Multifractal detrended moving average analysis of global temperature records

Provash Mali

Physics Department, North Bengal University, Siliguri 734013, India
E-mail: provashmali@gmail.com

Received 30 April 2015
Accepted for publication 12 November 2015
Published 26 January 2016

Abstract. The multifractal structure of global monthly mean temperature anomaly time series over the period 1850–2012 are studied in terms of the multifractal detrended moving average (MFDMA) analysis. We try to address the possible source(s) and the nature of multifractality in the time series data by comparing the results derived from the actual series with those from a set of shuffled and surrogate series. It is seen that the MFDMA method predicts a multifractal structure of the temperature anomaly records that is more or less similar to what was obtained from the multifractal detrended fluctuation analysis (MF DFA) for the same set of data. In our analysis the major contribution of multifractality in the data is found to be due to the long-range temporal correlation among the measurements, although the contribution of a fat-tail distribution function of the variables is not negligible. The existence of a long-range correlation is also confirmed by the constancy of the local slopes of the fluctuation function over a sufficient scale interval. The results of the moving average analysis are found to depend upon the location of the detrending window and tend to the observations of the MF DFA for a specific choice of the location of the detrending window.

Keywords: geophysical turbulence, statistical mechanics and climate
1. Introduction

The idea of fractal analysis was introduced by Mandelbrot in the late 1960s [1, 2]. Recently, with the advancement of computing facilities, the study of fractal and multifractal systems has gained an extra dimension in the field of nonlinear dynamics. Nowadays several methods of multifractal analysis have been furnished and, with their own merits and demerits they are applied to characterise the time series data of different variants. The simplest type of multifractal analysis is based on the standard partition function multifractal formalism, which has been developed for the multifractal characterisation of normalised stationary time series [3–5]. Unfortunately, this standard formalism does not give correct results for non-stationary time series that possess trends or that cannot be normalised. An improved multifractal formalism, namely the wavelet transform modulus maxima (WTMM) method, was developed in the early 1990s [6–11]. The WTMM method is based on wavelet analysis and hence involves tracing the maxima lines in the continuous wavelet transform over all scales. Later, in 2002 Kantelhardt et al [12] developed an alternative approach by generalising the detrended fluctuation analysis (DFA) [13, 14], which had already been efficiently used in atmospheric data analysis [15–18]. The multifractal version of DFA (MF DFA) does not require the modulus maxima procedure, and hence does not involve more effort in computer programming than the conventional DFA. Another important usefulness of MF DFA is that it is not affected by the underlying trends of the data. Probably because of the simplicity and efficiency of MF DFA, the spectrum of applicability of the method has covered all the possible fields of time series analysis within just a decade of its introduction. Usually, the empirical time series data are mostly affected by non-stationarities which have to be well distinguished from the intrinsic fluctuations of the record, in
order to find out the correct scaling behaviour of the data. Unfortunately, very often we do not know the underlying trends in the data, and even worse we do not know the scales of the underlying trends. Recently a novel technique, known as the multifractal detrended moving average (MFDMA) method, was proposed by Gu and Zhou [19] for the multifractal characterisation of time series data. The MFDMA method is a generalisation of the DMA method of Alessio et al [20], where the local trends of the analysing signal is filtered out (detrended) by subtracting the local means. The MFDMA method can easily describe the multifractal nature of non-stationary series without any assumption. So far the MFDMA method has been used in a limited area of time series analysis [21–26], although empirical studies suggest that under certain circumstances the performance of MFDMA is slightly better than MFDFA [23–25].

The global temperature is a crucial thermodynamic parameter of the atmosphere, and its rising trend, as illustrated in figure 1(a), has become a global issue in climate research [15–18, 27–30]. Moreover, the variations in daily and/or monthly mean temperatures measured over their smooth average are found to be so random that the data always require extra attention in order to characterise the underlying dynamics. In an early analysis of daily maximum temperature fluctuations from their average values [27] it was claimed that the fluctuating pattern follows a monofractal scaling relation with time lags, which means a single parameter called the scaling exponent is enough to interpret a sequence of observations. Later, a study by Weber and Talkner [28] revealed that the value of the exponent depends upon the altitude of the meteorological station. On several occasions the DFA method has been applied to analyse the temperature record data of different variant [15, 16, 18]. The ultimate finding of these analyses is that the fluctuation functions derived from a temperature record series of any form is found to follow a power-law type of scaling relation with time lags, and it was interpreted as an effect of the long-memory process. However, in [16, 18] it was demonstrated that the power-law scaling of the fluctuation functions cannot be regarded a priori, but should be established in conjunction with the investigation of the local slopes of the log–log plots of the fluctuation functions. According to [16], the comparison of a long-memory process with a short-memory model does not specify the existence of long-range correlations from the application of DFA on a finite data set, and hence scaling cannot be concluded from a log–log straight line fit to the fluctuation function. Recently, we analysed the global monthly mean temperature anomaly time series data in terms of the MFDFA method [30]. In this analysis we found that the time series records exhibit a rich multifractal structure which originates from two possible sources, namely long-range temporal correlation and the fat-tailed probability function of the values. Further, we found that the time series can be more or less described by a computer-generated series based on the generalised binomial multifractal algorithm [31].

