Magnetic exchange interaction mediated by a superconductor including vortices and impurities

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We theoretically investigate the magnetic exchange interaction between two ferromagnets coupled by a superconductor using a tight-binding lattice model. The main purpose of this study is to determine how the self-consistently determined superconducting state influences the exchange interaction, including the roles of superconducting vortices and impurity scattering. We find that superconducting state eliminates RKKY-like oscillations for a sufficiently large superconducting gap, making the antiparallel orientation the ground state of the system. Interestingly, the superconducting gap is larger in the parallel configuration than in the antiparallel configuration, even when the preferred ground state is antiparallel. With an applied external field, the generation of vortices in the superconductor enables a switching between parallel and antiparallel ground-state configurations. Finally, we show that increasing the impurity concentration in the superconductor causes the exchange interaction to decrease, likely due to an increasing localization of the mediating quasiparticles in the superconductor.

I. INTRODUCTION

The Ruderman–Kittel–Kasuya–Yosida (RKKY) is an indirect exchange interaction between localized spins mediated by itinerant electrons in metals. This interaction played an important role in the discovery of giant magnetoresistance (GMR) and has been studied in numerous materials.

The combination of magnetic and superconducting materials has been widely studied due to interesting features which cannot be observed in separate materials. Recently, the influence of superconductivity on the magnetic state was experimentally studied in a superconducting spin valve (SSV), GdN-Nb-GdN. On the basis of the de Gennes model, it was shown that the superconductor promoted an antiparallel configuration as the ground-state configuration. In the de Gennes model, a superconductor in an antiparallel SSV has a higher critical temperature than in the parallel orientation, leading to a larger superconducting gap in the antiparallel configuration.

The interaction between localized magnetic moments through dirty s-wave superconductors has previously been found to contain two contributions. One contribution is from the usual RKKY interaction and a second contribution from a long-ranged interaction (decaying over the superconducting coherence length $\xi$) which favors an antiferromagnetic alignment. Later, the interaction through a $d$-wave superconductor with an anisotropic order parameter was studied. It was shown on the basis of analytical approximations that this interaction, similarly to the s-wave case, contains one oscillatory term and one term favoring an antiparallel configuration. The oscillations occur when the length of the superconductor ($L_S$) is smaller than the coherence length ($\xi$) while the term favoring an anti–ferromagnetic configuration of the system occurs when $L_S > \xi$. The latter term was found to be proportional to the superconducting gap. Very recently, it was experimentally shown that in a d-wave SSV, the antiparallel ground-state was favored for some specific lengths of the superconducting system and that nodal quasiparticles likely played a central role in mediating the magnetic coupling.

In this study, we address numerically and, importantly, self-consistently the effect of conventional s-wave singlet superconductors on the indirect exchange coupling ($J$) between two ferromagnetic contacts in a SSV. Due to the proximity effect between the superconductors and the ferromagnet, the superconducting gap can be strongly affected by the magnetic configuration, and thus requires a self-consistent calculation, unlike Refs. that considered isolated magnetic impurities. In a singlet superconductor, electrons with zero total spin and opposite momentum constitute the Cooper pairs: $(\mathbf{k} \uparrow, -\mathbf{k} \downarrow)$. These Cooper pairs can penetrate into a weak ferromagnet (FM) which has been brought in contact with the superconductors in an oscillatory fashion. Bringing another ferromagnetic layer in contact with this bilayer makes the SSV.

We first briefly reproduce the well-known RKKY-like oscillations of an F-N-F system to contrast these results with what happens in the superconducting state. We consider a finite size system in two dimensions, meaning that we do not assume periodic boundary conditions in any direction. Then, by substituting the central part with a singlet type-II superconductor which leads to a F-S-F structure (Fig. 1), we demonstrate that two types of behaviour take place. For thin superconductors, $J$ oscillates around zero whereas for thick ones the coupling takes values $J \geq 0$, favoring the AP configuration, and reduces monotonically as the length is further increased. When the central part is a superconductor with small gap connected to two weakly polarized ferromagnets, we only find RKKY-like oscillations mediated by quasiparticles in the superconductor. In contrast, when the superconducting gap is large or if the exchange field in the ferromagnet

\[ J = J_0 \sin \left( \frac{4\pi L_S}{\xi} \right) \]
is strong, \( J > 0 \) and the interaction displays either a pure monotonic decay or with superimposed oscillations.

