] (1.2)

where \( \Delta \) is a complex number to be determined minimizing the ground state energy (that is \( \Delta \) solves the BCS gap equation). It has been argued in several papers, starting from [BR],[B],[H], that in the limit \( V \rightarrow \infty \) the reduced BCS model (1.1) and the mean field model (1.2) have the same correlation functions; this seems quite natural also by analogy with lattice classical statistical mechanics in which infinite range interaction gives mean field behavior in the thermodynamic limit. Indeed many arguments has been given to support this claim in the last fifty years but, as far as I known, a rigorous proof is still lacking; aim of this paper is show that a simple proof of this claim can be given at least if the instantaneous interaction in the reduced BCS model (1.1) is replaced with a long (but finite) range time interaction.

We consider then a generalization of the reduced BCS model in which fermions are on on a cubic lattice with step 1 and a time-dependent interaction between Cooper pairs is considered; indeed, as
stressed for instance in [CEKO], a realistic model for superconductivity should include a bosonic Hamiltonian describing phonons and a boson-fermion interaction, which can be written in a purely fermionic model only if *time dependent* interaction between fermions is included. The two point Schwinger function of the reduced BCS model on a lattice with time dependent interaction can be written as Grassmann functional integral in the following way

\[
< \tilde{\psi}_{k,\sigma}^+ \tilde{\psi}_{k',\sigma'}^+ >_{L,\beta,M} = \frac{\int P(\psi^e) e^{-\beta H} \sum_{\sigma} \int dx \psi_{k,\sigma}^e \psi_{k',\sigma'}^e \psi_{k,\sigma}^+ \psi_{k',\sigma'}^+}{\int P(\psi^e) e^{-\beta H} \sum_{\sigma} \int dx \psi_{k,\sigma}^e \psi_{k',\sigma'}^e} 
\]

(1.3)

where \( \int dx = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dx_0 \sum_{x \in \Lambda} \) and \( \Lambda \) is a \( d \)-dimensional lattice with step 1 and

\[
\mathcal{V} = \frac{\lambda}{L^d} \sum_{\sigma,\sigma'} \int dx \int dy v(x_0 - y_0) \psi_{x,\sigma}^+ \psi_{y,\sigma}^- \psi_{x,\sigma'}^+ \psi_{y,\sigma'}^- 
\]

(1.4)

In the above expression \( \{ \psi_{k,\sigma}^\pm \} \) is a set of Grassmannian variables, \( k \in \mathcal{D}_{L,\beta} \) where \( \mathcal{D}_{L,\beta} = \mathcal{D}_L \times \mathcal{D}_\beta \), with \( \mathcal{D}_L = \{ k = 2\pi n / L, n \in \mathbb{Z}^d, -[L/2] \leq n_i \leq [(L - 1)/2] \} \) and \( \mathcal{D}_\beta = \{ k_0 = 2(n + 1/2)\pi/\beta, n \in \mathbb{Z}, -M \leq n \leq M - 1 \} \), and \( P(\psi^e) \) is a linear functional on the generated Grassmann algebra such that

\[
\int P(\psi^e) \psi_{k_1,\sigma_1}^+ \psi_{k_2,\sigma_2}^- = L^d \delta_{k_1,k_2} \delta_{\sigma_1,\sigma_2} \hat{g}(\mathbf{k}_1), \quad \hat{g}(\mathbf{k}) = \frac{1}{-i k_0 + \varepsilon(\mathbf{k}) - \mu} 
\]

(1.5)

where

\[
\varepsilon(\mathbf{k}) = \sum_{i=1}^d (1 - \cos k_i) 
\]

(1.6)

is the dispersion relation and \( \mu \) is the chemical potential. We define also Grassmannian field \( \psi_{x,\sigma}^\pm \) is defined by, if \( x = (x_0, x) \) with \( x_0 \in (-\frac{\beta}{2}, \frac{\beta}{2}) \) and \( x = (x_1, ..., x_d) \) with \( i = 1, 2, ..., L, \)

\[
\psi_{x,\sigma}^\pm = \frac{1}{L^d \beta} \sum_{\mathbf{k} \in \mathcal{D}_{L,\beta}} \hat{\psi}_{\mathbf{k},\sigma}^\pm e^{\pm i \mathbf{k} \cdot \mathbf{x}} 
\]

(1.7)

The external field \( h \) is introduced to break the number symmetry and which will be removed after the thermodynamic limit \( L \to \infty \) will be taken and \( v(x_0 - y_0) \) is a *Kac potential* with a long but finite range potential \( \kappa^{-1} \); for definiteness we choose

\[
v(t) = \frac{1}{\beta} \sum_{k_0 = \frac{\pi n_0}{\beta}, n_0 = 0, \pm 1, \pm 2, ..., \pm M} e^{-ik_0 t} \frac{\kappa^2}{k_0^2 + \kappa^2} 
\]

(1.8)

Finally \( M \) is an ultraviolet cutoff in the time direction introduced to make the Grassmann integral well defined, and and the limit \( M \to \infty \) must be taken before the thermodynamic limit \( V \to \infty \).

As we mentioned above, it was claimed in [BR] that the reduced BCS model (not solvable) should be equivalent (in the sense that the Schwinger functions coincide) to the mean field BCS model (solvable) in the limit \( L \to \infty \). In [BZT] indeed it was shown at each order of the perturbative expansion that the difference of the correlation functions between the reduced and the BCS model goes as \( O(V^{-1}) \) at each order, but to make this argument rigorous one has to prove the uniformity of the convergence of the perturbative expansion. A similar perturbative argument in a more modern (RG) language has been given in [SHML], in which it is pointed out the similarity of the perturbative expansion of the reduced BCS model with the so called \( O(1/V) \) expansion. In [B] and [H] the proof of such equivalence was based on the idea that the spatial averages of field operators like \( V^{-1} \int d\mathbf{x} a_0^+ a_{\sigma,\mathbf{x}}^+ \) may be substituted by numbers in the thermodynamic limit, since commutators with them has an extra one volume factor. Such a proof was criticized by several authors; for instance in [TW] it was shown that the convergence of the reduced BCS to the mean
field model is true only in a rather small subspace and not in general. In [M] a new proof of the equivalence based on a functional integral approach was given, but in the analysis involves unjustified exchange of the $L \to \infty$ limit with the $M \to \infty$ limit. Finally in [T] a correct proof of such equivalence was given, but only under that rather unrealistic assumption the the dispersion relation is a constant (degenerate BCS model).

It is apparently surprising the difficulty in proving that a infinite range interaction interaction like the one in (1.1) leads to a mean field behavior in the thermodynamic limit; indeed in classical statistical mechanics for spin lattice systems the proof of a similar statement is a two line computation. The difficulty in the quantum case can be clearly understood in the functional integral formulation (1.3); in such a representation the interaction $\mathcal{V}$ is not factorized contrary to what happen in the Hamiltonian formulation, and this make the model not exactly solvable. Of course by replacing $v(x_{0} - y_{0})$ in (1.3) with a constant (that is we consider an infinite range interaction $\kappa^{-1} = \infty$) the interaction in the functional integral is factorized and the model is exactly solvable; mean field behavior in the thermodynamic limit is then easily established, by performing a saddle point analysis essentially identical to the one for long range spin systems, see [L].

