NORMAL FORM, SYMMETRY AND INFINITE DIMENSIONAL LIE ALGEBRA FOR SYSTEM OF ODE'S

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ABSTRACT. The normal form for a system of ode's is constructed from its polynomial symmetries of the linear part of the system, which is assumed to be semi-simple. The symmetries are shown to have a simple structure such as invariant function times symmetries of degree one called basic symmetries. We also show that the set of symmetries naturally forms an infinite dimensional Lie algebra graded by the degree of invariant polynomials. This implies that if this algebra is non-commutative then the method of multiple scales with more than two scaling variables fails to apply.

1.

The method of normal form is one of the most important tools in the study of nonlinear differential equations [1]. It classifies the vector field near the critical point, and the normal form consists of the resonant vectors characterized by the linear part of the vector field.

Here we study the normal form on $\mathbb{C}^N$ where the linear part of the vector field is assumed to be semi-simple (diagonalizable) for a simplicity (the general case will be discussed elsewhere). The main purpose of this letter is to give a method to determine the normal form in terms of the symmetries of the linear part of the system. We also show that for certain cases the set of the symmetries forms an infinite dimensional Lie algebra graded by the degree of invariant polynomials. As a consequence of this algebra structure, we see that the method of multiple scales with more than two variables fails in general.

2.

Let us consider a system of equations for $x \in \mathbb{C}^N$,

$$ \frac{dx}{dt} = X_F(x) = F(x). \quad (1) $$

where the vector field $X_F$ is given by

$$ X_F = \sum_{i=1}^N f_i(x) \frac{\partial}{\partial x_i} := F(x) \cdot \frac{d}{dx} \quad (2) $$

The components of $X_F$, denoted by the lower-case letters $\{f_i(x)\}_{i=1}^N$, are assumed to be

$$ f_i(x) \in C^\infty(\Omega) \quad (3) $$
for some neighborhood $\Omega$ of a critical point $x_0$. We assume $x_0 = 0$, i.e., $X_0(0) = 0$. In order to classify the critical point, we write the system in the series near the critical point,
\[
\frac{dx}{dt} = Ax + F^{(2)}(x) + \cdots,
\]
where $F^{(\ell)}(x)$ is a vector valued function of homogeneous polynomial of degree $\ell$. Here we also assume that the matrix $A \in M_N(\mathbb{C})$ of the linear part is diagonalizable with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$. This case is referred to be semi-simple. Now we consider the normal form of (4) near $x = 0$. In his thesis, Poincaré showed:

**Theorem.** (Poincaré [1]) If there is no eigenvalue $\lambda_s$ satisfying
\[
\lambda_s = \sum_{i=1}^N n_i \lambda_i
\]
for some $n_i \in \mathbb{Z}_{\geq 0}$ and $|n| := \Sigma n_i \geq 2$ (non-resonant condition), then there exists a formal change of variables $x = \phi(y)$ with $\phi(y)$ being polynomial, such that the transformed system for $y$ is just the linear part of (4),
\[
\frac{dy}{dt} = Ay.
\]

Thus the nonlinear parts in (4) can be eliminated by the non-resonant condition, and the solution of (4) near critical point can be described by that of the linear equation. The equation for $y$ is called the **normal form** of (4). For the case including the resonances, we have:

**Theorem.** (Poincaré-Dulac [1]) The normal form of (4) is given by
\[
\frac{dy}{dt} = Ay + G^{(2)}(y) + \cdots,
\]
where $G^{(\ell)}(y)$ is a vector valued function of homogeneous polynomial of degree $\ell$, and is a solution of
\[
L_AG^{(\ell)}(y) = \left\{ I_N(Ay) \cdot \frac{d}{dy} - A \right\} G^{(\ell)}(y) = 0,
\]
where $I_N$ is the $N \times N$ identity matrix.

