NEUMANN SYSTEM AND HYPERELLPTIC AL
FUNCTIONS

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Abstract. This article shows that the Neumann dynamical sys-

tem is described well in terms of the Weierstrass hyperelliptic al

functions.

1. Introduction

The Neumann dynamical system is a well-known integrable nonlinear
dynamical system, whose Lagrangian for \((q, \dot{q}) \in \mathbb{R}^{2g+2}\) is given by,

\[
L = \frac{1}{2} \sum_{i=1}^{g+1} \dot{q}_i^2 - \frac{1}{2} \sum_{i=1}^{g+1} a_i q_i^2,
\]

with a holonomic constraint,

\[
\Phi(q) = 0, \quad \Phi(q) := \sum_{i=1}^{g+1} q_i^2 - 1.
\]

This is studied well in frameworks of the dynamical system [Mo], of

the symplectic geometry [GS], of the algebraic geometry [Mu], of the

representation of the infinite Lie algebra [AHP, S].

Mumford gives explicit expressions of the Neumann system in terms

of hyperelliptic functions based upon classical and modern hyperelliptic

function theories. This article gives more explicit expressions of the

Neumann system using Weierstrass hyperelliptic al functions.

In the case of elliptic functions theory, Weierstrass \(\wp\) functions and
Jacobi \(\text{sn, cn, dn}\) functions play important roles in the theory even

though they are expressed by the \(\theta\) functions and all relations among

them are rewritten by the \(\theta\) functions. The expressions of Weierstrass

\(\wp\) functions and Jacobi \(\text{sn, cn, dn}\) functions make the theory of elliptic

functions fruitful and reveal the essentials of elliptic functions.

Unfortunately in the case of higher genus case, such studies are not

enough though Klein and Weierstrass defined hyperelliptic versions of

these \(\wp\) functions [Kl] and \(\text{sn, cn, dn}\) functions [W]. Thus several

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authors devote themselves to reinterpretations of the modern theory of hyperelliptic functions in terms of these functions and developing studies of these functions as special functions [BEL, Ma] and their references. In this article, we also proceed with such a project. We will show that the Weierstrass al functions give natural descriptions of the Neumann dynamical system: As in theorem 4.1, the configuration \( q^i \) of \( i \)-th particle (or coordinate) is directly given by the al function,

\[
q^i(t) = a_i(t),
\]

Here \( a_i(t) \) are defined in Definition 3.1, which was originally defined by Weierstrass as a generalization of Jacobi sn, cn, and dn functions over an elliptic curve to that over a hyperelliptic curve. As Jacobi sn, cn, dn functions are associated with several nonlinear phenomena and these relations enable us to recognize the essentials of the phenomena [1], we expect that this expression also plays a role in hyperelliptic function case. In fact the description in terms of the al functions makes several properties of the Neumann system rather simple. For examples, an essential property of the Neumann system \( \sum_{i=1}^{g+2} (q^i(t))^2 = 1 \) is interpreted as a hyperelliptic version of \( \text{sn}^2(u) + \text{cn}^2(u) = 1 \). Its hamiltonian is given as a manifestly constant quantity in Theorem 4.1 (3). Due to the description, proofs in this article basically need only primitive residual computations.

We will give our plan of this article. §2 gives a short review of the Neumann system. In §3, we introduce the hyperelliptic al functions and hyperelliptic \( \wp \) functions. There we also give a short review of their basic properties following [Ba, BEL, W]. §4 is our main section, where we give our main theorem. There al function naturally describes the Neumann system.

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2. NEUMANN SYSTEM

We will shortly review the Neumann system \((q, \dot{q}) \in \mathbb{R}^{2g+1}\) whose Lagrangian and constraint condition are given (1.1) and (1.2) in Introduction. The constraint (1.2) means \( \Phi(q) = 0 \),

\[
\sum_{i=1}^{g+1} \dot{q}_i q_i = 0.
\]
The canonical momentum $p_i$ to $q_i$ is given as
$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \dot{q}_i.$$

Purely kinematic investigations lead the following proposition \[Mu\].

