Kaluza-Klein-Kerr-Gödel Black Holes

Kaluza-Klein Black Holes with Rotations of Black Hole and Universe

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Applying squashing transformation to the Kerr-Gödel black hole solution, we present a new type of a rotating Kaluza-Klein black hole solution in the five-dimensional minimal supergravity. The new solution generated by the squashing transformation has no closed timelike curve outside the black hole horizons. The spacetime is asymptotically locally flat. One of the remarkable features is that the solution has two independent rotation parameters along an extra dimension associated with the black hole’s rotation and Gödel’s rotation. The spacetime also admits the existence of two disconnected ergoregions, an inner ergoregion and an outer ergoregion. These two ergoregions can rotate in opposite directions as well as in the same direction.

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§1. Introduction

The study of the five-dimensional Einstein-Maxwell theory with a Chern-Simons term plays an important role in elucidating the structure of the string theory since it is the bosonic sector of the minimal supergravity. All bosonic supersymmetric solutions of minimal supergravity in five dimensions have been classified by Gauntlett et al.1) Many interesting supersymmetric (BPS) solutions to the five-dimensional Einstein-Maxwell equations (with a Chern-Simons term) have been found by several authors. Following this classification of the five-dimensional supersymmetric solutions, they have been constructed on hyper-Kähler base spaces. The first asymptotically flat supersymmetric black hole solution, the Breckenridge-Myers-Peet-Vafa (BMPV) solution, was constructed on the four-dimensional Euclid space.2) A supersymmetric black hole solution with a compactified extra dimension on the Euclidean self-dual Taub-NUT base space was constructed by Gaiotto et al.3) It was extended to a multi-black hole solution with the same asymptotic structure.4) One of the most interesting properties is that the possible spatial topology of the horizon of each black hole is the lens space \(L(n; 1) = S^3/\mathbb{Z}_n\) \((n: \text{natural numbers})\) in addition to \(S^3\). Similarly, a black hole solution on the Eguchi-Hanson space5) was also constructed. Some supersymmetric black ring solutions have also been found on the basis of construction of the solutions by Gauntlett et al.1) Elvang et al. found...
the first supersymmetric black ring solution with asymptotic flatness on the four-
dimensional Euclidean base space, which is specified by three parameters: mass and
two independent angular momentum components. Gauntlett and Gutowski also
constructed a multi-black ring solution on the same base space. Black rings with
three arbitrary charges and three dipole charges on the flat space were also con-
structed in 9) and 10). Black ring solutions on the Taub-NUT base space and the Eguchi-Hanson space were constructed.

In recent years, some non-BPS black hole solutions have also been found in
addition to supersymmetric black hole solutions. Although no one has found higher-
dimensional Kerr-Newman solutions in the Einstein-Maxwell theory yet, Cvetič et al. found a nonextremal, charged and rotating black hole solution with asymptotic
flatness in the five-dimensional Einstein-Maxwell theory with a Chern-Simons term.
In the neutral case, the solution reduces to the same angular momentum case of
the Myers-Perry black hole solution. Exact solutions of non-BPS Kaluza-Klein
black hole solutions are found in a neutral case and a charged case. These
solutions have a nontrivial asymptotic structure, i.e., the spacetime is asymptotically
locally flat and approaches a twisted $S^1$ metric over a four-dimensional Minkowski
spacetime, topologically not a direct product. The horizons are deformed owing to
this nontrivial asymptotic structure and have a shape of a squashed $S^3$, where $S^3$ is
regarded as an $S^1$ bundle over an $S^2$ base space. The ratio of the radius $S^2$ to that
of $S^1$ is always larger than one.

As was proposed by Wang, a type of Kaluza-Klein black hole solution can be
generated by squashing transformation from black holes with asymptotic flatness. In
fact, he regenerated the five-dimensional Kaluza-Klein black hole solution found by Dobiasch and Maison from the five-dimensional Myers-Perry black hole solution with two equal angular momenta (The solution generated by Wang coincides
with that in Refs. 18) and 19). ). In a previous work, applying squashing trans-
formation to Cvetič et al.'s charged rotating black hole solution, we obtain a new
Kaluza-Klein black hole solution in the five-dimensional Einstein-Maxwell theory
with a Chern-Simons term. This is the generalization of the Kaluza-Klein black hole
solutions in Refs. 18–20). This solution has four parameters: mass, the angular
momentum in the direction of an extra dimension, electric charge and the size of the
extra dimension. The solution describes a physical situation in which, in general,
a non-BPS black hole is boosted in the direction of the extra dimension. As the
interesting feature of the solution, unlike that of the static solution, the horizon
admits a prolate shape in addition to a round $S^3$. The solution has limits to the
supersymmetric black hole solution, a new extreme non-BPS black hole solution and
a new rotating black hole solution with a constant $S^1$ fiber.

