Implications of a frame dependent gravitational effective action for perturbations on the Robertson-Walker Metric

Stephen L. Adler

Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA.

In earlier work we showed that a frame dependent effective action motivated by the postulates of three-space general coordinate invariance and Weyl scaling invariance exactly mimics a cosmological constant in Robertson-Walker (RW) spacetimes. Here we study the implications of this effective action for small fluctuations around a spatially flat RW background geometry. The equations for the conserving extension of the modified stress-energy tensor can be integrated in closed form, and involve only the metric perturbation $h_{00}$. Hence the equations for tensor and vector perturbations are unmodified, but there are Hubble scale additions to the scalar perturbation equations, which nonetheless admit no propagating wave solutions. Consequently, there are no modifications to standard gravitational wave propagation theory, but there may be observable implications for cosmology. We give a self-contained discussion, including an analysis of the restricted class of gauge transformations that act when a frame dependent effective action is present.

I. INTRODUCTION AND SUMMARY

The experimental observation of an accelerated expansion of the universe has been interpreted as evidence for a cosmological term in the gravitational action of the usual form

$$S_{\text{cosm}} = -\frac{\Lambda}{8\pi G} \int d^4x (g)^{1/2} ,$$

with $\Lambda = 3H_0^2\Omega_\Lambda$ in terms of the Hubble constant $H_0$ and the cosmological fraction $\Omega_\Lambda \approx 0.72$. This functional form incorporates the usual assumption that gravitational physics is four-space general coordinate invariant, with no frame dependence in the fundamental action.

In a series of papers [1]-[3], motivated by the frame dependence of the cosmological background radiation, we have studied the implications of the assumption that there is an induced gravitation effective action that is three-space general coordinate and Weyl scaling invariant, but is not four-space general coordinate invariant. For the special class of diagonal metrics for which $g_{0i} = 0$, these assumptions imply that the term in the induced effective action with no metric derivatives

*Electronic address: adler@ias.edu
has the form

\[ S_{\text{eff}} = A_0 \int d^4 x \left( \frac{g}{g_{00}} \right)^{1/2} (g_{00})^{-2} \]  

(2)

with \( A_0 \) a constant. When \( A_0 \) is given the value

\[ A_0 = -\frac{\Lambda}{8\pi G} \]  

(3)

this effective action becomes

\[ S_{\text{eff}} = -\frac{\Lambda}{8\pi G} \int d^4 x \left( \frac{g}{g_{00}} \right)^{1/2} (g_{00})^{-2} \]  

(4)

and in Robertson-Walker (RW) spacetimes where \( g_{00} = 1 \) exactly mimics the cosmological constant effective action of Eq. (1).

The paper [2] studied the implications of the effective action of Eq. (4) for spherically symmetric solutions of the Einstein equations, and showed that in a static, spherically symmetric Schwarzschild-like geometry it modifies the black hole horizon structure within microscopic distances of the nominal horizon, in such a way that \( g_{00} \) never vanishes. This could have important implications, yet to be investigated, for the black hole “information paradox”. In the present paper we turn to studying the implications of the effective action of Eq. (4) for the equations governing small perturbations around a spatially flat RW geometry. We find that the equations for tensor perturbations governing gravitational waves are unchanged, as are the equations for vector perturbations. However, the equations governing scalar perturbations receive Hubble scale corrections, which could have implications, again yet to be investigated, for structure formation in the early universe.

To set up a phenomenology for testing for the difference between the cosmological actions of Eq. (1) and Eq. (4), we make the Ansatz that the observed cosmological constant arises from a linear combination of these two actions of the form

\[ S_\Lambda = (1 - f)S_{\text{cosm}} + fS_{\text{eff}} = -\frac{\Lambda}{8\pi G} \int d^4 x \left( \frac{g}{g_{00}} \right)^{1/2} [1 - f + f(g_{00})^{-2}] \]  

(5)

so that \( f = 0 \) corresponds to only a standard cosmological constant, and \( f = 1 \) corresponds to only an apparent cosmological constant arising from a frame dependent effective action. Our results for the modifications to the equations governing scalar perturbations will thus contain the parameter \( f \).

As noted in [1] and again in [3], while the Einstein-Hilbert action and the particulate matter action are four-space general coordinate invariant, the frame-dependent effective action \( S_{\text{eff}} \) is
invariant only under the subset of general coordinate transformations that act on the spatial coordinates \( \vec{x} \), while leaving the time coordinate \( t \) invariant. Consequently, the stress-energy tensor obtained by varying \( S_\Lambda \) with respect to the full metric \( g_{\mu\nu} \) will not satisfy the covariant conservation condition, and thus cannot be used as a source for the full spacetime Einstein equations. However, it is consistent to include \( S_\Lambda \) in the source for the spatial components of the Einstein equations in the preferred rest frame of the action of Eq. (4), which we identify as the rest frame of the cosmological background radiation, giving the following rules:

1. The spatial components of the Einstein equations are obtained by varying the full action with respect to \( g_{ij} \), giving

\[
G^{ij} + 8\pi G(T^{ij}_\Lambda + T^{ij}_{\text{pm}}) = 0 ,
\]

with \( T^{ij}_{\text{pm}} \) the spatial components of the usual particulate matter stress-energy tensor, and with \( T^{ij}_\Lambda \) given by

\[
\delta S_\Lambda = -\frac{1}{2} \int d^4x (4) g^{1/2} T^{ij}_\Lambda \delta g_{ij} .
\]

2. The components of the Einstein tensor \( G^{0i} = G^{i0} \) and \( G^{00} \) are obtained from the Bianchi identities with \( G^{ij} \) as input, and from them we can infer the conserving extensions \( T^{i0}_\Lambda \) and \( T^{00}_\Lambda \) of the spatial stress-energy tensor components \( T^{ij}_\Lambda \).

