The Distributive Full Lambek Calculus with Modal Operators

Daniel Rogozin
Lomonosov Moscow State University
Moscow, Russia
Serokell OÜ
Tallinn, Estonia

Abstract
In this paper, we study logics of bounded distributive residuated lattices with modal operators considering □ and ◇ in a noncommutative setting. We introduce relational semantics for such noncommutative modal logics. We prove that any canonical logic is Kripke complete via discrete duality and canonical extensions. That is, we show that a modal extension of the distributive full Lambek calculus is the logic of its frames if its variety of algebras is closed under canonical extensions. After that, we establish a Priestley-style duality between residuated distributive modal algebras and certain topological Kripke structures based on Priestley spaces.

Keywords: The Lambek calculus, canonical extensions, bounded distributive lattice expansions, Priestley-style duality, residuated lattices

1 Introduction
A substructural logic is a logic lacking some of the well-known structural rules such as contraction, weakening, or permutation. Algebraically, substructural logics are often logics of ordered residuated algebras [16]. Modalities in substructural logics are studied at least in two perspectives. The first perspective has a connection with philosophical problems such as a relevant necessity. The second perspective is more applied and related to such issues as resource management. Here, linear logic use proved its efficiency in computer science and linguistics.

The computer science applications are more related to type theory, resource-sensitive computation and related applications in computation and type theory [1] [25].

The noncommutative version of linear logic, the Lambek calculus [35], proposed for linguistic issues, i.e., proof-theoretical characterisation of inference in Lambek grammars, the equivalent version of context-free grammars [39].

1 The research is supported by the Presidential Council, research grant MK-430.2019.1.
Modalities in those logics (the !-modality, if more precisely) introduce lacking structural rules in a restricted way as follows:

\[
\begin{align*}
\Gamma, \Delta & \Rightarrow B \\
\Gamma, !A, \Delta & \Rightarrow B \\
\Gamma, !A, !A, \Delta & \Rightarrow B \\
\Gamma, !A, !A, B, \Delta & \Rightarrow C \\
\Gamma, !A, B, !A, \Delta & \Rightarrow C \\
\Gamma, !A, B, \Delta & \Rightarrow C
\end{align*}
\]

Modal extensions of linear logic have an interpretation within the context of a resource management based on the phase semantics proposed by Girard [24]. Algebraically, exponential modalities were studied by Ono as additional exponential operators on FL algebras [36]. The Lambek calculus and their modal extensions also have the cover semantics proposed by Goldblatt [28] [29]. The abstract polymodal case of such an extension of the full Lambek calculus was recently studied by Kanovich, Kuznetsov, Scedrov, and Nigam [32]. In this paper, (sub)exponential modalities are considered from proof-theoretical and complexity perspectives.

The Lambek calculus characterises inference in categorial grammars. There are several approaches to consider categorial grammars from a broader modal point of view. Those approaches were overviewed by van Benthem in this paper [50], the example given.

From a semantical point of view, the basic Lambek calculus is, e.g., complete with respect to so-called language models, residual semigroups on subsets of free semigroup [40]. The Lambek calculus with product and residuals is also complete with respect to subsets of a transitive relation [2]. The relational semantics for the basic Lambek calculus was obtained by Dunn, Gehrke, Palmigiano, and other authors using the canonical extensions technique [7] [12]. Alternatively, one may consider bi-approximation semantics for substructural logic studied by Suzuki [47].

In this paper, we study the distributive version of the full Lambek calculus extended with normal modal operators $\Box$ and $\Diamond$ to consider a broader class of noncommutative modalities (as an abstraction of storage operators in noncommutative linear logic) in a distributive setting. We introduce noncommutative Kripke frames, relational structures for the distributive Lambek calculus extended with binary modal relations. We establish a discrete duality between such Kripke frames and perfect distributive residuated modal algebras developing approach proposed in, the example is given, [22]. After that, we overview canonical extensions of related modal algebras applying techniques provided in [12] [17] [21] to show that any canonical residuated distributive modal logic is Kripke complete. We also prove that the subexponential modality axioms are canonical ones. Thus, we show that the corresponding logics enriched with subexponentials are Kripke complete. Finally, we extend the obtained duality to topological duality between residuated distributive modal algebras and
special topological Kripke spaces based on bDRL-spaces, Priestley spaces with a ternary relation dual algebras of which are bounded distributive residuated lattices [15]. We also use some ideas from positive modal and intuitionistic modal logics [5] [38].

The text contains a short appendix with the brief survey on canonical extensions and duality for bounded distributive lattices to keep the paper self-contained.

2 The distributive Lambek calculus with modal operators

In this section, we formulate the Hilbert-style the distributive full Lambek calculus enriched with normal modal operators. The language extends the language of the full Lambek calculus with modal operators $\Box$ and $\Diamond$ as follows:

\[
\phi, \psi ::= \bot | \top | p | (\phi \cdot \psi) | (\phi \setminus \psi) | (\phi \lor \psi) | (\phi \land \psi) | (\phi \Rightarrow \psi).
\]

By a residuated distributive normal modal logic, we mean some set of sequents that have the form $\vartheta \Rightarrow \psi$, where $\vartheta, \psi$ are formulae, according to the following definition:

**Definition 2.1** A residuated normal distributive modal logic is the set of sequents $\Lambda$ that contains the axioms (1)-(14) and closed under the following inference rules:

1. $\bot \Rightarrow p$
2. $p \Rightarrow \top$
3. $p_i \Rightarrow p_1 \lor p_2, i = 1, 2$
4. $p_1 \land p_2 \Rightarrow p_i, i = 1, 2$
5. $p \land (q \lor r) \Rightarrow (p \land q) \lor (p \land r)$
6. $p \cdot (q \cdot r) \Rightarrow p \cdot (q \cdot r)$
7. $\varphi \Rightarrow \psi \Rightarrow \theta \Rightarrow \psi$
8. $\varphi \lor \theta \Rightarrow \varphi$
9. $\varphi \cdot \theta \Rightarrow \psi$
10. $\theta \Rightarrow \varphi \setminus \psi$
11. $\theta \cdot \varphi \Rightarrow \psi$
12. $\theta \Rightarrow \psi / \varphi$
13. $\varphi \Rightarrow \psi \land \theta$
14. $\varphi \Rightarrow \psi \Rightarrow \theta$

A residuated normal distributive modal logic extends normal distributive normal modal logic with residuals, product, and the connection axiom. The logic of bounded distributive lattices with modal operators was studied in depth in [22], where one of the generalisations of the Salqvist theorem is proved.

To define relational semantics we introduce ternary Kripke frames with the

\[
\varphi(p) \Rightarrow \psi(p)
\]

\[
\varphi[p := \psi] \Rightarrow \psi[p := \gamma]
\]

\[
\varphi \Rightarrow \psi \land \theta
\]

\[
\theta \Rightarrow \varphi \setminus \psi
\]

\[
\varphi \Rightarrow \psi / \varphi
\]

\[
\theta \Rightarrow \psi / \varphi
\]

\[
\theta \cdot \varphi \Rightarrow \psi
\]
additional binary modal relations. Such a ternary frame might be considered as a noncommutative generalisation of a relevant Kripke frame described, e.g., here [46]. As it is usual in the relational semantics of substructural logic, product and residuals have the ternary semantics as in, e.g., [2] [11] [43].

**Definition 2.2** A Kripke frame is a tuple $\mathcal{F} = (W, \leq, R, R_\diamond, R_\lozenge, O)$, where $\langle W, \leq \rangle$, $R \subseteq W^3$, $R_\diamond, R_\lozenge \subseteq W^2$ such that for all $\forall u, v, w, u', v', w' \in W$

(i) $Ruuvw & wR_\diamond w' \Rightarrow \exists x, y \in W\ Rxyw' & uR_\diamond x & vR_\diamond y$

(ii) $\exists x \in W\ (Ruux & Rxwv') \Leftrightarrow \exists y \in W\ (Ruwy & Ruyv')$

(iii) $Ruuvw & u' \leq u \Rightarrow Ru'vw$, $Ruuv \& \& u' \leq v \Rightarrow Ruvw', Ruvw \& w \leq w' \Rightarrow Ruvw'$

(iv) $\forall o \in O\ Ruow \Rightarrow Ruow$

(v) $v \leq w \Leftrightarrow \exists o \in ORuwv$

(vi) $O$ is upwardly closed

(vii) $u \leq v \& vR_\diamond w \Rightarrow uR_\diamond w$ and $u \leq v \& uR_\diamond w \Rightarrow vR_\diamond w$

A Kripke model is a Kripke frame with equipped a valuation function that maps each propositional variable to $\leq$-upwardly closed subset of worlds.

