A String and M-theory Origin for the Salam-Sezgin Model

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ABSTRACT

An M/string-theory origin for the six-dimensional Salam-Sezgin chiral gauged supergravity is obtained, by embedding it as a consistent Pauli-type reduction of type I or heterotic supergravity on the non-compact hyperboloid $H^{2,2}$ times $S^1$. We can also obtain embeddings of larger, non-chiral, gauged supergravities in six dimensions, whose consistent truncation yields the Salam-Sezgin theory. The lift of the Salam-Sezgin (Minkowski)$_4 \times S^2$ ground state to ten dimensions is asymptotic at large distances to the near-horizon geometry of the NS5-brane.

† Research supported in part by DOE grant DE-FG02-95ER40893, NATO grant 97061, NSF grant INTO-03585 and the Fay. R. and Eugene L. Langberg Chair.

‡ Research supported in full by DOE grant DE-FG02-95ER40899

† Research supported in part by DOE grant DE-FG03-95ER40917.
1 Introduction

It is well known that six-dimensional gauged Einstein-Maxwell supergravity (contained within the theories in [1]) admits a supersymmetric 2-sphere compactification to four-dimensional Minkowski spacetime [2]. Motivated in part by recent phenomenological interest in six-dimensional models, there has recently been considerable activity investigating the properties of this remarkable model of Salam and Sezgin [3–7]. From a theoretical point of view, two important properties are that the theory admits a consistent Pauli reduction on $S^2$ [6], and that the supersymmetric (Minkowski)4 × $S^2$ ground state is the unique non-singular solution with maximal four-dimensional spacetime symmetry [7]. An important and hitherto unsolved problem is whether or not the model can be derived from M/string-theory. Since the model has a scalar field with a positive potential, it is natural to suppose, because of the results in [8, 9], that any non-singular internal space must be non-compact.

In this paper we shall show that one can obtain the Salam-Sezgin theory from a dimensional reduction of ten-dimensional type I supergravity, and that indeed the internal space is non-compact. The construction can be described as follows:

1. Perform a Pauli dimensional reduction1 of eleven-dimensional supergravity on $S^4$, to give maximal $SO(5)$ gauged supergravity in $D = 7$. The complete details of this reduction, including the explicit demonstration of its consistency, were given in [11,12].

2. Take a singular limit of the seven-dimensional theory, in which the $SO(5)$ gauge group is In"{o}n"{u}-Wigner contracted to $SO(4)$. In terms of the $S^4$ reduction, this corresponds to an infinite “stretching” the $S^4$ along one axis, so that it limits to $S^3 \times \mathbb{R}$. The resulting seven-dimensional theory can now be viewed as a Pauli $S^3$ reduction of type IIA supergravity [13,14].

3. Perform a (consistent) truncation of this $SO(4)$ gauged supergravity, in which all the fields associated with the Ramond-Ramond sector of the type IIA supergravity are set to zero. The resulting theory can be interpreted as a consistent Pauli $S^3$ reduction of ten-dimensional type I supergravity, or of the heterotic theory.

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1The terminology of “Pauli reduction” was introduced in [10] to describe the situation, first envisaged by Pauli, where a dimensional reduction on a coset space yields a lower-dimensional theory that includes all the gauge bosons associated with the isometry group of the coset manifold. Such reductions are generically inconsistent, and it is only in exceptional cases, such as the one considered here, that a consistent reduction is possible.
4. Pass from the compact $SO(4)$ form for the gauge group to the non-compact form $SO(2, 2)$. In terms of the dimensional reduction, this corresponds [15] to replacing the $S^3$ specified in its ground state by $\sum_{A=1}^{4} \delta_{AB} \mu^A \mu^B = 1$, where $\mu^A$ are coordinates on $\mathbb{R}^4$, by the hyperboloid $\mathcal{H}^{2, 2}$ specified by $\eta_{AB} \mu^A \mu^B = 1$, where $\eta = \text{diag} (1, 1, -1, -1)$. Note that the metric induced on $\mathcal{H}^{2, 2}$ is positive definite, and that consequently even in its ground state, for which $ds_3^2 = \sum_{A=1}^{4} d\mu^A d\mu^A$, the metric on $\mathcal{H}^{2, 2}$ is inhomogeneous. Despite the non-compact gauging, the seven-dimensional theory still has positive-energy kinetic terms for all its fields.

5. Perform a standard Kaluza reduction of the seven-dimensional theory on $S^1$. This results in a six-dimensional non-chiral $N = (1, 1)$ supergravity, whose bosonic sector comprises the metric, the gauge fields of $SO(2, 2) \times U(1)$, an additional 2-form field strength, a 3-form field strength, and 17 scalar fields.

6. Perform a supersymmetric (consistent) truncation of this theory, to obtain the chiral $N = (1, 0)$ gauged supergravity of the Salam-Sezgin model. Its bosonic sector comprises the metric, a $U(1)$ gauge field, a 3-form field strength, and a dilaton.

Although, in order to make contact with previous work, we have described the procedure initially in terms of seven-dimensional supergravities with compact $SO(5)$ and $SO(4)$ gauge groups, one can of course begin directly with the theory obtained by reducing type I supergravity on the non-compact space $\mathcal{H}^{2, 2}$. Thus, in summary, our construction yields an embedding of the Salam-Sezgin theory as a consistent Pauli-type reduction of type I or heterotic supergravity on the non-compact internal space $\mathcal{H}^{2, 2}$ times $S^1$. A further lifting on $S^1$ allows us to embed the Salam-Sezgin theory in eleven-dimensional supergravity.

2 Seven-Dimensional $SO(2, 2)$ Gauged Supergravity

2.1 Summary of the $N = 4$, $SO(5)$ gauged supergravity

We begin by summarising the salient features of the the maximal $SO(5)$ gauged supergravity in seven dimensions. Specifically, we shall present the Lagrangian for the bosonic sector, together with the supersymmetry transformation rules. For the complete details, the reader is referred to [12, 16].

In the notation and normalisation that we shall use, the bosonic sector of the Lagrangian can be written as

$$L_7 = R \ast 1 - *P_{ij} \wedge P^{ij} - \frac{1}{2} \Pi^i_A \Pi^j_B \Pi^c_C \Pi^d_D *F_{(2)}^{AB} \wedge F_{(2)}^{CD} - \frac{1}{2} \Pi^{-1, i}_A \Pi^{-1, B}_i *S_{(3)}^A \wedge S_{(3)}^B$$
\[ + \frac{1}{2g} \eta^{AB} S_{(3)}^A \wedge DS_{(3)}^B - \frac{1}{8g} \epsilon_{AC_1 \cdots C_4} \eta^{AB} S_{(3)}^A \wedge F_{C_1 C_2}^C \wedge F_{C_3 C_4}^C - \frac{1}{g} \Omega - V \ast \mathbb{1}, \]

where

\[ F_{(2)A}^B = dA_{(1)A}^B + g A_{(1)A}^C \wedge A_{(1)C}^B, \]

\[ D S_{(3)A} = dS_{(3)A} + g A_{(1)A}^B \wedge S_{(3)B}, \]

\[ V = \frac{1}{2g^2} (2T_{ij} T_{ij} - (T_{ii})^2), \quad T_{ij} = \Pi^{-1}_i A \Pi^{-1}_j B \eta_{AB}, \]

\[ \Pi^{-1}_i A (\delta A^B d + g A_{(1)A}^B) \Pi B^k \delta_{kj} = P_{ij} + Q_{ij}; \quad P_{ij} = P_{(ij)}, \quad Q_{ij} = Q_{[ij]}, \] (2)

and by definition we have

\[ A_{(i)}^{AB} \equiv \eta^{AC} A_{(i)CB}, \quad \text{with} \quad A_{(i)}^{AB} = -A_{(i)}^{BA}. \] (3)

