Finite temperature Casimir effect for massive scalars in a magnetic field

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The finite temperature Casimir effect for a charged, massive scalar field confined between very large, perfectly conducting parallel plates is studied using the zeta function regularization technique. The scalar field satisfies Dirichlet boundary conditions at the plates and a magnetic field perpendicular to the plates is present. Four equivalent expressions for the zeta function are obtained, which are exact to all orders in the magnetic field strength, temperature, scalar field mass, and plate distance. The zeta function is used to calculate the Helmholtz free energy of the scalar field and the Casimir pressure on the plates, in the case of high temperature, small plate distance, strong magnetic field and large scalar mass. In all cases, simple analytic expressions of the zeta function, free energy and pressure are obtained, which are very accurate and valid for practically all values of temperature, plate distance, magnetic field and mass.

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I. INTRODUCTION

In the Casimir effect, an attractive force is observed between perfectly conducting and electrically neutral parallel plates in vacuum. Casimir’s theoretical prediction of the effect was achieved by calculating the attractive force between two neutral and parallel conducting plates caused by the electromagnetic field quantum fluctuations in vacuum [1]. A repulsive Casimir effect exists too, and was theoretically predicted by Boyer some time later when he showed that, in the case of a perfectly conducting sphere, the quantum fluctuations of the electromagnetic field produce a repulsive force on the wall of the sphere [2]. The first experimental proof of the Casimir force was obtained by Sparnaay [3], with many and more precise experimental observations reported since then. A comprehensive review of these experiments is presented in Refs. [4, 5].

The interdisciplinary character of the Casimir effect is well known, since it is relevant not only in QED, but also in condensed matter physics, theories with compactified extra dimensions, gravitation and cosmology, mathematical physics, and nanotechnology and nanotubes. Therefore, a large effort has gone into studying this effect and its generalization to quantum fields other than the electromagnetic field: fermions [6, 7] and especially scalar fields have been investigated extensively [4].

Casimir forces are very sensitive to the quantum field boundary conditions on the plates and, in the case of scalar fields, the most frequently used boundary conditions are Dirichlet and Neumann, while for fermion fields [8] or vector fields [9] bag boundary conditions are used. Here we use the simplest boundary conditions, Dirichlet, to constrain a scalar field between two perfectly conducting parallel plates.

Scalar fields, either massive or massless, appear everywhere in physics. The Higgs field, responsible for spontaneous symmetry breaking in the Standard Model, is a massless scalar before the SU(2) gauge symmetry is broken and a massive scalar after the symmetry is broken. Scalar fields are found within superstring theories as dilaton fields, breaking the conformal symmetry of the string [10]. Scalar fields are used to cause inflation, helping to solve the horizon problem and giving a reason for the non-vanishing cosmological constant. Massless fields are used in this context as inflatons, and massive ones (e.g. Higgs-like fields) are also used [11]. Scalar fields are used to explain Landau diamagnetism [12, 13], and in other areas of condensed matter physics. It has been shown that the electromagnetic Casimir force between parallel plates is obtained by simply doubling the Casimir force on the plates due to a massless scalar field, where the factor of two accounts for the two polarization states of the photon. Therefore the Casimir force between parallel plates caused by a charged scalar field will be the same, apart from a multiplicative factor, as the force due to a charged vector field such as the W-field or the gluon field.

The Casimir effect due to a charged scalar field in the presence of a magnetic field has been studied in vacuum [14] and at finite temperature [15]. These authors use the Schwinger proper time method to calculate the effective action, but are only able to obtain the free energy as an infinite sum of modified Bessel functions. In a recent paper [16], we used a different method, the zeta function technique, to study the finite temperature Casimir effect for a massless and
charged scalar field in the presence of a magnetic field. We obtained simple analytic forms for the free energy and Casimir pressure, valid for practically all values of the three parameters involved. In this paper we conduct a similar investigation of the Casimir effect at finite temperature, but we focus on a massive and charged scalar field in the presence of a magnetic field. We obtain the corrections to the results of our recent paper [16] due to a “light” scalar mass and obtain new results for the cases of “intermediate” mass, where the scalar mass is larger than only some of the parameters, and “large” mass, where the scalar mass is the largest parameter.

Casimir effect calculations generally follow Casimir’s definition of the vacuum energy, which requires a regularization recipe for its implementation. Many regularization techniques have been used for these calculations, such as the cutoff method in various piston configurations [17, 18], the world-line technique [19], the multiple-scattering method [20, 21], the zeta function technique [22–24], and others. As we stated above, in this paper we use the zeta function technique, a regularization technique used also in the computation of effective actions [25, 26]. We calculate the free energy and Casimir pressure due to a massive scalar field, of mass \( M \), confined between two very large, perfectly conducting parallel plates, at a distance \( a \) from each other. The scalar field satisfies Dirichlet boundary conditions on the plates and our system is in thermal equilibrium with a heat reservoir at finite temperature \( T \). We use the imaginary time formalism of finite temperature field theory, which is suitable for a system in thermal equilibrium. A uniform magnetic field \( \vec{B} \) is present in the region between the plates and is perpendicular to the plates.

In Sec. II, we present four equivalent expressions of the zeta function for this system, exact to all orders in \( B, T, M, \) and \( a \), and obtain simple analytic expressions for the zeta function in the case of high temperature, small plate distance, strong magnetic field and large scalar mass. In Sec. III we use this zeta function to calculate the Helmholtz free energy of the scalar field and the pressure on the plates, and we obtain simple analytic expressions for these quantities in the case of high temperature, small plate distance, strong magnetic field and large mass. We discuss our results in Sec. IV.

II. ZETA FUNCTION EVALUATION

We investigate a scalar field \( \phi(x, \tau) \) of mass \( M \) and charge \( e \) in three-dimensional space and Euclidean time \( \tau \), confined by two large, square, perfectly conducting parallel plates perpendicular to the \( z \) axis and located at \( z = 0 \) and \( z = a \). We impose Dirichlet boundary conditions that constrain the scalar field to vanish at the plates

\[
\phi(x, y, 0, \tau) = \phi(x, y, a, \tau) = 0,
\]

and use finite temperature field theory to take into account temperature effects on our system, which we assume to be in thermal equilibrium at temperature \( T \). The imaginary time formalism of finite temperature field theory allows only field configurations satisfying the following boundary conditions

\[
\phi(x, y, z, \tau) = \phi(x, y, z, \tau + \beta),
\]

for any \( \tau \), where \( \beta = 1/T \) is the periodic length in the Euclidean time axis. In the slab region there is also a uniform magnetic field pointing in the \( z \) direction, \( \vec{B} = (0, 0, B) \), and therefore the charged scalar field interacts with \( \vec{B} \).

The Helmholtz free energy \( F \) for the scalar field is

\[
F = \beta^{-1} \log \det (D_E | F_a),
\]

where the symbol \( F_a \) indicates the set of functions satisfying boundary conditions [11] and [12], and the operator \( D_E \) is defined as:

\[
D_E = -\partial_\tau^2 + p_z^2 + (\vec{p} - e\vec{A})_\perp^2 + M^2,
\]

where the subscript \( E \) indicates Euclidean time, \( \vec{A} \) is the electromagnetic vector potential, and we use the notation \( \vec{p}_\perp = (p_x, p_y, 0) \).

