A Continuum of Extinction Rates for the Fast Diffusion Equation

Marek Fila \(^*\)
Juan Luis Vázquez \(^\dagger\)
Michael Winkler \(^\ddagger\)

Abstract

We find a continuum of extinction rates for solutions \(u(y, \tau) \geq 0\) of the fast diffusion equation \(u_\tau = \Delta u^m\) in a subrange of exponents \(m \in (0, 1)\). The equation is posed in \(\mathbb{R}^n\) for times up to the extinction time \(T > 0\). The rates take the form \(\|u(\cdot, \tau)\|_\infty \sim (T - \tau)^\theta\) for a whole interval of \(\theta > 0\). These extinction rates depend explicitly on the spatial decay rates of initial data.

1 Introduction

We consider the Cauchy problem for the fast diffusion equation:

\[
\begin{align*}
    u_\tau &= \Delta(u^m/m), \\
    u(y, 0) &= u_0(y) \geq 0,
\end{align*}
\]

where \(m \in (0, 1)\) and \(T > 0\). The factor \(1/m\) is not essential; it is inserted into the equation for normalization so that it can also be written as \(u_\tau = \nabla \cdot (u^{m-1} \nabla u)\). In that way, it is readily seen that the diffusion coefficient \(c(u) = u^{m-1} \to \infty\) as \(u \to 0\) if \(m < 1\), hence the name Fast Diffusion Equation (but notice that \(c(u) \to 0\) as \(u \to \infty\)). Furthermore, it is known that for \(m\) below a critical exponent \(m_c = (n - 2)/n\) all solutions with initial data in some convenient space, like \(L^p(\mathbb{R}^n)\) with \(p = n(1 - m)/2\), extinguish in finite time. We will always work in this range, \(m < m_c\), and consider solutions which vanish in a finite time. The purpose of this paper is to study the rates of extinction of such solutions. Our main contribution is to provide a continuum of rates of extinction for fixed \(m\). Technical reasons imply that \(m\) must be in the range \(0 < m < m_* = (n - 4)/(n - 2), n \geq 5\), for the construction to work. This restriction may be essential.

\(^*\)fila@fmph.uniba.sk  
\(^\dagger\)juanluis.vazquez@uam.es  
\(^\ddagger\)michael.winkler@uni-due.de
Let us review the state of the question from a broader perspective. The description of the asymptotic behaviour of the global in time solutions of (1.1) as $\tau \to \infty$ for $m \geq m_c$ is a very active subject, and the study has been extended in recent times to the behaviour near extinction for $m < m_c$, both in bounded domains or in the whole space. In the former case, the rate of decay for bounded solutions is universal, of the form $\| u(\cdot, \tau) \|_\infty = O((T - \tau)^{1/(1 - m)})$ when $m > m_s = (n - 2)/(n + 2)$, cf. [1] [6], but the question is more complicated when $m \leq m_s$.

In the case of the whole space, which is the one of interest here, the book [10] contains a general description of the phenomenon of extinction, where it is explained that not only the occurrence of extinction depends on the size of the initial data, but also that different initial data may give rise to different extinction rates, even for the same extinction time; this may happen for all $0 < m < m_c$. It is also proved, cf. [10] and quoted references, that the size of the initial data at infinity (the tail of $u_0$) is very important in determining both the extinction time and the decay rates.

Special attention has been given recently to particular classes of data that produce definite estimates. This happens in the case of data with the maximal decay rate compatible with extinction in finite time, which is

$$u_0(y) \sim A|y|^{-\mu}, \quad \mu := 2/(1 - m)$$

as $|y| \to \infty$. Note that $\mu < n$ for $m < m_c$ so these data are not integrable. Thus, the papers [5] [2] [4] are concerned with the stabilization as $\tau \to T$ of general solutions towards some special self-similar solutions $U_{D,T}$ known as the generalized Barenblatt solutions, given by the formula

$$U_{D,T}(y, \tau) := \frac{1}{R(\tau)^n} \left( D + \frac{\beta(1 - m)}{2} \frac{|y|}{R(\tau)} \right)^{-\frac{1}{1 - m}}, \quad (1.3)$$

where for $m < m_c$ we put $R(\tau) := (T - \tau)^{-\beta}$, and

$$\beta := \frac{1}{n(1 - m) - 2} = \frac{1}{n(m_c - m)} = \frac{\mu}{2(n - \mu)}.$$ 

Here $T \geq 0$ (extinction time) and $D > 0$ are free parameters. Note that $R$ depends on $T$. It has been proved that the corresponding Barenblatt solutions with exponent $m > m_c$ play the role of the Gaussian solution of the linear diffusion equation in describing the asymptotic behaviour of a very wide class of nonnegative solutions, i.e., those with initial data in $L^1(\mathbb{R}^n)$, cf. [11]. To some extent, the solutions (1.3) play a similar role for $m < m_c$ but their basin of attraction may be much smaller. This is precisely described in [4], with results on the basin of attraction of the family of generalized Barenblatt solutions; it establishes the optimal rates of convergence of the solutions of (1.1) towards a unique attracting limit state in that family. All of these solutions will have a decay rate near extinction of the form $\| u(\cdot, \tau) \|_\infty = O((T - \tau)^{n\beta})$, and it is clear that $n\beta > 1/(1 - m)$.

A very interesting limit case occurs if we take $D = 0$ in formula (1.3), and we find the singular solution

$$U_{0,T}(y, \tau) := k_*(T - \tau)^{\mu/2}|y|^{-\mu}, \quad (1.4)$$
whose attracting properties have not been studied. Note the value $k_\ast = (2(n - \mu))^{\mu/2}$.

The question that we address here is the following: Can we obtain different decay rates near extinction for bounded data $u_0(y)$ that behave at infinity in first approximation like the singular solution, i.e., $u_0(y) \sim A|y|^{-\mu}$? We will show that the answer is yes, and actually we will obtain a whole continuum of rates.

**Theorem 1.1** Let $u \geq 0$ be a solution of Problem (1.1), assume that

$$n \geq 5 \quad \text{and} \quad m \in \left(0, \frac{n - 4}{n - 2}\right),$$

and let the initial function $u_0$ be continuous, bounded, and satisfy the conditions:

$$0 \leq u_0(y) \leq A|y|^{-\mu} \quad \text{for all } y \neq 0$$

and

$$A|y|^{-\mu} - c_1|y|^{-l} \leq u_0(y) \leq A|y|^{-\mu} - c_2|y|^{-l} \quad \text{for } |y| \geq 1$$

for some $A, c_1, c_2 > 0$, and

$$\mu + 2 < l \leq L = \mu + \sqrt{2(n - \mu)}.$$  

Then the solution has complete extinction precisely at the time $T = (A/k_\ast)^{1-m} > 0$, and there are positive constants $K_1, K_2$ such that for $0 < \tau < T$ we have

$$K_1(T - \tau)^\theta \leq \|u(\cdot, \tau)\|_{\infty} \leq K_2(T - \tau)^\theta,$$

where $\theta = \frac{n\mu - \gamma}{2(n - \mu)} > 0$, $\gamma = \frac{\mu(l - \mu - 2)(n - \ell)}{l - \mu}$.

It is easy to check that under the above assumptions $\theta$ covers an interval $[\theta_{\min}, \theta_{\max})$ with $0 < \theta_{\min} < \theta_{\max} = \mu n/2(n - \mu) = n\beta$. This is the precise range of extinction rates of these solutions, to be compared with the standard extinction rate $(T - \tau)^n\beta$ of the Barenblatt examples.

As a precedent to this result, the existence of different rates was established in Theorem 7.4 of [10] for all $m < m_c$ by means of the construction of self-similar solutions of the form $u(y, \tau) = (T - \tau)\alpha f(y(T - \tau)^\beta)$. In this way a whole interval $(\alpha, \infty)$ is covered, which extends the scope of our present theorem. However, $\alpha$ (the anomalous exponent) is not explicit, we obtain only one solution for each time-decay rate and the dependence of $\alpha$ on the spatial behavior of the data is not analyzed. Theorem 1.1 clarifies these aspects, explaining the delicate relationship between both limits, $|y| \to \infty$ for $u_0$ and $\tau \to T$ for $u(y, \tau)$.

The proof of the theorem needs techniques that are only natural after rescaling the problem. In fact, the rescaled problem allows us to formulate and prove a more precise result about the dependence of the rate on the tail of the data and the convergence of the spatial shapes. We devote the next section to the presentation of the rescaling.
transformation, the resulting rescaled equation and the asymptotic convergence plus grow-
up result in that context. Sections 3–5 will be concerned with proving the result for the
rescaled problem. The last section is devoted to comments and open problems.

Notations. Throughout the rest of the paper and unless mention to the contrary, we
keep the conditions \( n \geq 5 \) and \( m < m_* \). The exponent \( m_* \) also plays a big role in the
asymptotic results of \([2, 3, 4]\). We also keep the above symbols and variables. In particular,
\( \mu = 2/(1-m) \) so that \( m < m_c \) means \( \mu < n \) and \( m < m_* \) means \( \mu + 2 < n \).

