OPENLY FACTORIZABLE SPACES AND COMPACT EXTENSIONS OF TOPOLOGICAL SEMIGRAPHS

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Abstract. We prove that the semigroup operation of a topological semigroup $S$ extends to a continuous semigroup operation on its Stone-Čech compactification $\beta S$ provided $S$ is a pseudocompact openly factorizable space, which means that each map $f : S \to Y$ to a second countable space $Y$ can be written as the composition $f = g \circ p$ of an open map $p : X \to Z$ onto a second countable space $Z$ and a map $g : Z \to Y$. We present a spectral characterization of openly factorizable spaces and establish some properties of such spaces.

This paper was motivated by the problem of detecting topological semigroups that embed into compact topological semigroups. One of the ways to attack this problem is to find conditions on a topological semigroup $S$ guaranteeing that the semigroup operation of $S$ extends to a continuous semigroup operation on the Stone-Čech compactification $\beta S$ of $S$. A crucial step in this direction was made by E. Reznichenko [15] who proved that the semigroup operation on a pseudocompact topological semigroup $S$ extends to a separately continuous semigroup operation on $\beta S$. In this paper we show that the extended operation on $\beta S$ is continuous if the space $S$ is separable and openly factorizable, which means that each continuous map $f : S \to Y$ to a second countable space $Y$ can be written as the composition $f = g \circ p$ of an open continuous map $p : X \to Z$ onto a second countable space $Z$ and a continuous map $g : Z \to Y$. The class of openly factorizable spaces turned to be interesting by its own so we devote Sections 2, 3 to studying such spaces.

We recall that the Stone-Čech compactification of a Tychonov space $X$ is a compact Hausdorff space $\beta X$ containing $X$ as a dense subspace so that each continuous map $f : X \to Y$ to a compact Hausdorff space $Y$ extends to a continuous map $\overline{f} : \beta X \to Y$.

Replacing in this definition compact Hausdorff spaces by real complete spaces we obtain the definition of the Hewitt completion $\nu X$ of $X$. We recall that a topological space $X$ is real complete if $X$ is homeomorphic to a closed subspace of some power $\mathbb{R}^n$ of the real line. Thus a Hewitt completion of a Tychonov space $X$ is a real complete space $\nu X$ containing $X$ as a dense subspace so that each continuous map $f : X \to Y$ to a real complete space $Y$ extends to a continuous map $\nu f : \nu X \to Y$.

By [6, 3.11.16], the Hewitt completion $\nu X$ can be identified with the subspace

$$\{ x \in \beta X : G \cap X \neq \emptyset \text{ for any } G_{\delta}\text{-set } G \subset \beta X \text{ with } x \in G \}$$

of the Stone-Čech compactification $\beta X$ of $X$.

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The Hewitt completion $\nu X$ of a Tychonov space $X$ coincides with its Stone-Čech compactification $\beta X$ if and only if the space $X$ is pseudocompact in the sense that each continuous real-valued function on $X$ is bounded, see [3 §3.11]. On the other hand, if a Tychonov space $Z$ is real complete, then $\nu Z = Z$, see [3 3.11.12].

The problem of extending the group operation from a (para)topological group $G$ to its Stone-Čech or Hewitt extensions have been considered in [15], [2], [14], [16]. In this paper we address a similar problem for topological semigroups. All topological spaces appearing in this paper are Tychonov.

1. Semigroup compactifications of topological semigroups

In this section we recall some information on semigroup compactifications of a given (semi)topological semigroup $S$.

By a semitopological semigroup we understand a topological space $S$ endowed with a separately continuous semigroup operation $*: S \times S \to S$. If the operation is jointly continuous, then $S$ is called a topological semigroup.

Let $C$ be a class of compact Hausdorff semitopological semigroups. By a $C$-compactification of a semitopological semigroup $S$ we understand a pair $(C(S), \eta)$ consisting of a compact semitopological semigroup $C(S) \in C$ and a continuous homomorphism $\eta: S \to C(S)$ (called the canonic homomorphism) such that for each continuous homomorphism $h: S \to K$ to a semitopological semigroup $K \in C$ there is a unique continuous homomorphism $\bar{h}: C(S) \to K$ such that $h = \bar{h} \circ \eta$.

It follows that any two $C$-compactifications of $S$ are topologically isomorphic. We shall be interested in $C$-compactifications for the following classes of semigroups:

- WAP of compact semitopological semigroups;
- AP of compact topological semigroups;
- SAP of compact topological groups.

The corresponding $C$-compactifications of a semitopological semigroup $S$ will be denoted by $\text{WAP}(S)$, $\text{AP}(S)$, and $\text{SAP}(S)$. The notation came from the abbreviations for weakly almost periodic, almost periodic, and strongly almost periodic function rings that determine those compactifications, see [17 §III.2].

The inclusions of the classes $\text{SAP} \subset \text{AP} \subset \text{WAP}$ induce canonical continuous homomorphisms

$$\eta: S \to \text{WAP}(S) \to \text{AP}(S) \to \text{SAP}(S)$$

for each semitopological semigroup $S$. Since the space $\text{WAP}(S)$ is compact, the canonical map $\eta: S \to \text{WAP}(S)$ uniquely extends to a continuous map $\beta \eta: \beta S \to \text{WAP}(S)$ defined on the Stone-Čech compactification of $S$.