The zeta function technique allows us to evaluate \( F \) using the eigenvalues of \( D_E \). The Dirichlet boundary conditions [14] are satisfied if the \( z \) component of the momentum is only allowed to take the values

\[
p_z = \frac{n \pi}{a},
\]

where \( n = 0, 1, 2, 3, \ldots \). The eigenvalues of the operator \( (\vec{p} - e\vec{A})_\perp^2 \) are the Landau levels

\[
2eB \left( l + \frac{1}{2} \right),
\]
with $l = 0, 1, 2, 3, \ldots$, and therefore the eigenvalues of $D_E$ whose eigenfunctions satisfy (1) and (2) are:

$$\frac{\pi^2}{a^2}n^2 + \frac{4\pi^2}{\beta^2}m^2 + eB(2l + 1) + M^2,$$

(3)

where $n, l = 0, 1, 2, 3, \ldots$ and $m = 0, \pm 1, \pm 2, \pm 3, \ldots$. We use this set of eigenvalues to construct the zeta function of the operator $D_E$

$$\zeta(s) = L^2 \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \left( \frac{eB}{2\pi} \right)^{2s} \left( \frac{\pi}{a^2}n^2 + \frac{4\pi^2}{\beta^2}m^2 + eB(2l + 1) + M^2 \right)^{-s},$$

(4)

where $L^2$ is the area of the plates, the factor $eB/2\pi$ takes into account the degeneracy per unit area of the Landau levels and the arbitrary parameter $\mu$ with the dimension of a mass has been introduced to keep $\zeta(s)$ dimensionless for all values of $s$. Once we obtain $\zeta(s)$, we use the zeta function technique and easily find the free energy by taking a simple derivative

$$F = -\beta^{-1}\zeta'(0).$$

(5)

The following identities

$$x^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1}e^{-xt},$$

$$\sum_{l=0}^{\infty} e^{-(2l+1)x} = \frac{1}{2 \sinh x},$$

(6)

where $\Gamma(s)$ is the Euler gamma function, allow us to rewrite $\zeta(s)$ as

$$\zeta(s) = \frac{L^2 \beta^{2s}}{4\pi \Gamma(s)} \int_0^\infty dt \ t^{s-2}e^{-M^2t} \ \frac{eBt}{\sinh eBt} \left( \sum_{n=0}^{\infty} e^{-\frac{x^2}{a^2}n^2t} \right) \left( \sum_{m=-\infty}^{\infty} e^{-\frac{4\pi^2}{\beta^2}m^2t} \right).$$

(7)

It is not possible to evaluate (7) in closed form for any value of the four quantities $B$, $M$, $a$ and $T$, but it is possible to obtain simple expressions of $\zeta(s)$ when one or some of these quantities are small or large. We will use these expressions of the zeta function in closed form to easily obtain the free energy using (5).

First we evaluate the zeta function in the high temperature limit, when $T \gg a^{-1}, M, \sqrt{eB}$, and apply Poisson resummation formula (27) to the $n$ sum in (7), to obtain

$$\zeta(s) = a \left[ \zeta_M(s) + \zeta_{M,B}(s) + \tilde{\zeta}_{M,B}(s) + \zeta_{M,B,T}(s) + \zeta_{M,B,a,T}(s) \right],$$

(8)

where

$$\zeta_M(s) = \frac{L^2 \mu^{2s}}{8\pi \Gamma(s)} \int_0^\infty dt \ t^{s-2} \left( \frac{1}{\sqrt{\pi}t} + a^{-1} \right) e^{-M^2t} \ \frac{eBt}{\sinh eBt},$$

(9)

$$\tilde{\zeta}_{M,B}(s) = \frac{L^2 \mu^{2s}}{4\pi \Gamma(s)} \sum_{m=1}^{\infty} \int_0^\infty dt \ t^{s-2} \left( \frac{1}{\sqrt{\pi}t} + a^{-1} \right) e^{-M^2t} \ \frac{eBt}{\sinh eBt} e^{-\frac{4\pi^2}{\beta^2}m^2t},$$

(10)

$$\zeta_{M,B,a}(s) = \frac{L^2 \mu^{2s}}{4\pi^{3/2} \Gamma(s)} \sum_{n=1}^{\infty} \int_0^\infty dt \ t^{s-5/2} \ e^{-M^2t} \ \frac{eBt}{\sinh eBt} e^{-\frac{n^2}{a^2}t},$$

(11)

$$\zeta_{M,B,a,T}(s) = \frac{L^2 \mu^{2s}}{2\pi^{3/2} \Gamma(s)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^\infty dt \ t^{s-5/2} \ e^{-M^2t} \ \frac{eBt}{\sinh eBt} e^{-\frac{(n^2/a^2)t + 4\pi^2m^2t}{\beta^2}}.$$

(12)
We use (6) to rewrite (9) as

\[ \zeta_{M,B}(s) = \frac{L^2(eB)^{\frac{s}{2}}}{4\pi^2} \left( \frac{\mu^2}{eB} \right)^s \int_0^\infty dt \, t^{-\frac{s}{2}} e^{-zt} \sum_{l=0}^\infty e^{-(2l+1)t}, \]  

where \( z = M^2/eB \) and we dropped the term proportional to \( a^{-1} \), since it only contributes a constant independent of the plate distance to the free energy. After changing the integration variable from \( t \) to \( \frac{t}{2\pi(eB)^{\frac{s}{2}}} \), we find

\[ \zeta_{M,B}(s) = \frac{L^2(eB)^{\frac{s}{2}}}{4\pi^2} \left( \frac{\mu^2}{eB} \right)^s \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \left[ \zeta_H(s-\frac{1}{2}, z) - 2^{\frac{3}{2}-s} \zeta_H(s-\frac{1}{2}, \frac{1}{2}) \right], \]

which is exact for all values of \( s, B \) and \( M \) and where

\[ \zeta_H(s, z) = \sum_{l=0}^\infty (l+z)^{-s} \]

is the Hurwitz zeta function. To calculate the free energy, we only need to know \( \zeta(s) \) for \( s \to 0 \). For small \( s \) we find

\[ x^s \zeta_H(s-\frac{1}{2}, z) \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} = -2\sqrt{\pi} \zeta_H(-\frac{1}{2}, z)s + O(s^2), \]

and therefore

\[ \zeta_{M,B}(s) = \frac{L^2(eB)^{\frac{s}{2}}}{2\pi} \left[ \sqrt{2}\zeta_H(-\frac{1}{2}, \frac{1}{2}) - \zeta_H(-\frac{1}{2}, z) \right] s, \]

when \( s \) is small. In addition to Eq. (15), we have obtained two simpler expressions of \( \zeta_{M,B}(s) \), one valid in the small mass limit, \( M^2 \ll eB \), and the other in the large mass limit, \( M^2 \gg eB \). For \( M^2 \ll eB \) we take \( e^{-zt} \approx 1 - zt + O(z^2) \) in (13), integrate and find

\[ \zeta_{M,B}(s) = \frac{L^2(eB)^{\frac{s}{2}}}{4\pi} \left( \frac{\mu^2}{eB} \right)^s \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \left[ \zeta_R(s-\frac{1}{2}) - z \frac{\Gamma(s+\frac{1}{2})}{\sqrt{\pi} \Gamma(s)} \left( 1 - 2^{-\frac{3}{2}-s} \right) \zeta_R(s+\frac{1}{2}) + O(z^2) \right], \]

