Consensus of Network of Homogeneous Agents with General Linear Dynamics

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Abstract

This work addresses the synchronization/consensus problem for identical multi-agent system (MAS). The dynamics of the agent is a general linear system that can be unstable. This work uses a gain matrix in a dynamic compensator setting and shows that under mild conditions, synchronization/consensus is achieved when the gain is sufficiently large. The proposed controller structure can be seen as a special case of existing MAS structures but offers consensus conditions that are simpler than the existing results. It can be applied to the following communication settings: fixed network, switching among undirected and connected networks and switching among directed and connected networks. An adaptive scheme to realize the gain matrix for the case of undirected and fixed network is also shown and illustrated via an example.

Key words: Consensus, Multi-agent system, Network System.

1 Introduction

The study of multi-agent system (MAS) has been an active area of research and one well-studied topic in MAS is the consensus or synchronization problem among agents connected via a communication network. Early studies of this problem deal with agents that are scalar systems [1–3] and recent focus is on agents that have general linear dynamics [4–7]. These agents have a general expression of $\dot{x}_i = Ax_i + Bu_i$ where $x_i \in \mathbb{R}^n$ is the state and $A, B$ are respectively the state and input matrices. Various configurations of the systems have been studied, they varied from fixed communication networks [6–11] to switching networks [4,12,13] that are undirected or directed, from discrete-time [8,9,11,12,14] to continuous-time systems [6,7,10,13,15], from homogeneous [4–15] to heterogeneous agents [16–18] and from stable [4,14,15] to exponentially unstable $A$ [6–13].

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In most of these approaches, the assumption of $B$ being full row rank is made and the controller structure is of the form

$$u_i(t) = cK \sum_{j=1}^{m} \alpha_{ij}(t)(x_j(t) - x_i(t)), \ i \in \mathbb{Z}^m$$

where $m$ is the number of agents, $\alpha_{ij}(t)$ (or $\alpha_{ij}$ when graph is fixed) being the weighting coefficients for some choices of $c \in \mathbb{R}$ and $K \in \mathbb{R}^{n \times n}$. The various approaches deal with the design procedure for $c$ and $K$ under some chosen configurations of the communication network with $K$ often obtained from some modified form of Riccatti equation.

In the most fundamental case of a fixed and connected network, the use of (1) results in the consensus conditions of the matrices $(A + \lambda^iBK)$ for all $i = 2, \cdots, m$ having stable eigenvalues [6,11]. Here, $\lambda^i$ is the $i^{th}$ eigenvalue of the Laplacian matrix. Such a result is hard to check/design in practice for two reasons. The first is that knowledge of all eigenvalues of the Laplacian matrix is needed. The second is that the choice of a common $K$ has to be such that the eigenvalues of all $m - 1$ matrices be stable.

This paper addresses the problem of achieving consensus for agents described by general linear system for the case where $A$ is unstable in a leaderless network. It uses a gain matrix of a special structure before the diffusive term. In this sense, it is similar to the $K$ matrix of (1) except for the special structure which has several useful features. For one, the matrix $B$ need not be full row rank. In the case of a fixed graph, the gain matrix can be obtained constructively and is guaranteed to exist under mild conditions, resulting in a set of more amiable conditions. Specifically, only the second largest eigenvalue of the Laplacian matrix is needed in the conditions. The proposed approach also allows extensions to time-varying graphs, where the MAS switches among connected graphs that are undirected or directed. These results establish the existence of gain matrices that achieve consensus. The mechanization of a control scheme to achieve the required gain matrix adaptively is also discussed for the case of a fixed communication network.

To the best of our knowledge, this structure has not been attempted in the MAS literature. For completeness, this work shows the conditions needed for achieving consensus under three different communication settings: the network is fixed, the network is switching among connected, undirected graphs and the network is switching among connected, directed graphs.

The rest of this paper is organized as follows. This section ends with a description of the notations used. Section 2 reviews standard definitions, graph and matrix properties. Section 3 discusses the solutions of some special classes of agents. The intention is to simplify the subsequent results by avoiding the consideration of non-diagonalizable $A$. Section 4 shows the structure of the proposed controller. Section 5 shows the necessary and sufficient conditions for consensus for the case of a fixed network MAS. Section 6 considers the case where the network switches among undirected and connected graphs. While section 7 is for directed and connected graphs. The realization of the gain matrix that achieves consensus for a fixed network is given in section 8 with a corresponding example given in Section
9. The work concludes in Section 10.

The notations used in this paper are standard. Non-negative and positive integer sets are $\mathbb{Z}_{0+}$ and $\mathbb{Z}^+$ respectively. Selected ranges of the integer set are $\mathbb{Z}^N = \{1, \cdots, N\}$ and $\mathbb{Z}_k^j = \{\ell, \ell + 1, \cdots, k\}$ with $k > \ell$. Similarly, the sets of real numbers, $n$-dimensional real vectors and $n$ by $m$ real matrices are $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n \times m}$ respectively. For a square matrix $Q$, $Q > (\geq) 0$ means $Q$ is positive definite (semi-definite), $Q'$ is its transpose and $\text{spec}(Q)$ is its set of eigenvalues. The 2-norm of $x \in \mathbb{R}^n$ is $\|x\|$, $I_n$ is the $n \times n$ identity matrix, $1_m$ ($\hat{1}_m$) is the $m$-vector of all 1 ($\frac{1}{\sqrt{m}}$), $\mathbb{C}^-$ ($\mathbb{C}^-$) and $\mathbb{C}^+$ ($\mathbb{C}^+$) are the open (closed) left half and the open (closed) right half of the complex plane respectively. Given $A \in \mathbb{R}^{n \times m}, B \in \mathbb{R}^{q \times r}, A \otimes B \in \mathbb{R}^{nq \times mr}$ is the Kronecker product of $A$ and $B$. Given $x_i \in \mathbb{R}^n, i \in \mathbb{Z}^N, x_i^j \in \mathbb{R}$ is the $j^{th}$ element of $x_i$, the bold-faced symbols $x^q = [x_1^q x_2^q \cdots x_M^q]' \in \mathbb{R}^M, q \in \mathbb{Z}^n$ and $\hat{x} = [(x^1)' \cdots (x^n)']' = [(x_1^1 x_2^1 \cdots x_M^1), (x_1^2 x_2^2 \cdots x_M^2), \cdots (x_1^n x_2^n \cdots x_M^n)]' \in \mathbb{R}^{mn}$. Block diagonal matrix with $m$ diagonal blocks is denoted as $\text{diag}_m\{d_1, \cdots, d_m\}$ with $d_i \in \mathbb{R}^{n_i \times n_i}$, including the case where $n_i = 1$. $\mathbb{N}^r \in \mathbb{R}^{r \times r}$ is the canonical nilpotent matrix having zero elements everywhere except the value of 1 at the $(i, i+1)$ elements, $i \in \mathbb{Z}^{r-1}$.

Additional notations are introduced when required.

2 Preliminaries

As in standard MAS, the communication among agents is given by the graph $G(t) = (\mathcal{V}, \mathcal{E}(t))$ with vertex set $\mathcal{V} = \{1, 2, \cdots, m\}$ and edge set $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$. The notation $(i, j) \in \mathcal{E}(t)$ means that $j$ is a in-neighbor of $i$ at time $t$.

This network is described by a $m \times m$ Laplacian matrix $\mathcal{L}(G(t))$ with elements $\mathcal{L}_{ij}(G(t)) = -\alpha_{ij}(t)$ if $(i, j) \in \mathcal{E}(t)$ and $i \neq j, \mathcal{L}_{ii}(G(t)) = \sum_{j=1}^m \alpha_{ij}(t)$ for all $i \in \mathbb{Z}^m$ and $\mathcal{L}_{ij}(G(t)) = 0$ otherwise. Note that $\alpha_{ij}(t)$ is nonnegative for all $i, j \in \mathcal{V} \times \mathcal{V}$ and, if nonzero, is greater than $\bar{\alpha}$, for some $\bar{\alpha} > 0$. A common definition of connectivity of $G$ is reviewed and stated next, together with a summary of well-known properties of $\mathcal{L}$.

Definition 1 A graph, $G$, is connected if there exists a base node such that all other nodes can be reached from the base node via the edges of the graph.

Several well-known properties of $\mathcal{L}(G)$ include

(D1) The sum of each row of $\mathcal{L}$ is 0.

(D2) Smallest eigenvalue of $\mathcal{L}$ is 0 with right eigenvector of $1_m$.

(D3) When $G$ is connected, the smallest eigenvalue of $\mathcal{L}(G)$ is simple.

(D4) All eigenvalues of $\mathcal{L}$ lie in the closed right half of the complex plane. In the case where $G$ is undirected, $\mathcal{L}(G)$ is symmetric and hence has all real eigenvalues.