Aim of this paper is to prove that even if the range $\kappa^{-1}$ in (1.8) is finite, so that the interaction is not factorized and the model not solvable, the BCS model (1.3) is equivalent to the mean field BCS model if $\kappa$ is large enough, in the limit $V \to \infty$; that is the BCS model has a phase transition into a superconducting state described by the BCS theory.

Our main result is the following.

**Theorem** Assume $\mu < 2$ and $\lambda > 0$; there exist $\beta_{c}(\lambda)$ and $\kappa_{0}(\beta) > 0$ such that for $\beta \geq \beta_{c}(\lambda)$ and $0 < \kappa < \kappa_{0}(\beta)$ the Schwinger functions (1.3) with interaction (1.8) are such that

\[
\lim_{h \to 0^+} \lim_{L \to \infty} \langle \psi^{-}_{\mathbf{k}, \sigma} \psi^{+}_{\mathbf{k}, \sigma} \rangle = \frac{-i\varepsilon_{\mathbf{k}} - \mu}{k^{2} + (\varepsilon_{\mathbf{k}} - \mu)^{2} + \lambda|\Delta|^{2}}
\]

\[
\lim_{h \to 0^+} \lim_{L \to \infty} \langle \psi^{+}_{\mathbf{k}, \sigma} \psi^{+}_{-\mathbf{k}, \sigma} \rangle = \frac{\sqrt{\lambda\Delta}}{k^{2} + (\varepsilon_{\mathbf{k}} - \mu)^{2} + \lambda|\Delta|^{2}}
\]

where $\Delta \equiv \Delta(\beta)$ is the positive solution of the BCS gap equation

\[
1 = \lambda \int \frac{d^{d}k}{(2\pi)^{d}} \tanh(\frac{\beta}{2} \sqrt{E^{2}(\mathbf{k}) + \lambda\Delta^{2}}) \over 2((\varepsilon_{\mathbf{k}} - \mu)^{2} + \lambda\Delta^{2})
\]

and $\beta_{c}(\lambda)$ is the minimal $\beta$ for which (1.11) admits a solution.

The above Theorem ensures that, at a fixed temperature $\beta$ and for range $\kappa^{-1}$ large enough, the BCS model has the same behavior of the mean field model; in particular for $\beta \geq \beta_{c}$ a gap is generated and the particle number symmetry is broken as $\langle \psi^{+}_{\mathbf{k}, \sigma} \psi^{+}_{-\mathbf{k}, \sigma} \rangle$ is different from zero; this means that there is a phase transition into a superconducting phase for temperatures low enough. As an immediate corollary, it follows that for an interaction like (1.8) with an exponentially large range $O(e^{x})$ a gap is generated and the particle number symmetry is broken for temperatures small enough.

The proof of the above statement is by perturbation theory about the mean field theory, using as a perturbative parameter the inverse range $\kappa$ of the Kac potential (1.8); this is a classical approach in classical statistical mechanics to prove phase transition beyond mean field theory, see for instance [LMP]. We will show that the correction with respect to the mean field Schwinger function is expressed by a convergent series expansion (uniformly in the volume) and each order is $O(V^{-1})$; uniform convergence is established via determinant bounds for fermions. We can prove convergence only for small $\kappa$, as it turns out that $\kappa_{0} = O(\beta^{-2})$; of course it would be very interesting to prove convergence up to $\kappa_{0} = O(1)$ or even for any $\kappa$, so obtaining a real solution of the BCS model with instantaneous interaction.
2. Partial Hubbard-Stratonovich transformation

In momentum space we can write the interaction $V$ in the following way

$$
V = - \frac{\lambda}{(\beta V)^3} \sum_{k,k',p_0} \sum_{n_0 \in \mathbb{Z}} \sum_{\sigma,\sigma'} v(p_0) \psi_{\alpha,k}^+ \psi_{-\sigma,-k+p_0}^- \psi_{\sigma',k'}^- \psi_{-\sigma',-k'+p_0}^-
$$

where we have used that $p_0 = k_{0,1} + k_{0,2} = \frac{2\pi}{L}(n_{0,1} + n_{0,2} + 1)$. We split the interaction $V$ as sum over two terms

$$
V = \tilde{V} + \hat{V}_2
$$

$$
\tilde{V} = - \frac{\lambda}{(\beta V)^3} \sum_{k,k',p_0} \sum_{n_0 \geq \frac{2\pi}{L}} \sum_{\sigma,\sigma'} v(p_0) \psi_{\alpha,k}^+ \psi_{-\sigma,-k}^- \psi_{\sigma',k'}^- \psi_{-\sigma',-k'}^-
$$

Note that $\tilde{V}$ can be written as, $\varepsilon = \pm$

$$
\tilde{V} = - \Delta^+ \Delta^- \quad \Delta^\varepsilon = \frac{\sqrt{\lambda}}{(\beta V)^{1/2}} D^\varepsilon = \frac{\sqrt{\lambda}}{(\beta V)^{1/2}} \sum_{\sigma} \sum_{k} \psi_{k,\sigma}^\varepsilon \psi_{-k,\sigma}^\varepsilon
$$

that is can be written as the product of the total number of Cooper pairs. Let us consider the generating function of the Schwinger functions

$$
e^{S_{L,\beta,h}(J)} = \int P(d\psi) e^{2\Delta^+ \Delta^- - \int \frac{2\pi}{\beta V} \Delta^+ - h \Delta^- - \int \frac{2\pi}{\beta V} \Delta^-} e^{\int dx \sum_{\sigma} [J_{x,\sigma}^+ \psi_{x,\sigma}^- + \psi_{x,\sigma}^+ J_{x,\sigma}^-]}$$

where $J^\pm$ are external Grassmann field, so that

$$
< \psi_{x,\sigma}^\varepsilon \psi_{y,\sigma}^{\varepsilon'} > = \frac{\partial^2}{\partial J_{x,\sigma}^\varepsilon \partial J_{y,\sigma}^{\varepsilon'}} S(J)|_{J=0}
$$

By using the identity (Hubbard-Stratonovich transformation) ($\phi = u + iv$, $\bar{\phi} = u - iv$, $u, v \in R$)

$$
e^{2ab} = \frac{1}{2\pi} \int_{R^2} dudv e^{-\frac{1}{2}||\phi||^2} e^{a\phi + b\bar{\phi}}$$

we can rewrite the above expression as

$$
e^{S_{L,\beta,h}(J)} = \frac{1}{2\pi} \int_{R^2} dudv e^{-\frac{1}{2}||\phi||^2} \int P(d\psi) e^{-\int \frac{2\pi}{\beta V} \Delta^+ - (\phi - h \Delta^-)(\phi - h \Delta^-)} e^{\int dx \sum_{\sigma} [J_{x,\sigma}^+ \psi_{x,\sigma}^- + \psi_{x,\sigma}^+ J_{x,\sigma}^-]}$$

