Capacities and 1-strict subsets in metric spaces

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Abstract
In a complete metric space that is equipped with a doubling measure and supports a Poincaré inequality, we study strict subsets, i.e. sets whose variational capacity with respect to a larger reference set is finite. Relying on the concept of fine topology, we give a characterization of those strict subsets that are also sets of finite perimeter, and then we apply this to the study of condensers as well as BV capacities. We also apply the theory to prove a pointwise approximation result for functions of bounded variation.

1 Introduction
In potential theory, a set $A$ is said to be a $p$-strict subset of a set $D$ if the variational capacity $\text{cap}_p(A, D)$ is finite, or equivalently if there exists a Sobolev function $u$ with $u = 1$ in $A$ and $u = 0$ outside $D$. In the case $1 < p < \infty$, this concept has been considered in Euclidean spaces in [25] and in the setting of more general metric measure spaces in [7]. The typical assumptions on a metric space, which we make also in the current paper, are that the space is complete, equipped with a doubling measure, and supports a Poincaré inequality.

In the case $p = 1$, 1-strict subsets were studied, analogously to [7], in [30]. However, these papers left largely open the question of how to detect which sets are strict subsets. In the current paper we give a characterization of those 1-strict subsets that are also sets of finite perimeter, that is, their characteristic functions are of bounded variation (BV). The characterization involves the concepts of 1-fine interior and closure, and the measure-theoretic interior $I_E$ of the set $E$; see Section 2 for definitions.

**Theorem 1.1.** Let $D \subset X$ and let $E \subset X$ be a bounded set of finite perimeter with $I_E \subset D$. Then $\text{cap}_1(I_E, D) < \infty$ if and only if

$$\text{Cap}_1(I_E^{-1} \setminus \text{fine-int } D) = 0.$$  

Moreover, then $\text{cap}_1(I_E, D) \leq C_a \lambda^1(E, X)$ for a constant $C_a$ that depends only on the doubling constant of the measure and the constants in the Poincaré inequality.

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In Example 4.4 we demonstrate that without the assumption of finite perimeter, the theorem is not true. After considering some preliminary results in Section 3, we study 1-strict subsets in Section 4 and then we apply the theory to the study of condensers as well as BV versions of the variational capacity in Section 5. These concepts have been studied previously in e.g. [20]. Perhaps the most important contribution of the current paper lies in our careful analysis of the 1-fine topology and the closely related notion of quasiopen sets. These have recently proved to be very useful concepts (see especially [34]) and we expect that a solid understanding of their properties will contribute to future research as well.

As another application of our theory of 1-strict subsets, in Section 6 we prove the following theorem on the approximation of BV functions by means of Sobolev functions (often called Newton-Sobolev functions in the metric space setting).

**Theorem 1.2.** Let \( \Omega \subset X \) be an open set with \( \mu(\Omega) < \infty \) and let \( u \in BV(\Omega) \). Then there exists a sequence \((w_i) \subset N^{1,1}(\Omega)\) such that \( w_i \to u \) in \( L^1(\Omega) \),

\[
\limsup_{i \to \infty} \int_{\Omega} g_{w_i} \, d\mu \leq \|Du\|_1(\Omega) + C_a \|Du\|_j^2(\Omega),
\]

where each \( g_{w_i} \) is the minimal 1-weak upper gradient of \( w_i \) in \( \Omega \), and \( w_i(x) \to u^\vee(x) \) and \( w_i(x) \to u^\wedge(x) \) for every \( x \in \Omega \).

Here the constant \( C_a \) is the same as in Theorem 1.1. In Example 6.22 we show that the term \( C_a \|Du\|_j^2(\Omega) \) involving the jump part of the variation measure of \( u \) is necessary. Very recently, essentially the same result was proved in Euclidean spaces in [13, Proposition 7.3], based on an earlier result [12, Theorem 3.3]. In Euclidean spaces the term \( C_a \|Du\|_j^2(\Omega) \) is not needed, but for us the existence of this term makes it necessary to use rather different techniques in the proof, as we will discuss in Remark 6.23.

## 2 Notation and definitions

In this section we introduce the notation, definitions, and assumptions that are employed in the paper.

Throughout this paper, \((X,d,\mu)\) is a complete metric space that is equipped with a metric \( d \) and a Borel regular outer measure \( \mu \) satisfying a doubling property, meaning that there exists a constant \( C_d \geq 1 \) such that

\[
0 < \mu(B(x,2r)) \leq C_d \mu(B(x,r)) < \infty
\]

for every ball \( B(x,r) := \{y \in X : d(y,x) < r\} \). We assume that \( X \) consists of at least 2 points. Given a ball \( B = B(x,r) \) and \( \beta > 0 \), we sometimes abbreviate \( \beta B := B(x,\beta r) \); note that in a metric space, a ball (as a set) does not necessarily have a unique center point and radius, but these will be prescribed for all the balls that we consider. When we want to state that a constant \( C \) depends on the parameters \( a, b, \ldots \), we write \( C = C(a,b,\ldots) \). When a property holds outside a set of \( \mu \)-measure zero, we say that it holds almost everywhere, abbreviated a.e.
All functions defined on $X$ or its subsets will take values in $[-\infty, \infty]$. As a complete metric space equipped with a doubling measure, $X$ is proper, that is, closed and bounded sets are compact. Given a $\mu$-measurable set $A \subset X$, we define $L^1_{\text{loc}}(A)$ to be the class of functions $u$ on $A$ such that for every $x \in A$ there exists $r > 0$ such that $u \in L^1(A \cap B(x, r))$. Other local spaces of functions are defined analogously. For an open set $\Omega \subset X$, a function is in the class $L^1_{\text{loc}}(\Omega)$ if and only if it is in $L^1(\Omega')$ for every open $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that $\Omega'$ is a compact subset of $\Omega$.

For any set $A \subset X$ and $0 < R < \infty$, the restricted Hausdorff content of codimension one is defined by

$$H^R(A) := \inf \left\{ \sum_{j=1}^{\infty} \frac{\mu(B(x_j, r_j))}{r_j} : A \subset \bigcup_{j=1}^{\infty} B(x_j, r_j), r_j \leq R \right\}.$$ 

We also allow finite coverings by interpreting $\mu(B(x, 0))/0 = 0$. The codimension one Hausdorff measure of $A \subset X$ is then defined by

$$\mathcal{H}(A) := \lim_{R \to 0} H^R(A).$$

By a curve we mean a rectifiable continuous mapping from a compact interval of the real line into $X$. The length of a curve $\gamma$ is denoted by $\ell_\gamma$. We will assume every curve to be parametrized by arc-length, which can always be done (see e.g. [18, Theorem 3.2]). A nonnegative Borel function $g$ on $X$ is an upper gradient of a function $u$ on $X$ if for all nonconstant curves $\gamma$, we have

$$|u(x) - u(y)| \leq \int_\gamma g \, ds := \int_0^{\ell_\gamma} g(\gamma(s)) \, ds,$$

where $x$ and $y$ are the end points of $\gamma$. We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|$, $|u(y)|$ is infinite. We also express inequality (2.1) by saying that the pair $(u, g)$ satisfies the upper gradient inequality on the curve $\gamma$. Upper gradients were originally introduced in [23].

The 1-modulus of a family of curves $\Gamma$ is defined by

$$\text{Mod}_1(\Gamma) := \inf \int_X \rho \, d\mu$$

where the infimum is taken over all nonnegative Borel functions $\rho$ such that $\int_\gamma \rho \, ds \geq 1$ for every curve $\gamma \in \Gamma$. A property is said to hold for 1-almost every curve if it fails only for a curve family with zero 1-modulus. If $g$ is a nonnegative $\mu$-measurable function on $X$ and (2.1) holds for 1-almost every curve, we say that $g$ is a 1-weak upper gradient of $u$. By only considering curves $\gamma$ in a set $A \subset X$, we can talk about a function $g$ being a (1-weak) upper gradient of $u$ in $A$.

Given a $\mu$-measurable set $H \subset X$, we let

$$\|u\|_{N^{1,1}(H)} := \|u\|_{L^1(H)} + \inf \|g\|_{L^1(H)},$$
where the infimum is taken over all 1-weak upper gradients \( g \) of \( u \) in \( H \). Then we define the Newton-Sobolev space

\[
\dot{N}^{1,1}(H) := \{ u : \| u \|_{\dot{N}^{1,1}(H)} < \infty \},
\]

which was first introduced in [43]. When \( H \) is an open subset of \( \mathbb{R}^n \), then for any \( u \in \dot{N}^{1,1}(H) \) the quantity \( \| u \|_{\dot{N}^{1,1}(H)} \) agrees with the classical Sobolev norm, see e.g. [5, Corollary A.4]. For any \( \mu \)-measurable function \( u \) on a \( \mu \)-measurable set \( H \), we also let

\[
\| u \|_{\dot{D}^{1}(H)} := \inf \| g \|_{L^1(H)},
\]

where the infimum is taken over all 1-weak upper gradients \( g \) of \( u \) in \( H \), and then we define the Dirichlet space

\[
\dot{D}^{1}(H) := \{ u : \| u \|_{\dot{D}^{1}(H)} < \infty \}.
\]

We understand Newton-Sobolev and Dirichlet functions to be defined at every \( x \in H \) (even though \( \| \cdot \|_{\dot{N}^{1,1}(H)} \) is then only a seminorm). It is known that for any \( u \in \dot{D}^{1}_{\text{loc}}(H) \) there exists a minimal 1-weak upper gradient of \( u \) in \( H \), always denoted by \( g_u \), satisfying \( g_u \leq g \) a.e. in \( H \) for any 1-weak upper gradient \( g \in L^1_{\text{loc}}(H) \) of \( u \) in \( H \), see [5, Theorem 2.25].

For any \( D, H \subset X \), with \( H \) \( \mu \)-measurable, the space of Newton-Sobolev functions with zero boundary values is defined as

\[
N^{1,1}_0(D, H) := \{ u \mid_{D \cap H} : u \in N^{1,1}(H) \text{ and } u = 0 \text{ in } H \setminus D \}. \tag{2.2}
\]

This space is a subspace of \( N^{1,1}(D \cap H) \) when \( D \) is \( \mu \)-measurable, and it can always be understood to be a subspace of \( N^{1,1}(H) \). If \( H = X \), we omit it from the notation. Similarly, the space of Dirichlet functions with zero boundary values is defined as

\[
\dot{D}^{1}_0(D, H) := \{ u \mid_{D \cap H} : u \in \dot{D}^{1}(H) \text{ and } u = 0 \text{ in } H \setminus D \}.
\]

We will assume throughout the paper that \( X \) supports a \((1,1)\)-Poincaré inequality, meaning that there exist constants \( C_P > 0 \) and \( \lambda \geq 1 \) such that for every ball \( B(x, r) \), every \( u \in L^1_{\text{loc}}(X) \), and every upper gradient \( g \) of \( u \), we have

\[
\frac{1}{\mu(B(x, r))} \int_{B(x, r)} u | \nabla u | d\mu \leq C_P \int_{B(x, \lambda r)} g d\mu, \tag{2.3}
\]

where

\[
uu(B(x, r)) := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u d\mu.
\]

The 1-capacity of a set \( A \subset X \) is defined by

\[
\text{Cap}_1(A) := \inf \| u \|_{N^{1,1}(X)},
\]

where the infimum is taken over all functions \( u \in N^{1,1}(X) \) satisfying \( u \geq 1 \) in \( A \). We know that \( \text{Cap}_1 \) is an outer capacity, meaning that

\[
\text{Cap}_1(A) = \inf \{ \text{Cap}_1(W) : W \supset A, W \text{ is open} \}.
\]
for any $A \subset X$, see e.g. [5, Theorem 5.31].

The variational 1-capacity of a set $A \subset D$ with respect to a set $D \subset X$ is defined by

$$\text{cap}_1(A, D) := \inf \int_X g_u \, d\mu,$$

where the infimum is taken over functions $u \in N^{1,1}_0(D)$ satisfying $u \geq 1$ in $A$, and $g_u$ is the minimal 1-weak upper gradient of $u$ (in $X$). For basic properties satisfied by capacities, such as monotonicity and countable subadditivity, see e.g. [5].

If a property holds outside a set $A \subset X$ with $\text{Cap}_1(A) = 0$, we say that it holds 1-quasieverywhere, or 1-q.e. If $H \subset X$ is $\mu$-measurable, then

$$v = 0 \text{ 1-q.e. in } H \text{ implies } \|v\|_{N^{1,1}(H)} = 0,$$

see [5, Proposition 1.61]. In particular, in the definition (2.2) of the class $N^{1,1}_0(D, H)$, we can equivalently require $u = 0$ 1-q.e. in $H \setminus D$, and in the definition of the variational 1-capacity we can require $u \geq 1$ 1-q.e. in $A$.

By [19, Theorem 4.3, Theorem 5.1] we know that for any $A \subset X$,

$$\text{Cap}_1(A) = 0 \text{ if and only if } \mathcal{H}(A) = 0. \quad (2.5)$$

We will use this fact numerous times in the paper.

**Definition 2.6.** We say that a set $U \subset X$ is 1-quasiopen if for every $\varepsilon > 0$ there exists an open set $G \subset X$ such that $\text{Cap}_1(G) < \varepsilon$ and $U \cup G$ is open.

Given a set $H \subset X$, we say that a function $u$ is 1-quasicontinuous on $H$ if for every $\varepsilon > 0$ there exists an open set $G \subset X$ such that $\text{Cap}_1(G) < \varepsilon$ and $u|_{H \setminus G}$ is finite and continuous.

It is a well-known fact that Newton-Sobolev functions are quasicontinuous on open sets, see [11, Theorem 1.1] or [5, Theorem 5.29]. The following more general fact is a special case of [10, Theorem 1.3].

**Theorem 2.7.** Let $U \subset X$ be 1-quasiopen and let $u \in N^{1,1}_{\text{loc}}(U)$. Then $u$ is 1-quasicontinuous on $U$.

