Canonical quantization of a dissipative system interacting with an anisotropic non-linear absorbing environment

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Abstract

A canonical quantization scheme is represented for a quantum system interacting with a nonlinear absorbing environment. The environment is taken anisotropic and the main system is coupled to its environment through some coupling tensors of various ranks. The nonlinear response equation of the environment against the motion of the main system is obtained. The nonlinear Langevin-schrödinger equation is concluded as the macroscopic equation of motion of the dissipative system. The effect of nonlinearity of the environment is investigated on the spontaneous emission of an initially excited two level-atom imbedded in such an environment.

Keywords: Dissipative system, Absorbing environment, nonlinear response equation, Coupling tensor, Susceptibility tensor, Canonical quantization, Nonlinear Langevin-schrödinger equation, Spontaneous emission.

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1 Introduction

The simplest way of describing a damped system in classical dynamics is by adding a resisting force, generally velocity-dependent, to the equation of motion of the system. Frequently the magnitude of the resisting force may be closely presented, over a limited range of velocity, by the law $f_d = av^n$, where $v$ is the velocity of the damped system and $a$ and $n$ are constants. For example for the friction force $n = 0$, viscous force $n = 1$ and for high speed motion $n = 2$ [1]. Such an approach is no longer possible in quantum mechanics, because one cannot find a unitary time evolution operator for both the states and the observables, consistently.

In order to take into account the dissipation in a quantum system, there are usually two approaches. The first approach is a phenomenological way, by which the effect of dissipation is taken into account by constructing a suitable Lagrangian or Hamiltonian for the system [2, 3]. Following this method the first Hamiltonian was proposed by Caldirola [4] and Kanai [5] and afterward by others [6, 7]. There are difficulties about the quantum mechanical solutions of the Caldirola-Kanai Hamiltonian. For example quantization using this way violates the uncertainty relations or canonical commutation rules. The uncertainty relations vanishes as time tends to infinity [8]-[11].

The second approach is based on the assumption that the damping forces is caused by an irreversible transfer of energy from the system to a reservoir [12, 13]. In this method, modeling the absorptive environment by a collection of harmonic oscillators and choosing a suitable interaction between the system and the oscillators, a consistent quantization is achieved for both the main system and the environment [14]-[26]. In the Heisenberg picture, one can obtain the linear Langevin-Schrödinger equation, as the macroscopic equation of motion of the main system [14, 15].

In the present work, following the second approach, a fully canonical quantization is introduced for a system moving in an anisotropic non-linear absorbing environment. The dissipative system is the prototype of some important problems which the present approach can be applied to cover such problems straightforwardly.

The paper is organized as follows: In section 2, a Lagrangian for the total system (the main system and the environment) is proposed and a classical treatment of the dissipative system is achieved. In section 3, the Lagrangian introduced in the section 2 is used for a canonical quantization of both the
main system and the non-linear environment. In section 4, the present
quantization is used to investigate the effect of the nonlinearity of the environment
on the spontaneous decay rate of an initially excited two-level atom embedded
in the absorbing environment. Finally, the paper is closed with a summary
and some concluding remarks in section 5.

2 Three-dimensional quantum dissipative systems

When an absorbing environment responds non-linearly against the motion of
a system, the non-linear Langevin-Schrödinger equation is usually appeared
as the macroscopic equation of motion of the system. As an example, when
the electromagnetic field is propagated in an absorbing non-linear polarizable
medium, the vector potential satisfies the non-linear Langevin-Schrödinger
equation. In this section, the motion of a three-dimensional system in the
presence of an anisotropic non-linear absorbing environment is classically
treated. For this purpose, the environment is modeled by a continuum of
three dimensional harmonic oscillators labeled by a continuous parameter $\omega$.
The total Lagrangian is proposed as

$$L(t) = L_e + L_s + L_{int}. \quad (1)$$

which is the sum of three pars. The part $L_e$ is the Lagrangian of the environment

$$L_e(t) = \int_0^\infty d\omega \left[ \frac{1}{2} \dot{X}(\omega, t) \cdot \dot{X}(\omega, t) - \frac{1}{2} \omega^2 X(\omega, t) \cdot X(\omega, t) \right]. \quad (2)$$

where $X(\omega, t)$ is the dynamical variable of the oscillator labeled by $\omega$. The
second part $L_s$ in (1) is the Lagrangian of the main system. Taking the
system as a particle with mass, $m$, moving under an external potential $V(q)$,
one can write

$$L_s = \frac{1}{2}m \dot{q}(t) \cdot \dot{q}(t) - V(q). \quad (3)$$

The last part $L_{int}$ in the total Lagrangian (1) is the interaction term between
the system and its absorbing environment and includes both the linear and
nonlinear contributions as follows

\[ L_{int} = \int_0^\infty d\omega f^{(1)}_{ij}(\omega) \dot{q}^i(t) X^j(\omega, t) \]
\[ + \int_0^\infty d\omega \int_0^\infty d\omega' f^{(2)}_{ijk}(\omega, \omega') \dot{q}^i(t) X^j(\omega, t) X^k(\omega', t) \]
\[ + \int_0^\infty d\omega \int_0^\infty d\omega' \int_0^\infty d\omega'' f^{(3)}_{ijkl}(\omega, \omega', \omega'') \dot{q}^i(t) X^j(\omega, t) X^k(\omega', t) X^l(\omega'', t) + \ldots \ldots \]