**Definition 2.3** Let $\mathcal{F} = \langle W, \leq, R, R_\diamond, R_\lozenge, O \rangle$ be a Kripke frame, a Kripke model is a pair $\mathcal{M} = (\mathcal{F}, \vartheta)$, where $\vartheta : PV \rightarrow Up(W, \leq)$. Here, $Up(W, \leq)$ is the collection of all upwardly closed sets. The connectives have the following semantics:

(i) $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$

(ii) $\mathcal{M}, w \models \top, \mathcal{M}, w \vdash \bot, \mathcal{M}, w \models 1 \Leftrightarrow w \in O$

(iii) $\mathcal{M}, w \models \varphi \cdot \psi \Leftrightarrow \exists u, v \in W\ Ruuvw & \mathcal{M}, u \models \varphi \& \mathcal{M}, v \models \psi$

(iv) $\mathcal{M}, w \models \varphi \setminus \psi \Leftrightarrow \forall u, v \in W\ Ruuvw & \mathcal{M}, u \models \varphi \Rightarrow \mathcal{M}, v \models \psi$

(v) $\mathcal{M}, w \models \psi/\varphi \Leftrightarrow \forall u, v \in W\ Ruuvw & \mathcal{M}, u \models \varphi \Rightarrow \mathcal{M}, v \models \psi$

(vi) $\mathcal{M}, w \models \varphi \& \psi \Leftrightarrow \mathcal{M}, w \models \varphi \& \mathcal{M}, w \models \psi$

(vii) $\mathcal{M}, w \models \varphi \lor \psi \Leftrightarrow \mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$

(viii) $\mathcal{M}, w \models \Box \varphi \Leftrightarrow \forall v \in R_\diamond(w)\ \mathcal{M}, v \models \varphi$

(ix) $\mathcal{M}, w \models \Diamond \varphi \Leftrightarrow \exists v \in R_\lozenge(w)\ \mathcal{M}, v \models \varphi$

(x) $\mathcal{M}, w \models \varphi \Rightarrow \psi \Leftrightarrow \mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}, w \models \psi$

(xi) $\mathcal{M} \models \varphi \Rightarrow \psi \Leftrightarrow \forall w \in W\ \mathcal{M}, w \models \varphi \Rightarrow \psi$

The following definitions are also standard ones.

**Definition 2.4** Let $\mathcal{F}$ a Kripke frame

(i) Let $\varphi \Rightarrow \psi$ be a sequent, then $\mathcal{F} \models \varphi \Rightarrow \psi$ iff for all valuation $\vartheta (\mathcal{F}, \vartheta) \models \varphi \Rightarrow \psi$

(ii) $\text{Log}(\mathcal{F}) = \{ \varphi \Rightarrow \psi \mid \mathcal{F} \models \varphi \Rightarrow \psi \}$
Let $\mathcal{F}$ be a class of Kripke frames, then $\text{Log}(\mathcal{F}) = \bigcap_{\mathcal{F} \in \mathcal{F}} \text{Log}(\mathcal{F})$

Let $\Lambda$ be a residual normal modal logic, then $\text{Frames}(\mathcal{L}) = \{ \mathcal{F} \mid \mathcal{F} \models \mathcal{L} \}$ and $\mathcal{L}$ is complete iff $\mathcal{L} = \text{Log}(\text{Frames}(\mathcal{L}))$

By $\mathcal{L}_K$, we mean the minimal residual distributive normal modal logic, the minimal set of sequents that contains the axioms above and closed under the required inference rules.

The soundness theorem is the standard one.

**Theorem 2.5** Let $\mathcal{F}$ be a class of Kripke frames, then $\text{Log}(\mathcal{F})$ is a residual distributive normal modal logic.

**Proof.** Let us show that $\mathcal{F} \models \Box p \rightarrow \Box (p \cdot q)$, where $\mathcal{F} = \langle W, R, R_C, R_C, \mathcal{O} \rangle$ is an arbitrary frame. Let $\vartheta$ a valuation and $\mathcal{M} = \langle \mathcal{F}, \vartheta \rangle$ a model such that $\mathcal{M}, w, w' \models \Box p \cdot q$. Then there exists $u, v$ such that $\mathcal{R}_{uvw} \cdot \mathcal{M}, u \models \Box p, \mathcal{M}, v \models \Box q$. On the other hand, let $w' \in R_C(w) \cdot R_{uw} \cdot w_C \cdot w'$ implies $\exists x, y \in W. R_{xyw'} \cdot uR_Cx \cdot vR_Cy$. $\mathcal{M}, u \models \Box p, \mathcal{M}, v \models \Box q$ implies that for each $x \in R_C(u) \cdot \mathcal{M}, x \models p$ and for each $y \in R_C(v) \cdot \mathcal{R}_{xyw'}$ implies $\mathcal{M}, w' \models p \cdot q$. Thus, $\mathcal{M}, w \models \Box (p \cdot q)$ \hfill $\square$

One may extend the notion of a bounded morphism for the relevant case to have homomorphisms between observed Kripke frames and models that preserve truth.

**Definition 2.6** Let $\mathcal{F}_1, \mathcal{F}_2$ be Kripke frames, a map $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is a bounded morphism, if:

(i) $R_{1uvw} \Rightarrow R_{2f(u)f(v)f(w)}$

(ii) $R_{2f(u)v}w' \Rightarrow \exists v, w' \leq f(v) \& f(w) \leq w' \& R_{uwv}$

(iii) $R_{2u'}f(v)w' \Rightarrow \exists u, w' \leq f(u) \& f(w) \leq w' \& R_{uwv}$

(iv) $R_{2u'v}f(w) \Rightarrow \exists u, v, w' \leq f(u) \& w' \leq f(v) \& R_{uwv}$

(v) $f([R_{\mathcal{F}_1}(x)]) = (R_{\mathcal{F}_2}(f(x))), \nabla = \Box, \Diamond$

(vi) $f^{-1}[O_2] = O_1$

By the notation $\mathcal{F}_1 \rightarrow \mathcal{F}_2$ we mean that there exists a surjective bound morphism $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ and we call such a map p-morphism. Let $\mathcal{F}_1, \mathcal{F}_2$ be Kripke frames, $\vartheta_1, \vartheta_2$ valuations on $\mathcal{F}_1$ and $\mathcal{F}_2$ correspondingly, and $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ a bounded morphism. Then $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a bounded morphism of models, if $\mathcal{M}_1, w \models p_i \Leftrightarrow \mathcal{M}_2, f(w) \models p_i$ for each propositional variable $p_i$.

The following lemma is proved standardly, one may relativize the standard proof given in [4] [6].

**Lemma 2.7**

(i) $\mathcal{M}_1, w \models \varphi \Leftrightarrow \mathcal{M}_2, f(w) \models \varphi$

(ii) $\mathcal{F}_1 \rightarrow \mathcal{F}_2$ implies $\text{Log}(\mathcal{F}_1) \subseteq \text{Log}(\mathcal{F}_2)$
(iii) $F_1 \cong F_2$ implies $\text{Log}(F_1) = \text{Log}(F_2)$

3 Residuated distributive modal algebras

In this section, we study algebraic semantics and canonical extensions for observed logics. The required lattice-theoretic and canonical extensions definitions and notations are explained in the appendix. Let us define a residuated lattice [30].

Definition 3.1 A residuated lattice is an algebra $\mathcal{R} = \langle L, \cdot, \backslash, /, \varepsilon \rangle$, where $L$ is a bounded lattice, $\cdot$ is a binary associative monotone operation, $\varepsilon$ is a multiplicative identity. $\backslash$ and $/$ are residuals, that is, the following equivalence holds for all $a, b, c \in L$:

$$b \leq a \backslash c \iff a \cdot b \leq c \iff a \leq c / b$$

Note that the class of all residuated lattices forms a variety since the quasi-identities above might be equivalently reformulated as identities, see [30], Lemma 2.3.

A residuated lattice is called bounded distributive if its lattice reduct is a bounded distributive lattice. A residuated lattice morphism is a map $f : L_1 \to L_2$ that commutes with all operations in a usual way.

Let us recall the essential facts about (prime) filters on bounded distributive residuated lattices, see [15] [49]. As a matter of fact, these statements hold for an arbitrary distributive lattice ordered semigroup since those properties of filters and their products don’t depend on residuals.

Lemma 3.2 Let $\mathcal{L}$ be a bounded distributive residuated lattice. Let $X, Y \subseteq \mathcal{L}$ and $X \cdot Y = \{ x \cdot y \mid x \in X, y \in Y \}$. Let us define $X \blacklozenge Y = \uparrow X \cdot Y$, then

(i) If $Z \subseteq \mathcal{L}$ is a filter, then $X \cdot Y \subseteq Z$ iff $X \blacklozenge Y \subseteq Z$

(ii) If $X, Y \subseteq \mathcal{L}$ are filters, then $X \blacklozenge Y$ is a filter

(iii) Let $X, Y, Z$ be filters in $\mathcal{L}$ and $Z$ is prime and $XY \subseteq Z$, then there exist prime filters $X', Y'$ such that $X \subseteq X'$, $Y \subseteq Y'$, and $X'Y' \subseteq Z$

A residuated distributive modal algebra is a distributive bounded residuated lattice extended with the operators $\Box$ and $\Diamond$ that distribute over finite infima and suprema correspondingly. One may also consider such algebras as full Lambek algebras [36] [37] reducts of which are bounded distributive lattices. Here, modalities are merely the $K$-like operators without any additional requirements except the connection between $\Box$ and $\cdot$. We require that $\Box$ is also “normal” with respect to the product. Such a “normality” corresponds to the promotion principle which is widespread in linear logic. This principle often has the $K4$ form, where formulae in the premise are boxed from the left. This version of the promotion is rather the $K$-rule than the $K4$ one:

$$\varphi_1 \bullet \cdots \bullet \varphi_n \Rightarrow \psi$$

$$\Box \varphi_1 \bullet \cdots \bullet \Box \varphi_n \Rightarrow \Box \psi$$

The inference rule also allows one to obtain the Kripke axioms formulated in terms of residuals as, e.g., $\Box (\varphi \backslash \psi) \Rightarrow \Box \varphi \backslash \Box \psi$ and $\Box (\psi / \varphi) \Rightarrow \Box \psi / \Box \varphi$. 
This “normality” requirement is introduced as the additional inequation, more precisely:

**Definition 3.3** A residuated distributive modal algebra (RDMA) is an algebra $\mathcal{M} = \langle R, \Box, \Diamond \rangle$ with the following conditions for each $a, b \in R$:

(i) $\Box(a \land b) = \Box a \land \Box b$, $\Box \top = \top$
(ii) $\Diamond(a \lor b) = \Diamond a \lor \Diamond b$, $\Diamond \bot = \bot$
(iii) $\Box a \cdot \Box b \leq \Box(a \cdot b)$

An RDMA homomorphism is a bounded distributive residuated lattice homomorphism $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ such that $f(\Box a) = \Box(f(a))$ and $f(\Diamond a) = \Diamond(f(a))$.

