The Georgi Algorithms of Jet Clustering

Shao-Feng Ge *
KEK Theory Center, Tsukuba 305-0801, Japan
September 2, 2014

Abstract
We reveal the direct link between the jet clustering algorithms recently proposed by Howard Georgi and parton shower kinematics, providing sound support from the theoretical side. The kinematics of this class of elegant algorithms is explored systematically and the jet function is generalized to $J^{(n)}_{\beta}$ with a jet function index $n$. Based on three basic requirements that the result of jet clustering is process-independent, for softer subjets the inclusion cone is larger, and that the cone size cannot be too large in order to avoid mixing different jets, we derive constraints on the jet function index $n$ and the jet function parameter $\beta$ which are closely related to phase space boundaries. Finally, we demonstrate that the jet algorithm is boost invariant.

1. Introduction
Due to confinement, partons can not be observed directly. They first experience shower process and then fragment into low-energy hadrons which are the objects that can be experimentally measured. The information carried by partons is then buried in sprays of hadrons and needs to be reconstructed. Jet is a very useful tool for this purpose [1] and various algorithms have been proposed, including the longitudinally invariant $k_t$ algorithm [2, 3], the Cambridge/Aachen (C/A) algorithm [4, 5, 6], the anti-$k_t$ algorithm [7], and the Durham algorithm [8], with different features. For details, please also refer to comprehensive reviews [9, 10, 11, 12, 13].

Recently, a new class of algorithms have been proposed [14]. In these Georgi algorithms of jet clustering, a jet function is defined in terms of the jet momentum $P_\alpha = (E_\alpha, P_\alpha) = \sum_{i \in \alpha} p_i$ as,

$$J_{\beta}(P_\alpha) \equiv E_\alpha - \beta \frac{P_\alpha^2}{E_\alpha},$$

with $\beta > 1$ and we call the prefactor $E_\alpha$ the clustering scale. Note that the jet subscript $\alpha$ is a set of subjets. The basic motivation behind this definition is that jet clustering tends to increase energy $E_\alpha$ but $P_\alpha^2/E_\alpha$ does not increase that much. The former is due to energy conservation while the later can find support in parton shower models. For both massless [15] and massive [16] parton shower, virtuality can be reconstructed as,

$$p_\alpha^2 - m_\alpha^2 = \frac{p_j^2 - m_j^2}{z} + \frac{p_{\alpha-j}^2 - m_{\alpha-j}^2}{1 - z} + z(1 - z)t,$$

*gesf02@gmail.com
where \( P_{a,j} \equiv P_a - p_j \). The massless case can be restored by setting the parton masses \( m_a, m_j \), and \( m_{a,j} \) to be zero. The parton shower process is controlled by two parameters, the evolution scale \( t \) and the energy fraction,

\[
z \equiv \frac{E_i}{E_a} = \frac{E_a - E_{a-i}}{E_a},
\]

(1.3)
taken away by one of the two child partons\(^1\). For final-state parton shower, the evolution scale \( t \) and virtualities are positive. An immediate conclusion is that the parent parton has larger virtuality than the child partons. In other words, \( P^2_{a,i} \) increases by clustering which is the reverse of parton shower. Nevertheless, parton shower tends to emit soft partons, \( z \to 0 \). The second term in (1.1) does not increase as much as the first. When clustering the child partons, the relative energy increase is proportional to \( z \) as shown in (1.3). On the other hand, the relative increase in the second term of (1.1) is suppressed even more,

\[
\frac{1}{E_a} \left[ \frac{P^2_{a} - m^2_{a}}{E_a} - \frac{P^2_{a-j} - m^2_{a-j}}{E_{a-j}} \right] = \frac{p^2_{j} - m^2_{j}}{zE^2_{a}} + z(1-z)\frac{t}{E^2_{a}}.
\]

(1.4)

Since the parton masses are very small, they can be omitted for convenience. For a soft emission, the parton with index \( j \) tends to be a final-state particle, \( P^2_{j} \to 0 \), making the first term vanish. In addition, the evolution scale \( t \) decreases much faster than energy because of angular ordering \[17, 18\], \( t_i < (1 - z_{i-1})^2 t_{i-1} \), where the indices stand for the sequence of parton shower. Note that \( t_i \) has very small chance of being close to the starting scale \((1 - z_{i-1})^2 t_{i-1} \) due to suppression by the so-called Sudakov factor. Consequently, the order of the second term in (1.4) is lower than \( O(z) \), and the increase in virtuality \( P^2_{a,i} \) is expected to be smaller than the increase in \( E^2_{a,i} \). In total, the square bracket in (1.1) is roughly constant. This can be made apparent in the expanded form,

\[
J_{\beta}(P_{a}) = E_a \left[ (1 - \beta) + \beta v^2_{a} \right].
\]