Next we present the definition and basic properties of functions of bounded variation on metric spaces, following [41]. See also e.g. [3, 14, 15, 17, 44] for the classical theory in the Euclidean setting. Given an open set $\Omega \subset X$ and a function $u \in L^1_{\text{loc}}(\Omega)$, we define the total variation of $u$ in $\Omega$ by

$$\|Du\|(\Omega) := \inf \left\{ \liminf_{i \to \infty} \int_{\Omega} g_{u_i} \, d\mu : u_i \in N^{1,1}_{\text{loc}}(\Omega), u_i \to u \text{ in } L^1_{\text{loc}}(\Omega) \right\},$$

where each $g_{u_i}$ is the minimal 1-weak upper gradient of $u_i$ in $\Omega$. (In [41], local Lipschitz constants were used in place of upper gradients, but the theory can be developed similarly with either definition.) We say that a function $u \in L^1(\Omega)$ is of
bounded variation, and denote \( u \in \text{BV}(\Omega) \), if \( \|Du\|(\Omega) < \infty \). For an arbitrary set \( A \subset X \), we define

\[
\|Du\|(A) := \inf\{\|Du\|(W) : A \subset W, \ W \subset X \text{ is open}\}.
\]

In general, we understand the expression \( \|Du\|(A) < \infty \) to mean that there exists some open set \( W \supset A \) such that \( u \) is defined in \( W \) with \( u \in L_1^\text{loc}(W) \) and \( \|Du\|(W) < \infty \).

If \( u \in L_1^\text{loc}(\Omega) \) and \( \|Du\|(\Omega) < \infty \), then \( \|Du\|(-) \) is a Radon measure on \( \Omega \) by [41, Theorem 3.4]. A \( \mu \)-measurable set \( E \subset X \) is said to be of finite perimeter if \( \|D\chi_E\|(X) < \infty \), where \( \chi_E \) is the characteristic function of \( E \). The perimeter of \( E \) in \( \Omega \) is also denoted by

\[
P(E, \Omega) := \|D\chi_E\|.(\Omega).
\]

The measure-theoretic interior of a set \( E \subset X \) is defined by

\[
I_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} = 0 \right\}, \tag{2.8}
\]

and the measure-theoretic exterior by

\[
O_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} = 0 \right\}. \tag{2.9}
\]

The measure-theoretic boundary \( \partial^* E \) is defined as the set of points \( x \in X \) at which both \( E \) and its complement have strictly positive upper density, i.e.

\[
\limsup_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} > 0.
\]

Note that the space \( X \) is always partitioned into the disjoint sets \( I_E, O_E, \) and \( \partial^* E \). For an open set \( \Omega \subset X \) and a \( \mu \)-measurable set \( E \subset X \) with \( P(E, \Omega) < \infty \), we know that for any Borel set \( A \subset \Omega \),

\[
P(E, A) = \int_{\partial^* E \cap A} \theta_E d\mathcal{H}, \tag{2.10}
\]

where \( \theta_E : \Omega \to [\alpha, C_d] \) with \( \alpha = \alpha(C_d, C_P, \lambda) > 0 \), see [2, Theorem 5.3] and [4, Theorem 4.6]. It follows that for any set \( A \subset \Omega \),

\[
\alpha \mathcal{H}(\partial^* E \cap A) \leq P(E, A) \leq C_d \mathcal{H}(\partial^* E \cap A). \tag{2.11}
\]

The following coarea formula is given in [41, Proposition 4.2]: if \( \Omega \subset X \) is an open set and \( u \in L_1^\text{loc}(\Omega) \), then

\[
\|Du\|(\Omega) = \int_{-\infty}^\infty P\{u > t\}, \Omega) \, dt, \tag{2.12}
\]

where we abbreviate \( \{u > t\} := \{x \in \Omega : u(x) > t\} \). If \( \|Du\|(\Omega) < \infty \), the above holds with \( \Omega \) replaced by any Borel set \( A \subset \Omega \). By [35, Proposition 3.8] this is true also for every 1-quasiopen set \( A \subset \Omega \).
If $\Omega \subset X$ is open and $u, v \in L^1_{\text{loc}}(\Omega)$, then

$$\|D \min\{u, v\}\|_1(\Omega) + \|D \max\{u, v\}\|_1(\Omega) \leq \|Du\|_1(\Omega) + \|Dv\|_1(\Omega);$$

(2.13)

for a proof see e.g. [41, Proposition 4.7].

The BV-capacity of a set $A \subset X$ is defined by

$$\text{Cap}_{\text{BV}}(A) := \inf \left(\|u\|_{L^1(X)} + \|Du\|_1(X)\right),$$

where the infimum is taken over all $u \in \text{BV}(X)$ such that $u \geq 1$ in a neighborhood of $A$. As noted in [19, Theorem 4.3], for any $A \subset X$ we have

$$\text{Cap}_{\text{BV}}(A) \leq \text{Cap}_1(A).$$

(2.14)

The lower and upper approximate limits of a function $u$ on an open set $\Omega$ are defined respectively by

$$u^\wedge(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x, r) \cap \{u < t\})}{\mu(B(x, r))} = 0 \right\}$$

(2.15)

and

$$u^\vee(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x, r) \cap \{u > t\})}{\mu(B(x, r))} = 0 \right\}$$

(2.16)

for $x \in \Omega$. The jump set of $u$ is then defined by

$$S_u := \{u^\wedge < u^\vee\}.$$

Since we understand $u^\wedge$ and $u^\vee$ to be defined only on $\Omega$, also $S_u$ is understood to be a subset of $\Omega$. It is straightforward to check that $u^\wedge$ and $u^\vee$ are always Borel functions.

Unlike Newton-Sobolev functions, we understand BV functions to be $\mu$-equivalence classes. To consider fine properties, we need to consider the pointwise representatives $u^\wedge$ and $u^\vee$.

Recall that Newton-Sobolev functions are quasicontinuous; BV functions have the following quasi-semicontinuity property, which follows from [36, Corollary 4.2], which in turn is based on [38, Theorem 1.1]. The result was first proved in the Euclidean setting in [12, Theorem 2.5].

**Proposition 2.17.** Let $u \in \text{BV}(\Omega)$ and let $\varepsilon > 0$. Then there exists an open set $G \subset \Omega$ such that $\text{Cap}_1(G) < \varepsilon$ and $u^\wedge|_{\Omega \setminus G}$ is finite and lower semicontinuous and $u^\vee|_{\Omega \setminus G}$ is finite and upper semicontinuous.

By [4, Theorem 5.3], the variation measure of a BV function can be decomposed into the absolutely continuous and singular part, and the latter into the Cantor and jump part, as follows. Given an open set $\Omega \subset X$ and $u \in \text{BV}(\Omega)$, we have for any Borel set $A \subset \Omega$

$$\|Du\|_1(A) = \|Du\|^a_1(A) + \|Du\|^c_1(A) = \|Du\|^a_1(A) + \|Du\|^c_1(A) + \|Du\|^j_1(A) = \int_A \theta d\mu + \|Du\|^c_1(A) + \int_{A \cap S_u} \int_{u^\wedge(x)} \int_{u^\vee(x)} \theta_{\{u > t\}}(x) dt d\mathcal{H}(x),$$

(2.18)
where $a \in L^1(\Omega)$ is the density of the absolutely continuous part and the functions $\theta_{\{u > t\}} \in [\alpha, C_d]$ are as in (2.10). Moreover, $\|Du\|_{c}(A) = 0$ for any set $A$ of finite $\mathcal{H}$-measure.

Next we define the fine topology in the case $p = 1$. For the analogous definition and theory in the case $1 < p < \infty$, see e.g. the monographs [1, 22, 39] for the Euclidean case, as well as [5, 7, 8, 9] for the metric space setting.

**Definition 2.19.** We say that $A \subset X$ is 1-thin at the point $x \in X$ if

$$\lim_{r \to 0} \frac{\cap_1(A \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} = 0.$$ 

We also say that a set $U \subset X$ is 1-finely open if $X \setminus U$ is 1-thin at every $x \in U$. Then we define the 1-fine topology as the collection of 1-finely open sets on $X$.

We denote the 1-fine interior of a set $H \subset X$, i.e. the largest 1-finely open set contained in $H$, by $\text{fine-int} H$. We denote the 1-fine closure of a set $H \subset X$, i.e. the smallest 1-finely closed set containing $H$, by $\overline{H}^1$. The 1-fine boundary of $H$ is $\partial^1 H := \overline{H}^1 \setminus \text{fine-int} H$. The 1-base $b_1 H$ is defined as the set of points where $H$ is not 1-thin.

We say that a function $u$ defined on a set $U \subset X$ is 1-finely continuous at $x \in U$ if it is continuous at $x$ when $U$ is equipped with the induced 1-fine topology on $U$ and $[-\infty, \infty]$ is equipped with the usual topology.

See [31, Section 4] for discussion on this definition, and for a proof of the fact that the 1-fine topology is indeed a topology. Using [5, Proposition 6.16], we see that a set $A \subset X$ is 1-thin at $x \in X$ if and only if

$$\lim_{r \to 0} \frac{\cap_1(A \cap B(x, r), B(x, 2r))}{\cap_1(B(x, r), B(x, 2r))} = 0.$$ 

Now we list some known facts concerning the 1-fine topology. It is stated in [29, Corollary 3.5] that for any $A \subset X$,

$$\overline{A}^1 = A \cup b_1 A.$$ 

(2.20)

By [29, Lemma 3.1] we have for any set $A \subset X$

$$I_A \cup \partial^* A \subset b_1 A \subset \overline{A}^1.$$ 

(2.21)

Note that by Lebesgue’s differentiation theorem (see e.g. [21, Chapter 1]), for $\mu$-measurable $E \subset X$ we have $\mu(I_E \Delta E) = 0$, where $\Delta$ denotes the symmetric difference. Thus the above implies

$$I_E \cup \partial^* E \subset b_1 I_E \subset \overline{E}^1.$$ 

(2.22)

By [29, Proposition 3.3],

$$\text{Cap}_1(\overline{A}^1) = \text{Cap}_1(A) \quad \text{for any } A \subset X.$$ 

(2.23)
Theorem 2.24 ([37, Corollary 6.12]). A set $U \subset X$ is 1-quasiopen if and only if it is the union of a 1-finely open set and a $\mathcal{H}$-negligible set.

Theorem 2.25 ([30, Theorem 5.1]). A function $u$ on a 1-quasiopen set $U$ is 1-quasicontinuous on $U$ if and only if it is finite 1-q.e. and 1-finely continuous 1-q.e. in $U$.

Throughout this paper we assume that $(X, d, \mu)$ is a complete metric space that is equipped with the doubling measure $\mu$ and supports a $(1, 1)$-Poincaré inequality.

3 Preliminary results

In this section we prove some preliminary results.

By [5, Corollary 2.21] we know that if $H \subset X$ is a $\mu$-measurable set and $v, w \in N_{1,1}(H)$, then $g_v = g_w$ a.e. in $\{ x \in H : v(x) = w(x) \}$, (3.1)

where $g_v$ and $g_w$ are the minimal 1-weak upper gradients of $v$ and $w$ in $H$.

The following lemma is a special case of [5, Lemma 1.52].

Lemma 3.2. Let $u_i, i \in \mathbb{N}$, be functions on a $\mu$-measurable set $H \subset X$ with 1-weak upper gradients $g_i$. Let $u := \sup_{i \in \mathbb{N}} u_i$ and $g := \sup_{i \in \mathbb{N}} g_i$, and suppose that $\mu(\{ u = \infty \}) = 0$. Then $g$ is a 1-weak upper gradient of $u$ in $H$.

Lemma 3.3. Let $G \subset X$ and let $\varepsilon > 0$. Then there exists an open set $W \supset G$ such that $\text{Cap}_1(W) < C \text{Cap}_1(G) + \varepsilon$ and $P(W, X) < C \text{Cap}_1(G) + \varepsilon$, for a constant $C = C(C_d, C_P, \lambda)$.

Recall that $C_d$, $C_P$, and $\lambda$ are the doubling constant of $\mu$ and the constants in the Poincaré inequality (2.3).

Proof. By (2.14) we have $\text{Cap}_{BV}(G) \leq \text{Cap}_1(G)$. By [19, Lemma 3.2] (which is simply an application of Cavalieri’s principle and the coarea formula (2.12)) we find a set $E \subset X$ containing a neighborhood of $G$ such that $\mu(E) + P(E, X) < \text{Cap}_{BV}(G) + \varepsilon \leq \text{Cap}_1(G) + \varepsilon$. (3.4)

By a suitable boxing inequality, see [19, Lemma 4.2], we find balls $\{ B(x_j, r_j) \}_{j=1}^{\infty}$ with $r_j \leq 1$ covering the measure-theoretic interior $I_E$, and thus also the set $G$, such that

$$\sum_{j=1}^{\infty} \frac{\mu(B(x_j, r_j))}{r_j} \leq C_B(\mu(E) + P(E, X))$$

for some constant $C_B = C_B(C_d, C_P, \lambda)$. For each $j \in \mathbb{N}$, by applying the coarea formula (2.12) to the function $u(y) = d(x_j, y)$, we find a number $s_j \in [r_j, 2r_j]$ such that

$$P(B(x_j, s_j), X) \leq C_d \frac{\mu(B(x_j, r_j))}{r_j}. \quad (3.5)$$
Define $1/s_j$-Lipschitz functions

$$
\eta_j(\cdot) := \max\left\{0, 1 - \frac{\text{dist}(\cdot, B(x_j, s_j))}{s_j}\right\}, \quad j \in \mathbb{N},
$$

so that $0 \leq \eta_j \leq 1$ on $X$, $\eta_j = 1$ in $B(x_j, s_j)$ and $\eta_j = 0$ in $X \setminus B(x_j, 2s_j)$. Let $\eta := \sup_{j \in \mathbb{N}} \eta_j$. By (3.1), $\chi_{B(x_j, 2s_j)/s_j}$ is a 1-weak upper gradient of $\eta_j$. Hence by Lemma 3.2 the minimal 1-weak upper gradient of $\eta$ satisfies $g_\eta \leq \sum_{i=1}^{\infty} \chi_{B(x_j, 2s_j)/s_j}$.