Equivalently, we can infer these conserving extensions by imposing the covariant conservation condition on the tensor \( T^{\mu\nu}_\Lambda \), with \( T^{ij}_\Lambda \) as input, and this is how we proceed in Section 2. In Sec. 3 we analyze the residual gauge invariance when there is a frame dependent effective action. In Sec. 4 we give the modified equations for the scalar perturbations, and in Sec. 5 prove that they do not lead to scalar wave propagation. Sec. 6 gives a brief conclusion. In Appendix A we summarize our notational conventions, and give their relation to those of the text of Weinberg [4]. In Appendix B we give the formulas for the zeroth and first order inverse metric, affine connections, and Ricci tensor components. In Appendix C we show the equivalence between two different forms of the gauge invariance constraint on the matter perturbations, and derive formulas used in setting up the scalar perturbation equations. In Appendix D we show that our dark energy model cannot be tested using the effective field theory framework of Gubitosi et al. [5]. In Appendix E, we show that using the more general effective action that applies when the metric is not diagonal (as is the case for RW perturbations) does not alter the first order perturbation equations.
II. CONSERVING EXTENSION $T^{\mu \nu}_\Lambda$ OF $T^{ij}_\Lambda$

Adding a small perturbation $h_{\mu \nu}$ to the zeroth order spatially flat RW metric, we have for the total metric $g_{\mu \nu}$

\[
\begin{align*}
g_{00} &= 1 + h_{00} \\
g_{i0} &= g_{0i} = h_{i0} \\
g_{ij} &= -a^2(t)\delta_{ij} + h_{ij}.
\end{align*}
\]

(8)

The inverse metric, affine connection, and Ricci tensor corresponding to Eq. (8) are given through first order terms in $h_{\mu \nu}$ in Appendix B.

Varying the spatial metric components $g_{ij}$, we find from Eqs. (7) and (5) that

\[
T^{ij}_\Lambda = \frac{\Lambda}{8\pi G} \left[ (1 - f)g^{ij} + f \frac{g^{ij}}{g_{00}} \right] = \frac{\Lambda}{8\pi G} \left[ (1 - f)g^{ij} + f \frac{g^{ij}}{(1 + h_{00})^2} \right].
\]

(9)

Expanding through first order in $h_{00}$ and using $g^{ij} = -\delta_{ij}/a^2(t)$ in the term proportional to $h_{00}$, this becomes

\[
T^{ij}_\Lambda = \frac{\Lambda}{8\pi G} \left[ g^{ij} + ft^{ij} \right],
\]

\[
t^{ij} = \frac{2\delta_{ij}h_{00}}{a^2(t)}.
\]

(10)

Since the metric is covariantly conserved, $D_\nu g^{\mu \nu} = 0$, with both $D_\mu$ and $g^{\mu \nu}$ accurate to first order in the perturbation $h_{\mu \nu}$, the conserving extension of $g^{ij}$ is $g^{\mu \nu}$, including both zeroth and first order terms. Thus our task is to find a conserving extension $t^{\mu \nu}$ of $t^{ij}$ that obeys the covariant conservation condition $D_\nu t^{\mu \nu} = 0$. Since $t^{ij}$ is already first order in the perturbation $h_{00}$, it suffices to solve this equation using the zeroth order covariant derivative constructed by using the affine connections of Eq. (B2) with the first order perturbation terms omitted.

Expanding $D_\nu t^{\mu \nu} = \partial_\nu t^{\mu \nu} + \Gamma^{\mu \nu}_\alpha t^{\alpha \nu} + \Gamma^{\nu \mu}_\alpha t^{\alpha \nu}$ in terms of temporal and spatial index values, we get

\[
0 = D_\nu t^{\ell \nu} = \partial_\nu t^{\ell 0} + \partial_\ell t^{\nu 0} + \Gamma^{\ell \nu}_0 t^{00} + 2\Gamma^{\ell j}_0 T^{0j} + \Gamma^{\ell mn} t^{mn} + \Gamma^{\ell 0\alpha} t^{\alpha 0} + \Gamma^{\nu \ell j} t^{ij},
\]

\[
0 = D_\nu t^{0 \nu} = \partial_\nu t^{0 0} + \partial_0 t^{\nu 0} + \Gamma^{0 \nu}_0 t^{00} + 2\Gamma^{0 j}_0 T^{0j} + \Gamma^{0 mn} t^{mn} + \Gamma^{0 \nu 0} t^{\nu 0} + \Gamma^{\nu \nu j} t^{ij},
\]

(11)
which on substituting the zeroth order affine connections from Eq. (B2) becomes

\[ 0 = D_\nu t^{\nu} = \partial_0 t^{0_0} + \partial_j t^{j_\ell} + \frac{\dot{a}}{a} t^{0_0} + \frac{\dot{\dot{a}}}{a^2} \]

\[ = \partial_0 t^{0_0} + \partial_j t^{j_\ell} + 5 \frac{\dot{a}}{a} t^{0_0} , \]

\[ 0 = D_\nu t^{0_\nu} = \partial_0 t^{0_0} + \partial_j t^{j_0} + a \dot{a} t^{m_\ell m} + \frac{\dot{\dot{a}}}{a} t^{0_0} . \]  

(12)

These differential equations are readily integrated, to give

\[ t^{ij}(\vec{x}, t) = 2 \delta_{ij} a^{-2}(t) h_{00}(\vec{x}, t) , \]

\[ t^{0_0}(\vec{x}, t) = -2 a^{-5}(t) \int_{L_1}^t du a^3(u) \partial_\ell h_{00}(\vec{x}, u) , \]

\[ t^{0_0}(\vec{x}, t) = -a^{-3}(t) \int_{L_2}^t du a^3(u) \left[ \partial_\ell t^{0_\ell}(\vec{x}, u) + 6 \frac{\dot{a}(u)}{a(u)} h_{00}(\vec{x}, u) \right] . \]  

(13)

The lower limits \( L_1 \) and \( L_2 \) are arbitrary constants of integration; if we add an initial condition that all perturbations should be bounded at the initial time \( t_{\text{init}} \) where \( a(t_{\text{init}}) = 0 \), then we should take \( L_1 = L_2 = t_{\text{init}} \).

