On Certain Admissible Embeddings of L-groups

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Abstract

Let $F$ be a local field and $E/F$ be a separable extension of degree $n$. Regard $T = \text{Res}_{E/F} G_m$ as an elliptic maximal torus of $G = GL_n$. We can construct an admissible embedding of $L$-groups $^L T \hookrightarrow ^L G$ using Langlands-Shelstad $\chi$-data. Such embedding gives rise to an induced representation of the Weil group $W_F$ of $F$ from a character of $W_E$. The relation between induced representations and admissible embeddings provides a different interpretation of the work of Bushnell-Henniart on the essentially tame local Langlands correspondence.

1 Introduction

Let $F$ be a local field and $W_F$ be the Weil group of $F$. Let $G$ be $GL_n$ as a reductive algebraic group over $F$. Let $E$ be a field extension of degree $n$. We may regard $G(F)$ as the automorphism group of the $F$-vector space $E$ by choosing an $F$-basis of $E$. Write $E^\times$ as the $F$-point of the algebraic torus $T = \text{Res}_{E/F} G_m$. Therefore the multiplicative action of $T$ on $E$ gives rise to an $F$-embedding $T \hookrightarrow G$.

We consider the dual problem as follows. Let $G$ and $T$ be the dual groups of $G$ and $T$. Let $^L G$ and $^L T$ be the corresponding $L$-groups. By fixing a maximal torus $T$ in $G$, we ask whether there exists an admissible embedding $^L T \hookrightarrow ^L G$, an injective morphism of groups that maps $T$ bijectively onto $T$ and the $W_F$-component of $^L T$ identically to that of $^L G$. The answer is affirmative and a construction is given by LSS.

We introduce the idea briefly as follows. The problem can be shown to be equivalent to ask whether the exact sequence $1 \to T \to ^L T \to W_F \to 1$ splits, and is therefore equivalent to ask whether the cohomology class $t = t(^L T) \in H^2(W_F, T)$ defined by $^L T$ is trivial or not. We can construct a splitting for the class $t$ using a collection of characters $\{\chi_\lambda\}_\lambda$ called $\chi$-data. Here $\lambda$ runs through $\mathcal{R}(G, T)$ the root system of the maximal torus $T$ in $G$, and the character $\chi_\lambda$ is defined on the multiplicative group of a field extension $E_\lambda$ over some Galois conjugate of $E$. We shall go over the properties of $\chi$-data and construct the corresponding admissible embedding in Section 3.

The first main result of this article is a relation between admissible embedding and induced representation of Weil group. The 1-cohomology group $H^1(W_F, ^L T) = \text{Int}(T) \backslash \text{Hom}_{W_F}(W_F, ^L T \times W_F)$ is isomorphic to $\text{Hom}(W_E, C^\times)$ naturally. (This fact is known as Shapiro’s Lemma.) Using this fact and Proposition 2.3 the set $\text{Int}(T) \backslash \text{AE}(^L T, ^L G)$ of $\text{Int}(T)$-equivalence classes of admissible embeddings $^L T \to ^L G$ is a $\text{Hom}(W_E, C^\times)$-torsor. By specifying suitable isomorphism between these bijective sets, we have the following.

**Theorem 1.1** (Proposition 2.5). Suppose $\tilde{\xi} \in \text{Hom}_{W_F}(W_F, ^L T \times W_F)$ and $\chi \in \text{AE}(^L T, ^L G)$ correspond to characters $\xi$ and $\mu$ in $\text{Hom}(W_E, C^\times)$ respectively. The composition $\chi \circ \tilde{\xi}$ when projected to $G = GL_n(C)$ is isomorphic to $\text{Ind}_{W_E}^{W_F}(\xi \mu)$ as representations of $W_F$.

The second main result is a converse of the first one. Suppose now $\chi \in \text{AE}(^L T, ^L G)$ is defined by a collection of $\chi$-data $\{\chi_\lambda\}$. We can recover the character $\mu$, in terms of $\{\chi_\lambda\}$, that induces the representation $\text{Ind}_{W_E}^{W_F}(\xi \mu) \cong \chi \circ \tilde{\xi}$.

**Theorem 1.2** (Proposition 1.2). Suppose the admissible embedding $\chi : ^L T \to ^L G$ is defined by $\chi$-data $\{\chi_\lambda\}$. The character $\mu$ in Theorem 1.1 can be taken to be

$$
\mu = \prod_{\text{certain } \lambda} \text{Res}_{E/F}^E \chi_\lambda.
$$
The proof comes from a comparison between the expression of $\chi$ given by the recipe in [LS87] and the matrix coefficients of $\text{Ind}_{W_E}^{W_F}(\xi \mu)$. What missing in Theorem 1.2 is the set through which $\lambda$ runs in the product form of $\mu$. As suggested by the expression of each factor, we take those roots $\lambda$ whose corresponding field extensions $E_{\lambda}$ contain $E$. These roots form a set of representatives of the $W_F$-orbits of the root system $\mathcal{R}(G,T)$. We will be more specific on these representatives using a double coset expression in Proposition 4.1. We emphasize that such $\mu$ cannot be arbitrary. For example, $\mu$ satisfies $\mu|_{F^*} = \det \text{Ind}_{W_E}^{W_F} 1_{W_E}$, as shown in Proposition 4.3.

Finally we give an application on a particular case of the local Langlands correspondence, known as the essentially tame case, established in [BH05, BH10]. Let $\xi \in \Omega(G,T)$ be a non-Archimedean local field of characteristic 0, and $E$ be a tamely ramified extensions over $F$ of degree $n$. For each admissible character $\xi$ of $E^\times$, we introduce a character $\chi_{\mu E}$ of $E^\times$, called the rectifier of $\xi$. Its purpose is to measure the difference between a ‘naive’ version of the local Langlands Correspondence [BH05] and the essentially tame one. In a subsequent article [Tam], we prove that the rectifier admits a factorization of the form in Theorem 1.2 with canonical choices of the characters $\{\chi_{\lambda}\}$. In other words, we can express the essentially tame local Langlands Correspondence by admissible embeddings constructed by $\chi$-data.