where \( \zeta_R(s) \) is the Riemann zeta function of number theory. We use

\[ x^s \zeta_R(s+\frac{1}{2}) \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)} = \sqrt{\pi} \zeta_R(\frac{1}{2}) s + O(s^2), \]

and (14), to obtain the small mass and small \( s \) limit of \( \zeta_{M,B}(s) \)

\[ \zeta_{M,B}(s) = \frac{L^2(eB)^{\frac{s}{2}}}{2\pi} (\sqrt{2} - 1) \left[ \zeta_R(\frac{1}{2}) - \frac{M^2}{2\pi eB} \zeta_R(\frac{1}{2}) \right] s, \]

where \( \zeta_R(\frac{1}{2}) = -0.2079 \) and \( \zeta_R(\frac{1}{2}) = -1.4603 \). When \( M^2 \gg eB \), we take

\[ \frac{eBt}{\sinh eBt} \approx 1 - \frac{1}{6} (eBt)^2 + O(e^4 B^4) \]

inside Eq. (16), neglect again the term proportional to \( a^{-1} \), integrate and find

\[ \zeta_{M,B}(s) = \frac{L^2 M^3}{8\pi^2 \Gamma(s)} \left( \frac{\mu^2}{M} \right)^s \left[ \Gamma(s-\frac{1}{2}) - \frac{e^2 B^2 \Gamma(s-\frac{1}{2})}{6M^4} + O \left( \frac{e^4 B^4}{M^8} \right) \right], \]

which, for \( s \to 0 \), becomes

\[ \zeta_{M,B}(s) = \frac{L^2 M^3}{6\pi} \left( 1 - \frac{e^2 B^2}{8M^4} \right) s. \]
Next we evaluate $\zeta_{M,B,T}(s)$ for $eB \ll 4\pi^2 T^2 + M^2$. We substitute (18) into (10), neglect once more the term proportional to $a^{-1}$, integrate, and find

$$
\zeta_{M,B,T}(s) = \frac{L^2 \mu^{2s}}{4\pi^2 \Gamma(s)} \left[ \Gamma(s - \frac{3}{4}) E_{1/4}^M(s - \frac{3}{4}; 4\pi^2 T^2) - \frac{e^2 B^2}{6} \Gamma(s + \frac{1}{4}) E_{1/4}^M(s + \frac{1}{4}; 4\pi^2 T^2) \right],
$$

(20)

where we use Epstein functions [24, 28, 29] which, for any positive integer $N$, are defined as

$$
E_N^M(s; a_1, a_2, \ldots, a_N) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_N=1}^{\infty} \frac{1}{(a_1 n_1^2 + a_2 n_2^2 + \cdots + a_N n_N^2 + M^2)^s}.
$$

Eq. (20) is valid for any value of $T$ and $M$, as long as $T \gg eB$. Since $T \gg M$, a simple analytic expression can be obtained for $\zeta_{M,B,T}(s)$, not involving Epstein functions

$$
\zeta_{M,B,T}(s) = \frac{2\pi^2 L^2 T^3}{\Gamma(s)} \left( \frac{\mu}{2\pi T} \right)^2 \left[ \Gamma(s - \frac{3}{4}) \zeta_R(2s - 3) - \frac{M^2}{4\pi^2 T^2} \Gamma(s - \frac{1}{4}) \zeta_R(2s - 1) - \frac{e^2 B^2}{96\pi^2 T^4} \Gamma(s + \frac{1}{4}) \zeta_R(2s + 1) \right]
$$

which, for $s \to 0$, becomes

$$
\zeta_{M,B,T}(s) = L^2 T^3 \left[ \frac{\pi^2}{45} - \frac{M^2}{12T^2} - \frac{e^2 B^2}{48\pi^2 T^4} \left( \ln \frac{\mu}{4\pi T} + \gamma_E + \frac{1}{2s} \right) \right] s,
$$

(21)

where $\gamma_E = 0.5772$ is the Euler Mascheroni constant.

In the high temperature limit, it could happen that $M \gg \sqrt{eB}, a^{-1}$, or $\sqrt{eB} \gg M, a^{-1}$, or $a^{-1} \gg M, \sqrt{eB}$, and therefore we need to evaluate $\zeta_{M,B,a}(s)$ for each of these three different possibilities. When $M \gg \sqrt{eB}, a^{-1}$ we change the integration variable from $t$ to $\frac{na}{M}$ in Eq. (11) and find

$$
\zeta_{M,B,a}(s) = \frac{L^2 \mu^{2s}}{4\pi^2 \Gamma(s)} \sum_{n=1}^{\infty} \left( \frac{na}{M} \right)^{s-1/2} \int_0^{\frac{tna}{M}} dt t^{-5/2} \frac{e^{Bt}}{\sinh(\frac{2Bt}{na})} e^{-naM(t+1)}.
$$

Since $aM \gg 1$, only the term with $n = 1$ contributes significantly to the sum and, using the saddle point method, we evaluate the integral and find

$$
\zeta_{M,B,a}(s) = \frac{L^2 eB}{4\pi a \Gamma(s)} \left( \frac{aM}{M} \right)^s \frac{e^{-2aM}}{\sinh(\frac{2Bn}{M})},
$$

(22)

which, for small $s$, becomes

$$
\zeta_{M,B,a}(s) = \frac{L^2 eB}{4\pi a} \frac{e^{-2aM}}{\sinh(\frac{2Bn}{M})}.
$$

When $\sqrt{eB} \gg M, a^{-1}$ we use (9) into (11) and change the integration variable from $t$ to $\frac{tna}{\sqrt{M^2 + (2l+1)eB}}$, to find

$$
\zeta_{M,B,a}(s) = \frac{L^2 eB \mu^{2s}}{2\pi^3 \Gamma(s)} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \left( \frac{na}{\sqrt{M^2 + (2l+1)eB}} \right)^{s-1/2} \int_0^{\frac{tna}{\sqrt{M^2 + (2l+1)eB}}} dt t^{-3/2} e^{-na\sqrt{M^2 + (2l+1)eB(t+1)}}.
$$

Since $a\sqrt{eB} \gg 1$, only the term with $n = 1$ and $l = 0$ contributes significantly to the double sum and, using the saddle point method to evaluate the integral, we find

$$
\zeta_{M,B,a}(s) = \frac{L^2 eB}{2\pi a \Gamma(s)} \left( \frac{am}{\sqrt{M^2 + eB}} \right)^s e^{-2a\sqrt{M^2 + eB}},
$$

and therefore, for small $s$

$$
\zeta_{M,B,a}(s) = \frac{L^2 eB}{2\pi a} e^{-2a\sqrt{M^2 + eB}} s.
$$

(23)
When $a^{-1} \gg M, \sqrt{eB}$, we use $e^{-M^2 t} \simeq 1 - M^2 t$ and \cite{18}, integrate, and find
\[
\zeta_{M,B,a}(s) = \frac{L^2(\mu a)^{2s}}{4\pi^2 a^3 \Gamma(s)} \left[ \Gamma\left(\frac{3}{2} - s\right) \zeta_R(2s) - M^2 a^2 \Gamma\left(\frac{3}{2} - s\right) \zeta_R(1-2s) - \frac{e^2 B^2 a^4}{6} \Gamma(-\frac{3}{2} - s) \zeta_R(-1-2s) \right],
\]
which, in the small $s$ limit, becomes
\[
\zeta_{M,B,a}(s) = \frac{L^2}{8\pi a^3} \left[ \zeta_R(3) + M^2 a^2 \left( 2 \ln 2\mu a + \frac{1}{s} \right) - \frac{e^2 B^2 a^4}{18} \right] s,
\]
where $\zeta_R(3) = 1.201$.

