THE HOMOLOGY OF SIMPLICIAL COMPLEMENT AND THE COHOMOLOGY OF THE MOMENT-ANGLE COMPLEXES

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Abstract. A simplicial complement \( P = \{\sigma_1, \sigma_2, \ldots, \sigma_s\} \) is a sequence of subsets of \( [m] \) and the simplicial complement \( P \) corresponds to an unique simplicial complex \( K_P \) with vertices in \([m]\). In this paper, we defined the homology of a simplicial complement \( H_*(\Lambda^*[P], d) \) over a principle ideal domain \( k \) and proved that \( H_*(\Lambda^*[P], d) \) is isomorphic to the \( Tor \) of the corresponding face ring \( k(K_P) \) by the Taylor resolution. As applications, we give methods to compute the ring structure of \( Tor^{k[x]}_{i,j}(k(K_P), k) \), \( \text{link}_{K_P} \sigma \), \( \text{star}_{K_P} \sigma \) and the cohomology modules of the generalized moment-angle complexes.

1. Introduction and statement of results

The moment-angle complexes have been studied by topologists for many years (cf. [19] [15]). In 1990’s Davis and Januszkiewicz [8] introduced quasi-toric manifolds which were being studied intensively by algebraic geometers. They observed that every quasi-toric manifold is the quotient of a moment-angle complex by the free action of a real torus, here the moment-angle complex is denoted by \( Z_K \) corresponding to an abstract simplicial complex \( K \). The topology of \( Z_K \) is complicated and getting more attentions by topologists lately (cf. [11] [14] [4] [18] [10]). Recently a lot of work has been done on generalizing the moment-angle complex \( Z_K = Z_K(D^2, S^1) \) to pairs of spaces \((X, A)\) (cf. [2] [3] [12] [16]). In this paper we study the cohomology of the generalized moment-angle complexes \( Z_K(X, A) \), corresponding to the pairs of spaces \((X, A)\) with inclusions \( A_i \hookrightarrow X_i \) being homotopic to constant for all \( i \).

Classically the homological algebra aspect of the Stanley-Reisner face ring plays an important role in the cohomology of \( Z_K \) (cf. [5] [20]). Let \( K \) be an abstract simplicial complex with \( m \) vertices. Choose a ground ring \( k \) with unit (we are mostly interested in a principal ideal domain). Let \( k[x] \) be the \( \mathbb{N}^m \) graded polynomial algebra over \( k \) on \( m \) indeterminates \( x = \{x_1, x_2, \ldots, x_m\} \). The Stanley-Reisner face ring is the quotient ring \( k(K) = k[x]/I_K \), where \( I_K \) is the Stanley-Reisner ideal generated by the monomials \( x_{\sigma} \) corresponding to non-faces \( \sigma \notin K \).

\[ I_K = \left\langle x_{\sigma} = x_{i_1}x_{i_2} \cdots x_{i_n} | \sigma = \{i_1, i_2, \ldots, i_n\} \notin K \right\rangle. \]

It is well known that the cohomology of \( Z_K \) is isomorphic to the \( Tor \) of \( k(K) \) over \( k[x] \), that is

\[ H^2(Z_K, k) \cong \bigoplus_{2j-i=q} Tor^{k[x]}_{i,j}(k(K), k) \]

and

\[ Tor^{k[x]}_{i,j}(k(K), k) \cong Tor^{k[x]}_{i,j}(k, k(K)). \]

There are several methods to compute the \( Tor \) of \( k(K) \). For example, one can use the Koszul resolution \( \Lambda[u_1, u_2, \ldots, u_n] \otimes k[x] \) of \( k \), then by applying the functor \( \otimes_{k[x]} k(K) \), the homology

2000 Mathematics Subject Classification. Primary 13F55, 18G15, Secondary 16E05, 55U10.

Key words and phrases. Stanley-Reisner face ring, moment-angle complex, \( Tor \) algebra, homology of simplicial complement.

The authors were supported by NSFC grant No. 10771105 and No. 11071125.
of $\Lambda[u_1,u_2, \cdots, u_n] \otimes k(K)$ is $\text{Tor}_{i,j}^{k[x]}(k, k(K))$. On the other hand, one can construct a free resolution $(R^*, d)$ of $k(K)$, then by applying the functor $\otimes_{k[x]} k$, the homology of $R^* \otimes_{k[x]} k$ is $\text{Tor}_{i,j}^{k[x]}(k(K), k)$. In this paper, we will follow the second method to give a free resolution of $k(K)$ and compute the homology of the resulting chain complex $R^* \otimes_{k[x]} k$.

Let $P = \{\sigma_1, \sigma_2, \cdots, \sigma_s\}$ be a sequence of subsets of $[m] = \{1, 2, \cdots, m\}$, which is called a simplicial complement in this paper. It is easy to see that

$$I_P = \langle x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_s} \rangle$$

is an ideal generated by square-free monomials. Thus there is an unique simplicial complex $K_P$ such that $I_{K_P} = I_P$.

Let $\Lambda[P]$ be the exterior algebra over $k$ generated by $P$. Given a generator $u = \sigma_{k_1} \sigma_{k_2} \cdots \sigma_{k_q}$ in $\Lambda[P]$, let $S_u = \sigma_{k_1} \cup \sigma_{k_2} \cup \cdots \cup \sigma_{k_q}$ be the total subset of $u$. The bi-degree of $u$ is defined as $\text{bideg}(u) = (q, \text{S}_u)$, and the monomial $x_{S_u}$ determines an unique $\mathbb{N}^m$ vector $a_u$ such that $x_{S_u} = x^a_u$.

Let

$$\partial_i(u) = \sigma_{k_1} \cdots \hat{\sigma}_{k_i} \cdots \sigma_{k_q} = \sigma_{k_{i+1}} \cdots \sigma_{k_{k_q}}$$

and for a generator $u \otimes x^a \in \Lambda[P] \otimes_k k[x]$ define

$$\bar{d}(u \otimes x^a) = \sum_i (-1)^i \partial_i(u) \otimes x_{(S_u \setminus S_{\partial_i(u)})} x^a,$$

where $x_{(S_u \setminus S_{\partial_i(u)})}$ denote the monomial of $k[x]$ corresponding to the subset $S_u \setminus S_{\partial_i(u)}$.

It is known that $(\Lambda[P] \otimes_k k[x], \bar{d})$ is a free $k[x]$-resolution of $k[x]/I_P$ called the Taylor resolution $[17]$. Then applying the functor $\otimes_{k[x]} k$, we get

$$(\Lambda[P] \otimes_k k[x] \otimes_{k[x]} k, \bar{d} \otimes 1) = (\Lambda^{*,*}[P], d)$$

and

**Theorem 2.7** Let $P = \{\sigma_1, \sigma_2, \cdots, \sigma_s\}$ be a simplicial complement and $K_P$ be the corresponding simplicial complex such that $I_{K_P} = I_P$. Let $(\Lambda^{*,*}[P], d)$ be the chain complex induced from the bi-graded exterior algebra on $P$. The differential $d : \Lambda^{*,*}[P] \to \Lambda^{q-1,*}[P]$ is given by

$$d(u) = \sum_i (-1)^i \partial_i(u) \cdot \delta_{\partial_i(u)}$$

where $\delta_{\partial_i(u)} = 1$ if the total subset $S_u = S_{\partial_i(u)}$ and $\delta_{\partial_i(u)} = 0$ if $S_u \neq S_{\partial_i(u)}$. Then the homology of $(\Lambda^{*,*}[P], d)$ is

$$H_{q,*}(\Lambda^{*,*}[P], d) = \text{Tor}^{k[x]}_{q,*}(k[x]/I_P, k) = \text{Tor}^{k[x]}_{q,*}(k(K_P), k),$$

which is called the homology of the simplicial complement $P$.

Fix a $\sigma \in 2^m$ and let $\Lambda^{*,*}[P]$ be the submodule generated by $u$ with total subset $S_u = \sigma$. From the formula for $d(u)$, we see that $(\Lambda^{*,*}[P], d)$ is a sub-complex of $(\Lambda^{*,*}[P], d)$. Then from a theorem of Baskakov, we see that

$$H_{q,*}(\Lambda^{*,*}[P], d) = H_{q,*}(k(K_P), k) = \text{Tor}^{k[x]}_{q,*}(k(k(K_P)), k)$$

where $K_P \cap \sigma$ is the full sub-complex of $K_P$ consisting of all simplices of $K_P$ which have all of their vertices in $\sigma$, that is $K_P \cap \sigma = \{\tau \cap \sigma | \tau \in K_P\}$.

Notice that both $k(K_P)$ and $k[x]$ algebras, $\text{Tor}^{k[x]}_{q,*}(k(K_P), k)$ has natural algebraic structure and the following isomorphism of algebras holds.

$$H^*(Z_{K_P}, k) \cong \text{Tor}^{k[x]}_{q,*}(k(K_P), k).$$
In section 3, we proved that

**Theorem 3.6** The algebraic structure in $H_{\ast,\ast}(\Lambda^\ast \ast \mathbb{P}, d) \cong \text{Tor}_{\ast,\ast}^k(k(\mathbb{P}), k)$ is given by

$$[c] \times [c'] = [c \cdot c'] \cdot \delta_{\sigma,\sigma'}$$

where $[c] \in H_{q,\ast}(\Lambda^\ast \ast \mathbb{P}, d)$, $[c'] \in H_{j,\ast}(\Lambda^\ast \ast \mathbb{P}, d)$, the cycle $c \cdot c'$ is the product of $c$ and $c'$ in the exterior algebra $\Lambda^\ast \ast \mathbb{P}$ and $\delta_{\sigma,\sigma'} = 1$ if $\sigma \cap \sigma' = \phi$, $\delta_{\sigma,\sigma'} = 0$ if $\sigma \cap \sigma' \neq \phi$.