It should be mentioned that the canonic homomorphism $\eta: S \to \text{WAP}(S)$ needs not be injective. For example, for the group $H_+[0,1]$ of orientation-preserving homeomorphisms of the interval the WAP-compactification is a singleton, see [12]. However, for pseudocompact semitopological semigroups the situation is more optimistic. The following two results are due to E.Reznichenko [15]. They allow us to identify the WAP-compactification $\text{WAP}(S)$ of a (countably compact) pseudo-compact topological (semi)semigroup $S$ with the Stone-Čech compactification $\beta S$ of $S$. We recall that a topological space $X$ is countably compact if each countable open cover of $X$ has a finite subcover.
Theorem 1.1 (Reznichenko). For any countably compact semitopological semigroup $S$ the semigroup operation $S \times S \to S$ extends to a separately continuous semigroup operation $\beta S \times \beta S \to \beta S$, which implies that the canonic map $\beta \eta : \beta S \to \text{WAP}(S)$ is a homeomorphism.

The same conclusion holds for pseudocompact topological semigroups.

Theorem 1.2 (Reznichenko). For any pseudocompact topological semigroup $S$ the semigroup operation $S \times S \to S$ extends to a separately continuous semigroup operation $\beta S \times \beta S \to \beta S$, which implies that the canonic map $\beta \eta : \beta S \to \text{WAP}(S)$ is a homeomorphism.

If a topological semigroup $S$ has pseudocompact square, then its Stone-Čech compactification $\beta S$ coincides with its AP-compactification.

Theorem 1.3. For any topological semigroup $S$ with pseudocompact square $S \times S$ the semigroup operation $S \times S \to S$ extends to a continuous semigroup operation $\beta S \times \beta S \to \beta S$, which implies that the canonic maps $\beta S \to \text{WAP}(S) \to \text{AP}(S)$ are homeomorphisms.

Proof. By Theorem 1.2 the semigroup operation $\mu : S \times S \to S$ of $S$ extends to a separately continuous semigroup operation $\bar{\mu} : \beta S \times \beta S \to \beta S$ on $\beta S$. On the other hand, the operation $\mu : S \times S \to S \subset \beta S$ extends to a continuous map $\beta \mu : \beta(S \times S) \to \beta S \times \beta S$. By the Glicksberg Theorem [6, 3.12.20(c)], the pseudocompactness of the square $S \times S$ implies that the Stone-Čech extension $\beta i : \beta(S \times S) \to \beta S \times \beta S$ of the inclusion map $i : S \times S \to \beta S \times \beta S$ is a homeomorphism. Observe that the maps $\beta \mu$ and $\bar{\mu} \circ \beta i$ coincide on the dense subset $S \times S$ of $\beta S \times \beta S$. It is an easy exercise to check that those maps coincide everywhere, which implies that the map $\bar{\mu} = \beta \mu \circ (\beta i)^{-1}$ is continuous. This means that $\beta S$ is a compact topological semigroup and hence the canonic map $\beta \eta : \beta S \to \text{AP}(S)$ has continuous inverse. \[\square\]

It should be mentioned that for a pseudocompact topological semigroup $S$ the canonic map $\eta : S \to \text{AP}(S)$ needs not be a topological embedding. The following counterexample is constructed in [3].

Example 1.4. If there is a Tkachenko-Tomita group, then there is a countably compact topological semigroup $S$ for which the canonic homomorphism $\eta : S \to \text{AP}(S)$ is not injective.

By a Tkachenko-Tomita group we understand a commutative torsion-free countably compact topological group without non-trivial convergent sequences. The first example of such a group was constructed by M.Tkachenko [21] under the Continuum Hypothesis, which was later weakened to some forms of the Martin Axiom by A.Tomita et al. [22], [11], [7], [13]. We do not know if a Tkachenko-Tomita group exists in ZFC.

Example 1.4 shows that one should impose rather strong restrictions on a topological semigroup $S$ to guarantee that the canonic homomorphism $S \to \text{AP}(S)$ (or $S \to \text{SAP}(S)$) is an embedding.

Observe that for every semitopological semigroup $S$ its SAP-compactification $\text{SAP}(S)$ is a compact topological group. It is well-known that a semitopological semigroup $S$ is topologically isomorphic to a subgroup of a compact topological group if and if $S$ is a totally bounded topological group. We recall that a topological
group $G$ is called \textit{totally bounded} if for every non-empty open subset $U \subset G$ there is a finite subset $F \subset G$ such that $G = FU = UF$.

The following important result can be found in [17, III.3.3].

\textbf{Theorem 1.5} (Ruppert). \textit{For each totally bounded topological group $G$ the canonic homomorphisms $WAP(G) \to AP(G) \to SAP(G)$ are homeomorphisms and the canonic map $\eta : G \to SAP(G)$ is a topological embedding.}

The same conclusion holds for Tychonov pseudocompact topological semigroups that contain dense totally bounded topological subgroups.

\textbf{Theorem 1.6.} \textit{If a pseudocompact topological semigroup $S$ contains a totally bounded topological group $H$ as a dense subgroup, then the canonic maps $\beta S \to WAP(S) \to AP(S) \to SAP(S)$ are homeomorphisms.}

\textbf{Proof.} The embedding $H \subset S$ induces a continuous homomorphism $h : WAP(H) \to WAP(S)$. We claim that this homomorphism is surjective. Indeed, by Theorem 1.5, $WAP(H)$ is a compact topological group, containing $H$ as a dense subgroup. By Theorem 1.2, the Stone-\v{C}ech compactification $\beta S$ of $S$ can be identified with the WAP-compactification $WAP(S)$ of $S$. Then the image $h(WAP(H))$ contains the dense subset $H$ of $\beta S = WAP(S)$ and hence coincides with $\beta S$ being a compact dense subset of $\beta S$. The compact semitopological semigroup $WAP(S)$, being a continuous homomorphic image of the compact topological group $WAP(H)$, is a compact topological group. This implies that the canonic homomorphism $WAP(S) \to SAP(S)$ is a topological isomorphism. Consequently, the maps $\beta S \to WAP(S) \to AP(S) \to SAP(S)$ all are homeomorphisms. □

This theorem implies another one of the same spirit.

