Connectivity of Inhomogeneous Random K-Out Graphs

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Abstract—We propose inhomogeneous random K-out graphs \( H(n; \mu, K_n) \), where each of the \( n \) nodes is assigned to one of \( r \) classes independently with a probability distribution \( \mu = \{\mu_1, \ldots, \mu_r\} \). In particular, each node is classified as class \( i \) with probability \( \mu_i > 0 \), independently. With \( K_n = (K_{1,n}, \ldots, K_{r,n}) \), each class \( i \) node selects \( K_{i,n} \) distinct nodes uniformly at random from among all other nodes. A pair of nodes are adjacent in \( H(n; \mu, K_n) \) if at least one selects the other. Without loss of generality, we assume that \( K_{1,n} \leq K_{2,n} \leq \ldots \leq K_{r,n} \). Earlier results on homogeneous random K-out graphs \( H(n; K_n) \), where all nodes select the same number \( K_n \) of other nodes, reveal that \( H(n; \mu, K_n) \) is connected with high probability (whp) if \( K_n \geq 2 \) which implies that \( H(n; \mu, K_n) \) is connected whp if \( K_{1,n} \geq 2 \). In this paper, we investigate the connectivity of inhomogeneous random K-out graphs \( H(n; \mu, K_n) \) for the special case when \( K_{1,n} = 1 \), i.e., when each class 1 node selects only one other node. We show that \( H(n; \mu, K_n) \) is connected whp if \( K_{r,n} \) is chosen such that \( \lim_{n \to \infty} K_{r,n} = \infty \). However, any bounded choice of the sequence \( K_{r,n} \) gives a positive probability of \( H(n; \mu, K_n) \) being asymptotically not connected in the limit of large \( n \). Simulation results are provided to validate our results in the finite node regime.

Index Terms—Random graphs, inhomogeneous random k-out graphs, connectivity, security.

I. INTRODUCTION

The study of random graphs in their own right dates back to 1959 with the seminal work of Paul Erdős, Alfréd Rényi, and Edgar Gilbert. In particular, Erdős and Rényi [3] introduced the random graph model \( G(n; M) \), representing a graph selected uniformly at random from the collection of all graphs with \( n \) nodes and \( M \) edges. In the same year, Gilbert [4] independently introduced the random graph model \( G(n; p) \), where each pair of vertices is connected (respectively, not connected) by an edge independently with probability \( p \) (respectively, \( 1 - p \)). Since then, random graphs have received great attention in their own right and as a modeling framework for a wide class of real-world networks including social networks, communication networks, biological networks, among others [5]–[7].

Of particular interest to this paper is a commonly studied class of random graphs known as random K-out graphs \( H(n; K) \) [8]–[12] that are constructed as follows. Each of the \( n \) nodes selects \( K \) other nodes uniformly at random from among all other nodes. An undirected edge is assigned between nodes \( u \) and \( v \) if \( u \) selects \( v \), or \( v \) selects \( u \), or both; see [9] for details. Random K-out graphs have recently received great interest for their role in modeling secure connectivity of wireless sensor networks utilizing the random pairwise key predistribution scheme of Yağan and A. M. Makowski [9], [14], Chan et al. [13]. In this scheme, each of the \( n \) sensor nodes is paired (offline) with \( K \) distinct nodes which are randomly selected from among all other nodes. If nodes \( i \) and \( j \) were paired during the node-pairing stage, a unique (pairwise) key is generated and stored in the memory modules of each of the paired sensors together with both their IDs. After deployment, a secure link can be established between two communicating nodes if they have at least one pairwise key in common. Furthermore, a structure similar to random K-out graphs was recently proposed in [15, Algorithm 1] for generating anonymity graphs that facilitate the diffusion of transaction information, hence making the crypto-currency network robust to de-anonymization attacks. The connectivity of random K-out graphs was studied in [9], [10], where it was shown that

\[
\lim_{n \to \infty} P [H(n; K) \text{ is connected}] = \begin{cases} 
0 & \text{if } K = 1, \\
1 & \text{if } K \geq 2.
\end{cases}
\]

Hence, it is sufficient to set \( K = 2 \) to have a connected network with high probability in the limit of large network size. In fact, it was shown in [9] that the probability of \( H(n; 2) \) being connected exceeds 0.99 with as little as \( n = 25 \) nodes.

Observe that random K-out graphs are inherently homogeneous due to the uniform treatment of all vertices. In particular, each vertex selects the same number \( K \) of other vertices to be paired to. Hence, the modeling framework provided by random K-out graphs is limited to the cases when all network entities behave similarly. However, real-world complex networks are essentially composed of heterogeneous entities [5], [16], inducing the need for an inhomogeneous
variant of random K-out graphs. Emerging wireless sensor networks represent a pronounced example of heterogeneous networks that consist of different nodes with different levels of resources (for communication, computation, storage, power, etc.) and possibly a varying level of security and connectivity requirements [17]–[20]. Clustered sensor networks [21] represent another example of heterogeneous networks where different nodes could select different number of other nodes as compared to typical sensor nodes. As a result, when such heterogeneous networks utilize the random pairwise scheme, it may no longer be sensible to assign the same number \( K \) of selections to all sensors. Indeed, an inhomogeneous variant of random K-out graphs, where different nodes could select different number of other nodes to be paired to, would offer a modeling framework that captures the emerging heterogeneity in real-world complex networks secured by the random pairwise key predistribution scheme. It is worth noting that the literature on random graphs is already shifting towards inhomogeneous models initiated by the seminal work of Bollobás et al. on inhomogeneous Erdős–Rényi graph [22] (see also [23]).

In this paper, we propose inhomogeneous random K-out graphs \( \mathbb{H}(n; \mu, K_n) \), where each of the \( n \) nodes is assigned to one of \( r \) classes according to a probability distribution \( \mu = \{ \mu_1, \ldots, \mu_r \} \) where \( \mu_i > 0 \) for \( i = 1, \ldots, r \). With \( K_n = (K_{1,n}, \ldots, K_{r,n}) \), each class \( i \) node selects \( K_{i,n} \) distinct nodes uniformly at random from among all other nodes. Two nodes \( u \) and \( v \) are connected by an edge if \( u \) selects \( v \) or \( v \) selects \( u \), or both. Without loss of generality, we assume that \( K_{1,n} \leq K_{2,n} \leq \ldots \leq K_{r,n} \). We let \( K_{\text{avg},n} = \sum_{i=1}^r \mu_i K_{i,n} \) denote the expected number of selections. Inhomogeneous random K-out graphs generalize standard random K-out graphs to heterogeneous setting where different nodes make different number of selections depending on their corresponding classes. As a result, it might be expected that inhomogeneous K-out graphs would serve as a more natural model in many of the envisioned applications of K-out graphs including pairwise key predistribution in sensor networks and anonymous transactions in cryptocurrency networks.

By an easy monotonicity argument, we see from (1) that \( \mathbb{H}(n; \mu, K_n) \) is connected with high probability if \( K_{1,n} \geq 2 \). Of particular interest to our paper is the special case when \( K_{1,n} = 1 \), i.e., when each of the nodes belonging to class 1 selects only one other node to be paired to. One could reasonably conjecture that setting \( K_{\text{avg},n} \) to any finite number larger than or equal to two would be sufficient to ensure the connectivity of \( \mathbb{H}(n; \mu, K_n) \), in resemblance to (1). Our results reveal that such a conjecture does not hold and that the connectivity of \( \mathbb{H}(n; \mu, K_n) \) under the special case when \( K_{1,n} = 1 \) cannot be inferred from (1).

In this paper, we study the connectivity of \( \mathbb{H}(n; \mu, K_n) \) when \( K_{1,n} = 1 \). More precisely, we seek conditions on \( K_{2,n}, K_{3,n}, \ldots, K_{r,n} \) and \( \mu \) such that the resulting graph is connected with high probability. Our main results (see Theorems 1 and 2) show that \( \mathbb{H}(n; \mu, K_n) \) is connected with high probability if and only if \( K_{r,n} = \omega(1) \), i.e., \( \lim_{n \to \infty} K_{r,n} = \infty \). In other words, if \( K_{r,n} \) grows unboundedly large as \( n \to \infty \), then the probability that \( \mathbb{H}(n; \mu, K_n) \) is connected approaches one in the same limit. However, any bounded choice of \( K_{r,n} \) gives a positive probability of \( \mathbb{H}(n; \mu, K_n) \) being not connected in the limit of large \( n \). Comparing our results with (1) sheds the light on a striking difference between inhomogeneous random K-out graphs \( \mathbb{H}(n; \mu, K_n) \) and their homogeneous counterpart \( \mathbb{H}(n; K) \). In particular, the flexibility of organizing the nodes into several classes with different characteristics (with \( K_{1,n} = 1 \)) comes at the expense of requiring \( \lim_{n \to \infty} K_{r,n} = \infty \), in contrast to the homogeneous case where having \( K = 2 \) was sufficient to ensure connectivity.