**Proposition 2.1.** The Hamiltonian of this system is given by
$$H := \frac{1}{2} \sum_{i=1}^{g+1} \dot{q}_i^2 + \frac{1}{2} \sum_{i=1}^{g+1} a_i q_i^2,$$
and the Hamiltonian vector field is given by
$$D_H = \sum \dot{q}_i \frac{\partial}{\partial q_i} - \sum a_i q_i \frac{\partial}{\partial \dot{q}_i} + \left( \sum (a_i q_i^2 - q_i^2) \right) \sum q_i \frac{\partial}{\partial \dot{q}_i}.$$
The equation of motion is given by
$$\dot{q}_i = \dot{q}_i, \quad \ddot{q}_i = -(2L + a_i) q_i.$$

3. Hyperelliptic Functions

In this article, we will consider a hyperelliptic curve $C_g$ given by an affine equation \[Mu\] \[DRVW\],
$$y^2 = f(x), \quad f(x) = A(x) Q(x),$$
$$A(x) := (x - a_1)(x - a_2) \cdots (x - a_{g+1}),$$
$$Q(x) := (x - c_1)(x - c_2) \cdots (x - c_g),$$
where $a_i$'s and $c_i$'s are complex numbers. Let $b_i := a_i \ (i = 1, \cdots, g+1)$ and $b_{g+1} := c_i \ (i = 1, \cdots, g)$.

In this article, we deal with $(x_1, x_2, \cdots, x_g)$ belonging to $g$ symmetric product $\text{Sym}^g(C_g)$ of $C_g$.

Let us introduce the canonical coordinate $u := (u_1, \cdots, u_g)$ in the Jacobian $J_g$ related to $C_g$ \[BEL\],
$$u_i := \sum_{a=1}^{g} \int_{\infty}^{(x_a, y_a)} x^{i-1} dx \frac{1}{2y}.$$  
Here $u_- := (u_1, \cdots, u_{g-1})$, $u = (u_-, u_g)$.

Due to Abel theorem \[H\], the following proposition holds.

**Proposition 3.1.** $(u_1, u_2, \cdots, u_g)$ are linearly independent in $\mathbb{C}^g$. In other words, there are paths in $\text{Sym}^g(C_g)$ so that $\{u_g\}$ is equal to $\mathbb{C}$ with fixing $u_-$.  

As Mumford studied the Neumann system using UVW-expression of the hyperelliptic functions $[\mu]$, we will give $U$, $V$ and $W$ functions $[\mu]$.

\begin{align*}
U(x) &=: (x - x_1) \cdots (x - x_g), \\
V(x) &=: \sum_{a=1}^{g} \frac{y_a U(x)}{U'(x_a)(x - x_a)}, \\
W(x) &=: \frac{f(x) + V(x)^2}{U(x)}.
\end{align*}

In this article, we will express the system in terms of the hyperelliptic $\wp$ functions and al functions. Let us introduce these functions,

**Definition 3.1.** The hyperelliptic $\wp_{gi}$ ($i = 1, 2, \cdots, g$) functions of $u$'s are defined by

\begin{equation}
U(x) = x^g + \sum_{i=1}^{g} (-1)^i \wp_{gi} x^{g-i},
\end{equation}

e.g., $\wp_{gg} := x_1 + \cdots + x_g$.