To write the Einstein equations in terms of the Ricci tensor, as in Eq. (A5), we need \( t_{ij} \), \( t_{0_0} \), \( t_{0_0} \), and the trace \( t^{0_\alpha} \). These can be written as (taking the integration limits now as \( t_{\text{init}} \))

\[ t_{ij}(\vec{x}, t) = 2 \delta_{ij} a^{-2}(t) h_{00}(\vec{x}, t) , \]

\[ t_{0_0}(\vec{x}, t) = a^{-3}(t) \int_{t_{\text{init}}}^t du a^3(u) \partial_\ell h_{00}(\vec{x}, u) , \]

\[ t_{0_0}(\vec{x}, t) = a^{-3}(t) \int_{t_{\text{init}}}^t du [a(u) \partial_\ell t^{0_\ell}(\vec{x}, u) - 6 a^2(u) a(u) h_{00}(\vec{x}, u)] , \]

\[ \partial_\ell t^{0_0}(\vec{x}, t) = 2 a^{-3}(t) \int_{t_{\text{init}}}^t du a^3(u) \nabla^2 h_{00}(\vec{x}, u) , \]

\[ t^{0_\alpha}(\vec{x}, t) = t_{0_0}(\vec{x}, t) - 6 h_{00}(\vec{x}, t) . \]  

(14)

Substituting these equations into Eq. (A5) gives our final result for the modified Einstein equations, in Ricci tensor form.

Let us now separate all terms of Eq. (A5) into zeroth and first order parts. Writing

\[ R_{\mu\nu} = R_{\mu\nu}^{(0)} + R_{\mu\nu}^{(1)} , \]  

(15)
we read off from Eq. (B3) that
\[ R^{(0)}_{00} = 3 \frac{\ddot{a}}{a}, \]
\[ R^{(0)}_{0i} = 0, \]
\[ R^{(0)}_{ij} = - \delta_{ij} [a \ddot{a} + 2(\dot{a})^2] . \]
(16)

We make a similar splitting for \( T_{\mu\nu}^{(0)} \), taking the zeroth order stress energy tensor to be
\[ T^{(0)}_{\mu\nu} = (p + \rho) u_{\mu} u_{\nu} - pg_{\mu\nu}, \]
\[ T^{(0)}_{\alpha\beta} = \rho - 3p, \]
(17)

with \( \rho \) and \( p \) the particulate matter density and pressure, and \( u_{\mu} \) the four velocity with \( u_0 = 1 \), \( u_i = 0 \). The zeroth order part of Eq. (A5) gives the standard equations governing RW cosmology,
\[ \frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2 = \Lambda + 8\pi G \frac{1}{2}(\rho - p), \]
\[ 3\frac{\ddot{a}}{a} = \Lambda - 8\pi G \frac{1}{2}(\rho + 3p). \]
(18)

The first order part of Eq. (A5) is
\[ R^{(1)}_{\mu\nu} - \Lambda h_{\mu\nu} = -8\pi G [T_{\mu\nu}^{(0)} - \frac{1}{2}g_{\mu\nu} T^{\alpha}_{\alpha \beta}]^{(1)} - \Lambda f [t_{\mu\nu} - \frac{1}{2}g_{\mu\nu} t^{\alpha}_{\alpha}] \],
(19)

with
\[ [T_{\mu\nu}^{(0)} - \frac{1}{2}g_{\mu\nu} T^{\alpha}_{\alpha \beta}]^{(1)} = T^{(1)}_{\mu\nu} - \frac{1}{2} [g_{\mu\nu} T^{\alpha}_{\alpha \beta} + \Lambda f] . \]
(20)

Rewriting Eq. (19) with all terms on the same side of the equation, it is
\[ 0 = R^{(1)}_{\mu\nu} - \Lambda h_{\mu\nu} + 8\pi G [T_{\mu\nu}^{(0)} - \frac{1}{2}g_{\mu\nu} T^{\alpha}_{\alpha \beta}]^{(1)} + \Lambda f [t_{\mu\nu} - \frac{1}{2}g_{\mu\nu} t^{\alpha}_{\alpha}] . \]
(21)

III. RESIDUAL GAUGE INVARIANCE WITH A FRAME DEPENDENT EFFECTIVE ACTION

Since the effective action of Eq. (4) is not four-space general coordinate invariant, but only three-space invariant, the gauge invariance group of the first order perturbation equations will be reduced. Let us consider the infinitesimal transformation
\[ x^{\alpha} = x'^{\alpha} - \epsilon^{\alpha} (x') , \]
(22)
with \( \epsilon^0 = 0 \), so that \( t = t' \).\(^1\) The metric tensor transforms according to

\[
g'_{\mu'\nu'}(x') = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial x^\nu}{\partial x'^{\nu'}} = g_{\mu\nu}(x) \left[ \delta^\mu_{\mu'} - \frac{\partial \epsilon^\mu}{\partial x'^{\mu'}} \right] \left[ \delta^\nu_{\nu'} - \frac{\partial \epsilon^\nu}{\partial x'^{\nu'}} \right].
\]

(23)

Treating \( \epsilon \) as a first order perturbation, dropping second order terms, and using the fact that because of spatial homogeneity of the RW metric the zeroth order metric has no dependence on the difference between \( x' \) and \( x \), Eq. (23) reduces to a gauge transformation formula for the difference \( \delta g_{\mu\nu} \equiv g'_{\mu\nu} - g_{\mu\nu} \),

\[
\begin{align*}
\delta g_{ij} &= a^2(t) \left( \partial_j \epsilon^i + \partial_i \epsilon^j \right), \\
\delta g_{i0} &= a^2(t) \partial_0 \epsilon^i, \\
\delta g_{00} &= 0.
\end{align*}
\]

(24)

Since the zeroth order particulate matter stress-energy tensor is also spatially homogeneous, the same reasoning applies to calculating the gauge variation of the first order particulate matter stress-energy tensor, and we find using Eq. (17) that (see [4] for further details)

\[
\delta g T^{(1)}_{\mu\nu pm} = -p \delta g_{\mu\nu}
\]

(25)

Comparing this with \( \delta g \) applied to Eq. (A7), we learn that the perturbed quantities \( p^{(1)}, u^{(1)}, \rho^{(1)}, \pi_S, \pi_V^I, \pi_T^I, u^V_i \) are all invariant under the gauge transformation of Eq. (24).