(1.5)
as a function of the jet velocity \( v_a \equiv |P_a|/E_a \). For an energetic shower, the partons are highly relativistic, \( v_a \approx 1 \). Nevertheless, \( v_a \) can have slight decrease by clustering since \( P^2_{a,i}/E^2_{a,i} = 1 - v^2_{a,i} \) increases as indicated in (1.4). The jet function increases when reversing the parton shower chain, and hence can serve as a natural measure for reconstructing the parton shower history, mainly because of the increase in the clustering scale \( E_a \).

In the Georgi algorithms, one extra requirement is that, the jet function \( J_{\beta}(P_{a}) \) is positive. This imposes a constraint on the jet velocity,

\[
1 \geq v^2 \geq 1 - \frac{1}{\beta} \equiv v^2_{min}.
\]

(1.6)

Note that \( v^2_{min} \) is just a notation. It can take any value, even negative values, depending on the value of \( \beta \). Note that \( \beta \) should not be too large. Otherwise, the phase space will be highly suppressed and only highly relativistic jets can have a positive jet function. For \( \beta < 1 \), the whole phase space can be covered by a positive jet function.

In [14], the kinematic properties of the Georgi algorithms have been explored analytically for massless partons. We try to generalize the results to the massive case in Sec. 2 and provide further generalization of the jet function definition (1.1) in Sec. 3. Our conclusion is summarized in Sec. 5.

2. Clustering with Massive Subjets

Jet algorithm is designed to visualize the parton shower process. It is essential to provide a geometrical picture of the clustering process. An intuitive choice is the cone size. With energy and momentum magnitude fixed, those subjets contained within a certain cone are clustered. This property has already

\(^1\)For convenience, we have adopted the parton shower notation of energy fraction, which is different from the original notation \( E_j/E_a \equiv r_j \) used in [14] where \( z \) is used to denote the angle between \( P_a \) and \( p_j \), \( \cos \theta \), instead.
be implicitly incorporated in the jet definition (1.1) which only depends on the angle \( \theta \) between the jet 3-momentum \( P_\alpha \) and the 3-momentum \( p_j \) of the subjet,

\[
J_\beta(P_\alpha + p_j) = (E_\alpha + E_j) \left[ (1 - \beta) + \beta \frac{|P_\alpha|^2 + 2|P_\alpha||p_j| \cos \theta + |p_j|^2}{(E_\alpha + E_j)^2} \right]. \tag{2.1}
\]

The jet clustering condition that the jet function increases, \( J_\beta(P_\alpha + p_j) > J_\beta(P_\alpha) \), constrains the inclusion cone size, as will be explored in detail below.

Nevertheless, there are three extra properties that needs to be imposed. The first is that the result of jet clustering should be independent of the clustering sequence. For two subjets with the same energy \( E_j \), the same 3-momentum magnitude \( |p_j| \), and the same angle \( \theta \) with respect to the jet 3-momentum \( P_\alpha \), both of them should be clustered. Otherwise, the result is process-dependent for a sequential clustering.

In other words, the cone size should not shrink after swallowing a subjet. This requirement is consistent with the angular ordering \([17, 18]\) feature of parton shower. The opening angle between child partons keeps decreasing along the parton shower chain. When reversing the process during jet clustering, the inclusion cone size should increase in order to accommodate all partons branched from the same chain.

The other property comes from the tendency of parton shower to emit soft partons \([15, 16]\). Actually, softer partons tend to be produced at the earlier stage of the parton shower process with larger opening angle. Consequently, to make jet clustering approach the real parton shower process, it is necessary to have a larger cone size for softer subjet. We will show that these first two properties can be parameterized with a same quantity. Together, they eliminate the parameter region \( \beta < 0 \). The final property is that the inclusion cone cannot be too large. For the simplest case of \( e^+e^- \rightarrow jj \) at LEP, the inclusion cone should not be larger than half sphere. Otherwise, the two jets cannot be well separated. We will show that this gives a more stringent limit \( \beta > 1 \).

