Research article

MBJ-neutrosophic hyper $BCK$-ideals in hyper $BCK$-algebras

Abdelaziz Alsubie$^{1,*}$ and Anas Al-Masarwah$^{2}$

$^{1}$ Department of Basic Sciences, College of Science and Theoretical Studies, Saudi Electronic University, Saudi Arabia
$^{2}$ Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor DE, Malaysia

* Correspondence: Email: a.alsubie@seu.edu.sa; Tel: +966506098942.

Abstract: In 2018, Takallo et al. introduced the concept of an MBJ-neutrosophic structure, which is a generalization of a neutrosophic structure, and applied it to a $BCK/BCI$-algebra. The aim of this study is to apply the notion of an MBJ-neutrosophic structure to a hyper $BCK$-algebra. The notions of the MBJ-neutrosophic hyper $BCK$-ideal, the MBJ-neutrosophic weak hyper $BCK$-ideal, the MBJ-neutrosophic s-weak hyper $BCK$-ideal and the MBJ-neutrosophic strong hyper $BCK$-ideal are introduced herein, and their relations and properties are investigated. These notions are discussed in connection with the MBJ-neutrosophic level cut sets.

Keywords: hyper $BCK$-algebra; MBJ-neutrosophic hyper $BCK$-ideal; MBJ-neutrosophic weak (s-weak) hyper $BCK$-ideal; MBJ-neutrosophic strong hyper $BCK$-ideal

Mathematics Subject Classification: 03G25, 06F35, 08A72

1. Introduction

$BCK$-algebras entered into pure mathematics in 1966 through the work of Imai and Iséki [1], and were applied to various mathematical fields, such as functional analysis, group theory, topology and probability theory, etc. The hyperstructure theory was born in 1934 when Marty introduced hypergroups. In this side, he explored and applied their properties to groups and other algebraic structures [2]. Algebraic hyperstructures reflect a natural generalization of classical algebraic structures. In an algebraic hyperstructure, the composition of two elements is a set, while in a classical algebraic structure, the composition of two elements is an element. As an extension of a $BCK$-algebra, Jun et al. [3] introduced an algebraic hyperstructure called a hyper $BCK$-algebra. They studied hyper $BCK$-ideals in hyper $BCK$-algebras. Saeid and Zahedi [4] studied quotient hyper $BCK$-algebras and in [5] Saeid et al. introduced weak implicative and implicative hyper $K$-ideals of
hyper $K$-algebras. After that, many books and several articles have been published on hyper $BCK$-algebras and other hyper algebraic structures.

Zadeh [6] introduced fuzzy set theory in 1965 and in 1986 this concept has been generalized to intuitionistic fuzzy set theory by adding a non-membership function by Atanassov [7]. As a generalization of the classical set and (intuitionistic) fuzzy set theory, Smarandache [8, 9] launched a significant topic, that deals with indeterminacy, called neutrosophic set theory. In [10], Takallo et al. presented the notion of an MBJ-neutrosophic set as generalization of a neutrosophic set and they applied it to $BCK/BCI$-algebras. In an MBJ-neutrosophic set, the indeterminacy membership function is generalized to interval valued membership function. Next, Jun and Roh [11] introduced and studied the concept of an MBJ-neutrosophic ideal in $BCK/BCI$-algebras. In B-algebras, Manokaran and Prakasam [12] introduced the MBJ-neutrosophic subalgebra and Khalid et al. [13] defined and studied the MBJ-neutrosophic T-ideal. The notions of (intuitionistic) fuzzy sets, neutrosophic sets and other extensions of fuzzy sets have been applied to algebraic structures, decision making problems, etc. For algebraic structures, see [14–22] and for decision making problems, see [23, 24]. In an algebraic hyperstructure, Jun and Xin [25] discussed the topic of fuzzy set theory of hyper $BCK$-ideals in hyper $BCK$-algebras and in [26] Bakhshi et al. studied fuzzy (positive, weak) implicative hyper $BCK$-ideals. In 2004, Borzooei and Jun [27] studied the intuitionistic fuzzy set theory of hyper $BCK$-ideals in hyper $BCK$-algebras. In addition, Khademan et al. [28] studied neutrosophic set theory of hyper $BCK$-ideals in hyper $BCK$-algebras.

As no studies have been reported so far to generalize the above mentioned concepts, so the aim of this present article is:

1. To apply the notion of an MBJ-neutrosophic structure to a hyper $BCK$-algebra.
2. To define and study the notions of MBJ-neutrosophic (weak, s-weak, strong) hyper $BCK$-ideals of hyper $BCK$-algebras.
3. To discuss MBJ-neutrosophic (weak, strong) hyper $BCK$-ideals in relation to MBJ-neutrosophic level cut sets.

To do so, the rest of the article is structured as follows: In Section 2, we review some elementary notions. In Section 3, we introduce the notions of the MBJ-neutrosophic hyper $BCK$-ideal, the MBJ-neutrosophic weak hyper $BCK$-ideal, the MBJ-neutrosophic s-weak hyper $BCK$-ideal and the MBJ-neutrosophic strong hyper $BCK$-ideal and investigate several properties. We discuss MBJ-neutrosophic (weak, strong) hyper $BCK$-ideal in relation to MBJ-neutrosophic level cut sets. Finally, in Section 4, we present the conclusion and future works of the study.

2. Preliminaries

In the current section, we remember some of the basic notions of hyper $BCK$-algebras which will be very helpful in further study of the paper. Let $\mathcal{H}$ be a hyper $BCK$-algebra in what follows, unless otherwise stated.

Let $\mathcal{H}$ be a non-empty set and let “$\circ$” be a mapping

$$\circ : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{P}(\mathcal{H})\setminus\{\emptyset\}$$

which is said to be hyperoperation. For any two subsets $\mathcal{K}$ and $\mathcal{F}$, denote by $\mathcal{K} \circ \mathcal{F}$, the set $\bigcup\{\varrho \circ \tau \mid \varrho \in \mathcal{K}, \tau \in \mathcal{F}\}$. We shall use $\varrho \circ \tau$ instead of $\{\varrho\} \circ \tau$, $\varrho \circ \{\tau\}$, or $\{\varrho\} \circ \{\tau\}$.
By a hyper $BCK$-algebra $\mathcal{H}$ (see [3]), we mean a non-empty set $\mathcal{H}$ with a special element 0 and a hyperoperation $\circ$, for all $\varrho, \tau, \eta \in \mathcal{H}$, that satisfies the following axioms:

(HI) $(\varrho \circ \eta) \circ (\tau \circ \eta) = \varrho \circ \tau$,
(HII) $(\varrho \circ \tau) \circ \eta = (\varrho \circ \eta) \circ \tau$,
(HIII) $\varrho \circ \mathcal{H} \triangleleft \{\varrho\}$,
(HIV) $\varrho \ll \tau$ and $\tau \ll \varrho$ imply $\varrho = \tau$,

for all $\varrho, \tau, \eta \in \mathcal{H}$, where $\varrho \ll \tau$ is defined by $0 \in \varrho \circ \tau$ and for any $\mathcal{K}, \mathcal{F} \subseteq \mathcal{H}, \mathcal{K} \ll \mathcal{F}$ is defined by $\forall r \in \mathcal{K}, \exists t \in \mathcal{F}$ such that $r \ll t$.

In a hyper $BCK$-algebra $\mathcal{H}$ the axiom (HIII) is equivalent to the following axiom:

(HV) $\varrho \circ \tau \triangleleft \{\varrho\}$ for all $\varrho, \tau \in \mathcal{H}$.

**Proposition 2.1.** [3] Every hyper $BCK$-algebra $\mathcal{H}$ satisfies the following conditions, for all $\varrho, \tau, \eta \in \mathcal{H}$ and for any non-empty subsets $\mathcal{K}, \mathcal{F}, \mathcal{G}$ of $\mathcal{H}$,

(1) $\varrho \circ 0 \triangleleft \{\varrho\}, 0 \circ \varrho = \{0\}, 0 \circ 0 = \{0\}$,
(2) $0 \ll \varrho, \varrho \ll \varrho, \varrho \in \varrho \circ 0$,
(3) $\varrho \circ 0 \triangleleft \{\tau\} \Rightarrow \varrho \ll \tau$,
(4) $\tau \ll \eta \Rightarrow \varrho \circ \eta \ll \varrho \circ \tau$,
(5) $\varrho \circ \tau = \{0\} \Rightarrow \varrho \circ \eta \ll \tau \circ \eta, (\varrho \circ \eta) \circ (\tau \circ \eta) = \{0\}$,
(6) $\mathcal{K} \subseteq \mathcal{F} \Rightarrow \mathcal{K} \ll \mathcal{F}$,
(7) $\mathcal{K} \ll \{0\} \Rightarrow \mathcal{K} = \{0\}$,
(8) $\mathcal{K} \ll \mathcal{K}, \mathcal{K} \circ \mathcal{F} \ll \mathcal{K}, (\mathcal{K} \circ \mathcal{F}) \circ \mathcal{G} = (\mathcal{K} \circ \mathcal{G}) \circ \mathcal{F}$,
(9) $\mathcal{K} \circ \{0\} = \{0\} \Rightarrow \mathcal{K} = \{0\}$.