Fix a subset $\omega \subset [m]$, there is the $\omega$-compression $E_\omega$ defined on the simplicial complement $\mathbb{P}$ given by

$$E_\omega \mathbb{P} = \{\sigma_1 \setminus \omega, \sigma_2 \setminus \omega, \ldots, \sigma_s \setminus \omega\}.$$ 

For the subset $\omega$, the link and star of the simplicial complex $K_\mathbb{P}$ corresponding to $\omega$ are the sub-complexes

$$\text{star}_{K_\mathbb{P}} \omega = \{\tau \in K_\mathbb{P}| \tau \cup \omega \subseteq K_\mathbb{P}\}$$

$$\text{link}_{K_\mathbb{P}} \omega = \{\tau \in K_\mathbb{P}| \tau \cup \omega \subseteq K_\mathbb{P}; \tau \cap \omega = \phi\}.$$ 

In section 4 we proved that:

**Theorem 4.7** Let $\mathbb{P}$ be a simplicial complement and $K_\mathbb{P}$ be the simplicial complex corresponding to $\mathbb{P}$. Then:

1. The star, $\text{star}_{K_\mathbb{P}} \omega = K_{E_\omega \mathbb{P}}$ is the simplicial complex corresponding to the simplicial complement $E_\omega \mathbb{P}$. Thus

$$\text{Tor}_{q,\ast}^{k(\mathbb{P})}(k(\text{star}_{K_\mathbb{P}} \omega), k) = H_q,\ast(\Lambda^\ast \ast [E_\omega \mathbb{P}], d),$$

2. The link, $\text{link}_{K_\mathbb{P}} \omega = K_{E_\mathbb{P}} \cap ([m] \setminus \omega)$ and

$$H_{q,[m] \setminus \omega}(\Lambda^\ast \ast [E_\mathbb{P}], d) = \widetilde{H}_{m-|\omega|-q-1}(\text{link}_{K_\mathbb{P}} \omega, k).$$

Let $(\mathbb{X}, \mathcal{A}) = \{(X_i, A_i, x_i)|i \in [m]\}$ denote a set of triples of CW-complexes with base points $x_i \in A_i$ and $K_\mathbb{P}$ be an abstract simplicial complex. The generalized moment-angle complex determined by $(\mathbb{X}, \mathcal{A})$ and $K_\mathbb{P}$ denoted by $Z_{K_\mathbb{P}}(\mathbb{X}, \mathcal{A})$ is defined to be the colimit

$$Z_{K_\mathbb{P}}(\mathbb{X}, \mathcal{A}) = \bigcup_{\sigma \in K_\mathbb{P}} D(\omega),$$

where

$$D(\omega) = Y_1 \times Y_2 \times \cdots \times Y_m$$

and

$$Y_i = \begin{cases} X_i & \text{if } i \in \omega \\ A_i & \text{if } i \notin \omega \end{cases}.$$ 

Based on Theorem 4.7 and the decomposition of $\Sigma Z_{K_\mathbb{P}}(\mathbb{X}, \mathcal{A})$ given by Bahri, Bendersky, Cohen and Gitler (cf. [21, 32]), we proved in section 5

**Theorem 5.7** Let $K_\mathbb{P}$ be an abstract simplicial complex corresponding to a simplicial complement $\mathbb{P}$ and let $(\mathbb{X}, \mathcal{A}) = \{(X_i, A_i, x_i)|i \in [m]\}$ denote $m$ choices of connected, pointed pairs of CW-complexes, with the inclusion $A_i \hookrightarrow X_i$ homotopic to constant for all $i$. If $\tilde{H}^\ast(X_i, k)$ and $\tilde{H}^\ast(A_i, k)$ are free $k$-modules for all $i$, then the cohomology of $Z_{K_\mathbb{P}}(\mathbb{X}, \mathcal{A})$ is isomorphic to

$$H^\ast(Z_{K_\mathbb{P}}(\mathbb{X}, \mathcal{A}), k) = \bigoplus_{\omega \in K_\mathbb{P}} \left( \bigoplus_{\tau \subset [m] \setminus \omega} H_{\ast,\tau}(\Lambda^\ast \ast [E_\omega \mathbb{P}], d) \otimes \left( \bigotimes_{i \in \omega} \tilde{H}^\ast(X_i, k) \right) \otimes \left( \bigotimes_{j \in \tau} \tilde{H}^\ast(A_j, k) \right) \right)$$

as $k$-modules, where

$$H_{\ast,\tau}(\Lambda^\ast \ast [E_\omega \mathbb{P}], d) \cong \text{Tor}_{\ast,\tau}^{k}{k(\text{star}_{K_\mathbb{P}} \omega), k}.$$
As applications, we also consider the cohomology of \( Z_{K_p}(X, A) \) for some special triples of CW-complexes \((X, A)\) including all the \( X_i \) are contractible; all the \( A_i \) are contractible and \((X, A) = (S^2, S^1)\).

Acknowledgements The authors are grateful to Z. Lü for his introduction of this topic. The authors are indebted to V. M. Buchstaber for his comments and suggestions, especially for the name of definition simplicial complement which is originally called partition. The authors also thank S. Gitler for his helpful suggestions.

2. THE HOMOLOGY OF SIMPLICIAL COMPLEMENT AND THE Tor OF FACE RING

Let \([m] = \{1, 2, \cdots, m\}\) and \(2^m\) denote the power set of \([m]\). Choose a ground ring \(k\) with unit (We usually suppose that \(k\) is a principle ideal domain). Let \(k[x]\) be the \(\mathbb{N}^m\) graded polynomial algebra over \(k\) on \(m\) indeterminates \(x = \{x_1, x_2, \cdots, x_m\}\). The monomial of \(k[x]\) is expressed as
\[
x^n = x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}
\]
for a unique vector \(a = (a_1, a_2, \cdots, a_m) \in \mathbb{N}^m\). For a subset \(\sigma = \{i_1, i_2, \cdots, i_n\}\) of \([m]\), let \(x_\sigma = x_{i_1} x_{i_2} \cdots x_{i_n}\) be the corresponding monomial in \(k[x]\), while \(x_\emptyset = 1\). Then the monomial \(x_\sigma\) can be expressed as \(x_\sigma = x_{\sigma'}\) for a unique vector \(a_{\sigma'} \in \mathbb{N}^m\).

Definition 2.1 A sequence \(\mathbb{P} = \{\sigma_1, \sigma_2, \cdots, \sigma_s\}\) of subsets of \([m]\) is called a simplicial complement. Given a simplicial complement \(\mathbb{P} = \{\sigma_1, \sigma_2, \cdots, \sigma_s\}\), let \(I_\mathbb{P}\) denote the ideal of \(k[x]\) generated by \(x_{\sigma_i}\)’s
\[
I_\mathbb{P} = \langle x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_s} \rangle.
\]
Two simplicial complements \(\mathbb{P}\) and \(\mathbb{Q}\) are called equivalent if they generate the same ideal of \(k[x]\), that is \(I_\mathbb{P} = I_\mathbb{Q}\).

Given an abstract simplicial complex \(K\), let \(\mathbb{P}_K\) denote the simplicial complement consists of all the non-faces \(\sigma \notin K\) or equivalently consists of all the missing faces in the sense that it is not a simplex of \(K\) but all of its proper subsets are simplices of \(K\). One can easily see from its definition that \(I_{\mathbb{P}_K} = I_{K}\).

Furthermore, given a simplicial complement \(\mathbb{P}\), noticed that \(I_\mathbb{P}\) is an ideal generated by square-free monomials, there is a unique simplicial complex \(K_\mathbb{P}\) such that \(I_\mathbb{P} = I_{K_\mathbb{P}}\).

Definition 2.2 Define \(K_\mathbb{P}\) to be the simplicial complex corresponding to \(\mathbb{P}\) such that the non-faces \(2^m \setminus K_\mathbb{P}\) is the full subset of \(2^m\) consisting of all subsets \(\sigma \in 2^m\) which contain a subset \(\sigma_i\) in \(\mathbb{P}\), that is
\[
2^m \setminus K_\mathbb{P} = \{\sigma \in 2^m \mid \text{there is a } \sigma_i \in \mathbb{P} \text{ such that } \sigma_i \subset \sigma\}.
\]

Notice that, \(x_\sigma = x_{\sigma_i} \cdot x_{\sigma_i \setminus \sigma_i}\) if \(\sigma_i \subset \sigma\), we have \(I_\mathbb{P} = I_{K_\mathbb{P}}\). The Stanley-Reisner face ring is
\[
k(K_\mathbb{P}) = k[x]/I_{K_\mathbb{P}} = k[x]/I_\mathbb{P}.
\]

Proposition 2.3 If \(\mathbb{P}\) and \(\mathbb{Q}\) are equivalent simplicial complements, then
\[
k(K_\mathbb{P}) = k[x]/I_\mathbb{P} = k[x]/I_\mathbb{Q} = k(K_\mathbb{Q})
\]
and the corresponding simplicial complexes \(K_\mathbb{P} = K_\mathbb{Q}\). \(\square\)

Remark: Notice that if the empty set \(\emptyset\) is an element of the simplicial complement \(\mathbb{P}\), then \(I_\mathbb{P} = (x_\emptyset = 1) = k[x]\), the simplicial complex \(K_\mathbb{P} = \emptyset\), which is different from the empty simplicial complex \(K = \{\emptyset\}\).

Definition 2.4 Given a simplicial complement \(\mathbb{P} = \{\sigma_1, \sigma_2, \cdots, \sigma_s\}\), let \(\Lambda[\mathbb{P}]\) be the exterior algebra over \(k\) on the generators \(\sigma_1, \sigma_2, \cdots, \sigma_s\). Define an \(\mathbb{N} \times 2^m\)-graded \(k\)-module structure on \(\Lambda[\mathbb{P}]\) as follows. For a generator \(u = \sigma_{k_1} \sigma_{k_2} \cdots \sigma_{k_q}\) of \(\Lambda^{\ast \ast}[\mathbb{P}]\), let
\[
S_u = \sigma_{k_1} \cup \sigma_{k_2} \cup \cdots \cup \sigma_{k_q}
\]
and
\[
S_1 = \emptyset
\]
denote the union of the subsets $\sigma_{k_j}$’s, which is called the total subset of $u$. Define the degree of $u = \sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_q}$ as

$$\text{bideg}(u) = (q, S_u)$$

so that $\Lambda[P]$ becomes an $\mathbb{N} \times 2^{|m|}$ graded $k$–module and $\Lambda^{*,*}[P] = \bigoplus_{q,\sigma} \Lambda^{q,\sigma}[P]$.

**Remark:** $x_{S_u}$ is the least common multiple of the monomials $x_{\sigma_{k_1}}, x_{\sigma_{k_2}}, \ldots, x_{\sigma_{k_q}}$ and $\Lambda^{*,*}[P]$ corresponding to the $x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_q}$ labeled standard $s - 1$ simplex $\Delta^{s-1}$ (cf. [17] Chapter 4).

Let $R^{*,*} = \Lambda^{*,*}[P] \otimes_k k[x]$ denote the free $k[x]$-module generated by $\Lambda^{*,*}[P]$. For a generator $u = \sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_q}$ of $\Lambda^{*,*}[P]$, the monomial in $k[x]$ corresponding to the total subset $S_u$ is expressed as $x_{S_u} = x^{a_u}$ for an unique $\mathbb{N}^m$ vector $a_u \in \mathbb{N}^m$. We define the $\mathbb{N} \times \mathbb{N}^m$ degree of $u$ as

$$\text{bideg}(u) = (q, a_u).$$

Then $R^{*,*}$ becomes an $\mathbb{N} \times \mathbb{N}^m$ graded module over $k$.

Given a generator $u = \sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_q}$ of $\Lambda^{*,*}[P]$, denote that

$$\partial_i(u) = \partial_i(\sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_q}) = \sigma_{k_1}\cdots\hat{\sigma}_i\cdots\sigma_{k_q} = \sigma_{k_1}\cdots\sigma_{k_{i-1}}\sigma_{k_{i+1}}\cdots\sigma_{k_q}.$$

Notice from Definition 2.4 that, the total subset $S_u = S_{\partial_i(u)} \cup (S_u \setminus S_{\partial_i(u)})$, we define a $k[x]$-module homomorphism

$$\tilde{d} : \Lambda^{q,*}[P] \otimes_k k[x] \rightarrow \Lambda^{q-1,*}[P] \otimes_k k[x]$$

by setting

$$\tilde{d}(u \otimes x^a) = \sum_{i=1}^q (-1)^i \partial_i(u) \otimes x_{(S_u \setminus S_{\partial_i(u)})} x^a,$$

where $x_{(S_u \setminus S_{\partial_i(u)})} = \frac{x_{S_u}}{x_{S_{\partial_i(u)}}}$ is the monomial of $k[x]$ corresponding to the subset $S_u \setminus S_{\partial_i(u)}$.

