ABSTRACT. Let $p$ be an unramified prime in a totally real field $L$ such that $h^+(L) = 1$. Our main result shows that Hilbert modular newforms of parallel weight two for $\Gamma_0(p)$ can be constructed naturally, via classical theta series, from modules of isogenies of superspecial abelian varieties with real multiplication on a Hilbert moduli space. This can be viewed as a geometric reinterpretation of the Eichler Basis Problem for Hilbert modular forms.

Keywords: Superspecial abelian varieties; Eichler Basis Problem; Hilbert modular forms; Theta series; Eichler orders.

MSC 2000: 11G10, 11G18, 11F27, 11F41, and 14G35.
left ideal classes \([I_1], \ldots, [I_H]\) of \(\text{End}_{\mathbb{F}_p}(E)\) and isomorphism classes of supersingular elliptic curves \(E_1, \ldots, E_H\) over \(\overline{\mathbb{F}}_p\), given functorially by the tensor map

\[ [I_i] \mapsto [E \otimes_{\text{End}_{\mathbb{F}_p}(E)} I_i], \]

where the brackets \([-\] represent the respective isomorphism classes.

We are now in position to give a geometric interpretation of Eichler’s original Basis Problem in terms of modules of isogenies of supersingular elliptic curves.

**Proposition 1.1.** The span of the theta series coming from the modules

\[ \text{Hom}(E_i, E_j) \cong I_j^{-1}I_i \]

equipped with the quadratic degree map, includes the vector space \(S_2(\Gamma_0(p))\).

The bulk of the paper deals with the generalization of the above geometric interpretation to Hilbert modular forms, by using modules of isogenies of superspecial points on the Hilbert moduli space of dimension \(g\), endowed naturally with a quadratic structure via a so-called \(O_L\)-degree map coming from the polarizations. Recall that for a base scheme \(S\), the Hilbert moduli space classifies abelian varieties \(A/S\) with real multiplication by \(O_L\) satisfying the Rapoport condition. Let \(k\) be an algebraically closed field of characteristic \(p\). An abelian variety \(A/k\) is superspecial if and only if \(A \cong_k E^g\), for \(E\) some supersingular elliptic curve defined over \(k\). By a theorem of Deligne (see [18, Thm. 3.5]), there is a unique superspecial abelian variety for \(g \geq 2\) (this is of course false in general for \(g = 1\)), and it follows that the superspecial locus on the Hilbert moduli space is finite.

We state our geometric reinterpretation of the classical Eichler Basis Problem for Hilbert modular forms, in terms of \(O_L\)-modules of \(O_L\)-isogenies of superspecial abelian varieties with \(O_L\)-action succinctly (it follows from results of Section 5 and Cor. 6.2):

**Theorem 1.2.** Let \(h^+(L) = 1\), and \(p\) unramified in \(L\). Let \(A_m, A_n\) run through all superspecial points of the Hilbert moduli space over \(\overline{\mathbb{F}}_p\). The quadratic modules \(\text{Hom}_{O_L}(A_m, A_n)\) equipped with the \(O_L\)-degree map give all quadratic forms of level \(p\) attached to \(B_{p,\infty} \otimes L\). It follows that the space \(S_2^{\text{new}}(\Gamma_0((p)))\) is contained in the span of the theta series stemming from the quadratic modules \(\text{Hom}_{O_L}(A_m, A_n)\).

We streamline and improve slightly upon the results of [14]. In particular, we demonstrate that this geometric reinterpretation of the Eichler Basis Problem in terms of abelian varieties is possible even for levels not found on the Hilbert moduli space. That is, Cor. 5.16 and Cor. 6.2 imply that the space \(S_2(\Gamma_0(1))\) is contained in the span of theta series stemming from quadratic modules \(\text{Hom}_{O_L}(A, A')\), for suitable abelian varieties \(A, A'\) with RM with maximal endomorphism orders.

The paper is organized as follows. In Section 2, we prove a variant of Tate’s theorem for supersingular abelian varieties with real multiplication. In Section 3, we review some well-known material on Dieudonné modules and give an ad hoc proof of a very special case of the classification theorem for those modules. In Section 4, we compute the orders of endomorphisms of the corresponding abelian varieties which turn out to be superspecial. In Section 5, we explain the link between the arithmetic of hereditary orders in quaternion algebras and the geometry of abelian varieties with real multiplication. In Section 6, we recall how the Eichler Basis Problem for Hilbert modular forms of squarefree level follows from the Jacquet-Langlands correspondence.
2. A variant of a theorem of Tate

Let \( p \) be a rational prime and let \( B_{p,\infty} \) be the rational quaternion algebra ramified only at \( p \) and \( \infty \). Let \( L \) be a totally real field of degree \( g \) over \( \mathbb{Q} \). We suppose throughout that \( p \) is unramified in \( L \). Let \( B_{p,L} \) denote the quaternion algebra \( B_{p,\infty} \otimes_{\mathbb{Q}} L \) over \( L \).

Let \( A \) be an abelian variety over a perfect field \( k \) of characteristic \( p \). For a rational prime \( \ell \), we let

\[
T_{\ell}(A) = \begin{cases} 
\lim_{n \to \infty} A[\ell^n] & \ell \neq p \\
D(A) & \ell = p.
\end{cases}
\]

Here \( D(A) \) denotes the covariant Dieudonné module. Recall that a Dieudonné module is a left \( W(k)[F,V] \)-module, free of finite rank over the Witt vectors \( W(k) \) such that \( D/FD \) has finite length over \( W(k) \). In fact, for \( D(A) \) this length is precisely \( \text{dim}(A) \). Recall also that the operators \( F,V \) satisfy the relations \( FV = VF = p, F\lambda = \lambda F, AV = V\lambda^p \) for \( \lambda \in W(k) \). The morphisms between \( T_{\ell}(A_1) \) and \( T_{\ell}(A_2) \) are always \( \mathbb{Z}_p \)-linear if \( \ell \neq p \), and \( W(k)[F,V] \)-linear if \( \ell = p \). If \( A \) is defined over a scheme \( S \), we use the notation \( \text{End}(A) \) to denote its endomorphisms over \( S \); we emphasize this by using the notation \( \text{End}(A/S) \). The notation \( \text{End}^0(A) \) stands for \( \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \).

We shall consider the following type of abelian variety \( A \) with additional endomorphism structure:

**Real Multiplication (RM):** The abelian variety \( A \) over \( \mathbb{F}_p \) is supersingular, and is equipped with RM by \( \iota : \mathcal{O}_L \to \text{End}(A) \). By [2, Lemma 6], the centralizer \( \text{Cent}_{\text{End}^0(A)}(L) \cong B_{p,L} \), and thus \( \text{Cent}_{\text{End}(A)}(\mathcal{O}_L) \) is isomorphic to an order in \( B_{p,L} \).

