Simplicial volume and essentiality of manifolds fibered over spheres

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Abstract
We study the question when a manifold that fibers over a sphere can be rationally essential, or have positive simplicial volume. More concretely, we show that mapping tori of manifolds (whose fundamental groups can be quite arbitrary) of dimension $2n + 1 \geq 7$ with non-zero simplicial volume are very common. This contrasts the case of fiber bundles over a sphere of dimension $d \geq 2$: we prove that their total spaces are rationally inessential if $d \geq 3$, and always have simplicial volume 0. Using a result by Dranishnikov, we also deduce a surprising property of macroscopic dimension, and we give two applications to positive scalar curvature and characteristic classes, respectively.

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1 | INTRODUCTION

For a topological space $X$, M. Gromov defined in [13] the $\ell_1$ norm of a singular homology class $\alpha \in H_d(X; \mathbb{R})$ as

$$\|\alpha\| := \inf \left\{ \sum_{j \in J} |\lambda_j| \middle| \alpha \text{ is represented by the singular cycle } \sum_{j \in J} \lambda_j \sigma_j \right\}.$$ 

The simplicial volume of a closed connected oriented smooth manifold $M$ is then defined as $\|M\| := \|[M]\|$, where $[M]$ denotes the fundamental class of $M$. All manifolds in this paper are assumed to have the properties just mentioned, unless explicitly stated otherwise.
Recall the notion of smooth manifold bundles, meaning locally trivial fibrations whose transition maps are diffeomorphisms. A classical result by Gromov \cite{gromov1981} implies that for any oriented manifold bundle $M \to E \to B$ such that $\pi_1(M, *)$ is amenable (and $M$ has positive dimension), $||E||$ vanishes. However, no similar condition concerning the base $B$ ensures that the simplicial volume $||E||$ vanishes; for instance, a mapping torus of a pseudo-Anosov surface automorphism is a hyperbolic 3-manifold \cite{pollicott1984, pseudogrowth}, and thus has non-zero simplicial volume by a result by Gromov (again see \cite{gromov1981}). This observation leads to the following concise question that inspired this work.

**Question 1.1.** For positive integers $d, m$ and an oriented manifold bundle $M \to E \to S^d$ whose fiber $M$ is $m$-dimensional, which values can the simplicial volume of $E$ attain?

The main purpose of this note is to address Question 1.1 by proving the following:

- For $d = 1$ and $m \geq 6$ even, the set of values is dense. (Corollary B)
- For $d > 1$, the simplicial volume of the total space always vanishes. (Theorem C)

A vital step along the way to prove Corollary B is the following result which could be of independent interest and perhaps also useful beyond the applications in this note.

**Theorem A** (Mapping tori representing group homology classes). Let $\Gamma$ be a finitely presented group and let $\alpha \in H_{2n+1}(\Gamma; \mathbb{Q})$, where $2n + 1 \geq 7$, denote a non-zero homology class. Then there exists a closed connected oriented $2n$-manifold $S$ with $\pi_1(S, *) \cong \Gamma$ and a diffeomorphism $\Phi$ of $S$ that fixes a point $* \in S$ and acts trivially on $\pi_1(S, *)$, such that the resulting mapping torus $T_\Phi$ admits a map $f : T_\Phi \to B\Gamma$ which induces the projection to $\Gamma \times \mathbb{Z} \to \Gamma$ on fundamental groups and satisfies $f_*([T_\Phi]) = \lambda \alpha$ for some non-zero integer $\lambda$.

The main ingredient in the proof of Theorem A is a classical but perhaps a little overlooked observation by Lawson: every odd-dimensional manifold can be represented as a twisted double. From Theorem A, we easily deduce the following answer to Question 1.1.

**Corollary B** (Values of simplicial volumes of mapping tori). Let $2n + 1 \geq 7$ and let $\nu \in \mathbb{R}_{\geq 0}$ be such that there exists a manifold $M$ of dimension $2n + 1$ such that $||M|| = \nu$. Then there exists a mapping torus $T$ of some $2n$-manifold such that $||T|| = \nu$. Moreover, the possible values for the simplicial volume of mapping tori of a fixed dimension $2n + 1 \geq 7$ lie densely in $\mathbb{R}_{\geq 0}$.

Here the denseness result follows from work by Heuer and Löh in \cite{heuerloeh}. For $d > 1$ we have the following vanishing result.

**Theorem C** (Vanishing simplicial volume of total spaces). Let $M \to E \to S^d$ be a smooth fiber bundle with $d \geq 2$. Then the simplicial volume $||E||$ is zero.

### 1.1 Essentiality

A notion—also introduced by Gromov and linked to manifold ideas in geometry — that is, related to simplicial volume is the following.
**Definition 1.2.** A $d$-manifold $N$ is called essential if the classifying map of its universal covering, $f : N \to B\pi_1(N, \ast)$, satisfies

$$0 \neq f_\ast([N]) \in H_d(B\pi_1(N, \ast); \mathbb{Z}),$$

otherwise it is called inessential. If the analogous statement with $\mathbb{Q}$ coefficients holds, $N$ is called rationally essential; otherwise, it is called rationally inessential.

Since the classifying map of the universal covering induces an isomorphism on bounded cohomology $[13, 18]$ and thus — equivalently — on $\ell^1$ homology, the simplicial volume of a manifold $N$ only depends on the homology class $f_\ast([N]) \in H_d(B\pi_1(N, \ast); \mathbb{Q})$. Thus, it becomes evident that rational essentiality is a necessary condition for the simplicial volume to be positive. This leads to the following related question, a generalization of which we answer up to the Novikov conjecture and with the exception of base $S^2$.

