INFINITESIMAL CRITERION FOR FLATNESS OF PROJECTIVE MORPHISM OF SCHEMES

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Abstract. A generalization of the well-known criterion for flatness of a projective morphism of Noetherian schemes involving the Hilbert polynomial is given for the case of a nonreduced base of the morphism.

Introduction

We start with some classical notation. Let \( \mathbb{P}^N_T \) be a relative projective space of dimension \( N \) over a scheme \( T \), and let \( \mathcal{O}(1) \) be a line bundle on \( \mathbb{P}^N_T \) generated by a hyperplane section. It is very ample relative to \( T \). Also, if \( f : X \to T \) is a morphism of schemes and \( t \in T \) a closed point with residue field \( k(t) \), then \( X_t := f^{-1}(t) \) is a closed fiber of \( f \) over \( t \).

Our goal in this paper is to generalize the following well-known criterion for flatness of a projective morphism of Noetherian schemes (see [1, Chapter III, Theorem 9.9]).

Theorem 1. Let \( T \) be an integral Noetherian scheme and \( X \subset \mathbb{P}^N_T \) some closed subscheme. For each closed point \( t \in T \), take the Hilbert polynomial \( P_t \in \mathbb{Q}[m] \) of the fiber \( X_t \) viewed as a closed subscheme in \( \mathbb{P}^N_T \). Then the subscheme \( X \) is flat over \( T \) if and only if the Hilbert polynomial \( P_t \) does not depend on the choice of \( t \).

This theorem applies to any projective morphism of schemes \( f : X \to T \) with integral base scheme \( T \) if one reformulates it as follows.

Suppose that a projective morphism of Noetherian schemes \( f : X \to T \) with integral scheme \( T \) fits into the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i} & \mathbb{P}^N_T \\
\downarrow{f} & & \downarrow{f} \\
T & & T
\end{array}
\]

where \( i \) is a closed immersion. It is flat if and only if for an invertible \( \mathcal{O}_X \)-sheaf \( \mathcal{L} \) that is very ample relative to \( T \) and such that \( \mathcal{L} = i^* \mathcal{O}(1) \), and for every closed point \( t \in T \), the Hilbert polynomial of the fiber \( P_t(m) = \chi(\mathcal{L}^m|_{X_t}) \) does not depend on the choice of \( t \).

The proof of this theorem, as presented in [1], allows one to deduce flatness of any coherent \( \mathcal{O}_X \)-sheaf \( \mathcal{F} \) over an integral scheme \( T \) if the Hilbert polynomial \( \chi(\mathcal{F} \otimes \mathcal{L}^m|_{X_t}) \) of its restriction to the fiber \( X_t \) over each point \( t \in T \) does not depend on the choice of \( t \).

The above criterion is not applicable in the case where the scheme \( T \) has nonreduced scheme structure (\S1). As we shall show below in \S2, this inconvenience can be eliminated.

2010 Mathematics Subject Classification. Primary 14B25; Secondary 13D10, 14A15.

Key words and phrases. Noetherian algebraic schemes, projective morphism, nonreduced scheme structure, flat morphism, coherent sheaf of modules.

The author was partially supported by the Institute of Mathematics “Simion Stoilow” of Romanian Academy (IMAR) (partnership IMAR – BITDEFENDER) during the author’s stay as invited professor, June – July 2011.

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if the Hilbert polynomial is replaced by some other function. At the closed points, this function coincides with the Hilbert polynomial of fibers. We need some notation. If \( t \in T \) is a closed point of the scheme \( T \) and this point corresponds to a sheaf of maximal ideals \( m_t \subset \mathcal{O}_X \), then the symbol \( \ell^{(n)} \) stands for the \( n \)th infinitesimal neighborhood of the point \( t \in T \). The \( n \)th infinitesimal neighborhood is the subscheme determined by the sheaf of ideals \( m_t^{n+1} \) in \( T \). In our consideration, \( T \) is assumed to be a Noetherian scheme of finite type over a field; then for each \( t \) function coincides with the Hilbert polynomial of fibers. We need some notation. If the Hilbert polynomial is replaced by some other function. At the closed points, this function in \( T \) is of finite length equal to length \( \ell^{(n)} = \chi (\mathcal{O}_{\ell^{(n)}}) \). It is clear that this is a positive integer depending on \( t \) and \( n \). If the point \( t = \text{Supp} \ell^{(n)} \) is known and fixed, we denote the length of the \( n \)th infinitesimal neighborhood \( \ell^{(n)} \) by the symbol \( (n+1) \) (in accordance with the power of the maximal ideal corresponding to the subscheme \( \ell^{(n)} \)).

