CONVERGENCE TO NONLINEAR DIFFUSION WAVES FOR A HYPERBOLIC-PARABOLIC CHEMOTAXIS SYSTEM MODELLING VASCULOGENESIS

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Abstract. In this paper, we are concerned with a quasi-linear hyperbolic-parabolic system of persistence and endogenous chemotaxis modelling vasculogenesis in $\mathbb{R}$. Under some suitable structural assumption on the pressure function, we first predict the system admits a nonlinear diffusion wave in $\mathbb{R}$ based on the empirical results in the literature. Then we show that the solution of the concerned system will locally and asymptotically converges to this nonlinear diffusion wave if the wave strength is small. By using the time-weighted energy estimates, we further prove that the convergence rate of the nonlinear diffusion wave is algebraic.

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1. Introduction

In order to depict the key characteristics of the in vitro experiment of blood vessels, showing that the cells randomly scattered on the gel matrix will automatically organize into a network of connected blood vessels, Ambrosi et al. in [1, 13] proposed the following quasi-linear hyperbolic-parabolic chemotaxis system

$$\begin{cases}
\partial_t \rho + \nabla \cdot (\rho u) = 0, \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p(\rho) = \mu \rho \nabla \phi - \alpha \rho u, \\
\partial_t \phi = D \Delta \phi + a \rho - b \phi.
\end{cases}$$  (1.1)

Here the unknowns $\rho = \rho(x,t) \geq 0$ and $u = u(x,t) \in \mathbb{R}^n (n \geq 1)$ denote the density and velocity of endothelial cells, respectively, and $\phi = \phi(x,t) \geq 0$ denotes the concentration of the chemoattractant secreted by the endothelial cells. The convection term $\nabla \cdot (\rho u \otimes u)$ models the cell movement persistence (inertial effect), $p(\rho)$ is the cell-density dependent pressure function accounting for the fact that closely packed cells resist to compression due to the impenetrability of cellular matter, the parameter $\mu > 0$ measures the intensity of cell response to the chemoattractant concentration gradient and $-\alpha \rho u$ corresponds to a damping (friction) force with coefficient $\alpha > 0$ as a result of the interaction between cells and the underlying substratum; $D > 0$ is the diffusivity of the chemoattractant, the positive constants $a$ and $b$ denote the secretion and death rates of the chemoattractant, respectively.

The hyperbolic or hyperbolic-parabolic chemotaxis models with different structures from (1.1) have been studied in the literature (cf. [16, 25, 31, 41]). Nevertheless the mathematical structure of (1.1) is analogous to the well-known damped Euler-Poisson system where $\phi$ is called the flow potential satisfying a Poisson (elliptic) equation: $-\Delta \phi = \rho$, which has numerous essential applications such as propagation of electrons in semiconductor devices (cf. [30]) and the transport of
ions in plasma physics (cf. [3]) when \( \mu < 0 \) and \( \alpha \geq 0 \), as well as the collapse of gaseous stars due to self-gravitation [4] when \( \mu > 0 \) and \( \alpha = 0 \). Here the sign of \( \mu \) corresponds to attractive or repulsive forces similar to the attractive or repulsive chemotaxis (cf. [23]). Due to the essential difference of analysis between elliptic and parabolic equations, the analytical tools developed for the damped Euler-Poisson system is not directly applicable to the system (1.1). Up to date, there are not many analytical results available to the system (1.1). When the initial value \((\rho_0(x), u_0(x), \phi_0(x)) \in H^s(\mathbb{R}^n)(s > n/2 + 1)\) is a small perturbation of the constant ground state (i.e. equilibrium) \((\rho_c, 0, \phi_c)\) with \(\rho_c > 0\) sufficiently small, it was shown in [42, 43] that the system (1.1) admits global strong solutions without vacuum converging to \((\rho_c, 0, \phi_c)\) in \(L^2\)-norm with an algebraic decay rate. As \(\alpha \to \infty\) (strong damping), it was formally derived in [3] by the asymptotic analysis and subsequently justified in [12] that the solution of (1.1) converges to that of a parabolic-elliptic Keller-Segel type chemotaxis system. The asymptotic behavior of solutions to (1.1) and its limiting Keller-Segel system was further compared numerically in [35].

A typical form of \(p\) fulfilling (1.2) is \(p(\rho) = \frac{K}{2} \rho^2\) with \(K > \frac{\alpha}{\mu}\). The stationary solutions of one dimensional (1.1) with vacuum (bump solutions) in a bounded interval with zero-flux boundary condition were constructed in [2, 3, 34]. The model (1.1) with \(p(\rho) = \rho\) and periodic boundary conditions in one dimension was numerically explored in [11]. Recently the stability of transition layer stationary solutions of (1.1) on \(\mathbb{R}_+ = [0, \infty)\) was established in [15].

It is well known that damping is a factor triggering diffusion waves in many hyperbolic system such as the \(p\)-system or damped Euler equations (cf. [17, 21, 22, 28, 32, 33, 36, 39, 46] without vacuum and [7, 18, 29, 40] with vacuum), as well as the Euler-Poisson system of semiconductors [14], bipolar Euler-Maxwell equation [8], Timoshenko system [20] and the radiating gas model [27]. Motivated by the structural analogue between the Euler-Poisson system and (1.1) with dampings, the authors have shown recently in [26] that the hyperbolic-parabolic system (1.1) admits linear diffusion waves in \(\mathbb{R}^3\) which are locally asymptotically stable by the Fourier and spectral analysis. Under the framework of [26], the time decay of solutions decreases with respect to the space dimension and the energy estimates will lose time integrability in \(\mathbb{R}^n(n = 1, 2)\). Therefore whether the hyperbolic-parabolic chemotaxis system (1.1) admits stable diffusion waves in \(\mathbb{R}\) or \(\mathbb{R}^2\) remains unknown. The purpose of this paper is to show that the system (1.1) in \(\mathbb{R}\) admits nonlinear diffusion waves which are stable against a small perturbation by using the technique of taking antiderivative, unlike the framework of [26].

To demonstrate our ideas, we set \(m = \rho u\) for convenience, namely \(m\) denotes the momentum of cells, and recast the system (1.1) for \((x, t) \in \mathbb{R} \times \mathbb{R}^+\) as

\[
\begin{align*}
\rho_t + m_x &= 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
m_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_x &= \mu \rho \phi_x - \alpha m, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
\phi_t &= D \phi_{xx} + a \rho - b \phi, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
\end{align*}
\]  

where we prescribe the initial data

\[
U|_{t=0} = U_0 := (\rho_0, m_0, \phi_0) \longrightarrow (\rho_\pm, 0, \phi_\pm), \quad \text{as} \quad x \to \pm \infty, 
\]

with \(\rho_+ > 0, \rho_- \neq \rho_+ \) and \(\phi_\pm = \frac{\pm}{2} \rho_\pm\), where we assume \(\rho_\pm > 0\) to avoid possible vacuum. Moreover we impose the following conditions on the pressure function

\[
p(\rho) \in C^3(\mathbb{R}^+), \quad p'(\rho) - \frac{\alpha \mu}{b} \rho > 0, \quad \text{for any} \quad \rho > 0. 
\]
Due to the external damping (frictional) force, one may expect that the inertial term in the momentum equation of (1.3) decays to zero faster than other terms so that the pressure gradient force is balanced by the frictional force plus the potential force. Hence we may predict that the solution \((\rho, m, \phi)\) of (1.3)", which converges to the shifted diffusion wave \((\bar{\rho}, \bar{m}, \bar{\phi})\) as \(x \to \pm \infty\).

\[
\begin{aligned}
\rho_t + m_x &= 0, \\
(p(\bar{\rho}))_x &= \mu \bar{\rho} \bar{\phi}_x - \alpha \bar{m}, \\
a \bar{\rho} &= b \bar{\phi},
\end{aligned}
\]  
(1.6)

or equivalently, by denoting \(q(\bar{\rho}) = p(\bar{\rho}) - \frac{a \mu}{2b} \bar{\rho}^2\),

\[
\begin{aligned}
\bar{\rho}_t - \left(\frac{1}{\alpha} q(\bar{\rho})\right)_{xx} &= 0, \\
\bar{m} &= -\frac{1}{\alpha} \left[p(\bar{\rho}) - \frac{a \mu}{2b} \bar{\rho}^2\right]_x, \\
\bar{\phi} &= \frac{a}{b} \bar{\rho},
\end{aligned}
\]  
(1.7a)

(1.7b)

with the following asymptotic states at far fields

\[
\begin{pmatrix}
\bar{\rho}(x, t), \\
\bar{m}(x, t), \\
\bar{\phi}(x, t)
\end{pmatrix} \to \begin{pmatrix}
\rho_{\pm}, 0, \\
0, \\
0
\end{pmatrix}, \quad \text{as} \quad x \to \pm \infty.
\]  
(1.8)

In our paper, without loss of generality, we assume \(\rho_- < \rho_+\). As shown in [10], the equation (1.7a) admits unique nonlinear diffusion wave \(\bar{\rho}(x, t) = \varphi(\xi)\) with \(\xi = \frac{x}{\sqrt{1 + t}}\) under the condition (1.5) such that \(\varphi(\xi) = \rho_\pm\). Then substituting this into (1.7b) and (1.7c), we find a unique solution \((\bar{\rho}, \bar{m}, \bar{\phi})(x, t)\) of diffusion wave for (1.7) with

\[
\bar{\rho}(x, t) = \varphi \left(\frac{x}{\sqrt{1 + t}}\right), \quad \bar{m}(x, t) = -\frac{1}{\alpha} \left[p(\bar{\rho}) - \frac{a \mu}{2b} \bar{\rho}^2\right]_x, \quad \bar{\phi}(x, t) = \frac{a}{b} \bar{\rho}.
\]  
(1.9)

The aim of this paper is to show if the initial value \((\rho_0, m_0, \phi_0)\) satisfying (1.4) is a small perturbation of \((\bar{\rho}(x, 0), \bar{m}(x, 0), \bar{\phi}(x, 0))\), then the system (1.3) with (1.5) admits a unique solution whose asymptotic profile is the nonlinear diffusion wave \((\bar{\rho}, \bar{m}, \bar{\phi})(x, t)\) given by (1.9). To state our main results, we first define the perturbation of \((\bar{\rho}(x, 0), \bar{m}(x, 0), \bar{\phi}(x, 0))\) as follows

\[
\begin{aligned}
V_0(x) &= \int_{-\infty}^x (\rho_0(y) - \bar{\rho}(y + x_0, 0)) dy, \\
M_0(x) &= m_0(x) - \bar{m}(x + x_0, 0), \\
\Phi_0(x) &= \phi_0(x) - \bar{\phi}(x + x_0, 0),
\end{aligned}
\]  
(1.10)

where \(x_0\) is a constant uniquely determined (see section 2.2) such that the initial perturbation from the spatially shifted diffusion waves with shift \(x_0\) is of integral zero, namely \(V_0(\infty) = 0\). Above we define \(V_0(x)\) in a form of anti-derivative of the perturbation because the first equation of (1.3) is a conservation law for which the technique of taking anti-derivative is usually invoked (cf. [44]). By the method of weighted energy estimates, we shall prove the following results in this paper.

**Theorem 1.1.** Let (1.5) hold. Then there exists a constant \(\epsilon > 0\), such that if \((V_0, M_0, \Phi_0) \in H^3(\mathbb{R}) \times H^2(\mathbb{R}) \times H^4(\mathbb{R})\) satisfies

\[
\|V_0\|_{H^3(\mathbb{R})}^2 + \|M_0\|_{H^2(\mathbb{R})}^2 + \|\Phi_0\|_{H^4(\mathbb{R})}^2 + |\rho_+ - \rho_-| \leq \epsilon^2,
\]

where \((V_0, M_0, \Phi_0)\) is defined in (1.10), the system (1.3) - (1.4) possesses a unique global classical solution \((\rho, m, \phi)(x, t)\) which converges to the shifted diffusion wave \((\bar{\rho}, \bar{m}, \bar{\phi})(x + x_0, t)\) solving (1.7).
and (1.8) in $L^\infty(\mathbb{R})$ with algebraic decay rates:

$$
\| \partial_x^k (\rho - \bar{\rho}) (t) \|_{L^\infty(\mathbb{R})} \leq C \epsilon (1 + t)^{-3/4 - k/2}, \quad k = 0, 1, \\
\| \partial_x^k (m - \bar{m}) (t) \|_{L^\infty(\mathbb{R})} \leq C \epsilon (1 + t)^{-5/4 - k/2}, \quad k = 0, 1, \\
\| \partial_x^k (\phi - \bar{\phi}) (t) \|_{L^\infty(\mathbb{R})} \leq C \epsilon (1 + t)^{-3/4 - k/2}, \quad k = 0, 1,
$$

(1.11)

where $C > 0$ is a constant independent of $t$.

The rest of the paper is organized as follows. In Section 2, we present some known results on the diffusion wave solution $\bar{\rho}(x, t)$ of (1.7) with far field states (1.8), and reformulate the original equation (1.3) against a suitable perturbation. In Section 3, we derive the uniform $a$-priori estimates and hence establish the existence of global solutions of reformulated problem. In Section 4, we show the algebraic time asymptotic rate of solutions convergent to the nonlinear diffusion wave and prove the main Theorem 1.1.

2. Reformulation of the problem

In this section, we shall prove the global existence of solutions to (1.6). We first introduce some notations frequently throughout the paper.

Notations. In the sequel, $C$ denotes a generic positive constant where $C$ may vary in the context. For two quantities $a$ and $b$, $a \sim b$ means $\lambda a \leq b \leq \frac{1}{\lambda} a$ for some constant $0 < \lambda < 1$. For any integer $m \geq 0$, we use $H^m$ to denote the usual Sobolev space $H^m(\mathbb{R})$. For simplicity, the norm of $H^m$ is denoted by $\| \cdot \|_m$ with $\| \cdot \| = \| \cdot \|_0$, and we set

$$
\|[A, B]\|_X = \|A\|_X + \|B\|_X.
$$

Without confusion, we shall abbreviate $\| \cdot \|_{L^p(\mathbb{R})}$ as $\| \cdot \|_{L^p}$ for $1 \leq p \leq \infty$ in the sequel.

