Abstract. In part I it was shown that for each \( k \geq 1 \) the generalized Sato–Levine invariant detects a gap between \( k \)-quasi-isotopy of link and peripheral structure preserving isomorphism of the finest quotient \( G_k \) of its fundamental group, 'functorially' invariant under \( k \)-quasi-isotopy. Here we show that Cochran’s derived invariant \( \beta^k \), provided \( k \geq 3 \), and a series of \( \mu \)-invariants, starting with \( \mu(111112122) \) for \( k = 3 \), also fall in this gap. In fact, all \( \mu \)-invariants where each index occurs at most \( k+1 \) times, except perhaps for one occurring \( k+2 \) times, can be extracted from \( G_k \), and if they vanish, \( G_k \) is the same as that of the unlink.

We also study the equivalence relation on links (called 'fine \( k \)-quasi-isotopy') generated by ambient isotopy and the operation of interior connected sum with the \( (k+1) \)-th Milnor’s link, where the complement to its first component is embedded into the link complement. We show that the finest quotient of the fundamental group, functorially invariant under fine \( k \)-quasi-isotopy, is obtained from the fundamental group by forcing all meridians to be \( (k+2) \)-Engel elements. We prove that any group generated by two 3-Engel elements has lower central series of length \( \leq 5 \).

1. Introduction

Two links \( L_1, L_2: mS^1 \hookrightarrow S^3 \), where \( mS^1 = S^1 \sqcup \cdots \sqcup S^1_m \), are called \( k \)-quasi-isotopic if they are PL homotopic in the class of \( k \)-quasi-embeddings with at most one double point, defined as follows. Any PL embedding is set to be a \( k \)-quasi-embedding, for arbitrary \( k \). A PL map \( f: mS^1 \to S^3 \) with a single double point \( f(p) = f(q) \) is said to be a \( k \)-quasi-embedding if \( p, q \) lie in the same component \( S^1_i \) and (in the case \( k > 0 \) ) moreover, in addition to the singleton \( P_0 = \{ f(p) \} \), there exist subpolyhedra \( P_1 \subset \cdots \subset P_k \subset S^3 \setminus \bigcup_{j \neq i} f(S^1_j) \) and arcs \( J_0 \subset \cdots \subset J_{k-1} \subset S^1_i \) such that \( f(J_j) \subset P_{j+1} \) and \( f^{-1}(P_j) \subset J_j \), moreover the inclusion \( P_j \cup f(J_j) \subset P_{j+1} \) is null-homotopic in \( P_{j+1} \), for each \( j = 0, \ldots, k-1 \). (Beware that in [MM] \( k \)-quasi-isotopy was what we call strong \( k \)-quasi-isotopy now.) The \( k \)-inessential lobe of the \( k \)-quasi-embedding \( f \) is \( f(J_0) \), where \( J_0 \) is the subarc of \( J_0 \) such that \( \partial J_0 = \{ p, q \} \).

This definition was suggested by certain well-known higher dimensional constructions, in particular the Penrose–Whitehead–Zeeman–Irwin trick and the Casson handles, see [MR1] for details and references. The importance of these equivalence
relations lies in that they all resemble link homotopy in satisfying the following property: given a link in \( S^3 \), possibly wild, there exists an \( \varepsilon = \varepsilon(k) > 0 \) such that all \( \varepsilon \)-close PL links are \( k \)-quasi-isotopic to each other [MR1].

It was observed in [MR1] that the \( k \)th Milnor’s link [Mi2; Fig. 1] (see Fig. 5 below, where \( k = 4 \)) is \((k-1)\)-quasi-isotopic to the unlink but not \( k \)-quasi-isotopic to it. Also the \( k \)th Whitehead link \( \mathcal{W}_k \) (i.e. the \( k \)-fold untwisted left-handed Whitehead double of the Hopf link, where the doubling is performed on either one of the components at each stage) is \((k-1)\)-quasi-isotopic to the unlink (see Figures 4 and 5 in [MR1]). We do not presently know any invariants that could tell us whether the boundary link \( \mathcal{W}_k \) is not \( k \)-quasi-isotopically trivial for \( k \geq 2 \). Also it is still not clear to us whether connected sum ‘accumulates complexity’ up to \( k \)-quasi-isotopy, \( k \geq 1 \), in the sense of Problem 0.2 from [MR1], which was the initial motivation of the present series of papers.

It was proved in [MR1], [MR2] that the following are invariant under \( k \)-quasi-isotopy: Cochran’s derived invariants \( \beta^i \), \( i \leq k \); finite type invariants of type \( \leq k \) (either in the usual sense, or more generally in the sense of Kirk–Livingston [KL]) that are well-defined up to PL isotopy; and Milnor’s \( \bar{\mu} \)-invariants of length \( \leq 2k+3 \). It follows that any linear combination of Conway’s polynomial coefficients at the powers \( \leq k + m - 1 \), \( m \) invariant under PL isotopy, as well as the first non-vanishing coefficient of the Conway polynomial, if it occurs at a power less than \((2k+3)(m-1)\), are invariant under \( k \)-quasi-isotopy. In this paper we also show that all \( \bar{\mu} \)-invariants with no more than \( k + 1 \) entries of each index are invariant under \( k \)-quasi-isotopy.

The referee suggested us to mention a connection with capped half-gropes, and conjectured a version of the part (a) of the following proposition. Let \( M \) be an oriented 2-manifold; if \( M \) is a disk, it will be omitted from the notation. An \( M \)-like capped half-grope of class 1 is a polyhedral pair \((G_1, \partial G_1)\) obtained from \((M, \partial M)\) by choosing an even number of disjoint disks \( D^1_1, \ldots, D^2_n \) in the interior of \( M \) and identifying each \( D^2_{2i-1} \) with \( D^2_{2i} \) by an orientation-preserving homeomorphism \( h_i \). The images of \( D_i \) under the identification \( M \to G_1 \) are the caps, the complement in \( G_1 \) to their interiors is the surface, and the images in \( G_1 \) of any collection of \( n \) disjoint paths in \( M \setminus (D_1 \cup \cdots \cup D_{2n}) \), connecting a point \( p_i \in \partial D_{2i-1} \) to \( h_i(p_i) \in \partial D_{2i} \) for each \( i \), will be called an expansion base for \( G_1 \). An \( M \)-like capped half-grope of class \( k+1 \) is a polyhedral pair \((G_{k+1}, \partial G_{k+1})\) obtained from \((G_1, \partial G_1)\) by attaching a (disk-like) capped half-grope \( G_k(i) \) of class \( k \) along its boundary \( \partial G_k(i) \) to each circle \( S^1_j \) in some expansion base for \( G_1 \). Caps and surfaces of \( G_{k+1} \) are those of \( G_1 \) and of each \( G_k(i) \).

**Proposition 1.1.** a) Two \( m \)-component links related by \( k \)-quasi-isotopy cobound a link map of \( m \) annulus-like capped half-gropes of class \( k+1 \) into \( S^3 \times I \), which embeds the bottom stage surface and the union of all other surfaces of each grope.

b) The \( k \)-inessential lobe of a \( k \)-quasi-embedding \( f : mS^1 \to S^3 \) with a single double point on the \( j \)th component bounds an immersion of a capped half-grope of class \( k \) into \( S^3 \setminus \bigcup_{j \neq 1} f(S^1_j) \) which embeds each surface of the grope, moreover any two surfaces from the same stage meet transversely in their interiors.

The converse to (a) is false, since concordance does not imply 1-quasi-isotopy [MR1].

**Proof.** To prove (b), convert the track of the null-homotopy of the \( k \)-inessential lobe in \( P_1 \) into an embedded surface (by tricks II and IV from Appendix A of Cochran’s
memoire [Co]), then (if \( k > 1 \)) choose an expansion base so that any two circles have zero linking number (this can be achieved by dragging one end of a handle between the ends of another handle), and use the same tricks to convert the tracks of their transverse null-homotopies in \( P_2 \) into embedded surfaces, meeting only in their interiors. This process can be continued; in order to choose expansion bases for different class 1 capped half-gropes from the same stage so that circles from one have trivial linking numbers with that from the other, it may be necessary to go back to lower stages and use handle slides. To prove (a), convert the immersed annuli, forming the image of the \( k \)-quasi-isotopy in \( S^3 \times I \), into surfaces by resolving self-intersections, and then perturb the surfaces of each grope from (b) in the fourth coordinate, so as to place them consecutively in an order of increasing stage number.

We define \( G_k(L) \) to be the quotient of the fundamental group \( \pi(L) \) by the normal subgroup

\[
\mu_k = \left\{ [m, m^g] \mid m \in M, g \in \langle m \rangle^{\langle m \rangle_{\pi(L)}} \right\},
\]

where \( \langle m \rangle^H \) denotes the normal closure of \( m \) in the subgroup \( H \), and \( M \) denotes the set of all meridians.\(^1\) Notice that for \( k = 0 \) this coincides with Milnor’s link group \( G(L) \) [Mi1] which is not surprising as 0-quasi-isotopy is just link homotopy. Equivalently, \( \mu_k \) can be described as the normal closure of

\[
\bigcup_{i=1}^{m} \left\{ [m_i, m_i^g] \mid g \in \langle m_i \rangle^{\langle m \rangle_{\pi(L)}} \right\},
\]

where each \( m_i \) is a fixed meridian to the \( i \)th component and \( H_k^G \) denotes \( H^H_k \), where \( H_0^G = G \). In §2 we will give more descriptions of \( \mu_k \). It was shown in [MR1, Theorems 3.2 and 3.7] that \( G_k \) is the finest quotient of \( \pi(L) \), invariant under \( k \)-quasi-isotopy functorially, i.e. so that for any links \( L_0, L_1 \) obtained one from another by an allowed crossing change with \( L_s \) being the intermediate singular link, the diagram

\[
\begin{array}{ccc}
\pi_1(S^3 \setminus \text{regular neighborhood of } L_s(mS^1)) & \xrightarrow{i_*} & \pi_1(S^3 \setminus \text{regular neighborhood of } L_s(mS^1)) \\
\downarrow & & \downarrow \\
\pi(L_0) & \xrightarrow{p_{L_0}} & \pi(L_1) \\
\downarrow & & \downarrow p_{L_1} \\
G_k(L_0) & \xrightarrow{\cong} & G_k(L_1)
\end{array}
\]

