ON AN ALGEBRO-GEOMETRIC DISCRETIZATION OF KP HIERARCHY

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Abstract. This paper studies a certain completely integrable discretization of the KP hierarchy. This was constructed by Gieseker in [Gie1], from certain algebro-geometric data. This paper has the dual aim of showing that this construction is generically invertible, and obtaining explicit expressions for the flow equations. A subsequent article will discuss the Hamiltonian structure of this system.

1. The Relation Between the Discretization and Algebro-Geometric Data

1.1. Description of the system. We assume that $N$ and $M$ are positive integers such that $\gcd(N, M) = 1$.

Gieseker describes a family of lattice equations parameterized by $(N, M)$ in [Gie1]. These are two dimensional generalizations of the periodic Toda lattice equations and reduce to the latter when $M = 1$. The solutions of the KP equation coming from Riemann $\Theta$ functions can be approximated by solutions of the lattice equations, taking $M = N^2 + 1$ and $N \to \infty$.

Here, we are going to define the system in a roundabout way. Consider the following problem: We are looking for functions $\Psi(n, m, t)$ where $(n, m, t) \in \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z} \times \mathbb{C}$ that are almost periodic in the two space directions of the lattice, i.e.:

\begin{equation}
\Psi(n + N, m) = \alpha \Psi(n, m) \\
\Psi(n, m + M) = \beta \Psi(n, m)
\end{equation}

where $\alpha, \beta$ are independent of $(n, m)$. Moreover, we require that $\Psi(n, m + 1)$ (suppressing the time variable $t$) is expressible in terms of three of the $\Psi(k, m)$, more specifically that $\Psi(n, m + 1)$ is of the form:

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\[ \Psi(n, m + 1) = \Psi(n + 1, m) - A(n, m)\Psi(n, m) - B(n, m)\Psi(n - 1, m) \]

where \( A(n, m) \) and \( B(n, m) \) are periodic in both space entries, with respective periods \( N \) and \( M \). Given such a set of \( A(n, m), B(n, m) \) the presence of a nontrivial solution for \( \Psi \) forces an algebraic relation between \( \alpha \) and \( \beta \). There exists a matrix \( W \) such that the conditions above translate as \( \Psi \in \ker(W) \). To get this \( W \), order \( \Psi(n, m) \) keeping the second index more significant than the first (i.e. use the order \((\Psi(1, 1), \Psi(2, 1), \ldots, \Psi(N, 1); \Psi(1, 2), \ldots)\)). \( W \) then becomes an \( NM \) by \( NM \) matrix. We present it in block form with \( N \) by \( N \) blocks:

\[
W = \begin{bmatrix}
-\beta \ast I_N(1) & 0_N & 0_N & X(M) \\
X(1) & -I_N(2) & 0_N & 0_N \\
0_N & X(2) & -I_N(3) & 0_N \\
& \ddots & & \ddots \\
0_N & 0_N & X(M - 1) & -I_N(M)
\end{bmatrix}
\]

\( W \) is in block circulant form. It has two nonzero circulants. Block \((1, 1)\) of \( W \) is \( -\beta \ast I_N(1) \), and for \( i \neq 1 \), block \((i, i)\) is \( -I_N(i) \). Block \((i + 1, i)\) of \( W \) is \( X(i) \) for all \( i \). (Here, regard \( i \) in \( \mathbb{Z}/M\mathbb{Z} \).) \( I_N(i) \) and \( 0_N \) represent the \( N \) by \( N \) identity and zero matrices respectively. The sole purpose of indexing \( I_N \)’s is making references possible. \( X(m) \) is:

\[
X(m) = \begin{bmatrix}
-A(1, m) & 1 & 0 & 0 & -B(1, m)/\alpha \\
-B(2, m) & -A(2, m) & 1 & 0 & 0 \\
0 & -B(3, m) & -A(3, m) & 1 & 0 \\
& \ddots & & \ddots & \ddots \\
\alpha & 0 & 0 & -B(N, m) & -A(N, m)
\end{bmatrix}
\]

\( X(m) \) is a circulant matrix, this time with three nonzero circulants. The \((i, i)\) entry is \(-A(i, m)\), the \((i, i - 1)\) entry is \(-B(i, m)\), and the \((i, i + 1)\) entry is 1 for all \( i \) (Here regard \( i \) in \( \mathbb{Z}/N\mathbb{Z} \)).

We label entries of \( W \) with two pairs of numbers. Entry \(((n, m), (k, l))\), where \( 1 \leq n, k \leq N \) and \( 1 \leq m, l \leq M \), means the entry \((n, k)\) of block \((m, l)\) of \( W \). For instance \( W((n, m), (n, m-1)) \) is \(-A(n, m-1)\), whereas \( W((n, m), (n-1, m-1)) \) is \(-B(n, m-1)\).
In order for $W\Psi = 0$ and $\Psi$ be nontrivial, $\det(W)$ should be 0. Given a set of $A(n, m), B(n, m)$, this is the defining equation of a plane algebraic curve in the variables $\alpha$ and $\beta$. We look at a generic element of this family of curves. Such a curve has a certain definite behaviour at the loci $\beta = 0, \beta = \infty, \alpha = 0$ and $\alpha = \infty$ which we will see in a moment. Normalize $\Psi$ such that $\Psi(0, 0) = 1$. Then, apart from finitely many points of the curve $\det(W) = 0$, $\Psi(n, m)$ is a rational function in $\alpha$ and $\beta$. We realize each $\Psi(n, m)$ as sections of certain line bundles on the normalization $X$ of the curve. Under this connection, the line bundle corresponding to $\Psi(0, 0)$ turns out to be of degree $g =$ genus of the curve. We view this line bundle, $\mathcal{L}$, as a point of the lifted Jacobian parameterizing line bundles of degree $g$ on $X$.

The interesting thing about this correspondence is that it is possible to move the $A$’s and $B$’s in certain ways keeping the curve fixed. This can be seen simply by counting dimensions on either side. On the other hand, under the correspondence, these degrees of freedom precisely correspond to moving $\mathcal{L}$ in the Jacobian. From a given curve and a line bundle one can retrieve $A$’s and $B$’s.

The discrete KP hierarchy is the set of flows corresponding to moving $\mathcal{L}$ in linear directions on the Jacobian. Since the curve is fixed for these flows, one can immediately deduce that there are many conserved quantities: the coefficients of the curve equation. Since the flows on the Jacobian are linear, the discrete KP flows have to commute. This is a completely integrable system. For a discussion of these matters, see [Gie1].

1.2. The correspondence. We now look at the implications of $\det(W) = 0$. We are going to derive most of our results from information about monomials $f_{i,j}(A, B)\alpha^i\beta^j$ that appear in $\det(W)$. First of all, we state the correspondence between the algebro-geometric data and the discrete KP data. $\mathcal{X}$ denotes the normalization of the curve $\det W = 0$.

**Theorem 1.1.** There is a natural correspondence between the following sets of data:

1) A generic smooth curve $\mathcal{X}$ of genus $g$ which possesses points $P, Q$ such that $N(P - Q) = \text{div}(\alpha)$, an additional list of points $R_i, S_i$, $i = 1, \ldots, M$ so that $M(P + Q) - \sum(R_i + S_i) = \text{div}(\beta)$, where $\alpha, \beta$ are meromorphic functions on $\mathcal{X}$; and a line bundle $\mathcal{L}$ of degree $g$ on $\mathcal{X}$ such that

\begin{equation}
H^0(\mathcal{X}, \mathcal{L}((n + m - 1)P + (m - n)Q - \sum_{i=1}^{m}(R_i + S_i)) = 0
\end{equation}
for all \((n, m)\).