We operate in the category of Noetherian schemes over a field \( k \). This field is assumed to be algebraically closed. This assumption is essential in those parts of the argument where we use filtrations (and cofiltrations) of Artinian algebras for counting the dimension. Namely, these are the proofs of the Claim and Proposition 4. If \( A/I \) is an Artinian algebra over an algebraically closed field, then length \( A/I = \dim_k A/I \). Since all vector spaces appearing in this paper are defined over the field \( k \), the lower index in the notation of dimension is omitted.

To judge whether a morphism \( f \) is flat, we need to examine the preimages \( f^{-1}(\ell^{(n)}) := X \times_T \ell^{(n)} \) of infinitesimal neighborhoods of reduced points \( t \in T \). This provides data on the behavior of \( f \) over the nonreduced scheme structure of \( T \). Since \( T \) is of finite type, the power \( n \) to be examined for the given morphism \( f \) is bounded from above (and is at most the maximal index of the nilpotent elements in \( O_T \) minus 1). In this paper we prove the following results (Theorem 2 is a particular case of Theorem 3, so we only prove Theorem 3).

**Theorem 2.** Suppose that a projective morphism of Noetherian schemes of finite type \( f: X \to T \) fits into the commutative diagram (1). It is flat if and only if for an invertible \( \mathcal{O}_X \)-sheaf \( \mathcal{L} \) very ample relative to \( T \) and such that \( \mathcal{L} = i^* \mathcal{O}(1) \), and for any closed point \( t \in T \), the function

\[
\varpi^{(n)}_t (\mathcal{O}_X, m) = \frac{\chi (\mathcal{L}^m | f^{-1}(\ell^{(n)}))}{\chi (\mathcal{O}_{\ell^{(n)}})}
\]

does not depend on the choice of \( t \in T \) and \( n \in \mathbb{N} \).

**Theorem 3.** Suppose that a projective morphism of Noetherian schemes of finite type \( f: X \to T \) fits into the commutative diagram (1). The coherent sheaf of \( \mathcal{O}_X \)-modules \( \mathcal{F} \) is flat with respect to \( f \) (i.e., flat as a \( \mathcal{O}_T \)-module) if and only if for an invertible \( \mathcal{O}_X \)-sheaf \( \mathcal{L} \) very ample relative to \( T \) and such that \( \mathcal{L} = i^* \mathcal{O}(1) \), and for any closed point \( t \in T \), the function

\[
\varpi^{(n)}_t (\mathcal{F}, m) = \frac{\chi (\mathcal{F} \otimes \mathcal{L}^m | f^{-1}(\ell^{(n)}))}{\chi (\mathcal{O}_{\ell^{(n)}})}
\]

does not depend on the choice of \( t \in T \) and \( n \in \mathbb{N} \).

In case where \( f \) is a finite morphism, the function in Theorem 2 takes the form

\[
\varpi^{(n)}_t (\mathcal{O}_X, m) = \frac{\text{length}(f^{-1}(\ell^{(n)}))}{\text{length}(\ell^{(n)})}.
\]

If the scheme \( T \) is reduced, it suffices to examine only the case of \( n = 0 \). This corresponds to the classical situation where \( \varpi^{(0)}_t (\mathcal{O}_X, m) = P_t (m) \) (Theorem 1).
§1. Examples

Example 1. Consider the nonreduced scheme \( T = \text{Spec} \, k[x]/(x^2) \) of length 2 and a morphism \( f : X \rightarrow T \) of immersion of a unique closed point \( X = \text{Spec} \, k \). Since both schemes are supported at a point, we replace the study of the morphism of structure sheaves \( f^* : \mathcal{O}_T \rightarrow f_* \mathcal{O}_X \) by that of the homomorphism \( f^* : k[x]/(x^2) \rightarrow k \) of the corresponding Artinian algebras. It is clear that \( f^* \) is an epimorphism onto the quotient ring over the nilradical: \( f^* : k[x]/(x^2) \rightarrow (k[x]/(x^2))/\text{Nil} = k \). We use the criterion for flatness of a ring homomorphism formulated in [2] Chapter 1, Proposition 2.1.