In the following analysis, we derive the most general form of the Georgi algorithms \([14]\) by keeping the subjet velocity \( v_j = |p_j|/E_j \) without simplification. The massless case can be restored by replacing \( v_j \) with 1.

### 2.1. The Inclusion Cone

Let us start with the clustering of the first subjet with momentum \( p_j \). For convenience of comparison with the clustering of the second subjet, which will be explored in Sec. 2.2, we parameterize the clustering condition on a common ground \( P_\alpha \) where the first has already been clustered, but the second has not. In other words, \( p_j \) is a part of \( P_\alpha \), to be exact \( j \in \alpha \) where \( \alpha \) is a set of subjet indices, and the jet momentum before clustering is \( P_\alpha - p_j \). Jet clustering condition requires that the jet function (1.1) increases,

\[
J_\beta(P_\alpha) > \max \{ J_\beta(P_\alpha - p_j), J_\beta(p_j) \}. \tag{2.2}
\]

Note that the jet function increases with respect to both subjets, because in reality it is impossible to distinguish the two sources. Using the expanded form (1.5), these two constraints can be expressed as,

\[
\begin{align*}
(1 - \beta) + \beta v_\alpha^2 &> (1 - z) \left[ (1 - \beta) + \frac{\beta}{(1 - z)^2} (v_\alpha^2 + z^2 v_j^2 - 2z \cos \theta v_\alpha v_j) \right], \tag{2.3a} \\
(1 - \beta) + \beta v_\alpha^2 &> z \left[ (1 - \beta) + \beta v_j^2 \right]. \tag{2.3b}
\end{align*}
\]

We can see that the second inequality (2.3b) gives a limit on the clustered jet velocity,

\[
v_\alpha^2 v_j^2 > z (v_j^2 - v_{\min}^2). \tag{2.4}
\]

This simply indicates that if \( J_\beta(p_j) \) is positive, \( J_\beta(P_\alpha) \) is also positive. The phase space is even enlarged after clustering for \( \beta > 1 \), due to the \( z \) factor in (2.4) which originates from the enhancement contributed by the clustering scale \( E_\alpha \) in the jet definition (1.1).

On the other hand, (2.3a) limits the jet cone size,

\[
\cos \theta > \cos \theta_{in} = \frac{\left[ (1 - z) v_{\min}^2 + z v_j^2 \right] + v_\alpha^2}{2 v_\alpha v_j} \geq \sqrt{(1 - z) \frac{v_{\min}^2 v_j^2}{v_j^2} + z}. \tag{2.5}
\]
Note that the second inequality happens on the boundary (2.4) of \( v_\alpha \) if \( \beta > 1 \). For massive subjet, \( v_j^2 < 1 \), the maximal inclusion cone is larger than the massless limit. The most interesting feature is the dependence on the energy fraction \( z \). We can decompose the cone size \( \cos \theta_{in} \) (2.5) as a series of \( z \),

\[
\cos \theta_{in} = \frac{1}{2v_\alpha v_j} \left[ (v_\alpha^2 + v_j^2) + (v_j^2 - v_{min}^2) \right].
\]  

(2.6)

This indicates that the cone increases with decreasing \( z \). In other words, the cone is larger for softer subjet if (1.6) is satisfied, which is exactly what required by parton shower. It provides a strong support for the requirement on the positiveness of the jet function.

The inclusion region (2.6) can also be expressed in terms of \( \sin(\theta_{in}/2) \) which increases with the cone size,

\[
2 \sin^2 \left( \frac{\theta_{in}}{2} \right) = \frac{1}{2v_\alpha v_j} \left[ (1 - z) \left( v_j^2 - v_{min}^2 \right) - (v_\alpha - v_j)^2 \right] > \frac{v_j - v_\alpha}{v_j},
\]  

(2.7)

where the inequality comes from (2.4). Since virtuality increases when reversing the parton shower history according to (1.2), the velocity decreases, \( v_j > v_\alpha \). This indicates that the inclusion cone cannot be too small.