**Theorem 2.6 (Taylor resolution)** ($\Lambda^{*,*}[P] \otimes_k k[x]$, $\tilde{d}$) is a free resolution of $k[x]/I_P$. Thus

$$H_{q,a} \left( (\Lambda^{*,*}[P] \otimes_k k[x]) \otimes_{k[x]} k, \tilde{d} \otimes 1 \right) = Tor_{q,a}^{k[x]}(k[x]/I_P,k).$$

**Proof.** One may find a proof of this Theorem in Chapter 4 of [17], where it is given by the cellular resolution. Here we give a proof by induction on the cardinal number $|P| = s$.

Notice that, $\partial_i \partial_j = \partial_j \partial_i$ for $i < j$, an standard argument shows that $\tilde{d}$ is a derivation, that is $\tilde{d} \cdot \tilde{d} = 0$.

For $\Lambda^{0,*}[P] \otimes_k k[x] = k[x]$, we define the augmentation

$$\varepsilon : \Lambda^{0,*}[P] \otimes_k k[x] \rightarrow k[x]/I_P$$

to be the quotient map. Furthermore from $\tilde{d}(\sigma_i) = -x_{\sigma_i}$, we see that

$$\Lambda^{1,*}[P] \otimes_k k[x] \xrightarrow{\tilde{d}} \Lambda^{0,*}[P] \otimes_k k[x] \xrightarrow{\varepsilon} k[x]/I_P \rightarrow 0$$

is exact. Thus we only need to prove that $H_{q,*}(\Lambda^{*,*}[P] \otimes_k k[x], \tilde{d}) = 0$ for all $q > 0$, and this will be done by induction on the cardinal number $|P| = s$. 


If $|P| = 1$, that is $P = \{\sigma_1\}$, it is easy to see that
\[
0 \rightarrow k[\sigma_1] \otimes_k k[x] \xrightarrow{\tilde{d}} k[x] \xrightarrow{\varepsilon} k[x]/I_P
\]
is exact and then $H_{1,*}(\Lambda^{*,*}[\{\sigma_1\}] \otimes_k k[x], \tilde{d}) = 0$.

Inductively suppose that for any simplicial complement $P$ with cardinal number $|P| = s$, that is $P = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ contains $s$ subsets, the homology $H_{q,*}(\Lambda^{*,*}[P] \otimes_k k[x], d) = 0$ for $q > 0$. Then for a simplicial complement $Q$ with $|Q| = s + 1$, that is
\[
Q = \{\sigma_1, \sigma_2, \ldots, \sigma_s, \sigma\} = P \cup \{\sigma\},
\]
$\Lambda^{*,*}[P] \otimes_k k[x]$ is a sub-complex of $\Lambda^{*,*}[Q] \otimes_k k[x]$, then the short exact sequence
\[
0 \rightarrow \Lambda^{*,*}[P] \otimes_k k[x] \xrightarrow{i} \Lambda^{*,*}[Q] \otimes_k k[x] \xrightarrow{\partial} (\Lambda^{*,*}[Q]/\Lambda^{*,*}[P]) \otimes_k k[x] \rightarrow 0
\]
induces a long exact sequence in homologies
\[
\cdots \rightarrow H_{q,*}(\Lambda^{*,*}[Q] \otimes_k k[x]) \xrightarrow{\partial} H_{q,*}(\Lambda^{*,*}[P] \otimes_k k[x]) \xrightarrow{\partial} H_{q,*}((\Lambda^{*,*}[Q]/\Lambda^{*,*}[P]) \otimes_k k[x]) \rightarrow \cdots
\]
The quotient complex $(\Lambda^{q+1,*}[Q]/\Lambda^{q+1,*}[P]) \otimes_k k[x]$ is expressed as $\Lambda^{q,*}[P] \otimes_k k[x]$ with generators $u \cdot \sigma \otimes x^n$ and the induced differential
\[
\tilde{d} : \Lambda^{q,*}[P] \otimes_k k[x] \rightarrow \Lambda^{q-1,*}[P] \otimes_k k[x]
\]
is given by
\[
\tilde{d}(u \cdot \sigma \otimes x^n) = \sum_{i=1}^{q} (-1)^i (\partial_i(u)) \cdot \sigma \otimes x_{S_i} x^n
\]
where $S_i = S_{u \cdot \sigma} \setminus S_{(\partial_i(u)) \cdot \sigma} = (\sigma_{k_1} \cup \ldots \cup \sigma_{k_q} \cup \sigma) \setminus (\sigma_{k_1} \cup \ldots \cup \hat{\sigma}_{k_i} \cup \ldots \cup \sigma_{k_q} \cup \sigma)$.

By the induction hypothesis, we know that $H_{q,*}(\Lambda^{*,*}[P] \otimes_k k[x], d) = 0$ for $q > 0$. Thus we only need to prove that $H_{q,*}((\Lambda^{*,*}[Q]/\Lambda^{*,*}[P]) \otimes_k k[x]) = 0$ for $q > 1$ and
\[
0 \rightarrow H_{1,*}(\Lambda^{*,*}[Q]/\Lambda^{*,*}[P] \otimes_k k[x]) \xrightarrow{\partial} H_{0,*}(\Lambda^{*,*}[P] \otimes_k k[x]) = k[x]/I_P
\]
is exact or equivalently $\partial$ is a monomorphism. To do so, let $P' = \{\sigma_1', \sigma_2', \ldots, \sigma_s' \} = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$.

Define a homomorphism of $k[x]$-modules $f : \Sigma \Lambda^{P'} \otimes_k k[x] \rightarrow (\Lambda^{Q}/\Lambda^{P}) \otimes_k k[x]$ by
\[
f(\sigma_{k_1}', \sigma_{k_2}', \ldots, \sigma_{k_q}') = \sigma_{k_1} \sigma_{k_2} \ldots \sigma_{k_q} \sigma.
\]
Notice that,
\[
(\sigma_{k_1} \cup \ldots \cup \sigma_{k_q} \cup \sigma) \setminus (\sigma_{k_1} \cup \ldots \cup \hat{\sigma}_{k_i} \cup \ldots \cup \sigma_{k_q} \cup \sigma)
= (\sigma_{k_1} \setminus \sigma) \cup \ldots \cup (\sigma_{k_q} \setminus \sigma) \cup (\sigma_{k_1} \setminus \sigma) \cup \ldots \cup (\sigma_{k_q} \setminus \sigma)
= (\sigma_{k_1}' \cup \ldots \cup \hat{\sigma}_{k_i}' \cup \ldots \cup \sigma_{k_q}') \cup (\sigma_{k_1}' \cup \ldots \cup \hat{\sigma}_{k_i}' \cup \ldots \cup \sigma_{k_q}')
\]
we see that for a monomial $u' = \sigma_{k_1}' \sigma_{k_2}' \ldots \sigma_{k_q}' \in \Sigma \Lambda^{*,*}[P']$, $S_{u'} \setminus S_{(\partial_i(u)) \cdot \sigma} = S_{u \cdot \sigma} \setminus S_{(\partial_i(u)) \cdot \sigma}$. Thus
\[
f : \Sigma \Lambda^{P'} \otimes_k k[x] \rightarrow (\Lambda^{Q}/\Lambda^{P}) \otimes_k k[x]
\]
is an isomorphism of chain complexes. By induction on $s$ we see that
\[
H_{q+1,*}(\Lambda^{*,*}[Q]/\Lambda^{*,*}[P] \otimes_k k[x], \tilde{d}) \cong H_{q,*}(\Lambda^{P'} \otimes_k k[x], d) = 0
\]
Proof. The connecting homomorphism
\[ \partial : H_{1,*} \left( \left( \Lambda[Q]/\Lambda[P] \right) \otimes_k k[x], \tilde{d} \right) \to H_{0,*} \left( \Lambda[P] \otimes_k k[x], \tilde{d} \right) = k[x]/I_P \]
is given by \( \partial([\sigma \otimes x^a]) = -[x_\sigma x^a] \in k[x]/I_P \).

Fix an \( m \)-vector \( \mathbf{c} \), we see that \( H_{1,*} \left( \left( \Lambda[Q]/\Lambda[P] \right) \otimes_k k[x], \tilde{d} \right) \) is generated by \( [\sigma \otimes x^a] \) with an unique \( m \)-vector \( a \) such that \( x_\sigma x^a = x^c \). If
\[ \partial([\sigma \otimes x^a]) = -[x_\sigma x^a] = 0 \in H_{0,*} \left( \Lambda[P] \otimes_k k[x], \tilde{d} \right) = k[x]/I_P, \]
then in the \( m \)-graded \( k \)-module \( k[x] \), the monomial \( x_\sigma x^a \in I_P \). This implies that there is a \( \sigma_k \in P \) such that \( x_\sigma x^a = x_{\sigma_k} x^b \). Thus \( x_\sigma x^a \) could be expressed as \( x_{\sigma_i \sigma_j} x^c \). Noticed that \( \sigma \cup \sigma_k = \sigma \cup \sigma_k' \) where \( \sigma_k' = \sigma_k \setminus \sigma \), one has \( x_\sigma x^a = x_{\sigma_i} x^c \) and
\[ \tilde{d}(\sigma_i' \sigma \otimes x^c) = -\sigma \otimes x_{\sigma_i} x^c = -\sigma \otimes x^a. \]
This implies that \( [\sigma \otimes x^a] = 0 \in H_{1,*} \left( \left( \Lambda[Q]/\Lambda[P] \right) \otimes_k k[x], \tilde{d} \right) \) and the connecting homomorphism \( \partial \) is a monomorphism. The theorem follows. \( \square \)

To describe the differential
\[ d : \Lambda^{q,*}[P] \otimes_k k[x] \otimes_{k[x]} k \to \Lambda^{q-1,*}[P] \otimes_k k[x] \otimes_{k[x]} k \]
let \( \sigma \) be a subset of \([m]\) and let \( \Lambda^{*,*}[P] \) be the submodules of \( \Lambda[P] \) generated by the monomials \( \mathbf{u} = S_\mathbf{u} = \sigma. \) Then by Definition 2.4, \( \Lambda[P] \) becomes an \( \mathbb{N} \times 2^{[m]} \) graded module over \( k \) and
\[ \Lambda^{*,*}[P] = \bigoplus_{\mathbf{i} \in 2^{[m]}} \Lambda^{*,*}[P]. \]

Theorem 2.7 Let \( P = \{\sigma_1, \sigma_2, \ldots, \sigma_s\} \) be a simplicial complement and \( K_P \) be the corresponding simplicial complex by Definition 2.2. Let \((\Lambda^{*,*}[P], d)\) be the chain complex induced from the bi-graded exterior algebra on \( P \), the differential \( d : \Lambda^{*,*}[P] \to \Lambda^{q-1,*}[P] \) is given by
\[ d(\mathbf{u}) = \sum_{i} (-1)^i \partial_i(\mathbf{u}) \cdot \delta_{\partial_i(\mathbf{u})}, \]
where \( \delta_{\partial_i(\mathbf{u})} = 1 \) if the total subset \( S_\mathbf{u} = S_{\partial_i(\mathbf{u})} \) and \( \delta_{\partial_i(\mathbf{u})} = 0 \) if \( S_\mathbf{u} \neq S_{\partial_i(\mathbf{u})} \). Then the homology of \( (\Lambda^{*,*}[P], d) \) is
\[ H_{*,\sigma}(\Lambda^{*,*}[P], d) = Tor_{*,\sigma}^{k[x]}(k[x]/I_P, k) = Tor_{*,\sigma}^{k[x]}(k(K_P), k) \]
which is called the homology of the simplicial complement \( P \).