2 The rescaled flow

As we have just said, it is very convenient to rescale the flow and rewrite (1.1) in self-similar
variables by introducing the time-dependent change of variables

\[
t := \frac{1-m}{2} \log \left( \frac{R(\tau)}{R(0)} \right) \quad \text{and} \quad x := \sqrt{\frac{\beta(1-m)}{2}} \frac{y}{R(\tau)},
\]

(2.1)

with \( R \) as above, and the rescaled function

\[
v(x, t) := R(\tau)^n u(y, \tau).
\]

(2.2)

In these new variables, the generalized Barenblatt functions \( U_{D,T}(y, \tau) \) are transformed
into generalized Barenblatt profiles \( V_D(x) \), which are stationary:

\[
V_D(x) := (D + |x|^2)^{-m_*}, \quad x \in \mathbb{R}^n.
\]

(2.3)

If \( u \) is a solution to (1.1), then \( v \) solves the rescaled fast diffusion equation

\[
v_t = \Delta(v^{m}/m) + \mu \nabla \cdot (x v), \quad t > 0, \quad x \in \mathbb{R}^n,
\]

(2.4)

which is a nonlinear Fokker-Planck equation (NLFP). We put as initial condition \( v_0(x) := R(0)^{-n} u_0(y) \), where \( x \) and \( y \) are related according to (2.1) with \( \tau = 0, x = cy \). Roughly
speaking, \( v_0 \) is a rescaling of \( u_0 \) depending only on \( T \). We have taken the precise form
of this transformation from [4]. Note also that the factors \( 1/m \) and \( \mu \) in equation (2.4)
can be eliminated by manipulating the change of variables, but then the expression of
the Barenblatt solutions would contain new constants. Thus, in our scaling the singular
solution becomes

\[
V_0(x) = |x|^{-\mu}, \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

(2.5)

2.1 Main result for the NLFP equation

In the following sections we consider the \( v \)-equation (2.4) with initial data given by a
bounded function \( 0 \leq v_0 \leq V_0 \), and such that the difference \( V_0 - v_0 \) has a tail controlled
by a power rate. This is our detailed result about asymptotic behaviour of the solution
whose initial data \( v_0(x) \) are perturbations of the steady state \( V_0(x) \).
Theorem 2.1 Assume that $n$ and $m$ are as in (1.5). Suppose that $v_0$ is continuous, bounded and nonnegative, and fulfils

$$|x|^{-\mu} - c_1 |x|^{-l} \leq v_0(x) \leq |x|^{-\mu} - c_2 |x|^{-l} \quad \text{for } |x| \geq 1,$$  \hspace{1cm} (2.6)

where $l$ is as in (1.6) and $c_1, c_2 > 0$. Assume also that $v_0(x) \leq |x|^{-\mu}$ for all $x \neq 0$. Let $v$ denote the solution of (2.4). Then:

(i) There exist $K_1, K_2 > 0$ such that for $t \geq 1$ we have

$$K_1 e^{\gamma t} \leq \|v(\cdot, t)\|_{\infty} \leq K_2 e^{\gamma t}, \quad \gamma = \gamma(l) = \frac{\mu(l - \mu - 2)(n - l)}{l - \mu}. \hspace{1cm} (2.7)$$

(ii) For each $r_0 > 0$ one can find $C_1, C_2 > 0$ such that for $t \geq 1$ and $|x| \geq r_0$ the following holds

$$C_1 e^{-\lambda t} \leq |x|^{-\mu} - v(x, t) \leq C_2 e^{-\lambda t}, \quad \lambda = \lambda(l) = (l - \mu - 2)(n - l). \hspace{1cm} (2.8)$$

Let us comment on the contents and scope of the result.

1. First of all, it states the two main aspects of the convergence of the solution $v(\cdot, t)$ towards the singular steady state $V_0$: (2.8) establishes the uniform convergence of $v(\cdot, t)$ towards $V_0$ in the complement of a ball centered at the origin, with a precise rate that depends explicitly on the tail decay exponent $l$. On the other hand, estimate (2.7) gives the exact rate of growth of the solutions as $t \to \infty$ to account for the approach to the singular value $V_0(0) = +\infty$.

2. An important feature of the result is the existence of a continuum of grow-up rates for $\|v(\cdot, t)\|_{\infty}$, and a corresponding continuum of stabilization rates of $v(\cdot, t)$ towards $V_0$ in the outer region. Note furthermore that as $l$ approaches the lower value $\mu + 2$, the rates go to zero. This limit case is on the other hand easier and does not produce any convergence, since we can consider the example of the generalized Barenblatt solutions $V_D$ given in (2.3). Indeed, they satisfy $0 < V_D < V_0$ and

$$V_0(x) - V_D(x) = C|x|^{-(\mu+2)} + o \left( |x|^{-(\mu+2)} \right) \quad \text{as } |x| \to \infty.$$ 

Since they are stationary, no convergence to $V_0$ holds in this case.

3. The conditions on $l$ imply that the perturbation $V_0 - v_0$ is never integrable, contrary to the usual assumptions made in variational methods. Let us now examine the maximal grow-up rate that we have achieved. Note first that $\gamma(\mu + 2) = \gamma(n) = 0$. The maximum of $\gamma$ in (2.7) is attained at $l = L$, and

$$\gamma(L) = \mu \left( n + 2 - \mu - 2\sqrt{2(n - \mu)} \right).$$

This is lower than the maximal growth rate of any bounded solution that is given by the growth of the spatially homogeneous solution $\tilde{v}(t) = ce^{\mu t}$. We conjecture that $\gamma(L)$ is the largest exponent that can be achieved by the solutions under the conditions of the
theorem, even if we allow \( l \) to be larger than \( L \). The bound from below follows immediately from the lower bound in \( (2.7) \), but to obtain the corresponding bound from above is still an open problem.

4. As \( m \to m_* \) we have \( \mu + 2 \to n \) and the interval \( (\mu + 2, L] \) shrinks to the empty set while the admissible values of the exponents \( \gamma \) and \( \lambda \) go to zero.

5. Results similar as in Theorem \( 2.1 \) were obtained for the standard Fujita equation

\[ u_t = \Delta u + u^p, \quad x \in \mathbb{R}^n, \quad n > 10, \quad p > \frac{(N - 2)^2 - 4N + 8\sqrt{N - 1}}{(N - 2)(N - 10)}, \]

in \([7, 8, 9]\).

6. Finally, we apply the results of Theorem \( 2.1 \) (i) to prove Theorem \( 1.1 \). Notice that under the assumptions of Theorem \( 1.1 \) if we take the prescribed value of \( T \) then \( v_0 \) satisfies the hypotheses of Theorem \( 2.1 \) so that the solution \( v \) is global in time and stabilizes to \( V_0 \); this means that the extinction time of \( u \) is precisely \( T \). The extinction rate of \( u \) is obtained by rewriting the bounds in \( (2.7) \). Recall that

\[ \|u(\cdot, \tau)\|_{\infty} = R(\tau)^{-\beta}\|v(\cdot, t)\|_{\infty} \sim (T - \tau)^{\beta} e^{\gamma t}, \]

and \( T - \tau = T e^{-2(n-\mu)t} \). The conclusion follows.

3 Auxiliary results for the rescaled problem

After the previous transformation, in the radially symmetric case we end up with the problem

\[
\begin{aligned}
\mathcal{P}v := v_t - \frac{1}{m} (v^m)_{rr} + \frac{n-1}{r} (v^m)_r - \mu rv_r - \mu nv &= 0, \quad r > 0, \quad t > 0, \\
v(r, 0) = v_0(r), \quad r \geq 0.
\end{aligned}
\]

(3.1)

An important role is played by the quadratic equation

\[ \alpha^2 - (n - 2 - \mu - \kappa)\alpha + 2\kappa = 0, \]

(3.2)

where

\[ \kappa = \frac{(l - \mu - 2)(n - l)}{l - \mu} \]

(3.3)

is positive if \( \mu + 2 < l < n \). The roots \( \alpha_- \) and \( \alpha_+ \) of \( (3.2) \) are given by

\[ \alpha_{\pm} = \frac{n - 2 - \mu - \kappa \pm \sqrt{(n - 2 - \mu - \kappa)^2 - 8\kappa}}{2}, \]

(3.4)

and the following way to rewrite \( \alpha_{\pm} \) indicates why the value \( l = \mu + \sqrt{2(n-\mu)} \) plays an important role in the sequel (cf. Section \( 5 \)).
Lemma 3.1 Assume (1.5), and that \( l \in (\mu + 2, n) \). Then the roots \( \alpha_\pm \) of (3.2) can be expressed as follows:

i) If \( \mu + 2 < l \leq \mu + \sqrt{2(n-\mu)} \) then
\[
\alpha_- = l - \mu - 2 \quad \text{and} \quad \alpha_+ = \frac{2(n-l)}{l-\mu}.
\]

ii) If \( \mu + \sqrt{2(n-\mu)} \leq l < n \) then
\[
\alpha_- = \frac{2(n-l)}{l-\mu} \quad \text{and} \quad \alpha_+ = l - \mu - 2.
\]

