A Unified Analysis of Balancing Domain Decomposition by Constraints for Discontinuous Galerkin Discretizations

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Abstract. The BDDC algorithm is extended to a large class of discontinuous Galerkin (DG) discretizations of second order elliptic problems. An estimate of $C(1 + \log(H/h))^2$ is obtained for the condition number of the preconditioned system where $C$ is a constant independent of $h$ or $H$ or large jumps in the coefficient of the problem. Numerical simulations are presented which confirm the theoretical results. A key component for the development and analysis of the BDDC algorithm is a novel perspective presenting the DG discretization as the sum of element-wise “local” bilinear forms. The element-wise perspective allows for a simple unified analysis of a variety of DG methods and leads naturally to the appropriate choice for the subdomain-wise local bilinear forms. Additionally, this new perspective enables a connection to be drawn between the DG discretization and a related continuous finite element discretization to simplify the analysis of the BDDC algorithm.

Key words. Discontinuous Galerkin, Domain Decomposition, BDDC

AMS subject classifications. 65M55, 65M60, 65N30, 65N55

1. Introduction. Domain decomposition (DD) methods provide efficient parallel preconditioners for solving large system of equations arising from the discretization of partial differential equations. The development of domain decompositions methods for the solution of elliptic problems using conforming finite element methods has matured significantly over the past 20 years. Toselli and Widlund provide a detailed overview of domain decomposition methods in [22]. In this paper we consider a class of non-overlapping domain decomposition methods based on the Neumann-Neumann methods originally introduced by Bourgat et al. [6]. These methods were improved by introducing a coarse space based on the null-space of the local Schur complement problems, leading to the Balancing Domain Decomposition (BDD) method of Mandel [17]. Dohrmann extended the BDD method by selecting a coarse space formed by enforcing continuity of a small set of primal degrees of freedom [11]. This Balancing Domain Decomposition by Constraints (BDDC) method was later proven by Mandel et al. [18] to have a condition number bound of $\kappa \leq C(1 + \log(H/h))^2$ for preconditioned system of a continuous finite element discretization of second order elliptic problems. Further analysis of BDDC methods as well as the relationship between BDDC methods and dual-primal Finite Element Tearing and Interconnecting (FETI-DP) methods has been presented in [16, 7, 19].

In this paper we present a BDDC method for the solution of a discontinuous Galerkin (DG) discretization of a second-order elliptic problem. While domain decomposition methods have been widely studied for continuous finite element discretizations, relatively little work has been performed for discontinuous Galerkin discretizations. Feng and Karakashian presented a two-level Schwarz preconditioner for an interior penalty DG discretization of the Poisson problem [15]. Feng and Karakashian considered both overlapping and non-overlapping preconditioners and obtained condition number bounds of $O(H/\delta)$ and $O(H/h)$ respectively. Antonietti and Ayuso considered additive and multiplicative Schwarz preconditioners for a large class of DG discretizations of elliptic problems in [1, 2, 3]. Antonietti and Ayuso employed the unified framework of Arnold et al. [4] to analyze these DG methods and showed that condition number bounds of order $O(H/h)$ could be obtained with these preconditioners for symmetric DG schemes.

In the context of Neumann-Neumann type methods for DG discretizations, Dryja et al. employed a conforming finite
element discretization on each subdomain while using an interior penalty method across non-conforming subdomain boundaries [14, 12, 13]. Using this discretization Dryja et al were able to leverage results from the continuous finite element analysis to obtain condition number bounds of $\kappa \leq C(1 + \log(H/h))^2$ for particular BDD and BDDC methods. In this work we present a BDDC method applied to a large class of DG methods considered in the unified analysis of Arnold et al. [4]. A key component for the development and analysis of the BDDC algorithm is a novel perspective presenting the DG discretization as the sum of element-wise “local” bilinear forms. The element-wise perspective leads naturally to the appropriate choice for the subdomain-wise local bilinear forms. Additionally, this new perspective enables a connection to be drawn between the DG discretization and a related continuous finite element discretization. By exploiting this connection, we prove a condition number bound of $\kappa \leq C(1 + \log(H/h))^2$ for the BDDC preconditioned system for a large class of conservative and consistent DG methods.

In Section 2 we give a classical presentation of the DG discretization. In Section 3 we present our new perspective on the DG discretization. In Sections 4 and 5, respectively, we discuss our domain decomposition strategy and present the BDDC algorithm. The analysis of the BDDC algorithm in presented in Section 6, while in Section 7 we present numerical results confirming the analysis.

2. DG Discretization. We consider the following second order elliptic equation in a domain $\Omega \subset \mathbb{R}^n, n = 2, 3$.

\[-\nabla \cdot (\rho \nabla u) = f \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega \tag{2.1}\]

with positive $\rho > 0 \in L^\infty(\Omega), f \in L^2(\Omega)$. Following [4] we may rewrite (2.1) in mixed form in order to motivate the DG formulation. In practice, the fluxes are locally eliminated to obtain the DG discretization in primal form. The mixed form of (2.1) is given by:

\[\rho^{-1} \mathbf{q} + \nabla u = 0 \quad \text{in } \Omega,\]
\[\nabla \cdot \mathbf{q} = f \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega. \tag{2.2}\]

Prior to introducing the exact form of the discrete equations, we introduce the functional setting and notation. Denote by $T_h$ the family of triangulations obtained by partitioning $\Omega$ into triangles or quadrilaterals (if $n = 2$) or tetrahedra or hexahedra (if $n = 3$) of characteristic element size $h$. We make the usual assumption that the family of triangulations $T_h$ is shape-regular, and quasi-uniform [22]. Define $\mathcal{E}$ to be the union of edges (if $n = 2$) or faces (if $n = 3$) of elements $\kappa$. Additionally, define $\mathcal{E}^i \subset \mathcal{E}$ and $\mathcal{E}^{\partial} \subset \mathcal{E}$ to be the set of interior, respectively boundary edges. We note that any edge $e \in \mathcal{E}^i$ is shared by two adjacent elements $\kappa^+$ and $\kappa^-$ with corresponding outward pointing normal vectors $\mathbf{n}^+$ and $\mathbf{n}^-$. Let $\mathcal{P}(\kappa)$ denote the space of polynomials of order at most $p$ on $\kappa$ and define $\mathcal{P}(\kappa) := [\mathcal{P}(\kappa)]^n$. Given the triangulation $T_h$ define the following finite element spaces

\[W_h^p := \{ w_h \in L^2(\Omega) : w_h|_\kappa \in \mathcal{P}(\kappa) \quad \forall \kappa \in T_h \} \tag{2.3}\]
\[V_h^p := \{ v_h \in L^2(\Omega) : v_h|_\kappa \in \mathcal{P}(\kappa) \quad \forall \kappa \in T_h \}. \tag{2.4}\]

