An analytic formula determining quantum pointer basis and its application to a flying bullet

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Abstract

We propose a general scheme determining the quantum pointer basis of a macroscopic object interacting with environment. Requiring the best robustness of the object state gives an analytic formula for what we call pointer Hamiltonian whose eigenbasis corresponds to the pointer basis. Our pointer basis does not rely on master equations, and so is easily calculable since it has a time-local analytic formula. We show that, as a convincing example, our pointer basis gives an satisfactory answer to a macroscopic flying body such as a bullet whose pointer states should have a well-defined momentum as well as a highly-localized position. We also apply our scheme to a spring-mass system (harmonic oscillator) interacting with air molecules, and obtain a satisfactory pointer basis. Our pointer basis guarantees energy conservation before and after measurement if the measured ensemble is decomposed by our pointer basis.

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I. INTRODUCTION

Quantum mechanics describes a state of a physical system as a vector in a Hilbert space. A vector can be represented in infinitely many ways because there are infinitely many bases that are equivalently legal in quantum mechanics. However, the classical world we experience in everyday life seems to prefer a certain basis. Classical objects are always observed as if they prefer a spatially localized state, they never appear as a superposition state of here and there [1]. Why classical objects seem to prefer a certain basis among abundance of quantum possibilities has been one of the critical issues in the so-called quantum measurement problem [2–5]. In this paper we use the term “pointer basis” [4] rather than preferred basis.

There have been several proposals to determine the pointer basis. In [6] the concept of the “rate of deseparation” which measures how fast a quantum system becomes entangled with environment was introduced, and they suggested that the pointer basis are the states generating minimum rate of the deseparation. Another proposal was the “predictability sieve” strategy [7, 8]. It requires the pointer basis to have the least entropy production rate during the coupling to the environment. Another one which is based on the concept of “quantum state diffusion” was proposed suggesting that the pointer basis is just the quantum state diffusion basis [9]. Despite these previous efforts, no unique method has so far been adopted to calculate the pointer basis in the general case. Furthermore, none of them has given a satisfactory solution to a very familiar situation such as a flying bullet in the air. The pointer states of a flying bullet should have a well-defined momentum as well as a sharply localized position. To our knowledge, however, no previous research gave a successful answer to this problem.

The purpose of this paper is to provide a general method to determine the pointer basis. While previous research tried to find states that are robust against entanglement (i.e., decoherence), we will try to find states that are robust against the change of the system state itself. We will show that this strategy naturally derives an analytic formula for a Hamiltonian, which we call pointer Hamiltonian, whose eigenbasis is just the pointer basis. Then we will challenge the flying bullet problem and show that our pointer basis gives a satisfactory answer. We will also see that the answer to the bullet problem is still valid for a harmonic oscillator. Finally, it will be shown that energy is conserved before and after measurement if the ensemble is decomposed with our pointer basis.
II. DERIVING POINTER BASIS

The concept “pointer basis” applies to macroscopic objects such as a measuring apparatus or a baseball when we want to treat them quantum mechanically. A microscopic system is well described by quantum mechanical unitary evolution until it meets a macroscopic object. This paper deals with a closed system which consists of a macroscopic object and a huge number of environmental particles interacting with the object. We treat the total system as a bipartite system separated by a macroscopic object and the environment. In this paper, we denote the macroscopic object by $A$ (from the word apparatus), and the environment by $E$. The general form of the total Hamiltonian for a bipartite system is written as

$$\hat{H} = \hat{H}_A + \hat{H}_{\text{int}} + \hat{H}_E.$$  (1)

An arbitrary interaction Hamiltonian $\hat{H}_{\text{int}}$ can be written in the form of a diagonal decomposition of object and environmental operators $\hat{A}_\alpha$ and $\hat{E}_\alpha$, respectively. In other words, $\hat{H}_{\text{int}}$ can be expressed as

$$\hat{H}_{\text{int}} = \sum_\alpha \hat{A}_\alpha \otimes \hat{E}_\alpha,$$  (2)

where $\hat{A}_\alpha$ and $\hat{E}_\alpha$ are unitary in general, not necessarily Hermitian (p. 78 of [1]).

