GEOMETRY OF THE SPACE OF SECTIONS OF TWISTOR SPACES WITH CIRCLE ACTION

FLORIAN BECK, INDRANIL BISWAS, SEBASTIAN HELLER, AND MARKUS RÖSER

Abstract. We study the holomorphic symplectic geometry of (the smooth locus of) the space of holomorphic sections of a twistor space with rotating circle action. The twistor space has a meromorphic connection constructed by Hitchin. We give an interpretation of Hitchin’s meromorphic connection in the context of the Atiyah–Ward transform of the corresponding hyperholomorphic line bundle. It is shown that the residue of the meromorphic connection serves as a moment map for the induced circle action, and its critical points are studied. Particular emphasis is given to the example of Deligne–Hitchin moduli spaces.

Contents

Introduction 2
Acknowledgements 4
1. Geometry of the space of holomorphic sections of a twistor space 4
1.1. Twistor space 4
1.2. Space of holomorphic sections as a complexified hyperkähler manifold 5
1.3. Alternative description of the holomorphic symplectic form \( \Omega_x \) on \( S' \) 6
2. The hyperholomorphic line bundle and the energy functional 10
2.1. Rotating circle actions and the hyperholomorphic line bundle 10
2.2. The \( S^1 \)-action on \( S \) and the energy functional 12
2.3. Critical Points of \( E \) 14
2.4. The Atiyah–Ward transform of \( L_Z \) 15
2.5. The Atiyah–Ward transform and the meromorphic connection 17
2.6. An alternative construction of Hitchin’s meromorphic connection 19
3. Space of holomorphic sections of the Deligne–Hitchin moduli space 21
3.1. Hitchin’s self-duality equations 22
3.2. The Deligne–Hitchin moduli space 23
3.3. The line bundle on \( \mathcal{M}_{\text{DH}}^{\text{irr}} \) 25
3.4. Irreducible and admissible sections 25
4. Energy functional on sections of the Deligne–Hitchin moduli space 26
4.1. The energy as a moment map 26
4.2. Explicit description of some \( \mathbb{C}^* \)-fixed sections 28
4.3. The energy of a \( \mathbb{C}^* \)-fixed section 31
4.4. The second variation of the Energy at a \( \mathbb{C}^* \)-fixed section 33
4.5. Sections and the degree of the hyperholomorphic line bundle 34
References 35

2010 Mathematics Subject Classification. 53C26, 53C28, 53C43, 14H60, 14H70.
Key words and phrases. Deligne–Hitchin twistor space, self-duality equation, connection, circle action, hyperholomorphic bundle.
Introduction

For a compact Riemann surface $\Sigma$ of genus at least two, and a complex semisimple Lie group $G$, the moduli space of $G$-Higgs bundles on $\Sigma$ has a hyperkähler structure [20]. The corresponding twistor space $\mathcal{M}_{DH}(\Sigma, G)$ is known as the Deligne–Hitchin moduli spaces [44]. These spaces $\mathcal{M}_{DH}(\Sigma, G)$ have been the topic of many papers in recent years; see for example [10] and references therein.

In [6] the last three authors initiated a detailed investigation of the space $\mathcal{S}$ of holomorphic sections of the natural projection $\mathcal{M}_{DH}(\Sigma, G) \to \mathbb{C}P^1$. Our aim here is to go deeper into the study of $\mathcal{S}$. There are two distinct sources of motivation for doing this: One is the integrable systems approach to the theory of harmonic maps from Riemann surfaces into symmetric spaces, the other being the hyperkähler geometry.

We start by explaining the latter source. Let $K$ be a compact real form of a complex semisimple Lie group $G$ with Lie algebra $\mathfrak{g}$. Hitchin’s self-duality equations on $\Sigma$ are given by

\[ F^\nabla + [\Phi \wedge \Phi^\ast] = 0 = \nabla^\nabla \Phi \tag{0.1} \]

[20]. Here $\nabla$ is a $K$-connection and $\Phi \in \Omega^{0,1}(\Sigma, \mathfrak{g})$, which is called a Higgs field, while $\Phi \mapsto \Phi^\ast$ is given by the negative of the Cartan involution on $\mathfrak{g}$ associated with the real form $K \subset G$. Each solution of (0.1) gives rise to a family of flat $G$-connections parametrized by $\mathbb{C}^\ast$

\[ \nabla^\lambda = \nabla + \lambda^{-1}\Phi + \lambda\Phi^\ast \]

that satisfies a reality condition which is determined by the symmetric pair $(G, K)$. The nonabelian Hodge correspondence [40, 26, 12, 11] allows us to invert this process, meaning any (reductive) flat $G$-connection occurs in a suitable $\mathbb{C}^\ast$-family of flat $G$-connections given by a solution of the self-duality equations in (0.1).

Recall from [41] that the Deligne–Hitchin moduli space $\mathcal{M}_{DH}(\Sigma, G)$ is obtained by gluing the Hodge moduli spaces $\mathcal{M}_{Hod}(\Sigma, G)$ and $\mathcal{M}_{Hod}(\Sigma, G)$ of $\lambda$-connections via the Riemann Hilbert correspondence, where $\Sigma$ is the conjugate of $\Sigma$ (see § 3 for details). It enters the picture via the interpretation of the $\mathbb{C}^\ast$-family of flat $G$-connections associated with a solution of (0.1) as a family of $\lambda$-connections, which extends to all of $\mathbb{C}P^1$. From this point of view, we may think of the above $\mathbb{C}^\ast$-family as a holomorphic section of the fibration $\mathcal{M}_{DH}(\Sigma, G) \to \mathbb{C}P^1$ satisfying an appropriate reality condition. This observation fits naturally into the twistor theory of hyperkähler manifolds, as we shall explain next.

The moduli space $\mathcal{M}_{SD}(\Sigma, G)$ of solutions of (0.1) is a (typically singular) hyperkähler manifold which comes equipped with a certain rotating circle action given by $(\nabla, \Phi) \mapsto (\nabla, e^{i\theta}\Phi)$, $\theta \in \mathbb{R}$. Here rotating means that the circle action is isometric, it preserves one of the three Kähler forms while rotating the other two (see § 2.1 and § 3.1 for details). The Deligne–Hitchin moduli space $\mathcal{M}_{DH}(\Sigma, G)$ is the twistor space associated with $\mathcal{M}_{SD}(\Sigma, G)$. The $\mathbb{C}^\ast$-family of flat connections associated with a solution to the self-duality equations can then be interpreted as the twistor line corresponding to the point in $\mathcal{M}_{SD}(\Sigma, G)$ represented by $(\nabla, \Phi)$. In this way, we can view $\mathcal{M}_{SD}(\Sigma, G)$ in a natural way as a subset of the space $\mathcal{S}$ of holomorphic sections of the twistor family $\mathcal{M}_{DH}(\Sigma, G) \to \mathbb{C}P^1$. In fact, $\mathcal{S}$ may be interpreted as a natural complexification of the real analytic hyperkähler manifold $\mathcal{M}_{SD}(\Sigma, G)$. More generally, if $Z$ is the twistor space of a hyperkähler manifold $M$, the twistor construction of hyperkähler metrics produces a hyperkähler metric on the space $M'$ of all real holomorphic sections (with appropriate normal bundle) of the fibration $Z \to \mathbb{C}P^1$. From this point of view, it is a natural question whether $M = M'$ [44]. In the context of the self-duality equations, this question translates into whether every real holomorphic section of $\mathcal{M}_{DH}(\Sigma, G) \to \mathbb{C}P^1$ is actually obtained from a solution to the self-duality equations. In [6] an answer to this question was given in the case $G = \text{SL}(2, \mathbb{C})$ by showing that in general $\mathcal{M}_{SD}(\Sigma, G)$ is strictly contained in the space of real sections. Furthermore, in [6] the real holomorphic sections belonging to $\mathcal{M}_{SD}(\Sigma, G)$ were characterized in terms of a certain $\mathbb{Z}/2\mathbb{Z}$-valued invariant that can be associated with any real section.

Hyperkähler manifolds with rotating circle action are an active field of research in hyperkähler geometry, especially their connection with the well-known hyperkähler/quaternion Kähler correspondence [23, 11, 29, 30]. Haydys [23] has observed that if $(M, \omega_1, \omega_J, \omega_K)$ is a hyperkähler manifold with rotating circle action such that $\omega_I$ is integral, then there exists a complex line bundle $L \to M$ with a unitary connection whose curvature is $\omega_I + d\lambda_I \mu$, where $\mu$ is the moment map for the $S^1$-action with respect to the Kähler form $\omega_I$. This connection is hyperholomorphic, in the sense that its curvature is of type $(1, 1)$ with respect to every complex structure in the family of Kähler structures on $M$ parametrized by $S^2$. The moment map facilitates
a lift of the $S^1$-action to an $S^1$-action on the total space of the principal $S^1$-bundle $P \to M$ associated with $L_Z$. Haydys has shown that the quotient $Q = P/S^1$ for this action carries a quaternionic Kähler metric. The natural $S^1$-action on the principal bundle $S^1$-bundle $P$ descends to an isometric circle action on $Q$, and $M$ can be recovered from it as a hyperkähler quotient of the Swann bundle of $Q$ by the lift of this isometric circle action. This is a brief outline of what is known as the hyperkähler/quaternion Kähler (HK/QK) correspondence. Hitchin in [29, 30] has described the HK/QK correspondence from a purely twistorial point of view. A natural starting point for him is the observation that, by the Atiyah–Ward correspondence, the hyperholomorphic line bundle on $M$ produces a holomorphic line bundle $L_Z$ on the twistor space $Z$ associated to $M$. Hitchin has shown how to construct $L_Z$ directly on the twistor space $Z$ and explained how the circle action determines a distinguished meromorphic connection on $L_Z$. This meromorphic connection plays a key role in the construction, from $Z$, of the twistor space $Z_Q$ of the associated quaternionic Kähler manifold $Q$ in the sense that it determines the contact distribution on $Z_Q$. Therefore, in order to make progress towards the construction of the quaternionic Kähler manifold associated with $\mathcal{M}_{SD}(\Sigma, G)$, it is clearly important to obtain information on this meromorphic connection.

As mentioned earlier, our other motivation comes from the integrable system approach to harmonic maps from $\Sigma$ into symmetric spaces. As is well known, Hitchin’s self-duality equations (0.1) can be interpreted as the gauge-theoretic equations for a (twisted) harmonic map $\Sigma \to G/K$ (a so-called harmonic metric).

One observes that the twisted harmonic maps into other (pseudo-Riemannian) symmetric spaces of the form $G/G_\lambda$ and their duals have also analogous gauge-theoretic interpretations and they also give rise to $\mathbb{C}^*$-families of flat $G$-connections satisfying different reality conditions; the reality condition depends on the target of the harmonic map. For example, if $G = \text{SL}(2, \mathbb{C})$ then $K = \text{SU}(2)$, and the harmonic maps into the symmetric spaces $\text{SL}(2, \mathbb{C})/\text{SU}(2)$, $\text{SL}(2, \mathbb{C})/\text{SU}(1,1)$, $\text{SU}(2)$, $\text{SU}(1,1)$ and also into the hyperbolic disc $\text{SU}(1,1)/\text{U}(1)$ can all be encoded in $\mathbb{C}^*$-families of flat $\text{SL}(2, \mathbb{C})$ connections satisfying appropriate reality conditions; see for instance [22, 47, 27, 13]. Considering these $\mathbb{C}^*$-families as families of $\lambda$-connections with $\lambda \in \mathbb{C}P^1$ we can — at least formally — interpret twisted harmonic maps from $\Sigma$ into various symmetric spaces associated with real forms of $G$ as holomorphic sections of $\mathcal{M}_{DH}(\Sigma, G)$; see [6]. In this way we may view the space of holomorphic sections of the fibration $\mathcal{M}_{DH}(\Sigma, G) \to \mathbb{C}P^1$ as a master space for the moduli spaces of twisted harmonic maps from $\Sigma$ into various (pseudo-Riemannian) symmetric spaces associated with $G$.

This interaction with the theory of harmonic maps has provided useful guidance and intuition for the predecessor [6] and [4] of this project. For example, the new real holomorphic sections for the $\text{SL}(2, \mathbb{C})$-case constructed in [22] are closely related to twisted harmonic maps $\Sigma \to \text{SL}(2, \mathbb{C})/\text{SU}(1,1)$.

In [4], it is proved that on the space $\mathcal{S}$ of holomorphic sections of the twistor family $\mathcal{M}_{DH}(\Sigma, G) \to \mathbb{C}P^1$ there exists an interesting and useful holomorphic functional, called the energy functional, whose evaluation on a real section coming from a harmonic map is (a constant multiple of) the Dirichlet energy of the associated harmonic map. This functional is also intimately related to the Willmore energy of certain immersions of $\Sigma$ into the 3-sphere. It was then shown that this functional has a natural interpretation from the hyperkähler point of view. In fact it can be interpreted as a holomorphic extension of the moment map, associated with the rotating circle action, from the space $M$ of twistor lines to the whole of $\mathcal{S}$. Essentially, it is given by associating to any $s \in \mathcal{S}$ the residue of the meromorphic connections along $s$.

In this article, we continue our study of this setup in the context of the twistor space $Z$ of a general hyperkähler manifold $M$ with a rotating circle action. The natural geometric structure on the space $\mathcal{S}$ of holomorphic sections of $Z \to \mathbb{C}P^1$, viewed as a complexification of the hyperkähler manifold $M$, has been elucidated by Jardim and Verbitsky [35, 36]. They show that $\mathcal{S}$ comes equipped with a certain family of closed two-forms called a trisymplectic structure. In particular, the complexifications of the Kähler forms on $M$ are contained in this family. We describe in this paper how the picture gets enriched by the presence of a rotating circle action. It turns out that the energy functional $\mathcal{E}$ is a moment map for the circle action on $\mathcal{S}$ induced from the circle action on $M$ with respect to a natural holomorphic symplectic form $\Omega_0$ which is a part of the canonical trisymplectic structure on $\mathcal{S}$ (see Theorem 2.3). In particular, the critical points of $\mathcal{E}$ are exactly the fixed points of the circle action.

We also explain how to use the Atiyah–Ward transform to obtain a natural holomorphic extension of the line bundle $L \to M$ to $\mathcal{L} \to \mathcal{S}$, and how the meromorphic connection on $L_Z$ can be described naturally in terms of $\mathcal{L}$ and the data of the circle action (see Theorems 2.14 and § 2.6). This is implicit in Hitchin’s
work [29], but we believe our point of view in terms of the geometry of the space $S$ of holomorphic sections might shed a new light onto his constructions. Moreover, our results allow us to make conclusions about the global structure of $S$ (see Theorem \ref{t:main}).

We then apply this general framework to the space of holomorphic sections of $M_{\mathcal{DH}}(\Sigma, \text{SL}(n, \mathbb{C}))$. Even though $S$ is expected to be singular in this case, the general theory tells us that the critical points of the energy functional are closely related to the fixed points of the $\mathbb{C}^*$-action on $S$. We analyze these $\mathbb{C}^*$-invariant sections in detail, building on \cite{10} (see Proposition \ref{p:critical}). We obtain explicit formulas for the energy of a $\mathbb{C}^*$-invariant section (Proposition \ref{p:energy}) and we describe the second variation of the energy at such a section (Proposition \ref{p:second}). Moreover, we find an explicit formula for the degree of the hyperholomorphic line bundle restricted to a $\mathbb{C}^*$-invariant section (Proposition \ref{p:degree}), from which we are able to show that there exist sections along which the hyperholomorphic line bundle has non-zero degree. Using this it can be deduced that $S$ is in general not connected (Theorem \ref{t:nonconnected}).

The paper is organized as follows. In § 1 we discuss the twistor fibration $\varpi : Z \to \mathbb{C}P^1$ associated with a hyperkähler manifold $M$ and describe the geometric structure induced on the space $S$ of holomorphic sections of $\varpi$ in a way that is suitable for our purpose. We then go on in § 2 to explain how the geometry of $S$ is enriched by the presence of a rotating circle action on $M$ and hence on $Z$. In particular, we explain how the energy functional $E$ ties up naturally with the holomorphic symplectic geometry on $S$ discussed in § 1 and with the meromorphic connection on the hyperholomorphic line bundle. We have added a large amount of background material in § 1 and § 2 to keep the paper somewhat self-contained. The particular example of the Deligne–Hitchin moduli space is described in § 3. In § 4 the abstract geometric framework developed in the first two sections is illustrated in the context of the Deligne–Hitchin moduli space $M_{\mathcal{DH}}(\Sigma, G)$ and the results mentioned above are proved.

**Acknowledgements.** F.B. is supported by the DFG Emmy Noether grant AL 1407/2-1. I.B. is partially supported by a J. C. Bose Fellowship. S.H. is supported by the DFG grant HE 6829/3-1 of the DFG priority program SPP 2026 Geometry at Infinity.

1. Geometry of the space of holomorphic sections of a twistor space

In this section we collect some aspects of the geometry of holomorphic sections of a hyperkähler twistor space that will be used later. Useful references are \cite{31, 35, 36}. See also \cite{37} for the similar, but different, quaternionic Kähler case.

1.1. Twistor space. Let $(M, g_0, I, J, K)$ be a hyperkähler manifold of complex dimension $2d$, where $I, J, K$ are almost complex structures and $g_0$ a Riemannian metric on $M$. The associated Kähler forms are $\omega_L = g_0(L, \cdot, \cdot)$, $L \in \{I, J, K\}$. For convenience, the complex manifolds $(M, I)$ and $(M, -I)$ will sometimes be denoted by simply $M$ and $\overline{M}$ respectively; it will be ensured that this abuse of notation does not create any confusion.

There is a family of Kähler structures on $M$ with complex structures

$$\{I_x := x_1 I + x_2 J + x_3 K \mid x := (x_1, x_2, x_3) \in \mathbb{R}^3\}$$

parametrized by the sphere $S^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$. The twistor space $Z = Z(M)$ of $(M, g_0, I, J, K)$ is a complex manifold whose underlying smooth manifold is $S^2 \times M$ (\cite{31} § 3 (F))). The almost complex structure $I_Z$ of $Z$ at any point $(x, m) \in S^2 \times M$ is

$$I_Z|_{(x, m)} = (I_{\mathbb{C}P^1}|_{T_x S^2}) \oplus (I_x|_{T_m M}),$$

where $I_{\mathbb{C}P^1}$ is the standard almost complex structure on $S^2 = \mathbb{C}P^1$ and $I_x$ is the almost complex structure in (\ref{e:almost}). Here we identify $S^2$ with $\mathbb{C}P^1$ using the stereographic projection from $(-1, 0, 0)$ to the plane in $\mathbb{R}^3$ spanned by the $x_2$ and $x_3$ axes. In particular, $(1, 0, 0) \in S^2$ corresponds to $0 \in \mathbb{C}P^1$ and $(-1, 0, 0)$ corresponds to $\infty \in \mathbb{C}P^1$. Throughout we shall use an affine coordinate $\lambda$ on $\mathbb{C}P^1$, so that $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$.

The hyperkähler structure on $M$ is encoded in the following complex-geometric data on $Z$. The natural projection $S^2 \times M \to S^2$ corresponds to a holomorphic submersion

$$\varpi : Z \to \mathbb{C}P^1$$

(1.2)
with fibers
\[ \varpi^{-1}(\lambda) =: Z_\lambda = (M, I_\lambda). \] (1.3)
Here \( I_\lambda \) is the complex structure on \( M \) in \( \mathbb{C}P^1 \) corresponding to \( \lambda \in \mathbb{C}P^1 \cong S^2 \). For any complex vector bundle \( V \) on \( Z \) and any integer \( m \), we use the notation \( V(m) := V \otimes \varpi^*\mathcal{O}_{\mathbb{C}P^1}(m) \). Let
\[ T_\varpi = T_\varpi Z := (\ker d\varpi) \subset TZ \]
be the relative holomorphic tangent bundle (also called the vertical tangent bundle) for the projection \( \varpi \) in (1.2). Then \( Z \) carries the twisted relative holomorphic symplectic form \( \omega \in H^0(Z, (\Lambda^2 T_\varpi^*)^2) \) given by
\[
\omega = (\omega_J + i\omega_K + 2i\lambda \omega_I + \lambda^2 (\omega_J - i\omega_K)) \otimes \frac{\partial}{\partial \lambda} \in H^0(Z, (\Lambda^2 T_\varpi^*)^2). \] (1.4)
Moreover, \( Z \) carries an anti-holomorphic involution (or real structure)
\[ \tau_Z : Z \rightarrow Z \] (1.5)
given by the map \( S^2 \times M \rightarrow S^2 \times M, (x, m) \mapsto (-x, m) \), using the diffeomorphism \( Z \cong S^2 \times M \). Note that \( \tau_Z \) covers the antipodal map
\[ \tau_{\mathbb{C}P^1} : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1, \lambda \mapsto -\overline{\lambda}^{-1}. \] (1.6)
The relative twisted symplectic form \( \omega \) is real with respect to \( \tau \) in the sense that \( \tau^*\varpi = \omega \).

Let \( S \) denote the space of all holomorphic sections of the projection \( \varpi \) in (1.2); this space is discussed in detail in Section 1.2. We have the embedding
\[ \iota : M \hookrightarrow \mathcal{S} \] (1.7)
that sends any \( m \in M \) to the constant section \( x \mapsto (x, m) \in S^2 \times M \), which is called the twistor line \( s_m \) associated to \( m \). By [31] § 3, (F) it is known that the normal bundle of a twistor line is isomorphic to \( \mathcal{O}_{\mathbb{C}P^1}(1)^{\oplus 2d} \). The space \( \mathcal{S} \) has a real structure defined by
\[ \tau : \mathcal{S} \rightarrow \mathcal{S}, s \mapsto \tau_Z \circ s \circ \tau_{\mathbb{C}P^1}, \] (1.8)
where \( \tau_Z \) and \( \tau_{\mathbb{C}P^1} \) are the involutions in (1.5) and (1.6) respectively; we note that while the section \( \tau(s) \) is holomorphic for fixed \( s \), the map \( \tau \) itself is anti-holomorphic. The manifold \( M \), considered as space of twistor lines, is a component of the fixed point locus \( S^7 \subset \mathcal{S} \), also known as the space of real sections.

We have described the complex-geometric data on \( Z \) induced by the hyperkähler structure on \( M \). Conversely, suppose we have a complex manifold \( M \) with a holomorphic submersion \( \varpi : Z \rightarrow \mathbb{C}P^1 \), a twisted relative symplectic form \( \omega \in H^0(Z, (\Lambda^2 T_\varpi^*)^2) \) and an antiholomorphic involution \( \tau : Z \rightarrow Z \) covering the antipodal map on \( \mathbb{C}P^1 \) satisfying \( \tau^*\varpi = \omega \). Then the parameter space \( M \) of real sections of \( \varpi \) with normal bundle isomorphic to \( \mathcal{O}_{\mathbb{C}P^1}(1)^{\oplus 2d} \) is a (pseudo-)hyperkähler manifold of real dimension \( 4d \), if it is non-empty; see [31] Theorem 3.3].

1.2. Space of holomorphic sections as a complexified hyperkähler manifold. To examine the local structure of \( \mathcal{S} \), for any \( s \in \mathcal{S} \) let \( N_s = s^*TZ/ds(T\mathbb{C}P^1) \) be the normal bundle of \( s(\mathbb{C}P^1) \subset Z \). Since \( s \) is a section of \( \varpi \), we have a canonical isomorphism
\[ s^*T_\varpi Z \cong N_s, \]
where \( T_\varpi Z \subset TZ \) as before is the kernel of the differential \( d\varpi \) of the map \( \varpi \). The following proposition is well-known, however a proof of it is given because parts of the proof will be used later.

**Proposition 1.1.** Let \( Z \) be the twistor space of a hyperkähler manifold \( M \) of complex dimension \( 2d \). Then the set \( \mathcal{S} \) of holomorphic sections of \( \varpi : Z \rightarrow \mathbb{C}P^1 \) is a complex space in a natural way. The tangent space of \( s \in \mathcal{S} \) is \( H^0(\mathbb{C}P^1, N_s) \), and \( \mathcal{S} \) is smooth at a point \( s \in \mathcal{S} \) if \( H^1(\mathbb{C}P^1, N_s) = 0 \). If \( H^1(\mathbb{C}P^1, N_s) = 0 \), then \( \dim T_s\mathcal{S} = 4d \).