Outlines of the proofs of these theorems are as follows:

Under the change of coordinates, $x = \phi(y)$, the vector field $X_F$ becomes $X_G$ such that
\[
X_\phi \circ X_F = X_G \circ X_\phi.
\]
Then expanding $F, G$ and $\phi$ as
\[
F(y) = Ay + F^{(2)}(y) + \cdots,
G(y) = Ay + G^{(2)}(y) + \cdots,
\phi(y) = y + \phi^{(2)}(y) + \cdots
\]
(9) gives, on the \( n \)th degree polynomial,
\[
L_A \phi^{(n)} = \tilde{F}^{(n)}(y) - G^{(n)}(y),
\]
(11)
where \( \tilde{F}^{(n)}(y) \) is determined successively, e.g.,
\[
\tilde{F}^{(2)}(y) = F^{(2)}(y),
\]
\[
\tilde{F}^{(3)}(y) = F^{(3)}(y) + \phi^{(2)}(y) \cdot \frac{d}{dy} F^{(2)}(y) - G^{(2)}(y) \cdot \frac{d}{dy} \phi^{(2)}(y),
\]
and so on. In the case of \( A = \text{diag}(\lambda_1, \ldots, \lambda_N) \), the operator \( L_A \) in (8) is
\[
L_A = I_N \left( \sum_{i=1}^{N} \lambda_i y_i \frac{\partial}{\partial y_i} \right) - A.
\]
(12)
Then we first note that the eigenvectors of \( L_A \) are given by a monomial of degree \( |n| = \sum n_i \),
\[
\psi^{(n)}_{\ell}(y) = e_{\ell} \prod_{i=1}^{N} y_i^{n_i} \quad \text{for } \ell = 1, \ldots, N
\]
(13)
with the eigenvalue \( \Lambda_{\ell}^{n} \), \( L_A \psi^{(n)}_{\ell} = \Lambda_{\ell}^{n} \psi^{(n)}_{\ell} \),
\[
\Lambda_{\ell}^{n} = \left( \sum_{i=1}^{N} \lambda_i n_i \right) - \lambda_{\ell},
\]
(14)
where \( n_i \in \mathbb{Z}_{\geq 0} \) and \( e_{\ell} \) is an eiegenvector of \( A \) with eigenvalue \( \lambda_{\ell} \), \( A e_{\ell} = \lambda_{\ell} e_{\ell} \). If the \( \lambda_{\ell} \) satisfies the resonant condition (5), that is \( \Lambda_{\ell}^{n} = 0 \), then the vector (13) is called the resonant vector. Therefore, if there is no resonance (\( \ker L_A = \{0\} \)), then one chooses \( G^{(n)}(y) = 0 \), while for the case of resonances, we choose \( G^{(n)}(y) \in \ker L_A \cap \tilde{F}^{(n)}(y) \), which is the resonant vector (13) of degree \( n \). In the next section, we relate the resonant vector (13) to a symmetry of the linear part of the system (4).

3.

Let us first define the symmetry of the system (1).

**Definition.** A vector field \( X_S = S(x) \cdot \frac{d}{dx} \) is a symmetry of (1), if it commutes with the vector field \( X_F \), i.e.
\[
[X_S, X_F] = X_S \circ X_F - X_F \circ X_S
\]
\[
= \left\{ S(x) \cdot \frac{dF(x)}{dx} - F(x) \cdot \frac{dS(x)}{dx} \right\} \cdot \frac{d}{dx} = 0.
\]
(15)

The condition (15) implies that the vector function \( S(x) \) is a solution of the linearized equation of (1). The symmetry can be found directly from the general solution of (1), when the system is integrable. Namely we have:
Proposition 1. Let \( x(t) \) be the general solution of (1),
\[
    x(t) = U(t; C_1, \ldots, C_N)
\]  
(16)
with \( N \) arbitrary parameters \( C_i \). Then the symmetries are obtained by
\[
    S_i(x) = \frac{\partial U}{\partial C_i}(t; C_1, \ldots, C_N). \tag{17}
\]

Proof.
\[
    \frac{dS_i}{dt} = \frac{dx}{dt} \cdot \frac{dS_i}{dx} = F(x) \cdot \frac{dS_i}{dx} = \frac{\partial}{\partial C_i} F = \frac{\partial U}{\partial C_i} \cdot \frac{dF}{dx} = S_i(x) \cdot \frac{dF}{dx}. \tag*{\qed}
\]

Thus the symmetry is a generator of the transformation group with \( N \) parameters of the solution. In (16), some of the parameters may be determined by the level of the invariant functions of (1). For example, an integrable Hamiltonian system with \( d \) degree of freedom has \( d \) integrals in involution, and the \( d \) parameters in the general solution are given as the levels of these integrals. With an invariant function of (1), we also have:

Proposition 2. Let \( \xi(x) \) be an invariant function, i.e., \( \frac{d\xi}{dt} = 0 \), and \( X_S \) be a symmetry of (1). Then \( \varphi(\xi(x))X_S \) with an arbitrary scalar function \( \varphi \) of \( \xi \) is also a symmetry of (1).