### 2.2. The Exclusion Cone

After including the first subjet with certain energy \( (z) \) and 3-momentum magnitude \( (v_j) \), the jet momentum changes from \( P_\alpha - p_j \) to \( P_\alpha \). If the jet clustering is self-consistent, a second subjet inside the inclusion region (2.5) with the same energy and 3-momentum magnitude should also be clustered. Corresponding, the jet momentum changes from \( P_\alpha \) to \( P_\alpha + p'_j \), where \( p'_j(z, v_j, \theta') \) is the momentum of the second subjet. Suppose this second subjet can not be clustered, in other words, the jet function (1.1) decreases if so,

\[
J_\beta(P_\alpha) > \max \left\{ J_\beta(P_\alpha + p'_j), J_\beta(p'_j) \right\}.
\]  

(2.8)

From these two constraints we can derive the exclusion cone. Using the expanded form (1.5), we can get two inequalities,

\[
(1 - \beta) + \beta v_\alpha^2 > (1 + z) \left( (1 - \beta) + \frac{\beta}{(1 + z)^2} \left( v_\alpha^2 + z^2 v_j^2 + 2z \cos \theta' v_\alpha v_j \right) \right),
\]

\[
(1 - \beta) + \beta v_\alpha^2 > z \left[ (1 - \beta) + \beta v_j^2 \right].
\]  

(2.9a)

(2.9b)

Note that (2.9b) is exactly (2.3b), leading to the same constraint (2.4) on \( v_\alpha^2 \). But the cone limit,

\[
\cos \theta' < \cos \theta_{ex} \equiv \frac{(1 + z) v_{min}^2 - z \beta v_j^2 + v_\alpha^2}{2v_\alpha v_j},
\]  

(2.10)

is different from (2.5). This difference can be traced back to the sign difference of \( p_j \) and \( p'_j \) in the jet functions \( J_\beta(P_\alpha - p_j) \) and \( J_\beta(P_\alpha + p'_j) \), respectively, leading to an effective replacement \( z \to -z \). Note that the inclusion cone (2.5) and the exclusion cone (2.10) are well separated due to the lower limit (1.6) on \( v_j \),

\[
\cos \theta_{in} - \cos \theta_{ex} = \frac{z}{v_\alpha v_j} \left( v_j^2 - v_{min}^2 \right) > 0.
\]  

(2.11)

The subjet within the inclusion cone (2.5) with the same energy and 3-momentum magnitude can be readily clustered. This eliminates the possibility of \( v_{min}^2 > 1 \), or equivalently \( \beta < 0 \), since the jet velocity is bounded by the speed of light from above, \( v_j^2 \leq 1 \).

It should be emphasized that the only difference between the inclusion cone (2.5) and the exclusion cone (2.10) is a sign difference associated with \( z \). For the soft region, \( z \to 0 \), the difference between the two cones actually also characterizes the ability of accommodating softer subjet. This can be explicitly seen by comparing \( \cos \theta_{in} - \cos \theta_{ex} \) in (2.11) and the linear term of \( z \) in (2.6). The inclusion region should
expand during the jet clustering process in order to accommodate softer subjet while it is the opposite for the exclusion region, approaching each other.

Similar to (2.7), the exclusion cone size is bounded by,

\[ 2 \sin^2 \left( \frac{1}{2} \theta_{ex} \right) = \frac{1}{2v_\alpha v_j} \left[ (1 + z) \left( v_j^2 - v_{\text{min}}^2 \right) - (v_\alpha - v_j)^2 \right] < 1 - \frac{v_{\text{min}}^2}{v_\alpha v_j}. \]  

(2.12)

The inequality comes from (2.4). For \( \beta \geq 1 \), the exclusion cone is smaller than half sphere, \( \theta_{ex} < 90^\circ \). For a two-jet event in the center-of-mass frame, for example \( e^+ e^- \to jj \) at LEP, jet-clustering should reconstruct two jets that are back-to-back. On the other hand, the two jets can be mixed with each other if \( \beta < 1 \), which is a not good choice. A direct consequence is,

\[ 2 \sin^2 \left( \frac{1}{2} \theta_{ex} \right) < 1 - v_{\text{min}}^2 = \frac{1}{\beta}, \]  

(2.13)

since \( v_\alpha, v_j \leq 1 \). This can serve as a guide for choosing a reasonable value for \( \beta \). To recognize an event with more primary jets, like those events at a hadron collider, \( \beta \) should be larger. Note that this limit is independent of \( z \).

