Scattering approach for calculating one-loop effective action and vacuum energy

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Abstract: We propose an approach for calculating one-loop effective actions and vacuum energies in quantum field theory. Spectral functions are functions defined by the eigenvalues of an operator. One-loop effective actions and vacuum energies in quantum field theory, as well as scattering phase shifts and scattering amplitudes in quantum mechanics, are all spectral functions. If a transformation between different spectral functions is identified, we can obtain a spectral function from another through the transformation. In this paper, we convert quantum mechanical methods for calculating scattering phase shifts and scattering amplitudes into quantum field theory methods for calculating one-loop effective actions and vacuum energies. As examples, the Born approximation and the WKB approximation in quantum mechanics are converted into quantum field theory methods. We also calculate the one-loop effective action and vacuum energy of scalar fields in the Schwarzschild spacetime and the Reissner-Nordström spacetime as examples. Some integral representations of the Bessel function are given in appendices.

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1 Introduction

Physical information is embedded within physical operators, with classical mechanics and classical field theory extracting classical information from Hamiltonians, while quantum mechanics and quantum field theory extract quantum information. For instance, the same Hamiltonian can be used to construct both the Hamiltonian equation in classical mechanics and the Schrödinger equation in quantum mechanics, with the difference between them being the method of extracting information.

The eigenproblem of a Hamiltonian,
\[ H \phi_n = \lambda_n \phi_n, \]  
(1.1)
contains all of its information. If the eigenfunction \( \phi_n \) and corresponding eigenvalue \( \lambda_n \) are known, the operator \( H \) can be fully determined. In the spectral representation, the operator can be reconstructed by its eigenvalues and eigenfunctions:
\[ H (x', x) = \sum_n \lambda_n \phi_n^* (x') \phi_n (x). \]
Therefore, the eigenvalues and eigenfunctions contain all of the operator’s information.

The interesting question arises as to what information can be obtained solely from knowledge of the eigenvalues without knowledge of the corresponding eigenfunctions. This problem can be reformulated as follows: from the eigenvalues of a Hamiltonian, what physical quantities can be constructed, and what information can be extracted.

Spectral functions refer to physical quantities that can be constructed from eigenvalues. The global heat kernel, also known as the partition function, \( K (t) = \sum_n e^{-\lambda_n t} \), is an example of a spectral function that is defined by the eigenvalue spectrum \( \{ \lambda_n \} \). Other significant spectral functions include scattering phase shifts, one-loop effective actions, and vacuum energies.

The famous mathematical problem, "Can one hear the shape of a drum?" originally posed by Kac [1], asks how much information can be extracted from an eigenvalue spectrum. Essentially, Kac’s question inquires whether the Hamiltonian can be reconstructed solely from its eigenvalues. The answer, however, is no [2]. (Note that in Kac’s original problem, the information of the Hamiltonian is reflected in the boundary condition, specifically the shape of the drum.)

Since the eigenvalues alone do not provide all of the information of an operator, the problem shifts towards determining "what information can be extracted from eigenvalues."
Various spectral functions can be defined using eigenvalues, and different spectral functions can be transformed into one another. Presently, the known transformations between spectral functions include transformation among global heat kernels (partition functions), one-loop effective actions, vacuum energies [3], the transformation between global heat kernels and spectral counting functions [4, 5], and transformation between scattering phase shifts and global heat kernels [6, 7]. In this paper, we present the transformations between scattering phase shifts and amplitudes, one-loop effective actions, and vacuum energies.

The one-loop effective action and vacuum energy are quantities that arise in quantum field theory, whereas the scattering phase shift and scattering amplitude are quantities in quantum mechanics. All of these are considered spectral functions. In this paper, we present a methodology for calculating the one-loop effective action and vacuum energy in non-relativistic quantum field theory using the scattering phase shift and scattering amplitude. This approach enables the conversion of a quantum field theory problem into a quantum mechanical problem or, equivalently, the translation of a quantum mechanical method into a quantum field theory method. This methodology can be extended to a relativistic theory by substituting the non-relativistic scattering theory with a relativistic scattering theory.

Concretely, in Refs. [4, 5], we found the relation between the global heat kernel and the spectral counting function that counts the eigenstates whose eigenvalues are less than a certain number. In Ref. [6], we found the relation between the scattering phase shift and the partial-wave global heat kernel, and in Ref. [7], we found the relation between the scattering phase shift and the local heat kernel, when we discussed the relationship between the heat kernel method [3] and the spectral method [8] in quantum field theory. As an application, Ref. [9] proposes a method for calculating the scattering phase shift based on the Seeley-DeWitt expansion in heat kernel theory. Based on these previous works, in this paper, we give the relation between the scattering phase shift and amplitude and the one-loop effective action and the vacuum energy.

The relation between scattering phase shifts and amplitudes and one-loop effective actions and vacuum energies converts quantum mechanical methods into quantum-field-theory methods. By this relation, various methods for scattering in quantum mechanics can be converted into methods for calculating one-loop effective actions and vacuum energies in quantum field theory. As examples, in the following, we convert two quantum mechanical methods, the Born approximation and the WKB approximation, into methods for calculating one-loop effective actions and vacuum energies.

The heat kernel serves as a bridge in our method, which bridges quantum field theory and quantum mechanics. The heat kernel expansion is an important method in quantum field theory. There are two heat kernel expansions: the covariant perturbation theory [10–16] and the Schwinger-DeWitt technique [3, 17]. Various methods are developed for heat kernel approaches, such as calculating heat kernel traces by the path integral [18], the Green function approach [19], the technique of labeled operators [20], and heat kernel diagrammatic equations [21]. The heat kernel of higher-order differential operators is considered [22]. Heat kernel expansions are very important, such as the Schwinger-DeWitt expansion in the induced gravity on the AdS background [23]. The heat kernel method applies to calculate effective actions [24], such as the effective field theory in curved spacetime [25, 26],
the heat kernel expansion and the one-loop effective action in QCD [27], the Seeley-DeWitt expansion for the one-loop effective action in the Einstein-Maxwell theory [28], the one-loop effective action for the modified Gauss-Bonnet gravity [29] and in dS2 and AdS2 spacetime [30], $\varphi^4$-fields [31], and various operators [32–34]. The heat kernel method also applies to calculate vacuum energies, such as Casimir energies in curved spacetime [35–38] and in spherically symmetric backgrounds [39]. The vacuum energy is also calculated by the spectral functions [40, 41]. Applying scattering theory to calculate vacuum energy is pioneered in Refs. [8, 42–46]. Various methods in scattering theory can be found in Refs. [47–50]. Reviews on one-loop effective action can be found in Refs. [51–55]. The one-loop calculation on different backgrounds [56–58] and the two-loop effective action in quantum gravity [59] are considered.

In section 2, we give a brief review of various spectral functions. In section 3, we calculate global heat kernels, one-loop effective actions, and vacuum energies from scattering phase shifts. In sections 4 and 5, we convert the Born approximation method into a method for calculating one-loop effective actions and vacuum energies in three and $n$ dimensions, respectively. In section 6, we convert the WKB approximation method into a method for calculating one-loop effective actions and vacuum energies. In section 7, we calculate the one-loop effective action and vacuum energy of scalar fields in the Schwarzschild spacetime and the Reissner-Nordström spacetime. In section 8, we calculate global heat kernels, one-loop effective actions, and vacuum energies from scattering amplitudes. The conclusion is given in section 9. In Appendix A, we give some integral representations for the Bessel function.

## 2 Heat kernel, one-loop effective action, vacuum energy, and scattering phase shift: brief review

In this section, we briefly introduce the scattering phase shift, the heat kernel, the one-loop effective action, and the vacuum energy. They are spectral functions determined by the eigenvalue.

### 2.1 Scattering phase shift

We consider the elastic scattering of a plane wave on a spherically symmetric potential.

If there is no scattering, namely $V (r) = 0$, the wave function is a plane wave [60, 61]:

$$
\psi_0 (r, \theta) = e^{ikr \cos \theta} = \sum_{l=0}^{\infty} (2l + 1) i^l \frac{1}{2} \left[ h^{(2)}_l (kr) + h^{(1)}_l (kr) \right] P_l (\cos \theta),
$$

(2.1)

where $k$ is the incident momentum, $\theta$ is the scattering angle. For scattering, we need to impose scattering boundary condition. For short-range potentials, the scattering boundary condition is[48]

$$
\psi (r, \theta) = e^{ikr \cos \theta} + f (\theta) \frac{e^{ikr}}{r}.
$$

- 3 -
Expanding the wave function into partial waves gives [60, 61]

$$\psi(r, \theta) = \sum_{l=0}^{\infty} (2l + 1) i^l \frac{1}{2} \left[ h_{l}^{(2)}(kr) + e^{2i\delta_l} h_{l}^{(1)}(kr) \right] P_l(\cos \theta), \quad (2.2)$$

where $h_{l}^{(1)}(kr)$, the first kind spherical Hankel function, describes the outgoing wave, and $h_{l}^{(2)}(kr)$, the second kind spherical Hankel function, describes the incoming wave. Comparing $\psi_0(r, \theta)$ and $\psi(r, \theta)$, we can see that the effect of scattering is to multiply the outgoing wave by a phase factor $e^{2i\delta_l}$. Scattering causes a phase shift $\delta_l$ on the outgoing wave function, called the scattering phase shift.

If the observer is far from the target, one can take the large-distance asymptotic approximation [60]:

$$h_{l}^{(1,2)}(kr) \sim (\mp) i^{l+1} \frac{e^{\pm ikr}}{kr}. \quad (2.3)$$

Under the large-distance asymptotic approximation, the incident plane wave (2.1) becomes

$$\psi_0(r, \theta) \sim \sum_{l=0}^{\infty} (2l + 1) i^l \frac{\sin (kr - l\pi/2)}{kr} P_l(\cos \theta), \quad (2.4)$$

and the scattering wave (2.2) becomes

$$\psi(r, \theta) \sim \sum_{l=0}^{\infty} (2l + 1) i^l e^{i\delta_l} \frac{\sin (kr - l\pi/2 + \delta_l(k))}{kr} P_l(\cos \theta). \quad (2.5)$$

The corresponding radial wave functions, under the large-distance asymptotic approximation, are

$$R_{l}^0(r) \sim \frac{\sin (kr - l\pi/2)}{kr}, \quad (2.6)$$

and

$$R_{l}(r) \sim \frac{\sin (kr - l\pi/2 + \delta_l(k))}{kr}. \quad (2.7)$$

2.2 Global Heat kernel, one-loop effective action, and vacuum energy

For operator $D$, the spectral function is defined by its eigenvalues $\{\lambda_n\}$. Formally, the global heat kernel is [3]

$$K(t) = \sum_{n} e^{-\lambda_n t}, \quad (2.8)$$

the one-loop effective action is [3, 62]

$$W = \sum_{n} \ln \sqrt{\lambda_n}, \quad (2.9)$$

and the vacuum energy is [63]

$$E_0 = \frac{1}{2} \sum_{n} \lambda_n. \quad (2.10)$$

Physical operators, e.g., the Hamiltonian, are lower-bounded. The global heat kernel (2.8) is well-defined. Nevertheless, the one-loop effective action (2.9) and the vacuum energy
(2.10) are not well-defined, for they diverge for upper unbounded spectra. In order to obtain finite one-loop effective actions and vacuum energies, one introduces the regularized one-loop effective action and the regularized vacuum energy [63].

By inspecting the formal expressions of global heat kernels, one-loop effective actions, and vacuum energies, Eqs. (2.8), (2.9), and (2.10), we can see that they are related by the relations

\[ W = -\frac{1}{2} \int_0^\infty \frac{1}{t} K(t) \, dt \quad \text{and} \quad E_0 = \frac{1}{2} \Gamma(1-\frac{1}{2}) \int_0^\infty K(t) t^{-2} \, dt. \]

However, it is obvious that these two relations also diverge. To remove divergence, by using the well-defined global heat kernel, one introduces the regularized one-loop effective action \[ W(s) = -\frac{1}{2} \tilde{\mu}^{2s} \int_0^\infty K(t) t^{s-1} \, dt \] (2.11) and the regularized vacuum energy

\[ E_0(\epsilon) = \frac{1}{2} \tilde{\mu}^{2\epsilon} \frac{1}{\Gamma\left(-\frac{1}{2} + \epsilon\right)} \int_0^\infty K(t) t^{-\frac{1}{2} + \epsilon - 1} \, dt, \] (2.12)

where \( \tilde{\mu} \) is a constant of the dimension of mass introduced to keep the proper dimension [3]. When \( s = 0 \) and \( \epsilon = -1/2 \), the regularized one-loop effective action and vacuum energy, \( W(s) \) and \( E_0(\epsilon) \), recover one-loop effective action and vacuum energy, \( W \) and \( E_0 \), but, of course, such a substitution must undergo a regularization process. Moreover, Ref. [64] considers the vacuum energy in static and unbounded spacetime and provides a finite definition of vacuum energy using the minimal subtraction scheme. Various techniques for subtracting divergences can also be found in Ref. [65].

