Not-All-Equal and 1-in-Degree Decompositions: Algorithmic Complexity and Applications

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Abstract
A Not-All-Equal (NAE) decomposition of a graph $G$ is a decomposition of the vertices of $G$ into two parts such that each vertex in $G$ has at least one neighbor in each part. Also, a 1-in-Degree decomposition of a graph $G$ is a decomposition of the vertices of $G$ into two parts $A$ and $B$ such that each vertex in the graph $G$ has exactly one neighbor in part $A$. Among our results, we show that for a given graph $G$, if $G$ does not have any cycle of length congruent to 2 mod 4, then there is a polynomial time algorithm to decide whether $G$ has a 1-in-Degree decomposition. In sharp contrast, we prove that for every $r$, $r \geq 3$, for a given $r$-regular bipartite graph $G$ determining whether $G$ has a 1-in-Degree decomposition is \textbf{NP}-complete. These complexity results have been especially useful in proving \textbf{NP}-completeness of various graph related problems for restricted classes of graphs. In consequence of these results we show that for a given bipartite 3-regular graph $G$ determining whether there is a vector in the null-space of the 0,1-adjacency matrix of $G$ such that its entries belong to \{±1, ±2\} is \textbf{NP}-complete. Among other results, we introduce a new version of Planar 1-in-3 \textsc{sat} and we prove that this version is also \textbf{NP}-complete. In consequence of this result, we show that for a given planar (3,4)-semiregular graph $G$ determining whether there is a vector in the null-space of the 0,1-incidence matrix of $G$ such that its entries belong to \{±1, ±2\} is \textbf{NP}-complete.

Keywords: Not-All-Equal decomposition; 1-in-Degree decomposition; Total perfect dominating set; Zero-sum flow; Zero-sum vertex flow.

1 Introduction

Studying the structure of the null-space of the incidence matrix of any undirected graph is an active area in linear algebra and computer science. For instance, Villarreal \cite{48} proved that the null-space

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of the incidence matrix of every graph has a basis whose elements have entries in \{-2, -1, 0, 1, 2\} (for more information see [26, 35, 40, 38]). A zero-sum \(k\)-flow for a graph \(G\) is a vector in the nullspace of the 0,1-incidence matrix of \(G\) such that its entries belong to \{\pm 1, \ldots, \pm (k - 1)\}. Recently, zero-sum flows have been studied extensively by several authors, for example see [12, 13, 17, 41, 51, 52, 49, 50]. Zero-sum flows are interesting to study, because of their connections to Bouchet’s 6-flow conjecture (every bidirected graph that has a nowhere-zero bidirected flow admits a nowhere-zero bidirected 6-flow [7]) and Tutte’s 5-flow conjecture (every bridgeless graph has a nowhere-zero 5-flow [47]). For more information about these two conjectures, see [28, 45, 54, 53, 56, 58]. Recently, it was shown that it is \(\text{NP}\)-complete to determine whether a given graph \(G\) has a zero-sum 3-flow [17]. In this work, we improve the previous hardness result and show that it is \(\text{NP}\)-complete to determine whether a given planar graph \(G\) has a zero-sum 3-flow.

On the other hand, eigenspaces of the adjacency matrix of graphs have been studied for many years [18, 36, 40]. This is especially the case for the null-space of the adjacency matrix of graphs, which has been studied for a number of graph classes. A simply structured basis is a basis vector that contains only entries from the set \{-1, 0, 1\}. Simply structured bases have been shown to exist for a number of graph classes. Mainly, attention is restricted to the graph kernel. The existing literature features results on trees [35], line graphs of trees [29, 42], unicyclic graphs [33, 39], bipartite graphs [15], and cographs [37]. A zero-sum vertex \(k\)-flow is a vector in the null-space of the 0,1-adjacency matrix of \(G\) such that its entries belong to \{\pm 1, \ldots, \pm (k - 1)\}. In this paper, we show that for a given 3-regular bipartite graph \(G\) determining whether \(G\) has a zero-sum vertex 3-flow is \(\text{NP}\)-complete.

In this work, in order to study zero-sum vertex flows, we introduce the concept of Not-All-Equal (NAE) decomposition and the concept of 1-in-Degree decomposition of graphs. A graph \(G\) has a NAE decomposition if the vertices of the graph \(G\) can be partitioned into two total dominating sets. In other words, a NAE decomposition of a graph \(G\) is a decomposition of the vertices of \(G\) into two parts such that each vertex in the graph \(G\) has at least one neighbor in each part. Also, a graph \(G\) has a 1-in-Degree decomposition if the graph \(G\) has a total perfect dominating set. In other words, a 1-in-Degree decomposition of a graph \(G\) is a decomposition of the vertices of \(G\) into two sets \(A\) and \(B\) such that each vertex in the graph \(G\) has exactly one neighbor in part \(A\). The computational complexity of determining whether a given graph \(G\) has a total perfect dominating set has been studied by several authors, for instance see [8, 9, 11, 25, 27, 44]. It was shown that the problem of deciding whether a planar bipartite graph of maximum degree three has any total perfect dominating set is \(\text{NP}\)-complete [44]. On the other hand, about the partitioning of a graph into two total dominating sets, Zelinka [57] showed that the large minimum degree is not sufficient to guarantee the existence of a partition of a graph into two total dominating sets. Moreover, Calkin and Dankelmann [10] and Feige et al. [21] showed that if the maximum degree is not too large relative to the minimum degree, then sufficiently large minimum degree does suffice.

1.1 Our Results

Planar 1-in-3 SAT is a well-known \(\text{NP}\)-complete problem in computational complexity.
Planar 1-in-3 SAT.
INSTANCE: A 3SAT formula $\Psi = (X, C)$ such that the bipartite graph obtained by linking a variable and a clause if and only if the variable appears in the clause, is planar.
QUESTION: Is there a truth assignment for $X$ such that each clause in $C$ has exactly one true literal?

Also, it is well-known that Planar 1-in-3 SAT is NP-complete, even if we add the backbone between the variable vertices [32]. In this work, we introduce a new practical version of Planar 1-in-3 SAT which has a “tree” between the clause vertices and we prove that this version is also NP-complete. We call this problem Monotone Planar Tree-like 1-in-3 SAT.

Monotone Planar Tree-like 1-in-3 SAT.
INSTANCE: A 3SAT formula $\Psi = (X, C)$ such that
(i) every variable appears in exactly three clauses,
(ii) there is no negation in the formula, and
(iii) the following graph obtained from $\Psi$ is planar.
The graph has one vertex for each variable, one vertex for each clause, each clause vertex is connected by an edge to the variable vertices corresponding to the literals present in the clause, and some clause vertices are connected to each other such that the induced subgraph on the set of clause vertices forms a tree.
QUESTION: Is there a truth assignment for $X$ such that each clause in $C$ has exactly one true literal?

In order to show that the above problem is NP-complete, we reduce the following problem to our problem. Moore and Robson [30] proved that the following version of the 1-in-3 SAT problem is NP-complete.

Cubic Planar 1-in-3 SAT.
INSTANCE: A 3SAT formula $\Psi = (X, C)$ such that each variable appears in exactly three clauses, there is no negation in the formula, and the bipartite graph obtained by linking a variable and a clause if and only if the variable appears in the clause, is planar.
QUESTION: Is there a truth assignment for $X$ such that each clause in $C$ has exactly one true literal?

Next, we focus on an application of this hardness result. A zero-sum $k$-flow for a graph $G$ is a vector in the null-space of the 0,1-incidence matrix of $G$ such that its entries belong to $\{\pm 1, \ldots, (k-1)\}$. In other words, a zero-sum $k$-flow for a graph $G$ is a labeling of its edges from the set $\{\pm 1, \ldots, \pm (k-1)\}$ such that the sum of the labels of all edges incident with each vertex is zero. In 2009, Akbari et al. posed the following interesting conjecture about the zero-sum flows [2].

Conjecture A[Zero-Sum Conjecture (ZSC)] [2]: If $G$ is a graph with a zero-sum flow, then the graph $G$ admits a zero-sum 6-flow.