Note that traces of functions $u_h \in W_h^p$ are in general double valued on each edge, $e \in \mathcal{E}^i$, with values $u_h^+$ and $u_h^-$ corresponding to traces from elements $\kappa^+$ and $\kappa^-$ respectively. On $e \in \mathcal{E}^{\partial}$, associate $u_h^+$ with the trace taken from the element, $\kappa^+ \in T_h$, neighbouring $e$. The DG discretization of (2.1) obtains a solution $u_h \in W_h^p$ such that for all $\kappa \in T_h$,

\[(\rho \nabla u_h, \nabla w_h)_\kappa - (\rho^+(u_h^+ - \hat{u}_h) \mathbf{n}^+, \nabla w_h^+|_{\partial \kappa}) + (\mathbf{q}_h, w_h^+ \mathbf{n}^+|_{\partial \kappa}) = (f, w_h)_\kappa \quad \forall w_h \in \mathcal{P}(\kappa) \tag{2.5}\]
where \((\cdot, \cdot)_\kappa := \int_\kappa\) and \((\cdot, \cdot)_{\partial \kappa} := \int_{\partial \kappa}\). Superscript \(^+\) is used to explicitly denote values on \(\partial \kappa\), taken from \(\kappa\). For all \(w_h \in W^p_h, \dot{w}_h = w_h(w_h^+ \dot{w}_h^+, w_h^-)\) is a single valued numerical trace on \(\kappa \in E^i\), while \(\dot{w}_h = 0\) for \(\kappa \in E^\partial\). Note that \(\dot{u}_h = 0\) on \(\kappa \in E^\partial\), corresponds to weakly enforced homogeneous boundary conditions on \(\partial \Omega\). Similarly \(q_h = q_h(\nabla u_h^+, \nabla u_h^-, u_h^+, u_h^-, \rho^+, \rho^-)\) is a single valued numerical flux on \(\kappa \in E\). Summing (2.5) over all elements gives the complete DG discretization: Find \(u_h \in W^p_h\) such that

\[
a(u_h, w_h) = (f, w_h)_\Omega \quad \forall w_h \in W^p_h
\]

(2.6)

Following [4], a piecewise discontinuous numerical approximation of the flux, \(q_h^+\), may be evaluated locally as:

\[
q_h = -\left(\rho \nabla u_h - \rho^+ r_e(\rho^+ (u_h^+ - \dot{u}_h) n^+)\right),
\]

(2.7)

where \(r_e(\phi) \in \mathbb{P}^p(\kappa)\) is defined by:

\[
(r_e(\phi), \nu_h)_\kappa = \langle \phi, \nu_h^+ \rangle_{\partial \kappa} \quad \forall \nu_h \in \mathbb{P}^p(\kappa)
\]

(2.8)

We note that while \(\nabla u_h\) and \(r_e(\rho^+ (u_h^+ - \dot{u}_h) n^+)\) lie in the polynomial space \(\mathbb{P}^p(\kappa)\), \(q_h\), in general, does not when \(\rho\) varies within an element \(\kappa\). The DG discretizations presented in this paper lift \(\rho^+ \nabla u\) (as opposed to \(\nabla u\) or \(\rho \nabla u\)) to ensure that the discretization is symmetric for any \(\rho \in L^\infty(\Omega)\). In the case of piecewise constant \(\rho\) the DG formulations lifting \(\nabla u\), \(\rho^+ \nabla u\) or \(\rho \nabla u\) are identical.

The choice of the numerical trace \(\dot{u}_h\) and flux \(q_h\) defines the particular DG method considered. Table 2.1 lists the numerical traces and fluxes for the DG methods considered in this paper. In the definition of the different DG methods, the following average and jump operators are used to define the numerical trace and flux on \(\kappa \in E^i\):

\[
\{u_h\} = \frac{1}{2}(u_h^+ + u_h^-) \quad \text{and} \quad \llbracket u_h \rrbracket = u_h^+ n^+ + u_h^- n^-
\]

(2.9)

Additionally we define a second set of jump operators involving the numerical trace \(\dot{u}\):

\[
\llbracket u_h \rrbracket^+ = u_h^+ n^+ + \dot{u}_h^- n^- \quad \text{and} \quad \llbracket u_h \rrbracket^- = \dot{u}_h^+ n^- + u_h^- n^-
\]

(2.10)

such that we may express \(q_h\) as:

\[
q_h = -\left(\rho \nabla u_h - \rho^+ r_e(\rho^+ \llbracket u_h \rrbracket^+)\right)
\]

(2.11)

| Method          | \(u_h\)                          | \(q_h\)                          |
|-----------------|----------------------------------|----------------------------------|
| Interior Penalty| \(\{u_h\}\)                      | \(-\{\rho \nabla u_h\} + \frac{\rho}{h} \{\rho \llbracket u_h \rrbracket^+\}\) |
| Bassi and Rebay [5] | \(\{u_h\}\)                      | \(-\{\rho \nabla u_h\} + \eta_e \{\rho^+ r_e(\rho^+ \llbracket u_h \rrbracket^+)\}\) |
| Brezzi et al. [8] | \(\{u_h\}\)                      | \(\{q_h\} + \eta_e \{\rho^+ r_e(\rho^+ \llbracket u_h \rrbracket^+)\}\) |
| LDG [9]         | \(\{u_h\} - \beta \cdot \llbracket u_h \rrbracket\) | \(\{q_h\} + \beta \{\llbracket q_h \rrbracket\} + \frac{2 \eta_e}{\alpha} \{\rho \llbracket u_h \rrbracket^+\}\) |
| CDG [20]        | \(\{u_h\} - \beta \cdot \llbracket u_h \rrbracket\) | \(\{q_h^\alpha\} + \beta \{\llbracket q_h^\alpha \rrbracket\} + \frac{2 \eta_e}{\alpha} \{\rho \llbracket u_h \rrbracket^+\}\) |

Table 2.1

Numerical fluxes for different DG methods

We note that in the definition of the different DG methods, \(\eta_e\) is a penalty parameter defined on each edge in \(E\), while \(r_e(\phi) \in \mathbb{P}^p(\kappa)\) is a local lifting operator defined by:

\[
(r_e(\phi), \nu_h)_\kappa = \langle \phi, \nu_h^+ \rangle_{\kappa_e} \quad \forall \nu_h \in \mathbb{P}^p(\kappa)
\]

(2.12)
Additionally $q^e$ is given by:

$$q_h^e = - (\rho \nabla u_h - \rho^e r_e (\rho^{\frac{1}{2}} [u]^+))$$  \hspace{1cm} (2.13)