In this paper, we suggest an approach for calculating the heat kernel, the regularized one-loop effective action, and the regularized vacuum energy from the scattering phase shift and the scattering amplitude.

### 3 Calculating global heat kernel, one-loop effective action, and vacuum energy from scattering phase shift

In the section, we calculate the global heat kernel, the regularized one-loop effective action, and the regularized vacuum energy from the scattering phase shift.

The local heat kernel \( K(t; r, r') \) is the Green function of the initial-value problem for the operator \( D = -\nabla^2 + V(r) \). The local partial-wave heat kernel \( K_l(t; r, r') \) is the Green function of the initial-value problem for the radial operator \( D_l = -\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{\ell(\ell+1)}{r^2} + V(r) \). The global heat kernel \( K(t) \) is the trace of \( K(t; r, r') \), and the global partial-wave heat kernel \( K_l(t) \) is the trace of \( K_l(t; r, r') \).

The global partial-wave heat kernel can be divided into three parts: \( K_l(t) = K_l^s(t) + K_l^{\text{free}}(t) + K_l^{\text{bound}}(t) \), where \( K_l^s(t) \) is the scattering-state global partial-wave heat kernel, i.e.

\[ K_l^s(t) = \int_0^\infty \rho(\lambda) e^{-\lambda t} \, dt, \] (3.1)

where \( \rho(\lambda) \) is the state density. The summation over the continuum spectrum of scattering states becomes an integral. \( K_l^{\text{free}}(t) = (4\pi t)^{-3/2} \) is the free global partial-wave heat kernel,
and $K_{l}^{\text{bound}}(t)$ is the bound-state global partial-wave heat kernel, i.e.

$$K_{l}^{\text{bound}}(t) = \sum_{\text{all bound states}} e^{-\lambda_{n}t},$$

(3.2)

where the summation runs over all bound states. The three-dimensional free global partial-wave heat kernel $K_{l}^{\text{free}}(t) = R/\sqrt{4\pi t} - (l + 1/2)/2$. The bound-state global partial-wave heat kernel $K_{l}^{\text{bound}}(t)$ needs to work out the sum over all discrete bound states. In this paper, we care only about the scattering states, for most quantum-field-theory problems are related to scattering states. In Ref. [6], as a by-product of the discussion of the relation between the heat kernel method and the spectral method in quantum field theory, we find a relation between the partial-wave phase shift and the global partial-wave heat kernel of scattering states:

$$K_{l}^{s}(t) = \frac{2}{\pi} t \int_{0}^{\infty} \delta_{l}(k) e^{-k^{2}t} kdk - \frac{\delta_{l}(0)}{\pi},$$

(3.3)

where the superscript $s$ denotes the scattering-state heat kernel. According to the Levinson theorem [66, 67], $\delta_{l}(0) = n_{l}\pi$, where $n_{l}$ the number of bound states with the angular momentum $l$. If there exists the half-bound state (the half-bound state may occur only when $l = 0$), $\delta_{l}(0) = (n_{l} + 1/2)\pi$. If only considering scattering states, $\delta_{l}(0)$ does not contribute.

The relation between the global heat kernel $K^{s}(t)$ and the global partial-wave heat kernel $K_{l}^{s}(t)$ of scattering states is

$$K^{s}(t) = \sum_{l=0}^{\infty} D_{l} K_{l}^{s}(t),$$

(3.4)

where $D_{l}$ is the degeneracy. Then, by Eqs. (3.3) and (3.4), we can obtain the relation between the global heat kernel and the scattering phase shift:

$$K^{s}(t) = \frac{t}{\pi} \sum_{l=0}^{\infty} D_{l} \int_{0}^{\infty} dk^{2} e^{-k^{2}t} \delta_{l}(k).$$

(3.5)

The relation between the regularized one-loop effective action and the global heat kernel can be obtained by Eqs. (2.11) and (3.5):

$$W^{s}(s) = -\frac{1}{2\pi} \tilde{\mu}^{2s} \int_{0}^{\infty} t^{s} dt \int_{0}^{\infty} dk^{2} e^{-k^{2}t} \sum_{l=0}^{\infty} D_{l} \delta_{l}(k)$$

$$= -\frac{1}{2\pi} \tilde{\mu}^{2s} \Gamma(s + 1) \sum_{l=0}^{\infty} D_{l} \int_{0}^{\infty} dk^{2} \frac{\delta_{l}(k)}{(k^{2})^{s+1}}.$$  

(3.6)

The relation between the regularized vacuum energy and the global heat kernel can be obtained by Eqs. (2.12) and (3.5):

$$E_{0}^{s}(\epsilon) = \frac{1}{2\pi} \tilde{\mu}^{2\epsilon} \frac{1}{\Gamma(-\frac{1}{2} + \epsilon)} \int_{0}^{\infty} t^{-\frac{1}{2} + \epsilon} dt \int_{0}^{\infty} dk^{2} e^{-k^{2}t} \sum_{l=0}^{\infty} D_{l} \delta_{l}(k)$$

$$= \frac{1}{2\pi} \tilde{\mu}^{2\epsilon} \frac{\Gamma(\frac{1}{2} + \epsilon)}{\Gamma(-\frac{1}{2} + \epsilon)} \sum_{l=0}^{\infty} D_{l} \int_{0}^{\infty} dk^{2} \frac{\delta_{l}(k)}{(k^{2})^{1/2+\epsilon}}.$$  

(3.7)
The relations (3.6) and (3.7) convert the method for calculating scattering phase shifts in quantum mechanics into a method for calculating one-loop effective actions and vacuum energies in quantum field theory. That is, it converts a quantum-field-theory problem into a quantum-mechanical problem.

In Eqs. (3.5), (3.6), and (3.7), there is a sum should be worked out. In order to deal with this sum, in Appendix (A) we give some integral representations for the Bessel function.

Ref. [8] provides the relation between the scattering phase shift and the density of states $\rho_l(k)$ and the Jost function $F_l(k)$:

$$\frac{1}{\pi} \frac{d \delta_l}{dk} = \rho_l(k) - \rho_l^{(0)}(k)$$ (3.8)

and

$$e^{2i\delta_l} = \frac{F_l(-k)}{F_l(k)}.$$ (3.9)

This allows us to calculate the density of states and the phase of the Jost function from the scattering phase shift.

4 Born approximation: three-dimensional case

The integral equation method is a fundamental approach for scattering in quantum mechanics and in quantum field theory. It can also be applied to the scattering problem in curved spacetime [68, 69]. By the Green function, the integral equation method converts the differential equation defined by the operator $D$, e.g., the eigenequation of the Hamiltonian, into an integral equation. This integral equation can be solved as a perturbation series by the iterative method. The leading-order contribution of the perturbation solution is the Born approximation. The Born approximation is the most important and mature method in the perturbation theory of scattering.

In the following, we convert the Born approximation of calculating scattering phase shifts in quantum mechanics into a method of calculating one-loop effective actions and vacuum energies in quantum field theory.

Due to the high accuracy of Born approximation at large $k$, for the regularization of ultraviolet divergences (divergences at large $k$), we can directly employ subtraction of the first- or second-order Born approximation scattering phase shift to improve the convergence of the integral. Therefore, considering the results of the first two orders of Born approximation is meaningful for the regularization of ultraviolet divergences [8, 43].

4.1 First-order Born approximation

The first-order Born approximation of the scattering phase shift for a spherically symmetric potential $V(r)$ in three dimensions is [70]

$$\delta_l^{(1)} = -\frac{\pi}{2} \int_0^\infty J_{l+1/2}^2(kr) V(r) r dr,$$ (4.1)

where $J_\nu(z)$ is the Bessel function.
4.1.1 Heat kernel

The first-order approximation of the heat kernel can be obtained by substituting Eq. (4.1) and the degeneracy in three dimensions, $D_l = 2l + 1$, into Eq. (3.5):

$$K^{s(1)}_s(t) = -\frac{t^2}{2} \int_0^\infty V(r) r dr \int_0^\infty dk^2 e^{-k^2 t} \sum_{l=0}^\infty (2l + 1) J_{l+1/2}^2 (kr).$$  \hspace{1cm} (4.2)

By the sum rule [71]

$$\sum_{l=0}^\infty (2q + 2l) \frac{\Gamma (2q + l)}{\Gamma (l + 1)} J_{q+l}^2 (z) = \frac{\Gamma (2q + 1)}{\Gamma (q + 1)} \left( \frac{z}{2} \right)^{2q},$$  \hspace{1cm} (4.3)

we arrive at

$$\sum_{l=0}^\infty (2l + 1) J_{l+1/2}^2 (kr) = \frac{2}{\pi} kr.$$  \hspace{1cm} (4.4)

Thus, we have

$$K^{s(1)}_s(t) = -\frac{t^2}{2} \int_0^\infty V(r) r dr \int_0^\infty dk^2 e^{-k^2 t} \frac{2kr}{\pi}.$$  \hspace{1cm} (4.5)

Working out the integral that is a Laplace transform gives the first-order Born approximation for heat kernels:

$$K^{s(1)}_s(t) = -\frac{1}{\sqrt{4\pi t}} \int_0^\infty V(r) r^2 dr.$$  \hspace{1cm} (4.6)

4.1.2 One-loop effective action

The first-order one-loop effective action can be obtained by substituting the first-order phase shift (4.1) into Eq. (3.6):

$$W^{(1)}(s) = \frac{1}{4} \tilde{\mu}^{2s} \Gamma (s + 1) \int_0^\infty V(r) r dr \int_0^\infty (k^2)^{-s-1} \left[ \sum_{l=0}^\infty (2l + 1) J_{l+1/2}^2 (kr) \right] dk^2.$$  \hspace{1cm} (4.7)

Using the sum rule (4.4), we arrive at

$$W^{(1)}(s) = \frac{1}{2\pi} \tilde{\mu}^{2s} \Gamma (s + 1) \int_0^\infty V(r) r^2 dr \int_0^\infty (k^2)^{-s-1} k dk^2.$$  \hspace{1cm} (4.8)

Here the integral of $k$ may diverge. According to Ref. [31], we rewrite $(k^2)^{-s-1}$ as $(k^2 + m^2)^{-s-1}$:

$$W^{(1)}(s) = \frac{1}{2\pi} \tilde{\mu}^{2s} \Gamma (s + 1) \int_0^\infty V(r) r^2 dr \int_0^\infty (k^2 + m^2)^{-s-1} k dk^2.$$  \hspace{1cm} (4.9)

Working out the integral, we obtain the first-order Born approximation of the one-loop effective action:

$$W^{(1)}(s) = \frac{\tilde{\mu}^{2s} \Gamma \left( s - \frac{4}{2} \right)}{4\sqrt{\pi}} \left( m^2 \right)^{\frac{1}{2} - s} \int_0^\infty V(r) r^2 dr.$$  \hspace{1cm} (4.10)
4.1.3 Vacuum energy

The first-order vacuum energy can be obtained by substituting the first-order phase shift (4.1) into Eq. (3.7):

\[
E^{(1)}_0 (\nu) = -\frac{1}{4} \mu^2 \sqrt{\frac{1}{\pi} \frac{\Gamma \left( \frac{1}{2} + \epsilon \right)}{\Gamma \left( \frac{1}{2} + \epsilon \right)}} \int_0^\infty V (r) r dr \int_0^\infty (k^2)^{-1/2-\epsilon} \left[ \sum_{l=0}^{\infty} (2l+1) J_{l+1/2}^2 (kr) \right] dk^2. \tag{4.11}
\]

Using the sum rule (4.4), we arrive at

\[
E^{(1)}_0 (\nu) = -\frac{1}{2 \pi} \mu^2 \sqrt{\frac{1}{\pi} \frac{\Gamma \left( \frac{1}{2} + \epsilon \right)}{\Gamma \left( \frac{1}{2} + \epsilon \right)}} \int_0^\infty V (r) r^2 dr \int_0^\infty (k^2)^{-1/2-\epsilon} k dk^2. \tag{4.12}
\]

The integral of \( k \) may diverge. According to Ref. [31], we rewrite \((k^2)^{-1/2-\epsilon}\) as \((k^2 + m^2)^{-1/2-\epsilon}\):

\[
E^{(1)}_0 (\nu) = -\frac{1}{2 \pi} \mu^2 \sqrt{\frac{1}{\pi} \frac{\Gamma \left( \frac{1}{2} + \epsilon \right)}{\Gamma \left( \frac{1}{2} + \epsilon \right)}} \int_0^\infty V (r) r^2 dr \int_0^\infty (k^2 + m^2)^{-1/2-\epsilon} k dk^2. \tag{4.13}
\]

Working out the integral, we obtain the first-order Born approximation of the vacuum energy:

\[
E^{(1)}_0 (\nu) = -\frac{\mu^2 \epsilon}{4 \sqrt{\pi}} \frac{\Gamma \left( \epsilon - 1 \right)}{\Gamma \left( \frac{1}{2} + \epsilon \right)} (m^2)^{1-\epsilon} \int_0^\infty V (r) r^2 dr. \tag{4.14}
\]

4.2 Second-order Born approximation

The second-order Born approximation of the scattering phase shift for a spherically symmetric potential \( V (r) \) is [70]

\[
\delta_1^{(2)} = -\int_0^\infty \int_0^\infty k j_l (kr) n_l (kr) V (r) r^2 dr \int_0^r k j_l^2 (kr') V (r') r'^2 dr'
-\int_0^\infty \int_0^\infty k j_l^2 (kr) \int_0^\infty \int_0^\infty k j_l (kr') n_l (kr') V (r') r'^2 dr', \tag{4.15}
\]

where \( j_{\nu} (z) \) and \( n_{\nu} (z) \) are the spherical Bessel functions of the first kind and of the second kind, respectively.

