Dynamics of Momentum Distribution and Structure Factor in a Weakly Interacting Bose Gas with a Periodical Modulation

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In this paper, we study dynamical structure factor and momentum distribution of a weakly interacting many-body Bose gas whose interaction is periodically modulated in terms of time by time-dependent Bogoliubov theory. The evolution equation which derived from the canonical motion equations for operator in quadrature representation is solvable Mathieu equation in a periodical interacting condition. We identify the condition of periodical momentum distribution and dynamical structure factor is a series of relations of kinetic energy and scatter length, and give expressions of periodical evolution. Furthermore, we also show that both stable and unstable time evolution of momentum distribution and structure factor, which only depend on the parameters we choose. We find that the stable peaks are quite similar indicating the strong relation between momentum distribution and structure factor. There is no such clear relation in the unstable dynamics due to parametric resonance. In addition, we discuss the possibility of experimental test of those dynamics.

I. INTRODUCTION

In recent decades, the advance of experiment of ultracold atoms gases has an unprecedented influence on the understand of many-body systems.\(^1\) In particular, due to Feshbach resonance, we can arbitrarily regulate the magnitude of the interaction between atom gases even can change the sign of scatter length which describes the property of interaction\(^2\). It is a pure and precise platform to study the dynamics of many-body systems.

The time evolution of the core quantities is one of the key problems in ultracold gases, especially, in the case of interacting quenches. Many methods have employed to analyze quench-like dynamics in weakly interacting Bose gas\(^3\),\(^4\),\(^5\). Recently, a simple Bogoliubov treatment was developed where the evolution matrix connect quasiparticles operators and particle operators at different times. Moreover, some exactly solvable models are calculated\(^6\).

The few- and many-body problems with a periodically-modulated potential attract many physicists\(^7\),\(^8\). One of the most prominent example is Bloch oscillation, which has been observed in artificial semiconductors with structure of super lattice periods\(^9\). Impressively, the theoretical progress suggest even if in the condition of weak interaction between atoms, the density profiles can change tempestuously through a periodically modulated coupling constant\(^10\). It is worth noting that the study of the dynamics of harmonically trapped Tonks-Girardeau gas with a sinusoidal frequency, which shows that the solution of this problem can be mapped to the solution of Mathieu equation. The dynamics are calculated by analyzing this solvable differential equation\(^11\). However, it is impossible for Tonks-Girardeau gas to emerge Bose-Einstein condensation (BEC) due to the strong interaction of the one-dimension Bosons. More recently, a statistical analysis of one-dimension BEC in periodical modulation has been discussed\(^12\).

In this paper, We consider the time evolution of the weakly interacting Bose gas with a periodical time-dependent coupling constant. The creation and annihilation operators in time-dependent Hamilton of Bose system can be substituted by time-dependent harmonic operator. Therefore, the motion equation of time-dependent harmonic operator with a cosine-varied coupling constant can be found, namely Mathieu equation. Therefore, we can give the different kinds of time evolutions of distribution of momentum and structure factor, i.e. the stable and unstable dynamics with the choices of some particular experimentally-modulated parameters.

II. BOGOLIUBOV TREATMENT FOR EVOLUTION EQUATION

In this section, we briefly introduce a time-dependent Bogoliubov theory, and the time-dependent harmonic oscillator method (TDHO) which used to calculate the time-propagated Bogoliubov weights\(^6\). We focus on the dynamical equation of evolution function which is the key equation of the whole theory.

In a weakly interacting Bose gas, the condensate wave function obeys a nonlinear Schrödinger equation,

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + g(t)|\Psi(t)|^2 \right] \Psi(t). \quad (1)$$

The above Gross-Pitaevskii equation without regard to external traps, where \(g(t)\) is a time-dependent coupling constant.