3. Generalized Jet Function

We note that \( v_\alpha^2 \) appears on both sides of (2.3a) and also of (2.9a). It is possible to make a complete cancellation if the prefactor \( 1 - z \) is replaced by \( (1 - z)^2 \). The same trick can be used to remove the factor \( 1 - \beta \) in (2.3) and (2.9) by removing the prefactor \( 1 - z \). Nevertheless, the first observation can become true but the later is not realistic as will be shown in detail below.

Since the power of the \( 1 - z \) prefactor can be traced back to the power of the clustering scale \( E_\alpha \), to achieve the small tricks we need to generalize the jet function (1.1) as follows,

\[ J_\beta^{(n)}(P_\alpha) \equiv E_\alpha^n \left( 1 - \beta \frac{P_\alpha^2}{E_\alpha^2} \right) = E_\alpha^n \left[ (1 - \beta) + \beta v_\alpha^2 \right]. \]  

(3.1)

Accordingly, the jet function (1.1) can be renamed as \( J_\beta(P_\alpha) \equiv J_\beta^{(1)}(P_\alpha) \). For generality, we keep the jet function index \( n \) as a free parameter in the following derivations. Note that \( n \) needs not to be an integer and can serve as a jet function parameter as \( \beta \). Its value is constrained by kinematics. As we have argued that the jet function increases mainly because of the increase in the prefactor \( E_\alpha^n \). The jet function index \( n \) cannot be arbitrarily small if the jet function can increase fast enough. To be exact, \( n > 0 \). We will show further constraints in the following analysis.

With this generalized jet definition, the limit on jet velocity (1.6) from the requirement that the jet function has to be positive is still the same. As expected, the \( 1 - z \) and \( z \) prefactors in the inclusion (2.3) and exclusion (2.9) criteria receives a nontrivial power \( n \),

\[ (1 - \beta) + \beta v_\alpha^2 > (1 \mp z)^n \left( 1 - \beta \right) + \frac{\beta}{(1 \mp z)^2} \left( v_\alpha^2 + z^2 v_j^2 \mp 2z \cos \theta v_\alpha v_j \right), \]  

(3.2a)

\[ (1 - \beta) + \beta v_\alpha^2 > z^n \left[ (1 - \beta) + \beta v_\alpha^2 \right], \]  

(3.2b)

with the sign \( \mp \) corresponding to inclusion and exclusion, respectively. From the second inequality, we can get a generalized form of the phase space constraint (2.4),

\[ v_\alpha^2 - v_{\text{min}}^2 > z^n \left( v_j^2 - v_{\text{min}}^2 \right). \]  

(3.3)

Similarly, \( v_\alpha^2 \) is contained within the positive jet function region (1.6) if \( v_j^2 \) already satisfies (1.6). The phase space of the \( \alpha \)-set becomes larger than the phase space of the subjet. In soft jet approximation, \( z \to 0 \), the difference can be significant.
The inclusion cone (2.5) and the exclusion cone (2.10) are constrained by the first inequality (3.2a),
\[
\cos \theta_{in}^{(n)} = \frac{1}{2z v_{a} v_{j}} \left\{ \left[ 1 - (1 - z)2^{2-n} \right] (v_{a}^2 - v_{min}^2) + z^2 (v_{j}^2 - v_{min}^2) + 2z v_{min}^2 \right\}, \tag{3.4a}
\]
\[
\cos \theta_{ex}^{(n)} = \frac{1}{2z v_{a} v_{j}} \left\{ \left[ (1 + z)^2 - 1 \right] (v_{a}^2 - v_{min}^2) - z^2 (v_{j}^2 - v_{min}^2) + 2z v_{min}^2 \right\}. \tag{3.4b}
\]

Then, we can explore the difference between them,
\[
\cos \theta_{in}^{(n)} - \cos \theta_{ex}^{(n)} = \frac{2 - (1 - z)^2}{2z v_{a} v_{j}} \left( (2 - n) \left( 1 - \frac{1 - n}{2} z \right) (v_{a}^2 - v_{min}^2) + z (v_{j}^2 - v_{min}^2) + 2v_{min}^2 \right) \tag{3.5a}
\]
\[
\geq \frac{1}{2v_{a} v_{j}} \left\{ z^{n-1} \left[ 2 - (1 - z)^2 - (1 + z)^2 \right] + 2z \right\} (v_{j}^2 - v_{min}^2). \tag{3.5b}
\]

If the clustering algorithm is self-consistent, the inclusion cone expands after clustering a subjet with the same energy and 3-momentum magnitude with the only difference in opening angle. This property is something like the inequality in (3.5a), which is satisfied for \( \alpha < 2 \). The self-consistency requirement of jet clustering provides an upper limit on the jet function index \( n \). Note that, for both \( n = 1 \) and \( n = 2 \), (3.5b) reduces to (2.11).

