Efficient Plasma Heating by Radiofrequency

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We propose an efficient mechanism to heat a plasma by an intense microwave field solving the equation of ion motion in a wave field and a constant magnetic field in a large coupling regime. The mechanism does not rely explicitly on stochastic motion and is able to increase the ion velocity by several magnitude orders. Known thresholds for the onset of stochastic motion are also obtained.

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The possibility to realize confined nuclear fusion is linked to the ability to let ions reach enough high energies to ignite the process. Heating of such plasmas can be obtained by different means of which is by radiofrequency directly injected through the confining system that is generally a tokamak.

Such a heating exploits directly the interaction between a charged particle and an electromagnetic mode of a field propagating inside the magnetically confined plasma. This kind of interaction, being ruled by a strongly non-linear equation, is rather involved and a lot of analysis have been performed in the latest thirty years to properly understand the process.

The most exploited effect that happens in such wave-particle interaction is the onset of a stochastic behavior according to Kolmogorov-Arnold-Moser (KAM) theorem. The availability of a large portion of phase space to the ions, when a threshold is overcome, and the consequent entering into a higher energy regime is termed "stochastic heating" and it proves to be a very efficient method to heat a plasma by radiofrequency.

Stochastic heating is commonly analyzed through a well-known model that, with the proper modifications, is able to give an adequate understanding of the physics at hand. Indeed, for an ion with charge $q$ and mass $m$ with a constant magnetic field $B = (0, 0, B_0)$ and a wave $E = (0, E_0, 0) \cos(\kappa y - \omega t)$, one has the equation

$$\ddot{y}(t) + \Omega^2 y(t) = \frac{qE_0}{m_0} \cos(ky - \omega t)$$

being $\Omega = qB_0/m_0$ the cyclotron frequency. We make this equation adimensional by introducing $y(t) \rightarrow ky(t), t' = \Omega t$, $\alpha = E_0k/B_0\Omega$ and $\nu = \omega/\Omega$ so that, with the above redefinition of $y$,

$$\ddot{y}(t') + y(t') = \alpha \cos(y(t') - \nu t')$$

where we recognize that two critical parameters fix the physics for this problem, $\alpha$ and $\nu$. This is a Hamiltonian system. Indeed, it known that for stochastic heating produced by a lower hybrid wave the stochasticity threshold, assuming $\nu > 1$, is given by

$$\alpha \approx \frac{1}{4\nu^2}$$

while for ion cyclotron heating this condition becomes $\alpha \approx 1$. Dissipation of the energy of the wave is due to Landau damping that, for the presence of the magnetic field, makes the absorption irreversible.

The approach that is normally adopted to cope with this kind of problems is to work out canonical perturbation theory that holds for small $\alpha$. Then, Chirikov overlap criterion for resonances and comparison with numerical results are used to determine the threshold of the onset of stochasticity to see when effective heating starts to set in. In this situation physics is fairly well-known and a diffusion equation can be derived describing the behavior of the plasma under such conditions. The overall mark of stochastic heating mechanism is the fact that the system under study is not integrable and KAM theorem does apply.

Anyhow, we note that the opposite limit $\alpha \rightarrow \infty$ gives us the interesting result that the system is ruled by the equation

$$\ddot{y}_0(t') = \alpha \cos(y_0(t') - \nu t')$$

that is an integrable one. Indeed, the solution to this equation is straightforward to write down as

$$y_0(t') = \frac{\pi}{2} + \nu t' + 2 \arcsin(msn(\sqrt{\alpha}t', m))$$
being sn the Jacobi snoidal elliptic function and $m$ its modulus. We notice that if we take $y(0) = \pi/2$ and $\dot{y}(0) = kv_0/\Omega$ it easy to see that