Afterwards, we turn to the influence of vortices induced by an external magnetic field \( B \) and impurities in the superconductors. We show that since the vortices weaken superconductivity in the mixed state, oscillations reappear in the regime where only a monotonic reduction is present in the field-free case, \( B = 0 \). Moreover, we show that the ground-state configuration oscillates between \( P \) and \( AP \) as a function of the applied flux. Finally, we consider the effect of strong impurities on \( J \) in the F-S-F spin valve. When considering the impurity average \( \langle J \rangle_{\text{imp}} \) for a large number of realizations with random impurity configurations, we find that increasing the impurity concentration in the superconductor causes the exchange interaction to decrease. This is likely due to an increasing localization of the mediating quasiparticles in the superconductor.\(^{26,27}\)

![Diagram](image)

**FIG. 1:** (a) Parallel configuration and (b) antiparallel configuration for the superconducting spin valve.

## II. THEORY

The indirect exchange interaction between the ferromagnets in F-N-F or F-S-F structures is defined by \( J = F_{\uparrow \uparrow} - F_{\downarrow \downarrow} \). Here, \( F_{\uparrow \uparrow} \) is the free energy when the ferromagnetic contacts have a parallel (P) orientation and \( F_{\downarrow \downarrow} \) is the free energy when they have an antiparallel (AP) orientation. The free energy of such a system is defined by

\[
F = H_0 - \frac{1}{\beta} \sum_n \ln(1 + e^{-\beta E_n}/2).
\]

Here, \( \beta = \frac{1}{k_B T} \) and \( k_B \) is the Boltzmann constant and \( T \) is the temperature. \( H_0 \) is a constant term to be specified later, which consists of a superconducting constant term \( (H_0^S) \) and chemical potential constant term \( (H_0^\mu) \). \( H_0^S \) arises as a result of performing a mean-field approximation while \( H_0^\mu \) is due to a symmetrization of the Hamiltonian. Moreover, \( E_n \) is the nth eigenvalue and will be calculated by means of diagonalizing a tight-binding Hamiltonian for the structure of interest. The Hamiltonian is as follows,

\[
H = -\sum_{\langle i,j \rangle, \alpha} t_{ij}(A)c_{i\alpha}^\dagger c_{j\alpha} - \sum_i \mu_i n_{i\alpha} - \sum_i U_i n_{i\uparrow} n_{i\downarrow} + \sum_{i,\alpha\beta}(h_i \cdot \sigma)_{\alpha\beta} c_{i\alpha}^\dagger c_{i\beta}.
\]

(2)

Here, \( c_{i\alpha}^\dagger \) (\( c_{i\alpha} \)) creates (annihilates) an electron with spin \( \alpha \) at site \( i \) with \( t_{ij} = 1, \ldots N_x \) and \( y_j = 1, \ldots N_y \). Also, \( t_{ij}(A) \) is the hopping integral between nearest-neighbor sites as a function of vector potential \( A \). When \( A = 0 \), the hopping integral is considered to take a constant value \( t \) while it acquires a phase in the case of \( A \neq 0 \) (we will discuss this briefly later). \( \mu_i \) is the chemical potential at site \( i \) while \( n_{i\alpha} = c_{i\alpha}^\dagger c_{i\alpha} \) is the number operator. The fourth term in the Hamiltonian represents the local exchange interaction with \( h_i \) being the strength of this field and \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) the Pauli matrices. We consider a singlet type-II superconductor for the central part, modelling the interaction as an on-site attractive \( U \) as a third term in the Hamiltonian. \( U_i = U > 0 \) is the local attractive interaction which creates Cooper pairs in the superconductor while it is zero elsewhere. We treat the interaction term by a mean-field approximation to simplify the problem,

\[
-\sum_i U_i n_{i\uparrow} n_{i\downarrow} = -\sum_i U_i \left( c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger \langle c_{i\downarrow} c_{i\uparrow} \rangle + c_{i\downarrow} c_{i\uparrow} \langle c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger \rangle \right)
-\langle c_{i\downarrow} c_{i\uparrow} \rangle \langle c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger \rangle.
\]