Suppose $A \in \mathbb{R}^{r \times n}, B \in \mathbb{R}^{p \times q}$ and $C, D$ are of appropriate dimensions. Then

(P1) $A \otimes B = (A \otimes I_p)(I_n \otimes B) = (I_r \otimes B)(A \otimes I_q)$
(P2) $AB \otimes CD = (A \otimes C)(B \otimes D)$

If $A, B \in \mathbb{R}^{n \times n}$, then

(P3) $e^{(A+B)t} = e^{At}e^{Bt}$ if and only if $AB = BA$.

(P4) $e^{(A \otimes I_m)t} = e^{At} \otimes I_m$.

(P5) $e^{(I_m \otimes A)t} = I_m \otimes e^{At}$.

(P6) If $(\lambda, v)$ is the eigenvalue and eigenvector of $A$, then $(e^{\lambda t}, v)$ is the corresponding eigenvalue and eigenvector of $e^{At}$.

3 Special Classes of Agents

Before stating the most general form of the proposed control law, some special cases are first considered. These can be represented as

$$\dot{x}_i(t) = Ax_i(t) + \sum_{j=1}^{m} \alpha_{ij}(t)(x_j(t) - x_i(t)), \quad i \in \mathbb{Z}^m$$  \hspace{1cm} (2)

In addition, the notations for switching communication network is established. Let

(A1) The communication graph $G(t)$ switches as instants $t_0, t_1, \cdots$ and that $t_{k+1} - t_k \geq h$ for some $h > 0$.

Three simple results are first shown for the system of (2), to be followed by results for the case of a general $A$. The first is for the case where $A = 0$ in (2) such that

$$\dot{x}_i(t) = \alpha \sum_{j=1}^{m} \alpha_{ij}(t)(x_j(t) - x_i(t)), \quad i \in \mathbb{Z}^m$$

where $\alpha > 0$ is some constant. This, when expressed as $\dot{\mathbf{x}} = [(\mathbf{x}^1)' \cdots (\mathbf{x}^n)']' \in \mathbb{R}^{nm}$ becomes

$$\dot{\mathbf{x}}(t) = I_n \otimes (-\gamma \mathcal{L}(t))\hat{\mathbf{x}}(t)$$  \hspace{1cm} (3)

with $\mathcal{L}(t) := \mathcal{L}(G(t))$. Under (A1), it is clear that $G(t) = G(t_k)$ for $t \in [t_k, t_{k+1})$. Hence,

$$\mathcal{L}(t) := \mathcal{L}_k \text{ for } t \in [t_k, t_{k+1}) \text{ and } h_k := t_{k+1} - t_k.$$  \hspace{1cm} (4)

Then the solution, using (P5), is

$$\hat{\mathbf{x}}(t_{k+1}) = (I_n \otimes e^{-\gamma \mathcal{L}_kh_k})\hat{\mathbf{x}}(t_k).$$  \hspace{1cm} (5)
The second case is for $\dot{x}_i(t) = J_n x_i(t), i \in \mathbb{Z}^m$ where $J_n$ is an $n$–order Jordan block with eigenvalue $\lambda$ in the sense that $J_n = \lambda I_n + \mathcal{N}^n$ where $\mathcal{N}^n$ is the $n$-order canonical nilpotent matrix. Similarly, the MAS can be expressed as

$$\dot{x}(t) = (J_n \otimes I_m)\dot{x}(t)$$

which, since $\lambda I_n$ and $\mathcal{N}^n$ commute, has a solution under (P3) and (P4) as

$$\dot{x}(t) = ((e^{J_n t}) \otimes I_m)\dot{x}(0) = e^{\lambda t} e^{N^t} \otimes I_m \dot{x}(0).$$

The third result is one that combines the above two, stated as a lemma for easy reference.

**Lemma 1** Let $J_n = \lambda I_n + \mathcal{N}^n$. The MAS

$$\dot{x}_i(t) = J_n x_i(t) + \gamma I_n \sum_{j=1}^{m} \alpha_{ij}(t)(x_j(t) - x_i(t)), i \in \mathbb{Z}^m$$

reaches consensus if the MAS

$$\dot{x}_i(t) = \lambda I_n x_i(t) + \gamma I_n \sum_{j=1}^{m} \alpha_{ij}(t)(x_j(t) - x_i(t)), i \in \mathbb{Z}^m$$

reaches consensus exponentially for all $x_i(0), i \in \mathbb{Z}^m$.

**Proof:** Using (3), the system of (9) can be written as $\dot{x}(t) = (\lambda I_n \otimes I_m - I_n \otimes \gamma \mathcal{L}(t))\dot{x}(t)$. Since $\lambda I_n \otimes I_m$ and $I_n \otimes \gamma \mathcal{L}(t)$ commute,

$$\dot{x}(t_{k+1}) = e^{\lambda I_n \otimes I_m h_k} e^{I_n \otimes (-\gamma \mathcal{L} h_k)} \dot{x}(t_k) \quad \text{(using (P3) and notations of (4))}$$

$$= (e^{\lambda h_k} \otimes I_m)(I_n \otimes e^{-\gamma \mathcal{L} h_k})\dot{x}(t_k) \quad \text{(using (P4) and (P5))}$$

$$= (e^{\lambda h_k} \otimes e^{-\gamma \mathcal{L} h_k})\dot{x}(t_k) \quad \text{(using (P1))}$$

$$= (e^{\lambda h_k} I_n \otimes e^{-\gamma \mathcal{L} h_k})\dot{x}(t_k)$$

$$= e^{\lambda h_k} \text{diag}_n\{e^{-\gamma \mathcal{L} h_k}, \ldots, e^{-\gamma \mathcal{L} h_k}\}\dot{x}(t_k)$$

Clearly, the above is a block-diagonal system and can be expressed as collection of $n$ sets of $m$ equations. For each $j \in \mathbb{Z}^n$ with $x^j = [x^j_1 \ldots x^j_m]^T$, each of the $n$ sets can be expressed as

$$x^j(t_{k+1}) = e^{\lambda h_k} e^{-\gamma \mathcal{L} h_k} x^j(t_k)$$

$$= e^{\lambda h_k} e^{-\gamma \mathcal{L} h_k} e^{\lambda h_k-1} e^{-\gamma \mathcal{L} h_k-1} x^j(t_{k-1})$$

$$:= e^{\lambda h_k} \mathcal{W}(k) \mathcal{W}(k-1) \ldots \mathcal{W}(0) x^j(t_0)$$

$$:= e^{\lambda h_k} \mathcal{W}_0^k x^j(t_0)$$

(11)
where

\[ W(j) := e^{-\gamma L_j} \text{ and } W_0^k := W(k-1) \cdots W(0). \]  

(12)

If (9) reaches consensus exponentially for all \( x^j(0) \), this means, from (11), that

\[ W_0^k = 1_m \psi' + \delta(t_{k+1}) \text{ with } \|\delta(t)\| < \delta_0 e^{-\mu t} \]  

(13)

for some \( \psi \in \mathbb{R}^m, \mu, \delta_0 > 0 \) and \( \lim_{k \to \infty} W_0^k = 1_m \psi' \).

Now, rewriting (8) using (3) and (6) yields

\[ \dot{x}(t) = (J_n \otimes I_m - \gamma I_n \otimes \mathcal{L}(t))x(t) \]  

(14)

Note that \((J_n \otimes I_m)(\gamma I_n \otimes \mathcal{L}(t)) = \gamma J_n \otimes \mathcal{L}(t) = (\gamma I_n J_n) \otimes (\mathcal{L}(t)I_m) = (\gamma I_n \otimes \mathcal{L}(t))(J_n \otimes I_m)\) using (P2). This commutative property means that (P3) can be used in (14) and (5) giving

\[ \dot{x}(t_{k+1}) = e^{J_n \otimes I_m h_k}e^{-\gamma I_n \otimes \mathcal{L}_k h_k}x(t_k) \]
\[ = (e^{J_n h_k} \otimes I_m)(I_n \otimes e^{-\gamma \mathcal{L}_k h_k})x(t_k) \]
\[ = (e^{J_n h_k} \otimes e^{-\gamma \mathcal{L}_k h_k})x(t_k) \]
\[ = e^{\lambda h_k} e^{N^n h_k} \otimes e^{-\gamma \mathcal{L}_k h_k}x(t_k) \]
\[ = e^{\lambda h_k} (e^{N^n h_k} \otimes e^{-\gamma \mathcal{L}_k h_k})e^{\lambda h_{k-1}}(e^{N^n h_{k-1}} \otimes e^{-\gamma \mathcal{L}_{k-1} h_{k-1}})x(t_{k-1}) \]
\[ = e^{\lambda(h_k + h_{k-1})}(e^{N^n h_k} \otimes e^{-\gamma \mathcal{L}_k h_k})(e^{N^n h_{k-1}} \otimes e^{-\gamma \mathcal{L}_{k-1} h_{k-1}})x(t_{k-1}) \]
\[ = e^{\lambda(h_k + h_{k-1})}(e^{N^n (h_k + h_{k-1})} \otimes e^{-\gamma \mathcal{L}_k h_k} e^{-\gamma \mathcal{L}_{k-1} h_{k-1}})x(t_{k-1}) \text{ using (P2)} \]
Using the same notation for $\mathcal{W}(j)$ and $\mathcal{W}_0^k$ of (12), repeated application of the above yields

$$
\dot{z}(t_{k+1}) = e^{\lambda t}e^{N^\alpha(t_{k+1})} \otimes \mathcal{W}(k)\mathcal{W}(k-1) \cdots \mathcal{W}(0)\dot{z}(t_0) = e^{\lambda t_{k+1}}
$$

where

$$
\begin{bmatrix}
I_m & t_{k+1}I_m & \cdots & \frac{t_{k+1}^{n-1}}{(n-1)!}I_m \\
0 & I_m & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & I_m & \vdots \\
\end{bmatrix}
\begin{bmatrix}
\mathcal{W}_0^k\mathbf{x}_1(t_0) \\
\vdots \\
\vdots \\
\mathcal{W}_0^k\mathbf{x}_n(t_0) \\
\end{bmatrix}
= e^{\lambda t_{k+1}}
$$