(4)

where \( f^{(1)}, f^{(2)}, f^{(3)}, \ldots \) are the coupling tensors of the main system and its environment. As it is seen from (4) the coupling tensor \( f^{(1)} \) describes the linear contribution of the interaction part and the sequence \( f^{(2)}, f^{(3)}, \ldots \) describe, respectively, the first order of the non-linear interaction part, the second order of the non-linear interaction part and so on. The interaction Lagrangian (4) is the generalization of the Lagrangian that previously has been applied to quantize the electromagnetic field in the presence of anisotropic linear magnetodielectric media [27].

The coupling tensors \( f^{(1)}, f^{(2)}, f^{(3)}, \ldots \) in (4) are the key parameters of this quantization scheme. As it will be seen, in the next section, the susceptibility tensors of the environment (of the various ranks) are expressed in terms of the coupling tensors. Also the noise forces are obtained in terms of the coupling tensors and the dynamical variables of the environment at \( t = -\infty \).

2.1 The classical Lagrangian equations

The classical equations of motion of the total system can be obtained using the principle of the Hamilton's least action, \( \delta \int dt L(t) = 0 \). These equations are the Euler-Lagrange equations. For the dynamical variables, \( X(\omega, t) \), the
Euler-Lagrange equations are as
\[
\frac{d}{dt} \left( \frac{\delta L}{\delta (\dot{X}_i(\omega, t))} \right) - \frac{\delta L}{\delta (X_i(\omega, t))} = 0 \quad i = 1, 2, 3
\]
⇒ \[\ddot{X}_i(\omega, t) + \omega^2 X_i(\omega, t) = \dot{q}_j(t) f_{ji}^{(1)}(\omega) + \int_0^\infty d\omega' \dot{q}_j(t) \left[ f_{jk}^{(2)}(\omega, \omega') + f_{jk}^{(2)}(\omega', \omega) \right] X_k(\omega', t) + \int_0^\infty d\omega' \int_0^\infty d\omega'' \dot{q}_j(t) \left[ f_{ijkl}^{(3)}(\omega, \omega', \omega'') + f_{jki}^{(3)}(\omega', \omega, \omega'') \right] X_k(\omega', t) X_l(\omega'', t) + \cdots \tag{5}\]
Also the Lagrange equations for the freedom degrees of the main system are obtained as follows
\[
\frac{d}{dt} \left( \frac{\delta L}{\delta (\dot{q}_i(t))} \right) - \frac{\delta L}{\delta (q_i(t))} = 0 \quad i = 1, 2, 3
\]
⇒ \[m\ddot{q}_i(t) + \nabla V(q) = -\dot{R}(t) \tag{6}\]
where
\[
R_i(t) = \int_0^\infty d\omega f_{ij}^{(1)}(\omega) X_j(\omega, t) + \int_0^\infty d\omega \int_0^\infty d\omega' f_{ij}^{(2)}(\omega, \omega') X_j(\omega, t) X_k(\omega', t) + \int_0^\infty d\omega \int_0^\infty d\omega' \int_0^\infty d\omega'' f_{ijkl}^{(3)}(\omega, \omega', \omega'') X_j(\omega, t) X_k(\omega, t) X_l(\omega'', t) + \cdots \tag{7}\]
In Eq. (6) \(-\dot{R}(t)\) is the force exerted on the main system due to its motion inside the absorbing environment. It will be seen that the force \(-\dot{R}(t)\) can be separated into two parts. One part is the damping force which is dependent on the various powers of the velocity of the main system. The second part is the noise forces which has sinusodial time dependence. Both the damping and the noise forces are necessary for a consistent quantization of a dissipative system. Without the noise forces the quantization of a dissipative system encounter inconsistency. According to the fluctuation- dissipation theorem the absence of any of these two parts leads to the vanishing of the other part.
3 Canonical quantization