**Question 1.3.** Is every manifold $E$ which fibers over a simply connected sphere rationally inessential?

Recall that a manifold $N$ is called flexible if there exists a map $N \to N$ of degree $d \not\in \{0, \pm 1\}$.

**Theorem D** (Non-essentiality of total spaces). Let $M$ be a manifold whose fundamental group $\pi_1(M, \ast)$ satisfies the Novikov conjecture, let $B$ be a flexible 2-connected manifold, and let $M \to E \to B$ be an oriented manifold bundle. Then $E$ is rationally inessential.

The purpose of the later sections (3–5) is to present three independent applications.

### 1.2  Application to positive scalar curvature

Let $R^{psc}(M)$ denote the space of Riemannian metrics on $M$ whose scalar curvature is strictly positive at every point [“psc”]. Questions about the existence and uniqueness of such metrics mean in practice to decide whether this space is non-empty or contractible, respectively. Numerous papers cover these topics, for instance, see [2, 3, 8, 15, 30].

To address such questions systematically, one can employ the Stolz exact sequence [27] to obtain results on $R^{psc}(M)$; they will depend on the second stage of the Moore–Postnikov factorization of the map $M \to BO$ classifying the tangent bundle of $M$. In particular, such results become simpler if $M$ admits a spin structure.

As far as the authors know, the literature contains no concrete example of a manifold $M$ whose universal cover has no spin structure that supports infinitely many non-isotopic psc metrics. In Section 4 we provide such an example, building on the ideas we developed for the proof of Theorem A and Schoen–Yau’s hypersurface method. These insights are independent from the results by Stolz.

**Theorem E** (A non-spin manifold with infinitely many non-isotopic psc metrics). Let $f$ be the Morse function on $T^7 = (S^1)^7 \subset \mathbb{C}^7$ that is defined as the sum of real parts of the complex coordinates and let $S = f^{-1}(0)$. Then $\pi_0(R^{psc}(S\#(\mathbb{C}P^2 \times S^2)))$ is infinite.
1.3 Application to macroscopic dimension

In Section 5 we combine our results on the simplicial volume of mapping tori with a theorem by Dranishnikov to conclude a surprising fact about macroscopic dimension. This notion is defined for any metric space \( X \) as the smallest non-negative integer \( k \) such that there exists a Lipschitz map \( f : X \to Y \) so that \( Y \) is a \( k \)-dimensional simplicial complex and there is a uniform bound on the diameters of pre-images \( f^{-1}(y) \), \( y \in Y \). We use Theorem A to prove that in any dimension \( 2n + 1 \geq 7 \), there exist two manifolds whose universal coverings are not only quasi-isometric but also diffeomorphic, yet they have different macroscopic dimensions.

**Theorem F** (Twin manifolds with different macroscopic dimensions). *In every dimension \( 2n + 1 \geq 7 \) and for any finitely presented amenable group \( \Gamma \) such that \( H_{2n+1}(\Gamma;\mathbb{Q}) \neq 0 \), there exist two closed \((2n + 1)\)-manifolds \( M_1 \) and \( M_2 \) such that (for any choice of Riemannian metrics)

(i) The fundamental groups \( \pi_1(M_1) \) and \( \pi_1(M_2) \) are isomorphic to \( \Gamma \times \mathbb{Z} \).

(ii) The universal coverings of \( \tilde{M}_1 \) of \( M_1 \) and \( \tilde{M}_2 \) of \( M_2 \) are diffeomorphic and quasi-isometric (via two different maps).

(iii) \( \dim mc(\tilde{M}_1) < 2n + 1 = \dim mc(\tilde{M}_2) \).

1.4 Application to characteristic classes

In Section 6, we use Theorem D to study certain characteristic classes, akin to generalized Miller–Morita–Mumford classes, but pertaining to the cohomology of the fundamental group of the fiber. In recent years, the study of generalized Milller–Morita–Mumford classes has seen spectacular advances, in particular the seminal work by Galatius–Randal-Williams that culminated in [10], which showed that up to stabilization in many cases the cohomology rings of diffeomorphism groups are generated by these classes. Our result takes the fundamental group into account and has some similarity with the vanishing result proven in [16].

Suppose that \( M \) is an oriented manifold of dimension \( d \) such that \( \Gamma := \pi_1(M) \) is non-trivial. The space \( \text{maps}(M, B\Gamma) \) of all maps \( M \to B\Gamma \) admits a right action by the group \( \text{Diff}(M) \) of orientation-preserving diffeomorphisms of \( M \), and the homotopy quotient \( B\text{Diff} B\Gamma(M) := \text{maps}(M, B\Gamma) \land \text{Diff}(M) \land \text{EDiff}(M) \) is the base of a smooth \( M \)-bundle

\[ \pi^{B\Gamma} : M \to \text{maps}(M, B\Gamma) \land \text{Diff}(M) \land \text{EDiff}(M) \to B\text{Diff} B\Gamma(M), \]

whose total space maps to \( B\Gamma \) via evaluation. Now for any class \( \alpha \in H^*(B\Gamma; \mathbb{Q}) \) we may form the ‘\( \kappa \)-class’

\[ \kappa_\alpha := \int_{\pi^{B\Gamma}} \text{ev}^* \alpha \in H^{*-d}(B\text{Diff} B\Gamma(M); \mathbb{Q}). \]

**Theorem G** (Asphericity of \( \kappa \)-like classes from the fundamental group). *Let \( d \geq 6 \), let \( \Gamma \) be a finitely presented group that satisfies the Novikov conjecture, and let \( \alpha \in H^{d+j}(B\Gamma; \mathbb{Q}) \) be non-zero, where \( j \geq 0 \).