Then

$$
\int_X g_\eta \, d\mu \leq \sum_{j=1}^{\infty} \frac{\mu(B(x_j, 2s_j))}{s_j} \leq C_d^{\infty} \sum_{j=1}^{\infty} \frac{\mu(B(x_j, r_j))}{r_j} \leq C_d^{\infty} C_B(\mu(E) + P(E, X)) < C_d^{\infty} C_B(Cap_1(G) + \varepsilon)
$$

Similarly we show that $\|\eta\|_{L^1(X)} \leq C_d^{\infty} C_B(Cap_1(G) + \varepsilon)$. Let $W := \bigcup_{j=1}^{\infty} B(x_j, s_j)$. Since $\eta = 1$ in $W$, we get the estimate

$$
\text{Cap}_1(W) \leq \|\eta\|_{N^{1,1}(X)} \leq 2C_d^{\infty} C_B(Cap_1(G) + \varepsilon).
$$

Using the lower semicontinuity of perimeter with respect to $L^1$-convergence, as well as (2.13), we get

$$
P(W, X) \leq \sum_{j=1}^{\infty} P(B(x_j, s_j), X) \leq C_d^{\infty} \sum_{j=1}^{\infty} \frac{\mu(B(x_j, r_j))}{r_j} \leq C_d^{\infty} C_B(\mu(E) + P(E, X)) \leq C_d C_B(Cap_1(G) + \varepsilon).
$$

Next we note that Federer’s characterization of sets of finite perimeter holds also in metric spaces.

**Theorem 3.6** ([34, Theorem 1.1]). Let $\Omega \subset X$ be open, let $E \subset X$ be $\mu$-measurable, and suppose that $\mathcal{H}(\partial^*E \cap \Omega) < \infty$. Then $P(E, \Omega) < \infty$.

The converse holds by (2.11).

Recall the definitions of the measure-theoretic interior and exterior from (2.8) and (2.9).

**Proposition 3.7** ([29, Proposition 4.2]). Let $\Omega \subset X$ be open and let $E \subset X$ be $\mu$-measurable with $P(E, \Omega) < \infty$. Then $I_E \cap \Omega$ and $O_E \cap \Omega$ are 1-quasiopen sets.

Now we generalize this proposition to quasiope domains.

**Proposition 3.8.** Let $U \subset X$ be 1-quasiopen and let $E \subset X$ be $\mu$-measurable with $\mathcal{H}(\partial^*E \cap U) < \infty$. Then $I_E \cap U$ and $O_E \cap U$ are 1-quasiopen sets.
Proof. We find a sequence of open sets $G_j \subset X$ such that $U \cup G_j$ is open for each $j \in \mathbb{N}$ and $\Cap_1(G_j) \to 0$ as $j \to \infty$. By Lemma 3.3 we can assume that also $P(G_j, X) \to 0$, and so $\mathcal{H}(\partial^* G_j) \to 0$ by (2.11). It is straightforward to check that for each $j \in \mathbb{N}$

$$\partial^*(E \cup G_j) \cap (U \cup G_j) \subset (\partial^* E \cap U) \cup \partial^* G_j.$$

Then

$$\mathcal{H}(\partial^*(E \cup G_j) \cap (U \cup G_j)) \leq \mathcal{H}(\partial^* E \cap U) + \mathcal{H}(\partial^* G_j) < \infty$$

for each $j \in \mathbb{N}$. By Theorem 3.6 we conclude that $P(E \cup G_j, U \cup G_j) < \infty$. Thus each $I_{E \cup G_j} \cap (U \cup G_j)$ is 1-quasiopen by Proposition 3.7. By (2.23) and the fact that $\Cap_1$ is an outer capacity, we can take open sets $G_j' \supset \overline{G_j}$ such that still $\Cap_1(G_j') \to 0$. By (2.21) we have $I_{G_j} \cup \partial^* G_j \subset \overline{G_j}$, and so

$$I_E \setminus G_j = I_{E \cup G_j} \setminus G_j'.$$

Using this, we get

$$(I_E \cap U) \cup G_j' = (I_E \cap (U \cup G_j)) \cup G_j' = (I_{E \cup G_j} \cap (U \cup G_j)) \cup G_j',$$

which is a union of a 1-quasiopen and an open set for each $j \in \mathbb{N}$, and thus 1-quasiopen. It follows that $I_E \cap U$ is also 1-quasiopen. Similarly we show that $O_E \cap U$ is 1-quasiopen.

Recall the definitions concerning curves and 1-modulus from page 3.

**Proposition 3.9.** Let $U \subset X$ be 1-quasiopen and let $E \subset X$ be $\mu$-measurable with $\mathcal{H}(\partial^* E \cap U) < \infty$. Then for 1-a.e. curve $\gamma$ in $U$ with $\gamma(0) \in I_E$ and $\gamma(\ell_\gamma) \in O_E$, there exists $t \in (0, \ell_\gamma)$ such that $\gamma(t) \in \partial^* E$.

Proof. By Proposition 3.8 we know that $I_E \cap U$ and $O_E \cap U$ are 1-quasiopen sets. By [42, Remark 3.5] they are also 1-path open, meaning that for 1-a.e. curve $\gamma$, $\gamma^{-1}(I_E \cap U)$ and $\gamma^{-1}(O_E \cap U)$ are relatively open subsets of $[0, \ell_\gamma]$. Let $\gamma$ be such a curve in $U$, with $\gamma(0) \in I_E$ and $\gamma(\ell_\gamma) \in O_E$. Since $\gamma^{-1}(I_E \cap U)$ and $\gamma^{-1}(O_E \cap U)$ are nonempty disjoint relatively open subsets of the connected set $[0, \ell_\gamma]$, there necessarily exists a point $t \in (0, \ell_\gamma)$ with

$$t \notin \gamma^{-1}(I_E \cap U) \cup \gamma^{-1}(O_E \cap U),$$

and so $t \in \gamma^{-1}(\partial^* E)$. 

In [34, Example 5.4] it is shown that the assumption $\mathcal{H}(\partial^* E \cap U) < \infty$ cannot be removed.

**Lemma 3.10.** Let $E \subset X$ be $\mu$-measurable. Then fine-int $I_E^c = \text{fine-int} O_E$.

Proof. Note that $O_E \subset \overline{I_E^c}$ and so fine-int $O_E \subset \text{fine-int} I_E^c$. Conversely, we have fine-int $I_E^c = X \setminus \overline{I_E}$, and by (2.22) we have $X \setminus \overline{I_E} \subset X \setminus (I_E \cup \partial^* E) = O_E$. Thus fine-int $I_E^c \subset O_E$, and so fine-int $I_E^c \subset \text{fine-int} O_E$. 



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By using a Lipschitz cutoff function like in the proof of Lemma 3.3, it is easy to show that for any ball $B(x, r)$ with $r \leq 1$,

$$\text{Cap}_1(B(x, r)) \leq 2\frac{\mathcal{H}(B(x, 2r))}{r}.$$  \hspace{1cm} (3.11)

It follows that for any $A \subset X$,

$$\text{Cap}_1(A) \leq 2Cd\mathcal{H}(A).$$  \hspace{1cm} (3.12)

**Lemma 3.13.** Let $H \subset X$ be a Borel set with $\mathcal{H}(H) < \infty$. Then $X \setminus H$ is a 1-quasiopen set.

**Proof.** Let $\varepsilon > 0$. We find a closed set $K \subset H$ such that $\mathcal{H}(H \setminus K) < (2Cd)^{-1}\varepsilon$ (see e.g. [24, Proposition 3.3.37]). By (3.12), $\text{Cap}_1(H \setminus K) < \varepsilon$, and then since $\text{Cap}_1$ is an outer capacity, we find an open set $G \supset H \setminus K$ such that $\text{Cap}_1(G) < \varepsilon$. Now $(X \setminus H) \cup G = (X \setminus K) \cup G$ is an open set. \hfill \Box

Given a closed set $F \subset X$, one can of course always find open sets $W_1 \supset W_2 \supset \ldots \supset F$ such that $\bigcap_{j=1}^{\infty} W_j = F$. For 1-quasiopen sets we have the following analog of this fact.

**Lemma 3.14.** Let $F \subset X$ such that $X \setminus F$ is 1-quasiopen. Then there exist open sets $W_1 \supset W_2 \supset \ldots \supset F$ such that

$$\text{Cap}_1 \left( \bigcap_{j=1}^{\infty} W_j \setminus F \right) = 0.$$

By Lemma 3.13, $F$ can in particular be any Borel set of finite $\mathcal{H}$-measure.

**Proof.** For each $j \in \mathbb{N}$ we find an open set $G_j \subset X$ such that $F \setminus G_j$ is a closed set, and $\text{Cap}_1(G_j) \to 0$. Then for each $j \in \mathbb{N}$ we find open sets

$$V_{j_1} \supset V_{j_2} \supset \ldots \supset F \setminus G_j$$

such that $F \setminus G_j = \bigcap_{i=1}^{\infty} V_{j_i}$. Define $W_j := \bigcap_{k=1}^{j}(V_{kj} \cup G_k)$ for each $j \in \mathbb{N}$. These form a decreasing sequence of open sets containing $F$, and for each $N \in \mathbb{N}$,

$$\text{Cap}_1 \left( \bigcap_{j=1}^{\infty} W_j \setminus F \right) \leq \text{Cap}_1 \left( \bigcap_{j=N}^{\infty} W_j \setminus F \right) \leq \text{Cap}_1 \left( \bigcap_{j=N}^{\infty} (V_{Nj} \cup G_N) \setminus F \right) \leq \text{Cap}_1(G_N),$$

since $\bigcap_{j=N}^{\infty} V_{Nj} = F \setminus G_N$. Letting $N \to \infty$, we get the result. \hfill \Box

Finally we prove the following absolute continuity.
Lemma 3.15. Let $H \subset X$ with $\mathcal{H}(H) < \infty$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \subset X$ with $\text{Cap}_1(A) < \delta$, then $\mathcal{H}(H \cap A) < \varepsilon$.

Proof. Suppose by contradiction that there exists $\varepsilon > 0$ and a sequence of sets $A_j \subset X$, $j \in \mathbb{N}$, such that $\text{Cap}_1(A_j) < 2^{-j}$ but $\mathcal{H}(H \cap A_j) \geq \varepsilon$. Since $\text{Cap}_1$ is an outer capacity, we can assume that the sets $A_j$ are open. Then defining

$$A := \bigcap_{k=1}^{\infty} \bigcup_{j \geq k} A_j,$$

we have $\text{Cap}_1(A) = 0$. However, since the sets $\bigcup_{j \geq k} A_j$ constitute a decreasing sequence of Borel sets and since the restriction $\mathcal{H}|_{H(\cdot)} := \mathcal{H}(H \cap \cdot)$ is a Borel outer measure (see e.g. [24, Lemma 3.3.13]), we have

$$\mathcal{H}(H \cap A) = \lim_{k \to \infty} \mathcal{H}(H \cap \bigcup_{j \geq k} A_j) \geq \varepsilon,$$

which is a contradiction by (2.5).

4 Strict subsets

In this section we study 1-strict subsets.

Definition 4.1. A set $A \subset D$ is a 1-strict subset of $D$ if there is a function $\eta \in N_{1,1}^{\infty}(D)$ such that $\eta = 1$ in $A$.

By [29, Proposition 3.3] we know that if $A$ is a 1-strict subset of $D$, then $\text{Cap}_1(\overline{A} \setminus \text{fine-int} D) = 0$. Now we show that this holds also with the ambient space $X$ replaced by a more general quasiopen set $U$. Note that 1-quasiopen sets are $\mu$-measurable by [6, Lemma 9.3].

Proposition 4.2. Let $A \subset D$ and let $U \subset X$ be a 1-quasiopen set, and suppose that there exists $\rho \in N_0^{1,1}(D,U)$ with $\rho = 1$ in $A \cap U$. Then

$$\text{Cap}_1((\overline{A} \setminus \text{fine-int} D) \cap U) = 0. \quad (4.3)$$

Proof. The function $\rho$ is 1-quasicontinuous on $U$ by Theorem 2.7, and thus 1-finely continuous (with respect to the induced 1-fine topology on $U$) at 1-q.e. point in $U$ by Theorem 2.25. Now for 1-q.e. $x \in \overline{A} \cap U$ we have either $x \in A$ or $x \in b_1 A$ by (2.20), and also $x \in \text{fine-int} U$ by Theorem 2.24. Then either $x \in A$ or $x \in b_1 (A \cap U)$. If $\rho$ is 1-finely continuous at $x$, it follows that $\rho(x) = 1$. In conclusion, $\rho = 1$ 1-q.e. in $\overline{A} \cap U$.

Analogously, from the fact that $\rho = 0$ in $U \setminus D$ we get $\rho = 0$ 1-q.e. in $\overline{X \setminus D} \cap U = U \setminus \text{fine-int} D$, and then (4.3) follows.

Now we note that the converse to Proposition 4.2 is not true.
Example 4.4. Let \( X = \mathbb{R}^2 \) (unweighted, i.e. equipped with the usual 2-dimensional Lebesgue measure). We will choose a compact subset \( K \) of a 1-finely open set \( D \) such that \( K \) is not a 1-strict subset of \( D \) (note that \( K = \overline{K}^1 \)). First denote the unit square by \( Q := [0, 1] \times [0, 1] \). Define the following “gratings” that are compact subsets of \( Q \):

\[
H_j := \bigcup_{k=0}^{2^j} \{k2^{-j}\} \times [0, 1], \quad j \in \mathbb{N}.
\]

Given any set \( A \subset \mathbb{R}^2 \) and \( a > 0, b \in \mathbb{R}^2 \), scaling and translation are given by

\[
aA + b := \{ax + b : x \in A\}.
\]

Now consider the complement of the union of scaled and shifted “gratings”

\[
D := \mathbb{R}^2 \setminus \bigcup_{j=1}^{\infty} (2^{-2j} H_{2j} + (2^{-j}, 0))
\]

All points in \( D \) are interior points except the origin \( 0 \). We note that for every \( r > 0 \) and every set \( 2^{-2j} H_{2j} + (2^{-j}, 0) \) that intersects \( B(0, r) \), we have

\[
\text{cap}_1(2^{-2j} H_{2j} + (2^{-j}, 0), B(0, 2r)) \leq \text{cap}_1(2^{-2j} Q, B(0, 2r)) = 2^{-4j}.
\]

It follows that for every \( 0 < r < 1/4 \) (\( \mathcal{L}^2 \) denotes the 2-dimensional Lebesgue measure, and \( \lfloor a \rfloor \) is the largest integer at most \( a \in \mathbb{R} \))

\[
r \frac{\text{cap}_1(B(0, r) \setminus D, B(0, 2r))}{\mathcal{L}^2(B(0, r))} \leq \sum_{j=\lfloor -\log r/\log 2 \rfloor - 1}^{\infty} \text{cap}_1(2^{-2j} H_{2j} + (2^{-j}, 0), B(0, 2r)) \leq 2^{-4j} \sum_{j=\lfloor -\log r/\log 2 \rfloor - 1}^{\infty} 2^{-4j} \leq \frac{2r}{\mathcal{L}^2(B(0, r))} 2^{-4[\lfloor -\log r/\log 2 \rfloor] + 4} \leq \frac{2^9 r}{\mathcal{L}^2(B(0, r))} r^4 \to 0 \quad \text{as } r \to 0.
\]

Thus the set \( D \) is 1-finely open. Now define

\[
K_j := \bigcup_{k=0}^{2^j-1} \{(k + 1/2)2^{-j}\} \times [0, 1], \quad j \in \mathbb{N},
\]

which are also compact subsets of the unit square, and

\[
K := \bigcup_{j=1}^{\infty} (2^{-2j} K_{2j} + (2^{-j}, 0)) \cup \{0\},
\]

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which is a compact subset of $D$. Let $u \in N^{1,1}_0(D)$ be a function with $u = 1$ in $K$, and let $g$ be any upper gradient of $u$. Now for every $j \in \mathbb{N}$,

$$\|u\|_{N^{1,1}(2^{-2j}Q+(2^{-j},0))} \geq \int_{2^{-2j}Q+(2^{-j},0)} g \, d\mathcal{L}^2 \geq 2^{-2j} \cdot 2 \cdot (2^{2j} - 1) \geq 1.$$ 

Thus $\|u\|_{N^{1,1}(\mathbb{R}^2)} = \infty$, and so $K$ is not a 1-strict subset of $D$.

