Laplace operators on Sasaki-Einstein manifolds

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We decompose the de Rham Laplacian on Sasaki-Einstein manifolds as a sum over mostly positive definite terms. An immediate consequence are lower bounds on its spectrum. These bounds constitute a supergravity equivalent of the unitarity bounds in dual superconformal field theories. The proof uses a generalization of Kähler identities to the Sasaki-Einstein case.

A textbook result in Kähler geometry relates the de Rham with the Dolbeault Laplacian, \( \Delta = 2\Delta_{\bar{\partial}} \). The main result of this note is a similar identity for Sasaki-Einstein manifolds:

\[
\Delta = 2\Delta_{\bar{\partial}} - L_{\bar{\partial}}^2 - 2i(n-d)\partial_L \xi + 2L_{A} + 2(n-d)^2L_{\bar{\partial}}A_{\bar{\partial}} + 2i(L_{\bar{\partial}}\bar{\partial}B - \bar{\partial}B\partial_A).
\]

The right hand side of equation (1) features the Lefschetz operator, the action of the Reeb vector, the tangential Cauchy-Riemann operator as well as their adjoints. Full definitions will be given in section \( \text{I} \). The equation \( \Delta = 2\Delta_{\bar{\partial}} \) can be derived from the Kähler identities, commutators between the Dolbeault and Lefschetz operators and their adjoints \( \text{I, II} \). Our proof of (1) will follow a similar route; we will obtain Kähler-like identities that are valid on Sasaki-Einstein manifolds. These are summarized in appendix \( \text{A} \).

This note is written with two audiences in mind: physicists working in gauge/string duality or supergravity and mathematicians interested in Sasaki-Einstein geometry. Therefore we split the discussion into two separate parts, giving a proof of both \( \text{I} \) and the identities in section \( \text{II} \) while discussing their motivation by and relevance to physics in section \( \text{III} \). Readers who want to focus on the mathematical aspects can ignore section \( \text{II} \) and summarized in section \( \text{A} \). Note that we will refer to transverse forms as horizontal.

In what follows, we will give a full proof of \( \text{II} \) after setting the stage by giving all necessary definitions. Since the proof is based on the equivalent considerations in the Kähler case, our discussion will follow \( \text{I, II} \) very closely.

### A. Exterior calculus on Sasaki-Einstein manifolds

Consider a \( d = 2n + 1 \) dimensional Sasaki-Einstein manifold \( S \). Given the Reeb vector \( \xi \) and the contact form \( \eta \), the tangent bundle splits as \( TS = D \oplus L_{\xi} \). Furthermore, there is a two-form \( J = \frac{1}{2}d\eta \) with \( i_{\xi}J = 0 \). \( J \) defines an endomorphism on \( TS \) which satisfies \( J^2 = -1 + \xi \otimes \eta \). Since \( \eta(D) = 0 \), one can decompose the complexified tangent bundle as \( T_{\mathbb{C}}S = (\mathbb{C} \otimes D)^{1,0} \oplus (\mathbb{C} \otimes D)^{0,1} \oplus (\mathbb{C} \otimes \xi) \). This in turn induces a corresponding decomposition on the complexified cotangent bundle

\[
T_{\mathbb{C}}^{\ast}S = \Omega_{\mathbb{C}}^{1,0} \oplus \Omega_{\mathbb{C}}^{0,1} \oplus (\mathbb{C} \otimes \eta),
\]

which also extends to the exterior algebra

\[
\Omega_{\mathbb{C}}^{\ast} \cong \bigoplus_{p,q} \Omega^{p,q} \wedge (1 \oplus \eta).
\]

---

1. Our notation is non-standard. The basic differential is often denoted as \( \partial_{\bar{\partial}} \), with the tangential Cauchy-Riemann operator being \( \partial_{\bar{\partial}} \).

2. \( L_{\xi} \) is the line tangent to \( \xi \). In what follows we will set \( L_{\xi} = \xi \) and \( L_{\xi}^\ast = \eta \). See section 1 of [10] for a review of Sasaki-Einstein geometry.
Elements of $\Omega^c$ that vanish under the action of $i_\xi$ are called horizontal, while those annihilated by $\eta \wedge$ are vertical.

