ON THE HYDROSTATIC APPROXIMATION OF THE NAVIER-STOKES EQUATIONS IN A THIN STRIP

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Abstract. In this paper, we first prove the global well-posedness of a scaled anisotropic Navier-Stokes system and the hydrostatic Navier-Stokes system in a 2-D striped domain with small analytic data in the tangential variable. Then we justify the limit from the anisotropic Navier-Stokes system to the hydrostatic Navier-Stokes system with analytic data.

Keywords: Incompressible Navier-Stokes Equations, Hydrostatic approximation, Radius of analyticity.

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1. Introduction

This paper is concerned with the study of the Navier-Stokes system in a thin-striped domain and the hydrostatic approximation of these equations when the depth of the domain and the viscosity converge to zero simultaneously in a related way. This is a classical model in geophysical fluid dynamics where the vertical dimension of the domain is very small compared with the horizontal dimension of the domain. In this case, the viscosity is not isotropic and we have to use the anisotropic Navier-Stokes system with a “turbulent” viscosity. The formal limit thus obtained is the hydrostatic Navier-Stokes equations which are currently used as a standard model to describes the atmospheric flows and also oceanic flows in oceanography (see [18, 19]).

When we consider Dirichlet boundary conditions on the top and the bottom of a 2-D striped domain, we are able to prove the global well-posedness of both the anisotropic Navier-Stokes system and the hydrostatic/Prandtl approximate equations when the initial data is small and analytic in the tangential variable. This should be regarded as a global Cauchy-Kowalevskaya theorem for small analytic data, which originates from [5]. The proof of this type of results requires the control of the loss of the radius of the analyticity of the solution. Taking the advantage of the Poincaré inequality in the the strip, we are able to control the analyticity of the solution globally in time. We also rigorously prove the convergence of the anisotropic Navier-Stokes system to the hydrostatic/Prandtl equations in the natural framework of the analytic data in the tangential variable. We now present a precise description of the problem that we shall investigate.

We consider two-dimensional incompressible Navier-Stokes equations in a thin strip: $S^\varepsilon \overset{\text{def}}{=} \{(x,y) \in \mathbb{R}^2 : 0 < y < \varepsilon \}$,

\begin{equation}
\begin{cases}
\partial_t U + U \cdot \nabla U - \varepsilon^2 \Delta U + \nabla P = 0 & \text{in } S^\varepsilon \times ]0, \infty[, \\
\text{div } U = 0,
\end{cases}
\end{equation}

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where \( U(t, x, y) \) denotes the velocity of the fluid and \( P(t, x, y) \) denotes the scalar pressure function which guarantees the divergence free condition of the velocity field \( U \). We complement the system (1.1) with the non-slip boundary condition
\[
U|_{y=0} = U|_{y=\varepsilon} = 0,
\]
and the initial condition
\[
U|_{t=0} = \left( u_0(x, \frac{y}{\varepsilon}), \varepsilon v_0(x, \frac{y}{\varepsilon}) \right) = U_0^\varepsilon \quad \text{in} \quad S^\varepsilon.
\]

As in [21, 12], we write
\[
U(t, x, y) = \left( u^\varepsilon(t, x, \frac{y}{\varepsilon}), \varepsilon v^\varepsilon(t, x, \frac{y}{\varepsilon}) \right) \quad \text{and} \quad P(t, x, y) = p^\varepsilon(t, x, \frac{y}{\varepsilon}).
\]

Let \( S \) defined as \( \{(x, y) \in \mathbb{R}^2 : 0 < y < 1\} \). Then the system (1.1) becomes the following scaled anisotropic Navier-Stokes system:
\[
\left\{
\begin{aligned}
\partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon - \varepsilon^2 \partial_x^2 u^\varepsilon - \partial_y p^\varepsilon = 0 \quad \text{in} \quad S \times ]0, \infty[, \\
\varepsilon^2 \partial_t v^\varepsilon + u^\varepsilon \partial_x v^\varepsilon + v^\varepsilon \partial_y v^\varepsilon - \varepsilon^2 \partial_x^2 v^\varepsilon - \partial_y p^\varepsilon = 0, \\
\partial_x u^\varepsilon + \partial_y v^\varepsilon = 0, \\
(u^\varepsilon, v^\varepsilon)|_{t=0} = (u_0, v_0),
\end{aligned}
\right.
\]
(1.3)

with the boundary condition
\[
(u^\varepsilon, v^\varepsilon)|_{y=0} = (u^\varepsilon, v^\varepsilon)|_{y=1} = 0.
\]

Formally taking \( \varepsilon \to 0 \) in the system (1.3), we obtain the hydrostatic Navier-Stokes/Prandtl equations:
\[
\left\{
\begin{aligned}
\partial_t u + u \partial_x u + v \partial_y u - \partial_y p = 0 \quad \text{in} \quad S \times ]0, \infty[, \\
\partial_y p = 0, \\
\partial_x u + \partial_y v = 0, \\
u|_{t=0} = u_0,
\end{aligned}
\right.
\]
(1.5)

with the boundary condition
\[
(u, v)|_{y=0} = (u, v)|_{y=1} = 0.
\]

The goal of this paper is to justify the limit from the system (1.3) to the system (1.5). The first step is to establish the well-posedness of the two system. Similar to the Prandtl equation, the nonlinear term \( v \partial_y u \) in (1.5) will lead to one derivative loss in the \( x \) variable in the process of energy estimates. Thus, it is natural to work with analytic data in order to overcome this difficulty if we don’t impose extra structural assumptions on the initial data [9, 20]. Indeed, for the data which is analytic in \( x, y \) variables, Sammartino and Caflisch [21] established the local well-posedness result of (1.5) in the upper half space. Later, the analyticity in \( y \) variable was removed by Lombardo, Cannone and Sammartino in [13]. The main argument used in [21, 13] is to apply the abstract Cauchy-Kowalewskaya (CK) theorem. We also mention a well-posedness result of Prandtl system for a class of data with Gevrey regularity [10]. Lately, for a class of convex data, Gérard-Varet, Masmoudi and Vicol [11] proved the well-posedness of the system (1.5) in the Gevrey class.

Now let us state our main results.

The first result is the global well-posedness of the system (1.3) with small analytic data in \( x \) variable. The main interesting point is that the smallness of data is independent of \( \varepsilon \) and there holds the global uniform estimate (1.8) with respect to the parameter \( \varepsilon \).
Theorem 1.1. Let $a > 0$. We assume that the initial data satisfies
\begin{equation}
\|e^{a|D_x|}(u_0, \varepsilon v_0)\|_{\mathcal{B}^\frac{3}{2}} \leq c_0 a
\end{equation}
for some $c_0$ sufficiently small. Then the system (1.5) has a unique global solution $(u, v)$ so that
\begin{equation}
\|e^{\mathcal{R}t}(u_\Psi^\varepsilon, \varepsilon v_\Psi^\varepsilon)\|_{L^\infty([0, t^*]; \mathcal{B}^\frac{3}{2})} + \|e^{\mathcal{R}t} \partial_y (u_\Psi^\varepsilon, \varepsilon v_\Psi^\varepsilon)\|_{L^2([0, t^*]; \mathcal{B}^\frac{3}{2})} \leq C \|e^{a|D_x|}(u_0, \varepsilon v_0)\|_{\mathcal{B}^\frac{3}{2}},
\end{equation}
where $(u_\Psi^\varepsilon, v_\Psi^\varepsilon)$ will be given by (3.1) and the constant $\mathcal{R}$ is determined by Poincaré inequality on the strip $S$ (see (3.6)), and the functional spaces will be presented in Section 2.

The second result is the global well-posedness of the hydrostatic Navier-Stokes system (1.5) with small analytic data in $x$ variable. We remark that similar global result seems open for the Prandtl equation, where only a lower bound of the lifespan to the solution was obtained (see [22]).

Theorem 1.2. Let $a > 0$. We assume that the initial data satisfies
\begin{equation}
\|e^{a|D_x|}u_0\|_{\mathcal{B}^\frac{3}{2}} \leq c_1 a
\end{equation}
for some $c_1$ sufficiently small and there holds the compatibility condition $\partial_x \int_0^1 u_0 dy = 0$. Then the system (1.5) has a unique global solution $u$ so that
\begin{equation}
\|e^{\mathcal{R}t}u_\Phi\|_{L^\infty([0, t^*]; \mathcal{B}^\frac{3}{2})} + \|e^{\mathcal{R}t} \partial_y u_\Phi\|_{L^2([0, t^*]; \mathcal{B}^\frac{3}{2})} \leq C \|e^{a|D_x|}u_0\|_{\mathcal{B}^\frac{3}{2}},
\end{equation}
where $u_\Phi$ will be determined by (1.3). Furthermore, if $e^{a|D_x|}u_0 \in \mathcal{B}^\frac{3}{2}, e^{a|D_x|}\partial_y u_0 \in \mathcal{B}^\frac{3}{2}$ and
\begin{equation}
\|e^{a|D_x|}u_0\|_{\mathcal{B}^\frac{3}{2}} \leq \frac{c_2 a}{1 + \|e^{a|D_x|}u_0\|_{\mathcal{B}^\frac{3}{2}}}
\end{equation}
for some $c_2$ sufficiently small, then there exists a positive constant $C$ so that for $\lambda = C^2(1 + \|e^{a|D_x|}u_0\|_{\mathcal{B}^\frac{3}{2}})$ and $1 \leq s \leq \frac{3}{2}$, one has
\begin{align}
\|e^{\mathcal{R}t}u_\Phi\|_{L^\infty([0, t^*]; \mathcal{B}^s)} + \|e^{\mathcal{R}t} \partial_y u_\Phi\|_{L^2([0, t^*]; \mathcal{B}^s)} &\leq C \|e^{a|D_x|}u_0\|_{\mathcal{B}^s},
\|e^{\mathcal{R}t}(\partial_t u)_\Phi\|_{L^2([0, t^*]; \mathcal{B}^s)} + \|e^{\mathcal{R}t} \partial_y^2 u_\Phi\|_{L^2([0, t^*]; \mathcal{B}^s)} &\leq C \|e^{a|D_x|}\partial_y u_0\|_{\mathcal{B}^s} + \|e^{a|D_x|}u_0\|_{\mathcal{B}^\frac{3}{2}}.
\end{align}

The third result is concerning the convergence from the scaled anisotropic Navier-Stokes system (1.3) to the hydrostatic Navier-Stokes system (1.5).

Theorem 1.3. Let $a > 0$ and $(u_0^\varepsilon, v_0^\varepsilon)$ satisfy (1.7). Let $u_0$ satisfy $e^{a|D_x|}u_0 \in \mathcal{B}^\frac{3}{2} \cap \mathcal{B}^\frac{3}{2}, e^{a|D_x|}\partial_y u_0 \in \mathcal{B}^\frac{3}{2}$, and there holds (1.11) for some $c_2$ sufficiently small and the compatibility condition $\partial_x \int_0^1 u_0 dy = 0$. Then we have
\begin{align}
\|\!(w_1^\varepsilon, v_2^\varepsilon)\!\|_{\mathcal{L}^\infty([0, t^*]; \mathcal{B}^\frac{3}{2})} + \|\!(\partial_y (w_1^\varepsilon, v_2^\varepsilon))\!\|_{\mathcal{L}^2([0, t^*]; \mathcal{B}^\frac{3}{2})} + \varepsilon \|\!(w_1^\varepsilon, v_2^\varepsilon)\!\|_{\mathcal{L}^1([0, t^*]; \mathcal{B}^\frac{3}{2})}
&\leq C \left( \|e^{a|D_x|}(u_0^\varepsilon - u_0, \varepsilon (v_0^\varepsilon - v_0))\|_{\mathcal{B}^\frac{3}{2}} + M \varepsilon \right).
\end{align}
Here $w_1^\varepsilon \overset{\text{def}}{=} u^\varepsilon - u$, $w_2^\varepsilon \overset{\text{def}}{=} v^\varepsilon - v$ and $v_0$ is determined from $u_0$ via $\partial_x u_0 + \partial_y v_0 = 0$ and $v_0|_{y=0} = v_0|_{y=1} = 0$, and $(w_1^\varepsilon, v_2^\varepsilon)$ will be given by (5.3).

We remark that without the smallness conditions (1.7) and (1.11), we can prove the convergence of the system (1.3) to the system (1.5) on a fixed time interval $[0, T]$. 


We end this introduction by the notations that will be used in all that follows. For $a \lesssim b$, we mean that there is a uniform constant $C$, which may be different on different lines, such that $a \leq C b$. We denote by $(a|b)_{L^2}$ the $L^2(S)$ inner product of $a$ and $b$. We designate by $L^p_T(L^q_h (L^r_y))$ the space $L^p([0, T]; L^q_S (\mathbb{R}_x; L^r_y(\mathbb{R}_y)))$. Finally, we denote by $(d_k)_{k \in \mathbb{Z}}$ (resp. $(d_k(t))_{k \in \mathbb{Z}}$) to be a generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{k \in \mathbb{Z}} d_k = 1$ (resp. $\sum_{k \in \mathbb{Z}} d_k(t) = 1$).

2. Littlewood-Paley theory and functional framework

In the rest of this paper, we shall frequently use Littlewood-Paley decomposition in the horizontal variable $x$. Let us recall from \[1\] that

\begin{equation}
\Delta_h^k a = F^{-1}(\varphi(2^{-k}|\xi|)\hat{a}), \quad S_h^k a = F^{-1}(\chi(2^{-k}|\xi|)\hat{a}),
\end{equation}

where $Fa$ and $\hat{a}$ denote the partial Fourier transform of the distribution $a$ with respect to $x$ variable, that is, $\hat{a}(\xi, y) = F_{x \rightarrow \xi}(a)(\xi, y)$, and $\chi(\tau)$, $\varphi(\tau)$ are smooth functions such that

- $\text{Supp } \varphi \subset \{ \tau \in \mathbb{R} / 3/4 \leq |\tau| \leq 8/3 \}$ and $\forall \tau > 0$, $\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1$,
- $\text{Supp } \chi \subset \{ \tau \in \mathbb{R} / |\tau| \leq 4/3 \}$ and $\chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1$.

Let us also recall the functional spaces we are going to use.

**Definition 2.1.** Let $s$ in $\mathbb{R}$. For $u$ in $S'_h(S)$, which means that $u$ belongs to $S'(S)$ and satisfies $\lim_{k \to -\infty} \| S_h^k u \|_{L^\infty} = 0$, we set

$$
\| u \|_{B^s} \overset{\text{def}}{=} \| (2^{ks} \| \Delta_h^k u \|_{L^2})_{k \in \mathbb{Z}} \|_{\ell^1(\mathbb{Z})},
$$

- For $s \leq 1/2$, we define $B^s(S) = \{ u \in S'_h(S) \mid \| u \|_{B^s} < \infty \}$.
- If $k$ is a positive integer and if $1/2 + k < s \leq 3/2 + k$, then we define $B^s(S)$ as the subset of distributions $u$ in $S'_h(S)$ such that $\partial_x^k u$ belongs to $B^{s-k}(S)$.

In order to obtain a better description of the regularizing effect of the diffusion equation, we need to use Chemin-Lerner type spaces $\tilde{L}^p_T(B^s(S))$.

**Definition 2.2.** Let $p \in [1, +\infty]$ and $T \in (0, +\infty]$. We define $\tilde{L}^p_T(B^s(S))$ as the completion of $C([0, T]; S(S))$ by the norm

$$
\| a \|_{\tilde{L}^p_T(B^s)} \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}} 2^{ks} \left( \int_0^T \| \Delta_h^k a(t) \|_{L^2}^p \, dt \right)^{1/p}
$$

with the usual change if $p = \infty$.

In order to overcome the difficulty that one can not use Gronwall type argument in the framework of Chemin-Lerner space, we need to use the time-weighted Chemin-Lerner norm, which was introduced by the first two authors in \[15\].