Here, the metric \( \eta_{AB} \) for the gauge group is just \( \delta A_B \) in the compact \( SO(5)_g \) gauging. Later, we shall consider non-compact gaugings, for which \( \eta_{AB} \) will have indefinite signature. Note, however, that even in the case of non-compact gaugings, the composite group \( SO(5)_c \) will retain its compact form, with its metric \( \delta_{ij} \). The quantity \( \Omega \) denotes the Chern-Simons terms for the Yang-Mills fields, with \( d\Omega \sim (\text{tr} F_{(2)}^2)^2 + \text{tr} F_{(2)}^4 \) (see \([16]\) for details). The formulation of the supergravity theory that we are using here is the one of \([12,16]\), in which the “vielbein” \( \Pi A^i \) for the scalar coset manifold \( SL(5, \mathbb{R})/SO(5)_c \) is in \( SL(5, \mathbb{R}) \), and \( Q_{ij} \) is a composite connection. The Yang-Mills fields transform under the gauge group \( SO(5)_g \). The \( SO(5)_c \) and \( SO(5)_g \) groups of this compact case become equivalent if one imposes the symmetric gauge \( \Pi = \Pi^i \).

For some purposes, it is useful to introduce a tensor \( T^{AB} \) in place of \( T_{ij} \), in order to parameterise the scalar fields:

\[ T^{AB} \equiv \Pi^{-1}_i A \Pi^{-1}_j B. \] (4)

In terms of \( T^{AB} \), the scalar potential is given by

\[ V = \frac{1}{2g^2} (T^{AB} T^{CD} \eta_{AC} \eta_{BD} - (T^{AB} \eta_{AB})^2). \] (5)

The fermionic sector of the \( SO(5) \) gauged theory comprises the gravitini \( \psi^I_\mu \) and gaugini \( \lambda^i_I \), where the \( I \) index denotes the four-dimensional spinor of \( SO(5)_c \). Their supersymmetry transformation rules are \([16]\)

\[ \delta \psi_\mu = D_\mu \epsilon - \frac{1}{20} g T_{ii} \Gamma_\mu \epsilon - \frac{1}{40\sqrt{2}} (\Gamma_\mu^{\nu \rho} - 8 \delta_\mu^{\nu} \Gamma^{\rho}) \gamma_i \epsilon \Pi A^i \Pi B^j F_{\nu \rho}^{AB} \]

\[ - \frac{1}{10} (\Gamma_\mu^{\nu \rho \sigma} - \frac{9}{4} \delta_\mu^{\nu} \Gamma^{\rho \sigma} \gamma_j \epsilon \Pi^{-1}_i A S_{\nu \rho \sigma, A}, \]

\[ \delta \lambda_i = \frac{1}{10 \sqrt{2}} T^{\mu \nu} (\gamma_{k \ell} \gamma_i - \frac{1}{8} \gamma_{i} \gamma_{k \ell}) \epsilon \Pi A^k \Pi B^\ell F_{\mu \nu}^{AB} - \frac{1}{120} \Gamma^{\mu \nu \rho} (\gamma_i \gamma_j - 4 \delta_i^j) \epsilon \Pi^{-1}_j A S_{\mu \nu \rho, A} \]

\[ + \frac{1}{2} g (T_{ij} - \frac{1}{3} T_{kk} \delta_{ij}) \gamma^j \epsilon + \frac{1}{2} \Gamma_\mu \gamma^j \epsilon P_{\mu, ij}. \] (6)
Here $\gamma_i$ denotes the “internal” Dirac matrices of $SO(5)_c$, and the associated spinor indices $I, J, \ldots$ have been suppressed. The spinors $\lambda_i$ are subject to the constraint
\[ \gamma^i \lambda_i = 0. \tag{7} \]
The seven-dimensional Dirac matrices are denoted by $\Gamma^\mu$. The covariant derivative $D^\mu$ acting on spinors is given by
\[ D \epsilon = d \epsilon + \frac{1}{4} \omega_{ab} \Gamma_{\mu}^{ab} \epsilon + \frac{1}{2} Q_{ij} \gamma^{ij} \epsilon. \tag{8} \]

The supersymmetry transformation rules for the bosonic fields are given by
\[ \delta e^a_\mu = \frac{1}{2} \bar{\epsilon} \Gamma^a \psi_\mu, \]
\[ \Pi_A^i \Pi_B^j \delta A^{AB}_\mu = \frac{1}{3 \sqrt{2}} \bar{\epsilon} \gamma^{ij} \psi_\mu + \frac{1}{4 \sqrt{2}} \bar{\epsilon} \Gamma_\mu \gamma^i \lambda_k, \]
\[ \Pi^{-1}_{i} A^A \delta \Pi_A^j = \frac{1}{2} (\bar{\epsilon} \gamma_i \lambda^j + \bar{\epsilon} \gamma^j \lambda_i), \tag{9} \]
\[ \delta S_{\mu\nu\rho A} = -\frac{3}{4 \sqrt{2}} \Pi_A^i (2 \bar{\epsilon} \gamma_{ijk} \psi_{[\mu} + \bar{\epsilon} \Gamma_{[\mu} \gamma^i \gamma_{\rho]} \lambda_{\nu]} \Pi_B^j \Pi_C^k F^{BC}_{\nu]} \]
\[ -\frac{3}{2} \delta_{ij} \Pi_B^j D_{[\mu} (2 \bar{\epsilon} \Gamma_{[\nu} \gamma^i \psi_{\rho]} + \bar{\epsilon} \Gamma_{[\nu} \lambda_{\rho]} \gamma^j) \]
\[ + \frac{1}{2} \delta_{AB} \Pi^{-1}_{i} (3 \bar{\epsilon} \Gamma_{[\mu \nu} \gamma^i \psi_{\rho]} - \bar{\epsilon} \Gamma_{\mu \nu \rho} \lambda^i). \]

### 2.2 The $N = 2$ gauged $SO(4)$ limit of $N = 4, D = 7$ supergravity

We next consider the Inönu-Wigner group contraction limit of the $SO(5)_g$ gauged theory in seven dimensions, to obtain a maximal supergravity with $SO(4)$ gauging. The procedure was described in a truncated system in [13], and implemented in the bosonic sector of the full theory in [14]. The idea is to decompose the $SO(5)$ vector indices in a $4 + 1$ split, and make appropriate rescalings of the resulting $SO(4)$-valued fields, such that in a singular limit the gauge group degenerates to $SO(4)$. For the $SO(5)_c$ and $SO(5)_g$ indices we make the decompositions
\[ i = (0, \alpha), \quad A = (0, \bar{A}) \tag{10} \]
respectively, where $\alpha$ and $\bar{A}$ run over the values 1, 2, 3, 4.

