Local and Global Hartogs-Bochner Phenomenon in Tubes

Abstract

A generalization of the Hartogs theorem is proved for a class of Tube structures \((M, G, \mathcal{V})\). We assume that the intervening commutative Lie algebra \(G\) admits at least \(\text{codim} \mathcal{V}\) globally solvable generators. We give necessary and sufficient conditions for triviality of the first cohomological group with compact support associated to the Tube structure to be trivial. A such global result was previously obtained only when \(M = \mathbb{R}^n \times \mathbb{R}^m\) with \(\partial/\partial x_j\) for \(j = 1, \ldots, m\) generating a Lie subalgebra of \(G\).
Introduction.

We start recalling the so called Bochner’s extension theorem ([Bo1,2]). It states that if $u$ is a holomorphic function defined in an open connected set $\mathbb{R}^m + i\Omega \subset \mathbb{C}^n$ then it extends as a holomorphic function to the linear convex envelope $\mathbb{R}^m + i\hat{\Omega}$ of $\mathbb{R}^m + i\Omega$ (one year before that Stein ([St]) proved this result for $n = 2$). A kind of local version of the Bochner extension theorem is found in Komatsu ([Ko]) where $\mathbb{R}^m$ is replaced by a ball $B_R$ centered at the origin with radius $R$. Later Andronikof ([An]) precise the dependence between $R$ and the size domain of the extension, namely:

Let $\Omega \subset \mathbb{R}^n$ be a convex bounded set of dimension $> 2$; if

$$R - \rho > \sqrt{2}\text{diameter}(\Omega)$$

then each function holomorphic on a neighborhood of the tube $B_R \times \partial\Omega$ has a unique holomorphic extension to a neighborhood of the tube $B_\rho \times \Omega$. An example of Ye ([Ye]) shows that $R$ is necessarily bigger than $(1/2)\text{diameter}(\Omega)$, leaving the question of finding the sharp constant in the interval $(1/2, \sqrt{2}]$.

Another classical extension theorem is due to Hartogs([Har1,2]) and it asserts that a holomorphic function in $\mathbb{C}^n \setminus \Omega$, where $\Omega$ is a bounded open domain with connected boundary $\partial\Omega$ extends itself to all of $\mathbb{C}^n$ as a holomorphic function. The Bochner extension theorem implies the Hartogs’s one as we see now; let
\( C^n \xrightarrow{\Pi} \mathbb{R}^n \) be the projection into the imaginary part; \( \Pi(x + it) = t \). Then

\[
C^n \setminus \overline{\Omega} \supset \mathbb{R}^n + i\mathbb{R}^n \setminus \Pi(\overline{\Omega})
\] (0.1)

Since the convex envelope of \( \mathbb{R}^n \setminus \Pi(\overline{\Omega}) \) is \( \mathbb{R}^n \) the Hartogs extension theorem follows for pairs \((\Omega, K)\) with \( \Omega \setminus K \) connected. Only four years after Fichera ([Fi]) published his work reducing the amount of CR data to \( \partial \Omega \) under certain regularity constrains, Ehrenpreis ([Eh]) gave a new proof of the Hartogs extension theorem. The proof of Ehrenpreis was remarkable simple and its main idea is a cohomological vanishing argument. The same idea applied by Hounie & Tavares ([HT]) to gives necessary and sufficient conditions for the validity of the Fichera’s version of the Hartogs extension theorem for a smooth globally integrable Tubes structures in \( \mathbb{R}^m \times \mathbb{R}^n \). By a smooth globally integrable Hypoanalytic Tubes structures in \( \mathbb{R}^m \times \mathbb{R}^n \), we mean a subbundle \( \mathcal{L} \subset C \otimes T(\mathbb{R}^m \times \mathbb{R}^n) \) such that \( \mathcal{L}_p = \text{Ker} \, dZ(p) \) where \( Z : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{C}^m \) is a smooth function function \( Z(x, t) = x + i\Phi(t) \). It extends the concept of the Cauchy-Riemann system in \( \mathbb{C}^m \). By a hypoanalytic structure we understand as a pair \((M, \mathcal{L})\) consisting of a smooth manifold \( M \) and a subbundle \( \mathcal{L} \subset C \otimes T M \) endowed with a associated hypoanalytic atlas \((U_\alpha, Z_\alpha)\). We mean \( \cup_\alpha U_\alpha = M \) and the maps

\[
Z_\alpha : U_\alpha \to \mathbb{C}^m \text{ with } m = \dim M - \dim \mathcal{L}
\] (0.2)

are smooth and \( \det dZ_\alpha \neq 0 \) and if \( p \in U_\alpha \) then \( \mathcal{L}_p = \ker Z_\alpha(p) \). Finally the etymology comes from the constrain that \( Z_\beta = Z_\alpha \circ H_{\alpha,\beta} \) in an neighborhood every point \( p \in U_\alpha \cap U_\beta \) where \( H_{\alpha,\beta} \) is a biholomorphism in some open neighborhood of \( Z_\beta(p) \). It is well known that \textit{fibers} of the hypoanalytic structure defined by the germs