Proof. Consider the chain complex \((\Lambda^{*,*}[P] \otimes_k k[x] \otimes_{k[x]} k, \tilde{d} \otimes \mathbf{1})\). From (2.5), we see that for a generator \( \mathbf{u} \otimes 1 \) of \( \Lambda^{*,*}[P] \otimes_k k[x] \),
\[ \tilde{d}(\mathbf{u} \otimes 1) = \sum_{i=1}^{q} (-1)^i \partial_i(\mathbf{u}) \otimes x_{(S_\mathbf{u} \setminus S_{\partial_i(\mathbf{u})})} \]
and $x_i(s_{u \setminus S_{\partial_i(u)}}) = 1$ if and only if $S_u = S_{\partial_i(u)}$. Notice that, $k[x]$ acts on $k$ by sending all $x_i$'s to 0, the theorem follows from

$$d(u \otimes_k 1 \otimes_k [x]) = \frac{d(u \otimes_k 1 \otimes_k [x])}{1 = \sum_{i=1}^{q} (-1)^i \partial_i(u) \otimes_k x(s_{u \setminus S_{\partial_i(u)}}) \otimes_k [x] 1$$

and

$$\partial_i(u) \otimes_k x(s_{u \setminus S_{\partial_i(u)}}) \otimes_k [x] 1 = 0 \in \Lambda^{q-1,*}[P] \otimes_k k[x] \otimes_k [x] k$$

if $x_{u \setminus S_{\partial_i(u)}} \neq 1$, that is $S_{\partial_i(u)} \neq S_u$. \hfill $\Box$

Fix a $\sigma \in 2^{[m]}$ and let $\Lambda^{*,\sigma}[P]$ be the submodule generated by the elements $u$ with total subset $S_u = \sigma$. From the definition of $\delta_{\partial_i(u)}$ we see that $\delta_i(u) \cdot \delta_{\partial_i(u)} \in \Lambda^{*,\sigma}[P]$, otherwise $\delta_i(u) = 0$. Then $(\Lambda^{*,\sigma}[P], d)$ is a sub-complex of $\Lambda^{*,*}[P]$ and

$$Tor_{\ast,\sigma}^{k[x]}(k[x]/I_{K_{P}}, k) = H_{q}(\Lambda^{*,\sigma}[P], d)$$

A precise expression of the Hochster theorem (cf. [5]) is:

**Theorem:** (Baskakov) There are isomorphisms

$$Tor_{\ast,\sigma}^{k[x]}(k, k[x]/I_{K_{P}}, k) \cong H^{q(\Lambda^{*,\sigma}[P]), d} \cong H^{m-q(\Lambda^{*,\sigma}[P]), d}$$

where $K_{P} \cap \sigma$ is the full subcomplex of $K_{P}$ consisting of all simplices of $K_{P}$ which have all of their vertices in $\sigma$, that is $K_{P} \cap \sigma = \{ \tau \cap \sigma | \tau \in K_{P} \}$.

Thus we have the following theorem:

**Theorem 2.8** (Combinatorial Hochster theorem) Let $P$ and $K_{P}$ be as above, then

$$H_{q,\sigma}(\Lambda^{*,\sigma}[P], d) \cong H^{q(\Lambda^{*,\sigma}[P]), d} \cong H^{m-q(\Lambda^{*,\sigma}[P]), d}$$

Furthermore the cohomology module of the classical moment-angle complex is

$$H^{2j-q(\mathbb{Z}_{K_{P}}, k)} = \bigoplus_{\sigma \subset [m], |\sigma| = j} H_{q,\sigma}(\Lambda^{*,\sigma}[P], d).$$

**Remark:** Here we use the agreement that the cohomology of the empty simplicial complex $\{\phi\}$ is $H^{-1}(\{\phi\}, k) = k$.

**Proof.** From Theorem 2.7, we see that $H_{q,\sigma}(\Lambda^{*,\sigma}[P], d) = Tor_{q,\sigma}^{k[x]}(k(K_{P}), k)$. Then from

$$Tor_{q,\sigma}^{k[x]}(k(K_{P}), k) = Tor_{q,\sigma}^{k[x]}(k, k(K_{P}))$$

we get the first one. The second one follows from the Hochster Theorem (cf. [5] for example). \hfill $\Box$

3. The algebraic structure on $Tor_{q,\sigma}^{k[x]}(k(K_{P}), k)$

Let $A$ and $B$ be $k[x]$ algebras with structure maps

$$\varphi : A \otimes_k [x] A \to A,$$

$$\varphi' : B \otimes_k [x] B \to B.$$

Choose a free resolution $P$ of $A$,

$$P : \cdots \longrightarrow P_{q} \longrightarrow P_{q-1} \longrightarrow \cdots \longrightarrow P_{1} \longrightarrow P_{0} \longrightarrow A.$$
The tensor product $P \otimes P$ is a free resolution of $A \otimes_{k[x]} A$. The structure map $\varphi : A \otimes_{k[x]} A$ gives a chain map $\varphi : P \otimes_{k[x]} P \to P$ which is unique up to chain homotopy.

Applying the functor $\otimes_{k[x]} (B \otimes_{k[x]} B)$, we get a natural chain map

$$
\begin{array}{c}
\varphi \\
\downarrow \\
P = \cdots \to (P \otimes P)_{q} \to (P \otimes P)_{q-1} \to \cdots \to (P \otimes P)_{1} \to (P \otimes P)_{0} \to A, \\
\end{array}
$$

On the other hand, there are natural maps

$$
\begin{array}{c}
\varphi' \\
\downarrow \\
P = \cdots \to P_{q} \to P_{q-1} \to \cdots \to P_{1} \to P_{0} \to A.
\end{array}
$$

Applying the natural chain map $\times : (P \otimes_{k[x]} B) \otimes_{k[x]} (P \otimes_{k[x]} B) \to P \otimes_{k[x]} B$, we get the algebraic structure

$$
\begin{array}{c}
\varphi \otimes 1 \\
\downarrow \\
P \otimes_{k[x]} (B \otimes_{k[x]} B) \to P \otimes_{k[x]} B.
\end{array}
$$

Notice that both $k(K_{P})$ and $k$ are $k[x]$ algebras, $Tor^{k[x]}_{i,*}(k(K_{P}), k)$ is an algebra in a natural way and the following isomorphism of algebras holds:

$$
H^{*}(Z_{K_{P}}, k) \cong Tor^{k[x]}_{i,*}(k(K_{P}), k).
$$

To describe the algebraic structure of $Tor^{k[x]}_{i,*}(k(K_{P}), k)$, consider the free $k[x]$-resolution of $k(K_{P})$ given by Theorem 2.6, which is denoted by $R^{*,*}$

$$
R^{*,*} = \Lambda^{*,*}[P] \otimes_{k} k[x].
$$

The tensor product

$$
R^{*,*} \otimes_{k[x]} R^{*,*} \cong (\Lambda^{*,*}[P] \otimes_{k} \Lambda^{*,*}[P]) \otimes_{k} (k[x] \otimes_{k[x]} k[x]) \cong (\Lambda^{*,*}[P] \otimes_{k} \Lambda^{*,*}[P]) \otimes_{k} k[x]
$$

gives a free resolution of $k(K_{P}) \otimes_{k[x]} k(K_{P}) \cong k(K_{P})$. The algebraic structure of $k(K_{P})$ is given by

$$
k(K_{P}) \otimes_{k[x]} k(K_{P}) \cong k(K_{P}) \to \cdots \to k(K_{P}).
$$
Thus we need to construct a chain map \( \varphi : (\Lambda^\ast \ast \mathbb{P}) \otimes_k \Lambda^\ast \ast \mathbb{P}) \otimes_k k[x] \to \Lambda^\ast \ast \mathbb{P}) \otimes_k k[x] \) that makes the following diagram commute:

\[
\begin{array}{ccc}
(\Lambda^\ast \ast \mathbb{P}) \otimes_k \Lambda^\ast \ast \mathbb{P}) \otimes_k k[x] & \xrightarrow{\varphi} & \Lambda^\ast \ast \mathbb{P}) \otimes_k k[x] \\
\varepsilon & & \varepsilon \\
k(K_P) \otimes_k k[x] & \xrightarrow{1} & k(K_P)
\end{array}
\]

**Construction 3.4** The exterior algebra \( \Lambda^\ast \ast \mathbb{P}) \) has natural product structure. Given two generators \( u \) and \( v \) of \( \Lambda^\ast \ast \mathbb{P}) \) the total subset of \( u \cdot v \) is \( S_{u \cdot v} = S_u \cup S_v \) if \( u \cdot v \neq 0 \) in \( \Lambda^\ast \ast \mathbb{P}) \). Thus the monomial \( x_{S_{u \cdot v}} \) is a factor of \( x_{S_u} \cdot x_{S_v} \) in \( k[x] \). We define the product

\[
\varphi : (\Lambda^\ast \ast \mathbb{P}) \otimes_k \Lambda^\ast \ast \mathbb{P}) \otimes_k k[x] \to \Lambda^\ast \ast \mathbb{P}) \otimes_k k[x]
\]

to be the \( k[x] \)-module map induced by

\[
(3.5) \quad u \times v = u \cdot v \otimes \frac{x_{S_u} \cdot x_{S_v}}{x_{S_{u \cdot v}}}.
\]

It is apparent that so defined product keeps the second degree \((\mathbb{N}^m)\) degree).

**Theorem 3.6** The product defined in Construction 3.4 is a chain map. Thus the algebraic structure in \( H_\ast \ast (\Lambda^\ast \ast \mathbb{P}), d \) \( \cong \text{Tor}_{k[x]}^k(k(K_P), k) \) is given by

\[
[c] \times [c'] = [c \cdot c'] \cdot \delta_{\sigma, \sigma'},
\]

where \([c] \in H_{q, \sigma}(\Lambda^\ast \ast \mathbb{P}), d \), \([c'] \in H_{j, \sigma'}(\Lambda^\ast \ast \mathbb{P}), d \), the cycle \( c \cdot c' \) is product of \( c \) and \( c' \) in \( \Lambda^\ast \ast \mathbb{P}) \)

and \( \delta_{\sigma, \sigma'} = 1 \) if \( \sigma \cap \sigma' = \phi \), \( \delta_{\sigma, \sigma'} = 0 \) if \( \sigma \cap \sigma' \neq \phi \).

