Supplementary Material for the paper

“Pairwise likelihood inference for nested hidden Markov chain models for multilevel longitudinal data”

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In the Supplementary Material we collect Table S-1 and Table S-2 related to the order of complexity of the algorithm for computing the manifest distribution and the pairwise likelihood discussed in Sections 4.2 and 5.1. We also include Tables S-3, S-4, S-5, S-6, S-7, and S-8 showing the results of the simulation study described in Section 6. Finally, we represent simulated trajectories from the estimated latent processes, at individual and cluster-level, in Figures S-1 and S-2.
Table S-1: Order of the numerical complexity required to compute the manifest distribution in (6) by recursion (2) for different values of $k_1$, $k_2$, $n_h$, and $T$.

| $k_1$ | $k_2$ | $n_h = 5$ | $n_h = 10$ | $n_h = 20$ | $n_h = 5$ | $n_h = 10$ | $n_h = 20$ |
|------|------|-------|-------|-------|-------|-------|-------|
| 1    | 1    | 25    | 50    | 100   | 50    | 100   | 200   |
| 1    | 2    | 100   | 200   | 400   | 200   | 400   | 800   |
| 1    | 3    | 225   | 450   | 900   | 450   | 900   | 1800  |
| 1    | 4    | 400   | 800   | 1600  | 800   | 1600  | 3200  |
| 1    | 5    | 625   | 1250  | 2500  | 1250  | 2500  | 5000  |
| 2    | 1    | 800   | 1600  | 3200  | 51200 | 102400| 204800|
| 2    | 2    | 3200  | 6400  | 12800 | 204800| 409600| 819200|
| 2    | 3    | 7200  | 14400 | 28800 | 460800| 921600| 1843200|
| 2    | 4    | 12800 | 25600 | 51200 | 819200| 1638400| 3276800|
| 2    | 5    | 20000 | 40000 | 80000 | 1280000| 2560000| 5120000|
| 3    | 1    | 6075  | 12150 | 24300 | 2952450| 5904900| 11809800|
| 3    | 2    | 24300 | 48600 | 97200 | 11809800| 23619600| 47239200|
| 3    | 3    | 54675 | 109350| 218700| 26572050| 53144100| 106288200|
| 3    | 4    | 97200 | 194400| 388800| 47239200| 94478400| 188956800|
| 3    | 5    | 151875| 303750| 607500| 73811250| 147622500| 295245000|
| 4    | 1    | 25600 | 51200 | 102400| 52428800| 104857600| 209715200|
| 4    | 2    | 102400| 204800| 409600| 209715200| 419430400| 838860800|
| 4    | 3    | 230400| 460800| 921600| 471859200| 943718400| 1887436800|
| 4    | 4    | 409600| 819200| 1638400| 838860800| 1677721600| 3355443200|
| 4    | 5    | 640000| 1280000| 2560000| 1310720000| 2621440000| 52428800000|
| 5    | 1    | 78125 | 156250| 312500| 488281250 | 976562500 | 1953125000|
| 5    | 2    | 312500| 625000| 1250000| 1953125000 | 3906250000 | 7812500000|
| 5    | 3    | 703125| 1406250| 2812500| 4394531250 | 8789062500 | 17578125000|
| 5    | 4    | 1250000| 2500000| 5000000| 7812500000 | 15625000000 | 31250000000|
| 5    | 5    | 1953125| 3906250| 7812500| 12207031250 | 24414062500 | 48828125000|
Table S-2: Order of the numerical complexity required to compute the pairwise manifest probability (12) for all units in the same cluster $h$ and different values of $k_1$, $k_2$, $n_h$, and $T$.