Outline of the Article In Section 2 we study the induced representation of Weil group by admissible embedding of $L$-groups and prove Theorem 1.1. To construct an admissible embedding in general we need $\chi$-data, whose definition and properties are discussed in Section 3. We prove the main result Theorem 1.2 and some related facts in Section 4. Finally in Section 5 we describe providently how admissible embedding is related to the essentially tame local Langlands Correspondence.

Notations We fix our notations throughout the article. Let $H$ be a group and $K$ be a subgroup of $H$. The normalizer of $K$ in $H$ is denoted by $N_H(K)$. Suppose $H$ acts on a set $X$. For $h \in H$ and $x \in X$, we write $hx$ or simply $hx$, for the action of $h$ on $x$. The $H$-orbit of $x \in X$ is denoted by $Hx$. The collection of all $H$-orbits of $X$ is denoted by $H \backslash X$. The set of fixed points is denoted by $X^H$. If $f$ is a map whose domain is $X$, we write $hf(x) = f^{h^{-1}}(x) = f^{h^{-1}}x$. If $X$ is an abelian group, we denote the set of $j$-cocycle of $H$ with values in $X$ by $Z^j(H,X)$, and the $j$-cohomology group by $H^j(H,X)$.

Given a field extension $E/F$ and the corresponding Weil groups $W_E \subseteq W_F$, we denote induction $\text{Ind}_{W_E}^{W_F}$ by $\text{Ind}_{E/F}$ and restriction $\text{Res}_{W_E}^{W_F}$ by $\text{Res}_{E/F}$. For $G = \text{GL}_n$ as an $F$-group we define

$$\hat{G} = \text{GL}_n(\mathbb{C}) \quad \text{and} \quad {}^LG = \text{GL}_n(\mathbb{C}) \times W_F,$$

namely the dual-group and the $L$-group of $G$. Given a field extension $E/F$, let $T$ be the $F$-torus $T = \text{Res}_{E/F}^G \mathbb{G}_m$ with

$$\hat{T} = \text{Ind}_{E/F} \mathbb{C}^\times = ((\mathbb{C}^\times)^{[E/F]} \quad \text{and} \quad {}^LT = \hat{T} \times W_F$$

as its dual-group and $L$-group. Denote the root system of $T$ in $G$ by $\mathcal{R}(G,T)$ and the corresponding Weyl group by $\Omega(G,T)$.

2 Induction and Admissible Embedding

Let $G$ be a connected reductive algebraic group defined and quasi-split over $F$. Let $T$ be a maximal torus of $G$ also defined over $F$.

Definition 2.1. An admissible embedding from $^LT$ to $^LG$ is a morphism of groups $\chi : ^LT \to ^LG$ of the form

$$\chi(t \times w) = \iota(t) \tilde{\chi}(w) \times w$$

for some injective morphism $\iota : \hat{T} \to \hat{G}$ and some map $\tilde{\chi} : W_F \to \hat{G}$.

By expanding $\chi(s \times v)(t \times w) = \chi((s \times v)(t \times w))$, we can show that

$$\left(\text{Int} \tilde{\chi}(v)\right)^{\iota(v) \iota(t)} \iota(vt) \text{ and } \tilde{\chi}(vw) = \tilde{\chi}(v) \iota(\tilde{\chi}(w))$$

(1)
for all \( t \in \hat{T} \) and \( v, w \in W_F \). Hence \( \bar{\chi} \) has image in \( N_G(\iota(\hat{T})) \). Conversely if \( \iota \) and \( \bar{\chi} \) satisfy \( 1 \), then the map in Definition 2.1 is an admissible embedding. We can rephrase \( 1 \) as follows. Let \( N_G(\iota(\hat{T})) \times W_F \) acts on \( \hat{T} \) by \( x \times w t = \iota^{-1}(\text{Int}(x)(w \circ \iota(t))) \), then the morphism \( W_F \to \text{Aut}(\hat{T}), w \mapsto w_F \) factors through

\[
W_F \to N_G(\iota(\hat{T})) \times W_F, \ w \mapsto \bar{\chi}(w) \times w.
\]

Let \( \mathcal{H} \) be a subgroup of \( \hat{G} \). Two admissible embeddings \( \chi_1, \chi_2 \) are called \( \text{Int}(\mathcal{H}) \)-equivalent if there is \( x \in \mathcal{H} \) such that \( \chi_1(t \times w) = (x \times 1) \chi_2(t \times w)(x \times 1)^{-1} \) for all \( t \times w \in t^T \). Using \( 1 \) we can show that this condition is equivalent to require an \( x \in \mathcal{H} \) giving \( \chi_1(w) = x \chi_2(w)^{w(x)^{-1}} \) for all \( w \in W_F \).

Let’s provide more preliminary setup. By taking a conjugate of \( T \) in \( G \), which is still denoted by \( T \) for brevity, let \( T \) be contained in a Borel subgroup \( B \) defined over \( F \). Choose an \( W_F \)-invariant splitting \((\mathcal{T}, \mathcal{B}, \{\hat{X}_a\})\) of \( \hat{G} \) and an isomorphism \((\hat{T}, \hat{B}) \to (\mathcal{T}, \mathcal{B})\) whose restriction on \( \hat{T} \) is \( \iota \). For notation convenience we usually omit \( \iota \) and write \( t = \iota(t) \in \mathcal{T} \) for \( t \in \hat{T} \), but bear in mind that \( \hat{T} \) and \( \mathcal{T} \) may have inequivalent \( W_F \)-actions.

**Remark 2.2.** We choose splittings on \( G \) and \( \hat{G} \) so that we have a duality on the bases of \( T \) and \( \mathcal{T} \) for explicit computations. For example, we choose a basis of \( \mathcal{T} \) for the construction of the Steinberg section (see Section 2.1 of [LS87]). Our main results would be independent of these choices. For instance, the \( \hat{G} \)-conjugacy class of an admissible embedding is independent of the choices of the Borel subgroup \( B \) containing \( T \) and the splitting \((\mathcal{T}, \mathcal{B}, \{\hat{X}_a\})\) of \( \hat{G} \) (see [LS87] (2.6.1) and (2.6.2)).