\[
\mathcal{F}(p) = C_{Z_\alpha}(p) = \{ Z_\alpha = Z_\alpha(p) \}
\] (0.3)

are hypoanalytic invariants of the structure. The Sussmann’s orbit \( O_\mathcal{L}(p) \) (named after Sussmann ([Su])) is the minimal smooth submanifold contain-
ing $p$ which supports $\mathcal{L}$ in its complexified tangent space. We say that a smooth germ of function $u$ at $p$ is \textit{hypoanalytic} if $du$ is a germ of a section of $\mathcal{L}^\perp$. If $\mathcal{O}_\mathcal{L}(p)$ is compact then the trace of a hipoanalytic function in the orbit must be constant otherwise.

An Tube structure $(M, \mathcal{L}, \mathcal{G})$ is a hypoanalytic structure endowed with a commutative Lie algebra $\mathcal{G} \subset T M$ which verifies the conditions:

1. if $A_p \subset T_p M$ is the span of $\mathcal{G}_p$ then $\dim A_p \geq \text{codim} \mathcal{L}$ for all $p \in M$,
2. $\mathcal{L} + \mathcal{C} \otimes \mathcal{G}_p = \mathcal{C} \otimes T_p M$ for all $p \in M$,
3. $[\mathcal{L}, \mathcal{G}] \subset \mathcal{L}$.

It follows from 1. that $m = \dim \mathcal{G}$ is well defined and greater or equal to

$$\text{codim} \mathcal{C} \mathcal{L} = \dim M - \dim \mathcal{C} \mathcal{L}.$$ 

Under these hypothesis one can always find an hypoanalytic atlas $(U_\alpha, Z_\alpha)$ such that $Z_\alpha(x, t) = x + \Phi(t)$ for suitable coordinates where $\{ \partial/\partial x_1, \ldots, \partial/\partial x_m \}$ is a subset of generators of $\mathcal{A}$ over $U_\alpha$.

Let us denote by $\mathcal{F}_Z(p)$ the germ of the closed set $\{ Z = Z(p) \}$ for an arbitrary hypoanalytic function $Z$ at $p \in M$. For arbitrary Tubes structures $(M, \mathcal{L}, \mathcal{G})$ we have the following characterization of the \textit{local} Hartogs property:

\textbf{Theorem A.} A Tube $(M, \mathcal{L}, \mathcal{G})$ has the \textit{local} Hartogs property if and only if $\mathcal{F}_Z(p)$ is connected for all hypoanalytic germs $Z$ at $p$ for all $p \in M$.

\textbf{Remark.} Recently was established in work of Henkin & Michel ([HM]) for abstract real analytic (CR)-structures $(M, \mathcal{L})$ that the \textit{local} Hartogs phenomenon is equivalent to $(M, \mathcal{L})$ be nowhere strictly pseudoconvex with $\dim M \geq 3$. Actually the concept of pseudoconvexity is belongs to the larger class of structures called hypoanalytic structures. The Levi form $\Xi_\theta(p)$ is a hypoanalytic invariant defined in $\mathcal{L}_p \times \mathcal{L}_p$ for every $\theta \in \Sigma_p = \mathcal{L}^\perp \cap T^*_p M$ by

$$\Xi^\theta_p(v, w) = \theta \left( [\text{Re} L_0, \text{Im} L_1] \right)(p).$$
Here \( L_0, L_1 \) are germs sections of \( V \) satisfying \( L_0(p) = v \) and \( L_1(p) = w \) and \( \Sigma_p \) is the characteristic set of \( (M, \mathcal{L}) \). Actually it is an well defined object of the Sussmann orbit \( \mathcal{O}_\mathcal{L}(p) \) of \( V \). We say that \( (M, \mathcal{L}) \) is strictly pseudoconvex at \( p \in M \) if \( \Xi_\theta(p) \) is non degenerated with all the eigenvalues with a same sign. Consequently \( L_p \subset \mathbb{C} \otimes T_p \mathcal{O}_\mathcal{L}(p) \) and when \( \Xi_\theta(p) \) is nondegenerated with all eigenvalues of a same sign we say that \( \mathcal{L} \) is strictly pseudoconvex at \( \theta \in \Sigma_p \).

When this happens we can always find a germ of a hypoanalytic function at \( p \) such that \( \{ Z = Z(p) \} = \{ p \} \). Thus being nowhere strictly pseudoconvex is necessary condition for a hypoanalytic structure verify the local Hartogs property.

