Existence result under flatness condition for a nonlinear elliptic equation with Sobolev exponent

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Abstract. In this paper, we consider the following nonlinear elliptic equation with Dirichlet boundary condition: $-\Delta u = K(x)u^{\frac{n+2}{n-2}}$, $u > 0$ in $\Omega$, $u = 0$ on $\partial\Omega$, where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 4$, and $K$ is a $C^1$-positive function in $\bar{\Omega}$. Under the assumption that the order of flatness at each critical point of $K$ is $\beta \in [n-2, n]$, we give precise estimates on the losses of the compactness, and we prove an existence result through an Euler-Hopf type formula.

Key words : Elliptic equation, critical Sobolev exponent, loss of compactness, variational method, critical points at infinity
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1 Introduction and main result

In this work, we look for solution for the following nonlinear problem under the Dirichlet boundary condition

$$\begin{cases} -\Delta u = K(x)u^{\frac{n+2}{n-2}} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded smooth domain of $\mathbb{R}^n$, $n \geq 4$, and $K$ is a $C^1$-positive function in $\bar{\Omega}$.

One motivation to study this equation comes from its resemblance to the prescribed scalar curvature problem in conformal geometry, which consists on finding suitable conditions on a given function $K$ defined on $M$ to be the scalar curvature of a metric $\tilde{g}$ conformally equivalent to $g$, where $(M, g)$ is an $n$-dimensional Riemannian manifold without boundary. The special nature of the problem (1.1) appears when we consider it from the variational viewpoint. Indeed, although this problem enjoys a variational structure in the sense that its solutions can be interpreted as critical points of some functional, its associated Euler-Lagrange functional does not satisfy the Palais-Smale condition. This means that there exist noncompact sequences along which the functional is bounded and its gradient goes to zero. This is due to the non compactness of the embedding $H^1_0(\Omega)$ into $L^{\frac{2n}{n-2}}(\Omega)$.

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In the case of manifolds without boundary, this problem has been widely studied in various works, see, for example, the monograph [1] and the references therein. In contrast to the extensive literature regarding the prescribed scalar curvature problem on manifolds without boundary, in particular on spheres, there are few known results of (1.1); see, for example, [13] and [7] for $n = 4$, [12] for $n > 4$, $\Omega$ is a ball and $K = K(\|x\|)$, [7] for $n \geq 4$.

One group of existence results have been obtained under hypotheses involving $\Delta K$ at the critical points $y$ of $K$. For example, in [7] it is assumed that $K$ is a Morse function and $\Delta K(y) \neq 0$, $\forall y \in K$, $\forall y \in \{x \in \Omega / \nabla K(x) = 0\}$.

(1.2)

Under the condition (1.2), an Euler-Hopf criterion for $K$ was provided to find solution for the problem (1.1). In [7] we were explained that this global criterion is not satisfied for higher dimensional case $n \geq 5$. Naturally one may ask a similar question for all dimensions $n \geq 4$, namely, which function $K(x)$ on $\Omega$ arise a solution for the problem (1.1) under a global Euler-Hopf criterion?

In order to state our main result, we need to introduce some notations, and state the assumptions that we are using in our paper. We denote by $G$ the Green’s function and by $H$ its regular part, that is for each $x \in \Omega$,

$$
\begin{align*}
G(x, y) &= |x - y|^{-(n-2)} - H(x, y) \quad \text{in } \Omega \\
\Delta H(x, .) &= 0 \quad \text{in } \Omega \\
G(x, .) &= 0 \quad \text{on } \partial \Omega.
\end{align*}
$$

(1.3)

Let $K : \Omega \to \mathbb{R}$ be a $C^1$ positive function.

(A1) Assume that, for each $x \in \partial \Omega$, we have $\frac{\partial K(x)}{\partial \nu} < 0$, where $\nu$ is the outward normal vector on $\partial \Omega$.

Let

$$
\mathcal{K} := \{x \in \Omega / \nabla K(x) = 0\}
$$

the set of the critical points of $K$ in $\Omega$.

Throughout this paper, we assume that $K$ satisfies the following flatness condition:

(\text{f})_\beta$ for each critical point $y$ of $K$, there exist $\beta := \beta(y) \in ]n - 2, n[$, and $\eta > 0$ such that in some local coordinates system centered at $y$, we have

$$
K(x) = K(y) + \sum_{i=1}^{n} b_k |(x - y)_k|^\beta + R(x), \forall x \in B(y, \eta),
$$

where $b_k := b_k(y) \neq 0$, $\forall k = 1, \ldots, n$, and $R(z)$ is $C^1$ near 0 with

$$
\lim_{z \to 0} |R(z)||z|^{-\beta} = 0 \quad \text{and} \quad \lim_{z \to 0} |\nabla R(z)||z|^{1-\beta} = 0.
$$

Notice the following.
The \((f)_\beta\)-assumption was used widely as a standard assumption to guarantee the existence of solution to the scalar curvature problem on closed manifolds. However, for technical reason, it is assumed more regularity for the function \(R\) near 0, with the following assumption

\[
\lim_{z \to 0} \sum_{s=0}^{[\beta]} |\nabla^s R(z)| |z|^{s-\beta} = 0,
\]

where \([\beta]\) denotes the integer part of \(\beta\); see, for example, [14]. Thus, the \((f)_\beta\)-assumption mentioned above can be seen as a refined condition.

For each \(y_i \in \mathcal{K}\), we will denote, if necessary, by \(\beta_i\) for its order of flatness.

For each \(s\)-tuple, \(s \geq 1\), of distinct points \(\tau_s := (y_{i_1}, \ldots, y_{i_s})\) such that \(y_{i_k} \in \mathcal{K}\), \(\forall k = 1, \ldots, s\), we define a \(s \times s\) symmetric matrix \(M(\tau_s) = (m_{ij})\) by

\[
m_{jj} := \frac{H(y_{i_j}, y_{i_j})}{(K(y_{i_j}))^{n-2}}, \quad \forall 1 \leq j \leq s \tag{1.4}
\]

\[
m_{jk} := -\frac{G(y_{i_j}, y_{i_k})}{(K(y_{i_j})K(y_{i_k}))^{n-4}}, \quad \forall k \neq j, 1 \leq k, j \leq s.
\]

Let \(\rho(\tau_s)\) be the least eigenvalue of \(M(\tau_s)\), \(\forall s \in \mathbb{N}^*\).

(A2) Assume that \(\rho(\tau_s) \neq 0\) for each distinct points \(y_{i_1}, \ldots, y_{i_s} \in \mathcal{K}\).

We denote by

\[
C_\infty := \{ \tau_p := (y_{i_1}, \ldots, y_{i_p}), p \geq 1, \text{ s.t. } y_{i_j} \in \mathcal{K}, \forall j = 1, \ldots, p, y_{i_j} \neq y_{i_k}, \forall j \neq k, \text{ and } \rho(\tau_p) > 0 \}
\]

and define an index

\[
i : C_\infty \to \mathbb{Z}, \quad (y_{i_1}, \ldots, y_{i_p}) \mapsto i(y_{i_1}, \ldots, y_{i_p}) := p - 1 + \sum_{j=1}^{p} n - \tilde{i}(y_{i_j}),
\]

where \(\tilde{i}(y_{i_j}) := \#\{1 \leq k \leq n, \text{ such that } b_k(y_{i_j}) < 0\}\).

The main result of this paper is the following

**Theorem 1.1** Let \(\Omega \subset \mathbb{R}^n\), \(n \geq 4\), be a smooth bounded domain, and \(0 < K \in \mathcal{C}^1(\overline{\Omega})\) satisfying the assumptions \((f)_\beta\), (A1) and (A2).

If

\[
\sum_{\tau_p \in C_\infty} (-1)^i(\tau_p) \neq 1,
\]

then the problem (1.1) has a solution.
Our argument uses a careful analysis of the lack of compactness of the Euler Lagrange functional $J$ associated to the problem (1.1). Namely, we study the noncompact orbits of the gradient flow of $J$ the so called critical points at infinity following the terminology of A. Bahri [2]. With respect to the closed case, new difficulties here arise. For example, in [1], T. Aubin showed that if $\frac{\partial K}{\partial \nu} \geq 0$ on $\partial \Omega$, then we have a possibility of any concentration points on $\partial \Omega$ of a sequence of subcritical solutions (see proposition 6.44 of [1]). Using the assumption $(A_1)$ we can prove in our situation that the boundary does not make any contribution to the existence of a critical point at infinity. The critical points at infinity of our problem (1.1) can be treated as usual critical points once a Morse lemma at infinity is performed from which we can derive just as in the classical Morse theory the difference of topology induced by these noncompact orbits and compute their Morse index. Such a Morse lemma at infinity is obtained through the construction of a suitable pseudo-gradient for which the Palais-Smale condition is satisfied along the decreasing flow lines, as long as these flow lines do not enter the neighborhood of a finite number of critical points $y_{i_1}, \ldots, y_{i_p}$ of $K$ such that $(y_{i_1}, \ldots, y_{i_p}) \in C_{\infty}$.

The remainder of the paper is organized as follows. In the second section, we set up the variational structure and we recall some well known facts. In section three, we characterize the critical points at infinity of our problem. Section four is devoted to the proof of the main result.

2 Variational structure and lack of compactness

Our problem (1.1) enjoys a variational structure. Indeed, solutions of (1.1) correspond to positive critical points of the functional

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{n-2}{2n} \int_{\Omega} K|u|^{\frac{2n}{n-2}}$$

(2.1)

defined on $H^1_0(\Omega)$.

Let $\Sigma := \{ u \in H^1_0(\Omega), \text{s.t. } \|u\|^2 = \int_{\Omega} |\nabla u|^2 = 1 \}$, $\Sigma^+ := \{ u \in \Sigma, \ u \geq 0 \}$. Instead of working with the functional $I$ defined above, it is more convenient here to work with the functional

$$J(u) = \frac{\int_{\Omega} |\nabla u|^2}{\left( \int_{\Omega} K|u|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}}$$

(2.2)

defined on $\Sigma$. One can easily verify that if $u$ is a critical point of $J$ on $\Sigma^+$, then $J(u)^\frac{n}{2n} u$ is a solution of (1.1).

The variational viewpoint is delicate to be studied, because the functional $J$ does not satisfy the Palais-Smale condition ($(P - S)$ for short). Which means that there exist sequences along which $J$ is bounded, its gradient goes to zero and which is not convergent. The analysis of the sequences failing $(P - S)$ condition can be realized following the ideas introduced in [8] and [15]. For $a \in \Omega$, $\lambda > 0$, let

$$\delta_{a,\lambda}(x) = c_n \left( \frac{\lambda}{1 + \lambda^2|x - a|^2} \right)^{\frac{n-2}{2}},$$

(2.3)
where \( c_n \) is a positive constant chosen such that \( \delta_{a,\lambda} \) is the family of solutions of the following problem

\[- \Delta u = |u|^{\frac{4}{n-2}} u, \ u > 0 \text{ in } \mathbb{R}^n. \tag{2.4}\]

Let \( P \) be the projection from \( H^1(\Omega) \) on to \( H^1_0(\Omega) \); that is, \( u := Pf \) is the unique solution of

\[\Delta u = \Delta f \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega. \tag{2.5}\]

We define now the set of potential critical points at infinity associated to the functional \( J \). Let, for \( \varepsilon > 0, p \in \mathbb{N}^* \),

\[ V(p, \varepsilon) = \{ u \in \Sigma^+ \ s.t \ \exists a_i \in \Omega, \lambda_i > \frac{1}{\varepsilon}, \alpha_i > 0 \text{ for } 1 \leq i \leq p, \text{ with} \]

\[ \| u - \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} \| < \varepsilon, \varepsilon_{ij} < \varepsilon, \ \forall i \neq j, \]

\[ \lambda_id_i > \frac{1}{\varepsilon}, \frac{\lambda_i^{\frac{n+2}{2}} K(a_i)}{\lambda_j^{\frac{n-2}{2}} K(a_j)} - 1 < \varepsilon \ \forall i, j = 1, \ldots, p \}, \]

where \( d_i = d(a_i, \partial \Omega) \) and \( \varepsilon_{ij} = (\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2)^{-\frac{n+2}{2}} \).