| $k_1$ | $k_2$ | $n_h = 5$ | $n_h = 10$ | $n_h = 20$ | $T = 5$    | $n_h = 5$ | $n_h = 10$ | $n_h = 20$ | $T = 10$    |
|-------|-------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 1     | 1     | 50        | 225       | 950       | 100       | 450       | 1900      |           |           |
| 1     | 2     | 800       | 3600      | 15200     | 1600      | 7200      | 30400     |           |           |
| 1     | 3     | 4050      | 18225     | 76950     | 8100      | 36450     | 153900    |           |           |
| 1     | 4     | 12800     | 57600     | 243200    | 25600     | 115200    | 486400    |           |           |
| 1     | 5     | 31250     | 140625    | 593750    | 62500     | 281250    | 1187500   |           |           |
|       |       |           |           |           |           |           |           |           |           |
| 2     | 1     | 200       | 900       | 3800      | 400       | 1800      | 7600      |           |           |
| 2     | 2     | 3200      | 14400     | 60800     | 6400      | 2880      | 121600    |           |           |
| 2     | 3     | 16200     | 72900     | 307800    | 32400     | 145800    | 615600    |           |           |
| 2     | 4     | 51200     | 230400    | 972800    | 102400    | 460800    | 1945600   |           |           |
| 2     | 5     | 125000    | 562500    | 2375000   | 2500000   | 1125000   | 4750000   |           |           |
|       |       |           |           |           |           |           |           |           |           |
| 3     | 1     | 450       | 2025      | 8550      | 900       | 4050      | 17100     |           |           |
| 3     | 2     | 7200      | 32400     | 136800    | 14400     | 64800     | 273600    |           |           |
| 3     | 3     | 36450     | 164025    | 692550    | 72900     | 328050    | 1385100   |           |           |
| 3     | 4     | 115200    | 518400    | 2188800   | 230400    | 1036800   | 4377600   |           |           |
| 3     | 5     | 281250    | 1265625   | 5343750   | 562500    | 2531250   | 10687500  |           |           |
|       |       |           |           |           |           |           |           |           |           |
| 4     | 1     | 800       | 3600      | 15200     | 1600      | 7200      | 30400     |           |           |
| 4     | 2     | 12800     | 57600     | 243200    | 25600     | 115200    | 486400    |           |           |
| 4     | 3     | 64800     | 291600    | 1231200   | 129600    | 583200    | 2462400   |           |           |
| 4     | 4     | 204800    | 921600    | 3891200   | 409600    | 1843200   | 7782400   |           |           |
| 4     | 5     | 500000    | 2250000   | 9500000   | 1000000   | 4500000   | 19000000  |           |           |
|       |       |           |           |           |           |           |           |           |           |
| 5     | 1     | 1250      | 5625      | 23750     | 2500      | 11250     | 47500     |           |           |
| 5     | 2     | 20000     | 90000     | 380000    | 40000     | 180000    | 760000    |           |           |
| 5     | 3     | 101250    | 455625    | 1923750   | 202500    | 911250    | 3847500   |           |           |
| 5     | 4     | 320000    | 1440000   | 6080000   | 640000    | 2880000   | 12160000  |           |           |
| 5     | 5     | 781250    | 3515625   | 14843750  | 1562500   | 7031250   | 29687500  |           |           |
Table S-3: Simulation results for $k_1 = k_2 = 2$ and $T = 5$.

| true value | $\rho_1$ | $\rho_2$ | $\tau_1$ | $\tau_2$ | $\delta_0$ | $\delta_1$ | $\delta_2$ | $\alpha_2$ | $\beta_2$ |
|------------|----------|----------|----------|----------|------------|------------|------------|------------|----------|
|            | -2.197   | -2.197   | -2.197   | -2.197   | -2.000     | 1.000      | -1.000     | 2.000      | 2.000    |

### unweighted pairwise likelihood

| $H = 200, U(1; 10)$ | bias  | sd    | rmse   | mean-se |
|---------------------|-------|-------|--------|---------|
| bias                | -0.070| 0.371 | 0.380  | 0.404   |
| sd                  | -0.027| 0.416 | 2.185  | 1.314   |
| rmse                | -0.496| 2.118 | 2.325  | 1.444   |
| mean-se             | 0.096 | 0.080 | 0.083  | 0.081   |

| $H = 200, U(1; 20)$ | bias  | sd    | rmse   | mean-se |
|---------------------|-------|-------|--------|---------|
| bias                | -0.031| 0.282 | 0.285  | 0.294   |
| sd                  | -0.015| 0.252 | 0.253  | 0.291   |
| rmse                | -0.047| 0.518 | 0.544  | 0.563   |
| mean-se             | -0.054| 0.318 | 0.324  | 0.314   |

| $H = 400, U(1; 10)$ | bias  | sd    | rmse   | mean-se |
|---------------------|-------|-------|--------|---------|
| bias                | -0.020| 0.235 | 0.237  | 0.257   |
| sd                  | -0.027| 0.540 | 0.554  | 0.653   |
| rmse                | -0.161| 0.284 | 0.285  | 0.295   |
| mean-se             | -0.005| 0.053 | 0.053  | 0.051   |

| $H = 400, U(1; 20)$ | bias  | sd    | rmse   | mean-se |
|---------------------|-------|-------|--------|---------|
| bias                | -0.009| 0.192 | 0.193  | 0.197   |
| sd                  | -0.081| 0.324 | 0.335  | 0.358   |
| rmse                | -0.041| 0.179 | 0.181  | 0.197   |
| mean-se             | 0.020 | 0.031 | 0.032  | 0.036   |