In 2010, Akbari et al. proved that Bouchet’s Conjecture (every bidirected graph that has a
nowhere-zero bidirected flow admits a nowhere-zero bidirected 6-flow \[7\]) and Zero-Sum Conjecture are equivalent [1]. Regarding the computational complexity of zero-sum flows, it was shown that it is NP-complete to determine whether a given (3, 4)-semiregular graph has a zero-sum 3-flow [17]. Here, we improve the previous complexity result for the class of planar graphs. We show that for a given planar (3, 4)-semiregular graph \(G\) determining whether there is a vector in the null-space of the 0,1-incidence matrix of \(G\) such that its entries belong to \(\{\pm 1, \pm 2\}\) is NP-complete.

1-in-3 SAT \(\leq_p^m\) Cubic Planar 1-in-3 SAT
\(\leq_p^m\) Monotone Planar Tree-like 1-in-3 SAT
\(\leq_p^m\) Finding a Zero-Sum 3-Flow.

A graph \(G\) has a NAE decomposition if the vertices of the graph \(G\) can be partitioned into two total dominating sets. In other words, a NAE decomposition of a graph \(G\) is a decomposition of the vertices into two parts such that each vertex in the graph \(G\) has at least one neighbor in each part. Also, a graph \(G\) has a 1-in-Degree decomposition if the graph \(G\) has a total perfect dominating set. In other words, a 1-in-Degree decomposition of a graph \(G\) is a decomposition of the vertices into two sets \(A\) and \(B\) such that each vertex in the graph \(G\) has exactly one neighbor in part \(A\). Since many problems in graph theory and computer science have the symmetric structures, NAE and 1-in-Degree decompositions have many applications to prove the NP-completeness of the problems.

Next, we consider the computational complexity of the determining whether a graph has a 1-in-Degree decomposition. We prove that for every \(r \geq 3\), for a given \(r\)-regular bipartite graph \(G\) which \(G\) has a Not-All-Equal decomposition determining whether \(G\) has a 1-in-Degree decomposition is NP-complete. Moreover, we show that if \(G\) is a bipartite graph and does not have any cycle of length congruent to 2 mod 4, then there is a polynomial time algorithm to decide whether the graph \(G\) has a 1-in-Degree decomposition.

On the other hand, regarding the NAE decomposition, we show that if \(G\) is an \(r\)-regular bipartite graph and \(r \geq 4\), then the graph \(G\) always has a NAE decomposition. A summary of results on the NAE and 1-in-Degree decompositions is shown in Table 1.

| \(r\)-regular bipartite graph | The existence of 1-in-Degree decomposition | The existence of NAE decomposition |
|-------------------------------|------------------------------------------|----------------------------------|
| \(r = 3\)                    | NP-complete                              | Open                             |
| \(r \geq 4\)                 | NP-complete                              | \(P\)                            |

As we can see in Table 1, for the case \(r = 3\) the complexity of the existence of NAE decomposition is unknown. Thus, the computational complexities of the following two interesting problems remain unsolved.

\textit{Cubic Bipartite NAE Decomposition Problem.}
\textbf{Instance:} A 3-regular bipartite graph \(G\).
QUESTION: Is there a NAE decomposition for $G$?

*Cubic Monotone NAE 3SAT.*

**Instance:** Set $X$ of variables, collection $C$ of clauses over $X$ such that for each clause $c \in C$ we have $|c| = 3$, every variable appears in exactly three clauses and there is no negation in the formula.

**Question:** Is there a truth assignment for $X$ such that each clause in $C$ has at least one true literal and at least one false literal?

Note that there is a simple polynomial time reduction from *Cubic Bipartite NAE Decomposition Problem* to *Cubic Monotone NAE 3SAT*. Thus if *Cubic Bipartite NAE Decomposition Problem* is NP-complete, then, *Cubic Monotone NAE 3SAT* is NP-complete. Here, we ask the following question.

**Problem A.** Is there any polynomial time reduction from *Cubic Monotone NAE 3SAT* to *Cubic Bipartite NAE Decomposition Problem*?

Next, we focus on the applications of NAE and 1-in-Degree decompositions.

An application of the 1-in-Degree decomposition: A zero-sum vertex $k$-flow is a vector in the null-space of the 0,1-adjacency matrix of $G$ such that its entries belong to $\{\pm 1, \ldots, \pm (k - 1)\}$. It was shown that for a given bipartite $(2,3)$-graph $G$, it is NP-complete to decide whether the graph $G$ has a zero-sum vertex 3-flow [18]. We use from our hardness results on 1-in-Degree decompositions and prove that for a given 3-regular bipartite graph $G$ determining whether $G$ has a zero-sum vertex 3-flow is NP-complete.

An application of the NAE decomposition: For a given graph $G$, *The Minimum Edge Deletion Bipartition Problem* is to determine the minimum number of edges of $G$ such that their removal leads to a bipartite graph $H$. It was shown that this problem is NP-hard even if all vertices have degrees 2 or 3 [20]. As an application of the NAE decomposition, we show that if *Cubic Bipartite NAE Decomposition Problem* is NP-complete, then for a given 3-regular graph $G$, *The Minimum Edge Deletion Bipartition Problem* is NP-hard.

*Cubic Bipartite NAE Decomposition Problem* $\leq^P_m$ *Cubic Monotone NAE 3SAT* $\leq^P_m$ *The Minimum Edge Deletion Bipartition Problem*.

Next, we generalize the decompositions for vertex-weighted graphs. A vertex-weighted graph is a graph in which each vertex has been assigned a weight. Let $G$ be a vertex-weighted graph and $w : V(G) \to \mathbb{Z}$ be its weight function. A 1-in-Degree coloring for the graph $G$ is a function $f : V(G) \to \{0, 1\}$ such that for each vertex $v \in V(G)$, $\sum_{u \in N(v)} f(u)w(u) = 1$. We show that 1-in-Degree coloring in vertex-weighted graphs is harder than 1-in-Degree decomposition in simple graphs. For instance, although we show that if the given graph $G$ is a bipartite graph and does not have any cycle of length congruent to 2 mod 4, then there is a polynomial time algorithm to decide whether the graph $G$ has a 1-in-Degree decomposition, but for a given vertex-weighted bipartite
graph $G$, determining whether the graph $G$ has a 1-in-Degree coloring is strongly $\text{NP}$-complete, even if the given graph $G$ does not have any cycle of length congruent to 2 mod 4.

Finally, we study the edge version of the 1-in-Degree and the NAE decompositions. An edge coloring $f : E(G) \to \{0, 1\}$ for a graph $G$ is called 1-in-Degree edge coloring if and only if for every vertex $v$, $\sum_{e \ni v} f(e) = 1$. The graph $G$ has a 1-in-Degree edge coloring if and only if $G$ has a perfect matching. Now, consider the edge version of the NAE. An edge coloring $f : E(G) \to \{\text{red, blue}\}$ of a graph $G$ is called NAE edge coloring if and only if for every vertex $v$, there are edges $e$ and $e'$ incident with $v$, such that $f(e) \neq f(e')$. In this work, we show that for a given connected graph $G$ with $\delta(G) > 1$, the graph $G$ has a NAE edge coloring if and only if $G$ is not an odd cycle.

1.2 The organization of the paper

The remainder of this paper is organized as follows. Basic definitions and notation are provided in Subsection 1.3. In Section 2, we introduce a new version of Planar 1-in-3 SAT and we prove that this version is also $\text{NP}$-complete. In consequence of this result, we show that for a given planar (3, 4)-semiregular graph $G$ determining whether there is a vector in the null space of the 0,1-incidence matrix of $G$ such that its entries belong to $\{\pm 1, \pm 2\}$ is $\text{NP}$-complete. Next, in Section 3 we focus on the NAE and 1-in-Degree decompositions of graphs. In this section we introduce these decompositions. Afterwards, we study their computational complexities. In the next section, some applications of these decompositions in computational complexity are presented. In subsection 4.1, we consider zero-sum vertex flows and we prove that for a given bipartite 3-regular graph $G$ determining whether $G$ has a zero-sum vertex 3-flow is $\text{NP}$-complete. Also, In subsection 4.2, as another application of the 1-in-Degree decomposition, we consider The Minimum Edge Deletion Bipartition Problem and prove its hardness. Next, in Section 5, we consider the NAE and 1-in-Degree decompositions for weighted graphs and finally in Section 6, the edge versions of the NAE and 1-in-Degree decompositions are studied. The paper is concluded with some remarks in Section 7.