Now let us take a look at the soft region. If the subjet is soft, \( z \to 0 \), the inclusion and exclusion cones (3.4) can be approximated by an expansion up to the linear order of \( z \),
\[
\cos \theta_{in}^{(n)} \approx \frac{1}{2v_{a} v_{j}} \left\{ (2 - n) \left( 1 - \frac{1 - n}{2} z \right) (v_{a}^2 - v_{min}^2) + z (v_{j}^2 - v_{min}^2) + 2v_{min}^2 \right\}, \tag{3.6a}
\]
\[
\cos \theta_{ex}^{(n)} \approx \frac{1}{2v_{a} v_{j}} \left\{ (2 - n) \left( 1 + \frac{1 - n}{2} z \right) (v_{a}^2 - v_{min}^2) - z (v_{j}^2 - v_{min}^2) + 2v_{min}^2 \right\}. \tag{3.6b}
\]

The difference (3.5) between the inclusion and exclusion cones is roughly,
\[
\cos \theta_{in}^{(n)} - \cos \theta_{ex}^{(n)} \approx \frac{1}{2v_{a} v_{j}} (2 - n)(1 - n)z (v_{a}^2 - v_{min}^2) + \frac{z}{v_{a} v_{j}} (v_{j}^2 - v_{min}^2), \tag{3.7}
\]
which is highly suppressed. Nevertheless, overlapping can still happen. To avoid this tiny chance, the following relation between \( v_{a}^2 \) and \( v_{j}^2 \) has to be satisfied,
\[
(v_{j}^2 - v_{min}^2) \geq \frac{(2 - n)(1 - n)}{2} (v_{a}^2 - v_{min}^2). \tag{3.8}
\]
Together with (3.3), we can get,
\[
\left[ z \frac{1}{(2 - n)(1 - n)} - \frac{(2 - n)(1 - n)}{2} \right] (v_{a}^2 - v_{min}^2) \geq 0, \tag{3.9}
\]
which is always true for \( n > 0 \). The jet-clustering self-consistency in the soft region also imposes a lower limit on the jet function index \( n \). Since self-consistency in the soft region is directly related to the requirement that soft emission has a larger inclusion cone, in order to make the jet algorithm approach the parton shower evolution, this lower limit can also be treated as a requirement of this feature.

To see the boundary on the exclusion cone, we need to first check the sign of \( z \) in the expanded form (3.6b),
\[
\cos \theta_{ex}^{(n)} \approx \frac{1}{2v_{a} v_{j}} \left\{ (2 - n) (v_{a}^2 - v_{min}^2) + 2v_{min}^2 + \left[ \frac{(2 - n)(1 - n)}{2} (v_{a}^2 - v_{min}^2) - (v_{j}^2 - v_{min}^2) \right] z \right\}. \tag{3.10}
\]
We can see that, with the help of (3.3) the exclusion cone can have a bound like (2.13), which is independent of \( z \), if the coefficient of \( z \) in (3.10) is negative. This can be guaranteed for \( 1 \leq n \leq 2 \),
\[
\cos \theta_{ex}^{(n)} \approx \frac{1}{2v_{a} v_{j}}[(2 - n) (v_{a}^2 - v_{min}^2) + 2v_{min}^2]
\]
\[
\geq \frac{1}{2v_{a} v_{j}}[(2 - n) (v_{a}^2 - v_{min}^2) + 2v_{min}^2]
\]
\[ + \frac{1}{2v_\alpha v_j} \left[ \frac{(2-n)(1-n)}{2} (v_\alpha^2 - v_{\text{min}}^2) - (v_j^2 - v_{\text{min}}^2) \right] \left( \frac{v_\alpha^2 - v_{\text{min}}^2}{v_j^2 - v_{\text{min}}^2} \right)^{1/n}. \] (3.11)