In [30, Theorem 4.3] it is shown that when $A$ is a point in fine-int $D$, then $A$ is a 1-strict subset of $D$. Now our goal will be to show that despite Example 4.4, there are many other 1-strict subsets $A$ of $D$. Our first result in this direction is the following.

**Lemma 4.5.** Let $W \subset X$ be an open set, let $H \subset W$ with $\mathcal{H}(H) < \infty$, and let $\varepsilon > 0$. Then there exists $\eta \in N^{1,1}_0(W)$ with $0 \leq \eta \leq 1$ on $X$, $\eta = 1$ in a neighborhood of $H$, and

$$\int_X \eta \, d\mu < \varepsilon \quad \text{and} \quad \int_X g \eta \, d\mu < C_d \mathcal{H}(H) + \varepsilon.$$

Moreover, Cap$_1(\{\eta > 0\}) < 2C_d^2(\mathcal{H}(H) + \varepsilon)$ and Cap$_1(\{\eta > 0\} \setminus W) = 0$.

**Proof.** Let $W_\delta := \{x \in W : \text{dist}(x, X \setminus W) > \delta\}$, $\delta > 0$.

Let $V_1 := W_{2^{-1}}$ and for each $j = 2, 3, \ldots$, let $V_j := W_{2^{-j}} \setminus W_{2^{-j+1}}$. Then $W = \bigcup_{j=1}^\infty V_j$. For each $j \in \mathbb{N}$, take a collection of balls $\{B_{jk} = B(x_{jk}, r_{jk})\}_{k=1}^\infty$ covering $H \cap V_j$ with $r_{jk} \leq 2^{-j-2}\varepsilon(C_d \mathcal{H}(H) + \varepsilon + 1)^{-1}$ and

$$\sum_{k=1}^\infty \frac{\mu(B_{jk})}{r_{jk}} < \mathcal{H}(H \cap V_j) + \frac{2^{-j} \varepsilon}{C_d}. \quad (4.6)$$

We can assume that $B_{jk} \cap (H \cap V_j) \neq \emptyset$ for all $k \in \mathbb{N}$. Define Lipschitz functions

$$\eta_{jk} := \max \left\{0, 1 - \frac{\text{dist}(\cdot, B_{jk})}{r_{jk}}\right\}, \quad j, k \in \mathbb{N},$$

so that $\eta_{jk} = 1$ in $B_{jk}$ and $\eta_{jk} = 0$ in $X \setminus 2B_{jk}$. By (3.1), each $\eta_{jk}$ has a 1-weak upper gradient $\chi_{2B_{jk}}/r_{jk}$. Let $\eta := \sup_{j,k \in \mathbb{N}} \eta_{jk}$. Then $\eta = 1$ in a neighborhood of $H$ and $\eta = 0$ in $X \setminus W$. By Lemma 3.2,

$$\int_X g \eta \, d\mu \leq \sum_{j,k=1}^\infty \int_X g \eta_{jk} \, d\mu \leq \sum_{j,k=1}^\infty \frac{\mu(2B_{jk})}{r_{jk}} \leq C_d \sum_{j=1}^\infty \left(\mathcal{H}(H \cap V_j) + \frac{2^{-j} \varepsilon}{C_d}\right) \quad \text{by (4.6)} \quad (4.7)$$

as desired. Similarly,

$$\int_X \eta \, d\mu \leq \sum_{j,k=1}^\infty \mu(2B_{jk}) \leq \varepsilon(C_d \mathcal{H}(H) + \varepsilon + 1)^{-1} \sum_{j,k=1}^\infty \frac{\mu(2B_{jk})}{r_{jk}} < \varepsilon.$$
In conclusion, \( \eta \in N^{1,1}(X) \) and then in fact \( \eta \in N^{1,1}_0(W) \).

We can define a function \( \rho \) analogously to \( \eta \), but using the collections of balls \( \{2B_{jk}\}_{k=1}^{\infty} \) in place of \( \{B_{jk}\}_{k=1}^{\infty} \). We obtain \( \rho = 1 \) in \( \{\eta > 0\} \) and then

\[
\text{Cap}_1(\{\eta > 0\}) \leq \int_X \rho \, d\mu + \int_X g_\rho \, d\mu \leq 2C_d^2(\mathcal{H}(H) + \varepsilon).
\]

Moreover, by the characterization of the fine closure (2.20), we get \( \{\eta > 0\} \setminus W \subset \bigcup_{j=N}^{\infty} \bigcup_{k=1}^{\infty} 2B_{jk} \) for any \( N \in \mathbb{N} \), and so

\[
\text{Cap}_1(\{\eta > 0\} \setminus W) \leq \text{Cap}_1\left(\bigcup_{j=N}^{\infty} \bigcup_{k=1}^{\infty} 2B_{jk}\right) \leq 2 \sum_{j=N}^{\infty} \sum_{k=1}^{\infty} \frac{\mu(4B_{jk})}{2r_{jk}} \text{ by (3.11)}
\]

\[
\rightarrow 0
\]

as \( N \to \infty \), since we had \( \sum_{j,k=1}^{\infty} \mu(2B_{jk})/r_{jk} < \infty \) by (4.7). \( \square \)

**Lemma 4.8** ([35, Lemma 3.3]). Let \( G \subset X \) and \( \varepsilon > 0 \). Then there exists an open set \( G' \supset G \) with \( \text{Cap}_1(G') < C_1(\text{Cap}_1(G) + \varepsilon) \) and a function \( \rho \in N^{1,1}_0(G') \) with \( 0 \leq \rho \leq 1 \) on \( X \), \( \rho = 1 \) in \( G \), and \( \|\rho\|_{N^{1,1}(X)} < C_1(\text{Cap}_1(G) + \varepsilon) \), for some constant \( C_1 = C_1(C_d, C_P, \lambda) \geq 1 \).

The following proposition says that a subset of finite Hausdorff measure of a 1-quasiopen set is always a 1-strict subset.

**Proposition 4.9.** Let \( U \subset X \) be 1-quasiopen and let \( F \subset U \) with \( \mathcal{H}(F) < \infty \). Let \( 0 < \varepsilon < 1 \). Then there exists \( \eta \in N^{1,1}_0(U) \) with \( 0 \leq \eta \leq 1 \) on \( X \), \( \eta = 1 \) in a 1-quasiopen set containing \( F \), and

\[
\int_X \eta \, d\mu < \varepsilon \quad \text{and} \quad \int_X g_\eta \, d\mu < C_d \mathcal{H}(F) + \varepsilon.
\]

Moreover, \( \eta = 0 \) in a 1-quasiopen set containing \( X \setminus U \).

**Proof.** For each \( j \in \mathbb{N} \), by Lemma 3.15 there exists \( 0 < \delta_j < 1 \) such that if \( A \subset X \) with \( \text{Cap}_j(A) < \delta_j \), then \( \mathcal{H}(F \cap A) \leq 2^{-j-3}\varepsilon/C_d^2 \). For each \( j \in \mathbb{N} \) we find an open set \( G_j \subset X \) such that \( U \cup G_j \) is open and \( \text{Cap}_j(G_j) < 2^{-j-1}\varepsilon\delta_j/C_1 \). By Lemma 4.8 we then find an open set \( G'_j \supset G_j \) with \( \text{Cap}_j(G'_j) < 2^{-j-1}\varepsilon\delta_j \) and a function \( \rho_j \in N^{1,1}_0(G'_j) \) such that \( 0 \leq \rho_j \leq 1 \) on \( X \), \( \rho_j = 1 \) in \( G_j \), and \( \|\rho_j\|_{N^{1,1}(X)} < 2^{-j-1}\varepsilon\delta_j \).

By (2.23), also \( \text{Cap}_j(G'_j) < 2^{-j-1}\varepsilon\delta_j \) for each \( j \in \mathbb{N} \), and so \( \mathcal{H}(F \cap G'_j) < 2^{-j-3}\varepsilon/C_d^2 \). Let also \( G'_0 := X \). For each \( j \in \mathbb{N} \), apply Lemma 4.5 with the choices
$H = F \cap \overline{G_{j-1}}$ and $W = U \cup G_j$ to find a function $\eta_j \in N_{0}^{1,1}(U \cup G_j)$ such that $0 \leq \eta_j \leq 1$ on $X$, $\eta_j = 1$ in an open set $W_j \supset F \cap \overline{G_{j-1}}$, and

$$
\int_X \eta_j \, d\mu < 2^{-j} \varepsilon \quad \text{and} \quad \int_X g_{\eta_j} \, d\mu < C_d \mathcal{H}(F \cap \overline{G_{j-1}}) + 2^{-j-2} \varepsilon.
$$

Moreover, Lemma 4.5 further gives for $j = 2, 3, \ldots$ (note that the $\varepsilon$ in that lemma can be chosen as small as needed)

$$
\text{Cap}_1(\{\eta_j > 0\}) < 2C_d^2 \mathcal{H}(F \cap \overline{G_{j-1}}) + 2^{-j-2} \varepsilon < 2^{-j} \varepsilon + 2^{-j-1} \varepsilon = 2^{-j} \varepsilon.
$$

Now

$$
\int_X g_{\eta_1} \, d\mu < C_d \mathcal{H}(F) + 2^{-3} \varepsilon
$$

and for $j = 2, 3, \ldots$,

$$
\int_X g_{\eta_j} \, d\mu < C_d \mathcal{H}(F \cap \overline{G_{j-1}}) + 2^{-j-2} \varepsilon < 2^{-j} \varepsilon + 2^{-j-2} \varepsilon = 2^{-j-1} \varepsilon.
$$

Then let $\eta'_j := \eta_j(1 - \rho_j)$ for each $j \in \mathbb{N}$. Now we have

$$
\eta'_j = 1 \text{ in } W_j \setminus G'_j \supset F \cap \overline{G_{j-1}} \setminus G'_j, \quad \eta'_j = 0 \text{ in } X \setminus U, \text{ and by the Leibniz rule [5, Theorem 2.15]},
$$

$$
\int_X g_{\eta'_1} \, d\mu \leq \int_X g_{\eta_1} \, d\mu + \int_X g_{\rho_1} \, d\mu < C_d \mathcal{H}(F) + 2^{-1} \varepsilon
$$

and for $j = 2, 3, \ldots$

$$
\int_X g_{\eta'_j} \, d\mu \leq \int_X g_{\eta_j} \, d\mu + \int_X g_{\rho_j} \, d\mu < 2^{-j-1} \varepsilon + 2^{-j-1} \varepsilon = 2^{-j} \varepsilon.
$$

Also,

$$
\|\eta'_j\|_{L^1(X)} \leq \|\eta_j\|_{L^1(X)} < 2^{-j} \varepsilon
$$

for all $j \in \mathbb{N}$. Then $\eta'_j \in N_{0}^{1,1}(X)$ for all $j \in \mathbb{N}$. Let $\eta := \sup_{j \in \mathbb{N}} \eta'_j$. By (4.11) we have $\eta = 1$ in the set $\bigcup_{j=1}^\infty (W_j \setminus \overline{G'_j})$, which is 1-finely open and contains 1-quasi all of $F$ since we had $\text{Cap}_1(\overline{G'_j}) \to 0$. Then by (2.4) we can redefine $\eta = 1$ in $F$; by Theorem 2.24 we now have that $\eta = 1$ in a 1-quasiopen set containing $F$. Moreover, by Lemma 3.2, (4.12), and (4.13) we find that

$$
\int_X g_\eta \, d\mu \leq \sum_{j=1}^\infty \int_X g_{\eta'_j} \, d\mu < C_d \mathcal{H}(F) + \varepsilon,
$$

and we also have

$$
\int_X \eta \, d\mu \leq \sum_{j=1}^\infty \int_X \eta'_j \, d\mu < \varepsilon.
$$
Thus $\eta \in N^{1,1}(X)$. Clearly also $\eta = 0$ in $X \setminus U$, so that $\eta \in N_0^{1,1}(U)$.