In order to represent a canonical quantization, the canonical conjugate momenta corresponding to the dynamical variables $X(\omega, t)$ and $q$ should be computed using the Lagrangian (11). These momenta are as follows

$$Q_i(\omega, t) = \frac{\delta L}{\delta (\dot{X}_i(\omega, t))} = \dot{X}_i(\omega, t) \quad i = 1, 2, 3$$

$$p_i(t) = \frac{\delta L}{\delta (\dot{q}_i)} = m\dot{q}_i + R_i(t) \quad i = 1, 2, 3.$$ 

Having the canonical momenta, both the dissipative system and the environment can be quantized in a standard fashion by imposing the following equal-time commutation rules

$$[q_i(t), p_j(t)] = i\hbar \delta_{ij}$$

$$[X_i(\omega, t), Q_j(\omega', t)] = i\hbar \delta_{ij} \delta(\omega - \omega')$$

Using the Lagrangian (11) and the expressions for the canonical momenta given by (8) and (9), the Hamiltonian of the total system clearly can be written as

$$H(t) = \left[\frac{p(t) - R(t)}{2m}\right]^2 + V(q) + \frac{1}{2} \int_0^\infty d\omega \left[Q^2(\omega, t) + \omega^2 X^2(\omega, t)\right]$$

where the cartesian components of $R(t)$ is defined by (7). The Hamiltonian (12) is the counterpart of the Hamiltonian of the quantized electromagnetic field in the presence of magnetodielectric media[27]-[29]. Using the commutation relations (10), (11) and applying the total Hamiltonian (12), it can be shown that the combination of the Heisenberg equations of motion of the canonical variables $X(\omega, t)$ and $Q(\omega, t)$ leads to the Eq.(5). Similarly, one can obtain Eq.(6) as the equation of motion of $q(t)$ in the Heisenberg picture.

Let us introduce the annihilation and creation operators of the environment as follows

$$b_i(\omega, t) = \sqrt{\frac{1}{2\hbar\omega}} [\omega X_i(\omega, t) + iQ_i(\omega, t)].$$

From the commutation relations (11) it is clear that the ladder operators $b_i(\omega, t)$ and $b_i^\dagger(\omega, t)$ obey the commutation relations

$$[b_i(\omega, t), b_j^\dagger(\omega', t)] = \delta_{ij} \delta(\omega - \omega')$$
The Hamiltonian (12) can be rewritten in terms of the creation and annihilation operators $b_i(\omega, t)$ and $b_i^\dagger(\omega, t)$ as follows

$$H = \frac{(p - R(t))^2}{2m} + V(q) + H_m$$

where

$$H_m = \sum_{i=1}^{3} \int d\omega \hbar \omega \ b_i^\dagger(\omega, t)b_i(\omega, t)$$

is the Hamiltonian of the absorbing environment in the normal ordering form and

$$R_i(t) = \int_0^\infty d\omega \sqrt{2\omega} f_{ij}^{(1)}(\omega) \left[ b_j(\omega, t) + b_j^\dagger(\omega, t) \right]$$

$$+ \int_0^\infty d\omega \int_0^\infty d\omega' \sqrt{2\omega\omega'} f_{ijk}^{(2)}(\omega, \omega') \left[ b_j(\omega, t)b_k(\omega', t) + b_j^\dagger(\omega, t)b_k^\dagger(\omega', t) + b_j(\omega, t)b_k^\dagger(\omega', t) + b_j^\dagger(\omega, t)b_k(\omega', t) \right] + \cdots$$

are the cartesian components of the operator $R$, where the summation should be done over the repeated indices.

### 3.1 The response equation of the environment

The response equation of the absorbing environment is the base of separating the force $-\dot{\mathbf{R}}(t)$, in the right hand of (6), into two parts, that is, the damping force and the noise force. If Eq. (5) is solved for $\mathbf{X}(\omega, t)$ and then, the obtained solution is substituted into the definition of $\mathbf{R}(t)$ given by (7), one can obtain the response equation of the environment. The differential equations (5) are a continuous collection of coupled non-linear differential equations for the dynamical variables $\mathbf{X}(\omega, t)$. The exact solution of this equation is impossible unless an iteration method to be used. For simplicity here we apply the first order of approximation and neglect the terms containing the coupling tensors $f^{(2)}, f^{(3)}, \ldots$ in the right hand of (5) and write the solution of Eq. (5), approximately, as

$$\mathbf{X}(\omega, t) = \mathbf{X}_N(\omega, t) + \int_{-\infty}^t dt' \frac{\sin(\omega(t - t'))}{\omega} \left( f^{(1)}(\omega) \right)^\dagger(\omega) \cdot \dot{\mathbf{q}}(t'),$$

(18)
where \((f^{(1)})_{ij}^\dagger(\omega) = f_{ji}^{(1)}(\omega)\) and \(X_N(\omega, t)\) is the solution of homogeneous equation \(\ddot{X}_N(\omega, t) + \omega^2 X_N(\omega, t) = 0\). In fact \(X_N(\omega, t)\) is asymptotic form of \(X(\omega, t)\) for very large negative times and can be written as

\[
X_{Ni}(\omega, t) = \sqrt{\frac{\hbar}{2\omega}} \left[ b_{in}^\dagger(\omega)e^{-i\omega t} + b_{in}(\omega)e^{i\omega t} \right]
\]

where \(b_{in}(\omega)\) and \(b_{in}^\dagger(\omega)\) are some time independent annihilation and creation operators which obviously satisfy the same commutation relations (14). The approximated solution (18) yields the response equation of the environment, such that, the susceptibility tensors appearing in it, satisfy the various symmetry properties reported by the literature [30].

