In this paper, the effect of halo substructures on galaxy rotation curves is investigated using a simple model of dark matter clustering. A dark matter halo density profile is developed based only on the scale-free nature of clustering that leads to a statistically self-similar distribution of the substructures at the galactic scale. A semi-analytical method is used to derive rotation curves for such a clumpy dark matter density profile. It is found that the halo substructures significantly affect the galaxy velocity field. Based on the fractal geometry of the halo, this self-consistent model predicts a Navarro–Frenk–White-like rotation curve and a scale-free power spectrum of the rotation velocity fluctuations.

**Key words:** dark matter – galaxies: dwarf – galaxies: general – galaxies: ISM – galaxies: structure – ISM: kinematics and dynamics

**Online-only material:** color figures
Sommer-Larsen & Dolgov 2001; Burkert & D’Onghia 2004) or the “missing satellite problem” (Klypin et al. 1999; Moore et al. 1999), which are yet to be addressed.

Here, we have developed a self-consistent model of the dark matter halo substructure distribution at the galactic scale to explain observed NFW-like rotation curves. Support for the existence of dark matter substructures has mainly come from numerical simulations (Giocoli et al. 2008; Madau et al. 2008; Springel et al. 2008; Elahi et al. 2009a; Ludlow et al. 2009). But there are strong observational hints such as flux anomalies and time delays in gravitational lensing (Chen 2009; Keeton & Moustakas 2009; Vegetti & Koopmans 2009; Xu et al. 2009), or enhanced gamma rays and leptonic cosmic rays (Elahi et al. 2009b; Pinzke et al. 2009), indicating the presence of substructures. The present model is based on the assumption of a scale-free nature of the dark matter clustering that leads to a statistically self-similar distribution of the halo substructures at the galactic scale. It is shown that a simple fractal model of the dark matter halo substructure predicts an NFW-like rotation curve. Such a model also predicts a scale-free power spectrum of the rotation velocity fluctuations. The model is described in Section 2, and the results are presented in Section 3. Possible limitations of our analysis are discussed in Section 4. Finally, we summarize and present our conclusions in Section 5.

2. THE MODEL

2.1. Assumptions, Parameters, and Constraints

Assuming that the dark matter clustering has a scale-free nature (i.e., there exist halo substructures of a wide range of mass), the density profile can be described as a combination of a smooth radial profile $\rho(r/r_c)$ and a stochastic part $\delta\rho$. Here $r_c$ is the characteristic “core” radius, and $\vec{r}$ contains the information of the density variation at scales larger than or comparable to $r_c$. For the purpose of this work, we have used a simpler model of this density distribution which, however, retains all relevant key features. In this simplified model, we have assumed that each dark matter clump has a number of smaller clumps around it. Each of these smaller clumps is in turn just a scaled down version with even smaller clumps around them. As a result, the whole structure has an approximate spherical symmetry and a statistical self-similarity. It is assumed here that all these clumps have a non-singular isothermal density profile where the central density ($\rho_0$), the core radius ($r_c$), the cutoff radius ($r_{max}$), and the halo to subhalo distance ($D$) are scaled down accordingly. However, this specific density profile is not a crucial assumption in our model, and the individual substructures may have any non-singular density profile $\rho(r/r_c)$, where $r_c$ is some characteristic radius. Essentially, this is a fractal structure with three parameters: (1) $n$, the number of small clumps around any clump, (2) $f_x$, the spatial scaling factor for core radius, cutoff radius, and distance, and (3) $f_\rho$, the central density scaling factor between any clump and its next smallest clumps. A fractal is a fragmented and irregular geometrical shape with exact or stochastic self-similar structures at all scales (Mandelbrot 1982). Independent of the value of $f_x$ and $n$, the (Hausdorff) fractal dimension of such a structure is 3 for $n f_x^2 < 1$. However, since at each iteration, the linear size of clumps scales by a factor of $f_x$ and the mass scales by a factor of $n$, the local mass dimension for any substructure level is $D_m = -\log(n)/\log(f_x)$ over a certain range of scales. A mass dimension of $D_m$ for a medium implies that the mass enclosed in a sphere of radius $r$ in such a medium will be $M(r) = k r^{D_m}$. So, for the $N$th substructure level, $M_N(r) = k N^{D_m}$ over a range of scales depending on $N$, $n$, $f_x$, $D$, and $r_{max}$. Note that this range is different for different substructure levels. Hence, for the complete structure, the total mass $M(r)$, which is the sum of $M_N(r)$ of all the substructure levels, will not have a simple power-law radial dependence. But the dark matter halo mass function will still be a power law, $N(m) \propto m^{-\alpha}$, where the power-law index $\alpha$ is $-\log(n)/\log(f_x^2 f_\rho)$ for $N$ is a more physically motivated parameter of this model and can be constrained from theoretical and numerical analyses of dark matter structure formation (Gao et al. 2004; Zemp et al. 2009). The density distribution can be written as