Further on, we will impose stringent requirements on the abelian variety \((A,\iota)\): in particular, \( A \) will be superspecial and the Eichler order \( \text{End}_{\mathcal{O}_L}(A) \) will be of prescribed level. Otherwise explicitly mentioned, we consider abelian varieties that satisfy Rapoport’s condition (see [16]) or equivalently (since \( p \) is unramified, see [4, Cor. 2.9]), the Deligne-Pappas condition (cf. [4]). Recall that those are moduli conditions: \((A,\iota)\) over \( S \) satisfies the Rapoport condition if the Lie algebra of \( A \) is a locally free \( \mathcal{O}_L \otimes_{\mathbb{Z}} \mathcal{O}_S \)-module of rank 1. We recall that the Hilbert moduli space \( \mathcal{M} \) decomposes as: \( \mathcal{M} = \bigcup_{(\mathfrak{A}, \mathfrak{A}^+)} C_{\mathfrak{A}^+}^{\mathfrak{A}}(L) \cdot \mathcal{M}(\mathfrak{A}, \mathfrak{A}^+) \), where \((\mathfrak{A}, \mathfrak{A}^+)\) is an \( \mathcal{O}_L \)-module with a notion of positivity. Under the hypothesis that the narrow class number \( h^+(L) \) is 1, \( \mathcal{M} = \mathcal{M}(\mathcal{O}_L, \mathcal{O}_L) \), the component of principally polarized points (see [4, §2.6]). Denote by \( A^t \) the dual abelian variety. Recall that a principal polarisation \( \lambda \) of an abelian scheme \( A \) is a symmetric isomorphism: \( \lambda : A \to A^t \), coming from an ample line bundle on geometric points; we require the polarisation to be \( \mathcal{O}_L \)-linear. Denote by \( \text{Hom}_{\mathcal{O}_L}(A, A^t)^{sym} \) the invertible \( \mathcal{O}_L \)-module of symmetric \( \mathcal{O}_L \)-linear homomorphisms from \( A \) to \( A^t \). If \( A \) over a field \( k \) satisfies the Rapoport condition and \( \text{Hom}_{\mathcal{O}_L}(A, A^t)^{sym} \cong \mathcal{O}_L \), then principal \( \mathcal{O}_L \)-polarisations are in bijection with \( \mathcal{O}_L^{sym} \). Furthermore, under \( h^+(L) = 1 \), \( \mathcal{O}_L^{sym} = (\mathcal{O}_L^+)^2 \), so if \( \lambda_1 \) and \( \lambda_2 \) are principal \( \mathcal{O}_L \)-polarisations, \((\iota, A, \lambda_1) \cong (\iota, A, \lambda_2)\) as polarized \( \mathcal{O}_L \)-abelian varieties, by multiplication by a suitable unit. We point out that \( h^+(L) = 1 \) by itself does not imply that all abelian varieties with RM over \( \mathbb{F}_p \) admit a principal polarisation. On the other hand, a principally polarized abelian variety with RM automatically satisfies the Rapoport condition.
We start our analysis by establishing a variant of Tate’s theorem on endomorphisms of abelian varieties taking into account the real multiplications. This is well-known but we provide a proof for completeness.

**Theorem 2.1.** Let $A_i, i = 1, 2$ be supersingular abelian varieties over $\mathbb{F}_p$ with RM by $\mathcal{O}_L$ as above, $\iota_i : \mathcal{O}_L \rightarrow \text{End}_{\mathbb{F}_p}(A_i)$. Then for $\ell \neq p$,

$$\text{Hom}_{\mathcal{O}_L}(A_1, A_2) \otimes \mathbb{Z}_\ell \cong \text{Hom}_{\mathcal{O}_L \otimes \mathbb{Z}_p}(T_\ell(A_1), T_\ell(A_2)),$$

and at $\ell = p$,

$$\text{Hom}_{\mathcal{O}_L}(A_1, A_2) \otimes \mathbb{Z}_p \cong \text{Hom}_{\mathcal{O}_L \otimes W(k)[F, V]}(\mathbb{D}(A_1), \mathbb{D}(A_2)).$$

**Proof.** We prove only the claim for $\ell \neq p$, as the proof is similar for $\ell = p$. Choose a finite extension $\mathbb{F}_q \supseteq \mathbb{F}_p$ such that all endomorphisms are defined over $\mathbb{F}_q$.

We can assume $A_1 = A_2 = A$ by an easy reduction (as in the proof of the original theorem). Let $G := \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_q)$. Tate’s theorem ([20], [23]) implies that:

$$\text{End}(A/\mathbb{F}_q) \otimes \mathbb{Z}_\ell \cong \text{End}(T_\ell(A))^G.$$ 

Let $M_1 := \{f \in \text{End}(A/\mathbb{F}_q) \otimes \mathbb{Z}_\ell | f r = r f, \forall r \in \mathcal{O}_L\}$, which we may identify with $\{f \in \text{End}(T_\ell(A))^G | f T_\ell(r) = T_\ell(r)f, \forall r \in \mathcal{O}_L\}$, by Tate’s theorem. Let $M_2 = \{f \in \text{End}(A/\mathbb{F}_q) | f r = r f\} \otimes \mathbb{Z}_\ell$. It is clear that $M_2 \subseteq M_1$ and that $M_1/M_2$ is torsion-free. By comparing ranks, the modules coincide. We next use the fact that the endomorphism ring is large to eliminate the mention of $G$ in the statement of the theorem. We know that the action of $G$ commutes with the action of $\text{End}(A) \otimes \mathbb{Q}_\ell$ in $\text{End}(T_\ell(A)) \otimes \mathbb{Q}_\ell$. By assumption, $A$ is supersingular and therefore $\text{End}(A/\mathbb{F}_q) \otimes \mathbb{Q}_\ell \cong M_2(\mathbb{Q}_\ell)$; the action of $G$ is thus given by scalars in $\mathbb{Q}_\ell$ and thus $\text{End}_{\mathcal{O}_L}(A) \otimes \mathbb{Z}_\ell \cong \text{End}_{\mathcal{O}_L}(T_\ell(A))$. □

3. **Classification of (some) Dieudonné modules with multiplications**

Dieudonné modules have been classified up to isomorphism by Manin [13]. For any fixed $g$, it turns out that there is a unique superspecial Dieudonné module of rank $2g$. We consider here Dieudonné modules with RM and impose some additional conditions. These conditions are natural from the point of view of abelian varieties. Note that we rely crucially on the arithmetic assumption that the prime $p$ is unramified. For $p$ ramified in $L$, the ideas involved are more subtle as there is more than one isomorphism class of superspecial Dieudonné modules with RM (see [14, 15]).

Let $k$ be a perfect field of characteristic $p$. We concentrate on rank $2g$ Dieudonné modules with RM.

**Definition 3.1.** A Dieudonné module $\mathbb{D}$ with RM is an $(\mathcal{O}_L \otimes W(k))[F, V]$-module of rank 2 over $\mathcal{O}_L \otimes W(k)$, where $F, V$ commute with $\mathcal{O}_L$.