The Weierstrass $al_i$ and $at_i$ ($i = 1, 2, \cdots, g$) functions are defined by $[\mathbf{Bal}]$ $[\mathbf{W}]$,

\begin{align*}
al_r(u) &=: \gamma_r al_r(u), \quad al_r(u) := \sqrt{U(a_r)(u)}, \\
\end{align*}

where we set $\gamma_r = 1/\sqrt{A'(a_r)}$ in this article. We write

\begin{align*}
al^{[i]}(u) &=: \frac{\partial}{\partial u_i} al_r(u), \quad at^{[i]}(u) := \frac{\partial}{\partial u_i} at_r(u).
\end{align*}

The sn function is defined by $1/sn(u) := \sqrt{x - a_3}/\sqrt{A'(a_3)}$ and $sn(u) = 1/sn(u + \Omega)$, al functions should be recognized as an extension of sn function. As sn function has the relations

\begin{align*}
k^2 sn^2(u) + dn^2(u) &= 1, \quad sn^2(u) + cn^2(u) = 1.
\end{align*}

The $al$ functions are also has similar relations, which were studied in $[\mathbf{M}]$ as a generalization of Frobenius identity.

**Proposition 3.2.**

\begin{align*}
\sum_{i=1}^{g+1} al_i^2(u) &= 1, \quad \sum_{i=1}^{g+1} \frac{1}{a_i} [at_i^{[g]}]^2(u) = 1,
\end{align*}
Proof. The left hand side is given by
\[ \sum_{i=1}^{g+1} \frac{U(a_i)}{A'(a_i)} = \frac{1}{2} \sum_{i=1}^{g+1} \text{res}(a_i,0) \frac{U(x)}{A(x)}, \]
since around the finite ramified point \((a_i,0)\) of the curve \(C_g\), we have a local parameter \(t^2 = (x - a_i)\) and
\[ \text{res}(a_i,0) \frac{U(x)}{A(x)} dx = \text{res}(a_i,0) \frac{2U(t^2 + a_i)tdt}{(t^2 + a_i - a_1) \cdots t^2 \cdots (t^2 + a_i - a_{g+1})}. \]

Let us consider an integral over a boundary of polygon expression \(C_0\) of \(C_g\),
\[ \oint_{\partial C_0} \frac{U(x)}{A(x)} dx = 0, \]
which gives the relation,
\[ \sum_{i=1}^{g+1} \text{res}(a_i,0) \frac{U(x)}{A(x)} dx = -\text{res}_\infty \frac{U(x)}{A(x)} dx. \]

At \(\infty\), a local parameter \(t\) of \(C_g\) is given by \(x = 1/t^2\):
\[ \text{res}_\infty \frac{U(x)}{A(x)} dx = \text{res}_\infty \frac{1}{tg^{g}(1 - x_1 t^2) \cdots (1 - x_g t^2) \cdots} \frac{-2}{t^{g+2}} \cdots (1 - a_{g+1} t^2) dt = -2. \]

Hence it is proved. Similarly we obtain the relations for \(a_i^{[g]}\) though we should evaluate \(W(x)/xA(x)\). \(\square\)

There is a natural relation between \(a_i\) function and \(\wp_{gg}\) function

**Proposition 3.3.**
\[ \frac{\partial}{\partial u^2} a_i(u) = \frac{\partial}{\partial u} a_i^{[g]}(u) = \left( 2g+1 \sum_{j=1, b_i \neq a_i} b_j - 2\wp_{gg}(u) \right) a_i. \]

**Proof.** This is directly obtained if we assume the following Lemma 3.1 and 3.2. \(\square\)

We have primitive relations between differentials of \(a_i\) functions and \(UVW\) expressions:

**Lemma 3.1.**
\[ a_i^{[g]}(u) = \frac{V(a_i)(u)}{a_i(u)}, \quad a_i^{[g]}(u) = \frac{V(a_i)(u)}{a_i(u)A'(a_i)}. \]
Proof. Noting \( \frac{\partial}{\partial u_g} = \sum_{a=1}^{g} \frac{2y_a}{U'(x_a)} \frac{\partial}{\partial x_a} \) and \( \frac{\partial}{\partial x_a} U(x) = -\frac{U(x)}{(x-x_a)} \), we find that \( \frac{1}{2} \frac{\partial}{\partial u_g} U(x) = V(x) \), which directly gives the relations. \(\square\)