commutes. Let \( P_k \) be the finest peripheral structure in \( G_k(L) \), invariant under \( k \)-quasi-isotopy, more precisely, \( P_k \) is the collection of \( m \) pairs \( (\bar{m}_i, \Lambda_i) \), each defined up to simultaneous conjugation, where \( \bar{m}_i \in G_k(L) \) is the coset of a meridian \( m_i \in G_k(L) \) to the \( i \)th component of \( L \), and \( \Lambda_i \subset G_k(L) \) is the set of cosets \( \bar{l}_{i\alpha} \in G_k(L_{\alpha}) \simeq G_k(L) \) of the longitudes \( l_{i\alpha} \) corresponding to some representatives \( m_{i\alpha} \) of

\[^1\]We use the left normed notation \( h^g = g^{-1}hg, [g, h] = g^{-1}h^{-1}gh \) throughout the paper.
$\tilde{m}_i$ in the fundamental groups of all links $L_\alpha$, $k$-quasi-isotopic to $L$ (that $m_{\alpha i}$ are meridians follows from functoriality). It follows from [MR1; proofs of Theorems 3.2 and 3.7] that the set $\Lambda_i$ is the right coset $N_k(m_i)\bar{l}_i$ of the subgroup

$$N_k(m_i) = \left\langle [g^{-1}, g^m] \mid g \in \langle m \rangle_k \right\rangle$$

containing the class $\bar{l}_i$ of the longitude $l_i$ corresponding to $m_i$.

**Theorem 1.2.**

a) [MR1] If $k \geq 1$, the generalized Sato–Levine invariant is invariant under $k$-quasi-isotopy but cannot be extracted from $(G_k, P_k)$.

b) If $k \geq 3$, Cochran’s derived invariants $\beta^{[k/2]+2}_{\pm}, \ldots, \beta^k_{\pm}$ (see [Co]) are invariant under $k$-quasi-isotopy but cannot be extracted from $(G_k, P_k)$.

c) If $k \geq 3$, a nonzero $\bar{\mu}$-invariant of 2-component links of length $\leq 2k + 3$ with at least $k + 3$ entries of one of the indices is invariant under $k$-quasi-isotopy but cannot be extracted from $(G_k, P_k)$.

*Remark.* Of course, $k \geq 3$ is no real restriction in (c), since there are no nontrivial $\bar{\mu}$-invariants of two-component links of length 5 or 7, and those of length 4 or 6 must contain at least two entries of each index. Notice also that the only invariants of length 8 that satisfy the conditions of (c) with $k = 3$ are $\bar{\mu}(11111122)$ and $\bar{\mu}(11222222)$ [Mi2], which in the case of zero linking number coincide with the residue classes of Cochran’s $\beta^3_{\pm}$ mod g.c.d. of the Sato-Levine invariant and $\beta^2_{\pm}$ [Co]. But there are 4 linearly independent $\bar{\mu}$-invariants of 2-component links of length 9 [Orr], and a computation using Proposition 2.13 below along with the second Witt formula (for the number of basic commutators with prescribed entries) [Hall; eq. (11.2.4)], [MKS] shows that one of these has 6 entries of the index “1”, thus satisfying the hypothesis of (c) with $k = 3$. Using Milnor’s symmetry relations, it is easy to verify that this invariant is $\frac{1}{2}\bar{\mu}(112111122) = \frac{1}{4}\bar{\mu}(111112122)$.

*Proof of 1.2.* The part (c) follows from Theorem 2.12c and [MR1; Corollary 3.5]. The invariance in part (b) was proved in [MR2], and since $\beta^k$ is an integer lifting of one of the invariants from (c), it cannot be extracted. □
The following argument is a modification of [MR1; proof of Corollary 1.3].

Example 1.3. The infinite family of links $\mathcal{M}_n$, depicted on the right-hand side of Fig. 1, has the same $(G_k, P_k)$ as the Hopf link (for each $k$), yet no two links of this family are 1-quasi-isotopic.

To see the latter, let us recall the axiomatic definition of the generalized Sato–Levine invariant $\tilde{\beta}$, constructed independently by Polyak–Viro, Kirk–Livingston, and Akhmet’ev (see [KL], [AR], [AMR]).

Given two links $L_0, L_1: S^1 \sqcup S^1 \rightarrow \mathbb{R}^3$, related by a single crossing change in one of the components, of the type shown on Fig. 2a, so that the two lobes of the singular component of the intermediate singular link have linking numbers $n$ and $l - n$ with the other component, where $l$ is the linking number of $L_0$, one postulates the formula $\tilde{\beta}(L_0) - \tilde{\beta}(L_1) = n(l - n)$. In addition, it is assumed that $\beta(H_n) = 0$, where $H_n$ is shown on Fig. 2b. It was proved in [AR], [KL] that these two axioms yield a well-defined invariant of ambient isotopy. Moreover, it is easy to see that this invariant $\tilde{\beta}$ is unchanged under 1-quasi-isotopy.

We note that $\mathcal{M}_n$ can be obtained from $\mathcal{M}_{n-1}$ by one crossing change, where the jump of $\tilde{\beta}$ is $-2$. Hence $\tilde{\beta}(\mathcal{M}_n) = -2n$ which proves that these links are pairwise non-1-quasi-isotopic.

On the other hand, it is not hard to see from the Wirtinger presentation that each fundamental group $\pi(\mathcal{M}_n)$ is generated by two meridians, say $m_1$ and $m_2$. Then $[m_1, l_1] = 1$ in $\pi = \pi(\mathcal{M}_n)$, where $l_1$ is the longitude corresponding to $m_1$. Since the linking number is 1, we have $l_1 = cm_2$ for some $c \in [\pi, \pi]$. Hence $1 = [m_1, cm_2] = [m_1, m_2][m_1, c]^{m_2}$ and so $[m_1, m_2] = [c, m_1]^{m_2} \in [\pi, \pi]$.

Thus $\gamma_2 \pi = \gamma_3 \pi$ and it follows that $\pi$ has length 2, i.e. $\gamma_k \pi = \gamma_2 \pi$ for any $k \geq 2$. (We recall that the finite lower central series of $\pi$ is defined inductively by $\gamma_1 \pi = \pi$ and $\gamma_{n+1} \pi = [\gamma_n \pi, \pi]$.) But for every link $L$, the group $G_k(L)$ is nilpotent by [MR1] (or by Theorem 2.1 below). Therefore $G_k(\mathcal{M}_n)$ is Abelian, more precisely it is $\mathbb{Z} \oplus \mathbb{Z}$. In particular, $P_k$ coincides with the image of the peripheral structure in $\pi$ under the quotient map $\pi \rightarrow G_k$. □

Example 1.4. Let us point out certain multi-component analogues of the generalized Sato–Levine invariant, well-defined up to 1-quasi-isotopy.
For each rational vector $s = (s_1, \ldots, s_m) \in \mathbb{Q}^m$ and any choice of a base link in each link homotopy class of $m$-component links, the following crossing change formula yields a well-defined link invariant $\beta(L, s)$ [J2]:

$$
\beta(L_1, s) - \beta(L_0, s) = \begin{pmatrix}
  s_1 & \ldots & a_{1,x-1} & q_1 & a_{1,x+1} & \ldots & a_{1m} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{x-1,1} & \ldots & s_{x-1} & q_{x-1} & a_{x-1,x+1} & \ldots & a_{x-1,m} \\
  p_1 & \ldots & p_{x-1} & \lambda & p_{x+1} & \ldots & p_{m} \\
  a_{x+1,1} & \ldots & a_{x+1,x-1} & q_{x+1} & s_{x+1} & \ldots & a_{x+1,m} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{m1} & \ldots & a_{m,x-1} & q_m & a_{m,x+1} & \ldots & s_m 
\end{pmatrix}
$$

Here $L_0$ and $L_1$ differ by a crossing change in the $x^{\text{th}}$ component, of the type shown on Fig. 2a. $a_{ij}$ are the linking numbers between the components, $p_i$ (resp. $q_i$) are the linking numbers between the first (resp. second) lobe of the singular $x^{\text{th}}$ component and the other components (so that $p_i + q_i = a_{xi}$), and $\lambda$ is the linking number between the two smoothed lobes.

(In fact, let $M_s$ denote the 3-manifold obtained from $S^3$ by the surgery on $(L, s)$, and $A = (a_{ij})$ denote the surgery matrix, where $a_{ii} = s_i$. Then up to a constant depending on the choice of base links, $\beta(L, s)$ is the Casson–Walker–Lescop invariant of $M_s$ divided by $(-1)^bq_1 \ldots q_m$ where $b$ is the number of negative eigenvalues of $A$ and $s_i = \frac{b}{q_i}$ with $(p_i, q_i) = 1$, $q_i > 0$ [J2]. Also if det $A \neq 0$, then up to the same constant, $\beta(L, s)$ is the Casson–Walker invariant of the rational homology sphere $M_s$ divided by $\frac{2}{\text{det } A}$ [J1].)

We note that $\beta(L, s)$ is invariant under 1-quasi-isotopy if $s$ is such that for each $i$ the minor $\det A_i = 0$, where $A_i$ is obtained from $A_i$ by deleting the $i^{\text{th}}$ line and the $i^{\text{th}}$ column. If $m = 2$ this means that $s = (0, 0)$, thus the only invariant of 1-quasi-isotopy yielded by this construction for 2-component links is the generalized Sato–Levine invariant (up to a constant). If $m = 3$ and all $a_{ij} = 0$, again it is not hard to see that if $\beta(L, s)$ is invariant under 1-quasi-isotopy, it is a linear function with constant coefficients of the generalized Sato–Levine invariant of a two-component sublink.

Let us consider the case $m = 3$, all $a_{ij} \neq 0$ in more detail. Clearly, there are only two ways of choosing $s_i$’s so that all minors $\det A_i = 0$, namely, $s_i = \pm \frac{a_{ii}a_{ik}}{a_{jk}}$. The
positive sign implies \( \det A = 0 \), while the negative sign leads to \( \det A \neq 0 \). Thus we obtain two invariants \( \beta_+, \beta_- \) of 1-quasi-isotopy of 3-component links whose two-component sublinks have nonzero linking number. A straightforward computation shows that each of these two invariants distinguishes two links, shown on Fig. 3, which have their 2-component sublinks ambient isotopic.