Following [14], we decompose and rescale the bosonic fields, and the gauge coupling constant, as follows:
\[ g = k^2 \bar{g}, \quad A_{(i)}^0 = k^3 \bar{A}_{(i)}^0, \quad A_{(i)}^{AB} = k^{-2} \bar{A}_{(i)}^{AB}, \]
\[ S_{(3)}^0 = k^{-4} \bar{H}_{(3)}, \quad S_{(3)}^A = k \bar{S}_{(3)}^A, \]
\[ T_{ij} = \begin{pmatrix} k^8 \Phi^{-1} + k^8 \chi \gamma \chi & -k^3 \chi^\alpha \\ -k^3 \chi^\alpha & k^{-2} \Phi^{1/4} M_{\alpha \beta} \end{pmatrix}. \tag{11} \]
where $M_{\alpha\beta}$, like the original scalar matrix $T_{ij}$, is unimodular. In the fermionic sector, no scalings by $k$ are required.

After taking the limit $k \rightarrow 0$, one obtains the group-contracted $SO(4)$-gauged theory. The complete results for the bosonic sector can be found in [14]. Note that after taking the limit, $\bar{H}_{(3)}$, which is the rescaled 3-form $S^0_{(3)}$, acquires the interpretation of a field-strength for a 2-form potential, rather than being a fundamental field in its own right (see [14]).

At this point no truncation of fields has been performed, and the $SO(4)$-gauged theory can be interpreted as a consistent Pauli reduction of type IIA supergravity on $S^3$ [14]. It has $N = 4$ (i.e. maximal) supersymmetry. Since the seven-dimensional theory at this stage is quite complicated, and we shall not require its detailed form for our purposes, we shall not repeat expressions found in [14], and their fermionic counterparts, here. Rather, our interest at this stage is in performing a consistent truncation of the seven-dimensional theory to one with $N = 2$ supersymmetry, which can be interpreted as a consistent Pauli reduction of ten-dimensional type I supergravity, or equivalently, the heterotic theory.

To make this truncation, in the bosonic sector we set the four scalars $\chi^\alpha$, the four gauge bosons $A_0^{(1)}$, and the four 3-form fields $S_3^\alpha$ to zero. It is easily seen from the formulæ in [14] that this truncation is a consistent one, in the sense that it is compatible with the equations of motion for the truncated fields. Of course since we wish to obtain a supersymmetric truncated theory it is also necessary to set the appropriate fermionic superpartners to zero, and to check that this fermionic truncation is consistent with the supersymmetry transformation rules.

We find that the appropriate fermionic truncation can be achieved by first projecting all the spinors into eigenstates of $\gamma_0$, which is the chirality operator with respect to the $SO(4)$ contracted subgroup of the original $SO(5)$ internal group. Writing $\epsilon = \epsilon^+ + \epsilon^-$, $\epsilon^\pm = \pm \gamma_0 \epsilon^\pm$, etc, we then make the following truncation:

$$\epsilon^- = 0, \quad \psi^-_\mu = 0, \quad \lambda^-_0 = 0, \quad \lambda^+_\alpha = 0. \quad (12)$$

The constraint (7) now leads to the relation

$$\lambda^+_0 = -\gamma^\alpha \lambda^-_\alpha, \quad (13)$$

and so the independent fermionic fields that survive the truncation can be taken to be just $\psi^+_\mu$ and $\lambda^-_\alpha$.

After some algebra, it can be verified from the original transformation rules (6) and (9) that the bosonic and fermionic truncations described above are fully consistent with supersymmetry.
At this stage, it is useful to summarise the details of the bosonic contraction and truncation, in terms of the scalar vielbein $\Pi_A^i$. Thus we have

$$
\Pi_0^0 = k^{-4} \Phi^{1/2}, \quad \Pi_A^\alpha = k \Phi^{-1/8} \pi_A^\alpha, \\
g = k^2 \tilde{g}, \quad A_{(1)}^{\hat{A}\hat{B}} = k^{-2} \tilde{A}_{(1)}^{\hat{A}\hat{B}}, \quad S_{(3)} = k^{-4} \tilde{H}_{(3)},
$$

(14)

where $\det(\pi_A^\alpha) = 1$. As described in [14], although $S_{(3)}^0$ was itself a fundamental field, subject to an “odd-dimensional self-duality equation” in the original $SO(5)$-gauged theory, it now acquires the interpretation of being a 3-form field strength for a 2-form potential in the contraction limit. In what follows we shall drop the tildes that were previously used to distinguish between the original fields and the rescaled fields of the In"on"u-Wigner contraction limit. The Lagrangian for the bosonic sector of the $N = 2$ gauged $SO(4)$ supergravity is thus given by (see [14])

$$
\mathcal{L}_7 = R + \frac{5}{32} \Phi^{-2} * d\Phi \wedge d\Phi - p_{\alpha \beta} \wedge p^{\alpha \beta} - \frac{1}{2} \Phi^{-1} * H_{(3)} \wedge H_{(3)}
- \frac{1}{2} \Phi^{-1/2} \pi_{\hat{A}}^\alpha \pi_{\hat{B}}^\beta \pi_{\hat{C}}^\gamma \pi_{\hat{D}}^\alpha F_{(2)}^{\hat{A}\hat{B}} \wedge F_{(2)}^{\hat{C}\hat{D}} - \frac{1}{g} \Omega - V * 1, \tag{15}
$$

where $p_{\alpha \beta}$ is defined analogously to $P_{ij}$ in (2), namely

$$
p_{\alpha \beta} = \pi^{-1}(\alpha \hat{A} [\bar{\delta}_{\hat{A}} \hat{B} d + g A_{(1)}^{\hat{B}}] \bar{\pi}_{\hat{B}}^\gamma \delta_{\beta})^\gamma.
$$

Thus $p_{\alpha \beta}$ is traceless, and it describes the derivatives of the 9 scalar fields in the unimodular $M_{\alpha \beta}$. The scalar potential is given by

$$
V = \frac{1}{2} g^2 \Phi^{1/2} (2M_{\alpha \beta} M_{\alpha \beta} - (M_{\alpha \alpha})^2). \tag{17}
$$

Note that the invariant tensor $\eta_{\hat{A}\hat{B}}$ for the group of the gauge symmetry appears in the expressions

$$
M_{\alpha \beta} = \pi^{-1} \hat{A} \pi^{-1} \hat{B} \eta_{\hat{A}\hat{B}}, \quad A_{(1)}^{\hat{A}\hat{B}} = \eta \hat{A}\hat{C} A_{(1)}^{\hat{C}\hat{B}}. \tag{18}
$$

For now, since we are discussing the compact case with $SO(4)$ gauging, we have $\eta_{\hat{A}\hat{B}} = \delta_{\hat{A}\hat{B}}$. Note that the $\alpha, \beta$ indices are always raised and lowered with a Kronecker delta, regardless of whether the gauge group is compact or non-compact.

