One loop radiative corrections to the translation-invariant noncommutative Yukawa Theory

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Abstract
We elaborate in this paper a translation-invariant model for fermions in 4-dimensional noncommutative Euclidean space. The construction is done on the basis of the renormalizable noncommutative translation-invariant $\phi^4$ theory introduced by Gurau et al. 2009 Commun. Math. Phys. 287 275–90. We combine our model with a scalar model, in order to study the noncommutative pseudo-scalar Yukawa theory. After we derive the Feynman rules of the theory, we perform an explicit calculation of the quantum corrections at one loop level for the propagators and vertices.

Keywords: loop corrections, noncommutative field theory, renormalization

1. Introduction

In the last two decades, a lot of work has been devoted to the study of noncommutative quantum field theories. The main idea behind these theories is that at the Planck scale space–time is no longer commutative: this fact makes the noncommutative geometry an essential ingredient when probing space–time structure at very small distances [1, 2]. The original motivation for investigating such theories was the hope of solving the problem in infinities of quantum field theory and for the possible formulation of consistent quantum gravity.
Despite the collective effort made by physicists, none of these goals has yet been reached.

In fact, instead of the elimination of ultraviolet infinities, the use of noncommutative geometry in quantum field theories gives rise to a new set of problems and makes the short distance behavior of those theories more ambiguous [5–7]. The conventional theories became non-renormalizable due to the infamous ultraviolet/infrared mixing. Many attempts have been made to overcome this UV/IR mixing, but in general, the problem persists.

However, there are a few models in which renormalizability was restored. It was achieved by adding a suitable term to the initial action of the theory. The procedure was first used by Grosse and Wulkenhaar [8–11] to solve the UV/IR mixing of the noncommutative Euclidean $\varphi^4$ theory. In their model they added a harmonic oscillator term which depends explicitly on the Moyal space coordinates $\hat{x}_i \hat{\varphi}^2$, where $\hat{x}_i = (\theta^{-1})_{\mu \nu} x^\mu$. It turns out that the model is covariant under Langmann–Szabo duality [12] but also breaks the translation invariance of the action.

Another approach, using the same method, was proposed by R. Gurau et al [13]. This model preserves the translation invariance of the noncommutative $\varphi^4$ theory. The term added to the action is in fact a non-local counter-term of the form $\frac{1}{m^2} \varphi^2$, which is written in momentum space as $\frac{1}{p^2}$. This model is known as the translation-invariant $1/p^2$-model. The UV/IR mixing problem was solved by the elimination of the quadratic IR divergence of non-planar diagrams. Both of these scalar models were constructed on the Moyal space and were proven to be renormalizable to all orders in perturbation theory.

The noncommutative fermion theory was also formulated in the case of the Gross–Neveu model [14]. Following the same procedure as the Grosse and Wulkenhaar model, the term added to the action is $i \bar{\psi} \gamma^\mu (\theta^{-1})_{\mu \nu} \psi$. This model was proven to be renormalizable to all orders in perturbation theory, but unlike the noncommutative $\varphi^4$ models, it still presents a UV/IR mixing even after renormalization. In fact, the Gross–Neveu model is renormalizable even without adding an extra term.

Motivated by the renormalizable noncommutative translation-invariant $1/p^2$-model, and since it has not been extended to fermions, we propose the construction of its fermionic version. It is well known in ordinary quantum field theory that the scalar propagator is perceived as the square of the Dirac propagator; indeed we have

$$\hat{G}(p^2) = \frac{1}{p^2 + m^2} = \frac{1}{i \not{p} + m} \times \frac{1}{-i \not{p} + m},$$

(1)

this means also that the scalar propagator appears naturally in the expression of the Dirac propagator

$$\hat{D}(p) = \hat{G}(p^2)(i \not{p} + m),$$

(2)

where $\hat{D}(p)$ is the Dirac propagator and $\hat{G}(p^2)$ is the scalar propagator, here expressed in their Euclidean forms.

It seems reasonable to impose condition (1) in the noncommutative case if we want to have a consistent theory that involves both scalar and fermion fields. Thus, our starting point is the construction of a model in which the modified scalar and fermion propagators are correlated in the same way as in the ordinary quantum field theory. The extra term in the fermionic action is chosen accordingly.

The consistency of our model relies on the fulfillment of condition (1), but this does not guarantee its renormalizability. This is why we apply it, in addition to the scalar model, to study the noncommutative pseudo-scalar Yukawa theory. We recall that the Yukawa
interaction between a pseudo-scalar field \( \varphi \) and a Dirac field \( \psi \) is represented in Euclidean space by the action

\[
S[\psi, \bar{\psi}, \varphi] = \int \text{d}^4x \bar{\psi} \gamma^5 \psi \varphi;
\]  

(3)

this interaction is used in the standard model to describe the coupling of Higgs particle with fermions. The calculation of the quantum corrections at one loop level enables us to test the consistency of the whole model and its renormalizability. Furthermore, it reveals more about the behavior of these modified models and allows us to improve them if necessary.