In this article, applying this squashing transformation to Kerr-Gödel black hole
solution, we construct a new type of rotating Kaluza-Klein black hole solution in
the five-dimensional minimal supergravity. We also investigate the features of the
solution. Although the Gödel black hole solution has closed timelike curves in the
region away from the black hole, the new Kaluza-Klein black hole solution generated
by squashing transformation have no closed timelike curve everywhere outside the
black hole horizons. At the infinity, the spacetime is asymptotically a Kaluza-Klein
spacetime. The solution has four independent parameters: mass, the size of an extra dimension and two kinds of rotations parameters in the same direction of the extra dimension. These two independent parameters are associated with the rotations of the black hole and the universe. In the case of the absence of a black hole, the solution describes the Gross-Perry-Sorkin (GPS) monopole, which is boosted in the direction of an extra dimension and has an ergoregion due to the effect of the rotation of the universe.

The rest of this article is organized as follows. In §2, we present a new Kaluza-Klein black hole solution in the five-dimensional Einstein-Maxwell theory with a Chern-Simons term. In §3, we study the special cases of our solution. In §4, we investigate the basic features of the solution. In §5, we summarize the results in this article. In the Appendix, we will present a more general solution.

§2. Squashed Kerr-Gödel black hole

2.1. Solution

First, we present the metric of a new Kaluza-Klein black hole solution to the five-dimensional Einstein-Maxwell theory whose action is given by

$$S = \frac{1}{16\pi} \int \left( R - 1 - 2F \land *F - \frac{8}{3\sqrt{3}} F \land F \land A \right).$$

The metric and gauge potential are given by

$$ds^2 = -f(r)dt^2 - 2g(r)\sigma_3 dt + h(r)\sigma_3^2 + \frac{k(r)r^2 dr^2}{V(r)} + \frac{r^2}{4} [k(r)(\sigma_1^2 + \sigma_2^2) + \sigma_3^2],$$

and

$$A = \frac{\sqrt{3}}{2} jr^2 \sigma_3,$$

respectively, where the functions in the metric are

$$f(r) = 1 - \frac{2m}{r^2},$$
$$g(r) = jr^2 + \frac{ma}{r^2},$$
$$h(r) = -j^2 r^2 (r^2 + 2m) + \frac{ma^2}{2r^2},$$
$$V(r) = 1 - \frac{2m}{r^2} + \frac{8jm(a + 2jm)}{r^2} + \frac{2ma^2}{r^4},$$
$$k(r) = \frac{V(r\infty)r_{\infty}^4}{(r^2 - r_{\infty}^2)^2}$$

and the 1-forms on $S^3$ are given by

$$\sigma_1 = \cos \psi d\theta + \sin \psi \sin \theta d\phi,$$
$$\sigma_2 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi,$$
$$\sigma_3 = d\psi + \cos \theta d\phi.$$
The coordinates \( r, \theta, \phi \) and \( \psi \) run the ranges of \( 0 < r < r_{\infty}, 0 \leq \theta < \pi, 0 \leq \phi < 2\pi \) and \( 0 \leq \psi < 4\pi \), respectively. \( m, a, j \) and \( r_{\infty} \) are constants. The spacetime has the timelike Killing vector field \( \partial_t \) and two spatial Killing vector fields with closed orbits, \( \partial_{\phi} \) and \( \partial_{\psi} \). In the limit of \( k(r) \rightarrow 1 \), i.e., \( r_{\infty} \rightarrow \infty \) with the other parameters kept finite, the metric coincides with that of the Kerr-G"odel black hole solution \(^{23}\) with CTCs. In this article, we assume that the parameters \( j, m, a \) and \( r_{\infty} \) appearing in the solution satisfy the following inequalities:

\[
\begin{align*}
    m &> 0, \\
    \frac{r_{\infty}^2}{m} &> 1 - 4j(a + 2jm) > \sqrt{\frac{2}{m}|a|}, \\
    r_{\infty}^4 - 2m(1 - 4j(a + 2jm))r_{\infty}^2 + 2ma^2 &> 0, \\
    -4j^2r_{\infty}^6 + (1 - 8j^2m)r_{\infty}^4 + 2ma^2 &> 0.
\end{align*}
\]

(12)–(15)

As will be explained later, these are the necessary and sufficient condition that there should be two horizons and no CTCs outside the horizons. Equations (12)–(14) are conditions for the presence of two horizons, and Eq. (15) is the condition for the absence of CTCs outside the horizons. It is noted that in the limit of \( r_{\infty} \rightarrow \infty \) with the other parameters finite, Eq. (15) cannot be satisfied. Thus, applying the squashing transformation to the Kerr-Gödel black hole solution, we can obtain such a Kaluza-Klein black hole solution without CTCs everywhere outside the black hole.