$$\rho(r) = \rho_{bg} + \sum_{i=0}^{\infty} \sum_{j=1}^{n_f} \rho_i(\rho_{0i}, r_{ci}, r_{max,i}, \vec{r}_{ij}),$$

(1)

where $\rho_{bg}$ is the background density and $\rho_i(\rho_{0i}, r_{ci}, r_{max,i}, \vec{r}_{ij})$ is the density profile of individual substructure centered at $\vec{r}_{ij}$ with central density ($\rho_{0i}$), core radius ($r_{ci}$), and cutoff radius ($r_{max,i}$). Considering the self-similarity of this model,

$$\rho_{0i} = f_\rho \times \rho_{0,i-1}$$

$$r_{ci} = f_x \times r_{ci-1}$$

$$r_{max,i} = f_x \times r_{max,i-1}$$

$$\vec{r}_{ij} = \vec{r}_{i-1,k} + (f_x \times D_{i-1} \times \vec{d}),$$

(2)

where $k = n^{i-1}$, $\vec{d}$ is a unit length vector with random orientation, and the initial set of parameters $\rho_{00}, r_{c0}, r_{max0} = \rho_0, r_c, r_{max}$ is for the largest subhalo centered at the origin. In principle, $\rho_{bg}$ may be a smooth function of $r$. But since we are assuming it to be a small background density threshold, its effect on the final rotation curve is not very significant. So, for simplicity, we have assumed $\rho_{bg}$ to be constant over the radius of our interest. This structure is shown schematically (without any randomness for the sake of clarity) in the left panel of Figure 1. After introducing randomness in the angular position of the subhalos, one realization of such a structure with $n = 7$ and $f_x = 0.33$ is shown with two and four substructure levels in the middle and right panels, respectively. This model can be considered as a simplified representation of the scale-free, clumpy density structure of dark matter above a small threshold density at the galactic scale.

We note that the parameters for this model are constrained to a good extent by various physical considerations. Assuming that the structure is extended down to the infinitely small scale, to avoid a divergence of the total mass, the quantity $n f_x^2 f_\rho$ must be less than unity. Similarly, for a halo with cutoff radius $r_{max}$, halo to subhalo distance $D$ must be greater than or equal to $(1 - f_x) r_{max}/(1 - 2 f_x)$ to avoid any overlap. Also, note that the density of any subhalo at the cutoff radius $\rho(r_{max})$ scales as $f_\rho$. So, we adopt $f_\rho = 1.0$ for further analysis to ensure a constant density at the cutoff radius for all the substructures. In this simplified model, we have assumed that all the parameters such as $D$, $\rho_0$, $r_{max}$, etc., are identical for all the subhalos for a particular substructure level and $n$, $f_x$, and $f_\rho$ remain constant for all the levels. In a realistic situation, however, all these parameters may have some random variation. As a result of these fluctuations, the halo mass function, the radial mass distribution, and, in turn, the rotation curves, are expected to be somewhat smoother than that predicted from this analysis.
2.2. Tidal Stability

