Cyclotron Damping along a Uniform Magnetic Field

Xixia Ma *

Abstract. We prove cyclotron damping for the collisionless Vlasov-Maxwell equations on $\mathbb{T}^2 \times \mathbb{R}^3$ under the assumption that the electric induction is zero. Our proof is based on a new observation from Faraday Law of Electromagnetic induction and Lenz’s Law. On the basis of it, we use the improved Newton iteration scheme to show the damping mechanism.

1. Introduction

In this paper, it is assumed that the plasma system is collisionless, nonrelativistic, and hot. Cyclotron damping describes the phenomenon that a plasma with a prescribed zero-order distribution function, imbedded in a uniform magnetic field, which is assumed to be perturbed by an electromagnetic wave propagating parallel to the field. A number of treatments of the problem of cyclotron damping have appeared in the literature[18,20,24,25], but there are few rigorous mathematical results except the recent results of Bedrossian and Wang [7]. The usual method that deals with this phenomenon is via the Vlasov-Maxwell equations. In this paper we study cyclotron damping at the level of kinetic description based on the Vlasov-Maxwell equations from the mathematical viewpoint. First, we analyze the Vlasov-Maxwell equations from perspective of both equilibrium and stability theories.

Now we give a detailed description of the Vlasov-Maxwell equations. We denote the particles distribution function by $f = f(t, x, v)$, and the electric and magnetic fields by $E(t, x)$ and $B(t, x)$, respectively. Then the Vlasov equation says

$$\partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f = 0. \tag{0.1}$$

The electric and magnetic fields $E(t, x)$ and $B(t, x)$ in Eq.(0.1) are determined from Maxwell’s equations:

$$\nabla \cdot E(t, x) = \int_{\mathbb{R}^3} f(t, x, v)dv, \quad \nabla \times B(t, x) = \int_{\mathbb{R}^3} vf(t, x, v)dv + \frac{\partial E(t, x)}{\partial t},$$

$$\frac{\partial B(t, x)}{\partial t} = -\nabla \times E(t, x), \quad \nabla \cdot B(t, x) = 0. \tag{0.2}$$

Note that Eq.(0.1) is nonlinear since $E(t, x)$ and $B(t, x)$ are determined in terms of $f(t, x, v)$ from Maxwell’s equations (0.2).

An equilibrium analysis of Eq.(0.1) and Eqs.(0.2) proceeds by setting $\frac{\partial f}{\partial t} = 0$ and looking for stationary solutions, $f^0(x, v), E^0(x), B^0(x)$, that satisfy the equations

$$v \cdot \nabla_x f^0(x, v) + (E^0 + v \times B^0) \cdot \nabla_v f^0(x, v) = 0, \quad \nabla \cdot B^0(x) = 0,$$

$$\nabla \times B^0(x) = \int_{\mathbb{R}^3} vf^0(x, v)dv, \quad \nabla \cdot E^0(x) = \int_{\mathbb{R}^3} f^0(x, v)dv. \tag{0.3}$$

An analysis of Eq.(0.3) reduces to a determination of the particle constants of the motion in the equilibrium fields $E^0(x)$ and $B^0(x)$. In this paper, we assume that $E^0(x) = 0$, namely, $\int_{\mathbb{R}^3} f^0(x, v)dv = 0$. This implies that there are no deviations from charge neutrality in equilibrium, $B^0(x)$ is produced by external current sources as well as any equilibrium plasma currents.

A stability analysis based on Eq.(0.1) and Eqs.(0.2) proceeds in the following manner. The quantities $f(t, x, v), E(t, x)$, and $B(t, x)$ are expressed as the sum of their equilibrium values plus a time-dependent perturbation:

$$f(t, x, v) = f^0(x, v) + \delta f(t, x, v), \quad E(t, x) = E^0(x) + \delta E(t, x), \quad B(t, x) = B^0(x) + \delta B(t, x). \tag{0.4}$$

The quantities $f^0(x, v), E^0(x)$ and $B^0(x)$ satisfy (0.3). The time development of the perturbations $\delta f(t, x, v), \delta E(t, x)$, and $\delta B(t, x)$ is studied by using Eq.(0.1) and Eqs.(0.2). For small-amplitude perturbations, the Vlasov-Maxwell

*Yau Mathematical Sciences Center, Tsinghua University. E-mail addresses:kfmaxixia@tsinghua.edu.cn
equations are linearized about the equilibrium \( f^0(x, v), E^0(x) \) and \( B^0(x) \). This gives

\[
\begin{align*}
\frac{\partial \delta f}{\partial t} + v \cdot \nabla_x \delta f(t, x, v) + (E^0 + v \times B^0) \cdot \nabla_v \delta f(t, x, v) &= -(\delta E + v \times \delta B) \cdot \nabla_v f^0(t, x, v), \\
\nabla \cdot \delta B(t, x) &= 0, \\
\nabla \cdot \delta E(t, x) &= \int_{\mathbb{R}^3} v \delta f(t, x, v) dv + \frac{\partial \delta E}{\partial t},
\end{align*}
\]

(0.5)

If the perturbations \( \delta f(t, x, v), \delta E(t, x) \), and \( \delta B(t, x) \) grow, then the equilibrium distribution \( f^0(x, v) \) is unstable. Otherwise, the perturbations damp, so the system returns to equilibrium and is stable. We assume that the equilibrium \( f^0 \) is independent of space, namely, \( f^0(x, v) = f^0(v) \).

From the above analysis and the form of the Vlasov-Maxwell equations, it is obvious that when \( B \equiv 0 \), cyclotron damping is reduced to Landau damping. Hence, the method used is similar to that employed by Mouhot and Villani [23]. However, compared with the electric field, a static magnetic field introduces a fascinating complication into the motion of charged particles. And the particles trajectories become helices, spiraling around the magnetic lines of force. This severe alteration of the orbits tends to inhibit transport across the magnetic field. The mechanism of Landau damping does depend on the transfer of electric field energy to particles moving in phase with the wave. However, for cyclotron damping in electromagnetic plasmas, the electric field of the wave is perpendicular to the direction of the magnetic field and the particle drifts and accelerates the particle perpendicular to the drift direction.

In the following we recall Landau damping through gathering lots of physical literature and results of mathematical articles. The existence of a damping mechanism by which plasma particles absorb wave energy was found by L.D. Landau at the linear level, under the condition that the plasma is not cold and the velocity distribution is of finite extent. Next in linear case, many works from mathematical aspects found in [9,15,25,28] gave rigorous proofs under different assumptions. Later, a ground-breaking work for Landau damping was made by Mouhot and Villani in the nonlinear case. They gave the first and rigorous proof of nonlinear Landau damping under different assumptions. Later, a ground-breaking work for Landau damping was made by Mouhot and Villani in the nonlinear case. They gave the first and rigorous proof of nonlinear Landau damping under the assumption of the electric field. In this paper we will extend their results and prove that cyclotron damping in electromagnetic fields still occurs.

Now we will make a brief statement about the connection and difference between the results of [23] and ours. First, in electric field case, Mouhot and Villani proved the existence of Landau damping under assumption of the (L) condition, that is expressed as follows:

(\text{L}) \quad \text{There are constants } C_0, \lambda, \kappa > 0 \text{ such that } |\hat{f}^0(\eta)| \leq C_0 e^{-2\pi \lambda |\eta|} \text{ for any } \eta \in \mathbb{R}^d; \text{ and for any } \xi \in \mathbb{C} \text{ with } 0 \leq \Re \xi < \lambda,

\[
\inf_{k \in \mathbb{Z}^d} |\mathcal{L}(\xi, k) - 1| \geq \kappa,
\]

(0.6)

where we define a function \( \mathcal{L}(\xi, k) = -4\pi^2 \int_0^\infty e^{2\pi t|\xi|} |\hat{W}(k)\hat{f}^0(kt)|^2 dt, \) and \( \xi^* \) is the complex conjugate to \( \xi \).

To some extent, (0.6) of the (L) condition is similar to the “Small Denominators” condition in KAM theory in [1], but is a uniform bound from below. Here we will consider the condition of cyclotron damping from a totally different perspective, in detail, we will give a physical condition that we call it the “Stability Condition”, which is stated in the following form (here we assume the background magnetic field \( B_0 \) along the \( z \) direction):

\text{Stability condition} : for any component velocity in the \( z \) direction \( v_z \in \mathbb{R} \), there exists some positive constant \( v_{\tau z} \) such that if \( v_3 = \frac{v_z}{c}, \omega, k \) are frequencies of time and space \( t, x \), respectively, then \( |v_3| \gg v_{\tau z} \).

The above Stability condition tells us that the number of particles in which the wave velocity greatly exceeds their velocity is much larger than the number of particles’ velocity slower than the wave velocity. And we will show that cyclotron damping occurs under the above conditions, and that isn’t only consistent with the physical observation, but also is the same with the “Small Denominators” condition in KAM theory in [1] in some sense.

Second, compared with the electric field case, it is easy to find a new term \( v \times B \) in the electromagnetic field setting. And this brings many difficulties because of the unboundedness of \( v \). Based on the physical facts of Faraday Law of Electromagnetic induction and Lenz’s Law, we know that \( \delta B \) generates the force that inhibits the change of the electric field. This helps us estimate such term. And the above fact leads us to study the following dynamics of the particles trajectory:

\[
\begin{cases}
\frac{d}{d t} X_{\tau, \tau}(x, v) = V_{\tau, \tau}(x, v), \\
\frac{d}{d t} V_{\tau, \tau}(x, v) = V_{\tau, \tau}(x, v) \times B_0 + E[f](t, X_{\tau, \tau}(x, v)), \\
X_{\tau, \tau}(x, v) = x, \\
V_{\tau, \tau}(x, v) = v.
\end{cases}
\]

(0.7)

where \( B = B_0 + \delta B \). In other words, we reduce inhomogeneous dynamical system to homogeneous dynamical system. Hence, based on the above dynamical system (0.7), we call the adopted Newton iteration as the improved Newton iteration.

Indeed, there are many papers that contribute to Landau damping. Here we only list some results, except the above mentioned results. Bedrossian, Masmoudi, and Mouhot [4] provided a new, simple and short proof of nonlinear Landau damping on \( \mathbb{T}^d \) in only electric field case that nearly obtains the “critical” Gevrey-\( \frac{1}{2} \) regularity predicted in [23]. Although their proofs have lots of the same ingredients as the proof in [23] from
a physical point of view, at a mathematical level, the two proofs are quite different, they “mod out” by the characteristics of free transport and work in the coordinates \( z = x - vt \) with \( (t, x) \to (t, z, v) \). The evolution equation (0.1) \((B = 0)\) becomes

\[
\partial_t f + E(t, z + vt) \cdot (\nabla_v - t\nabla_z)f + E(t, z + vt) \cdot \nabla_v f^0 = 0.
\]

From this formula, it is easy to see the phase mixing mechanism. And this coordinate shift is related to the notion of “gliding regularity” used in [23]. One of the main ingredients of their proof is to split nonlinear terms into the transport structure term and “reaction” term in [23] by using paradifferential calculus. Bedrossian, Masmoudi [5] also used this method to prove the inviscid damping and asymptotic stability of 2-D Euler equations and later also proved the stability threshold for the 3D Couette flow in Sobolev regularity in [5] and so on. They [6] also proved Landau damping for the collisionless Vlasov equation with a class of \( L^1 \) interaction potentials on \( \mathbb{R}^3_+ \times \mathbb{R}^3 \) for localized disturbances of infinite, inhomogeneous background. Also, there are counterexamples that can be found in [2,10,13], showing that there is in general no exponential decay without analyticity and confining. Bedrossian stated one of these counterexamples by proving that the theorem of Mouhot and Villani on Landau proved Landau damping for the collisionless Vlasov equation with a class of transport structure term and “reaction” term in [23] by using paradifferential calculus. Bedrossian, Masmoudi of “gliding regularity” used in [23]. One of the main ingredients of their proof is to split nonlinear terms into the

**Theorem 0.1**. For any \( \eta, v \in \mathbb{R}^3, k \in \mathbb{N}_0^3 \), we assume that the following conditions hold in equations (2.1).

(i) \( \hat{W}(k_1, k_2, 0) = 0, |\hat{W}(k)| \leq \frac{1}{1+|k|} \gamma, \gamma > 1 \), where \( W(x) = (W_1(x), W_2(x), 0) \);

(ii) \( \left| \hat{f}(\eta) \right| \leq C e^{-2\pi \lambda_0 \eta_0}, \left| \partial_{\eta_0} \hat{f}(\eta) \right| \leq C e^{-2\pi \lambda_0 \eta_0}, \left| f(0, v, 3) \right| \leq C e^{-2\pi \alpha_0 \alpha_3} \), for some constants \( \lambda_0, \alpha_0, C > 0 \);

(iii) \( \left| \hat{f}_0(k, \eta) \right| \leq C e^{-2\pi \lambda_0 \eta_0} \), for some constant \( C > 0 \), where \( \lambda_0 \) is defined in (ii);

(iv) for any component velocity in the \( \tilde{z} \) direction \( v_3 \in \mathbb{R} \), there exists some positive constant \( v_{\text{Tc}} \in \mathbb{R} \) such that if \( v_3 = \frac{\pi}{k}, \) then \( |v_3| \gg v_{\text{Tc}} \).

Then for any fixed \( \eta_k, k_3 \), and for any \( \lambda'_0 < \lambda_0 \), we have

\[
\left| \hat{f}(t, k, \eta) - \hat{f}_0(k, \eta) \right| \leq e^{-2\pi \lambda'_0 |\eta_1 + k_3 t|}, \left| \hat{\rho}(t, k) - \hat{\rho}_0 \right| \leq e^{-2\pi \lambda'_0 |\eta_1|} e^{-2\pi \lambda'_0 |\eta_2|} e^{-\lambda_0 |k_3| t},
\]

\[
\left| \hat{E}(t, k) \right| \leq e^{-2\pi \lambda'_0 |\eta_1|} e^{-2\pi \lambda'_0 |\eta_2|} e^{-2\pi \lambda'_0 |k_3| t}, \left| \hat{B}(t, k) \right| \leq e^{-2\pi \lambda'_0 |\eta_1|} e^{-2\pi \lambda'_0 |\eta_2|} e^{-2\pi \lambda'_0 |k_3| t},
\]

where \( \rho_0 = \int_{\mathbb{R}^3} f_0(x, v) dv dx, \eta_{k_1} = \frac{1}{\pi}(-k_2 \cos \Omega t + k_2 - k_1 \sin \Omega t), \eta_{k_2} = \frac{1}{\pi}(-k_2 \sin \Omega t - k_1 + k_1 \cos \Omega t), v_h = |\eta_{k_1}| + |\eta_{k_2}| + |k_3| t). \)

**Remark 0.2**. In the linear case, from (0.8) and (2.24)-(2.26), it is easy to observe that cyclotron damping is almost the same with Landau damping when \( B_0 \) tends to zero. However, when \( B_0 \) is fixed, in the horizontal direction there is no damping and the motion of particles is a circle; in the \( \tilde{z} \) direction, the damping still occurs. Therefore, there is no immediate tendency toward trapping. This is the crucial point for cyclotron damping. And in the latter case, the motion of plasma particles moves along spiral trajectories.

Now we state our main result as follows.
Theorem 0.3 Let $f^0 : \mathbb{R}^3 \to \mathbb{R}^+$ be an analytic velocity profile, and assume $W(x) = (W_1(x), W_2(x), 0) : \mathbb{T}^3 \to \mathbb{R}^3$ satisfying
\[ W(k_1, k_2, 0) = 0, \quad |\hat{W}(k)| \leq \frac{1}{1 + |k|^\gamma}, \quad \gamma > 1. \]

Further we assume that, for some constant $\lambda_0$ such that $\lambda_0 - B_0 > 0$,
\[ \sup_{\eta \in \mathbb{R}^3} e^{2\pi(\lambda_0 - B_0)|\eta|} |\hat{f}(\eta)| \leq C_0, \quad \sum_{n \in \mathbb{N}_0^3} \frac{(\lambda_0 - B_0)^n}{n!} \| \nabla_v f^0 \|_{L^2_{\kappa_0}} \leq C_0 < \infty. \quad (0.10) \]

And we consider the following system,
\[
\begin{aligned}
\partial_t f + v \cdot \nabla_x f + \frac{e}{m} (v \times B_0) \cdot \nabla_v f &= - \frac{e}{m} (E + v \times B) \cdot \nabla_v f, \\
\partial_t B &= \nabla_x \times E,
\end{aligned}
\]
\[ E = W(x) \ast \rho(t, x), \quad \rho(t, x) = \int_{\mathbb{R}^3} f(t, x, v) dv,
\]
\[ f(0, x, v) = f_0(x, v) = f_0(x, v_1, v_2, v_3), \quad f^0(v) = f^0(v_1, v_2, v_3), \]

there is $\varepsilon = \varepsilon(\lambda_0, \mu, \beta, \gamma, \lambda_0', \rho_0')$ verifying the following property: $f_0 = f_0(x, v)$ is an initial data such that
\[ \sup_{k \in \mathbb{Z}^3, \eta \in \mathbb{R}^3} e^{2\pi(\lambda_0 - B_0)|\eta|} e^{2\pi |k|} |f^0 - f_0| + \int_{\mathbb{T}^3} \| f^0 - f_0 \| e^{\beta |v|} dv dx \leq \varepsilon, \quad (0.12) \]

where any $\beta > 0$, $\lambda_0 > \lambda_0', \mu > 0$, $\mu > \mu' > 0$.

In addition, we also assume that the following stability condition holds:

\textbf{Stability condition: } for any component velocity in the $z$ direction $v_3 \in \mathbb{R}$, there exists some positive constant $\varepsilon_0$ such that $v_3 = \frac{e}{\lambda_0^eta}$, where $\omega, k$ are frequencies of time and space $t, x$, respectively, then $|v_3| \gg |v_{\tau_0}|

Then there exists a unique classical solution $(f(t, x, v), E(t, x), B(t, x))$ to the non-linear Vlasov system (0.11). Moreover, for any fixed $n_3, k_3, \forall r \in \mathbb{N}_0$, as $|t| \to \infty$, we have
\[ |\hat{f}(t, k, \eta) - \hat{f}_0(k, \eta)| \leq e^{-\gamma(\lambda_0' - B_0)|n_3 + k_3 t|}, \quad \| \rho(t, \cdot) - \rho_0 \|_{C^r(T^3)} = O(e^{-2\pi(\lambda_0' - B_0)|t|}), \]
\[ \| E(t, \cdot) \|_{C^r(T^3)} = O(e^{-2\pi(\lambda_0' - B_0)|t|}), \quad \| B(t, \cdot) \|_{C^r(T^3)} = O(e^{-2\pi(\lambda_0' - B_0)|t|}), \quad (0.13) \]

where $\rho_0 = \int_{\mathbb{R}^3} f_0(x, v) dv dx$.

Now we simply analyze the relation among the Vlasov-Poisson equations, the Vlasov-Maxwell equations, and our model. If we assume that both the electric induction and the magnetic field are zero, then Vlasov-Maxwell equations reduce to the Vlasov-Poisson equations; if we only assume that the electric induction is zero, then the Vlasov-Maxwell equations reduce to our case. In other words, our case is the generalized case of the Vlasov-Poisson equations, and provides a new observation from the physical viewpoint to solve the corresponding problem of the Vlasov-Maxwell equations. However, we cannot still solve the Vlasov-Maxwell equations completely only through this new observation. Therefore, the stability’s or un-stability’s problem of the Vlasov-Maxwell equations is still open.

In the following we sketch the difficulties and methods in our paper’s setting. The crucial estimates of cyclotron damping include the following two inequalities:

- a control of $\rho = \int_{\mathbb{R}^3} f dv$ in $L^{3, \gamma + \mu}$ norm, that is, $\sup_{\gamma \geq 0} \| \rho \|_{L^{3, \gamma + \mu}} < \infty$,
- a control of $f \circ \Omega_{s, t}$ in $L^{\frac{3}{2}}(\mathbb{T}^3)$ norm, where $\lambda' < \lambda, \mu' < \mu$.

However, during the iteration scheme, for cyclotron damping, from the view of the original Newton iteration, the characteristics are not only determined by the density $\rho$, but also related with the velocity at stage $n$. However, $\rho^n$ is independent of the velocity and the key difficulty is that the velocity is unbounded. This makes that we obtain the estimates of the associated deflection $\Omega^n$ more difficult, but this difficulty doesn’t exist for Landau damping in [23]. To overcome this difficulty, on the basis of a new observation from Lenz’s Law, we reduce the classical dynamical system to the improved dynamical system (0.7). And the corresponding equation of the density $\rho[h^{n+1}]$ at stage $n + 1$ becomes
\[ \rho[h^{n+1}] (t, x) = \int_0^t \int_{\mathbb{R}^3} \left[ (E[h^{n+1}] \circ \Omega_{s, t}^n (x, v) \cdot G_{s, t}^n) - (B[h^{n+1}] \circ \Omega_{s, t}^n (x, v) \cdot G_{s, t}^{n, v}) \right. \
- (B[f^0] \circ \Omega_{s, t}^n (x, v)) \cdot (\nabla_v h^{n+1} \times V_{s, t}^0 (x, v)) \circ \Omega_{s, t}^n (x, v) \right] (s, \chi_{s, t}^n (x, v), V_{s, t}^0 (x, v)) dv ds + \text{(terms from stage } n). \]

From the above equation, we see that, comparing with that in [23], there is a new term $(B[f^0] \circ \Omega_{s, t}^n (x, v)) \cdot (\nabla_v h^{n+1} \times V_{s, t}^0 (x, v) \circ \Omega_{s, t}^n (x, v))$ that have the information of the stage $n + 1$. Of course, this is due to the reason
that we regard the perturbation of the magnetic field as a negligible term. To get a self-consistent estimate, we have to deal with this term and have little choice but to come back the equation of \(h^{n+1}\). This leads to different kinds of resonances (in term of different norms), for example, \(v\) have to deal with this term and have little choice but to come back the equation of that we regard the perturbation of the magnetic field as a negligible term. To get a self-consistent estimate, we then the solution of Eq.(1.1) is obtained as follows:

\[
\tau = \hat{\lambda}^{\tau} + \beta^* \eta \tau.
\]

Remark 0.4 \(\gamma > 1\) of Theorems 0.1 and 0.3 can be extended to \(\gamma > 1\), the difference between \(\gamma > 1\) and \(\gamma = 1\) is the proof of the growth integral in section 7. The proof of \(\gamma = 1\) is similar to section 7 in [23], here we omit this case.

Remark 0.5 From the physical viewpoint, we should write \(E = \nabla W(x) * \rho(t,x)\), where \(W(x) : T^3 \to \mathbb{R}\) is a scalar function, the corresponding condition \(\gamma > 1\) becomes \(\gamma > 2\). But to simplify the writing, we denote \(E = W(x) * \rho(t,x)\), where \(W(x)\) is a vector-valued function, this has no influence on the mathematical difficulty.

Remark 0.6 From the physics viewpoint, the condition of the damping is that the number of particles that the wave velocity greatly exceeds their velocity is much larger than the number of particles whose velocity is slower than the wave velocity. The stability condition in above theorem is in consistent with the statement of the physics viewpoint. In fact, the stability condition and (0.9) imply that the number of resonant particles is exponentially small and their effect corresponding is weak. That is, under this condition, most particles absorb energy from the wave, and then the damping occurs.

Remark 0.7 From the definition of the hybrid analytic norms and the proof of the following sections, there is still the phenomena of cyclotron damping for the nonlinear Vlasov-Maxwell equations, which is the same to that in the nonlinear Vlasov-Poisson equations but only in \(\hat{z}\) direction, the position of the corresponding resonances translates \(\theta\) into \(B_0\), and in the horizon direction, the action of the particles moves along circle that is the same to the linear case.

1 Notation and Hybrid analytic norm

Now we introduce some notations. We denote \(T^3 = \mathbb{R}^3 / \mathbb{Z}^3\). For function \(f(x,v)\), we define the Fourier transform \(\hat{f}(k,\eta)\), where \((k,\eta) \in \mathbb{Z}^3 \times \mathbb{R}^3\), via

\[
\hat{f}(k,\eta) = \int_{T^3 \times \mathbb{R}^3} e^{-2\pi i k \cdot x} e^{-2\pi i \eta \cdot v} f(x,v) dx dv,
\]

\[
\hat{f}(\omega,k,\eta) = \int_{\mathbb{R}^+} e^{2\pi i \omega t} \int_{T^3 \times \mathbb{R}^3} e^{-2\pi i k \cdot x} e^{-2\pi i \eta \cdot v} f(x,v) dx dv dt.
\]

We also write

\[
k = (k_1, k_2, k_3) = (k_\perp \cos \varphi, k_\perp \sin \varphi, k_3), \quad \eta = (\eta_1, \eta_2, \eta_3) = (\eta_\perp \cos \gamma, \eta_\perp \sin \gamma, \eta_3).
\]

Now we define some notations

\[
\hat{f} = \hat{f}(\omega,k,\eta), \quad \hat{f} = \hat{f}(t,k,\eta), \quad \hat{p} = \hat{p}(\omega,k), \quad \hat{\rho} = \hat{\rho}(t,k).
\]

In order to prove our result in this paper, we have to introduce the hybrid analytic norm that is one of the cornerstones of our analysis, because they will connect well to both estimates in \(x\) on the force field and uniform estimates in \(v\). We give a description on the motion of charged particles in the following way:

\[
\frac{dx'}{dt} = v', \quad \frac{dv'}{dt} = \frac{q}{m} v' \times B_0, \quad (1.1)
\]

where \(B_0 = B_0 \hat{z}\).