The proof is straight-forward. This proposition is useful for constructing higher order symmetries as we will see below.

For the case of linear system, \( F(x) = Ax \) with \( A \in M_N(\mathbb{C}) \), the symmetry is just a solution of the system. If we assume the eigenvalues of \( A \) to be all distinct, i.e., \( \lambda_i \neq \lambda_j \) for \( i \neq j \), then we have:

Proposition 3. The vectors \( A^n x \), for \( n = 0, \ldots, N-1 \), give \( N \) linearly independent symmetries of the linear system \( F(x) = Ax \) in (1) with the above condition for \( A \).

Proof. Let \( e_\ell \) be the eigenvectors of \( A \) with eigenvalue \( \lambda_\ell \), i.e., \( Ae_\ell = \lambda_\ell e_\ell \). Since \( A \) is diagonalizable, \( e_\ell \)'s are linearly independent. The equations \( \sum_{i=1}^N a_i A_i^{-1} e_\ell = 0 \), for \( \ell = 1, \ldots, N \), give
\[
    \begin{bmatrix}
        1 & \lambda_1 & \ldots & \lambda_1^{N-1} \\
        1 & \lambda_2 & \ldots & \lambda_2^{N-1} \\
        \vdots & \vdots & \ddots & \vdots \\
        1 & \lambda_N & \ldots & \lambda_N^{N-1} 
    \end{bmatrix}
    \begin{bmatrix}
        a_1 \\
        a_2 \\
        \vdots \\
        a_N 
    \end{bmatrix} = 0. \tag{18}
\]

This equation has only trivial solution \( a_i = 0 \) for all \( i \) since the determinant of the coefficient matrix is the Vandermonde determinant \( \prod_{i>j} (\lambda_i - \lambda_j) \) which is not zero by the assumption. \tag*{\qed}

The (linear) symmetries given by polynomial of degree one are called the basic symmetries. The Proposition 3 then implies that any linear symmetry can be
written as a linear combination of the basic symmetries. Namely, for a matrix with distinct eigenvalues, we have

\[ \text{Span}\{A^n : 0 \leq n \leq N - 1\} = \{B \in M_N(\mathbb{C}) : [A, B] = 0\}. \]  

(19)

In order to find nonlinear symmetry, especially of polynomial type, we look for an invariant function of the system and use the Proposition 2, as will be shown below.

4.

Hereafter we consider the system (4) with the linear part having distinct eigenvalues. Note that the set of matrix with distinct eigenvalues is generic in \( M_N(\mathbb{C}) \). In this case, we first note that the resonant condition (5) leads to an invariant function of the linear system:

**Proposition 4.** If there exists a resonance, i.e., \( \exists \lambda_s = \sum_{i=1}^{N} n_i \lambda_i \) for some \( n_i \in \mathbb{Z}_{\geq 0} \) with \( |n| \geq 2 \), then a function

\[ \xi(x) = x_s^{-1} \prod_{i=1}^{N} x_i^{n_i}, \]  

(20)

with the coordinates \( x_i \) on which \( A = \text{diag}(\lambda_1, \ldots, \lambda_N) \), is an invariant function.