(3)

If we define superconducting gap as \( \Delta_i = -U_i \langle c_{i\downarrow} c_{i\uparrow} \rangle \), then

\[
-\sum_i U_i n_{i\uparrow} n_{i\downarrow} = \sum_i (c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger \Delta_i + c_{i\downarrow} c_{i\uparrow} \Delta_i^* ) + H_0^S,
\]

(4)

where we have defined

\[
H_0^S = \sum_i \frac{|\Delta_i|^2}{U_i}.
\]

(5)

We proceed to explain how the eigenvalues \( E_n \) are obtained. Our Hamiltonian Eq. \( [2] \) is bilinear in the fermion operators and can be diagonalized. Choosing the following basis,

\[
W_i = \left[ D_{i1}^\dagger \ D_{i2}^\dagger \ D_{i3}^\dagger \ ... \ D_{iN_{xy}}^\dagger \right],
\]

(6)

where we have defined

\[
D_{iy}^\dagger = \left[ B_{(1,iy)}^\dagger \ B_{(2,iy)}^\dagger \ B_{(3,iy)}^\dagger \ ... \ B_{(N_x,iy)}^\dagger \right]
\]

(7)
The eigenfunctions for the Hamiltonian by introducing a new basis gives a Hamiltonian that can be diagonalized numerically. Note that the Hamiltonian may now be written as

\[ H = H_0 + \frac{1}{2} W^\dagger S W = H_0 + \frac{1}{2} \sum_{ij} B_{ij} h_i B_j. \]  

Here, \( H_0 \) is the constant term that we discussed previously, and

\[
S = \begin{bmatrix} S_{11} & \cdots & S_{1,N_y} \\ \vdots & \ddots & \vdots \\ S_{N_y,1} & \cdots & S_{N_y,N_y} \end{bmatrix}
\]

with

\[
S_{i_y,j_y} = \begin{bmatrix} h_{(1,i_y)(1,j_y)} & \cdots & h_{(1,i_y)(N_y,j_y)} \\ \vdots & \ddots & \vdots \\ h_{(N_y,i_y)(1,j_y)} & \cdots & h_{(N_y,i_y)(N_y,j_y)} \end{bmatrix}.
\]

Finally, the 4 x 4 matrix for interaction between sites \( i \) and \( j \) is

\[
h_{ij} = -\left[ \frac{t}{2} (\delta_{i_x,j_x-1} + \delta_{i_x,j_x+1}) + \mu_i \delta_{i_x,j_x} \right] \delta_{i_y,j_y} \tau_3 \sigma_0 \\
- \left[ \frac{t}{2} (\delta_{i_y,j_y-1} + \delta_{i_y,j_y+1}) \right] \tau_3 \sigma_0 \\
+ \left[ h_{i_x}^2 \tau_3 \sigma_x + h_{i_y}^2 \tau_3 \sigma_y + h_{i_x}^2 \tau_0 \sigma_y \right] \\
+ \Delta_{i_x,j_y} i \tau^+ \sigma_y - \Delta_{i_x,j_y}^* i \tau^- \sigma_y \delta_{i_y,j_y}. \]

Here, \( \tau_m \sigma_l = \tau_m \otimes \sigma_l \) and \( \tau^\pm = \frac{1}{2} (\tau_1 \pm i \tau_2) \). \( S \) is Hermitian and can be diagonalized numerically. Note that we are considering a finite size 2D system without any periodic boundary conditions. Diagonalizing the Hamiltonian by introducing a new basis gives

\[ H = H_0 + \frac{1}{2} \sum_{n} E_n \gamma_n^\dagger \gamma_n. \]

The eigenfunctions for \( S \) are

\[
\Phi_n^\dagger = \left[ \phi_{1,n}^\dagger \phi_{2,n}^\dagger \cdots \phi_{N_y,n}^\dagger \right],
\]

where we have defined

\[
\phi_{i_y,n}^\dagger = \left[ \varphi_{(1,i_y),n} \varphi_{(2,i_y),n} \cdots \varphi_{(N_y,i_y),n} \right],
\]

\[
\varphi_{(i_x,i_y),n} = \left[ \nu_{i_x,i_y}^* \omega_{i_x,i_y}^* \chi_{i_x,i_y}^* \right].
\]

