Effect of shear and rotatory inertia on the bending vibration method without weighing specimens

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Abstract
This study examines the effect of shear and rotatory inertia on the accuracy of the vibration method with additional mass (VAM). Bending vibration tests were performed for rectangular bars with a width of 30 mm, thicknesses ranging from 5 to 60 mm, and a length of 300 mm, small round bars with diameters in the range of 6–36 mm, and lengths of 150–300 mm, and cross beams for timber guardrails with and without a concentrated mass under a free–free condition. The estimation accuracy of the VAM and the effect of shear and rotatory inertia increased and decreased, respectively, as the length/thickness ratio of the rectangular bar and length/diameter ratio of the round bar increased. The estimation accuracy of the VAM decreased with an increase in the effect of shear and rotatory inertia, and it could be corrected. The range of the effect of shear and rotatory inertia for the sufficient estimation accuracy of the VAM was obtained.

Keywords: Bending vibration, Rotatory inertia, Shear, Specimen dimension, Vibration method with additional mass

Introduction
The vibration test is widely used because it enables Young's modulus to be measured simply, quickly, and non-destructively. Density is a necessary input for calculating the Young's modulus. Weighing specimens can be challenging for certain materials, such as each piled lumber and each cross-beam of timber guardrails. Therefore, a measuring method that does not require specimens to be weighed when measuring Young's modulus would represent a significant practical improvement.

A method for measuring mass, density, and Young's modulus without needing to weigh the specimen was developed based on a frequency equation that incorporates the effect of an additional mass attached to a wooden bar [1–6]. This method is referred to as the vibration method with additional mass (VAM) in this study. This method utilizes decreases in the resonance frequency by attaching additional mass. The ratio of the resonance frequency with the additional mass to that without it is used for the frequency equation, incorporating the effect of the concentrated mass attached on a specimen and its position. The mass ratio (concentrated mass/specimen) is subsequently calculated from the frequency equation.

To analyze the potential application of the VAM for practical purposes, a series of parameters were investigated, including the suitable mass ratio (additional mass/specimen) [7], the connection between the additional mass and specimen [7], the crossers' position for the piled lumber [8], the specimen moisture content [9], and the bending vibration generation method [10]. The VAM could perhaps be used to assess the deterioration of the cross beams of timber guardrails [11].

Although the VAM in conjunction with the bending vibration may be effective for piled lumber [12] and the cross beams of timber guardrails [11], the apparent deflection in the bending vibration consists of deflections due to shear and rotatory inertia, as well as pure bending deflection. Hence, this study aims to investigate the effect...
of shear and rotatory inertia on the estimation accuracy of the VAM.

**Theory**

**Vibration testing method without weighing the specimen**

Here, bending vibrations under free–free conditions are considered. In the case of a thin beam with constant cross section, the effect of deflections due to shear and rotatory inertia involved in the bending vibrational deflection (hereafter, SR effect) is negligible, and the Euler–Bernoulli elementary theory of bending can be applied to the bending vibration.

The resonance frequency, represented by $f_{n0}$ ($n$: resonance mode number, 0: value without the additional mass), is expressed as follows:

$$f_{n0} = \frac{1}{2\pi} \left(\frac{m_{n0}}{l}\right)^2 \sqrt{\frac{EI}{\rho A}},$$

where $l$, $E$, $\rho$, $I$, and $A$ are the specimen length, Young’s modulus, density, the second moment of area, and the cross-sectional area, respectively. $m_{n0}$ is a constant that depends on the end conditions and is expressed as follows:

$$m_{n0} = \frac{1}{2}(2n + 1)\pi(n > 3).$$

The measured resonance frequencies $f_{n0}$ and $f_n$ are substituted into Eq. (4) to calculate $m_n$, and the calculated $m_n$ is substituted into Eq. (5) to calculate $\mu$. The specimen mass and density can be obtained by substituting the calculated $\mu$, the concentrated mass, and the dimensions of a bar into Eq. (6). Young’s modulus can be calculated by substituting the estimated density, the resonance frequency without the concentrated mass, and the dimensions of a bar into Eq. (1) [1–6].