### weighted pairwise likelihood

| $H = 200, U(1; 10)$ | bias  | sd    | rmse   | mean-se |
|---------------------|-------|-------|--------|---------|
| bias                | -0.052| 0.358 | 0.364  | 0.380   |
| sd                  | -0.056| 1.378 | 1.412  | 1.439   |
| rmse                | -0.291| 1.559 | 1.593  | 1.643   |
| mean-se             | -0.116| 0.417 | 0.435  | 0.447   |

| $H = 200, U(1; 20)$ | bias  | sd    | rmse   | mean-se |
|---------------------|-------|-------|--------|---------|
| bias                | -0.036| 0.259 | 0.263  | 0.262   |
| sd                  | -0.009| 0.509 | 0.513  | 0.518   |
| rmse                | -0.031| 0.482 | 0.485  | 0.489   |
| mean-se             | -0.003| 0.284 | 0.287  | 0.288   |

| $H = 400, U(1; 10)$ | bias  | sd    | rmse   | mean-se |
|---------------------|-------|-------|--------|---------|
| bias                | -0.015| 0.223 | 0.225  | 0.240   |
| sd                  | -0.025| 0.581 | 0.601  | 0.604   |
| rmse                | -0.142| 0.482 | 0.485  | 0.496   |
| mean-se             | -0.098| 0.284 | 0.287  | 0.290   |

| $H = 400, U(1; 20)$ | bias  | sd    | rmse   | mean-se |
|---------------------|-------|-------|--------|---------|
| bias                | -0.004| 0.180 | 0.181  | 0.178   |
| sd                  | -0.019| 0.330 | 0.341  | 0.332   |
| rmse                | -0.079| 0.305 | 0.310  | 0.330   |
| mean-se             | -0.048| 0.174 | 0.176  | 0.186   |

### full likelihood

| $H = 200, U(1; 10)$ | bias  | sd    | rmse   | mean-se |
|---------------------|-------|-------|--------|---------|
| bias                | -0.029| 0.290 | 0.293  | 0.290   |
| sd                  | -0.022| 0.824 | 0.835  | 0.835   |
| rmse                | -0.110| 0.325 | 0.359  | 0.359   |
| mean-se             | -0.060| 0.064 | 0.066  | 0.066   |

| $H = 200, U(1; 20)$ | bias  | sd    | rmse   | mean-se |
|---------------------|-------|-------|--------|---------|
| bias                | -0.011| 0.213 | 0.214  | 0.206   |
| sd                  | 0.001 | 0.380 | 0.383  | 0.383   |
| rmse                | 0.026 | 0.389 | 0.392  | 0.392   |
| mean-se             | 0.022 | 0.380 | 0.383  | 0.383   |

| $H = 400, U(1; 10)$ | bias  | sd    | rmse   | mean-se |
|---------------------|-------|-------|--------|---------|
| bias                | -0.003| 0.172 | 0.173  | 0.174   |
| sd                  | -0.016| 0.382 | 0.389  | 0.392   |
| rmse                | -0.065| 0.366 | 0.372  | 0.372   |
| mean-se             | -0.005| 0.190 | 0.191  | 0.191   |

| $H = 400, U(1; 20)$ | bias  | sd    | rmse   | mean-se |
|---------------------|-------|-------|--------|---------|
| bias                | -0.001| 0.136 | 0.137  | 0.147   |
| sd                  | -0.001| 0.282 | 0.288  | 0.288   |
| rmse                | -0.047| 0.274 | 0.278  | 0.278   |
Table S-4: Simulation results for $k_1 = 2$ and $k_2 = 3$ and $T = 5$.