2.2. Parameter region

Here, we derive the parameter region of Eqs. (12)–(15) from the demands of the presence of two horizons and the absence of CTC outside the horizons. As will be shown later, the Killing horizons are located at

\[
\begin{align*}
    r_{\pm}^2 &= \pm(1 - 4j(a + 2jm)) \pm \sqrt{m^2(1 - 4j(a + 2jm))^2 - 2ma^2},
\end{align*}
\]

(16)

which are the roots of the equation \( V(r) = 0 \), i.e., the quadratic equations with respect to \( r^2 \):

\[
\begin{align*}
    v(r^2) := r^4 - 2m(1 - 4j(a + 2jm))r^2 + 2ma^2 &= 0.
\end{align*}
\]

(17)

This equation has two different real roots within the range of \((0, r_{\infty}^2)\) if and only if the parameters in the metric obey the inequalities

\[
\begin{align*}
    v(0) &= 2ma^2 > 0, \\
    v(r_{\infty}^2) &= r_{\infty}^4 - 2m(1 - 4j(a + 2jm))r_{\infty}^2 + 2ma^2 > 0, \\
    r_{\infty}^2 &= m|1 - 4j(a + 2jm)| > 0, \\
    m^2|1 - 4j(a + 2jm)|^2 - 2ma^2 &= 0.
\end{align*}
\]

(18)–(21)

Under the condition \( a \neq 0 \), these inequalities are equivalent to Eqs. (12)–(14).

Next, in order to avoid the existence of CTCs outside the horizons, the parameters are chosen so that the two-dimensional \((\psi, \phi)\)-part of the metric is positive-definite everywhere outside the outer horizon. To do so, it is sufficient to require
\[ g_{\psi\psi} > 0 \] since \( g_{\phi\phi} = g_{\psi\psi} \cos^2 \theta + r^2 k(r) \sin^2 \theta / 4 > 0 \) is automatically ensured if it is satisfied. Hence, the following inequality must be satisfied everywhere in the region \([r_+^2, r_\infty^2]\):

\[
g_{\psi\psi} = h(r) + \frac{r^2}{4} > 0 \iff u(r^2) := -4j^2r^6 + (1 - 8j^2m)r^4 + 2ma^2 > 0. \quad (22)
\]

Here, it should be noted that it is sufficient to impose \( u(r_\infty^2) > 0 \), which gives the equality (15). Consequently, the condition for the absence of CTCs outside the horizons is equal to the demand for the absence of CTCs at the infinity.

In order to plot the parameter region of Eqs. (12)–(15) in a two-plane, we normalize the parameters \( a, j \) and \( r_\infty \) as \( A = a/\sqrt{m}, J = \sqrt{m} j \) and \( R_\infty = r_\infty / \sqrt{m} \), respectively, furthermore, we fix \( R_\infty \).

Then, in the cases of \( R_\infty^2 < 2, R_\infty^2 = 2 \) and \( R_\infty^2 > 2 \), the quadratic curve \( R_\infty^4 - 2(1 - 4J(A + 2J))R_\infty^2 + 2A^2 = 0 \) in the condition (14) becomes an ellipse, a line and a hyperbola, respectively. The curve \( R_\infty^2 = 1 - 4J(A + 2J) \) in the condition (13) has different shapes in the cases of \( R_\infty^2 < 1, R_\infty^2 = 1 \) and \( R_\infty^2 > 1 \). Hence, we consider the cases of (i) \( 0 < R_\infty^2 < 1 \), (ii) \( R_\infty^2 = 1 \), (iii) \( 1 < R_\infty^2 < 2 \), (iv) \( R_\infty^2 = 2 \) and (v) \( R_\infty^2 > 2 \). The shaded regions in Figs. 1–5 show the parameter regions of Eqs. (12)–(15) for a given \( R_\infty \) in all cases of (i)–(v), respectively.