4.2.1 Heat kernel

The second-order approximation of the heat kernel can be obtained by substituting the second-order phase shift (4.15) into Eq. (3.5):

\[
K^{(2)} (t) = -\frac{t}{\pi} \int_0^\infty e^{-k^2 t} k^2 dk^2 \int_0^\infty V (r) r^2 dr \int_0^r dr' r'^2 V (r') \Sigma_1 (k; r, r') \nonumber
-\frac{t}{\pi} \int_0^\infty e^{-k^2 t} k^2 dk^2 \int_0^\infty V (r) r^2 dr \int_r^\infty dr' r'^2 V (r') \Sigma_2 (k; r, r'), \tag{4.16}
\]

where

\[
\Sigma_1 (k; r, r') = \sum_{l=0}^{\infty} (2l+1) j_l (kr) n_l (kr) j_l^2 (kr'), \tag{4.17}
\]

\[
\Sigma_2 (k; r, r') = \sum_{l=0}^{\infty} (2l+1) j_l^2 (kr) j_l (kr') n_l (kr'). \tag{4.18}
\]
In the following, we deal with the above sums.

To perform these sums, we give an integral representation of \( j_l^2 (kr) \) in Appendix A.1:

\[
j_l^2 (kr) = \frac{1}{2} \int_0^\pi \frac{\sin qr}{qr} P_l (\cos \theta) \, d \cos \theta
\]

and an integral representation of \( j_l (kr) n_l (kr) \) in Appendix A.2:

\[
j_l (kr) n_l (kr) = -\frac{1}{2} \int_0^\pi \frac{\cos qr}{qr} P_l (\cos \theta) \, d \cos \theta.
\]

Substituting the above two integral representations into Eq. (4.17) gives

\[
\Sigma_1 (k; r, r') = -\frac{1}{4} \int_0^\pi \frac{\cos qr}{qr} d \cos \theta \int_0^\pi d \cos \theta' \frac{\sin q'r'}{q'r'} \sum_{l=0}^\infty (2l + 1) P_l (\cos \theta) P_l (\cos \theta')
\]

where \( q = 2k \sin \frac{\theta}{2} \) and \( q' = 2k \sin \frac{\theta'}{2} \). Using the relation [72]

\[
\sum_{l=0}^\infty (2l + 1) P_l (\cos \theta) P_l (\cos \theta') = 2 \delta (\cos \theta - \cos \theta')
\]

and performing the integral, we have

\[
\Sigma_1 (k; r, r') = -\frac{1}{2} \int_0^\pi \frac{\cos qr}{qr} d \cos \theta \int_0^\pi d \cos \theta' \frac{\sin q'r'}{q'r'} \delta (\cos \theta - \cos \theta')
\]

\[
= -\frac{1}{2} \int_0^\pi \frac{\cos qr \sin q'r'}{qr} d \cos \theta
\]

\[
= \frac{\text{Si} (2kr - 2kr') - \text{Si} (2kr + 2kr')}{4k^2 rr'},
\]

where \( \text{Si} (z) \) is the Sine integral function.

Similarly, we obtain

\[
\Sigma_2 (k; r, r') = -\frac{\text{Si} (2kr + 2kr') + \text{Si} (2kr - 2kr')}{4k^2 rr'}.
\]

Substituting Eqs. (4.23) and (4.24) into Eq. (4.16) gives the second-order global heat kernel:

\[
K^{(2)} (t) = -\frac{t}{4\pi} \int_0^\pi V (r) \, r \, dr \int_r^\infty V (r') \, r' \, dr' \int_0^\infty dk^2 e^{-k^2 t} \left[ \text{Si} (2kr - 2kr') - \text{Si} (2kr + 2kr') \right]
\]

\[
+ \frac{t}{4\pi} \int_0^\pi V (r) \, r \, dr \int_r^\infty V (r') \, r' \, dr' \int_0^\infty dk^2 e^{-k^2 t} \left[ \text{Si} (2kr + 2kr') + \text{Si} (2kr - 2kr') \right].
\]

The integral of \( k^2 \) is a Laplace transform. Performing the Laplace transform gives the second-order Born approximation of heat kernels:

\[
K^{(2)} (t) = -\frac{1}{8} \int_0^\infty V (r) \, r \, dr \left\{ \int_r^\infty dr' r' V (r') \left[ \text{erf} \left( \frac{r - r'}{\sqrt{t}} \right) - \text{erf} \left( \frac{r + r'}{\sqrt{t}} \right) \right] - \int_r^\infty dr' V (r') \, r' \left[ \text{erf} \left( \frac{r' - r}{\sqrt{t}} \right) + \text{erf} \left( \frac{r + r'}{\sqrt{t}} \right) \right] \right\}
\]

where \( \text{erf} (z) \) is the Error function [72].
4.2.2 One-loop effective action

We next calculate the second-order Born approximation of the one-loop effective action by directly using the relation between heat kernels and one-loop effective actions, Eq. (2.11), though, in principle, we can also obtain this result by substituting the second-order phase shift (4.15) into Eq. (3.6).

Substituting Eq. (4.26) into Eq. (2.11) gives the second-order Born approximation of the one-loop effective action:

\[
W^{(2)}(s) = \frac{1}{16} \bar{\mu}^2 s \int_{0}^{\infty} dr V(r) \left\{ \int_{0}^{r} dr' r' V(r') \left[ \text{erf} \left( \frac{r - r'}{\sqrt{t}} \right) - \text{erf} \left( \frac{r + r'}{\sqrt{t}} \right) \right] t^{s-1} dt \right. \\
- \left. \int_{r}^{\infty} dr' r' V(r') \int_{0}^{\infty} \left[ \text{erf} \left( \frac{r' - r}{\sqrt{t}} \right) + \text{erf} \left( \frac{r + r'}{\sqrt{t}} \right) \right] t^{s-1} dt \right\}.
\]

Performing the integral, we have

\[
W^{(2)}(s) = \frac{\bar{\mu}^2 \Gamma \left( \frac{1}{2} - s \right)}{16 \sqrt{\pi} s} \int_{0}^{\infty} dr V(r) \left\{ \int_{0}^{r} dr' r' V(r') \left[ (r - r')^{2s} - (r + r')^{2s} \right] \\
- \int_{r}^{\infty} dr' r' V(r') \left[ (r - r')^{2s} + (r + r')^{2s} \right] \right\}.
\]

4.2.3 Vacuum energy

Similarly, using the relation between the heat kernel and the vacuum energy, Eq. (2.12), we can obtain the second-order Born approximation of the vacuum energy. Substituting Eq. (4.26) into Eq. (2.12) gives

\[
E_{0}^{(2)}(\epsilon) = -\frac{\bar{\mu}^2 \Gamma \left( \frac{1}{2} - \epsilon \right)}{16 \sqrt{\pi} \Gamma \left( \frac{1}{2} + \frac{1}{2} \epsilon \right)} \int_{0}^{\infty} dr V(r) \left\{ \int_{0}^{r} dr' r' V(r') \left[ (r - r')^{2\epsilon - 1} + (r + r')^{2\epsilon - 1} \right] \\
- \int_{r}^{\infty} dr' r' V(r') \left[ (r - r')^{2\epsilon - 1} + (r + r')^{2\epsilon - 1} \right] \right\}.
\]

5 Born approximation: \textit{n}-dimensional case

In this section, we consider the \textit{n}-dimensional Born approximation. The \textit{n}-dimensional results can be used to perform the dimensional renormalization. The dimensional renormalization can remove the divergence in the Born approximation [60].

5.1 First-order Born approximation

The first-order Born approximation of scattering phase shifts in \textit{n} dimensions is [71]

\[
\delta_{l}^{(1)}(k) = -\frac{\pi}{2} \int_{0}^{\infty} J_{l+\frac{1}{2}}^{2} (kr) V(r) r dr.
\]

5.1.1 Heat kernel

The first-order approximation of the heat kernel can be obtained by substituting Eq. (5.1) into Eq. (3.5):

\[
K^{s(1)}(t) = -\frac{t}{2} \int_{0}^{\infty} V(r) r dr \int_{0}^{\infty} dk^2 e^{-k^2 t} \sum_{l=0}^{\infty} D_l J_{l+\frac{1}{2}}^{2} (kr).
\]
The $n$-dimensional degeneracy for spherically symmetric potentials is [71]

$$D_l = \frac{(n + 2l - 2) \Gamma (n + l - 2)}{\Gamma (n - 1) \Gamma (l + 1)}. \quad (5.3)$$

Taking $q = \frac{n}{2} - 1$ in the sum rule [71],

$$\sum_{l=0}^{\infty} \frac{(2q + 2l) \Gamma (2q + l)}{\Gamma (l + 1)} J_{q+l}^2 (z) = \frac{\Gamma (2q + 1)}{\Gamma (q + 1)^2} \left( \frac{z}{2} \right)^{2q}, \quad (5.4)$$

gives

$$\sum_{l=0}^{\infty} \frac{\Gamma (n + l - 2) (n + 2l - 2)}{\Gamma (n - 1) \Gamma (l + 1)} J_{\frac{n}{2}+l-1}^2 (kr) = \frac{1}{\Gamma (\frac{n}{2} - 1)^2} \left( \frac{kr}{2} \right)^{n-2}. \quad (5.5)$$

Substituting Eq. (5.5) into Eq. (5.2) gives the first-order Born approximation of the heat kernel:

$$K^{(1)} (s) (t) = -t \frac{2^{1-n} \mu^2}{\Gamma (\frac{n}{2}) ^{n/2 - 1}} \int_0^\infty V(r) r^{n-1} dr. \quad (5.6)$$

### 5.1.2 One-loop effective action

The first-order $n$-dimensional one-loop effective action can be obtained by substituting the first-order $n$-dimensional phase shift (5.1) into Eq. (3.6):

$$W^{(1)} (s) = \frac{1}{4} \mu^2 \Gamma (s + 1) \int_0^\infty V(r) r dr \int_0^\infty dk^2 (k^2)^{-s-1} \sum_{l=0}^{\infty} D_l J_{\frac{n}{2}+l-1}^2 (kr). \quad (5.7)$$

Using the sum rule (5.5), we arrive at

$$W^{(1)} (s) = \frac{1}{2^n \Gamma (\frac{n}{2}) ^{n/2 - 1}} \mu^2 \Gamma (s + 1) \int_0^\infty V(r) r^{n-1} dr \int_0^\infty dk^2 (k^2)^{-s-1} k^{n-2}. \quad (5.8)$$

Here the integral of $k$ may diverge. According to Ref. [31], we rewrite $(k^2)^{-s-1}$ as $(k^2 + m^2)^{-s-1}$:

$$W^{(1)} (s) = \frac{1}{2^n \Gamma (\frac{n}{2}) ^{n/2 - 1}} \mu^2 \Gamma (s + 1) \int_0^\infty V(r) r^{n-1} dr \int_0^\infty dk^2 (k^2 + m^2)^{-s-1} k^{n-2}. \quad (5.9)$$

Working out the integral, we obtain the first-order Born approximation of the one-loop effective action:

$$W^{(1)} (s) = \frac{\mu^2 \Gamma (s + 1 - \frac{n}{2})}{2^n \Gamma (\frac{n}{2}) ^{n/2 - 1-s}} \int_0^\infty V(r) r^{n-1} dr. \quad (5.10)$$
5.1.3 Vacuum energy

The first-order $n$-dimensional vacuum energy can be obtained by substituting the first-order $n$-dimensional phase shift (5.1) into Eq. (3.7):

$$E^{(1)}_0 (\epsilon) = -\frac{1}{2n} \mu^{2n} \Gamma \left( \frac{n}{2} + \epsilon \right) \int_0^\infty V (r) r dr \int_0^\infty dk^2 \left( k^2 \right)^{-1/2-\epsilon} \sum_{l=0}^\infty D_l J^2_{\frac{n}{2}+l-1} (kr). \tag{5.11}$$

Using the sum rule (5.5), we arrive at

$$E^{(1)}_0 (\epsilon) = -\frac{1}{2n} \mu^{2n} \Gamma \left( \frac{n}{2} + \epsilon \right) \int_0^\infty V (r) r^{n-1} dr \int_0^\infty dk^2 \left( k^2 \right)^{-1/2-\epsilon} k^{n-2}. \tag{5.12}$$

The integral of $k$ may diverge. According to Ref. [31], we rewrite $(k^2)^{-1/2-\epsilon}$ as $(k^2 + m^2)^{-1/2-\epsilon}$:

$$E^{(1)}_0 (\epsilon) = -\frac{1}{2n} \mu^{2n} \Gamma \left( \frac{n}{2} + \epsilon \right) \int_0^\infty V (r) r^{n-1} dr \int_0^\infty dk^2 \left( k^2 + m^2 \right)^{-1/2-\epsilon} k^{n-2}. \tag{5.13}$$

Working out the integral, we obtain the first-order Born approximation of the $n$-dimensional vacuum energy:

$$E^{(1)}_0 (\epsilon) = -\frac{1}{2n} \mu^{2n} \Gamma \left( \epsilon + \frac{1}{2} - \frac{n}{2} \right) \int_0^\infty V (r) r^{n-1} dr. \tag{5.14}$$

In performing the integral in the Born approximation, one may encounter divergences that do not come from the usual divergence in quantum field theory. Such divergences can be removed by the procedure given in Ref. [60].