(i) If \( d = 2n \) and \( \Gamma \) has the finiteness property \( F_{n} \), there exists a closed connected smooth \( d \)-manifold \( M \) such that \( \pi_1 M \cong \Gamma \) and \( \kappa_\alpha \in H(B\text{Diff} B\Gamma(M); \mathbb{Q}) \) is non-zero.*
(2) Assume \( j \geq 3 \). Then for any closed connected smooth \( d \)-manifold \( M \) with \( \pi_1 M \cong \Gamma, \kappa_\alpha \) evaluates trivially on the image of the Hurewicz homomorphism

\[
\pi_j\left( \text{BDiff}^B \Omega(M) \right) \to H_j\left( \text{BDiff}^B \Omega(M); \mathbb{Q} \right).
\]

## 2 TWISTED DOUBLES, MAPPING TORI, AND GROUP HOMOLOGY

In this section we prove Theorem A. In order to represent any (rational) odd-dimensional group homology class by a mapping torus, we first sketch the proof of a general structure result for odd-dimensional manifolds, due to Lawson [19]. To state it, we need the following definition.

**Definition 2.1.** Given an oriented, closed manifold \( S \) together with an oriented nullbordism \( W \) of \( S \) and an orientation-preserving diffeomorphism \( \Phi : S \to S \), we call \( W \cup \Phi(-W) \) the \( \Phi \)-twisted double of \( W \). Here \( -W \) stands for \( W \) with the opposite orientation. Note that the requirement that \( \Phi \) is orientation-preserving ensures that the twisted double is again oriented. Lawson proved that any manifold of odd dimension at least 7 can be represented as a twisted double. Moreover the proof in [19] yields an even stronger result given below, which was not stated in his work. Since we need this strengthening, we will provide a sketch of Lawsons proof to justify it.

**Theorem 2.2** (Twisted double theorem). Given a closed, orientable manifold \( M \) of dimension \( 2m + 1 \geq 7 \), there exists a closed orientable manifold \( S \), a pointed orientation-preserving diffeomorphism \( \Phi : S \to S \), and an oriented nullbordism \( W \) of \( S \) such that:

(i) \( \Phi \) induces the identity on the fundamental group of \( S \);
(ii) the \( \Phi \)-twisted double of \( W \) is diffeomorphic to \( M \);
(iii) the inclusion \( S \hookrightarrow W \) induces an isomorphisms on fundamental groups.

**Proof.** We start with a handlebody decomposition of \( M \): we denote the union of all handles of index less or equal to \( m \) by \( W_1 \) and the union of all handles of index bigger or equal than \( m + 1 \) by \( W_2 \). These two submanifolds with boundary intersect in their common boundary \( S \). The goal is to find an \( m \)-dimensional subcomplex \( K \subset S \) such that \( K \to S \to W_i \) is a simple homotopy equivalence. For a general handlebody decomposition this is not necessarily possible, but it always becomes possible if one adds a sufficient number of canceling \( m \) and \( m + 1 \)-handles (or in other words ‘replaces \( M \) by \( M \# S^{2m+1} \)'), which is proven in [19]. Let us assume that the handlebody decomposition has been stabilized like that.

Let \( V \) denote a regular neighborhood of \( K \subset S \). Since the codimension of \( K \subset S \) is at least 3, we conclude that \( \pi_1(\partial V, \ast) \) is isomorphic to the fundamental group of \( K \). Fix a tubular neighborhood \( \partial V \times I \subset S \) and denote the complement of \( V \cup \partial V \times I \) by \( V' \). Now \( W_i \) yields a relative bordism \( (W_i, V, V', \partial V \times I) \).

Using the Seifert–Van Kampen theorem one can conclude that the inclusion \( V' \to S \) induces an isomorphism on fundamental groups and by Poincaré–Lefschetz duality for twisted coefficients one concludes that the inclusion \( V' \to W_i \) is in fact a homotopy equivalence. Hence the aforementioned relative bordism is a relative s-cobordism and hence by the relative s-cobordism theorem isomorphic to \( V \times I \); furthermore, the diffeomorphism between \( W_i \) and \( V \times I \) can be chosen to be...
the identity on $V$ and hence induces the identity on $\pi_1$. Note that if we fix an orientation on $S$ and fix the induced orientation on $V$, then only one of the above diffeomorphisms can be orientation preserving and the other one is going to be orientation reversing. All in all this implies that $M$ is isomorphic to $W_1 \cup_\Phi -W_1$ and that this representation has the required properties. In particular the last point follows, because both inclusion $S \to W_1$ and $W_1 \to M$ induce an isomorphism on fundamental groups.

Using this theorem we proceed by proving Theorem A.

**Proof of Theorem A.** By Thom’s solution of the rational Steenrod problem [28], there exists a $0 \neq \lambda \in \mathbb{Z}$ such that $\lambda \alpha$ is representable by a closed orientable manifold $M$, that is, there exists a map $f' : M \to B\Gamma$ such that $f'_*(\{M\}) = \lambda \alpha$. Since the dimension of $M$ is at least 5, by performing surgery on $M$, we can assume that $f'$ induces an isomorphism on fundamental groups.