Assume from now on that $k$ is algebraically closed. In order to classify Dieudonné modules with RM, we introduce useful decompositions. A Dieudonné module $\mathbb{D}$ with RM decomposes according to the decomposition of $p$ in $\mathcal{O}_L$:

$$\mathbb{D} = \bigoplus_{p | \ell} \mathbb{D}_p,$$

where $\mathbb{D}_p$ are Dieudonné modules with $\mathcal{O}_{L_p}$-action. Furthermore, letting $\Phi_p = \text{Hom}(\mathcal{O}_{L_p}, W(k))$, each $\mathbb{D}_p$ decomposes as an $\mathcal{O}_L \otimes W(k)$-module as $\bigoplus_{\phi \in \Phi_p} \mathbb{D}_p(\phi)$.
Superspecial Varieties and Modular Forms

5

where $D_p(\phi)$ is the summand where $O_{L_p}$ acts via $\Phi_p$. Altogether, we obtain the decomposition

$$D \cong \bigoplus_{p \mid p} \bigoplus_{\phi \in \Phi_p} W(k)^2_{\phi},$$

as $O_L \otimes W(k)$-modules.

Recall that an $O_L$-linear polarisation $A \overset{\lambda}{\rightarrow} A^t$ with degree prime to $p$ defines a principal quasi-polarisation with RM, i.e., a skew-symmetric form

$$D(A) \times D(A) \rightarrow W(k) \otimes O_L,$$

which is a perfect pairing over $W(k) \otimes O_L$, and such that $F$ and $V$ are adjoint.

An important consequence of $p$ being unramified in $L$ is the following fact, whose use pervades the whole text. Recall that a Dieudonné module $D$ is superspecial if

$FD = VD$.

**Fact 3.2.** Let $k$ be algebraically closed. Then there exists a unique superspecial (principally quasi-polarized) Dieudonné module with RM.

**Proof.** Recall that Dieudonné modules decompose according to primes; since $p$ in unramified, the claim follows from the inert case (see [9, Thm 5.4.4]). \[\square\]

Moreover, since for any abelian variety $A$ over an algebraically closed field $k$, $T_\ell(A) \cong (O_L \otimes \mathbb{Z}_\ell)^2$ for $\ell \neq p$ (cf. [16, Lem. 1.3]), the behaviour of $A$ at the prime $p$ is pivotal.

From the decomposition of $D$ according to primes, we may in the rest of this section assume that $p$ is inert. The Dieudonné module $D$ is then an $O_{L_p} \otimes \mathbb{Z}_p W(k)$-module of rank two, where $O_{L_p}$ is unramified of degree $g$. Put $R = O_{L_p}$. Up to a choice of an embedding $\tau$ of $R$ in $W(k)$, we can decompose the ring:

$$R \otimes W(k) = \bigoplus_{\lambda \in \text{Hom}(R, W(k))} W(k)_{\lambda} = \bigoplus_{i=1}^g W(k)_{i},$$

$$r \otimes \lambda \mapsto \left(\tau(r)\lambda, \sigma \circ \tau(r)\lambda, \ldots, \sigma^{g-1} \circ \tau(r)\lambda\right),$$

where $\sigma$ denotes the Frobenius. To get a uniform notation, put $\sigma_i := \sigma^{i-1} \circ \tau$. Then

$$\left(\tau(r)\lambda, \sigma \circ \tau(r)\lambda, \ldots, \sigma^{g-1} \circ \tau(r)\lambda\right) = \left(\sigma_1(r)\lambda, \sigma_2(r)\lambda, \ldots, \sigma_g(r)\lambda\right).$$

In turn, this implies that in the decomposition $D = \bigoplus_{i=1}^g D_i$, the action of $R$ is given by:

$$r \cdot d_i = \sigma^{i-1} \circ \tau(r) \cdot d_i = \sigma_i(r)d_i,$$

for $d_i \in D_i$, $r \in R$.

Note that $F : D_i \rightarrow D_{i+1}$, $V : D_{i+1} \rightarrow D_i$, $FV = VF = p$ and

$$pD_{i+1} = F(V D_{i+1}) \subseteq F(D_i) \subseteq D_{i+1}$$

for all $i$.

There are some constraints on how these pieces fit together. In particular,

$$\bigoplus_i \dim_{k}(D_{i+1}/FD_i) = g,$$

with $\phi_i := \dim_{k}(D_{i+1}/FD_i) \in \{0, 1, 2\}$.

**Definition 3.3.** We call $\Phi = \{\phi_i\}$ an admissible set of indices if $\phi = 1$ for all $i$ or if $\phi \in \{0, 2\}$ for all $i$.

**Proposition 3.4.** Let $\Phi = \{\phi_i\}$ be an admissible set of indices. Let $D = \bigoplus_{i=1}^g D_i$ be a supersingular Dieudonné module with RM with invariants $\Phi$. Then $D$ is uniquely determined e.g., by the elementary divisors of the $\sigma^g$-linear map $F^g : D_1 \rightarrow D_1$.  


Proof. We only give all the details of the proof supposing that $\phi \in \{0, 2\}$ for all $i$. The other case occurs when the Rapoport condition holds, and is shown in [19, §3] to be superspecial in which case the result follows by uniqueness (see Fact 3.2). Suppose that $D'$ is a rank 2 module over $W(k)$, equipped with the action of a $\sigma$-linear operator that we call $F^\sigma$. By [19, p.419], if we pick a basis $\{\alpha, \beta\}$ for $D'$, we can represent $F^\sigma$ by a matrix $M$ with elementary divisors i.e., according to whether $g$ is even or odd, there exists a matrix $N \in \text{GL}_2(W(k))$ such that

$$N^{-1}M^\sigma N = \begin{pmatrix} p^\mathbb{Z} & 0 \\ 0 & p^\mathbb{Z} \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ p^\mathbb{Z} & 0 \end{pmatrix}.$$

We now explain how to choose the bases in a normalized way that is, starting from a basis $\{\alpha_1, \beta_1\}$ of $\mathbb{D}_1$, we show how to define a basis for $\mathbb{D}_2$, and then for $\mathbb{D}_3$, etc. back to the normalized map $F^\sigma : \mathbb{D}_1 \rightarrow \mathbb{D}_1$.

Consider the map $\mathbb{D}_i \xrightarrow{F} \mathbb{D}_{i+1}$, starting, say, with $i = 1$. If $\phi_{i+1} = 0$, then $F$ is an isomorphism, and for any basis $\{\alpha_i, \beta_i\}$, define $\alpha_{i+1} := F(\alpha_i)$, and $\beta_{i+1} := F(\beta_i)$. If $\phi_{i+1} = 2$, then $F$ is $p$ times an isomorphism, and for any basis $\{\alpha_i, \beta_i\}$, define $\alpha_{i+1} := \frac{1}{p}F(\alpha_i)$, and $\frac{1}{p}\beta_{i+1} := F(\beta_i)$. It follows from the construction that any change of basis to $\{\alpha_1, \beta_1\}$ uniquely gives all $\{\alpha_i, \beta_i\}$'s, and we conclude based on the uniqueness of the normal presentation with elementary divisors.