The above steps represent the calculation procedure for the VAM. Weighing the specimen is not required for the calculations.

**Goens–Hearmon regression method based on the Timoshenko theory of bending (TGH method)**

Young’s and shear moduli can be obtained simultaneously using only the bending vibration test without a torsional vibration test by the following the Goens–Hearmon regression method based on the Timoshenko theory of bending (TGH method) [13–15].

The apparent deflection in the bending vibration consists of deflections due to shear and rotatory inertia, as well as pure bending deflection. Timoshenko added the terms deflections of shear and rotatory inertia to the Euler–Bernoulli elementary theory of bending and developed the following differential equation of bending [13]:

$$(\cos m_n \cosh m_n - 1) - \frac{1}{2} \mu m_n \left\{ (\cos am_n \cosh am_n + 1) (\sin bm_n \cosh bm_n - \cos bm_n \sinh bm_n) + (\cos bm_n \cosh bm_n + 1) \sin am_n \cosh am_n - \cos am_n \sinh am_n \right\} = 0,$$
\[
\frac{EI}{\rho A} \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} - I \left( 1 + \frac{sE}{G} \right) \frac{\partial^4 y}{\partial x^4} + \frac{\rho sI}{GA} \frac{\partial^4 y}{\partial t^4} = 0, \quad (7)
\]

where \( G \) is the shear modulus, \( y \) is the lateral deflection, and \( t \) is the time.

\[
m_{n0}^4 \frac{k_{n0}^4}{k_{n0}^4} = \frac{E}{E_{an}} = T = 1 + \frac{I}{l^2 A} \left\{ m_{n0}^2 F^2(m_{n0}) + 6m_{n0}F(m_{n0}) \right\} + \frac{I}{l^2 A} \frac{sE}{G} \left\{ m_{n0}^2 F^2(m_{n0}) - 2m_{n0}F(m_{n0}) \right\} - \frac{4\pi^2 \rho s f_{n0}^2}{GA}, \quad (13)
\]

where Goens approximated Eqs. (9) and (10) using a Taylor series into the following formula [14]:

\[
F(m_{n0}) = 0.9825, F(m_{20}) = 1.0008, F(m_{n0}) = 1 (n > 2). \quad (14)
\]

When Eq. (7) is solved under the free–free condition, the resonance frequency corresponding to the \( n \)th mode \( f_{gn0} \) can be written as follows:

\[
T = 1 + \frac{I}{l^2 A} \left\{ m_{n0}^2 F^2(m_{n0}) + 6m_{n0}F(m_{n0}) \right\} + \frac{sE}{G} \left\{ m_{n0}^2 F^2(m_{n0}) - 2m_{n0}F(m_{n0}) \right\}.
\]

Approximately,

\[
f_{gn0} = \frac{1}{2\pi} \left( \frac{k_{n0}}{l} \right)^2 \sqrt{\frac{EI}{\rho A}} = \frac{1}{2\pi} \left( \frac{m_{n0}}{l} \right)^2 \sqrt{\frac{E_{an} l}{\rho A}}, \quad (8)
\]

Hearmon calculated \( E \) and \( G \) using the following procedure after separating Eq. (13) as follows [15]:

\[
Y = E_{an} \left[ 1 + \frac{I}{l^2 A} \left\{ m_{n0}^2 F^2(m_{n0}) + 6m_{n0}F(m_{n0}) \right\} - \frac{4\pi^2 \rho s f_{n0}^2}{GA} \right], \quad (16)
\]

where \( E_{an} \) is Young’s modulus from the Euler–Bernoulli elementary theory of bending using the resonance frequency of the \( n \)th mode.