2) Generic functions \(A(n, m), B(n, m)\), periodic in the two space directions with periods \(N\) and \(M\).

The relation between \(g, N, M\) is \(g = (N - 1)M\). The \(\Psi(n, m)\) obtained from \(W\Psi = 0\), if properly normalized, are holomorphic sections of the line bundles \(L((n + m)P + (m - n)Q - \sum_{i=1}^{m} (R_i + S_i))\).

The construction of (2) from (1), and hence the introduction of this correspondence is the subject of [Gie1]. It will be described here later. [Gr] studies the reverse construction as well, from the point of view of deformation theory. Our approach is more combinatorial, more in the spirit of [vM-M].

The equation of the plane curve is \(\det(W) = 0\). We want to analyze the behavior of this curve when \(\beta\) or \(\alpha\) tend to \(\infty\). We make some linguistic conventions: A “monomial” will mean a full multiplicative expression involving \(\alpha, \beta, A, B\). We distinguish one specific piece of a given monomial: the “coefficient” of the monomial is the piece formed by \(A\)'s and \(B\)'s, in accordance with our treatment of \(\alpha, \beta\) as variables.

Expand the determinant using all permutations of \(n\) letters. A monomial is said to “appear” in the expansion of \(\det(W)\) if there exists a permutation \(\pi\) of \(NM\) letters so that the product associated to \(\pi\) in the expansion of \(\det(W)\) is a nonzero multiple of this monomial.

Lemma 1.1. A monomial appearing in the expansion of \(\det(W)\) with a nonconstant coefficient cannot cancel another monomial with the same properties in the evaluation of \(\det(W)\).

Proof: It will be sufficient to prove that a monomial with these properties cannot be associated to more than one permutation. Suppose that \(\pi\) is one permutation that such a monomial \(m\) is associated to. Remove the rows and columns of \(W\) that the \(A\)'s and \(B\)'s in \(m\) are on. We obtain a certain minor \(H\) of \(W\). Assign all other \(A, B\)'s the value 0, and call the new matrix \(H_0\). Clearly, \(H_0\) has at most two nonzero elements in each row or column. Suppose that there is more than one permutation on \(H_0\) that picks no zero entries. Then, there exists a row where these two permutations differ, hence collectively they pick both nonzero entries of that row. It follows that they also have to pick different entries on the columns of these entries, and so on. The complete set of entries picked by one permutation but not the other form at least one closed loop that can be traversed by changing either the row or column but not both at one move. But this is impossible. Indeed, without loss of generality suppose that the diagonal entry \(((b, a), (b, a))\) belongs in such a loop. Then, the only other nonzero entry of \(W\) in this column, \(((b - 1, a + 1), (b, a))\) should also belong
in the loop. Going one step further, the only other nonzero entry in this new row, \((b - 1, a + 1), (b - 1, a + 1)\) should belong in the loop. Continued, this list contains \((b - k, a + k), (b - k, a + k)\) for all \(k\), which exhausts the diagonal of \(W\) before coming back to the initial point since \(\gcd(N, M) = 1\). Therefore \(H = W\), which is contradictory to the assumption that \(\pi\) picks at least one \(A\) or \(B\).

The lemma says that the list of \(A, B\)'s in \(m\) determines the associated permutation \(\pi\) uniquely, if such a permutation exists.

**Definition 1.1.** We assign degrees \(d\) to multiplicative expressions in \(\alpha, \beta, A, B\) as follows:

\[
\begin{align*}
(i) & \quad d(\alpha) = N \\
(ii) & \quad d(\beta) = M \\
(iii) & \quad d(A) = 1 \\
(iv) & \quad d(B) = 2 \\
v) & \quad d(c) = 0, c \in \mathbb{C}
\end{align*}
\]

(1.6)

and the degree of a product is the sum of the degrees.

The following lemma suggests that this degree assignment is natural:

**Lemma 1.2.** If \(m\) is a nonzero monomial appearing in \(\det(W)\), then \(d(m) = NM\).

**Proof:** According to the definition, \(d((n, 1), (n, 1)) = M\), and \(d((n, m), (n, m)) = 0\) for \(m \neq 1\). Observe that, for nonzero entries of an \(X(i)\):

\[
d((n_1, m), (n_2, m - 1)) = n_1 - n_2 + 1
\]

(1.7)

Suppose that \(m\) is associated to the permutation \(\pi\). Also, suppose that the number of \(\beta\)'s that \(\pi\) picks is \(k\). Let \(S_1\) be the multiset (i.e., a collection of elements, where elements may be listed more than once) of \(n_1\) such that \(\pi((n_1, m)) = (n_2, m - 1)\) for some value of \(m\) and for some \(n_2\). Let \(S_2\) be the multiset of \(n_2\)'s that appear on the right hand side of such an equation for some \(n_1\). If there are \(t\) copies of \(n_1\) in \(S_1\), then \(\pi(n_1, m) = (n_2, m - 1)\) for \(t\) values of \(m\). But for the remaining \(N - t\) values of \(m\), the only possibility that remains is \(\pi(n_1, m) = (n_1, m)\). Now, this implies that \(n_1\) appears on the right hand sides of \(N - t\) equations of the latter form, but then \(n_1\) satisfies an equation of the form \(\pi((k, m + 1)) = (n_1, m)\) for precisely the remaining \(t\) values of \(m\). This implies:
Using this and the previous degree calculations:

\[
d(m) = \sum_{n_1 \in S_1} n_1 - \sum_{n_2 \in S_2} n_2 + M(N - k) + Mk
\]

\[(1.9)\]

\[= NM\]

\[\square\]

**Lemma 1.3.** A monomial of the form \(f(A, B)\alpha^k \beta^j\) appears in \(\det(W)\) if and only if a monomial of the form \(g(A, B)\alpha^{-k} \beta^j\) also does.

**Proof:** Suppose that \(f(A, B)\alpha^k \beta^j\) is associated to the permutation \(\pi\). By Lemma 1.1 we know that no combination of monomials in the expansion cancel. Thus it suffices to display a permutation that produces a monomial of the form \(g(A, B)\alpha^{-k} \beta^j\). We claim that there exists a unique permutation \(\pi'\) subject to the following two conditions:

- \(\pi'\) picks the entry \(((n, m), (n, m))\) of \(W\) iff \(\pi\) picks the entry \(((n, M - m + 1), (n, M - m + 1))\), and \(\pi'\) picks \(((k, m + 1), (l, m))\) iff \(\pi'\) picks \(((l, M - m + 1), (k, M - m))\). It is elementary to verify that \(\pi'\) is a permutation that doesn’t pick any zeroes. Furthermore, \(\pi\) and \(\pi'\) pick an equal number of elements from each diagonal block, in particular an equal number of \(\beta\)’s. \(\pi'\) picks \(\alpha^j\) from block \((m + 1, m)\) iff \(\pi\) picks \(\alpha^{-j}\) from block \((M - m + 1, M - m)\). Thus \(\pi'\) produces a monomial of the form \(g(A, B)\alpha^{-k} \beta^j\) \(\square\).