The present analysis attempts to investigate the existence of the intrinsic scaling properties of the global temperature anomaly records over the period 1850–2012 [32] by using the MFDMA method [19]. We establish the power-law scaling of the MFDMA fluctuation functions from the constancy nature of the local slopes of the log–log plots of fluctuation functions. The investigation shows that the power-law scaling, as also observed in [30], is an outcome of the long-range correlations of the time series records. The MFDMA results are supplemented by the autocorrelation function analysis, which provides a preliminary idea about the correlation pattern present in the data. This
paper is organised as follows: in section 2 we present the data characteristics. In section 3 we provide the details of our analysis, where under two different subsections the aspects of the autocorrelation analysis and MFDMA analysis are discussed. We conclude the paper in section 4.

2. Data

The temperature anomaly time series data used here are taken from the database of the Climatic Research Unit, the University of East Anglia [32]. The global monthly mean temperature anomaly time series from 1850 to 2012 are shown in figure 1. Note that the anomalies are relative to the mean over the reference period 1961–1990. Apparently the original series with an overall upward trend contains nonstationarities. According to Environmental Protection Agency (EPA) reports, the Earth’s temperature has increased by 0.8 °C over the past century and more than half of this increase has happened in the last 25 years. Although apparently it is not visible from the diagram (figure 1(a)), the time series data contain a periodic seasonal trend. In order to eliminate this seasonal trend, we calculate the departures $T_i = t_i - \bar{t}_i$ from the mean monthly record $\bar{t}_i$. The monthly mean $\bar{t}_i$ is calculated for each calendar month $i$, e.g. January, by averaging over all 162 years in the records. The seasonal detrended series illustrated in figure 1(b) also shows a more or less similar upward trend as the original series. Therefore, the long-term trend (appearing as a periodic one) is filtered out by subtracting the smooth

Figure 1. Global monthly mean temperature anomaly time series from 1850 to 2012. The anomalies are relative to the mean of the reference period of 1961–1990. (a) The original series with a background trend (red line) best determined by a polynomial of degree 7, (b) the seasonal detrended series (see the text for details) and (c) the polynomial detrended (residue) series corresponding to the series shown in (a). The dotted line represents the 0 °C reference level of the respective series.
Multifractal detrended moving average analysis...

background, the best fitted polynomial to the original series. The smooth (red) line in diagram 1(a) represents the polynomial of degree 7, which gives the best figure of merit. The polynomial detrended anomalies are shown in figure 1(c), where the long-term periodic trend is no longer present. The subsequent analysis is carried out by using the polynomial detrended anomaly values 1(c), although in the text it is said to be the original series. In order to specify the statistics of the series variables, we construct the cumulative distribution function (CDF) for the (polynomial detrended) series variables. The CDF is shown in figure 2. The inset in the diagram magnifies the tail region of the CDF. The tail exponent $\alpha_{\text{tail}}$ is evaluated by a power-law regression that gives $\alpha_{\text{tail}} \approx 4$. As we know that the limit $\alpha_{\text{tail}} \sim 3$ is taken as the onset of a fat-tailed distribution. Accordingly, the underlying probability distribution function for the data is a fat-tailed one, and hence the distribution function may in principle be a source of multifractality.

3. Analysis and results

3.1. Autocorrelation function

Autocorrelation function for the time series data provides the correlation between the $i$th measurement, with that of the $(i+s)$th one for different values of time lag $s$. Consider a time series $\{x_i : i = 1, 2, \cdots, N\}$, here the index $i$ corresponds to the time of a measurement $x_i$. In order to remove the constant offset of the series (if any), the mean of the series $\langle x \rangle = (1/N) \sum_{i=1}^{N} x_i$ is subtracted: $\bar{x}_i = x_i - \langle x \rangle$. Then the auto-covariance between any two $\bar{x}$’s separated by $s$ steps (or lag) is defined as

$$ C'(s) = \langle \bar{x}_i \bar{x}_{i+s} \rangle = \frac{1}{N-s} \sum_{i=1}^{N-s} \bar{x}_i \bar{x}_{i+s}. \quad (1) $$

Figure 2. Cumulative distribution function for the polynomial detrended temperature anomaly time series. The inset implies that the tail region of the distribution function can be well fitted to a power-law formula with a tail exponent $\alpha_{\text{tail}} \approx 4$, indicating the distribution is a fat-tailed one.
When the above $C'(s)$-function is normalised by the variance $\langle \tilde{x}_i^2 \rangle$, the function is called the autocorrelation function $C(s)$. If the series $\{x_i\}$ are uncorrelated, $C(s)$ is zero for any $s > 0$. The $\{x_i\}$s are said to be short-range correlated, if $C(s)$ declines exponentially: $C(s) \propto \exp(-s/s_0)$ for $s \to \infty$. On the other hand, for a long-range correlated series, $C(s)$ declines as a power-law: $C(s) \propto s^{-\gamma}$ for $s \to \infty$ with exponent $0 < \gamma < 1$. A direct calculation of $C(s)$ is usually not appropriate due to the noise superimposed on the series $x_i$ and due to the underlying trends of some unknown origin, and hence the exponent $\gamma$ is extracted indirectly. In this analysis we employ the MFDMA technique to capture the nature of correlation present in the temperature anomaly records. However, for a time series analysis the autocorrelation function analysis is always appreciable, as it may provide an elementary idea about the type of correlation in the data. Note that the pattern of correlation for a stationary time series may also be studied in terms of the so-called power spectrum $E(f)$ at frequency $f$: $E(f) \sim f^{\beta}$. For stationary time series the exponent $\beta$ is related to $\gamma$ through $\gamma = 1 - \beta$.