We assume that \(x'(t,x,v) = (x_1, x_2, x_3, v'_1(t), v'_2(t), v'_3(t), v_3) = (v_\perp \cos \theta, v_\perp \sin \theta, v_3)\) at \(t = \tau\), then the solution of Eq.(1.1) is obtained as follows:

\[
v'_2(t) = v_\perp \cos(\theta + \Omega t), \quad v'_3(t) = v_\perp \sin(\theta + \Omega t), \quad v'_3 = v_3;
\]

\[
x'_1(t) = x_1 + \frac{v_\perp}{\Omega} [\sin(\theta + \Omega (t - \tau)) - \sin \theta],
\]

\[
x'_2(t) = x_2 - \frac{v_\perp}{\Omega} [\cos(\theta + \Omega (t - \tau)) - \cos \theta], \quad x'_3(t) = x_3 + v_3(t - \tau), \quad (1.2)
\]
where \( \Omega = \frac{qB}{m}, \) \( v_{\perp} = \sqrt{v_{1}^{2} + v_{2}^{2}}. \) From now on, without loss of generality, we assume \( m = q = 1. \) Also through simple computation, it is easy to get

\[
\begin{bmatrix}
v'_{1} \\
v'_{2} \\
v'_{3}
\end{bmatrix} = \begin{bmatrix}
\cos \Omega (t - \tau) & -\sin \Omega (t - \tau) & 0 \\
\sin \Omega (t - \tau) & \cos \Omega (t - \tau) & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
v_{1} \\
v_{2} \\
v_{3}
\end{bmatrix} \triangleq \mathcal{R}(t - \tau) v,
\]

(1.3)

\[
\begin{bmatrix}
x'_{1} \\
x'_{2} \\
x'_{3}
\end{bmatrix} = \begin{bmatrix}
x_{1} \\
x_{2} \\
x_{3}
\end{bmatrix} + \begin{bmatrix}
\frac{i}{\pi} \sin \Omega (t - \tau) & \frac{i}{\pi} \cos \Omega (t - \tau) - \frac{1}{\Omega} & 0 \\
0 & \frac{i}{\pi} \cos \Omega (t - \tau) + \frac{1}{\Omega} & \frac{i}{\pi} \sin \Omega (t - \tau) \\
0 & 0 & 1
\end{bmatrix} (t - \tau)
\]

\[
\triangleq \begin{bmatrix}
x_{1} \\
x_{2} \\
x_{3}
\end{bmatrix} + \mathcal{M}(t - \tau) \begin{bmatrix}
v_{1} \\
v_{2} \\
v_{3}
\end{bmatrix},
\]

(1.4)

and

\[
\nabla_{v'} x' = \left( \frac{\sin \Omega (t - \tau)}{\Omega}, \frac{\sin \Omega (t - \tau)}{\Omega}, (t - \tau) \right).
\]

(1.5)

From the above equality (1.5), we can observe clearly the connection and difference between Landau damping and cyclotron damping.

**Definition 1.1** From the system (1.1), we can define the corresponding transform

\[
S_{t, \tau}^{0}(x, v) \triangleq (x + \mathcal{M}(t - \tau) v, \mathcal{R}(t - \tau) v),
\]

where \( \mathcal{M}(t - \tau), \mathcal{R}(t - \tau) \) are defined in (1.3)-(1.4).

**Remark 1.2** From the definition 1.1, it is known that \( S_{t, \tau}^{0} \) satisfies

\[
S_{t_{2}, t_{3}}^{0} \circ S_{t_{1}, t_{2}}^{0} = S_{t_{1}, t_{3}}^{0}.
\]

To estimate solutions and trajectories of kinetic equations, maybe we have to work on the phase space \( T_{x}^{\mathbb{R}} \times R_{v}^{\mathbb{R}}. \) And we also use the following three parameters: \( \lambda \) (gliding analytic regularity), \( \mu \) (analytic regularity in \( x \)) and \( \tau \) (time-shift along the free transport semigroup). From Remark 1.2, we know that the linear Vlasov equation has the property of the free transport semigroup. This property is crucial to our analysis. In this paper, one of the cornerstones of our analysis is to compare the solution of the nonlinear case at time \( \tau \) with the solution of the linear case.

Now we start to introduce the very important tools in our paper. These are time-shift pure and hybrid analytic norms. They are similar with those in the paper [23] written by Mouhot and Villani.

We denote the notation \( S_{t, \tau}^{0}(x, v) \) that represents the third component of \( S_{t, \tau}^{0}(x, v), \) that is, \( S_{t, \tau}^{0}(x, v) = (0, 0, (S_{t, \tau}^{0}(x, v))_{3}). \)

**Definition 1.3** (Hybrid analytic norms)

\[
\| f \|_{C^{\lambda, \mu}} = \sum_{m, n \in \mathbb{N}^{3}} \frac{\lambda^{n} \mu^{m}}{n! m!} \| \nabla_{x}^{m} \nabla_{v}^{n} f \|_{L^{\infty}(T_{x}^{\mathbb{R}} \times R_{v}^{\mathbb{R}})}, \quad \| f \|_{F^{\lambda, \mu}} = \sum_{k \in Z^{3}} \int_{R^{3}} |\tilde{f}(k, \eta)| e^{2\pi \nu |\eta|} e^{2\pi \nu |k|} d\eta,
\]

\[
\| f \|_{Z^{\lambda, \mu}} = \sum_{l \in Z^{3}} \sum_{n \in \mathbb{N}^{3}} \frac{\lambda^{n}}{n!} e^{2\pi \nu |l|} \| \nabla_{x}^{l} f(l, v) \|_{L^{\infty}(R^{3})}.
\]

**Definition 1.4** (Time-shift pure and hybrid analytic norms) For any \( \lambda, \mu \geq 0, p \in [1, \infty], \) we define

\[
\| f \|_{C^{\lambda, \mu}} = \| f \circ S_{t, \tau}^{0}(x, v) \|_{C^{\lambda, \mu}} = \sum_{m, n \in \mathbb{N}^{3}} \frac{\lambda^{n} \mu^{m}}{n! m!} \| \nabla_{x}^{m} \nabla_{v}^{n} f \|_{L^{\infty}(T_{x}^{\mathbb{R}} \times R_{v}^{\mathbb{R}})},
\]

\[
\| f \|_{F^{\lambda, \mu}} = \| f \circ S_{t, \tau}^{0}(x, v) \|_{F^{\lambda, \mu}} = \sum_{k_{1} \in Z^{3}, k_{2} \in Z^{3}} \int_{R^{3}} |\tilde{f}(k, \eta)| e^{2\pi \nu |\eta|} e^{2\pi \nu |k_{1} + \eta|} e^{2\pi \nu |k|} d\eta,
\]

\[
\| f \|_{Z^{\lambda, \mu}} = \| f \circ S_{t, \tau}^{0}(x, v) \|_{Z^{\lambda, \mu}} = \sum_{l_{1} \in Z^{3}, l_{2} \in Z^{3}} \sum_{n \in \mathbb{N}^{3}} \frac{\lambda^{n}}{n!} e^{2\pi \nu |l_{1}|} \| \nabla_{x}^{l_{1}} \nabla_{v}^{l_{2}} f(l_{1}, v) \|_{L^{\infty}(R^{3})},
\]

\[
\| f \|_{Y^{\lambda, \mu}} = \| f \|_{Y^{\lambda, \mu}} = \sup_{k \in Z^{3}, \eta \in R^{3}} e^{2\pi \nu |\eta|} e^{2\pi \nu |k_{1} + \eta|} |\tilde{f}(k, \eta)|.
\]
From the above definitions, we can state some simple and important propositions and the related proofs can be found in [23], so we remove the proofs. From (1.5), we know that the damping occurs only in the $\dot{z}$ direction. Therefore, without loss of generality, we assume $(x, v) = (x_3, v_3) \in \mathbb{T} \times \mathbb{R}$.

**Proposition 1.5** For any $\tau \in \mathbb{R}, \lambda, \mu \geq 0$,

(i) if $f$ is a function only of $x$, then $\|f\|_{C^0_{\lambda+\mu}} = \|f\|_{C^0_{\lambda+\mu}}$, $\|f\|_{\mathbb{P}_{\lambda+\mu}} = \|f\|_{\mathbb{P}_{\lambda+\mu}}$;

(ii) if $f$ is a function only of $v$, then $\|f\|_{C^0_{\lambda+\mu}} = \|f\|_{C^0_{\lambda+\mu}}$, $\|f\|_{\mathbb{P}_{\lambda+\mu}} = \|f\|_{\mathbb{P}_{\lambda+\mu}}$;

(iii) for any $\lambda > 0$, then $\|f \circ (I + G)\|_{\mathbb{P}_{\lambda}} \leq \|f\|_{\mathbb{P}_{\lambda}}$.

Proof. Here we only give the proof of (v). By the invariance under the action of free transport, it is sufficient to do the proof for $t = 0$. Applying the Fourier transform formula, we have

$$\nabla^m_k (v f)(k, \eta) = \int_{\mathbb{R}} \partial_k f(k, \eta) (2i\pi \eta)^m e^{2i\pi \eta \cdot v} d\eta,$$

and therefore

$$\sum_{m \in \mathbb{N}_0} \lambda^m \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \partial_k f(k, \eta) (2i\pi \eta)^m e^{2i\pi \eta \cdot v} d\eta \right| \leq \sum_{m \in \mathbb{N}_0} \lambda^m \sup_{\eta \in \mathbb{R}} (2i\pi \eta)^m \partial_k f(k, \eta) \int_{\mathbb{R}} \left| e^{2i\pi \eta \cdot v} d\eta \right| dv,$$

$$\leq C(\lambda, \tilde{\lambda}) \sum_{m \in \mathbb{N}_0} \lambda^m \sup_{\eta \in \mathbb{R}} (2i\pi \eta)^m \tilde{f}(k, \eta) \leq C(\lambda, \tilde{\lambda}) \sum_{m \in \mathbb{N}_0} \lambda^m \int_{\mathbb{R}} \left| \nabla^m_k f(k, v) \right| dv,$$

where the last second inequality uses the property (iv), then we have

$$e^{2\pi i [k]} \sum_{m \in \mathbb{N}_0} \frac{\lambda^m}{m!} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \partial_k f(k, \eta) (2i\pi \eta)^m e^{2i\pi \eta \cdot v} d\eta \right| dv \leq e^{2\pi i [k]} \sum_{m \in \mathbb{N}_0} \frac{\lambda^m}{m!} \int_{\mathbb{R}} \left| \nabla^m_k f(k, v) \right| dv.$$

This establishes (v).

**Proposition 1.6** For any $X \in \{C, F, Z\}$ and any $t, \tau \in \mathbb{R}$,

$$\|f \circ S^0_t\|_{X_{\lambda+\mu}} = \|f\|_{X_{\lambda+\mu}}.$$

**Lemma 1.7** Let $\lambda, \mu \geq 0, t \in \mathbb{R}$, and consider two functions $F, G : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{T} \times \mathbb{R}$. Then there is $\varepsilon \in (0, \frac{1}{2})$ such that if $F, G$ satisfy

$$\|\nabla (F - Id)\|_{Z^0_{\lambda+\mu}} \leq \varepsilon,$$

where $\lambda = \lambda + 2\|F - G\|_{Z^0_{\lambda+\mu}}$, $\mu = \mu + 2(1 + |\tau|)|F - G|_{Z^0_{\lambda+\mu}}$, then $F$ is invertible and

$$\|F^{-1} - Id\|_{Z^0_{\lambda+\mu}} \leq 2\|F - G\|_{Z^0_{\lambda+\mu}}.$$

**Proposition 1.8** For any $\lambda, \mu \geq 0$ and any $p \in [1, \infty], \tau \in \mathbb{R}, \sigma \in \mathbb{R}, a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$, we have

$$\|f(x + bv) - F(x, v)\|_{Z^0_{\lambda+\mu}} \leq \|a - \frac{v}{2}\|_{Z^0_{\lambda+\mu}},$$

where $\alpha = \lambda |a| + |V|_{Z^0_{\lambda+\mu}}, \beta = \mu + \lambda |b + \tau - a| + \|X - \sigma V\|_{Z^0_{\lambda+\mu}}$.

**Lemma 1.9** Let $G = G(x, v)$ and $R = R(x, v)$ be valued in $\mathbb{R}$, and $\beta(x) = \int_{\mathbb{R}} (G \cdot R)(x', v') dv'$. Then for any $\lambda, \mu, t \geq 0$ and any $b > -1$, we have

$$\|\beta\|_{Z^0_{\lambda+\mu}} \leq 3\|G\|_{Z^0_{\lambda+\mu}} \|R\|_{Z^0_{\lambda+\mu}}.$$
2 Linear Cyclotron Damping

In this section, let us consider the following linear Vlasov equations:

\[
\begin{align*}
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \frac{q}{m} (v \times B_0) \cdot \nabla_v f &= -\frac{q}{m} (E + v \times B) \cdot \nabla_v f^0, \\
\frac{\partial B}{\partial t} &= \nabla_x \times E, \\
E &= W(x) \ast \rho(t, x), \\
\rho(t, x) &= \int_{\mathbb{R}^3} f(t, x, v) dv,
\end{align*}
\]  
\( (2.1) \)

where the distribution function \( f = f(t, x, v) : \mathbb{R}^+ \times \mathbb{T}^3 \times \mathbb{R}^3 \to \mathbb{R}, W(x) : \mathbb{T}^3 \to \mathbb{T}^3, B_0 \) is a constant magnetic field along the \( z \) direction, \( E = E(t, x) \) is the electric field, \( B_0 + B \) is the magnetic field.

Before proving Theorem 0.1, we give a key lemma.

**Lemma 2.1** Under the assumptions of Theorem 0.1, let \( \Phi(t, k) = e^{2\pi i \lambda_0 |k|} e^{2\pi i \lambda_1 |k|} e^{2\pi i \lambda_2 |k|} \tilde{\rho}(t, k), A(t, k) = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{-2\pi i x \cdot k} f_0(x'(0, x, v'), 0, x, v) dv' dx, \) where \( \lambda_0 > \lambda_1 > \lambda_2 > 0 \), \( \lambda_0 \) is defined in Theorem 0.1, we have

\[
||\Phi||_{L^\infty(dt)} \leq \left( 1 + \frac{C(k, W, \Omega)}{(\lambda_0 - \lambda_1)^2} \right) ||e^{\lambda_0 v_k^0} A||_{L^\infty(dt)},
\]

\( (2.2) \)

where \( \eta_1 = \frac{1}{\pi} (-k_2 \cos \Omega t + k_3 - k_1 \sin \Omega t), \eta_2 = \frac{1}{\pi} (-k_2 \sin \Omega t - k_1 + k_1 \cos \Omega t), v_k = |\eta_1| + |\eta_2| + |k_3| t. \)

**Proof.** First, we consider the case that \( \omega \neq 0, k_3 \neq 0, \)

\[
\tilde{\rho}(\omega, k) = \int_{\mathbb{R}^+} \left( \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{2\pi i \omega t} e^{-2\pi i x \cdot k} f(t, x, v) dv \right) dx dt
\]

\[
= \int_{\mathbb{R}^+} \left( \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{2\pi i \omega t} e^{-2\pi i x \cdot k} f_0(x'(0, x, v), 0, x, v) dv dx dt \right) - \frac{q}{m} \int_{\mathbb{R}^+} \left( \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{0}^{t} \int_{\mathbb{R}^3} e^{2\pi i \omega t} e^{-2\pi i x \cdot k} [(E + v' \cdot x, v) \times B] \cdot \nabla_v f_0^0(t, x, v) dx dv dt dt dt.
\]

\( (2.3) \)

Recall \( E(t, x) = W(x) \ast \rho(t, x), \partial_t B(t, x) = \nabla_x \times E(t, x), \) then taking the Fourier transform in the variables \( t, x, \)

\[
\tilde{E}(\omega, k) = \tilde{W}(k) \tilde{\rho}(\omega, k), \quad \omega \tilde{B}(\omega, k) = k \times \tilde{E}(\omega, k).
\]

Furthermore, we get

\[
v \times \tilde{B}(\omega, k) = \frac{1}{\omega} [v \times (k \times \tilde{E}(\omega, k))] 
\]

\[
= \frac{1}{\omega} \left( v_2(k_1 \tilde{W}_2 - k_2 \tilde{W}_1) - v_3 k_3 \tilde{W}_1, -v_1(k_1 \tilde{W}_2 - k_2 \tilde{W}_1) + v_3 k_3 \tilde{W}_2, v_1 k_3 \tilde{W}_1 + v_2 k_3 \tilde{W}_2 \right) \tilde{\rho}(\omega, k).
\]

\( (2.4) \)

Combining (1.3)-(1.4) and (2.3)-(2.4), and note that \( dv \to dv' \) preserves the measure, we can change between \( dv \) and \( dv' \) whenever we need, but in order to simply the notations, we don’t differ the notations \( dv \) and \( dv' \) in this paper, so we have

\[
\tilde{\rho}(\omega, k) = \int_{\mathbb{R}^+} \left( \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{2\pi i \omega t} e^{-2\pi i x \cdot k} f_0(x'(0, x, v), 0, x, v) dv dx dt \right) 
\]

\[
+ \frac{q}{m} \tilde{\rho}(\omega, k) \frac{1}{\omega} \int_{\mathbb{R}^+} \left( \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{2\pi i \omega t} e^{-2\pi i k_3 v_3 t} e^{-2\pi i k_3 v_3 t} e^{-2\pi i k_1 v_1 t} e^{-2\pi i k_1 v_1 t} (k_3 v_3) (k_1 v_1) (k_1 \tilde{W}_2 - k_2 \tilde{W}_1) \right) dv dt 
\]

\[
+ \frac{q}{m} \tilde{\rho}(\omega, k) \frac{1}{\omega} \int_{\mathbb{R}^+} \left( \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{2\pi i \omega t} e^{-2\pi i k_3 v_3 t} e^{-2\pi i k_3 v_3 t} e^{-2\pi i k_1 v_1 t} e^{-2\pi i k_1 v_1 t} (k_1 \tilde{W}_2 - k_2 \tilde{W}_1) \right) dv dt
\]

\[
\cdot (v_2 \partial_{v_1} f^0 - v_1 \partial_{v_1} f^0) dv dt - \frac{q}{m} \tilde{\rho}(\omega, k) \frac{1}{\omega} \int_{\mathbb{R}^+} \left( \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{2\pi i \omega t} e^{-2\pi i k_3 v_3 t} e^{-2\pi i k_3 v_3 t} e^{-2\pi i k_1 v_1 t} e^{-2\pi i k_1 v_1 t} (k_3 v_3) \right) dv dt 
\]

\[
\cdot (v_1 \tilde{W}_1 + v_2 \tilde{W}_2) k_3 \partial_{v_3} f^0 dv dt - \frac{q}{m} \tilde{\rho}(\omega, k) \frac{1}{\omega} \int_{\mathbb{R}^+} \left( \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{2\pi i \omega t} e^{-2\pi i k_3 v_3 t} e^{-2\pi i k_3 v_3 t} e^{-2\pi i k_1 v_1 t} e^{-2\pi i k_1 v_1 t} (k_3 v_3) \right) dv dt 
\]

\[
\cdot (\tilde{W}_1 \partial_{v_3} f^0 + \tilde{W}_2 \partial_{v_3} f^0) dv dt,
\]

\( (2.5) \)

where \( \eta_3 = \frac{1}{\pi} (-k_2 \cos \Omega t + k_3 - k_1 \sin \Omega t), \eta_2 = \frac{1}{\pi} (-k_2 \sin \Omega t - k_1 + k_1 \cos \Omega t). \)
Let

$$\hat{L}(\omega, k) = \frac{q}{m \omega} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} e^{2 \pi i t \omega} e^{-2 \pi i k_3 v_3 t} e^{-2 \pi i \eta_1 v'_1} (k_3 v_3) (\hat{W}_1 \partial_{v'_1} f^0 - \hat{W}_2 \partial_{v'_1} f^0) dv dt$$

$$- \frac{q}{m} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} e^{2 \pi i t \omega} e^{-2 \pi i k_3 v_3 t} (\hat{W}_1 \partial_{v'_1} f^0 + \hat{W}_2 \partial_{v'_1} f^0) \cdot e^{-2 \pi i \eta_1 v'_1} dv dt$$

$$- \frac{q}{m} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} e^{2 \pi i t \omega} e^{-2 \pi i k_3 v_3 t} e^{-2 \pi i \eta_1 v'_1} (v_1 \hat{W}_1 + v_2 \hat{W}_2) k_3 \partial_{v'_1} f^0 dv dt$$

$$+ \frac{q}{m} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} e^{2 \pi i t \omega} e^{-2 \pi i k_3 v_3 t} e^{-2 \pi i \eta_1 v'_1} (k_1 \hat{W}_2 - k_2 \hat{W}_1) (v_2 \partial_{v'_1} f^0 - v_1 \partial_{v'_1} f^0) dv dt,$$

(2.6)

hence

$$\hat{\rho}(\omega, k) = \hat{A}(\omega, k) + \hat{\rho}(\omega, k) \hat{L}(\omega, k).$$

(2.7)

Then taking the inverse Fourier transform in time $t$, we get $\hat{\rho}(t, k) = \hat{A}(t, k) + \hat{\rho}(t, k) * \hat{L}(t, k),$ and

$$e^{2 \pi i \eta_1 v_1} e^{2 \pi i \eta_2 v_2} e^{2 \pi i \eta_3 v_3} \hat{\rho}(t, k) = e^{2 \pi i \eta_1 v_1} e^{2 \pi i \eta_2 v_2} e^{2 \pi i \eta_3 v_3} \hat{A}(t, k) + e^{2 \pi i \eta_1 v_1} e^{2 \pi i \eta_2 v_2} e^{2 \pi i \eta_3 v_3} \hat{\rho}(t, k) * \hat{L}(t, k).$$

(2.8)

Let $\Phi(t, k) = e^{2 \pi i \eta_1 v_1} e^{2 \pi i \eta_2 v_2} e^{2 \pi i \eta_3 v_3} \hat{\rho}(t, k), \ A(t, k) = e^{2 \pi i \eta_1 v_1} e^{2 \pi i \eta_2 v_2} e^{2 \pi i \eta_3 v_3} \hat{A}(t, k), K^0(t, k) = e^{2 \pi i \eta_1 v_1} e^{2 \pi i \eta_2 v_2} e^{2 \pi i \eta_3 v_3} \hat{L}(t, k),$ then from (2.8), we have $\Phi(\omega, k) = \hat{A}(\omega, k) + \Phi(\omega, k) K^0(\omega, k).$

Then

$$\|\Phi(t, k)\|_{L^2(dt)} = \|\Phi(\omega, k)\|_{L^2} \leq \|\hat{A}(\omega, k)\|_{L^2} + \|\Phi(\omega, k)\|_{L^2}\|K^0(\omega, k)\|_{L^\infty}$$

$$\leq \|e^{2 \pi i \eta_1 v_1} \hat{A}(t, k)\|_{L^2} + \|e^{2 \pi i \eta_1 v_1} \hat{\rho}(t, k)\|_{L^2}\|K^0(\omega, k)\|_{L^\infty},$$

(2.9)

where $\nu_k = |\eta_1| + |\eta_2| + |k_3| t.$

Next we have to estimate $\|K^0(\omega, k)\|_{L^\infty}.$

Indeed,

$$\|K^0(\omega, k)\|_{L^\infty} \leq \sup_{\omega} \frac{q}{m} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} e^{2 \pi i t \omega} e^{-2 \pi i k_3 v_3 t} e^{2 \pi i \eta_1 v_1} e^{-2 \pi i \eta_1 v'_1} (k_3 v_3) (\hat{W}_1 \partial_{v'_1} f^0 - \hat{W}_2 \partial_{v'_1} f^0) dv dt$$

$$- \hat{W}_2 \partial_{v'_1} f^0 dv dt$$

$$+ \frac{q}{m} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} e^{2 \pi i t \omega} e^{-2 \pi i k_3 v_3 t} e^{2 \pi i \eta_1 v_1} \cdot (\hat{W}_1 \partial_{v'_1} f^0 + \hat{W}_2 \partial_{v'_1} f^0) \cdot e^{-2 \pi i \eta_1 v'_1}$$

$$e^{-2 \pi i \eta_1 v'_1} dv dt$$

$$+ \frac{q}{m} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} e^{2 \pi i t \omega} e^{2 \pi i \eta_1 v_1} e^{-2 \pi i k_3 v_3 t} e^{-2 \pi i \eta_1 v'_1} (v_1 \hat{W}_1 + v_2 \hat{W}_2) k_3$$

$$\partial_{v'_1} f^0 dv dt$$

$$+ \frac{q}{m} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} e^{2 \pi i t \omega} e^{-2 \pi i k_3 v_3 t} e^{-2 \pi i \eta_1 v'_1} e^{2 \pi i \eta_1 v'_1} \cdot (v_1 \hat{W}_1 - v_2 \hat{W}_1)$$

$$(v_2 \partial_{v'_1} f^0 - v_1 \partial_{v'_1} f^0) dv dt$$

$$= I + II + III + IV.$$

(2.10)

In fact, we only need to estimate one term of (2.10) because of similar processes of other terms. Without loss of generality, we give an estimate for $I$. In the same way, we only estimate one term of $I$, here we still denote $I$.

$$I = \sup_{\omega} \frac{q}{m} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} e^{2 \pi i t \omega} (2 \pi i \eta_1) e^{2 \pi i \eta_1 v_1} e^{2 \pi i \eta_1 v_1} e^{-2 \pi i k_3 v_3 t} e^{2 \pi i \eta_1 v'_1} (k_3 v_3) f^0(\eta_1, \eta_2, v_3) dv_3 dt$$

$$= \sup_{\omega} \frac{q}{m} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} e^{2 \pi i t \omega} (2 \pi i \eta_1) e^{2 \pi i \eta_1 v_1} e^{2 \pi i \eta_1 v_1} e^{-2 \pi i k_3 v_3 t} \cdot \sum_{n} \frac{2 \pi i \eta_1 k_3 |k_3|^n}{n!} v_3 f^0(\eta_1, \eta_2, v_3) dv_3 dt$$

$$= \sup_{\omega} \frac{q}{m} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} (2 \pi i \eta_1) e^{2 \pi i \eta_1 v_1} e^{2 \pi i \eta_1 v_1} e^{-2 \pi i k_3 v_3 t} (v_3 f^0(\eta_1, \eta_2, v_3) dv_3 dt$$

$$= \sup_{\omega} \frac{q}{m} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} (2 \pi i \eta_1) e^{2 \pi i \eta_1 v_1} e^{2 \pi i \eta_1 v_1} e^{-2 \pi i k_3 v_3 t} (v_3 f^0(\eta_1, \eta_2, v_3) dv_3 dt$$

$$\leq \frac{q}{m} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} e^{-\eta_0 v_3 t}.$$
where in the last inequality we use the facts that if \( v_3 = \frac{v_2}{k_3} \), then \( v_3 \gg v_{Te} \), and the assumption (i) and (iv).

Then there exists some constant \( 0 < \kappa < 1 \) such that \( \| \tilde{K}_0(\omega, k) \| \leq \kappa \).