Performing the change of variables $(u, v) \rightarrow \sqrt{\beta V} (u, v)$ we obtain

$$
e^{S_{L,\beta,h}(J)} = \frac{\sqrt{\beta V}}{2\pi} \int_{R^2} dudv e^{-\frac{\sqrt{\beta V}}{2}(v^2 + (u + \frac{h}{\sqrt{\beta V}}))^2} e^{-\beta V \tilde{F}_{L,\beta,h}(u,v) + \beta V L_{\beta,h}(u,v,\phi)}$$

where

$$
e^{-\beta V \tilde{F}_{L,\beta,h}(u,v) + \beta V L_{\beta,h}(u,v,\phi)} = \int P(d\psi) e^{-\int \sqrt{\beta V} \phi^2 D^+ + \sqrt{\beta V} \phi^2 D^-} e^{\int dx \sum_{\sigma} [J_{x,\sigma}^+ \psi_{x,\sigma}^- + \psi_{x,\sigma}^+ J_{x,\sigma}^-]}$$
and by definition $B_{L,β,h}(u,v,J)$ is vanishing for $J = 0$ so that $F_{L,β,h}(u,v)$ is given by

$$e^{-β V F_{L,β,h}(u,v)} = \int P(dψ)e^{-\sqrt{β} \phi D^+ + \sqrt{β} φ D^-}$$

(2.12)

We are interested in computing the two point Schwinger function, given by (2.7)

$$\langle \psi_{k,σ}^{ε,ε'} | \psi_{-k,-σ}^{ε,ε'} \rangle = \frac{1}{Z_{L,β,h}} \int_{R^2} dudv e^{\frac{β}{2}(v^2 + u^2 + \frac{1}{4}κ^2)} e^{-β V F_{L,β,h}(u,v)} S_{L,β}^{ε,ε'}(k,u,v)$$

(2.13)

where $S_{L,β,h}(u,v) = \partial J_2 \partial J'_2 B(J, u, v)|_{J=0}$ and

$$Z_{L,β,h} = \int_{R^2} dudv e^{\frac{β}{2}(v^2 + u^2 + \frac{1}{4}κ^2)} e^{-β V F_{L,β,h}(u,v)}$$

(2.14)

We will show in the following section that

$$F_{L,β,h}(u,v) = t_{BCS} + \tilde{F}_{L,β,h}(u,v)$$

(2.15)

where, if $E(k) = ε(k) - μ$

$$t_{BCS} = \frac{1}{V} \sum_k 2β^{-1} \log \frac{\cosh(\frac{2}{β} \sqrt{E^2(k) + λφ^2})}{\cosh\frac{2}{β} E(k)}$$

(2.16)

is the free energy in the mean field BCS model [BCS] and $\tilde{F}_{L,β,h}$ is the perturbation to the mean field; we will show in the following section that, for $0 < κ < κ_0(β)$, $κ_0(β) = C^{-1} β^{-d+6}$, for a suitable constant $C$

$$|\tilde{F}_{L,β,h}(u,v)| \leq C \frac{λ}{V} (κ^2 β^3) β^{d+2}$$

(2.17)

hence $V \tilde{F}_{L,β,h}(u,v)$ it is uniformly bounded as $V \to \infty$; it is more convenient to call $V \tilde{F}_{L,β,h}(u,v) \equiv \tilde{F}_{L,β,h}(u,v)$ and we can write the two point Schwinger functions as

$$\frac{1}{Z_{L,β,h}} \int_{R^2} dudv e^{\frac{β}{2}(v^2 + u^2 + \frac{1}{4}κ^2 + t_{BCS}(u,v))} e^{-β \tilde{F}_{L,β,h}(u,v)} S_{L,β}^{ε,ε'}(k,u,v)$$

(2.18)

By the saddle point Theorem, for $β$ large enough

$$\lim_{L \to \infty} \frac{e^{-β L(v^2 + u^2 + \frac{1}{4}κ^2 + t_{BCS}(u,v))}}{\int dudv e^{-β L(v^2 + u^2 + \frac{1}{4}κ^2 + t_{BCS}(u,v))}} = δ(u)δ(ν - ν_0)$$

(2.19)

where $ν_0$ is given by the negative (for $h > 0$) solution of

$$ν_0 [λ \int \frac{dκ}{(2π)^d} tanh(\frac{2}{β} \sqrt{E^2(k) + λν_0^2})] = 2|h|$$

(2.20)

In the limit $h \to 0$ it reduces to the BCS equation (1.11). Moreover we will show in the following section that $S_{L,β}^{ε,ε'} - S_{L,β}^{ε,ε'}_{BCS}$ is $O(V^{-1})$ so that the Theorem follows.

### 3. Convergence of series expansion

#### 3.1 The partition function
We can “absorb” the quadratic fermion term in the free interaction

\[
\int P(d\psi)e^{\sqrt{\lambda}\phi D^+ + \sqrt{\lambda}\phi D^-}e^{-\hat{V}(\psi)} = e^{-\beta V_{BCS}} \int P_\sigma(d\psi)e^{-\hat{V}(\psi)}
\]  

(3.1)

where

\[
\hat{V}(\psi) = -\frac{\lambda}{V} \sum_{\sigma,\sigma'} \int dx dy \tilde{v}(x_0 - y_0) \psi^+_{\sigma,\chi}(x) \psi^+_{\sigma',\chi}(y) \psi^-_{\sigma',\chi}(y)
\]

(3.2)

and

\[
\tilde{v}(x_0 - y_0) = \frac{1}{\beta} \sum_{k \neq 0} e^{ik_{0}t} \frac{k^2}{k^2 + k_0^2}
\]

(3.3)

and, if \( \sigma = \sqrt{\lambda} \phi \)

\[
P_\sigma(d\psi) = \prod_k \frac{d\hat{\psi}_k^+ d\hat{\psi}_k^-}{\mathcal{N}(k)} \left\{ -\frac{1}{V\beta} \sum_{k'} \sum_{\varepsilon,\varepsilon' = \pm} \hat{\psi}_{\varepsilon\varepsilon'k} \bar{T}_{\varepsilon\varepsilon'} \hat{\psi}_{\varepsilon\varepsilon'k} \right\}
\]

(3.4)

where \( \mathcal{N}(k) \) is the normalization of \( P_\sigma(d\psi) \) and

\[
t_{BCS} = -\frac{1}{V\beta} \sum_k \log \frac{k_0^2 + E^2(k) + |\sigma|^2}{k_0^2 + E^2(k)}
\]

(3.5)

and the \( 2 \times 2 \) matrix \( T(k') \) is given by

\[
T(k) = \begin{pmatrix}
-ik_0 + E(k) & \sigma \\
\bar{\sigma} & -ik_0 - E(k)
\end{pmatrix}
\]

(3.6)

We can equivalently write, see for instance \([L]\), \( t_{BCS} \) as (2.16) and of course \( t_1 \leq \sqrt{|\lambda|C[1 + |\phi|]} \).