\textbf{Theorem 1.7.} \textit{If a topological semigroup $S$ contains a dense subgroup and has countably compact square $S \times S$, then the canonic maps $\beta S \to WAP(S) \to AP(S) \to SAP(S)$ are homeomorphisms.}

\textbf{Proof.} Let $H$ be a dense subgroup of $S$ and let $e$ be the idemponent of $H$. Let $H_e = \{x \in S : \exists x^{-1} \in S \text{ with } xx^{-1} = x^{-1}x = e, xe = ex = x, x^{-1}e = ex^{-1} = x^{-1}\}$ be the maximal subgroup of $S$ containing the idemponent $e$. Our theorem will follow from Theorem 1.6 as soon as we check that $H_e$ is a totally bounded topological group. For this observe that $H_e$ coincides with the projection of the closed subset $A = \{(x, y) \in S \times S : xy = yx = e, xe = ex = x, ye = ey = y\}$ of $S \times S$ onto the first factor. The countable compactness of $S \times S$ implies that of $A$ and of its projection $H_e$. The paratopological group $H_e$, being a Tychonov countably compact paratopological group, is a totally bounded topological group according to [15, 2.7]. □

Our final result concerns the AP-compactifications of pseudocompact openly factorizable topological semigroups. Those are pseudocompact topological semigroups whose topological spaces are \textit{openly factorizable}.

We define a topological space $X$ to be \textit{openly factorizable} if for each continuous map $f : X \to Y$ to a second countable space $Y$ there are a continuous open map
$p : X \to Z$ onto a second countable space $Z$ and a continuous map $g : Z \to Y$ such that $f = g \circ p$. Openly factorizable spaces will be studied in details in the next two sections. Now we present our main extension result for which we need the notion of a weakly Lindelöf space.

We call a topological space $X$ weakly Lindelöf if each open cover $U$ of $X$ contains a countable subcollection $V \subset U$ whose union $\bigcup V$ is dense in $X$. It is clear that the class of weakly Lindelöf spaces includes all Lindelöf spaces and all countably cellular (in particular, all separable) spaces.

**Theorem 1.8.** For any openly factorizable topological semigroup $S$ having weakly Lindelöf square $S \times S$, the semigroup operation $S \times S \to S$ extends to a continuous semigroup operation $\upsilon S \times \upsilon X \to \upsilon S$ defined on the Hewitt completion $\upsilon S$ of $S$.

**Proof.** By Theorem 3.3 below the semigroup operation $\mu : S \times S \to S$ extends to a continuous map $\bar{\mu} : \upsilon S \times \upsilon S \to \upsilon S$ thought as a continuous binary operation on $\upsilon S$. This operation is associative on $S$ and by the continuity remains associative on $\upsilon S$. □

This theorem implies another one:

**Theorem 1.9.** For each pseudocompact openly factorizable topological semigroup $S$ with weakly Lindelöf square the canonic maps $\beta S \to \operatorname{WAP}(S) \to \operatorname{AP}(S)$ are homeomorphisms.

**Proof.** By Theorem 1.8 the semigroup operation $\mu : S \times S \to S$ extends to a continuous semigroup operation $\bar{\mu} : \upsilon S \times \upsilon S \to \upsilon S$ turning the Hewitt completion $\upsilon S$ of $S$ into a topological semigroup that contains $S$ as a dense subsemigroup. Since the space $S$ is pseudocompact, its Hewitt completion coincides with its Stone-Čech compactification $\beta S$ [6, §3.11]. Consequently, $\beta S$ is a compact topological semigroup, which implies that the canonic map $\beta \eta : \beta S \to \operatorname{AP}(S)$ has a continuous inverse and consequently, the maps

$\beta S \to \operatorname{WAP}(S) \to \operatorname{AP}(S)$

are homeomorphisms. □

2. SOME ELEMENTARY PROPERTIES OF OPENLY FACTORIZABLE SPACES

In this section we establish some elementary properties of openly factorizable spaces. First we prove a helpful lemma.

**Lemma 2.1.** Let $p : X \to Z$ be a map from a Tychonov space to a second countable space and let $vp : \upsilon X \to Z$ be its continuous extension to the Hewitt completion of $X$. The map $vp$ is surjective (open) if and only if so is the map $p$.

**Proof.** Endow the second countable space $Z$ with a metric generating the topology of $Z$.

If the map $p$ is surjective, then $vp$ is surjective too because

$Z = p(X) \subset vp(\upsilon X) \subset Z$.

Now assume conversely that the map $vp$ is surjective but $p$ is not. Then we can find a point $z_0 \in Z \setminus p(X)$ and consider the continuous function $f : \upsilon X \to [0, +\infty)$, $f : x \mapsto \operatorname{dist}(p(x), z_0)$. It follows from $z_0 \notin p(X)$ that $f(X) \subset (0, +\infty)$. The function $f|X : X \to (0, +\infty)$ has a unique continuous extension $\bar{f} : \upsilon X \to (0, \infty)$. Since $f$ also extends $f|X$, we get $\bar{f} = f$ and hence $f(\upsilon X) = \bar{f}(\upsilon X) \subset (0, \infty)$ which
is not possible because \( f(x_0) = 0 \) for any point \( x_0 \in p^{-1}(z_0) \). Hence the map 
\( p|X : X \to Z \) is surjective.