**Definition 2.2.** Let $(\mathcal{H}, \circ)$ be a hyper $BCK$-algebra. A subset $\mathcal{K}$ of $\mathcal{H}$ is called:

- A hyper $BCK$-ideal of $\mathcal{H}$ (see [3]) if
  (1) $0 \in \mathcal{K}$,
  (2) $\varrho \circ \tau \ll \mathcal{K}, \tau \in \mathcal{K} \Rightarrow \varrho \in \mathcal{K}, \forall \varrho, \tau \in \mathcal{H}$.

- A weak hyper $BCK$-ideal of $\mathcal{H}$ (see [3]) if it satisfies (1) and
  (3) $\varrho \circ \tau \subseteq \mathcal{K}, \tau \in \mathcal{K} \Rightarrow \varrho \in \mathcal{K}, \forall \varrho, \tau \in \mathcal{H}$.

- A strong hyper $BCK$-ideal of $\mathcal{H}$ (see [29]) if it satisfies (1) and
  (4) $(\varrho \circ \tau) \cap \mathcal{K} \neq \emptyset, \tau \in \mathcal{K} \Rightarrow \varrho \in \mathcal{K}, \forall \varrho, \tau \in \mathcal{H}$.

By an interval $\bar{u}$ we mean an interval $\bar{u} = [u^-, u^+]$, where $0 \leq u^- \leq u^+ \leq 1$. The set of all closed intervals $I$ is denoted by $[I]$. The interval $[u, u]$ is identified with the number $u$.

For two intervals $\bar{u}_1 = [u^+_1, u^-_1]$ and $\bar{u}_2 = [u^+_2, u^-_2]$, we define

$$r \max \{\bar{u}_1, \bar{u}_2\} = [\max \{u^-_1, u^+_2\}, \max \{u^+_1, u^-_2\}],$$
$$r \min \{\bar{u}_1, \bar{u}_2\} = [\min \{u^-_1, u^+_2\}, \min \{u^+_1, u^-_2\}],$$

Furthermore, we have

(1) $\bar{u}_1 \geq \bar{u}_2 \iff u^-_1 \geq u^-_2, u^+_1 \geq u^+_2$,.
Definition 2.3. Let $M$ be a false membership functions, respectively. where $D$ is an indeterminate interval-valued membership function.

Let $H$ be a nonempty set. A function $\tilde{D} : H \to [I]$ is said to be an interval-valued fuzzy set over a universe $H$. Let $[I]^H$ stands for the set of all interval-valued fuzzy sets $H$. For any $\tilde{D} \in [I]^H$ and $\varrho \in H$, $\tilde{D} = [D^-(\varrho), D^+(\varrho)]$ is called the degree of membership of an element $\varrho$ to $\tilde{D}$, where $D^-(\varrho) : H \to I$ and $D^+(\varrho) : H \to I$ are fuzzy sets over a universe $H$ which are called a lower fuzzy set and an upper fuzzy set over $H$, respectively. For simplicity, we denote $\tilde{D} = [D^-, D^+]$.

Let $H$ be a nonempty set. A neutrosophic set over a universe $H$ (see [9]) is a structure of the form:

$$D = \{(\varrho; D_T(\varrho), D_I(\varrho), D_F(\varrho)) \mid \varrho \in H\},$$

where $D_T$, $D_I$ and $D_F$ are fuzzy sets over a universe $H$, which are called a truth, an indeterminate and a false membership functions, respectively.

For the sake of simplicity, we shall use the symbol $D = (D_T, D_I, D_F)$ for the neutrosophic set

$$D = \{(\varrho; D_T(\varrho), D_I(\varrho), D_F(\varrho)) \mid \varrho \in H\}.$$

In [10], Takallo et el. introduced the idea of an MBJ-neutrosophic set as follows:

**Definition 3.1.** An MBJ-neutrosophic set $D$ on $H$ is called an MBJ-neutrosophic set over a universe $H$, we mean a structure of the form:

$$D = \{(\varrho; M_T(\varrho), M_I(\varrho), M_F(\varrho)) \mid \varrho \in H\},$$

where $M_T$ and $M_I$ are fuzzy sets over a universe $H$, which are called a truth and a false membership functions, respectively, and $\tilde{M}_T$ is an interval-valued fuzzy set over a universe $H$ which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $D = (M_T, \tilde{M}_T, \tilde{M}_I)$ for the MBJ-neutrosophic set

$$D = \{(\varrho; M_T(\varrho), \tilde{M}_T(\varrho), \tilde{M}_I(\varrho)) \mid \varrho \in H\}.$$

Given an MBJ-neutrosophic set $D = (M_T, \tilde{M}_T, \tilde{M}_I)$ over a universe $H$, we consider the following sets:

$$U(M_T, \alpha) = \{\varrho \in H \mid M_T(\varrho) \geq \alpha\},$$

$$U(\tilde{M}_T, \tilde{\beta}) = \{\varrho \in H \mid \tilde{M}_T(\varrho) \geq \tilde{\beta}\},$$

$$L(\tilde{M}_T, \gamma) = \{\varrho \in H \mid \tilde{M}_T(\varrho) \leq \gamma\},$$

where $\alpha, \gamma \in [0, 1]$ and $\tilde{\beta} = [\beta^-, \beta^+] \in [I]$.

3. MBJ-neutrosophic hyper $BCK$-ideals

**Definition 3.1.** An MBJ-neutrosophic set $D$ on $H$ is called an MBJ-neutrosophic hyper $BCK$-ideal of $H$ if it satisfies:

1. $$(\forall \varrho, \tau \in H) \left(\varrho \ll \tau \Rightarrow M_T(\varrho) \geq M_T(\tau), \tilde{M}_T(\varrho) \geq \tilde{M}_T(\tau), \tilde{M}_I(\varrho) \leq \tilde{M}_I(\tau)\right),$$
Example 3.1. Let \( \mathcal{H} = \{0, \varrho, \tau\} \) be a set with the hyperoperation \( \varrho \circ \tau \), which is given by Table 1.

\[
\begin{array}{c|c|c|c}
\varrho \circ \tau & 0 & \varrho & \tau \\
0 & [0] & [0] & [0] \\
\varrho & [\varrho] & [0, \varrho] & [0, \varrho] \\
\tau & [\tau] & [\varrho, \tau] & [0, \varrho, \tau] \\
\end{array}
\]

Then, \( \mathcal{H} \) is a hyper \( BCK \)-algebra (see [3]). Let \( \mathcal{D} = (M_\varrho, \tilde{B}_\varrho, \mathcal{J}_\varrho) \) be an MBJ-neutrosophic set over \( \mathcal{H} \) given by Table 2.

\[
\begin{array}{c|c|c|c}
\mathcal{D} & M_\varrho & \tilde{B}_\varrho & \mathcal{J}_\varrho \\
0 & 0 & \frac{1}{3}, 0.71 & \frac{2}{3} \\
\varrho & \frac{1}{3} & \frac{1}{2}, 0.51 & \frac{1}{2} \\
\tau & \frac{1}{3} & \frac{1}{2}, 0.21 & \frac{1}{2} \\
\end{array}
\]

It is routine to check that \( \mathcal{D} = (M_\varrho, \tilde{B}_\varrho, \mathcal{J}_\varrho) \) is an MBJ-neutrosophic hyper \( BCK \)-ideal of \( \mathcal{H} \).

Proposition 3.2. Let \( \mathcal{D} = (M_\varrho, \tilde{B}_\varrho, \mathcal{J}_\varrho) \) be an MBJ-neutrosophic hyper \( BCK \)-ideal of \( \mathcal{H} \). Then,

(i) \( (\forall \varrho \in \mathcal{H}) \left( M_\varrho(0) \geq M_\varrho(\varrho), \tilde{B}_\varrho(0) \geq \tilde{B}_\varrho(\varrho), \mathcal{J}_\varrho(0) \leq \mathcal{J}_\varrho(\varrho) \right) \),

(ii) If \( \mathcal{D} = (M_\varrho, \tilde{B}_\varrho, \mathcal{J}_\varrho) \) satisfies

\[
(\forall \mathcal{K} \subseteq \mathcal{H})(\exists q_\varrho, \tau_\varrho, \eta_\varrho \in \mathcal{K}) \text{ such that } \\
\begin{aligned}
M_\varrho(q_\varrho) &= \inf \{M_\varrho(\varrho) \mid \varrho \in \mathcal{K} \} \\
\tilde{B}_\varrho(\tau_\varrho) &= \inf \{\tilde{B}_\varrho(\tau) \mid \tau \in \mathcal{K} \} \\
\mathcal{J}_\varrho(\eta_\varrho) &= \sup \{\mathcal{J}_\varrho(\eta) \mid \eta \in \mathcal{K} \} \\
\end{aligned}
\]

then

\[
(\forall \varrho, \tau \in \mathcal{H})(\exists u, v, w \in \varrho \circ \tau) \text{ such that } \\
\begin{aligned}
M_\varrho(u) &= \min \{M_\varrho(u), M_\varrho(\tau)\} \\
\tilde{B}_\varrho(v) &= \min \{\tilde{B}_\varrho(v), \tilde{B}_\varrho(\varrho)\} \\
\mathcal{J}_\varrho(w) &= \max \{\mathcal{J}_\varrho(w), \mathcal{J}_\varrho(\varrho)\} \\
\end{aligned}
\]

Proof. (i) Since \( 0 \ll \varrho \) for all \( \varrho \in \mathcal{H} \), it follows from Definition 3.1(1) that \( M_\varrho(0) \geq M_\varrho(\varrho), \tilde{B}_\varrho(0) \geq \tilde{B}_\varrho(\varrho) \) and \( \mathcal{J}_\varrho(0) \leq \mathcal{J}_\varrho(\varrho) \) for all \( \varrho \in \mathcal{H} \).

