Uniformly Generating Distribution Functions for Discrete Random Variables

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Abstract

An algorithm is presented which, with optimal efficiency, solves the problem of uniform random generation of distribution functions for an $n$-valued random variable.

1 Introduction

In the general framework of Probabilistic Inference the case occurs that either for experimental or empirical validation purposes one needs to generate unbiased collections of distribution functions for some discrete random variable [2]. In this note, an algorithm is presented which efficiently solves the problem.

2 The Problem

Let $x$ be a discrete random variable whose outcomes belong to a finite set of elementary events $\Omega$, and let $n$ indicate the cardinality of $\Omega$. Let $\mathcal{I}_\Omega$ be the totality of distribution functions for $x$.

Problem: find an algorithm to sample $\mathcal{I}_\Omega$ uniformly and independently of $n$.

In the following, we shall rely on the existence of a subroutine, $\mathcal{A}$, able to return series of pseudo random numbers uniformly distributed in the interval $[0,1]$. Existence of such subroutine is thoroughly discussed in [1].

Let us start by observing that $\mathcal{I}_\Omega$ is naturally parametrized by $n$ numbers, $x_1, \ldots, x_n$, satisfying the conditions:

$$0 \leq x_i \leq 1, \quad \forall i \in \{1, \ldots, n\}, \quad \text{(1)}$$

and

$$\sum_{i=1}^{n} x_i = 1. \quad \text{(2)}$$
\( I_\Omega \) is therefore the \((n-1)\)-simplex, \( S^{n-1} \), and our problem in equivalent to finding an algorithm for the uniform sampling of \( S^{n-1} \). It may be worth reminding that the \( n \)-volume of the \( n \)-simplex tends to zero (super)-exponentially in \( n \). This implies that the naïve sampling strategy consisting in generating points within the unit \( n \)-cube, and discarding those falling outside \( S^n \) is virtually inapplicable – even for very small values of \( n \). Other approaches such as that of generating points within the unit \( n \)-cube, and rescale them as to satisfy condition \( \sum_{i=1}^{n} x_i = 1 \) are plainly wrong.

3 Solution

Here is the basic idea: for each sample point to be generated on the \((n-1)\)-simplex, and for each of its first \( n-1 \) coordinates, \( x_1, \ldots, x_{n-1} \), randomly sample interval \([0,1]\) according to a density function able – in average – to assign to each \( x_i \) “just its fair share” of the total amount \( \sum_{i=1}^{n} x_i = 1 \). It does not take much to get convinced that such density function indeed exist for any component \( x_j \): it is the marginal distribution of \( x_j \) over the simplex \( S^{n-1} \), given the outcomes of \( x_1, \ldots, x_{j-1} \).

The proposed algorithm therefore runs as it follows:

1. set \( r_1 = 1 \);
2. set \( j = 1 \);
3. until \( j = n - 1 \)
   3.1. randomly extract \( x_j \) from \([0, r_j]\) according to the marginal distribution of \( x_j \) over the simplex \( S^{n-1} \), given outcomes \( x_1 = \bar{x}_1, \ldots, x_{j-1} = \bar{x}_{j-1} \), that is according to:
      \[
      \psi(x) = \text{Prob} \ (x_j = x \mid x_1 = \bar{x}_1, x_2 = \bar{x}_2, \ldots, x_{j-1} = \bar{x}_{j-1}) ;
      \]
   3.2. set \( r_{j+1} = r_j - x_j \);
   3.3. set \( j = j + 1 \);
4. set \( \bar{x}_n = r_n \);
5. output \((\bar{x}_1, \ldots, \bar{x}_n)\).

Step 3.1 is the crucial one. To perform it, we need to: (1) determine \( \psi \) for any \( n \) and any set of outcomes, \( \bar{x}_1, \ldots, \bar{x}_{j-1} \); (2) sample interval \([0, r_j]\) according to \( \psi \).

3.1 Determining \( \psi(x) \)

Let us start by observing that the marginal distribution of \( x_1 \) must be proportional to the \((n-2)\)-volume of the subset of \( R^n \) defined by \( x_2 + x_3 + \ldots + x_n = 1 - x_1 \). Let us indicate such subset with \( S_{x_1} \). For any \( x_1, S_{x_1} \) is just a rescaling of the \((n-2)\)-simplex, and its volume is therefore proportional to \((1 - x_1)^{n-2} \) (see Fig. 1a). The marginal distribution of \( x_1 \) can
Figure 1: (a) The 3-dimensional case: for any outcome of $x_1$ rescaling of the 1-simplex is determined. The marginal distribution of $x_1$ is therefore proportional to the 1-volume of $S^1$. (b) 5000 samples of $S^2$ as obtained by applying the proposed algorithm.

Therefore be written in the form $\psi(x_1) = \alpha (1 - x_1)^{n-2}$, where factor $\alpha$ is determined via the normalization condition:

$$\alpha \int_0^1 (1 - x_1)^{n-2} dx_1 = 1.$$  

This yields $\alpha = n - 1$, and then:

$$\psi(x_1) = (n - 1)(1 - x_1)^{n-2}.$$  

The same process can be iterated for all the other components $x_j$, $1 < j < n$, accounting for the fact that any $x_j$ is now to be limited to the range $[0, r_j]$. This implies that $\psi(x_j)$ must be proportional to $(r_j - x_j)^{n-2}$, and it is an interesting fact that dependence of $\psi(x_j)$ from outcomes $\bar{x}_1, \ldots, \bar{x}_{j-1}$ is just contained in their sum $1 - r_j$. Thus, we can write:

$$\beta \int_0^{r_j} (r_j - x_j)^{n-2} dx_j = 1,$$

which yields $\beta = \frac{n-1}{r_j^{n-1}}$, and, finally

$$\psi(x_j) = \frac{n-1}{r_j^{n-1}}(1 - x_j)^{n-2}.$$
The cumulative function of the marginal distribution of $x_j$ is then:

$$
\Psi(x_j) = 1 - \left( \frac{r_j - x_j}{r_j} \right)^{n-1}.
$$

(8)

### 3.2 Sampling $[0, s]$ according to $\psi(x)$

As it is well known [3], sampling a random variable $x$ according to a given distribution function, $\psi(x)$, is readily obtained once that the inverse of the cumulative function of $x$, $\Psi^{-1}$, is known. Indeed, Eq. 8 guarantees that, for any $n$ and $j$:

$$
\Psi^{-1}(\xi) = r_j[1 - (1 - \xi) \frac{1}{n-1}].
$$

(9)

### 4 Efficiency

The proposed algorithm is optimally efficient: in dimension $n$ it requires just $n - 1$ runs of subroutine $A$, plus $n - 2$ calls of function $\Psi^{-1}$, whose complexity is constant in $n$.

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**References**

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