Now substituting \(X(\omega, t)\) from (18) in (7), the response equation of the nonlinear absorbing environment is found as follows

\[
R(t) = R^{(1)} + R^{(2)} + ....
\]

\[
R^{(1)}_i(t) = \int_{-\infty}^{+\infty} dt \chi^{(1)}_{ij}(t-t') \dot{q}_j(t') + R^{(1)}_{Ni}(t)
\]

\[
R^{(2)}_i(t) = \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} dt'' \chi^{(2)}_{ijk}(t-t', t-t'') \dot{q}_j(t') \dot{q}_k(t'') + R^{(2)}_{Ni}(t)
\]

where \(\chi^{(1)}\) is the susceptibility tensor of the environment in the linear regime and is defined by

\[
\chi^{(1)}_{ij}(t) = \begin{cases} 
\int_0^{+\infty} d\omega \frac{\sin \omega t}{\omega} f_{in}^{(1)}(\omega) f_{jn}^{(1)}(\omega) & t > 0 \\
0 & t \leq 0
\end{cases}
\]

and \(\chi^{(2)}_{ijk}\) causes the first order of the nonlinearity of the response equation, where for for \(t_1, t_2 \geq 0\) are given by

\[
\chi^{(2)}_{ijk}(t_1, t_2) = \int_0^{+\infty} d\omega_1 \int_0^{+\infty} d\omega_2 \frac{\sin \omega_1 t_1}{\omega_1} \frac{\sin \omega_2 t_2}{\omega_2} \chi^{(2)}_{imn}(\omega_1, \omega_2) f_{jn}^{(1)}(\omega_1) f_{km}^{(1)}(\omega_2)
\]

and \(\chi^{(2)}_{ijk}\) is zero for \(t_1, t_2 < 0\). In (21) and (22) the summation should be done over the repeated indices \(m, n\). From the definition (21) it is clear...
that $\chi^{(1)}$ is a symmetric tensor, $\chi^{(1)}_{ij} = \chi^{(1)}_{ji}$. There are also some symmetry features for the non-linear susceptibility tensors of the various orders. These symmetry properties can be satisfied by imposing some conditions on the coupling tensors $f^{(2)}, f^{(3)}, \ldots$. For example the susceptibility tensor $\chi^{(2)}$, should satisfy the symmetry property \[30\]

$$
\chi^{(2)}_{ijk}(t_1, t_2) = \chi^{(2)}_{ikj}(t_2, t_1)
$$

where is fulfilled provided that the coupling tensor $f^{(2)}$ obey the symmetry condition

$$
f^{(2)}_{ijk}(\omega, \omega') = f^{(2)}_{ikj}(\omega', \omega), \tag{24}
$$

Similarly inserting the approximated solution $X(\omega, t)$ from (18) into (7), one can obtain the $(n-1)'$th susceptibility tensor of the environment in the non-linear regime for $t_1, t_2, \ldots t_n \geq 0$ as the following

$$
\chi^{(n)}_{i_1 \ldots i_n}(t_1, t_2, \ldots, t_n) = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \ldots \int_0^\infty d\omega_n \sin \omega_1 t_1 \sin \omega_2 t_2 \ldots \sin \omega_n t_n \times f^{(n)}_{i_1 j_1 \ldots j_n}(\omega_1, \omega_2, \ldots, \omega_n) f^{(1)}_{j_1 \ldots j_1}(\omega_1) f^{(1)}_{j_2 \ldots j_2}(\omega_2) \ldots f^{(1)}_{j_n j_n}(\omega_n) \tag{25}
$$

and $\chi^{(n)}_{i_1 \ldots i_n}(t_1, t_2, \ldots, t_n)$ is identically zero for $t_1, t_2, \ldots t_n < 0$. The Susceptibility tensor $\chi^{(n)}_{i_1 \ldots i_n}(t_1, t_2, \ldots, t_n)$ should satisfy the symmetry relations \[30\]