We investigate the robustness of the object state during the evolution by the total Hamiltonian. We assume that the initial state of the total system is a product state and the environmental state is given as $|E_0\rangle$. We want to find an object state $|A_i\rangle$ so as that the state $|A_i\rangle |E_0\rangle$ exhibits robustness on $|A_i\rangle$ during the evolution, allowing the change of $|E_0\rangle$. A perfectly robust object state is desired to satisfy

$$|A_i\rangle |E_0\rangle \xrightarrow{t} |A_i\rangle |E_0^{(i)}(t)\rangle.$$  (3)

If the state $|A_i\rangle$ was a common eigenstate of $\hat{H}_A$ and $\hat{A}_\alpha$’s in Eq. (1) and (2), Eq. (3) could be fulfilled. However, as in the general case, if $\hat{H}_A$ and $\hat{A}_\alpha$’s were not commute, there hardly exists a common eigenstate of them. In most cases, there does not exist $|A_i\rangle$ satisfying Eq. (3). Hence, for general Hamiltonians of Eq. (1), we could at most demand best robustness instead of the perfect robustness.

We define the pointer basis based on the best robustness of the object state. By robustness, we do not mean the robustness against decoherence or entanglement, but we mean the robustness of the object state itself. We again assume an environmental state $|E_0\rangle$ is...
given. Then our task here is to find an object basis \{ |A_i⟩ \} such that the evolution of each |A_i⟩ |E_0⟩ exhibits the best robustness on |A_i⟩. When the initial state is |A_i⟩ |E_0⟩, transition into other states are governed by the matrix elements of the total Hamiltonian. Suppose the matrix elements are represented by the tensor-product basis made of the object basis \{ |A_i⟩ \} and the environmental basis \{ |E_m⟩ \} in which \( m = 0 \) corresponds to |E_0⟩. We want to prohibit the transition from |A_i⟩ |E_0⟩ to |A_j⟩ |E_0⟩. This prohibition can be accomplished if the matrix elements responsible for the transition are zero, i.e.,

\[ \langle E_0 | \langle A_j | \hat{H} | A_i⟩ |E_0⟩ = 0. \]  

(4)

Inserting Eq. (1) into Eq. (4), we obtain a more explicit formula

\[ \langle A_j \neq i | \hat{H} | A_i⟩ |E_0⟩ = 0. \]  

(5)

Noticing that \( \langle E_0 | \hat{H}_E | E_0⟩ \) is a number, Eq. (4) and (5) are satisfied if \{ |A_i⟩ \} is the eigenbasis of \( \hat{H}_P \equiv \hat{H}_A + \langle E_0 | \hat{H}_{\text{int}} | E_0⟩ \),

which we call pointer Hamiltonian. We define the pointer basis as the eigenbasis of \( \hat{H}_P \). If \( \hat{H}_{\text{int}} \) was expressed in the form of Eq. (2), \( \hat{H}_P \) would become

\[ \hat{H}_P \equiv \hat{H}_A + \sum_\alpha \langle \hat{E}_\alpha | \hat{A}_\alpha, \]  

(7)

where \( \langle \hat{E}_\alpha | \equiv \langle E_0 | \hat{E}_\alpha | E_0⟩ \). \( \hat{H}_P \) is Hermitian as far as the total Hamiltonian is Hermitian. Thus the pointer basis is guaranteed that it is an orthogonal complete set.

To see more rigorously that our pointer states are the most robust states that minimize the change of the object state, let us consider infinitesimal time evolution of the total system whose initial state is |A_0⟩ |E_0⟩ (we use \( \hbar = 1 \)).

\[ |\Psi(\Delta t)⟩ = e^{-i\hat{H} \Delta t} |A_0⟩ |E_0⟩ \]

\[ \simeq (1 - i \Delta t \hat{H}) |A_0⟩ |E_0⟩ \]

\[ = |A_0⟩ |E_0⟩ - i \Delta t \sum_j \sum_m \langle A_j | \langle E_m | \hat{H} | A_0⟩ |E_0⟩ \cdot |A_j⟩ |E_m⟩. \]

(8)

In order to quantify the robustness of the object state, we obtain the reduced density matrix of the object alone. To trace out the environment side, we rearrange Eq. (8) as the following
we use the notation $H_{j,0,0}^{j,m} \equiv \langle A_j | \langle E_m | \hat{H} | A_0 \rangle | \langle E_0 | for brevity).