**Proof.** For any \( s \in \mathcal{S} \), the sufficiently small deformations of the complex submanifold \( s(\mathbb{C}P^1) \subset Z \) continue to be the image of a section of \( \varpi \). Consequently, \( \mathcal{S} \) is an open subset of the corresponding Douady space of rational curves in \( Z \); see also [10] Theorem 2. In particular, \( T_s\mathcal{S} = H^0(\mathbb{C}P^1, N_s) \) for all \( s \in \mathcal{S} \).
The complex structure of \( S \) around a point \( s \in S \) is constructed in the following way. There are open neighbourhoods \( V_s \subset H^0(\mathbb{C}P^1, N_s) \) and \( U_s \subset S \) of \( 0 \in H^0(\mathbb{C}P^1, N_s) \) and \( s \in S \) respectively, as well as a holomorphic map

\[
k_s : H^0(\mathbb{C}P^1, N_s) \to H^1(\mathbb{C}P^1, N_s),
\]

which is sometimes called the Kuranishi map. Then there is a natural isomorphism

\[
U_s \cong V_s \cap k^{-1}_s(0) \tag{1.9}
\]

taking \( s \) to 0. The complex structure on \( U_s \) is given by the complex structure of \( V_s \cap k^{-1}_s(0) \) using the isomorphism in (1.9).

The statement on the smoothness of \( S \) follows from (1.9), because \( k_s \) is the zero map if \( H^1(\mathbb{C}P^1, N_s) = 0 \). The dimension of \( T_sS \cong H^0(\mathbb{C}P^1, N_s) \) is computed from the Riemann–Roch theorem applied to \( N_s \):

\[
h^0(\mathbb{C}P^1, N_s) = \deg(N_s) + \text{rank}(N_s) = 2d + 2d = 4d.
\]

Here we have used the fact \( \deg N_s = 2d \) for any holomorphic section \( s : \mathbb{C}P^1 \to Z \) (cf. [18, 39]). To see this, note that \( N_s = s^*T_\mathbb{C}P^1 \) and that the twisted relative symplectic form \( \omega \in H^0(Z, (\mathcal{N}^*T_\mathbb{C}P^1)(2)) \) induce an isomorphism \( s^*\omega : N_s \to N_s^* \otimes \mathcal{O}_{\mathbb{C}P^1}(2) \), and thus \( (\det N_s)^{\otimes 2} \cong \mathcal{O}_{\mathbb{C}P^1}(4d) \). \( \square \)

As mentioned before, the normal bundle \( N_{s_m} \) of any twistor line \( s_m, m \in M \), is isomorphic to \( \mathcal{O}_{\mathbb{C}P^1}(1)^{\otimes 2d} \) so that \( H^1(\mathbb{C}P^1, N_{s_m}) = 0 \). Consequently, by Proposition 1.1, the image \( M \subset S \) of the embedding in (1.7) is contained in the smooth locus of \( S \). Define the complex manifold

\[
S' := \{ s \in S \mid N_s \cong \mathcal{O}_{\mathbb{C}P^1}(1)^{\otimes 2d} \} \subset S
\]

which is of complex dimension \( \text{dim} S' = 4d = 2 \dim M \) by Proposition 1.1. Since the vector bundle \( \mathcal{O}_{\mathbb{C}P^1}(1)^{\otimes 2d} \) is semistable, from the openness of the semistability condition, \([38, \text{p. 635, Theorem 2.8(B)}] \), it follows immediately that \( S' \) is an open subset of \( S \).

We want to transfer geometric objects from \( Z \) to \( S \). To this end, it is useful to introduce the correspondence space

\[
\mathcal{F} = \mathbb{C}P^1 \times S' \tag{1.10}
\]

as well as \( \mathcal{F}' = \mathbb{C}P^1 \times S' \).

**Remark 1.2.** The correspondence space is usually described as the space of pairs \((z, \ell)\), where \( \ell \) is a complex line in \( Z \) passing through \( z \in Z \). In our work, the lines \( \ell \) in \( Z \) are treated as sections, meaning \( \ell \in S \). Then we obtain an isomorphism from this space of pairs \((z, \ell)\) to \( \mathcal{F} \) in (1.10) by mapping any \((z, \ell)\) to \((\varpi(z), \ell)\).

Consider the evaluation map

\[
ev : \mathcal{F} \to Z, \quad \ev(\lambda, s) = s(\lambda).
\]

For a fixed \( \lambda \in \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} \) define the map

\[
ev_\lambda : S \to Z_\lambda := \varpi^{-1}(\lambda), \quad \ev_\lambda(s) = \ev(\lambda, s). \tag{1.11}
\]

We will denote the restrictions of \( \ev \) to \( \mathcal{F}' \) and of \( \ev_\lambda \) to \( S' \) by \( \ev \) and \( \ev_\lambda \) respectively.

**Lemma 1.3.** The evaluation map \( \ev : \mathcal{F}' \to Z \) is a surjective holomorphic submersion.

**Proof.** The surjectivity of \( \ev : \mathcal{F}' \to Z \) is clear, since there is a twistor line through each point of \( Z = \mathbb{C}P^1 \times M \). If \((l, V) \in T_{(\lambda, s)} \mathcal{F}' = T_{(\lambda)} \mathbb{C}P^1 \oplus T_{(s)} S' = T_{(\lambda)} \mathbb{C}P^1 \oplus H^0(s^* T_\mathbb{C}P^1) \), then we have

\[
(\ev_\lambda(l, V)) = (ds)_\lambda(l) + V(\lambda).
\]

Now, since the evaluation map \( H^0(\mathcal{O}(1)) \to \mathcal{O}(1) \) at \( \lambda \) is surjective, \( \ev_{\lambda}(s) \) is surjective as well. \( \square \)

We thus obtain a commutative diagram in which each arrow is a holomorphic submersion:

\[
\begin{array}{ccc}
\mathbb{C}P^1 & \xrightarrow{\pi_1} & S \\
\downarrow \pi & & \\
\mathcal{F} = \mathbb{C}P^1 \times S & \xrightarrow{\ev} & Z
\end{array} \tag{1.12}
\]
If $V \to \mathcal{F}'$ is a holomorphic vector bundle such that $H^q((\pi_2^{-1}(s), V), V) = 0$ for all $s \in S'$ and $q' \neq q$, then the $q$-th direct image $(\pi_2)^q V$ is a holomorphic vector bundle on $S'$ (see [21, Ch. 10]) with fibers $((\pi_2), V)_s = H^q(\pi_2^{-1}(s), V)$. Hence the sheaves

$$\mathcal{V} := (\pi_2)_*\ev^*(T_{\pi}Z_{(-1)}), \quad \mathcal{H} := (\pi_2)_*\pi^*O_{\mathcal{C}P^1}(1)$$

(1.13)

are holomorphic vector bundles over $S'$ because $s^\ast T_{\pi}Z_{(-1)} \cong O_{\mathcal{C}P^1}^{2d}$, for any $s \in S'$. Note that $(\pi_2)_*\pi^*W$ is in fact trivial with fiber $H^0(\mathcal{C}P^1, W)$.

**Lemma 1.4.** There is a canonical isomorphism $\mathcal{V} = \mathcal{V} \otimes \mathcal{H}$ (defined in (1.13)), and the bundles $\mathcal{V}$ and $\mathcal{H}$ carry natural holomorphic symplectic forms $\omega_\mathcal{V}$ and $\omega_\mathcal{H}$. Thus, $S'$ comes naturally equipped with the holomorphic Riemannian metric $g = \omega_\mathcal{V} \otimes \omega_\mathcal{H}$.

**Proof.** To prove the first statement observe that

$$T_sS' = H^0(\mathcal{C}P^1, s^\ast T_{\pi}Z) = H^0(\mathcal{C}P^1, s^\ast T_{\pi}Z_{(-1)}) \otimes H^0(\mathcal{C}P^1, O_{\mathcal{C}P^1}(1)) = \mathcal{V}_s \otimes \mathcal{H}_s$$

for every $s \in S'$. This produces the identification $TS' = \mathcal{V} \otimes \mathcal{H}$.

To obtain the symplectic forms note that the twisted relative symplectic structure $\omega \in H^0(Z, (\lambda^2 T_{\pi}Z)(2))$ induces a natural symplectic form on the vector bundle $T_{\pi}Z_{(-1)}$. Since $\ev^\ast T_{\pi}Z_{(-1)}$ is trivial on each fiber $\pi_2^{-1}(s) = \{s\} \times \mathcal{C}P^1$, $s \in S'$, the pull-back $\ev^\ast \omega$ induces a symplectic form $\omega_\mathcal{V}_s$ on

$$\mathcal{V}_s = H^0((\pi_2^{-1}(s), \ev^\ast T_{\pi}Z_{(-1)})) = H^0(\mathcal{C}P^1, s^\ast T_{\pi}Z_{(-1)}).$$

Finally, the symplectic form $\omega_\mathcal{H}_s$ on $\mathcal{H}_s = H^0(\mathcal{C}P^1, O_{\mathcal{C}P^1}(1))$ is induced from the Wronskian on $\pi_1^*O_{\mathcal{C}P^1}(1)$. More explicitly, let $\psi_1, \psi_2 \in H^0(\mathcal{C}P^1, O_{\mathcal{C}P^1}(1)) = \mathcal{H}_s$, and denote by $d\psi_i$ the derivative defined in terms of local trivialisations. Then we set

$$\omega_\mathcal{H}_s(\psi_1, \psi_2) := \psi_1 \otimes (d\psi_2) - (d\psi_1) \otimes \psi_2$$

(1.14)

which is a well-defined element of $H^0(\mathcal{C}P^1, K_{\mathcal{C}P^1} \otimes O_{\mathcal{C}P^1}(1) \otimes 2) = H^0(\mathcal{C}P^1, O_{\mathcal{C}P^1}) = \mathbb{C}$. □

The restriction to $M \subset S'$ of $g$ in Lemma 1.4 coincides with the Riemannian metric $g_0$ on $M$.

The above considerations yield natural distributions $T_{\ev}F' := \ker(\ev)$ and $T_{\ev}S' = \ker(\ev_x)$, $x \in \mathcal{C}P^1$, on $\mathcal{F}$ and $S'$ respectively. Their associated leaves are the fibers

$$\mathcal{F}_z := \ev^{-1}(z) \cong S_z := \ev^{-1}(z), \quad z \in Z,$$

where $x = \varpi(z) \in \mathcal{C}P^1$.

**Lemma 1.5.** For any $x \in \mathcal{C}P^1$, the above integrable distribution $T_{\ev}S'$ is maximally isotropic with respect to $g$ in Lemma 1.4. For two distinct points $x \neq y \in \mathcal{C}P^1$,

$$T_{\ev}S' \cap T_{\ev}S' = \{0\}.$$

**Proof.** We have $T_{\ev}S'|_s = \{V \in H^0(s^\ast T_{\pi}Z) \mid V(x) = 0\}$, and this subbundle is of rank $2d = \frac{1}{2} \dim S'$. Now note that $\ev^\ast T_{\pi}(-1)$ is trivial on $\pi_2^{-1}(s) = \{s\} \times \mathcal{C}P^1$. If we choose an affine coordinate $\lambda$ on $\mathcal{C}P^1$ such that $\lambda(x) = 0$ and view $\lambda$ as an element of $H^0(\mathcal{C}P^1, O_{\mathcal{C}P^1}(1))$, we may therefore write any $V \in T_{\ev}S'|_s$ in the form $V = v \otimes \lambda$. Thus, if we evaluate $g(V_1, V_2)$ at $x \in \mathcal{C}P^1$ it follows from the formula for $\omega_\mathcal{H}$ (see (1.14)) that we get zero.

Recall that $T_sS' = H^0(\mathcal{C}P^1, s^\ast T_{\pi}Z) = H^0(\mathcal{C}P^1, O_{\mathcal{C}P^1}(1) \otimes 2d)$. Any holomorphic section of $O_{\mathcal{C}P^1}(1)$ vanishing at two distinct points of $\mathcal{C}P^1$ must be identically zero. This implies the second part of the lemma. □

Given any $x \in \mathcal{C}P^1$ we can thus define an associated non-degenerate holomorphic two-form $\Omega_x$ on $S'$ by taking the natural skew-form on

$$TS' = T_{\ev}S' + T_{\ev \tau_{\mathcal{C}P^1}(x)}S'$$

(1.15)

(recall that $\tau_{\mathcal{C}P^1}(x) \neq x$) induced from the metric $g$. Write $V, W \in TS'$ as $V = V_x + V_{\tau_{\mathcal{C}P^1}(x)}$, $W = W_x + W_{\tau_{\mathcal{C}P^1}(x)}$ with respect to the splitting in (1.15), and put

$$\Omega_x(V, W) := -i(g(V_x, W_{\tau_{\mathcal{C}P^1}(x)}) - g(V_{\tau_{\mathcal{C}P^1}(x)}, W_x)).$$

(1.16)

---

1We identify a holomorphic vector bundles with its locally free analytic sheaf of sections.
We may now define an endomorphism $I_x \in H^0(S', \text{End}(T_S'))$ via
\begin{equation}
I_x = \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix} \tag{1.17}
\end{equation}
again with respect to the splitting in (1.15). Then we see immediately that $I_x$ is orthogonal with respect to $g$ and satisfies the equation
\[\Omega_x = g(I_x - , -).\]

Next consider the map
\[\phi_x := ev_x \times ev_{\tau_{CP^1}(x)} : S \rightarrow Z_x \times Z_{\tau_{CP^1}(x)} = Z_x \times \overline{Z_x}. \tag{1.18}\]

**Proposition 1.6.** The map $\phi_x$ in (1.18) restricts to a local biholomorphism $\phi_x : S' \rightarrow Z_x \times \overline{Z_x}$.

**Proof.** Clearly, the spaces have the common complex dimension $4d$. The kernel of the differential $d\phi_x$ is given by $T_{ev_x}S' \cap T_{ev_{\tau_{CP^1}(x)}}S'$, and $T_{ev_x}S' \cap T_{ev_{\tau_{CP^1}(x)}}S' = \{0\}$, by Lemma 1.5. The proposition follows. \(\square\)

Now we discuss how the above data interact with the real structure $\tau$ on $S'$ defined in (1.2). Let
\[M' := (S')^\tau\]
be the space of real sections so that we have an embedding $M \hookrightarrow M'$ induced by (1.7). Hence $S'$ is a natural complexification of the real analytic smooth manifolds $M'$. Note that in some examples $M$ is all of $M'$ (e.g. for the standard flat hyperkähler manifolds $\mathbb{CP}^d$) but not always (see Example 4.11).

For any $s \in M'$ in (1.9), the differential $d\tau : T_xS' \rightarrow T_xS'$ is $\mathbb{C}$-antilinear, involutive and satisfies the equation
\[d\tau(T_{ev_x}S') = T_{ev_{\tau_{CP^1}(x)}}S'.\]

Indeed, we have for $V \in T_xS' = H^0(s^*T_{CP^1}Z)$ the formula
\[d\tau(V)(x) = d\tau_2(V(\tau_{CP^1}(x))).\]

Thus, if $V(x) = 0$, then $d\tau(V)(\tau_{CP^1}(x)) = 0$. This implies in particular that if $s \in M'$, and $V \in (T_xS')^\tau$ is real, then the section $V \in H^0(\mathbb{CP}^1, s^*T_{CP^1}Z)$ is either identically zero or it is nowhere vanishing (a nonzero holomorphic section of $O_{\mathbb{CP}^1}(1)$ cannot vanish at two distinct points of $\mathbb{CP}^1$). As a consequence, the map $ev_x$ in (1.11) gives a local diffeomorphism from $M'$ to $Z_x = (M, I_x)$. Moreover, $I_x$ is real in the sense that $d\tau \circ I_x = I_x \circ d\tau$ and therefore preserves $TM' \cong (TS')^\tau$, so it defines an almost complex structure on $M'$. Consequently, we obtain a hypercomplex structure $\{(ev_x)^*I_x \mid x \in \mathbb{CP}^1\}$ on $M'$.

**Remark 1.7.** The differential of the inclusion map $M \hookrightarrow M'$ is a $\mathbb{R}$-linear isomorphism $T_mM \xrightarrow{\sim} T_{sM}M'$ whose complexification yields an identification $T_{sM}S' \cong T_{sM}M \otimes \mathbb{C}$. Under this identification, the decomposition in (1.2) is mapped to the natural decomposition $T_{sM}M \otimes \mathbb{C} \cong T_{sM}^0M \oplus T_{sM}^{1,0}M$ with respect to the complex structure $I_x$.

The real tangent vectors at $s \in M'$ can be described as follows: Let $V \in T_{ev_x}S'|_s$. Then the tangent vector
\[V + \tau(V) \in T_{ev_x}S'|_s \oplus T_{ev_{\tau_{CP^1}(x)}}S'|_s\]
is obviously real and we have
\[g(V + \tau(V), V + \tau(V)) = 2g(V, \tau(V)).\]

Since the twisted relative symplectic form $\omega$ on $Z$ satisfies the condition $\tau^*\omega = \overline{\omega}$, it follows, by working through the definition of $g$, that $g$ is real in the sense that $\tau^*g = \overline{g}$. Hence $g$ induces a real-valued pseudo-Riemannian metric on $M'$. Note that this immediately forces the restriction of $\Omega_x$ to $M'$ to be real as well. Pulled back to $(M, I_x)$, the form $\Omega$ is just the Kähler form associated with the pseudo-Riemannian metric and the hermitian almost complex structure $I_x$.

In summary, we have obtained the following result.

**Proposition 1.8.** For each $x \in \mathbb{CP}^1$ the form $\Omega_x$ constructed in (1.17) defines a holomorphic symplectic form on each component of $S'$ that intersects $M' = (S')^\tau$. On $M \subset M'$ it induces the Kähler form $\omega_x$. 
Remark 1.9. So far we have not shown that $\Omega_x$ is actually closed. One way to show that is to use the Atiyah–Ward transform which we will discuss in detail in Section 2.4. The bundle $\mathcal{V}$ can be seen to arise from this transform applied to the bundle $T_x Z(-1)$. As such it carries a natural connection. Its tensor product with the trivial connection on the trivial bundle $\mathcal{H}$ can be shown to give the Levi-Civita connection of the holomorphic Riemannian manifold $(TS', g)$. The form $\Omega_x$ can then be shown to be parallel, from which its closedness follows. In the next subsection we will give another, more direct proof that $\Omega_x$ is closed.

1.3. Alternative description of the holomorphic symplectic form $\Omega_x$ on $S'$. Fix a point $x \in \mathbb{C}P^1$. We now give an alternative description of $\Omega_x$ which is better suited for computations. This alternative description also shows that $\Omega_x$ can be extended to a holomorphic two-form on $S$, which is typically strictly larger than $S'$.

Consider the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{ev} & F = \mathbb{C}P^1 \times S \\
\pi_1 & \downarrow & \pi_2 \\
\mathbb{C}P^1 & \xrightarrow{\pi} & S.
\end{array}
$$

We observe that for any $k \in \mathbb{Z}$ the bundle $(\pi_2)_* \pi_1^* O_{\mathbb{C}P^1}(k)$ on $S$ is trivial with fiber $H^0(\mathbb{C}P^1, O_{\mathbb{C}P^1}(k))$.

Starting with the twisted relative symplectic form $\omega \in H^0(Z, (\Lambda^2 T_x Z)(2))$, its pull-back $ev^* \omega$ defines a holomorphic section of $\Lambda^2 \pi_2^*(T^* S)(2)$. Invoking push-forward to $S$ we obtain a vector-valued holomorphic two-form

$$\Omega \in H^0(S, \Lambda^2 T^* S) \otimes H^0(\mathbb{C}P^1, O_{\mathbb{C}P^1}(2)).$$

Now note that $H^0(\mathbb{C}P^1, O_{\mathbb{C}P^1}(2))$ is the space of holomorphic vector fields on $\mathbb{C}P^1$ and therefore has the structure of a Lie algebra isomorphic to $sl_2(\mathbb{C})$. Fix an affine coordinate $\lambda$ on $\mathbb{C}P^1$ such that $\lambda(x) = 0$ and $\tau(\lambda) = -\lambda^{-1}$. Then we obtain the following basis of $H^0(\mathbb{C}P^1, O_{\mathbb{C}P^1}(2))$:

$$e = \frac{\partial}{\partial \lambda}, \quad h = -2\lambda \frac{\partial}{\partial \lambda}, \quad f = -\lambda^2 \frac{\partial}{\partial \lambda},$$

which satisfies the standard relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$ 

We can now use the Killing form $\kappa$ on $sl_2(\mathbb{C}) = H^0(\mathbb{C}P^1, O_{\mathbb{C}P^1}(2))$ to define

$$\tilde{\Omega}_x := \frac{1}{2\pi} \kappa(\Omega, h) \in H^0(S, \Lambda^2 T^* S). \quad (1.20)$$

Note that $\tilde{\Omega}_x$ in (1.20) is independent of the affine coordinate $\lambda$ at $x$. Using $\kappa(h, h) = \text{tr}(\text{ad}_h \circ \text{ad}_h) = 8$ and $\kappa(h, e) = 0 = \kappa(h, f)$ we may rewrite this as follows. A general element $A \in H^0(\mathbb{C}P^1, O_{\mathbb{C}P^1}(2))$ is of the form

$$A = A_e e + A_h h + A_f f = (A_e - 2\lambda A_h - \lambda^2 A_f) \frac{\partial}{\partial \lambda} =: A(\lambda) \frac{\partial}{\partial \lambda},$$

with $A_e, A_h, A_f \in \mathbb{C}$. Then

$$\frac{1}{2\pi} \kappa(A, h) = -i A_h = \frac{1}{2\pi} \kappa|_{\lambda=0} A(\lambda).$$

With this setup in place, we may therefore write for $s \in S'$, and tangent vectors

$$V_s, W_s \in T_s S' = H^0(\mathbb{C}P^1, s^* T_\pi Z)$$

the following:

$$\tilde{\Omega}_x|_s(V, W) = \frac{1}{2\pi} \kappa(\text{ev}^* (\omega(\pi_2^* V, \pi_2^* W), h) = \frac{i}{2\pi} \kappa|_{\lambda=0} (\omega(\lambda) (V_s(\lambda), W_s(\lambda))). \quad (1.21)$$

Recall the non-degenerate two-form $\Omega_x$ defined in (1.10).

Theorem 1.10. Let $x \in \mathbb{C}P^1$.

a) The two-form $\tilde{\Omega}_x \in H^0(S, \Lambda^2 T^* S)$ defined in (1.21) restricts to a holomorphic symplectic form on $S'$ which is real with respect to $\tau$.

b) Over the open subset $S' \subset S$,

$$\Omega_x|_{S'} = \tilde{\Omega}_x|_{S'}.$$ 

In particular, $(S', \Omega_0)$ is a complexification of the (real analytic) Kähler manifold $(M', \omega_I)$, where $\omega_I$ is the Kähler form associated to $0 \in \mathbb{C}P^1$.

c) The distributions $T_{ev_x} S'$ and $T_{ev_{\mathbb{C}P^1(x)}} S'$ are Lagrangian with respect to $\tilde{\Omega}_x$. 
Remark 1.11. Let $\omega$ be a Kähler form on $M$ such that the image of the map $ev : U \times B_x \to Z$ is contained in a relative Darboux chart $U'' \subset Z$ for $\omega$ around $s(x)$. Let $(\lambda, v_i, \xi_i)$, $i \in \{1, \cdots, d\}$, be these Darboux coordinates. In the following we restrict to the case $d = 1$ because the general case works exactly the same way. Thus we write $v = v_1$, $\xi = \xi_1$. Note that in these coordinates $\omega = d\pi v \wedge d\pi \xi$, where $d\pi$ is the relative differential with respect to $\pi$.