We note here that the method used in the renormalizable models gave an alternative approach to construct noncommutative field theories free of UV/IR mixing. So, it was natural to extend these models to noncommutative gauge field theory, hoping to have the same success. However, unfortunately this method failed to solve the UV/IR mixing problem, although several promising approaches were made [15–21]. Currently, there is no explicit procedure to deal with this problem.

This paper is organized as follows: in the next section we define our model and derive its Feynman rules. In section 3 we perform explicit Feynman graph calculations at one loop level in order to evaluate the radiative contributions to the scalar and the fermion propagators and the Yukawa and \( \varphi^4 \) vertices. Section 4 is devoted to remarks and conclusions.

2. The model

The realization of the noncommutative modified \( \varphi^4 \) models cited above was achieved by the substitution of the ordinary product between fields by the Weyl–Moyal star \( \star \) product [22]

\[
f(x) \star g(x) \equiv e^{i\theta_{\mu\nu} \frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial y^\nu} f(x)g(y)|_{x=y}.
\]  

(4)

This approach is considered to be the simplest way to construct a noncommutative field theory; the coordinates fulfill the commutation relation

\[
[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu},
\]  

(5)

where \( (\theta^{\mu\nu}) \) is the deformation matrix, which is assumed to have a simple block-diagonal form

\[
(\theta^{\mu\nu}) = \theta
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\]  

(6)

Here, \( \theta \) is the deformation parameter; it is taken to be real and gives a measure of the strength of noncommutativity. Throughout this paper we use the Euclidean metric, the Feynman convention \( \alpha = \gamma^a a^a \) and the notation \( \bar{\partial}^\mu = (\theta^{\mu\nu}) a^\nu \).

The free scalar action of the translation-invariant \( 1/p^2 \)-model is [13]

\[
S_\varphi[\varphi] = \int_{\mathbb{R}^4} \text{d}^4x \frac{1}{2} \left[ \bar{\partial}^\mu \varphi \star \partial^\mu \varphi + M^2 \varphi \star \varphi - a'^2 \varphi \star \frac{1}{\theta^2} \varphi \right],
\]  

(7)

in Euclidean space. In the expression (7), the parameter \( a' \) is a real dimensionless constant. The modified scalar propagator in momentum space is then
\[ G'(p^2, M, a') = \frac{1}{p^2 + M^2 + \frac{a'}{\theta^2}}. \]  

In order to recover the modified scalar propagator from the square of the fermion propagator, as in the commutative theory (1), we propose the modification of the free fermion action in the following way

\[ S_f[\psi, \bar{\psi}] = \int dx^4 \left[ \bar{\psi} \gamma^5 \psi + m \bar{\psi} \psi - b' \bar{\psi} \gamma^5 \psi + \frac{\bar{\psi} \gamma^5 \psi}{\theta^2} \right], \]  

where \( b' \) is a real dimensionless constant. We have added an extra term \( b' \bar{\psi} \gamma^5 \psi \) to the original fermion action which reads in momentum space \( \sim \frac{p}{\theta^2} \).

The Yukawa theory in four dimensional Euclidean space includes the \( \varphi^4 \) self interaction in order to be renormalized; the noncommutative interaction action is thus

\[ S^{\text{int}}_4 = \int dx^4 \left[ \left( c_1 \bar{\psi} \gamma^5 \psi \right) \varphi + c_2 \bar{\psi} \gamma^5 \psi + c_3 \bar{\psi} \gamma^5 \psi \right] \]  

with the use of the trace property of the star product \[23\]

\[ \int (f \star g \star h)(x) d^4x = \int (h \star f \star g)(x) d^4x = \int (g \star h \star f)(x) d^4x, \]  

the pseudo scalar Yukawa action reduces to

\[ S^{\text{int}}_4[\psi, \bar{\psi}, \varphi] = \int dx^4 \left[ g_1 \bar{\psi} \gamma^5 \psi \varphi + g_2 \bar{\psi} \gamma^5 \varphi \psi \right], \]  

where \( g_1 = c_1 + c_2 \) and \( g_2 = c_3 \).

The total action of our model reads

\[ S^\text{tot}_4[\psi, \bar{\psi}, \varphi] = S_f[\varphi] + S_f[\psi, \bar{\psi}] + S^{\text{int}}_4[\psi, \bar{\psi}, \varphi] + S^\text{ct}_4[\psi, \bar{\psi}, \varphi], \]  

where \( \varphi \) and \( \psi \) are the dressed fields and \( \psi_0 \) and \( \varphi_0 \) are the bare fields; we used as usual the substitution

\[ \psi_0 = \sqrt{Z_\psi} \psi \quad \text{and} \quad \varphi_0 = \sqrt{Z_\varphi} \varphi. \]  