Notice that, by Lemma 1.2, if a monomial is of the form \(f(A, B)\alpha^k \beta^l\), then \(f(A, B)\) is of degree \(NM - kN - lM\). For another pair of exponents \((k', l')\), suppose \(NM - kN - lM = NM - k'N - l'M\). Then \((k - k')N = (l' - l)M\). Since \(\gcd(N, M) = 1\), this implies \(N|(l' - l)\). But \(0 \leq l, l' \leq N\). Thus either \(l = l'\), in which case \(k = k'\), or \(l' = N\) and \(l = 0\). But if a permutation picks \(N\) \(\beta\)’s, it has to be the identity permutation. Thus \(k' = 0\) and so \(k = M\). We conclude that except for the terms \(\alpha^0 \beta^N\) and \(\alpha^M \beta^0\) which have constant coefficients, the degree of \(f(A, B)\) determines \((k, l)\).

The following Corollary follows from Lemma’s 1.2 and 1.3.

**Corollary 1.1.** A monomial with a coefficient of degree \(d\) cannot appear in \(\det(W)\) unless \(d\) is among the following list of numbers:
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\[ 0, N, N - M, \ldots \]
\[ 2N, 2N - M, 2N - 2M, \ldots \]
\[ \ldots \]
\[ (1.10) \quad NM, NM - M, NM - 2M, NM - 3M, \ldots, M, 0 \]
\[ \ldots \]
\[ 2NM - 2N, 2NM - 2N - M, 2NM - 2N - 2M, \ldots \]
\[ 2NM - N, 2NM - N - M, \ldots \]
\[ 2NM \]

For \( 1 \leq k \leq M + 1 \), row \( k \) of this list contains the numbers \((k - 1)N - iM\) for \(0 \leq i \leq \left\lfloor \frac{(k - 1)N}{M} \right\rfloor\) which are all nonnegative. Row \( M + 1 + k \) contains \((M + k)N - iM\) for \(0 \leq i \leq \left\lfloor \frac{(k - 1)N}{M} \right\rfloor\), i.e. it has the same number of entries as row \( M + 1 - k \).

Notice that the numbers in (1.10) are situated symmetrically across the middle row. We make a definition:

**Definition 1.2.** Define \( i(k) \) to be the number symmetric to \( k \) across the middle row in the list (1.10).

Notice that \( i(k) \equiv k \mod 2 \), since by definition \( i(k) - k \) is \( 2lN \) where \( l \) is the distance between \( k \) and the middle row.

(1.10) shows the degrees of coefficients of the monomials. The corresponding \( \alpha^i \beta^j \) are:

\[
\begin{align*}
&\alpha^M \\
&\alpha^{M-1}, \alpha^{M-1} \beta^1, \ldots \\
&\alpha^{M-2}, \alpha^{M-2} \beta^1, \alpha^{M-2} \beta^2, \ldots \\
&\ldots \\
&(1.11) \quad \alpha^0, \alpha^0 \beta^1, \alpha^0 \beta^2, \alpha^0 \beta^3, \ldots, \alpha^0 \beta^{N-1}, \alpha^0 \beta^N \\
&\ldots \\
&\alpha^{-M+2}, \alpha^{-M+2} \beta^1, \alpha^{-M+2} \beta^2, \ldots \\
&\alpha^{-M+1}, \alpha^{-M+1} \beta^1, \ldots \\
&\alpha^{-M} \\
\end{align*}
\]

It turns out that each of these terms appear in \( \det(W) \) for generic \( A, B \).
We inspect the curve \( \det(W) = 0 \) when \( \alpha \) or \( \beta \) tend to \( \infty \). \( \alpha^M \) and \( \beta^N \) appear in \( \det(W) \) with nonzero constant coefficients. \( \alpha^{-M} \) has coefficient \( \Pi_{n,m}B(n,m) \); we assume that \( B(n,m) \neq 0 \) for any \( (n,m) \) so that this coefficient is not zero. If \( \alpha \) is finite and nonzero, it is impossible for \( \beta \to \infty \) on the curve, since the \( \beta^N \) term dominates the others. \( \beta = \infty \) implies \( \alpha = \infty \) or \( \alpha = 0 \), so there are two points of the curve at \( \beta = \infty \). Call these points \( \bar{P} \) and \( \bar{Q} \) respectively. Similarly, if \( \beta \) is finite it is impossible to have \( \alpha = 0 \) or \( \alpha = \infty \).

The next thing we wish to show is that the behaviour of \( \det(W) = 0 \) at \( \bar{P} \) and \( \bar{Q} \) is locally identical to the behaviour of \( \alpha^M + \beta^N = 0 \) and \( \alpha^{-M} + \beta^N = 0 \) at these points respectively. More precisely, in the normalization \( X \) of \( \det(W) = 0 \), there is only one point above either of \( \bar{P} \) and \( \bar{Q} \) (these will be denoted by \( P \) and \( Q \)). This will imply \( \text{div}(\alpha) = N(P - Q) \), and \( \text{div}_\infty(\beta) = M(P + Q) \) where \( \alpha, \beta \) on \( X \) mean composition of \( \alpha, \beta \) on \( \det(W) = 0 \) and projection from \( X \) to \( \det(W) = 0 \).

To prove this claim, we present algorithms to blow the curve up at the points \( \bar{P} \) and \( \bar{Q} \). These two algorithms are almost identical, so we explain the procedure for \( \bar{Q} \) only. Set \( \hat{\beta} = 1/\alpha \). So \( \bar{Q} \) is the point \( \alpha = \hat{\beta} = 0 \). To make all exponents nonnegative, multiply the curve equation by \( \alpha^M \hat{\beta}^N \). The terms in the curve equation are:

\[
\begin{align*}
\alpha^2 \hat{\beta}^N,
\alpha^2 \hat{\beta}^{N-1},
\alpha^2 \hat{\beta}^{N-2},
\ldots
\end{align*}
\]

This curve is singular at \( \bar{Q} \) iff both \( M \) and \( N \) are greater than 1. If there is no singularity, there is nothing to prove. So suppose that the curve is singular at \( \bar{Q} \). We blow the curve up at \( \bar{Q} \), set \( v = \alpha/\beta \) or \( v = \hat{\beta}/\alpha \) depending on whether \( N > M \) or \( M > N \) in the respective order. Without loss of generality, we assume that \( N > M \). So we eliminate
α from the equation. In the proper transform of the curve, the terms are:

\[
\begin{align*}
v^{2M} \hat{\beta}^{N+M} \\
v^{2M-1} \hat{\beta}^{N+M-1}, v^{2M-1} \hat{\beta}^{N+M-2}, \ldots \\
v^{2M-2} \hat{\beta}^{N+M-2}, v^{2M-2} \hat{\beta}^{N+M-3}, v^{2M-2} \hat{\beta}^{N+M-4}, \ldots \\
\vdots \\
v^{M} \hat{\beta}^{N}, v^{M} \hat{\beta}^{N-1}, v^{M} \hat{\beta}^{N-2}, v^{M} \hat{\beta}^{N-3}, \ldots, v^{M} \hat{\beta}^{1}, [v^{M}] \\
\vdots \\
v^{2} \hat{\beta}^{N-M+2}, v^{2} \hat{\beta}^{N-M+1}, v^{2} \hat{\beta}^{N-M}, \ldots \\
v^{1} \hat{\beta}^{N-M+1}, v^{1} \hat{\beta}^{N-M}, \ldots \\
[\hat{\beta}^{N-M}]
\end{align*}
\]

Every term except the two boxed terms contain both \( v \) and \( \hat{\beta} \) with positive exponents (this will be proven below). Hence whether or not this blow-up is singular is completely determined by whether or not \( N - M \) or \( M \) is 1. Since we assumed that \( M > 1 \), it is determined by \( N - M \) only. If \( N - M \) is not 1, we blow-up again by setting \( w = \frac{v}{\hat{\beta}} \) or \( w = \frac{1}{v} \), depending on \( N - M > M \) or \( N - M < M \), and proceed like this until one of the two exponents in the boxed terms is one.