Throughout the paper, all statements involving limits, including asymptotic equivalences, are understood with \( n \) going to infinity. The cardinality of any discrete set \( S \) is denoted by \(|S|\). The random variables (rvs) under consideration are all defined on the same probability triple \((\Omega, \mathcal{F}, \mathbb{P})\). Probabilistic statements are made with respect to this probability measure \(\mathbb{P}\), and we denote the corresponding expectation operator by \(\mathbb{E}\). We say that an event holds with high probability (whp) if it holds with probability 1 as \( n \to \infty \). In comparing the asymptotic behaviors of the sequences \(\{a_n\}, \{b_n\}\), we use \(a_n = o(b_n)\), \(a_n = \omega(b_n)\), and \(a_n = O(b_n)\) with their meaning in the standard Landau notation. We write \(\mathbb{N}_0\) to denote the set of natural numbers excluding zero.

The rest of the paper is organized as follows. In Section II, we introduce inhomogeneous random K-out graphs \( \mathbb{H}(n; \mu, K_n) \) and derive the basic quantities associated with the model. In Section III, we present our main results on the connectivity of \( \mathbb{H}(n; \mu, K_n) \). Theorems 1 and 2 show that \( \mathbb{H}(n; \mu, K_n) \) is connected with high probability if and only if \( K_{r,n} = \omega(1) \). However, any bounded choice of \( K_{r,n} \) gives a positive probability of \( \mathbb{H}(n; \mu, K_n) \) being not connected in the limit of large \( n \). Theorem 3 reveals conditions on the parameters \((\mu, K_n)\) under which the probability of connectivity of \( \mathbb{H}(n; \mu, K_n) \) remains strictly larger than zero as \( n \) gets large, meaning that a zero-law does not exist. We close Section III by providing numerical results to validate the upper bound given by Theorem 1 and presenting Conjecture 1 that gives a tighter upper bound on the probability of connectivity as compared to the bound given by Theorem 1. In Section IV, we provide few preliminaries that will be needed throughout the paper. In Sections V, VI, and VII, we establish Theorems 1, 2, and 3, respectively. In Section VIII, we provide the intuition behind Conjecture 1. Finally, Section IX concludes the paper.

II. INHOMOGENEOUS RANDOM K-OUT GRAPHS

The inhomogeneous random K-out graph, denoted \( \mathbb{H}(n; \mu, K_n) \), is constructed on the vertex set \( V = \{1, 2, \ldots, n\} \) as follows. First, each node is assigned a class \( i \in \{1, \ldots, r\} \) independently according to a probability distribution \( \mu = \{\mu_1, \ldots, \mu_r\} \); i.e., \( \mu_i \) denotes the probability that a node is class \( i \) and we have \( \sum_{i=1}^r \mu_i = 1 \). We assume \( \mu_i > 0 \) for all \( i = 1, 2, \ldots, r \) and that \( r \) is a fixed integer.
that does not scale with \( n \). With \( \mathbf{K}_n = (K_{1,n}, \ldots, K_{r,n}) \), each class \( i \) node \( v \) selects \( K_{i,n} \) distinct nodes uniformly at random from \( \mathcal{V} \setminus \{v\} \) and an undirected edge is assigned between a pair of nodes if at least one selects the other. Formally, each node \( v \) is associated (independently from others) with a subset \( \Gamma_{n,v}(\mu, \mathbf{K}_n) \) (whose size depends on the class of node \( v \)) of nodes selected uniformly at random from \( \mathcal{V} \setminus \{v\} \). Specifically, for any \( A \subseteq \mathcal{V} \setminus \{v\} \), we have

\[
P(\Gamma_{n,v}(\mu, \mathbf{K}_n) = A \mid t_v = i) = \begin{cases} 
(n-1)^{-1} & \text{if } |A| = K_{i,n} \\
0 & \text{otherwise}
\end{cases}
\]

where \( t_v \) denotes the class of node \( v \). Then, vertices \( u \) and \( v \) are said to be adjacent in \( H(n; \mu, \mathbf{K}_n) \), written \( u \sim v \), if at least one selects the other; i.e.,

\[
u \sim v \iff u \in \Gamma_{n,u}(\mu, \mathbf{K}_n) \lor v \in \Gamma_{n,u}(\mu, \mathbf{K}_n).
\]

When \( r = 1 \), all vertices belong to the same class and thus select the same number (say, \( K \)) of other nodes, leading to the homogeneous random K-out graph \( H(n; K) \) [8]–[10].

Throughout, we set

\[
K_{\text{avg},n} = \sum_{i=1}^{r} \mu_i K_{i,n}.
\]

For any distinct nodes \( u, v \in \mathcal{V} \), we have

\[
P(u \sim v) = 1 - P(u \not\in \Gamma_{n,u}(\mu, \mathbf{K}_n) \cap v \not\in \Gamma_{n,v}(\mu, \mathbf{K}_n))
\]

\[
= 1 - \left( \sum_{i=1}^{r} \mu_i \left( \frac{n-1}{K_i} \right)^2 \right) \frac{n}{n-1}
\]

\[
= 1 - \left( 1 - \frac{K_{\text{avg},n}}{n} \right)^2.
\]

### III. MAIN RESULTS

We refer to any mapping \( \mathbf{K} : \mathbb{N}_0 \to \mathbb{N}_0^r \) as a scaling provided it satisfies the condition

\[
K_{1,n} \leq K_{2,n} \leq \ldots \leq K_{r,n} < n, \quad n = 2, 3, \ldots
\]

Our main technical results, given next, characterize the connectivity of inhomogeneous random K-out graphs. Throughout, it will be convenient to use the notation

\[
P(n; \mu, \mathbf{K}_n) := P[H(n; \mu, \mathbf{K}_n) \text{ is connected}]
\]

and

\[
C(\mu, \mathbf{K}_n) = \frac{1}{1 + \frac{2}{\mu_1^2} K_{\text{avg},n}}
\]

with \( 0 < \mu_1 < 1 \) and \( K_{\text{avg},n} \) as defined in (4).

The following result establishes an upper bound on the probability of connectivity of the inhomogeneous random K-out graphs when the sequence \( K_{r,n} \) is bounded, i.e., \( K_{r,n} = O(1) \).

**Theorem 1:** Consider a scaling \( \mathbf{K} : \mathbb{N}_0 \to \mathbb{N}_0^r \) and a probability distribution \( \mu = \{\mu_1, \mu_2, \ldots, \mu_r\} \) with \( \mu_i > 0 \). If \( K_{r,n} = O(1) \), then

\[
\lim_{n \to \infty} P(n; \mu, \mathbf{K}_n) < 1.
\]

More precisely, we have

\[
P(n; \mu, \mathbf{K}_n) \leq 1 - C(\mu, \mathbf{K}_n) + o(1).
\]

The following result establishes a one-law for connectivity for the inhomogeneous random K-out graph.

**Theorem 2:** Consider a scaling \( \mathbf{K} : \mathbb{N}_0 \to \mathbb{N}_0^r \) and a probability distribution \( \mu = \{\mu_1, \mu_2, \ldots, \mu_r\} \) with \( \mu_i > 0 \). If \( K_{r,n} = \omega(1) \), then

\[
\lim_{n \to \infty} P(n; \mu, \mathbf{K}_n) = 1.
\]

Theorems 1 and 2 state that \( H(n; \mu, \mathbf{K}_n) \) is connected with high probability if \( K_{r,n} \) is chosen such that \( K_{r,n} = \omega(1) \). On the other hand, if \( K_{r,n} = O(1) \), then the probability of connectivity of \( H(n; \mu, \mathbf{K}_n) \) is strictly less than one in the limit of large network size. In other words, any bounded choice for \( K_{r,n} \) gives rise to a positive probability of \( H(n; \mu, \mathbf{K}_n) \) being not connected. Observe that (7) follows from (8) by virtue of the fact that \( K_{\text{avg},n} = O(1) \) when \( K_{r,n} = O(1) \). The proofs of Theorems 1 and 2 are given in Section V and Section VI, respectively.

### A. Discussion

Connectivity results in the literature of random graphs are usually presented in the form of zero-one laws, where the probability of connectivity (in the limit as \( n \to \infty \)) exhibits a sharp transition between two different regimes. In the first (respectively, second) regime, the probability tends to zero (respectively, one) as \( n \) tends to infinity. One example of such results is given by (1) where the probability that \( H(n; K) \) is connected tends to zero when \( K = 1 \) and tends to one when \( K \geq 2 \). Other examples include the connectivity results on random key graphs [24], Erdős-Rényi graphs [3], etc. Indeed, Theorem 1 states that the probability of connectivity is strictly less than one whenever \( K_{r,n} = O(1) \) but it does not specify whether or not a zero-law exists in this case.

In other words, Theorem 1 does not reveal whether or not \( \lim_{n \to \infty} P(n; \mu, \mathbf{K}_n) = 0 \) when \( K_{r,n} = O(1) \). Such a zero-law, if exists, would complement the one-law given by Theorem 2.

Next, we present a result that provides conditions on \( \mu \) and \( K_{r,n} \) for which the probability of connectivity of \( H(n; \mu, \mathbf{K}_n) \) is strictly larger than zero in the limit of large \( n \). In other words, we provide conditions under which a zero-law does not hold. Henceforth, we set \( \bar{\mu} = \sum_{i=1}^{r-1} \mu_i \). If \( \mu_i > 0 \) for all \( i = 1, \ldots, r \), then we clearly have \( 0 < \bar{\mu} < 1 \).