The proof to show \( d\xi(x)/dt = 0 \) is straightforward. Here we remark that if \( n_s \neq 0 \), then (20) gives a monomial of degree \( |n| - 1 \). Among all the invariant functions of the form (20) for each independent \( n \in \mathbb{Z}_{\geq 0}^N \), the invariant with the minimum \( |n| \) is called primitive. From the Proposition 2, we have:

**Proposition 5.** The vector fields defined by

\[ X^m_{\ell} = S^m_{\ell}(x) \cdot \frac{d}{dx} := \xi^m(x)(A^\ell x) \cdot \frac{d}{dx}, \]  

(21)

are symmetries of the linear system. Here \( d \) is the total number of primitive invariant functions, \( \{\xi_i(x)\}_{i=1}^{d} \), and in the coordinates where \( A = \text{diag}(\lambda_1, \ldots, \lambda_N) \), the invariant function \( \xi(x) \) is given by

\[ \begin{align*}
\xi_j(x) &= x_s^{-1} \prod_{i=1}^{N} x_i^{n_i^{(j)}} := \prod_{i=1}^{N} x_i^{n_i^{(j)}} \\
\xi(x) &= \prod_{j=1}^{d} (\xi_j(x))^{m_j} := \prod_{i=1}^{N} x_i^{M_i}
\end{align*} \]  

(22)

where \( \lambda_{s_j} = \sum_{i=1}^{N} n_i^{(j)} \lambda_i \) with \( \sum_{i=1}^{N} n_i^{(j)} \geq 2 \) (resonant condition), and \( M_i = \sum_{j=1}^{d} n_j^{(j)} m_j \) with \( n_j^{(j)} = n_i^{(j)} - \delta_{ij} \).

Note that in the symmetries (21) the number \( d \) is just the total number of resonances with independent \( n \in \mathbb{Z}_{\geq 0}^N \). In the formula (21), we remark that if \( n_s^{(j)} \neq 0 \), \( X^m_{\ell}(x) \) gives a polynomial symmetry for any \( m \in \mathbb{Z}_{\geq 0}^d \) and \( \ell \). However, if \( n_s^{(j)} = 0 \), \( X^m_{\ell}(x) \) is not a polynomial, but a certain linear combination of \( X^m_{\ell}(x) \) with \( m_j = 1 \) becomes a polynomial. In the latter case, there is no higher degree polynomial symmetries with respect to the index \( m_j \) (i.e., \( \xi_j(x)x_{s_j} \) is polynomial, but \( \xi_j^{m_j}x_{s_j} \) for \( m_j > 1 \) is not). We now show that any polynomial symmetry can be given by a linear combination of (21);
Lemma 6. If $X_S = S(x) \cdot \frac{d}{dx}$ is a polynomial symmetry, then there exists an invariant function $\xi(x)$ of (20) and

$$S(x) = \xi(x) \sum_{i=0}^{N-1} C_i A^i x$$

for some constants $C_i$, for $i = 0, \ldots, N-1$.

Proof. We first note that $S(x)$ is a solution of the linear equation, i.e.,

$$\frac{d}{dt} S(x) = A S(x). \tag{24}$$

Since $A$ is diagonalizable, we take $A = \text{diag}(\lambda_1, \ldots, \lambda_N)$ without loss of generality. Then (24) leads to the equations for the component $s_j(x)$ of $S(x)$,

$$\frac{d}{dt} s_j(x) = \frac{dx}{dt} \cdot \frac{d}{dx} s_j(x) = (Ax) \cdot \frac{d}{dx} s_j(x)$$

$$= \sum_{i=1}^{N} \lambda_i x_i \frac{\partial}{\partial x_i} s_j(x)$$

$$= \lambda_j s_j(x), \text{ for } j = 1, \ldots, N.$$

This implies that if $s_j(x)$ is a polynomial, it is a monomial in the form,

$$s_j(x) = \prod_{i=1}^{N} x_i^{n_i^{(j)}}. \tag{25}$$

Substituting (25) into (24), we obtain

$$\left( \sum_{i=1}^{N} n_i^{(j)} \lambda_i - \lambda_j \right) s_j(x) = 0.$$

This is just the resonant condition (6). Namely the polynomial symmetry results from the resonant condition. Then, from the Proposition 4, there exists an invariant function of the form (20), from which we have

$$s_j(x) = \xi_j(x) x_j = \left( x_j^{-1} \prod_{i=1}^{N} x_i^{n_i^{(j)}} \right) x_j. \tag{26}$$

Since the set of the basic symmetries $\{A^n x\}_{n=0}^{N-1}$ gives a basis of $\mathbb{C}^N$, a symmetry $S(x) = (0, \ldots, 0, s_j(x), 0, \ldots, 0)^t$ can be expressed as a linear combination of the basic symmetries. This completes the proof.