Proof. Since it can easily be checked that \( \alpha_1 := l - \mu - 2 < \frac{2(n-l)}{l-\mu} =: \alpha_2 \) if and only if \( l < \mu + \sqrt{2(n-\mu)} \), we only need to check that both \( \alpha_1 \) and \( \alpha_2 \) solve (3.2). As to \( \alpha_1 \), this follows from
\[
\alpha_1^2 - (n-2-\mu-\kappa) + 2\kappa = (l-\mu-2)^2 - (n-2-\mu-\kappa)(l-\mu-2) + 2\kappa
\]
\[
= (l-\mu-2)(l-\mu-2-(n-2-\mu-\kappa)) + 2\kappa
\]
\[
= (l-\mu-2)(l-n) + (l-\mu)\kappa
\]
and the fact that \( (l-\mu)\kappa = (l-\mu-2)(n-l) \). Using \( \alpha_2 + 2 = \frac{2(n-l)}{l-\mu} \), we moreover compute
\[
\alpha_2^2 - (n-2-\mu-\kappa)\alpha_2 + 2\kappa = \frac{2(n-l)}{l-\mu}^2 - (n-2-\mu)\frac{2(n-l)}{l-\mu} + (\alpha + 2)\kappa
\]
\[
= \frac{2(n-l)}{(l-\mu)^2}(2(n-l)-(n-2-\mu)(l-\mu)+(n-\mu)(l-\mu-2)),
\]
from which we immediately find that also \( \alpha_2 \) solves (3.2). \( \Box \)

The following two lemmata apply to parameters \( n, m \) and \( \kappa \) more general than required in (1.5) and (3.3).

Lemma 3.2 Let \( n \geq 1, m > 0, \kappa > 0 \) and \( \sigma_0 > 0 \), and set
\[
\sigma(t) := \sigma_0 e^{\mu t}, \quad t \geq 0,
\]
and
\[
\xi(r,t) := \sigma_\frac{1}{\sqrt{t}}(t)r, \quad r \geq 0, \quad t \geq 0.
\]
Suppose that \( \psi : [0, \infty) \to [0, \infty) \) is twice continuously differentiable in \( (\xi_0, \xi_1) \) with some \( \xi_0 \) and \( \xi_1 \) satisfying \( 0 \leq \xi_0 < \xi_1 \). Then for
\[
v(r,t) := \sigma(t)\left(\xi^2(r,t) + \psi(\xi(r,t))\right)^{-\frac{\kappa}{4}}, \quad r \geq 0, \quad t \geq 0,
\]
we have the identity
\[ \mathcal{P}v(r, t) = \frac{\mu}{2} \sigma(t) \left( \frac{\xi^2(r, t) + \psi(\xi(r, t))}{2} \right)^{-\frac{\mu}{2}} A\psi(\xi(t, r)) \] (3.5)
for all \((r, t) \in S := \{(\rho, \tau) \in (0, \infty)^2 \mid \xi(\rho, \tau) \in (\xi_0, \xi_1)\}\), where
\[ A\psi(\xi) := (\xi^2 + \psi) \left( \psi_{\xi\xi} + \frac{n-1}{\xi} \psi_{\xi} \right) + 2\kappa\psi - (\mu + \kappa)\psi_{\xi} - \frac{\mu}{2} \psi_{\xi}^2 \] (3.6)
for \(\xi \in (\xi_0, \xi_1)\).

**Proof.** Using \(\xi_t = \frac{1}{\mu} \sigma^{\frac{1}{2}} \xi_t \sigma = \frac{1}{\mu} \sigma^{\frac{1}{2}} \xi \) and \(\sigma_t = \mu \kappa \sigma\), we compute
\[
v_t = \sigma_t \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu}{2} - 1} \left( \frac{2}{\mu} \xi + \psi(\xi) \right) \xi_t
= \frac{\mu}{2} \sigma \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu}{2} - 1} \left\{ \frac{2}{\mu} \frac{\sigma_t}{\sigma} \left( \xi^2 + \psi(\xi) \right) - \frac{1}{\mu} \sigma_t \left( \xi^2 + \psi(\xi) \right) \right\}
= \frac{\mu}{2} \sigma \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu}{2} - 1} \left\{ 2\kappa \psi(\xi) - \kappa \xi \psi_{\xi}(\xi) \right\} \tag{3.7}
\]
whenever \((r, t) \in S\). Since \(\xi_r = \sigma^{\frac{1}{2}} \xi^2\), moreover have
\[
(v^m)_r = \left( \sigma^m \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu m}{2}} \right)_r = -\mu \frac{m}{\sigma} \sigma^{m+\frac{1}{2}} \sigma \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu m}{2} - 1} \left( \xi^2 + \psi(\xi) \right)
\]
as well as
\[
(v^m)_{rr} = -\mu \frac{m}{\sigma} \sigma^{m+\frac{1}{2}} \sigma \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu m}{2} - 1} \left( 2 + \psi_{\xi\xi}(\xi) \right)
+ \mu \frac{m}{2} \left( \frac{m}{2} + 1 \right) \sigma^{m+\frac{3}{2}} \sigma \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu m}{2} - 2} \left( \xi^2 + \psi(\xi) \right)^2
\]
at such points. Thus, in view of the identity \(m + \frac{2}{\mu} = 1\) and, equivalently, \(\frac{m}{2} = \frac{\mu}{2} - 1\), we find that
\[
\frac{1}{m} \left( (v^m)_{rr} + \frac{n-1}{r} (v^m)_r \right) = -\frac{\mu}{2} \sigma \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu}{2} - 1} \left( 2 + \psi_{\xi\xi}(\xi) \right)
+ \left( \frac{\mu}{2} \right)^2 \sigma \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu}{2} - 1} \left( 2 + \psi_{\xi\xi}(\xi) \right)^2
-\frac{\mu}{2} \frac{n}{r} \sigma^{1-\frac{1}{\mu}} \sigma \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu}{2} - 1} \left( 2 + \psi_{\xi\xi}(\xi) \right)
\]
\[
= \frac{\mu}{2} \sigma \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu}{2} - 1} \left\{ -\left( \xi^2 + \psi(\xi) \right) \left( \psi_{\xi\xi} + \frac{n-1}{\xi} \psi_{\xi}(\xi) \right)
+ \frac{\mu}{2} \left( 2 + \psi_{\xi\xi}(\xi) \right)^2 - \frac{n-1}{\xi} \left( \xi^2 + \psi(\xi) \right) \left( 2 + \psi_{\xi}(\xi) \right) \right\}
= \frac{\mu}{2} \sigma \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu}{2} - 1} \left\{ -\left( \xi^2 + \psi(\xi) \right) \left( \psi_{\xi\xi}(\xi) + \frac{n-1}{\xi} \psi_{\xi}(\xi) \right)
-2(n-\mu)\xi^2 - 2n\psi(\xi) + 2\mu\psi_{\xi}(\xi) + \frac{\mu}{2} \psi_{\xi}(\xi) \right\} \tag{3.8}
\]
if \((r,t) \in S\). As \(r \xi = \xi\), we finally have

\[
\mu rv_r = \mu \left\{ -\frac{\mu}{2} \sigma \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu}{2} - 1} \left( 2\xi + \psi(\xi) \right) \xi_r \right\} \\
= \frac{\mu}{2} \sigma \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu}{2} - 1} \left\{ -2\mu \xi^2 - \mu \psi(\xi) \right\}
\]

and therefore obtain from (3.7) and (3.8) that

\[
P_v = \frac{\mu}{2} \sigma \left( \xi^2 + \psi(\xi) \right)^{-\frac{\mu}{2} - 1} \left( 2\kappa \psi(\xi) - \frac{\mu}{2} \xi^2 \right) + 2(\xi^2 + \psi(\xi))(\psi(\xi) + n) - 2n(\xi^2 + \psi(\xi)) \}
\]

which after a straightforward rearrangement yields (3.5).

**Lemma 3.3** Let \(n \geq 1, m > 0\) and \(\varepsilon > 0\). Then

\[
\psi(\xi) := 1 - \varepsilon \xi^2, \quad \xi \geq 0,
\]

satisfies

\[
A \psi(\xi) = 2(\kappa - n \varepsilon) - 2(\kappa - \mu) \varepsilon (1 - \varepsilon) \xi^2 \quad \text{for all } \xi > 0, \quad (3.9)
\]

where \(A\) is defined by (3.6).