One may associate with an arbitrary residual normal modal logic its variety of RDMA as follows:

**Definition 3.4** Let $\Lambda$ be a residual normal modal logic, then $\mathcal{V}_{\Lambda}$ is a variety defined by the set inequation \{ $\varphi \leq \psi \mid \Lambda \vdash \varphi \Rightarrow \psi$ \}

Note that $\varphi, \psi$ are terms of the signature $\langle \land, \lor, \bot, \top, \cdot, \kappa, /, \varepsilon, \Box, \Diamond \rangle$ in such inequations as $\varphi \leq \psi$. One has an algebraic completeness for each residual distributive normal modal logic as usual.

**Theorem 3.5** Let $\Lambda$ be a residual normal modal logic, then there exists an RDMA $\mathcal{R}_{\Lambda}$ such that $\Lambda \vdash \varphi \Rightarrow \psi$ iff $\mathcal{R}_{\Lambda} \models \varphi \leq \psi$

Such an RDMA is a free countably generated algebra in the variety $\mathcal{V}_{\Lambda}$, the Lindenbaum-Tarski algebra up to isomorphism.

The following statement also holds according to the general technique [26]:

**Lemma 3.6** Let $\Lambda$ be a residuated distributive normal modal logic, then the map $\Lambda \mapsto \mathcal{V}_{\Lambda}$ is the isomorphism between the lattice of residuated distributive normal modal logics and the lattice of varieties of RDMA.

We define a completely distributive residuated perfect lattice as a distributive version of a residuated perfect one defined in [12].

**Definition 3.7** A distributive residuated lattice $\mathcal{L} = \langle L, \cdot, \kappa, /, \varepsilon \rangle$ is called perfect distributive residuated lattice, if:

- Its lattice reduct is perfect distributive
- $\cdot, \kappa, \land, \lor$ are binary operations on $L$ such that $\land$ and $\lor$ right and left residuals of $\cdot$, respectively. $\cdot$ is a complete operator on $\mathcal{L}$, and $\cdot : \mathcal{L} \times \mathcal{L}^\delta \rightarrow \mathcal{L}$, $\land : \mathcal{L}^\delta \times \mathcal{L} \rightarrow \mathcal{L}$ are complete dual operators, where $\mathcal{L}^\delta$ is a dual lattice.

Here we formulate canonical extensions for bounded distributive lattices with a residuated family in the fashion of [18], where canonical extensions for Heyting algebras were studied. Here we take a generalised version of that construction formulated for residuated lattices, see [17].

**Lemma 3.8** Let $\mathcal{L} = \langle L, \cdot, \kappa, /, \varepsilon \rangle$ be a bounded distributive residuated lattice, then $\mathcal{L}^\sigma = \langle L^\sigma, \cdot^\sigma, \kappa^\sigma, /^\sigma, \varepsilon \rangle$ is a perfect distributive residuated lattice.
Instead of proof that repeats this one [17], we just define \( \cdot^\sigma, \setminus^\sigma, \text{ and } /^\sigma \) explicitly. Here we note that the canonical extension of a lattice reduct is a perfect distributive lattice [20].

Let \( a, a' \in \mathcal{F}(\mathcal{L}^\sigma) \) and \( b \in \mathcal{F}(\mathcal{L}^\sigma) \), then

(i) \( a \setminus^\sigma b = \bigvee \{ x \setminus y \mid a \leq x \in \mathcal{L} \land y \leq b \} \) and similarly for the right residual

(ii) \( a \cdot^\sigma a' = \bigwedge \{ x \cdot x' \mid a \leq x \in \mathcal{L} \land a \leq x' \in \mathcal{L} \} \)

Let \( a, b \in \mathcal{L}^\sigma \), then.

(i) \( a \cdot^\sigma b = \bigvee \{ x \cdot^\sigma y \mid a \geq x \in \mathcal{F}(\mathcal{L}^\sigma) \land b \geq y \in \mathcal{F}(\mathcal{L}^\sigma) \} \)

(ii) \( a \setminus^\sigma b = \bigwedge \{ x \setminus^\sigma y \mid a \geq x \in \mathcal{F}(\mathcal{L}^\sigma) \land b \leq y \in \mathcal{I}(\mathcal{L}^\sigma) \} \) and \( b/a \) is defined similarly

The residuality property follows from the meet-density of \( \mathcal{F}(\mathcal{L}^\sigma) \) and join-density of \( \mathcal{I}(\mathcal{L}^\sigma) \) in \( \mathcal{L}^\sigma \). Thus \( \mathcal{L}^\sigma \) is a perfect distributive residuated lattice.

Let us describe the discrete duality for perfect distributive residuated lattices. Here we concretise the construction that establishes the discrete duality between perfect residuated lattices and perfect posets with ternary relation in [12] within a distributive setting. We piggyback the Raney representation of perfect distributive lattices as algebras of downsets of completely join-irreducible elements [42] that generalise Birkhoff representation for finite lattices [3]. We just recall that any perfect distributive lattice \( \mathcal{L} \) is isomorphic to the lattice \( \text{Down}(\mathcal{J}^\infty(\mathcal{L})) \) mapping \( a \in \mathcal{L} \) to \( \mathcal{J}^\infty(\mathcal{L}) \cap \downarrow a \).

This representation might be extended to the duality between the categories of perfect distributive lattices and posets.

Let \( \mathcal{L} \) be a perfect distributive residuated lattice. We define the relation \( R \subseteq \mathcal{J}^\infty(\mathcal{L}) \times \mathcal{J}^\infty(\mathcal{L}) \times \mathcal{J}^\infty(\mathcal{L}) \) as \( Rabc \Leftrightarrow a \cdot b \leq c \). Let us put \( \mathcal{O} = \uparrow \varepsilon \), where \( \varepsilon \) is a multiplicative identity. The structure \( \mathcal{L}_+ = (\mathcal{J}^\infty(\mathcal{L}), \leq, R, \mathcal{O}) \) is the dual frame of a perfect distributive residuated lattice \( \mathcal{L} \).

Let \( (W, \leq) \) be a poset and \( R \subseteq W^3 \), \( \mathcal{O} \) with the conditions (ii)-(vi) from Definition 2.4. Let us define the following operations on \( \text{Up}(W, \leq) \):

- \( A \setminus B = \{ w \in W \mid \forall u, v \in W \text{ with } Ruvu \text{ and } u \in A \Rightarrow v \in B \} \)
- \( B/A = \{ w \in W \mid \forall u, v \in W \text{ with } Ruvu \text{ and } v \in A \Rightarrow v \in B \} \)
- \( A \cdot B = \{ w \in W \mid \exists u, v \in W \text{ with } Ruvu \text{ and } u \in A \land v \in B \} \)

These operations are clearly well defined. Let us check \( A \setminus B \in \text{Up}(W, \leq) \), if \( A, B \in \text{Up}(W, \leq) \). Let \( x \in A \setminus B \) and \( x \leq y \), then \( A \cdot \{ x \} \subseteq B \). Let \( w \in \{ y \} \cdot A \), then there exists \( v \in A \text{ such that } Ruvw \), then \( Rxaw \) by the item (iii), Definition 2.4. Thus, \( \{ y \} \cdot A \subseteq B \). Similarly, \( B/A \in \text{Up}(W, \leq) \). The residuality property for these operations holds immediately.

The following theorem establishes the discrete duality between perfect distributive residuated lattices and posets with a ternary relation that encodes the product. Let us call such a poset with the relation as above a ternary Kripke frame.

**Theorem 3.9**
(i) Let $\mathcal{R}$ be a perfect distributive residuated lattice, then $\mathcal{R} \cong (\mathcal{R}_+)^+$

(ii) Let $\mathcal{F}$ be a ternary Kripke frame, then $\mathcal{F} \cong (\mathcal{F}^+_\oplus)$

Proof.

(i) Let $R$ be a lattice reduct of $\mathcal{R}$. According to the Raney representation, 
\[ \eta : R \cong \text{Up}(\mathcal{J}^\infty(R), \leq) \] such that $\eta : a \mapsto \{ b \in \mathcal{J}^\infty(R) \mid a \leq b \}$, where $\leq$ is a dual order on $\mathcal{J}^\infty(R)$. Let us ensure that this isomorphism also preserves products and residuals.

Let $z \in \eta(a) \cdot \eta(b)$, then there exists $x \in \eta(a)$ and $y \in \eta(b)$ such that $x \cdot y \leq z$. That is, $a \leq x$ and $b \leq y$, so $a \cdot b \leq z$, then $z \in \eta(a \cdot b)$.

