Edge Expansion of Cubical Complexes

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Abstract

In this paper we show that graphs of “neighbourly” cubical complexes – cubical complexes in which every pair of vertices spans a (unique) cube – have good expansion properties, using a technique based on multicommodity flows. By showing that graphs of stable set polytopes are graphs of neighbourly cubical complexes we give a new proof that graphs of stable set polytopes have edge expansion 1.

1 Introduction

0/1-polytopes arise naturally in a great variety of contexts, yet only few combinatorial properties are well understood (for an introduction see [11]). Special attention was paid to polytope classes that correspond to some key combinatorial optimization problems.

One of these properties is the edge expansion of the graph of 0/1-polytopes, where the edge expansion $\chi(G)$ of a graph $G = (V, E)$ is defined as

$$\chi(G) = \min_{X \subseteq V} \frac{|\delta(X)|}{|X|}$$

where $\delta(X)$ is the set of edges with one endpoint in $X$ and the other endpoint in $V \setminus X$. Mihail and Vazirani conjectured $\chi(G) \geq 1$ for all graphs arising as 1-skeleton of 0/1-polytopes [7][9]. This lower bound was verified

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for a number of polytope classes, among them matroid basis polytopes of balanced matroids [4] and graphs of stable set polytopes [7][9]. Lower bounds on the expansion of the 1-skeleton of polytopes are of particular interest because they correspond to upper bounds on the mixing time of the natural random walk on the graph of the polytope (where a transition between two vertices has positive probability if and only if the vertices are connected by an edge). This was successfully used to obtain algorithms for random sampling and approximate counting of the vertices in the case of balanced matroids [5].

In this paper we will consider the edge expansion of neighbourly (abstract) cubical complexes:

**Definition 1.1.** An (abstract) cubical complex is a pair $K = (V, C)$ of a set $V$ and a subset $C$ of the power set of $V$ (“cubes”) such that

1. $c_1, c_2 \in C \Rightarrow c_1 \cap c_2 \in C$

2. For every $c \in C$ there is a $d \geq 0$ and a bijection $\phi_c$ between $\{0, 1\}^d$ and $c$ such that the image $\phi_c(s)$ of a subset $s \subset \{0, 1\}^d$ is a member of $C$ if and only if $s$ is a face of $\{0, 1\}^d$.

A graph $G = (V, E)$ is the graph of an abstract cubical complex if there is a cubical complex $(V, C)$ such that $\{v_1, v_2\} \in E \Leftrightarrow \{v_1, v_2\} \in C$.

A cubical complex is **neighbourly** if for every pair of vertices $v, w$ there is a cube $c \in C$ such that $v$ and $w$ are vertices in $c$.

If $G$ is the graph of a cubical complex then the induced subgraph $G[c]$ is a cube graph for every $c \in C$. Cubes have edge expansion 1, which extends to neighbourly cubical complexes:

**Theorem 1.2.** Let $G = (V, E)$ be the graph of a neighbourly cubical complex $K = (V, C)$. Then the edge expansion of $G$ satisfies $\chi(G) \geq 1$.

We will prove Theorem 1.2 in Section 2.

In Section 3 we will turn to the structure of stable set polytopes, where the stable set polytope $\text{STAB}(G)$ of a graph $G = (V, E)$ is defined as the convex hull of the incidence vectors of all stable sets in $G$. $\text{STAB}(G)$ Every stable set polytope $\text{STAB}(G)$ corresponds to a neighbourly cubical complex $(V(G_{\text{STAB}}), C)$ in a very natural way, which yields a new proof for the edge expansion of stable set polytopes. ($G_{\text{STAB}}$ is the graph of $\text{STAB}(G)$.)