From (15) it is easy to see that $\mathbf{x}_j(t)$ reaches consensus for all $j \in \mathbb{Z}_n$. Specifically, $\lim_{k \to \infty} \mathbf{x}_j(t_{k+1}) = \lim_{k \to \infty} \mathcal{W}_0^k\mathbf{x}_j(t_0) + t_{k+1}\mathcal{W}_0^k\mathbf{x}_j(t_0) + \cdots + \frac{t_{k+1}^{n-1}}{(n-1)!}\mathcal{W}_0^k\mathbf{x}_j(t_0) = 1_m(\alpha_j + t_{k+1}\alpha_{j+1} + \cdots + \frac{t_{k+1}^{n-1}}{(n-1)!}\alpha_n)$ since $\mathcal{W}_0^k$ is given by (13) and $\alpha_j = \psi^j\mathbf{x}_j(t_0)$. Repeating this for all $j \in \mathbb{Z}_n$ shows that $\dot{z}$ reaches consensus $\Box$.

4 The proposed controller and the closed-loop structure

This section considers the general case of unstable $A$. For notational simplicity, let

$$
\dot{z}_i(t) = A z_i(t) + B_z u_i(t)
$$

be the open-loop dynamics of the $i$-th agent with $B_z \in \mathbb{R}^{n \times n_B}, n_B < n$ and assume that

(A2) $(A, B_z)$ is stabilizable.

(A3) All states are measurable.

Assumption (A2) is a standard requirement. (A3) is needed since values of $z_i$ is needed for the dynamic compensator of (17). This assumption can be relaxed using an observer and that is given in Remark 2. However, the observer-based design is not pursued here as the focus is on the more novel aspects of the work.

The controller is a dynamical compensator of the form

$$
\eta_i = (A + B_z K)\eta_i + \Phi \sum_j \alpha_{ij}(t)(\eta_j - \eta_i + z_i - z_j) \\
u_i = K \eta_i
$$
where $\Phi \in \mathbb{R}^{n \times n}$ is the controller gain, $\eta_i \in \mathbb{R}^n$ is the state of the compensator and $K \in \mathbb{R}^{n_B \times n}$ is such that $(A + B_z K)$ is Hurwitz. Let $s_i := z_i - \eta_i$ and it follows from the combined system of (16)-(18) that

$$
\dot{s}_i = \dot{z}_i - \dot{\eta}_i = A z_i + B_z u_i - (A + B_z K) \eta_i - \Phi \sum_j \alpha_{ij}(t)(s_i - s_j) = A s_i + \Phi \sum_j \alpha_{ij}(t)(s_j - s_i)
$$

(19)

If $s_i$ reaches consensus in (19), $\eta_i$ approaches 0 from (17) which implies that $z_i$ reaches consensus. Hence, the important result is to achieve consensus for the MAS having a structure of the form given by (19). In view of this, the focus is on consensus conditions for the system of

$$
\dot{z}_i(t) = A z_i(t) + \Phi \sum_j \alpha_{ij}(t)(z_j(t) - z_i(t)), \quad i \in \mathbb{Z}^m
$$

(20)

where $\Phi$ is an appropriate matrix. Clearly, the above expression has the same structure as (19). It also becomes the standard system of (2) when $\Phi = I_n$.

The most general result is to express $A$ via appropriate transformation into its Jordan form. However, due to the result of Lemma 1, this is not necessary. Instead, the following assumption is made:

(A4) $A$ is diagonalizable and has $r$ eigenvalues in $\mathbb{C}^+$. Under this assumption, the expression of (20) can be transformed via $z_i = Q x_i, i \in \mathbb{Z}^m$ where $Q \in \mathbb{R}^{n \times n}$ contains column-wise eigenvectors of $A$ as

$$
\dot{x}_i(t) = S x_i(t) + \Gamma \sum_{j=1}^m \alpha_{ij}(t)(x_j(t) - x_i(t)), \quad i \in \mathbb{Z}^m
$$

(21)

where $\Gamma = Q^{-1} \Phi Q$ and

$$
S = Q^{-1} A Q = \text{diag}_n \{\lambda_1^A, \cdots, \lambda_r^A, \cdots, \lambda_n^A\} \text{ with } Re(\lambda_1^A) \geq \cdots \geq Re(\lambda_r^A) \geq \cdots \\
\{\lambda_1^A, \cdots, \lambda_r^A\} \in \mathbb{C}^+ \text{ and } \{\lambda_{r+1}^A, \cdots, \lambda_n^A\} \in \overline{\mathbb{C}^-}.
$$

(22)

(23)

Hereafter, all MASs are assumed to have the form of (21) for simplicity in presentation. Clearly, if (21) reaches consensus, so does system (20). While $\Gamma$ can be a general matrix, the proposed approach is to consider the case where

$$
\Gamma = \text{diag}_n \{\gamma^1, \cdots, \gamma^n\}
$$

(24)
for all agents. Then, since both $S$ and $\Gamma$ are diagonal, the MAS can be written as

$$
\dot{\hat{x}}(t) = (S \otimes I_m - \Gamma \otimes L(t))\hat{x}(t)
= \text{diag}\{\lambda_A^1 I_m - \gamma^1 L(t), \lambda_A^2 I_m - \gamma^2 L(t), \ldots, \lambda_A^n I_m - \gamma^n L(t)\}\hat{x}(t)
$$

The above is a block-diagonal system consisting of $n$ blocks with the $i$-block being

$$
\dot{\hat{x}}^i(t) = (\lambda_A^i I_m - \gamma^i L(t))\hat{x}^i(t), \quad i \in \mathbb{Z}^n
$$

The next few sections discuss the results for this system under different switching conditions.

**Remark 1** Similar to the approach of [4], the system of (21) can also be obtained from (17) using $u_i = B'_z(B_zB'_z)^{-1}\Phi \sum_{j=1}^m \alpha_{ij}(t)(z_j(t) - z_i(t))$ in the case where $B_z$ has full row rank.

**Remark 2** In the case where some of the states are not measurable, an observer-based dynamic compensator can be used. The design is quite standard and, hence, only the essential steps are presented. Obviously, the additional assumption that $(A, C)$ is detectable is needed besides $(A2)$. The dynamic compensator in this case is given by

$$
\dot{\tilde{z}}_i = A\tilde{z}_i + Bu_i + H(\hat{y}_i - y_i), \quad \hat{y}_i = C\tilde{z}_i
$$

$$
\dot{\eta}_i = (A + BK)\eta_i + \Phi \sum_{j=1}^m \alpha_{ij}(t)(\eta_j - \eta_i + \tilde{z}_j - \tilde{z}_i)
$$

$$
u_i = K\eta_i
$$

Letting $s_i := \tilde{z}_i - \eta_i$ and $e_i := z_i - \tilde{z}_i$, it is easy to show that

$$
\dot{s}_i = As_i + \Phi \sum_{j=1}^m \alpha_{ij}(t)(s_j - s_i) - HCe_i
$$

$$
\dot{e}_i = (A + HC)e_i
$$

Clearly, when $H$ is chosen such that $(A + HC)$ is Hurwitz, $e_i$ goes to zero and the last term of the $\dot{s}_i$ equation above goes to zero. When that happens, the expression of $s_i$ becomes that of (20) and $s_i$ reaches consensus if (21) does.

5 Consensus under fixed graph

This section considers the case when the communication network is fixed.

