The application of successive overrelaxation method for the solution of linearized half-sweep finite difference approximation to two-dimensional porous medium equation

Jackel Chew Vui Lung¹, Jumat Sulaiman², Andang Sunarto³

¹Faculty of Computing and Informatics, Universiti Malaysia Sabah Labuan International Campus, 87000, F.T. Labuan, Malaysia
²Faculty of Science and Natural Resources, Universiti Malaysia Sabah, 88400, Kota Kinabalu, Sabah, Malaysia
³IAIN Bengkulu, Jl Raden Fatah Kota Bengkulu, Bengkulu, Indonesia

*jackelchew93@ums.edu.my

Abstract. Successive overrelaxation or S.O.R. method is a widely known parameter-based iterative method that can regulate a large and sparse system of equations so that the number of iterations required to solve the system can be reduced. Many researchers have applied the S.O.R. method to get the solution to the mathematical problem efficiently. This paper extends the application of the S.O.R. method to solve one of the nonlinear partial differential equation, which is the two-dimensional porous medium equation. The S.O.R. method is incorporated into an iterative method that is formulated based on a half-sweep finite difference approximation, and the Newton-type linearization solves the nonlinear term that presents in the equation. The numerical experiment that uses this innovative numerical method to solve several two-dimensional porous medium equation problems shows significant improvement to the percentage of reduction in the number of iterations and computation time.

1. Introduction

Successive overrelaxation (S.O.R.) method is a widely known parameter-based iterative method that has been introduced by [1]. The S.O.R. method can regulate a large and sparse system of equations in order to reduce the number of iterations required to solve the system. The S.O.R. method can be considered as one of the efficient ways to accelerate the convergence of primary iterative methods such as Jacobi and Gauss-Seidel methods. Since the introduction of the S.O.R. method, several other modifications have been made to refine the performance of the S.O.R. method in solving the system of equations. For instance, the modified S.O.R. (M.S.O.R.) method was introduced by [2] for the solution of a system of equations that has a red-black coefficient matrix. Another two-parameter-based SOR-like method has been presented as the Accelerated Overrelaxation (A.O.R.) method by [3]. Besides that, two different versions of the S.O.R. method, which are called the K.S.O.R. [4] and M.K.S.O.R. [5] methods, respectively, have been proposed.

Since the system of equations generated from the discretization of a partial differential equation using the finite difference method deal with a large number of equations, S.O.R. method is required to approximate the solutions efficiently. Several researchers have applied the S.O.R. method to get the
solutions of their mathematical models. From our short literature review, [6] proposed a preconditioned version of the S.O.R. method to solve the multilinear systems with an $\mathcal{M}$-tensor efficiently. [7] examined the performance of another preconditioned version of the S.O.R. method for solving the time-fractional diffusion equation. Then, [8] applied the S.O.R. method to solve the normalized system of equations from the mathematical model of a Maxwell-Cattaneo’s magnetohydrodynamic fluid flow. On the other hand, [9] studied the performance of the S.O.R. method with the combination of wave variable transformation and second-order central difference scheme for solving the one-dimensional advection-diffusion equation. And then, [10] considered the effectiveness of the S.O.R. method with the Crank-Nicolson scheme to solve the diffusion equation.

In this paper, we extend the application of the S.O.R. method to solve one of the nonlinear partial differential equations, which is the two-dimensional porous medium equation. We adopt the S.O.R. method in the formulation of an iterative method which is based on a half-sweep finite difference approximation. For the discretization, we use the implicit finite difference scheme together with the half-sweep technique from [11]. The implicit finite difference scheme is chosen because of its unconditionally stable and has a lower computation cost compared to the Crank-Nicolson scheme. And then, the half-sweep technique is used as a complexity reduction approach as we attempt to simulate the solutions of the two-dimensional porous medium equation at the finer mesh. The Newton-type linearization solves the nonlinear term that presents in the finite difference approximation.