We find that the supersymmetry transformation rules for the fermions in the $N = 2$ gauged $SO(4)$ theory are given by

$$
\delta \psi_{\mu} = D_{\mu} \epsilon - \frac{1}{40} g M_{\alpha \alpha} \Phi^{1/4} \Gamma_{\mu} \epsilon - \frac{1}{16 \sqrt{2}} (\Gamma_{\mu}^\nu \rho_{\rho} - 8 \epsilon^\nu \rho_{\rho} \gamma_{\alpha \beta} \gamma_{\alpha} \pi_{\beta}^\rho) \epsilon \Phi^{-1/4} \pi_{\beta}^\rho F_{\mu \rho}^{\hat{A}\hat{B}}
- \frac{1}{60} (\Gamma_{\mu}^\nu \rho_{\rho} - 2 \epsilon^\nu \rho_{\rho} \epsilon) \Phi^{-1/2} H_{\mu \nu \rho},
$$

$$
\delta \lambda_{\alpha} = \frac{1}{2} \Gamma^\mu \gamma_{\alpha} \epsilon F_{\mu \alpha \beta} + \frac{1}{10 \sqrt{2}} \Gamma^{\mu \nu} (\gamma_{\alpha \gamma} \gamma_{\alpha} - \frac{5}{3} \epsilon \gamma_{\alpha \gamma}) \epsilon \Phi^{-1/4} \pi_{\beta}^\rho \pi_{\gamma}^\rho F_{\mu \rho}^{\hat{A}\hat{B}} +
- \frac{1}{120} \Gamma^{\mu \nu} \gamma_{\alpha} \epsilon \Phi^{-1/2} H_{\mu \nu \rho} + \frac{1}{6} g (M_{\alpha \beta} - \frac{1}{5} M_{\gamma \gamma} \delta_{\alpha \beta}) \Phi^{1/4} \gamma_{\alpha} \epsilon, \tag{19}
$$

(19)
where we are now suppressing the internal $SO(4)$ chirality superscripts on $\epsilon^+, \psi^+_{\mu}$ and $\lambda^-_{\alpha}$.

Here $P_{\alpha\beta} = p_{\alpha\beta} - \frac{1}{8} \Phi^{-1} d\Phi \delta_{\alpha\beta}$.

The supersymmetry transformations rules for the bosonic fields take the form

$$
\delta \epsilon^a_{\mu} = \frac{1}{2} \epsilon^a \Gamma^a \psi^a_{\mu},
\pi^A_{\alpha} \pi^B_{\beta} \delta A^{AB}_{\mu} = \frac{1}{2\sqrt{2}} \bar{\epsilon} \gamma^{\alpha\beta} \psi^a_{\mu} + \frac{1}{2\sqrt{2}} \epsilon \Gamma^a (\gamma^\beta \lambda^\alpha - \gamma^\alpha \lambda^\beta),
\pi^{-1} \alpha \delta \pi^A_{\beta} = \frac{1}{4} (\bar{\epsilon} \gamma_{\alpha} \lambda^\beta + \bar{\epsilon} \gamma^\beta \lambda_{\alpha}), \quad \Phi^{-1} \delta \Phi = -\bar{\epsilon} \gamma^\alpha \lambda_{\alpha},

\delta H_{\mu\nu\rho} = -\frac{3}{2\sqrt{2}} \Phi^{1/4} (\bar{\epsilon} \gamma_{\alpha\beta} \psi^a_{\mu} - \bar{\epsilon} \Gamma_{[\mu} \gamma_{\alpha\beta] \gamma} \lambda^\gamma) \pi^B_{\alpha} \pi^C_{\beta} F^{BC}_{\nu\rho}
- \frac{3}{2} \Phi^{1/2} D_{[\mu} (2\bar{\epsilon} \Gamma_{\nu]} \psi^a_{\rho]} + \bar{\epsilon} \Gamma_{\mu\rho} \gamma^\alpha \lambda_{\alpha}) + \frac{1}{2} \Phi^{-1/2} (3\bar{\epsilon} \Gamma_{[\mu\nu} \psi^a_{\rho]} + \bar{\epsilon} \Gamma_{\mu\nu\rho} \gamma^\alpha \lambda_{\alpha}).
$$

2.3 The $N = 2$ gauged $SO(2, 2)$ theory in $D = 7$

Our discussion so far in this section has focussed on the $N = 4$ compact $SO(5)$ gauging in seven dimensions, its contraction limit to an $SO(4)$ gauged supergravity, again with $N = 4$, and then the truncation to an $N = 2$ gauged theory, again with $SO(4)$ Yang-Mills fields.

In this subsection, we turn to the consideration of a non-compact gauging for the $N = 2$ theory, with $SO(2, 2)$ as gauge group. (See [15], and references therein, for a discussion of the higher-dimensional origins of non-compact gaugings in supergravities.)

The main idea is to take $\eta_{AB} = \text{diag}(+1, +1, -1, -1)$. The gauge group becomes $SO(2, 2)$, but the global $SO(4)$ group is reduced to its intersection with the gauge group, namely $SO(2) \times SO(2)$. The internal manifold is the hyperboloid $\mathcal{H}^{2,2}$, specified by

$$
\mu_1^2 + \mu_2^2 - \mu_3^2 - \mu_4^2 = 1,
$$

embedded in Euclidean space $\mathbb{E}^4$ with the standard metric $ds^2 = d\mu_1^2 + d\mu_2^2 + d\mu_3^2 + d\mu_4^2$.

The hyperboloid is invariant under the action of the non-compact group $SO(2, 2)$, and may be identified with the symmetric space $SO(2, 2)/SO(2, 1)$. However, the metric induced from the Euclidean metric is not the standard homogeneous metric on $SO(2, 2)/SO(2, 1)$, which has signature $(1, 2)$, but rather an inhomogeneous metric of cohomogeneity one, whose isometry group is $SO(2) \times SO(2)$. Concretely, it is convenient to parameterise the $\mathbb{E}^4$ coordinates as

$$
\mu_1^1 + i \mu_2^2 = \cosh \rho e^{i\alpha}, \quad \mu_3^3 + i \mu_4^4 = \sinh \rho e^{i\beta},
$$

where $0 \leq \rho < \infty$, $0 \leq \alpha < 2\pi$, $0 \leq \beta < 2\pi$, so that the constraint (21) is satisfied.

In the ground state, where $M_{AB} = \delta_{AB}$, the metric on $\mathbb{E}^4$ induces the metric

$$
ds_3^2 = \cosh 2\rho d\rho^2 + \cosh^2 \rho d\alpha^2 + \sinh^2 \rho d\beta^2
$$

(23)
on $\mathcal{H}^{2,2}$. Because the length of the $\alpha$ circle never vanishes, the topology of $\mathcal{H}^{2,2}$ is $\mathbb{R}^2 \times S^1$, where $\rho$ and $\beta$ parameterise the $\mathbb{R}^2$ and $\alpha$ parameterises the $S^1$.

By inspection, the Lagrangian (15), with this choice $\eta_{AB} = \text{diag}(1,1,-1,-1)$, has $SO(2,2)$ local gauge invariance and positive kinetic energies for all the fields. For the scalars, this is because the $\alpha$ and $\beta$ indices in $-\ast P_{\alpha\beta} \wedge P^{\alpha\beta}$ are always raised and lowered with the positive-definite metric $\delta_{\alpha\beta}$, whether or not the gauge group is compact. Likewise, the gauge-field kinetic energies are all positive, as can be seen by defining $F^{\alpha\beta}_{(2)} = \pi_A^\alpha \pi_B^\beta F^{AB}_{(2)}$, so that one has $-\frac{1}{2} \Phi^{-1/2} \ast F^{\alpha\beta}_{(2)} \wedge F^{\alpha\beta}_{(2)}$.