The following Lemma explains why we can go on, by establishing that the two boxes are the deciding terms in every step of the blowing up.

**Lemma 1.4.** At any stage of the blow-up algorithm, all of the terms except the boxed terms (the term on the last row and the term on the rightmost column) contain both of the variables of that stage with positive exponents. Therefore whether or not there is a singularity is completely determined by the exponents of the boxed terms.

**Proof:** The proof uses a degree argument. Since we are using the local coordinate \( \hat{\beta} \) rather than \( \beta \), we use a degree definition slightly different from \( d \). Notice, by table (1.10) and (1.12), if we set \( \hat{d}(\alpha) = N \) and \( \hat{d}(\hat{\beta}) = M \), then the boxed terms have degree \( NM \), and all the other terms have higher degrees. Whenever a new variable is introduced at some step of the algorithm, define its degree naturally as the difference of the degrees of the variables that it is a quotient of. Notice that all of these degrees are positive because the boxed terms have equal total degree, and the comparison of exponents ensures that the new
variable is the quotient of the higher degree variable by the lower degree variable. Furthermore, notice that, with each proper transform, we are decreasing the degrees of all terms in the list by a fixed number. So the boxed terms always preserve their significance of being the only terms of lowest degree. Now suppose that the variables at some step are $v$ and $w$, boxed terms are $v^k$ and $w^l$ (so we should have $kd(v) = ld(w)$), and $k > l$. The new variable of the subsequent step is $y = \frac{w}{v}$ and we eliminate $w$ from the equation. Notice that $w$'s are replaced by $y$'s with the same exponent, so by induction they survive with a positive exponent away from the boxed terms. Contrary to the claim, suppose $v^i w^j$ is replaced by $v^s y^j$ where $s \leq 0$. But then $i + j - l = s \leq 0$. Thus $i + j \leq l$. Since $d(v) < d(w)$ this implies that $d(v^i w^j) < d(w^l)$. But this is a contradiction since $w^l$ has minimal degree among the terms. \hfill \qed

This lemma shows that we may keep blowing up until one of the exponents of the boxed terms becomes 1. But then, suppose the sequence of variables gotten in the blowing ups is $\alpha, \hat{\beta}, v_1, v_2, ..., v_k$. $\hat{\beta} = 0$ implies $v_1 = 0$, since in the curve equation at the relevant stage, all but one term contains $\hat{\beta}$. Repeating this, we have $v_2 = 0, v_3 = 0, ..., v_k = 0$. But the last one is a nonsingular point, and $v_k$ has a simple zero there. Thus there is one point $Q$ over $\bar{Q}$ in the normalization. Tracing back from the last step, from the defining relations of each new variable, it is elementary to see that $\alpha$ has a zero of order $N$ and $\hat{\beta}$ has a zero of order $M$ at $Q$. As we remarked, by a similar analysis at $\bar{P}$, there is one point $P$ above $\bar{P}$ in the normalization of the curve, $\alpha$ has a pole of order $N$, and $\beta$ has a pole of order $M$ at $P$.

Now, choose a local parameter $z$ at $P$. By the discussion above $(\alpha, \beta) = (z^{-N} + ..., z^{-M} + ...)$, where the Laurent expansions are written starting from the lowest degree terms in $z$. Similarly, at $Q$, again in a local parameter $z$, $(\alpha, \beta) = (z^N + ..., z^{-M} + ...)$. Here, again, the lowest degree terms in the expansion are shown.

We now know the behaviour of the curve at $\beta = \infty$. Notice that, when $\beta = 0$, the curve is defined by the vanishing of a polynomial of degree $2M$ in $\alpha$. Thus there are $2M$ points (counted with multiplicity) at $\beta = 0$. Call these points $R_i, S_i, i = 1, ..., M$. We will describe why they are indexed like this later. It turns out that the $2M$ coefficients of the equation are algebraically independent, hence almost any set of $2M$ points can be gotten by an appropriate choice of $A, B$.

In order to calculate the genus, we project $X$ to the $\beta$ axis, and use the Riemann-Hurwitz formula. The ramification index at $\beta = \infty$ is $M - 1$ at each of the two points, so the total is $2(M - 1)$. Let $f(\alpha, \beta) =$
\[ \frac{d}{d\alpha}(\alpha^M \det(W)) \]. Then the ramification points for finite \( \beta \) values are the preimages of points of intersection of \( f = 0 \) with \( \det(W) = 0 \). We need to calculate the degree of this divisor only. Since \( \det(W) = 0 \) is projective, this number is certainly equal to the intersection number of \( \det(W) = 0 \) and \( f = \infty \). \( f \) can be \( \infty \) only if \( \alpha \) or \( \beta \) is \( \infty \). The only points of \( \det(W) = 0 \) at this locus are \( \bar{P} \) and \( \bar{Q} \). Using the given local parameters, we inspect the behaviour of \( f \) at these points. First of all, the terms that appear in \( f \) are the \( \alpha \) derivatives of the product of \( \alpha^M \) and the terms in figure (1.11), i.e.

\[ \begin{align*}
2M & \alpha^{2M-1} \\
(2M - 1) & \alpha^{2M-2}, (2M - 1)\alpha^{M-2} \beta^1, ...
\end{align*} \]

\[ \begin{align*}
(2M - 2) & \alpha^{2M-3}, (2M - 2)\alpha^{M-3} \beta^1, (2M - 2)\alpha^{2M-3} \beta^2, ...
\end{align*} \]

\[ \begin{align*}
M & \alpha^{M-1} \beta^1, M\alpha^{M-1} \beta^2, M\alpha^{M-1} \beta^3, ..., M\alpha^{M-1} \beta^{N-1}, M\alpha^{M-1} \beta^N \\
2 & \alpha^1, 2\alpha^1 \beta^1, 2\alpha^1 \beta^2, ...
\end{align*} \]

\[ \alpha^0, \alpha^0 \beta^1, ...
\]

Therefore at \( P \), the only dominating terms are \( \alpha^{2M-1} \) and \( \beta^N \alpha^{M-1} \), and in the local parameter \( z \) both have Laurent expansions starting with \( z^{-N(2M-1)} \). These two poles cannot cancel each other, because the \( \alpha^{2M} \) and \( \alpha^M \beta^N \) terms cancel in \( \det(W) \), but different constants drop in front with \( \alpha \) differentiation.

Similarly, at \( Q \), the term \( \beta^N \alpha^{M-1} \) dominates by itself, using figure (1.14) again. Its local expansion is \( z^{-N^2M+(M-1)N} = z^{-N} + .... \)

Therefore, the intersection number of \( f = \infty \) and \( \det W = 0 \), which is the total ramification index of \( \mathcal{X} \) at finite points, equals \( 2MN - N + N = 2MN \).