2. Methodology
To formulate the approximation equation using the implicit finite difference and half-sweep, let us consider the following two-dimensional porous medium equation [12]:

$$\frac{\partial w}{\partial t} = a \frac{\partial}{\partial x} \left( w^{n} \frac{\partial w}{\partial x} \right) + b \frac{\partial}{\partial y} \left( w^{n} \frac{\partial w}{\partial y} \right).$$  \hspace{1cm} (1)

When $a = b = n = 1$, Eq. (1) is called the Boussinesq equation. Eq. (1) is also one of the many important equations that arise in the theory of nonlinear heat transfer. Then, with the arbitrary values $a = b$ and $n = -1$, we get the equation that is used to model the long van der Waals interactions in thin films of a fluid spreading on a solid surface.

The numerical solution of Eq. (1) that is labelled as $\psi(x, y, t)$ can be approximated by subjecting Eq. (1) with the initial value $w(x, y, 0) = w_0$ for $0 \leq x, y \leq L$, and the boundary values are set into $w(0, y, t) = w_{left}, w(1, y, t) = w_{right}, w(x, 0, t) = w_{bottom}$, and $w(x, 1, t) = w_{top}$ for $0 \leq t \leq T$. Then, we denote $\psi_{p,q,r} = \psi(x_p, y_q, t_r), 1 \leq p, q \leq M - 1, 1 \leq r \leq R$ and the size of both space and time steps are $h = \frac{1}{M}$ and $k = \frac{1}{R}$ respectively. Using the concept of half-sweep to derive the implicit finite difference operators for Eq. (1), we have a backward difference at the time $t_{r+1}$ that is

$$\frac{\partial w}{\partial t} = \psi_{p,q,r+1} - \psi_{p,q,r},$$  \hspace{1cm} (2)

and the second-order rotated central difference at position $(x_p, y_q)$ for the first and second derivatives that are

$$\frac{\partial w}{\partial x} = \frac{\psi_{p+1,q,r+1} - \psi_{p-1,q,r+1}}{2\Delta x},$$ \hspace{1cm} (3)

$$\frac{\partial w}{\partial y} = \frac{\psi_{p,q+1,r+1} - \psi_{p,q-1,r+1}}{2\Delta y},$$ \hspace{1cm} (4)

$$\frac{\partial^2 w}{\partial x^2} = \frac{\psi_{p+1,q+1,r+1} - 2\psi_{p,q+1,r+1} + \psi_{p-1,q+1,r+1}}{2\Delta x^2},$$ \hspace{1cm} (5)

$$\frac{\partial^2 w}{\partial y^2} = \frac{\psi_{p,q+1,r+1} - 2\psi_{p,q+1,r+1} + \psi_{p,q-1,r+1}}{2\Delta y^2}. \hspace{1cm} (6)

Applying the chain rule of calculus on Eq. (1) and then substitute Eq. (2) – (6) into each of the derivatives yields a nonlinear approximation as follows,

$$f_{p,q,r+1} = \psi_{p,q,r+1} - \rho_1 \psi_{p,q,r+1}^n \psi_{p+1,q+1,r+1} + 2 \rho_1 \psi_{p,q,r+1}^{n+1} - \rho_1 \psi_{p,q+1,r+1}^n \psi_{p-1,q+1,r+1} - \rho_2 \psi_{p,q,r+1}^{n+1} \psi_{p+1,q+1,r+1} + 2 \rho_2 \psi_{p,q+1,r+1} \psi_{p-1,q+1,r+1} - \rho_2 \psi_{p,q+1,r+1}^n \psi_{p+1,q+1,r+1} - \rho_3 \psi_{p,q,r+1}^n \psi_{p-1,q+1,r+1} + 2 \rho_3 \psi_{p,q+1,r+1} \psi_{p-1,q+1,r+1}$$