**Proof.** Notice that \( \Lambda^0 \ast \ast \mathbb{P}) = k \), it is apparent that

\[
(\Lambda^0 \ast \ast \mathbb{P}) \otimes_k \Lambda^0 \ast \ast \mathbb{P}) \otimes_k k[x] \xrightarrow{\varepsilon} k(K_P) \otimes_k k[x] \xrightarrow{1} k(K_P)
\]

commute.

To prove that \( \varphi \) defined by Construction 3.4 is a chain map, we only need to check that

\[
d(u \times v) = d(u) \times v + (-1)^q u \times d(v)
\]

or equivalently to check that \( \Lambda^\ast \ast \mathbb{P}) \otimes_k k[x] \) is a differential graded algebra, because the differential in \( \Lambda^\ast \ast \mathbb{P}) \otimes_k \Lambda^\ast \ast \mathbb{P}) \otimes_k k[x] \) is given by \( d(u \otimes v) = d(u) \otimes v + (-1)^q u \otimes d(v) \) if \( u \in \Lambda^q \ast \ast \mathbb{P}) \). 

Let \( V \) be the free \( k \)-module generated by \( \mathbb{P} = \{ \sigma_1, \sigma_2, \ldots, \sigma_s \} \) and let \( T(V) \) be the tensor algebra of \( V \), that is \( T(V) = \bigoplus_{q \geq 0} V \otimes^q \). We still use

\[
u = \sigma_{k_1} \sigma_{k_2} \cdots \sigma_{k_q} \quad \text{and} \quad S_u = \sigma_{k_1} \cup \sigma_{k_2} \cup \cdots \cup \sigma_{k_q}
\]

to denote the generator of \( T(V) \) and the total subset of \( u \) respectively. The exterior algebra

\[
\Lambda^\ast \ast \mathbb{P}) = T(V)/(uv - (-1)^qvu, uu)
\]

where \((uv - (-1)^qvu, uu)\) is the ideal generated by \(uv - (-1)^qvu\) and \(uu\) for any generators \( u \in T^q \ast \ast (V)\) and \( v \in T^p \ast (V)\).
Similarly, the monomial $x_{s_u} \in k[x]$ corresponding to the total subset $s_u$ is expressed as $x_{s_u} = x^{a_u}$ for a unique $\mathbb{N}^m$ vector $a_u$. We also define

$$bideg(u) = (q, a_u)$$
$$\partial_i(u) = \sigma_{k_1} \cdots \sigma_{k_i} \cdots \sigma_{k_q}.$$  

Consider the $\mathbb{N} \times \mathbb{N}^m$ graded free $k[x]$-module $T(V) \otimes_k k[x]$, we define a differential

$$\tilde{d} : T^{q,*}(V) \otimes_k k[x] \to T^{q-1,*}(V) \otimes_k k[x]$$

by setting

$$\tilde{d}(u \otimes x^a) = \sum_i (-1)^i \partial_i(u) \otimes \frac{x_{s_u}}{x_{s_{\partial_i(u)}}} \cdot x^a$$

as (2.5).

Similar to Construction 3.4, $T(V) \otimes_k k[x]$ has algebraic structure given by

$$u \times v = u \cdot v \otimes \frac{x_{s_u} \cdot x_{s_v}}{x_{s_{uv}}}.$$  

From

$$\tilde{d}(u \times v) = \tilde{d}(uv) \otimes \frac{x_{s_u} \cdot x_{s_v}}{x_{s_{uv}}}$$

$$= \sum_i (-1)^i \partial_i(u) \otimes \frac{x_{s_{uv}}}{x_{s_{\partial_i(u)v}}} \cdot \frac{x_{s_u} \cdot x_{s_v}}{x_{s_{uv}}} + (-1)^q \sum_i (-1)^i u \partial_i(v) \otimes \frac{x_{s_{uv}}}{x_{s_{uv}}} \cdot \frac{x_{s_u} \cdot x_{s_v}}{x_{s_{v}}},$$

and

$$\tilde{d}(u) \times v + (-1)^q u \times \tilde{d}(v)$$

$$= \sum_i (-1)^i \partial_i(u) \times v \otimes \frac{x_{s_u}}{x_{s_{\partial_i(u)}}} + (-1)^q \sum_i (-1)^i u \partial_i(v) \otimes \frac{x_{s_u}}{x_{s_{\partial_i(v)}}}$$

$$= \sum_i (-1)^i \partial_i(u) \otimes \frac{x_{s_{uv}}}{x_{s_{\partial_i(u)v}}} \cdot \frac{x_{s_u}}{x_{s_{\partial_i(u)}}} \cdot \frac{x_{s_v}}{x_{s_{\partial_i(v)}}} + (-1)^q \sum_i (-1)^i u \partial_i(v) \otimes \frac{x_{s_u} \cdot x_{s_{\partial_i(v)}}}{x_{s_{uv}}} \cdot \frac{x_{s_v}}{x_{s_{\partial_i(v)}}}$$

we see that $\tilde{d}(u \times v) = \tilde{d}(u) \times v + (-1)^q u \times \tilde{d}(v)$ and $T(V) \otimes_k k[x]$ is a differential graded algebra. It is easy to check that the ideal $(uv - (-1)^q v u, uu)$ is invariant under the differential $\tilde{d}$. Thus $\{T(V) \otimes_k k[x], \tilde{d}\}$ induces the differential graded algebraic structure on $\Lambda^*[\mathbb{P}] \otimes_k k[x]$ as desired.

To prove the second part of the Theorem, applying the functor $\otimes_k k[x]$, we see that in $(\Lambda^*[\mathbb{P}] \otimes_k k[x]) \otimes_k k$,

$$(u \times v \otimes k_1) \otimes_k k_1 = (u \cdot v \otimes k_1) \otimes_k k_1 \frac{x_{s_u} \cdot x_{s_v}}{x_{s_{uv}}} = 0$$

if the monomial $\frac{x_{s_u} \cdot x_{s_v}}{x_{s_{uv}}} \neq 1$ or equivalently $s_u \cap s_v \neq \phi$.  

**Example:** Compute the $Tor^k_{\mathbb{P}}(k[K], k)$ and the cohomology of $Z_K$ corresponding to $K$ (See Figure 1)

There is the simplicial complement $\mathbb{P} = \{\sigma_1 = \{1, 5\}, \sigma_2 = \{2, 4\}, \sigma_3 = \{1, 2, 3\}, \sigma_4 = \{3, 4, 5\}\}$ corresponding to the simplicial complex $K = K_{\mathbb{P}}$.  

From Theorem 2.7, we compute the differentials in $\Lambda^*[\mathbb{P}]$:
\[
d(\sigma_1 \sigma_2 \sigma_3 \sigma_4) = -\sigma_2 \sigma_3 \sigma_4 + \sigma_1 \sigma_3 \sigma_4 - \sigma_1 \sigma_2 \sigma_4 + \sigma_1 \sigma_2 \sigma_3;
\]
\[
d(\sigma_1 \sigma_3 \sigma_4) = -\sigma_3 \sigma_4 .
\]
And these are all the non-trivial differentials. Thus the $Tor_{\ast, \ast}^k(k(K), k)$ is a free $k$-module generated by
\[
\begin{cases}
\sigma_1 \sigma_2 & 1 \\
\sigma_1 \sigma_3 & 2 \\
\sigma_1 \sigma_4 & 3 \\
\sigma_2 \sigma_4 & 4
\end{cases}
\]

Consider the algebraic structure of $Tor_{\ast, \ast}^k(k(K), k)$, we see from Theorem 3.6 that
\[
\sigma_1 \times \sigma_2 = \sigma_1 \sigma_2
\]
is the only non-trivial product. By the Hochester Theorem, we see that $H^*(\mathbb{Z}_K, k)$ is a free $k$-module with Poincaré series $1 + 2x^3 + 2x^5 + 5x^6 + 2x^7$.

4. THE SIMPLICIAL COMPLEMENTS AND THE SIMPLICIAL COMPLEXES

In [7] X. Cao and Z. Lü introduced a $\mathbb{Z}/2$-algebra
\[
2^{[m]^*} = \{ f : 2^{[m]} \to \mathbb{Z}/2 = \{0, 1\} \}
\]
consists of all $\mathbb{Z}/2$ valued functions on the power set $2^{[m]}$ with addition $(f + g)(\sigma) = f(\sigma) + g(\sigma)$ and multiplication $(f \cdot g)(\sigma) = f(\sigma) \cdot g(\sigma)$.

Let $f : 2^{[m]} \to \mathbb{Z}/2$ be a function in $2^{[m]^*}$. We define
\[
supp(f) = \{ \sigma \in 2^{[m]} \mid f(\sigma) = 1 \} = f^{-1}(1) \subset 2^{[m]}
\]
which is called the support of $f$.

On the other hand, given a simplicial complex $K$, there is the characteristic function $f_K \in 2^{[m]^*}$ of $K$ defined by
\[
f_K(\sigma) = \begin{cases} 1 & \text{if } \sigma \in K \\ 0 & \text{otherwise} \end{cases},
\]
with $supp(f_K) = K$.

Given a simplex $\sigma \in 2^{[m]}$, there are functions $\delta_\sigma$ and $\mu_\sigma \in 2^{[m]^*}$ defined as
\[
\begin{align*}
\delta_\sigma(\tau) & = \begin{cases} 1 & \text{if } \tau = \sigma \\ 0 & \text{otherwise} \end{cases}, \\
\mu_\sigma(\tau) & = \begin{cases} 1 & \text{if } \sigma \subset \tau \\ 0 & \text{otherwise} \end{cases}
\end{align*}
\]
From (4.2), one can easily show that any \( f \in 2^{|m|}^* \) can be expressed as
\[
f = \sum_{\sigma \in 2^{|m|}} f(\sigma)\delta_\sigma = \sum_{\sigma \in \text{supp}(f)} \delta_\sigma.
\]
Thus \( \{\delta_\sigma \mid \sigma \in 2^{|m|}\} \) forms a basis of \( 2^{|m|}^* \).

**Theorem 4.3** Let \( \mathbb{P} = \{\sigma_1, \sigma_2, \cdots, \sigma_s\} \) be a simplicial complement, \( K_\mathbb{P} \) be the simplicial complex corresponding to \( \mathbb{P} \) by Definition 2.2 and \( f_{K_\mathbb{P}} \) be the characteristic function corresponding to \( K_\mathbb{P} \) by (4.1). Then
\[
f_{K_\mathbb{P}} = \prod_{i=1}^s (1 + \mu_{\sigma_i}) = \sum_{u \in \Lambda[\mathbb{P}]} \mu_{S_u} = \sum_{\sigma \in 2^{|m|}} \left( \sum_i \text{dim}_k \Lambda^{i,\sigma}[\mathbb{P}] \right) \mu_\sigma,
\]
where the first sum runs over all the \( k \)-module generators \( u \) of \( \Lambda^{*,*}[\mathbb{P}] \).