The propagator of \( P_\sigma(d\psi) \) is given by

\[
\int P_\sigma(d\psi)\bar{\psi}_{\chi,\sigma} \psi_{y',\sigma'} \equiv g_{\varepsilon,\varepsilon'}(x, y) = \frac{1}{V\beta} \sum_k e^{-ik(x-y)} [T^{-1}(k)]_{\varepsilon,\varepsilon'}
\]

(3.7)

We decompose the free propagator \( \hat{g}_k \) into a sum of two propagators supported in the regions of \( k_0 \) “large” and “small”, respectively. The regions of \( k_0 \) large and small are defined in terms of a smooth support function \( H_0(t), \ t \in \mathbb{R} \), such that

\[
H_0(t) = \begin{cases} 
1 & \text{if } t < 1/\gamma , \\
0 & \text{if } t > 1 , 
\end{cases}
\]

(3.8)

with \( \gamma > 1 \). We define \( h(k_0) = H_0(|k_0|) \) so that we can rewrite \( \hat{g}_k \) as:

\[
\hat{g}_k = \hat{g}^{(u.v.)}(k) + \hat{g}^{(i.r.)}(k)
\]

(3.9)

where

\[
g^{(i.r.)}_{\varepsilon,\varepsilon'}(x, y) = \frac{1}{V\beta} \sum_k e^{-ik(x-y)} h(k_0) [T^{-1}(k)]_{\varepsilon,\varepsilon'}
\]

(3.10)

\[
g^{(u.v.)}_{\varepsilon,\varepsilon'}(x, y) = \frac{1}{V\beta} \sum_k e^{-ik(x-y)} (1 - h(k_0)) [T^{-1}(k)]_{\varepsilon,\varepsilon'}
\]

(3.11)

In the Appendix we show that

\[
\int P_\sigma(d\psi^{u.v.})e^{-\hat{V}(\psi^{i.r.} + \psi^{u.v.})} = e^{-\hat{V}(\psi^{i.r.})}
\]

(3.12)
with
\[ \mathcal{V}^0 = \frac{1}{V} \sum_{\sigma, \sigma'} \int dx dy \tilde{v}(x_0 - y_0) \tilde{\psi}^+_{x, \sigma} \tilde{\psi}^+_{x, -\sigma} \psi^-_{y, \sigma'} \psi^-_{y, -\sigma'} + \sum_{n=1}^{\infty} \int dx_1 \ldots \int dx_2n W_2^n(x_1, \ldots, x_{2n}) \prod_{i=1}^{2n} \psi^i_{x_i, \sigma_i}. \] (3.13)

with
\[ \frac{1}{V} \int dx_1 \ldots dx_2n |W_n(x_1, \ldots, x_{2n})| \leq C |\lambda|^{\max(1, n-1)} (\kappa^2 \beta^3)^{\max(1, n-1)} \] (3.14)

3.2 Convergence of the infrared integration

We define a distance \( d(x, y)_{L, \beta} = (d_{\beta}(x_0, y_0), d_{L}(x_1, y_1), \ldots, d_{L}(x_n, y_n) \) as
\[ d_{\beta}(x_0, y_0) = \frac{\beta}{\pi} \sin \left( \frac{\pi}{\beta} (x_0 - y_0) \right), \quad d_{L}(x_i, y_i) = \frac{L}{\pi} \sin \left( \frac{\pi}{L} (x_i - y_i) \right) \] (3.15)

In order to perform the infrared integration we need the large distances behaviour of the infrared propagator.

**Lemma** For any integer \( N \) the following bounds hold
\[ |g^{(i, r)}_{\varepsilon, \varepsilon'}(x, y)| \leq \frac{C_N}{1 + [\beta^{-1} d(x, y)]^N} \] (3.16)
\[ |g^{(i, r)}_{\varepsilon, \varepsilon'}(x, y)| \leq \frac{\sqrt{\lambda |\phi|}}{\sqrt{\lambda |\phi|} + \beta^{-1} d(x, y) + 1} \frac{C_N}{1 + [\beta^{-1} d(x, y)]^N} \] (3.17)

**Proof.** The above bounds follows by integrating by parts. Consider integers \( N_0, N_1, \ldots, N_d \) note that, \( i = 1, \ldots, d \)
\[ d_{L}(x_i, y_i)_{N_i} d_{\beta}(x_0, y_0)^{N_0} g^{(i, r)}_{\varepsilon, \varepsilon'}(x - y) = e^{-i \pi (x^L_{-N_i} y^L_{-N_0})} \frac{1}{V \beta} \sum_k e^{-i k (x - y)} \partial^N_k \partial^N_{0_k} \left( (1 - h(k_0))[T_0^{-1}(k')]_{\varepsilon, \varepsilon'} \right), \] (3.18)

where \( \partial_k \) and \( \partial_{0_k} \) denote the discrete derivatives. The bound then easily follows noting that \( T_0^{-1}(k')_{\varepsilon, \varepsilon'} \) is bounded by \( C \beta \) and each derivative over it is bounded by an extra \( \beta \). The non diagonal term has an extra \( \frac{|\phi|}{\sqrt{\lambda |\phi|}} \) in the bound, from which we see that the bound is uniform in \( \sigma \).\( \blacksquare \)

We can write
\[ \int P(d\psi^{(i, r)}) e^{-\mathcal{V}^0(\psi^{i, r})} = e^{\sum_{n=0}^{\infty} (-1)^n \mathcal{E}^n(\psi^{i, r} ; \psi^{i, r})} \] (3.19)
where \( \mathcal{E}^n \) are the fermionic truncated expectations
\[ \mathcal{E}^n(\psi ; n) = \frac{\partial^n}{\partial \lambda^n} \log \int P(d\psi) e^{\lambda \mathcal{X}(\psi)} \bigg|_{\lambda=0} \] (3.20)

We write (3.13) as
\[ \sum_P \int dP P W(x_P) \tilde{\psi}(P) \] (3.21)
where \( P \) is the set of field labels appearing in (3.13), \( W(x_P) \) are the kernels in (3.13), that is \( \mathcal{V}^{-1}(x_0 - y_0) \) or \( W(x_1, \ldots, x_{2n}) \) and
\[ \tilde{\psi}(P) = \prod_{f \in P} \psi^f_{x(f), \sigma(f)} \] (3.22)
Then we get
\[ \mathcal{E}^T(T^0; n) = \sum_{P_1, \ldots, P_n} \int dx_{P_1} \ldots \int dx_{P_n} W(x_{P_1}) \ldots W(x_{P_n}) \mathcal{E}^T(\tilde{\psi}(P_1), \ldots, \tilde{\psi}(P_n)) \] (3.23)

The fermionic truncated expectations can be bounded by the formula (see [GM] for example), if \( s > 1 \),
\[ \tilde{\mathcal{E}}^T(\tilde{\psi}(P_1), \ldots, \tilde{\psi}(P_n)) = \sum_{T} \prod_{t \in T} g_{t_{\iota'}, t_{\iota}}(x_t - y_t) \int dP_T(t) \det G_T(t), \] (3.24)
where
\[ \tilde{\psi}(P) = \prod_{f \in P} \psi_{x(f), \sigma(f)}^{\zeta(f)} (3.25) \]

and

a) \( T \) is a set of lines forming an anchored tree between the cluster of poinst \( P_1, \ldots, P_s \) i.e. \( T \) is a set of lines which becomes a tree if one identifies all the points in the same clusters.

c) \( t = \{ t_{i', \iota'} \in [0, 1], 1 \leq i, i' \leq s \} \), \( dP_T(t) \) is a probability measure with support on a set of \( t \) such that \( t_{i', \iota'} = u_i \cdot u_{i'} \) for some family of vectors \( u_i \in \mathbb{R}^s \) of unit norm.

d) \( G_T(t) \) is a \((N - s + 1) \times (N - s + 1)\) matrix, \( 2N = |P_1| + \ldots + |P_s| \) whose elements are given by \( G_{ij, i'j'}^T = t_{i', j} g_{\omega_{ij} - \omega_{i'j'}}(x_{ij} - y_{i'j'}) \) with \((f_{ij}, f_{i'j'})\) not belonging to \( T \).