In figure 3 we illustrate the autocorrelation function $C(s)$ with time lags $s$ for the temperature record data (actually for the polynomial detrended sequences, figure 1(c)). The autocorrelation function is found to follow the power-law scaling $C(s) \sim s^{-\gamma}$ in the scale interval $1 \leq s \leq 12$ with exponent $\gamma \approx -0.4$. The exact trend for a $C(s)$-function with exponent $-0.4$ is shown in the diagram by the dashed line. The discontinuities and huge fluctuations at large $s$ might be due to the limited statistics of the data, and/or it might be an indication of the fact that the correlation does not hold at a large lag. The hypothesis of studying the autocorrelation function at scale $s \to \infty$ is an ideal concept only. For observational data some limitations of measurement always restrict the analysis within a time domain; the minimum scale of which is specified by the sampling interval $\Delta \tau$, whereas the maximum scale is determined by the length of the series $N$. It is not certain if an autocorrelation function with a maximum time lag of about 12 months can be used to identify the nature of correlation in time series data, especially when the long-range correlation is suspected. However, the estimated $\gamma$ value ambiguously gives a preliminary indication of the existence of a long-range correlation in the temperature anomaly series studied here.

Figure 3. Autocorrelation function for the global monthly mean temperature anomalies. The dashed straight line with slope $-0.4$ is for the visual reference only.
3.2. Multifractal detrended moving average analysis

The MFDMA methodology is well described in [19]. For the sake of completeness, we briefly outline the procedure step by step in the following subsection, although we do not claim the originality of [19].

Let \( \{ x_i \colon i = 1, 2, \ldots, N \} \) be a time series of length \( N \). The MFDMA procedure consists of the following few steps:

(i) Construct a sequence of cumulative sums

\[
y(i) = \sum_{k=1}^{i} x_k, \quad i = 1, 2, \ldots, N.
\]

In the subsequent steps the above sequence is considered as the signal.

(ii) Calculate the moving average function \( \tilde{y}(i) \) in a moving window of size \( n \)

\[
\tilde{y}(i) = \frac{1}{n} \sum_{k=-[(n-1)\theta]}^{[(n-1)(1-\theta)]} y(i-k),
\]

where \( [\xi] \) is the largest integer not larger than \( \xi \) and \( [\xi] \) is the smallest integer not smaller than \( \xi \). Here \( \theta \) is a parameter \( \in [0, 1] \) that specifies the position of the moving window. In general the moving average function includes \( [(n-1)(1-\theta)] \) data points in the past and \( [(n-1)\theta] \) data points in the future. We consider three different values of \( \theta = 0, 0.5 \) and 1. For \( \theta = 0 \) the moving average function \( \tilde{y}(i) \) is calculated over all the past \( (n-1) \) data points of the signal, and hence it refers to the backward moving average. In the case of \( \theta = 0.5 \) the function \( \tilde{y}(i) \) includes half past and half future information in each window, and it is said to be the central moving average. In the third option \( \theta = 1 \), where the moving average function \( \tilde{y}(i) \) is calculated over all the \( (n-1) \) data points in the future, is known as the forward moving average.

(iii) Detrend the sequences \( y(i) \) by subtracting the moving average function \( \tilde{y}(i) \) and obtain the residue series

\[
e(i) = y(i) - \tilde{y}(i),
\]

where \( i \) satisfies the criterion: \( n - [(n-1)\theta] \leq i \leq N - [(n-1)\theta] \).

(iv) Divide the residue series \( e(i) \) into \( N = \lceil N/n-1 \rceil \) non-overlapping segments of equal length \( n \). Let the segments be denoted by \( e_v \), so that \( e_v(i) = e(l+i) \) for \( 1 \leq i \leq n \) with \( l = (v-1) \) \( n \). For an arbitrary segment \( v \) the mean-square fluctuation function \( F_v^2(n) \) is calculated as a function of \( n \) through

\[
F_v^2(n) = \frac{1}{n} \sum_{i=1}^{n} (e_v(i))^2.
\]
(v) The $q$th order overall fluctuation function $F_q(n)$ is then determined as

$$F_q(n) = \left\{ \frac{1}{N_0} \sum_{v=1}^{N_0} (F^2_{(n)})^{1/2} \right\}^{1/q} \text{ for all } q \neq 0,$$

$$F_q(n) = \exp \left\{ \frac{1}{2N_0} \sum_{v=1}^{N_0} \ln[F^2_{(n)}] \right\} \text{ for } q = 0. \quad (6)$$

(vi) The scaling behaviour of $F_q(n)$ is examined for several different values of the exponent $q$. For a multifractal series $F_q(n)$ for large values of $n$ would follow a power-law type of scaling relation, such as

$$F_q(n) \sim n^{h(q)}, \quad (8)$$

and the exponent $h(q)$ would be a function of $q$.