Now we want to show that the projection \( \mathcal{X} \to (\beta \text{ axis}) \) for a generic choice of \( A,B \) does not have any other branch points. It is enough to show that generic \( \mathcal{X} \) is nonsingular at finite points. We will prove that \( \det(W) = 0 \) is nonsingular at finite points for generic \( A,B \). By upper semicontinuity theorem, it is sufficient to exhibit one such curve, and with no new singularities at \( \infty \). Let \( \eta \) be a primitive \( MN \)-th root of unity. Since \( \gcd(M, N) = 1 \), \( (1,1) \) generates the additive group
\(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}\). Therefore, each \((a, a)\) corresponds to a unique representative \((a, a)\) on the toroidal grid for \(0 \leq a < MN\). Let

\[
B((a, a)) = \eta^a \\
A(n, m) = 0
\]

Then, first of all \(\Pi_{n, m} B(n, m) = \eta^s\) for some \(s\). Any coefficient in the curve equation containing an \(A\) vanishes. Furthermore, suppose that the product \(c_{(i_1, j_1), \ldots, (i_k, j_k)} = B(i_1, j_1) \ldots B(i_k, j_k)\) appears as a coefficient in some monomial. Since \(W\) has toroidal symmetry, \(c_{(i_1+1, j_1+1), \ldots, (i_k+1, j_k+1)}\) also has to appear as part of the coefficient associated to the same \(\alpha^i \beta^j\). Unless \(c_{(i_1, j_1), \ldots, (i_k, j_k)}\) is the product of all \(B\)'s, these two coefficients are different. When \(B\)’s assume the values in (1.15), the ratio of the second coefficient to the first is \(\eta^k\). But \(\sum_{n=1}^{NM} \eta^{kn} = 0\) for any \(k < NM\). Thus the entire sum vanishes for such \(k\). We deduce that the curve equation for this assignment of values is:

\[
\alpha^{2M} \pm \alpha^M \beta^N \pm \eta^s = 0
\]

It easy to see that (1.16) has no singular points other than \(P\) and \(Q\) with the multiplicities described, so the claim is established.

Therefore, for the generic curve, the total branching is the calculated minimum, \(2NM + 2M - 2\), not more. By Riemann-Hurwitz formula,

\[
g = \frac{2NM + 2M - 2}{2} - 2M + 1 \\
= (N - 1)M
\]

Next, we look at the solutions \(\Psi\) of \(W\Psi = 0\). By linear algebra, the components \(\Psi(k, l)\) of \(\Psi\) should obey

\[
\frac{\Psi(k, l)}{\Psi(a, b)} = \frac{\det(W(i, j), (k, l))}{\det(W(i, j), (a, b))}
\]

for any row \((i, j)\) (recall that we use a pair of indices to indicate a row or column). Here, \(W(i, j), (k, l)\) denotes the minor of \(W\) obtained by removing row \((i, j)\) and column \((k, l)\).

Once and for all, we normalize \(\Psi\) such that \(\Psi(0, 0) = 1\). Then \(\Psi(k, l)\) is a meromorphic function on \(X - \{P, Q\}\) by (1.18). We first want to compare the behaviour of the \(\Psi(k, l)\)'s at the \(\beta\)-infinite points \(P\) and \(Q\) of \(X\). We look at \(\frac{\Psi(k+1, l)}{\Psi(k, l)}\), and \(\frac{\Psi(k, l+1)}{\Psi(k, l)}\). We make a careful choice of
(i, j) for these analyses: Use (i, j) = (k, l + 1) for both comparisons. Repeating the formula above, we have:

\[
\frac{\Psi(k + 1, l)}{\Psi(k, l)} = \frac{\det(W^{(k,l+1),(k+1,l)})}{\det(W^{(k,l+1),(k,l)})}
\]

(1.19)

and

\[
\frac{\Psi(k, l + 1)}{\Psi(k, l)} = \frac{\det(W^{(k,l+1),(k,l+1)})}{\det(W^{(k,l+1),(k,l)})}
\]

(1.20)

Although it is definitely difficult to find the determinants on the right hand side explicitly, it is much easier to determine their behaviour at \(P\) and \(Q\). Notice that the element \(((k, l + 1), (k, l))\) of \(W\) is \(A(k, l)\). This immediately implies that the expansion of \(\det(W^{(k,l+1),(k,l)})\) contains an \(\alpha^a\beta^b\) term iff the total coefficient of \(\alpha^a\beta^b\) in \(\det(W)\) contains \(A(k, l)\). The coefficients in \(\det(W)\) are toroidally symmetric, therefore we may deduce that this happens iff the coefficient of \(\alpha^a\beta^b\) contains any \(A\). From this observation, one can easily make a list of the \(\alpha^a\beta^b\) that appear in \(\det(W^{(k,l+1),(k,l)})\), but the following result will be enough for us:

**Lemma 1.5.** In the expansion of \(\det(W^{(k,l+1),(k,l)})\), the terms \(\alpha^M\), \(\alpha^{-M}\) and \(\beta^N\) are absent, the terms with coefficients of degree 1 and degree \(i(1)\) (see definition 1.2) are present.

**Proof:** The absence result is clear. The coefficients of \(\alpha^M\), \(\alpha^{-M}\) or \(\beta^N\) never contain an \(A\). For the presence result, note that 1 and \(i(1)\) are both odd numbers. Therefore a monomial consisting in only \(B\)'s, which evidently is of even degree, cannot give them. \(\square\)

We make a similar analysis for \(\det(W^{(k,l+1),(k,l+1)})\). The element \(((k, l + 1), (k, l + 1))\) of \(W\) is a diagonal entry. Hence the terms that appear in this determinant are precisely those whose coefficients in the expansion of \(\det(W)\) contain a diagonal entry, and consequently contain a \(\beta\). Unless the \(((k, l + 1), (k, l + 1))\) term is itself \(\beta\) (i.e. unless \(l = 0\)) the complete list of terms in the expansion of \(\det(W^{(k,l+1),(k,l+1)})\) is:
\[
\begin{align*}
\alpha^{M-1}\beta^1, \\
\alpha^{M-2}\beta^1, \alpha^{M-2}\beta^2, \\
\vdots \\
\alpha^0\beta^1, \alpha^0\beta^2, \alpha^0\beta^3, \ldots, \alpha^0\beta^{N-1}, \alpha^0\beta^N \\
\vdots \\
\alpha^{-M+2}\beta^1, \alpha^{-M+2}\beta^2, \\
\alpha^{-M+1}\beta^1, \\
\end{align*}
\] (1.21)

The last determinant that we need to look at is \(\det(W^{(k,l+1),(k+1,l)})\). Similarly, look at terms whose coefficients in \(\det(W)\) contain \(((k,l+1),(k+1,l))\). This is an off-diagonal 1. Again, we are going to state what we need:

**Lemma 1.6.** In the expansion of \(\det(W^{(k,l+1),(k+1,l)})\), the \(\alpha^{-M}\) and degree \(i(1)\) terms are absent, the \(\alpha^{M}\) and degree \(i(2)\) terms are present.