The decomposition \[ d = \partial B + \bar{\partial} B + \mathcal{L}_\xi \eta \wedge . \] (4)\] induces a decomposition of the exterior derivative, $\partial B$ is the tangential Cauchy-Riemann operator. \( \partial B \) and \( \bar{\partial} B \) satisfy \( \{ \partial B, \bar{\partial} B \} = -2 J \wedge \mathcal{L}_\xi \) as well as \( \partial^2 B = \bar{\partial}^2 B = 0 \). The sequence
\[
\ldots \partial B \Omega^{p,q+1} \Omega^{p,q} \partial B \Omega^{p,q-1} \partial B \ldots
\]
gives rise to the Kohn-Rossi cohomology groups $H^{p,q}_c(S)$. Continuing with the theme of generalizing concepts from Kähler geometry to Sasaki-Einstein manifolds, we define the Lefschetz operator $L : \Omega^k_c \to \Omega^{k+2}_c$ via $\alpha \mapsto J \wedge \alpha$ and the Reeb operator $L_\eta : \Omega^k_c \to \Omega^{k+1}_c$ as $\alpha \mapsto \eta \wedge \alpha$.

Introducing the Hodge star $\star$:
\[
\star \alpha \wedge \beta = \frac{1}{p!} \alpha^{m_1 \ldots m_p} \beta_{m_1 \ldots m_p} \text{vol} = \langle \alpha, \beta \rangle \text{vol},
\]
allows us to define adjoints for the above operators when acting on $\Omega^c$:
\[
d^* = (-1)^k \star d \star,
\]
\[
\partial^*_B = (-1)^k \star \partial B \star,
\]
\[
\bar{\partial}^*_B = (-1)^k \star \bar{\partial} B \star,
\]
\[
\Lambda = L^* = \star L \star = J \mathcal{L},
\]
\[
\Lambda_\eta = L^*_\eta = (-1)^{k+1} \star L_\eta \star = i_\xi,
\]
\[
(L_\eta \mathcal{L})^* = -L_\eta \mathcal{L} \xi.
\]
Recall that on odd-dimensional manifolds $\star$ satisfies $\star^2 = 1$.

When restricted to $D^*$, the action of $J$ becomes that of an almost complex structure $\mathcal{I}$ which acts as $\mathcal{I}(\alpha) = J m \alpha dx^m$ and $\mathcal{I}(X) = X m J m \partial_n$. Of course $\Omega^{1,0} = \{ \alpha \in \Omega^1 | \mathcal{I}(\alpha) = 0 \}$. We also define
\[
\mathcal{I} = \sum_{p,q} (-1)^{p-q} P^p q,
\]
which makes use of the projection $\mathcal{P}^{p,q} : \Omega^c \to \Omega^{p,q}$.

It will turn out useful to distinguish between the rank of a form on $\Omega^c$ and on $\wedge^k D^*$. Hence we define the operator $d^0$ on $\Omega^c$ via
\[
d^0 |_{\wedge^k D^* \wedge (1 \oplus \eta)} = k \cdot \text{id}.
\]

By definition, $d^0$ commutes with $L_\eta$. A first example of the uses of $d^0$ is given by the notion of primitive forms. $\alpha \in \Omega^c$, with $d^0 \alpha = \eta$ is primitive if and only if $\Lambda_\eta = 0$. Essentially, the idea of primitivity on $\wedge^k D^*$ is the same as on Kähler manifolds, the contact one-form just comes along for the ride and there is in no difference between horizontal and vertical forms. We define $P^k$ as the set of primitive elements of $\wedge^k D^*$.

Next we introduce an orthonormal frame $e^i$ on $D^*$. Defining $z^i = e^{2i-1} + ie^{2i}$ and imposing $\mathcal{I}(z^i) = 1$, consistency requires that $\mathcal{I}(e^{2i-1}) = -e^{2i}$ and $\mathcal{I}(e^{2i}) = e^{2i-1}$. Then
\[
J = \sum_i e^{2i-1} \wedge e^{2i} = \frac{i}{2} \sum_i z^i \wedge \bar{z}^i.
\]

Defining $e^{2n+1} = \eta$, one finds vol $= \text{vol}_D \wedge e^{2n+1} = \frac{1}{n!} J^n \wedge \eta$.