$$g(t) = \frac{4\pi \hbar^2 a(t)}{m}. \quad (2)$$

where \(a(t)\) is time-vary scatter length of \(s\) wave. As usual,
In order to calculate the Bogoliubov weights, the Hamiltonian is
\[ H = E_0 + \sum_{k \neq 0} (\epsilon_k + g(t)\rho) a_k^\dagger a_k + \frac{g(t)\rho}{2} \sum_{k \neq 0} (a_k^\dagger a_{-k} + a_k a_{-k}) , \]
where \( E_0 \) is mean-field ground energy, also dependent on \( g(t) \). \( \epsilon_k = \hbar^2 k^2 / 2m \). Then we can obtain Heisenberg equation of \( a_k, a_{-k} \) from \( H \)
\[ i\hbar \frac{d}{dt} \begin{pmatrix} a_k \\ a_{-k} \end{pmatrix} = \begin{pmatrix} \epsilon_k + g(t)\rho & g(t)\rho \\ -g(t)\rho & -(\epsilon_k + g(t)\rho) \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k} \end{pmatrix} , \]
i.e.
\[ i\hbar \dot{A}_k = H_A(t) A_k . \]
Introduce Bogoliubov transformation, \( a_k = u_k(t) b_k + v_k(t) b_{-k}^\dagger \), where
\[ u_k(t) \pm v_k(t) = \frac{\epsilon_k}{\hbar \omega_k(t)} \left( \pm \frac{1}{2} \right), \quad \left( \hbar \omega_k(t) \right)^2 = \epsilon_k(\epsilon_k + 2g(t)\rho) , \]
Similarly, define \( B = (b_k, b_{-k}^\dagger)^T \). The connection of the two new operators is expressed as
\[ A_k = W(t, t_0) B_k(t_0) , \]
where \( W(t, t_0) \) is evolution matrix,
\[ W(t, t_0) = \begin{pmatrix} U(t, t_0) & V^\dagger(t, t_0) \\ V(t, t_0) & U^\dagger(t, t_0) \end{pmatrix} . \]
In order to calculate the Bogoliubov weights, we introduce the time-dependent harmonic oscillator method (TDHO). We define harmonic operators
\[ q_k = \frac{1}{\hbar} (a_k + a_{-k}^\dagger) , \quad \frac{\hbar k}{2\hbar} a_{-k} - a_k^\dagger . \]
Therefore, rewrite the Hamilton and find that \( q_k \) obeys the dynamical equation
\[ \dot{q}_k = \frac{p_k}{m} , \quad \dot{p}_k = -m \omega_k^2(t) q_k = 0 . \]
If we know the initial value \( q_k \), construct the solution of the equation from \( t_0 \) to \( t \),
\[ q_k(t) = \gamma_1(t, t_0) q_k(t_0) + \frac{\gamma_2(t, t_0)}{m \epsilon_k / \hbar} p_{-k}(t_0) , \]
\[ p_{-k}(t) = \gamma_1(t, t_0) q_k(t_0) + \frac{\gamma_2}{\epsilon_k / \hbar} p_{-k}(t_0) . \]
We find the relation between the Bogoliubov weights and the evolution functions \( \gamma_1, \gamma_2 \),
\[ U(t, t_0) + V(t, t_0) = \sqrt{\frac{\epsilon_k}{\hbar \omega_k(t_0)}} \gamma_k(t, t_0) , \]
\[ U(t, t_0) - V(t, t_0) = \sqrt{\frac{\hbar \omega_k(t_0)}{\epsilon_k}} \gamma_k(t, t_0) . \]
where
\[ \gamma_k(t, t_0) = \gamma_1 - i \frac{\hbar \omega_k(t_0)}{\epsilon_k} \gamma_2 , \quad \gamma_k'(t, t_0) = \frac{i \gamma_1}{\omega_k(t_0)} . \]

The evolution function \( \gamma_k \) fulfills the equation,
\[ \gamma_k'' + \omega_k^2(t) \gamma_k = 0 . \]
with initial conditions
\[ \gamma_k(t_0, t_0) = 1 , \quad \gamma_k(t_0, t_0) = i \omega_k(t_0) . \]
Therefore, the Eq. \ref{eq:17} is the key equation of the dynamical theory. In the next section, we calculated the Eq. \ref{eq:17} in the case of the periodical modulation and find the evolution of the momentum distribution and structure factor.