\[
k_{n0} \text{ in Eq. (8) is obtained by transcendental equations represented as follows:}
\]

\[
\tan \frac{k_{n0}}{2} \sqrt{\frac{B_{1k_{n0}^4}}{B_{1k_{n0}^4} + 1 + A_1 k_{n0}^2}} = -\sqrt{\frac{B_{1k_{n0}^4}}{B_{1k_{n0}^4} + 1 + A_1 k_{n0}^2}} \sqrt{\frac{B_{2k_{n0}^4}}{B_{2k_{n0}^4} + 1 - B_1 k_{n0}^2}} \quad \text{(for symmetric modes),} \quad (9)
\]

\[
\tan \frac{k_{n0}}{2} \sqrt{\frac{B_{1k_{n0}^4}}{B_{1k_{n0}^4} + 1 - A_1 k_{n0}^2}} = \sqrt{\frac{B_{1k_{n0}^4}}{B_{1k_{n0}^4} + 1 - A_1 k_{n0}^2}} \sqrt{\frac{B_{2k_{n0}^4}}{B_{2k_{n0}^4} + 1 + B_1 k_{n0}^2}} \quad \text{(for asymmetric modes),} \quad (10)
\]

\[
\quad \beta = E, \quad (19)
\]

and then from Eqs. (16)–(19):

\[
Y = -\alpha X + \beta, \quad (20)
\]

Therefore, the linear regression between \( X \) and \( Y \) provides the \( E \) and \( G \) values. This is the Goens–Hearmon regression method based on the Timoshenko theory of
bending (TGH method). The value of $s$ is 6/5 theoretically [16] and 1.18 experimentally [17] for the rectangular cross section and 10/9 [16] for the circular cross section.

**Materials and methods**

**Specimens**

Sitka spruce (*Picea sitchensis* Carr.) rectangular bars with a width of 30 mm (radial direction, R), thicknesses of 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, and 60 mm (tangential direction, T), and a length of 300 mm (longitudinal direction, L) were used as the specimens. Japanese cedar (*Cryptomeria japonica* D. Don) small round bars without a pith with varying diameters of 6, 9, 20, 25, 30, and 36 mm and a length of 300 mm, and those with a diameter of 36 mm and varying lengths of 150, 200, and 250 mm were used as the specimens. Three specimens were made for each dimension. Three Japanese cedar cross beams for timber guardrails with a pith having a diameter of 200 mm and a length of 1980 mm [11] were also used as the specimens. Considering the variation in the results of the vibration test, the number of specimens was three.

The specimens were conditioned at 20 °C and 65% relative humidity. All tests were conducted under the same conditions.

**Vibration tests**

Free–free bending vibration tests were conducted to measure the resonance frequencies of the first to fifth resonance modes using the following procedure: the test bar without a concentrated mass was suspended by two threads at the nodal positions of the free–free vibration, corresponding to each resonance mode. The bending vibration was subsequently generated by tapping the LR-plane of the rectangular bars and the small round bars in the T-direction and it was generated by tapping the LT-plane of the cross beams for timber guardrails in the R-direction, using a wooden hammer. The motion of the specimen was detected with a microphone. The signal was processed through a fast Fourier transform (FFT) digital signal analyzer (Multi-Purpose FFT Analyzer CF-5220, Ono-Sokki, Co., Ltd., Yokohama, Japan) to yield the high-resolution resonance frequencies. A diagram of the experimental setup is provided in Fig. 2.

The free–free bending vibration tests were also performed on the specimens with the concentrated mass using the procedure described above and the resonance frequency of the first mode was measured. Iron plates, thumbtacks, and wood screws were used as the concentrated mass. The position of the concentrated mass was $x = 0.5 \ l$, considering the on-site quality evaluation of cross beams of wooden guardrails. The concentrated masses and specimens are shown in Tables 1 and 2.

**Results and discussion**

The mean (standard deviation) density and Young’s modulus of the specimens were 449 (14) kg/m$^3$, 14.23 (0.45) GPa (sitka spruce rectangular bars; TGH method), and 351 (16) kg/m$^3$, 10.05 (1.04) GPa (Japanese cedar small round bars; TGH method), respectively.