Now assume that the map \( p \) is open. To show that the map \( vp \) is open, take any open subset \( U \subset vX \). We claim that \( vp(U) = p(U \cap X) \). In the opposite case, we can find a point \( y \in vp(U) \setminus p(U \cap X) \). Choose any point \( x_0 \in U \) with 
\( vp(x_0) = y \) and find a continuous function \( g : vX \to [0,1] \) such that \( g^{-1}(0) \) is a neighborhood of \( x_0 \) and \( g^{-1}(0,1) \subset U \). Consider the continuous function \( f : vX \to [0,\infty) \) defined by \( f(x) = g(x) + \text{dist}(vp(x),y) \) and note that \( f(x_0) = 0 \) while \( f(x) \in (0,1] \) for all \( x \in X \). Indeed, if \( x \in X \cap U \), then \( f(x) \geq \text{dist}(p(x),y) > 0 \) because \( y \notin p(U \cap X) \). If \( x \in X \setminus U \), then \( f(x) \geq g(x) = 1 > 0 \). Since \( vX \) is a Hewitt completion of \( X \), the function \( f|X : X \to (0,\infty) \) admits a unique continuous extension \( \bar{f} : vX \to (0,\infty) \). Since \( X \) is dense in \( vX \), we get \( f = \bar{f} \) and thus \( 0 = f(x_0) = \bar{f}(x_0) \in (0,\infty) \). This is a contradiction showing that the set 
\( vp(U) = p(U \cap X) \) is open and hence the map \( vp \) is open.

Now assume that the map \( vp \) is open. To show that \( p \) is open, fix any non-empty open set \( U \subset X \) and find an open set \( V \subset vX \) such that \( U = V \cap X \). To prove that the image \( p(U) \) is open, take any point \( y_0 \in p(U) \) and find a point \( x_0 \in U \) with \( p(x_0) = y_0 \). Since the space \( vX \) is Tychonov, there is a continuous function \( f : vX \to [0,1] \) such that \( W = f^{-1}(0) \) is a neighborhood of \( x_0 \) in \( vX \) while \( f^{-1}(0,1) \subset V \). Since the map \( vp \) is open, the image \( vp(W) \) is an open neighborhood of \( y_0 \) in \( Z \). We claim that \( vp(W) \subset p(V \cap X) = p(U) \). Assume conversely that there is a point \( y \in vp(W) \setminus p(U) \) and consider the continuous function 
\( g : vX \to [0,\infty), \quad g(x) = f(x) + \text{dist}(p(x),y). \)
It follows that \( g(X) \subset (0,\infty) \) and hence \( g(vX) \subset (0,\infty) \) too. On the other hand, for any point \( x \in W \) with \( p(x) = y \) we get \( g(x) = 0 \), which is a contradiction showing that \( p \) is open. \( \square \)

**Proposition 2.2.** The Hewitt completion \( vX \) of a Tychonov space \( X \) is openly factorizable if and only if so is the space \( X \).

**Proof.** Assume that a Tychonov space \( X \) is openly factorizable. To show that the Hewitt completion \( vX \) is openly factorizable, take any continuous map \( f : vX \to Y \) to a second countable space \( Y \). Since \( X \) is openly factorizable, there are an open surjective continuous map \( p : X \to Z \) to a second countable space \( Z \) and a continuous map \( g : Z \to Y \) such that \( f|X = g \circ p \). The space \( Z \), being second countable, is real complete [6, 3.11.12]. Consequently, the map \( p \) admits a continuous extension \( vp : vX \to Z \). It follows that \( f = g \circ vp \). By Lemma 2.1, the map \( vp \) is open and surjective, witnessing that \( vX \) is openly factorizable.

Now assume that \( vX \) is openly factorizable. To show that \( X \) is openly factorizable, take any continuous function \( f : X \to Y \) to a second countable space \( Y \). Since \( Y \) is real complete [6, 3.11.12], the map \( f \) extends to a continuous map \( vf : vX \to Y \). Since \( vX \) is openly factorizable, there are an open surjective continuous map \( p : vX \to Z \) to a second countable space \( Z \) and a continuous map \( g : Z \to Y \) such that \( f = g \circ p \). Then \( f|X = g \circ p|X \) and the map \( p|X : X \to Z \) is open and surjective by Lemma 2.1. \( \square \)

**Proposition 2.3.** The Stone-Čech compactification \( \beta X \) of a Tychonov space \( X \) is openly factorizable if and only if \( X \) is pseudocompact and openly factorizable.
Proof. If $X$ is pseudocompact and openly factorizable, then the Hewitt completion $vX$ is openly factorizable by Proposition 2.2. Since $X$ is pseudocompact, its Hewitt completion coincides with the Stone-Čech compactification $\beta X$. So, $\beta X$ is openly factorizable.

Now assume conversely that $\beta X$ is openly factorizable. We claim that $X$ is pseudocompact. If the opposite case, we could find a continuous unbounded function $f : X \to [0, \infty)$. Let $\beta f : \beta X \to [0, \infty]$ be the Stone-Čech extension of the map $f$ to the one-point compactification of the half-line $[0, \infty)$. Since $\beta X$ is openly factorizable, there are a continuous open surjective map $p : \beta X \to Z$ onto a metrizable compact space $Z$ and a continuous map $g : Z \to [0, \infty]$ such that $f = g \circ p$.