4. Going global: Eichler orders

In this section, we show how to construct abelian varieties with real multiplication that are exotic from the point of view of moduli. In particular, their endomorphism orders will be maximal in $B_{p,L}$ e.g., level one may occur. Our starting point, though, is the superspecial locus of the Hilbert moduli space.

We suppose that $A$ is a superspecial abelian variety with RM over $k = \mathbb{F}_p$ satisfying the Rapoport condition i.e., the Lie$(A)$ is a free $O_L \otimes \mathbb{Z}$ $k$-module. Then, as we will see shortly, $R = \text{Cent}_{\text{End}(A)}(O_L)$ is an Eichler order of level $p$ in $B_{p,L}$. That is,

$$R \otimes_{O_L} O_{L_{p_i}} \cong \begin{cases} \text{maximal order of } B_{p,L} \otimes L_{p_i} & f(p_i/p) = 1 \mod 2 \\ \begin{pmatrix} * & * \\ p * & * \end{pmatrix} & : = \Gamma_0(p_i) \subseteq M_2(O_{p_i}) \text{ otherwise.} \end{cases}$$

Recall that by local class field theory (see [17]), the condition $f(p_i/p)$ odd implies that $B_{p,L} \otimes L_{p_i}$ is a division algebra.

Proposition 4.1. Let $A$ be a principally polarized superspecial abelian variety with RM over $\mathbb{F}_p$. The endomorphism order $\text{End}_{O_L}(A) := \text{Cent}_{\text{End}(A)}(O_L)$ is an Eichler order of level $p$ in $B_{p,L}$.

Proof. The $O_L$-version of Tate’s theorem allows us to perform all local computations on $T_L(A)$, which are uniquely determined (this is clear for $\ell \neq p$, and at $\ell = p$, it follows from Fact 3.2). For this local computation, pick $A = E \otimes_{\mathbb{Z}} O_L$ for $E$ a supersingular elliptic curve. It follows that $\text{End}_{O_L}(A) = \text{End}(E) \otimes O_L$ and the level is clearly $pO_L$. Since $p$ is unramified, it is squarefree, and thus the order is Eichler by [1, Prop. 1.2].

From now on, we set $A = E \otimes_{\mathbb{Z}} O_L$; as we are interested in local considerations in this section, this entails no loss of generality and computations are more transparent. Note that $\mathbb{D}(A) \cong \mathbb{D}(E) \otimes O_L$. For $p_i$ above $p$ such that $f(p_i/p) = 0 \mod 2$,
let $R_p$ be the Eichler order $\Gamma_0(p)_{\mathbb{Z}}$. Put $R_p^+ := M_2(\mathcal{O}_{L_p})$. We can associate to $R_p^+$ a (supersingular) abelian variety $A^+$ with RM (depending on $p$) and an isogeny $f : A \to A^+$. The isogeny $f$ (and thus $A^+$) is determined uniquely by $f^*\mathbb{D}(A^+)$. We describe the Dieudonné module more precisely:

$$\mathbb{D}_{p_i}(A^+) := \begin{cases} \mathbb{D}(E) \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_{p_i}}, & p_i|p, f(p_i/p) = 1 \mod 2 \\ p \cdot R^+ \cdot \left( \mathbb{D}(E) \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_{p_i}} \right) & \text{otherwise.} \end{cases}$$

Without loss of generality, we can assume that $p\mathcal{O}_L = p$ is inert. Our goal is to describe $R_p^+ \cdot (\mathbb{D}(E) \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p})$, in order to characterize the abelian variety $A^+$ arising from such a construction. We can thus suppose that $g$ is even, as it follows from the condition: $f(p) = 0 \mod 2$.

The Dieudonné module $\mathbb{D}(E)$ is identified with $W(k)^2$, with $F = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ and $V = (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix})$. The maximal order $\mathcal{O} = \text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ has the following presentation:

$$\mathcal{O} = \{(a, b) : a, b \in W(\mathbb{F}_{p^2})\}.$$ With this notation, $R_p \cong \mathcal{O} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}$.

Choose an isomorphism $\mathcal{O}_{L_p} \cong W(\mathbb{F}_{p^2})$ (i.e., choose an embedding $\mathbb{F}_{p^2} \subseteq \mathbb{F}_{p^2}$). We thus get an $\mathcal{O} \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^2})$-action on $W(k)^2 \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^2}) \cong \otimes_{j=1}^g W(k)^2$. On the other hand, $\mathcal{O} \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^2}) = \left( \left( \mathcal{O} \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^2}) \right) \otimes_{W(\mathbb{F}_{p^2})} W(\mathbb{F}_{p^2}) \right)$ and note that $W(\mathbb{F}_{p^2})$ already splits $\mathcal{O} \otimes \mathbb{Q}$.

For $p \neq 2$, one can explicitly pick a unit $b$ such that $b^2 = -b$, and the element

$$t := \begin{pmatrix} 0 & b \\ pb^2 & 0 \end{pmatrix} \otimes 1/2pb^2 + \begin{pmatrix} 0 & 1/b \\ 1 & 0 \end{pmatrix} \otimes 1,$$

yields an embedding

$$\mathcal{O} \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^2}) \hookrightarrow M_2(W(\mathbb{F}_{p^2})).$$

by sending $t \mapsto (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$. Under this embedding, the image of $\mathcal{O} \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^2})$ is the chosen Eichler order of level $p$, so that the lattice $R_p^+ \cdot \mathbb{D}(E) \otimes \mathcal{O}_L$ is:

$$\{(v_1), (v_2), \ldots, (v_n) | v_i, w_i \in W(k) \text{ for all } i, j \text{ and } v_i, w_i \in 1/p \cdot W(k), \text{ for } i \text{ odd}\}.$$ There remains the task to compute explicitly the Frobenius action. This follows from the previous presentation of $R_p^+ \cdot \mathbb{D}(E) \otimes \mathcal{O}_L$. Recall that the Frobenius acts via the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The Frobenius $Fr \otimes 1$ maps the bases for the various $\mathbb{D}_i$ in the following way (with the notation $Fr_i := Fr \otimes 1_{W(k)}$):

$$i \text{ odd } \begin{cases} \mathbb{D}_i = \langle (1/p, 0), (0, 1) \rangle & \langle Fr_i, (0, 1) \rangle > \langle (0, 1), (1, 0) \rangle \\ \mathbb{D}_{i+1} = \langle (1, 0), (0, 1) \rangle & \langle Fr_{i+1}, (0, 1) \rangle > \langle (0, 1), (1, 0) \rangle \end{cases},$$ and similarly for $i$ even i.e., $Fr_i$ is again given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. From this, we can compute the dimensions $\phi_i$ of the various quotients $\mathbb{D}_{i+1}/Fr_i(\mathbb{D}_i) = \mathbb{D}_{i+1}/p\mathbb{D}_{i+1}$. In short, $\phi_i = \begin{cases} 0 & i \text{ even} \\ 2 & i \text{ odd}. \end{cases}$

