MULTIPLICATIVITY OF CONNES’ CALCULUS

PARTHA SARATHI CHAKRABORTY AND SATYAJIT GUIN

Abstract. We consider the quadruples \((\mathcal{A}, \mathcal{V}, D, \gamma)\) where \(\mathcal{A}\) is a unital, associative \(\mathbb{K}\)-algebra represented on the \(\mathbb{K}\)-vector space \(\mathcal{V}\), \(D \in \text{End}(\mathcal{V})\), \(\gamma \in \text{End}(\mathcal{V})\) is a \(\mathbb{Z}_2\)-grading operator which commutes with \(\mathcal{A}\) and anticommutes with \(D\). We prove that the collection of such quadruples, denoted by \(\text{Spec}_D\), is a monoidal category. We consider the monoidal subcategory \(\text{Spec}^\text{sub}_D\) of objects of \(\text{Spec}_D\) for which \(\gamma \in \pi(\mathcal{A})\). We show that there is a covariant functor \(\mathcal{G} : \text{Spec}_D \rightarrow \text{Spec}^\text{sub}_D\). Let \(\Omega^\star_D\) be the differential graded algebra defined by Connes \((\text{Con}^2)\) and \(\text{DGA}\) denotes the category of differential graded algebras over the field \(\mathbb{K}\). We show that \(\mathcal{F} : \text{Spec}^\text{sub}_D \rightarrow \text{DGA}\), given by \((\mathcal{A}, \mathcal{V}, D, \gamma) \mapsto \Omega^\star_D(\mathcal{A})\), is a monoidal functor. To show that \(\mathcal{F} \circ \mathcal{G}\) is not trivial we explicitly compute it for the cases of compact manifold and the noncommutative torus along with the associated cohomologies.

1. Introduction

A noncommutative differential structure on an associative algebra \(\mathcal{A}\) over a field \(\mathbb{K}\) is the specification of a differential graded algebra(dga), which is interpreted as the space of differential forms. Study of differential calculus in noncommutative geometry appears in early 80’s through the invention of noncommutative differential geometry \((\text{Con}^1)\), and to search for its examples \((\text{Wor}^1), (\text{Wor}^2)\). Since then quite a lot of works have been done involving differential calculus in various noncommutative contexts for e.g. \((\text{Pod}), (\text{BMa1}), (\text{BMa2}), (\text{Maj1}), (\text{Maj2}), (\text{BGM})\) and references therein. In his spectral formulation of the subject, Connes unified various treatments in noncommutative geometry in terms of a \(K\)-cycle \((\mathcal{A}, \mathcal{H}, D)\). He defined a canonical dga \(\Omega^\star_D(\mathcal{A})\) associated to a \(K\)-cycle \((\mathcal{A}, \mathcal{H}, D)\) and extended several classical notions including connection, curvature, Yang-Mills action functional etc. to the noncommutative framework. It is also shown in \((\text{Con}^2)\) that using this dga one can produce Hochschild cocycle and cyclic cocycle (under certain assumption) for Poincaré dual algebras which establishes \(\Omega^\star_D\) worth studying. Since it is possible to multiply even \(K\)-cycles

\[(A_1, H_1, D_1, \gamma_1) \otimes (A_2, H_2, D_2, \gamma_2) := (A_1 \otimes A_2, H_1 \otimes H_2, D_1 \otimes 1 + \gamma_1 \otimes D_2, \gamma_1 \otimes \gamma_2),\]

natural question strikes regarding the behaviour of \(\Omega^\star_D\) under this multiplication \((\text{Mad})\) and this is the content of this paper. This question was investigated earlier in \((\text{KaT})\) and main out-turn was that \(\Omega^\star_D(A_1 \otimes A_2) \not\cong \Omega^\star_D(A_1) \otimes \Omega^\star_D(A_2)\) in general. Here \(\Omega^\star_D(A_1) \otimes \Omega^\star_D(A_2)\) denotes the tensor product of two differential graded algebras. We call it multiplicativity property of \(\Omega^\star_D\) and hence, in this language, the result in \((\text{KaT})\) states that \(\Omega^\star_D\) is in general not multiplicative.

Since the output of \((\text{KaT})\) is not conclusive we reinvestigate this question. To define \(\Omega^\star_D\) one does not use self-adjointness and compactness of the resolvent of \(D\). We cast Connes definition in a slightly more general algebraic framework. We consider the quadruple \((\mathcal{A}, \mathcal{V}, D, \gamma)\) where \(\mathcal{A}\) is an associative, unital algebra over \(\mathbb{K}\), represented on a vector space \(\mathcal{V}\), \(D \in \text{End}(\mathcal{V})\), \(\gamma \in \text{End}(\mathcal{V})\) is a \(\mathbb{Z}_2\)-grading operator

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which commutes with \( A \) and anticommutes with \( D \). We show that the collection of such quadruple

\((A, V, D, \gamma)\)

is a monoidal category and denote it by \( \widehat{\text{Spec}} \). We identify a smaller subcategory \( \widehat{\text{Spec}}_{\text{sub}} \) and show that there is a covariant functor \( \mathcal{G} : \widehat{\text{Spec}} \to \widehat{\text{Spec}}_{\text{sub}} \). Moreover, \( \widehat{\text{Spec}}_{\text{sub}} \) becomes a monoidal subcategory of \( \widehat{\text{Spec}} \). Next we consider the category \( \text{DGA} \) of differential graded algebras over a field \( K \) and show that the association \( F : (A, V, D, \gamma) \to \Omega^*_{D}(A) \) gives a covariant functor from \( \widehat{\text{Spec}} \) to \( \text{DGA} \).

In this category theoretic language, article \([KaT]\) says that this functor is in general not monoidal. We show that restricted to \( \widehat{\text{Spec}}_{\text{sub}} \), \( F \) becomes a monoidal functor. To validate the nontriviality of this functor, i.e. the associated dga \( \Omega^*_{D} \) is not trivial, we explicitly compute \( F \circ \mathcal{G} \) for the cases of compact manifold and the noncommutative torus. We also compute the associated cohomologies in each case and it turns out that the resulting dga \( \Omega^*_{D} \) in these two cases is cohomologically also not trivial.

Organization of this paper is as follows. In section (2) we first define algebraic spectral triple and go through the definition of Connes’ calculus \( \Omega^* \). Then we formulate the category \( \widehat{\text{Spec}} \) and prove that it is a monoidal category and \( F : (A, V, D, \gamma) \to \Omega^*_{D}(A) \) is a covariant functor. Next we identify the subcategory \( \widehat{\text{Spec}}_{\text{sub}} \) and obtain the covariant functor \( \mathcal{G} : \widehat{\text{Spec}} \to \widehat{\text{Spec}}_{\text{sub}} \). Finally we show that the functor \( F \) restricted to \( \widehat{\text{Spec}}_{\text{sub}} \) is a monoidal functor between the monoidal categories \( \widehat{\text{Spec}}_{\text{sub}} \) and \( \text{DGA} \). Sections (3) and (4) have been devoted to the computation for the cases of compact manifold and the noncommutative torus respectively.

## 2. Multiplicativity of Connes’ Calculus

**Definition 2.1.** An algebraic spectral triple \((A, V, D)\), over an unital associative \( K \)-algebra \( A \), consists of the following things:

1. A representation \( \pi \) of \( A \) on a \( K \)-vector space \( V \),
2. A linear operator \( D \) acting on \( V \).

It is said to be an even algebraic spectral triple if there exists a \( \mathbb{Z}_2 \)-grading \( \gamma \in \text{End}(V) \) such that \( \gamma \) commutes with each element of \( A \) and anticommutes with \( D \). It will be assumed that \( A \) is unital and the unit \( 1 \in A \) acts as the identity operator on \( V \).

**Definition 2.2.** Let \( \Omega^*_{k}(A) = \bigoplus_{k=0}^{\infty} \Omega^k_{k}(A) \) be the reduced universal differential graded algebra over \( A \). Here \( \Omega^k(A) := A \otimes \mathbb{A}^k \), \( \mathbb{A} = A/K \). The graded product is given by

\[
\left( \sum_{k} a_{0k} \otimes \bar{a}_{1k} \otimes \ldots \otimes \bar{a}_{mk} \right) \cdot \left( \sum_{k'} b_{0k'} \otimes \bar{b}_{1k'} \otimes \ldots \otimes \bar{b}_{nk'} \right)
\]

\[
= \sum_{k,k'} a_{0k} \otimes (\otimes_{j=1}^{m} \bar{a}_{jk}) \otimes \bar{a}_{mk} b_{0k'} \otimes (\otimes_{i=1}^{n} \bar{b}_{ik')}
\]

\[
+ \sum_{i=1}^{m-1} (-1)^{i} a_{0k} \otimes \bar{a}_{1k} \otimes \ldots \otimes \bar{a}_{m-i,k} \otimes a_{m-i+1,k} \otimes \ldots \otimes \bar{a}_{mk} \otimes (\otimes_{i=0}^{n} \bar{b}_{ik'})
\]

\[
+ (-1)^{m} a_{0k} \otimes \bar{a}_{1k} \otimes (\otimes_{j=2}^{m} \bar{a}_{jk}) \otimes (\otimes_{i=1}^{n} \bar{b}_{ik'})
\]

for \( \sum_{k} a_{0k} \otimes \bar{a}_{1k} \otimes \ldots \otimes \bar{a}_{mk} \in \Omega^m(A) \) and \( \sum_{k'} b_{0k'} \otimes \bar{b}_{1k'} \otimes \ldots \otimes \bar{b}_{nk'} \in \Omega^n(A) \). There is a differential \( d \) acting on \( \Omega^*_{k}(A) \) given by

\[
d(a_{0} \otimes \bar{a}_{1} \otimes \ldots \otimes \bar{a}_{k}) = 1 \otimes \bar{a}_{0} \otimes \bar{a}_{1} \otimes \ldots \otimes \bar{a}_{k} \forall a_{j} \in A,
\]

and it satisfies the relations
Consider the reduced universal dgas

\( \Omega^\bullet(A) \),

where the description of \( \Omega^\bullet(A) \) is given by the following even algebraic spectral triple

\[
\Omega^\bullet = \{ J^{(k)} = \{ \omega \in \Omega^k : \pi(\omega) = 0 \} \text{ and } J' = \bigoplus J^{(k)} \}. \quad \text{But } J' \text{ is not a differential graded ideal. We consider } \quad \Omega^\bullet = \bigoplus J^{(k)} \text{ where } J^{(k)} = J^{(k)} + dJ^{(k-1)}. \text{ Then } J^\bullet \text{ becomes a differential graded two-sided ideal and hence the quotient } \Omega^\bullet_{J^\bullet} = \Omega^\bullet / J^\bullet \text{ becomes a differential graded algebra. The representation } \pi \text{ gives an isomorphism,}
\]

\[
(2.1) \quad \Omega^k_D \cong \pi(\Omega^k)/\pi(dJ^{k-1}) \quad \forall k \geq 1.
\]

The abstract differential \( d \) induces a differential \( d \) on the complex \( \Omega^\bullet_D(A) \) so that we get a chain complex \( (\Omega^\bullet_D(A), d) \) and a chain map \( \pi_D : \Omega^\bullet(A) \to \Omega^\bullet_D(A) \) such that the following diagram

\[
\begin{array}{ccc}
\Omega^\bullet(A) & \xrightarrow{\pi_D} & \Omega^\bullet_D(A) \\
\downarrow & & \downarrow \\
\Omega^{\bullet+1}(A) & \xrightarrow{\pi_D} & \Omega^{\bullet+1}_D(A) \\
\end{array}
\]

commutes. This makes \( \Omega^\bullet_D \) a differential graded algebra.

Let \( (A_1, V_1, D_1, \gamma_1) \) and \( (A_2, V_2, D_2, \gamma_2) \) be two even algebraic spectral triples. The product of these is given by the following even algebraic spectral triple

\[
(A_1, V_1, D_1, \gamma_1) \otimes (A_2, V_2, D_2, \gamma_2) := (A_1 \otimes A_2, V_1 \otimes V_2, D_1 \otimes 1 + \gamma_1 \otimes D_2, \gamma_1 \otimes \gamma_2).
\]

One can consider two dgas \( \Omega^\bullet_{D_1}(A_1) \) and \( \Omega^\bullet_{D_2}(A_2) \). The product of these two dgas is given by

\[
\Omega^\bullet_{D_1}(A_1) \otimes \Omega^\bullet_{D_2}(A_2) := \bigoplus_{n \geq 0} \bigoplus_{i+j=n} \Omega^i_{D_1}(A_1) \otimes \Omega^j_{D_2}(A_2).
\]

It is natural to ask how \( \Omega^\bullet_{D(n)} \) behaves under this multiplication, i.e. whether

\[
\Omega^\bullet_{D(n)}(A_1 \otimes A_2) \cong \bigoplus_{i+j=n} \Omega^i_{D_1}(A_1) \otimes \Omega^j_{D_2}(A_2) \quad \forall n \geq 0.
\]

Article \[ \text{[Kall]} \] deals with this investigation and does not lead to a final conclusion. However, using the universality of \( \Omega^\bullet(A_1 \otimes A_2) \), a useful outcome was that for all \( n \geq 0 \)

\[
(2.3) \quad \Omega^\bullet_{D(n)}(A_1 \otimes A_2) \cong \Omega^\bullet_{D}(A_1, A_2),
\]

where the description of \( \Omega^\bullet_{D}(A_1, A_2) \) is given by the following definition.

**Definition 2.3.** Consider the reduced universal dgas \( (\Omega^\bullet(A_1), d_1) \) and \( (\Omega^\bullet(A_2), d_2) \), associated with the algebraic spectral triples \( (A_1, V_1, D_1, \gamma_1) \) and \( (A_2, V_2, D_2, \gamma_2) \) respectively. Consider the product dga

\[
\big( \Omega^\bullet(A_1) \otimes \Omega^\bullet(A_2), \bar{d} \big)
\]

where

\[
(2.4) \quad (\omega_i \otimes u_j) \cdot (\omega_p \otimes u_q) := (-1)^{ip} \omega_i \omega_p \otimes u_j u_q,
\]

\[
(2.5) \quad \bar{d}(\omega_i \otimes u_j) := d_1(\omega_i) \otimes u_j + (-1)^{i} \omega_i \otimes d_2(u_j),
\]
for $\omega, \psi \in \Omega^*(A_1)$ and $u, v \in \Omega^*(A_2)$. One can define a representation $\tilde{\pi}$ of $\Omega^*(A_1) \otimes \Omega^*(A_2)$ by

$$\tilde{\pi}(\omega \otimes u) := \pi_1(\omega) \gamma_1^I \otimes \pi_2(u).$$

Let

$$\tilde{J}_0^k := \text{Ker} \left\{ \tilde{\pi} : \bigoplus_{i+j=k} \Omega^i(A_1) \otimes \Omega^j(A_2) \to \text{End}(V_1 \otimes V_2)^k \right\},$$

and $\tilde{J}_n = \tilde{J}_0^n + \tilde{d} \tilde{J}_0^{n-1}$. Define $\tilde{\Omega}_D^j(A_1, A_2) := \bigoplus_{i+j=n} \Omega^i(A_1) \otimes \Omega^j(A_2)$ if and only if

$$\tilde{J}_n(A_1, A_2) \cong \bigoplus_{i+j=n} J^i(A_1) \otimes \Omega^j(A_2) + \Omega^i(A_1) \otimes J^j(A_2).$$

But it is in general not true. This is the prime investigation of this article. We propose a category theoretic construction of even algebraic spectral triples, which satisfies (2.8).