### 2.3. Mass and angular momenta

In the coordinate system \((t, r, \theta, \phi, \psi)\), the metric diverges at \( r = r_\infty \) but we see that this is an apparent singularity and corresponds to the spatial infinity. To investigate the asymptotic structure of the solution, we introduce a new radial coordinate defined by

\[
\rho = \rho_0 \frac{r^2}{r_\infty^2 - r^2}, \quad (23)
\]

where the positive constant \( \rho_0 \) is given by

\[
\rho_0^2 = \frac{r_\infty^2}{4} V(r_\infty). \quad (24)
\]

Moreover, we introduce the coordinates \((\tilde{t}, \tilde{\psi})\) so that the metric is in a rest frame at the infinity

\[
\tilde{t} = \frac{1}{C} t, \quad \tilde{\psi} = \psi - \frac{D}{C} t, \quad (25)
\]

where two constants, \( C \) and \( D \), are chosen as

\[
C = \sqrt{\frac{2a^2m + r_\infty^4(1 - 4j^2(2m + r_\infty^2))}{2a^2m + 8ajmr_\infty^2 + r_\infty^2(-2m + 16j^2m^2 + r_\infty^2)}}, \quad (26)
\]

\[
D = \frac{4(am + jr_\infty^4)}{\sqrt{2a^2m + 8ajmr_\infty^2 + r_\infty^2(-2m + 16j^2m^2 + r_\infty^2)}} \times \frac{1}{\sqrt{2a^2m + r_\infty^4(1 - 4j^2(2m + r_\infty^2))}}. \quad (27)
\]
Then, for $\rho \to \infty$, the metric behaves as

$$ds^2 \simeq -dt^2 + d\rho^2 + \rho^2(\sigma_1^2 + \sigma_2^2) + L^2\sigma_3^2,$$

(28)

where the angular coordinate $\psi$ in $\sigma_3$ is replaced with $\bar{\psi}$ and the size of an extra dimension, $L$, is given by

$$L^2 = \frac{2a^2m + r_\infty^4(1 - 4j^2(2m + r_\infty^2))}{4r_\infty^2}.$$  

(29)
Fig. 3. Parameter region in the $(J, A)$-plane in the case of $1 < R^2_\infty < 2$.

Fig. 4. Parameter region in the $(J, A)$-plane in the case of $R^2_\infty = 2$.

The Komar mass and Komar angular momenta at the spatial infinity are given by

$$M_K = \frac{\pi}{\sqrt{2a^2m + 8ajmr^2_\infty + r^2_\infty(-2m + 16J^2m^2 + r^2_\infty)}}$$
$$\times \frac{1}{\sqrt{2a^2m + r^2_\infty(1 - 4j^2(2m + r^2_\infty))}}$$
$$\times \left[-2a^2m^2 + mr^4_\infty + 32aj^3mr^4_\infty(m + r^2_\infty) + 64j^4m^2(m + r^2_\infty)r^4_\inftyight.$$
$$+ 4ajm(2a^2m - r^4_\infty) + 2j^2(-8m^2r^4_\infty - 4mr^6_\infty + r^8_\infty)$$
respectively. Therefore, the spacetime has only an angular momentum in the direction of the extra dimension. In particular, note that $J_\psi = \pi ma/2$ in the case of $j = 0$ and $J_\psi = -\pi r_\infty^6 j^3$ in the case of $a = 0$. Therefore, $(a, j)$ with opposite signs indicates rotations in the same direction, and $(a, j)$ with the same sign means rotations in the opposite directions.

2.4. Regularity

Here, we investigate regularity on the horizon. As is mentioned previously, at places where $V(r) = 0$, i.e., at $r = r_\pm$, the metric seems to be apparently singular but these correspond to Killing horizons. To see this, we introduce new coordinates $(v, \psi')$ defined by

$$
t = v + \int \frac{2\sqrt{h(r) + r^2/4k(r)}}{rV(r)} dr, \quad (33)
$$

$$
\psi = \psi' + \int \frac{2k(r)g(r)}{rV(r)\sqrt{h(r) + r^2/4}} dr + \frac{g(r_\pm)}{h(r_\pm) + r_\pm^2/4} v. \quad (34)
$$

In the neighborhood of $r = r_\pm$, the metric behaves as

$$
 ds^2 \simeq -\frac{r_\pm k(r_\pm)}{2\sqrt{h(r_\pm) + r_\pm^2/4}} dv dr + \left[ \frac{r_\pm^2}{4} k(r_\pm)(\sigma_1^2 + \sigma_2^2) + \left( h(r_\pm) + \frac{r_\pm^2}{4} \right) \sigma_3^2 \right]. \quad (35)
$$
where the angular coordinate \( \psi \) in \( \sigma_3 \) is replaced by \( \psi' \). This means that the hypersurfaces, \( r = r_{\pm} \), are Killing horizons since the Killing vector field \( \partial_\psi \) becomes null on \( r = r_{\pm} \), furthermore, it is hypersurface-orthogonal, i.e., \( V_\mu dx^\mu = g_{\nu r} dr \). We should also note that in the coordinate system \((v, \phi, \psi', r, \theta)\), each component of the metric form is analytic on and outside the black hole horizon. Hence, the spacetime has no curvature singularity on and outside the black hole horizon.