Langlands and Shelstad constructed a particular form of admissible embedding using \( \chi \)-data (see [LS87] (2.5)). We will give the construction in Section 3. Let’s assume such construction for a moment, and denote \( \text{AE}(\mathcal{T}, \mathcal{B}, \{\hat{X}_a\}) \) of \( \hat{G} \). Our main results would be independent of these choices. For instance, the \( \hat{G} \)-equivariant splitting \((\mathcal{T}, \mathcal{B}, \{\hat{X}_a\})\) of \( \hat{G} \) and an isomorphism \((\hat{T}, \hat{B}) \to (\mathcal{T}, \mathcal{B})\) whose restriction on \( \hat{T} \) is \( \iota \). For notation convenience we usually omit \( \iota \) and write \( t = \iota(t) \in \mathcal{T} \) for \( t \in \hat{T} \), but bear in mind that \( \hat{T} \) and \( \mathcal{T} \) may have inequivalent \( W_F \)-actions.

**Proposition 2.3.** The set \( \text{AE}(\mathcal{T}, \mathcal{B}, \hat{G}) = Z^1(W_F, \hat{T}) \)-torsor, and the set of its \( \text{Int}(\mathcal{T}) \)-equivalence classes \( \text{Int}(\mathcal{T}) \backslash \text{AE}(\mathcal{T}, \mathcal{B}, \hat{G}) \) is an \( H^1(W_F, \hat{T}) \)-torsor.

**Proof.** We fix an embedding \( \chi_0 \in \text{AE}(\mathcal{T}, \mathcal{B}, \hat{G}) \) and take \( \bar{\chi}_0 : W_F \to \hat{G} \) as in Definition 2.1. Then for each \( \chi \in \text{AE}(\mathcal{T}, \mathcal{B}, \hat{G}) \), the difference \( \bar{\chi} \chi_0^{-1} \) is a 1-cocycle of \( W_F \) valued in \( T \), i.e. \( \bar{\chi} \chi_0^{-1} \in Z^1(W_F, T) \). Indeed for a fixed \( w \in W_F \) both \( \bar{\chi}(w) \) and \( \chi_0(w) \) project to the same element in \( \Omega(\mathcal{G}, \mathcal{T}) = N_{\hat{G}}(\mathcal{T})/\mathcal{T} \). Using \( 1 \) we have that

\[
\bar{\chi} \chi_0^{-1}(vw) = \bar{\chi}(v) \bar{\chi}(w) \chi_0(w)^{-1} \chi_0(v)^{-1} = \bar{\chi}(v) \chi_0(v)^{-1} \chi_0(w)^{-1} \chi_0(w)^{-1},
\]

for all \( v, w \in W_F \). We can readily verify that the map

\[
\text{AE}(\mathcal{T}, \mathcal{B}, \hat{G}) \to Z^1(W_F, \hat{T}), \ \chi \mapsto \bar{\chi} \chi_0^{-1}
\]

is bijective. From the equality \( t\bar{\chi}(v) \chi_0(v)^{-1} t^{-1} = t\bar{\chi}(v) \chi_0^{-1}(v) w t^{-1} \) for all \( t \in \mathcal{T} \), we know that two embeddings are \( \text{Int}(\mathcal{T}) \)-equivalent if and only if the corresponding 1-cocycles differ by a coboundary in \( Z^1(W_F, \hat{T}) \).

**Remark 2.4.** For \( G = \text{GL}_n \) we can construct an explicit embedding \( T \to L G \). Choose \( \mathcal{T} \) to be the diagonal subgroup of \( \hat{G} \). We embed \( \hat{T} \) into \( \hat{G} \) with image \( \mathcal{T} \) and define

\[
W_F \to N_{\hat{G}}(\hat{T}), \ w \mapsto N(w)
\]

the permutation matrix whose assignment is according to the \( W_F \)-action on \( \hat{T} \cong C^{[E/F]} \), i.e. \( \text{Int}(N(v)) t = v t \) for all \( t \in \hat{T} \). Clearly the map \( T \to L G, t \times w \mapsto t N(w) \times w \) defines an admissible embedding.

For \( G = \text{GL}_n \) and \( T = \text{Res}_{E/F} \text{GL}_m \) we give the main result Proposition 2.5 of this section after the following setup. By Shapiro’s Lemma (see the Exercise in [Ser79] VII §5.), we have a special case of Langlands Correspondence for torus

\[
\text{Hom}(E^x, C^x) = H^1(W_F, \hat{T}).
\]
The precise correspondence is given as follows. Suppose \( \xi \) is a character of \( E^\times \). We regard \( \xi \) as a character of \( W_E \) by class field theory \cite{Tat79}. Take a collection of coset representatives \( \{g_1, \ldots, g_n\} \) of \( W_E \backslash W_F \). Define for each \( g_i \) a map \( u_{g_i} : W_F \to W_E \) given by
\[
g_i w = u_{g_i}(w)g_i(w) \quad \text{for } g_i(w) \in \{g_1, \ldots, g_n\}.
\]
Then define
\[
\tilde{\xi} : W_F \to \hat{T} \cong \mathbb{C}^n, \ w \mapsto (\xi(u_{g_1}(w)), \ldots, \xi(u_{g_n}(w))) .
\]
It can be checked that \( \tilde{\xi} \) is a 1-cocycle in \( Z^1(W_F, \hat{T}) \), and different choices of coset representatives give cocycles different from \( \tilde{\xi} \) by a 1-cohomobdary. Hence the 1-cohomology class of \( \tilde{\xi} \) is defined. By abusing of language, we call \( \tilde{\xi} \) a Langlands parameter of \( \xi \). Moreover, combining Proposition 2.3 and (4) we have
\[
\text{Hom}(E^\times, \mathbb{C}^\times) = \text{Int}(\mathcal{T}) \backslash \text{AE}(L^T, L^G).
\]
Explicitly, if we have a character \( \mu \) of \( E^\times \), then define
\[
\chi : L^T \to L^G, \ t \times w \mapsto t \begin{pmatrix}
\mu(u_{g_1}(w)) \\
\vdots \\
\mu(u_{g_n}(w))
\end{pmatrix} N(w) \times w .
\]
Here \( N(w) \) is the permutation matrix as introduced in Remark 2.4. Notice that this bijection is non-canonical. Write \( \text{proj} : L^G \to \hat{G}, \ g \times w \mapsto g \), which is a morphism of groups because \( G = \text{GL}_n \) splits over \( F \). Combining the bijections (2) and (4), we have the following result.