It should be noted that the symmetry given by (25) with zeros for other component, $S(x) = (0, \ldots, 0, s_j(x), 0, \ldots, 0)^t$, is nothing but a resonant vector of (8) for $A = \text{diag}(\lambda_1, \ldots, \lambda_N)$. Therefore, we have the main theorem of this paper on the normal form of (4):
Theorem 7. The normal form (7) can be expressed as
\[
\frac{dy}{dt} = Ay + \sum_{|m| \geq 1} \sum_{0 \leq \ell \leq N-1} C^m_\ell S^m_\ell (y),
\]
(27)
where \(S^m_\ell (y)\) for \(m \in \mathbb{Z}^d_\geq 0\) and \(\ell = 0, 1, 2, \ldots, N - 1\) are given by (21), \(C^m_\ell\) are constants determined uniquely from the original system (4), and \(|m| = \sum_{i=1}^d m_i\).

Let us illustrate the theorem by taking a nonlinear system of \(\mathbb{R}^2\) near an elliptic point as a simple example. Namely we consider the system,
\[
\frac{d}{dt}(x_1 x_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + F^{(2)}(x_1, x_2) + \cdots.
\]
(28)
The primitive invariant polynomial of the linear part of this sytem is \(\xi(x_1, x_2) = x_1^2 + x_2^2\), and the symmetries (21) are given by
\[
S^m_0(x) = (x_1^2 + x_2^2)^m \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{and} \quad S^m_1(x) = (x_1^2 + x_2^2)^m \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}.
\]
(29)
Introducing the complex variable \(z = x_1 + ix_2\), the normal form (27) can be written in the following familiar form [1],
\[
\frac{dz}{dt} = iz + \sum_{m=1}^\infty C^m |z|^{2m} z
\]
(30)
where \(C^m = C^m_0 - iC^m_1\).

5.

We now discuss the algebra generated by the symmetries (21). A set of vector fields naturally forms an infinite dimensional Lie algebra. Here we show that the symmetry algebra gives an infinite dimensional graded Lie subalgebra of the vector fields. To do this, we compute the commutator of \(X^m_\ell\) and \(X^{m'}_{\ell'}\) by writing these in the coordinates with \(A = \text{diag}(\lambda_1, \ldots, \lambda_N)\), i.e.,
\[
X^m_\ell = \left( \prod_{i=1}^N x_i^{M_i} \right) \sum_{i=1}^N \lambda_i^\ell x_i \frac{\partial}{\partial x_i}.
\]
(31)
Then it is immediate that we have:

Theorem 8. The set of the symmetries (21) of the linear system forms an infinite dimensional Lie algebra with the commutator,
\[
[X^m_\ell, X^{m'}_{\ell'}] = \left( \sum_{i=1}^N M'_i \lambda_i^\ell \right) X^{m+m'}_{\ell'} - \left( \sum_{i=1}^N M_i \lambda_i^\ell' \right) X^{m+m'}_{\ell'},
\]
(32)
where \(M'_i = \sum_{j=1}^d \overline{m}^{(j)} m'_j\).

We have several remarks on the relation (32):


Remarks. a) This algebra is graded by the integer $m \in \mathbb{Z}_{\geq 0}$ of the degree of the invariant functions $\xi_j(x)$; $\mathbb{Z}^d$-gradation.

b) Since $X^0_1 = (Ax) \cdot \frac{d}{dx}$ is the original vector field, $X^0_1$ gives a center of this algebra (recall $\sum_{i=1}^N M_i \lambda_i = 0$).

c) The set of $X^m_0$ forms a classical Virasoro algebra (Witt algebra), i.e.

$$[X^m_0, X^{m'}_0] = \left( \sum M'_i - \sum M_i \right) X^{m+m'}_0$$

$$= \left( \sum_{j=1}^d N^{(j)} (m'_j - m_j) \right) X^{m+m'}_0,$$  \hspace{1cm} (33)

where $N^{(j)} := \sum_{i=1}^N \pi^{(j)}_i$.