5.2 Second-order Born approximation

The $n$-dimensional second-order Born approximation of the scattering phase shift for a spherically symmetric potential $V (r)$ is [70]

$$\delta^{(2)} (r) = -\frac{\pi^2}{4} \int_0^r J^2_{\frac{n}{2}+l-1} (kr) Y^2_{\frac{n}{2}+l-1} (kr) V (r) r dr \int_0^r J^2_{\frac{n}{2}+l-1} (kr') V (r') r' dr'$$

$$-\frac{\pi^2}{4} \int_0^\infty J^2_{\frac{n}{2}+l-1} (kr) V (r) r dr \int_0^\infty J^2_{\frac{n}{2}+l-1} (kr') Y^2_{\frac{n}{2}+l-1} (kr') V (r') r' dr', \tag{5.15}$$

where $Y_\nu (z)$ is the Bessel function of the second kind.

5.2.1 Heat kernel

The second-order approximation of the $n$-dimensional heat kernel can be obtained by substituting the second-order phase shift (5.15) into Eq. (3.5):

$$K^{(2)} (t) = -\frac{\pi t}{4} \int_0^\infty e^{-k^2 t} dk^2 \int_0^\infty V (r) r dr \int_0^r dr' V (r') r' \Sigma_1 (k; r, r')$$

$$-\frac{\pi t}{4} \int_0^\infty e^{-k^2 t} dk^2 \int_0^\infty V (r) r dr \int_r^\infty dr' V (r') r' \Sigma_2 (k; r, r'), \tag{5.16}$$

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where
\[ \Sigma_1 (k; r, r') = \sum_{l=0}^{\infty} \frac{(n+2l-2) \Gamma (n+l-2)}{\Gamma (n-1) \Gamma (l+1)} J_{\frac{2}{l}+l-1}^2 (kr) Y_{\frac{2}{l}+l-1}^2 (kr) J_{\frac{2}{l}+l-1}^2 (kr') , \quad (5.17) \]
\[ \Sigma_2 (k; r, r') = \sum_{l=0}^{\infty} \frac{(n+2l-2) \Gamma (n+l-2)}{\Gamma (n-1) \Gamma (l+1)} J_{\frac{2}{l}+l-1}^2 (kr) J_{\frac{2}{l}+l-1}^2 (kr') Y_{\frac{2}{l}+l-1}^2 (kr') . \quad (5.18) \]

To perform these sums, we give an integral representation of \( J_{\frac{1}{l}+\mu}^2 (kr) \) in Appendix A.3,
\[ J_{\frac{1}{l}+\mu}^2 (kr) = \frac{2^{\mu-1} \Gamma (l+1) \Gamma (\mu)}{\pi \Gamma (2\mu+l)} (kr)^{2\mu} \int_0^{\pi} \frac{J_{\mu} (qr)}{(qr)^{2\mu}} C_{l}^{\mu} (\cos \theta) \sin^{2\mu} \theta \sin \theta \sin \theta , \quad (5.19) \]
and an integral representation of \( J_{\frac{1}{l}+\mu} (kr) Y_{\frac{1}{l}+\mu} (kr) \) in Appendix A.4,
\[ J_{\frac{1}{l}+\mu} (kr) Y_{\frac{1}{l}+\mu} (kr) = \frac{2^{\mu-1} \Gamma (l+1) \Gamma (\mu)}{\pi \Gamma (2\mu+l)} (kr)^{2\mu} \int_0^{\pi} \frac{J_{\mu} (qr)}{(qr)^{2\mu}} C_{l}^{\mu} (\cos \theta) \sin^{2\mu} \theta \sin \theta \sin \theta , \quad (5.20) \]
where \( C_{l}^{\mu} (\cos \theta) \) is the Gegenbauer polynomial. Substituting the above two integral representations into Eq. (5.17) gives
\[ \Sigma_1 (k; r, r') = \sum_{l=0}^{\infty} \frac{(2\mu+2l) \Gamma (2\mu+l)}{\Gamma (2\mu+1) \Gamma (l+1)} J_{\frac{1}{l}+\mu} (kr) Y_{\frac{1}{l}+\mu} (kr) J_{\frac{1}{l}+\mu}^2 (kr') \]
\[ = \sum_{l=0}^{\infty} \frac{(2\mu+2l) \Gamma (2\mu+l)}{\Gamma (2\mu+1) \Gamma (l+1)} \left[ \frac{2^{\mu-1} \Gamma (l+1) \Gamma (\mu)}{\pi \Gamma (2\mu+l)} (kr)^{2\mu} \int_0^{\pi} \frac{J_{\mu} (qr)}{(qr)^{2\mu}} C_{l}^{\mu} (\cos \theta) \sin^{2\mu} \theta \sin \theta \right] \]
\[ \times \frac{2^{\mu-1} \Gamma (l+1) \Gamma (\mu)}{\pi \Gamma (2\mu+l)} (kr')^{2\mu} \int_0^{\pi} \frac{J_{\mu} (qr')}{(qr')^{2\mu}} C_{l}^{\mu} (\cos \theta') \sin^{2\mu} \theta' \sin \theta' \]
\[ = \frac{2^{2\mu-1} \Gamma^2 (\mu)}{\pi^2 \Gamma (2\mu+1)} (kr)^{2\mu} (kr)^{2\mu} \int_0^{\pi} d\theta \frac{J_{\mu} (qr)}{(qr)^{2\mu}} \sin^{2\mu} \theta (kr')^{2\mu} \int_0^{\pi} d\theta' \frac{J_{\mu} (qr')}{(qr')^{2\mu}} \sin^{2\mu} \theta' , \quad (5.21) \]
where \( \mu = \frac{\beta}{2} - 1, q = 2k \sin \theta, \) and \( q' = 2k \sin \theta' \). Using the relation [72]
\[ \sum_{l=0}^{\infty} \frac{\Gamma (l+1) (2\mu+2l)}{\Gamma (2\mu+1)} C_{l}^{\mu} (\cos \theta) C_{l}^{\mu} (\cos \theta') = \frac{2^{2\mu-2\mu \pi}}{\Gamma^2 (\mu)} (\sin \theta)^{1-2\mu} (\sin \theta')^{1-2\mu} \delta (\cos \theta - \cos \theta') , \quad (5.22) \]
we arrive at
\[ \Sigma_1 (k; r, r') = \frac{(kr)^{2\mu} (kr')^{2\mu}}{\pi^2 \Gamma (2\mu+1)} \int_0^{\pi} \frac{J_{\mu} (qr)}{(qr)^{2\mu}} \sin^{2\mu-1} \theta \cos \theta \int_0^{\pi} \frac{J_{\mu} (qr')}{(qr')^{2\mu}} \sin^{2\mu-1} \theta' \cos \theta' \sin \theta' \sin \theta' \]
\[ = \frac{(kr)^{2\mu} (kr')^{2\mu}}{\pi^2 \Gamma (2\mu+1)} \int_0^{\pi} \frac{J_{\mu} (qr)}{(qr)^{2\mu}} J_{\mu} (qr') \sin^{2\mu-1} \theta \sin \theta . \quad (5.23) \]
Similarly, we have
\[ \Sigma_2 (k; r, r') = \frac{(kr)^{2\mu} (kr')^{2\mu}}{\pi^2 \Gamma (2\mu+1)} \int_0^{\pi} \frac{J_{\mu} (qr')}{(qr')^{2\mu}} J_{\mu} (qr) \sin^{2\mu-1} \theta \sin \theta . \quad (5.24) \]
Substituting Eqs. (5.23) and (5.24) into Eq. (5.16) gives the $n$-dimensional second-order global heat kernel:

$$K^{\text{odd}} (t) = -\frac{t}{4\Gamma (n-1)} \int_0^\infty e^{-k^2 t} \text{d}k \int_0^\infty \text{d}rr V (r) (kr)^{n-2}$$

$$\times \left\{ \int_0^r \text{d}rr' V (r') \left[ (kr')^{n-2} \int_0^\pi Y_{n-1} (qr) J_{n-1} (q r') (kr')^{n-1} \sin^{n-3} \theta \text{d} \cos \theta \right] \right.$$  

$$+ \int_r^\infty \text{d}rr' V (r') \left[ (kr')^{n-2} \int_0^\pi Y_{n-1} (qr) J_{n-1} (q r') (kr')^{n-1} \sin^{2n-1} \theta \text{d} \cos \theta \right] \right\}. \quad (5.25)$$

The integral encountered in Eq. (5.25) is difficult. The odd-dimensional case and even-dimensional case are very different [73], for the odd-dimensional Bessel polynomial is a polynomial but the even-dimensional case is not [60]. In the following we only consider the odd-dimensional case.

For odd-dimensional cases, the integral representation given in Appendix A.4 with $\mu = \frac{1}{2}$ and $l = \frac{n}{2} - \frac{1}{2} \ (n = 3, 5, 7, \ldots)$, becomes

$$Y_{\frac{n}{2}-1} (qr) J_{\frac{n}{2}-1} (q r') = \frac{q \sqrt{rr'}}{\sqrt{2 \pi}} \int_0^\pi Y_{1/2} \left( \frac{q \sqrt{r^2 + r'^2 - 2rr' \cos \phi}}{(q \sqrt{r^2 + r'^2 - 2rr' \cos \phi})^{1/2}} \right) P_{\frac{n}{2} - \frac{1}{2}} (\cos \phi) \text{d} \cos \phi$$

$$= -\frac{q \sqrt{rr'}}{\pi} \int_0^\pi \cos \left( \frac{q \sqrt{r^2 + r'^2 - 2rr' \cos \phi}}{q \sqrt{r^2 + r'^2 - 2rr' \cos \phi}} \right) P_{\frac{n}{2} - \frac{1}{2}} (\cos \phi) \sin \phi \text{d} \phi. \quad (5.26)$$

Then the integral over $\theta$ in Eq. (5.25) reads

$$\int_0^\pi Y_{\frac{n}{2}-1} (2kr \sin \frac{\theta}{2}) J_{\frac{n}{2}-1} (2kr' \sin \frac{\theta}{2}) \sin^{n-3} \theta \text{d} \cos \theta$$

$$= \int_0^\pi \frac{1}{(2kr \sin \frac{\theta}{2})^{n-1} (2kr' \sin \frac{\theta}{2})^{n-1}}$$

$$\times \left\{ -\frac{q \sqrt{rr'}}{\pi} \int_0^\pi \cos \left( \frac{q \sqrt{r^2 + r'^2 - 2rr' \cos \phi}}{q \sqrt{r^2 + r'^2 - 2rr' \cos \phi}} \right) P_{\frac{n}{2} - \frac{1}{2}} (\cos \phi) \sin \phi \text{d} \phi \right\} \sin^{n-3} \theta \text{d} \cos \theta \quad (5.27)$$

with $q = 2k \sin \frac{\theta}{2}$. We first perform the integral over $\theta$:

$$\int_0^\pi \cos \left( \frac{2k \sin \frac{\theta}{2} \sqrt{r^2 + r'^2 - 2rr' \cos \phi}}{\sin \frac{\theta}{2}} \right) \frac{\sin^{n-3} \theta \text{d} \cos \theta}{\sin^{n-2} \frac{\theta}{2}}$$

$$= -2^{n-2} \sqrt{\pi} \Gamma \left( \frac{n}{2} - \frac{1}{2} \right) \frac{J_{\frac{n}{2}-1} \left( \frac{2k \sqrt{r^2 + r'^2 - 2rr' \cos \phi}}{k \sqrt{r^2 + r'^2 - 2rr' \cos \phi}} \right)}{\left( k \sqrt{r^2 + r'^2 - 2rr' \cos \phi} \right)^{n-1}}. \quad (5.28)$$
Then we have
\[
\int_0^\pi \frac{Y_{n-1}(qr) J_{n-1}(qr')}{(qr)^{\frac{n}{2}-1} (qr')^{\frac{n}{2}-1}} \sin^{n-3} \theta \, d\theta \cos \theta
\]
\[
= \frac{2 \pi \Gamma \left( \frac{n}{2} - \frac{1}{2} \right)}{k^{n-3} \sqrt{\pi} (rr')^{\frac{n}{2} - \frac{1}{2}}} \int_0^\pi J_{n-1} \left( \frac{2k \sqrt{r^2 + r'^2 - 2rr' \cos \phi}}{2k \sqrt{r^2 + r'^2 - 2rr' \cos \phi}} \right) P_{\frac{n}{2} - \frac{1}{2}} (\cos \phi) \, d\cos \theta. \quad (5.29)
\]