**Theorem 3:** Consider a scaling \( \mathbf{K} : \mathbb{N}_0 \to \mathbb{N}_0^r \) and a probability distribution \( \mu = \{\mu_1, \mu_2, \ldots, \mu_r\} \) with \( \mu_i > 0 \). Let \( \bar{\mu} = \sum_{i=1}^{r-1} \mu_i \). For any \( 0 < \bar{\mu} < 1 \), there exists \( K^*(\bar{\mu}) \) such that

\[
\lim_{n \to \infty} P(n; \mu, \mathbf{K}_n) > 0 \text{ if } K_{r,n} \geq K^*(\bar{\mu}) \text{ for all } n.
\]

More precisely, \( K^*(\bar{\mu}) \) is given as follows:

a) For all \( 0 < \bar{\mu} \leq 0.8 \), we have \( K^*(\bar{\mu}) = 3 \).

b) For all \( 0.8 < \bar{\mu} < 1 \), \( K^*(\bar{\mu}) \) is given by (9), shown at the bottom of the next page.

In Table I, we provide the values of \( K^*(\bar{\mu}) \) corresponding to selected values of \( \bar{\mu} \). Note that whether or not a zero-law holds
for the cases when i) $\tilde{\mu} \leq 0.8$ with $K_{r,n} = 2$ or ii) $\tilde{\mu} > 0.8$ with $2 \leq K_{r,n} < K^*(\tilde{\mu})$ is currently unknown and can be a topic for future work. A graphical illustration of Table I is given in Figure 1, where we show the parameter regime for which a zero-law does not hold by virtue of Theorem 3. The proof of Theorem 3 is given in Section VII.

Theorems 1 and 2 reveal a striking difference between inhomogeneous random K-out graphs and their homogeneous counterpart. In particular, we see from (1) that it is sufficient to set $K = 2$ to make $\mathbb{H}(n; K)$ connected with high probability in the limit of large network size. When the network size $n$ is fixed, Sood and Yağan [25] recently showed that

$$\mathbb{P}[\mathbb{H}(n; 2) \text{ is connected}] \geq 1 - \frac{155}{n^2} e^{-\frac{3+\tilde{\mu}}{4}} \sqrt{n} \sqrt{n-3}, \quad n \geq 16,$$

indicating that the probability of connectivity exceeds 0.999 for as little as $n = 16$ nodes (with $K = 2$). As a result, random K-out graphs $\mathbb{H}(n; K)$ can be connected with orders of magnitude fewer links, in total, as compared to most other random graph models such as Erdős–Rényi graphs [3], random key graphs [24], and inhomogeneous random key graphs [26], where the mean degree (respectively, the minimum mean degree in inhomogeneous random key graphs) has to be on the order of $\log n$ to ensure connectivity. In contrast, the mean degree of $\mathbb{H}(n; K)$ is of order $2K$, i.e., a mean degree of 4 is sufficient to ensure connectivity of $\mathbb{H}(n; K)$.

In contrast, inhomogeneous random K-out graphs (with $K_{1,n} = 1$) require $K_{r,n}$ to grow unboundedly large as $n \to \infty$ so that the probability of connectivity approaches one in the same limit. In other words, the flexibility of arranging nodes into classes comes at the expense of sparsity. In particular, the mean degree of $\mathbb{H}(n; \mu, K)$ has to grow unboundedly large as $n \to \infty$ to ensure the connectivity of the graph. Fortunately, Theorem 2 does not specify a particular growth rate function for the sequence $K_{r,n}$, other than $K_{r,n} = \omega(1)$. Hence, one can set $K_{r,n} = \log \log \ldots \log n$ to meet the requirements of Theorem 2. As a result, inhomogeneous random K-out graphs $\mathbb{H}(n; \mu, K)$ can be connected with orders of magnitude fewer links, in total, as compared to most other random graph models as mentioned above.

### B. Numerical Study

The objective of this subsection is to validate the upper bound given by Theorem 1 in the finite-node regime using computer simulations. In Figure 2, we consider an inhomogeneous random K-out graph with three classes. Namely, we set $\mu = \{0.9, 0.06, 0.04\}$ and $K = (1, 2, K_3)$, i.e., each node is classified as class 1 with probability 0.9, class 2 with probability 0.06, and class 3 with probability 0.04. Nodes belonging to class 1 (respectively, class 2) select only one (respectively, two) other node(s) to be paired to. We vary $K_3$ from 5 to 20 and observe how the empirical probability of connectivity varies in accordance. In particular, for each value of $K_3$, we run $10^5$ independent experiments for each data point and count the number of times (out of $10^5$) when the resulting graph is connected. Dividing this number by $10^5$ gives the empirical probability of connectivity.

Note that as $K_3$ varies, $\tilde{K}_{avg}$ varies as well according to (4). We can then use (6) to plot the theoretical upper bound given by $1 - C(\mu, K)$. The results given in Figure 2 confirm the validity of Theorem 1 but also reveals its shortcomings. Observe that the bound appears to be loose for small values of $K_3$, yet it becomes tighter as $K_3$ increases. The reasoning behind this phenomenon would become apparent in Section V as we outline our approach in establishing Theorem 1. At a high level, our approach is based on bounding the probability of connectivity by the probability of not observing isolated components of size two, i.e., components formed by two class

| $\tilde{\mu}$ | $K^*(\tilde{\mu})$ | $\tilde{\mu}$ | $K^*(\tilde{\mu})$ |
|--------------|-----------------|--------------|-----------------|
| 0.1          | 3               | 0.6          | 3               |
| 0.2          | 3               | 0.7          | 3               |
| 0.3          | 3               | 0.8          | 3               |
| 0.4          | 3               | 0.9          | 6               |
| 0.5          | 3               | 0.95         | 12              |

Fig. 1. A graphical illustration of Table I. Theorem 3 predicts the absence of a zero-law in the region above (and including) the blue curve. Whether or not a zero-law exists in the red region is currently unknown and can be a topic for future work.

\[
K^*(\tilde{\mu}) = \min \left\{ K \in \{3, 4, \ldots\} : \frac{\tilde{\mu}^2 \exp \left( -2 (1 - \tilde{\mu}) \left( \frac{K-1.375}{1.2} - \frac{1}{\tilde{\mu}} \left( \frac{0.375}{K-1} \right)^{K-1} \right) \right)}{1 - \tilde{\mu} \exp \left( - (1 - \tilde{\mu}) \left( \frac{K-1.375}{1.2} - \frac{1}{\tilde{\mu}} \left( \frac{0.375}{K-1} \right)^{K-1} \right) \right)} < 1 \right\}. \tag{9}
\]
1 nodes $u$ and $v$ such that $u$ has selected $v$, $v$ has selected $u$, and none of the other $n - 2$ nodes has either selected $u$ or $v$. When $K_3$ is large, the probability of observing isolated components of sizes larger than two (i.e., three, four, etc.) will be small. Hence, the probability of connectivity in this regime would be tightly bounded by the probability of not observing isolated components of size two. However, in the regime where $K_3$ is small, isolated components of sizes other than two are more likely to be formed, as compared to the case when $K_3$ is large (see Figure 5). Since our approach does not consider such components, our bound becomes slightly loose in this regime.

Note that it is possible to enhance the upper bound given by Theorem 1 by incorporating isolated components of sizes larger than two. However, including all higher order components will likely render the analysis intractable. In what follows, we present a conjecture that provides a tighter upper bound on the probability of connectivity when $K_{r,n} = O(1)$ by incorporating isolated components of size three as well. In Conjecture 1, the probability of connectivity is bounded by the probability of not observing isolated components of size two and size three, where in both cases we focus on components consisting only of type-1 nodes. In Section VIII, we provide the intuition behind Conjecture 1.

**Conjecture 1:** Consider a scaling $K : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ and a probability distribution $\mu = \{\mu_1, \mu_2, \ldots, \mu_r\}$ with $\mu_i > 0$. If $K_{r,n} = O(1)$, we have

$$P(n; \mu, K_n) \leq 1 - \tilde{C}(\mu, K_n) + o(1), \quad (10)$$

where

$$\tilde{C}(\mu, K_n) = \left(1 + \frac{\mu^3}{2^3} e^{-2 K_{\text{avg}, n}} + \frac{4\mu^2}{3^3} e^{-3 K_{\text{avg}, n}}\right)^{-1}.$$

In Figure 3, we plot the theoretical upper bound predicted by Conjecture 1, i.e., $1 - \tilde{C}(\mu, K_n)$. Observe that the bound given by Conjecture 1 is tighter than the one given by Theorem 1 due to the inclusion of isolated components of size three.

This confirms our reasoning as to why $1 - C(\mu, K_n)$ appears to be loose for small values of $K_{\text{avg}, n}$. Indeed, tighter bounds are expected upon the inclusion of isolated components of higher order.