The counter-terms action is then

\[ S^\text{ct}_4[\psi, \bar{\psi}, \varphi] = \int dx^4 \left[ \left( \frac{1}{2} \delta_\nu \partial^\mu \varphi \star \partial_\mu \varphi + \delta M \varphi \star \varphi - \frac{1}{4!} \varphi \star \varphi \right) \right] \]  

\[ + \int dx^4 \left[ \delta_\nu \bar{\psi} \gamma^5 \bar{\psi} \star \delta_\mu \gamma^5 \varphi + \delta_\nu \psi \gamma^5 \varphi + \delta_\mu \bar{\psi} \gamma^5 \psi - \frac{1}{4!} \varphi \star \varphi \star \varphi \right]. \]
where the renormalization factors are
\[
\begin{align*}
\delta \varphi &= Z_\varphi - 1, & \delta \psi &= Z_\psi - 1, \\
\delta M &= M_0^2 Z_\varphi^2 - M^2, & \delta m &= m_0 Z_\psi - m, \\
\delta \lambda &= \lambda_0 Z_\varphi^2 - \lambda, & \delta g_i &= g_{0i} Z_\psi Z_{\varphi}^{1/2} - g_i, \\
\delta a^2 &= a_0^2 Z_\varphi^2 - a^2, & \delta b' &= b_0' Z_\psi - b'.
\end{align*}
\]
(16)

The different actions written above are used next to derive the Feynman rules for the propagators and vertices.

2.1. Propagators

The noncommutative free theory is the same as the commutative one [23], the action remains unchanged, and this is due to the relation
\[
\int (f \ast g)(x) d^4x = \int (f \cdot g)(x) d^4x.
\]
(17)

Even when the actions are modified by adding some extra terms, the propagators are calculated using the same techniques as for ordinary quantum field theory. The modified scalar propagator in momentum space (8) is written as
\[
\tilde{G}'(p^2, M, a) = \frac{1}{p^2 + M^2 + \frac{a^2}{4}}.
\]
(18)

where \(a = \frac{a'}{2}\). It is possible to rewrite this propagator, in a more suitable form [24], in order to evaluate the Feynman integrals by the use of the usual mathematical techniques
\[
\frac{1}{p^2 + M^2 + \frac{a^2}{4}} = \frac{1}{2} \sum_{\zeta = \pm 1} \frac{1 + \zeta \frac{M^2}{2A^2}}{p^2 + \frac{M^2}{2A^2} + \zeta A^2},
\]
(19)

where \(A^2 = \frac{M^2}{2} - a^2\). If we use Schwinger's exponential parametrization, with \(M > 0\) and \(a \neq 0\), the propagator is then
\[
\tilde{G}'(p^2, M, a) = \frac{1}{2} \sum_{\zeta = \pm 1} \left(1 + \frac{M^2}{2A^2}\right) \int_0^\infty e^{-\left(p^2 + \frac{M^2}{2A^2} + \zeta A^2\right)a_0} da_0.
\]
(20)

The modified fermion propagator is calculated from action (9), and we obtain
\[
\tilde{D}'(p, m, b) = \frac{1}{-i p + m + i b \frac{p^2}{2p^2}},
\]
(21)

where \(b = \frac{b'}{2}\). This propagator fulfills condition (1), or in this case
\[
\frac{1}{p^2 + m^2 + \frac{b^2}{4}} = \frac{1}{-i p + m + i b \frac{p^2}{2p^2}} \times \frac{1}{i p + m + i b \frac{p^2}{2p^2}}.
\]
(22)

thereafter, the modified scalar propagator is naturally recovered in the expression of \(\tilde{D}'\)

\[
\tilde{D}'(p, m, b) = \tilde{G}'(p^2, m, b) \left(\frac{1}{-i p + m + i b \frac{p^2}{4p^2}}\right).
\]
(23)

as a consequence, the fermion propagator reproduces the same ‘damping’ behavior for vanishing momentum As the modified scalar propagator [24]
\[ \lim_{p \to 0} \tilde{D}'(p, m, b) = 0. \]  

(24)

### 2.2. Vertices

The Feynman rule in momentum space for the \( \varphi^4 \) self interaction vertex is given by [25]

\[ \mathcal{V}(p_1, p_2, p_3, p_4) = V_{\lambda}(p_1, p_2, p_3, p_4), \]

(25)

where \( V_{\lambda}(p_1, p_2, p_3, p_4) = -\frac{\lambda}{4} (\cos \frac{p_1 \theta}{2} \cos \frac{p_2 \theta}{2} + \cos \frac{p_3 \theta}{2} \cos \frac{p_4 \theta}{2} + \cos \frac{p_1 \theta}{2} \cos \frac{p_4 \theta}{2}) \). We notice the factor that appears in the noncommutative case, however in the commutative limit \( \theta \to 0 \), it vanishes and we recover the ordinary \( \varphi^4 \) vertex: \( V_{\lambda} \to -\lambda \).