Let $z \in \eta(a \cdot b)$, then $z$ is a join-irreducible element such that $a \cdot b \leq z$. Then $\uparrow z$ is a prime filter since $R$ is perfect distributive. It is clear that $\uparrow a \cdot \uparrow b \subseteq \uparrow b$. By Lemma 3.2, there exists prime filters $A$ and $B$ such that $\uparrow a \subseteq A$, $\uparrow b \subseteq B$, $\uparrow a \cdot \uparrow b \subseteq \uparrow z$ and $\uparrow a \cdot B \subseteq \uparrow z$. $R$ is completely distributive, so there exists $a', b' \in \mathcal{J}^\infty(R)$ such that $\uparrow a' = A$ and $\uparrow b' = B$. Moreover, $\uparrow a \subseteq \uparrow a'$ and $\uparrow b \subseteq \uparrow b'$ implies $a \leq a'$ and $b \leq b'$, so $a' \in \eta(a)$ and $b' \in \eta(b)$. So $a' \cdot b' \leq z$, and, thus, $z \in \eta(a) \cdot \eta(b)$.

$\eta$ preserves left and right residuals similarly to [15], Lemma 6.10. that was shown for arbitrary bounded residuated residuated lattices and the extended Priestley embedding.

(ii) $\varepsilon : (W, \leq) \to (\mathcal{J}^\infty(W, \leq), \leq)$ is a poset isomorphism such that $\varepsilon : a \mapsto \uparrow a$.

This isomorphism might be extended to the frame isomorphism via the frame conditions that connect a ternary relation with the partial order.

\[ \square \]

4 Discrete duality and completeness

In this section, we establish a discrete duality between the categories of all Kripke frames and the category of all perfect residuated distributive modal algebras. We show that the Thomason’s theorem [48] holds for normal residuated distributive modal logics.

Definition 4.1 Let $\mathcal{L}$ be a perfect distributive residuated lattice and $\Box, \Diamond$ unary operators on $\mathcal{L}$, then $\mathcal{M} = \langle \mathcal{L}, \Box, \Diamond \rangle$ is called a perfect distributive residuated modal algebra, if for each where $A \subseteq \mathcal{L}$ and $a, b \in A$

\begin{itemize}
  \item $\Box \land A = \land \{ \Box a \mid a \in A \}$
  \item $\Diamond \lor A = \lor \{ \Diamond a \mid a \in A \}$
  \item $\Box a \cdot \Diamond b \leq \Box (a \cdot b)$
\end{itemize}

Given $\mathcal{M}, \mathcal{N}$ perfect residuated distributive modal algebras, a map $\mathcal{M} \to \mathcal{N}$ is a homomorphism if $f$ is a complete lattice homomorphism that preserves product, residuals, modal operators, and the multiplicative identity.

Let us show that the variety of all RDMAs is closed under canonical extensions.

Lemma 4.2 Let $\mathcal{R}$ be a perfect distributive residuated lattice and $\mathcal{M} = \langle \mathcal{R}, \Box, \Diamond \rangle$ an RDMA, then $\mathcal{M}^\sigma = \langle \mathcal{R}, \Box^\sigma, \Diamond^\sigma \rangle = \langle \mathcal{R}, \Box^\sigma, \Diamond^\sigma \rangle$ is a perfect
DRMA.

Proof. The lattice reduct of $\mathcal{R}$ is a perfect distributive lattice, [20]. In fact, one needs to show that the inequation $\Box a \cdot \Diamond b \leq \Diamond(a \cdot b)$ is canonical. Firstly, let us suppose that $a, b \in C(\mathcal{M}^\sigma)$. Note that $\Box^\sigma a \cdot \Diamond^\sigma b = \bigwedge\{\Box x \cdot \Diamond y \mid a \leq x \in \mathcal{M}, b \leq y \in \mathcal{M}\}$ that follows from the definition of a filter element, the fact that $\Box^\sigma$ preserves all infima and $\cdot$ is an order-preserving operation. Then:

$$\Box^\sigma a \cdot \Diamond^\sigma b = \bigwedge\{\Box x \cdot \Diamond y \mid a \leq x \in \mathcal{L}, x \leq x \in \mathcal{L}\} \leq \bigwedge\{\Box(x \cdot y) \mid a \leq x \leq \mathcal{L} \& b \leq x \in \mathcal{L}\} = \bigwedge\{\Box^\sigma(x \cdot y) \mid a \leq x \in \mathcal{L} \& b \leq x \in \mathcal{L}\} = \bigwedge\{\Box^\sigma b \mid a \leq x \in \mathcal{L} \& b \leq x \in \mathcal{L}\}$$

Let $a, b \in \mathcal{L}^\sigma$, then

$$\Box^\sigma a \cdot \Diamond^\sigma b = \bigvee\{\Box^\sigma x \cdot \Diamond^\sigma y \mid a \leq x \in \mathcal{C}(\mathcal{L}^\sigma) \& b \leq y \in \mathcal{C}(\mathcal{L}^\sigma)\} \leq \bigvee\{\Box^\sigma(x \cdot y) \mid a \leq x \in \mathcal{C}(\mathcal{L}^\sigma) \& b \leq y \in \mathcal{C}(\mathcal{L}^\sigma)\} \leq \Box^\sigma a \cdot \Diamond^\sigma b$$

Definition 4.3 A residual normal modal logic $\mathcal{L}$ is called canonical, if $\mathcal{V}_\mathcal{L}$ is closed under canonical extensions

The complex algebra of a Kripke frame $\mathcal{F} = (W, \preceq, R, R_0, R_0, O)$ is a complex algebra of the underlying residuated frame $\mathcal{F}^+$ with the modal operators defined as $[R_0]A = \{u \in W \mid \forall w (uR_0w \Rightarrow w \in A)\}$ and $[R_0] = \{u \in W \mid \exists w (uR_0w \& w \in A)\}$. Here $A$ is upwardly closed subset. These operations are well-defined. The dual frame of a perfect RDMA $\mathcal{M} = (\mathcal{M}, \bigvee, \bigwedge, \Box, \Diamond, \land, \lor, \varepsilon)$ is the dual frame $\mathcal{M}^+$ of an underlying perfect distributive residuated lattice with binary relations on completely join irreducible elements introduced as $aR_0b \Leftrightarrow \Box \kappa(a) \leq \kappa(b)$ and $aR_0 b \Leftrightarrow a \leq \Diamond b$. Here, $\kappa$ is an order isomorphism between between $\mathcal{F}^\infty(\mathcal{M})$ and $\mathcal{M}^\infty(\mathcal{M})$.

Logically, Kripke frames and their complex algebras are connected with each other as follows:

Proposition 4.4 Let $\mathcal{F}$ be a Kripke frame, then $\log(\mathcal{F}) = \log(\mathcal{F}^+) = \{\varphi \Rightarrow \psi \mid \mathcal{F}^+ \models \varphi \leq \psi\}$

The following discrete duality theorem is merely a combination of Theorem 3.9 and the similar fact proved for distributive modal algebras and frames for distributive modal logics [22].

Theorem 4.5

(i) Let $\mathcal{F}$ be a Kripke frame, then $\mathcal{F} \cong (\mathcal{F}^+)_+$

(ii) Let $\mathcal{M}$ be a perfect DRMA, then $\mathcal{M} \cong (\mathcal{M}_+)^+$

(iii) Functors $(\_)_+ : pDRMA \rightleftharpoons KF : (\_)_+^*$ establish a dual equivalence between the categories of all Kripke frames and all perfect RDMA.

Proof. It is easy to check that if $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ a bounded morphism is a bounded morphism of Kripke frames, then $f^+ : \mathcal{F}_2^+ \rightarrow \mathcal{F}_1^+$ such that $f^+ : A \rightarrow f^{-1}[A]$ is a perfect DRMA morphism. It is immediate that $h_+ : \mathcal{M}_2^+ \rightarrow \mathcal{M}_1^+$ is a bounded morphism, where $h : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a perfect DRMA.
morphism. Thus, the dual equivalence follows from the previous two items and
the lemma that claims that $(.)_+$ and $(.)^+$ are contravariant functors.

The discrete duality established above together with canonical extensions
of residuated distributive modal algebras provides the following consequence:

**Theorem 4.6** Let $L$ be a canonical residual distributive modal logic, then $L$ is
Kripke complete.

**Proof.** The proof is similar to the analogous fact proved in [22], but we repro-
duce a sketch.

Let $\varphi \Rightarrow \psi \in L$. Then $M_L \models \varphi \leq \psi$ by Theorem 3.5. But $M_L \models \varphi \leq \psi$ iff
and only $(M_L)^\sigma \models \varphi \leq \psi$ since $M_L$ is a subalgebra of $(M_L)^\sigma$ by Lemma 4.2
and the condition according to which $\mathcal{V}_L$ is a canonical variety. By discrete
duality, Theorem 4.5, $(M_L)^\sigma+ \models \varphi \leq \psi$. By Proposition 4.4, $(M_L)^\sigma+ \models
\varphi \Rightarrow \psi$. In fact, $(M_L)^\sigma+ \models \varphi \Rightarrow \psi$. Let us show that $(M_L)^\sigma+ \models L.

As a consequence, the minimal normal residual distributive modal logic is
complete with respect to the class of all Kripke frames.