In a gauge theory without scalars, it is not possible to construct gauge-invariant kinetic terms for the gauge bosons that have positive kinetic energy. It is the presence of the scalar fields $\pi_{iA}$, transforming non-trivially under the gauge group, that allows the existence of gauge-invariant kinetic terms of positive energy. Under this gauge transformation, the scalar kinetic term is unchanged because $P_{\alpha\beta}$ is invariant. Of course in the ground state, where $M_{AB} = \delta_{AB}$, the non-compact gauge group $SO(2,2)$ is broken down to its maximal compact subgroup, $SO(2) \times SO(2)$.

## 3 Kaluza Reduction to Six Dimensions

In this section, we shall show how the $N = (1,0)$ chiral Einstein-Maxwell gauged supergravity in six dimensions can be obtained by performing a Kaluza reduction of the seven-dimensional $N = 2$ gauged $SO(2,2)$ theory of section 2.3 to give an $N = (1,1)$ supergravity, and then performing a consistent chiral truncation of this non-chiral theory. We begin by summarising the general formalism for the Kaluza reduction, and then we implement it, together with the chiral truncation, in the subsequent subsection.

### 3.1 Kaluza reduction to the $N = (1,1)$ theory

The procedure for performing a Kaluza reduction on $S^1$ is well established, and here we shall just review the essential points, in order to establish our notation. We shall now place hats on the seven-dimensional fields, and take the seven-dimensional coordinate and tangent-frame indices to be $\hat{\mu} = (\mu, z)$ and $\hat{a} = (a,7)$ respectively.

The metric is reduced according to

$$ds_7^2 = e^{2a\varphi} \, ds_6^2 + e^{-8a\varphi} \left( dz + A_{(1)} \right)^2,$$

(24)
where \( \alpha = 1/(2\sqrt{10}) \), for which we choose the natural vielbein basis
\[
\hat{e}^a = e^{\alpha \varphi} e^a, \quad \hat{e}^7 = e^{-4\alpha \varphi} (dz + A_{(1)}).
\]

The dilaton couplings in the above are chosen so that an Einstein-Hilbert Lagrangian in \( D = 7 \) reduces to give an Einstein-Hilbert term in \( D = 6 \), and so that the kinetic term for \( \varphi \) has its canonical normalisation. A \( p \)-form potential is reduced according to \( \hat{A}_p = A_p + A_{(p-1)} \wedge dz \). The associated field strength is reduced according to

\[
\hat{F}_{(p+1)} = F_{(p+1)} + F_p \wedge (dz + A_{(1)}),
\]

where the lower-dimensional field strengths are defined by

\[
F_{(p+1)} = dA_{(p)} - dA_{(p-1)} \wedge A_{(1)}, \quad F_p = dA_{(p-1)}.
\]

The Kaluza reduction of the fermions is determined by the requirement that the lower-dimensional kinetic terms, like those in the higher dimension, should have no scalar prefactors. Thus we take

\[
\hat{\lambda} = e^{-\frac{1}{2} \alpha \varphi} \lambda, \quad \hat{\psi}_a = e^{-\frac{1}{2} \alpha \varphi} \psi_a, \quad \hat{\psi}_7 = e^{-\frac{1}{2} \alpha \varphi} \psi_7.
\]

(Note that the reduction for the gravitino is expressed in terms of vielbein components for the vector index.) In order to get a canonical transformation rule for the lower-dimensional gravitino, \( \delta \psi_a = \nabla_a \epsilon + \cdots \), we should reduce \( \hat{\epsilon} \) according to

\[
\hat{\epsilon} = e^{\frac{1}{2} \alpha \varphi} \epsilon.
\]

Applying the above reduction procedures to the \( N = 2 \) gauged \( SO(2, 2) \) theory obtained in section 2.3 is a purely mechanical exercise, and since the complete details of the resulting non-chiral \( N = (1, 1) \) supergravity in six dimensions are not needed for our present purposes, we shall just give an outline of the result here. We find the bosonic Lagrangian in six dimensions is given by

\[
L_6 = R \ast 1 - \frac{5}{16} \Phi^{-2} \ast d\Phi \wedge d\Phi - \ast P_{\alpha \beta} \wedge P^{\alpha \beta} - \frac{1}{2} \ast d\epsilon \wedge d\varphi - \frac{1}{2} e^{-10\alpha \varphi} \ast \mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)}
\]
\[\quad - \frac{1}{2} \Phi^{-1} e^{-4\alpha \varphi} \ast H_{(3)} \wedge H_{(3)} - \frac{1}{2} \Phi^{-1} e^{6\alpha \varphi} \ast H_{(2)} \wedge H_{(2)} - \frac{1}{2} \Omega
\]
\[\quad - \frac{1}{2} \Phi^{-1/2} \pi_A^\alpha \pi_B^\beta \pi_C^\alpha \pi_D^\beta \left( e^{-2\alpha \varphi} \ast F_{(2)}^{AB} \wedge F_{(2)}^{CD} + e^{8\alpha \varphi} \ast F_{(1)}^{AB} \wedge F_{(1)}^{CD} \right) - V \ast 1,
\]

where the scalar potential is now given by

\[
V = \frac{1}{2} g^2 \Phi^{1/2} e^{2\alpha \varphi} (2M_{\alpha \beta} M_{\alpha \beta} - (M_{\alpha \alpha})^2).
\]
The fields $H_{(2)}$ and $F^{\bar{A}\bar{B}}_{(1)}$ come from the reductions of $H_{(3)}$ and $F_{(2)}^{\bar{A}\bar{B}}$ respectively.

The associated six-dimensional fermionic fields are $\psi_\mu$, $\psi_7$, and $\lambda_\alpha$, still carrying suppressed internal $SO(4)$ spinor indices as well as Spin(1,5) spacetime spinor indices. The internal $SO(4)$ chiralities, as in $D = 7$, are $\gamma_0 \psi_\mu = + \psi_\mu$, $\gamma_0 \psi_7 = + \psi_7$, and $\gamma_0 \lambda_\alpha = - \lambda_\alpha$.

Of course, as usual in a dimensional reduction, it is convenient to make a redefinition of the gravitino, of the form $\psi_\mu' = \psi_\mu - \frac{1}{4} \Gamma_\mu \psi_7$ here, in order to obtain diagonalised kinetic terms for the gravitini $\psi_\mu'$ and the spin-$\frac{1}{2}$ fields $\psi_7$ and $\lambda_\alpha$.

The supersymmetry transformation rules for the six-dimensional fields can be straightforwardly read off by applying the Kaluza reduction procedure described above to the transformation rules in section 2.3.

### 3.2 The truncation to the Salam-Sezgin $N = (1, 0)$ theory

We are now in a position to implement the final stage of our reduction procedure, in which we perform a truncation of the six-dimensional $N = (1, 1)$ supergravity described in section 3.1 to a chiral $N = (1, 0)$ supergravity. Of course a crucial point about this truncation, as with the previous ones we have implemented, is that it must be consistent with both the equations of motion and the supersymmetry transformation rules of the fields that are being set to zero.

