Magnetic translation group and classification of states of an itinerant electron

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Abstract. We consider an itinerant electron on two-dimensional finite square lattice in a magnetic field. A magnetic translation group (MTG) for this system with the periodic Born-Karman conditions has been introduced. The irreducible representation of MTG is used for classification of energy levels of electron states for this model.

1. Introduction
The motion of a free electron in a plane perpendicular to a magnetic field is periodic. The corresponding states are quantized according to the Landau rule [1]. For an electron in both magnetic and periodic electric potential, an additional quantum effect appears - the Bohm-Aharanov effect. It implies that a jump from one node of the lattice to another one is accompanied by a phase. In this way the translation group of the planar lattice is no longer a symmetry group of the Hamiltonian of the electron. Brown [2] and Zak [3] introduced operators which commute with the Hamiltonian. They can form a group - the magnetic translation group (MTG) - which provides a restriction on the magnitude of the magnetic field. The magnetic flux through the cells of the planar lattice is quantized and the number \( \eta \) of quanta through the elementary cell should be the rational number \( p/q \), where \( p \) and \( q \) are coprime integers [2, 3, 4, 5].

The important part in the description of electron dynamics is played by gauge symmetry \( C \) (Weyl-Heisenberg group), according to which the vector potential of magnetic field is changing along translation. This group, together with the translational symmetry \( T \), builds the magnetic translation group, which noncommutativity is in connection with quantum character of magnetic interaction. The group theoretical analysis of the MTG has been done by Lulek [6], Kuzma [7], Florek [8], and Geyler [9].

The Hamiltonian of an electron in a magnetic field can be reduced to the tight-binding Hamiltonian which depends on the phase factors associated with each edge of the planar lattice [10]. Such a system was the subject of studies during last years in context of integer and fractional Hall effects, high-\( T_C \) superconductors and even quantum computation [11, 12, 13].

2. The hopping Hamiltonian
The Hamiltonian of an electron in a periodic electric potential and homogeneous magnetic field is given by

\[
\mathcal{H} = \frac{1}{2m} \left( \mathbf{p} + \frac{e}{c} \mathbf{A} \right)^2 + V(\mathbf{r}),
\]
where the vector potential can be chosen in the form \( A = \frac{1}{2} \mathbf{H} \times \mathbf{r} \). In the case of a two-dimensional finite \( N \times N \) square lattice \( \Lambda \), for the tight-binding approximation this Hamiltonian can be reduced to the form:

\[
H_l = \sum_{\lambda, \mu \in \Lambda} \exp(2\pi i \vartheta_{\lambda \mu}) c^+_{\mu} c_{\lambda}
\]

where \( \lambda, \mu \) are vertices of the lattice \( \Lambda \) and \( c^+_{\mu} \) is the creation operator on the vertex \( \mu \). The summation includes all the nearest-neighbor sites (4 in case of a square lattice). Periodic boundary conditions are applied to the system.

The Hamiltonian is expressed by the phase factors \( \vartheta_{\lambda \mu} \) which are defined for each edge. They fulfill the condition:

\[
\vartheta_{\lambda \mu} = -\vartheta_{\mu \lambda}
\]

and are determined mod 1. The sum of these factors along the closed sequence of points (circuit) \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n, \lambda_1 \) gives the magnetic flux through the area bounded by this circuit:

\[
\phi = \sum_{j=1}^{n} \vartheta_{\lambda_j \lambda_{j+1}}, \quad \text{where} \quad \lambda_{n+1} = \lambda_1.
\]

The action of the Hamiltonian on the nodes of the lattice, numbered by two indices \((j_1, j_2)\) can be described as follows:

\[
H|j_1, j_2\rangle = e^{2\pi i \vartheta_{j_1 j_2}} |j_1 + 1, j_2\rangle + e^{2\pi i \vartheta_{j_2 j_1}} |j_1 - 1, j_2\rangle + e^{2\pi i \vartheta_{j_1 j_2}} |j_1, j_2 + 1\rangle + e^{2\pi i \vartheta_{j_1 j_2}} |j_1, j_2 - 1\rangle
\]

This action should commute with the action of the magnetic translation group.

### 3. Magnetic translation group

The elements of the MTG group, denoted hereafter as \( G \), have the form \((t, \xi)\), where \( t \in T \) is a translation by a vector \( t \) and \( \xi \) is an element of the gauge group \( C \):

\[
C = \{ \xi^0 | n \in \mathbb{Z} \}, \quad \xi_0 = e^{i\pi \eta}.
\]

The multiplication rule for the group \( G \) is given by

\[
(t, \xi)(t', \xi') = (t + t', \xi \xi' e^{i\pi \eta(t't_y - t_y t_x)}),
\]

the exponent in the right hand side of the equation denotes the magnetic flux through the rectangle spanned by vectors \( t \) and \( t' \). The group is generated by two elements \( a = (10,1) \) and \( b = (01,1) \), which correspond to translation, accompanied by the phase “1”, in \( x \) and \( y \)-direction of the planar lattice, respectively (the magnetic field is applied in \( z \)-direction).