In conclusion, we have \( \| e^{2\pi \lambda_0^{v_a} \hat{\rho}(t, k)} \|_{L^\infty} \leq \frac{\| e^{2\pi \lambda_0^{v_a} \hat{\rho}(t, k)} \|_{L^2(\mathbb{R}^3)}}{\kappa} \).

Then we get

\[
\| \Phi \|_{L^\infty(\mathbb{R}^3)} \leq \| e^{2\pi \lambda_0^{v_a} \hat{\rho}(t, k)} \|_{L^\infty(\mathbb{R}^3)} \leq \frac{\| e^{2\pi \lambda_0^{v_a} \hat{\rho}(t, k)} \|_{L^2(\mathbb{R}^3)}}{\kappa} \]

By simple computation, and using the conditions of Theorem 0.1,

\[
\| e^{2\pi \lambda_0^{v_a} \hat{\rho}(t, k)} \|_{L^2(\mathbb{R}^3)} \leq C(W, k, \Omega) \int_0^\infty e^{-4\pi |\lambda_0 - \lambda_0^v| k_3 t} C(W, k, \Omega) dt \leq \frac{C(W, k, \Omega)}{(\lambda_0 - \lambda_0^v)}.
\]

Now we estimate \( \| e^{2\pi \lambda_0^{v_a} \hat{\rho}(t, k)} \|_{L^\infty(\mathbb{R}^3)} \) as the above process,

\[
\| e^{2\pi \lambda_0^{v_a} \hat{\rho}(t, k)} \|_{L^\infty(\mathbb{R}^3)} \leq \frac{C(\Omega)}{(\lambda_0 - \lambda_0^v)^\frac{1}{2}} \| e^{2\pi \lambda_0^{v_a} \hat{\rho}(t, k)} \|_{L^2(\mathbb{R}^3)}.
\]

Now we consider \( k_3 = 0 \). In fact,

\[
\hat{\rho}(t, k_1, k_2, 0) = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{-2\pi i(x_1, x_2) \cdot (k_1, k_2)} f_0(x' \cdot (0, x, v), v' \cdot (0, x, v)) dv dx_1 dx_3
\]

\[
- \frac{q}{m} \int_0^t \int_{\mathbb{R}^3} e^{-2\pi i(v_3 - v_2 + v_1) \cdot (k_1, k_2)} (\hat{E}(\tau, k_1, k_2, 0) + v' \times \hat{B}(\tau, k_1, k_2, 0)) \cdot \nabla v' f_0'(\tau, x, v) dv' d\tau.
\]

Then assume \( \hat{W}(k_1, k_2, 0) = 0 \), we get \( \int_{\mathbb{R}^3} e^{-2\pi i(v_3 - v_2 + v_1) \cdot (k_1, k_2)} (\hat{E}(\tau, k_1, k_2, 0) + v' \times \hat{B}(\tau, k_1, k_2, 0)) \cdot \nabla v' f_0'(\tau, x, v) dv' d\tau = 0 \).

Finally, we get \( \hat{\rho}(t, k_1, k_2, 0) = \int_{\mathbb{R}^3} e^{-2\pi i(x_1, x_2) \cdot (k_1, k_2)} f_0 dx_1 dx_3 \cdot \nabla v' f_0' \).

Then by Lemma 2.1, we can study the asymptotic behavior of the electric field and the magnetic field \( E(t, x), B(t, x) \).

**Corollary 2.2** Under the assumptions of Theorem 0.1, and \( E(t, x), B(t, x) \) satisfy the Maxwell equations of (2.1), then we have

\[
| \hat{\tilde{E}}(t, k) | \leq e^{-2\pi \lambda_0^{v_a} \hat{\rho}(t, k, 1)} e^{-2\pi \lambda_0^{v_a} \hat{\rho}(t, k, 2)} e^{-2\pi \lambda_0^{v_a} \hat{\rho}(t, k, 3)}.
\]

**Proof.** Since \( \partial_t \tilde{B}(t, k) = k \times \hat{E}(t, k), \hat{E}(t, k) = \hat{W}(k) \hat{\rho}(t, k) = (\hat{W}_1(k), \hat{W}_2(k), 0) \hat{\rho}(t, k), \) then \( \partial_t \tilde{B}(t, k) = e^{-2\pi \lambda_0^{v_a} \hat{\rho}(t, k)} \cdot \nabla v' f_0' \).

Then, from \( \hat{B}(0, k) = 0 \), we have \( \| \tilde{B}(t, k) \| \leq e^{-2\pi \lambda_0^{v_a} \hat{\rho}(t, k)} e^{-2\pi \lambda_0^{v_a} \hat{\rho}(t, k)} e^{-2\pi \lambda_0^{v_a} \hat{\rho}(t, k)} \).

**Proof of Theorem 0.1.** From (2.1), we have

\[
f(t, x, v) = f_0(x' \cdot (0, x, v), v' \cdot (0, x, v)) - \frac{q}{m} \int_0^t [(E + v' \times B) \cdot \nabla v' f_0' ](\tau, x', (\tau, x, v), v' \cdot (\tau, x, v)) d\tau.
\]
Taking the Fourier-Laplace transform in variables $x, v, t$, we find

$$\tilde{f}(\omega, k, \eta) = \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{2\pi i t \omega} e^{-2\pi i x \cdot k} e^{-2\pi i v \cdot k} f_0(x'(0, x, v), v'(0, x, v)) dv dx dt$$

$$= \frac{q}{m} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{2\pi i t \omega} e^{-2\pi i x \cdot k} e^{-2\pi i v \cdot k} \int_0^t \left[ \partial \cdot \nabla' \times \left( f_0^0(t') \right) \right] (\tau, x'(\tau, x, v), v'(\tau, x, v)) d\tau dv dx dt$$

$$= \frac{q}{m} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{2\pi i t \omega} e^{-2\pi i x \cdot k} e^{-2\pi i v \cdot k} \int_0^t \left[ E \cdot \nabla' f_0^0(t) \right] (\tau, x'(\tau, x, v), v'(\tau, x, v)) d\tau dv dx dt$$

$$= I + II + III.$$  \quad (2.17)

$$I = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{2\pi i t \omega} e^{-2\pi i v \cdot k} f_0(x, v') \exp(-2\pi i [x + (\frac{1}{\Omega} (v_2' - v_2) - \frac{1}{\Omega} (x' - v_1)], v_3(t)] \cdot k) dv dx dt$$

$$= \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{2\pi i t \omega} e^{-2\pi i v_3(\eta + k t)} e^{-2\pi i v_2'(\eta + v_2)} e^{-2\pi i v_1'(\eta + v_1)} f_0(k, v') dv dt$$  \quad (2.18)

As the above process, we have

$$II = -\frac{q}{m} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{2\pi i t \omega} e^{-2\pi i x \cdot k} e^{-2\pi i v_3(\eta + k t)} e^{-2\pi i v_2'(\eta + v_2)} e^{-2\pi i v_1'(\eta + v_1)}$$

$$\cdot \int_0^t (\tilde{B}(\tau, k) \cdot (v_3 \partial_{v_3} f_0^0 - v_2' \partial_{v_2} f_0^0, v_2' \partial_{v_2} f_0^0 - v_3 \partial_{v_3} f_0^0, v_2' \partial_{v_2} f_0^0 - v_1' \partial_{v_1'} f_0^0)) d\tau dv dx dt$$

$$= -\frac{q}{m} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{2\pi i t \omega} \int_{\mathbb{R}^3} e^{-2\pi i x \cdot k} e^{-2\pi i v_3(\eta + k t)} e^{-2\pi i v_2'(\eta + v_2)} e^{-2\pi i v_1'(\eta + v_1)}$$

$$\cdot (v_3 \partial_{v_3} f_0 - v_2' \partial_{v_2} f_0^0, v_2' \partial_{v_2} f_0^0 - v_3 \partial_{v_3} f_0^0, v_2' \partial_{v_2} f_0^0 - v_1' \partial_{v_1'} f_0^0) dv dx dt$$

$$\cdot \int_0^t (\tilde{B}(\tau, k) \cdot (v_3 \partial_{v_3} f_0 - v_2' \partial_{v_2} f_0^0, v_2' \partial_{v_2} f_0^0 - v_3 \partial_{v_3} f_0^0, v_2' \partial_{v_2} f_0^0 - v_1' \partial_{v_1'} f_0^0)) d\tau dv dx dt.$$  \quad (2.19)

Since $\tilde{W}(k) = (\tilde{W}_1(k), \tilde{W}_2(k), 0)$, then $k \times \tilde{W}(k) = (-k_3 \tilde{W}_2, k_3 \tilde{W}_1, k_1 \tilde{W}_2 - k_2 \tilde{W}_1)$, and

$$II = \frac{q}{m} \rho(\omega, k) \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{2\pi i t \omega} \int_{\mathbb{R}^3} e^{-2\pi i x \cdot k} e^{-2\pi i v_3(\eta + k t)} e^{-2\pi i v_2'(\eta + v_2)} e^{-2\pi i v_1'(\eta + v_1)}$$

$$\cdot (k \times \tilde{W}(k) \cdot (v_3 \partial_{v_3} f_0 - v_2' \partial_{v_2} f_0^0, v_2' \partial_{v_2} f_0^0 - v_3 \partial_{v_3} f_0^0, v_2' \partial_{v_2} f_0^0 - v_1' \partial_{v_1'} f_0^0)) dv dx dt$$

$$= \frac{q}{m} \rho(\omega, k) \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{2\pi i t \omega} \int_{\mathbb{R}^3} e^{-2\pi i x \cdot k} e^{-2\pi i v_3(\eta + k t)} e^{-2\pi i v_2'(\eta + v_2)}$$

$$e^{-2\pi i v_1'(\eta + v_1)} (\tilde{W}_1 \partial_{v_1'} f_0^0 + \tilde{W}_2 \partial_{v_2'} f_0^0) dv dx dt.$$  \quad (2.20)

Similarly,

$$III = \frac{q}{m} \rho(\omega, k) \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{2\pi i t \omega} \int_{\mathbb{R}^3} e^{-2\pi i x \cdot k} e^{-2\pi i v_3(\eta + k t)} e^{-2\pi i v_2'(\eta + v_2)}$$

$$e^{-2\pi i v_1'(\eta + v_1)} (\tilde{W}_1 \partial_{v_1'} f_0^0 + \tilde{W}_2 \partial_{v_2'} f_0^0) dv dx dt.$$  \quad (2.21)

Therefore, from (2.17)-(2.21), we have

$$\omega \tilde{f}(\omega, k, v', \eta) = \omega (I + II + III)$$  \quad (2.22)

Furthermore,

$$\partial_t \tilde{f}(t, k, \eta) = \partial_t \int_{\mathbb{R}^3} e^{-2\pi i x \cdot k} e^{-2\pi i v_3(\eta + k t)} e^{-2\pi i v_2'(\eta + v_2)} e^{-2\pi i v_1'(\eta + v_1)} f_0(k, v') dv$$

$$- \frac{q}{m} \rho(t, k) \int_{\mathbb{R}^3} e^{-2\pi i x \cdot k} e^{-2\pi i v_3(\eta + k t)} e^{-2\pi i v_2'(\eta + v_2)} e^{-2\pi i v_1'(\eta + v_1)} \cdot (k_3 \tilde{W}_2(-v_3 \partial_{v_3} f_0^0$$

$$+ v_2' \partial_{v_2} f_0^0) + k_3 \tilde{W}_1(v_3 \partial_{v_3} f_0^0 - v_3 \partial_{v_3} f_0^0) + (k_1 \tilde{W}_2 - k_2 \tilde{W}_1)(v_2' \partial_{v_2} f_0^0 - v_1' \partial_{v_1'} f_0^0)) dv dx dt$$

$$- \frac{q}{m} \rho(t, k) \int_{\mathbb{R}^3} e^{-2\pi i x \cdot k} e^{-2\pi i v_3(\eta + k t)} e^{-2\pi i v_2'(\eta + v_2)} e^{-2\pi i v_1'(\eta + v_1)}$$

$$\cdot (k_1 \tilde{W}_1 \partial_{v_1'} f_0^0 + \tilde{W}_2 \partial_{v_2'} f_0^0) dv dx dt.$$  \quad (2.23)
Now we estimate IV, V, VI, respectively.

\[ IV = \partial_t \int_{\mathbb{R}^3} e^{-2\pi i (k_3 t + \eta_3)} e^{-2\pi i \nu'_1 (\eta_1 + \eta_3)} f_0(k, v') dv = \partial_t f_0(k, \eta_1 + \eta_2 + \eta_3, k_3 t + \eta_3) \]
\[ \leq C(\Omega, \kappa) e^{-2\pi \lambda_0 |\eta_1 + \eta_2 + \eta_3|} e^{-2\pi \lambda_0 |k_3 t + \eta_3|}, \]  
(2.24)

here we use the assumption that \(|\hat{f}_0(k, \eta)| \leq C_0 e^{-2\pi \lambda_0 |\eta_1|} e^{-2\pi \lambda_0 |\eta_2|} e^{-2\pi \lambda_0 |\eta_3|} \).

\[ V = -\frac{m}{\hbar} \hat{p}(t, k) \star \int_{\mathbb{R}^3} e^{-2\pi i (k_3 t + \eta_3)} v_3 e^{-2\pi i \nu'_1 (\eta_1 + \eta_3)} \left( k_3 \hat{W}_2 - v_3 \hat{\partial}_v f^0 \right) \hat{\partial}_v f^0 dv \]
\[ + v_3 \hat{\partial}_v f^0 + k_3 \hat{W}_1 (v_1' \hat{\partial}_v f^0 - v_3 \hat{\partial}_v f^0) + (k_1 \hat{W}_2 - k_2 \hat{W}_1) (v_1' \hat{\partial}_v f^0 - v_3 \hat{\partial}_v f^0) \hat{\partial}_v f^0 \]
\[ = -\frac{m}{\hbar} \hat{p}(t, k) \star \left[ k_3 \hat{W}_2 (v_1' \hat{\partial}_v f^0 - v_3 \hat{\partial}_v f^0) (\eta_1 + \eta_2, \eta_2 + \eta_3) \right] \]
\[ \leq C \int_0^t \int_{\mathbb{R}^3} e^{-2\pi \lambda_0 |\eta_1 + \eta_2 + \eta_3|} e^{-2\pi \lambda_0 |\eta_1 + \eta_2 + \eta_3|} e^{-2\pi \lambda_0 |k_3 (t - \tau) + \eta_3|} e^{-2\pi \lambda_0 |k_3 (t - \tau) + \eta_3|} d\tau \]
\[ \leq C \int_0^t \int_{\mathbb{R}^3} e^{-2\pi \lambda_0 |\eta_1 + \eta_2 + \eta_3|} e^{-2\pi \lambda_0 |\eta_1 + \eta_2 + \eta_3|} e^{-2\pi \lambda_0 |\eta_1 + \eta_2 + \eta_3|} e^{-2\pi \lambda_0 |k_3 (t - \tau) + \eta_3|} e^{-2\pi \lambda_0 |k_3 (t - \tau) + \eta_3|} d\tau \]
\[ \leq C \left( \int_0^t e^{-2\pi \lambda_0 |\eta_1 + \eta_2 + \eta_3|} e^{-2\pi \lambda_0 |\eta_1 + \eta_2 + \eta_3|} e^{-2\pi \lambda_0 |k_3 (t - \tau) + \eta_3|} e^{-2\pi \lambda_0 |k_3 (t - \tau) + \eta_3|} d\tau \right)^\frac{1}{2} e^{-2\pi \lambda_0 |k_3 t + \eta_3|} \]
\[ \leq C \left( \int_0^t e^{-2\pi \lambda_0 |\eta_1 + \eta_2 + \eta_3|} e^{-2\pi \lambda_0 |\eta_1 + \eta_2 + \eta_3|} e^{-2\pi \lambda_0 |k_3 (t - \tau) + \eta_3|} e^{-2\pi \lambda_0 |k_3 (t - \tau) + \eta_3|} d\tau \right)^\frac{1}{2} e^{-2\pi \lambda_0 |k_3 t + \eta_3|} , \]
(2.25)

where \( \lambda_0' = \lambda_0 - \frac{1}{2} (\lambda_0 - \lambda_0) \), tan \( \varphi_1 = \frac{k_1}{k_2} \), tan \( \varphi_2 = \frac{k_2}{k_1} \).

\[ VI = -\frac{m}{\hbar} \hat{p}(t, k) \partial_t \left( \hat{W}_1 \hat{\partial}_v f^0 + \hat{W}_2 \hat{\partial}_v f^0 \right) (\eta_1 + \eta_2 + \eta_3) \]
\[ \leq C \left( \int_0^t e^{-2\pi \lambda_0 |\eta_1 + \eta_2 + \eta_3|} e^{-2\pi \lambda_0 |\eta_1 + \eta_2 + \eta_3|} e^{-2\pi \lambda_0 |\eta_1 + \eta_2 + \eta_3|} e^{-2\pi \lambda_0 |\eta_1 + \eta_2 + \eta_3|} d\tau \right)^\frac{1}{2} e^{-2\pi \lambda_0 |k_3 t + \eta_3|} , \]
(2.26)

From (2.23)-(2.26) and Corollary 2.2, the results of Theorem 0.1 are obvious.

3 Nonlinear Cyclotron damping

We next give the proof of the main Theorem 0.3, stating the primary steps as propositions which are proved in subsections.
3.1 The improved Newton iteration

The first idea which may come to mind is a classical Newton iteration as done by Mouhot and Villani [23]: Let

\[ f^0 = f^0(v) \]

and

\[ f^n = f^0 + h^1 + \ldots + h^n, \]

where

\[
\begin{aligned}
\partial_t h^1 + v \cdot \nabla_v h^1 + v \times B_0 \cdot \nabla_v h^1 + (E[h^1] + v \times B[h^1]) \cdot \nabla_v f^0 &= 0, \\
h^1(0, x, v) &= f_0 - f^0,
\end{aligned}
\]

and now we consider the Vlasov equation in step \( n + 1, n \geq 1 \),

\[
\begin{aligned}
\partial_t h^{n+1} + v \cdot \nabla_v h^{n+1} + v \times B_0 \cdot \nabla_v h^{n+1} + E[f^n] \cdot \nabla_v h^{n+1} + v \times B[f^n] \cdot \nabla_v h^{n+1} \\
&= -E[h^{n+1}] \cdot \nabla_v f^n - v \times B[h^{n+1}] \cdot \nabla_v f^n - E[h^n] \cdot \nabla_v h^n - v \times B[h^n] \cdot \nabla_v h^n,
\end{aligned}
\]

the corresponding dynamical system is described by the equations: for any \((x, v) \in \mathbb{T}^3 \times \mathbb{R}^3\), let \((X^n_{\tau, \tau}, V^n_{\tau, \tau})\) as the solution of the following ordinary differential equations

\[
\begin{aligned}
\frac{d}{dt} X^n_{\tau, \tau}(x, v) &= V^n_{\tau, \tau}(x, v), \\
X^n_{\tau, \tau}(x, v) &= x, \\
\frac{d}{dt} V^n_{\tau, \tau}(x, v) &= V^n_{\tau, \tau}(x, v), \\
V^n_{\tau, \tau}(x, v) &= v.
\end{aligned}
\]

At the same time, we consider the corresponding linear dynamics system as follows,

\[
\begin{aligned}
\frac{d}{dt} X^0_{\tau, \tau}(x, v) &= V^0_{\tau, \tau}(x, v), \\
X^0_{\tau, \tau}(x, v) &= x, \\
\frac{d}{dt} V^0_{\tau, \tau}(x, v) &= V^0_{\tau, \tau}(x, v) \times B_0, \\
V^0_{\tau, \tau}(x, v) &= v.
\end{aligned}
\]

It is easy to check that

\[
\Omega^n_{\tau} - Id \triangleq (\delta X^n_{\tau, \tau}, \delta V^n_{\tau, \tau}) \circ (X^0_{\tau, \tau}, V^0_{\tau, \tau}) = (X^n_{\tau, \tau} \circ (X^0_{\tau, \tau}, V^0_{\tau, \tau}) - Id, V^n_{\tau, \tau} \circ (X^0_{\tau, \tau}, V^0_{\tau, \tau}) - Id).
\]

Therefore, in order to estimate \((X^n_{\tau, \tau} \circ (X^0_{\tau, \tau}, V^0_{\tau, \tau}) - Id, V^n_{\tau, \tau} \circ (X^0_{\tau, \tau}, V^0_{\tau, \tau}) - Id)\), we only need to study \((\delta X^n_{\tau, \tau}, \delta V^n_{\tau, \tau}) \circ (X^0_{\tau, \tau}, V^0_{\tau, \tau})\).

From Eqs.(3.3) and (3.4),

\[
\begin{aligned}
\frac{d}{dt} \delta V^n_{\tau, \tau}(x, v) &= \delta V^n_{\tau, \tau}(x, v), \\
\frac{d}{dt} \delta X^n_{\tau, \tau}(x, v) &= 0,
\end{aligned}
\]

and \(|\delta V^n_{\tau, \tau}(x, v)| = |v|\). Since \(v \in \mathbb{R}^3\) and \(B[f^n](t, X^n_{\tau, \tau}(x, v))\) independent of \(v\), there is almost no hope to get a "good" estimates of \(\Omega^n_{\tau} - Id\).

To circumvent these difficulties, we recall the basic physical Law on Lenz’s Law:

*The direction of current induced in a conductor by a changing magnetic field due to induction is such that it creates a magnetic field that opposes the change that produced it.*

According to the statement of Lenz’s Law and Maxwell equations, based on the approximation equations (3.2), it is easy to know that we only need to consider the following dynamical system

\[
\begin{aligned}
\frac{d}{dt} X^n_{\tau, \tau}(x, v) &= V^n_{\tau, \tau}(x, v), \\
\frac{d}{dt} V^n_{\tau, \tau}(x, v) &= V^n_{\tau, \tau}(x, v) \times B_0 + E[f^n](t, X^n_{\tau, \tau}(x, v)),
\end{aligned}
\]

then we get

\[
\begin{aligned}
\frac{d}{dt} \delta X^n_{\tau, \tau}(x, v) &= \delta V^n_{\tau, \tau}(x, v), \\
\delta X^n_{\tau, \tau}(x, v) &= 0, \\
\frac{d}{dt} \delta V^n_{\tau, \tau}(x, v) &= \delta V^n_{\tau, \tau}(x, v) \times B_0 + E[f^n](t, X^n_{\tau, \tau}(x, v)),
\end{aligned}
\]

and we write the approximation equations (3.2) into the following form,

\[
\begin{aligned}
\frac{d}{dt} h^{n+1} + v \cdot \nabla_v h^{n+1} + v \times B_0 \cdot \nabla_v h^{n+1} + E[f^n] \cdot \nabla_v h^{n+1} \\
&= -E[h^{n+1}] \cdot \nabla_v f^n - v \times B[h^{n+1}] \cdot \nabla_v f^n - E[h^n] \cdot \nabla_v h^n - v \times B[h^n] \cdot \nabla_v h^n, \\
h^{n+1}(0, x, v) &= 0.
\end{aligned}
\]
3.2 Main challenges

Integrating (3.8) in time and \( h^{n+1}(0, x, v) = 0 \), we get

\[
h^{n+1}(t, X^n_{t,0}(x, v), V^n_{t,0}(x, v)) = \int_0^t \Sigma^{n+1}(s, X^n_{s,0}(x, v), V^n_{s,0}(x, v)) \, ds,
\]

where

\[
\Sigma^{n+1}(t, x, v) = -E[h^{n+1}] \cdot \nabla_v f^n - v \times B[h^{n+1}] \cdot \nabla_v f^n - E[h^n] \cdot \nabla_v h^n - v \times B[h^n] \cdot \nabla_v h^n - v \times B[f^n] \cdot \nabla_v h^{n+1}.
\]

By the definition of \( (X^n_{t, \tau}(x, v), V^n_{t, \tau}(x, v)) \), we have

\[
h^{n+1}(t, x, v) = \int_0^t \Sigma^{n+1}(s, X^n_{s,t}(x, v), V^n_{s,t}(x, v)) \, ds
\]

Since the unknown \( h^{n+1} \) appears on both sides of (3.9), we hope to get a self-consistent estimate. For this, we have little choice but to integrate in \( v \) and get an integral equation on \( \rho[h^{n+1}] = \int_{\mathbb{R}^3} h^{n+1} \, dv \), namely

\[
\rho[h^{n+1}](t, x) = \int_0^t \int_{\mathbb{R}^3} \left( \Sigma^n \circ X^n_{s,t}(x, v), V^n_{s,t}(x, v) \right) \, dv \, ds
\]

where

\[
\Sigma^n = E[h^n] \cdot \nabla_v f^n - v \times B[h^n] \cdot \nabla_v f^n - E[h^n] \cdot \nabla_v h^n - v \times B[h^n] \cdot \nabla_v h^n - v \times B[f^n] \cdot \nabla_v h^{n+1}.
\]

It is obvious that Eq.(3.10) is not a closed equation, while \( \rho[h^{n+1}](t, x) \) satisfies a closed equation written in [23] with the electric field case. To obtain a self-consistent estimate, we go back the Vlasov equations (3.9), composing with \( (X^n_{0, \tau}(x, v), V^n_{0, \tau}(x, v)) \), where \( 0 \leq \tau \leq t \), this gives

\[
h^{n+1}(t, X^n_{t, \tau}(x, v), V^n_{t, \tau}(x, v)) = \int_0^t \Sigma^{n+1}(s, X^n_{s,t}(x, v), V^n_{s,t}(x, v)) \, ds.
\]

In order to achieve the goal, we have to combine Eq.(3.10) and Eq.(3.11) to form a iteration, then we obtain a self-consistent estimate on \( \rho[h^{n+1}] \). At the technical level, it is more difficult than that in Mouhot and Villani’s paper [23], since the new term \( v \times B[f^n] \cdot \nabla_v h^{n+1} \) in Eq.(3.9)-(3.10) brings different kinds of resonances (in terms of different norms).

3.3 Inductive hypothesis

For \( n = 1 \), from (3.1), it is known that (3.1) is a linear Vlasov equation. From section 2, we know that the conclusions of Theorem 0.3 hold.