In the bosonic sector, the truncation consists of setting to zero the 9 scalar fields parameterised by $\pi_\alpha^A$; the Kaluza-Klein vector $A_{(1)}$; the 2-from $H_{(2)}$; the 1-form field strengths $F^{\bar{A}\bar{B}}_{(1)}$; one combination of the seven-dimensional scalar field $\Phi$ and the Kaluza-Klein scalar $\varphi$; and, finally, setting to zero all except one vector within the $SO(2, 2)$ Yang-Mills sector. Thus we set

$$
\begin{align*}
\pi_\alpha^A &= \delta_\alpha^A, \\
A_{(1)} &= 0, \\
F^{\bar{A}\bar{B}}_{(1)} &= 0, \\
\Phi &= e^{16\alpha \varphi} = e^{-\frac{1}{5} \varphi}, \\
A_{(1)}^{12} &= - A_{(1)}^{34} = \frac{1}{2} A_{(1)}.
\end{align*}
$$

We also, for convenience, define a rescaled gauge coupling,

$$
\bar{g} = \frac{g}{\sqrt{2}}.
$$

We first note that this truncation leads to the six-dimensional bosonic Lagrangian

$$
\mathcal{L}_6 = R \ast 1 - \frac{1}{4} * d \Phi \wedge d\Phi - \frac{1}{2} e^{\frac{1}{2} \Phi} * F_{(2)} \wedge F_{(2)} - \frac{1}{2} e^{\Phi} * H_{(3)} \wedge H_{(3)} - 8\bar{g}^2 e^{-\frac{1}{2} \Phi} \ast 1,
$$

where $dH_{(3)} = \frac{1}{2} F_{(2)} \wedge F_{(2)}$. It is straightforward to verify that this truncation is indeed consistent with the bosonic equations of motion. The Lagrangian (34) precisely describes the
bosonic sector of the six-dimensional Salam-Sezgin gauged Einstein-Maxwell supergravity.\(^2\)

At the same time as truncating the bosonic sector, we must also set to zero appropriate fermionic fields, in order to obtain a supersymmetric \(N = (1,0)\) theory. To do this, we first decompose the six-dimensional fermionic fields into eigenstates of \(\Gamma_7\), which plays the role of the chirality operator in \(D = 6\). Thus we write \(\epsilon = \epsilon^+ + \epsilon^-\), where \(\Gamma_7 \epsilon^\pm = \pm \epsilon^\pm\), etc. Note that these chiralities are quite independent of the \(SO(4)\) internal chiralities under \(\gamma_0\), which we introduced in the truncation to \(N = 2\) supersymmetry in \(D = 7\). We then suppressed the \(SO(4)\) chirality labels, in order to avoid a proliferation of \(\pm\) superscripts in this final stage of the construction.

We find that the appropriate truncation in the fermionic sector is obtained by setting

\[
\epsilon^- = 0, \quad \psi^-_\mu = 0, \quad \psi^+_7 = 0, \quad \lambda^-_\alpha = \gamma_0 \lambda^-, \quad \lambda^+_\alpha = \eta_{\alpha\beta} \gamma_0 \lambda^+.
\]

Thus the remaining independent fermionic degrees of freedom are described by the fields \((\psi^+_\mu, \lambda^-, \lambda^+),\) and the supersymmetry parameter is \(\epsilon^+\). As well as their explicitly-indicated six-dimensional chiralities, they are also subject to the internal chirality constraints

\[
\gamma_0 \psi^+_\mu = +\psi^+_\mu, \quad \gamma_0 \chi^- = -\chi^-, \quad \gamma_0 \lambda^+ = -\lambda^+.
\] (36)

As before, to obtain diagonal kinetic terms we should define

\[
\psi^+_\mu = \psi^+_\mu' + \frac{1}{4} \Gamma_\mu \psi^-_7.
\] (37)

It is convenient also to introduce rescaled fermionic fields, which will have canonically-normalised kinetic terms. Tracing back through the sequence of reductions and truncations, we find that the original fermion kinetic terms \(\mathcal{L}_F = \bar{\lambda}^i \Gamma^\mu D_\mu \lambda_i + \bar{\psi}_{\mu} \Gamma^{\mu\rho} D_\nu \psi^\rho\) of the seven-dimensional \(SO(5)\) gauged theory [16] give rise, after our reduction and truncation to the \(N = (1,0)\) Salam-Sezgin supergravity, to the six-dimensional fermionic kinetic terms \(\mathcal{L}_F = 25 \bar{\lambda}^- \Gamma^\mu D_\mu \lambda^- + 4 \bar{\lambda}^+ \Gamma^\mu D_\mu \lambda^+ + \bar{\psi}_{\mu}^{+'} \Gamma^{\mu\rho} \psi^{+'}_\rho\). Thus if we define

\[
\lambda^- = \frac{1}{5} \chi, \quad \lambda^+ = \frac{1}{7} \lambda,
\]

then \((\psi^{+'}_\mu, \chi, \lambda)\) describe the canonically-normalised fermionic fields of the Salam-Sezgin theory.

\(^2\)The bosonic sector has also been obtained via a generalised dimensional reduction of ungauged seven-dimensional supergravity [17]. It is not clear whether that construction would allow an extension to include the fermionic sector.
It is now a straightforward matter to show that the truncation of the bosons and fermions described above is consistent with the supersymmetry transformation rules that descend from seven dimensions. It is worth remarking that two crucial ingredients in establishing the consistency are that

\[ M_{\alpha\alpha} = 0, \quad F_{(2)}^{\alpha\beta} \gamma_{\alpha\beta} \epsilon^+ = 0. \]  

(39)

The first of these equations follows because we made the transition to the $SO(2,2)$ non-compact gauging, whilst the second follows from the internal $SO(4)$ chirality condition $\gamma_0 \epsilon^+ = \epsilon^+$, which implies that $\gamma_{12} \epsilon^+ = \gamma_{34} \epsilon^+$ (we choose conventions where $\gamma_0 = -\gamma_{1234}$).

We find that the supersymmetry transformation rules for the fermions that remain in the truncated theory are given by

\[
\begin{align*}
\delta \psi_\mu^+ & = D_\mu \epsilon^+ + \frac{1}{32} e^{\frac{1}{2}\phi} H_{\nu\rho\sigma} \Gamma^{\nu\rho\sigma} \Gamma_\mu \epsilon^+, \\
\delta \chi & = \frac{1}{4} \partial_\mu \phi \Gamma_\mu - \frac{1}{8} e^{\frac{1}{2}\phi} \Gamma^{\mu
u\rho} H_{\mu
u\rho} \epsilon^+, \\
\delta \lambda & = -\frac{1}{4\sqrt{2}} e^{\frac{1}{2}\phi} F_{\mu\nu} \Gamma_{\gamma_{12}} - 8 \bar{g} e^{-\frac{1}{4}\phi} \epsilon^+, \\
\end{align*}
\]

(40)

where

\[
D_\mu \epsilon^+ = \nabla_\mu \epsilon^+ + \bar{g} A_\mu \gamma_{12} \epsilon^+. 
\]  

(41)

The transformation rules (40) are precisely those for the fermions in the Salam-Sezgin theory. We see from (41) that the fermions all carry charge $\bar{g}$ under the $U(1)$ gauge field $A_\mu$. One can pass to a complex notation, in which one takes $\gamma_{12} \epsilon^+ = -i \epsilon^+$, or else keep $\gamma_{12}$ explicitly, which corresponds to working with a 2-component chiral $SO(4)$ representation for symplectic Majorana spinors in $D = 6$.