**Theorem 1** Suppose $(A4)$ holds and $G$ is a fixed connected communication graph such that $L(G(t)) = L(G) = L$ for all $t$. The MAS of (21) with conditions (22), (23) and (24) reaches consensus if and only if $\gamma^i > \frac{\text{Re}(\lambda_A^i)}{\text{Re}(\lambda_L^i)}$ for $i \in \mathbb{Z}^r$ and $\gamma^{r+1} = \cdots = \gamma^n = 1$ where $\lambda_L^2$ is the second smallest eigenvalue of $L$. 
Proof: \((\Rightarrow)\) Since \(S\) and \(\Gamma\) are diagonal, system (21) can be expressed as (26). The solution of (26), following the fact that \(\lambda^i_A I_m\) and \(\gamma^i \mathcal{L}\) commute, is

\[
x^i(t) = e^{(\lambda^i_A I_m - \gamma^i \mathcal{L})t} x^i(0) = e^{\lambda^i_A t} e^{-\gamma^i \mathcal{L} t} x^i(0) \\
= e^{\lambda^i_A t} \left( \sum_{j=1}^{m} \varphi_j \xi_j e^{-\gamma^j \mathcal{L} t} \right) x^i(0) \\
= e^{\lambda^i_A t} 1_m \xi_i x^i(0) + \sum_{j=2}^{m} e^{(\lambda^i_A - \gamma^j \mathcal{L})t} \varphi_j \xi_j x^i(0) \quad (27)
\]

where the eigen-decomposition of \(e^{-\gamma^i \mathcal{L} t}\) is used and \(\lambda^j_L\) is the \(j^{th}\) eigenvalue of \(\mathcal{L}\) with right and left eigenvectors \(\varphi_j\) and \(\xi_j\) respectively. The last equality on the right above follows because \(\mathcal{L}\) is a connected graph which by (D3) means that \(\varphi_1 = 1_m, \lambda^1_L = 0\) and \(\text{Re}(\lambda^j_L) > 0\) for \(j \in \mathbb{Z}_2^n\). When \(\gamma^i > \frac{\text{Re}(\lambda^i_A)}{\text{Re}(\lambda^j_L)}\), \(\text{Re}(\lambda^i_A - \gamma^j \lambda^j_L) < 0\) for all \(j \in \mathbb{Z}_2^m\) which implies that

\[
\lim_{t \to \infty} x^i(t) = e^{\lambda^i_A t} \xi_i x^i(0) 1_m \quad (28)
\]

The above is for the single block of \(x^i\). Using the same development and noting that \(\lambda^j_A\) is the largest eigenvalue of \(S\) and that \(\lambda^2_L\) is the second smallest eigenvalue of \(\mathcal{L}\), the expression is applicable for all \(i \in \mathbb{Z}^r\) as \(\lambda^1_A, \ldots, \lambda^r_A\) are unstable eigenvalues. For \(i \in \mathbb{Z}_r^{n+1}\), it follows from \(\text{Re}(\lambda^i_A) < 0\) that \(\gamma^i = 1 > \frac{\text{Re}(\lambda^i_A)}{\text{Re}(\lambda^j_L)}\) and \(\lim_{t \to \infty} x^i = 0\). This shows \(\hat{x}\) reaches consensus.

\((\Leftarrow)\) Suppose one of the \(\gamma^i\) where \(i \in \mathbb{Z}^r\) is such that \(\gamma^i \leq \frac{\text{Re}(\lambda^i_A)}{\text{Re}(\lambda^j_L)}\), then it is clear from (27) that consensus is not achieved for this \(x^i(t)\). \(\Box\)

The above uses \(\Gamma = \text{diag}_n\{\gamma^1, \ldots, \gamma^n\}\) with \(\gamma^i > \frac{\text{Re}(\lambda^i_A)}{\text{Re}(\lambda^j_L)}\) for \(i \in \mathbb{Z}^r\) and \(\gamma^{r+1} = \cdots = \gamma^n = 1\). A useful choice of \(\gamma^i\) that satisfies the consensus condition is to let \(\gamma^1 = \cdots = \gamma^n > \frac{\text{Re}(\lambda^i_A)}{\text{Re}(\lambda^j_L)}\). That this choice of \(\gamma\) is sufficient to enforce consensus is easy to see from the proof of Theorem 1. Hence, the following corollary is stated without proof.

Corollary 1 Suppose all conditions and variables of Theorem 1 hold. The MAS of (21) with conditions (22), (23) and (24) reaches consensus if \(\gamma^1 = \cdots = \gamma^n > \frac{\text{Re}(\lambda^i_A)}{\text{Re}(\lambda^j_L)}\).

Remark 3 The corollary above has an interesting implication. If \(\Gamma = \gamma^1 I_n\), then \(\Phi = Q \gamma^1 I_n Q^{-1} = \gamma^1 I_n\). In the original \(z\) coordinates, the \(i\) agent can be written as

\[
\dot{z}_i = A z_i + \gamma^1 \sum \alpha_{ij}(z_j - z_i) = A z_i + \sum \beta_{ij}(z_j - z_i) \quad (29)
\]

where \(\beta_{ij} = \gamma^1 \alpha_{ij}\). Then a Laplacian matrix \(\mathcal{L}_\beta\) exists with \([\mathcal{L}_\beta]_{ij} = \beta_{ij}\) such that (29) reaches consensus. That \(\mathcal{L}_\beta\) is a valid Laplacian matrix is easy to verify given that \(\mathcal{L}\) is a Laplacian matrix and \(\gamma^1 > 0\).
Remark 4 The expression of the $z_i$ when $\dot{x}$ reaches consensus is of interest. Since there are $r$ unstable eigenvalues in $A$, it follows from (28) that only the first $r$ of $x^i$ are non-trivial. Using $z_j = Qx_j$ and (28), the asymptotic consensus value is

$$\lim_{t \to \infty} z_j(t) = \sum_{i=1}^{r} q_i x_j^i(t) = \sum_{i=1}^{r} q_i e^{\lambda_i t} \mu_i(0)$$

$$= \sum_{i=1}^{r} q_i e^{\lambda_i t} \zeta_i (I_m \otimes p_i') \hat{z}(0), \quad j \in \mathbb{Z}_1^m \tag{30}$$

where $q_i$ is the $i$-th column of $Q$, $\mu_i(0) = \zeta_i x^i(0)$ and $x^i(0)$ is the initial condition of the $i$-th row of $x$. $\hat{z}(0)$ is the initial condition of $\hat{z}$. 

Remark 5 The above expression of (30) is the general expression of the consensus values for system (20) having $r$ unstable eigenvalues. If each $z_i$ is a scalar system (in which case, $z_i = x_i$ and $r = 1$) with unstable eigenvalue $\lambda_A^*$ such that $\dot{x}_i = \lambda_A^* x_i + \gamma \sum_j c_{ij} (x_j - x_i)$, the final consensus value becomes $e^{\lambda_A^* t} \zeta_i x^i(0)$. In addition, when $\lambda_A^* = 0$, the consensus value of $x_i(t)$ becomes the well-known result of $\zeta_i x^i(0)$ for directed graph and $I_n \cdot x^i(0)$ for undirected graph.

6 Consensus under Undirected and Connected Graphs

This section deals with the MAS of (21) where $G(t)$ switches among a set of undirected and connected graphs at every $t_k$ as in (A1). Let the collection of these graphs be denoted by

$$\Omega := \{ L^i : i \in \mathbb{Z}^v \}. \tag{31}$$

Hence, the switching is such that $L(t) \in \Omega$ for all $t$. A typical representation of the switching process is the use of an indication function

$$\sigma : \mathbb{R}_0^+ \to \mathbb{Z}^v \tag{32}$$

such that $\sigma(t)$ indicates the index $i$ of $L^i \in \Omega$ at time $t$ and it is continuous on the left. In addition, (A5) Each $L^i \in \Omega$ corresponds to an undirected graph $G_i$, which is connected.

Theorem 2 Suppose (A1)-(A5) hold. Consider the system of (21) with (22), (23) and (24) holding and arbitrary switching of $G(t)$ such that $L(t) = L^\sigma(t) \in \Omega$ for all $t$. Suppose $\gamma^1 = \cdots = \gamma^r > \frac{\Re(\lambda_A^*)}{\Re(\lambda_2^*)}$ and $\gamma^{r+1} = \cdots = \gamma^n = 1$. Then (21) reaches consensus exponentially if and only if

$$\lambda_A^* - \gamma^1 \lambda_2^* < 0 \tag{33}$$

where $\lambda_2^2 := \min \{ \lambda_2^2, \lambda_3^2 \}$ is the second smallest eigenvalue of $L^i, i \in \mathbb{Z}^v$. 