By Theorem 2.2, the manifold $M$ is diffeomorphic to a $\Phi$-twisted double, where $\Phi : S \to S$ is a diffeomorphism and $W$ a nullbordism of $S$, with the properties listed in that theorem. In particular by (2.2) the fundamental group of the mapping torus $T_\Phi$ is isomorphic to $\Gamma \times \mathbb{Z}$. Let us consider the composition of the classifying map of the universal covering of $T_\Phi$ with the map induced by the projection $\Gamma \times \mathbb{Z} \to \Gamma$ and denote it by $f : T_\Phi \to B\Gamma$. We will show that $f'$ and $f$ are bordant, and hence, they map the respective fundamental class to the same homology class in $B\Gamma$. The mapping torus $T_\Phi$ is defined as $S \times [-1, 1]/\sim$ where we have $(-1, x) \sim (1, \Phi(x))$. The manifold

$$P = [0, 1] \times T_\Phi \cup_{\{1\} \times S \times [-0.5, 0.5]} W \times [-0.5, 0.5]$$

gives a bordism between $T_\Phi$ and $W \cup_\Phi W = M$. Furthermore an easy Seifert–Van-Kampen argument shows that the inclusion of $T_\Phi$ into $P$ induces an isomorphism on fundamental groups and hence $\pi_1(P, \ast)$ is isomorphic to $\Gamma \times \mathbb{Z}$. In particular, the composition of the classifying map for the universal covering of $P$ with the map corresponding to the projection $\Gamma \times \mathbb{Z} \to \mathbb{Z}$ on fundamental groups yields a bordism between $f'$ and $f$.

We conclude this section by deducing Corollary B.

**Proof of Corollary B.** The map $f$ in Theorem A induces an isometry in $\ell_1$-homology because $\mathbb{Z}$ is amenable. Hence the simplicial volume of the constructed mapping torus agrees with the $\ell_1$-norm of the homology class $\alpha \in H_\pi(\Gamma; \mathbb{R})$.

Similarly, given a manifold $M$, the simplicial volume of $M$ agrees with the $\ell_1$-norm of the image of the fundamental class in $H_\pi(B\pi \{M, \ast\}; \mathbb{R})$, because the classifying map of the universal covering induces the identity on fundamental groups. This concludes the first part of the corollary.

Letting $M$ vary over all $2n + 1$-manifolds, the main result of [17], which states that the spectrum of simplicial volume is dense in degrees at least 4, implies the second assertion.

**Remark 2.3.** Corollary B can be contrasted with recent work by Löh–Moraschini [21, Corollary 4.10] that provides vanishing results for the simplicial volume of mapping tori with assumptions on the amenable category (a generalization of the LS-category) of the fiber.

**Remark 2.4.** It should be noted that the manifolds $M$ in Corollary B can be assumed to be inessential, as the proof of Theorem A, upon which the proof of Corollary B is built, shows that these manifolds are nullbordant over $B\pi_1(M)$.
3 | VANISHING RESULTS

We next prove the two vanishing results outlined in the introduction, Theorems C and D, starting with the latter. For this, we make use of the following proposition from [31] (therein stated as the first corollary in Section 6) that we only state for connected manifolds and spell out in detail to make the notion of ‘rational isomorphism’ completely explicit.

**Proposition 3.1** (Rationally bijective self-maps of rationally essential manifolds). *Suppose that $M$ is an oriented, rationally essential manifold, and the Novikov conjecture holds for $\pi_1(M, \ast)$ and suppose that $\phi : M' \to M$, where $M'$ is also oriented, denotes a smooth map, satisfying:

- induces a map of positive sign on top homology (w.r.t. the fixed orientations),
- induces an isomorphism on fundamental groups, and
- induces an isomorphism on all higher rational homotopy groups (or equivalently on all rational homology groups of the universal covering).

Then $\phi$ necessarily has degree 1.*

**Proof of Theorem D.** We construct a self-map of $E$ that has degree bigger than 1 and satisfies all three assumptions of Proposition 3.1. This implies that $E$ is rationally inessential.

Let $\phi : B \to B$ denote a map of degree $d \neq -1, 0, 1$. By potentially replacing $\phi$ by $\phi \circ \phi$ we may assume that $d > 1$. We consider the following pullback diagram:

$$
\begin{array}{ccc}
\phi^*E & \xrightarrow{\phi_E} & E \\
\downarrow & & \downarrow \\
B & \xrightarrow{\phi} & B
\end{array}
$$

We prove below that the map labeled $\phi_E$ has the three labeled properties from Proposition 3.1 above. On the other hand it follows either from the Serre spectral sequence or by counting preimages of a regular value that this map has degree $d > 1$, so in conclusion it follows that $E$ cannot be rationally essential.

The first condition of Proposition 3.1 holds, since the orientations of the fibers of the two bundles agree and $\phi$ has positive degree.

Since $B$ is 2-connected, the long exact sequences in homotopy groups for both fiber bundles show that $\phi_E$ induces an isomorphism on fundamental groups as well.

For the third assumption of Proposition 3.1, consider the following diagram, obtained from the one above by taking universal coverings:

$$
\begin{array}{ccc}
\tilde{\phi}^*\tilde{E} & \xrightarrow{\tilde{\phi}_E} & \tilde{E} \\
\downarrow & & \downarrow \\
\tilde{B} & \xrightarrow{\tilde{\phi}} & \tilde{B}
\end{array}
$$

Since $B$ is 2-connected, we have that $\tilde{\phi}^*\tilde{E}$ is isomorphic to $\phi^* \tilde{E}$. Furthermore a map of non-zero degree has to be injective on rational cohomology, since the cup product pairing is non-singular. Hence $\phi$ induces an isomorphism on rational homology of $B$. Combined this implies that the maps...
in the second diagram induce an isomorphism between the $E^2$ pages of the corresponding fiber bundles. This implies that $\tilde{\varphi}_E$ induces an isomorphism on rational cohomology as required. □

**Remark 3.2.** Consider the real projectivization of the tautological quaternionic bundle over $S^4 \cong H\mathbb{P}^1$. The pullback of the fiber sequence $SU(2)/(\mathbb{Z}/2\mathbb{Z}) \rightarrow B\mathbb{Z}/2\mathbb{Z} \rightarrow BSU(2)$ along the inclusion $S^4 \rightarrow BSU(2)$ of the 4-skeleton yields the isomorphic smooth fiber bundle $\mathbb{R}P^3 \rightarrow \mathbb{R}P^7 \rightarrow S^4$. The total space of this bundle is clearly essential. Thus, the ‘rational’ in the conclusion of Theorem D cannot be omitted.