$$|\Psi(\Delta t)\rangle = \left( |A_0\rangle - i\Delta t \sum_j H_{j,0,0}^{j,0} |A_j\rangle \right) |\mathcal{E}_0\rangle - i\Delta t \sum_{m \neq 0} \left( \sum_j H_{j,0,0}^{j,m} |A_j\rangle \right) |\mathcal{E}_m\rangle$$

$$= \left( e^{-i\Delta t H_{0,0}^{0,0}} |A_0\rangle - i\Delta t \sum_{j \neq 0} H_{j,0,0}^{j,0} |A_j\rangle \right) |\mathcal{E}_0\rangle - i\Delta t \sum_{m \neq 0} \left( \sum_j H_{j,0,0}^{j,m} |A_j\rangle \right) |\mathcal{E}_m\rangle \tag{9}$$

Then the reduced density matrix of the object at the time $\Delta t$ is obtained by

$$\rho_A(\Delta t) = |A_0\rangle \langle A_0| + \Delta t \left( i \sum_{j \neq 0} e^{-i\Delta t H_{0,0}^{0,0}} H_{j,0,0}^{j,0} |A_0\rangle \langle A_j| + h.c. \right)$$

$$+ \Delta t^2 \sum_m \left( \sum_j \sum_k H_{j,m}^{j,m} H_{k,0,0}^{k,m} |A_j\rangle \langle A_k| \right) \tag{10}$$

Since the initial object state was $\rho_A(0) = |A_0\rangle \langle A_0|$, the change of the object state will be minimized if the second term (first-order term of $\Delta t$) vanishes. This could be achieved if $|A_0\rangle$ was one of the pointer states satisfying Eq. (4) because in that case all $H_{j,0,0}^{j,0}$ of the second term of Eq. (10) would be zero. Then only the third term (second-order term of $\Delta t$) survives. This means that the change of the object state is minimized when $|A_0\rangle$ is a member of the pointer basis $\{|A_i\rangle\}$.

The robustness of the pointer states can be viewed in another way using the “trace distance” which is a widely used distance measure between two quantum states. The trace distance between $\rho_A(0)$ and $\rho_A(\Delta t)$ is defined by [10]

$$D(\rho_A(0), \rho_A(\Delta t)) \equiv \frac{1}{2} \text{Tr} \left| \rho_A(0) - \rho_A(\Delta t) \right|. \tag{11}$$

If $|A_0\rangle$ was a pointer state, only the third term of Eq. (10) would involve in the trace distance as the following.

$$D(\rho_A(0), \rho_A(\Delta t)) = \frac{\Delta t^2}{2} \text{Tr} \left| \sum_m \left( \sum_j \sum_k H_{j,m}^{j,m} H_{k,0,0}^{k,m} |A_j\rangle \langle A_k| \right) \right| = \kappa \Delta t^2, \tag{12}$$

where $\kappa$ is a finite number. Now as a measure of the robustness of the object state, we define “trace distance rate”, $\dot{D}_A$,

$$\dot{D}_A \equiv \lim_{\Delta t \to 0} \frac{D(\rho_A(0), \rho_A(\Delta t))}{\Delta t}. \tag{13}$$

$\dot{D}_A$ should be zero when $|A_0\rangle$ is one of the pointer states giving Eq. (12). If $|A_0\rangle$ was not a pointer state, $\dot{D}_A$ would have a finite value because the second term of Eq. (10) (first-order
term of $\Delta t$) would survive. Thus the pointer states are the unique states that make $\dot{D}_A$ zero. The robustness of the object state is quantified by $\dot{D}_A$ where the best robustness corresponds to zero $\dot{D}_A$.

III. APPLYING TO SIMPLE EXAMPLES

From now on, we apply our pointer basis to several examples. First of all, let us consider the simplest case where no self Hamiltonian exists and the interaction Hamiltonian has a single product form, i.e.,

$$\hat{H} = \hat{A} \otimes \hat{\mathcal{E}}.$$  \hfill (14)

In this case, the pointer Hamiltonian of Eq. (6) becomes

$$\hat{H}_P = \langle \mathcal{E}_0 | \hat{\mathcal{E}} | \mathcal{E}_0 \rangle \hat{\mathcal{A}}.$$  \hfill (15)

Since $\langle \mathcal{E}_0 | \hat{\mathcal{E}} | \mathcal{E}_0 \rangle$ is a number, pointer basis is just the eigenbasis of $\hat{\mathcal{A}}$ regardless of environmental state $|\mathcal{E}_0\rangle$. In this simple example, our result is the same as what the previous methods have given 1. If the object state was an eigenstate of $\hat{\mathcal{A}}$, the state should be perfectly robust against both decoherence and the change of the object state itself.