Eq. (5.25), by Eq. (5.29), becomes
\[
K^{(2)}(t) = -\frac{t}{4\sqrt{\pi} \Gamma \left( \frac{n}{2} - \frac{1}{2} \right)} \int_0^\infty dr \frac{r^{n+1}}{2} V(r)
\times \left\{ \int_0^r dr' \left( \frac{n+1}{2} \right) \frac{V(r')}{r'} \int_0^\pi \int_0^\infty e^{-k^2 t} k^{n-1} J_{n-1} \left( \frac{2k \sqrt{r^2 + r'^2 - 2rr' \cos \phi}}{2k \sqrt{r^2 + r'^2 - 2rr' \cos \phi}} \right) \, dk^2 \right\}
\times P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) \, d\cos \theta
\]
\[
+ \int_0^r dr' \left( \frac{n+1}{2} \right) \frac{V(r')}{r'} \int_0^\pi \int_0^\infty e^{-k^2 t} k^{n-1} J_{n-1} \left( \frac{2k \sqrt{r^2 + r'^2 - 2rr' \cos \phi}}{2k \sqrt{r^2 + r'^2 - 2rr' \cos \phi}} \right) \, dk^2 \right\}
\times P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) \, d\cos \theta \right\}. \quad (5.30)
\]

Performing the integral over \( k \), which is a Laplace transform, gives
\[
K^{(2)}(t) = -\frac{1}{2\sqrt{2\pi} t^{\frac{n}{2} - \frac{1}{2}} \Gamma \left( \frac{n}{2} - \frac{1}{2} \right)} \int_0^\infty r \frac{r^{n+1}}{2} V(r) \, dr
\times \left\{ \int_0^r dr' \left( \frac{n+1}{2} \right) \frac{V(r')}{r'} \int_0^\pi \int_0^\infty K_{1/2} \left( \frac{r^2 + r'^2 - 2rr' \cos \phi}{t} \right) P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) \, d\cos \phi \right\}
\]
\[
+ \int_0^r dr' \left( \frac{n+1}{2} \right) \frac{V(r')}{r'} \int_0^\pi \int_0^\infty K_{1/2} \left( \frac{r^2 + r'^2 - 2rr' \cos \phi}{t} \right) P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) \, d\cos \phi \right\}. \quad (5.31)
\]

By \( \sqrt{r^2 + r'^2 - 2rr' \cos \phi} = |r - r'| \), we rewrite
\[
K^{(2)}(t) = -\frac{1}{2\sqrt{2\pi} t^{\frac{n}{2} - \frac{1}{2}} \Gamma \left( \frac{n}{2} - \frac{1}{2} \right)} \int_0^\infty r \frac{r^{n+1}}{2} V(r) \, dr
\times \left\{ \int_0^r dr' \left( \frac{n+1}{2} \right) \frac{V(r')}{r'} \int_0^\pi \int_0^\infty K_{1/2} \left( \frac{(r - r')^2}{t} \right) P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) \, d\cos \phi \right\}
\]
\[
+ \int_0^r dr' \left( \frac{n+1}{2} \right) \frac{V(r')}{r'} \int_0^\pi \int_0^\infty K_{1/2} \left( \frac{(r - r')^2}{t} \right) P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) \, d\cos \phi \right\}, \quad (5.32)
\]
or, equivalently,

\[
K^{s(2)}(t) = -\frac{1}{4\sqrt{\pi t^{n-1}}} \frac{\Gamma \left( \frac{n}{2} - \frac{1}{2} \right)}{\Gamma(n-1)} \int_0^\infty r^{n+1} V(r) \, dr \\
\times \left\{ \int_0^r dr' \left( r' \right)^{n+1} V(r') \int_0^\pi \frac{\exp \left( -\frac{(r-r')^2}{2} \right) P_{\frac{n}{2} - \frac{1}{2}} \left( \cos \phi \right) d \cos \phi}{|r - r'|} \right. \\
+ \int_r^\infty dr' \left( r' \right)^{n+1} V(r') \int_0^\pi \frac{\exp \left( -\frac{(r-r')^2}{2} \right) P_{\frac{n}{2} - \frac{3}{2}} \left( \cos \phi \right) d \cos \phi}{|r - r'|} \left. \right\},
\]

(5.33)

where \( K_\nu (z) \) is the modified Bessel function of the second kind [72].

In the heat-kernel theory, one often concentrates on the small \( t \) case, e.g., the Seeley-DeWitt expansion [3]. For small \( t \), we have the following expansion:

\[
K_{1/2} \left( \frac{(r-r')^2}{t} \right) \sim \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi}} + \cdots.
\]

(5.34)

Substituting into Eq. (5.32) gives

\[
K^{s(2)}(t) \sim -\frac{1}{4\sqrt{\pi t^{n-1}}} \frac{\Gamma \left( \frac{n}{2} - \frac{1}{2} \right)}{\Gamma(n-1)} \int_0^\infty r^{n+1} V(r) \, dr \\
\times \left\{ \int_0^r dr' \left( r' \right)^{n+1} V(r') \int_0^\pi \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi}} P_{\frac{n}{2} - \frac{1}{2}} \left( \cos \phi \right) d \cos \phi \\
+ \int_r^\infty dr' \left( r' \right)^{n+1} V(r') \int_0^\pi \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi}} P_{\frac{n}{2} - \frac{3}{2}} \left( \cos \phi \right) d \cos \phi \right\}.
\]

(5.35)

To perform the integral in Eq. (5.35), we use [72]

\[
\begin{align*}
\frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi}} &= \frac{1}{r} \sum_{l=0}^{\infty} \left( \frac{r'}{r} \right)^l P_l \left( \cos \phi \right), \quad r > r', \\
\frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi}} &= \frac{1}{r'} \sum_{l=0}^{\infty} \left( \frac{r}{r'} \right)^l P_l \left( \cos \phi \right), \quad r < r'.
\end{align*}
\]

(5.36)

In Eq. (5.35), the first term corresponds to \( r > r' \) and the second term corresponds to \( r < r' \). Then

\[
\int_0^\pi \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi}} P_{\frac{n}{2} - \frac{1}{2}} \left( \cos \phi \right) d \cos \phi = \frac{1}{r} \sum_{l=0}^{\infty} \left( \frac{r'}{r} \right)^l \int_0^\pi P_l \left( \cos \phi \right) P_{\frac{n}{2} - \frac{3}{2}} \left( \cos \phi \right) d \cos \phi \\
= \frac{1}{r} \sum_{l=0}^{\infty} \left( \frac{r'}{r} \right)^l \frac{2}{2l + 1} \delta_{l, \frac{n}{2} - \frac{3}{2}} \\
= \frac{2}{n-2} \frac{1}{r} \left( \frac{r'}{r} \right)^{\frac{n}{2} - \frac{3}{2}}, \quad (r > r').
\]

(5.37)
Similarly,

\[
\int_0^\infty \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi}} \frac{1}{\sqrt{n^2 - 2}} \cos \phi \, d \cos \phi = \frac{2}{n - 2} \frac{1}{r'} \left(\frac{r}{r'}\right)^{\frac{n}{2} - 1}, \quad (r < r').
\]

The second-order heat kernel in odd dimensions, by Eq. (5.35), reads

\[
K^{(2)}(t) \sim -\frac{1}{2(n-2)} \frac{\Gamma \left(\frac{n}{2} - \frac{1}{2}\right)}{\sqrt{\pi} \Gamma \left(\frac{n}{2} - 1\right)} \times \left[ \int_0^\infty drrV(r) \int_0^r dr' (r')^{n-1} V(r') + \int_0^{\infty} drr^{-1} V(r) \int_r^{\infty} dr' r' V(r') \right].
\]

5.2.2 One-loop effective action

Next by using the relation between the global heat kernel and the one-loop effective action, Eq. (2.11), and substituting Eq. (5.25) into Eq. (2.11), we obtain the second-order \(n\)-dimensional one-loop effective action:

\[
W^{(2)}(s) = \frac{\pi}{8\mu^2} \int_0^\infty V(r) r dr \\
\times \left\{ \int_0^r V(r') r' dr' \int_0^\infty dk^2 \left[ \frac{(kr)^{n-2} (kr')^{n-2}}{\pi \Gamma(2\mu + 1)} \int_0^{\infty} \frac{Y_{\frac{2}{2} - 1}(qr') J_{\frac{2}{2} - 1}(qr')}{(qr')^{\frac{n}{2} - 1}} \sin^{n-3} \theta d\cos \theta \right] \right. \\
\times \int_0^\infty e^{-k^2 t} e^s dt \\
+ \left[ \int_r^\infty V(r') r' dr' \int_0^\infty drr^{-1} \left[ \frac{(kr)^{n-2} (kr')^{n-2}}{\pi \Gamma(2\mu + 1)} \int_0^{\infty} \frac{Y_{\frac{2}{2} - 1}(qr') J_{\frac{2}{2} - 1}(qr')}{(qr')^{\frac{n}{2} - 1}} \sin^{n-3} \theta d\cos \theta \right] \right] \right\}.
\]

Performing the integral over \(t\), we have

\[
W^{(2)}(s) = \frac{\pi}{8\mu^2} \Gamma(s + 1) \int_0^\infty V(r) r dr \\
\times \left\{ \int_0^r V(r') r' dr' \int_0^\infty drr^{-1} \left[ \frac{(kr)^{n-2} (kr')^{n-2}}{\pi \Gamma(2\mu + 1)} \int_0^{\infty} \frac{Y_{\frac{2}{2} - 1}(qr') J_{\frac{2}{2} - 1}(qr')}{(qr')^{\frac{n}{2} - 1}} \sin^{n-3} \theta d\cos \theta \right] \right. \\
+ \left[ \int_r^\infty V(r') r' dr' \int_0^\infty drr^{-1} \left[ \frac{(kr)^{n-2} (kr')^{n-2}}{\pi \Gamma(2\mu + 1)} \int_0^{\infty} \frac{Y_{\frac{2}{2} - 1}(qr') J_{\frac{2}{2} - 1}(qr')}{(qr')^{\frac{n}{2} - 1}} \sin^{n-3} \theta d\cos \theta \right] \right\}.
\]

For odd-dimensional cases, substituting Eq. (5.33) into Eq. (2.11) gives the second-
order Born approximation of the one-loop effective action:

\[
W^{(2)}(s) = \frac{\tilde{\mu}^{2 s}}{8 \sqrt{\pi}} \frac{\Gamma \left( \frac{\nu}{2} - \frac{1}{2} \right)}{\Gamma(n - 1)} \int_0^\infty dr r^{n+1} V(r) \times \left\{ \int_0^r dr' (r')^{n+1} V(r') \int_0^\pi P_{\frac{n}{2}} - \frac{1}{2} \left( \cos \phi \right) d \cos \phi \times \int_0^\infty dt t^{s-1} \left[ t^{1-\frac{n}{2}} \exp \left( -\frac{1}{2} \left( r^2 + r'^2 - 2rr' \cos \phi \right) \right) \right] \right\}.
\]

Performing the integral over \( t \) gives

\[
W^{(2)}(s) = \frac{\tilde{\mu}^{2 s}}{8 \sqrt{\pi}} \frac{\Gamma \left( \frac{\nu}{2} - \frac{1}{2} \right)}{\Gamma(n - 1)} \int_0^\infty dr r^{n+1} V(r) \times \left\{ \int_0^r dr' (r')^{n+1} V(r') \int_0^\pi \left( r^2 + r'^2 - 2rr' \cos \phi \right)^{-s-2} P_{\frac{n}{2}} - \frac{1}{2} \left( \cos \phi \right) d \cos \phi \right\}.
\]

Around \( s = 0 \), for \( n \neq 1 \), we have

\[
W^{(2)} = \frac{1}{16} \frac{\Gamma \left( \frac{\nu}{2} - \frac{1}{2} \right)}{\Gamma(n - 1)} \int_0^\infty dr r^{n+1} V(r) \left\{ \int_0^r dr' (r')^{n+1} V(r') \int_0^\pi \frac{P_{\frac{n}{2}} - \frac{1}{2} \left( \cos \phi \right)}{\left( r^2 + r'^2 - 2rr' \cos \phi \right)^2} d \cos \phi \times \int_0^\pi \frac{P_{\frac{n}{2}} - \frac{1}{2} \left( \cos \phi \right)}{\left( r^2 + r'^2 - 2rr' \cos \phi \right)^2} d \cos \phi \right\}.
\]