1.3 Notation

Throughout this paper all graphs are finite, simple and we follow [22, 55] for terminology and notation not defined here. Also, throughout the paper we denote $\{1, 2, \cdots, k\}$ by $\mathbb{N}_k$. We denote the vertex set and the edge set of $G$ by $V(G)$ and $E(G)$, respectively. Also, we denote the maximum degree and minimum degree of $G$ by $\Delta(G)$ and $\delta(G)$, respectively. Furthermore, for every $v \in V(G)$ and $X \subseteq V(G)$, $d(v)$, $N(v)$ and $N(X)$ denote the degree of $v$, the neighbor set of $v$ and the set of vertices of $G$ which has a neighbor in $X$, respectively. The adjacency matrix of a simple graph $G$ is an $n \times n$ matrix $A = [a_{ij}]$ where $n$ is the number of vertices, such that $a_{ij} = 1$ if the vertex $v_i$ is adjacent with the vertex $v_j$ and 0 otherwise. The incidence matrix of a directed graph $G$ is an $n \times m$ matrix $B = [b_{ij}]$ where $n$ and $m$ are the number of vertices and edges respectively, such that $b_{ij} = -1$ if the edge $e_j$ leaves vertex $v_i$, 1 if it enters vertex $v_i$ and 0 otherwise. Similarly, for an undirected graph $G$, the incidence matrix is an $n \times m$ matrix $B = [b_{ij}]$ such that $b_{ij} = 1$ if the
vertex \( v_i \) and edge \( e_j \) are incident and 0 otherwise. The null-space (kernel) of a matrix \( A \) is the set of all vectors \( x \) for which \( Ax = 0 \). An eigenvector of a square matrix \( A \) is a non-zero vector \( v \) that, when multiplied with \( A \), yields a scalar multiple of \( A \). The scalar multiplier is often denoted by \( \lambda \) and we have \( Av = \lambda v \). The number \( \lambda \) is called the eigenvalue of \( A \) corresponding to \( v \). An eigenspace of a matrix \( A \) is the set of all eigenvectors with the same eigenvalue, together with the zero vector.

A graph \( G \) is a \((d, d+s)\)-graph if the degree of every vertex of the graph \( G \) lies in the interval \([d, d+s]\). A \((d, d+1)\)-graph is said to be semiregular. A total dominating set of a graph is a set of vertices such that all vertices in the graph have at least one neighbor in the dominating set. Also, a total perfect dominating set of a graph \( G = (V, E) \) is a subset \( S \) of \( V(G) \) such that every vertex \( v \in V(G) \) is adjacent to exactly one vertex of \( S \) (for every vertex \( v, \ |N(v) \cap S| = 1 \). A matching in a graph is a set of edges without common vertices and a perfect matching is a matching which matches all vertices of the graph. A trail in a graph is a walk without repeated edges and an Eulerian cycle is a trail in a graph which visits every edge exactly once. A graph \( G = [X, Y] \) is called bipartite, if the vertex set \( V(G) \) can be partitioned into two disjoint subsets \( X \) and \( Y \), i.e., \( V(G) = X \cup Y \) and \( X \cap Y = \emptyset \), such that every edge in \( E \) connects a vertex from \( X \) to a vertex from \( Y \).

2 The structure of the kernel of the incidence matrix

It is well-known that Planar 1-in-3 SAT is \( \text{NP}\)-complete, even if we add the backbone between the variable vertices [32]. In the proof of the next theorem (Theorem 1), we introduce a new version of Planar 1-in-3 SAT which has a "tree" between the clause vertices and we prove that this version is also \( \text{NP}\)-complete. In consequence of this result, we show that for a given planar \((3, 4)\)-semiregular graph \( G \) determining whether there is a vector in the null-space of the 0,1-incidence matrix of \( G \) such that its entries belong to \( \{\pm 1, \pm 2\} \) is \( \text{NP}\)-complete.

**Theorem 1** It is \( \text{NP}\)-complete to determine whether a given planar \((3, 4)\)-semiregular graph has a zero-sum 3-flow.

**Proof** Our proof consists of two steps. In the first step, we prove that Monotone Planar Tree-like 1-in-3 SAT is \( \text{NP}\)-complete, then in the next step, we reduce this version to our problem in polynomial time.

**Step 1.** We reduce Cubic Planar 1-in-3 SAT into Monotone Planar Tree-like 1-in-3 SAT in polynomial time. Let \( \Psi \) be an instance of Cubic Planar 1-in-3 SAT. Consider the following graph obtained from \( \Psi \). The graph has one vertex for each variable, one vertex for each clause and each clause vertex is connected by an edge to the variable vertices corresponding to the literals present in the clause. Note that this graph is planar. Step by step, put a new edge between the clause vertices until if you add another edge the graph becomes non-planar. Call the resultant graph \( G \). Since every edge in the graph \( G \) is between two clause vertices or between a variable vertex and a clause vertex, also, there is no edge between variable vertices, the induced subgraph on the
set of variable vertices is an independent set. In other words the set of variable vertices with the set of edges that incident with them forms a galaxy (a galaxy or a star forest is a union of vertex disjoint stars). So, between every two clause vertices there is a path that consists of clause vertices. Consequently, the induced subgraph on the set of clause vertices is connected (otherwise we can add more edges between the clause vertices). Thus we can remove some edges from the graph $G$ such that the induced subgraph on the set of clause vertices forms a tree. Call the resultant planar graph $H_\Psi$. Note that in the next step, when we say that the graph $H_\Psi$ is the graph which is obtained from the formula $\Psi$, we mean a graph which is obtained from the above-mentioned procedure form the formula $\Psi$. This completes the first step.

**Step 2.** In this step, we reduce *Monotone Planar Tree-like 1-in-3 SAT* to our problem in polynomial time. Let $\Psi$ be an instance of *Monotone Planar Tree-like 1-in-3 SAT* and $H_\Psi$ be the graph which is obtained from $\Psi$ (we explained how to obtain $H_\Psi$ from $\Psi$ in Step 1). For every clause $c$, $c \in C$, we denote the number of clause vertices which are adjacent with the clause vertex $c$ in $H_\Psi$ by $\gamma(c)$. Also, for a vertex $v$, we say that the zero-sum rule holds on the vertex $v$ when the sum of assignments of all edges incident with the vertex $v$ is zero. For a given formula $\Psi$, we construct a planar $(3, 4)$-semiregular graph $G_\Psi$ from $H_\Psi$, such that the formula $\Psi$ has a 1-in-3 SAT satisfying assignment if and only if the graph $G_\Psi$ has a zero-sum 3-flow. For every variable $x$, $x \in X$, we create a cycle of length six with the vertices $v_{x1}, w_{x1}, v_{x2}, w_{x2}, v_{x3}, w_{x3}$, in that order. Now, consider three copies $I(u_{x1}), I(u_{x2}), I(u_{x3})$ of the gadget shown in Figure 1 and connect each $u_i$ to $v_i$ for $i = 1, 2, 3$. Call the resultant gadget $A_x$. In the gadget $A_x$ call the set of vertices of degree two, free vertices.

![Figure 1: The gadget $I(u)$.](image)

For every clause $c$, $c \in C$, consider a cycle of length $2\gamma(c)+8$ with the vertices $v_{c1}, w_{c1}, \ldots, v_{c(\gamma(c)+4)}, w_{c(\gamma(c)+4)}$ in that order. Now, consider $\gamma(c)+4$ copies $I(u_{c1}), \ldots, I(u_{c(\gamma(c)+4)})$ of the gadget shown in Figure 1 and connect each vertex $u_i$ to the vertex $v_i$ for $i = 1, \ldots, \gamma(c)+4$. Also, put a vertex $b_c$ and join it to the vertex $w_{c(\gamma(c)+4)}$. Call the resultant gadget $A_c$. In the gadget $A_c$ call the set of vertices of degree two, *free vertices* of $A_c$ and call the set of edges incident with the free vertices, *free edges*. For each pair $(g, h)$, where $g, h \in V(H_\Psi)$ if $gh \in E(H_\Psi)$, then join one of the free vertices of $A_g$ to one of the free vertices of $A_h$, such that having done these procedures for all variables and all clauses, the resultant graph is planar and its maximum degree is three. After these procedures, for every clause $c \in C$, there are exactly three free vertices of $A_c$ that were joined to the free vertices of variables, call these vertices, *important vertices* and join them to the vertex $b_c$. Note that in the resultant graph the degree of the vertex $b_c$ is four and the graph is a planar $(3, 4)$-semiregular.
Finally, for every important vertex \( v \), choose one of the free edges incident with the vertex \( v \) (note that every important vertex was a free vertex), suppose that we choose \( e = vz \), remove the edge \( e \) from the graph and put a vertex \( f_v \). Also, put two copies \( I(uv), I(uv') \) of the gadget shown in Figure 1 and connect the vertex \( f_v \) to the vertices \( v, z, u_v, u_v' \). Call the resultant planar \((3,4)\)-semiregular graph \( G_{Ψ} \).