Proof: \((\Rightarrow)\) The proof begins with showing \(x^1(t)\) reaches consensus, to be followed by \(x^q(t)\) for \(q \in \mathbb{Z}^+_2\). Let \(L_k, h_k\) be as defined by (4). Since \(L_k\) satisfies (A5) for all \(k\), they satisfy (D1)-(D4). The same is true for \(\gamma L_k\) for any \(\gamma > 0\). Following from (D2) and (D3), \(e^{-\gamma L_k h_k}\) is a symmetric and doubly stochastic matrix with a simple largest eigenvalue of 1 with \(\hat{1}_m\) being the corresponding left and right eigenvectors. Following the same development from (21) to (26), it follows that

\[
\dot{x}(t+1) = (S \otimes I_m - \Gamma \otimes L_k)\bar{x}(t_k)
= \text{diag}_n\{\lambda^n_k I_m - \gamma I_L\}
\]

and has a solution for \(x^1\) as

\[
x^1(t+1) = e^{(\lambda^n_k I_m - \gamma I_L) h_k} x^1(t_k) = e^{\lambda^n_k h_k - \gamma L_k h_k} x^1(t_k)
= e^{\lambda^n_k h_k} e^{\lambda^n_k h_{k-1}} \cdots e^{\lambda^n_k h_0} W(k) W(k-1) \cdots W(0) x^1(t_0)
= e^{\lambda^n_k t_k+1} W^k_0 x^1(t_0)
\]

where \(W(j)\) and \(W^k_0\) are defined by (12). Consider the decomposition of \(W(j)\) by letting

\[
D(j) := W(j) - \hat{1}_m \hat{1}_m', \quad \text{where} \quad \hat{1}_m := \frac{1_m}{\sqrt{m}}
\]

and note that \(D(j) \hat{1}_m = (W(j) - \hat{1}_m \hat{1}_m') \hat{1}_m = 0\) and \(\hat{1}_m D(j) = 0\) since \(D(j)\) is symmetric. Using these expressions in equation (34) leads to

\[
x^1(t+1) = e^{\lambda^n_k t_k+1} (D(k) + \hat{1}_m \hat{1}_m') \cdots (D(1) + \hat{1}_m \hat{1}_m') (D(0) + \hat{1}_m \hat{1}_m') x^1(t_0)
= e^{\lambda^n_k t_k+1} (D(k) + \hat{1}_m \hat{1}_m') \cdots (D(1) D(0) + \hat{1}_m \hat{1}_m') x^1(t_0)
= e^{\lambda^n_k t_k+1} (D(k) D(k-1) \cdots D(1) D(0) + \hat{1}_m \hat{1}_m') x^1(t_0)
\]

where the last equality of (35) is the repeated application of the \((D(k) + \hat{1}_m \hat{1}_m')(D(k) + \hat{1}_m \hat{1}_m') = D(k+1) D(k) + \hat{1}_m \hat{1}_m'\) for all \(k\). Note that each \(L_j\) is connected and symmetric under (A5), it follows from (D4) and (P6) that \(\text{spec}(\gamma L_j) = \{0, \gamma \lambda^2_j, \ldots, \gamma \lambda^n_j\}\) and \(\text{spec}(W(j)) = \{1, e^{-\gamma \lambda^2_j h_j}, \ldots, e^{-\gamma \lambda^n_j h_j}\}\) since \(\text{spec}(L_j) = \{0, \lambda^2_j, \ldots, \lambda^n_j\}\) with \(0 < \lambda^2_j \leq \cdots \leq \lambda^n_j\). In addition, \(e^{-\gamma \lambda^2_j h_j} < 1\) for all \(q \in \mathbb{Z}^+_2\) and for all \(h_j > 0\) since \(e^{-\gamma L_j h_j}\) is a symmetric and doubly stochastic matrix. Similarly, \(\text{spec}(D(j)) = \{e^{-\gamma \lambda^2_j h_j}, e^{-\gamma \lambda^3_j h_j}, \ldots, e^{-\gamma \lambda^n_j h_j}\}\)
\[
\|D(j)\| = e^{-\gamma^2 j h_j}. \quad \text{Using this expression of } \|D(j)\|, \quad (35) \text{ can be further expressed as}
\]

\[
x^1(t_{k+1}) - e^{\lambda_A t_{k+1}} \hat{1}_m \hat{m}^T x^1(t_0) = e^{\lambda_A t_{k+1}} D(k) D(k-1) \cdots D(0) x^1(0)
\]

\[
= (e^{\lambda_A h_k} D(k))(e^{\lambda_A h_{k-1}} D(k-1)) \cdots (e^{\lambda_A h_0} D(0)) x^1(t_0)
\]

\[
\Rightarrow \|x^1(t_{k+1}) - e^{\lambda_A t_{k+1}} \hat{1}_m \hat{m}^T x^1(t_0)\| \leq e^{\lambda_A h_k} \|D(k)\| \cdots e^{\lambda_A h_0} \|D(0)\| \|x^1(t_0)\|
\]

\[
= e^{-\gamma^2 \lambda_A t_{k+1}} \|x^1(t_0)\|
\]

\[
\leq e^{-\gamma^2 \lambda_A t_{k+1}} \|x^1(t_0)\| = e^{-\gamma^2 \lambda_A t_{k+1}} \|x^1(t_0)\|
\]

where the inequality of (37) follows from that fact that \( \lambda_2^2 = \min_\Omega \{\lambda_2^2 : \mathcal{L} \in \Omega\} \). If \( \gamma^2 \lambda_2^2 > \lambda_A^1 \), then the right hand side of the above goes to zero exponentially and \( x^1(t) \) reaches consensus exponentially towards \( e^{\lambda_A t_{k+1}} 1_m \alpha(t_0) \) where \( \alpha(t_0) = \hat{1}_m^T x^1(t_0) \).

The above reasoning is for \( x^1(t) \). It can be extended directly for \( x^q(t), q \in \mathbb{Z}_2^q \) by noting that \( x^q(t_{k+1}) = e^{\lambda_A t_{k+1}} W^q x^q(t_0) \) form (34) and that \( \gamma^2 \lambda_2^q > \lambda_A^q \) implies that \( \gamma^2 \lambda_2^q > \lambda_A^q \) for \( q \in \mathbb{Z}_2^q \). Hence, \( x^q(t) \) reaches consensus for \( q \in \mathbb{Z}_2^q \) and that \( \dot{x}(t) \) reaches consensus exponentially.

\((\Leftarrow)\) The proof is to show that there exists a switching sequence and an initial state \( \dot{x}(0) \) such that consensus is not reached when \( \gamma^2 \lambda_2^0 \leq \lambda_A^1 \). Let \( i* = \arg \min_\mathcal{L}_i \{\lambda_2^i \in \mathcal{L}_i, i \in \mathbb{Z}^n\} \) and \( \xi_{i*} \) be the eigenvector of \( D(i*) \) corresponding to the eigenvalue of \( e^{-\gamma^2 \lambda_2^{i*}} \). Since the switching is arbitrary, choose the sequence \( \mathcal{L}_k = \mathcal{L}_{i*} \) for all \( k \) and \( x^1(t_0) = \xi_{i*} \), then following the same reasoning given in the proof of the "if" condition up to (36), noting that \( D(k) = D(k-1) = \cdots = D(0) = D(i*) \), it follows from \( D(k) \xi_{i*} = e^{-\gamma^2 \lambda_2^{i*} t_{k+1}} \xi_{i*} \) that

\[
x^1(t_{k+1}) - e^{\lambda_A t_{k+1}} \hat{1}_m \hat{m}^T \xi_{i*} = (e^{\lambda_A h_k} D(k))(e^{\lambda_A h_{k-1}} D(k-1)) \cdots (e^{\lambda_A h_0} D(0)) \xi_{i*} = e^{-\gamma^2 \lambda_2^{ij} - \lambda_A^1 t_{k+1}} \xi_{i*}
\]

When \( \lambda_A^1 = \gamma^2 \lambda_2^{i*} \), the right hand side is \( \xi_{i*} \). Since \( \xi_{i*} \) is not a scalar multiple of \( 1_m \) (eigenvector of \( e^{-\lambda_2^{i*} h_j} \) is orthogonal to \( 1_m \)), \( x^1(t) \) does not reach consensus. \( \square \)

### 7 Consensus under Directed and Connected Graphs

The result of the above is now extended to the case where the graphs are directed and connected. For this purpose, the set of \( \Omega = \{\mathcal{L}^1, \ldots, \mathcal{L}^n\} \) is such that

\( \mathbf{A6} \) Each \( \mathcal{L}^i \in \Omega \) corresponds to a directed graph \( G_i \), which is connected.