**Example 1.5.** It is not difficult to construct 2-component links with \( (\mathcal{G}_1, \mathcal{P}_1) \) being the same as that of the unlink, for each \( k \), but not obviously 1-quasi-isotopic to the unlink; see for example [MR1, Remark in the end of §1]. Here is a particular family of such candidates (one can verify that all \( \bar{\mu} \)-invariants of these links of length \( \leq 5 \) vanish, using the skein relation for the Sato–Levine invariant \( \bar{\mu}(1122) \)).

![Fig. 4](image)

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2. **The Structure of \( G_k(L) \)**

Let us fix some notation. Unless otherwise mentioned, we work in the PL category. \( F_m \) will denote the free group on \( m \) fixed generators, identified with a fixed set of meridians of the trivial \( m \)-component link. By \( \gamma_n G \) (or just \( \gamma_n \) if \( G \) is clear from the context) we denote the \( n \)th term of the finite lower central series of the group \( G \), defined inductively by \( \gamma_1 = G \) and \( \gamma_{n+1} = [\gamma_n, G] \). We use the left-normed convention for multiple commutators: \([a_1, \ldots, a_n] = [[[a_1, a_2], \ldots, a_{n-1}], a_n]\). We also use the notation \( g^{-h} = (g^h)^{-1} \).

It was proved in [MR1] that \( G_k(L) \) is nilpotent for any \( k \) and \( L \). The same approach, based on the Hirsch–Plotkin Theorem, can also be used to obtain an upper bound estimate of the nilpotent class of \( G_k(L) \) (depending on the minimal number of meridional generators of \( \pi(L) \)). However, this estimate turns out to be rather weak, for example, using this method, even with some improvements, we could obtain only class \( \leq 6 \) for \( G_1(L) \) where \( \pi(L) \) is generated by two meridians.

**Theorem 2.1.** If \( L \) has \( m \) components, \( G_k(L) \) is nilpotent of class \( m(k + 1) \).

This is immediate from the following three easy lemmas.
Lemma 2.2. [Chen], [MKS; Lemma 5.9] Given a choice of meridians, one to each component of \( L \), then every nilpotent quotient of \( \pi(L) \) is generated by their images.

(Indeed, any generator of the Wirtinger presentation is conjugate to one of the fixed meridians, so any element of \( \gamma_i \), which modulo \( \gamma_{i+1} \) is a product of commutators of weight \( i \) in the generators, can be expressed modulo \( \gamma_{i+1} \) in the commutators of weight \( i \) in the fixed meridians; this follows using the commutator identities \([ab, c] = [a, c][b, c], [a, bc] = [a, c][a, b]^c\) and \( x^y = x[y, x] \).)

Lemma 2.3. Let \( G \) be a group generated by \( x_1, \ldots, x_m \). Then \( \gamma_nG \) is the normal closure of all left-normed commutators \([x_{i_1}, \ldots, x_{i_n}]\).

The proof is similar to the proofs of [Hall; Theorem 10.2.3], [MKS; Theorem 5.4] and is given below for completeness.

Lemma 2.4. In the group \( \mathcal{G}_k(L) \), any left-normed commutator \([g_1, \ldots, g_n]\) with at least \( k + 2 \) entries \( g_1, \ldots, g_{i+2} \) occupied by certain meridian\(^2\), is trivial.

Proof. Notice that for any elements \( m, g \) of a group \( G \), the commutator \([m, g] = m^{-1}mg \) lies in the normal closure \( \langle m \rangle^G \), and the same holds for \([g, m] \). Moreover, even if \( m \) is replaced by some \( m' \in \langle m \rangle^G \), we still have \([m', g] \in \langle m \rangle^G \).

Consequently, if the meridian in question, which will also be denoted by \( m \), first occurs in position \( i_1 \), the commutator \([g_1, \ldots, g_{i_1}] = [g_1, \ldots, g_{i_1-1}, m] \) will be in \( \langle m \rangle^{\mathcal{G}_k} \), and moreover all the subsequent commutators \([g_1, \ldots, g_{i_1+j}], \) where \( j \geq 1 \), will also lie \( \langle m \rangle^{\mathcal{G}_k} \). Now if \( m \) occurs also in position \( i_2 \), the same reasoning ensures that the commutator \([g_1, \ldots, g_{i_2}] \) must be in \( \langle m \rangle^{\mathcal{G}_k} \), and in the same fashion it follows that \([g_1, \ldots, g_{i_k+1}] \in \langle m \rangle^{\mathcal{G}_k} \). But \( m \) is in the center of \( \langle m \rangle^{\mathcal{G}_k} \), according to [MRI; formula (3.1)]. Therefore the next bracket \([g_1, \ldots, g_{i_k+2}, m] \) is trivial, and so the whole commutator must be trivial. \( \square \)

Proof of 2.3. By the definition, any element of \( \gamma_nG \) can be presented as \( \prod_{i=1}^\infty [d_i, g_i] \) where \( d_i \in \gamma_{n-1}G \) and \( g_i \in G \). By induction, we can assume that each \( d_i \) is a product of conjugates of left-normed commutators of weight \( n - 1 \) in \( x_i \)'s and of their inverses, so that \([d_i, g_i] = [c_i^1, h_i, \ldots, c_i^k, h_i, g_i] = [c_i^1, v_i]^{c_i^2} \ldots [c_i^k, v_i]^{c_i^k} [c_i^k, v_i]^{-1} \), where one uses the commutator identity \([ab, c] = [a, c][b, c]\). Now for each \( j \) we have \([c_j^j, v_i] = [c_j^j, v_i^j]^{h_j} \), and if the sign is negative, we can eliminate it using the identity \([a^{-1}, b] = [a, b]^{-a^{-1}} \). Analogously, \([c_j, v_i^j]^{-1} = [c_j, \prod_{k=1}^t x_{i_k}]^{x_{i_k}} \) can be written as a product of conjugates of the commutators \([c_j, x_{i_k}] \). \( \square \)

Proposition 2.5. Any multiple commutator in the group \( \mathcal{G}_k(L) \) with at least \( k + 2 \) entries occupied by certain meridian, is trivial.

Proof. To each multiple commutator there corresponds a finite directed binary tree with vertices representing the bracket pairs and edges going in the direction of (the vertex representing) the next exterior bracket. Let \( L \) denote a directed path of maximal length in this graph, and let us define complexity of the commutator as the number of vertices not belonging to this path.

A commutator of complexity zero is one where no two brackets are separated by a comma, or in other words where the pairs of brackets form a chain. The proof of

\(^2\)The images of meridians in \( \mathcal{G}_k(L) \) will also be called meridians.
Lemma 2.4 works as well in this case (alternatively, one could reduce the case of complexity zero to the left-normed case similarly to what follows).

Let us assume that the assertion holds for all commutators of complexity $< c$, and let us consider a commutator of complexity $c > 0$ and weight $n > 3$. By downward induction on the weight (starting from the nilpotent class of $G_k$) we may assume also that the statement of the theorem is true for all commutators of weight $> n$. Since $c > 0$, there is a bracket not in the chain $L$, moreover we can assume that the next exterior bracket is in $L$. This must look like

$$\ldots[[a, b], c] \ldots \quad \text{or} \quad \ldots[c, [b, a]] \ldots$$

where $a$ and $b$ denote either commutators or single entries, and $c$ necessarily denotes a commutator since the exterior bracket lies in the chain $L$. In fact these two configurations are equivalent modulo commutators of higher weight but still containing at least $k + 2$ occurrences of our meridian, which follows using the commutator identities $[x^{-1}, y] = [x, y]^{-x}$, $x^y = x[x, y]$ and $[ab, c] = [a, c]b[b, c]$. Now in the situation of the first configuration, apply the Hall–Witt identity

$$[[a, b^{-1}], c][[c, a^{-1}], b]^n[[b, c^{-1}], a]^c = 1$$

which modulo commutators of higher weight can be simplified as

$$[[a, b], c] \sim [[a, c], b][[c, b], a].$$

Clearly, in our case these commutators of higher weight are going to retain the $\geq k + 2$ occurrences of the meridian, so our commutator of weight $n$ can be expressed as the product of two new commutators of weight $n$ where the expression $[[a, b], c]$ is substituted respectively by $[[a, c], b]$ and $[[c, b], a]$. Notice that in both cases the commutator $c$ has moved into a deeper bracket, so that the length of the chain $L$ has increased by one (in each of the two trees corresponding to the new commutators). It remains to apply the inductive hypothesis. □

**Lemma 2.6.** [Chan; Lemma 2] Let $G$ be a group and $g$ an element of $G$. Then

$$[g, g_n^G] = [G_{n+1}, g] \overset{\text{def}}{=} [[\ldots[G, g]\ldots], g].$$

As we could not find an English translation of [Chan], we provide a short proof for convenience of the reader.

**Proof.** By induction on $n$. It is immediate from the definition that

$$[[[\ldots[G, g]\ldots], g], g] = [g, [g, [g, G]\ldots]].$$

Therefore it remains to show that $[g, [g, g_{n-1}^G]] = [g, g_n^G]$. Indeed,

$$[g, [g, g_h^1]\ldots[g, h_r]^m] = [g, (g^{-1}g_h^1)^{\alpha_1}\ldots(g^{-1}g_r^m)^{\alpha_r}].$$

For some $m_0, \ldots, m_r \in \{-1, 0, 1\}$, the latter expression can be rewritten as

$$[g, g^{m_0}_0(g^{h_1}_1)^{\alpha_1}\ldots g^{m_{r-1}}(g^{h_r}_r)^{\alpha_r}g^{m_r}] = [g, g^{m_0}(g^{h_1}_1)^{\alpha_1}\ldots g^{m_{r-1}+m_r}(g^{h_r}_r)^{\alpha_r}]$$

$$= \ldots = [g, g^{m_0\cdots+m_r}(g^{h_1}g^{m_{r-1}+\cdots+m_r})^{\alpha_1}\ldots(g^{h_r})^{\alpha_r}] = [g, (g^{h_1}_1)^{\alpha_1}\ldots(g^{h_r}_r)^{\alpha_r}]$$

for some new $h'_1, \ldots, h'_r \in g_{n-1}^G$. □

Milnor’s original definition of $G(L)$ was based on the trivial case $k = 0$ of the following fact (compare [MR1; remark in the end of §3]).
Theorem 2.7. \( \mu_k = (\gamma_{k+2} A_1) \cdots (\gamma_{k+2} A_m) \) where \( A_i \) denotes the normal closure of a meridian to the \( i \)th component.