Now, suppose that the graph \( G_{Ψ} \) has a zero-sum 3-flow. First, we present some useful lemmas.

**Lemma 1** For every vertex \( t \) of degree three, the labels of the three edges incident with that vertex are \( 2, -1, -1 \) or \( -2, 1, 1 \).

**Proof.** For every vertex \( t \) of degree three, the zero-sum rule implies that not all three edges incident with that vertex can have odd labels. Since the label of each edge is from \( \{±1, ±2\} \), the label \( -2 \) or the label 2 should appear on exactly one of three edges incident the vertex \( t \). Thus, by the zero-sum rule the labels of the three edges incident with the vertex \( t \) are \( 2, -1, -1 \) or \( -2, 1, 1 \). ♠

Similarly, for every vertex \( t \) of degree four, the labels of the four edges incident with that vertex are \( 2, -2, 1, -1 \) or \( 2, -2, 2, -2 \) or \( 1, -1, 1, -1 \). By above-mentioned lemma we have the following important lemma.

**Lemma 2** Let \( t, t' \in V(G_{Ψ}) \) be two vertices of degree three and \( tt' \in E(G_{Ψ}) \). Then the labels of the three edges incident with the vertex \( t \) are 2, -1, -1 if and only if the labels of the three edges incident with the vertex \( t' \) are 2, -1, -1.

**Proof.** Since \( t \) and \( t' \) have a common edge, by Lemma 1, the proof is clear. ♠

**Lemma 3** The set of edges with labels 2 or -2, in the induced subgraph on the set of vertices of degree three forms a matching.

**Proof.** By Lemma 1, for every vertex \( t \) of degree three, the labels of the three edges incident with that vertex are \( 2, -1, -1 \) or \( -2, 1, 1 \). So, the set of edges with labels 2 or -2, in the induced subgraph on the set of vertices of degree three, forms a matching. ♠

Now, we study the main property of the gadget \( I(u) \) which is shown in Figure 1.

**Lemma 4** In each copy of the gadget \( I(u) \) in the graph \( G_{Ψ} \), the vertex \( u \) has exactly one neighbor other than its neighbors in \( I(u) \). Without loss of generality call that vertex \( t \). The label of the edge \( ut \) is 2 or -2.

**Proof.** By Lemma 1 and Lemma 3, the set of edges with labels 2 or -2, in the induced subgraph on the set of vertices \( V(I(u)) \cup \{t\} \) forms a matching that saturates all the vertices in \( V(I(u)) \). Since \( I(u) \) has five vertices, in every matching \( M \) for the induced subgraph on the set of vertices
\( V(I(u)) \cup \{t \} \) that saturates all the vertices in \( V(I(u)) \), we have \( ut \in M \). Thus, the label of the edge \( ut \) is 2 or \(-2\). ♠

Let \( S \subseteq V(G_\Psi) \) be a subset of vertices such that for every vertex \( v \in S \), we have \( d(v) = 3 \) and the induced subgraph on the set of vertices \( S \) is connected. The zero-sum rule implies that not all three edges incident with a vertex in \( S \) can have odd labels, so -2 or 2 should appear on exactly one of them. Also, since the induced subgraph on the set of vertices \( S \) is connected, without loss of generality we can suppose that every vertex in \( S \) is incident with exactly one edge with label 2 or every vertex in \( S \) is incident with exactly one edge with label \(-2\). Therefore, for every \( x \in X \), in the subgraph \( A_x \), every vertex in \( A_x \) is incident with exactly one edge with label 2 or every vertex in \( A_x \) is incident with exactly one edge with label \(-2\). Similarly, for each variable \( x \in X \), by Lemma 4 and Lemma 2 the set of labels of the edges between \( V(A_x) \) and \( V(G_\Psi) \setminus V(A_x) \) is \( \{2\} \) or \( \{-2\} \). On the other hand, since in the graph \( H_\Psi \), the induced subgraph on the set of clause vertices forms a tree (is connected), by Lemma 1 and Lemma 4, in the graph \( G_\Psi \), the set of labels of edges \( \{w^c_{\gamma(c)+4}b_c|c \in C\} \) is \( \{2\} \) or \( \{-2\} \). Without loss of generality, suppose that \( \{w^c_{\gamma(c)+4}b_c|c \in C\} = \{-2\} \) (Fact 1).

See Figure 2. In this subgraph by Lemma 4 and lemma 2, The labels of colored edges are \( \pm 2 \) and the labels of black edges are \( \pm 1 \).

Figure 2: The labels of blue edges are 2, the labels of red edges are \(-2\) and the labels of black edges are \( \pm 1 \). Note that for every important vertex \( v \), because of the vertex \( f_v \), the labels of the four edges incident with the vertex \( v \) are 2, \(-2\), 1, \(-1\).

Now, we present a 1-in-3 SAT satisfying assignment for \( \Psi \). For every \( x \in X \), if every vertex in \( A_x \) is incident with exactly one edge with label 2, put \( \Gamma(x) = \text{true} \) and if every vertex in \( A_x \) is
incident with exactly one edge with label $-2$, put $\Gamma(x) = \text{false}$. For every clause $c \in C$, the vertex $b_c$ is incident with the edges with labels from $\{-2, 2\}$. By Fact 1, the label of edge $w^c_{\gamma(c)+4}b_c$ is $-2$, thus the zero-sum rule implies that the labels of edges incident with $b_c$, other than $w^c_{\gamma(c)+4}b_c$ are exactly $2, 2, -2$. Thus $\Gamma$ is a 1-in-3 SAT satisfying assignment for $\Psi$. Conversely, if $\Psi$ has a 1-in-3 SAT satisfying assignment $\Gamma$, we may assign labels $-2$ to the set of edges $\{w^c_{\gamma(c)+4}b_c \mid c \in C\}$.

Next, for every variable $x$, if $\Gamma(x) = \text{true}$ (respectively, $\text{false}$), label the edges of $A_x$ such that each vertex in $A_x$ is incident with exactly one edge with label 2 (respectively, $-2$). It is easy to extent this labeling to a zero-sum 3-flow. This completes the proof.

\[\square\]

3 Not-All-Equal and 1-in-Degree decompositions

In the next theorem, we consider the computational complexity of the determining whether a graph has a 1-in-Degree decomposition.

**Theorem 2**

(i) For every $r \geq 3$, for a given $r$-regular bipartite graph $G$ which $G$ has a Not-All-Equal decomposition determining whether $G$ has a 1-in-Degree decomposition is $\text{NP}$-complete.

(ii) If $G$ is a bipartite graph and does not have any cycle of length congruent to 2 mod 4, then there is a polynomial time algorithm to decide whether the graph $G$ has a 1-in-Degree decomposition.

**Proof** (i) The problem is in $\text{NP}$. We reduce Cubic Planar 1-in-3 SAT to our problem. Moore and Robson [30] proved that the following problem is $\text{NP}$-complete.

**Cubic Planar 1-in-3 SAT.**

**Instance:** A 3SAT formula $\Phi = (X, C)$ such that every variable appears in exactly three clauses, there is no negation in the formula, and the bipartite graph obtained by linking a variable and a clause if and only if the variable appears in the clause, is planar.

**Question:** Is there a truth assignment for $X$ such that each clause in $C$ has exactly one true literal?

On the other hand, Moret [31] proved that Planar NAE 3SAT is in $\text{P}$ by an interesting reduction to a known problem in $\text{P}$, namely Planar MaxCut. We also use Moret’s result in our proof.

**Planar NAE 3SAT.**

**Instance:** A 3SAT formula $(X, C)$ such that the following graph obtained from 3SAT is planar. The graph has one vertex for each variable, one vertex for each clause; all variable vertices are connected in a simple cycle and each clause vertex is connected by an edge to variable vertices corresponding to the literals present in the clause (note that positive and negative literals are treated exactly alike).