(ii) Suppose that \( \mathcal{D} = (M_\varrho, \tilde{B}_\varrho, \mathcal{J}_\varrho) \) satisfies the condition (3.1). For any \( \varrho, \tau \in \mathcal{H} \), there exist \( u_\varrho, v_\varrho, w_\varrho \in \varrho \circ \tau \) such that \( M_\varrho(u_\varrho) = \inf \{M_\varrho(u) \mid u \in \varrho \circ \tau\} \), \( \tilde{B}_\varrho(v_\varrho) = \inf \{\tilde{B}_\varrho(v) \mid v \in \varrho \circ \tau\} \) and \( \mathcal{J}_\varrho(w_\varrho) = \sup \{\mathcal{J}_\varrho(w) \mid w \in \varrho \circ \tau\} \). It follows from Definition 3.1(2) that

\[
M_\varrho(\varrho) \geq \min \{ \inf \{M_\varrho(z) \mid z \in \varrho \circ \tau\}, M_\varrho(\tau) \} = \min \{M_\varrho(u_\varrho), M_\varrho(\tau)\},
\]
\[ \mathcal{B}_\mathcal{Z}(\varrho) \geq r \min \{ \inf \{ \mathcal{B}_\mathcal{Z}(v) | v \in \varrho \circ \tau \}, \mathcal{B}_\mathcal{Z}(\tau) \} = r \min \{ \mathcal{B}_\mathcal{Z}(v), \mathcal{B}_\mathcal{Z}(\tau) \} \]

and

\[ \mathcal{F}_\mathcal{Z}(\varrho) \leq \max \{ \sup \{ \mathcal{F}_\mathcal{Z}(w) | w \in \varrho \circ \tau \}, \mathcal{F}_\mathcal{Z}(\tau) \} = \max \{ \mathcal{F}_\mathcal{Z}(w), \mathcal{F}_\mathcal{Z}(\tau) \}. \]

This completes the proof. \(\square\)

**Corollary 3.3.** In a finite hyper BCK-algebra, every MBJ-neutrosophic hyper BCK-ideal \( \mathcal{D} = (\mathcal{M}_\mathcal{Z}, \mathcal{B}_\mathcal{Z}, \mathcal{F}_\mathcal{Z}) \) over \( \mathcal{H} \) satisfies the condition (3.2).

**Lemma 3.4** ([30]). Let \( \mathcal{K} \) be a subset of a hyper BCK-algebra \( \mathcal{H} \). If \( I \) is a hyper BCK-ideal of \( \mathcal{H} \) such that \( \mathcal{K} \ll I \), then \( \mathcal{K} \) is contained in \( I \).

**Theorem 3.5.** An MBJ-neutrosophic set \( \mathcal{D} = (\mathcal{M}_\mathcal{Z}, \mathcal{B}_\mathcal{Z}, \mathcal{F}_\mathcal{Z}) \) over \( \mathcal{H} \) is an MBJ-neutrosophic hyper BCK-ideal of \( \mathcal{H} \) if and only if the nonempty sets \( \mathcal{U}(\mathcal{M}_\mathcal{Z}, \alpha), \mathcal{U}(\mathcal{B}_\mathcal{Z}, [\beta^-, \beta^+]) \) and \( \mathcal{L}(\mathcal{F}_\mathcal{Z}, \gamma) \) are hyper BCK-ideals of \( \mathcal{H} \) for all \((\alpha, \gamma) \in [0, 1] \times [0, 1] \) and \([\beta^-, \beta^+] \in I\).

**Proof.** Assume that \( \mathcal{D} = (\mathcal{M}_\mathcal{Z}, \mathcal{B}_\mathcal{Z}, \mathcal{F}_\mathcal{Z}) \) is an MBJ-neutrosophic hyper BCK-ideal of \( \mathcal{H} \). Let \((\alpha, \gamma) \in [0, 1] \times [0, 1] \) and \([\beta^-, \beta^+] \in I\) be such that \( \mathcal{U}(\mathcal{M}_\mathcal{Z}, \alpha), \mathcal{U}(\mathcal{B}_\mathcal{Z}, [\beta^-, \beta^+]) \) and \( \mathcal{L}(\mathcal{F}_\mathcal{Z}, \gamma) \) are nonempty sets. It is easy to see that \( 0 \in \mathcal{U}(\mathcal{M}_\mathcal{Z}, \alpha) \), \( 0 \in \mathcal{U}(\mathcal{B}_\mathcal{Z}, [\beta^-, \beta^+]) \) and \( 0 \in \mathcal{L}(\mathcal{F}_\mathcal{Z}, \gamma) \) by proposition 3.2(i).

Let \( \varrho, \tau, u, v, a, b \in \mathcal{H} \) be such that \( \varrho \circ \tau \in \mathcal{U}(\mathcal{M}_\mathcal{Z}, \alpha), \tau \in \mathcal{U}(\mathcal{M}_\mathcal{Z}, \alpha), u \circ v \in \mathcal{U}(\mathcal{B}_\mathcal{Z}, [\beta^-, \beta^+]), v \in \mathcal{U}(\mathcal{B}_\mathcal{Z}, [\beta^-, \beta^+]), a \circ b \in \mathcal{L}(\mathcal{F}_\mathcal{Z}, \gamma), \) and \( b \in \mathcal{L}(\mathcal{F}_\mathcal{Z}, \gamma) \). Then, \( \varrho \circ \tau \ll \mathcal{U}(\mathcal{M}_\mathcal{Z}, \alpha), \tau \in \mathcal{U}(\mathcal{M}_\mathcal{Z}, \alpha), u \circ v \ll \mathcal{U}(\mathcal{B}_\mathcal{Z}, [\beta^-, \beta^+]), v \ll \mathcal{U}(\mathcal{B}_\mathcal{Z}, [\beta^-, \beta^+]), a \ll \mathcal{L}(\mathcal{F}_\mathcal{Z}, \gamma), \) and \( b \ll \mathcal{L}(\mathcal{F}_\mathcal{Z}, \gamma) \). It follows that

\[(\forall x \in \varrho \circ \tau)(\exists x_0 \in \mathcal{U}(\mathcal{M}_\mathcal{Z}, \alpha) \text{ such that } x \ll x_0) \text{ and so } \mathcal{M}_\mathcal{Z}(x) \geq \mathcal{M}_\mathcal{Z}(x_0),\]

\[(\forall y \in u \circ v)(\exists y_0 \in \mathcal{U}(\mathcal{B}_\mathcal{Z}, [\beta^-, \beta^+]) \text{ such that } y \ll y_0) \text{ and so } \mathcal{B}_\mathcal{Z}(y) \geq \mathcal{B}_\mathcal{Z}(y_0)\]

and

\[(\forall z \in a \circ b)(\exists z_0 \in \mathcal{L}(\mathcal{F}_\mathcal{Z}, \gamma) \text{ such that } z \ll z_0) \text{ and so } \mathcal{F}_\mathcal{Z}(z) \leq \mathcal{F}_\mathcal{Z}(z_0),\]

which imply that \( \mathcal{M}_\mathcal{Z}(x) \geq \alpha, \mathcal{B}_\mathcal{Z}(y) \geq [\beta^-, \beta^+] \) and \( \mathcal{F}_\mathcal{Z}(z) \leq \gamma \) for all \( x \in \varrho \circ \tau, y \in u \circ v \) and \( z \in a \circ b \).

Hence, \( \inf \{ \mathcal{M}_\mathcal{Z}(x) | x \in \varrho \circ \tau \} \geq \alpha, \inf \{ \mathcal{B}_\mathcal{Z}(y) | y \in u \circ v \} \geq [\beta^-, \beta^+] \) and \( \sup \{ \mathcal{F}_\mathcal{Z}(z) | z \in a \circ b \} \leq \gamma \), and so

\[ \mathcal{M}_\mathcal{Z}(\varrho) \geq \min \{ \inf \{ \mathcal{M}_\mathcal{Z}(x) | x \in \varrho \circ \tau \}, \mathcal{M}_\mathcal{Z}(\tau) \} \geq \min(\alpha, \alpha) = \alpha,\]

\[ \mathcal{B}_\mathcal{Z}(u) \geq r \min \{ \inf \{ \mathcal{B}_\mathcal{Z}(y) | y \in u \circ v \}, \mathcal{B}_\mathcal{Z}(\tau) \} \geq r \min([\beta^-, \beta^+], [\beta^-, \beta^+]) = [\beta^-, \beta^+]\]

and

\[ \mathcal{F}_\mathcal{Z}(a) \leq \max \{ \sup \{ \mathcal{F}_\mathcal{Z}(z) | z \in a \circ b \}, \mathcal{F}_\mathcal{Z}(b) \} \leq \max(\gamma, \gamma) = \gamma.\]

Thus, \( \varrho \in \mathcal{U}(\mathcal{M}_\mathcal{Z}, \alpha), u \in \mathcal{U}(\mathcal{B}_\mathcal{Z}, [\beta^-, \beta^+]) \) and \( a \in \mathcal{L}(\mathcal{F}_\mathcal{Z}, \gamma) \) and therefore \( \mathcal{U}(\mathcal{M}_\mathcal{Z}, \alpha), \mathcal{U}(\mathcal{B}_\mathcal{Z}, [\beta^-, \beta^+]) \) and \( \mathcal{L}(\mathcal{F}_\mathcal{Z}, \gamma) \) are hyper BCK-ideals of \( \mathcal{H} \).