In order to prove Theorem C we will need the notion of diameter of a simplex. If $X$ is a metric space and $\sigma = \sum \sigma_i$ is a chain in $X$, then we call $\sup_i \text{diam}(\sigma_i)$ the diameter of $\sigma$, where $\text{diam}(\sigma_i)$ denotes the diameter of the image of $\sigma_i$. We will use the following lemma from [32].

**Lemma 3.3 [32, Lemma 1].** Let $X$ denote a compact metrized polyhedron, $m$ a non-negative integer, and $\varepsilon$ a positive number. Then there exist positive constant $\delta$ and $C$ such that: If a singular cycle $z \in C_m(X)$ is homologous to zero and its diameter is bounded by $\delta$, then there exists a chain $w \in C_{m+1}(X)$ such that $\delta w = z$, the diameter of $w$ is bounded by $\varepsilon$ and $||w|| \leq C||z||$.

**Proof of Theorem C.** In a first step we will replace the bundle by one with the same simplicial volume which fibers over $S^2$. Then we will leverage that the clutching function is defined over $S^1$, which will allow us to construct very efficient cycles.

Let $M \rightarrow E \rightarrow S^d$ denote a bundle, then by trivializing the bundle on the upper and lower hemisphere one sees that the total space is a twisted double of $D^d \times M$ twisted along a clutching function $\varphi_E : S^{d-1} \rightarrow \text{Diff}_0(M)$. By abuse of notation, we also denote by $\varphi_E$ the adjoint diffeomorphism from $S^d \times M$ to itself. Completely analogous to the proof of Theorem 2.2, the manifold $T_{\varphi_E} \times [0,1] \cup D^d \times M \times [0,1]$, where $D^d \times M \times [0,1]$ is glued to the $[\frac{1}{3}, \frac{2}{3}]$ part of the mapping torus, yields a bordism from $E$ to $T_{\varphi_E}$ with fundamental group $\pi_1(M,*) \times \mathbb{Z}$. Hence post-composing the classifying map of its universal covering with the $\ell^1$-norm preserving projection $\pi(M,*) \times \mathbb{Z} \rightarrow \pi(M,*)$ shows that $T_{\varphi_E}$ has the same simplicial volume as $E$. Since $\varphi_E$ is the identity on the sphere component, the mapping torus fibers over $S^{d-1}$. By repeating this process we get a bundle over $S^2$, with the same simplicial volume as $E$.

So let us now assume that we have a bundle $M \rightarrow E \rightarrow S^2$. In this case we have a clutching function $S^1 \rightarrow \text{Diff}_d(M)$ and a representative of the fundamental class of $E$ is given by a representative $\sigma$ of the fundamental class of $D^2 \times M$ on the upper hemisphere and a chain that bounds $\varphi_d(\partial \sigma)$ on the lower hemisphere $D^2 \times M$. We will construct representatives of $E$ whose $\ell^1$-norm is arbitrarily small.

Fix some metric on $M$ and let us denote the maximal Lipschitz constant of $\varphi_d(\cdot)$ by $L$. By Lemma 3.3 (applied with an arbitrary but overall fixed $\varepsilon$) there exists some $\delta$ and a constant $C$ such that any nullhomologous cycle $\sigma$ in $D^2 \times M$ with diameter less than $\delta$ bounds a chain whose $\ell^1$-norm is bounded by $C||\sigma||$. Now pick some arbitrary representative $\sigma_M$ of the fundamental class of $M$ such that the diameter of all simplices is bounded by $\frac{\delta}{100L}$, and some representative $\sigma_S$ of the fundamental class of $S^1$. Let $f^d : S^1 \times M \rightarrow S^1 \times M$ denote the degree $d$-covering map given by $f^d((z,x)) = (z^d, x)$ and let $\tau_d : C_\ast(S^1 \times M, \mathbb{R}) \rightarrow C_\ast(S^1 \times M, \mathbb{R})$ denote the corresponding transfer map, normalized such that $f_{d*} \circ \tau_{d*}$ is the identity on homology. The transfer map preserves the $\ell^1$ norm of a class while decreasing the diameter in $S^1$-direction of all simplices. Because the topology of $\text{Diff}_d(M)$ is generated by the metric of uniform convergence, one can
choose a big enough $d$ such that $\tau_{d}(\varphi_{E}(\sigma_{S1} \times \sigma_{M}))$ has diameter less than $\delta$, and hence it is bounded by a cycle $\rho$ such that

$$||\rho|| \leq C||\tau_{d}(\varphi_{E}(\sigma \times \sigma_{M}))|| = C\left(\frac{m+1}{1}\right)||\sigma_{S1}||||\sigma_{M}||,$$

where $m$ denotes the dimension of the fiber. Here the binomial coefficient stems from the standard triangulation of the product of two simplices. Now $f^{d}(\rho)$ bounds $\varphi_{E}(\sigma_{S1} \times \sigma_{M})$ and has norm bounded by $C\left(\frac{m+1}{1}\right)||\sigma_{S1}||||\sigma_{M}||$. By taking the cone of $\sigma_{S1}$ we end up with a representative of the fundamental class of $D^{2}$ denoted by $\sigma_{D^{2}}$. Hence

$$\sigma_{D^{2}} \times \sigma_{M} + f^{d}(\rho)$$

(here the sum means that we take the first summand on the upper hemisphere and the second summand on the lower hemisphere) is a representative of the fundamental class of $E$ whose norm is bounded by

$$C'||\sigma_{S1}||||\sigma_{M}||.$$

Since the norm of $\sigma_{S1}$ can be chosen to be arbitrarily small, this concludes the proof.