Let us now calculate the variations of the first order Ricci tensor components given in Eq. (B3) under the gauge transformation of Eq. (24). After a lengthy calculation, in which many terms cancel, we find for the \( \mu\nu = ij, i0, 00 \) cases

\[
\delta g R_{\mu\nu} = \left[ \frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 \right] \delta g_{\mu\nu}.
\]

(26)

Since \( \delta g t_{\mu\nu} = 0 \), the gauge variation of Eq. (A5) becomes, using Eq. (18)

\[
\begin{align*}
\delta g \left[ R^{(1)}_{\mu\nu} - \Lambda h_{\mu\nu} \right] &= \left[ \frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 - \Lambda \right] \delta g_{\mu\nu} = 8\pi G \frac{1}{2} (\rho - p) \delta g_{\mu\nu} = -8\pi G \delta g \left[ T_{\mu\nu pm} - \frac{1}{2} g_{\mu\nu} T^\alpha_{\alpha pm} \right]^{(1)}. \\
\end{align*}
\]

(27)

\(^1\) With a nonzero \( \epsilon^0(\vec{x}) \) that is independent of \( t \), \( \delta g_{00} \) remains zero (see [4], Eq. (5.3.7)). This additional gauge invariance can be used to impose a condition at only one time, and we do not use it in what follows.
Thus gauge invariance of the first order perturbation equations requires
\[ \frac{1}{2}(p - \rho)\delta g_{\mu\nu} = \delta g [T_{\mu\nu}^{(1)} - \frac{1}{2}g_{\mu\nu}T_{\alpha\beta}^{(1)}] \quad (28) \]
In Appendix C we show that Eq. (28) is implied by Eq. (25), and so the first order perturbation
equations are gauge invariant.

Having established gauge invariance of the perturbed equations, we are free to make a choice
of gauge to simplify the subsequent calculations. Taking
\[ \epsilon^i = \frac{1}{2}\partial_i B, \quad (29) \]
we get \( \delta g_{ij} = a^2\partial_i\partial_j B \), which cancels the \( B \) term in Eq. (A6), giving what one might term
“restricted Newtonian gauge”. However, one cannot also gauge \( F \) to zero as in full Newtonian
gauge, since this requires use of \( \epsilon_0 \). We also cannot attain synchronous gauge, since this again
requires use of \( \epsilon_0 \).

IV. THE MODIFIED SCALAR PERTURBATION EQUATIONS

Combining Eqs. (18), (21), (A6), (B3), and (C2), and choosing \( B = 0 \) gauge, we get the
following results for the modified scalar perturbation equations. They correspond to Eqs. (5.1.44)-
(5.1.47) of Weinberg [4], with the omission of the \( B \) terms, and the addition of the \( \Lambda ft_{\mu\nu} \) terms
arising from the frame dependent effective action.\(^2\) The \( ij \) part of the scalar perturbation can be
written as
\[
0 = \delta_{ij}X + \partial_i\partial_jY \\
X = 4\pi G a^2 (p^{(1)} - p^{(1)} - \nabla^2 \pi S) \\
+ [a\ddot{a} + 2(a\dot{a})^2]E + \frac{1}{2}a\dot{a}E + a\nabla^2 F - 3a\dot{a}\ddot{A} - \frac{1}{2}a^2\ddot{A} + \frac{1}{2}\nabla^2 A \\
+ \Lambda f a^2 (\frac{1}{2}t_{00} - E) \\
Y = 8\pi G a^2 \pi S \\
+ 2a\dot{F} + a\dot{F} + \frac{1}{2}(E + A) \quad (30)
\]
\(^2\) Note that \( \Lambda \) terms not multiplied by \( f \) have cancelled when Eqs. (18) and (C2) were used. That is why our \( f = 0 \)
equations in \( B = 0 \) gauge are identical to those of [4], which omits a cosmological constant.
The $i0$ part of the scalar perturbation is

$$0 = -8\pi G(p + \rho)\partial_i u^{(1)}$$

$$-\frac{\dot{a}}{a}\partial_i E + \partial_i \dot{A}$$

$$+ \Lambda f t_{0i}$$

(31)

and the $00$ part of the scalar perturbation is

$$0 = 4\pi G(\rho^{(1)} + 3p^{(1)} + \nabla^2 \pi^S)$$

$$-3\frac{\dot{a}}{a}E + 3\frac{\dot{a}}{a}\dot{A} + \frac{3}{2}\frac{\dot{a}}{a}E - \frac{\dot{a}}{a^2} \nabla^2 F - \frac{1}{a} \nabla^2 \dot{F} - \frac{1}{2a^2} \nabla^2 E$$

$$+ \Lambda f (\frac{1}{2} t_{00} + 3E)$$

(32)

We have displayed these equations with separate lines giving the matter perturbation source terms, the metric terms, and the additional pieces proportional to $\Lambda f t_{\mu\nu}$. The latter are given, we recall, by Eq. (14) of Sec. 2, with $h_{00} = E$.

Since the particulate matter stress-energy tensor $T_{\mu\nu pm}$ and the added stress-energy tensor $t_{\mu\nu}$ are separately covariantly conserved, $D^\mu T_{\mu\nu pm} = 0$, $D^\mu t_{\mu\nu} = 0$, the momentum and energy conservation equations given in Eqs. (5.1.48) and (5.1.49) of [4] are unmodified. Also, since $t_{\mu\nu}$ involves only the scalar $h_{00}$, the vector and tensor perturbation equations are unmodified.