$$m = \frac{1}{2\sqrt{\alpha}} \left( \frac{kv_0}{\Omega} - \nu \right)$$

and then the modulus of the Jacobi function depends on the initial velocity of the ion. Besides, $T_{tr} = 4K(m)/\sqrt{\alpha}$, with $K(m) = \int_0^\frac{\pi}{2} \frac{d\phi}{\sqrt{1-m^2\sin^2\phi}}$, is the bounce time of a ion trapped inside a well of the wave. From this exact solution we can recognize two regimes. One has the ions moving uniformly with the phase velocity of the wave and the other is just the contribution of the trapped motion. If the first effect prevails one can have a net increase of the energy of the ion that, for the presence of the magnetic field, is irreversibly absorbed. Indeed, this can be assured by taking $m \ll 1$ so that one can expand the snoidal function to have

$$y_0(t') \approx \frac{\pi}{2} + \nu t' + 2m \sin(\sqrt{\alpha} t') + O(m^2)$$

and an efficient heating mechanism is granted. In fact, the averaged velocity on a bounce time will be given by $\langle \dot{y}_0 \rangle = \nu$. This means that the ion is moving with the phase velocity of the wave that is assumed much larger of the initial ion velocity. Similarly, one has $\langle \dot{y}_0^2 \rangle = \nu^2$ and a diffusion coefficient can be estimated dividing by the bounce time as $D = \langle \dot{y}_0^2 \rangle / 2T_{tr} \approx \nu^2 / \sqrt{\alpha}$.

This scenario would be consistent if we would be able to develop an asymptotic perturbation series in the limit $\alpha \to \infty$ to compute higher order corrections due to the presence of the magnetic field. Indeed this can be accomplished by the duality principle in perturbation theory. This principle can be stated by saying that interchanging the perturbation terms in an equation gives two perturbation series having a development parameter that in a series is the inverse of the other. That is, in our case one would have (weak perturbation, $\alpha \to 0$) $y = y_0 + \alpha y_1 + \alpha^2 y_2 + \ldots$ and for the dual series $y = y_0 + \alpha^{-1} y_1 + \alpha^{-2} y_2 + \ldots$ that holds in the opposite limit $\alpha \to \infty$. In order to get a non trivial set of perturbation equations, one need to rescale time because, as already seen above, this is the proper scaling of the leading order solution. So, we take

$$\tau = \sqrt{\alpha} t'$$
$$y_0 = y_0 + \alpha^{-1} y_1 + \alpha^{-2} y_2 + \ldots$$

into the motion equation and the following working set of equations is easily obtained

$$\ddot{y}_0(\tau) - \cos(\frac{y_0(\tau)}{\sqrt{\alpha} \nu}) = 0$$
$$\ddot{y}_1(\tau) + \sin(y_0(\tau) - \frac{\nu}{\sqrt{\alpha} \nu}) y_1(\tau) = -y_0(\tau)$$
$$\ddot{y}_2(\tau) + \sin(y_0(\tau) - \frac{\nu}{\sqrt{\alpha} \nu}) y_2(\tau) = -y_1(\tau) - \frac{1}{2} \cos(y_0(\tau) - \frac{\nu}{\sqrt{\alpha} \nu}) y_1^2(\tau)$$

Using the exact solution to the leading order the above set of equations takes the form

$$\ddot{y}_1(\tau) + [1 - 2m^2 \sin^2(\tau, m)] y_1 = -y_0(\tau)$$
$$\ddot{y}_2(\tau) + [1 - 2m^2 \sin^2(\tau, m)] y_2 = -y_1(\tau) + m \sin(\tau, m) \sqrt{1 - m^2 \sin^2(\tau, m)} y_1^2(\tau)$$

that in the limit $m \ll 1$ take the simpler form

$$\ddot{y}_1(\tau) + y_1 = -y_0(\tau)$$
$$\ddot{y}_2(\tau) + y_2 = -y_1(\tau) + m \sin(\tau) y_1^2(\tau)$$