Since the partons are quite relativistic, \( v_\alpha \approx v_j \approx 1 \), this limit reduces to,

\[ 2 \sin^2 \left( \frac{1}{2} \theta_\text{ex}^{(n)} \right) \lesssim \frac{n(5-n)}{4} \frac{1}{\beta}. \] (3.12)

The result (2.13) can be reproduced with \( n = 1 \). We can see that \( \beta \) is still directly related to the kinematic boundary. Since the cone size should not be larger than half sphere, \( \beta \) has a lower limit,

\[ \beta > \frac{4}{n(5-n)} \geq \frac{2}{3}. \] (3.13)

In the range \( 1 \leq n \leq 2 \), the coefficient \( n(5-n) \) decreases with \( n \). Consequently, \( \beta \) should increase with \( n \). For the original scheme, \( n = 1, \beta > 1 \), leading to \( 0 < v_{\text{min}}^2 < 1 \). Only part of the phase space can be covered which is especially true with more than 2 jets with \( \beta \) further enhanced. By generalization, \( n > 1 \), the parameter \( \beta \) can be smaller than 1, leading to a negative \( v_{\text{min}}^2 \) which can cover the whole phase space.

Similarly, there is a lower limit on the inclusion cone size,

\[ \cos \theta_\text{in}^{(n)} \lesssim \frac{1}{2v_\alpha v_j} \left[ (2-n) (v_\alpha^2 - v_{\text{min}}^2) + 2v_{\text{min}}^2 \right] \]
\[ - \frac{1}{2v_\alpha v_j} \left[ \frac{(2-n)(1-n)}{2} (v_\alpha^2 - v_{\text{min}}^2) - (v_j^2 - v_{\text{min}}^2) \right] \left( \frac{v_\alpha^2 - v_{\text{min}}^2}{v_j^2 - v_{\text{min}}^2} \right)^{1/n}, \] (3.14)

which reduces to,

\[ 2 \sin^2 \left( \frac{1}{2} \theta_\text{in}^{(n)} \right) \gtrsim \frac{n(n-1)}{4} \frac{1}{\beta}, \] (3.15)

in the relativistic limit. For \( n \geq 1 \), the inclusion cone cannot be arbitrarily small.

### 4. Lorentz Invariance

From the constraint on the jet velocity (1.6), we can see that \( \beta = 1/(1 - v_{\text{min}}^2) \) is actually the square of the corresponding boost factor \( \gamma_{\text{min}} = 1/\sqrt{1 - v_{\text{min}}^2} \). This indicates that \( \beta \) can be used to comply with Lorentz boost. It is important to check how Lorentz boost affect the jet algorithm, especially for highly boosted jets at hadron collider like LHC.

Under Lorentz boost the jet virtuality is invariant and the only change comes from the jet energy,

\[ P_\alpha^2 \to P_\alpha^2, \quad E_\alpha \to \gamma_B E_\alpha. \] (4.1)

Note that \( \gamma_B \) is not a universal boost factor, but varies from jet to jet. To comply with the effect of Lorentz boost, the change in the dimensionless part of (3.1) can be compensated by

\[ \beta \to \gamma_B^2 \beta, \] (4.2)

and the jet function should be redefined as,

\[ J_\beta^{(n)} \to \gamma_B^{-n} J_\beta^{(n)} \gamma_B^2 \beta, \] (4.3)

to retain the original jet function value. Consequently, the clustering sequence would not be affected. In this sense, the jet algorithm is essentially Lorentz invariant.
5. Conclusion

We reveal the direct link between the Georgi algorithms of jet clustering and the parton shower kinematics. The energy increases during jet clustering due to energy conservation while the ratio $P_\alpha^2/E_\alpha$ does not increase much because of the fact that parton shower tends to emit soft partons. Our observation provides a sound support for this elegant class of jet-clustering algorithms whose kinematic features are explored systematically for both massless and massive subjets. We also generalize the jet function definition to $J_\alpha^{(n)}(P_\alpha)$, with a free jet index $n$ which is constrained within the range $1 \leq n \leq 2$. Its upper limit comes from the self-consistency of the jet algorithm, while the lower one comes from the requirement that the exclusion cone cannot be arbitrarily large. The parameter $\beta$ of the jet function $J_\beta(P_\alpha)$ is found to have the meaning of phase space boundaries and is constrained to be $\beta > 4/n(5-n) \geq 2/3$. In this generalization, the original Georgi algorithms can be covered as special cases, $J_\beta(P_\alpha) = J_\beta^{(1)}(P_\alpha)$. This new class of jet algorithms are boost invariant.