**Definition 2.3.** Let $f(t) \in L^1_{koc}(\mathbb{R}_+)$ be a nonnegative function. We define

\begin{equation}
\| a \|_{\tilde{L}^p_{t,f}(B^s)} \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}} 2^{ks} \left( \int_0^t f(t') \| \Delta_h^k a(t') \|_{L^2}^p \, dt' \right)^{1/p}.
\end{equation}

For the convenience of the readers, we recall the following anisotropic Bernstein type lemma from \[1, 14\].
Lemma 2.1. Let $B_h$ be a ball of $\mathbb{R}_h$, and $C_h$ a ring of $\mathbb{R}_h$; let $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q \leq \infty$. Then there holds:

If the support of $\hat{a}$ is included in $2^k B_h$, then

$$\|\partial_x^k a\|_{L^{p_1}(L^q)} \lesssim 2^{k\left(\left|\alpha\right| + \left(\frac{1}{p_2} - \frac{1}{p_1}\right)\right)} \|a\|_{L^{p_2}(L^q)}.$$

If the support of $\hat{a}$ is included in $2^k C_h$, then

$$\|a\|_{L^{p_1}(L^q)} \lesssim 2^{-kN} \|\partial_x^N a\|_{L^{p_1}(L^q)}.$$

In the following context, we shall constantly use Bony’s decomposition (see [4]) for the horizontal variable:

$$fg = T^h f g + T^h f + R^h(f, g),$$

where

$$T^h f g \overset{\text{def}}{=} \sum_k S_{k-1}^h f \Delta_k^h g, \quad \text{and} \quad R^h(f, g) \overset{\text{def}}{=} \sum_k \Delta_k^h f \hat{\Delta}_k^h g$$

with $\hat{\Delta}_k^h g \overset{\text{def}}{=} \sum |k - k'| \leq 1 \Delta_k^h g$.

3. Global well-posedness of the system (1.3)

In this section, we establish the global well-posedness of the scaled anisotropic Navier-Stokes system (1.3) with small analytic data.

Proof of Theorem 1.1. As in [5, 6, 8, 16, 17, 22], for any locally bounded function $\Psi$ on $\mathbb{R}^+ \times \mathbb{R}$, we define

$$u_\Psi^\varepsilon(t, x, y) \overset{\text{def}}{=} F_{\xi \rightarrow x}^{-1} (e^{\Psi(t, \xi)} \hat{u}_\varepsilon(t, \xi, y)).$$

We introduce a key quantity $\eta(t)$ to describe the evolution of the analytic band of $u^\varepsilon$:

$$\left\{ \begin{aligned}
\dot{\eta}(t) &= \varepsilon \|\partial_x u_\Psi^\varepsilon(t)\|_{L^{2\frac{1}{2}}} + \|\partial_y u_\Psi^\varepsilon(t)\|_{L^{2\frac{1}{2}}}, \\
\eta(t=0) &= 0.
\end{aligned} \right.$$ 

Here the phase function $\Psi$ is defined by

$$\Psi(t, \xi) \overset{\text{def}}{=} (a - \lambda \eta(t))|\xi|.$$ 

In the rest of this section, we shall prove that under the assumption of (1.7), there holds the a priori estimate (1.8) for smooth enough solutions of (1.3), and neglect the regularization procedure. For simplicity, we shall neglect the script $\varepsilon$. Then in view of (1.3) and (3.1), we observe that $(u_\Psi, v_\Psi)$ verifies

$$\left\{ \begin{aligned}
\partial_t u_\Psi + \lambda \eta(t)|D_x|u_\Psi + (u \partial_x u)_\Psi + (v \partial_y u)_\Psi - \varepsilon^2 \partial_x^2 u_\Psi - \partial_y^2 u_\Psi + \partial_x p_\Psi &= 0, \\
\varepsilon^2 \left( \partial_t v_\Psi + \lambda \eta(t)|D_x|v_\Psi + (u \partial_x v)_\Psi + (v \partial_y v)_\Psi - \varepsilon^2 \partial_x^2 v_\Psi - \partial_y^2 v_\Psi \right) + \partial_y p_\Psi &= 0, \\
\partial_x u_\Psi + \partial_y v_\Psi &= 0 \quad \text{for} \quad (t, x, y) \in \mathbb{R}^+ \times \mathcal{S}, \\
(u_\Psi, v_\Psi) \bigr|_{y=0} &= (u_\Psi, v_\Psi) \bigr|_{y=1} = 0,
\end{aligned} \right.$$ 

where $|D_x|$ denotes the Fourier multiplier with symbol $|\xi|$.
By applying the dyadic operator $\Delta_k^h$ to (3.4) and then taking the $L^2$ inner product of the resulting equation with $(\Delta_k^h u, \Delta_k^h v)$, we find

$$
\frac{1}{2} \frac{d}{dt} \|\Delta_k^h (u, v)\|^2_{L^2} + \lambda \eta (|D_x| \Delta_k^h (u, v) \cdot \Delta_k^h (u, v))_{L^2}
+ \varepsilon^2 \|\partial_x \Delta_k^h (u, v)\|^2_{L^2} + \|\partial_y \Delta_k^h (u, v)\|^2_{L^2}
= - (\Delta_k^h (u \partial_x v) \cdot \Delta_k^h (u)_{L^2} - (\Delta_k^h (v \partial_y u) \cdot \Delta_k^h (v)_{L^2}
- \varepsilon^2 (\Delta_k^h (u \partial_x v) \cdot \Delta_k^h (v)_{L^2} - \varepsilon^2 (\Delta_k^h (v \partial_y v) \cdot \Delta_k^h (v)_{L^2},
$$

where we used the fact that $\partial_x u + \partial_y v = 0$, so that

$$
(\nabla \Delta_k^h| \Delta_k^h(u, v)\|^2_{L^2} = 0.
$$

While due to $(u, v)|_{y=0} = (u, v)|_{y=1} = 0$, by applying Poincaré inequality, we have

$$
\mathcal{R} \|\Delta_k^h (u, v)\|^2_{L^2} \leq \frac{1}{2} \|\partial_y \Delta_k^h (u, v)\|^2_{L^2}.
$$

Then by using Lemma 2.1 and by multiplying (3.5) by $e^{2 \mathcal{R} t}$ and then integrating the resulting inequality over $[0, t]$, we achieve

$$
\frac{1}{2} \left| e^{\mathcal{R} t} \Delta_k^h (u, v) \right|^2_{L^2} + \lambda^2 k \int_0^t \eta (t') \left| e^{\mathcal{R} t'} \Delta_k^h (u, v) (t') \right|^2_{L^2} dt'
+ \frac{1}{2} \int_0^t e^{2 \mathcal{R} t'} \left| \Delta_k^h \partial_x u \right|^2_{L^2} + \varepsilon^2 (2k^2 (\|\Delta_k^h u\|^2_{L^2} + \varepsilon^2 \|\Delta_k^h v\|^2_{L^2} + \|\Delta_k^h \partial_y v\|^2_{L^2}) \right) dt'
\leq \| a |D_x| \Delta_k^h (u, v) \|^2_{L^2} + \int_0^t \left| (e^{\mathcal{R} t'} \Delta_k^h (u \partial_x v) \cdot \Delta_k^h (u)_{L^2} \right| dt'
+ \int_0^t \left| (e^{\mathcal{R} t'} \Delta_k^h (v \partial_y u) \cdot \Delta_k^h (v)_{L^2} \right| dt'
+ \varepsilon^2 \int_0^t \left| (e^{\mathcal{R} t'} \Delta_k^h (v \partial_y v) \cdot \Delta_k^h (v)_{L^2} \right| dt'.
$$

In what follows, we shall always assume that $t < T^*$ with $T^*$ being determined by

$$
T^* \overset{\text{def}}{=} \sup \{ t > 0, \eta(t) < a/\lambda \}.
$$

So that by virtue of (3.3), for any $t < T^*$, there holds the following convex inequality

$$
\Psi(t, \xi) \leq \Psi(t, \xi - \eta) + \Psi(t, \eta) \quad \forall \ \xi, \eta \in \mathbb{R}.
$$

The estimate of (3.7) relies on the following lemmas.

**Lemma 3.1.** For any $s \in [0, 1]$ and $t \leq T^*$, there holds

$$
\int_0^t \left| \left( e^{\mathcal{R} t'} \Delta_k^h (u \partial_x w) \cdot \Delta_k^h (w)_{L^2} \right| dt' \leq d_k^2 2^{-2ks} \| e^{\mathcal{R} t'} w \|^2_{L^2, \eta(t)} (5t^* + \frac{1}{2} t^*).
$$

**Lemma 3.2.** For any $s \in [0, 1]$ and $t \leq T^*$, there holds

$$
\int_0^t \left| \left( e^{\mathcal{R} t'} \Delta_k^h (v \partial_y u) \cdot \Delta_k^h (u)_{L^2} \right| dt' \leq d_k^2 2^{-2ks} \| e^{\mathcal{R} t'} u \|^2_{L^2, \eta(t)} (5t^* + \frac{1}{2} t^*).
Lemma 3.3. For $t \leq T^*$, there holds

\begin{equation}
\varepsilon^2 \int_0^t \left( e^{\mathcal{R}t} \Delta_k^h (\nu \partial_y \nu) \right)_{L^2} \, dt' \lesssim d_k^2 2^{-k} \left\| e^{\mathcal{R}t} (u \nu, \varepsilon \nu) \right\|^2_{L^2_{t,0}(\Omega)}.
\end{equation}

Let us admit the above lemmas for the time being and continue our proof. Indeed, thanks to Lemmas 3.1 [3.3] we deduce from (3.7) that

\begin{align*}
\frac{1}{2} \left\| e^{\mathcal{R}t} \Delta_k^h (u \nu, \varepsilon \nu) \right\|^2_{L^2(L^2)} + \lambda^2 \int_0^t \eta(t') \left\| e^{\mathcal{R}t'} \Delta_k^h (u \nu, \varepsilon \nu)(t') \right\|^2_{L^2} \, dt' \\
+ \frac{c}{2} \int_0^t e^{2\mathcal{R}t'} \left( \left\| \Delta_k^h \partial_y (u \nu, \varepsilon \nu) \right\|^2_{L^2} + \varepsilon^2 2^{2k} \left\| \Delta_k^h (u \nu, \varepsilon \nu) \right\|^2_{L^2} \right) \, dt'
\leq & \left\| e^{\mathcal{R}[D_k]} \Delta_k^h (u_0, \varepsilon v_0) \right\|^2_{L^2_{t,0}(\Omega)} + C d_k^2 2^{-k} \left\| e^{\mathcal{R}t} (u \nu, \varepsilon \nu) \right\|^2_{L^2_{t,0}(\Omega)}.
\end{align*}

By multiplying the above inequality by $2^k$ and then taking square root of the resulting inequality, and finally by summing up the resulting ones over $k$, we find that for $t \leq T^*$

\begin{align*}
&\left\| e^{\mathcal{R}t} (u \nu, \varepsilon \nu) \right\|^2_{L^2_{t,\infty}(B_1^2)} + \lambda \left\| e^{\mathcal{R}t} (u \nu, \varepsilon \nu) \right\|^2_{L^2_{t,0}(B_1^2)} + c \left\| e^{\mathcal{R}t} \partial_y (u \nu, \varepsilon \nu) \right\|^2_{L^2_{t,0}(B_1^2)} \\
&+ \varepsilon^2 \left\| e^{\mathcal{R}t} (u \nu, \varepsilon \nu) \right\|^2_{L^2_{t,0}(B_1^2)} \leq \left\| e^{\mathcal{R}[D_k]} (u_0, \varepsilon v_0) \right\|^2_{B_1^2} + C \left\| e^{\mathcal{R}t} (u \nu, \varepsilon \nu) \right\|^2_{L^2_{t,0}(B_1^2)}
\end{align*}

Taking $\lambda = C^2$ in the above inequality leads to

\begin{align}
\left\| e^{\mathcal{R}t} (u \nu, \varepsilon \nu) \right\|^2_{L^2_{t,\infty}(B_1^2)} + c \left\| e^{\mathcal{R}t} \partial_y (u \nu, \varepsilon \nu) \right\|^2_{L^2_{t,0}(B_1^2)} \\
+ \varepsilon^2 \left\| e^{\mathcal{R}t} (u \nu, \varepsilon \nu) \right\|^2_{L^2_{t,0}(B_1^2)} \leq \left\| e^{\mathcal{R}[D_k]} (u_0, \varepsilon v_0) \right\|^2_{B_1^2} \quad \text{for } t \leq T^*.
\end{align}

Then for $t \leq T^*$, we deduce from (3.2) that

\begin{align*}
\eta(t) = & \int_0^t \left( \varepsilon \left\| \partial_x u \tilde{\nu}(t') \right\|^2_{B_1^2} + \left\| \partial_y u \tilde{\nu}(t') \right\|^2_{B_1^2} \right) \, dt' \\
\leq & \int_0^t e^{-2\mathcal{R}t'} \, dt' \left( \left\| e^{\mathcal{R}t'} \partial_x u \tilde{\nu}(t') \right\|^2_{B_1^2} + \left\| e^{\mathcal{R}t'} \partial_y u \tilde{\nu}(t') \right\|^2_{B_1^2} \right) \, dt' \\
\leq & C \left\| e^{\mathcal{R}t} (\varepsilon \partial_x u \tilde{\nu}, \partial_y u \tilde{\nu}) \right\|^2_{L^2_{t,0}(B_1^2)} \\
\leq & C \left\| e^{\mathcal{R}[D_k]} (u_0, \varepsilon v_0) \right\|^2_{B_1^2}.
\end{align*}

In particular, if we take $c_0$ in (1.7) to be so small that

\begin{equation}
C \left\| e^{\mathcal{R}[D_k]} (u_0, \varepsilon v_0) \right\|^2_{B_1^2} \leq \frac{\alpha}{2\lambda},
\end{equation}

we deduce by a continuous argument that $T^*$ determined by (3.5) equals $+\infty$ and (1.8) holds. This completes the proof of Theorem 1.1. $\square$

Now let us present the proof of Lemmas 3.1 to 3.3. Indeed, we observe that it amounts to prove these lemmas for $R = 0$. Without loss of generality, we may assume that $\tilde{u} \geq 0$ and $\tilde{v} \geq 0$ (and similar assumption for the proof of the product law in the rest of this paper, one may check [9] for detail).