**Proposition 2.5.** Suppose that \( \xi = \chi \) and \( \mu \) come from \( \tilde{\xi} \) and \( \chi \) by the bijections (2) and (4). The composition
\[
H^1(W_F, \hat{T}) \times \text{Int}(\mathcal{T}) \backslash \text{AE}(L^T, L^G) \to \text{Int}(\hat{G}) \backslash \text{Hom}_{W_F}(W_F, \hat{G})
\]
such that \( (\tilde{\xi}, \chi) \mapsto \chi \circ \tilde{\xi} \) gives an isomorphism \( \text{proj} \circ \chi \circ \tilde{\xi} \cong \text{Ind}_{E/F}(\xi \mu) \) as representations of \( W_F \).

**Proof.** Choose a suitable basis on the representation space of \( \text{Ind}_{E/F}(\xi \mu) \). For example, if we realize our induced representation by the subspace of functions
\[
\{ f : W_F \to \mathbb{C} | f(xg) = \xi \mu(x)f(g) \text{ for all } x \in W_E, g \in W_F \},
\]
then we choose those \( f_i \) determined by \( f_i(g_j) = \delta_{ij} \) (Kronecker delta) as basis vectors. The matrix coefficient of \( \text{Ind}_{E/F}(\xi \mu) \) is therefore
\[
\begin{pmatrix}
\mu \xi(u_{g_1}(w)) \\
\vdots \\
\mu \xi(u_{g_n}(w))
\end{pmatrix} N(w)
\]
does the same matrix as the image of \( \chi \circ \tilde{\xi} \).

**Remark 2.6.** We can recover \( \xi \) from \( \text{Ind}_{E/F}(\xi \mu) \) as follows. We choose the first \( k \) coset representatives \( g_1 = 1, g_2, \ldots, g_k \) to be those in the normalizer \( N_{W_E}(W_E) = \text{Aut}_F(E) \), and (by choosing suitable basis) consider the matrix coefficient of \( \text{Res}_{E/F} \text{Ind}_{E/F}(\xi) \). The first \( k \) diagonal entries are always non-zero and give the characters \( \xi^n \).

## 3 Langlands-Shelstad \( \chi \)-data

In this section we recall the construction of admissible embeddings \( L^T \to L^G \) given in Chapter 2 of \cite{LS87}. Here \( G \) is a connected reductive algebraic group defined and quasi-split over \( F \) and \( T \) is a maximal torus in \( G \) also defined over \( F \). Take a maximal torus \( T \) of \( \hat{G} \) and choose a splitting \( (\mathcal{T}, \mathcal{B}, \{X_\alpha\}) \) for \( \hat{G} \). We again emphasis that different choices yield \( \text{Int}(\hat{G}) \)-equivalent embeddings. For computational convenience we choose \( \mathcal{T} \) to be the diagonal group and \( \mathcal{B} \) to be the group of upper triangular matrices. The tori \( \hat{T} \) and \( T \) are isomorphic as groups but with different \( W_F \)-actions.
Remark 3.1. If we regard a group subgroup of \( N \), then the map \( n : W_F \to N_G(T) \times W_F, w \mapsto n(w) = \tilde{n}(w) \times w \).

The map \( n \) may not be a morphism of groups, yet \( \tilde{n} \) satisfies the first equation in (1) in place of \( \tilde{\chi} \).

We then define a collection of \( \chi \)-data, such that the action of \( \omega(w) \) on \( t \in T \) is the same as \( \omega \cdot t \). We recall in (2.1) of [LS87] the Steinberg section \( n_{St} : \Omega(G, T) \to N_G(T) \) and define

\[
\tilde{n}(w) \equiv \tilde{n} \left( \omega^r \tilde{n}(w) \right) \tilde{n}(w)^{-1} = \tilde{n}(w)^{r \omega \tilde{n}(w)} \tilde{n}(w)^{-1},
\]

(5) a 2-cocycle of \( W_F \), whose values are in \( \{ \pm 1 \}^n \subseteq T \) by Lemma 2.1.A of [LS87]. Hence the problem of seeking such \( \tilde{\chi} \) is equivalent to looking for a map \( r_b : W_F \to T \) that splits \( t_b^{-1} \), i.e.

\[
r_b(v, w)^{r_b(v) r_b(w)} = t_b(v, w)^{-1}.
\]

(6)

**Definition 3.2.** We define a collection of characters \( \{ \chi_\lambda : E_{\pm \lambda}^\times \to \mathbb{C}^\times | \lambda \in R \} \), called \( \chi \)-data, such that the following conditions hold.

(i) For each \( \chi \in R \), we have \( \chi_{-\lambda} = \chi_\lambda^{-1} \) and \( \chi_{w \lambda} = \chi_\lambda^{w^{-1}} \) for all \( w \in W_F \).

(ii) If \( \lambda \) is symmetric, then \( \chi|_{E_{\pm \lambda}^\times} \) equals the quadratic character \( \delta_{E_{\pm \lambda}/E_{\pm \lambda}} \) attached to the extension \( E_{\pm \lambda}/E_{\pm \lambda} \).

(7)

If we choose \( R_0 \) to be a subset of \( R \) consisting of representatives of \( W_F \backslash R_{sym} \) and \( W_F \backslash R_{asym} \), then by condition (4) it is enough to define a collection of \( \chi \)-data on \( R_0 \). For a chosen \( \chi \)-data \( \{ \chi_\lambda \}_{\lambda \in R_0} \), following (2.5) of [LS87] we define for each \( \lambda \in R_0 \) a map

\[
r_\lambda : W_F \to T, \quad w \mapsto \prod_{g \in W_{\pm \lambda} \setminus W_F} \chi_\lambda(v_1(u_{g_1}(w)))^{g_1^{-1} \lambda},
\]

(8) where \( u_{g_1} \) is the map (3) for \( W_{\pm \lambda} \setminus W_F \) and \( v_1 \) is defined similarly for \( W_{\pm \lambda} \setminus W_{\pm \lambda} \). We then define

\[
r_\rho = \prod_{\lambda \in R_0} r_\lambda.
\]
Such construction yields (Lemma 2.5.A of [LS87]) a 2-cocycle

\[ t_g(v, w) = r_g(v)^{r_g(w)} w r_g(v) w^{-1} \in Z^2(W_F, \{\pm 1\}^n). \]

(9)

In constructing the 2-cocycles [3] and [9] we implicitly used two different notions of gauges (defined just before Lemma 2.1.B of [LS87]) on the set \( \mathcal{R} \). To relate them we introduce a map (see (2.4) of [LS87])

\[ s = s_b/g : W_F \to \{\pm 1\}^n \text{ such that } \]

\[ s(v)^{r_g} s(w) s(vw)^{-1} = t_b(v, w) t_g(v, w)^{-1}. \]

(10)

Write \( r_b = s_b/g r_g \) and \( \tilde{\chi} = r_b \tilde{n} \).