$$
\chi^{(n)}_{i_1 \ldots i_k \ldots i_l \ldots i_n}(t_1, t_2, \ldots, t_k, \ldots t_l, \ldots t_n) = \chi^{(n)}_{i_1 \ldots i_l \ldots i_k \ldots i_n}(t_1, t_2, \ldots, t_k, \ldots t_l, \ldots t_n) \tag{26}
$$

where this symmetry relation is clearly fulfilled by imposing the symmetry conditions

$$
\begin{align*}
&f^{(n)}_{i_1 j_1 \ldots j_n}(\omega_1, \omega_2, \ldots, \omega_k, \ldots, \omega_l, \ldots, \omega_n) = f^{(n)}_{i_1 j_1 \ldots j_n}(\omega_1, \omega_2, \ldots, \omega_l, \ldots, \omega_k, \ldots, \omega_n) \tag{27}
\end{align*}
$$

on the $n'$th coupling tensor in the interaction Lagrangian (4).

In Eq. (20) $R^{(1)}_N(t)$ and $R^{(2)}_N(t)$ are the noise forces in the linear regime and the first order of non-linearity, respectively, and using the symmetry
relation (24) are obtained as

\[
R^{(1)}_{N\,i}(t) = \int_0^\infty d\omega \, f^{(1)}_{ij}(\omega)X^j_N(\omega, t)
\]

\[
R^{(2)}_{N\,i}(t) = \int_0^\infty d\omega \int_0^\infty d\omega' \, f^{(2)}_{ij}(\omega, \omega') \, X^j_N(\omega, t) \, x^k_N(\omega', t)
+ \int_0^\infty d\omega \int_0^\infty d\omega' \, f^{(2)}_{im}(\omega, \omega') \, f^{(1)}_{jm}(\omega') \int_{-\infty}^t dt' \, \frac{\sin \omega' (t - t')}{\omega'}
\times \left[ X^m_N(\omega, t) \dot{q}^j(t') + \dot{q}^j(t') \, X^m_N(\omega', t) \right]
\]  

where the summation should be done over the repeated indices and \(X^N_N(\omega, t)\) is the asymptotic solution (19).

It is remarkable that for some known susceptibility tensors \(\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(n)}\), the coupling tensors \(f^{(1)}, f^{(2)}, \ldots, f^{(n)}\) satisfying the definitions (21), (22) and (25) are not unique. In fact if the coupling tensors \(f^{(1)}, f^{(2)}, \ldots, f^{(n)}\) satisfy (21), (22) and (25) for the given susceptibility tensors, also the coupling tensors \(f^{(1)}', f^{(2)}', \ldots, f^{(n)}'\) defined by

\[
f^{(1)}_{ij} = f^{(1)}_{im} A_{jm}
\]

\[
f^{(n)}_{i_1i_2\ldots i_n} = f^{(n)}_{j_1j_2\ldots j_n} A_{i_1j_1} A_{i_2j_2} \ldots A_{i_nj_n}
\]  

(29)

satisfy (21), (22) and (25), where \(A\) is an orthogonal matrix \(A_{im}A_{mj} = \delta_{ij}\). The various choices of the coupling tensors \(f^{(1)}, f^{(2)}, \ldots, f^{(n)}\) which is related to each other by the orthogonal transformation (29) do not change the physical observables. The commutation relations between the dynamical variables of the total system remain unchanged under the orthogonal transformation (29). For example in the next section it is shown that the decay rate of an initially excited two-level atom, embedded in a non-linear absorbing environment, are independent of the various choices of the coupling tensors which is related to each other by the transformation (29).

Now combination of the response equation (20) and equation (6) yields the non-linear Langevin-Schrödinger equation

\[
m\ddot{q}_i(t) + \int_{-\infty}^{+\infty} dt' \chi^{(1)}_{ij}(t - t') \, \dot{q}^j(t') + \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} dt'' \chi^{(2)}_{ijk}(t - t', t - t'') \dot{q}^j(t') \, \dot{q}^k(t'') + \ldots
+ \frac{\partial V(\tilde{q})}{\partial q} = -\dot{R}^{(1)}_{N\,i}(t) - \dot{R}^{(2)}_{N\,i}(t) + \ldots
\]  

(30)
as the macroscopic equation of motion of the main system in the anisotropic non-linear absorbing environment. The velocity dependent terms in the left hand of this equation are the damping forces exerted on the main system. The noise forces $-\dot{R}_{N_1}^{(1)}, -\dot{R}_{N_1}^{(2)}, \ldots$ in the right hand of (30) are necessary for a consistent quantization of the dissipative system. As a realization, if this quantization method would be applied for the electromagnetic field in the presence of an absorbing non-linear dielectric medium, the vector potential would satisfy the equation (30). In that case, the tensors $\chi^{(1)}, \chi^{(2)}, \ldots$ would play the role of the electric susceptibility tensors and $-\dot{R}_{N_1}^{(1)}, -\dot{R}_{N_1}^{(2)}, \ldots$ would be the noise polarization densities of various orders.