Proposition 1. A homomorphism \( f : A \rightarrow B \) is flat if and only if the mappings \( (a \otimes b) \mapsto f(a)b) : I \otimes_A B \rightarrow B \) are injective for all ideals \( I \) of \( A \).

Thus, we need to test the homomorphism \( (x) \otimes_{k[x]/(x^2)} k \rightarrow k \) for injectivity. This homomorphism is induced by the inclusion of the ideal \( (x) \hookrightarrow k[x]/(x^2) \). The tensor product \( (x) \otimes_{k[x]/(x^2)} k \) is nonzero and is a \( k \)-linear span of the element \( x \otimes 1, \, x^2 = 0 \). The element \( x \otimes 1 \) is taken to 0 under the mapping to \( k \). Thus, the ring homomorphism in question and the corresponding scheme morphism are not flat.

The same result is given by Theorem 2. The Hilbert polynomial of the fiber \( f \) over a unique closed point \( t \) of the scheme \( T \) is equal to \( P_t(m) = 1 \). The function \( \varpi \) computed for the 1st infinitesimal neighborhood of the closed point on the base \( T \) (it coincides with the entire scheme \( T \)) is equal to \( \varpi_t(1)(m) = 1/2 \). This differs from the value \( \varpi_t^{(0)}(m) = P_t(m) = 1 \).

Example 2. Let \( A = k[x]/(x^3) \), and let \( B = k[x]/(x^2) \) be the \( A \)-module in question. It is clear that \( B \) is finitely generated (and has one generator) over \( A \). \( A \) is a local \( k \)-algebra with the maximal ideal \( (x) \) and residue field \( k \). Since \( B \) is not free as an \( A \)-module, it follows that \( B \) is not flat as an \( A \)-module. On the other hand, the group \( \text{Tor}_1^A(k, B) \) fits into the exact sequence induced by tensoring the triple

\[
0 \rightarrow (x^2) \rightarrow A \rightarrow B \rightarrow 0
\]

by \( \otimes_A k \):

\[
0 \rightarrow \text{Tor}_1^A(k, B) \rightarrow (x^2) \otimes_A k \rightarrow A \otimes_A k \rightarrow B \otimes_A k \rightarrow 0
\]

This sequence implies that \( \text{Tor}_1^A(k, B) = (x^2) \otimes_A k = k \). This also shows that \( B \) is not flat as an \( A \)-module. Computing the function \( \varpi \), we get \( \varpi_t^{(0)}(m) = \varpi_t^{(1)}(m) = 1, \, \varpi_t^{(2)}(m) = 2/3 \).

The following two examples illustrate applications of Theorem 2.

Example 3. Let \( T = \text{Spec} \, k[x, y]/(xy, y^2) \), \( X = \text{Spec} \, k[x, y, z]/(xy, y^2, z^2) \); the morphism \( f : X \rightarrow T \) is given by projection to the \((XOy)\)-plane parallel to the \( z \)-axis. Since the morphism \( f \) is finite, the function \( \varpi_t^{(n)} \) takes the form

\[
\varpi_t^{(n)}(\mathcal{O}_X, m) = \frac{\text{length}(f^{-1}(t^{(n)}))}{\text{length}(t^{(n)})}.
\]

Also, the Hilbert polynomial of a fiber \( f^{-1}(t) \) takes the form

\[
P_t(m) = \text{length}(f^{-1}(t))
\]

for every closed point \( t \in T \). For \( t \neq O \) we have \( P_t(m) = \varpi_t^{(n)}(m) = 2 \). We turn to the neighborhood of the point \( t = O \):

\[
P_O(m) = \varpi_O^{(0)}(m) = 2, \quad \varpi_O^{(1)}(m) = 6/3.
\]