**Proposition 3.3.** The map \( \chi \) defines an admissible embedding \( L^T \to L^G \).

**Proof.** It suffices to show that \( \tilde{\chi} \) satisfies the two conditions in [1]. The first condition is just from the definition of \( n(w) \), while the second condition is a straightforward calculation using [3], [6], [9], and [10]. \( \square \)

## 4 The Main Results

Let \( G = GL_n \) and \( T = \text{Res}_{E/F} GL_n \), both regarded as algebraic groups over \( F \). Any root in the root system \( \mathcal{R} = \mathcal{R}(G, T) \) can be expressed as \( [g/h] \) for some \( g, h \in W_F \), with

\[ [g/h](t) = g_{t^{-1}} t^{-1} \]

for all \( t \in E^\times \). The \( W_F \)-action on \( \mathcal{R} \) is given by \( w [g/h] = \left[ gw^{-1} h_{w^{-1}} \right] \) for \( w \in W_F \). Notice that such action factors through the one of the Weyl group \( \Omega(G, T) \). If we choose a collection of coset representatives \( \{g_1 = 1, g_2, \ldots, g_n\} \) of \( WE_W \), then we can write \( \lambda = [g_i/g_j] \) with \( i \neq j \). It is clear that each orbit \( W_F \lambda \) contains a root of the form \( [g_j] \) for some \( g \in \{g_2, \ldots, g_n\} \).

**Proposition 4.1.** The set \( W_F \mathcal{R} \) of \( W_F \)-orbits of the root system \( \mathcal{R} \) is bijective to the collection of non-trivial double cosets in \( WE_W \), by

\[ W_F \mathcal{R} \to \{WE_W\} - \{WE_E\}, \quad W_F \lambda = [g] \mapsto W_E g W_E. \]

**Proof.** The set of roots \( \mathcal{R} \) can be identified with the set of off-diagonal elements of \( WE_W \times WE_W \), with \( W_F \)-action by \( \theta(WE_g, WE_h) = (W_E g^{-1} E W_E g^{-1}) \). By elementary group theory, we know that the orbits are bijective to the non-trivial double cosets in \( WE_W \).

We denote \( (WE_W) \) the collection of non-trivial double cosets, and \( [g] \) the double coset \( W_E W_E \). We call \( g \in W_F \) symmetric if \( [g] = [g^{-1}] \) and asymmetric otherwise. Clearly such symmetry descends to an analogous property on \( (WE_W) \). By Proposition 4.1 the symmetry of \( (WE_W) \) is equivalent to the symmetry of \( W_F \mathcal{R} \). Let

(i) \( (WE_W)_{sym} \) be the set of symmetric non-trivial double cosets,

(ii) \( (WE_W)_{asym} \) be the set of asymmetric non-trivial double cosets, and

(iii) \( (WE_W)_{asym/+\} \) be the set of equivalent classes of \( (WE_W)_{asym} \) by identifying \( [g] \) with \( [g^{-1}] \).

Let \( \mathcal{D} \) be a set of representatives in \( g \in WE_W \) of \( (WE_W)_{sym} \) and \( (WE_W)_{asym/+\} \). If \( \lambda = [g] \), we write \( \chi_\lambda \) as \( \chi_g \) and \( E_\lambda \) as \( E_g \) which equals \( E_g = g^{-1} E E \). By condition [6] of Definition 3.2 a collection of \( \chi \)-data \( \{\chi_\lambda\} \) depends only on its sub-collection \( \{\chi_g\} \) for \( g \in \mathcal{D} \). We also call such sub-collection \( \chi \)-data.

Given \( \chi \)-data \( \{\chi_g\} \) let \( \chi \) be the admissible embedding defined by \( \{\chi_g\} \) in Proposition 3.3. Let \( \xi \) be a character of \( E^\times \) and \( \xi \in Z^1(W_F, T) \) be a Langlands parameter of \( \xi \). (The choice of \( \xi \) is known to be irrelevant.) In Proposition 2.5 we described an induced representation \( \text{Ind}_{E/F} \xi \) as certain embedding of the image of \( \xi \) into \( GL_n(\mathbb{C}) \). Here we have the reverse.
Proposition 4.2. Given \( \chi \)-data \( \{ \chi_g \} \), define
\[
\mu = \mu(\chi_g) = \prod_{[g] \in (W_E \setminus W_F / W_E)'} \Res^{E^\times}_{E^\times} \chi_g.
\]
Let \( \chi \) be the admissible embedding defined by \( \{ \chi_g \} \). Then for all character \( \xi \) of \( E^\times \), the composition
\[
W_F \xrightarrow{\xi} T \ltimes W_F \xrightarrow{\chi} \hat{G} \times W_F \xrightarrow{\text{proj}} \GL_n(\mathbb{C})
\]
is isomorphic to \( \text{Ind}_{E/F}(\xi \mu) \) as representations of \( W_F \).

Remark 4.3. Notice that the product in Proposition 4.2 is uniquely determined by \( \{ \chi_\lambda \} \), i.e. independent of the representative \( g \in \mathcal{D} \), which is itself a coset representative of \( W_E \setminus W_F \), of the double coset \([g]\). Indeed if \( x[g] = [h] \) for some \( x \in W_F \), then \( xE_g = E_h \) and so \( \Res^{E^\times}_{E^\times} \chi_h = \Res^{E^\times}_{E^\times} \chi_g \) by condition 4 of Definition 3.2.