Proof. The inclusion ‘\( \subset \)’ follows from [MR1; formula (3.1)] and Lemma 2.6. To prove the reverse inclusion, notice that any \((k+2)\)-fold commutator in elements of \( A_i \) can be rewritten, using the commutator identities, as a product of commutators where a fixed meridian \( m_i \in A_i \) occurs at least \( k+2 \) times. By Proposition 2.5, these commutators lie in \( \mu_k \). \( \square \)

For \( k = 0 \) the following was proved by Levine [Le]:

Theorem 2.8. For \( i = 1, \ldots, m \), fix a meridian \( m_i \) to the \( i \)th component of \( L \). Then \( \mu_k \) is the subgroup generated by \( \gamma_{m(k+1)+1} \) and the basic commutators of weight \( \leq m(k+1) \) in \( m_i \)'s with at least \( k+2 \) occurrences of some \( m_j \).

We refer to [Hall] for a concise treatment of basic commutators (see also [MKS]).

Proof. A half of the statement is contained in Theorem 2.1 and Proposition 2.5. To prove the other half, by [MR1; formula (3.1)], Lemma 2.6 and a straightforward induction on weight (using that \( \gamma_r / \gamma_{r+2} \) is Abelian for \( r > 1 \)) it suffices to show that for each \( r \leq m(k+1) \), any element \( c \) of the subgroup \( \langle \pi(L), m \rangle \cap \gamma_r \pi(L) \), where \( m_0 \) is a meridian, is expressible modulo \( \gamma_{r+1} \pi(L) \) in basic commutators of weight \( r \) in \( m_i \)'s with at least \( k+2 \) occurrences of some \( m_j \).

We may assume without loss of generality that \( \pi(L) \) is the free group \( F_m \). If \( m_0 \) is conjugate to some \( m_i^{\pm 1} \), then \( m_0 \) is a product of \( m_i^{\pm 1} \) with some element of \( \langle m_j, F_m \rangle \), so the following Lemma 2.9 implies that the Magnus expansion \( M(m_0) \) contains at least one occurrence of \( x_j \) in every term of positive degree. Now apply the same lemma inductively to the elements of \( \langle F_{m_i}, m_0 \rangle \) to obtain that for any \( c \in \langle F_{m, k+2}, m_0 \rangle \), every term of \( M(c) - 1 \) contains \( x_j \) with total degree \( \geq k+2 \).

On the other hand, it is not hard to verify by induction that any commutator \( c_0 \) of weight \( r \) in \( m_1, \ldots, m_m \) has its Magnus expansion \( M(c_0) \) of the form \( 1 + \rho + \xi \) where \( \rho = \rho(c_0) \) is a homogenous polynomial of degree \( r \) and every term of \( \xi \) has degree \( > r \), moreover each \( x_i \) appears in every term of \( \rho \) with total degree equal to the number of occurrences of \( m_i \) in \( c_0 \). Now by the P. Hall Basis Theorem [Hall], each \( c \in \gamma_r F_m \) can be expressed modulo \( \gamma_{r+1} F_m \) as a monomial \( c = c_1^{e_1} \cdots c_t^{e_t} \) in basic commutators of weight \( r \).

Assume on the contrary that \( c \in \langle F_{m, k+2}, m_0 \rangle \) but some \( c_j \) in the above expression, with \( e_i \neq 0 \), contains less than \( k+2 \) occurrences of \( m_j \). We have \( \rho(c) = e_i \rho(c_i) + \cdots + e_s \rho(c_s) \), and since \( \rho(c_1), \ldots, \rho(c_s) \) are linearly independent [Hall], \( \rho(c) \) contains nonzero terms where the total degree of \( x_j \) is less than \( k+2 \). On the other hand, since \( c \in \langle F_{m, k+2}, m_0 \rangle \), we have shown above that every term of \( \rho(c) \) contains \( x_j \) with total degree \( \geq k+2 \). This contradiction proves the assertion. \( \square \)

Lemma 2.9. Let \( F_m = \langle m_1, \ldots, m_m \rangle \) be the free group, \( P \) be the ring of formal power series with integer coefficients in noncommuting variables \( x_1, \ldots, x_m \), and \( M : F_m \to P \) be the Magnus expansion defined by \( m_i \mapsto 1 + x_i \) (so that \( m_i^{-1} \mapsto 1 - x_i + x_i^2 - x_i^3 + \ldots \)). Let \( a, b \in F_m \) and \( i \in \{1, \ldots, m\} \). If each term of \( M(a) - 1 \) has total degree \( \geq d_1 \) in \( x_i \) and each term of \( M(b) - 1 \) has total degree \( \geq d_2 \) in \( x_i \), then

a) each term of \( M(a^{-1}) - 1 \) has total degree \( \geq d_1 \) in \( x_i \),

b) each term of \( M(ab) - 1 \) has total degree \( \geq \min(d_1, d_2) \) in \( x_i \).
c) each term of \(M([a,b]) - 1\) has total degree \(\geq d_1 + d_2\) in \(x_i\).

**Proof of (c).** Denote \(l_x = M(x) - 1\), then \(M([a,b])\) can be rewritten, using the identities \((l_{a-1} + 1)(l_{a} + 1) = 1\) and \((l_{b-1} + 1)(l_{b} + 1) = 1\), as

\[
(l_{a-1} + 1)(l_{b-1} + 1)(l_{a+1})(l_{b+1}) = (l_{a-1} + 1)l_{b-1}(l_{a} + 1)l_{b} + (l_{a-1} + 1)l_{b-1}(l_{a} + 1)l_{b} + 1
\]

\[
= l_{a-1}l_{b-1}l_{a}l_{b} + l_{a-1}l_{b-1}l_{a}l_{b} + l_{a-1}l_{b-1}l_{a}l_{b} + l_{a-1}l_{b-1}l_{a}l_{b} + l_{a-1}l_{b-1}l_{a}l_{b} + l_{a-1}l_{b-1}l_{a}l_{b} + 1
\]

\[
= l_{a-1}l_{b-1}l_{a}l_{b} + l_{a-1}l_{b-1}l_{a}l_{b} + l_{a-1}l_{b-1}l_{a}l_{b} + l_{a-1}l_{b-1}l_{a}l_{b} + 1
\]

\[\square\]

Theorem 2.8 together with the Hall Basis Theorem yield the following

**Corollary 2.10.** Every element \(\alpha\) of the \('k\)-quasi-free group \(F_m/\mu_k\) (isomorphic to \(G_k(m\text{-component unlink})\)) can be uniquely represented as \(\alpha = C_1^{i_1} \ldots C_r^{i_r}, e_i \in \mathbb{Z}\), where \(C_1, \ldots, C_r\) are the basic commutators in the fixed set of free generators of \(F_m\) with at most \(k + 1\) entries of each generator.

For the rest of this section, let \(m_i\) denote a fixed meridian to the \(i\)th component of \(L\) and \(l_{i}\) the corresponding longitude, and let \(l_{i}\) be a word in \(m_1, \ldots, m_m\) representing by Lemma 2.2 the class of \(l_{i}\) in \(\pi(L)/\gamma_q\) for some fixed \(q\). Note that it may be impossible to use the same word \(l_{i}\) = \(l_{i}(q)\) for each positive integer \(q\).

**Corollary 2.11.** If \(q > m(k + 1)\), \(G_k(L)\) can be presented as

\[
G_k(L) = \langle m_1, \ldots, m_m, [m_1, l_{i_1}], \ldots, [m_m, l_{i_m}], c_1, \ldots, c_r, d_1, \ldots, d_s \rangle
\]

where \(c_1, \ldots, c_r\) are all basic commutators of weight \(\leq m(k + 1)\) in \(m_i^{j_i}\)'s such that at least \(k + 2\) entries are occupied by the same symbol, and \(d_1, \ldots, d_s\) are all left-normed\(^3\) commutators in \(m_i^{j_i}\)'s of weight \(m(k + 1) + 1\).

This follows from 2.8, 2.3 and [Mi2; Theorem 4] where it was proved that \(\pi(L)/\gamma_q\) can be presented as \(\langle m_1, \ldots, m_m, [m_1, l_{i_1}], \ldots, [m_m, l_{i_m}], \gamma_q \rangle\). The case \(k = 0\) of Corollary 2.11 improves on a result of Levine [Le: Proposition 2.11].

**Remark.** In general, a presentation for \(G_k(L)\) can not be obtained by adding the relators \(c_1, \ldots, c_r, d_1, \ldots, d_s\) to a presentation of \(\pi(L)\), for \(\mu_k\pi(L)\) need not coincide with the normal closure \(N\) of \(\phi_{\mu_k}F_m\) in \(\pi(L)\), where \(\phi: F_m \to \pi(L)\) sends the generators to the fixed meridians \(m_i\). The latter is obvious e.g. when \(L\) is a split link with knotted components: the forgetful homomorphism into the fundamental group of a component sends \(\mu_k\pi(L)\) onto the commutator subgroup (since this is the case for the unlink, which is \(k\)-quasi-isotopic to \(L\)), while any \([m_i, m_j]\), hence \(N\), clearly goes to 1.

We recall that the invariant \(\bar{\mu}(i_1 \ldots i_s)\) [Mi2], where \(s < q\), is the coefficient \(\mu(i_1 \ldots i_s)\) at \(x_{i_1} \ldots x_{i_s}\) in the Magnus expansion (see 2.9) of \(l_{i}\), reduced modulo the greatest common divisor \(\Delta(i_1 \ldots i_s)\) of all \(\bar{\mu}(j_1 \ldots j_t)\) where \(j_1, \ldots, j_t\) runs over all ordered subsequences of \(i_1, \ldots, i_s\) of length \(2 \leq t \leq s\).