The estimation accuracy of the VAM, which is expressed by the mass ratio (estimated specimen mass through the VAM/measured specimen mass), increased as the length/thickness ratio of the rectangular bar and the length/diameter ratio of the small round bar increased, as shown in Fig. 3. Conversely, the effect of deflections due to shear and

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**Table 1** Concentrated masses (iron plates) attached to rectangular bars

| Concentrated mass | Specimens | Dimension |
|-------------------|-----------|-----------|
| 2 x 2 x 30        | 1.0       | 30 x 5 x 300 | 0.0453 |
| 4 x 2 x 30        | 2.0       | 30 x 10 x 300 | 0.0474 |
| 7 x 2 x 30        | 3.2       | 30 x 15 x 300 | 0.0504 |
| 9 x 2 x 30        | 4.1       | 30 x 20 x 300 | 0.0492 |
| 11 x 2 x 30       | 5.1       | 30 x 25 x 300 | 0.0524 |
| 13 x 2 x 30       | 6.0       | 30 x 30 x 300 | 0.0518 |
| 14 x 2 x 30       | 7.0       | 30 x 35 x 300 | 0.0512 |
| 17 x 2 x 30       | 7.9       | 30 x 40 x 300 | 0.0506 |
| 20 x 2 x 30       | 9.4       | 30 x 45 x 300 | 0.0515 |
| 21 x 2 x 30       | 10.3      | 30 x 50 x 300 | 0.0510 |
| 24 x 2 x 30       | 11.3      | 30 x 55 x 300 | 0.0506 |
| 26 x 2 x 30       | 12.2      | 30 x 60 x 300 | 0.0507 |

Unit of dimensions: mm, unit of mass: g
rotatory inertia involved in the bending vibration deflection (SR effect) shown by \( T \) (\( E_{n0} \) is calculated from the first resonance mode in this study) in Eq. (13) decreased as the length/thickness and length/diameter ratios increased, as shown in Fig. 3. Thus, the estimation accuracy of the VAM is discussed from the aspect of \( T \).

When the SR effect is taken into consideration, the resonance frequency without the concentrated mass is expressed by Eq. (8), and the resonance frequency with the concentrated mass is expressed as follows:

\[
k_n = \frac{f_{gn0}}{f_{gn0}} k_{n0} = \sqrt{\frac{f_{gn0}}{f_{gn0}^2}} \frac{m_{n0}}{\sqrt{T}}.
\]  

When the SR effect can be ignored, the \( m_n \) decreases monotonically with \( \mu \) for the free–free vibration [5]. Hence, the specimen mass estimated by the VAM is low for a low \( m_n \) from Eq. (6). Analogizing this, when \( k_n \) decreases, \( \mu \) increases, and the specimen mass decreases, which causes the estimation accuracy of the VAM to decrease. Equation (22) shows that \( k_n \) decreases with an increasing \( T \). Therefore, when \( T \) increases with the decreases in the length/thickness and length/diameter ratios, \( k_n \) decreases, causing the estimation accuracy of the VAM to decrease.

| Concentrated mass | Specimens |
|-------------------|-----------|
| Type              | Head diameter | Nominal diameter | Length | Mass | Diameter, length |
| Thumbtack         | 11          | 1               | 8      | 0.6  | 6, 300          |
|                   |             |                 |        |      | 9, 300          |
| Wood screw        | 6           | 4               | 32     | 2.1  | 20, 300         |
|                   |             |                 |        |      | 25, 300         |
|                   |             |                 |        |      | 36, 150         |
| Wood screw        | 9           | 5               | 37     | 3.6  | 30, 300         |
|                   |             |                 |        |      | 36, 200         |
|                   |             |                 |        |      | 36, 250         |
| Wood screw        | 8           | 4               | 65     | 3.9  | 36, 300         |

Units of diameter and length: mm, unit of mass: g

Table 2 Concentrated masses (thumbtacks, wood screws) attached to small round bars

Fig. 3 Estimation accuracy of the VAM and the effect of deflections due to shear and rotatory inertia involved in the bending vibrational deflection (SR effect) at various length/thickness and length/diameter ratios
As the SR effect is not taken into consideration in Eq. (5), \(k_n\) cannot be directly substituted for Eq. (5). In other words, \(f_n/f_{n0}\) needed to be used instead of \(f_{gn}/f_{gn0}\) for Eq. (4). With the assumption that the SR effect changes \(f_n/f_{n0}\) to \(f_{gn}/f_{gn0}\), the relationship between \((f_{gn}/f_{gn0})/(f_n/f_{n0})\) and \(T\) was investigated. For this purpose, \((f_{gn}/f_{gn0})/(f_n/f_{n0})\) was plotted against \(T\).