The original creation and annihilation operators \( \{ c^\dagger, c \} \) now can be expressed with new quasiparticle operators,

\[
c_{i\tau} = \sum_n \nu_{i,n}^* \gamma_n, \quad c_{i\bar{\tau}} = \sum_n \nu_{i,n} \gamma_n,
\]

\[
c_{i\tau}^\dagger = \sum_n \omega_{i,n}^* \gamma_n, \quad c_{i\bar{\tau}}^\dagger = \sum_n \chi_{i,n} \gamma_n.
\]

Using these, we obtain a self-consistency equation for \( \Delta_i \),

\[
\Delta_i = -U_i \sum_n \nu_{i,n} \omega_{i,n}^* (1 - f(E_n/2)).
\]

The local density of states (LDOS) is the density of states at one site and in our model it can be calculated for \( T = 0 \). The number of charges at site \( i \) is given by

\[
\rho_i = \sum_{\alpha} \langle c_{i\alpha}^\dagger c_{i\alpha} \rangle.
\]

At an arbitrary temperature, the number of charges at site \( i \) is

\[
\rho_i = \int_{-\infty}^{+\infty} N_i(E) f(E) dE.
\]

Here, \( N_i(E) \) is the local density of states at site \( i \) and \( f(E) \) is the Fermi-Dirac distribution with energy \( E \) measured relative the chemical potential. When \( T = 0 \), we know that \( f(E) = 1 \) for \( E < 0 \) and \( f(E) = 0 \) when \( E > 0 \). Therefore, the LDOS takes the form:

\[
N_i(E) = \sum_{n} (|\nu_{i,n}|^2 + |\nu_{i,n}|^2) \delta(E_n/2 - E).
\]

#### III. F-N-F JUNCTION, BRIEFLY REVISITED

The main purpose of this paper is to investigate the indirect exchange coupling between two ferromagnets mediated by a superconductor using a 2D square lattice model. Before considering the superconducting case, it is worth considering briefly a three layer F-N-F structure as shown in Fig. 1. We include this treatment so that the reader can more easily contrast the normal and superconducting cases. We choose a representative set of parameters as \( L_y = 10 \), \( L_x = 2 \), \( \mu_N = 0.8t \), \( \mu_F = 0.9t \), and \( k_B T = 0.01t \). The length of the ferromagnetic part has little influence on the final results in the F-N-F case and also does not change the results qualitatively in the F-S-F case. Therefore, we have chosen a small value for \( L_x \) to reduce the required time of the numerical simulations. Both ferromagnetic contacts have the same exchange field strength and the magnetization is directed along \( \hat{y} \) (\( |h_{i_y}^L| = |h_{i_y}^R| = h_i \)). As the length of the normal part increases, the amplitude of the well-known RKKY-like oscillations in the F-N-F structure decreases as shown in Fig. 2. These oscillations indicate a switching between P
and AP configurations as the ground-state of the junction: $J > 0$ corresponds to an AP configuration, while $J < 0$ corresponds to a P configuration.

![FIG. 2: The indirect exchange interaction $J$ between the two ferromagnetic contacts mediated by a normal material (F-N-F structure) when $L_y = 10, L_xF = 2, \mu_N = 0.8t, \mu_F = 0.9t, |h_y^L| = |h_y^R| = h_i = 1t$ and $k_B T = 0.01t$.]

![FIG. 3: Indirect exchange interaction as a function of the exchange field strength $h_i$ for several different normal region lengths. The oscillations stem from the fact that the eigenstates for the quasiparticle excitations in the system interfere constructively or destructively at the ferromagnetic contacts, depending on the length $L_xN$ and the exchange field $h_i$ since both these quantities determine the phase-change of an eigenstate as one moves across the normal metal.]

Fig. 2 shows $J$ as a function of the exchange field strength in the ferromagnets ($h_i$) for several different normal region lengths. It demonstrates that $J$ not only oscillates as a function of $L_xN$, but also as a function of $h_i$. The oscillations stem from the fact that the eigenstates for the quasiparticle excitations in the system interfere constructively or destructively at the ferromagnetic contacts, depending on the length $L_xN$ and the exchange field $h_i$ since both these quantities determine the phase change of an eigenstate as one moves across the normal metal.