**Definition 2.4.** The objects of the category $\tilde{\text{Spec}}$ are even algebraic spectral triples $(A, V, D, \gamma)$. Given two such objects $(A_i, V_i, D_i, \gamma_i)$, with $i = 1, 2$, a morphism between them is a pair $(\phi, \Phi)$ where $\phi : A_1 \to A_2$ is unital algebra morphism between the algebras $A_1, A_2$ and $\Phi \in \text{End}(V_1, V_2)$ is surjective which intertwines the representations $\pi_1, \pi_2 \circ \phi$ and the operators $D_1, D_2$ or equivalently the following diagrams commute for every $x \in A_1$:

\[
\begin{array}{ccc}
V_1 & \xrightarrow{\phi} & V_2 \\
D_1 \downarrow & & \downarrow D_2 \\
V_1 & \xrightarrow{\Phi} & V_2 \\
\end{array}
\quad
\begin{array}{ccc}
V_1 & \xrightarrow{\Phi} & V_2 \\
\pi_1(x) \downarrow & & \downarrow \pi_2 \circ \phi(x) \\
V_1 & \xrightarrow{\Phi} & V_2 \\
\end{array}
\]

and $\Phi$ also intertwines the grading operators $\gamma_1, \gamma_2$.

**Remark 2.5.** This definition is essentially from [BCL1]. However, our requirement demands the extra condition on surjectivity of $\Phi$. This is in line with [Eps], [Ter].

**Proposition 2.6.** The category $\tilde{\text{Spec}}$ is a monoidal category.

**Proof.** Define the identity object ‘1’ of monoidal category as follows

$$1 := (K, K, 0, 1).$$

Define the functor tensor product ‘$\otimes$’ on objects as

$$(A_1, V_1, D_1, \gamma_1) \otimes (A_2, V_2, D_2, \gamma_2) := (A_1 \otimes A_2, V_1 \otimes V_2, D_1 \otimes D_2 + \gamma_1 \otimes D_2, \gamma_1 \otimes \gamma_2)$$
and on morphisms

\[(\phi, \Phi) : (A, \mathbb{V}, D, \gamma) \longrightarrow (\tilde{A}, \tilde{\mathbb{V}}, \tilde{D}, \tilde{\gamma})\]

\[(\phi', \Phi') : (A', \mathbb{V}', D', \gamma') \longrightarrow (\tilde{A}', \tilde{\mathbb{V}}', \tilde{D}', \tilde{\gamma}')\]

by \((\phi \otimes \phi', \Phi \otimes \Phi')\), where \(\phi \otimes \phi'\) is the usual tensor product of two algebra morphisms and \(\Phi \otimes \Phi'\) is the usual tensor product of two linear maps. Now one can easily verify all the conditions of a monoidal category. \(\square\)

Let \(DGA\) be the category of differential graded algebras over field \(\mathbb{K}\). We will only consider nonnegatively graded algebras in this article.

**Lemma 2.7.** There is a covariant functor \(F : \tilde{\text{Spec}} \longrightarrow DGA\) given by \((A, \mathbb{V}, D, \gamma) \longrightarrow \Omega^\bullet_D(A)\).

**Proof.** Consider two objects \((A_1, \mathbb{V}_1, D_1, \gamma_1), (A_2, \mathbb{V}_2, D_2, \gamma_2) \in \text{Ob}(\tilde{\text{Spec}})\) and suppose there is a morphism \((\phi, \Phi) : (A_1, \mathbb{V}_1, D_1, \gamma_1) \longrightarrow (A_2, \mathbb{V}_2, D_2, \gamma_2)\). Define

\[\Psi : \Omega_{D_1}^\bullet(A_1) \longrightarrow \Omega_{D_2}^\bullet(A_2) \quad \left[\sum a_0 \prod_{i=1}^n [D_1, a_i]\right] \longmapsto \left[\sum \phi(a_0) \prod_{i=1}^n [D_2, \phi(a_i)]\right]\]

for all \(a_j \in A_j, n \geq 0\). To show \(\Psi\) is well-defined we must show that \(\Psi(\pi(d_1J_0^n)) \subseteq \pi(d_2J_0^n)\) for all \(m \geq 1\), where \(d_1, d_2\) are the universal differentials for \(\Omega^\bullet(A_1), \Omega^\bullet(A_2)\) respectively. Observe that

\[(2.9) \quad \Phi \circ \left(\sum a_0 \prod_{i=1}^n [D_1, a_i]\right) = \left(\sum \phi(a_0) \prod_{i=1}^n [D_2, \phi(a_i)]\right) \circ \Phi.\]

Consider arbitrary element \(\xi \in \pi(d_1J_0^n)\). By definition, \(\xi = \sum \prod_{i=0}^n [D_1, a_i] \in \pi(d_1J_0^n)\) such that \(\sum a_0 \prod_{i=1}^n [D_1, a_i] = 0\). Now using equation (2.9) and surjectivity of \(\Phi\), we have

\[\sum \phi(a_0) \prod_{i=1}^n [D_2, \phi(a_i)] = 0.\]

This shows well-definedness of \(\Psi\). Now it is easy to check that \(\Psi\) is a differential graded algebra morphism. \(\square\)

**Remark 2.8.** This is the only place where we needed the stronger assumption on surjectivity of the map \(\Phi\) and because of this reason we differ from (HCL).

Now consider \((A, \mathbb{V}, D, \gamma) \in \text{Ob}(\tilde{\text{Spec}})\) such that \(\gamma \in \pi(A)\). Let \(\tilde{\text{Spec}}_{\text{sub}}\) be the subcategory of \(\tilde{\text{Spec}}\), objects of which are \((A, \mathbb{V}, D, \gamma)\) with \(\gamma \in \pi(A)\). Clearly \(\tilde{\text{Spec}}_{\text{sub}}\) is a monoidal subcategory of \(\tilde{\text{Spec}}\). Now suppose \((A, \mathbb{V}, D, \gamma) \in \text{Ob}(\tilde{\text{Spec}})\) and \(\gamma \notin \pi(A)\). Consider the vector space \(A \oplus A\) with the product rule

\[(a, b) \star (\tilde{a}, \tilde{b}) := (a\tilde{a} + b\tilde{b}, a\tilde{b} + b\tilde{a}).\]

The algebra \((A \oplus A, \star)\) becomes unital with unit \((1, 0)\). The map \((a, b) \longrightarrow (a + b, a - b)\) gives a unital algebra isomorphism between the algebra \((A \oplus A, \star)\) and the direct sum algebra \(A \oplus \mathbb{A}\) where the multiplication is defined as co-ordinatewise. Now the map

\[(a, b) \longrightarrow \pi(a) + \gamma \pi(b) \in \mathcal{E}nd(\mathbb{V})\]

gives a representation of the unital algebra \((A \oplus A, \star)\) on the vector space \(\mathbb{V}\). Since \((0, 1) \longrightarrow \gamma \in \mathcal{E}nd(\mathbb{V})\) we have \(\gamma \in \pi((A \oplus A, \star))\) and hence \(((A \oplus A, \star), \mathbb{V}, D, \gamma) \in \text{Ob}(\tilde{\text{Spec}}_{\text{sub}})\).
Proposition 2.9. The association $\mathcal{G} : (A, V, D, \gamma) \mapsto ((A \oplus A, \ast), V, D, \gamma)$ gives a covariant functor from $\text{Spec}$ to $\text{Spec}_{\text{sub}}$.

Proof. For a morphism $(\phi, \Phi) : (A, V, D, \gamma) \rightarrow (A', V', D', \gamma')$, define

$$(\tilde{\phi}, \tilde{\Phi}) : ((A \oplus A, \ast), V, D, \gamma) \rightarrow ((A' \oplus A', \ast), V', D', \gamma')$$

by taking $\tilde{\Phi} := \Phi$ and

$$\tilde{\phi} : A \oplus A \rightarrow A' \oplus A'$$

$$(a, b) \longmapsto (\phi(a), \phi(b)).$$

It is easy to check that $(\tilde{\phi}, \tilde{\Phi})$ defines a morphism in $\text{Spec}_{\text{sub}}$. □

In the next two sections we will see that the functor $\mathcal{F} \circ \mathcal{G}$ is not trivial. Throughout the rest of this article we will reserve the notation $\mathcal{F}$ and $\mathcal{G}$ to mean the functors in Lemma 2.7 and Proposition 2.9 respectively.

Theorem 2.10. Restricted to the monoidal subcategory $\text{Spec}_{\text{sub}}$ of $\text{Spec}$, the covariant functor $\mathcal{F} : \text{Spec}_{\text{sub}} \rightarrow DGA$ defined in Lemma 2.7 is a monoidal functor.

Proof. Only nontrivial part is to prove that

$$\Omega^1_D(A_1 \otimes A_2) \cong \bigoplus_{i+j=n} \Omega^i_D(A_1) \otimes \Omega^j_D(A_2).$$

where $D = D_1 \otimes 1 + \gamma_1 \otimes D_2$. We break the proof into two lemmas.

Lemma 2.11. For any $a \in A$, $[D^2, a] \in \pi(dJ^1_0)$.

Proof. Consider $p = (1 + \gamma)/2$ and $q = (1 - \gamma)/2$. Then $pq = 0$ and $pDp = qDq = 0$. Consider $a \in A$ and $\eta \in A$ be such that $\pi(\eta) = \gamma$. Now consider $\omega = \frac{1}{4}(1 + \eta)d(a)(1 + \eta) + \frac{1}{4}(1 - \eta)d(a)(1 - \eta)$ in $\Omega^1(A)$. Then,

$$\pi(\omega) = p[D, ap] - pa[D, p] + q[D, aq] - qa[D, q]$$

$$= pDap - papD - paDp + papD + qDq - qaqD - qaDq + qaqD$$

$$= 0; \text{ since } pap = pa = ap; qaq = qa = aq.$$
Now observe that \( pD^2q = pD(p+q)Dq = 0 \) and \( qD^2p = 0 \). Hence,

\[
[D^2, a] = [(p+q)D^2(p+q), (p+q)a] \\
= [pD^2p + qD^2q, pa + qa] \\
= [pD^2p, pa] + [qD^2q, qa] \\
= \pi(d\omega)
\]

This proves that \( [D^2, a] \in \pi(dJ^n(A)) \).

**Lemma 2.12.** We have

\[
\bar{J}^n(A_1, A_2) = \bigoplus_{i+j=n} J^i(A_1) \otimes \Omega^j(A_2) + \Omega^i(A_1) \otimes J^j(A_2),
\]

where definition of \( \bar{J}^n \) is provided in definition (2.3).

**Proof.** Let \( \omega = \bar{\pi}(\tilde{d}\omega') \) where \( \omega' \in \bar{J}^{-1}_0 \). Suppose \( \omega' = \sum_k \bigoplus_{i+j=n-1} v^i_{1,k} \otimes v^j_{2,k} \), where \( v^i_{1,k} \in \Omega^i(A_1) \) and \( v^j_{2,k} \in \Omega^j(A_2) \). Hence we have the following equation,

\[
(2.10) \quad \sum_{k} \sum_{i+j=n-1} \pi_1(v^i_{1,k}) \gamma_i^j \otimes \pi_2(v^j_{2,k}) = 0.
\]

Let

\[
v^i_{1,k} = \sum a^{(i)}_{0,k} \prod_{r=1}^i d_1(a^{(i)}_{r,k})
\]

\[
v^j_{2,k} = \sum b^{(j)}_{0,k} \prod_{s=1}^j d_2(b^{(j)}_{s,k})
\]

for \( a^{(i)}_{r,k} \in A_1 \) and \( b^{(j)}_{s,k} \in A_2 \). Then equation (2.10) becomes

\[
(2.11) \quad \sum_{k} \sum_{i+j=n-1} \sum \left( a^{(i)}_{0,k} \prod_{r=1}^i [D_1, a^{(i)}_{r,k}] \gamma_i^j \right) \otimes \left( b^{(j)}_{0,k} \prod_{s=1}^j [D_2, b^{(j)}_{s,k}] \right) = 0.
\]

Now since \( \omega' = \sum_k \bigoplus_{i+j=n-1} v^i_{1,k} \otimes v^j_{2,k} \),

\[
\tilde{d}(\omega') = \sum_k \sum_{i+j=n-1} d_1(v^i_{1,k}) \otimes v^j_{2,k} + (-1)^i v^i_{1,k} \otimes d_2(v^j_{2,k})
\]

and hence,

\[
\bar{\pi}(\tilde{d}\omega') = \sum_k \sum_{i+j=n-1} \pi_1(d_1(v^i_{1,k})) \gamma_i^j \otimes \pi_2(v^j_{2,k}) + (-1)^i \pi_1(v^i_{1,k}) \gamma_i^j \otimes \pi_2(d_2(v^j_{2,k})).
\]

Using equation (2.11) we get,

\[
\sum_{k} \sum_{i+j=n-1} \pi_1(d_1(v^i_{1,k})) \gamma_i^j \otimes \pi_2(v^j_{2,k})
\]

\[
= \sum_{k} \sum_{i+j=n-1} \sum \left( -a^{(i)}_{0,k} D_1 \prod_{r=1}^i [D_1, a^{(i)}_{r,k}] \gamma_i^j \right) \otimes \left( b^{(j)}_{0,k} \prod_{s=1}^j [D_2, b^{(j)}_{s,k}] \right)
\]

\[
= -\sum \sum_{i+j=n-1} \sum \sum_{r=1}^i \left( (-1)^{r+1} a^{(i)}_{0,k} [D_1, a^{(i)}_{r,k}] \ldots [D_2, a^{(i)}_{r,k}] \ldots [D_1, a^{(i)}_{1,k}] \gamma_i^j \right)
\]

\[
\otimes \left( b^{(j)}_{0,k} \prod_{s=1}^j [D_2, b^{(j)}_{s,k}] \right) - \left( (-1)^j a^{(i)}_{0,k} \prod_{r=1}^j [D_1, a^{(i)}_{r,k}] \gamma_i^j \right) \otimes \left( b^{(j)}_{0,k} \prod_{s=1}^j [D_2, b^{(j)}_{s,k}] \right).
\]
This term is contained in $\sum_{i+j=n} \pi_1(J^1) \gamma^i_1 \otimes \pi_2(\Omega^j)$, since we have seen that $[D^i_j, a^i_{r,k}]$ is in $J^2$ for each $1 \leq r \leq i$ (Lemma 2.11). Finally,

$$\sum_k \sum_{i+j=n-1} (-1)^i \pi_1(v^1_{1,k}) \gamma^{j+1} \otimes \pi_2(d_2(v^2_{1,k}))$$

$$= \sum_k \sum_{i+j=n-1} \sum [\gamma_i \otimes D_2, \left( a^i_{0,k} \prod_{t=1}^i (D_1, a^j_{t,k}) \gamma_1^1 \right) \otimes \left( b^{j+1}_{0,k} \prod_{s=1}^j (D_2, b^{j}_{s,k}) \right)]$$

$$+ \sum_{r=1}^j (-1)^{r+1} \left( a^i_{r,k} \prod_{t=1}^r (D_1, a^j_{t,k}) \gamma_1^1 \right) \otimes \left( b^{j+1}_{0,k} \prod_{s=1}^j (D_2, b^{j}_{s,k}) \right).$$

This term is in $\sum_{i+j=n} \pi_1(\Omega^1) \gamma^i_1 \otimes \pi_2(J^j)$, since $[D^i_j, b^j_{r,k}] \in J^2$ for each $1 \leq r \leq j$ (2.11). So we get

$$\bar{\pi}(\bar{J}^n) \subseteq \sum_{i+j=n} \pi_1(\Omega^1) \gamma^i_1 \otimes \pi_2(J^j) + \pi_1(J^j) \gamma^i_1 \otimes \pi_2(\Omega^j),$$

i.e. $\bar{\pi}(\bar{J}^n) \subseteq \bar{\pi} \left( \bigoplus_{i+j=n} (J^i \otimes \Omega^j) \right).$

Hence,

$$\frac{j^n}{\bigoplus_{i+j=n} (J^i \otimes \Omega^j)} \cong \frac{\bar{\pi}(\bar{J}^n)}{\bar{\pi}(\bigoplus_{i+j=n} (J^i \otimes \Omega^j))} = \{0\}$$

and our claim has been justified.