**Proof of Lemma 3.1.** We first get, by applying Bony’s decomposition (2.3) for the horizontal variable to $u \partial_x w$, that

\[ u \partial_x w = T^h_u \partial_x w + T^h_{\partial_x u} u + R^h(u, \partial_x w). \]

Accordingly, we shall handle the following three terms:
Estimate of \( \int_0^t (\Delta_k^h(T^h u) \partial_x w) \psi \ | \Delta_k^h w \psi)_{L^2} dt' \)

Considering the support properties to the Fourier transform of the terms in \( T^h u \partial_x w \), we infer

\[
\int_0^t \left| \left( \Delta_k^h(T^h u) \partial_x w) \psi \right| \Delta_k^h w \psi)_{L^2} \right| dt' \\
\lesssim \sum_{|k'-k| \leq 4} \int_0^t \left\| S^h_{k'-1} u \psi(t') \right\|_{L^\infty} \left\| \Delta_k^h \partial_x w \psi(t') \right\|_{L^2} \left\| \Delta_k^h w \psi(t') \right\|_{L^2} dt'.
\]

However, it follows from Lemma 2.1 and Poincaré inequality that

\[
\left\| \Delta_k^h u \psi(t) \right\|_{L^\infty} \lesssim 2^{\frac{k}{2}} \left\| \Delta_k^h u \psi(t) \right\|_{L^2(L^\infty)} \\
\lesssim 2^{\frac{k}{2}} \left\| \Delta_k^h u \psi(t) \right\|_{L^2} \left\| \Delta_k^h \partial_y u \psi(t) \right\|_{L^2} \\
\lesssim 2^{\frac{k}{2}} \left\| \Delta_k^h \partial_y u \psi(t) \right\|_{L^2} \lesssim d_j(t) \left\| \partial_y u \psi(t) \right\|_{L^2},
\]

so that

\[
\left\| S^h_{k'-1} u \psi(t) \right\|_{L^\infty} \lesssim \left\| \partial_y u \psi(t) \right\|_{L^{4}},
\]

which implies that

\[
\int_0^t \left| \left( \Delta_k^h(T^h u) \partial_x w) \psi \right| \Delta_k^h w \psi)_{L^2} \right| dt' \\
\lesssim \sum_{|k'-k| \leq 4} 2^{k'} \int_0^t \left\| \partial_y u \psi(t) \right\|_{B^\frac{1}{2}} \left\| \Delta_k^h w \psi(t) \right\|_{L^2} \left\| \Delta_k^h w \psi(t') \right\|_{L^2} dt'.
\]

Applying Hölder inequality and using Definition 2.3 gives

\[
\int_0^t \left| \left( \Delta_k^h(T^h u) \partial_x w) \psi \right| \Delta_k^h w \psi)_{L^2} \right| dt' \lesssim \sum_{|k'-k| \leq 4} 2^{k'} \left( \int_0^t \left\| \partial_y u \psi(t') \right\|_{B^\frac{1}{2}} \left\| \Delta_k^h w \psi(t') \right\|_{L^2} dt' \right)^\frac{1}{2} \\
\times \left( \int_0^t \left\| \partial_y u \psi(t') \right\|_{B^\frac{1}{2}} \left\| \Delta_k^h w \psi(t') \right\|_{L^2}^2 dt' \right)^\frac{1}{2} \\
\lesssim d_k 2^{-2ks} \left\| \psi \right\|_{L^2_{t,x} \psi}^2 \left( \sum_{|k'-k| \leq 4} d_k 2^{2(k-k')(s-\frac{1}{2})} \right) \\
\lesssim d_k^2 2^{-2ks} \left\| \psi \right\|_{L^2_{t,x} \psi}^2.
\]

Estimate of \( \int_0^t (\Delta_k^h(T^h \partial_x w) u) \psi \ | \Delta_k^h w \psi)_{L^2} dt' \)
Again considering the support properties to the Fourier transform of the terms in $T_{\partial_x w}^h u$ and thanks to \((3.15)\), we have

$$\int_0^t \left| \langle \Delta_k^h (T_{\partial_x w}^h u) \rangle_{\Psi} \mid \Delta_k^h w \Psi \rangle \right|_{L^2} \, dt'$$

$$\lesssim \sum_{|k'-k| \leq 4} \int_0^t \| S_{k'-1}^h \partial_x w \Psi(t') \|_{L^\infty(L^2)} \| \Delta_k^h u \Psi(t') \|_{L^2(L^\infty)} \| \Delta_k^h w \Psi(t') \|_{L^2} \, dt'$$

$$\lesssim \sum_{|k'-k| \leq 4} 2^{\frac{k'}{2}} \int_0^t \| S_{k'-1}^h \partial_x w \Psi(t') \|_{L^\infty(L^2)} \| \partial_y u \Psi(t') \|_{B_{\frac{3}{2}}} \| \Delta_k^h w \Psi(t') \|_{L^2} \, dt'$$

$$\lesssim \sum_{|k'-k| \leq 4} d_k 2^{\frac{k'}{2}} \left( \int_0^t \| S_{k'-1}^h \partial_x w \Psi(t') \|_{L^\infty(L^2)} \| \partial_y u \Psi(t') \|_{B_{\frac{3}{2}}} \, dt'\right)^{\frac{1}{2}}$$

$$\times \left( \int_0^t \| \Delta_k^h w \Psi(t') \|_{L^2} \| \partial_y u \Psi(t') \|_{B_{\frac{3}{2}}} \, dt'\right)^{\frac{1}{2}}.$$

Yet we observe from Definition \(2.3\) and \(s < 1\) that

$$\left( \int_0^t \| S_{k'-1}^h \partial_x w \Psi(t') \|_{L^\infty(L^2)} \| \partial_y u \Psi(t') \|_{B_{\frac{3}{2}}} \, dt'\right)^{\frac{1}{2}}$$

$$\lesssim \sum_{\ell \leq k'-2} 2^{\frac{\ell}{2}} \left( \int_0^t \| \Delta_k^h w \Psi(t') \|_{L^2}^2 \| \partial_y u \Psi(t') \|_{B_{\frac{3}{2}}} \, dt'\right)^{\frac{1}{2}}$$

$$\lesssim \sum_{\ell \leq k'-2} d_k 2^{\ell(1-s)} \| w \Psi \|_{\tilde{L}_{t,\infty(c)}(B^{s+\frac{1}{2}})}$$

$$\lesssim 2^{k'(1-s)} \| w \Psi \|_{\tilde{L}_{t,\infty(c)}(B^{s+\frac{1}{2}})}.$$

So that it comes out

$$\int_0^t \left| \langle \Delta_k^h (T_{\partial_x w}^h u) \rangle_{\Psi} \mid \Delta_k^h w \Psi \rangle \right|_{L^2} \, dt' \lesssim d_k 2^{-2k} \| w \Psi \|_{\tilde{L}_{t,\infty(c)}(B^{s+\frac{1}{2}})}.$$ 

**Estimate of \( \int_0^t \langle \Delta_k^h (R^h(u, \partial_x w)) \rangle_{\Psi} \mid \Delta_k^h w \Psi \rangle \right|_{L^2} \, dt' \)**

Again considering the support properties to the Fourier transform of the terms in $R^h(u, \partial_x w)$, we get, by applying lemma \(2.1\) and \((3.15)\), that

$$\int_0^t \left| \langle \Delta_k^h (R^h(u, \partial_x w)) \rangle_{\Psi} \mid \Delta_k^h w \Psi \rangle \right|_{L^2} \, dt'$$

$$\lesssim 2^{\frac{h}{2}} \sum_{k' \geq k-3} \int_0^t \| \Delta_k^h u \Psi(t') \|_{L^2(L^\infty)} \| \Delta_k^h \partial_x w \Psi(t') \|_{L^2} \| \Delta_k^h w \Psi(t') \|_{L^2} \, dt'$$

$$\lesssim 2^{\frac{h}{2}} \sum_{k' \geq k-3} 2^{k'} \int_0^t \| \partial_y u \Psi(t') \|_{B_{\frac{3}{2}}} \| \Delta_k^h w \Psi(t') \|_{L^2} \| \Delta_k^h w \Psi(t') \|_{L^2} \, dt'.$$
Applying Hölder inequality and using Definition 2.3 yields
\[
\int_0^t |(\Delta_h^k (R^h(u, \partial_x w))_\Psi | \Delta_h^k w_\Psi)_{L^2}| dt' \\
\lesssim 2^{s} \sum_{k' \geq k-3} 2^{k'} \left( \int_0^t \| \Delta_h^{k'} w_\Psi(t') \|_{L^2}^2 \| \partial_y u_\Psi(t') \|_{B^{s+\frac{1}{2}}} \right)^{\frac{1}{2}} \\
\times \left( \int_0^t \| \Delta_h^k w_\Psi(t') \|_{L^2}^2 \| \partial_y u_\Psi(t') \|_{B^{s+\frac{1}{2}}} dt' \right)^{\frac{1}{2}} \\
\lesssim d_k 2^{-2k s} \| w_\Psi \|_{L^2_{t, \eta(t)}(B^{s+\frac{1}{2}})}^2 \left( \sum_{k' \geq k-3} d_k 2^{(k-k') s} \right) \\
\lesssim d_k^2 2^{-2k s} \| w_\Psi \|_{L^2_{t, \eta(t)}(B^{s+\frac{1}{2}})},
\]
where we used the fact that \( s > 0 \) in the last step.

By summing up the above estimates, we conclude the proof of (3.10). \( \square \)

**Remark 3.1.** In the particular case when \( w = u \) in (3.10), (3.10) holds for any \( s > 0 \), that is
\[
(3.16) \quad \int_0^t \left| \langle e^{\delta t'} \Delta_h^k (u \partial_x u) \psi | e^{\delta t'} \Delta_h^k u_\Psi \rangle_{L^2} \right| dt' \lesssim d_k^2 2^{-2k s} \| e^{\delta t'} u_\Psi \|_{L^2_{t, \eta(t)}(B^{s+\frac{1}{2}})}^2.
\]

It follows from the proof of Lemma 3.1 that we only need to prove
\[
(3.17) \quad \int_0^t \left| \langle \Delta_h^k (T^h_{\partial_x u}) \psi | \Delta_h^k u_\Psi \rangle_{L^2} \right| dt' \lesssim d_k^2 2^{-2k s} \| u_\Psi \|_{L^2_{t, \eta(t)}(B^{s+\frac{1}{2}})}^2 \quad \text{for any} \ s > 0.
\]

Indeed in view of (3.16), we infer
\[
\int_0^t \left| \langle \Delta_h^k (T^h_{\partial_x u}) \psi | \Delta_h^k u_\Psi \rangle_{L^2} \right| dt' \\
\lesssim \sum_{|k' - k| \leq 4} \int_0^t \| S_{k', \eta(t)}^h \partial_x u_\Psi(t') \|_{L^\infty} \| \Delta_h^{k'} u_\Psi(t') \|_{L^2} \| \Delta_h^k u_\Psi(t') \|_{L^2} dt' \\
\lesssim \sum_{|k' - k| \leq 4} 2^{k'} \int_0^t \| \partial_y u_\Psi(t') \|_{B^{s+\frac{1}{2}}} \| \Delta_h^{k'} u_\Psi(t') \|_{L^2} \| \Delta_h^k u_\Psi(t') \|_{L^2} dt' \\
\lesssim \sum_{|k' - k| \leq 4} 2^{k'} \left( \int_0^t \| \Delta_h^{k'} u_\Psi(t') \|_{L^2}^2 \| \partial_y u_\Psi(t') \|_{B^{s+\frac{1}{2}}} \right)^{\frac{1}{2}} \\
\times \left( \int_0^t \| \Delta_h^k u_\Psi(t') \|_{L^2}^2 \| \partial_y u_\Psi(t') \|_{B^{s+\frac{1}{2}}} dt' \right)^{\frac{1}{2}},
\]
which leads to (3.17).

**Proof of Lemma 3.2.** We first get, by applying Bony’s decomposition 2.3 for the horizontal variable to \( v \partial_y u \), that
\[
v \partial_y u = T^h_v \partial_y u + T^h_{\partial_y u} v + R^h(v, \partial_y u).
\]

Accordingly, we shall handle the following three terms:

- **Estimate of** \( \int_0^t \langle \Delta_h^k (T^h_v \partial_y u) \psi | \Delta_h^k u_\Psi \rangle_{L^2} dt' \)**
We first observe that
\[
\int_0^t \left| \left( \Delta_k^h(T^h_v \partial_y u) \right)_L^2 \right| dt' 
\leq \sum_{|k' - k| \leq 4} \int_0^t \left| \sum_{L_0}^h \| S_{k' - 1}^h v(t') \|_{L^\infty} \| \Delta_k^h \partial_y u(t') \|_{L^2} \| \Delta_k^h u(t') \|_{L^2} dt' \right|
\leq \sum_{|k' - k| \leq 4} d_k 2^{-k' - \frac{3}{2}} \int_0^t \left| \sum_{L_0}^h \| S_{k' - 1}^h v(t') \|_{L^\infty} \| \partial_y u(t') \|_{B^\frac{1}{2}} \| \Delta_k^h u(t') \|_{L^2} dt' \right|
\]

Due to \( \partial_x u + \partial_y v = 0 \) and (1.4) we write \( v(t, x, y) = -\int_0^y \partial_x u(t, x, y') dy' \). Then we deduce from Lemma 2.1 that

\[
\| \Delta_k^h v(t) \|_{L^\infty} \leq \int_0^1 \left| \Delta_k^h \partial_x u(t, \cdot, y') \right|_{L^\infty} dy' \leq 2 \frac{\Delta_k}{\Delta_h} \int_0^1 \left| \Delta_k^h u(t, \cdot, y') \right|_{L^2} dy' \leq 2 \frac{\Delta_k}{\Delta_h} \| \Delta_k^h u(t) \|_{L^2},
\]

from which and \( s \leq 1 \), we infer

\[
\left( \int_0^t \left| \sum_{L_0}^h \| S_{k' - 1}^h v(t') \|_{L^\infty} \| \partial_y u(t') \|_{B^\frac{1}{2}} \right| dt' \right) \leq \sum_{|k' - k| \leq 4} d_k 2^{-k' - \frac{3}{2}} \left( \int_0^t \left| \sum_{L_0}^h \| \Delta_k^h u(t) \|_{L^2} \| \partial_y u(t') \|_{B^\frac{1}{2}} \right| dt' \right) \leq \sum_{|k' - k| \leq 4} d_k 2^{-k' - \frac{3}{2}} \left( \sum_{|k' - k| \leq 4} d_k 2^{-k' - \frac{3}{2}} \right) \leq 2 \frac{\Delta_k}{\Delta_h} \| \Delta_k^h u(t) \|_{L^2},
\]

Consequently, by virtue of Definition 2.3 we obtain

\[
\int_0^t \left| \left( \Delta_k^h(T^h_v \partial_y u) \right)_L^2 \right| dt' \leq \sum_{|k' - k| \leq 4} \int_0^t \left| \sum_{L_0}^h \| S_{k' - 1}^h v(t') \|_{L^\infty} \| \partial_y u(t') \|_{B^\frac{1}{2}} \right| dt' \times \left( \sum_{|k' - k| \leq 4} d_k 2^{-k' - \frac{3}{2}} \right) \leq d_k 2^{-2ks} \| \Delta_k^h u(t) \|_{L^2}^2 .
\]

- Estimate of \( \int_0^t \left( \Delta_k^h(T^h_v \partial_y u) \right)_L^2 dt' \)

Notice that

\[
\int_0^t \left| \left( \Delta_k^h(T^h_v \partial_y u) \right)_L^2 \right| dt' \leq \sum_{|k' - k| \leq 4} \int_0^t \left| \sum_{L_0}^h \| S_{k' - 1}^h \partial_y u(t') \|_{L^\infty} \| \Delta_k^h v(t') \|_{L^2} \| \Delta_k^h u(t') \|_{L^2} \right| dt'.
\]
which together with (3.18) ensures that

\[
\int_0^t \left| \left( \Delta_k^h(T_{\partial_{y,v}}^h v) \psi \mid \Delta_k^h u\psi \right)_{L^2} \right| dt' \\
\lesssim \sum_{|k'-k| \leq 4} 2^{k'} \int_0^t \left\| \partial_y u\psi(t') \right\|_{B^{1/2}} \left\| \Delta_k^h u\psi(t') \right\|_{L^2} dt' \\
\lesssim \sum_{|k'-k| \leq 4} 2^{k'} \left( \int_0^t \left\| \Delta_k^h u\psi(t') \right\|_{L^2}^2 \left\| \partial_y u\psi(t') \right\|_{B^{1/2}} dt' \right)^{1/2} \\
\times \left( \int_0^t \left\| \Delta_k^h u\psi(t') \right\|_{L^2}^2 \left\| \partial_y u\psi(t') \right\|_{B^{1/2}} dt' \right)^{1/2}.
\]

Then thanks to Definition 2.3, we arrive at

\[
\int_0^t \left| \left( \Delta_k^h(T_{\partial_{y,v}}^h v) \psi \mid \Delta_k^h u\psi \right)_{L^2} \right| dt' \lesssim d_k^2 2^{-2k_s} \| u\psi \|_{L^2_{t,s(t)}(B^{1/2})}.
\]

- **Estimate of \( \int_0^t \left( \Delta_k^h(R^h(v, \partial_{y,v}) \psi \mid \Delta_k^h u\psi \right)_{L^2} dt' \)**