**Proof:** The absence of the \(\alpha^{-M}\) term and the presence of \(\alpha^{M}\) term are clear. By the symmetry of Lemma 1.3, an off-diagonal 1 appears in the degree \(i(1)\) coefficient iff a \(B\) appears in the degree 1 coefficient. This is impossible since \(d(B) = 2\). Similarly, a 1 appears in the \(i(2)\) coefficient iff a \(B\) appears in the degree 2 coefficient. This always happens: Pick \(B(r,s)\), and now set all other \(A,B = 0\) in \(W\). Now there are two nonzero elements in every row and column except the ones that \(B(r,s)\) belongs to. The only nonzero element left in column \((r,s+1)\) is \(((r-1,s+2),(r,s+1))\). Pick this. Continuing, pick all \(((r-t,s+t+1),(r-t+1,s+t))\) until this sequence comes to \(((r-2,s+1),(r-1,s))\). This point appears before \(((r,s+1),(r+1,s))\) in the sequence because \(\gcd(N,M) = 1\). Pick all the diagonal elements for the remaining rows and columns. This gives a valid permutation of the type we want. □

By using Lemmas 1.3, 1.6 and figure (1.21), we can compare the behaviour of different \(\Psi(k,l)\) at \(P\) and \(Q\).

**Proposition 1.1.** \(\frac{\Psi^{(k+1,l)}}{\Psi^{(k,l)}}\) has one pole at \(P\) and one zero at \(Q\).

**Proof:** At \(P\), the dominating term of \(\det(W^{(k,l+1),(k+1,l)})\) is \(\alpha^M\). Expanded in the local parameter, this is \(z^{-MN} + \ldots\). The dominating term of \(\det(W^{(k,l+1),(k,l)})\) is the degree 1 term, and the local series is \(z^{-MN+1} + \ldots\). Thus the claim about \(P\) is established. At \(Q\), the dominating terms have degree \(i(2)\) and degree \(i(1)\) coefficients respectively.
Thus one gets $z^{-MN+2} + \ldots$ and $z^{-MN+1} + \ldots$ in the local parameter respectively. This finishes the proof of the claim about $Q$. □

Proposition 1.2. $\frac{\Psi(k+1,l)}{\Psi(k,l)}$ has one pole at $P$ and one pole at $Q$.

Proof: At $P$, the dominating terms of $\det(W(k,l+1)(k,l))$ and $\det(W(k,l)(k,l+1))$ are $\beta^N$ and the term with degree 1 coefficient respectively. Local expansions give $z^{-MN} + \ldots$ and $z^{-MN+1} + \ldots$. And at $Q$, the dominating terms are $\beta^N$ and the term with degree $i(1)$ coefficient respectively. These give $z^{-MN} + \ldots$ and $z^{-MN+1} + \ldots$ respectively. □

So we have obtained that ratios of $\Psi$ obey this fixed structure of poles at $\beta = \infty$. Now we look at $\beta = 0$. Notice that when $\beta = 0$, $\det(W)$ splits as $\det(X(1)) \ldots \det(X(M))$. Likewise, when $\beta = 0$ the $\det(W(k,l+1)(k,l))$ and $\det(W(k,l)(k,l+1))$ split as a product of block determinants. Observe that in these block expansions, all $\det(X(m))$ appear except for $\det(X(l))$. The remaining $N-1 \times N-1$ determinant has nonvanishing determinant at points $R_i, S_i$ of the curve for generic $A,B$. Therefore, these two determinants have the same zeros on the curve at $\beta = 0$. This shows that $\Psi(k,l)$ and $\Psi(k+1,l)$ have the same divisor at $\beta = 0$. On the other hand, we saw that $\det(L(k,l+1)(k,l+1))$ vanishes identically at $\beta = 0$. Therefore $\Psi(k,l+1)$ has two more zeros than $\Psi(k,l)$ at $\beta = 0$, namely the two solutions for $\alpha$ of $\det(X(l))$. Call these two points $R_l, S_l$.

The line bundle mentioned in the correspondence comes from a divisor $D$ encoding the totality of remaining poles of $\Psi$'s:

**Definition 1.3.** Let $D$ be a minimal effective divisor such that

$$D + \text{div}(\Psi(n,m)) + (n+m)P + (m-n)Q - \sum_{i=1}^{m} (R_i + S_i) \geq 0$$

for all $(n,m)$.

We remark that such a finite $D$ exists: Suppose the condition is satisfied for $0 \leq n < N$ and $0 \leq m < M$, then it is automatically satisfied for all positive $(n,m)$. This happens since $\Psi(n+kN,m+lM) = \alpha^k \beta^l \Psi(n,m)$. The new divisors introduced by the $\alpha$'s and $\beta$'s are exactly taken care of by the extra points in definition of $D$. Define the line bundle $L = \mathcal{O}(D)$. Furthermore, set

$$(1.22) \quad L_{n,m} = L((m+n)P + (m-n)Q - \sum_{i=1}^{m} (R_i + S_i))$$

The choice of $D$ says $\Psi(n,m)$ is a section of $L_{n,m}$.  

Now, fix $m, n$, and select $k$ large enough so that $\deg(D) + k > 2g - 2$. For this particular $k$, Riemann-Roch theorem gives

\[(1.23) \quad h^0(\mathcal{L}_{n,m}(kP)) = \deg(\mathcal{L}_{n,m}) + k - g + 1\]

because $h^1(\mathcal{L}_{n,m}(kP)) = 0$ by the choice of $k$.

**Proposition 1.3.** $\Psi(n + k, m)$ is a section of $\mathcal{L}_{n,m}(kP)$ but is not a section of $\mathcal{L}_{n,m}((k - 1)P)$

**Proof:** $\Psi(n + k, m)$ is a section of $\mathcal{L}_{n,m}(kP)$ by definition. Suppose it is a section of $\mathcal{L}_{n,m}((k - 1)P)$. Then by the pole comparisons at $P$, we can replace $k$ by any nonnegative number, and the hypothesis remains true. But then $\Psi(n, m)$ is a section of $\mathcal{L}_{n,m}(-P)$, and by pole comparisons again this holds for all $(n, m)$. In particular $\Psi(0, 0) = 1$ is a section of $\mathcal{L}(-P)$, therefore $D - P$ is effective. But these imply that we can replace $D$ by $D - P$, which contradicts the minimality of $D$. □

From the proposition we deduce that

\[(1.24) \quad h^0(\mathcal{L}_{n,m}(kP)) = \deg(D) + k - g + 1\]

for all $k$, in particular $h^0(\mathcal{L}_{n,m}) = \deg(D) - g + 1$. Indeed, increasing $k$ by one always gives a new section, therefore, $h^1$ should be zero at each step, because it is eventually zero.

The divisor $D$ is of degree $g$. For a proof of this, we refer the reader to [Gr].

This concludes the construction of the algebro-geometric data from the difference operator. We now summarize the construction of the KP data from the algebro-geometric data.