**Proof of Theorem 2.10:** Lemma 2.12 proves that the isomorphism in equation (2.8) holds i.e.

$$\bar{J}^n(A_1, A_2) \cong \bigoplus_{i+j=n} J^i(A_1) \otimes \Omega^j(A_2) + \Omega^i(A_1) \otimes J^j(A_2),$$

when we restrict ourselves to the subcategory $\widetilde{Spec}_{cmab}$. Hence the proof follows from the fact that $\Omega^0(B_1 \otimes B_2) \cong \Omega^1(B_1, B_2)$ for all $n \geq 0$ and for any unital algebras $B_1, B_2$ (see the isomorphism in 2.3).

**Corollary 2.13.** $\mathcal{F}(\mathcal{G}(A_1) \otimes \mathcal{G}(A_2)) \cong \mathcal{F} \circ \mathcal{G}(A_1) \otimes \mathcal{F} \circ \mathcal{G}(A_2)$.

However, we do not know whether $\mathcal{F} \circ \mathcal{G}(A_1 \otimes A_2) \cong \mathcal{F} \circ \mathcal{G}(A_1) \otimes \mathcal{F} \circ \mathcal{G}(A_2)$.

**3. Computation for Compact Manifold**

In this section we show that there exists a contravariant functor $\mathcal{P}$ from the category of manifolds with embeddings as morphisms to the category $\widetilde{Spec}$ and we show that $\mathcal{F} \circ \mathcal{G} \circ \mathcal{P}$ is not trivial.

Let $\mathcal{M}$ be a compact manifold of dimension $n$ with atlas $\{U_i, \phi_i\}_{i=1}^k$. Consider the complexified exterior bundle $\wedge^{*}\mathcal{M}$ over $\mathcal{M}$ and $(x^1, \ldots, x^n)$ denotes the local co-ordinates. Let $d$ be the exterior differentiation. If we consider the category of manifolds $\mathcal{M}$ with embeddings as morphisms ([Eps], [Ter]), then there is a contravariant functor from $\mathcal{M}$ to $\widetilde{Spec}$. To see this consider the following object

$$\left( C^\infty(\mathcal{M}), \Gamma(\wedge^*\mathcal{M}) \cong \Gamma(\wedge^{even}\mathcal{M}) \oplus \Gamma(\wedge^{odd}\mathcal{M}), D := \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}, \gamma := \text{parity} \right)$$

in $\widetilde{Spec}$, where ‘parity’ means the odd-even parity of a form in $\Gamma(\wedge^*\mathcal{M})$. Now for an embedding $\phi : \mathcal{M} \hookrightarrow \mathbb{N}$, we have

$$\phi : \left( C^\infty(\mathbb{N}), \Gamma(\wedge^*\mathbb{N}), \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}, \gamma \right) \rightarrow \left( C^\infty(\mathcal{M}), \Gamma(\wedge^*\mathcal{M}), \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}, \gamma \right),$$

but $\mathcal{F} \circ \mathcal{G} \circ \mathcal{P}$ is not trivial.
a morphism in $\tilde{\mathcal{S}}pec$. Moreover, the following commutative diagram

$$
\begin{array}{ccc}
\Gamma(\wedge^k T^* \mathbb{N}) & \xrightarrow{d} & \Gamma(\wedge^{k+1} T^* \mathbb{N}) \\
\phi^* & & \phi^* \\
\downarrow & & \downarrow \\
\Gamma(\wedge^k T^* \mathbb{M}) & \xrightarrow{d} & \Gamma(\wedge^{k+1} T^* \mathbb{M})
\end{array}
$$

where $\phi^*$ is the pullback of $\phi$, explains that the consideration of the quadruple $(C^\infty(\mathbb{M}), \Gamma(\wedge^k T^* \mathbb{M}), D, \gamma)$ is natural. Henceforth we will be dealing with $(C^\infty(\mathbb{M}), \Gamma(\wedge^k T^* \mathbb{M}), D, \gamma) \in \mathcal{Ob}(\tilde{\mathcal{S}}pec)$ in this section, where $D = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Notice that $D^2 = 0$. Since $\gamma \notin \pi(C^\infty(\mathbb{M}))$ we first apply the functor $\mathcal{G}$ of Proposition 2.9 and then compute $\mathcal{F} \circ \mathcal{G}$ along with the associated cohomologies.

**Notation :** $C^\infty(\mathbb{M}) := \mathcal{G}(C^\infty(\mathbb{M}), \Gamma(\wedge^k T^* \mathbb{M}), D, \gamma)$ where $\mathcal{G}$ is defined in Proposition 2.9 and $\text{dim}(\mathbb{M}) = n$ throughout this section.

**Lemma 3.1.** $\Omega^m_0 \left( C^\infty(\mathbb{M}) \right) \cong \pi \left( \Omega^m(\tilde{\mathcal{S}}pec) \right) \forall m \geq 0.$

**Proof.** Observe that $J_0^0 \left( C^\infty(\mathbb{M}) \right) = \{0\}$ in this case. We show that $\pi \left( dJ_0^m(\tilde{\mathcal{S}}pec) \right) = \{0\} \forall m \geq 1$. Note that

$$
\pi(dJ_0^m) = \{ \sum_{i=0}^m \prod [D, x_i] : x_i \in C^\infty(\mathbb{M}) ; \sum_{i=0}^m \prod [D, x_i] = 0 \}
$$

$$
= \{ - \sum x_0 D \prod_{i=1}^m [D, x_i] : x_i \in C^\infty(\mathbb{M}) ; \sum_{i=1}^m \prod [D, x_i] = 0 \}
$$

Now,

$$
\sum x_0 D \prod_{i=1}^m [D, x_i] = \sum x_0 D \prod_{i=1}^m (Dx_i - x_i D)
$$

$$
= - \sum x_0 D x_1 D \prod_{i=2}^m (Dx_i - x_i D)
$$

$$
= (-1)^m \sum x_0 D \prod_{i=1}^m D x_i D
$$

$$
= (-1)^m \left( \sum x_0 \prod_{i=1}^{m-1} [D, x_i] \right) D x_m D
$$

$$
= (-1)^m \left( \sum x_0 \prod_{i=1}^m [D, x_i] \right) D
$$

But $\sum x_0 \prod_{i=1}^m [D, x_i] = 0$ by assumption and hence we are done. \hfill \Box$

Let $1 \leq m \leq n$, where $n = \text{dim}(\mathbb{M})$. We define the following linear operator

$$
T_{a_0, \ldots, a_m} : \Gamma(\wedge^k T^* \mathbb{M}) \longrightarrow \Gamma(\wedge^k T^* \mathbb{M})
$$

$$
\omega \mapsto a_0 d a_1 \wedge \ldots \wedge d(a_m \omega)$$
where \( a_i \in C^\infty(M) \). Let \( \mathcal{M}_m = \text{span}\{T_{a_0,\ldots,a_m} : \Gamma(\bigwedge^\bullet T^*M) \rightarrow \Gamma(\bigwedge^\bullet T^*M) : a_i \in C^\infty(M)\} \). Then \( \mathcal{M}_m \) is a \( \mathbb{C} \)-vector space. Note that for \( a, b \in C^\infty(M) \)

\[
\begin{bmatrix}
D_i (a \ a) \\
0 \\
0 \\
0 \\
\end{bmatrix} = 
\begin{bmatrix}
0 \\
T_{1,a} - T_{b,1} \\
0 \\
\end{bmatrix}.
\]

Since elements of \( \pi \left( \Omega^m(C_\infty(M)) \right) \) are of the form

\[
\sum \left( a_0 \ 0 \ b_0 \right) \prod_{i=1}^m D_i \left( a_i \ 0 \ b_i \right) ; \ a_j, b_j \in C^\infty(M),
\]

it is easy to observe that \( \pi \left( \Omega^m(C_\infty(M)) \right) \) is a subspace of \( \mathcal{M}_m \oplus \mathcal{M}_m \). Moreover, using the equality

\[
(3.12) \quad \sum_k \left( a_{0k} \ 0 \ 0 \right) \left[ D_i (0 \ 0 \ 1) \left( -a'_{1k} \ 0 \ 0 \right) \right] = \sum_k \left( 0 \ T_{a_{0k} \cdot a_{1k}} \right).
\]

we see that for \( m \geq 3 \) odd

\[
\sum \left( 0 \ T_{a_{0k}, a_{1k}, \ldots, a_{m}} \right)
\]

\[
= \sum \left( T_{a_{0k} \cdot a_{1k}, a_{m}} \right) \prod_{i=2, i \text{ even}}^m \left(D_i \left( 0 \ 0 \ 1 \right) \left(-a'_{1i} \ 0 \ 0 \right) \right)
\]

\[
= \sum \left( 0 \ 0 \ a_{0k} \right) \prod_i^m \left(D_i \left( 0 \ 0 \ 1 \right) \left(-a'_{1i} \ 0 \ 0 \right) \right)
\]

and similarly for \( m \geq 2 \) even. Hence we conclude that \( \pi \left( \Omega^m(C_\infty(M)) \right) = \mathcal{M}_m \oplus \mathcal{M}_m \).

**Lemma 3.2.** Let \( \mathcal{V} \) be the vector space of all linear endomorphisms acting on \( \Gamma(\bigwedge^\bullet T^*M) \). We have the following subspaces of \( \mathcal{V} \)

\[
\mathcal{M}^{(1)}_m := \{ M_{\omega_{m-1}} \circ d : \Gamma(\bigwedge^\bullet T^*M) \rightarrow \Gamma(\bigwedge^\bullet T^*M) : \omega_{m-1} \in \Gamma(\bigwedge^{m-1} T^*M) \},
\]

\[
\mathcal{M}^{(2)}_m := \{ M_{\omega_m} : \Gamma(\bigwedge^\bullet T^*M) \rightarrow \Gamma(\bigwedge^\bullet T^*M) : \omega_m \in \Gamma(\bigwedge^m T^*M) \},
\]

where \( \xi \) denotes multiplication by \( \xi \). Then \( \mathcal{M}^{(1)}_m \cap \mathcal{M}^{(2)}_m = \{0\} \) and \( \mathcal{M}_m \subseteq \mathcal{M}^{(1)}_m \oplus \mathcal{M}^{(2)}_m \subseteq \mathcal{V} \).

**Proof.** Observe that \( T_{a_0,\ldots,a_m} (\omega) = (M_{a a m a \downarrow a \ldots a m_{-1}} \circ d + M_{a a} d a \ldots a m) (\omega) \), \( \forall \omega \in \Gamma(\bigwedge^\bullet T^*M) \).

Since \( d(1) = 0 \) and \( \wedge(1) = 1 \), we have the direct sum. \( \square \)

**Lemma 3.3.** For \( 1 \leq m \leq n \) define

\[
\Phi : \mathcal{M}_m \rightarrow \Omega^{m-1}M \oplus \Omega^m M
\]

\[
T_{a_0,\ldots,a_m} \rightarrow (a a m a a \ldots a m_{-1} \wedge a a \ldots a m)
\]

where \( \Omega^k M := \Gamma(\bigwedge^k T^*M) \) denotes the space of \( k \)-forms on \( M \). Then

\[
\Phi = (\Phi, \Phi) : \mathcal{M}_m \oplus \mathcal{M}_m \rightarrow \Omega^{m-1}M \oplus \Omega^m M \oplus \Omega^{m-1}M \oplus \Omega^m M
\]

is a linear bijection.
Proof. Observe that to prove well-definedness of $\Phi$, in view of Lemma 3.2, we only need to show that for $0 \leq k \leq n-1$, if $M_{ik} \circ \delta$ is zero then $\omega_k = 0$. In a co-ordinate neighbourhood around a point $p \in \mathbb{M}$, suppose $\omega_k = \sum f_{i_1 \ldots i_k} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$. Since $k \leq n-1$, there always exist $j \notin \{i_1, \ldots, i_k\}$ and we have $\omega_k \wedge dx^j = 0$ i.e. $\sum f_{i_1 \ldots j \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} = 0$ at each point of the co-ordinate neighbourhood around $p \in \mathbb{M}$. This will show that each $f_{i_1 \ldots j \ldots i_k}$ is zero showing $\omega_k = 0$. Injectivity of $\Phi$ easily follows from Lemma 3.2. To see surjectivity, choose $(\omega_{m-1}, \omega_m) \in \Omega^{m-1} \mathbb{M} \oplus \Omega^m \mathbb{M}$. Let in a co-ordinate neighbourhood

$$\omega_m = \sum_{i_1 < \ldots < i_m} f_{i_1 \ldots i_m} dx^{i_1} \wedge \ldots \wedge dx^{i_m}$$

$$\omega_{m-1} = \sum_{j_1 < \ldots < j_{m-1}} g_{j_1 \ldots j_{m-1}} dx^{j_1} \wedge \ldots \wedge dx^{j_{m-1}}$$

with support of $f_{i_1 \ldots i_m}, g_{j_1 \ldots j_{m-1}}$ in that neighbourhood. Then

$$\sum T_{g_{j_1 \ldots j_{m-1}, x_{i_1} \ldots x_{i_m}}} \mapsto (\omega_{m-1}, 0)$$

This shows that

$$\Phi^{-1}(\omega_{m-1}, \omega_m) = \sum T_{f_{i_1 \ldots i_m, x_{i_1} \ldots x_{i_m}}} + \sum T_{g_{j_1 \ldots j_{m-1}, x_{i_1} \ldots x_{i_m}}} - \sum T_{f_{i_1 \ldots i_m, x_{i_1} \ldots x_{i_m}}}$$

and containment of support of the functions $f_{i_1 \ldots i_m}$ and $g_{j_1 \ldots j_{m-1}}$ in the co-ordinate neighbourhood fulfills our claim. $\square$

Lemma 3.4. For all $m \geq n+1$, where $n = \dim(\mathbb{M})$, $\mathcal{M}_m = \{0\}$.