Without loss of generality, we may assume that under the isomorphism $N_s \cong s^* T\pi Z \cong \mathcal{O}_{CP^1}(1)^{\oplus 2}$, the sections $s^* \frac{\partial}{\partial v}$ and $s^* \frac{\partial}{\partial \xi}$ form a frame for the first and the second summand respectively. Any two holomorphic vector fields $V, W$ on $U' \subset H^0(\mathbb{C}P^1, N_s)$ are expressed as

$$V_s' = (a_1 + a_2 \lambda) s^* \frac{\partial}{\partial v} + (b_1 + b_2 \lambda) s^* \frac{\partial}{\partial \xi},$$

$$W_s' = (c_1 + c_2 \lambda) s^* \frac{\partial}{\partial v} + (d_1 + d_2 \lambda) s^* \frac{\partial}{\partial \xi},$$

for $s' \subset U$, where $a_i, b_i, c_i, d_i$ are holomorphic functions in $s' \subset U$. Note that in (1.22) we restrict the global sections $V_{s'}, W_{s'}$ of $N_s$ over $\mathbb{C}P^1$ to $B_x$. Now we compute

$$\omega_s(V_{s'}, W_{s'}) = dv \wedge d\xi \left( (a_1 + a_2 \lambda) \frac{\partial}{\partial v} + (b_1 + b_2 \lambda) \frac{\partial}{\partial \xi}, (a_1' + a_2' \lambda) \frac{\partial}{\partial v} + (b_1' + b_2' \lambda) \frac{\partial}{\partial \xi} \right)$$

$$= (a_1 + a_2 \lambda)(b_1' + b_2') - (a_1' + a_2' \lambda)(b_1 + b_2).$$

Thus we have

$$\tilde{\Omega}_x|_{s'}(V_{s'}, W_{s'}) = \frac{i}{2} \frac{\partial}{\partial \lambda}|_{\lambda=0} \omega_s(V_{s'}, W_{s'}) = \frac{i}{2} \left( (a_1 b_2' + b_1' a_2) - (a_1' b_2 + a_2' b_1) \right).$$

Here it is crucial that $\tilde{\Omega}$ is linear in functions on $U$. If we choose $(a_1, a_2, b_1, b_2)$ as coordinates on $U$, we see that

$$\tilde{\Omega}_x = \frac{i}{2} (da_1 \wedge db_2 + da_2 \wedge db_1).$$

Hence $\tilde{\Omega}_x$ is a holomorphic symplectic form on $S'$.

By explicitly comparing $\Omega|_{s}$ and $\tilde{\Omega}_x|_{s}$, we see that $\Omega_x = \tilde{\Omega}_x$. This implies the remaining claims because they have been established for $\Omega_x$.

Remark 1.11. Both definitions of $\Omega$ will be used. For example, (1.21) makes sense on all of $S$ and thus can be evaluated on any holomorphic section $s$, even if we do not know the normal bundle of $s$.

2. The hyperholomorphic line bundle and the energy functional

2.1. Rotating circle actions and the hyperholomorphic line bundle. Now assume that $M$ is equipped with an action of $S^1$ that preserves the Riemannian metric $g_0$ and also preserves the family of complex structures $\{I_x\}_{x \in S^3}$. We also assume that the resulting action of $S^1$ on $S^2$ is nontrivial. This implies that the $S^1$-action on $M$ preserves the associative algebra structure of $\mathbb{R} \cdot Id_M \oplus \mathbb{R} \cdot I \oplus \mathbb{R} \cdot J \oplus \mathbb{R} \cdot K$. Consequently, the action of $S^1$ on $S^2$ is a nontrivial rotation. This means that without loss of generality we may assume that the $S^3$-action on $M$ preserves $I$ (thus also preserves $-I$) and rotates the plane spanned by $J, K$ in the standard way. We therefore call this a rotating circle action. The Kähler forms $\omega_I, \omega_J, \omega_K$ and the Killing vector field $X$ on $M$ associated with this circle satisfy the following:

$$\mathcal{L}_X \omega_I = 0, \quad \mathcal{L}_X \omega_J = \omega_K, \quad \mathcal{L}_X \omega_K = -\omega_J.$$ 

Note that this condition implies that the Kähler forms $\omega_J, \omega_K$ are exact, and therefore the manifold $M$ must necessarily be non-compact.
The above $S^1$-action on $M$ evidently induces a holomorphic $S^1$-action on $Z$. In terms of the identification $Z = CP^1 \times M$ of the underlying smooth manifold, this action of $S^1$ is given by

$$\zeta(\lambda, m) = (\zeta(\lambda, \zeta.m), \zeta \in S^1).$$

(2.1)

Here $\lambda$ is an affine coordinate on $CP^1$ such that $I = I_0$. The $C^\infty$-vector field on $Z$ associated to the $S^1$-action in (2.1) will be denoted by $Y$. Note that $Y|_{Z_0 \cup Z_\infty}$ (see (4.3)) is actually a holomorphic vector field on the divisor $Z_0 \cup Z_\infty = (M, I) \cup (M, -I)$, because the $S^1$-action on $M$ preserves both $I$ and $-I$.

We normalize the affine coordinate $\lambda$ on $CP^1$ such that the antipodal map $S^2 \to S^2$, $x \mapsto -x$, corresponds to the map $CP^1 \to CP^1$, $\lambda \mapsto -\lambda^{-1}$. The coordinate $\lambda$ is then uniquely determined up to multiplication by a constant phase $e^{i\theta}$, $\theta \in [0, 2\pi]$.

Clearly, the $S^1$-action on $Z$ is compatible with the real structure $\tau_Z$ in (15) in the sense that

$$\tau_Z(\zeta, z) = \zeta^{-1}.\tau_Z(z) = \zeta.\tau_Z(z)$$

(2.2)

for all $z \in Z$ and $\zeta \in S^1$. It is straightforward to check that for any $\zeta \in S^1$ the twisted relative symplectic form $\omega$ in (14) satisfies the equation

$$\zeta^*\omega = \omega,$$

i.e., $\omega$ is $S^1$-invariant.

Let $\mu : M \to i\mathbb{R} = u(1)$ be a moment map with respect to $\omega_f$ for the $S^1$-action on $M$. Note that since our moment map is complex-valued, the moment map equation takes the form

$$d\mu(-) = i\omega_f(X, -).$$

(2.3)

Haydys has shown in [23] that the 2-form $\omega_f + i d\zeta_\lambda \mu$ is of type $(1, 1)$ with respect to every complex structure $I_\lambda$, $\lambda \in CP^1$. Thus, if $[\omega_f/2\pi] \in H^2(M, \mathbb{Z})$, then there exists a $C^\infty$-hermitian line bundle $(L_M, h_M) \to M$ with a compatible hermitian connection $\nabla_M$ whose curvature is $\omega_f + i d\zeta_\lambda \mu$. Consequently, if $\nabla_M^{(0,1)}$ is the $(0,1)$-part of $\nabla_M$ with respect to the complex structure $I_\lambda$, we obtain a holomorphic line bundle $(L_M, \nabla_M^{(0,1)}) \to (M, I_\lambda)$. This implies that $L_Z = (q^*L_M, (q^*\nabla_M)^{(0,1)}) \to Z$ is a holomorphic line bundle over the twistor space $Z$ of $M$. Here $q : Z \to M$ is the $C^\infty$ submersion given by the natural projection $S^2 \times M \to M$. Note that the Chern connection of the hermitian holomorphic line bundle $(L, e^{-\mu}h_M) \to (M, I)$ has curvature $\omega_f$, i.e., it is a prequantum line bundle on the Kähler manifold $(M, \omega_f)$.

In [28] Hitchin provided a twistorial description of the line bundle $L_Z$ exhibiting a natural meromorphic connection $\nabla$ on $L_Z$. To recall this, observe that the $S^1$-action on $Z$ covers the standard $S^1$-action on $CP^1$, and therefore the associated holomorphic vector field $Y$ on $Z$ is $\omega$-related to

$$\sigma := i\lambda \frac{\partial}{\partial \lambda}$$

on $CP^1$. Viewing $\sigma$ as a section of $\pi^*O_{CP^1}(2)$ which vanishes on the divisor $D := Z_0 \cup Z_\infty$, we have the short exact sequence

$$0 \to T^*Z \xrightarrow{\sigma} T^*Z(2) \to T^*Z(2)|_D \to 0$$

(2.4)

of sheaves on $Z$. Hitchin then constructs from the $S^1$-action a certain element $\varphi \in H^0(D, T^*Z(2)|_D)$. Explicitly, using the $C^\infty$-splitting $T^*Z = T^*M \oplus T^*CP^1$ we can write $\varphi$ in terms of the data on $M$ as

$$\varphi = \left(\frac{1}{2}(d\mu \partial_\mu + i d\zeta_\lambda \mu) \otimes \frac{\partial}{\partial \lambda}, (\mu \partial_\lambda) \otimes \frac{\partial}{\partial \lambda}\right) = \left(-\frac{1}{2}Y\omega_\lambda = 0, (\mu \partial_\lambda) \otimes \frac{\partial}{\partial \lambda}\right).$$

Note that $\varphi$ satisfies the equation

$$\varphi|_{T^*Z|_D} = \iota_Y\omega,$$

(2.5)

where $\omega$ is the relative symplectic form on $Z$. Note that since the vector field $Y|_D$ is vertical, the formula in (2.5) makes sense. From the long exact sequence of cohomologies associated to (2.4)

$$0 \to H^0(Z, T^*Z) \xrightarrow{\sigma} H^0(Z, T^*Z(2)) \to H^0(D, T^*Z(2)|_D) \xrightarrow{\delta} H^1(Z, T^*Z) \to \ldots,$$

we then obtain that

$$\alpha_L := \delta(\varphi) \in H^1(Z, T^*Z).$$
In fact, this element $\alpha_L$ lies in the image of $H^1(\mathbb{Z}, \Omega^1_{Z,cl})$, where $\Omega^1_{Z,cl}$ denotes the sheaf of closed 1-forms on $Z$. The above class $\alpha_L$ therefore defines an extension

$$0 \longrightarrow O \longrightarrow E \longrightarrow TZ \longrightarrow 0$$

such that $E \rightarrow TZ$ is a holomorphic Lie algebroid. If $\alpha_L$ is integral, this is the Atiyah algebroid of a line bundle $L_Z$. Thus, $\alpha_L$ is simply the Atiyah class of the line bundle $L_Z$ in this case. Explicitly, relative to some open cover $U = \{U_i\}$ of $Z$, we have

$$(\alpha_L)_{ij} = g^{-1}_{ij}dg_{ij}$$

and \{$g_{ij} \in H^0(U_i \cap U_j, \mathcal{O}_Z)$\} is a cocycle representing $L_Z$. Since on the other hand $\alpha_L = \delta(\phi)$, we may rewrite this using the definition of the connecting homomorphism $\delta$. Let $\phi$ be given by $\{\varphi_i \in H^0(U_i \cap D, T^*Z(2))\}$ and take extensions $\{\tilde{\varphi}_i \in H^0(U_i, T^*Z(2))\}$. Then a representative for $\alpha_L = \delta(\phi)$ is given by

$$(\alpha_L)_{ij} = \frac{\tilde{\varphi}_i - \tilde{\varphi}_j}{\sigma}.$$ 

Thus, $A_t = \frac{\tilde{\varphi}}{\sigma}$ are the local 1-forms of a meromorphic connection $\nabla$ on $L_Z$, which have a simple pole along the divisor $D$. Its residue along $D$ is given by $\varphi$.

Moreover, Hitchin shows that the curvature $F$ of the meromorphic connection $\nabla$ has the following properties:

- $\iota_Y F = 0$ and $Y$ spans the annihilator of $F$.
- $F = \frac{\omega}{\sigma}$ on $T\mathbb{P}^1|_{\mathbb{P}^1 \smallsetminus \{Z\}}$, where $\omega \in H^0((\Lambda^2 T\mathbb{P}^1)(2))$ is the relative symplectic form on $Z \rightarrow \mathbb{C}P^1$.

We may thus think of $(L_Z, \nabla)$ as a “meromorphic relative prequantum data” for the meromorphic relative symplectic form $\frac{\omega}{\sigma}$.

### 2.2. The $S^1$-action on $S$ and the energy functional

The $S^1$-action on $Z$ obtained from the $S^1$-action on $M$ produces an $S^1$-action on $S$, which is constructed as follows:

$$(\zeta, s)(\lambda) = \zeta, s(\zeta^{-1}\lambda)$$

for all $s \in S, \zeta \in S^1$ and $\lambda \in \mathbb{C}P^1$. The evaluation map $\text{ev} : \mathbb{C}P^1 \times S \rightarrow Z$ is evidently equivariant with respect to the diagonal action of $S^1$ on $\mathbb{C}P^1 \times S$ and the $S^1$-action on $Z$. Moreover, the $S^1$-action on $S$ is compatible with $\tau$ in the sense that

$$\tau(\zeta, s) = \zeta, \tau(s)$$

for all $s \in S$ and $\zeta \in S^1$. In particular, the $S^1$-action on $S$ preserves the subset $S^\tau$ fixed by $\tau$.

**Proposition 2.1.** The holomorphic two-form $\Omega_0$ on $S$ constructed in (1.19) is $S^1$-invariant.

**Proof.** We use the description of $\Omega_0$ given in §1.3. Recall that $\omega$ is $S^1$-invariant and $\text{ev} : F \rightarrow S$ is $S^1$-equivariant. Moreover, the Killing form on $\mathfrak{s}\mathfrak{u}_2 = H^0(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1}(2))$ and the element

$$h = 2\sigma \in H^0(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1}(2))$$

are also $S^1$-invariant. Hence $\Omega_0 = \frac{1}{2\pi}(\text{ev}^* \Omega, h)$ too must be $S^1$-invariant. \hfill $\square$

Let $Y$ be the vector field on $Z$ which is induced by the $S^1$-action. Since the $S^1$-action commutes with $\tau_Z$ (see (2.2)), $Y$ is $\tau_Z$-invariant, meaning $d\tau_Z(Y) = Y \circ \tau_Z$.

**Lemma 2.2.** Let $s \in S^\tau$, and let $X$ be the vector field associated to the $S^1$-action on $S^\tau$. Then $X_s \in T_s S^\tau \cong H^0(\mathbb{C}P^1, N_s)$ is given by

$$X_s(\lambda) = Y_s(\lambda) - i\lambda s(\lambda).$$

(2.6)

Note that $X_s$ is indeed vertical because

$$d\omega(Y_s(\lambda) - i\lambda s(\lambda)) = i\lambda \frac{\partial}{\partial \lambda} - i\lambda \frac{\partial}{\partial \lambda} = 0.$$
Proof. This is a direct computation: Let \( s \in S' \) and \( \lambda \in \mathbb{C}P^1 \). Then
\[
X_s(\lambda) = \frac{d}{dt} \bigg|_{t=0} (e^{it}s)(\lambda) = \frac{d}{dt} \bigg|_{t=0} e^{it} (s(e^{-it}\lambda)) = Y_s(\lambda) - i\lambda \dot{s}(\lambda).
\]
More invariantly, we can use the vector field \( \sigma \) for the standard \( S^1 \)-action on \( \mathbb{C}P^1 \), i.e., \( \sigma(\lambda)(\lambda) = i\lambda \frac{\partial}{\partial \lambda} \) to rewrite the relation of Lemma 2.2 as
\[
\text{dev} \circ X = Y \circ \text{ev} - \text{dev} \circ \sigma.
\]
The fundamental vector field \( Y_F \) for the diagonal \( S^1 \)-action on \( \mathcal{F} = \mathbb{C}P^1 \times S \) is given by
\[
Y_F(\lambda, s) = \pi_1^* \sigma(\lambda) + \pi_2^* X(s).
\]
Thus, the formulae (2.6) and (2.7) just become \( \text{dev} \circ Y_F = Y \circ \text{ev} \).

The recent paper [4] discusses the holomorphic energy function
\[
\mathcal{E} : S \longrightarrow \mathbb{C}, \quad \mathcal{E}(s) = \frac{i}{2\pi} \text{res}_{\lambda=0}(s^* D).
\]  
(2.8)

Note that \( \mathcal{E}(s) = s^* \varphi \in T^* \mathbb{C}P^1(2)|_{(0, \infty)} \cong \mathcal{O}_{\mathbb{C}P^1}|_{(0, \infty)} \), so this indeed is a complex number.

The function \( \mathcal{E} \) has the property that \( \iota^* \mathcal{E} = \mu \) for the natural inclusion \( \iota : M \longrightarrow S \) in (1.7). The residue formula in (2.8) implies that
\[
\tau^* \mathcal{E}(s) = \mathcal{E}(s) + \deg(s^* L_Z).
\]  
(2.9)

To give an explicit formula for the energy \( \mathcal{E} \), we recall the section \( \varphi_0 = \varphi|_{Z_0} \in \Gamma(Z_0, T^*_Z(2)|_{Z_0}) \) defined above. Contracting \( d\lambda \otimes \frac{\partial}{\partial \lambda} \), we therefore obtain that
\[
\mathcal{E}(s) = s^* \varphi_0 = -\frac{1}{2} \iota_Y \omega (\dot{s}(0) - \dot{s}_s(0)) + \mu(s(0))
\]
for \( \dot{s}(0) = d\varphi S\frac{\partial}{\partial \lambda} \) etc. The difference \( \dot{s}(0) - \dot{s}_s(0) \) accounts for the fact that we are working with the \( C^\infty \)-splitting induced by twistor lines; see [4] for the details.

Theorem 2.3. The energy function \( \mathcal{E} : S' \longrightarrow \mathbb{C} \) is a momentum map for the natural \( S^1 \)-action and holomorphic symplectic structure \( \Omega_0 \) on \( S' \), i.e.,
\[
d\mathcal{E}(-) = i \Omega_0(X, -).
\]
In particular, the \( S^1 \)-fixed points in \( S' \) are precisely the critical points of the function \( \mathcal{E} \).

Since \( \Omega_0|_M = \omega_I \), this is a holomorphic extension of (2.3).

Proof. Take any \( s \in S' \), and set \( m := s(0) \). We first compute \( d_s \mathcal{E}(V) \) for
\[
V \in T_{ev_s}S'|_s \subset T_s S' = H^0(\mathbb{C}P^1, N_s).
\]
By definition \( V(0) = 0 \), so \( V(\lambda) = v \otimes \lambda \) for some \( v \in H^0(\mathbb{C}P^1, s^* T_{ev_s}Z(-1)) \) (see Lemma [1,4]). Further \( V \) is representable by a family \( s_t \) of sections with \( s_t(0) = m \) and \( \partial_{t=0}s_t(\lambda) = V(\lambda) \). Note that \( \partial_{\lambda=0} V(\lambda) \) is well-defined because \( V(0) = 0 \). Then a local computation shows that
\[
\partial_{t=0} \dot{s}_t(0) = \partial_{\lambda=0} V(\lambda) = v \in T_{m} M.
\]
Hence we conclude that
\[
d_s \mathcal{E}(V) = \frac{d}{dt} \bigg|_{t=0} \left( -\frac{1}{2} \iota_Y \omega (\dot{s}_t(0) - \dot{s}_m(0)) + \mu(m) \right) = -\frac{1}{2} \iota_Y \omega(v).
\]
Next we consider \( X_s \in H^0(\mathbb{C}P^1, N_s) \), the fundamental vector field \( X \) evaluated at \( s \). Since \( Y(m) = X_s(0) \) (see Lemma [2,2]), we conclude that
\[
\Omega_0|_s(X_s, V) = \frac{1}{2} \iota_Y \omega_s(\lambda)(X_s(\lambda), V(\lambda)) = \omega(Y(0), v) = \frac{1}{2} \iota_Y \omega(v).
\]
It remains to prove the claim for \( V \in T_{ev_s}S'|_s \). To this end, observe that \( d(\tau^* \mathcal{E}) = d\mathcal{E} \) by (2.8) because \( s \mapsto \deg(s^* L_Z) \) is locally constant. Further we know \( d\tau(T_{ev_s}S') = \tau^* T_{ev_s}S' \), so that any \( V \in T_{ev_s}S'|_s \).
is of the form $d\tau_{\tau(s)}(W)$ for $W \in \overline{T_{ev_{0}} S'}_{\tau(s)}$. Then we compute using the previous result and $\tau^*\Omega = \overline{\Omega}$,

$$
\begin{align*}
    d\mathcal{E}(V) &= \tau^*d\mathcal{E}(V) \\
          &= d\mathcal{T}_{\tau(s)}(W) \\
          &= \overline{\Omega}_{\tau(s)}(X_{\tau(s)}, d\tau_{\tau(s)}(V)) \\
          &= (\tau^*\overline{\Omega})(X_s, V) \\
          &= \Omega_s(X_s, V).
\end{align*}
$$

This completes the proof. \qed

2.3. Critical Points of $\mathcal{E}$. In this subsection we assume that the $S^1$-action on $Z$ extends to a holomorphic action of $\mathbb{C}^*$. That is, the vector field $I_Z Y$ on $Z$ is complete. Since $\mathcal{E}$ is holomorphic and $S^1$-invariant, it follows that it is in fact $\mathbb{C}^*$-invariant in this case. Thus the critical points of $\mathcal{E}$ are the $\mathbb{C}^*$-fixed points in $S'$. We first examine the $\mathbb{C}^*$-fixed points in $S$. Any $\mathbb{C}^*$-fixed point $s \in S$ is characterized by

$$
    s(\zeta \lambda) = \zeta.s(\lambda)
$$

for all $\zeta \in \mathbb{C}^*$ and $\lambda \in \mathbb{C}P^1$. In particular, $s \in S^{\mathbb{C}^*}$ is determined by its value at $\lambda = 1$. Indeed, $s(\lambda) = \lambda.s(1)$ for $\lambda \in \mathbb{C}^*$, and by continuity we have

$$
    s(0) = \lim_{\lambda \to 0} \lambda.s(1), \quad s(\infty) = \lim_{\lambda \to \infty} \lambda.s(1).
$$

(2.10)

Hence the closures of the $\mathbb{C}^*$-orbits in $Z$ lying over $\mathbb{C}^* \subset \mathbb{C}P^1$ correspond precisely to the $\mathbb{C}^*$-fixed points in $S$. Conversely, any point $z \in Z_1$ potentially determines a $\mathbb{C}^*$-invariant section $s^z : \mathbb{C}^* \to Z$ of $\mathbb{C}^*$ as follows. For $\lambda \in \mathbb{C}^*$, set

$$
    s^z(\lambda) = \lambda.z
$$

which is clearly a section over $\mathbb{C}^*$. If the limits

$$
    \lim_{\lambda \to 0} \lambda.z, \quad \lim_{\lambda \to \infty} \lambda.z,
$$

exist in $Z$, cf. (2.10), then the section extends to a $\mathbb{C}^*$-invariant section $s^z : \mathbb{C}P^1 \to Z$ of $\mathbb{C}^*$. The existence of these limits has been investigated in detail in [45] for $\mathcal{M}_{1}^{1}$; see also Section 4.2 below.

Clearly, for any fixed point $s \in S^{\mathbb{C}^*}$, we have $s(0) \in Z_0 = M$ and $s(\infty) \in Z_\infty = \overline{M}$ are fixed points of the $\mathbb{C}^*$-actions on $M$ and $\overline{M}$ respectively. The following gives the converse on $S' \subset S$ (also see [18, 19, 20]).

**Proposition 2.4.** Let $s \in S'$ be such that $s(0) \in Z_0^{\mathbb{C}^*}$ and $s(\infty) \in Z_\infty^{\mathbb{C}^*}$. Then $s \in (S')^{\mathbb{C}^*}$.

**Proof.** We see from $X_s(\lambda) = Y_s(\lambda) - i\lambda s(\lambda)$ that $X_s(0) = 0 = X_s(\infty)$. Since $N_s \cong O_{\mathbb{C}P^1}(1)^{\oplus 2d}$, this implies that $X_s = 0$, and thus $s$ is a fixed point. \qed

**Proposition 2.4** has the following consequence.

**Corollary 2.5.** Let $s \in S$ be a section such that $s(0) \in Z_0^{\mathbb{C}^*}$ and $s(\infty) \in Z_\infty^{\mathbb{C}^*}$. If $s$ is not a fixed point of the $\mathbb{C}^*$-action, then $s$ cannot be contained in $S'$, i.e., the normal bundle of $s$ is not isomorphic to $O_{\mathbb{C}P^1}^{\oplus 2d}$.