We get, by applying lemma 2.11 and (3.18), that

\[
\int_0^t \left| \left( \Delta_k^h(R^h(v, \partial_{y,v}) \psi \mid \Delta_k^h u\psi \right)_{L^2} \right| dt' \\
\lesssim 2^{\frac{k}{2}} \sum_{k' \geq k - 3} \int_0^t \left\| \Delta_k^h u\psi(t') \right\|_{L^2_{t,s(t)}} \left\| \Delta_k^h \partial_y u\psi(t') \right\|_{L^2} \left\| \Delta_k^h u\psi(t') \right\|_{L^2} dt' \\
\lesssim 2^{\frac{k}{2}} \sum_{k' \geq k - 3} 2^{k'} \int_0^t \left\| \Delta_k^h u\psi(t') \right\|_{L^2} \left\| \partial_y u\psi(t') \right\|_{B^{1/2}} \left\| \Delta_k^h u\psi(t') \right\|_{L^2} dt' \\
\lesssim 2^{\frac{k}{2}} \sum_{k' \geq k - 3} 2^{k'} \left( \int_0^t \left\| \Delta_k^h u\psi(t') \right\|_{L^2}^2 \left\| \partial_y u\psi(t') \right\|_{B^{1/2}} dt' \right)^{1/2} \\
\times \left( \int_0^t \left\| \Delta_k^h u\psi(t') \right\|_{L^2}^2 \left\| \partial_y u\psi(t') \right\|_{B^{1/2}} dt' \right)^{1/2},
\]

which together with Definition 2.3 and \( s > 0 \) ensures that

\[
\int_0^t \left| \left( \Delta_k^h(R^h(v, \partial_{y,v}) \psi \mid \Delta_k^h u\psi \right)_{L^2} \right| dt' \lesssim d_k^2 2^{-2k_s} \| u\psi \|_{L^2_{t,s(t)}(B^{1/2})} \left( \sum_{k' \geq k - 3} d_k^2 2^{(k-k')s} \right) \\
\leq d_k^2 2^{-2k_s} \| u\psi \|_{L^2_{t,s(t)}(B^{1/2})}.
\]

By summing up the above estimates, we achieve (3.11). \( \square \)

**Proof of Lemma 3.3.** We first get, by applying Bony’s decomposition (2.3) for the horizontal variable to \( v\partial_y v \), that

\[
v\partial_y v = T_v^h \partial_y v + T_{\partial_y v}^h v + R^h(v, \partial_y v).
\]

Let us handle the following three terms:

- **Estimate of \( \int_0^t \left( \Delta_k^h(T_{\partial_y v}^h v) \psi \mid \Delta_k^h u\psi \right)_{L^2} dt' \)**
Due to $\partial_y v = -\partial_x u$, one has
\[
\varepsilon^2 \int_0^t |(\Delta^h_k(T^h_y \partial_y v)\psi | \Delta^h_k v\psi)_L^2 | dt' \\
\lesssim \varepsilon^2 \sum_{|k' - k| \leq 4} \int_0^t ||S^h_{k'-1} v\psi(t')||_{L^\infty \varepsilon} ||\Delta^h_k \partial_y v\psi(t)||_{L^2} ||\Delta^h_k v\psi(t')||_{L^2} dt'
\lesssim \varepsilon \sum_{|k' - k| \leq 4} 2^{-k} \int_0^t ||S^h_{k'-1} v\psi(t')||_{L^\infty} ||\partial_x u\psi(t')||_{B^5} ||\Delta^h_k v\psi(t')||_{L^2} dt'
\lesssim \varepsilon \sum_{|k' - k| \leq 4} 2^{-k} \left( \int_0^t ||S^h_{k'-1} v\psi(t')||^2_{L^\infty} ||\partial_x u\psi(t')||_{B^5} dt' \right)^{\frac{1}{2}}
\times \left( \int_0^t ||\Delta^h_k v\psi(t')||^2_{L^2} ||\partial_x u\psi(t')||_{B^5} dt' \right)^{\frac{1}{2}}.
\]
Yet we get, by a similar derivation of (3.19), that
\[
\left( \int_0^t ||S^h_{k'-1} v\psi(t')||^2_{L^\infty} ||\partial_x u\psi(t')||_{B^5} dt' \right)^{\frac{1}{2}} \lesssim d_k 2^{-k} ||u\psi||_{L^2_{t;\mathfrak{b}(t)}(B^1)}.
\]
Hence we deduce from Definition 2.3 that
\[
\varepsilon^2 \int_0^t |(\Delta^h_k(T^h_y \partial_y v)\psi | \Delta^h_k v\psi)_L^2 | dt' \leq d_k 2^{-k} ||u\psi||_{L^2_{t;\mathfrak{b}(t)}(B^1)} ||v\psi||_{L^2_{t;\mathfrak{b}(t)}(B^1)}.
\]
- Estimate of $\int_0^t (\Delta^h_k(T^h_y \partial_y v)\psi | \Delta^h_k v\psi)_L^2 | dt'$
  Notice that
  \[
  \int_0^t |(\Delta^h_k(T^h_y \partial_y v)\psi | \Delta^h_k v\psi)_L^2 | dt'
  \lesssim \sum_{|k' - k| \leq 4} \int_0^t ||S^h_{k'-1} \partial_x u\psi(t')||_{L^\infty} ||\Delta^h_k v\psi(t)||_{L^2} ||\Delta^h_k v\psi(t')||_{L^2} dt',
  \]
  which together with (3.15) ensures that
  \[
  \int_0^t |(\Delta^h_k(T^h_y \partial_y v)\psi | \Delta^h_k v\psi)_L^2 | dt'
  \lesssim \sum_{|k' - k| \leq 4} 2^{k'} \int_0^t ||\partial_y u\psi(t')||_{B^5} ||\Delta^h_k v\psi(t)||_{L^2} ||\Delta^h_k v\psi(t')||_{L^2} dt'
  \lesssim \sum_{|k' - k| \leq 4} 2^{k'} \left( \int_0^t ||\Delta^h_k v\psi(t)||^2_{L^2} ||\partial_y u\psi(t')||_{B^5} dt' \right)^{\frac{1}{2}}
  \times \left( \int_0^t ||\Delta^h_k v\psi(t')||^2_{L^2} ||\partial_y u\psi(t')||_{B^5} dt' \right)^{\frac{1}{2}}
  \]
  Then thanks to Definition 2.3 we arrive at
  \[
  \int_0^t |(\Delta^h_k(T^h_y \partial_y v)\psi | \Delta^h_k v\psi)_L^2 | dt' \lesssim d_k 2^{-k} ||v\psi||_{L^2_{t;\mathfrak{b}(t)}(B^1)}.
  \]
- Estimate of $\int_0^t (\Delta^h_k(R^h(v, \partial_y v))\psi | \Delta^h_k v\psi)_L^2 | dt'$

Due to $\partial_x u + \partial_y v = 0$, we get, by applying lemma 2.1 and (3.18), that
\[
\int_0^t \left| (\Delta_h^b (R^h(v, \partial_y v))_\Psi \mid \Delta_h^b v_\Psi )_{L^2} \right| dt' \\
\leq 2^{\frac{b}{2}} \sum_{k' \geq k-3} \left( \int_0^t \| \Delta_h^b v_\Psi (t') \|_{L^2} \| \Delta_h^b \partial_x u_\Psi (t') \|_{L^2} \| \Delta_h^b v_\Psi (t') \|_{L^2} dt' \right)
\leq 2^{\frac{b}{2}} \sum_{k' \geq k-3} \left( \int_0^t \| \Delta_h^b v_\Psi (t') \|_{L^2} \| \partial_y u_\Psi (t') \|_{L^2} \| \Delta_h^b v_\Psi (t') \|_{L^2} dt' \right)
\leq 2^{\frac{b}{2}} \sum_{k' \geq k-3} \left( \int_0^t \| \Delta_h^b v_\Psi (t') \|_{L^2} \| \partial_y u_\Psi (t') \|_{L^2} dt' \right)^{\frac{1}{2}}
\times \left( \int_0^t \| \Delta_h^b v_\Psi (t') \|_{L^2} \| \partial_y u_\Psi (t') \|_{L^2} dt' \right)^{\frac{1}{2}},
\]
which together with Definition 2.3 and $s > 0$ ensures that
\[
\int_0^t \left| (\Delta_h^b (R^h(v, \partial_y v))_\Psi \mid \Delta_h^b u_\Psi )_{L^2} \right| dt' \leq 2^{\frac{b}{2}} 2^{-k} \| v_\Psi \|_{L^2}^{2} \| p_\Psi \|_{L^2}^{2} \| u \|_{L^2}^{2}. \]

By summing up the above estimates, we obtain (3.12). This concludes the proof of Lemma 3.3.

\[ \square \]

4. Global well-posedness of the system (1.5)

In this section, we study the global well-posedness of the hydrostatic approximate equations (1.5) with small analytic data.

Due to the compatibility condition $\partial_x \int_0^1 u_0 dy = 0$, we deduce from $\partial_x u + \partial_y v = 0$ that
\[
\partial_x \int_0^1 u(t, x, y) dy = 0
\]
so that by integrating the equation $\partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p = 0$ for $y \in [0,1]$ and using the fact that $\partial_y p = 0$, we obtain
\[
\partial_x^2 p = \partial_x \left( \partial_y u(t, x, 1) - \partial_y u(t, x, 0) - \partial_x \int_0^1 u^2(t, x, y) dy \right).
\]

We define
\[
u_\Phi(t, x, y) \text{ def } F_{\xi \rightarrow x}^{-1} \left( e^{\Phi(t, \xi)} \mathbf{\hat{u}}(t, \xi, y) \right) \quad \text{with} \quad \Phi(t, \xi) \text{ def } (a - \lambda \theta(t)) |\xi|,
\]
where the quantity $\theta(t)$ describes the evolution of the analytic band of $u$, which is determined by
\[
\dot{\theta}(t) = \| \partial_y u_\Phi(t) \|_{H^\frac{1}{2}} \quad \text{with} \quad \theta|_{t=0} = 0.
\]

Proof of Theorem 4.3. In view of (1.5) and (4.3), we observe that $u_\Phi$ verifies
\[
\partial_t u_\Phi + \lambda \dot{\theta}(t) |D_x| u_\Phi + (u \partial_x u)_\Phi + (v \partial_y u)_\Phi - \partial_y^2 u_\Phi + \partial_x p_\Phi = 0,
\]
where $|D_x|$ denotes the Fourier multiplier with symbol $|\xi|$.
By applying $\Delta_h^k$ to (4.5) and taking $L^2$ inner product of the resulting equation with $\Delta_h^k u_\Phi$, we find

$$
\frac{1}{2} \frac{d}{dt} \| \Delta_h^k u_\Phi(t) \|^2_{L^2} + \lambda \theta(\| D_x \Delta_h^k u_\Phi \|_{L^2}^2) \| \Delta_h^k \partial_x u_\Phi \|_{L^2}^2 + \| \Delta_h^k \partial_y u_\Phi \|_{L^2}^2
= - (\Delta_h^k (u \partial_x u_\Phi) \Delta_h^k u_\Phi)_{L^2} - (\Delta_h^k (v \partial_y u_\Phi) \Delta_h^k u_\Phi)_{L^2} - (\Delta_h^k \partial_x p_\Phi \Delta_h^k u_\Phi)_{L^2}.
$$

(4.6)

Thanks to (1.6) and $\partial_x u + \partial_y v = 0$, we get, by using integration by parts, that

$$
(\Delta_h^k \partial_x p_\Phi \Delta_h^k u_\Phi)_{L^2} = -(\Delta_h^k p_\Phi \Delta_h^k \partial_x u_\Phi)_{L^2}
= (\Delta_h^k p_\Phi \Delta_h^k \partial_y v_\Phi)_{L^2} = -(\Delta_h^k \partial_y p_\Phi \Delta_h^k v_\Phi)_{L^2} = 0.
$$

Then by using Lemma 2.1 (3.6) and by multiplying (4.6) by $e^{2rt}$ and then integrating the resulting inequality over $[0, t]$, we achieve

$$
\frac{1}{2} \| e^{2rt} \Delta_h^k u_\Phi \|^2_{L^2(\mathcal{L}^2)} + \lambda \int_0^t \theta(t') \| e^{2rt} \Delta_h^k u_\Phi(t') \|^2_{L^2} dt' + \frac{1}{2} \| e^{2rt} \Delta_h^k \partial_y u_\Phi \|^2_{L^2(\mathcal{L}^2)}
\leq \| e^{2rt} \Delta_h^k u_0 \|^2_{L^2} + \int_0^t \| e^{2rt} \Delta_h^k (u \partial_x u_\Phi) e^{2rt} \Delta_h^k u_\Phi \|_{L^2} dt'
+ \int_0^t \| e^{2rt} \Delta_h^k (v \partial_y u_\Phi) e^{2rt} \Delta_h^k u_\Phi \|_{L^2} dt'.
$$

(4.7)

In what follows, we shall always assume that $t < T^*$ with $T^*$ being determined by

$$
T^* \overset{\text{def}}{=} \sup \{ \ t > 0, \ \theta(t) < a/\lambda \}.
$$

So that by virtue of (4.3), for any $t \leq T^*$, there holds the following convex inequality

$$
\Phi(t, \xi) \leq \Phi(t, \xi - \eta) + \Phi(t, \eta) \quad \text{for} \quad \forall \ \xi, \eta \in \mathbb{R}.
$$

(4.9)

Then we deduce from Lemma 3.3 that for any $s \in [0, 1]$ and $t \leq T^*$

$$
\int_0^t \| e^{2rs} \Delta_h^k (u \partial_x u_\Phi) e^{2rs} \Delta_h^k u_\Phi \|_{L^2} dt' \lesssim d_k^2 2^{-2ks} \| e^{2rs} u_\Phi \|^2_{L^2_{t, \theta(t)}(B^s + \frac{1}{4})}.
$$

Whereas it follows from Lemma 3.4 that for any $s \in [0, 1]$ and $t \leq T^*$

$$
\int_0^t \| e^{2rs} \Delta_h^k (v \partial_y u_\Phi) e^{2rs} \Delta_h^k u_\Phi \|_{L^2} dt' \lesssim d_k^2 2^{-2ks} \| e^{2rs} u_\Phi \|^2_{L^2_{t, \theta(t)}(B^s + \frac{1}{4})}.
$$

Inserting the above estimates into (4.7) gives rise to

$$
\frac{1}{2} \| e^{2rt} \Delta_h^k u_\Phi \|^2_{L^2(\mathcal{L}^2)} + \lambda \theta \int_0^t \theta(t') \| e^{2rt} \Delta_h^k u_\Phi(t') \|^2_{L^2} dt' + \frac{1}{2} \| e^{2rt} \Delta_h^k \partial_y u_\Phi \|^2_{L^2(\mathcal{L}^2)}
\leq \| e^{2rt} \Delta_h^k u_0 \|^2_{L^2} + C d_k^2 2^{-2ks} \| e^{2rs} u_\Phi \|^2_{L^2_{t, \theta(t)}(B^s + \frac{1}{4})}.
$$

Then for any $s \in [0, 1]$, by multiplying the above inequality by $2^{2ks}$ and then taking square root of the resulting inequality, and finally by summing up the resulting ones over $\mathbb{Z}$, we obtain

$$
\| e^{2rs} u_\Phi \|_{L^\infty_t(B^s)}^2 + \sqrt{\lambda} \| e^{2rs} u_\Phi \|_{L^2_{t, \theta(t)}(B^s + \frac{1}{4})}^2 + \| e^{2rs} \partial_y u_\Phi \|_{L^2_t(B^s)}^2
\leq \| e^{2rs} u_0 \|_{B^s}^2 + C \| e^{2rs} u_\Phi \|_{L^2_{t, \theta(t)}(B^s + \frac{1}{4})}^2.
$$
Taking $\lambda = C^2$ in the above inequality leads to
\begin{equation}
\|e^{\bar{R}t}u_\Phi\|_{\bar{L}^\infty(B^*)} + \|e^{\bar{R}t}\partial_y u_\Phi\|_{\bar{L}^2(B^*)} \leq \|e^{a|Dz|}u_0\|_{B^*} \quad \text{for } s \in [0, 1] \text{ and } t \leq T^*.
\end{equation}
In particular, we deduce from (4.10) for $s = \frac{1}{2}$ and (4.4) that
\[
\begin{align*}
\theta(t) &= \int_0^t \|\partial_y u_\Phi(t')\|_{B^{\frac{3}{2}}} dt' \\
&\leq \left(\int_0^t e^{-2\bar{R}t'} dt'\right)^{\frac{1}{2}} \left(\int_0^t \|e^{\bar{R}t'}\partial_y u_\Phi(t')\|_{B^{\frac{3}{2}}}^2 dt'\right)^{\frac{1}{2}} \\
&\leq C \|e^{\bar{R}t'}\partial_y u_\Phi\|_{\bar{L}^2(B^*)} \leq C \|e^{a|Dz|}u_0\|_{B^{\frac{3}{2}}}.
\end{align*}
\]
Then if we take $c_1$ in (1.9) to be so small that
\begin{equation}
C \|e^{a|Dz|}u_0\|_{B^{\frac{3}{2}}} \leq \frac{a}{2\lambda},
\end{equation}
we deduce by a continuous argument that $T^*$ determined by (4.8) equals $+\infty$ and (1.10) holds. Then Theorem 1.2 is proved provided that we present the proof of Proposition 4.1, which requires the following propositions.