Proof. Note that for any $\omega \in \Gamma(\wedge^\mathbb{C} T^* \mathbb{M})$,

$$T_{a_0, \ldots, a_m}(\omega) := (M_{a_0 a_m} \ldots a_{m-1} \omega \wedge \ldots \wedge da_m) \circ \delta + M_{a_0 a_1 \ldots \wedge da_m}(\omega).$$

Since $m \geq n+1$, it follows that $\mathcal{M}_m = \{0\}$ because $\Omega^k \mathbb{M} = \Gamma(\wedge^k T^* \mathbb{M}) = \{0\}$ for all $k > n$. $\square$

Proposition 3.5. $\Omega^{n-1} \mathbb{M} \oplus \Omega^n \mathbb{M} \oplus \Omega^{m-1} \mathbb{M} \oplus \Omega^m \mathbb{M}$ has a $\mathcal{C}^\infty(\mathbb{M})$-bimodule structure given by,

$$\begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix} \cdot (\omega_{m-1}, \omega_m, \bar{\omega}_{m-1}, \bar{\omega}_m) := (\phi \omega_{m-1}, \phi \omega_m, \psi \bar{\omega}_{m-1}, \psi \bar{\omega}_m)$$

$$\begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix} \cdot (\omega_{m-1}, \omega_m, \bar{\omega}_{m-1}, \bar{\omega}_m) := \begin{cases} (\phi \omega_{m-1}, \phi \omega_m - d\phi \wedge \omega_{m-1}, \psi \bar{\omega}_{m-1}, \psi \bar{\omega}_m - d\psi \wedge \bar{\omega}_{m-1}) & \text{if } m \text{ even} \\ (\psi \omega_{m-1}, \psi \omega_m + d\psi \wedge \omega_{m-1}, \phi \bar{\omega}_{m-1}, \phi \bar{\omega}_m + d\phi \wedge \bar{\omega}_{m-1}) & \text{if } m \text{ odd} \end{cases}$$

Proof. In co-ordinate chart,

$$\omega_{m-1} = \sum_{j_1 < \ldots < j_{m-1}} g_{j_1 \ldots j_{m-1}} dx^{j_1} \wedge \ldots \wedge dx^{j_{m-1}}$$

$$\omega_m = \sum_{i_1 < \ldots < i_m} f_{i_1 \ldots i_m} dx^{i_1} \wedge \ldots \wedge dx^{i_m}$$
\[
\tilde{\omega}_{m-1} = \sum_{j_1 \ldots < j_{m-1}} g_{j_1 \ldots j_{m-1}} \, dx^{j_1} \land \ldots \land dx^{j_{m-1}}
\]
\[
\tilde{\omega}_m = \sum_{i_1 \ldots < i_m} f_{i_1 \ldots i_m} \, dx^{i_1} \land \ldots \land dx^{i_m}
\]
Alos let
\[
\xi = \sum T_{f_{i_1 \ldots i_m}} x^{i_1}, \ldots, x^{i_m} + \sum T_{g_{j_1 \ldots j_{m-1}}} x^{j_1}, \ldots, x^{j_{m-1}, 1} - \sum T_{f_{i_1 \ldots i_m}} x^{i_1}, \ldots, x^{i_m-1, 1}
\]
and
\[
\tilde{\xi} = \sum T_{f_{i_1 \ldots i_m}} x^{i_1}, \ldots, x^{i_m} + \sum T_{g_{j_1 \ldots j_{m-1}}} x^{j_1}, \ldots, x^{j_{m-1}, 1} - \sum T_{f_{i_1 \ldots i_m}} x^{i_1}, \ldots, x^{i_m-1, 1}
\]
Define,
\[
\begin{pmatrix}
\phi & 0 \\
0 & \psi
\end{pmatrix}
\cdot
(\omega_{m-1}, \omega_m, \tilde{\omega}_{m-1}, \tilde{\omega}_m)
\]
\[
:= \Phi
\begin{pmatrix}
\phi & 0 \\
0 & \psi
\end{pmatrix}
\cdot
\Phi^{-1}(\omega_{m-1}, \omega_m, \tilde{\omega}_{m-1}, \tilde{\omega}_m)
\]
\[
= \begin{cases}
\Phi
\begin{pmatrix}
\phi & 0 \\
0 & \psi
\end{pmatrix}
\cdot
\begin{pmatrix}
0 & \xi \\
0 & 0
\end{pmatrix}
; \text{ if } m \text{ even} \\
\Phi
\begin{pmatrix}
0 & \phi \\
0 & \xi
\end{pmatrix}
\cdot
\begin{pmatrix}
0 & \xi \\
0 & 0
\end{pmatrix}
; \text{ if } m \text{ odd}
\end{cases}
\]
\[
= (\phi \omega_{m-1}, \phi \omega_m, \psi \tilde{\omega}_{m-1}, \psi \tilde{\omega}_m) ; \text{ for both even and odd } m .
\]
and
\[
(\omega_{m-1}, \omega_m, \tilde{\omega}_{m-1}, \tilde{\omega}_m)
\cdot
\begin{pmatrix}
\phi & 0 \\
0 & \psi
\end{pmatrix}
\]
\[
:= \Phi
\begin{pmatrix}
\phi & 0 \\
0 & \psi
\end{pmatrix}
\cdot
\Phi^{-1}(\omega_{m-1}, \omega_m, \tilde{\omega}_{m-1}, \tilde{\omega}_m)
\]
\[
= \begin{cases}
\Phi
\begin{pmatrix}
0 & \xi \\
0 & \xi
\end{pmatrix}
\cdot
\begin{pmatrix}
\phi & 0 \\
0 & \psi
\end{pmatrix}
; \text{ if } m \text{ even} \\
\Phi
\begin{pmatrix}
0 & \xi \\
0 & \xi
\end{pmatrix}
\cdot
\begin{pmatrix}
\phi & 0 \\
0 & \psi
\end{pmatrix}
; \text{ if } m \text{ odd}
\end{cases}
\]
\[
= \begin{cases}
\Phi
\begin{pmatrix}
\xi \phi & 0 \\
0 & \xi \psi
\end{pmatrix}
; \text{ if } m \text{ even} \\
\Phi
\begin{pmatrix}
0 & \xi \phi \\
0 & \xi \psi
\end{pmatrix}
; \text{ if } m \text{ odd}
\end{cases}
\]
where $\Phi$ is the map defined in Lemma 3.3. Now

$$\xi \phi = \sum T_{f_{1i\ldots im}} x^{i1\ldots im} \phi + \sum T_{g_{1i\ldots jm}} x^{j1\ldots jm} \phi - \sum T_{f_{1i\ldots im}} x^{i1\ldots im} \phi$$

and

$$\tilde{\xi} \psi = \sum T_{f_{1i\ldots im}} x^{i1\ldots im} \psi + \sum T_{g_{1i\ldots jm}} x^{j1\ldots jm} \psi - \sum T_{f_{1i\ldots im}} x^{i1\ldots im} \psi$$

So

$$\Phi \left( \begin{array}{cc} \xi \phi & 0 \\ 0 & \tilde{\xi} \psi \end{array} \right) = \left( \sum f_{1i\ldots im} x^{i1\ldots im} \phi dx^{i1} \wedge \ldots \wedge dx^{im-1} - f_{1i\ldots im} x^{i1\ldots im} \phi dx^{i1} \wedge \ldots \wedge dx^{im-1} \\
+ \phi \omega_{m-1} \sum f_{1i\ldots im} x^{i1\ldots im} dx^{i1} \wedge \ldots \wedge dx^{im-1} \wedge d(x^{im} \phi) + \omega_{m-1} \wedge d\phi \\
- f_{1i\ldots im} x^{i1\ldots im} dx^{i1} \wedge \ldots \wedge dx^{im-1} \wedge d\phi, \sum f_{1i\ldots im} x^{i1\ldots im} \psi dx^{i1} \wedge \ldots \wedge dx^{im-1} \\
- f_{1i\ldots im} x^{i1\ldots im} \psi dx^{i1} \wedge \ldots \wedge dx^{im-1} + \psi \omega_{m-1}, \\
\sum f_{1i\ldots im} dx^{i1} \wedge \ldots \wedge dx^{im-1} \wedge d(x^{im} \psi) + \omega_{m-1} \wedge d\psi \\
- f_{1i\ldots im} dx^{i1} \wedge \ldots \wedge dx^{im-1} \wedge d\psi \right)
$$

Similarly one can prove that

$$\Phi \left( \begin{array}{cc} 0 & \xi \psi \\ \tilde{\xi} \phi & 0 \end{array} \right) = (\psi \omega_{m-1}, \psi \omega_{m} + \omega_{m-1} \wedge d\psi, \phi \omega_{m-1}, \phi \omega_{m} + \omega_{m-1} \wedge d\phi)$$

This is clearly a bimodule structure since it is induced by that on $\Omega^m_D \left( \widetilde{\Omega^\infty(M)} \right)$.

**Notation:** $\widetilde{\Omega^m_D} := \Omega^{m-1} M \oplus \Omega^m M \oplus \Omega^{m-1} M \oplus \Omega^m M$, $1 \leq m \leq n$, until the end this section, where $\Omega^* M$ denotes the space of forms on $M$.

**Theorem 3.6.** $\Omega^m_D \left( \widetilde{\Omega^\infty(M)} \right) \cong \Omega^{m-1} M \oplus \Omega^m M \oplus \Omega^{m-1} M \oplus \Omega^m M$, for all $1 \leq m \leq n$, and $\Omega^m_D \left( \widetilde{\Omega^\infty(M)} \right) = \{0\}$ for $m > n$. This isomorphism is a $\widetilde{\Omega^\infty(M)}$-bimodule isomorphism.

**Proof.** We have for all $1 \leq m \leq n$,

$$\Omega^m_D \left( \widetilde{\Omega^\infty(M)} \right) \cong \pi \left( \Omega^m \left( \widetilde{\Omega^\infty(M)} \right) \right) \quad \text{by Lemma 3.1}
$$

$$\cong \Omega^{m-1} M \oplus \Omega^m M \oplus \Omega^{m-1} M \oplus \Omega^m M \quad \text{by Lemma 3.3}.$$

Lemma 3.4 proves that $\Omega^m_D \left( \widetilde{\Omega^\infty(M)} \right) = \{0\}$ for $m > n$. Finally Proposition 3.3 proves that this isomorphism is $\widetilde{\Omega^\infty(M)}$-bimodule isomorphism for all $1 \leq m \leq n$. 

□
Now we will turn $\Omega^\bullet_D$ into a chain complex. To avoid confusion we denote the induced differential $d : \Omega^\bullet_D(A) \rightarrow \Omega^{\bullet+1}_D(A)$ of diagram 2.2 by $\bar{d}$ in this section so that it should not be confused with the exterior differentiation $d$.

**Lemma 3.7.** The differential $\bar{d} : \Omega^n_D(C^\infty(M)) \rightarrow \Omega^{n+1}_D(C^\infty(M))$ of diagram 2.2 has the following action:

1. For $m \geq 1$ odd,
   $$\bar{d} : \left( \begin{array}{c} 0 \\ T_{a_0,\ldots,a_m} \end{array} \right) \mapsto \left( \begin{array}{c} T_{1,a_0',\ldots,a_m'} + T_{a_0,\ldots,a_m,1} \\ 0 \\ T_{1,a_0,\ldots,a_m} + T_{a_0',\ldots,a_m',1} \end{array} \right)$$

2. For $m \geq 2$ even,
   $$\bar{d} : \left( \begin{array}{c} 0 \\ T_{a_0,\ldots,a_m} \end{array} \right) \mapsto \left( \begin{array}{c} 0 \\ 0 \\ T_{1,a_0,\ldots,a_m} - T_{a_0',\ldots,a_m',1} \end{array} \right)$$

**Proof.** We first note that

\begin{equation}
\left[ D, \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \right] = \left( \begin{array}{c} 0 \\ -T_{1,1} \\ 0 \end{array} \right)
\end{equation}

\begin{equation}
\left( \begin{array}{c} 0 \\ T_{1,1} \\ 0 \\ -T_{1,1} \end{array} \right) \left( \begin{array}{c} -a_1' \\ 0 \\ 0 \\ a_1 \end{array} \right) = \left( \begin{array}{c} 0 \\ T_{1,a_1} \\ 0 \end{array} \right)
\end{equation}

\begin{equation}
\left( \begin{array}{c} a_0 \\ 0 \\ a_0' \\ 0 \end{array} \right) \left( \begin{array}{c} 0 \\ T_{1,a_1} \\ 0 \end{array} \right) = \left( \begin{array}{c} 0 \\ T_{a_0,a_1} \\ 0 \end{array} \right)
\end{equation}

Hence combining these three we get,

\begin{equation}
\left( \begin{array}{c} a_0 \\ 0 \\ a_0' \end{array} \right) \left[ D, \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \right] \left( \begin{array}{c} -a_1' \\ 0 \\ a_1 \end{array} \right) = \left( \begin{array}{c} 0 \\ T_{a_0,a_1} \\ 0 \end{array} \right)
\end{equation}

**Case 1:** Let $m \geq 3$ be odd. Observe that

$$\left( \begin{array}{c} 0 \\ T_{a_0,a_1,\ldots,a_m} \end{array} \right) = \left( \begin{array}{c} 0 \\ T_{a_0,a_1} \end{array} \right) \prod_{i=2,i \text{ even}}^m \left( \left( \begin{array}{c} 0 \\ T_{1,a_1} \end{array} \right) \left( \begin{array}{c} 0 \\ T_{1,a_{i+1}} \end{array} \right) \right)$$

$$= \left( \begin{array}{c} a_0 \\ 0 \\ a_0' \end{array} \right) \left[ D, \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \right] \left( \begin{array}{c} -a_1' \\ 0 \\ a_1 \end{array} \right) \cdot \prod_{i=2,i \text{ even}}^m \left( \left[ \left( \begin{array}{c} D, \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \right) \left( \begin{array}{c} -a_i \\ 0 \\ a_i' \end{array} \right) \right] \left[ \left( \begin{array}{c} D, \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \right) \left( \begin{array}{c} a_i \\ 0 \\ a_i' \end{array} \right) \right] \right)$$

Consider the expression $\eta = x_0 \prod_{i=1}^m (\bar{d}(b)x_i)$ where,

$$x_0 = \left( \begin{array}{c} a_0 \\ 0 \\ a_0' \end{array} \right) ; b = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) ; \bar{d}(y) = \left[ D, \left( \begin{array}{c} y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{array} \right) \right]$$

$$x_i = \left( \begin{array}{c} -a_i' \\ 0 \\ a_i \end{array} \right) \text{ for } 1 \leq i \leq m, \text{ odd} ; \quad x_j = \left( \begin{array}{c} -a_j \\ 0 \\ a_j' \end{array} \right) \text{ for } 1 \leq j \leq m, \text{ even.}$$
One should note that \( \tilde{d} \circ \tilde{d}(b) = 0 \) because \( d^2 = 0 \), \( d \) being the exterior differentiation. Now for the differential \( \tilde{d} : \Omega^m_D \left( C^\infty(M) \right) \rightarrow \Omega^{m+1}_D \left( C^\infty(M) \right) \) of diagram 2.2 we get,

\[
\tilde{d}\eta = \tilde{d}(x_0) \prod_{i=1}^m \{\tilde{d}(b)x_i\} + x_0 \tilde{d}\left(\tilde{d}(b)x_1 \prod_{i=2}^m \{\tilde{d}(b)x_i\}\right) \\
= \tilde{d}(x_0) \prod_{i=1}^m \{\tilde{d}(b)x_i\} + \sum_{k=2}^{m} (-1)^{k-1} \prod_{j=0}^{k-2} \{x_j \tilde{d}(b)\} \tilde{d}(x_{k-1}) \left( \prod_{i=k}^m \{\tilde{d}(b)x_i\} \right) \\
+ (-1)^m \left( \prod_{i=0}^{m-1} \{x_i \tilde{d}(b)\} \right) \tilde{d}(x_m) \\
= \left[ D_i \begin{pmatrix} a_0 & 0 \\ 0 & a_0' \end{pmatrix} \right] \left[ D_i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \left( -a_i' & 0 \\ 0 & a_i \right) \prod_{i=2, i \text{ even}}^m \left( 0 & T_{1,a_i} \\ T_{1,a_i} & 0 \right) \left( 0 & T_{1,a_{i+1}} \\ T_{1,a_{i+1}} & 0 \right) \\
+ \sum_{k=2}^{m} (-1)^{k-1} \prod_{j=0}^{k-2} \{x_j \tilde{d}(b)\} \tilde{d}(x_{k-1}) \left( \prod_{i=k}^m \{\tilde{d}(b)x_i\} \right) \\
+ (-1)^m \left( \prod_{i=1, i \text{ odd}}^{m-1} \{x_i \tilde{d}(b)\} \right) \\
\left( -a_i+1 & 0 \\ 0 & a_i+1 \right) \left[ D_i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \left( -a_m' & 0 \\ 0 & a_m \right) \left[ D_i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \\
\left( -a_i' & 0 \\ 0 & a_i \right) \left[ D_i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \\
\left( -a_m' & 0 \\ 0 & a_m \right) \left[ D_i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \right]
\]