Now for any \( i \leq n, i \in \mathbb{N}_0 \), we assume that the following estimates hold,

\[
\sup_{t \geq 0} \| \rho[h^i](t, \cdot) \|_{\mathcal{F}^{(\lambda_i - \beta_0 + \mu_1) \cdot \mu_i}},
\]

\[
\sup_{0 \leq \tau \leq t} \| h^i_\tau \circ \Omega^{i-1}_{\tau} \|_{\mathcal{F}^{(\lambda_i - \beta_0)(1+b) \cdot \mu_i}, 1} \leq \delta_i,
\]

\[
\sup_{0 \leq \tau \leq t} \| (h^i_\tau v) \circ \Omega^{i-1}_{\tau} \|_{\mathcal{F}^{(\lambda_i - \beta_0)(1+b) \cdot \mu_i}, 1} \leq \delta_i,
\]

then we have the following inequalities, denote \( (E^n) \):

\[
\sup_{t \geq 0} \| E[h^i](t, \cdot) \|_{\mathcal{F}^{(\lambda_i - \beta_0 + \mu_1) \cdot \mu_i}}, \quad \sup_{t \geq 0} \| B[h^i](t, \cdot) \|_{\mathcal{F}^{(\lambda_i - \beta_0 + \mu_1) \cdot \mu_i}}, \leq \delta_i,
\]

\[
\sup_{0 \leq \tau \leq t} \| \nabla_x ((h^i_\tau v) \circ \Omega^{i-1}_{\tau}) \|_{\mathcal{F}^{(\lambda_i - \beta_0)(1+b) \cdot \mu_i}, 1} \leq \delta_i,
\]

\[
\sup_{0 \leq \tau \leq t} \| \nabla_x ((h^i_\tau v) \circ \Omega^{i-1}_{\tau}) \|_{\mathcal{F}^{(\lambda_i - \beta_0)(1+b) \cdot \mu_i}, 1} \leq \delta_i.
\]
where the third term of the right-hand inequality uses (5.2) of Theorem 5.1 in the following section 5. First, we
need to show that the other equalities of (E^n) hold under Maxwell equations and (3.12), so we
need to show that the other equalities of (E^n) also hold, the related proofs are found in section 4.

3.4 Local time iteration

Before working out the core of the proof of Theorem 0.3, we shall show a short time estimate, which will
play a role as an initial data layer for the Newton scheme. The main tool in this section is given by the following
lemma, which is through the direct computation from the definition of the corresponding norms. Therefore, we
omit the proof.

Lemma 3.1 Let f be an analytic function, \( \lambda(t) = \lambda - Kt \) and \( \mu(t) = \mu - Kt \), \( K > 0 \), let \( T > 0 \) be so small
that \( \lambda(t) > \lambda'(t) > 0, \mu(t) > \mu'(t) > 0 \) for \( 0 \leq t \leq T \). Then for any \( \tau \in [0,T] \) and any \( p \geq 1 \),
\[
\frac{d^+}{dt} \| f \|_{\mathcal{L}^p(\tau)} \leq -K \frac{1}{2(1+\tau)} \| \nabla f \|_{\mathcal{L}^p(\tau)} - K \frac{1}{2(1+\tau)} \| v \nabla f \|_{\mathcal{L}^p(\tau)},
\]
where \( d^+ \) stands for the upper right derivative.

For \( n \geq 1 \), now let us solve
\[
\partial_t h^{n+1} + v \cdot \nabla h^{n+1} + v \times B_0 \cdot \nabla v h^{n+1} = \tilde{\Sigma}^{n+1},
\]
where
\[
\tilde{\Sigma}^{n+1} = -E[f^n] \cdot \nabla v h^{n+1} - E[h^{n+1}] \cdot \nabla f^n - v \times B[h^{n+1}] \cdot \nabla v h^{n} - v \times B[f^n] \cdot \nabla v h^{n+1}.
\]
Hence
\[
\| h^{n+1} \|_{\mathcal{L}^p(\tau)} \leq \int_0^T \| \tilde{\Sigma}^{n+1} \| \mathcal{S}^0_{(t-\tau)} \| z_{\lambda_n+1(t),\mu_{n+1}(t)} \| dt \leq \int_0^T \| \tilde{\Sigma}^{\tau} \| \| z_{\lambda_n+1(\tau),\mu_{n+1}(\tau)} \| d\tau,
\]
then by Lemma 3.1,
\[
\frac{d^+}{dt} \| h^{n+1} \|_{\mathcal{L}^p(\tau)} \leq -K \frac{1}{2} \| \nabla h^{n+1} \|_{\mathcal{L}^p(\tau)} - K \frac{1}{2} \| v \nabla h^{n+1} \|_{\mathcal{L}^p(\tau)} + \int_0^T \| E[f^n] \|_{\mathcal{L}^p(\tau)} \| \tilde{\Sigma} \|_{\mathcal{L}^p(\tau)} \| z_{\lambda_n+1(t),\mu_{n+1}(t)} \| dt,
\]
where the third term of the right-hand inequality uses (5.2) of Theorem 5.1 in the following section 5. First, we
easily get \( \| E[h^n] \|_{\mathcal{L}^p(\tau)} \leq C \| \nabla h^n \|_{\mathcal{L}^p(\tau)} \| z_{\lambda_n+1(t),\mu_{n+1}(t)} \| \), \( \| B[h^n] \|_{\mathcal{L}^p(\tau)} \leq C \| \nabla h^n \|_{\mathcal{L}^p(\tau)} \| z_{\lambda_n+1(t),\mu_{n+1}(t)} \| \). Moreover,
\[
\| \nabla v f^n \|_{\mathcal{L}^p(\tau)} \leq \sum_{i=1}^n \| \nabla v h^i \|_{\mathcal{L}^p(\tau)} \| z_{\lambda_n+1,\mu_{i+1}} \| \leq C \sum_{i=1}^n \| h^i \|_{\mathcal{L}^p(\tau)} \| z_{\lambda_n+1,\mu_{i+1}} \| \| \mu_{i+1} \| \| \lambda_{i+1} \| \| \mu_n \| \| \lambda_n \| \| \mu_{n+1} \| \| \mu_i \| \| \lambda_i \| \| \mu_{i+1} \|.n
We gather the above estimates, 

\[
\frac{d}{dt} \left\| h^{n+1} \right\|_{L^2(t, \tau; \cdot, \cdot)} \leq \left( C \sum_{i=1}^{n} \delta_i \min\{\lambda_i - \lambda_{n+1}, \mu_i - \mu_{n+1}\} \right) \left\| \nabla h^{n+1} \right\|_{L^2(t, \tau; \cdot, \cdot)} + \left( C \sum_{i=1}^{n} \delta_i \min\{\lambda_i - \lambda_{n+1}, \mu_i - \mu_{n+1}\} \right) \left\| \nabla h^{n+1} \right\|_{L^2(t, \tau; \cdot, \cdot)} + \left( C \sum_{i=1}^{n} \delta_i \min\{\lambda_i - \lambda_{n+1}, \mu_i - \mu_{n+1}\} \right) \left\| h^{n+1} \right\|_{L^2(t, \tau; \cdot, \cdot)}.
\]

We may choose 

\[
\delta_{n+1} = \frac{\delta_n^2}{\min\{\lambda_n - \lambda_{n+1}, \mu_n - \mu_{n+1}\}},
\]

if 

\[
C \max \left\{ \sum_{i=1}^{n} \delta_i \min\{\lambda_i - \lambda_{n+1}, \mu_i - \mu_{n+1}\}, \sum_{i=1}^{n} \delta_i \min\{\lambda_i - \lambda_{n+1}, \mu_i - \mu_{n+1}\} \right\} \leq \frac{K}{2}
\]

holds.

We choose \( \lambda_i - \lambda_{i+1} = \mu_i - \mu_{i+1} = \frac{A}{n} \), where \( A > 0 \) is arbitrarily small. Then for \( i \leq n, \lambda_i - \lambda_{n+1} \geq \frac{A}{\sqrt{n}} \), and \( \delta_{n+1} \leq \frac{\delta_n^2 n^2}{A} \). Next we need to check that \( \sum_{i=1}^{\infty} \delta_i n^2 \leq \infty \). In fact, we choose \( K \) large enough and \( T \) small enough such that \( \lambda_0 - KT > \lambda_i - \lambda_{n+1}, \mu_0 - KT > \mu_i - \mu_{n+1} \), and (3.13) hold.

If \( \delta_1 = \delta \), then \( \delta_n = n^{2}A^{n} \left( \frac{2}{A} \right)^{n-rac{1}{2}} \). This is an example of a small enough, by induction that \( \delta_n \leq z^{n} \), where \( z \) small enough and \( a \in (1, 2) \). We claim that the conclusion holds for \( n + 1 \). Indeed, \( \delta_{n+1} \leq \frac{\delta_n^2}{\lambda} \leq z^{n+1} \). If \( z \) is so small that \( z^{2-a} \leq \frac{1}{A} \) for all \( n \in N \), then \( \delta_{n+1} \leq \frac{\delta_n^2}{\lambda} \), this concludes the local-time argument.

**Remark 3.2** It is worthy to note that there are resonances to occur in local time which are caused by the action of the magnetic field, in detail, the resonances are from the term \( v \times B[f] \cdot \nabla_v h^{n+1} \). This phenomenon is very different from Landau damping in [23] in local time.

### 3.5 Global time iteration

Based on the estimates of local-time iteration, without loss of generality, sometimes we only consider the case \( \tau \geq \frac{1}{10a} \), where \( a \) is small enough.

First, we give deflection estimates to compare the free evolution with the true evolution from the particles trajectories.

**Proposition 3.3** Assume for any \( i \in N, 0 < i \leq n, \)

\[
\sup_{t \geq 0} \{ \| E[h^i](t, \cdot) \|_{L^2(\cdot, \cdot)} \} \leq \delta_i, \quad \sup_{t \geq 0} \{ \| B[h^i](t, \cdot) \|_{L^2(\cdot, \cdot)} \} \leq \delta_i.
\]

And there exist constants \( \lambda_i > B_0, \mu_i > 0 \) such that \( \lambda_0 - B_0 > \lambda_i - B_0 > \cdots > \lambda_i - B_0 > \lambda_{n+1} - B_0 > \cdots > \lambda_{n+1} - B_0 > \mu_i - B_0 > \cdots > \mu_i - B_0 > \mu_{n+1} - B_0 > \cdots > \mu_{n+1} - B_0 \).

Then we have 

\[
\| \delta X_{t, \tau}^{n+1} \|_{L^2(\cdot, \cdot)} \|_{L^2(\cdot, \cdot)} \leq C \sum_{i=1}^{n} \delta_i e^{-\pi (\lambda_i - \lambda_i') \tau} \min \left\{ \frac{(t - \tau)^2}{2 \pi (\lambda_i - \lambda_i')}, 1 \right\},
\]

\[
\| \delta V_{t, \tau}^{n+1} \|_{L^2(\cdot, \cdot)} \|_{L^2(\cdot, \cdot)} \leq C \sum_{i=1}^{n} \delta_i e^{-\pi (\lambda_i - \lambda_i') \tau} \min \left\{ \frac{(t - \tau)^2}{2 \pi (\lambda_i - \lambda_i')}, 1 \right\},
\]

for \( 0 < \tau < t, b = b(t, \tau) \) sufficiently small.

**Remark 3.4** From Proposition 3.3, it is easy to know that the gliding analytic regularity \( \lambda > B_0 \) in the cyclotron damping, comparing with \( \lambda > 0 \) in Landau damping, here \( B_0 \) is called cyclotron frequency. In other words, resonances in cyclotron damping occur at the cyclotron frequency, not zero frequency (Landau resonances occur at zero frequency).

**Proposition 3.5** Under the assumptions of Proposition 3.3, then 

\[
\left\| \nabla X_{t, \tau}^{n+1} - (I, 0) \right\|_{L^2(\cdot, \cdot)} \leq C_1^n, \quad \left\| \nabla V_{t, \tau}^{n+1} - (0, I) \right\|_{L^2(\cdot, \cdot)} \leq C_2^n,
\]

where \( C_1^n = C \sum_{i=1}^{n} \frac{e^{-\pi \lambda_i (\lambda_i' - \lambda_i') \tau}}{2 \pi (\lambda_i - \lambda_i')} \min \left\{ \frac{(t - \tau)^2}{2 \pi (\lambda_i - \lambda_i')}, 1 \right\}, C_2^n = C \sum_{i=1}^{n} \frac{e^{-\pi \lambda_i (\lambda_i' - \lambda_i') \tau}}{2 \pi (\lambda_i - \lambda_i')} \min \left\{ t - \tau, 1 \right\}. \)
Proposition 3.6 Under the assumptions of Proposition 3.3, then
\[ \left\| \Omega^I_{t, \tau} - \Omega^n_{t, \tau} \right\|_{\mathcal{V}((\lambda_n - \nu_0)(1-b), \nu_h)} < C_1^{i,n}, \]
\[ \left\| \Omega^V_{t, \tau} - \Omega^n_{V_{t, \tau}} \right\|_{\mathcal{V}((\lambda_n - \nu_0)(1-b), \nu_h)} < C_2^{i,n}, \]
where \( C_1^{i,n} = C \sum_{j=1}^n e^{-\tau_j(\lambda_j - \lambda_j')} \min \left\{ \frac{(t-\tau)^2}{2}, 1 \right\}, \)
\( C_2^{i,n} = C \sum_{j=1}^n e^{-\tau_j(\lambda_j - \lambda_j')} \min \left\{ t - \tau, 1 \right\}. \)

Remark 3.7 Note that \( C_1^{i,n}, C_2^{i,n} \) decay fast as \( \tau \to \infty, i \to \infty \), and uniformly in \( n \geq i \), since the sequence \( \{\delta_n\}_{n=1}^{\infty} \) has fast convergence. Hence, if \( r \in \mathbb{N} \) is given, we shall have
\[ C_1^{i,n} \leq \omega_{i,n}^{r,1} \quad \text{and} \quad C_2^{i,n} \leq \omega_{i,n}^{r,2}, \quad \text{all} \ r \geq 1, \]
(3.14)
with \( \omega_{i,n}^{r,1} = C \sum_{j=1}^n \frac{\delta_j}{2^{r(\lambda_j - \lambda_j')^2}} \frac{R_{j,n}}{(t-\tau)^2 + \delta_j} \) and \( \omega_{i,n}^{r,2} = C \sum_{j=1}^n \frac{\delta_j}{2^{r(\lambda_j - \lambda_j')^2}} \frac{R_{j,n}}{(t-\tau)^2 + \delta_j} \), for some absolute constant \( C_\omega \) depending only on \( r \).

Proposition 3.8 Under the assumptions of Proposition 3.3, then
\[ \left\| (\Omega^I_{t, \tau})^{-1} \circ \Omega^n_{t, \tau} - 1d \right\|_{\mathcal{V}((\lambda_n - \nu_0)(1-b), \nu_h)} < C_1^{i,n} + C_2^{i,n}. \]

To give a self-consistent estimate, we have to control each term of Eq.\( (3.10) \): I, II, III, IV, V. And the most difficult terms are I, II, V, respectively, because there is some resonance phenomena occurring in these terms that makes the propagated wave away from equilibrium.

Let us first consider the first two terms I, II, because they have the same proofs.

\[ I^{n+1,n}(t, x) + II^{n+1,n}(t, x) = \int_0^t \int_{\mathbb{R}^3} (\mathcal{C}^{n+1} \cdot G_{s,t}^n)(s, X_{s,t}^0(x, v), V_{s,t}^0(x, v)) \]
\[ - (F_{s,t}^{n+1} \cdot G_{s,t}^n)(s, X_{s,t}^0(x, v), V_{s,t}^0(x, v))dvds. \]  
(3.15)

To handle these terms, we start by introducing
\[ G_{s,t}^n = \nabla_v f^0 + \sum_{i=1}^n \nabla_v (h^i \circ \Omega_{s,t}^{-1}), \quad G_{s,t}^{n,v} = \nabla_v \times (f^0 v) + \sum_{i=1}^n \nabla_v \times ((h^i \circ \Omega_{s,t}^{-1}), \]
and the error terms \( \mathcal{R}_0, \tilde{\mathcal{R}}_0, \mathcal{R}_1, \tilde{\mathcal{R}}_1 \) are defined by
\[ \mathcal{R}_0 = \int_0^t \int_{\mathbb{R}^3} (B[h^{n+1}] \circ \Omega_{s,t}^n(x, v) - B[h^{n+1}]) \cdot G_{s,t}^{n,v}(s, X_{s,t}^0(x, v), V_{s,t}^0(x, v))dvds, \]  
(3.17)
\[ \tilde{\mathcal{R}}_0 = \int_0^t \int_{\mathbb{R}^3} (B[h^{n+1}] \circ \Omega_{s,t}^n(x, v) - B[h^{n+1}]) \cdot (G_{s,t}^n - G_{s,t}^{n,v})(s, X_{s,t}^0(x, v), V_{s,t}^0(x, v))dvds, \]  
(3.18)
\[ \mathcal{R}_1 = \int_0^t \int_{\mathbb{R}^3} (E[h^{n+1}] \circ \Omega_{s,t}^n(x, v) - E[h^{n+1}]) \cdot G_{s,t}^{n,v}(s, X_{s,t}^0(x, v), V_{s,t}^0(x, v))dvds, \]  
(3.19)
\[ \tilde{\mathcal{R}}_1 = \int_0^t \int_{\mathbb{R}^3} (E[h^{n+1}] \circ \Omega_{s,t}^n(x, v) - E[h^{n+1}]) \cdot (G_{s,t}^n - G_{s,t}^{n,v})(s, X_{s,t}^0(x, v), V_{s,t}^0(x, v))dvds, \]  
(3.20)
then we can decompose
\[ I^{n+1,n} = I^{n+1,n} + \mathcal{R}_1 + \tilde{\mathcal{R}}_1, \quad II^{n+1,n} = II^{n+1,n} + \mathcal{R}_0 + \tilde{\mathcal{R}}_0. \]

Because dealing with the first term I is the same to the second term II, to simply the proof, here we only prove the second term II. Now first we consider \( II^{n+1,n} \), which we decompose as
\[ II^{n+1,n} = II_0^{n+1,n} + \sum_{i=1}^n II_i^{n+1,n}, \]
where
\[ II_0^{n+1,n}(t, x) = \int_0^t \int_{\mathbb{R}^3} B[h^{n+1}](\tau, x', \tau, x, v', \tau, x, v) \cdot (\nabla_v \times (f^0 v)) v'dvdt, \]
\[ I_{t}^{n+1}(t, x) = \int_{0}^{t} \int_{\mathbb{R}^3} B[h^{n+1}](\tau, x') (v'(\tau, x, v), v'(\tau, x, v)) \cdot (\nabla'_v \times (h'_v v) \circ \Omega_{\tau}^{\lambda}) (\tau, x', \tau, x, v, v'(\tau, x, v)) d\tau d\tau. \]

Since both \( B[h^{n+1}](\tau, x', \tau, x, v, v'(\tau, x, v)) \) and \( (\nabla'_v \times (h'_v v) \circ \Omega_{\tau}^{\lambda}) (\tau, x', \tau, x, v, v'(\tau, x, v)) \) have the variable \( \tau \), then applying Fourier transform in \( \tau \), we get
\[
\|[B[h^{n+1}]] \cdot (\nabla'_v \times (h'_v v) \circ \Omega_{\tau}^{\lambda}) (x', \tau, x, v)\|^2(k) = \left| \sum_{l} B[h^{n+1}](k - l) (\nabla'_v \times (h'_v v) \circ \Omega_{\tau}^{\lambda}) (l, \tau) \right|.
\]

It is easy to see that Eq.(3.21) has two waves of distinct frequencies \( k - l, l \), which may interact. When interacting at certain particular times, the influence of the waves becomes very strong: this is known in plasma physics as plasma echo (we explain it in detail in next section), and can be thought of as a kind of resonance. It is the key point in our paper. Based on the iteration scheme different from that in Mouhot and Villani’s paper [23], we have to deal with the new term \( v \times B[h^{n+1}] \cdot \nabla h^{n+1} \) which also generates resonances.

**Proposition 3.9 (Main term I) Assume \( b(t, \tau, \Omega) \geq 0, \eta > 0 \) small. And there exist constants \( \lambda > B_{0}, \mu > 0 \) such that \( \lambda_{0} - B_{0} > \lambda_{i} - B_{0} > \lambda_{0} - B_{0} > \ldots > \lambda_{0} - B_{0} > \ldots > \lambda_{0} - B_{0} = \mu + 1 = \mu_{1} > \ldots > \mu_{i} > \mu_{i} > \ldots > \mu_{i} > \ldots > \mu_{i} \).

We have
\[
\|[I_{t}^{n+1}](\lambda_{0} - B_{0})^{i+1} + \nu_{i}\|_{\mathcal{F}_{\nu_{i}}} \leq C \int_{t_{0}}^{t} \sup_{k_{1}, k_{2} \in \mathbb{Z}} \|\nabla_{v}' \times ((h_{v}' v) \circ \Omega_{\nu_{i+1}}^{\lambda_{0} - B_{0}}) - (\nabla_{v}' \times ((h_{v}' v) \circ \Omega_{\nu_{i}}^{\lambda_{0} - B_{0}}))\|_{\mathcal{F}_{\nu_{i}}} \int_{t_{0}}^{t} \|B[h^{n+1}]\|_{\mathcal{F}_{\nu}} d\tau,
\]

where \( \nu = \max \left\{ \frac{1}{2}(\lambda - B_{0})\tau + \mu_{i}, \frac{1}{2}(\lambda - B_{0})\tau + \mu_{i} \right\} \).

**Lemma 10** We have \( \frac{1}{2} \|[B[h^{n+1}]\|_{\mathcal{F}_{\nu}} \leq C \sup_{0 \leq s \leq t} \|\rho[h^{n+1}]\|_{\mathcal{F}_{\nu_{i}}}, \quad \|h^n\|_{\mathcal{F}_{\nu_{i}}} \leq C \|[h^n]_{\mathcal{F}_{\nu_{i}}} \leq C \|[h^n]_{\mathcal{F}_{\nu_{i}}}

\]

**Corollary 3.11** From the above statement, we have
\[
\|[I_{t}^{n+1}](\lambda_{0} - B_{0})^{i+1} + \nu_{i}\|_{\mathcal{F}_{\nu_{i}}} \leq \int_{t_{0}}^{t} K_{0}^{i+1}(t, \tau) \delta_{i} \|[h^{n+1}]\|_{\mathcal{F}_{\nu_{i}}} d\tau,
\]

where \( K_{0}^{i+1}(t, \tau) = e^{-\pi(\lambda - \lambda_{0})\tau + \nu_{i}} d\tau \).

**Proposition 3.12 (Error term I)**
\[
\|[R_{0}](\cdot, \cdot)\|_{\mathcal{F}_{\nu_{i}}} \leq C \left( C_{0} + \sum_{i=1}^{n} \delta_{i} \right) \left( \sum_{i=1}^{n} \frac{\delta_{i}}{\lambda_{i}} \right) \int_{0}^{t} \|[h^{n+1}]\|_{\mathcal{F}_{\nu_{i}}} d\tau.
\]

**Proposition 3.13 (Error term II)**
\[
\|[\tilde{R}_{0}](\cdot, \cdot)\|_{\mathcal{F}_{\nu_{i}}} \leq \left( C_{0} + \sum_{i=1}^{n} \delta_{i} \right) \left( \sum_{i=1}^{n} \frac{\delta_{i}}{2\pi(\lambda_{i} - \lambda_{i})} \right) \int_{0}^{t} \|[h^{n+1}]\|_{\mathcal{F}_{\nu_{i}}} d\tau.
\]

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Proposition 3.14 (Main term V)

\[ \|V\|_{L^k(\lambda - n_0) + \mu'_n} \leq \int_0^t e^{-\pi(\lambda - n_0)(k(t-s)+1)} \left( \sum_{i=1}^n \delta_i \right) \|h^{n+1} \circ \Omega_{\tau,x}^{n+1}(\eta_1, \eta_2) ds. \]

Proposition 3.15 (Main term III)

\[ \sup_{0 \leq s \leq 1} \|h^{n+1} \circ \Omega_{\tau,x}^{n+1}(\lambda - n_0(1 + \frac{\delta}{k}), \mu'_n) \leq \delta_n + \left( \sum_{i=1}^n \delta_i \right) \sup_{0 \leq s \leq t} \|h^{n+1} \circ \Omega_{\tau,x}^{n+1}(\lambda - n_0(k) + \mu'_n). \]

3.6 The proof of main theorem

Step 2. Note if \( \varepsilon \in (0,1) \) is small enough, up to slightly lowering \( \lambda_1 \), we may choose all parameters in such a way that \( \lambda_1 - B_0, \lambda'_1 - B_0 \to \lambda_{\infty} - B_0 > \Delta - B_0 \) and \( \mu_k, \mu'_k \to \mu_\infty > \mu \) as \( k \to \infty \); then we pick up \( D > 0 \) such that \( \mu_\infty > \lambda_{\infty}(1 + D) \geq \mu_\infty > \mu \), and we let \( b(t) = \frac{D}{1 + t} \). From the iteration, we have, for all \( k \geq 2 \),

\[ \sup_{0 \leq s \leq t} \|h^k \circ \Omega_{\tau,x}^{k-1}(\lambda - n_0(1 + \frac{\delta}{k}), \mu'_n) \leq \delta_k, \]

where \( \sum_{k=2}^\infty \delta_k \leq C \delta \). Choosing \( \tau = t \) in (3.22) yields \( \sup_{0 \leq s \leq t} \|h^k \circ \Omega_{\tau,x}^{k-1}(\lambda - n_0(1 + \frac{\delta}{k}), \mu'_n) \leq \delta_k \). This implies that

\[ \sup_{t \geq 0} \|h^k \circ \Omega_{\tau,x}^{k-1}(\lambda - n_0(1 + \frac{\delta}{k}), \mu'_n) \leq C \delta. \]

Summing up over \( k \) yields for \( f = f^0 + \sum_{k=1}^\infty h^k \) the estimate

\[ \sup_{t \geq 0} \|f(t, \cdot) - f^0 \circ \Omega_{\tau,x}^{k-1}(\lambda - n_0, \mu'_n) \leq C \delta. \]

From (viii) of Proposition 1.5, we can deduce from (3.21) that

\[ \sup_{t \geq 0} \|f(t, \cdot) - f^0 \circ \Omega_{\tau,x}^{k-1}(\lambda - n_0, \mu'_n) \leq C \delta. \]

Moreover, \( \rho = \int f dv \) satisfies similarly \( \sup_{t \geq 0} \|\rho(t, \cdot) \|_{L^k(\lambda - n_0) + \mu'_n} \leq C \delta \). It follows that \( |\dot{\rho}(t,k)| \leq C \delta e^{-2\pi(\lambda - B_0) t + \mu k} \) for any \( k \neq 0 \). On the one hand, by Sobolev embedding, we deduce that for any \( r \in \mathbb{N} \),

\[ \|\rho(t, \cdot) - (\rho)_{T(t)}\|_{C_r(\mathbb{R}^d)} \leq C e^{2\pi(\lambda - B_0) t}; \]

on the other hand, multiplying \( \dot{\rho} \) by the Fourier transform of \( W \), and \( \partial_t B = \nabla \times E \), we see that the electric and magnetic fields \( E, B \) satisfy

\[ |\dot{E}(t,k)| \leq C \delta e^{-2\pi(\lambda - B_0) t} + |\dot{B}(t,k)| \leq C \delta e^{-2\pi(\lambda - B_0) t} + |\dot{B}(t,k)|, \]

for some \( \lambda_0 > \lambda > \lambda, \mu_0 > \mu' > \mu \).