2.1. Nonlinear diffusion waves. In this subsection, we give some properties of nonlinear diffusion wave profile (1.9) that will be used in the paper. Let

$$
\bar{\rho}(x, t) = \varphi \left( \frac{x}{\sqrt{1 + t}} \right) = \varphi(\xi), \quad -\infty < \xi < \infty.
$$

Substituting the above self-similar structure form into (1.7a), we find that $\varphi(\xi)$ satisfies

$$
\begin{cases}
\varphi''(\xi) + \frac{\alpha \xi + q''(\varphi(\xi))\varphi'(\xi)}{q'(\varphi(\xi))}\varphi'(\xi) = 0, \\
\varphi(\pm \infty) = \rho_{\pm},
\end{cases}
$$

where $q'(\bar{\rho}) = p'(\bar{\rho}) - \frac{\alpha b}{b} \bar{\rho} > 0$. The existence of unique solution of the above equations has been shown in [17]. For any $\xi_0$, it further follows that

$$
\varphi'(\xi) = \frac{\varphi'(\xi_0)q'(\varphi(\xi_0))}{q'(\varphi(\xi))} \exp \left( - \int_{\xi_0}^{\xi} \frac{\alpha \eta}{2q'(\varphi(\eta))} d\eta \right).
$$

The solution $\varphi(\xi)$ is increasing if $\rho_- < \rho_+$ and decreasing if $\rho_- > \rho_+$, and satisfies

$$
\sum_{k=1}^{6} \frac{d^k}{d\xi^k} \varphi(\xi) + |\varphi(\xi) - \rho_+|_{\xi > 0} + |\varphi(\xi) - \rho_-|_{\xi < 0} \leq C|\rho_+ - \rho_-| \exp(-c\alpha \xi^2),
$$

where $c$ is a positive constant independent of $x$ and $t$. Moreover, we can obtain following $L^p$-estimates of the derivatives of $\varphi$ (cf. [13, 19, 45, 46]).
Lemma 2.1. Let \( \bar{\rho}(x,t) \) be the self-similar solution of (1.7a) and (1.8). Then for \( p \in [2, +\infty] \) and \( 1 \leq l + k \leq 6 \), we have

\[
\left\| \partial_l^k \bar{\rho}(\cdot, t) \right\|_{L^p} \leq C|\rho_+ - \rho_-|((1 + t)^{-\frac{1}{2} - \frac{l}{2}} + \frac{1}{2p}).
\]

2.2. Reformulation of the problem. Inspired by the work [17], we set the perturbation function around the diffusion wave \( (\bar{\rho}, \bar{m}, \bar{\phi}) \) as

\[
\begin{align*}
V(x, t) &= f_{-\infty}^{x}(\rho(y, t) - \bar{\rho}(y + x_0, t))dy, \\
M(x, t) &= m(x, t) - \bar{m}(x + x_0, t), \\
\Phi(x, t) &= \phi(x, t) - \bar{\phi}(x + x_0, t),
\end{align*}
\]

where \( x_0 \) is a constant uniquely determined by

\[
\int_{-\infty}^{+\infty} (\rho(x, 0) - \bar{\rho}(x + x_0, 0))dx = 0,
\]

namely,

\[
x_0 = \frac{1}{\rho_+ - \rho_-} \int_{-\infty}^{+\infty} (\rho_0(y) - \bar{\rho}(y, 0)) dy.
\]

Define the initial perturbation function as

\[
\begin{align*}
V_0(x) &= V(x, 0) = f_{-\infty}^{x}(\rho_0(y) - \bar{\rho}(y + x_0, 0))dy, \\
M_0(x) &= M(x, 0) = m_0(x) - \bar{m}(x + x_0, 0), \\
\Phi_0(x) &= \Phi(x, 0) = \phi_0(x) - \bar{\phi}(x + x_0, 0).
\end{align*}
\]

Then upon the substitution of (2.1), we reformulate our problem (1.3)-(1.4) as

\[
\begin{align*}
V_t + M &= 0, \\
M_t + \left( \frac{(M + \bar{m})^2}{V_x + \bar{\rho}} \right)_x + [p(V_x + \bar{\rho}) - p(\bar{\rho})]_x &= \mu V_x \Phi_x + \mu V_x \bar{\phi}_x + \mu \bar{\rho} \Phi_x - \alpha M - \bar{m}_t, \\
\Phi_t &= D\Phi_{xx} + aV_x - b\Phi - \bar{\phi}_t + D\bar{\phi}_{xx},
\end{align*}
\]

with initial data \((V_0, M_0, \Phi_0)\) satisfying

\[
(V(x, 0), M(x, 0), \Phi(x, 0)) = (V_0(x), M_0(x), \Phi_0(x)) \to 0, \quad \text{as } x \to \pm\infty. \tag{2.3}
\]

Rewrite (2.2)-(2.3) as

\[
\begin{align*}
V_{tt} - (p'(\bar{\rho})V_x)_x + \alpha V_t + \mu V_x \Phi_x + \mu V_x \bar{\phi}_x + \mu \bar{\rho} \Phi_x + h_x + f_x &= 0, \\
\Phi_t &= D\Phi_{xx} + aV_x - b\Phi + g,
\end{align*}
\]

with initial data

\[
(V(x, 0), V_t(x, 0), \Phi(x, 0)) = (V_0(x), -M_0(x), \Phi_0(x)) \to 0, \quad \text{as } x \to \pm\infty, \tag{2.5}
\]

where

\[
\begin{align*}
h &= -\frac{(V_t + \frac{1}{\alpha}q(\bar{\rho})_x)^2}{V_x + \bar{\rho}}, \\
f &= \frac{1}{\alpha}(q(\bar{\rho}))_t - [p(V_x + \bar{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})V_x], \\
g &= -\bar{\phi}_t + D\bar{\phi}_{xx}.
\end{align*}
\]

For system (2.2)-(2.3), we shall establish the following results.
Proposition 2.2. Let $\delta_0 = |\rho_+ - \rho_-|$ and \((1.5)\) hold. If \((V_0, M_0, \Phi_0) \in H^3 \times H^2 \times H^3\) and
\[
||[V_0, \Phi_0]|^2_3 + ||M_0||^2_2 + \delta_0 \leq \varepsilon_0^2,
\]
for some sufficiently small constant $\varepsilon_0 > 0$, then problem \((2.2)-(2.3)\) admits a unique global classical solutions \((V, M, \Phi)\) satisfying
\[
(V(x, t), M(x, t), \Phi(x, t)) \in L^\infty \left([0, \infty); H^3 \times H^2 \times H^3\right),
\]
and
\[
\sup_{t \geq 0} \|V(t)\|_3^2 + \|V_t(t)\|_2^2 + \|\Phi(t)\|_3^2 \leq C \left(\|V_0\|_3^2 + \|M_0\|_2^2 + \|\Phi_0\|_3^2 + \delta_0\right) \leq C \varepsilon_0^2.
\]
Moreover, if there exists a constant $\epsilon$ such that
\[
\delta_0 + \|V_0\|_3^2 + \|M_0\|_2^2 + \|\Phi_0\|_4^2 \leq \epsilon^2,
\]
then the solution \((V, M, \Phi)\) has the following decay
\[
\begin{align*}
||\partial_x^k V_x(t)||_{L^2} &\leq C \epsilon (1 + t)^{-(k+1)/2}, \ k = 0, 1, 2, \\
||\partial_x^k M(t)||_{L^2} &\leq C \epsilon (1 + t)^{-(k+2)/2}, \ k = 0, 1, 2, \\
||\partial_x^k \Phi(t)||_{L^2} &\leq C \epsilon (1 + t)^{-(k+1)/2}, \ k = 0, 1, 2.
\end{align*}
\]
where $C > 0$ is a positive constant independent of $t$.

In view of \((2.1)\), Theorem \((1.1)\) is a direct consequence of Proposition \((2.2)\). Hence we will focus on the proof of Proposition \((2.2)\). Before proceeding, we briefly outline the ideas of proving Proposition \((2.2)\) where part of them are inspired from works \([36, 38]\). First we establish the global existence of smooth solutions to \((2.2)-(2.3)\), and then we derive time decay rates of the solution toward diffusion waves. Although such procedures are routine, the desired results are not easy to be achieved due to the coupling of $\Phi$ and $V$. Since we can not expect the exponential decay of $\Phi$ like the electronic field $E$ as in \([14]\) or do not want to impose smallness assumption on the constant equilibrium as in \([43]\) either, some new ideas need to be developed in order to control the linear terms $\mu \rho \Phi_x$ in the first equation of \((2.1)\) and $a V_x$ in the second equation of \((2.1)\). To this end, we take up the assumption \((1.5)\) which implies that the following matrix
\[
\left(\begin{array}{cc}
p'(\bar{\rho}) & -\mu \bar{\rho} \\
-\mu \bar{\rho} & b \mu \bar{\rho}/a
\end{array}\right) =: A(\bar{\rho}),
\]
is positive definite. Then for any \((x_1, x_2)\), we have
\[
\begin{align*}
\min\{\lambda_1(\bar{\rho}), \lambda_2(\bar{\rho})\} (x_1^2 + x_2^2) \leq p'(\bar{\rho}) x_1^2 - 2\mu \bar{\rho} x_1 x_2 + \frac{b \mu \bar{\rho}}{a} x_2^2 &\leq \max\{\lambda_1(\bar{\rho}), \lambda_2(\bar{\rho})\} (x_1^2 + x_2^2),
\end{align*}
\]
where $\lambda_1(\bar{\rho}) > 0$ and $\lambda_2(\bar{\rho}) > 0$ are the eigenvalues of $A(\bar{\rho})$. Since $\rho_- < \bar{\rho} < \rho_+$, there exist two constants $C_1 > 0$ and $C_2 > 0$ such that
\[
C_1 (x_1^2 + x_2^2) \leq p'(\bar{\rho}) x_1^2 - 2\mu \bar{\rho} x_1 x_2 + \frac{b \mu \bar{\rho}}{a} x_2^2 \leq C_2 (x_1^2 + x_2^2). \tag{2.8}
\]
Under the condition \((2.8)\), the two linear terms $\mu \rho \Phi_x$ and $a V_x$ can be absorbed or eliminated in the energy estimates. Due to the coupling effect, we find that the decay properties of $\Phi$ and $V_x$ can be obtained synchronously. Moreover the term $g$ in the second equation of \((2.1)\) will complicated the weighted energy estimates in the sequel.

For later use, we recall a Sobolev inequality about the $L^p$ estimate on products of any two or several terms with the sum of the order of their derivatives equal to a given integer (cf. \([4]\)).
Lemma 2.3. Let \( \alpha^1 = (\alpha_1^1, \cdots, \alpha_n^1) \) and \( \alpha^2 = (\alpha_1^2, \cdots, \alpha_n^2) \) be two multi-indices with \( |\alpha^1| = k_1, |\alpha^2| = k_2 \) and set \( k = k_1 + k_2 \). Then, for \( 1 \leq p, q, r \leq \infty \) with \( 1/p = 1/q + 1/r \), we have

\[
\|\partial^{\alpha^1} u_1 \partial^{\alpha^2} u_2\|_{L^p(\mathbb{R}^n)} \leq C \left( \|u_1\|_{L^q(\mathbb{R}^n)} \|\nabla^k u_2\|_{L^r(\mathbb{R}^n)} + \|u_2\|_{L^q(\mathbb{R}^n)} \|\nabla^k u_1\|_{L^r(\mathbb{R}^n)} \right),
\]

where \( C \) is a positive constant.

3. Global existence. Proof of Proposition 2.2

The existence of local-in-time solutions to (2.4)-(2.5) can be readily established by the standard iteration argument and hence will be stated without proof details.

Proposition 3.1 (Local existence). Let the conditions of Proposition 2.2 hold. Then there exists a positive constant \( T_0 \) depending on \( \varepsilon_0 \) such that the problem (2.4)-(2.5) admits a unique solution \( (V(x,t), M(x,t), \Phi(x,t)) \in L^\infty \left([0,T_0); H^3 \times H^2 \times H^3 \right) \) satisfying

\[
\sup_{t \in [0,T_0]} \left( \|V\|_3^2 + \|M\|_2^2 + \|\Phi\|_3^2 \right) \leq 2\varepsilon_0^2.
\]

To extend local solutions to be global in time, it suffices to derive the uniform a priori estimates of solutions to (2.4)-(2.5) where \( V_0 = -M \). For any given \( T > 0 \), we denote the solution space for the Cauchy problem (2.4)-(2.5) by

\[
X(T) = \{ (V, V_t, \Phi) \in H^3 \times H^2 \times H^3, \ 0 \leq t \leq T \}.
\]

Assume that the following a priori assumption holds:

\[
N(T) = \sup_{0 < t < T} \{ \|V(t)\|_3 + \|V_t(t)\|_2 \} \ll 1. \tag{3.1}
\]

By the Sobolev inequality and \( \rho_- < \bar{\rho} < \rho_+ \), it follows from (3.1) that

\[
\frac{1}{2}\bar{\rho}^- \leq V_0 + \bar{\rho} \leq \frac{3}{2}\rho_+.
\]

The we prove the following uniform a priori estimate.

Proposition 3.2 (A priori estimate). Let the conditions of Proposition 2.2 hold and \( (V, V_t, \Phi) \in X(T) \) be a smooth solution of (2.4)-(2.5). Then there exists a constant \( C > 0 \) independent of \( t \) such that

\[
\|V\|_3^2 + \|V_t\|_2^2 + \|\Phi\|_3^2 + \int_0^T \left( \|V_x\|_2^2 + \|\Phi_x\|_2^2 + \|V_t\|_2^2 + \|V_{tt}\|_2^2 \right) dt \leq C \varepsilon_0^2. \tag{3.3}
\]

Proof. We divide the proof into three steps.