We now adress the question of whether the global Hartogs property holds for all pairs \((K,U)\) of compact sets \( K \subset U \) where \( U \subset M \) is open. Hopefully we answer the question of Nacinovtch and Hill about the example of the CR-structure on the hypersurface \( |z_1|^2 + |z_2|^2 - |z_3|^2 = 1 \) in \( \mathbb{C}^3 \) where the zero of the restriction of \( z_3 \) becomes compact failing the global Hartogs property but curiously holding the local one. Such hypersurface is actually a zero of a homogenous solution of a Tube structure globally defined by the map

\[
Z: \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{C} \times \mathbb{C} \times \mathbb{C} \\
\]

where \( Z(v_1, v_2, v_3, v_4) = (z_1, z_2, z_3, x + i \Phi(|v_1|, |v_2|, |v_3|)) \) with \( dz_1 \wedge dz_2 \wedge dz_3 \wedge dx + i d \Phi \neq 0 \) on \( \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \). Finally it is taken \( \Phi(\xi_1, \xi_2, \xi_3) = \xi_1^2 + \xi_2^2 - \xi_3^2 \) and the zeroes of \( \{ x + i \Phi(|v_1|, |v_2|, |v_3|) \} \) becomes CR-substructures which are actually also Tubes. By means of a right biholomorphism we find \( Z(v_1, v_2, v_3, v_4) = (z_1, z_2, z_3, x + i \Phi(\text{Im} z_1, \text{Im} z_2, \text{Im} z_3)) \) thus a tube according the Definition VI.9.2 in Treves([Tr1]) and the embedded CR-submanifolds \( \{ x + i \Phi(|v_1|, |v_2|, |v_3|) = \text{constant} \} \) are Tubes with the restriction of \( Z_0 = (z_1, z_2, z_3) \) as a global integral. After a unitary linear right composition the intersection
zeroes \( \{x + i \Phi(\text{Im } z_1, \text{Im } z_2, \text{Im } z_3) = z_3 = a + i b\} \) have the expression

\[
\{ \mathbb{R}^2 \times \{(t_1, t_2) \in \mathbb{R}^2 : \text{Im}^2 z_1 + \text{Im}^2 z_2 = c + b^2\} \times \{a + i b\} \times \{a + i c\} \}
\]

and it is empty if \( c < -b^2 \). When \( c = b^2 \) it is the plane

\[
\{ \mathbb{R}^2 \times \{(0, 0)\} \times \{a + i b\} \times \{a + i c\} \}
\]

becoming homeomorphic to \( \mathbb{R}^2 \times S \) for all \( c > -b^2 \). It happens that the function

\[
\text{Re } [i z_3 + \kappa(z_1^2 + z_2^2)] = (1 - \kappa)[\text{Im}^2 z_1 + \text{Im}^2 z_2] + \kappa[\text{Re}^2 z_1 + \text{Re}^2 z_2]
\]

has for \( 0 < \kappa < 1 \) compact zeros homeomorphic to \( S^3 \subset \mathbb{R}^2 \times \mathbb{R}^2 \times \{a + i b\} \times \{a + i \Phi(\text{Im } z_1, \text{Im } z_2, b) = a + i c\} \) and noncompact zeros for \( \kappa \geq 1 \).

Let us now denote by \( C_Z(p) \) the closed set \( \{\Re Z \leq \Re Z(p)\} \) for an arbitrary hypoanalytic function \( Z \). We will say that a Tube structure \((M, \mathcal{L}, \mathcal{G})\) verifies the global Hartogs condition \((H)\) if:

\begin{itemize}
  \item \(4\) \( \mathcal{G} \) admits at least codim \( \mathcal{L} \) globally solvable generators,
  \item \(5\) \( C_Z(p) \subset M \) does not have compact components for all hypoanalytic function \( Z \) and \( p \in M \),
  \item \(6\) \( \mathcal{O}_\mathcal{L}(p) \subset M \) is never compact for all \( p \in M \).
\end{itemize}

The example of Hill & Nacinovitch ([HN]) will show that condition \( \bullet_4 \) is necessary. The condition \( \bullet_5 \) is obviously needed for global Hartogs property holds. Otherwise any open set containing one compact component would fail the Hartogs property. Finally we may consider the quotient space \( \mathcal{O}_\mathcal{L} \) defined by the equivalent relation \( \sim \), where \( p \sim q \) in \( M \) if and only if \( p, q \in \mathcal{O}_\mathcal{L}(p) \). Then every \( u \in C(\mathcal{O}_\mathcal{L}) \) can be lifted to a function in \( C(M) \) which is a weak solution of \( \mathcal{L} \) and \( (u - u(p))^{-1} \) fails the global hartogs property showing that \( \bullet_6 \) is also necessary.