Conversely, assume that the nonempty sets \( \mathcal{U}(\mathcal{M}_\mathcal{Z}, \alpha), \mathcal{U}(\mathcal{B}_\mathcal{Z}, [\beta^-, \beta^+]) \) and \( \mathcal{L}(\mathcal{F}_\mathcal{Z}, \gamma) \) are hyper BCK-ideals of \( \mathcal{H} \) for all \((\alpha, \gamma) \in [0, 1] \times [0, 1] \) and \([\beta^-, \beta^+] \in I\). Let \( \varrho, \tau, u, v, a, b \in \mathcal{H} \) be such that \( \varrho \ll \tau, \ldots\)
$u \ll v$, $a \ll b$, $\mathbb{M}_\mathbb{E}(\tau) = \alpha$, $\tilde{B}_\mathbb{E}(v) = [\beta^-, \beta^+]$ and $\mathcal{J}_\mathbb{E}(b) = \gamma$. Then, $\tau \in U(\mathbb{M}_\mathbb{E}, \alpha)$, $v \in U(\tilde{B}_\mathbb{E}, [\beta^-, \beta^+])$ and $b \in L(\mathcal{J}_\mathbb{E}, \gamma)$, and so $\{q\} \ll U(\mathbb{M}_\mathbb{E}, \alpha)$, $\{u\} \ll U(\tilde{B}_\mathbb{E}, [\beta^-, \beta^+])$ and $\{a\} \ll L(\mathcal{J}_\mathbb{E}, \gamma)$. It follows that from Lemma 3.4 that $\{q\} \subseteq U(\mathbb{M}_\mathbb{E}, \alpha)$, $\{u\} \subseteq U(\tilde{B}_\mathbb{E}, [\beta^-, \beta^+])$ and $\{a\} \subseteq L(\mathcal{J}_\mathbb{E}, \gamma)$, i.e., $\rho \in U(\mathbb{M}_\mathbb{E}, \alpha)$, $u \in U(\tilde{B}_\mathbb{E}, [\beta^-, \beta^+])$ and $a \in L(\mathcal{J}_\mathbb{E}, \gamma)$. Hence, $\mathbb{M}_\mathbb{E}(\rho) \geq \alpha = \mathbb{M}_\mathbb{E}(\tau)$, $\tilde{B}_\mathbb{E}(u) \geq [\beta^-, \beta^+] = \tilde{B}_\mathbb{E}(v)$ and $\mathcal{J}_\mathbb{E}(a) \leq \gamma = \mathcal{J}_\mathbb{E}(b)$. For any $\rho, \tau, u, v, a, b \in \mathcal{H}$, let

$$\alpha = \min \left\{ \inf \{ \mathbb{M}_\mathbb{E}(t_1) \mid t_1 \in \rho \circ \tau, \mathbb{M}_\mathbb{E}(\tau) \} \right\},$$

$$[\beta^-, \beta^+] = r \min \left\{ \inf \{ \tilde{B}_\mathbb{E}(t_2) \mid t_2 \in u \circ v, \tilde{B}_\mathbb{E}(v) \} \right\}$$

and

$$\gamma = \max \left\{ \sup \{ \mathcal{J}_\mathbb{E}(t_3) \mid t_3 \in a \circ b, \mathcal{J}_\mathbb{E}(b) \} \right\}.$$ 

Then, $\tau \in U(\mathbb{M}_\mathbb{E}, \alpha)$, $v \in U(\tilde{B}_\mathbb{E}, [\beta^-, \beta^+])$, $b \in L(\mathcal{J}_\mathbb{E}, \gamma)$ and

$$\mathbb{M}_\mathbb{E}(t_4) \geq \inf \{ \mathbb{M}_\mathbb{E}(t_1) \mid t_1 \in \rho \circ \tau \} \geq \min \left\{ \inf \{ \mathbb{M}_\mathbb{E}(t_1) \mid t_1 \in \rho \circ \tau, \mathbb{M}_\mathbb{E}(\tau) \} \right\} = \alpha,$$

$$\tilde{B}_\mathbb{E}(t_5) \geq \inf \{ \tilde{B}_\mathbb{E}(t_2) \mid t_2 \in u \circ v \} \geq r \min \left\{ \inf \{ \tilde{B}_\mathbb{E}(t_2) \mid t_2 \in u \circ v, \tilde{B}_\mathbb{E}(v) \} \right\} = [\beta^-, \beta^+]$$

and

$$\mathcal{J}_\mathbb{E}(t_6) \leq \sup \{ \mathcal{J}_\mathbb{E}(t_3) \mid t_3 \in a \circ b \} \leq \max \left\{ \sup \{ \mathcal{J}_\mathbb{E}(t_3) \mid t_3 \in a \circ b, \mathcal{J}_\mathbb{E}(b) \} \right\} = \gamma.$$ 

for all $t_4 \in \rho \circ \tau$, $t_5 \in u \circ v$ and $t_6 \in a \circ b$, i.e, $t_4 \in U(\mathbb{M}_\mathbb{E}, \alpha)$, $t_5 \in U(\tilde{B}_\mathbb{E}, [\beta^-, \beta^+])$ and $t_6 \in L(\mathcal{J}_\mathbb{E}, \gamma)$. Thus, $\rho \circ \tau \subseteq U(\mathbb{M}_\mathbb{E}, \alpha)$, $u \circ v \subseteq U(\tilde{B}_\mathbb{E}, [\beta^-, \beta^+])$ and $a \circ b \subseteq L(\mathcal{J}_\mathbb{E}, \gamma)$, which imply from Proposition 2.1(6) that $\rho \circ \tau \ll U(\mathbb{M}_\mathbb{E}, \alpha)$, $u \circ v \ll U(\tilde{B}_\mathbb{E}, [\beta^-, \beta^+])$ and $a \circ b \ll L(\mathcal{J}_\mathbb{E}, \gamma)$. Since $U(\mathbb{M}_\mathbb{E}, \alpha)$, $U(\tilde{B}_\mathbb{E}, [\beta^-, \beta^+])$ and $L(\mathcal{J}_\mathbb{E}, \gamma)$ are hyper BCK-ideals of $\mathcal{H}$, we have $\rho \in U(\mathbb{M}_\mathbb{E}, \alpha)$, $u \in U(\tilde{B}_\mathbb{E}, [\beta^-, \beta^+])$ and $a \in L(\mathcal{J}_\mathbb{E}, \gamma)$, which imply that

$$\mathbb{M}_\mathbb{E}(\rho) \geq \alpha = \min \left\{ \inf \{ \mathbb{M}_\mathbb{E}(t_1) \mid t_1 \in \rho \circ \tau, \mathbb{M}_\mathbb{E}(\tau) \} \right\},$$

$$\tilde{B}_\mathbb{E}(u) \geq [\beta^-, \beta^+] = r \min \left\{ \inf \{ \tilde{B}_\mathbb{E}(t_2) \mid t_2 \in u \circ v, \tilde{B}_\mathbb{E}(v) \} \right\}$$

and

$$\mathcal{J}_\mathbb{E}(a) \leq \gamma = \max \left\{ \sup \{ \mathcal{J}_\mathbb{E}(t_3) \mid t_3 \in a \circ b, \mathcal{J}_\mathbb{E}(b) \} \right\}.$$ 

Therefore, $\mathcal{D} = (\mathbb{M}_\mathbb{E}, \tilde{B}_\mathbb{E}, \mathcal{J}_\mathbb{E})$ is an MBJ-neutrosophic hyper BCK-ideal of $\mathcal{H}$. □

Now, we define and study the notions of an MBJ-neutrosophic weak (s-weak) hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$.

**Definition 3.6.** An MBJ-neutrosophic set $\mathcal{D} = (\mathbb{M}_\mathbb{E}, \tilde{B}_\mathbb{E}, \mathcal{J}_\mathbb{E})$ over $\mathcal{H}$ is called:

1. An MBJ-neutrosophic weak hyper BCK-ideal of $\mathcal{H}$ if it satisfies Proposition 3.2(i) and Definition 3.1(2).
2. An MBJ-neutrosophic s-weak hyper BCK-ideal of $\mathcal{H}$ if it satisfies Proposition 3.2(i) and (3.2).
Theorem 3.7. Every MBJ-neutrosophic hyper BCK-ideal is an MBJ-neutrosophic weak hyper BCK-ideal.