For the Local Discontinuous Galerkin (LDG) and Compact Discontinuous Galerkin (CDG) methods, $\beta$ is a vector which is defined on each edge/face in $E^i$ as

$$\beta = \frac{1}{2} \left( S_{\kappa^+}^- n^+ + S_{\kappa^-}^+ n^- \right)$$  \hspace{1cm} (2.14)

where $S_{\kappa^+}^- \in \{0, 1\}$ is a switch defined on each face of element $\kappa^+$ shared with element $\kappa^-$, such that

$$S_{\kappa^+}^- + S_{\kappa^-}^+ = 1.$$

3. The DG discretization from a new perspective. A key component, required for the development and analysis of the algorithms presented, is to express the global bilinear form $a(u_h, w_h)$ as the sum of element-wise contributions $a_{\kappa}(u_h, w_h)$ such that

$$a(u_h, w_h) = \sum_{\kappa \in T_h} a_{\kappa}(u_h, w_h)$$  \hspace{1cm} (3.1)

where $a_{\kappa}(u_h, w_h)$ is a symmetric, positive semi-definite “local bilinear form”. In particular, we wish the local bilinear form to have a compact stencil, such that $a_{\kappa}(u_h, w_h)$ is a function of only $u_h$, $\nabla u_h$ in $\kappa$, and $u_h^\pm$, $\nabla u_h^\pm$, $\rho^-$ and $\rho^+$ on $\partial \kappa$. In particular, we note that in (2.5), which is summed over all elements to give $a(u_h, w_h)$, $q$ depends in general upon $u^+, u^-$, $\nabla u^+, \nabla u^-$, $\rho^+$ and $\rho^-$. We write that local bilinear form as:

$$a_{\kappa}(u_h, w_h) = (\rho \nabla u_h, \nabla w_h)_\kappa - \left( \rho^+(u_h^+ - \hat{u}_h) n^+, \nabla w_h^+ \right)_{\partial \kappa} + \left( \tilde{q}_h^+, (w_h^+ - \tilde{w}_h) n^+ \right)_{\partial \kappa}$$

$$= (\rho \nabla u_h, \nabla w_h)_\kappa - \left( \rho^+ [u]^+, \nabla w_h^+ \right)_{\partial \kappa} + \left( \tilde{q}_h^+, [w]^+ \right)_{\partial \kappa}$$  \hspace{1cm} (3.2)

where $\tilde{q}_h^+ = \tilde{q}_h^+(\nabla u_h^+, u_h^+, \hat{u}_h^+, \rho^+)$ is a “local numerical flux”. In particular, in order to recover the original global bilinear form, $\tilde{q}_h^+$ must satisfy the following relationship on each edge, $e$:

$$\tilde{q}_h^+[w_h] = \tilde{q}_h^+[w_h]^+ + \tilde{q}_h^-[w_h]^- \hspace{1cm} \forall w_h \in W_h^p$$  \hspace{1cm} (3.3)

Table 3.1 lists the numerical traces and local fluxes for the DG methods considered, while Table 3.2 lists the corresponding local bilinear forms. It is simple to verify that (3.3) holds for each of the DG methods considered by using the identities:

$$[u_h] = [u_h]^+ + [u_h]^-$$.  \hspace{1cm} and  \hspace{1cm} and  \hspace{1cm} and

$$\begin{cases} [u_h]^+ = [u_h]^- = 1 & \text{if } \hat{u}_h = \{u_h\} \\
[u_h]^+ = [u_h], [u_h]^+ = 0 & \text{if } \hat{u}_h = \{u_h\} - \beta [u_h] \text{ and } S_{\kappa^+}^- = 1 \\
[u_h]^+ = 0, [u_h]^+ = 1 & \text{if } \hat{u}_h = \{u_h\} - \beta [u_h] \text{ and } S_{\kappa^+}^- = 0 \\
\end{cases}$$  \hspace{1cm} (3.4)

We now make an observation on the degrees of freedom involved in the local bilinear form, $a_{\kappa}(u_h, w_h)$. We consider using a nodal basis on each element $\kappa$ to define $W_h^p$. For the Interior Penalty (IP) method and the methods of Bassi and Rebay, and Brezzi et al., the numerical trace $\hat{u}_h$ on an edge/face depends on both $u_h^+$ and $u_h^-$. Hence the local bilinear form corresponds to all nodal degrees of freedom defining $u_h$ on $\kappa$ as well as nodal values on all edges/faces of $\partial \kappa \cap E^i$ corresponding to the trace of $u_h$ from elements neighbouring $\kappa$. On the other hand, for the LDG and CDG methods, the numerical trace $\hat{u}_h$ takes on the value of $u_h^+$ if $S_{\kappa^+}^- = 0$ or $u_h^-$ if $S_{\kappa^+}^- = 1$. Hence the local bilinear form corresponds only to degrees of freedom defining $u_h$ on $\kappa$ and nodal values corresponding to the trace of $u_h$ on neighbouring elements across edge/faces of $\partial \kappa \cap E^i$ for which $S_{\kappa^+}^- = 1$. 


In order to prove the result for the interior penalty method we employ the following result from Arnold et al. [4]:

\[ u_h = 0 \] though here we consider the contribution of a single element.

Proof

We denote by \( K \) the average value of \( \rho(\mathbf{x}) \) on each element \( \kappa \) and assume that the variation of \( \rho(\mathbf{x}) \) within an element is uniformly bounded as:

\[ c_\rho \rho_\kappa \leq \rho(\mathbf{x}) \leq C_\rho \rho_\kappa \quad \forall \mathbf{x} \in \kappa, \quad \forall \kappa \tag{3.5} \]

where the constants \( c_\rho \) and \( C_\rho \) are independent of \( \rho_\kappa \).

We now give the following lemma regarding the local bilinear form \( a_\kappa(u_h, w_h) \).

**Lemma 3.1.** The element-wise bilinear form \( a_\kappa(u_h, w_h) \) satisfies

\[ a_\kappa(u_h, w_h) \geq 0 \tag{3.6} \]

with \( a_\kappa(u_h, w_h) = 0 \) iff \( u_h = \hat{u}_h = K \) for some constant \( K \).

**Proof.** We proceed to show that Lemma 3.1 holds for all of the DG methods considered. The proof of Lemma 3.1 closely follows the proof of boundedness and stability of the different DG methods presented in Arnold et al. [4], though here we consider the contribution of a single element.

For each of the DG methods considered we can show \( u_h = \hat{u}_h = K \Rightarrow a_\kappa(u_h, w_h) = 0 \) by recognizing \( u_h = K \Rightarrow \nabla u_h = 0 \) and substituting into the different bilinear forms. It remains to prove \( a_\kappa(u_h, u_h) \geq 0 \) and \( a_\kappa(u_h, u_h) = 0 \) imply \( u_h = \hat{u}_h = K \).