The Feynman rule for the Yukawa interaction vertex in momentum space is calculated from the Yukawa action (12) using the Fourier transformation of the fields

\[ \psi(x) = \int \frac{d^4p}{(2\pi)^4} \tilde{\psi}(p)e^{-ipx}, \quad \tilde{\psi}(x) = \int \frac{d^4q}{(2\pi)^4} \tilde{\psi}(q)e^{ipx}, \quad \varphi(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{\varphi}(k)e^{-ikx}, \]

thus

\[ \mathcal{V}_{\psi}(p, p') = -\gamma V_{\varphi}(p', p), \]

(27)

where \( V_{\varphi}(p', p) \) is a phase factor

\[ V_{\varphi}(p', p) = \left[ g_1e^{ip' \theta} + g_2e^{-ip' \theta} \right] = \sum_{\sigma = \pm 1} g_{\sigma} e^{i\sigma \theta p' \theta}. \]

(28)

In the last expression we used the notation \( g_2 \equiv g_{-1} \). The Yukawa interaction in our model is represented by two coupling constants \( g_1 \) and \( g_2 \); the commutative coupling constant is recovered when \( \theta \to 0 \): \( V_{\varphi} \to g \) where \( g = g_1 + g_2 \).

We note, finally, that the modification of the ordinary commutative vertices is a natural consequence of the introduction of the Moyal star product, unlike the propagators which are modified artificially by adding the extra terms to the actions.

### 2.3. Counter-terms

The renormalized Feynman rules can be deduced easily from the counter-terms action (15), and they are written in momentum space as

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \]

where

\[ \mathcal{L}_1 = i\gamma [ \delta_{\lambda} e^{ip' \theta} + \delta_{\lambda} e^{-ip' \theta} ], \]

\[ \mathcal{L}_2 = \frac{\lambda}{\lambda} V_{\lambda}(p_1, p_2, p_3, p_4). \]

(29)

We note from the counter-terms that the constants \( a \) and \( b \) could receive corrections in order to eliminate IR divergences of the form \( \frac{1}{p} \) and \( \frac{1}{p'} \), respectively. The Feynman rules for propagators and vertices are now established, and they are used in the next section to evaluate
the one loop quantum corrections. The Feynman rules for the counter-terms are also given; they will be used in the renormalization process which will be discussed in a forthcoming paper.

3. One loop corrections

We are going to determine, in this section, the relevant corrections for the 1PI two-point functions, for the scalar and fermion field, and the three and four-point functions at one loop level using the dimensional regularization method. We use the results of multiscale analysis [13] to eliminate the subleading logarithmic singularities \( \ln \tilde{p}^2 \) of non-planar graphs for vanishing momentum, because they represent a mild divergence [24]. Therefore, we keep in our results only the UV divergences of the planar integrals and the leading quadratic IR divergences of the non-planar integrals.

3.1. Two-point function \( \Gamma^{(2)} \)

3.1.1. Scalar propagator. The diagrammatic expansion of \( \Gamma^{(2)}_{\phi} \) represents the quantum corrections for the scalar field propagator; at one loop level we have the tadpole and the fermion loop graphs to evaluate, and the first one is represented by the integral

\[
- \frac{1}{6} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + M^2 + \frac{a^2}{k^2}} \ln \left( \frac{p^2}{2} \right),
\]

where the integration is in a \( D \)-dimension Euclidian space. We evaluate the divergent part of the planar and non-planar integral using the dimensional regularization method, the result is

\[
I_{\text{tadpole}} = - \frac{1}{6} \frac{1}{(4\pi)^2} \sum_{\zeta} \pm 1 \left( 1 + \zeta \frac{M^2}{2A^2} \right) \left( \frac{M^2}{2} + \zeta A^2 \right)^{-1} \Gamma \left( \frac{2 - D}{2} \right)
\]

\[
- \frac{1}{6} \frac{1}{(4\pi)^2} \sum_{\zeta = \pm 1} \left( 1 + \zeta \frac{M^2}{2A^2} \right) \left( \frac{M^2}{2} + \zeta A^2 \right) K_{\frac{1}{2}} \left( \sqrt{\frac{M^2}{2} + \zeta A^2} \right),
\]

where \( K_{\frac{1}{2}} \) is the modified Bessel function. Thereafter, we put \( D = 4 - \epsilon \), where \( \epsilon \to 0 \); this reveals the UV divergence of the planar part

\[
I_{\text{tadpole}}^p = \frac{2\lambda M^2}{3(4\pi)^2 \epsilon^2}.
\]

The non-planar integral depends on external momentum and it is finite for \( \tilde{p}^2 \neq 0 \), however it reveals a leading quadratic IR for \( \tilde{p}^2 \to 0 \)

\[
I_{\text{tadpole}}^{\text{NP}} = - \frac{2\lambda}{3(4\pi)^2} \frac{1}{\theta^2 \tilde{p}^2}.
\]
The total divergence of the tadpole integral is then
\[ I_{\text{tadpole}} = \frac{2}{3} \lambda M^2 - \frac{2}{3} \frac{\lambda}{(4\pi)^2} \theta^2 \mathcal{P} + O(\lambda^2). \] (34)

We note that the UV divergence is different by a numeric factor \(\frac{2}{3}\) from the commutative case. This difference is due to the scalar vertex \(V_1\) which adds a factor \(\frac{1}{3}\) (see reference [25]) and the extra term \(\frac{a}{\mathcal{P}}\) in the propagator numerator which adds a factor 2.