The exponent $h(q)$, known as the generalised Hurst exponent, is an important parameter for a multifractal analysis. For $q = 2$ the $h(q)$ exponent is related to the correlation exponent $\gamma$ and the power-spectrum exponent $\beta$ through $h(2) = 1 - \gamma/2 = (1 + \beta)/2$. For stationary time series such as the fGn (fractional Gaussian noise), $h(q = 2) = H$, the well-known Hurst exponent, and also the exponent satisfies the criterion $0 \leq h(q = 2) < 1.0 \quad [33]$. In the case of a non-stationary signal, e.g. the fBm (fractional Brownian motion), $h(q = 2) = H + 1$ and for such signals $h(q = 2) > 1 \quad [13, 34]$. For a monofractal series with a compact support, on the other hand, $h(q)$ is independent of $q$. Knowing $h(q)$ one can easily derive the multifractal scaling exponents $\tau(q)$ through

$$\tau(q) = qh(q) - 1. \quad (9)$$

A nonlinear $\tau(q)$ spectrum signals the existence of the multifractal nature of the data. For a monofractal process $\tau(q)$ is a linear function of $q$. The generalised multifractal dimensions are given as

$$D(q) \equiv \frac{\tau(q)}{q-1} = \frac{qh(q) - 1}{q-1}. \quad (10)$$

For a monofractal time series, though, $h(q)$ is independent of $q$, $D(q)$ depends on $q$. Another important variable of a multifractal analysis is the multifractal singularity spectrum $f(\alpha)$, which is related to $\tau(q)$ via a Legendre transformation $[35, 4]$: $\alpha = \partial \tau(q)/\partial q$. The multifractal spectrum $f(\alpha)$ is defined as

$$f(\alpha) = q\alpha - \tau(q). \quad (11)$$

Here $\alpha$ is the singularity strength or the Hölder exponent. For a monofractal structure only one $\alpha$ exponent is expected to describe the system and the corresponding $f(\alpha)$-spectrum would appear as a delta function. On the other hand, for a multifractal structure a spectrum of $\alpha$ is observed, which leads to the existence of a $f(\alpha)$ spectrum.
The parameters $f(\alpha)$ and $\tau(q)$ can also be used to provide a thermodynamical description of a random chaotic system \[36, 37\]. In this approach $\tau(q)$ is analogous to the free energy and its Legendre transformation $\alpha$ is analogous to the entropy of the system. In general, the function $\tau(q)$ exhibits two different realms which are separated by a ‘critical value’ of $q = q_c < 0$. In the thermodynamical interpretation of multifractality this is called two distinct phases of the system and turning $q$ over its critical value is said to a ‘phase transition’. Here the parameter $q_c$ plays the role of the inverse of transition temperature.

It is to be noted that the moving average method shares many ideas with the detrended fluctuation analysis, but an added advantage in the former method probably makes it more sophisticated over the latter one. The advantage in MFDMA analysis is that it gives us the freedom to choose the location of the detrending window with respect to the measurement to be detrended. For instance, for a computer-generated series based on the dynamical random cascade model with log-Poisson distribution \[12\] the MF DFA method is not found to be very sensitive. For this particular series the MF DFA estimated values of $h(q)$ for $q < 0$ grossly deviate from their analytic values, but the MFDMA estimated values of the parameter offer a reasonable agreement with their analytic values. Both methods are affected by statistical limitations. Note that so far there has been no systematic comparative study between these two methods available in the literature, although on a few occasions it has been claimed that the MFDMA method is somewhat superior to the other one \[23, 24, 26\]. An opposite observation was also reported in \[38\] but it is not on the multifractal form of the methods.

### 3.3. Results of the MFDMA analysis

We calculate the MFDMA fluctuation functions $F_q(n)$ as a function of window size $n$ (scale parameter) for three different choices of the window parameter $\theta = 0, 0.5$ and 1. The scale parameter $n$ is varied from 10 to $N/10$ and the exponent $q$ is varied from $-4$ to $+4$ in steps of 0.25. The tail exponent of a CDF $\alpha_{\text{tail}}$ usually sets the limits on $q$. In general, for $\alpha_{\text{tail}} \geq 3$ the underlying distribution function is classified as a fat-tailed distribution and the exponent $q$ in the interval $\pm 3$ provides the desired information of a fractal measurement. Beyond the specified limits the multifractal variables, such as the exponents $h(q)$, $D(q)$ etc, are expected to show a linear asymptotic behaviour, since the limit of the $p$-norm (or Hölder norm) of a vector $\mathbf{x}$ of components $\{x_i\}$ as $p$ goes to infinity is the infinity norm, i.e. the supreme of the absolute values of the function $x_\infty = \max_{1 \leq i \leq n} |x_i|$ \[39\]. As a consequence, the $q$-moment of any variable is rapidly dominated by $|x_\infty|^q$. Corresponding to each of the $\theta$ values the scaling pattern of $F_q(n)$ for $q = 0, \pm 2, \pm 4$ are shown in figure 4. We also repeat the analysis for a set of ten randomly shuffled series as well as ten surrogate series. The lower panel in figure 4 illustrates the $F_q(n)$ functions calculated from an arbitrarily shuffled series. The scaling of the surrogate series generated $F_q$ functions that are apparently similar to those obtained from the shuffled series, and hence they are not pictorially shown. The importance of analysing the shuffled and surrogate series are discussed below. The statistical error bars for the fluctuation functions are invisibly small in this plot. The 95% confidence bands are calculated for all the $F_q(n)$ functions but in the figure the confidence band is shown only for the $F_{+4}(n)$ functions calculated from the original series with $\theta = 0$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Scaling patterns of $F_q(n)$ for $q = 0, \pm 2, \pm 4$ with different window sizes $n$ and window parameters $\theta = 0, 0.5, 1$. The lower panel shows the $F_q(n)$ functions calculated from an arbitrarily shuffled series. The 95% confidence bands are calculated for all the $F_q(n)$ functions but in the figure the confidence band is shown only for the $F_{+4}(n)$ functions calculated from the original series with $\theta = 0$.}
\end{figure}
Multifractal detrended moving average analysis...