In order to prove the result for the interior penalty method we employ the following result from Arnold et al [4]:

\[ c \left( r_e(\rho^{\frac{2}{3}} w), r_e(\rho^{\frac{2}{3}} w) \right)_\kappa \leq \frac{1}{\beta} \left( \langle \rho w, w \rangle \right)_\kappa \leq C \left( r_e(\rho^{\frac{2}{3}} w), r_e(\rho^{\frac{2}{3}} w) \right)_\kappa \quad \forall w \in W_h^p \tag{3.7} \]
where $c$ and $C$ are constants which depend only upon the minimum angle of $\kappa$ the polynomial order $p$ and the constants in (3.5). Hence, choosing $\eta_c$ sufficiently large for the interior penalty method we have

$$a_{k,IP}(u_h, u_h) \geq a_{k, BR2}(u_h, u_h) \quad (3.8)$$

and hence it is sufficient to show that Lemma 3.1 holds for the method of Bassi and Rebay [5]. Specifically, $\eta_c$ may be chosen for the interior penalty method as described in Shahbazi [21]. For the method of Bassi and Rebay,

$$a_{k, BR2}(u_h, u_h) = (\rho \nabla u_h, \nabla u_h)_\kappa - 2 \left( \rho \nabla u_h, [u_h]^+ \right)_{\partial \kappa} + \sum_{e \in \partial \kappa} \eta_c \left( r_e(\rho \frac{1}{2}^- [u_h]^+), r_e(\rho \frac{1}{2}^- [u_h]^+) \right)_\kappa$$

$$= (\rho \nabla u_h, \nabla u_h)_\kappa - 2 \left( \rho \frac{1}{2}^- \nabla u_h, r_e(\rho \frac{1}{2}^- [u_h]^+) \right)_\kappa + \sum_{e \in \partial \kappa} \eta_c \left( r_e(\rho \frac{1}{2}^- [u_h]^+), r_e(\rho \frac{1}{2}^- [u_h]^+) \right)_\kappa$$

$$\geq \sum_{e \in \partial \kappa} \frac{1}{N_e} \left( \rho \frac{1}{2}^- \nabla u_h - r_e(\rho \frac{1}{2}^- [u_h]^+), \rho \frac{1}{2}^- \nabla u_h - r_e(\rho \frac{1}{2}^- [u_h]^+) \right)_\kappa$$

$$+ \sum_{e \in \partial \kappa} \left( \eta_c - N_e \right) \left( r_e(\rho \frac{1}{2}^- [u_h]^+), r_e(\rho \frac{1}{2}^- [u_h]^+) \right)_\kappa \geq 0 \quad (3.9)$$

given $\eta_c > N_c$, where $N_c$ is the number of edge/faces of $\kappa$. In order to show $a_{k, BR2}(u_h, u_h) = 0 \Rightarrow u_h = \hat{u}_h = K$, we note $a_{k, BR2}(u_h, u_h) = 0$ implies

$$\sum_{e \in \partial \kappa} \frac{1}{N_e} \left( \rho \frac{1}{2}^- \nabla u_h - r_e(\rho \frac{1}{2}^- [u_h]^+), \rho \frac{1}{2}^- \nabla u_h - r_e(\rho \frac{1}{2}^- [u_h]^+) \right)_\kappa + \sum_{e \in \partial \kappa} \left( \eta_c - N_e \right) \left( r_e(\rho \frac{1}{2}^- [u_h]^+), r_e(\rho \frac{1}{2}^- [u_h]^+) \right)_\kappa = 0$$

$$= 0 \quad (3.10)$$

Hence $r_e(\rho \frac{1}{2}^- [u_h]^+) = 0$ and $\rho \frac{1}{2}^- \nabla u_h - r_e(\rho \frac{1}{2}^- [u_h]^+) = 0$ which implies $\hat{u}_h = u_h^+$ on $\partial\kappa$ and $\nabla u_h = 0$ in $\kappa$.

Proof of the method of Brezzi et al. [8] follows in a similar manner. Namely:

$$a_{k, Brezzi et al.}(pu_h, u_h) = (\rho \nabla u_h, \nabla u_h)_\kappa - 2 \left( \rho \nabla u_h, [u_h]^+ \right)_{\partial \kappa} + \sum_{e \in \partial \kappa} \eta_c \left( r_e(\rho \frac{1}{2}^- [u_h]^+), r_e(\rho \frac{1}{2}^- [u_h]^+) \right)_\kappa$$

$$\geq \left( \rho \frac{1}{2}^- \nabla u_h - r_\kappa(\rho \frac{1}{2}^- [u_h]^+), \rho \frac{1}{2}^- \nabla u_h - r_\kappa(\rho \frac{1}{2}^- [u_h]^+) \right)_\kappa + \sum_{e \in \partial \kappa} \eta_c \left( r_e(\rho \frac{1}{2}^- [u_h]^+), r_e(\rho \frac{1}{2}^- [u_h]^+) \right)_\kappa \geq 0 \quad (3.11)$$

provided $\eta > 0$. In order to show $a_{k, Brezzi et al.}(u_h, u_h) = 0 \Rightarrow u_h = \hat{u}_h = K$, we note $a_{k, Brezzi et al.}(u_h, u_h) = 0$ implies

$$\left( \rho \frac{1}{2}^- \nabla u_h - r_\kappa(\rho \frac{1}{2}^- [u_h]^+), \rho \frac{1}{2}^- \nabla u_h - r_\kappa(\rho \frac{1}{2}^- [u_h]^+) \right)_\kappa + \sum_{e \in \partial \kappa} \eta_c \left( r_e(\rho \frac{1}{2}^- [u_h]^+), r_e(\rho \frac{1}{2}^- [u_h]^+) \right)_\kappa = 0 \quad (3.12)$$

Hence $r_e(\rho \frac{1}{2}^- [u_h]^+) = 0$ and $\rho \frac{1}{2}^- \nabla u_h - r_\kappa(\rho \frac{1}{2}^- [u_h]^+) = 0$ which implies $\hat{u}_h = u_h^+$ on $\partial\kappa$ and $\nabla u_h = 0$ in $\kappa$. 

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Finally for the CDG method, we again use (3.7) and note that if \( \eta \) then we have
\[
W \text{e note that constants } c \text{ and } \eta \text{ will, unless explicitly stated otherwise, depend on the polynomial order .}
\]

We now parameterize the space \( W_h^p \) using a standard nodal basis defined at nodes \( x \) on each element \( \kappa \). The following lemmas show that the bilinear form is equivalent to a quadratic form based on the value of \( u_h \) at the nodes \( x \).