If \( u \) is a function in \( V(p, \varepsilon) \), one can find an optimal representation of \( u \) following the ideas introduced in [3] and [4], namely we have

**Proposition 2.1** For any \( p \in \mathbb{N}^* \), there is \( \varepsilon_p > 0 \) such that if \( \varepsilon < \varepsilon_p \) and \( u \in V(p, \varepsilon) \), then the following minimization problem

\[ \min \{ \| u - \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} \|, \alpha_i > 0, \lambda_i > 0, a_i \in \Omega \} \tag{2.6} \]

has a unique solution \((\bar{\alpha}, \bar{a}, \bar{\lambda})\) (up to permutation). Thus we can write \( u \) uniquely as follows (we drop the bar):

\[ u = \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} + v, \tag{2.7} \]

where \( v \) satisfies

\[ (V_0) : \langle v, \phi_i \rangle = 0, \text{ for } i = 1, \ldots, p, \text{ and } \phi_i = P\delta_i, \frac{\partial P\delta_i}{\partial \lambda_i}, \frac{\partial P\delta_i}{\partial a_i} \]

Here, \( P\delta_i := P\delta_{a_i,\lambda_i} \), and \( \langle, \rangle \) denotes the scalar product defined on \( H^1_0(\Omega) \) by

\[ \langle u, v \rangle = \int_{\Omega} \nabla u \nabla v. \]

In the next we will say that \( v \in (V_0) \) if \( v \) satisfies \((V_0)\).

The failure of the \((P - S)\) condition can be described following the ideas developed in [8], [15] and [17]. Such a description is by now standard and reads as follows: let \( \partial J \) be the gradient of \( J \).

**Proposition 2.2** Let \((u_j)_j \subset \Sigma^+ \) be a sequence such that \( \partial J \) tends to zero and \( J(u_j) \) is bounded. Then there exists an integer \( p \in \mathbb{N}^* \), a sequence \( \varepsilon_j > 0, \varepsilon_j \to 0 \), and an extracted subsequence of \( u_j \)'s, again denoted by \( u_j \), such that \( u_j \in V(p, \varepsilon_j) \).
Proposition 2.4 Let \( \bar{v} \) be defined in proposition 2.3. We then have the following estimate: there exists a change of variables \( v \rightarrow \bar{v} \) such that \( J \) reads in \( V(p, \varepsilon) \) as

\[
J\left( \sum_{i=1}^{p} \alpha_i P \delta_{a_i, \lambda_i} + v \right) = J\left( \sum_{i=1}^{p} \alpha_i P \delta_{a_i, \lambda_i} + \bar{v} \right).
\]

Moreover, there exists a small positive constant such that the following holds

\[
\|\bar{v}\| = O\left( \sum_{i=1}^{p} \frac{\|\nabla K(a_i)\|}{\lambda_i^n} + \frac{1}{\lambda_i^{\beta_i}} + \frac{(\log \lambda_i)^{\frac{n+2}{2n}}}{\lambda_i^{\frac{n+2}{2n}}} \right)
\]

\[
+ \left\{ \begin{array}{ll}
O\left( \sum_{i \neq j} \varepsilon_{ij}^2 \frac{(\log \varepsilon_{ij})^{\frac{n+2}{2n}}}{\lambda_i^{\frac{n+2}{2n}}} \right) , & \text{if } n < 6 \\
O\left( \sum_{i \neq j} \varepsilon_{ij}^2 \frac{(\log \varepsilon_{ij})^{\frac{n+2}{2n}}}{(\lambda_i \lambda_j)^{\frac{n+2}{2n}}} \right) , & \text{if } n \geq 6.
\end{array} \right.
\]

**Proof.** Following the proof of proposition 5.3 of [2], it remains to estimate \( \int_{\Omega} K(x) \delta_{\lambda_i}^{\frac{n+2}{2n}} v dx \), and then we need to prove the following claim:

\[
\int_{\Omega} K(x) \delta_{\lambda_i}^{\frac{n+2}{2n}} v dx = O\left( \|v\| \sum_{i=1}^{p} \frac{\|\nabla K(a_i)\|}{\lambda_i^n} + \frac{1}{\lambda_i^{\beta_i}} + \frac{(\log \lambda_i)^{\frac{n+2}{2n}}}{\lambda_i^{\frac{n+2}{2n}}} \right).
\]

For this, we distinguish two cases: let \( \rho > 0 \) a small positive constant such that the condition (f) holds in \( B(y, 4\rho) \), \( \forall \ y \in K \).

**Case 1.** If \( a_i \not\in \cup_{y \in K} B(y, \rho) \). Let \( \mu > 0 \) a positive constant small enough. Using the fact that \( v \in (V_0) \), we obtain

\[
\int_{\Omega} K(x) \delta_{\lambda_i}^{\frac{n+2}{2n}} v dx = \int_{B(a_i, \mu)} (K(x) - K(a_i)) \delta_{\lambda_i}^{\frac{n+2}{2n}} v dx + O\left( \|v\| \right)
\]

\[
\leq c \frac{\|v\|}{\lambda_i} \left( \sum_{i=1}^{p} \frac{\|\nabla K(a_i)\|}{\lambda_i^n} + \frac{1}{\lambda_i^{\beta_i}} + \frac{(\log \lambda_i)^{\frac{n+2}{2n}}}{\lambda_i^{\frac{n+2}{2n}}} \right).
\]

\[
\leq c \frac{\|v\| \cdot \|\nabla K(a_i)\|}{\lambda_i}, \text{ since } \nabla K(x) \neq 0, \ \forall \ x \not\in \cup_{y \in K} B(y, \rho).
\]
Case 2. If $a_i \in B(y_i, \rho)$, $y_i \in \mathcal{K}$. Let $C > 0$ a positive constant large enough.
If $\lambda_i |a_i - y_i| \leq C$, let $B_i := B(a_i, \rho)$, then, by using the condition ($f^\beta_i$) in $B_i$ and the fact that $v \in (V_0)$, we obtain

$$
\int_{\Omega} K(x)\delta_i^{n+2} v dx = O\left(\int_{B_i} (|x - a_i|^\beta_i + |y_i - a_i|^\beta_i)\delta_i^{n+2} v dx\right) + O\left(\frac{\|v\|}{\lambda_i^{\frac{n-2}{2}}}\right)
$$

$$
= O\left(\|v\| + \frac{\|v\| \cdot (\log \lambda_i)^{\frac{n+2}{2n}}}{\lambda_i^{\frac{n+2}{2}}}\right).
$$

(2.11)

If $\lambda_i |a_i - y_i| \geq C$, let $B_i := B(a_i, \frac{|a_i - y_i|}{2})$ and $B_{y_i} := B(y_i, 2\rho)$, then, by using the condition ($f^\beta_i$) in $B_{y_i}$ and the fact that $v \in (V_0)$, we obtain

$$
\int_{\Omega} K(x)\delta_i^{n+2} v dx = \int_{B_i} [K(x) - K(a_i)]\delta_i^{n+2} v dx
$$

$$
+ \int_{B_{y_i}\setminus B_i} [K(x) - K(a_i)]\delta_i^{n+2} v dx + O\left(\frac{\|v\|}{\lambda_i^{\frac{n-2}{2}}}\right)
$$

$$
= O\left(\sup_{B_i} |\nabla K(x)| \cdot \int_{B_i} |x - a_i|\delta_i^{n-2} |v| dx\right)
$$

$$
+ \int_{B_{y_i}\setminus B_i} (|a_i - y_i|^\beta_i + |x - a_i|^\beta_i)\delta_i^{n+2} |v| dx + O\left(\frac{\|v\|}{\lambda_i^{\frac{n+2}{2}}}\right).
$$

Since $|\nabla K(x)| \sim |x - y_i|^{\beta_i - 1}$ in $B_i$, then we obtain

$$
\int_{\Omega} K(x)\delta_i^{n+2} v dx = \|v\| \cdot O\left(\frac{|\nabla K(a_i)|}{\lambda_i^{\beta_i}} + \frac{1}{\lambda_i^{\beta_i}} + \frac{(\log \lambda_i)^{\frac{n+2}{2n}}}{\lambda_i^{\frac{n+2}{2}}}\right).
$$

(2.12)

Combining (2.10), (2.11) and (2.12), the claim (2.9) follows.

Following A. Bahri [2], we introduce the following definition:

**Definition 2.1** A critical point at infinity of $J$ in $\Sigma^+$ is a limit of a flow line $u(s)$ of the equation

$$
\begin{cases}
\frac{\partial u}{\partial s} = -\partial J(u) \\
u(0) = u_0 \in \Sigma^+
\end{cases}
$$

such that $u(s)$ remains in $V(p, \varepsilon(s))$, for $s \geq s_0$.

Here, $\varepsilon(s)$ is some function tending to zero when $s \to +\infty$. Using proposition 2.1, $u(s)$ can be written as

$$
u(s) = \sum_{i=1}^{p} \alpha_i(s)P\delta_{a_i(s),\lambda_i(s)} + v(s).
$$
Denoting by $a_i := \lim a_i(s)$ and $\alpha_i := \lim \alpha_i(s)$, we denote by
\[
(a_1, \ldots, a_p)_{\infty} \text{ or } \sum_{i=1}^{p} \alpha_i P\delta_{a_i, \infty}
\]
such a critical point at infinity.

For such a critical point at infinity there are associated stable and unstable manifolds. These manifolds can be easily described once a Morse type reduction is performed (see [3], pages 356-357).

3 Characterization of the critical points at infinity

This section is devoted to the characterization of the critical points at infinity, associated to the problem $(1.1)$, in $V(p, \varepsilon)$, $p \geq 1$. This characterization is obtained through the construction of a suitable pseudo-gradient at infinity in $V(p, \varepsilon)$. The construction is based on very delicate expansion of the gradient of the associated Euler-Lagrange functional $J$ near infinity. In the second subsection, we will characterize the critical points at infinity in $V(p, \varepsilon)$, $p \geq 1$.

Using proposition $2.3$, we can write, for $u = \sum_{i=1}^{p} \alpha_i P\delta_{a_i, \lambda_i} + \bar{v} \in V(p, \varepsilon)$,
\[
J(u) = J(\sum_{i=1}^{p} \alpha_i P\delta_{a_i, \lambda_i} + \bar{v}) + \|\bar{v}\|^2.
\]
In the $V-$variable, we define a pseudo-gradient by setting
\[
\frac{\partial V}{\partial s} = -\mu V
\]
where $\mu$ is a very large constant. Then, at $s = 1$, $V(1) = e^{-\mu}V(0)$ will be very small, as we wish. This shows that, in order to define our deformation, we can work as if $V$ was zero. The deformation will extend immediately, with the same properties, to a neighborhood of zero in the $V$ variable. Therefore we need to define a pseudo-gradient in \( \{ \sum_{i=1}^{p} \alpha_i P\delta_{a_i, \lambda_i} + \bar{v} \in V(p, \varepsilon) \} \).