Figure 4. Scaling behaviour of the MFDMA fluctuation functions $F_q(n)$ for $q = 0, \pm 2, \pm 4$ for three different choices of $\theta$. The upper panel represents the original series and the lower panel represents an arbitrary shuffled series corresponding to the original one. In diagram (a) the 95% confidence band of $F_{\pm 0}(n)$ is indicated by the shadowed region.

From figure 4 one can infer that the functions $F_q$ nicely respect the scaling relation (8) but mainly in the scale interval $10 \leq n \leq 50$. Above $n \sim 50$ the $F_{q<0}$ functions are highly fluctuating and a nonlinearity is also visible at large scale. One can also notice that the backward ($\theta = 0$) and forward ($\theta = 1$) moving average schemes result in almost similar $F_q$ functions, while the central ($\theta = 0.5$) moving average scheme produces slightly stiffer and more closely spaced $F_q$ in comparison with the other two schemes. The observations do not require any explanation at this point of our analysis. However, it is clear that the central moving averaging creates a residual series $\epsilon(i)$ (equation (4)) which is a less correlated one than that produces in the forward/backward moving averaging.

The importance of analysing a shuffled series for any given empirical time series data is that, a direct comparison between the results obtained from an original series and a shuffled series gives insight into the nature of the multifractality present (if any) in the data [12, 34]. A simple way to check whether the correlations in the data
produce any kind of scaling is to shuffle the data. Random shuffling removes temporal correlation and any scaling that remains must be due to the probability distribution of the variables, since the shuffling procedure does not affect the probability distribution function (PDF). If a time series contains multifractality that stems from both correlation and PDF, the corresponding shuffled series will exhibit weaker multifractality than the actual series. The surrogate (also called the phase randomised) series analysis, on the other hand, is a numerical technique of testing nonlinearity in time series data [40, 41]. The aim here is to test whether the dynamics are consistent with some linearly filtered noise or a nonlinear dynamical system. The basic idea of the surrogate data method is to first specify some kind of linear stochastic process that mimics the ‘linear properties’ of the original data. If the predictions (statistics) of the original data are significantly different from those of the surrogate series, we may consider the presence of some higher order temporal correlations, that is the presence of dynamic nonlinearities. In this analysis we use the amplitude-adjusted fourier transform (AAFT) algorithm [41, 42], a well-known method for surrogate data.

The generalised Hurst exponent \( h(q) \) is calculated by fitting a linear function, like

\[
\ln F_q(n) = h(q) \ln n + \zeta,
\]

(12)
to the \( \ln F_q \) versus \( \ln n \) data points, but within a limited scale interval: \( 10 \leq n \leq 50 \). Note that the \( F_q \) functions are more or less linear and do not possess significant statistical fluctuation within those limits. The \( h(q) \) values are plotted against their order number \( q \) in figure 5 (upper panel) for all three choices of \( \theta = 0, 0.5 \) and 1. In our fitting procedure Pearson’s \( R^2 \) coefficient, which measures the goodness of fit, is found in the interval \( 0.93 < R^2 < 1 \). The \( R^2 \) values ensure that the fit quality is quite good. The errors associated with \( h(q) \) (shown in figure 5) are of statistical origin. The shadowed region describes the 95% confidence bands for the original series estimated \( h(q) \) spectrum. The bands are measured as \( \bar{h}(q) \pm 1.96 \sigma_{h(q)} \), where \( \bar{h}(q) \) and \( \sigma_{h(q)} \) are the mean and variance of the Gaussian function fitted to the \( h(q) \) distribution, which is obtained by a similar procedure as mentioned before. The lower panel of figure 5 represents the spectra of the multifractal scaling exponent: \( \tau(q) = q h(q) - 1 \). Since the \( \tau(q) \) exponents are directly calculated from the \( h(q) \) exponents, we do not put any additional emphasis on this parameter. All the \( h(q) \) and \( \tau(q) \) spectra are supplemented by their respective shuffled and surrogate counterparts. In order to optimise the randomness of the shuffled/surrogate series generated \( h(q) \) (and \( \tau(q) \)) spectra, an average value of these spectra over ten independent calculations is considered. The figure reflects the nonlinear nature of the \( h(q) \) and \( \tau(q) \) spectra. It is seen that for the original and surrogate series the central moving averaging yields a significantly larger values of \( h(q) \), and the degree of nonlinearity in the \( h(q) \) spectrum is also weaker than the forward and backward moving averaging. According to the theory of multifractals, the \( h(q) \) and \( \tau(q) \) spectra carry a clear signal of multifractality in the global monthly mean temperature records, but the spectra greatly depend upon the location of the moving window. The forward and the backward moving methods possess some kind of similarity to each other. From this observation one cannot say which the detrending window (backward, central or forward) is suited better for the time series data analysed here. For this purpose the results of the detrended moving average analysis have to be systemically compared with those of other known multifractal methods, as well as with various model computations. From figure 5 it is
also seen that the AAFT surrogate series-generated spectra to some extent take care of their empirical values but the shuffled series-generated spectra are underestimated by the corresponding original/surrogate series. Further, the shuffled series calculated $h(q)$ values are all located at $\sim 0.5–0.6$ and are almost linear in $q$. These observations indicate that the correlation present in the actual series is probably destroyed by the random shuffling, and therefore the shuffled series shows a weak multifractal pattern which stems from the distribution function (fat-tailed) of the series variables.

