Inverse spherical Bessel functions generalize Lambert $W$ and solve similar equations containing trigonometric or hyperbolic subexpressions or their inverses

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Abstract

A strict integer Laurent polynomial in a variable $x$ is 0 or a sum of one or more terms having integer coefficients times $x$ raised to a negative integer exponent. Equations that can be transformed to certain such polynomials times $\exp(-x) = \text{constant}$ are exactly solvable by inverses of modified spherical Bessel functions of the second kind $k_n(x)$ where $n$ is the order, generalizing the Lambert $W$ function when $n > 0$. Equations that can be converted to certain such polynomials times $\cos(x)$ or such polynomials times $\sin(x)$ or a sum thereof $= \text{constant}$ are exactly solvable by inverses of spherical Bessel functions $y_n(x)$ or $j_n(x)$. Such equations include $\cos(x)/x = \text{constant}$, for which the solution inverse$_1(y_0)(-\text{constant})$ is the Dottie number when $\text{constant} = 1$, where subscript 1 is the branch number. Equations that can be converted to certain strict integer Laurent polynomials times $\sinh(x)$ and possibly also plus such a polynomial times $\cosh(x)$ are exactly solvable by inverses of modified spherical Bessel functions of the first kind $i_n(x)$.

These discoveries arose from the AskConstants program surprisingly proposing the explicit exact closed form solution inverse$_1(y_0)(-1)$ for the approximate input $0.739085133215160642$, because no explicit exact closed form representation was known for this Dottie number from approximately 1865 to 2022. This article includes descriptions of how to implement these spherical Bessel functions and their multi-branched real inverses.

1 Introduction

Professor Dottie noticed that whenever she typed any real number $x_0$ into her calculator in radian mode then repeatedly pressed the cosine button, the result always converged to the value

$$\cos \cos \ldots \cos x_0 \to 0.73908513321516.$$

This rounded value of the Dottie number (OEIS A003957, [1]) is the only real fixed point of

$$x = \cos x.$$ (1)
This transcendental number was known at least as long ago as 1865 [1]. Hansha [6] nicely summarizes some facts about it.

Figure 1 plots the intersecting two sides of Dottie’s fixed-point equation (1).

Figure 1: The one and only real intersection of $x$ with $\cos x$

Mathematica can express this number as $\text{Root}\left[\{#1 - \cos #1\&, 0.73908513321516\}\right]$. This representation can be considered exact because Mathematica can do some exact computations with it, such as

$$\text{With}\left[\{x = \text{Root}\left[\{#1 - \cos #1\&, 0.73908513321516\}\right]\}, \text{FullSimplify}\left[x - \cos x\right]\right],$$

which returns exact integer 0 rather than merely a floating-point residual that has small magnitude compared to 0.73908513321516. For brevity, floating-point numbers are hereinafter called floats.

Moreover, $\text{N}\left[\text{Root}\left[#1 - \cos #1\&, 0.73908513321516\right], \text{precision}\right]$ returns a float having the requested value of precision despite the float in the Root expression having only 14 significant digits. For example on my 4 gigahertz computer,

$$\text{N}\left[\text{Root}\left[#1 - \cos #1\&, 0.73908513321516\right], 1000\right]$$

returns 1000 digits in 0.002 seconds. In contrast, $\text{N}\left[0.73908513321516, 1000\right]$ returns 0.73908513321516 unchanged because there is insufficient information in 0.73908513321516 alone to extend it. The finite-precision float in the Root expression is merely to indicate the closest root with sufficient precision to guarantee that the same root can be extended to any finite precision up to the limits of your patience and computer memory.

Although such Root expressions can be considered exact, they are implicit rather than explicit representations, making them aesthetically less satisfying than solutions of the form $x = \text{constant}$ where constant is composed of a finite number of rational numbers, symbolic constants such as $\pi$, operators, and functions having known names, including
subexpressions of the form $\text{inverse}(f)(\text{constant})$, where $f$ is a known name, such as erf. Functional forms such as $\int \ldots \, dx$ and $\sum \ldots \,$ do not occur, having been replaced by explicit closed-form equivalents.

As a test, I copied 18 digits of the approximate value of the Dottie number into Version 5.0 of my AskConstants application [13] and [14]. I merely hoped that it did not propose an impostor exact closed form and misleadingly assess it a high likelihood of being a correct limit as the precision approaches $\infty$. However, I was surprised and delighted to discover that the application returned a simple candidate that is easily proved to be a previously unknown explicit exact closed form solution. Figure 2 shows that AskConstants proposes

$$\text{RealInverseSphericalBesselY} \ [0, -1, 1]$$

as the limit. Such constant-“recognition” programs generate conjectures rather than proofs. Section 2 contains a proof, and the ordinate of the large dot in the scatter plot encouragingly indicates that this candidate agrees with the input to all 18 of the entered significant digits.

1Just as nouns enjoy the economy of concisely representing descriptive phrases, explicitly named inverses such as arctan or Mathematica’s InverseErf are mentally more economical than subexpressions containing compositions such as $\text{inverse}(f)$, which are more economical than natural language descriptions such as “the least positive solution to the equation . . .”.

“What’s in a name?”
– Shakespeare

2$\text{RealInverseSphericalBesselY} \ [n, t, b]$ is one of about 45 AskConstants functions that implement real inverses of Mathematica functions that have no built-in inverse counterparts. Here $n = 0$ is the order of the inverted function $y_n$, $t = -1$ is the value of $y_n$, and $b = 1$ is the branch number of the countably infinite branch numbers of the inverse. The Real prefix is because the implementation is designed only for real values of $t$ where the inverse is also real, which is all that is necessary for AskConstants (and for many other applications).
Figure 2: AskConstants proposes an explicit exact closed form for the Dottie number

However, it is easy to match any number of float digits with a sufficiently complicated non-float expression. For example, all floats are exactly representable as rational numbers – albeit often with many digits in the numerator and/or denominator. Therefore AskConstants also assesses Agreement discounted by a complexity measure that also has digits as units: The scatter plot also shows that the abscissa Entropy10 complexity measure of the proposed candidate is approximately 4.0. The Entropy10 of an exact expression is the sum of the base 10 logarithms of the absolute values of the nonzero integers in numerators and denominators in the expression, plus an average of approximately 1.0 per operator, function or symbolic constant. The Likelihood of a proposed exact candidate expression being the true limit increases with

$$\text{Margin} := \text{Agreement} - \text{Entropy10}.$$  

The Margin is thus approximately $18.0 - 4.0 = 14.0$ digits, which together with the Agreement loss of approximately 0.0 digits makes the candidate very likely to be a true limit\(^3\).