3.1 Expansion of the gradient of the functional

**Proposition 3.1** Let $n \geq 4$. For $\varepsilon$ small enough and $u = \sum_{i=1}^{p} \alpha_i P\delta_i \in V(p, \varepsilon)$, we have the following expansion:

\[
\langle \partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \rangle_{H^1_0} = 2c_2 J(u) \left[ -\frac{n-2}{2} \alpha_i \frac{H(a_i, a_i)}{\lambda_i^{n-2}} - \sum_{j \neq i} \alpha_j \left( \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-2}{2} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \right) \right] \times (1 + o(1)) + o \left( \frac{1}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} + \sum_{k \neq j} \varepsilon_{kj}^{n-1} + \sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^{n-1}} \right),
\]
where $c_2 = c_n^{2\nu} \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{n+\frac{2}{2}}}$. 
(ii) If \( a_i \in B(y_{j_i}, \rho) \), with \( y_{j_i} \in \mathcal{K} \), and \( \rho \) is a positive constant small enough so that \((f)_{\beta}\) holds in \( B(y_{j_i}, 4\rho) \), we have

\[
\langle \partial J(u), \lambda_i \partial P \delta_i \rangle_{H_0^1} = 2c_2 J(u) \left[ -\frac{n-2}{2} \alpha_i \frac{H(a_i, a_i)}{\lambda_i^{n-2}} + \sum_{j \neq i} \alpha_j \left( \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-2}{2} \frac{H(a_j, a_j)}{(\lambda_i \lambda_j)^{n-2}} \right) \right] \times \\
(1 + O(1)) + o \left( \sum_{i \neq j} \varepsilon_{ij} + \sum_{k \neq j} \varepsilon_{k^2} + \sum_{k=1}^p \frac{1}{(\lambda_k \delta_k)^{n-1}} \right) \\
+ (\text{if } \lambda_i |a_i - y_{j_i}| \geq C) o \left( \frac{\nabla K(a_i)}{\lambda_i} \right),
\]

where \( C \) is a positive constant large enough.

**Proof.** Claim (i) is immediate from [7]. Concerning claim (ii), regarding the estimates used to prove claim (i), we need to estimate the quantity \( \int_{\Omega} K(x) \delta_i^{n-2} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \, dx \). Let \( C \) a positive constant large enough.

If \( \lambda_i |a_i - z_j| \leq C \), let \( B_i := B(a_i, \rho) \), then

\[
\int_{\Omega} K(x) \delta_i^{n-2} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \, dx = \int_{B_i} [K(x) - K(z)] \delta_i^{n-2} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \, dx + o \left( \frac{1}{\lambda_i^{n-2}} \right)
\]

\[
= \frac{n-2}{2} \left( \frac{1}{\lambda_i^2} \right) \sum_{j=1}^n b_j \int_{B_i} |y_j + \lambda_i (a_i - z_j)|^{\beta} \frac{|y_j|^{(1-|y_j|^2)} ((|y_j|^2 + 1)^{n+1})}{\lambda^{n+1}} \, dy + \frac{1}{\lambda_i^{n-2}}, \quad \text{since } \beta > n - 2.
\]

If \( \lambda_i |a_i - y_{j_i}| \geq C \), let \( M > 0 \) a positive constant large enough, \( B_{i,k} := B(a_i, \frac{|a_i - y_{j_i}|}{2M}) \), \( \forall \ k = 1, \ldots, n \), and \( B_{y_{j_i}} := B(y_{j_i}, 2\rho) \), then

\[
\int_{\Omega} K(x) \delta_i^{n-2} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \, dx = \int_{B_{y_{j_i}}} [K(x) - K(a_i)] \delta_i^{n-2} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \, dx + o \left( \frac{1}{\lambda_i^{n-2}} \right).
\]

By the condition \((f)_{\beta}\), we get

\[
\int_{B_{y_{j_i}}} [K(x) - K(a_i)] \delta_i^{n-2} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \, dx \\
= \sum_{k=1}^n b_k \int_{B_{y_{j_i}}} |(a_i - y_{j_i})_k - (a_i - x)_k|^{\beta} \delta_i^{n-2} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \, dx - |(a_i - y_{j_i})_k|^{\beta} \int_{B_{y_{j_i}}} \delta_i^{n-2} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \, dx \\
+ o \left( \frac{1}{\lambda_i^{n-2}} + \left| \frac{a_i - z_k}{\lambda_i} \right|^{\beta-1} \right).
\]

(3.2)
Observe that
\[
\int_{B_{y_j} \setminus B_{i,k}} |(a_i - y_j)_k - (a_i - x)_k|^\beta \delta_i^{n+2} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \, dx
= O\left( \int_{B_{y_j} \setminus B_{i,k}} \left( |(a_i - y_j)_k|^\beta + |(a_i - x)_k|^\beta \delta_i^{2n} \right) \, dx \right) = o\left( \frac{1}{\lambda_i^{n-2}} \right). \tag{3.3}
\]

However, by elementary calculation, we obtain
\[
\int_{B_{i,k}} |(a_i - y_{j})_k - (a_i - x)_k|^\beta \frac{\partial \delta_i}{\partial \lambda_i} \, dx - |(a_i - y_j)_k|^\beta \int_{B_{i,k}} \frac{n+2}{\delta_i^{n-2}} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \, dx
= o\left( \frac{1}{\lambda_i^{n-2}} + \frac{|a_i - y_j|^\beta}{\lambda_i} \right). \tag{3.4}
\]

Combining (3.1), (3.2), (3.3) and (3.4), we get
\[
\int_{\Omega} K(x) \delta_i^{n+2} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \, dx = o\left( \frac{1}{\lambda_i^{n-2}} + \frac{|a_i - y_j|^\beta}{\lambda_i} \right).
\]

We remark from the condition (f)\(\beta\) that, for \(\rho\) small enough, \(|\nabla K(a_i)| \sim |(a_i - y_j)|^{\beta-1}\). Then we can write
\[
\int_{\Omega} K(x) \delta_i^{n+2} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \, dx = o\left( \frac{1}{\lambda_i^{n-2}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right).
\]

This finishes the proof of claim (ii).

**Proposition 3.2** Let \(n \geq 4\). For \(\varepsilon\) small enough and \(u = \sum_{i=1}^{p} \alpha_i P\delta_i \in V(p, \varepsilon)\), we have

(i)\[
\langle \partial J(u), \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \rangle_{H^1} = 2J(u) - \alpha_i \frac{n-2}{n} c_4 J^{n-1} \frac{\lambda_i}{\lambda_i^{n-1}} \frac{\nabla K(a_i)}{\lambda_i} + \alpha_i \frac{\partial H(a_i, a_i)}{\partial a_i}
- c_2 \sum_{j \neq i} \alpha_j \left( \frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} - \frac{1}{\lambda_i \lambda_j} \frac{\partial H(a_i, a_j)}{\partial a_i} \right) (1 + o(1))
+ o\left( \frac{1}{\lambda_i^{n-2}} + \sum_{k=1}^{p} \frac{1}{\lambda_i k^{n-2}} \varepsilon_{ij} \right) + O\left( \lambda_i^2 |a_i - a_j|^{\frac{n+2}{n-1}} \right),
\]
where \(c_4 = c_n^{n-1} \int_{\mathbb{R}^n} \frac{|x|^2}{(1 + |x|^2)^n+1} \).

(ii) If \(a_i \in B(y_j, \rho)\), with \(y_j \in \mathcal{K}\), and \(\rho\) is a positive constant small enough so that (f)\(\beta\) holds in \(B(y_j, 4\rho)\), then the above estimate can be improved. Let \(C\) a positive constant large
enough. If \( \lambda_i \| (a_i - z_{j_i})_k \| \geq C \), we get

\[
\langle \partial J(u), \frac{1}{\lambda_i} \partial (a_i)_k \rangle_{H^1_0} = -2(n - 2)J^{2n-4}_\alpha \, \partial L \frac{\lambda_i^{\beta-1} (a_i - z_{j_i})_k}{\lambda_i} b_k c_5 \\
+ o\left( \frac{1}{\lambda_i^{n-2}} + \frac{\nabla K(a_i)}{\lambda_i} + \sum_{j \neq i} \varepsilon_{ij} + \sum_{k \neq j} \varepsilon_{kj} + \sum_{k=1}^p \frac{1}{\lambda_i d_k (n-1)} \right) \\
+ O\left( \sum_{j \neq i} \frac{1}{\lambda_i} \partial a_i \right).
\]

Here, \( c_5 = \frac{2n-4}{n} \frac{1}{\lambda_i} \int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^{n+1}} \, dx, k \in \{1, \ldots, n\} \), and \( (a_i)_k \) denotes the \( k^{th} \) component of \( a_i \) in some local coordinates system.

**Proof.** Claim (i) is immediate from [7]. Concerning claim (ii), arguing as in the proof of proposition [3.1], claim (ii) is proved under the following estimate: let \( C \) a positive constant large enough and \( a_i \in B(y_{j_i}, \rho) \), where \( \rho \) is a positive constant small enough so that (f) \( \beta \) holds in \( B(y_{j_i}, 4\rho) \). If \( \lambda_i \| (a_i - z_{j_i})_k \| \geq C \), then we have

\[
\int_{\Omega} K(x) \delta_i^{\beta-1} \frac{1}{\lambda_i} \partial (a_i)_k \, dx = (n - 2) \text{sgn}[(a_i - z_{j_i})_k] \frac{\lambda_i^{\beta-1} (a_i - z_{j_i})_k}{\lambda_i} b_k c_5 + o\left( \frac{1}{\lambda_i^{n-2}} + \frac{\nabla K(a_i)}{\lambda_i} \right).
\]

This finishes the proof of claim (ii).

### 3.2 Critical points at infinity

This subsection is devoted to the characterization of the critical points at infinity, associated to the problem (1.1), in \( V(p, \varepsilon) \), \( p \geq 1 \). This characterization is obtained through the construction of a suitable pseudo-gradient at infinity for which the Palais-Smale condition is satisfied along the decreasing flow lines as long as these flow lines do not enter in the neighborhood of finite number of critical points \( y_{j_i}, j = 1, \ldots, p \), of \( K \) such that \( (y_{i_1}, \ldots, y_{i_p}) \in C_\infty \). Now, we introduce the following main result:

**Theorem 3.1** Let \( n \geq 4 \). There exists a pseudo-gradient \( W \) so that the following holds. There is a constant \( c > 0 \) independent of \( u = \sum_{i=1}^p \alpha_i P \delta_{a_i, \lambda_i} \in V(p, \varepsilon) \) so that:

(i) \[
\langle \partial J(u), W(u) \rangle \leq -c \left( \sum_{i=1}^p \frac{\| \nabla K(a_i) \|}{\lambda_i} + \frac{1}{\lambda_i^{n-2}} + \frac{1}{\lambda_i d_i (n-1)} + \sum_{i \neq j} \varepsilon_{ij}^{n-2} \right).
\]

(ii) \[
\langle \partial J(u + \varphi), W(u) + \frac{\partial \varphi}{\partial (\alpha, a, \lambda)}(W) \rangle \leq -c \left( \sum_{i=1}^p \frac{\| \nabla K(a_i) \|}{\lambda_i} + \frac{1}{\lambda_i^{n-2}} + \frac{1}{\lambda_i d_i (n-1)} + \sum_{i \neq j} \varepsilon_{ij}^{n-2} \right).
\]

(iii) The minimal distance to the boundary, \( d_i(t) := d(a_i(t), \partial \Omega) \), only increases if it is small enough.
small enough such that

Observe that $u$ is a fixed positive constant. Then we have the following propositions:

**Proposition 3.3** In $V_{d_0}(p, \varepsilon) := \{ u = \sum_{i=1}^{p} \alpha_i P \delta_i \in V(p, \varepsilon), d(a_i, \partial \Omega) \geq d_0, \forall 1 \leq i \leq p \}$, there exists a pseudo-gradient $W_1$ so that the following holds: There is a constant $c > 0$ independent of $u \in V_{d_0}(p, \varepsilon)$ so that

$$\langle \partial J(u), W_1(u) \rangle \leq -c \left( \sum_{i}^{p} \left| \frac{\nabla K(a_i)}{\lambda_i} \right| + \frac{1}{\lambda_i^{n-2}} \right) + \sum_{i \neq j} \varepsilon_{ij}.$$ 

**Proposition 3.4** In $V_b(p, \varepsilon) := \{ u = \sum_{i=1}^{p} \alpha_i P \delta_i \in V(p, \varepsilon), d(a_i, \partial \Omega) \leq 2d_0, \forall 1 \leq i \leq p \}$, there exists a pseudo-gradient $W_2$ so that the following holds: There is a constant $c > 0$ independent of $u \in V_b(p, \varepsilon)$ so that

$$\langle \partial J(u), W_2(u) \rangle \leq -c \left( \sum_{i}^{p} \left( \frac{1}{\lambda_i} + \frac{1}{(\lambda_i d_i)^{n-1}} \right) \right) + \sum_{i \neq j} \varepsilon_{ij}.$$ 

**Proof of theorem 3.1** We divide the set $\{1, \ldots, p\}$ into two sets. The first contains the indices of the points near the boundary $\partial \Omega$, and the second contains the indices of the points far away from $\partial \Omega$. Let us define

$$B := \{ 1 \leq i \leq p \text{ s.t } d_i \geq 2d_0 \}.$$ 

$$B_1 := B \cup \{ i \not\in B \text{ s.t } \exists (i_1, \ldots, i_r) \text{ with } i_1 = i, i_r \in B \text{ and } |a_{i_k} - a_{i_{k+1}}| < \frac{d_0}{p}, \forall k \leq r \}.$$ 

$$B_2 := \{ 1, \ldots, p \} \setminus B_1.$$ 

Observe that

$$(O_1) \ d_i := d(a_i, \partial \Omega) \leq 2d_0, \forall i \in B_2.$$ 

$$(O_2) \ The \ advantage \ of \ B_1 \ is \ that \ if \ i \in B_1 \ and \ j \not\in B_1, \ then \ |a_i - a_j| \geq \frac{d_0}{p}.$$ 

Now we write $u$ as

$$u := u_1 + u_2, u_k := \sum_{i \in B_k} \alpha_i P \delta_i \ (1 \leq k \leq 2).$$ 

Observe that $u_1 \in V_{d_0}(\text{card}(B_1), \varepsilon)$. Then we use the previous construction as in proposition 3.3 to $u_1$, which means we apply the previous construction to the sub-pack of functions $u :=$
We will define the pseudo-gradient depending on the sets $V_i$. Let $W_1(u_1)$ be the vector field thus defined. The same argument can be repeated for $u_2$, which is in $V_2(\text{card}(B_2), \varepsilon)$, and we will denote by $W_2(u_2)$ the vector field thus defined. Define $W$ as $W(u) = W_1(u_1) + W_2(u_2)$. Thus we have

$$\langle \partial J(u), W(u) \rangle = \langle \partial J(u_1), W_1(u_1) \rangle + \langle \partial J(u_2), W_2(u_2) \rangle + O\left( \sum_{i \in B_1, j \in B_2} \varepsilon_{ij} \right).$$

Observe that, for $i \in B_1$ and $j \in B_2$,

$$\varepsilon_{ij} = o\left( \frac{1}{\lambda_i^{n-2}} + \frac{1}{\lambda_j} \right).$$

So claim (i) of theorem 3.1 follows. Now, arguing as in appendix 2 of [3], claim (ii) follows from (i) and proposition 2.4. The conditions (iii) and (iv) are satisfied by the definition of the vector field $W$.

**Proof of proposition 3.3.** Let $\eta > 0$ a fixed constant small enough with $|y_i - y_j| > 2 \eta \forall i \neq j$. We divide the set $V(p, \varepsilon, d_0)$ into three sets:

$$V_1(p, \varepsilon, d_0) := \{ u = \sum_{i=1}^{p} \alpha_i P \delta_i \in V_0(p, \varepsilon) \text{ s.t. } a_i \in B(y_j, \eta), \ y_j, \in \mathcal{K}, \ \forall i = 1, \ldots, p, \$$

with $y_{j_i} \neq y_{j_k} \forall i \neq k$, and $\rho(y_{j_i}, \ldots, y_{j_p}) > 0 \}.$

$$V_2(p, \varepsilon, d_0) := \{ u = \sum_{i=1}^{p} \alpha_i P \delta_i \in V_0(p, \varepsilon) \text{ s.t. } a_i \in B(y_j, \eta), \ y_j, \in \mathcal{K}, \ \forall i = 1, \ldots, p, \$$

with $y_{j_i} \neq y_{j_k} \forall i \neq k$, and $\rho(y_{j_i}, \ldots, y_{j_p}) < 0 \}.$

$$V_3(p, \varepsilon, d_0) := \{ u = \sum_{i=1}^{p} \alpha_i P \delta_i \in V_0(p, \varepsilon) \text{ s.t. } a_i \in B(y_j, \eta), \ y_j, \in \mathcal{K}, \ \forall i = 1, \ldots, p, \$$

and $\exists i \neq k \text{ s.t. } y_{j_i} = y_{j_k} \}.$

$$V_4(p, \varepsilon, d_0) := \{ u = \sum_{i=1}^{p} \alpha_i P \delta_i \in V_0(p, \varepsilon) \text{ s.t. } \exists i \neq \cup_{y \in \mathcal{K}} B(y, \eta) \}.$$

We will define the pseudo-gradient depending on the sets $V_i(p, \varepsilon)$, $i = 1 - 4$, to which $u$ belongs.

**Lemma 3.1** In $V_2(p, \varepsilon, d_0)$, there exists a pseudo-gradient $\tilde{W}_2$ so that the following holds: There is a constant $c > 0$ independent of $u \in V_2(p, \varepsilon, d_0)$ so that

$$\langle \partial J(u), \tilde{W}_2 \rangle \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \frac{\nabla K(a_i)}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \right).$$

**Proof.** Let $\rho$ be the least eigenvalue of $M$. Then there exists an eigenvector $e = (e_1, \ldots, e_p)$ associated to $\rho$ such that $\|e\| = 1$ with $e_i > 0$, $\forall i = 1, \ldots, p$. Indeed, let $e = (e_1, \ldots, e_p)$ an
eigenvector associated to $\rho$, with $\|e\| = 1$. By elementary calculation, we obtain, since $m_{ij} < 0$, $\forall \ i \neq j$,
\[
e_i > 0, \forall \ i = 1, \ldots, p, \text{ or } e_i < 0, \forall \ i = 1, \ldots, p.
\]
(3.5)
Let $\gamma > 0$ such that for any $x \in B(e, \gamma) := \{y \in S^{p-1}/\|y - e\| \leq \gamma\}$ we have $T x \cdot M \cdot x < (1/2)\rho$.
Two cases may occur:
Case 1: $|\Lambda|^{-1}\Lambda \in B(e, \gamma)$. Since, for any $i \neq j$, $|a_i - a_j| \geq c$, then
\[
\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\varepsilon_{ij}(1 + o(1)) = -\frac{1}{(\lambda_i \lambda_j |a_i - a_j|^2)^{\frac{n-2}{2}}}(1 + o(1))
\]
(3.6)
and
\[
\frac{1}{\lambda_i \partial a_i} = o(\varepsilon_{ij}).
\]
(3.7)
We define $\tilde{W}_2^1$ by
\[
\tilde{W}_2^1 = -\sum_{i=1}^{p} \alpha_i \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i}.
\]
From proposition 3.1 and 3.6, we obtain
\[
\langle \partial J(u), \tilde{W}_2^1 \rangle = -64\pi^2 J(u) \left[ \sum_{i=1}^{p} \alpha_i^2 \frac{H(a_i)}{\lambda_i^{n-2}} - \sum_{j \neq i} \alpha_j \alpha_i \lambda_i \frac{G(a_i, a_j)}{\left(\lambda_i \lambda_j \right)^{\frac{n-2}{2}}} \right] \times (1 + o(1))
\]
\[
+ \sum_{i=1}^{p} (\text{if } \lambda_i |a_i - y_j| \geq C) o\left(\frac{|\nabla K(a_i)|}{\lambda_i}\right).
\]
Observe that, since $u \in V(p, \varepsilon)$, we have $J(u) \frac{4}{\lambda_i^2} \alpha_i^{\frac{n}{2}} K(a_i) = (1 + o(1))$. Thus we derive that
\[
\langle \partial J(u), \tilde{W}_2^1 \rangle = -c^{\top} \left[ T \Lambda \cdot M \cdot \Lambda \right] (1 + o(1)) + \sum_{i=1}^{p} (\text{if } \lambda_i |a_i - y_j| \geq C) o\left(\sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i}\right)
\]
\[
\leq -c\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \sum_{i=1}^{p} (\text{if } \lambda_i |a_i - y_j| \geq C) o\left(\sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i}\right)
\]
(3.8)
\[
\leq -c\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} (\text{if } \lambda_i |a_i - y_j| \geq C) o\left(\sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i}\right),
\]
where $M$ is the matrix defined by (1.4), and $\Lambda := T (\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_p})$.
Case 2: $|\Lambda|^{-1}\Lambda \notin B(e, \gamma)$. In this case, we define
\[
\tilde{W}_2^2 = -\sum_{i=1}^{p} |\Lambda| \beta_i \alpha_i \lambda_i^\frac{n}{2} \frac{\partial P \delta_i}{\partial \lambda_i},
\]
where $\beta_i = \frac{|\Lambda| |e - \Lambda|}{\|y(0)\|} - \frac{y(0)}{\|y(0)\|} (y(0), |\Lambda| e - \Lambda)$ and $y(t) = (1 - t)\Lambda + t |\Lambda| e$. Define
\[
\Lambda(t) := y(t)/\|y(t)\|.
\]
Using proposition 3.11, we derive

\[
\langle \partial J(u), \widetilde{W}_2 \rangle = cJ(u)\|\Lambda\| \left[ \sum_{i=1}^{p} \alpha_i^2 \beta_i \frac{H(a_i, a_i)}{\lambda_i^{n-2}} - \sum_{j \neq i} \alpha_j \alpha_i \beta_i \frac{G(a_i, a_j)}{\lambda_j^{n-2}} \right] \\
+ o\left( \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} \right) + \sum_{i=1}^{p} (\text{if } \lambda_i |a_i - y_{ji}| \geq C) \circ \left( \frac{\nabla K(a_i)}{\lambda_i} \right) \\
= \frac{c}{J(u)^{n-2}} |\Lambda|^2 \left[ T \Lambda'(0) \cdot M \cdot \Lambda(0) \right] \\
+ o\left( \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} \right) + \sum_{i=1}^{p} (\text{if } \lambda_i |a_i - y_{ji}| \geq C) \circ \left( \frac{\nabla K(a_i)}{\lambda_i} \right), \text{ since } |a_i - a_j| \geq c, \forall i \neq j.
\]

We claim that

\[
\frac{\partial}{\partial t} (T \Lambda(t) \cdot M \cdot \Lambda(t)) < -c, \text{ for } t \text{ near } 0,
\]