By Proposition 3.4, we know that all Dieudonné modules with invariants $\{\phi_i\}$ as above form a unique isomorphism class. On the other hand, from the above computations, the $a$-number of $A^+$ is easily seen to be $g$, so $A^+$ is also a superspecial abelian variety. Note that it does not satisfy the Rapoport condition. This is in a posteriori agreement with the classification of superspecial crystals of [25].
5. LEFT IDEAL CLASSES AND SUPERSPECIAL POINTS

From this section on, we suppose that $h^+(L) = 1$; we recall that any point on
the Hilbert moduli space is thence principally polarized. Abelian varieties and
their morphisms are assumed to be defined over $\mathbb{F}_p$, in particular, $\text{Hom}_{\mathcal{O}_L}(A_1, A_2) = \text{Hom}_{\mathcal{O}_L, \mathbb{F}_p}(A_1, A_2)$ for $A_1, A_2$ abelian varieties with RM.

We establish a link between the arithmetic of Eichler orders of level $p$ in the
quaternion algebra $B_{p,\infty} \otimes L$ and the superspecial locus of the Hilbert moduli
space. The first step is to relate left ideals of an Eichler order of level $p$ in $B_{p,L}$ and
modules of $\mathcal{O}_L$-isogenies between superspecial abelian varieties. Let $\mathcal{O}$ be an order
in a quaternion algebra. An $\mathcal{O}$-ideal $I$ is projective if and only if $I$ is locally principal
([1, Prop. 1.1]). Thus, the following is a direct corollary of the supersingular $\mathcal{O}_L$-
version of Tate’s theorem (see Theorem 2.1):

**Corollary 5.1.** Let $A_1, A_2$ be two superspecial abelian varieties with RM satisfying
the Rapoport condition. Let $\mathcal{O} = \text{End}_{\mathcal{O}_L}(A_1)$. Then $\text{Hom}_{\mathcal{O}_L}(A_1, A_2)$ is a projective
$\mathcal{O}$-module of rank one.

**Proof.** Since $T_\ell(A_i) \cong (\mathcal{O}_L \otimes \mathbb{Z}_\ell)^2$ for $\ell \neq p$, it follows from Thm. 2.1 that

$$\text{Hom}_{\mathcal{O}_L}(A_1, A_2) \otimes \mathbb{Z}_\ell \cong \text{Hom}_{\mathcal{O}_L \otimes \mathbb{Z}_\ell}(T_\ell(A_1), T_\ell(A_2)) \cong \text{End}_{\mathcal{O}_L}(A_1) \otimes \mathbb{Z}_\ell,$$

and for $\ell = p$, we use Fact 3.2. \qed

Further, we show that this geometric construction recovers all ideal classes by
establishing a bijection between left ideals classes and superspecial points. The
classical proof using the concept of kernel ideals (see [22, Thm. 3.15]) applies equally
well to hereditary orders (e.g., Eichler orders of squarefree level), and thus can be
extended to our setting (see below for a definition of kernel ideals). Unfortunately,
the classical reference for the original result of Nehrkorn is: [5, Satz 27, p.106],
whose statement concerns maximal orders. One has thus to check that the proof
uses only the fact that ideals of hereditary orders are locally principal and through
the implication that all locally principal ideals are kernel ideals, one obtains the
desired characterization. For this reason, we indicate an alternate proof not relying
on [5].

Let $A$ be a superspecial abelian variety with RM satisfying the Rapoport condi-
tion. Let $\mathcal{O} = \text{End}_{\mathcal{O}_L}(A)$. We give an explicit bijection by using the Serre tensor
construction $A \otimes_{\mathcal{O}} -$ (see [3, §7] for a description of this formalism).

**Lemma 5.2.** Let $A$ be a superspecial abelian variety with RM satisfying the Rapo-
port condition. Let $I$ be a projective rank one $\mathcal{O}$-module. Then $A \otimes_{\mathcal{O}} I$ is also
superspecial abelian variety with RM satisfying the Rapoport condition.

**Proof.** The tensor construction respects the Rapoport condition. Since $I$ is isomor-
phic to an ideal of $\mathcal{O}$, it is always possible to choose a representative that makes it
locally isomorphic to $\mathcal{O}$ at any given prime. By Theorem 2.1, this implies that the
corresponding Tate modules (resp. Dieudonné modules) are isomorphic, and thus
the abelian variety is superspecial. \qed

An argument bypassing Nehrkorn’s theorem by using Tate’s theorem is provided
by the sequence of three lemmas:
Lemma 5.3. For any two superspecial abelian varieties with \( RM A_1 \) and \( A_2 \) satisfying the Rapoport condition, there exists an \( \mathcal{O}_L \)-isogeny \( f : A_1 \rightarrow A_2 \) of degree prime to \( p \).

Proof. The idea is to approximate the isomorphism between the Dieudonné modules \( p \)-adically. All morphisms are \( \mathcal{O}_L \)-morphisms, and we drop the mention of \( \mathcal{O}_L \) from the notation. Choose \( q \) big enough so that \( A_1, A_2 \) are defined over \( \mathbb{F}_q \).

Fix an isomorphism: \( \phi \in \text{Hom}(\mathbb{D}(A_1), \mathbb{D}(A_2))^\text{Gal}(\mathbb{F}_q/\mathbb{F}_p) \cong \text{Hom}_{\mathbb{F}_q}(A_1, A_2) \otimes \mathbb{Z}_p \).

Since \( \text{Hom}_{\mathbb{F}_q}(A_1, A_2) \) is a free \( \mathbb{Z} \)-module, \( \text{Hom}_{\mathbb{F}_q}(A_1, A_2) \) dense image. Thus, there exists \( f \in \text{Hom}_{\mathbb{F}_q}(A_1, A_2) \) such that \( p^n|f - \mathbb{D}^{-1}(\phi) \), and \( p^n|f - \mathbb{D}^{-1}(\phi) \). The degree of \( \phi \) is equal to \( \text{deg}(\mathbb{D}(A_2)/\phi(\mathbb{D}(A_1))) \). Let us switch to matrix representatives: Since \( \phi \) is an isomorphism (i.e., \( \text{deg}_p(\phi) = 1 \)), the determinant of its matrix \( M(\phi) \in M_{2g}(W(\overline{\mathbb{F}_p})) \) is a unit. Call \( N_f \) the matrix representative of \( f \). Since \( p^n|N_f - M(\phi) \), it follows that \( \det(N_f) \) is a \( p \)-adic unit if we take \( r \) big enough, and thus \( \text{deg}_p(f) = 1. \)

We can associate an \( \mathcal{O}_L \)-subgroup scheme to an ideal \( J \) by taking the scheme-theoretic intersection \( A[J] := \cap_{f \in J} A[f] \). Let \( I(H) \) be the ideal of \( \mathcal{O} \) associated to a finite \( \mathcal{O}_L \)-group scheme \( H \) i.e.,

\[
I(H) := \{ f \in \mathcal{O} | f(H) = 0 \}.
\]

We say that an ideal \( J \) is a kernel ideal if \( I(A[J]) = J \).