Now we consider a slightly complicated example where self Hamiltonian is present. Suppose the total Hamiltonian is given by

$$\hat{H} = \hat{H}_A + \hat{A} \otimes \hat{\mathcal{E}},$$  \hfill (16)

where interaction Hamiltonian has still a single product form. Environmental self Hamiltonian has nothing to do with pointer basis in our method as shown in Eq. (6). Even this slight complication makes the previous methods suffer from determining pointer basis unless $\hat{H}_A$ and $\hat{A}$ were commute. In contrast, our pointer basis is easily determined because the eigenbasis of $\hat{H}_P = \hat{H}_A + \langle \hat{\mathcal{E}} \rangle \hat{\mathcal{A}}$ is just the pointer basis. Although the environment consists of a huge number of particles, we only need the expectation value $\langle \hat{\mathcal{E}} \rangle$ of the environmental state $|\mathcal{E}_0\rangle$. To be more specific, let us consider a spin-$1/2$ system interacting with many environmental spins simultaneously, where the total Hamiltonian is given by

$$\hat{H} = \alpha \hat{\sigma}_x^A + \hat{\sigma}_z^A \otimes \sum_{k=1}^{N} g_k \hat{\sigma}_z^{\mathcal{E}_k}.$$  \hfill (17)
FIG. 1. Bloch sphere representation of the pointer basis of a spin-1/2 system interacting with environment. The total Hamiltonian primarily narrows down the possible pointer bases onto the thick curve, and the environmental state finally pinpoints the pointer basis as two dots on the curve.

In this case, the pointer Hamiltonian becomes

\[ \hat{H}_P = \alpha \hat{\sigma}_x^A + \langle \hat{\mathcal{E}} \rangle \hat{\sigma}_z^A, \]  

where \( \langle \hat{\mathcal{E}} \rangle = \langle \mathcal{E}_0 | \sum_{k=1}^{N} g_k \hat{\sigma}_z^{E_k} | \mathcal{E}_0 \rangle \). Since \( \hat{H}_P \) is of a two-dimensional Hilbert space, it is very easy to obtain the pointer basis as far as \( \langle \hat{\mathcal{E}} \rangle \) can be calculated.

In our method, the role of the total Hamiltonian and that of the environmental state turn out to be separated in determining pointer basis. The total Hamiltonian primarily narrows down the possible pointer bases from the abundance of quantum possibilities. In the example of Eq. (17), the total Hamiltonian restricts the form of the pointer Hamiltonian to the linear combination of \( \hat{\sigma}_x^A \) and \( \hat{\sigma}_z^A \). This means that, in the Bloch sphere representation, possible pointer bases are restricted to a curve as shown in Fig. 1 (thick red curve). Then the environmental state \( |\mathcal{E}_0\rangle \) gives the value \( \langle \hat{\mathcal{E}} \rangle \) fixing the pointer Hamiltonian and finally pinpoints the pointer basis as two diametric points on the curve (blue dots in Fig. 1). This separated role between the total Hamiltonian and the environmental state is a generic feature of our method.
FIG. 2. Interaction potential function $V$ between the object (bullet) and a single environmental particle (air molecule). $x_A$ and $x_i$ are the coordinates of the center-of-mass of the object and the $i_{th}$ particle, respectively. The size of the object is assumed to be $2a$.

IV. DERIVING THE POINTER BASIS OF A FLYING BULLET

Now we apply our pointer basis to a realistic situation of a flying bullet. Although we stressed that our pointer basis is easier to calculate than the previous ones, the easiness alone cannot justify that our one is the actual pointer basis in the real world. Therefore, as a convincing example, we will show that our method derives the pointer basis of a flying bullet agreeing with the everyday observation.

For simplicity, we establish a model about the bullet, environment, and the interaction between them. We assume that the bullet has a cubic shape of a size $2a$ and a mass $m$, and we consider only one-dimensional center-of-mass degree $x$. We further assume that environmental particles are only air molecules, and the bullet interacts with each of them individually (i.e., two-body interaction). The interaction potential between the bullet and a molecule should depends only on the distance between them. Let us denote the position operator of the center-of-mass of the bullet by $\hat{x}_A$ and that of the $i_{th}$ molecule $\hat{x}_i$. Then the individual interaction Hamiltonian with the $i_{th}$ molecule should be given as a potential function of the distance operator $\hat{d}^{(i)} \equiv (\hat{x}_A - \hat{x}_i)$. Let us denote the potential function by $V(\hat{d}^{(i)})$. We assume $V$ has a shape (in the position-basis representation) as shown in Fig. 2. All air molecules are supposed to interact with the bullet in the same manner.