Now it is straightforward computation to observe that

\[
\prod_{i=k}^m \{\tilde{d}(b)x_i\} = \prod_{i=k}^m \left( \begin{pmatrix} 0 & T_{1,1} \\ -T_{1,1} & 0 \end{pmatrix} x_i \right) \\
= \begin{cases} 
\begin{pmatrix} T_{1,a_k',...,a_m'} & 0 \\
0 & T_{1,a_k,...,a_m}\end{pmatrix} & ; \text{ if } k \text{ even} \\
\begin{pmatrix} 0 & T_{1,a_k,...,a_m} \\
T_{1,a_k',...,a_m'} & 0 \end{pmatrix} & ; \text{ if } k \text{ odd}
\end{cases}
\]

and

\[
\prod_{j=0}^{k-2} \{x_j \tilde{d}(b)\} \tilde{d}(x_{k-1}) = \begin{cases} 
\begin{pmatrix} -T_{a_0,...,a_{k-1},1} & 0 \\
0 & -T_{a_0',...,a_{k-1},1}\end{pmatrix} & ; \text{ if } k \text{ even} \\
\begin{pmatrix} 0 & -T_{a_0,...,a_{k-1},1} \\
-T_{a_0',...,a_{k-1},1} & 0 \end{pmatrix} & ; \text{ if } k \text{ odd}
\end{cases}
\]

The fact \( d^2 = 0 \) will now ensure that only the first and last term in the expression for \( \tilde{d}\eta \) survive. Hence,

\[
\tilde{d}\eta = \left( T_{1,a_0'} - T_{a_0,1} \right) T_{1,a_1',...,a_m'} \begin{pmatrix} 0 & 0 \\
0 & T_{1,a_0} - T_{a_0',1} \end{pmatrix} \left( T_{1,a_1,...,a_m} \right) + (-1)^m \left( -T_{a_0,...,a_{m-1},1} & 0 \\
0 & -T_{a_0',...,a_{m-1},1} \right)
\]

\[
= \left( T_{1,a_0',...,a_m'} + T_{a_0,...,a_m} \right) \begin{pmatrix} 0 & 0 \\
0 & T_{1,a_0,...,a_m} + T_{a_0',...,a_m,1} \end{pmatrix} \left( T_{1,a_0,...,a_m} + T_{a_0',...,a_m,1} \right).
\]

**Case 2.** Let \( m \) be even.
One can prove in exact similar manner like the ‘odd’ case. The only difference in this case is a negative sign and it appears because of the presence of \((-1)^m\) at the last term in the expression for \(\tilde{d}\).

**Case 3.** Let \(m = 1\). Recall from equation (3.10),
\[
\begin{pmatrix}
  0 & T_{a_0, a_1} \\
  T_{a_0', a_1'} & 0
\end{pmatrix} = \begin{pmatrix}
  a_0 & 0 \\
  0 & a_0'
\end{pmatrix} \begin{pmatrix}
  D_1 & 0 & 0 \\
  0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
  -a_1' & 0 \\
  0 & a_1
\end{pmatrix}
\]
and hence,
\[
\tilde{d} : \begin{pmatrix}
  0 & T_{a_0, a_1} \\
  T_{a_0', a_1'} & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
  T_{1, a_0', a_1'} + T_{a_0, a_1, 1} & 0 \\
  0 & T_{1, a_0, a_1} + T_{a_0', a_1', 1}
\end{pmatrix}
\]

Using the isomorphism in Theorem 3.6 we can transfer the differential \(\tilde{d} : \Omega_D^m \longrightarrow \Omega_D^m \left(\widetilde{\mathcal{M}}\right)\) to the differential \(\delta : \Omega_D^m \longrightarrow \Omega_D^{m+1}\). This will turn \(\Omega_D^m\) into a chain complex and then we will be able to compute the cohomologies of the complex \((\Omega_D^m, \delta)\).

**Proposition 3.8.** For \(1 \leq m \leq n\), the map
\[
\delta : \Omega_D^m \longrightarrow \Omega_D^{m+1}
\]
\[(\omega_{m-1}, \omega_m, \widetilde{\omega}_{m-1}, \widetilde{\omega}_m) \mapsto (d\omega_{m-1} + (-1)^m (\widetilde{\omega}_m - \omega_m), d\widetilde{\omega}_m, d\omega_m + (-1)^m (\omega_m - \widetilde{\omega}_m), d\widetilde{\omega}_m)\]

makes the following diagram
\[
\begin{array}{ccc}
\Omega_D^m \left(\widetilde{\mathcal{M}}\right) & \xrightarrow{\tilde{d}} & \Omega_D^{m+1} \left(\widetilde{\mathcal{M}}\right) \\
\downarrow \cong & & \downarrow \cong \\
\Omega_D^m & \xrightarrow{\delta} & \Omega_D^{m+1}
\end{array}
\]

commutative.

**Proof.** For \(1 \leq m \leq n\) take \((\omega_{m-1}, \omega_m, \widetilde{\omega}_{m-1}, \widetilde{\omega}_m) \in \Omega_D^m\). In terms of local co-ordinates
\[
\omega_{m-1} = \sum_{j_1 < \ldots < j_{m-1}} g_{j_1 \ldots j_{m-1}} dx^{j_1} \wedge \ldots \wedge dx^{j_{m-1}}
\]
\[
\omega_m = \sum_{i_1 < \ldots < i_m} f_{i_1 \ldots i_m} dx^{i_1} \wedge \ldots \wedge dx^{i_m}
\]
\[
\widetilde{\omega}_{m-1} = \sum_{j_1 < \ldots < j_{m-1}} \tilde{g}_{j_1 \ldots j_{m-1}} dx^{j_1} \wedge \ldots \wedge dx^{j_{m-1}}
\]
\[
\widetilde{\omega}_m = \sum_{i_1 < \ldots < i_m} \tilde{f}_{i_1 \ldots i_m} dx^{i_1} \wedge \ldots \wedge dx^{i_m}
\]
Using Lemma 3.3 we see that, isomorphic image of this element in \(\Omega_D^m \left(\widetilde{\mathcal{M}}\right)\) is
\[
\begin{cases}
  \left(\begin{array}{c}
  \xi \\
  0
\end{array}\right) & \text{if } m \text{ even} \\
  \left(\begin{array}{c}
  0 \\
  \xi
\end{array}\right) & \text{if } m \text{ odd}
\end{cases}
\]
where,
\[
\xi = \sum T_{f_{i_1 \ldots i_m},x^{i_1}, \ldots, x^{i_m}} - T_{f_{i_1 \ldots i_m}, x^{i_1}, x^{i_1}, \ldots, x^{i_{m-1}}} + \sum T_{g_{j_1 \ldots j_{m-1}}, x^{j_1}, \ldots, x^{j_{m-1}}},
\]
and
\[
\tilde{\xi} = \sum T_{\tilde{f}_{j_1 \ldots j_{m-1}}, x^{j_1}, \ldots, x^{j_m}} - T_{\tilde{f}_{j_1 \ldots j_{m-1}}, x^{j_1}, \ldots, x^{j_{m-1}}} + \sum T_{\tilde{g}_{j_1 \ldots j_{m-1}}, x^{j_1}, \ldots, x^{j_{m-1}}}.
\]
By Lemma 3.7, we see that the differential \(d : \Omega^m_D(\widetilde{C^\infty(M)}) \to \Omega^{m+1}_D(\widetilde{C^\infty(M)})\) sends this element to
\[
\begin{cases}
(d_\xi + \xi T_{1,1} & 0) ; \text{if } m \text{ odd} \\
0 & d_\xi + \xi T_{1,1} ; \text{if } m \text{ even} \\
0 & d_\xi - \tilde{\xi} T_{1,1} ; \text{if } m \text{ even}
\end{cases}
\]
where
\[
d_\xi = \sum T_{1, f_{i_1 \ldots i_m}, x^{i_1}, \ldots, x^{i_m}} - T_{1, f_{i_1 \ldots i_m}, x^{i_1}, x^{i_1}, \ldots, x^{i_{m-1}}}
+ \sum T_{1, g_{j_1 \ldots j_{m-1}}, x^{j_1}, \ldots, x^{j_{m-1}}}
\]
and
\[
d_{\tilde{\xi}} = \sum T_{1, \tilde{f}_{j_1 \ldots j_{m-1}}, x^{j_1}, \ldots, x^{j_m}} - T_{1, \tilde{f}_{j_1 \ldots j_{m-1}}, x^{j_1}, x^{j_1}, \ldots, x^{j_{m-1}}}
+ \sum T_{1, \tilde{g}_{j_1 \ldots j_{m-1}}, x^{j_1}, \ldots, x^{j_{m-1}}}
\]
Isomorphic image of this element in \(\widetilde{\Omega^{m+1}_D}\), under the map \(\Phi\) of Lemma 3.3, is
\[
\begin{cases}
(d\tilde{\omega}_m + \omega_m + \omega_m, d\tilde{\omega}_m, d\omega_m - \omega_m + \tilde{\omega}_m, d\omega_m) ; \text{if } m \text{ odd} \\
(d\tilde{\omega}_m + \omega_m - \omega_m, d\tilde{\omega}_m, d\omega_m - \omega_m + \omega_m - \tilde{\omega}_m, d\omega_m) ; \text{if } m \text{ even}
\end{cases}
\]
i.e. \((d\tilde{\omega}_m - (-1)^m(\tilde{\omega}_m - \omega_m), d\tilde{\omega}_m, d\omega_m - (-1)^m(\omega_m - \tilde{\omega}_m), d\omega_m)\).

\[\square\]

**Remark 3.9.** Notice that \(\delta = \Phi \circ \tilde{d} \circ \Phi^{-1}\), and hence \(\delta^2 = 0\). Thus \((\Omega^*_D, \delta)\) is a chain complex. Furthermore, the graded algebra structure on \(\Omega^*_D(\widetilde{C^\infty(M)})\) will induce the same on \((\Omega^*_D, \delta)\) through the commutative diagram of Proposition 3.8. So we get \((\Omega^*_D(\widetilde{C^\infty(M)}), \tilde{d}) \cong (\Omega^*_D, \delta)\) as differential graded algebras and Theorem 3.6 gives \(\widetilde{C^\infty(M)}\)-bimodule isomorphism at each term of these chain complexes.

**Theorem 3.10.** The cohomologies \(\widetilde{H^*}(M)\) of the chain complex \((\Omega^*_D, \delta)\) are given by,
\[
\widetilde{H^m}(M) \cong H^{m-1}(M) \oplus H^m(M) ; \text{for } 0 \leq m \leq \dim(M),
\]
where \(H^*(M)\) denotes the de-Rham cohomologies of \(M\).
Proof. (1) Let $m = 0$. Recall that for $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \in C^\infty(M)$,
\[
D_\ast \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} 0 & T_{1,f} - T_{f,1} \\ T_{1,g} - T_{g,1} & 0 \end{pmatrix}.
\]
The isomorphism of Lemma 3.3 sends this element to $(g - f, dg, f - g, df)$. Hence
\[
\tilde{H}^0(M) = \{ \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} : df = 0, f \in C^\infty(M) \} \cong H^0(M).
\]

(2) Let $1 \leq m \leq \text{dim}(M)$. Consider $\delta^{m-1} : \tilde{\Omega}_{D}^{m-1} \rightarrow \tilde{\Omega}_{D}^{m}$ and $\delta^m : \tilde{\Omega}_{D}^{m} \rightarrow \tilde{\Omega}_{D}^{m+1}$. Then
\[
\delta^{m-1}(v_{m-2}, v_{m-1}, \tilde{v}_{m-2}, \tilde{v}_{m-1}) = (d\tilde{v}_{m-2} + (-1)^{m-1}(\tilde{v}_{m-1} - v_{m-1}), d\tilde{v}_{m-1},
\]
d$\tilde{v}_{m-2} + (-1)^{m-1}(v_{m-1} - \tilde{v}_{m-1}), d\tilde{v}_{m-1})$
for all $(v_{m-2}, v_{m-1}, \tilde{v}_{m-2}, \tilde{v}_{m-1}) \in \tilde{\Omega}_{D}^{m-1}$. Let $\zeta = (w_{m-1}, w_0, \tilde{w}_{m-1}, \tilde{w}_0) \in \text{Ker}(\delta^m)$. Then we have the following
\[
\begin{dcases}
  d(w_{m-1}) = 0 & d(\tilde{w}_{m-1}) + (-1)^m(w_{m-1} - \tilde{w}_{m-1}) = 0 \\
  d(w_0) = 0 & d(\tilde{w}_0) + (-1)^m(w_0 - \tilde{w}_0) = 0
\end{dcases}
\]
Define
\[
\Psi : \frac{\text{Ker}(\delta^m)}{\text{Im}(\delta^{m-1})} \rightarrow H^m(M) \oplus H^{m-1}(M)
\]
\[
[\zeta] \mapsto \bigl( [w_m + w_0], [w_{m-1} + \tilde{w}_{m-1}] \bigr).
\]
This map is well-defined (because of equation (3.18)) and linear. Now define
\[
\Phi : H^m(M) \oplus H^{m-1}(M) \rightarrow \frac{\text{Ker}(\delta^m)}{\text{Im}(\delta^{m-1})}
\]
\[
([w_m], [w_{m-1}]) \mapsto \left[ \frac{1}{2} w_{m-1}, \frac{1}{2} w_m, \frac{1}{2} \tilde{w}_{m-1}, \frac{1}{2} \tilde{w}_m \right].
\]
Using equation (3.17) one can check that $\Phi$ is well-defined and linear. Now observe that $\Psi \circ \Phi = I_d$, and
\[
\Phi \circ \Psi ([\zeta]) = \left[ \left( \frac{1}{2}(w_{m-1} + \tilde{w}_{m-1}), \frac{1}{2}(w_m + \tilde{w}_m), \frac{1}{2}(\tilde{w}_{m-1} + w_{m-1}), \frac{1}{2}(\tilde{w}_m + w_m) \right) \right].
\]
If we can show that
\[
\xi = \left( \frac{1}{2}(w_{m-1} - w_{m-1}), \frac{1}{2}(\tilde{w}_m - w_m), \frac{1}{2}(w_{m-1} - \tilde{w}_{m-1}), \frac{1}{2}(w_m - \tilde{w}_m) \right) \in \text{Im}(\delta^{m-1}),
\]
then $\Phi \circ \Psi$ will also be identity. Observe that
\[
\delta^{m-1} \left( 0, \frac{(-1)^m}{4}(w_{m-1} - \tilde{w}_{m-1}), 0, \frac{(-1)^m}{4}(w_m - \tilde{w}_m) \right) = \xi
\]
using equation (3.18), and hence (2) follows. \qed
4. Computation for the Noncommutative Torus

In this section our objective is to show that the functor $F \circ G$ is not trivial for the case of noncommutative torus, one of the most fundamental and widely studied example in noncommutative geometry. We recall the definition of noncommutative torus from $[\text{Rfl}]$. Let $\theta$ be a real number. Denote by $A_\theta$, the universal $C^*$-algebra generated by unitaries $U, V$ satisfying $UV = e^{-2\pi i \theta} VU$. Throughout this section $i$ will stand for $\sqrt{-1}$. On $A_\theta$, the Lie group $G = T^2$ acts as follows:

$$\alpha_{(z_1, z_2)}(U) = z_1 U \quad \text{and} \quad \alpha_{(z_1, z_2)}(V) = z_2 V.$$ 

The smooth subalgebra of $A_\theta$, is given by

$$A_{\theta}^\infty := \left\{ \sum a_{r_1, r_2} U^{r_1} V^{r_2} : \{a_{r_1, r_2}\} \in S(\mathbb{Z}^2), r_1, r_2 \in \mathbb{Z} \right\}$$

where $S(\mathbb{Z}^2)$ denotes vector space of multisequences $(a_{r_1, r_2})$ that decay faster than the inverse of any polynomial in $r = (r_1, r_2)$. This subalgebra is equipped with a unique $G$-invariant tracial state, given by $\tau(a) = a_{0,0}$. The Hilbert space obtained by applying the G.N.S. construction to $\tau$ can be identified with $l^2(\mathbb{Z}^2)$ $([\text{Rfl}])$ and we have $A_{\theta}^\infty \subseteq l^2(\mathbb{Z}^2)$ as subspace. We have the following derivations acting on $A_{\theta}^\infty$,

$$\tilde{\delta}_j(\sum_{r_1, r_2} a_{r_1, r_2} U^{r_1} V^{r_2}) := \sqrt{-1} \sum_{r_1, r_2} r_j a_{r_1, r_2} U^{r_1} V^{r_2} \quad \text{for} \ j = 1, 2.$$

Let $\delta_j := -\sqrt{-1} \tilde{\delta}_j, \ j = 1, 2$. It is known that $([\text{Con2}])$

$$(A_{\theta}^\infty, l^2(\mathbb{Z}^2) \otimes \mathbb{C}^2, D := \begin{pmatrix} 0 & \delta_1 - i \delta_2 \\ \delta_1 + i \delta_2 & 0 \end{pmatrix}, \gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$$

forms an even spectral triple on $A_{\theta}^\infty$. Let $H^\infty := \bigcap_{k \geq 1} \text{Dom}(D^k)$. Then $H^\infty = A_{\theta, s}^\infty \otimes \mathbb{C}^2$.