(3.10)

where \( c > 0 \) is a constant independent of \(|\Lambda|^{-1} \Lambda \in (B(e, \gamma))^c\). Indeed, we have

\[
T \Lambda(t) \cdot M \cdot \Lambda(t) = \rho + \frac{(1-t)^2}{\|y(t)\|^2} [T \Lambda \cdot M \cdot \Lambda - \rho |\Lambda|^2].
\]

(3.11)

Equation (3.11) implies

\[
\frac{\partial}{\partial t} (T \Lambda(t) \cdot M \cdot \Lambda(t)) = 2 \frac{(1-t)}{\|y(t)\|^4} \left[ T \Lambda \cdot M \cdot \Lambda - \rho |\Lambda|^2 \right] \left[ (1-t)|\Lambda|(e, \Lambda) - t|\Lambda|^2 \right] \\
= 2 \frac{(1-t)|\Lambda|^4}{\|y(t)\|^4} \left[ \frac{1}{|\Lambda|^4} \left( T \Lambda \cdot M \cdot \Lambda - \rho |\Lambda|^2 \right) \left( -|\Lambda|(e, \Lambda) \right) + o(1) \right]
\]

(3.12)

By using the observation (3.11), we derive that there exists \( c > 0 \) (\( c \) independent of \(|\Lambda|^{-1} \Lambda \in (B(e, \gamma))^c\)) such that

\[
T \Lambda \cdot M \cdot \Lambda - \rho |\Lambda|^2 \geq c|\Lambda|^2.
\]

(3.13)

Also, observe that

\[
|\Lambda|(e, \Lambda) \geq \alpha |\Lambda|^2, \text{ where } \alpha := \inf \{ \varepsilon_i, 1 \leq i \leq p \}.
\]

(3.14)

Combining (3.12), (3.13) and (3.14), the claim (3.10) follows. Now, by combining (3.9) and (3.10), we obtain

\[
\langle \partial J(u), \widetilde{W}_2 \rangle \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j} \varepsilon_{ij} \right) + \sum_{i=1}^{p} (\text{if } \lambda_i |a_i - y_{ji}| \geq C) \circ \left( \frac{\nabla K(a_i)}{\lambda_i} \right).
\]

(3.15)

We define \( \widetilde{W}_3 \) as a convex combination of \( \widetilde{W}_1 \) and \( \widetilde{W}_2 \). Combining (3.8) and (3.15), we obtain

\[
\langle \partial J(u), \widetilde{W}_3 \rangle \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j} \varepsilon_{ij} \right) + \sum_{i=1}^{p} (\text{if } \lambda_i |a_i - y_{ji}| \geq C) \circ \left( \frac{\nabla K(a_i)}{\lambda_i} \right).
\]

(3.16)
Let $\Psi$ a positive cut-off function defined by $\Psi(t) = 1$, if $t \leq C$ and $\Psi(t) = 0$, if $t \geq 2C$, where $C$ is a positive constant large enough. To make appear $\sum_{i=1}^{p} \frac{\nabla K(a_i)}{\lambda_i}$, we define, for each $i = 1, \ldots, p$,

$$
\Xi_i = \sum_{k=1}^{n} \left[ 1 - \Psi(\lambda_i |(a_i - y_{ji}), k|) \right] b_k \cdot \sgn((a_i - y_{ji})_k) \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_k}.
$$

Using proposition 3.2 and (3.7), we obtain

$$
\langle \partial J(u), \Xi_i \rangle = -c \sum_{k=1}^{n} \left[ 1 - \Psi(\lambda_i |(a_i - y_{ji}), k|) \right] b_k \frac{(a_i - y_{ji})_k^{\beta_{ji} - 1}}{\lambda_i} + o\left( \sum_{i=1}^{p} \frac{\nabla K(a_i)}{\lambda_i} \right) \leq -c \sum_{k=1}^{n} \left[ 1 - \Psi(\lambda_i |(a_i - y_{ji}), k|) \right] \frac{\nabla K(a_i)}{\lambda_i} + o\left( \sum_{i=1}^{p} \frac{\nabla K(a_i)}{\lambda_i} \right),
$$

where $k_i$ denotes the index such that $|(a_i - y_{ji}), k_i| = \max_{1 \leq k \leq n} |(a_i - y_{ji}), k|$. Combining (3.16) and (3.17), we obtain

$$
\langle \partial J(u), \tilde{W}_2^3 + \sum_{i=1}^{p} \Xi_i \rangle \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} + \left[ 1 - \Psi(\lambda_i |(a_i - y_{ji}), k|) \right] \frac{\nabla K(a_i)}{\lambda_i} \right) + o\left( \sum_{i=1}^{p} \frac{\nabla K(a_i)}{\lambda_i} \right).
$$

If $\Psi \leq \frac{1}{2}$, then $\frac{\nabla K(a_i)}{\lambda_i}$ appears in the upper bound of (3.18). However, if $\Psi > \frac{1}{2}$, then we have $\frac{\nabla K(a_i)}{\lambda_i} = o\left( \frac{1}{\lambda_i^{n-2}} \right)$, and so we can make appear $\frac{\nabla K(a_i)}{\lambda_i}$ from $\frac{1}{\lambda_i^{n-2}}$. From this discussion, the estimate (3.18) becomes

$$
\langle \partial J(u), \tilde{W}_2^3 + \sum_{i=1}^{p} \Xi_i \rangle \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \frac{\nabla K(a_i)}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \right).
$$

The pseudo-gradient $\tilde{W}_2 := \tilde{W}_2^3 + \sum_{i=1}^{p} \Xi_i$ satisfies the claim of lemma 3.1.

**Lemma 3.2** In $V_1(p, \varepsilon, d_0)$, there exists a pseudo-gradient $\tilde{W}_1$ so that the following holds: There is a constant $c > 0$ independent of $u \in V_1(p, \varepsilon, d_0)$ so that

$$
\langle \partial J(u), \tilde{W}_1(u) \rangle \leq -c \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \frac{\nabla K(a_i)}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij}.
$$

**Proof.** Let $\delta > 0$ a fixed constant small enough, and denote, for each $\beta_{ji}, \alpha_{ji} := \frac{n-3}{\beta_{ji}}$. We distinguish two cases:
Case 1: \( \max_{1 \leq i \leq p} \{ \lambda_i^{\alpha_i} |a_i - y_{ji}| \} \leq \delta \). In this case, we define
\[
Y_1 := \sum_{i=1}^{p} \alpha_i \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i}.
\]
Arguing as in the proof of the estimate \( \text{(3.8)} \), and using the fact \( \rho(y_{j1}, \ldots, y_{jp}) > 0 \), we obtain
\[
\langle \partial J(u), Y_1 \rangle \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j} \varepsilon_{ij} \right) + \sum_{i=1}^{p} (\text{if } \lambda_i |a_i - y_{ji}| \geq C) \bigg( \sum_{i=1}^{p} \frac{\big| \nabla K(a_i) \big|}{\lambda_i} \bigg).
\] (3.20)
Observe that \( \bigg| \nabla K(a_i) \bigg| = o\left( \frac{1}{\lambda_i^{n-2}} \right), \forall 1 \leq i \leq p. \) Thus, from (3.20), we get
\[
\langle \partial J(u), Y_1 \rangle \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j} \varepsilon_{ij} \right) + \sum_{i=1}^{p} (\text{if } \lambda_i |a_i - y_{ji}| \geq C) \bigg( \sum_{i=1}^{p} \frac{\big| \nabla K(a_i) \big|}{\lambda_i} \bigg).
\] (3.21)
Case 2: \( \max_{1 \leq i \leq p} \{ \lambda_i^{\alpha_i} |a_i - y_{ji}| \} > \delta \). Let \( i_1 := \min\{1 \leq i \leq p, \text{ s.t } \lambda_i^{\alpha_i} |a_i - y_{ji}| > \delta \} \).
Without loss of generality, we suppose \( \lambda_1 \leq \cdots \leq \lambda_p \). Let \( M > 0 \) a fixed constant large enough. We set
\[
I := \{i_1\} \cup \{i < i_1, \text{ s.t } \lambda_{j-1} \geq \frac{1}{M} \lambda_j, \forall i < j \leq i_1 \}
=: \{i_0, \ldots, i_1\}.
\] (3.22)
We distinguish two subcases:
Subcase 2.1: \( i_0 > 1 \). Let \( \tilde{u} := \sum_{i<i_0} \alpha_i P \delta_i \).
\( \tilde{u} \) has to satisfy the case 1 or \( \tilde{u} \in V_2(i_0 - 1, \varepsilon) \). Then, we define \( Z_1(\tilde{u}) \) the corresponding vector field, and we get
\[
\langle \partial J(u), Z_1 \rangle \leq -c \left( \sum_{i<i_0} \frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j, i<i_0} \varepsilon_{ij} \right) + \sum_{i<i_0, j \geq i_0} \frac{1}{\lambda_i \lambda_j \frac{n-2}{2}}. \] (3.23)
Observe that \( \lambda_i = o(\lambda_j), \forall i < i_0, \forall j \geq i_0. \)
Thus (3.23) becomes
\[
\langle \partial J(u), Z_1 \rangle \leq -c \left( \sum_{i<i_0} \frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j, i<i_0} \varepsilon_{ij} \right).
\] (3.24)
Observe that all the \( \frac{1}{\lambda_i} \)'s, \( i_0 \leq i \leq p \), appear in the upper bound (3.24). Thus (3.24) becomes

\[
\langle \partial J(u), Z_1 \rangle \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \sum_{i<i_0} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=j, i < i_0} \varepsilon_{ij} \right)
\]

(3.25)

\[
\langle \partial J(u), Z_1 + \sum_{i \geq i_0} X_i \rangle \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \sum_{i<i_0} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=j, i \leq i_0} \varepsilon_{ij} \right).
\]

(3.26)

Now, arguing as in the proof of lemma 3.1, we get

\[
\langle \partial J(u), Z_1 + \sum_{i \geq i_0} X_i \rangle \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \sum_{i<i_0} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=j, i \leq i_0} \varepsilon_{ij} \right).
\]

(3.27)

The vector field \( \tilde{W}_1 := Z_1 + \sum_{i \geq i_0} X_i \) satisfies lemma 3.2.

**Subcase 2.2:** \( i_0 = 1 \). Recall that

\[
\langle \partial J(u), \sum_{i=1}^{p} X_i \rangle \leq -c \sum_{i=1}^{p} \left[ 1 - \Psi(\lambda_i |(a_i - y_{j_i})_n|) \right] \frac{|\nabla K(a_i)|}{\lambda_i}
\]

\[
+ o \left( \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^{n-2}} \right),
\]

(3.28)

where \( k_i \) denotes the index such that \( |(a_i - y_{j_i})_n| = \max_{1 \leq k \leq n} |(a_i - y_{j_i})_k| \). Now, observe that in this case 2, we have

\[
\frac{1}{\lambda_i^{n-2}} \leq c \frac{1}{\lambda_i^{3 \beta_i - 1}} \frac{|\nabla K(a_{i_1})|}{\lambda_{i_1}}
\]

(3.29)

Combining (3.27) and (3.29), we obtain

\[
\langle \partial J(u), \sum_{i=1}^{p} X_i \rangle \leq -c \left( \sum_{i=1}^{p} \frac{|\nabla K(a_{i_1})|}{\lambda_{i_1}} + \sum_{i=1}^{p} \left[ 1 - \Psi(\lambda_i |(a_i - y_{j_i})_n|) \right] \frac{|\nabla K(a_i)|}{\lambda_i} \right)
\]

\[
+ o \left( \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^{n-2}} \right).
\]

(3.30)

From the observation (3.28), we can make appear \( \frac{1}{\lambda_{i_1}} \) in the upper bound (3.30), and then all
The vector field \( \sum_{i=1}^{p} X_i \) satisfies lemma 3.2. The pseudo-gradient \( \tilde{W}_1 \) required in lemma 3.2 will be defined by convex combination of \( Y_1, \tilde{W}_1 \) and \( \sum_{i=1}^{p} \bar{X}_i \).