However, the natural random walk on stable set polytopes cannot be used to construct an approximate counting algorithm for the number of stable sets
of a graph $G$: Even though there exist fully polynomial approximation algorithms if $G$ has maximal degree up to 4 using a different random walk 

Dyer, Frieze and Jerrum \cite{2} proved that it is not possible to approximate the number of stable sets in polynomial time if the degree of $G$ is larger than 25, unless RP=NP. Since $\chi(G_{\text{STAB}}) \geq 1$ for any graph $G$ we know that a random walk on this graph would be mixing in time polynomial in $\log(|V(G_{\text{STAB}})|)$ and it follows from the previous remarks that it is not possible to use this neighbourhood structure of $G_{\text{STAB}}$ for a rapidly mixing random walk – which isn’t surprising since the number of neighbours of a stable set (in this neighbourhood structure) can be exponential in the size of $G$, which makes it difficult to design a random walk $X$ that converges to the uniform distribution in polynomial time.

## 2 Edge Expansion of cubical complexes

In order to find a lower bound on the edge expansion of neighbourly cubical complexes we use a flow method that may be viewed as a generalization of canonical paths (see \cite{6}) and was made explicit in \cite{7} \cite{10}: Define for each ordered pair of vertices \(v, w \in V\) a flow $f_{vw}: E \to \mathbb{R}$ that sends one unit of flow from $v$ to $w$. We obtain the desired result by giving an upper bound on the total directed flow $F := \sum_{v, w \in V} f_{vw}$ on every edge: Suppose the maximal directed flow on any edge is $\mu$. For every subset $X \subset V$, $|X| \leq \frac{1}{2}|V|$ we have by definition a total flow from $X$ to $V \setminus X$ of at least $|X||V \setminus X|$. Since every cut edge contributes at most $\mu$ to this flow we have $|\delta(X)||\mu \geq |X||V \setminus X|$ and therefore

$$\frac{|\delta(X)|}{|X|} \geq \frac{|V \setminus X|}{\mu} \geq \frac{|V|}{2\mu}.$$  

In particular, if $\mu \leq \frac{1}{2}|V|$ then $\chi(G) \geq 1$. 

![Figure 1: $P_3$ and the graph of $G_{\text{STAB}}(P_3)$](image)
It will be convenient to consider $G$ as a directed graph (every undirected edge is replaced by two directed edges), so we just have to give a bound for the maximal flow on any directed edge.

Since $G$ is the graph of a cubical complex there is a canonical way to define a flow $f_{vw}$ between two vertices $v, w$ that are in some cube $c \in \mathcal{C}$: Let $c_{vw} \in \mathcal{C}$ be the subcube spanned by $v$ and $w$ (so $v$ and $w$ are antipodal in $c_{vw}$) and distribute the flow equally over all shortest paths from $v$ to $w$ in the (directed) cube graph $G'[c_{vw}]$.

Now Theorem 1.2 follows easily from a lemma that may be interesting in its own right:

**Lemma 2.1.** Let $K = (V, \mathcal{C})$ be any abstract cubical complex and $e$ a 1-dimensional cube of the complex. Let $C_e = \{ c \in \mathcal{C} \mid e \subset c \}$ be the set of cubes that contain $e$. Define a flow $f_{vw}$ as described above for every pair of vertices $v, w \in V$ that span a cube in $\mathcal{C}$ and let $F = \sum_{v, w \in V} f_{vw}$ (where $f_{vw}$ is zero if there is no cube $c$ such that $v, w \in c$). Then

$$F(e) = |C_e| \leq \frac{1}{2}|V|.$$

**Proof:** First observe that for any $d$-dimensional cube $c \in \mathcal{C}$ the flow (on directed edges)

$$f_c(e) := \sum_{v, w \in c} f_{vw}(e)$$

induced by antipodal pairs of vertices in $c$ is sent using edges of $G'[c]$ and adds up to 1 for every edge $e \in G'[c]$. To see this, check that there are $2^{d-1}$ antipodal pairs of vertices and every pair generates a total flow of $2d$, so the flow on each of the $d2^d$ edges is 1. So we have