Remark 3.4. For hyperbolic groups, all cohomology classes are bounded [23] and hence any rationally essential manifold has positive simplicial volume. Thus, in case $\Gamma$ is hyperbolic, Theorem C implies that Theorem D also holds for base $S^{2}$.

4 | POSITIVE SCALAR CURVATURE

In this section we prove Theorem E by showing that the manifold $S$ in the Theorem supports a psc-metric, whereas a certain mapping torus of $S$ does not. We will prove the latter by constructing a map of non-zero degree from this mapping torus to the 7-torus, following the line of thought of the proof of Theorem 2.2, and employing the well-known machinery due to Schoen–Yau [25, 26]. The idea how to conclude the desired statement appeared in recent work by Frenck [9].

Proof of Theorem E. We regard $T^{7} = (S^{1})^{7}$ as a subset of $C^{7}$. The function $f : T^{7} \rightarrow \mathbb{R}, (z_{1}, ..., z_{7}) \mapsto \sum_{i} \text{Re}(z_{i})$ is a minimal Morse function. Its critical points have the coordinates $(\pm 1, ..., \pm 1)$, and the index of each of these is the number of $+1$’s. In particular, the critical values of $f$ are the odd numbers between $-7$ and $7$.

Let us denote $f^{-1}([-7, 0])$ by $W$ and $f^{-1}(\{0, 7\})$ by $W'$. Evidently $W$ contains all critical points (and hence all handles) of index $\leq 3$ and $W'$ contains all critical points (and all handles) of index $\geq 4$. Since 0 is a regular value of $f$, we conclude that $S = f^{-1}(\{0\})$ is a smooth submanifold with trivial normal bundle separating $W$ and $W'$.

Let us investigate the geometric properties of $S$: The manifolds $S$ and $W$ inherit spin structures from $T^{7}$, and hence $S$ is spin null-bordant in $T^{7}$. Since $T^{7} = B\mathbb{Z}^{7} = B\pi_{1}(S, \ast)$, this implies that $S$ carries a psc-metric by [14]. (The theorem we need is formulated as [7, Theorem 2.1].) Since $S^{2} \times \mathbb{C}P^{2}$ is not spin, it follows that the manifold $M = S\#(S^{2} \times \mathbb{C}P^{2})$ does not carry a spin structure as well. Nevertheless $S^{2} \times \mathbb{C}P^{2}$ carries a psc-metric; by [14, Theorem A] the connected sum carries a psc-metric as well.
In order to show that $R^{psc}(M)$ has infinitely many components, we carry out the constructions in the proof of Theorem 2.2 by hand.

We need to construct a particular diffeomorphism between $W$ and $W'$. In order to do that let us denote $S \cap \{ (z_1, \ldots, z_7) \in T^7 \mid \sum_i \text{Im}(z_i) \leq 0 \}$ by $V$. This is a manifold with boundary, denoted by $V_0$, consisting of the points such that $\sum_i z_i = 0$.

Let $\Psi : V \times S^1 \to T^7$ denote the restriction of the diagonal action to $V$. Note that this map is surjective, but it fails to be injective. Namely two points $(x, t)$ and $(x', t')$ map to the same point in $T^7$ if and only if $x, x' \in V_0$ and $t' \cdot x' = t \cdot x$. Let us denote by $\sim$ the equivalence relation generated by $(x, t) \sim (t x, 1)$ for all $x \in V_0$. Then $\Psi$ descends to a homeomorphism $\bar{\Psi} : V \times S^1 / \sim \to T^7$ and by construction $W \cong V \times S^1 / \sim$ and $W' \cong V \times S^1_+ / \sim$, where $S^1_\pm$ denotes the points in $S^1$ with non-negative and, respectively, non-positive imaginary part.

Fix a tubular neighborhood $V_0 \times [0, 1] \subset V$ and let us define a map

$$\Phi : V \times S^1_+ / \sim \to V \times S^1_+ / \sim$$

$$[x, t] \mapsto \begin{cases} [x, \tilde{t}] & x \in (V_0 \times [0, 1])^c \\ [e^{2 \arg(t)(1-s)} x, s, \tilde{t}] & [x, s, t] \in V_0 \times [0, 1] \times S^1_\sim \\ 

\end{cases}.$$

It is easy to verify that this yields a well-defined diffeomorphism. In conclusion, this yields that the torus is diffeomorphic to a twisted double of $V \times S^1_+ / \sim$ and the gluing is given by the restriction of $\Phi$ to $S \cong V \times \{-1, 1\} / \sim$; we denote this restriction by $\Phi$. Since $\Phi$ is the identity on $(V_0 \times [0, 1])^c$, $\Phi$ induces the identity on fundamental groups.

We will construct a particular diffeomorphism $\Phi$ of $S$ that is the identity on some open set and such that the mapping torus $T_\Phi$ has a map $g$ of non-zero degree to $T^7$.