A much stronger constraint on the parameters comes from the consideration of the stability of this structures preventing tidal disruption by invoking self-gravity. Considering a rigid object of mass $m$ and radius $r$ at a distance $d$ from a bigger object of mass $M$ and radius $R$, from the standard Roche limit consideration, $D$ should be $\sim r(2M/m)^{1/3}$ to avoid tidal disruption of the smaller body. In the case of any halo and its immediate subhalo, $m/M = f_s^3$ and $r/R = f_s$ imply $d \approx 1.25R$. For even smaller subhalo structures with $r/R = f_s^k$, the mass is scaled accordingly, $m/M = f_s^{3k}$ and $d \approx 1.25R$ assures stability. Hence, a distance

$$D \geq 1.25x \frac{1}{1-2f_s} r_{\text{max}}$$

(3)

will make the whole structure stable. Here, $x$ is a fudge factor, and we use the value of $x = 1.1$ to accommodate non-rigid density clumps. For a given value of $D$, the number of substructures $n$ is also constrained to be

$$n \leq \frac{4\pi D^2}{\pi (f_sD/1-f_s)^2} = \frac{4(1-f_s)^2}{f_s^2}$$

(4)

to avoid any possible overlap of subhalo structures.

2.3. Virial Stability

A detailed virial stability analysis requires the numerical simulation of the dynamics of such a density distribution to get the time-averaged dynamical quantities. But a simple ensemble average virial scaling analysis may be used to constrain the central density $\rho_0$ for a set of model parameter. As the whole structure is assumed to have an approximate spherical symmetry, average kinetic and potential energies, $\langle T \rangle$ and $\langle V \rangle$, for thin spherical shells of radius $r$ and thickness $dr$ will be

$$\langle T \rangle = \frac{1}{2} m \sigma_{\text{DM}}^2 = \frac{4\pi}{2} r^2 \rho(r) \sigma_{\text{DM}}^2$$

$$\langle V \rangle = -\frac{GM(r)m}{r} = -v_c^2(r)4\pi r^2 \rho(r),$$

(5)

where $\sigma_{\text{DM}}$ is the dark matter velocity dispersion, $M(r)$ is the total mass within radius $r$, and $v_c(r) = (GM/r)^{1/2}$ is the scale-dependent virial velocity equivalent of the “circular velocity” for rotating disk. Since the rotation curve has a roughly constant value $v_0$ at large radius, the ratio $2\langle T \rangle/\langle V \rangle$ will tend to the equilibrium value of 1 at large radius if $\sigma_{\text{DM}} \approx v_0$. Using the minimum value for $D$ from Equation (3), the maximum extent of the structure $R_{\text{max}}$ will be $D/(1-f_s)$, and the average density will be

$$\langle \rho \rangle = \frac{3(1-2f_s)^2(s - \tan^{-1}s)}{(1.25xs)^3(1-16f_s^3)} \rho_c,$$

(6)

where $s = r_{\text{max}}/r_c$. Note, however, that this is a fractal structure with significant porosity. So, the average density of any individual clump is higher than $\langle \rho \rangle$ by a factor of $(1-nf_s^2)/(R_{\text{max}}/r_{\text{max}})^2$. Now, for global stability of the whole structure, $R_{\text{max}}$ should be less than or equal to the radius within which the virial equilibrium is maintained. Average density within this virial radius $r_{\text{vir}}$ or $r_{200}$ should be $\approx 200$ times more than the critical density $\rho_{\text{cr}} = 3M_H^2/8\pi G$. For a choice of model parameters $n, f_s, s$, and $x$, this will constrain the lower limit of the central density $\rho_0$ so that $\langle \rho \rangle > 200\rho_{\text{cr}}$. For individual substructures, both mass and volume scale as $f_s^3$, keeping the average density constant. This implies that stability for one substructure level ensures stability for all other levels.

3. RESULTS

3.1. Rotation Curve

Since the halo density distribution is significantly clumpy, the velocity field for such a system is also expected to have fluctuations at all scales. However, due to the approximately spherical symmetry of the clump distribution, the average rotation velocity over a spherical shell at radius $r$ will still be $\langle v_c(r) \rangle \approx (GM/r)^{1/2}$, where $M_r$ is the total mass within this radius. As pointed out in Section 2.3 using the virial stability argument, the virial velocity or, equivalently, the “rotation” velocity is expected to be approximately same as the local velocity dispersion. This derived rotation curve for the fractal model is found to be NFW-like at a large radial distance. At small radius, by construction, the rotation curve is obviously

Figure 1. Left: schematic representation of the halo substructure with three subhalos around any big halo. Note that both the cutoff radius ($r_{\text{max}}$) and the halo to subhalo distance ($D$) are scaled down by a factor of $f_s$ for each substructure level. Middle and right: halo substructure with seven subhalos around any big halo and with $f_s = 0.33$. The middle and right panels show the structure with two and four substructure levels, respectively.