Next, we perform the angle integral. Rewrite the angle integral as

\[
\int_0^\pi \frac{P_{\frac{n}{2}} - \frac{1}{2} \left( \cos \phi \right)}{\left( r^2 + r'^2 - 2rr' \cos \phi \right)^2} d \cos \phi = \frac{1}{4 (rr')^2} \int_0^\pi \frac{P_{\frac{n}{2}} - \frac{1}{2} \left( \cos \phi \right)}{\left( R - \cos \phi \right)^2} d \cos \phi,
\]

where \( R = \frac{r^2 + r'^2}{2rr'} \). It can be checked that

\[
\frac{1}{\left( R - \cos \phi \right)^2} = \frac{d}{dR} \frac{1}{R - \cos \phi}.
\]
so the integral (5.45) becomes

\[
\int_0^\pi \frac{P_{\frac{3}{2}-\frac{3}{2}}(\cos \phi)}{(r^2 + r'^2 - 2rr' \cos \phi)^2} d\cos \phi \\
= -\frac{1}{4} \frac{d}{4(r')^2 dR} \int_0^\pi \frac{P_{\frac{3}{2}-\frac{3}{2}}(\cos \phi)}{R - \cos \phi} d\cos \phi \\
= -\frac{1}{2} \frac{d}{(r')^2 dR} Q_{\frac{3}{2}-\frac{3}{2}} (R) \\
= -2 \left( \frac{n - \frac{3}{2}}{\frac{3}{2}} \right) Q_{\frac{3}{2}-\frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) - \frac{r^2 + r'^2}{2rr'} \left( \frac{n - \frac{3}{2}}{\frac{3}{2}} \right) Q_{\frac{3}{2}-\frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) \right),
\]

(5.47)

where \(Q_n(\varpi)\) is the Legendre function of the second kind: \(\Pi \int_0^\pi \frac{d}{R - \cos \varpi} P_{\frac{3}{2}-\frac{3}{2}}(\cos \varphi) d\cos \varphi = 2Q_{\frac{3}{2}-\frac{3}{2}} (R) [72]\). Then the second-order one-loop effective action reads

\[
W^{(2)} \sim -\frac{1}{8} \frac{\Gamma \left( \frac{n}{2} - \frac{1}{2} \right)}{\Gamma (n - 1)} \int_0^\infty dr \frac{n+1}{2} V (r) \\
\times \left\{ \int_0^r dr' \left( \frac{n+1}{2} V (r') \right) \left( \frac{n - \frac{3}{2}}{\frac{3}{2}} \right) Q_{\frac{3}{2}-\frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) - \frac{r^2 + r'^2}{2rr'} \left( \frac{n - \frac{3}{2}}{\frac{3}{2}} \right) Q_{\frac{3}{2}-\frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) \right) \\
\times + \int_r^\infty \int_0^\infty dr' \left( \frac{n+1}{2} V (r') \right) \left( \frac{n - \frac{3}{2}}{\frac{3}{2}} \right) Q_{\frac{3}{2}-\frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) - \frac{r^2 + r'^2}{2rr'} \left( \frac{n - \frac{3}{2}}{\frac{3}{2}} \right) Q_{\frac{3}{2}-\frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) \right) \\
\right\}.
\]

(5.48)

5.2.3 Vacuum energy

Similarly, using the relation between the heat kernel and the vacuum energy, Eq. (2.12), we can obtain the second-order \(n\)-dimensional Born approximation of the vacuum energy. Substituting Eq. (5.25) into Eq. (2.12) gives

\[
E_0^{(2)} (\epsilon) = -\frac{\pi}{8} \mu^2 \epsilon \frac{1}{\Gamma (-\frac{1}{2} + \epsilon)} \int_0^\infty V (r) r dr \\
\times \left\{ \int_0^r V (r') r' dr' \int_0^\infty dk^2 \left[ \frac{(kr)^{n-2} (kr')^{n-2}}{\pi \Gamma (2\mu + 1)} \int_0^\pi Y_{\frac{n}{2}-1} (qr) J_{\frac{n}{2}-1} (qr') \sin^{n-3} \theta d\cos \theta \right] \\
\times + \int_r^\infty V (r') r' dr' \int_0^\infty dk^2 \left[ \frac{(kr)^{n-2} (kr')^{n-2}}{\pi \Gamma (2\mu + 1)} \int_0^\pi Y_{\frac{n}{2}-1} (qr) J_{\frac{n}{2}-1} (qr') \sin^{n-3} \theta d\cos \theta \right] \\
\right\}.
\]

(5.49)
Performing the integral, we have

\[ E_0^{(2)} (\epsilon) = -\frac{\mu^2 \epsilon}{8 \pi} \frac{\Gamma \left( \frac{3}{2} - \epsilon \right)}{\Gamma \left( -\frac{1}{2} + \epsilon \right) \Gamma (n - 1)} \int_0^\infty r \, V(r) \, rdr \times \left\{ \int_0^r \left( \frac{kr}{2} \right)^{-\frac{1}{2} + \epsilon} V(r') \, dr' \int_0^\pi P_{\frac{3}{2} - \frac{1}{2}} (\cos \phi) \, d \cos \phi \right. \right. \\
+ \left. \int_0^\infty V(r') \, dr' \int_0^\pi \left( \frac{kr'}{2} \right)^{-\frac{1}{2} + \epsilon} V(r') \, dr' \int_0^\pi P_{\frac{3}{2} - \frac{1}{2}} (\cos \phi) \, d \cos \phi \right\}. \]

(5.50)

For odd-dimensional cases, substituting Eq. (5.33) into Eq. (2.12) gives the odd-dimensional second-order Born approximation of the vacuum energy:

\[ E_0^{(2)} (\epsilon) = -\frac{\mu^2 \epsilon}{8 \pi} \frac{\Gamma \left( \frac{3}{2} - \epsilon \right)}{\Gamma \left( -\frac{1}{2} + \epsilon \right) \Gamma (n - 1)} \int_0^\infty r \frac{n+1}{2} V(r) \, dr \times \left\{ \int_0^r \left( \frac{kr}{2} \right)^{-\frac{1}{2} + \epsilon} V(r') \, dr' \int_0^\pi P_{\frac{3}{2} - \frac{1}{2}} (\cos \phi) \, d \cos \phi \right. \\
+ \left. \int_0^\infty V(r') \, dr' \int_0^\pi \left( \frac{kr'}{2} \right)^{-\frac{1}{2} + \epsilon} V(r') \, dr' \int_0^\pi P_{\frac{3}{2} - \frac{1}{2}} (\cos \phi) \, d \cos \phi \right\}. \]

(5.51)

Performing the integral over \( t \) gives

\[ E_0^{(2)} (\epsilon) = -\frac{\mu^2 \epsilon}{8 \pi} \frac{\Gamma (2 - \epsilon)}{\Gamma (n - 1)} \int_0^\infty r \, V(r) \, dr \times \left\{ \int_0^r \left( \frac{kr}{2} \right)^{-\frac{1}{2} + \epsilon} V(r') \, dr' \int_0^\pi \left( \frac{kr'}{2} + \frac{1}{2} \right) V(r') \, dr' \int_0^\pi P_{\frac{3}{2} - \frac{1}{2}} (\cos \phi) \, d \cos \phi \right. \\
+ \left. \int_0^\infty \left( \frac{kr}{2} \right)^{-\frac{1}{2} + \epsilon} V(r') \, dr' \int_0^\pi \left( \frac{kr'}{2} + \frac{1}{2} \right) V(r') \, dr' \int_0^\pi P_{\frac{3}{2} - \frac{1}{2}} (\cos \phi) \, d \cos \phi \right\}. \]

(5.52)

Around \( \epsilon = -\frac{1}{2} \), for \( n \neq 1 \), we have

\[ E_0^{(2)} = -\frac{3 \mu^{-1} \Gamma \left( \frac{3}{2} - \frac{1}{2} \right)}{16 \pi^2} \int_0^\infty dr' \frac{n+1}{2} V(r') \left\{ \int_0^r \left( \frac{kr'}{2} \right)^{\frac{n+1}{2}} V(r') \, dr' \int_0^\pi \frac{P_{\frac{3}{2} - \frac{1}{2}} (\cos \phi)}{(r'^2 + r^2 - 2rr' \cos \phi)} \, d \cos \phi \right. \\
+ \left. \int_0^\infty \left( \frac{kr'}{2} \right)^{\frac{n+1}{2}} V(r') \, dr' \int_0^\pi \frac{P_{\frac{3}{2} - \frac{1}{2}} (\cos \phi)}{(r'^2 + r^2 - 2rr' \cos \phi)} \, d \cos \phi \right\}. \]

(5.53)
A similar treatment gives

\[
\int_{0}^{\pi} \frac{P_{\frac{n}{2} + \frac{3}{2}}(\cos \phi)}{(r^2 + r'^2 - 2rr' \cos \phi)^2} d\cos \phi
\]

\[
= \frac{1}{8 (rr')^2 dR^2} \int_{0}^{\pi} \frac{P_{\frac{n}{2} + \frac{3}{2}}(\cos \phi)}{R - \cos \phi} d\cos \phi
\]

\[
= \frac{1}{8 (rr')^2 dR^2} Q_{\frac{n}{2} + \frac{3}{2}}(R)
\]

\[
= \frac{1}{8 (rr')^2} \left\{ \left( \frac{n}{2} + \frac{1}{2} \right) \left[ \left( \frac{r^2 + r'^2}{2r r'} \right)^2 \left( \frac{n}{2} + \frac{3}{2} \right) + 1 \right] Q_{\frac{n}{2} + \frac{3}{2}}(\frac{r^2 + r'^2}{2rr'}) \right. \\
- \left. \left( \frac{n}{2} + \frac{1}{2} \right) (n + 4) \frac{r^2 + r'^2}{2rr'} Q_{\frac{n}{2} + \frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) \right\}.
\]

Then the second-order vacuum energy reads

\[
E_0^{(2)} \sim -\frac{3\mu^{-1}}{256} \frac{\Gamma(\frac{n}{2} + \frac{3}{2})}{\Gamma(n - 1)}
\]

\[
\times \int_{0}^{\infty} dr' r'^{n-\frac{3}{2}} V(r') \int_{0}^{\pi} d\cos \phi \left\{ \left( \frac{n}{2} + \frac{1}{2} \right) \left[ \left( \frac{r^2 + r'^2}{2r r'} \right)^2 \left( \frac{n}{2} + \frac{3}{2} \right) + 1 \right] Q_{\frac{n}{2} + \frac{3}{2}}(\frac{r^2 + r'^2}{2rr'}) \right. \\
- \left. \left( \frac{n}{2} + \frac{1}{2} \right) (n + 4) \frac{r^2 + r'^2}{2rr'} Q_{\frac{n}{2} + \frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) \right\}.
\]

6 WKB approximation

In this section, we convert the WKB approximation method into a method for calculating one-loop effective actions and vacuum energies.
The WKB approximation of the scattering phase shift is [74]

\[ \delta_W^\text{WKB}(k) \sim -\frac{1}{2k} \int_0^\infty \frac{rV(r)}{\sqrt{r^2 - \left(\frac{l+1/2}{k}\right)^2}} dr. \]  

(6.1)

Substituting the phase shift (6.1) into Eq. (3.5) gives the WKB approximation of the heat kernel:

\[ K_W^\text{WKB}(t) \sim -\frac{t}{2\pi} \int_0^\infty dk^2 \frac{1}{k^2} e^{-kt} \sum_{l=0}^\infty (2l + 1) \int_{(l+1/2)/k}^\infty \frac{rV(r)}{\sqrt{r^2 - \left(\frac{l+1/2}{k}\right)^2}} dr. \]  

(6.2)

Substituting the phase shift (6.1) into Eq. (3.6) gives the WKB approximation of the one-loop effective action:

\[ W_s^\text{WKB} \sim \frac{1}{4\pi} \mu^{2s} \Gamma(s+1) \int_0^\infty dk^2 \frac{1}{(k^2)^{s+3/2}} \sum_{l=0}^\infty (2l + 1) \int_{(l+1/2)/k}^\infty \frac{rV(r)}{\sqrt{r^2 - \left(\frac{l+1/2}{k}\right)^2}} dr. \]  

(6.3)

Substituting the phase shift (6.1) into Eq. (3.7) gives the WKB approximation of the vacuum energy:

\[ E_0^\text{WKB}(\epsilon) \sim -\frac{1}{4\pi} \epsilon^{2s} \frac{\Gamma\left(s + \frac{1}{2} + \epsilon\right)}{\Gamma\left(s + \frac{1}{2}\right)} \int_0^\infty dk^2 \frac{1}{(k^2)^{1+s}} \sum_{l=0}^\infty (2l + 1) \int_{(l+1/2)/k}^\infty \frac{rV(r)}{\sqrt{r^2 - \left(\frac{l+1/2}{k}\right)^2}} dr. \]  

(6.4)

6.1 Example: \( V(r) = \frac{\alpha}{(r^2 + r_0^2)^2} \)

In this section, we consider the potential

\[ V(r) = \frac{\alpha}{(r^2 + r_0^2)^2} \]  

(6.5)

as an example.