By the bordism construction in the proof of Theorem A, we conclude that the mapping torus $T_\Phi$ has a map $g$ to $B\mathbb{Z}^7$ representing a non-zero homology class: in other words, $g$ is a map of non-zero degree to $T^7$. The restriction of $\Phi$ to $S$ is the identity on an open subset; hence, $\Phi$ can be extended to a diffeomorphism $\bar{\Phi}$ of $M := S \# (\mathbb{C}P^2 \times S^2)$. Since $\Phi$ is the identity on a neighborhood of the gluing of the connected sum, there exists a map $g' : T_\Phi \to T_\Phi$ of degree 1, and hence a map $g \circ g' : T_\Phi \to T^7$ of non-zero degree. Using the Schoen–Yau minimal surface method (see [11, p. 41f]), we conclude that $T_\Phi$ supports no psc-metric. Furthermore, by the same argument no finite covering of $T_\Phi$ carries a psc-metric, which implies that the subgroup of $\text{Diff}(M)$ generated by $\bar{\Phi}$ acts freely on the path-components of $R^{psc}(M)$; indeed, if for some $k \geq 1$ the pull-back of the metric on $M$ along $\Phi^k$ was isotopic to the original metric, then it would follow that the $k$-sheeted covering $T_{\Phi^k}$ of $T_\Phi$ admits a psc-metric, which we ruled out above. We thus deduce that $\pi_0(R^{psc}(M))$ is infinite. The proof for $S$ instead of $M$ is completely analogous. \qed

Note that the same line of argument would have worked with any other $S \subset T^7$ that is compatible with the constructions of Theorem 2.2.

5 | MACROSCOPIC DIMENSION

We have seen above that rational group homology classes in odd degrees at least 7 can be represented by mapping tori. In this section, we combine this insight with a result by Dranishnikov to construct examples of two closed manifolds with isomorphic fundamental groups whose
universal coverings are diffeomorphic, but have different macroscopic dimensions. We begin by recalling this important concept.

**Definition 5.1** (Uniformly cobounded). A continuous map \( f : X \to Y \) between metric spaces is called uniformly cobounded if there exists \( s > 0 \) such that the diameter of any pre-image \( f^{-1}(y), y \in Y \), is bounded by \( s \) from above.

**Definition 5.2** (Macroscopic dimension). For a Riemannian manifold \((M, g)\), the macroscopic dimension of the universal cover, denoted as \( \dim_{mc}(\widetilde{M}, g) \), is the smallest \( k \geq 0 \) such that there exists a uniformly cobounded Lipschitz map from \( M \) to a \( k \)-dimensional simplicial complex.

**Remark 5.3.**

(a) In fact, for a closed manifold \( M \) the macroscopic dimension \( \dim_{mc}(\widetilde{M}, g) \) is independent of the Riemannian metric \( g \) on \( M \), so we will simply write \( \dim_{mc}(\widetilde{M}) \) henceforth.

(b) One could also try to define the macroscopic dimension of any metric space \( X \), but to get a working theory it then seems vital to assume that \( X \) is a uniform simplicial complex, and require that the map \( f \) from Definition 5.2 is simplicial after iterated barycentric subdivisions, cf. [4, Definition 1.1]. One way to ensure that this holds is restricting to smooth manifolds and requiring the maps to be Lipschitz. For elaborations on several related but somewhat inequivalent notions of macroscopic dimension, see [6].

**Example 5.4.** If \( M \) is the \( n \)-torus \((S^1)^n\), \( \dim_{mc}(\widetilde{M}) \) is equal to \( n \).

**Proposition 5.5.** An inessential \( d \)-manifold \( M \) satisfies \( \dim_{mc}(\widetilde{M}) < d \).

**Proof.** Let \( \pi_1(M) \) be denoted by \( \Gamma \). Obstruction theory allows us to deform the map \( M \to B\Gamma \) classifying the universal covering of \( M \) along the inclusion of the \((d-1)\)-skeleton \( B\Gamma^{(d-1)} \), see [1, Proposition 3.2] for more details.

Then for any choice of a Riemannian metric on \( M \), the lift of this map to universal coverings, \( \widetilde{M} \to E\Gamma^{(d-1)} \), is uniformly cobounded, hence \( \dim_{mc}(\widetilde{M}) \leq d - 1 \). 

**Remark 5.6.** Note that the conclusion of the previous proposition fails under the weaker assumption that \( M \) is just rationally inessential [22].

The converse of this statement, that is, the assertion that any rationally essential manifold has maximal macroscopic dimension, was conjectured by Gromov [12] and proven by Dranishnikov for amenable fundamental groups [4], but shortly thereafter disproven in general [5].

**Proof of Theorem F.** We start by applying Theorem A to a rational non-zero homology class of \( \Gamma \) in degree \( 2n + 1 \): let \( M_1 := S \times S^1, M_2 := T_\varphi \). Then (i) is clear, (ii) holds as both universal coverings are diffeomorphic. That they are quasi-isometric follows from the Milnor–Swarc lemma. Note that there cannot exist a continuous quasi-isometry, since this would force the macroscopic dimensions to agree.

From the construction in the proof of Theorem A it is clear that the map \( S \to B\Gamma \) classifying the universal covering of \( S \) is nullbordant, hence \( S \) is inessential. By Proposition 5.5, this implies that \( \dim_{mc}(\widetilde{S}) < 2n \) and hence \( \dim_{mc}(\widetilde{M}_1) < 2n + 1 \), as we can cross any cobounded map \( \widetilde{S} \to K^{(2n-1)} \) with the identity on \( \mathbb{R} \). The mapping torus \( M_2 = T_\varphi \), however, is rationally essential.
by construction. Recall Gromov’s conjecture stating that the universal cover of any rationally essential manifold is of maximal macroscopic dimension which was verified for amenable fundamental groups by Dranishnikov [4]. Applied to the situation at hand we deduce that $M_2$ satisfies $\dim_{mc}(\tilde{M}_2) = 2n + 1$.

Remark 5.7. A non-abelian example of $\Gamma$ as in Theorem F is the integral Heisenberg group of rank $4n + 1$, which is finitely presented and amenable. Its cohomology, calculated in [20], satisfies $H_{2n+1}(G; \mathbb{Q}) \neq 0$ and $H_j(\Gamma; \mathbb{Q}) = 0$ for all $0 < j \leq 2n$.