(A color version of this figure is available in the online journal.)
Figure 2. Predicted rotation curve for the fractal substructure model (black) and the best-fit NFW profile to that (red). This is for $n=35$, $f_r=0.25$, $f_p=1.0$, and $r_{max}=6.0r_c$. Rotation curves for the non-singular isothermal sphere with and without a cutoff (green and blue curves, respectively) as well as for an NFW halo (magenta curve) are also shown for comparison. Radial distance and rotation velocity are scaled by $r_c$ and $v_0$ (rotation velocity at the furthest radial distance), respectively.

(A color version of this figure is available in the online journal.)

Figure 3. Predicted rotation curve for the fractal substructure model for different background density thresholds $\rho_{bg}$. All other parameters are same as in Figure 2. Different curves are for $\rho_{bg}/\rho_{r_{max}}=0.00, 0.03, 0.10, 0.30$ (black, red, green, and blue curves, respectively).

(A color version of this figure is available in the online journal.)

The general NFW-like nature of the rotation curve and the radial fluctuations remain unchanged. But depending on the value of $\rho_{bg}$, the rotation curve at large radius may be rising, flat, or declining. Note that the derived rotation curves shown in Figure 3 are with the simple model of a constant $\rho_{bg}$. In reality, $\rho_{bg}$ is expected to be decreasing with increasing $r$, giving rise to a rotation curve somewhat intermediate between the extremes shown in Figure 3.

3.2. Velocity Fluctuations

The predicted rotation curve due to the clustering of dark matter subhalos is very similar to the observed rotation curve and the empirical NFW rotation curve. However, unlike the NFW model with a smooth radial density distribution, the present model predicts a significant fluctuation of rotation velocity in both angular and radial directions. Since the underlying density field, which gives rise to these velocity fluctuations, is scale free, the velocity fluctuation power spectrum is expected to be a power law. Though the fractal model has many free parameters, the only relevant parameters for the index of this power law are $n$ and $f_r$, while the rest of them will just introduce different multiplicative scaling. This prediction can be easily verified from high spatial and spectral resolution observation of neutral hydrogen of normal galaxies. Note that part of these fluctuations will cancel out for the spherically averaged rotation curve, and hence it is important to use the full velocity field to search for such scale-free fluctuations. It is also important to note that fluctuations of the velocity field of the hydrogen gas will have contributions from the local density perturbations of the disk. But the scale dependence of these perturbations may make it possible to disentangle the scale-free perturbations due to the halo substructures. There is some indication of such a power-law scaling of the rotation velocity fluctuation power spectra from direct observation and analysis of the H I 21 cm velocity field of some nearby galaxy (P. Dutta et al. 2010, in preparation).

We leave a more detailed treatment of this aspect to a future work.

4. DISCUSSION

Throughout this analysis, we have assumed that the clump distribution in this model has an approximately spherical symmetry. This assumption is most likely to be untrue in reality. Due to small count, bigger substructures are more likely to cause departure from spherical symmetry. In this situation, rotation velocity field will also have strong azimuthal asymmetry. Observationally, about half of the spiral galaxies show some degree of kinematical lopsidedness (Richter & Sancisi 1994; Haynes et al. 1998; Swaters et al. 1999; Sofue & Rubin 2001; Jog & Combes 2009) which may originate from tidal distortion partly due to deviation from the spherical symmetry of the underlying gravitational potential of the lopsided dark matter halo (Chakrabarti & Blitz 2009; Saha et al. 2009). Due to their large number, smaller substructures are likely to have relatively more symmetrical distribution. So, the velocity fluctuations spectrum is expected to show a power-law scaling at large spatial frequency (large $k$, i.e., small physical scale), and a departure from the power law at small $k$.