In Figure 4, we recall (6) and explore the interplay between $\mu$ and $K$ as governed by $C(\mu, K)$. We focus on the case where there are only two classes with $\mu = \{\mu, 1 - \mu\}$ and $K = (1, K)$. Our objective is to show the effect of increasing $\mu$ on the minimum value of $K$ required to achieve a target probability of connectivity. More precisely, for each value of $\mu$, we seek the minimum value of $K$ such that the probability of connectivity is at least 0.999. This will help derive guidelines in choosing the parameters of the scheme to achieve desired network properties. A second goal of this figure is to check whether the upper bound (8), although not very tight as discussed above, can be useful in practice, e.g., to estimate the minimum $K$ value to have a probability of connectivity at least 0.999. Empirical values of $K$ shown in Figure 4 denote the minimum $K$ for which the empirical probability of connectivity hits 0.999. This is obtained by running 3000 independent experiments for each $K$ value, and marking the minimum $K$ for which at least 2997 of the experiments resulted in a connected network. The solid line utilizes (8) to get the smallest $K$ for which there is any hope to have a probability of connectivity greater than or equal to 0.999. This is achieved by finding the largest $K$ for which (8) gives $P(n; \mu, K) < 0.999$, then adding one to this value of $K$. Namely, solid line shows

$$1 + \max\{K : C(\mu, K) > 0.001\} = 1 + \max\{K : P(n; \mu, K) < 0.999\}.$$
upper bound (8) having some practical usefulness, even though it may not indicate the exact probability of connectivity.

IV. PRELIMINARIES

Throughout, we will make use of the following results.

Fact 1 ([27, Fact 2]): For $0 \leq x < 1$, and $y = 0, 1, 2, \ldots$, we have

$$1 - xy \leq (1 - x)^y \leq 1 - xy + \frac{1}{2}x^2y^2.$$  

Fact 2 ([27, Fact 4]): Let integers $x$ and $y$ be both positive functions of $n$, where $y \geq 2x$. For $z = 0, 1, \ldots, x$, we have

$$\frac{(y-z)}{x} \geq 1 - \frac{zx}{y-z}$$

and

$$\frac{(y^z)}{x} = 1 - \frac{xz}{y} \pm O\left(\frac{x^4}{y^2}\right).$$  

Fact 3: For $r = 1, \ldots, \lfloor \frac{y}{x} \rfloor$ and $n = 1, 2, \ldots$, we have

$$\left(\frac{n}{r}\right) \leq \left(\frac{n}{r}\right) r \leq \left(\frac{n}{n-r}\right)^{n-r}.$$  

Proof: The following bound, established in [28], is valid for all $x = 1, 2, \ldots$

$$\sqrt{2\pi x^{x+0.5}e^{-x}} < x! < \sqrt{2\pi x^{x+0.5}e^{-x}} e^{\frac{1}{12x}}.$$  

Observe that

$$\sqrt{2\pi e^{\frac{1}{12x}}} \leq e$$

for all $x \geq 2$ and

$$e^{\frac{1}{12x}} \geq 1.$$  

Hence, (14) can be written as

$$\sqrt{2\pi x^{x+0.5}e^{-x}} < x! < e^{x+0.5}e^{-x}.$$  

Using (15), we get

$$\left(\frac{n}{r}\right) = \frac{n!}{r!(n-r)!} e^{n+0.5}e^{-n} \leq \frac{1}{2\pi\pi^{n+r+0.5}e^{-r}\pi^{n-r+0.5}e^{-(n-r)}}$$

$$= e^{\frac{1}{12x}} \frac{1}{2\pi\pi^{n+r+0.5}e^{-r} \pi^{n-r+0.5}e^{-(n-r)}}$$

$$\leq \left(\frac{n}{r}\right)^r \left(\frac{n}{n-r}\right)^{n-r}$$

as we use the crude bounds $r \geq 1$ and $r \leq n/2$ in the third step. \[Q.E.D.\]

For $0 \leq K \leq y$, we have

$$\left(\frac{x}{y}\right) = \prod_{\ell=0}^{K-1} \left(1 - \frac{e^{-x}}{\ell!}\right) \leq \left(\frac{x}{y}\right)^K$$

since $\frac{e^{-x}}{\ell!}$ decreases as $\ell$ increases from $0$ to $\ell = K - 1$. Moreover, we have

$$1 \pm x \leq e^{\pm x}, \quad 0 \leq x \leq 1$$

and

$$1 - e^{-x} \geq \frac{x}{2}, \quad 0 \leq x \leq 1.$$  

Throughout, we set

$$\left(\frac{x}{y}\right) = 0$$

whenever $x < y$.

V. PROOF OF THEOREM 1

In what follows, we establish (8) whenever $K_{r,n} = O(1)$. In particular, with each class 1 node selecting only one other node, we will show that whenever each class $r$ node gets paired to a bounded number of nodes, there will be a positive probability that the graph is not connected. Note that if the sequence $K_{r,n}$ is bounded, then so are the sequences $K_{i,n}$ for $i = 2, \ldots, r - 1$ by virtue of (5). Put differently

$$K_{r,n} = O(1) \Rightarrow K_{r,n} = O(1), \quad i = 2, \ldots, r - 1$$

Observe that when a positive fraction of the nodes, each, gets paired with only one node, the graph may contain isolated components consisting of two class 1 nodes, say $i$ and $j$, that were paired with each other, i.e., $\Gamma_{i,j}(\mu, K_{n}) = \{i,j\}$, $\Gamma_{i,j}(\mu, K_{n}) = \{i\}$, and $\Gamma_{i,j}(\mu, K_{n}) \subseteq N \setminus \{i,j, \ell\}$ for all $\ell \in N \setminus \{i,j\}$. Indeed, these isolated components render the graph disconnected. A graphical illustration is given in Figure 5. Our approach in establishing Theorem 1 relies on the method of second moment applied to a variable that counts the number of isolated components that contain two class 1 vertices.

Recall that $t_i$ denotes the class of node $i$. Let $U_{ij}(n; \mu, K_n)$ denote the event that nodes $i$ and $j$ are both class 1 and are...
show that when it contains two isolated components, highlighted in red and green, respectively. The first isolated component consists of two nodes, while the second isolated component consists of three nodes. We set \( n = 100 \) and \( \mu = \{0.9, 0.05, 0.05\} \). The size of each node corresponds to its degree.

forming an isolated component, i.e.,

\[
U_{ij}(n; \mu, K_n) = \left( \bigcap_{\ell \in N \setminus \{i, j\}} \Gamma_{n,\ell}(\mu, K_n) \subseteq N \setminus \{i, j, \ell\} \right) \cap [\Gamma_{n,i}(\mu, K_n) = \{j\}] \cap [\Gamma_{n,j}(\mu, K_n) = \{i\}] \cap [t_i = 1] \cap [t_j = 1].
\]

Next, let

\[
\chi_{ij}(n; \mu, K_n) = 1 \cdot U_{ij}(n; \mu, K_n)
\]

and

\[
Y(n; \mu, K_n) = \sum_{1 \leq i < j \leq n} \chi_{ij}(n; \mu, K_n).
\]

Clearly, \( Y(n; \mu, K_n) \) gives the number of isolated components in \( \mathbb{H}(n; \mu, K_n) \) that contain two class 1 vertices. We will show that when \( K_{r,n} = O(1) \), we have

\[
P \left[ Y(n; \mu, K_n) = 0 \right] \leq 1 - C(\mu, K_n) + o(1)
\]

where \( C(\mu, K_n) \) is given by (6). Recall that if \( \mathbb{H}(n; \mu, K_n) \) is connected, then it does not contain any isolated component. In particular, \( \mathbb{H}(n; \mu, K_n) \) would consist of a single component of size \( n \). However, the absence of isolated components of size two does not necessarily mean that \( \mathbb{H}(n; \mu, K_n) \) is connected, as it may contain isolated components of other sizes (see Figure 5). It follows that

\[
P(\mu, K_n) \leq P \left[ Y(n; \mu, K_n) = 0 \right].
\]

Hence, establishing (8) is equivalent to establishing

\[
P \left[ Y(n; \mu, K_n) = 0 \right] \leq 1 - C(\mu, K_n) + o(1).
\]

By applying the method of second moments \([29, \text{Remark 3.1, p. 55}]\) on \( Y(n; \mu, K_n) \), we get

\[
P \left[ Y(n; \mu, K_n) = 0 \right] \leq 1 - \frac{(E[Y(n; \mu, K_n)])^2}{E[Y^2(n; \mu, K_n)]}
\]

where

\[
E[Y(n; \mu, K_n)] = \sum_{1 \leq i < j \leq n} E[\chi_{ij}(n; \mu, K_n)] = \left( \begin{array}{c} n \vspace{1mm} \\ 2 \end{array} \right) E[\chi_{12}(n; \mu, K_n)]
\]

and

\[
E[Y^2(n; \mu, K_n)] = E \left[ \sum_{1 \leq i < j \leq n} \sum_{1 \leq m < n} \chi_{ij}(n; \mu, K_n) \chi_{m}(n; \mu, K_n) \right]
\]

by exchangeability and the binary nature of the random variables \( \{\chi_{ij}(n; \mu, K_n)\}_{1 \leq i < j \leq n} \). Observe that

\[
E[\chi_{12}(n; \mu, K_n)\chi_{13}(n; \mu, K_n)] = 0,
\]

since \( [U_{12}(n; \mu, K_n) \cap U_{13}(n; \mu, K_n)] = \emptyset \) by definition. Hence,

\[
E[Y^2(n; \mu, K_n)] = \left( \begin{array}{c} n \vspace{1mm} \\ 2 \end{array} \right) E[\chi_{12}(n; \mu, K_n)]
\]

Using (24) and (25), we get

\[
\frac{E[Y^2(n; \mu, K_n)]}{(E[Y(n; \mu, K_n)])^2} = \frac{1}{\left( \begin{array}{c} n \vspace{1mm} \\ 2 \end{array} \right) E[\chi_{1,2}(n; \mu, K_n)]} + \frac{\left( \begin{array}{c} n \vspace{1mm} \\ 2 \end{array} \right) E[\chi_{1,2}(n; \mu, K_n)\chi_{3,4}(n; \mu, K_n)]}{\left( \begin{array}{c} n \vspace{1mm} \\ 2 \end{array} \right) E[\chi_{1,2}(n; \mu, K_n)]^2}
\]

The next two results will help establish (22).