to the first order in $m$. These equations can be solved straightforwardly taking at the leading order eq. 7.
The solutions contain secular terms. These terms can be removed in many ways. The one we prefer is the renormalization group method as presented in \[13, 14, 15\]. This technique assumes that the perturbation solution should be computed to an initial time \(\tau_0\) as to have \(y(\tau, \tau_0) = y_0(\tau, \tau_0) + \alpha^{-1}y_1(\tau, \tau_0) + \alpha^{-2}y_2(\tau, \tau_0) + \ldots\) and the initial constants depend on the initial time, \(\phi = \phi(\tau_0)\) and \(m = m(\tau_0)\). The requirement that the perturbation series should not depend on the choice of the initial point translates into the condition

\[
\frac{dy(\tau, \tau_0)}{d\tau_0} \bigg|_{\tau_0 = \tau} = 0 \tag{12}
\]

that gives back renormalization group equations for the constants \(\phi\) and \(m\) that now turn out to depend on \(\tau\). Then, the condition \(y(\tau, \tau_0)|_{\tau_0 = \tau}\) produces the correct perturbation series without secular terms. It is interesting to point out that all we have done is just to compute an envelope \[15\].

After resummation we get

\[
y_0(\tau) = \frac{\pi}{2} + \frac{\nu}{\sqrt{\alpha}}\tau + 2m(\tau)\sin(\tau + \phi(\tau)) \tag{13}
\]

\[
y_1(\tau) = \frac{\pi}{2}[\cos(\tau + \phi(\tau)) - 1] - m(\tau)\sin(\tau + \phi(\tau))
\]

\[
y_2(\tau) = \frac{\pi}{2}[1 - \cos(\tau + \phi(\tau))] + \left(\frac{\pi^2}{48} + \frac{3}{4}\right)m(\tau)\sin(\tau + \phi(\tau)) - \frac{\pi^2}{32}m(\tau)\sin(\tau + \phi(\tau))\cos^2(\tau + \phi(\tau)) + \frac{\pi^2}{6}m(\tau)\sin(\tau + \phi(\tau))\cos(\tau + \phi(\tau))
\]

where now the modulus \(m\) depends on \(\tau\) and the phase \(\phi(\tau)\) contains a frequency shift as from the renormalization group equations

\[
\frac{dm}{d\tau} = -\frac{\pi}{8\alpha^2} + O(\alpha^{-3}) \tag{14}
\]

\[
\frac{d\phi}{d\tau} = \frac{1}{2\alpha} - \frac{1}{4\alpha^2}\left(\frac{5\pi^2}{8} + 3\right) + O(\alpha^{-3}).
\]

Having the modulus depending linearly on \(\tau\) implies that if one waits a time long enough our initial approximation \(m \ll 1\) in absolute value may fail. We discuss this point below about stochasticity thresholds. We have also found higher order corrections to the bounce time. Finally, we note the renormalization constant \(1 - 1/\alpha + 1/\alpha^2 + \ldots = \alpha/(1 + \alpha)\) to the \(\pi/2\) term.

We now discuss the stochasticity threshold in the case of a ion cyclotron wave (ICW) \[1\] and lower hybrid wave (LHW) \[2, 3\]. The mechanism of heating does work only if the particle moves freely at the phase velocity of the wave. Instead, if bouncing of the ion in a well of the wave prevails, heating is not effective. For a ICW one has \(\nu = 1\) and the only guaranty of a fully consistency of the above discussion is \(\alpha \gg 1\) in agreement with the result given in \[1\]. For a LHW, when \(\alpha\) is not so large but greater than one and \(\nu \gg 1\) as discussed in \[2, 3\], it is fundamental that the modulus \(m\) does not prevail on the free particle term forcing the ion behavior to a simple bouncing in the wells of the wave. This is granted taking \(1/\nu > > \pi/8\alpha^2\) giving us the improved threshold for this wave

\[
\alpha \gg \frac{\left(\pi\nu\right)^2}{4} \tag{15}
\]

and the numerical factor is now about 0.54. In this way we have proved fully consistency of our approach with respect to previous ones on stochastic heating obtaining a deeper understanding of the involved physics.