We have also assumed in this analysis that the baryons will not significantly affect the galactic dynamics. However,
some recent numerical studies (e.g., Weinberg et al. 2008; Romano-Díaz et al. 2008, 2009) have shown that the presence of baryons has effects such as flattening the central cusp, reducing the halo triaxiality, and introducing bias in clustering with environment density. Effectively, baryon dissipation may destroy the similarity between different scales of clustering. However, Romano-Díaz et al. (2010) have found that the phenomenon such as “efficient feedback from stellar evolution and the central supermassive black holes” may counterbalance this effect to some extent. On the other hand, Knebe et al. (2010) have found no significant effect of baryonic physics on properties such as shape and radial alignment of substructures. Numerical simulations also suggest that, even in the presence of baryons, the subhalo mass function remains a power law, though the power-law index changes from $-0.99$ to $-1.13$ (Romano-Díaz et al. 2010). It is, hence, important to keep in mind that, in a realistic situation, baryon dissipation may alter the velocity fluctuations causing a significant departure from the scale-free velocity fluctuations.

We note that the key result of this analysis, that is, an NFW-like rotation curve for the fractal model, is not crucially dependent on the exact density profile of individual halos. A variety of density profile without any central singularity (variant of the non-singular isothermal sphere) will lead to a similar NFW-like rotation curve. This strongly suggests that the clustering properties of the dark matter particles dominantly govern the radial density profile of the halo. Finally, these assembly of substructures will give rise to a flat NFW-like rotation curve for normal galaxies. But for more dark-matter-dominated low-mass dwarf galaxies and low surface brightness galaxies, depending on the exact form of the non-singular density profile, the dominant contribution of the central big halo may make the rotation curve for such galaxies intrinsically different which is consistent with observational results (de Blok & Bosma 2002; de Blok 2005).

5. CONCLUSIONS
1. The dark matter halo substructures at the galactic scale are found to significantly affect the rotation curve.
2. A self-consistent model of statistically self-similar hierarchical dark matter substructures predicts an NFW-like rotation curve, though each clump has a non-singular isothermal density profile. This NFW-like rotation curve emerges out of the fractal geometry and is independent of the specific density profile of individual clumps.
3. The model predicts a scale-free power spectrum of the rotation velocity fluctuations which can be observationally verified.
4. The model also provides some plausible explanation of the observed intrinsic difference between the dark matter halo density profile of normal galaxies and dark-matter-dominated low-mass dwarf galaxies and low surface brightness galaxies.