As will be explained later, the ergosurfaces are located at \( r \) such that \( g_{tt} = F(r^2)/r^4 \) vanishes, where \( F(r^2) \) is the cubic equation with respect to \( r^2 \) given by

\[
F(r^2) = -16j^2(am + jr^4) - 4(am + jr^4)[4a^2jm + am(-1 + 8j^2m) + jr^4(1 - 8j^2(m + r^2))] - 2mr^8[1 - 2j(a + 2j(2m + r^2))].
\]

§3. Special cases

In this section, we focus on a few simple cases of the solution.

3.1. Rotating Gross-Perry-Sorkin (GPS) monopole

First, we consider the simplest case of \( m = a = q = 0 \), where the solution has no event horizon. In terms of the coordinate \( \rho \), the metric can be written as

\[
ds^2 = -\left[ dt + 4j\rho^2 \left(1 + \frac{\rho_0}{\rho}\right)^{-1} \sigma_3 \right]^2 + \left(1 + \frac{\rho_0}{\rho}\right) \left[d\rho^2 + \rho^2(\sigma_1^2 + \sigma_2^2)\right] + \rho_0^2 \left(1 + \frac{\rho_0}{\rho}\right)^{-1} \sigma_3^2, \tag{37}\]

where it should be noted that the requirement, (15) for the absence of CTCs imposes the parameters on the inequality \( j^2 < 1/(16\rho_0^2) \). In the case of \( j = 0 \), Eq. (37) exactly coincides with the GPS monopole solution in Ref. 24). Note that the point of \( \rho = 0 \) is a fixed point of the Killing vector field \( \partial_\psi \), and corresponds to a Kaluza-Klein monopole. This is often called a nut. In the presence of the parameter \( j \), \( \rho = 0 \) is also a fixed point of the Killing vector field \( \partial_\psi \) and the metric is analytic at the point. Hence, the presence of the parameter \( j \) means that although the spacetime has no black hole, it is rotating along the direction \( \partial_\psi \) of the extra dimension. For \( \rho \to \infty \), the metric behaves as

\[
ds^2 \simeq -dt^2 + d\rho^2 + \rho^2(\sigma_1^2 + \sigma_2^2) + \rho_0^2 \left[1 - 16j^2\rho_0^2\right] \sigma_3^2, \tag{38}\]

where it should be noted that CTCs at the infinity in the Gödel universe vanish via squashing transformation.

Although this rotating GPS monopole solution has no event horizon, we can confirm the presence of an ergoregion. The \( \bar{t}\bar{t} \)-component of the metric in the rest frame at the infinity takes the form

\[
g_{\bar{t}\bar{t}} = -\left[ \left(C + \frac{1 - C^2}{C} \left(1 + \frac{\rho_0}{\rho}\right)^{-1}\right)^2 - \frac{1 - C^2}{C^2} \left(1 + \frac{\rho_0}{\rho}\right)^{-1} \right], \tag{39}\]
Typical behavior of $g_{tt}$ in the case of $\sqrt{3}/8 < |j|\rho_0 < 1/4$. There exists an ergoregion in the
region such that $g_{tt} > 0$.

where the constant $C$ can be written in the form of $C = \sqrt{1 - 16j^2\rho_0^2}$ from Eq. (26).

As shown in Fig. 6, in the case of $\sqrt{3}/8 < |j|\rho_0 < 1/4$, $g_{tt}$ becomes positive in the
region of $\gamma_- < \rho < \gamma_+$, where

$$\gamma_{\pm} := \rho_0 \frac{1 - 3C^2 \pm (1 - C^2)\sqrt{1 - 4C^2}}{2C^2}. \quad \text{(40)}$$

This means that there is an ergoregion, although there is no black hole horizon in
the spacetime and in the neighborhood of the nut, the ergoregion vanishes.

3.2. Squashed Schwarzschild-Gödel black hole

Next, we consider the case of $a = q = 0$. In this case, the solution is obtained
via the squashing transformation for the Schwarzschild-Gödel black hole solution in
Ref. 23). The parameter region in which there is a black hole horizon and there is
no CTC outside the horizon becomes

$$m > 0, \quad \text{(41)}$$
$$1 - 8j^2m > 0, \quad \text{(42)}$$
$$-2m + \frac{1}{4j^2} > r_\infty^2 > 2m(1 - 8j^2m). \quad \text{(43)}$$

The horizon is located at $r_{\mathcal{H}}$ satisfying

$$r_{\mathcal{H}}^2 = 2m(1 - 8j^2m). \quad \text{(44)}$$

The shaded region in Fig. 7 shows the region satisfying the inequalities, (41)–(43).