Note that in view of cyclic symmetry [Mi2; formula (21)] we may always assume that if some index occurs ‘too many’ times in a \(\bar{\mu}\)-invariant, one of its entries is in the rightmost place.

\(^3\)See comments to Problem 3.15.
Theorem 2.12. Suppose \( k \geq 1 \).

a) If for some \( z \) the sequence \( i_1, \ldots, i_n \) contains at most \( k + 1 \) entries of each index, \( \bar{\mu}(i_1, \ldots, i_n) \) can be extracted from \( (G_k, P_k) \); in particular, it is invariant under \( k \)-quasi-isotopy.

b) An \( m \)-component link \( L \) has the same \( (G_k, P_k) \) as the \( m \)-component unlink iff all \( \bar{\mu} \)-invariants with \( \leq k + 2 \) entries of one of the indices and \( \leq k + 1 \) entries of each of the remaining indices vanish.

c) A \( \bar{\mu} \)-invariant with at least one index occupying more than \( k + 2 \) entries or at least two indices occupying more than \( k + 1 \) entries each cannot be extracted from \( (G_k, P_k) \), unless identically trivial.

Proof. (a). By Theorem 2.8 and Proposition 2.13(i) below, the value of the \( \bar{\mu} \)-invariant on any link can be determined from \( G_k \) together with the images \( (m_i, l_i) \) of meridians and longitudes. It remains to verify that this value is unchanged under multiplication of \( l_i \) on the left by an element of \( N_k(m_i) \). But this follows from Lemma 2.9 (similarly to [Mi2; proof of (25)]) using that \( 2k + 1 > k + 1 \), which accounts for the restriction \( k > 0 \). \( \square \)

(b). The ‘only if’ part follows from (a). The ‘if’ part follows from Proposition 2.5 and Proposition 2.13(ii) below. \( \square \)

c) Let \( \bar{\mu}(i_1, \ldots, i_n) \) be the invariant in question. By [Orr] and Proposition 2.13(iii) below it is possible to construct a link \( L_n \) which has a nontrivial \( \bar{\mu}(i_1, \ldots, i_n) \), trivial \( \bar{\mu} \)-invariants of length \( < n \), and \( \bar{\mu}(j_1, \ldots, j_n) = 0 \) whenever \( (j_1, \ldots, j_n) \) is not a permutation of \( (i_1, \ldots, i_n) \). By Proposition 2.13(iv), since cyclic symmetry holds for \( \bar{\mu} \)-invariants of arbitrary length [Mi2], we see that the \( \bar{\mu} \)-invariants of \( L_n \) of length \( n + 1 \) satisfy all the relations that the \( \bar{\mu} \)-invariants of length \( n + 1 \) of a link with vanishing \( \bar{\mu} \)-invariants of length \( < n + 1 \) have to satisfy. Therefore by [Orr] there exists a link \( L'_n + 1 \) with vanishing \( \bar{\mu} \)-invariants of length \( < n + 1 \) and exactly the same \( \bar{\mu} \)-invariants of length \( n + 1 \) as those of \( L_n \), but with the opposite sign. Then by [Kr] any connected sum \( L_{n+1} = L_n + L'_n + 1 \) has the same \( \bar{\mu} \)-invariants of length \( \leq n \) as \( L_n \), moreover

\[
\bar{\mu}_{L_{n+1}}(j_1, \ldots, j_{n+1}) = 0 \pmod{\Delta_{L_n}(j_1, \ldots, j_{n+1})}
\]

for any multi-index \( (j_1, \ldots, j_{n+1}) \).

Iterating this process, we obtain a link \( L_{m(k+1)+1} \) which has \( \bar{\mu}(i_1, \ldots, i_n) \neq 0 \), and all \( \bar{\mu} \)-invariants with \( \leq k + 2 \) entries of one of the indices and \( \leq k + 1 \) entries of each of the remaining indices trivial. The assertion now follows from the ‘if’ part of (b). \( \square \)

Proposition 2.13. Let \( \mathcal{I} = (i_1, \ldots, i_n) \), where \( n + 1 < q \), denote an unordered multi-index (with possible repeats of entries), \( 1 \leq i_j \leq m \), let \( \mathcal{I}^\sigma \) denote \( \mathcal{I} \) with a fixed order, and let \( \mathcal{I}^c \) denote a basic commutator in the alphabet \( m_1, \ldots, m_m \) whose entries are \( m_{i_1}, \ldots, m_{i_n} \) (in some order). Let \( i \) be a fixed index, \( 1 \leq i \leq m \), and let \( E(c_1; w), \ldots, E(c_r; w) \) denote the powers of the basic commutators \( c_1, \ldots, c_r \) of weight \( < q \) in the decomposition of a word \( w \) in the free nilpotent group \( F_m/\gamma_q \).

(i) The invariants \( \bar{\mu}(\mathcal{I}^\sigma i) \) for all orderings \( \sigma \) of \( \mathcal{I} \) can be expressed in the commutator numbers \(^4 E(\mathcal{I}^c; l_i) \) where \( c \) runs over all bracketings of \( \mathcal{I} \).

\(^4\) Beware that these integers are not, in general, invariant even under ambient isotopy of \( L \); see [Le] for a discussion of their indeterminacies in the case where all indices are distinct.
(ii) Conversely, if \( \Delta(I^\sigma;i) = 0 \) for all orderings \( \sigma \) of \( I \), the commutator numbers \( E(I^\sigma;I_1) \) for all bracketings \( c \) of \( I \) can be expressed in the invariants \( \bar{\mu}(I^\sigma;i) \).

(iii) Furthermore, if all \( \bar{\mu} \)-invariants of length \( \leq n \) vanish, then a complete set of relations between the integer invariants \( E(J^c;J_j) \), where \( J^c \) runs over all basic commutators of weight \( n \) and \( j \) runs over \( 1, \ldots, m \), is given by

\[
\sum_{(J,j) = \mathcal{K}^c} \sum_{E(J^c, J_j; m_j, J^c)} E(J^c; J_j) = 0
\]

for each basic commutator \( \mathcal{K}^c \) of weight \( n+1 \).

(iv) Moreover, relations (*) are equivalent to cyclic symmetry of \( \bar{\mu} \)-invariants of length \( n+1 \).

(In the last assertion it is understood that the commutator numbers \( E(J^c;J_j) \) and the \( \bar{\mu} \)-invariants are calculated from arbitrary fixed words \( I_1 = i_1(m_1, \ldots, m_m) \) rather than those representing the longitudes, and it is assumed that among these formal \( \bar{\mu} \)-invariants, all of length \( \leq n \) vanish. Of course, there are other symmetries of \( \bar{\mu} \)-invariants, but we see that, at least in the case where all \( \bar{\mu} \)-invariants of length \( \leq n \) vanish, all relations between \( \bar{\mu} \)-invariants of length \( n+1 \), except for cyclic symmetry, come from nontriviality of the cokernel of the Magnus expansion rather than from geometry.)

Proof. In the ring \( P \) of Lemma 2.9, let \( D_i \) denote the subset, which is easily seen to be a two-sided ideal, consisting of all \( \sum \mu_{h_1 \ldots h_r} x_{h_1} \ldots x_{h_r} \) such that either \( s \geq q \) or \( s < q \) and \( \mu_{h_1 \ldots h_s} \equiv 0 \pmod{\Delta(h_1, \ldots, h_s;i)} \).

It is not hard to see that for every \( c \in F_m \), every monomial of \( M(c) - \rho(c) - 1 \) can be obtained from some monomial of \( \rho(c) \), where \( \rho(c) \) denotes the leading part of \( M(c) \) (that is, \( M(c) = 1 + \rho(c) + \text{terms of higher degree}, \) where \( \rho(c) \) is homogenious) by insertion of at least one letter and multiplication by a constant. It follows that

\[
M(I_i) = M(c_1^{e_1} \ldots c_r^{e_r}) \equiv e_1 \rho(c_1) + \cdots + e_r \rho(c_r) \quad (\text{mod } D_i).
\]

Since the total degree of \( x_i \) in every term of \( \rho(c_j) \) equals the number of entries of \( m_i \) in \( c_j \), and the residue classes \( \bar{\mu}(I^\sigma;i) \) are well-defined up to congruence of \( M(I_i) \) modulo \( D_i \), this proves the first assertion.

By the above, we have \( \sum_{I_1} E(I^\sigma;I_1) \rho(I^\sigma) = \sum_{I} \mu(I^\sigma) x_{I^\sigma} \pmod{D_i} \), hence absolutely, and since the leading parts of the Magnus expansions of the basic commutators of weight \( n \) are linearly independent [Hall], this proves the second assertion.

It is well-known [Rol] that one of the relations in the Wirtinger presentation of a link is redundant. It follows from the precise form of this redundancy (see [Mi2; proof of Lemma 5] or use the above argument with the tangle) that a product of certain conjugates of the relators \( [m_i, l_i] \) of Milnor’s presentation for \( \pi(L)/\gamma_q \) equals 1 in \( F_m/\gamma_q \). In our situation, where all \( \bar{\mu} \)-invariants of length \( \leq n \) are trivial, and so all longitudes lie in \( \gamma_n \), this implies

\[
[m_1, l_1][m_2, l_2] \ldots [m_m, l_m] = 1 \quad (\text{mod } \gamma_{n+2}F_m).
\]

This identity means that the words \( l_1, \ldots, l_m \) cannot be chosen independently of each other, and it follows that their commutator numbers satisfy relations (*).

From the definition \( \gamma_{n+1}F_m = [F_m, \gamma_nF_m] \) it follows that each of relations (*) contains at least one nonzero coefficient \( E(\mathcal{K}^d; [J^c, m_j]) \). Therefore we have \( N_{n+1} \)}
of nontrivial linear relations imposed on \( mN_n \) of integer indeterminates, where \( N_n \) denotes the number of basic commutators of weight \( n \) (in the alphabet \( m_1, \ldots, m_m \)). But it was proved in [Orr] that if all \( \bar{\mu} \)-invariants of length \( \leq n \) vanish, then there are exactly \( mN_n - N_{n+1} \) linearly independent \( \bar{\mu} \)-invariants of length \( n + 1 \). Thus relations (*) are all that hold.