**Proposition 1:** Consider a scaling \( K : N_0 \to N_0^+ \) and a probability distribution \( \mu = \{\mu_1, \mu_2, \ldots, \mu_r\} \) with \( \mu_i > 0 \). If \( K_{r,n} = O(1) \), then

\[
\left( \begin{array}{c} n \vspace{1mm} \\ 2 \end{array} \right) E[\chi_{1,2}(n; \mu, K_n)] = (1 + o(1)) \frac{\mu_i^2}{2} \exp \left( -2 K_{\text{avg},n} \right).
\]

**Proof:** Note that under \( U_{12}(n; \mu, K_n) \), we have

\[
\Gamma_{n,1}(\mu, K_n) = \{2\} \quad \text{and} \quad \Gamma_{n,2}(\mu, K_n) = \{1\}.
\]

Moreover, we have

\[
\Gamma_{n,i}(\mu, K_n) \subseteq N \setminus \{1, 2, i\}, \quad i = 3, 4, \ldots, n.
\]
Recall that each of the other \( n - 2 \) nodes is class \( i \) with probability \( \mu_i \) and that the random variables \( \Gamma_{n,1}(\mathbf{K}_n), \Gamma_{n,2}(\mathbf{K}_n), \ldots, \Gamma_{n,n}(\mathbf{K}_n) \) are mutually independent. Hence, we have

\[
\mathbb{E} \left[ \chi_{12}(n; \mathbf{K}_n) \right] = \mathbb{P} \left[ U_{12}(n; \mathbf{K}_n) \right] = \mu_1^2 \left( \frac{1}{n-1} \right)^2 \left( \sum_{i=1}^{r} \mu_i \left( \frac{n-3}{K_{i,n}} \right) \right)^{n-2}.
\]

Then, we have

\[
\binom{n}{2} \mathbb{E} \left[ \chi_{12}(n; \mathbf{K}_n) \right] = \mu_1^2 \left( \frac{n}{n-1} \right) \left( \sum_{i=1}^{r} \mu_i \left( \frac{n-3}{K_{i,n}} \right) \right)^{n-2} = \mu_1^2 \left( \frac{n}{n-1} \right) \cdot \left( \sum_{i=1}^{r} \mu_i \left( \frac{n-1}{K_{i,n}} \right) (n-2) \right)^{n-2} = \mu_1^2 \left( \frac{2}{n-1} \right) \cdot \left( \sum_{i=1}^{r} \mu_i (n-1)(n-2) \right)^{n-2} \leq \mu_1^2 \left( \frac{2}{n-1} \right) \cdot \exp \left( -2 \left( \frac{n-1}{n-2} \right) \sum_{i=1}^{r} \mu_i K_{i,n} + \frac{1}{n-1} \sum_{i=1}^{r} \mu_i K_{i,n}^2 \right) = (1 + o(1)) \mu_1^2 \frac{n}{n-1} e^{-2} \kappa_{\text{avg},n},
\]

where the last equality follows since \( K_{r,n} = O(1) \).

**Proposition 2:** Consider a scaling \( K : N_0 \rightarrow N_0 \) and a probability distribution \( \mathbf{u} = \{\mu_1, \mu_2, \ldots, \mu_r\} \) with \( \mu_i > 0 \). If \( K_{r,n} = O(1) \), then

\[
\mathbb{E} \left[ \chi_{12}(n; \mathbf{K}_n) \chi_{34}(n; \mathbf{K}_n) \right] = \mathbb{P} \left[ U_{12}(n; \mathbf{K}_n) \cap U_{34}(n; \mathbf{K}_n) \right] = 1 + o(1).
\]

**Proof:** Note that an immediate consequence of Fact 2 is that

\[
\binom{n}{2} \left( \frac{n-2}{2} \right) = 1 + o(1).
\]

Observe that under \( U_{12}(n; \mathbf{K}_n) \cap U_{34}(n; \mathbf{K}_n) \), we have

\[
\Gamma_{n,1}(\mathbf{K}_n) = \{2\} \quad \text{and} \quad \Gamma_{n,2}(\mathbf{K}_n) = \{1\}
\]

and

\[
\Gamma_{n,3}(\mathbf{K}_n) = \{4\} \quad \text{and} \quad \Gamma_{n,4}(\mathbf{K}_n) = \{3\}.
\]

Moreover, we have

\[
\Gamma_{n,i}(\mathbf{K}_n) \subseteq \mathcal{N} \setminus \{1, 2, 3, 4, i\}, \quad i = 5, 6, \ldots, n.
\]

Invoking Fact 2, we get

\[
\binom{n}{2} \left( \frac{n-2}{2} \right) \mathbb{E} \left[ \chi_{12}(n; \mathbf{K}_n) \chi_{34}(n; \mathbf{K}_n) \right] = \mathbb{P} \left[ U_{12}(n; \mathbf{K}_n) \cap U_{34}(n; \mathbf{K}_n) \right] = 1 + o(1) \cdot \left( \sum_{i=1}^{r} \mu_i \left( \frac{n-3}{K_{i,n}} \right)^{n-4} \right).
\]

The main result (8) now follows by virtue of (22) and (23) as we combine (26), (27), and (28). Recall that (7) follows from (8) since \( \kappa_{\text{avg},n} = O(1) \) when \( K_{r,n} = O(1) \).

**VI. PROOF OF THEOREM 2**

In what follows, we establish that

\[
\lim_{n \to \infty} P(n; \mathbf{K}_n) = 1
\]

whenever \( K_{r,n} = \omega(1) \).

Observe that for any non-empty subset \( S \) of nodes, i.e., \( S \subseteq \mathcal{N} \), we say that \( S \) is isolated in \( \mathbb{H}(n; \mathbf{K}_n) \) if there are no edges in \( \mathbb{H}(n; \mathbf{K}_n) \) between the nodes in \( S \) and the nodes in the complement \( S^c = \mathcal{N} \setminus S \). This is characterized by the event \( B_n(\mathbf{K}_n; S) \) given by

\[
B_n(\mathbf{K}_n; S) = \bigcap_{i \in S} \bigcap_{j \in S^c} \left( \{i \notin \Gamma_{n,j}(\mathbf{K}_n) \} \cap [j \notin \Gamma_{n,i}(\mathbf{K}_n)] \right).
\]

Note that if \( \mathbb{H}(n; \mathbf{K}_n) \) is not connected, then there must exist a non-empty proper subset \( S \) of nodes which is isolated.
Recall that each node in $\mathbb{H}(n; \mu, K_n)$ is class $i$ with probability $\mu_i$ and that $K_{1,n} = 1$. Thus, we may observe isolated sets in $\mathbb{H}(n; \mu, K_n)$ of cardinality $\ell = 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor$. Thus, with $D_n(\mu, K_n)$ denoting the event that $\mathbb{H}(n; \mu, K_n)$ is connected, we have the inclusion

$$D_n(\mu, K_n) \subseteq \bigcup_{S \in \mathcal{P}_n : 2 \leq |S| \leq \lfloor \frac{n}{2} \rfloor} B_n(\mu, K_n; S) \quad (30)$$

where $\mathcal{P}_n$ stands for the collection of all non-empty subsets of $\mathcal{N}$. A standard union bound argument immediately gives

$$\mathbb{P}[D_n(\mu, K_n)] \leq \sum_{S \in \mathcal{P}_n : 2 \leq |S| \leq \lfloor \frac{n}{2} \rfloor} \mathbb{P}[B_n(\mu, K_n; S)] = \sum_{\ell = 2}^{\lfloor \frac{n}{2} \rfloor} \left( \sum_{S \in \mathcal{P}_n, |S| = \ell} \mathbb{P}[B_n(\mu, K_n; S)] \right) \quad (31)$$

where $\mathcal{P}_{n, \ell}$ denotes the collection of all subsets of $\mathcal{N}$ with exactly $\ell$ elements.