Since the function \( \varpi_t^{(n)}(m) \) is preserved, Theorem 2 shows that \( f \) is a flat morphism.
Example 4. As before, let $T = \text{Spec} \ k[x, y]/(xy, y^2)$, but the scheme $X$ is supplied with another structure: $X = \text{Spec} \ k[x, y, z]/(xy, y^2, yz, z^2)$; the morphism $f: X \to T$ is again the $z$-parallel projection onto the $(xOy)$-plane. The Hilbert polynomial of a fiber $f^{-1}(t)$ is equal to $P_t(m) = \omega_O^{(1)}(m) = 2$ for any closed point $t \in T$. However, $\omega_O^{(1)}(m) = 5/3$, and the morphism $f$ is nonflat by Theorem 2.

§2. Algebraic version

We shall need the following criterion for flatness [3, Chapter 1, Theorem 7.8].

Proposition 2. An $A$-module $M$ is flat if and only if $\text{Tor}_1^A(A/I, M) = 0$ for any finitely generated ideal $I \subset A$.

Convention. Let $A$ be a local Noetherian $k$-algebra with residue field $k = \overline{k}$, and let $I \subset A$ be an ideal such that $A/I$ is an Artinian $k$-algebra of length $n$, i.e., $\dim A/I = n$. Then the ideal $I$ is said to be of colength $n$ and we write $I_n$ instead of $I$.

Proposition 3. Let $M$ be a finitely generated module over the local Noetherian $k$-algebra $A$ with residue field $k$. The module $M$ is free if and only if for all $n > 0$ and all ideals $I_n \subset A$ of colength $n$ we have

$$\frac{\dim(M \otimes_A A/I_n)}{n} = \dim(M \otimes_A k).$$

Proof. Observe that $M/I_nM = M \otimes_A A/I_n$ and $M/I_nM \otimes_{A/I_n} k = M \otimes_A A/I_n \otimes_{A/I_n} k = M \otimes_A k$. Similarly,

$$M/I_nM \otimes_{A/I_n} A/I_{n-1} = M \otimes_A A/I_{n-1} = M/I_{n-1}M.$$

We have an exact triple of $A$-modules (and of $A/I_n$-modules)

$$0 \to \mathfrak{m}_n \to A/I_n \to k \to 0.$$  

For the maximal ideal $\mathfrak{m}_n \subset A/I_n$, we have $M \otimes_A \mathfrak{m}_n = M \otimes_A A/I_n \otimes_{A/I_n} \mathfrak{m}_n = M/I_nM \otimes_{A/I_n} \mathfrak{m}_n$. Tensoring of (4) by $M/I_nM \otimes_{A/I_n}$ yields the exact sequence

$$0 \to \text{Tor}_1^{A/I_n}(M/I_nM, k) \to M/I_nM \otimes_{A/I_n} \mathfrak{m}_n \to M/I_nM \to M/I_nM \otimes_{A/I_n} k \to 0.$$  

Left exactness is guaranteed by $\text{Tor}_1^{A/I_n}(M/I_nM, A/I_n) = 0$ because any ring is flat over itself.

Claim. Identity (3) implies the exactness of the sequence

$$0 \to M/I_nM \otimes_{A/I_n} \mathfrak{m}_n \to M/I_nM \to M/I_nM \otimes_{A/I_n} k \to 0.$$  

The proof of this claim will be presented below, after the proof of Proposition 3. This claim and the exact sequence (5) imply that $\text{Tor}_1^{A/I_n}(M/I_nM, k) = 0$. By Proposition 2, $M/I_nM$ is flat as an $A/I_n$-module. Now we may consider not all possible ideals of finite colength but the powers of the maximal ideal $\mathfrak{m} \subset A$ only. Passing to the $\mathfrak{m}$-adic completions $\widehat{A}$ and $\widehat{M}$ of the ring $A$ and of the module $M$, respectively, we conclude that $\widehat{M}$ is a flat $\widehat{A}$-module, see [3, proof of Theorem 22.4(ii)].