Proof. Straightforward. □

The converse of Theorem 3.7 is not true in general, as seen in the following example.

Example 3.2. Let $\mathcal{H} = \{0, \varrho, \tau\}$ be a hyper BCK-algebra as in Example 3.1. Let $\mathfrak{D} = (M_\mathfrak{D}, \mathcal{J}_\mathfrak{D}, \mathcal{F}_\mathfrak{D})$ be an MBJ-neutrosophic set over $\mathcal{H}$ given by Table 3.

| $\mathfrak{D}$ | $M_\mathfrak{D}$ | $\mathcal{J}_\mathfrak{D}$ |
|--------------|------------------|------------------|
| 0            | 0.8              | 0.3              |
| $\varrho$    | 0.2              | 0.6              |
| $\tau$       | 0.6              | 0.5              |

Then, $\mathfrak{D} = (M_\mathfrak{D}, \mathcal{J}_\mathfrak{D}, \mathcal{F}_\mathfrak{D})$ is an MBJ-neutrosophic weak hyper BCK-ideal of $\mathcal{H}$. Note that $\varrho \ll \tau$,

$M_\mathfrak{D}(\varrho) = 0.2 < 0.6 = M_\mathfrak{D}(\tau),$

$\mathcal{J}_\mathfrak{D}(\varrho) = 0.6 > 0.5 = \mathcal{J}_\mathfrak{D}(\tau)$

and

$\mathcal{F}_\mathfrak{D}(\varrho) = 0.6 > 0.5 = \mathcal{F}_\mathfrak{D}(\tau).$

Hence, $\mathfrak{D} = (M_\mathfrak{D}, \mathcal{J}_\mathfrak{D}, \mathcal{F}_\mathfrak{D})$ is not an MBJ-neutrosophic hyper BCK-ideal of $\mathcal{H}$. 

Theorem 3.8. In a hyper BCK-algebra, every MBJ-neutrosophic s-weak hyper BCK-ideal is an MBJ-neutrosophic weak hyper BCK-ideal.

Proof. Let $\mathfrak{D} = (M_\mathfrak{D}, \mathcal{J}_\mathfrak{D}, \mathcal{F}_\mathfrak{D})$ be an MBJ-neutrosophic s-weak hyper BCK-ideal of over $\mathcal{H}$ and let $\varrho, \tau, u, v, a, b \in \mathcal{H}$. Then, there exist $z_1 \in \varrho \circ \tau$, $z_2 \in u \circ v$ and $z_3 \in a \circ b$ such that $M_\mathfrak{D}(\varrho) \geq \min \{M_\mathfrak{D}(z_1), M_\mathfrak{D}(\tau)\}$, $\mathcal{J}_\mathfrak{D}(u) \geq \inf \{\mathcal{J}_\mathfrak{D}(z_2), \mathcal{J}_\mathfrak{D}(v)\}$ and $\mathcal{F}_\mathfrak{D}(a) \leq \max \{\mathcal{F}_\mathfrak{D}(z_3), \mathcal{F}_\mathfrak{D}(b)\}$ by the condition (ii) of Proposition 3.2. Since $M_\mathfrak{D}(z_1) \geq \inf \{M_\mathfrak{D}(z_2) \mid z_2 \in \varrho \circ \tau\}$, $\mathcal{J}_\mathfrak{D}(z_2) \geq \inf \{\mathcal{J}_\mathfrak{D}(z_3) \mid z_3 \in u \circ v\}$ and $\mathcal{F}_\mathfrak{D}(z_3) \leq \sup \{\mathcal{F}_\mathfrak{D}(z_4) \mid z_4 \in a \circ b\}$, it follows that

$M_\mathfrak{D}(\varrho) \geq \min \{\inf \{M_\mathfrak{D}(z_2) \mid z_2 \in \varrho \circ \tau\}, M_\mathfrak{D}(\tau)\},$

$\mathcal{J}_\mathfrak{D}(u) \geq r \min \{\inf \{\mathcal{J}_\mathfrak{D}(z_3) \mid z_3 \in u \circ v\}, \mathcal{J}_\mathfrak{D}(v)\}$

and

$\mathcal{F}_\mathfrak{D}(a) \leq \max \{\sup \{\mathcal{F}_\mathfrak{D}(z_4) \mid z_4 \in a \circ b\}, \mathcal{F}_\mathfrak{D}(b)\}.$

Therefore, $\mathfrak{D} = (M_\mathfrak{D}, \mathcal{J}_\mathfrak{D}, \mathcal{F}_\mathfrak{D})$ is an MBJ-neutrosophic weak hyper BCK-ideal of over $\mathcal{H}$. □

Question 1. Is the converse of Theorem 3.8 true?

It is not easy to find an example of an MBJ-neutrosophic weak hyper BCK-ideal which is not an MBJ-neutrosophic s-weak hyper BCK-ideal. However, we give the following theorem.
An MBJ-neutrosophic set

Definition 3.11. Hyper

Proof. It is similar to the proof of Theorem 3.5.

Theorem 3.10. An MBJ-neutrosophic set $\mathcal{D} = (M_\mathcal{E}, \tilde{B}_\mathcal{E}, J_\mathcal{E})$ over $\mathcal{H}$ is an MBJ-neutrosophic s-weak hyper $BCK$-ideal of $\mathcal{H}$ if and only if the nonempty sets $U(M_\mathcal{E}, \alpha), U(\tilde{B}_\mathcal{E}, \beta^-, \beta^+)$ and $L(J_\mathcal{E}, \gamma)$ are weak hyper $BCK$-ideals of $\mathcal{H}$ for all $(\alpha, \gamma) \in [0, 1] \times [0, 1]$ and $[\beta^-, \beta^+] \in [1]$. 

Proof. It is similar to the proof of Theorem 3.5.

The following definition presents the concept of an MBJ-neutrosophic strong hyper $BCK$-ideal of a hyper $BCK$-algebra $\mathcal{H}$. Next, we study some properties of this concept.

Definition 3.11. An MBJ-neutrosophic set $\mathcal{D} = (M_\mathcal{E}, \tilde{B}_\mathcal{E}, J_\mathcal{E})$ over $\mathcal{H}$ is called an MBJ-neutrosophic strong hyper $BCK$-ideal of $\mathcal{H}$ if it satisfies:

1. $(\forall \varrho, \tau \in \mathcal{H}) \left( \inf \{ M_\mathcal{E}(z_1) \mid z_1 \in \varrho \odot \varrho \} \geq M_\mathcal{E}(\varrho) \right)$

2. $(\forall \varrho, \tau \in \mathcal{H}) \left( \inf \{ \tilde{B}_\mathcal{E}(z_2) \mid z_2 \in \varrho \odot \tau \} \geq \tilde{B}_\mathcal{E}(\varrho) \right)$

3. $(\forall \varrho, \tau \in \mathcal{H}) \left( \inf \{ J_\mathcal{E}(z_4) \mid z_4 \in \varrho \odot \tau \} \geq J_\mathcal{E}(\varrho) \right)$

Example 3.3. Let $\mathcal{H} = \{0, \varrho, \tau\}$ be a set with the hyperoperation $\odot$, which is given by Table 4.

Table 4. Tabular representation of the hyperoperation $\odot$.

| $\odot$ | 0   | $\varrho$ | $\tau$ |
|---------|-----|-----------|--------|
| 0       | 0   | 0         | 0      |
| $\varrho$ | $\varrho$ | $\varrho$ | $\varrho$ |
| $\tau$   | $\tau$ | $\tau$    | 0, $\tau$ |
Then, $\mathcal{H}$ is a hyper $BCK$-algebra (see [3]). Let $\mathcal{D} = (\mathcal{M}_\varrho, \bar{\mathcal{B}}_\varrho, \mathcal{J}_\varrho)$ be an MBJ-neutrosophic set over $\mathcal{H}$ given by Table 5.

| $\varrho$ | $\bar{\mathcal{B}}_\varrho$ | $\mathcal{J}_\varrho$ |
|---|---|---|
| 0.63 | $[\frac{1}{3}, 0.71]$ | 0.30 |
| 0.43 | $[\frac{1}{5}, 0.51]$ | 0.50 |
| 0.32 | $[\frac{1}{6}, 0.31]$ | 0.70 |

Table 5. Tabular representation of $\mathcal{D} = (\mathcal{M}_\varrho, \bar{\mathcal{B}}_\varrho, \mathcal{J}_\varrho)$.

It is routine to check that $\mathcal{D} = (\mathcal{M}_\varrho, \bar{\mathcal{B}}_\varrho, \mathcal{J}_\varrho)$ an MBJ-neutrosophic strong hyper $BCK$-ideal of $\mathcal{H}$.