Finally we show that $\eta = 0$ in a 1-quasiopen set containing $X \setminus U$. Fix $\delta > 0$. From Lemma 4.5 we had that for every $j \in \mathbb{N}$,

$$\text{Cap}_1(\{\eta_j' > 0\} \setminus (U \cup G_j)) \leq \text{Cap}_1(\{\eta_j > 0\} \setminus (U \cup G_j)) = 0,$$

and then since $\eta_j' = 0$ in the open set $G_j$,

$$\text{Cap}_1(\{\eta_j' > 0\} \setminus U) = 0. \tag{4.14}$$

For $N = 2, 3, \ldots$ we have by (4.10)

$$\sum_{j=N}^{\infty} \text{Cap}_1(\{\eta_j' > 0\}) \leq \sum_{j=N}^{\infty} \text{Cap}_1(\{\eta_j > 0\}) \leq \sum_{j=N}^{\infty} 2^{-j} \varepsilon = 2^{-N+1} \varepsilon < \delta$$

for large enough $N$. Then by (2.23), also $\text{Cap}_1(\bigcup_{j=N}^{\infty} \{\eta_j' > 0\}) < \delta$. Now by the characterization (2.20), we see that

$$\{\eta > 0\} \cap (U \setminus G_j) \subset \bigcup_{j=1}^{N-1} \{\eta_j' > 0\} \cup \bigcup_{j=N}^{\infty} \{\eta_j' > 0\},$$

and so by (4.14),

$$\text{Cap}_1(\{\eta > 0\} \setminus U) \leq \text{Cap}_1(\bigcup_{j=N}^{\infty} \{\eta_j' > 0\}) < \delta.$$

Since $\delta > 0$ was arbitrary, we have $\text{Cap}_1(\{\eta > 0\} \setminus U) = 0$. The set $X \setminus \{\eta > 0\}$ is 1-finely open, and then by Theorem 2.24, $X \setminus \{\eta > 0\} \cap U$ is 1-quasiopen. Thus $\eta = 0$ in a 1-quasiopen set containing $X \setminus U$. \qed

Now we wish to show, essentially, that the converse to Proposition 4.2 holds when $A$ is a set of finite perimeter. Note that a set of finite perimeter can be perturbed in any set of $\mu$-measure zero without changing the perimeter. However, a set of $\mu$-measure zero may well have an effect on Newton-Sobolev norms; in Example 4.4 we have $L^2(K) = 0$ and so $P(K, \mathbb{R}^2) = 0$, but $K$ was not a 1-strict subset of $U$.

For this reason, we always need to consider a reasonable representative of a set of finite perimeter $E$. We choose this representative to be the measure-theoretic interior $I_E$, as defined in (2.8); note that by Lebesgue’s differentiation theorem, we indeed have $\mu(I_E \Delta E) = 0$.

The proof of the following lemma can be found e.g. in [5, Lemma 1.34].

**Lemma 4.15.** If $\Gamma$ and $\Gamma'$ are families of curves such that for every $\gamma \in \Gamma$ there exists a subcurve $\gamma' \in \Gamma'$ of $\gamma$, then $\text{Mod}_1(\Gamma) \leq \text{Mod}_1(\Gamma')$.

Recall the definition of the Dirichlet spaces $D^1(\cdot)$ from page 4.
Theorem 4.16. Let $D, U \subset X$ with $U$ 1-quasiopen and let $E \subset X$ be $\mu$-measurable with $\mathcal{H}(\partial^* E \cap U) < \infty$. Suppose that also

$$\text{Cap}_1(U \cap (I_E \cup \partial^* E) \setminus \text{fine-int } D) = 0.$$ 

Let $\varepsilon > 0$. Then there exists $\rho \in D_0^1(D, U)$ such that $\rho = 1$ 1-a.e. in $(I_E \cup \partial^* E) \cap U$, $\|\rho - \chi_E\|_{L^1(U)} < \varepsilon$, and

$$\int_U g_\rho \, d\mu < C_d \mathcal{H}(\partial^* E \cap U) + \varepsilon.$$ 

Note that the condition $\mathcal{H}(\partial^* E \cap U) < \infty$ is satisfied by any set of finite perimeter $E$, more precisely if $P(E, \Omega) < \infty$ for some open set $\Omega \supset U$. Note also that if $\chi_E \in L^1(U)$, then $\rho \in N_0^{1,1}(D, U)$.

Proof. The set fine-int $D$ is 1-quasiopen by Theorem 2.24. By Proposition 4.9 we find a function $\eta \in N_0^{1,1}(\text{fine-int } D) \subset N_0^{1,1}(\text{fine-int } D, U)$ such that $\eta = 1$ in $\partial^* E \cap \text{fine-int } D \cap U$, $\|\eta\|_{L^1(X)} < \varepsilon$, and

$$\int_X g_\eta \, d\mu < C_d \mathcal{H}(\partial^* E \cap \text{fine-int } D \cap U) + \varepsilon. \quad (4.17)$$ 

Define

$$\rho := \begin{cases} \eta & \text{in } U \setminus (D \cap I_E), \\ 1 & \text{in } U \cap D \cap I_E. \end{cases}$$

Since $\text{Cap}_1(U \cap (I_E \cup \partial^* E) \setminus \text{fine-int } D) = 0$, we have $\rho = 1$ 1-a.e. in $(I_E \cup \partial^* E) \cap U$, as desired. Also,

$$\|\rho - \chi_E\|_{L^1(U)} \leq \|\eta\|_{L^1(U)} < \varepsilon$$

as desired. Now we show that in the set $U$ we have $g_\rho \leq g_\eta$, where $g_\rho$ is the minimal 1-weak upper gradient of $\rho$ in $U$. Choose a curve $\gamma$ in $U$. If $\gamma$ lies entirely in $U \setminus (D \cap I_E)$, then $\rho = \eta$ on this curve and so the pair $(\rho, g_\eta)$ satisfies the upper gradient inequality on 1-a.e. such curve $\gamma$. If $\gamma$ lies entirely in $D \cap I_E$, then $\rho = 1$ on the curve and so again the upper gradient inequality is satisfied.

Assume then that $\gamma$ intersects both $D \cap I_E$ and $U \setminus (D \cap I_E)$; by splitting $\gamma$ into two subcurves and reversing direction, if necessary, we can assume that $\gamma(0) \in D \cap I_E$ and $\gamma(t) \in U \setminus (D \cap I_E)$. Since we had $\text{Cap}_1(U \cap (I_E \cup \partial^* E) \setminus \text{fine-int } D) = 0$, by [5, Proposition 1.48] we know that 1-a.e. curve avoids $U \cap (I_E \cup \partial^* E) \setminus \text{fine-int } D$. Thus we can assume that $\gamma(t) \in U \setminus I_E$, and then by Proposition 3.9 we can assume that there is $t \in (0, \ell_\gamma)$ such that $\gamma(t) \in \partial^* E \cap \text{fine-int } D$; note that here we use also Lemma 4.15. We can also assume that the pair $(\eta, g_\eta)$ satisfies the upper gradient inequality on $\gamma$. Then

$$|\rho(0) - \rho(t)| = |t - \eta(t)| = |t - \eta(\ell_\gamma)| \leq \int_\gamma g_\eta \, ds.$$ 

In total, we have established that $g_\rho \leq g_\eta$ in $U$. Thus by $(4.17)$,

$$\int_U g_\rho \, d\mu < C_d \mathcal{H}(\partial^* E \cap U) + \varepsilon,$$

as desired. Now $\rho \in D_0^1(D, U)$. \qed
5 Applications in the study of capacities

In this section we apply the results of the previous section to the study of variational capacities. We begin with the proof of the first theorem in the introduction.

Proof of Theorem 1.1. To prove the “only if” direction, assume that \( \text{cap}_1(I_E, D) < \infty \). Thus there exists \( u \in N^{1,1}_0(D) \) with \( u = 1 \) in \( I_E \). By applying Proposition 4.2 with the choices \( A = I_E \) and \( U = X \), we obtain that \( \text{Cap}_1(I_E \setminus \text{fine-int } D) = 0 \).

To prove the “if” direction, let \( \varepsilon > 0 \). We note that \( H(\partial^* E) < \infty \) by (2.11), and \( \text{Cap}_1((I_E \cup \partial^* E) \setminus \text{fine-int } D) = 0 \) by (2.22). Thus by Theorem 4.16 we find a function \( \rho \in D^{1,0}_0(D) \) such that \( \rho = 1 \) \( 1 \)-q.e. in \( I_E \cup \partial^* E \), \( \| \rho - \chi_E \|_{L^1(X)} < \varepsilon \), and

\[
\int_X g_\rho \, d\mu < C_d P(E, X) + \varepsilon,
\]

using also (2.11). Since we assume \( E \) to be bounded, \( \rho \in L^1(X) \) and so in fact \( \rho \in N^{1,1}_0(D) \). Hence

\[
\text{cap}_1(I_E, D) \leq \int_X g_\rho \, d\mu \leq C_d \alpha^{-1} P(E, X) + \varepsilon,
\]

so that letting \( \varepsilon \to 0 \) we get the conclusion. \( \square \)

Now we define the variational 1-capacity in more general (ambient) sets than the entire space \( X \).

Definition 5.1. Let \( A \subset D \) and let \( U \subset X \) be \( \mu \)-measurable. We define

\[
\text{cap}_1(A, D, U) := \inf \int_U g_u \, d\mu,
\]

where the infimum is taken over functions \( u \in N^{1,1}_0(D, U) \) such that \( u = 1 \) in \( A \cap U \), and \( g_u \) is the minimal 1-weak upper gradient of \( u \) in \( U \).

Sometimes \( (A, D^c, U) \) is called a condenser and \( \text{cap}_1(A, D, U) \) is called the capacity of the condenser, see e.g. [20], as well as [23, 26, 40] where \( \text{cap}_p \) for more general \( p \geq 1 \) is considered. Note that by (2.4) we can equivalently require that \( u = 1 \) \( 1 \)-q.e. in \( A \cap U \).

Now we can show that the capacity of a condenser that consists of two sets of finite perimeter is finite if and only if the sets do not “touch” each other.

Theorem 5.2. Let \( \Omega \subset X \) be open and bounded and let \( E, F \subset \Omega \) with \( P(E, \Omega) < \infty \), \( P(F, \Omega) < \infty \), and \( E \cap F = \emptyset \). Then \( \text{cap}_1(I_E, I_F, \Omega) < \infty \) if and only if \( H(\partial^* E \cap \partial^* F \cap \Omega) = 0 \). Moreover, then \( \text{cap}_1(I_E, I_F, \Omega) \leq C \min \{ P(E, \Omega), P(F, \Omega) \} \) for a constant \( C = C(C_d, C_P, \lambda) \).

Proof. The set \( \Omega \cap O_F \) is \( 1 \)-quasiopen by Proposition 3.7, and thus by Theorem 2.24,

\[
H(\Omega \cap O_F \setminus \text{fine-int } O_F) = 0.
\]
For sets \( A_1, A_2 \subset X \), we write \( A_1 \overset{\mathcal{H}}{\cap} a.e. \) \( A_2 \) if \( \mathcal{H}(A_1 \Delta A_2) = 0 \). Now we get

\[
\Omega \cap (I_E \cup \partial^* E) \setminus \text{fine-int } I_F^c = \Omega \cap (I_E \cup \partial^* E) \setminus \text{fine-int } O_F \quad \text{by Lemma 3.10}
\]

\[
\mathcal{H} \overset{\text{a.e.}}{=} \Omega \cap (I_E \cup \partial^* E) \setminus O_F \quad \text{by (5.3)}
\]

\[
= \Omega \cap (I_E \cup \partial^* E) \cap (I_F \cup \partial^* F)
\]

\[
= \Omega \cap \partial^* E \cap \partial^* F \quad \text{since } E \cap F = \emptyset.
\]

If \( \text{cap}_1(I_E, I_F^c, \Omega) < \infty \), then by Proposition 4.2 and (2.5) we know that \( \mathcal{H}(\Omega \cap \overline{I_E}^\infty \setminus \text{fine-int } I_F^c) = 0 \). Then also \( \mathcal{H}(\Omega \cap (I_E \cup \partial^* E) \setminus \text{fine-int } I_F^c) = 0 \) by (2.22), and so \( \mathcal{H}(\Omega \cap \partial^* E \cap \partial^* F) = 0 \).

Conversely, if \( \mathcal{H}(\partial^* E \cap \partial^* F \cap \Omega) = 0 \), then \( \mathcal{H}(\Omega \cap (I_E \cup \partial^* E) \setminus \text{fine-int } I_F^c) = 0 \) and so \( \text{Cap}_1(\Omega \cap (I_E \cup \partial^* E) \setminus \text{fine-int } I_F^c) = 0 \) by (2.5). Let \( \varepsilon > 0 \). Since we also have \( \mathcal{H}(\partial^* E \cap \Omega) < \infty \) by (2.11), we can apply Theorem 4.16 with the sets \( E, D = I_F^c \), and \( U = \Omega \), to find a function \( \rho \in N_0^{1,1}(I_F^c, \Omega) \) such that \( \rho = 1 \) \( \mu \)-a.e. in \( (I_E \cup \partial^* E) \cap \Omega \) and

\[
\int_{\Omega} g_\rho \, d\mu < C_d \mathcal{H}(\partial^* E \cap \Omega) + \varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, using also (2.11) we now obtain

\[
\text{cap}_1(I_E, I_F^c, \Omega) \leq C_d \mathcal{H}(\partial^* E \cap \Omega) \leq \alpha^{-1} C_d P(E, \Omega).
\]

By the exactly analogous reasoning, we get \( \text{cap}_1(I_F, I_E^c, \Omega) \leq \alpha^{-1} C_d P(F, \Omega) \), and since clearly \( \text{cap}_1(I_F, I_E^c, \Omega) = \text{cap}_1(I_E, I_F^c, \Omega) \), we get the conclusion

\[
\text{cap}_1(I_E, I_F^c, \Omega) \leq C \min\{P(E, \Omega), P(F, \Omega)\} < \infty
\]

with \( C = \alpha^{-1} C_d \).

It is perhaps interesting that the quantity \( \text{cap}_1(I_E, I_F^c, \Omega) \) can never take a large finite value; it is either at most of the order \( \min\{P(E, \Omega), P(F, \Omega)\} \), or else it is infinite. The analogous \( p \)-capacity for \( p > 1 \) typically becomes arbitrarily large as the sets \( E \) and \( F \) get closer to each other.

Now we define two different BV-versions of the variational 1-capacity. Recall the definitions of the approximate limits \( u^\wedge \) and \( u^\vee \) from (2.15) and (2.16); by Lebesgue’s differentiation theorem, \( u = u^\wedge = u^\vee \) a.e. for any locally integrable function \( u \). In the case \( u = \chi_E \) with \( E \subset X \), we have \( x \in I_E \) if and only if \( u^\wedge(x) = u^\vee(x) = 1 \), \( x \in O_E \) if and only if \( u^\wedge(x) = u^\vee(x) = 0 \), and \( x \in \partial^* E \) if and only if \( u^\wedge(x) = 0 \) and \( u^\vee(x) = 1 \).