**Proof.** We directly compute

\[
A \psi(\xi) = (\xi^2 + 1 + \varepsilon \xi^2)(-2\varepsilon - (n - 1)2\varepsilon) + 2\kappa(1 - \varepsilon \xi^2) + (\mu + \kappa)2\varepsilon \xi^2 - \frac{\mu}{2}(-2\varepsilon \xi^2) \\
= (1 - \varepsilon)\xi^2 + 1)(-2n\varepsilon) + 2\kappa - 2\varepsilon \xi^2 + 2\kappa \xi^2 + 2\mu \varepsilon^2 \xi^2 - 2\mu \varepsilon^2 \xi^2 \\
= -2n\varepsilon(1 - \varepsilon)\xi^2 - 2\varepsilon + 2\kappa + 2\mu \varepsilon^2 - 2\mu \varepsilon^2 \xi^2 \quad \text{for } \xi > 0,
\]

and thereby immediately obtain (3.9).

**4 Lower bound**

Once we have the preparatory material, we proceed next to establish the lower bound for the solutions mentioned in Theorem 2.1. This is the content of Proposition 4.2. Section 5 will contain the proof of the corresponding upper bound, Proposition 5.3.

**Lemma 4.1** Suppose that condition (1.5) on \(n\) and \(m\) holds, and that \(l \in (\mu + 2, n)\) and \(\alpha \in [\alpha_-, \alpha_+]\) with \(\alpha_\pm\) given by (3.3) and \(\kappa\) as in (3.3). Then, there exist \(a > 0, \xi_0 > 0\) and a positive \(\psi \in C^0([0, \infty)) \cap C^2((0, \infty) \setminus \{\xi_0\})\) satisfying

\[
A \psi \leq 0 \quad \text{in } (0, \infty) \setminus \{\xi_0\} \quad (4.1)
\]
and
\[
\liminf_{\xi \nearrow \xi_0} \psi_\xi(\xi) > \limsup_{\xi \searrow \xi_0} \psi_\xi(\xi),
\]
and such that
\[
\xi_0^\alpha \psi(\xi) \to a \quad \text{as } \xi \to \infty.
\]

Here, \(A\) is the operator defined in (3.6).

**Proof.** We let \(\varepsilon := \frac{\kappa}{n}\) and fix \(a > 0\) small such that
\[
\left\{ \left( \frac{2\varepsilon}{\alpha} \right)^{\frac{\alpha}{\alpha+2}} + \varepsilon \left( \frac{\alpha}{2\varepsilon} \right)^{\frac{\alpha}{\alpha+2}} \right\} a^{\frac{2}{\alpha+2}} < 1.
\]
Then the function \(\varphi\) defined by
\[
\varphi(\xi) := a\xi^{-\alpha} - 1 + \varepsilon\xi^2, \quad \xi > 0,
\]
has a unique local minimum at the point at which \(\varphi_\xi(\xi) = -\alpha a\xi^{-\alpha-1} + 2\varepsilon\xi = 0\), that is, at the point
\[
\xi_{\text{min}} = \left( \frac{a\alpha}{2\varepsilon} \right)^{\frac{1}{\alpha+2}},
\]
with corresponding minimum value
\[
\varphi(\xi_{\text{min}}) = a \left( \frac{a\alpha}{2\varepsilon} \right)^{-\frac{\alpha}{\alpha+2}} - 1 + \varepsilon \left( \frac{a\alpha}{2\varepsilon} \right)^{\frac{2}{\alpha+2}} = \left\{ \left( \frac{2\varepsilon}{\alpha} \right)^{\frac{\alpha}{\alpha+2}} + \varepsilon \left( \frac{\alpha}{2\varepsilon} \right)^{\frac{\alpha}{\alpha+2}} \right\} a^{\frac{2}{\alpha+2}} - 1 < 0
\]
by (4.4). Therefore,
\[
\xi_0 := \inf\{ \xi > 0 \mid \varphi(\xi) \leq 0 \}
\]
lies in \((0, \xi_{\text{min}})\) and we have
\[
\varphi_\xi(\xi_0) < 0.
\]
Accordingly,
\[
\psi(\xi) := \begin{cases} 1 - \varepsilon\xi^2 & \text{if } \xi \in [0, \xi_0], \\ a\xi^{-\alpha} & \text{if } \xi > \xi_0, \end{cases}
\]
defines a positive continuous function on \([0, \infty)\) which, by (4.5), satisfies (4.2) and clearly also fulfils (4.3). Moreover, using \(l > \mu + 2\) and (3.3) we find
\[
(l - \mu)(n - \kappa) = (l - \mu)n - (l - \mu - 2)(n - 1) = l^2 - (\mu + 2)l + 2n
\]
\[
= \left( l - \frac{\mu + 2}{2} \right)^2 - \frac{(\mu + 2)^2}{4} + 2n > \left( \mu + 2 - \frac{\mu + 2}{2} \right)^2 - \frac{(\mu + 2)^2}{4} + 2n
\]
\[
= 2n > 0,
\]
hence $\varepsilon \equiv \frac{\kappa}{n} < 1$. Thus, recalling Lemma 3.3 and the fact that $\mu < n$, we obtain

$$A_{\psi}(\xi) = -2(n - \mu)\varepsilon(1 - \varepsilon)\xi^2 < 0 \quad \text{for all } \xi \in (0, \xi_0). \quad (4.6)$$

As to large $\xi$, we compute

$$A_{\psi}(\xi) = \left(\xi^2 + a\xi^{-\alpha}\right)\left(\alpha(\alpha + 1)a\xi^{-\alpha - 2} - (n - 1)\alpha a\xi^{-\alpha - 2}\right) + 2\kappa a\xi^{-\alpha} - (\mu + \kappa)\alpha a\xi^{-\alpha} - \frac{\mu}{2}\alpha^2 a^2 \xi^{-2\alpha - 2} \quad (4.11)$$

$$= \left\{\alpha(\alpha + 1) - (n - 1)\alpha + 2\kappa - (\mu + \kappa)\alpha\right\}a\xi^{-\alpha} + \left\{\alpha(\alpha + 1) - (n - 1)\alpha - \frac{\mu}{2}\alpha^2\right\}a^2 \xi^{-2\alpha - 2} - \left\{\frac{\mu}{2}\alpha^2 + (n - 2)\alpha\right\}a^2 \xi^{-2\alpha - 2} \quad \text{for } \xi > \xi_0.$$  

Since $\alpha^2 - (n - 2 - \mu - \kappa)\alpha + 2\kappa \leq 0$ due to our assumption $\alpha \in [\alpha_-, \alpha_+]$, and since $\mu = \frac{2}{1-m} > 2$ and $n > 2$, from this we immediately infer that

$$A_{\psi}(\xi) < 0 \quad \text{for all } \xi > \xi_0,$$

which combined with (4.6) proves (4.1). \(\square\)

**Proposition 4.2** Assume again condition (1.5). Suppose that $v_0 \in C^0([0, \infty))$ is positive and fulfills

$$v_0(r) \geq r^{-\mu} - c_0 r^{-l} \quad \text{for all } r \geq 1 \quad (4.7)$$

with some $c_0 > 0$ and $l$ as in (1.6). Let $v$ denote the solution of (3.1). Then:

(i) There exists $C_1 > 0$ such that

$$v(0, t) \geq C_1 e^{\frac{\mu(l - \mu - 2)(n - l)t}{1 - \mu}} \quad \text{for all } t \geq 0. \quad (4.8)$$

(ii) For each $r_0 > 1$ one can find $C_2 > 0$ fulfilling

$$v(r, t) \geq r^{-\mu} - C_2 e^{-(l - \mu - 2)(n - l)t} \quad \text{for all } r \geq r_0 \text{ and } t \geq 0. \quad (4.9)$$

**Proof.** (i) As $l$ is as in (1.6), we can apply Lemma 4.1 to $\alpha_- = l - \mu - 2$ and obtain $\xi_0 > 0$ and a function $\psi$ with the properties provided by that lemma. Since $v_0$ is positive, there exists $c_1 > 0$ such that