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REFERENCES

Begeman, K. G., Broeils, A. H., & Sanders, R. H. 1991, MNRAS, 249, 523
Bekenstein, J., & Milgrom, M. 1984, ApJ, 286, 7
Bertone, G., Hooper, D., & Silk, J. 2005, Phys. Rept., 405, 279
Browstein, I. R., & Moffat, J. W. 2006, MNRAS, 367, 527
Burkert, A. 1995, ApJ, 447, L25
Burkert, A. M., & D’Onghia, E. 2004, in Astrophys. Space Sci. Libr. 319, Penetrating Bars through Masks of Cosmic Dust: the Hubble Tuning Fork Strikes a New Note, ed. D. L. Block et al. (Dordrecht: Kluwer), 341
Chakrabarti, S., & Blitz, L. 2009, MNRAS, 399, L118
Chen, J. 2009, A&A, 498, 49
Clowe, D., Bradač, M., Gonzalez, A. H., Markevitch, M., Randall, S. W., Jones, C., & Zaritsky, D. 2006, ApJ, 648, L109
de Blok, W. J. G. 2005, ApJ, 634, 227
de Blok, W. J. G., & Bosma, A. 2002, A&A, 385, 816
de Blok, W. J. G. 2007, in Island Universe, ed. R. S. de Jong (Dordrecht: Springer), 89
Elahi, P., Thacker, R. J., Widrow, L. M., & Scannapieco, E. 2009a, MNRAS, 395, 1950
Elahi, P. J., Widrow, L. M., & Thacker, R. J. 2009b, Phys. Rev. D, 80, 123513
El-Zant, A., Hoffman, Y., Primack, J., Combes, F., & Shlosman, I. 2004, ApJ, 607, L75
Faber, S. M., & Jackson, R. E. 1976, ApJ, 204, 668
Fahr, H. J. 1990, A&A, 236, 86
Fukushige, T., & Makino, J. 2001, ApJ, 575, 533
Gao, L., White, S. D. M., Jenkins, A., Stoehr, F., & Springel, V. 2004, MNRAS, 355, 819
Giocoli, C., Toren, G., & van den Bosch, F. C. 2008, MNRAS, 386, 2135
Giovanelli, R., Haynes, M. P., van Zee, L., Hogg, D. E., Roberts, M. S., & Maddalena, R. J. 1999, AJ, 118, 62
Jing, Y. P. 2000, ApJ, 535, 30
Jog, C. J., & Combes, F. 2009, Phys. Rep., 471, 75
Keeton, C. R., & Moustakas, L. A. 2009, ApJ, 699, 1720
Klypin, A., Kravtsov, A. V., Valenzuela, O., & Prada, F. 1999, ApJ, 522, 520
Knebe, A., Libeskind, N. I., Knollmann, S. R., Yepes, G., Gottlüber, S., & Hoffman, Y. 2010, MNRAS, 405, 1119
Kuzio de Naray, R., McGaugh, S. S., & de Blok, W. J. G. 2008, ApJ, 676, 920
Ludlow, A. D., Navarro, J. F., Springel, V., Jenkins, A., Frenk, C. S., & Helmi, A. 2009, ApJ, 692, 931
Madau, P., Diemand, J., & Kuhlen, M. 2008, ApJ, 679, 1260
Mandelbrot, B. 1982, The Fractal Geometry of Nature (New York: Freeman)
Massey, R., et al. 2007, Nature, 445, 286
Milgrom, M. 1983, ApJ, 270, 365
Moore, B., Ghigna, S., Governato, F., Lake, G., Quinn, T., Stadel, J., & Tozzi, P. 1999, ApJ, 524, L19
Navarro, J. F., Frenk, C. S., & White, S. D. M. 1996, ApJ, 462, 563
Navarro, J. F., Frenk, C. S., & White, S. D. M. 1997, ApJ, 490, 493
Pinzke, A., Pfrommer, C., & Bergström, L. 2009, Phys. Rev. Lett., 103, 181302
Richter, O.-G., & Sancisi, R. 1994, A&A, 920, L19
Romano-Díaz, E., Shlosman, I., Heller, C. H., & Hoffman, Y. 2008, ApJ, 685, L105
Romano-Díaz, E., Shlosman, I., Heller, C. H., & Hoffman, Y. 2010, ApJ, 716, 1095
Romano-Díaz, E., Shlosman, I., Hoffman, Y., & Heller, C. H. 2009, ApJ, 702, 1250
Rubin, V., & Ford, W. K., Jr. 1970, ApJ, 159, 379
Rubin, V., Ford, W. K., Jr., & Thomann, N. 1980, ApJ, 238, 471
Saha, K., Levine, E. S., Jog, C. I., & Blitz, L. 2009, ApJ, 697, 2015
Sanders, R. H. 1986, MNRAS, 223, 539
Sanders, R. H. 1997, ApJ, 480, 492
Shlosman, I. 2010, in ASP Conf. Ser. 419, Galaxy Evolution: Emerging Insights and Future Challenges, ed. S. Jogee et al. (San Francisco, CA: ASP), 39
Sofue, Y. 1996, ApJ, 458, 120
Sofue, Y., & Rubin, V. 2001, A&A, 39, 137
Sommer-Larsen, J., & Dolgov, A. 2001, ApJ, 551, 608
Sommer-Larsen, J., Gelato, S., & Vedel, H. 1999, ApJ, 519, 501
Spano, M., Marcelin, M., Amram, P., Carignan, C., Epinat, B., & Hernandez, O. 2008, \textit{MNRAS}, 383, 297
Springel, V., et al. 2008, \textit{MNRAS}, 391, 1685
Swaters, R. A., Schoenmakers, R. H. M., Sancisi, R., & van Albada, T. S. 1999, \textit{MNRAS}, 304, 330
Taylor, J. E., & Navarro, J. F. 2001, \textit{ApJ}, 563, 483
Vegetti, S., & Koopmans, L. V. E. 2009, \textit{MNRAS}, 400, 1583
Weinberg, D. H., Colombi, S., Davé, R., & Katz, N. 2008, \textit{ApJ}, 678, 6
Xu, D. D., et al. 2009, \textit{MNRAS}, 398, 1235
Zemp, M., Diemand, J., Kuhlen, M., Madau, P., Moore, B., Potter, D., Stadel, J., & Widrow, L. 2009, \textit{MNRAS}, 394, 641
Zhao, H. 1996, \textit{MNRAS}, 278, 488