V. ABSENCE OF PROPAGATING SCALAR GRAVITATIONAL WAVES

We turn now to the question of whether the modified scalar perturbation equations of the preceding section admit propagating wave solutions. Thus, we investigate whether the homogeneous equations, obtained by dropping the matter source terms, have solutions. Since these equations have space-independent coefficients, we can Fourier analyze with respect to the coordinate $\vec{x}$, and it suffices to keep a generic mode $e^{iE\vec{x}}$. Once this is done, the $X$ and $Y$ terms in Eq. (30) decouple, since the tensors $\delta_{ij}$ and $k_i k_j$ are linearly independent. So we are left with four coupled equations to solve. We can now anticipate the answer, before doing detailed arithmetic: Since gauge invariance, which was used to set $B = 0$, has reduced the number of unknowns from four to the set of three comprising $A$, $F$, and $E$, the homogeneous equations are overdetermined. So unless there is a hidden linear dependence (which we shall see there is not) there are no solutions, other than the
trivial solution $A = F = E = 0$. This turns out to be the case independent of the coefficient $f$ of the added term $\Lambda f t_{\mu\nu}$.

To start our calculation, we rescale $F \rightarrow F/a$, after which all terms in the scalar perturbation equations with equal numbers of spatial derivatives have the same power of $a$ as coefficient. Since the gravitational waves of interest for binary pulsars or black hole mergers have much shorter periods than the Hubble time scale on which $a(t)$ changes, we make the approximation of treating $a$, $\dot{a}$, and $\ddot{a}$ as constants. The homogeneous scalar equations then become differential equations for the evolution of $A$, $F$, and $E$ with constant coefficients, which we solve by making an Ansatz of $e^{-i\omega t}$ time dependence (where $\omega$ has a small negative imaginary part so the wave vanishes at $t = -\infty$) and $e^{i\vec{k} \cdot \vec{x}}$ spatial dependence (with $a$ treated here as time-independent).\(^3\) We write $\dot{a} \equiv Ha$, $\ddot{a} \equiv H^2 Qa$, with constant $H$, $Q$, and $a$, replace $\partial_j$ by $iak_j$, $\partial_t$ by $-i\omega$, and $\int dt$ by $(-i\omega)^{-1}$, and factor away the uniform powers of $a$ in all terms of the equations (which is equivalent to setting $a = 1$). The time integrations of Eq. (14) now give

\[
\begin{align*}
t_{i0} &= -\frac{k_i}{\omega} E, \\
t_{00} &= 2 \left(\frac{(\vec{k})^2}{\omega^2} - 6i \frac{H}{\omega}\right) E \\
\end{align*}
\]

(33)

Using these, we then get the following set of four coupled equations for the now constant unknowns $A$, $F$, and $E$: From $Y$ in the $ij$ equation of Eq. (30) we get

\[
0 = HF - i\omega F + \frac{1}{2} (E + A) ;
\]

(34)

from the $i0$ equation of Eq. (31) we get (after factoring out $k_i$),

\[
0 = -iHE + \omega A - \frac{2\Lambda f}{\omega} E ;
\]

(35)

from $X$ in the $ij$ equation of Eq. (30) we get

\[
0 = (Q+2)H^2 E - \frac{1}{2} i\omega HE - (\vec{k})^2 HF + 3i\omega HA + \frac{1}{2} \omega^2 A - \frac{1}{2} (\vec{k})^2 A + \Lambda f \left[\frac{(\vec{k})^2}{\omega^2} - 3i \frac{H}{\omega} - 1\right] E ;
\]

(36)

and from the $00$ equation of Eq. (32) we get

\[
0 = -3QH^2 E - 3i\omega HA - \frac{3}{2} \omega^2 A + \frac{3}{2} i\omega HE - i\omega(\vec{k})^2 F + \frac{1}{2} (\vec{k})^2 E + \Lambda f \left[\frac{(\vec{k})^2}{\omega^2} - 3i \frac{H}{\omega} + 3\right] E .
\]

(37)

\(^3\) Regarding the $a$ in the spatial wave as $a(t_0)$ for some late time $t_0$, our approximation thus consists of neglecting terms of order $H(t - t_0)$, $[H(t - t_0)]^2$, etc. in the Taylor expansion of the coefficient functions $a(t)$, $\dot{a}(t)$, etc. at $t = t_0$. When applied to the tensor perturbation equation of Eq. (5.1.53) of [4], this procedure leads to the equation $\nabla^2 D_{ij} - D_{ij} - 3HD_{ij} = 0$, which describes a weakly damped propagating wave.
Eqs. (34) and (35) can be solved for $F$ and $A$ in terms of $E$, giving

$$F = \frac{1}{2i} \frac{1 + \frac{iH}{\omega} + \frac{2\Lambda f}{\omega^2} E}{1 + \frac{iH}{\omega}} ,$$

$$A = \left( \frac{iH}{\omega} + \frac{2\Lambda f}{\omega^2} \right) E .$$

(38)

Substituting these into Eqs. (36) and (37), we can write the results respectively in the form

$$0 = E[(\vec{k})^2 \alpha(\omega) + \beta(\omega)] ,$$

$$0 = E[(\vec{k})^2 \gamma(\omega) + \delta(\omega)] ,$$

(39)

with

$$\alpha(\omega) = \gamma(\omega) = \left( \frac{i\Lambda f H}{\omega^3} \right)/(1 + \frac{iH}{\omega}) ,$$

$$\beta(\omega) = (Q - 1)H^2 + 3 \frac{i\Lambda f H}{\omega} ,$$

$$\delta(\omega) = - 3\beta(\omega) .$$

(40)

Evidently the two equations in Eq. (39) are inconsistent, and so can only be solved by $E = 0$.

One caveat to this analysis is that the algebra leading to Eqs. (40) involves multiple cancellations of leading terms, and so there could be significant corrections when the time dependence of $a$, $\dot{a}$, $\ddot{a}$ is taken into account. But there is no reason for these corrections to conspire to make the two equations of Eq. (39) consistent.