In what follows, we will make use of two results concerning the Hodge star. To begin, assume that $(\langle \cdot, \cdot \rangle)$ is a Euclidean vector space admitting a decomposition $V = W_1 \oplus W_2$ that is compatible with the metric $\langle \cdot, \cdot \rangle$. For simplicity we assume that $\dim \text{dim}_R W_i = 2 \mathbb{Z}$. The metrics $\langle \cdot, \cdot \rangle_i$ induce Hodge star operators $\star_i, i = 1, 2$. Then $\wedge^k V^* = \wedge^k W_1^* \otimes \wedge^k W_2^*$, and for $\alpha_i \in \wedge^k W_i^*$, the Hodge dual on $\wedge^k V^*$, $\star$, threads as
\[
\star (\alpha_1 \otimes \alpha_2) = (-1)^{k_1 k_2} \alpha_1 \alpha_2 \tan 2, (6)
\]
since $\beta_i \in W_i$.

One can use identical considerations to decompose the action of $\star$ on $\Omega^c$ into separate operations on $D^*$ and $\eta$. Introducing a Hodge dual $\star$ on $D^*$, one finds
\[
\star |_{\wedge^k D^* \wedge (1 \oplus \eta)} = \star |_{\wedge^k D^* \wedge \eta} = \star |_{\wedge^k D^* \wedge (1 \oplus \eta)} = (-1) \Lambda \eta.
\]

B. Lefschetz decomposition

The starting point for our discussion of Lefschetz decomposition is the commutator
\[
[L, \Lambda] = (d^0 - n).
\]

The proof is via induction in $n$. Consider $d = 3, n = 1$. Then $\Omega^c$ is spanned by $\{ \eta, \mu_1, J, J \wedge \eta \}$ where $\mu_i \in \Omega^{1,0} \oplus \Omega^{0,1}$ and both $\mu_i$ are annihilated by $L$ and $\Lambda$. Then $\Lambda J = 1$ and thus $[L, \Lambda]|_{\Omega^2} = -1$, $[L, \Lambda]|_{\eta} = -1$, $[L, \Lambda]|_{D^* = 0}$, $[L, \Lambda]|_{\Omega^{1,1}} = 1$, and $[L, \Lambda]|_{\Omega^c} = 1$. Hence
\[
[L, \Lambda]|_{\wedge^k D^* \wedge (1 \oplus \eta)} = (k - 1), \quad k = 0, 1, 2,
\]
as claimed. The induction then proceeds as if it  
\[
(L^i, \Lambda)|_{\wedge^k D^* \wedge (1 \oplus \eta)} = i(k - n + i - 1) L^{i-1}.
\]
Again the proof is a copy of that in [1].

To proceed we follow [2]. Restricting to $\bigwedge^k D^*$ one can copy all results from proposition 6.20 to lemma 6.24. The most important of these results is Lefschetz decomposition. Given $\alpha \in \bigwedge^k D^*$, there is a unique decomposition

$$\alpha = \sum_r L^r \alpha_r, \quad \alpha \in P^{k-2r}.$$ 

The decomposition is compatible with the bidigree decomposition and with the decomposition into horizontal and vertical components. Moreover,

$$L^{n-k} : \bigwedge^k D^* \to \bigwedge^{2n-k} D^*$$

is an isomorphism and the primitivity condition is equivalent to $L^{n-k+1} \alpha = 0$.

The Lefschetz decomposition becomes incredibly useful when used together with the Bidigree decomposition, equation (7) and the identity

$$\forall \alpha \in P^k, \quad \cdot L\alpha = F(n, j, k) L^{n-k-j} I(\alpha), \quad F(n, j, k) = (-1)^{k(k-1)/2-j!/(n-k-j)!}$$

Since no differential operators are involved and $\alpha \in \bigwedge^k D^*$, one can copy the proof in [1] after adjusting for conventions. Once the dust settles, the only difference is in the $k$-dependent prefactor.

C. Kähler-like identities

We are finally in a position to make use of the previous results and calculate the (anti-) commutators. The results are in summarized in table 1. A number of identities are fairly obvious:

$$0 = [\partial_B, L] = [\partial_B, \Lambda] = [\partial_B^*, \Lambda] = [\partial_B^*, \Lambda] = [L, \Lambda] = [\Lambda, \Lambda] = [L, \Lambda].$$

One finds $\{L_\eta, \Lambda_\eta\} = 1$ by direct calculation using the decomposition $\alpha = \alpha_H + \Lambda_\eta \alpha_V$. Finally, $[\partial^0, \partial_B] = \partial_B + L_\eta \Lambda_\eta$.