III. THE DYNAMICS WITH PERIODICAL MODULATION

After deriving the equation of evolution function, we let \( g(t) = g_0 \cos \Omega t \), \( g_0 \) is a time-independent constant. The Eq. \ref{eq:17} becomes
\[ \frac{d^2 \gamma_k}{d\tau^2} + (\lambda - 2q \cos 2\tau) \gamma_k = 0 , \]
where
\[ \tau = \Omega t / 2 , \quad \lambda = \frac{4\epsilon_k^2}{\Omega^2 \hbar^2} , \quad q = - \frac{4\rho g_0 \epsilon_k}{\Omega^2 \hbar^2} . \]
This equation is known as the canonical form of Mathieu equation when the value of \( \Omega \) is equal to \( 2^{[13,14,15]} \). The Mathieu equation have periodical and aperiodic solutions. The condition of periodical solutions is that the parameters \( \lambda \) and \( q \) meet a series of relations. It implies that if we tune \( g_0 \) to satisfy the particular relation between \( \lambda \) and \( q \), the periodical solutions obtained. It’s obvious that the momentum distribution and structure factor are also periodical.

Mathieu functions(thesolutions of Mathieu equations) have many applications with periodic potential in various areas in physics. For instance, in designing the Paul ion trap, we have to consider the stability of the device from the stability of the solutions of Mathieu equations. What’s more, the dynamical problems of a simple ion in the trap can be obtained[17,18]. The similar example is the motion of atom gases in optical lattice[19,20].

The expressions of momentum distribution \( n_k(t) \) and the structure factor \( S(k, t) \) are,
\[ n_k(t, t_0) = |V_k(t, t_0)|^2 + |U_k(t, t_0)|^2 N_k(t_0) , \]
\[ S(k, t) = \frac{\hbar \omega_k(t_0)}{\epsilon_k} |U_k(t, t_0) + V_k(t, t_0)|^2 S(k, t_0) . \]
solution of Mathieu equation (blue line). Any curve of eigenvalue $b_i$ corresponds to the odd periodical solution (purple line). The solution of Mathieu equation is stable or unstable in blank areas or green areas.

where

$$N_k(t_0) = \langle b_k(t_0)b_k(t_0) \rangle,$$ (23)

$$S(k, t_0) = \frac{\epsilon_k}{\hbar \omega_k(t_0)} [2N_k(t_0) + 1].$$ (24)

We can also consider the above expressions at zero temperature, $N_k(t_0) = 0$ at initial time $t_0$. Then

$$n_k(t, t_0) = |V_k(t, t_0)|^2,$$ (25)

$$S(t, t_0) = |U_k(t, t_0) + V_k(t, t_0)|^2.$$ (26)

The expressions become much simpler at zero temperature. The system is equilibrium state at initial time $t_0$, then $N_k(t_0)$ is Bose-Einstein statistics. The shape of evolution graph of the system would not be affected whether or not at finite temperature in initial time $t_0$, just a difference of coefficients depend on the ratio of scaling energy $k_BT$ and kinetic energy $\epsilon_k$. Therefore, we just consider the weakly-interacting Bose gas at zero temperature at initial time.

A. stable dynamics

1. periodical condition and periodical dynamics

To study the periodical dynamics, We primarily obtain the periodical condition from the boundary curves which separate stable and unstable areas in Fig.1. In other words, the condition of periodical dynamics can be shown in terms of two experimentally-modulated quantities: kinetic energy and $g_0$ in Fig 2.

We find the expressions of periodical momentum distribution and structure factor from the periodical solutions of Eq. (19)

$$\gamma_k(t, t_0) = \begin{cases} se_{n, k}(t, t_0), & n = 1, 2, \ldots \\ ce_{m, k}(t, t_0), & m = 0, 1, 2, \ldots, \end{cases}$$ (27)

where $se_{n, k}$ (or $ce_{m, k}$) are odd (even) periodical solution. Therefore, evolution matrix elements can be expressed as

$$U = \frac{1}{2} \sqrt{\frac{\epsilon_k}{\hbar \omega_k(t_0)}} \left[ se_{n, k}(t, t_0) + \frac{i\hbar}{\epsilon_k} se'_{n, k}(t, t_0) \right],$$ (28)

$$V = \frac{1}{2} \sqrt{\frac{\epsilon_k}{\hbar \omega_k(t_0)}} \left[ se_{n, k}(t, t_0) - \frac{i\hbar}{\epsilon_k} se'_{n, k}(t, t_0) \right].$$ (29)

where $se'_{n, k}(t, t_0)$ is the derivative of $se_{n, k}(t, t_0)$ in term of time $t$. One has

$$n_k(t, t_0) = \frac{1}{4 \hbar \omega_k(t_0)} [se^2(t, t_0) + \frac{\hbar^2}{\epsilon_k^2} se'^2(t, t_0)] (1 + 2N_k(t_0)),$$ (30)

$$S(k, t) = se^2(t, t_0) S(k, t_0).$$ (31)