**Theorem 3** Suppose \( \mathbf{A1} \)-(\( \mathbf{A4} \)) and \( \mathbf{A6} \) hold. Consider the system of (21) with (22), (23) and (24) holding. Suppose \( \tilde{\gamma} := \gamma^1 = \cdots = \gamma^r \) and \( \tilde{\gamma}^{r+1} = \cdots = \gamma^n = 1 \) when \( \gamma^1 \) is as defined by (24). Then, (21) reaches consensus exponentially when \( \tilde{\gamma} \) is sufficiently large.
such that $q$ with $\xi_j^i \hat{1}_m = 1$ is the left eigenvector of $\mathcal{W}(j)$ corresponding to eigenvalue 1. Then (34) can be written as

$$x^1(t_{k+1}) = e^{\lambda_{lk} t_{k+1}} (B(k) + \hat{1}_m \xi_j^i) \cdots (B(1) + \hat{1}_m \xi_j^i)(B(0) + \hat{1}_m \xi_j^i) x^1(t_0)$$

$$= e^{\lambda_{lk} t_{k+1}} (B(k) + \hat{1}_m \xi_j^i) \cdots (B(1)B(0) + \hat{1}_m (\xi_j^i B(0) + \xi_j^i)) x^1(t_0)$$

since $\mathcal{W}(1)\hat{1}_m = \hat{1}_m$ and $(B(1) + \hat{1}_m \xi_j^i)(B(0) + \hat{1}_m \xi_j^i) = B(1)B(0) + \hat{1}_m \xi_j^i B(0) + (\mathcal{W}(1) - \hat{1}_m \xi_j^i) \hat{1}_m \xi_j^i = B(1)B(0) + \hat{1}_m \xi_j^i B(0) + \hat{1}_m \xi_j^i B(0) + \hat{1}_m (\xi_j^i B(0) + \xi_j^i)$. Repeating this process for $B(0)$ to $B(k)$ yields

$$x^1(t_{k+1}) - e^{\lambda_{lk} t_{k+1}} \hat{1}_m \xi_j^i x^1(0) = e^{\lambda_{lk} t_{k+1}} (B(k)B(k - 1) \cdots B(0)) x^1(t_0)$$

$$= (e^{\lambda_{hk_l} B(k)})(e^{\lambda_{hk_l - 1} B(k - 1) \cdots e^{\lambda_{h_0} B(0)}) x^1(t_0)$$

where $\xi_j^i = \xi_j^i B(0) + \xi_j^i B(1)B(0) + \cdots + \xi_j^i B(k - 1)B(k - 2) \cdots B(0)$. The above also implies that

$$\|x^1(t_{k+1}) - e^{\lambda_{lk} t_{k+1}} \hat{1}_m \xi_j^i x^1(0)\| \leq \|e^{\lambda_{hk_l} B(k)}\| \cdots \|e^{\lambda_{h_0} B(0)}\| \|x^1(t_0)\|$$

(38)

Consider the eigen-decomposition of $\mathcal{W}(j)$ in the form of

$$\mathcal{W}(j) = \sum_{i=1}^{m} q_i p_i e^{-\gamma^1 \lambda_{j_l}^{i} h_j^i}$$

(39)

where $q_i, p_i$ are the right and left eigenvectors of $\mathcal{W}(j)$ corresponding to eigenvalue $e^{-\gamma^1 \lambda_{j_l}^{i} h_j^i}$. It follows from (39) that $B(j) = \sum_{i=2}^{m} q_i p_i e^{-\gamma^1 \lambda_{j_l}^{i} h_j^i}$. Hence,

$$\|e^{\lambda_{h_l} B(j)}\| \leq \sum_{i=2}^{m} \|q_i p_i\| e^{-\gamma^1 \lambda_{j_l}^{i} - \lambda^1_h} h_j^i$$

Since $\|q_i p_i\|$ is a finite quantity, $\|e^{\lambda_{h_l} B(j)}\| \to 0$ when $\gamma^1 \to \infty$. This holds also for all $\|e^{\lambda_{h_l} B(i)}\|, i = 0, \cdots, k$. Hence, the RHS of (38) approaches 0 and $x^1(t_{k+1}) \to e^{\lambda_{lk} t_{k+1}} \hat{1}_m \xi_j^i x^1(0)$. Repeating the above to $x^q(t_{k+1})$ for $q = 2, \cdots, n$ completes the proof. □

It is noteworthy that the results of Theorem 2 are both necessary and sufficient when $\Gamma$ is a diagonal matrix. In contrast, results of Theorem 3 is only sufficient.

With the results of Theorem 3 and Lemma 1, it is possible to state the result for the general case of $A$ having repeated eigenvalues. The result given below is also applicable to the case of a fixed graph by letting $\Omega$ to be a singleton set.

**Theorem 4** Suppose (A1)-(A4) hold, $\Omega$ is a finite set and $\ell_i(t_k) \in \Omega$ for all $k$. Let $Q$ be the transformation matrix such that $J = Q^{-1} A Q = \text{diag}\{J_{11}(\lambda_1^A) \cdots J_{1n}(\lambda_1^A) \cdots \cdots J_{n1}(\lambda_1^A) \cdots J_{nn}(\lambda_1^A)\}$ where $J_{ni}(\lambda_1^A) = \lambda_1^A I_{n_i} + A^{n_i} \in \mathbb{R}^{n_i \times n_i}$
and \( \{\lambda_A^1, \ldots, \lambda_A^f, \ldots, \lambda_A^p\} \) is the set of eigenvalues of \( A \) where \( \lambda_A^i \) are not necessarily distinct (since geometric multiplicities of an eigenvalue can be greater than 1) and \( \text{Re}(\lambda_A^1) \geq \cdots \geq \text{Re}(\lambda_A^f) \geq \cdots \geq \text{Re}(\lambda_A^p) \), \( \{\lambda_A^1, \ldots, \lambda_A^f\} \in \mathbb{C}^+ \), \( \{\lambda_{A_{i+1}}^1, \ldots, \lambda_A^p\} \in \mathbb{C}^- \) such that \( n_1 + \cdots + n_\ell = r \) where \( r \) is the number of unstable eigenvalues in \( A \) and \( n_1 + \cdots + n_p = n \). The MAS with agents given by

\[
\dot{z}_i(t) = A z_i(t) + \Phi \sum \alpha_{ij}(t) (z_j(t) - z_i(t)), \quad i \in \mathbb{Z}^m
\]

reaches consensus if the system

\[
\dot{z}_i(t) = A_D z_i(t) + \Phi \sum \alpha_{ij}(t) (z_j(t) - z_i(t)), \quad i \in \mathbb{Z}^m
\]

reaches consensus where \( A_D = Q \text{diag} \{ \lambda_A^1, \ldots, \lambda_A^f, \ldots, \lambda_A^p \} Q^{-1}, \Phi = Q \text{diag} \{ \gamma^1 I_{n_1}, \ldots, \gamma^\ell I_{n_\ell}, \ldots, \gamma^p I_{n_p} \} Q^{-1} \) for some \( \gamma^1, \ldots, \gamma^p \).

**Proof:** Let \( z_i = Q x_i \) for all \( i \in \mathbb{Z}^m \). The agent dynamics of (40) can be written as

\[
\dot{x}_i(t) = J x_i + \Gamma \sum \alpha_{ij}(t) (x_j(t) - x_i(t)), \quad i \in \mathbb{Z}^p
\]

Rearrange the above in terms of \( x_i = [x_i^{11} \cdots x_i^{1p}] \) where \( x_i^{11} \in \mathbb{Z}^{n_1} \) is the first \( n_1 \) elements of \( x_i \), \( x_i^{12} \in \mathbb{Z}^{n_2} \) is the second and so on. Collecting the \( x_i^{11} \) from the \( m \) agents and denoting \( \mathbf{x}^{11} = [(x_1^{11})'(x_2^{11})'(\cdots(x_m^{11}))']' \), it follows that

\[
\mathbf{x}^{11} = (J_{n_1}(\lambda_A^1) \otimes I_m - \gamma^1 I_{n_1} \otimes L(t))\mathbf{x}^{11}
\]

which is similar to the expression of (25) except for \( S \) being replaced by \( J_{n_1}(\lambda_A^1) \). Using the result Lemma 1, the above result reaches consensus if

\[
\mathbf{x}^{11} = (\lambda_A^1 I_{n_1} \otimes I_m - \gamma^1 I_{n_1} \otimes L(t))\mathbf{x}^{11}
\]

reaches consensus. Repeating this for \( \mathbf{x}^{1q} \), \( q \in \mathbb{Z}_2^p \) completes the proof. \( \Box \).

### 8 Realization of the Controller for Fixed Graph

Theorems 1 to 3 are existence results. They show that consensus of (21) is achieved when \( \Gamma \) is sufficiently large. However, the necessary value of \( \Gamma \) needed may not be known apriori. A more practical approach is to start with some choice of \( \Gamma \) and increment their values when consensus is not reached. This section discusses the implementation of such a scheme for the case where the graph is fixed and undirected.