| $H = 200, U(1; 10)$ | $H = 400, U(1; 10)$ | $H = 200, U(1; 20)$ | $H = 400, U(1; 20)$ | $H = 200, U(1; 10)$ | $H = 400, U(1; 10)$ |
|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| **true value**      | -2.197 -2.197 -2.079 -2.079 -3.000 | 1.000 -1.000 2.000 2.000 4.000 |
| **unweighted pairwise likelihood** | | |
| bias                | 0.000 -0.039 0.416 0.887 0.384 -0.597 0.070 -0.072 0.192 0.416 1.039 |
| sd                  | 0.472 0.451 1.667 3.010 1.709 1.126 0.121 0.137 0.288 1.118 1.928 |
| rmse                | 0.475 0.454 1.726 3.153 1.760 1.279 0.140 0.156 0.347 1.198 2.198 |
| mean-se             | 0.485 0.502 0.749 1.181 0.645 0.995 0.115 0.116 0.293 0.993 1.457 |
| **weighted pairwise likelihood** | | |
| bias                | -0.036 -0.001 0.098 0.445 0.084 -0.216 0.017 -0.023 0.051 0.230 0.257 |
| sd                  | 0.339 0.346 0.451 2.163 0.824 1.057 0.068 0.082 0.171 1.077 1.093 |
| rmse                | 0.343 0.347 0.464 2.219 0.833 1.084 0.071 0.086 0.180 1.107 1.128 |
| mean-se             | 0.355 0.349 0.512 0.770 0.967 0.738 0.070 0.071 0.166 0.708 0.951 |
| **full likelihood** | | |
| bias                | -0.051 -0.016 0.053 0.329 0.079 0.031 0.064 0.068 0.047 0.174 0.186 |
| sd                  | 0.315 0.333 0.537 1.492 0.544 0.640 0.063 0.068 0.159 0.569 0.699 |
| rmse                | 0.319 0.335 0.555 1.531 0.548 0.663 0.066 0.072 0.167 0.595 0.727 |
| mean-se             | 0.336 0.339 0.530 0.836 0.555 0.625 0.068 0.086 0.158 0.650 0.822 |

$p_k = 1$, $\tau = 5$, $\delta = 0$ and $\alpha = 5$, $\beta = 3$. 

Table S-4: Simulation results for $k_1 = 2$ and $k_2 = 3$ and $T = 5$. 

| $H = 200, U(1; 10)$ | $H = 400, U(1; 10)$ | $H = 200, U(1; 20)$ | $H = 400, U(1; 20)$ |
|---------------------|---------------------|---------------------|---------------------|
| **true value**      | -2.197 -2.197 -2.079 -2.079 -3.000 | 1.000 -1.000 2.000 2.000 4.000 |
| **unweighted pairwise likelihood** | | |
| bias                | 0.001 0.005 0.108 0.164 0.166 -0.262 0.036 -0.036 0.085 0.362 0.424 |
| sd                  | 0.288 0.312 0.637 0.761 0.854 0.739 0.089 0.095 0.208 1.485 0.814 |
| rmse                | 0.289 0.314 0.649 0.782 0.874 0.788 0.096 0.102 0.226 1.536 0.922 |
| **weighted pairwise likelihood** | | |
| bias                | -0.016 -0.014 0.018 0.023 0.002 -0.042 0.006 -0.010 0.008 0.018 0.091 |
| sd                  | 0.241 0.242 0.333 0.451 0.368 0.318 0.056 0.061 0.123 0.246 0.388 |
| rmse                | 0.242 0.243 0.336 0.454 0.370 0.323 0.057 0.062 0.124 0.248 0.400 |
| **full likelihood** | | |
| bias                | -0.009 -0.006 0.063 0.075 0.039 -0.073 0.018 -0.013 0.026 0.044 0.147 |
| sd                  | 0.228 0.227 0.346 0.407 0.434 0.395 0.056 0.055 0.128 0.340 0.489 |
| rmse                | 0.229 0.228 0.353 0.416 0.438 0.403 0.059 0.057 0.132 0.344 0.513 |

$p_k = 2$, $\tau = 20$, $\delta = 0$ and $\alpha = 5$, $\beta = 3$. 

Table S-4: Simulation results for $k_1 = 2$ and $k_2 = 3$ and $T = 5$. 

| $H = 200, U(1; 10)$ | $H = 400, U(1; 10)$ | $H = 200, U(1; 20)$ | $H = 400, U(1; 20)$ |
|---------------------|---------------------|---------------------|---------------------|
| **true value**      | -2.197 -2.197 -2.079 -2.079 -3.000 | 1.000 -1.000 2.000 2.000 4.000 |
| **unweighted pairwise likelihood** | | |
| bias                | -0.001 -0.003 0.005 -0.012 0.004 -0.032 -0.002 -0.002 -0.005 0.033 0.053 |
| sd                  | 0.152 0.176 0.216 0.281 0.190 0.223 0.039 0.035 0.069 0.169 0.294 |
| rmse                | 0.152 0.176 0.216 0.281 0.190 0.225 0.039 0.035 0.069 0.172 0.299 |
**Table S-5: Simulation results for \( k_1 = 3 \) and \( k_2 = 2 \) and \( T = 5 \).**