The values of the second-order generalised Hurst exponent $h(2)$ obtained for $\theta = 0$, 0.5 and 1 are given in table 1. In addition, we also show the $h(2)$ values for $\theta = 0.7$, the reason for which will be discussed latter. The table shows that the original series calculated values of $h(2)$ are very close to their AAFT surrogate series estimation, although the $h(q)$ values obtained from the shuffled series are ($\sim 0.5$) much shorter than their original/surrogate values. The degree of goodness of the measurement of $h(2)$ through straight fit to the $\ln F_2(n)$ versus $\ln n$ data is specified by Pearson’s $R^2$ coefficient. The $R^2$ values here are found to be quite satisfactory. It should be noted that the accuracy of a time series analysis depends upon the length of the series [25, 43, 44]. Here we find about a 5% increase in the $h(2)$ values when the series length $N$ is truncated by 500 points (measurements). The dependence of the confidence bandwidth on $N$ is found to be more pronounced. For instance, when the $h(2)$ exponent is

Figure 5. Upper panel: Generalised Hurst exponent spectra for the backward ($\theta = 0$), central ($\theta = 0.5$) and forward ($\theta = 1$) moving averaging schemes. The shadowed regions imply the 95% confidence bands. Lower panel: multifractal exponent spectra for the three choices of $\theta$. In all the cases the original series estimated spectra are compared with their respective shuffled and surrogate predictions.

\[ \text{doi:10.1088/1742-5468/2016/01/013201} \]
increased by approximately 5% because of the truncation of the series by 500 points, its confidence band becomes wider by more than 12%.

We can now compare the results of the MFDMA analysis with those of the autocorrelation analysis. Recall the autocorrelation exponent value for the data $\gamma \approx 0.4$. This yields $h(2) = 1 - \gamma / 2 \approx 0.8$. Thus, the autocorrelation function estimated value of $h(2)$ roughly matches the forward and backward moving average estimated values ($\sim 0.76$). Moreover, for the original series the moving average method with $\theta = 0.7$ estimates $h(2)$ value that is very close to the value obtained from the MFDFA (first order) analysis [30]. From the $h(q)$ exponents/spectra we understood that, in the case of an uncorrelated (AAFT-surr) series, where a weak multifractal structure might appear due to the fat-tailed PDF (linear correlations), the two detrended analysis methods differ significantly. Considering the values of $\gamma$ and $h(2)$ exponents, one can argue that the global temperature records behave more or less like a stationary time series for which $0 < h(q = 2) < 1.0$ [33], although the series is not a stationary one. This implies that the autocorrelation function is not a suitable tool for characterising time series data.

As argued in [16] that the power-law scaling of the detrended fluctuation functions should not be taken as an evidence of long-range correlations, rather it has to be established from the constancy of local slopes $\kappa$ of the fluctuation functions over a sufficient scale range, although the extent of the range cannot be defined yet [45]. Figure 6 illustrates the local slopes $\kappa$ of the MFDMA fluctuation functions $F_{q=2}(n)$ versus the scale $n$. In diagram (a), where the central moving scheme ($\theta = 0.5$) is used, the variation of $\kappa$ with $n$ is shown for two different values of window size: $w = 12$ and 48 months. It is seen that for $w = 12$ the $\kappa$ values are very chaotic, whereas for $w = 48$ and onward they possess very few fluctuations and the values are, within the 95% confidence bands, approximately constant over the scale range $n \approx 500-1400$. Beyond the specified limits of $n$ the local slope values decline very slowly with increasing $n$. In the case of $\theta = 0$ (b) and 1 (c) the constancy interval of $\kappa$ is shifted towards the low $n$ region: $n \leq 1000$. The shuffled series estimated $\kappa$ values always strongly fluctuate at about $\kappa = \kappa_0 \approx 0.55$. The observation supports the possibility of a long-term memory process in the data.

Next, we calculate the multifractal singularity spectrum $f(\alpha)$ for the analysed time series data. The importance of this in connection with a multifractal analysis is that

| Method          | Original $\theta = 0$ | Shuffled $\theta = 0$ | Surrogate $\theta = 0$ |
|-----------------|------------------------|------------------------|-------------------------|
| MFDM (\(\theta = 0\)) | 0.769 ± 0.007          | 0.522 ± 0.002          | 0.756 ± 0.008           |
|                 | (0.981)                 | (0.991)                | (0.943)                 |
| MFDM (\(\theta = 0.5\)) | 1.008 ± 0.011          | 0.497 ± 0.002          | 1.006 ± 0.007           |
|                 | (0.985)                 | (0.973)                | (0.933)                 |
| MFDM (\(\theta = 1\)) | 0.767 ± 0.007          | 0.521 ± 0.002          | 0.753 ± 0.008           |
|                 | (0.945)                 | (0.973)                | (0.963)                 |
| MFDM (\(\theta = 0.7\)) | 0.919 ± 0.067          | 0.523 ± 0.006          | 0.919 ± 0.017           |
|                 | (0.943)                 | (0.991)                | (0.993)                 |
| MFDF (Linear)   | 0.907 ± 0.011           | 0.515 ± 0.007          | 0.767 ± 0.003           |
|                 | (0.931)                 | (0.989)                | (0.981)                 |