If \( s = 1 \) the sum over \( T \) is empty, but we can still use the above equation by interpreting the r.h.s. as \( 1 \) if \( P_1 \) is empty, and \( \det G_T(P_1) \) otherwise.

We bound the determinant using the well known Gram-Hadamard inequality, stating that, if \( M \) is a square matrix with elements \( M_{ij} \) of the form \( M_{ij} = \langle A_i, B_j \rangle \), where \( A_i, B_j \) are vectors in a Hilbert space with scalar product \( \langle \cdot, \cdot \rangle \), then
\[ |\det M| \leq \prod_i ||A_i|| \cdot ||B_i||. \] (3.26)

where \( || \cdot || \) is the norm induced by the scalar product.

Let \( \mathcal{H} = \mathbb{R}^s \otimes \mathcal{H}_0 \), where \( \mathcal{H}_0 \) is the Hilbert space of complex four dimensional vectors \( F(k) = (F_1(k), \ldots, F_4(k)) \), \( F_i(k) \) being a function on the set \( \mathcal{D}_{-,-} \), with scalar product
\[ \langle F, G \rangle = \sum_{i=1}^{4} \frac{1}{L\beta} \sum_k F_i^*(k) G_i(k). \] (3.27)

and one checks that
\[ G_{ij, i'j'}^T = t_{i', j} g_{\omega_{ij} - \omega_{i'j'}}(x_{ij} - y_{i'j'}) = \langle u_i \otimes A_{\omega_{ij} - \omega_{i'j'}}, u_{i'} \gamma \otimes B_{\omega_{ij} - \omega_{i'j'}} \rangle >, \] (3.28)
where \( u_i \in \mathbb{R}^s, i = 1, \ldots, s \), are the vectors such that \( t_{i', \iota'} = u_i \cdot u_{i'} \), and
\[ A_{\omega, \omega}(k) = e^{i k x} \frac{\sqrt{\hbar(k_0)}}{\sqrt{k_0^2 + E^2 + |\sigma|^2}} \begin{cases} (-i k_0 + E(k), 0, 0, 1), & \text{if } \omega = +1, \\ (0, \sigma, 0, 1), & \text{if } \omega = -1, \end{cases}, \] (3.29)
\[ B_{\omega, \omega}(k) = e^{i k y} \frac{\sqrt{\hbar(k_0)}}{\sqrt{k_0^2 + E^2 + |\sigma|^2}} \begin{cases} (1, 1, 0, 0), & \text{if } \omega = +1, \\ (0, 0, 1, -i k_0 - E(k)), & \text{if } \omega = -1. \end{cases} \]

Hence from (3.26), as \( ||A|| \leq C \) and \( ||B|| \leq \beta \) we find
\[ |G_{ij, i'j'}^T| \leq C_1^{N-s+1} \beta^{N-s+1}. \] (3.30)
where $C_1$ is an $O(1)$ constant.

By using the above formula in (3.23) we get

$$|\mathcal{C}^T(V^0,\ldots,V^0)| \leq \sum_{P_1,\ldots,P_n} \beta \|P_1|+\ldots|P_n\| \int dx_{P_1} \ldots \int dx_{P_n} |W(x_{P_1})|\ldots|W(x_{P_n})| \sum_{T \in \mathcal{T}} |\beta^{-1} g_{\varepsilon,e'}(x_t-y_l)|$$

where we have used that $\int dP_T(t) = 1$. The number of addends in $\sum_T$ is bounded by $n!C^2$.

In order to bound the integration over propagators we use antiperiodicity

$$\int_{-\beta}^{\beta} dx_0 g_{\varepsilon,e'}(\vec{x},x_0) = \int_{-\beta/2}^{\beta/2} dx_0 g_{\varepsilon,e'}(\vec{x},x_0) + \int_{|x_0| \geq \beta/2} dx_0 g_{\varepsilon,e'}(\vec{x},x_0) = \int_{-\beta}^{\beta} dx_0[g_{\varepsilon,e'}(\vec{x},x_0)+g_{\varepsilon,e'}(x_0-\beta)]$$

The tree $T$ realizes a connection between all the $V$, and we get the bound

$$\int \prod_{i=1}^n dx_i \frac{1}{n!} \sum_{\mathcal{T} \in \mathcal{P}} |\beta^{-1} g_{\varepsilon,e'}(x_i-y_l)| \leq (\beta V)^{(n-1)(d+1)}$$

In order to perform the integration over the remaining coordinates we note that if $W(x_P) = V^{-1}\bar{v}(x_0)$ then

$$|\bar{v}(x_0)| = \frac{1}{\beta} \sum_{k_0 \neq 0} \frac{\kappa^2}{\kappa^2 + k_0^2} \leq \frac{1}{\beta} \sum_{n \neq 0} \frac{\kappa^2 \beta^2}{n^2} \leq \beta^{-1}(\kappa\beta)^2$$

so that

$$\int dxV^{-1}|\bar{v}(x_0)| \leq C(\kappa\beta)^2$$

if $C$ is a suitable constant. On the other hand if $W(x_P) = W(x_1,\ldots,x_{2n})$ we use the bound (3.14); then we get, assuming $\kappa\beta \leq C^{-1}$ in order to sum over $P_i$

$$|\mathcal{C}^T(V^0,n)| \leq n! \prod_{i=1}^n C P_i |\lambda|^{\text{max}(1,|P_i|/2-1)}(\kappa^2 \beta^3 \text{max}(1,|P_i|/2-1)) (\beta V)^{(n-1)(d+1)} \beta^{2n} \leq$$

$$(\beta V)n!C^2 \lambda^n (\kappa^2 \beta^3)^n \beta^{(d+3)n} \beta^{-(d+1)n} \leq (\beta V)n!C \lambda(\kappa^2 \beta^{d+6})^n \beta^{-(d+1)n} \beta$$

Finally the following bound can be found, calling $\mathcal{F}_{L,\beta,h} = t_{BCS} + \mathcal{F}_{L,\beta,h}$

$$|\mathcal{F}_{L,\beta,h}| \leq C\lambda(\kappa^2 \beta^{d+6})\beta^{-(d+1)}$$

assuming that $\kappa \leq C^{-1}\beta^{-\frac{d+6}{2}} = \kappa_0(\beta)$ to assuring the convergence of the sum over $n$.