| \( \rho_1 \) | \( \rho_2 \) | \( \rho_3 \) | \( \tau_1 \) | \( \tau_2 \) | \( \delta_0 \) | \( \delta_1 \) | \( \delta_2 \) | \( \alpha_1 \) | \( \alpha_2 \) | \( \alpha_3 \) | \( \beta_2 \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| true value | -2.079 | -2.890 | -2.079 | -2.197 | -2.197 | -3.000 | 1.000 | -1.000 | 2.000 | 4.000 | 2.000 |
| \( H = 200, U(1; 10) \) | | | | | | | | | | | | |
| bias | 0.100 | 0.027 | 0.034 | -0.370 | -0.355 | -0.288 | 0.029 | -0.025 | 0.103 | 0.221 | 0.318 |
| sd | 0.476 | 0.427 | 0.460 | 1.991 | 1.980 | 0.760 | 0.113 | 0.116 | 0.366 | 0.536 | 0.550 |
| rmse | 0.488 | 0.430 | 0.463 | 2.035 | 2.022 | 0.816 | 0.117 | 0.120 | 0.382 | 0.582 | 0.637 |
| mean-se | 0.548 | 0.531 | 0.454 | 1.168 | 1.231 | 0.884 | 0.095 | 0.096 | 0.417 | 0.626 | 0.558 |
| \( H = 200, U(1; 20) \) | | | | | | | | | | | | |
| bias | 0.012 | 0.018 | 0.021 | -0.170 | -0.116 | -0.073 | 0.017 | -0.018 | 0.041 | 0.083 | 0.136 |
| sd | 0.338 | 0.322 | 0.309 | 1.179 | 1.167 | 0.441 | 0.061 | 0.056 | 0.219 | 0.339 | 0.308 |
| rmse | 0.340 | 0.325 | 0.311 | 1.197 | 1.179 | 0.449 | 0.064 | 0.059 | 0.224 | 0.350 | 0.338 |
| mean-se | 0.366 | 0.332 | 0.359 | 0.887 | 0.851 | 0.508 | 0.065 | 0.065 | 0.252 | 0.366 | 0.357 |
| \( H = 400, U(1; 10) \) | | | | | | | | | | | | |
| bias | 0.043 | -0.017 | 0.003 | -0.237 | -0.224 | -0.081 | 0.010 | -0.008 | 0.041 | 0.069 | 0.099 |
| sd | 0.314 | 0.305 | 0.303 | 0.951 | 0.970 | 0.436 | 0.055 | 0.057 | 0.215 | 0.349 | 0.254 |
| rmse | 0.319 | 0.307 | 0.305 | 0.985 | 1.000 | 0.445 | 0.056 | 0.058 | 0.220 | 0.357 | 0.274 |
| mean-se | 0.371 | 0.302 | 0.338 | 0.804 | 0.802 | 0.450 | 0.057 | 0.057 | 0.231 | 0.337 | 0.302 |
| \( H = 400, U(1; 20) \) | | | | | | | | | | | | |
| bias | 0.051 | 0.003 | 0.012 | -0.108 | -0.067 | -0.034 | -0.003 | -0.003 | 0.015 | 0.047 | 0.053 |
| sd | 0.226 | 0.187 | 0.234 | 0.416 | 0.399 | 0.278 | 0.038 | 0.039 | 0.137 | 0.199 | 0.185 |
| rmse | 0.232 | 0.188 | 0.235 | 0.432 | 0.406 | 0.281 | 0.038 | 0.039 | 0.139 | 0.205 | 0.193 |
| mean-se | 0.239 | 0.212 | 0.290 | 0.458 | 0.458 | 0.289 | 0.040 | 0.040 | 0.152 | 0.252 | 0.200 |
| \( H = 200, U(1; 10) \) | | | | | | | | | | | | |
| bias | 0.099 | 0.013 | 0.010 | -0.130 | -0.084 | -0.074 | 0.013 | -0.017 | 0.038 | 0.075 | 0.114 |
| sd | 0.296 | 0.296 | 0.280 | 0.954 | 0.953 | 0.396 | 0.054 | 0.055 | 0.204 | 0.332 | 0.268 |
| rmse | 0.297 | 0.297 | 0.281 | 0.968 | 0.962 | 0.405 | 0.056 | 0.057 | 0.208 | 0.342 | 0.293 |
| mean-se | 0.348 | 0.311 | 0.327 | 0.854 | 0.859 | 0.451 | 0.055 | 0.055 | 0.221 | 0.328 | 0.290 |
| \( H = 200, U(1; 20) \) | | | | | | | | | | | | |
| bias | 0.052 | -0.006 | 0.004 | -0.144 | -0.084 | -0.068 | 0.011 | -0.008 | 0.037 | 0.071 | 0.106 |
| sd | 0.294 | 0.272 | 0.290 | 0.516 | 0.534 | 0.428 | 0.049 | 0.050 | 0.215 | 0.327 | 0.329 |
| rmse | 0.300 | 0.274 | 0.291 | 0.539 | 0.543 | 0.436 | 0.050 | 0.051 | 0.219 | 0.336 | 0.263 |
| mean-se | 0.297 | 0.304 | 0.290 | 0.625 | 0.616 | 0.432 | 0.052 | 0.052 | 0.234 | 0.345 | 0.293 |
| \( H = 400, U(1; 10) \) | | | | | | | | | | | | |
| bias | 0.043 | -0.004 | 0.011 | -0.099 | -0.062 | -0.019 | -0.003 | -0.001 | 0.003 | 0.040 | 0.039 |
| sd | 0.210 | 0.179 | 0.213 | 0.397 | 0.382 | 0.266 | 0.036 | 0.036 | 0.128 | 0.191 | 0.178 |
| rmse | 0.215 | 0.180 | 0.214 | 0.411 | 0.389 | 0.268 | 0.036 | 0.036 | 0.129 | 0.196 | 0.183 |
| mean-se | 0.213 | 0.192 | 0.207 | 0.421 | 0.423 | 0.271 | 0.037 | 0.037 | 0.142 | 0.215 | 0.185 |