By the same [3, Theorem 2.2.4(ii)] we conclude that the $A$-module $M$ is flat.

The proof of the reverse implication is trivial. If a finitely generated module over the local ring is flat, then it is free, i.e., $M \cong A^q$. This implies identities of the form (3) for all $n > 0$ and all ideals $I_n \subset A$. This completes the proof of the proposition. 

\[ \square \]
Proof of the Claim. To organize induction on $n$, consider exact diagrams of the form

\begin{align*}
0 & \rightarrow k \rightarrow A/I_n \rightarrow A/I_{n-1} \rightarrow 0 \\
0 & \rightarrow k \rightarrow m_n \rightarrow m_{n-1} \rightarrow 0
\end{align*}

(7)

Such a diagram of $A$-modules (and of $k$-algebras) can be built up for any $n > 0$ and for any ideal $I_n \subset A$ of colength $n$. For $n = 2$, we have $m_n = m_2 \cong k$, $m_{n-1} = m_1 = 0$.

Suppose (3) is true. Then

$$
\dim M \otimes_A A/I_n = \dim M \otimes_A A/I_{n-1} + \dim M \otimes_A k.
$$

This implies that the triple $0 \rightarrow M \otimes_A k \rightarrow M \otimes_A A/I_n \rightarrow M \otimes_A A/I_{n-1} \rightarrow 0$ is exact.

Remark. A priori, the exactness of this triple does not imply that

$$
\Tor^A_1(M, A/I_{n-1}) = \Tor^A_1(M, A/I_n) = \Tor^A_1(M, k) = 0.
$$

This result follows from Proposition 3.

Tensoring the diagram (7) by $M \otimes_A$ leads to the diagram

\begin{align*}
0 & \rightarrow M \otimes_A k \rightarrow M \otimes_A A/I_n \rightarrow M \otimes_A A/I_{n-1} \rightarrow 0 \\
0 & \rightarrow M \otimes_A k \rightarrow M \otimes_A m_n \rightarrow M \otimes_A m_{n-1} \rightarrow 0
\end{align*}

(8)

Let $K := \ker(M \otimes_A m_n \rightarrow M \otimes_A m_{n-1})$. Then the isomorphism $M \otimes_A k \cong M \otimes_A k$ factors as $M \otimes_A k \rightarrow K \rightarrow M \otimes_A k$. This implies that $K \cong M \otimes_A k$, and the lower horizontal triple in (8) is left-exact. Consequently, $\dim M \otimes_A m_n = \dim M \otimes_A m_{n-1} + \dim M \otimes_A k$. Applying induction on $n$, we see that $\dim M \otimes_A m_n = (n-1) \dim M \otimes_A k$. This implies that the triple $M \otimes_A m_n \rightarrow M \otimes_A A/I_n \rightarrow M \otimes_A A/I_{n-1} \rightarrow 0$ (which is equivalent to the triple (6)) is left-exact. This proves the claim. \qed

Proposition 4. Identities (3) are valid for all $n > 0$ and all $I_n \subset A$ if and only if similar identities, namely,

$$
\dim M \otimes_A A/m^n = \dim A/m^n \; \dim M \otimes_A k,
$$

hold true for $m^n$ for all $n > 0$. 

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Proof. The “only if” part is obvious; it remains to prove the “if” part. We denote by the symbol \((n)\) the length of the quotient algebra \(A/m^n\), i.e., \(n := \dim A/m^n\). Using descending induction on the lengths of quotient algebras, we write exact sequences of the form

\[
0 \to k \to A/m^n \to A/I_{(n)-1} \to 0,
0 \to k \to A/I_{(n)-1} \to A/I_{(n)-2} \to 0,
\]

\[\cdots\]

\[
0 \to k \to A/I_{(n')-1} \to A/m^{n'} \to 0
\]

for appropriate \(n' < n\). Tensoring by \(M \otimes_A\) and dimension counting yield the sequence of inequalities

\[
\dim M \otimes_A A/m^n \leq \dim M \otimes_A k + \dim M \otimes_A A/I_{(n)-1},
\]

\[
\dim M \otimes_A A/I_{(n)-1} \leq \dim M \otimes_A k + \dim M \otimes_A A/I_{(n)-2},
\]

\[\cdots\]

\[
\dim M \otimes_A A/I_{(n')-1} \leq \dim M \otimes_A k + \dim M \otimes_A A/m^{n'}.
\]

Continuing descent till \(n' = 1\) and applying (9), we conclude that the inequalities in (10) are in fact identities.