We can now state that

**Theorem B.** Let \( M \) be simply connected and \( (M, \mathcal{L}, \mathcal{G}) \) be a Tube structure. Then \( (M, \mathcal{L}, \mathcal{G}) \) verify \((H)\) if and only if global Hartogs property holds.
Remark. This gives an explanation for the embedded example of Hill & Nacinovitch (see [HN]) which gives an example of a Tube structure which verify the local Hartogs phenomena but not the global one. The Tube structure in question is defined by given by the map

\[ Z : \mathbb{C}^3 \times \mathbb{R} \rightarrow \mathbb{C}^4 \]

defined by \( Z(z, y) = (z, y + i(|z_1^2| + |z_2^2| - |z_3^2|)) \). By means of an biholomorphism in \( \mathbb{C}^4 \) we may rewrite \( Z : \mathbb{C}^3 \times \mathbb{R} \rightarrow \mathbb{C}^4 \) as

\[ Z(z, x) = (x_1 + i t_1, x_2 + i t_2, x_3 + i t_3, x + i(t_1^2 + t_2^2 - t_3^2)). \]

In this case there exists only one orbit and every the Hartogs global phenomena holds. On the other hand the zero \( C_Z(p) \) with \( Z(p) = i \) is a hypoanalytic submanifold which happens to be a globally integrable. The global integral in question is the restriction of \((x_1 + i t_1, x_2 + i t_2, x_3 + i t_3)\) to the zero

\[ C_Z(p) = \{ x + i(t_1^2 + t_2^2 - t_3^2) = 1 \}. \]

Thus it is a Tube which enjoy the the local Hartogs phenomena but not the global one. It happens that \((x_3 + i t_3)^{-1}\) is well defined in \( C_Z(p) \) with \( Z(p) = i \) except by its with intersection with \( x_3 + i t_3 = 0 \). The latter intersection is a set homeomorphic to the cylinder \( \mathbb{R}^2 \times S^1 \) and one can check that the function

\[ Z_\kappa = x_3 + i t_3 + \kappa ((x_1 + i t_1)^2 + (x_2 + i t_2)^2) \]

for some small \( \kappa < 1 \) has a compact zero inside a Torus contained in \( \mathbb{R}^2 \times S^1 \subset \mathbb{R}^2 \times i \mathbb{R}^2 \simeq \mathbb{C}^2 \) violating the condition \( \bullet_5 \) in Theorem B. The characterization given in [HT] for the global Hartogs phenomena in here stands for \( \bullet_5 \) one of the global condition in (H). Thus the Theorem B is a generalization of the result presented there.

2. Proofs of Theorem A and B
Proof of Theorem A. It follows from the main result in [HT] that a Tube structure \((M, \mathcal{L}, \mathcal{G})\) enjoys the local Hartogs property if and only if the germ \(C_Z(p)\) of a hypoanalytic function \(Z\) with \(dZ(p) \neq 0\) at \(p\) is connected and equivalent to nowhere strictly pseudoconvexity for Tubes structures \((M, \Lambda, \mathcal{L})\). Observe that \(C \otimes T_pM = \mathcal{L}_p + C \otimes \mathcal{A}_p \subset C \otimes T_p\mathcal{O}_\mathcal{L} + C \otimes \mathcal{A}_p\) and consequently \(C \otimes N^\mathcal{O}_\mathcal{L}(p) \subset \mathcal{L}^\perp\). If the local Hartogs phenomena occurs for this hypoanalytic structure then the germ \(C_Z(p)\) for a hypoanalytic function with \(dZ(p) \neq 0\) must be connected. Otherwise it will display some compact component or a denumerable set of components. In the first case \(Z^{-1}\) would fails the Hartogs phenomena for some pair \((U, C_Z(p) \cap U)\) and in the second is void, otherwise \(dZ(p) = 0\). By means of a complex linear transformation one may assume that for a hypoanalytic chart \((Z_\alpha, U)\) with \(p \in U\) that \(Z_\alpha(p) = 0\) and \(dZ_\alpha(p) = I\). Then \(N^\mathcal{O}_\mathcal{L}(p)\) will necessarily have a basis among the differentials \(\{d\text{Re}Z_{1,\alpha}(p), \ldots, d\text{Re}Z_{m,\alpha}(p)\}\). This implies with \(Z_\alpha^2 = Z_{1,\alpha}^2 + \cdots + Z_{m,\alpha}^2\) and large \(\kappa\) that the germ \(C_{Z+\kappa Z_2}(p) \cap \mathcal{O}_\mathcal{L}(p)\) must be connected if \(C_Z(p)\) is. Consequently for Tubes a necessary and sufficient condition the validity of local Hartogs phenomena is translated on germs \(F_Z(p) = C_Z(p) \cap \mathcal{O}_\mathcal{L}(p)\) by the condition; \(F_Z(p)\) is connected for all hypoanalytic germs \(Z\) with \(\text{Ker}dZ(p) = \{0\}\) (P).