For each $\ell = 1, \ldots, n$, we simplify the notation by writing $B_{n, \ell}(\mu, K_n) = B_n(\mu, K_n; \{1, \ldots, \ell\})$. Under the enforced assumptions, exchangeability implies

$$\mathbb{P}[B_n(\mu, K_n; S)] = \mathbb{P}[B_{n, \ell}(\mu, K_n)], \quad S \in \mathcal{P}_{n, \ell}$$

and the expression

$$\sum_{S \in \mathcal{P}_{n, \ell}} \mathbb{P}[B_n(\mu, K_n; S)] = \binom{n}{\ell} \mathbb{P}[B_{n, \ell}(\mu, K_n)] \quad (32)$$

follows since $|\mathcal{P}_{n, \ell}| = \binom{n}{\ell}$. Substituting into (31) we obtain the bounds

$$\mathbb{P}[D_n(\mu, K_n)] \leq \sum_{\ell = 2}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{\ell} \mathbb{P}[B_{n, \ell}(\mu, K_n)] \quad (33)$$

For each $\ell = 2, \ldots, \lfloor \frac{n}{2} \rfloor$, it is easy to check that

$$\mathbb{P}[B_{n, \ell}(\mu, K_n)] = \left( \sum_{i=1}^{\ell} \mu_i \left( \frac{n-1}{K_{i,n}} \right) \right)^{\ell} \left( \sum_{i=1}^{\ell} \mu_i \left( \frac{n-1}{K_{i,n}} \right) \right)^{-n-\ell} \quad (34)$$

To see why this last relation holds, recall that for nodes $\{1, \ldots, \ell\}$ to be isolated in $\mathbb{H}(n; \mu, K_n)$, we need that (i) none of the sets $\Gamma_{n,1}(\mu, K_n), \ldots, \Gamma_{n,\ell}(\mu, K_n)$ contains an element from the set $\{\ell + 1, \ldots, n\}$; and (ii) none of the sets $\Gamma_{n,\ell+1}(\mu, K_n), \ldots, \Gamma_{n,n}(\mu, K_n)$ contains an element from $\{1, \ldots, \ell\}$. More precisely, we must have

$$\Gamma_{n,i}(\mu, K_n) \subseteq \{1, \ldots, \ell\} \setminus \{i\}, \quad i = 1, \ldots, \ell$$

and

$$\Gamma_{n,j}(\mu, K_n) \subseteq \{\ell + 1, \ldots, n\} \setminus \{j\}, \quad j = \ell + 1, \ldots, n.$$ Hence, the validity of (34) is now immediate from (2) and the mutual independence of the rvs $\Gamma_{n,1}(\mu, K_n), \ldots, \Gamma_{n,n}(\mu, K_n)$.

1Note that if vertices $S$ form an isolated set then so do vertices $\mathcal{N} - S$, hence the sum need to be taken only until $\lfloor \frac{n}{2} \rfloor$.

We now establish that under the enforced assumptions of Theorem 2, we have

$$\lim_{n \to \infty} \sum_{\ell = 2}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{\ell} \mathbb{P}[B_{n, \ell}(\mu, K_n)] = 0$$

which in turn establishes Theorem 2 by virtue of (33).

Note that the quantities

$$\left( \frac{\ell - 1}{K_{\ell,n}} \right)^{\ell} \left( \frac{n-\ell}{K_{n-\ell,n}} \right)^{n-\ell}$$

are monotonically decreasing in $K_{\ell,n}$. We use (17) and (34) to get

$$\mathbb{P}[B_{n, \ell}(\mu, K_n)] \leq \left( \sum_{i=1}^{\ell-1} \mu_i \left( \frac{n-1}{K_{i,n}} \right) \right) \left( \sum_{i=1}^{\ell-1} \mu_i \left( \frac{n-1}{K_{i,n}} \right) \right)^{-n-\ell} \quad (35)$$

where $\tilde{\mu} = \sum_{i=1}^{\ell-1} \mu_i$ and $1 - \tilde{\mu} = \mu_r$.

Observe that the bound appearing in (35) resembles the case where each node belongs to one of two classes. Namely, a node could either be class 1 (with probability $\tilde{\mu}$) or class $r$ (with probability $1 - \tilde{\mu}$). We further use (18) to get

$$\mathbb{P}[B_{n, \ell}(\mu, K_n)] \leq \left( \tilde{\mu} \left( \frac{\ell}{n} \right) \right)^{\ell} \left( 1 - \left( \frac{\ell}{n} \right) \right)^{n-\ell} \left( 1 - \tilde{\mu} \left( \frac{\ell}{n} \right) \right)^{n-\ell} \quad (36)$$

\[ \cdot \left( 1 - \left( \frac{\ell}{n} \right) \right)^n \left( 1 - \tilde{\mu} \left( \frac{\ell}{n} \right) \right)^n \]
where (39) follows from the facts that

\[ (1 - \hat{\mu}) (n - \ell) \left( 1 - e^{-\ell \left( \frac{K_{r,n} - 1}{n} \right)} \right) \]

Combining (13) with (36), we conclude that

\[ \mathbb{P} [ D_n(\mathbf{\mu}, \mathbf{K}_n)^c] \leq \sum_{\ell = 2}^{\left\lceil \frac{n}{\ell} \right\rceil} \left( \frac{n}{\ell} \right)^{\frac{n}{\ell}} \mathbb{P} [ B_{n,\ell}(\mathbf{\mu}, \mathbf{K}_n)] = \sum_{\ell = 2}^{\left\lceil \frac{n}{\ell} \right\rceil} \hat{\mu}^\ell A_{n,\ell} \]

(37)

with \( 2 \leq \ell \leq n/2 \).

Next, our goal is to derive an upper bound on \( A_{n,\ell} \) that is valid for all \( n \) sufficiently large and \( \ell = 2, \ldots, \left\lceil \frac{n}{2} \right\rceil \), and show that this bound tends to zero as \( n \) gets large. Fix \( n \) and note that for each \( \ell = 2, \ldots, \left\lceil \frac{n}{2} \right\rceil \), either one of the following should hold

\[ \frac{\ell (K_{r,n} - 1)}{n} \leq 1 \quad \text{and} \quad \frac{\ell (K_{r,n} - 1)}{n} > 1. \]

If it holds that \( \frac{\ell (K_{r,n} - 1)}{n} \leq 1 \), then we use (19) to get \( 1 - e^{-\ell \left( \frac{K_{r,n} - 1}{n} \right)} \geq \frac{\ell (K_{r,n} - 1)}{2n} \). Using this in (38) yields

\[ A_{n,\ell} \]

\[ \leq \exp \left( \frac{1 - \hat{\mu}}{\hat{\mu}} \left( \frac{\ell}{n} \right) K_{r,n} - 1 \right) - (1 - \hat{\mu}) (n - \ell) \left( 1 - e^{-\ell \left( \frac{K_{r,n} - 1}{n} \right)} \right) \]

\[ \leq \exp \left( \frac{1 - \hat{\mu}}{\hat{\mu}} \left( \frac{1}{2} \right) K_{r,n} - 1 \right) - (1 - \hat{\mu}) \left( 1 - \frac{\ell (K_{r,n} - 1)}{4} \right) \]

\[ = \exp \left( - (1 - \hat{\mu}) \ell \left( \frac{K_{r,n} - 1}{4} - \frac{(0.5)K_{r,n} - 1}{\hat{\mu}} \right) \right) \]

\[ \leq \exp \left( -2 (1 - \hat{\mu}) \left( \frac{K_{r,n} - 1}{4} - \frac{(0.5)K_{r,n} - 1}{\hat{\mu}} \right) \right) \]

(40)

where (39) follows from the facts that \( n - \ell \geq n/2 \) and \( \ell/n \leq 0.5 \) on the specified range for \( \ell \), and (40) follows for all \( K_{r,n} \) sufficiently large such that \( K_{r,n} \geq \left\lceil 4 \left( \frac{(0.5)K_{r,n} - 1}{\hat{\mu}} \right) + 1 \right\rceil \) upon noting that \( \ell \geq 2 \).

If, on the other hand, it holds that \( \frac{\ell (K_{r,n} - 1)}{n} > 1 \), we see that \( 1 - e^{-\ell \left( \frac{K_{r,n} - 1}{n} \right)} \geq 1 - e^{-1} \). Reporting this into (38) and using \( \ell \leq n/2 \), we get

\[ A_{n,\ell} \]

\[ \leq \exp \left( \frac{1 - \hat{\mu}}{\hat{\mu}} \left( \frac{\ell}{n} \right) K_{r,n} - 1 \right) - (1 - \hat{\mu}) (n - \ell) \left( 1 - e^{-1} \right) \]

\[ \leq \exp \left( \frac{1 - \hat{\mu}}{\hat{\mu}} \left( \frac{n}{2} \right) (0.5)K_{r,n} - 1 - (1 - \hat{\mu}) \frac{n}{2} \left( 1 - e^{-1} \right) \right) \]

\[ = \exp \left( - (1 - \hat{\mu}) \frac{n}{2} \left( 1 - e^{-1} - \frac{(0.5)K_{r,n} - 1}{\hat{\mu}} \right) \right) \]

(41)

Combining (40) and (41) we see that \( A_{n,\ell} \leq \Psi(n, \mathbf{\mu}, \mathbf{K}_n) \) for all \( n \) sufficiently large and all \( \ell = 2, \ldots, \left\lceil \frac{n}{2} \right\rceil \), where

\[ \Psi(n, \mathbf{\mu}, \mathbf{K}_n) \]

\[ = \max \left\{ \begin{array}{c} \exp \left( -2 (1 - \hat{\mu}) \left( \frac{K_{r,n} - 1}{4} - \frac{(0.5)K_{r,n} - 1}{\hat{\mu}} \right) \right) \right\} \]

\[ \exp \left( - (1 - \hat{\mu}) \frac{n}{2} \left( 1 - e^{-1} - \frac{(0.5)K_{r,n} - 1}{\hat{\mu}} \right) \right) \right\} \]