Step 1. We claim the following inequality hold

\[
\frac{d}{dt} \sum_{0 \leq k \leq 2} \left\{ \frac{\alpha}{2} \|\partial_x^k V\|^2 + \int_R \partial_x^k V \partial_x^k V \, dx + \frac{\mu}{2a} \int_R \bar{\rho} \left( \partial_x^k \Phi \right)^2 \, dx \right\} + \frac{\mu D}{a} \sum_{0 \leq k \leq 2} \int_R \bar{\rho} \left( \partial_x^k \Phi \right)^2 \, dx + \sum_{0 \leq k \leq 2} \left\{ \int_R p'(\bar{\rho}) \left( \partial_x^k V \right)^2 \, dx - 2\mu \int_R \bar{\rho} \partial_x^k \Phi \partial_x^k V \, dx + \frac{b\mu}{a} \int_R \bar{\rho} \left( \partial_x^k \Phi \right)^2 \, dx \right\} \leq -\frac{\mu}{b} \int_R \bar{\rho} x V \, dx + \|V_t\|_2^2 + C(N(T) + \bar{\delta}_0)(\|V_x\|_2^2 + \|V_t\|_2^2 + \|\Phi\|_3^2) + C\delta_0(1 + t)^{-\frac{1}{2}}. \tag{3.4}
\]
To this end, we rewrite (2.4) as

\[
\begin{aligned}
V_t - (p'(\bar{\rho})V_x)_x + \alpha V_t + \mu(\bar{\rho}\Phi)_x &= -\mu V_x \Phi_x - \mu V_x \bar{\phi}_x + \mu \bar{\rho}_x \Phi - h_x - f_x, \\
\Phi_t - D \Phi_{xx} + b \Phi - a V_x &= g.
\end{aligned}
\]  

(3.5)

Applying \(\partial^k_x\) for \(0 \leq k \leq 2\) to the first equation of (3.5) and multiplying the result by \(\partial^k_x V\), one has

\[
\begin{aligned}
\partial^k_x V_t \partial^k_x V &= \frac{d}{dt} (\partial^k_x V_t \partial^k_x V) - |\partial^k_x V_t|^2, \\
-\partial^k_x (p'(\bar{\rho})V_x)_x \partial^k_x V &= - [\partial^k_x (p'(\bar{\rho})V_x)_x \partial^k_x V]_x + \partial^k_x (p'(\bar{\rho})V_x) \partial^k_x V_x \\
&= - [\partial^k_x (p'(\bar{\rho})V_x)_x \partial^k_x V]_x + p'(\bar{\rho}) \left( \partial^k_x V_x \right)^2 + \sum_{\ell < k} C^\ell_k \partial^{k-\ell}_x (p'(\bar{\rho})) \partial^\ell_x V_x \partial^k_x V_x, \\
\mu \partial^k_x (\bar{\rho}\Phi)_x \partial^k_x V &= \mu [\partial^k_x (\bar{\rho}\Phi)_x \partial^k_x V]_x - \mu \partial^k_x (\bar{\rho}\Phi) \partial^k_x V_x \\
&= \mu [\partial^k_x (\bar{\rho}\Phi)_x \partial^k_x V]_x - \mu \bar{\rho} \partial^k_x \Phi \partial^k_x V_x - \mu \sum_{\ell < k} C^\ell_k \partial^{k-\ell}_x \bar{\rho} \partial^\ell_x \Phi \partial^k_x V_x, \\
\mu \partial^k_x (V_x \bar{\phi}_x - \bar{\rho}_x \Phi) \partial^k_x V &= \frac{\mu}{b} \partial^k_x [\bar{\rho}_x (a V_x - b \Phi)] \partial^k_x V,
\end{aligned}
\]  

(3.6)

By simple calculations, we have

\[
\begin{aligned}
\partial^k_x V_t \partial^k_x V &= \frac{d}{dt} (\partial^k_x V_t \partial^k_x V) - |\partial^k_x V_t|^2, \\
-\partial^k_x (p'(\bar{\rho})V_x)_x \partial^k_x V &= - [\partial^k_x (p'(\bar{\rho})V_x)_x \partial^k_x V]_x + \partial^k_x (p'(\bar{\rho})V_x) \partial^k_x V_x \\
&= - [\partial^k_x (p'(\bar{\rho})V_x)_x \partial^k_x V]_x + p'(\bar{\rho}) \left( \partial^k_x V_x \right)^2 + \sum_{\ell < k} C^\ell_k \partial^{k-\ell}_x (p'(\bar{\rho})) \partial^\ell_x V_x \partial^k_x V_x, \\
\mu \partial^k_x (\bar{\rho}\Phi)_x \partial^k_x V &= \mu [\partial^k_x (\bar{\rho}\Phi)_x \partial^k_x V]_x - \mu \partial^k_x (\bar{\rho}\Phi) \partial^k_x V_x \\
&= \mu [\partial^k_x (\bar{\rho}\Phi)_x \partial^k_x V]_x - \mu \bar{\rho} \partial^k_x \Phi \partial^k_x V_x - \mu \sum_{\ell < k} C^\ell_k \partial^{k-\ell}_x \bar{\rho} \partial^\ell_x \Phi \partial^k_x V_x, \\
\mu \partial^k_x (V_x \bar{\phi}_x - \bar{\rho}_x \Phi) \partial^k_x V &= \frac{\mu}{b} \partial^k_x [\bar{\rho}_x (a V_x - b \Phi)] \partial^k_x V,
\end{aligned}
\]

where we have used the relationship between \(\bar{\phi}\) and \(\bar{\rho}\), i.e., \(\bar{\phi} = \frac{\xi}{b} \bar{\rho}\). Substituting the above equality into (3.6) and integrating the equation over \(\mathbb{R}\), we get

\[
\begin{aligned}
\frac{d}{dt} \left\{ \frac{\alpha}{2} ||\partial^k_x V||^2 + \int_{\mathbb{R}} \partial^k_x V_t \partial^k_x V dx \right\} + \int_{\mathbb{R}} p'(\bar{\rho}) \left( \partial^k_x V_x \right)^2 dx &= - \mu \int_{\mathbb{R}} \bar{\rho} \partial^k_x \Phi \partial^k_x V x dx \\
= ||\partial^k_x V_t||^2 - \sum_{\ell < k} C^\ell_k \int_{\mathbb{R}} \partial^{k-\ell}_x (p'(\bar{\rho})) \partial^\ell_x V_x \partial^k_x V dx + \mu \sum_{\ell < k} C^\ell_k \int_{\mathbb{R}} \partial^{k-\ell}_x \bar{\rho} \partial^\ell_x \Phi \partial^k_x V dx \\
&- \int_{\mathbb{R}} \mu \partial^k_x (V_x \bar{\phi}_x) \partial^k_x V dx - \frac{\mu}{b} \int_{\mathbb{R}} \partial^k_x [\bar{\rho}_x (a V_x - b \Phi)] \partial^k_x V dx \\
&- \int_{\mathbb{R}} \partial^k_x h_x \partial^k_x V dx - \int_{\mathbb{R}} \partial^k_x f_x \partial^k_x V dx.
\end{aligned}
\]  

(3.7)

Applying \(\partial^k_x\) for \(0 \leq k \leq 2\) to the second equation of (3.5), multiplying the result by \(\frac{\mu}{a} \partial^k_x \Phi\), and integrating the resulting equation with respect to \(x\) give

\[
\begin{aligned}
\frac{\mu}{2a} \int_{\mathbb{R}} \bar{\rho} \left( \partial^k_x \Phi \right)^2 dx + \frac{\mu D}{a} \int_{\mathbb{R}} \bar{\rho} \left( \partial^k_x \Phi \right)^2 dx + \frac{b \mu}{a} \int_{\mathbb{R}} \bar{\rho} \left( \partial^k_x \Phi \right)^2 dx &= - \mu \int_{\mathbb{R}} \partial^k_x V_x \bar{\rho} \partial^k_x \Phi dx \\
= \frac{\mu}{2a} \int_{\mathbb{R}} \bar{\rho} \left( \partial^k_x \Phi \right)^2 dx - \frac{\mu D}{a} \int_{\mathbb{R}} \bar{\rho} \partial^k_x \Phi _x \partial^k_x \Phi dx + \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho} \partial^k_x \Phi \partial^k_x g dx.
\end{aligned}
\]  

(3.8)
Taking summation of (3.7) and (3.8) and using integration by parts for the last two terms of (3.7), we have

\[
\frac{d}{dt} \left\{ \frac{\alpha}{2} \| \partial_x^k V \|^2 + \int \rho \partial_x^k \Phi \partial_x Vdx + \frac{\mu}{2a} \int \bar{\rho} \left( \partial_x^k \Phi \right)^2 dx \right\} + \frac{\mu D}{a} \int \bar{\rho} \left( \partial_x^k \Phi \right)^2 dx \\
+ \int \rho \left( \partial_x^k V \right)^2 dx - 2\mu \int \bar{\rho} \partial_x^k \Phi \partial_x V dx + \frac{b\mu}{a} \int \bar{\rho} \left( \partial_x^k \Phi \right)^2 dx \\
= \| \partial_x^k V \|^2 - \int \mu \partial_x^k (V \Phi) \partial_x V dx - \frac{\mu}{b} \int \partial_x^k \left[ \bar{\rho}_x (aV - b\Phi) \right] \partial_x^k V dx \\
+ \int \partial_x^k h \partial_x^k V dx + \int \partial_x^k f \partial_x^k V dx + \frac{\mu}{2a} \int \bar{\rho} \left( \partial_x^k \Phi \right)^2 dx - \frac{\mu D}{a} \int \bar{\rho}_x \partial_x^k \Phi \partial_x^k \Phi dx \\
+ \frac{\mu}{a} \int \bar{\rho} \partial_x^k \Phi \partial_x^k gdx + \sum \mathcal{C}_k \mathcal{I}_{k,\ell} \\
=: \| \partial_x^k V \|^2 + \sum_{j=1}^7 I_j + \sum_{\ell<k} \mathcal{C}_k \mathcal{I}_{k,\ell},
\]

with

\[
\mathcal{I}_{k,\ell} = - \int \partial_x^k \partial_x^1 \left( \rho \partial_x^k \Phi \right) \partial_x^k V \partial_x \partial_x^k V dx + \mu \int \partial_x^k \partial_x^1 \rho \partial_x^k \Phi \partial_x^k V dx,
\]

where the fact \( \ell < k \) has been used.

When \( k = 0 \), we estimate \( I_1 - I_7 \) as follows. By using the Cauchy-Schwartz inequality, Sobolev inequality and (3.1), \( I_1 \) can be estimated as follows,

\[
I_1 \leq C \| V \|_{L^\infty} \| V \| \| \Phi_x \| \leq CN(T) (\| V \|^2 + \| \Phi_x \|^2).
\]

For \( I_2 \), we have from the second equation of (3.5) and Lemma 2.1 that

\[
I_2 = - \frac{\mu}{b} \int \bar{\rho}_x (aV - b\Phi) V dx \\
= - \frac{\mu}{b} \int \bar{\rho}_x (\Phi_x - D \Phi_{xx} - g) V dx \\
= - \frac{\mu}{b} \int \bar{\rho}_x \Phi V dx + \frac{\mu}{b} \int \bar{\rho}_x \Phi V dx + \frac{\mu}{b} \int \bar{\rho}_x V dx \\
- \frac{\mu D}{b} \int \bar{\rho}_x \Phi V dx - \frac{\mu D}{b} \int \bar{\rho}_x \Phi V dx + \frac{\mu}{b} \int \bar{\rho}_x (-\phi_t + D\phi_{xx}) V dx \\
\leq - \frac{\mu}{b} \int \bar{\rho}_x \Phi V dx + C (||\bar{\rho}_x||^2 + ||\bar{\rho}_x||^2) + C \| V \|_{L^\infty}^2 (||\Phi \|^2 + ||\Phi_x \|^2) \\
+ C||\bar{\rho}_x||_{L^\infty}^2 (||\Phi \|^2 + ||V \|^2 + ||\Phi_x \|^2 + ||V_x \|^2) + C||V \|| \|\bar{\rho}_x \||_{L^\infty} (||\phi_t \|| + ||\bar{\phi}_{xx}||) \\
\leq - \frac{\mu}{b} \int \bar{\rho}_x \Phi V dx + C\delta_0 (1 + t)^{-\frac{3}{2}} + CN(T) \delta_0 (1 + t)^{-\frac{5}{4}} \\
+ C(N(T) + \delta_0) (||\Phi \|^2 + ||V \|^2 + ||\Phi_x \|^2 + ||V_x \|^2).
\]

Recall the definition of \( h \) and \( f \) in (2.6) and (3.2), we have

\[
h \sim V_t^2 + \bar{\rho}_x^2, \quad f \sim \bar{\rho}_t + V_x^2.
\]
which implies that
\[ I_3 + I_4 \leq C \int_{\mathbb{R}} |V_t|^2 + \bar{\rho}_x^2 |V_x| dx + C \int_{\mathbb{R}} |\bar{\rho}_t + V_x|^2 |V_x| dx \]
\[ \leq C(N(T) + \delta_0) (\|V_t\|^2 + \|V_x\|^2) + C\delta_0 (1 + t)^{-\frac{3}{2}}. \]

Moreover, by using Cauchy-Schwartz inequality and Lemma 2.1, the other three terms can be estimated as
\[ I_5 + I_6 + I_7 \leq C(N(T) + \delta_0) (\|\Phi\|^2 + \|\Phi_x\|^2) + C\delta_0 (1 + t)^{-\frac{3}{2}}. \]

When \(1 \leq k \leq 2\), we estimate \(I_1-I_7\) term by term. By Lemma 2.3 it is easy to have
\[ I_1 \leq C \left( \|V_x\|_{L^\infty} \|\partial_x^k \Phi_x\| + \|\Phi_x\|_{L^\infty} \|\partial_x^k V_x\| \right) \|\partial_x^k V\| \]
\[ \leq C N(T) \left( \|\partial_x^k \Phi_x\|^2 + \|\partial_x^k V_x\|^2 + \|\partial_x^k V\|^2 \right) \]
and
\[ I_2 = -\frac{\mu}{\beta} \int_{\mathbb{R}} \partial_x^k [\bar{\rho}_x (aV_x - b\Phi)] \partial_x^k V dx \]
\[ \leq C \left( \|\bar{\rho}_x\|_{L^\infty} \|aV_x - b\Phi\| + \|aV_x - b\Phi\|_{L^\infty} \|\partial_x^k \bar{\rho}_x\| \right) \|\partial_x^k V\| \]
\[ \leq C\delta_0 \left( \|\Phi\|^2_2 + \|V_x\|^2_2 \right). \]

For \(I_3, I_4\), recalling the definition of \(h\) and \(f\) in (2.6) and using (3.1), Cauchy-Schwartz inequality and Lemma 2.1 we can derive that
\[ I_3 + I_4 \leq C(N(T) + \delta_0) \left( \|V_t\|^2_2 + \|V_x\|^2_2 \right) + C\delta_0 (1 + t)^{-\frac{3}{2} - k}. \]

Due to the properties of \(\bar{\rho}_t\) and \(\bar{\rho}_x\) in Lemma 2.1, \(I_5, I_6\) and \(I_7\) can be estimated as follows.
\[ I_5 + I_6 + I_7 \leq C\delta_0 \left( \|\partial_x^k \Phi\|^2 + \|\partial_x^k \Phi_x\|^2 \right) + C\delta_0 (1 + t)^{-\frac{3}{2} - k}. \]

In order to complete step 1, we still need to estimate the last term \(I_{k,\ell}\). It follows from (2.9), (3.1) and Lemma 2.1 that
\[ |I_{k,\ell}| = \left| -\int_{\mathbb{R}} \partial_x^{k-1-\ell} (p''(\bar{\rho})\bar{\rho}_x) \partial_x^\ell V_x \partial_x^k V_x dx + \mu \int_{\mathbb{R}} \partial_x^{k-1-\ell} \bar{\rho}_x \partial_x^\ell \Phi \partial_x^k V_x dx \right| \]
\[ \leq C(\|\bar{\rho}_x\|_{L^\infty} \|\partial_x^{k-1} V_x\| + \|V_x\|_{L^\infty} \|\partial_x^{k-1} \bar{\rho}_x\|) \|\partial_x^k V_x\| \]
\[ + C(\|\bar{\rho}_x\|_{L^\infty} \|\partial_x^{k-1} \Phi\| + \|\Phi\|_{L^\infty} \|\partial_x^{k-1} \bar{\rho}_x\|) \|\partial_x^k V_x\| \]
\[ \leq C\delta_0 \left( \|V_x\|^2_2 + \|\Phi\|^2_2 \right). \]