To prove the last assertion, we note that it was shown in [Mi2; proof of Lemma 5] that cyclic symmetry of formal \( \bar{\mu} \)-invariants of length \( n + 1 \) is equivalent to the relation \( \rho([m_1, l_1] \ldots [m_m, l_m]) = 0 \) (mod \( D \)), where \( D \) is the ideal formed by all polynomials \( \sum \nu^{h_1 \ldots h_s} x_{h_1} \ldots x_{h_s} \) such that either \( s \geq q \) or \( s < q \) and \( \nu^{h_1 \ldots h_s} \equiv 0 \) (mod formal \( \Delta(h_1, \ldots, h_s) \)). Now if all formal \( \bar{\mu} \)-invariants of length \( \leq n \) vanish, we have \( \rho([m_1, l_1] \ldots [m_m, l_m]) = 0 \) absolutely, which is equivalent to \( [m_1, l_1] \ldots [m_m, l_m] = 1 \) (mod \( \gamma_{n+2}F_m \)).

3. Milnor’s links and bounded Engel elements

Let \( \mathcal{M}: S^1_1 \sqcup S^1_2 \hookrightarrow S^3 \) denote the \( (k+1) \)th Milnor link [Mi2; Fig. 1] (see Fig. 5, where \( k = 3 \)) where \( S^1_1 \) is standardly embedded, and let \( T \cong S^1 \times D^2 \) be the exterior of a regular neighborhood of \( \mathcal{M}(S^1_1) \) in \( S^3 \setminus \mathcal{M}(S^1_2) \). Let us call the embedding \( \mathcal{M}|_{S^1_1}: S^1_1 \to T \) the \( k \)th Milnor curve in the solid torus \( T \) (so that the zeroth is the well-known Whitehead curve).

We say that an \( m \)-component link \( L' \) is obtained from \( L \) by an elementary \( k \)-move if there is a solid torus \( T \subset S^3 \setminus L(mS^1) \) with the \( k \)th Milnor curve \( C \subset T \), so that \( L' \) is the interior connected sum of \( L \) and \( C \) along some band \( b: I \times (I, \partial I) \hookrightarrow (S^3, L(mS^1) \cup C) \) joining the arc \( b(I \times 0) \subset C \) with the arc \( b(I \times 1) \subset L(mS^1) \) and meeting \( \partial T \) in an arc, say, \( b(I \times I/2) \). More precisely,

\[
L'(mS^1) = (L(mS^1) \cup C) \setminus h(I \times \partial I) \cup h(\partial I \times I).
\]

Then fine \( k \)-quasi-isotopy is the equivalence relation generated by elementary \( k \)-moves and ambient isotopy. It is straightforward that fine \( k \)-quasi-isotopy implies \( k \)-quasi-isotopy, and fine 0-quasi-isotopy coincides with 0-quasi-isotopy (i.e. link homotopy).
**Proposition 3.1.** If two links are fine $k$-quasi-isotopic, then one can be deformed into another by Habiro moves [Hab], corresponding to the graph with $k+2$ univalent vertices, shown on Fig. 6.

![Fig. 6](image)

**Proof.** It suffices to consider links that differ by an elementary $k$-move. In this case our claim is clear from Fig. 7, which demonstrates unlinking of the $(k+1)^{th}$ Milnor link by means of Habiro moves, corresponding to the unitrivalent graphs above. 

![Fig. 7](image)

This result should be considered in the context of statements on Engel elements in groups, which we are to discuss next. The informal idea here is that Habiro
First let us notice that

\[ \text{Proof.} \]

Then \( \delta_k \) is normal and coincides with the subgroup of \( \pi(L) \) generated by

\[ \{ [m, m^g] \mid m \in M, g \in \pi(L) \} , \]

and the quotient group \( G_k^+(L) = \pi(L)/\delta_k \) is the finest quotient of \( \pi(L) \), functorially invariant under fine \( k \)-quasi-isotopy.

**Proof.** First let us notice that \([m, m^g] = [m, [m, g]] = [m, [m, g]]\), hence

\[ [m, \underbrace{m, \ldots, m}_{k + 1 \text{ of } m's}, m^g] = [m, \underbrace{[m, \ldots, [m, g], \ldots]}_{k + 2}]. \]

Also

\[ [m, [m, \ldots, m, g, \ldots]] = [[m, m]^{-1}, \ldots, m]^{-1}, m]^{-1} = ([[[[m^k + 2, m]^{-1}, m]^{-1}, \ldots, m]^{-1}], m]^{-1} \]

This means, using the notation \( m_0^g = g, m_{k+1}^g = m_0^g \), that \( \delta_k \) is generated by

\[ \{ [m, m_{k+1}^g] \mid m \in M, g \in \pi(L) \} . \]

A standard computation based on [MR1; proof of Theorem 3.2] shows that \( G_k^+ \) is functorially invariant under fine \( k \)-quasi-isotopy.

So (cf. [MR1; proof of Theorem 3.7]) it remains to show that for any link \( L_0 \), any meridian \( m \) of \( L_0 \), and any \( g \in \pi(L_0) \) there exists a fine \( k \)-quasi-isotopy \( L_t \) with a single self-intersection of a component such that the link \( L_{s-t} \) immediately preceding the singular link \( L_s \), has meridians \( m \) and \( m_{k+1}^g \) sharing the same stem and winding around the arcs \( L_{s-t}(l_1) \) and \( L_{s-t}(l_2) \) respectively, where \( l_1 \cup l_2 \subset S^1_j \)

is a small neighborhood of the preimage of the double point of \( L_s \).

To see this, first in the special case where \( L_0 \) is the two-component unlink with image in the plane \( \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3 \) and \( m, g \) are the Wirtinger generators of \( \pi(L_0) \), let us consider the evident straight line homotopy with a single self-intersection from the Milnor link \( M_{k+1} \) to the link \( M_{k+1} \), which is isotopic to the unlink and differs from \( M_{k+1} \) only in that the clasp is 'open'. One verifies that the meridians \( m \) and \( m_{k+1}^g \) of the link \( M_{k+1} \) are exactly as required, and our special case follows.

The general case follows by applying the special case locally. We leave details to the reader. \( \square \)

In fact, the difference between the subgroups \( \mu_k \) and \( \delta_k \) is similar to the difference between subgroups \( \nu_{k+2} \) and \( \varepsilon_{k+2} \) defined below, which are of some importance in combinatorial group theory.
**Engel groups and \( \varepsilon_n \).** A group \( G \) is called an *Engel group* if

\[
[[\ldots, [g, h], \ldots, h], h] = 1
\]

for each \( g, h \in G \) and some \( n = n(g, h) \), and an *\( n \)-Engel group* if moreover \( n \) can be chosen independent of \( g \) and \( h \) [Gr1], [Ku], [KMe], [Rob]. An element \( h \) of an arbitrary group \( G \) is called a *(left) Engel element* if the equality above holds for each \( g \in G \) and some \( n = n(g, h) \), and a *(left) \( n \)-Engel element* if in addition \( n \) can be chosen the same for all \( g \) [Gr2], [Ku], [Rob]. For any group \( G \) we denote by \( \varepsilon_n G \), \( n \geq 1 \), its \( n \)th *Engel subgroup*, generated by the set

\[
\{[[\ldots, [g, h], \ldots, h], h] \mid g, h \in G \}.
\]

Since \( [g, h]^f = [g^f, h^f] \), this subgroup is normal. It is easy to see that a group \( G \) is \( n \)-Engel iff \( \varepsilon_n G = 1 \). Clearly, \( \varepsilon_1 G = \gamma_2 G \) for any \( G \). Notice that \( \varepsilon_2 G \) is generated by \( \{[h, h^g] \mid g, h \in G \} \) since \( \varepsilon_2 G \) is normal and

\[
[[g, h], h] = [[h, g]^{-1}, h] = [h, [h, g]]^{[g, h]} = [h, h[h, g]]^{[g, h]} = [h, h^g]^{[g, h]}.
\]

In this form \( \varepsilon_2 \) looks more similar to \( \gamma_2 \) rather than \( \gamma_3 \).

**Remark.** The above observation that \( [g, h, h] \) is conjugate with \( [h, h^g] \) can be pushed one step further, that is, in any group, \( [g, h, h, h] \) is conjugate with \( [h, h^{h^g}]^{-1} \).

Indeed, notice that \( [h, h^g] = [h, h[h, g]] = [h, [h, g]], \) hence \( [h, h^{h^g}] = [h, [h, g]] \).

So

\[
[g, h, h, h] = [[h, [h, g]], [h, g]]^{[g, h]} = [[h, [h, g]], h[h, g]]^{[g, h]} = [h, h[h, g]]^{[g, h]} = [h, h[h, g]]^{[g, h]}^{-1}.
\]

**Example 3.3.** Here are two simple cases where \( \varepsilon_2 G \neq \gamma_2 G \). The group

\[
Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = (ijk)^{-1} \rangle
\]

is the union of its subgroups \( \langle i \rangle \), \( \langle j \rangle \), \( \langle k \rangle \) so that any two elements, conjugate in \( Q_8 \), lie in one of them, which means that \( \varepsilon_2 Q_8 = 1 \), whereas \( \gamma_2 Q_8 = \langle ijk \rangle \).