Substituting the potential (6.5) into Eq. (6.2) gives

\[ K_W^\text{WKB}(t) \sim -\frac{t}{2\pi} \int_0^\infty dk^2 \frac{1}{k^2} e^{-kt} \sum_{l=0}^\infty (2l + 1) \int_{(l+1/2)/k}^\infty \frac{\pi \alpha}{\sqrt{r_0^2 + \left(\frac{l+1/2}{k}\right)^2}^{3/2}} dr. \]

\[ = -\frac{t}{2\pi} \int_0^\infty \frac{\alpha k^2 \pi}{\sqrt{1 + 4r_0^2 k^2}} e^{-kt} dk^2 \]

\[ = -\frac{\alpha}{8r_0^2} + \frac{\alpha \sqrt{\pi} e^{-t/r_0} \text{erfc}\left(\frac{\sqrt{t}}{2r_0}\right)}{16r_0^3} - \frac{\alpha \sqrt{\pi} e^{-t/r_0} \text{erfc}\left(\frac{\sqrt{t}}{2r_0}\right)}{8r_0 \sqrt{t}}, \]  

(6.6)

where \( \text{erfc}(z) \) is the complementary error function [72].
Similarly, by Eqs. (6.3) and (6.4), we can obtain the one-loop effective action and the vacuum energy:

\[ W_s^{\text{WKB}} \sim \frac{\alpha \tilde{\mu}^{2s} r_0^{2s-2}}{4} s \Gamma (1 - s) \Gamma (2s - 1) \]  

(6.7)

and

\[ E_0^{\text{WKB}} (\epsilon) \sim - \frac{4^{\epsilon - 3} \alpha \tilde{\mu}^{2s} r_0^{2s-3}}{\sqrt{\pi}} (2\epsilon - 1) \Gamma \left( \frac{3}{2} - \epsilon \right) \Gamma (\epsilon - 1). \]  

(6.8)

### 6.2 WKB approximation and Born approximation: comparison

Taking the heat kernel as an example, we compare the WKB approximation with the Born approximation.

The heat kernel given by the first-order Born approximation can be obtained by substituting the potential (6.5) into Eq. (4.6):

\[ K_s^{(1)} (t) \sim - \frac{\alpha \sqrt{\pi}}{8r_0 \sqrt{t}}. \]  

(6.9)

Expanding the heat kernel obtained by the WKB approximation, Eq. (6.6), we have

\[ K_l^{\text{WKB}} (t) \sim - \frac{\alpha \sqrt{\pi}}{8r_0 \sqrt{t}} + \frac{\alpha (2r_0 + \sqrt{\pi} - 2)}{16r_0^3}. \]  

(6.10)

The leading order contribution of these two methods are the same.

It should be noted that both WKB approximation and Born approximation are valid for short-range potentials. The short-range potential at \( r \to 0 \) satisfies [75, 76]

\[ \int_0^a |V (r)| r dr < \infty, \quad a < \infty, \]  

(6.11)

and at \( r \to \infty \) satisfies

\[ \int_b^\infty |V (r)| dr < \infty, \quad b > 0. \]  

(6.12)

For long-range potentials, divergence may be encountered, and the renormalization procedure is required to remove the divergence. The treatment of removing divergence can refer to Ref. [60].

### 7 One-loop effective action and vacuum energy of scalar field in curved spacetime

#### 7.1 Schwarzschild spacetime

Ref. [68] gives a first-order scattering phase shift of a scalar field on a Schwarzschild black hole:

\[ \delta_l^{(1)} = - \text{arctan} \left( \frac{\int_1^\infty \sin^2 (2M \eta [\rho + \ln (\rho - 1)]) \frac{1}{\rho^{\epsilon - 1}} V_l^{\text{eff}} (\rho) \rho d \rho}{\frac{2M}{\sqrt{\pi}} + \frac{1}{2} \int_1^\infty \sin (4M \eta [\rho + \ln (\rho - 1)]) \frac{1}{\rho^{\epsilon - 1}} V_l^{\text{eff}} (\rho) \rho d \rho} \right), \]  

(7.1)

where \( V_l^{\text{eff}} (\rho) = \left( 1 - \frac{1}{\rho^2} \right) \left[ \frac{(l+1)}{\rho^2} + \frac{1}{\rho^3} \right] - \frac{(2M \rho)^2}{\rho} \) with \( \rho = r/2M \), \( M \) the mass of the black hole, and \( \mu \) the mass of the scalar field.
Consider a massless scalar field \((\mu = 0)\) in a Schwarzschild spacetime with \(l = 0\), we have \(\eta = \sqrt{k^2 - \mu^2} = k\). Then
\[
\delta_0^{(1)} = - \frac{1}{2Mk} \int_1^\infty \sin^2 (2Mk [\rho + \ln(\rho - 1)]) \frac{1}{\rho^3} d\rho.
\] (7.2)
Substituting Eq. (7.2) into Eq. (3.3) and performing the integral over \(k\) give the heat kernel,
\[
K_0^\eta (t) = \frac{\sqrt{t}}{4M \sqrt{\pi}} \left( \int_1^\infty \frac{1}{\rho^3} e^{-\frac{4M^2}{3} (\rho + \ln(\rho - 1))^2} d\rho - \frac{1}{2} \right).
\] (7.3)
The one-loop effective action for \(l = 0\) by Eq. (2.11) reads
\[
W_0 (s) = -\bar{\mu}^{2s} \frac{2^{2(s-1)} M^{2s}}{\sqrt{\pi}} \Gamma \left( -s - \frac{1}{2} \right) \int_1^\infty \frac{1}{\rho^3} [\rho + \ln(\rho - 1)]^{2s+1} d\rho.
\] (7.4)
The regularized vacuum energy for \(l = 0\) by Eq. (2.12) reads
\[
E_0 (\epsilon) = \bar{\mu}^{2s} \frac{\Gamma (-\epsilon)}{\Gamma (-\frac{s}{2} + \epsilon)} \frac{2^{2-3s}}{\sqrt{\pi}} M^{2s-1} \int_1^\infty \frac{1}{\rho^3} [\rho + \ln(\rho - 1)]^{2s} d\rho.
\] (7.5)

### 7.2 Reissner-Nordström spacetime

Ref. [69] gives a first-order scattering phase shift of a scalar field on a Reissner-Nordström black hole:
\[
\delta_0^{(1)} = - \arctan \left[ \frac{1}{\eta} \int_{r_+}^\infty \sin^2 (2\eta r_+) \frac{dr}{dr} V_l^{\text{eff}} dr + (r_+ - r_-) \eta \ln \frac{r_+ - r_-}{r_+ + r_-} \right],
\] (7.6)
where \(V_l^{\text{eff}} = (1 - \frac{r_+}{r}) (1 - \frac{r_-}{r}) \left[ \frac{l(l+1)}{r^2} + \left( \frac{r_+ + r_-}{r^3} - \frac{2r_+ r_-}{r^4} \right) \right] \), \(r_\pm = M \pm \sqrt{M^2 - Q^2}\) with \(M\) the mass and \(Q\) the charge of the spacetime, and the tortoise coordinate \(r_* = r + \frac{r^2}{r_+ - r_-} \ln \left( \frac{r_+}{r_-} - 1 \right) \). Consider a massless scalar field \((\mu = 0)\) in a Reissner-Nordström spacetime with \(l = 0\), we have \(\eta = \sqrt{k^2 - \mu^2} = k\). Then
\[
\delta_0^{(1)} = - \frac{1}{k} \int_{r_+}^\infty \sin^2 (kr_+) \left( \frac{2M}{r^3} - \frac{2Q^2}{r^4} \right) dr + Mk \ln \left( 1 - \frac{Q^2}{M^2} \right).
\] (7.7)
Substituting into Eq.(3.3) and performing the integral over \(k\) give the heat kernel,
\[
K_0^\eta (t) = - \frac{\sqrt{t}}{2\sqrt{\pi}} \int_{r_+}^\infty e^{-\frac{r^2}{4t}} \left( \frac{r^2}{r_+} - 1 \right) \left( \frac{2M}{r^3} - \frac{2Q^2}{r^4} \right) dr - \frac{M}{2\sqrt{\pi} t} \ln \left( 1 - \frac{Q^2}{M^2} \right).
\]
The one-loop effective action for \(l = 0\) by Eq. (2.11) reads
\[
W_0 (s) = -\bar{\mu}^{2s} \frac{\Gamma \left( -s - \frac{1}{2} \right)}{\Gamma (-\frac{s}{2} + \epsilon)} \int_{r_+}^\infty r_*^{2s+1} \left( \frac{2M}{r^3} - \frac{2Q^2}{r^4} \right) dr.
\] (7.8)
The regularized vacuum energy for \(l = 0\) by Eq. (2.12) reads
\[
E_0 (\epsilon) = -\bar{\mu}^{2s} \frac{\Gamma \left( -s - \frac{1}{2} + \epsilon \right)}{\Gamma (-\frac{s}{2} + \epsilon)} \int_{r_+}^\infty r_*^{2s} \left( \frac{2M}{r^3} - \frac{2Q^2}{r^4} \right) dr.
\] (7.9)
8 Calculating global heat kernel, one-loop effective action, and vacuum energy from scattering amplitude

In the above, we calculate the heat kernel, the vacuum energy, and the one-loop effective action from the scattering phase shift. In this section, we suggest a method that calculates them from the scattering amplitude.

The scattering wave function is \[ \psi (r, \theta) = e^{ikr \cos \theta} + \sum_{l=0}^{\infty} a_l (\theta) h_l^{(1)} (kr), \tag{8.1} \]
where
\[ a_l (\theta) = (2l + 1) \frac{i}{2} \left( e^{2i\delta_l} - 1 \right) P_l (\cos \theta) \tag{8.2} \]
is the partial scattering amplitude.

Under the large-distance approximation, by the asymptotics of the Hankel function \( h_l^{(1)} (kr) \), Eq. (2.3), we have
\[ \psi (r, \theta) = e^{ikr \cos \theta} + f (\theta) \frac{e^{ikr}}{r}, \tag{8.3} \]
where
\[ f (\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1) \delta_l P_l (\cos \theta) \left( e^{2i\delta_l} - 1 \right) \tag{8.4} \]
is the scattering amplitude, for the differential scattering cross section \( \sigma (\theta) = |f (\theta)|^2 \).

For small phase shifts, we approximate \( e^{2i\delta_l} \simeq 1 + 2i\delta_l \) in Eq. (8.4):
\[ f (\theta) \simeq \frac{1}{k} \sum_{l=0}^{\infty} (2l + 1) \delta_l P_l (\cos \theta). \tag{8.5} \]

Then the forward-scattering amplitude, the scattering amplitude in the direction \( \theta = 0 \), is
\[ f (0) = \frac{1}{k} \sum_{l=0}^{\infty} (2l + 1) \delta_l. \tag{8.6} \]
Noting that \( D_l = 2l + 1 \) is the degeneracy, we have
\[ \sum_{l=0}^{\infty} (2l + 1) \delta_l = kf (0). \tag{8.7} \]
Substituting into Eq. (3.5) gives
\[ K (t) = \frac{t}{\pi} \int_0^{\infty} dk^2 e^{-k^2 t} \sum_{l=0}^{\infty} (2l + 1) \delta_l \]
\[ = \frac{2}{\pi} t \int_0^{\infty} f (0) e^{-k^2 t} k^2 dk. \tag{8.8} \]
The heat kernel is now expressed by the forward scattering amplitude.
Similarly, by Eq. (2.11) we can express the one-loop effective action by the forward scattering amplitude,

\[
W(s) = -\frac{1}{2} \tilde{\mu}^2 s \int_0^\infty dt t^{-1} \left[ \frac{2}{\pi} t \int_0^\infty f(0) e^{-k^2 t} k^2 dk \right] 
= -\tilde{\mu}^2 s \Gamma(s+1) \int_0^\infty f(0) (k^2)^{-s} dk. \tag{8.9}
\]

By Eq. (2.12), we can express the vacuum energy by the forward scattering amplitude,

\[
E_0(\epsilon) = \frac{1}{2} \tilde{\mu}^2 \frac{1}{\Gamma(-\frac{1}{2} + \epsilon)} \int_0^\infty dt t^{-\frac{1}{2} + \epsilon - 1} \left[ \frac{2}{\pi} t \int_0^\infty f(0) e^{-k^2 t} k^2 dk \right] 
= \tilde{\mu}^2 \frac{1}{\pi \Gamma(\epsilon + \frac{1}{2})} \int_0^\infty f(0) (k^2)^{\frac{1}{2} - \epsilon} dk. \tag{8.10}
\]