**Lemma 3.2.** There exist constants \( c \) and \( C \) independent of \( h \) and \( \rho_\kappa \) such that for all \( u_h \in W_h^p \)
\[
ca_\kappa(u_h, u_h) \leq \rho_\kappa h_\kappa^{n-2} \sum_{x_i, x_j \in \kappa \cup \kappa'} (u_h(x_i) - u_h(x_j))^2 \leq Ca_\kappa(u_h, u_h) \tag{3.16}
\]
where \( x_i, x_j \) are the nodes on \( \kappa \) defining the basis for \( u_h \) and nodes on \( \partial \kappa' \) defining a basis for the trace \( u_h \) from neighbours \( \kappa' \) of \( \kappa \). (We note that for the LDG and CDG methods nodes \( x_i, x_j \) include nodes defining a basis for \( u_h \) only on faces for which \( S_\kappa^{-} = 1 \).)

**Proof.** Lemma 3.2 is a direct consequence of Lemma 3.1 and a scaling argument. See [10] Lemma 4.3 for the equivalent proof for a mixed finite element discretization.

We note that constants \( c \) and \( C \) in Lemma 3.2 depend, in general, on the polynomial order \( p \). Throughout this paper all generic constants will, unless explicitly stated otherwise, depend on the polynomical order \( p \).

**Lemma 3.3.** Consider a region \( \omega \subset \Omega \) composed of elements in \( \mathcal{T}_h \). Denote by \( \rho_\omega \) the average value of \( \rho \) on \( \omega \) and suppose that \( \rho \) is uniformly bounded on \( \omega \) such that exists constants \( c_\rho \) and \( C_\rho \) independent of \( \rho_\omega \)
\[
c_\rho \rho_\omega \leq \rho \leq C_\rho \rho_\omega. \tag{3.17}
\]

Then there exist different constants \( c \) and \( C \) independent of \( h \), \( |\omega| \) and \( \rho_\omega \) such that for all \( u_h \in W_h^p \)
\[
ca_\omega(u_h, u_h) \leq \rho_\omega h^{n-2} \sum_{\kappa \in \omega} \sum_{x_i, x_j \in \kappa \cup \kappa'} (u_h(x_i) - u_h(x_j))^2 \leq C a_\omega(u_h, u_h) \tag{3.18}
\]
where we write the discrete form of (2.6) as:

\[ R \rho \]

We denote by \( \Gamma_z \) the space of discrete nodal values on \( \Gamma \). We note that for the Interior penalty method, and the methods of Bassi and Rebay, and Brezzi et al. we next define the local interface \( \Gamma_i \) to each element \( \Omega_i \) with edge/face \( \Gamma_i \) includes for each edge/face \( \Gamma_i \) from \( \Omega_i \). The presentation of the BDDC algorithm follows that presented in [16] for the case of continuous finite elements. We consider a DG discretization and derive a Schur complement problem for the interfaces between subdomains. The presentation will depend in general upon \( C_h, c_\rho \) and \( C_\rho \).

4. Domain Decomposition. In this section we present a domain decomposition of the discrete form of the DG discretization and derive a Schur complement problem for the interfaces between subdomains. The presentation of the BDDC algorithm follows that presented in [16] for the case of continuous finite elements. We consider a DG discretization and derive a Schur complement problem for the interfaces between subdomains. The presentation will depend in general upon \( C_h, c_\rho \) and \( C_\rho \).

We define the local interface \( \Gamma_i \) as local to a particular substructure we may locally eliminate them to obtain a system

\[ \hat{S}_\Gamma u_\Gamma = g_\Gamma \]
where
\[
\tilde{S}_\Gamma = A_{\Gamma \Gamma} - A_{\Gamma I} A_{II}^{-1} A_{I \Gamma}^T \\
g = b_{\Gamma \Gamma} - A_{\Gamma I} A_{II}^{-1} b_{\Gamma I}
\] (4.5)
Additionally we note that \(\tilde{S}_\Gamma\) and \(g\) may be formed by a direct assembly:
\[
\tilde{S}_\Gamma = \sum_{i=1}^{N} R_{\Gamma i}^T S_{\Gamma i} R_{\Gamma i} \\
g = \sum_{i=1}^{N} R_{\Gamma i}^T g_{\Gamma i}
\] (4.6)
where
\[
S_{\Gamma i} = A_{\Gamma i} - A_{\Gamma i} A_{II}^{-1} A_{\Gamma i}^T \\
g_{\Gamma i} = b_{\Gamma i} - A_{\Gamma i} A_{II}^{-1} b_{\Gamma i}.
\] (4.7)
Here \(A_{II}, A_{I I}^{(i)}, A_{\Gamma I}^{(i)}\) and \(b_{\Gamma i}, b_{\Gamma i}^{(i)}\) correspond to the contributions of a single substructure to the global system (4.3). We may also write \(\tilde{S}_\Gamma\) as
\[
\tilde{S}_\Gamma = R_{\Gamma}^T S_{\Gamma} R_{\Gamma}
\] (4.8)
where
\[
S_{\Gamma} = \begin{bmatrix}
S_{\Gamma 1}^{(i)} \\
\vdots \\
S_{\Gamma N}^{(i)}
\end{bmatrix}
\] (4.9)

5. BDDC method. In this section we introduce the BDDC preconditioner for the Schur complement problem given in (4.4). In order to define the BDDC preconditioner we reparameterize \(W_{\Gamma}^{(i)}\) into two orthogonal spaces \(W_{\Pi}^{(i)}\) and \(W_{\Delta}^{(i)}\). The primal space \(W_{\Pi}^{(i)}\) is the space of discrete unknowns corresponding to functions with a constant value of \(\tilde{u}\) on each edge (face if \(n = 3\)) \(F^{(j)}\) of substructure \(\Omega_{i}\). The dual space, \(W_{\Delta}^{(i)}\), is the space of discrete unknowns corresponding to functions which have zero mean value of \(\tilde{u}\) on \(\Gamma_{i}\). We note that the reparameterization to obtain \(W_{\Pi}^{(i)}\) and \(W_{\Delta}^{(i)}\) may be performed locally on each subdomain as described in [16]. We next define the partially assembled space
\[
\tilde{W}_{\Gamma} = W_{\Pi} \oplus \left( \bigoplus_{i=1}^{N} W_{\Delta}^{(i)} \right)
\] (5.1)
where \(\tilde{W}_{\Pi}\) is the assembled global primal space, single valued on \(\Pi\), which is formed by assembling the local primal spaces, \(W_{\Pi}^{(i)}\). We define additional local operators \(\tilde{R}_{\Gamma i} : \tilde{W}_{\Gamma} \to W_{\Gamma}^{(i)}\) which extract the degrees of freedom in \(\tilde{W}_{\Gamma}\) corresponding to \(\Gamma_{i}\). The global operator \(\tilde{R}_{\Gamma} : \tilde{W}_{\Gamma} \to W_{\Gamma}\) is formed by a direct assembly of \(\tilde{R}_{\Gamma i}\). We also define the global operator \(\tilde{R}_{\Gamma} : \tilde{W}_{\Gamma} \to W_{\Gamma}\). We now define the partially assembled Schur complement matrix \(\tilde{S}_{\Gamma}\), given by:
\[
\tilde{S}_{\Gamma} = \sum_{i=1}^{N} \tilde{R}_{\Gamma i}^T S_{\Gamma i} \tilde{R}_{\Gamma i}
\] (5.2)
We note that we may also write \(\tilde{S}_{\Gamma}\) as \(\tilde{S}_{\Gamma} = \tilde{R}_{\Gamma i}^T S_{\Gamma} \tilde{R}_{\Gamma}\), where \(S_{\Gamma}\) is given in (4.9). In order to complete the definition of the BDDC preconditioner we define a positive scaling factor \(\delta_{\Gamma i}^{\gamma}\) defined for each nodal degree of freedom on \(\partial\Omega_{i} \cap \partial\Omega_{j}\), corresponding to \(W_{\Gamma}^{(i)}\) by
\[
\delta_{\Gamma i}^{\gamma} = \frac{\rho_{\Gamma i}^{\gamma}}{\rho_{\Gamma i}^{\gamma} + \rho_{\Gamma j}^{\gamma}} \\
\gamma \in [1/2, \infty)
\] (5.3)
where $N_x$ is the set of indices of subdomains which share that particular degree of freedom. We define the scaled operator $\hat{R}_{D,\Gamma} : \hat{W}_\Gamma \rightarrow \hat{W}_\Gamma$ which is obtained by multiplying the entries of $\tilde{R}_\Gamma$ corresponding to $W^{(i)}_\Delta$ by $\delta_i(x)$. Using $\tilde{R}_\Gamma$ and $\hat{R}_{D,\Gamma}$ we define the interface averaging operator $E_D : \hat{W}_\Gamma \rightarrow \hat{W}_\Gamma$ as