6 | CHARACTERISTIC CLASSES

In this final section we prove Theorem G. We begin by recalling that for an oriented smooth manifold bundle $M \to E \to B$, the vertical tangent bundle vector bundle $T_\pi$ over the total space $E$ whose dimension $d$ agrees with the dimension of the fiber. For any characteristic class of oriented vector bundles, $c \in H^*(BSO(d); \mathbb{Q})$, the class obtained from fiber integration $\kappa_c := \int c(T_\pi) \in H^{*-d}(B; \mathbb{Q})$

is known as the tautological class or generalized Milller–Morita–Mumford or simply $\kappa$-class of $c$. These classes are characteristic classes of smooth fiber bundles, so they naturally live in the cohomology of $BDiff(M)$.

Galatius–Randal-Williams generalized the concept of $\kappa$-classes to allow for tangential structures and proved a higher-dimensional analogue of the Madsen–Weiss theorem [10]. Their theorems show that the cohomology rings of diffeomorphism groups are, up to stabilization, generated by certain cohomology classes defined on the level of MT-spectra; the classes $\kappa_\alpha$ can be understood in this context.

Proof of Theorem G. For (1), we invoke the work by Galatius–Randal-Williams. We first construct a manifold with boundary that is weakly equivalent to $B\Gamma^{(n)}$ and has trivial tangent bundle. From the finiteness assumption, we may assume that $B\Gamma^{(n)}$ is a finite CW-complex, hence it admits a local embedding into $\mathbb{R}^{2n}$. More precisely, we may assume that this local embedding has only finitely many self-intersections, all contained in top cells. We then consider a ‘tubular’ open neighborhood of the image of $B\Gamma^{(n-1)}$, and glue on a tubular neighborhood $C \times D^n$ for each $n$-cell $C$ of $B\Gamma^{(n)}$. This yields a manifold $W$ with non-empty boundary as desired.

The $n$th stage of the Moore–Postnikov filtration of the classifying map of its tangent bundle relative to $\partial W$ is

$$W \to BO(2n)(n) \times B\Gamma^n \xrightarrow{\theta} BO(2n),$$

where $\theta = \theta^n \circ \text{pr}_1$ and $\theta^n : BO(2n)(n) \to BO(2n)$ denotes the $n$-connected cover of $BO(2n)$. The classifying map $W \to BO(2n)$ of the tangent bundle of $W$ admits a lift along $\theta$, so we can fix a $\theta$ structure $\theta_W$ on $W$, that is, a bundle map $\ell : TW \to \theta^n \gamma$.

Now let $\text{MT} \Theta^n$ denote the Madsen–Tilmann spectrum of $BO(2n) \to BO(2n)$, as in [10]. Then the relevant Thom spectrum for $W$ is $\text{MT} \Theta^n \wedge B\Gamma_+$. By [10, Proposition 7.11], we can now find a universal end $(K, \ell_K)$ for $\theta_W$; sphericity is guaranteed by the results in [10, Section 5.1]. Then
[10, Theorem 1.8] states that

\[ \text{hocolim}_{i \to \infty} \mathcal{N}^{\theta_W}(K_i, \ell_{K_i}) \to \Omega^\infty \mathbf{MT}^{\theta'} \]

is a weak equivalence, where \( \theta' \) refers to the \( n \)-th stage of the Moore–Postnikov factorization of \( K \to \mathbf{MT}^{\theta^n} \wedge B\Gamma_+ \), and the left-hand side refers to the moduli spaces of manifolds as introduced in [10]. Note however, that by [10, Addendum 1.9], the map \( K \to \mathbf{MT}^{\theta^n} \wedge B\Gamma_+ \) is already guaranteed to be \( n \)-connected, so \( \mathbf{MT}^{\theta'} \) agrees with \( \mathbf{MT}^{\theta^n} \wedge B\Gamma_+ \). After rationalizing, that is, taking the smash product with \( H\mathbb{Q} \), this spectrum has \( H\mathbb{Q} \wedge \Sigma^{\infty-d}_{+} B\Gamma \) as a retract. This spectrum is the natural home of \( \kappa_{\alpha} \), so we can deduce that \( \kappa_{\alpha} \in H^\infty(\Omega^\infty \mathbf{MT}^{\theta'}; \mathbb{Q}) \) is non-zero. As the homotopy colimit of moduli spaces on the left is built out of classifying spaces of diffeomorphism groups, we deduce that there exists some finite \( j \) so that

\[ 0 \neq \kappa_{\alpha} \in H^j(B\text{Diff}_3(W \cup K_j); \mathbb{Q}). \]

To conclude, let \( M \) be the double of \( W \cup K_j \). Consider the map

\[ B\text{Diff}_3(W \cup K_j) \to B\text{Diff}^B \Gamma(M), \]

which is given by trivially extending a diffeomorphism on the second half of the double. Then the class \( \kappa_{\alpha} \) in the cohomology of the second space (as defined in the Theorem G) pulls back to the non-zero class that was considered before. This finishes the proof of (1).

Assertion (2) can be easily deduced from Theorem D: we need to prove that for any \( M \)-bundle \( \pi : E \to S^d \) with \( d \geq 3 \) and \( \alpha \in H^{d+2}(B\Gamma; \mathbb{Q}) \), we have \( \kappa_{\alpha}(\pi) = 0 \). Since

\[ \langle \kappa_{\alpha}(\pi), [S^d] \rangle = \langle \alpha, [E] \rangle = 0, \]

as \( E \) is rationally inessential, this is indeed the case. \( \square \)

Remark 6.1. Note that by applying Theorem C, we see that the second part of Theorem G also holds for bounded classes \( \alpha \in H^{d+2}(B\Gamma; \mathbb{Q}) \). In particular, it holds for all homology classes in degree \( d + 2 \) if \( \Gamma \) is hyperbolic (compare Remark 3.4).

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