**Question:** Is there a NAE truth assignment for $X$?
By a simple argument we can see that the reduction holds also for the following problem. So the following problem is in \( \mathbf{P} \) (for more information see [3, 16]).

**Planar NAE 3SAT Type 2.**

**INSTANCE:** A 3SAT formula \((X, C)\) such that the following graph obtained from 3SAT is planar. The graph has one vertex for each variable, one vertex for each clause and each clause vertex is connected by an edge to variable vertices corresponding to the literals present in the clause.

**QUESTION:** Is there a NAE truth assignment for \( X \)?

Let \( \Phi \) be an instance of **Cubic Planar 1-in-3 SAT**. We can check in polynomial time whether \( \Phi \) has a NAE SAT satisfying assignment. If \( \Phi \) does not have any NAE SAT satisfying assignment, then it does not have any 1-in-3 SAT satisfying assignment. So, suppose that \( \Phi \) has a NAE SAT satisfying assignment. Let \( r \geq 3 \) be a fixed number. For a given formula \( \Phi \) we construct an \( r \)-regular bipartite graph \( G \) such that the formula \( G \) has a 1-in-Degree decomposition if and only if the formula \( \Phi \) has a 1-in-3 SAT satisfying assignment. Our proof consists of two steps.

**Step 1.** Without loss of generality suppose that the number of clauses in \( \Phi \) is \( s \). For each \( j, j \in \mathbb{N}_{(r-3)s} \), consider the following \( r \) clauses:

- For every \( i, i \in \mathbb{N}_{r-1} \), consider the clause \((\varepsilon_i^j \lor \bigvee_{k=1}^{r-1} \alpha_k^j)\).
- The clause \((\alpha_1^j \lor \bigvee_{k=1}^{r-1} \varepsilon_k^j)\).

Assume that the set of above-mentioned clauses has a 1-in-\( r \) SAT satisfying assignment \( \Gamma \). If there is an index \( i \) such that \( \Gamma(\varepsilon_i^j) = \text{true} \), then \( \Gamma(\alpha_1^j) = \Gamma(\alpha_2^j) = \cdots = \Gamma(\alpha_{r-1}^j) = \text{false} \). So \( \Gamma(\varepsilon_i^j) = \cdots = \Gamma(\varepsilon_{r-1}^j) = \text{true} \). Thus, \( \Gamma \) is not a 1-in-\( r \) SAT satisfying assignment for the clause \((\alpha_1^j \lor \bigvee_{k=1}^{r-1} \varepsilon_k^j)\). This is a contradiction. Similarly, if there is an index \( i > 1 \) such that \( \Gamma(\alpha_i^j) = \text{true} \), then \( \Gamma \) is not a 1-in-\( r \) SAT satisfying assignment for the clause \((\alpha_1^j \lor \bigvee_{k=1}^{r-1} \varepsilon_k^j)\). Hence, we have \( \Gamma(\alpha_1^j) = \Gamma(\alpha_2^j) = \cdots = \Gamma(\alpha_{r-1}^j) = \Gamma(\varepsilon_1^j) = \cdots = \Gamma(\varepsilon_{r-1}^j) = \text{false} \). Call the set of above clauses \( \Upsilon \) and their variables \( X' \). Now, consider \( r(r-2) \) copies of \( \Phi \) and a copy of \( \Upsilon \). For each clause \( c \) with 3 variables, add \( r-3 \) variables from \( X' \) to \( c \), such that in the resultant formula no variable appears in more then \( r \) clauses. Call the resultant formula \( \Phi' \). Each clause in \( \Phi' \) has \( r \) variables and every variable appears in exactly \( r \) clauses. Since for each \( j \), \( \Gamma(\alpha_1^j) = \cdots = \Gamma(\alpha_{r-1}^j) = \Gamma(\varepsilon_1^j) = \cdots = \Gamma(\varepsilon_{r-1}^j) = \text{false} \), the formula \( \Phi' \) has a 1-in-\( r \) SAT satisfying assignment if and only if the formula \( \Phi \) has a 1-in-3 SAT satisfying assignment. This completes the first step.

**Step 2.** Let \( \Phi' \) be a given formula with the set of variables \( X \) and the set of clauses \( C \). Define the following graph.

\[
V(G') = \{\alpha_{i,j,k}^k, \delta_{i,j,x}^k, x_j^k | 0 \leq s \leq 3, 0 \leq j, k \leq r - 1, x \in X \} \\
\cup \{\alpha_{i,j,x}^k, \varepsilon_{i,j,x}^k, \zeta_{i,j,x}^k | 0 \leq i \leq 3, 0 \leq j, k \leq r - 1, c \in C, x \in X \}.
\]

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$E(G') = \{ x_j^k \alpha_{i,j,x} | 0 \leq i \leq r - 3, 0 \leq j, k \leq r-1, x \in X \}
\cup \{ \alpha_{i,j,x} \beta_{i',j,x}, \beta_{i',j,x} \alpha_{i,j,x} | 0 \leq i \leq r - 3, 0 \leq i' \leq r - 2, 0 \leq j, k \leq r - 1, x \in X \}
\cup \{ \delta_{i,j,x}^k x_{(j+1) \mod r} | 0 \leq j, k \leq r - 1, x \in X \}
\cup \{ \varepsilon_{j,x}^k \beta_{i,j,x}^k | 0 \leq i \leq r - 2, 0 \leq j, k \leq r - 1, x \in X \}
\cup \{ \varepsilon_{j,x}^k \zeta_{j,x} | 0 \leq j, k \leq r - 1, c \in C, x \in X \}.

Now, we discuss the basic properties of the graph $G'$. Remove the set of vertices $\{ \zeta_{j,x} | 0 \leq j \leq r - 1, x \in X \}$ from the graph $G'$ and call the resultant graph $G''$. The vertices of the graph $G''$ can be partitioned into $r$ parts such that there is no edge between two parts and each part is a copy of the gadget $J(z)$ which is shown in 3 (it is enough to partition the vertices based on parameter $k$). In the graph $J(z)$, $z$ is a fixed number. We have the following lemma about the graph $J(z)$.

Lemma 5 For every copy of the gadget $J(z)$, define the following function $g : V(J(z)) \rightarrow \{ \text{true, false} \}$ for the vertices of $J(z)$ such that $g(v) = \text{true}$ if the vertex $v$ is a white vertex, $g(v) = \text{false}$ if the vertex $v$ is a black vertex and for each two vertices $v$ and $u$, if $v$ is a red vertex and $u$ is a blue vertex, then $g(v) \neq g(u)$. In the function $g$ each vertex of degree $r$ has exactly one neighbor with label true.

**Proof.** In the graph $J(z)$, each vertex of degree $r$ has exactly one white neighbor or a blue neighbor and a red neighbor. Thus, in the function $g$ each vertex of degree $r$ has exactly one neighbor with label true. ♠

Assume that the function $g$ in each copy of the gadget $J(z)$, labels the set of red vertices with true. Define $S_1 = \{ \varepsilon_{j,x}^{-1}, c^{-1}, \zeta_{j,x} | 0 \leq j \leq r - 1, c \in C, x \in X \}$ and $S_2 = \{ \delta_{i,j,x}^{-1} | 0 \leq j \leq r - 1, x \in X \}$. Now, consider the graph $G'$ and using $S_1, S_2$ define the function $h : V(G') \rightarrow \{ \text{true, false} \}$:

$$h(v) = \begin{cases} 
\text{true,} & \text{if } v \in S_1, \\
\text{false,} & \text{if } v \in S_2, \\
g(v) & \text{otherwise.}
\end{cases}$$

Figure 3: The bipartite graph $J(z)$. 

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In the graph $G'$ the degree of each vertex except the set of vertices $\{x^k_j, c^k | 0 \leq j, k \leq r - 1, c \in C, x \in X\}$ is $r$. We have the following lemma for the function $h$.

**Lemma 6** In the function $h$ each vertex of degree $r$ in the graph $G'$ has exactly one neighbor with label `true`.

**Proof.** Since in the function $g$ each vertex of degree $r$ in $G''$ has exactly one neighbor with label `true`, in the function $h$ each vertex of degree $r$ in the graph $G'$ has exactly one neighbor with label `true`. ♠

See Figure 4, to see the structure of bipartite graph $G'$, where $r = 3$, $C = \{c\}$ and $X = \{x\}$.