Lemma 5.4. Every ideal \( J \) of \( \mathcal{O} \) of prime-to-\( p \) norm is a kernel ideal i.e., \( J = I(A[J]) \).

Proof. Under the prime-to-\( p \) condition, this is a straightforward computation with Tate modules. Since this is meant to replace the appeal to Nehrkhorn's theorem, we give details. First, note that \( J = I(A[J]) \) is an ideal. We compute locally: we need to show that \( \tilde{J}_q = J_q \) for \( q \nmid p \), and \( \tilde{J}_q = \mathcal{O}_q \) for \( q \mid p \). Since \( J \) is an integral ideal, we can fix isomorphisms \( \mathcal{O}_q \cong M_2(\mathcal{O}_{L_q}) \) for \( q \nmid p \), and thus since \( J \) is locally principal, there are \( m_q \in M_2(\mathcal{O}_{L_q}) \) such that \( J_q = \mathcal{O}_q m_q \). Second, write

\[
I(A[J])_q = \tilde{J} \otimes_{\mathcal{O}_L} \mathcal{O}_{L_q} \subseteq \mathcal{O} \otimes_{\mathcal{O}_L} \mathcal{O}_{L_q} = \text{End}_{\mathcal{O}_{L_q}}(T_q(A)),
\]

where \( T_q(A) \) is defined by replacing \( \ell \) by \( q \) in the definition of the Tate module.

Note that for \( H = A[J] \), \( I(A[J])_q = \{ f \in \text{End}_{\mathcal{O}_{L_q}}(T_q(A)) | f(T_q(A)) \subseteq \Lambda_q(H) \} \), where \( \Lambda_q(H) := \pi_H^{-1}(T_q(A/H))/T_q(A) \) for \( \pi_H : T_q(A) \rightarrow T_q(A/H) \). Next, we check that \( A[J]_q \cong T_q(A)/\sum_{f \in J_q} f(T_q(A)) = T_q(A)/m_q T_q(A) \) for \( q \nmid p \). Using the identifications \( \mathcal{O}_q \cong M_2(\mathcal{O}_{L_q}) \), we check easily that if \( n_q \in \text{GL}_2(\mathcal{O}_{L_q}) \) and \( n_q T_q(A) \subseteq m_q T_q(A) \), then \( n_q \in M_2(\mathcal{O}_{L_q}) m_q \). This implies that \( \tilde{J}_q \subseteq J_q \) and thus \( \tilde{J}_q = J_q \). To finish the proof, we need to check the remaining equality at \( q \mid p \): since \( A[J]_q = \{ 0 \} \), \( J_q = \mathcal{O}_q \); we get that:

\[
\tilde{J}_q = \{ f \in \text{End}_{\mathcal{O}_{L_q}}(T_q(A)) | f(T_q(A)) \subseteq T_q(A) \} = \mathcal{O}_q.
\]

Lemma 5.5. Any \( \mathcal{O}_L \)-subgroup \( H \) of \( A \) of prime-to-\( p \) order is of the form \( A[I] \) for some integral ideal \( I \) of \( \mathcal{O} \).
Proof. This is clear, as any sublattice (of finite index) in $O_L^2$ has the form $m_qO_L^2$ for some $m_q \in M_2(O_{L_q})$, $q \nmid p$. □

Note that $A/A[I] \cong \text{Hom}_O(I, A) \cong A \otimes O^{-1}$.

**Theorem 5.6.** Let $h^+(L) = 1$. Let $A$ be a superspecial abelian variety with RM satisfying the Rapoport condition. The map $I \mapsto A \otimes O I$ induces a functorial bijection from left ideal classes of $O$ to superspecial abelian varieties with RM satisfying the Rapoport condition.

Proof. Since all necessary ideals are kernel ideals, the map is injective by [22, Prop. 3.11]. Functoriality follows from the definition of the tensor construction. Surjectivity follows from the general formalism: let $A'$ be another superspecial abelian variety with RM. Then the natural map:

$$A \otimes O \text{Hom}_O(L(A, A')) \xrightarrow{\psi} A', \ a \otimes \phi \mapsto \phi(a),$$

is a well-defined isomorphism of abelian varieties with RM, and considering the projective $O$-module $\text{Hom}_O(L(A, A'))$ gives the desired preimage of $A'$. □

**Corollary 5.7.** All Eichler orders of level $p$ in the quaternion algebra $B_{p,L}$ arise from geometry.

Proof. Since Eichler orders of level $p$ are locally isomorphic ([1, Prop. 5.3]), the set of right orders of a complete set of representatives of left, projective ideal classes of any Eichler order of level $p$ represent all isomorphism classes by [21, Lem. 4.10, p.26] and [21, Cor. 5.5, p.88]. By Theorem 5.6, it is enough to consider the right orders $\text{End}_{O_L}(A_m)$ of $\text{Hom}_{O_L}(A, A_m)$ for varying $m$. □

The next step is to show that the quadratic module structure coming from the norm of the quaternion algebra can be defined geometrically. As for supersingular elliptic curves, a necessary ingredient is the existence of two isogenies of coprime degree between any two superspecial points.

Let $A$ be a fixed principally polarized superspecial abelian variety with RM. Let $G$ be the group scheme over $\text{Spec}(\mathbb{Z})$ whose group of $R$-points, for any commutative ring $R$, is:

$$G(R) = \{\phi \in (\text{End}_{O_L}(A) \otimes R)^{\times} | \phi' \phi = 1\},$$

where $\phi \mapsto \phi'$ is the Rosati involution induced by the polarisation of $A$.

Using Fact 3.2, the superspecial locus $\Lambda$ is parametrized by double cosets by a theorem of Yu ([24, Thm. 10.5]). More precisely, the set $\Lambda$ is in natural bijection with the adelic double cosets $G(\mathbb{Q}) \backslash G(k_f)/G(\overline{\mathbb{Q}})$. As in the elliptic case, a standard application of the strong approximation theorem (cf. [21, p.81]) then shows that for $\ell \neq p$, the $\ell$-power Hecke orbit of a superspecial point on the Hilbert moduli space is the whole superspecial locus (giving incidentally a stronger result than Lemma 5.3). Summing up, we get the desired:

**Corollary 5.8.** Let $A_1, A_2$ be two principally polarized superspecial abelian varieties with RM. Then for any prime $\ell \neq p$, there exists an $\ell$-power isogeny between $A_1$ and $A_2$. In particular, the module $\text{Hom}_{O_L}(A_1, A_2)$ contains two isogenies which are of coprime degrees.
There remains to give to the module $\text{Hom}_O(A_1, A_2)$ a quadratic structure by defining an associated quadratic form geometrically. The presence of polarisations and the totally real field $L$ introduces some ambiguity, so we give details.