In this section our candidate for the even algebraic spectral triple is the following quadruple

$$E := \left( A_{\theta}^\infty, H^\infty \otimes \mathbb{C}^2, D := \begin{pmatrix} 0 & \delta_1 - i \delta_2 \\ \delta_1 + i \delta_2 & 0 \end{pmatrix}, \gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right).$$

Since $\gamma \notin \pi(C^\infty(M))$ we first apply the functor $G$ of Proposition $[\text{2.3}]$ and then compute $F \circ G$ along with the associated cohomologies. We will only work with the smooth subalgebra $A_{\theta, s}^\infty$ and hence denote it by $A_{\theta}$ for notational brevity. Note that,

$$\delta_1(U) = U, \quad \delta_1(V) = 0, \quad \delta_2(U) = 0, \quad \delta_2(V) = V.$$

We denote $d := \delta_1 - i \delta_2$ and $d^* := \delta_1 + i \delta_2$. Hence,

$$d(U) = U, \quad d^*(U) = U, \quad d(V) = -iV, \quad d^*(V) = iV,$$

$$d(U^*) = -U^*, \quad d^*(U^*) = -U^*, \quad d(V^*) = iV^*, \quad d^*(V^*) = -iV^*.$$

**Notation :** $\widehat{A_\theta} = G(E)$ throughout this section where $G$ is as defined in Proposition $[\text{2.3}]$.

Note that $J_0^0(\widehat{A_\theta}) = \{0\}$ in this case. Now observe that

$$\begin{pmatrix} D, \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & db - ad \\ 0 & d^*a - bd^* \end{pmatrix},$$

and hence each element of $\pi\left(\Omega^1(\widehat{A_\theta})\right)$ is linear span of following elements :
Proof. If we define
\[
\left( \begin{array}{cc} 0 & cde \\ e'd' e' & 0 \end{array} \right) \text{ such that } c, e, e', e' \in \mathcal{A}_\Omega.
\]
For \( b, c \in \mathcal{A}_\Omega \) consider the linear operator
\[
cdb : \mathcal{A}_\Omega \to \mathcal{A}_\Omega
\]
\[
e \mapsto cd(be)
\]
Let \( \mathcal{M}_1 := \text{span}\{cdb : \mathcal{A}_\Theta \to \mathcal{A}_\Theta : b, c \in \mathcal{A}_\Theta\} \). Then \( \mathcal{M}_1 \) is a \( \mathbb{C} \)-vector space and using equation 4.19 we see that \( \pi \left( \Omega^1(\widetilde{\mathcal{A}}_\Theta) \right) \subseteq \mathcal{M}_1 \oplus \mathcal{M}_1 \). Now the following equality
\[
\left( \begin{array}{cc} 0 & cde \\ e'd' e' & 0 \end{array} \right) = \left( c 0 \\ 0 -c' \right) \left[ D, \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \right] \left( e' 0 \\ 0 e \right)
\]
proves that \( \pi \left( \Omega^1(\widetilde{\mathcal{A}}_\Theta) \right) = \mathcal{M}_1 \oplus \mathcal{M}_1 \).

Lemma 4.1. Let \( \mathcal{V} \) be the vector space of linear endomorphisms acting on \( \mathcal{A}_\Theta \). Let \( M_\xi \) denote multiplication by \( \xi \). The vector subspaces \( \{M_{\sum c_i d_i(b_i)} : \mathcal{A}_\Theta \to \mathcal{A}_\Theta : c_i, b_i \in \mathcal{A}_\Theta\} \) and \( \{M_e \circ d : \mathcal{A}_\Theta \to \mathcal{A}_\Theta : e \in \mathcal{A}_\Theta\} \) of \( \mathcal{V} \) has trivial intersection and \( \mathcal{M}_1 \subseteq \{M_{cd(b)} : \mathcal{A}_\Theta \to \mathcal{A}_\Theta\} \oplus \{M_e \circ d : \mathcal{A}_\Theta \to \mathcal{A}_\Theta\} \).

Proof. Observe that \( cd(e) = (M_{cd(b)} + M_e \circ d)(e) \) for any \( e \in \mathcal{A}_\Theta \). Since \( d(1) = 0 \) we have the direct sum. \( \square \)

Let \( T_{a,b} = (ad(b), ab) \in \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \) and \( \overline{T}_{a,b} = (ad^*(b), ab) \in \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \). Define \( \Phi : \pi \left( \Omega^1(\widetilde{\mathcal{A}}_\Theta) \right) \to \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \)
\[
\left( \begin{array}{cc} 0 & adb \\ a'd' b' & 0 \end{array} \right) \mapsto (T_{a,b}, \overline{T}_{a', b'})
\]

Lemma 4.2. \( \Phi \) is a linear bijection.

Proof. To prove \( \Phi \) is well-defined, let \( \sum a_i d_i b_i = 0 \). Acting it on \( 1 \in \mathcal{A}_\Theta \) and \( U \in \mathcal{A}_\Theta \) respectively, we see that both \( \sum a_i d_i b_i \) and \( \sum a_i b_i \) are zero. Similarly for the case of \( \sum a'_i d'_i b'_i = 0 \). This proves well-definedness and Lemma 4.1 proves injectivity. To see surjectivity, observe that
\[
\Phi \left( \begin{array}{cc} 0 & aU^*dU + bd1 - ad1 \\ a'U^*d^*U + b'd^*1 - a'd^*1 & 0 \end{array} \right) \mapsto (a, b, a', b').
\]
\( \square \)

Proposition 4.3. \( \mathcal{A}_\Theta \otimes \mathbb{C}^4 \) is a \( \widetilde{\mathcal{A}}_\Theta \)-bimodule where the module action is specified by
\[
\left( \begin{array}{cc} f & 0 \\ 0 & g \end{array} \right). (a, b, a', b') = \left( \begin{array}{cc} f' & 0 \\ 0 & g' \end{array} \right)
\]
\[
:= (fa^*g' + fbd(g'), fbg', ga'f' + gb'd^*(f'), gb'f').
\]

Proof. If we define
\[
\left( \begin{array}{cc} f & 0 \\ 0 & g \end{array} \right). (a, b, a', b') := \Phi \left( \left( \begin{array}{cc} f & 0 \\ 0 & g \end{array} \right) \Phi^{-1}(a, b, a', b') \right),
\]
where \( \Phi \) is in Lemma 4.2, then it is clearly a left module structure induced by that on \( \Omega^1_D(\widetilde{\mathcal{A}}_\Theta) \). Now one can check that
Now consider the following linear operators

\[
\begin{pmatrix}
f \\ 0 \\ 0 \\ g
\end{pmatrix} \cdot (a, b, a', b') = (fa, fb, ga', gb')
\]

Similarly for the right module structure, we define

\[
(a, b, a', b'). \begin{pmatrix}
f' \\ 0 \\ 0 \\ g'
\end{pmatrix} := \Phi \left( \Phi^{-1} (a, b, a', b'). \begin{pmatrix}
f' \\ 0 \\ 0 \\ g'
\end{pmatrix} \right)
\]

and it equals to \((ag' + bd(g'), bg', a'f' + b'd'(f'), b'f')\). □

**Proposition 4.4.** \(\Omega^2(D(\widetilde{A}_\Theta)) \cong A_\Theta \otimes \mathbb{C}^4\) as \(\widetilde{A}_\Theta\)-bimodule.

**Proof.** The \(\widetilde{A}_\Theta\)-bimodule action on right hand side is given by Proposition 4.3 and \(\Phi\) of Lemma 4.2 becomes a bimodule isomorphism under this action. □

Since elements of \(\pi\left(\Omega^2(\widetilde{A}_\Theta)\right)\) are linear sum of

\[
\begin{pmatrix}
a_0 \\ 0 \\ 0 \\ b_0
\end{pmatrix} \begin{pmatrix}
D_1 \\ a_1 \\ 0 \\ b_1
\end{pmatrix} \begin{pmatrix}
D_2 \\ a_2 \\ 0 \\ b_2
\end{pmatrix}
\]

they are of the form \(\sum \begin{pmatrix}
adbd^*c \\ 0 \\ a'd'b'dc'
\end{pmatrix}\) for \(a, b, a', b' \in A_\Theta\). This shows that \(\pi\left(\Omega^2(\widetilde{A}_\Theta)\right) \subseteq M_2 \oplus \widetilde{M}_2\), where

\[
M_2 := \text{span}\{adbd^*c : A_\Theta \rightarrow A_\Theta\},
\]

\[
\widetilde{M}_2 := \text{span}\{a'd'b'dc' : A_\Theta \rightarrow A_\Theta\}.
\]

To see equality use equation 4.19 and observe that

\[
\sum \begin{pmatrix}
adbd^*c \\ 0 \\ a'd'b'dc'
\end{pmatrix}
\]

\[
= \sum \begin{pmatrix}
a \\ 0 \\ -a'
\end{pmatrix} \begin{pmatrix}
D_1 \\ 0 \\ 0 \\ 1
\end{pmatrix} \begin{pmatrix}
b' \\ 0 \\ b \\ 0
\end{pmatrix} \begin{pmatrix}
D_2 \\ 0 \\ 0 \\ 1
\end{pmatrix} \begin{pmatrix}
-c \\ 0 \\ 0 \\ c'
\end{pmatrix}.
\]

Now consider the following linear operators

\[
T_{a, b, c} := adbd^*c : A_\Theta \rightarrow A_\Theta
\]

\[
e \mapsto ad(bd^*(ce))
\]

\[
\widetilde{T}_{a', b', c'} := a'd'b'dc' : A_\Theta \rightarrow A_\Theta
\]

\[
e \mapsto a'd^*(b'd'(ce))
\]

Then,

\[
(T_{a, b, c}) = M_{ad(b)d^*(c)} + M_{abd^*(c)} + M_{abd^*(c)} \circ d + M_{ad(bc)} \circ d^* + M_{abc \circ d \circ d^*}
\]

\[
\widetilde{T}_{a', b', c'} = M_{a'd'r'(b'd'(c'))} + a'b'd'(d'(c')) + M_{a'b'd'(c') \circ d} + M_{a'b'd'(c') \circ d^*} + M_{a'b'd'(c') \circ d^* \circ d}
\]

where \(M_\xi\) denotes multiplication by \(\xi\).

**Lemma 4.5.** \(\{f \circ d : A_\Theta \rightarrow A_\Theta\} \cap \{g \circ d^* : A_\Theta \rightarrow A_\Theta\} = \{0\}\)
Proof. Let \( \{e_k : k \in \mathbb{Z}^2\} \) denotes the standard orthonormal basis of \( \ell^2(\mathbb{Z}^2) \). Here \( \ell^2(\mathbb{Z}^2) \) represents the G.N.S. Hilbert space and \( \mathcal{A}_\Theta \subseteq \ell^2(\mathbb{Z}^2) \). Any element from the intersection must satisfy
\[
\langle e_\alpha, M_f \circ d(e_\beta) \rangle = \langle e_\alpha, M_g \circ d^*(e_\beta) \rangle \quad \forall \alpha, \beta \in \mathbb{Z}^2.
\]
\[
\Rightarrow \sum_k \hat{g}_k^* e_{k+\alpha}, d(e_\beta) = \sum_k \hat{g}_k^* e_{k+\alpha}, d^*(e_\beta)
\]
\[
\Rightarrow \sum_k \hat{g}_k^* e_{k+\alpha}, e_\beta)(\beta_1 - i\beta_2) = \sum_k \hat{g}_k^* e_{k+\alpha}, e_\beta)(\beta_1 + i\beta_2)
\]
So,
\[
\hat{f}_{\beta-\alpha}^*(\beta_1 - i\beta_2) = \hat{g}_{\beta-\alpha}^*(\beta_1 + i\beta_2)
\]
for all \( \alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{Z}^2 \), i.e.
\[
(4.21) \quad \hat{f}_\gamma^*(\alpha_1 + \gamma_1 - i\alpha_2 - i\gamma_2) = \hat{g}_\gamma^*(\alpha_1 + \gamma_1 + i\alpha_2 + i\gamma_2)
\]
where \( \beta - \alpha = \gamma \in \mathbb{Z}^2 \). In order to have nontrivial intersection, equation (4.21) must have nontrivial solution for all \( \alpha, \gamma \in \mathbb{Z}^2 \). Let \( \hat{f}_\gamma^* = x \) and \( \hat{g}_\gamma^* = y \). We get
\[
(4.22) \quad x(1 + \gamma_1 - i\gamma_2) = y(1 + \gamma_1 + i\gamma_2)
\]
\[
(4.23) \quad x(2 + \gamma_1 - 2i\gamma_2) = y(2 + \gamma_1 + 2i\gamma_2)
\]
(4.23) - (4.22) implies
\[
(4.24) \quad x(1 - i) = y(1 + i)
\]
Again (4.21) gives,
\[
(4.25) \quad x(1 + \gamma_1 - i\gamma_2) = y(1 + \gamma_1 + i\gamma_2)
\]
(4.22) and (4.25) together implies \( x = -y \). Hence from (4.24) we get \( x = y = 0 \), i.e. \( \hat{f}_\gamma^* = 0 \) for all \( \gamma \), which proves triviality of the intersection. \( \square \)

Lemma 4.6. \( \{M_a + M_b \circ d + M_c \circ d^* : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta\} \bigcap \{M_f \circ dd^* : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta\} = \{0\} \)