Assume $\mathcal{X}$ is a smooth curve of genus $g$, $P, Q, R_i, S_i$ points on $\mathcal{X}$ such that $\text{div}(\alpha) = N(P - Q)$ and $\text{div}(\beta) = M(P + Q) - \sum_{i=1}^{M}(R_i + S_i)$, $\mathcal{L}$ a line bundle of degree $g$ on $\mathcal{X}$ such that

\[(1.25) \quad h^0(\mathcal{X}, \mathcal{L}((n + m - 1)P + (m - n)Q - \sum_{i=1}^{m}(R_i + S_i))) = 0\]

for all $m, n$. Let

\[(1.26) \quad \mathcal{L}_{n,m} = \mathcal{L}((n + m)P + (m - n)Q - \sum_{i=1}^{m}(R_i + S_i))\]

as before. Then,

**Proposition 1.4.** $h^0(\mathcal{X}, \mathcal{L}_{n,m}) = 1$ for each $n, m$. 
Proof: By Riemann-Roch theorem, $h^0(\mathcal{X}, \mathcal{L}_{n,m}) \geq g + 1 - g = 1$. On the other hand, it is impossible to have strict inequality because of the regularity assumption. (Removing one $P$ prohibits at most one section.) □

Say $\Psi(n, m)$ is a nonzero section of $\mathcal{L}_{n,m}$.

Proposition 1.5. The set of sections $\{\Psi(n + 1, m), \Psi(n, m), \Psi(n - 1, m), \Psi(n, m + 1)\}$ is linearly dependent.

Proof: Notice that all four of them are sections of $\mathcal{L}_{n,m}(P + Q)$. On the other hand, by Riemann-Roch theorem, we get $h^0(\mathcal{L}_{n,m}(P + Q)) \geq 3$, and by the regularity assumption again, equality has to be realized. □

The ratio $f_{n,m} = \frac{\Psi(n, m)}{\Psi(0, 0)}$ has a pole of order precisely $n + m$ at $P$.

Suppose $z$ is a uniformizing parameter at $P$. We can normalize $\Psi(n, m)$ so that $f_{n,m}z^{n+m} = 1$ at $P$. We can also normalize the meromorphic functions $\alpha, \beta$ such that $\alpha = z^{-N} + ..., \beta = z^{-M} + ....$

After these normalizations it is elementary to show that the linear dependence between the four sections in Prop 1.5 has to be of the form

\[
(1.27) \\
\Psi(n + 1, m) - \Psi(n, m + 1) - A(n, m)\Psi(n, m) - B(n, m)\Psi(n, m) = 0
\]

for some functions $A, B$ on $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$.

Thus we get back to $A,B$, having started from the algebro-geometric data. Next, we wish to calculate the flow equations explicitly by computing the effect of moving $\mathcal{L}$ in the Jacobian in certain directions.

2. The flow equations

2.1. Some functions on $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$. As before, suppose that $N, M \in \mathbb{Z}$, and $\gcd(N, M) = 1$. Let $S$ denote the set of functions $f: \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z} \rightarrow \{-1, 0, 1\}$.

Proposition 2.1. There is a unique function $\kappa$ in $S$ that satisfies the following conditions:

\[
(i)\kappa(0, 0) - \kappa(1, -1) = -1 \\
(ii)\kappa(0, 1) - \kappa(1, 0) = 1 \\
(iii)\kappa(-1, 0) - \kappa(0, -1) = 1 \\
(iv)\kappa(-1, 1) - \kappa(0, 0) = -1
\]

(2.1)

and except for these four values of $(i, j)$, $\kappa(i - 1, j + 1) = \kappa(i, j)$
Proof: Uniqueness is easy to prove, because if two such functions exist, their difference has to be a constant. But by (i) and (iv) the only possibility for $\kappa(0, 0)$ is 0. Therefore the constant is zero.

To prove existence, we note that since $\gcd(N, M) = 1$, $(-1, 1)$ is a generator. Look at the sequence

$$(2.2)\quad (-1, 1), (-2, 2), ..., (-a, a), ...$$

We will distinguish the two cases below:

1) Suppose in the sequence (2.2), $(1, 0)$ appears before $(-1, 0)$. Then we declare

$$\kappa(-1, 1) = \kappa(-2, 2) = ... = \kappa(1, 0) = -1$$

and $\kappa(a, b) = 0$ if $(a, b)$ is not in these lists.

2) Suppose $(-1, 0)$ appears before $(1, 0)$. We declare

$$\kappa(-1, 1) = \kappa(-2, 2) = ... = \kappa(0, 1) = 1$$

and $\kappa(a, b) = 0$ if $(a, b)$ is not in these lists.

One can check that $\kappa$ satisfies the conditions that we asked for. $\blacksquare$

Remark: One may wonder which case happens when. It turns out that the deciding quantity is the parity of the number of steps in the Euclidean algorithm for the ordered pair $(N, M)$. In particular we have alternate cases for $(N, M)$ and $(M, N)$.

Note that $\kappa(n, m) = -\kappa(-n, -m)$.

We introduce another function $\rho$ in $S$:

**Definition 2.1.**

$$(2.5)\quad \rho(n, m) = \kappa(n + 1, m) + \kappa(n, m) + \delta_{(n, m), (0, 0)} - \delta_{(n, m), (-1, 0)}$$

$$(2.6)\quad \phi(n, m) = -\rho(-n - 1, -m) - \rho(-n, -m)$$

$(\delta$ is the Kronecker delta function, i.e. $\delta_{X,Y} = 1$ if $X = Y$, and 0 otherwise, etc.)

Then the following holds:

$$\rho(n - 1, m + 1) - \rho(n, m) = (\kappa(n, m + 1) - \kappa(n + 1, m))$$

$$+ (\kappa(n - 1, m + 1) - \kappa(n, m))$$

$$+ \delta_{(n-1,m+1), (0, 0)} - \delta_{(n-1,m+1), (-1,0)}$$

$$- \delta_{(n,m), (0,0)} + \delta_{(n,m), (-1,0)}$$

$$= (2.7)$$
Therefore $\rho$ is the unique function in $S$ satisfying the following conditions:

\[
\begin{align*}
(i) \rho(-2, 0) - \rho(-1, -1) &= 1 \\
(ii) \rho(0, 1) - \rho(1, 0) &= 1 \\
(iii) \rho(-1, 0) - \rho(0, -1) &= -1 \\
(iv) \rho(-1, 1) - \rho(0, 0) &= -1 \\
\end{align*}
\]

and $\rho(n - 1, m + 1) = \rho(n, m)$ for all other $(n, m)$.

2.2. Derivation of the discrete KP equations. We would like to compute what happens to $A$ and $B$ when we keep $X$ and the points fixed, but deform the line bundle $L$ linearly in the Jacobian. By choosing the direction of deformation carefully, the computation of the first few evolution equations for $A$ and $B$ becomes feasible.

$X$ is mapped inside its Jacobian by the Abel-Jacobi map. We are going to deform the line bundle in the tangent direction to the curve at $P$. Assume again that $z$ is a local parameter at $P$. For small $t \in \mathbb{C}$, define

\[
L_{n,m,t} = L_{n,m}(P - T)
\]

where $T = z^{-1}(t)$. First, notice that $h^0(L_{n,m,t}(-P)) = 0$ by upper semicontinuity for small $t$, since $h^0(L_{n,m}(-P)) = 0$. Thus $h^0(L_{n,m,t}) = 1$. Suppose $\Psi(n, m, t)$ is a nonzero section.