VI. CONCLUSION

We have shown that cosmological action of Eq. (5) has potentially testable consequences for the equations governing scalar perturbations around the RW metric. This may make it possible to distinguish between the standard and the frame dependent actions, both of which give rise to an effective cosmological constant.

VII. ACKNOWLEDGEMENTS

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**Appendix A: Notational conventions**

Since many different notational conventions are in use for gravitation and cosmology, we summarize here the notational conventions used in this paper and in \[1]-\[3\].

1. The Lagrangian in flat spacetime is \( L = T - V \), with \( T \) the kinetic energy and \( V \) the potential energy, and the flat spacetime Hamiltonian is \( H = T + V \).

2. We use a \((1, -1, -1, -1)\) metric convention, so that in flat spacetime, where the metric is denoted by \( \eta_{\mu\nu} \), the various 00 components of the stress energy tensor \( T_{\mu\nu} \) are equal, \( T_{00} = T^0_0 = T^{00} \).

3. The affine connection, curvature tensor, contracted curvatures, and the Einstein tensor, are given by

\[
\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (g_{\sigma\nu, \mu} + g_{\sigma\mu, \nu} - g_{\mu\nu, \sigma}) ,
\]

\[
R^\lambda_{\mu\tau\nu} = \Gamma^\lambda_{\mu\tau, \nu} - \Gamma^\lambda_{\mu\nu, \tau} + \text{quadratic terms in } \Gamma ,
\]

\[
R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = \Gamma^\lambda_{\mu\lambda, \nu} - \Gamma^\lambda_{\mu\nu, \lambda} + \text{quadratic terms in } \Gamma ,
\]

\[
R = g^{\mu\nu} R_{\mu\nu} ,
\]

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R .
\]

(A1)

4. The Einstein-Hilbert gravitational action and its variation with respect to the metric \( g_{\mu\nu} \) are

\[
S_{\text{EH}} = \frac{1}{16\pi G} \int d^4 x (4g)^{1/2} R ,
\]

\[
\delta S_{\text{EH}} = - \frac{1}{16\pi G} \int d^4 x (4g)^{1/2} G^{\mu\nu} \delta g_{\mu\nu} .
\]

(A2)
(5) The particulate matter action and its variation with respect to the metric $g_{\mu\nu}$ are

$$S_{pm} = \int dt L = \int d^4x (g^{(4)} g)^{1/2} \mathcal{L}(x),$$

$$\delta S_{pm} = -\frac{1}{2} \int d^4x (g^{(4)} g)^{1/2} T_{\mu\nu}^{pm} \delta g_{\mu\nu}. \quad (A3)$$

(6) The Einstein equations with cosmological constant $\Lambda$ are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu}^{pm} + \Lambda f t_{\mu\nu} = 0, \quad (A4)$$

with the final term the additional term arising from the frame dependent effective action, which we have split off by writing $T_{\mu\nu}^{\Lambda} = \frac{\Lambda}{8\pi G} (g_{\mu\nu} + ft_{\mu\nu})$. Eq. (A4) can equivalently be written as

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = -8\pi G [T_{\mu\nu}^{pm} - \frac{1}{2} g_{\mu\nu} T_\alpha^{(0)\alpha}] - \Lambda f [t_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T_\alpha^{(0)\alpha}], \quad (A5)$$

with $g_{\mu\nu}^{(0)}$ the unperturbed RW metric.

(7) To compare with Weinberg [4], our metric $g_{\mu\nu}$ and metric perturbation $h_{\mu\nu}$ are opposite in sign to his, while our relations between the Ricci tensor $R_{\mu\nu}$, the affine connection $\Gamma^\lambda_{\mu\nu}$, and the metric are the same. Since the affine connection is an even function of the metric, this means that the zeroth order parts of the affine connection and the Ricci tensor are the same as in [4], while the first order metric perturbations in these quantities are opposite in sign. In defining the scalar, vector, and tensor parts of the metric perturbations, we introduce an extra minus sign relative to [4],

$$h_{00} = E, \quad h_{i0} = -a (\partial_i F + G_i), \quad h_{ij} = -a^2 (A \delta_{ij} + \partial_i \partial_j B + \partial_i C_j + \partial_j C_i + D_{ij}) \quad (A6)$$

with $\partial_i C_i = \partial_i G_i = \partial_i D_{ij} = D_{ii} = 0$. The quantities $E$, $F$, $G_i$, $A$, $B$, $C_i$, $D_{ij}$ then are the same as in [4]. Our energy momentum tensor sign is the same as in Weinberg, but reflecting our opposite sign of $h_{\mu\nu}$ we define the tensorial decomposition of the perturbed stress-energy tensor $T_{\mu\nu}^{(1)}$ pm by

$$T_{ij}^{(1)} pm = - p h_{ij} + a^2 [\delta_{ij} p^{(1)} + \partial_i \partial_j \pi^S + \partial_i \pi^V + \partial_j \pi^V + \pi^T],$$

$$T_{i0}^{(1)} pm = - p h_{i0} - (p + \rho) (\partial_i u^{(1)} + u_i^V),$$

$$T_{00}^{(1)} pm = \rho h_{00} + \rho^{(1)}. \quad (A7)$$
with \( \partial_i \pi^V_i = \partial_i u^V_i = \partial_i \pi^T_{ij} = \pi^T_{ii} = 0 \). The quantities \( p^{(1)}, u^{(1)}, \rho^{(1)} \), and \( u^V_i \) are then the same as Weinberg’s \( \delta p, \delta u, \delta \rho, \) and \( \delta u^V_i \), and the quantities \( \pi^S, \pi^V_i, \) and \( \pi^T_{ij} \) are the same as his similarly labeled quantities.