Observe that the variation of the maximum of the \( \lambda'_i \)s occurs only under the condition
\[
\lambda_i^{\alpha_{ji}} |a_i - y_j| \leq \delta, \quad \forall 1 \leq i \leq p,
\]
for \( \delta > 0 \) a fixed constant small enough. In this case all the \( \lambda'_i \)s increase and go to \( +\infty \) along the flow-lines generated by \( \tilde{W}_1 \).

**Lemma 3.3** In \( V_3(p, \varepsilon, d_0) \), there exists a pseudo-gradient \( \tilde{W}_3 \) so that the following holds: There is a constant \( c > 0 \) independent of \( u \in V_3(p, \varepsilon, d_0) \) so that
\[
\langle \partial J(u), \tilde{W}_3(u) \rangle \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \right).
\]

**Proof.** For each critical point \( y_k \) of \( K \), we set \( B_k := \{ j, \; a_j \in B(y_k, \eta) \} \). Without loss of generality, we can assume \( y_1, \ldots, y_q \) are the critical points such that \( \text{card}(B_k) \geq 2, \; \forall k = 1, \ldots, q \). Let \( \chi \) be a smooth cut-off function such that \( \chi \geq 0, \; \chi = 0 \text{ if } t \leq \gamma, \) and \( \chi = 1 \text{ if } t \geq 1, \) where \( \gamma \) is a small constant. Set \( \bar{\chi}(\lambda_j) = \sum_{i \neq j, i, j \in B_k} \chi(\lambda_j / \lambda_i) \). Define
\[
W_3 = -\sum_{k=1}^{q} \sum_{j \in B_k} \alpha_j \bar{\chi}(\lambda_j) \lambda_j \frac{\partial P \delta_i}{\partial \lambda_j}.
\]
Using proposition 3.1, we derive that

\[
\langle \partial J(u), W_3 \rangle = 2c_2 J(u) \sum_{k=1}^{q} \sum_{j \in B_k} \alpha_j \chi(\lambda_j) \left[ \frac{n - 2}{2} \alpha_j \frac{H(a_j, a_j)}{\lambda_j^{n-2}} \right. \\
+ \sum_{i \neq j} \alpha_i (\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \frac{n - 2}{2} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{n-2}}) \\
\left. + o \left( \sum_{i \neq k} \varepsilon_{ik} + \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} \right) + \sum_{k=1}^{q} \sum_{i \in B_k, \chi(\lambda_i) \neq 0} \varepsilon_{ij} \right] (if \lambda_i |a_i - y_j| \geq C) o(\frac{\nabla K(a_i)}{\lambda_i}).
\]

For \( j \in B_k \), with \( k \leq q \), if \( \chi(\lambda_j) \neq 0 \), then there exists \( i \in B_k \) such that \( \lambda_j^{-1} = o(\lambda_i^{-1}) \) or \( \lambda_j^{2-n} = o(\varepsilon_{ij}) \) (for \( \eta \) small enough). Furthermore, for \( j \in B_k \), if \( i \notin B_k \) or \( i \in B_k \), with \( \gamma < \lambda_i/\lambda_j < 1/\gamma \), then we have \( \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} = -\varepsilon_{ij} (1 + o(1)) \). In the case where \( i \in B_k \) and \( \lambda_i/\lambda_j \leq \gamma \), we have \( \chi(\lambda_j) - \chi(\lambda_i) \geq 1 \) and \( \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \leq -\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \). Thus

\[
\chi(\lambda_j) \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \chi(\lambda_i) \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \leq \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} = -\varepsilon_{ij} (1 + o(1))
\]

Thus we derive that

\[
\langle \partial J(u), W_3 \rangle \leq -c \sum_{k=1}^{q} \sum_{j \in B_k, \chi(\lambda_j) \neq 0} \left( \sum_{i \neq j} \varepsilon_{ij} \right) + o \left( \sum_{i \neq k} \varepsilon_{ik} + \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \frac{\nabla K(a_i)}{\lambda_i} \right) \\
\leq -c \sum_{k=1}^{q} \sum_{j \in B_k, \chi(\lambda_j) \neq 0} \left( \frac{1}{\lambda_j^{n-2}} + \sum_{i \neq j} \varepsilon_{ij} \right) + o \left( \sum_{i \neq k} \varepsilon_{ik} + \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} \right) \\
+ \sum_{k=1}^{q} \sum_{i \in B_k, \chi(\lambda_i) \neq 0} \varepsilon_{ij} \right] (if \lambda_i |a_i - y_j| \geq C) o(\frac{\nabla K(a_i)}{\lambda_i}).
\]

Observe that \( \{ j \in B_k, \chi(\lambda_j) = 0 \} \) contains at most one index. Thus we obtain

\[
\langle \partial J(u), W_3 \rangle \leq -c \sum_{k=1}^{q} \sum_{j \in B_k, \chi(\lambda_j) \neq 0} \frac{1}{\lambda_j^{n-2}} + \sum_{i \neq j, j \in \cup_{k=1}^{q} B_k} \varepsilon_{ij} + o \left( \sum_{i \neq k} \varepsilon_{ik} + \sum_{i=1}^{p} \frac{1}{\lambda_i} \right) \\
+ \sum_{k=1}^{q} \sum_{i \in B_k, \chi(\lambda_i) \neq 0} \varepsilon_{ij} \right] (if \lambda_i |a_i - y_j| \geq C) o(\frac{\nabla K(a_i)}{\lambda_i}).
\]

This upper bound does not contains all the indices. We need to add some terms. Let

\[
\lambda_{i_0} = \inf \{ \lambda_i, i = 1, \ldots, p \}.
\]

Two cases may occur:

**Case 1:** If \( i_0 \in \cup_{k=1}^{q} B_k \), with \( \chi(\lambda_{i_0}) \neq 0 \), then we can make appear in the last upper bound

\[
\lambda_{i_0} = \inf \{ \lambda_i, i = 1, \ldots, p \}.
\]
\( i_0 \), and therefore all the \( \frac{1}{\lambda_i} \), and so \( \varepsilon_{ik}, 1 \leq i, k \leq p \).

**Case 2:** \( i_0 \in \{ i \in \bigcup_{k=1}^{q} B_k, \overline{\lambda}(\lambda_i) = 0 \} \cup (\bigcup_{k=1}^{q} B_k)^c \). In this case, we define

\[
D = (\{ i \in \bigcup_{k=1}^{q} B_k, \overline{\lambda}(\lambda_i) = 0 \} \cup (\bigcup_{k=1}^{q} B_k)^c) \cap \{ 1 \leq i \leq p, \lambda_i/\lambda_{i_0} < 1/\gamma \}.
\]

It is easy to see that for \( i, j \in D, i \neq j \), we have \( a_i \in B(y_{ki}, \eta) \) and \( a_j \in B(y_{kj}, \eta) \) with \( k_i \neq k_j \).

Let

\[
u_1 = \sum_{i \in D} \alpha_i P \delta(a_i, \lambda_i).
\]

\( u_1 \) has to satisfy one of the two above cases, that is, \( u_1 \in V_i(\text{card}(D), \varepsilon, d_0) \) for \( i = 1, 2 \). Thus we can apply the associated vector field which we will denote \( W_3' \), and we have the following estimate:

\[
\langle \partial J(u), W_3' \rangle \leq -c \left( \sum_{i \in D} \frac{1}{\lambda_i^{n-2}} + \frac{1}{\lambda_i \gamma} \right) + \sum_{i \neq j, i,j \in D} \varepsilon_{ij} + O\left( \sum_{r \in \partial D, k \in D} \varepsilon_{rk} + \sum_{i \in D} \frac{1}{\lambda_i^{n-2}} \right).
\]

Observe that for \( k \in D \) and \( r \not\in D \), we have either \( r \in \bigcup_{k=1}^{q} B_k, \overline{\lambda}(\lambda_r) \neq 0 \) (in this case we have \( \varepsilon_{kr} \) in the upper bound (3.33)) or no, and in this last case we observe that \( a_i \in B(y_{ji}, \eta) \), for \( i = r, k \), with \( j_r \neq j_k \). Thus

\[
\varepsilon_{kr} \leq \frac{c}{(\lambda_k \lambda_r)^{\frac{n-2}{2}}} \leq \frac{c^2}{\lambda_{i_0}^{n-2}} = o\left( \frac{1}{\lambda_{i_0}^{n-2}} \right).
\]

We get the same observation for \( \lambda_i^{2-n}, i \not\in D \). Now we define

\[
Y_2 = C W_3 + W_3',
\]

where \( C \) is a large positive constant. We obtain

\[
\langle \partial J(u), Y_2 \rangle \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j} \varepsilon_{ij} \right) + o\left( \sum_{i=1}^{p} \frac{\nabla K(a_i)}{\lambda_i} \right).
\]

We define \( W_2'' \) as a convex combination of \( W_3 \) and \( Y_2 \). Then the pseudo-gradient

\[
\tilde{W}_3 := W_2'' + \sum_{i=1}^{p} \overline{\lambda}_i
\]

satisfies the claim of lemma 3.3.

**Lemma 3.4** In \( V_4(p, \varepsilon, d_0) \), there exists a pseudo-gradient \( \tilde{W}_4 \) so that the following holds: There is a constant \( c > 0 \) independent of \( u \in V_4(p, \varepsilon, d_0) \) so that

\[
\langle \partial J(u), \tilde{W}_4(u) \rangle \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \frac{\nabla K(a_i)}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \right).
\]
Proof. Without loss of generality, we suppose $\lambda_1 \leq \cdots \leq \lambda_p$.
We denote by $i_1$ the index satisfying $a_{i_1} \not\in \bigcup_{\nabla K(y)=0} B(y, \eta)$ and $a_i \in B(y_j, \eta), \forall i < i_1$. Let
\[ \tilde{u} = \sum_{i<i_1} \alpha_i P \delta_i. \]
Observe that $\tilde{u} \in V_i(i_1 - 1, \varepsilon), i = 1, 2$ or $3$. Then we define $Z_4'(\tilde{u})$ the corresponding vector field and we have
\[ \langle \partial J(u), Z_4'(\tilde{u}) \rangle \leq -c \left( \sum_{i<i_1} \frac{1}{\lambda_i^{n-2}} + \frac{\nabla K(a_i)}{\lambda_i} + \sum_{i \neq j, i,j < i_1} \varepsilon_{ij} \right) + O \left( \sum_{i<i_1, j \geq i_1} \varepsilon_{ij} + \sum_{i \geq i_1} \frac{1}{\lambda_i^{n-2}} \right). \]
Let now
\[ Z_4 = \frac{1}{\lambda_i} \frac{\nabla K(a_{i_1})}{|\nabla K(a_{i_1})|} \frac{\partial \delta_{a_{i_1}, \lambda_{i_1}}}{\partial a_{i_1}} - M_3 \sum_{i \geq i_1} 2^i \lambda_i \frac{\partial \delta_{a_i, \lambda_i}}{\partial \lambda_i}, \]
where $M_3 > 0$ is a fixed constant large enough. From propositions 3.1 and 3.2, we obtain, since $\nabla K(a_{i_1}) \neq 0$,
\[ \langle \partial J(u), Z_4(u) \rangle \leq -c \frac{\lambda_{i_1}}{\lambda_i} + O \left( \sum_{j \neq i_1} \lambda_j |a_{i_1} - a_j| \varepsilon_{i_1,j}^3 \right) - M_3 c \sum_{i \geq i_1, j \neq i} \varepsilon_{ij}. \]
Observe that $\lambda_j |a_{i_1} - a_j| \varepsilon_{i_1,j}^3 = O(\varepsilon_{i_1,j}), \forall j \neq i_1$. Thus
\[ \langle \partial J(u), Z_4(u) \rangle \leq -c \frac{\lambda_{i_1}}{\lambda_i} + O \left( \sum_{j \neq i_1} \varepsilon_{i_1,j} \right) - M_3 c \sum_{i \geq i_1, j \neq i} \varepsilon_{ij}. \]
We choose $M_3$ large enough so that $O \left( \sum_{j \neq i_1} \varepsilon_{i_1,j} \right)$ is absorbed by $M_3 c \sum_{i \geq i_1, j \neq i} \varepsilon_{ij}$. We deduce
\[ \langle \partial J(u), Z_4(u) \rangle \leq -c \left( \frac{1}{\lambda_{i_1}} + \sum_{i \geq i_1, j \neq j} \varepsilon_{ij} \right). \quad (3.34) \]
Also $\frac{1}{\lambda_{i_1}}$ makes appear $\sum_{i \geq i_1} \frac{1}{\lambda_i}$ in the upper bound of (3.34). Taking $M$ a positive constant large enough and let
\[ \tilde{W}_4(u) = M Z_4 + Z_4'. \]
Thus we derive
\[ \langle \partial J(u), \tilde{W}_4(u) \rangle \leq -c \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j} \varepsilon_{ij}. \]
The claim of lemma 3.4 follows.