$$+ \rho_3 \psi_{p,q+1,r+1}^n \psi_{p+1,q+1,r+1} - \rho_4 \psi_{p,q,r+1}^{n+1} \psi_{p+1,q+1,r+1} + 2 \rho_4 \psi_{p,q+1,r+1} \psi_{p-1,q+1,r+1} - \rho_4 \psi_{p,q+1,r+1}^n \psi_{p+1,q+1,r+1} - \rho_5 \psi_{p,q,r+1}^n \psi_{p-1,q+1,r+1} + 2 \rho_5 \psi_{p,q+1,r+1} \psi_{p-1,q+1,r+1}$$

$$+ \rho_5 \psi_{p,q+1,r+1}^n \psi_{p+1,q+1,r+1} - \rho_6 \psi_{p,q,r+1}^{n+1} \psi_{p+1,q+1,r+1} + 2 \rho_6 \psi_{p,q+1,r+1} \psi_{p-1,q+1,r+1} - \rho_6 \psi_{p,q+1,r+1}^n \psi_{p+1,q+1,r+1} - \rho_7 \psi_{p,q,r+1}^n \psi_{p-1,q+1,r+1} + 2 \rho_7 \psi_{p,q+1,r+1} \psi_{p-1,q+1,r+1}$$

$$+ \rho_7 \psi_{p,q+1,r+1}^n \psi_{p+1,q+1,r+1} - \rho_8 \psi_{p,q,r+1}^{n+1} \psi_{p+1,q+1,r+1} + 2 \rho_8 \psi_{p,q+1,r+1} \psi_{p-1,q+1,r+1} - \rho_8 \psi_{p,q+1,r+1}^n \psi_{p+1,q+1,r+1} - \rho_9 \psi_{p,q,r+1}^n \psi_{p-1,q+1,r+1} + 2 \rho_9 \psi_{p,q+1,r+1} \psi_{p-1,q+1,r+1}$$

$$+ \rho_9 \psi_{p,q+1,r+1}^n \psi_{p+1,q+1,r+1} - \rho_{10} \psi_{p,q,r+1}^{n+1} \psi_{p+1,q+1,r+1} + 2 \rho_{10} \psi_{p,q+1,r+1} \psi_{p-1,q+1,r+1} - \rho_{10} \psi_{p,q+1,r+1}^n \psi_{p+1,q+1,r+1}$$

$$- \rho_{11} \psi_{p,q,r+1}^n \psi_{p-1,q+1,r+1} + 2 \rho_{11} \psi_{p,q+1,r+1} \psi_{p-1,q+1,r+1} - \rho_{11} \psi_{p,q+1,r+1}^n \psi_{p+1,q+1,r+1} - \rho_{12} \psi_{p,q,r+1}^{n+1} \psi_{p+1,q+1,r+1} + 2 \rho_{12} \psi_{p,q+1,r+1} \psi_{p+1,q+1,r+1}$$
\[-\rho_3 \psi^n_{p,q,r+1} \psi_{p+1,q-1,r+1} - \rho_4 n \psi^n_{p,q,r+1} \psi^2_{p-1,q+1,r+1} + 2 \rho_4 n \psi^n_{p,q,r+1} \psi_{p+1,q+1,r+1} \psi_{p+1,q-1,r+1} - \rho_4 n \psi^n_{p,q,r+1} \psi^2_{p+1,q-1,r+1} - \psi_{p,q,r}, \tag{7}\]

where \( \rho_1 = \rho_3 = ak/2h^2 \), \( \rho_2 = \rho_4 = ak/h^2 \), and also, we let the constant \( \alpha = a = b \).

Since we consider \( M - 1 \) mesh points for the simulation, then there are \( M - 1 \) nonlinear equations to be solved iteratively. With \( M - 1 \) times of Eq. (7), we have a system of nonlinear equations that is

\[ F(\Psi) = 0, \tag{8}\]

where \( F(.) = \left( f_{1,1,r+1}(.), ..., f_{M-1,M-1,r+1}(.) \right)^T \) and \( \Psi = (\psi_{1,1,r+1}, ..., \psi_{M-1,M-1,r+1}) \).