![Figure 4: The bipartite graph $G'$ for $r = 3$, $C = \{c\}$ and $X = \{x\}$](image)

Now, consider a copy of the graph $G'$. In the graph $G'$ the degree of each vertex except the set of vertices $\{x^k_j, c^k | 0 \leq j, k \leq r - 1, c \in C, x \in X\}$ is $r$. For every clause $c$ and variable $x$ if the variable $x$ appears in the clause $c$, then for each $k$, $0 \leq k \leq r - 1$, join the vertex $c^k$ to exactly one of the vertices $x^k_j$ such that in the resultant graph the degree of each vertex is $r$. Call the resultant graph $G$. The graph $G$ is an $r$-regular bipartite graph. First, assume that $\Gamma : X \rightarrow \{\text{true}, \text{false}\}$ is a 1-in-$r$ SAT satisfying assignment for $\Phi'$. Now, we present a 1-in-Degree decomposition for the graph $G$. Define the function $f$ as follows.

$$f(x^k_j) = \text{true}, \text{ if and only if } \Gamma(x) = \text{true},$$

$$f(\alpha^k_{i,j,x}) = \text{false},$$

$$f(\beta^k_{i,j,x}) = \text{false}, \text{ for } i \neq 0,$$

$$f(\beta^k_{0,j,x}) = \text{false}, \text{ if and only if } \Gamma(x) = \text{true},$$

$$f(\varepsilon^k_{j,x}) = \text{false} \text{ and } f(\delta^k_{1,j,x}) = \text{true}, f(c^k) = \text{false}, \text{ for } k \neq r - 1,$$
\[ f(x_{j,x}^{r-1}) = \text{true} \text{ and } f(\delta_{j,x}^{r-1}) = \text{false}, \text{ } f(c^{r-1}) = \text{true}, \]
\[ f(\zeta_{j,x}) = \text{true}, \text{ if and only if } \Gamma(x) = \text{true}. \]

The function \( f \) is similar to the function \( h \) except that
- For each \((j, k)\), \( f(x_{j}^{k}) = \text{true} \) and \( f(\beta_{j,x}^{k}) = \text{false} \) if and only if \( \Gamma(x) = \text{true} \).
- For each \((j, x)\), \( f(\zeta_{j,x}) = \text{true}, \text{ if and only if } \Gamma(x) = \text{true}. \)

Thus, by Lemma 6, the function \( f \) is a 1-in-Degree decomposition for the graph \( G \) (note that the graph \( G \) always has a NAE decomposition).

Next, suppose that the graph \( G \) has a 1-in-Degree decomposition \( f : V(G) \to \{\text{true, false}\}. \) For every variable \( x \in X \), we have the following important lemma.

**Lemma 7** For every variable \( x \in X \), \( f(x_{0}^{r}) = f(x_{1}^{r}) = \cdots = f(x_{r-1}^{r}) \).

**Proof.** In the graph \( G \), the red vertices are connected through some paths of length four. Let \( k_{1} \) and \( k_{2} \) be two red vertices and \( k_{1}s_{1}s_{2}s_{3}k_{2} \) be a path of length four. First, assume that the label of \( k_{1} \) is \text{true}. By this assumption, the labels of all the neighbors of the vertex \( s_{1} \) except the vertex \( k_{1} \) are \text{false}. Since \( N(s_{3}) = \{k_{2}\} \cup N(s_{1}) \setminus \{k_{1}\} \), the label of the vertex \( k_{2} \) is \text{true}. Similarly, if the label of the vertex \( k_{1} \) is \text{false}, one can see that the label of the vertex \( k_{2} \) is \text{false}. \( \diamondsuit \)

Now we present a 1-in-\( r \) SAT satisfying assignment for \( \Phi' \). Define the function \( \Gamma : X \to \{\text{true, false}\} \) such that for each variable \( x \) \( \Gamma(x) = \text{true} \) if and only if \( f(x_{0}^{r}) = \text{true} \). By Lemma 7 and since \( f \) is a symmetric 1-in-\( d \) SAT satisfying assignment, \( \Gamma \) is a 1-in-\( r \) SAT satisfying assignment for \( \Phi' \). This completes the proof.

\((ii)\) Let \( G \) be a given graph. Since the graph \( G \) is a bipartite graph, the following integer linear program determines whether the graph \( G \) has a 1-in-Degree decomposition:

\[
\begin{align*}
\text{minimize} & \quad 1 \\
\text{subject to} & \quad \sum_{u \in N(v)} f(u) \geq 1, \quad \forall v \in V(G) \\
& \quad \sum_{u \in N(v)} -f(u) \geq -1, \quad \forall v \in V(G) \\
& \quad f(v) \in \{0, 1\}, \quad \forall v \in V(G)
\end{align*}
\]

The above integer linear program is feasible if and only if the graph \( G \) has a 1-in-Degree decomposition. When an integer linear program has all-integer coefficients and the matrix of coefficients is totally unimodular, then the optimal solution of its relaxation is integral [34]. Hence, it can be obtained in polynomial time. A signed bipartite graph is a bipartite graph in which every edge is given a label of \(-1\) or \(+1\). The weight of a cycle in a signed bipartite graph is the sum of the labels of its edges. A signed bipartite graph is called restricted unimodular if the weight of any cycle in the graph \( G \) is divisible by 4. There is a correspondence between a signed bipartite graph \( G \) and its \([-1, 0, 1]\) adjacency matrix \( D \), where the rows and columns of \( D \) are indexed by the vertices of the graph \( G \), with \( D_{ij} = -1, 0, 1 \), according as the edge between vertices \( v_{i} \) and \( v_{j} \) has label \(-1\), is absent, or has label \(+1\), respectively. Commoner [14] proved that a
signed bipartite graph that is restricted unimodular has a totally unimodular adjacency matrix. If the graph $G$ is a bipartite graph, does not have any cycle of length congruent to 2 mod 4 and each edge label is 1, then the graph $G_\Phi$ is restricted unimodular and has a totally unimodular adjacency matrix. Consequently, there is polynomial time algorithm to determine whether the above-mentioned integer linear program is feasible. This completes the proof.

□

A Hypergraph is a pair $H = (V, E)$ such that $E$ is a subset of the power set of $V$. The set $V$ is the set of vertices and $E$ is the set of edges. A proper $l$-coloring of a hypergraph $H$ is a function $c : V(H) \to \mathbb{N}_l$ in which there is no monochromatic edge in $H$. We say that a hypergraph $H$ is $t$-colorable if there is a proper $t$-coloring of it. If $E$ contains only sets of size $k$ then $H$ is said to be $k$-uniform and if every vertex appears in exactly $r$ edges then $H$ is said to be $r$-regular. The main result by Thomassen [46] implies that every $r$-uniform, $r$-regular hypergraph is 2-colorable for all $r \geq 4$ (for more information see [4]). Therefore, we have the following result:

**Theorem A [46]** If $G$ is an $r$-regular bipartite graph and $r \geq 4$, then the graph $G$ has a NAE decomposition.

There are 3-uniform 3-regular hypergraphs that are not 2-colorable. For instance, consider the Fano Plane. The Fano Plane is a hypergraph with seven vertices $\mathbb{Z}_7$ and seven edges $\{(i, i+1, i+3) : i \in \mathbb{Z}_7\}$. The Fano Plane is not 2-colorable and is minimal with respect to this property [6].

**Remark 1** There is another interesting problem which is related to the NAE decomposition. A vertex-labeling $f$ is a gap labeling if

$$c(v) = \begin{cases} 1 & \text{if } d(v) = 0, \\ f(u)_{uv \in E(G)} & \text{if } d(v) = 1, \\ \max_{uv \in E(G)} f(u) - \min_{uv \in E(G)} f(u) & \text{otherwise,} \end{cases}$$

is a proper vertex coloring [19]. Every bipartite graph $G = [X, Y]$ has a gap labeling, label the set of vertices $X$ by one and label the set of vertices $Y$ by different even numbers. In [19] it was asked, to determine the computational complexity of deciding whether a given 3-regular bipartite graph $G$ have a gap labeling from $\mathbb{N}_2$. In other words, given a formula $\Phi$ in conjunctive normal form, such that each clause contains three literals and each literal appears in exactly three clauses and the formula is monotone, determine the computational complexity of deciding whether there exists a truth assignment to the variables and clauses so that each clause has at least one true literal and at least one false literal or each variable appears in at least one true clause and at least one false clause.