The smaller scatter plot dots in Figure 2 are the best of many rejects, and mousing over them invokes tooltips with their candidate formulas. For example, the leftmost small dot above the upper dashed horizontal line is for the candidate

$$\frac{2(55103 + 19462 \times \text{Khinchin})}{290541}$$

\(^3\)And entering all 108 digits from OEIS A003957, \[1\] gives the same explicit closed-form result agreeing to 108 digits, which is much stronger evidence.
with Khinchin representing his constant $\approx 2.68545$. Its Agreement is also approximately all 18 digits, but with Entropy10 approximately 17.0, giving it an agreement Margin of only approximately 1.0. As indicated in the plot legend, a plot point must be in a color band near the pink end of the spectrum in the upper left corner of the plot to have a high likelihood of being the limit of the float as its Precision approaches infinity.

Bill Gosper expressed similar surprise when he subsequently obtained the same candidate exact value for the Dottie number. This article is an explanation together with some additional discoveries made while formulating the explanation. More specifically:

Section 2 shows how the inverses of spherical Bessel functions $y_n(x)$ can solve some equations transformable to the form

$$\left(\sum_{\ell=1}^{n+1} \frac{c_\ell}{x^{\ell}}\right) \cos x + \left(\sum_{\ell=1}^{n} \frac{s_\ell}{x^{\ell}}\right) \sin x = c_0$$

for finite $n \geq 0$ where $c_\ell$ and $s_\ell$ are particular integers and $c_0$ is any constant of any kind in the range of the left side. Dottie’s equation [10] can be transformed to this form for $n = 0$.

Section 3 shows how the inverses of spherical Bessel functions $j_n(x)$ can similarly solve some equations transformable to the form

$$\left(\sum_{\ell=1}^{n+1} \frac{s_\ell}{x^{\ell}}\right) \sin x + \left(\sum_{\ell=1}^{n} \frac{c_\ell}{x^{\ell}}\right) \cos x = c_0$$

where $x^{n+1}$ divides $\sin x$ rather than $\cos x$.

Section 4 shows how the inverses of modified spherical Bessel function $i_n(x)$ can solve some analogous equations transformable to the form

$$\left(\sum_{\ell=1}^{n+1} \frac{s_\ell}{x^{\ell}}\right) \sinh x + \left(\sum_{\ell=1}^{n} \frac{c_\ell}{x^{\ell}}\right) \cosh x = c_0.$$ 

Section 5 shows how the inverses of modified spherical Bessel function $k_n(x)$ can solve some analogous equations transformable to the form

$$\left(\sum_{\ell=1}^{n+1} \frac{c_\ell}{x^{\ell}}\right) e^{-x} = c_0,$$

for which Lambert $W$ also solves the special case $n = 0$.

2 Inverses of $y_n(x)$ solve some equations containing $\cos x/x^\ell$ and possibly also $\sin x/x^\ell$

As with most mathematics software, Mathemtica implements principal-branch inverses for all of its elementary functions, but implements inverses for very few of its many special functions. Perhaps this is because almost all mathematical functions in Mathemtica
work for nonreal as well as real values of all their arguments, including orders and other parameters that are usually real or integer; and implementing inverse special functions for such multivariate complex domains would be a daunting task. However, AskConstants needs only real inverses that are much easier to implement. Moreover, it is relatively easy to implement all or at least many real branches for most special functions. Therefore AskConstants includes such multi-branched real inverses for many of the Mathematica special functions, including spherical Bessel functions $y_n(x)$ and $j_n(x)$.

**Definition.** A strict univariate Laurent polynomial is either 0 or a sum of one or more terms that are a numeric coefficient times a negative integer power of the variable.

This article is concerned with equations that can be transformed to one of the forms

$$p(x) e^{-x} = c_0,$$

$$p(x) \cos x + q(x) \sin x = c_0,$$

$$p(x) \cosh x + q(x) \sinh x = c_0,$$

where $c_0$ is real and where $p(x)$ and $q(x)$ are strict Laurent polynomials in variable $x$ with integer coefficients.

The spherical Bessel function of the second kind $y_n(x)$ is defined as a specific solution to the ordinary differential equation

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} + \left(x^2 - n(n + 1)\right)y = 0.$$

This function is less concisely representable as

$$y_n(x) = \sqrt{\frac{n}{2x}} J_{n+\frac{1}{2}}(x),$$

where $J_{n+\frac{1}{2}}(x)$ is an ordinary (cylindrical) Bessel function of the second kind.

For integer orders $n$, the Mathematica `FunctionExpand` function transforms $y_n(x)$ into an exact closed form Laurent cos sin representation partially listed in Table 1. These table entries can be computed from a formula of Lord Rayleigh’s for $x \neq 0$, known for over 100 years:

$$y_n(x) := -(-x)^n \left(\frac{1}{x} \frac{d}{dx}\right)^n \cos x \quad (2)$$

or from $y_0(x)$, $y_1(x)$, and the recurrence

$$y_{n+1}(x) := \frac{2n+1}{x} y_n(x) - y_{n-1}(x). \quad (3)$$

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4Mathematica has a general purpose function named `InverseFunction`, but it offers no branch choice, and for versions through 12.1 it can jump between branches or omit segments, as illustrated by executing, for example

```mathematica
Plot[InverseFunction[ExpIntegralEi][y], {y, -4, 1}],
```

which plots **nothing**.
Table 1: Exact strict Laurent cos sin expansions of spherical Bessel functions $y_n(x)$

| $n$ | Partially expanded $y_n(x) = c_0$ | Solutions $\forall$ branches $b$ containing $c_0$ |
|-----|----------------------------------|---------------------------------|
| 0   | $-\frac{\cos x}{x} = c_0$       | $x = \text{inverse}_b(y_0)(c_0)$ |
| 1   | $-\frac{\cos x}{x^2} - \frac{\sin x}{x} = c_0$ | $x = \text{inverse}_b(y_1)(c_0)$ |
| 2   | $\left( -\frac{3}{x^3} + \frac{1}{x} \right) \cos z - \frac{3}{x^2} \sin x = c_0$ | $x = \text{inverse}_b(y_2)(c_0)$ |
| 3   | $\left( -\frac{15}{x^4} + \frac{6}{x^2} \right) \cos x + \left( -\frac{15}{x^3} + \frac{1}{x} \right) \sin x = c_0$ | $x = \text{inverse}_b(y_3)(c_0)$ |
| ... | ...                             | ...                             |

The strict Laurent polynomials in Table 1 are Bessel polynomials in $1/x$, which are a special case of Lommel polynomials \[3\]. Thus, for example, equations such as

$$-\frac{\cos x}{x^2} - \frac{\sin x}{x} = c_0$$

for constant $c_0$ in the range of the left side are solvable by $\text{inverse}_b(y_1)(c_0)$ for the branches $b$ that contain $c_0$. Published equations that can be transformed to this form most often have nonnegative powers of $x$, making it necessary to divide all the terms on both sides by the largest power of $x$. Published equations for $n > 0$ often have $\tan x$ or $\cot x$ rather than $\cos x$ and $\sin x$, requiring multiplication by $\cos x$ or $\sin x$.