**Corollary 4.7** $L_K$ is Kripke-complete.

Now we show that the following sequents that describe modalities as storage
operators are canonical ones. This lemma partially repeats the Propositions
6.7 – 6.10 here [12].

**Lemma 4.8** The following sequents are canonical:

(i) $\Box p \cdot q \Leftrightarrow q \cdot \Box p$
(ii) $\Box p \Rightarrow \Box p \cdot \Box p$
(iii) $\Box p \cdot q \Rightarrow \Box p \cdot q \cdot \Box p$, $q \cdot \Box p \Rightarrow \Box p \cdot q \cdot \Box p$
(iv) $\Box p \leq \varepsilon$
(v) $\Box (p \land q) \Leftrightarrow \Box p \cdot \Box q$
(vi) $\Box (p \lor q) \Rightarrow \Box p \lor \Box q$ and $\Box (p / q) \Rightarrow \Box (p / q)$

**Proof.** Let us check only the third sequent. The rest sequents might be checked
similarly.

Let $K_{\Gamma}$ be $K \oplus \Box a \bullet b \Rightarrow \Box a \bullet b \bullet a$ and $V_{\Gamma_{K_{\Gamma}}}$ its variety. Let us show that
$V_{\Gamma_{K_{\Gamma}}}$ is closed under canonical extensions.

Let $M \in V_{\Gamma_{K_{\Gamma}}}$, one needs to check that $M^\sigma \in V_{\Gamma_{K_{\Gamma}}}$. Let $a,b \in C(M^\sigma)$.

\[ \square^\sigma a \cdot \sigma b = \bigwedge \{ \Box x \mid a \leq x \in M \} \cdot \sigma \bigwedge \{ y \mid b \leq y \in M \} = \]
\[ \bigwedge \{ \Box x \cdot y \mid a \leq x \in M \land b \leq y \in M \} \leq \]
\[ \bigwedge \{ \Box x \cdot y \mid a \leq x \in M \land b \leq y \in M \} = \square^\sigma a \cdot \sigma b \cdot \sigma a \]

Let $a,b \in M^\sigma$, then

\[ \square^\sigma a \cdot \sigma b = \bigvee \{ \Box^\sigma x \cdot \sigma y \mid a \geq x \in C(M^\sigma) \land b \geq y \in C(M^\sigma) \} \leq \]
\[ \bigvee \{ \Box^\sigma x \cdot \sigma y \mid a \geq x \in C(M^\sigma) \land b \geq y \in C(M^\sigma) \} = \square^\sigma a \cdot \sigma b \cdot \sigma a \]

The following completeness theorems follows from Theorem 4.6, Corol-
ary 4.7, and Lemma 4.8.
Corollary 4.9 Let $\Gamma$ be the set of all sequents from the lemma above and $\Delta \subseteq \Gamma$. Then $\mathcal{L} = L_K \oplus \Delta$ is Kripke complete.

One may also consider the subexponential polymodal case similar to [32]. Let us define a subexponential signature:

Definition 4.10 A subexponential signature is an ordered quintuple: $\Sigma = \langle I, \preceq, W, C, E \rangle$, where $\langle I, \preceq \rangle$ is a preorder. $W, C, E$ are upwardly closed subsets of $I$ and $W \cap C \subseteq E$.

Let us define the following axioms:

- $\Box_s p \cdot \Box_s q \Rightarrow \Box_s (p \cdot q)$, $s \preceq s_1, s_2$
- $\Box_s p \Rightarrow p$
- $\Box_s p \Rightarrow \Box_s p$
- $\Box_s p \cdot q \Rightarrow \Box_s p \cdot q \cdot \Box_s p \Rightarrow \Box_s p \cdot q \cdot \Box_s p$, where $s \in C$
- $\Box_s p \cdot q \Leftrightarrow p \cdot \Box_s q$, $s \in E$
- $\Box_s p \Rightarrow 1$, where $s \in W$

The logic DSMALC$_\Sigma$ is a residual distributive polymodal logic with modal axioms as above plus $\Box_s (p \land q) \Leftrightarrow \Box_s p \land \Box_s q$ and $\Box_s \top \Leftrightarrow \top$ for each $s \in \Sigma$. For simplicity, let us put the diamond axioms $\Diamond_s (p \lor q) \Leftrightarrow \Diamond_s p \lor \Diamond_s q$ and $\Diamond_s \bot \Leftrightarrow \bot$ with the dual S4 axioms $p \Rightarrow \Diamond_s p$ and $\Diamond_s \Diamond_s p \Rightarrow \Diamond_s p$ for each $s \in \Sigma$ axioms without any additional postulates.

Theorem 4.11 DSMALC$_\Sigma$ is canonical and, thus, Kripke complete.

Proof. One may show that the variety $Y_{DSMALC}_\Sigma$ is canonical showing that $\Box_s p \cdot \Box_s q \leq \Box_s (p \cdot q)$ for $s \preceq s_1, s_2$ is a canonical inequation similarly to Lemma 4.2. The whole statement follows from the observation above and Theorem 4.6, Lemma 4.8, and Corollary 4.9. \qed

5 Topological duality

In this section, we characterise a topological duality for residuated distributive modal algebras in the same fashion as it is provided for Boolean modal or intuitionistic logics described, e. g., in [14] [44]. That is, we consider topological Kripke frames, ternary Kripke frames defined on Priestley spaces with binary modal relations, the category of which is dually equivalent to the category of all RDMAs. Alternatively, one may characterise such a duality in terms of general descriptive frames following the Goldblatt’s approach [27]. See the Appendix to have an explanation the Priestley duality related definitions, terms, and notations.

Firstly, we consider a Priestley-style duality for residuated distributive bounded lattices as follows. Here we piggyback the construction observed by Galatos in his Ph.D. thesis [15]. This construction is a noncommutative generalisation of relevant spaces, the dual spaces of relevant algebras studied by Urquhart [49]. In fact, bDRL-spaces and their extensions with modal relations are the instances of relational Priestley spaces [26].
Definition 5.1 Let $\mathcal{X} = (X, \tau, \leq)$ be a Priestley space, $R \subseteq X^3$ and $E \subseteq X$. A bDRL-space is a tuple $\mathcal{X} = (X, \tau, \leq, R, E)$ such that:

(i) For all $x, y, z, w \in X$ there exists $u \in X$ such that $R(x, y, u)$ and $R(u, z, w)$ iff there exists $v \in X$ such that $R(y, z, v)$ and $R(x, v, w)$

(ii) For all $x, y, z, w \in X$ if $x \leq y$ then $R(y, u, v)$ implies $R(x, u, v)$, $R(u, v, y)$ implies $R(u, x, v)$, and $R(u, v, y)$ implies $R(u, v, x)$

(iii) Let $A, B \subseteq X$ be upwardly closed clopens then $R[A, B, \bot]$, $\{ z \in X \mid R[z, B, \bot] \subseteq A \}$, and $\{ z \in X \mid R[B, z, \bot] \}$ are also clopens

(iv) For all $x, y, z \in X$ $\sim R(x, y, z)$ iff there exists upwardly closed clopens $A, B \subseteq X$ such that there exist $x \in A$ and $y \in B$ such that $z \notin R[A, B, \bot]$

(v) $E$ is upwardly closed clopen such that if $A$ is clopen then $R[E, A, \bot] = R[A, E, \bot] = A$

Here $R[A, B, \bot]$ denotes $\{ c \in X \mid \exists a \in A \exists b \in B Rabc \}$.

We note that such a space is totally disconnected with respect to a ternary relation according to the fourth condition. We introduce desired topological Kripke frames as bDRL-spaces with modal relations.

Definition 5.2 A modal bDRL-space is a structure $\mathcal{X} = (X, \tau, \leq, R, R_\bot, R_\circ, E)$, where $(X, \tau, \leq, R, E)$ is a bDRL space and the following conditions hold:

(i) If $A$ is upwardly closed clopen, then $R_\bot(A), R_\circ(A)$ are upwardly closed clopens

(ii) For each $x \in X$, $R_\bot(x)$ and $R_\circ(x)$ are closed

(iii) $Ruvw \& uR_\bot w \Rightarrow \exists x, y \in W \text{ } Rxyw' \& uR_\circ x \& vR_\circ y$

(iv) $u \leq v \& vR_\circ w \Rightarrow uR_\circ w$

(v) $u \leq v \& uR_\circ w \Rightarrow vR_\circ w$

Given a residuated distributive modal algebra $\mathcal{M} = (\mathcal{R}, \diamond, \Box)$ on a bounded distributive lattice $\mathcal{R}$, we define the set of all prime filters $\text{PF}(\mathcal{L})$ and a map $\phi$ similarly to the bounded distributive lattice case described in the paper appendix. Let us define a ternary relation $R \subseteq \text{PF}(\mathcal{L}) \times \text{PF}(\mathcal{L}) \times \text{PF}(\mathcal{L})$ as $R(A, B, C) \equiv A \bullet B \subseteq C$ and $E = \phi(e)$. We also define binary relations $R_\bot$ and $R_\circ$ on $\text{PF}(\mathcal{L})$ as $AR_\bot B \equiv b \in B \Rightarrow \Box a \in A \Rightarrow a \in B$. $AR_\circ B \equiv \exists a \in A \Rightarrow a \in B$.

Standardly, the subbasis of the topology $\tau$ is defined with by the sets $\phi(a)$ and $-\phi(a)$, $a \in \mathcal{M}$. Then the structure $\langle \text{PF}(\mathcal{M}), \tau, E, R, R_\bot, R_\circ, E \rangle$ is the dual space of a residuated distributive modal algebra $\mathcal{M}$.