Thus nonlinear system having resonances naturally carries an infinite dimensional Lie algebra of the symmetries, while linear system having no-resonance has only a finite dimensional Abelian algebra of the basic symmetries.

For the example (28), we have an algebra of Kac-Moody ($\tilde{u}(1)$) and Virasoro (Witt) type as the symmetry algebra,

$$[X^m_0, X^{m'}_0] = 2(m' - m)X^{m+m'}_0,$$

$$[X^m_0, X^m_1] = 2m'X^{m+m'}_1,$$

$$[X^m_1, X^{m'}_1] = 0.$$  \hspace{1cm} (34)

6.

As a consequence of the previous result, we here show that the method of multiple scales with more than two scaling variables fails in general.

Let $t_n$ be the scaling variables defined as

$$t_n = \epsilon^n t, \hspace{0.5cm} for \hspace{0.5cm} n = 0, 1, 2, \ldots,$$  \hspace{1cm} (35)

where $\epsilon > 0$ is a small scaling parameter for the variable $x$. Then, changing $x$ to $\epsilon x$, (4) becomes,

$$\frac{\partial x}{\partial t_0} - Ax = \sum_{n=1}^{\infty} \epsilon^n \left( F^{(n+1)}(x) - \frac{\partial x}{\partial t_n} \right).$$  \hspace{1cm} (36)

Now, expanding $x$ in the power series of $\epsilon$, equivalently in the homogeneous polynomials, i.e.

$$x = x^{(1)} + \epsilon x^{(2)} + \epsilon^2 x^{(3)} + \cdots,$$  \hspace{1cm} (37)

where $x^{(\ell)}$ is the homogeneous polynomial of degree $\ell$. Substituting (37) into (36), we have at order 1,

$$\frac{\partial x^{(1)}}{\partial t_0} - Ax^{(1)} = 0$$  \hspace{1cm} (38)

which is the linearized equation of (1), and in the case of $A$ having distinct eigenvalues, all the polynomial solutions of (28) are given by the symmetries (21).
At order $\epsilon$, we have
\[ \frac{\partial x^{(2)}}{\partial t_0} - Ax^{(2)} = F^{(2)}(x^{(1)}) - \frac{\partial x^{(1)}}{\partial t_1}. \] (39)

Now as a standard procedure of perturbation method, we eliminate secular terms in $F^{(2)}$ by choosing an appropriate equation for $\frac{\partial x^{(1)}}{\partial t_1}$. The secular term is given by the homogeneous solution of (39), that is, the symmetry of (38). Thus we choose $\frac{\partial x^{(1)}}{\partial t_1}$ to be
\[ \frac{\partial x^{(1)}}{\partial t_1} = S^{(2)}(x^{(1)}) = F^{(2)}(x^{(1)}) \cap \ker L_A. \] (40)

In a similar manner, we have $\frac{\partial x^{(1)}}{\partial t_n} = S^{(n+1)}(x^{(1)}) \in \ker L_A$, and the equation for $x^{(1)}$ in the original time variable becomes
\[ \frac{dx^{(1)}}{dt} = \sum_{n=0}^{\infty} \epsilon^n \frac{\partial x^{(1)}}{\partial t_n} = Ax^{(1)} + \sum_{n=1}^{\infty} \epsilon^n S^{(n+1)}(x^{(1)}). \] (41)

This is nothing but the normal form (27) with $y = x^{(1)}$. It is however important to note that the equations of higher orders, $\frac{\partial x^{(1)}}{\partial t_n} = S^{(n+1)}(x^{(1)})$, are not compatible in general, as was shown before, i.e. $[X_{S^{(n+1)}}, X_{S^{(m+1)}}] \neq 0$, or equivalently
\[ \frac{\partial^2 x^{(1)}}{\partial t_n \partial t_m} \neq \frac{\partial^2 x^{(1)}}{\partial t_m \partial t_n}, \quad \text{for} \quad n \neq m. \] (42)

This implies that the method of multiple scales, where the scaling variables $t_n$ are considered to be independent, fails in general. Note however that the normal form (27) makes sense.

As a final remark, we mention that a similar result can be obtained for a Hamiltonian system where the symmetry algebra is given by a Poisson algebra over resonant polynomials.

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References

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