Turning now to the supersymmetry transformations of the bosonic fields, we find

\[
\begin{align*}
\delta e_\mu^a & = \frac{1}{2} \epsilon^+ \Gamma^a \psi_\mu^+ - \frac{1}{20} \epsilon^+ \Gamma_\mu \chi, \\
\delta \phi & = \bar{\epsilon}^+ \chi, \\
\delta A_\mu & = -\frac{1}{\sqrt{2}} e^{-\frac{1}{4}\phi} \epsilon^+ \Gamma_\mu \gamma_{12} \lambda. \\
\end{align*}
\]

(42)

The second term on the right-hand side of the vielbein transformation rule can be removed by performing a compensating local Lorentz transformation, with parameters $\Lambda^a_{\ b} = \frac{1}{20} \epsilon^+ \Gamma^a_{\ b} \chi$. It is also straightforward to write down the transformation rule for the 3-form $H_{\mu
u\rho}$, and from this, one can deduce the transformation rule for $B_{\mu\nu}$. 

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4 Embedding of the Salam-Sezgin Theory in Ten Dimensions

Now that we have established how the Salam-Sezgin theory can be embedded into a non-compact gauged supergravity in seven dimensions, we can straightforwardly lift it back to ten dimensions, by making use of previously-established results obtained in [14], which themselves are based on the consistent $S^4$ reduction of eleven-dimensional supergravity found in [11, 12].

From the results in [14], adapted to our conventions and the situation where the $SO(4)$ gauge group considered there is allowed to become non-compact, the bosonic reduction ansatz from $D = 10$ to $D = 7$ becomes

\[
\begin{align*}
\hat{d}s^2_{10} &= \Phi^{3/16} \Delta^{1/4} (ds^2_7 + \frac{1}{2g} \Phi^{-1/2} \Delta^{-1} M^{-1}_{AB} D\mu^A D\mu^B), \\
\hat{F}_{(3)} &= \frac{1}{4} \Phi^{-2} \Delta^{-2} \epsilon_{A_1 \ldots A_4} \mu^{A_1} \mu^{B_1} \eta^{A_1 C_1} \eta^{A_2 C_2} M_{C_1 B_1} M_{C_2 B_2} \wedge D\mu^{A_3} \wedge D\mu^{A_4} \\
&\quad - \frac{1}{2g} \Phi^{-2} \Delta^{-2} W + \frac{1}{2g} \Delta^{-1} \epsilon_{A_1 \ldots A_4} M_{A_1 B} \mu^B F_{(2)}^{A_2 A_3} \wedge D\mu^{A_4} + H_{(3)}, \\
\hat{e}^\phi &= \Phi^{5/8} \Delta^{-1/2},
\end{align*}
\]

where $\mu^A$ are coordinates on $\mathbb{R}^4$, subject to the constraint

\[
\eta_{AB} \mu^A \mu^B = 1, \tag{44}
\]

and

\[
\begin{align*}
D\mu^A &= d\mu^A + 2\bar{g} A^A_{(1)B} \mu^B, \\
\Delta &= M_A \mu^A \mu^B, \quad U = 2M_{AB} M_{CD} \mu^A \mu^C \eta^{BD} - \Delta M_{AB} \eta^{AB}, \\
W &= \frac{1}{g} \epsilon_{A_1 \ldots A_4} \mu^{A_1} \mu^{A_2} \wedge D\mu^{A_3} \wedge D\mu^{A_4}. \tag{45}
\end{align*}
\]

(We are now suppressing the “bar” that we placed on the $SO(4)$ gauge indices $\bar{A}$ in the previous sections.) The ansatz (43) describes the embedding of the truncated $N = 2$ gauged seven-dimensional theory, with gauge group $SO(4)$ or $SO(2, 2)$, depending upon the choice made for $\eta_{AB}$.

Let us now specialise to the truncation that gave us the Salam-Sezgin theory in six dimensions. Using the coordinates introduced in equation (22), the quantities $\Delta$, $U$ and $W$ defined in (45) become

\[
\begin{align*}
\Delta &= \cosh 2\rho, \quad U = 2, \\
W &= -\frac{1}{2} \sinh 2\rho \, d\rho \wedge (d\alpha - \bar{g} A) \wedge (d\beta + \bar{g} A). \tag{46}
\end{align*}
\]
Combining the $D = 10$ to $D = 7$ reduction with the Kaluza reduction to $D = 6$ given in section 3, we therefore arrive at the following reduction ansatz that describes the embedding of the Salam-Sezgin theory into ten-dimensional type I supergravity:

\[ ds_{10}^2 = (c\cosh 2\rho)^{1/4} \left[ e^{-\frac{4}{3}\phi} ds_{6}^2 + e^{\frac{4}{3}\phi} dz^2 \
+ \frac{1}{2} g^{-2} e^{\frac{4}{3}\phi} (dp^2 + \frac{c^2}{\cosh 2\rho} (d\alpha - \bar{g} A)^2 + \frac{s^2}{\cosh 2\rho} (d\beta + \bar{g} A)^2) \right], \]

\[ \hat{F}_{(3)} = \frac{sc}{g^2 (c\cosh 2\rho)^2} dp \wedge (d\alpha - \bar{g} A_{(1)}) \wedge (d\beta + \bar{g} A_{(1)}) \]

\[ + \frac{1}{2\bar{g} \cosh 2\rho} F_{(2)} \wedge [c^2 (d\alpha - \bar{g} A_{(1)}) - s^2 (d\beta + \bar{g} A_{(1)})] + H_{(3)}, \]

\[ e^{\tilde{\phi}} = (c\cosh 2\rho)^{-1/2} e^{-\frac{1}{2}\phi}, \] (47)

where we have defined $c = \cosh \rho$, $s = \sinh \rho$.

It is straightforward to verify, by direct substitution of the reduction ansatz (47) into the equations of motion following from the Lagrangian

\[ \mathcal{L}_{10} = \hat{R} \hat{*} \mathbf{1} - \frac{1}{2} \hat{*} d\hat{\phi} \wedge d\hat{\phi} - \frac{1}{2} e^{-\hat{\phi}} \hat{*} \hat{F}_{(3)} \wedge \hat{F}_{(3)} \] (48)

for the bosonic sector of ten-dimensional type I supergravity, that one indeed obtains the bosonic equations of motion for the Salam-Sezgin six-dimensional theory, which follow from (34).

The metric reduction ansatz in (47) takes a somewhat more elegant form in the string frame, related to the Einstein frame by $ds_{str}^2 = e^{\frac{1}{2}\tilde{\phi}} ds_{10}^2$:

\[ ds_{str}^2 = e^{-\frac{1}{2}\tilde{\phi}} ds_{6}^2 + dz^2 + \frac{1}{2} g^{-2} \left( dp^2 + \frac{c^2}{\cosh 2\rho} (d\alpha - \bar{g} A)^2 + \frac{s^2}{\cosh 2\rho} (d\beta + \bar{g} A)^2 \right). \] (49)

The ten dimensional string coupling constant is given by $g_s = e^{\tilde{\phi}}$, and so it goes to zero at large distances $\rho$ in the internal directions. Naively at least, the ratio $G_{10}/G_6$ of the Newton constants in ten and seven dimensions is given by

\[ \frac{G_{10}}{G_6} = \frac{\pi^2}{\sqrt{2g^3}} \int dz \int_0^{\infty} d\rho \sinh 2\rho, \] (50)

and so the diverging $\rho$ integration leads to a vanishing six-dimensional gravitational constant. This is the customary feature that one encounters when the internal space has infinite volume [18].