The most involved calculation is that of the commutator

$$[\Lambda, \partial_B] = -i\partial_B^* + iL_\eta \Lambda + (n - d^0) \Lambda_\eta.$$  

Before we turn to the proof, let us try to interpret this result as a generalization of the Kähler case $[\Lambda, \partial] = -i\partial^\ast$. The naive guess $[\Lambda, \partial_B] = -i\partial_B^\ast$ cannot be correct since the left hand side maps $[\Lambda, \partial_B] : \bigwedge^{r} D^* \to \bigwedge^{r} D^*$ while $\partial_B^* : \bigwedge^{r} D^* \to \bigwedge^{r} D^\ast \wedge (1 \oplus \eta)$. Similarly, the right hand side annihilates $\eta$ while the left hand side does not. One can guess the correct result by considering the action of both sides on $J$ and $\eta$, adding suitable terms on the right hand side to achieve equality.

The proof of (12) is once again an elaboration on the proof for Kähler manifolds in [1]. Let us first consider horizontal forms. Here, it is sufficient to explicitly evaluate the action of (12) $L\alpha$ for $\alpha \in P^k$: the result will generalize for generic elements of $\bigwedge^k D^*$ due to Lefschetz decomposition. Furthermore one applies Lefschetz decomposition to $\partial \alpha = \alpha_0 + L_\alpha \Lambda_0 + L_2 \alpha_2 + \ldots$. We have $\alpha \in P^k$ and thus $0 = \sum_j L^{n-k+1+j} \alpha_j$ and finally $L^{n-k+1+j} \alpha_j = 0$. Using equation (10) it follows that most of the $\alpha_j$ vanish and $\partial \alpha = \alpha_0 + L_\alpha$. Using (9) one finds

$$[\Lambda, \partial_B] L\alpha = -(i-1)^{k} \alpha_0 - (k + i - n - 1) L\alpha_1.$$ 

Similarly, using $\partial_B \alpha = iI(\partial_B \alpha)$ and $I^2(\bigwedge^k D^*) = (-1)^k$ as well as (7) and (11)

$$\ast \partial_B \ast L\alpha = i(-1)^k [\Lambda, \partial_B] L\alpha - (-1)^k L_\eta \bigwedge L^1, \Lambda \alpha.$$ 

Finally,

$$[\Lambda, \partial_B] \bigwedge^k D^* = -i\partial_B^* + iL_\eta \Lambda.$$ 

To study vertical forms, we consider $L_\eta L\alpha$. Again $\alpha \in P^k$ and $\partial_B \alpha = \alpha_0 + L_\alpha$. Then

$$[\Lambda, \partial_B] L_\eta L\alpha = iL_\eta L\alpha_0 + (k + i - n - 1) L_\eta L\alpha_1 + [n - (2i + k)] L\alpha_1.$$ 

Note that $2i + k$ is the degree of $L\alpha$. Furthermore,

$$\ast \partial_B \ast L_\eta L\alpha = (1)^{k^2+1} \times [iL_\eta L\alpha_0 + (k + i - n - 1) L_\eta L\alpha_1].$$ 

In total,

$$[\Lambda, \partial_B] (L_\eta L\alpha) = \{ -i\partial_B^* + [n - (2i + k)] \Lambda_\eta \}(L_\eta L\alpha).$$ 

Since $L_\eta \Lambda(L_\eta L\alpha) = 0$, we can add or subtract $iL_\eta \Lambda$.

Therefore it is consistent to combine the results on horizontal and vertical forms into the overall result (12). An identical calculation or complex conjugation give $[\Lambda, \partial_B]$. This completes the proof.