Let $\mathcal{X} = (X, \tau, \leq, R, R_\bot, R_\circ, E)$ be a modal bDRL-space and $\text{CIUp}(\mathcal{X})$ the set of all upwardly closed clopens of $\mathcal{X}$. We define product as a binary operation on $\text{CIUp}(\mathcal{X})$ as $A \circ B = R[A, B, \bot]$. Residuals are $A \setminus B = \{ c \in X \mid A \circ c \subseteq B \}$ and $B/A = \{ c \in X \mid c \circ A \subseteq B \}$. Modal operators $[R_\bot], [R_\circ]$ are defined as $[R_\bot]A = \{ a \in X \mid \forall b \in X(aR_\bot b \Rightarrow b \in A) \}$ and $[R_\circ]A = \{ a \in X \mid \exists b \in X(aR_\circ b \& b \in A) \}$. The structure $\langle \text{CIUp}(\mathcal{X}), \cup, \cap, \emptyset, X, \circ, \setminus, /, E, [R_\bot], [R_\circ] \rangle$ is
Theorem 5.5

(i) The dual algebra of a modal bDRL-space \(X\) is an RDMA.

(ii) The dual space of a residuated distributive modal algebra \(M\) is a modal bDRL-space.

Proof.

(i) Let \(A, B\) be upwardly closed clopens, let us show that \([R_\square]A \supseteq [R_\square]B \subseteq [R_\square](A \bullet B)\). Let \(c \in [R_\square]A \supseteq [R_\square]B\), then \(c \in R[R_\square]A, [R_\square]B, \square\). This denotes that \(R(a, b, c)\) for some \(a \in R_\square A\) and \(b \in R_\square B\). Let \(c' \in X\) such that \(cR_\square c'\), let us show that \(c' \in A \bullet B\). \(R(a, b, c)\) and \(cR_\square c'\) implies that \(R(a', b', c')\) for some \(a' \in R_\square(a)\) and \(b' \in R_\square(b)\) by the definition of a modal bDRL-space. Thus, \(a' \in A\) and \(b' \in B\), so \(c' \in A \bullet B\).

(ii) We check the condition that connect the ternary relation with the \(\square\)-relation. Let \(A, B\) be prime filters. Let us suppose that \(RABC\) and \(CR_\square C'\), where \(A, B, C, C'\) are prime filters. Let us show that \(RA'BC'\) for some prime filters \(A', B'\). Let us put \(A_1 := \{a \in X \mid \square a \in A\}\) and \(B_1 := \{b \in X \mid \square b \in B\}\). \(A_1\) and \(B_1\) are clearly filters. Now let us show that \(A_1 \cdot B_1 \subseteq C'\). Let \(a \cdot b \in A_1 \cdot B_1\), then \(\square a \in A\) and \(\square b \in B\). So \(\square a \cdot \square b \in A \bullet B\), so \(\square (a \cdot b) \in A \bullet B\). Hence \(\square (a \cdot b) \in C\) and \(a \cdot b \in C'\). By Lemma 3.2, there exist prime filters \(A' \supseteq A_1\) and \(B' \supseteq B_1\) such that \(A''B'' \subseteq C'\).

Let \(A\) be a prime filter, then \(R_\square(A)\) and \(R_\square(A)\) are closed that might be shown as the well-known Esakia’s lemma [13]. Let \(A\) be an upwardly closed clopen, \(R_\square(A)\) and \(R_\square(A)\) are upwardly closed clopen. The last two facts might be proved similarly to the intutionistic modal logic case [38].

\(\Box\)

Definition 5.4 Given modal bDRL spaces \(X, Y\), a contnuous bounded is a map \(f : X \rightarrow Y\) such that \(f\) is a Priestley map that preserves ternary and binary relations as a bounded morphism.

As usual, the previous lemma allows one to claim that PF and ClUp are contravariant functors. The following theorem establish a desired topological duality itself.

Theorem 5.5

(i) Let \(M\) be a residuated distributive modal algebra, then \(\text{ClUp}(\text{PF}(M)) \cong M\)

(ii) Let \(X\) be a modal dBRL-space, then \(\text{PF}(\text{ClUp}(X)) \cong X\)
Contravariant functors $PF : RDMA \rightleftarrows TKF : ClUp$ constitute a dual equivalence between the category of all residual distributive modal algebras and the category of all modal dBRL-spaces.

Proof.

(i) The isomorphism is map $a \mapsto \phi(a)$ that commutes with products, residuals and modal operators as discussed above.

(ii) A homeomorphism is a map $g : x \mapsto \{A \in ClUp \mid x \in A\}$. As it is shown by Galatos, $R_X(x, y, z) \Leftrightarrow R_{PF(ClUp(X))}(g(x), g(y), g(z))$. One may immediately extend this homeomorphism and show that this map commutes with binary modal relations.

(iii) Follows from the previous two items and the previous theorem. Let us ensure briefly that these functors behave as expected with morphisms. Let $h : M_1 \rightarrow M_2$ be an RDMA homomorphism, then a map $PF(h) : PF(M_2) \rightarrow PF(M_1)$ such that $PF(h) : F \mapsto h^{-1}[F]$ is a Priestley map, where $F$ is a prime filter. This map also satisfies the monotonicity and lifting properties for a ternary relation, see [15]. One may show that $PF(h)$ has the monotonicity and lifting properties for $\lhd$ and $\rhd$ similarly to the intuitionistic modal logic case. On the other hand, $ClUp(f) : ClUp(Y) \rightarrow ClUp(X)$ such that $ClUp(f) : A \mapsto f^{-1}[A], A \in ClUp(Y)$ is a lattice homomorphism. $ClUp(f)$ also preserves product and residuals. $ClUp(f)$ preserves $[R_G]$ and $\langle R_O \rangle$ similarly to the intuitionistic modal logic case.

Hence, we obtained the Priestley-style duality for the category of residuates distributive modal algebras and the category of all topological Kripke frames introduced by us.

6 Further work

In this paper, we overviewed canonicity for the distributive full Lambek calculus and its modal extensions within a “usual” Kripkean semantics. The further questions that should be solved are Sahlqvist and Goldblatt-Thomason theorems for such semantics and its non-distributive generalisation to study canonicity and modal definability for noncommutative modal logic with residuals in depth. One may consider for these purposes the frameworks described in [9] and [8].

One may also consider a Kleene star as a modal operator [34], but such a modality is neither $\Box$ nor $\Diamond$. The (non)canonicity of the variety of residuated Kleene lattices also should be studied and explored considering a Kleene residuated lattice as a sort of bounded lattice with operators.

We took the Lambek calculus with additive connections as the underlying logic requiring the lacking distributivity principle which is unprovable in the full Lambek calculus. A Priestley-style topological duality provided in this paper might be extended considering dual spaces for non-distributive residuated modal algebras using the canonical extensions technique studied by Gehrke and van Gool, e.g., here [23].

The distributive Lambek calculus also has the cut-free hypersequent cal-
The Distributive Full Lambek Calculus with Modal Operators

culus and FMP [33], but the same issues for the modal extensions are not investigated yet.

Appendix
In this section, we recall the required background related to bounded distributive lattice canonical extensions and topological duality. Such notions as lattice, distributive lattice, filter, and prime filters are supposed to be known. We refer the reader to these textbooks [10] [45].

Let us recall Priestley duality, the dual equivalence between the category of bounded distributive lattices and the category of Priestley spaces [41]. First of all, we define a Priestley space.

Definition .1

(i) A Priestley space is a triple \( \mathcal{X} = (X, \tau, \leq) \), where \( (X, \tau) \) is a compact topological space, \( (X, \leq) \) is a bounded partial order with the additional Priestley separation axiom:

\[ x \not\leq y \implies \text{there exists a clopen } U \subseteq X \text{ such that } x \in U \text{ and } y \notin U \]

(ii) Let \( \mathcal{X}_1, \mathcal{X}_2 \) be Priestley spaces and \( f : \mathcal{X}_1 \to \mathcal{X}_2 \), then \( f \) is a Priestley map, if \( f \) is continuous, order-preserving and preserves bounds.

Let \( \mathcal{L} \) be a bounded distributive lattice and PF the set of prime filters in \( \mathcal{L} \). Let us define a map \( \phi : \mathcal{L} \to \mathcal{P}(\text{PF}(\mathcal{L})) \) such that \( \phi : a \mapsto \{ F \in \text{PF}(\mathcal{L}) \mid a \in F \} \). The sets \( \phi(a) \) and \( -\phi(a) \) form a subbasis of topology on \( \text{PF}(\mathcal{L}) \), where \( a \in \mathcal{L} \). The structure \( (\text{PF}(\mathcal{L}), \tau, \subseteq) \) is a Priestley space, where \( \tau \) is generated by the subbasis above.

Let \( \mathcal{X} = (X, \tau, \leq) \) be a Priestley space and ClUp(\( \mathcal{X} \)) the set of all clopen upwardly closed subsets of \( \mathcal{X} \). The dual algebra of a Priestley space is an algebra \( (\text{ClUp}(\mathcal{X}), \cap, \cup, \emptyset, X) \), which is a bounded distributive lattice.