The pseudo-gradient $W_1$, required in proposition 3.3 will be defined by a convex combination of the vector fields $\tilde{W}_1(u), \tilde{W}_2(u), \tilde{W}_3(u)$ and $\tilde{W}_4(u)$.
Existence Result

Proof of Proposition 3.4. We will introduce some technical lemmas for the proof of proposition 3.4. Without loss of generality, we suppose \( \lambda_1 d_1 \leq \cdots \leq \lambda_p d_p \). Let \( c_1 > 0 \) a fixed constant small enough. We define

\[
I_2 := \{1\} \cup \{1 \leq i \leq p, \text{ s.t } c_1 \lambda_k d_k \leq \lambda_{k-1} d_{k-1} \leq \lambda_k d_k, \forall k \leq i\}.
\]

In \( I_2 \), we order the \( \lambda_i \)'s: \( \lambda_{i_1} \leq \cdots \leq \lambda_{i_s} \). For \( c_2 > 0 \) a fixed constant small enough, we define

\[
I_{\lambda_{i_s}} := \{i_s\} \cup \{1 \leq k \leq s, \text{ s.t } c_2 \lambda_{i_{j+1}} \leq \lambda_{i_j} \leq \lambda_{i_{j+1}}, \forall j \geq k\}.
\]

For \( u = \sum_{i=1}^{p} \alpha_i P \delta_i \in V_{b}(p, \varepsilon) \), we introduce the following condition: for \( i \in \{1, \ldots, p\} \),

\[
\frac{1}{2p+1} \sum_{k \neq i} \varepsilon_{ki} \leq \sum_{j=1}^{p} \frac{H_{ij}}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}}.
\]  (3.35)

We divide the set \( \{1, \ldots, p\} \) into \( T_1 \cup T_2 \), where

\[
T_1 = \{1 \leq i \leq p, \text{ s.t } i \text{ satisfies (3.35)}\}
\]

and

\[
T_2 = \{1, \ldots, p\} \setminus T_1.
\]

Lemma 3.5 There exists a vector field \( X_1 \) such that

\[
\langle \partial J(u), X_1 \rangle \leq -c \left( \frac{1}{\lambda_{i_s}} \sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^{n-1}} + \sum_{k \in T_2, 1 \leq j \leq p} \varepsilon_{kj} + \sum_{j \neq k, j \in T_1, 1 \leq k \leq p} \varepsilon_{kj} \right) + o \left( \sum_{i=1}^{p} \frac{1}{\lambda_i} \right).
\]

Proof. We define the vector field \( Y_1^{2} \) by

\[
Y_1^{2} := \frac{1}{\lambda_{i_s}} \sum_{i \in I_{\lambda_{i_s}}} \frac{\partial P \delta_i}{\partial a_i} (-\alpha_i \nu_i).
\]

From proposition 3.2 we obtain

\[
\langle \partial J(u), Y_1^{2} \rangle \leq -c \left( \frac{1}{\lambda_{i_s}} + \frac{1}{(\lambda_i d_i)^{n-1}} \right) + \frac{1}{\lambda_{i_s}} O \left( \sum_{k,j \in I_{\lambda_{i_s}}} \lambda_k \lambda_j |a_k - a_j| \nu_k - \nu_j |\varepsilon_{kj}^{\frac{n-2}{2}} \right)
\]

\[
+ \frac{1}{\lambda_{i_s}} O \left( \sum_{k \in I_{\lambda_{i_s}}, j \notin I_{\lambda_{i_s}}} \lambda_k \lambda_j |a_k - a_j| \varepsilon_{kj}^{\frac{n-2}{2}} \right) + o \left( \sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^{n-1}} \right).
\]  (3.36)
Thus, after appearing \( \sum_{i=1}^{p} \frac{1}{(\lambda_id_i)^{n-1}} \), we have existence of \( \sum_{j \neq k; j \in T_1, 1 \leq k \leq p} \frac{n-1}{\varepsilon_{jk}} \) in the same upper bound. Observe that, for \( k, j \in I_{\lambda_i} \), \( |\nu_k - \nu_j| = O(|a_k - a_j|) \). So, we get

\[
\frac{1}{\lambda_i} \lambda_k \lambda_j |a_k - a_j| \varepsilon_{jk} = \frac{1}{\lambda_i} O(\varepsilon_{k,j}) = o\left(\frac{1}{\lambda_i}\right).
\]

Thus, after appearing \( \sum_{i=1}^{p} \frac{1}{(\lambda_id_i)^{n-1}} \), we have existence of \( \sum_{j \neq k; j \in T_1, 1 \leq k \leq p} \frac{n-1}{\varepsilon_{jk}} \) in the same upper bound. Observe that, for \( k, j \in I_{\lambda_i} \), \( |\nu_k - \nu_j| = O(|a_k - a_j|) \). So, we get

\[
\frac{1}{\lambda_i} \lambda_k \lambda_j |a_k - a_j| \varepsilon_{jk} = \frac{1}{\lambda_i} O(\varepsilon_{k,j}) = O\left(\varepsilon_{k,j}\right).
\]

We are left for the estimate of \( \frac{1}{\lambda_i} \lambda_k \lambda_j |a_k - a_j| \varepsilon_{jk} \) with \( j \notin I_{\lambda_i} \) and \( k \in I_{\lambda_i} \).

If \( k \in T_2 \) or \( j \in T_2 \), we get

\[
\frac{1}{\lambda_i} \lambda_k \lambda_j |a_k - a_j| \varepsilon_{jk} = O\left(\frac{\lambda_j}{\lambda_k} |a_k - a_j| \varepsilon_{jk}\right) = O\left(\frac{\lambda_j}{\lambda_k} \varepsilon_{jk}\right) = O\left(\varepsilon_{k,j}\right).
\]

If \( k, j \in T_1 \). In this case, we observe that

\[
\frac{1}{\lambda_i} \lambda_k \lambda_j |a_k - a_j| \varepsilon_{jk} = o\left(\sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^{n-1}}\right).
\]

As a conclusion of the last observations, we obtain

\[
\langle \partial J(u), Y_2 \rangle \leq -c \left(\frac{1}{\lambda_i} + \sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^{n-1}} + \sum_{j \neq k; j \in T_1, 1 \leq k \leq p} \frac{n-1}{\varepsilon_{jk}}\right)
+ O\left(\sum_{k \in T_2, 1 \leq j \leq p} \varepsilon_{k,j}\right).
\]

Let \( T_2 = \{i_1, \ldots, i_r\} \), with \( \lambda_{i_1} \leq \cdots \leq \lambda_{i_r} \). We define

\[
Y_2 := -\sum_{k=1}^{r} 2^k \alpha_{ik} \lambda_{ik} \frac{\partial P \delta_{ik}}{\partial \lambda_{ik}}.
\]

Using proposition 3.1, we obtain

\[
\langle \partial J(u), Y_2 \rangle = 2c_2 J(u) \sum_{k=1}^{r} \left[ \sum_{j \neq i_k} 2^k \alpha_j \alpha_{ik} \lambda_{ik} \frac{\partial \varepsilon_{jk}}{\partial \lambda_{ik}} + \sum_{j=1}^{n-2} \frac{p}{2} 2^k \alpha_j \alpha_{ik} \frac{H(a_j, a_{ik})}{(\lambda_j \lambda_{ik})^{n-2}} \right] \times (1 + o(1))
+ O\left(\sum_{i \neq j, i \in T_2} \varepsilon_{ij} + \sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^{n-1}} + \frac{1}{\lambda_k}\right).
\]
We have
\[ \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\frac{n-2}{2} \varepsilon_{ij} \left( 1 - 2 \frac{\lambda_j}{\lambda_i} \varepsilon_{ij}^{\frac{n-2}{2}} \right), \forall i \neq j. \]

Observe also that
\[ \lambda_j \varepsilon_{ij}^{\frac{n-2}{2}} = o(1), \forall i \neq j, j \in T_1. \]

Thus we derive
\[ \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \leq -\frac{3(n-2)}{8} \varepsilon_{ij}, \forall i \neq j, j \in T_1. \quad (3.38) \]

However, for \( j \in T_2 \) and \( \lambda_j \leq \lambda_i \), we obtain
\[ 2\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \leq -\frac{3(n-2)}{8} \varepsilon_{ij}, \forall i \neq j. \quad (3.39) \]

Combining (3.37), (3.38) and (3.39), we derive that
\[ \langle \partial J(u), Y_2^2 \rangle \leq (n-2)c_2 J(u) \sum_{k \in T_2} \left[ -\frac{3}{4} \sum_{j \neq k} \varepsilon_{jk} + 2^{p} \sum_{j=1}^{p} \frac{H(a_j, a_k)}{\lambda_j \lambda_k^{\frac{n-2}{2}}} \right] \]
\[ + o \left( \sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^{n-1}} + \frac{1}{\lambda_k} \right) \]
\[ \leq - \sum_{j \neq k, k \in T_2} \varepsilon_{jk} + o \left( \sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^{n-1}} + \frac{1}{\lambda_k} \right). \quad (3.40) \]

Taking \( m > 0 \) a fixed constant large enough, the vector field
\[ X_1 = Y_1^2 + m \]

satisfies the claim of lemma 3.5.