$$v_0(r) \geq c_1 \quad \text{for all } r \in [0, 1]. \quad (4.10)$$

Let us pick $c_2 > 0$ large such that with $c_0$ as in (4.7) we have

$$(1 + c_2)^{-\frac{n}{2}} \leq 1 - c_0, \quad (4.11)$$
and then choose $\hat{\xi} > 0$ small fulfilling
\[
\xi^{-2} \psi(\xi) \geq c_2 \quad \text{for all } \xi \in (0, \hat{\xi}), \tag{4.12}
\]
which is possible since $\psi$ is positive on $[0, \infty)$. We next define
\[
z_0 := \max_{\xi \in [\hat{\xi}, \infty)} \xi^{-2} \psi(\xi) \tag{4.13}
\]
and let $c_3 > 0, c_4 > 0$ and $c_5 > 0$ be small enough such that
\[
(1 + z)^{-\frac{\mu}{2}} \leq 1 - c_3 z \quad \text{for all } z \in [0, z_0] \tag{4.14}
\]
and
\[
\xi^2 + \psi(\xi) \geq c_4 \quad \text{for all } \xi \geq 0 \tag{4.15}
\]
as well as
\[
\psi(\xi) \geq c_5 \xi^{-\alpha} \quad \text{for all } \xi \geq \hat{\xi}, \tag{4.16}
\]
where we make use of (4.3) and, again, the positivity of $\psi$. Finally, we take a small number $\sigma_0 > 0$ satisfying
\[
\sigma_0 \leq c_1 c_4^\frac{\mu}{4} \tag{4.17}
\]
and define
\[
\sigma(t) := \sigma_0 e^{\kappa t}, \quad t \geq 0,
\]
with $\kappa = \frac{(l-\mu-2)(n-l)}{l-\mu}$ as in (3.3). Then
\[
\psi(r, t) := \sigma(t) \left( \xi^2(r, t) + \psi(\xi(r, t)) \right)^{-\frac{\mu}{2}}, \quad r \geq 0, t \geq 0,
\]
with $\xi(r, t) := \sigma^\frac{1}{\mu}(t)r$, is continuous in $[0, \infty)^2$ and smooth at each point $(r, t) \in [0, \infty)^2$ where $r \neq r_0(t) := \xi_0 \sigma^{-\frac{1}{\mu}}(t)$. An application of Lemma \ref{lemma:3.2} and Lemma \ref{lemma:4.1} shows that with $\mathcal{A}$ as in (3.6),
\[
\mathcal{P}_\nu = \frac{\mu}{2} \sigma(t) \left( \xi^2(r, t) + \psi(\xi(r, t)) \right)^{-\frac{\mu}{2} - 1} \mathcal{A}(\psi(\xi(r, t))) \leq 0 \quad \text{whenever } r \neq r_0(t),
\]
which implies that $\nu$ is a subsolution of (3.1) in the Nagumo sense, because from (4.12) we infer that
\[
\limsup_{r \nearrow r_0(t)} \nu_+(r, t) < \liminf_{r \searrow r_0(t)} \nu_-(r, t) \quad \text{for all } t \geq 0.
\]
Accordingly, if we can show that $\nu$ does not exceed $\nu$ initially then the comparison principle will tell us that
\[
\nu \geq \nu \quad \text{in } [0, \infty)^2 \tag{4.19}
\]
and, in particular,
\[ v(0, t) \geq v(0, t) = \sigma(t) = \sigma_0 e^{\mu t} \quad \text{for all } t \geq 0, \]

which will yield (4.8) in view of the definition of \( \kappa \). It thus remains to show that
\[ \underline{v}(r, 0) \leq v_0(r) \quad \text{for all } r \geq 0. \] (4.20)

To this end, we first consider the case \( r \leq 1 \), when (4.15), (4.17) and (4.10) imply
\[
\underline{v}(r, 0) = \sigma_0 \left( \xi^2(r, 0) + \psi(\xi(r, 0)) \right)^{-\frac{\mu}{2}} \\
\leq \sigma_0 c^{-\frac{\mu}{2}} \leq c_1 \leq v_0(r) \quad \text{for all } r \in [0, 1]. \] (4.21)

Next, if \( r \geq 1 \) is such that \( r \geq \hat{\xi} \sigma_0^{-\frac{1}{\mu}} \) then \( \xi(r, 0) = \sigma_1 r \geq \hat{\xi} \), and hence from (4.13), (4.14) and (4.16) we obtain
\[
\underline{v}(r, 0) = r^{-\mu} \left( 1 + \xi^{-2}(r, 0) \psi(\xi(r, 0)) \right)^{-\frac{\mu}{2}} \\
\leq r^{-\mu} \left( 1 - c_3 \xi^{-2}(r, 0) \psi(\xi(r, 0)) \right) = r^{-\mu} - c_3 c_5 \sigma_0^{-\frac{2+2}{\mu}} r^{-\mu-\alpha-2},
\]
which in view of our choice \( \alpha = l - \mu - 2 \), (4.18) and (4.7) gives
\[
\underline{v}(r, 0) \leq r^{-\mu} - c_3 c_5 \sigma_0^{-\frac{2+2}{\mu}} r^{-l} \leq r^{-\mu} - c_0 r^{-l} \\
\leq v_0(r) \quad \text{for all } r \geq \min \left\{ 1, \hat{\xi} \sigma_0^{-\frac{1}{\mu}} \right\}. \] (4.22)

Finally, if \( r \geq 1 \) is such that \( r \leq \hat{\xi} \sigma_0^{-\frac{1}{\mu}} \) then we use (4.12) and (4.11) to see that
\[
\underline{v}(r, 0) = r^{-\mu} \left( 1 + \xi^{-2}(r, 0) \psi(\xi(r, 0)) \right)^{-\frac{\mu}{2}} \\
\leq r^{-\mu} (1 + c_2)^{-\frac{\mu}{2}} \\
\leq r^{-\mu} (1 - c_0) \quad \text{for all } r \leq \hat{\xi} \sigma_0^{-\frac{1}{\mu}},
\]
whereas by (4.7),
\[ v_0(r) \geq r^{-\mu} \left( 1 - c_0 r^{-(l-\mu)} \right) \geq r^{-\mu} (1 - c_0) \quad \text{for all } r \geq 1,
\]
so that \( \underline{v}(r, 0) \leq v_0(r) \) also holds if \( 1 \leq r \leq \hat{\xi} \sigma_0^{-\frac{1}{\mu}} \). Together with (4.21) and (4.22) this establishes (4.20).

(ii) To see (4.9), we observe that in view of (4.3) there exists \( c_6 > 0 \) such that
\[
\psi(\xi) \leq c_6 \xi^{-\alpha} \quad \text{for all } \xi > 0.
\]
where still \( \alpha = \alpha_- = l - \mu - 2 \). Then by (4.19) and the convexity of \( z \mapsto (1 + z)^{\frac{\mu}{2}} \) for \( z \geq 0 \) we obtain

\[
\begin{align*}
v(r, t) & \geq v(r, t) = r^{-\mu} \left( 1 + \xi^{-2} \psi(\xi) \right)^{-\frac{\mu}{2}} \geq r^{-\mu} - \frac{\mu}{2} r^{-\mu} \xi^{-2} \psi(\xi) \\
& \geq r^{-\mu} - \frac{\mu}{2} c_0 r^{-\mu} \xi^{-\alpha - 2} = r^{-\mu} - \frac{\mu}{2} c_0 \sigma^{-\alpha - 2} \xi^{-\mu - \alpha - 2} \\
& = r^{-\mu} - \frac{\mu}{2} c_0 \sigma_0^{-\frac{l-\mu}{\mu}} r^{-\mu} e^{-(l-\mu-2)(n-l)t} \quad \text{for all } r > 0 \text{ and } t > 0.
\end{align*}
\]

Given \( r_0 > 0 \), this easily yields (4.9) upon an obvious choice of \( C_2 \). \( \square \)

5 Upper bound and proof of Theorem 2.1

Lemma 5.1 Assume (1.3), and let \( l \in (\mu + 2, n) \) and \( \alpha_- \) be as defined in (3.4) with \( \kappa \) given by (3.3). Then there exist \( \beta > \alpha_- \) and \( C_\beta > 0 \) with the following property: Suppose that \( A > 0 \) and \( B > 0 \) are such that

\[
B^{\alpha_- + 2} A^{\beta + 2} \geq C_\beta, 
\]

and let

\[
\xi_1 := \left( \frac{\beta B}{\alpha_- A} \right)^{\frac{1}{\beta - \alpha_-}}. 
\]

Then the function \( \psi_{\text{out}} \) defined by

\[
\psi_{\text{out}}(\xi) := A\xi^{-\alpha_-} - B\xi^{-\beta}, \quad \xi > 0,
\]

satisfies

\[
(\psi_{\text{out}})_\xi(\xi_1) = 0 \quad \text{and} \quad (\psi_{\text{out}})_\xi(\xi) < 0 \quad \text{for all } \xi > \xi_1,
\]

and moreover we have

\[
A\psi_{\text{out}}(\xi) \geq 0 \quad \text{for all } \xi > \xi_1
\]

with \( A \) given by (3.6).