$$E_D = \tilde{R}_\Gamma \hat{R}_{D,\Gamma}$$

(5.4)

The BDDC preconditioner $M^{-1}_{\text{BDDC}} : \hat{W}_\Gamma \rightarrow \hat{W}_\Gamma$ is given by:

$$M^{-1}_{\text{BDDC}} = \tilde{R}_{D,\Gamma}^T \hat{S}_\Gamma^{-1} \hat{R}_{D,\Gamma}$$

(5.5)

We note that this preconditioner can be efficiently implemented in parallel, as the only globally coupled degrees of freedom of $\hat{S}$ are those associated with the primal space $W_\Pi$. Additionally, in the following section we will show that this preconditioner is quasi-optimal in that the condition number of the preconditioned system, $M^{-1}_{\text{BDDC}}\hat{S}$, is independent of the number of subdomains and depends only weakly upon the number of degrees of freedom on each subdomain.

6. Analysis. In the following section we present the technical tools required to obtain the condition number bound. The analysis presented in the section closely follows that presented in [23] for mixed finite element methods, which in turn builds upon [10]. In particular, we note that all of the results presented in this section are simply the DG equivalents of similar results presented in [23] or [10]. The innovation which allows us to extend these results to DG discretizations is the new perspective presented in Section 3.

The main tools developed in this section connect the DG discretization to a related continuous finite element discretization on a subtriangulation of $T_h$. Using these tools we are able to leverage the theory for continuous finite element to obtain the desired condition number bound. In order to define the related continuous finite element discretization we consider a special reparameterization of the space $W^p_h$ on each subdomain $\Omega_i$. Specifically, a nodal basis is employed on each element using a special set of nodal locations on each element $\kappa$. Specifically, on elements, $\kappa$, which do not touch $\partial\Omega_i$ nodal locations are chosen strictly interior to $\kappa$. On elements $\kappa$ which touch $\partial\Omega_i$ nodal location are chosen on $\partial\kappa \cap \partial\Omega_i$ such that $\hat{u}|_{\partial\kappa \cap \partial\Omega_i}$ is uniquely defined by nodal values on $\partial\kappa$, while remaining nodal location are chosen interior to $\kappa$. We use this reparameterization so that each node defining the basis corresponds to a unique coordinate $\hat{\mathbf{x}}$, and $\hat{u}|_{\partial\Omega_i}$ is determined by nodal values on $\partial\Omega_i$. The following Lemma connects the two different parameterizations of the space $W^p_h$.

**Lemma 6.1.** There exist constants $c$ and $C$ independent of $h$ such that for each element $\kappa$.

$$c \sum_{\mathbf{x}_i \in \kappa} \phi(\mathbf{x}_i)^2 \leq \sum_{\hat{\mathbf{x}}_i \in \kappa} \phi(\hat{\mathbf{x}}_i)^2 \leq C \sum_{\mathbf{x}_i \in \kappa} \phi(\mathbf{x}_i)^2 \quad \forall \phi \in P^p(\kappa)$$

(6.1)

and

$$c \sum_{\mathbf{x}_i, \mathbf{x}_j \in \kappa} (\phi(\mathbf{x}_i) - \phi(\mathbf{x}_j))^2 \leq \sum_{\hat{\mathbf{x}}_i, \hat{\mathbf{x}}_j \in \kappa} (\phi(\hat{\mathbf{x}}_i) - \phi(\hat{\mathbf{x}}_j))^2 \leq C \sum_{\mathbf{x}_i, \mathbf{x}_j \in \kappa} (\phi(\mathbf{x}_i) - \phi(\mathbf{x}_j))^2 \quad \forall \phi \in P^p(\kappa)$$

(6.2)

**Proof.** Proof of Lemma 6.1 follows directly from the fact that using either nodes $\mathbf{x}$ or $\hat{\mathbf{x}}$ we can form a Lagrange basis for $\phi \in P^p(\kappa)$, with basis function bounded as in [22] Lemma B.5. \(\square\)

We now define the subtriangulation $\hat{T}_h$ of $T_h$ by considering each element $\kappa \in T_h$. The subtriangulation on each element $\kappa$ consists of the primary vertices used to define $W^p_h$, and secondary vertices corresponding to nodes on $\partial\kappa \cap \partial\Omega_i$ required to form a quasi-uniform triangulation of $\kappa$. We note that such a subtriangulation may be obtained on the reference element $\hat{\kappa}$ then mapped to $T_h$. As an example, Figure 6.1 shows the nodes defining the reparameterization as well as the subtriangulation for a $p = 1$ triangular element.
Define $U_h(\Omega)$ to be the continuous linear finite element space defined on the triangulation $\hat{T}_h$. Additionally we define $U_h(\Omega_i)$ and $U_h(\partial \Omega_i)$, as the restriction of $U_h(\Omega)$ to $\Omega_i$ and $\partial \Omega_i$ respectively. We now define a mapping $I^\Omega_{\partial \Omega_i}$ from any function $\phi$ defined at the primary vertices in $\Omega_i$ to $U_h(\Omega_i)$ as

$$I^\Omega_{\partial \Omega_i} \phi(x) = \begin{cases} 
\phi(x), & \text{if } x \text{ is a primary vertex;} \\
\frac{1}{n} \sum_{\xi \in \Omega_i} \phi(\xi), & \text{if } x \text{ is a secondary vertex on } \partial \Omega_i; \\
\frac{1}{n} \sum_{\xi \in \Omega_i} \phi(\xi), & \text{if } x \text{ is a secondary vertex in the interior of } \Omega_i; \\
\text{the linear interpolation of the vertex values,} & \text{if } x \text{ is not a vertex of } T_h.
\end{cases}$$