Let \( \mathcal{L} \) be a bounded distributive lattice, then \( \eta_L : \mathcal{L} \to \text{ClUp}(\text{PF}(\mathcal{L})) \) is a lattice isomorphism. Given a Priestley space \( \mathcal{X} \), then \( \varepsilon_X : \mathcal{X} \to \text{PF}(\text{ClUp}(\mathcal{X})) \) is a Priestley homeomorphism. Moreover, if \( h : \mathcal{L}_1 \to \mathcal{L}_2 \) is a bounded lattice homomorphism, then \( \text{PF}(h) : \text{PF}(\mathcal{L}_1) \to \text{PF}(\mathcal{L}_2) \) is a Priestley map, where \( \text{PF}(h) : A \mapsto h^{-1}(A) \). If \( f : \mathcal{X}_1 \to \mathcal{X}_2 \) is a Priestley map, then \( \text{ClUp}(f) : \text{ClUp}(\mathcal{X}_2) \to \text{ClUp}(\mathcal{X}_1) \) is a bounded lattice homomorphism, where \( \text{ClUp}(f) : A \mapsto f^{-1}(A) \). The facts called above establish Priestley duality:

**Theorem .2** The functors \( \text{PF} : \text{BDistr} \rightleftharpoons \text{Priest} : \text{ClUp} \) constitute a dual equivalence between the category of all bounded distributive lattices and the category of all Priestley spaces.

Canonical extensions were introduced by Jonsson and Tarski to extend a Stone representation to Boolean algebras with operators [31]. Let us overview canonical extensions of distributive lattice expansions. We refer the reader to these paper [20] [21] to have a more detailed picture of bounded distributive lattice expansions and canonical extensions for them.
Given a complete lattice $\mathcal{L}$, $a \in \mathcal{L}$ is called completely join irreducible, if $a = \bigvee_{i \in I} a_i$ implies that $a = a_i$ for some $i \in I$. Completely meet irreducible elements are defined dually. By $J^\infty(\mathcal{L})$ ($M^\infty(\mathcal{L})$) we denote the set of all completely join (meet) irreducible elements. There is an order isomorphism $\kappa : J^\infty(\mathcal{L}) \to M^\infty(\mathcal{L})$ such that $\kappa : u \mapsto \bigvee (- \uparrow u)$ [20].

A complete lattice $\mathcal{L}$ is completely distributive [10], if for each doubly indexed subset $\{x_{ij}\}_{i \in I, j \in J}$ of $\mathcal{L}$ one has
$$\bigwedge_{i \in I} (\bigvee_{j \in J} x_{ij}) = \bigvee_{\alpha : I \to J} (\bigwedge_{i \in I} x_{i\alpha(i)})$$

A perfect distributive lattice is a doubly algebraic completely distributive lattice, that is:

**Definition 3** Let $\mathcal{L}$ be a bounded distributive lattice, then $\mathcal{L}$ is called perfect distributive lattice, if it is completely distributive and every $x \in \mathcal{L}$ has the form $x = \bigvee \{j \in J^\infty(\mathcal{L}) \mid j \leq x\}$ and $x = \bigwedge \{m \in M^\infty(\mathcal{L}) \mid x \leq m\}$. That is, $J^\infty(\mathcal{L})$ and $M^\infty(\mathcal{L})$ are join-dense and meet-dense in $\mathcal{L}$ correspondingly.

Given a lattice $\mathcal{L}$, by the completion of $\mathcal{L}$ we mean an embedding $\mathcal{L} \hookrightarrow \mathcal{L}'$, where $\mathcal{L}'$ is a complete lattice. For simplicity, we assume that $\mathcal{L}'$ contains $\mathcal{L}$ as a sublattice. The definition of a canonical extension is the standard one:

**Definition 4** Let $\mathcal{L}$ be a bounded distributive lattice, a canonical extension of $\mathcal{L}$ is a completion $\mathcal{L} \hookrightarrow \mathcal{L}$, where $\mathcal{L}$ is a complete lattice and the following conditions hold:

(i) (Density) Every element of $\mathcal{L}$ is both a join of meets and meets of joins of elements from $\mathcal{L}$

(ii) (Compactness) Let $S, T \subseteq \mathcal{L}$ such that $\bigwedge S \leq \bigvee T$ in $\mathcal{L}$, then there exist finite subsets $S' \subseteq S$ and $T' \subseteq T$ such that $\bigwedge S' \leq \bigvee T'$.

Now we define filter and ideal elements, or, closed and open elements, according to the alternative terminology.

**Definition 5** Let $\mathcal{L}$ be a bounded distributive lattice and $\mathcal{L}$ a canonical extension of $\mathcal{L}$. Let us define the following sets:

(i) $\mathcal{F}(\mathcal{L}') = \{x \in \mathcal{L}' \mid x$ is a meet of elements from $\mathcal{L}\}$, the set of filter elements

(ii) $\mathcal{I}(\mathcal{L}') = \{x \in \mathcal{L}' \mid x$ is a join of elements from $\mathcal{L}\}$, the set of ideal elements

It is known that the poset $F(\mathcal{L})$ is isomorphic to the poset $\text{Filt}(\mathcal{L})$, the set of all filters of $\mathcal{L}$ and the similar statement holds for $\mathcal{I}(\mathcal{L}')$ and the of all ideals of $\mathcal{L}$. We recall that a canonical extension of a bounded lattice $\mathcal{L}$ is unique up to isomorphism that fixes $\mathcal{L}$, see, e. g., [19]. For each canonical extension $\mathcal{L} \hookrightarrow \mathcal{L}'$, the poset $F(\mathcal{L}') \cup I(\mathcal{L}')$ is uniquely defined by $\mathcal{L}$. The uniqueness of a canonical extension of $\mathcal{L}$ up to an isomorphism fixing $\mathcal{L}$ follows from this observation.

In the case of bounded distributive lattices, one has the following fact [20]:

**Proposition 6**
Let $\mathcal{L}$ be a bounded distributive lattice, then $\mathcal{L}^\sigma$ is a perfect distributive lattice.

We also note that canonical extensions commute with dual order and Cartesian product. That is, $(\mathcal{L}^{op})^\sigma \cong (\mathcal{L}^\sigma)^{op}$ and $(\mathcal{L}^\sigma)^{op} \cong (\mathcal{L}^{op})^\sigma$. Now we overview bounded distributive lattices expansions, that is, bounded distributive lattices enriched with the additional family of operators, and their canonical extensions [19].

Let $\mathcal{K}$, $\mathcal{L}$ be bounded distributive lattices and $f : \mathcal{K} \rightarrow \mathcal{L}$. Let us define maps $\mathcal{f}^\sigma$, $\mathcal{f}^\pi : \mathcal{K}^\sigma \rightarrow \mathcal{L}^\sigma$ as follows:

(i) $\mathcal{f}^\sigma (u) = \bigwedge \{ \mathcal{V} \{ f(a) \mid a \in \mathcal{K}, a \leq y \} \mid \mathcal{F}(\mathcal{K}^\sigma) \ni x \leq u \leq y \in \mathcal{I}(\mathcal{K}^\sigma) \}$

(ii) $\mathcal{f}^\pi (u) = \bigvee \{ \mathcal{V} \{ f(a) \mid a \in \mathcal{K}, x \leq a \leq y \} \mid \mathcal{F}(\mathcal{K}^\sigma) \ni x \leq u \leq y \in \mathcal{I}(\mathcal{K}^\sigma) \}$

Every element of $\mathcal{K}$ is a filter element and an ideal element of $\mathcal{L}^\sigma$, then $\mathcal{f}^\sigma$ and $\mathcal{f}^\pi$ both extend $f$. It is clear that $\mathcal{f}^\sigma \leq \mathcal{f}^\pi$ in a pointwise order. We formulate the following fact about extensions of maps on bounded distributive lattices that we are going to use further. If the original map $f$ is order preserving, then one may simplify the definitions of $\mathcal{f}^\sigma$ and $\mathcal{f}^\pi$ as follows, e.g. [22] [19]:

**Proposition 7**

Let $\mathcal{K}$, $\mathcal{L}$ be bounded distributive lattices and $f : \mathcal{K} \rightarrow \mathcal{L}$. Then:

(i) $\mathcal{f}^\sigma (u) = \bigwedge \{ \mathcal{V} \{ f(a) \mid a \in \mathcal{K}, a \leq y \} \mid u \leq y \in \mathcal{I}(\mathcal{K}^\sigma) \}$, where $u \in \mathcal{K}^\sigma$

(ii) $\mathcal{f}^\pi (u) = \bigvee \{ \mathcal{V} \{ f(a) \mid a \in \mathcal{K}, x \leq a \leq y \} \mid u \geq x \in \mathcal{F}(\mathcal{K}^\sigma) \}$, where $u \in \mathcal{K}^\sigma$

(iii) $\mathcal{f}^\sigma (x) = \mathcal{f}^\pi (x) = \bigwedge \{ f(a) \mid x \leq a \in \mathcal{K} \}$, where $x \in \mathcal{F}(\mathcal{K}^\sigma)$

(iv) $\mathcal{f}^\sigma (y) = \mathcal{f}^\pi (y) = \bigvee \{ f(a) \mid a \in \mathcal{K}, a \leq y \}$, where $y \in \mathcal{I}(\mathcal{K}^\sigma)$

As a consequence, for each $u \in \mathcal{K}^\sigma$ one has $\mathcal{f}^\sigma (u) = \bigvee \{ f(x) \mid u \geq x \in \mathcal{F}(\mathcal{K}^\sigma) \}$. The third and fourth items denote that $\mathcal{f}^\sigma$ and $\mathcal{f}^\pi$ send filter (ideal) elements to filter (ideal) ones. A map $f : \mathcal{K} \rightarrow \mathcal{L}$ is called smooth, if $\mathcal{f}^\sigma (u) = \mathcal{f}^\pi (u)$ for each $u \in \mathcal{K}^\sigma$. In particular, a map is smooth if it preserves or reverses finite joins or meets. For instance, modal operators $\Box$ and $\Diamond$ are smooth and their smoothness since $\Box \leq \Diamond$ and $\Diamond \leq \Box$.