Figure 3 superimposes plots of $y_n(x)$ for order $n = 0$ through 3, with each order a different color. As suggested by Table 1 and Figure 3:

- $y_n(x)$ has odd symmetry for even $n$ and even symmetry for odd $n$;
- for even $n$, $y_n(x) = c_0$ always has at least one real solution for all real $c_0$;
- for odd $n$, $y_n(x) = c_0$ always has at least two real solutions for all real $c_0 \leq y_n(x_{n,1})$, where $x_{n,1}$ is the least positive abscissa for a local maximum of $y_n$;
- a pole of order $n + 1$ dominates as $x \to 0$;
- $y_n(x) \to (\cos x)/(-x)^{n+1}$ for even $n$ or $y_n(x) \to (\sin x)/(-x)^n$ for odd $n$ as $x \to \pm \infty$. 

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All of the real inverse functions for special functions $f(x)$ implemented in AskConstants work by partitioning the portion of real $x$ where $f(x)$ is real into maximally monotonic intervals terminated by the refinable float local infima and suprema of $f$. Table 2 lists six digits of the abscissas and ordinates of some of those interval endpoints for $y_0(x)$, which AskConstants can compute for a particular infimum or supremum number of $y_n(x)$ to precision $p$ by entering, for example,

$$ \text{N[SphericalBesselYInfimumOrSupremumAbscissa}[n, \text{infsupumNumber}], p] $$

and

$$ \text{N[SphericalBesselYInfimumOrSupremumOrdinate}[n, \text{infsupumNumber}], p] $$

The infimum or supremum having the least positive abscissa is number 1, with any to its right successively numbered 2, 3, ... and any to its left successively numbered 0, -1, ... .

Table 2: Some of the countably infinite local infima and suprema of $y_0(x)$

| infsupum number $m$ | abscissa $x_{0,m}$ | ordinate $y_0(x_{0,m})$ |
|---------------------|---------------------|--------------------------|
| :                  | :                   | :                        |
| -2                 | -6.12125            | 0.161228                 |
| -1                 | -2.79839            | -0.336508                |
| 0                  | 0.00000             | $\begin{cases} -\infty, & \text{Direction} \rightarrow \text{“FromAbove”}; \\ \infty, & \text{Direction} \rightarrow \text{“FromBelow”} \\ \text{ComplexInfinity}, & \text{otherwise} \end{cases}$ |
| 1                  | 2.79839             | 0.336508                 |
| 2                  | 6.12125             | -0.161228                |
| :                  | :                   | :                        |
SphericalBesselYInfimumOrSupremumOrdinate has an optional argument that appropriately returns either $\infty$ or $-\infty$ instead of ComplexInfinity when entered as Direction $\rightarrow$ “FromAbove” or Direction $\rightarrow$ “FromBelow”.

Float values of nontrivial stationary infsupum abscissas of $f(x)$ are computed by using the iterative Mathematica function invocation

\[
\text{FindRoot}[f'(x) = 0, \text{guess}, \text{lowerBound}, \text{upperBound}, \text{WorkingPrecision} \rightarrow \ldots],
\]

using a guess that is the value correct to 16 significant digits.

Each maximally monotonic interval of $f(x)$ corresponds to a real branch of the inverse function. Branch 1 is the branch whose right endpoint is least positive. Branches for successively more positive right endpoints are numbered 2, 3, etc., whereas branches for successively smaller right endpoints are numbered 0, -1, etc. The left endpoint belongs to the branch. The right endpoint belongs to the branch only for the rightmost branch.

Float values of inverse$_b(f)(c_0)$ are calculated by

\[
\text{FindRoot}[f(x) = c_0, \text{guess}, \text{lowerBound}, \text{upperBound}, \text{WorkingPrecision} \rightarrow \ldots]
\]

with bounds that are the bounding pair of adjacent infimum or supremum abscissas. The guess is usually piecewise, using invertible truncated series approximations to $f(x)$ at the end abscissas and for some functions also at a point between. The guess is usually within 20% of the converged value, typically achieving convergence within about 5 to 20 iterations.

Dividing both sides of Dottie’s fixed-point equation \(1\) by nonzero $-x$ then referring to Table 1 gives

\[-1 = \frac{-\cos x}{x} = y_0(x).\]

Therefore Figure 3 also plots a horizontal line through ordinate $-1$, which intersects $y_0(x)$ at and only at abscissa Dottie number $x \approx 0.73908513321516$. This number is between and only between infsupum abscissas $x_0 = 0$ and $x_1 \approx 2.79839$ in Table 2 which delimit branch 1. Therefore the Dottie number is

\[\text{inverse}_1(y_0)(-1).\]

Figure 4 superimposes plots of a vertical line through abscissa $-1$ and the order 0 AskConstants function RealInverseSphericalBesselY \([0, t, b]\) for branches $b = -2$ through branch 3, with each branch a different color. Here branch 1 and only branch 1 of the inverse function intersects the vertical line at the Dottie number $\approx 0.73908513321516$.

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5Rather than using FindRoot, accelerated truncated reverted series and other methods more specific to each implemented function could be faster with more control over the resulting accuracy. However, I wanted to implement quickly about 45 inverse functions, many of which have parameters such as an order and/or several or a countably infinite number of branches.
Figure 4 shows the four lines through the origin tangent to \( \cos x \) closest to the origin. The tangent points are at abscissas and ordinates in Table 2 except for the pole at infsupum \( m = 0 \). These tangent lines and similar ones for larger \(|m|\) suggest how the number of real solutions to the generalized Dottie equation

\[
\cos x = c_0 x
\]  

changes as the value of \( c_0 \) decreases from \(+\infty\): The one intersection abscissa increases from \( x = 0^+ \) through the Dottie number until a negative solution also appears at the negative tangency abscissa nearest the origin. Then that negative solution abscissa bifurcates into two that increase in separation. Then another positive solution appears at the positive tangency abscissa nearest the origin. Then that solution bifurcates into two that increase in separation. This process alternates at tangency points increasingly far from the origin until \( c_0 = 0 \), making the solutions all be the zeros of \( \cos x \). Then pairs of points coalesce then disappear alternately from the solutions furthest from the origin in the positive and negative directions until there is just one solution at \( x = 0^- \).
Figure 5: Some lines through the origin tangent to \( \cos x \) at extrema of \( y_0(x \neq 0) \)

Constant recognition programs generate conjectures. It is the responsibility of the user to prove or disprove them. For the AskConstants conjecture that the Dottie number is exactly \( \text{inverse}_1(y_0)(-1) \), it is easy to prove a much more general result:

**Proposition 1.** For real \( x \) and an equation of the form

\[
p(x) \cos x + q(x) \sin x = c_0
\]

from row \( n \) of the countably infinite number of rows partially listed in Table 2 that can be extended by recurrence \([2]\), let \( x_{n,m} \) denote the \( m \)th infsupum abscissa of the left side and let \( B \) denote the set of all real branches \( b \) of \( \text{inverse}_b(y_n) \) for which the interval

\[
[y_n(x_{n,b-1}), y_n(x_{n,b})]
\]

contains constant \( c_0 \).

Then all of the real solutions to equation \([5]\), if any, are

\[
x = \text{inverse}_b(y_n)(c_0)
\]

where branches \( b \) are all elements of set \( B \).