**Proof.** Notice that, \( 1(\tau) = 1 \) for any \( \tau \in 2^{|m|} \) and \( (1 + \mu_\sigma)(\tau) = 0 \) if and only if \( \sigma \subseteq \tau \), we see that for any \( \tau \in 2^{|m|} \)
\[
f_{K_\mathbb{P}}(\tau) = (1 + \mu_{\sigma_1})(\tau) \cdot (1 + \mu_{\sigma_2})(\tau) \cdots (1 + \mu_{\sigma_s})(\tau) = 0
\]
if and only if there exists an \( i \) such that \( (1 + \mu_{\sigma_i})(\tau) = 0 \). This is equivalent to that there is a \( \sigma_i \in \mathbb{P} \) such that \( \sigma_i \subseteq \tau \). By Definition 2.2, \( \tau \) is a non-face of \( K_\mathbb{P} \). Thus \( f_{K_\mathbb{P}} = \prod_{i=1}^s (1 + \mu_{\sigma_i}) \).

For the second part of the theorem, notice that for any subset \( \omega \in 2^{|m|} \), \( (\mu_\sigma \cdot \mu_\tau)(\omega) = \mu_\sigma(\omega) \cdot \mu_\tau(\omega) = 1 \) if and only if \( \sigma \subseteq \omega \) and \( \tau \subset \omega \iff \sigma \cup \tau \subset \omega \). One has
\[
\mu_\sigma \cdot \mu_\tau = \mu_{\sigma \cup \tau}.
\]
Thus
\[
\prod_{i=1}^s (1 + \mu_{\sigma_i}) = 1 + \sum_{\{i_1, i_2 \cdots i_k\} \subset \{s\}} \mu_{\sigma_{i_1}} \mu_{\sigma_{i_2}} \cdots \mu_{\sigma_{i_k}} = \sum_{u \in \Lambda[\mathbb{P}]} \mu_{S_u}
\]
by Definition 2.4.

Fix a \( \sigma \in 2^{|m|} \), \( \Lambda^{*,*}[\mathbb{P}] \) is the \( k \)-submodule generated by \( u \) with total subset \( S_u = \sigma \),
\[
\sum_{S_u = \sigma} \mu_{S_u} = \left( \sum_i \text{dim}_k \Lambda^{1,\sigma}[\mathbb{P}] \right) \mu_\sigma.
\]
The third part of the Theorem follows. \( \square \)

**Definition 4.4** Fix a subset \( \omega \in \{m\} \), there is the map \( \varepsilon_\omega : 2^{|m|} \to 2^{|m|} \) defined by \( \varepsilon_\omega(\tau) = \tau \cup \omega \). The \( \omega \)-compression \( E_\omega : 2^{|m|}^* \to 2^{|m|}^* \) is defined by
\[
E_\omega(f) = f \circ \varepsilon_\omega.
\]
For the simplicial complement \( \mathbb{P} = \{\sigma_1, \sigma_2, \cdots, \sigma_s\} \), we define the \( \omega \)-compression of \( \mathbb{P} \) by
\[
E_\omega \mathbb{P} = \{\sigma_1 \setminus \omega, \sigma_2 \setminus \omega, \cdots, \sigma_s \setminus \omega\}.
\]
From \( (f + g) \circ \varepsilon_\omega = f \circ \varepsilon_\omega + g \circ \varepsilon_\omega \) and \( (f \circ g) \circ \varepsilon_\omega = (f \circ \varepsilon_\omega) \cdot (g \circ \varepsilon_\omega) \), we see that \( E_\omega : 2^{|m|}^* \to 2^{|m|}^* \) is a homomorphism of \( Z/2 \)-algebra.
Lemma 4.5 Let \( P \) be a simplicial complement and \( f_{K_P} = \prod_{i=1}^s (1 + \mu_i) \) be the characteristic function of the simplicial complex \( K_P \) by Theorem 4.3. Then
\[
E_\omega(f_{K_P}) = \prod_{i=1}^s (1 + \mu_i | \omega) = f_{K_{E_P}},
\]
i.e. \( E_\omega(f_{K_P}) \) is the characteristic function of the simplicial complex \( K_{E_P} \) corresponding to the simplicial complement \( E_\omega P \).

Proof. The \( \omega \)-compression \( E_\omega : 2^m \to 2^m \) is a homomorphism of \( Z/2 \)-algebra. Thus
\[
E_\omega(f_{K_P}) = E_\omega \left( \prod_{i=1}^s (1 + \mu_i) \right) = \prod_{i=1}^s (E_\omega(1) + E_\omega(\mu_i)).
\]
For the function \( \mu \), take a \( \tau \in 2^m \), we have
\[
E_\omega(\mu)(\tau) = 1 \iff \mu(\varepsilon(\tau)) = 1 \iff \sigma \subset \tau \cup \omega \iff (\sigma \setminus \omega) \subset \tau.
\]
This implies that
\[
(4.6) \quad E_\omega(\mu) = \mu(\sigma \setminus \omega).
\]
Then from \( E_\omega(1) = 1 \), we see that
\[
E_\omega(f_{K_P}) = \prod_{i=1}^s (1 + E_\omega(\mu_i)) = \prod_{i=1}^s (1 + \mu_i | \omega) = f_{K_{E_P}}. \quad \Box
\]

For an arbitrary simplex \( \omega \in K \), define its link and star as the sub-complexes
\[
\text{star}_K \omega = \{ \tau \in K | \omega \cup \tau \in K \}; \quad \text{link}_K \omega = \{ \tau \in K | \omega \cup \tau \in K, \omega \cap \tau = \emptyset \}.
\]

Theorem 4.7 Let \( P \) be a simplicial complement and \( K_P \) be the simplicial complex corresponding to \( P \). Then:

1. The star, \( \text{star}_{K_P} \omega = K_{E_{P^*}} \), is the simplicial complex corresponding to the simplicial complement \( E_{\omega P} \). Thus
\[
H_{q,\sigma}(\Lambda^{*,\sigma}[E_{\omega P}], d) = T_{o}_{q,\sigma}[k](\text{star}_{K_P} \omega), k),
\]
2. The link, \( \text{link}_{K_P} \omega = K_{E_{P^*} \cap [m] \setminus \omega} \) and
\[
H_{q,\sigma}[m \setminus \omega](\Lambda^{*,\sigma}[E_{\omega P}], d) = \widetilde{H}^{m-|\omega|-q-1}(\text{link}_{K_P} \omega, k).
\]

Proof. Let \( f_{K_P} \) and \( f_{K_{E_P}} \) be the characteristic functions of \( K_P \) and \( K_{E_P} \) respectively. Then a simplex \( \tau \in \text{star}_{K_P} \omega \) if and only if
\[
\tau \cup \omega \in K_P \iff f_{K_P} (\tau \cup \omega) = 1 \iff f_{K_P} \circ e_\omega (\tau) = 1 \iff E_\omega(f_{K_P})(\tau) = f_{K_{E_P}}(\tau) = 1
\]
Thus the characteristic function of \( \text{star}_{K_P} \omega \) is \( f_{K_{E_P}} \) and
\[
\text{star}_{K_P} \omega = \{ \tau \in K_P | \omega \cup \tau \in K_P \} = K_{E_{P^*}},
\]
\[
\text{link}_{K_P} \omega = \{ \tau \in K_P | \omega \cup \tau \in K_P, \tau \cap \omega = \emptyset \} = K_{E_{P^*} \cap \{m\} \setminus \omega}.
\]
From Theorem 2.7 we see that
\[
T_{o}_{q,\sigma}[k](\text{star}_{K_P} \omega, k) = T_{o}_{q,\sigma}[k](K_{E_{P^*}}), k) = H_{q,\sigma}(\Lambda^{*,\sigma}[E_{\omega P}], d)
\]
and from Theorem 2.8 we get
\[
\widetilde{H}^{m-|\omega|-q-1}(\text{link}_{K_P} \omega, k) = \widetilde{H}^{m-|\omega|-q-1}(K_{E_{P^*}} \cap \{m\} \setminus \omega), k) = H_{q,\sigma}[m \setminus \omega](\Lambda^{*,\sigma}[E_{\omega P}], d).
\]
The theorem follows. \( \Box \)
5. THE COHOMOLOGY OF THE GENERALIZED MOMENT-ANGLE COMPLEX

In this section we consider the cohomology module of the generalized moment-angle complex. Recall from [2][3].

Definition 5.1 Let \( (X, A) = \{(X_i, A_i, x_i) | i \in [m]\} \) denote a set of triples of CW-complexes with base points \( x_i \in A_i \) and \( K_P \) be an abstract simplicial complex. The generalized moment-angle complex determined by \( (X, A) \) and \( K_P \) denoted by \( Z_{K_P}(X, A) \) is defined using the functor

\[ D : K_P \rightarrow CW_\ast \]

as follows: For every \( \omega \in K_P \), let

\[ D(\omega) = Y_1 \times Y_2 \times \cdots \times Y_m \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in \omega \\ A_i & \text{if } i \notin \omega. \end{cases} \]

The generalized moment-angle complex is

\[ Z_{K_P}(X, A) = \bigcup_{\omega \in K_P} D(\omega). \]

Let \( Y_1 \wedge Y_2 \wedge \cdots \wedge Y_m \) be the smash product given by the quotient space

\[ Y_1 \times Y_2 \times \cdots \times Y_m / S(Y_1 \times Y_2 \times \cdots \times Y_m) \]

where \( S(Y_1 \times Y_2 \times \cdots \times Y_m) \) is the subspace of the product with at least one coordinate given by the base-point \( x_i \in Y_i \). The generalized smash moment-angle complex is defined to be the image of \( Z_{K_P}(X, A) \) in \( X_1 \wedge X_2 \wedge \cdots \wedge X_m \), that is

\[ \hat{Z}_{K_P}(X, A) = \bigcup_{\omega \in K_P} \hat{D}(\omega) \]

where

\[ \hat{D}(\omega) = Y_1 \wedge Y_2 \wedge \cdots \wedge Y_m \quad \text{subject to} \quad Y_i = \begin{cases} X_i & \text{if } i \in \omega \\ A_i & \text{if } i \notin \omega. \end{cases} \]

Given a non-empty subset \( I = \{i_1, i_2, \ldots, i_k\} \subset [m] \) and a family of pairs \( (X_I, A_I) = \{(X_{i_j}, A_{i_j}) | i_j \in I\} \) which is the subfamily of \( (X, A) \) determined by \( I \). It is known from [2][3] that:

\[ \sigma : \Sigma(Z_{K_P}(X, A)) \rightarrow \Sigma \left( \bigvee_{I \subset [m]} \hat{Z}_{K_P \cap I}(X_I, A_I) \right) \]

is a natural pointed homotopy equivalence. To describe it more precisely, let \( \omega \subset I \) and

\[ \hat{D}_I(\omega) = Y_{i_1} \wedge Y_{i_2} \wedge \cdots \wedge Y_{i_k} \quad \text{with} \quad Y_{i_j} = \begin{cases} X_{i_j} & \text{if } i_j \in \omega \\ A_{i_j} & \text{if } i_j \notin \omega. \end{cases} \]

Then

\[ \hat{Z}_{K_P \cap I}(X_I, A_I) = \bigcup_{\omega \in K_P \cap I} \hat{D}_I(\omega). \]

Associated to a simplicial complex \( K_P \), there is a partial ordered set (poset) \( K_P \) with point \( \sigma \in K_P \) corresponding to a simplex \( \sigma \in K_P \) and order given by reverse inclusion of simplices. Thus \( \sigma_1 \leq \sigma_2 \) in \( K_P \) if and only if \( \sigma_2 \subset \sigma_1 \) in \( K_P \). Given an \( \omega \in K_P \) there are further posets given by

\[ K_P < \omega = \{ \tau \in K_P | \tau \subset \omega \} \quad \text{and} \quad K_P \subseteq \omega = \{ \tau \in K_P | \tau \subseteq \omega \}. \]
Given a poset $P$, the order complex $\Delta(P)$ is the simplicial complex with vertices given by set of points of $P$ and $k$-simplices given by ordered $(k + 1)$-tuples $(p_0, p_1, \ldots, p_k)$ with $p_0 < p_1 < \cdots < p_k$. It follows that $\Delta(K) < \phi = K_\phi$ is the barycentric subdivision of $K$.