Incrementing the values of \( \Gamma \) can only be done by individual agents. To allow for such an updating scheme, a more
general set of notations is needed. Specifically, (21) is replaced by

$$\dot{x}_i(t) = Sx_i(t) + \Gamma_i \sum_{j=1}^{m} a_{ij}(t)(x_j(t) - x_i(t)), \quad i \in \mathbb{Z}^m$$

(42)

where $\Gamma_i := \text{diag}_n\{\gamma_1^i, \cdots, \gamma_n^i\}$

(43)

and $\gamma_i^\ell$ is the $\ell$ diagonal element of $\Gamma_i$. In this setting, $\gamma_i^\ell$ is not necessary the same as $\gamma_j^\ell$ when updates are performed by different agents. Collecting the $\ell$ components over all agents yields

$$\dot{\mathbf{x}}^\ell(t) = (\lambda^\ell A_m - \Upsilon^\ell \mathbf{L})\mathbf{x}^\ell(t), \quad \ell \in \mathbb{Z}^n$$

(44)

where $\Upsilon^\ell := \text{diag}_m\{\gamma_1^\ell, \cdots, \gamma_m^\ell\}$

(45)

The next theorem, which is a generalization of Theorem 1, shows the properties of the MAS under this setting.

**Theorem 5** Suppose (A4) holds and $G$ is an undirected and connected graph. The MAS of (42) with conditions (22), (23) and (43) reaches consensus if $\gamma^\ell > \frac{\text{Re}(\lambda^\ell)}{\lambda^2}$ for all $\ell \in \mathbb{Z}^n$ where $\gamma^\ell := \min\{\gamma_1^\ell, \cdots, \gamma_m^\ell\}$.

**Proof:** Following the development from (42) till (44) and that $\lambda^\ell A_m$ and $\Upsilon^\ell \mathbf{L}$ commute,

$$\mathbf{x}^\ell(t) = e^{(\lambda^\ell A_m - \Upsilon^\ell \mathbf{L})t}\mathbf{x}^\ell(0) = e^{\lambda^\ell t}e^{-\Upsilon^\ell \mathbf{L}t}\mathbf{x}^\ell(0) = e^{\lambda^\ell t}1_m\xi_1^\ell\mathbf{x}^\ell(0) + \sum_{j=2}^{m} e^{(\lambda^\ell - \lambda_j^\ell)t}\varphi_j^\ell \xi_j^\ell\mathbf{x}^\ell(0)$$

(46)

where $\lambda_j^\ell$ is the $j$th eigenvalue of $\Upsilon^\ell \mathbf{L}$ with $\varphi_j$ and $\xi_j$ being the right and left eigenvectors respectively. Note that $\Upsilon^\ell \mathbf{L}$ is a Laplacian matrix of a connected graph since its connectivity is the same as that of $\mathbf{L}$ and hence has $1_m$ as its right eigenvector for eigenvalue 0. It can also be expressed as $\Upsilon^\ell \mathbf{L} = \text{diag}_m\{\gamma_1^\ell, \cdots, \gamma_m^\ell\} \gamma^\ell \mathbf{L}$ where $\gamma^\ell := \min\{\gamma_1^\ell, \cdots, \gamma_m^\ell\}$. Suppose $\gamma^\ell > \frac{\text{Re}(\lambda^\ell)}{\lambda^2}$, then it follows from lemma 2, $\mathbf{L}$ being symmetric and the structure of $\Upsilon^\ell$ that

$$\text{Re}(\lambda^2_{1\mathbf{L}}) \geq \lambda^2_{1\mathbf{L}} = \gamma^\ell \lambda^2_{\mathbf{L}} > \text{Re}(\lambda^\ell)$$

where the equality follows from $\text{spec}(\gamma^\ell \mathbf{L}) = \gamma^\ell \text{spec}(\mathbf{L})$. In addition, $\text{Re}(\lambda^2_{j\mathbf{L}}) \geq \text{Re}(\lambda^2_{1\mathbf{L}})$ for $j \in \mathbb{Z}^n$. This implies clearly that the second term on the right of equation (46) approaches 0 exponentially if $\text{Re}(\lambda^2_{1\mathbf{L}}) > \text{Re}(\lambda^\ell)$ and $\mathbf{x}^\ell(t)$ approaches $e^{\lambda^\ell t}1_m\xi_1^\ell\mathbf{x}^\ell(0)$. That this property holds for all $\ell \in \mathbb{Z}^n$ implies that the agents reach consensus. □

With Theorem (5), it is now possible to discuss the controller design. The basic idea is to update the values of $\Gamma_i, i \in \mathbb{Z}^m$ periodically by agent $i$. In between the updates, the values of $\Gamma_i$ remains unchanged. Let the updating period be $\tau > 0$ (see Remark 6 below). Hence, $\Gamma_i(k\tau) := \text{diag}_n\{\gamma_1^i(k\tau), \cdots, \gamma_n^i(k\tau)\}$ change values at every increment of $k$. The increment value of $\gamma_i^\ell$ is based on the amount of divergence observed among the agents for each $\ell$ in $x_i^\ell$. 
over each $\tau$ period. Specifically, for each agent $i$ and for all $\ell \in \mathbb{Z}^m$ set up the system

$$\rho_i^\ell(t) = r(\omega_i^\ell(t))^2 \text{ where } \omega_i^\ell(t) = \sum_{j=1}^{m} \alpha_{ij}(x_j^\ell(t) - x_i^\ell(t))$$  \hspace{1cm} (47)

$$\phi_i^\ell(t) = \sum_{j=1}^{m} \alpha_{ij}(\phi_j^\ell(t) - \phi_i^\ell(t))$$  \hspace{1cm} (48)

with $r > 0$ and $\phi_i^\ell(0) = 1$ and $\rho_i^\ell(0) = 0$. Here, $\phi_i^\ell$ is a proxy variable for $\gamma_i^\ell$ - it stores the changes needed for $\gamma_i^\ell$ during each period and updates $\gamma_i^\ell$ at the end of each. The workings of these variables are best explained in the pseudo code below. Note that since $\gamma_i^\ell$ is updated periodically, its value at $k\tau$ is different before and after the update. These two quantities, as well as other appropriate variables, are denoted by $\gamma_i^\ell(k\tau^-)$ and $\gamma_i^\ell(k\tau^+)$ respectively.

**Algorithm 1** Controller for agent $i$

1: Input: $r > 0, \tau > 0$.
2: Initialization: $\rho_i^\ell(0) = 0$, $\phi_i^\ell(0) = \gamma_i^\ell(0) = 1$ for all $i \in \mathbb{R}^m, \ell \in \mathbb{R}^n, k = 1$
3: Obtain $\phi_i^\ell(k\tau)$ and $\rho_i^\ell(k\tau)$ over the $k$ period from (47) and (48) respectively with initial values $\phi_i^\ell((k-1)\tau^-)$ and $\rho_i^\ell((k-1)\tau^-)$.
4: At $t = k\tau$
5: $\gamma_i^\ell(k\tau^+) \leftarrow \phi_i^\ell(k\tau^-) + \rho_i^\ell(k\tau) - \rho_i^\ell((k-1)\tau^-)$
6: $\phi_i^\ell(k\tau^+) \leftarrow \gamma_i^\ell(k\tau^-)$
7: set $k = k + 1$
8: goto step 3

**Theorem 6** Suppose (A4) holds and $G$ is an undirected and connected graph. The MAS of (42), (43) with conditions (22) and (23) reaches consensus under the adaptive control law given by (47)-(48) with $\Gamma_i(t)$ and $\phi_i^\ell(t)$ updated periodically according to Algorithm 1. Moreover, $\lim_{t \to \infty} \gamma_i^\ell(t) = \lim_{t \to \infty} \gamma_j^\ell(t)$ for all $i, j \in \mathbb{Z}^m$.

**Proof:** The proof of the first part is by contradiction. Suppose MAS (42) does not reach consensus, it follows from (46) of Theorem 5 that $Re(\lambda_i^\ell L) < Re(\lambda_j^\ell L)$ for some $\ell \in \mathbb{Z}^n$, which implies that $x^\ell(t) - e^{\lambda_i^\ell t}1_m\xi_i^\ell x^0(t)$ diverges exponentially as $t \to \infty$. Since $\xi_2$ is linearly independent from $1_m$, it follows from (46) that there exists at least one agent, $i^*$, such that $|x_{i^*}^\ell(t) - x_j^\ell(t)| > \zeta e^{0.5\beta t}$ for some $\beta, \zeta > 0$ where $j$ is a neighbor of $i^*$. This implies that $r(w_{i^*}^\ell(t))^2 > r\zeta^2 e^{\beta t}$ in (47) and

$$\int_{(k-1)\tau}^{k\tau} \rho_{i^*}^\ell(t)dt > \int_{(k-1)\tau}^{k\tau} r\zeta^2 e^{\beta t}dt = \frac{r\zeta^2}{\beta}(1 - e^{-\beta\tau})e^{\beta k\tau} := \bar{\zeta} e^{\beta k\tau}$$

with $\bar{\zeta} > 0$. Steps 5 and 6 of Algorithm 1 show that $\gamma_{i^*}^\ell(k\tau^-)$ and $\phi_{i^*}^\ell(k\tau^-)$ are incremented at each $k$ by a value that is greater than $\bar{\zeta} e^{\beta k\tau}$. This is exactly the settings for Lemma 3 which shows that $\bar{\phi}(k\tau) > \delta e^{\beta k\tau}$ for some $\delta > 0$ where $\bar{\phi}() := min_i\{\phi_i() : i \in \mathbb{Z}^m\}$. This implies that $\phi_i^\ell(k\tau)$ is an exponentially increasing function of $k$ for all $i \in \mathbb{Z}^m$ and all $\ell \in \mathbb{Z}^n$. For sufficiently large $k$, $\bar{\phi}^\ell(k\tau) > \frac{Re(\lambda_j^\ell L)}{\lambda_2^\ell}$ and it follows from Theorem 5 that MAS (42)
reaches consensus, which leads to a contradiction.