**Definition 5.4.** Let \( A \subset D \) and let \( U \subset X \) be \( \mu \)-measurable. We define the variational BV-capacity by

\[
\text{cap}_{\text{BV}}(A, D, U) := \inf \|Du\|(U),
\]

where the infimum is taken over functions \( u \in L^1(U) \) such that \( u^\wedge = u^\vee = 0 \) \( \mathcal{H} \)-a.e. in \( U \setminus D \) and \( u^\wedge \geq 1 \) \( \mathcal{H} \)-a.e. in \( A \cap U \).
We define an alternative version of the variational BV-capacity by
\[
\text{cap}^{\lor}_{\text{BV}}(A, D, U) := \inf \|Du\|(U),
\]
where the infimum is taken over functions \(u \in L^1(U)\) such that \(u^\wedge = u^\lor = 0\) \(\mathcal{H}\text{-a.e.}\) in \(U \setminus D\) and \(u^\lor \geq 1\) \(\mathcal{H}\text{-a.e.}\) in \(A \cap U\). If \(U = X\), we omit it from the notation.

By truncation we see that it is enough to consider test functions \(0 \leq u \leq 1\).

Note that the condition \(\|Du\|(U) < \infty\) implicitly means that \(\|Du\|(\Omega) < \infty\) for some open \(\Omega \supset U\). It is obvious that always \(\text{cap}^{\lor}_{\text{BV}}(A, D, U) \leq \text{cap}_{\text{BV}}(A, D, U)\), and in [32, Eq. (4.2)] it was noted that also
\[
\text{cap}_{\text{BV}}(A, D, U) \leq \text{cap}^1_{\text{BV}}(A, D, U)
\]
whenever \(U\) is open.

In [12] it was shown, with rather different methods compared to ours and with slightly different definitions, that in Euclidean spaces one has \(\text{cap}^{\lor}_{\text{BV}}(A, \mathbb{R}^n) = \text{cap}_{\text{BV}}(A, \mathbb{R}^n)\) for every \(A \subset \mathbb{R}^n\). A definition similar to \(\text{cap}^{\lor}_{\text{BV}}(A, D)\) was also studied (in metric spaces) in [20], in the case where \(A\) is a compact subset of an open set \(D\); it follows from [20, Theorem 4.5, Theorem 4.6] that
\[
\text{cap}_{\text{BV}}(A, D) \leq C \text{cap}^{\lor}_{\text{BV}}(A, D).
\]
The constant \(C \geq 1\) is indeed necessary in the metric space setting, see [20, Example 4.4]. It is of interest to study capacities for more general sets; in [32] it was shown that
\[
\text{cap}_{\text{BV}}(A, D) = \text{cap}^1_{\text{BV}}(A, D)
\]
when \(A \subset D\) and both \(D\) and \(X \setminus A\) are 1-quasiopen, and this was used to prove an approximation result for BV functions. Now we wish to complement these results with the following theorem on the capacity \(\text{cap}^{\lor}_{\text{BV}}\); note that \(A\) is now a completely general set and so the theorem greatly strengthens (5.6).

**Theorem 5.7.** Let \(A \subset V\) and \(U \subset X\) such that \(V\) and \(U\) are 1-quasiopen. Then
\[
\text{cap}^1_{\text{BV}}(A, V, U) \leq C \text{cap}^{\lor}_{\text{BV}}(A, V, U)
\]
for a constant \(C = C(C_d, C_P, \lambda) \geq 1\). In particular, if \(U\) is open,
\[
\text{cap}_{\text{BV}}(A, V, U) \leq C \text{cap}^{\lor}_{\text{BV}}(A, V, U).
\]

**Proof.** We can assume that \(\text{cap}^{\lor}_{\text{BV}}(A, V, U) < \infty\). Let \(\varepsilon > 0\). Pick an open set \(U_\varepsilon \supset U\) (we can assume that \(\mu(U_\varepsilon \setminus U) < \infty\)) and a function \(u \in L^1(U_\varepsilon)\) such that \(0 \leq u \leq 1\) in \(U_\varepsilon\), \(u^\wedge = u^\lor = 0\) \(\mathcal{H}\text{-a.e.}\) in \(U_\varepsilon \setminus V\), \(u^\lor = 1\) \(\mathcal{H}\text{-a.e.}\) in \(A \cap U_\varepsilon\), and
\[
\|Du\|(U_\varepsilon) < \text{cap}^{\lor}_{\text{BV}}(A, V, U) + \varepsilon.
\]
By the coarea formula (2.12), we find \(t \in (0, 1)\) such that \(E := \{u > t\}\) satisfies \(P(E, U_\varepsilon) \leq \|Du\|(U_\varepsilon)\). Then
\[
\mathcal{H}(A \cap U \setminus (I_E \cup \partial^* E)) \leq \mathcal{H}(A \cap U \setminus \{u^\lor = 1\}) = 0
\]
(5.8)
and similarly
\[ \mathcal{H}(U \cap (I_E \cup \partial^* E) \setminus V) \leq \mathcal{H}(U \cap \{u^+ > 0\} \setminus V) = 0. \]

Now by Theorem 2.24 and (2.5),
\[ \mathcal{H}(U \cap (I_E \cup \partial^* E) \setminus \text{fine-int } V) = \mathcal{H}(U \cap (I_E \cup \partial^* E) \setminus V) = 0. \]

Moreover, by (2.11), \( \mathcal{H}(\partial^* E \cap U_\epsilon) < \infty \). Now by Theorem 4.16 we find a function \( \rho \in D_0^1(V, U) \) such that \( \rho = 1 \) 1-q.e. in \( (I_E \cup \partial^* E) \cap U \), \( \|\rho - \chi_E\|_{L^1(U)} < \epsilon \) (and then in fact \( \rho \in N_{0,1}^1(V, U) \)), and \( \int_U g_\rho \, d\mu < C_d \mathcal{H}(\partial^* E \cap U) + \epsilon \). Then by (2.11),
\[ \int_U g_\rho \, d\mu < C_d \mathcal{H}(\partial^* E \cap U) + \epsilon \leq C_d \mathcal{H}(\partial^* E \cap U_\epsilon) + \epsilon \leq C_d \alpha^{-1} P(E, U_\epsilon) + \epsilon. \]

Since \( \rho = 1 \) 1-q.e. in \( (I_E \cup \partial^* E) \cap U \), also \( \rho = 1 \) 1-q.e. in \( A \cap U \) by (5.8) and (2.5). Thus
\[ \text{cap}_1(A, V, U) \leq \int_U g_\rho \, d\mu < C_d \alpha^{-1} P(E, U_\epsilon) + \epsilon \leq C_d \alpha^{-1}(\text{cap}_1^A(V, U) + \epsilon) + \epsilon. \]

Letting \( \epsilon \to 0 \), we get the first claim. The second claim then follows from (5.5). \( \square \)

Even though \( A \) is allowed to be an arbitrary set, the assumption that \( V \) is 1-quasiopen cannot be removed, as demonstrated by the following example.

**Example 5.9.** Let \( X = \mathbb{R} \) (unweighted) and let \( A = V = [0,1] \) and \( U = \mathbb{R} \). Then
\[ \text{cap}_1^A(V, U) \leq \|D\chi_{[0,1]}\|(\mathbb{R}) = 2, \]
but \( \text{cap}_{BV}(A, V, U) = \infty \) since there are no admissible functions.

### 6 An approximation result for BV functions

In this section we apply our theory of 1-strict subsets to prove a pointwise approximation result for BV functions, given in Theorem 1.2 in the introduction.

**Lemma 6.1.** Let \( S_1, \ldots, S_n \subset X \) be pairwise disjoint Borel sets that are of finite \( \mathcal{H} \)-measure. Then there exist pairwise disjoint 1-quasiopen sets \( U_j \supset S_j \), \( j = 1, \ldots, n \).

**Proof.** By Lemma 3.13 the set \( X \setminus \bigcup_{k=2}^n S_k \) is 1-quasiopen, and contains \( S_1 \). Thus by Proposition 4.9 we find a function \( \eta_1 \in N_{0,1}^1(X \setminus \bigcup_{k=2}^n S_k) \) with \( \eta_1 = 1 \) in \( S_1 \). By the quasicontinuity of Newton-Sobolev functions, it is straightforward to check that \( \{\eta_1 > 1/2\} \) and \( X \setminus \{\eta_1 \geq 1/2\} \) are 1-quasiopen sets (see e.g. [10, Proposition 3.4]).

We can do the same for each set \( S_1, \ldots, S_n \). Then define for each \( j = 1, \ldots, n \)
\[ U_j := \{\eta_j > 1/2\} \setminus \bigcup_{k=1}^n \{\eta_k \geq 1/2\}. \]

Now each set \( U_j \) contains \( S_j \) and is a 1-quasiopen set by the fact that every finite intersection of 1-quasiopen sets is 1-quasiopen (see e.g. [16, Lemma 2.3]). \( \square \)
Next we prove the following Leibniz rule.

**Lemma 6.2.** Let $\Omega \subset X$ be open and let $U_1, \ldots, U_n \subset \Omega$ be pairwise disjoint 1-quasiopen sets. For each $j = 1, \ldots, n$ let $\eta_j \in N_{1}^{1,1}(U_j)$ with $0 \leq \eta_j \leq 1$ on $X$, $\eta_j = 0$ in a 1-quasiopen set containing $X \setminus U_j$, and such that there is a 1-quasiopen set $V_j \subset \{ \eta_j = 1 \}$, $j = 1, \ldots, n$. Let $U_0$ be another 1-quasiopen set with $U_0 \cup V_1 \cup \ldots \cup V_n = \Omega$ and let $\eta := \sum_{j=1}^{n} \eta_j$, and finally suppose that $v \in N_{1}^{1,1}(U_0)$ and $\rho_j \in N_{1}^{1,1}(U_j)$ for each $j = 1, \ldots, n$. Then

$$w := \sum_{j=1}^{n} \eta_j \rho_j + (1 - \eta)v$$

has a 1-weak upper gradient (in $\Omega$)

$$g := \sum_{j=1}^{n} \eta_j g_{\rho_j} + (1 - \eta)g_v + \sum_{j=1}^{n} g_{\eta_j} |\rho_j - v|,$$

where each $g_{\rho_j}$ is the minimal 1-weak upper gradient of $\rho_j$ in $U_j$, and $g_v$ is the minimal 1-weak upper gradient of $v$ in $U_0$.

Note that $g_{\eta_j} = 0$ outside $U_j \cap U_0$ by (3.1), and so the function $g$ is well defined in the whole of $\Omega$.

**Proof.** Using the fact that $\eta_j = 0$ in a 1-quasiopen set containing $X \setminus U_j$, and the fact that finite intersections of 1-quasiopen sets are 1-quasiopen, we find a 1-quasiopen set $V \subset \Omega$ containing $\Omega \setminus \bigcup_{j=1}^{n} U_j$ but not intersecting any of the sets $\{ \eta_j > 0 \}$. By [42, Remark 3.5] we know that $V$ is 1-path open, meaning that for 1-a.e. curve $\gamma$, the set $\gamma^{-1}(V)$ is a relatively open subset of $[0, \ell_{\gamma}]$. The same holds for each of the sets $U_0 \cap U_j$ and $V_j$. Let $\gamma$ be a curve such that this property for preimages holds for all subcurves of $\gamma$. In the set $V$ we know that $g_v = g$ is 1-weak upper gradient of $v = w$. In each $V_j$, the function $g_{\rho_j} = g$ is a 1-weak upper gradient of $\rho_j = w$. Finally, by the Leibniz rule given in [5, Lemma 2.18], in each set $U_j \cap U_0$ the function

$$\eta_j g_{\rho_j} + (1 - \eta_j)g_v + g_{\eta_j} |\rho_j - v| = \eta_j g_{\rho_j} + (1 - \eta)g_v + g_{\eta_j} |\rho_j - v| = g$$

is a 1-weak upper gradient of $\eta_j u + (1 - \eta_j)v = w$. Assume further that the pair $(w, g)$ satisfies the upper gradient inequality on each subcurve of $\gamma$ lying either in $V$, in one of the sets $U_j \cap U_0$, or in one of the sets $V_j$. By Lemma 4.15 these properties are satisfied by 1-a.e. curve $\gamma$.

Note that we can write the entire $\Omega$ as the union of 1-quasiopen sets

$$\Omega = V \cup \bigcup_{j=1}^{n} U_j = V \cup \bigcup_{j=1}^{n} (U_0 \cap U_j) \cup V_j.$$

Since $[0, \ell_{\gamma}]$ is a compact set, the curve $\gamma$ can be broken into a finite number of subcurves each of which lies either in $V$, or in one of the sets $U_0 \cap U_j$, or in $V_j$. Summing up the subcurves, we find that the pair $(w, g)$ satisfies the upper gradient inequality on $\gamma$, and thus $g$ is a 1-weak upper gradient of $w$ in $\Omega$. □
Proposition 6.3 ([33, Proposition 3.6]). Let \( U \subset \Omega \subset X \) be such that \( U \) is 1-quasiopen and \( \Omega \) is open, and let \( u \in BV(\Omega) \) and \( \beta > 0 \) such that \( u^v - u^w < \beta \) in \( U \). Then there exists a sequence \((u_i) \subset N^{1,1}(U)\) such that \( u_i \to u \) in \( L^1(U) \), sup\( |u_i - u^v| \leq 9\beta \) for all \( i \in \mathbb{N} \), and

\[
\lim_{i \to \infty} \int_U g_{u_i} \, d\mu = \|Du\|(U),
\]

where each \( g_{u_i} \) is the minimal 1-weak upper gradient of \( u_i \) in \( U \).