We end this subsection with the observation that $\mathbb{C}^*$-fixed points $s \in S$ are also the fixed points under the twisting procedure that was introduced in [4]; see also [6]. Recall that a section $s \in S$ is twistable if the section $\tilde{s}(\lambda) := \lambda^{-1}.s(\lambda^2)$ over $\mathbb{C}^*$ extends to a section on all of $\mathbb{C}P^1$.

**Proposition 2.6.** Let $s$ be a fixed point of the $\mathbb{C}^*$-action, then $s$ is twistable, and the twist $\tilde{s}$ satisfies the equation $\tilde{s} = s$.

**Proof.** If $s$ is fixed, then $s(\lambda) = \lambda.s(1)$ for all $\lambda \in \mathbb{C}^*$. If follows that

$$
    \tilde{s}(\lambda) = \lambda^{-1}.s(\lambda^2) = \lambda^{-1}.\lambda^2.s(1) = \lambda.s(1) = s(\lambda),
$$

completing the proof. \qed
2.4. The Atiyah–Ward transform of \( L_Z \). Let \( S^0 \subset S' \) for the space of all sections \( s \in S \) such that

- the normal bundle is isomorphic to \( \mathcal{O}_{\mathbb{C}P^1}(1)^{\oplus 2d} \), and
- \( s^*L_Z \) trivial.

Since \( s^*L_Z \) trivial if and only if \( \deg s^*L_Z = 0 \), we conclude that \( S^0 \) is an open subset of \( S \), and it is a union of some connected components of \( S' \). Consider the space \( F^0 = \mathbb{C}P^1 \times S^0 \) and restrict the diagram \(^2\) to \( F^0 \):

\[
\begin{array}{ccc}
Z & \xrightarrow{ev} & F^0 = \mathbb{C}P^1 \times S^0 \\
\pi \downarrow & & \downarrow \pi_2 \\
\mathbb{C}P^1 & \xrightarrow{\pi_1} & S^0,
\end{array}
\]

Clearly, we have the identification

\[ T_F^0 = \pi_1^*T\mathbb{C}P^1 \oplus \pi_2^*TS^0. \]

Since \( \text{dev} : \pi_2^*TS' \to ev^*T_{\mathbb{C}P^1}Z \) is surjective by Proposition \(^1\) we obtain the following commutative diagram with exact rows

\[
\begin{array}{ccccccc}
0 & \to & T_{ev}F^0 & \xrightarrow{id} & T_F^0 & \xrightarrow{\text{dev}} & ev^*T_{\mathbb{C}P^1}Z & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & T_{ev}F^0 & \xrightarrow{\pi_2^*TS^0} & \pi_2^*TS^0 & \xrightarrow{\text{dev}} & ev^*T_{\mathbb{C}P^1}Z & \to & 0.
\end{array}
\]

We next describe the Atiyah–Ward transform of \( L_Z \) \(^2\) and study how it interacts with the meromorphic connection on \( L_Z \). The Atiyah–Ward construction is of course valid in a more general context, but for the convenience of the reader we spell out the details relevant for our discussion in the case of a line bundle.

The Atiyah–Ward transform of \( L_Z \) is a holomorphic line bundle \( \mathcal{L} \) over \( S^0 \) with a holomorphic connection \( \nabla^{AW} \). To construct this line bundle, recall that \( L_Z \) is trivial along \( s(\mathbb{C}P^1) \) for any \( s \in S^0 \). Hence \( ev^*L_Z \) is trivial along \( \mathbb{C}P^1 \times \{ s \} = \pi_2^{-1}(s) \), and the (0-th) direct image construction yields the line bundle

\[ \mathcal{L} := (\pi_2)_*ev^*L_Z. \]

We observe that there is a natural isomorphism \( \pi_2^*\mathcal{L} \cong ev^*L_Z \). Over \( (\lambda, s) \in F^0 \) it is given by evaluating an element of \( \pi_2^*\mathcal{L}(\lambda, s) = H^0(\mathbb{C}P^1, s^*L_Z) \) at \( \lambda \).

We next equip \( \mathcal{L} \) with a holomorphic connection. First consider the relative exterior differential \( d_{ev} : \mathcal{O}_{F^0} \to T_{ev}(F^0) \) which is defined as the composition

\[ d_{ev} : \mathcal{O}_{F^0} \xrightarrow{d} T^*F^0 \xrightarrow{\text{ev}^*} T_{ev}(F^0). \]

By construction, we have \( ev^*\mathcal{O}_Z \subset \ker d_{ev} \), and for any \( f \in \mathcal{O}_{F^0} \):

\[ d_{ev}f = df|_{T_{ev,F^0}}. \]

Next we tensor the diagram defining \( d_{ev} \) with \( ev^*L_Z \) and observe that this gives a well-defined "relative connection"

\[ \nabla_{ev} : ev^*L_Z \to T_{ev,F^0} \oplus ev^*L_Z, \quad \nabla_{ev}(f \cdot ev^*\nu) := (d_{ev}f) \otimes ev^*\nu, \]

for a locally defined trivializing section \( \nu \) of \( L_Z \) and \( f \in \mathcal{O}_{F^0} \). This operator is well-defined because the transition functions for \( ev^*L_Z \) lie in \( ev^*\mathcal{O}_Z \) and hence in the kernel of \( d_{ev} \). The next lemma will enable us to construct \( \nabla^{AW} \) from \( \nabla_{ev} \).

**Lemma 2.7.** There is a natural isomorphism of vector bundles over \( S^0 \):

\[ T^*S^0 \cong (\pi_2)_*T_{ev,F^0}^*, \quad A \mapsto \pi_2^*A|_{T_{ev,F^0}}. \]

In particular, there is an isomorphism

\[ T^*S^0 \otimes \mathcal{L} \cong (\pi_2)_*(T_{ev,F^0}^* \otimes ev^*L_Z). \]

\(^2\)Here we slightly abuse notation since the restriction of the maps will be evident in the following.
Proof. Dualizing the second row of (2.11) yields

\[ 0 \longrightarrow ev^*T^*_\mathbb{P} \longrightarrow \pi_2^*T^*_S \longrightarrow T^*_0 \longrightarrow 0. \]

Here \( r \) denotes the restriction of 1-forms to \( T^*_0 \). For this short exact sequence, consider the corresponding long exact sequence of direct image sheaves with respect to \( \pi_2 \):

\[ 0 \longrightarrow (\pi_2)_*(ev^*(T^*_\mathbb{P})) \longrightarrow (\pi_2)_*(\pi_2^*T^*_S) \longrightarrow (\pi_2)_*(T^*_0) \longrightarrow (\pi_2)_*(ev^*(T^*_\mathbb{P})) \longrightarrow \cdots . \]

The fibers of the vector bundles \( (\pi_2)_*(ev^*(T^*_\mathbb{P})) \), \( q = 0, 1 \), over any \( s \in S^0 \) satisfy

\[ H^q(\mathbb{C}P^1, s^*T^*_\mathbb{P}) \cong H^q(\mathbb{C}P^1, \mathcal{O}(-1)^{\oplus 2d}) = 0. \]

Thus the above long exact sequence of cohomologies gives the isomorphism

\[ \pi_2_* r : (\pi_2)_*(\pi_2^*T^*_S) \cong (\pi_2)_*(T^*_0) \]

of vector bundles on \( S^0 \).

By the projection formula (see for example [22, Chapter 3]) or direct computation, we have

\[ (\pi_2)_*(\pi_2^*S^0 \otimes \mathcal{O}_{F0}) \cong T^*S^0 \otimes (\pi_2)_*\mathcal{O}_{F0} \cong T^*S^0. \]

The last isomorphism follows from \((\pi_2)_*\mathcal{O}_{F0} \cong \mathcal{O}_{S^0}\), which in turn follows from the fact that the fibers of \( \pi_2 \) are connected. Since \( ev^*L^\mathbb{P} \cong \pi_2^*L \) over \( S^0 \), the second statement in the lemma is again derived from the projection formula.

\[ \square \]

Proposition 2.8. The operator

\[ \nabla^{AW} = (\pi_2)_*(\nabla_{ev}) : \mathcal{L} \longrightarrow T^*S^0 \otimes \mathcal{L} \cong (\pi_2)_*(T^*_0 \otimes ev^*L) \]

induces a natural holomorphic connection on \( \mathcal{L} \). This connection is trivial on the submanifolds \( S^0_\mathbb{C} = \pi_2(ev^{-1}(z)) \) for all \( z \in \mathbb{Z} \). This property determines \( \nabla^{AW} \) completely as long as \( S^0_\mathbb{C} \) is connected.

We call the above connection \( \nabla^{AW} \) on \( \mathcal{L} \) the Atiyah–Ward connection.

Proof. The \( \mathcal{O}_{S^0} \)-module structure on the sheaf of sections of the vector bundle \( \mathcal{L} = (\pi_2)_*ev^*L \) is as follows.

Let \( S \subseteq S^0 \) be an open subset, and take \( f \in \mathcal{O}_{S^0}(S) \). Then \( f \) acts on \( \mathcal{L}(S) = H^0(\mathbb{C}P^1 \times S, ev^*L) \) by multiplication with \( \pi^*_2 f \). Hence for any \( \psi \in \mathcal{L}(S) \) we have

\[ \nabla^{AW}(f \psi) = \nabla_{ev}(\pi^*_2 f \psi) = d_{ev} \pi^*_2 f \otimes \psi + f\nabla_{ev}\psi. \]

We see that \( d_{ev} \pi^*_2 f \otimes \psi + f\nabla_{ev}\psi \) is obtained from \( \pi^*_2(df \otimes \psi + f\nabla^{AW}\psi) \) by restricting to the subbundle \( T^*_0 \). Note that this is uniquely determined by Lemma 2.7, so we conclude that

\[ \nabla^{AW}(f \psi) = df \otimes \psi + f\nabla\psi. \]

Therefore, \( \nabla^{AW} \) is indeed a connection.

To show that this connection is trivial on \( S^0_\mathbb{C} \) for each \( z \in \mathbb{Z} \), observe that any section of the form \( ev^*\psi \) is actually parallel (covariant constant) for the relative connection \( \nabla_{ev} \) by the formula

\[ \nabla_{ev}(f \cdot ev^*\psi) = (d_{ev} f) \otimes ev^*\psi. \]

Now the fibers of \( ev \) are of the form

\[ ev^{-1}(z) = \{ \varpi(z) \} \times S^0_\mathbb{C}. \]

Then we get a frame \( \psi \) of \( ev^*L_{|ev^{-1}(z)} \) by putting

\[ \psi(\varpi(z), s) = l \]

for any fixed \( 0 \neq l \in L_z \). Note that this construction makes sense because \( ev(\varpi(z), s) = s(\varpi(z)) = z \) for all \( s \in S^0_\mathbb{C} \). The frame \( \psi \) induces a natural trivialization \( \psi^\mathbb{C} \) of \( \mathcal{L} \) along \( S^0_\mathbb{C} \). Since \( \nabla_{ev}\psi = 0 \), we have \( \nabla^{AW}\psi^\mathbb{C} = 0 \) along \( S^0_\mathbb{C} \). Thus \( \nabla^{AW} \) is trivial on \( S^0_\mathbb{C} \).

Now let \( \tilde{\nabla} = \nabla^{AW} + A \) be another holomorphic connection on \( S^0 \) which is trivial along each \( S^0_\mathbb{C} \). Then the holomorphic 1-form \( A \) restricted to \( S^0_\mathbb{C} \) is exact for all \( z \in \mathbb{Z} \). A result of Buchdahl in [9] implies that
we can find $f \in \mathcal{O}(F^0)$ such that $d\sigma f = (\pi_2^*A)|_{T_{\sigma}F^0}$. Note that $f$ is constant along the compact fibers of $\pi_2$. Therefore there exists $\tilde{f} \in \mathcal{O}(S^0)$ such that

$$f = \tilde{f} \circ \pi_2$$

if $S_z^0$ is connected. It now follows that

$$\pi_2^*(A - d\tilde{f})|_{T_{\sigma}F^0} = 0,$$

and this implies that $A = d\tilde{f}$. Thus, $\nabla^{AW}$ and $\nabla = \nabla^{AW} + d\tilde{f}$ are gauge-equivalent. \hfill \Box

**Corollary 2.9.** On the real submanifold $M \subset S^0$ the connection $\nabla^{AW}$ coincides with the hyperholomorphic connection on $L_M$. Thus its curvature $F^{AW}$ of $\nabla^{AW}$ satisfies

$$F^{AW} = \Omega_0 + i dI_0 d\mathcal{E}$$

on every component of $S^0$ that meets $M$.

**Proof.** The evaluation map $ev : F^0 \to Z$ restricts to a diffeomorphism $\epsilon : M \times \mathbb{C}P^1 \to Z$. Thus we have a natural map $q = \pi_2 \circ \epsilon^{-1} : Z \to M$. Note that

$$q^*\mathcal{L} = q^*(\pi_2^*), ev^*L_Z = (\epsilon^{-1})^*\pi_2^*(\pi_2^*), ev^*L_Z = (\epsilon^{-1})^*ev^*L_Z = L_Z.$$

On the other hand, we have, by definition, that $q^*L_M = L_Z$.

Now the hyperholomorphic connection $\nabla_M$ is real analytic. Thus we can complexify and extend it to a holomorphic connection $\nabla$ on $L \to S^0$ at least locally in a neighbourhood of $M \subset S^0$. Note that its curvature is $\Omega_0 + i dI_0 d\mathcal{E}$, i.e., the complexification of the curvature of $\nabla_M$.

Since there is a unique twistor line through each point in $z$, we see that $S_z^0$ intersects $M$ in a unique point, namely $q(z)$. Moreover, as the twistor line $q(z)$ through $z$ also passes through $\tau_z(z)$, we see that $q(z) = S_z^0 \cap S_{\tau_z(z)} \cap M$. We obtain a splitting

$$T_{q(z)}S^0 = T_{q(z)}S_z^0 \oplus T_{q(z)}S_{\tau(z)}^0,$$

which can be identified with the splitting $T_{q(z)}M \otimes \mathbb{C} = T^{1,0,\lambda}M \oplus T^{0,1,\lambda}M$. Since $\nabla_M$ has curvature of type $(1,1)$ on $(M, I_\lambda)$, (here $\lambda = \omega(z)$), its complexification $\nabla$ is flat along $S_z^0$ (and $S_{\tau(z)}^0$) for all $z \in Z$. By Proposition, the connections $\nabla$ and $\nabla$ must be gauge equivalent in a neighborhood of $M$ in $S^0$.

The curvature $F^{AW}$ is a holomorphic two-form on $S^0$ which is compatible with the real structure. On the other hand, $\Omega_0 + i dI_0 d\mathcal{E}$ is also holomorphic and real and coincides with $F^{AW}$ on the real submanifold $M$. Thus, they must coincide on every component of $S^0$ that meets $M$. \hfill \Box

**Corollary 2.10.** The curvature $F^{V_0}$ of the holomorphic connection $\nabla^0 = \nabla^{AW} - i I_0 d\mathcal{E}$ satisfies the equation $F^{V_0} = \Omega_0$ on every component of $S^0$ that intersects $M$.

### 2.5. The Atiyah–Ward transform and the meromorphic connection

We want to describe the relationship between the meromorphic connection $\nabla$ on $L_Z$ and the Atiyah–Ward transform $(\mathcal{L}, \nabla^{AW})$ of $L_Z$. As the first step, we observe that the Atiyah class $ev^*\mathcal{A}_L$ of $ev^*L_Z \to F^0$ vanishes because $ev^*L_Z = \pi_2^*\mathcal{L}$ admits the holomorphic connection $\pi_2^*\nabla^{AW}$. This has the following implications: on $F^0$ consider

$$\hat{\sigma} = ev^*\sigma \in H^0(\pi_2^*\mathcal{O}(2))$$

which vanishes on the divisor

$$\hat{D} = ev^{-1}(D) = \{(0) \times S^0 \} \cup \{\infty \} \times S^0.$$

We have the short exact sequence of sheaves

$$0 \to T^*F^0 \to T^*F^0(2) \to T^*F^0(2)|\hat{D} \to 0$$

Note that there exists a neighbourhood $M \subset U \subset S$ such that $S_z \cap U$ is connected for each $z \in Z$, because $M \cap S_z = \{q(z)\}$. 

---

Note that there exists a neighbourhood $M \subset U \subset S$ such that $S_z \cap U$ is connected for each $z \in Z$, because $M \cap S_z = \{q(z)\}$. 

---

Note that there exists a neighbourhood $M \subset U \subset S$ such that $S_z \cap U$ is connected for each $z \in Z$, because $M \cap S_z = \{q(z)\}$. 

---

Note that there exists a neighbourhood $M \subset U \subset S$ such that $S_z \cap U$ is connected for each $z \in Z$, because $M \cap S_z = \{q(z)\}$. 

---

Note that there exists a neighbourhood $M \subset U \subset S$ such that $S_z \cap U$ is connected for each $z \in Z$, because $M \cap S_z = \{q(z)\}$. 

---

Note that there exists a neighbourhood $M \subset U \subset S$ such that $S_z \cap U$ is connected for each $z \in Z$, because $M \cap S_z = \{q(z)\}$.
with the corresponding long exact sequence

\[
0 \rightarrow H^0(F^0, T^*F^0) \xrightarrow{\phi} H^0(F^0, T^*F^0(2)) \rightarrow H^0(\hat{D}, T^*F^0(2)\big|_{\hat{D}}) \xrightarrow{\delta} H^1(F^0, T^*F^0) \rightarrow \cdots
\]

(2.12)

The construction of \(L_Z\) implies that \(ev^*\alpha_L = \hat{\delta}(ev^*\varphi)\), which vanishes by the previous observation. Hence there exists \(\phi \in H^0(F^0, T^*F^0(2))\) such that \(|\phi|_{\hat{D}} = ev^*\varphi\) by the long exact sequence in (2.12).

**Proposition 2.11.** The connection \(ev^*\nabla - \frac{\varphi}{\sigma}\) on \(ev^*L_Z = \pi_2^*\mathcal{L}\) is holomorphic.

**Proof.** From the construction of \(ev^*\nabla\), we know that it has connection 1-forms

\[
\hat{A}_i = \frac{ev^*\varphi_i}{\hat{\sigma}}
\]

with respect to the open covering \(ev^*\mathcal{U} = \{ev^{-1}(U_i)\}\). Here \(\mathcal{U} = \{U_i\}\) and \(\varphi_i\) are as in the construction of \(\nabla\) in Section 2.1. The connection \(ev^*\nabla - \frac{\varphi}{\sigma}\) has the connection 1-forms

\[
\hat{A}_i - \frac{\phi_i}{\sigma}.
\]

But \(\phi_i|_{\hat{D}}\) and \(ev^*\varphi_i\) coincide on \(ev^{-1}(U_i) \cap \hat{D}\) and the forms \(\hat{A}_i\) have no other poles. Hence the proposition follows. \(\square\)

We next explicitly construct such a section \(\phi \in H^0(F^0, T^*F^0(2))\) and describe the difference form \(B\) between the holomorphic connections \(ev^*\nabla - \frac{\varphi}{\sigma}\) and \(\pi_2^*\nabla^{AW}\). To do so, we first observe that

\[
T^*F^0(2) \cong (\pi_1^*T^*\mathbb{C}P^1 \oplus \pi_2^*T^*S^0)(2) \cong \mathcal{O}_{\pi^0} \oplus \pi_2^*T^*S^0(2).
\]

(2.13)

We describe \(\phi\) component-wise, i.e., \(\phi = (\beta, \gamma)\) with respect to the splitting (2.13). Given any \(V \in T_\omega S^0\), construct the following \(\gamma \in H^0(F^0, \pi_2^*T^*S^0(2))\)

\[
\gamma_{(s,\lambda)}(V) := \omega(X_\lambda(s), V(\lambda)) = (\iota_X ev^*\omega)_{(s,\lambda)}(V).
\]

The first component \(\beta\) of \(\phi\) is

\[
\beta := \mathcal{E}(s)\text{id}_{\pi_1^*\mathcal{O}(2)} \in H^0(F^0, \pi_1^*T^*\mathbb{C}P^1(2)) = H^0(F^0, \text{End}(\pi_1^*\mathcal{O}(2))) \cong H^0(F^0, \mathcal{O}_{\pi^0}).
\]

**Lemma 2.12.** The element \(\phi = (\beta, \gamma) \in H^0(F^0, T^*F^0(2))\) constructed above has the desired properties. Moreover, \(\phi|_{T_\omega,F^0} = 0\).

**Proof.** We may write \(\beta = \mathcal{E}d\lambda \otimes \frac{\partial}{\partial \lambda}\) on \(\{\lambda \neq \infty\}\), and hence at \(\lambda = 0\):

\[
ev^*\varphi_{(s,0)}(\frac{\partial}{\partial \lambda}, 0) = \varphi_{s(0)}(\delta(0)) = \mathcal{E}(s)\frac{\partial}{\partial \lambda} = \beta_{s,0}(\frac{\partial}{\partial \lambda}).
\]

Similarly, at \(\lambda = \infty\),

\[
ev^*\varphi_{(s,\infty)}(\frac{\partial}{\partial \lambda}, 0) = \varphi_{s(\infty)}(\delta(\infty)) = \mathcal{E}(s)\frac{\partial}{\partial \lambda} = \beta_{s,\infty}(\frac{\partial}{\partial \lambda}).
\]

Now we check the component \(\gamma\). Let \(V \in T_\omega S^0\). Then, by (2.10),

\[
ev^*\varphi_{(s,0)}(V) = \varphi_{s(0)}(V(0)) = (\iota_X\omega)_{s(0)}(V(0)).
\]

On the other hand, by (2.10),

\[
(\iota_X ev^*\omega)_{s(0)}(V) = \omega(X(0), V(0)) = \omega(Y(0), V(0)) = \iota_Y\omega(V(0)) = ev^*\varphi_{(s,0)}(V).
\]

Similar considerations apply at \(\lambda = \infty\). Since \(\gamma = \iota_X ev^*\omega|_{T^*S^0}\) and \(T_{ev}F^0 \subset \pi_2^*TS^0\), it follows that \(\phi|_{T_\omega,F^0} = 0\). \(\square\)

Hence the connection

\[
\hat{\nabla} := ev^*\nabla - \frac{\varphi}{\sigma}
\]

is holomorphic. To understand its relationship with the Atiyah–Ward connection, we need the following:

**Lemma 2.13.** There is a unique holomorphic connection \(\nabla^\mathcal{L}\) on \(\mathcal{L}\) such that \(\pi_2^*\nabla^\mathcal{L} = \hat{\nabla}\) on \(\pi_2^*\mathcal{L} = ev^*L_Z\).
Proof. We will prove the more general statement that any holomorphic connection on \( \pi_2^* L \) is pulled back from a unique holomorphic connection on \( L \). Let \( F \) be the curvature of such a connection which is a holomorphic two-form on \( F^0 \). Further let \( V \) be a vector field on \( S^0 \) which we view as a vector field on \( F^0 = \mathbb{C}P^1 \times S^0 \). Then \( \iota_V F^0 \) pulls back to a holomorphic 1-form on \( \mathbb{C}P^1 \times \{ s \} \), which must vanish, since there are no non-trivial holomorphic 1-forms on \( \mathbb{C}P^1 \). This shows that \( F \) is purely horizontal with respect to the map \( \pi_2 : \mathbb{C}P^1 \times S^0 \rightarrow S^0 \).