\[ \frac{f_{gn}/f_{gn0}}{f_n/f_{n0}} = -0.0084T + 1.01 \quad (r = 0.972^{**}, n = 1) \text{ for the rectangular bar,} \]  

\[ \frac{f_{gn}/f_{gn0}}{f_n/f_{n0}} = -0.030T + 1.03 \quad (r = 0.956^{**}, n = 1) \text{ for the small round bar.} \]  

The estimation accuracy of the VAM is corrected with the following method. \(f_{n}/f_{n0}\) was estimated by substituting the measured \(T = E/E_{an}\) (TGH method/Euler–Bernoulli elementary theory) and measured \(f_{gn}/f_{gn0}\) for Eqs. (23) and (24). The estimated \(f_{n}/f_{n0}\) was used for Eq. (4), and \(m_n\) was subsequently obtained. The specimen mass was estimated by substituting the obtained \(m_n\) for Eq. (5). The results showed that the estimation accuracy of the VAM was improved, as shown in Fig. 5. The estimation accuracy of the VAM could not be improved in several cases. In such cases, \((f_{gn}/f_{gn0})/(f_n/f_{n0})\) was out of lines that are shown by Fig. 4 and Eqs. (23) and (24).

As mentioned above, \(T = E/E_{an}\) is necessary to estimate \(f_{n}/f_{n0}\) from \(f_{gn}/f_{gn0}\). In other words, the bending Young’s modulus based on the Euler–Bernoulli elementary theory, which can be calculated by measuring one resonance mode, and the “true” Young’s modulus without the SR effect are required. The true Young’s modulus can be measured through the TGH method or the longitudinal vibration test. As the resonance frequencies of plural vibration modes are required, obtaining \(T = E/E_{an}\) on-site, where simplicity and speed are needed, is challenging.

Since the estimation accuracy of the VAM was high when \(T\) was small, as shown in Fig. 3, \(T\) that provides a sufficiently high estimation accuracy of VAM is discussed
here. Based on the coefficient of regression of the TGH method in our previous study, the SR effect was experimentally low in the range of $T \leq 1.2$ [18]. In the range of $T \leq 1.2$, the estimation accuracy of the VAM for 11 out of 11 specimens (100%) for rectangular bars and that for 15 out of 17 specimens (88%) for small round bars were from 0.9 to 1.1. Hence, it can be said that the estimation accuracy of the VAM is sufficiently high in the range of $T \leq 1.2$. The estimation accuracies of the VAM for cross beams of wooden guardrails were 0.89, 0.91, and 1.06 [11] for $T = 1.13$, 1.11, and 1.17, respectively. The dimensions of a specimen that give $T \leq 1.2$ can be calculated by substituting $T = 1.2$, Young’s modulus and shear modulus published in the literature for Eq. (15).

Conclusions

Bending vibrations were undertaken for rectangular bars and round bars with various dimensions with and without the concentrated mass, and the following results were obtained:

1. The estimation accuracy of the VAM and the SR effect increased and decreased, respectively, as the length/thickness ratio of the rectangular bar and the length/diameter ratio of the round bar increased.
2. The estimation accuracy of the VAM decreased as the SR effect increased.
3. The SR effect on the estimation accuracy of the VAM could be corrected.

4. The range of $T$ for obtaining a sufficiently high accuracy of VAM was $T \leq 1.2$.

Abbreviations

VAM: Vibration method with additional mass; SR: Shear and rotatory inertia; TGH: Timoshenko–Goens–Hearmon; L: Longitudinal direction; R: Radial direction; T: Tangential direction; FFT: Fast Fourier transform.

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Authors' contributions

All authors designed the experiments. YK performed the experiments, analyzed the data, and was a major contributor in writing the manuscript. All authors contributed to interpretation and discussed results. All authors read and approved the final manuscript.

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Availability of data and materials

All data generated or analyzed during this study are included in this published article.

Competing interests

The authors declare that they have no competing interests.

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