Now, from (3.22), we have, for any \((k_3, \eta_3) \in \mathbb{Z} \times \mathbb{R} \) and any \( t \geq 0 \),

\[ \|f(t, k, \eta_k t, \eta_2 + k_3 t) - f^0(\eta) \| \leq C \delta e^{-2\pi(\lambda - B_0) t} e^{-2\pi \mu' |k|}, \]

this finishes the proof of Theorem 0.3.

4 Dynamical behavior of the particles’ trajectory

To prove Proposition 3.3, by the classical Picard iteration, we only need to consider the following equivalent equations

\[
\begin{align*}
\frac{\partial}{\partial t} \delta X_{t,r}^{n+1}(x, v) &= \delta V_{r, t}^{n+1}(x, v), \\
\frac{\partial}{\partial t} \delta V_{t,r}^{n+1}(x, v) &= \delta W_{t, r}^{n+1}(x, v) + E[f^n](t, \delta X_{t,r}^{n}(x, v) + X_{t,r}^{0}(x, v)), \\
\delta X_{t,r}^{n+1}(x, v) &= 0, \delta V_{t,r}^{n+1}(x, v) = 0.
\end{align*}
\]
It is easy to check that
\[ \Omega_{t,\tau}^{n+1} - Id \triangleq (\delta X_{t,\tau}^{n+1}, \delta V_{t,\tau}^{n+1}) \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0}) = (X_{t,\tau}^{n+1} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0}) - Id, V_{t,\tau}^{n+1} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0}) - Id). \]

Therefore, in order to estimate \((X_{t,\tau}^{n+1} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0}) - Id, V_{t,\tau}^{n+1} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0}) - Id\), we only need to study \((\delta X_{t,\tau}^{n+1}, \delta V_{t,\tau}^{n+1}) \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0})\). Now we give a detailed proof of Proposition 3.3.

Proof. If \(n = 0\), it is trivial that \(\delta V_{t,\tau}^{0}(x, v) = 0\), then Eqs.(4.1) reduces to the following equations
\[
\left\{ \begin{array}{l}
\frac{d}{dt}\delta X_{t,\tau}^{0}(x, v) = \delta V_{t,\tau}^{0}(x, v), \quad \frac{d}{dt}\delta V_{t,\tau}^{0}(x, v) = \delta V_{t,\tau}^{0}(x, v) \times B_0 + E[f^0](t, X_{0,\tau}^{0}(x, v)), \\
\delta X_{t,\tau}^{0}(x, v) = 0, \quad \delta V_{t,\tau}^{0}(x, v) = 0.
\end{array} \right.
\] (4.2)

Then we have
\[
\delta X_{t,\tau}^{n+1} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0})(x, v) = \int_{\tau}^{t} \delta V_{s,\tau}^{n+1} \circ (X_{s,\tau}^{0}, V_{s,\tau}^{0})(x, v)ds,
\]
\[
\delta V_{t,\tau}^{n+1} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0})(x, v) = \int_{\tau}^{t} e^{B_0(t-s)} E[f^0](s, X_{s,\tau}^{0}(x, v))ds.
\]

By the definition of \(E[f^0]\), we know that \(|E[f^0](s, \cdot)|_{L^{(\lambda'_0-\lambda_0)(1+b), \nu'_0}} = 0\), it is trivial that \(\|\delta V_{t,\tau}^{n+1} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0})\|_{L^{(\lambda'_0-\lambda_0)(1+b), \nu'_0}} \leq C_0 \delta^2 e^{-2\pi(\lambda_0-\lambda'_0)\tau} \min \left\{ \frac{(t-\tau)^2}{2}, \frac{1}{2\pi(\lambda_0-\lambda'_0)^2} \right\} \).

Similarly, \(\|\delta X_{t,\tau}^{n} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0})\|_{L^{(\lambda'_0-\lambda_0)(1+b), \nu'_0}} \leq C_0 \delta^2 e^{-2\pi(\lambda_0-\lambda'_0)\tau} \min \left\{ \frac{(t-\tau)^2}{2}, \frac{1}{2\pi(\lambda_0-\lambda'_0)^2} \right\} \).

Suppose for \(n > 0\), both
\[
\|\delta X_{t,\tau}^{n} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0})\|_{L^{(\lambda'_0-\lambda_0)(1+b), \nu'_0}} \leq C \sum_{i=1}^{n-1} \delta_i e^{-2\pi(\lambda_i-\lambda'_i)\tau} \min \left\{ \frac{(t-\tau)^2}{2}, \frac{1}{2\pi(\lambda_i-\lambda'_i)^2} \right\},
\]
and
\[
\|\delta V_{t,\tau}^{n} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0})\|_{L^{(\lambda'_0-\lambda_0)(1+b), \nu'_0}} \leq C \sum_{i=1}^{n-1} \delta_i e^{-2\pi(\lambda_i-\lambda'_i)\tau} \min \left\{ \frac{(t-\tau)^2}{2}, \frac{1}{2\pi(\lambda_i-\lambda'_i)^2} \right\}.
\]

Then for \(n + 1\), since \((\delta X_{t,\tau}^{n+1}, \delta V_{t,\tau}^{n+1})\) satisfy
\[
\left\{ \begin{array}{l}
\frac{d}{dt}\delta X_{t,\tau}^{n+1}(x, v) = \delta V_{t,\tau}^{n+1}(x, v), \\
\frac{d}{dt}\delta V_{t,\tau}^{n+1}(x, v) = \delta V_{t,\tau}^{n+1}(x, v) \times B_0 + E[f^n](t, X_{t,\tau}^{n}(x, v) + X_{0,\tau}^{0}(x, v)),
\end{array} \right. \] (4.3)

Then we have \(\delta V_{t,\tau}^{n+1} = \int_{\tau}^{t} e^{B_0(t-s)} E[f^n](s, \delta X_{s,\tau}^{n}(x, v) + X_{0,\tau}^{0}(x, v))ds\), and \(\delta V_{t,\tau}^{n+1} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0}) = \int_{\tau}^{t} e^{B_0(t-s)} E[f^n] \circ (\delta X_{s,\tau}^{n} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0}))(s, X_{s,\tau}^{0}(x, v))ds\).

Hence
\[
\|\delta V_{t,\tau}^{n+1} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0})\|_{L^{(\lambda'_0-\lambda_0)(1+b), \nu'_0}} \leq \int_{\tau}^{t} e^{B_0(t-s)} \|E[f^n] \circ (\delta X_{s,\tau}^{n} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0}))(s, X_{s,\tau}^{0}(x, v))\|_{L^{(\lambda'_0-\lambda_0)(1+b), \nu'_0}} ds \leq \int_{\tau}^{t} \|E[f^n](s, \cdot)\|_{L^{(\lambda'_0-\lambda_0)(1+b), \nu'_0}} ds \leq \sum_{i=1}^{n} \int_{\tau}^{t} \|E[f^n](s, \cdot)\|_{L^{(\lambda'_0-\lambda_0)(1+b), \nu'_0}} ds,
\]
where \(\nu'_n = (\lambda'_n - B_0)|s - b(t-s)| + \mu'_n + \|\delta X_{0,\tau}^{n} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0})\|_{L^{(\lambda'_0-\lambda_0)(1+b), \nu'_0}}\).

Note that \(\|\delta X_{s,\tau}^{n} \circ (X_{0,\tau}^{0}, V_{0,\tau}^{0})\|_{L^{(\lambda'_0-\lambda_0)(1+b), \nu'_0}} \leq C_n^0\), if \(s \geq \frac{B_0}{1+b}\), then \(\nu'_n \leq (\lambda'_n - B_0)|s + \mu'_n| + C_n^0 \leq (\lambda_0 - B_0)s + \mu_0 - (\lambda'_0 - \lambda'_n)s\) as soon as \(C_n^0 \leq \frac{B_0}{1+b}((1+b)s + C_1^0 \leq \lambda'_0 - B_0)D + \mu_0 - (\lambda'_0 - \lambda'_n)s\) as soon as \(C_n^0 \leq \frac{B_0}{1+b}((1+b)s + C_1^0 \leq \lambda'_0 - B_0)D + \mu_0 - (\lambda'_0 - \lambda'_n)s\) as soon as \(C_n^0 \leq \frac{B_0}{1+b}((1+b)s + C_1^0 \leq \lambda'_0 - B_0)D + \mu_0 - (\lambda'_0 - \lambda'_n)s\). In order to the feasibility of the conditions (I) and (II), we only need to check that the following assumption (I) holds
\[
2C_0 \left( \sum_{i=1}^{n} \frac{\delta_i}{(2\pi(\lambda_i-\lambda'_i))^2} \right) \leq \min \left\{ \frac{(\lambda_0 - B_0)|t - s|}{6}, \frac{\mu_0 - \mu'_n}{2} \right\},
\]
since \(C_1^0 \leq \omega_{1,2}^0 = 2C_0 \left( \sum_{i=1}^{n} \frac{\delta_i}{(2\pi(\lambda_i-\lambda'_i))^2} \right) \leq \min \left\{ \frac{(\lambda_0 - B_0)|t - s|}{6}, \frac{\mu_0 - \mu'_n}{2} \right\}.
\]
We can obtain the following conclusion,

$$\|\delta V^{n+1}_{t,\tau} \circ (X^0_{t,\tau}, V^0_{t,\tau})\| \leq C \sum_{i=1}^{n} \delta_i \int_{\tau}^{t} e^{-2\pi(l_i-l_{i-1})s} ds \leq C \sum_{i=1}^{n} \delta_i e^{-2\pi(l_i-l_{i-1})\tau} \min \left\{ \frac{(t-\tau)^2}{2}, \frac{1}{2\pi(l_i-l_{i-1})^2} \right\};$$

then we have

$$\|\delta X^{n}_{t,\tau} \circ (X^0_{t,\tau}, V^0_{t,\tau})\| \leq C \sum_{i=1}^{n} \delta_i \int_{\tau}^{t} e^{-2\pi(l_i-l_{i-1})s} ds \leq C \sum_{i=1}^{n} \delta_i e^{-2\pi(l_i-l_{i-1})\tau} \min \left\{ \frac{(t-\tau)^2}{2}, \frac{1}{2\pi(l_i-l_{i-1})^2} \right\}. $$

We finish the proof of Proposition 3.3.

In the following we estimate $$\nabla_{\tau} \Omega^0_{t,\tau} - Id.$$ In fact, we write $$(\Omega^0_{t,\tau} - Id)(x,v) = (\delta X^n_{t,\tau}, \delta V^n_{t,\tau}) \circ (X^0_{t,\tau}, V^0_{t,\tau}),$$ we get by differentiation $$\nabla_{\tau} \Omega^{n+1}_{t,\tau} -(I,0) = \nabla_{\tau} (\delta X^n_{t,\tau} \circ (X^0_{t,\tau}, V^0_{t,\tau}), \delta V^n_{t,\tau} \circ (X^0_{t,\tau}, V^0_{t,\tau})), \nabla_{\tau} \Omega^0_{t,\tau} - (0, I) = (\nabla_{\tau} + \mathcal{M}(t-\tau)\nabla_{x})(\delta X^n_{t,\tau} \circ (X^0_{t,\tau}, V^0_{t,\tau}), \delta V^n_{t,\tau} \circ (X^0_{t,\tau}, V^0_{t,\tau})).$$

Thus, the process in the proof of Proposition 3.3, we can obtain Proposition 3.5.

To establish a control of $$\Omega^0_{t,\tau} - \Omega^0_{t,\tau}$$ in norm $$\mathcal{Z}(\lambda'_0-B_0(1+b), \mu'_0),$$ we start again from the differential equation satisfied by $$\delta V^n_{t,\tau}$$ and $$\delta V^n_{t,\tau}$$ :

$$\frac{d}{dt}(\delta V^n_{t,\tau} - \delta V^n_{t,\tau})(x,v) = \delta V^n_{t,\tau}(x,v) - \delta V^n_{t,\tau}(x,v), \quad \delta V^n_{t,\tau}(x,v), \quad \delta V^n_{t,\tau}(x,v), \quad \delta V^n_{t,\tau}(x,v) = B_0 + E[f^{n+1}](t, \delta X^n_{t,\tau}(x,v) + X^0_{t,\tau}(x,v)) \quad \delta V^n_{t,\tau}(x,v), \quad \delta V^n_{t,\tau}(x,v) = B_0 + E[f^{n+1}](t, \delta X^n_{t,\tau}(x,v) + X^0_{t,\tau}(x,v)) \quad \delta V^n_{t,\tau}(x,v), \quad \delta V^n_{t,\tau}(x,v) = B_0 + E[f^{n+1}](t, \delta X^n_{t,\tau}(x,v) + X^0_{t,\tau}(x,v)) \quad \delta V^n_{t,\tau}(x,v), \quad \delta V^n_{t,\tau}(x,v) = B_0 + E[f^{n+1}](t, \delta X^n_{t,\tau}(x,v) + X^0_{t,\tau}(x,v)).$$

So from (4.5), $$\delta V^n_{t,\tau} - \delta V^n_{t,\tau}$$ satisfies the equation:

$$\frac{d}{dt}(\delta V^n_{t,\tau} - \delta V^n_{t,\tau})(x,v) = \delta V^n_{t,\tau}(x,v) - \delta V^n_{t,\tau}(x,v) = B_0 + E[f^{n+1}](t, \delta X^n_{t,\tau}(x,v) + X^0_{t,\tau}(x,v)) \quad \delta V^n_{t,\tau}(x,v), \quad \delta V^n_{t,\tau}(x,v) = B_0 + E[f^{n+1}](t, \delta X^n_{t,\tau}(x,v) + X^0_{t,\tau}(x,v)) \quad \delta V^n_{t,\tau}(x,v), \quad \delta V^n_{t,\tau}(x,v) = B_0 + E[f^{n+1}](t, \delta X^n_{t,\tau}(x,v) + X^0_{t,\tau}(x,v)).$$

Under the assumption (I), we can use the similar proof of Proposition 3.3 to finish Proposition 3.6.

Let $$\varepsilon$$ be the small constant appearing in Lemma 1.7. If

$$3C_1^0 + C_2^0 \leq \varepsilon, \quad \text{for all} \quad i \geq 1, \quad (I)$$

then $$\|\nabla^i \Omega^0_{t,\tau}\|_{\mathcal{Z}(\lambda'_0-B_0(1+b), \mu'_0)} \leq \varepsilon;$$ if in addition

$$2(1+\tau)(1+B)(3C_1^0 + C_2^0) \leq \varepsilon, \quad \text{for all} \quad i \in \{1, \ldots, n-1\} \quad \text{and all} \quad t \geq \tau,$$

then Lemma 1.7 and (4.6) yield Proposition 3.8.

As a corollary of Proposition 3.8 and Proposition 1.8, under the assumption (IV) :

$$4(1+\tau)(C_1^0 + C_2^0) \leq \min\{\lambda'_0 - \lambda'_0, \mu'_0 - \mu'_0\},$$

for all $$i \in \{1, \ldots, n\}$$ and all $$\tau \in [0, t],$$ we have

**Corollary 4.1** under the assumption (3.12), we have

$$\|h^i_{\tau} \circ \Omega^0_{t,\tau}\|_{\mathcal{Z}(\lambda'_0-B_0(1+b), \mu'_0)} \leq \delta_i, \quad \text{sup}_{0 \leq \tau \leq t} \|h^i_{\tau} \circ \Omega^0_{t,\tau}\|_{\mathcal{Z}(\lambda'_0-B_0(1+b), \mu'_0)} \leq \delta_i,$$

$$\sup_{0 \leq \tau \leq t} \|h^i_{\tau} \circ \Omega^0_{t,\tau}\|_{\mathcal{Z}(\lambda'_0-B_0(1+b), \mu'_0)} \leq \delta_i, \quad \text{sup}_{0 \leq \tau \leq t} \|h^i_{\tau} \circ \Omega^0_{t,\tau}\|_{\mathcal{Z}(\lambda'_0-B_0(1+b), \mu'_0)} \leq \delta_i,$$

$$\|h^i_{\tau} \circ \Omega^0_{t,\tau}\|_{\mathcal{Z}(\lambda'_0-B_0(1+b), \mu'_0)} \leq \delta_i, \quad \|h^i_{\tau} \circ \Omega^0_{t,\tau}\|_{\mathcal{Z}(\lambda'_0-B_0(1+b), \mu'_0)} \leq \delta_i,$$
5 The estimates of main terms

In order to estimate these terms $I, II, V$, we have to make good understanding of plasma echoes. Therefore, we will try to explain plasma echoes and give the relevant mathematical results.

5.1 Plasma echoes

This section is one of the key sections in our paper. And from Theorem 5.1 in this section, we can see that the influence of the magnetic field, which is regarded as an error term, is reasonable. First we plan to briefly explain plasma echoes though a simple example from [17], then we control plasma echoes (or obtain plasma echoes) in time-shift pure and hybrid analytic norms.

The unusual non-linear phenomena that results from the undamped oscillations of the distribution function $f$ satisfying the nonlinear Vlasov equation is called plasma echo. Let a perturbation be specified at the initial instant, such that the distribution function $\delta f$ is the perturbation of that of Maxwellian plasma $f^0(v) \sim \exp(-\alpha v^2), \alpha > 0$ is a constant and varies periodically in the $x-$direction. Without loss of generality, we assume $\delta f = A_1 f^0(v) \cos k_1 x$ at $t = 0$; in this section, $A_i$ denotes the amplitude and $k_i$ denotes the wave number for $i = 1, 2$. The perturbation of the density, i.e. the integral $\int \delta f dv$, varies in the same manner in the $x-$direction at $t = 0$. Subsequently, the perturbation of the distribution function varies at time $t$ according to $\delta f = A_1 f^0(v) \cos k_1 (x - vt)$, which corresponds to a free movement of each particle in the $x-$direction with its own speed $v$. But the density perturbation is damped (in a time $\sim \frac{1}{k_1 v}$), because $\int \delta f dv$ is made small by the speed-oscillatory factor $\cos k_1 (x - vt)$. The asymptotic form of the damping at times $t \gg \frac{1}{k_1 v}$ is given by

$$\delta \rho = \int \delta f dv \propto \exp(-\frac{1}{2} k_1^2 v^2 t^2),$$

(5.1)

where the proof of (5.1) can be found in [17].

Now let the distribution function be again modulates at a time $t = \tau \gg \frac{1}{k_2 v}$, with amplitude $A_2$ and a new wave number $k_2 > k_1$. The resulting density perturbation is damped in time $t \sim \frac{1}{k_2 v}$, but reappears at a time $\tau' = \frac{k_2 \tau}{k_2 - k_1}$, since the second modulation creates in the distribution function for $t = \tau$ a second-order term of the form

$$\delta f^{(2)} = A_1 A_2 f^0(v) \cos k_1 (x - v \tau) \cos k_2 x,$$

whose further development at $t > \tau$ changes it into

$$\delta f^{(2)} = A_1 A_2 f^0(v) \cos k_1 (x - v \tau) \cos k_2 [x - v(t - \tau)]$$

$$= A_1 A_2 f^0(v) [\cos((k_1 - k_2)(x - vt) + k_2 v \tau) + \cos((k_1 + k_2)(x - vt) + k_2 v \tau)].$$

We see that at $t = \tau'$ the oscillatory dependence of the first term on $v$ disappears, so that this term makes a finite contribution to the perturbation of the density with wave number $k_2 - k_1$. The resulting echo is then damped in a time $\sim \frac{1}{k_2 v}$ and the final stage of this damping follows a law similar to (5.1).

From the above physical point of view, under the assumption of the stability condition, we are discovering that, even in magnetic field case, echoes occurring at distinct frequencies are asymptotically well separated. In the following, through complicated computation, we give a detailed description by using mathematical tool. The same to Section 1, since resonances only occur in the $\hat{z}$ direction, in order to simplify the statement of the proof of the following theorem, we assume $(x, v) = (x_3, v_3) \in \mathbb{T} \times \mathbb{R}$

**Theorem 5.1** Let $\lambda, \bar{\lambda}, \mu, \bar{\mu}, \bar{\lambda}$ be such that $2\lambda \geq \bar{\lambda} > \lambda > 0, \bar{\mu} \geq \mu > \bar{\mu} > 0$, and let $b = b(t,s) > 0$, $R = R(t,s), G = G(t,s)$ and assume $\bar{G}(t,k_1,k_2,0,v) = 0$, we have if

$$\sigma(t,x,v) = \int_0^t R(s,x + M(t-s)v)G(s,x + M(t-s)v,v)ds,$$

$$\sigma_1(t,x) = \int_0^t \int_{\mathbb{R}} R(s,x + M(t-s)v)G(s,x + M(t-s)v,v)vdvds.$$

then

$$\|\sigma(t,\cdot)\|_{L_t^{1+\frac{1}{\bar{\lambda}}} L_v^{1+\frac{1}{\bar{\lambda}}}} \leq \int_0^t \sup_{k_1,k_2 \in \mathbb{Z}} e^{-2\pi(\bar{\mu} - \mu) |k_1|} e^{-2\pi(\bar{\lambda} - \lambda) |k_2|} \|R\|_{F_{\lambda+\bar{\mu}}} \|G\|_{F_{\lambda+\bar{\lambda}}} ds,$$

(5.2)

$$\|\sigma(t,\cdot)\|_{L_t^{1+\frac{1}{\bar{\lambda}}} L_v^{1+\frac{1}{\bar{\lambda}}}} \leq \int_0^t \sup_{k_1,k_2 \in \mathbb{Z}} e^{-2\pi(\bar{\mu} - \mu) |k_1|} e^{-2\pi(\bar{\lambda} - \lambda) |k_2|} \|R\|_{F_{\lambda+\bar{\mu}+\lambda(t-s)}} \|G\|_{F_{\lambda+\bar{\lambda}+\lambda(t-s)}} ds,$$

(5.3)
\[
\|\sigma_1(t, \cdot)\|_{\mathcal{F}^{\lambda+\mu}} \leq \int_0^t \sup_{k, l \in \mathbb{Z}_+} e^{-\pi(\mu - \mu')|l|} e^{-\pi(\lambda - \lambda')|k(t-s)|+|l|} e^{-2\pi[\mu' - \mu + \lambda(t-s)]|k-l|} \| R \|_{\mathcal{F}^{\lambda+\mu', \lambda(t-s)}} \| G \|_{\mathcal{Z}^{\lambda(1+b), \mu, 1}} ds,
\]
(5.4)
\[
\|\sigma_1(t, \cdot)\|_{\mathcal{F}^{\lambda+\mu}} \leq \int_0^t \sup_{k \neq l, k, l \in \mathbb{Z}_+} e^{-2\pi(\lambda - \lambda')|k-l|s} \| R \|_{\mathcal{F}^{\lambda+\mu', \lambda(t-s)}} \| G \|_{\mathcal{Z}^{\lambda(1+b), \mu, 1}} ds.
\]
(5.5)

**Proof.**

\[
\|\sigma(t, x, \cdot)\|_{\mathcal{Z}^{\lambda, \mu, 1}} \leq \int_0^t \| (RG) \circ S^0_{t-s}(s, \cdot) \|_{\mathcal{Z}^{\lambda, \mu, 1}} ds = \int_0^t \| (RG)(s, \cdot) \|_{\mathcal{Z}^{\lambda, \mu, 1}} ds.
\]

Let \( s' = s + b(t-s), b = \frac{D_s}{(1+t)} \), where some constant \( D > 0 \) small enough. Note that

\[
\| (RG)(s, \cdot) \|_{\mathcal{Z}^{\lambda, \mu, 1}} = \sum_{k \in \mathbb{Z}_+} \sum_{n \in \mathbb{N}_0} \lambda^n\frac{e^{2\pi n k}}{n!} \left\| \left[ \nabla_v + 2i\pi ks \right]^n (RG)(s, k, v) \right\|_{L^1_{dv}}
\]

\[
= \sum_{k \in \mathbb{Z}_+} \sum_{n \in \mathbb{N}_0} \lambda^n\frac{e^{2\pi n k}}{n!} \left\| e^{2i\pi n k(t-s)} \left[ \nabla_v + 2i\pi ks \right]^n (RG)(s, k, v) \right\|_{L^1_{dv}}
\]

\[
= \sum_{k \in \mathbb{Z}_+} \sum_{n \in \mathbb{N}_0} \lambda^n\frac{e^{2\pi n k}}{n!} \left\| \left[ 2i\pi k(t-s) + 2i\pi ks \right]^n e^{2i\pi n k(t-s)} (RG)(s, k, v) \right\|_{L^1_{dv}}
\]

\[
= \sum_{k \in \mathbb{Z}_+} \sum_{n \in \mathbb{N}_0} \lambda^n\frac{e^{2\pi n k}}{n!} \left\| 2i\pi k(t-s) + 2i\pi k(s' - s) + 2i\pi (k - (l-1)b)s + 2i\pi ls(1-b) \right\|^n e^{2i\pi n k(t-s)} (RG)(s, k, v)
\]

\[
\leq \sum_{k \in \mathbb{Z}_+} \sum_{n \in \mathbb{N}_0} \lambda^n\frac{e^{2\pi n k}}{n!} \left\| 2i\pi k(t-s) + 2i\pi ls(1-b) \right\|^{n-\gamma} e^{2i\pi n k(t-s)} (RG)(s, k, v)
\]

\[
\leq \sum_{k \in \mathbb{Z}_+} \sum_{n \in \mathbb{N}_0} \lambda^n\frac{e^{2\pi n k}}{n!} \left\| 2i\pi k(t-s) + 2i\pi ls(1-b) \right\|^{n-\gamma} e^{2i\pi n k(t-s)} (RG)(s, k, v)
\]

\[
\leq \sum_{k \in \mathbb{Z}_+} \sum_{n \in \mathbb{N}_0} \lambda^n\frac{e^{2\pi n k}}{n!} \left\| 2i\pi k(t-s) + 2i\pi ls(1-b) \right\|^{n-\gamma} e^{2i\pi n k(t-s)} (RG)(s, k, v)
\]

\[
\leq \sum_{k \in \mathbb{Z}_+} \sum_{n \in \mathbb{N}_0} \lambda^n\frac{e^{2\pi n k}}{n!} \left\| 2i\pi k(t-s) + 2i\pi ls(1-b) \right\|^{n-\gamma} e^{2i\pi n k(t-s)} (RG)(s, k, v)
\]

Now we will divide \( k, l \) into the following cases:

Case 1. \( \min\{|k|, |l|\} > k - l > 0 \). We still decompose this case into two steps.