**unweighted pairwise likelihood**

**weighted pairwise likelihood**
Table S-6: Simulation results for $k_1 = k_2 = 2$ and $T = 10$.

| true value | $p_1$ | $p_2$ | $r_1$ | $r_2$ | $d_0$ | $d_1$ | $d_2$ | $a_2$ | $b_2$ |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|            | -2.197 | -2.197 | -2.197 | -2.197 | -2.000 | 1.000 | -1.000 | 2.000 | 2.000 |

| H = 200, U(1; 10) | bias  | sd    | rmse  | mean-se |
|--------------------|-------|-------|-------|---------|
| unweighted pairwise likelihood | -0.001 | 0.207 | 0.208 | 0.239 |
| sd                 | 0.014 | 0.223 | 0.224 | 0.243 |
| rmse               | -0.024 | 0.354 | 0.356 | 0.298 |
| mean-se            | -0.056 | 0.315 | 0.317 | 0.310 |

| H = 200, U(1; 20) | bias  | sd    | rmse  | mean-se |
|--------------------|-------|-------|-------|---------|
| weighted pairwise likelihood | -0.031 | 0.198 | 0.201 | 0.199 |
| sd                 | -0.012 | 0.229 | 0.231 | 0.243 |
| rmse               | -0.016 | 0.195 | 0.196 | 0.248 |
| mean-se            | -0.018 | 0.197 | 0.198 | 0.241 |

| H = 400, U(1; 10) | bias  | sd    | rmse  | mean-se |
|--------------------|-------|-------|-------|---------|
| unweighted pairwise likelihood | -0.020 | 0.155 | 0.157 | 0.166 |
| sd                 | -0.028 | 0.168 | 0.171 | 0.165 |
| rmse               | -0.051 | 0.194 | 0.202 | 0.211 |
| mean-se            | -0.004 | 0.193 | 0.194 | 0.202 |

| H = 400, U(1; 20) | bias  | sd    | rmse  | mean-se |
|--------------------|-------|-------|-------|---------|
| weighted pairwise likelihood | -0.003 | 0.143 | 0.144 | 0.138 |
| sd                 | -0.011 | 0.141 | 0.142 | 0.138 |
| rmse               | -0.013 | 0.141 | 0.141 | 0.141 |
| mean-se            | -0.015 | 0.145 | 0.143 | 0.140 |

| H = 200, U(1; 10) | bias  | sd    | rmse  | mean-se |
|--------------------|-------|-------|-------|---------|
| unweighted pairwise likelihood | -0.006 | 0.198 | 0.199 | 0.218 |
| sd                 | -0.014 | 0.328 | 0.330 | 0.279 |
| rmse               | -0.004 | 0.307 | 0.308 | 0.291 |
| mean-se            | -0.050 | 0.221 | 0.228 | 0.214 |