Then for any \(I_i, l > 0\) there exists \(n > 0\) such that \(A/m^n \to A/I_i\). In this case there is a cofiltration \(A/m^n \to A/I_{(n)-1} \to \cdots \to A/I_1 \to \cdots \to k \to 0\) of length \(n\) with kernels isomorphic to \(k\) and such that it contains \(A/I_i\). Counting the dimensions of the vector spaces \(M \otimes_A A/I_j, j = 1, \ldots, l\), and applying identities (10), we arrive at the required formula \(\dim M \otimes_A A/I_i = l \dim M \otimes_A k\).

\[\Box\]

§3. PROOF FOR A COHERENT \(\mathcal{O}_T\)-MODULE

Since the assertion of the theorem is local in \(T\), we may assume that \(T = \text{Spec} \ A\) for a local Noetherian \(k\)-algebra \(A\). In the text below we shall use the notation \(\mathcal{F}(m) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^m\). The group \(H^0(\text{Spec} \ A, f_* \mathcal{F}(m)) = H^0(X, \mathcal{F}(m))\) carries the structure of a finitely generated \(A\)-module. Before all, we prove that for \(m \gg 0\) the \(A\)-module \(H^0(X, \mathcal{F}(m))\) is flat. For this, we consider a finite presentation of the quotient ring \(A/I_n\) (it exists because the ideal \(I_n\) is finitely generated):

\[
A^g \to A \to A/I_n \to 0.
\]

The quotient ring \(A/I_n\) fixes a zero-dimensional subscheme \(Z \subset T\) of length \(n\). The presentation (11) induces the triple

\[
\mathcal{F}(m)^g \to \mathcal{F}(m) \to \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Z \to 0.
\]

Passing to global sections leads to the complex

\[
H^0(X, \mathcal{F}(m))^g \to H^0(X, \mathcal{F}(m)) \to H^0(X, \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Z) \to 0.
\]

By Serre’s theorem [11, Chapter III, Theorem 5.2], for \(m \gg 0\) this complex is exact. Tensoring the presentation (11) by \(H^0(X, \mathcal{F}(m)) \otimes_A\) leads to the right-exact triple

\[
H^0(X, \mathcal{F}(m))^g \to H^0(X, \mathcal{F}(m)) \to H^0(X, \mathcal{F}(m)) \otimes_A A/I_n \to 0.
\]

Comparison of (12) and (13) yields isomorphism

\[
H^0(X, \mathcal{F}(m)) \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Z = H^0(f^{-1}Z, \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Z) \cong H^0(X, \mathcal{F}(m)) \otimes_A A/I_n.
\]

We suppose that

\[
\dim H^0(f^{-1}Z, \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Z) = n \dim H^0(f^{-1}t, \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^*k_t),
\]

\[\square\]
where \( t \) is a unique closed point of the scheme \( T \). By (14), we have
\[
H^0(f^{-1}Z, \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Z) \cong H^0(X, \mathcal{F}(m)) \otimes_A A/I_n,
\]
\[
H^0(f^{-1}t, \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^*k_t) \cong H^0(X, \mathcal{F}(m)) \otimes_A k.
\]

Substituting these isomorphisms in (13), we conclude that the following relations are true for all \( n > 0 \) and all \( I_n \subset A \):
\[
\dim H^0(X, \mathcal{F}(m)) \otimes_A A/I_n = n \dim H^0(X, \mathcal{F}(m)) \otimes_A k,
\]
This allows us to apply Proposition 8 to \( H^0(X, \mathcal{F}(m)) \). Hence, \( H^0(X, \mathcal{F}(m)) \) is a flat \( A \)-module.