**Remark** The above analysis immediately imply a bound for the effective potential $\int P(d\psi^{(i,r)}) e^{-\psi^0(\psi^{i,r} + \phi)}$ where $\phi$ is an external fermionic field. The kernels of the effective potential $W^{(n)}(x_1,\ldots,x_n)$ at order $n$ obey to the bound

$$\frac{1}{V^\beta} \int dx_1 \ldots dx_n |W_n(x_1,\ldots,x_n)| \leq C^n \lambda^n (\kappa^2 \beta^{d+6})^n \beta \beta^{-(d+1)}$$

as now the propagators are $2n - \frac{n}{2}$

### 3.3 Extracting a volume factor

The above analysis says that $\mathcal{F}_{L,\beta,h}$, which is the correction to the mean field, is given by a convergent expansion for sufficiently long range interaction $0 < \kappa < \kappa_0(\beta)$. We prove now that we can improve the above bound by a factor $V^{-1}$. 
Consider first the case in which in (3.23) there is at least a $W(P_1)$ associated to $\tilde{v}$. We can write, by using that for the fields in $\mathcal{E}^T$ holds the rule $\psi = \int d\xi' g(\xi - \xi') \frac{d}{d\xi}$

\[
\frac{1}{n!} \frac{1}{\beta V} \mathcal{E}^T(y; n) = \frac{1}{n!} \frac{1}{\beta V} \left( \frac{\lambda}{V} \int dx_1 dy_1 dx_a dx_b dx_c dx_d \tilde{v}(x_{01} - y_{01})
\right.
\]

\[g_{+,+a}(x_1 - x_a)g_{+,+b}(x_1 - x_b)g_{-,-a}(y_1 - x_a)g_{-,-b}(y_1 - x_b)H^{(4,n)}_{\epsilon,\epsilon, \epsilon, \epsilon, \epsilon}(x_1, x_b, x_c, x_d)
\]

\[+ 2 \frac{\lambda}{V} \int dx_1 dy_1 dx_a dx_b \tilde{v}(x_{01} - y_{01})g_{+,+a}(x_1 - x_a)g_{+,+b}(y_1 - x_b)H^{(4,n)}_{\epsilon,\epsilon, \epsilon, \epsilon, \epsilon}(x_1, x_b, x_c, x_d) + \tag{3.36}
\]

\[
\sum_{\epsilon = \pm} \frac{\lambda}{V} \int dx_1 dy_1 dx_a dx_b g_{-,+a}(x_1 - y_{01})g_{+,+b}(x_1 - x_a)g_{-,+b}(y_1 - x_b)H^{(2,n)}_{\epsilon,\epsilon, \epsilon, \epsilon, \epsilon}(x_1, x_b)
\]

where

\[
H^{(4,n)}_{\epsilon,\epsilon, \epsilon, \epsilon, \epsilon}(x_1, x_b, x_c, x_d) = \frac{\partial}{\partial \psi^a_{x_1}} \frac{\partial}{\partial \psi^c_{x_1}} \frac{\partial}{\partial \psi^d_{x_1}} \frac{\partial}{\partial \psi^b_{x_1}} \mathcal{E}^T(\tilde{v}; n - 1)
\]

\[
H^{(2,n)}_{\epsilon,\epsilon, \epsilon, \epsilon, \epsilon}(x_1, x_b) = \frac{\partial}{\partial \psi^a_{x_1}} \frac{\partial}{\partial \psi^b_{x_1}} \mathcal{E}^T(\tilde{v}; n - 1)
\]

Note that the last addend in (3.36) (corresponding to a tadpole contribution) is vanishing; in fact it can be written in momentum space as

\[
\frac{\lambda}{V} \mathcal{E}^T(0) \frac{1}{\beta} \sum_{p_0 \neq 0} \delta(p_0, 0) \tilde{v}(p_0) \frac{1}{\beta V} \sum_{k'} g_{-,+a}(k')g_{+,+b}(k') + p_0)H^{(2,n)}_{\epsilon,\epsilon, \epsilon, \epsilon, \epsilon}(k', p_0) = 0
\]

The first addend in (3.36) can be bounded in the following way, remembering that $|g_{-,+}(x, y)| \leq \beta$

\[
\leq C \frac{\lambda}{V} \beta^{-1}(\kappa \beta)^2 \beta^d \sup_{x_a} \left| \int dx_1 |g_{+,+a}(x_1 - x_a)| \right| \sup_{x_b} \left| \int dy_1 |g_{-,+b}(y_1 - x_b)| \right| \frac{1}{n!} \frac{1}{\beta V} \left| \int dx_a dx_b dx_c dx_d |H^{(4,n)}_{\epsilon,\epsilon, \epsilon, \epsilon, \epsilon}(x_1, x_b, x_c, x_d)| \right| \tag{3.39}
\]

The bound for the last integral is given by (3.35) with $n^\epsilon = 4$; hence the first addend in (3.36) obeys to the following bound

\[
C^n \frac{\lambda^n}{V} \beta^{-1}(\kappa \beta)^2 \beta^d \beta^{2(d+1)}(\kappa \beta^{2d+6}) \leq n \beta^{-2} \beta^{-(d+1)} \tag{3.40}
\]

so that summing over $n$ we have the bound $\frac{1}{V}(\kappa \beta)^2 \beta^{d+2}$.

Finally the second addend in (3.36) can be bounded by

\[
\frac{\lambda}{V} (\kappa \beta)^2 \beta \sup_{x_a} \left| \int dx_1 |g_{+,+a}(x_1 - x_a)| \right| \sup_{x_b} \left| \int dy_1 |g_{+,+b}(y_1 - x_b)| \right| \frac{1}{n!} \frac{1}{\beta V} \left| \int dx_a dx_b |H^{(2,n)}_{\epsilon,\epsilon, \epsilon, \epsilon, \epsilon}(x_a, x_b)| \right| \tag{3.41}
\]

and again using (3.35) with $n^\epsilon = 2$ we get that hence the second addend in (3.36) obeys to the following bound

\[
C^n \frac{\lambda^n}{V} \beta^{-1}(\kappa \beta)^2 \beta^3 \beta^{2(d+1)}(\kappa \beta^{2d+6}) \leq n \beta^{-1} \beta^{-(d+1)} \tag{3.42}
\]

Of course there is no $\tilde{v}$, we can apply the same reasoning to one of the kernel $W(x_1, \ldots x_n)$. Summing over $n$ we have the bound $\frac{1}{V}(\kappa \beta^{2d+3}) \beta^{d+2}$; then, for $k \leq k_0$ we get the better bound

\[
|\tilde{F}_{L,\beta,h}| \leq C \frac{\lambda}{V}(\kappa \beta^{2d+3}) \beta^{d+2} \tag{3.43}
\]
3.4 The integration of $S$