Since the function $f$ is unbounded, we can choose a sequence $\{x_n\}_{n \in \omega} \subset X$ such that the sequence $\{f(x_n)\}_{n \in \omega} \subset [0, \infty)$ is strictly increasing and unbounded. Passing to a subsequence, if necessary, we can assume that the sequence $\{p(x_n)\}_{n \in \omega} \subset Z$ converges to some point $z_\infty \in Z$. It follows from $f = g \circ p$ that $g(z_\infty) = \infty$ and the points $z_\infty, p(x_n), n \in \omega$, all are distinct. So each point $p(x_n)$ has a neighborhood $U_n \subset Z \setminus \{z_\infty\}$ such that the family $\{U_n : n \in \omega\}$ is disjoint. Moreover, we can assume that the sequence $(U_n)$ converges to $z_\infty$ in the sense that each neighborhood $O(z_\infty)$ contains all but finitely many sets $U_n$. Since the sequence $\{f(x_n)\}_{n \in \omega}$ is closed and discrete in $[0, \infty)$, to each point $f(x_n)$ we can assign an open neighborhood $V_n \subset [0, \infty)$ such that the family $\{V_n : n \in \omega\}$ is discrete in $[0, \infty)$ (in the sense that each point has a neighborhood that meets at most one set $V_n$). Now for every $n \in \omega$ consider the open neighborhood $W_n = f^{-1}(V_n) \cap p^{-1}(U_n)$ of the point $x_n$ in $X$. Since the family $\{V_n\}_{n \in \omega}$ is discrete in $[0, \infty)$, the family $\{W_n\}_{n \in \omega}$ is discrete in $X$. Let $x_\infty \in \beta X$ be any accumulation point of the sequence $\{x_{2n}\}_{n \in \omega}$.

Since the space $X$ is Tychonov and $\{W_{2n}\}_{n \in \omega}$ is discrete, we can construct a continuous function $\varphi : X \to [0, 1]$ such that

$$\{x_{2n}\}_{n \in \omega} \subset \varphi^{-1}(1) \subset \varphi^{-1}(0, 1) \subset \bigcup_{n \in \omega} W_{2n}. $$

Let $\beta \varphi : \beta X \to [0, 1]$ be the Stone-Čech extension of $\varphi$. It follows from the continuity of $\beta \varphi$ that $\beta \varphi(x_\infty) = 1$. Then the set $W = (\beta \varphi)^{-1}(1/2, 1]$ is an open neighborhood of $x_\infty$ in $\beta X$ with

$$W \cap X \subset \overline{W} \cap X \subset \varphi^{-1}[1/2, 1] \subset \bigcup_{n \in \omega} W_{2n}. $$

It follows that $p(W \cap X) \subset V$ where $V = \bigcup_{n \in \omega} V_{2n}$ and consequently,

$$p(W) \subset p(W \cap X) \subset p(W \cap X) \subset \nabla. $$

Since $\nabla \subset X \setminus \bigcup_{n \in \omega} V_{2n+1}$ and $V_{2n+1} \to z_\infty$, the set $\nabla$ contains no neighborhood of the point $z_\infty = p(x_\infty)$. Consequently, the set $p(W)$ cannot be open. This contradiction completes the proof of the pseudocompactness of $X$.

In this case the Stone-Čech compactification $\beta X$ coincides with the Hewitt completion $vX$ of $X$. Applying Proposition 2.2 we conclude that $X$ is openly factorizable.

\[ \square \]

3. Spectral characterization of openly factorizable spaces

In this section we shall present a spectral characterization of openly factorizable topological spaces. First we remind some information related to inverse spectra, see [5, §3.1] and [6, §2.5].
A partially ordered set \((A, \leq)\) is called

- **directed** if for every \(a, b \in A\) there exists \(c \in A\) with \(c \geq a, c \geq b\);
- **\(\omega\)-directed** if for any countable subset \(C \subset A\) has an upper bound in \(A\) (which a point \(a \in A\) such that \(a \geq c\) for every \(c \in C\));
- **\(\omega\)-complete** if \(A\) each countable subset \(C \subset A\) has the smallest upper bound

For example, the ordinal \(\omega_1\) endowed with the natural order is a well-ordered \(\omega\)-complete set.

By a spectrum over a directed set \((A, \leq)\) we understand a collection \(S = \{X_\alpha, \pi_\alpha, A\}\) consisting of Tychonov spaces \(X_\alpha, \alpha \in A\), and continuous surjective maps \(\pi_\alpha : X_\gamma \to X_\alpha\) for \(\alpha \leq \gamma\) from \(A\) such that \(\pi_\alpha^\gamma = \pi_\alpha^\beta \circ \pi_\beta^\gamma\) for every elements \(\alpha \leq \beta \leq \gamma\) of \(A\). Let

\[
\lim S = \{(x_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} X_\alpha : \forall \alpha, \beta \in A \quad \alpha \leq \beta \Rightarrow x_\alpha = \pi_\alpha^\beta(x_\beta)\} \subset \prod_{\alpha \in A} X_\alpha
\]

denote the limit space of the spectrum \(S\).

For a directed subset \(B\) of \(A\) by \(S|B\) we denote the subspectrum \(S|B = \{X_\alpha, \pi_\alpha, B\}\) of \(S\), consisting of the spaces \(X_\alpha\) and the projections \(\pi_\alpha^\gamma\) for which \(\alpha, \gamma \in B\). Given a collection \(\{f_\alpha : X \to X_\alpha\}_{\alpha \in A}\) of maps from a space \(X\) into the spaces of the spectrum \(S\) such that \(\pi_\alpha^\gamma \circ f_\alpha = f_\alpha\) for every \(\alpha \leq \gamma\) in \(B\) by \(\lim f_\alpha : X \to \lim S\) we denote the induced map into the limit space of \(S\).

