Critical behavior of spin and polymer models with aperiodic interactions

T. A. S. Haddad and S. R. Salinas
Instituto de Física
Universidade de São Paulo
Caixa Postal 66318
05315-970, São Paulo, SP, Brazil

March 3, 2022

Abstract

We review and extend some recent investigations of the effects of aperiodic interactions on the critical behavior of ferromagnetic $q$-state Potts models. By considering suitable diamond or necklace hierarchical lattices, and assuming a distribution of interactions according to a class of two-letter substitution rules, the problem can be formulated in terms of recursion relations in parameter space. The analysis of stability of the fixed points leads to an exact criterion to gauge the relevance of geometric fluctuations. For irrelevant fluctuations, the critical behavior remains unchanged with respect to the uniform systems. For relevant fluctuations, there appears a two-cycle of saddle-point character in parameter space. A scaling analysis, supported by direct numerical thermodynamic calculations, shows the existence of novel critical universality classes associated with relevant geometric fluctuations. Also, we show that similar qualitative results are displayed by a simple model of two directed polymers on a diamond hierarchical structure with aperiodic bond interactions.

PACS numbers: 64.60.Ak; 05.50.+q; 61.44.Br; 83.80.Rs

1 Introduction

In some recent publications[1, 2, 3, 4], we investigated the critical behavior of a $q$-state ferromagnetic Potts model on a class of hierarchical diamond lattices with aperiodic interactions. Taking advantage of the lattice structure,
and assuming a (layered) distribution of exchange interactions according to suitable two-letter substitution rules, we were able to write exact recursion relations in order to characterize the critical behavior under the influence of geometric fluctuations. For disordered ferromagnetic systems, the well-known Harris criterion predicts a change in critical behavior if the underlying uniform system displays a positive specific heat critical exponent. An extension of this heuristic criterion has been proposed by Luck[5] in order to gauge the relevance of geometric fluctuations associated with aperiodicity. We derived an exact expression of an analog of Luck’s criterion on a hierarchical lattice. For weak fluctuations, there is a nontrivial symmetric fixed point in parameter space that leads to the same critical behavior of the uniform model. For strong enough fluctuations, this symmetric fixed point becomes fully unstable, and there appears a cycle-two in parameter space, that is shown to lead to a novel class of (aperiodic) critical behavior[2,3]. As the systems are quite simple, it has been feasible to perform a direct thermodynamic analysis of the free energy to check the predictions of scaling arguments.

Now we review and extend these calculations. In Section II, we show that the same results are obtained for a $q$-state Potts model on a necklace hierarchical lattice. Although the critical temperature may change, critical exponents associated with either the symmetric fixed point or the cycle-two attractor do display universal features that do not depend on the particular choice of diamond or necklace hierarchical structures (and on particular values of the interaction energies). In Section III, we consider a model of two directed polymers on a diamond lattice[6,7] with a (layered) distribution of aperiodic interactions, according to some two-letter substitution rules. The exact recursion relations in parameter space turn out to be very simple, but with a similar structure as in the case of the aperiodic Potts model. We then present some examples to show that, for relevant geometric fluctuations, a symmetric fixed point becomes fully unstable and there is again a novel cycle-two attractor of saddle-point character in parameter space.

## 2 Potts model with aperiodic interactions

The successive application of the period-doubling two-letter rule,

\[ A \rightarrow AB, \quad B \rightarrow AA, \]

on an initial letter $A$, produces the sequence

\[ A \rightarrow AB \rightarrow ABAA \rightarrow ABAAABAB \rightarrow .... \]
At each stage of this construction, the numbers $N_A$ and $N_B$, of letters $A$ and $B$, can be obtained from the recursion relations

$$
\begin{pmatrix}
N'_A \\
N'_B
\end{pmatrix} = \mathbf{M} \begin{pmatrix}
N_A \\
N_B
\end{pmatrix} = \begin{pmatrix}
1 & 2 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
N_A \\
N_B
\end{pmatrix},
$$