Then (3.4) is obtained by substituting all the estimates \(I_1-I_7\) and \(I_{k,\ell}\) into (3.9) and then taking summation over \(0 \leq k \leq 2\).
Step 2. It is easy to check that

\[
\begin{align*}
\frac{d}{dt} \sum_{0 \leq k \leq 2} \left\{ \frac{1}{2} \| \partial_x^k V_t \|^2 + \frac{\mu D}{2a} \int_R \rho \left( \partial_x^k \Phi_x \right)^2 dx \right\} + \alpha \| V_t \|^2_2 + \frac{\mu}{a} \sum_{0 \leq k \leq 2} \int_R \rho \left( \partial_x^k \Phi_t \right)^2 dx \\
+ \frac{1}{2} \frac{d}{dt} \sum_{0 \leq k \leq 2} \left\{ \int_R p'(\bar{\rho}) \left( \partial_x^k V_x \right)^2 dx - 2\mu \int_R \rho \partial_x^k \Phi \partial_x^k V_x dx + \frac{\mu b}{a} \int_R \rho \left( \partial_x^k \Phi \right)^2 dx \right\} \\
\leq \frac{1}{2} \frac{d}{dt} \sum_{0 \leq k \leq 2} \left\{ \int_R (V_t + \frac{1}{a} (q(\bar{\rho}))_{x})^2 \left( \partial_x^k V_x \right)^2 dx \right\} - \int_R [p'(V_x + \bar{\rho}) - p'(\bar{\rho})] \left( \partial_x^k V_x \right)^2 dx \\
+ C(\delta_0 + N(T)) \left( \| V_x \|^2_2 + \| V_t \|^2_2 + \| \Phi \|^2_3 + \| \Phi_t \|^2_2 \right) + C\delta_0 (1 + t)^{-\frac{3}{2}}.
\end{align*}
\]

(3.10)

Applying \( \partial_x^k \) for \( 0 \leq k \leq 2 \) to the first equation of (3.5) and multiplying it by \( \partial_x^k V_t \), we get

\[
\partial_x^k V_t \partial_x^k V_t = \frac{d}{dt} \| \partial_x^k V_t \|^2,
\]

\[
-\partial_x^k (p'(\bar{\rho})V_x) \partial_x^k V_t = -\partial_x^k (p''(\bar{\rho})\bar{\rho}_x V_x) \partial_x^k V_t - \partial_x^k (p'(\bar{\rho})V_{xx}) \partial_x^k V_t \] 

\[
= - \partial_x^k (p''(\bar{\rho})\bar{\rho}_x V_x) \partial_x^k V_t - p''(\bar{\rho}) \partial_x^k V_x \partial_x^k V_t - \sum_{\ell < k} C_{\ell} \partial_x^{(k-\ell)} (p'(\bar{\rho})) \partial_x^\ell V_{xx} \partial_x^k V_t \\
= - \left( p'(\bar{\rho}) \partial_x^k V_x \partial_x^k V_t \right)_x + p''(\bar{\rho}) \bar{\rho}_x \partial_x^k V_x \partial_x^k V_t + p'(\bar{\rho}) \partial_x^k V_x \partial_x^k V_t \\
- \partial_x^k (p''(\bar{\rho})\bar{\rho}_x V_x) \partial_x^k V_t - \sum_{\ell < k} C_{\ell} \partial_x^{(k-\ell)} (p'(\bar{\rho})) \partial_x^\ell V_{xx} \partial_x^k V_t \\
= - \left( p'(\bar{\rho}) \partial_x^k V_x \partial_x^k V_t \right)_x + p''(\bar{\rho}) \bar{\rho}_x \partial_x^k V_x \partial_x^k V_t + \frac{1}{2} \frac{d}{dt} \left( p'(\bar{\rho}) (\partial_x^k V_x)^2 \right) \\
- \frac{1}{2} p''(\bar{\rho}) \bar{\rho}_t \left( \partial_x^k V_x \right)^2 - \partial_x^k (p''(\bar{\rho})\bar{\rho}_x V_x) \partial_x^k V_t - \sum_{\ell < k} C_{\ell} \partial_x^{(k-\ell)} (p'(\bar{\rho})) \partial_x^\ell V_{xx} \partial_x^k V_t,
\]

\[
\mu \partial_x^k (\bar{\rho} \Phi)_x \partial_x^k V_t = \mu [\partial_x^k (\bar{\rho} \Phi)]_x - \mu \partial_x^k (\bar{\rho} \Phi) \partial_x^k V_t \\
= \mu [\partial_x^k (\bar{\rho} \Phi)]_x - \mu \partial_x^k (\bar{\rho} \Phi) \partial_x^k V_t - \mu \sum_{\ell < k} C_{\ell} \partial_x^{(k-\ell)} \bar{\rho} \partial_x^\ell \Phi \partial_x^k V_t \\
= \mu [\partial_x^k (\bar{\rho} \Phi)]_x - \mu \frac{d}{dt} (\bar{\rho} \partial_x^k \Phi \partial_x^k V_t) + \mu \bar{\rho}_t \partial_x^k \Phi \partial_x^k V_t + \mu \partial_x^k (\bar{\rho} \Phi) \partial_x^k V_x \\
- \mu \sum_{\ell < k} C_{\ell} \left( \partial_x^{(k-\ell)} \bar{\rho}_x \partial_x^\ell \Phi \partial_x^k V_t \right) + \mu \sum_{\ell < k} C_{\ell} \left( \partial_x^{(k-\ell)} \bar{\rho}_x \partial_x^\ell \Phi \partial_x^k V_t + \partial_x^{(k-\ell)} \bar{\rho}_x \partial_x^\ell \Phi \partial_x^k V_t \right),
\]

\[
\mu \partial_x^k (V_x \bar{\rho}_x - \bar{\rho}_x \Phi) \partial_x^k V_t = \frac{\mu}{b} \partial_x^k [\bar{\rho}_x (aV_x - b\Phi)] \partial_x^k V_t.
\]
Substituting the above equalities into (3.11) and integrating the equation over \( \mathbb{R} \), we have
\[
\frac{d}{dt} \left\{ \frac{1}{2} \| \partial_x^k V_t \|^2 + \frac{1}{2} \int_{\mathbb{R}} p'(\bar{\rho}) \left( \partial_x^k V_t \right)^2 dx - \mu \int_{\mathbb{R}} \bar{\rho} \partial_x^k \Phi \partial_x^k V_t dx \right\} + \alpha \| \partial_x^k V_t \|^2 + \mu \int_{\mathbb{R}} \bar{\rho} \partial_x^k \Phi \partial_x^k V_t dx
= \frac{1}{2} \int_{\mathbb{R}} p''(\bar{\rho}) \rho_t \left( \partial_x^k V_t \right)^2 dx
- \mu \int_{\mathbb{R}} p''(\bar{\rho}) \rho_x \partial_x^k V_t \partial_x^k V_t dx + \int_{\mathbb{R}} \partial_x^k \left( p''(\bar{\rho}) \rho_x V_t \right) \partial_x^k V_t dx
+ \sum_{\ell < k} C_{k, \ell}^k \int_{\mathbb{R}} \partial_x^{k-\ell} \left( p'(\bar{\rho}) \right) \partial_x^\ell V_t \partial_x^k V_t dx
- \mu \int_{\mathbb{R}} \rho_t \partial_x^k \Phi \partial_x^k V_t dx
- \int_{\mathbb{R}} \partial_x^k h_x \partial_x^k V_t dx
- \int_{\mathbb{R}} \partial_x^k f_x \partial_x^k V_t dx.
\]
(3.12)

Applying \( \partial_x^k \) for \( 0 \leq k \leq 2 \) to the second equation of (3.15), multiplying the resultant equation by \( \frac{\mu}{a} \rho \partial_x^k \Phi_t \) and taking integration in \( x \) give
\[
\frac{d}{dt} \left\{ \frac{1}{2} \| \partial_x^k V_t \|^2 + \frac{\mu D}{2a} \int_{\mathbb{R}} \bar{\rho} \left( \partial_x^k \Phi_t \right)^2 dx \right\} + \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho} \left( \partial_x^k \Phi_t \right)^2 dx
= \frac{\mu b}{2a} \int_{\mathbb{R}} \rho_t \left( \partial_x^k \Phi_t \right)^2 dx + \frac{\mu D}{2a} \int_{\mathbb{R}} \bar{\rho} \left( \partial_x^k \Phi_t \right)^2 dx - \frac{\mu D}{a} \int_{\mathbb{R}} \rho_x \partial_x^k \Phi_t \partial_x^k \Phi_t dx + \frac{\mu}{a} \int_{\mathbb{R}} \rho \partial_x^k \Phi_t \partial_x^k \Phi_t dx.
\]
(3.13)

Combining (3.12) with (3.13) yields
\[
\frac{d}{dt} \left\{ \frac{1}{2} \| \partial_x^k V_t \|^2 + \frac{\mu D}{2a} \int_{\mathbb{R}} \bar{\rho} \left( \partial_x^k \Phi_t \right)^2 dx \right\} + \alpha \| \partial_x^k V_t \|^2 + \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho} \left( \partial_x^k \Phi_t \right)^2 dx
+ \frac{1}{2} \frac{d}{dt} \left\{ \int_{\mathbb{R}} p'(\bar{\rho}) \left( \partial_x^k V_t \right)^2 dx - 2\mu \int_{\mathbb{R}} \bar{\rho} \partial_x^k \Phi \partial_x^k V_t \partial_x^k V_t dx + \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho} \left( \partial_x^k \Phi_t \right)^2 dx \right\}
= \mathcal{J}^{(k)}_1 + \mathcal{J}^{(k)}_2 + \mathcal{J}^{(k)}_3 + \mathcal{J}^{(k)}_4 + \sum_{\ell < k} C_{k, \ell}^k \mathcal{J}_{k, \ell},
\]
(3.14)

where
\[
\begin{align*}
\mathcal{J}^{(k)}_1 &= \frac{1}{2} \int_{\mathbb{R}} p''(\bar{\rho}) \rho_t \left( \partial_x^k V_t \right)^2 dx
- \mu \int_{\mathbb{R}} p''(\bar{\rho}) \rho_x \partial_x^k V_t \partial_x^k V_t dx
+ \frac{\mu b}{2a} \int_{\mathbb{R}} \rho_t \left( \partial_x^k \Phi_t \right)^2 dx + \frac{\mu D}{2a} \int_{\mathbb{R}} \bar{\rho} \left( \partial_x^k \Phi_t \right)^2 dx - \frac{\mu D}{a} \int_{\mathbb{R}} \rho_x \partial_x^k \Phi_t \partial_x^k \Phi_t dx,
\mathcal{J}^{(k)}_2 &= \int_{\mathbb{R}} \partial_x^k \left( p''(\bar{\rho}) \rho_x V_t \right) \partial_x^k V_t dx
- \mu \int_{\mathbb{R}} \partial_x^k \Phi_t \partial_x^k V_t \partial_x^k V_t dx - \frac{\mu}{b} \int_{\mathbb{R}} \partial_x^k \left( \rho_x \left( aV_x - b \Phi_x \right) \right) \partial_x^k V_t dx,
\mathcal{J}^{(k)}_3 &= - \int_{\mathbb{R}} \partial_x^k h_x \partial_x^k V_t dx
- \int_{\mathbb{R}} \partial_x^k f_x \partial_x^k V_t dx = \mathcal{J}^{(k)}_{3h} + \mathcal{J}^{(k)}_{3f},
\mathcal{J}^{(k)}_4 &= \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho} \partial_x^k \Phi_t \partial_x^k \Phi_t dx.
\end{align*}
\]

and
\[
J_{k, \ell} = \int_{\mathbb{R}} \partial_x^{k-\ell} \left( p'(\bar{\rho}) \right) \partial_x^k V_t \partial_x^k V_t dx
- \mu \int_{\mathbb{R}} \partial_x^{k-\ell} \rho_x \partial_x^k \Phi_t \partial_x^k V_t dx
- \mu \int_{\mathbb{R}} \partial_x^{k-\ell} \rho \partial_x^k \Phi_t \partial_x^k V_t dx.
\]
It follows from the Cauchy-Schwartz inequality and Lemma 2.1 that
\[ J_1^{(k)} \leq C \delta_0 \left( \| \partial^k_x V_x \|^2 + \| \partial^k_x V_t \|^2 + \| \partial^k_x \Phi \|^2 + \| \partial^k_x \Phi_t \|^2 \right). \]

Due to Lemma 2.3, we can estimate \( J_2^{(k)} \) and \( J_{k, \ell} \) as follows,
\[
J_2^{(k)} \leq C \left( \| \tilde{p}_x \|_{L^\infty} \| \partial^k_x [V_x, \Phi] \| + \| [V_x, \Phi] \|_{L^\infty} \| \partial^k_x \tilde{p}_x \| \right) \| \partial^k_x V_t \|
+ \left( \| V_x \|_{L^\infty} \| \partial^k_x \Phi_x \| + \| \Phi_x \|_{L^\infty} \| \partial^k_x V_t \| \right) \| \partial^k_x V_t \|
\leq C(N(T) + \delta_0)(\| V_x \|^2 + \| \partial^k_x V_t \|^2 + \| \Phi \|^3).
\]

Thanks to \( \ell < k \), \( J_{k, \ell} \) can be rewritten in the following form
\[
J_{k, \ell} = -\int_\mathbb{R} \partial_x^{k-1-\ell} \left( (p''(\tilde{\rho}) \tilde{\rho}_x) \partial^k_x V_{xx} \partial^k_x V_x \right) dx - \mu \int_\mathbb{R} \partial_x^{k-1-\ell} \tilde{p}_x \partial^k_x \Phi \partial^k_x V_x dx - \mu \int_\mathbb{R} \partial_x^{k-1-\ell} \tilde{p}_x \partial^k_x \Phi \partial^k_x V_x dx
\leq C \left( \| \tilde{p}_x \|_{L^\infty} \| \partial_x^{k-1} [V_{xx} \Phi_x] \| + \| [V_{xx}, \Phi_x] \|_{L^\infty} \| \partial_x^{k-1} \tilde{p}_x \| \right) \| \partial^k_x V_t \|
+ \left( \| \tilde{p}_x \|_{L^\infty} \| \partial^k_x \Phi \| + \| \Phi_x \|_{L^\infty} \| \partial^k_x \tilde{p}_x \| \right) \| \partial^k_x V_t \|
\leq C \delta_0 (\| V_x \|^2 + \| \partial^k_x V_t \|^2 + \| \Phi \|^3).
\]