Proof of Theorem B.(1st version via Ehrenpreis argument)

We will prove that the first cohomological group of the complex induced by \(\mathcal{L}\) is trivial reviving the original idea of Ehrenpreis [Eli] and giving a stronger version of Theorem B. We select \(m = \text{codim}\mathcal{L}\) globally integrable vector fields from \(\mathcal{G}\) and assume without loss of generality that \(\dim\mathcal{G} = m\). It follows that there exist smooth manifold \(N\) such that \(M = \mathbb{R}^m \times N\). For \(m = 1\) it is the statement of Theorem 6.4.2 (f) in [DH]. When \(m\) is bigger than one we proceed by induction taking advantage of the commutative property of the fields. As a consequence we get an open projection \(\Pi_\Lambda : M \to N\) having as fibers the the \(m\)–dimensional submanifolds \(A \subset M\) verifying \(T_pA = \mathcal{A}_p\) if \(p \in A\), that is
\[ A = \mathbb{R}^m \times \{ \Pi_A(p) \}. \] Now follows from the characterization of tubes structures found in VI.8 Partial Local Group Structures([Tr1]) that one can construct a hypoanalytic atlas \((U_\alpha, Z_\alpha)\) such that \(Z_\alpha(x, t) = x + i\Phi(t)\) where the first coordinates \(x\) are first integrals of the chosen \(m\) globally integrable vector fields in \(\mathcal{G}\). Since a pair of hypoanalytic charts \((U_\alpha, Z_\alpha), (U_\alpha', Z_\alpha')\) changes by a biholomorphism and they have identical real parts in \(U_\alpha \cap U_\alpha'\) they must agree there. It follows that \(\mathcal{L}\) has a global integral \(Z\) and the topological space \(M/\sim\mathcal{L}\), where \(\sim\mathcal{L}\) is the equivalence of being in a same fiber of \(\mathcal{L}\), is globally defined. Let \(Z = (Z^1, ..., Z^m)\) be a global integral for \((M, \mathcal{L}, \lambda)\), that is a map from \(M \rightarrow \mathbb{C}^k\) with \(k = \text{codim } \mathcal{L}\). Let \(\omega = \sum_{j=1}^n f_j dt_j\) be a smooth closed class in the first cohomological group with compact support induced by the differential complex associated to \(\mathcal{L}\). We mean that

\[ d\omega \wedge dZ = 0 \]

where \(dZ = dZ_1 \wedge ... \wedge dZ_m\). The same steps in [HT] by performing Fourier transform of \(\omega \wedge dZ\) in the linear fibers \(\{t\} \times \mathbb{R}^m\) to find \(\hat{\omega} \in \wedge^1 \mathcal{T}^*(N)\) such that

\[ d_t e^{\Phi \cdot \xi} \hat{\omega} = 0 \text{ for all } \xi \in \mathbb{R}^{m*}. \]

Since \(M\) is simply connected so it is \(N\) which enables us to define \(v(\xi, t)\) by

\[ d_t v(\xi, t) = e^{\Phi \cdot \xi} \hat{\omega}, \quad v(\xi, t_0) = 0 \]

where \(t_0 \in N \setminus \Pi_A(\text{supp } \omega)\). Now set \(\hat{u}(\xi, t) = e^{-\Phi \cdot \xi} v(\xi, t)\) which vanishes outside \(\Pi_A(\text{supp } \omega)\). It remains to prove that \(\hat{u}\) is indeed the fiber Fourier transform of a function \(u \in C_c^\infty(M)\) to finishes the proof. It follows from \(\bullet_5\) that the sublevels

\[ \mathcal{C}_{-i\xi}(0, t) = \{ s \in N : \Phi(s) \cdot \xi \leq \Phi(t) \cdot \xi \} \]

does not have compact components. Since it is a closed set it implies that \(N\) can not be compact. We now cover \(N\) by charts \((\chi_\beta, W_\beta)\) associated to the maximal
atlas of $N$ such that each one maps $W_\beta$ onto $Q_0 = [0, 1]^n$. Also we may assume that $\{W_\beta\}$ is a locally finite covering. Now we consider a subdivision of $Q_0$ in $2^{nk}$ cubes $Q_k$ of side length $2^{-k}$. Then any polygonal line inside $Q_0$ which intercepts each division cube in a unique line segment will have a length bounded by $\sqrt{n}2^{-k}2^{nk} = \sqrt{n}2^{(n-1)k}$. In particular the image of a such polygonal line by $\chi_\beta^{-1}$ into $W_\beta$ will have length bounded by $C_\beta\sqrt{n}2^{(n-1)k}$ for some metric in $N$ which is equivalent to the euclidian metric of $Q_0$ via any $\chi_\beta$. We now consider only cubes $Q_k$ such that $\chi_\beta(Q_k)$ meets the connected component of $C_{-i2}(0, t)$ which contains $t$ for some $\beta$. It entails that $\bigcup_\beta \chi_\beta(Q_k)$ ⊃ $C_{-i2}(0, t)$ is a connected set and we can find a curve differentiable by parts $\gamma$ linking $t$ to an arbitrary point in $\bigcup_\beta \chi_\beta(Q_k)$ such that $\chi_\beta(\gamma)$ is a polygonal curve in $[0, 1]^n$ which meets any $Q_k$ in a line segment for all $\beta$. Now every $s \in \gamma$ is at a distance (for the chosen metric) comparable with $\sqrt{n}2^{-k}$ from the component of $C_{-i2}(0, t)$ which contains $t$. Let $t'$ a point of the component within this range and apply the mean value theorem to obtain