By performing the change of variables, if $\psi = (\psi^+, \psi^-)$ and $g$ is the matrix propagator of $P_\epsilon(d\psi)$, $\psi_k \to \psi_k + g\psi_k$, we get for two point Schwinger function the formula

$$S_{\epsilon,\epsilon'}(k, u, v) = g_{\epsilon,\epsilon'}(k) + \sum_{\epsilon',\epsilon''} g_{\epsilon,\epsilon''}(k)V_{2,\epsilon',\epsilon''}(k)g_{\epsilon'',\epsilon'}(k)$$

(3.44)

where $V_{2,\epsilon',\epsilon''}(k)$ is the kernel of the effective potential with two external fields; it can be bounded by

$$|V_{2,\epsilon,\epsilon'}(k)| \leq \frac{1}{\beta V} \int dx \int dy |V_{2,\epsilon,\epsilon'}(x, y)|$$

(3.45)

By using (3.35) we get that

$$|V_{2,\epsilon,\epsilon'}(k)| \leq C^n \lambda^n (\kappa^2 \beta^{d+6})^n \beta^{-1}(d+1)$$

(3.46)

By can improve the above bound as described in §3.3. If there is at least a $W = V^{-1} \tilde{v}$, we get

$$V_{2,\epsilon,\epsilon'}(x, y) = \frac{\lambda}{V} \int dz dz' dx_1 dx_2 dx_3 dx_4 g(z-x_1)g(z-x_2)g(z'-x_3)g(z'-x_4)$$

$$\int dx dx_1 dx_2 \int dz dz' g(z-z')g(z-x_1)g(z'-x_2)V_1(x, y, x_1, x_2, x_3, x_4)$$

where we have used that the tadpoles contributions is vanishing. The the integral over $x, y$ times $(\beta V)^{-1}$ of the first addend can be bounded, using also that $\sup |g| \leq \beta$

$$\frac{\lambda}{\beta V} (\kappa \beta)^2 \frac{1}{\beta V} \int dx \int dy dx_1 dx_2 dx_3 dx_4 \int dz dz'$$

$$|g(z-x_1)g(z-x_2)g(z'-x_3)g(z'-x_4)||S(x, y, x_1, x_2, x_3, x_4)| \leq$$

$$\frac{\lambda}{\beta V} (\kappa \beta)^2 \beta^2 (\sup_{x_1} |dz g(z-x_1)|) (\sup_{x_3} |dz' g(z'-x_3)|) \frac{1}{\beta V} \int dx \int dy dx_1 dx_2 dx_3 dx_4 |S(x, y, x_1, x_2, x_3, x_4)|$$

(3.47)

By using (3.35) we get for (3.44) the bound $\frac{\lambda}{\beta V} (\kappa \beta)^2 \beta^{d+1}$. On the other hand the second addend is bounded by

$$\frac{1}{\beta V^2} \beta (\sup_{x_1} |dz g(z-x_1)|) (\sup_{x_3} |dz' g(z'-x_3)|) \int dx \int dy dx_1 dx_2 |V_2(x, y, x_1, x_2)|$$

from again by using (3.35) we get the same bound $\frac{\lambda}{\beta V} (\kappa \beta)^2 \beta^{d+1}$. If there are no vertex $\tilde{v}$, we can repect the above argument on the kernels of (3.14). Hence from (3.44)

$$|S_{\epsilon,\epsilon'}(k, u, v) - g_{\epsilon,\epsilon'}(k)| \leq \frac{\lambda}{V} (\kappa^2 \beta^3)^{\beta^{d+1}}$$

(3.48)

Appendix A1. The ultraviolet integration

The integration of the ultraviolet part (3.12) can be done by a multiscale analyses; it is quite standard and we refer to §3 of [BM] for details in a similar case. It is convenient to introduce an ultraviolet cut-off $N$ by writing

$$g^{[1,N]}(x) = \sum_{k=1}^{N} g^{(k)}(x)$$

(3.1)

where

$$g^{(k)}(x) = \frac{1}{V^\beta} \sum_{k \in D_{\beta, L}} h_k(k_0) \frac{e^{-i k x}}{-i k_0 + \epsilon(k) - \mu}$$

(3.2)
with \( h_k(k_0) = H_0(\gamma^{-k}|k_0|) - H_0(\gamma^{-k+1}|k_0|) \). Note that \( \lim_{N \to \infty} g^{[1,N]}(x) = g^{(u,v)}(x) \) and that, for any integer \( K \geq 0 \), \( g^{(k)}(x) \) satisfies the bound, for any integer \( K \)

\[
|g^{(k)}(x)| \leq \frac{C_K}{1 + (\gamma^k|x| + |x|^2)^K} \tag{3.3}
\]

We associate to any propagator \( g^{(k)}(x) \) a Grassmann field \( \psi^{(k)} \) and a Gaussian integration \( P(d\psi^{(k)}) \) with propagator \( g^{(k)}(x) \). We can rewrite \( \mathcal{V}^{(0)} \) as:

\[
\mathcal{V}^{(0)}(\phi) + V\beta E_1 = -\lim_{N \to \infty} \log \int P(d\psi^{(0)})P(d\psi^{(1)}) \cdots P(d\psi^{(N)}) e^{-V(\psi^{[1,N]}+\phi)} \tag{3.4}
\]

We can integrate iteratively the fields on scale \( N, N-1, \ldots, k+1 \) and after each integration, using iteratively an identity like (3.19), we can rewrite the r.h.s. of (3.4) in terms of a new effective potential \( \mathcal{V}^{[1,k]} \):

\[
(3.4) = \lim_{N \to \infty} \left\{ V\beta \sum_{j=k+1}^{N} E_j - \log \int P(d\psi^{(1,0)})P(d\psi^{(1,1)}) \cdots P(d\psi^{(1,k)}) e^{-V^{[1,k]}(\psi^{[1,k]}+\phi)} \right\} \tag{3.5}
\]

with \( \mathcal{V}^{[k,N]}(\psi^{[1,k]}) \) admitting a representation in terms of trees defined in the following way:

1) Let us consider the family of all trees which can be constructed by joining a point \( r \), the root, with an ordered set of \( n \geq 1 \) points, the endpoints of the unlabeled tree, so that \( r \) is not a branching point. \( n \) will be called the order of the unlabeled tree and the branching points will be called the non trivial vertices. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol \(<\) to denote the partial order.

Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with \( n \) end-points is bounded by \( 4^n \).

We shall consider also the labeled trees (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

2) We associate a label \( k \geq 0 \) with the root and we denote \( T_{(k,N),n} \) the corresponding set of labeled trees with \( n \) endpoints. Moreover, we introduce a family of vertical lines, labeled by an integer taking values in \([k,N]\), and we represent any tree \( \tau \in T_{(k,N),n} \) so that, if \( v \) is an endpoint or a non trivial vertex, it is contained in a vertical line with index \( h_v > k \), to be called the scale of \( v \), while the root is on the line with index \( k \). There is the constraint that, if \( v \) is an endpoint, \( h_v = N + 1 \).

The tree will intersect in general the vertical lines in set of points different from the root, the endpoints and the non trivial vertices; these points will be called trivial vertices. The set of the verticals will be the union of the endpoints, the trivial vertices and the non trivial vertices.