Remark 4.4. Suppose we have fixed a character \( \xi \) of \( E^\times \). Take a subset \( \{ g_1, g_2, \ldots, g_k \} \) of coset representatives of \( W_E \setminus W_F \) in the normalizer \( N_{W_F}(W_E) = \text{Aut}_F(E) \) and write \( \mu_1 = \mu(\chi_g) \) as in Proposition 4.2. Then all other characters \( \mu_k \) such that \( \text{Ind}_{E/F}(\xi \mu_k) \cong \text{Ind}_{E/F}(\xi \mu_1) \) are of the form \( \mu_k = \xi^g \mu_1 \). This character \( \mu_k \) also has a factorization in Proposition 4.2 with the same \( \chi \)-data of \( \mu \), except when \( g = g_k \) the character \( \chi_g \) is changed according to the following.

(i) If \( g \) is symmetric, then \( \chi_g \) is replaced by \( \xi^g \chi_g \).

(ii) If \( g \) is asymmetric, then \( \chi_g \) is replaced by \( \chi_g \) and so \( \chi_g^{-1} \) by \( \xi^{-1} \chi_g^{-1} \).

Proof. (of Proposition 4.2) We first abbreviate \( H = W_F \) and \( K = W_E \). For each \( \lambda = [g] \) we denote \( K_g = K \cap g^{-1}Kg \), which equals \( W_{\pm \lambda} \). If \( [g] \in (K \setminus H / K)_{\pm} \), then because \( KgK = Kg^{-1}K \) we can replace \( g \) by an element in \( Kg \) such that \( g^2 \in K \). Subsequently we have \( g \in W_{\pm \lambda} \) and \( g^2 \in K_g = W_{\mp \lambda} \). We denote \( K_{\pm g} \) the group generated by \( K \cap g^{-1}Kg \) and \( g \), which equals \( W_{\pm \lambda} \). By condition 4 of Definition 3.2, we rewrite the product in Proposition 4.2 as
\[
\prod_{[g] \in (K \setminus H / K)_{\pm}} \Res^{E^\times}_{E^\times} \chi_g (\Res^{E^\times}_{E^\times} \chi_g)^{-1} \prod_{[g] \in (K \setminus H / K)_{\pm}} \Res^{E^\times}_{E^\times} \chi_g.
\]
(11)
Recall that our dual group \( \mathcal{T} \) is the diagonal subgroup. In order to check \( \chi \) gives rise to a character \( \mu \) as (11) it is enough to consider the first entry of \( r_\xi \) (see 7) and the discussion in Remark 2.4. From 7 we have
\[
r_\xi(w) = \left( \prod_{[g] \in (K \setminus H / K)_{\pm}} \chi_g(u_{g_1}(w))^{[g_1][g_1]} \right) \left( \prod_{[g] \in (K \setminus H / K)_{\pm}} \chi_g(v_{u_{g_1}}(w))^{[g_1][g_1]} \right).
\]
By restricting \( w \in W_E \), we get the first entry of \( r_\xi(w) \), namely
\[
r_\xi(w)_1 = \left( \prod_{g \in (K \setminus H / K)_{\pm}} \chi_g(u_{g_1}(w)) \right) \left( \prod_{g_1 \in K_{g_1} \setminus H} \chi_g(u_{g_1}(w))^{g_1} \right) \left( \prod_{g \in (K \setminus H / K)_{\pm}} \chi_g(v_{u_{g_1}}(w)) \right) \left( \prod_{g_1 \in K_{g_1} \setminus H} \chi_g(v_{u_{g_1}}(w))^{g_1} \right).
\]
(12)
We now analyze the products in (12) and match them to those in (11). First, for \( g \in (K/H/K)_{\text{sym/}} \), the first product of (12) 
\[
\prod_{g_i \in K_g \backslash K} \chi_{g}(u_{g_i}(w)), \quad w \in K
\]
is the transfer map \( T^K_{K_g} : K^{ab} \to (K_g)^{ab} \). By class field theory (see [Tat79]), it corresponds to the inclusion \( E^x \hookrightarrow E^x_g \). Therefore 
\[
\prod_{g_i \in K_g \backslash K} \chi_{g}(u_{g_i}(w)) = \text{Res}_{E^x}^{E^x_g} \chi_g(w),
\]
which is the first factor in (11). Next we consider (the inverse of) the second product of (12) 
\[
\prod_{g_i \in K_g \backslash H} \chi_{g}(u_{g_i}(w)), \quad w \in K
\]
For \( \chi g \in K_g \backslash H \) such that \( gg_i \in K \), we can write \( g_i = g_i^\chi x_i \) for some \( x_i \) running through a set in \( K \) of representatives of \( K \cap gKg^{-1} \backslash K \). If \( u_{x_i} \) is the map (3) for \( K \cap gKg^{-1} \backslash K \) then we have 
\[
g^{-1}(x_i w) = g^{-1}(u_{x_i}(w)x_j(x_i, w)), \quad (13)
\]
where \( u_{x_i}(w) \in K \cap gKg^{-1} \). On the other hand, by regarding \( g^{-1}x_i \in K_g \backslash H \) we have 
\[
g^{-1}x_i w = u_{g^{-1}x_i}(w)g_j(g^{-1}x_i, w) \quad (14)
\]
where \( u_{g^{-1}x_i}(w) \in K_g \) and \( g_j(g^{-1}x_i, w) \) is of the form \( g^{-1}x_j \) for some \( j \). By comparing (13) and (14) we have 
\[
g^{-1}u_{x_i}(g) = u_{g^{-1}x_i}(w) \quad (15)
\]
and hence 
\[
\prod_{g_i \in K_g \backslash H} \chi_{g}(u_{g_i}(w)) = \chi_{g}^{g^{-1}}(T^K_{K \cap gKg^{-1}}(w)) = (\text{Res}_{E^x}^{E^x_g} \chi_g^{g^{-1}})(w)
\]
which is (the inverse of) the second factor in (11). Finally, for \( g \in (K/H/K)_{\text{sym}} \), we choose coset representatives \( g_1, \ldots, g_k, gg_1, \ldots, gg_k \) for \( K_g \backslash H \) such that \( g_1, \ldots, g_k \) are those of \( K_{\pm g} \backslash H \). Moreover we can assume that 
\[
g_1, \ldots, g_h, ggh_1, \ldots, ggh_2h \in K.
\]
Hence the third product in (12) is 
\[
\prod_{g_i \in K_g \backslash H} \chi_{g}(v_1(u_{g_i}(w))), \quad w \in K
\]
Here \( v_i g \) is the map (3) for \( K \backslash H \) and so \( v_1 u_{g_i} \) is the one for \( K_g \backslash H \). For the fourth product in (12), because \( \chi_g^g = \chi_g^{-1} \) (by condition (11) of Definition 3.2) and \( g(v_1(u_{g_i}(w)))g^{-1} = v_1(u_{gg_i}(w)) \), we have indeed 
\[
\prod_{g_i \in K_{\pm g} \backslash H, gg_i \in K} \chi_{g}(v_1 u_{g_i}(w))^{-1} = \prod_{i=h+1}^{2h} \chi_{g}(v_1(u_{gg_i}(w))).
\]
Therefore the product of (15) and (16) is \( \chi_{g}(T^K_{K_g}(w)) = (\text{Res}_{E^x}^{E^x_g} \chi_g)(w) \) which is the last factor of (11). \( \square \)
The product $\mu = \mu_{\{\chi_g\}}$ in Proposition 4.2 as $\{\chi_g\}$ runs through all $\chi$-data, does not produce arbitrary character of $E^\times$. Its restriction on $F^\times$ has a specific form by Proposition 4.7. First recall the following known results. Given a group $H$ we write $1_H$ the trivial representation of $H$. If $K$ is a subgroup of $H$ of finite index, denote $T^K_H : H^{ab} \rightarrow K^{ab}$ the transfer morphism. For any $g \in H$, we write $gKg^{-1} = gKg^{-1}$.