4 The effect of nonlinearity of the environment on the spontaneous emission of a two-level atom imbedded in an absorbing environment

In this section the effect of non-linearity of the absorbing environment is investigated on the spontaneous emission of a two-level atom embedded in such an environment. To calculate the spontaneous decay rate of an initially excited two-level atom, the quantization scheme in the preceding section is used and the theory of damping based on the density operator method is applied [31]. Neglecting the second power of the operator $R$ in (15) the Hamiltonian of the total system can be written as

$$H = H_0 + H'$$

$$H_0 = H_S + H_m = \frac{p^2}{2m} + V(q) + \sum_{i=1}^{3} \int d\omega \frac{\hbar}{\omega} b_i(\omega)b_i(\omega)$$

$$H' = -p \cdot R$$

(31)

Let us suppose the main system is a one electron atom with two eigenstates $|1\rangle$ and $|2\rangle$ correspond to two eigenvalues $E_1$ and $E_2$, respectively ($E_2 > E_1$).
the Hamiltonian (31) can now be rewritten as 31, 32.

\[
H = H_0 + H'
\]

\[
H_0 = \hbar \omega_0 \sigma^\dagger \sigma + \sum_{i=1}^{3} \int d\omega \hbar \omega b_i^\dagger(\omega) b_i(\omega)
\]

\[
H' = i \hbar \omega_0 \mathbf{R} \cdot \left[ d\sigma - d^* \sigma^\dagger \right]
\]

(32)

where \( m \) is the electron mass of the atom, \( \sigma = |1\rangle\langle 2| \), \( \sigma^\dagger = |2\rangle\langle 1| \), are the Pauli operators and \( d = \langle 1| \mathbf{r} |2\rangle \), where \( \mathbf{r} \) is the position vector of the electron with respect to the center of mass of the atom. Dropping the energy nonconserving terms correspond to rotating wave approximation and regarding the relation (17), the interaction term \( H' \) up to the first order of nonlinearity in the interaction picture is expressed as

\[
H'_1(t) = e^{\frac{iH_0 t}{\hbar}} H'(0) e^{-\frac{iH_0 t}{\hbar}}
\]

\[
= i m \omega_0 \int_0^\infty d\omega \sqrt{\frac{\hbar}{2\omega}} f^{(1)}_{ij}(\omega) \left[ d_i \sigma b_j^\dagger(\omega) e^{i(\omega - \omega_0)t} - H.C. \right]
\]

\[
+ i m \omega_0 \int_0^\infty d\omega \int_0^\infty d\omega' \frac{\hbar}{2\sqrt{\omega\omega'}} f^{(2)}_{ijk}(\omega, \omega') \left[ d_i \sigma b_j^\dagger(\omega) b_k^\dagger(\omega') e^{-i(\omega_0 - \omega) t} + d_i \sigma b_j(\omega) b_k(\omega') e^{-i(\omega_0 - \omega + \omega') t} + d_i^* \sigma^\dagger b_j^\dagger(\omega) b_k^\dagger(\omega') e^{i(\omega_0 - \omega - \omega') t} - d_i^* \sigma^\dagger b_j(\omega) b_k(\omega') e^{i(\omega_0 - \omega + \omega') t} - d_i \sigma b_j^\dagger(\omega) b_k^\dagger(\omega') e^{i(\omega_0 + \omega - \omega') t} - d_i \sigma^\dagger b_j^\dagger(\omega) b_k^\dagger(\omega ') e^{i(\omega_0 + \omega - \omega') t} \right]
\]

(33)

where the symmetry relation (23) has been used. Let the combined density operator of the atom together with the environment is denoted by \( \rho_{SR} \) in the interaction picture. Then, the reduced density operator of the atom alone, denoted by \( \rho_S \), is obtained by taking the trace of \( \rho_{SR} \) with respect to the coordinates of the environment, that is \( \rho_S = Tr_R[\rho_{SR}] \). Since it is assumed that \( H'_1(t) \) is sufficiently small, according to the density operator approach for the damping theory 31, the time evolution of the reduced density operator \( \rho_S \) is the solution of equation

\[
\dot{\rho}_s(t) = -\frac{i}{\hbar} Tr_R[H'_1(t), \rho_s(0) \otimes \rho_R(0)]
\]

\[
-\frac{1}{\hbar^2} Tr_R \int_0^t dt' [H'_1(t'), \rho_s(t') \otimes \rho_R(0)]]
\]

(34)
up to order of $H^2$, where $\rho_R(0)$ is the density operator of the environment at $t = 0$. In this formalism the environment is taken in equilibrium. Also the Markovian approximation has been applied replacing $\rho_S(t')$ by $\rho_S(t)$ in the integrand in Eq. (34).