We can compute the commutator of the adjoints ($\alpha \in P^k$):

$$[L, \partial_B^\ast] \alpha = (-1)^p \ast [-i \ast \partial_B^\ast \ast + i \ast L_\eta \Lambda \ast \ast \ast (n - d^0) \Lambda_\eta \ast] \alpha_p.$$ 

With $\ast (n - d^0) \ast = (d^0 - n)$, $\ast \partial_B^\ast \ast \ast = (-1)^{p+1} \partial_B \alpha$, and $\ast L_\eta \Lambda \ast \ast = (-1)^{p+1} L_\eta \Lambda$ one finds

$$[L, \partial_B^\ast] = i\partial_B - \Lambda_\eta L + (d^0 - n) L_\eta,$$

$$[L, \partial_B^\ast] = -i\partial_B + \Lambda_\eta L + (d^0 - n) L_\eta.$$ 

The calculation of the anticommutator $\{ \Lambda_\eta, \partial_B \}$ is considerably simpler. Consider again $\alpha \in P^k$ with $\partial_0 \alpha = \alpha_0 + L_\alpha$. Then $\Lambda_\eta \partial_B \alpha = 0$ and $\partial_B \Lambda_\eta \alpha = 0$. The next step is only slightly more complicated: $\Lambda_\eta \partial_B L_\eta \alpha =$
This concludes the calculation of the identities. The (anti-) commutators allow us to express $\Delta = d^* d + dd^*$ in terms of $\Delta_{\partial B} = \bar{\partial} B \bar{\partial} B + \bar{\partial} B^2$. The decomposition yields

$$\Delta = \Delta_{\partial B} + \Delta_{\bar{\partial} B} + \{\partial B, \bar{\partial} B\} + \{\bar{\partial} B, \partial B\} = -\mathcal{L}_{\xi}^2.$$  

Then, using $\{\partial B, \bar{\partial} B\} = \{\partial B, L_\eta \Lambda\} + i\omega B \Lambda$, one finds that

$$\Delta_{\partial B} = \Delta_{\bar{\partial} B} = 2i(n-d^0)\mathcal{L}_{\xi} + \{\partial B, L_\eta \Lambda\} - i(\partial B + \bar{\partial} B) \Lambda,$$

which leads to

$$\Delta = 2\Delta_{\partial B} - 2i(n-d^0)\mathcal{L}_{\xi} - \mathcal{L}_{\xi}^2 + 2\{\partial B, L_\eta \Lambda\} - 2i\bar{\partial} B \Lambda.$$

Application of $\{\partial B, L_\eta \Lambda\} = iL_\eta \bar{\partial} B + (n-d^0)L_\eta \Lambda + L \Lambda$ completes the proof.

D. Beyond Kähler identities?

Since we found Sasaki-Einstein equivalents of both $\Delta = 2\Delta_{\partial B}$ and the Kähler identities, it is tempting to ask how much more of Kähler geometry can be generalized. For example, since $\Delta_{\partial B}$ is self-adjoint and elliptic, one can show that $\Omega^k_{\partial B} = H^k \oplus \Delta(\Omega^k_{\partial B})$ which implies Hodge’s theorem. Similarly, the relation between the de Rham and Hodge Laplacians allows for an isomorphism between the respective spaces of harmonic forms. However, it turns out that $\Delta_{\partial B}$ is not elliptic. We will sketch the calculation leading to this result. Recall that $\Delta_{\partial B}$ is elliptic if the symbol $\sigma_{\Delta_{\partial B}} \in \text{Hom}(\Omega^k_{\partial B}, \Omega^k_{\partial B}) \otimes S^2(T^* S)$ maps any non-zero $\omega \in T^* S$ to an automorphism on $\Omega^k_{\partial B}$. When calculating the symbol one essentially keeps only those terms of $\Delta_{\partial B}$ that are of highest order in derivatives. In the context of the tangential Cauchy-Riemann operator, this means that $\partial B$ and $\bar{\partial} B$ can be taken to be anti-commuting and that the overall result is essentially the same as for the symbol of the Dolbeault Laplacian on a Kähler manifold, provided one substitutes $\partial \omega \mapsto \partial \omega - \eta(\partial \omega) \mathcal{L}_{\xi}$. Therefore, $\sigma_{\Delta_{\partial B}}(\xi) = 0$ and $\Delta_{\partial B}$ is not elliptic. Tievsky’s discussion of a transverse Laplacian $\Delta_T$ arrives at a similar result. In that case, it turns out that $\Delta_T - (\Lambda_\eta d^0)^2$ is elliptic. A similar result should hold here, possibly after replacing $\Lambda_\eta d^0$ with $\mathcal{L}_{\xi}^2$. El Kacimi-Alaoui has studied elliptic operators acting on basic forms.