Last we evaluate $\zeta_{M,B,a,T}(s)$. After changing the integration variable from $t$ to $\frac{t}{\sqrt{\pi}a^3 T} + M^2 a^2$ in \cite{12}, we obtain
\[
\zeta_{M,B,a,T}(s) = \frac{L^2}{2\pi a^3/2 \Gamma(s)} \sum_{n=1}^\infty \sum_{m=1}^\infty \left( \frac{na}{\sqrt{\pi}a^3 T^2 + M^2} \right)^{s-1/2} \frac{e^{Bt}}{\sinh\left( \frac{eBt}{\sqrt{\pi}a^3 T^2 + M^2} \right)} \int_0^\infty dt \, t^{s-\frac{1}{2}} e^{-\frac{eBt}{\sqrt{\pi}a^3 T^2 + M^2} (t+t^{-1})}.
\]
Since $aT \gg 1$, only the term with $n = m = 1$ contributes significantly to the double sum and, using the saddle point method, we evaluate the integral for $eB \ll 4\pi^2 T^2 + M^2$ to obtain
\[
\zeta_{M,B,a,T}(s) = \frac{L^2 eB}{2\pi a^3} \left( \frac{\alpha T}{\sqrt{4\pi^2 T^2 + M^2}} \right)^s \frac{e^{-2\alpha a \sqrt{4\pi^2 T^2 + M^2}}}{\sinh\left( \frac{eB}{\sqrt{4\pi^2 T^2 + M^2}} \right)},
\]
which, for small $s$, is
\[
\zeta_{M,B,a,T}(s) = \frac{L^2 eB}{2\pi a^3} \frac{e^{-2\alpha a \sqrt{4\pi^2 T^2 + M^2}}}{\sinh\left( \frac{eB}{\sqrt{4\pi^2 T^2 + M^2}} \right)} s.
\]
We add \cite{19, 21, 22} and \cite{24}, and find the high temperature and small $s$ limit of $\zeta(s)$, when $T \gg M \gg \sqrt{eB}, a^{-1}$
\[
\zeta(s) = L^2 a \left[ \frac{\pi^2 T^3}{45} - \frac{M^2 T}{12} + \frac{M^3}{6\pi} - \frac{e^2 B^2}{48\pi^2 T} \left( \frac{\pi T}{M} + \ln \frac{\mu}{4\pi T} + \gamma_E \right) \right] s + \frac{L^2 eB}{4\pi} \left[ \frac{e^{-2\alpha a \sqrt{4\pi^2 T^2 + M^2}}}{\sinh\left( \frac{eB}{\sqrt{4\pi^2 T^2 + M^2}} \right)} \right] s,
\]
where we dropped a term independent of $s$ that does not contribute to the free energy. We add \cite{17, 21, 23} and \cite{25}, and find $\zeta(s)$ when $T \gg \sqrt{eB} \gg M, a^{-1}$
\[
\zeta(s) = L^2 a \left[ \frac{\pi^2 T^3}{45} + \frac{(eB)^{\frac{3}{2}}}{2\pi} (\sqrt{2} - 1) \zeta_R\left(-\frac{3}{2}\right) - \frac{M^2 T}{12} - \frac{\sqrt{eB} M^2}{4\pi} \left( 1 - \frac{1}{\sqrt{2}} \right) \zeta_R\left(\frac{1}{2}\right) - \frac{e^2 B^2}{48\pi^2 T} \left( \ln \frac{\mu}{4\pi T} + \gamma_E \right) \right] s
\]
\[
+ \frac{L^2 eB}{2\pi} \left[ \frac{e^{-2\alpha a \sqrt{4\pi^2 T^2 + M^2}}}{\sinh\left( \frac{eB}{\sqrt{4\pi^2 T^2 + M^2}} \right)} \right] s,
\]
where we took again the small $s$ limit and dropped a term independent of $s$. Last we add \cite{15, 21, 24} and \cite{20} to find the zeta function when $T \gg a^{-1} \gg \sqrt{eB}, M$
\[
\zeta(s) = \frac{L^2 a (eB)^{\frac{3}{2}}}{2\pi} \left[ \sqrt{2} \zeta_H\left(-\frac{3}{2}, \frac{M^2}{\pi T} \right) - \zeta_H\left(-\frac{1}{2}, \frac{M^2}{\pi T} \right) \right] s
\]
\[
+ L^2 a \left[ \frac{\pi^2 T^3}{45} - \frac{M^2 T}{12} - \frac{e^2 B^2}{48\pi^2 T} \left( \ln \frac{\mu}{4\pi T} + \gamma_E \right) \right] s
\]
\[
+ \frac{L^2}{8\pi} \left[ \frac{\zeta_R(3)}{a^2} + 2 M^2 \ln (2\mu a) - \frac{e^2 B^2 a^2}{18} \right] s
\]
\[
+ \frac{L^2 eB}{2\pi} \frac{e^{-2\alpha a \sqrt{4\pi^2 T^2 + M^2}}}{\sinh\left( \frac{eB}{\sqrt{4\pi^2 T^2 + M^2}} \right)} s,
\]
where we took the small $s$ limit and dropped two terms independent of $s$.

To evaluate $\zeta(s)$ in the small plane distance limit, $a^{-1} \gg T, M, \sqrt{eB}$, we apply the Poisson resummation formula to the $m$ sum in \cite{7}, and obtain
\[
\zeta(s) = \frac{\beta}{2} \left[ \tilde{\zeta}_{M,B}(s) + \tilde{\zeta}_{M,B,a}(s) + \zeta_{M,B,T}(s) + \tilde{\zeta}_{M,B,a,T}(s) \right],
\]
where \cite{26}.
\[ \tilde{\zeta}_{M,B}(s) = \frac{L^2 \mu^{2s}}{4\pi^{3/2} \Gamma(s)} \int_0^\infty dt t^{-5/2} e^{-M^2 t} \frac{e^{Bt}}{\sinh e^{Bt}}, \]  
\[ \tilde{\zeta}_{M,B,a}(s) = \frac{L^2 \mu^{2s}}{4\pi^{3/2} \Gamma(s)} \sum_{n=1}^\infty \int_0^\infty dt t^{-5/2} e^{-M^2 t} \frac{e^{Bt}}{\sinh e^{Bt}} e^{-\pi^2 n^2 t/a^2}, \]  
\[ \zeta_{M,B,T}(s) = \frac{L^2 \mu^{2s}}{2\pi^{3/2} \Gamma(s)} \sum_{m=1}^\infty \int_0^\infty dt e^{-5/2} e^{-M^2 t} \frac{e^{Bt}}{\sinh e^{Bt}} e^{-\frac{m^2 \beta^2}{4t}}, \]  
\[ \tilde{\zeta}_{M,B,a,T}(s) = \frac{L^2 \mu^{2s}}{2\pi^{3/2} \Gamma(s)} \sum_{n=1}^\infty \sum_{m=1}^\infty \int_0^\infty dt t^{-5/2} e^{-M^2 t} \frac{e^{Bt}}{\sinh e^{Bt}} e^{-\left(\pi^2 n^2 t/a^2 + m^2 \beta^2 /4t\right)}, \]  
Comparing Eqs. (30) - (33) to (9) - (12), it is evident that \( \tilde{\zeta}_{M,B}(s) = 2\tilde{\zeta}_{M,B}(s) \), since the part of \( \zeta_{M,B}(s) \) proportional to \( a^{-1} \) is negligible, that \( \zeta_{M,B,a}(s) \) and \( \zeta_{M,B,a,T}(s) \) are equal to \( \tilde{\zeta}_{M,B,T}(s) \) and \( \zeta_{M,B,a,T}(s) \) respectively, once we replace \( a \) with \( \beta/2 \) and \( \beta \) with \( 2a \), and that \( \zeta_{M,B,a}(s) \) equals twice \( \zeta_{M,B,a}(s) \) once we make the same replacement. Therefore we find that, in the small \( s \) limit and when \( a^{-1} \gg M \gg T, \sqrt{eB} \), the zeta function is given by