The second contribution to the scalar two-point function comes from the fermionic loop. After performing a trace over the fermion loop, the integral representing the second graph reads
\[ -\int \frac{d^D k}{(2\pi)^D} = -4 \int \frac{d^D k}{(2\pi)^D} \frac{g_1^2 + g_2^2 + 2g_1g_2 \cos(p\mathcal{P})}{(k^2 + m^2 + b^2)(m^2 + b^2 + \theta^2/(k^2 + p^2)) \left(\theta/(k + p) + b \frac{k^2 p^2}{k^2 + p^2}\right)} \times \left[m^2 + k^2 p^2 + k^2 - b \frac{k^2 p^2}{k^2 + p^2}\right] \] (35)

This can be divided, as usual, into planar and non-planar integrals, following the same terms order as the last expression, we have
\[ I_{\text{fermion-loop}}^P = -4 \left( g_1^2 + g_2^2 \right) \left[m^2 I_1 + p^2 I_2^I + I_3 + b \frac{\bar{p} \mu}{\theta} I_3^I + b^2 I_5 + b \frac{\bar{p} \mu}{\theta} I_6^I + b^2 p^2 I_7^I, \right] \] (36)
and
\[ I_{\text{fermion-loop}}^{\text{NP}} = -8 g_1 g_2 \left[m^2 I_1 + p^2 I_2^I + I_3 + b \frac{\bar{p} \mu}{\theta} I_3^I + b^2 I_5 + b \frac{\bar{p} \mu}{\theta} I_6^I + b^2 p^2 I_7^I \right]. \] (37)

where each integral \(I_i\) and \(J_i\) is evaluated separately. For the planar integrals, we have the following results:
- the integrals \(I_1, I_2\) and \(I_3\) present UV divergence, their divergent parts give the contribution \(\frac{4g_1^2 + g_2^2}{(4\pi)^2} (p^2 + 2m^2)\).
- the integrals \(I_4\) and \(I_5\) are finite, but after integration the products \(\bar{p} \mu I_4^I\) and \(\bar{p} \mu I_5^I\) are proportional to \(\bar{p}^\mu p^\mu\) and then vanish.
- the integrals \(I_6\) and \(I_7\) are finite for \(\theta \neq 0\).

The non-planar integrals are finite for \(\bar{p}^2 \neq 0\), but they could reveal IR divergence when \(\bar{p}^2 \to 0\), we have the following results:
- the integrals \(J_1, J_2\) and \(J_3\) present an IR divergence, their divergent parts give the contribution \(-\frac{4g_1 g_2}{(4\pi)^2} \frac{1}{\theta p^2}\).
- the integrals \(J_4\) and \(J_5\) are finite, but after integration the products \(\bar{p} \mu I_4^I\) and \(\bar{p} \mu I_5^I\) are proportional to \(\bar{p} \mu p^\mu\) and then vanish.
- the integrals \(J_6\) and \(J_7\) are finite for \(\theta \neq 0\).
The fermion contribution to the scalar two-point function reads

\[ I_{\text{fermion-loop}} = \frac{4}{(4\pi)^2\varepsilon} \left( g_1^2 + g_2^2 \right) (p^2 + 2m^2) - \frac{32g_1g_2}{(4\pi)^2\theta^2p^2} + (f_1 + f_2p^2) + O\left(\frac{g^3}{\theta^2p^2}\right), \tag{38} \]

where \( f_i \) denote functions that result from the finite integrals; they are analytic for \( \theta \neq 0 \), and this notation is used hereafter. We note here that the UV divergence in (38) is the same as in the commutative case where \( g^2 = g_1^2 + g_2^2 \).

Thus, the total one loop contribution to the scalar field propagator is

\[ \Gamma^{(2)}_{\phi - \text{1-loop}} = I_{\text{tadpole}} + I_{\text{fermion-loop}}. \tag{39} \]

\( I_{\text{tadpole}} \) and \( I_{\text{fermion-loop}} \) are given in the expressions (34) and (38), respectively. There is, as expected, a leading quadratic IR divergence \( \sim \frac{1}{\theta^2p^2} \) resulting from the non-planar integrals besides the ordinary UV divergence. The additional term \( f_1 + f_2p^2 \) is finite for \( \theta \neq 0 \).