Lemma 3.2.
\[
\frac{1}{2} \frac{\partial}{\partial u_g} V(a_i) = U(a_i) \left( \sum_{i=1}^{2g+1} b_i - a_i - 2 \sum_{a=1}^{g} x_a \right) - \frac{1}{U(a_i)} V(a_i)^2.
\]

Proof. Here we will check the left hand side.

\[
\frac{\partial}{\partial u_g} V(a_i) = \sum_{a,b=1}^{g} \frac{y_a}{U'(x_a)} \frac{\partial}{\partial x_a} \frac{y_b U(a_i)}{U'(x_b)(a_i - x_b)}
\]

\[
= \sum_{a=1}^{g} \frac{2y_a}{U'(x_a)} \frac{\partial}{\partial x_a} \frac{2y_a U(a_i)}{U'(x_b)(a_i - x_b)} + \sum_{a \neq b} \frac{y_a}{U'(x_a)} \frac{\partial}{\partial x_a} \frac{2y_b U(a_i)}{U'(x_b)(a_i - x_b)}
\]

\[
= U(a_i) \sum_{a=1}^{g} \frac{1}{2} \left[ \frac{1}{U'(x_a)} \frac{\partial}{\partial x_a} \left( \frac{f(x) U(a_i)}{U'(x)(a_i - x)} \right) \right]_{x=x_a}
\]

\[
+ U(a_i) \sum_{a \neq b} \frac{f(x_a)}{U'(x_a)^2(a_i - x_a)^2}
\]

\[
+ U(a_i) \sum_{a \neq b} \frac{2y_a}{U'(x_a)} \frac{y_b}{U'(x_b)(a_i - x_a)(a_i - x_b)}
\]

\[
= \sum_{i=1}^{2g+1} b_i - a_i - 2 \sum_{a=1}^{g} x_a + U(a_i) \left( \sum_{a \neq b} \frac{2y_a}{U'(x_a)(a_i - x_a)} \right)^2.
\]

Here we used the following relations.

\[
\left. \frac{\partial}{\partial x_a} U'(x_a) = \frac{1}{2} \frac{\partial}{\partial x} U(x) \right|_{x=x_a},
\]

\[
\left[ \frac{1}{U'(x)} \frac{\partial}{\partial x} \left( \frac{f(x) U(a_i)}{U'(x)(a_i - x)} \right) \right]_{x=x_a} = \text{res}_{x_a,y_a} \frac{f(x)}{U(x)^2(a_i - x)} dx,
\]

and

\[
\sum_{i=1}^{2g+1} b_i - a_i - 2 \sum_{a=1}^{g} x_a = \sum_{a=1}^{g} \text{res}_{x_a,y_a} \frac{f(x)}{U(x)^2(a_i - x)} dx.
\]

The third relation is obtained by an evaluation of the integral

\[
\oint_{\partial C_0} \frac{f(x)}{U(x)^2(a_i - x)} dx.
\]

\(\square\)
Remark 3.1. The Klein hyperelliptic $\wp$ function obeys the KdV equations \cite{BEL, Ma}. On the other hand, $\frac{\partial}{\partial u_g} \log a_i$ is a solution of the MKdV equation \cite{Ma}. The relation in Proposition 3.3 means so-called Miura transformation,

$$\left( \frac{\partial}{\partial u_g} \log a_i \right)^2 + \frac{\partial^2}{\partial u_g^2} \log a_i = (L - a_i),$$

where $L := \frac{1}{2} \left( 2\wp_{gg} - \sum_{i=1}^{2g+1} b_i \right)$.

4. Neumann system and hyperelliptic AL functions

This section gives our main theorem as follows.

Theorem 4.1. Suppose that configurations of $(x_1, \cdots, x_g) \in \text{Sym}^g(C_g)$ are given so that $(a_i)$ belongs to $\mathbb{R}^{g+1}$, $u_g \in \mathbb{R}$ fixing $u_- \in \mathbb{R}^{g-1}$.