In order to complete our analysis we give here some numerical studies that are essential to support the above scenario. We have considered two cases for \(\alpha, \nu\) and \(\dot{y}(0)\). We have not taken \(\alpha\) too large because otherwise the plots would not be much interesting as the analytical solution hits the exact one as does the numerical one. The goodness of the approximation we have applied is evident provided that \(m\) is kept small and the thresholds given above are sufficiently overcome. So, fig.\[1\] corresponds to have \(|m| \approx 0.45\) but \(\alpha = 10\) is not so large. The same happens in fig.\[2\] that has \(|m| \approx 0.022\). The worst situation is seen in fig.\[3\] where \(|m| \approx 2.3\) and the thresholds not properly overcome. So due to the value of \(\alpha = 10\) and the corresponding problems in \(m\) and thresholds the agreement goes from relatively good to bad. The relevance of the threshold is due both to the largeness of \(\alpha\) and to avoid that \(m\) increasing in time may overcome the initial condition \(m \ll 1\) from which we started. Things improve, as should be, with \(\alpha = 100\) where the agreement is very satisfactory. An interesting situation is seen in fig.\[4\] where the solution given in eq. \[5\]
is better than the perturbation series. This is due to the fact that, although the thresholds are properly overcome, one has $m \approx 0.75$ very near one, where our fundamental assumption to derive the perturbation series starts to fail. As expected, in some cases the perturbation series performs much better than eq. (5).

So, we can conclude that a satisfactory perturbation approach has been presented here to treat problem of strong particle-wave interaction in a plasma. A heating scenario for ions has been given showing that it can increase the ion velocity of several magnitude orders making it move to the phase velocity of the wave. Known thresholds have been derived in a fully analytical and rigorous way forming a complete scenario for plasma heating by radiofrequency. A lot of problems having a longstanding numerical treatment could be faced now in an analytical way by the method of dual perturbation theory giving an improved insight into physics.

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[1] G. R. Smith and A. N. Kaufman, Phys. Rev. Lett. 34, 1613 (1975).
[2] C. F. F. Karney and A. Bers, Phys. Rev. Lett. 39, 550 (1977).
FIG. 3: Numerical, analytical and eq. 15 for $\alpha = 10$, $\nu = 0.15$ and $\dot{y}(0) = 3$.

FIG. 4: Numerical, analytical and eq. 15 for $\alpha = 100$, $\nu = 15$ and $\dot{y}(0) = 0.01$.

[3] C. F. F. Karney, Phys. Fluids 21, 1584 (1978).
[4] C. F. F. Karney, Phys. Fluids 22, 2188 (1979).
[5] F. Skiff, F. Anderegg and M. Q. Tran, Phys. Rev. Lett. 58, 1430 (1987).
[6] R. Z. Sagdeev, D. A. Usikov, G. M. Zaslavsky, Nonlinear Physics, (Harwood, Philadelphia, 1988).
[7] B. V. Chirikov, Phys. Rep. 52, 263 (1979).
[8] M. Frasca, Phys. Rev. A 58, 3439 (1998).
[9] M. Frasca, Phys. Rev. A 60, 573 (1999).
[10] M. Frasca, hep-th/0508246.
[11] M. Frasca, hep-th/0509125.
[12] M. Frasca, hep-th/0511008.
[13] L. Y. Chen, N. Goldenfeld and Y. Oono, Phys. Rev. Lett. 73, 1311 (1994).
[14] L. Y. Chen, N. Goldenfeld and Y. Oono, Phys. Rev. E 54, 376 (1996).
[15] T. Kunihoro, Prog. Theor. Phys. 94, 503 (1995); (Erratum), Prog. Theor. Phys. 95, 835 (1996).
FIG. 5: Numerical, analytical and eq. (5) for $\alpha = 100$, $\nu = 0.15$ and $\dot{y}(0) = 0.01$.

FIG. 6: Numerical, analytical and eq. (5) for $\alpha = 100$, $\nu = 0.15$ and $\dot{y}(0) = 3$. 