Alternatively, let \( D_4 \) be the 2-Sylow subgroup of the symmetric group \( S_4 \), generated by the permutations (12) and (13)(24). Then \( D_4 \) is the union of its subgroups

\[
H_1 = \{(12), (34), (12)(34), 1\},
\]

\[
H_2 = \{(13)(24), (14)(23), (12)(34), 1\},
\]

\[
H_3 = \{(1324), (1423), (12)(34), 1\}.
\]

Any two permutations, conjugate in \( D_4 \), should lie in one of these Abelian subgroups. On the other hand, \( [D_4, D_4] = [H_i, H_j] = \{(12)(34)\}, i \neq j \).
Example 3.4. Let us see that $\varepsilon_2 F_m \neq \gamma_3 F_m$ for $m \geq 3$, where $F_m = \langle a_1, \ldots, a_m \rangle$ denotes the free group. It suffices to prove that $\varepsilon_2 F_3/\gamma_4 F_3 \neq \gamma_3 F_3/\gamma_4 F_3$. The basic commutators of weight 3 in $F_3$ are

$$[[b, a], [b, a], [c, a], [c, a], [c, a], [c, b], [c, b], [c, b]].$$

Let us consider the quotient $Q$ of $\gamma_3 F_3/\gamma_4 F_3$ by the subgroup

$$(\gamma_3 \langle a, b \rangle / \gamma_4 F_3) \oplus (\gamma_3 \langle a, c \rangle / \gamma_4 F_3) \oplus (\gamma_3 \langle b, c \rangle / \gamma_4 F_3).$$

Now $[[h, g], g]$ can be written modulo $\gamma_4 F_3$ as $[[a^{b^i c^j}, a^{b^m c^n}], a^{b^m c^n}]$ for some $i, j, k, l, m, n \in \mathbb{Z}$, and

$$[[a^{b^i c^j}, a^{b^m c^n}], a^{b^m c^n}] = [[a, b^{m c^n}], a^{b^i c^j}][[b, a^{b^i c^j}], a^{b^m c^n}] = Q$$

$$[[a, b], c]^{i n m} = [[a, b], c]^{i n m}[[b, a], c]^{j l m}[[c, a], b]^{k l m}[[c, b], a]^{k l m} = Q$$

$$[[b, c], a]^{- i n m} = [[b, c], a]^{- i n m}[[b, a], c]^{- j l m}[[c, a], b]^{- k l m}[[c, b], a]^{k l m},$$

where the last line uses the Hall–Witt identity $[[a, b^{-1}], c]^{b[[b, c^{-1}], a]^{-1}}[c, a^{-1}], b]^{a} = 1$ in the form $[[a, b], c][[b, c], a]^{a}[[c, a], b]_{\gamma_4} = 1$.

Since $Q$ is generated by two basic commutators, the quotient $Q'$ of $Q$ by the subgroup $\langle [[a, b], c][[c, a], b] \rangle$ is infinite cyclic generated by $[[c, a], b]_{Q'} = [[b, a], c]^{-1}$.

From the above we see that the image of $\varepsilon_2 F_3/\gamma_4 F_3$ in $Q'$ is the subgroup $3Q' \neq Q'$, which proves the assertion.

Remarks. (i). Notice that $\varepsilon_2 F_2 = \gamma_3 F_2$ since $\varepsilon_2 F_2$ contains $\gamma_4 F_2$ [Rob], meanwhile both basic commutators of weight 3 in $F_2$ lie in $\varepsilon_2 F_2$, namely:

$$[[b, a], a] \in \varepsilon_2 F_2$$

$$[[b, a], b] = [[a, b]^{-1}, b] = [[[a, b], b]^{-1}]^{b, a} \in \varepsilon_2 F_2.$$

(ii). It is not hard to show, using the above methods, that $\varepsilon_3 F_2/\gamma_5 F_2$ is a subgroup of index two in $\gamma_4 F_2/\gamma_5 F_2$, and therefore $\varepsilon_3 F_2 \neq \gamma_4 F_2$.

(iii). From the above argument we also see that the product of two 2-Engel elements is not necessarily a 2-Engel element.

Notice that to prove that a group, generated by $(k + 2)$-Engel elements (in particular, $G^k_2 \langle L \rangle$ for any link $L$) may be non-nilpotent, one has to deal with at least one of the following two group-theoretic problems, dating back to 1950s [Gr1], [Ba2], [Pl1], which to the best of the authors’ knowledge are still unsolved (cf. [KMc], [Rob], [BM], [Va], [Ko], [Ba2], [Pl1]):

(i) whether every finitely generated group where each element is bounded Engel, must be nilpotent, and

(ii) whether the set of bounded Engel elements of every group is a subgroup.

These are modifications of the two famous problems: whether finitely generated $n$-Engel groups are nilpotent, and whether the set of Engel elements is always a subgroup. There are simple examples of non-nilpotent infinitely generated $n$-Engel groups [Rob; Ex. 12.3.1], [KMc] and an example, due to Golod (1964), of a non-nilpotent finitely generated Engel group [KMc].
Theorem 3.5.  

a) Consider $G_k^{++}(L) = \pi(L)/\varepsilon_{k+2}$, the 'homogenious' quotient of $G_k^+(L)$. Then $G_k^{++}$ is nilpotent, all epimorphic images of $G_k^+$ of exponent 5 are nilpotent, and for any $k$, all residually finite epimorphic images of $G_k^{++}$ are nilpotent.

b) Any epimorphic image of $G_k^+(L)$ with all Abelian subgroups finitely generated or with all normal closures of single elements finitely generated, is Engel.

c) Any solvable or Noetherian (in particular, any finite) epimorphic image of $G_k^+(L)$ is nilpotent.

Recall that a group is Noetherian if all its subgroups are finitely generated.

Proof. (a). By the definition, $G_k^{++}$ is a $(k+2)$-Engel group. Now finitely generated 3-Engel groups are known to be nilpotent [Hei]. The second and third assertions follow from the results of Vaughan-Lee [Va] and Wilson [Wi] that all 4-Engel groups of exponent 5 and all finitely generated residually finite $l$-Engel groups are nilpotent.

Remark. For finite, rather then residually finite, epimorphic image it suffices to use the easier result that any finite Engel group is nilpotent (see [Rob] and references in [Gr1] and [Ku]).

(c). The image $\bar{m}_i$ of each meridian $m_i$ in the quotient $G_k^+(L)$ is a $(k+2)$-Engel element. This holds if we proceed further to the given solvable (or Noetherian) quotient $Q$ of $G_k^+(L)$, that is, $\bar{m}_i \in Q$ is a $(k+2)$-Engel element. Since $Q$ is solvable (resp. Noetherian), by [Gr2] (resp. by [Ba2]) all its Engel elements form a normal subgroup. But the $m_i$'s normally generate $\pi(L)$, hence the $\bar{m}_i$'s normally generate the whole $S$, thus $Q$ is an Engel group. Now every Noetherian (resp. finitely generated solvable) Engel group is nilpotent by [Ba2] (resp. by [Gr1], see also [Rob]) which completes the proof.

(b). This is analogous to the above argument, using the result of Plotkin [Pl2] that in a group with all Abelian subgroups (or all normal closures of elements) finitely generated the set of Engel elements is a normal subgroup.

Baer groups and $\nu_n$. Let $\nu_n G$, $n \geq 1$, denote the subgroup generated by

$$\bigcup_{g \in G} \langle \ldots \langle g \rangle \ldots \langle g \rangle, \langle g \rangle \rangle,$$

then $\nu_n G$ is normal in $G$. The group $G$ is called a Baer group [Ba1], [Rob], if for each $g \in G$, the cyclic subgroup $H = \langle g \rangle$ is subnormal\(^5\) in $G$. By Lemma 2.6, $\nu_n G$ coincides with $\langle \cup_{g \in G} \langle \langle g \rangle \rangle \rangle$, so $\nu_1 G = 1$ implies $\langle g \rangle^G = \langle g \rangle$ (but not vice versa, as seen, say, for $n = 1$ and the group $D_4$). In particular, if $\nu_n G = 1$ for some finite $n$, then $G$ is a Baer group.

\(^5\) A subgroup $H$ of a group $G$ is said to be subnormal in $G$ if there exists a finite chain of subgroups $H = H_0 \subset H_1 \subset \ldots \subset H_d = G$ such that each $H_i$ is normal in $H_{i+1}$. 

Theorem 3.6. \( \nu_n G = 1 \) iff \( \langle g \rangle^G \) is nilpotent of class \( n-1 \) for each \( g \in G \).

Proof. The ‘if’ part holds by the definition, since \( [G, \langle g \rangle] \subset \langle g \rangle^G \). To prove the converse, we may assume that \( G \) is finitely generated. (Indeed, we need to show that for any \( g \in G \), any commutator \( c = [h_1, \ldots, h_{n-1}] \), where each \( h_i \) is a product of conjugates of \( g \) by some \( h_{ij} \in G \) and their inverses, lies in \( \nu_n G \). But \( c \in \gamma_{n-1} G \) and \( \nu_n H \subset \nu_n G \), where \( H \) is the subgroup generated by \( g \) and all \( g_{ij} \).) Now the proof is analogous to the proof of the inclusion ‘⊃’ in Theorem 2.7, taking into account that finitely generated Baer groups are nilpotent [Ba1] or, alternatively, that \( \nu_n F_m \supset \mu_n - F_m \supset \gamma_{m(n-1)+1} \) by Theorem 2.1. □

Evidently, \( \varepsilon_1 G = \nu_1 G = \gamma_2 G \) and in general \( \varepsilon_n G \subset \nu_n G \subset \gamma_{n+1} G \). However, it is not obvious that \( \nu_n G \) may differ from \( \varepsilon_n G \) for some \( n \) and \( G \). The possible difference between \( \varepsilon_2 G \) and \( \nu_2 G \) could be that the latter contains all the elements of the form

\[
[[h_1, g^{m_1}], [h_2, g^{m_2}], \ldots, [h_r, g^{m_r}], g^m]
\]

where all \( \alpha_i = \pm 1 \), \( m_i, m \in \mathbb{Z} \), \( h_i \in G \) and \( g \in G \), while the former does not obviously contain them, unless \( r = 1 \) and \( \alpha_1 = m_1 = m = 1 \). However it turns out that \( \varepsilon_2 G \) and \( \nu_2 G \) happen to be identical. Indeed, since \( \varepsilon_2 G \) is normal, it contains all elements of type \( [[[h, g], g]^{\pm 1}]^c \), where \( c \) is a product of commutator identities \( [ab, c] = [a, c]^b [b, c] \) and \( [x^{-1}, y] = ([x, y]^{-1})^{x^{-1}} \). Notice that, using the identity \( x^\nu = x[x, g] \), the argument above generalizes to prove that \( \nu_n G / \gamma_{n+2} G = \varepsilon_n G / \gamma_{n+2} G \) (cf. [Mo]).

Furthermore, it was shown in [KK] that \( \varepsilon_3 = \nu_3 \) in any group (here and below \( \nu_n \) is understood to be the subgroup generated by the \( (n-1) \)th lower central terms of the normal closures of elements). Around 1980, Gupta, Heineken and Levin constructed groups with \( \varepsilon_4 \neq \nu_4 \) [GL]. For \( k \geq 5 \) there are also examples of \( k \)-Engel groups with \( \nu_k \neq 1 \) for any finite \( n \) [GL], but no such finitely generated groups seem to be known. The groups in these examples are solvable, although \( \varepsilon_n G = \nu_n G \) for metabelian \( G \) [KMo] (see [Mo] for further results in this direction).