$$F(e) = \sum_{c \in C_e} f(c) = |C_e|.$$

But the number of cubes in $C_e$ is limited by $\frac{1}{2}|V|$: Given any $c \in C_e$ there exists an unique edge $e_a(c)$ that is antipodal to $e$ in $c$. Since $e$ and $e_a(c)$ span $c$ ($c$ is the unique inclusion-minimal cube that contains both edges) we know that different cubes $c, \hat{c}$ containing $e$ are mapped to vertex-disjoint different edges $e_a(c), e_a(\hat{c})$. So the lemma follows from the fact that there exist $|C_e|$ pairwise vertex-disjoint edges in $\mathcal{C}$.

**Proof of Theorem 1.2:** Since every pair of vertices spans a cube, the flow constructed in Lemma 2.1 satisfies that every vertex sends one commodity of flow to every other vertex. The flow on any edge $e$ is at most $\frac{1}{2}|V|$ and therefore the edge expansion of $G$ is at least 1.
3 Graphs of stable set polytopes

The idea of using cubic subgraphs for expansion of 0/1 polytopes was used by Milena Mihail in [9] to show that the perfect matching polytope of any graph has edge expansion 1. Volker Kaibel generalized this concept in [7] to polytopes with “cube-spanned walls” which includes stable set polytopes and matching polytopes. The proof presented here is similar but exploits the fact that the graph of a stable set polytope is the graph of a neighbourly cubical complex.

Lemma 3.1. For any graph \( G \) there exists a neighbourly cubical complex \( K = (V(G_{\text{stab}}), C) \) such that \( G_{\text{stab}} \) is the graph of \( K \).

Every arbitrary graph \( H = (V_H, E_H) \) corresponds to the trivial cubical complex \( K_H = (V_H, C) \) with \( C = V_H \cup \{ \{x, y\} | (x, y) \in E_H \} \). This complex may be extended to a neighbourly cubical complex if \( H \) can be covered by a set of suitable cube graphs that is closed under taking intersections. Figure 2 shows the graph of the 4-dimensional polytope \( P \) arising as the product of a triangle and a square quadrilateral. This graph on 12 vertices that can be decomposed as the union of the 3-dimensional cubes induced by the vertices \( \{1, 2, 5, 6, 7, 8, 11, 12\} \), \( \{2, 3, 4, 5, 8, 9, 10, 11\} \) and \( \{1, 3, 4, 6, 7, 9, 10, 11\} \). By adding these three cubes and all their faces to the trivial cubical complex we obtain a neighbourly cubical complex. Note that \( P \) is the stable set polytope.
of the simple graph on 4 vertices with exactly one edge.

For arbitrary stable set polytopes we can explicitly construct the cubes for all pairs of vertices:

**Proof of Lemma 3.1**: For \( \tilde{V} \subset V \) let \( G[\tilde{V}] \) be the induced subgraph on the vertices \( \tilde{V} \).

We will define a neighbourly cubical complex such that \( G_{\text{STAB}} \) is the graph of this complex.

Chvátal [1] proved that vertices corresponding to stable sets \( s, t \) are adjacent in \( G_{\text{STAB}} \) if and only if \( G[s \triangle t] \) is connected. (We need only the “if” part in this proof.) Fix arbitrary stable sets \( s, t \) and let \( A_1, \ldots, A_k \subseteq V \) be the vertex sets of the connected components of the bipartite graph \( G[s \triangle t] \). In this graph we can choose \( 2^k \) maximal independent sets by choosing independently for each \( j \) either all the vertices of \( A_j \cap s \) or all the vertices of \( A_j \cap t \). Any pair of these independent sets is connected by an edge of \( G_{\text{STAB}} \) if and only if they differ in exactly one component, so the subgraph induced by these \( 2^k \) sets is the graph of a \( k \)-dimensional cube and the faces of the cube correspond to the stable sets where we fix the choice in a number of components.