II. MOTIVATION AND APPLICATIONS

Both equation 11 as well as the identities in appendix A find application in the AdS/CFT correspondence. Freund-Rubin compactification on Sasaki-Einstein manifolds yields supergravity duals of superconformal field theories 12,13. The AdS/CFT dictionary links the conformal energy of SCFT operators to the spectrum of $\Delta$, their $R$-charge to that of the Lie-derivative along the Reeb vector, $\mathcal{L}_\xi$. The conformal energy, $R$-charge, and spin of any SCFT operator have to satisfy the unitarity bounds 14, which should be reflected on the supergravity side in the spectrum of $\Delta$. We will argue shortly that equation 11 allows us to re-derive the unitarity bounds from supergravity when considered in conjunction with the calculations in 18,19.

First, note that the Kähler-like identities allow for a study of the eigenmodes of $\Delta$. In the case where the Sasaki-Einstein manifold has a coset structure, this has been done using harmonic analysis 20. 18,19 obtained the structure of the Kaluza-Klein spectrum of generic Sasaki-Einstein manifolds using a construction similar to that in 21, which can be nicely summarized in terms of the identities in appendix A. Given any eigen $k$-form $\omega$ of $\Delta$, one diagonalizes the action of $\Delta$ on the $k+1$-forms $\{\partial B \omega, \bar{\partial} B \omega, \Lambda \omega, L_\eta \omega, \partial B \Lambda \omega, \ldots\}$. The resulting eigenstates fill out representations of the superconformal algebra, equivalence classes in Kohn-Rossi cohomology groups $H^p_q(S\text{E})$ correspond to short multiplets. Whereas the original calculations were based on a rather tiresome direct approach, the methods developed in this note are expected to simplify that kind of analysis considerably.

With this in mind, we turn to the spectral problem for $\Delta$. Consider a $k$-form $\omega$ with $L_\eta \omega = q \omega$, $q \geq 0$, and $d^0 \leq n$. All terms on the right hand side of 11 are positive definite except for the mixed term $M = i(\partial B \partial B - \bar{\partial} B \Lambda_\eta) = N + N^*$. $M$ is self-adjoint and its spectrum is real. Moreover, $N^2 = 0$ and $N(\Lambda^* D^*) \subset \Lambda^* D^* \Lambda$ and $N(\Lambda^* D^* \eta) = 0$. That is, $N$ maps horizontal to vertical forms and annihilates the latter. $N^*$ behaves accordingly and it follows that $\langle \omega, M \omega \rangle$ vanishes if $\omega$ is horizontal or vertical. This is also the case if $\omega$ is neither horizontal nor vertical yet holomorphic.4 As long as we restrict to one of these cases, 11 takes the form of a bound on the spectrum of $\Delta$.

This was conjectured and partially shown in the context of the calculations of the superconformal index 22,23 in 18,19. Here, the spectrum was constructed from primitive elements of $\Omega^p_q$. For such forms, 11 implies

$$\Delta \geq q^2 + 2q(n-d^0)$$

with equality if and only if $\partial B \omega = \bar{\partial} B \omega = 0$. In the Kähler case, the latter of these is implied by transversality — $d^0 \omega = 0$. Here however, $d^* \omega = 0$ leads only to the vanishing of the horizontal component of $\bar{\partial} B \omega$. Indeed,

$$\partial B \omega = iL_\eta \Lambda \omega, \quad \bar{\partial} B \omega = -iL_\eta \Lambda \omega.$$  

4 In the remainder of this discussion, the term holomorphic is meant in respect to the tangential Cauchy-Riemann operator $\partial B$. 
which vanishes since $\omega$ was assumed to be primitive. Assuming that every element of $H^{p,q}_{\partial_B}(S)$ has a representative closed under $\overline{\partial}_B$, the bound \([13]\) is saturated on the elements of $H^{p,q}_{\partial_B}(S)$. These are the forms that correspond to the short multiplets in the SCFT, and \([13]\) together with the expressions for the derived eigenmodes of $\Delta$ given in \([18, 19]\) allows to recover the unitarity bounds from supergravity. Note that \([13]\) and a precursor to \([11]\) were already conjectured in those references. Furthermore, the appendix of \([19]\) contains an argument that every element of $H^{p,q}_{\partial_B}(S)$ is either primitive, carrying zero charge, or both. For the cases of interest in the context of that paper it turned out that all elements are primitive.