3.1.2. Fermion propagator. The quantum corrections for the fermion field propagator at one loop level are given by the integral

\[ \mathcal{F} = \int \frac{d^4k}{(2\pi)^4} \left( \delta^2 + \frac{2g_1s_1\cos(\theta k)}{(p+k)^2 + m^2 + \frac{b^2}{(p+k)^2} \left( k^2 + M^2 + \frac{a^2}{k^2} \right)} \right) \]

this can be divided into planar and non-planar integrals. Following the same terms order as the last expression, we have

\[ I^P = -\left( g_1^2 + g_2^2 \right) \left[ (i\not{p} - m)I_1 + i\gamma^5\not{I}_3 + ib\not{\theta}f_1 + ib\gamma^5\not{I}_3 \right], \tag{41} \]

and

\[ I^{NP} = -2g_1s_2 \left[ (i\not{p} - m)J_1 + i\gamma^5\not{J}_3 + ib\not{\theta}f_3 + ib\gamma^5\not{J}_3 \right]. \tag{42} \]

We calculate each integral separately and using the same procedure as in the tadpole integral, we obtain

\[ \Gamma^{(2)}_{\psi - \text{1-loop}} = -\frac{i}{\theta} \left( g_1^2 + g_2^2 \right) \left( \not{p} + 2Im \right) + f_3 \frac{\not{\theta}}{\theta} + O\left(\frac{g^3}{\theta^2p^2}\right). \tag{43} \]

We note that the UV divergence in the last relation is the same as in the commutative theory with \( g^2 = g_1^2 + g_2^2 \). We have also here an additional term \( f_3 \frac{\not{\theta}}{\theta} \) which results from the integral

\[ -\frac{b}{\theta} \int \frac{d^4k}{(2\pi)^4} \left( g_1^2 + g_2^2 + 2g_1s_1\cos(\theta k) \right) \not{\theta} + k \right) \left( p + k \right)^2 + m^2 + \frac{b^2}{(p+k)^2} \left( k^2 + M^2 + \frac{a^2}{k^2} \right) \left( p + k \right)^2. \tag{44} \]
Hence the function $f_3$ is obtained for $p^2 \to 0$

$$f_3 = \frac{-i\hbar (g_1 + g_2)^2}{4(4\pi)^2} \sum_{\rho, \lambda = \pm 1} \left( 1 + \frac{M^2}{2\lambda^2} \right) \left( 1 + \frac{m^2}{2B^2} \right) \frac{1}{2\mu^2 - \mu^2_\rho} \ln \left( \frac{\mu^2}{\mu^2_\rho} \right) \right).$$

(45)

where $\mu^2 = \frac{(M^2 + \rho A^2)}{2}$, $\mu^2_\lambda = \frac{(m^2 + \lambda B^2)}{2}$, $A^2 = \frac{M^2}{2} - a^2$ and $B^2 = \frac{m^2}{2} - b^2$. The term $f_3 \frac{p^2}{\theta}$ is a result of the fermionic extra term, thus it vanishes for $b = 0$.

3.2. Three-point function $\Gamma^{(3)}$

The one loop quantum corrections to the Yukawa vertex are given by only one graph, namely

$$\Gamma^{(3)}_{\text{loop}} = \gamma^5 \int \frac{d^4k}{(2\pi)^4} \frac{F(k, q, \sigma) N(k, q)}{\left( k^2 + m^2 + \frac{b^2}{k^2} \right) \left( k - \frac{p^2}{\theta(k-q)^2} \right) \left( k^2 + m^2 + \frac{a^2}{(k-p)^2} \right)}.$$

(46)

where $F(k, q, \sigma)$ represents the product of the phase factors of the Yukawa vertices

$$F(k, q, \sigma) = \sum_{\sigma = \pm 1} \left( g_\sigma \bar{\epsilon}^{(\sigma)k} \sum_{\sigma = \pm 1} g_\sigma \bar{\epsilon}^{(\sigma)q} \sum_{\sigma = \pm 1} g_\sigma \bar{\epsilon}^{(\sigma)p} \right).$$

(47)

The first term is independent of $k$, therefore it appears as a factor of the planar integrals while the other terms enter in the non-planar integrals. The function $N(k, q)$ represents the product of the fermion propagator numerators with the $\gamma^5$ matrix

$$N(k, q) = \left[ i\not{k} + m + ib\frac{\not{q}}{\theta(k-q)^2} \right] \gamma^5 \left[ i\not{k} - \not{q} + m + ib\frac{\not{q}}{\theta(k-q)^2} \right] \gamma^5.$$

(48)

The use of dimensional power counting reveals that all the terms of this function contribute in convergent integrals except for the one with $k^2$; the resulting divergence, from the planar and non-planar integrals, is then logarithmic. Thus, the evaluation of the divergent parts of the integral (46) gives

$$\Gamma^{(3)}_{\text{loop}} = \gamma^5 \int \left( \frac{2g_1 g_2}{(4\pi)^2} + f_3 \right) V_\theta(p, p') + f_5 + O(g^4).$$

(49)

where $f_3$ are analytic functions for $\theta \neq 0$ resulting from the finite integrals. In order to recover the results for the UV divergence of the commutative case we have to make the substitution $g_1 g_2 (g_1 + g_2) = g^3$. 