| H = 200, U(1; 20) | bias  | sd    | rmse  | mean-se |
|--------------------|-------|-------|-------|---------|
| weighted pairwise likelihood | -0.027 | 0.169 | 0.172 | 0.177 |
| sd                 | -0.011 | 0.180 | 0.183 | 0.178 |
| rmse               | -0.026 | 0.193 | 0.194 | 0.189 |
| mean-se            | -0.009 | 0.148 | 0.149 | 0.190 |

| H = 400, U(1; 10) | bias  | sd    | rmse  | mean-se |
|--------------------|-------|-------|-------|---------|
| unweighted pairwise likelihood | -0.030 | 0.143 | 0.147 | 0.151 |
| sd                 | -0.030 | 0.155 | 0.158 | 0.151 |
| rmse               | -0.052 | 0.178 | 0.186 | 0.199 |
| mean-se            | 0.000  | 0.140 | 0.189 | 0.190 |

| H = 400, U(1; 20) | bias  | sd    | rmse  | mean-se |
|--------------------|-------|-------|-------|---------|
| weighted pairwise likelihood | -0.003 | 0.126 | 0.127 | 0.123 |
| sd                 | -0.016 | 0.122 | 0.124 | 0.123 |
| rmse               | -0.007 | 0.135 | 0.136 | 0.133 |
| mean-se            | -0.012 | 0.145 | 0.146 | 0.131 |
Table S-7: Simulation results for $k_1 = 2$ and $k_2 = 3$ and $T = 10$.

|     | $\rho_1$ | $\rho_2$ | $\tau_1$ | $\tau_2$ | $\tau_3$ | $\delta_0$ | $\delta_1$ | $\delta_2$ | $\alpha_2$ | $\beta_2$ | $\beta_3$ |
|-----|----------|----------|----------|----------|----------|-----------|-----------|-----------|-----------|----------|----------|
| true value | -2.197 | -2.197 | -2.079 | -2.079 | -2.079 | -3.000 | 1.000 | -1.000 | 2.000 | 2.000 | 4.000 |

### Unweighted pairwise likelihood

|     | $H = 200$, $U(1; 10)$ | $H = 200$, $U(1; 20)$ | $H = 400$, $U(1; 10)$ | $H = 400$, $U(1; 20)$ |
|-----|------------------------|------------------------|------------------------|------------------------|
| bias | 0.002 0.023 0.067 0.015 -0.002 -0.136 0.007 0.005 0.039 0.091 0.197 | -0.043 -0.002 0.032 -0.019 -0.017 -0.055 0.001 -0.002 0.010 0.045 0.104 | -0.044 -0.033 0.021 -0.003 0.005 -0.091 0.001 -0.008 0.023 0.081 0.129 | 0.002 -0.011 -0.008 0.016 0.033 -0.006 0.001 0.000 0.006 0.012 0.036 |
| sd  | 0.279 0.258 0.415 0.386 0.419 0.511 0.059 0.056 0.132 0.360 0.487 | 0.253 0.244 0.255 0.340 0.264 0.329 0.034 0.039 0.084 0.245 0.328 | 0.187 0.210 0.247 0.339 0.264 0.341 0.038 0.041 0.087 0.274 0.392 | 0.166 0.153 0.199 0.256 0.177 0.243 0.028 0.028 0.064 0.158 0.265 |
| rmse | 0.280 0.260 0.422 0.388 0.421 0.532 0.060 0.056 0.138 0.373 0.527 | 0.258 0.246 0.258 0.342 0.266 0.335 0.034 0.039 0.085 0.251 0.346 | 0.193 0.214 0.249 0.341 0.265 0.355 0.038 0.042 0.090 0.287 0.414 | 0.167 0.154 0.200 0.258 0.181 0.245 0.028 0.028 0.064 0.159 0.269 |
| mean-se | 0.303 0.306 0.357 0.513 0.350 0.492 0.058 0.057 0.142 0.401 0.604 | 0.235 0.236 0.262 0.334 0.245 0.333 0.040 0.040 0.094 0.260 0.379 | 0.211 0.208 0.256 0.300 0.230 0.289 0.040 0.040 0.093 0.259 0.383 | 0.160 0.162 0.179 0.230 0.184 0.229 0.028 0.028 0.063 0.169 0.259 |