We can thus find local frames for \( \pi_2^* L \) which are parallel along the fibers of \( \pi_2 \) (which are \( \mathbb{C}P^1 \)). More precisely, there exists an open cover of \( \mathbb{C}P^1 \times S^0 \) by open subsets of the form \( \mathbb{C}P^1 \times U \), where \( U \subset S^0 \) is open, together with holomorphic frames for \( \pi_2^* L \) over \( \mathbb{C}P^1 \times U \) that are parallel (covariant constant) in the \( \mathbb{C}P^1 \)-direction. The transition functions with respect to these local frames are holomorphic functions \( \mathbb{C}P^1 \times (U \cap V) \rightarrow \mathbb{C}^*, \) which are thus constant along the \( \mathbb{C}P^1 \)-fibers. Hence the transition functions are of the form \( g_{U \cap V} \circ \pi_2 \), where \( g_{U \cap V} \) define the line bundle \( L \) on \( S \). Moreover, the holomorphic connection uniquely descends to \( L \), concluding the proof. \( \square \)

**Theorem 2.14.** The holomorphic connections \( \nabla^L \) and \( \nabla^{AW} \) on \( L \) coincide.

**Proof.** Clearly, \( \nabla^L = \nabla^{AW} + B \) for a unique holomorphic 1-form \( B \) on \( S^0 \). Consider the pulled back connection

\[
\nabla = \pi_2^* \nabla^L - \pi_2^* \nabla^{AW} + \pi_2^* B.
\]

(2.14)

Let \( U \subset Z \) be an open subset such that \( L_Z \) has a local frame \( \nu \) on \( U \). Hence \( \psi := \text{ev}^* \nu \) is a local frame of \( \text{ev}^* L_Z \) on \( \hat{U} := \text{ev}^{-1}(U) \). Observe that along \( S^0_Z \), for \( z \in U \), the frame \( \psi(z, \cdot) \) coincides with the parallel frame (with respect to \( \nabla^{AW} \)) constructed in the proof of Proposition 2.8. Since \( \psi^{-1}(z) = \{ \psi(z) \} \times S^0_z \), it follows that \( \pi_2^* \nabla^{AW} \psi \) restricted to \( T_{ev} F^0 = \ker \text{dev} \) vanishes on all of \( \hat{U} \).

Next let \( (\lambda, s) \in \hat{U} - \hat{D} \), and \( X \in (T_{ev,F^0}(\lambda,s)) \). Then we compute

\[
\hat{\nabla}_X \psi = (\text{ev}^* \nabla)_X \psi - \dot{\psi}(X) \psi = 0.
\]

Here we made use of the definition of \( T_{ev,F^0} \) and Lemma 2.12. By continuity, the restriction of \( \hat{\nabla} \) to \( T_{ev,F^0} \) vanishes on all of \( \hat{U} \) as well. Since \( F^0 \) can be covered by open subsets \( \hat{U} \) as above, from (2.14) it follows that \( \pi_2^* B_{T_{ev,F^0}} = 0 \). But the isomorphism \( T^* S^0 \cong \pi_2^* T_{ev,F^0} \) is given by \( A \mapsto \pi_2^* A_{T_{ev,F^0}} \) (cf. Lemma 2.7), so we have \( B = 0 \).

The existence of a holomorphic connection on \( S^0 \) with curvature \( \Omega_0 \) together with Theorem 1.10 allow us to obtain the following results on the global structure of \( S^0 \). Denote

\[
S^0_z = \text{ev}^{-1}_{\pi(\cdot)}(z) \cap S^0
\]

for \( z \in Z \).

**Theorem 2.15.** Suppose \( 0 \neq [\omega_1] \in H^2(M) \), and let \( z \in Z_\infty \). Then \( S^0_z \) cannot be isomorphic to \( Z_0 \). In particular, the local diffeomorphism \( \text{ev}_0 \times \text{ev}_\infty : S^0 \rightarrow Z_0 \times Z_\infty \) cannot extend to a global diffeomorphism.

**Proof.** Since \( TS^0 \) is Lagrangian for the symplectic form \( \Omega_0 \), the Atiyah-Ward connection \( \nabla^{AW} \) pulls back to a flat connection on \( L^{(z)} := L|_{ev^{-1}_{\pi}(z)} \). Consequently, \( c_1(L^{(z)}, \mathbb{C}) = 0 \in H^2(\text{ev}^{-1}_{\pi}(z)) \). But the first Chern class of \( L \) is \( [\omega_1] \neq 0 \in H^2(Z_0, \mathbb{C}) \).

**Remark 2.16.** If \( Z_0 = M \) is not simply-connected, then \( \text{ev}_0 : S^0 \rightarrow Z_0 \) might be a covering map. In fact, this happens for the rank 1 Deligne–Hitchin moduli space; see [5]. It would be interesting to understand what happens if \( Z_0 \) is simply-connected, for example in the case of \( \text{SL}(2, \mathbb{C}) \)-Deligne-Hitchin moduli spaces.

### 2.6. An alternative construction of Hitchin’s meromorphic connection

We can reverse the above constructions to obtain the meromorphic Hitchin connection on a hyperholomorphic bundle over the twistor space \( Z \) of a hyperkähler manifold with a rotating circle action. We only sketch the steps, as the technical details are already explained above. The necessary twistor data are:

- the twistor space \( Z \) of a hyperkähler manifold with a rotating circle action;
- the twisted relative holomorphic symplectic form \( \omega \in H^0(Z, (A^2 T^*_w)^{(2)}) \).
Adding the additional structure of a “prequantum” hyperholomorphic line bundle $L$, we obtain from the Atiyah-Ward construction that $ev^*L \to S^0$ is equipped with the holomorphic connection $\pi_2^*\nabla^{AW}$. From Theorem 2.14 we have
\[
  ev^*\nabla = \pi_2^*\nabla^{AW} + \hat{\phi}.
\]
Note that not only $ev^*\nabla$ is the pull-back of a meromorphic connection on the twistor space but also $\pi_2^*\nabla^{AW}$ and the meromorphic 1-form $\frac{\hat{\phi}}{2}$ are pull-backs from $Z$ as well. Clearly, the corresponding objects on $Z$ are neither a holomorphic connection nor a meromorphic 1-form but only real analytic objects.

We shall give a local construction of the holomorphic connection $\nabla^{AW}$ on $S^0$ which only involves the above listed twistor data: As explained in Section 1, we obtain from the twist or data the holomorphic symplectic form $\Omega_0$. Define the bundle automorphism
\[
  \eta : T^*(Z_0 \times Z_\infty) \to T^*(Z_0 \times Z_\infty)
\]
that acts on $T^*Z_0$ (respectively, $T^*Z_\infty$) as multiplication by $+1$ (respectively, $-1$). It is evident that there are locally defined holomorphic 1-forms $\alpha_U$ with $\eta(\alpha_U) = \alpha_U$ and
\[
  d\alpha_U = \Omega_0 + idI_0dE. \tag{2.15}
\]
We may think of the forms $\alpha_U$ as holomorphic connection 1-forms. Note that the corresponding cocycle is the pull-back of a cocycle on $Z_0$.

By Corollary 2.10 the connection defined by (2.15) is flat on $S_z$ for every $z \in Z$, and therefore it is locally trivial. Hence, we recover the Atiyah–Ward connection at least locally. By adding the 1-form $\frac{\hat{\phi}}{2}$, which is constructed from the circle action and from the twisted symplectic form via Lemma 2.12, we obtain (the pullback of) the meromorphic connection on $Z$.

2.6.1. $\tau$-sesquilinear forms and a generalized Chern connection. It seems appropriate to put the above local construction into a global context. We need an additional structure on $L \to S$, which can be regarded as the complexification of the hermitian metric $h_M$ on $L_M$.

**Definition 2.17.** Let $L \to S^0$ be a holomorphic line bundle, and let $\tau : S^0 \to S^0$ be an anti-holomorphic involution. A non-degenerate pairing
\[
  \langle \cdot, \cdot \rangle : L \times \tau^*\overline{L} \to \mathbb{C}
\]
is called a **holomorphic $\tau$-sesquilinear form** on $L$ if for all local holomorphic sections $v, w \in \Gamma(L, L)$ defined on some $\tau$-invariant open subset $U \subset S$, the function $\langle \langle v, w \rangle \rangle$ on $U$ defined by
\[
  \langle \langle v, w \rangle \rangle(s) := \langle v(s), \tau^*w(s) \rangle
\]
is holomorphic and it satisfies the identity
\[
  \tau^*\langle \langle v, \alpha \rangle \rangle = \langle \langle v, \alpha \rangle \rangle.
\]

Let $M' \subset S^0$ the set of real points. It follows that $\langle v_s, v_s \rangle \in \mathbb{R}$ for all $s \in M'$ and $v_s \in L_s$. Note that for functions $f, g : U \to \mathbb{C}$, we have
\[
  \langle \langle fv, gv \rangle \rangle = f(\tau^*g) \langle \langle v, v \rangle \rangle.
\]
Thus, up to sign, a holomorphic $\tau$-sesquilinear form is a complexification of a hermitian metric over the locus of real points.

Assume that there is a holomorphic involution
\[
  \eta \in H^0(S^0, \text{End}(T^*S^0))
\]
such that
\[
  \eta(\tau^*\alpha) = -\tau^*\eta(\alpha) \quad \text{for any} \quad \alpha \in \Omega^{1,0}(S^0). \tag{2.16}
\]
Further, assume that the holomorphic line bundle $L \to S^0$ is given by a cocycle of the form
\[
  \{ f_{i,j} : \mathcal{U}_{i,j} \to \mathbb{C}^* \} \tag{2.17}
\]
where the derivatives $df_{i,j}$ are in the $+1$ eigenspaces of $\eta$, i.e.,
\[
  df_{i,j} = \frac{1}{2}(df_{i,j} + \eta(df_{i,j})).
\]
Under these assumptions we have the following:
Lemma 2.18. There is a unique holomorphic Chern-connection $\nabla$ on $\mathcal{L} \longrightarrow S^0$ with
\[ d\,\langle\langle \sigma, \sigma \rangle\rangle = \langle\langle \nabla \sigma, \sigma \rangle\rangle + \langle\sigma, \nabla \sigma \rangle \tag{2.18} \]
for any holomorphic section $\sigma \in \Gamma(\mathcal{U}, \mathcal{L})$ and any $\tau$-invariant open subset $\mathcal{U} \subset S^0$.

Proof. Take a collection of local trivializations by nowhere-vanishing holomorphic sections $\sigma_i \in \Gamma(\mathcal{U}_i, \mathcal{L})$ satisfying the condition that locally $\sigma_i = \text{ev}_i \mathcal{L}$, where each open subset $\mathcal{U}_i$ is $\tau$-invariant, such that the transition functions satisfy $df_{i,j} = \eta(df_{i,j})$, where $f_{i,j} = \frac{\sigma_i}{\sigma_j}$. Then, writing
\[ \nabla \sigma_i = A_i \sigma_i, \]
we have $\eta(A_i) = A_i$. The condition in (2.18) then implies that
\[ d\log\langle\langle \sigma_i, \sigma_i \rangle\rangle = (A_i + \tau^* A_i)\langle\langle \sigma_i, \sigma_i \rangle\rangle. \]
By (2.16), we have $\eta(\tau^* A_i) = -\tau^* A_i$ and thus
\[ A_i = \frac{1}{2}(d\log\langle\langle \sigma_i, \sigma_i \rangle\rangle + \eta d\log\langle\langle \sigma_i, \sigma_i \rangle\rangle). \]
This shows the uniqueness of the connection.

For the existence, set
\[ \nabla \sigma_i = \frac{1}{2}(d\log\langle\langle \sigma_i, \sigma_i \rangle\rangle + \eta d\log\langle\langle \sigma_i, \sigma_i \rangle\rangle) \otimes \sigma_i. \]
Arguing analogously as for the usual Chern connection, we see that this defines a connection with the desired properties. \qed

The main example we have in mind is given by the space $S^0$ of sections of a twistor space $Z$ of a hyperkähler manifold $M$ equipped with a hyperholomorphic hermitian line bundle $L \longrightarrow M$. Assume that there exists a global $\tau$-sesquilinear form on $\mathcal{L} \longrightarrow S^0$. Consider the decomposition in (1.15) for $x = 0 \in \mathbb{C}P^1$, and the involution which is $-\text{Id}$ on $T_{ev_0}S^0$ and $+\text{Id}$ on $T_{ev_\infty}S^0$. Denote its dual endomorphism by $\eta$. It satisfies (2.16). Clearly, $\mathcal{L}$ admits a cocycle as in (2.17). Applying (2.18) we obtain a natural connection, which can be shown to be the Atiyah–Ward connection. It would be interesting to find natural expressions for the holomorphic $\tau$-sesquilinear form in concrete examples such as the Deligne–Hitchin moduli spaces; see also Section 3.3 below and the following remark.

Remark 2.19. The existence of a $\tau$-sesquilinear form on $\mathcal{L} \longrightarrow S^0$ follows almost automatically from the existence of an Atiyah–Ward type connection $\nabla$ on $\mathcal{L} \longrightarrow S^0$: assume that there exists a hermitian metric $h$ on $L|_M \longrightarrow M$ over the real points $M \subset S^0$. Consider a local holomorphic frame $\sigma \in \Gamma(\mathcal{U}, \mathcal{L})$ on a simply-connected $\tau$-invariant open subset $\mathcal{U} \subset S^0$ such that there exists $s \in M \cap \mathcal{U}$. Write $\nabla \sigma = \alpha \otimes \sigma$. Then
\[ \alpha + \tau^* \overline{\alpha} \]
is closed; its exterior derivative is a $(2,0)$-form which vanishes on $M$, hence on its complexification $S^0$. Integrating this closed form on the simply connected set $\mathcal{U}$ produces the $\tau$-sesquilinear form via
\[ \langle\langle (\mu_1 \sigma(p), \mu_2 \sigma(\tau(p))) \rangle\rangle = \mu_1 \overline{\mu_2} h(\sigma(p), \sigma(p)) \exp \left( \int_s^p \alpha + \tau^* \overline{\alpha} \right) \]
for all $\mu_1, \mu_2 \in \mathbb{C}$ and $p \in \mathcal{U}$.

3. Space of holomorphic sections of the Deligne–Hitchin moduli space

In this section we illustrate the general theory described in § 1 and § 2 for the Deligne–Hitchin twistor space of the moduli space of Higgs bundles on a compact Riemann surface.
3.1. Hitchin’s self-duality equations. Let \( \Sigma \) be a compact Riemann surface of genus at least two; denote its holomorphic cotangent bundle by \( K_\Sigma \). Consider a smooth complex rank \( n \) vector bundle \( E \) of degree zero with structure group in \( SU(n) \).

Denote by \( \mathcal{A}(E) \) the space of \( SU(n) \)-connections on \( E \). Sending any \( \nabla \in \mathcal{A}(E) \) to its \( (0, 1) \)-part \( \hat{\nabla} \) we may identify \( \mathcal{A} \) with the space of holomorphic structures on \( E \) inducing the trivial holomorphic structure \( \hat{\nabla} \) on the determinant bundle \( \text{det} \ E = \bigwedge^n E \) of \( E \), i.e., \( (\text{det} \ E, \hat{\nabla}) = (\mathcal{O}_\Sigma, \hat{\nabla}) \). Thus, \( \mathcal{A} \) is an affine space modelled on \( \Omega^{0,1}(\Sigma, \mathfrak{s}(E)) \), where \( \mathfrak{s}(E) \) is the bundle of trace-free endomorphisms of \( E \). The product

\[
T^* \mathcal{A} = \mathcal{A} \times \Omega^{1,0}(\Sigma, \mathfrak{s}(E))
\]

can be thought of as the cotangent bundle of \( \mathcal{A} \). We will denote its elements by \( (\hat{\nabla}, \Phi) \) and call \( \Phi \) the Higgs field of the pair \( (\hat{\nabla}, \Phi) \). Formally, \( T^* \mathcal{A} \) carries a flat hyperkähler structure that can be described as follows. The Riemannian metric is given by the \( L^2 \)-inner product on \( \Omega^{1,0}(\Sigma, \mathfrak{s}(E)) \). The almost complex structures \( I, J, K = IJ \) act on a tangent vector \( (\alpha, \phi) \in T_{(\hat{\nabla}, \Phi)} T^* \mathcal{A} = \Omega^{0,1}(\Sigma, \mathfrak{s}(E)) \oplus \Omega^{1,0}(\Sigma, \mathfrak{s}(E)) \) as

\[
I(\alpha, \phi) = (i\alpha, i\phi) \quad J(\alpha, \phi) = (-\phi^*, \alpha^*).
\]

The Kähler forms are given by

\[
\omega_I((\alpha, \phi), (\beta, \psi)) = \int_\Sigma \text{tr}(\alpha^* \wedge \beta - \beta^* \wedge \alpha + \phi \wedge \psi^* - \psi \wedge \phi^*)
\]

\[
(\omega_I + i\omega_K)((\alpha, \phi), (\beta, \psi)) = 2i \int_\Sigma \text{tr}(\phi \wedge \alpha - \phi \wedge \beta).
\]

The group \( \mathcal{G} := \Gamma(\Sigma, SU(E)) \) of unitary gauge transformations of \( E \) acts (on the right) on \( T^* \mathcal{A} \) as

\[
(\hat{\nabla}, \Phi).g = (g^{-1} \circ \hat{\nabla} \circ g, g^{-1} \Phi g).
\]

This action preserves the flat hyperkähler structure, and, formally, the vanishing condition for the associated hyperkähler moment map yields Hitchin’s self-duality equations:

\[
F^\nabla + [\Phi \wedge \Phi^*] = 0
\]

\[
\bar{\nabla} \Phi = 0.
\]

The second equation implies that \( \Phi \in H^0(\Sigma, \mathfrak{s}(E) \otimes K_\Sigma) \) with respect to the holomorphic structure \( \hat{\nabla} \).

Note that these equations imply that the connection \( \nabla + \Phi + \Phi^* \) is flat.

Let \( \mathcal{H} \subset T^* \mathcal{A} \) be the set of solutions of (3.3). The moduli space of solutions to the self-duality equations is the quotient

\[
\mathcal{M}_{\text{SD}} := \mathcal{M}_{\text{SD}}(\Sigma, SL(n, \mathbb{C})) := \mathcal{H}/\mathcal{G}.
\]

Formally, it is the hyperkähler quotient of \( T^* \mathcal{A} \) by the action of \( \mathcal{G} \).

A solution \( (\hat{\nabla}, \Phi) \) to the self-duality equations (3.3) is called irreducible if its stabilizer in \( \mathcal{G} \) is the center of \( SU(n) \). We write \( \mathcal{H}^{\text{irr}} \subset \mathcal{H} \) for the set of irreducible solutions. It is known that

\[
\mathcal{M}_{\text{SD}}^{\text{irr}} = \mathcal{H}^{\text{irr}}/\mathcal{G}
\]

is the smooth locus of \( \mathcal{M}_{\text{SD}} \). It is equipped with a hyperkähler metric induced by the flat hyperkähler structure on \( T^* \mathcal{A} \) described above. Moreover, there is a rotating circle action on \( \mathcal{M}_{\text{SD}} \) given by

\[
\zeta(\hat{\nabla}, \Phi) = (\hat{\nabla}, \zeta \Phi),
\]

\( \zeta \in S^1 \). It is Hamiltonian with respect to \( \omega_I \), and the map

\[
\mu_I : \mathcal{M}_{\text{SD}} \to i\mathbb{R}, \quad \mu_I(\hat{\nabla}, \Phi) = -\int_\Sigma \text{tr}(\Phi \wedge \Phi^*)
\]

restricts to a natural moment map on \( \mathcal{M}_{\text{SD}}^{\text{irr}} \).

The complex manifold \( (\mathcal{M}_{\text{SD}}^{\text{irr}}, I) \) can be described as follows. An \( SL(n, \mathbb{C}) \)-Higgs bundle is a pair consisting of a holomorphic vector bundle \( (E, \bar{\nabla}_E) \), such that \( \text{det}(E, \bar{\nabla}_E) = \mathcal{O}_\Sigma \), together with a Higgs field \( \Phi \in H^0(\Sigma, \mathfrak{s}(E) \otimes K_\Sigma) \). The group \( \mathcal{G}_C = \Gamma(\Sigma, SL(E)) \) acts on the set of Higgs bundles by the same rule as in (3.2). A Higgs bundle \( (\bar{\nabla}_E, \Phi) \) is called stable if every \( \Phi \)-invariant holomorphic subbundle \( E' \) has
negative degree, and it is called \textit{polystable} if it is a direct sum of stable Higgs bundles of degree zero. We denote by \( \mathcal{M}_{Higgs}^{st} := \mathcal{M}_{Higgs}^s(\Sigma, \text{SL}(n, \mathbb{C})) \) and \( \mathcal{M}_{Higgs}^{ps} := \mathcal{M}_{Higgs}^{ps}(\Sigma, \text{SL}(n, \mathbb{C})) \) the moduli spaces of stable and polystable Higgs bundles respectively. It is known that the map \( \overline{\partial}^n, \Phi \mapsto ((E, \overline{\partial}^n), \Phi) \) induces a biholomorphism

\[
(\mathcal{M}_{SD}^{st}, I) = \mathcal{M}_{Higgs}^{st}
\]

that extends to a homeomorphism

\[
\mathcal{M}_{SD} = \mathcal{M}_{Higgs}^{ps}.
\]

The circle action described in (3.4) extends to a holomorphic \( \mathbb{C}^* \)-action on \( \mathcal{M}_{Higgs} \):

\[
\zeta \cdot (\overline{\partial}^n, \Phi) = (\overline{\partial}^n, \zeta \Phi).
\]

While the complex structure \( I \) is induced from the complex structure on \( \Sigma \), the complex structure \( J \) has a more topological origin. Consider the space \( \mathcal{A}_C \) of \( \text{SL}(n, \mathbb{C}) \) connections on \( E \). The group \( \mathcal{G}_C \) acts on \( \mathcal{A}_C \) as

\[
\nabla \cdot g = g^{-1} \circ \nabla \circ g.
\]

A connection \( \nabla \in \mathcal{A}_C \) is called \textit{irreducible} if its stabilizer in \( \mathcal{G}_C \) is the center of \( \text{SL}(n, \mathbb{C}) \), and \( \nabla \) is called \textit{reductive} if it is isomorphic to a direct sum of irreducible connections, i.e., if any \( \nabla \)-invariant subbundle \( E' \subset E \) admits a \( \nabla \)-invariant complement.

Let

\[
\mathcal{M}_{dr} := \mathcal{M}_{dr}(\Sigma, \text{SL}(n, \mathbb{C}))
\]

be the moduli space of reductive flat \( \text{SL}(n, \mathbb{C}) \)-connections on \( E \). Its smooth locus is given by \( \mathcal{M}_{dr}^{st} \), the moduli space of irreducible flat \( \text{SL}(n, \mathbb{C}) \)-connections. Then the map \( (\overline{\partial}^n, \Phi) \mapsto \nabla + \Phi + \Phi^* \) is a biholomorphism

\[
(\mathcal{M}_{SD}^{st}, J) \cong \mathcal{M}_{dr}^{st}
\]

that extends to a homeomorphism

\[
\mathcal{M}_{SD} \cong \mathcal{M}_{dr}.
\]

The Riemann–Hilbert correspondence gives a biholomorphism

\[
\mathcal{M}_{dr} \cong \text{Hom}(\pi_1(\Sigma), \text{SL}(n, \mathbb{C}))^{\text{red}}/\text{SL}(n, \mathbb{C}) =: \mathcal{M}_B(\Sigma, \text{SL}(n, \mathbb{C})) =: \mathcal{M}_B;
\]

the space \( \mathcal{M}_B \) is known as the Betti moduli space.

3.2. The Deligne–Hitchin moduli space. We now recall Deligne’s construction of the twistor space \( Z(\mathcal{M}_{SD}) \) via \( \lambda \)-connections; see [24].

Let \( \lambda \in \mathbb{C} \). A holomorphic \( \text{SL}(n, \mathbb{C}) \) \( \lambda \)-connection on the \( \text{SU}(n) \)-vector bundle \( E \rightarrow \Sigma \) is a pair \((\overline{\partial}, D)\), where \( \overline{\partial} \) is a holomorphic structure on \( E \) and \( D: \Gamma(\Sigma, E) \rightarrow \Omega^{1,0}(\Sigma, E) \) is a differential operator such that

- for any \( f \in \mathcal{C}^\infty(\Sigma) \) we have \( D(fs) = \lambda s \otimes \partial f + f Ds \),
- \( D \) is holomorphic, i.e., \( \overline{\partial} \circ D + D \circ \overline{\partial} = 0 \), and
- as a holomorphic vector bundle we have \( \Lambda^n E, \overline{\partial} = (\mathcal{O}_\Sigma, \overline{\partial}_\Sigma) \), and the holomorphic differential operator on \( \Lambda^n E \) induced by \( D \) coincides with \( \lambda \partial_E \).