One more thing we should note is that the distance between two objects depends on the relative velocity of them due to the relativistic length-contraction effect. Although we deal with nonrelativistic quantum mechanics and a slowly flying bullet, we will see that the relativistic effect should be taken into account in determining pointer basis because the
negligible effect accumulates and becomes considerable due to the huge number of environmental particles. If we denote the velocity operator of the bullet by $\hat{v}_A$ and that of the $i_{th}$ molecule $\hat{v}_i$, then the relativistic distance operator $\hat{d}_\text{rel}^{(i)}$ of the two objects should be given by

$$\hat{d}_\text{rel}^{(i)} = \sqrt{1 - \frac{(\hat{v}_A - \hat{v}_i)^2}{c^2}} (\hat{x}_A - \hat{x}_i). \quad (19)$$

In this consideration, the argument of the potential function $V$ (see Fig. 2) is replaced by $\hat{d}_\text{rel}^{(i)}$ from $\hat{d}^{(i)} \equiv (\hat{x}_A - \hat{x}_i)$.

Now we can make a mathematical model for the total Hamiltonian. We do not concern about the self Hamiltonian of the environment because it has nothing to do with the pointer basis (see Eq. (6)). Since the self Hamiltonian of the bullet is obviously $(1/2m) \hat{p}_A^2$, the total Hamiltonian is written by

$$\hat{H} = \frac{1}{2m} \hat{p}_A^2 + \sum_i V \left( \hat{d}_\text{rel}^{(i)} \right). \quad (20)$$

In order to obtain the pointer Hamiltonian $\hat{H}_P$, we need the environmental state $|\mathcal{E}_0\rangle$. Let us assume that each air molecule has its own state $|\epsilon_i\rangle$ and is in a product state, i.e.,

$$|\mathcal{E}_0\rangle = |\epsilon_1\rangle |\epsilon_2\rangle \cdots |\epsilon_N\rangle. \quad (21)$$

Now we consider the fact that the presence of the bullet constrains the possible environmental states. For example, the air molecules cannot reside in the region occupied by the bullet. Suppose the center-of-mass of the bullet was at $x_0$. Then it would be reasonable to assume that the number density of the air molecules $n(x)$ is like Fig. 3. We further consider that the ambient air molecules can be in a bulk average motion at a velocity $v_0$ although individual molecules have random velocities. This bulk motion can arise by possibly two reasons. The first one is obviously winds. The second one is the velocity of the bullet itself. Since the bullet is surrounded by air molecules, the ambient molecules cannot help moving at the nearly same velocity of the bullet. In summary, we specify the environmental state as a product state where air molecules are absent in the region occupied by the object and have an average velocity $v_0$.

We are at the position to obtain the explicit expression of the pointer Hamiltonian $\hat{H}_P$. 9
FIG. 3. One-dimensional number density distribution of air molecules surrounding the object (bullet). Air molecules are absent in the region occupied by the object whose center-of-mass locates at \( x_0 \).

Equation (6), (20) and (21) lead to the following pointer Hamiltonian.

\[
\hat{H}_P = \frac{1}{2m} \hat{p}_A^2 + \langle \mathcal{E}_0 \rangle \sum_i V \left( \hat{d}_{rel}^{(i)} \right) |\mathcal{E}_0\rangle \\
= \frac{1}{2m} \hat{p}_A^2 + \sum_i \langle \epsilon_i | V \left( \sqrt{1 - \frac{(\hat{v}_A - \hat{v}_i)^2}{c^2}} (\hat{x}_A - \hat{x}_i) \right) |\epsilon_i\rangle.
\]

(22)