\[ \zeta(s) = L^2 \beta \left[ \frac{\pi^2}{720a^3} - \frac{M^2}{48a} + \frac{M^3}{6} - \frac{e^2 B^2 a}{48a^2} \right] \left( \frac{\pi}{6a} + \ln \left( \frac{\mu a}{2\pi} + \gamma_E \right) \right) s + \frac{L^2 eB}{2\pi} \left[ \frac{e^{-\beta M}}{\sinh \left( \frac{eB \beta}{2M} \right)} + \frac{e^{-\beta \sqrt{\pi^2 a^{-2} + M^2}}}{\sinh \left( \frac{eB \beta}{2\pi a^{-2} + M^2} \right)} \right] s, \]

when \( a^{-1} \gg \sqrt{eB} \gg T, M \), the zeta function is

\[ \zeta(s) = L^2 \beta \left[ \frac{\pi^2}{720a^3} + \frac{eB}{2\pi} \left( \sqrt{2} - 1 \right) \zeta(\frac{1}{2}) - \frac{e^2 B^2 a}{48a^2} \right] \left( 1 - \frac{1}{\sqrt{2}} \right) \zeta(\frac{1}{2}) - \frac{e^2 B^2 a}{48a^2} \left( \ln \left( \frac{\mu a}{2\pi} + \gamma_E \right) \right) s \]

and \( \zeta(s) = L^2 \beta \left( eB \right) \left[ \frac{\zeta_H(-\frac{1}{2}, \frac{M^2}{\pi^2})}{2\pi} - \zeta_H(-\frac{1}{2}, \frac{M^2}{\pi^2}) \right] s + \frac{L^2 eB}{2\pi} \left[ \frac{\pi^2}{720a^3} - \frac{M^2}{48a} - \frac{e^2 B^2 a}{48a^2} \right] \left( \ln \left( \frac{\mu a}{2\pi} + \gamma_E \right) \right) s \]

Next we evaluate \( \zeta(s) \) in the strong magnetic field limit, \( \sqrt{eB} \gg T, a^{-1}, M \), and apply the Poisson resummation formula to both the \( n \) and \( m \) sums in (7), to find

\[ \zeta(s) = a\beta [\zeta_W(s) + \tilde{\zeta}(s)] \]

where

\[ \zeta_W(s) = \frac{L^2 \mu^{2s}}{16\pi^2 \Gamma(s)} \int_0^\infty dt t^{-3} \left( 1 + a^{-1} \sqrt{\pi t} \right) e^{-M^2 t} \frac{e^{Bt}}{\sinh e^{Bt}}, \]

and

\[ \tilde{\zeta}(s) = \frac{L^2 \mu^{2s}}{16\pi^2 \Gamma(s)} \int_0^\infty dt t^{-3} e^{-M^2 t} \frac{e^{Bt}}{\sinh e^{Bt}} \left( \sum_{n,m=-\infty}^{\infty} e^{-\frac{1}{2}a^2 n^2 t} e^{-\frac{1}{2}m^2 \beta^2 /4t} - 1 \right). \]
Eq. (38), once we neglect the term proportional to $a^{-1}$, yields the zeta function of the one-loop Weisskopf effective Lagrangian for massive scalar QED (30). In the strong magnetic field limit, we set $e^{-M^2t} \approx 1 - M^2t$, integrate, and obtain

$$
\zeta_W(s) = \frac{L^2(eB)^2}{8\pi^2 \Gamma(s)} \left( \frac{\mu^2}{eB} \right)^s \left[ (1 - 2^{1-s})\Gamma(s - 1) - (1 - 2^{-s})\frac{M^2}{eB} \Gamma(s) \zeta_R(s) \right],
$$

and, for small $s$, we find

$$
\zeta_W(s) = \frac{L^2(eB)^2}{96\pi^2} \left( \ln \frac{eB}{3\mu^2} - \frac{1}{2} - s^{-1} + \frac{M^2}{eB} 6 \ln 2 \right) s,
$$

where we used the interesting numerical fact (31) (32)

$$
\frac{6}{\pi^2} \zeta_R(2) - \log \pi - \gamma_E = -2.2918 \approx -\ln 6 - \frac{1}{2}.
$$

Once we compare Eq. (40) to the well known result for the Weisskopf Lagrangian (30), (31), we realize that we must take the arbitrary parameter $\mu = M$, and we'll do that in all our results containing $\mu$. We evaluate $\zeta(s)$ by using

$$
\frac{1}{\sinh eBt} \approx 2e^{-eBt},
$$

and changing integration variable from $t$ to $\sqrt{\frac{n^2a^2 + m^2\beta^2}{eB + M^2}}$ in Eq. (39), to find

$$
\tilde{\zeta}(s) = \frac{L^2 \mu^2 eB}{8\pi^2 \Gamma(s)} \sum_{n,m=-\infty}^{\infty} \left( \frac{n^2a^2 + m^2\beta^2/4}{eB + M^2} \right)^{\frac{s}{2}} \int_0^{\infty} dt \, t^{s-2} e^{-\frac{1}{2}(t + t^{-1})\sqrt{(eB + M^2)(4n^2a^2 + m^2\beta^2)}},
$$

where the term with $m = n = 0$ is excluded and only terms with $n = 0, \pm 1$ and $m = 0, \pm 1$ contribute significantly to the double sum. We integrate using the saddle point method and, using (37) and (40), obtain the zeta function in the strong magnetic field and small $s$ limit,

$$
\zeta(s) = \frac{L^2aeB(eB + M^2)}{96\pi^2 T} \left( \ln \frac{eB}{3M^2} - \frac{1}{2} \right) s + \frac{L^2aeB(eB + M^2)}{4\pi^2 T} \left[ e^{-2\sqrt{eB + M^2}} \frac{2\lambda e^{-\beta\sqrt{eB + M^2}}}{a^2 + \frac{\beta^2}{2}} + 2 \lambda^2 e^{-\beta\sqrt{eB + M^2}} \frac{2e^{-\beta\sqrt{eB + M^2}}}{\sqrt{\frac{\beta^2}{2}}} \right] s,
$$

where we neglected higher order terms in $\frac{eB}{M}$ and terms that do not depend on $s$.