**Proof.**

- The spherical Bessel functions \( y_n(x) \) are exactly equivalent to the left sides of these equations by the Raleigh formula \([2]\) for \( y_n(x) \).
- \( \text{inverse}_b(y_n)(c_0) \) is defined as the one real solution to \( y_n(x) = c_0 \) in a maximally monotonic continuous interval number \( b \) of \( x \) containing \( c_0 \).
- The set \( B \) of all such branches thus contains all of the solutions. \( \square \)

**Corollary.** An exact closed form for the Dottie number is \( \text{inverse}_1(y_0)(-1) \).

**Proof.**

- Divide both sides of Dottie’s equation \([1]\) by nonzero \(-x\), then transpose the two sides giving the first equation in Table \( \Pi \) with \( c_0 = -1 \), for which the possible solutions are \( \text{inverse}_b(y_0)(-1) \) by Proposition \( \Pi \).
Referring to Table 2, ordinate $-1$ occurs between and only between infsupum numbers 0 and 1, making the only branch $b = 1$, which includes the Dottie number at abscissa $\approx 0.739085$.

**Exercise.** Determine both a float value and an exact explicit closed form for the Dottie number in degree mode.

The initial reason for implementing the real inverse functions $\text{inverse}_b(f)$ was to apply them to the AskConstants float input $\tilde{x}$ giving a transformed float $\tilde{t}$ to help determine if the application could propose an exact constant candidate $t$ for $\tilde{t}$. If so, then a candidate for the input float is $f(t)$. However, as illustrated by this example, a bonus of these real inverses is to enable recognition of them too, which is often applicable to proposing exact candidates for float equation solutions.

Generalized Dottie equations occur rather often in the literature. For example, despite containing only about 2500 constants, Table 1 in Robinson and Potter [12] lists 22 such constants.

That table also lists eight constants containing $\tan x$ or $\cot x$ that can be converted to the form

$$-\frac{\cos x}{x^2} - \frac{\sin x}{x} = y_1(x) = c_0$$

and are therefore exactly solvable by $\text{inverse}_b(y_1)(c_0)$.

The expansions illustrated in Table 1 are also valid in the complex domain. However, it might require substantial effort to extend $\text{RealInverseSphericalBesselY}$ to the complex domain efficiently and robustly – particularly if the order is also generalized from non-negative integers to the complex domain. Meanwhile, Fettis [4] lists numeric values of five complex solutions to the generalized Dottie equation (4) for each of about 30 values of $c_0$.

The general technique that enabled the exact solutions discussed in this section and subsequent ones is to recognize when the left side of an equation in the form

$$\text{expression} = \text{constant}$$

is an exact transformation of some named $f(x)$ for which we can compute and concisely represent inverses.

**Remark.** Gaidash [5] derives a different explicit exact closed-form solution to Dottie’s equation:

$$\arcsin \left( 1 - 2 I_{1/2}^{-1} \left( \frac{1}{2}, \frac{3}{2} \right) \right)$$

where $I_{1/2}^{-1}$ is the inverse regularized incomplete beta function.

### 2.1 Extending computer algebra Solve functions

Mathematica permits users to enhance the Solve or the more thorough Reduce function with rewrite rules. Therefore you could enhance those functions to transform suitable equations to strict Laurent $\cos \sin$ form, then express their exact solutions using $\text{inverse}_b(y_n)(x)$. Or you could instead encourage the authors of your computer algebra systems to implement those enhancements, including numeric $\text{inverse}_b(y_n)(x)$. 

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2.2 Fixed points of \( f(x) \) are also fixed points of inverse \( b(f)(x) \)

For any function \( f(x) \), if \( x = f(x) \) for a particular \( x = x_0 \), then also \( x_0 = \text{inverse}_b(f)(x_0) \)
for an appropriate branch \( b \), as proved in [7]. Thus, for example, Table 3 reveals that the
Dottie number, inverse_1(y_0)(-1), is also the real solution to the equation

\[
\arccos x = x,
\]

where \( \arccos \) denotes the principal branch of the multi-branched inverse cosine function.

2.3 The stability of a fixed point is irrelevant

The stability of a fixed point is irrelevant for this article, because the AskConstants
RealInverse functions use FindRoot rather than fixed point iteration, and FindRoot uses
faster more robust iterations.

3 Inverses of \( j_n(x) \) solve some different equations containing \( \sin x/x^\ell \) and possibly also \( \cos x/x^\ell \)

After proving then generalizing the AskConstants conjecture for the Dottie number, I
decided to investigate some analogous results for some other inverse spherical Bessel
functions. As illustrated by Table B, spherical Bessel \( j_n(x) \) also has exact closed form
representations similar to those for \( y_n(x) \), except that \( 1/x^{n+1} \) multiplies \( \sin x \) rather than \( \cos x \). The expansion for \( n = 0 \) is also known as the sinc function. These table entries
can be computed for nonzero \( x \) from another Rayleigh formula for nonnegative integer \( n \)
that is supplemented here to be correct also for \( x = 0 \):

\[
\begin{align*}
j_n(x) := \begin{cases} 
(-x)^n \left( \frac{1}{x} \frac{d}{dx} \right)^n \frac{\sin x}{x}, & x \neq 0; \\
1, & n = 0; \\
0, & n \neq 0; 
\end{cases} 
& \quad x = 0,
\end{align*}
\]

or from \( j_0(x) \), \( j_1(x) \), and the recurrence

\[
j_{n+1}(x) := \frac{2n + 1}{x} j_n(x) - j_{n-1}(x).
\]

(6)
Table 3: Exact strict Laurent sin cos expansions of spherical Bessel functions \( j_n(x) \)

| \( n \) | Partially expanded \( j_n(x) = c_0 \) | Solutions \( \forall \) branches \( b \) containing \( c_0 \) |
|---|---|---|
| 0 | \( \begin{cases} \frac{\sin x}{x}, & x \neq 0; \\ 1, & x = 0 \end{cases} = c_0 \) | \( x = \text{inverse}_b(j_0)(c_0) \) |
| 1 | \( \begin{cases} \frac{\sin x}{x^2} - \frac{\cos x}{x}, & x \neq 0; \\ 0, & x = 0 \end{cases} = c_0 \) | \( x = \text{inverse}_b(j_1)(c_0) \) |
| 2 | \( \begin{cases} \left( \frac{3}{x^3} - \frac{1}{x^2} \right) \sin x - \frac{3}{x^2} \cos x, & x \neq 0; \\ 0, & x = 0 \end{cases} = c_0 \) | \( x = \text{inverse}_b(j_2)(c_0) \) |
| 3 | \( \begin{cases} \left( \frac{-15}{x^4} + \frac{6}{x^2} \right) \sin x + \left( \frac{-15}{x^4} + \frac{1}{x} \right) \cos x, & x \neq 0; \\ 0, & x = 0 \end{cases} = c_0 \) | \( x = \text{inverse}_b(j_3)(c_0) \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) |

Thus a procedure similar to that described in Section 2 can determine exact closed-form solutions to some equations that are transformable to one of the equations in the middle column of Table 3.