Calculating the second term is the crucial part of obtaining the pointer Hamiltonian. Since the second term is the interaction part of the pointer Hamiltonian, let us denote it by \( \hat{H}_P^{(I)} \). \( |\epsilon_i\rangle \) applies to \( \hat{v}_i \) and \( \hat{x}_i \), but we only concern about the averaged result after summing over all the individual results of \( |\epsilon_i\rangle \)'s. Putting the velocity effect on hold for a while, let us first investigate the averaged result of the term \( (\hat{x}_A - \hat{x}_i) \) only. Figure 4(a) illustrates this. In the position-basis representation, if \( x_A \) deviates from \( x_0 \), the air molecules in the shaded region of Fig. 4(a) come into the high potential energy region of \( \eta \). The number of the air molecules in the shaded region is \( n |x_A - x_0| \). Hence the averaged result of the term \( (\hat{x}_A - \hat{x}_i) \) contributing to \( \hat{H}_P^{(I)} \) is

\[
\eta n |\hat{x}_A - x_0|.
\]

(23)

If \( |x_A - x_0| \geq 2a \), \( \hat{H}_P^{(I)} \) saturates to the value \( 2\eta na \). Now we investigate the velocity effect. Even when \( x_A = x_0 \), some air molecules can come into the high potential energy region due to the relativistic length contraction as shown in Fig. 4(b). Here we assume that all air molecules have the same velocity \( v_0 \). This assumption is safe for further calculation because it underestimates the contribution of the velocity effect, since variations in \( v_i \) would not cancel out their contributions due to \( \hat{v}_i^2 \) term. As Fig. 4(b) describes, the length of the shaded region is \( 2a \left( 1 - \sqrt{1 - (v_A - v_0)^2/c^2} \right) \). We further note that the number density of air molecules increases due to the length-contraction effect, so that \( n \) is replaced by...
FIG. 4. Averaged effect of the second term of Eq. (22) which is the interaction part of the pointer Hamiltonian. (a) Position effect. If the object (bullet) coordinate $x_A$ was shifted from $x_0$, air molecules in the shaded region come into the high potential energy region. (b) Velocity effect. $v_A$ and $v_0$ are the velocity of the object and the average velocity of the ambient air, respectively. If the relative velocity $(v_A - v_0)$ deviates from zero, the object feels air molecules closer due to the relativistic length-contraction effect. Hence air molecules in the shaded region come into the high potential energy region. Also the number density of air molecules $n$ increases due to the length-contraction effect.

\[ n/\sqrt{1 - (v_A - v_0)^2/c^2}. \]

Hence the averaged result of the term $(\hat{v}_A - \hat{v}_i)$ contributing to $\hat{H}_P^{(I)}$ is

\[ \frac{1 - \sqrt{1 - (\hat{v}_A - \hat{v}_i)^2/c^2}}{2\eta n a} \frac{1 - \sqrt{1 - (v_A - v_0)^2/c^2}}{\sqrt{1 - (\hat{v}_A - v_0)^2/c^2}}. \] (24)

The second term of Eq. (22), i.e., $\hat{H}_P^{(I)}$, after summing over, should be a function of $(\hat{v}_A - v_0)$ and $(\hat{x}_A - x_0)$. When $(\hat{v}_A - v_0) = 0$, $\hat{H}_P^{(I)}$ becomes Eq. (23). Similarly when $(\hat{x}_A - x_0) = 0$, $\hat{H}_P^{(I)}$ becomes Eq. (24). Moreover, $\hat{H}_P^{(I)}$ goes to minimum when both $(\hat{v}_A - v_0) = 0$ and
\[ (\hat{x}_{A} - x_0) = 0. \] Hence, near the minimum, \( \hat{H}_P^{(I)} \) can be expressed as
\[
\hat{H}_P^{(I)} = \hat{H}_P^{(I)} ((\hat{v}_A - v_0), (\hat{x}_{A} - x_0)) \\
\approx \hat{H}_P^{(I)} ((\hat{v}_A - v_0) = 0, (\hat{x}_{A} - x_0)) + \hat{H}_P^{(I)} ((\hat{v}_A - v_0), (\hat{x}_{A} - x_0) = 0) \\
= \eta m |\hat{x}_{A} - x_0| + \frac{2 \eta a}{\sqrt{1 - (\hat{v}_A - v_0)^2/c^2}}. \tag{25}
\]

After doing Taylor series expansion of the velocity term up to the first non-zero order, we finally get the explicit formula of the pointer Hamiltonian,
\[
\hat{H}_P = \frac{m}{2} \hat{v}_A^2 + \frac{\eta ma}{c^2} (\hat{v}_A - v_0)^2 + \eta m |\hat{x}_{A} - x_0|, \tag{26}
\]
where the self Hamiltonian is expressed by \( \hat{v}_A \) instead of \( \hat{p}_A \) using the relation \( \hat{p}_A = m\hat{v}_A \).