**Theorem 3** If $G$ does not have any cycle of length congruent to 2 mod 4, then there is a polynomial time algorithm to decide whether $G$ has a NAE decomposition.

**Proof** Let $G$ be a graph. The following integer linear program determines whether $G$ has a NAE decomposition:
minimize 1
subject to
\[ \sum_{u \in N(v)} f(u) \geq 1, \quad \forall v \in V(G) \]
\[ \sum_{u \in N(v)} -f(u) \geq -d(v) + 1, \quad \forall v \in V(G) \]
\[ f(v) \in \{0,1\}, \quad \forall v \in V(G) \]

The above-mentioned integer linear program is feasible if and only if \( G \) has a NAE decomposition. The other parts of the proof are similar to the proof of Theorem 2, Part (ii).

\[\square\]

4 Applications of the 1-in-Degree and NAE decompositions

4.1 The structure of the kernel of the adjacency matrix

A zero-sum vertex flow of a graph \( G \) is an assignment of non-zero integer numbers to the vertices of \( G \) such that the sum of the labels of all vertices adjacent with each vertex is zero. Let \( k \) be a natural number. A zero-sum vertex \( k \)-flow is a zero-sum vertex flow with labels from the set \( \{\pm 1, \ldots, \pm (k-1)\} \). The first natural problem about the computational complexity of the existence of zero-sum vertex flow in graphs is the following problem.

Problem B. [18] Is there a polynomial time algorithm to decide whether a given graph \( G \) has a zero-sum vertex flow?

It was shown that for a given bipartite \((2,3)\)-graph \( G \), it is \( \text{NP} \)-complete to decide whether the graph \( G \) has a zero-sum vertex 3-flow [18]. Here, we improve the previous complexity result. A given 3-regular bipartite graph \( G \) has a zero-sum vertex 3-flow if and only if the graph \( G \) has a 1-in-Degree decomposition. Therefore, by Theorem 2, we have the following:

Corollary 1 For a given 3-regular bipartite graph \( G \) determining whether \( G \) has a zero-sum vertex 3-flow is \( \text{NP} \)-complete.

4.2 The Minimum Edge Deletion Bipartition Problem

For a given graph \( G \), The Minimum Edge Deletion Bipartition Problem is to determine the minimum number of edges of \( G \) such that their removal leads to a bipartite graph \( H \). It was shown that the minimum edge deletion bipartition problem is \( \text{NP} \)-hard even if all vertices have degrees 2 or 3 [20]. Here, we show the following:

Theorem 4 If Cubic Bipartite NAE Decomposition Problem is \( \text{NP} \)-complete, then for a given 3-regular graph \( G \), The Minimum Edge Deletion Bipartition Problem is \( \text{NP} \)-hard.
Proof Suppose that Cubic Bipartite NAE Decomposition Problem is NP-complete. There is a simple polynomial time reduction from Cubic Bipartite NAE Decomposition Problem to Cubic Monotone NAE 3SAT, thus, Cubic Monotone NAE 3SAT is NP-complete. We reduce Cubic Monotone NAE 3SAT to our problem in polynomial time. The proof, given below, follows the same approach as [20].

Let $\Phi$ be a given formula with the set of variables $X$ and the set of clauses $C$. Without loss of generality, suppose that $|C| = k$. We construct a graph $G$ such that the minimum number of edges of $G$ such that their elimination leads to a bipartite graph is exactly $k$ if and only if the formula $\Phi$ has a NAE SAT satisfying assignment. For every variable $x$, put a vertex $r_x$ in the graph. Also, for every pair $(x, c)$, where $x \in X$ and $c \in C$, if $x$ appears in $c$ then put a vertex $c_x$ in the graph and join the vertex $c_x$ to the vertex $r_x$. Finally, for every clause $c = (x \lor y \lor z)$, join the vertex $c_x$ to the vertex $c_y$ and $c_z$, also, join the vertex $c_y$ to the vertex $c_z$. Call the resultant 3-regular graph $G$. Since the graph $G$ contains at least $k$ edge disjoint triangles, the minimum number of edges of the graph $G$ such that their removal leads to a bipartite graph is at least $k$. First, suppose that the instance $\Phi$ has a NAE assignment $\Gamma$. For each clause $c = (x \lor y \lor z)$, the triple $(x, y, z)$ in the formula $\Gamma$ has either two trues and a false, or two false and a true. From the three edges $c_x c_y, c_y c_z, c_z c_x$, remove the edge joining the two trues or the two false. Now, we show that the remaining graph is bipartite. Consider the following partition for the vertices of the graph. For every variable $x$, put the vertex $r_x$ in $F$ if and only if $\Gamma(x) = true$. Also, for every vertex $c_x$ put the vertex $c_x$ in $F$ if and only if $\Gamma(x) = false$. One can see that $(F, \overline{F})$ is a partitioning for the vertices of the graph such that every edge in the remaining graph connects a vertex from $F$ to a vertex from $\overline{F}$. Thus, the remaining graph is bipartite. Next, assume that $G$ has a solution that removes exactly one edge per triangle to obtain a bipartite graph $H$. Label the two sides of the bipartition of $H$ by 1 and $-1$ respectively. Call this labeling $\ell$. For every pair $(x, c)$, where $x \in X$ and $c \in C$, if $x$ appears in $c$, by the structure of $H$, we have $\ell(c_x) = -\ell(r_x)$. For every variable $x$, put $\Gamma(x) = true$ if and only if $\ell(r_x) = -1$. One can see that the function $\Gamma$ is a NAE SAT satisfying assignment for the formula $\Phi$.

\[\square\]

5 1-in-Degree decomposition in vertex-weighted graphs

A vertex-weighted graph is a graph in which each vertex has been assigned a weight. Let $G$ be a vertex-weighted graph and $w : V(G) \rightarrow Z$ be its weight function. A 1-in-Degree coloring for the graph $G$ is a function $f : V(G) \rightarrow \{0, 1\}$ such that for each vertex $v \in V(G)$, $\sum_{u \in N(v)} f(u)w(u) = 1$. Although, if the graph $G$ is a bipartite graph and does not have any cycle of length congruent to 2 mod 4, then there is a polynomial time algorithm to decide whether the graph $G$ has a 1-in-Degree decomposition, but for a given vertex-weighted bipartite graph $G$, determining whether the graph $G$ has a 1-in-Degree coloring is strongly NP-complete, even if the graph $G$ does not have any cycle of length congruent to 2 mod 4.
**Theorem 5** For a given vertex-weighted bipartite graph $G$, determining whether $G$ has a 1-in-Degree coloring is strongly NP-complete, even if the graph $G$ does not have any cycle of length congruent to 2 mod 4.

**Proof** It was shown that 3-Partition is NP-complete in the strong sense [22].

3-Partition.

**Instance**: A positive integer $k \in \mathbb{Z}^+$ and $3n$ positive integers $a_1, \ldots, a_{3n} \in \mathbb{Z}^+$ such that for each $1 \leq i \leq 3n$, $k/4 < a_i < k/2$. Also $\sum_{i=1}^{3n} a_i = nk$.

**Question**: Can $\{a_1, \ldots, a_{3n}\}$ be partitioned into $n$ disjoint sets $A_1, \ldots, A_n$ such that for each $i, i \in \mathbb{N}_n$, $\sum_{a \in A_i} a = k$?