Last we use (37) to evaluate $\zeta(s)$ in the large mass limit, $M \gg T, a^{-1}, \sqrt{eB}$. We first obtain $\zeta_W(s)$ for $M \gg \sqrt{eB}$ and small $s$

$$
\zeta_W(s) = \frac{L^2 M^4}{16\pi^2} \left( \frac{3}{4} + \ln \frac{\mu}{M} + \frac{1}{2s} - \frac{e^2B^2}{6M^4} \right) s.
$$

To evaluate $\tilde{\zeta}(s)$, we change the integration variable from $t$ to $\sqrt{\frac{n^2a^2 + m^2\beta^2}{M^2}}$ in Eq. (39), retain only terms with $n = 0, \pm 1$ and $m = 0, \pm 1$ in the double sum, use the saddle point method to integrate and, for small $s$, obtain

$$
\zeta(s) = \frac{3L^2a\beta M^4}{64\pi^2} s + \frac{L^2a\beta eB\sqrt{M}}{8\pi^{3/2}} \left[ e^{-2aM/a^{3/2}} \sinh \left( \frac{eB}{4M} \right) + \frac{2a}{\beta^2} e^{-\beta M/a^{3/2}} \sinh \left( \frac{eB\beta}{2M} \right) \right] s,
$$

where we set $\mu = M$, and neglected higher order terms in $\frac{eB}{M}$ and terms that do not depend on $s$.

III. FREE ENERGY AND CASIMIR PRESSURE

We use Eq. (33) to calculate the free energy and are able take the derivative of the zeta function easily, by using the fact that the derivative of $G(s)/\Gamma(s)$ at $s = 0$ is simply $G(0)$, if $G(s)$ is a well behaved function. Using (37) and
our other results for the zeta function [3, 29], and [37], we are able to write four expressions of the free energy, all equivalent to each other,

\[ F = -\frac{L^2}{4\pi\beta} \int_0^\infty dt \, t^{-2} e^{-Mt^2} \frac{e^{Bt}}{\sinh eBt} \left( \sum_{n=0}^{\infty} e^{-\frac{\pi^2}{a^2}n^2} + \sum_{m=-\infty}^{\infty} e^{-\frac{\pi^2}{a^2}m^2} \right), \] (43)

\[ F = -\frac{L^2}{8\pi^{3/2}} \int_0^\infty dt \, t^{-5/2} e^{-Mt^2} \frac{e^{Bt}}{\sinh eBt} \left( \sum_{n=0}^{\infty} e^{-\frac{\pi^2}{a^2}n^2} + \sum_{m=-\infty}^{\infty} e^{-\frac{\pi^2}{a^2}m^2} \right), \] (44)

better suited for a high temperature expansion \((2Ta \gg 1, 2T \gg \sqrt{eB/\pi} \text{ and } T \gg M)\),

\[ F = -\frac{L^2}{16\pi} \int_0^\infty dt \, t^{-3} e^{-Mt^2} \frac{e^{Bt}}{\sinh eBt} \left( \sum_{n=0}^{\infty} e^{-\frac{\pi^2}{a^2}n^2} + \sum_{m=-\infty}^{\infty} e^{-\frac{\pi^2}{a^2}m^2} \right) \] (45)

better suited for a small plate distance expansion \((2Ta \ll 1, a^{-1} \gg \sqrt{eB/\pi} \text{ and } a^{-1} \gg M)\), and

\[ F = -\frac{L^2}{8\pi^{3/2}} \int_0^\infty dt \, t^{-5/2} e^{-Mt^2} \frac{e^{Bt}}{\sinh eBt} \left( \sum_{n=0}^{\infty} e^{-\frac{\pi^2}{a^2}n^2} + \sum_{m=-\infty}^{\infty} e^{-\frac{\pi^2}{a^2}m^2} \right) \] (46)

better suited for strong magnetic field or large mass expansion. The last equation has been obtained by other authors [13] in a very similar form, and used by them to write the free energy as an infinite sum of modified Bessel functions.

None of the four expressions, (43) - (46), can be evaluated in closed form for arbitrary values of the four quantities \(M, B, a\) and \(T\), but it is possible to use them to evaluate numerically the free energy for any values of these four relevant quantities. However, using our results from Sec. II, we found simple analytic expressions for the free energy when one or some of those four quantities are small or large. To obtain the free energy in the high temperature limit, we use (26) - (28) and find

\[ F = -V \left[ \frac{\pi^2 T^4}{45} - \frac{M^2 T^2}{12} + \frac{e^2 B^2}{48\pi^2} \left( \frac{\pi T}{M} + \ln \frac{M}{4\pi T} + \gamma_E \right) \right] - \frac{L^2 TeB}{2\pi} \left[ \frac{e^{-2aM}}{2 \sinh(eB/a)} + \frac{e^{-2a\sqrt{4\pi^2 T^2 + M^2}}}{\sinh(eB/a)} \right], \] (47)

valid for \(T \gg M \gg \sqrt{eB}, a^{-1}\) and where \(V = L^2 a\) is the volume of the slab,

\[ F = -V \left[ \frac{\pi^2 T^4}{45} + \frac{(eB)^2 T}{2\pi} (\sqrt{2} - 1) \zeta_R(-\frac{1}{2}) - \frac{M^2 T^2}{12} - \sqrt{eBM^2T} \right] \left( 1 - \frac{1}{2\sqrt{2}} \right) \zeta_R(\frac{3}{2}) - \frac{e^2 B^2}{48\pi^2} \left( \ln \frac{M}{4\pi T} + \gamma_E \right) \] (48)

valid for \(T \gg \sqrt{eB} \gg M, a^{-1}\), and

\[ F = -V \left[ \frac{\pi^2 T^4}{45} - \frac{M^2 T^2}{12} - \frac{e^2 B^2}{48\pi^2} \left( \ln \frac{M}{4\pi T} + \gamma_E \right) \right] - \frac{VT(eB)^2}{2\pi} \left[ \sqrt{5} \zeta_H(-\frac{3}{2}, \frac{a^2}{M^2}) - \zeta_H(-\frac{3}{2}, \frac{eB^2}{M^2}) \right] \] (49)

valid when \(T \gg a^{-1} \gg \sqrt{eB}, M\). Notice that Eqs. (48) and (49), once we set \(M = 0\), are in full agreement with the results of Ref. [14], where we studied the Casimir effect for a massless scalar field. In addition to reproducing the results of our previous paper, Eqs. (48) and (49) contain the corrections to those results due to a “light” scalar mass. Eq. (47) gives the free energy in the case of an “intermediate” scalar mass, since it is valid when \(M\) is much smaller than the temperature and much larger than the inverse plate distance and \(\sqrt{eB}\). Eqs. (47) - (49) show that, in the high temperature limit, the dominant term in the free energy is the Stefan-Boltzmann term \(-\frac{\pi^2}{46}VT^4\), as expected. Terms in the free energy that have a linear dependence on \(a\), such as this one, are uniform energy density term. If the medium outside the plates is at the same temperature \(T\) and has the same magnetic field present as the medium
between the plates, uniform energy density terms do not contribute to the Casimir pressure. Only if there is not a magnetic field and the temperature is zero outside the plates, uniform energy density terms contribute a constant pressure. In this work we assume that the same magnetic field is present between and outside the plates, and that the medium outside the plates is at the same temperature as the one between the plates, therefore we will neglect contributions to the Casimir pressure from uniform energy density terms.