To calculate the spontaneous emission of the atom, the initial states of the atom and the environment are taken as

$$\rho_R(0) = |0\rangle\langle 0| \quad \rho_S(0) = |2\rangle\langle 2|$$  \hspace{1cm} (35)

where $|0\rangle$ is the vacuum state of the environment. Now substituting $H^t(t)$ from (33) into (34) and regarding (35) the time evolution of the reduced density operator $\rho_S$ is obtained as

$$\dot{\rho}_S = \frac{m\omega_0}{2} \int_0^\infty \frac{d\omega}{\omega} f_{ij}^{(2)}(\omega, \omega)[d_i \sigma e^{-i\omega t} + H.C]$$

$$-\frac{m^2\omega_0^2}{2\hbar} \int_0^\infty \frac{d\omega}{\omega} d_i^* f_{ij}^{(1)}(\omega) f_j^{(1)}(\omega) d_t \left[ \int_0^t dt' e^{-i(\omega-\omega_0)(t-t')} \sigma^\dagger \sigma \rho_S(t) + H.C \right]$$

$$+ \frac{m^2\omega_0^2}{\hbar} \int_0^\infty \frac{d\omega}{\omega} d_i^* f_{ij}^{(1)}(\omega) f_j^{(1)}(\omega) d_i \sigma \rho_S(t) \sigma^\dagger \int_0^t dt' \cos(\omega-\omega_0)(t-t')$$

$$-\frac{m^2\omega_0^2}{2} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \frac{1}{\omega_1\omega_2} d_i \ f_{i_1,j_1j_1}(\omega_1, \omega_1) f_{i_2,j_2j_2}(\omega_2, \omega_2) d_{i_2}$$

$$\times \left[ \int_0^t dt' \sigma \rho_S(t) \sigma e^{-i\omega_0(t+t')} + H.C \right]$$

$$+ \frac{m^2\omega_0^2}{2} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \frac{1}{\omega_1\omega_2} d_i^* f_{i_1,j_1j_1}(\omega_1, \omega_1) f_{i_2,j_2j_2}(\omega_2, \omega_2) d_{i_2}$$

$$\times \left[ \sigma^\dagger \rho_S(t) \sigma + \sigma \rho_S(t) \sigma^\dagger \right] \int_0^t dt' \cos \omega_0(t-t')$$

$$+ m^2\omega_0^2 \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \frac{1}{\omega_1\omega_2} d_i^* f_{i_1j_1j_2}(\omega_1, \omega_2) f_{i_2j_2j_2}(\omega_1, \omega_2) d_{i_2}$$

$$\times \sigma \rho_S(t) \sigma^\dagger \int_0^t dt' \cos(\omega_0 - \omega_1 - \omega_2)(t-t')$$

$$-\frac{m^2\omega_0^2}{4} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \frac{1}{\omega_1\omega_2} d_i^* f_{i_1j_1j_1}(\omega_1, \omega_1) f_{i_2j_2j_2}(\omega_2, \omega_2) d_{i_2}$$

$$\times \left[ \int_0^t dt' e^{-i\omega_0(t-t')} \sigma \sigma^\dagger \rho_S(t) + H.C \right]$$
where the repeated indices implies that the summation should be done over them. Then, the equation of motion of the matrix elements $\rho_{S11} = \langle 1 | \rho_S | 1 \rangle$, $\rho_{S22} = \langle 2 | \rho_S | 2 \rangle$ and $\rho_{S12} = \rho_{S21} = \langle 1 | \rho_S | 2 \rangle$ now is obtained as

\[
\dot{\rho}_{S11} = \frac{m^2 \omega_0^2}{h} \int_0^\infty \frac{d\omega}{\omega} \int_0^\infty \frac{d\omega_2}{\omega_1\omega_2} \ d_i^* f_{i_1 j_1 j_1}^{(1)}(\omega) f_{i_j}^{(1)}(\omega) \ dt \ \rho_{S22}(t) \int_0^t dt' \ \cos(\omega - \omega_0)(t - t') \\
+ \frac{m^2 \omega_0^2}{2} \int_0^\infty \frac{d\omega}{\omega} \int_0^\infty \frac{d\omega_2}{\omega_1\omega_2} \ d_i^* f_{i_1 j_1 j_1}^{(2)}(\omega_1, \omega_2) f_{i_2 j_1 j_2}^{(2)}(\omega_1, \omega_2) \ dt \ \rho_{S22}(t) \\
\times [\rho_{S22}(t) - \rho_{S11}(t)] \int_0^t dt' \ \cos(\omega_0)(t - t')
\] (36)