**Proposition 3.12.** Every MBJ-neutrosophic strong hyper $BCK$-ideal $\mathcal{D} = (\mathcal{M}_\varrho, \bar{\mathcal{B}}_\varrho, \mathcal{J}_\varrho)$ over $\mathcal{H}$ satisfying the following conditions:

1. $(\forall \varrho \in \mathcal{H}) \left( \mathcal{M}_\varrho(0) \geq \mathcal{M}_\varrho(\varrho), \bar{\mathcal{B}}_\varrho(0) \geq \bar{\mathcal{B}}_\varrho(\varrho), \mathcal{J}_\varrho(0) \leq \mathcal{J}_\varrho(\varrho) \right)$.
2. $(\forall \varrho, \tau \in \mathcal{H}) \left( \varrho \ll \tau \Rightarrow \mathcal{M}_\varrho(\varrho) \geq \mathcal{M}_\varrho(\tau), \bar{\mathcal{B}}_\varrho(\varrho) \geq \bar{\mathcal{B}}_\varrho(\tau), \mathcal{J}_\varrho(\varrho) \leq \mathcal{J}_\varrho(\tau) \right)$.
3. $(\forall z, \varrho, \tau \in \mathcal{H}) \left( z \in \varrho \circ \tau \Rightarrow \right) \begin{cases} \mathcal{M}_\varrho(\varrho) \geq \min \{ \mathcal{M}_\varrho(z), \mathcal{M}_\varrho(\tau) \} \\ \bar{\mathcal{B}}_\varrho(\varrho) \geq \min \{ \bar{\mathcal{B}}_\varrho(z), \bar{\mathcal{B}}_\varrho(\tau) \} \\ \mathcal{J}_\varrho(\varrho) \leq \max \{ \mathcal{J}_\varrho(z), \mathcal{J}_\varrho(\tau) \} \end{cases}.$

**Proof.** (1) Since $0 \in \varrho \circ \varrho \ \forall \varrho \in \mathcal{H}$, we have

\[
\mathcal{M}_\varrho(0) \geq \inf \{ \mathcal{M}_\varrho(z_1) \mid z_1 \in \varrho \circ \varrho \} \geq \mathcal{M}_\varrho(\varrho), \\
\bar{\mathcal{B}}_\varrho(0) \geq \inf \{ \bar{\mathcal{B}}_\varrho(z_2) \mid z_2 \in \varrho \circ \varrho \} \geq \bar{\mathcal{B}}_\varrho(\varrho)
\]

and

\[
\mathcal{J}_\varrho(0) \leq \sup \{ \mathcal{J}_\varrho(z_3) \mid z_3 \in \varrho \circ \varrho \} \leq \mathcal{J}_\varrho(\varrho)
\]

for all $\varrho \in \mathcal{H}$.

(2) Let $\varrho, \tau \in \mathcal{H}$ be such that $\varrho \ll \tau$. Then, $0 \in \varrho \circ \tau$ and thus $\sup \{ \mathcal{M}_\varrho(z_1) \mid z_1 \in \varrho \circ \tau \} \geq \mathcal{M}_\varrho(0)$, $\sup \{ \bar{\mathcal{B}}_\varrho(z_2) \mid z_2 \in \varrho \circ \tau \} \geq \bar{\mathcal{B}}_\varrho(0)$ and $\inf \{ \mathcal{J}_\varrho(z_3) \mid z_3 \in \varrho \circ \tau \} \leq \mathcal{J}_\varrho(0)$. It follows from Definition 3.11 and (1) that

\[
\mathcal{M}_\varrho(\varrho) \geq \min \{ \sup \{ \mathcal{M}_\varrho(z_1) \mid z_1 \in \varrho \circ \tau \}, \mathcal{M}_\varrho(\tau) \} \\
\geq \min \{ \mathcal{M}_\varrho(0), \mathcal{M}_\varrho(\tau) \} \\
= \mathcal{M}_\varrho(\tau),
\]

\[
\bar{\mathcal{B}}_\varrho(\varrho) \geq r \min \{ \sup \{ \bar{\mathcal{B}}_\varrho(z_2) \mid z_2 \in \varrho \circ \tau \}, \bar{\mathcal{B}}_\varrho(\tau) \} \\
\geq \min \{ \bar{\mathcal{B}}_\varrho(0), \bar{\mathcal{B}}_\varrho(\tau) \} \\
= \bar{\mathcal{B}}_\varrho(\tau),
\]
and
\[ J_\mathcal{E}(\rho) \leq \max\{\inf\{J_\mathcal{E}(z_3) \mid z_3 \in \varrho \circ \tau\}, J_\mathcal{E}(\tau)\} \]
\[ \leq \min\{J_\mathcal{E}(0), J_\mathcal{E}(\tau)\} \]
\[ = J_\mathcal{E}(\tau), \]
i.e., \( M_\mathcal{E}(\rho) \geq M_\mathcal{E}(\tau) \), \( \tilde{B}_\mathcal{E}(\rho) \geq \tilde{B}_\mathcal{E}(\tau) \) and \( J_\mathcal{E}(\rho) \leq J_\mathcal{E}(\tau) \) for all \( \rho, \tau \in \mathcal{H} \) with \( \rho \ll \tau \).

(3) Let \( z, \rho, \tau \in \mathcal{H} \) be such that \( z \in \varrho \circ \tau \). Then,
\[ M_\mathcal{E}(\rho) \geq \min\{\sup\{M_\mathcal{E}(z_1) \mid z_1 \in \varrho \circ \tau\}, M_\mathcal{E}(\tau)\}, \]
\[ \tilde{B}_\mathcal{E}(\rho) \geq r \min\{\sup\{\tilde{B}_\mathcal{E}(z_2) \mid z_2 \in \varrho \circ \tau\}, \tilde{B}_\mathcal{E}(\tau)\}, \]
and
\[ J_\mathcal{E}(\rho) \leq \max\{\inf\{J_\mathcal{E}(z_3) \mid z_3 \in \varrho \circ \tau\}, J_\mathcal{E}(\tau)\} \]
\[ \leq \max\{J_\mathcal{E}(z), J_\mathcal{E}(\tau)\} \]
for all \( z, \rho, \tau \in \mathcal{H} \) with \( z \in \varrho \circ \tau \).

\[\square\]

**Corollary 3.13.** If \( \mathcal{D} = (M_\mathcal{E}, \tilde{B}_\mathcal{E}, J_\mathcal{E}) \) is an MBJ-neutrosophic strong hyper BCK-ideal over \( \mathcal{H} \), then the condition (2) of Definition 3.1 is valid.

**Proof.** Note that \( M_\mathcal{E}(\rho) \geq \inf\{M_\mathcal{E}(z) \mid z \in \varrho \circ \tau\}, \tilde{B}_\mathcal{E}(\rho) \geq \inf\{\tilde{B}_\mathcal{E}(z) \mid z \in \varrho \circ \tau\} \) and \( J_\mathcal{E}(\rho) \leq \sup\{J_\mathcal{E}(z) \mid z \in \varrho \circ \tau\} \) for all \( z, \rho, \tau \in \mathcal{H} \) with \( z \in \varrho \circ \tau \). Hence, the condition (2) of Definition 3.1 follows from Proposition 3.12(2).

\[\square\]

**Theorem 3.14.** Every MBJ-neutrosophic strong hyper BCK-ideal is an MBJ-neutrosophic hyper BCK-ideal.

**Proof.** Straightforward.

\[\square\]

The converse of Theorem 3.14 is not true in general. That is, an MBJ-neutrosophic hyper BCK-ideal may not be an MBJ-neutrosophic strong hyper BCK-ideal.

**Example 3.4.** Let \( \mathcal{H} = \{0, \rho, \tau\} \) be a hyper BCK-algebra as in Example 3.1. Let \( \mathcal{D} = (M_\mathcal{E}, \tilde{B}_\mathcal{E}, J_\mathcal{E}) \) be an MBJ-neutrosophic set over \( \mathcal{H} \) given by Table 6.

**Table 6.** Tabular representation of \( \mathcal{D} = (M_\mathcal{E}, \tilde{B}_\mathcal{E}, J_\mathcal{E}) \).

| \( \mathcal{D} \) | \( M_\mathcal{E} \) | \( \tilde{B}_\mathcal{E} \) | \( J_\mathcal{E} \) |
|-------------|-------------|-------------|-------------|
| 0           | 0.63        | [\( \frac{1}{6}, \frac{1}{3} \)] | 0.21        |
| \( \rho \)  | 0.63        | [\( \frac{1}{6}, \frac{1}{3} \)] | 0.32        |
| \( \tau \)  | 0.32        | [\( \frac{1}{12}, \frac{1}{6} \)] | 0.39        |
Then, $\mathcal{D} = (\mathcal{M}_6, \mathcal{B}_6, \mathcal{J}_6)$ is an MBJ-neutrosophic hyper $BCK$-ideal of $\mathcal{H}$, but it is not an MBJ-neutrosophic strong hyper $BCK$-ideal of $\mathcal{H}$, since

$$M_6(\tau) = 0.32 < 0.63 = M_6(\varrho) = \min \left\{ \sup \{M_6(z) \mid z \in \tau \circ \varrho \}, M_6(\varrho) \right\},$$

$$\mathcal{B}_6(\tau) = \left[ \frac{1}{12}, \frac{1}{6} \right] < \left[ \frac{1}{9}, \frac{1}{5} \right] = \mathcal{B}_6(\varrho) = r \min \left\{ \sup \{\mathcal{B}_6(z) \mid z \in \tau \circ \varrho \}, \mathcal{B}_6(\varrho) \right\}$$

and

$$\mathcal{J}_6(\tau) = 0.39 > 0.32 = \mathcal{J}_6(\varrho) = \max \left\{ \inf \{\mathcal{J}_6(z) \mid z \in \tau \circ \varrho \}, \mathcal{J}_6(\varrho) \right\}.$$

**Theorem 3.15.** Let $\mathcal{D} = (\mathcal{M}_6, \mathcal{B}_6, \mathcal{J}_6)$ be an MBJ-neutrosophic set over $\mathcal{H}$. If $\mathcal{D} = (\mathcal{M}_6, \mathcal{B}_6, \mathcal{J}_6)$ is an MBJ-neutrosophic strong hyper $BCK$-ideal of $\mathcal{H}$, then the nonempty sets $U(M_6, \alpha), U(\mathcal{B}_6, [\beta^-, \beta^+])$ and $L(\mathcal{J}_6, \gamma)$ are strong hyper $BCK$-ideals of $\mathcal{H}$ for all $\alpha, \gamma \in [0, 1] \times [0, 1]$ and $[\beta^-, \beta^+] \in [I]$. 