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3.3. **Four-point function** $\Gamma^{(4)}$

In order to evaluate the four-point function at the one loop level, we have to include all the contributions that give $O(\lambda^2)$ and $O(\lambda^3)$ corrections to the $\phi^4$ vertex.

3.3.1. $\Gamma^{(4)}$ with $\phi^4$ coupling. The scalar one loop contributions to the $\phi^4$ vertex come from the following graphs

\[
\Gamma^{(4)}_{\phi^4-\text{1loop}} = \frac{1}{2} \left( \frac{\lambda}{p_1^2} \right) ,
\]

which are evaluated from these integrals

\[
\Gamma^{(4)}_{\phi^4-\text{1loop}} = \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} F_1(k, p_i) \left[ \frac{1}{(p_1 + p_2 - k)^2 + M^2 + \frac{a^2}{(p_1 + p_2 - k)^2}} \right] \left[ \frac{1}{k^2 + M^2 + \frac{a^2}{k^2}} \right] + \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} F_2(k, p_i) \left[ \frac{1}{(k + p_3 - p_1)^2 + M^2 + \frac{a^2}{(k + p_3 - p_1)^2}} \right] \left[ \frac{1}{k^2 + M^2 + \frac{a^2}{k^2}} \right] + \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} F_3(k, p_i) \left[ \frac{1}{(k + p_1 - p_3)^2 + M^2 + \frac{a^2}{(k + p_1 - p_3)^2}} \right] \left[ \frac{1}{k^2 + M^2 + \frac{a^2}{k^2}} \right] ,
\]

where $F_i(k, p_i)$ is the product of the two $\phi^4$ vertices.

These integrals were evaluated in [24] by introducing a cut-off; we find equivalent results using the dimensional regularization method. These diagrams present a logarithmic UV divergence

\[
\Gamma^{(4)}_{\phi^4-\text{1loop}} = - \frac{2\lambda}{(4\pi)^2} V_\Lambda(p_1, p_2, p_3, p_4) + O(\lambda^3) ,
\]

which is different by a numeric factor $\frac{2}{3}$ from the commutative case where $V_\Lambda \rightarrow -\lambda$.

3.3.2. $\Gamma^{(4)}$ with Yukawa coupling. The contributions to the $\phi^4$ vertex come in this case from the Yukawa interaction. It represents the fermion corrections to the scalar $\phi^4$ coupling constant. In order to have an effective contribution to the $\phi^4$ vertex, from the fermion loop, we need to recover in our final result the $\phi^4$ extra-factor of $V_\Lambda$ from the product of the Yukawa phase factors $V_{\phi}$. If we consider only the permutations of external momenta, as in the commutative theory, then we will have only six diagrams to evaluate, namely
However, in the noncommutative case, there are also the phase factors coming from the Yukawa vertices which depend explicitly on internal momenta. This means that when we expand the four-point function at the one loop level we will have more diagrams to evaluate. In fact there are only two different permutations of internal momentum for each one of the last six diagrams, which can be represented as follows

\[ \Gamma^{(6)}_{\text{1-loop}} = \text{diagrams} \].

(53)

As a result, the diagrammatic expansion of the four-point function is represented by twelve diagrams, and the integral corresponding to each diagram has the generic form

\[
I_j^{(4)} = (-1) \int \frac{d^4k}{(2\pi)^4} \frac{F'(k, p_i)}{(k^2 + m^2 + \frac{b^2}{k^2})} \left( (k + p_1)^2 + m^2 + \frac{b^2}{(k + p_1)^2} \right) \quad (55)
\]

where \( F'(k, p_i) = \prod_{i=1}^{4} V_{ij} \) is the product of the phase factors of the four Yukawa vertices. It can be written as

\[
F'(k, p_i) = \sum_{\sigma=\pm 1} \sum_{\sigma_0=\pm 1} \gamma_1^2 e^{i\sigma_0 k} e^{\sigma_0 \frac{i}{2}(\epsilon_{-\sigma_0} + \epsilon_{\sigma_0} + \epsilon_{-\sigma_0} + \epsilon_{\sigma_0})} + \sum_{\sigma=\pm 1} \Gamma_{\alpha=\pm 2} \gamma_1^2 e^{i\sigma_0 k} e^{\sigma_0 \frac{i}{2}(\epsilon_{-\sigma_0} + \epsilon_{\sigma_0} + \epsilon_{-\sigma_0} + \epsilon_{\sigma_0})},
\]