As an example, OEIS constant A199460 [11] is the one positive solution to

\[
\sin x = \frac{1}{2} x, 
\]

which is equivalent to the equation

\[
\frac{\sin x}{x} = j_0(x) = \frac{1}{2}
\]

for our nonzero \( x \), making \( x = \text{inverse}_b(j_0)(1/2) \approx 1.89549 \) for the appropriate branch \( b \).

Figure 6 plots the two sides of equation (7). Figure 7 plots \( j_0(x) \) through \( j_3(x) \) with each order a different color, together with a horizontal line through ordinate 1/2. Comparing these two figures, notice how dividing both sides of equation (7) by \( x \) annihilated the unsought solution \( x = 0 \).
Figure 6: Real solutions to $\sin x = x/2$

Figure 7: Spherical Bessel $j_0(x)$ through $j_3(x)$ and ordinate $1/2$

Table 4 lists six digits of the abscissas and ordinates of some infima and suprema of $j_0(x)$. 

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Table 4: Some of the countably infinite number of infima and suprema of $j_0(x)$

| infsupum number $m$ | abscissa $x_{0,m}$ | ordinate $j_0(x_{0,m})$ |
|---------------------|---------------------|-------------------------|
| :                  | :                   | :                       |
| -2                 | -7.72525            | 0.128375                |
| -1                 | -4.49341            | -0.217234               |
| 0                  | 0.00000             | 1.00000                 |
| 1                  | 4.49341             | -0.217234               |
| 2                  | 7.72525             | 0.128375                |
| :                  | :                   | :                       |

Figure 8 superimposes plots of a vertical line through abscissa $1/2$ and the order zero AskConstants function RealInverseSphericalBesselJ $[0, t, b]$ for branches $b = -2$ through 3, with each branch a different color. Here branches 0 and 1 of the inverse function intersect the vertical line through abscissa $t = 1/2$ only at ordinates inverse $0(j_0)(1/2) \approx -1.89549$ and inverse $1(j_0)(1/2) \approx 1.89549$.

Figure 8: RealInverseSphericalBesselJ $[0, 1/2, 1]$ solves $\sin x = \frac{1}{2}x$.

Figure 9 shows the three lines through the origin tangent to $\sin x$ closest to the origin. The tangent points are at abscissas $x_{0,m}$ and ordinates $\sin x_{0,m}$ in Table 4. These tangent lines and similar ones for larger $|m|$ suggest how the number of real solutions to the generalized equation

$$\sin x = c_0 x$$

(8)

increases from 1 to 3, then 5, etc. as the absolute value of $c_0$ decreases from $+\infty$ to 0 where there are a countably infinite number of solutions, then decreases to 1 as $c_0$ further decreases toward $-\infty$.

Fettis [4] lists numeric values of five complex solutions to the generalized equation (8) for about 30 values of $c_0$. 

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To summarize for spherical Bessel $j_n(x)$:

**Proposition 2.** For real $x$ and an equation of the form

$$p(x) \sin x + q(x) \cos x = c_0$$  \hspace{1cm} (9)

from row $n$ of the countably infinite number of rows partially listed in Table 3 that can be extended by recurrence (6), let $x_{n,m}$ denote the $m^{\text{th}}$ infimum abscissa of the left side and let $B$ denote the set of all real branches $b$ of inverse $b(y_n)$ for which the interval

$$[j_n(x_{n,b-1}), j_n(x_{n,b})]$$

contains constant $c_0$.

Then all of the real solutions to equation (9), if any, are

$$x = \text{inverse}_b(j_n)(c_0)$$

where branches $b$ are all elements of set $B$.

**Proof.** Similar to that of Proposition 1.

As an example for higher-order Bessel $j_n$, OEIS A115365 lists the float value 4.49341 of the least positive solution to

$$\tan x - x = 0,$$  \hspace{1cm} (10)

which is the abscissa of the least positive local minimum of $j_0(x) = \text{sinc}(x)$. Multiplying both sides of equation (10) by $(\cos x)/x^2$ gives for $x \neq 0$

$$\frac{\sin x}{x^2} - \frac{\cos x}{x} = j_1(x) = 0$$  \hspace{1cm} (11)
from Table 3. Figure 7 reveals that the least positive solution to equation (11) is in branch $b = 2$ of $\text{inverse}_b(j_1)(0)$, making $x = \text{inverse}_2(j_1)(0) \approx 4.49341$. Because of the identity

$$j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+1/2}(z)$$

and $c_0 = 0$, Vladimir Reshetnikov [11, A115365] could express the solution as the Mathematica function invocation $\text{BesselJZero}[3/2, 1]$.

3.1 Kepler’s equation for elliptic orbits is a near miss

“Kepler’s equation is central to orbital mechanics.”

– me, being whimsical.

A common form of Kepler’s equation for elliptic orbits is

$$M = E - e \sin E \quad (12)$$

where $0 \leq e < 1$ is the eccentricity, $-\pi < M \leq \pi$ is the mean anomaly, and $E$ is the eccentric anomaly. This equation is widely regarded as having no currently known explicit exact closed form solution when $E$ is the unknown, although a few authors regard solutions containing raw integrals or infinite series as closed forms.

Rearranging terms and factors of equation (12) into

$$\sin E = \frac{1}{e} E - \frac{M}{e},$$

which matches the related equation form for $n = 0$ in Table 3 with slope $c_0 = 1/e$ only when $M = 0$. However, $1/e \geq 1$, and Figure 9 illustrates that the only solution is then $E = 0$.

But it makes me wonder if there might be an infinite series of the form

$$E = M + \sum_{n=1}^{\infty} a_m j_n(ne) \sin(nM)$$

with appropriate coefficients $a_m$ that has better properties than the well known series representation

$$E = M + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne) \sin(nM)$$

for $e < 1$ and $-\pi \leq M \leq \pi$, where $J_n$ is the ordinary Bessel $J$ function. Spherical Bessel functions seem more appropriate than the ordinary (cylindrical) Bessel functions for the inverse square law of Newtonian gravity.
4 Inverses of $i_n(x)$ solve some equations containing \( \sinh x/x^\ell \) and possibly also \( \cosh x/x^\ell \)