Note that $g_{tt}(r = r_{\mathcal{H}}) > 0$ and $g_{tt}(r = r_\infty) < 0$ always hold. Hence, the
ergosurfaces are always located at $r$ such that $g_{tt} = 0$. In particular, when $F(\alpha^2) < 0$
and $r_{\mathcal{H}} < \alpha$, remarkably there are two ergoregions, $r_{\mathcal{H}} < r < r_1$ and $r_2 < r < r_3$,
outside the black hole horizon, where $\alpha^2$ and $\beta^2$ ($0 < \alpha < \beta$) are the roots of the
quadratic equation with respect to $r^2$, $\partial_{r^2} F(r^2) = 0$, and $r_i^2 (i = 1, 2, 3, \ r_1 < r_2 < r_3)$
are the roots of the cubic equation with respect to $r^2$, $F(r^2) = 0$. The small dark
region in Fig. 8 denotes a set of solutions that admit two ergoregions outside the horizon.
Fig. 7. Parameter region of the squashed Schwarzschild-Gödel black hole solution in the $(jr_\infty, m/r_\infty^2)$-plane.

Fig. 8. In the small dark parameter region in the figure above, there are two ergoregions. The figure below shows a closeup of the dark region in the figure above.

§4. Gödel parameter versus Kerr parameter

4.1. Angular momentum

The squashed Kerr-Gödel black hole solution has two independent rotation parameters $a$ and $j$, where both $a$ and $j$ denote the rotation along the Killing vector $\partial_\psi$ (or $-\partial_\psi$) associated with a black hole and the (squashed) Gödel universe, i.e.,
the rotating GPS monopole as the background, respectively. In this article, we call these parameters \textit{Kerr parameter} and \textit{Gödel parameter}, respectively. Hence, if the black hole is rotating in the inverse direction of the rotation of the (squashed) Gödel universe, the effect of their rotations can cancel out the Komar angular momentum. In fact, the total angular momentum with respect to the Killing vector field $\partial_\psi$, $J_\psi$, vanishes if the parameters satisfy

$$2a^2 jm + 2j^3 r_\infty^6 + m(-1 + 2j^2(4m + 3r_\infty^2))a = 0. \quad (45)$$

As is shown in Figs. 9–13, the solid curve denotes a set of solutions such that the total angular momentum $J_\psi$ vanishes for various values of $R_\infty$. In each case, as is expected, the curve of $J_\psi = 0$ is located in the regions $A < 0, J < 0$ or $A > 0, J > 0$. On the other hand, the Komar angular momentum $J_\psi(r)$ over the $r (r < r_\infty)$ constant surface is obtained as

$$J_\psi(r) = -\frac{\pi}{2}[2a^2 jm + 2j^3 r_\infty^6 + m(-1 + 2j^2(4m + 3r^2))a]. \quad (46)$$

The dotted curves in Figs. 9–13 denote the curves of $J_\psi(R_+)$ = 0. The parameter region is decomposed into four regions, $\Sigma_1$, $\Sigma_\Pi$, $\Sigma_{III}$ and $\Sigma_{IV}$, by the two curves $J_\psi(R_\infty) = 0$ and $J_\psi(R_+)$ = 0 as follows:

$$\Sigma_1 = \{(J, A) \mid J_\psi(R_\infty) > 0, J_\psi(R_+) > 0\}, \quad (47)$$

$$\Sigma_\Pi = \{(J, A) \mid J_\psi(R_\infty) > 0, J_\psi(R_+) < 0\}, \quad (48)$$

$$\Sigma_{III} = \{(J, A) \mid J_\psi(R_\infty) < 0, J_\psi(R_+) > 0\}, \quad (49)$$

$$\Sigma_{IV} = \{(J, A) \mid J_\psi(R_\infty) < 0, J_\psi(R_+) < 0\}. \quad (50)$$

In the solutions within $\Sigma_1$ and $J < 0$ or $\Sigma_{IV}$ and $J > 0$, $J_\psi(R_\infty)$ and $J_\psi(R_+)$ have the same sign since the black hole is rotating in the same direction as the squashed Gödel universe (rotating GPS monopole). In the solutions within $\Sigma_1$ and $J > 0$ or $\Sigma_{IV}$ and $J < 0$, although two parameters $(J, A)$ have the same sign, $J_\psi(R_\infty)$ and $J_\psi(R_+)$ have the same sign because the effect of the black hole’s rotation exceeds that of the Gödel rotation.

On the other hand, in the solutions within $\Sigma_\Pi$ or $\Sigma_{III}$, the angular momenta $J_\psi(R_\infty)$ and $J_\psi(R_+)$ also have opposite signs as well as $J$ and $A$. Therefore, there is a surface $R_0$ between the horizon $R = R_+$ and the infinity $R = R_\infty$ where $J_\psi(R_0)$ vanishes. In fact, as is shown in Fig. 16, the angular momentum vanishes between the horizon and the infinity. It is expected that the effect of the Gödel rotation exceeds that of the black hole’s rotation at a large distance.