Therefore, an \( \text{SL}(n, \mathbb{C}) \) \( 0 \)-connection is an \( \text{SL}(n, \mathbb{C}) \)-Higgs bundle, and an \( \text{SL}(n, \mathbb{C}) \) \( 1 \)-connection is an usual holomorphic \( \text{SL}(n, \mathbb{C}) \)-connection. More generally, the operator \( \overline{\partial}_E + \lambda^{-1} D(\lambda) \) is a holomorphic connection for every \( \lambda \neq 0 \).

The group \( \mathcal{G}_C \) of complex gauge transformations of \( E \) acts on the set of holomorphic \( \lambda \)-connections in the usual way:

\[
(\overline{\partial}, D) \cdot g = (g^{-1} \circ \overline{\partial} \circ g, g^{-1} \circ D \circ g).
\]

We again call \((\overline{\partial}, D)\) stable if any \( D \)-invariant holomorphic subbundle \( E' \subset E \) has negative degree, and \((\overline{\partial}, D)\) is called polystable if it is isomorphic to a direct sum of stable \( \lambda \)-connections of degree zero (if \( \lambda \neq 0 \), then the degree of any \( \lambda \)-connection is automatically zero). Equivalently, \((\overline{\partial}, D)\) is stable if it is irreducible, i.e., its stabilizer in \( \mathcal{G}_C \) given by the center of \( \text{SL}(n, \mathbb{C}) \). The Hodge–moduli space \( \mathcal{M}_{Hod} := \mathcal{M}_{Hod}(\Sigma, \text{SL}(n, \mathbb{C})) \) is the moduli space of polystable holomorphic \( \lambda \)-connections, where \( \lambda \) varies over \( \mathbb{C} \):

\[
\mathcal{M}_{Hod} := \mathcal{M}_{Hod}(\Sigma, \text{SL}(n, \mathbb{C})) = \{ (\overline{\partial}, D, \lambda) \mid \lambda \in \mathbb{C}, (\overline{\partial}, D) \text{ polystable holomorphic } \lambda \text{-connection} \}/\mathcal{G}_C.
\]

It has a natural holomorphic projection to \( \mathbb{C} \) given by

\[
\varpi: \mathcal{M}_{Hod} \rightarrow \mathbb{C}, \quad (\overline{\partial}, D, \lambda) \mapsto \lambda.
\]

(3.7)
The smooth locus $M_{\text{Hod}}^{\text{irr}}$ of the Hodge moduli space $M_{\text{Hod}}$ coincides with the locus of irreducible $\lambda$-connections.

The $\mathbb{C}^*$-action in (3.6) extends to a natural $\mathbb{C}^*$-action on $M_{\text{Hod}}$ covering the standard $\mathbb{C}^*$-action on $\mathbb{C}$ by $\zeta \cdot (\overline{\partial}, D, \lambda) = (\overline{\partial}, \zeta D, \zeta \lambda)$.

The map $(\overline{\partial}, D, \lambda) \mapsto (\overline{\partial} + \lambda^{-1} D, \lambda)$ induces a biholomorphism $\varpi^{-1}(\mathbb{C}^*) \cong M_{\text{dR}} \times \mathbb{C}^* \cong M_{\text{B}} \times \mathbb{C}^*$, where $\varpi$ is the map in (3.7).

The Deligne–Hitchin moduli space $M_{\text{DH}} := M_{\text{DH}}(\Sigma, \text{SL}(n, \mathbb{C}))$ is obtained by gluing $M_{\text{Hod}}$ and $\overline{M_{\text{Hod}}} \cong M_{\text{Hod}}(\overline{\Sigma}, \text{SL}(n, \mathbb{C}))$ over $\mathbb{C}^*$ via the Riemann–Hilbert correspondence:

$$M_{\text{DH}} := M_{\text{DH}}(\Sigma, \text{SL}(n, \mathbb{C})) = (M_{\text{Hod}}(\Sigma, \text{SL}(n, \mathbb{C})) \cup M_{\text{Hod}}(\overline{\Sigma}, \text{SL}(n, \mathbb{C}))) / \sim,$$

where $[(\overline{\partial}, D, \lambda)] \sim [(\lambda^{-1} D, \lambda^{-1} \overline{\partial}, \lambda^{-1} \bar{\lambda})]$ for any $[(\overline{\partial}, D, \lambda)] \in M_{\text{Hod}}$ with $\lambda \neq 0$.

The projections from the respective Hodge moduli spaces to $\mathbb{C}$ glue to give a holomorphic projection $\varpi : M_{\text{DH}} \rightarrow \mathbb{C}P^1$.

The smooth locus of $M_{\text{DH}}$ coincides with the locus $M_{\text{Hod}}^{\text{irr}}$ of irreducible $\lambda$-connections which in turn coincides with the twistor space $Z(M_{\text{SD}}^{\text{irr}})$ of the hyperkähler manifold $M_{\text{SD}}^{\text{irr}}$.

The space $M_{\text{B}} \times \mathbb{C}^*$ admits an anti-holomorphic involution $\tau_{M_{\text{B}}}$ covering the antipodal involution $\lambda \mapsto -\bar{\lambda}^{-1}$ of $\mathbb{C}^*$ which is constructed as follows. For any $\rho \in \text{Hom}(\pi_1(\Sigma), \text{SL}(n, \mathbb{C}))$, define $\tau_{M_{\text{B}}}(\rho, \lambda) = (\rho^{-1} T, -\bar{\lambda}^{-1})$.

This produces the following antiholomorphic involution $\tau_{M_{\text{DH}}}$ of $M_{\text{DH}}$:

$$\tau_{M_{\text{DH}}} : M_{\text{DH}} \rightarrow M_{\text{DH}}, \quad [(\overline{\partial}, D, \lambda)] \mapsto [(\bar{\lambda}^{-1} D^*, -\bar{\lambda}^{-1} \overline{\partial^*}, -\bar{\lambda})] = [(\overline{\partial^*}, -D^*, -\bar{\lambda})].$$

This involution $\tau_{M_{\text{DH}}}$ is compatible with the $\mathbb{C}^*$-action in the following sense. For $\zeta \in \mathbb{C}^*$,

$$\tau_{M_{\text{DH}}}((\zeta, (\overline{\partial}, D, \lambda))) = \overline{\zeta}^{-1} \cdot \tau_{M_{\text{DH}}}((\overline{\partial}, D, \lambda)).$$

The $\mathbb{C}^*$-action on $M_{\text{Hod}}$ extends to a $\mathbb{C}^*$-action on $M_{\text{DH}}$. For any $\zeta \in \mathbb{C}^*$, we have $\zeta \cdot (\overline{\partial}, D(\lambda), \lambda) = [(\overline{\partial}, D(\lambda), \zeta \lambda)]$.

Clearly, this action covers the natural $\mathbb{C}^*$-action on $\mathbb{C}P^1$. Therefore the $\mathbb{C}^*$-fixed points of $M_{\text{DH}}$ are given by

$$M_{\text{DH}}^{\mathbb{C}^*} \cong M_{\text{Higgs}}(\Sigma, \text{SL}(n, \mathbb{C}))^{\mathbb{C}^*} \sqcup M_{\text{Higgs}}(\overline{\Sigma}, \text{SL}(n, \mathbb{C}))^{\mathbb{C}^*},$$

i.e., by the locus of complex variations of Hodge structures on $\Sigma$ and $\overline{\Sigma}$ (see [46]).

The twisted relative symplectic form on $M_{\text{DH}}^{\mathbb{C}^*}$ can be described as follows. Let $V_i = (\overline{\partial}_i, \dot{D}_i)$, $i = 1, 2$, be a pair of tangent vectors to the fiber $\varpi^{-1}(\lambda)$. Then

$$\omega_\lambda(V_1, V_2) = 2i \int_{\Sigma} \text{tr}(\dot{D}_2 \wedge \overline{\partial}_1 - \dot{D}_1 \wedge \overline{\partial}_2). \quad (3.8)$$

Note that at $\lambda = 0$ this exactly resembles $\omega_f + i\omega_K$ as defined in (3.1).
3.3. The line bundle on $\mathcal{M}^{\text{Higgs}}_{\tau}$. Let us describe the holomorphic line bundle $L_Z$ for $Z = \mathcal{M}^{\text{Higgs}}_{\tau}$ along each fiber of $\varpi$. First consider the moduli space of holomorphic $SL(n, \mathbb{C})$-connections $\mathcal{M}_{\text{dR}}$. Let $\mathcal{M}^{\text{Higgs}}_{\text{dR}}$ be the Zariski open subset of $\mathcal{M}_{\text{dR}}$ such that the underlying holomorphic bundle is stable. So we have a map $f : \mathcal{M}^{\text{Higgs}}_{\text{dR}} \to \mathcal{N}$, where $\mathcal{N}$ is the moduli space of stable $SL(n, \mathbb{C})$-bundles, that sends any $((E, \mathcal{F}_E), \nabla^E)$ to $(E, \mathcal{F}_E)$.

The pullback map $f^* : \text{Pic}(\mathcal{N}) \to \text{Pic}(\mathcal{M}^{\text{Higgs}}_{\text{dR}})$ is an isomorphism. On the other hand, $\text{Pic}(\mathcal{M}^{\text{Higgs}}_{\text{dR}}) = \mathbb{Z}$, and holomorphic line bundles on $\mathcal{M}^{\text{Higgs}}_{\text{dR}}$ are uniquely determined by their first class. Also the restriction map $\text{Pic}(\mathcal{M}^{\text{Higgs}}_{\text{dR}}) \to \text{Pic}(\mathcal{M}_{\text{dR}})$ is an isomorphism, because the codimension of the complement of $\mathcal{M}^{\text{Higgs}}_{\text{dR}}$ is at least two. Therefore, we have $\text{Pic}(\mathcal{M}^{\text{Higgs}}_{\text{dR}}) = \text{Pic}(\mathcal{N}) \cong \mathbb{Z}$, i.e., the holomorphic line bundles on $\mathcal{M}^{\text{Higgs}}_{\text{dR}}$ are uniquely determined by their first Chern class.

Let $L_1$ denote the restriction of $L_Z$ to $Z_1 = \mathcal{M}^{\text{Higgs}}_{\text{dR}}$. On $L_1$, the meromorphic connection $\nabla$ on $L_Z$ induces a holomorphic connection and its curvature on $Z_1$ is the holomorphic symplectic form $2\omega_f + 2\omega_f$, which is cohomologous to $2\omega_f$, since $\omega_f$ is exact. So by Chern–Weil theory, we have $c_1(L_1) = [\omega_f]$, the cohomology class of $\omega_f$.

It follows that $L_1$ is the holomorphic line bundle on $Z_1$ determined by $[\omega_f]$. To calculate $[\omega_f]$, note that the above projection $f$ has a $C^\infty$-section that sends any stable bundle $(E, \mathcal{F}_E)$ to the unique unitary flat connection on $E$. The restriction of $\omega_f$ to the image of that section coincides with the standard Kähler form on $\mathcal{N}$. On the other hand, the first Chern class of the determinant line bundle $\xi$ on $\mathcal{N}$ is the Kähler form. In fact, the curvature for the Quillen metric on $\xi$ is the Kähler form.

We conclude that $L_1$ is holomorphically isomorphic to the determinant bundle on $Z_1$, which coincides with the pullback of the determinant line bundle on $\mathcal{N}$ because the determinant line bundle is functorial. Similarly, the line bundles on the moduli space of stable Higgs bundles, namely $Z_0 = \mathcal{M}^{\text{Higgs}}_{\text{irr}}$, are uniquely determined by their first Chern class. Let $L_0$ denote the restriction of $L_Z$ to $Z_0$. Since $L_1$ is isomorphic to the determinant line bundle, and since the first Chern class of the family of line bundles $L_t : Z_t \to \mathbb{C}P^1$, is independent of $t$, it follows that $L_0$ is isomorphic to the determinant line bundle. For a description of $L_Z$ on all of $Z = \mathcal{M}_{\text{dR}}$, see [29 § 3.7].

3.4. Irreducible and admissible sections. In this subsection we recall some concepts and definitions from [6] and [4] on sections of the twistor projection $\varpi : \mathcal{M}_{\text{dR}} \to \mathbb{C}P^1$. We write $\mathcal{S}_{\mathcal{M}_{\text{dR}}}$ for the space of holomorphic sections. Since $\mathcal{M}_{\text{dR}}$ is a complex space, a similar argument as in Proposition 1.13 shows that $\mathcal{S}_{\mathcal{M}_{\text{dR}}}$ is a complex space as well. It is equipped with an antiholomorphic involution $\tau$ defined in (1.3), with $\tau_{\mathcal{M}_{\text{dR}}}$ playing the role of $\tau_Z$.

Definition 3.1. A holomorphic section $s \in \mathcal{S}_{\mathcal{M}_{\text{dR}}}$ is irreducible if the image of $s$ is contained in $\mathcal{M}^{\text{Higgs}}_{\text{dR}}$. These are precisely the sections of the twistor space of $\mathcal{M}^{\text{Higgs}}_{\text{dR}}$.

If $s : \mathbb{C}P^1 \to \mathcal{M}_{\text{dR}}$ is an irreducible section, then by [6] Lemma 2.2] (see also [4] Remark 1.11]) it admits a holomorphic lift

$$\tilde{s}(\lambda) = (\overline{\nabla}(\lambda), D(\lambda), \lambda) = (\overline{\nabla} + \sum_{j=1}^{\infty} \lambda^j \nabla_j, \lambda \partial + \Phi + \sum_{j=2}^{\infty} \lambda^j \Phi_j, \lambda), \quad \lambda \in \mathbb{C}$$

(3.9)

to the space of holomorphic $\lambda$-connections of class $C^k$. Here $\Psi, \Psi_k \in \Omega^{0,1}(\mathfrak{sl}(E)), \Phi, \Phi_k \in \Omega^{1,0}(\mathfrak{sl}(E))$. There is also a lift $\tilde{s}$ on both $\mathbb{C}P^1 \setminus \{0\}$.

The lifts $\tilde{s}$, $\tilde{s}$ over $\mathbb{C}$ and $\mathbb{C}P^1 \setminus \{0\}$ respectively, allow us to interpret $s|_{\mathbb{C}^*}$ as a $C^*$-family of flat connections

$$+ \nabla \lambda = \overline{\nabla}(\lambda) + \lambda^{-1} D(\lambda) = \lambda^{-1} \Phi + \nabla + \ldots,$$

and $- \nabla^\lambda$ defined similarly over $\mathbb{C}P^1 \setminus \{0, \infty\}$. Here we write $\nabla = \overline{\nabla} + \partial$ using the notation of equation (3.3). There exists a holomorphic $C^*$-family $g(\lambda)$ of $\text{GL}(n, \mathbb{C})$-valued gauge transformations, unique up to multiplication by a holomorphic scalar function, such that $+ \nabla g(\lambda) = - \nabla(\lambda)$ (see [6]).

Definition 3.2. We call a holomorphic section $s \in \mathcal{S}_{\mathcal{M}_{\text{dR}}}$ admissible if it admits a lift $\tilde{s}$ on $\mathbb{C}$ of the form

$$\tilde{s}(\lambda) = (\overline{\nabla} + \lambda \Psi, \lambda \partial + \Phi, \lambda)$$

for a Dolbeault operator $\overline{\partial}$ of type $(0,1)$, a Dolbeault operator $\partial$ of type $(1,0)$, a $(1,0)$-form $\Phi$ and a $(0,1)$-form $\Psi$, such that $(\overline{\partial}, \Phi)$ and $(\partial, \Psi)$ are semi-stable Higgs pairs on $\Sigma$ and $\Sigma$ respectively.
Now suppose that \( s \in \mathcal{S}^\mathrm{Mdh}_{\Sigma} \) is a real section, i.e., \( s = \tau_{\text{Mdh}} \circ s \circ \tau_{\mathbb{C}P^1} \). If we have a lift \( \nabla^\lambda \) on \( \mathbb{C} \subset \mathbb{C}P^1 \) of \( s \), then for every \( \lambda \in \mathbb{C}^* \) there is a gauge transformation \( g(\lambda) \in \mathcal{G}_\mathbb{C} \) such that
\[
\nabla^\lambda g(\lambda) = \nabla^{-\lambda^{-1}}.
\]

**Remark 3.3.** If \( (\partial^\mathbb{C} , \Phi) \) is a solution to the self-duality equations, then the associated twistor line is given by the \( \mathbb{C}^* \)-family of flat \( \text{SL}(n, \mathbb{C}) \)-connections
\[
\nabla^\lambda = \lambda^{-1} \Phi + \nabla + \lambda \Phi^{*\lambda}.
\]

By the non-abelian Hodge correspondence, this family gives rise to an equivariant harmonic map \( f : \tilde{\Sigma} \to \text{SL}(n, \mathbb{C})/\text{SU}(n) \) from the universal cover. If the solution \( (\partial^\mathbb{C} , \Phi) \) is irreducible, then so is the associated section in \( \mathcal{S}^\mathrm{Mdh}_{\Sigma} \).

In the next example we describe a large class of irreducible sections for \( n = 2 \).

**Example 3.4.** Let \( \nabla \) be an irreducible flat \( \text{SL}(2, \mathbb{C}) \)-connection on the rank two bundle \( V \to \Sigma \). We assume that \( \partial^\mathbb{C} \) is not strictly semi-stable, but either stable or strictly unstable. We also assume the analogous condition for the holomorphic structure \( \partial^\Sigma \) on \( \Sigma \). Due to [7] this assumption does not hold for all irreducible flat \( \text{SL}(n, \mathbb{C}) \)-connections. Under the assumption, we obtain a section \( s = s^\mathbb{C} \) as follows. If \( \partial^\mathbb{C} \) is stable
\[
\lambda \mapsto (\lambda, \partial^\mathbb{C}, \lambda \partial^\mathbb{C})
\]
is an irreducible section over \( \mathbb{C} \subset \mathbb{C}P^1 \). If \( \partial^\mathbb{C} \) is unstable, we consider its destabilizing subbundle \( L \subset V \) of positive degree. The connection induces a nilpotent Higgs field \( \Phi \) on the holomorphic vector bundle \( L \oplus (V/L) = L \oplus L^* \) via
\[
\Phi = \pi^{V/L} \circ \nabla|_L.
\]
This is a special case of [15] and can be interpreted from a gauge theoretic point of view (see also [4, § 4] for details): Consider a complementary bundle \( \bar{L} \subset V \) of \( L \), and the family of gauge-transformations
\[
g(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}.
\]
The family
\[
\lambda \mapsto (\lambda, \partial^\mathbb{C}, g(\lambda), \lambda \partial^\mathbb{C}, g(\lambda))
\]
extends to an irreducible (stable) Higgs pair at \( \lambda = 0 \) which identifies with \( (L \oplus L^*, \Phi) \).

4. Energy functional on sections of the Deligne–Hitchin moduli space

**4.1. The energy as a moment map.** It was proven in [3] Corollary 3.11] that the energy of an irreducible section \( s \) with lift \( \hat{\sigma} \) as in (3.19) is given by
\[
\mathcal{E}(s) = \frac{1}{2\pi i} \int_\Sigma \text{tr}(\Phi \wedge \Psi).
\]
In particular, this integral is independent of the lift \( \hat{\sigma} \). The reader should be aware of the different prefactors in (4.1) and in (3.5). In particular, if we think of \( \mathcal{E} \) as the energy of a harmonic map, it should be real-valued, while we want a moment map for the \( S^2 \)-action to be \( \mathbb{R} \)-valued. Working with the prefactor \( \frac{1}{2\pi i} \) also has the advantage that we get fewer factors of \( 2\pi i \) in the statements of the results below.

**Remark 4.1.** As pointed out in [6] Remark 2.3], the energy in the present example is defined for all local sections around \( \lambda = 0 \) which admit a lift as in (3.3).

Let us write again \( S' = S'_{\text{Mdh}} \) for the space of irreducible sections whose normal bundle is isomorphic to \( \mathcal{O}_{\mathbb{C}P^1(1)} \oplus 2d \). Take any \( s \in S' \). In terms of lifts of sections, a tangent vector \( V \in T_s S' \) is expressed as follows. Let \( \hat{\sigma} \) be a lift of \( s \) as in (3.19), and denote the curvature of the connection \( \partial + \partial^\mathbb{C} \) by \( F^{\partial + \partial^\mathbb{C}} = \partial\partial + \partial\partial^\mathbb{C} \). Expanding the integrability condition
\[
\overline{\partial}(\lambda)D(\lambda) + D(\lambda)\overline{\partial}(\lambda) = 0
\]
\[(4.2)\]
in powers of $\lambda$, the zeroth and first order coefficients yield
\[ \mathcal{F}_\Phi = 0 \quad (4.3) \]
\[ F^\partial + \mathcal{F} = 0. \]
Consider a family of sections $(s_t \in S_{\text{Moh}})_t$ with $s = s_0$ which represents $V \in T_\lambda S'$. The corresponding (lifted) infinitesimal variation $\hat{s} = (\hat{\mathcal{F}}(\lambda), \hat{D}(\lambda), \lambda)$ satisfies the linearisation of (4.2), i.e.,
\[ \mathcal{F}(\lambda)(\hat{D}(\lambda)) + D(\lambda)(\mathcal{F}(\lambda)) = 0. \quad (4.4) \]
Expanding $\hat{s}$ into a power series
\[ \hat{s}(\lambda) = \left( \sum_{k=0}^\infty \psi_k \lambda^k, \sum_{k=0}^\infty \varphi_k \lambda^k, \lambda \right), \]
for $\varphi_k \in \Omega^{1,0}(\mathfrak{sl}(E)), \psi_k \in \Omega^{0,1}(\mathfrak{sl}(E))$, the linearisation of (4.3) becomes
\[ \mathcal{F}_\varphi_0 + [\psi_0 \wedge \Phi] = 0 \]
\[ \mathcal{F}_\varphi_1 + \partial \psi_0 + [\varphi_0 \wedge \Psi] + [\Phi \wedge \psi_1] = 0. \]
Variations along the gauge orbit of $\hat{s}$ are determined by infinitesimal gauge transformations $\mathbb{C} \ni \lambda \mapsto \xi(\lambda) \in \Gamma(\Sigma, \mathfrak{sl}(E))$ and are of the form
\[ (\overline{\mathcal{F}}(\lambda)\xi(\lambda), D(\lambda)\xi(\lambda), \lambda). \quad (4.5) \]
By expanding $\xi(\lambda) = \sum_{k=0}^\infty \xi_k \lambda^k$, we get with (3.4) and (3.9)
\[ \mathcal{F}(\lambda)\xi(\lambda) = \mathcal{F}_0 + (\overline{\mathcal{F}}(\xi_1) + [\Phi, \xi_0])\lambda + O(\lambda^2) \]
\[ D(\lambda)\xi(\lambda) = [\Phi, \xi_0] + (\partial \xi_0 + [\Phi, \xi_1])\lambda + O(\lambda^2). \]
Now let $s \in S'$ with lift $\hat{s}$ over $\mathbb{C}$, and consider $V_j \in T_\lambda S'$, $j = 1, 2$, represented by
\[ \hat{s}_j = (\overline{\mathcal{F}}(\lambda), \hat{D}(\lambda), \lambda) = (\psi^{(j)} + \varphi^{(j)} \lambda, \varphi^{(j)} \lambda, \lambda) + O(\lambda^2). \]
Then we define, recalling the definition of $\omega_\lambda$ given in (3.8),
\[ \hat{\Omega}_2(V_1, V_2) = -\frac{i}{2} \frac{\partial}{\partial \lambda} |_{\lambda=0} \omega_\lambda(V_1(\lambda), V_2(\lambda)) \]
\begin{align*}
&= -\frac{i}{2} \frac{\partial}{\partial \lambda} |_{\lambda=0} 2 \int_{\Sigma} \text{tr} \left( -\hat{D}_1(\lambda) \wedge \mathcal{F}_2(\lambda) + \hat{D}_2(\lambda) \wedge \mathcal{F}_1(\lambda) \right) \\
&= \int_{\Sigma} \text{tr} \left( -\varphi^{(1)}(\psi^{(2)} + \varphi^{(2)} \wedge \psi^{(1)} - \varphi^{(1)} \wedge \psi^{(2)} + \varphi^{(2)} \wedge \psi^{(1)} \right). \\
\end{align*}

We view $\hat{\Omega}$ as a two-form on the infinite-dimensional space of germs of sections of $\varpi$ at $\lambda = 0$. Note that the formula for $\hat{\Omega}$ is exactly (1.21) in the present context.