We can also provide the spectral zeta function expressed in terms of the scattering amplitude. By the relation between \(\delta(q)\) and the spectral zeta function \([39]\),

\[
\zeta(s) = \frac{2s}{\pi} \int_0^\infty q \left( q^2 + m^2 \right)^{-s-1} \delta(q) dq, \tag{8.11}
\]

and

\[
\delta(q) = -\frac{\pi}{2\pi i} \int_{C-i\infty}^{C+i\infty} q^{-s} \zeta_0 \left( \frac{-\frac{q}{2}}{s} \right) ds, \tag{8.12}
\]

we can connect the spectral zeta function to the scattering amplitude:

\[
\zeta(s) = \frac{2s}{\pi} \int_0^\infty k \left( k^2 + m^2 \right)^{-s-1} f(0) k^2 dk. \tag{8.13}
\]

As a verification, substituting the first-order Born approximation for the scattering amplitude \([77]\),

\[
f^{(1)}_{\text{Born}}(\theta) = -\frac{1}{q} \int_0^\infty dr r V(r) \sin(qr), \tag{8.14}
\]

where \(q = 2k \sin \frac{\theta}{2}\), into Eq. (8.8) gives

\[
K(t) = \frac{2}{\pi} t \int_0^\infty \left[ -\frac{1}{q} \int_0^\infty dr r V(r) \sin(qr) \right] e^{-k^2 t} k^2 dk. \tag{8.15}
\]

For \(\theta = 0\), \(\frac{\sin(qr)}{q} = r\). Substituting into Eq. (8.15) give

\[
K(t) = -\frac{2}{\pi} t \int_0^\infty \int_0^\infty dr r^2 V(r) e^{-k^2 t} k^2 dk 
= -\frac{1}{\sqrt{4\pi t}} \int_0^\infty V(r) r^2 dr. \tag{8.16}
\]

This agrees with the result given by Eq. (4.6).

Moreover, similar calculations give the one-loop effective action,

\[
W(s) = \frac{\tilde{\mu}^2 \Gamma(s - \frac{1}{2})}{4\sqrt{\pi}} \left( m^2 \right)^{\frac{1}{2} - s} \int_0^\infty V(r) r^2 dr \tag{8.17}
\]

and the vacuum energy

\[
E_0(\epsilon) = -\frac{\tilde{\mu}^2 \Gamma(\epsilon - 1)}{4\sqrt{\pi \Gamma(\epsilon - \frac{1}{2})}} \left( m^2 \right)^{1-\epsilon} \int_0^\infty V(r) r^2 dr. \tag{8.18}
\]
9 Conclusion

One-loop effective actions and vacuum energies in quantum field theory, scattering phase shifts and scattering amplitudes in quantum mechanics, partition functions, and various thermodynamic quantities in statistical mechanics are all spectral functions. By identifying the relationship between these spectral functions, it becomes possible to translate the methodology of calculating one spectral function into the methodology of calculating another. In this paper, we demonstrate the conversion of the scattering method in quantum mechanics into the corresponding method in quantum field theory. Specifically, we convert the Born approximation and the WKB approximation into methodologies for calculating the one-loop effective action and vacuum energy. Theoretically, all methodologies for calculating scattering phase shifts and amplitude in quantum mechanics, such as the eikonal approximation, can be converted into methodologies for calculating effective actions and vacuum energies.

This approach can calculate various spectral functions across different physical domains. Spectral functions in quantum field theory, quantum mechanics, and statistical mechanics can be transformed into one another using their corresponding relations. As such, methods utilized in one physical domain can be converted into methods in another. For instance, determining the energy spectrum of an interacting many-body system is a fundamental problem in statistical mechanics. Eigenvalues are the most basic spectral functions and can be calculated from other spectral functions. In Ref. [5], for example, the energy eigenvalue of an interacting many-body system is calculated using the partition function. Various statistical mechanics methods have been developed to calculate partition and grand partition functions, such as the cluster expansion method, field theory method [78], and some mathematical methods [79, 80]. Both the eigenvalue and partition function are spectral functions. Methods for calculating partition functions, such as the cluster expansion method, can be transformed into methods for calculating the energy spectrum of interacting gases [81].

In quantum-mechanical scattering theory, the focus is primarily on short-range scattering, although the Born approximation can handle certain long-range potentials, such as the Coulomb potential. For long-range potential scattering, the scattering phase shift can be uniformly treated using the tortoise coordinate [76]. Future works will delve into this approach.

A duality exists in classical and quantum mechanics, as well as in field theory [82–84]. In quantum mechanics, this duality pertains to the relationship between various eigenproblems, whereas in field theory, it pertains to the relationship between different fields. In this paper, we establish a link between spectral functions in quantum mechanics and quantum field theory. Future works will delve further into the relationship between these problems.

In summary, this paper proposes a methodology for transforming one spectral function problem into another spectral function problem through the use of spectral function transformations. This approach enables the conversion of methods from different areas of physics, such as quantum field theory, quantum mechanics, and statistical mechanics, into one another.
A Integral representation of Bessel function

In this appendix, we give some integral representations for the Bessel function.

A.1 Integral representation of $j_l^2(kr)$

Taking $|u| = |v| = kr$ in the expansion [85]

$$\frac{\sin w}{w} = \sum_{l=0}^{\infty} (2l + 1) j_l(v) j_l(u) P_l(\cos \theta), \quad (A.1)$$

where $w = \sqrt{u^2 + v^2 - 2uv \cos \theta}$ and $\theta$ is the angle between $u$ and $v$, gives

$$\frac{\sin qr}{qr} = \sum_{l=0}^{\infty} (2l + 1) j_l^2(kr) P_l(\cos \theta) \quad (A.2)$$

with $w = qr = 2kr \sin \frac{\theta}{2}$. Multiplying $P_l(\cos \theta)$ on both sides of Eq. (A.2) and then integrating from 0 to $\pi$ give

$$\int_0^{\pi} \frac{\sin qr}{qr} P_l(\cos \theta) \sin \theta d\theta = \int_0^{\pi} \sum_{l=0}^{\infty} (2l + 1) j_l^2(kr) P_l(\cos \theta) P_l(\cos \theta) \sin \theta d\theta. \quad (A.3)$$

By

$$\int_0^{\pi} P_l(\cos \theta) P_l(\cos \theta) \sin \theta d\theta = \frac{2}{2l + 1} \delta_{l'}, \quad (A.4)$$

we have

$$\int_0^{\pi} \frac{\sin qr}{qr} P_l(\cos \theta) \sin \theta d\theta = 2 j_{l'}^2(kr). \quad (A.5)$$

This gives an integral representation of $j_l^2(kr)$:

$$j_l^2(kr) = \frac{1}{2} \int_0^{\pi} \frac{\sin qr}{qr} P_l(\cos \theta) \sin \theta d\theta, \quad (A.6)$$

where $l$ is an integer.

A.2 Integral representation of $j_l(kr) n_l(kr)$

Taking $|u| = |v| = kr$ in the expansion [85]

$$\frac{\cos w}{w} = -\sum_{l=0}^{\infty} (2l + 1) j_l(v) n_l(u) P_l(\cos \theta), \quad (A.7)$$

where $w = \sqrt{u^2 + v^2 - 2uv \cos \theta}$ and $\theta$ is the angle between $u$ and $v$, gives

$$\frac{\cos qr}{qr} = -\sum_{l=0}^{\infty} (2l + 1) j_l(kr) n_l(kr) P_l(\cos \theta) \quad (A.8)$$
with \( w = qr = 2kr \sin \frac{\theta}{2} \). Multiplying \( P_\nu (\cos \theta) \) on both sides of Eq. (A.8) and then integrating from 0 to \( \pi \) give

\[
\int_0^\pi \frac{\cos qr}{qr} P_\nu (\cos \theta) \sin \theta \, d\theta = - \sum_{l=0}^{\infty} (2l + 1) j_l (kr) n_l (kr) P_1 (\cos \theta) P_\nu (\cos \theta) \sin \theta \, d\theta
\]

\[
= -2 j_\nu (kr) n_\nu (kr). 
\]  

(A.9)

This gives an integral representation of \( j_l (kr) n_l (kr) \):

\[
j_l (kr) n_l (kr) = - \frac{1}{2} \int_0^\pi \frac{\cos qr}{qr} P_1 (\cos \theta) \sin \theta \, d\theta,
\]  

(A.10)

where \( l \) is an integer.

**A.3 Integral representation of \( J_{l+\mu}^2 (kr) \)**

Taking \( |u| = |v| = kr \) in the expansion [85]

\[
\frac{J_\mu (w)}{w^\mu} = 2^\mu \Gamma (\mu) \sum_{l=0}^{\infty} (l + \mu) J_{l+\mu} (u) J_{l+\mu} (v) C_l^\mu (\cos \theta), \quad (u > v),
\]  

(A.11)

where \( w = \sqrt{u^2 + v^2 - 2uv \cos \theta} = qr = 2kr \sin \frac{\theta}{2} \) and \( C_l^\mu (z) \) is the Gegenbauer polynomial [72], multiplying \( C_l^\mu (\cos \theta) \) on both sides of Eq. (A.11), and integrating from 0 to \( \pi \) give

\[
\int_0^\pi J_\mu (qr) q^{\mu+\mu} C_l^\mu (\cos \theta) \sin^{2\mu} \theta \, d\theta
\]

\[
= 2^\mu \Gamma (\mu) \sum_{l=0}^{\infty} (l + \mu) \frac{J_{l+\mu}^2 (kr)}{(kr)^{2\mu}} \int_0^\pi C_l^\mu (\cos \theta) C_l^\mu (\cos \theta) \sin^{2\mu} \theta \, d\theta.
\]  

(A.12)

By [72]

\[
\int_0^\pi C_l^\mu (\cos \theta) C_l^\mu (\cos \theta) \sin^{2\mu} \theta \, d\theta = \frac{2^{1-2\mu} \pi \Gamma (l + 2\mu)}{\Gamma (l + 1) (l + \mu)} \Gamma (\mu)^2 \delta_{ll'},
\]  

(A.13)

we arrive at an integral representation of \( J_{l+\mu}^2 (kr) \):

\[
J_{l+\mu}^2 (kr) = \frac{2^{\mu-1} \Gamma (l + 1) \Gamma (\mu)}{\pi \Gamma (2\mu + l)} (kr)^{2\mu} \int_0^\pi J_\nu (qr) q^{\mu+\mu} C_l^\mu (\cos \theta) \sin^{2\mu} \theta \, d\theta,
\]  

(A.14)

where \( l \) is an integer and \( \mu \) is a real number.

**A.4 Integral representation of \( J_{l+\nu} (kr) Y_{l+\nu} (kr) \)**

Taking \( |u| = |v| = kr \) in the expansion [85]

\[
\frac{Y_\mu (w)}{w^\mu} = 2^\mu \Gamma (\mu) \sum_{l=0}^{\infty} (l + \mu) Y_{l+\mu} (u) J_{l+\mu} (v) C_l^\mu (\cos \theta), \quad (u > v),
\]  

(A.15a)
where \( w = \sqrt{u^2 + v^2 - 2uv \cos \theta} = qr = 2kr \sin \frac{\theta}{2} \) and, multiplying \( C_l^\mu (\cos \theta) \) on both sides of Eq. (A.15a) and integrating from 0 to \( \pi \) give

\[
\int_0^{\pi} Y_\mu(qr) C_l^\mu(\cos \theta) \sin^{2\mu} \theta d\theta = 2^\mu \Gamma(\mu) \sum_{l=0}^{\infty} (l+\mu) \frac{J_{l+\mu}(kr) Y_{l+\mu}(kr)}{(kr)^{2\mu}} \int_0^{\pi} C_l^\mu(\cos \theta) C_l^\mu(\cos \theta) \sin^{2\mu} \theta d\theta. \tag{A.16}
\]

By [72]

\[
\int_0^{\pi} C_l^\mu(\cos \theta) C_l^\mu(\cos \theta) \sin^{2\mu} \theta d\theta = \frac{2^{1-2\mu} \pi \Gamma(l+2\mu) (l+\mu) \Gamma^2(\mu)}{\Gamma(l+1) (l+\mu) \Gamma^2(\mu)} \delta_{l\ell}, \tag{A.17}
\]

we arrive at an integral representation of \( J_{l+\mu}(kr) Y_{l+\mu}(kr) \):

\[
J_{l+\mu}(kr) Y_{l+\mu}(kr) = 2^{\mu-1} \frac{\Gamma(l+1) \Gamma(\mu)}{\pi \Gamma(2\mu+1)} (kr)^{2\mu} \int_0^{\pi} \frac{Y_\mu(qr)}{q^{\mu+\mu}} C_l^\mu(\cos \theta) \sin^{2\mu} \theta d\theta, \tag{A.18}
\]

where \( l \) is an integer and \( \mu \) is a real number.

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