Proof. Throughout the proof, let us abbreviate \( \alpha := \alpha_- \) for convenience. Then since \( l \in (\mu + 2, n) \), we have \( 0 < \alpha < \alpha_+ \), and

\[
p(\beta) := \beta^2 - (n - 2 - \mu) \beta + 2 \kappa, \quad \beta \in \mathbb{R},
\]

has the properties

\[
p(\alpha) = 0 \quad \text{and} \quad p(\beta) < 0 \quad \text{for all } \beta \in (\alpha, \alpha_+).
\]

As furthermore

\[
q(\beta) := \beta(\beta + 2 - n) - \alpha(\alpha + 2 - n) + \mu \alpha \beta, \quad \beta \in \mathbb{R},
\]

14
evidently satisfies \( q(\alpha) = \mu \alpha^2 > 0 \), by continuity we can choose some \( \beta > \alpha \) such that
\[
\beta \leq 2\alpha + 2, \quad p(\beta) < 0 \quad \text{and} \quad q(\beta) > 0.
\]
(5.7)

With this value of \( \beta \) fixed, we pick \( C_\beta > 0 \) large such that
\[
\frac{2c_2}{c_1} \left( \frac{\alpha}{\beta} \right)^{2+\beta-\alpha} \left( \frac{1}{C_\beta} \right)^{\frac{1}{\beta-\alpha}} \leq 1
\]
and
\[
\frac{2c_3}{c_1} \left( \frac{\alpha}{\beta} \right)^{3+\beta-\alpha} \left( \frac{1}{C_\beta} \right)^{\frac{1}{\beta-\alpha}} \leq 1,
\]
(5.8)
(5.9)

where
\[
c_1 := -p(\beta), \quad c_2 := \frac{\mu - 2}{2} \alpha^2 + (n - 2)\alpha \quad \text{and} \quad c_3 := \frac{\mu - 2}{2} \beta^2 + (n - 2)\beta
\]
(5.10)

are all positive according to (5.7) and the inequalities \( \mu = \frac{2}{1-m} > 2 \) and \( n > 2 \). Then, given \( A > 0 \) and \( B > 0 \) fulfilling (5.11), we let \( \bar{\psi}_{\text{out}} \) be defined by (5.3) and compute
\[
\begin{align*}
(\bar{\psi}_{\text{out}})_{\xi} (\xi) &= -\alpha A\xi^{-\alpha-1} + \beta B\xi^{-\beta-1} \quad \text{and} \\
(\bar{\psi}_{\text{out}})_{\xi} (\xi) &= \alpha (\alpha + 1) A\xi^{-\alpha-2} - \beta (\beta + 1) B\xi^{-\beta-2}
\end{align*}
\]
(5.11)

for \( \xi > 0 \). From this it can easily be deduced that in fact \( \bar{\psi}_{\text{out}} \) attains its maximum at \( \xi = \xi_1 \) and decreases on \((\xi_1, \infty)\), where \( \xi_1 \) is as in (5.2). Using (5.11) we furthermore obtain
\[
A\bar{\psi}_{\text{out}} (\xi) = (\xi^2 + A\xi^{-\alpha} - B\xi^{-\beta}) \left\{ (\alpha (\alpha + 1) A\xi^{-\alpha-2} - \beta (\beta + 1) B\xi^{-\beta-2}
\right.
\left. - (n - 1)\alpha A\xi^{-\alpha-2} + (n - 1)\beta B\xi^{-\beta-2} \right\}
\]
\[
+ 2\alpha A\xi^{-\alpha} - 2\alpha A\xi^{-\beta} - (\mu + \kappa)\alpha A\xi^{-\alpha} + (\mu + \kappa)\beta B\xi^{-\beta}
\]
\[
- \frac{\mu}{2} \left( \alpha^2 A^2\xi^{-\alpha-2} - 2\alpha\beta AB\xi^{-\alpha-2} + \beta^2 B^2\xi^{-\beta-2} \right)
\]
\[
= \left\{ \alpha (\alpha + 1) - (n - 1)\alpha + 2\kappa - (\mu + \kappa)\alpha \right\} A\xi^{-\alpha}
\]
\[
+ \left\{ - \beta (\beta + 1) + (n - 1)\beta - 2\kappa + (\mu + \kappa)\beta \right\} B\xi^{-\beta}
\]
\[
+ \left\{ \alpha (\alpha + 1) - (n - 1)\alpha - \frac{\mu}{2} \alpha^2 \right\} A^2\xi^{-2\alpha-2}
\]
\[
+ \left\{ \beta (\beta + 1) - (n - 1)\beta - \frac{\mu}{2} \beta^2 \right\} B^2\xi^{-2\beta-2}
\]
\[
+ \left\{ - \alpha (\alpha + 1) + (n - 1)\alpha + \beta (\beta + 1) - (n - 1)\beta + \mu \alpha \beta \right\} AB\xi^{-\alpha-\beta-2}
\]
\[
= p(\alpha) A\xi^{-\alpha} - p(\beta) B\xi^{-\beta} - \left\{ \frac{\mu - 2}{2} \alpha^2 + (n - 2)\alpha \right\} A^2\xi^{-2\alpha-2}
\]
\[
- \left\{ \frac{\mu - 2}{2} \beta^2 + (n - 2)\beta \right\} B^2\xi^{-2\beta-2} + q(\beta) AB\xi^{-\alpha-\beta-2}
\]
for all \( \xi > 0 \).
Now (5.6) and (5.7) imply that the first term on the right vanishes and that the last is nonnegative, because \( A \) and \( B \) are positive. Hence, recalling (5.10) we arrive at the inequality

\[
A_{\text{out}}(\xi) \geq c_1 B\xi^{-\beta} - c_2 A^2 \xi^{-2\alpha - 2} - c_3 B^2 \xi^{-2\beta - 2} \quad \text{for all } \xi > 0.
\]  (5.12)

Now if \( \xi \geq \xi_0 \) then (5.8) along with (5.1) and our restriction \( \beta \leq 2\alpha + 2 \) ensures that

\[
\frac{c_2 A^2 \xi^{-2\alpha - 2}}{c_1 B \xi^{-\beta}} = \frac{2c_2 A^2}{c_1 B} \xi^{\beta - 2\alpha - 2} \leq \frac{2c_2 A^2}{c_1 B} \left( \frac{\beta B}{\alpha A} \right)^{\beta - 2\alpha - 2} \frac{1}{\beta} \leq 1 \quad \text{for all } \xi \geq \xi_1.
\]  (5.13)

Moreover, for such \( \xi \) we find

\[
\frac{c_3 B^2 \xi^{-2\beta - 2}}{c_1 B \xi^{-\beta}} = \frac{2c_3 B^2}{c_1 B} \xi^{\beta - 2\beta - 2} \leq \frac{2c_3 B^2}{c_1 B} \left( \frac{\beta B}{\alpha A} \right)^{\beta - 2\beta - 2} \frac{1}{\beta} \leq 1 \quad \text{for all } \xi \geq \xi_1
\]

by (5.9). Together with (5.13) and (5.12), this shows that indeed \( A_{\text{out}} \geq 0 \) for all \( \xi \geq \xi_1 \), as claimed.

**Lemma 5.2** Suppose that (1.3) holds. Let \( l \in (\mu, n) \) and \( \alpha_- \) be as in (3.4) with \( \kappa \) given by (3.3). Then there exist \( A > 0 \), \( \xi_1 > 0 \) and a positive function \( \bar{\psi} \in C^0([0, \infty)) \cap C^2([0, \infty) \setminus \{\xi_1\}) \) such that with \( A \) as in (3.5),

\[
A \bar{\psi} \geq 0 \quad \text{in } (0, \infty) \setminus \{\xi_1\}
\]  (5.14)

and

\[
\limsup_{\xi \nearrow \xi_1} \psi_\xi(\xi) < \liminf_{\xi \searrow \xi_1} \psi_\xi(\xi)
\]  (5.15)

as well as

\[
\xi^{\alpha_-} \bar{\psi}(\xi) \to A \quad \text{as } \xi \to \infty.
\]  (5.16)

**Proof.** Again we write \( \alpha := \alpha_- \) for simplicity. Since \( \mu < n \), it is possible to fix \( c_1 \in (0, 1) \) such that

\[
c_1 \leq \sqrt{\frac{\kappa}{2(n - \mu)}},
\]  (5.17)

and since \( l \in (\mu + 2, n) \), there exist \( \beta > \alpha \) and \( C_\beta > 0 \) such that the conclusion of Lemma 5.1 holds. We now define

\[
c_2 := \left( \frac{\alpha}{\beta} \right)^{\alpha_-} - \left( \frac{\alpha}{\beta} \right)^{\beta_-}.
\]  (5.18)
which is positive because $0 < \alpha < \beta$, and

$$K := \left(\frac{1 - c_1^2}{c_2}\right)^{\beta - \alpha}. \tag{5.19}$$

Next, we pick $A > 0$ large fulfilling

$$A \geq \left\{ \frac{2n}{\kappa} c_1^2 \left(\frac{\alpha}{\beta}\right) \frac{2}{\beta - \alpha} K \frac{2}{\alpha(\beta - \alpha)} \right\}^{\frac{\alpha}{\beta - \alpha}} \tag{5.20}$$

and

$$A \geq \left( C \beta K^{\frac{\alpha + 2}{\alpha}} \right)^{\frac{1}{\beta - \alpha}} \tag{5.21}$$

and let

$$B := \left( \frac{A^\beta}{K} \right)^{\frac{1}{\alpha}} \tag{5.22}$$

and

$$\varepsilon := c_1^2 \left(\frac{\alpha A}{\beta B}\right)^{\frac{2}{\beta - \alpha}} \tag{5.23}$$

as well as

$$\xi_1 := \left(\frac{\beta B}{\alpha A}\right)^{\frac{1}{\beta - \alpha}}, \tag{5.24}$$

so that

$$\varepsilon \xi_1^2 = c_1^2 \left(\frac{\alpha A}{\beta B}\right)^{\frac{2}{\beta - \alpha}} \left(\frac{\beta B}{\alpha A}\right)^{\frac{2}{\beta - \alpha}} = c_1^2. \tag{5.25}$$