Inverses of the modified spherical Bessel function of the first kind, $i_n(x)$, can solve equations analogous to those of Section 3 but containing $\sinh$ and $\cosh$ rather than $\sin$ and $\cos$. These functions have analogous exact expansions partially displayed in Table 5. These table entries can be computed for $x \neq 0$ and nonnegative $n$ from a Rayleigh-type formula that is supplemented here to be correct also for $x = 0$:

$$i_n(x) := \begin{cases} 
  x^n \left( \frac{1}{x \, dx} \right)^n \frac{\sinh x}{x}, & x \neq 0; \\
  1, & n = 0; \\
  0, & n > 0, 
\end{cases}$$

or from $i_0(x)$, $i_1(x)$, and the recurrence

$$i_{n+1}(x) := i_{n-1}(x) - \frac{2n + 1}{x} i_n(x). \quad (13)$$

| $n$ | $i_n(x)$  | Solutions $∀$ branches $b$ containing $c_0$ |
|-----|-----------|---------------------------------------------|
| 0   | $\begin{cases} 
  \frac{\sinh x}{x}, & x \neq 0; \\
  1, & x = 0 
\end{cases}$ = $c_0$ | $x = \text{inverse}_b(i_0)(c_0)$ |
| 1   | $\begin{cases} 
  \frac{\sinh x}{x^2} + \frac{\cosh x}{x}, & x \neq 0; \\
  0, & x = 0 
\end{cases}$ = $c_0$ | $x = \text{inverse}_b(i_1)(c_0)$ |
| 2   | $\begin{cases} 
  \left( \frac{3}{x^3} + \frac{1}{x} \right) \sinh x - \frac{3}{x^2} \cosh x, & x \neq 0; \\
  0, & x = 0 
\end{cases}$ = $c_0$ | $x = \text{inverse}_b(i_2)(c_0)$ |
| 3   | $\begin{cases} 
  \left( -\frac{15}{x^4} - \frac{6}{x^2} \right) \sinh x + \left( -\frac{15}{x^3} + \frac{1}{x} \right) \cosh x, & x \neq 0; \\
  0, & x = 0 
\end{cases}$ = $c_0$ | $x = \text{inverse}_b(i_3)(c_0)$ |
| ... | ...       | ...                                         |

Mathematica through version 12.1 has no built-in spherical Bessel function $i_n$ or its inverses, but I have implemented them for inclusion in the next version of AskConstants after 5.0, using the techniques described in Section 2.

Figure 10 shows a plot of spherical Bessel $i_0(x)$ through $i_3(x)$.

As suggested by Figure 10 and Table 5.
1. The even-order functions have even symmetry and the odd-order functions have odd symmetry.

2. The even order functions only have real branches numbered 0 and 1.

3. The only real branch of odd order functions is branch 1.

4. For real $c_0$, any equation transformable to a form partially listed in the rightmost column of Table 5 has
   - one real solution for odd $n$,
   - one solution for $n = 0$ and $c_0 = 1$ or two solutions for $n = 0$ and $c_0 > 1$,
   - one solution for even $n \geq 2$ and $c_0 = 0$ or two solutions for even $n \geq 2$ and $c_0 > 0$.

To summarize for spherical Bessel $i_n(x)$:

**Proposition 3.** For real $x$ and an equation of the form
\[ p(x) \sinh x + q(x) \cosh x = c_0 \] (14)

from row $n$ of the countably infinite number of rows partially listed in Table 5 that can be extended by recurrence (13), let $x_{n,m}$ denote the $m^{\text{th}}$ infsupum abscissa of the left side and let $B$ denote the set of all real branches $b$ of inverse $b(y_n)$ for which the interval
\[ [i_n(x_{n,b-1}), i_n(x_{n,b})] \]
contains $c_0$.

Then all of the real solutions to equation (14), if any, are
\[ x = \text{inverse}_b(i_n)(c_0) \] (15)

where branches $b$ are all elements of set $B$. 

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Proof. Similar to that of Proposition 1.

I have not found an example of such equations published in paper or on-line form, refereed or not. I welcome published examples and unpublished ones that have a natural application.

Because a fixed point of $f(x)$ is a fixed point of $\text{inverse}_b(f)$, $\text{inverse}_1(i_0)(c_0)$ can also solve equations transformable to the form

$$\frac{\text{arcsinh } x}{x} = c_0,$$

but I have not found a published examples for equations equivalent to that either.

4.1 Kepler’s equation for hyperbolic paths is also a near miss

A common form of Kepler’s equation for hyperbolic paths is

$$M = e \sinh H - H$$  \hspace{1cm} (16)

where $e > 1$ is the eccentricity, $M$ is the mean anomaly, and $H$ is the hyperbolic eccentric anomaly. This equation is also widely regarded as currently having no known explicit closed form solution when $H$ is the unknown.

The terms and factors of equation (16) can be rearranged into

$$\sinh H = 1/e H + M/e,$$

which matches the related equation form for $n = 0$ in Table 5 with $c_0 = 1/e$ only when $M = 0$. However, line slope $1/e < 1$ for hyperbolic paths, and Figure 11 illustrates that the only solution is then $H = 0$.

Figure 11: inverse $(i_0)$ does not generally solve the hyperbolic Kepler equation

But it makes me wonder if there might be a useful infinite series of the form

$$H = M + \sum_{n=1}^{\infty} a_m i_n(ne) \sin(nM)$$

with appropriate coefficients $a_m$. 

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5 $\text{inverse}_b(k_n)$ generalizes Lambert $W$

Some equations transformable to the form $\left(\sum_{\ell=1}^{n} c_{\ell} x^{\ell}\right) e^{-x} = c_0$ can be solved by the inverse spherical Bessel function of the second kind, $\text{inverse}_b\left(k_n\right)(c_0)$, where $c_{\ell}$ are certain integer constants and $c_0$ is any nonzero constant in the range of $k_n(x)$.

For nonnegative integer $n$, $k_n(x)$ can be expressed in strict Laurent negative exponential form, as listed in Table 6, which is adapted from Weisstein [13], who also lists a recurrence for the rational factors $q_n(x)$ multiplying $e^{-x}$:

$$q_n(x) = q_{n-1}(x) + \frac{2n-1}{x} q_{n-2},$$

which can be used to extend the table, computing $k_n(x)$ for larger $n$ from $k_0(x)$ and $k_1(x)$.

Alternatively, a Rayleigh-type formula is

$$k_n(x) = (-x)^n \left( \frac{1}{x} \frac{d}{dx} \right)^n e^{-x} \frac{x}{x}.$$ 

Table 6: Exact exponential expansions of spherical Bessel functions $k_n(x)$

| $n$ | Partially expanded $k_n(x) = c_0$ | Solutions $\forall$ branches $b$ containing $c_0$ |
|-----|---------------------------------|---------------------------------------------|
| 0   | $\frac{1}{x} e^{-x} = c_0$      | $x = \text{inverse}_b(k_0)(c_0)$            |
| 1   | $\left( \frac{1}{x^2} + \frac{1}{x} \right) e^{-x} = c_0$ | $x = \text{inverse}_b(k_1)(c_0)$            |
| 2   | $\left( \frac{3}{x^3} + \frac{3}{x^2} + \frac{1}{x} \right) e^{-x} = c_0$ | $x = \text{inverse}_b(k_2)(c_0)$            |
| 3   | $\left( \frac{15}{x^4} + \frac{15}{x^3} + \frac{6}{x^2} + \frac{1}{x} \right) e^{-x} = c_0$ | $x = \text{inverse}_b(k_3)(c_0)$            |
| 4   | $\left( \frac{105}{x^5} + \frac{105}{x^4} + \frac{45}{x^3} + \frac{10}{x^2} + \frac{1}{x} \right) e^{-x} = c_0$ | $x = \text{inverse}_b(k_4)(c_0)$            |
| ... | ...                            | ...                                         |

If a given equation contains some positive powers of the unknown $x$, then divide by the largest such power of $x$. If a given strict Laurent polynomial multiplies $e^x$ rather than $e^{-x}$, then the substitution $x \mapsto -s$ transforms that product to a strict Laurent negative exponential form.