We will show that the first term of Eq. (26) is so small compared to the second term that the first term can be neglected. Let us compare the coefficients of the first term and the second term. While \( \eta \) and \( c \) have nothing to do with \( m, n \) and \( a \) can relate to \( m \). If the density of the bullet is \( \rho \), then \( a = (1/2)(m/\rho)^{1/3} \). Next, since \( n \) is the one-dimensional number density of air molecules, \( n \) is proportional to the cross section of the bullet \( 4a^2 \). If the volumetric number density of air molecules was \( n_0 \), then \( n = n_0(4a^2) = n_0(m/\rho)^{2/3} \). As a consequence, the coefficient of the second term turns out to be proportional to \( m \) such that
\[
\frac{\eta a}{c^2} = \frac{\eta n_0}{c^2 \rho} \frac{m}{2}. \tag{27}
\]

We estimate the dimensionless quantity \( \eta n_0/c^2 \rho \) by putting some realistic values to each symbol. \( \eta \) is the potential barrier in Fig. 2, and we put \( \eta = 1 \) J. For \( n_0 \), we put the number density of ideal gases at 0°C, 1 atm, i.e., \( n_0 = 2.7 \times 10^{25} \) m\(^{-3} \). For \( \rho \), we use the density of steel \( \rho = 7.8 \times 10^3 \) kg/m\(^3 \). \( c \) is the speed of light. Then the dimensionless quantity becomes \( \eta n_0/c^2 \rho = 3.9 \times 10^4 \) which means the coefficient of the second term is much larger than that of the first term. Hence we can safely ignore the first term of Eq. (26).

Now let us obtain the pointer basis, i.e., the eigenbasis of the approximated pointer Hamiltonian,
\[
\hat{H}_P = \frac{\eta ma}{c^2} (\hat{v}_A - v_0)^2 + \eta n |\hat{x}_{A} - x_0|. \tag{28}
\]

Since \( \hat{v}_A = \hat{p}_A/m \), Eq. (28) is rewritten in the position-basis representation,
\[
\hat{H}_P = \frac{\eta m a h^2}{m^2 c^2} \left[ (\frac{-i}{\hbar} \frac{\partial}{\partial x} - \frac{p_0}{\hbar})^2 + b|x - x_0| \right], \tag{29}
\]
where
\[ b \equiv \frac{m^2 c^2}{a h^2}, \]  
and \( p_0 = \frac{v_0}{m} \). Above, we dropped the suffix \( A \) for brevity. Equation (29) is the well known “vee potential” Hamiltonian and has analytic eigenfunctions [11]. Its ground state eigenfunction is
\[ \psi_0(x) = e^{i\frac{p_0}{\hbar} x} \text{Ai} \left[ 2^{1/3} b^{-2/3} \left( b |x - x_0| - 0.808614 \cdot b^{2/3} \right) \right], \]  
where \( \text{Ai} \) is the Airy function. This is a localized wave packet at \( x_0 \) having the momentum \( p_0 \) (i.e., the velocity \( v_0 \)). The size of the wave packet is in an order of \( b^{-1/3} \). If the bullet is made of steel and weighs \( m = 10 \) g, then \( a = 5.4 \) mm. In that case, the size of the wave packet is in an order of \( 2 \times 10^{-28} \) m. Consequently, we have succeeded to show that our pointer basis naturally gives a convincing pointer basis of a flying bullet which agrees with everyday experience.

The above example can also be applied to a harmonic oscillator surrounded by air molecules. The only difference between the flying bullet and the harmonic oscillator is that there is an additional quadratic potential in the self Hamiltonian part. Hence the pointer Hamiltonian of Eq. (26) would be modified to
\[ \hat{H}_P = \frac{m}{2} \dot{v}_A^2 + \frac{\eta m}{c^2} (\dot{v}_A - v_0)^2 + \frac{1}{2} k \dot{x}_A^2 + \eta m |\dot{x}_A - x_0| \cdot \]  
Just as the first term was neglected because of its smallness compared to the second term, the third term can also be neglected owing to the last term. Let us compare the last two terms in a realistic situation. We consider a spring-mass system whose natural frequency is 10 Hz. If mass was 1 kg, then \( k = 4 \times 10^3 \) N/m. However, \( \eta m = 7 \times 10^{22} \) J/m provided that the mass was made of steel. Therefore, the quadratic potential can safely be neglected in any practical situation, and the pointer Hamiltonian becomes nearly the same as the bullet case. Through this consideration, we conclude that the self Hamiltonian part does almost nothing to do in determining pointer basis when the system interacts with a huge number of environmental particles. In this respect, our method is different from the previous research where they argued that the pointer states of a harmonic oscillator is a coherent state [6, 7, 9] which is defined by the self Hamiltonian.