Given a simplicial complex $K$, we use $|K|$ to denote its geometric realization. The symbol $|K| * Y$ denotes the join of $K$ and $Y$. If $Y$ is a pointed CW-complex, then $|K| * Y$ has the homotopy type of $\Sigma|K| \wedge Y$.

**Theorem (Bahri, Bendersky, Cohen and Gitler [2, 3] 2.12)** Let $K$ be an abstract simplicial complex and $\overline{K}$ its associated poset. Let $(X, A) = \{(X_i, A_i, x_i)i \in [m]\}$ denote $m$ choices of connected, pointed pairs of CW-complexes, with the inclusion $A_i \hookrightarrow X_i$ homotopic to constant for all $i$. Then there is a homotopy equivalence

$$\tilde{Z}_K(X, A) \longrightarrow \bigvee_{\omega \in K} |\Delta(K \cap \omega)| \ast \tilde{D}(\omega).$$

From (5.2), we see that if $A_i \hookrightarrow X_i$ are null-homotopic for all $i$, then

$$\Sigma(\tilde{Z}_K(X, A)) \simeq \Sigma \left( \bigvee_{I \subset [m]} \tilde{Z}_{K \cap I}(X_I, A_I) \right) \simeq \Sigma \left( \bigvee_{I \subset [m]} \left( \bigvee_{\omega \in K \cap I} |\Delta(K \cap \omega)| \ast \tilde{D}(\omega) \right) \right).$$

Fix an $\omega \in K$, it is easy to see that $\omega \in K \cap I$ if and only if $\omega \subset I$. Thus $\Sigma(\tilde{Z}_K(X, A))$ has the homotopy type of

$$\left( \bigvee_{I \subset [m]} \left( \bigvee_{\omega \in K \cap I} |\Delta(K \cap \omega)| \ast \tilde{D}(\omega) \right) \right) = \left( \bigvee_{\omega \in K} \left( \bigvee_{\omega \subset I \subset [m]} |\Delta(K \cap \omega)| \ast \tilde{D}(\omega) \right) \right) \simeq \left( \bigvee_{\omega \in K} \left( \bigvee_{\omega \subset I \subset [m]} \Sigma|\Delta(K \cap \omega)| \wedge \tilde{D}(\omega) \right) \right).$$

**Remark 5.4** If $\omega$ is a maximum face of $K \cap I$ in the sense that $\omega \in K \cap I$ but it is not a proper subset of any other simplices, then $K \cap I_{<\omega} = \{\tau \in K \cap I \mid \omega \supset \tau\} = \emptyset$. The simplicial complex $\Delta(K \cap I_{<\omega}) = \{\emptyset\}$ is the empty simplicial complex. Here we use the agreement

$$|\{\phi\}| \ast \tilde{D}(\omega) = \tilde{D}(\omega) \quad \text{and} \quad \Sigma|\{\phi\}| \wedge \tilde{D}(\omega) = \tilde{D}(\omega).$$

Combine with the agreement in Theorem 2.8, $\tilde{H}^{-1}(|\{\phi\}|, k) = k$, we have

$$\tilde{H}^0(\Sigma|\{\phi\}|, k) = k \quad \text{and} \quad \tilde{H}^*(\Sigma|\{\phi\}| \wedge \tilde{D}(\omega), k) = \tilde{H}^*(\tilde{D}(\omega), k).$$

Consider the reduced cohomology of $\tilde{Z}_K(X, A)$, we see from (5.3) that:

$$\tilde{H}^* (\tilde{Z}_K(X, A), k) = \bigoplus_{\omega \in K} \left( \bigoplus_{\omega \subset I \subset [m]} \tilde{H}^* (\Sigma|\Delta(K \cap \omega)| \wedge \tilde{D}(\omega), k) \right).$$

Furthermore suppose that there is no Tor problem in the Künneth formulae for the cohomology of $|\Delta(K \cap \omega)| \wedge \tilde{D}(\omega)$ (For example, take $k$ to be a field or suppose that $\tilde{H}^*(X_i, k)$ and
$\tilde{H}^*(A_i, k)$ are free $k$-modules for all $i$. Then from
\[
\tilde{D}_I(\omega) = Y_i \wedge Y_{i_2} \wedge \cdots \wedge Y_{i_k} \simeq \left( \bigwedge_{i \in \omega} X_i \right) \wedge \left( \bigwedge_{j \in \mathbb{N} \setminus \omega} A_j \right),
\]
\[
H^*(\tilde{D}_I(\omega), k) = \left( \bigotimes_{i \in \omega} \tilde{H}^*(X_i, k) \right) \otimes \left( \bigotimes_{j \in \mathbb{N} \setminus \omega} \tilde{H}^*(A_j, k) \right)
\]
which is a free $k$-module, we have
\[
H^*\left(\Sigma|\Delta(K_P \cap I_{<\omega})| \cap \tilde{D}_I(\omega), k\right) = H^*\left(\Sigma|\Delta(K_P \cap I_{<\omega})|, k\right) \otimes \tilde{H}^*\left(\tilde{D}_I(\omega), k\right)
\]
then we have
\[
H^*\left(\Sigma|\Delta(K_P \cap I_{<\omega})|, k\right) = \tilde{H}^*\left(\Sigma|\Delta(K_P \cap I_{<\omega})|, k\right) \otimes \left( \bigotimes_{i \in \omega} \tilde{H}^*(X_i, k) \right) \otimes \left( \bigotimes_{j \in \mathbb{N} \setminus \omega} \tilde{H}^*(A_j, k) \right)
\]
\[(5.6) \quad \tilde{H}^*\left(\Sigma|\Delta(K_P \cap I_{<\omega})| \cap \tilde{D}_I(\omega), k\right) = \tilde{H}^*\left(\Sigma|\Delta(K_P \cap I_{<\omega})|, k\right) \otimes \tilde{H}^*\left(\tilde{D}_I(\omega), k\right)
\]
\[
= \tilde{H}^*\left(\Sigma|\Delta(K_P \cap I_{<\omega})|, k\right) \otimes \left( \bigotimes_{i \in \omega} \tilde{H}^*(X_i, k) \right) \otimes \left( \bigotimes_{j \in \mathbb{N} \setminus \omega} \tilde{H}^*(A_j, k) \right).
\]

**Theorem 5.7** Let $K_P$ be an abstract simplicial complex corresponding to a simplicial complement $\mathbb{P}$ and let $(X, A) = \{(X_i, A_i, x_i)| i \in [m]\}$ denote $m$ choices of connected, pointed pairs of CW-complexes, with the inclusion $A_i \hookrightarrow X_i$ homotopic to constant for all $i$. If $\tilde{H}^*(X_i, k)$ and $\tilde{H}^*(A_i, k)$ are free $k$-modules for all $i$, then the cohomology of $Z_{K_P}(X, A)$ is isomorphic to
\[
H^*(Z_{K_P}(X, A), k) = \bigoplus_{\omega \in K_P} \left( \bigoplus_{\tau} H_{*, \tau}(\Lambda^*, [E_\omega \mathbb{P}], d) \otimes \left( \bigotimes_{i \in \omega} \tilde{H}^*(X_i, k) \right) \otimes \left( \bigotimes_{j \in \mathbb{N} \setminus \omega} \tilde{H}^*(A_j, k) \right) \right)
\]
as $k$-modules, where
\[
H_{*, \tau}(\Lambda^*, [E_\omega \mathbb{P}], d) \cong Tor^{k[\mathbb{P}]}_{*, \tau}(k(\text{star}_{K_P} \omega), k)
\]
subject to $\tau \subset [m] \setminus \omega$.

**Proof.** Given an $\omega \subset I$, from its definition we see that the posets
\[
\text{link}_{K_P \cap I} \omega = \{\tau \in K_P \cap I| \omega \subset \tau \} = \{\tau \in K_P|\omega \subset \tau \subset I\}
\]
and
\[
\text{link}_{K_P \cap I} \omega = \{\tau' \in K_P \cap I| \omega \cup \tau' \in K_P \cap I, \tau' \cap \omega = \emptyset\}
\]
\[
= \{\tau' \in K_P|\omega \cup \tau' \in K_P, \tau' \cap \omega = \emptyset, \text{and } \tau' \subset I\} = (\text{link}_{K_P} \omega) \cap I_{<\emptyset}.
\]
There is a one to one correspondence between the posets $\Psi: K_P \cap I_{<\omega} \to (\text{link}_{K_P} \omega) \cap I_{<\emptyset}$ given by $\Psi(\tau) = \tau \setminus \omega$. Thus the order complex $\Delta(K_P \cap I_{<\omega}) = (\text{link}_{K_P} \omega) \cap I_{<\emptyset}$ is the barycentric subdivision of $(\text{link}_{K_P} \omega) \cap I$ and then
\[
(5.8) \quad \tilde{H}^*\left(\Sigma|\Delta(K_P \cap I_{<\omega})|, k\right) = \tilde{H}^{*-1}\left(\Sigma|\Delta(K_P \cap I_{<\omega})|, k\right) = \tilde{H}^{*-1}(\text{link}_{K_P} \omega) \cap I, k).
\]

Recall from Theorem 4.7 that $\text{link}_{K_P} \omega = K_{E_p \mathbb{P}} \cap ([m] \setminus \omega)$, we have
\[
(\text{link}_{K_P} \omega) \cap I = K_{E_p \mathbb{P}} \cap ([m] \setminus \omega) \cap I = K_{E_p \mathbb{P}} \cap (I \setminus \omega).
\]
Then from Theorem 2.8 and (5.8), we have
\[
H_{q, I \setminus \omega}(\Lambda^*, [E_\omega \mathbb{P}], d) = \tilde{H}^{I \setminus \omega}_{q-1}\left(\text{link}_{K_P} \omega \cap I, k\right) = \tilde{H}^{I \setminus \omega}_{q-1}\left(\Sigma|\Delta(K_P \cap I_{<\omega})|, k\right).
\]
From the definition of $E_\omega \mathbb{P} = \{\sigma_1 \setminus \omega, \sigma_2 \setminus \omega, \cdots, \sigma_s \setminus \omega\}$, we see that total subset $S_u$ of any generator $u \in \Lambda^* \mathbb{P}_I$ is contained in $[m] \setminus \omega$. Thus the homology of the simplicial complement $H_{q, \tau}(\Lambda^*, [E_\omega \mathbb{P}], d)$ is concentrated in $\tau \subset [m] \setminus \omega$. Denote the non-empty set $I$ by $\tau \cup \omega$ for any $\omega \subset I \subset [m]$, we have
\[
H_{q, \tau}(\Lambda^*, [E_\omega \mathbb{P}], d) = \tilde{H}^{I \setminus \omega}_{q-1}\left(\Sigma|\Delta(K_P \cap (I \cup \omega)_{<\omega})|, k\right).
\]
Apply this formula to (5.6) and (5.5) we get

\[
\widetilde{H}^*(Z_{K_P}(\underline{X}, \underline{A}), k) = \bigoplus_{\omega \in K_P} \left( \bigoplus_{\tau \subset \omega} \tilde{H}^* \left( \Sigma \Delta(K_P \cap (\omega \cup \tau) \rangle \right) \right)
\]

where \( I = \omega \cup \tau \neq \phi \).