**Proposition 4.1.** Under the assumption of (1.11), for any $s > 0$, there exists a positive constant $C$ so that for $\lambda = C^2 (1 + \|e^{a|Dz|}u_0\|_{B^{\frac{3}{2}}})$, there holds
\begin{equation}
\|e^{\bar{R}t}u_\Phi\|_{\bar{L}^\infty(R_+, B^*)} + \|e^{\bar{R}t}\partial_y u_\Phi\|_{\bar{L}^2(R_+, B^*)} \leq C \|e^{a|Dz|}u_0\|_{B^*}.
\end{equation}

**Proposition 4.2.** Under the assumption of (1.11), for any $s > 0$, there exists a positive constant $C$ so that for $\lambda = C^2 (1 + \|e^{a|Dz|}u_0\|_{B^{\frac{3}{2}}})$, there holds
\begin{equation}
\|e^{\bar{R}t}\partial_y u_\Phi\|_{\bar{L}^\infty(B^*)} + \|e^{\bar{R}t}\partial_y^2 u_\Phi\|_{\bar{L}^2(B^*)} \leq C \left(\|e^{a|Dz|}\partial_y u_0\|_{B^s} + \|e^{a|Dz|}u_0\|_{B^{s+1}}\right).
\end{equation}

We admit the above propositions for the time being and continue our proof of Theorem 1.2.

As a matter of fact, it remains to present the estimate of $\|e^{\bar{R}t}(\partial_t u_\Phi)\|_{\bar{L}^2(R_+, B^{\frac{3}{2}})}$. Indeed, by applying $\Delta_k^h$ to (1.5) and then taking $L^2$ inner product of resulting equation with $e^{2\bar{R}t}\Delta_k^h(\partial_t u_\Phi)$, we obtain
\[
\|e^{\bar{R}t}\Delta_k^h(\partial_t u_\Phi)\|_{L^2}^2 = e^{2\bar{R}t}(\Delta_k^h\partial_y^2 u_\Phi|\Delta_k^h(\partial_t u_\Phi))_{L^2} \\
- e^{2\bar{R}t}(\Delta_k^h(\partial_\nu u_\Phi)\Delta_k^h(\partial_t u_\Phi))_{L^2} - e^{2\bar{R}t}(\Delta_k^h(v\partial_y u_\Phi)\Delta_k^h(\partial_t u_\Phi))_{L^2},
\]
from which, we deduce that
\[
\|e^{\bar{R}t}\Delta_k^h(\partial_t u_\Phi)\|_{L^2} \leq C \left(\|e^{\bar{R}t}\Delta_k^h\partial_y^2 u_\Phi\|_{L^2} + \|e^{\bar{R}t}(u\partial_x u_\Phi)\|_{L^2} + \|e^{\bar{R}t}(v\partial_y u_\Phi)\|_{L^2}\right).
\]
This gives rise to
\begin{equation}
\|e^{\bar{R}t}(\partial_t u_\Phi)\|_{L^2(B^{\frac{3}{2}})} \leq C \left(\|e^{\bar{R}t}\partial_y^2 u_\Phi\|_{L^2} + \|e^{\bar{R}t}(u\partial_x u_\Phi)\|_{L^2} + \|e^{\bar{R}t}(v\partial_y u_\Phi)\|_{L^2}\right).
\end{equation}
Yet it follows from the law of product in anisotropic Besov space and Poincare inequality that
\[
\|e^{Rt'}(u \partial_x u)\|_{L^2_t(B^{s+\frac{1}{2}})} \lesssim \|u\|_{L^\infty_t(B^s)} \|e^{Rt'} \partial_y u\|_{L^2_t(B^{s+\frac{1}{2}})},
\]
\[
\|e^{Rt'}(v \partial_y u)\|_{L^2_t(B^{s+\frac{1}{2}})} \lesssim \|u\|_{L^\infty_t(B^s)} \|e^{Rt'} \partial_y u\|_{L^2_t(B^{s+\frac{1}{2}})} + \|u\|_{L^\infty_t(B^{s+\frac{1}{2}})} \|e^{Rt'} \partial_y u\|_{L^2_t(B^{s+\frac{1}{2}})}.
\]
Inserting the above estimates into (4.14) and then using (1.9), (1.10) and Proposition 4.1, we achieve
\[
\|e^{Rt'}(u \partial_x u)\|_{L^2_t(B^{s+\frac{1}{2}})} \lesssim \|e^{a|D_x|} \partial_y u\|_{L^2} + \|e^{a|D_x|} u\|_{L^2}.
\]
This completes the proof of Theorem 1.2. \qed

Now let us present the proof of the above two propositions.

**Proof of Proposition 4.4.** We first deduce from Remark 3.1 that for any \(s > 0\)
\[
\int_0^t |(\Delta_k^h (u \partial_x u)_{\phi} | \Delta_k^h u_{\phi})|_{L^2} | dt' \lesssim d_k^2 2^{-2ks} \|u\|_{L^2_t(B^{s+\frac{1}{2}})}^2.
\]
While it follows from the proof of Lemma 3.2 that
\[
\int_0^t |(\Delta_k^h (T_v \partial_y u + R^h(v, \partial_y u))_{\phi} | \Delta_k^h u_{\phi})|_{L^2} | dt' \lesssim d_k^2 2^{-2ks} \|u\|_{L^2_t(B^{s+\frac{1}{2}})}^2.
\]
In view of (3.18), we have
\[
\|\Delta_k^h u_{\phi}(t)\|_{L^\infty} \lesssim d_k(t) \frac{1}{2} \|u_{\phi}(t)\|_{L^2} \|\partial_y u_{\phi}(t)\|_{L^2}^2,
\]
so that there holds
\[
\int_0^t |(\Delta_k^h (T_v \partial_y u)_{\phi} | \Delta_k^h u_{\phi})|_{L^2} | dt' \lesssim \sum_{|k'| - k \leq 4} \int_0^t \|S_{k'-1}^h u_{\phi}(t')\|_{L^\infty} \|\Delta_{k'}^h \partial_y u_{\phi}(t)\|_{L^2} \|\Delta_k^h u_{\phi}(t')\|_{L^2} dt'
\]
\[
\lesssim \sum_{|k' - k| \leq 4} 2^{k'} \|u_{\phi}\|_{L^\infty_t(B^{s+\frac{1}{2}})} \|\Delta_{k'}^h \partial_y u_{\phi}\|_{L^2_t(B^s)} \left( \int_0^t \|\partial_y u_{\phi}(t')\|_{L^2} \|\Delta_k^h u_{\phi}(t')\|_{L^2} dt' \right)^{\frac{1}{2}}
\]
\[
\lesssim d_k^2 2^{-2ks} \|u_{\phi}\|_{L^\infty_t(B^{s+\frac{1}{2}})} \|\partial_y u_{\phi}\|_{L^2_t(B^s)} \|u_{\phi}\|_{L^2_t(B^{s+\frac{1}{2}})}.
\]
As a result, it comes out
\[
\int_0^t |(\Delta_k^h (v \partial_y u)_{\phi} | \Delta_k^h u_{\phi})|_{L^2} | dt' \lesssim d_k^2 2^{-2ks} \|u_{\phi}\|_{L^2_t(B^{s+\frac{1}{2}})} \times \left( \|u_{\phi}\|_{L^2_t(B^{s+\frac{1}{2}})} \|\partial_y u_{\phi}\|_{L^2_t(B^s)} + \|u_{\phi}\|_{L^\infty_t(B^{s+\frac{1}{2}})} \|\partial_y u_{\phi}\|_{L^2_t(B^s)} \right).
\]
By virtue of (4.15) and (4.17), we deduce from (4.7) that
\[
\frac{1}{2} \left\| e^{\tilde{R}^r} \Delta^h_k u\Phi \right\|^2_{L^\infty_t(L^2)} + \lambda 2k \int_0^t \dot{\theta}(t') \left\| e^{\tilde{R}^r} \Delta^h_k u\Phi(t') \right\|^2_{L^2} dt' + \frac{1}{2} \left\| e^{\tilde{R}^r} \Delta^h_k \partial_y u\Phi \right\|^2_{L^2_t(L^2)} \\
\leq \frac{1}{2} \left\| e^{a|D_x|} \Delta^h_k u_0 \right\|^2_{L^2} + C d_k^2 2^{-2k} \left\| e^{\tilde{R}^r} u\Phi \right\|^2_{L^2_t,\theta(t)}(B^{s+\frac{1}{2}}) \\
\times \left( \left\| e^{\tilde{R}^r} u\Phi \right\|^2_{L^2_t,\theta(t)}(B^{s+\frac{1}{2}}) + \left\| u\Phi \right\|^2_{L^2_t}(B^t) \right) \left\| e^{\tilde{R}^r} \partial_y u\Phi \right\|^2_{L^2_t(B^s)}.
\]
from which, we infer
\[
\left\| e^{\tilde{R}^r} u\Phi \right\|^2_{L^\infty_t(L^2)} + \sqrt{A} \left\| e^{\tilde{R}^r} u\Phi \right\|^2_{L^2_t,\theta(t)}(B^{s+\frac{1}{2}}) + \left\| e^{\tilde{R}^r} \partial_y u\Phi \right\|^2_{L^2_t(B^s)} \leq C \left( \left\| e^{a|D_x|} u_0 \right\|_{B^s} + \left\| e^{\tilde{R}^r} u\Phi \right\|^2_{L^2_t}(B^{s+\frac{1}{2}}) \right) \left( \left\| e^{\tilde{R}^r} u\Phi \right\|^2_{L^2_t}(B^t) \right) \left\| e^{\tilde{R}^r} \partial_y u\Phi \right\|^2_{L^2_t(B^s)}.
\]
Applying Young’s inequality yields
\[
C \left\| u\Phi \right\|^2_{L^\infty_t(B^s)} \left\| e^{\tilde{R}^r} \partial_y u\Phi \right\|^2_{L^2_t(B^{s+\frac{1}{2}})} + \frac{1}{2} \left\| e^{\tilde{R}^r} u\Phi \right\|^2_{L^2_t(B^s)} \leq C \left( \left\| e^{\tilde{R}^r} u\Phi \right\|^2_{L^2_t}(B^t) \right) \left\| e^{\tilde{R}^r} \partial_y u\Phi \right\|^2_{L^2_t(B^s)}.
\]
Therefore if we take
\[
\lambda \geq C^2 \left( 1 + \left\| u\Phi \right\|_{L^\infty_t(B^s)} \right),
\]
we obtain
\[
\left\| e^{\tilde{R}^r} u\Phi \right\|^2_{L^\infty_t(B^s)} + \left\| e^{\tilde{R}^r} \partial_y u\Phi \right\|^2_{L^2_t(B^s)} \leq C \left\| e^{a|D_x|} u_0 \right\|_{B^s},
\]
which in particular implies that under the condition (4.18), there holds
\[
\left\| u\Phi \right\|^2_{L^\infty_t(B^s)} \leq C \left\| e^{a|D_x|} u_0 \right\|_{B^s}.
\]
Then by taking \( \lambda = C^2 \left( 1 + \left\| e^{a|D_x|} u_0 \right\|_{B^s} \right) \), (4.18) holds. Therefore under the condition (1.11), both (4.11) and (4.18) hold, and thus (3.19) holds for any \( t > 0 \), which leads to (4.12). This completes the proof of the proposition. \( \square \)

**Proof of Proposition 4.2.** Due to \( \partial_x u + \partial_y v = 0 \), we get, by applying \( \partial_y \) to (1.5), that
\[
\partial_t \partial_y u + u \partial_x \partial_y u + v \partial_y^2 u - \partial_y^3 u + \partial_x p = 0,
\]
from which, we get, by using a similar derivation of (4.7), that
\[
\frac{1}{2} \left\| e^{\tilde{R}^r} \Delta^h_k \partial_y u\Phi \right\|^2_{L^\infty_t(L^2)} + \lambda 2k \int_0^t \dot{\theta}(t') \left\| e^{\tilde{R}^r} \Delta^h_k \partial_y u\Phi(t') \right\|^2_{L^2} dt' + \frac{1}{2} \left\| e^{\tilde{R}^r} \Delta^h_k \partial_y^2 u\Phi \right\|^2_{L^2_t(L^2)} \\
\leq \frac{1}{2} \left\| e^{a|D_x|} \Delta^h_k \partial_y u_0 \right\|^2_{L^2} + \int_0^t \left( \left| e^{\tilde{R}^r} \Delta^h_k (u \partial_x \partial_y u) \Phi \right| + \left| e^{\tilde{R}^r} \Delta^h_k \partial_y u\Phi \right| \right)_L dt' \\
+ \int_0^t \left( \left| e^{\tilde{R}^r} \Delta^h_k (v \partial_y^2 u) \Phi \right| + \left| e^{\tilde{R}^r} \Delta^h_k \partial_y u\Phi \right| \right)_L dt'.
\]
It follows from the proof of Lemma 3.1 that for any \( s > 0 \)
\[
\int_0^t \left( \left| \Delta^h_k (T^h u \partial_x \partial_y u + R^h (u, \partial_x \partial_y u)) \Phi \right| \Delta^h_k \partial_y u\Phi \right)_L dt' \lesssim d_k 2^{-2k} \left\| \partial_y u\Phi \right\|^2_{L^2_t(\theta(t))(B^{s+\frac{1}{2}})}.
\]
While we deduce from Lemma 2.4 and Definition 2.3 that
\[
\int_0^t \left| \left( \Delta_k^h T^h_d, \partial_y u \right) \right|_{L^2} \, dt' \\
\leq \sum_{|k' - k| \leq 4} \int_0^t \left\| S_{k', k}^h \partial_y u \right\|_{L^2(L^2)} \left\| \Delta_k^h \partial_y u \right\|_{L^2(L^2)} \left\| \Delta_k^h \partial_y u \right\|_{L^2(L^2)} dt'
\]
\[
\leq \sum_{|k' - k| \leq 4} 2^k \int_0^t \left\| \partial_y u \right\|_{L^2(B^n_{1/2})} \left\| \Delta_k^h \partial_y u \right\|_{L^2} \left\| \Delta_k^h \partial_y u \right\|_{L^2} dt'
\]
\[
\leq \sum_{|k' - k| \leq 4} 2^k \left( \int_0^t \left\| \partial_y u \right\|_{L^2(B^n_{1/2})} \left\| \Delta_k^h \partial_y u \right\|_{L^2} dt' \right)^{1/2}
\]
\[
\times \left( \int_0^t \left\| \partial_y u \right\|_{L^2(B^n_{1/2})} \left\| \Delta_k^h \partial_y u \right\|_{L^2} dt' \right)^{1/2}
\]
\[
\leq \frac{\Delta_k^h 2^{-ks} \left\| \partial_y u \right\|_{L^2(t, \theta(t))}^{1/2}}{2^{s + 1/2}}.
\]
As a result, it comes out that for any \( s > 0 \),
\[
\int_0^t \left| \left( \Delta_k^h (u \partial_x \partial_y u) \right) \right|_{L^2} \, dt' \leq \frac{\Delta_k^h 2^{-ks} \left\| \partial_y u \right\|_{L^2(t, \theta(t))}^{1/2}}{2^{s + 1/2}}.
\]
On the other hand, we deduce from Lemma 2.4 and (3.48) that for any \( s > 0 \)
\[
\int_0^t \left| \left( \Delta_k^h (R^h(v, \partial_y^2 u)) \right) \right|_{L^2} \, dt' \leq \frac{\Delta_k^h 2^{-ks} \left\| \partial_y u \right\|_{L^2(t, \theta(t))}^{1/2}}{2^{s + 1/2}}.
\]
And the proof of (4.15) ensures that
\[
\int_0^t \left| \left( \Delta_k^h (T^h_d \partial_y^2 u) \right) \right|_{L^2} \, dt' \leq \frac{\Delta_k^h 2^{-ks} \left\| \partial_y u \right\|_{L^2(t, \theta(t))}^{1/2}}{2^{s + 1/2}}.
\]
Finally, by using integration by parts, we have
\[
\int_0^t \left| \left( \Delta_k^h (T^h_d \partial_y u) \right) \right|_{L^2} \, dt' \leq \int_0^t \left| \left( \Delta_k^h (T^h_d \partial_y u) \right) \right|_{L^2} \, dt + \int_0^t \left| \left( \Delta_k^h (T^h_d \partial_y u) \right) \right|_{L^2} \, dt.
\]
Due to $\partial_x u + \partial_y v = 0$, we deduce from a similar derivation of (4.21) that

$$
\int_0^t \left| \langle \Delta_h^k (T_{\partial u}^h \partial_y v) \rangle \Phi, \Delta_h^k \partial_y u \Phi \rangle \right|_{L^2} \, dt \\
\lesssim \sum_{|k' - k| \leq 4} \int_0^t \| S_h^{k' - 1} \partial_y u \Phi(t') \|_{L^{\infty} \left( L^2 \right)} \| \Delta_h^{k'} \partial_y u \Phi(t') \|_{L^2 \left( L^{\infty} \right)} \| \Delta_h^{k'} \partial_y u \Phi(t') \|_{L^2} \, dt' \\
\lesssim \sum_{|k' - k| \leq 4} 2^{k'} \int_0^t \| \partial_y u \Phi(t') \|_{L^2} \| \Delta_h^{k'} \partial_y u \Phi(t') \|_{L^2} \| \Delta_h^{k'} \partial_y u \Phi(t') \|_{L^2} \, dt' \\
\lesssim d_k^2 2^{-2(k s)} \| \partial_y u \Phi \|_{L^2 \left( B^s \right)}^2 \| u \Phi \|_{L^\infty \left( B^{s+1} \right)} \| \partial_y^2 u \Phi \|_{L^2 \left( B^s \right)}.
$$