Appendix B: Inverse metric, affine connections, and Ricci curvature tensor for the perturbed RW metric

The inverse perturbed RW metric corresponding to Eq. (8) is given, through first order terms, by

\[
\begin{align*}
\tilde{g}^{00} &= 1 - h_{00}, \\
\tilde{g}^{0i} &= \tilde{g}^{i0} = h_{i0}/a^2(t), \\
\tilde{g}^{ij} &= -\delta^{ij}/a^2(t) - h_{ij}/a^4(t).
\end{align*}
\]

(B1)

The perturbed RW affine connection, through first order terms, is

\[
\begin{align*}
\Gamma^0_{00} &= \frac{1}{2} \partial_0 h_{00}, \\
\Gamma^0_{0i} &= \frac{1}{2} \partial_i h_{00} - \frac{\dot{a}}{a} h_{0i}, \\
\Gamma^0_{ij} &= a\dot{a}(1 - h_{00})\delta_{ij} + \frac{1}{2}(\partial_j h_{0i} + \partial_i h_{0j} - \partial_0 h_{ij}), \\
\Gamma^\ell_{00} &= -\frac{1}{a^2}(\partial_0 h_{\ell 0} - \frac{1}{2} \partial_0 h_{00}), \\
\Gamma^\ell_{0i} &= \frac{\dot{a}}{a}\delta_{\ell i} - \frac{1}{2a^2}(\partial_0 h_{\ell i} + \partial_\ell h_{0i} - \partial_\ell h_{0i}) + \frac{\dot{a}}{a^3} h_{\ell i}, \\
\Gamma^\ell_{ij} &= \frac{\dot{a}}{a} h_{\ell 0}\delta_{ij} - \frac{1}{2a^2}(\partial_j h_{\ell i} + \partial_i h_{\ell j} - \partial_\ell h_{ij}).
\end{align*}
\]

(B2)
The corresponding Ricci tensor components are

\[ R_{00} = \frac{\ddot{a}}{a} + \frac{\dot{a}}{a^2} \partial_0 h_{mm} + \frac{1}{2a^2} \left( \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 \right) h_{mm} - \frac{3}{2} \frac{\dot{a}}{a} \partial_0 h_{00} + \frac{1}{a^3} \partial_0 \partial_m h_{m0} - \frac{1}{2a^2} \left( \partial_0^2 h_{mm} + \nabla^2 h_{00} \right) , \]

\[ R_{0i} = \frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 h_{0i} - \frac{1}{a^2} \partial_i h_{00} + \frac{\dot{a}}{a^3} \left[ \partial_i h_{mm} - \partial_m h_{mi} \right] + \frac{1}{2a^2} \left( \partial_0 \partial_0 h_{mm} + \partial_0 \partial_m h_{m0} - \nabla^2 h_{0i} - \partial_0 \partial_i h_{mm} \right) , \]

\[ R_{ij} = - \delta_{ij} [ \dot{a} + 2 (\dot{a})^2 ] (1 - \dot{h}_{00}) + \delta_{ij} \left[ \frac{1}{2} \frac{\dot{a} a}{a^2} \partial_0 h_{00} - \frac{\dot{a}}{a} \partial_m h_{m0} + \frac{1}{2a^2} \partial_0 \partial_m h_{mm} - \left( \frac{\dot{a}}{a} \right)^2 h_{mm} \right] \]

\[- \frac{1}{2} \left( \partial_0 \partial_j h_{0i} + \partial_i \partial_j h_{00} - \delta_{ij} \partial_0 h_{00} - \partial_0 \partial_j h_{0i} \right) + \frac{1}{2a^2} \left( \partial_m \partial_j h_{mi} + \partial_m \partial_i h_{mj} - \nabla^2 h_{ij} - \partial_i \partial_j h_{mm} \right) \]

\[- \frac{\dot{a}}{2a} (\partial_0 h_{ij} + \partial_j h_{0i} + \partial_i h_{0j}) + 2 \left( \frac{\dot{a}}{a} \right)^2 h_{ij} . \]

(B3)

Appendix C: Equivalence of two forms of the gauge variation of the first order particulate matter stress-energy tensor

We proceed to show that Eq. (28) is implied by Eq. (25). Expanding

\[ [T_{\mu\nu}^{(1)} - \frac{1}{2} g_{\mu\nu} T^\alpha_{\alpha \mu \nu}] = T_{\mu\nu}^{(1)} - \frac{1}{2} h_{\mu \nu} T_{\alpha \mu \nu}^{(0)} - \frac{1}{2} h_{\mu \nu} T_{\alpha \mu \nu}^{(0)} , \]

\[ T_{\alpha \mu \nu}^{(1)} = g^{\mu \nu} T_{\mu \nu}^{(1)} + g^{\mu \nu} T_{\mu \nu}^{(0)} , \]

(C1)

and substituting the decompositions of Eq. (A7), we get

\[ T_{\alpha \mu \nu}^{(0)} = \rho^{(1)} - 3 \rho^{(1)} - \nabla^2 \pi^S , \]

\[ [T_{\mu \nu}^{(1)} - \frac{1}{2} g_{\mu \nu} T_{\alpha \mu \nu}^{(1)}] = \frac{1}{2} (3 \rho + \rho) h_{00} + \frac{1}{2} (\rho^{(1)} + 3 \rho^{(1)}) + \nabla^2 \pi^S , \]

\[ [T_{0 \nu}^{(1)} - \frac{1}{2} g_{0 \nu} T_{\alpha \mu \nu}^{(1)}] = \frac{1}{2} (\rho - \rho) h_{00} - (p + \rho) (\delta_{ij} u_{ij}^{(1)} + u_i^V) , \]

\[ [T_{ij}^{(1)} - \frac{1}{2} g_{ij} T_{\alpha \mu \nu}^{(1)}] = \frac{1}{2} (p - \rho) h_{ij} + a^2 \delta_{ij} \frac{1}{2} (\rho^{(1)} - p^{(1)}) - \nabla^2 \pi^S + a^2 (\partial_i \partial_j \pi^S + \partial_i \pi^j + \partial_j \pi^i + \pi^T_{ij}) . \]

(C2)

Applying \( \delta_g \) to these equations, and using \( 0 = \delta_g h_{00} = \delta_g \rho^{(1)} = \delta_g p^{(1)} = \delta_g u_{ij}^{(1)} = \delta_g \pi^S = \delta_g \pi^V = \delta_g \pi^T = \delta_g u_i^V \), we obtain Eq. (28). The detailed decompositions of Eq. (C2) are used in writing down the detailed form of the scalar perturbation equations in Sec. 4.
Appendix D: Relation between our dark energy model and the effective field theory of dark energy [5]

We show here that although our model can be mapped to the framework of [5], the underlying physics is different. In particular, their method cannot be used to test the frame-dependent model because their procedure leads to a non-covariantly conserved dark energy stress-energy tensor, as we shall now show in detail.