Due to the oscillations for small exchange field strengths, $J$ monotonically decreases when $h_i$ becomes sufficiently large. This decay is likely related to the depletion in the number of available states around the Fermi level in the ferromagnetic part as shown in Fig. 4.

![FIG. 4: Local density of states (LDOS) for the (2,2) site inside the left ferromagnet as a function of energy for 4 different $h_i$. Here, $L_y = 10, L_xF = 2, L_xS = 8, \mu_N = 0.8t, \mu_F = 0.9t, k_B T = 0.01t$. The dashed box indicates an area around the Fermi energy.]

![FIG. 5: Indirect exchange interaction as a function of the exchange field strength of the ferromagnets for the F-N-F structure). Here, $L_y = 10, L_xF = 2, \mu_N = 0.8t, \mu_F = 0.9t$ and $k_B T = 0.01t$. The superconducting gap $\Delta$ tends to zero for a short superconductor with a weak superconducting interaction $U$. For the case of $U/t = 1$, $J$ is the same as the F-N-F case. This is simply because $\Delta$ is zero for short superconductors. For longer superconductors, the superconducting gap increases and dominates the indirect exchange interaction $J$. For short superconductors, the RKKY-like oscillations are approximately the same as in the F-N-F case because the gap is too small to block any significant fraction of the quasiparticles. For longer superconductors the gap increases and dominates the indirect exchange interaction $J$.

IV. F-S-F: NO FLUX

We now turn to the main topic of this manuscript, namely a study of how the exchange interaction between two ferromagnets is mediated by an $s$-wave type-II superconductor. In Fig. 5 we plot $J$ against the length of the superconducting region for three different on-site pairing interactions $U/t = 1, 1.5$ and 2. The superconducting gap $\Delta$ tends to zero for a short superconductor with a weak superconducting interaction $U$. For the case of $U/t = 1$, $J$ is the same as the F-N-F case. This is simply because $\Delta$ is zero for short superconductors. For longer superconductors, the RKKY-like oscillations are approximately the same as in the F-N-F case because the gap is too small to block any significant fraction of the quasiparticles. For longer superconductors, the superconducting gap increases and dominates the indirect exchange interaction $J$.

Fig. 6(a) and (b) show that for $L_xS = 4$ and $L_xS = 5$, the gap is finite in both the P and AP configuration, but still RKKY-like oscillations dominate as seen in Fig. 5 for $U/t = 1.5$. However, as $L_xS$ increases in (c) and (d), $\Delta$ becomes sufficiently large to block the oscillations caused by quasiparticles. Now, we see that $\Delta_P > \Delta_{AP}$ which leads to $J > 0$, favoring an AP magnetic configuration as the ground state. At first glance, this might seem strange since a larger $\Delta$ in the P configuration should give a larger superconducting condensation energy gain compared to the AP configuration. However, the configuration with the largest gap will also block the largest amount of quasiparticles that can mediate the interaction
between the ferromagnets and lower the free energy. In our numerical simulations, we find that when the gap is large enough in magnitude, it is the latter blocking effect that determines the ground-state of the system. Hence, $\Delta_P > \Delta_{AP}$ causes $J > 0$.

It is often assumed in the literature that the AP configuration in a superconducting spin valve should give the largest superconducting gap. The rationale behind this assumption is that the induced magnetization in the superconducting region of the F-S-F structure is weakest in the AP configuration, leading to the least amount of pair-breaking. However, as we will discuss below, this is a simplified picture which neglects a key process in the spin valve: crossed Andreev reflection. The effect on $\Delta$ of various pair-breaking processes in equilibrium F-S-F structures has been studied previously, but primarily in layers with monoatomic thickness. In Ref. 32, it was stated that $\Delta_P < \Delta_{AP}$ at any temperature for sufficiently large thicknesses. In our work, we instead find that the opposite inequality holds for sufficiently large thicknesses of the superconductor.