For \( J_4^{(k)} \), with the definition of \( g \) in (2.6), using the Cauchy-Schwartz inequality and Lemma 2.1, we have
\[
J_4^{(k)} \leq C \delta_0 \| \partial^k_x \Phi \|^2 + \frac{C}{\delta_0} \| \partial^k_x g \|^2 \leq C \delta_0 \| \partial^k_x \Phi_t \|^2 + C \delta_0 (1 + t)^{-\frac{3}{2} - k}.
\]

In order to estimate \( J_3^{(k)} \), we calculate \( h_x \) and \( f_x \) first. Note that
\[
h_x = -\left( \frac{(V_t + \frac{1}{\alpha} q(\tilde{\rho})_x)^2}{V_x + \tilde{\rho}} \right)_x = \left( \frac{(V_t + \frac{1}{\alpha} q(\tilde{\rho}))_x^2}{V_x + \tilde{\rho}} \right)_x - \frac{2(V_t + \frac{1}{\alpha} q(\tilde{\rho}))_x}{V_x + \tilde{\rho}} V_{xt} \]
+ \frac{(V_t + \frac{1}{\alpha} q(\tilde{\rho}))_x}{(V_x + \tilde{\rho})^2} \tilde{p}_x - \frac{2(V_t + \frac{1}{\alpha} q(\tilde{\rho}))_x}{V_x + \tilde{\rho}} \frac{1}{\alpha} q(\tilde{\rho})_{xx}, \tag{3.15}
\]
and
\[
f_x = \frac{1}{\alpha} (q(\tilde{\rho})_x)_x - \left[ p(V_x + \tilde{\rho}) - p(\tilde{\rho}) - p'(\tilde{\rho}) V_x \right]_x \]
= \frac{1}{\alpha} (q(\tilde{\rho})_x)_x - \left[ p'(V_x + \tilde{\rho}) - p'(\tilde{\rho}) V_{xx} - (p'(V_x + \tilde{\rho}) - p'(\tilde{\rho}) - p''(\tilde{\rho}) V_x) \tilde{p}_x \right]. \tag{3.16}
Then, for $J_{3h}^{(k)}$, we focus on the term which contains the highest-order derivative $V_{xx}$ and $V_{xt}$,

$$J_{3h}^{(k)} = - \int_R \partial_x^k h_x \partial_x^k V_t dx$$

$$= - \int_R \frac{(V_t + \frac{1}{\alpha}(q(\bar{\rho}))_x)^2}{(V_x + \bar{\rho})^2} \partial_x^k V_{xx} \partial_x^k V_t dx + \int_R \frac{2(V_t + \frac{1}{\alpha}(q(\bar{\rho}))_x)^2}{V_x + \bar{\rho}} \partial_x^k V_{xt} \partial_x^k V_t dx$$

$$- \int_R (\text{O.T.H.}) \partial_x^k V_t dx$$

$$= \int_R \left( \frac{V_t + \frac{1}{\alpha}(q(\bar{\rho}))_x^2}{(V_x + \bar{\rho})^2} \right) \partial_x^k V_{xx} \partial_x^k V_t dx + \int_R \frac{1}{2} \frac{d}{dt} \int_R \left( \frac{V_t + \frac{1}{\alpha}q(\bar{\rho})_x}{V_x + \bar{\rho}} \right)^2 \left( \partial_x^k V_t \right)^2 dx$$

$$- \frac{1}{2} \int_R \left( \frac{V_t + \frac{1}{\alpha}(q(\bar{\rho}))_x^2}{(V_x + \bar{\rho})^2} \right)_t \left( \partial_x^k V_t \right)^2 dx - \int_R \left( \frac{V_t + \frac{1}{\alpha}q(\bar{\rho})_x}{V_x + \bar{\rho}} \right)_x \left( \partial_x^k V_t \right)^2 dx$$

$$- \int_R (\text{O.T.H.}) \partial_x^k V_t dx,$$

where (O.T.H.) as an abbreviation for “other terms of $\partial_x^k h_x$” reading as

$$\text{(O.T.H.)} = \sum_{\ell<k} C^\ell_k \partial_x^{k-\ell} \left( \frac{V_t + \frac{1}{\alpha}(q(\bar{\rho}))_x^2}{(V_x + \bar{\rho})^2} \right) \partial_x^\ell V_{xx} - \sum_{\ell<k} C^\ell_k \partial_x^{k-\ell} \left( \frac{2(V_t + \frac{1}{\alpha}q(\bar{\rho})_x)}{V_x + \bar{\rho}} \right) \partial_x^\ell V_{xt}$$

$$+ \partial_x^k \left( \frac{V_t + \frac{1}{\alpha}(q(\bar{\rho}))_x^2}{(V_x + \bar{\rho})^2} \right)_x \bar{\rho} - \frac{2(V_t + \frac{1}{\alpha}q(\bar{\rho})_x)}{V_x + \bar{\rho}} \frac{1}{\alpha}q(\bar{\rho})_{xx} .$$

Similarly, for $J_{3f}^{(k)}$, we focus on the term which contains the highest-order derivative $V_{xx}$,

$$J_{3f}^{(k)} = - \int_R \partial_x^k f_x \partial_x^k V_t dx = \int_R [p'(V_x + \bar{\rho}) - p'(\bar{\rho})] \partial_x^k V_{xx} \partial_x^k V_t dx - \int_R (\text{O.T.F.}) \partial_x^k V_t dx,$$

$$= \int_R [p'(V_x + \bar{\rho}) - p'(\bar{\rho})] \partial_x^k V_{xx} \partial_x^k V_t dx - \int_R [p'(V_x - \bar{\rho}) - p'(\bar{\rho})] \partial_x^k V_{xx} \partial_x^k V_t dx$$

$$- \int_R (\text{O.T.F.}) \partial_x^k V_t dx$$

$$= \int_R [p'(V_x + \bar{\rho}) - p'(\bar{\rho})] \left( \partial_x^k V_t \right)^2 dx + \int_R [p'(V_x + \bar{\rho}) - p'(\bar{\rho})] \left( \partial_x^k V_t \right)^2 dx$$

$$- \int_R [p'(V_x + \bar{\rho}) - p'(\bar{\rho})] \partial_x^k V_{xx} \partial_x^k V_t dx - \int_R (\text{O.T.F.}) \partial_x^k V_t dx,$$

with

$$\text{(O.T.F.)} = - \sum_{\ell<k} C^\ell_k \partial_x^{k-\ell} [p'(V_x + \bar{\rho}) - p'(\bar{\rho})] \partial_x^\ell V_{xx}$$

$$+ \partial_x^k \left( \frac{1}{\alpha}q(\bar{\rho})_{xt} - (p'(V_x + \bar{\rho}) - p'(\bar{\rho}) - p''(\bar{\rho})V_x)_{xx} \right) .$$

Here we use (O.T.F.) as an abbreviation for “other terms of $\partial_x^k f_x$”. Clearly the total order of spatial derivatives for terms which are the product between $V_x$ and $V_t$ is not greater than 3 in (O.T.H.) and (O.T.F.), and the highest order of derivatives for $V_x$ and $V_t$ is not greater than 2. Then we can use Lemma 2 and Cauchy-Schwartz inequality to get

$$\int_R (\text{O.T.H.}) \partial_x^k V_{tx} dx + \int_R (\text{O.T.F.}) \partial_x^k V_{tx} dx \leq C(\delta_0 + N(T)) \left( \|V_x\|_2^2 + \|V_t\|_2^2 \right) + C\delta_0(1 + t)^{-\frac{3}{2} - k} ,$$
where we have used the following fact
\[ |V_t| \leq C \left( |V_{xx}| + |V_t| + |V_x| |\Phi_x| + (|\tilde{\rho}_x| + |\tilde{\Phi}_x|) |V_x| + |V_{xt}| + |\Phi_x| + |\tilde{\rho}_x|^3 + |\tilde{\Phi}_x| |\tilde{\rho}_x| \right) \leq C(N(T) + \delta_0). \]

Based on the above calculations for \( J_{3h}^{(k)} \) and \( J_{3f}^{(k)} \), we can estimate \( J_3^{(k)} \) as follows,
\[
J_3^{(k)} \leq \frac{1}{2} \frac{d}{dt} \int_\mathbb{R} \frac{(V_t + \frac{1}{\alpha}(q(\tilde{\rho}))_x)^2}{(V_x + \tilde{\rho})^2} \left( \partial_x^k V_x \right)^2 \, dx - \frac{1}{2} \frac{d}{dt} \int_\mathbb{R} \left| p'(V_x + \tilde{\rho}) - p'(|\tilde{\rho}|) \right| \left( \partial_x^k V_x \right)^2 \, dx \\
+ C(\delta_0 + N(T)) \left( \| V_x \|_2^2 + \| V_t \|_2^2 \right) + C\delta_0 (1 + t)^{-\frac{3}{2} - k}.
\]

Feeding (3.14) on all the estimations of \( J_1^{(k)} - J_4^{(k)} \) and \( J_{k,t} \), and taking summation of (3.14) over \( 0 \leq k \leq 2 \), one has (3.10).

**Step 3.** First, under the assumptions \( p'(\tilde{\rho}) - \frac{\mu}{\bar{\rho}} \tilde{\rho} > 0 \) and \( \rho_- < \tilde{\rho} < \rho_+ \), we have
\[
\int_\mathbb{R} p'(\tilde{\rho}) \left( \partial_x^k V_x \right)^2 \, dx - 2\mu \int_\mathbb{R} \tilde{\rho} \partial_x^k \Phi \partial_x^k V_x \, dx + \frac{\mu b}{a} \int_\mathbb{R} \tilde{\rho} \left( \partial_x^k \Phi \right)^2 \, dx \sim \| [\partial_x^k \Phi, \partial_x^k V_x] \|^2.
\]

Second, adding (3.3) with (3.10) multiplied by a constant \( K \) which will be determined later, it follows that
\[
\frac{d}{dt} \mathcal{E}(t) + C \left( \| V_x \|_2^2 + \| \Phi \|_3^2 + \| \Phi_t \|_2^2 \right) + (\alpha K - 1) \| V_t \|_2^2 \\
\leq 2C\delta_0 (1 + t)^{-\frac{3}{2}} + C(N(T) + \delta_0) \left( \| V_x \|_2^2 + \| V_t \|_2^2 + \| \Phi \|_3^2 + \| \Phi_t \|_2^2 \right).
\]

where \( \alpha K > 1 \), \( \mathcal{E}(t) \) is given by
\[
\mathcal{E}(t) = \sum_{0 \leq k \leq 2} \left\{ \frac{\alpha}{2} \| \partial_x^k V \|^2 + \int_\mathbb{R} \partial_x^k V_t \partial_x^k V \, dx + \frac{K}{2} \| \partial_x^k V_t \|^2 \right\} \\
+ \frac{K}{2} \sum_{0 \leq k \leq 2} \left\{ \int_\mathbb{R} p'(\tilde{\rho}) \left( \partial_x^k V_x \right)^2 \, dx - 2\mu \int_\mathbb{R} \tilde{\rho} \partial_x^k \Phi \partial_x^k V_x \, dx + \frac{\mu b}{a} \int_\mathbb{R} \tilde{\rho} \left( \partial_x^k \Phi \right)^2 \, dx \right\} \\
+ \sum_{0 \leq k \leq 2} \left\{ \frac{\mu}{2a} \int_\mathbb{R} \tilde{\rho} \left( \partial_x^k \Phi \right)^2 \, dx + \frac{\mu DK}{2a} \int_\mathbb{R} \tilde{\rho} \left( \partial_x^k \Phi_t \right) \left( \partial_x^k V_x \right) \, dx \right\} \\
+ \frac{\mu}{b} \int_\mathbb{R} \tilde{\rho}_x \Phi Vx \, dx - \frac{K}{2} \sum_{0 \leq k \leq 2} \int_\mathbb{R} \frac{(V_t + \frac{1}{\alpha}(q(\tilde{\rho}))_x)^2}{(V_x + \tilde{\rho})^2} \left( \partial_x^k V_x \right)^2 \, dx \\
+ \frac{\mu}{b} \int_\mathbb{R} \tilde{\rho}_x \Phi Vx \, dx - \frac{K}{2} \sum_{0 \leq k \leq 2} \int_\mathbb{R} \left| p'(V_x + \tilde{\rho}) - p'(|\tilde{\rho}|) \right| \left( \partial_x^k V_x \right)^2 \, dx.
\]

On the one hand, we can choose \( K \) large enough such that
\[
\frac{\alpha}{2} \| \partial_x^k V \|^2 + \int_\mathbb{R} \partial_x^k V_t \partial_x^k V \, dx + \frac{K}{2} \| \partial_x^k V_t \|^2 \geq C \left( \| \partial_x^k V_t \|^2 + \| \partial_x^k V \|^2 \right)
\]
for some constant \( C > 0 \) independent of \( t \). On the other hand, with the Cauchy-Schwartz inequality, Lemma 2.1 and a priori assumption, we have
\[
\int_\mathbb{R} \tilde{\rho}_x \Phi Vx \, dx \leq C\delta_0 \left( \| \Phi \|_2^2 + \| V \|_2^2 \right),
\]
\[
\int_\mathbb{R} \frac{(V_t + \frac{1}{\alpha}(q(\tilde{\rho}))_x)^2}{(V_x + \tilde{\rho})^2} \left( \partial_x^k V_x \right)^2 \, dx \leq C(N(T) + \delta_0) \| \partial_x^k V_x \|^2
\]
and
\[ \int_{\mathbb{R}} [p'(V_x + \bar{\rho}) - p'(\bar{\rho})] \left( \partial_x^k V_x \right)^2 \, dx \leq CN(T) \| \partial_x^k V_x \|^2. \]
Choosing \( N(T) \) and \( \delta_0 \) small enough, one gets from (3.19) and (3.21) that
\[ \mathcal{E}(t) \sim \| V \|_3^2 + \| V_t \|_2^2 + \| \Phi \|_3^2. \]
With the above equivalence for \( \mathcal{E}(t) \), after integrating (3.20) over \((0, t)\), and taking \( N(T) \) and \( \delta_0 \) sufficiently small, we get (3.3). Thus, the proof of Proposition 3.2 is completed.

Finally, we need to verify that the a priori assumption (3.1) is achievable. Since under the a priori assumption (3.1), we have proved that (3.3) holds true when \( \varepsilon_0 \) is small enough, (3.1) is ensured by (3.3). As such under the conditions of Proposition 2.2, we close the a priori assumption (3.1).