$$|\Phi(s) - \Phi(t') \cdot \xi| \leq \sup_{t' \in Q_k} |\nabla \Phi(t')|\sqrt{n}2^{-n}|\xi|.$$ 

It is also true that $\Phi_t(s) - \Phi_t(s') \cdot \xi \leq 0$ for every $\xi \in \{t\} \times \mathbb{R}^{m*}$ with $t \in N$ and we can choose $\gamma$ such that $\partial Q_k \cap \chi_\beta(W_\beta \cap \gamma)$ oriented set $\{t_0, t_1\}$ obeying $|t_0 - t|$ and $|t_1 - t|$ are minimum and maximum of $|s - t|$ with $s \in \gamma \cap Q_k$. Now we may estimate $\hat{u}$ as $|\hat{u}(\xi, t)| =$

$$\left|\int_\gamma e^{(\Phi(s) - \Phi(t)) \cdot \xi} \hat{\omega} \right| \leq \left|\int_\gamma e^{(\Phi(t') - \Phi(t)) \cdot \xi} \hat{\omega} \right| \leq \sqrt{n}2^{(n-1)k}e^{C|\xi|2^{-k}} \sup |\hat{\omega}|$$

where the supreme of $|\hat{\omega}|$ is uniformly bounded in $\Pi_A(supp \omega)$ by multiples of arbitrary powers of $(1 + |\xi|)^{-1}$. Choosing $2^{-k}$ comparable with $(1 + |\xi|)^{-1}$ we may find constants such that

$$|\hat{u}(\xi, t)| \leq C_l(1 + |\xi|)^{-l} \text{ for } t \in \Pi_A(supp \omega), \xi \in \mathbb{R}^m, l \in \mathbb{N}.$$
This happens because \( \hat{u}(\xi, t) \) is uniformly bounded in the Schwartz space \( \mathcal{S}(\mathbb{R}^m) \) for every \( t \in \mathbb{N} \). It entails that \( u(x, t) \), the Fourier inverse transform of \( u(\xi, t) \) is indeed a function in \( C^\infty(M) \). Compactness of \( \text{supp} u \) follows from a theorem of propagations of zeroes of solutions for the sections of \( \mathcal{L} \). It states that solutions which vanishes in a neighborhood of a point \( p \in \mathcal{O}_\mathcal{L}(p) \) must vanishes in all orbit (see Theorem 1.1 in [HP]). In our case we consider the structure \((M \setminus \text{supp} u, \mathcal{L}, A)\) to apply the cited theorem. Uniqueness of the solution \( u \) follows in a similar argument.

**Proof of Theorem B. (2nd version via Arens-Royden theorem)** We select \( m = \text{codim} \mathcal{L} \) globally integrable vector fields from \( \mathcal{G} \) and assume without loss of generality that \( \dim \mathcal{G} = m \). It follows that there exist smooth manifold \( N \) such that \( M = \mathbb{R}^m \times N \). For \( m = 1 \) it is the statement of Theorem 6.4.2 (f) in [DH]. When \( m \) is bigger than one we proceed by induction taking advantage of the commutative property of the fields. As a consequence we get an open projection \( \Pi_A : M \to N \) having as fibers the \( m \)-dimensional submanifolds \( A \subset M \) verifying \( T_p A = \mathcal{A}_p \) if \( p \in A \), that is \( A = \mathbb{R}^m \times \{ \Pi_A(p) \} \). Now follows from the characterization of tubes structures found in VI.8 Partial Local Group Structures ([Tr1]) that one can construct a hypoanalytic atlas \((U_\alpha, Z_\alpha)\) such that \( Z_\alpha(x, t) = x + i\Phi(t) \) where the first coordinates \( x \) are first integrals of the chosen \( m \) globally integrable vector fields in \( \mathcal{G} \). Since a pair of hypoanalytic charts \((U_\alpha, Z_\alpha), (U_{\alpha'}, Z_{\alpha'})\) changes by a biholomorphism and they have identical real parts in \( U_\alpha \cap U_{\alpha'} \) they must agree there. It follows that \( \mathcal{L} \) has a global integral \( Z \) and the topological space \( M/\sim_{\mathcal{F}} \), where \( \sim_{\mathcal{F}} \) is the equivalence of being in a same fiber of \( \mathcal{L} \), is globally defined.