(42)

Observing that the bound derived on \( A_{n,\ell} \) is independent on \( \ell \), we get from (42) and (37)

\[ \sum_{\ell = 2}^{\left\lceil \frac{n}{2} \right\rceil} \left( \frac{n}{\ell} \right)^{\frac{n}{\ell}} \mathbb{P} [ B_{n,\ell}(\mathbf{\mu}, \mathbf{K}_n)] \leq \Psi(n, \mathbf{\mu}, \mathbf{K}_n) \sum_{\ell = 2}^{\infty} \hat{\mu}^\ell \]

\[ = \frac{\hat{\mu}^2}{1 - \hat{\mu}} \Psi(n, \mathbf{\mu}, \mathbf{K}_n). \]

(43)

Letting \( n \) go to infinity, it is now easy to see that

\[ \lim_{n \to \infty} \Psi(n, \mathbf{\mu}, \mathbf{K}_n) = 0, \quad 2 \leq \ell \leq n/2 \]

under the enforced assumption that \( \lim_{n \to \infty} K_{r,n} = \infty \). Hence, the conclusion

\[ \lim_{n \to \infty} \sum_{\ell = 2}^{\left\lceil \frac{n}{2} \right\rceil} \left( \frac{n}{\ell} \right)^{\frac{n}{\ell}} \mathbb{P} [ B_{n,\ell}(\mathbf{\mu}, \mathbf{K}_n)] = 0 \]

immediately follows since \( 0 < \hat{\mu} < 1 \). This establishes Theorem 2.

VII. PROOF OF THEOREM 3

In this section, we consider the case when \( K_{r,n} \) is fixed and does not scale with \( n \). Recall that \( \hat{\mu} = \sum_{i=1}^{\infty} \mu_i \). We will show that when \( K_r \geq K^*(\hat{\mu}) \), the probability of connectivity of \( \mathbb{H}(n; \mathbf{\mu}, \mathbf{K}_n) \) is strictly larger than zero in the limit of large \( n \), hence a zero-law does not hold in this case. Indeed, an easy coupling argument implies that a zero-law does not hold for any scaling \( \mathbf{K} : \mathbb{N}_0 \to \mathbb{N}_0^d \) such that \( K_{r,n} \geq K^*(\hat{\mu}) \).

Recall (37) and (38). Fix \( n = 2, 3 \), sufficiently large. If it holds that \( \frac{\ell (K_{r,n} - 1)}{n} \leq 0.375 \), then we use the fact that

\[ 1 - e^{-x} \geq x/1.2 \quad 0 \leq x \leq 0.375 \]
to get $1 - e^{-\ell(K_r-1)} \geq \frac{\ell(K_r-1)}{1.2n}$. Using this in (38) yields
\[
A_{n,\ell} \leq \exp \left( \frac{1 - \tilde{\mu}}{\mu} \ell \left( \frac{K_r-1}{n} - (1 - \tilde{\mu})(n - \ell) \frac{\ell(K_r-1)}{1.2n} \right) \right)
\leq \exp \left( \frac{1 - \tilde{\mu}}{\mu} \ell \left( \frac{0.375}{K_r-1} \right) \right)
\leq \exp \left( - (1 - \tilde{\mu}) \ell \left( \frac{K_r - 1.375}{1.2} \right) \right)
\leq \exp \left( - (1 - \tilde{\mu}) \ell \left( \frac{0.375}{K_r-1} \right) \right)
\] (44)

(45)

where (44) follows from the facts that $n - \ell \geq n \left(1 - \frac{0.375}{K_r-1}\right)$ and $\ell/n \leq \frac{0.375}{K_r-1}$ on the specified range for $\ell$.

If, on the other hand, it holds that $\frac{\ell(K_r-1)}{n} > 0.375$, we see that $1 - e^{-\ell(\frac{0.375}{n})} \geq 1 - e^{-0.375\ell}$. Reporting this into (38) and using the fact that $\ell \leq n/2$, we get
\[
A_{n,\ell} \leq \exp \left( - (1 - \tilde{\mu}) \frac{n}{2} \left(1 - e^{-0.375\ell} - (0.5\frac{K_r-1}{\tilde{\mu}})\right) \right).
\] (46)

Assume that $K_r$ is large enough so that
\[
\left(1 - e^{-0.375\ell} - (0.5\frac{K_r-1}{\tilde{\mu}})\right) > 0 \quad \text{and} \quad \left(\frac{K_r - 1.375}{1.2} - \frac{0.375}{\tilde{\mu}(K_r-1)}\right) > 0.
\] (47)

Using (37), (45), and (46), we could then write
\[
\mathbb{P}[D_n(\mu, K)] \leq \sum_{\ell = 2}^{\left\lfloor \frac{n}{\ell} \right\rfloor} \binom{n}{\ell} \mathbb{P}[B_{n,\ell}(\mu, K)]
\leq \sum_{\ell = 2}^{\left\lfloor \frac{n}{\ell} \right\rfloor} \mu^\ell A_{n,\ell}
= \sum_{\ell = 2}^{\left\lfloor \frac{n}{\ell} \right\rfloor} \tilde{\mu}^\ell A_{n,\ell} + \sum_{\ell = \left\lfloor \frac{n}{0.375} \right\rfloor + 1} \tilde{\mu}^\ell A_{n,\ell}
\]
leading to (48), shown at the bottom of the next page.

Observe that in the limit of large $n$, the second term appearing in (48) vanishes. Since $P(n; \mu, K) = 1 - \mathbb{P}[D_n(\mu, K)]$, we then get
\[
P(n; \mu, K) \geq 1 - \tilde{\mu}^2 \cdot \exp \left( - 2(1 - \tilde{\mu}) \left(\frac{0.375}{K_r-1} \right)\right)
\leq 1 - \tilde{\mu} \cdot \exp \left( - (1 - \tilde{\mu}) \left(\frac{0.375}{K_r-1} \right)\right) - o(1)
\] (49)

whenever (47) holds. Let $K^*(\tilde{\mu})$ denote the smallest value of $K_r$ such that (47) holds and
\[
\tilde{\mu}^2 \cdot \exp \left( - 2(1 - \tilde{\mu}) \left(\frac{0.375}{K_r-1} \right)\right)
\leq 1 - \tilde{\mu} \cdot \exp \left( - (1 - \tilde{\mu}) \left(\frac{0.375}{K_r-1} \right)\right) - 1 < 1.
\] (50)

Then, in view of (49), we have the desired result
\[
\liminf_{n \to \infty} P(n; \mu, K) > 0 \quad \text{whenever} \quad 0 < \tilde{\mu} \leq 0.8 \quad \text{and} \quad K_r \geq 3.
\] (51)

This can be shown by the following coupling argument. Consider a graph $\mathbb{H}_A(n; \mu, K_n)$ with $\mu = \{\mu_1, \ldots, \mu_r\}$ and $K = (K_1, \ldots, K_r)$. Assume that $\mu_r \geq 0.2$ and hence $\tilde{\mu} = \mu_1 + \ldots + \mu_{r-1} \leq 0.8$ and $K_r \geq 3$. To form the coupling, consider the graph $\mathbb{H}_B(n; \tilde{\mu}, K)$ with $\tilde{\mu} = \{\tilde{\mu}_1, \ldots, \tilde{\mu}_r\}$ constructed as
\[
\tilde{\mu}_1 = \mu_1 + (\mu_r - 0.2); \quad \tilde{\mu}_r = 0.2; \quad \tilde{\mu}_i = \mu_i, \quad i = 2, \ldots, r-1.
\]

Since $\tilde{\mu}_r = 0.2$, we have $\tilde{\mu} = \tilde{\mu}_1 + \ldots + \tilde{\mu}_{r-1} = 0.8$. Also, since $K_r \geq 3$, we have from the preceding discussion that
\[
\liminf_{n \to \infty} \mathbb{P}[\mathbb{H}_B(n; \tilde{\mu}, K) \text{ is connected}] > 0.
\] (53)

We will show the existence of a coupling such that $\mathbb{H}_B(n; \tilde{\mu}, K)$ is a spanning subgraph of $\mathbb{H}_A(n; \mu, K_n)$, whence
\[
\liminf_{n \to \infty} \mathbb{P}[\mathbb{H}_A(n; \mu, K) \text{ is connected}] \geq \liminf_{n \to \infty} \mathbb{P}[\mathbb{H}_B(n; \tilde{\mu}, K) \text{ is connected}].
\] (54)