4. The time-decay rate of solutions

In this section, we are devoted to establishing the decay rate of the solution \((V, M, \Phi)\) or \((V, V_t, \Phi)\) to (2.4). First, we make the following a priori assumption on \((V, V_t, \Phi)\)
\[ \sum_{k=0}^{2} (1 + t)^{k+1} \left\| \partial_x^k [V_x, \Phi, \Phi_x] \right\|^2 + \sum_{k=0}^{2} (1 + t)^{k+2} \left\| \partial_x^k V_t \right\|^2 + \sum_{k=0}^{1} (1 + t)^{k+3} \left\| \partial_x^k \Phi_t \right\|^2 \ll 1. \quad (4.1) \]
Then we turn to prove the time-decay rate of \((V, V_t, \Phi)\) which is indicated by the following energy estimates.

**Proposition 4.1.** Under the assumption of Proposition 2.2, we have
\[ \sum_{k=0}^{2} (1 + t)^{k+1} \left\| \partial_x^k [V_x, \Phi, \Phi_x] \right\|^2 + \sum_{k=0}^{2} \int_0^t (1 + s)^k \left\| \partial_x^k [V_x, \Phi, \Phi_x] (\cdot, s) \right\|^2 \, ds \]
\[ \leq C \left( \| V_0 \|_3^2 + \| M_0 \|_2^2 + \| \Phi_0 \|_4^2 + \delta_0 \right) \leq C \varepsilon^2 \quad (4.2) \]
and
\[ \sum_{k=0}^{2} (1 + t)^{k+2} \left\| \partial_x^k V_t \right\|^2 + \sum_{k=0}^{2} \int_0^t (1 + s)^{k+1} \left\| \partial_x^k V_t (\cdot, s) \right\|^2 \, ds \]
\[ + \sum_{k=0}^{1} (1 + t)^{k+3} \left\| \partial_x^k \Phi_t, \Phi_x \right\|^2 + \sum_{k=0}^{1} \int_0^t (1 + s)^{k+2} \left\| \partial_x^k \Phi_t, \Phi_x \right\|^2 \, ds \]
\[ \leq C \left( \| V_0 \|_3^2 + \| M_0 \|_2^2 + \| \Phi_0 \|_4^2 + \delta_0 \right) \leq C \varepsilon^2. \]

Similar to the proof for the global existence, with the help of the assumption \( bp'(\rho) - a \mu \rho > 0 \), the decay rates of \( V_x \) and \( \Phi \) can be obtained. The main ideas of the proof for Proposition 4.1 come from [36] and [38], but additional efforts are needed to deal with the term \( g \) in (2.4)\(_2\). The time-weighted energy estimates (4.2) for \( V_x \) and \( \Phi \) follows from Lemmas 4.2, 4.4, and a coarse decay rate for \( V_t \) can also be obtained simultaneously. The refined decay rate of \( V_t \) follows from Lemma 4.5. Here, due to the similarity, we only give the detailed proof of Lemmas 4.2, 4.4 to see how we deal with the coupling of \( V \) and \( \Phi \).

**Lemma 4.2.** Under the assumption of Proposition 2.2, we have
\[ (1 + t) \left\| [V_x, V_t, \Phi, \Phi_x] \right\|^2 + \int_0^t (1 + s)(\| V_t \|^2 + \| \Phi_t \|^2) \, ds \leq C \left( \| V_0 \|_3^2 + \| M_0 \|_2^2 + \| \Phi_0 \|_4^2 + \delta_0 \right). \]
Proof. Let \( k = 0 \) in (3.14), we have
\[
\frac{d}{dt} \left\{ \frac{1}{2} \| V_t \|^2 + \frac{\mu D}{2a} \int \rho \Phi_x^2 dx \right\} + \alpha \| V_t \|^2 + \frac{\mu}{a} \int \rho \Phi_t^2 dx
\]
\[
+ \frac{d}{dt} \left\{ \frac{1}{2} \int p'(\rho) V_x^2 dx - \mu \int \rho \Phi_x V_t dx + \frac{\mu b}{2a} \int \rho \Phi_t^2 dx \right\}
\]
\[
= J_1^{(0)} + J_2^{(0)} + J_3^{(0)} + J_4^{(0)},
\]
with
\[
J_1^{(0)} = \frac{1}{2} \int p''(\rho) \rho_t V_x^2 dx - \int p''(\rho) \rho_x V_x V_t dx - \mu \int \rho_t \Phi_x V_t dx
\]
\[
+ \frac{\mu b}{2a} \int \rho \Phi_x^2 dx + \frac{\mu D}{2a} \int \rho \Phi_x^2 dx - \frac{\mu D}{a} \int \rho_x \Phi_x \Phi_t dx,
\]
\[
J_2^{(0)} = \int p''(\rho) \rho_x V_x V_t dx - \int \mu V_x \Phi_x V_t dx - \frac{\mu}{b} \int \rho_t (aV_x - \Phi) V_t dx,
\]
\[
J_3^{(0)} = - \int h_x V_t dx - \int f_x V_t dx, \quad J_4^{(0)} = \frac{\mu}{a} \int \rho \Phi_t g dx.
\]

Now we deal with the terms of the right hand side of (4.3) one by one. It follows from Lemma 24.1 that
\[
J_1^{(0)} \leq C \delta_0(1 + t)^{-1} \int [V_x^2 + \Phi^2 + \Phi_x^2] dx + C \delta_0 \int [V_t^2 + \Phi_t^2] dx,
\]
where we have used that
\[
- \int p''(\rho) \rho_x V_x V_t dx - \frac{\mu D}{a} \int \rho_x \Phi_x \Phi_t dx
\]
\[
\leq \delta_0 \int [V_t^2 + \Phi_t^2] dx + \frac{C \delta_0}{\delta_0} \| \rho_x \|_{L^\infty} \int [V_x^2 + \Phi_x^2] dx
\]
\[
\leq \delta_0 \int [V_t^2 + \Phi_t^2] dx + C \delta_0(1 + t)^{-1} \int [V_x^2 + \Phi_x^2] dx.
\]

Similarly, we can estimate the first and the third term of \( J_2^{(0)} \), and for the second term, by using the \( a \ priori \) assumption (4.1), the Sobolev and Cauchy-Schwartz inequalities, we have
\[
- \mu \int V_x \Phi_x V_t dx \leq \frac{\alpha}{4} \int V_t^2 dx + C \| \Phi_x \|_{L^\infty} \int V_x^2 dx
\]
\[
\leq \frac{\alpha}{4} \int V_t^2 dx + C \| \Phi_x \| \| \Phi_x \| \int V_x^2 dx
\]
\[
\leq \frac{\alpha}{4} \int V_t^2 dx + C (1 + t)^{-\frac{5}{2}} \int V_x^2 dx,
\]
which implies that
\[
J_2^{(0)} \leq C (1 + t)^{-1} \int [V_x^2 + \Phi^2] dx + C \left( \frac{\alpha}{4} + \delta_0 \right) \int V_t^2 dx.
\]
Recalling (3.18) and (3.17) when \( k = 0 \), we can estimate \( J_3^{(0)} \) as

\[
J_3^{(0)} \leq - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} [p'(V_x + \bar{\rho}) - p'(\bar{\rho})] V_x^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \frac{(V_t + \frac{1}{\alpha}(q(\bar{\rho}))_x)^2}{(V_x + \bar{\rho})^2} V_x^2 dx
\]

\[
+C\varepsilon_0 \int_{\mathbb{R}} V_t^2 dx + C(1 + t)^{-1} \int_{\mathbb{R}} V_x^2 dx + C\delta_0(1 + t)^{-\frac{5}{4}},
\]

where we have used (4.1), Lemma 2.1, (2.4) and the following estimates:

\[
\left\| \left( \frac{(V_t + \frac{1}{\alpha}(q(\bar{\rho}))_x)^2}{(V_x + \bar{\rho})^2} \right)_x \right\|_{L^\infty} + \left\| \left( \frac{(V_t + \frac{1}{\alpha}(q(\bar{\rho}))_x)^2}{(V_x + \bar{\rho})^2} \right)_t \right\|_{L^\infty} \leq C(1 + t)^{-1}.
\]

Noticing that

\[
\|g\|^2 \leq C \left( \|\bar{\phi}_t\|^2 + \|\bar{\phi}_{xx}\|^2 \right) \leq C\delta_0^2(1 + t)^{-\frac{5}{4}}
\]

is insufficient to warrant the decay rate \( (1 + t)^{-1} \) of \( \|\Phi\|^2 \), we estimate \( J_4^{(0)} \) as follows

\[
J_4^{(0)} = \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho}\Phi_t g dx = \frac{d}{dt} \left( \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho}\Phi g dx \right) - \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho}_t\Phi g dx - \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho}\Phi_t g dx
\]

\[
\leq \frac{d}{dt} \left( \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho}\Phi g dx \right) + C(1 + t)^{-1} \int_{\mathbb{R}} \Phi^2 dx + C(1 + t) \int_{\mathbb{R}} [g_t^2 + \bar{\rho}_t^2 g^2] dx
\]

\[
\leq \frac{d}{dt} \left( \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho}\Phi g dx \right) + C(1 + t)^{-1} \int_{\mathbb{R}} \Phi^2 dx + C\delta_0(1 + t)^{-\frac{5}{4}}.
\]

The estimation from \( J_1^{(0)} \) to \( J_3^{(0)} \) updates (4.3) as

\[
\frac{d}{dt} \left\{ \frac{1}{2} \int_{\mathbb{R}} p'(\bar{\rho}) V_x^2 dx - \mu \int_{\mathbb{R}} \bar{\rho}_t V_x dx + \frac{\mu b}{2a} \int_{\mathbb{R}} \bar{\rho}_t^2 dx \right\}
\]

\[
+ \frac{d}{dt} \left\{ \frac{1}{2} \|V_t\|^2 + \frac{\mu D}{2a} \int_{\mathbb{R}} \bar{\rho}_x dx \right\} + \frac{\alpha}{2} \|V_t\|^2 + \frac{\mu}{2a} \int_{\mathbb{R}} \bar{\rho}\Phi^2 dx
\]

\[
\leq C(1 + t)^{-1} \int_{\mathbb{R}} [V_x^2 + \Phi_t^2 + \Phi_x^2] dx + \frac{d}{dt} \left( \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho}\Phi g dx \right) + C\delta_0(1 + t)^{-\frac{5}{4}}
\]

\[
- \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} [p'(V_x + \bar{\rho}) - p'(\bar{\rho})] V_x^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \frac{(V_t + \frac{1}{\alpha}(q(\bar{\rho}))_x)^2}{(V_x + \bar{\rho})^2} V_x^2 dx.
\]

(4.4)
Multiplying (4.3) with \((1 + t)\), we have
\[
\begin{align*}
\frac{d}{dt} \left( (1 + t) \int \left[ \frac{1}{2} p' \langle \rho \rangle V^2_x - \mu \bar{\rho} \Phi V_x + \frac{\mu b}{2a} \bar{\rho} \Phi^2 \right] dx \right) \\
+ \frac{d}{dt} \left( (1 + t) \int \left[ \frac{1}{2} V^2_t + \frac{D \mu}{2a} \bar{\rho} \Phi^2 \right] dx \right) + \frac{1}{2} (1 + t) \left( \int \alpha V^2_t dx + \frac{\mu}{a} \int \bar{\rho} \Phi_t^2 dx \right) \\
= \int \left[ \frac{1}{2} p' \langle \rho \rangle V^2_x - \mu \bar{\rho} \Phi V_x + \frac{\mu b}{2a} \bar{\rho} \Phi^2 \right] dx + \int \left[ \frac{1}{2} V^2_t + \frac{D \mu}{2a} \bar{\rho} \Phi^2 \right] dx \\
&+ C \int [V^2_x + \Phi^2 + \Phi^2_x] dx + \frac{d}{dt} \left( (1 + t) \int \frac{\mu}{a} \int \bar{\rho} \Phi g dx \right) - \frac{\mu}{a} \int \bar{\rho} \Phi g dx + C \delta_0 (1 + t)^{-\frac{3}{2}} \\
&+ \frac{1}{2} \frac{d}{dt} \left( (1 + t) \int \frac{(V_t + \frac{1}{a} (q(\rho)) x^2)}{(V_x + \bar{\rho})^2} V^2_x dx \right) - \frac{1}{2} \int \frac{(V_t + \frac{1}{a} (q(\rho)) x^2)}{(V_x + \bar{\rho})^2} V^2_x dx \\
&- \frac{1}{2} \frac{d}{dt} \left( (1 + t) \int [p'(V_x + \bar{\rho}) - p'(\bar{\rho})] V^2_x dx \right) + \frac{1}{2} \int [p'(V_x + \bar{\rho}) - p'(\bar{\rho})] V^2_x dx.
\end{align*}
\]
(4.5)

Integrating (4.5) over \([0, t]\) and using Proposition 3.2 give us that
\[
(1 + t) \| [V_x, V_t, \Phi, \Phi_x] \|^2 + \int_0^t (1 + s) (\| V_t \|^2 + \| \Phi_t \|^2) ds \leq C \left( \| V_0 \|^2_3 + \| M_0 \|^2_2 + \| \Phi_0 \|^2_3 + \delta_0 \right),
\]
where we have used the following estimates
\[
(1 + t) \frac{\mu}{a} \int \bar{\rho} \Phi g dx \leq C \delta_0 (1 + t) \int \Phi^2 dx + \frac{C}{\delta_0} (1 + t) \int g^2 dx \leq \delta_0 (1 + t) \int \Phi^2 dx + C \delta_0
\]
and
\[
\frac{\mu}{a} \int_0^t \int \bar{\rho} \Phi g dx dt \leq C \int_0^t \int \Phi^2 dx dt + \int_0^t \int g^2 dx dt \leq C \left( \| V_0 \|^2_3 + \| M_0 \|^2_2 + \| \Phi_0 \|^2_3 + \delta_0 \right).
\]
Thus, the proof of Lemma 4.2 is completed. \(\square\)

**Lemma 4.3.** Under the assumption of Proposition 2.2, we have
\[
(1 + t)^2 \| [V_{xt}, V_{xx}, \Phi_x, \Phi_{xx}] \|^2 + \int_0^t [(1 + s)^2 \| V_{xt} \|^2 + (1 + s) \| V_{xx}, \Phi_x, \Phi_{xx} \|^2] ds \leq C \left( \| V_0 \|^2_3 + \| M_0 \|^2_2 + \| \Phi_0 \|^2_3 + \delta_0 \right).
\]
(4.6)