In order to avoid misconceptions that might arise, it would be worth noting two aspects of our scheme. First, in Fig. 4, the penetration of air molecules into the bullet is fictitious.
The penetration is not of an actual interaction, but of a mere calculating process according to Eq. (6). Second, our calculation about the flying bullet is not of a circular argument even though we assumed that the bullet initially locates at $x_0$ with a velocity $v_0$. The assumption would not be necessary if we were given an environmental state. In a realistic situation, the most plausible environmental state is such that air molecules are empty in the region that the bullet occupies and have some average velocity due to either winds or the motion of the bullet itself. Thus, what we assumed is the environmental state rather than the bullet state. The essential message of the bullet case is that, with a plausible environmental state, our pointer basis naturally arises as highly localized position states having also a well-defined momentum.

V. IMPLICATIONS FOR THE QUANTUM MEASUREMENT PROBLEM

The critical issues in the quantum measurement problem are two-fold: how pointer basis is determined, and how and when a single outcome appears. This paper touches the basis problem, and we will address the single outcome problem in another paper elsewhere soon. In this paper, we would like to further mention a rather specific but still important issue in the measurement problem, i.e., energy conservation in an ensemble decomposed with pointer states. It is well known that an arbitrarily decomposed ensemble does not satisfy energy conservation [12]. Also in the objective collapse model [13], energy conservation has been a critical issue.

We will show that the energy expectation value of the total system before the measurement would be conserved after the measurement if the measured ensemble was decomposed with our pointer basis. Suppose, before the measurement, the total system was given an arbitrary product state $|A_0\rangle |\mathcal{E}_0\rangle$. Then the energy expectation value of the total system before the measurement should be $\langle \mathcal{E}_0 | \langle A_0 | \hat{H} | A_0 \rangle | \mathcal{E}_0 \rangle$. Now let us decompose the total state with our pointer basis $\{|A_i\rangle\}$ such that $|A_0\rangle |\mathcal{E}_0\rangle = \sum_i c_i |A_i\rangle |\mathcal{E}_0\rangle$. Keeping in mind
that our pointer basis fulfills Eq. (4), we can show the following

$$
\langle E_0 | \langle A_0 | \hat{H} | A_0 \rangle | E_0 \rangle = \sum_i \sum_j c_i c_j^* \langle E_0 | \langle A_j | \hat{H} | A_i \rangle | E_0 \rangle = \sum_i |c_i|^2 \langle E_0 | \langle A_i | \hat{H} | A_i \rangle | E_0 \rangle.
$$

(33)

The last line is exactly the energy expectation value of the classical ensemble after the measurement. Thus our pointer basis is a special basis that guarantees energy conservation in ensemble level.

VI. CONCLUSIONS

In conclusion, we proposed a general scheme determining pointer basis based on the robustness of the object state. As a measure of the robustness, we adopted trace distance rate, and we required that the pointer basis should exhibit zero trace distance rate. The requirement naturally gave the concept of pointer Hamiltonian whose eigenbasis corresponds to the pointer basis. After applying our pointer basis to the simple models, we showed that our pointer basis gives a satisfactory answer to the problem of not only a flying bullet but also a harmonic oscillator.

It would be worth summarizing the conceptual differences between previous research and our one. First, while previous approaches tried to find robust states against entanglement, our pointer basis was derived to minimize the change of the object state itself allowing entanglement. Second, while predictability sieve strategy [8], which is one of the previous methods, has a time-integrated expression (p. 35 of [14]), our method has a time-local expression that determines the pointer basis at an instant. Third, in order to deal with large environmental degrees, previous research relied on master equations [7, 9] which are generally hard to solve, while our method provides a directly calculable analytic formula. Fourth, our method reveals the separated roles between the total Hamiltonian and the environmental state. The total Hamiltonian firstly narrows down the possible pointer bases, and the environmental state secondly pinpoints a certain pointer basis. Lastly, our pointer basis guarantees energy conservation in an ensemble level while most previous ones did not consider it.
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