### Weighted pairwise likelihood

| bias | 0.002 0.023 0.067 0.015 -0.002 -0.136 0.007 0.005 0.039 0.091 0.197 | -0.043 -0.002 0.032 -0.019 -0.017 -0.055 0.001 -0.002 0.010 0.045 0.104 | -0.044 -0.033 0.021 -0.003 0.005 -0.091 0.001 -0.008 0.023 0.081 0.129 | 0.002 -0.011 -0.008 0.016 0.033 -0.006 0.001 0.000 0.006 0.012 0.036 |
| sd  | 0.279 0.258 0.415 0.386 0.419 0.511 0.059 0.056 0.132 0.360 0.487 | 0.253 0.244 0.255 0.340 0.264 0.329 0.034 0.039 0.084 0.245 0.328 | 0.187 0.210 0.247 0.339 0.264 0.341 0.038 0.041 0.087 0.274 0.392 | 0.166 0.153 0.199 0.256 0.177 0.243 0.028 0.028 0.064 0.158 0.265 |
| rmse | 0.280 0.260 0.422 0.388 0.421 0.532 0.060 0.056 0.138 0.373 0.527 | 0.258 0.246 0.258 0.342 0.266 0.335 0.034 0.039 0.085 0.251 0.346 | 0.193 0.214 0.249 0.341 0.265 0.355 0.038 0.042 0.090 0.287 0.414 | 0.167 0.154 0.200 0.258 0.181 0.245 0.028 0.028 0.064 0.159 0.269 |
| mean-se | 0.303 0.306 0.357 0.513 0.350 0.492 0.058 0.057 0.142 0.401 0.604 | 0.235 0.236 0.262 0.334 0.245 0.333 0.040 0.040 0.094 0.260 0.379 | 0.211 0.208 0.256 0.300 0.230 0.289 0.040 0.040 0.093 0.259 0.383 | 0.160 0.162 0.179 0.230 0.184 0.229 0.028 0.028 0.063 0.169 0.259 |

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Table S-8: Simulation results for $k_1 = 3$ and $k_2 = 2$ and $T = 10$.  

| \( H = 200, \ U(1; 10) \) | \( H = 200, \ U(1; 20) \) | \( H = 400, \ U(1; 10) \) | \( H = 400, \ U(1; 20) \) | \( H = 200, \ U(1; 10) \) | \( H = 200, \ U(1; 20) \) | \( H = 400, \ U(1; 10) \) | \( H = 400, \ U(1; 20) \) |
|---|---|---|---|---|---|---|---|
| true value | \(-2.079\) | \(-2.079\) | \(-2.197\) | \(-2.197\) | \(-3.000\) | \(1.000\) | \(-1.000\) | \(2.000\) | \(4.000\) | \(2.000\) |

### unweighted pairwise likelihood

| \( H = 200, \ U(1; 10) \) | \( H = 200, \ U(1; 20) \) | \( H = 400, \ U(1; 10) \) | \( H = 400, \ U(1; 20) \) |
|---|---|---|---|
| bias | \(0.024\) | \(0.007\) | \(-0.012\) | \(0.007\) |
| sd  | \(0.226\) | \(0.170\) | \(0.171\) | \(0.171\) |
| rmse | \(0.229\) | \(0.209\) | \(0.210\) | \(0.220\) |
| mean-se | \(0.241\) | \(0.210\) | \(0.220\) | \(0.210\) |

### weighted pairwise likelihood

| \( H = 200, \ U(1; 10) \) | \( H = 200, \ U(1; 20) \) | \( H = 400, \ U(1; 10) \) | \( H = 400, \ U(1; 20) \) |
|---|---|---|---|
| bias | \(0.026\) | \(0.002\) | \(-0.008\) | \(-0.008\) |
| sd  | \(0.192\) | \(0.193\) | \(0.157\) | \(0.157\) |
| rmse | \(0.194\) | \(0.194\) | \(0.194\) | \(0.194\) |
| mean-se | \(0.225\) | \(0.192\) | \(0.192\) | \(0.192\) |
Figure S-1: Representation of 1,000 trajectories randomly generated from the estimated distribution of the cluster-level latent process; values are randomly perturbed in order to avoid overlapping and make the trajectories distinguishable.

Figure S-2: Representation of 1,000 trajectories randomly generated from the estimated distribution of the individual-level latent process; values are randomly perturbed in order to avoid overlapping and make the trajectories distinguishable.