**Proof.** The proof consists of the following three steps.

**Step 1.** Letting \(k = 1\) in (3.14), we have
\[
\begin{align*}
\frac{d}{dt} \left\{ \frac{1}{2} \| V_{xt} \|^2 + \mu \frac{D}{2a} \int \bar{\rho} \Phi_{xx}^2 dx \right\} + \alpha \| V_{xt} \|^2 + \frac{\mu}{a} \int \bar{\rho} \Phi_{xt}^2 dx \\
+ \frac{d}{dt} \left\{ \frac{1}{2} \int p'(\langle \rho \rangle) V^2_{xx} dx - \mu \int \bar{\rho} \Phi_x V_{xx} dx + \frac{\mu b}{2a} \int \bar{\rho} \Phi^2_{xx} dx \right\} \\
= J_1^{(1)} + J_2^{(1)} + J_3^{(1)} + J_4^{(1)} + J_{1,0},
\end{align*}
\]
(4.7)
with
\[ J_1^{(1)} = \frac{1}{2} \int_{\mathbb{R}} p''(\bar{\rho}) \bar{\rho}_x V_{xx}^2 dx - \int_{\mathbb{R}} p''(\bar{\rho}) \bar{\rho}_x V_{xt} V_{tx} dx - \mu \int_{\mathbb{R}} \bar{\rho}_t \Phi_x V_{xx} dx \]
\[ + \frac{\mu b}{2a} \int_{\mathbb{R}} \bar{\rho}_t \Phi_x^2 dx + \frac{\mu D}{2a} \int_{\mathbb{R}} \bar{\rho}_t \Phi_x^2 dx - \frac{\mu D}{a} \int_{\mathbb{R}} \bar{\rho}_x \Phi_x \Phi_{xt} dx, \]
\[ J_2^{(1)} = \int_{\mathbb{R}} (p''(\bar{\rho}) \bar{\rho}_x V_{xx}^2) V_{xt} dx - \int_{\mathbb{R}} \mu (V_x \Phi_x)_x V_{xt} dx - \frac{\mu}{b} \int_{\mathbb{R}} [\bar{\rho}_x (a V_x - b \Phi)]_x V_{xt} dx, \]
\[ J_3^{(1)} = - \int_{\mathbb{R}} h_{xx} V_{xt} dx - \int_{\mathbb{R}} f_{xx} V_{xt} dx, \]
\[ J_4^{(1)} = \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho} \Phi_{xt} g_x dx \]
and
\[ J_{1,0} = \int_{\mathbb{R}} p''(\bar{\rho}) \bar{\rho}_x V_{xx} V_{xt} dx - \mu \int_{\mathbb{R}} \bar{\rho}_x \Phi_x V_{xx} dx - \mu \int_{\mathbb{R}} \bar{\rho}_x \Phi_x V_{xt} dx. \]

Similar to Lemma 4.2, the term \( J_4^{(1)} \) can be estimated as
\[ J_4^{(1)} = \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho} \Phi_{xt} g_x dx = \frac{d}{dt} \left( \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho} \Phi_x g_x dx \right) - \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho}_t \Phi_x g_x dx - \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho}_x \Phi_t g_x dx \]
\[ \leq \frac{d}{dt} \left( \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho} \Phi_x g_x dx \right) + C(1 + t)^{-1} \int_{\mathbb{R}} \Phi_x^2 dx + C(1 + t) \int_{\mathbb{R}} [g_{xt} + \bar{\rho}^2 g_{xx}] dx \]
\[ \leq \frac{d}{dt} \left( \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho} \Phi_x g_x dx \right) + C(1 + t)^{-1} \int_{\mathbb{R}} \Phi_x^2 dx + C \delta_0 (1 + t)^{-\frac{7}{2}}. \]

Recalling (3.17) and (3.18), we have
\[ J_3^{(1)} = - \int_{\mathbb{R}} h_{xx} V_{xt} dx - \int_{\mathbb{R}} f_{xx} V_{xt} dx \]
\[ \leq - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} [p'(V_x + \bar{\rho}) - p'(\bar{\rho})] V_{xx}^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left( V_t + \frac{1}{a} (q(\bar{\rho})) x \right)_x^2 V_{xx}^2 dx + C \varepsilon_0 \int_{\mathbb{R}} V_{xt}^2 dx \]
\[ + C(1 + t)^{-1} \int_{\mathbb{R}} V_{xx}^2 dx + C(1 + t)^{-2} \int_{\mathbb{R}} [V_x^2 + V_t^2] dx + C \delta_0 (1 + t)^{-\frac{7}{2}}. \]

Substituting the above inequalities into (4.7) leads to
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} V_{xt}^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left( p'(\bar{\rho}) V_{xx}^2 dx - 2 \mu \bar{\rho} \Phi_x V_{xx} + \frac{\mu b \bar{\rho}}{a} \Phi_x^2 dx \right) \]
\[ + \frac{\mu D}{2a} \frac{d}{dt} \int_{\mathbb{R}} \bar{\rho} \Phi_x^2 dx + \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho} \Phi_x^2 dx + \alpha \int_{\mathbb{R}} V_{xt}^2 dx \]
\[ \leq \frac{1}{2} \int_{\mathbb{R}} \left( \frac{\mu \bar{\rho}}{a} \Phi_{xt} + \alpha V_{xt}^2 \right) dx + C(1 + t)^{-1} \int_{\mathbb{R}} [V_{xx}^2 + \Phi_x^2 + \Phi_{xx}^2] dx \]
\[ + C \delta_0 (1 + t)^{-\frac{7}{2}} + C(1 + t)^{-2} \int_{\mathbb{R}} [V_x^2 + V_t^2 + \Phi^2] dx + \frac{d}{dt} \left( \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho} \Phi_x g_x dx \right) \]
\[ + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left( V_t + \frac{1}{a} (q(\bar{\rho})) x \right)_x^2 V_{xx}^2 dx - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} [p'(V_x + \bar{\rho}) - p'(\bar{\rho})] V_{xx}^2 dx. \]
Step 2. Multiplying (2.4) with $-V_{xx}$ and (2.4) with $-\frac{\mu \rho}{a} \Phi_{xx}$, integrating the result over $\mathbb{R}$, and using the integration by parts, we obtain

\[
\int_{\mathbb{R}} \left( \frac{\alpha}{2} V_x^2 + V_{xt} V_x \right) dx + \frac{\mu}{2a} \int_{\mathbb{R}} \bar{\rho} \Phi_x^2 dx \\
+ \int_{\mathbb{R}} \left( p'(\bar{\rho}) V_{xx}^2 - 2\mu \bar{\rho} V_x V_{xx} + \frac{b \mu \bar{\rho}}{a} \Phi_x^2 \right) dx + \frac{\mu D}{a} \int_{\mathbb{R}} \bar{\rho} \Phi_{xx}^2 dx \\
= \int_{\mathbb{R}} V_{xx}^2 dx - \int_{\mathbb{R}} p''(\bar{\rho}) V_x V_{xx} dx + \mu \int_{\mathbb{R}} V_x \Phi_x V_{xx} dx \\
+ \mu \int_{\mathbb{R}} \bar{\rho} \Phi_x^2 dx - \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho} \Phi_x \Phi_x dx - \frac{b \mu}{a} \int_{\mathbb{R}} \bar{\rho} \Phi_x \Phi dx - \int_{\mathbb{R}} \frac{\mu \bar{\rho}}{a} g \Phi_{xx} dx.
\]

Integrating the last term by parts twice leads to

\[
- \int_{\mathbb{R}} \frac{\mu \bar{\rho}}{a} g \Phi_{xx} dx = - \frac{\mu}{a} \int_{\mathbb{R}} \bar{\rho} g_{xx} \Phi dx \\
\leq (1 + t)^{-1} \int_{\mathbb{R}} \Phi^2 dx + C(1 + t) \int_{\mathbb{R}} \left( g_{xx}^2 + \bar{\rho}_{xx}^2 g_x^2 + \bar{\rho}_x^2 g_x^2 \right) dx \\
\leq (1 + t)^{-1} \int_{\mathbb{R}} \Phi^2 dx + C \delta_0 (1 + t)^{-\frac{5}{2}}.
\]

Thus we arrive at

\[
\int_{\mathbb{R}} \left( \frac{\alpha}{2} V_x^2 + V_{xt} V_x \right) dx + \frac{\mu}{2a} \int_{\mathbb{R}} \bar{\rho} \Phi_x^2 dx \\
+ \int_{\mathbb{R}} \left[ p'(\bar{\rho}) V_{xx}^2 - 2\mu \bar{\rho} V_x V_{xx} + \frac{b \mu \bar{\rho}}{a} \Phi_x^2 \right] dx + \frac{\mu D}{a} \int_{\mathbb{R}} \bar{\rho} \Phi_{xx}^2 dx \\
\leq 2 \int_{\mathbb{R}} V_{xx}^2 dx + C(1 + t)^{-1} \int_{\mathbb{R}} \left[ V_x^2 + \Phi^2 + \Phi_t^2 + V_t^2 \right] dx \\
+ C \delta_0 (1 + t)^{-\frac{5}{2}} + C \varepsilon_0 \int_{\mathbb{R}} \left[ V_{xx}^2 + \Phi_x^2 \right] dx.
\]

Here, we have used the expressions of $h_x$ and $f_x$ in (3.15) and (3.16).

Step 3. Multiplying $(1 + t)$ to $\left[ 2 \times (4.8) + \frac{\alpha}{4} \times (4.9) \right]$ and integrating the result over $[0, t]$, we get

\[
(1 + t) \left[ \| V_x \|_{L^2}^2 + \| V_{xt} \|_{L^2}^2 + \| V_{xx} \|_{L^2}^2 + \| \Phi_x \|_{L^2}^2 \right] + \int_0^t (1 + s) \left[ \| V_{xt} \|_{L^2}^2 + \| V_{xx} \|_{L^2}^2 + \| \Phi_x \|_{L^2}^2 \right] ds \\
\leq C \left( \| V_0 \|_{H^3}^2 + \| M_0 \|_{L^2}^2 + \| \Phi_0 \|_{L^2}^2 + \delta_0 \right).
\]

Similarly, multiplying (4.8) by $(1 + t)^2$ and integrating the resulting equation over $[0, t]$, we have from (4.10) that

\[
(1 + t)^2 \left[ \| V_{xx} \|_{L^2}^2 + \| \Phi_x \|_{L^2}^2 \right] + \int_0^t (1 + s)^2 \left[ \| V_{xx} \|_{L^2}^2 + \| \Phi_x \|_{L^2}^2 \right] ds \leq C \left( \| V_0 \|_{H^3}^2 + \| M_0 \|_{L^2}^2 + \| \Phi_0 \|_{L^2}^2 + \delta_0 \right).
\]
Lemma 4.4. Under the assumption of Proposition 2.2, we have

\[(1 + t) \frac{\mu}{\alpha} \int \bar{\rho} \Phi_x g_x dx \leq C \delta_0 (1 + t) \int \Phi_x^2 dx + \frac{C}{\delta_0} (1 + t) \int g_x^2 dx \leq \delta_0 (1 + t) \int \Phi_x^2 dx + C \delta_0,
\]

with the following estimates

\[
(1 + t) \frac{\mu}{\alpha} \int \bar{\rho} \Phi_x g_x dxds \leq C \int \Phi_x^2 dx + C \int g_x^2 dx \leq C \left( \|V_0\|_2^2 + \|M_0\|_2^2 + \|\Phi_0\|_3^2 + \delta_0 \right),
\]

\[
(1 + t) \frac{\mu}{\alpha} \int \bar{\rho} \Phi_x g_x dx \leq C \delta_0 (1 + t)^2 \int \Phi_x^2 dx + \frac{C}{\delta_0} (1 + t)^2 \int g_x^2 dx \leq \delta_0 (1 + t)^2 \int \Phi_x^2 dx + C \delta_0
\]

Then the combination of (4.10) and (4.11) gives (4.8). The proof of Lemma 4.3 is completed.

Lemma 4.4. Under the assumption of Proposition 2.2, we have

\[(1 + t)^3 \| [V_{xxx}, V_{xxt}, \Phi_{xx}, \Phi_{xxx}] \|^2 + \int (1 + s)^3 \| [V_{xxt}, \Phi_{xxt}] \|^2 + (1 + t)^2 \| [V_{xxx}, \Phi_{xx}, \Phi_{xxx}] \|^2 ds
\]

\[
\leq C \left( \|V_0\|_3^2 + \|M_0\|_2^2 + \|\Phi_0\|_3^2 + \delta_0 \right).
\]

Proof. The proof consists of three steps.

Step 1. Let \( k = 2 \) in (3.14), we have

\[
\frac{d}{dt} \int V_{xxt}^2 dx + \frac{d}{dt} \int \left[ p' (\bar{\rho}) V_{xxx}^2 - 2 \mu \bar{\rho} \Phi_{xxx} V_{xxx} + \frac{\mu b \bar{\rho}}{a} \Phi_{xxx}^2 \right] dx
\]

\[
+ \frac{D}{2a} \frac{d}{dt} \int \bar{\rho} \Phi_{xxx}^2 dx + \frac{\alpha}{2} \int V_{xxt}^2 dx + \frac{\mu}{a} \int \bar{\rho} \Phi_{xxx}^2 dx
\]

\[
= J_1^{(2)} + J_2^{(2)} + J_3^{(2)} + J_4^{(2)} + \sum_{\ell < 2} J_{2, \ell},
\]

with

\[
J_1^{(2)} = \frac{1}{2} \int p'' (\bar{\rho}) \bar{\rho} V_{xxx}^2 dx - \int p'' (\bar{\rho}) \bar{\rho} V_{xxx} V_{xxx} dx - \mu \int \bar{\rho} \Phi_{xxx} V_{xxx} dx
\]

\[
+ \frac{\mu b}{2a} \int \bar{\rho} \Phi_{xxx}^2 dx + \frac{\mu D}{2a} \int \bar{\rho} \Phi_{xxx}^2 dx - \frac{\mu D}{a} \int \bar{\rho} \Phi_{xxx} \Phi_{xxx} dx,
\]

\[
J_2^{(2)} = \int (p'' (\bar{\rho}) \bar{\rho} V_{xxx}^2)_x V_{xxx} dx - \int (V_{x} \Phi_{x})_{xxx} V_{xxx} dx - \mu b \int \bar{\rho} (a V_x - b \Phi)_x V_{xxx} dx,
\]

\[
J_3^{(2)} = - \int h_{xxx} V_{xxx} dx - \int f_{xxx} V_{xxx} dx,
\]

and

\[
J_{2, \ell} = \int \bar{\rho} \Phi_{xxx}^2 (p' (\bar{\rho})) \Phi_x^2 \Phi_{xxx} V_{xxx} dx - \mu \int \bar{\rho} \Phi_{xxx} \Phi_x \Phi_{xxx} \Phi_{xxx} V_{xxx} dx - \mu \int \bar{\rho} \Phi_{xxx} \Phi_x \Phi_{xxx} V_{xxx} dx.
\]
It is similar to estimate $J_4^{(2)}$ as in Lemma 4.2 that
\[
\frac{\mu}{a} \int_R \tilde{\rho} \Phi_{xxt} g_{xx} dx = \frac{d}{dt} \left( \frac{\mu}{a} \int_R \tilde{\rho} \Phi_{xx} g_{xx} dx \right) - \frac{\mu}{a} \int_R \tilde{\rho} \Phi_{xxt} g_{xx} dx - \frac{\mu}{a} \int_R \tilde{\rho} \Phi_{xxx} g_{xxt} dx
\]
\[
\leq \frac{d}{dt} \left( \frac{\mu}{a} \int_R \tilde{\rho} \Phi_{xx} g_{xx} dx \right) + C(1 + t)^{-1} \int_R \Phi_{xx}^2 dx + C(1 + t) \int_R \left[ g_{xxt}^2 + \tilde{\rho}^2 g_{xx}^2 \right] dx
\]
\[
\leq \frac{d}{dt} \left( \frac{\mu}{a} \int_R \tilde{\rho} \Phi_{xx} g_{xx} dx \right) + C(1 + t)^{-1} \int_R \Phi_{xx}^2 dx + C\delta_0(1 + t)^{-\frac{9}{2}}.
\]