Now that we have obtained the explicit formulae describing the embedding of the Salam-Sezgin supergravity in ten-dimensional supergravity, we can uplift any solution of the Salam-Sezgin theory. In fact since it was shown in [6] that there exists a consistent Pauli reduction
of the Salam-Sezgin theory on \( S^2 \), to give a four-dimensional chiral \( N = 1 \) supergravity, it follows that by combining this with our new results we can obtain a consistent embedding of this four-dimensional theory in the ten-dimensional type I supergravity. A solution of particular interest is the Salam-Sezgin (Minkowski) \( 4 \times S^2 \) ground state, which is given by [2]

\[
\begin{align*}
    ds_6^2 &= dx^\mu \, dx_\mu + \frac{1}{8g^2} (d\theta^2 + \sin^2 \theta \, d\varphi^2), \\
    A_{(1)} &= -\frac{1}{2g} \cos \theta \, d\varphi, \quad H_{(3)} = 0, \quad \phi = 0.
\end{align*}
\]

Using the ten-dimensional string frame ansatz (49), this lifts to give the ten-dimensional solution

\[
\begin{align*}
    ds_{str}^2 &= dx^\mu \, dx_\mu + \frac{1}{8g^2} (d\theta^2 + \sin^2 \theta \, d\varphi^2) + dz^2 \\
    &\quad + \frac{1}{2g^2} \left( d\rho^2 + \frac{c^2}{\cosh 2\rho} (d\alpha + \frac{1}{2} \cos \theta \, d\varphi)^2 + \frac{s^2}{\cosh 2\rho} (d\beta - \frac{1}{2} \cos \theta \, d\varphi)^2 \right). \quad (52)
\end{align*}
\]

The ten-dimensional 3-form and dilaton are given by (47).

In the large-\( \rho \) limit, the solution approaches

\[
\begin{align*}
    ds_{str}^2 &= dx^\mu \, dx_\mu + dz^2 + \frac{d\rho^2}{2g^2} \\
    &\quad + \frac{1}{8g^2} \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 + (d\alpha - d\beta + \cos \theta \, d\varphi)^2 + (d\alpha + d\beta)^2 \right), \\
    \hat{F}_{(3)} &= \frac{1}{8g^2} \sin \theta \, d\theta \wedge d\varphi \wedge (d\alpha - d\beta + \cos \theta \, d\varphi). \quad (53)
\end{align*}
\]

This asymptotic limit is a well-known exact solution of string theory, sometimes called the linear-dilaton vacuum, which arises as the near-horizon geometry of the NS5-brane. Specifically, the solution is defined on \( \mathbb{R}^{3,1} \times S^1 \times S^1 \times \mathbb{R} \times S^3 \), with coordinates \((x^\mu, z, \alpha + \beta, \rho, \theta, \varphi, \alpha - \beta)\) respectively. The NS-NS 3-form \( \hat{F}_{(3)} \) is proportional to the volume form of the \( S^3 \), and the dilaton \( \hat{\phi} \) is a linear function of the coordinate \( \rho \), namely \( \hat{\phi} \to -\rho \). The coordinates \((x^\mu, z, \alpha + \beta)\) span the NS5-brane world-volume (which is therefore wrapped over a \( T^2 \)), and the coordinates \((\rho, \theta, \varphi, \alpha - \beta)\) cover the transverse space. Note, however, that in the case of the NS5-brane, moving towards the horizon corresponds to approaching the strong-coupling region (i.e. \( \hat{\phi} \) increasing), whereas in the asymptotic limit we are considering here, the coupling decreases to zero. In the NS5-brane case, decreasing coupling corresponds to moving away from the horizon. In the exact solution we have obtained here, \( e^{\hat{\phi}} \) is bounded above by 1.

Although we arrived at the embedding of the Salam-Sezgin theory in ten dimensions via an eleven-dimensional and type IIA supergravity reduction, we can equally well view it as a
reduction of type IIB supergravity, since this shares the same type I supergravity common sector. We can then perform an S-duality transformation, so that the 3-form $\tilde{F}_{(3)}$ in (47) becomes the R-R 3-form of the type IIB theory. In the process, the sign of the dilaton in (47) will be reversed.

Within this type IIB framework, the asymptotic geometry of the solution (53) becomes the near-horizon geometry of the D5-brane. In this limit the coupling now becomes strong rather than weak.

5 Conclusions

In this paper we have shown that the six-dimensional $N = (1,0)$ gauged Einstein-Maxwell supergravity of Salam and Sezgin can be obtained via a consistent reduction from type I supergravity in ten dimensions. The embedding involves a reduction on the three-dimensional non-compact space $H^{2,2}$ times $S^1$, where $H^{2,2}$ is the quadric

$$\mu_1^2 + \mu_2^2 - \mu_3^2 - \mu_4^2 = 1$$

in Euclidean space $E^4$, followed by a reduction on $S^1$. The metric on $H^{2,2}$ is positive definite, and is conformal to the metric induced from the Euclidean metric on $E^4$. The reduction ansatz for the bosonic fields is given by (47). Upon substitution into the equations of motion of ten-dimensional type I supergravity, one obtains the equations of motion of the Salam-Sezgin theory.

The reduction procedure involved performing consistent truncations as well as the consistent reductions. If one elects not to perform the truncations, then one obtains larger, and non-chiral, gauged supergravities in six dimensions, which contain the Salam-Sezgin theory upon truncation. In particular, we exhibited an $SO(2,2)$ gauged $N = (1,1)$ supergravity in six dimensions, that arose from the $S^1$ reduction of seven-dimensional $N = 2$ gauged $SO(2,2)$ supergravity. We could also obtain an $N = (2,2)$ gauged supergravity in six dimensions if we did not make the truncation from $N = 4$ to $N = 2$ in seven dimensions.

Since the Salam-Sezgin theory is embedded in ten-dimensional type I supergravity, it follows that our construction can equally well be viewed as a reduction of the heterotic theory, or, via an S-duality transformation, as a reduction of type IIB supergravity in which the R-R rather than the NS-NS 3-form is non-vanishing. We can also, of course, trivially lift the embedding (viewed as type I within type IIA) to an embedding within eleven-dimensional supergravity.
Having obtained the Salam-Sezgin theory via a consistent dimensional reduction, it follows that any of its solutions can be lifted to give a solution in $D = 10$ or $D = 11$. It is striking that the lift of the $(\text{Minkowski})_4 \times S^2$ ground state to type I supergravity is asymptotic to the exact linear-dilaton solution of string theory, i.e. the near-horizon geometry of the NS5-brane. By S-duality, it can be viewed instead as the near-horizon geometry of the D5-brane.

**Acknowledgments**

We are grateful to Klaus Behrndt, Rahmi Güven, Jim Liu, Hong Lü, Krzysztof Pilch, Toine van Proyen, Fernando Quevedo, Ergin Sezgin and Paul Townsend for discussions. We thank the Benasque Center for Science, and M.C. and C.N.P. thank the Cambridge Relativity and Gravitation Group and the organisers of the *Cosmological Perturbations on the Brane* workshop, for hospitality during the course of this work.

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