Beware that the Digital Library of Mathematical Functions [10] defines $k_n(x)$ as $\pi/2$ times these definitions. I am interested in opinions about if I should use that definition instead.

"The good thing about standards is that there are so many to choose from."

– Andrew S. Tanenbaum
Mathematica through version 12.1 has no builtin spherical Bessel function $k_n$, but the AskConstants application has one implemented one using these expansions, and AskConstants also has a function

\[
\text{RealInverseSphericalBesselK}[n, t, \text{branch}]
\]

implemented using the techniques described in Section 2.

Figure 12 plots $k_0(x)$ through $k_3(x)$ with each order a different color. From that Figure and Table 6, notice how

- a pole of order $n + 1$ dominates as $x \to 0$;
- $k_n(x) \sim e^{-x}/x$ for large $|x|$, making $k_n(x) \to 0$ as $x \to +\infty$ and $k_n(x) \to -\infty$ as $x \to -\infty$;
- $k_n(x)$ has only branches 0 and 1 for odd $n$;
- $k_n(x)$ has only branches $-1, 0$ and 1 for even $n$;
- For odd $n$, $k_n(x) = c_0$ has two solutions when $c_0 > 0$ or one solution when $c_0 \leq 0$;
- For even $n$, $k_n(x) = c_0$ has one solution when $c_0 > 0$ or when $c_0 = k_n(x_{n, -1})$, versus two solutions when $c_0 < k_n(x_{n, -1})$, where $x_{n, -1}$ is the abscissa of the local maximum at infsupum $-1(k_n)$.

Figure 12: Modified spherical Bessel function of the second kind

For a constant $d_0$ and an unknown $x$, solutions to the equation

\[
x e^x = d_0
\]  

are $x = W_b(d_0)$ for all branches $b$ of Lambert $W$ that satisfy equation (18). This function was standardized, analyzed and named by Corless, Gonnet, Hare, Jeffrey, and Knuth [2]. Its history extends back hundreds of years and Bessel functions history is even
longer, but what has apparently not been recognized before is that Lambert W can be
expressed in terms of the inverse modified spherical Bessel function $k_0(x)$ and vice versa:
For nonzero $d_0$ and real $x$, the first equation in Table 6 can be converted to equation (18)
by reciprocating both sides then using the substitution $1/c_0 \rightarrow d_0$. Thus

$$\text{inverse}_1(k_0)(x) \mid x > 0 \equiv \frac{1}{W_0(x)}, \quad (19)$$

$$\text{inverse}_0(k_0)(x) \mid -\frac{1}{e} \leq x < 0 \equiv \frac{1}{W_0(x)}, \quad (20)$$

$$\text{inverse}_{-1}(k_0)(x) \mid -\frac{1}{e} < x < 0 \equiv \frac{1}{W_{-1}(x)}. \quad (21)$$

From Table 6 and Figure 12 it is evident that $\text{inverse}_0(k_0)(0)$ has infinite magnitude, as
does $1/W_0(0)$. This together with equations (19) through (21) imply that

$$W_0(x) \equiv \begin{cases} 
\text{inverse}_1(k_0)(x), & x > 0; \\
0, & x = 0; \\
\text{inverse}_0(k_0)(x), & -\frac{1}{e} \leq x < 0;
\end{cases}$$
and

$$W_{-1}(x) \mid -\frac{1}{e} \leq x < 0 \equiv \frac{1}{\text{inverse}_{-1}(k_0)(x)}. \quad (21)$$

Thus for nonzero $x$, the mutual inverse $xe^x$ of $W_0(x)$ and $W_{-1}(x)$ is also the mutual
inverse of the corresponding three branches of the reciprocal of $1/k_0(x)$. The first-order
pole at $k_0(0)$ versus none for $xe^x$ split branch 0 of Lambert W into branches 0 and
1 of inverse spherical Bessel $k_0(x)$. However, all nonzero real solutions expressible in
terms of Lambert $W_0(x)$ or $W_{-1}(x)$ are also expressible in terms of $\text{inverse}_b(k_0)(x)$ with
$b \in \{-1, 0, 1\}$.

Because of this relationship between Lambert W and $\text{inverse}_b(k_0)$, we can regard
$\text{inverse}_b(k_n)$ for integer $n > 0$ as generalizations of Lambert W.

To summarize for spherical Bessel $k_n(x)$:

**Proposition 4.** For real $x$ and an equation of the form

$$p(x)e^{-x} = c_0 \quad (22)$$

from row $n$ of the countably infinite number of rows partially listed in Table that can be
extended by recurrence (17), let $x_{n,m}$ denote the $m^{th}$ infsupum abscissa of the, left side
and let $B$ denote the set of all real branches $b$ of $\text{inverse}_b(k_n)(x)$ for which the interval

$$[k_n(x_{n,b-1}), k_n(x_{n,b})]$$
contains $c_0$.

Then all of the real solutions to equation (22), if any, are

$$x = \text{inverse}_b(k_n)(c_0)$$

where branches $b$ are all elements of set $B$. 

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**Proof.** Similar to that of Proposition $\square$

Knowing the correspondence between Lambert $W_0(x)$, $W_{-1}(x)$ and inverse$_b(k_0)(x)$ might enable us to exploit identities for $k_0$ to transform more equations to a form solvable by an explicit closed form expression. It also might similarly facilitate integration, summation of infinite series, and other operations with these functions.

However, a potentially greater benefit of these closed-form representations of $k_n(x)$ is that for $n > 0$, if we can transform any equation to the form

\[
\text{any table entry for } k_n(x) = \text{any constant in the range of that entry},
\]

then we can express a solution in terms of branch $b$ of inverse$_b(k_n)(x)$, where $n$ is the corresponding entry in the column labeled $n$ and $b$ is a branch that contains the float approximation of the constant.

References $[8, 9]$ describe how nested instances of Lambert $W$ can be used to express explicit exact closed-form real solutions to equations transformable to

\[e^{-cx} = a_0(x - r_1)^{m_1}(x - r_1)^{m_2} \cdots (x - r_n)^{m_n}\]

where $c$, $a_0$ and $r_1$ through $r_n$ are exact real constants, with $m_1$ through $m_n$ integer. They also describe some physics applications. This complements the inverse spherical Bessel $k_n(x)$ solutions discussed in this section, which are not limited to equations exactly factorable into linear factors having all real zeros, but are limited to specific integer coefficients after factoring out an appropriate unit times the gcd of the coefficients.