A spectrum \(S = \{X_\alpha, \pi_\alpha, A\}\) is defined to be

- **continuous** if for every chain \(B \subset A\) having supremum \(\beta = \sup B\) the map \(\lim_{\alpha \in B} \pi_\alpha^\beta : X_\beta \to \lim S|B\) is a homeomorphism;
- **open** if the projections \(\pi_\alpha^\gamma : X_\gamma \to X_\alpha\) are open and surjective for all \(\alpha \leq \gamma\) in \(A\);
- **\(\omega\)-directed** (resp. **\(\omega\)-complete**) provided so is its index set \(A\);
- a **\(\omega\)**-**spectrum** if it is \(\omega\)-directed and each space \(X_\alpha, \alpha \in A\), is second countable;
- **factorizable** if every continuous map \(f : \lim S \to \mathbb{R}\) can be written as \(f = f_\alpha \circ \pi_\alpha\) for some \(\alpha \in A\) and some continuous map \(f_\alpha : X_\alpha \to \mathbb{R}\).

According to [5] 3.1.5 a continuous \(\omega\)-complete spectrum \(S\) with surjective bonding maps is factorizable if and only if every bounded continuous map \(f : \lim S \to \mathbb{R}\) can be written as \(f = f_\alpha \circ \pi_\alpha\) for some \(\alpha \in A\) and some bounded continuous map \(f_\alpha : X_\alpha \to \mathbb{R}\). By another result of [5] 3.1.7 a continuous \(\omega\)-complete open spectrum \(S = \{X_\alpha, \pi_\alpha, A\}\) is factorizable provided the limit space \(\lim S\) is countably cellular (that is contains no uncountable family of disjoint open sets).

In fact, the proof of Proposition 3.1.7 of [5] can be modified to get the following more general statement, cf. [4] 3.2.

**Proposition 3.1.** Suppose \(S = \{X_\alpha, \pi_\alpha, A\}\) is a \(\omega\)-spectrum and \(X \subset \lim S\) is a weakly Lindelöf subspace of its limit such that the restrictions \(\pi_\alpha|X : X \to X_\alpha, \alpha \in A\), of the limit projections is open and surjective. Then every map \(f : X \to Y\) to a second countable space \(Y\) can be written as \(f = f_\alpha \circ \pi_\alpha|X\) for some \(\alpha \in A\) and some map \(f_\alpha : X_\alpha \to Y\). In particular, \(X\) is C-embedded into \(\lim S\) and hence \(\lim S\) is a Hewitt completion of \(X\).

We recall that a subspace \(X\) of a topological space \(Y\) is **C-embedded** in \(Y\) if each continuous functions \(f : X \to \mathbb{R}\) extends to a continuous function \(\tilde{f} : Y \to \mathbb{R}\).
The following theorem gives a spectral characterization of openly factorizable spaces.

**Theorem 3.2.** A (weakly Lindelöf) topological space \( X \) is openly factorizable (if and) only if \( X \) is a dense subspace of the limit space \( \lim S \) of an open \( \omega \)-spectrum \( S = \{ X_\alpha, \pi_\alpha^\gamma, A \} \) such that for every \( \alpha \in A \) the restriction \( \pi_\alpha : X : X \to X_\alpha \) of the limit projection is open and surjective.

**Proof.** The “if” part follows immediately from Proposition 3.1. To prove the “only if” part, assume that a Tychonov space \( X \) is weakly Lindelöf, then it is openly factorizable. Let \( A' \) be a set of all open continuous surjective maps \( \alpha : X \to X_\alpha \) with \( X_\alpha \subset \mathbb{R}^\omega \). The set \( A \) is partially preordered by the relation: \( \alpha \leq \gamma \) if there is a continuous map \( \pi_\alpha^\gamma : X_\gamma \to X_\alpha \) such that \( \alpha = \pi_\alpha^\gamma \circ \gamma \). This map \( \pi_\alpha^\gamma \) is necessarily open and surjective because the map \( \alpha \) is open and surjective while \( \gamma \) is continuous. Also the map \( \pi_\alpha^\gamma \) is uniquely determined, which implies that \( \pi_\alpha^\gamma \circ \pi_\beta^\gamma = \pi_\alpha^\gamma \) for any \( \alpha \leq \beta \leq \gamma \) in \( A' \). This means that the relation \( \leq \) on \( A' \) is transitive. The preorder \( \leq \) induces the equivalence relation \( \equiv \) on \( A' \): \( \alpha \equiv \gamma \) if \( \alpha \leq \gamma \) and \( \gamma \leq \alpha \). Let \( A \) be a subset of \( A' \) intersecting each equivalence class in a single point. Then \( A \) becomes a partially ordered set with respect to the order \( \leq \).

Let us show that the set \( (A, \leq) \) is \( \omega \)-directed. Given a countable subset \( C \subset A \) consider the diagonal product \( f = \Delta_{\gamma \in C} \gamma : X \to \prod_{\gamma \in C} X_\gamma \). Taking into account that \( \prod_{\gamma \in C} X_\gamma \) is second countable and \( X \) is openly factorizable, find an open surjective map \( \alpha : X \to X_\alpha \) onto a second countable space \( X_\alpha \) and a continuous map \( g : X_\alpha \to \prod_{\gamma \in C} X_\gamma \) such that \( g \circ \alpha = f \). We can assume that \( X_\alpha \subset \mathbb{R}^\omega \) and thus \( \alpha \in A' \). Moreover, we can replace \( \alpha \) by an equivalent map and assume that \( \alpha \in A \). Let us show that \( \alpha \geq \beta \) for each \( \beta \in C \). Consider the projection \( \pi_\beta : \prod_{\gamma \in C} X_\gamma \to X_\beta \) and observe that the equality \( g \circ \alpha = f \) implies \( (\pi_\beta \circ g) \circ \alpha = \pi_\beta \circ f = \beta \), which means that \( \alpha \geq \beta \).