The proof of the flatness of \( \mathcal{F} \) as an \( \mathcal{O}_T \)-module copies the proof of the implication (ii)\( \Rightarrow \) (i) in [1, Chapter III, proof of Theorem 9.9] word for word. This part of the cited proof remains valid also for a nonreduced scheme \( T \). By the projectivity of the morphism \( f \), we can restrict ourselves to the case where \( f \) is a structure morphism of some projective bundle \( f: \text{Proj} \ A[x_0 : \cdots : x_n] \to \text{Spec} \ A \) and consider the graded \( A[x_0 : \cdots : x_n] \)-module \( M = \bigoplus_{m \geq m_0} H^0(X, \mathcal{F}(m)) \). The integer \( m_0 \) is chosen so large that the \( A \)-modules \( H^0(X, \mathcal{F}(m)) \) are free for all \( m \geq m_0 \). By the proof of the flatness of the \( A \)-modules \( H^0(X, \mathcal{F}(m)) \) for \( m \gg 0 \) presented above, such a choice is possible. Then \( \mathcal{F} = \tilde{M} \), where \( \tilde{\ } \) denotes formation of a coherent sheaf of \( \mathcal{O}_{\text{Spec} \ A} \)-modules that is associated with the finitely generated \( A \)-module \( M \) (“sheafification”). In this case, \( M \) and \( \bigoplus_{m \geq 0} H^0(X, \mathcal{F}(m)) \) are equal for all \( m \geq m_0 \), and hence, see [1, Chapter II, Proposition 5.15], \( \tilde{M} = \bigoplus_{m \geq 0} H^0(X, \mathcal{F}(m)) \). Since \( M \) is a free (and, consequently, flat) \( A \)-module, \( \mathcal{F} \) is flat over \( A \) (and hence over \( T = \text{Spec} \ A \)).

The proof of the reverse implication repeats that of the implication (i)\( \Rightarrow \) (ii) in [1, Chapter III, proof of Theorem 9.9]. Let \( \mathcal{F} \) be a flat \( \mathcal{O}_T \)-module; we reduce our consideration to the case where \( X = \text{Proj} \ A[x_0 : \cdots : x_n], \ T = \text{Spec} \ A \) for a Noetherian local ring \( A \). We compute \( H^i(X, \mathcal{F}(m)) \) as the Čech cohomology of the standard open affine covering \( \mathcal{U} \) of the space \( X \). Namely, \( H^i(X, \mathcal{F}(m)) = H^i(C^*(\mathcal{U}, \mathcal{F}(m))) \). Since the sheaf \( \mathcal{F} \) is flat, the term \( C^i(\mathcal{U}, \mathcal{F}(m)) \) is a flat \( A \)-module for all \( i \geq 0 \). If \( i > 0 \), then for \( m \gg 0 \) we have \( H^i(X, \mathcal{F}(m)) = 0 \). Then the Čech complex provides a right resolution for the \( A \)-module \( H^0(X, \mathcal{F}(m)) \), and the sequence
\[
0 \to H^0(X, \mathcal{F}(m)) \to C^0(\mathcal{U}, \mathcal{F}(m)) \to \cdots \to C^n(\mathcal{U}, \mathcal{F}(m)) \to 0
\]
is exact. Since all terms of the Čech complex are flat \( A \)-modules, cutting this exact sequence into exact triples shows that the \( A \)-module \( H^0(X, \mathcal{F}(m)) \) is flat. Then Proposition 8 yield the following identities for all \( n > 0 \) and all \( I_n \subset A \):
\[
\dim H^0(X, \mathcal{F}(m)) \otimes_A A/I_n = n \dim H^0(X, \mathcal{F}(m)) \otimes_A k.
\]
By the isomorphism (14), which was proved independently, the assertions of Theorem 8 are fulfilled.

**ACKNOWLEDGMENTS**

I express my deep and sincere gratitude to Professor Dr. Vasile Brinzanescu (IMAR, Bucharest, Romania) for drawing my attention to the question discussed in this paper. Also, I thank the Institute of Mathematics of the Romanian Academy (IMAR), where part of this work was done, for hospitality and support.
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Received 18/DEC/2012

Translated by THE AUTHOR