Step 1. \( \min\{|k|, |l|\} > k - l > 0, k < 0 \).

\[
\sum_{\gamma=0}^{n} \lambda^n\frac{\gamma!}{\gamma!} \left[ 2i\pi k(s' - s) + 2i\pi (k - (l-1)b)s \right]^{\gamma}
\]

\[
= \sum_{\gamma=0}^{n} \lambda^n\frac{\gamma!}{\gamma!} \left[ 2\pi (k-l)s + 2i\pi kb(t-s) + 2\pi lbs \right]^{\gamma} = \sum_{\gamma=0}^{n} \lambda^n\frac{\gamma!}{\gamma!} \left[ 2\pi (k-l)s(1-b) + 2\pi kbt \right]^{\gamma}.
\]

If \( s \geq \frac{kD_s}{(1+t)(k-l)(1-b)} \), we have \( \sum_{\gamma=0}^{n} \lambda^n\frac{\gamma!}{\gamma!} \left[ 2i\pi k(s' - s) + 2i\pi (k - (l-1)b)s \right]^{\gamma} = \sum_{\gamma=0}^{n} \lambda^n\frac{\gamma!}{\gamma!} \left[ 2\pi (k-l)s(1-b) + 2\pi kbt \right]^{\gamma}. \)
Then
\[
\sum_{n \in \mathbb{N}_0} e^{2\pi \mu |k|} \sum_{l} |\hat{R}(s, k-l)| \sum_{\gamma=0}^{n} \frac{\lambda^\gamma}{\gamma!} \left[ 2i\pi k(s'-s) + 2i\pi (k-l(1-b))s \right] \gamma \frac{\lambda^{n-\gamma}(1-b)^{n-\gamma}}{(n-\gamma)!} \|\nabla_v + 2i\pi ls\|^{n-\gamma} \hat{G}(s, l, v) \|_{L^2_v}
\]
\leq \sum_{l} \sum_{n \in \mathbb{N}_0} e^{2\pi \mu |k|} e^{2\pi (k-l)s(1-b)+2\pi klb} |\hat{R}(s, k-l)| \cdot \frac{\lambda^{n}(1-b)^n}{n!} \|\nabla_v + 2i\pi ls\|^{n} \hat{G}(s, l, v) \|_{L^2_v}
\]
\leq \sum_{l} \sum_{n \in \mathbb{N}_0} e^{2\pi (\mu + \lambda b t)|k-l|s(1-b)} |\hat{R}(s, k-l)| \cdot e^{2\pi (\mu - \lambda b t)|l|} \frac{\lambda^{n}(1-b)^n}{n!} \|\nabla_v + 2i\pi ls\|^{n} \hat{G}(s, l, v) \|_{L^2_v}.
\]
If \( s \leq \frac{k-D}{(1+l)(k-l)-l} \leq t, \) for some constant \( 0 < \epsilon_0 < \frac{k}{l}, \)
\[
\sum_{\gamma=0}^{n} \frac{\lambda^\gamma}{\gamma!} \left[ 2i\pi k(s'-s) + 2i\pi (k-l(1-b))s \right] \gamma = \sum_{\gamma=0}^{n} \frac{\lambda^\gamma}{\gamma!} \left[ -2\pi (k-l)s(1-b) - 2\pi klb \right] \gamma
\]
\[
\leq \sum_{\gamma=0}^{n} \frac{\lambda^\gamma}{\gamma!} \left[ (2 + \epsilon_0)\pi klb + \epsilon_0 klb \right] \gamma = \sum_{\gamma=0}^{n} \frac{\lambda^\gamma}{\gamma!} \left[ (2 + \epsilon_0)\pi k \left| \frac{D_s}{l(1+t)} \right| l + \epsilon_0 klb \right] \gamma
\]
\[
= \sum_{\gamma=0}^{n} \frac{\lambda^\gamma}{\gamma!} \left[ (2 + \epsilon_0)\pi |k-l| \left| \frac{D_s}{l(1+t)} \right| l - \epsilon_0 |\epsilon_0| l \left| \frac{D_s}{l(l+1)} \right| l \right] \gamma
\]
\[
\leq \sum_{\gamma=0}^{n} \frac{\lambda^\gamma}{\gamma!} \left[ (2 + \epsilon_0)\pi |k-l| \left| \frac{B_s}{l(1-t)} \right| l - \epsilon_0 |\epsilon_0| l \left| \frac{D_s}{l(l+1)} \right| l \right] \gamma
\]
so we get
\[
\sum_{n \in \mathbb{N}_0} e^{2\pi \mu |k|} \sum_{l} |\hat{R}(s, k-l)| \sum_{\gamma=0}^{n} \frac{\lambda^\gamma}{\gamma!} \left[ 2i\pi k(s'-s) + 2i\pi (k-l(1-b))s \right] \gamma \frac{\lambda^{n-\gamma}(1-b)^{n-\gamma}}{(n-\gamma)!} \|\nabla_v + 2i\pi ls\|^{n-\gamma} \hat{G}(s, l, v) \|_{L^2_v}
\]
\[
\leq \sum_{l} \sum_{n \in \mathbb{N}_0} e^{2\pi \mu |k-l|} e^{2\pi \lambda |k-l|s(1-b)} |\hat{R}(s, k-l)| \cdot e^{2\pi \lambda t |l|} \frac{\lambda^{n}(1-b)^n}{n!} \|\nabla_v + 2i\pi ls\|^{n} \hat{G}(s, l, v) \|_{L^2_v}.
\]
Step 2. \( \min\{|\mu|,|l|\} > k-l > 0, k > 0 \). Indeed, we only consider \( k-l > 0, l > 0, \) since \( \min\{|\mu|,|l|\} > k-l > -\min\{|\mu|,|l|\} \).

It is easy to check that
\[
\sum_{\gamma=0}^{n} \frac{\lambda^\gamma}{\gamma!} \left[ 2i\pi k(s'-s) + 2i\pi (k-l(1-b))s \right] \gamma = \sum_{\gamma=0}^{n} \frac{\lambda^\gamma}{\gamma!} [2\pi (k-l)s' - 2\pi klb] \gamma.
\]
This can be reduced to case 1, here we omit the details. We can obtain
\[
\sum_{n \in \mathbb{N}_0} e^{2\pi \mu |k|} \sum_{l} |\hat{R}(s, k-l)| \sum_{\gamma=0}^{n} \frac{\lambda^\gamma}{\gamma!} \left[ 2i\pi k(s'-s) + 2i\pi (k-l(1-b))s \right] \gamma \frac{\lambda^{n-\gamma}(1-b)^{n-\gamma}}{(n-\gamma)!} \|\nabla_v + 2i\pi ls\|^{n-\gamma} \hat{G}(s, l, v) \|_{L^2_v}
\]
\[
\leq \sum_{l} \sum_{n \in \mathbb{N}_0} e^{2\pi \mu |k-l|} e^{2\pi \lambda |k-l|s(1-b)} |\hat{R}(s, k-l)| \cdot e^{2\pi \lambda t |l|} \frac{\lambda^{n}(1-b)^n}{n!} \|\nabla_v + 2i\pi ls\|^{n} \hat{G}(s, l, v) \|_{L^2_v}.
\]
Case 2. \( -\min\{|\mu|,|l|\} < k-l < 0 \). The method of this case is the same to Case 1.
\[
\sum_{n \in \mathbb{N}_0} e^{2\pi \mu |k|} \sum_{l} |\hat{R}(s, k-l)| \sum_{\gamma=0}^{n} \frac{\lambda^\gamma}{\gamma!} \left[ 2i\pi k(s'-s) + 2i\pi (k-l(1-b))s \right] \gamma \frac{\lambda^{n-\gamma}(1-b)^{n-\gamma}}{(n-\gamma)!} \|\nabla_v + 2i\pi ls\|^{n-\gamma} \hat{G}(s, l, v) \|_{L^2_v}
\]
\[
\leq \sum_{l} \sum_{n \in \mathbb{N}_0} e^{2\pi \mu |k-l|} e^{2\pi \lambda |k-l|s(1-b)} |\hat{R}(s, k-l)| \cdot e^{2\pi \lambda t |l|} \frac{\lambda^{n}(1-b)^n}{n!} \|\nabla_v + 2i\pi ls\|^{n} \hat{G}(s, l, v) \|_{L^2_v}.
\]
Case 3. \( k-1 > \min\{|\mu|,|l|\}, \) or \( k-l < -\min\{|\mu|,|l|\} \). We only need to consider one of two cases. Without loss of generality, we assume \( k-l > \min\{|\mu|,|l|\} \).
\[
\sum_{\gamma=0}^{n} \frac{\lambda^\gamma}{\gamma!} \left[ 2i\pi k(s'-s) + 2i\pi (k-l(1-b))s \right] \gamma = \sum_{\gamma=0}^{n} \frac{\lambda^\gamma}{\gamma!} [2\pi (k-l)s(1-b) + 2\pi klb] \gamma.
\]
In general, we assume that $\min\frac{|k|}{n}$.

Now we estimate the second inequality of Theorem 5.1

$$\sum_{n \in \mathbb{N}_0} e^{2\pi \mu[k]} |\hat{R}(s, k-l)| \sum_{\gamma=0}^{n} \frac{\lambda^n}{\gamma!} \left[ 2\pi k (s' - s) + 2i\pi (k-l)(1-b)s \right] \left[ \frac{\lambda^{n-\gamma}(1-b)^{n-\gamma}}{(n-\gamma)!} \right] \|\nabla v + 2i\pi ls \|^n \hat{G}(s, l, v) \|^L_{\Delta_k}.$$ 

$$\leq \sum_{l} e^{2\pi \mu[k]} |\hat{R}(s, k-l)| \sum_{\gamma=0}^{n} \frac{\lambda^n}{\gamma!} \left[ 2\pi k (s' - s) + 2i\pi (k-l)(1-b)s \right] \left[ \frac{\lambda^{n-\gamma}(1-b)^{n-\gamma}}{(n-\gamma)!} \right] \|\nabla v + 2i\pi ls \|^n \hat{G}(s, l, v) \|^L_{\Delta_k}.$$ 

Case 4. $t < \frac{|k|}{n} \frac{\lambda}{(1+t)(1-b)} \leq \frac{|k|}{n} \frac{\lambda}{1-t}$ with $\min \{|k|, |l|\} > k-l > -\min \{|k|, |l|\}, k \neq l$. Without loss of generality, we assume that $\min \{|k|, |l|\} > k-l > 0$. 

$$\sum_{n \in \mathbb{N}_0} \frac{\lambda^n}{\gamma!} \left[ 2i\pi k (s' - s) + 2i\pi (k-l)(1-b)s \right] \leq \sum_{\gamma=0}^{n} \frac{\lambda^n}{\gamma!} \left[ 2i\pi (k-l)s' - 2\pi lbt \right].$$ 

If $k-l > 0, l > 0$, 

$$\sum_{\gamma=0}^{n} \frac{\lambda^n}{\gamma!} \left[ 2i\pi k (s' - s) + 2i\pi (k-l)(1-b)s \right] \leq \sum_{\gamma=0}^{n} \frac{\lambda^n}{\gamma!} \left[ 2i\pi (k-l)s' - 2\pi lbt \right].$$ 

If $k-l > 0, l < 0$, this is equal to $k-l > 0, k < 0$, since $\min \{|k|, |l|\} > k-l > -\min \{|k|, |l|\}$.

$$\sum_{\gamma=0}^{n} \frac{\lambda^n}{\gamma!} \left[ 2i\pi k (s' - s) + 2i\pi (k-l)(1-b)s \right] \leq \sum_{\gamma=0}^{n} \frac{\lambda^n}{\gamma!} \left[ 2i\pi (k-l)s' - 2\pi lbt \right].$$ 

In summary, 

$$\sum_{k,l \in \mathbb{Z}, n \in \mathbb{N}_0} e^{2\pi \mu[k]} |\hat{R}(s, k-l)| \sum_{\gamma=0}^{n} \frac{\lambda^n}{\gamma!} \left[ 2i\pi k (s' - s) + 2i\pi (k-l)(1-b)s \right] \left[ \frac{\lambda^{n-\gamma}(1-b)^{n-\gamma}}{(n-\gamma)!} \right] \|\nabla v + 2i\pi ls \|^n \hat{G}(s, l, v) \|^L_{\Delta_k} \leq \sup_{k \neq l, k, l \in \mathbb{Z}_0} e^{-2\pi (\mu - \mu)|k-l|} e^{-2\pi (\lambda - \lambda)|k-l|} R \|\hat{G} \|_{\mathbb{L}^{\lambda-\mu, 1}}.$$ 

then 

$$\|\sigma(t, \cdot)\|_{\mathbb{L}^{\mu, 1}} \leq \int_0^t \sup_{k \neq l, k, l \in \mathbb{Z}_0} e^{-2\pi (\mu - \mu)|k-l|} e^{-2\pi (\lambda - \lambda)|k-l|} R \|\hat{G} \|_{\mathbb{L}^{\lambda-\mu, 1}} ds.$$ 

Now we estimate the second inequality of Theorem 5.1

$$\|\sigma(t, x, v)\|_{\mathbb{L}^{\mu, 1}} \leq \int_0^t \|(RG) \circ S_{s-t}(s, \cdot)\|_{\mathbb{L}^{\mu, 1}} ds = \int_0^t \|(RG)(s, \cdot)\|_{\mathbb{L}^{\mu, 1}} ds.$$ 

Note that 

$$\|(RG)(s, \cdot)\|_{\mathbb{L}^{\mu, 1}} = \sum_{k \in \mathbb{Z}, n \in \mathbb{N}_0} \frac{\lambda^n}{n!} e^{2\pi \mu[k]} \|\nabla v + 2i\pi ks \|^n \hat{G}(s, k, v) \|^L_{\Delta_k} \leq \sum_{k \in \mathbb{Z}, n \in \mathbb{N}_0} \frac{\lambda^n}{n!} e^{2\pi \mu[k]} \|\nabla v + 2i\pi ks \|^n \hat{G}(s, k, v) \|^L_{\Delta_k}.$$ 

$$= \sum_{k \in \mathbb{Z}, n \in \mathbb{N}_0} \frac{\lambda^n}{n!} e^{2\pi \mu[k]} \|\nabla v + 2i\pi ks \|^n \hat{G}(s, k, v) \|^L_{\Delta_k}.$$
The proof of the inequality (5.3) is as follows, here we sketch the main steps. We let \( s' = s - b(t-s) \) and write
\[
e^{2\pi i(\lambda t + \mu t)} |k| \leq e^{-2\pi i(\bar{\mu} - \mu t)} |k| e^{-2\pi i(\lambda x - s')} |k| e^{-2\pi i(\mu t - \lambda x + s')} |k| e^{-2\pi i(\lambda t - \lambda x + s')} |k|
\]
then we can deduce that
\[
\| |\sigma(t,-)\|_{\mathbb{Z}^{n+1}} \leq \int_0^t \sup_{k,l \in \mathbb{Z}^n} e^{-2\pi i(\bar{\mu} - \mu t)} |k| e^{-2\pi i(\lambda x - s')} |k| e^{-2\pi i(\mu t - \lambda x + s')} |k| e^{-2\pi i(\lambda t - \lambda x + s')} |k| |l| ds.
\]
In the following we estimate the norm \( \mathcal{F}^{\lambda t + \mu t} \) of the function \( \sigma_1(t,x) \),
\[
\sigma_1(t,x) = \int_0^t \int_{\mathbb{R}} R(s,x + M(t-s)v) G(s,x + M(t-s)v,v) dv ds.
\]
We write
\[
\hat{\sigma}_1(t,k) = \int_0^t \sum_{l \in \mathbb{Z}^n} \int_{\mathbb{R}} \hat{R}(s,k-l) \hat{G}(s,l,v) e^{2\pi i v \cdot k(t-s)} dv ds,
\]
\[
|\hat{\sigma}_1(t,k)| \leq \int_0^t \left( \sum_{l \in \mathbb{Z}^n} \int_{\mathbb{R}} |\hat{G}(s,l,v) e^{2\pi i v \cdot k(t-s)}| dv \right) |\hat{R}(s,k-l)| ds.
\]
Next,
\[
e^{2\pi i(\lambda t + \mu t)} |k| \leq e^{2\pi i(\lambda t + \mu t)} |k| e^{-2\pi i(\lambda x - s')} |k| e^{-2\pi i(\mu t - \lambda x + s')} |k| e^{-2\pi i(\lambda t - \lambda x + s')} |k| e^{-2\pi i(\lambda t - \lambda x + s')} |k| e^{-2\pi i(\lambda t - \lambda x + s')} |k| |l| ds.
\]
Hence
\[
\| \sigma_1(t,-) \|_{\mathcal{F}^{\lambda t + \mu t}} \leq \int_0^t \sup_{k \neq l, k,l \in \mathbb{Z}^n} e^{-2\pi i(\bar{\mu} - \mu t)} |k| e^{-2\pi i(\lambda x - s')} |k| e^{-2\pi i(\mu t - \lambda x + s')} |k| e^{-2\pi i(\lambda t - \lambda x + s')} |k| |l| ds,
\]
where \( \bar{\mu} = \mu + \lambda b(t-s) \).

### 5.2 Estimates of main terms

In the following we estimate \( \Pi^{n+1,1}_t(t,x) \). Note that their zero modes vanish. For any \( n \geq i \geq 1 \),
\[
\Pi^{n+1,1}_t(t,k) = \int_0^t \int_{\mathbb{R}^3} e^{-2\pi i k \cdot x} \left( B[h^{n+1}] \cdot ((\nabla_q \times ((h_q^i \circ \Omega_{1,q}^{-1}) \right) \right) (\tau, x', (\tau, x, v), v' (\tau, x, v)) d\tau dx dv.
\]
\[
|\Pi^{n+1,1}_t(t,k)| \leq \int_0^t \left( \sum_{l \in \mathbb{Z}^n} \left| \int_{\mathbb{R}^3} e^{-2\pi i k \cdot \cdot (\nabla_q \times ((h_q^i \circ \Omega_{1,q}^{-1}) d\tau
\right),
\]
\[
\cdot (\nabla_q \times ((h_q^i \circ \Omega_{1,q}^{-1})) (\tau, l, v, v') d\tau) |B[h^{n+1}](\tau, k-l)|) d\tau = \int_0^t \left( \sum_{l \in \mathbb{Z}^n} \int_{\mathbb{R}^3} e^{-2\pi i k \cdot \cdot (\nabla_q \times ((h_q^i \circ \Omega_{1,q}^{-1}) (\tau, l, v, v') d\tau) d\tau \cdot (\nabla_q \times ((h_q^i \circ \Omega_{1,q}^{-1})) (\tau, l, v, v') d\tau)
\]
\[
(5.6)
\]
From (5.3) of Theorem 5.1 and (5.6), we can get Proposition 3.6.
To finish Propositions 3.11-3.12, we shall again use the Vlasov equation. We rewrite it as
\[
h^{n+1}(t, X^n_{\tau,q}, (x, v), V^n_{\tau,q}(x, v)) = \int_0^t \Sigma^{n+1}(s, X^n_{\tau,q}(x, v), V^n_{\tau,q}(x, v)) ds.
\]
Then we get
\[ \| H^{n+1}(t, X^n_{t+1}(x, v), V^n_{t+1}(x, v)) \|_{Z_{t+1}^{(\lambda'_n - \lambda_{n0})(1+b), \mu'_n}} (\eta_k_1, \eta_k_2) \]
\[ \leq \int_0^t \| \Sigma^{n+1}(s, X^n_{t+1}(x, v), V^n_{t+1}(x, v)) \|_{Z_{t+1}^{(\lambda'_n - \lambda_{n0})(1+b), \mu'_n}} (\eta_k_1, \eta_k_2) ds \]
\[ = \int_0^t \| \Sigma^{n+1}(s, \Omega^n_{t+1}(x, v)) \|_{Z_{t+1}^{(\lambda'_n - \lambda_{n0})(1+b), \mu'_n}} (\eta_k_1, \eta_k_2) ds \]
\[ \leq \int_0^t \sup_{k, l \in \mathbb{Z}_2} e^{-\pi (\mu'_n - \mu_k)(l_1 + \kappa)(s + \frac{\mu'_n}{\mu_k})} e^{-\pi (\mu'_n - \mu_k)(k(t-s) + l_1)} e^{-2\pi (\mu'_n - \mu_k)(s - \lambda'_n) b(t-s)(t-s)} ds \]
\[ \leq \int_0^t \sup_{k, l \in \mathbb{Z}_2} e^{-\pi (\mu'_n - \mu_k)(l_1 + \kappa)(s + \frac{\mu'_n}{\mu_k})} e^{-\pi (\mu'_n - \mu_k)(k(t-s) + l_1)} e^{-2\pi (\mu'_n - \mu_k)(s - \lambda'_n) b(t-s)(t-s)} ds \]
\[ \leq C \left( C' + \sum_{i=1}^n \delta_i \right) \sup_{0 \leq s \leq t} \| h^{n+1} \|_{Z_{t+1}^{(\lambda'_n - \lambda_{n0})(1+b), \mu'_n}} ds + \frac{\delta^2_n}{(\lambda_n - \lambda'_n)^2} \]
\[ \leq C \left( C' + \sum_{i=1}^n \delta_i \right) \sup_{0 \leq s \leq t} \| h^{n+1} \|_{Z_{t+1}^{(\lambda'_n - \lambda_{n0})(1+b), \mu'_n}} + \frac{\delta^2_n}{(\lambda_n - \lambda'_n)^2} \]
Therefore, we obtain
\[ \sup_{0 \leq s \leq t} \| h^{n+1} \|_{Z_{t+1}^{(\lambda'_n - \lambda_{n0})(1+b), \mu'_n}} (\eta_k_1, \eta_k_2) \leq \delta^2_n + \left( \sum_{i=1}^n \delta_i \right) \sup_{0 \leq s \leq t} \| h^{n+1} \|_{Z_{t+1}^{(\lambda'_n - \lambda_{n0})(1+b), \mu'_n}}, \] (5.7)
this is the conclusion of Proposition 3.11.

Finally, we estimate the last term
\[ V(t, x) = -\int_0^t \int_{\mathbb{R}^3} \left[ (B[f^n] \circ \Omega^n_{t+1}(x, v)) \cdot H^{n+1}_{t+1}(s, X^n_{t+1}(x, v), V^n_{t+1}(x, v)) \right] dvds \]
\[ = -\int_0^t \int_{\mathbb{R}^3} \left[ (B[f^n] \circ \Omega^n_{t+1}(x, v)) \cdot (\nabla_v h^{n+1} \times v) \circ \Omega^n_{t+1}(x, v) \right] dvds \]
\[ = -\int_0^t \int_{\mathbb{R}^3} \left[ (B[f^n] \circ \Omega^n_{t+1}(x, v)) \cdot \left( (\nabla_v \times (h^{n+1}v)) \circ \Omega^n_{t+1}(x, v) \right) \right] dvds \]
\[ = \left( (\nabla_v \times \left( h^{n+1}v \right)) \circ \Omega^n_{t+1}(x, v) \right) \cdot \left( (\nabla_v \times \left( h^{n+1}v \right)) \circ \Omega^n_{t+1}(x, v) \right) \]
\[ - \int_0^t \int_{\mathbb{R}^3} \left[ (B[f^n] \circ \Omega^n_{t+1}(x, v)) \cdot \left( (\nabla_v \times \left( h^{n+1}v \right)) \circ \Omega^n_{t+1}(x, v) \right) \right] dvds \]
We claim that for \( \varepsilon > 0 \) sufficiently small,
\[ \| (\nabla \Omega^n_{t+1})^{-1} - Id \|_{Z_{t+1}^{(\lambda'_n - \lambda_{n0})(1+b), \mu'_n}} \leq \varepsilon. \]
If the claim holds, then
\[
\|V\|_{\mathcal{F}(\lambda_n - \nu_0, \nu_0')} \leq \int_{0}^{t} \sup_{k \neq l, k', l' \in \mathbb{Z}} e^{-2\pi(\lambda_n - \lambda_n')|k-l|s} \|B[f^n]\|_{L^\infty} \cdot \left( \left\| (\nabla_v \times (h^{n+1}v)) \circ \Omega_{\nu_0}^n(x, v) \right\|_{L^\infty} + \left\| (\nabla_v \times (h^{n+1}v)) \circ \Omega_{\nu_0'}^n(x, v) \right\|_{L^\infty} \right) ds + \int_{0}^{t} \sup_{k \neq l, k', l' \in \mathbb{Z}} e^{-2\pi(\lambda_n - \lambda_n')|k-l|s} \|B[f^n]\|_{L^\infty} \cdot \left\| (\nabla_v \times (h^{n+1}v)) \circ \Omega_{\lambda, \nu}^n(x, v) \right\|_{L^\infty} ds + \int_{0}^{t} \sup_{k \neq l, k', l' \in \mathbb{Z}} e^{-2\pi(\lambda_n - \lambda_n')|k-l|s} \|B[f^n]\|_{L^\infty} \cdot \left\| (\nabla_v \times (h^{n+1}v)) \circ \Omega_{\lambda, \nu}^n(x, v) \right\|_{L^\infty} ds,
\]
where
\[
\nu_0' = (\lambda_n' - B_0)s + \mu_n' + (\lambda_n' - B_0)b(t-s) + \|\Omega_{\nu_0}^n - Id\|_{L^\infty} \leq \|\lambda_n' - B_0\|_{L^\infty} + \|\lambda_n' - B_0\|_{L^\infty} \cdot \|\nu_0\|_{L^\infty} + \|\Omega_{\nu_0}^n - Id\|_{L^\infty}.
\]

So
\[
\|V\|_{\mathcal{F}(\lambda_n - \nu_0, \nu_0')} \leq \int_{0}^{t} e^{-\pi(\lambda_n - \lambda_n')|k-l|s} \left( \sum_{i=1}^{n} \lambda_i \right) \|\Omega_{\nu_0}^n(x, v)\|_{L^\infty} ds.
\]

We finish the proof of Proposition 3.12.