For $U/t = 2$ in Fig. 5 one observes a monotonic decrease of $J$ as a function of $L_{xS}$. This behavior occurs both for a strong pairing interaction $U$ or when the exchange field $h_i$ is large. We have already explained why it occurs for strong $U_i$, leading to a large gap. To explain why it occurs for a large exchange field, we consider the behaviour of $\Delta$ as a function of exchange field strength: this is shown in Fig. 7 for $U/t = 1$ and $U/t = 1.5$.

In both cases, for large enough $h_i$, there exists a specific $h_i$ value which marks the transition from $\Delta_P < \Delta_{AP}$ to $\Delta_P > \Delta_{AP}$. The reason for this transition can be explained in terms of a competition between the pair-breaking influence of the induced exchange field in the superconductor and inverse crossed Andreev reflection (CAR), which we proceed to explain.

In a superconducting spin valve, a magnetization is induced inside the superconductor. The corresponding induced exchange field is stronger in the case of P orientation than the AP one. As a result of this induced exchange field, the opposite spin electrons in the Cooper pair accumulate different phases in the superconductor, ultimately leading to a loss of phase coherence and Cooper-pair breaking. As the induced exchange field is stronger in the P case, the destructive effect of the ferromagnets is more severe in the P orientation than the AP. If this were the only mechanism, the superconducting gap should always be smaller in the P orientation ($\Delta_P < \Delta_{AP}$).

On the other hand, inverse crossed Andreev reflection is another pair breaking mechanism in competition with the pair breaking effect of the induced exchange field. What happens here is that spin up and down electrons in a Cooper pair move into separate ferromagnets. This is in contrast to the usual proximity effect mediated by local Andreev reflection, where both electrons (and thus the entire pair) leak into a single material. Crossed Andreev reflection is thus a non-local process.
orientation, the electrons tunnel into the spin majority band of the two ferromagnets while in the P orientation one spin goes to a majority band while the other one goes to the minority band of the other ferromagnet.

As the exchange field becomes stronger, CAR becomes less probable to occur in the P configuration since the minority band involved in the process gradually vanishes. In the half-metallic limit, there is no longer any conducting minority band to enable CAR in the P configuration. Therefore, the destructive effect of CAR is stronger in the AP case, thus making the gap smaller ($\Delta_{AP} < \Delta_P$).

The configuration giving the largest $\Delta$ then depends on which of the two described effects that dominates. From Fig. 7, the fact that $\Delta_P$ overtakes $\Delta_{AP}$ in magnitude at a critical value for the exchange field $h_i$ indicates that crossed Andreev reflection dominates in this regime. This reduces the leakage of superconductivity into the ferromagnets, and enhances the gap.

In Fig. 8, we show how $J$ in a superconducting spin valve behaves with respect to exchange field. The interaction between the ferromagnets weakens the longer the superconductor is. Despite of a small region where the P orientation is the ground state, it is clearly seen that AP is mostly the dominating ground state, especially as $h_i$ becomes large. As we mentioned previously, this is as a result of $\Delta_P$ exceeding the magnitude of $\Delta_{AP}$. Similarly to the F-N-F case, for high enough exchange field $h_i$ the number of available conduction electron states near the Fermi level that can become spin-polarized and mediate the interaction monotonically decreases, leading to a corresponding reduction of the indirect exchange interaction.

V. F-S-F: SUPERCONDUCTING VORTICES

We now turn to an investigation of how the presence of vortices in the superconducting region influences the interaction between the ferromagnets in a superconducting spin valve.

A. Theory of Peierls phases

We apply a constant magnetic field along $\hat{z}$ to the system ($B = (0, 0, B_z)$). Therefore, the Hamiltonian is no longer translationally invariant. This is due to the fact that despite uniform magnetic field, the vector potential $A$ in $B = \nabla \times A$ is not translationally invariant. This leads to hopping integrals including a Peierls phase:

$$H_{\text{hop}} = \sum_{\langle ij \rangle, \alpha} \bar{t}_{ij} e^{i\Phi_{ij}} c_{i\alpha}^\dagger c_{j\alpha},$$

where

$$\Phi_{ij} = \frac{2\pi}{\Phi_0} \int_{R_i} A(r) \cdot dr.$$  \hspace{1cm} (22)