Now we see that \( S = \{ X_\alpha, \pi_\alpha^\gamma, A \} \) is an open \( \omega \)-spectrum. Let \( \pi_\alpha : \lim S \to X_\alpha \), \( \alpha \in A \), be the limit projections of this spectrum. The open surjective maps \( \alpha \in A \) determine a map
\[
A : X \to \lim S, \ A : x \mapsto (\alpha(x))_{\alpha \in A}
\]
such that \( p_\alpha \circ A = \alpha \) for every \( \alpha \in A \). The surjectivity of the maps \( \alpha \in A \) imply that the map \( A : X \to \lim S \) has dense image \( A(X) \subset \lim S \). Let us show that \( A \) is a topological embedding. Given a point \( x \in X \) and an open set \( O(x) \subset X \) we should find an open set \( U \subset \lim S \) such that \( A(x) \in U \cap A(X) \subset A(O(x)) \). Since \( X \) is Tychonov, there is a map \( f : X \to [0,1] \) such that \( x \in f^{-1}(0,1) \subset O(x) \). The choice of the set \( A \) guarantees that there is a map \( \alpha : X \to X_\alpha \in A \) and a continuous map \( g : X_\alpha \to (0,1) \) such that \( g \circ \alpha = f \). Then the set \( V = g^{-1}(0,1) \) is open in \( X_\alpha \) and hence \( U = \pi_\alpha^{-1}(V) \) is open in \( \lim S \). It is easy to check that this set \( U \) has the required property: \( A(x) \in U \cap A(X) \subset A(O(x)) \).

We apply the spectral characterization of openly factorizable spaces to derive the following main result of this paper.

**Theorem 3.3.** Let \( X, Y \) be two openly factorizable spaces. If the product \( X \times Y \) is weakly Lindelöf, then

1. the product \( X \times Y \) is openly factorizable;
2. each continuous map \( f : X \times Y \to Z \) to a Tychonov space \( Z \) extends to a continuous map \( \tilde{f} : \nu X \times \nu Y \to \nu Z \).
Proof. By Theorem 3.2, $X$ is a dense subspace of the limit space $\lim S_X$ of an open $\omega$-spectrum $S_X = \{X_\alpha, \pi'_\alpha, A\}$ such that the restrictions $\pi'|X : X \to X_\alpha$, $\alpha \in A$, of the limit projections are open and surjective. By Proposition 3.1, the limit space $\lim S_X$ is a Hewitt completion of $X$.

By the same reason, the Hewitt completion $\nu Y$ of $Y$ can be identified with the limit space $\lim S_Y$ of an open $\omega$-spectrum $S_Y = \{Y_\alpha, p_\alpha, B\}$ such that the restrictions $p_\alpha|Y : Y \to Y_\alpha$, $\alpha \in B$, of the limit projections are open and surjective.

On the product $A \times B$ consider the partial order: $(\alpha, \beta) \leq (\alpha', \beta')$ if $\alpha \leq \alpha'$ and $\beta \leq \beta'$. It is easy to see that the partially order set $A \times B$ is $\omega$-directed. It follows that $X \times Y$ is a subspace of the limit space $\lim S_X \times \lim S_Y$ of the open $\omega$-spectrum

$$S = \{X_\alpha \times Y_\beta, \pi'_\alpha \times p_\beta, A \times B\}$$

such that for every $(\alpha, \beta) \in A \times B$ the restriction $\pi_\alpha \times p_\beta : X \times Y \to X_\alpha \times Y_\beta$ is open and surjective. Since the product $X \times Y$ is weakly Lindelöf, we may apply Proposition 3.1 and Theorem 3.2 and conclude that the product $X \times Y$ is openly factorizable and $\lim S_X \times \lim S_Y = \nu X \times \nu Y$ is a Hewitt completion of $X \times Y$.

Now take any continuous map $f : X \times Y \to Z$ to a second countable space $Z$. By Proposition 3.1, there is an index $(\alpha, \beta) \in A \times B$ and a continuous map $f_{(\alpha, \beta)} : X_\alpha \times Y_\beta \to Z$ such that $f = f_{(\alpha, \beta)} \circ (\pi_\alpha \times p_\beta)|X \times Y$. Then $\tilde{f} = f_{(\alpha, \beta)} \circ (\pi_\alpha \times p_\beta)$ is a continuous extension of the map $f$ onto the product $\lim S_X \times \lim S_Y = \nu X \times \nu Y$.

Finally take any continuous map $f : X \times Y \to Z$ to any Tychonov space $Z$. Identify the Hewitt completion $\nu Z$ of $Z$ with a closed subspace of $\mathbb{R}^\kappa$ for a suitable cardinal $\kappa$. The preceding case insures that the map $f$ extends to a continuous map $\tilde{f} : \nu X \times \nu Y \to \mathbb{R}^\kappa$. It follows that

$$\tilde{f}(\nu X \times \nu Y) = \tilde{f}(\nu X \times \nu Y) \subset \tilde{f}(\nu X \times \nu Y) \subset \nu Z \subset \mathbb{R}^\kappa.$$

So $\tilde{f}$ is a continuous map into $\nu Z$. \qed

4. Some comments and open problems

In this section we discuss the relation of the class of openly factorizable compact spaces to other known classes of compact spaces and pose some open problems. The survey [18] provided the necessary information on various classes of compact spaces.