While we observe that

$$
\int_0^t \left| \langle \Delta_h^k (T_{\partial u}^h v) \rangle \Phi, \Delta_h^k \partial_y u \Phi \rangle \right|_{L^2} \, dt \\
\lesssim \sum_{|k' - k| \leq 4} \int_0^t \| S_h^{k' - 1} \partial_y u \Phi(t') \|_{L^{\infty} \left( L^2 \right)} \| \Delta_h^{k'} \partial_y u \Phi(t') \|_{L^2 \left( L^{\infty} \right)} \| \Delta_h^{k'} \partial_y u \Phi(t') \|_{L^2} \, dt' \\
\lesssim \sum_{|k' - k| \leq 4} 2^{k'} \int_0^t \| \partial_y u \Phi(t') \|_{L^2} \| \Delta_h^{k'} \partial_y u \Phi(t') \|_{L^2} \| \Delta_h^{k'} \partial_y u \Phi(t') \|_{L^2} \, dt' \\
\lesssim d_k^2 2^{-2(k s)} \| \partial_y u \Phi \|_{L^2 \left( B^s \right)}^2 \| u \Phi \|_{L^\infty \left( B^{s+1} \right)} \| \partial_y^2 u \Phi \|_{L^2 \left( B^s \right)}.
$$

This gives rise to

$$
\int_0^t \left| \langle \Delta_h^k \left( T_{\partial u}^h v \right) \rangle \Phi, \Delta_h^k \partial_y u \Phi \rangle \right|_{L^2} \, dt' \lesssim d_k^2 2^{-2(k s)} \left( \| \partial_y u \Phi \|_{L^2 \left( B^{s+\frac{1}{2}} \right)}^2 + \| \partial_y u \Phi \|_{L^2 \left( B^{s+\frac{1}{2}} \right)} \| u \Phi \|_{L^\infty \left( B^{s+1} \right)} \| \partial_y^2 u \Phi \|_{L^2 \left( B^{s} \right)} \right).
$$

By summarizing the above estimates, we obtain

$$
\int_0^t \left| \langle \Delta_h^k (v \partial_y^2 u) \rangle \Phi, \Delta_h^k \partial_y u \Phi \rangle \right|_{L^2} \, dt' \\
\lesssim d_k^2 2^{-2(k s)} \left( \| u \Phi \|_{L^\infty \left( B^{s+\frac{1}{2}} \right)}^2 \| \partial_y u \Phi \|_{L^2 \left( B^{s+\frac{1}{2}} \right)} \| \partial_y^2 u \Phi \|_{L^2 \left( B^{s} \right)} \\
+ \| \partial_y u \Phi \|_{L^2 \left( B^{s+\frac{1}{2}} \right)} \| u \Phi \|_{L^\infty \left( B^{s+1} \right)} \| \partial_y^2 u \Phi \|_{L^2 \left( B^{s} \right)} \right).
$$

By inserting (4.22) and (4.23) into (4.20) and then repeating the last step of the proof of Proposition 1.1, we obtain

$$
\| e^{\partial'_x} \partial_y u \Phi \|_{L^2 \left( B^s \right)} + \sqrt{\lambda} \| e^{\partial'_x} \partial_y u \Phi \|_{L^2 \left( t, \theta \right) \left( B^{s+\frac{1}{2}} \right)} + \| e^{\partial'_x} \partial_y^2 u \Phi \|_{H^2 \left( B^s \right)} \\
\leq \| e^{a |D_x|} \partial_y u_0 \|_{B^s} + C \left( \| e^{\partial'_x} u \Phi \|_{L^2 \left( t, \theta \right) \left( B^{s+\frac{1}{2}} \right)} + \left( \| u \Phi \|_{L^\infty \left( B^{s+\frac{1}{2}} \right)} + \| e^{\partial'_x} \partial_y u \Phi \|_{L^2 \left( B^{s+\frac{1}{2}} \right)} \right) \right) \right) \| e^{\partial'_x} \partial_y^2 u \Phi \|_{L^2 \left( B^s \right)}.
$$

where $\| e^{a |D_x|} \partial_y u_0 \|_{B^s}$ is a constant.
Applying Young's inequality yields
\[
\|e^{\alpha t} \partial_y u_\Phi\|_{L^\infty(B^t)} + \sqrt{\lambda} \|e^{\alpha t} \partial_y u_\Phi\|_{L^2_{t,\delta(t)}}(B^{t+rac{1}{2}}) + \|e^{\alpha t} \partial_y^2 u_\Phi\|_{L^2_{t,\delta(t)}}(B^{t+rac{1}{2}})
\]
\[
\leq \|e^{\alpha |D_x|} \partial_y u_0\|_{B^s} + C \left( (1 + \|u_\Phi\|_{L^\infty(B^t)}^2) \|e^{\alpha t} u_\Phi\|_{L^2_{t,\delta(t)}}(B^{t+rac{1}{2}}) \right)
\]
\[
+ \|\partial_y u_\Phi\|_{L^2_{t,\delta(t)}}(B^{t+rac{1}{2}}) \|e^{\alpha t} u_\Phi\|_{L^\infty(B^{s+1})} + \frac{1}{2} \|e^{\alpha t} \partial_y^2 u_\Phi\|_{L^2_{t,\delta(t)}}(B^{t+rac{1}{2}}),
\]
from which, (1.9), (1.10) and Proposition 1.1 we infer
\[
\|e^{\alpha t} \partial_y u_\Phi\|_{L^\infty(B^t)} + \sqrt{\lambda} \|e^{\alpha t} \partial_y u_\Phi\|_{L^2_{t,\delta(t)}}(B^{t+rac{1}{2}}) + \|e^{\alpha t} \partial_y^2 u_\Phi\|_{L^2_{t,\delta(t)}}(B^{t+rac{1}{2}})
\]
\[
\leq \|e^{\alpha |D_x|} \partial_y u_0\|_{B^s} + C \left( (1 + \|e^{\alpha |D_x|} u_0\|_{B^s}) \|e^{\alpha t} u_\Phi\|_{L^2_{t,\delta(t)}}(B^{t+rac{1}{2}}) + \|e^{\alpha |D_x|} u_0\|_{B^{s+1}} \right).
\]
Taking \( \lambda = C^2 (1 + \|e^{\alpha |D_x|} u_0\|_{B^s}) \) in the above inequality leads to (4.13). This completes the proof of Proposition 1.2. \( \square \)

5. THE CONVERGENCE TO THE HYDROSTATIC NAVIER-STOKES SYSTEM

In this section, we justify the limit from the scaled anisotropic Navier-Stokes system to the hydrostatic Navier-Stokes system in a 2-D striped domain. To this end, we introduce

\[
\begin{align*}
  w_1^\varepsilon &\overset{\text{def}}{=} u^\varepsilon - u, \quad w_2^\varepsilon = v^\varepsilon - v, \quad q_\varepsilon = p^\varepsilon - p.
\end{align*}
\]

Then \((w_1^\varepsilon, w_2^\varepsilon, q_\varepsilon)\) verifies

\[
\begin{align*}
  \partial_t w_1^\varepsilon - \varepsilon^2 \partial_x^2 w_1^\varepsilon - \partial_y^2 w_1^\varepsilon + \partial_x q_\varepsilon &= R_1^\varepsilon \quad \text{in } S \times [0, \infty],
  \\
  \varepsilon^2 (\partial_t w_2^\varepsilon - \varepsilon^2 \partial_x^2 w_2^\varepsilon - \partial_y^2 w_2^\varepsilon) + \partial_y q_\varepsilon &= R_2^\varepsilon,
\end{align*}
\]

\[ (1.1) \]

\[
\begin{align*}
  \partial_x w_1^\varepsilon + \partial_y w_2^\varepsilon &= 0, \\
  (w_1^\varepsilon, w_2^\varepsilon)|_{y=0} &= (w_1^\varepsilon, w_2^\varepsilon)|_{y=1} = 0, \\
  (w_1^\varepsilon, w_2^\varepsilon)|_{t=0} &= (u_0^\varepsilon - u_0, v_0^\varepsilon - v_0),
\end{align*}
\]

where \( v_0 \) is determined from \( u_0 \) via \( \partial_x u_0 + \partial_y v_0 = 0 \) and \( v_0|_{y=0} = v_0|_{y=1} = 0 \), and

\[
\begin{align*}
  R_1^\varepsilon &= \varepsilon^2 \partial_x u - \left( u^\varepsilon \partial_x u^\varepsilon - u \partial_x u \right) - \left( v^\varepsilon \partial_y u^\varepsilon - v \partial_y u \right),
  \\
  R_2^\varepsilon &= -\varepsilon^2 \left( \partial_t v - \varepsilon^2 \partial_x^2 v - \partial_y^2 v + u^\varepsilon \partial_x v^\varepsilon + v^\varepsilon \partial_y v^\varepsilon \right).
\end{align*}
\]

Let us define

\[
(5.3) \quad u_\Theta(t, x, y) \overset{\text{def}}{=} F_{\xi^{-1}}(e^{\Theta(t, \xi)} \hat{u}(t, \xi, y)) \quad \text{and} \quad \Theta(t, \xi) \overset{\text{def}}{=} (a - \mu \zeta(t))|\xi|,
\]

where \( \mu \geq \lambda \) will be determined later, and \( \zeta(t) \) is given by

\[
\zeta(t) = \int_0^t \left( ||(\partial_y u_\Phi, \varepsilon \partial_x u_\Phi)(t')||_{B^2} + ||\partial_y u_\Phi(t')||_{B^2} \right) dt'.
\]

Similar notation for \((w_2^\varepsilon)_\Theta\) and so on.

It is easy to observe that if we take \( c_0 \) in (1.7) and \( c_1 \) in (1.9) small enough, then \( \Theta(t) \geq 0 \) and

\[
\Theta(t, \xi) \leq \min(\Psi(t, \xi), \Phi(t, \xi)).
\]

Thanks to Theorem 1.2 we deduce that

\[
(5.4) \quad \|u_\Phi\|_{L^\infty(\mathbb{R}^+, B^2)} + \|u_\Phi\|_{L^\infty(\mathbb{R}^+, B^2)} + \|\partial_y u_\Phi\|_{L^2(\mathbb{R}^+, B^2)} + \|\partial_y u_\Phi\|_{L^2(\mathbb{R}^+, B^2)} \leq M,
\]

\[
\|u_\Phi\|_{L^\infty(\mathbb{R}^+, B^2)} + \|u_\Phi\|_{L^\infty(\mathbb{R}^+, B^2)} + \|\partial_y u_\Phi\|_{L^2(\mathbb{R}^+, B^2)} + \|\partial_y u_\Phi\|_{L^2(\mathbb{R}^+, B^2)} \leq M,
\]
where $u_\varepsilon$ and $u_\phi$ are determined respectively by (3.1) and (4.3) and $M \geq 1$ is a constant independent of $\varepsilon$.

In what follows, we shall neglect the subscript $\varepsilon$ in $(w_\varepsilon^1, w_\varepsilon^2)$.

**Proof of Theorem 5.3** In view of (5.1), we get, by using a similar derivation of (3.7), that

\[
\| \Delta_k^h (w_\Theta^1, \varepsilon w_\Theta^2) \|^2_{L^\infty_t(L^2)} + \mu^2 \int_0^t \| \Delta_k^h (w_\Theta^1, \varepsilon w_\Theta^2) \|^2_{L^2} \, dt' \\
+ \int_0^t \left( \| \Delta_k^h \partial_y (w_\Theta^1, \varepsilon w_\Theta^2) \|^2_{L^2} + \varepsilon^2 \| \Delta_k^h (w_\Theta^1, \varepsilon w_\Theta^2) \|^2_{L^2} \right) \, dt' \\
\leq \| e^{aDx} (u_0^\varepsilon) - u_0, \varepsilon (v_0^\varepsilon - v_0) \|^2_{L^2} \\
+ \int_0^t \left( |(\Delta_k^h R_\Theta^1| \Delta_k^h w_\Theta^1)_{L^2} \right) \, dt' + \int_0^t \left( |(\Delta_k^h R_\Theta^2| \Delta_k^h w_\Theta^2)_{L^2} \right) \, dt'.
\]  

(5.5)

We now claim that

\[
\int_0^t \left( |(\Delta_k^h R_\Theta^1| \Delta_k^h w_\Theta^1)_{L^2} \right) \, dt' \leq d_k^2 2^{-k} \left( \varepsilon \| \partial_y u_\Theta \|_{L^2_t(B^{\frac{3}{2}})} \| \varepsilon w_\Theta^1 \|_{L^2_t(B^{\frac{3}{2}})} \\
+ \| u_\Theta \|_{L^\infty_t(B^{\frac{3}{2}})} \| \partial_y w_\Theta^1 \|_{L^2_t(B^{\frac{3}{2}})} \| w_\Theta^1 \|_{L^2_t(B^{1})} + \| w_\Theta^2 \|_{L^2_t(B^{1})} \right),
\]  

(5.6)

and

\[
\int_0^t \left( |(\Delta_k^h R_\Theta^2| \Delta_k^h w_\Theta^2)_{L^2} \right) \, dt' \leq d_k^2 2^{-k} \left( \| (w_\Theta^1, \varepsilon w_\Theta^2) \|_{L^2_t(B^{1})} + \varepsilon^2 \| (\partial_y w_\Theta^2, \varepsilon \partial_x w_\Theta^2) \|_{L^2_t(B^{\frac{3}{2}})} \\
\times \left( \| \partial_y u_\Theta \|_{L^2_t(B^{1})} + \| \partial_y u_\Theta \|_{L^2_t(B^{1})} + \varepsilon \| \partial_y u_\Theta \|_{L^2_t(B^{\frac{3}{2}})} \right) \\
+ \varepsilon^2 \| w_\Theta^2 \|_{L^2_t(B^{1})} \left( \| w_\Theta^2 \|_{L^2_t(B^{1})} + \| w_\Theta^2 \|_{L^2_t(B^{1})} \| \partial_y w_\Theta^2 \|_{L^2_t(B^{\frac{3}{2}})} + \| \partial_y u_\Theta \|_{L^2_t(B^{\frac{3}{2}})} \right) \\
+ \| u_\Theta \|_{L^\infty_t(B^{\frac{3}{2}})} \left( \| \partial_y w_\Theta^2 \|_{L^2_t(B^{\frac{3}{2}})} + \| \partial_y u_\Theta \|_{L^2_t(B^{\frac{3}{2}})} \right) \right).
\]  