*Note: The estimate of the MFDF (first order) [30] is also given for comparison. The errors are statistical only. Pearson’s $R^2$ coefficients are quoted under the parenthesis.*
the parameter itself gives a direct and quantitative measure of the degree of multifractality present in the data. The width and (a)symmetry parameters of the spectrum are closely connected to the chaotic/fractal nature of the data: a wider and asymmetric singularity spectrum roughly implies that the time series is more chaotic (rich structure) compared to a series that produces a narrower and more symmetric singularity spectrum. Also, the location of the spectrum provides other important information. For an uncorrelated series the mean of the spectrum is usually spotted at \( \alpha \sim 0.5 \), but for a long-range correlated series the mean is expected to be shifted at large \( \alpha \). In figure 7 the singularity spectra of our analysis are plotted against the singularity (Hölder) exponent \( \alpha \). Separate diagrams are shown for the three choices of \( \theta \): (a) backward (\( \theta = 0 \)), (b) forward (\( \theta = 1 \)) and (c) central (\( \theta = 0.5 \)). In diagram (d) the singularity spectrum of the MFDFA (first order) method [30] is compared with that of the MFDMA method with \( \theta = 0.7 \). We find that (i) the original series for all the choices of \( \theta \) results in a stable and wider singularity spectrum, (ii) the surrogate spectrum for \( \theta = 1 \) more or less matches the empirical values, otherwise the surrogate spectra are mostly unstable, (iii) in all the cases the spectra for the shuffled series are located at \( \sim 0.5 \) and they are narrower than their original series generated counterpart and (iv) the prediction of the MFDMA analysis with \( \theta = 0.7 \) is approximately identical to that of the MF DFA technique. All these observations are related to the fact that the degree of chaoticity/multifractality in the actual series is higher than their shuffled and/or surrogate partner. Once again we observe the effect of random shuffling in the singularity spectra. The weak
Multifractal detrended moving average analysis...

Figure 7. (a)–(c) Multifractal spectra, respectively for the backward ($\theta = 0$), central ($\theta = 0.5$) and forward ($\theta = 1$) moving window. (d) The multifractal spectrum of MFDFA [30] is compared with that of the MFDMA ($\theta = 0.7$) analysis.

multifractal effect visible in the shuffled series generated $f(\alpha)$ spectra probably arises from the fat-tailed distribution function of the series values.

At the end of this section we tie up a comparative study between the MFDMA and MFDFA techniques of time series analysis with a reference to the global temperature anomaly time series data analysed here. Note that by doing so we do not mean that the MFDFA method is a standard method of multifractal time series analysis, though it has been extensively applied to various fields of stochastic data analysis [1]. In this analysis we try to adjust the location of the detrending window, in order to minimise the deviation between the $f(\alpha)$ spectra of MFDMA and MFDFA (first order) [30] methods. In this process we find the best match at $\theta = 0.7$. The comparison is shown in figure 7(d) and the corresponding $h(2)$ values are quoted in table 1. The superiority of any one of the methods over the other in connection with real data has not yet been thoroughly studied. However, there exists some evidence where the MFDMA analysis method is found to be more useful than the other one [19, 23, 24]. In the MFDMA analysis the parameter $\theta = 0.7$ implies that a measurement in the records is to be detrended by a window composed of 30% backward and 70% forward memories. In reality the forward memory of a time series may not be a convenient concept. But one may think in this way: any measurement $x_i$ in a time series which already carries a past memory/persistence of about 30% might influence the $x_{i+1}$ measurement by at best 70%. In that sense, the global temperature anomaly time series is highly long-range correlated and the correlation itself might be the main source of the observed multifractality.

1 The list of references is too long to cite. To get a comprehensive idea follow [12, 30, 34, 46–49] and the references therein.

doi:10.1088/1742-5468/2016/01/013201
4. Conclusions

In this article we present the multifractal detrended moving average analysis of global monthly mean temperature anomaly time series over the period 1850–2012. Various observables related to (multi)fractals, namely the generalised Hurst exponent $h(q)$, the multifractal exponent $\tau(q)$ and the multifractal singularity spectra are calculated for the temperature anomaly records. We find that the global monthly mean temperature records are of a multifractal nature and the main source of this is the long-range correlation in the measurements. The multifractal signature of the time series is also obtained from autocorrelation function analysis. The results of this analysis are found to be comparable with those of the MFDFA (first-order) method provided the detrending moving window for an arbitrary measurement is constructed out of 30% backward and 70% forward memories with respect to the measurement. To date MFDMA has not been widely applied to analysing time series data of different variants. Therefore, a systematic comparative study between MFDMA analysis and other known methods as well as various multifractal models would be a highly encouraging exercise which might help us to visualise the predictability and hence applicability of the MFDMA method in time series analysis.

Acknowledgment

The author thanks S Sarkar of the Department of Physics, NBU for drafting some of the figures shown in this article.