See for instance [11, 30, 31, pp.7] for similar applications of the coupling argument. To establish the coupling under which $\mathbb{H}_B(n; \tilde{\mu}, K) \subseteq \mathbb{H}_A(n; \mu, K_n)$ under the enforced conditions, we start with an arbitrary realization of $\mathbb{H}_B(n; \mu, K_n)$. Then, we let each type-1 node in $\mathbb{H}_B(n; \tilde{\mu}, K)$ select an additional set of $K_r - K_1$ distinct nodes that it did not initially pick with probability $(\tilde{\mu}_1 - \mu_1)/\mu_1 = (\mu_r - 0.2)/\mu_1 \geq 0$ (and no extra choices with the complementary probability $\mu_1/\mu_1$). Then, we draw an undirected edge between each pair of nodes where at least one picked the other. Clearly, this process creates a graph whose edge set is a superset of the edge set of the given realization of $\mathbb{H}_B(n; \mu, K_n)$. In addition, in the new graph, the probability of a node picking only one other node (i.e., being type-1) is given by $\tilde{\mu}_1\mu_1/\mu_1 = \mu_1$. Also, probability of choosing $K_2$ neighbors (i.e., being type-$2$) is given by $0.2 + \mu_1(\mu_r - 0.2)/\mu_1 = \mu_r$. Since the probability of other node types already satisfy $\mu_j = \mu_j$ for all
The values of $K^*(\tilde{\mu})$ corresponding to different values for $\tilde{\mu}$. When $K_{r,n} \geq K^*(\tilde{\mu})$, we have\[ \lim_{n \to \infty} \mathbb{P} [\mathbb{H}_n(n;\mu, K_n) \text{ is connected}] > 0; \]

hence a zero-law does not exist when $K_{r,n} \geq K^*(\tilde{\mu})$.

| $\tilde{\mu}$ | $K^*(\tilde{\mu})$ | $\tilde{\mu}$ | $K^*(\tilde{\mu})$ |
|--------------|-------------------|--------------|-------------------|
| 0.1          | 6                 | 0.6          | 4                 |
| 0.2          | 5                 | 0.7          | 4                 |
| 0.3          | 5                 | 0.8          | 3                 |
| 0.4          | 4                 | 0.9          | 6                 |
| 0.5          | 4                 | 0.95         | 12                |

$j = 2, \ldots, r - 1$, we conclude that the graph obtained by the above realization constitutes a realization of $\mathbb{H}_A(n;\mu, K_n)$. Since, the initial realization of $\mathbb{H}_B(n;\mu, K)$ was arbitrary, this establishes the desired coupling argument and we conclude that (54) holds. From (53) this in turn gives\[ \lim_{n \to \infty} \mathbb{P} [\mathbb{H}_A(n;\mu, K) \text{ is connected}] > 0 \quad (55) \]
establishing (52).

VIII. INTUITION BEHIND CONJECTURE 1

In Section V, we used a second moment argument to show that when $K_{r,n} = O(1)$ we have\[ P(n;\mu, K_n) \leq \mathbb{P}[Y(n;\mu, K_n) = 0] \leq 1 - \frac{1}{\mathbb{E}[Y(n;\mu, K_n)]} + o(1) \]
where $Y(n;\mu, K_n)$ gives the number of isolated components of size two.

Conjecture 1 is based on calculating the first moment of the number of isolated components of size two and size three, where in both cases we focus on components consisting only of type-1 nodes. In particular, with $Z(n;\mu, K_n)$ denoting the number of isolated components (consisting only of type-1 nodes) of size two and size three, we conjecture that\[ P(n;\mu, K_n) \leq \mathbb{P}[Z(n;\mu, K_n) = 0] \leq 1 - \frac{1}{\mathbb{E}[Z(n;\mu, K_n)]} + o(1). \quad (56) \]

We expect, as in the case with size two components (see Section V), that a second moment analysis would lead to a proof of this conjecture. In the remainder of this section, we calculate the first moment of $Z(n;\mu, K_n)$ to establish (10) under the conjecture (56).

Recall that $t_i$ denotes the class of node $i$. Let $\tilde{\nu}_{ijk}(n;\mu, K_n)$ denote the event that nodes $i$, $j$, and $k$ are all class 1 and are forming an isolated component, i.e.,\[ \tilde{\nu}_{ijk}(n;\mu, K_n) \]

Next, let $\tilde{\chi}_{ijk}(n;\mu, K_n) = 1 (\tilde{\nu}_{ijk}(n;\mu, K_n))$ and $\tilde{Y}(n;\mu, K_n) = \sum_{1 \leq i < j < k \leq n} \tilde{\chi}_{ijk}(n;\mu, K_n)$.

Let $Z(n;\mu, K_n) = Y(n;\mu, K_n) + \tilde{Y}(n;\mu, K_n)$, where $Y(n;\mu, K_n)$ gives the number of isolated components of size two (see Section V). It follows that\

\[ \mathbb{E}[Z(n;\mu, K_n)] = \mathbb{E}[Y(n;\mu, K_n)] + \mathbb{E}[\tilde{Y}(n;\mu, K_n)] \]

The next result establishes (10) under the conjecture (56).

**Proposition 3:** Consider a scaling $K : \mathbb{N}_0 \to \mathbb{N}_0$ and a probability distribution $\mu = \{\mu_1, \mu_2, \ldots, \mu_r\}$ with $\mu_i > 0$. If $K_{r,n} = O(1)$, then\

\[ \mathbb{E}[Z(n;\mu, K_n)] = \frac{n}{2} \mathbb{E}[\chi_{12}] + \left( \frac{n}{3} \right) \mathbb{E}[\tilde{\chi}_{123}] \]

\[ = (1 + o(1)) \left( \frac{\mu_1^2}{2} \exp(-2K_{avg,n}) + \frac{4\mu_1^3}{3} \exp(-3K_{avg,n}) \right). \quad (59) \]
Proof: In Proposition 1, we established that
\[
\frac{n}{2} \mathbb{E}[\chi_{12}] = (1 + o(1)) \mu_1 \frac{2}{n-1} \exp(-2 K_{avg,n}).
\]
Hence, Proposition 3 will be established upon showing that
\[
\frac{n}{3} \mathbb{E}[\chi_{123}] = (1 + o(1)) \frac{4\mu_3^3}{3} \exp(-3 K_{avg,n}).
\]
The proof is in the same vein with Proposition 1, but we give it below for completeness. Note that we have
\[
\mathbb{E}[\chi_{123}] = \mathbb{P}(U_{123}(n; \mu, K_n)) = \mu_3^3 \left( \frac{2}{n-1} \right)^3 \left( \sum_{i=1}^{r} \mu_i \left( \frac{n-1}{K_{i,n}} \right) \right)^{n-3}.
\]
Then, it follows that
\[
\frac{n}{3} \mathbb{E}[\chi_{123}] = 4\mu_3^3 \frac{(n(n-2))}{(n-1)^2} \left( \sum_{i=1}^{r} \mu_i \frac{(n-1-K_{i,n})(n-2-K_{i,n})(n-3-K_{i,n})}{(n-1)(n-2)(n-3)} \right)^{n-3}
\]
\[
= 4\mu_3^3 \frac{(n(n-2))}{(n-1)^2} \left( \sum_{i=1}^{r} \mu_i \frac{1}{n-3} \left( 1 - \frac{K_{i,n}}{n-3} \right) \right)^{n-3}
\]
\[
= 4\mu_3^3 \frac{(n(n-2))}{(n-1)^2} \left( \sum_{i=1}^{r} \mu_i \left( 1 - \frac{K_{i,n}}{n-3} \right) \right)^{n-3}
\]
\[
= 4\mu_3^3 \frac{(n(n-2))}{(n-1)^2} \left( \sum_{i=1}^{r} \mu_i \left( 1 - \frac{K_{i,n}}{n-3} \right) \right)^{n-3}
\]
\[
= 4\mu_3^3 \frac{(n(n-2))}{(n-1)^2} \left( \sum_{i=1}^{r} \mu_i \left( 1 - \frac{K_{i,n}}{n-3} \right) \right)^{n-3}
\]
\[
\leq 1 + o(1)) \frac{4\mu_3^3}{3} e^{-3 K_{avg,n}}
\]
where the last inequality follows by virtue of (18) and that \( K_{r,n} = O(1) \).

IX. CONCLUSION

In this paper, we have proposed inhomogeneous random K-out graphs \( \mathbb{H}(n; \mu, K_n) \) where nodes are arranged into \( r \) disjoint classes and the number of selections made by a node is dependent on its class. In particular, we consider the case where each node is classified as class \( i \) with probability \( \mu_i > 0 \) for \( i = 1, \ldots, r \). A class \( i \) node selects \( K_{i,n} \) other nodes uniformly at random to be paired to. Two nodes are deemed adjacent if at least one selects the other. Without loss of generality, we assumed that \( K_{1,n} \leq K_{2,n} \leq \cdots \leq K_{r,n} \).

Earlier results on homogeneous random K-out graphs (where all nodes select \( K \) other nodes) suggest that the graph is connected w.h.p if \( K \geq 2 \). Hence, \( \mathbb{H}(n; \mu, K_n) \) is trivially connected whenever \( K_{1,n} \geq 2 \). We investigated the connectivity of \( \mathbb{H}(n; \mu, K_n) \) in the particular case when \( K_{1,n} = 1 \). Our results revealed that when \( K_{1,n} = 1 \), \( \mathbb{H}(n; \mu, K_n) \) is connected with high probability if and only if \( K_{r,n} = \omega(1) \). Any bounded choice of \( K_{r,n} \) is shown to yield a positive probability of \( \mathbb{H}(n; \mu, K_n) \) being not connected, and an explicit upper bound on the probability of connectivity is provided. We also presented various numerical results to verify our theory in the finite node regime and illustrate the tightness of the derived bound. Finally, we presented a result that provides conditions on \( \mu \) and \( K_{r,n} \) for which the probability of connectivity of \( \mathbb{H}(n; \mu, K_n) \) is strictly larger than zero in the limit of large \( n \).

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