(5.7)

By virtue of (5.4), (5.6) and (5.7), we infer

\[
\sum_{i=1}^{2} \int_0^t \left( |(\Delta_k^h R_\Theta^i| \Delta_k^h w_\Theta^i)_{L^2} \right) \, dt' \leq d_k^2 2^{-k} \left( M \varepsilon \| (\varepsilon \partial_x (w_\Theta^1, \varepsilon w_\Theta^2), \varepsilon \partial_y w_\Theta^2) \|_{L^2_t(B^{\frac{3}{2}})} \\
+ M \varepsilon \| \partial_y (w_\Theta^1, \varepsilon w_\Theta^2) \|_{L^2_t(B^{\frac{3}{2}})} \| (w_\Theta^1, \varepsilon w_\Theta^2) \|_{L^2_t(B^{1})} \\
+ M \varepsilon \| \varepsilon w_\Theta^2 \|_{L^2_t(B^{1})} + \| (w_\Theta^1, \varepsilon w_\Theta^2) \|_{L^2_t(B^{1})} \right),
\]
from which and (5.5), we deduce that
\[
\begin{align*}
&\|(w_\Theta^1, \varepsilon w_\Theta^2)\|_{L^2_t(\mathbb{B}^2)} + \mu^\frac{1}{2} \|(w_\Theta^1, \varepsilon w_\Theta^2)\|_{\tilde{L}^2_{t, \xi(t)}(\mathbb{B}^1)} + \|\partial_y (w_\Theta^1, \varepsilon w_\Theta^2)\|_{L^2_t(\mathbb{B}^2)} \\
&+ \varepsilon \|(w_\Theta^1, \varepsilon w_\Theta^2)\|_{\tilde{L}^2_{t, \xi(t)}(\mathbb{B}^1)} \leq C \|e^{a|Dx^1|} (u_0^\varepsilon - u_0, \varepsilon (v_0^\varepsilon - v_0))\|_{\mathbb{B}^2} \\
&+ C \left( \sqrt{M} \varepsilon \right) \left( \varepsilon \|\partial_x (w_\Theta^1, \varepsilon w_\Theta^2)\|_{\tilde{L}^2_{t, \xi(t)}(\mathbb{B}^1)} + \|\partial_y (w_\Theta^1, \varepsilon w_\Theta^2)\|_{L^2_t(\mathbb{B}^2)} \right) \\
&+ M^\frac{1}{2} \|\partial_y (w_\Theta^1, \varepsilon w_\Theta^2)\|_{L^2_t(\mathbb{B}^2)} ^\frac{1}{2} \|(w_\Theta^1, \varepsilon w_\Theta^2)\|_{L^2_{t, \xi(t)}(\mathbb{B}^1)} ^\frac{1}{2} \\
&+ M^\frac{3}{2} \varepsilon \|\varepsilon w_\Theta^2\|_{L^2_t(\mathbb{B}^1)} + \|(w_\Theta^1, \varepsilon w_\Theta^2)\|_{\tilde{L}^2_{t, \xi(t)}(\mathbb{B}^1)} ) .
\end{align*}
\]
(5.8)

Applying Young’s inequality gives rise to
\[
\begin{align*}
&\|(w_\Theta^1, \varepsilon w_\Theta^2)\|_{L^2_t(\mathbb{B}^2)} + \mu^\frac{1}{2} \|(w_\Theta^1, \varepsilon w_\Theta^2)\|_{\tilde{L}^2_{t, \xi(t)}(\mathbb{B}^1)} + \|\partial_y (w_\Theta^1, \varepsilon w_\Theta^2)\|_{L^2_t(\mathbb{B}^2)} \\
&+ \varepsilon \|\partial_x (w_\Theta^1, \varepsilon w_\Theta^2)\|_{\tilde{L}^2_{t, \xi(t)}(\mathbb{B}^1)} \leq C \left( \|e^{a|Dx^1|} (u_0^\varepsilon - u_0, \varepsilon (v_0^\varepsilon - v_0))\|_{\mathbb{B}^2} + M \left( \varepsilon + \|(w_\Theta^1, \varepsilon w_\Theta^2)\|_{\tilde{L}^2_{t, \xi(t)}(\mathbb{B}^1)} \right) \right) .
\end{align*}
\]
Taking $\mu = C^2 M^2$ leads to (1.11). This completes the proof of the theorem. \hfill \Box

Now let us present the proof of (5.6) and (5.7).

**Proof of (5.6).** According to (5.2), we write
\[
R_\varepsilon^1 = \varepsilon^2 \partial_x^2 u - (u^\varepsilon \partial_x w^1 + w^1 \partial_x u) - (v^\varepsilon \partial_y w^1 + w^2 \partial_y u).
\]

We first observe that
\[
\varepsilon^2 \int_0^t \left| \left( \Delta^h_k \partial_x^2 u \big| \Delta^h_k w_\Theta^1 \right) \right|_{L^2} dt' \leq C d^2_k 2^{-k} \varepsilon \|\partial_y u_\Theta\|_{\tilde{L}^2_{t, \xi(t)}(\mathbb{B}^1)} \left\| \varepsilon w_\Theta^1 \right\|_{L^2_t(\mathbb{B}^2)} .
\]
(5.9)

- The estimate of \int_0^t \left| \left( \Delta^h_k (u^\varepsilon \partial_x w^1 + w^1 \partial_x u) \big| \Delta^h_k w_\Theta^1 \right) \right|_{L^2} dt'.

It follows from Lemma 3.1 that
\[
\int_0^t \left| \left( \Delta^h_k (u^\varepsilon \partial_x w^1) \big| \Delta^h_k w_\Theta^1 \right) \right|_{L^2} dt' \lesssim d^2_k 2^{-k} \left\| w_\Theta^1 \right\|_{\tilde{L}^2_{t, \xi(t)}(\mathbb{B}^1)}^2 .
\]
(5.10)

By applying Bony’s decomposition (2.3) for the horizontal variable to $w^1 \partial_x u$, we obtain
\[
w^1 \partial_x u = T^h_{\partial_x u} + T^h_{\partial_y u} w^1 + R^h (w^1, \partial_x u).
\]

Notice that
\[
\left\| \Delta^h_k \partial_x u_\Theta (t') \right\|_{L^2_{t, \xi(t)}} \lesssim d^k(t) \left\| u_\Theta (t') \right\|_{L^2_{\mathbb{B}^2}} \left\| \partial_y u_\Theta (t') \right\|_{L^2_{\mathbb{B}^2}} .
\]
we infer

\[
\int_0^t \left| (\Delta^h_k (T^{w_1}_{h, \partial_x u}) \mid \Delta^h_k w_\Theta) \right|_{L^2} \, dt' \\
\lesssim \sum_{|k' - k| \leq 4} \int_0^t \|S^h_{k' - 1} w_\Theta(t')\|_{L^\infty_t(L^2)} \|\Delta^h_{k'} \partial_x u_\Theta(t')\|_{L^2_t(L^\infty)} \|\Delta^h_k w_\Theta(t')\|_{L^2} \, dt'
\]

\[
\lesssim \sum_{|k' - k| \leq 4} d_{k'}^{\frac{3}{2}} \|w_\Theta\|_{L^\infty_t(L^2)} \|\partial_y u_\Theta(t')\|_{B^2} \|\Delta^h_k w_\Theta(t')\|_{L^2} \, dt' \lesssim d_{k'}(t) 2^{k'} \|\partial_y u_\Theta(t')\|_{B^2},
\]

we deduce

\[
\int_0^t \left| (\Delta^h_k (T^{w_1}_{h, \partial_x u}) \mid \Delta^h_k w_\Theta) \right|_{L^2} \, dt' \\
\lesssim \sum_{|k' - k| \leq 4} \int_0^t \|S^h_{k' - 1} \partial_x u_\Theta(t')\|_{L^\infty} \|\Delta^h_{k'} w_\Theta(t')\|_{L^2} \|\Delta^h_k w_\Theta(t')\|_{L^2} \, dt'
\]

\[
\lesssim \sum_{|k' - k| \leq 4} 2^{k'} \left( \int_0^t \|\Delta^h_{k'} w_\Theta(t')\|_{L^2} \|\partial_y u_\Theta(t')\|_{B^2} \, dt' \right)^{\frac{3}{2}}
\]

\[
\lesssim d_{k'}^{2} 2^{-k} \|w_\Theta\|_{L^2_t(B^1)}.
\]

Along the same line, we have

\[
\int_0^t \left| (\Delta^h_k (R^h(w_1, \partial_x u)) \mid \Delta^h_k w_\Theta) \right|_{L^2} \, dt' \\
\lesssim 2^{\frac{h}{2}} \sum_{k' \geq k - 3} \int_0^t \|\Delta^h_{k'} w_\Theta(t')\|_{L^2} \|\Delta^h_{k'} \partial_x u_\Theta(t')\|_{L^2_t(L^\infty)} \|\Delta^h_k w_\Theta(t')\|_{L^2} \, dt'
\]

\[
\lesssim 2^\frac{h}{2} \sum_{k' \geq k - 3} 2^{k'} \left( \int_0^t \|\Delta^h_{k'} w_\Theta(t')\|_{L^2} \|\partial_y u_\Theta(t')\|_{B^2} \, dt' \right)^{\frac{1}{2}}
\]

\[
\times \left( \int_0^t \|\Delta^h_k w_\Theta(t')\|_{L^2} \|\partial_y u_\Theta(t')\|_{B^2} \, dt' \right)^{\frac{1}{2}}
\]

\[
\lesssim d_{k'}^{2} 2^{-k} \|w_\Theta\|_{L^2_t(B^1)}.
\]
As a result, it comes out

\[
\int_0^t \left| \left( \Delta_h^k (w^1 \partial_x u) \mid \Delta_h^k w_\Theta^1 \right)_{L^2} \right| dt' \leq d_k^2 2^{-k} \| w_\Theta^1 \|_{L^2_{t,\zeta}(B^1)} \left( \| u_\Theta^1 \|_{L^2_{t,\zeta}(B^1)} + \| u_\Theta^1 \|_{L^\infty(\mathbb{R}^3)} \| \partial_y w_\Theta^1 \|_{L^2_{t,\zeta}(B^1)} \right).
\]

(5.11)

- The estimate of \( \int_0^t \left| \left( \Delta_h^k (v^\zeta \partial_y w^1) \mid \Delta_h^k w_\Theta^1 \right)_{L^2} \right| dt' \).

We write

\[
v^\zeta \partial_y w^1 = w^2 \partial_y w^1 + v \partial_y w^1.
\]

We first deduce from Lemma [3.2] that

\[
\int_0^t \left| \left( \Delta_h^k (w^2 \partial_y w^1) \mid \Delta_h^k w_\Theta^1 \right)_{L^2} \right| dt' \leq d_k^2 2^{-k} \| w_\Theta^1 \|_{L^2_{t,\zeta}(B^1)}^2.
\]

(5.12)

Whereas by applying Bony’s decomposition [2.3] for the horizontal variable to \( v \partial_x w^1 \), we find

\[
v \partial_x w^1 = T^h_v \partial_y w^1 + T^h_{\partial_y w^1} v + R^h(v, \partial_y w^1).
\]

It follows from (5.15) that

\[
\| S^h_{k' - 1} v_\Theta(t') \|_{L^\infty} \leq \sum_{\ell \leq k' - 2} 2^{\ell \gamma} \| \Delta_h^\ell u_\Theta(t') \|_{L^2} \| \Delta_h^\ell \partial_y u_\Theta(t') \|_{L^2}^{1/2} \| \partial_y u_\Theta(t') \|_{L^2}^{1/2} \]

\[
\lesssim d_k'(t) 2^{\gamma} \| u_\Theta(t') \|_{B^{3\gamma}} \| \partial_y u_\Theta(t') \|_{L^2}^{1/2},
\]

from which we infer

\[
\int_0^t \left| \left( \Delta_h^k (T^h_v \partial_y w^1) \mid \Delta_h^k w_\Theta^1 \right)_{L^2} \right| dt' \]

\[
\lesssim \sum_{|k' - k| \leq 4} \int_0^t \| S^h_{k' - 1} v_\Theta(t') \|_{L^\infty} \| \Delta_h^k \partial_y w_\Theta^1(t') \|_{L^2} \| \Delta_h^k w_\Theta^1(t') \|_{L^2} dt' \]

\[
\lesssim \sum_{|k' - k| \leq 4} 2^{\gamma} \| u_\Theta \|_{L^\infty(B^{3\gamma})} \| \Delta_h^k \partial_y w_\Theta^1(t') \|_{L^2} \left( \int_0^t \| \partial_y u_\Theta(t') \|_{B^{3\gamma}} \| \Delta_h^k w_\Theta^1(t') \|_{L^2} dt' \right)^{1/2} \]

\[
\lesssim d_k^2 2^{-k} \| u_\Theta \|_{L^\infty(B^{3\gamma})} \| \partial_y w_\Theta^1 \|_{L^2(B^{3\gamma})} \| \partial_y w_\Theta^1 \|_{L^2_{t,\zeta}(B^1)} \| w_\Theta^1 \|_{L^2_{t,\zeta}(B^1)} \| \partial_y w_\Theta^1 \|_{L^2_{t,\zeta}(B^1)}.
\]

Whereas thanks to (5.11), we get

\[
\int_0^t \left| \left( \Delta_h^k (T^h_{\partial_y w^1}) \mid \Delta_h^k w_\Theta^1 \right)_{L^2} \right| dt' \]

\[
\lesssim \sum_{|k' - k| \leq 4} \int_0^t \| S^h_{k' - 1} \partial_y w_\Theta^1(t') \|_{L^\infty(L^2)} \| \Delta_h^k v_\Theta(t') \|_{L^2(L^\infty)} \| \Delta_h^k w_\Theta^1(t') \|_{L^2} dt' \]

\[
\lesssim \sum_{|k' - k| \leq 4} d_k' \| S^h_{k' - 1} \partial_y w_\Theta^1 \|_{L^2(L^\infty)} \| u_\Theta \|_{L^\infty(B^{3\gamma})} \left( \int_0^t \| \Delta_h^k w_\Theta^1(t') \|_{L^2} \| \partial_y u_\Theta(t') \|_{B^{3\gamma}} dt' \right)^{1/2} \]

\[
\lesssim d_k^2 2^{-k} \| u_\Theta \|_{L^\infty(B^{3\gamma})} \| \partial_y w_\Theta^1 \|_{L^2(B^{3\gamma})} \| w_\Theta^1 \|_{L^2(B^1)} \| \partial_y w_\Theta^1 \|_{L^2_{t,\zeta}(B^1)} \| w_\Theta^1 \|_{L^2_{t,\zeta}(B^1)}. \]
Along the same line, we obtain
\[
\int_0^t \left| \left( \Delta_h^k (R^h (v, \partial_y w^1)) \right)_\Theta \frac{\Delta_h^k w^1_\Theta}{L^2} \right| dt'
\leq 2^{k_1} \sum_{k \geq k_1} \int_0^t \left| \Delta_h^k v_\Theta (t') \right|_{L^\infty (L^2)} \left| \Delta_h^k \partial_y w^1_\Theta (t') \right|_{L^2} \left| \Delta_h^k w^1_\Theta (t') \right|_{L^2} dt'
\lesssim 2^{k_1} \sum_{k' \geq k_1} \left| u_\Theta \right|_B \left| \Delta_h^k \partial_y w^1_\Theta \right|_{L^2 (L^2)} \left( \int_0^t \left| \Delta_h^k w^1_\Theta (t') \right|_{L^2} dt' \left| \partial_y u_\Theta (t') \right|_{L^2} \right)^{\frac{1}{2}}
\lesssim d_k^2 2^{-k} \left| u_\Theta \right|_{L^2 (B_t^2)} \left| \partial_y w^1_\Theta \right|_{L^2 (B_t^2)} \left| w^1_\Theta \right|_{L^2 (B_t^2 (B^1))}.
\]