**Proposition 4.2.** The two-form $\hat{\Omega}$ descends to a holomorphic two-form on the space of irreducible sections, which on $S'_{\text{Moh}}$ coincides with the holomorphic symplectic form $\Omega_0$ defined in (1.21).

**Proof.** We will show that $\hat{\Omega}_2$ is degenerate along the gauge orbits. To this end, let $\hat{s}$ be a germ of a section near $\lambda = 0$, and let $\xi(\lambda) = \sum_{k=0}^\infty \xi_k \lambda^k$ be an infinitesimal gauge transformation. The corresponding tangent vector $V_1$ is represented by
\[ \hat{s}_1 = (\overline{\mathcal{F}}(\lambda)\xi(\lambda), D(\lambda)\xi(\lambda), \lambda). \]
Then for an arbitrary tangent vector $V_2$ represented by $\hat{s}_2 = (\overline{\mathcal{F}}(\lambda), \hat{D}(\lambda), \lambda)$, we find
\[ \hat{\Omega}_2(V_1, V_2) = \frac{\partial}{\partial \lambda} |_{\lambda=0} \int_{\Sigma} \text{tr} \left( -\hat{D}(\lambda) \wedge \mathcal{F}(\lambda) + D(\lambda)\mathcal{F}(\lambda) \wedge \mathcal{F}(\lambda) \right) \\
\text{(Stokes)} = \frac{\partial}{\partial \lambda} |_{\lambda=0} \int_{\Sigma} \text{tr} \left( \overline{\mathcal{F}(\lambda)}(\hat{D}(\lambda)) + D(\lambda)(\overline{\mathcal{F}(\lambda)})\xi(\lambda) \right) = 0; \]
we used (4.4). This shows that $\hat{\Omega}$ descends to $S'$. \qed
Theorem \[2.3\] thus allows us to make the following conclusion.

**Corollary 4.3.** The restriction of $2\pi i \mathcal{E} : S'_{\text{Mdh}} \rightarrow \mathbb{C}$ is a holomorphic moment map for the natural $\mathbb{C}^\ast$-action on $S'_{\text{Mdh}}$ with respect to the holomorphic symplectic form $\Omega_0$. In particular, the $\mathbb{C}^\ast$-orbits in $S'_{\text{Mdh}}$ are exactly the critical points of $\mathcal{E}|_{S'_{\text{Mdh}}}$.

4.2. **Explicit description of some $\mathbb{C}^\ast$-fixed sections.** Corollary 4.3 shows a close relationship between $\mathbb{C}^\ast$-orbits in $S_{\text{Mdh}}$ and the energy functional. We therefore examine the $\mathbb{C}^\ast$-orbits more closely in this section. Before explicitly determining the $\mathbb{C}^\ast$-fixed sections, we first observe:

**Lemma 4.4.** The set $S^\ast_{\text{Mdh}}$ of all $\mathbb{C}^\ast$-fixed sections is in a natural bijection with $\mathcal{M}_{\text{dR}}$, the moduli space of flat completely reducible $\text{SL}(n, \mathbb{C})$-connections.

In particular, the critical points of $\mathcal{E} : S'_{\text{Mdh}} \rightarrow \mathbb{C}$ correspond to an open subset of $\mathcal{M}^\ast_{\text{dR}}$, the moduli space of flat irreducible $\text{SL}(n, \mathbb{C})$-connections.

**Proof.** Let $\nabla \in \mathcal{M}_{\text{dR}}$. As in Section 2.3 we obtain the following $\mathbb{C}^\ast$-invariant section $s_\mathcal{V} : \mathbb{C}^\ast \rightarrow \mathcal{M}_{\text{DH}}$:

$$ s_\mathcal{V}(\lambda) = [(\overline{\partial} \mathcal{V}, \lambda \overline{\partial} \mathcal{V}, \lambda)], \quad \overline{\partial} \mathcal{V} = \nabla^{1,0}, \quad \overline{\partial} \mathcal{V} = \nabla^{0,1}. \tag{4.6} $$

By a crucial result of Simpson (\[44\] for existence and \[45\] for a more explicit approach), the limits of $s_\mathcal{V}(\lambda)$ for $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$ always exist in $\mathcal{M}_{\text{Higgs}}(\Sigma, \text{SL}(n, \mathbb{C}))$ and $\mathcal{M}_{\text{Higgs}}(\overline{\Sigma}, \text{SL}(n, \mathbb{C}))$ respectively. The resulting section, also denoted by $s_\mathcal{V} \in \mathcal{M}_{\text{DH}}$, is $\mathbb{C}^\ast$-invariant by continuity. Evaluation of sections $s : \mathbb{C}P^1 \rightarrow \mathcal{M}_{\text{DH}}$ at $\lambda = 1$ gives the inverse of the map $\nabla \mapsto s_\mathcal{V}$.

The last statement in the lemma is a direct consequence of Theorem 2.3 and Corollary 4.3. \[\square\]

We next determine explicitly the $\mathbb{C}^\ast$-fixed sections $s \in S_{\text{Mdh}}$ such that $s$ is irreducible over $\mathbb{C}$, by using some results of \[10\]. In terms of Lemma 4.4, these are precisely the sections $s_\mathcal{V}$ such that $s_\mathcal{V}(0)$ is stable. Indeed, since irreducibility is an open condition, $s_\mathcal{V}(\lambda)$ is an irreducible $\lambda$-connection for $\lambda$ close to $0$. Using the $\mathbb{C}^\ast$-invariance, we see that $s(\lambda)$ is irreducible for every $\lambda \in \mathbb{C}$.

For any $\mathbb{C}^\ast$-fixed sections $s_\mathcal{V}$, its values at $0$ and $\infty$ are $\mathbb{C}^\ast$-fixed Higgs bundles on $\Sigma$ and $\overline{\Sigma}$ respectively. These are called complex variations of Hodge structures (VHS). Let $(\overline{\partial}, \Phi)$ be any VHS on $\Sigma$. The fact that $(\overline{\partial}, \Phi)$ is a $\mathbb{C}^\ast$-fixed point yields a splitting

$$ E = \bigoplus_{j=1}^l E_j \tag{4.7} $$

into a direct sum of holomorphic bundles. With respect to this splitting, $\overline{\partial}$ and $\Phi$ are given in the following block form

$$ \overline{\partial} = \begin{pmatrix} \overline{\partial} E_1 & 0 & \cdots & \cdots & 0 \\ 0 & \overline{\partial} E_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \overline{\partial} E_j & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \Phi^{(1)} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & \Phi^{(l-1)} \end{pmatrix}. \tag{4.8} $$

where $\Phi^{(j)} \in H^0(\Sigma, \text{Hom}(E_j, E_{j+1}) \otimes K_\Sigma)$. The sheaf $\mathfrak{s}(E)$ of trace-free holomorphic endomorphisms of $E$ further decomposes into

$$ \mathfrak{s}(E) = \bigoplus_{k \in \mathbb{Z}} \mathfrak{s}(E)_k, \quad \mathfrak{s}(E)_k = \{ \psi \in \mathfrak{s}(E) \mid \psi(E_k) \subset E_{k+1} \}. $$

By construction, $\Phi \in H^0(\Sigma, K_\Sigma \otimes \mathfrak{s}(E)_{-1})$. To define the next notion, let

$$ N_+ = \bigoplus_{k>0} \mathfrak{s}(E)_k, \quad N_- = \bigoplus_{k<0} \mathfrak{s}(E)_k, \quad L = \mathfrak{s}(E)_0. \tag{4.9} $$

Note that $N_+$ (respectively, $N_-$) is the subspace of $\mathfrak{s}(E)$ consisting of endomorphisms of $E$ that are strictly upper (respectively, lower) block-triangular with respect to the splitting (4.7), while $L$ is the space of block-diagonal elements of $\mathfrak{s}(E)$. 


Now let \((\overline{\partial}, \Phi) \in \mathcal{M}_{\text{Higgs}}(\text{SL}(n, \mathbb{C}))\) be a stable complex variation of Hodge structures. Then the BB-slice \([10\text{ Definition 3.7}]\) through \((\overline{\partial}, \Phi)\) is defined by

\[
\mathcal{B}^+_{(\overline{\partial}, \Phi)} = \{ (\beta, \phi) \in \Omega^{0, 1}(N_+^1) \oplus \Omega^{1, 0}(L \oplus N_+) \mid D''(\beta, \phi) + [\beta \wedge \phi] = 0, \quad D'(\beta, \phi) = 0 \}. \tag{4.10}
\]

Here we denote by

\[
D := \overline{\partial} + \partial^h + \Phi + \Phi^* h
\]

the non-abelian Hodge connection attached to \((\overline{\partial}, \Phi)\) with harmonic metric \(h\), and

\[
D'' := \overline{\partial} + \Phi, \quad D' := \partial^h + \Phi^* h.
\]

Hence the equations in \((4.10)\) are explicitly given by

\[
D''(\beta, \phi) + [\beta \wedge \phi] = \overline{\partial}\phi + [(\Phi + \phi) \wedge \beta] = 0, \quad D'(\beta, \phi) = \partial^h \beta + [\Phi^* h \wedge \phi] = 0. \tag{4.11}
\]

Note that \(\mathcal{B}^+_{(\overline{\partial}, \Phi)}\) is a finite-dimensional affine space. Then, \([10\text{ Theorem 1.4 (3)]}\) states that the map

\[
p : \mathcal{B}^+_{(\overline{\partial}, \Phi)} \times \mathbb{C} \longrightarrow \mathcal{M}_{\text{Hod}}, \quad ((\beta, \phi), \lambda) \longmapsto [\lambda, \overline{\partial} + \lambda\Phi^* + \beta, \lambda \partial^h + \Phi + \phi]
\]

is a holomorphic embedding onto the “attracting set”

\[
W(\overline{\partial}, \Phi) = \{ m \in \mathcal{M}_{\text{Hod}}^{\text{irr}} \mid \lim_{\zeta \to 0} \zeta \cdot m = (\overline{\partial}, \Phi) \}
\]

and is compatible with the obvious projections to \(\mathbb{C}\). In particular, if \(W^\lambda(\overline{\partial}, \Phi)\) denotes the intersection of \(W(\overline{\partial}, \Phi)\) with the fiber \(\overline{\omega}^{-1}(\lambda)\), then \(W^\lambda(\overline{\partial}, \Phi)\) is biholomorphic to the affine space \(\mathcal{B}^+_{(\overline{\partial}, \Phi)}\) via the map \(p_\lambda := p(\bullet, \lambda)\). Thus, \(\mathcal{M}_{\text{Hod}}^{\text{irr}}\) is stratified by affine spaces.

Given \((\beta, \phi) \in \mathcal{B}^+_{(\overline{\partial}, \Phi)}\), we can use Lemma \([4.4]\) and \([10.6]\) to define the \(\mathbb{C}^*\)-fixed section

\[
s_{(\beta, \phi)} := s_{p_1(\beta, \phi)} \in \mathcal{S}_{\mathcal{M}_{\text{Hod}}}.
\]

As observed earlier, \(s_{(\beta, \phi)}\) is an irreducible section over \(\mathbb{C} \subset \mathbb{C}P^1\) but not necessarily over all of \(\mathbb{C}P^1\).

**Proposition 4.5.** Over \(\mathbb{C}\), the \(\mathbb{C}^*\)-fixed section \(s_{(\beta, \phi)}\) may be expressed as

\[
s_{(\beta, \phi)}(\lambda) = \left[ \lambda, \overline{\partial} + \lambda(\Phi^* h + \beta_1) + \sum_{j=2}^{l} \lambda^j \beta_j, \Phi + \lambda \partial^h + \sum_{j=0}^{l} \lambda^{j+1} \phi_j \right], \tag{4.12}
\]

where \(\beta = \sum_{j=1}^{l} \beta_j\), with \(\beta_j \in \Omega^{0, 1}(\mathfrak{sl}(E)_j)\) and \(\phi = \sum_{j=0}^{l} \phi_j\) with \(\phi_j \in \Omega^{1, 0}(\mathfrak{sl}(E)_j)\).

**Proof.** Let \(\nabla = p_1(\beta, \phi) = [D + \beta + \phi]\) so that

\[
\overline{\nabla} = \overline{\partial} + \Phi^* h + \beta, \quad \nabla = \partial^h + \Phi + \phi.
\]

Hence \(s_{(\beta, \phi)} = s_{\overline{\nabla}}\) is given by

\[
s_{(\beta, \phi)}(\lambda) = \left[ \lambda, \overline{\partial} + (\Phi^* h + \beta), \lambda \partial^h + \lambda \Phi + \lambda \phi \right]
\]

for \(\lambda \in \mathbb{C}^*\) (see \([10.6]\)). This does not give a lift of \(s_{(\beta, \phi)}\) over all of \(\mathbb{C}\), unless the holomorphic bundle \((E, \overline{\partial})\) is stable, in which case we must have \(\beta = 0\) and \(\Phi = 0\).

To construct a lift over all of \(\mathbb{C}\) we use the \(\mathbb{C}^*\)-family of gauge transformations

\[
g(\lambda) = \lambda^m \begin{pmatrix} \lambda^{1-l} \text{id}_{E_1} & 0 & \cdots & \cdots & 0 \\ 0 & \lambda^{2-l} \text{id}_{E_2} & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \lambda^{l} \text{id}_{E_l} \end{pmatrix}, \tag{4.13}
\]

where \(m = \frac{1}{n} \sum_{j=1}^{l} (l - j) \text{rk}(E_j)\) in order to ensure \(\det g(\lambda) = 1\). Then any \(\xi \in \mathfrak{sl}(E)_j\) satisfies

\[
g(\lambda)^{-1} \xi g(\lambda) = \lambda^l \xi.
\]
Let $\beta = \sum_{j=1}^l \beta_j$, with $\beta_j \in \Omega^{0,1}(\mathfrak{sl}(E)_j)$, and similarly $\phi = \sum_{j=0}^l \phi_j$ with $\phi_j \in \Omega^{1,0}(\mathfrak{sl}(E)_j)$. Then using $\Phi \in H^0(K \otimes \mathfrak{sl}(E)_{-1})$ and $\Phi^* \in \Omega^{0,1}(K \otimes \mathfrak{sl}(E)_1)$, we get that

$$(\bar{\partial} + (\Phi^* + \beta), \lambda \partial^h + \lambda \Phi + \lambda \phi) \cdot g(\lambda) = \left( \bar{\partial} + \lambda(\Phi^* + \beta_1) + \sum_{j=2}^{l} \lambda^j \beta_j, \Phi + \lambda \partial^h + \sum_{j=0}^{l} \lambda^{j+1} \phi_j \right).$$

The result follows. □

We next discuss the implications for the $\mathbb{C}^*$-fixed leaves of the foliation $F^+$ on $S' = S_{\mathcal{M}_{2n}}$. Recall that these leaves consist, in particular, of irreducible sections (on all of $\mathbb{C}P^1$) by definition. We denote by $S'_{(\Omega, \Phi)}$ all sections in $S'$ which pass through the stable complex variation of Hodge structure $(\bar{\partial}, \Phi) \in \mathcal{M}_{\text{Higgs}}$ at $\lambda = 0$.

**Proposition 4.6.** The $\mathbb{C}^*$-fixed point locus $(S'_{(\bar{\partial} + \Omega, \Phi)})^{\mathbb{C}^*}$ is isomorphic to an open and non-empty subset of the affine space $B^+(\bar{\partial}, \Phi)$.

**Proof.** Consider the section $s_{(\beta, \phi)} : \mathbb{C}P^1 \rightarrow \mathcal{M}_{\text{DH}}$ for $(\beta, \phi) \in B^+(\bar{\partial}, \Phi)$ which is irreducible over $\mathbb{C}$. Since the complement of $\mathcal{M}_{\text{Higgs}}(\Sigma, \text{SL}(n, \mathbb{C}))$ in $\mathcal{M}_{\text{Higgs}}(\Sigma, \text{SL}(n, \mathbb{C}))$ is closed and of codimension at least two (cf. [17]), it follows that $s_{(\beta, \phi)}$ is an irreducible section for $(\beta, \phi) \in B^+(\bar{\partial}, \Phi)$ in an open and dense subset of $B^+(\bar{\partial}, \Phi)$. Note that $(\beta, \phi) = (0, 0)$ corresponds to the twistor line $s_{(\bar{\partial}, \Phi)}$ through $(\bar{\partial}, \Phi)$, which lies in $S'$. Since $S'$ is open and non-empty in the space of all irreducible sections, we therefore see that the irreducible and $\mathbb{C}^*$-fixed section $s_{(\beta, \phi)}$ has the desired normal bundle for $(\beta, \phi)$ in an open and non-empty subset $U \subset B^+(\bar{\partial}, \Phi)$.

Altogether we obtain the isomorphism $p_1^{-1} \circ \text{ev}_1 : (S'_{(\bar{\partial}, \Phi)})^{\mathbb{C}^*} \xrightarrow{\cong} U$. □

From Theorem 2.3, we immediately obtain:

**Corollary 4.7.** The locus of critical points $s \in S'$ of $E : S' \rightarrow \mathbb{C}$ is isomorphic to an open and non-empty subset in $\mathcal{M}^{\mathbb{C}^*}_{\text{DH}}$. It is foliated by leaves which are isomorphic to open and non-empty subsets of affine spaces.

**Proof.** The first statement follows, by a genericity argument, from Lemma 4.4. The second one is a consequence of Proposition 4.6. □

**Remark 4.8.** Let $s : \mathbb{C}P^1 \rightarrow \mathcal{M}_{\text{DH}}$ be a $\mathbb{C}^*$-fixed section such that $s(0) = (\bar{\partial}, \Phi)$ and $s(\infty) = (\bar{\partial}, \tilde{\Psi})$ are stable VHS on $\Sigma$ and $\Sigma$ respectively, with respective splittings of the underlying smooth bundle $E$ of the form

$$E = \bigoplus_{j=1}^{l} E_j, \quad E' = \bigoplus_{j=1}^{l'} E'_j.$$

With respect to these splittings the respective holomorphic structures are diagonal and the Higgs fields $\Phi$ and $\tilde{\Psi}$ are lower triangular as in (4.8). Then we have the BB-slices $B^+(\bar{\partial}, \Phi)(\Sigma)$ and $B^+(\bar{\partial}, \tilde{\Phi})(\Sigma)$. By Proposition 4.6 and its analog on $\Sigma$ we see that, on the one hand, $s$ corresponds to $(\beta, \phi) \in B^+(\bar{\partial}, \Phi)(\Sigma)$, and on the other hand to $(\tilde{\beta}, \tilde{\phi}) \in B^+(\bar{\partial}, \tilde{\Phi})(\Sigma)$. Therefore, we obtain two distinguished lifts of $s$ over $\Sigma$ and $\Sigma \cup \{\infty\}$ of the form

$$s(\lambda) = [\lambda, \hat{s}_{(\beta, \phi)}(\lambda)]_{\Sigma} = \left[ \lambda, \bar{\partial} + \lambda(\Phi^* + \beta_1) + \sum_{j=2}^{l} \lambda^j \beta_j, \Phi + \lambda \partial^h + \sum_{j=0}^{l} \lambda^{j+1} \phi_j \right]_{\Sigma},$$

$$s(\lambda) = [\lambda^{-1}, \hat{s}_{(\tilde{\beta}, \tilde{\phi})}(\lambda^{-1})]_{\Sigma} = \left[ \lambda^{-1}, \bar{\partial} + \lambda^{-1}((\Psi^* + \tilde{\beta}_1) + \sum_{j=2}^{l'} \lambda^{-j} \tilde{\beta}_j, \Psi + \lambda^{-1} \partial^{h} + \sum_{j=0}^{l'} \lambda^{-(j+1)} \phi_j \right]_{\Sigma}.$$

Let $g_0$ be a gauge transformation such that

$$(\bar{\partial} + \Psi + \Psi^* + \tilde{\beta} + \tilde{\phi}) = g_0 = \bar{\partial} + \partial^h + \beta + \phi.$$
Going through the proof of Proposition 4.8 and writing \( g(\lambda) \) and \( \bar{g}(\lambda^{-1}) \) for the respective \( \mathbb{C}^* \)-families of gauge transformations we get that
\[
\hat{s}_{(\beta, \phi)}(\lambda) = \hat{s}_{(\bar{\beta}, \bar{\phi})}(\lambda^{-1}) \cdot \bar{g}(\lambda^{-1})^{-1} g_0 g(\lambda)
\]
for any \( \lambda \in \mathbb{C}^* \).

In general, starting only with the lift \( \hat{s}_{(\beta, \phi)} \) over \( \mathbb{C} \) obtained above, it seems hard to determine explicitly the lift \( \hat{s}_{(\beta, \phi)}(\lambda^{-1}) \) over \( \mathbb{C} P^1 \setminus \{0\} \) or even the limiting VHS \( s_{(\beta, \phi)}(\infty) \). The next two examples discuss some situations in which the limit can be computed.

**Example 4.9.** Suppose the holomorphic structure \( \partial^h + \Phi + \phi \) is stable on \( \Sigma \). Then we can argue as follows. For \( \lambda \in \mathbb{C}^* \) we can write, using the Deligne gluing:
\[
s_{(\beta, \phi)}(\lambda) = [\lambda, \bar{\theta} + \Phi^* + \beta, \lambda(\partial^h + \Phi + \phi)]_{\Sigma} = [\lambda^{-1}, \partial^h + \Phi + \phi, \lambda^{-1}(\bar{\theta} + \Phi^* + \beta)]_{\Sigma}
\]
Under our assumption that \( \partial^h + \Phi + \phi \) is stable, this allows us to conclude \( s_{(\beta, \phi)}(\infty) = (\partial^h + \Phi + \phi, 0) \). We will see in the proof of Theorem 4.17 that this situation does in fact occur, at least for rank 2 bundles.

**Example 4.10.** Consider the rank two case, \( n = 2 \). If \( s \) is the twistor line through a VHS \( (\bar{\theta}, \Phi) \) on \( \Sigma \), then we have \( E = V \oplus V^* \), where \( V \) is a line bundle with \( 0 < \deg V \leq g - 1 \) and \( V^* = \ker \Phi \). Then \( s(\infty) = (\partial^h, \Phi^* + \beta) \) and the corresponding splitting is \( E = V^* \oplus V \). Note that, since \( \Sigma \) and \( \Sigma \) come with opposite orientations, we have \( \deg V^* > 0 \), as a bundle on \( \Sigma \). Then \( \bar{g}(\lambda^{-1}) = g(\lambda) \) in this case, as the order is reversed. The associated lifts are thus just the lifts of \( s \) over \( \mathbb{C} \) and \( \mathbb{C}^* \) given by the harmonic metric, i.e. the associated solution of the self-duality equations.

**Example 4.11 (Grafting sections).** In [24] a special class of \( \mathbb{C}^* \)-invariant sections of \( \mathcal{M}_{\text{DH}}(\Sigma, \text{SL}(2, \mathbb{C})) \), called *grafting sections*, have been constructed by using grafting of projective structures on \( \Sigma \). We recover them from the previous proposition as follows.