Let \( Z \) be a global integral for \((M, \mathcal{L}, A)\), that is a map from \( M \to \mathbb{C}^k \) with \( k = \text{codim} \mathcal{L} \). Now, under the hypothesis \((P)\) the closed set \( Z(p) = \{ Z = Z(p) \} \) is locally connected and the map \( \Pi_{\mathcal{L}} : M/\sim_{\mathcal{L}} \to Z(M) \) is relatively open and locally injective. Here \( \sim_{\mathcal{L}} \) represents the equivalence relation of two points of
M being in a same component of the closed set $\mathcal{Z}(p)$, that is the set $\mathcal{Z}(p)$ agree locally with $\mathcal{F}_p$ implying that $\mathcal{Z}_p \equiv \bigcup_{p \in \mathcal{Z}(p)} \mathcal{F}_p$, thus invariantly defined. We call $\mathcal{M}/\sim_L$ the reduced manifold by $L$ which makes any automorphism commute with a homeomorphism of $\mathcal{M}/\sim_L$ via the canonical projection. Such subgroup of homeomorphisms is a hypoanalytic invariant since the germ of the fiber $\mathcal{F}(p)$ propagates through $\mathcal{Z}(p)$. Thus we may say that the fiber of $L$ is globally defined and $\mathcal{M}/\sim_L$ is invariant under automorphism of the structure. We mean by global diffeomorphisms of $\mathcal{M}$ which leaves $L$ invariant in the sense that its differential is an automorphism of $L_p$ for every $p \in \mathcal{M}$. We say that an open subset $U \subset \mathcal{M}$ is a domain for $L$ if the canonical projection $\Pi_L : \mathcal{M} \rightarrow \mathcal{M}/\sim_L$ is injective in $U$. If the intersection $\mathcal{C}_Z(p) \cap \mathcal{O}_L(p)$ is relatively open in $\mathcal{O}_L(p)$ then by uniqueness (see [Tr1]) $\mathcal{O}_L(p) \subset \mathcal{C}_Z(p)$ and the germ propagates into the orbit $\mathcal{O}_L(p)$. Despite the discreteness of fibers of the canonical projection $\Pi_{\sim_L}$, one can not expect that $\mathcal{M}/\sim_L$ evenly covers $\mathcal{Z}(\mathcal{M})$ and in this way not necessarily a covering space. Now, for any compact subset $K \subset \mathcal{M}/\sim_L$ we we consider its $L$-convex envelope $\hat{K} \subset \mathcal{M}/\sim_L$ with respect to the finitely generated Banach algebra $\mathcal{A}_L(K)$ of continuous functions $u$ of $K$ which are uniform limits in $K$ of polynomials in $Z$. Such continuous are of course also defined in $\hat{K}$ the polynomial convex envelope of $K$ and $Z(\hat{K}) = Z(K)$. Thus $Z(\hat{K})$ indeed agree with the maximal ideal space of the algebra $\mathcal{A}_L(K)$ (see Theorem 3.1.15 in ([Ho])). If the first Cech cohomology group of $\mathcal{H}^1(K/\sim_L)$ is not trivial it follows from the Arens-Royden theorem (see [Ar], [Ro]) that we can find a hypoanalytic polynomial $Z_0$ such that $dZ_0/Z_0 \not= 0$ is well defined in $\mathcal{M}$ and a representant of a non trivial class in $\mathcal{H}^1_\partial (\mathcal{M})$ (the first cohomological DeRham group of the complex defined by the exterior derivative $d$). If the DeRham cohomology group $\mathcal{H}^1_\partial (U \setminus K)$ is trivial then $K \subset \mathcal{M}$ is an irremovable singularity of the ring $\mathcal{A}(\mathcal{M})$ because in this case Log $Z_0$ will be a hypoanalytic function which is defined in $\mathcal{M} \setminus K$ which cannot be extended for all $\mathcal{M}$ failing the Hartogs phenomena for
the pair \((M, K)\). On the other hand it follows from Poincaré duality that

\[
\varinjlim_{K \subset \subset M} H^p(M, M \setminus K) \simeq H^p_{d,c}(M) \simeq H^\dim M - p(M).
\]