(6.3)

Since $(I^\Omega_{\partial \Omega_i} \phi)|_{\partial \Omega_i}$ is uniquely defined by $\phi|_{\partial \Omega_i}$, we may define the map $I^\Omega_{\partial \Omega_i}$ from a function defined on the primary vertices on $\partial \Omega_i$ to $U_h(\partial \Omega_i)$ such that $I^\Omega_{\partial \Omega_i} \phi|_{\partial \Omega_i} = (I^\Omega_{\partial \Omega_i} \phi)|_{\partial \Omega_i}$. We define $\hat{U}_h(\Omega_i) \subset U_h(\Omega_i)$ and $\hat{U}_h(\partial \Omega_i) \subset U_h(\partial \Omega_i)$ as the range of $I^\Omega_{\partial \Omega_i}$ and $I^\partial_{\Omega_i}$ respectively.

We now connect the original DG discretization to the continuous finite element discretization on $\hat{T}_h$ by showing that both discretizations are equivalent to a quadratic form in terms of the nodal values on $\hat{T}_h$. The following lemmas and theorems are the equivalent of similar theorems for mixed finite element discretizations presented in [10] and [23]. These results are a direct consequence of Lemma 3.1, which is the DG equivalent of Lemma 4.2 of [10]. We list the relevant results from [10] and [23] and refer to these papers for the proofs.

**Lemma 6.2.** For $\Omega_i$ composed of elements $\kappa$ in $T_h$, there exist constants $c$ and $C$ independent of $h$, $H$ and $\rho_\kappa$ such that for all $u_h \in W_h^p$

$$ca_i(u_h, u_h) \leq \sum_{\kappa \in \Omega_i} \rho_\kappa h^{n-2} \sum_{\tilde{x}_i, \tilde{x}_j \in \kappa \cup \kappa'} (u_h(\tilde{x}_i) - u_h(\tilde{x}_j))^2 \leq Ca_i(u_h, u_h)$$

(6.4)

**Proof.** Lemma 6.2 follows directly from Lemmas 3.3 and 6.1. \[\square\]
Lemma 6.3. There exists a constant \( C > 0 \) independent of \( h \) and \( H \) such that
\[
\left| I_h^{\Omega_i} \phi \right|_{H^1(\Omega_i)} \leq C \left| \phi \right|_{H^1(\Omega_i)} \quad \forall \phi \in U_h(\Omega_i),
\]
\[
\| I_h^{\Omega_i} \phi \|_{L^2(\Omega_i)} \leq C \| \phi \|_{L^2(\Omega_i)} \quad \forall \phi \in U_h(\Omega_i),
\]
(6.5)
(6.6)

Proof. See [10] Lemma 6.1. \( \Box \)

We define the following scaled norms:
\[
\| \phi \|_{H^1(\Omega_i)}^2 = |\phi|_{H^1(\Omega_i)}^2 + \frac{1}{H_i} \| \phi \|_{L^2(\Omega_i)}^2
\]
\[
\| \phi \|_{H^{1/2}(\partial \Omega_i)}^2 = |\phi|_{H^{1/2}(\partial \Omega_i)}^2 + \frac{1}{H_i} \| \phi \|_{L^2(\partial \Omega_i)}^2
\]
(6.7)
(6.8)

Lemma 6.4. There exist constants \( c, C > 0 \) independent of \( h \) and \( H \) such that for any \( \hat{\phi} \in \hat{U}_h(\partial \Omega_i) \):
\[
c \| \hat{\phi} \|_{H^{1/2}(\partial \Omega_i)} \leq \inf_{\phi \in \hat{U}_h(\partial \Omega_i)} \| \phi \|_{H^1(\Omega_i)} \leq C \| \hat{\phi} \|_{H^{1/2}(\partial \Omega_i)}
\]
\[
c \left| \hat{\phi} \right|_{H^{1/2}(\partial \Omega_i)} \leq \inf_{\phi \in \hat{U}_h(\partial \Omega_i)} |\phi|_{H^1(\Omega_i)} \leq C \left| \hat{\phi} \right|_{H^{1/2}(\partial \Omega_i)}
\]
(6.9)
(6.10)

Proof. See [10] Lemma 6.2. \( \Box \)

Lemma 6.5. There exists a constant \( C > 0 \) independent of \( h \) and \( H \) such that
\[
\| I_h^{\Omega_i} \hat{\phi} \|_{H^{1/2}(\partial \Omega_i)} \leq C \| \hat{\phi} \|_{H^{1/2}(\partial \Omega_i)} \quad \forall \hat{\phi} \in \hat{U}_h(\partial \Omega_i)
\]
(6.11)

Proof. See [10] Lemma 6.3. \( \Box \)

Lemma 6.6. There exist constants \( c \) and \( C \) independent of \( h, H \) and \( \rho_i \) such that for all \( u^{(i)} \in W^{(i)}_\Gamma \),
\[
c \rho_i \left| I_h^{\Omega_i} u^{(i)} \right|_{H^{1/2}(\partial \Omega_i)}^2 \leq |u^{(i)}|_{S^{(i)}_\Gamma}^2 \leq C \rho_i \left| I_h^{\Omega_i} u^{(i)} \right|_{H^{1/2}(\partial \Omega_i)}^2
\]
(6.12)

Proof. See [10] Theorem 6.5. \( \Box \)

Lemma 6.7. There exist constants \( c \) and \( C \) independent of \( h \) and \( H \) such that for all \( u \Gamma \in \hat{W}_\Gamma \)
\[
|E_{D\Gamma} u|_{S^{(i)}_\Gamma}^2 \leq C (1 + \log(H/h))^2 |u|_{S^{(i)}_\Gamma}^2
\]
(6.13)

Proof. The proof of Lemma 6.7 closely follows that of [23] Lemma 5.5. We note Assumption 4.1 is essential for this result. In particular, if Assumption 4.1 were not valid then \( (E_{D\Gamma})_j \) the restriction of \( E_{D\Gamma} u \) to degrees of freedom on \( \Omega_j \) would necessarily depend on degrees of freedom \( u_k \) corresponding to a subdomain \( \Omega_k \) which does not neighbour \( \Omega_j \) however are connected through the element \( \kappa \) in \( \Omega_i \) which has edges/faces on both \( \partial \Omega_i \cap \partial \Omega_j \) and \( \partial \Omega_i \cap \partial \Omega_k \). \( \Box \)
We now give the main theoretical result of the paper.