Consider the \( \mathbb{C}^* \)-fixed stable Higgs bundle \((\bar{\theta}, \Phi)\) with
\[
E = K_{\Sigma}^+ \oplus K_{\Sigma}^- \quad \Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]
where \( K_{\Sigma}^+ \) is a square root of the canonical bundle \( K_{\Sigma} \). To determine (4.9) in this example, we define \( E_1 := K_{\Sigma}^+, E_2 := K_{\Sigma}^- \). Then we see that
\[
N_+ \cong K_{\Sigma}, \quad N_- = K_{\Sigma}^{-1}, \quad L \cong O_{\Sigma}
\]
By (1111), \((0, \phi) \in B^+_{(\bar{\theta}, \Phi)} \) if and only if \( \bar{\theta}\phi = 0 \) and \([\phi \wedge \Phi^* + \beta] = 0 \). Hence \( \phi \) is of the form
\[
\phi = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}, \quad q \in H^0(\Sigma, K_{\Sigma}^\otimes 2),
\]
with respect to the splitting \( E = E_1 \oplus E_2 \). For those \( q \) such that the monodromy of the corresponding flat connection at \( \lambda = 1 \) is real, the sections \( s_{(0, \phi)} \) are precisely the grafting sections of [24] §2.1. Since \( \beta = 0 \) in this case, we see that the energy of a grafting section is the same as the energy of the twistor line associated with the stable Higgs pair \((\bar{\theta}, \Phi)\). If the monodromy of the corresponding flat connection is real, then [24] shows that the section \( s_{(0, \phi)} \) is real and defines an element of \( (S'_{\text{M}_{\text{DH}}})^\tau \), in particular it has the correct normal bundle \( O_{\mathbb{C} P^1}(1)^\otimes 2d \). But the section \( s_{(0, \phi)} \) is not admissible and thus cannot correspond to a solution of the self-duality equations. This shows that \( \mathcal{M}_{\text{SD}}(\Sigma, \text{SL}(2, \mathbb{C})) \subseteq (S'_{\text{M}_{\text{DH}}})^\tau \).

### 4.3. The energy of a \( \mathbb{C}^* \)-fixed section

Proposition 4.8 gives concrete formulas for all \( \mathbb{C}^* \)-fixed points \( s \in S'_{\text{M}_{\text{DH}}} \) such that \( s(0) \) is a stable VHS. We next compute the energy of such sections.

**Proposition 4.12.** Let \((\bar{\theta}, \Phi)\) be a stable \( \mathbb{C}^* \)-fixed \( \text{SL}(n, \mathbb{C}) \)-Higgs bundle, and let \( s_{(\beta, \phi)} \) be the \( \mathbb{C}^* \)-fixed section corresponding to \((\beta, \phi) \in B^+_{(\bar{\theta}, \Phi)} \). Its energy is given by
\[
\mathcal{E}(s_{(\beta, \phi)}) = \mathcal{E}(s_0) = \sum_{k=2}^l (k - 1) \deg(E_k),
\]
where \( s_0 \) is the twistor line through \((\bar{\theta}, \Phi)\).
Proof. Write \( s(\beta, \phi) \) in a form as in (4.12). Then the definition of \( \mathcal{E} \) immediately implies that
\[
\mathcal{E}(s(\beta, \phi)) = \mathcal{E}(s_0) + \frac{1}{2\pi} \int_{\Sigma} \text{tr}(\Phi \wedge \beta_1).
\]
Next we will show that \( \int_{\Sigma} \text{tr}(\Phi \wedge \beta_1) = 0 \). To this end, let us write
\[
\Phi = \sum_{k=1}^{l-1} \Phi^{(k)}, \quad \beta_1 = \sum_{k=1}^{l-1} \beta^{(k)},
\]
where \( \Phi^{(k)} \in \Omega^{1,0}(\text{Hom}(E_k, E_{k+1})) \), \( \beta^{(k)} \in \Omega^{0,1}(\text{Hom}(E_{k+1}, E_k)) \); see the block form in (4.8). It follows that
\[
\text{tr}(\Phi \wedge \beta_1) = \sum_{k=1}^{l} \text{tr}_{E_k}(\Phi^{(k-1)} \wedge \beta^{(k-1)}).
\]
Note that each summand \( \Phi^{(k-1)} \wedge \beta^{(k-1)} \) belongs to \( \Omega^{1,1}(\text{End}(E_k)) \) and we have adopted the convention that \( \Phi^{(k)} = 0 = \beta^{(k)} \) if \( k = 0, l \).

Now, equation (4.11) implies that
\[
\overline{\partial} \phi_0 + [\Phi \wedge \beta_1] = 0,
\]
and we can write
\[
[\Phi \wedge \beta_1] = \sum_{k=1}^{l-1} \Phi^{(k-1)} \wedge \beta^{(k-1)} + \beta^{(k)} \wedge \Phi^{(k)}.
\]
Thus, for each \( k = 1, \ldots, l \),
\[
\overline{\partial} \phi_0^{(k)} + \Phi^{(k-1)} \wedge \beta^{(k-1)} + \beta^{(k)} \wedge \Phi^{(k)} = 0.
\]
Consider the case of \( k = l \):
\[
\overline{\partial} \phi_0^{(l)} + \Phi^{(l-1)} \wedge \beta^{(l-1)} = 0.
\]
Taking the trace of this equation and integrating over \( \Sigma \), we find, using Stokes’ theorem, that
\[
\int_{\Sigma} \text{tr}_{E_l}(\Phi^{(l-1)} \wedge \beta^{(l-1)}) = 0.
\]
Now assume that \( \int_{\Sigma} \text{tr}_{E_{k+1}}(\Phi^{(k)} \wedge \beta^{(k)}) = 0 \) for all \( k \geq k_0 \). Then we have
\[
\overline{\partial} \phi_0^{(k_0)} + \Phi^{(k_0-1)} \wedge \beta^{(k_0-1)} + \beta^{(k_0)} \wedge \Phi^{(k_0)} = 0.
\]
Taking the trace and integrating yields
\[
0 = \int_{\Sigma} \text{tr}_{E_{k_0}}(\Phi^{(k_0-1)} \wedge \beta^{(k_0-1)} + \beta^{(k_0)} \wedge \Phi^{(k_0)})
= \int_{\Sigma} \text{tr}_{E_{k_0}}(\Phi^{(k_0-1)} \wedge \beta^{(k_0-1)}) - \int_{\Sigma} \text{tr}_{E_{k_0+1}}(\Phi^{(k_0)} \wedge \beta^{(k_0)})
= \int_{\Sigma} \text{tr}_{E_{k_0}}(\Phi^{(k_0-1)} \wedge \beta^{(k_0-1)}).
\]
It follows inductively that \( \int_{\Sigma} \text{tr}(\Phi \wedge \beta_1) = 0 \).

It remains to compute the energy of the twistor line \( s_0 \). To this end, we observe that
\[
\mathcal{E}(s_0) = \frac{1}{2\pi} \int_{\Sigma} \text{tr}(\Phi \wedge \Phi^* s) = \frac{1}{2\pi} \sum_{k=2}^{l} \int_{\Sigma} \text{tr}_{E_k}(\Phi^{(k-1)} \wedge (\Phi^{(k-1)*)^s}) = \sum_{k=2}^{l} \mathcal{E}_k(s_0),
\]
where we put \( \mathcal{E}_k(s_0) = \frac{1}{2\pi} \int_{\Sigma} \text{tr}_{E_k}(\Phi^{(k-1)} \wedge (\Phi^{(k-1)*)^s}) \) for \( k \geq 2 \). The equation \( F^{\nabla^h} + [\Phi \wedge \Phi^* s] = 0 \) is block-diagonal with respect to the splitting \( E = \bigoplus_{k=1}^{l} E_k \), with components
\[
F^{\nabla^h}_{E_k} + (\Phi^{(k-1)})^s + (\Phi^{(k-1)*)^s} \wedge \Phi^{(k)} = 0.
\]
This gives the following recursive relations:
\[
\mathcal{E}_k(s_0) = \frac{1}{2\pi i} \int_{\Sigma} \text{tr} E_k (\Phi^{(k-1)} \wedge (\Phi^{(k-1)})^* \kappa)
\]
\[
= \frac{1}{2\pi i} \int_{\Sigma} \text{tr} E_k (F\nabla^h_k) + \frac{1}{2\pi i} \int_{\Sigma} \text{tr} E_{k+1} (\Phi^{(k)} \wedge (\Phi^{(k)})^* \kappa)
\]
\[
= \deg(E_k) + \mathcal{E}_{k+1}(s_0).
\]
Thus, if \( k = l \), we find that
\[
\mathcal{E}_l(s_0) = \deg(E_l),
\]
and for general \( k \) we get that
\[
\mathcal{E}_k(s_0) = \sum_{j=k}^{l-1} \deg(E_j) + \mathcal{E}_l(s_0) = \sum_{j=k}^{l} \deg(E_j).
\]
Therefore,
\[
\mathcal{E}(s_0) = \sum_{k=2}^{l} \mathcal{E}_k(s_0) = \sum_{k=2}^{l} \sum_{j=k}^{l} \deg(E_j) = \sum_{k=2}^{l} (k-1) \deg(E_k),
\]
and this completes the proof. \(\square\)

4.4. The second variation of the Energy at a \( \mathbb{C}^* \)-fixed section. Next we study the second variation of the energy functional \( \mathcal{E} \) at a \( \mathbb{C}^* \)-fixed point.

Examining the proof of Proposition 4.3 we can check explicitly that the sections \( s_{(\beta,\phi)} \) satisfy for any \( \zeta \in \mathbb{C}^* \) the relation
\[
\zeta \hat{s}_{(\beta,\phi)} = \hat{s}_{(\beta,\phi)} \cdot g(\zeta)^{-1}.
\]
Moreover, if we use the notation of equation (4.13) and put
\[
\xi = \begin{pmatrix}
(m + 1 - l)\text{id}_{E_2} & 0 & \ldots & \ldots & 0 \\
0 & (m + 2 - l)\text{id}_{E_2} & \ddots & \ldots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots \\
0 & \ldots & \ldots & 0 & \text{mid}_{E_1}
\end{pmatrix},
\]
then \([\xi, \cdot]\) acts as multiplication by \( k \) on \( \mathfrak{sl}(E) \) and we see that
\[
-i\lambda \frac{d}{d\lambda} \overline{D}(\lambda) = \overline{D}(\lambda) \xi(\lambda), \quad iD(\lambda) - i\lambda \frac{d}{d\lambda} D(\lambda) = D(\lambda) \xi(\lambda)
\]
with
\[
(\overline{D}(\lambda), D(\lambda)) = \left( \overline{\beta} + \lambda (\Phi^* \kappa + \beta_1) + \sum_{j=2}^{l} \lambda^j \beta_j, \Phi + \lambda \partial \beta + \sum_{j=0}^{l} \lambda^{j+1} \phi_j \right).
\]
For \( \xi(\lambda) = \sum_{k=0}^{\infty} \xi_k \lambda^k \) we deduce from (4.14) the following equations
\[
0 = \overline{\xi}_0, \quad \Phi = [\Phi, \xi_0], \quad 0 = [\Phi, \xi_1] + \partial \xi_0.
\]
(4.15)
We can now compute the second variation of \( \mathcal{E} \) at such fixed points.

**Proposition 4.13.** The second variation of \( \mathcal{E} \) at a \( \mathbb{C}^* \)-fixed point \( s \) with lift \( \hat{s} \) as in (3.9) is given by
\[
d^2\mathcal{E}(s) = \frac{1}{2\pi i} \int_{\Sigma} \text{tr} (\psi_0 \wedge [\varphi_1, \xi] + \varphi_1 \wedge [\psi_0, \xi] + \psi_1 \wedge [\varphi_0, \xi]) + \varphi_0 \wedge [\psi_1, \xi] + 2 \varphi_0 \wedge \psi_1).
\]

**Proof.** Let \( (s_t) \) be a family of sections with \( s_0 = s \). We compute, using the notation for \( \hat{s} \) and \( \hat{s} \) as in Section 4
\[
2\pi i \frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{E}(s_t) = \int_{\Sigma} \text{tr}(\Phi \wedge \psi_1 + \varphi_0 \wedge \Psi + 2 \varphi_0 \wedge \psi_1).
\]
There exist irreducible sections

Proof.

Proposition 4.12 allows us to compute

Proposition 4.16.

Proposition 4.13 shows that the second variation is closely related to the infinitesimal $\mathbb{C}^*$-action on the tangent space. The following proposition is obtained.

Proposition 4.14. Let $\dot{s}(\lambda) = (\partial \lambda, \dot{\lambda}(\lambda)) = (\sum_{k=0}^{\infty} \psi_k \lambda^k, \sum_{k=0}^{\infty} \varphi_k \lambda^k, \lambda)$ be an infinitesimal deformation of the critical point $s \in S$. Suppose that $\dot{s}$ satisfies

$$\dot{[\psi_0, \xi]} = m_0 \psi_0, \quad \dot{[\psi_1, \xi]} = n_1 \psi_1, \quad \dot{[\varphi_0, \xi]} = m_0 \varphi_0, \quad \dot{[\varphi_1, \xi]} = m_1 \varphi_1$$

for some $m_i, n_i \in \mathbb{Z}$. Then

$$d^2 \mathcal{E}(s) = \frac{1}{2\pi i} \int_{\Sigma} \text{tr} ( (m_1 + n_0) \psi_0 \wedge \varphi_1 + (m_0 + n_1 + 2) \psi_1 \wedge \varphi_0 ).$$

Remark 4.15. Note that this resembles the discussion surrounding Eq. (8.10) in [28]. In fact, it does reproduce Hitchin’s result in the case that $s$ is the twistor line corresponding to a $\mathbb{C}^*$-fixed point in $M_{\text{Higgs}}$ and the deformation $\dot{s}$ is real, so that $\psi_1 = \varphi_0$, $\psi_0 = -\varphi_1^*$.

4.5. Sections and the degree of the hyperholomorphic line bundle. Our previous results together with the energy can be used to show that the space of irreducible sections is not connected. We begin with the following

Proposition 4.16. Let $(\overline{\mathcal{H}}, \Phi)$ be a stable $\mathbb{C}^*$-fixed Higgs bundle and let $s_{(\beta, \phi)}$ be a $\mathbb{C}^*$-fixed section corresponding to $(\beta, \phi) \in B^+_0(\overline{\mathcal{H}}, \Phi)$. If $s_{(\beta, \phi)}(\infty)$ is given by a VHS on $\Sigma$ with underlying holomorphic bundle $E = \bigoplus_{k=1}^{l'} E_k^l$, then we have

$$\deg(s_{(\beta, \phi)} L_Z) = \sum_{k=1}^{l} (k - 1) \deg(E_k) + \sum_{k=1}^{l'} (k - 1) \deg(E_k').$$

Proof. Proposition 4.12 allows us to compute $\mathcal{E}_0(s)$ and $\mathcal{E}_\infty(s)$. The assertion now follows from the formula

$$\deg(s_{(\beta, \phi)} L_Z) = \mathcal{E}(s_{(\beta, \phi)}) + \mathcal{E}_\infty(s_{(\beta, \phi)}).$$

Theorem 4.17. There exist irreducible sections $s$ of $\varpi : M_{\text{DH}}(\Sigma, \text{SL}(2, \mathbb{C})) \rightarrow \mathbb{C}P^1$ such that the pullback $s^* L_Z$ of the holomorphic line bundle $L_Z \rightarrow M_{\text{DH}}(\Sigma, \text{SL}(2, \mathbb{C}))$ has non-zero degree. In particular, the space of irreducible sections is not connected.

Proof. Let $K^*_\Sigma$ be a square-root of the canonical line bundle $K_\Sigma$. Consider the uniformization (Fuchsian) flat connection

$$\nabla_{\text{Fuchs}} = \begin{pmatrix} \nabla K^*_\Sigma & 1^* \\ 1 & \nabla K^*_\Sigma \end{pmatrix}$$
on the rank two bundle $K^\frac{1}{2}_\Sigma \oplus K^{-\frac{1}{2}}_\Sigma$. For generic holomorphic quadratic differential $q \in H^0(\Sigma, K^2_\Sigma)$, the anti-holomorphic structure 
\[ \begin{pmatrix} \partial_{K^\frac{1}{2}_\Sigma} & q \\ 1 & \partial_{K^{-\frac{1}{2}}_\Sigma} \end{pmatrix} \]
is stable (i.e., it defines a stable holomorphic bundle on $\Sigma$). Then, 
\[ \nabla := \nabla^{Fuchs} + \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \]
gives a $\mathbb{C}^*$-invariant section $s_\Sigma \in \mathcal{S}_{\text{MHH}}$ by the construction of Lemma 4.4. In view of Proposition 4.12, the energy at $\lambda = 0$ is given by 
\[ \deg K^{-\frac{1}{2}}_\Sigma = 1 - g \neq 0. \]
By assumption, $\partial \nabla$ is stable, so the anti-Higgs field of $s$ at $\lambda = \infty$ vanishes, and the energy at $\lambda = \infty$ is given by $\mathcal{E}_\infty = 0$. Finally, we have 
\[ \deg(s^*L_2) = \mathcal{E}(s) + \mathcal{E}_\infty(s) \neq 0 \]
by the residue formula for the pull-back under $s$ of the meromorphic connection to $\mathbb{CP}^1$ (see Section 3 of [4]).

Given an irreducible section $s \in \mathcal{S}_{\text{MHH}}$, it is in general very difficult to compute its normal bundle $N_s$. However, by using the methods of [24], it can be shown that the $\mathbb{C}^*$-fixed points considered in the proof of Theorem 4.17 do not have normal bundles of generic type, i.e., their normal bundles admit holomorphic sections with double zeros.

References

[1] D. V. Alekseevsky, V. Cortés and T. Mohaupt, Conification of Kähler and hyper-Kähler manifolds, Comm. Math. Phys. 324 no. 2, (2013), 637 – 655.
[2] M. F. Atiyah and R. S. Ward, Instantons and algebraic geometry, Comm. Math. Phys. 55 no. 2, (1977), 117 – 124.
[3] R. J. Baston and M. G. Eastwood, The Penrose transform: its interaction with representation theory, Oxford Mathematical Monographs, Clarendon Press, Oxford (1989).
[4] F. Beck, S. Heller and M. Röser, Energy of sections of the Deligne–Hitchin twistor space, Math. Ann. (2020), https://doi.org/10.1007/s00208-020-02042-0.
[5] I. Biswas and S. Heller, On the Automorphisms of a Rank One Deligne–Hitchin Moduli Space, SIGMA 13 (2017), https://doi.org/10.3842/SIGMA.2017.072.
[6] I. Biswas, S. Heller and M. Röser, Real holomorphic sections of the Deligne–Hitchin twistor space, Comm. Math. Phys. 366 (2019), 1099–1133.
[7] I. Biswas and S. Dumitrescu, and S. Heller, Irreducible flat $\text{SL}(2,\mathbb{R})$-connections on the trivial holomorphic bundle, Jour. Math. Pures Appl. (to appear), arXiv:2003.06997.
[8] I. Biswas and N. Raghavendra, Line bundles over a moduli space of logarithmic connections on a Riemann surface, Geom. Funct. Anal. 15 (2005), 780–808.
[9] N. Buchdahl, On the relative de Rham sequence Proc. AMS 87, No. 2 (1983), 363–366.
[10] B. Collier and R. Wentworth, Conformal limits and the Bialynicki-Birula stratification of the space of $\lambda$-connections, Adv. Math. 350 (2019), 1193–1225.
[11] K. Corlette, Flat $G$-bundles with canonical metrics, J. Diff. Geom. 28 (1988), 361 – 382.
[12] S. K. Donaldson, Twisted Harmonic Maps and the self-duality equations, , Proc. London Math. Soc. (3) 55, no. 1, (1987), 127 – 131.
[13] J. Dorfmeister, F. Pedit and H. Wu, Weierstrass type representation of harmonic maps into symmetric spaces, Comm. Anal. Geom. 6 (1998), no. 4, 633–668.
[14] J.-M. Drézet and M. S. Narasimhan, Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques, Invent. Math. 97 (1989), 53–94.
[15] O. Dumitrescu, L. Fredrickson, G. Kydonakis, M. Mulase and A. Neitzke, From the Hitchin section to opers through Nonabelian Hodge, Journal of Differential Geometry, 117, No. 2, (2021), 223 – 253.
[16] O. Dumitrescu and M. Mulase, Interplay between opers, quantum curves, WKB analysis, and Higgs bundles, preprint [arXiv:1702.00511v2 [math.AG]]
[17] G. Faltings, Stable $G$-bundles and projective connections, Journal of Algebraic Geometry 2 (1993), No.3, 507–568.
[18] B. Feix, Hyperkähler metrics on cotangent bundles, J. Reine Angew. Math. 532 (2001), 33 – 46.
[19] B. Feix, Hypercomplex manifolds and hyperholomorphic bundles, Math. Proc. Camb. Philos. Soc. 133, (2002), 443–457.
[20] B. Feix, Twistor spaces of hyperkähler manifolds with $S^1$-actions, Differential Geom. Appl. 19 (2003), 15–28.
[21] H. Grauert and R. Remmert, Coherent analytic sheaves, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 265, Springer-Verlag, Berlin, 1984.
[22] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, 52, New York: Springer-Verlag, (1977).
[23] A. Haydys, HyperKähler and quaternionic Kähler manifolds with $S^1$-symmetries, *J. Geom. Phys.* **58** (2008), 293–306.
[24] S. Heller, Real projective structures on Riemann surfaces and new hyper-kähler manifolds. [arXiv:1906.10350](https://arxiv.org/abs/1906.10350) (2019).
[25] L. Heller and S. Heller, Higher solutions of Hitchin’s self-duality equations, *Journal of Integrable Systems*, Volume 5, Issue 1, (2020).
[26] N. J. Hitchin, The self-duality equations on a Riemann surface, *Proc. London Math. Soc.* (3) **55**, no. 1, (1987) 59 –126.
[27] N. J. Hitchin, Harmonic maps from a 2-torus to the 3-sphere, *J. Differential Geom.* **31**, no. 3, (1990), 627 – 710.
[28] N. J. Hitchin, Lie groups and Teichmüller space, *Topology* **31** (1992), 449 – 473.
[29] N. J. Hitchin, On the hyperkähler/quaternion Kähler correspondence, *Comm. Math. Phys.* **324** (2013), 77 – 106.
[30] N. J. Hitchin, The hyperholomorphic line bundle, in Algebraic and Complex Geometry. In honour of Klaus Hulek’s 60th birthday, Springer Publishing (2014).
[31] N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, Hyper-Kähler metrics and supersymmetry, *Comm. Math. Phys.* **108** (1987), 535–589.
[32] Z. Hu and P. Huang, Flat $\lambda$-Connections, Mochizuki Correspondence and Twistor Spaces, [arXiv:1905.10765](https://arxiv.org/abs/1905.10765).
[33] P. Huang, Non-Abelian hodge theory and related topics, *SIGMA* **16** (2020), https://doi.org/10.3842/SIGMA.2020.029.
[34] S. A. Huggett and S. A. Merkulov, Twistor Transform of vector bundles, *Math. Scand.* Vol. 85, No. 2 (1999), pp. 219 – 244.
[35] M. Jardim and M. Verbitsky, Trihyperkähler reduction and instanton bundles on $CP^3$, *Compositio Math.* **150** (2014), 1836–1868.
[36] M. Jardim and M. Verbitsky, Moduli spaces of framed instanton bundles on $CP^3$ and twistor sections of moduli spaces of instantons on $C^4$ *Adv. Math.*, **227** (2011), 1526–1538.
[37] C. LeBrun, Quaternionic-Kähler manifolds and conformal geometry. *Math. Ann.* **284**, 353–376 (1989).
[38] M. Maruyama, Openness of a family of torsion free sheaves, *Jour. Math. Kyoto Univ.* **16** (1976), 627–637.
[39] M. Mayrand, Hyperkähler metrics near Lagrangian submanifolds and symplectic groupoids, [arXiv:2011.09282](https://arxiv.org/abs/2011.09282).
[40] M. S. Narasimhan and C. S. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, *Ann. of Math.* **82** (1965), 540–567.
[41] K. Pohlmeyer, Integrable Hamiltonian systems and interactions through quadratic constraints, *Comm. Math. Phys.* 46, no. 3, 1976.
[42] D. G. Quillen, Determinants of Cauchy–Riemann operators over a Riemann surface, *Funct. Anal. Appl.* **19** (1985), 31–34.
[43] C. Simpson, The Hodge filtration on nonabelian cohomology, *Algebraic geometry—Santa Cruz 1995*, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 217–281.
[44] C. Simpson, Iterated destabilizing modifications for vector bundles with connection, *Contemporary Math.* **522**, 2010 (Proceedings of the Ramanan Conference, Madrid, 2008.), 2008.
[45] K. Uhlenbeck, Harmonic maps into Lie groups (classical solutions of the chiral model), *J. Diff. Geom.*, Vol. 30, pages 1–50, 1989.