**Proposition 4.5.** Let $\sigma$ and $\pi$ be finite dimensional representations of $K$ and $H$ respectively. We have the following formulae.

(i) (Mackey’s Formula)
\[
\text{Res}^H_K \text{Ind}^H_K \sigma \cong \bigoplus_{[g] \in K \backslash H / K} \text{Ind}^K_{Kg} \text{Res}^g_{Kg} \sigma.
\]

(ii) $\text{det Ind}^H_K \sigma \cong (\text{det Ind}^H_K 1_K)^{\dim \sigma} \otimes (\text{det} \circ T^K_H)$.

(iii) $(\text{det} \text{Res}^H_K \pi) \circ T^K_H = (\text{det} \pi)^{|H/K|}$.

**Proof.** Formulae (i) and (ii) are well-known, for example (i) is proved in [Ser77] 7.3, and (ii) can be found in the Exercise in [Ser79] VII §8. Formula (iii) is direct from (ii) if we take $\sigma = \text{Res}^H_K \pi$. \(\square\)

In particular, if $\chi$ is a character of $K$, then by (ii) we have
\[
\chi \circ T^K_H \cong \left( \text{det Ind}^H_K \chi \right) \left( \text{det Ind}^H_K 1_K \right).
\]

**Lemma 4.6.** We have the formula
\[
\text{det Ind}^H_K 1_K = \prod_{[g] \in K \backslash H / K'} \text{det Ind}^H_{Kg} 1_{Kg}.
\]

Notice that $\text{det Ind}^H_K 1_{Kg}$ is independent of the choice of representative $g$ of the double coset $[g]$ if we interpret the character as the sign of the canonical $H$-action on $H/K_g$. For all representatives of $[g]$, the corresponding actions are equivalent each other.

**Proof.** (of Lemma 4.6) Applying Mackey’s formula on $\sigma = 1_K$ we obtain
\[
\text{Res}^H_K \text{Ind}^H_K 1_K \cong \bigoplus_{[g] \in K \backslash H / K} \text{Ind}^K_{Kg} 1_{Kg}.
\]

We take determinant and then transfer morphism $T^K_H$ on both sides. By (ii) and (iii) of Proposition 4.5 we obtain
\[
\left( \text{det Ind}^H_K 1_K \right)^{|H/K|} = \prod_{[g] \in K \backslash H / K} \left( \text{det Ind}^H_{Kg} 1_{Kg} \right) \left( \text{det Ind}^H_K 1_K \right)^{|K/K_g|}.
\]

Because the sum of $|K/K_g|$ for $[g]$ runs through $K \backslash H / K$ is $|H/K|$, the factor $(\text{det Ind}^H_K 1_K)$ on both sides vanish. What remains gives the desired formula. \(\square\)

**Proposition 4.7.** For all $\chi$-data $\{\chi_g\}$, if $\mu$ is the character of $E^\times$ defined by $\{\chi_g\}$ as in Proposition 4.2, then $\mu_{|F^\times} \cong \text{det Ind}_{E/F} 1_{W_E}$.

**Proof.** We first abbreviate $H = W_F$, $K = W_E$. For each $\lambda = [\chi_g]$ we denote $K_\lambda = K \cap gK = W_{+\lambda}$ and $K_{\pm \lambda} = W_{\pm \lambda}$. The isomorphism in Proposition 4.7 can be rewritten as
\[
\prod_{[g] \in (K \backslash H / K)'} \chi_g \circ T^K_{Kg} = \text{det Ind}^H_K 1_K.
\]

By Lemma 4.6 we have to show that
\[
\prod_{[g] \in (K \backslash H / K)'} \chi_g \circ T^K_{Kg} = \prod_{[g] \in (K \backslash H / K)'} \text{det Ind}^H_{Kg} 1_{Kg}.
\]

By comparing termwise, we claim that
If \([g] \in (K \setminus H/K)_{\text{asym}}\), then
\[
\left( \chi_g \circ T_{K_g}^H \right) \left( \chi_{g^{-1}} \circ T_{K_{g^{-1}}}^H \right) = \left( \det \text{Ind}_{K_g}^H 1_{K_g} \right) \left( \det \text{Ind}_{K_{g^{-1}}}^H 1_{K_{g^{-1}}} \right) \equiv 1.
\]

(ii) If \([g] \in (K \setminus H/K)_{\text{sym}}\), then \(\chi_g \circ T_{K_g}^H = \det \text{Ind}_{K_g}^H 1_{K_g}\).