**Proof of Theorem 1.2.** We begin by decomposing the jump set \( S_u = \{u^w < u^v\} \) into the pairwise disjoint sets \( S_1 := \{u^v - u^w \geq 1\} \) and \( S_j := \{1/j \leq u^v - u^w < 1/(j-1)\} \) for \( j = 2, 3, \ldots \) (all understood to be subsets of \( \Omega \)). By the decomposition (2.18), we have \( \mathcal{H}(S_j) < \infty \) for every \( j \in \mathbb{N} \). Applying Lemma 3.14, for each \( j \in \mathbb{N} \) we find open sets \( W_{ji} \supset W_{j2} \supset \cdots \supset S_j \) such that

\[
\text{Cap}_1 \left( \bigcap_{i=1}^\infty W_{ji} \setminus S_j \right) = 0. \tag{6.4}
\]

We can also assume that these are subsets of \( \Omega \) and that \( \|Du\|(W_{ji}) < \|Du\|(S_j) + 1/i^2 \). By Lemma 6.1, for each \( i \in \mathbb{N} \) and \( j \leq i \) we find 1-quasiopen sets \( V_{ji} \supset S_j \) such that \( V_{1i}, \ldots, V_{ii} \) are pairwise disjoint. Moreover, note that the sets \( \{u^v - u^w < 1/(j-1)\} \) are 1-quasiopen by Proposition 2.17. Thus the sets \( U_{ji} := W_{ji} \cap V_{ji} \cap \{u^v - u^w < 1/(j-1)\} \) are 1-quasiopen. Now for each \( i \in \mathbb{N} \), \( U_{1i}, \ldots, U_{ii} \) are pairwise disjoint sets with

\[
\|Du\|(U_{ji}) < \|Du\|(S_j) + \frac{1}{i^2} \tag{6.5}
\]

for all \( j = 1, \ldots, i \).

Using Proposition 4.9, take functions \( \eta_{ji} \in N^{1,1}_0(U_{ji}) \) such that \( 0 \leq \eta_{ji} \leq 1 \) on \( X \), \( \eta_{ji} = 1 \) in a 1-quasiopen set containing \( S_j \), and \( \eta_{ji} = 0 \) in a 1-quasiopen set containing \( X \setminus U_{ji} \).

Let \( u_i := \max\{-i, u\} \) for each \( i \in \mathbb{N} \). By Lemma 3.13, each \( \Omega \setminus (S_1 \cup \ldots \cup S_i) \) is a 1-quasiopen set in which \( u_i^v - u_i^w \leq u^v - u^w < 1/i \). Thus by Proposition 6.3 we find a function \( v_i \in N^{1,1}(\Omega \setminus (S_1 \cup \ldots \cup S_i)) \) such that

\[
|v_i - u_i^v| \leq 9/i \quad \text{in} \quad \Omega \setminus (S_1 \cup \ldots \cup S_i) \quad \text{and} \quad \|v_i - u_i\|_{L^1(\Omega \setminus (S_1 \cup \ldots \cup S_i))} < 1/i \tag{6.6}
\]

and

\[
\int_{\Omega \setminus (S_1 \cup \ldots \cup S_i)} g_{v_i} \, d\mu < \|Du\|(\Omega \setminus (S_1 \cup \ldots \cup S_i)) + \frac{1}{i}, \tag{6.7}
\]

where \( g_{v_i} \) is the minimal 1-weak upper gradient of \( v_i \) in \( \Omega \setminus (S_1 \cup \ldots \cup S_i) \). It is easy to show that \( \mu(S_1 \cup \ldots \cup S_n) = 0 \) since the \( S_j \)'s are sets of finite \( \mathcal{H} \)-measure (see e.g. [27, Lemma 6.1]). Since \( L^1 \)-convergence implies pointwise convergence a.e. for a subsequence, by Lebesgue’s dominated convergence theorem (using (6.6)) we can also assume that

\[
\int_{\Omega} g_{\eta_{ji}}|v_i - u_i| \, d\mu < \frac{1}{i^2} \tag{6.8}
\]
for each \( j = 1, \ldots, i \). Also, for each \( i \in \mathbb{N} \) let
\[
\alpha_i := \frac{1}{t^2} \min \left\{ 1, \left( \int_X g_{\eta_i} \, d\mu \right)^{-1}, \ldots, \left( \int_X g_{\eta_i} \, d\mu \right)^{-1} \right\}.
\]
Now fix \( i \in \mathbb{N} \) and \( j \in \{1, \ldots, i\} \). For all \( k \in \mathbb{N} \) pick
\[
\beta_{jik} \in ((k-1)\alpha_i, k\alpha_i)
\]
such that (in what follows, we work with the function \( u_i + i \) since it is nonnegative)
\[
P\{u_i + i > \beta_{jik}, \Omega\} < \infty
\]
and
\[
\alpha_i P\{u_i + i > \beta_{jik}, U_{ji}\} \leq \int_{(k-1)\alpha_i}^{k\alpha_i} P\{u_i + t > \beta_{jik}, U_{ji}\} \, dt; \quad (6.9)
\]
note that this choice is possible since \( P\{u_i + i > t, \Omega\} < \infty \) for a.e. \( t \in \mathbb{R} \) by the coarea formula \((2.12)\). Now we will apply Theorem 4.16 with the choices
\[
E = \{u_i + i > \beta_{jik}\}, \quad D = X \setminus \left( S_j \cap \{u_i + i > \beta_{jik}\} \right), \quad \text{and} \quad U = U_{ji}.
\]
Note that if \( x \in I_E \cup \partial^* E \), then \((u_i + i)^\vee(x) \geq \beta_{jik}\). Thus \( I_E \cup \partial^* E \subset D \). Also note that \( D \) is 1-quasiopen by Lemma 3.13, and so \( \mathcal{H}(D \setminus \text{fine-int } D) = 0 \) by Theorem 2.24. Thus
\[
\mathcal{H}(\{I_E \cup \partial^* E \setminus \text{fine-int } D) = 0
\]
as required in Theorem 4.16. Clearly also
\[
\{u_i + i)^\vee > \beta_{jik}\} \subset I_{\{u_i + i > \beta_{jik}\}} \cup \partial^* \{u_i + i > \beta_{jik}\}. \quad (6.10)
\]
Now Theorem 4.16 gives functions
\[
\rho_{jik} \in N_0^{1,1}(X \setminus (S_j \cap \{(u_i + i)^\vee < \beta_{jik}\}), U_{ji})
\]
with \( 0 \leq \rho_{jik} \leq 1 \) in \( U_{ji} \), \( \rho_{jik} = 1 \) 1-q.e. in \( \{u_i + i > \beta_{jik}\} \cup \partial^* \{u_i + i > \beta_{jik}\}\cap U_{ji} \) (and by redefining, we can leave out the “1-q.e.”) and thus in \( \{u_i + i)\vee > \beta_{jik}\} \cap U_{ji} \) by \((6.10)\),
\[
\|\rho_{jik} - \chi_{\{u_i + i \geq \beta_{jik}\}}\|_{L^1(U_{ji})} < \frac{2^{-k}}{t^2}, \quad (6.11)
\]
and
\[
\int_{U_{ji}} g_{\rho_{jik}} \, d\mu < C_d \mathcal{H}(\partial^* \{u_i + i > \beta_{jik}\} \cap U_{ji}) + \frac{2^{-k}}{t^2}. \quad (6.12)
\]
Since we can choose the norm \( \|\rho_{jik} - \chi_{\{u_i + i \geq \beta_{jik}\}}\|_{L^1(U_{ji})} \) to be as small as we like and since \( L^1\)-convergence implies pointwise convergence for a subsequence, by Lebesgue’s dominated convergence theorem we can also assume that
\[
\int_{U_{ji}} g_{\eta_{ji}} |\rho_{jik} - \chi_{\{u_i + i \geq \beta_{jik}\}}| \, d\mu < \frac{2^{-k}}{t^2}. \quad (6.13)
\]
Then define two functions in the set $U_{ji}$ (both understood to be pointwise defined)

$$
\tilde{\rho}_{ji} := \alpha_i \sum_{k=1}^{\infty} \rho_{jik} - i \quad \text{and also} \quad \hat{\mu}_{ji} := \alpha_i \sum_{k=1}^{\infty} \chi_{\{u_i + i \cap U_j > \beta_{jik}\}} - i.
$$

Note that $\tilde{\rho}_{ji} \geq \hat{\mu}_{ji}$ and $\sup_{U_{ji}} |\tilde{\rho}_{ji} - u_i| \leq \alpha_i$, and so

$$
\tilde{\rho}_{ji} \geq u_i^\vee - \alpha_i \quad \text{in } U_{ji}.
$$

(6.14)

Moreover, since $\rho_{jik} = 0$ in $S_j \cap \{(u_i + i) \cap U_j > \beta_{jik}\}$, it follows that

$$
\tilde{\rho}_{ji} \leq (u_i + i) \vee + \alpha_i - i = \max\{u_i^\vee, -i\} + \alpha_i \quad \text{in } S_j.
$$

(6.15)

Additionally, by (6.11) and the fact that $\alpha_i \leq 1$,

$$
\|\tilde{\rho}_{ji} - \hat{\mu}_{ji}\|_{L^1(U_{ji})} \leq \sum_{k=1}^{\infty} \int_{U_{ji}} \alpha_i |\rho_{jik} - \chi_{\{u_i + i \cap U_j > \beta_{jik}\}}| \, d\mu < \frac{1}{i^2}
$$

and so

$$
\|\tilde{\rho}_{ji} - u_i\|_{L^1(U_{ji})} \leq \|\tilde{\rho}_{ji} - \hat{\mu}_{ji}\|_{L^1(U_{ji})} + \|\hat{\mu}_{ji} - u_i\|_{L^1(U_{ji})}
$$

$$
< \frac{1}{i^2} + \alpha_i \mu(\Omega) \leq \frac{1}{i^2}(1 + \mu(\Omega))
$$

(6.16)

(recall that we assume $\mu(\Omega) < \infty$).

Using Lemma 3.2 we get (note that $\alpha$ and $\alpha_i \leq 1$ denote different quantities)

$$
\int_{U_{ji}} g_{\tilde{\rho}_{ji}} \, d\mu \leq \alpha_i \sum_{k=1}^{\infty} \int_{U_{ji}} g_{\rho_{jik}} \, d\mu
$$

$$
< \alpha_i \sum_{k=1}^{\infty} \left( C_d \mathcal{H}(\partial^* \{u_i + i \cap U_{ji}\} \cap U_{ji}) + \frac{2^{-k}}{i^2} \right) \quad \text{by (6.12)}
$$

$$
\leq \alpha_i \sum_{k=1}^{\infty} \left( C_d \alpha^{-1} P(\{u_i + i \cap U_{ji}\}, U_{ji}) + \frac{2^{-k}}{i^2} \right) \quad \text{by (2.11)}
$$

$$
\leq \sum_{k=1}^{\infty} \left( C_d \alpha^{-1} \int_{(k-1) \alpha_i}^{k \alpha_i} P(\{u_i + i > t\}, U_{ji}) dt + \frac{2^{-k}}{i^2} \right) \quad \text{by (6.9)}
$$

$$
= C_d \alpha^{-1} \int_{-i}^{\infty} P(\{u_i > t\}, U_{ji}) dt + \frac{1}{i^2}
$$

$$
= C_d \alpha^{-1} \|Du_i\|_{L^1(U_{ji})} + \frac{1}{i^2}
$$

by the coarea formula (2.12). Also, by the fact that $\|\hat{\mu}_{ji} - u_i\|_{L^\infty(U_{ji})} \leq \alpha_i$ and the choice of $\alpha_i$, we have

$$
\int_{U_{ji}} g_{\hat{\mu}_{ji}} |\tilde{\rho}_{ji} - u_i| \, d\mu \leq \int_{U_{ji}} g_{\hat{\mu}_{ji}} |\tilde{\rho}_{ji} - \hat{\mu}_{ji}| \, d\mu + \int_{U_{ji}} g_{\hat{\mu}_{ji}} |\hat{\mu}_{ji} - u_i| \, d\mu
$$

$$
\leq \sum_{k=1}^{\infty} \int_{U_{ji}} g_{\hat{\mu}_{ji}} |\rho_{jik} - \chi_{\{u_i + i \cap U_j > \beta_{jik}\}}| \, d\mu + \frac{1}{i^2}
$$

(6.17)

$$
< \frac{2}{i^2} \quad \text{by (6.13)}.
$$
Since \( u_i^\vee - u_i^\wedge \leq u_i^\vee - u^\wedge < 1/(j-1) \) in \( U_{ji} \), by Proposition 6.3 we also find a function \( v_{ji} \in N^{1,1}(U_{ji}) \) such that

\[
    u_i^\vee \leq v_{ji} \leq u_i^\vee + 18/(j-1) \quad \text{in} \quad U_{ji} \quad \text{and} \quad \int_{U_{ji}} g_{v_{ji}} \, d\mu < \|Du_i\|(U_{ji}) + \frac{1}{i^2}.
\]

Let \( \rho_{ji} := \min\{\tilde{\rho}_{ji}, v_{ji}\} \). Then

\[
    \int_{U_{ji}} g_{\rho_{ji}} \, d\mu \leq \int_{U_{ji}} g_{\tilde{\rho}_{ji}} \, d\mu + \int_{U_{ji}} g_{v_{ji}} \, d\mu < (C_d \alpha^{-1} + 1)\|Du_i\|(U_{ji}) + \frac{2}{i^2}. \tag{6.18}
\]

Thus \( \rho_{ji} \in N^{1,1}(U_{ji}) \). Also,

\[
    \rho_{ji} \leq u_i^\vee + 18/(j-1) \quad \text{and} \quad \rho_{ji} \geq u_i^\vee - \alpha_i \quad \text{in} \quad U_{ji} \quad \text{by (6.14),} \tag{6.19}
\]

and by (6.16),

\[
    \|\rho_{ji} - u_i\|_{L^1(U_{ji})} \leq \|\tilde{\rho}_{ji} - u_i\|_{L^1(U_{ji})} \leq \frac{1}{i^2}(1 + \mu(\Omega)). \tag{6.20}
\]

Moreover, by (6.17) we have

\[
    \int_{U_{ji}} g_{\eta_{ji}} |\rho_{ji} - u_i| \, d\mu \leq \int_{U_{ji}} g_{\eta_{ji}} |\tilde{\rho}_{ji} - u_i| \, d\mu < \frac{2}{i^2}.
\]

Then using also (6.8), we get

\[
    \int_{U_{ji}} g_{\eta_{ji}} |\rho_{ji} - u_i| \, d\mu < \frac{3}{i^2}. \tag{6.21}
\]