To solve the system of nonlinear equations (8), we apply the Newton-type linearization that is

\[ J_{r+1} = \begin{bmatrix} D_{1,r} & U_{1,r} \\ L_{1,r} & U_{2,r} \\ \vdots & \vdots \\ D_{M-1,r} & U_{M-1,r} \end{bmatrix}, \tag{9}\]

where the diagonal block has the form of

\[ D_{i,r} = \begin{bmatrix} \frac{\partial f_{1,q}}{\partial \psi_{1,q}} \\ \vdots \\ \frac{\partial f_{M-1,q}}{\partial \psi_{M-1,q}} \end{bmatrix}, \tag{10}\]

the lower and upper blocks respectively have the form of

\[ L_{i,q} = \begin{bmatrix} \frac{\partial f_{i,q}}{\partial \psi_{i,q}} \\ \vdots \\ \frac{\partial f_{M-1,q}}{\partial \psi_{M-1,q}} \end{bmatrix}, \tag{11}\]

and

\[ U_{i,q} = \begin{bmatrix} \frac{\partial f_{i,q}}{\partial \psi_{i+1,q}} \\ \vdots \\ \frac{\partial f_{M-1,q}}{\partial \psi_{M-1,q}} \end{bmatrix}. \tag{12}\]

As a result of the linearization, we get the linearized system of equations at time \( r + 1 \) that is

\[ J_{r+1} X_{r+1} = -F(\Psi), \tag{13}\]

where \( X_{r+1} = \psi_{r+1}^{(l+1)} - \psi_{r+1}^{(l)} \) and \( l \) is the iteration index.

To formulate the S.O.R. method for solving the linearized system (13), we let

\[ J_{r+1} = D_{r+1} + L_{r+1} + U_{r+1}, \tag{14}\]

and using Eq. (14), the S.O.R. method that we propose to solve Eq. (1) becomes

Method 1:

\[ X_{r+1}^{(l+1)} = (1 - \omega)X_{r+1}^{(l)} + \omega (D_{r+1} + L_{r+1})^{-1} (-U_{r+1} X_{r+1}^{(l)} - F(\psi_{r+1}^{(l)})), \tag{15}\]

\[ \psi_{r+1}^{(l+1)} = X_{r+1}^{(l+1)} + \psi_{r+1}^{(l)}. \tag{16}\]

The algorithm for the implementation of the S.O.R. method for solving Eq. (1) is as follows:
i. Set the initial value $w_0$ and the values of a squared boundary $w_{left}, w_{right}, w_{bottom},$ and $w_{top}$. 

ii. Set the parameter $\omega \in (1, 2)$, 

iii. For $1 \leq r \leq R$, set $\psi_r^{(0)} = 1.0$ and $X_r^{(0)} = 0$, 

iv. For $l = 1, 2, \ldots$, iterate Eq. (15), 

v. If the correctors converge, compute the solutions using Eq. (16), 

vi. Check the convergence criterion, display the numerical outputs.

3. **Result and Discussion**

For the numerical experiment, Method 1 is tested against another version of Eq. (15) with the parameter $d = 1$ and the Newton-Gauss-Seidel method [13] as a control. The number of iterations required to obtain the converged solutions $c$ and the computation time required to complete the computer program is the important criteria to be observed during the numerical experiment. For the accuracy control, the maximum absolute error ($\epsilon_{max}$) is observed.

The two-dimensional porous medium equation problems that are used for the experiment can be described as follows. For the first problem, by referring to Eq. (1), we let $a = b = 1/5$ and consider a Boussinesq ($n = 1$) problem. The solutions are bounded within $0 \leq x, y \leq 1$ and the time interval is $0 \leq t \leq 1$. The exact solution that is given by [12] is $u(x, y, t) = x + y + 0.4t$. For the second problem, the constants that we chose are $a = b = 1/5$ and $n = 2$. The solution domain is the same as Example 1, and the exact solution is $u(x, y, t) = \sqrt{5x + 5y + 5t}$ [12]. For the third and last problem, we fix the constant $a = b = 1$ and increase the value of $n$ to 5 as discussed in [14]. The provided exact solution is $u(x, y, t) = \sqrt[4]{0.8x + 0.8y + 1.6t}$.