As an example, Figure $13$ plots both sides of the equation

\[
\left(\frac{1}{x} + \frac{3}{x^2} + \frac{3}{x^3}\right) e^{-x} = \frac{3}{3 \log 2 - \pi} \approx -2.82446.
\]

Figure 13: Both sides of the equation \(\left(\frac{1}{x} + \frac{3}{x^2} + \frac{3}{x^3}\right) e^{-x} = \frac{3}{3 \log 2 - \pi}\)
This equation has two exact closed-form real solutions:

\[
x = \text{inverse}_b(k_2) \left( \frac{3}{3 \log 2 - \pi} \right)
\]

with branch \( b = -1 \) for \( x \leq -1.78324 \) where the monotonicity changes and branch \( b = 0 \) for \(-1.78324 < x < 0 \). (There are no solutions for branch \( b = 1 \) where \( x > 0 \).)

I am unaware of a published example solvable by \( \text{inverse}_b(k_n) \) for \( n > 0 \), and I would greatly appreciate learning of some.

6 Conclusions

1. It is pleasantly surprising that four classes of equations long-thought to have no exact closed form solutions actually do:

   (a) I found about 180 published examples solvable by \( \text{inverse}_b(y_n) \).

   (b) I found about 150 published examples solvable by \( \text{inverse}_b(j_n) \).

   (c) Although I have not yet found published examples of strict Laurent sinh cosh equation solvable by \( \text{inverse}_b(i_n) \), I expect that there are some.

   (d) Although the many published examples solvable by \( \text{inverse}_b(k_0) \) are more conveniently solvable by \( W_0 \) or \( W_{-1} \) and I have not yet found published examples solvable by \( \text{inverse}_b(k_n) \) for \( n > 0 \), I expect that there are some.

2. The general technique that enabled the exact solutions discussed in this article was to recognize when the left side of an equation in the form

\[\text{expression} = \text{constant}\]

is an exact transformation of some named \( f(x) \) for which we can compute and concisely represent inverses. This recognition is often difficult for humans and more difficult to implement in computer algebra systems than transforming \( f(x) \) to a cryptic equivalent. For example, Mathematica 12.1 has \text{EllipticNomeQ} and \text{InverseEllipticNomeQ} functions, and

\[
\text{FunctionExpand} \left[ \text{EllipticNomeQ} \left[ x \right] \right] \mapsto e^{-\pi \frac{\text{EllipticK} [1 - x]}{\text{EllipticK} [x]}},
\]

but neither the \text{Solve} nor the more comprehensive \text{Reduce} function can determine that a solution to

\[
e^{-\pi \frac{\text{EllipticK} [1 - x]}{\text{EllipticK} [x]}} = \frac{1}{5}
\]

is

\[
\text{InverseEllipticNomeQ} \left[ \frac{1}{5} \right]. \quad (23)
\]
However, executing

\[
\text{FindRoot} \left[ -\frac{\pi \text{EllipticK} [1 - x]}{e^{\text{EllipticK}[x]}} = \frac{1}{5}, \{x, 0.5\}, \text{WorkingPrecision} \rightarrow 16 \right]
\]

returns \( \{x \rightarrow 0.9658521935950790\} \); and entering that constant into AskConstants returns the candidate expression \(23\) with an excellent assessment of being the limit as the precision approaches infinity. Such constant-recognition software is best at recognizing low-complexity constants such as expression \(23\), and low complexity exact closed forms are the most useful. Therefore routine use of such tools can lead to surprising sought or unsought discoveries – so much so that I would like to find a programmer who could arrange for several such tools to run in the background, automatically checking all of the floats in my top-level results from all of my mathematical software as a low priority background task. For example, I could imagine having an optionally open window that logged floats and their session location paired with highly likely exact limits thereof – even if entries appeared noticeably after I had moved on to further immediate calculations.

3. The utility of computer algebra systems would benefit greatly from more multi-branched inverse special functions, even if it initially necessitates limiting the domain to the reals and limiting orders to appropriate integers.

4. Judging from the number of different published equations currently known to me that were widely thought to have no explicit exact closed-form solution but do using the techniques described in this article:

(a) The inverses of \(y_n(x)\) for strict Laurent cosines and sines has the most impact, followed by those of \(j_n(x)\) for strict Laurent sines and cosines

(b) I suspect that there are published examples solvable by inverses of \(k_n\) for \(n > 0\) and by \(i_n\), but the fact that I have not yet found any suggests that these solutions are significantly less frequent.

The current AskConstants version 5.0 \(13\) contains implementation of the multi-branched inverse spherical Bessel functions \(y_n\) and \(j_n\). The files are ASCII text files that can be viewed with any ASCII text editor. I plan for the next version of AskConstants to contain implementations of the spherical Bessel functions \(i_n\) and \(k_n\) together with their inverses. Perhaps AskConstants will lead me to published examples for those functions too.

**Current Limitations**

This is a “truth in advertising” subsection intended to reduce the chance of misunderstanding:

---

\(^6\)The reason for using several is that most constant recognition programs can propose candidates that none of the other can propose.
1. Although spherical Bessel $k_n(x)$ and the concept of multi-branched inverse functions precede the name Lambert $W$, I do not propose replacing Lambert $W_0$ and $W_{-1}$ with $1/\text{inverse}_0(k_0)$, for which the reciprocal introduces an inconvenient pole, an extra branch, and a less aesthetic piecewise definition into the equivalent of $W_0(x)$. The benefit of inverse spherical $k_0(x)$ is merely that it provides an alternative viewpoint that might enable new identities to be applied. The benefit of other order inverse $\text{inverse}_n(k_n)(x)$ is that they can solve equations that appear to have no other known explicit exact closed form solution.

2. It is important to realize that although all of the inverse spherical Bessel functions discussed in this article can solve equations containing certain linear combinations of cofactors having a certain form, the techniques described here cannot solve equations having arbitrary linear combinations of such cofactors. The techniques described in this article exploit only one degree of freedom by transposing all terms depending on the unknown and $\cos x$, $\sin x$, $\cosh x$, $\sinh x$, or $e^{-x}$ to the left side and all other terms to the right side, then dividing both sides by an appropriate unit times the gcd of the coefficients on the left side, making the left side coefficients be integers; then matching all of the terms on the left but permitting one arbitrary constant $c_0$ on the right side.

3. Many applicable published equations contain positive rather than negative integer powers of the variable requiring multiplication by a power of $x$; and many equations contain functions that must be converted to $\cos x$, $\sin x$, $\cosh x$, $\sinh x$ and $e^{-x}$ to achieve a form partially listed in Tables 1, 3, 5, and 6. This requires some manual or semi-manual effort until such steps are implemented in your computer algebra system Solve function. Moreover, during those manual operations, it is important to beware of and account for introducing or annihilating solutions $x = 0$.

That said, I hope that you and others will devise ways to use the ideas here more flexibly.

Acknowledgments

Thank you Bill Gosper for nudging me for an explanation of the proposed closed form for the Dottie number. Thank you Rob Corless and Christophe Vignat for your encouragement and suggestions. I also thank all of the early pioneers of constant recognition tables and software, many of whom are named in [14]. They foresaw the value of and developed new ways to greatly increase the synergy between approximate and exact computation.

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