The pressure $P$ on the plates is

$$P = -\frac{1}{L^2} \frac{\partial F}{\partial a},$$

and therefore, for $T \gg M \gg \sqrt{eB}/a^{-1}$

$$P = -\frac{TeB}{\pi} \left[ \frac{M}{2} e^{-2aM} + \sqrt{4\pi^2T^2 + M^2} \frac{e^{-2a\sqrt{4\pi^2T^2 + M^2}}}{\sinh(\frac{eBe}{\sqrt{4\pi^2T^2 + M^2}})} \right],$$

for $T \gg \sqrt{eB} \gg M, a^{-1}$

$$P = -\frac{TeB}{\pi} \left[ \sqrt{M^2 + eBe^{-2a\sqrt{M^2 + eB}}} + \sqrt{4\pi^2T^2 + M^2} \frac{e^{-2a\sqrt{4\pi^2T^2 + M^2}}}{\sinh(\frac{eBe}{\sqrt{4\pi^2T^2 + M^2}})} \right],$$

and for $T \gg a^{-1} \gg M, \sqrt{eB}$

$$P = -\frac{T}{4\pi} \left[ \frac{\zeta_R(3)}{a^3} - \frac{M^2}{6\pi} + \frac{e^2B^2a}{48\pi^2} \left( \frac{\pi}{Ma} + \ln \frac{Ma}{2\pi} + \gamma_E \right) \right] - \frac{L^2TeB}{2\pi} \left[ \frac{e^{-\beta M}}{\sinh(\frac{eBe}{2M})} + \frac{e^{-\beta \sqrt{\pi^2a^{-2} + M^2}}}{\sinh(\frac{eBe}{2\sqrt{\pi^2a^{-2} + M^2}})} \right].$$

Notice that, in Eqs. (54) - (56), we left out some terms that are negligibly small.

We use (34) - (36) to obtain the free energy in the small plate distance limit

$$F = -L^2 \left[ \frac{\pi^2}{720a^5} - \frac{M^2}{48a} - \frac{e^2B^2a}{48\pi^2} \left( \frac{\pi}{Ma} + \ln \frac{Ma}{2\pi} + \gamma_E \right) \right] - \frac{L^2TeB}{2\pi} \left[ \frac{e^{-\beta M}}{\sinh(\frac{eBe}{2M})} + \frac{e^{-\beta \sqrt{\pi^2a^{-2} + M^2}}}{\sinh(\frac{eBe}{2\sqrt{\pi^2a^{-2} + M^2}})} \right]$$

for $a^{-1} \gg M \gg T, \sqrt{eB}$.

$$F = -L^2 \left[ \frac{\pi^2}{720a^5} + \frac{(eB)^2}{2\pi} (\sqrt{2} - 1) \zeta_R(-\frac{3}{2}) - \frac{M^2}{48a} - \frac{\sqrt{eB}M^2}{4\pi} \left( 1 - \frac{1}{\sqrt{2}} \right) \zeta_R(\frac{3}{2}) - \frac{e^2B^2a}{48\pi^2} \left( \ln \frac{Ma}{2\pi} + \gamma_E \right) \right]$$

$$- \frac{L^2TeB}{2\pi} \left[ 2e^{-\beta \sqrt{eB} + M^2} + \frac{e^{-\beta \sqrt{\pi^2a^{-2} + M^2}}}{\sinh(\frac{eBe}{2\sqrt{\pi^2a^{-2} + M^2}})} \right]$$

for $a^{-1} \gg \sqrt{eB} \gg T, M, a^{-1}$.

$$F = -\frac{L^2(eB)^2}{2\pi} \left[ \sqrt{2} \zeta_R(-\frac{3}{2}, \frac{M^2}{4\pi^2}) - \zeta_R(-\frac{3}{2}, \frac{M^2}{2\pi}) \right] - L^2 \left[ \frac{\pi^2}{720a^5} - \frac{M^2}{48a} - \frac{e^2B^2a}{48\pi^2} \left( \ln \frac{Ma}{2\pi} + \gamma_E \right) \right]$$

$$- \frac{L^2}{\pi} \left[ \zeta_R(3)T^2 + \frac{M^2}{2} \ln \left( \frac{M}{T} \right) - \frac{e^2B^2}{288T^2} \right] - \frac{L^2TeB}{2\pi} \left[ \frac{e^{-\beta \sqrt{\pi^2a^{-2} + M^2}}}{\sinh(\frac{eBe}{2\sqrt{\pi^2a^{-2} + M^2}})} \right]$$

when $a^{-1} \gg T \gg \sqrt{eB}/M$. Eqs. (54) and (55) contain the corrections to the results of Ref. 16 due to a “light” scalar mass, and Eq. (53) shows the free energy in the case of an “intermediate” scalar mass. The dominant term in (54) - (55) is $-\frac{L^2}{2\pi}$, which is the familiar vacuum Casimir energy for a complex scalar field, and for the photon field [1]. The Casimir pressure is the same in all three cases of small plate distance

$$P = -\frac{\pi^2}{240a^5} + \frac{\pi eB}{2a^3 \sqrt{\pi^2a^{-2} + M^2}} \frac{e^{-\beta \sqrt{\pi^2a^{-2} + M^2}}}{\sinh(\frac{eBe}{2\sqrt{\pi^2a^{-2} + M^2}})} - \frac{e^2B^2}{48\pi^2} \ln \frac{Ma}{2\pi} + 1,$$
and, again, we left out some negligibly small terms. For strong magnetic field, \( \sqrt{eB} \gg T, a^{-1}, M \), the free energy is found using \( 11 \)

\[
F = -\frac{V(eB)^2}{96\pi^2} \left( \ln \frac{eB}{3M^2} - \frac{1}{2} \right) - \frac{L^2 eB(eB + M^2)^{1/4}}{4\pi^{3/2}} \left[ \frac{e^{-2a \sqrt{eB + M^2}}}{\sqrt{a}} + \frac{2^{3/2} a \sqrt{\beta} e^{-\beta \sqrt{eB + M^2}}}{\beta^{3/2}} + \frac{2a e^{-\sqrt{(eB + M^2)(4a^2 + \beta^2)}}}{\left( a^2 + \frac{\beta^2}{4} \right)^{3/4}} \right],
\]

where the dominant term is the one-loop vacuum effective potential for scalar QED [31], proportional to the volume of the slab. If we set \( M = 0 \) in (57), we reproduce the result of Ref. [16] obtained for a massless scalar field in the presence of a strong magnetic field. The effective potential is a uniform energy density term and therefore, under our assumptions, does not contribute to the Casimir pressure. The pressure, in the strong magnetic field case, is given by

\[
P = -\frac{eB(eB + M^2)^{3/4}}{2\pi^{3/2}} \left[ \frac{e^{-2a \sqrt{eB + M^2}}}{\sqrt{a}} + \frac{2a^2 \sqrt{\beta} e^{-\beta \sqrt{eB + M^2}}}{\beta^{3/2}} \right],
\]

where we neglected uniform energy density terms and some smaller terms.