If \([g] \in (K \setminus H/K)_{\text{asym}}\), then we have \(K_{g^{-1}} = gK_g\), which is the stabilizer of the root \([g^{-1} = 1\) by condition (i) of Definition 3.2 we have
\[
\left( \chi_g \circ T_{K_g}^H \right) \left( \chi_{g^{-1}} \circ T_{K_{g^{-1}}}^H \right) = \left( \chi_{g^{-1}} \circ T_{K_{g^{-1}}}^H \right) \equiv 1.
\]

On the other hand, since the \(H\)-action on \(H [g^{-1} = 1\) is equivalent to that on \(H [g = 1\), we have \(\text{Ind}_{K_g}^H 1_{K_g} \cong \text{Ind}_{K_{g^{-1}}}^H 1_{K_{g^{-1}}}\). Therefore \(\left( \det \text{Ind}_{K_g}^H 1_{K_g} \right) \left( \det \text{Ind}_{K_{g^{-1}}}^H 1_{K_{g^{-1}}} \right) \equiv 1\). We have proved the first claim. If \([g] \in (K \setminus H/K)_{\text{sym}}\), then we have an isomorphism \(\text{Ind}_{K_g}^{K_{g^{-1}}} 1_{K_g} \cong 1_{K_{g^{-1}}} \oplus \delta_{K_{g^{-1}}/K_g}\) as representations of \(K_{g^{-1}}\). Here \(\delta_{K_{g^{-1}}/K_g}\) is the quadratic character of \(K_{g^{-1}}\). We denote this character by \(\delta\). Hence \(\text{Ind}_{K_g}^H 1_{K_g} \cong \text{Ind}_{K_{g^{-1}}}^H 1_{K_{g^{-1}}} \oplus \text{Ind}_{K_{g^{-1}}}^H \delta\) and
\[
\det \text{Ind}_{K_g}^H 1_{K_g} \cong (\det \text{Ind}_{K_{g^{-1}}}^H 1_{K_{g^{-1}}})(\det \text{Ind}_{K_{g^{-1}}}^H \delta)
\]
by taking determinant. Now condition (ii) of Definition 3.2, namely \(\chi_g \circ T_{K_g}^{K_{g^{-1}}} = \delta\), gives \(\chi_g \circ T_{K_g}^H = \delta \circ T_{K_g}^{K_{g^{-1}}}\).

By 17, this is just the right side of 19. We have proved the second claim and therefore Proposition 17. □

5 An application on Local Langlands Correspondence

We recall briefly the essentially tame local Langlands Correspondence, in the sense of [BH05 BH10]. Let \(F\) be a non-Archimedean local field of characteristic 0. Let \(G\) be GL\(_n\) as a reductive group over \(F\). Let \(\mathcal{A}_n^\text{et}(F)\) be the set of the isomorphism classes of irreducible essentially tame supercuspidal representations of \(G(F)\), and \(\mathcal{G}_n^\text{et}(F)\) be the set of the equivalence classes of essentially tame \(n\)-dimensional irreducible complex representations of \(W_F\). The two notions of essential tameness above are defined in [BH05]. These two sets are bijective, whose map
\[
\mathcal{L} = \mathcal{L}_n^\text{et} : \mathcal{G}_n^\text{et}(F) \to \mathcal{A}_n^\text{et}(F)
\]
is called the essentially tame local Langlands Correspondence.

We introduce a collection for describing \(\mathcal{L}\) explicitly. Let \(P_n(F)\) be the set of \(W_F\)-equivalence classes \((E, \xi)\) of admissible characters \([BH05]\) \(\xi\) of \(E^\times\) in which \(E/F\) is a tamely ramified extension of degree \(n\). By [BH05] we know that \(P_n(F)\) bijectively parameterizes both \(\mathcal{A}_n^\text{et}(F)\) and \(\mathcal{G}_n^\text{et}(F)\) simultaneously. Here the bijection \(\sigma : P_n(F) \to \mathcal{G}_n^\text{et}(F)\) is simply induction of representations, while the one \(\pi : P_n(F) \to \mathcal{A}_n^\text{et}(F)\) is constructed in [BK93] and [BH05]. The ‘naive’ Correspondence \(\pi \circ \sigma^{-1} : \mathcal{G}_n^\text{et}(F) \to P_n(F) \to \mathcal{A}_n^\text{et}(F)\) does not satisfy certain conditions of the essentially tame local Langlands Correspondence (see Theorem 3.1 of [BH05] or Remark 5.2). In other words, the composition
\[
\mu : P_n(F) \xrightarrow{\sigma} \mathcal{G}_n^\text{et}(F) \xrightarrow{\xi} \mathcal{A}_n^\text{et}(F) \xrightarrow{\pi^{-1}} P_n(F)
\]
does not give the identity map on \(P_n(F)\). In [BH10] it is proved that for each admissible character \(\xi\) of \(E^\times\), there is a character \(F \mu \xi\) of \(E^\times\), called the rectifier of \(\xi\), such that \(F \mu \xi\) is also admissible and \(\mu(E, \xi) = (E, F \mu \xi)\).

In terms of admissible embeddings \(L^T \to L^G\), it means that for each \(\xi\) we have to embed the image of the chosen Langlands parameter of \(\xi\) by not the canonical one defined by the Weyl group action (as in Remark 2.3) but the one twisted by its rectifier \(F \mu \xi\). The rectifier \(F \mu \xi\) is explicitly described in [BH10], and so is the Correspondence \(\mathcal{L}\). Using this description we prove the following result in [Tam].
**Theorem 5.1.** For each admissible $\xi$, the rectifier $F_{\mu \xi}$ has a factorization of the form as in Theorem 1.2, for some canonical choice of $\chi$-data.

The proof requires a substantial amount of new concepts and computations, so it is better to deal with these in a separated article. This Theorem suggests that the rectifier has more properties inherited from that of $\chi$-data. Indeed the symmetric structure of $F_{\mu \xi}$ form that of $\chi$-data $\{\chi_g\}$ is almost trivial because it is known [Tam] that those $\chi_g$ are of order at most 2 except for exactly one whose has order at most 4. We will have a closer look on this and also other inherited properties of $F_{\mu \xi}$ in [Tam]. The following is a property which can be stated with the knowledge of this article.

**Remark 5.2.** Suppose $\sigma \in G_E^n(F)$ and $\pi = \mathcal{L}(\sigma) \in A_E^n(F)$. Let $\omega_\pi$ be the central character of $\pi$. One of the conditions of Langlands Correspondence, namely $\omega_\pi = \det \sigma$, implies that $F_{\mu \xi}|_{F^\times} = \det \text{Ind}_{E/F}1_{W_E}$. This is a general fact about the restriction of the product of the characters in any $\chi$-data as in Proposition 4.7, if we have established Theorem 5.1 beforehand. \qed

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