## 6 Estimates of error terms

In the following we estimate one of the error terms $R_0$.

Recall
\[
R_0(t, x) = \int_{0}^{t} \int_{\mathbb{R}^3} \left( \left( B[h^{n+1}] \circ \Omega_{\nu_0}^n(x, v) - B[h^{n+1}] \right) \cdot G_{\nu_0}^{n+1} \right) (s, X^0_{\nu_0}(x, v), V^0_{\nu_0}(x, v)) dv ds.
\]

First,
\[
\|R_0(t, \cdot)\|_{\mathcal{F}(\lambda_n - \nu_0, \nu_0')} \leq \int_{0}^{t} \|B[h^{n+1}] \circ \Omega_{\nu_0}^n(x, v) - B[h^{n+1}]\|_{\mathcal{F}(\lambda_n - \nu_0, \nu_0')} ds.
\]

Next,
\[
\|B[h^{n+1}] \circ \Omega_{\nu_0}^n(x, v) - B[h^{n+1}]\|_{\mathcal{F}(\lambda_n - \nu_0, \nu_0')} \leq \|\nabla B[h^{n+1}]\|_{L^\infty} \left( \|\nabla^2 B[h^{n+1}]\|_{L^\infty} + \|\nabla^3 B[h^{n+1}]\|_{L^\infty} \right) ds.
\]

where
\[
\nu_0' = (\lambda_n' - B_0)(1+b) + \mu_n' + \|\Omega^n - Id\|_{L^\infty}.
\]

(6.1)
Here we only focus on the case $\tau \geq \frac{1}{1+t}$, then we need to show $\|\nabla B[h^{n+1}]\|_{F_{\lambda_n}^n,\gamma_n^{\theta}} \leq \|\rho[h^{n+1}]\|_{F_{\lambda_n}^n,\gamma_n^{\theta}}$. For that, we have to prove $\nu'_{\tau} \leq (\lambda'_{n} - B_{0})\tau + \mu'_{n} - \nu_{\tau}^{\theta} - \theta$, for some constant $\theta > 0$ sufficiently small.

Indeed,

$$\nu'_{\tau} \leq (\lambda'_{n} - B_{0})\tau + \mu'_{n} - (\lambda'_{n} - B_{0})b(t - \tau) + C\sum_{i=1}^{n} \delta_i e^{-\pi|k|_3 (\lambda_i - \lambda'_i)^2} \cdot \min \left\{ \frac{(t - \tau)^2}{2(1 + \tau)^{2}}, \frac{1}{2(\lambda_i - \lambda'_i)^2} \right\}$$

$$\leq (\lambda'_{n} - B_{0})\tau + \mu'_{n} - (\lambda'_{n} - B_{0})B(\frac{t - \tau}{1 + \tau}) + C\left( \sum_{i=1}^{n} \frac{\delta_i}{(\lambda_i - \lambda'_i)^2} \right) \min \left\{ \frac{(t - \tau)}{1 + \tau}, 1 \right\}$$

Note that $\frac{\min(t - \tau, 1)}{1 + \tau} \leq \frac{3(t - \tau)}{1 + \tau}$. In the following we also need to show that

$$C\sum_{i=1}^{n} \frac{\delta_i}{(\lambda_i - \lambda'_i)^2} \leq \frac{\lambda'_{n} - B_{0}}{3} - \nu_{\tau}^{\theta}, \quad (VI)$$

Now we assume that (6.1) holds, then

$$\|B[h^{n+1}] \circ \Omega_{t,s}^{n}(x, v) - B[h^{n+1}]\|_{F_{\lambda_n}^n,\gamma_n^{\theta}} \leq C\left( \sum_{i=1}^{n} \frac{\delta_i}{(\lambda_i - \lambda'_i)^2} \right) \frac{1}{(1 + \tau)^{\frac{3}{2}}} \|\rho[h^{n+1}]\|_{F_{\lambda_n}^n,\gamma_n^{\theta}}.$$

Since $G_{t,s}^{n,v} = (\nabla'_{\lambda} f_{n} \times V_{t,s}^{0}(x, v)) \circ \Omega_{t,s}^{n}(x, v),

$$\|G_{t,s}^{n,v} \|_{F_{\lambda_n}^n,\gamma_n^{\theta}} \leq C\left( \sum_{i=1}^{n} \frac{\delta_i}{(\lambda_i - \lambda'_i)^2} \right) \frac{1}{(1 + \tau)^{\frac{3}{2}}} \|\rho[h^{n+1}]\|_{F_{\lambda_n}^n,\gamma_n^{\theta}}.$$

We can conclude

$$\|R_{0}(t, \cdot)\|_{F_{\lambda_n}^n,\gamma_n^{\theta}} \leq C\left( \sum_{i=1}^{n} \frac{\delta_i}{(\lambda_i - \lambda'_i)^2} \right) \int_{0}^{t} \|\rho[h^{n+1}]\|_{F_{\lambda_n}^n,\gamma_n^{\theta}} \frac{dt}{(1 + \tau)^{\frac{3}{2}}}.$$
where we need to prove
\[
\|\Omega^n X_{t, \tau} - \Omega^{i-1} X_{t, \tau}\|_{\mathfrak{Z}_{(x_n - \eta_0)/1+b, \nu_n}^n(\eta_1, \eta_2)} \leq 2\mathcal{R}_{i-1}^n(t, \tau),
\]
\[
\|\Omega^n V_{t, \tau} - \Omega^{i-1} V_{t, \tau}\|_{\mathfrak{Z}_{(x_n - \eta_0)/1+b, \nu_n}^n(\eta_1, \eta_2)} \leq \mathcal{R}_{i-1}^n(t, \tau) + \mathcal{R}_{i-1}^n(t, \tau)
\]
with
\[
\mathcal{R}_{i-1}^n(t, \tau) = \left(\sum_{j=1}^n \delta_j e^{-2\pi (\lambda_j - \lambda'_j) t} \right) \min\{t, 1\}, \quad \mathcal{R}_{i-1}^n(t, \tau) = \left(\sum_{j=1}^n \delta_j e^{-2\pi (\lambda_j - \lambda'_j) t} \right) \min\{t, \tau\}.
\]

On the other hand, by the induction hypothesis, since \(Z_{\lambda, \mu}^n\) norms are increasing as a function of \(\lambda\) and \(\mu\),
\[
\sum_{i=1}^n \|\nabla_v h^i_{\tau} \circ \Omega_{i-1}^{t, \tau} - \nabla_v (h^i_{\tau} \circ \Omega_{i-1}^{t, \tau})\|_{\mathfrak{Z}_{(x_n - \eta_0)/1+b, \nu_n}^n(\eta_1, \eta_2)} \leq \left(\sum_{i=1}^n \delta_i\right) \frac{1}{(1 + \tau)^2}.
\]
So we have
\[
\|\hat{R}_0(t, \cdot)\|_{\mathfrak{Z}_{(x_n - \eta_0)/1+b, \nu_n}^n} \leq C_0 \left(C_0 + \sum_{i=1}^n \delta_i\right) \left(\sum_{i=1}^n \frac{\delta_i}{2\pi (\lambda_j - \lambda'_j) t}\right) + \sum_{i=1}^n \delta_i \int_0^t \|\rho\|_{\mathfrak{Z}_{(x_n - \eta_0)/1+b, \nu_n}^n} \frac{1}{(1 + \tau)^2} d\tau
\]
\[
= \int_0^t \hat{K}_{i-1}^{n, i+1} \|\rho\|_{\mathfrak{Z}_{(x_n - \eta_0)/1+b, \nu_n}^n} \frac{1}{(1 + \tau)^2} d\tau.
\]
(6.2)

Up to now, we finish the estimates of error terms.

### 7 Iteration

Now let us first deal with the source term
\[
III(t, \cdot) + IV(t, \cdot) = -\int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{-2\pi ik \cdot x} (\mathcal{E}_{x, s} H^n_{t, s})(s, X_0^0(x, v), V_{t, s}^0(x, v))
\]
\[
dvdxds - \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} e^{-2\pi ik \cdot x} (\mathcal{F}_{x, s} H^n_{t, s})(s, X_0^0(x, v), V_{t, s}^0(x, v)) dvdxds,
\]
(7.1)

then
\[
\|III(t, \cdot)\|_{\mathfrak{Z}_{(x_n - \eta_0)/1+b, \nu_n}^n} + \|IV(t, \cdot)\|_{\mathfrak{Z}_{(x_n - \eta_0)/1+b, \nu_n}^n}
\]
\[
\leq \int_0^t \|\mathcal{E}_{x, s}\|_{\mathfrak{Z}_{(x_n - \eta_0)/1+b, \nu_n}^n} \|H_{t, s}^n\|_{\mathfrak{Z}_{(x_n - \eta_0)/1+b, \nu_n}^n(\eta_1, \eta_2)} + \|\mathcal{F}_{x, s}\|_{\mathfrak{Z}_{(x_n - \eta_0)/1+b, \nu_n}^n(\eta_1, \eta_2)} d\tau
\]
\[
\leq \int_0^t \|\rho_{t, s}\|_{\mathfrak{Z}_{(x_n - \eta_0)/1+b, \nu_n}^n} (1 + \tau) d\tau \leq \int_0^t \|\rho_{t, s}\|_{\mathfrak{Z}_{(x_n - \eta_0)/1+b, \nu_n}^n} d\tau \leq 2C_0^2 \frac{\delta_n}{(\lambda_n - \lambda'_n)^2}.
\]
(7.2)

From Propositions 3.6-3.12, combining (3.10), we conclude
\[
\|\rho \|_{H_{n+1}^n(t, \cdot)} \leq C_0^2 \frac{\delta_n}{(\lambda_n - \lambda'_n)^2} + \int_0^t \|K_1^n(t, \tau)(1 + \tau) \sum_{i=1}^n \delta_i \|\rho \|_{H_{n+1}^n(t, \cdot)} d\tau + \int_0^t \|K_1^n(t, \tau)\|_{H_{n+1}^n(t, \cdot)} d\tau \leq C_0^2 \frac{\delta_n}{(\lambda_n - \lambda'_n)^2} + \int_0^t \|K_1^n(t, \tau)\|_{H_{n+1}^n(t, \cdot)} d\tau.
\]

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\[ \sum_{i=1}^{n} \delta_{i}\|\rho h^{n+1}\|_{\mathcal{F}(x_{n}-\eta_{n}+\nu_{n})} + \int_{0}^{t} \left( \bar{K}_{0}^{n+1} \right. \left. + \bar{K}_{1}^{n+1} \right) \frac{1}{(1 + \tau)^{2}} \|\rho h^{n+1}\|_{\mathcal{F}(x_{n}-\eta_{n}+\nu_{n})} d\tau, \]

where \( K_{0}^{n}(t, \tau) \) and \( K_{1}^{n}(t, \tau) \) are defined in Proposition 3.4, and

\[ \bar{K}_{0}^{n+1} \triangleq 2C \left( C_{0} + \sum_{i=1}^{n} \delta_{i} \right) \left( \sum_{i=1}^{n} \frac{\delta_{i}}{(2\pi(\lambda_{i} - \lambda_{i}^{0})^{2}} \right), \]

\[ \bar{K}_{1}^{n+1} \triangleq \left( C_{\omega} \left( C_{0} + \sum_{i=1}^{n} \delta_{i} \right) \left( \sum_{j=1}^{n} \frac{\delta_{j}}{2\pi(\lambda_{j} - \lambda_{j}^{0})^{2}} \right) + \sum_{i=1}^{n} \delta_{i} \right). \]

**Proposition 7.1** From the above inequality, we obtain the following integral inequality:

\[ \left\| \rho h^{n+1}(t, x) \right\| - \int_{0}^{t} \int_{\mathbb{R}^{3}} \left( E[h^{n+1}] + v \times B[h^{n+1}] \right) (\tau, x + M(t - \tau)v) \cdot \nabla_{v} f^{0} d\tau dt \]

\[ \leq \frac{C^{2}}{(\lambda_{n} - \lambda_{n}^{0})^{2}} + \int_{0}^{t} \left( \bar{K}_{1}(t, \tau) + K_{0}^{n}(t, \tau) + \frac{c_{n}^{0}}{(1 + \tau)^{2}} \right) \left\| \rho h^{n+1}(t, \cdot) \right\|_{\mathcal{F}(x_{n}-\eta_{n}+\nu_{n})} d\tau, \]  

(7.3)

where \( K_{1}^{n}(t, \tau) = \| K_{1}^{n}(t, \tau) \|_{L^{2}} \sum_{i=1}^{n} \delta_{i} \), \( K_{0}^{n}(t, \tau) = \| K_{0}^{n}(t, \tau) \|_{\mathcal{F}(x_{n}-\eta_{n})} \sum_{i=1}^{n} \delta_{i} \), \( c_{0}^{n} = \bar{K}_{0}^{n+1} + \bar{K}_{1}^{n+1} \).

### 7.1 Exponential moments of the kernel

Before doing the iteration, based on the analysis and computation in Sections 3-6, we explain the connection and difference in between electromagnetic field case and electric field case. First, we state the connection, from (5.3)-(5.4) of Theorem 5.1 in section 5.1, the kernel is the same under two different norms \( Z_{\lambda}^{p,1} \) and \( F^{\lambda \cdot p} \), in other words, there are no new echoes to generate in different norms \( Z_{\lambda}^{p,1} \) and \( F^{\lambda \cdot p} \), this is a key point for us, it implies that the first-order term of the magnetic field has no influence on the resonances of the plasma particles, that is, the first-order term of the magnetic field can be regarded as an error term and doesn’t play an important role in dynamical behavior of the particle’s trajectory. The difference is to add the new term \( V = -\int_{0}^{t} \int_{\mathbb{R}^{3}} \left( |B|^{1/2}_{\alpha} \Omega_{\alpha}^{\gamma}(x, v) \right) \cdot \left( \nabla_{v} h^{n+1}(x, v) \right) \right) \left( s, X_{\alpha}^{n}., V_{0}^{n}(x, v) \right) d\tau ds \) in the density on \( \rho h^{n+1} \). In order to deal with this term, we have to go back to the Vlasov equation on \( h^{n+1}(s, X_{\alpha}^{n}(x, v), V_{0}^{n}(x, v)), \) and then there are new echoes to appear during estimating \( h^{n+1}(s, X_{\alpha}^{n}(x, v), V_{0}^{n}(x, v)), \) but the echoes form is still the same with that generated by the Vlasov-Poisson equation, the details can be found in section 5.2. From the inequalities (5.3)-(5.4), we know that the reason is that the echoes don’t change as the norm change. In summary, in order to iterate on the density \( \rho h^{n+1} \), we only need to estimate the same kernel with Landau damping in electric field case. The following theorems are the same with the results in [23] and the proofs can also be found in Section 7 in [23], so we sketch the proof.

**Proposition 7.2** (Exponential moments of the kernel) Let \( \gamma \in (1, \infty) \) be given. For any \( \alpha \in (0, 1) \), let \( K^{(\alpha, \gamma)} \) be defined

\[ K^{(\alpha, \gamma)}(t, \tau) = (1 + \tau) \sup_{k, l \in \mathbb{Z}} e^{-\alpha|t-l|} e^{-\alpha|\tau-t|} e^{-\alpha|\tau-t-l|}, \]

Then for any \( \gamma \), there is \( \bar{a} = \alpha(\gamma) > 0 \) such that if \( \alpha \leq \bar{a} \) and \( \varepsilon \in (0, 1) \), then for any \( t > 0 ),

\[ e^{-\varepsilon t} \int_{0}^{t} K^{(\alpha, \gamma)}(t, \tau) e^{\tau} d\tau \leq C \left( \frac{1}{\alpha \varepsilon^{\gamma} t^{\gamma} - 1} + \frac{1}{\alpha \varepsilon^{\gamma} t^{\gamma} - 1} \log \frac{1}{\alpha} + \frac{1}{\alpha \varepsilon^{\gamma} t^{\gamma} - 1} \log \frac{1}{\alpha} \right) e^{-\bar{a} t} + e^{-\frac{\varepsilon^{\gamma} t}{\alpha^{2}}}, \]

where \( C = C(\gamma) \).

In particular, if \( \varepsilon \) is large, then \( e^{-\varepsilon t} \int_{0}^{t} K^{(\alpha, \gamma)}(t, \tau) e^{\tau} d\tau \leq \frac{C(\gamma)}{\alpha \varepsilon^{\gamma} t^{\gamma} - 1} \).

**Proof.** Without loss of generality, we shall set \( d = 1 \) and first consider \( \tau \leq \frac{1}{2} t \). We can write

\[ K^{(\alpha)}(t, \tau) \leq (1 + \tau) \sup_{k, l \in \mathbb{Z}} e^{-\alpha|t-l|} e^{-\alpha|\tau-t|} e^{-\alpha|\tau-t-l|}. \]

By symmetry, we may also assume that \( k > 0 \).

Explicit computations yield

\[ \int_{0}^{\frac{t}{2}} e^{-\alpha|\tau-t|+\tau+k} (1+\tau) d\tau \leq \begin{cases} \frac{1}{\alpha|t-k|} + \frac{1}{\alpha|t-k|^{2}}, & \text{if } k > 0, \\ \frac{e^{-\alpha k t} (1 + \frac{1}{\alpha|t-k|})}{2k}, & \text{if } k = 0, \\ \frac{e^{-\alpha k t} (1 + \frac{1}{\alpha|t-k|})}{2k}, & \text{if } k < 0. \end{cases} \]

(7.4)
and by simple computation, we get the explicit bounds

where $z = \sup_x xe^{-x} = e^{-1}$.

Using the bounds (for $\alpha \sim 0^+$)

we end up, for $\alpha \leq \frac{1}{4}$, with a bound like

Next we turn to the more delicate contribution of $\tau \geq \frac{1}{2}t$. We write

Without loss of generality, we restrict the supremum $l > 0$. The function $x \to (1 + |x - l|^\gamma)^{-1}e^{-\alpha|x(t - \tau) + l\gamma|}$ is decreasing for $x \geq l$, increasing for $x \leq -l\tau/(t - \tau)$; and on the interval $[-l\tau/(t - \tau), l]$, its logarithmic derivative goes from $\left(-\alpha + \frac{\gamma}{1 + \gamma}l\gamma/(1 + \gamma)\right)(t - \tau)$ to $-\alpha(t - \tau)$. It is easy to check that a given integer $k$ occurs in the supremum only for some times $\tau$ satisfying $k - 1 < -l\tau/(t - \tau) < k + 1$. We can assume $k \geq 0$, then $k - 1 \frac{t}{l\tau} < k + 1$ holds, and it is equivalent to $k\frac{t}{l\tau} < k\frac{t}{l\tau} + 1$. More importantly, $\tau \geq \frac{1}{2}t$ implies that $k \geq l$. Thus, for $t \geq \frac{1}{4}$, we have

For $t \leq \frac{1}{4}$, it is easy to check that $e^{-\alpha t} \int_0^l K^{(a)}(t, \tau) e^{\tau\gamma} d\tau$ is decreasing and the same change of variables as before gets the explicit bounds

Hence, (7.6) is bounded above by

We consider the first term $I(t)$ of (7.7) and change variables $(x, y) \to (x, u)$, where $u(x, y) = \frac{e^{-\alpha y}}{x+y}$, then we can find that

The same computation for the second integral of (7.7) yields

Finally, we estimate the last term of (7.7) that is the worst. It yields a contribution $\frac{1}{\alpha} \sum_{k=1}^{\infty} e^{-\alpha l} \sum_{k=1}^{\infty} e^{-\alpha(k+l)(k+l)\gamma/(1+\gamma)}$. We compare this with the integral $\frac{1}{\alpha} \int_0^\infty e^{-\alpha x} \int_x^\infty e^{-\alpha\gamma(x+y)} dy dx$, and the same change of variables as before equates this with

The proof of Proposition 7.2 follows by collecting all these bounds and keeping only the worst one. To finish the growth control, we have to prove the following result.
Proposition 7.3 With the same notation as in Proposition 7.2, for any $\gamma > 1$, we have

$$\sup_{\tau \geq 0} e^{\varepsilon \tau} \int_{\tau}^{\infty} e^{-\varepsilon t} K^{(\alpha)}(t, \tau) dt \leq C(\gamma) \left( \frac{1}{\alpha^2 \varepsilon} + \frac{1}{\alpha \varepsilon^7} \right)$$  \hspace{1cm} (7.8)

Proof. We first still reduce to $d = 1$, and split the integral as

$$e^{\varepsilon \tau} \int_{\tau}^{\infty} e^{-\varepsilon t} K^{(\alpha)}(t, \tau) dt = e^{\varepsilon \tau} \int_{2\tau}^{\infty} e^{-\varepsilon t} K^{(\alpha)}(t, \tau) dt + e^{\varepsilon \tau} \int_{\tau}^{2\tau} e^{-\varepsilon t} K^{(\alpha)}(t, \tau) dt = I_1 + I_2.$$  

For the first term $I_1$, we have $K^{(\alpha)}(t, \tau) \leq (1 + \tau) \sum_{k=2}^{\infty} \sum_{l \in \mathbb{Z}} e^{-\alpha|l|} e^{-\alpha|l|/2} \leq \frac{C(1 + \tau)^{1/2}}{\alpha^2}$, and thus $e^{\varepsilon \tau} \int_{\tau}^{\infty} e^{-\varepsilon t} K^{(\alpha)}(t, \tau) dt \leq \frac{C(1 + \tau)^{1/2}}{\alpha^2}.$

We treat the second term $I_2$ as in the proof of Proposition 7.2:

$$e^{\varepsilon \tau} \int_{\tau}^{\infty} e^{-\varepsilon t} K^{(\alpha)}(t, \tau) dt \leq e^{\varepsilon \tau} (1 + \tau) \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} e^{-\alpha l} \int_{\tau}^{(k+1)\tau} e^{-\alpha(k-l)\tau} + \frac{c_0}{1 + (k+l)\varepsilon} e^{-\varepsilon t} dt \leq \frac{C}{\alpha^2 \varepsilon^7},$$

where the last inequality is obtained by the same method in Proposition 7.2 with the change of variable $u = \frac{\varepsilon t}{\varepsilon}$.

7.2 Growth control

From now on, we will state the main result of this section that is the same with section 7.4 in [23]. We define $\|\Phi(t)\|_{s} = \sum_{k \in \mathbb{Z}^2} |\Phi(k, t)| e^{2\pi |k|}$.

Theorem 7.4 Assume that $f_0(f), W = W(x)$ satisfy the conditions of Theorem 0.3, and the Stability condition holds. Let $A \geq 0, \mu \geq 0$ and $\lambda \in (0, \lambda^*)$ with $0 < \lambda^* < \lambda_0$. Let $(\Phi(k, t))_{k \in \mathbb{Z}^2, t \geq 0}$ be a continuous functions of $t \geq 0$, valued in $C^{s, 2}$, such that for all $t \geq 0$,

$$\|\Phi(t) - \int_{0}^{t} K^0(t - \tau) \Phi(\tau) d\tau\|_{s + \mu} \leq A + \int_{0}^{t} (K_0(t, \tau) + K_1(t, \tau) + \frac{c_0}{1 + (\tau + \mu)^{1/2}}) \|\Phi(\tau)\|_{s + \mu} d\tau,$$  \hspace{1cm} (7.9)

where $c_0 \geq 0, m \geq 1$, and $K_0(t, \tau), K_1(t, \tau)$ are non-negative kernels. Let $\varphi(t) = \|\Phi(t)\|_{s + \mu}$. Then we have the following:

(i) Assume that $\gamma > 1$ and $K_1 = c K^{(\alpha)}$ for some $c > 0, \alpha \in (0, \alpha(\gamma))$, where $K^{(\alpha)}(\alpha) = \alpha(\gamma)$ are the same with that defined by Proposition 7.2. Then there are positive constants $C, \chi$, depending only on $\gamma, \lambda^*, \lambda_0, c, c_W$ and $m$, uniform as $\gamma \to 1$, such that if $\sup_{t \geq 0} \int_{0}^{1} K_0(t, \tau) d\tau \leq \chi$ and $\sup_{t \geq 0} (\int_{0}^{1} K_0(t, \tau)^2 d\tau)^{1/2} + \sup_{t \geq 0} \int_{0}^{\infty} K_0(t, \tau) dt \leq 1$, then for any $\varepsilon \in (0, \alpha)$, for all $t \geq 0$,

$$\varphi(t) \leq C \frac{1 + c_0}{\sqrt{\varepsilon}} e^{C \varepsilon (1 + T^2)} e^{\varepsilon t},$$  \hspace{1cm} (7.10)

where $T_{\varepsilon} = C \max \left\{ \left( \frac{\lambda}{\varepsilon \lambda^* \alpha} \right)^{1/2}, \left( \frac{\alpha}{\varepsilon \lambda^* \alpha} \right)^{1/2}, \left( \frac{\lambda}{\varepsilon \lambda^* \alpha} \right)^{1/2} \right\}$.

(ii) Assume that $K_1 = \sum_{j=1}^{N} c_j K^{(\alpha_j)}$ for some $\alpha_j \in (0, \alpha(\gamma))$, where $\alpha(\gamma)$ also appears in proposition 7.2; then there is a numeric constant $\Gamma > 0$ such that whenever $1 \geq \varepsilon \geq \Gamma \sum_{j=1}^{N} \frac{c_j}{\alpha_j}$, with the same notation as in (i), for all $t \geq 0$, one has,

$$\varphi(t) \leq C \frac{1 + c_0}{\sqrt{\varepsilon}} e^{C \varepsilon (1 + T^2)} e^{\varepsilon t},$$  \hspace{1cm} (7.11)

where $c = \sum_{j=1}^{N} c_j$ and $T = \max \left\{ \frac{1}{\varepsilon} \sum_{j=1}^{N} \frac{c_j}{\alpha_j}, \left( \frac{\lambda}{\varepsilon \lambda^* \alpha} \right)^{1/2} \right\}$.