**Proof.** Assume that $\mathcal{D} = (\mathcal{M}_6, \mathcal{B}_6, \mathcal{J}_6)$ is an MBJ-neutrosophic set over $\mathcal{H}$. Let $(\alpha, \gamma) \in [0, 1] \times [0, 1]$ and $[\beta^-, \beta^+] \in [I]$ such that $U(M_6, \alpha), U(\mathcal{B}_6, [\beta^-, \beta^+])$ and $L(\mathcal{J}_6, \gamma)$ are nonempty sets. Then there exist $r \in U(M_6, \alpha), t \in U(\mathcal{B}_6, [\beta^-, \beta^+])$ and $s \in L(\mathcal{J}_6, \gamma)$, and so $M_6(r) \geq \alpha, \mathcal{B}_6(t) \geq [\beta^-, \beta^+]$ and $\mathcal{J}_6(s) \leq \gamma$. Clearly, $0 \in U(M_6, \alpha), 0 \in U(\mathcal{B}_6, [\beta^-, \beta^+])$ and $0 \in L(\mathcal{J}_6, \gamma)$ by Proposition 3.12 (1). Let $\varrho, \tau, u, v, a, b \in \mathcal{H}$ be such that $\tau \in U(M_6, \alpha), (\varrho \circ \tau) \cap U(M_6, \alpha) \neq \emptyset, v \in U(\mathcal{B}_6, [\beta^-, \beta^+]), (u \circ v) \cap U(\mathcal{B}_6, [\beta^-, \beta^+]) \neq \emptyset, b \in L(\mathcal{J}_6, \gamma)$ and $(a \circ b) \cap L(\mathcal{J}_6, \gamma) \neq \emptyset$. Then, there exist $r_0 \in (\varrho \circ \tau) \cap U(M_6, \alpha), t_0 \in (u \circ v) \cap U(\mathcal{B}_6, [\beta^-, \beta^+])$ and $s_0 \in (a \circ b) \cap L(\mathcal{J}_6, \gamma)$. Hence, $M_6(r_0) \geq \alpha, \mathcal{B}_6(t_0) \geq [\beta^-, \beta^+]$ and $\mathcal{J}_6(s_0) \leq \gamma$. It follows that

$$M_6(\varrho) \geq \min \left\{ \sup \{M_6(r) \mid r \in \varrho \circ \tau \}, M_6(\tau) \right\} \geq \min \{M_6(r_0), M_6(\tau)\} \geq \alpha,$$

$$\mathcal{B}_6(u) \geq r \min \left\{ \sup \{\mathcal{B}_6(t) \mid t \in u \circ v \}, \mathcal{B}_6(v) \right\} \geq r \min \{\mathcal{B}_6(t_0), \mathcal{B}_6(v)\} \geq [\beta^-, \beta^+]$$

and

$$\mathcal{J}_6(a) \leq \max \left\{ \inf \{\mathcal{J}_6(s) \mid s \in a \circ b \}, \mathcal{J}_6(b) \right\} \leq \max \{\mathcal{J}_6(s_0), \mathcal{J}_6(b)\} \leq \gamma.$$

Hence, $\varrho \in U(M_6, \alpha), u \in U(\mathcal{B}_6, [\beta^-, \beta^+])$ and $a \in L(\mathcal{J}_6, \gamma)$. Therefore, $U(M_6, \alpha), U(\mathcal{B}_6, [\beta^-, \beta^+])$ and $L(\mathcal{J}_6, \gamma)$ are strong hyper $BCK$-ideals of $\mathcal{H}$.

**Theorem 3.16.** Let $\mathcal{D} = (\mathcal{M}_6, \mathcal{B}_6, \mathcal{J}_6)$ be an MBJ-neutrosophic set over $\mathcal{H}$ which satisfies the condition:

$$(\forall \mathcal{K} \subseteq \mathcal{H})(\exists \varrho_0, \tau_0, \eta_0 \in \mathcal{K}) \text{ such that } \begin{cases} M_6(\varrho_0) = \sup \{M_6(\varrho) \mid \varrho \in \mathcal{K} \} \\ \mathcal{B}_6(\tau_0) = \sup \{\mathcal{B}_6(\tau) \mid \tau \in \mathcal{K} \} \\ \mathcal{J}_6(\eta_0) = \inf \{\mathcal{J}_6(\eta) \mid \eta \in \mathcal{K} \} \end{cases} \tag{3.3}$$

If the nonempty sets $U(M_6, \alpha), U(\mathcal{B}_6, [\beta^-, \beta^+])$ and $L(\mathcal{J}_6, \gamma)$ are strong hyper $BCK$-ideals of $\mathcal{H}$ for all $(\alpha, \gamma) \in [0, 1] \times [0, 1]$ and $[\beta^-, \beta^+] \in [I]$, then $\mathcal{D} = (\mathcal{M}_6, \mathcal{B}_6, \mathcal{J}_6)$ is an MBJ-neutrosophic strong hyper $BCK$-ideal of $\mathcal{H}$.

**Proof.** Assume that the nonempty sets $U(M_6, \alpha), U(\mathcal{B}_6, [\beta^-, \beta^+])$ and $L(\mathcal{J}_6, \gamma)$ are strong hyper $BCK$-ideals of $\mathcal{H}$ for all $(\alpha, \gamma) \in [0, 1] \times [0, 1]$ and $[\beta^-, \beta^+] \in [I]$. Then, $\varrho \in U(M_6, \alpha), \tau \in U(\mathcal{B}_6, [\beta^-, \beta^+])$ and $\eta \in L(\mathcal{J}_6, \gamma)$ for some $\varrho, \tau, \eta \in \mathcal{H}$, and so $\varrho \circ \tau \ll \{\varrho\} \subseteq U(M_6, \alpha), \tau \circ \tau \ll \{\tau\} \subseteq U(\mathcal{B}_6, [\beta^-, \beta^+])$.
For any $w$ BCKAIMS Mathematics instance, hyper work, we will use the concept and results in this paper to study other hyper algebraic structures, for the concepts of the MBJ-neutrosophic hyper $BCK$-algebra $\{M, \circ, \bowtie, \wedge, \vee, \neg, \odot, \otimes, \Diamond, \heartsuit, \spadesuit\}$ and thus $\eta \circ \eta \approx \{\eta\} \subseteq L(F_B, \gamma)$. By Lemma 3.4, we have $\circ \subseteq U(M_v, \alpha), \tau \circ \tau \subseteq U(\tilde{B}_v, [\beta^-, \beta^+])$ and $\eta \circ \eta \subseteq L(F_B, \gamma)$. Thus, for any $a \in \circ \circ, b \in \tau \circ \tau$ and $c \in \eta \circ \eta$, we get $a \in U(M_v, \alpha), b \in U(\tilde{B}_v, [\beta^-, \beta^+]),$ and $c \in L(F_B, \gamma)$. Hence, $M_v(\alpha) \geq \alpha, B_v(b) \geq [\beta^-, \beta^+]$ and $F_v(c) \leq \gamma$. It follows that
\[
\inf\{M_v(a) \mid a \in \circ \circ\} \geq \alpha = M_v(\circ),
\]
\[
\inf\{\tilde{B}_v(b) \mid b \in \tau \circ \tau\} \geq [\beta^-, \beta^+] = \tilde{B}_v(\tau)
\]
\[
\sup\{F_v(c) \mid c \in \eta \circ \eta\} \leq \gamma = F_v(\eta).
\]

For any $\circ, \tau, u, v, w, z \in H$. Taking
\[
r = \min\{\sup\{M_v(a) \mid a \in \circ \circ, \tau\}, M_v(\tau)\},
\]
\[
\bar{t} = [\tau^-, \tau^+] = r \min\{\sup\{\tilde{B}_v(b) \mid b \in u \circ v\}, \tilde{B}_v(v)\},
\]
\[
s = \max\{\inf\{F_v(c) \mid c \in w \circ z\}, F_v(z)\}.
\]