When the action of Eq. (4) is expanded to first order in perturbations by writing $g_{00} = 1 + h_{00}$, it takes the form

$$S_{\text{eff}} = -\frac{\Lambda}{8\pi G} \int d^4x (g_{\mu\nu}^{(4)})^{1/2} [1 - 2h_{00}]$$

$$= -\frac{\Lambda}{8\pi G} \int d^4x (g_{\mu\nu}^{(4)})^{1/2} [-1 + 2g^{00}]$$

(D1)

where on the second line we have substituted (to first order accuracy) $g_{00}^{00} = 1 - h_{00}$. Taking $8\pi G = 1$, and comparing with Eq. (1) of [5], noting that Eq. (D1) is written in the Einstein frame, and that the sign convention for the metric of [5] is opposite to ours, we get the following identifications of their coefficient functions,

$$f(t) = 1 ~ \text{(Einstein frame)}$$

$$\Lambda(t) = -\Lambda$$

$$c(t) = -2\Lambda$$

(D2)

that is, their three coefficient functions are all constants, independent of time in our model.

Up to this point our model has fit into the framework of [5]; the difference shows up at their Eq. (14), where they vary with respect to the full $g^{\mu\nu}$ to get the Einstein equations. Since we have $f = 1$, their Eq. (14) becomes

$$G_{\mu\nu} M^2 + \left(c\delta g^{00} + \Lambda - c\right)g_{\mu\nu} - 2c\delta_{\mu}^{\nu} \delta_{\nu}^0 = T_{\mu\nu}$$

(D3)

With constant $c$ and $\Lambda$, this equation is inconsistent at zeroth order in perturbations, since $g_{\mu\nu}$, $G_{\mu\nu}$ and $T_{\mu\nu}$ are covariantly conserved on the RW background, while the term $c\delta_{\mu}^{\nu} \delta_{\nu}^0$ is not covariantly
conserved. To see this explicitly, switching to the upper index version of this equation we have

\[ D_\nu c \delta^\mu_\nu \delta^\alpha_0 \delta^\beta_0 = \partial_\nu c \delta^\mu_\nu \delta^\alpha_0 + \Gamma^\mu_\nu_\alpha \delta^\alpha_0 \delta^\beta_0 + \Gamma^\nu_\nu_\alpha \delta^\mu_0 \delta^\alpha_0 \]

\[ = [\dot{c} + 3 \frac{\dot{a}}{a} \delta^\mu_0, \] \hspace{1cm} (D4)

where in the second line we have substituted the RW affine connection \( \Gamma^\ell_0 i = \frac{\dot{a}}{a} \delta^\ell_i \). We see that when \( c \) is a constant, so \( \dot{c} = 0 \), covariant conservation fails. This problem is a direct reflection of the fact that our model is not four-space diffeomorphism invariant. This problem does not arise in the paper [5], because their \( c(t), \Lambda(t) \) etc. are obtained by transforming an underlying model that is diffeomorphism invariant to unitary gauge, leading to non-constant \( c(t), \Lambda(t) \) etc. which take values that guarantee covariant conservation of their dark energy stress-energy tensor.

For our frame-dependent model, the correct way to get a covariantly conserved dark energy stress-energy tensor is given in Sec. 2 above. First one varies with respect to the spatial components \( g_{ij} \) to generate the spatial components \( T^{ij} \) of the stress-energy tensor; then one integrates the covariant conservation equations to get the remaining components \( T^{i0} \) and \( T^{00} \). This procedure works both because the action of Eq. (D1) is three-space general coordinate invariant, and because in the ADM formulation [6] of general relativity, the \( g_{ij} \) are the fundamental degrees of freedom of the gravitational field.

**Appendix E: Off-diagonal metric terms in the effective action do not change the first order perturbation equations**

When the metric has nonzero off-diagonal terms, as is the case for RW perturbations, the effective action of Eq. (2) generalizes (1) to

\[ S_{\text{eff}} = \int d^4x (4g)^{1/2}(g_{00})^{-2} A(h_{0i} h_{0j} g^{ij}/g_{00}, D^i g_{ij} D^j/g_{00}, h_{0i} D^i/g_{00}) \] \hspace{1cm} (E1)

with \( D^i \) defined through the co-factor expansion of \( (4g) \) by \( (4g)/(3g) = g_{00} + h_{0i} D^i \), with \( A(x, y, z) \) a general function of its arguments, and where we have used \( g_{0i} = h_{0i} \). Evaluating \( D^i \) from its
definition, we find

\[ D^1 = -\frac{h_{10}}{g_{11}} + O(h^2), \]
\[ D^2 = -\frac{h_{20}}{g_{22}} + O(h^2), \]
\[ D^3 = -\frac{h_{30}}{g_{33}} + O(h^2). \]

(E2)

From this we see that \( D^i \) and \( \delta D^i / \delta g_{kl} \) are both \( O(h) \), as a consequence of which

\[ \delta A(h_{0i}, h_{0j}, g^{ij}/g_{00}, D^i g_{ij}, D^j/g_{00}, h_{0i} D^i/g_{00}) / \delta g_{kl} = O(h^2). \]  (E3)

Thus when we vary \( S_{\text{eff}} \) with respect to the spatial metric components \( g_{kl} \), the additions to \( T_{\Lambda}^{kl} \) coming from the nonzero arguments of the function \( A \) are second order corrections to the first order result of Eq. (11). Hence in a first order calculation, it suffices to take \( A \) as \( A_0 = A(0,0,0) \), as was done in the sections above.

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