Proof. Here we only prove (i), the proof of (ii) is similar. We decompose the proof into three steps.

Step 1. Crude pointwise bounds. From (7.9), we have

$$\varphi(t) = \sum_{k \in \mathbb{Z}^2} |\Phi(k, t)| e^{2\pi (\lambda + \mu)|k|} \leq A + \sum_{k \in \mathbb{Z}^2} \int_{0}^{t} |K^0(k, t - \tau)| e^{2\pi (\lambda + \mu)|k|} |\Phi(\tau)| d\tau$$

$$+ \int_{0}^{t} (K_0(t, \tau) + K_1(t, \tau) + \frac{c_0}{1 + (\tau + \mu)^{1/2}}) \varphi(\tau) d\tau.$$
\[
\leq A + \int_0^t (K_0(t, \tau) + K_1(t, \tau)) + \frac{c_0}{(1 + \tau)^m} + \sup_{k \in \mathbb{Z}_+^3} |K^0(k, t - \tau)|e^{2\pi \lambda(t - \tau)|k|} \varphi(\tau) d\tau.
\]

We note that for any \( k \in \mathbb{Z}_+^3 \) and \( t \geq 0 \),
\[
|K^0(k, t - \tau)|e^{2\pi \lambda|k|(t - \tau)} \leq 4\pi^2 |\hat{W}(k)|C_0 e^{-2\pi(\lambda_0 - \lambda)|k|t|k|^2} t \leq \frac{CC_0C_W}{\lambda_0 - \lambda},
\]
where (here and below) \( C \) stands for a numeric constant which may change from line to line. Assuming that \( \int_0^t K_0(t, \tau) d\tau \leq \frac{1}{2} \), we deduce that
\[
\varphi(t) \leq A + \frac{1}{2} \sup_{0 \leq \tau \leq t} \varphi(\tau) + C \int_0^t \left( \frac{C_0C_W}{\lambda_0 - \lambda} + c(1 + \tau) + \frac{c_0}{(1 + \tau)^m} \right) \varphi(\tau) d\tau,
\]
and, by Grönwall’s lemma,
\[
\varphi(t) \leq 2Ae^{C(C_0C_W t/(\lambda_0 - \lambda) + c(t + t^2) + c_0C_m)},
\]
where \( C_m = \int_0^\infty (1 + \tau)^{-m} d\tau \).

**Step 2.** \( L^2 \) bound. For all \( k \in \mathbb{Z}_+^3 \) and \( t \geq 0 \), we define \( \Psi_k(t) = e^{-\epsilon t} \Phi(k, t)e^{2\pi(\lambda t + \mu)|k|}K^0_k(t) = e^{-\epsilon t} K^0_k(t) \) \( e^{2\pi(\lambda t + \mu)|k|} \), \( R_k(t) = e^{-\epsilon t} (\Phi(k, t) - \int_0^t K^0_k(t, t - \tau) \Psi(k, \tau) d\tau)e^{2\pi(\lambda t + \mu)|k|} = (\Psi_k - \Psi_k * K^0_k)(t) \), and we extend all these functions by 0 for negative values of \( t \). Taking Fourier transform in the time-variable yields \( \hat{R}_k = (1 - \hat{K}^0_k)\hat{\Psi}_k \). Since the Stability condition implies that \( |1 - \hat{K}^0_k| \geq \kappa \), we can deduce that \( \|\Psi_k\|_{L^2} \leq \kappa^{-1}\|\hat{R}_k\|_{L^2} \), i.e., \( \|\Psi_k\|_{L^2} \leq \kappa^{-1}\|\hat{R}_k\|_{L^2} \). So we have
\[
\|\Psi_k - R_k\|_{L^2(dt)} \leq \kappa^{-1}\|K^0_k\|_{L^1(dt)} \|R_k\|_{L^2(dt)} \quad \text{for all} \quad k \in \mathbb{Z}_+^3.
\]
Then
\[
\|\varphi(t)e^{-\epsilon t}\|_{L^2(dt)} = \| \sum_{k \in \mathbb{Z}_+^3} |\Psi_k|\|_{L^2(dt)} \leq \| \sum_{k \in \mathbb{Z}_+^3} |R_k|\|_{L^2(dt)} + \| \sum_{k \in \mathbb{Z}_+^3} |R_k - \Psi_k|\|_{L^2(dt)} \leq \| \sum_{k \in \mathbb{Z}_+^3} |R_k|\|_{L^2(dt)} (1 + \frac{1}{\kappa}).
\]

Next, we note that
\[
\|K^0_k\|_{L^1(dt)} \leq 4\pi^2 |\hat{W}(k)| \int_0^\infty C_0 e^{-2\pi(\lambda_0 - \lambda)|k|t|k|^2} t dt \leq 4\pi |\hat{W}(k)| \frac{C_0}{(\lambda_0 - \lambda)^2},
\]
so
\[
\sum_{k \in \mathbb{Z}_+^3} |K^0_k|\|_{L^1(dt)} \leq 4\pi |\hat{W}(k)| \frac{C_0}{(\lambda_0 - \lambda)^2}. \quad \text{Furthermore, we get}
\]
\[
\|\varphi(t)e^{-\epsilon t}\|_{L^2(dt)} \leq \left(1 + \frac{CC_0C_W}{\kappa(\lambda_0 - \lambda)^2}\right) \left( \sum_{k \in \mathbb{Z}_+^3} \| R_k \|_{L^2(dt)} \right) \leq \left(1 + \frac{CC_0C_W}{\kappa(\lambda_0 - \lambda)^2}\right) \left( \int_0^\infty e^{-2\epsilon t} \left( A + \int_0^t \left( K_1 + K_0 + \frac{c_0}{(1 + \tau)^m} \right) \varphi(\tau) d\tau \right)^2 \right)^{\frac{1}{2}}.
\]

By Minkowski’s inequality, we separate (7.15) into various contributions which we estimate separately. First,
\[
\left( \int_0^\infty e^{-2\epsilon t} A^2 dt \right)^{\frac{1}{2}} = \frac{4}{\sqrt{2\epsilon}}. \quad \text{Next, for any} \ T \geq 1, \ \text{by Step 1 and} \ \int_0^T K_1(t, \tau) d\tau \leq \frac{C(1 + t)}{\alpha}, \ \text{we have}
\]
\[
\left( \int_0^T e^{-2\epsilon t} \left( \int_0^t K_1(t, \tau) \varphi(\tau) \right)^2 \right)^{\frac{1}{2}} \leq \left( \sup_{0 \leq \tau \leq T} \varphi(\tau) \right) \left( \int_0^T e^{-2\epsilon t} \left( \int_0^t K_1(t, \tau) \right)^2 dt \right)^{\frac{1}{2}} \leq C Ae^{C(C_0C_W T/(\lambda_0 - \lambda) + c(T + T^2)) \frac{C_0}{\alpha}} \left( \int_0^\infty e^{-2\epsilon t} (1 + t)^2 dt \right)^{\frac{1}{2}} \leq C A \frac{c}{\alpha} \left( C(C_0C_W T/(\lambda_0 - \lambda) + c(T + T^2)) \right).
\]
Similarly,

\[
\int_T^\infty e^{-2ct} \left( \int_0^t K_1(t, \tau) \varphi(\tau) \, d\tau \right)^2 \, dt \quad \leq \quad \left( \int_0^t K_1(t, \tau) \varphi(\tau) \, d\tau \right)^2 \quad \leq \quad \left( \int_0^\infty K_1(t, \tau) e^{-2\varepsilon(\tau-t)} \, d\tau \right)^2 \quad \leq \quad \left( \int_0^\infty K_1(t, \tau) e^{-2\varepsilon \varphi(\tau)^2} \, d\tau \right)^2 \quad \leq \quad \left( \int_0^\infty \varphi(\tau)^2 e^{-2\varepsilon \varphi(\tau)^2} \, d\tau \right)^2.
\]

(7.17)

The last term is also split, this time according to \( \tau \leq T \) or \( \tau > T \):

\[
\left( \int_0^\infty e^{-2ct} \left( \int_T^\infty \frac{\varphi(\tau)}{(1 + \tau)^m} \, d\tau \right)^2 \, dt \right)^2 \quad \leq \quad c_0 \left( \int_T^\infty e^{-2ct} \left( \int_T^\infty \frac{\varphi(\tau)}{(1 + \tau)^m} \, d\tau \right)^2 \, dt \right)^2 \quad \leq \quad c_0 \left( \int_T^\infty \frac{\varphi(\tau)}{(1 + \tau)^m} \, d\tau \right)^2 \quad \leq \quad \frac{C_A}{\sqrt{\varepsilon}} e^{C(C_0 \varepsilon + \lambda(t+T)^T)} C_m,
\]

(7.19)

and

\[
\left( \int_0^\infty e^{-2ct} \left( \int_T^\infty \frac{\varphi(\tau)}{(1 + \tau)^m} \, d\tau \right)^2 \, dt \right)^2 \quad \leq \quad c_0 \left( \int_0^\infty e^{-2ct} \varphi(t)^2 \, dt \right)^2 \quad \left( \int_0^\infty \frac{\varphi(\tau)}{(1 + \tau)^m} \, d\tau \right)^2 \quad \leq \quad \frac{C_A^2 c_0}{T^{m+1/2} \sqrt{\varepsilon}} \left( \int_0^\infty e^{-2ct} \varphi(t)^2 \, dt \right)^2.
\]

(7.20)

Gathering estimates (7.16)-(7.20), we deduce from (7.15) that

\[
\| \varphi(t) e^{-\varepsilon t} \|_{L^2(dt)} \leq \left( 1 + \frac{C_C C_W}{\kappa(\lambda_0 - \lambda)^2} \right) \frac{C_A}{\sqrt{\varepsilon}} \left( 1 + \frac{c}{\alpha \varepsilon} + c_0 C_m \right) e^{C(C_0 \varepsilon + \lambda(t+T)^T)}
\]

\[
+ \lambda \| \varphi(t) e^{-\varepsilon t} \|_{L^2(dt)},
\]

(7.21)

where

\[
a = \left( 1 + \frac{C_C C_W}{\kappa(\lambda_0 - \lambda)^2} \right) \left[ \left( \sup_{\tau \geq T} \int_0^t e^{-\varepsilon \tau} K_1(t, \tau) \, d\tau \right)^2 \right] \frac{a}{a \varepsilon} + c_0 C_m \right) e^{C(C_0 \varepsilon + \lambda(t+T)^T)}.
\]

(7.22)

Using Propositions 7.2 and 7.3, together with the assumptions of Theorem 7.4, we see that \( a \leq \frac{1}{\pi} \) for \( \chi \) sufficiently small. Then we have

\[
\| \varphi(t) e^{-\varepsilon t} \|_{L^2(dt)} \leq \left( 1 + \frac{C_C C_W}{\kappa(\lambda_0 - \lambda)^2} \right) \frac{C_A}{\sqrt{\varepsilon}} \left( 1 + \frac{c}{\alpha \varepsilon} + c_0 C_m \right) e^{C(C_0 \varepsilon + \lambda(t+T)^T)}.
\]

Step 3. For \( t \geq T \), using (7.9), we get

\[
e^{-\varepsilon t} \varphi(t) \leq A e^{-\varepsilon t} + \left[ \int_0^t \left( \sup_{k \in \mathbb{Z}} \left| K_0(t, k - \tau) \right| e^{2\varepsilon \lambda |k - \tau|} \right)^2 \, d\tau \right]^{1/2} + \left( \int_0^t K_0(t, \tau)^2 \, d\tau \right)^{1/2} + \left( \int_0^\infty \frac{c_0^2}{(1 + \tau)^2m} \, d\tau \right)^{1/2} \quad \leq \quad \left( \int_0^\infty \varphi(\tau) e^{-\varepsilon \tau} \, d\tau \right)^{1/2}.
\]

(7.22)
We note that, for any $k \in \mathbb{Z}^d_+$, $(\|K^0(t)\|e^{2\pi |k|^2})^2 \leq C\pi \| \tilde{K}(k) \|^2 \| \tilde{f}(kt) \|^2 |k|^{4j^2} \leq \frac{C C_0}{(\lambda_0 - \lambda)^2} C_0^2 e^{-2\pi(\lambda_0 - \lambda)^j}$, so we get $\int_0^t \left( \sup_{k \in \mathbb{Z}^d_+} |K^0(t, \tau)\|e^{2\pi |\tau|^2})^2 d\tau \leq \frac{C C_0^2 C_0^2}{(\lambda_0 - \lambda)^2}.$

From Propositions 7.2, (7.22), the conditions of Theorem 7.4 and Step 2, the conclusion is finished.

**Corollary 7.5** Assume that $f^0 = f^0(v)$, under the assumptions of Theorem 0.3, we pick up $\lambda' - B_0 < \lambda' - B_0$ such that $2\pi(\lambda' - \lambda'^2) \leq \alpha_n$, choose $\varepsilon = 2\pi(\lambda' - \lambda'^2)$; recalling that $\tilde{\rho}(0, 0) = 0$, our conditions imply an upper bound on $c_n$ and $c_n^0$, we have the uniform control,

$$\|\rho[h^{n+1}](t, x)\|_{\mathcal{P}(\lambda'^2 - B_0)^t + \varepsilon} \leq \frac{C_0^2 (1 + c_0^2)^2 \lambda_n^2 (1 + \frac{1}{\alpha_n(\lambda' - \lambda'^2)^2}) e^{CT_n^2}},$$

where $T_n = C \left( \frac{1}{\alpha_n(\lambda' - \lambda'^2)} \right)^{1/4}.$

**Proof.** From Propositions 7.1-7.3, we know that

$$\int_0^t K^0_0(t, \tau) d\tau \leq C \sum_{i=1}^n \frac{\delta_i}{\pi(\lambda_i - \lambda_i^2)}, \quad \int_0^\infty K^0_n(t, \tau) d\tau \leq C \sum_{i=1}^n \frac{\delta_i}{\pi(\lambda_i - \lambda_i^2)},$$

$$\left( \int_0^t K^0_n(t, \tau)^2 d\tau \right)^{1/2} \leq C \sum_{i=1}^n \frac{\delta_i}{\sqrt{\pi(\lambda_i - \lambda_i^2)}}.$$

Here $\alpha_n = \pi \min\{\mu_n - \mu', \lambda - \lambda'\}$, and assume that $\alpha_n$ is smaller than $\alpha(\gamma)$ in Theorem 7.4, and that

$$\left( C_0^2 \left( C_0 + \sum_{i=1}^n \delta_i + 1 \right) \left( \sum_{j=1}^n \frac{\delta_j}{2\pi(\lambda_j - \lambda_j^2)} \right) \right) \leq \frac{1}{8}, \quad (VII)$$

$$C \sum_{i=1}^n \frac{\delta_i}{\sqrt{2\pi(\lambda_i - \lambda_i^2)}} \leq \frac{1}{4}, \quad \sum_{i=1}^n \frac{\delta_i}{\pi(\lambda_i - \lambda_i^2)} \leq \max\{\lambda, \frac{1}{8}\}. \quad (VIII)$$

Applying Theorem 7.4, we can deduce that for any $\varepsilon \in (0, \alpha_n)$ and $t \geq 0,$

$$\|\rho[h^{n+1}](t, x)\|_{\mathcal{P}(\lambda'^2 - B_0)^t + \varepsilon} \leq e^{-2\pi(\lambda' - \lambda'^2)t} \|\rho[h^{n+1}](t, x)\|_{\mathcal{P}(\lambda'^2 - B_0)^t + \varepsilon}$$

$$\leq \frac{C_0^2 (1 + c_0^2)^2 \lambda_n^2 (1 + \frac{1}{\alpha_n(\lambda' - \lambda'^2)^2}) e^{CT_n^2}},$$

where $T_n = C \left( \frac{1}{\alpha_n(\lambda' - \lambda'^2)} \right)^{1/4}.$

### 7.3 Estimates related to $h^{n+1}(t, X_n^t(x, v), V_n^t(x, v))$

Next we show the control on $h^i$.

**Lemma 7.6** For any $n \geq i \geq 1,$

$$\|v \times ((h^i_0 v) \circ \Omega_{i-1}^{-1}) - (\nabla v \times ((h^i_0 v) \circ \Omega_{i-1}^{-1}))\|_{\mathcal{P}(\lambda' - B_0)^{i+1}(\lambda' - B_0)^{i+1}} \leq (1 + \tau) \delta_i.$$

**Proof.** First, consider $i = 1$.

In fact,

$$\|v \times ((h^1_0 v) - (\nabla v \times ((h^1_0 v))\|_{\mathcal{P}(\lambda' - B_0)^{1+1}(\lambda' - B_0)^{1+1}} \leq \frac{\|v \times (h^1_0 v)\|_{\mathcal{P}(\lambda' - B_0)^{1+1}(\lambda' - B_0)^{1+1}}}{\|v \times (h^1_0 v)\|_{\mathcal{P}(\lambda' - B_0)^{1+1}(\lambda' - B_0)^{1+1}}} \leq \|\nabla v \times ((h^1_0 v))\|_{\mathcal{P}(\lambda' - B_0)^{1+1}(\lambda' - B_0)^{1+1}} \leq (1 + \tau) \delta_1.$$
\[ = \| (\nabla' + \tau \nabla_x) \times (h^1 v) \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} (\eta_k, \eta_k) \leq \| (\nabla' + \tau \nabla_x) \times (h^1 v) \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} (\eta_k, \eta_k) \leq \delta_1, \]

where we use (vi) of Proposition 1.5.

Suppose that \( i = k \), the conclusion holds, that is,

\[
\left\| \nabla' \times ((h^k v) \circ \Omega^{k-1}_t) - (\nabla' \times ((h^k v) \circ \Omega^{k-1}_t)) \right\|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} (\eta_k, \eta_k) \leq (1 + \tau) \delta_k.
\]

We need to show that the conclusion still holds for \( i = k+1 \). We can get the estimate for \( h^{k+1} (t, X^k_t(x, v), V^k_t (x, v)) \) from (3.11).

Note that

\[
\begin{align*}
(\nabla h) \circ \Omega &= (\nabla h)^{-1} \nabla h \\
(\nabla^2 h) \circ \Omega &= (\nabla^2 h)^{-1} \nabla^2 h - (\nabla^2 h)^{-1} \nabla h (\nabla h)^{-1} \Omega.
\end{align*}
\]

(7.23)

Therefore, from (7.23), we get

\[
\begin{align*}
\| (\nabla h^{n+1} \circ \Omega^n_t) \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} (\eta_k, \eta_k) &\leq C(d) \| (\nabla h^{n+1} \circ \Omega^n_t) \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} (\eta_k, \eta_k) \\
&\leq \frac{C(d)(1 + \tau)}{\min\{\lambda_{n+1}^I - \lambda_{n+1}^I, \mu_{n+1} - \mu_{n+1}^I\}} \| h^{n+1} \circ \Omega^n_t \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} (\eta_k, \eta_k).
\end{align*}
\]

(7.24)

and

\[
\begin{align*}
\| (\nabla^2 h^{n+1} \circ \Omega^n_t) \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} (\eta_k, \eta_k) &\leq C(d) \left[ \| \nabla^2 (h^{n+1} \circ \Omega^n_t) \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} + \| \nabla^2 (h^{n+1} \circ \Omega^n_t) \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} \right] (\eta_k, \eta_k) \\
&\leq \frac{C(d)(1 + \tau)}{\min\{\lambda_{n+1}^I - \lambda_{n+1}^I, \mu_{n+1} - \mu_{n+1}^I\}} \| h^{n+1} \circ \Omega^n_t \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} (\eta_k, \eta_k) \\
&\leq \frac{C(d)(1 + \tau)}{\min\{\lambda_{n+1}^I - \lambda_{n+1}^I, \mu_{n+1} - \mu_{n+1}^I\}} \| h^{n+1} \circ \Omega^n_t \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} (\eta_k, \eta_k).
\end{align*}
\]

(7.25)

We first write

\[
\nabla (h^{n+1} \circ \Omega^n_t) - (\nabla h^{n+1} \circ \Omega^n_t) = \nabla (\Omega^n_t - Id) \cdot (\nabla h^{n+1} \circ \Omega^n_t),
\]

and get

\[
\| \nabla (h^{n+1} \circ \Omega^n_t) - (\nabla h^{n+1} \circ \Omega^n_t) \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} (\eta_k, \eta_k) \leq \| \nabla (\Omega^n_t - Id) \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} \| h^{n+1} \circ \Omega^n_t \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} (\eta_k, \eta_k).\]

\[
\leq \frac{C}{\min\{\lambda_n - \lambda_n^I, \mu_n - \mu_n^I\}} \| \nabla h^{n+1} \circ \Omega^n_t \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} (\eta_k, \eta_k) \leq \frac{CC^4}{\min\{\lambda_n - \lambda_n^I, \mu_n - \mu_n^I\}} \sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_k^I))^n} (1 + \tau)^{-2} \| h^{n+1} \circ \Omega^n_t \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} (\eta_k, \eta_k).
\]

the above inequality implies \( \nabla (h^{n+1} \circ \Omega^n_t) \approx (\nabla h^{n+1} \circ \Omega^n_t) \) as \( \tau \to \infty \).

Since

\[
\| \nabla (h^{n+1} \circ \Omega^n_t) \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} \leq C \left( \frac{1 + \tau}{\min\{\lambda_n - \lambda_n^I, \mu_n - \mu_n^I\}} \right) \| h^{n+1} \circ \Omega^n_t \|_{Z_l(\mathbb{R}^3) \cap H^r(\mathbb{R}^3)} (\eta_k, \eta_k).
\]

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and
\[ \| \nabla_x (h^{n+1}_\tau \circ \Omega^n_{t, \tau}) \|_{L^2((\lambda^n_k - \lambda^n_{k+1})/4, \lambda^n_{k+1})} + \| (\nabla_x + \tau \nabla_n) (h^{n+1}_\tau \circ \Omega^n_{t, \tau}) \|_{L^2((\lambda^n_k - \lambda^n_{k+1})/4, \lambda^n_{k+1})} \]
\[ \leq \frac{C}{\min \{ \lambda^n_k - \lambda^n_{k+1}, \mu^n_k - \mu^n_{k+1} \}} \| h^{n+1}_\tau \circ \Omega^n_{t, \tau} \|_{L^2((\lambda^n_k - \lambda^n_{k+1})/4, \lambda^n_{k+1})} \]
we have
\[ \| (\nabla_x h^{n+1}_\tau \circ \Omega^n_{t, \tau}) \|_{L^2((\lambda^n_k - \lambda^n_{k+1})/4, \lambda^n_{k+1})} + \| (\nabla_x + \tau \nabla_n) (h^{n+1}_\tau \circ \Omega^n_{t, \tau}) \|_{L^2((\lambda^n_k - \lambda^n_{k+1})/4, \lambda^n_{k+1})} \]
\[ \leq C \left( \frac{C^4}{\min \{ \lambda^n_k - \lambda^n_{k+1}, \mu^n_k - \mu^n_{k+1} \}^2} \left( \sum_{k=1}^{n} \delta_k \right)^{1/2} \left( \sum_{k=1}^{n} \frac{\delta_k}{(2\pi (\lambda^n_k - \lambda^n_{k+1}))^6} \right) + \frac{1}{\min \{ \lambda^n_k - \lambda^n_{k+1}, \mu^n_k - \mu^n_{k+1} \}} \| h^{n+1}_\tau \circ \Omega^n_{t, \tau} \|_{L^2((\lambda^n_k - \lambda^n_{k+1})/4, \lambda^n_{k+1})} \right) \]

7.4 The Choice of \( \delta_{n+1} \)

Now we see that the corresponding \( n+1 \)th step conclusion of the inductive hypothesis has all been established with
\[ \delta_{n+1} = C\epsilon (1 + C\epsilon) (1 + C^4 \epsilon CT^2) \max \left\{ \frac{1}{\sum_{i=1}^{n} \delta_i} \right\} \left( \frac{1}{\sum_{i=1}^{n} \frac{\delta_i}{(2\pi (\lambda^n_i - \lambda^n_{i+1}))^6}} \right) \delta^2_n. \]

For any \( n \geq 1 \), we set \( \lambda^n_k - \lambda^n_{k+1} = \lambda^n_k - \lambda^n_{k+1} = \mu^n_k - \mu^n_{k+1} = \mu^n_k - \mu^n_{k+1} = \frac{1}{\Lambda^2} \) for some \( \Lambda > 0 \). By choosing \( \Lambda \) small enough, we can make sure that the conditions \( 2\pi (\lambda^n_k - \lambda^n_{k+1}) < 1 \) and \( 2\pi (\mu^n_k - \mu^n_{k+1}) < 1 \) are satisfied for all \( k \), as well as the other smallness assumptions made throughout this section. We also have \( \lambda^n_k - \lambda^n_{k+1} \geq \frac{1}{\Lambda^2} \).

(I) - (VIII) will be satisfied if we choose constants \( \Lambda, \omega > 0 \) such that \( \sum_{i=1}^{\infty} i^{1/2} \delta_i \leq \Lambda \delta \).

Then we have that \( T_n \leq C \epsilon_n (\pi^2/\Lambda) \), so the induction relation on \( \delta_n \) gives \( \delta_1 \leq C\delta \) and \( \delta_{n+1} = C(\pi^2/\Lambda) \epsilon_n C(\pi^2/\Lambda) \delta_{n+1} \).

To make this relation hold, we also assume that \( \delta_n \) is bounded below by the error coming from the short-time iteration; but this follows easily by construction, since the constraints imposed on \( \delta_n \) are much worse than those on the short-time iteration.

Having fixed \( \Lambda \), we will check that for \( \delta \) small enough, the above relation for \( \delta_{n+1} \) holds and there is a fast convergence of \( \{ \delta_i \}_{i=1} \).

The details are similar to that of the local-time case, and it can also be found in [23], here we omit it.

Acknowledgements: I would like to thank professor Pin Yu in YMSC, Tsinghua University who gave me lots of advice. At the same time, I am deeply grateful to Professor Jinxin Xue in YMSC, Tsinghua University who spent lots of time to discuss with me.

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