Then the function $\overline{\psi} : [0, \infty) \to \mathbb{R}$ given by

$$\overline{\psi}(\xi) := \left\{ \begin{array}{cl} \overline{\psi}_{in}(\xi) := 1 - \varepsilon \xi^2 & \text{if } \xi \in [0, \xi_1], \\
\overline{\psi}_{out}(\xi) = A \xi^{-\alpha} - B \xi^{-\beta} & \text{if } \xi \in (\xi_1, \infty), \end{array} \right.$$ 

is continuous on $[0, \infty)$, because (5.25) ensures that

$$\overline{\psi}_{in}(\xi_1) = 1 - c_1^2,$$

whereas invoking (5.18), (5.22) and (5.19) we find

$$\overline{\psi}_{out}(\xi_1) = A \left(\frac{\beta B}{\alpha A}\right)^{-\frac{\alpha}{\beta - \alpha}} - B \left(\frac{\beta B}{\alpha A}\right)^{-\frac{\beta}{\beta - \alpha}} = c_2 A^{\frac{\beta}{\beta - \alpha}} B^{-\frac{\alpha}{\beta - \alpha}} = c_2 K^{\frac{1}{\beta - \alpha}} = 1 - c_1^2.$$

Note that since $c_1 < 1$, this also implies that $\overline{\psi}$ is positive on $[0, \infty)$. Next, from Lemma 3.3 and (5.25) we obtain

$$\overline{A} \overline{\psi}_{in}(\xi) = 2(\kappa - n \varepsilon) - 2(n - \mu) \varepsilon(1 - \varepsilon) \xi^2 \geq 2(\kappa - n \varepsilon) - 2(n - \mu) \varepsilon \xi_1^2 = 2(\kappa - n \varepsilon) - 2(n - \mu) c_1^2 \quad \text{for all } \xi \in (0, \xi_1),$$
where by (5.23), (5.22) and (5.20),

\[
\varepsilon = c_1^2 \left( \frac{\alpha A}{\beta (4^\alpha R^\alpha)} \right)^{2/\beta} = c_1^2 \left( \frac{\alpha}{\beta} \right)^{2/\beta} K^{2(\beta-\alpha)} A^{-2/\alpha} \leq \frac{\kappa}{2n}.
\]
Hence,

\[
\mathcal{A} \psi_{in}(\xi) \geq 2 \left( \kappa - \frac{\kappa}{2} \right) - 2(n - \mu) c_1^2 \geq 0 \quad \text{for all } \xi \in (0, \xi_1)
\]
according to (5.17).

Now the requirement (5.21) along with (5.22) guarantees that

\[
\frac{B^{\alpha+2}}{A^{\beta+2}} = K^{\alpha+2} A^{2(\beta-\alpha)} \geq C_\beta,
\]
so that Lemma 5.1 becomes applicable to tell us that

\[
\mathcal{A} \psi_{out}(\xi) \geq 0 \quad \text{for all } \xi > \xi_1
\]
as well as

\[
(\psi_{out})_\xi(\xi_1) = 0.
\]
As a consequence of (5.26) and (5.27), we see that (5.14) holds, while (5.28) combined with the fact that

\[
(\psi_{in})_\xi(\xi_1) = -2\varepsilon \xi_1^2 < 0
\]
yields (5.15). The assertion (5.16) immediately results from the definition of \( \bar{\psi} \). \( \square \)

**Proposition 5.3** Suppose that (1.5) holds, and that \( v \) is the solution of (3.1), where the initial data \( v_0 \in C^0([0, \infty)) \) are nonnegative and such that there exist \( l \) as in (1.6) and \( c_1 > 0 \) fulfilling

\[
v_0(r) \leq r^{-\mu} - c_1 r^{-l} \quad \text{for all } r \geq 1,
\]
and which in addition satisfies

\[
v_0(r) < r^{-\mu} \quad \text{for all } r > 0.
\]

i) There exists \( C_1 > 0 \) such that

\[
v(r, t) \leq C_1 e^{\mu((l-\mu-2)(n-l)t)} \quad \text{for all } r \geq 0 \text{ and } t \geq 0.
\]

ii) For all \( r_0 > 0 \) there exists \( C_2 > 0 \) with the property

\[
v(r, t) \leq r^{-\mu} - C_2 e^{-(l-\mu-2)(n-l) t} \quad \text{for all } r \geq r_0 \text{ and } t \geq 0.
\]
Proof. i) Since \( l \) is as in (1.6), the number \( \alpha := \alpha_\gamma \) satisfies \( \alpha = l - \mu - 2 \) by Lemma 3.1. Hence, applying Lemma 5.2 we find \( \xi_1 > 0 \) and a positive function \( \psi \in C^0([0, \infty)) \cap C^2([0, \infty) \setminus \{\xi_1\}) \) with the properties (5.14) and (5.15) and such that
\[
\xi^{\alpha} \psi(\xi) \leq c_2 \quad \text{for all} \quad \xi \geq 0
\] (5.33)
with some \( c_2 > 0 \). Taking \( c_3 > 0 \) large such that
\[
v_0(r) \leq c_3 \quad \text{for all} \quad r \geq 0,
\] (5.34)
we can find \( r_0 > 0 \) small enough fulfilling
\[
r_0 \leq (2c_3)^{-\frac{1}{\mu}}
\] (5.35)
and then, by (5.29) and (5.30), fix \( c_4 \in (0, c_1] \) such that
\[
v_0(r) \leq r^{-\mu} - c_4 r^{-1} \quad \text{for all} \quad r \geq r_0.
\] (5.36)
We pick \( \hat{\xi} > 0 \) and \( c_5 > 0 \) sufficiently large satisfying
\[
\hat{\xi} \geq (mc_2)^\frac{1}{\alpha + 2}
\] (5.37)
and
\[
\xi^2 + \hat{\psi}(\xi) \leq c_5 \quad \text{for all} \quad \xi \in [0, \hat{\xi}]
\] (5.38)
and finally choose a large number \( \sigma_0 > 0 \) with
\[
\sigma_0 \geq c_3 c_5^\frac{\beta}{2}
\] (5.39)
and
\[
\sigma_0 \geq \left( \frac{mc_2}{2c_4} \right)^\frac{\mu}{1-\mu}.
\] (5.40)
We now define
\[
\sigma(t) := \sigma_0 e^{\mu ct}, \quad t \geq 0,
\]
and
\[
\varpi(r, t) := \sigma(t)\left(\xi^2(r, t) + \psi(\xi(r, t))\right)^{-\frac{\beta}{2}}, \quad r \geq 0, \quad t \geq 0,
\]
again with \( \xi(r, t) := \sigma(t)^\frac{1}{\mu} t \), and claim that
\[
\varpi(r, 0) \geq v_0(r) \quad \text{for all} \quad r \geq 0.
\] (5.41)
Indeed, if \( r \leq \hat{\xi} \sigma_0^{-\frac{1}{\mu}} \) then \( \xi(r, 0) \leq \hat{\xi} \) and hence (5.38), (5.39) and (5.34) imply that
\[
\varpi(r, 0) = \sigma_0\left(\xi^2(r, 0) + \psi(\xi(r, 0))\right)^{-\frac{\beta}{2}} \geq \sigma_0 c_5^{-\frac{\beta}{2}} \geq c_3 \geq v_0(r) \quad \text{for} \quad r \leq \hat{\xi} \sigma_0^{-\frac{1}{\mu}}.
\] (5.42)
Next, in the case when $\hat{\xi} \sigma^{-\frac{1}{\mu}} \leq r \leq r_0$ we have $\xi(r, 0) \geq \hat{\xi}$, so that using the convexity of $0 \leq z \mapsto (1 + z)^{-\frac{\mu}{2}}$ along with (5.33), (5.37), (5.35) and (5.34), we can estimate

$$v(r, 0) = r^{-\mu} \left(1 + \xi^{-2}(r, 0) \psi(\xi(r, 0))\right)^{-\frac{\mu}{2}} \geq r^{-\mu} \left(1 - \frac{\mu}{2} \xi^{-2}(r, 0) \psi(\xi(r, 0))\right)$$

$$\geq r^{-\mu} \left(1 - \frac{\mu}{2} c_2 \xi^{-\alpha-2}(r, 0)\right) \geq r^{-\mu} \left(1 - \frac{\mu}{2} c_2 \sigma_0^{-\alpha+\frac{2}{\mu}} r^{-1}\right)$$

$$\geq \frac{1}{2} r_0^{-\mu} \geq c_3 \geq v_0(r) \quad \text{if } \hat{\xi} \sigma^{-\frac{1}{\mu}} \leq r \leq r_0.$$  (5.43)