References

[1] Mandelbrot B B and van Ness J W 1968 SIAM Rev. 10 422
[2] Mandelbrot B B and Wallis J R 1969 Water Resour. Res. 5 321
[3] Barabási A-L and Vicsek T 1991 Phys. Rev. A 44 2730
[4] Peitgen H-O, Jurgens H and Saupe D 1992 Chaos and Fractals (New York: Springer) (appendix B)
[5] Bacry E, Delour J and Muzy J F 2001 Phys. Rev. E 64 026103
[6] Muzy J F, Bacry E and Arneodo A 1991 Phys. Rev. Lett. 67 3515
[7] Muzy J F, Bacry E and Arneodo A 1994 Int. J. Bifurcation Chaos 4 245
[8] Arneodo A, Bacry E, Graves P V and Muzy J F 1995 Phys. Rev. Lett. 74 3293
[9] Ivanov P C, Amaral L A N, Goldberger A L, Havlin S, Rosenblum M G, Struzik Z R and Stanley H E 1999 Nature 399 461
[10] Amaral L A N, Ivanov P C, Aoyagi N, Hidaka I, Tomono S, Goldberger A L, Stanley H E and Yamamoto Y 2001 Phys. Rev. Lett. 86 6026
[11] Silchenko A and Hu C K 2001 Phys. Rev. E 63 041105
[12] Kantelhardt J W, Zschiegner S A, Koscielny-Bunde E, Havlin S, Bunde A and Stanley H E 2002 Physica A 316 87
[13] Peng C-K, Buldyrev S V, Havlin S, Simons M, Stanley H E and Goldberger A L 1994 Phys. Rev. E 49 1685
[14] Ossadnik S M, Buldyrev S B, Goldberger A L, Havlin S, Mantegna R N, Peng C-K, Simons M and Stanley H E 1994 Biophys. J. 67 64
[15] Eichner J F, Koscielny-Bunde E, Bunde A, Havlin S and Schellnhuber H J 2003 Phys. Rev. E 68 046133
[16] Maraun D, Rust H W and Timmer J 2004 Nonlinear Proc. Geophys. 11 495
[17] Varotsos C 2005 Int. J. Remote Sens. 26 3333
[18] Varotsos C A, Efstathiou M N and Cracknell A P 2013 Atmos. Chem. Phys. 13 5243
[19] Gu G-F and Zhou W X 2010 Phys. Rev. E 82 011136
[20] Alessio E, Carbone A, Castelli G and Frappietro V 2002 Eur. Phys. J. B 27 197
[21] Schumann A Y and Kantelhardt J W 2011 Physica A 390 2637

doi:10.1088/1742-5468/2016/01/013201
Multifractal detrended moving average analysis...

[22] Wang Y, Wu C and Pan Z 2011 Physica A 390 3512
[23] Ruan Y-P and Zhou W-X 2011 Physica A 390 1646
[24] Shao Y H et al 2012 Sci. Rep. 2 835
[25] Zhou W X 2012 Chaos Solitons Fractals 45 147
[26] Wang F, Wang L and Zou R-B 2014 Chaos 24 033127
[27] Koscielny-Bunde E, Bunde A, Havlin S, Roman H R, Goldreich Y and Schellnhuber H J 1998 Phys. Rev. Lett. 81 729
[28] Weber R O and Talkner P 2001 J. Geophys. Res. 106 20131
[29] Efstatthiou M N and Varotsos C A 2013 Meteorol. Appl. 20 72
[30] Mali P 2015 Theor. Appl. Climatol. 121 641
[31] Kantelhardt J W, Rybski D, Zschiegner S A, Braun P, Koscielny-Bunde E, Livina V, Havlin S and Bunde A 2003 Physica A 330 240
[32] Jones P D, Parker D E, Osborn T J and Briffa K R 2013 Trends: a Compendium of Data on Global Change, Carbon Dioxide Information Analysis Center (Oak Ridge National Laboratory, U.S. Department of Energy, Oak Ridge, Tenn USA)
[33] Chianca C V, Ticona A and Penna T J P 2005 Physica A 357 44754
[34] Movahed M S, Jafari G R, Ghasemi F, Rahvar S and Tabar M R R 2006 J. Stat. Mech. P02003
[35] Halsey T C, Jensen M H, Kadanoff L P, Procaccia I and Shraiman B I 1986 Phys. Rev. A 33 1141
[36] Bohr T and Jensen H M 1987 Phys. Rev. A 36 321
[37] Bohr T and Rand D 1987 Physica D 25 387
[38] Xu L, Ivanov P C, Hu K, Chen Z, Carbone A and Stanley H E 2005 Phys. Rev. E 71 051101
[39] Quarteroni A, Sacco R and Saleri F 2007 Numerical Mathematics (Berlin: Springer) p 20
[40] Uhlenbeck G E and Ornstein L S 1930 Phys. Rev. 36 823
[41] Schreiber T and Schmitz A 2000 Physica D 142 346
[42] Theiler J, Eubank S, Longtin A, Galdrikian B and Farmer J D 1992 Physica D 58 77
[43] Alì V, Cocetti F, Petri A and Pietronero L 2007 Euro. Phys. J. B 55 135
[44] Bashan A, Bartsch R, Kantelhardt J W and Havlin S 2008 Physica A 387 5080
[45] Avnir D, Birham O, Lindar D and Malcai O 1998 Science 279 39
[46] Gires A, Tchiguirinskaia I, Schertzer D and Lovejoy S 2013 Nonlinear Proc. Geophys. 20 343
[47] Yu Z-G, Leung Y, Chen Y D, Zhang Q, Anh V and Zhou Y 2014 Physica A 405 193
[48] Mali P and Mukhopadhyay A 2014 Physica A 413 361
[49] Mali P, Sarkar S, Ghosh S, Mukhopadhyay A and Singh G 2015 Physica A 424 25

doi:10.1088/1742-5468/2016/01/013201