Theorem 3.7. Any quotient of the metabelianized fundamental group, functorially invariant under \( k \)-quasi-isotopy, is functorially invariant under fine \( k \)-quasi-isotopy.

Speaking informally, the Alexander module “cannot distinguish” \( k \)-quasi-isotopy from fine \( k \)-quasi-isotopy.

Proof. For any elements \( b_1, \ldots, b_n \) of a metabelian group \( G \) and any \( a \in G' \), one has \( [a, b_1, \ldots, b_n] = [a, b_{\sigma(1)}, \ldots, b_{\sigma(n)}] \), where \( \sigma \) is any permutation, see e.g. [Ne]. As noted in the proof of Theorem 2.7, \( \mu_k \) is generated by the left-normed commutators where one of the fixed meridians \( m_1, \ldots, m_m \) occurs at least \( k+2 \) times as an entry. The above identity can be used to transform any such commutator, modulo \( \pi(L)^n \), to a commutator (of weight 1 perhaps) where one of the entries is either

\[
x := [g, m_i, m_i, \ldots, m_i]_{k+1} \\
y := [m_i, g, m_i, \ldots, m_i]_{k+1}
\]
for some $g \in \pi(L)$. Now notice that since $[a^{-1}, b] = [a, b]^{-1}([a, b]^{-1}, a^{-1})$, for any element $b$ of a metabelian group $G$ and any $a \in G$, one has $[a^{-1}, b] = [a, b]^{-1}$. Repeated use of this identity shows that $x = y^{-1}$ modulo $\pi(L)^n$. Thus the images of $\mu_k$ and $\delta_k$ in the metabelianization coincide. □

**Conjecture 3.8.** For $k \geq 2$, but not for $k = 1$, difference between $k$-quasi-isotopy and fine $k$-quasi-isotopy can be detected by the higher (non-commutative) Alexander modules of Cochran et al.

**Theorem 3.9.** A group $G$, generated by two 3-Engel elements (in particular, $G^+_5(L)$ for any 2-component link with $\pi(L)$ generated by two meridians), is of length $\leq 5$, i.e. $\gamma_5 G = \gamma_6 G = \ldots$

**Remark.** A group, generated by two 2-Engel elements, is nilpotent of class 2, since the two basic commutators of weight $3$ in $F_2 = \langle a, b \mid \rangle$ normally generate $\gamma_3 F_2$ (cf. Lemma 2.3), while they can be expressed as

$$[[b, a], a] = [b, a, a];$$

$$[[b, a], b] = [[a, b]^{-1}, b] = ([a, b]^{-1})^{[b, a]}.$$

**Proof.** It suffices to prove that the group

$$G = F_2/\delta_1 F_2 = \langle a, b \mid x, a, a, a = [x, b, b, b] = 1 \forall x \in G \rangle$$

has length $\leq 5$. In other words, the image of $\gamma_5 F_2$ in the quotient of $F_2$ by $(\gamma_6 F_2)(\delta_1 F_2)$ must be trivial. The abelian group $\gamma_5 F_2/\gamma_6 F_2$ is freely generated by the basic commutators

$$[b, a, a, a, a]; \quad [b, a, b, b]; \quad [b, a, a, b]; \quad [b, a, a, b]; \quad [b, a, a, [b, a]]; \quad [b, a, b, [b, a]].$$

It is immediate that the first three commutators are trivial modulo $\delta_1$.

**Claim 3.10.** In any group, $[h, g, h, h]$ is conjugate to $[g, h, h, h]^{-1}$.

**Proof.** Indeed,

$$[h, g, h, h] = [[h, g]^{-1}, h, h] = [[h, g, h], h] = [[h, [g, h]], h] = ([h, g, h], h)^{[h, g]} = ([h, g, h], h)^{[h, g]} = ([g, h, h]^{-1}, h)^{[g, h, h]} = ([g, h, h]^{-1}, h)^{[g, h, h]} = [g, h, h]^{-1}. \quad \square$$

**Claim 3.11.** In any group, $[g, h, h, [h, g]]$ and $[h, g, h, [h, g]]$ lie in the normal closure of two commutators $[g, h, h, h]$ and $[g^{-1}, h, h, h]$.

**Proof.** From 3.10, $[[g^{\pm 1}, h]^{-1}, h, h]$ lies in the normal closure of $[[g^{\pm 1}, h], h, h]$. Now let us modify:

$$[[g^{-1}, h]^{-1}, h, h] = [[g, h]^{\mp 1}, h, h] = [[g, h]^{\mp 1}, h, g, h]^{g^{-1}} = [[g, h]^{\mp 1}, [h, g], h]^{g^{-1}} = ([g, h]^{\mp 1}, [h, g])^{g^{-1}} = ([g, h]^{\mp 1}, h)g^{-h}.$$
Consequently \([ [g, h]^{\pm 1}, h, [h, g] ] = [[g, h]^{\pm 1}, h, h]^{-1} [[g, h]^{\pm 1}, h, h]^{h, h, -1} g \) and the statement follows. \(\square\)

**Observation 3.12.** The statement of Claim 3.11 also applies to \([g, h, h, [g, h]]\) and \([h, g, h, [g, h]]\), since they are conjugate to \([g, h, h, [h, g]]^{-1}\) and \([h, g, h, [h, g]]^{-1}\) respectively.

From 3.11 and 3.12 we conclude that the basic commutators \([b, a, a, [b, a]]\) and \([b, a, b, [b, a]]\) also lie in \(\delta_1\). Therefore it remains to prove the following.

**Lemma 3.13.** \([b, a, a, b, b, b] \in (\gamma_6 F_2)(\delta_1 F_2)\).

**Proof.** Let us modify:

\[
[a, b, b, b] = [a, a, a, b, b] = [a, b, a, b, b] = [(a, b, a, b, b)] = [a, b, a, b, b]^{a, b, a, b, a} = [(a, b, a, b, b)]^{a, b, a, b, a} = [(a, b, a, b, b)]^{a, b, a, b, a} = [(a, b, a, b, b)]^{a, b, a, b, a}.
\]

Using Claim 3.11,

\[
[a, b, a, b] = [(a, b, a, b)] = [(a, b, a), b] = [(a, b, a), b]^{b, a} = [(a, b, a), b]^{b, a} = [(a, b, a), b]^{b, a}.
\]

Using Observation 3.12,

\[
[(a, b, a), b] = [(a, b, a), b]^{a, b, a, b, a} = [(a, b, a), b]^{a, b, a, b, a} = [(a, b, a), b]^{a, b, a, b, a} = [(a, b, a), b]^{a, b, a, b, a}.
\]

Finally, we see that

\[
[a, b, a, b, b] = [(a, b, a), a, b] = [(a, b, a), a, b]^{a, b, a, b, a} = [(a, b, a), a, b]^{a, b, a, b, a} = [(a, b, a), a, b]^{a, b, a, b, a}.
\]

**Remark.** Let us sketch an alternative proof of Theorem 3.9 avoiding lengthy calculations of Lemma 3.13. By Fitting’s Theorem, \(G/(\gamma_2 A)(\gamma_2 B)\) is nilpotent, where \(A\) is the normal closure of \(a\) and \(B\) is the normal closure of \(b\). We want to show that, modulo \(\gamma_6\), every element of \((\gamma_2 A)(\gamma_2 B)\) lies in \(\delta_1\). It is easy to see that \((\gamma_2 A)(\gamma_2 B)\) is normally generated by the commutators \([m, n^a, n^b] \in \{a, b\}\) and \(g, h \in G\). Now \([a, a^g, a^h] \) is conjugate to \([a, g, a, a^h] = [a, g, a, a^{a, h}][a, g, a, a, h]\), and by Claim 3.10 the first factor is in \(\delta_1\), while the second factor can be written, modulo \(\gamma_6\), as a power of \([a, b, a, [a, b]]\) which by Claim 3.11 is also in \(\delta_1\).

**Remark.** An earlier version of this paper contained the conjecture that the quotient \(F_2/\delta_1\) is not nilpotent. We were informed by A. Abdollahi that he recently disproved this conjecture. In more detail, he proved that \(F_2/\delta_1\) is metabelian and deduced that it is nilpotent of class 4. This implication also follows from the above computations and the fact that \(\gamma_5 F_2\) is the normal closure of basic commutators of weight 5 [JGS], or alternatively from Theorems 3.7 and 2.1.
Conjecture 3.14. a) The group $F_k/\delta_1$ is nilpotent of class $2k$; 
b) The group $F_2/\delta_2$ is not nilpotent.

The fact that $F_2/\delta_1$ is nilpotent implies that $G_1^+(L)$ is not a complete invariant of fine 1-quasi-isotopy. Indeed, $G_1^+$ was shown to have length 2 for the links from Fig. 1, whence it is has to be isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Moreover, the alternative proof of Theorem 3.9 above shows that $G_1(L) = G_1^+(L)$ for any 2-component link whose fundamental group is generated by two meridians.

In conclusion we state a problem, which is of some interest in light of the computations throughout the paper.

Problem 3.15. Let $\langle X \mid R \rangle$ be a finite presentation of a group $G$, where $R$ is some collection of basic commutators (perhaps of different weights) in some Hall basis. Is it true that $G$ is residually nilpotent?

The affirmative answer implies (by the collecting process [Hall; (11.1.4)]) that $\gamma_n F_k$ coincides with the normal closure of the set of basic commutators of weight $n$, which is unsolved for $n > 5$ and for $n = 4, k > 2$ [Ko], [JGS]. Note that Problem 3.15 is also related to some problems from [Co], in particular, whether the fundamental group of the Whitehead link $W_1$ is residually nilpotent. It is not hard to show that the group $G$ is residually nilpotent in the simplest case $\langle a, b \mid [b, a, a] = 1 \rangle$, but already for the series of cases $\langle a, b \mid [b, a, a, \ldots, a] = 1 \rangle$ the problem is unsolved.

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