Here $\Phi_0 = h/e$ is the flux quantum. In order to simplify the calculations we use the Landau gauge ($\nabla \cdot A = 0$) and therefore set the vector potential to $A = (-By, 0, 0)$. We note that the above form of the Peierls phase is only valid for a weak magnetic field, so that the vector potential only varies very slowly on a lattice constant length-scale. By symmetrizing the above Hamiltonian, we get

$$H_{\text{hop}} = \sum_{\langle ij \rangle, \alpha} \left( \bar{t}_{ij} e^{i\Phi_{ij}} c_{i\alpha}^\dagger c_{j\alpha} - \bar{t}_{ij} e^{-i\Phi_{ij}} c_{i\alpha} c_{j\alpha}^\dagger \right).$$  \hspace{1cm} (23)

B. Results

As we have discussed, for a strong superconductor (high $U$) or strong ferromagnet (high $h_i$), the superconducting gap will block the interaction more efficiently in the P configuration, causing the system to favor the AP
orientation as the ground state. When the gap is sufficiently large, the RKKY-like oscillations are blocked and $J$ monotonically reduces as a function of $L_x S$.

We now apply a magnetic field of the strength $\Phi = \frac{B \times a_y^2}{\Phi_0} = 0.036$ where $a_y$ is the lattice constant in $y$-direction. As shown in Fig. 9, oscillations appear in $J$ versus $L_x S$ when a magnetic field is applied. This can be understood from the fact that in the mixed state of a type-II superconductor, the presence of vortices causes a mixture of normal and superconducting region. Inside a vortex, superconductivity is suppressed. This is shown in the inset of Fig. 11 where we plotted the spatial profile of $|\Delta|$ when $\Phi = 0.036$. $|\Delta|$ goes to zero close to the center of the vortex representing normal state at this region. We have also verified numerically (not shown) that the phase of the superconducting order parameter winds with $2\pi$ as one circulates the vortex, as it should.

![FIG. 9: $J$ as a function of length of the central part ($L_x S$). Monotonic reduction when magnetic flux through the system is zero (blue curve) and oscillations when magnetic flux through the system is non zero (pink curve). Here, $L_y = 10, L_x F = 2, \mu_S = 0.8t, \mu_F = 0.9t, k_B T = 0.01t, U/t = 1.5$ and $h_i = 2t$](image1)

To understand why the oscillations reappear when a flux is applied, we compare the behavior of the average gap value $\langle \Delta \rangle$ (averaged over lattice sites in the $y$-direction) for the case of no magnetic flux $\Phi = 0$ and a finite magnetic flux $\Phi \neq 0$ for several different lengths in Fig. 10. The left column plots are for $\Phi = 0$ while the right column plots are for $\Phi \neq 0$. When there is no flux through the system, $\Delta_P > \Delta_{AP}$ [Fig. 10(a), (c), (e)]. Moreover, the gap is sufficiently large to block out the RKKY-like oscillations. This causes a monotonic reduction of $J$ as the superconductor length is increased, as shown in the blue curve of Fig. 9.

As the flux is applied [Fig. 10(b), (d), (f)], vortices are generated and the oscillations reappear in $J$ versus $L_x S$. The reason for this is the suppression of $|\Delta|$ by the vortices. Due to the reduced gap magnitude, the system gets closer to the F-N-F case where RKKY-like oscillations appear. However, it is seen from Fig. 9 that the oscillations are damped more quickly in the mixed state of the superconductor due to the difference between $\Delta_P$ and $\Delta_{AP}$ quickly becoming negligible [see Fig. 10(f)].

![FIG. 10: Superconducting gap when $L_y = 10, L_x F = 2, \mu_S = 0.8t, \mu_F = 0.9t, k_B T = 0.01t, U/t = 1.5$ and $h_i = 2t$. Left column is for zero flux $\Phi = 0$, while the right column is for finite flux $\Phi \neq 0$. The specific parameter values are (a) $L_x S = 2$ and $\Phi = 0$ (b) $L_x S = 2$ and $\Phi = 0.036$ (c) $L_x S = 5$ and $\Phi = 0$ (d) $L_x S = 5$ and $\Phi = 0.036$ (e) $L_x S = 8$ and $\Phi = 0$ (f) $L_x S = 8$ and $\Phi = 0.036$](image2)