Since in paracompact differentiable manifolds Čech, singular and De Rahm cohomology agree and \(M/\sim_L\) inherits from the manifold \(M\) a CW-complex structure. It follows that the Čech and singular cohomology of \(M/\sim_L\) are well defined and agree. If \(\sim_L\) is proper then there exist a natural injection

\[
H^p_c(\sim_L) : H^p_c(M/\sim_L) \hookrightarrow H^p_c(M) \simeq \check{H}^p_c(M)
\]

given by the singular cohomology functor. With \(m + n = \dim M\) we have

\[
H^{m+1}_c(M/\sim_L) \simeq H_{n-1}(M/\sim_L)
\]

where \(n = \dim L\) by Poincaré duality. Now we have direct decomposition

\[
H^{m+1}_{d,c}(M) \simeq H^{m+1}_c(\sim_L)[H^{m+1}_c(M/\sim_L)] \oplus \ker \wedge \Omega
\]

where \(\Omega = d\zeta\) is a exact nonvanishing section of \(\wedge^m L^\perp\) and the

\[
\wedge \Omega : \wedge^1 T^*(M) \longrightarrow \wedge^{m+1} T^*(M)
\]

is defined for \(\omega \in \wedge^1 T^*(M)\) by \(\omega \wedge \Omega\) verifies \(\Omega \wedge d = d \wedge \Omega\) and induces homomorphism \(\wedge \Omega : H^1_{d,c}(M) \rightarrow H^{m+1}_{d,c}(M)\). We can represent \(H^1_{d,c}(M)\) (where \(d_L\) is the exterior derivative induced by \(d\) in the sections of \(C \otimes T^*M/L^\perp\)) as the kernel of the map \(\omega \mapsto \omega \wedge \Omega\) in \(H^1_{d,c}(M)\). Thus \(\omega \wedge \Omega\) represent a class in \(H^{m+1}_{d,c}(M)\) if \(\omega\) is represents a class in \(H^1_{d,c}(M)\). In this setting the Hartogs phenomena holds if and only if for all \(\omega \in H^1_{d,c}(M)\) there exist \(u \in C^\infty_c(M)\) such that

\[
du \wedge \Omega = \omega \wedge \Omega
\]

Solvability of \((*)\) assures the triviality of the intersection \(H^{m+1}_{d,c}(M) \cap H^1_{d,c}(M) = H^1_{d,c}(M)\) which in turn must represent some subgroup of the de Rham group.
$H_{d}^{m+1}(M) \cong H_{d}^{m+1}(M)$ via Poincaré duality. The existence of a Lie algebra $\Lambda$ oriented by $\Omega$ allows one to decompose $L \subset C \otimes \Lambda \oplus TB$ where $B = \Pi_{A}(M)$ is a real $n$–dimensional manifold obtained by identifying the fibers of $\Pi_{A}$ to points in $M$. It follows that every real section of $TB$ has a unique lifting to $L$. This enables us to define the connection

$$\nabla_T L(p) = T(L)(T(p)) - T_h(L(p)) \in C \otimes p$$

where $T\Pi_{A}(T_h) = T$ at $p$ when the fibers $A$ of $\Pi_{A}$ have an affine linear structure, and this is always the case for an open covering $U_\alpha$ of $M$ such that $\Lambda$ admits $m - 1$ globally solvable generators in $\Pi_{A}^{-1}(\Pi_{A}(U_\alpha))$, turning $M$ into a real vector bundle by defining local charts $\Pi_{A}^{-1}(\Pi_{A}(U_\alpha)) \cong \mathbb{R}^m \times \Pi_{A}(U_\alpha)$. Assume that $H_{d}^{m+1}(M) \cong H_{d}^{m+1}(M)$ verifies $H_{d}^{m+1}(M) = \{0\}$ which means that any section $\omega \wedge \Omega$ is automatically exact if it represents a class in $H_{d}^{m+1}(M)$. Then we can find a section $e \Omega + \lambda$ of $\wedge^{m}T^*(M)$ such that $d(e \Omega + \lambda) = \omega \wedge \Omega$. It follows from the Stoke’s Theorem that for for rectifiable $m + 1$–rectifiable chain of form

$$\sigma = \Pi_{A}^{-1}(\Pi_{A}(\sigma)) = \mathbb{R}^m \times \Pi_{A}(\sigma)$$

that

$$\int_{\partial \sigma} e \Omega = \int_{\partial \sigma} (e \Omega + \lambda) = \int_{\sigma} \omega \wedge d \Omega = \int_{t \in \Pi_{A}(\sigma)} \int_{\Pi_{A}^{-1}(t)} \omega \wedge \Omega = \int_{\Pi_{A}(\sigma)} \int_{\mathbb{R}^m} \omega \wedge \Omega$$

for all $\omega \in H_{d}^{m+1}(M)$. In particular $\sigma$ is invariant by the $\Lambda$–flow and the left side is finite if $\omega$ has compact support. Thus if $\sigma$ is a $m + 1$–rectifiable chain with boundary $\partial \sigma$ and $\Omega|_{\partial \sigma} \neq 0$ then locally $\Pi_{G}(\sigma)$ is a $1$–rectifiable. If we choose $\sigma$ such that $\Pi_{A}(\partial \sigma) = \{t\}$ then the left side above is a smooth function of $t$ which vanishes outside $\Pi_{A}(\text{supp} \omega)$. We finish the proof applying the Treves propagation of zeroes theorem as we did before.

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