Then, by a direct calculation, we have
\[
\frac{1}{2} \frac{d}{dt} \int_R V_{xxt}^2 dx + \frac{1}{2} \frac{d}{dt} \int_R \left[ p'(\tilde{\rho}) V_{xxx}^2 - 2 \mu \tilde{\rho} \Phi_{xxx} V_{xxx} + \frac{\mu b \tilde{\rho}}{a} \Phi_{xxx}^2 \right] dx
\]
\[
+ \frac{D\mu}{2a} \frac{d}{dt} \int_R \tilde{\rho} \Phi_{xx}^2 dx + \frac{\mu}{a} \int_R \tilde{\rho} \Phi_{xxt}^2 dx + \alpha \int_R V_{xxt}^2 dx
\]
\[
\leq \frac{1}{2} \int_R \left[ \alpha V_{xx}^2 + \frac{\mu b \tilde{\rho}}{a} \Phi_{xxx}^2 \right] dx + C(1 + t)^{-1} \int_R [V_{xx}^2 + \Phi_{xx}^2 + \Phi_{xxx}^2 + V_{xt}^2] dx
\]
\[
+ C(1 + t)^{-2} \int_R [V_{xx}^2 + \Phi_{xx}^2] dx + C(1 + t)^{-3} \int_R [V_{x}^2 + V_{t}^2 + \Phi_{xx}^2] dx
\]
\[
+ C\delta_0(1 + t)^{-\frac{9}{2}} + \frac{d}{dt} \left( \frac{\mu}{a} \int_R \tilde{\rho} \Phi_{xx} g_{xx} dx \right)
\]
\[
+ \frac{1}{2} \frac{d}{dt} \int_R \left( \frac{V_t + \frac{1}{2} (g(\tilde{\rho}))_t}{(V_x + \tilde{\rho})^2} \right) V_{xxx}^2 dx - \frac{1}{2} \frac{d}{dt} \int_R [p'(V_x + \tilde{\rho}) - p'(\tilde{\rho})] V_{xxx}^2 dx.
\]

Step 2. Differentiating $\Phi_{xt} - D\Phi_{xxx} - aV_{xx} + b\Phi_{x} = g_x.$

Multiplying $\Phi_{xt} - aV_{xx}$ and multiplying $\Phi_{xt} - D\Phi_{xxx} - aV_{xx} + b\Phi_{x} = g_x.$ with $-V_{xxx}$ and multiplying $\Phi_{xt} - D\Phi_{xxx} - aV_{xx} + b\Phi_{x} = g_x.$ with $-\frac{\mu \tilde{\rho}}{a} \Phi_{xxx},$ we have
\[
\frac{d}{dt} \int_R \left( \frac{\alpha}{2} V_{xx}^2 + V_{xxt}V_{xx} \right) dx + \frac{\mu}{2a} \frac{d}{dt} \int_R \tilde{\rho} \Phi_{xx}^2 dx
\]
\[
+ \int_R \left[ p'(\tilde{\rho}) V_{xxx}^2 - 2 \mu \tilde{\rho} \Phi_{xxx} V_{xxx} + \frac{b \mu \tilde{\rho}}{a} \Phi_{xxx}^2 \right] dx + \frac{\mu D}{a} \int_R \tilde{\rho} \Phi_{xxx}^2 dx
\]
\[
= \int_R V_{xxx}^2 dx - 2 \int_R p''(\tilde{\rho}) \tilde{\rho} V_{x} V_{xxx} dx - \int_R (p'(\tilde{\rho}))_{xx} V_{x} V_{xxx} dx
\]
\[
+ \mu \int_R (V_{x} \Phi_{x})_x V_{xxx} dx + \mu \int_R (V_{x} \tilde{\Phi}_{x})_x V_{xxx} dx + \mu \int_R \tilde{\rho} \Phi_{xxx} V_{xxx} dx
\]
\[
+ \mu \int_R (h + f)_{x} V_{xxx} dx + \frac{\mu}{2a} \int_R \tilde{\rho} \Phi_{xx}^2 dx - \frac{\mu}{a} \int_R \tilde{\rho} \Phi_{xxt} \Phi_{x} dx
\]
\[
+ \mu \int_R \tilde{\rho} \Phi_{x} V_{xxx} dx - \frac{b \mu}{a} \int_R \tilde{\rho} \Phi_{xx} \Phi_{xxx} dx - \frac{\mu}{a} \int_R \tilde{\rho} g_x \Phi_{xxx} dx.
\]
Integrating the last term by parts twice, and then we have
\[-\frac{\mu}{a} \int_R \bar{\rho} g \Phi_{xxx} dx = -\frac{\mu}{a} \int_R \bar{\rho} g x \Phi_{x} dx - \frac{\mu}{a} \int_R \bar{\rho} g x \Phi_{x} dx - \frac{2\mu}{a} \int_R \bar{\rho} g x \Phi_{x} dx\]
\[\leq C\delta_0 (1 + t)^{-1} \int_R \Phi_x^2 dx + \frac{C}{\delta_0} (1 + t) \int_R (\bar{\rho}^2 g^2 + \bar{\rho}^2 g^2 + \bar{\rho}^2 g^2) dx\]
\[\leq C\delta_0 (1 + t)^{-1} \int_R \Phi_x^2 dx + C\delta_0 (1 + t)^{-\frac{7}{2}},\]
which along with (4.14) yields
\[\frac{d}{dt} \int_R \left( \frac{\alpha}{2} V_{xx}^2 + V_{xxt} V_{xx} \right) dx + \frac{\mu}{2a} \frac{d}{dt} \int_R \bar{\rho} \Phi_{xx}^2 dx\]
\[+ \int_R \left[ p' (\bar{\rho}) V_{xxx}^2 - 2\mu \bar{\rho} \Phi_{xx} V_{xxx} + \frac{\mu}{a} \Phi_{xx}^2 \right] dx + \frac{\mu D}{a} \int_R \bar{\rho} \Phi_{xx}^2 dx\]
\[\leq 2 \int_R V_{xxt}^2 dx + C\varepsilon_0 \int_R [V_{xx}^2 + \Phi_{xx}^2] dx + C\delta_0 (1 + t)^{-\frac{7}{2}}\]
\[+ C(1 + t)^{-1} \int_R [V_{xx}^2 + \Phi_{xx}^2 + \Phi_{xx}^2 + V_{xx}^2] dx + C(1 + t)^{-2} \int_R [V_{xxt}^2 + V_{xxt}^2] dx.\]

**Step 3.** Multiplying \((1 + t)^j (j = 1, 2)\) to \([2 \times (4.14) + \frac{\alpha}{4} \times (4.12)]\) and integrate the result over \([0, t],\) we get by Lemma 4.2 and Lemma 4.3 that
\[\int_0^t (1 + s)^2 \| [V_{xx}, V_{xxt}, V_{xxt}, \Phi_{xx}, \Phi_{xxx}] \|^2 + \int_0^t (1 + s)^2 \| [V_{xx}, V_{xxt}, \Phi_{xx}, \Phi_{xxx}, \Phi_{xxt}] \|^2 ds\]
\[\leq C \left( \| V_0 \|_3^2 + \| M_0 \|_2^2 + \| \Phi_0 \|_3^2 + \delta_0 \right).\] (4.15)

Furthermore integrating (4.12) multiplied by \((1 + t)^{j+1} (j = 1, 2)\) over \([0, t],\) we obtain from (4.15) that
\[\int_0^t (1 + s)^3 \| [V_{xxt}, V_{xxx}, \Phi_{xx}, \Phi_{xxx}] \|^2 + \int_0^t (1 + s)^3 \| [V_{xxt}, \Phi_{xxt}] \|^2 ds\]
\[\leq C \left( \| V_0 \|_3^2 + \| M_0 \|_2^2 + \| \Phi_0 \|_3^2 + \delta_0 \right).\]

Here, for \(j = 1, 2,\) we have used the following estimates
\[\int_0^t (1 + s)^j \frac{\mu}{a} \int_R \bar{\rho} \Phi_{xx} g_{xxx} dx \leq C\delta_0 (1 + t)^j \int_R \Phi_{xx}^2 dx + \frac{C}{\delta_0} (1 + t)^j \int_R g_{xxx}^2 dx\]
\[\leq \delta_0 (1 + t)^j \int_R \Phi_{xx}^2 dx + C\delta_0 (1 + t)^{-\frac{7}{2} + j}\]
\[\leq \delta_0 (1 + t)^j \int_R \Phi_{xx}^2 dx + C\delta_0,\]
\[\frac{\mu}{a} \int_0^t (1 + s)^{j-1} \int_R \bar{\rho} \Phi_{xx} g_{xxx} dx ds \leq C \int_0^t (1 + s)^{j-1} \int_R \Phi_{xx}^2 dx + C \int_0^t (1 + s)^{j-1} \int_R g_{xxx}^2 dx ds\]
\[\leq C \int_0^t (1 + s)^{j-1} \int_R \Phi_{xx}^2 dx + C \int_0^t (1 + s)^{-\frac{7}{2} + j} ds\]
\[\leq C \left( \| V_0 \|_3^2 + \| M_0 \|_2^2 + \| \Phi_0 \|_3^2 + \delta_0 \right),\]
(1 + t)^{j+1}\frac{\mu}{a} \int_\mathbb{R} \partial_t \Phi_{xx} g_{xx} dx \leq C \delta_0 (1 + t)^{j+1} \int_\mathbb{R} \Phi_{xx}^2 dx + \frac{C}{\delta_0} (1 + t)^{j+1} \int_\mathbb{R} g_{xx}^2 dx \\
leq \delta_0 (1 + t)^{j+1} \int_\mathbb{R} \Phi_{xx}^2 dx + C \delta_0 (1 + t)^{-\frac{j}{2} + \delta_0}
leq \delta_0 (1 + t)^{j+1} \int_\mathbb{R} \Phi_{xx}^2 dx + C \delta_0

and

\frac{\mu}{a} \int_0^t (1 + s)^j \int_\mathbb{R} \partial_t \Phi_{xx} g_{xx} dx ds \leq C \int_0^t (1 + s)^j \int_\mathbb{R} \Phi_{xx}^2 dx + C \int_0^t (1 + s)^j \int_\mathbb{R} g_{xx}^2 dx ds \\
leq C \int_0^t (1 + s)^j \int_\mathbb{R} \Phi_{xx}^2 dx + C \int_0^t (1 + s)^{-\frac{j}{2} + \delta_0} ds \\
leq C \left( \|V_0\|^2 + \|M_0\|^2 + \|\Phi_0\|^2 + \delta_0 \right).

Thus, we complete the proof of Lemma 4.4.

Similar to Lemma 4.4, we can get the following higher-order estimates for which we only outline the procedures without details for brevity.

**Lemma 4.5.** Under the assumption of Proposition 2.2, we have

\[(1 + t)^3 \Vert [V_{tt}, V_{xt}, \Phi_t, \Phi_{xt}] \Vert^2 + (1 + t)^2 \Vert V_t \Vert^2 \]

\[+ \int_0^t (1 + s)^3 \Vert [V_{tt}, \Phi_{tt}] \Vert^2 + (1 + t)^2 \Vert [V_{xt}, \Phi_t, \Phi_t] \Vert^2 ds \]

\[\leq C \left( \|V_0\|^3_3 + \|M_0\|^2 + \|\Phi_0\|^2_3 + \delta_0 \right) \tag{4.16} \]

and

\[(1 + t)^4 \Vert [V_{xxt}, V_{xxt}, \Phi_{xt}, \Phi_{xxt}] \Vert^2 + (1 + t)^3 \Vert V_{xxt} \Vert^2 \]

\[+ \int_0^t (1 + s)^4 \Vert [V_{xxt}, \Phi_{xxt}] \Vert^2 ds + (1 + t)^3 \Vert [V_{xxt}, \Phi_{xt}, \Phi_{xxt}] \Vert^2 ds \]

\[\leq C \left( \|V_0\|^2_3 + \|M_0\|^2_2 + \|\Phi_0\|^2_4 + \delta_0 \right). \tag{4.17} \]

**Proof.** In fact, taking integration over \( \mathbb{R} \times (0, t) \) of equations \( (1 + t)^i \times \{2[\partial_t (3.5) \times V_t] + \partial_t (3.5) + \partial_t (3.5) \times \frac{\mu}{a} \Phi_{tt}\} \), and \( (1 + t)^{i+1} \times \{2[\partial_t (3.5) \times V_{tt} + \partial_t (3.5) \times \frac{\mu}{a} \Phi_{tt}]\} \) for \( i = 0, 1, 2 \), we get \( (4.16) \). Taking integration over \( \mathbb{R} \times (0, t) \) of equations \( (1 + t)^j \times \{2[\partial_t (3.5) \times V_{xxt} + \partial_t (3.5) \times \frac{\mu}{a} \Phi_{xxt}]\} + \frac{\delta_t}{4} [\partial_t (3.5) \times V_{xxt} + \partial_t (3.5) \times \frac{\mu}{a} \Phi_{xxt}] \) for \( j = 0, 1, 2, 3 \), we get \( (4.17) \). \( \square \)

Note Proposition 4.4 is a direct consequence of Lemmas 4.2, 4.5, shown above. Thus, we can close the \textit{a priori} assumptions (1.1) by taking \( \epsilon \) to be sufficiently small in Proposition 2.2.

Finally we are in a position to prove Proposition 2.2 and Theorem 1.1.

**Proof of Proposition 2.2.** The first part of Proposition 2.2 (global existence) is a consequence of Proposition 3.1 and Proposition 3.2. For the decay rate, we have from Proposition 4.4 directly.
Proof of Theorem 1.1. Notice that \( M(x, t) = -V_t(x, t) \) and the transformation (2.1), we get (1.11) from (2.7) by the Sobolev inequality \( \|f\|_{L^\infty(\mathbb{R})}^2 \leq 2\|f\|_{L^2(\mathbb{R})}\|f_x\|_{L^2(\mathbb{R})} \) and hence complete the proof.

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