(56)

where \( \epsilon_{0,1,2} = 1 \) and \( \epsilon_{3,4,5} = -1 \). The function \( F'(k, p_i) \) divides the integral (55) into planar and non-planar parts; the first term in (56) is just a factor of planar integrals while the others enter in the non-planar integrals. The indices \( m, n, r, s \) take different values from 1 to 4; this gives us twelve different phase factors for \( e^{i\frac{r}{2}(p_i + p_j)} \). These phase factors result from the permutations of \( p_i \), where each one of them corresponds to a different graph.
The trace of the fermionic loop is Tr $N'(k, p_i)$, where

$$N'(k, p_i) = \gamma^5 \left( i k + m + \frac{\tilde{\beta}}{\theta k^2} \right) \gamma^5 \left( i (k + p_1) + m + \frac{i \tilde{\beta}}{\theta (k + p_1)^2} \right)$$

$$\times \gamma^5 \left( i (k + p_1 + p_2) + m + \frac{i \tilde{\beta}}{\theta (k + p_1 + p_2)^2} \right)$$

$$\times \gamma^5 \left( i (k + p_3) + m + \frac{i \tilde{\beta}}{\theta (k + p_3)^2} \right).$$

When expanding the product $F'(k, p_i) \times Tr [N'(k, p_i)]$, the integral (55) is divided into numerous integrals; fortunately, most of them are finite. The divergent integrals are only those having $k^4$ in the numerator; the resulting divergence is then logarithmic. The summation over all the planar graphs allows us to recover the $\varphi^4$ extra factor of $V_\lambda$.

$$\sum_{\text{perms. of } p_i} e^{i (\tilde{p}_0 + \tilde{p}_4)} = 4 \left( \cos \frac{p_1 \tilde{p}_2}{2} \cos \frac{p_3 \tilde{p}_4}{2} + \cos \frac{p_1 \tilde{p}_3}{2} \cos \frac{p_2 \tilde{p}_4}{2} + \cos \frac{p_1 \tilde{p}_4}{2} \cos \frac{p_2 \tilde{p}_3}{2} \right).$$

thus, the fermionic contributions to the scalar coupling constant are

$$\Gamma_{\varphi^4-\text{loop}}^{(4)} = \sum_{j=1}^{12} \Gamma_{\varphi^4-\text{loop}}^{(4)} = \left( \frac{96 (g_1^4 + g_2^4)}{(4\pi)^2 \varepsilon} + f_0 \right) \frac{V_\lambda (p_1 \cdot p_2 \cdot p_3 \cdot p_4)}{\lambda} + f_2 + O (g^6),$$

where $f_0$ are analytic functions for $\theta \neq 0$ resulting from the finite integrals. Since twelve graphs are evaluated in the noncommutative case instead of six, the UV divergence in this case is twice that of the commutative theory, where $g^4 = g_1^4 + g_2^4$.

The total one loop contribution to the $\varphi^4$ vertex is then

$$\Gamma_{\varphi^4-\text{loop}}^{(4)} = \Gamma_{\varphi^4-\text{loop}}^{(4)} + \Gamma_{\varphi^4-\text{loop}}^{(4)}.$$  

We note here the importance of recovering the $\varphi^4$ vertex from the product of the Yukawa vertices because it can be seen as a consistency test for our model.

4. Conclusions and remarks

In this work we have constructed a translation-invariant noncommutative pseudo-scalar Yukawa model and calculated the quantum corrections at the one loop level up to the 1PI four-point function. The results obtained will be used hereafter to discuss the issue of renormalizability for this model and to adjust it if necessary. However the renormalization process at the one loop level will be discussed in a forthcoming paper.

The analytic functions $f_0$ that appear in some results of quantum corrections do not affect the renormalizability of our model in the noncommutative case. Moreover, they can contribute as noncommutative corrections to the fields, masses and coupling constants. However, the commutative limit could be problematic if $f_0 \to \infty$. In this case, one has to use the mechanism described in [26], which relies on the analysis of the UV/IR mixing in Feynman
graphs to recover the commutative theory. The commutative limit for the modified noncommutative models is not recovered simply by taking $\theta \longrightarrow 0$, even with this mechanism the limit is not smooth.

The presence of the term $\sim \frac{\theta}{\pi}$ instead of $\sim \frac{1}{\pi}$ in the fermion two-point function could add a divergence, in higher order corrections, that cannot be absorbed by any renormalization factor. This suggests adding, besides the term $\sim \frac{1}{\pi}$, another term of the form $\sim \frac{\theta}{\pi}$ to the fermion action. This fact can be explained by the existence of an inner derivative in Moyal space which does not appear in ordinary space \([27]\). We note that the analytic functions $f_i$ and the term $\sim \frac{\theta}{\pi}$ discussed above result from the fermionic extra term, thus they vanish for $b = 0$. In this case, the theory could be renormalizable at the one loop level but it doesn't fulfill the consistency condition (1).

Finally, this model can be extended to gauge field theory as has been done with scalar models; in this case one has to respect BRS symmetry. However the loop corrections are harder to evaluate due to the existence of the extra terms both in the scalar and fermion actions. The model can be also extended to supersymmetry, where this work can be included in the bosonic part of the theory \([28–32]\).

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