Last we examine the large mass limit, \( M \gg T, a^{-1}, \sqrt{eB} \) and, using \( 12 \), we obtain the free energy in this limit

\[
F = -\frac{3VM^4}{64 \pi^2} - \frac{L^2 eB \sqrt{M}}{8\pi^{3/2}} \left[ \frac{e^{-2a \sqrt{eB + M^2}}}{\sqrt{a} \sinh \left( \frac{eB}{M} \right)} + \frac{2^{3/2} a \sqrt{\beta} e^{-\beta \sqrt{eB + M^2}}}{\beta^{3/2} \sinh \left( \frac{eB}{2M} \right)} + \frac{2a e^{-M \sqrt{4a^2 + \beta^2}}}{\left( a^2 + \frac{\beta^2}{4} \right)^{3/4} \sinh \left( \frac{eB}{2M} \sqrt{4a^2 + \beta^2} \right)} \right].
\]

Also in this case the dominant term is the scalar QED effective potential and it will not contribute to the pressure, since it is a uniform energy density term. The Casimir pressure in the large mass limit is given by

\[
P = -\frac{eBM^{3/2}}{4\pi^{3/2}} \left[ \frac{e^{-2a \sqrt{eB + M^2}}}{\sqrt{a} \sinh \left( \frac{eB}{M} \right)} + \frac{2a^2 \sqrt{\beta} e^{-\beta \sqrt{eB + M^2}}}{\beta^{3/2} \sinh \left( \frac{eB}{2M} \sqrt{4a^2 + \beta^2} \right)} \right].
\]

IV. DISCUSSION AND CONCLUSIONS

In this work we used the zeta function method to investigate the finite temperature Casimir effect of a charged, massive scalar field confined between two perfectly conducting parallel plates and in the presence of a uniform magnetic field. We derived four expressions of the zeta function \( 7, 8, 29, \) and \( 37 \), which are exact to all orders in \( B, a, M, \) and \( T \), and used them to obtain expressions for the free energy of the scalar field and for the Casimir pressure on the plates in the case of high temperature \( (T \gg a^{-1}, \sqrt{eB}, M) \), small plate distance \( (a^{-1} \gg T, \sqrt{eB}, M) \), strong magnetic field \( (\sqrt{eB} \gg T, a^{-1}, M) \), and large mass \( (M \gg T, a^{-1}, \sqrt{eB}) \).

We numerically evaluated the free energy with very high precision, using three of the exact expressions we obtained, and we compared the exact numerical values of the free energy to the values obtained from our simple analytic expressions. In the high temperature case we found that, for \( T/2 \geq a^{-1}, M, \sqrt{eB} \), Eq. \( 17 \) is within 1.3% of the exact value of the free energy when \( M \geq a^{-1}, \sqrt{eB} \), Eq. \( 18 \) is within 2.9% of the exact value of the free energy when \( \sqrt{eB} \geq a^{-1}, M \), and Eq. \( 19 \) is within 4.2% of the exact value of the free energy when \( a^{-1} \geq \sqrt{eB}, M \). For \( T/4 \geq a^{-1}, M, \sqrt{eB} \), Eqs. \( 17 \)-\( 19 \) are within 0.5% or less of the exact value of the free energy, showing a very rapid convergence of Eqs. \( 17 \)-\( 19 \) to the exact values of the free energy. These three equations are a simple analytic expression of \( F \) in the high temperature limit, valid for all values of \( B, a, M, \) and \( T \), and have a discrepancy from the exact value of \( F \) that is not larger than 4.2%, as long as \( T/2 \geq a^{-1}, M, \sqrt{eB} \). An equally accurate expression of the Casimir pressure, valid for \( T/2 \geq a^{-1}, M, \sqrt{eB} \), is obtained immediately from \( 17 \)-\( 19 \), and is shown in \( 30 \)-\( 32 \).

When investigating the small plate distance limit we found that, for \( a^{-1}/4 \geq 2T, M, \sqrt{eB} \), Eq. \( 33 \) is within 2.9% of the exact value of the free energy when \( M \geq 2T, \sqrt{eB} \), Eq. \( 34 \) is within 5.3% of the exact value of the free energy when \( \sqrt{eB} \geq 2T, M \), and Eq. \( 35 \) is within 8.2% of the exact value of the free energy when \( 2T \geq M, \sqrt{eB} \). For \( a^{-1}/8 \geq 2T, M, \sqrt{eB} \), Eqs. \( 36 \)-\( 38 \) are within 1% or less of the exact value of the free energy, showing once more a rapid convergence of our analytical expressions to the exact value of the free energy. Eqs. \( 33 \)-\( 35 \) are a simple analytic expression of the free energy in the small plate distance limit, valid for all values of \( B, M, \) and \( T, \) and with
a discrepancy of no more than 8.2% from the exact value of $F$ when $a^{-1}/4 \geq 2T, M, \sqrt{eB}$. The pressure in the case of small plate distance, obtained immediately from (53) - (55) and shown in (56), is similarly accurate.

We find that, in the case of strong magnetic field or large mass, Eqs. (57) and (58) are even more accurate. For $\sqrt{eB}/2 \geq 2T, M, a^{-1}$, Eq. (57) is within 4.6% of the exact value of the free energy, and for $\sqrt{eB}/4 \geq 2T, M, a^{-1}$, Eq. (58) is within 0.0001% of the exact value of the free energy, showing an extremely rapid convergence to the exact value of $F$. The large mass limit of Eq. (59), for $M/2 \geq 2T, \sqrt{eB}, a^{-1}$, is within 0.05% of the exact value of the free energy.

If we set $T = 0$ in Eqs. (59) and (51), and eliminate terms that do not depend on $a$ and uniform energy density terms, we obtain the same quantity, which is the Casimir energy $E_C$ due to a massive and charged scalar field in a magnetic field in the limit of small plate distance ($a^{-1} \gg \sqrt{eB}, M$)

$$E_C = \frac{\pi^2}{720a^3} + \frac{M^2 + e^2B^2a}{48\pi^2} \ln \frac{Ma}{2\pi}$$

If we do the same in Eqs. (57) and (59), we find that, for strong magnetic field ($\sqrt{eB} \gg a^{-1}, M$)

$$E_C = -\frac{eB(eB + M^2)^{1/4}}{4\pi^{3/2}\sqrt{a}} e^{-2\sqrt{eB} + M^2},$$

and for large scalar mass ($M \gg a^{-1}, \sqrt{eB}$)

$$E_C = -\frac{eB\sqrt{M}}{8\pi^{3/2}} \frac{e^{-2aM}}{\sqrt{a\sinh \left( \frac{eB}{Ma} \right)}}.$$

Eqs. (61) - (63) show that scalar mass, as it grows, inhibits the Casimir energy. The situation is different for a growing magnetic field which, in the case of small plate distance, boosts the Casimir energy, as we can see from Eq. (61) where $\ln \frac{Ma}{2\pi}$ is negative because $aM \ll 1$. In the case of strong magnetic field and large scalar mass, Eqs. (62) and (63) show that magnetic field, as it grows, inhibits the Casimir energy as it is also shown in [14] and, in the case of a massless scalar, in [10]. Our results, simple analytic expressions for $E_C$, are more explicit than those of [14] where the magnetic field correction to the Casimir energy is presented as an infinite sum of integrals, and more general than those of [10], where only the case of a massless scalar field is examined. Notice that Eq. (62), once we set $M = 0$, agrees with the result of Ref. [16] and agrees with [14] on the dependence of $E_C$ from $a$ and $B$, but disagrees with this paper for the overall sign.

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