Consider the time after \( x_i(t) \) has reached consensus, the values of \( \omega_i(t) = 0 \) for every period after the consensus is reached. In addition, \( \gamma^f_i \) is updated by \( \phi^f_i \) in step 5 of Algorithm 1. Since \( \phi_i(t) \) reaches consensus following (48), \( \gamma^f_i = \gamma^f_j \) for all \( i,j \in \mathbb{Z}^m \). □

**Remark 6** The parameters \( r \) and \( \tau \) are used in Algorithm 1 and the proof of Theorem 6. Their values affect the rate at which consensus is reached but not the asymptotic results so long as \( r, \tau \geq \epsilon \) for some \( \epsilon > 0 \).

9 Numerical Example

The example shown in this section is for the case described in Theorem 6 with \( m = 4 \) and \( n = 2 \). A dynamic compensator of the form of (17),(18) is used for the system (16) with

\[
A = \begin{bmatrix} 0.1 & 0.1 \\ -0.1 & 0.3 \end{bmatrix}, \quad B = [1 \, 4]^T, \quad K = [-8.1 \, 1.3]
\]

and the transformation matrix \( Q = \begin{bmatrix} 1 & -2 \\ 1 & 8 \end{bmatrix} \). Initial conditions are \( z_1(0) = [-6 \, 3]^T, \ z_2(0) = [7 \, -15]^T, \ z_3(0) = [-18 \, 21]^T \), \( z_4(0) = [21 \, -2]^T \) and \( \eta_i(0) = 0.5z_i(0) \) for \( i = 1, \ldots, 4 \). Note that \( A \) is unstable and is similar to \( J = Q^{-1}AQ = 0.2I_2 + N^2 \). The communication network is fixed and given by

\[
\mathcal{L} = \begin{bmatrix}
0.7513 & -0.7513 & 0 & 0 \\
-0.7513 & 1.0064 & -0.2551 & 0 \\
0 & -0.2551 & 0.7611 & -0.5060 \\
0 & 0 & -0.5060 & 0.5060 \\
\end{bmatrix}.
\]

As stated in Theorem 6 and Theorem 4, the system of (16) reaches consensus if \( \gamma^f \) is sufficiently large. \( \gamma^f_0 \) is incremented from \( \gamma_0 \) following the procedure of Algorithm 1 in Section 8 with inputs to the algorithm being \( \gamma_0 = 1, \ r = 0.1, \ \tau = 4 \). Figure 1a shows the trajectories of the four agents in the state-space while Figures 2a and 2b show \( \gamma^1_i \) and \( \gamma^2_i \) increase in values from \( t = 0 \) till \( t = 4 \) sec. At \( t = 4 \), consensus has not been reached (see figure 1b). The values of \( \gamma^1_i \) and \( \gamma^2_i \) then move towards the consensus value from \( t = 8 \) seconds onwards due to the updating of \( \gamma^f_i \) by \( \phi^f_i \) and that \( \phi^f_i(t) \) reaches consensus following (48).

10 Conclusions

This work considers the consensus problem of multi-agent system with identical agents. Each agent is described by a general linear dynamical system with possibly repeated unstable eigenvalues. The approach to achieve consensus is
based on a dynamic compensator approach where additional design freedom is made available using a gain matrix. The design of the gain matrix is via a similarity transformation to a diagonal form. In the diagonal form, the consensus conditions can be easily checked and be made to achieve consensus when the diagonal matrix has entries that are sufficiently large. Three cases of the communication graphs are considered: (i) fixed; (ii) switching among connected and undirected; and (iii) switching among connected and directed. The results provided establish the existence of gain matrices that achieve consensus with specific conditions given in the first two cases. A more constructive controller based on an adaptive scheme is also established for the case of a fixed communication graph that is connected and undirected. It uses no global information of the network and relies on the periodic updates of the controller gain to ensure consensus.
Lemma 2 ([19]) Let $A \in \mathbb{R}^{m \times m}$ be a symmetric matrix and $B = \text{diag}_m\{b_1, b_2, \cdots, b_m\}$ with $b_i \geq 1$ for all $i$. Let the respective eigenvalues of $A$ and $BA$ be $\{\lambda_A^i\}_{i=1}^m$ and $\{\lambda_{BA}^i\}_{i=1}^m$ in increasing order. Then for all $i$,

$$\lambda_{BA}^i \geq \lambda_A^i \tag{49}$$

Proof: Found in [19].

Lemma 3 Consider the dynamics (48) and assume that there exists an $i^*$ such that $\phi_{i^*}'$ is periodically updated at every $k$ as given in steps 5 and 6 of Algorithm 1 such that

$$\phi_{i^*}'(k\tau^+) = \phi_{i^*}'(k\tau^-) + \bar{\zeta}e^{\beta k\tau} \tag{50}$$

where $\bar{\zeta}, \beta > 0$. Let, $\phi^\ell(\cdot) := \min_i \{\phi_i^\ell(\cdot) : i \in \mathbb{Z}^m\}$. Then $\phi^\ell(k\tau)$ an exponentially increasing function of $k$.

Proof: Since all discussions in this proof refers to a specific value of $\ell$, the superscript $\ell$ is dropped for notational convenience. The $m$ equations of (48) can be written as $\dot{\phi}(t) = -\mathcal{L}\phi(t)$ where $\phi = [\phi_1 \phi_2 \cdots \phi_m]'$ and $\phi_i \in \mathbb{R}$. Since $\mathcal{L}$ corresponds to a connected and undirected graph, the final consensus value, $\phi^c$, is given by the average of initial values. However, since $\phi_i(k\tau^+)$ is updated at every $k$, the consensus value $\phi^c$ also changes at $t = k\tau^+$ for every $k$, see Figure 3. Specifically, $\phi^c(t) := \frac{1}{m}\sum_{i=1}^m \phi_i(k\tau^+)$ for $t \in [k\tau^+, (k+1)\tau^-)$. Since $\phi^c(t)$ is a constant for $t \in [k\tau^+, (k+1)\tau^-)$, denote $\phi^c(k) := \phi^c(k\tau^+)$. Then,

$$\phi^c(k) := \frac{1}{m}\sum_{i=1}^m \phi_i(k\tau^+) = \frac{1}{m}\left( \sum_{i=1, i \neq i^*}^m \phi_i(k\tau^+) + \phi_{i^*}(k\tau^-) + \bar{\zeta}e^{\beta k\tau} \right) = \phi^c(k) + \frac{1}{m}\bar{\zeta}e^{\beta k\tau} \tag{51}$$

Since $\phi_i(t) \rightarrow \phi^c(k)$ exponentially for $t \in [k\tau^+, (k+1)\tau^-)$ for every $i \in \mathbb{Z}^m$ [20], it follows that $\phi(t) \rightarrow \phi^c(k)$ exponentially where $\phi(\cdot) := \min_i \{\phi_i(\cdot) : i \in \mathbb{Z}^m\}$. This means that there exists a $\mu > 0$ such that

$$\phi(t) \geq \phi^c(k) - \psi_k e^{-\mu(t-k\tau)} t \in [k\tau^+, (k+1)\tau^-)$$

where $\psi_k := \phi^c(k) - \phi(k\tau^+)$. Let $t = (k+1)\tau^-$, then

$$\phi((k+1)\tau^-) \geq \phi^c(k) - (\phi^c(k) - \phi(k\tau^+)) e^{-\mu\tau} = \phi^c(k)(1 - e^{-\mu\tau}) + \phi(k\tau^+) e^{-\mu\tau}$$

$$\Rightarrow \phi((k+1)\tau^+) \geq \phi((k+1)\tau^-) \geq \phi^c(k) \eta = (\phi^c(k) + \frac{1}{m}\bar{\zeta}e^{\beta k\tau}) \eta \geq \delta e^{\beta k\tau}$$

where $\eta = (1 - e^{-\mu\tau}) > 0$, $\delta = \frac{1}{m}\bar{\zeta}\eta$ and the equality follows from (51). The last inequality establishes the claim of the lemma. $\square$.
Fig. 3. Depiction of $\phi^c$, $\phi_i(k\tau^-)$ and $\phi_i(k\tau^+)$ within the period $k\tau$ to $(k+1)\tau$

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