4.2. Two counter-rotating ergoregions

Here, we examine the ergoregions of the squashed Kerr-Gödel black hole solution. The metric can be rewritten as

$$ds^2 = \left(\frac{h(r) + r^2}{4}\right) \left[\sigma_3 + \left(-\frac{g(r)}{h(r) + r^2}C + D\right) dt\right]^2 - \frac{r^2 V(r)}{h(r) + r^2} dt^2 + \frac{k(r)^2}{V(r)} dr^2 + \frac{r^2}{4} k(r)(\sigma_1^2 + \sigma_2^2). \quad (51)$$
Therefore, on the outer horizon \( r = r_+ \) and the infinity \( r = r_\infty \), the \( \bar{t}\bar{t} \)-component of the metric takes a non-negative form and a negative-definite form as

\[
g_{\bar{t}\bar{t}}(r = r_+) = \left( -\frac{g(r_+)}{h(r_+) + r_+^2/4} C + D \right)^2 \geq 0, \tag{52}
\]

\[
g_{\bar{t}\bar{t}}(r = r_\infty) = -\frac{r_\infty^2}{4} \frac{V(r_\infty)}{h(r_\infty) + r_\infty^2/4} C^2 < 0. \tag{53}
\]
Fig. 11. Curves of $J_\psi(R_\infty) = 0$ and $J_\psi(R_+) = 0$ in the case of $1 < R_\infty^2 < 2$.

Fig. 12. Curves of $J_\psi(R_\infty) = 0$ and $J_\psi(R_+) = 0$ in the case of $R_\infty^2 = 2$.

Hence, as is mentioned previously, the ergosurfaces are located at $r$ such that $g_{tt} = 0$, i.e., the cubic equation with respect to $r^2$, $F(r^2) = 0$. Similarly to the squashed Schwarzschild-Gödel black hole solution in the previous section, the spacetime admits the presence of two ergoregions outside the black hole horizon. The spike like dark region $\Sigma$ in Fig. 14 corresponds to the parameter region in which $F(r_+^2) > 0$, $F(r_\infty^2) < 0$, $F(\alpha^2) < 0$ and $r_+ < \alpha < \beta < r_\infty$ in the case of $R_\infty^2 = 50$. Note that in the case of $0 < R_\infty^2 \leq 2$, no such region appears within the parameter regions of
Fig. 13. Curves of $J_\psi(R_\infty) = 0$ and $J_\psi(R_+) = 0$ in the case of $R_\infty^2 = 3.0$. 

Eqs. (12)–(15).

Figure 15 shows a closeup of the dark region $\Sigma$ in Fig. 14. The dotted curve in Fig. 15 denotes $F(r_+^2) = 0$, where the ergoregion near the outer horizon vanishes. As is shown in Fig. 15, the curve of $F(r_+^2) = 0$ ($\Omega_+ = 0$) decomposes $\Sigma$ into two regions, $\Sigma_1$ and $\Sigma_2$. In the region of $\Sigma_1$, the black hole horizon is rotating in the same direction as the Gödel universe, while in the region of $\Sigma_2$, the horizon and the universe are rotating in opposite directions. Here, we introduce virtual ZAMOs (zero angular momentum observer) located in the ergoregions. The angular velocity of a ZAMO, $\omega_\psi$, is given by $\omega_\psi := -g_{t\bar{\psi}}/g_{\psi\bar{\psi}}$. On the horizon, note that $\omega_\psi|_{r=r_+} = \Omega_+$. The dashed curves in Fig. 16 denote the dependence on the radial coordinate $r$ of the angular velocity $\omega_\psi$ of ZAMOs in the case of $J = -0.061, A = -0.300$, and $R_\infty^2 = 50$ in $\Sigma_1$ and $J = -0.061, A = -0.500$, and $R_\infty^2 = 50$ in $\Sigma_2$, respectively. In the region of $\Sigma_1$, a ZAMO in the ergoregion $r_+ < r < r_1$ (ZAMO1) and a ZAMO in the ergoregion $r_2 < r < r_3$ (ZAMO2) is rotating in the same direction $-\partial_\psi$, while in the region of $\Sigma_2$, ZAMO1 and ZAMO2 are rotating in the directions of $\partial_\psi$ and $-\partial_\psi$, respectively.