6. Acknowledgements

SFG is grateful to Kaoru Hagiwara, Grisha Kirilin, and Junichi Kanzaki for discussions about parton shower and introduction to this field of research. The current work is supported by Grant-in-Aid for Scientific research (No. 25400287) from JSPS.

References

[1] George F. Sterman and Steven Weinberg. *Jets from Quantum Chromodynamics*. Phys.Rev.Lett., 39:1436, 1977.

[2] S. Catani, Yuri L. Dokshitzer, M.H. Seymour, and B.R. Webber. *Longitudinally invariant Kt clustering algorithms for hadron hadron collisions*. Nucl.Phys., B406:187–224, 1993.

[3] Stephen D. Ellis and Davison E. Soper. *Successive combination jet algorithm for hadron collisions*. Phys.Rev., D48:3160–3166, 1993, [arXiv:hep-ph/9305266].

[4] Yuri L. Dokshitzer, G.D. Leder, S. Moretti, and B.R. Webber. *Better jet clustering algorithms*. JHEP, 9708:001, 1997, [arXiv:hep-ph/9707323].

[5] M. Wobisch and T. Wengler. *Hadronization corrections to jet cross-sections in deep inelastic scattering*. 1998, [arXiv:hep-ph/9907280].

[6] M. Wobisch. *Measurement and QCD analysis of jet cross-sections in deep inelastic positron proton collisions at $\sqrt{s} = 300$-GeV*. 2000.

[7] Matteo Cacciari, Gavin P. Salam, and Gregory Soyez. *The Anti-k, jet clustering algorithm*. JHEP, 0804:063, 2008, [arXiv:0802.1189 [hep-ph]].

[8] S. Catani, Yuri L. Dokshitzer, M. Olsson, G. Turnock, and B.R. Webber. *New clustering algorithm for multi-jet cross-sections in e$^+$e$^-$ annihilation*. Phys.Lett., B269:432–438, 1991.

[9] Stefano Moretti, Leif Lonnblad, and Torbjorn Sjostrand. *New and old jet clustering algorithms for electron - positron events*. JHEP, 9808:001, 1998, [arXiv:hep-ph/9804296].

[10] Gerald C. Blazey, Jay R. Dittmann, Stephen D. Ellis, V. Daniel Elvira, K. Frame, et al. *Run II jet physics*. pages 47–77, 2000, [arXiv:hep-ex/0005012].

[11] S.D. Ellis, J. Huston, K. Hatakeyama, P. Loch, and M. Tonnesmann. *Jets in hadron-hadron collisions*. Prog.Part.Nucl.Phys., 60:484–551, 2008, [arXiv:0712.2447 [hep-ph]].
[12] Gavin P. Salam. Towards Jetography. Eur.Phys.J., C67:637–686, 2010, [arXiv:0906.1833 [hep-ph]].

[13] Ahmed Ali and Gustav Kramer. Jets and QCD: A Historical Review of the Discovery of the Quark and Gluon Jets and its Impact on QCD. Eur.Phys.J., H36:245–326, 2011, [arXiv:1012.2288 [hep-ph]].

[14] Howard Georgi. A Simple Alternative to Jet-Clustering Algorithms. 2014, [arXiv:1408.1161 [hep-ph]].

[15] S. Catani, L. Trentadue, G. Turnock, and B.R. Webber. Resummation of large logarithms in e+ e- event shape distributions. Nucl.Phys., B407:3–42, 1993.

[16] Stefan Gieseke, P. Stephens, and Bryan Webber. New formalism for QCD parton showers. JHEP, 0312:045, 2003, [arXiv:hep-ph/0310083].

[17] Alfred H. Mueller. On the Multiplicity of Hadrons in QCD Jets. Phys.Lett., B104:161–164, 1981.

[18] B.I. Ermolaev and Victor S. Fadin. Log-Log Asymptotic Form of Exclusive Cross-Sections in Quantum Chromodynamics. JETP Lett., 33:269–272, 1981.