\[
\dot{\rho}_{S22} = -\frac{m^2 \omega_0^2}{h} \int_0^\infty \frac{d\omega}{\omega} \int_0^\infty \frac{d\omega_2}{\omega_1\omega_2} \ d_i^* f_{i_1 j_1 j_1}^{(1)}(\omega) f_{i_j}^{(1)}(\omega) \ dt \ \rho_{S22}(t) \int_0^t dt' \ \cos(\omega - \omega_0)(t - t') \\
- \frac{m^2 \omega_0^2}{2} \int_0^\infty \frac{d\omega}{\omega} \int_0^\infty \frac{d\omega_2}{\omega_1\omega_2} \ d_i^* f_{i_1 j_1 j_1}^{(2)}(\omega_1, \omega_2) f_{i_2 j_1 j_1}^{(2)}(\omega_1, \omega_2) \ dt \ \rho_{S22}(t) \\
\times [\rho_{S22}(t) - \rho_{S11}(t)] \int_0^t dt' \ \cos(\omega_0)(t - t')
\] (37)
\[ \dot{\rho}_{S12} = \dot{\rho}^s_{S21} = \frac{m \omega_0}{2} \int_0^\infty \frac{d\omega}{\omega} f^{(2)}_{ijj}(\omega, \omega) \, d_i e^{-\omega_0 t} \]

\[ \quad - \frac{m^2 \omega_0^2}{2} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \frac{1}{\omega_1 \omega_2} d_{i_1} f^{(2)}_{i_1 i_1 j_1}(\omega_1, \omega_1) f^{(2)}_{i_2 j_2 j_2}(\omega_2, \omega_2) \, d_{i_2} \]

\[ \times \rho_{S22}(t) \int_0^t dt' e^{-\omega_0 (t+t')} \] (38)

For sufficiently large times the integrals appeared in the equations (36) and (37) can be approximated by

\[ \frac{1}{\pi} \int_0^t dt' \cos(\omega - \omega_0)(t - t') \sim \delta(\omega - \omega_0) \]

\[ \frac{1}{\pi} \int_0^t dt' \cos(\omega_0 - \omega_1 - \omega_2)(t - t') \sim \delta(\omega_0 - \omega_1 - \omega_2) \]

\[ \frac{1}{\pi} \int_0^t dt' \cos \omega_0(t - t') \sim \delta(\omega_0) = 0 \] (39)

Hence the time evolution of the matrix elements \( \rho_{S11} \) and \( \rho_{S22} \) for sufficiently large times is reduced to

\[ \dot{\rho}_{S11} = \Gamma \rho_{S22} \quad \dot{\rho}_{S22} = -\Gamma \rho_{S22} \] (40)

where

\[ \Gamma = \frac{\pi m \omega_0^2}{\hbar} d^2 f^{(1)}_{ij} (\omega) f^{(1)}_{ij} (\omega) d_i \]

\[ + \pi m^2 \omega_0^2 \int_0^\infty d\omega \frac{1}{\omega(\omega - \omega_0)} d_{i_1} f^{(2)}_{i_1 j_1 j_2}(\omega, \omega - \omega_0) f^{(2)}_{i_2 j_2 j_2}(\omega, \omega - \omega_0) \, d_{i_2} \] (41)

is the decay rate of the spontaneous emission of the initially excited two level atom up to the first order of nonlinearity. The first term in (41) is the decay rate in the absence of nonlinearity effects and the second term is the first contribution related to nonlinear effects of the environment. It may be noted from (40) that \( \dot{\rho}_{S11} + \dot{\rho}_{S22} = 0 \) which implies the conservation of the probability. An important point is that the decay rate \( \Gamma \) is invariant under the various coupling tensors which is related to each other by the transformation (29). This should be so, because the decay rate \( \Gamma \) is a physical observable.
5 Summary

A fully canonical quantization of a quantum system moving in an anisotropic non-linear absorbing environment was introduced. The main dissipative system was coupled with the environment through some coupling tensors of various ranks. The coupling tensors have an important role in this theory. Based on a response equation, the forces against the motion of the main system were resolved into two parts, the damping forces and the noise forces. The response equation of the environment was obtained using the Heisenberg equations describing the time evolution of the coordinates of the system and the environment. Some susceptibility tensors of various ranks were attributed to the environment. The susceptibility tensors in the linear and non-linear regimes were defined in terms of the coupling tensors of the system and its environment. It was shown that, by imposing some symmetry conditions on the coupling tensors, the susceptibility tensors obey the symmetry properties reported in the literature. A realization of this quantization method is the quantized electromagnetic field in the presence of a non-linear absorbing dielectric. Finally the effect of the nonlinearity of the environment was investigated on the spontaneous decay rate of a two-level atom imbedded in the non-linear environment.

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