Then by assumption, $U(M_v, r), U(\tilde{B}_v, [\tau^-, \tau^+])$ and $L(F_B, s)$ are strong hyper $BCK$-ideals of $H$. The condition (3.3) implies that there exist $a_0 \in \circ \circ, b_0 \in u \circ v$ and $c_0 \in w \circ z$ such that $M_v(a_0) = \sup\{M_v(a) \mid a \in \circ \circ, \tau\}, \tilde{B}_v(b_0) = \sup\{\tilde{B}_v(b) \mid b \in u \circ v\}$ and $F_v(c_0) = \inf\{F_v(c) \mid c \in w \circ z\}$. Hence,
\[
M_v(a_0) = \sup\{M_v(a) \mid a \in \circ \circ, \tau\} \geq \min\{\sup\{M_v(a) \mid a \in \circ \circ, \tau\}, M_v(\tau)\} = r,
\]
\[
\tilde{B}_v(b_0) = \sup\{\tilde{B}_v(b) \mid b \in u \circ v\} \geq r \min\{\sup\{\tilde{B}_v(b) \mid b \in u \circ v\}, \tilde{B}_v(v)\} = [\tau^-, \tau^+],
\]
\[
F_v(c_0) = \inf\{F_v(c) \mid c \in w \circ z\} \leq \max\{\inf\{F_v(c) \mid c \in w \circ z\}, M_v(z)\} = s.
\]

This imply that $a_0 \in U(M_v, r), b_0 \in U(\tilde{B}_v, [\tau^-, \tau^+])$ and $c_0 \in L(F_B, s)$. Hence, $(\circ \circ) \cap U(M_v, r) \neq \phi$, $(u \circ v) \cap U(\tilde{B}_v, [\tau^-, \tau^+]) \neq \phi$, and $w \circ z \cap L(F_B, s) \neq \phi$ and thus $\circ \subseteq U(M_v, r), u \subseteq U(\tilde{B}_v, [\tau^-, \tau^+])$ and $w \subseteq L(F_B, s)$. It follows that
\[
M_v(\circ) \geq r = \min\{\sup\{M_v(a) \mid a \in \circ \circ, \tau\}, M_v(\tau)\},
\]
\[
\tilde{B}_v(u) \geq [\tau^-, \tau^+] = r \min\{\sup\{\tilde{B}_v(b) \mid b \in u \circ v\}, \tilde{B}_v(v)\},
\]
\[
F_v(c) \leq s = \max\{\inf\{F_v(c) \mid c \in w \circ z\}, F_v(z)\}.
\]

Therefore, $\tilde{D} = (M_v, \tilde{B}_v, F_v)$ is an MBJ-neutrosophic strong hyper $BCK$-ideal of $H$. \[\Box\]

4. Conclusions

In this paper, we have applied the MBJ-neutrosophic set to hyper $BCK$-algebra. We have presented the concepts of the MBJ-neutrosophic hyper $BCK$-ideal, the MBJ-neutrosophic weak hyper $BCK$-ideal, the MBJ-neutrosophic s-weak hyper $BCK$-ideal and the MBJ-neutrosophic strong hyper $BCK$-ideal, and have discussed related properties and their relations. We have investigated MBJ-neutrosophic (weak, s-weak, strong) hyper $BCK$-ideals in relation to level cut sets. In the future work, we will use the concept and results in this paper to study other hyper algebraic structures, for instance, hyper $BCI$-algebra, hyper hoop, hyper $MV$-algebra and hyper $B$-algebra.
Acknowledgments

The authors are thankful to the editors and the anonymous reviewers for their valuable suggestions and comments on the manuscript.

Conflict of interest

We declare that we have no conflict of interest.

References

1. Y. Imai, K. Iséki, On axiom systems of propositional calculi, *Proc. Jpn. Acad.*, 42 (1966), 19–21.
2. F. Marty, Sur une generalization de la notion de groupe, *8th Congress Math. Scandenaves*, (1934), 45–49.
3. Y. B. Jun, M. M. Zahedi, X. L. Xin, R. A. Borzoei, On hyper BCK-algebras, *Ital. J. Pure Appl. Math.*, 8 (2000), 127–136.
4. A. B. Saeid, M. M. Zahedi, Quotient hyper BCK-algebras, *Quasigroups and Related Systems*, 12 (2004), 93–102.
5. A. B. Saeid, R. Borzooei, M. M. Zahedi, (Weak) Implicative hyper K-ideals, *B. Korean Math. Soc.*, 40 (2003), 123–137.
6. L. A. Zadeh, Fuzzy sets, *Inform. Control*, 8 (1965), 338–353.
7. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, 20 (1986), 87–96.
8. F. Smarandache, Neutrosophic set—a generalization of the intuitionistic fuzzy set, *Int. J. Pure Appl. Math.*, 24 (2005), 287–297.
9. F. Smarandache, *Unifying field in logics: neutrosophic logic*. Neutrosophy, Neutrosophic set, neutrosophic probability, American Reserch Press: Rehoboth, NM, USA, 1999.
10. M. M. Takallo, R. A. Borzooei, Y. B. Jun, MBJ-neutrosophic structures and its applications in BCK/BCI-algebras, *Neutrosophic Sets Syst.*, 23 (2018), 72–84.
11. Y. B. Jun, E. H. Roh, MBJ-neutrosophic ideals of BCK/BCI-algebras, *Open Math.*, 17 (2019), 588–601.
12. S. Manokaran, M. Prakasam, On MBJ-neutrosophic B-Subalgebra, *Neutrosophic Sets Syst.*, 28 (2019), 216–227.
13. M. Khalid, N. A. Khalid, R. Iqbal, MBJ-neutrosophic T-ideal on B-algebra, *International Journal of Neutrosophic Science*, 1 (2020), 29–39.
14. M. Abu Qamar, A. G. Ahmad, N. Hassan, On Q-neutrosophic soft fields, *Neutrosophic Sets Syst.*, 32 (2020), 80–93.
15. A. Al-Masarwah, A. G. Ahmad, Doubt bipolar fuzzy subalgebras and ideals in BCK/BCI-algebras, *J. Math. Anal.*, 9 (2018), 9–27.
16. A. Al-Masarwah, A. G. Ahmad, On some properties of doubt bipolar fuzzy H-ideals in BCK/BCI-algebras, *Eur. J. Pure Appl. Math.*, 11 (2018), 652–670.
17. A. Al-Masarwah, A. G. Ahmad, Novel concepts of doubt bipolar fuzzy H-ideals of BCK/BCI-algebras, *Int. J. Innov. Comput., Inf. Control*, 14 (2018), 2025–2041.

18. A. Al-Masarwah, A. G. Ahmad, m-polar fuzzy ideals of BCK/BCI-algebras, *J. King Saud Univ.-Sci.*, 31 (2019), 1220–1226.

19. A. Al-Masarwah, A. G. Ahmad, m-Polar (α, β)-fuzzy ideals in BCK/BCI-algebras, *Symmetry*, 11 (2019), 44.

20. R. A. Borzooei, X. H. Zhang, F. Smarandache, Y. B. Jun, Commutative generalized neutrosophic ideals in BCK-algebras, *Symmetry*, 10 (2018), 350.

21. Y. B. Jun, Neutrosophic subalgebras of several types in BCK/BCI-algebras, *Ann. Fuzzy Math. Inform.*, 14 (2017), 75–86.

22. Y. B. Jun, S. J. Kim, F. Smarandache, Interval neutrosophic sets with applications in BCK/BCI-algebra, *Axioms*, 7 (2018), 23.

23. M. Abu Qamar, N. Hassan, An approach toward a Q-neutrosophic soft set and its application in decision making, *Symmetry*, 11 (2019), 139.

24. M. Abu Qamar, N. Hassan, Entropy, measures of distance and similarity of Q-neutrosophic soft sets and some applications, *Entropy*, 20 (2018), 672.

25. Y. B. Jun, X. L. Xin, Fuzzy hyper BCK-ideals of hyper BCK-algebras, *Scientiae Mathematicae Japonicae*, 53 (2001), 353–360.

26. M. Bakhshi, M. M. Zahedi, R. A. Borzooei, Fuzzy (positive, weak) implicative hyper BCK-ideals, *Iran. J. Fuzzy Syst.*, 1 (2004), 63–79.

27. R. A. Borzooei, Y. B. Jun, Intuitionistic fuzzy hyper BCK-ideals of hyper BCK-ideals, *Iran. J. Fuzzy Syst.*, 1 (2004), 65–78.

28. S. Khademan, M. M. Zahedi, R. A. Borzooei, Y. B. Jun, Neutrosophic hyper BCK-ideals, *Neutrosophic Sets Syst.*, 27 (2019), 201–217.

29. Y. B. Jun, X. L. Xin, M. M. Zahedi, E. H. Roh, Strong hyper BCK-ideals of hyper BCK-algebras, *Math. Jpn.*, 51 (2000), 493–498.

30. Y. B. Jun, X. L. Xin, Scalar elements and hyper atoms of hyper BCK-algebras, *Sci. Math.*, 2 (1999), 303–309.

© 2021 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)