Let $A_1, A_2$ be supersingular abelian varieties with RM. Let $\lambda_i : A_i \to A'_i, i = 1, 2,$ be principal $O_L$-polarisations, and define, for $\phi \in \text{Hom}_O(A_1, A_2)$

\[
\begin{array}{c}
A'_2 \xrightarrow{\lambda_2} A_2 \\
|\phi| = |\phi|_{\lambda_1, \lambda_2} := \lambda_1^{-1} \circ \phi' \circ \lambda_2 \circ \phi,
\end{array}
\]

Then we obtain a function which we call the $O_L$-degree:

\[|| - || : \text{Hom}_O(A_1, A_2) \to \text{End}_O(A_1).\]

**Lemma 5.9.** The $O_L$-degree $|| - ||$ takes values in $O_L$ and is a totally positive $O_L$-integral quadratic form i.e.,

1. $||\phi|| = 0$ if and only if $\phi = 0$ and $||\phi|| \gg 0$ i.e., it is totally positive for all $\phi \neq 0$;
2. $< \phi, \psi>_O := ||\phi + \psi|| - ||\phi|| - ||\psi|| = \lambda_1^{-1} \psi' \phi + \lambda_1^{-1} \phi' \lambda_2 \psi$, is a symmetric $O_L$-bilinear form;
3. $||\ell \circ \phi|| = \ell^2||\phi||$, for $\ell \in O_L$.

**Proof.** The element $||\phi||$ is fixed by the Rosati involution $f \mapsto f' = \lambda_1^{-1} f^t \lambda_1$:

\[\lambda_1^{-1} \cdot (\lambda_1^{-1} \circ \phi' \circ \lambda_2 \circ \phi)^t \lambda_1 = \lambda_1^{-1} \circ \phi' \circ \lambda_2 \circ \phi.\]

The Rosati involution fixes $L$ in $\text{End}_O(A_1)$ that is, if $A_1$ and $A_2$ are supersingular abelian varieties, it follows from Albert’s classification that we are in the Type III situation: the quaternion algebra $\text{End}_O(A_1) \otimes \mathbb{Q}$ over the totally real field $L$ is totally definite, hence the Rosati involution is the canonical involution i.e., the conjugation map i.e., $x^\sigma = \text{Tr}(x) - x = \overline{x}$ on the quaternion algebra $B_{p,L}$. Since $\lambda_1$ is principal, all computations are integral, and the image of $|| - ||$ is $O_L$.

Let us check Assertion (1). Clearly, $||\phi|| = 0$ if and only if $\phi$ is the zero map (any non-zero $O_L$-homomorphism of abelian varieties is an isogeny). The total positivity follows from properties of the embedding of the Néron-Severi group $NS^0(A)$ in $\text{End}_O(A_1)^{sym}$ via the map $\mu \mapsto \lambda_1^{-1} \mu$: the polarisations are sent to positive symmetric elements. The remaining claims are straightforward computations. \hfill $\square$

It is easy to see that the $O_L$-degree $|| - ||$ is multiplicative: if $\psi \in \text{Hom}_O(A_2, A_3)$ and $\phi \in \text{Hom}_O(A_1, A_2)$, then $||\psi \circ \phi||_{\lambda_1, \lambda_3} = ||\psi||_{\lambda_2, \lambda_3} \cdot ||\phi||_{\lambda_1, \lambda_2}$, and this property defines the quadratic form up to a constant multiple.

**Lemma 5.10.** Suppose that $A_1$ and $A_2$ are superspecial. Let $\psi \in \text{Hom}_O(A_1, A_2)$. Let $O = \text{End}_O(A_1)$. We can use $\psi$ to embed $\text{Hom}_O(A_1, A_2)$ as an $O$-ideal:

\[
\text{Hom}_O(A_1, A_2) \xrightarrow{j_\psi} \text{End}_O(A_1),
\]

\[
\phi \mapsto \lambda_1^{-1} \circ \psi' \circ \lambda_2 \circ \phi.
\]

Then

\[\text{Norm}(j_\psi(\phi)) = ||\psi'|| \cdot ||\phi||.\]

Let $I_\psi$ be the $O$-ideal $j_\psi(\text{Hom}_O(A_1, A_2))$. Then the reduced norm $N(I_\psi)$ is equal to the ideal $(||\psi'||)$.
Proof. Compute the norm of $j_\psi(\phi)$:

\[
\text{Norm}(j_\psi(\phi)) = \bar{j_\psi(\phi)}j_\psi(\phi) \\
= [\lambda_1^{-1} \circ (\lambda_1^{-1} \circ \psi^t \circ \lambda_2 \circ \phi) \circ \lambda_1] \circ [\lambda_1^{-1} \circ \psi^t \circ \lambda_2 \circ \phi] \\
= [\lambda_1^{-1} \phi^t \lambda_2 \psi][\lambda_1^{-1} \psi^t \lambda_2 \phi] \\
= [\lambda_2 \psi \lambda_1^{-1} \psi^t][\lambda_1^{-1} \phi^t \lambda_2 \phi] \\
= ||\psi^t|| \cdot ||\phi||,
\]

since $\lambda_2 \psi \lambda_1^{-1} \psi^t \in \mathcal{O}_L$. It follows that the norm of $I_\psi$, being the greatest common denominator of the norms of the elements $||j_\psi(\phi)||$, is the greatest common denominator of all $||\psi^t|| \cdot ||\phi||$, for $\phi \in \text{Hom}_{\mathcal{O}_L}(A_1, A_2)$. Any two superspecial abelian varieties admit two isogenies $\phi_1, \phi_2$ of relatively coprime degree by Corollary 5.8, thence it follows that $N(I_\psi) = (||\psi^t||)$.

\[\square\]

Summarizing the previous discussion, we obtain the desired link between the norm map and the $\mathcal{O}_L$-degree.

Proposition 5.11. Let $A_1, A_2$ be principally polarized abelian varieties with RM. Let $\mathcal{O} := \text{End}_{\mathcal{O}_L}(A_1)$. Let $\text{Hom}_{\mathcal{O}_L}(A_1, A_2) \cong I$, $I$ an integral $\mathcal{O}$-ideal. Let $\phi_x \in \text{Hom}_{\mathcal{O}_L}(A_1, A_2)$ map to $x \in I$. Then, up to a unit, the following formula holds:

\[
||\phi_x|| = \frac{\text{Norm}(x)}{\text{Norm}(I)}.
\]

The indeterminacy between the $\mathcal{O}_L$-degree and the norm of an element is thus a totally positive element well-defined up to a totally positive unit.