(1) $a_i$ obey the Neumann system, i.e.,

$$q_i(t) = a_i(u_-, t), \quad \dot{q}_i = a_i^{[g]}(u_-, t),$$

where the time $t$ of the system is identified with $u_g$ and thus the Hamiltonian vector field is given by

$$D_H := \frac{d}{dt} \equiv \frac{\partial}{\partial u_g}.$$

(2) The hamiltonian \cite{Le} and the lagrangian \cite{Le} are given by

$$H = \frac{1}{2} \left( \sum_{i=1}^{g+1} a_i - \sum_{a=1}^{g} c_a \right), \quad L = \frac{1}{2} \left( 2\wp_{gg} - \sum_{i=1}^{2g+1} b_i \right).$$

(3) The conserved quantities are $c_i$ ($i = 1, \cdots, g$) and

$$m_i := q_i^2 + \sum_{i=1}^{g+1} \sum_{j=1, j \neq i}^{g+1} \frac{(q_i \dot{q}_j - q_j \dot{q}_i)^2}{a_i - a_j}, \quad (i = 1, \cdots, g + 1),$$

which obey relations,

$$m_i = Q(a_i) \ A'(a_i), \quad \sum_{i=1}^{g+1} m_i = 1, \quad \sum_{i=1}^{g+1} a_i m_i = H.$$

These relations were essentially proved in \cite{Mu} using UVW expression without AL functions. However from the viewpoint of studies of special functions, we will show them directly using nature of AL functions.
Proof. Assumptions are asserted by Proposition 3.1 (1): Due to Proposition 3.2, $a_i$’s obviously obeys the constraint condition $\Phi(a) = 0$ (1.2) and $\dot{\Phi}(a) = 0$ (2.1) by differentiating the both sides of the identity in $u_g$. We should check whether they obey the equation of motion (2.4), which are proved in Proposition 3.3 if we assume the form of the Lagrangian $L$ in (2). (2) is directly obtained by using the relations in Lemma 4.1. Finally (3) is proved in Remark 4.3. □

Remark 4.1. (1) The equation of motion (2.4) is directly related to Proposition 3.3 which is connected with the Miura transformation. Further the constraint (1.2) satisfies due to the identity of $a_i$ function as mentioned in Proposition 3.2. These exhibits essentials of $a_i$ functions. Hence the Neumann system should be expressed by the $a_i$ function as some dynamical systems are expressed by Jacobi sn, cn, dn functions [1].

(2) We remark that the hamiltonian depends only upon $a_i$’s and $c_i$’s which determines the hyperelliptic curve $C_g$. Thus it is manifest that it is invariant for the time $u_g$ development of the system.

(3) There are $2g$ degrees of freedom as a kinematic system because the constraints $\Phi$ and $\dot{\Phi}$ reduce $(2g + 2)$ ones to $2g$ ones. The independent conserved quantities $m_i$ are $g = g + 1 - 1$; ”$- 1” comes from $\sum m_i = 1$. Since the sum of $m_i$ gives hamiltonian $H$, $H$ is not linearly independent conserved quantities. Since there are other $g$ conserved quantities $c_i$ but their sum gives the hamiltonian $\sum m_i$, the dimensional of independent $c_i$ is $g - 1$. However $\sum_{i=1}^{g+1} \frac{q_i^2}{a_i} = 1$ compensates the lacking one. Hence the degrees of freedom of this system is equal to number of the conserved quantities.

(4) By the definition of $c_i$’s, $c_i$ depends upon the initial condition of the Neumann system whereas $a_i$ is fixed as coupling constants of the Neumann system. Thus $S_g := \{ C_g : y^2 = A(x)Q(x) \mid c_1, c_2, \cdots c_g \in \mathbb{C} \}$ corresponds to the solution space $N_g$ of the Neumann system if $u_g \in \mathbb{R}$ and $(a, a^{[g]}) \in \mathbb{R}^{2g+2}$. The $S_g$ is a subspace of the moduli $\mathcal{M}_g$ of hyperelliptic curves of genus $g$.