![Figure 3: The bipartite subgraph defined by two stable sets](image.png)

More formally, we take the set of these \( 2^k \) stable sets \( c_{st} \) as a cube in our cubical complex by defining

\[
c_{st} := \{ s \triangle \left( \bigcup_{j \in J} A_j \right) \mid J \subseteq [k] \}
\]
where \([k] = \{1, \ldots, k\}\). By construction any pair of stable sets defines a cube, so we can define the neighbourly cubical complex \(K = (V(G_{STAB}), C)\) where \(C = \{c_{st}\vert s, t\ \text{stable sets of } G\}\), provided that the intersection of two cubes \(c_1, c_2 \in C\) is again a cube in \(C\). But this is the case:

Let \(s, t\) be as above, set \(A := \{A_1, \ldots, A_k\}\) and pick another arbitrary pair of stable sets \(\tilde{s}, \tilde{t}\) with sets \(\tilde{A} := \{\tilde{A}_1, \ldots, \tilde{A}_k\}\) such that \(G[\tilde{A}_j]\) are the connected components of \(G[\tilde{s} \triangle \tilde{t}]\) for \(j \leq \tilde{k}\). Assume that the intersection of the cubes \(c_{st}\) and \(c_{\tilde{s}\tilde{t}}\) contains at least one stable set \(v\) (otherwise there is nothing to show). Let \(l\) be the number of sets \(A_j \in A\) that are identical to some \(\tilde{A}_j \in \tilde{A}\) (possibly \(l = 0\)) and assume w.l.o.g. \(A_j = \tilde{A}_j\) for all \(j \leq l\). We claim that

\[
    c_{st} \cap c_{\tilde{s}\tilde{t}} = \{v \triangle \left( \bigcup_{i \in I} A_i \right) \mid I \subset [l] \}
\]

which is a cube in \(C\) since it is spanned by the stable sets \(v\) and \(v \triangle \left( \bigcup_{i \in I} A_i \right)\). Clearly \(v \triangle \left( \bigcup_{i \in I} A_i \right) \in c_{st} \cap c_{\tilde{s}\tilde{t}}\) for every \(I \subset [l]\), so we have to show that for any \(w \in c_{st} \cap c_{\tilde{s}\tilde{t}}\) there exists an index set \(I \subset [l]\) such that \(w = v \triangle \left( \bigcup_{i \in I} A_i \right)\).

By definition we have sets \(J \subset [k], \tilde{J} \subset [\tilde{k}]\) such that

\[
    w = v \triangle \left( \bigcup_{j \in J} A_j \right) = v \triangle \left( \bigcup_{j \in \tilde{J}} \tilde{A}_j \right)
\]

But since the \(A_j\) (and \(\tilde{A}_j\) respectively) induce disconnected regions of \(G\), the condition \(\bigcup_{j \in J} A_j = \bigcup_{j \in \tilde{J}} \tilde{A}_j\) implies that there is a bijection \(\phi : J \to \tilde{J}\) such that \(A_j = \tilde{A}_{\phi(j)}\) for all \(j \in J\), therefore \(J \subset [l]\).

So \(G_{STAB}\) is the graph of the complex \(V, C\) and by Theorem 1.2

\[
    \chi(G_{STAB}) \geq 1.
\]

Remark: Let \(s, t, k\) and \(A_j\) for \(j \leq k\) be defined as above. Then the vertices of \(c_{st}\) do span a face \(F_{st}\) of \(STAB(G)\) (see [7], the face is defined by the equations \(x_j = 0\) for \(j \in V \setminus (s \cup t)\) and \(x_i = 1\) for \(i \in s \cap t\)). However, by choosing all vertices in \(s \cup t\) and adding suitable subsets of \(A_j \cap s\) or \(A_j \cap t\) for each \(j \leq k\) we can construct stable sets that are contained in \(F_{st}\) but not in \(c_{st}\), so the complex \((V(G_{STAB}), C)\) is not a subcomplex of \(STAB(G)\).

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