The $n_k(t)$ and $S(k, t)$ are both periodical as in Eq.(30) and Eq.(31) due to the periodic function of $se, se'$. In Fig 3, we show the periodical evolutions of $n_k(t)$ and $S(k, t)$ in the condition of $\lambda = \alpha_1(q)$ which corresponds to periodical solution $ce_1(t)$.

2. nonperiodic stable dynamics

We can obtain another dynamics that is stable but nonperiodic by fixing parameters of $\lambda$ and $q$ in the blank areas of Fig.1. In Fig 4, the momentum distribution have slight vibration around the initial value. In other words, They both emerge many finite peaks but have no period. However, the structure factor have more larger amplitude on its own intrinsic meaning of correlation compare to the momentum distribution $n_k(t)$.
FIG. 3. The periodical dynamics of momentum distribution (in blue) and structure factor (in purple) for $\lambda = 1.46677$, $q = 0.5$ which is on the curve of $a_1(q)$ in Fig.1. $\hbar = 1$.

FIG. 4. Same as in Fig.3, the nonperiodic stable dynamics for $\lambda = 2$, $q = 0.2$ at initial condition $\gamma_k(0) = 0$, $\dot{\gamma}_k = 0.5$.

In Fig. 4, we can find the peaks appear at same time, which indicate a strong relation between momentum distribution and structure factor. Such similar peak structure of single-particle property and two-body correlation shows that particularity of Bose-Einstein condensate. In general, however, there is no direct relationship between the properties of single particle and two-body correlation, as we can see in unstable dynamics.

B. Unstable dynamics

In Fig. 5, we show the unstable dynamics of the weakly-interacting Bose gas by fixing parameters in the green areas of Fig.1. As evolution of the system in such fixed parameters, the momentum distribution and structure factor become very large. The peaks are growing in exponential form because of the natural breathing modes of the Bose gas at the particular modulated parameters. It is due to the emergence of parametric resonances.

The unstable dynamics suggests that even in a weak interaction of Bose gas, the correlation $S(k, t)$ would be very large, so is $n_k(t)$. It is also a similar case with the result in the reference. As time goes on, the density of bosons amplifies at momentum space, the correlation between Bosons also become very large. It is analogous to Faraday wave emerged in real space of Bose-Einstein condensation.

Unlike the case of stable dynamics, although the increasing peaks of dynamical structure factor have clear shape, the peaks of momentum distribution are obscure. It shows that one-particle property have no clear relation of density-density correlation. This means that the Bose-Einstein condensate system has been destroyed.

IV. CONCLUSION

In this paper, we show that the time evolution of momentum distribution $n_k(t)$ and structure factor $S(k, t)$ in a periodical modulation of weak Bose gas. From a time-dependent Bogoliubov theory, we obtain the dynamics of the system directly with no need for calculating the wave function from Gross-Pitaevskii equation. With the cosine-modulated interaction of the system, The evolution equations become Mathieu equations which have three-type solutions, namely three-type dynamics. We note that the condition of periodical dynamics is a series of eigenfunctions (i.e., kinetic energy) of scatter length. It means that the condition are some particular values of scatter length. We also give stable dynamics that the fluctuations of $n_k(t)$ and $S(k, t)$ are small and many finite peaks but not the periodical. The unstable dynamics, similar to the instability of classical liquid interface which can emerge Faraday waves, are very interesting. The instability of system lies in the homogeneity of the Bose system corresponding to the linear Eq.(9).
Our results are valid in any dimension, not confined in one dimension. It is the virtue of Bogoliubov treatment. These results can be verified in ultracold gas experiment, such as the dynamical behavior of dynamical structure factor, can be verified in experiment of inelastic light scattering experiments of probe particles and Bose atomic gas.

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