Let us give a lemma and remarks as follows, which are parts of the proofs of the theorem.

Lemma 4.1. (1) $\sum_{i=1}^{g+1} [a_i^{[g]}(u)]^2 = \phi_{gg}(u) - \sum_{a=1}^{g} c_a$. 

Let us give a lemma and remarks as follows, which are parts of the proofs of the theorem.
\[(2) \sum_{i=1}^{g+1} a_i a_i(u)^2 = \sum_{i=1}^{g+1} a_i - \varphi_{g+1}(u).\]

**Proof.** 1) Due to Lemma 3.1, we deal with
\[\oint_{\partial C_0} V(x) \frac{U(x)^2}{A(x)} dx = 0\]
giving
\[2 \sum_{i=1}^{g+1} \frac{V(a_i)^2}{U(a_i)A'(a_i)} + \sum_{a=1, \epsilon=\pm}^{g} \text{res}(x_a, \epsilon y_a) \frac{V(x)^2}{U(x)A(x)} dx + \text{res}_\infty \frac{V(x)^2}{U(x)A(x)} dx = 0.\]

Whereas the third term vanishes, each element in the second term is given by
\[\text{res}(x_a, \epsilon y_a) \frac{V(x)^2}{U(x)A(x)} dx = Q(x_a).\]

Further we also evaluate an integral, \[\oint_{\partial C_0} Q(x_a) \frac{U(x)}{A(x)} dx = 0.\] The integrand has singularities at \((x_a, \epsilon y_a)\) and the infinity. Similar consideration leads us to the identities
\[\sum_{a=1, \epsilon=\pm}^{g} Q(x_a) = 2(c_1 + \cdots + c_g) - 2(x_1 + \cdots + x_g).\]

Due to these relations, we have the final equal.

2) Next we will consider an integral, \[\oint_{\partial C_0} x \frac{U(x)}{A(x)} dx = 0.\] A residual computation gives
\[\sum_{i=1}^{g+1} a_i \frac{U(a_i)}{A'(a_i)} = -\text{res}_\infty \frac{U(x)}{A(x)} dx.\]

The infinity term gives \(2((x_1 + \cdots x_g) - (a_1 + \cdots a_{g+1})\). Hence we also have the relation in (2). \(\square\)

**Remark 4.2.** Using the fact \(\frac{\partial x_a}{\partial u_g} = \frac{2y_a}{U'(x_a)}\), we obtain another form of
\[\sum_{i=1}^{g+1} a_i a_i^{[g]} = \sum_{a,b=1}^{g} g(x)_{a,b} \frac{\partial x_a}{\partial u_g} \frac{\partial x_b}{\partial u_g},\]
where \(g(x)_{a,b} := -\sum_{i=1}^{g+1} \frac{U(a_i)}{(a_i - x_a)(a_i - x_b)A'(a_i)}\) whose off-diagonal part does not vanish for the case genus \(g > 2\) in general.

**Remark 4.3.** (Proof of Theorem 4.1 3) Here we will give the conserved quantities of the Neumann system. Let us consider,
\[m_i(x) = q_i^2 + \sum_{i=1}^{g+1} \sum_{j=1, \neq i}^{g+1} \frac{(q_i q_j - q_j q_i)^2}{x - a_j}.\]
Then we have identities

\[
\frac{f(x)}{A(x)^2} = \frac{U(x)W(x) - V(x)^2}{A(x)^2} = \sum_{i=1}^{g+1} \frac{m_i(x)}{x - a_i},
\]

\[
m_i = \text{res}_{x=a_i} \frac{m_i(x)}{x - a_i} = q_i^2 + \sum_{i=1}^{g+1} \sum_{j=1, j \neq i}^{g+1} \frac{(q_i \dot{q}_j - q_j \dot{q}_i)^2}{a_i - a_j}.
\]

The direct computation gives the relations in Theorem when we deal with the integrals of differentials \( \frac{Q(x)}{A(x)} dx, \frac{xQ(x)}{A(x)} dx \).

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