Recall that so far we have kept \( i \in \mathbb{N} \) and \( j \in \{1, \ldots, i\} \) fixed. Now for each \( i \in \mathbb{N} \), let \( \eta_i := \max_{j \in \{1, \ldots, i\}} \eta_{ji} \). Define the functions

\[
    w_i := \sum_{j=1}^{i} \eta_{ji} \rho_{ji} + (1 - \eta_i) v_i + \frac{9}{i}, \quad i \in \mathbb{N}.
\]

Then by (6.20) and (6.6),

\[
    \|w_i - u_i\|_{L^1(\Omega)} \leq \sum_{j=1}^{i} \|\rho_{ji} - u_i\|_{L^1(U_{ji})} + \|v_i - u_i\|_{L^1(\Omega \setminus (S_1 \cup \cdots \cup S_i))} + \frac{9}{i} \mu(\Omega)
\]

\[
    \leq \frac{1}{i}(1 + \mu(\Omega)) + \frac{1}{i} \mu(\Omega) + \frac{9}{i} \mu(\Omega) = \frac{2}{i} + \frac{10}{i} \mu(\Omega).
\]

Clearly \( u_i \to u \) in \( L^1(\Omega) \) as \( i \to \infty \), and so \( w_i \to u \) in \( L^1(\Omega) \), as desired. We have by the Leibniz rule of Lemma 6.2, using also the fact that \( \|Du_i\| \leq \|Du\| \) as measures.
for each \( i \in \mathbb{N} \),
\[
\int_{\Omega} g_{w_i} \, d\mu \\
\leq \sum_{j=1}^{i} \int_{\Omega} \eta_{ji} g_{\rho_{ji}} \, d\mu + \int_{\Omega} (1 - \eta_i) g_{w_i} \, d\mu + \sum_{j=1}^{i} \int_{\Omega} g_{\rho_{ji}} |\rho_{ji} - v_i| \, d\mu \\
\leq \sum_{j=1}^{i} \int_{\Omega \setminus (S_1 \cup \ldots \cup S_i)} g_{\rho_{ji}} \, d\mu + \sum_{j=1}^{i} \int_{U_{ji}} g_{\rho_{ji}} |\rho_{ji} - v_i| \, d\mu \\
\leq \sum_{j=1}^{i} \left( (C_d \alpha^{-1} + 1) \|Du\|(U_{ji}) + \frac{2}{i^2} \right) \\
+ \|Du\|(\Omega \setminus (S_1 \cup \ldots \cup S_i)) + \frac{1}{i} + \frac{3}{i} \quad \text{by (6.18), (6.7), (6.21)} \\
\leq (C_d \alpha^{-1} + 1) \sum_{j=1}^{i} \left( \|Du\|(S_j) + \frac{3}{i^2} \right) + \|Du\|(\Omega \setminus (S_1 \cup \ldots \cup S_i)) + \frac{4}{i} \quad \text{by (6.5)} \\
\leq \|Du\|(\Omega) + C_d \alpha^{-1} \|Du\|(S_u) + \frac{7(C_d \alpha^{-1} + 1)}{i}. 
\]
Thus recalling the decomposition of the variation measure (2.18),
\[
\limsup_{i \to \infty} \int_{\Omega} g_{w_i} \, d\mu \leq \|Du\|(\Omega) + C_d \alpha^{-1} \|Du\|(S_u) = \|Du\|(\Omega) + C_d \alpha^{-1} \|Du\|^2(\Omega),
\]
as desired.
Since we had \(|v_i - u_i^\nu| \leq 9/i\) in \(\Omega \setminus (S_1 \cup \ldots \cup S_i)\) (recall (6.6)) and \(\rho_{ji} \geq u_i^\nu - \alpha_i \geq u_i^\nu - 1/i\) in \(U_{ji}\) by (6.19), it follows that \(w_i \geq u_i^\nu \geq u^\nu\) in \(\Omega\), as desired. Moreover, since by (6.15) we have
\[
\rho_{ji} \leq \tilde{\rho}_{ji} \leq \max\{u^\nu, -i\} + \alpha_i \quad \text{in} \ S_j,
\]
then also
\[
w_i = \rho_{ji} + \frac{9}{i} \leq \max\{u^\nu, -i\} + \alpha_i + \frac{9}{i} \leq \max\{u^\nu, -i\} + \frac{10}{i}
\]
in \(S_j\), for \(j = 1, \ldots, i\). Since \(S_u = \bigcup_{j=1}^{\infty} S_j\), we get \(w_i \to u^\nu\) in \(S_u\).
Finally consider \(x \in \Omega \setminus S_u\). Fix \(\varepsilon > 0\). Recall from (6.4) that we had \(U_{ji} \subset W_{ji}\) with \(W_{j1} \supset W_{j2} \supset \ldots \supset S_j\) such that
\[
\operatorname{Cap}_1 \left( \bigcap_{i=1}^{\infty} W_{ji} \setminus S_j \right) = 0.
\]
Denote these \(\operatorname{Cap}_1\)-negligible sets by \(H_j\), and \(H := \bigcup_{j=1}^{\infty} H_j\). Then assume that \(x \in \Omega \setminus (S_u \cup H)\).
Take \(M \in \mathbb{N}\) such that \(18/M < \varepsilon\). Since \(x \notin H\), for some \(N \in \mathbb{N}\) we have \(x \notin U_{1i} \cup \ldots \cup U_{Mi}\) for all \(i \geq N\). Now if for a given \(i \geq N\) we have \(x \in \Omega \setminus \bigcup_{j=1}^{i} U_{ji}\),
then \( w_i(x) = v_i(x) + 9/i \leq u_i^\vee(x) + 18/i \). If \( x \in \bigcup_{j=1}^j U_{ji} \), then \( x \in U_{ji} \) for some \( j > M \) and so by (6.19), \( \rho_{ji}(x) \leq u_i^\vee(x) + 18/(j - 1) < u_i^\vee(x) + \varepsilon \). Hence using (6.6) once more,
\[
w_i(x) = \eta_{ji}(x)\rho_{ji}(x) + (1 - \eta_{ji}(x))v_i(x) + 9/i \leq u_i^\vee(x) + \varepsilon + 18/i
\]
for all \( i \geq N \). Since \( \varepsilon > 0 \) was arbitrary, we get \( w_i(x) \to u^\vee(x) \). Thus we have the desired pointwise convergence 1-q.e., and then in fact we obtain it at every point by redefining the functions \( w_i \); recall (2.4).

In the next example we show that the term \( C_a \|Du\|^j(\Omega) \) in Theorem 1.2 is necessary.

**Example 6.22.** Let \( X = \mathbb{R} \) equipped with the Euclidean metric and the weighted Lebesgue measure \( d\mu := w \, d\mathcal{L}^1 \), with \( w = 1 \) in \([-1, 1]\) and \( w = 2 \) in \( \mathbb{R} \setminus [-1, 1] \). Clearly this measure is doubling and the space supports a \((1, 1)\)-Poincaré inequality. Let \( u := \chi_{[-1, 1]} \in BV(X) \). Let \( w \in \mathcal{N}^{1,1}(X) \) with \( w \geq u^\vee = \chi_{[-1, 1]} \) everywhere, and let \( g \) be an upper gradient of \( w \). Let \( \varepsilon > 0 \). For some \( x < -1 \) we have
\[
1 - \varepsilon < |w(x) - w(-1)| \leq \int_x^{-1} g \, ds.
\]
Similarly for some \( y > 1 \) we have
\[
1 - \varepsilon < |w(1) - w(y)| \leq \int_1^{y} g \, ds.
\]
Thus
\[
\int_X g \, d\mu \geq \int_x^{-1} g \, d\mu + \int_1^{y} g \, d\mu = 2 \int_x^{-1} g \, ds + 2 \int_1^{y} g \, ds > 4 - 4\varepsilon.
\]
Since \( \varepsilon > 0 \) was arbitrary, we get \( \|g\|_{L^1(X)} \geq 4 \). Then the minimal 1-weak upper gradient of \( u \) also satisfies \( \|g_u\|_{L^1(X)} \geq 4 \) (see e.g. [5, Lemma 1.46]). However, defining the Lipschitz functions \( u_i(x) := \min\{1, \max\{0, i - i|x|\}\} \), we have \( u_i \to u \) in \( L^1(X) \) and then
\[
\|Du\|(X) \leq \liminf_{i \to \infty} \int_X g_{u_i} \, d\mu = \liminf_{i \to \infty} \int_{[-1,1]} i\chi_{[-1,1]\setminus[-i-1/i,1-i/i]} \, d\mathcal{L}^1 = 2.
\]
This shows that the term \( C_a \|Du\|^j(\Omega) \) in Theorem 1.2 is necessary. Letting \( E := [-1, 1] \), this reasoning also shows that the constant \( C_a \) in Theorem 1.1 is necessary.

**Remark 6.23.** Recall that in the Euclidean setting, the term \( C_a \|Du\|^j(\Omega) \) is not needed (see [13, Proposition 7.3]). Having \( \lim_{i \to \infty} \int_{\Omega} g_{u_i} \, d\mu = \|Du\|(\Omega) \) is in fact used in [13] to prove the pointwise convergence, whereas in our setting it seems necessary to construct the approximations “by hand”, which makes the proof of Theorem 1.2 rather technically involved.

The assumption \( \mu(\Omega) < \infty \) in Theorem 1.2 is not necessary; it could be removed by using cutoff functions in a very similar way to [32, Lemma 3.2]. We refrain from repeating this rather technical argument here.
Lemma 6.24. Let $\Omega \subset X$ be open and let $u_i \to u$ in $N^{1,1}(\Omega)$. Then for every $\varepsilon > 0$ and every open set $\Omega' \Subset \Omega$ there exists an open set $G \subset \Omega$ such that $\text{Cap}_1(G) < \varepsilon$ and $u_i \to u$ uniformly in $\Omega' \setminus G$.

Proof. Let $\varepsilon > 0$ and let $\Omega' \Subset \Omega$ be open. Take $\eta \in \text{Lip}_c(\Omega)$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ in $\Omega'$. It is easy to check that $\eta u_i \to \eta u$ in $N^{1,1}(X)$, and then according to [5, Corollary 1.72], there exists an open set $G \subset X$ such that $\text{Cap}_1(G) < \varepsilon$ and $\eta u_i \to \eta u$ uniformly in $X \setminus G$. Since $\eta u_i = u_i$ and $\eta u = u$ in $\Omega'$, we have the result. \hfill \Box

In [28, Proposition 4.1] it is shown that for any $u \in BV(X)$, there exists a sequence $(v_i) \subset \text{Lip}_{\text{loc}}(X)$ such that $v_i \to u$ in $L^1(X)$,

$$\limsup_{i \to \infty} \int_{\Omega} g_{v_i} \, d\mu \leq C\|Du\|(\Omega),$$

and

$$(1 - \gamma)u^\wedge(x) + \gamma u^\vee(x) \leq \liminf_{i \to \infty} v_i(x) \leq \limsup_{i \to \infty} v_i(x) \leq \gamma u^\wedge(x) + (1 - \gamma)u^\vee(x)$$

for $\mathcal{H}$-a.e. $x \in X$. Here $C$ and $0 < \gamma \leq 1/2$ are constants that depend only on the doubling constant of the measure and the constants in the Poincaré inequality. The functions $v_i$ can be taken to be discrete convolution approximations of $u$; this is a natural approximation method but a drawback is that in the jump set one does not obtain pointwise convergence, but rather just the lower and upper bounds given above.

We can now give a similar result where we do obtain pointwise convergence $\mathcal{H}$-almost everywhere also in the jump set.

Corollary 6.25. Let $\Omega \subset X$ be open with $\mu(\Omega) < \infty$ and let $u \in BV(\Omega)$. Then there exists a sequence $(v_i) \subset \text{Lip}_{\text{loc}}(\Omega)$ such that $v_i \to u$ in $L^1(\Omega)$,

$$\limsup_{i \to \infty} \int_{\Omega} g_{v_i} \, d\mu \leq \|Du\|(\Omega) + C_a\|Du\|^2(\Omega),$$

and $v_i(x) \to u^\vee(x)$ for $\mathcal{H}$-a.e. $x \in \Omega$.

Here the constant $C_a$ is the same as in Theorem 1.2. The analogous fact naturally holds for $u^\wedge$ replaced by $u^\wedge$.

Proof. By Theorem 1.2 we find a sequence $(w_i) \subset N^{1,1}(\Omega)$ such that $w_i \to u$ in $L^1(\Omega)$,

$$\limsup_{i \to \infty} \int_{\Omega} g_{w_i} \, d\mu \leq \|Du\|(\Omega) + C_a\|Du\|^2(\Omega),$$

and $w_i(x) \to u^\vee(x)$ for every $x \in \Omega$. By [5, Theorem 5.47], for each $i \in \mathbb{N}$ we find $v_i \in \text{Lip}_{\text{loc}}(\Omega)$ such that $\|v_i - w_i\|_{N^{1,1}(\Omega)} < 1/i$. Thus $v_i \to u$ in $L^1(\Omega)$ and

$$\limsup_{i \to \infty} \int_{\Omega} g_{v_i} \, d\mu = \limsup_{i \to \infty} \int_{\Omega} g_{w_i} \, d\mu \leq \|Du\|(\Omega) + C_a\|Du\|^2(\Omega).$$
Take open sets $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega$ with $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$. By Lemma 6.24 we can assume that for each $i \in \mathbb{N}$ there is a set $G_i \subset \Omega$ such that $\text{Cap}_1(G_i) < 2^{-i}$ and $|v_i - w_i| < 1/i$ in $\Omega_i \setminus G_i$. Let $\varepsilon > 0$. For sufficiently large $N \in \mathbb{N}$ we have $\text{Cap}_1\left(\bigcup_{j=N}^{\infty} G_j\right) < \varepsilon$, and for every $x \in \Omega \setminus \bigcup_{j=N}^{\infty} G_j$ we have for all $i \geq N$ large enough that $x \in \Omega_i$,

$$|v_i(x) - \bar{u}(x)| \leq |v_i(x) - w_i(x)| + |w_i(x) - \bar{u}(x)| < 1/i + |w_i(x) - \bar{u}(x)| \to 0$$

as $i \to \infty$. Since $\varepsilon > 0$ was arbitrary, we have $v_i(x) \to \bar{u}(x)$ for 1-q.e. $x \in \Omega$ and then by (2.5), $\mathcal{H}$-a.e. $x \in \Omega$. 

\[\square\]

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