We recall that a compact space $X$ is called

- **Dugundji compact** if for each embedding $X \to Y$ to another compact space $Y$ there is a linear positive norm one operator $u : C(X) \to C(Y)$ extending continuous functions from $X$ to $Y$;
- **$AE(0)$-space** if each continuous map $f : B \to X$ defined on a closed subspace $B$ of a zero-dimensional compact space $A$ can extended to a continuous map $\tilde{f} : A \to X$;
- **openly generated** if $X$ is homeomorphic to the limit $\lim S$ of an open continuous $\omega$-complete $\omega$-spectrum $S = \{X_\alpha, p_\alpha, A\}$;
- **dyadic compact** if $X$ is a continuous image of the Cantor cube $\{0, 1\}^\kappa$ for some cardinal $\kappa$;
- **$\kappa$-adic** if $X$ is a continuous image of some $\kappa$-metrizable compact space;
- **$\kappa$-metrizable** if $X$ admits a $\kappa$-metric.
We recall that a \(\kappa\)-\emph{metric} on \(X\) is a function assigning to each point \(x \in X\) and a regular closed set \(F \subset X\) a non-negative number \(\rho(x, F)\) so that the following axioms hold:

1. \(\rho(x, F) = 0\) if and only if \(x \in F\);
2. \(\rho(x, F) \geq \rho(x, F')\) for any regular closed sets \(F \subset F'\) of \(X\);
3. For any regular closed set \(F\) the function \(\rho(\cdot, F) : x \mapsto \rho(x, F)\) is continuous with respect to the first argument;
4. For any point \(x \in X\) and a linearly ordered family \(F\) of regular closed subsets of \(X\), we get \(\rho(x, \bigcup F) = \inf_{F \in F} \rho(x, F)\).

By the classical result of Haydon [9], the classes of Dugundji and AE(0)-compacta coincide. By [19], the classes of openly generated and \(\kappa\)-metrizable compacta coincide. It is well-known that each compact topological group is Dugundji compact. Each Dugundji compact is openly generated and each openly generated compact space of weight \(\leq \aleph_1\) is Dugundji [19]. Each \(\kappa\)-adic compact space has countable cellularity [19]. The hyperspace \(\exp(\{0, 1\}^{\aleph_1})\) is openly generated but not Dugundji.

The spectral characterization of openly factorizable spaces from Theorem 3.2 implies that each openly generated compact space is openly factorizable. The simplest example of an openly factorizable compact space which is not openly generated is the ordinal space \([0, \omega_1]\). It is not openly generated because has uncountable cellularity. By the same reason, \([0, \omega_1]\) is not \(\kappa\)-adic.

Thus we have the following chain of implications:

\[
\text{compact topological group} \quad \Rightarrow \quad \text{Dugundji compact} \quad \Leftrightarrow \quad \text{AE(0)-compact} \quad \Rightarrow \quad \text{\(\kappa\)-metrizable} \quad \Leftrightarrow \quad \text{\(\kappa\)-adic} \quad \Rightarrow \quad \text{openly generated} \quad \Rightarrow \quad \text{openly factorizable}
\]

Let us observe that the classes of openly generated and openly factorizable compact spaces are preserved by open normal functors in the sense of Shchepin [19], see also [20]. This allows us to construct many openly factorizable compacta failing to be Dugundji compact.

There is another chain of important classes of compact spaces, that is “orthogonal” to the chain of classes considered above.

We recall that a compact space \(X\) of weight \(\kappa\) is

1. \emph{Corson compact} if \(X\) embeds into the \(\Sigma\)-product of lines
   \[\Sigma = \{(x_\alpha) \in \mathbb{R}^\kappa : |\{\alpha \in \kappa : x_\alpha \neq 0\}| \leq \aleph_0\} \subset \mathbb{R}^\kappa;\]
2. \emph{Eberlein compact} if \(X\) embeds into the subspace
   \[\Sigma_0 = \{(x_\alpha) \in \mathbb{R}^\kappa : \forall \varepsilon > 0 \left|\{\alpha \in \kappa : |x_\alpha| < \varepsilon\}\right| < \aleph_0\} \subset \mathbb{R}^\kappa;\]
3. \emph{Valdivia compact} if \(X\) embeds into \(\mathbb{R}^\kappa\) so that \(X \cap \Sigma\) is dense in \(X\).

Those properties relate as follows:

\begin{itemize}
  \item Eberlein compact \(\Rightarrow\) Corson compact \(\Rightarrow\) Valdivia compact.
  \item Each Eberlein compact with countable cellularity is metrizable [1]. So the classes of Eberlein compacta and \(\kappa\)-adic compacta intersect by the class of metrizable compacta.
\end{itemize}
Problem 4.1. Is each openly factorizable Eberlein (or Corson) compact space metrizable?

The openly factorizable space $[0, \omega_1]$ is known to be Valdivia compact while $[0, \omega_2]$ is not Valdivia [10].

Problem 4.2. Is each ordinal space $[0, \lambda]$ openly factorizable for each ordinal $\lambda$?

The ordinals segments are examples of both scattered and linearly ordered compacta. We recall that a topological space $X$ is scattered if each subspace of $X$ has an isolated point. Scattered spaces need not be openly factorizable. The simplest example is the one-point compactification $\alpha \mathbb{N}_1$ of a discrete space of cardinality $\mathbb{N}_1$. This space is Eberlein compact but not linearly ordered.

The simplest example of a linearly orderable scattered compact space that fails to be openly factorizable is the bouquet of the spaces $[0, \omega_1]$ and $[0, \omega]$ with points $\omega_1$ and $\omega$ glued together.

Problem 4.3. Characterize openly factorizable spaces in the class of scattered (compact) spaces; in the class of linearly ordered (compact) spaces.
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