Thus, $\Box^\sigma = \Box^\pi$ and $\Diamond^\sigma = \Diamond^\pi$.

A **Acknowledgements**

The author is grateful to Nick Bezhanishvili, Stepan Kuznetsov, Andre Scedrov, and Valentin Shehtman for remarkable conversations, advice, and consulting.

The author would like to thank Prof. Johan van Benthem especially for miscellaneous discussions on the Lambek calculus semantics and its consideration in a broader modal landscape.

References

[1] Abramsky, S., *Computational interpretations of linear logic*, Theoretical computer science 111 (1993), pp. 3–57.
[2] Andréea, H. and S. Mikuláš, _Lambek calculus and its relational semantics: Completeness and incompleteness_, Journal of Logic, Language and Information 3 (1994), pp. 1–37. URL https://doi.org/10.1007/BF01066355

[3] Birkhoff, G. et al., _Rings of sets_, Duke Mathematical Journal 3 (1937), pp. 443–454.

[4] Blackburn, P., M. d. Rijke and Y. Venema, “Modal Logic,” Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 2001.

[5] Celani, S. and R. Jansana, Priestley duality, a subleqst theorem and a goldblatt-thomason theorem for positive modal logic, Journal of IGPL 7 (1999), pp. 683–715.

[6] Chagrov, Z., A. Chagrov and M. Zakharyaschev, “Modal Logic,” Oxford logic guides, Clarendon Press, 1997. URL https://books.google.ru/books?id=dhgi5NF4RtcC

[7] Chernilovskaya, A., M. Gehrke and L. Van Rooijen, Generalized kripke semantics for the lambek–grishin calculus, Logic Journal of the IGPL 20 (2012), pp. 1110–1132.

[8] Conradie, W. and A. Palmigiano, Algorithmic correspondence and canonicity for non-distributive logics, Ann. Pure Appl. Log. 170 (2019), pp. 923–974. URL https://doi.org/10.1016/j.apal.2019.04.003

[9] Conradie, W., A. Palmigiano and A. Tzimoulis, Goldblatt-thomason for le-logics, arXiv preprint arXiv:1809.08225 (2018).

[10] Davey, B. A. and H. A. Priestley, “Introduction to lattices and order,” Cambridge university press, 2002.

[11] Doen, K., _A brief survey of frames for the lambek calculus_, Mathematical Logic Quarterly 38 (1992), pp. 179–187. URL https://onlinelibrary.wiley.com/doi/abs/10.1002/malq.19920380113

[12] Dunn, J. M., M. Gehrke and A. Palmigiano, Canonical extensions and relational completeness of some substructural logics, J. Symbolic Logic 70 (2005), pp. 713–740. URL https://doi.org/10.2178/jsl/1122038911

[13] Esakia, L., _Topological kripke models_, , 214, Russian Academy of Sciences, 1974, pp. 298–301.

[14] Esakia, L., “Duality Theory: Hybrids,” Springer International Publishing, Cham, 2019 pp. 41–75. URL https://doi.org/10.1007/978-3-030-12096-2_3

[15] Galatos, N., “Varieties of residuated lattices,” Ph.D. thesis (2003).

[16] Galatos, N., P. Jipsen, T. Kowalski and H. Ono, 151, Elsevier, 2007.

[17] Gehrke, M., _Topological duality and algebraic completions_.

[18] Gehrke, M., “Canonical Extensions, Esakia Spaces, and Universal Models,” Springer Netherlands, Dordrecht, 2014 pp. 9–41. URL https://doi.org/10.1007/978-94-017-8860-1_2

[19] Gehrke, M. and J. Harding, _Bounded lattice expansions_, Journal of Algebra 238 (2001), pp. 345 – 371. URL http://www.sciencedirect.com/science/article/pii/S0021869300986228

[20] Gehrke, M. and B. Jnsson, Boundned distributive lattices with operators, Math. Japonica 40 (1994), pp. 207–215.

[21] Gehrke, M. and B. Jnsson, _Bounded distributive lattice expansions_, Mathematica Scandinavica 94 (2004), pp. 13–45. URL http://www.mscand.dk/article/view/14428

[22] Gehrke, M., H. Nagahashi and Y. Venema, _A sahqvist theorem for distributive modal logic_, Annals of Pure and Applied Logic 131 (2005), pp. 65 – 102. URL http://www.sciencedirect.com/science/article/pii/S0168007204400880

[23] Girard, J.-Y., _Linear logic: its syntax and semantics_, London Mathematical Society Lecture Note Series (1995), pp. 1–42.

[24] Girard, J.-Y. and Y. Lafont, _Linear logic and lazy computation_, in: International Joint Conference on Theory and Practice of Software Development, Springer, 1987, pp. 52–66.

[25] Goldblatt, R., _Varieties of complex algebras_, Annals of Pure and Applied Logic 44 (1989), pp. 173 – 242. URL http://www.sciencedirect.com/science/article/pii/0168007289900328
20 The Distributive Full Lambek Calculus with Modal Operators

[27] Goldblatt, R., “Mathematics of modality,” 43, Center for the Study of Language (CSLI), 1993.

[28] Goldblatt, R., *Grishin algebras and cover systems for classical bilinear logic*, Studia Logica 99 (2011), p. 203.

[29] Goldblatt, R., *Cover systems for the modalities of linear logic*, arXiv preprint arXiv:1610.09117 (2016).

[30] Jipsen, P. and C. Tsinakis, *A survey of residuated lattices*, in: *Ordered algebraic structures*, Springer, 2002 pp. 19–56.

[31] Jonsson, B. and A. Tarski, *Boolean algebras with operators. part i*, American Journal of Mathematics 73 (1951), pp. 891–939. URL http://www.jstor.org/stable/2372123

[32] Kanovich, M., S. Kuznetsov, V. Nigam and A. Scedrov, *Subexponentials in non-commutative linear logic*, Mathematical Structures in Computer Science 29 (2019), pp. 1217–1249.

[33] Kozak, M., *Distributive full lambek calculus has the finite model property*, Studia Logica 91 (2009), pp. 201–216.

[34] Kuznetsov, S., *-continuity vs. induction: Divide and conquer.,* in: *Advances in Modal Logic*, 2018, pp. 493–510.

[35] Lambek, J., *The mathematics of sentence structure*, The American Mathematical Monthly 65 (1958), pp. 154–170.

[36] Ono, H., *Semantics for substructural logics*, Substructural logics (1993).

[37] Ono, H., “Modal and Substructural Logics,” Springer Singapore, Singapore, 2019 pp. 47–60. URL https://doi.org/10.1007/978-981-13-7997-0_4

[38] Palmigiano, A., *Dualities for some intuitionistic modal logics* (2004).

[39] Pentus, M., *Lambek grammars are context free*, in: [1993] Proceedings Eighth Annual IEEE Symposium on Logic in Computer Science, IEEE, 1993, pp. 429–433.

[40] Pentus, M., *Models for the lambek calculus*, Annals of Pure and Applied Logic 75 (1995), pp. 179 – 213, invited papers presented at the Conference on Proof Theory, Provability Logic, and Computation. URL http://www.sciencedirect.com/science/article/pii/0168007294000639

[41] Priestley, H. A., *Ordered topological spaces and the representation of distributive lattices*, Proceedings of the London Mathematical Society 3 (1972), pp. 507–530.

[42] Raney, G. N., *Completely distributive complete lattices*, Proceedings of the American Mathematical Society 3 (1952), pp. 677–680.

[43] Routley, R. and R. Meyer, *The semantics of entailment*, , 68, Elsevier, 1973 pp. 199–243.

[44] Sambin, G. and V. Vaccaro, *Topo}logy and duality in modal logic*, Annals of Pure and Applied Logic 37 (1988), pp. 249–296.

[45] Sankappanavar, H. P. and S. Burris, *A course in universal algebra*, Graduate Texts Math 78 (1981).

[46] Seki, T., *A sahjveist theorem for relevant modal logics*, Studia Logica 73 (2003), pp. 383–411. URL https://doi.org/10.1023/A:1023335229747

[47] Suzuki, T., *Bi-approximation semantics for substructural logic at work*, in: M. Kracht, M. de Rijke, H. Wansing and M. Zakharyaschev, editors, *Advances in Modal Logic*, CSLI Publications, 2010 pp. 411–433.

[48] Thomason, S. K., *Categories of frames for modal logic 1*, The journal of symbolic logic 40 (1975), pp. 439–442.

[49] Urquhart, A., *Duality for algebras of relevant logics*, Studia Logica: An International Journal for Symbolic Logic 56 (1996), pp. 263–276. URL http://www.jstor.org/stable/20015846

[50] van Benthem, J., *Categorial grammar at a crossroads*, in: *Resource-Sensitivity, Binding and Anaphora*, Springer, 2003 pp. 3–21.