Finally, for $r \geq \hat{\xi} \sigma^{-\frac{1}{\mu}}$ fulfilling $r \geq r_0$, by the same convexity argument in conjunction with the fact that $\alpha = l - \mu - 2$, from (5.40) and (5.36) we have

$$v(r, 0) \geq r^{-\mu} \left(1 - \frac{\mu}{2} c_2 \xi^{-\alpha-2}(r, 0)\right) \geq \frac{1}{2} r_0^{-\mu} = \frac{1}{2} c_4 r^{-l} \quad \text{for } r \leq \max \{r_0, \hat{\xi} \sigma^{-\frac{1}{\mu}}\}.$$  (5.44)

Together with (5.42) and (5.43), this proves (5.41). Since by Lemma 3.2 and Lemma 3.3, recalling (3.6) we have

$$\frac{\partial}{\partial r} \psi(r, t) = \sigma(t) \left(\xi^2(r, t) + \psi(\xi(r, t))\right)^{-\frac{\mu}{2} - 1} \leq 0 \quad \text{whenever } r \neq r_1(t) := \xi_1 \sigma^{-\frac{1}{\mu}}(t)$$

and

$$\liminf_{r \to r_1(t)} \psi(r, t) > \limsup_{r \to r_1(t)} \psi(r, t) \quad \text{for all } t \geq 0$$

according to (5.15), the comparison principle applies to yield

$$\overline{\psi} \geq v \quad \text{for all } r \geq 0 \text{ and } t \geq 0.$$  (5.44)

This immediately leads to (5.31).

ii) To obtain (5.32), we fix $r_0 > 0$ and first pick $c_5 > 0$ small enough fulfilling

$$(1 + z)^{-\frac{\mu}{2}} \leq 1 - c_5 z \quad \text{for all } z \in [0, 1],$$  (5.45)

and then fix $t_0 > 0$ large such that

$$I := c_2 \sigma_0^{-\frac{\mu}{\mu}} r_0^{-l-\mu} e^{-(l-\mu-2)(n-l)t_0} \leq 1,$$  (5.46)

where $c_2$ and $\sigma_0$ are as determined by (5.33), (5.39) and (5.40). Then for all $r \geq r_0$ and $t \geq t_0$, still writing $\alpha = \alpha_\pm = l - \mu - 2$ we have

$$c_2 \xi^{-\alpha-2}(r, t) \leq c_2 \xi^{-\alpha-2}(r_0, t_0) = I \leq 1$$
and thus
\[(1 + c_2 \xi^{-\alpha-2})^{-\frac{\mu}{\xi}} \leq 1 - c_2 c_5 \xi^{-\alpha-2}\]
by (5.45). Therefore, (5.44) and (5.33) entail that for such \(r\) and \(t\) we have
\[
v(r, t) \leq \Psi(r, t) = r^{-\mu} \left(1 + \xi^{-2} \psi(\xi)\right)^{-\frac{\mu}{2}}
\leq r^{-\mu} (1 + c_2 \xi^{-\alpha-2})^{-\frac{\mu}{2}} \leq r^{-\mu} (1 - c_2 c_5 \xi^{-\alpha-2})
= r^{-\mu} - c_2 c_5 \sigma \frac{\alpha+2}{\mu} r^{-\mu-\alpha-2} = r^{-\mu} - c_2 c_5 \sigma_0 \frac{-1}{l-1} r^{-l} e^{-(l-\mu-2)(n-l)t}.
\]
This shows that (6.32) is valid for some sufficiently large \(C_2 > 0\).

As we have said, Propositions 4.2 and 5.3 together imply Theorem 2.1.

6 Comments and open problems

1. The construction of the new extinction rates for \(m \in [m_\ast, m_c)\) is open. The relevance of \(m_\ast\) in the asymptotic analysis of stability of the Barenblatt solutions has been documented in [2, 3, 4].

2. We have not performed the analysis of positive perturbations of the tail of the singular solution \(V_0\). Preliminary calculations show that we can have in that case global grow-up if the perturbation is large, i.e., if \(l - \mu > 0\) is small. The case \(l = \mu\) is explicit; indeed, it is easy to check that the solution with initial value \(v_0(x) = A |x|^{-\mu}\) is
\[
v(x, t) = (C e^{2(n-\mu)t} + 1)^{1/(1-m)} |x|^{-\mu}, \quad C = A^{1-m} - 1.
\]
For \(A > 1\) this solution blows up everywhere as \(t \to \infty\) with rate \(O(e^{2(n-\mu)t})\), while for \(A < 1\) it vanishes in finite time.

3. The analysis of perturbations of the Barenblatt profiles, \(V_D\) with \(D > 0\), with large tails of the form \(v_0(x) - V_D(x) = O(|x|^{-l})\), is an interesting related problem. The difference with the above analysis is that the \(v\)-profile is regular, so no grow-up is expected if \(l > \mu + 2\). Since the behaviour of \(V_D\) at infinity is similar to the singular one, \(V_0\), and \(V_D\) is still stationary, we also expect a continuum of convergence rates depending on \(l\) from a certain range. In this case we have to mention that for \(l > n\) there is a variational theory developed in the recent papers [2, 3, 4] that proves convergence with rate using the techniques of entropies, linearization and functional inequalities.

4. We could have used another of the possible scaling options, which is not adapted to the Barenblatt profiles but is still adapted to the singular solution. The simplest choice is
\[
w(y, s) = [(1 - m)(T - \tau)]^{-1/(1-m)} u(y, \tau), \quad s = (1 - m) \log[(T - \tau)/T] = (2/\beta) t,
\]
\[\tag{6.2} \]
which leads to the equation
\[
\frac{\partial w}{\partial s} = \Delta (w^m/m) + w.
\] (6.3)

Putting \(w^m = Z\) and \(p = 1/m\) we get a variation of the Fujita equation
\[
\frac{\partial Z^p}{\partial s} = a\Delta Z + bZ^p.
\] (6.4)

Studying this equation is equivalent to the study of the \(v\) equation. It is interesting to translate the results we have obtained and to compare with the standard Fujita equation \(u_t = \Delta u + u^p\).

5. Our methods are not variational and our solutions do not belong to the usual spaces of that theory, like spaces of finite relative energy or finite relative mass.

ACKNOWLEDGMENT. Part of the work was done during visits of M. F. and M. W. to the Univ. Autónoma de Madrid and M. F. to the Univ. Duisburg–Essen. M. F. was partially supported by the Slovak Research and Development Agency under the contract No. APVV-0414-07 and by VEGA Grant 1/0465/09. J. L. V. was partially supported by Spanish Project MTM2008-06326-C02.

References

[1] J. G. Berryman, C. J. Holland. Stability of the separable solution for fast diffusion. Arch. Rat. Mech. Anal. 74 (1980), 379–388.

[2] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, J. L. Vázquez. Asymptotics of the fast diffusion equation via entropy estimates, Arch. Rat. Mech. Anal. 191 (2009), 347–385.

[3] M. Bonforte, J. Dolbeault, G. Grillo, J. L. Vázquez. Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities, Preprint 2009, submitted.

[4] M. Bonforte, G. Grillo, J. L. Vázquez. Special fast diffusion with slow asymptotics. Entropy method and flow on a Riemannian manifold, Arch. Rat. Mech. Anal. (to appear), published online: 1 July 2009; arXiv 0805.4750v1 [math.AP].

[5] J. Denzler, R. J. McCann. Fast diffusion to self-Similarity: Complete spectrum, long-time asymptotics, and numerology, Arch. Rat. Mech. Anal. 175 (2005), 301–342.

[6] E. Feiresl, F. Simondon. Convergence for semilinear degenerate parabolic equations in several space dimensions, J. Dyn. Diff. Eq. 12 (2000), 647–673.

[7] M. Fila, J. King, M. Winkler and E. Yanagida. Optimal lower bound of the grow-up rate for a supercritical parabolic equation, J. Diff. Equations 228 (2006), 339–356.
[8] M. Fila and M. Winkler. *Rate of convergence to a singular steady state of a supercritical parabolic equation*, J. Evol. Equations 8 (2008), 673–692.

[9] M. Fila, M. Winkler and E. Yanagida. *Grow-up rate of solutions for a supercritical semilinear diffusion equation*, J. Diff. Equations 205 (2004), 365-389.

[10] J. L. Vázquez. “*Smoothing and Decay Estimates for Nonlinear Diffusion Equations*”, vol. 33 of Oxford Lecture Notes in Maths. and its Applications, Oxford University Press, 2006.

[11] J. L. Vázquez. “*The Porous Medium Equation. Mathematical Theory*”, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2007.

**Addresses**

Marek Fila  
Department of Applied Mathematics and Statistics, Comenius University,  
84248 Bratislava, Slovakia

Juan Luis Vázquez  
Departamento de Matemáticas, Universidad Autónoma de Madrid,  
28049 Madrid, Spain

and  

Michael Winkler  
Fakultät für Mathematik, Universität Duisburg-Essen,  
45117 Essen, Germany

**Key words:** Fast diffusion, extinction in finite time, extinction rate, nonlinear Fokker-Planck equation, grow-up.

**MSC 2000:** 35K65, 35B40