The discussion so far has been for a specific magnitude of flux through the system. For instance, it is seen from Fig. 9 that for a flux $\Phi = 0.036$, $J$ is practically zero when the superconductor has a length $L_x S = 10$. As one varies the flux at a fixed length, $J$ in general oscillates. We show this in Fig. 11 where the preferred ground-state changes rapidly between a P and AP alignment as a function of $\Phi$. This occurs both in the F-N-F and F-S-F case, but the distinction between the two systems is most clear for the smallest values of $\Phi$. For such values, the superconducting gap is not yet strongly suppressed due to the formation of vortices. As the flux increases and vortices appear, the average gap $\langle \Delta \rangle$ is suppressed and the F-S-F case becomes similar to the F-N-F case. This corresponds to the region for larger fluxes in Fig. 11 where $J$ behaves the same in both systems.
we consider the impurity-averaged exchange interaction $J$ over a large set of different impurity configurations. We define $Z$ as the number of impurity configurations that we have averaged over. The Hamiltonian of the system including impurity scattering is as follows

$$H = -\sum_{i,j} t_{ij}(A)c_{i\alpha}^\dagger c_{j\alpha} + \sum_{i,\alpha} (V_i^{\text{imp}} - \mu_i)n_{i\alpha} + \sum_{i,\alpha} (\mathbf{h}_i \cdot \sigma)\delta_{\alpha\beta}c_{i\alpha}^\dagger c_{i\beta} - \sum_i U_i n_{i\uparrow} n_{i\downarrow}. \quad (24)$$

Here $V_i^{\text{imp}}$ is the potential describing the impurity strength at site $i$. In Fig. 12 we consider $J$ as a function of the number of impurities in the system for $U/t = 1.5$, $h_i = 1t$ and $V_i^{\text{imp}} = 2t$, averaging over $Z = 2000$ configurations. We see that $J$ decays in an oscillatory fashion as the number of impurities randomly placed in the superconductor increases.

To understand the behavior of $J$, we consider both how the magnitude and the LDOS changes for the F-S-F structure when comparing the clean case and the case with impurities. Consider first the case with zero impurities and zero magnetic field, shown in Fig. 13(a) and (b). The LDOS has its minimum value in the middle of structure while $|\Delta_{AP}|$ is maximal at the middle of structure, as expected. When adding impurities, in (c) and (d), $|\Delta_{AP}|$ will tend to zero around the impurity atoms. Their location is marked with white crosses. Interestingly, the average LDOS in the dirty F-S-F case [Fig. 13(d)] has increased in comparison to the clean F-S-F [Fig. 13(b)] case. At first glance, this might indicate that more available quasiparticle states are available to mediate the exchange interaction between the ferromagnets. This should lead to an increase in $J$ compared to the clean case $N_i = 0$. However, Fig. 12 shows the opposite: $J$ is reduced compared to the clean case. We attribute this decrease in $J$ with increasing impurity concentration to an increasing localization of quasiparticles. When the localization increases, the interaction $J$ should be reduced, as seen in Fig. 12.

**VII. CONCLUDING REMARKS**

In conclusion, we have considered the magnetic exchange interaction $J$ in a 2D superconducting spin valve with a type-II s-wave superconductor. We find that the qualitative dependence of $J$ on the separation distance
between the ferromagnets depends on the strength of the superconducting gap and the strength of the exchange field in the ferromagnets. RKKY-like oscillations are observed when the superconducting gap $\Delta$ is small, whereas a monotonic decay is observed when $\Delta$ is larger. In the latter case, the AP configuration is always preferred even though the gap is larger in the P configuration. We explain this in terms of a competition between a proximity-induced pair-breaking magnetization in the superconductor and crossed Andreev reflection.

We also considered the effect on $J$ by a magnetic field which induces vortices in the superconductor on exchange interaction between the ferromagnets. As the gap is suppressed due to the formation of vortices, RKKY-like oscillations reappear in $J$. For a fixed length of the system, the preferred ground-state oscillates rapidly as a function of the applied flux. Finally, we considered impurity scattering and found that $J$ decreases with increasing impurity concentration.

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