**Theorem 6.8.** The condition number of the preconditioner operator $M_{BDDC}^{-1} \hat{S}$ is bounded by $C(1 + \log(H/h))^2$ where $C$ is a constant independent of $h$, $H$ or $\rho_i$, though in general dependent upon the polynomial order $p$.

**Proof.** Theorem 6.8 follows directly from Lemma 6.7. (See for example [23] Theorem 6.1). □

### 7. Numerical Results

In this section we present numerical results for the BDDC preconditioner introduced in Section 5. For each numerical experiment we solve the linear system resulting from the DG discretization using a Preconditioned Conjugate Gradient (PCG) method. The PCG algorithm is run until the initial $l_2$ norm of the residual is decreased by a factor of $10^{10}$. We consider a domain $\Omega \in \mathbb{R}^2$ with $\Omega = (0, 1)^2$. We divide $\Omega$ into $N \times N$ square subdomains $\Omega_i$ with side lengths $H$ such that $N = \frac{H}{h}$. Each subdomain is the union of triangular elements obtained by bisecting squares of side length $h$, ensuring that Assumption 4.1 is satisfied. Thus each subdomain has $n_i$ elements, where $n_i = 2 \left( \frac{H}{h} \right)^2$.

In the first set of numerical experiments we solve (2.1) on $\Omega$ with $f$ chosen such that the exact solution is given by $u = \sin(\pi x) \sin(\pi y)$. We discretize using each of the DG methods discussed in Section 2 for polynomial orders $p = 1$, 3, and 5. Tables 7.1-7.5 show the corresponding number of PCG iteration required to converge for the considered DG methods. As predicted by the analysis the number of iterations is independent of the number of subdomains and only weakly dependent upon the number of elements per subdomain. In addition we note that the number of iterations also appears to be only weakly dependent on the solution order $p$. Finally, we note that the number of iterations required for the solution of the LDG and CDG discretizations is smaller than those of the other DG methods. For the LDG and CDG methods the Schur complement problem has approximately half the number of degrees of freedom as for the other DG methods, hence it is not surprising that a smaller number of iterations is required to converge.

In the second numerical experiment we examine the behaviour of the preconditioner for large jumps in the coefficient $\rho$. We partition the domain in a checkerboard pattern and set $\rho = 1$ on half of the subdomains and set $\rho = 1000$ in

---

**Table 7.1**

| $H$ | $p = 1$ | $p = 2$ | $p = 3$ |
|-----|---------|---------|---------|
| 2   | 16      | 29      | 31      |
| 4   | 30      | 31      | 31      |
| 8   | 34      | 31      | 32      |
| 16  | 35      | 35      | 33      |

**Table 7.2**

| $H$ | $p = 1$ | $p = 2$ | $p = 3$ |
|-----|---------|---------|---------|
| 2   | 16      | 29      | 31      |
| 4   | 30      | 31      | 31      |
| 8   | 34      | 31      | 32      |
| 16  | 35      | 35      | 33      |
the remaining subdomains. We solve (2.1) with \( f = 1 \). We discretize this problem using the CDG method. Initially we set \( \delta_i^1 = \frac{1}{|N_x|} \), where \( |N_x| \) is the number of elements in the set \( N_x \). We note that this choice of \( \delta_i^1 \) corresponds to setting \( \gamma = 0 \) in (5.3), which does not satisfy the assumption \( \gamma \in [1/2, \infty) \). Hence, we obtain poor convergence of the BDDC algorithm as shown in Table 7.5(a). Next we set \( \delta_i^1 \) as in (5.3) with \( \gamma = 1 \). With this choice of \( \delta_i^1 \) the good convergence properties of the BDDC algorithm are recovered as shown in Table 7.5(b).

8. Conclusions. We have extended the BDDC preconditioner to a large class of DG discretizations for second-order elliptic problems. The analysis shows that the condition number of the preconditioned system is bounded by \( C(1 + \log(H/h))^2 \), with constant \( C \) independent of \( h, H \) or large jumps in the coefficient \( \rho \). Numerical results confirm the theory.

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(a) $p = 1$

| $h$ | 2  | 4  | 8  | 16 | 32 |
|-----|----|----|----|----|----|
| 2   | 12 | 19 | 20 | 20 | 19 |
| 4   | 12 | 20 | 23 | 22 | 22 |
| 8   | 13 | 23 | 25 | 25 | 25 |
| 16  | 13 | 24 | 28 | 28 | 27 |

(b) $p = 3$

| $h$ | 2  | 4  | 8  | 16 | 32 |
|-----|----|----|----|----|----|
| 2   | 11 | 20 | 22 | 22 | 22 |
| 4   | 12 | 21 | 24 | 25 | 24 |
| 8   | 12 | 23 | 27 | 27 | 26 |
| 16  | 12 | 25 | 28 | 29 | 26 |

(c) $p = 5$

| $h$ | 2  | 4  | 8  | 16 | 32 |
|-----|----|----|----|----|----|
| 2   | 11 | 22 | 25 | 24 | 24 |
| 4   | 12 | 24 | 27 | 26 | 25 |
| 8   | 12 | 24 | 29 | 29 | 29 |
| 16  | 11 | 26 | 31 | 31 | 31 |

Table 7.5
Iteration count for BDDC preconditioner using the CDG method

(a) $\delta_i^p = 1$

| $p$ | 2  | 4  | 8  | 16 | 32 |
|-----|----|----|----|----|----|
| 0   | 17 | 69 | 118| 138| 147|
| 1   | 51 | 119| 179| 215| 232|
| 2   | 52 | 129| 192| 241| 252|
| 3   | 55 | 133| 207| 267| 316|
| 4   | 58 | 144| 226| 285| 304|
| 5   | 59 | 153| 242| 306| 361|

(b) $\delta_i^p = \frac{\rho_i}{\sum_j \rho_j}$

| $p$ | 2  | 4  | 8  | 16 | 32 |
|-----|----|----|----|----|----|
| 0   | 4  | 6  | 13 | 15 | 16 |
| 1   | 4  | 7  | 14 | 18 | 19 |
| 2   | 4  | 7  | 13 | 17 | 18 |
| 3   | 4  | 7  | 15 | 18 | 19 |
| 4   | 4  | 7  | 14 | 19 | 20 |
| 5   | 4  | 7  | 14 | 19 | 20 |

Table 7.6
Iteration count for BDDC preconditioner using the CDG method with $\rho = 1$ or 1000

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