**Proposition 2.2.** The set $\{\Psi(n, m, t), \Psi(n, m), \Psi(n + 1, m)\}$ is linearly dependent.

**Proof:** This follows from the observation that all three are sections of $L_{n,m,t}(T)$ and $h^0(L_{n,m,t}(T)) = 2$ by Riemann-Roch theorem. □

Thus suppose

\[
\begin{align*}
\Psi(n, m, t)(z) &= a_0(n, m, t)\Psi(n, m)(z) + a_1(n, m, t)\Psi(n + 1, m)(z) \\
\end{align*}
\]

where $a_0$ and $a_1$ are holomorphic in $t$. Now, $\Psi(n, m, t)$ vanishes at $z = t$ by definition. Therefore

\[
\begin{align*}
a_0(n, m, t)\Psi(n, m)(t) + a_1(n, m, t)\Psi(n + 1, m)(t) &= 0 \\
\end{align*}
\]

Now we can expand this expression in powers of $t$ and calculate the first few terms of the power series expansions of the holomorphic functions $a_i$ in terms of the Laurent coefficients of the function $\frac{\Psi(n, m)}{\Psi(n, 0)}$ whose pole behaviour at $P$ we know. Let’s give a name to these coefficients.
First define
\begin{equation}
  f(n, m) = \frac{\Psi(n, m)}{\Psi(0, 0)}
\end{equation}

Then let \( d_1(n, m) \) be such that
\begin{equation}
  f(n, m) = z^{-n-m} + d_1(n, m)z^{-n-m+1} + d_2(n, m)z^{-n-m+2} + ...
\end{equation}

Then since
\begin{equation}
  \Psi(n+1, m) = \Psi(n, m+1) + A(n, m)\Psi(n, m) + B(n, m)\Psi(n-1, m)
\end{equation}
we have
\begin{equation}
  d_1(n, m) - d_1(n, m + 1) = A(n, m)
\end{equation}

Now we can expand (2.11) in powers of \( t \). We define one more piece of notation for this purpose: say \( a_i(n, m, t) = \sum_j a_i,j(n, m)t^j \).

One sees immediately that \( a_{1, 1}(n, m) = -1 \), therefore we can normalize \( \Psi(n, m, t) \) in a neighborhood of \( P \), replacing it by \( \Psi(n, m, t)/a_1(n, m, t) \).

Then if we compute the coefficient of \( t^{-n-m+1} \) we get
\begin{equation}
  a_{0, 1}(n, m) = d_1(n + 1, m) - d_1(n, m)
\end{equation}

On the other hand, one can expand the relation
\begin{equation}
  \Psi(n + 1, m, t) = \Psi(n, m + 1, t) + A(n, m, t)\Psi(n, m, t) + B(n, m, t)\Psi(n - 1, m, t)
\end{equation}
using equation (2.10). One can further write all \( \Psi(k, l) \) in terms of \( \Psi(n + i, m) \) by reducing \( l \) to \( m \) using the difference relation repeatedly. Finally, one gets a time dependent linear relation between \( \Psi(k, m) \), where \( m \) is fixed. But these \( \Psi \) all have different order poles at \( P \) as we have seen, so they are linearly independent. Therefore many time dependent quantities (that one may calculate from the last equation) ought to vanish. The coefficients of \( t \) in front of \( \Psi(n, m) \) and \( \Psi(n-1, m) \) give evolution equations for \( A \) and \( B \):

\begin{align}
  \dot{A}(n, m) &= a_{1, 1}(n, m + 1)B(n + 1, m) + (a_{0, 1}(n, m + 1) - a_{0, 1}(n, m))A(n, m) \\
  &
  - a_{1, 1}(n - 1, m)B(n, m) \\
  \dot{B}(n, m) &= (a_{0, 1}(n, m + 1) - a_{0, 1}(n - 1, m))B(n, m)
\end{align}
Writing \( a \)'s in terms of \( d \)'s, equations become

\[
\dot{A}(n, m) = B(n, m) - B(n + 1, m) + \left( d_1(n + 1, m + 1) - d_1(n, m + 1) - d_1(n + 1, m) + d_1(n, m) \right) A(n, m)
\]

\[
\dot{B}(n, m) = (d_1(n + 1, m + 1) - d_1(n, m + 1) - d_1(n + 1, m) + d_1(n, m)) B(n, m)
\]

\( d_1(0, 0) = 0 \) by definition. Now, in the light of (2.15), the terms in both parentheses are linear polynomials in \( A \)'s.

\[
\dot{A}(n, m) = \sum g_{(n,m),(k,l)} A(k, l) A(n, m)
\]

\[
\dot{B}(n, m) = \sum h_{(n,m),(k,l)} A(k, l) B(n, m)
\]

By toroidal symmetry,

\[
g_{(n+a,n+b),(k+l,m)} = g_{(n,m),(k,l)}
\]

for any \((a, b) \in \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}\). Same result holds for \( h \). Thus, with some abuse of notation, we may write

\[
\dot{A}(n, m) = B(n, m) - B(n + 1, m) + \left( \sum g_{(k-n,l-m)} A(k, l) \right) A(n, m)
\]

\[
\dot{B}(n, m) = \left( \sum h_{(k-n,l-m)} A(k, l) \right) B(n, m)
\]

We subtract the terms of the first equation of (2.22) in parentheses for \((n, m)\) from the same terms for \((n + 1, m - 1)\):

\[
\left( \sum g_{(k-n-1,l-m+1)} A(k, l) \right) - \left( \sum g_{(k-n,l-m)} A(k, l) \right) =
\]

\[
\left( d_1(n + 2, m) - d_1(n + 1, m) - d_1(n + 2, m - 1) + d_1(n + 1, m - 1) \right)
\]

\[
- \left( d_1(n + 1, m + 1) - d_1(n, m + 1) - d_1(n + 1, m) + d_1(n, m) \right)
\]

\[
= A(n + 1, m) - A(n, m) - A(n + 1, m - 1) + A(n, m - 1)
\]

And we do the same for the second equation:

\[
\left( \sum h_{(k-n-1,l-m+1)} A(k, l) \right) - \left( \sum h_{(k-n,l-m)} A(k, l) \right) =
\]

\[
\left( d_1(n + 2, m) - d_1(n + 1, m) - d_1(n + 1, m - 1) + d_1(n, m - 1) \right)
\]

\[
- \left( d_1(n + 1, m + 1) - d_1(n, m + 1) - d_1(n, m) + d_1(n - 1, m) \right)
\]

\[
= A(n + 1, m) - A(n, m) - A(n, m - 1) + A(n - 1, m - 1)
\]
Equating the coefficients for all \((k, l)\) we obtain

\[
\begin{align*}
(i) & \quad g(0, 1) - g(1, 0) = 1 \\
(ii) & \quad g(-1, 1) - g(0, 0) = -1 \\
(iii) & \quad g(0, 0) - g(1, -1) = 1 \\
(iv) & \quad g(-1, 0) - g(0, -1) = 1
\end{align*}
\]

(2.25)

and \(g(n + 1, m - 1) = g(n, m)\) for any other \((n, m)\).

(2.26)

and \(h(n + 1, m - 1) = h(n, m)\) for any other \((n, m)\)

These are precisely the relations for \(\kappa\) and \(\rho\), so we have proven:

**Proposition 2.3.** The equations of evolution are

\[
\begin{align*}
\dot{A}(n, m) & = B(n, m) - B(n + 1, m) + \left( \sum \kappa(k-n, l-m)A(k, l) \right)A(n, m) \\
\dot{B}(n, m) & = \left( \sum \rho(k-n, l-m)A(k, l) \right)B(n, m)
\end{align*}
\]

(2.27)
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