A further application of \([11]\) is the stability analysis of Pilch-Warner solutions in \([24]\). In the absence of general theorems concerning Laplace operators on Sasaki-Einstein manifolds, the authors constructed explicitly examples of primitive, basic $(1,1)$-forms whose existence renders these solutions perturbatively unstable. Assuming that the calculations in \([24]\) generalize to generic transverse forms, our results might allow for a continuation of their analysis to manifolds where explicit constructions are not feasible.

Finally, it would be interesting to extend the results presented here beyond the Sasaki-Einstein case. As long as there is a dual SCFT, there is a unitarity bound meaning that there should be some equivalent of \([11]\) or at least \([13]\).

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**Appendix A: The identities**

Table \([1]\) summarizes the various (anti-) commutators. The more involved ones that do not fit in the table are listed in equation \([A1]\).

\[
[\partial_B, \Lambda] = -i\overline{\partial}_B^* + i\Lambda n_\eta (n - d^0)\Lambda_\eta,
\]

\[
[\overline{\partial}_B, \Lambda] = -i\overline{\partial}_B^* - i\Lambda n_\eta (n - d^0)\Lambda_\eta,
\]

\[
[\overline{\partial}_B, L] = -i\overline{\partial}_B^* + i\Lambda n_\eta - (d^0 - n)\Lambda_\eta,
\]

\[
\{\partial_B, \overline{\partial}_B^*\} = -i(L_\eta \partial_B + \overline{\partial}_B^* \Lambda_\eta) + (n - d^0)\Lambda n_\eta + L\Lambda.
\]

\[
\{\overline{\partial}_B, \overline{\partial}_B^*\} = -i(L_\eta \overline{\partial}_B + \overline{\partial}_B^* \Lambda_\eta) + (n - d^0)\Lambda n_\eta + L\Lambda.
\]

\[
(A1)
\]

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TABLE I. The Kähler-like identities

| $\bar{\partial} \{ \partial B, L \eta \}$ | $\Lambda \{ \partial B, \Lambda \eta \}$ | $\Lambda \{ \Lambda \eta, L \eta \}$ |
|---------------------------------|---------------------------------|---------------------------------|
| $L \{ \partial B, L \eta \} = 0$ | $L \{ \partial B, \Lambda \eta \} = 0$ | $L \{ \Lambda \eta, L \eta \} = 0$ |
| $\partial B \{ \partial B, L \eta \} = 0$ | $\partial B \{ \partial B, \Lambda \eta \} = 0$ | $\partial B \{ \Lambda \eta, L \eta \} = 0$ |
| $\Lambda \{ \partial B, L \eta \} = 0$ | $\Lambda \{ \partial B, \Lambda \eta \} = 0$ | $\Lambda \{ \Lambda \eta, L \eta \} = 0$ |
| $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, L \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \Lambda \eta, L \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, \Lambda \eta \} = 0$ |
| $\{ \partial B, \Lambda \eta \} \Lambda \{ \Lambda \eta, L \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, \Lambda \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \Lambda \eta, L \eta \} = 0$ |
| $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, L \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, \Lambda \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \Lambda \eta, L \eta \} = 0$ |
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| $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, L \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, \Lambda \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \Lambda \eta, L \eta \} = 0$ |
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| $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, L \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, \Lambda \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \Lambda \eta, L \eta \} = 0$ |
| $\{ \partial B, \Lambda \eta \} \Lambda \{ \Lambda \eta, L \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, \Lambda \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \Lambda \eta, L \eta \} = 0$ |
| $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, L \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, \Lambda \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \Lambda \eta, L \eta \} = 0$ |
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| $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, L \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, \Lambda \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \Lambda \eta, L \eta \} = 0$ |
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| $\{ \partial B, \Lambda \eta \} \Lambda \{ \Lambda \eta, L \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \partial B, \Lambda \eta \} = 0$ | $\{ \partial B, \Lambda \eta \} \Lambda \{ \Lambda \eta, L \eta \} = 0$ |