Proof. Let \( \{e_k : k \in \mathbb{Z}^2\} \) denotes the standard orthonormal basis of \( \ell^2(\mathbb{Z}^2) \). Any element from the intersection must satisfy
\[
\langle e_\alpha, (M_a + M_b \circ d + M_c \circ d^*)(e_\beta) \rangle = \langle e_\alpha, M_f \circ dd^*(e_\beta) \rangle \quad \forall \alpha, \beta \in \mathbb{Z}^2.
\]
\[
\Rightarrow \sum_k a_k^* e_{k+\alpha}, e_\beta) + \sum_k b_k^* e_{k+\alpha}, e_\beta) = \sum_k f_k^* e_{k+\alpha}, e_\beta)(\beta_1 - i\beta_2) + \sum_k \hat{g}_k^* e_{k+\alpha}, e_\beta)(\beta_1 + i\beta_2)
\]
So,
\[
\sum_k a_k^* e_{k+\alpha}, e_\beta) = \sum_k \hat{g}_k^* e_{k+\alpha}, e_\beta)(\beta_1 - i\beta_2) + \sum_k c_k^* e_{k+\alpha}, e_\beta)(\beta_1 + i\beta_2)
\]
for all \( \alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{Z}^2 \), i.e.
\[
(4.26) \quad a_\gamma + \hat{b}_\gamma^*(\alpha_1 + \gamma_1 - i\alpha_2 - i\gamma_2) + c^*_\gamma (\alpha_1 + \gamma_1 + i\alpha_2 + i\gamma_2
\]
\[
= \hat{f}_\gamma^*((\alpha_1 + \gamma_1)^2 + (\alpha_2 + \gamma_2)^2)
\]
where $\beta - \alpha = \gamma \in \mathbb{Z}^2$. In order to have nontrivial intersection, equation (4.26) must have nontrivial solution for all $\alpha, \gamma \in \mathbb{Z}^2$. Let $\overline{a}_{\gamma}^\ast = w, \overline{b}_{\gamma}^\ast = x, \overline{c}_{\gamma}^\ast = y,$ and $\overline{f}_{\gamma}^\ast = z$. So (4.28) turns to

\begin{equation}
(\text{4.27}) \quad w + x(\alpha_1 + \gamma_1 - i\alpha_2 - i\gamma_2) + y(\alpha_1 + \gamma_1 + i\alpha_2 + i\gamma_2) = z((\alpha_1 + \gamma_1)^2 + (\alpha_2 + \gamma_2)^2)
\end{equation}

\[z\text{From (4.27) we get}

\begin{equation}
(\text{4.28}) \quad w + x(1 + \gamma_1 - i\gamma_2) + y(1 + \gamma_1 + i\gamma_2) = z((1 + \gamma_1)^2 + (1 + \gamma_2)^2)
\end{equation}

\begin{equation}
(\text{4.29}) \quad w + x(2 + \gamma_1 - 2i - i\gamma_2) + y(2 + \gamma_1 + 2i + i\gamma_2) = z((2 + \gamma_1)^2 + (2 + \gamma_2)^2)
\end{equation}

\[\text{(4.29) - (4.28) gives,}

\begin{equation}
(\text{4.30}) \quad x(1 - i) + y(1 + i) = z((2 + \gamma_1)^2 + (2 + \gamma_2)^2 - (1 + \gamma_1)^2 - (1 + \gamma_2)^2)
\end{equation}

\[\text{again, (4.27) gives}

\begin{equation}
(\text{4.31}) \quad w + x(\gamma_1 - i\gamma_2) + y(\gamma_1 + i\gamma_2) = z(\gamma_1^2 + \gamma_2^2)
\end{equation}

\[\text{(4.32) - (4.31) gives,}

\begin{equation}
(\text{4.32}) \quad x(1 - i) + y(1 + i) = z((1 + \gamma_1)^2 + (1 + \gamma_2)^2 - \gamma_1^2 - \gamma_2^2)
\end{equation}

\text{Finally (4.30) - (4.32) gives } z = 0. \text{ Hence } \overline{f}_{\gamma}^\ast = 0 \text{ for all } \gamma \text{, i.e. intersection is trivial.} \quad \square

\textbf{Lemma 4.7.} \{M_a \circ d + M_b \circ d^\ast : \mathcal{A}_\Theta \rightarrow \mathcal{A}_\Theta\} \cap \{M_f : \mathcal{A}_\Theta \rightarrow \mathcal{A}_\Theta\} = \{0\}

\textit{Proof.} \text{Since } d(1) = d^\ast(1) = 0 \text{ for } 1 \in \mathcal{A}_\Theta, \text{ this follows trivially.} \quad \square

\textbf{Proposition 4.8.} \textit{The following map}

\[\Phi : \pi \left(\Omega^2(\overline{\mathcal{A}}_\Theta)\right) \rightarrow \mathcal{A}_\Theta \otimes \mathbb{C}\]

\[\Phi = (\overline{\Phi}, \overline{\Phi}^\ast)\]

\textit{where}

\[\overline{\Phi} : T_{a,b,c} \rightarrow (ad(b)d^\ast(c) + abd^\ast(c), abd^\ast(c), ad(bc), abc),\]

\textit{and}

\[\overline{\Phi}^\ast : T'_{a',b',c'} \rightarrow (a'd^\ast(b)d(c'), a'b'd^\ast(c'), a'd^\ast(b')d(c'), a'd^\ast(b)c'), a'd^\ast(b)c')\],

\textit{is a linear bijection, where } T_{a,b,c} = abd^\ast c \text{ and } T'_{a,b,c'} = a'd^\ast b'dc' \text{ as define in (4.20).}

\textit{Proof.} \text{Since } d(U) = d^\ast(U) = U \text{ and } UU^* = U^*U = I, \text{ Lemma 4.5, Lemma 4.6 and Lemma 4.7 proves well-definedness as well as injectivity of } \Phi. \text{ To see surjectivity observe that}

\[T_{aU^*,U} + T_{aV^*,V} - T_{-aV^*,V} - T_{-aU^*,U} + T_{-ib,V^*,V} - T_{-ib,U^*,U} + T_{cV^*,V} + T_{cU^*,U} + T_{cU^*,V} \rightarrow (a,b,c,e) \in \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta\]
and

\[ T_{a'U',1,U} - T_{a'U',U,1} - T_{-ia'V,V,\gamma} - T_{ia',1,1} + T_{-i\nu',V,V,\gamma} - T_{-i\nu',1,1} + T_{c'U',1,U} - T_{c',1,1} + T_{c',1,1} \xrightarrow{\tilde{\Phi}'} (a', b', c', e') \in \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \]

This completes the proof. \qed

**Proposition 4.9.** \( \pi \left( dJ_1^0(\tilde{\mathcal{A}_\Theta}) \right) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^6. \)

**Proof.** Elements of \( \pi \left( dJ_1^0(\tilde{\mathcal{A}_\Theta}) \right) \) looks like

\[
\sum [D, pa + qb] [D, pe + qf] \text{ such that } \sum (pa + qb) [D, pe + qf] = 0,
\]

where \( p = (1 + \gamma)/2 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \) and \( q = (1 - \gamma)/2 = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \) are the projections onto the eigenspaces of \( \gamma \). Expanding the commutators and simplifying we get

\[
\sum \begin{pmatrix} -add^* e + aed^* & 0 \\ 0 & bdf^* d - bdf d^* \end{pmatrix} \text{ s.t. } \sum a df = \sum a ed \quad \sum b df e = \sum b df^* \tag{4.33}
\]

Recall that \( T_{a,b,c} = adbd^* c \) and \( \tilde{T}_{a,b,c'} = a'd'^* b'd' \). So equation (4.33) becomes,

\[
\sum \begin{pmatrix} T_{ae,1,1} - T_{a,1,e} & 0 \\ 0 & \tilde{T}_{bf,1,1} - \tilde{T}_{b,1,f} \end{pmatrix} \text{ s.t. } \sum a df = \sum a ed \quad \sum b df e = \sum b df^* \tag{4.34}
\]

The bijection of Proposition 4.8 gives,

\[
\tilde{\Phi}(T_{ae,1,1} - T_{a,1,e}) = (-add^*(e), -ad^*(e), -ad(e), 0)
\]

\[
\tilde{\Phi}(\tilde{T}_{bf,1,1} - \tilde{T}_{b,1,f}) = (-bd^*(f), -bd(f), -bd^*(f), 0)
\]

To fulfill our claim it is enough to show that elements of the form

\[
(a add^*(e), ad^*(e), ad(e), bd^*(f), bd(f), bd^*(f))
\]

can generate \( \mathcal{A}_\Theta \otimes \mathbb{C}^6 \), where conditions in equation (4.33) hold. Choose any arbitrary element \((a_1, a_2, a_3, a'_1, a'_2, a'_3) \in \mathcal{A}_\Theta \otimes \mathbb{C}^6\). Observe that,

\[
(a_1 V^* dd^*(V), a_1 V^* d^*(V), a_1 V^* d(V), a'_1 V^* d(V), a'_1 V^* d^*(V)) + (a_1 V d^*(V^*), a_1 V d(V^*), a'_1 V d(V^*), a'_1 V d^*(V^*))
\]

\[
= (2a_1, 0, 0, 2a'_1, 0, 0)
\]

and conditions of (4.33) also satisfied. Hence \((a_1, 0, 0, a'_1, 0, 0) \in \pi \left( dJ_1^0(\tilde{\mathcal{A}_\Theta}) \right) \). Now,

\[
(i a_3 U^* dd^*(U^*), i a_3 U d^*(U^*), i a_3 U d(U^*), 0, 0, 0)
\]

\[
+ (a_3 V^* dd^*(V), a_3 V^* d^*(V), a_3 V^* d(V), 0, 0, 0)
\]

\[
+ \left( -\frac{1}{2}(a_3 + ia_3)V^* dd^*(V), \frac{1}{2}(a_3 + ia_3)V^* d^*(V), -\frac{1}{2}(a_3 + ia_3)V^* d(V), 0, 0, 0 \right)
\]

\[
+ \left( -\frac{1}{2}(a_3 + ia_3) V d^*(V^*), -\frac{1}{2}(a_3 + ia_3) V d(V^*), -\frac{1}{2}(a_3 + ia_3) V d(V), 0, 0, 0 \right)
\]

\[
= (0, 0, -2ia_3, 0, 0, 0)
\]
Theorem 4.11. \(\Omega^2(\mathcal{A}_\Theta)\) is also satisfied. Hence, \((0,0,a_3,0,0,0) \in \pi\left(dJ_1^1(\mathcal{A}_\Theta)\right)\). Finally
\[
\begin{align*}
(0,0,0,ia_2^cV^*d(V),ia_2^cV^*d(V)) \\
+ (0,0,0,a_2^cV^*d(V),a_2^cV^*d(V)) \\
+ (0,0,0,\frac{-1}{2}(a_2^c+ia_2^c)\bar{V}^*d(V),\frac{-1}{2}(a_2^c+ia_2^c)\bar{V}^*d(V)) \\
+ (0,0,0,\frac{-1}{2}(a_2^c+ia_2^c)V^*d(V),\frac{-1}{2}(a_2^c+ia_2^c)V^*d(V)) \\
= (0,0,0,0,-2ia_2^c,0)
\end{align*}
\]
and conditions of (4.33) also satisfied. Hence, \((0,0,0,a_2^c,0) \in \pi\left(dJ_1^1(\mathcal{A}_\Theta)\right)\). Thus combining we have \((0,0,a_3,0,a_2^c,0) \in \pi\left(dJ_1^1(\mathcal{A}_\Theta)\right)\). Similarly one can show that \((0,a_2,0,0,a_3^0) \in \pi\left(dJ_1^1(\mathcal{A}_\Theta)\right)\) and this completes the proof.

Proposition 4.10. The following action
\[
\begin{pmatrix}
x \\
0
\end{pmatrix}, a_2) := (xa_1, ya_2)
\]
\[
(a_1,a_2), \begin{pmatrix}
x \\
0
\end{pmatrix} := (a_1y, a_2x).
\]
defines an \(\mathcal{A}_\Theta\)-bimodule structure on \(\pi(\Omega^2(\mathcal{A}_\Theta)) \cong \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta\).

Proof. If we define
\[
\begin{pmatrix}
x \\
0
\end{pmatrix}, (a_1,a_2) := \Phi \begin{pmatrix}
x \\
0
\end{pmatrix} \Phi^{-1}(a_1,a_2)
\]
\[
(a_1,a_2), \begin{pmatrix}
x \\
0
\end{pmatrix} := \Phi^{-1}(a_1,a_2), \begin{pmatrix}
x \\
0
\end{pmatrix}
\]
where \(\Phi\) is as defined in Proposition 4.8, then clearly it is a bimodule action induced by that on \(\Omega^2_2(\mathcal{A}_\Theta)\). One can verify that these actions match with the ones defined in question.

Theorem 4.11. For noncommutative torus we have,
(1) \(\Omega_D^1(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta\), as \(\mathcal{A}_\Theta\)-bimodule.
(2) \(\Omega_n^1(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta\), for all \(n \geq 2\) as \(\mathcal{A}_\Theta\)-bimodule.

Proof. Proposition 4.4 gives part (1). Proposition 4.8 and 4.9 proves part (2) for \(n = 2\). The fact that the isomorphisms in Propositions 4.8, 4.9 are not only \(C\)-linear but also \(\mathcal{A}_\Theta\)-bimodule isomorphisms follows from the defining property of the bimodule action in Proposition 4.10.

We need to prove part (2) for \(n \geq 3\). For that purpose first note that
\[
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 2 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 2 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]

(4.35)
These matrices are the key role to compute \( \Omega^n_D(\widetilde{A}_\Theta) \) for all \( n \geq 3 \). Now for any unital algebra \( \mathcal{A} \),

\[
\Omega^n(\mathcal{A}) = \bigotimes_{\text{n times}} \underbrace{\mathcal{A} \cdots \mathcal{A}}_{\text{n times}} \mathcal{A}^1(\mathcal{A}).
\]

By Lemma 4.2 we have

\[
\pi \left( \Omega^1(\widetilde{A}_\Theta) \right) = (\mathcal{A}_\Theta \oplus \mathcal{A}_\Theta) \otimes \mathbb{C} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (\mathcal{A}_\Theta \oplus \mathcal{A}_\Theta) \otimes \mathbb{C} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

In view of Proposition 4.3 and using (4.35), (4.36) we get \( \pi \left( \Omega^n(\widetilde{A}_\Theta) \right) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^{2n} \bigoplus \mathcal{A}_\Theta \otimes \mathbb{C}^{2n} \) for all \( n \geq 3 \) (actually true for all \( n \geq 1 \) by part (1) and Proposition 4.8). We will show the following

\[
\pi \left( dJ_0^n(\widetilde{A}_\Theta) \right) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^{2n+1} \bigoplus \mathcal{A}_\Theta \otimes \mathbb{C}^{2n+1} \quad \forall \ n \geq 2.
\]

Recall from Lemma 2.11, \([D^2, a] \in \pi(dJ_0^n)\). It is then easy to prove that

\[
\pi(dJ_0^n) = \sum_{i=0}^{n-1} \pi \left( \Omega^i \otimes_\mathcal{A} J^2 \otimes_\mathcal{A} \Omega^{n-1-i} \right) \quad \text{for all } n \geq 2
\]

by writing down any arbitrary element of \( \pi(dJ_0^n) \) and then passing \( D \) through the commutators from left to right. Hence for all \( n \geq 2 \) odd,

\[
\begin{align*}
\pi \left( dJ_0^n(\widetilde{A}_\Theta) \right) &= \sum_{i=0, \text{ even}}^{n-1} \pi \left( \Omega^i \otimes J^2 \otimes \Omega^{n-1-i} \right) + \sum_{i=1, \text{ odd}}^{n-1} \pi \left( \Omega^i \otimes J^2 \otimes \Omega^{n-1-i} \right) \\
&= \sum_{i=0, \text{ even}}^{n-1} \left( \mathcal{A}_\Theta \otimes \mathbb{C}^{2i} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{A}_\Theta \otimes \mathbb{C}^{2i} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \\
&\quad + \mathcal{A}_\Theta \otimes \mathbb{C}^{3} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{A}_\Theta \otimes \mathbb{C}^{3} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad + \mathcal{A}_\Theta \otimes \mathbb{C}^{2(n-1-i)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{A}_\Theta \otimes \mathbb{C}^{2(n-1-i)} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \mathcal{A}_\Theta \otimes \mathbb{C}^{2n+1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{A}_\Theta \otimes \mathbb{C}^{2n+1} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&\cong \mathcal{A}_\Theta \otimes \mathbb{C}^{2n+1} \bigoplus \mathcal{A}_\Theta \otimes \mathbb{C}^{2n+1}.
\end{align*}
\]

Here the last equality uses (4.35) frequently. Similarly one can do for all \( n \geq 2 \) even. Hence, for all \( n \geq 3 \) we have \( \Omega^n_D(\widetilde{A}_\Theta) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^2 \) as \( \mathcal{A}_\Theta \)-bimodule where the bimodule action on \( \mathcal{A}_\Theta \otimes \mathbb{C}^2 \) will be specified by Proposition 4.10.