§5. Summary and discussion

Applying squashing transformation to the Kerr-Gödel black hole solution, we have constructed a new type of rotating Kaluza-Klein black hole solution to the five-dimensional Einstein-Maxwell theory with a Chern-Simons term. The features of the solutions have been investigated. Although the Gödel black hole solutions have closed timelike curves in the region away from the black hole, the new solutions...
Fig. 14. Small dark region $\Sigma$ shows the parameter region in which there are two ergo regions between the horizon and the infinity in the case of $R_\infty^2 = 50$.

generated via squashing transformation have no closed timelike curve everywhere outside the black hole horizons. As the infinity, the spacetime is asymptotically local a $S^1$ bundle over a four-dimensional Minkowski spacetime but the $S^1$ is not a direct bundle.

In the absence of a black hole, i.e., in the case of $m = a = 0$, the solution describes the rotating GPS (Gross-Perry-Sorkin) monopole solution, which is boosted in the direction of an extra dimension. Remarkably, in spite of the absence of a black hole, this spacetime can have an ergoregion due to the effect of the Gödel rotation. In the squashed Schwarzschild-Gödel black hole solution with the parameters $r_\infty, m$
and $j$, the spacetime has one black hole horizon. It has one ergoregion around the black hole horizon or two disconnected ergoregions, i.e., an inner ergoregion located around the horizon and an outer ergoregion located far away from it. On the other hand, the squashed Kerr-Gödel black hole solution with the parameters $r_\infty$, $m$, $a$ and $j$ describes a rotating black hole in the squashed Gödel universe (the rotating GPS monopole background). The spacetime has two horizons, i.e., an inner black hole horizon and an outer black hole horizon. The spacetime has two independent rotations along the direction $\partial_\psi$ associated with the black hole and the squashed Gödel universe. In the case of opposite rotations, the effect of Gödel’s rotation and the black hole’s rotation can cancel out the angular momentum at the infinity. In such a spacetime, the ergoregions have richer and more complex structures. Similarly to the squashed Schwarzschild-Gödel black hole solution, the spacetime also admits the existence of two ergoregions, an inner ergoregion and an outer ergoregion. These two ergoregions can rotate in opposite directions as well as in the same direction.

Note that in addition to mass and the angular momentum in the extra dimension, this spacetime carries an electric charge $Q$ given by

$$Q = 2\sqrt{3} \text{maj}.$$  \hspace{1cm} (54)

In particular, in the case of the rotating GPS monopole solution or squashed Schwarzschild-Gödel black hole solution, the electric charge vanishes. We can generalize our solution to a more general solution with an extra parameter $q$ by applying squashing transformation to the Kerr-Newman-Gödel black hole solution in Ref. 25). The solution is given in Appendix A. We leave the analysis of this solution for future study.

Several multi-black hole solutions on the Euclid base space, Taub-NUT base space and Eguchi-Hanson base space were also constructed in the five-dimensional Einstein-Maxwell theory (with a positive cosmological constant). In our forthcoming research, we will also study the squashed Kerr-Newman-Gödel multi-black hole solution in the five-dimensional minimal supergravity.
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Appendix A

--- Squashed Kerr-Newman-Gödel Black Holes ---

The metric and gauge potential of the squashed Kerr-Newman-Gödel black hole solution are given by

\[
\begin{align*}
\text{ds}^2 &= -f(r)dt^2 - 2g(r)\sigma_3 dt + h(r)\sigma_3^2 + \frac{k(r)^2 dr^2}{V(r)} \\
&\quad + \frac{r^2}{4}[k(r)(\sigma_1^2 + \sigma_2^2) + \sigma_3^2], \\
\end{align*}
\]

(A.1)

and

\[
A = \frac{\sqrt{3}}{2} \left[ \frac{q}{r^2} dt + \left(jr^2 + 2jq - \frac{qa}{2r^2}\right)\sigma_3 \right],
\]

(A.2)

respectively, where the metric functions are

\[
\begin{align*}
f(r) &= 1 - \frac{2m}{r^2} + \frac{q^2}{r^4}, \\
g(r) &= jr^2 + 3jq + \frac{(2m - q)a}{2r^2} - \frac{q^2a^2}{2r^4}, \\
h(r) &= -j^2r^2(r^2 + 2m + 6q) + 3jqa + \frac{(m - q)a^2}{2r^2} - \frac{q^2a^2}{4r^4}, \\
V(r) &= 1 - \frac{2m}{r^2} + \frac{8j(m + q)[a + 2j(m + 2q)]}{r^2} \\
&\quad + \frac{2(m - q)a^2 + q^2(1 - 16ja - 8j^2(m + 3q))}{r^4}, \\
k(r) &= \frac{V(r_\infty)r_\infty^4}{(r^2 - r_\infty^2)^2}.
\end{align*}
\]

(A.3) (A.4) (A.5) (A.6) (A.7)

In the limit of \( r_\infty \to \infty \), i.e., \( k(r) \to 1 \), the solution coincides with the Kerr-Newman-Gödel black hole solution in Ref. 25).

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