As a consequence, we arrive at
\[
(5.13) \quad \int_0^t \left| \left( \Delta_h^k (v \partial_y w^1) \right)_\Theta \frac{\Delta_h^k w^1_\Theta}{L^2} \right| dt' \lesssim d_k^2 2^{-k} \left| u_\Theta \right|_B \left| \partial_y w^1_\Theta \right|_{L^2 (B_t^2)} \left| w^1_\Theta \right|_{L^2 (B_t^2 (B^1))},
\]

**The estimate of \( \int_0^t \left| \left( \Delta_h^k (w^2 \partial_y u) \right)_\Theta \frac{\Delta_h^k w^1_\Theta}{L^2} \right| dt' \).**

By applying Bony’s decomposition (2.3) for the horizontal variable to \( w^2 \partial_y u \), we write
\[
w^2 \partial_y u = T^h_{w^2} \partial_y u + T^h_{\partial_y w^2} w^2 + R^h (w^2, \partial_y u).
\]

In view of (3.19), we have
\[
\left( \int_0^t \left| S^h_{k' - 1} w^2_\Theta (t') \right|_{L^\infty} \left| \partial_y u_\Theta (t') \right|_{B_t^2} dt' \right)^{\frac{1}{2}} \lesssim d_k^2 2^{-k} \left| u_\Theta \right|_{L^2 (B_t^2 (B^1))},
\]
so that we get, by applying Hölder’s inequality, that
\[
\int_0^t \left| \left( \Delta_h^k (T^h_{w^2} \partial_y u) \right)_\Theta \frac{\Delta_h^k w^1_\Theta}{L^2} \right| dt'
\lesssim \sum_{|k' - k| \leq 4} 2^{-\frac{k'}{2}} \int_0^t \left| S^h_{k' - 1} w^2_\Theta (t') \right|_{L^\infty} \left| \partial_y u_\Theta (t') \right|_{B_t^2} \left| \Delta_h^k w^1_\Theta (t') \right|_{L^2} dt'
\lesssim \sum_{|k' - k| \leq 4} 2^{-\frac{k'}{2}} \left( \int_0^t \left| S^h_{k' - 1} w^2_\Theta (t') \right|_{L^\infty} \left| \partial_y u_\Theta (t') \right|_{B_t^2} dt' \right)^{\frac{1}{2}}
\times \left( \int_0^t \left| \Delta_h^k w^1_\Theta (t') \right|_{L^2} \left| \partial_y u_\Theta (t') \right|_{B_t^2} dt' \right)^{\frac{1}{2}}
\lesssim d_k^2 2^{-k} \left| u_\Theta \right|_{L^2 (B_t^2 (B^1))}.
\]

While thanks to (3.19), we find
\[
\int_0^t \left| \left( \Delta_h^k (T^h_{\partial_y w^2} w^2) \right)_\Theta \frac{\Delta_h^k w^1_\Theta}{L^2} \right| dt'
\lesssim \sum_{|k' - k| \leq 4} \int_0^t \left| S^h_{k' - 1} \partial_y u_\Theta (t') \right|_{L^\infty (L^2)} \left| \Delta_h^k w^2_\Theta (t') \right|_{L^2 (L^\infty)} \left| \Delta_h^k w^1_\Theta (t') \right|_{L^2} dt'
\lesssim \sum_{|k' - k| \leq 4} 2^{k'} \int_0^t \left| \partial_y u_\Theta (t') \right|_{B_t^2} \left| \Delta_h^k w^1_\Theta (t') \right|_{L^2} \left| \Delta_h^k w^1_\Theta (t') \right|_{L^2} dt'
\lesssim d_k^2 2^{-k} \left| u_\Theta \right|_{L^2 (B_t^2 (B^1))}.
\]
Along the same line, we obtain

\[ \int_0^t \left| (\Delta_k^h(R^h(w^2, \partial_y u))_\Theta | \Delta_k^h(w^1_\Theta))_{L^2} \right| \, dt' \]

\[ \lesssim 2^{k'} \sum_{k' \geq k-3} \int_0^t \| \Delta_k^h \partial_y(w^2(t'))_{L^2} \| L_{\ell,(t)} \| \Delta_k^h \partial_y(u(t'))_{L^2} \| L_{\ell,(t)} \| \Delta_k^h(w^1_{\Theta}(t'))_{L^2} \, dt' \]

\[ \lesssim 2^{k'} \sum_{k' \geq k-3} 2^{k'} \int_0^t \| \Delta_k^h(w^1_{\Theta}(t'))_{L^2} \| L_{\ell,(t)} \| \partial_y(u(t'))_{L^2} \| L_{\ell,(t)} \| \Delta_k^h(w^1_{\Theta}(t'))_{L^2} \, dt' \]

\[ \lesssim d_k^{2-2k} \| w^1_{\Theta} \| L_{\ell,(t)}^2(B^1) \].

This gives rise to

\[ (5.14) \quad \int_0^t \left| (\Delta_k^h(w^2 \partial_y u) \Theta | \Delta_k^h(w^1_{\Theta}))_{L^2} \right| \, dt' \lesssim d_k^{2-2k} \| w^1_{\Theta} \| L_{\ell,(t)}^2(B^1) \].

By summing up (5.9)-(5.14), we conclude the proof of (5.6). □

**Proof of (5.7).** We first observe from \( \partial_x u + \partial_y v = 0 \) and Poincare inequality that

\[ \varepsilon^2 \int_0^t \left| (\Delta_k^h(\partial_x v) \Theta | \Delta_k^h(w^2_\Theta))_{L^2} \right| \, dt' \lesssim \varepsilon^2 d_k^{2-2k} \| (\partial_x u)_{\Theta} \| L_{\ell,(t)}^2(B^1) \| \partial_y(w^2_\Theta) \| L_{\ell,(t)}^2(B^1) \],

\[ (5.15) \]

\[ \varepsilon^2 \int_0^t \left| (\Delta_k^h(\partial_y v) \Theta | \Delta_k^h(w^2_\Theta))_{L^2} \right| \, dt' \lesssim \varepsilon^2 d_k^{2-2k} \| \partial_y(u)_{\Theta} \| L_{\ell,(t)}^2(B^1) \| \partial_y(w^2_\Theta) \| L_{\ell,(t)}^2(B^1) \],

\[ \varepsilon^4 \int_0^t \left| (\Delta_k^h(\partial_x v) \Theta | \Delta_k^h(w^2_\Theta))_{L^2} \right| \, dt' \lesssim \varepsilon^4 d_k^{2-2k} \| \partial_y(u)_{\Theta} \| L_{\ell,(t)}^2(B^1) \| w^2_\Theta \| L_{\ell,(t)}^2(B^1) \].

- The estimate of \( \int_0^t \left| (\Delta_k^h(u^\varepsilon \partial_x v^\varepsilon) \Theta | \Delta_k^h(w^1_{\Theta}))_{L^2} \right| \, dt' \).

We write

\[ u^\varepsilon \partial_x v^\varepsilon = u^\varepsilon \partial_x w^2 + u^\varepsilon \partial_x v. \]

It follows from Lemma 3.1 that

\[ (5.16) \quad \int_0^t \left| (\Delta_k^h(u^\varepsilon \partial_x w^2) \Theta | \Delta_k^h(w^1_{\Theta}))_{L^2} \right| \, dt' \lesssim d_k^{2-2k} \| w^2_{\Theta} \| L_{\ell,(t)}^2(B^1) \].

By applying Bony’s decomposition for the horizontal variable to \( u^\varepsilon \partial_x v \) gives

\[ u^\varepsilon \partial_x v = T_{w^\varepsilon} \partial_x v + T_{\partial_y v^\varepsilon} u^\varepsilon + R^h(u^\varepsilon, \partial_x v). \]

Due to

\[ \| S_k^h u^\varepsilon(t') \|_{L^\infty} \lesssim \| u^\varepsilon(t') \|_{L^\infty} \| \partial_y u^\varepsilon(t') \|_{L^\infty}^{ \frac{1}{2} }, \]
and (3.18), we have
\[
\int_0^t \left| \left( \Delta^h_k \left( T_{w^\varepsilon} \partial_x v \right) \right)_\Theta, \Delta^h_k w^{g_2}_\Theta \right|_{L^2} \, dt' \lesssim \sum_{|k' - k| \leq 4} \int_0^t \left| \left( \Delta^h_{k'} \left( S_{k-1} u^\varepsilon_\Theta(t) \right) \right)_\Theta, \Delta^h_k w^{g_2}_\Theta(t) \right|_{L^2} \, dt'
\]
\[
\lesssim \sum_{|k' - k| \leq 4} 2^{2k'} \| u^\varepsilon_\Theta \|_{L^2_k(B^2)} \| \Delta^h_k w^{g_2}_\Theta(t) \|_{L^2_k(B^2)} \left( \int_0^t \| \partial_y u^\varepsilon_\Theta(t) \|_{B^{1/2}} \| \Delta^h_k w^{g_2}_\Theta(t') \|_{L^2} \, dt' \right)^{1/2}
\]
\[
\lesssim d_k^2 2^{-k} \| u^\varepsilon_\Theta \|_{L^2_k(B^2)} \| \partial_y u_\Theta \|_{L^2_k(B^2)} \| w^{g_2}_\Theta \|_{L^2_k(C(t))}^2.
\]

While again thanks to (3.18), we find
\[
\| S_{k-1} \partial_x v_\Theta(t') \|_{L^\infty} \lesssim \frac{2^{k'}}{2} \| \partial_y u_\Theta(t') \|_{B^2},
\]
which leads to
\[
\int_0^t \left| \left( \Delta^h_k \left( T_{w^\varepsilon} \partial_x v \right) \right)_\Theta, \Delta^h_k w^{g_2}_\Theta \right|_{L^2} \, dt' \lesssim \sum_{|k' - k| \leq 4} \int_0^t \left| \left( \Delta^h_{k'} \left( S_{k-1} \partial_x v_\Theta(t') \right) \right)_\Theta, \Delta^h_k w^{g_2}_\Theta(t') \right|_{L^2} \, dt'
\]
\[
\lesssim \sum_{|k' - k| \leq 4} d_{k'} \| \partial_y u_\Theta \|_{L^2_k(B^2)} \| u^\varepsilon_\Theta \|_{L^2_k(B^2)} \left( \int_0^t \| \partial_y u^\varepsilon_\Theta(t') \|_{B^{1/2}} \| \Delta^h_k w^{g_2}_\Theta(t') \|_{L^2} \, dt' \right)^{1/2}
\]
\[
\lesssim d_k^2 2^{-k} \| u^\varepsilon_\Theta \|_{L^2_k(B^2)} \| \partial_y u_\Theta \|_{L^2_k(B^2)} \| w^{g_2}_\Theta \|_{L^2_k(C(t))}^2.
\]

Along the same line, we obtain
\[
\int_0^t \left| \left( \Delta^h_k \left( R^h(u^\varepsilon, \partial_x v) \right) \right)_\Theta, \Delta^h_k w^{g_2}_\Theta \right|_{L^2} \, dt' \lesssim 2^{k} \sum_{k' \geq k - 3} \int_0^t \| \Delta^h_{k'} u^\varepsilon_\Theta(t') \|_{L^2_k(B^2)} \| \tilde{\Delta}^h_k \partial_x v_\Theta(t') \|_{L^2_k(B^2)} \| \Delta^h_k w^{g_2}_\Theta(t') \|_{L^2} \, dt'
\]
\[
\lesssim 2^{k} \sum_{k' \geq k - 3} 2^{4k'} \| u^\varepsilon_\Theta \|_{L^2_k(B^2)} \| \Delta^h_k u_\Theta \|_{L^2_k(B^2)} \left( \int_0^t \| \partial_y u^\varepsilon_\Theta(t') \|_{B^{1/2}} \| \Delta^h_k w^{g_2}_\Theta(t') \|_{L^2} \, dt' \right)^{1/2}
\]
\[
\lesssim d_k^2 2^{-k} \| u^\varepsilon_\Theta \|_{L^2_k(B^2)} \| \partial_y u_\Theta \|_{L^2_k(B^2)} \| w^{g_2}_\Theta \|_{L^2_k(C(t))}^2.
\]

This gives rise to
\[
(5.17) \quad \int_0^t \left| \left( \Delta^h_k \left( u^\varepsilon \partial_x v \right) \right)_\Theta, \Delta^h_k w^{g_2}_\Theta \right|_{L^2} \, dt' \lesssim d_k^2 2^{-k} \| u^\varepsilon_\Theta \|_{L^2_k(B^2)} \| \partial_y u_\Theta \|_{L^2_k(B^2)} \| w^{g_2}_\Theta \|_{L^2_k(C(t))}.
\]

• The estimate of \( \int_0^t \left| \left( \Delta^h_k \left( v^\varepsilon \partial_y v^\varepsilon \right) \right)_\Theta, \Delta^h_k w^{g_1}_\Theta \right|_{L^2} \, dt' \).

We first note that
\[
v^\varepsilon \partial_y v^\varepsilon = v \partial_y w^2 + w^2 \partial_y w^2 + v \partial_y v + w^2 \partial_y v.
\]
We first deduce Lemma 3.3 that
\[ \varepsilon^2 \int_0^t \left| (\Delta^h_k (w^2 \partial_y w^2) \circ \Delta^h_k w^2) \right|_{L^2_x} \, dt' \lesssim d^2 k^{2-\varepsilon} \| (w^1, \varepsilon w^2) \|_{L^2_{t, \xi(t)} (B^1)}^2. \]
It follows from (5.13) that
\[ \int_0^t \left| (\Delta^h_k (v \partial_y w^2) \circ \Delta^h_k w^2) \right|_{L^2_x} \, dt' \lesssim d^2 k^{2-\varepsilon} \| u \|_{L^\infty_t (L^2_x)}^{\frac{3}{2}} \| \partial_y v \|_{L^2_{t, \xi(t)} (B^1)} \| w^2 \|_{L^2_{t, \xi(t)} (B^1)}. \]
And (5.11) ensures that
\[ \int_0^t \left| (\Delta^h_k (w^2 \partial_x u) \circ \Delta^h_k w^2) \right|_{L^2_x} \, dt' \lesssim d^2 k^{2-\varepsilon} \| u \|_{L^\infty_t (L^2_x)}^{\frac{3}{2}} \| \partial_y u \|_{L^2_{t, \xi(t)} (B^1)} \| w^2 \|_{L^2_{t, \xi(t)} (B^1)}. \]
We deduce from the proof of (5.13) that
\[ \int_0^t \left| (\Delta^h_k (v \partial_y v) \circ \Delta^h_k w^2) \right|_{L^2_x} \, dt' \lesssim d^2 k^{2-\varepsilon} \| u \|_{L^\infty_t (L^2_x)}^{\frac{3}{2}} \| \partial_y v \|_{L^2_{t, \xi(t)} (B^1)} \| w^2 \|_{L^2_{t, \xi(t)} (B^1)} \]
\[ \lesssim d^2 k^{2-\varepsilon} \| u \|_{L^\infty_t (L^2_x)}^{\frac{3}{2}} \| \partial_y u \|_{L^2_{t, \xi(t)} (B^1)} \| w^2 \|_{L^2_{t, \xi(t)} (B^1)}. \]
As a result, it comes out
\[ \varepsilon^2 \int_0^t \left| (\Delta^h_k (v \partial_y v) \circ \Delta^h_k w^1) \right|_{L^2_x} \, dt' \lesssim d^2 k^{2-\varepsilon} \left( \| (w^1, \varepsilon w^2) \|_{L^2_{t, \xi(t)} (B^1)}^2 \right) \]
\[ + \varepsilon^2 \| u \|_{L^\infty_t (L^2_x)}^{\frac{3}{2}} \left( \| \partial_y w^2 \|_{L^2_{t, \xi(t)} (B^1)} + \| \partial_y u \|_{L^2_{t, \xi(t)} (B^1)} \right) \| w^2 \|_{L^2_{t, \xi(t)} (B^1)} \].
Summing up (5.15, 5.18) gives rise to (5.7).

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