**Lemma 3.6** There exist a vector field \( X_2 \) such that
\[ \langle \partial J(u), X_2 \rangle \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i} \right) + O \left( \sum_{j \in T_2, 1 \leq k \leq p} \varepsilon_{kj} \right) + O \left( \sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^{n-1}} \right). \]

**Proof.** Without loss of generality, we suppose \( \lambda_1 \leq \cdots \leq \lambda_p \). We define, for \( M > 0 \) a fixed constant large enough,
\[ I'_1 := \{ 1 \leq i \leq p, \ \text{s.t} \ |a_i - a_1| \geq \frac{2}{M} d_1 \}, \]
\[ I''_1 := \{ i \in I'_1, \ \text{s.t} \ \exists (i_1, \ldots, i_r), \ \text{with} \ i_1 = i, i_r \in I'_1, \ \text{and} \ |a_{i_{k-1}} - a_{i_k}| < \frac{d_1}{p M}, \ \forall \ k \leq r \} \]
and
\[ I_1 := \{ 1, \ldots, p \} \setminus \{ I'_1 \cup I''_1 \}. \]
Observe that, for $k \in I_1$ and $j \notin I_1$, we have

$$|a_k - a_j| \geq \frac{1}{pM}d_1.$$ 

Let us define, for $c_2 > 0$ a fixed constant small enough,

$$I_{\lambda_1} := \{1\} \cup \{1 \leq j \leq p, \text{ s.t } c_2\lambda_{ik} \leq \lambda_{ik} - \lambda_{ij}, \forall k \leq j\}.$$ 

We set

$$X_2 := \frac{1}{\lambda_1} \sum_{i \in I_1 \cap I_{\lambda_1}} \frac{\partial P\delta_i}{\partial a_i}(-\alpha_i\nu_i).$$ 

Observe that $d_i \sim d_1$, for $i \in I_1 \cap I_{\lambda_1}$. From proposition 3.2, we obtain

$$\langle \partial J(u), X_2 \rangle \leq -c\left(\frac{1}{\lambda_1}\right) + \frac{1}{\lambda_1}O\left(\sum_{j \in I_1 \cap I_{\lambda_1}} \lambda_k\lambda_j|a_k - a_j|\nu_k - \nu_j \varepsilon_{kj}^{\frac{a}{2}}\right)$$

$$+ \frac{1}{\lambda_1}O\left(\sum_{k \in I_1 \cap I_{\lambda_1}, j \notin I_{\lambda_1}} \lambda_k\lambda_j|a_k - a_j|\nu_k - \nu_j \varepsilon_{kj}^{\frac{a}{2}}\right)$$

$$+ o\left(\sum_{k \neq j} \frac{\varepsilon_{kj}^{\frac{n-2}{n}}}{\lambda_1}\right) + o\left(\sum_{k = 1}^{p} \frac{1}{(\lambda_k d_k)^{n-1}}\right).$$ 

Observe that, for $k, j \in I_{\lambda_1} \cap I_1$, $|\nu_k - \nu_j| = O(|a_k - a_j|)$. From this, we deduce

$$\frac{1}{\lambda_1}\lambda_k\lambda_j|a_k - a_j|\nu_k - \nu_j \varepsilon_{kj}^{\frac{a}{2}} = \frac{1}{\lambda_1}O(\varepsilon_{kj}) = o\left(\frac{1}{\lambda_1}\right).$$

Now, we need to estimate the quantity $\frac{1}{\lambda_1}\lambda_k\lambda_j|a_k - a_j|\varepsilon_{kj}^{\frac{a}{2}}$, for $j \notin I_{\lambda_1} \cap I_1$ and $k \in I_{\lambda_1} \cap I_1$. We have two cases:

**Case 1:** $j \notin I_1$. In this case, we have $|a_k - a_j| \geq \frac{1}{pM}d_1$. From another side, since $k \in I_1$, we observe that $d_k \sim d_1$. Thus we deduce

$$\frac{1}{\lambda_1}\lambda_k\lambda_j|a_k - a_j|\varepsilon_{kj}^{\frac{a}{2}} = O\left(\frac{1}{\lambda_1d_1}\varepsilon_{kj}\right) = O\left(\frac{1}{(\lambda_1d_1)^{n-1}}\right) + o(\varepsilon_{kj}^{\frac{a-1}{2}}).$$

**Case 2:** $j \in I_1$. In this case, we observe that $d_j \sim d_k \sim d_1$ and $|a_k - a_j| \leq \frac{1}{pM}d_1$. If $j \notin T_1$. We use the fact that $H(a_j, a_j) \leq c_2\frac{1}{d_j^{n-1}}$, and $H(a_i, a_j) \leq c\frac{1}{(d_i d_j)^{\frac{n-1}{2}}}$ (see [16]), then we deduce

$$\frac{1}{\lambda_1}\lambda_k\lambda_j|a_k - a_j|\varepsilon_{kj}^{\frac{a}{2}} \leq \frac{1}{c_2}\lambda_j|a_k - a_j|\varepsilon_{kj}^{\frac{a}{2}}$$

$$\leq \frac{c}{c_2}\lambda_j|a_k - a_j|\sum_{i = 1}^{p} \frac{1}{(\lambda_id_i)^{\frac{n}{2}}}(\lambda_j d_j)^{\frac{n}{2}}$$

$$\leq \frac{c}{c_2} \frac{1}{M} \sum_{i = 1}^{p} \frac{1}{(\lambda_i d_i)^{\frac{n}{2}}}(\lambda_j d_j)^{\frac{n}{2}}.$$
We choose $\frac{1}{\epsilon_j^{n-2}} = o(1)$, and therefore
\[ \frac{1}{\lambda_1} \lambda_k \lambda_j |a_k - a_j|^{\frac{n}{n-2}} = O \left( \sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^{n-1}} \right). \]
If $j \in T_2$, we easily have
\[ \frac{1}{\lambda_1} \lambda_k \lambda_j |a_k - a_j|^{\frac{n}{n-2}} = O(\epsilon_{kj}). \]
We conclude that
\[ \langle \partial J(u), X_2 \rangle \leq -c \left( \frac{1}{\lambda_1} \right) + O \left( \sum_{j \in T_2, 1 \leq k \leq p} \epsilon_{kj} \right) + O \left( \frac{1}{(\lambda_1 d_1)^{n-1}} \right) + o \left( \sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^{n-1}} \right). \]
Such vector field $X_2$ satisfies the upper bound of lemma 3.6. Thus, for $m_1 > 0$ a fixed constant large enough, the pseudo-gradient
\[ W_2(u) := X_2 + m_1 X_1 \]
satisfies the claim of proposition 3.3.

**Corollary 3.1** Let $n \geq 4$. Assume that $K$ satisfies the condition $(f)_3$, Under the assumptions $(A_1)$ and $(A_2)$, The critical points at infinity of $J$ in $V(p, \varepsilon)$, $p \geq 1$, correspond to
\[ \sum_{j=1}^{p} \frac{1}{K(y_j)} \frac{1}{(\lambda_j^{n-2})} \delta_{(y_j, \infty)}, \]
where $(y_1, \ldots, y_p) \in C_\infty$. Moreover, such a critical point at infinity has an index equal to $p - 1 + \sum_{j=1}^{p} n - \sum_{j=1}^{p} \frac{1}{K(y_j)}$.

**Proof.** Using theorem 3.1 the only region where the $\lambda_i$’s are unbounded is the one where each $a_i$ is close to a critical point $y_{i'}$, where $y_{i'} \neq y_{j_k}$, for $i \neq k$, and $(y_{i_1}, \ldots, y_{i_p}) \in C_\infty$. In this region, arguing as in appendix 2 of 3, we can find a change of variables
\[ (a_1, \ldots, a_p, \lambda_1, \ldots, \lambda_p) \mapsto (\tilde{a}_1, \ldots, \tilde{a}_p, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_p) =: (\tilde{a}, \tilde{\lambda}) \]
such that
\[ J(\sum_{i=1}^{p} \alpha_i P \delta_i + \tilde{v}) = \frac{S_n^2}{(\sum_{i=1}^{p} \alpha_i^{n-2}) K(\tilde{a}_i)} \left( 1 + c \cdot T \Lambda M \right) =: \Psi(\alpha, \tilde{a}, \tilde{\lambda}), \]
where $M := M(y_{i_1}, \ldots, y_{i_p})$ is the matrix defined by (1.2), $T \Lambda := \left( \frac{1}{(\lambda_1)^{n-2}}, \ldots, \frac{1}{(\lambda_p)^{n-2}} \right)$, $S_n := \int_{\mathbb{R}^n} \delta_{\alpha, \lambda}^\frac{2n}{n-2} (x) dx$ and $c$ is a positive constant. Observe that the function $\Psi$ admits for the variables $\alpha_i$’s an absolute degenerate maximum with one dimensional nullity space. Then the index of such critical point at infinity is equal to $p - 1 + \sum_{i=1}^{p} n - \sum_{j=1}^{p} \frac{1}{K(y_{j_i})}$. The result of corollary 3.1 follows.
4 Proof of the main result

Proof of theorem 1.1 For technical reasons, we introduce, for $\varepsilon_0 > 0$ small enough, the following neighborhood of $\Sigma^+$:

$$V_{\varepsilon_0}(\Sigma^+) := \{ u \in \Sigma / \| u^- \|_{L_{n-2}} < \varepsilon_0 \},$$

where $u^- := \max(0, -u)$. Recall that in theorem 3.1 we construct a vector field $W$ defined in $V(p, \varepsilon)$, $p \geq 1$. Outside $\bigcup_{p \geq 1} V(p, \varepsilon^2)$ we will use $-\partial J$, and our global vector field $Z$ will be built using a convex combination of $W$ and $-\partial J$. Arguing as in the proof of lemma 4.1 [6], we can prove that $V_{\varepsilon_0}(\Sigma^+)$ is invariant under the flow lines generated by $-\partial J$, and therefore $V_{\varepsilon_0}(\Sigma^+)$ is invariant under the flow lines generated by $Z$. We will prove the existence result by contradiction. We suppose that $J$ has no critical points in $V_{\varepsilon_0}(\Sigma^+)$.

It follows from corollary 3.1 that the only critical points at infinity of $J$ in $V_{\varepsilon_0}(\Sigma^+)$ correspond to

$$\sum_{j=1}^p \frac{1}{K(y_{i_j})} n_{i_j} P\delta(y_{i_j}, \infty), \ p \geq 1, \text{ where } (y_{i_1}, \ldots, y_{i_p}) \in C_\infty.$$

Such a critical point at infinity has an index equal to $p - 1 + \sum_{j=1}^p n - \tilde{i}(y_{i_j})$.

Given the pseudo-gradient $Z$ for $J$ on $V_{\varepsilon_0}(\Sigma^+)$, we derive from the retraction theorem 8.2 [5] that $V_{\varepsilon_0}(\Sigma^+)$ retracts by deformation onto

$$X_\infty := \bigcup_{\tau_p \in C_\infty} W_u((\tau_p)_\infty),$$

where $W_u((\tau_p)_\infty)$ is the unstable manifold at infinity associated to the critical point at infinity $(\tau_p)_\infty$. Since $V_{\varepsilon_0}(\Sigma^+)$ is contractible, then we obtain

$$\chi(X_\infty) = \chi(V_{\varepsilon_0}(\Sigma^+)) = 1. \quad (4.1)$$

Now, we call back the following fact: let $M$ be a finite cw-complex in dimension $k$, then the Euler-Poincaré characteristic of $M$, $\chi(M)$, is given by

$$\chi(M) = \sum_{i=1}^k (-1)^i n(j), \quad (4.2)$$

where $n(j)$ is the number of cells of dimension $j$ in $M$ (see [11]). We apply this fact to our situation, where the cells of dimension an integer $j$ in $X_\infty$ are given by $W_u((\tau_p)_\infty)$ such that $i((\tau_p)_\infty) = j$. According to (4.2), we obtain

$$\chi(X_\infty) = \sum_{\tau_p \in C_\infty} (-1)^i(\tau_p), \quad (4.3)$$
Combining (4.3) and (4.1), we get
\[ \sum_{\tau_p \in C_\infty} (-1)^i(\tau_p) = 1 \]
which contradicts the assumption of our theorem. Thus there exists a critical point of \( J \) in \( V_{\varepsilon_0}(\Sigma^+) \). Now, since \( \varepsilon_0 \) is small enough, we derive by a standard argument that \( u^- = 0 \), and therefore \( u > 0 \) in \( \Omega \). This finishes the proof of our result.

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