**Table 1. Comparison in terms of number of iterations**

| Problem | Method | Mesh Size  |       |       |       |
|---------|--------|-----------|-------|-------|-------|
|         |        | $16 \times 16$ | $32 \times 32$ | $64 \times 64$ | $128 \times 128$ | $256 \times 256$ |
| 1       | Newton-Gauss-Seidel | 136 | 436 | 1525 | 5462 | 19404 |
|         | Method 1 with $\omega = 1$ | 80 | 245 | 829 | 2901 | 10313 |
|         | Method 1 with optimum $\omega$ | 70 | 150 | 607 | 1204 |
| 2       | Newton-Gauss-Seidel | 130 | 400 | 1380 | 4901 | 17458 |
|         | Method 1 with $\omega = 1$ | 77 | 221 | 738 | 2593 | 9243 |
|         | Method 1 with optimum $\omega$ | 74 | 155 | 621 | 1219 |
| 3       | Newton-Gauss-Seidel | 739 | 2630 | 9478 | 34908 | 121649 |
|         | Method 1 with $\omega = 1$ | 405 | 1402 | 5005 | 18002 | 64469 |
|         | Method 1 with optimum $\omega$ | 214 | 430 | 835 | 3178 |

**Table 2. Comparison in terms of computation time**

| Problem | Method | Mesh Size  |       |       |       |
|---------|--------|-----------|-------|-------|-------|
|         |        | $16 \times 16$ | $32 \times 32$ | $64 \times 64$ | $128 \times 128$ | $256 \times 256$ |
| 1       | Newton-Gauss-Seidel | 0.76 | 2.93 | 19.59 | 360.89 | 4586.69 |
|         | Method 1 with $\omega = 1$ | 0.40 | 2.68 | 9.12 | 201.13 | 2958.93 |
|         | Method 1 with optimum $\omega$ | 0.34 | 2.16 | 6.45 | 68.01 | 419.75 |
| 2       | Newton-Gauss-Seidel | 0.97 | 2.70 | 18.26 | 248.82 | 4243.79 |
|         | Method 1 with $\omega = 1$ | 0.55 | 1.90 | 8.56 | 183.43 | 1965.08 |
|         | Method 1 with optimum $\omega$ | 0.48 | 1.73 | 6.12 | 59.97 | 413.57 |
| 3       | Newton-Gauss-Seidel | 0.98 | 9.51 | 113.63 | 1653.85 | 29234.80 |
|         | Method 1 with $\omega = 1$ | 0.85 | 8.39 | 63.64 | 530.82 | 13038.50 |
|         | Method 1 with optimum $\omega$ | 0.61 | 2.36 | 14.06 | 125.73 | 1026.12 |
Based on the recorded numerical results and tabulated in Table 1, 2 and 3, we find that Method 1 has reduced significantly the number of iterations and the computation time required in solving the three problems. By comparison, Method 1, with the optimum value of $\omega$ needs about 61.45% lesser number of iterations than when the parameter $\omega = 1$ is used. Then, Method 1 needs about 78.48% lesser number of iterations when it is compared against the Newton-Gauss-Seidel method. In terms of computation time, Method 1 is much faster to complete the computer program by 50.57% and 68.61%, which corresponds to Method 1 with $\omega = 1$ and the control method, respectively. Moreover, the maximum absolute errors produced by Method 1 are smaller to the other two tested methods indicating the approximate solutions of the two-dimensional porous medium equation obtained by Method 1 are accurate.

4. Conclusion

In this paper, we extend the application of the S.O.R. method to solve the two-dimensional porous medium equation based on the half-sweep finite difference scheme and the Newton method. The developed iterative method is shown to be more efficient in terms of the number of iterations and the computation time compared to its different version with the parameter $\omega = 1$ and the Newton-Gauss-Seidel method. The developed iterative method also shows a great improvement to the percentage of reduction in the number of iterations and computation time. Hence, this innovative numerical method can be a good solver for the nonlinear parabolic type partial differential equations.

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