Let us denote by \( T(a) \) and by \( T(b) \) the operators describing the action of the elements \( a \) and \( b \) on the nodes of the lattice respectively. These actions are given by:

\[
T(a)|j_1, j_2\rangle = e^{i\pi \eta j_2} |j_1 - 1, j_2\rangle, \quad T(b)|j_1, j_2\rangle = e^{-i\pi \eta j_1} |j_1, j_2 - 1\rangle
\]

These operators commute with the Hamiltonian

\[
[H, T(a)] = [H, T(b)] = 0.
\]

The last equation provides, together with the property that the magnetic flux through an elementary cell is equal to \( \eta \), the conditions for phase factors associated with each edge. Their distribution is presented in figure 1.
4. **Classification of states of an electron**

Let the eigenvalues of the Hamiltonian be denoted by \( w_i \) and corresponding eigenvectors by \( f_{ij} \), the index \( j \) is member of the set \( \{1, 2, \ldots d\} \) where \( d \) is the degeneration of the energy level. The action of the magnetic group on the nodes is described as follows

\[
T(g)|j_1, j_2\rangle = T(t_1 t_2, \xi)|j_1, j_2\rangle = \xi e^{i\pi \eta \left(t_1 j_2 - t_2 j_1\right)}|j_1 - t_1, j_2 - t_2\rangle
\]  

where \( g \in G \). To classify the states, the following matrix should be determined for a given eigenvalue \( w_i \):

\[
<f_{ij'}|T(g)|f_{ij}>  
\]

where \( j, j' \in \{1, 2, \ldots d\} \). These matrices form the representation \( \Gamma' \) which can be decomposed into irreducible representations \( \Gamma \) of \( G \).

5. **Example:** \( N = 4, \eta = \frac{1}{2} \)

The magnetic translation group for this case has been described in previous author’s paper [14]. The irreducible representations of this group were also determined. There are four ”physical” [9] representations and they are two-dimensional. Their characters are presented in table 1.

| \( \Gamma \) | \( (00,1)(00,-1)(20,1)(20,-1)(02,1)(02,-1)(22,1)(22,-1) \) |
|---|---|
| \( \Gamma_6 \) | 2 | -2 | 2 | -2 | 2 | -2 | 2 | -2 |
| \( \Gamma_8 \) | 2 | -2 | 2 | -2 | 2 | -2 | 2 | -2 |
| \( \Gamma_{11} \) | 2 | -2 | -2 | 2 | 2 | -2 | -2 | 2 |
| \( \Gamma_{16} \) | 2 | -2 | -2 | 2 | -2 | 2 | 2 | -2 |

The eigenvalues of the Hamiltonian (2) for this case are the following: \( 2\sqrt{2} (2), -2\sqrt{2} (2), -2 (4), 2 (4) \) and \( 0 (4) \). The numbers in parentheses denote the degeneracy of the level. The...
representations \( \Gamma' \) calculated according to the equation (11) provide the classification of the energy levels of the electron. This classification is presented in table 2.

Table 2. The classification of energy levels of an electron. \( d \) is the degeneration of the state, \( \Gamma_i \) is the representation of MTG. For all other elements \( g \in G \), the characters are \( \chi(\Gamma'(g)) = 0 \)

| energy | \( d \) | the character of a representation | decomposition |
|--------|--------|----------------------------------|--------------|
| \( 2\sqrt{2} \) | 2 2 2 -2 2 -2 2 -2 2 -2 2 2 2 \( \Gamma_8 \) | | |
| \(-2\sqrt{2} \) | 2 2 2 -2 2 -2 2 -2 2 -2 2 2 2 \( \Gamma_8 \) | | |
| 2 4 4 -4 0 0 0 0 0 0 4 -4 \( \Gamma_{16} + \Gamma_6 \) | | |
| 4 4 4 -4 0 0 0 0 0 0 4 -4 \( \Gamma_{16} + \Gamma_6 \) | | |
| 0 4 4 -4 -4 4 4 -4 -4 4 -4 4 \( \Gamma_{14} + \Gamma_{14} \) | | |

6. Conclusion
For an electron in a constant magnetic field and periodic electric potential the eigenvalue problem on the \( 4 \times 4 \) lattice was solved. The energy states are determined by the phase factors \( \vartheta_{\lambda\mu} \). The distribution of these factors on the planar lattice, which guarantees the commutation relation for \( G \) and \( \mathcal{H} \) was determined. It should be noticed that periodic boundary conditions restrict the number of allowed size \( N \) of the lattice. Only those sizes are admissible for which the values of phase factors on the corresponding border are equivalent \( \text{(mod 1)} \). For the case \( \eta = \frac{1}{2} \) and \( N = 4 \) the electron states have been classified using the irreducible representations \( \Gamma \) of the magnetic translation group \( G \).

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References
[1] Landau L 1930 Z. Phys. 64 629
[2] Brown E 1964 Phys. Rev. 133 A1038
[3] Zak J 1964 Phys. Rev. 134 A1602; A1607
[4] Brown E (1968) Phys. Rev. 166 626
[5] Brown E and Meisel L V 1976 Phys. Rev. B 13 5271
[6] Lulek T 1994 Rep. Math. Phys. 34 71
[7] Kuzma M, Mortensen K, Lulek B and Lulek T 1997 Proc. Int. Conf. Symmetry and Structural Properties of Condensed Matter (Zajaczkowo) ed T Lulek W Florek at al (Singapore: World Scientific) p 484
[8] Florek W 1994 Rep. Math. Phys. 34 81
[9] Geyler V A and Popov I Yu 1995 Phys. Lett. A 201 359
[10] Lieb E H 1994 Phys. Rev. Lett. 73 2158
[11] Mikhailov S A 2001 Physica B 299 6
[12] Vourdas A 1996 Proc. Int. Conf. Symmetry and Structural Properties of Condensed Matter (Myczkowce) ed T Lulek W Florek at al (Singapore:World Scientific) p 310
[13] Vourdas A 2002 Proc. Int. Conf. Symmetry and Structural Properties of Condensed Matter (Myczkowce) ed T Lulek B Lulek at al (Singapore:World Scientific) p 55
[14] Wal A 2005 Phys. Stat. Sol. (b) 242 291