We recall the classical description of the theta series associated to a quadratic module. As we have seen, the $\mathcal{O}_L$-module $\text{Hom}_{\mathcal{O}_L}(A_1, A_2)$ becomes a quadratic module when equipped with the $\mathcal{O}_L$-degree. We associate to this quadratic module a theta series by the recipe:

\[
\theta_{\text{Hom}_{\mathcal{O}_L}(A_1, A_2)} := \sum_{\mathcal{O}_L \ni \nu \geq 0 \text{ or } \nu = 0} a_\nu q^\nu,
\]

where

\[a_\nu = | \{ \phi \in \text{Hom}_{\mathcal{O}_L}(A_1, A_2) \text{ such that } ||\phi|| = \nu \} |.
\]

Theorem 5.12. A theta series constructed from a quadratic $\mathcal{O}_L$-lattice $(M, q)$ of level $N$ yields a Hilbert modular form of weight 2 and quadratic character $\chi_M$ (modulo the level) given by a Gauss sum, which transforms under the group

\[
\text{SL}_2(\mathcal{O}_L \oplus N \cdot \text{Norm}(M)\mathcal{D}_L)
\]

defined as \{ \((\frac{a}{b}, \frac{c}{d}) \in \mathcal{O}_L^2 \mid \text{Norm}(M)\mathcal{D}_L(N\text{Norm}(M)\mathcal{D}_L)^{-1}) \mid ad - bc = 1\} \}, where $N$ is the level of the lattice $M$, and $\mathcal{D}_L$ is the different of $L$.

Proof. See [6, Th. 1].

\[\square\]

By specializing this result, we get the desired description of our theta series:

Corollary 5.13. Let $h^+(L) = 1$. Let $A_1, A_2$ be two principally polarized superspecial abelian varieties with RM. Then the theta series $\theta_{\text{Hom}_{\mathcal{O}_L}(A_1, A_2)}$ is a Hilbert modular form of parallel weight 2 and level $\Gamma_0(p)$ with trivial quadratic character.
All results of this section admit analogous versions for levels dividing \( p \), with the difference that the corresponding abelian varieties do not satisfy Rapoport’s condition. The essential point is that the endomorphism orders are also Eichler. We make this more precise for level one i.e., maximal orders. Let \( A^+ \) be a superspecial abelian variety with RM such that \( \mathcal{O}^+ := \text{End}_{\mathcal{O}_L}(A^+) \) is a maximal order of level one in \( B_{\infty,1}, \ldots, \infty \) \( = B_{p,\infty} \otimes L \), as constructed in Section 4. For short, we call such a superspecial abelian variety with RM an abelian variety of level one.

**Proposition 5.14.** Let \( h^+(L) = 1 \). The map \( I \mapsto A^+ \otimes_{\mathcal{O}^+} I \) induces a functorial bijection from left ideal classes of \( \mathcal{O}^+ \) to isomorphism classes of abelian varieties of level one.

**Corollary 5.15.** Any maximal order of level one in \( B_{p,\infty} \otimes L \) arises from geometry as the endomorphism order of an abelian variety of level one.

Moreover, the construction of theta series from two abelian varieties of level one goes through in the same way as in level \( p \).

**Corollary 5.16.** Let \( h^+(L) = 1 \). Let \( A_1, A_2 \) be two abelian varieties of level one. Then the theta series \( \theta_{\text{Hom}_{\mathcal{O}_L}(A_1, A_2)} \) is a Hilbert modular form of parallel weight 2 and level \( \Gamma_0(1) \) with trivial quadratic character.

## 6. The Eichler Basis Problem

The Jacquet-Langlands correspondence, as a special case of the theta correspondence, provides in a sense a solution to the Basis Problem. In classical terms, this translates in the statement that the subspace of Hilbert modular newforms can be embedded in a space of automorphic forms spanned by suitable theta series. In the setting of this paper, we want theta series coming from Eichler orders. Fortunately, this case follows readily from the literature (cf. [11, Chap. 9], where are mentioned thornier issues arising with other (special) orders). We sketch the argument. We write \( S_{2}(\mathfrak{N}, \mathbb{C}) \) for the space of quaternionic modular forms on \( B_{p,L} \) of level \( \Gamma_0(\mathfrak{N}) \) (as in [10, p.201]). The space \( S_{2}(\Gamma_0(\mathfrak{N}), \mathbb{C}) \) is the space of classical Hilbert cusp forms of weight 2 of level \( \Gamma_0(\mathfrak{N}) \).

**Theorem 6.1.** ([12, Thm. 16.1]) Suppose that \( \mathfrak{N} = \mathfrak{N}_0d(B_{p,L}) \) for an integral ideal \( \mathfrak{N}_0 \) prime to \( d(B_{p,L}) \). Then we have a Hecke-equivariant embedding \( S_{2}^{B_{p,L}}(\mathfrak{N}, \mathbb{C}) \hookrightarrow S_{2}(\Gamma_0(\mathfrak{N}), \mathbb{C}) \). The image of this embedding consists of cusp forms in \( S_{2}(\Gamma_0(\mathfrak{N}), \mathbb{C}) \) new at all primes dividing \( d(B_{p,L}) \).

It follows that if we let \( \mathcal{H}_\mathfrak{N} \) be the prime-to-\( \mathfrak{N} \) Hecke algebra in \( \text{End}(S_{2}^{B_{p,L}}(\mathfrak{N}, \mathbb{C})) \), for \( \mathfrak{N} \) squarefree, then \( S_{2}^{B_{p,L}}(\mathfrak{N}, \mathbb{C}) \) is free of rank 1 over \( \mathcal{H}_\mathfrak{N} \). We can define a map \( \Theta \) via classical theta series associated to Eichler orders (see [10, Eq. 7.9], cf. [8, p.294], [7, §10]):

\[
\Theta : S_{2}^{B_{p,L}}(\mathfrak{N}, \mathbb{C}) \otimes_{\mathcal{H}(\mathbb{C})} S_{2}^{B_{p,L}}(\mathfrak{N}, \mathbb{C}) \rightarrow S_{2}^{\text{new}}(\Gamma_0(\mathfrak{N}); \mathbb{C}), \quad (f, g) \mapsto \Theta(f, g),
\]

which is an isomorphism since \( S_{2}^{B_{p,L}}(\mathfrak{N}, \mathbb{C}) \) is free of rank 1. We state our conclusion.

**Corollary 6.2.** Let \( \mathfrak{N} \) be squarefree. The space \( S_{2}^{\text{new}}(\Gamma_0(\mathfrak{N})) \) is contained in the span of theta series coming from left ideals of an Eichler order of level \( \mathfrak{N} \) in the quaternion algebra \( B_{p,\infty} \otimes L \).
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