**Remark 4.12.** One can also consider
\[ D = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \text{ or } D = \begin{pmatrix} 0 & d^* \\ d^* & 0 \end{pmatrix}. \]

However, in that case one will get same answer as in Theorem 4.11. Since in noncommutative geometry it is customary to take \( D = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \), we provide computation with this value for \( D \).

**Notation:** \( \widetilde{\Omega}_D^1 := A_\Theta \otimes \mathbb{C}^4 \) and \( \widetilde{\Omega}_D^2 := A_\Theta \otimes \mathbb{C}^2 \) until the rest of this section.

Now we want to show that \( \Omega^\bullet_D(\widetilde{A}_\Theta) \) is cohomologically not trivial. For that purpose we use the isomorphism in Theorem 4.11 to compute the differentials on \( \widetilde{\Omega}_D^1 \) and \( \widetilde{\Omega}_D^2 \).

**Proposition 4.13.** The maps

\[ \delta: \widetilde{A}_\Theta \to \Omega_D^1 \]

\[ \begin{pmatrix} a \\ 0 \\ b \end{pmatrix} \mapsto (d(b), b - a, d^*(a), a - b) \]

and

\[ \delta: \Omega_D^1 \to \Omega_D^2 \]

\[ (a, b, c, e) \mapsto (b + e, b + e) \]

make the following diagrams

\[
\begin{array}{c}
\begin{array}{c}
\widetilde{A}_\Theta \\
\downarrow \delta
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Omega_D^1 \\
\downarrow \cong
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Omega_D^2(\widetilde{A}_\Theta) \\
\downarrow \cong
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Omega_D^1(\widetilde{A}_\Theta) \\
\downarrow \cong
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\widetilde{A}_\Theta \\
\downarrow \delta
\end{array}
\end{array}
\]

commutative, where \( \widetilde{d}: \Omega_D^1(\widetilde{A}_\Theta) \to \Omega_D^{*+1}(\widetilde{A}_\Theta) \) denotes the differential of Connes complex.

**Proof.** Use Lemma 4.2 to see commutativity of the first diagram. For the second, take any \((a, b, c, e) \in \Omega_D^1 \) and use \( \Phi^{-1} \) of Lemma 4.2 to get an element in \( \pi \left( \Omega^1(\widetilde{A}_\Theta) \right) \), which is

\[ (4.37) \begin{pmatrix} 0 \\ 0 \\ aU^*dU + bd1 - ad1 \\ cU^*dU + cd^*1 - cd^* \end{pmatrix}. \]

Use the fact

\[ \begin{pmatrix} -U^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D, & (U^2 & 0) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -U^* \end{pmatrix} \begin{pmatrix} D, & (0 & 0) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix} \]

to observe that (4.37) can be re-written as

\[
\begin{pmatrix} -aU^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D, & (U^2 & 0) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} aU^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D, & (U & 0) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -U^* \end{pmatrix} \begin{pmatrix} D, & (0 & 0) \\ 0 & 0 \end{pmatrix} + \\
\begin{pmatrix} 0 & 0 \\ 0 & cU^* \end{pmatrix} \begin{pmatrix} D, & (0 & 0) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} (a - b)U^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D, & (U & 0) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (c - e)U^* \end{pmatrix} \begin{pmatrix} D, & (0 & 0) \\ 0 & 0 \end{pmatrix}.
\]
Applying $\Phi \circ \tilde{d}$ (of Proposition 4.8 and $\tilde{d}: \Omega^1_D \to \Omega^2_D$), we get the following element
\[
\begin{align*}
(d(c) + 2c + 2a + b, & b + c, d(e) + a + b, b + e, d^*(a) + 2a + 2c + e, e + a, \\
&d^*(b) + c + e, e + b) + \pi \left( dJ^1_1(\tilde{\mathcal{A}_\Theta}) \right)
\end{align*}
\]
of $\Omega^2_D(\tilde{\mathcal{A}_\Theta})$. This element is equal to $(b + e, b + e) \in \Omega^2_D(\tilde{\mathcal{A}_\Theta})$ by Theorem 4.11. ⊳

**Remark 4.14.** Notice that $\delta = \Phi \circ \tilde{d} \circ \Phi^{-1}$, and hence $\delta^2 = 0$.

Before we proceed to show that $\Omega^n_D(\tilde{\mathcal{A}_\Theta})$ is cohomologically not trivial we first compute the cohomologies for $(\Omega^n_D(\mathcal{A}_\Theta), d)$, in order to notice the similarity. To do so recall Proposition 13, in the last chapter of ([Con2]).

**Proposition 4.15 ([Con2]).** For noncommutative torus $\mathcal{A}_\Theta$, we have

1. $\Omega^1_D(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta$,
2. $\Omega^2_D(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta$,
3. The differentials $\tilde{d}: \mathcal{A}_\Theta \to \Omega^1_D(\mathcal{A}_\Theta)$ and $\tilde{d}: \Omega^1_D(\mathcal{A}_\Theta) \to \Omega^2_D(\mathcal{A}_\Theta)$ are given by
   \[
   \tilde{d}: a \mapsto (\delta_1 a, \delta_2 a)
   \]
   \[
   \tilde{d}: (a_1, a_2) \mapsto \delta_2(a_1) - \delta_1(a_2)
   \]

**Remark 4.16.** For $n \geq 3$, the space of higher forms $\Omega^n_D(\mathcal{A}_\Theta)$ vanish. To see this first observe that $[D, a] = \delta_1(a) \otimes \sigma_1 + \delta_2(a) \otimes \sigma_2$ where $\sigma_1, \sigma_2$ are the spin matrices satisfying $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$. The isomorphism $\Omega^1_D(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^2$ is obtained using the linear independence of $\sigma_1, \sigma_2$ in $M_2(\mathbb{C})$. The isomorphism $\pi(\Omega^2_D(\mathcal{A}_\Theta)) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^2$ is obtained using the linear independence of $I_2$ and $\sigma_1 \sigma_2$ in $M_2(\mathbb{C})$ and in this way one will obtain that $\pi(dJ^1_1(\mathcal{A}_\Theta)) \cong \mathcal{A}_\Theta \otimes I_2$. Because of this reason $\Omega^2_D(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta$. Observe that $\pi(\Omega^3_D(\mathcal{A}_\Theta)) = \mathcal{A}_\Theta \otimes \sigma_1 + \mathcal{A}_\Theta \otimes \sigma_2$ and hence $\pi(\Omega^3_D(\mathcal{A}_\Theta)) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^2$. Now recall that $J^\bullet$ is a graded ideal in $\Omega^\bullet$ and hence we have
\[
\pi \left( \Omega^1(\mathcal{A}_\Theta) J^2(\mathcal{A}_\Theta) \right) \subseteq \pi \left( J^3(\mathcal{A}_\Theta) \right) \subseteq \pi \left( \Omega^3(\mathcal{A}_\Theta) \right)
\]
This shows that $\pi \left( J^3(\mathcal{A}_\Theta) \right) = \pi \left( dJ^1_1(\mathcal{A}_\Theta) \right) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^2$ i.e. $\Omega^2_D(\mathcal{A}_\Theta) = \{0\}$. Now note that $\Omega^n(\mathcal{A}) = \Omega^1(\mathcal{A}) \otimes_A \cdots \otimes_A \Omega^1(\mathcal{A})$ for any unital algebra $\mathcal{A}$. Hence, $\pi(\Omega^n(\mathcal{A}_\Theta)) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^2$ for all $n \geq 4$.

Finally, the inclusion
\[
\pi \left( \Omega^{n-2}(\mathcal{A}_\Theta) J^2(\mathcal{A}_\Theta) \right) \subseteq \pi \left( J^n(\mathcal{A}_\Theta) \right) \subseteq \pi \left( \Omega^n(\mathcal{A}_\Theta) \right)
\]
proves that $\Omega^2_D(\mathcal{A}_\Theta) = \{0\}$ for all $n \geq 4$. This is needed in the next Lemma.

**Lemma 4.17.** The cohomologies $H^\bullet(\mathcal{A}_\Theta)$ are given by,

1. $H^0(\mathcal{A}_\Theta) \cong \mathbb{C}$,
2. $H^1(\mathcal{A}_\Theta) \cong \mathbb{C} \oplus \mathbb{C}$,
3. $H^2(\mathcal{A}_\Theta) \cong \mathbb{C}$.

**Proof.** (1) We have
\[
H^0(\mathcal{A}_\Theta) = \{a \in \mathcal{A}_\Theta : \delta_1(a) = \delta_2(a) = 0\}
\]

$\cong \mathbb{C}$
(2) We have
\[ H^1(\mathcal{A}_\Theta) = \frac{\{(a, b) : a, b \in \mathcal{A}_\Theta; \delta_2(a) = \delta_1(b)\}}{\{(\delta_1(a), \delta_2(a)) : a \in \mathcal{A}_\Theta\}} \]

Let
\[ a = \sum_{m,n} \alpha_{m,n} U^m V^n - \alpha_{0,0} , \quad b = \sum_{p,q} \beta_{p,q} U^p V^q - \beta_{0,0} \]
i.e. \( a, b \notin \mathbb{C}1 \). Then \( \delta_2(a) = \delta_1(b) \) will imply
\[ \sum_{m \neq 0, n \neq 0} n\alpha_{m,n} U^m V^n + \sum_{n \neq 0} n\alpha_{0,n} V^n = \sum_{p \neq 0, q \neq 0} p\beta_{p,q} U^p V^q + \sum_{p \neq 0} p\beta_{p,0} U^p \]
If \{\epsilon_{mn}\}_{m,n \in \mathbb{Z}} be orthonormal basis of \( \ell^2(\mathbb{Z}^2) \) then we get
\[ \beta_{p,0} = 0 \forall p \neq 0 ; \quad \alpha_{0n} = 0 \forall n \neq 0 ; \quad n\alpha_{mn} = m\beta_{mn} \forall m \neq 0, n \neq 0. \]
Let
\[ c = \sum_{m \neq 0, n \neq 0} \gamma_{m,n} \frac{\epsilon_{mn}}{mn} U^m V^n + \sum_{m \neq 0} \frac{\alpha_{m,0}}{m} U^m + \sum_{n \neq 0} \frac{\beta_{0,n}}{n} V^n \]
For \( m \neq 0, n \neq 0 \), if we choose \( \gamma_{m,n} = n\alpha_{mn} \) then we get \( \delta_1(c) = a \) and \( \delta_2(c) = b \) which proves our claim.

(3) Finally,
\[ H^2(\mathcal{A}_\Theta) = \frac{\mathcal{A}_\Theta}{\{\delta_2(a) - \delta_1(b) : a, b \in \mathcal{A}_\Theta\}} \]
Let \( a \in \mathcal{A}_\Theta \) be s.t. \( a \notin \mathbb{C}1 \). Let \( a = \sum_{m \neq 0 \text{ or } n \neq 0} \alpha_{m,n} U^m V^n \). Then
\[ a = \sum_{m \in \mathbb{Z}, n \in \mathbb{Z} - \{0\}} \alpha_{m,n} U^m V^n + \sum_{m \in \mathbb{Z} - \{0\}} \frac{\alpha_{m,0}}{m} U^m \]
Consider \( b = \sum_{m \in \mathbb{Z}, n \in \mathbb{Z} - \{0\}} \frac{\alpha_{m,n}}{m} U^m V^n \) and \( c = -\sum_{m \in \mathbb{Z} - \{0\}} \frac{\alpha_{m,0}}{m} U^m \). Then \( \delta_2(b) - \delta_1(c) = a \), which proves our claim.

\[ \square \]

**Theorem 4.18.** If \( \widehat{\Omega}^* (\mathcal{A}_\Theta) \) denotes the cohomologies of the chain complex \( \left( \Omega^*_D (\mathcal{A}_\Theta), \delta \right) \), then we have

1. \( H^0(\mathcal{A}_\Theta) \cong \mathbb{C} \)
2. \( H^1(\mathcal{A}_\Theta) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathcal{A}_\Theta / \mathbb{C} \)

**Proof.**

1. We have
\[ H^0(\mathcal{A}_\Theta) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : d(b) = 0, d^*(a) = 0, a = b \right\} \]
\[ = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : \delta_1(a) = \delta_2(a) = 0 \right\} \]
\[ \cong \mathbb{C} \]

2. We have
\[ H^1(\mathcal{A}_\Theta) = \frac{\{(a, b, c, e) : b + e = 0\}}{\{(d(f), f - g, d^*(g), g - f)\}} \]
Let \( \mathcal{M} = \{(a, b, c, -b) : a, b, c \in \mathcal{A}_\Theta\} \) and \( \mathcal{N} = \{(d(f), f - g, d^*(g), g - f) : f, g \in \mathcal{A}_\Theta\} \). Clearly \( \mathcal{M} \cong \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \). Now define
\[ \psi: N \odot C \odot A_\Theta \odot C \oplus C \rightarrow M \]

\[ (d(f), f - g, d^*(g), g - f, \lambda_1, a, \lambda_2) \mapsto (d(f) + \lambda_1, f - g + a, d^*(g) + \lambda_2) \]

This map is \( C \)-linear and one-one. To see surjectivity take any \( (a, b, c) \in \mathcal{A}_\Theta^3 \). Suppose \( a = \sum \alpha_{m,n} U_m V_n \). If we choose \( f = \sum_{m \neq 0} \alpha_{m,0} U_m V_n \) then \( d(f) = a - \alpha_{0,0} \) and we see that

\[ (d(f), f, 0, -f, \alpha_{0,0}, -f, 0) \mapsto (a, 0, 0) \]

Now suppose \( b = \sum \beta_{m,n} U_m V_n \). If we choose \( f = \beta_{0,0}, g = 0 \) then

\[ (0, \beta_{0,0}, 0, -\beta_{0,0}, 0, b - \beta_{0,0}, 0) \mapsto (0, b, 0) \]

Finally let \( c = \sum \gamma_{m,n} U_m V_n \) and choose

\[ g = \sum_{m \neq 0} \gamma_{m,0} U_m V_n \]

then we see that

\[ (0, -g, c - \gamma_{0,0}, g, 0, \gamma_{0,0}) \mapsto (0, 0, c) \]

This shows that \( \psi \) is a linear isomorphism with \( \psi(N) = N \) and hence our claim has been justified.

This shows that the complex \( \Omega^*_\Theta(\mathcal{A}_\Theta) \) is cohomologically not trivial. \( \square \)

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The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113

E-mail address: parthac@imsc.res.in

The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113

E-mail address: gsatyajit@imsc.res.in