We reduce 3-Partition to our problem in polynomial time. For an instance $A = [a_1, \ldots, a_{3n}]$ and number $k$, define the bipartite graph $G$ with the weight function $w$ as follow:

$$V(G) = \{x_{i,j}, y^1_i, z^1_j | i \in \mathbb{N}_n, j \in \mathbb{N}_{3n}, t \in \mathbb{N}_2, l \in \mathbb{N}_5\}.$$  

$$E(G) = \{y^1_i x_{i,j}, y^2_i y^1_i, z^1_j x_{i,j}, z^2_j z^2_j, z^3_j z^3_j, z^4_j z^4_j, z^5_j z^5_j | i \in \mathbb{N}_n, j \in \mathbb{N}_{3n}\}.$$  

$$w(x_{i,j}) = a_j, \ w(y^1_i) = 1, \ w(y^2_i) = 1 - k, \ w(z^1_j) = w(z^2_j) = w(z^3_j) = w(z^4_j) = 1, \ w(z^5_j) = a_j.$$  

The graph $G$ does not have any cycle of length congruent to 2 mod 4. See Figure 5

![Figure 5: The graph $G$ for $n = 1$.](image)

Let $f : V(G) \to \{1, 0\}$ be a 1-in-Degree coloring for the graph $G$. For each $j, j \in \mathbb{N}_{3n}$ we have:
1 = \sum_{u \in N(z^j)} f(u)w(u) \\
= (1 - a_j)f(z^j) + \sum_i f(x_{i,j}) \\
= \sum_i f(x_{i,j}) \hspace{1cm} \text{Property 1,}

Also, for each \( i, i \in \mathbb{N}_n \) we have:

1 = \sum_{u \in N(y^l)} f(u)w(u) \\
= (1 - k)f(y^l) + \sum_j \left( f(x_{i,j}) \times a_j \right) \\
\Rightarrow \sum_j f(x_{i,j}) = 3 \hspace{1cm} \text{Property 2,}

Now, define the partition \( A_1, \ldots, A_n \), where for each \( l, l \in \mathbb{N}_n \), \( A_l = \{a_j : f(x_{i,j}) = 1\} \). By Property 1 and Property 2, \( A_1, \ldots, A_n \) are disjoint and each one has three members. Next, assume that \( \{a_1, \ldots, a_{3n}\} \) can be partitioned into \( n \) disjoint sets \( A_1, \ldots, A_n \) such that, for each \( i, i \in \mathbb{N}_n \), \( \sum_{a \in A_i} a = k \). Define the function \( f \) such that for each \( l, l \in \mathbb{N}_n \), \( A_l = \{a_j : f(x_{i,j}) = 1\} \). By Property 1 and Property 2, \( f \) is a 1-in-Degree coloring for the graph \( G \). Therefore, \( G \) has a 1-in-Degree coloring if and only if \( \{a_1, \ldots, a_{3n}\} \) can be partitioned into \( n \) disjoint sets \( A_1, \ldots, A_n \) such that, for each \( i, i \in \mathbb{N}_n \), \( \sum_{a \in A_i} a = k \). This completes the proof.

\( \square \)

6 The edge versions of 1-in-Degree and Not-All-Equal decompositions

An edge coloring \( f : E(G) \rightarrow \{0, 1\} \) for a graph \( G \) is called 1-in-Degree edge coloring if and only if for every vertex \( v \), \( \sum_{e \ni v} f(e) = 1 \). The graph \( G \) has a 1-in-Degree edge coloring if and only if \( G \) has a perfect matching. Now, consider the edge version of the NAE. An edge coloring \( f : E(G) \rightarrow \{\text{red, blue}\} \) of a graph \( G \) is called NAE edge coloring if and only if for every vertex \( v \), there are edges \( e \) and \( e' \) incident with \( v \), such that \( f(e) \neq f(e') \).

**Theorem 6** For a given connected graph \( G \) with \( \delta(G) > 1 \), \( G \) has a NAE edge coloring if and only if \( G \) is not an odd cycle.

**Proof** If \( G \) is an odd cycle, then the graph \( G \) does not have any NAE edge coloring. Also, for every even cycle \( C \) any proper edge coloring of \( C \) is a NAE edge coloring for its edges. Let \( G \) be a
connected graph with $\delta(G) \geq 2$ and $\Delta(G) \geq 3$. Two cases can be considered:

Case 1: $\Delta(G) \geq 4$. Let $v$ be a vertex of degree more than three. The number of vertices of odd degrees is even. Let $x$ and $y$ be two vertices of odd degrees in $G$, put a new vertex $t_{x,y}$ in the graph and join the vertex $t_{x,y}$ to the vertices $x$ and $y$. Repeat this procedure until the resultant graph does not have any vertex of odd degree. Call the resultant graph $G'$. $G'$ has an Eulerian cycle $C$. Start from $v$ and alternatively color the edges of $C$ by red and blue. Call this coloring $f$. Without loss of generality assume that $C = e_1e_2\ldots e_{|E(G')|}$. By the structure of the graph $G'$ and since the degree of vertex $v$ is at least four, for every vertex $u$, there is an index $i$, such that $e_i, e_{i+1} \in E(G)$ and the vertex $u$ is incident with $e_i$ and $e_{i+1}$. Thus, $f(e_i) \neq f(e_{i+1})$. Consequently, the function $f$ is a NAE edge coloring for $G$.

Case 2: $\Delta(G) = 3$. The proof of this case is by induction on the number of edges. There are two vertices $v$ and $u$ such that $d(v) = d(u) = 3$ and there exists a path $P = vx_1x_2\ldots x_iu$ such that $d(x_1) = \cdots = d(x_i) = 2$. Consider the subgraph $H = G \setminus \{x_j | j \in N_i\}$ (note that $u$ and $v$ can be adjacent, in this case we put $H = G \setminus \{e = vu\}$). $H$ contains at most two connected components. Start from $vx_1$ and alternatively color the edges of $P$ by red and blue. Call this partial coloring $\ell$. First, suppose that no connected component of $H$ is an odd cycle. By the inductive hypothesis, color each connected component of $H$ with red and blue. The resultant coloring is an NAE edge coloring for $G$. Next, suppose that some connected components of $H$ are odd cycles. Note that there is an edge coloring $f : E(C_{2k+1}) \rightarrow \{red, blue\}$ for the odd cycle $C_{2k+1}$ such that for every vertex $a \in V(C_{2k+1})$, except exactly one vertex $b \in V(C_{2k+1})$, there are edges $e$ and $e'$ incident with $a$, such that $f(e) \neq f(e')$. If for every edge $e$ incident with $b$, $f(e) = red$, we call this coloring, red coloring with respect to $b$. And if, for every edge $e$ incident with $b$, $f(e) = blue$, we call this coloring, blue coloring with respect to $b$. 

![Figure 6: Three subcases in the proof of Theorem 6.](image)

Now, suppose that some connected components of $H$ are odd cycles. Three subcases can be considered:

Subcase 2.1: If $H$ has exactly one nontrivial connected component and this connected component is an odd cycle. Then, without loss of generality suppose that, the connected component contains $v$, consider a blue coloring with respect to $v$ for $H$. The resultant coloring is an NAE edge coloring for $G$.

Subcase 2.2: If $H$ has two connected components and each connected component is an odd cycle.
Then, without loss of generality suppose that \( f(x,u) = \text{blue} \). Consider a blue coloring with respect to \( v \) for the connected component that contains \( v \) and a red coloring with respect to \( u \) for the connected component that contains \( u \). The resultant coloring is an NAE edge coloring for \( G \).

**Subcase 2.3**: If \( H \) has two connected components and exactly one of these connected components is an odd cycle. Then, without loss of generality suppose that the connected component that contains \( v \) is an odd cycle. Consider a blue coloring with respect to \( v \) for the connected component that contains \( v \) and by the inductive hypothesis, color the connected component of \( H \) that contains \( u \) with red and blue. The resultant coloring is a NAE edge coloring for \( G \).

\[\square\]

### 7 Concluding remarks

- In this work we introduced the concept of the NAE decomposition and the concept of 1-in-Degree decomposition of graphs. The NAE decomposition of a graph \( G \) is a decomposition of the vertices into two sets such that each vertex in the graph \( G \) has at least one neighbor in each part. Also, the 1-in-Degree decomposition of a graph \( G \) is a decomposition of the vertices of \( G \) into two sets \( A \) and \( B \) such that each vertex in the graph has exactly one neighbor in part \( A \). A summary of the results on the NAE and 1-in-Degree decompositions was shown in Table 1. The ultimate goal of determining the computational complexity of deciding whether a given 3-regular bipartite graph \( G \) has a NAE decomposition, remains outstanding for the moment. Any hardness result can be interesting to work on.

- In the NAE and 1-in-Degree decompositions, we partition the vertices of a graph into two parts with some properties, other types of these problems were considered by several authors, for instance, Gerber and Kobler introduced the problem of deciding if a given graph has a vertex partition into two nonempty parts such that each vertex has at least as many neighbors in its part as in the other part [23]. For more information about this problem see [5, 24, 43].

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