Note that, if \( v_1 \) and \( v_2 \) are two vertices and \( v_1 < v_2 \), then \( h_{v_1} < h_{v_2} \).

Moreover, there is only one vertex immediately following the root, which will be denoted \( v_0 \) and can not be an endpoint; its scale is \( k + 1 \).

3) With each endpoint \( v \) of scale \( h_v = N + 1 \) we associate \( \hat{\mathcal{V}} \) (3.2). Given a vertex \( v \), which is not an endpoint, \( x_e \) will denote the family of all space-time points associated with one of the endpoints following \( v \).

4) The trees containing only the root and an endpoint of scale \( k + 1 \) will be called the trivial trees.

5) We introduce a field label \( f \) to distinguish the field variables appearing in the terms \( \hat{\mathcal{V}} \) associated with the endpoints. The set of field labels associated with the endpoint \( v \) will be called \( I_v \). Analogously, if \( v \) is not an endpoint, we shall call \( I_v \) the set of field labels associated with the endpoints following the vertex \( v \); \( x(f) \), and \( \sigma(f) \) will denote the space-time point, the \( \sigma \) index and the \( \omega \) index, respectively, of the field variable with label \( f \).

The effective potential can be written in the following way:

\[
\mathcal{V}^{(h)}(\psi^{(h)}) + L\beta \hat{E}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in T_{(h,N),n}} \mathcal{V}^{(h)}(\tau, \psi^{(h)}) \tag{3.6}
\]
where, if \( v_0 \) is the first vertex of \( \tau \) and \( \tau_1, \ldots, \tau_s (s = s_{v_0}) \) are the subtrees of \( \tau \) with root \( v_0 \),

\[
V^{(h)}(\tau, \psi^{(\leq h)}) = \sum_{\mathbf{P} \in \mathcal{P}} V^{(h)}(\tau, \mathbf{P}) ;
\]

(3.8)

\( V^{(h)}(\tau, \mathbf{P}) \) can be represented as

\[
V^{(h)}(\tau, \mathbf{P}) = \int d\mathbf{x}_{v_0} \tilde{\psi}^{(\leq h)}(P_{v_0}) K^{(h+1)}_{\tau, \mathbf{P}}(\mathbf{x}_{v_0}) ,
\]

(3.9)

with \( K^{(h+1)}_{\tau, \mathbf{P}}(\mathbf{x}_{v_0}) \) defined inductively (recall that \( h_{v_0} = h + 1 \) by the equation, valid for any \( v \in \tau \) which is not an endpoint,

\[
K^{(h+1)}_{\tau, \mathbf{P}}(\mathbf{x}_{v_0}) = \frac{1}{s_{v_0}} \prod_{i=1}^{s_{v_0}} \left[ K^{(h+1)}_{\tau_i}(\mathbf{x}_{v_i}) \right] ;
\]

(3.10)

\[
\cdot \mathcal{E}^{T}_{h_{v_0}} \left[ \tilde{\psi}^{(h)}(P_{v_1} \setminus Q_{v_1}), \ldots, \tilde{\psi}^{(h)}(P_{v_{v_0}} \setminus Q_{v_{v_0}}) \right] ;
\]

where \( iv \) is an endpoint \( K^{(h+1)}_{\tau, \mathbf{P}}(\mathbf{x}_{v_0}) \) is the kernel \( \tilde{\psi}(\mathbf{x}) \). We call \( \chi \)-vertices the vertices \( v \) of \( \tau \) such that \( \mathcal{E}^{T}_{h_{v}} \) is not trivial.

By using the representation of the truncated expectation analogous to (3.20) and the Gram inequality we get that the contribution from a tree \( \tau \in \mathcal{T}_{h[N]} \) associated to a kernel with 2l external legs can be bounded as (see §3.14 [BM] for details in a similar case):

\[
\frac{1}{V} \int d\mathbf{x}_1 \cdots d\mathbf{x}_2 |W_{2l}^{(h,N)}(\tau; \mathbf{x}_1, \sigma_1, \varepsilon_1; \ldots; \mathbf{x}_2, \sigma_2l, \varepsilon_2l)| \leq C^n |\lambda(\kappa/\beta)|^{2n} |\gamma^{-k(n-1)} v \not= e.p. \int d\mathbf{y} \tilde{\psi}(x_{0} - y_{0}) \psi^{+}_{x, \sigma, x^{-}} \psi^{-}_{y, \sigma, y^+} g^{h+}_{x, y}(\mathbf{x}, \mathbf{y}) \]

(3.12)
where we have used that the contraction of $\psi^+\psi^+\psi^-$ in $\hat{V}$ is vanishing by momentum conservation (remember that $p_0 = 0$). In (3.11) $g^{h_v^+} h_v^-$ is bounded with a constant; however such bound can be improved by writing

$$g^{h_v^+} h_v^-(x, y) = g^{h_v^+} h_v^-(\bar{x}, \bar{y}, x_0) + (x_0 - y_0) \int_0^1 dt \partial_{\bar{y}} g^{h_v^+} h_v^-(\bar{x}, \bar{y}, y_0 + t(x_0 - y_0))$$  

and noting that

$$|g^{h_v^+} h_v^- (x, x)| \leq \frac{1}{V} \sum_k e^{-i\tilde{E}x} f_k(k) \frac{-iE}{k_0^2 + E^2(k) + |\tilde{E}|^2} + \frac{1}{V} \sum_k e^{-i\tilde{E}x} f_k(k) \frac{E\tilde{k}}{k_0^2 + E^2(\tilde{k}) + |\tilde{E}|^2} \leq C_{\gamma}^{-h_v}$$

Then the contribution to the kernel of (3.12) coming from the first addend in the r.h.s. of (3.3) is bounded by $C_{\gamma}^{-h_v}$; in the contribution coming from the second addend we bound the propagator by $C_{\gamma}^{-h_v}$ so getting a factor $V^{-1} \int d\tau ||\tilde{v}(\tau)|| \leq C\beta$, so that at the hand we get a bound for (3.12) $C_{\beta\gamma}^{-h_v}$.

Then we get the bound

$$\frac{1}{V} \int d\tau \cdots d\tau_2 |W^{(k,N)}_{2l}(\tau; \tau_1, \tau_2, \cdots; \tau_2, \tau_2, \sigma_2, \sigma_2)| \leq C_n |\lambda(\kappa\beta)|^2 \gamma^{-k(n-1+n^{\text{pad}})} \beta^n \prod_{v \text{ not e.p.}} \gamma^{-(h_v-h_v')}(n_v-1+z_v),$$

where $z_v = 1$ if $n_v = 1$ and 0 otherwise, and $n^{\text{pad}}$ is the total number of $c$-vertices $v$ with $n_v = 1$ such that its set of internal lines is not empty. Then, proceeding as in §3.14 of [BM], one can sum over $\tau$ and the bound (3.14) is proved.

Finally by proceeding as in §3.3 it is easy to see that we can extract a factor $O(V^{-1})$ from each kernel $W$.

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