The theorem follows from \( H^0(Z_{K_P}(\underline{X}, \underline{A}), k) = k \) and the agreement by taking \( I = \omega = \tau = \phi \) to be the empty set

\[
H_{0, \phi}(\Lambda^* \phi[E\omega P], d) \otimes \left( \bigotimes_{i \in \phi} \tilde{H}^*(X_i, k) \right) \otimes \left( \bigotimes_{j \in \phi} \tilde{H}^*(A_j, k) \right) = k.
\]

\[\Box\]

Remark: If the abstract simplicial complex \( K_P \) is given by an abstract simplicial complement \( P \), we might not know if a subset \( \omega \) is a simplex of \( K_P \) or not. This is not a problem, because while \( \omega \notin K_P \), the empty set \( \phi \) is an element of \( E\omega P \) and \( H_{r,*}(\Lambda^* \phi[E\omega P], d) = 0 \). Thus Theorem 5.7 could be written as

\[
H^*(Z_{K_P}(\underline{X}, \underline{A}), k)
\]

(5.9) = \[\bigoplus_{\omega \in [m]} \left( \bigoplus_{\tau \subset [m]} H_{r, \tau}(\Lambda^* \tau[E\omega P], d) \otimes \left( \bigotimes_{i \in \omega} \tilde{H}^*(X_i, k) \right) \otimes \left( \bigotimes_{j \in \tau} \tilde{H}^*(A_j, k) \right) \right) \]

Corollary 5.10 If all the \( A_i \) are contractible and \( \tilde{H}^*(X_i, k) \) are free \( k \)-modules. Then

\[
H^*(Z_{K_P}(\underline{X}, \underline{A}), k) = \bigoplus_{\omega \in K_P} \left( \bigotimes_{i \in \omega} \tilde{H}^*(X_i, k) \right)
\]

Proof. If all the \( A_i \) are contractible then \( \bigotimes_{j \in \tau} \tilde{H}^*(A_j, k) = 0 \) for any non-empty set \( \tau \). The corollary follows from \( H_{0, \phi}(\Lambda^* \phi[E\omega P], d) = k \) if \( \omega \notin K_P \). \[\Box\]

Corollary 5.11 If all the \( X_i \) are contractible and \( \tilde{H}^*(A_i, k) \) are free \( k \)-modules. Then

\[
H^*(Z_{K_P}(\underline{X}, \underline{A}), k) = \bigoplus_{\tau} \left( H_{r, \tau}(\Lambda^* \tau[P], d) \otimes \left( \bigotimes_{j \in \tau} \tilde{H}^*(A_j, k) \right) \right).
\]

Furthermore take \( X_i = D^2 \) and \( A_i = S^1 \) for all \( i \),

\[
H^*(Z_{K_P}, k) = H^2|_{\tau = q}(Z_{K_P}(D^2, S^1), k) = \bigoplus_{2|\tau - q = r} H_{q, \tau}(\Lambda^* \tau[P], d),
\]

where \( H_{q, \tau}(\Lambda^* \tau[P], d) = Tor^k_{q, \tau}(k(K_P), k) \).
Proof. If all the $X_i$ are contractible, then $\bigotimes_{i \in \omega} \tilde{H}^*(X_i, k) = 0$ for any non-empty set $\omega$. The corollary follows from

$$H^*(\mathcal{Z}_K(\mathcal{A}, \mathcal{A}), k) = \bigoplus_{\tau} \left( \bigoplus_{\omega \in K_{\mathcal{P}}} \left( \bigotimes_{i \in \omega} \tilde{H}^*(X_i, k) \right) \otimes \left( \bigotimes_{j \in \tau} \tilde{H}^*(A_j, k) \right) \right)$$

and the agreement $\bigotimes_{i \in \phi} \tilde{H}^*(X_i, k) = k$. \hfill \Box

**Proposition 5.12** Let $X_i = S^2$ and $A_i = S^1$ all $i$. Then

$$H^r(\mathcal{Z}_K(S^2, S^1), k) = \bigoplus_{\omega \in K_{\mathcal{P}}} \left( \bigoplus_{\tau \subset [m]} \tilde{H}^*(\sum_{|\omega| + |\tau| + 1} |\Delta(K_{\mathcal{P}} \cap (\omega \cup \tau)_{\leq \omega}|, k) \right).$$

Proof. Noticed that $\hat{D}_{\omega \cup \tau}(\omega)$ is the $2|\omega| + |\tau|$ sphere if $\omega \cap \tau = \phi$,

$$\hat{D}_{\omega \cup \tau}(\omega) = \left( \bigwedge_{i \in \omega} S^2 \right) \wedge \left( \bigwedge_{j \in \tau} S^1 \right) = S^{2|\omega| + |\tau|},$$

we see from (5.5) that

$$H^r(\mathcal{Z}_K(\mathcal{X}, \mathcal{A}), k) = \bigoplus_{\omega \in K_{\mathcal{P}}} \left( \bigoplus_{\tau \subset [m]} \tilde{H}^*(\sum_{|\omega| + |\tau| + 1} |\Delta(K_{\mathcal{P}} \cap (\omega \cup \tau)_{\leq \omega}|, k) \right)$$

including $\omega = \tau = \phi$. The result follows from

$$\tilde{H}^{r-q}(\sum_{|\Delta(K_{\mathcal{P}} \cap (\omega \cup \tau)_{\leq \omega})|}, k) = H_q(\Lambda^*[E_{\omega}^0], d).$$

\hfill \Box

We finish this paper by giving an example.

**Example 5.13** Let $m = 6$ and the simplicial complement $\mathcal{P} = \{\sigma_1 = \{1, 2\}, \sigma_2 = \{3, 4\}, \sigma_3 = \{5, 6\}\}$. The corresponding simplicial complex is a triangulation of the sphere $S^2$ (see Finger 2)

![Finger 2](image)

The cohomology ring of $\mathcal{Z}_K = \mathcal{Z}_K(\mathcal{P}, \mathcal{S}^1)$ could be easily got from the homology of the simplicial complement $\mathcal{P} = \{\sigma_1 = \{1, 2\}, \sigma_2 = \{3, 4\}, \sigma_3 = \{5, 6\}\}$,

$$H_*(\Lambda^*[\mathcal{P}], d) \cong \Lambda^*[\{\sigma_1, \sigma_2, \sigma_3\}]$$
as an exterior algebra over $k$. Thus the cohomology ring $H^*(\mathbb{Z}_K, k)$ is the exterior algebra on $\mathbb{P}$ with non-trivial products. The Poincaré series of $H^*(\mathbb{Z}_K, k)$ is $1 + 3x^3 + 3x^6 + x^9$ and the total Betti number of $\mathbb{Z}_K$ is 8.

Consider the cohomology of $\mathbb{Z}_{K_5}(S^2, S^1)$. We start from computing the homology of the simplicial complement $E_\omega \mathbb{P}$ with $\omega \in K$:

1. Take $\omega = \phi$, the homology of the simplicial complement $E_\phi \mathbb{P}$ is $\Lambda^{*}\{[\sigma_1, \sigma_2, \sigma_3]\}$. Thus the submodule

$$\bigoplus_{\tau} H_{*,\tau}(\Lambda^{*}\{E_\phi \mathbb{P}\}, d) = \Lambda^{*}\{[\sigma_1, \sigma_2, \sigma_3]\}$$

is a free $k$-module with Poincaré series $1 + 3x^3 + 3x^6 + x^9$.

2. Take $\omega = \{1\}$ (similarly, take any of its 6 0-simplex $\omega = \{i\}, i \in [6]$), the homology of the simplicial complement $E_{\{i\}} \mathbb{P}$ is $\Lambda^{*}\{E_{\{i\}} \mathbb{P}\}$. Thus the submodules

$$\bigoplus_{\tau} H_{*,\tau}(\Lambda^{*}\{E_{\{i\}} \mathbb{P}\}, d)$$

is the free $k$-module with Poincaré series $x^2(1 + x + 2x^3 + 2x^4 + x^6 + x^7)$.

3. Take $\omega = \{1, 3\}$ and similarly any of its 12 1-simplex $\omega = \{i_1, i_2\}$, the homology of the corresponding simplicial complement is $\Lambda^{*}\{E_{\{i_1, i_2\}} \mathbb{P}\}$. The submodule

$$\bigoplus_{\tau} H_{*,\tau}(\Lambda^{*}\{E_{\{i_1, i_2\}} \mathbb{P}\}, d)$$

is the free $k$-module with Poincaré series $x^4(1 + 2x + x^2 + x^3 + 2x^4 + x^5)$.

4. Take $\omega = \{1, 3, 5\}$ and similarly any of its 8 2-simplex $\omega = \{i_1, i_2, i_3\}$, the homology of the corresponding simplicial complement is $\Lambda^{*}\{E_{\{i_1, i_2, i_3\}} \mathbb{P}\}$. The submodule

$$\bigoplus_{\tau} H_{*,\tau}(\Lambda^{*}\{E_{\{i_1, i_2, i_3\}} \mathbb{P}\}, d)$$

is the free $k$-module with Poincaré series $x^6(1 + 3x + 3x^2 + x^3)$.

Thus

$$H^*(\mathbb{Z}_{K_5}(S^2, S^1), k) = \bigoplus_{\omega \in K_5} \left( \bigoplus_{\tau} H_{*,\tau}(\Lambda^{*}\{E_{\omega} \mathbb{P}\}, d) \right)$$

is a free $k$-module with Poincaré series

$$(1 + 3x^3 + 3x^6 + x^9)$$

$$+ 6x^2(1 + x + 2x^3 + 2x^4 + x^6 + x^7)$$

$$+ 12x^4(1 + 2x + x^2 + x^3 + 2x^4 + x^5)$$

$$+ 8x^6(1 + 3x + 3x^2 + x^3)$$

$$= 1 + 6x^2 + 9x^3 + 12x^4 + 36x^5 + 35x^6 + 36x^7 + 54x^8 + 27x^9$$

The total Betti number of $\mathbb{Z}_{K_5}(S^2, S^1)$ is 216.

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