(3)

where the eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = -1$, of the substitution matrix $\mathbf{M}$ govern most of the geometric properties of this sequence[8]. At order $n$, we can write

$$
N^{(n)}_A = \frac{2}{3} \lambda_1^n + \frac{1}{3} \lambda_2^n \quad \text{and} \quad N^{(n)}_B = \frac{1}{3} \lambda_1^n - \frac{1}{3} \lambda_2^n,
$$

(4)

from which we have the asymptotic expressions $N^{(n)}_A \sim \lambda_1^n$ and $\Delta N^{(n)}_A \sim \lambda_2^n$, for large values of $n$, where $\Delta N^{(n)}_A$ can be regarded as a fluctuation around an asymptotic value. As the total number of letters, at order $n$, is given by $N^{(n)} = N^{(n)}_A + N^{(n)}_B$, we have $\Delta N^{(n)}_A \sim (N^{(n)})^\omega$, with the wandering exponent

$$
\omega = \frac{\ln |\lambda_2|}{\ln \lambda_1}.
$$

(5)

The ferromagnetic $q$-state Potts model is given by the Hamiltonian

$$
H = -q \sum_{(i,j)} J_{i,j} \delta_{\sigma_i, \sigma_j},
$$

(6)

where $\sigma_i = 1, 2, ..., q$, at all sites of a lattice, $J_{i,j} > 0$, and the sum refers to nearest-neighbor pairs of sites (for $q = 2$, we regain the standard Ising model). We now consider a Potts model on a hierarchical necklace lattice, with $b = 2$ bonds and $m$ branches, and assume that the couplings can take just two values, $J_A$ and $J_B$, associated with a sequence of letters produced by the period-doubling substitution. In Fig. 1, we indicate some stages of this construction (for $b = 2$ and $m = 3$). Note that the period-doubling sequence is perfectly suitable for a hierarchical lattice with $b = 2$ bonds. Also, note that the choice of the same interactions along the branches turns out to mimic an aperiodic layered structure in the corresponding Bravais lattice.

Now we decimate the internal degrees of freedom of the necklace structure and write exact recursion relations for the reduced coupling parameters,

$$
x'_A = \frac{x_A^{m} x_B^{m} + q - 1}{x_A^{m} + x_B^{m} + q - 2},
$$

(7)

and

$$
x'_B = \frac{x_A^{2m} + q - 1}{2x_A^{m} + q - 2},
$$

(8)
Figure 1: Some generations of a hierarchical necklace lattice with $b = 2$ bonds and $m = 3$ branches. Letters $A$ and $B$ indicate the layered distribution of exchange interactions.

where $x_{A,B} = \exp (\beta J_{A,B})$ with $\beta = 1/k_B T$.

In parameter space, besides the trivial fixed points, associated with zero and infinite temperature, there is always a non-trivial symmetric fixed point, associated with the critical behavior of the underlying uniform model (see the sketch in Fig. 2a). The linearization of the recursion relations about the symmetric fixed point $x^*$ leads to the matrix form

$$
\begin{pmatrix}
\Delta x'_A \\
\Delta x'_B
\end{pmatrix} = C(x^*) \tilde{M} \begin{pmatrix}
\Delta x_A \\
\Delta x_B
\end{pmatrix},
$$

(9)

where $C(x^*)$ is a structure factor and $\tilde{M}$ is the transpose of the substitution matrix. The eigenvalues of this linear form are given by

$$
\Lambda_1 = \lambda_1 C(x^*) = 2 C(x^*),
$$

(10)

and

$$
\Lambda_2 = \lambda_2 C(x^*) = -C(x^*).
$$

(11)

It is easy to show that $\Lambda_1 > 1$. Note that we can write $\Lambda_1 = 2^{yt}$, with $y_T = D/(D - \alpha_u)$, where $D = \ln (2m)/\ln 2$ is the fractal dimension of the lattice and $\alpha_u$ is the specific heat critical exponent of the underlying uniform model. Also, it is easy to show that $|\Lambda_2| < 1$, for $q < q_c$, and $|\Lambda_2| > 1$, for $q > q_c$. In this last case, for $q > q_c$, the symmetric fixed point becomes fully unstable, and the geometric fluctuations are relevant (see the sketch in Fig. 2b). For $b = m = 2$, both for necklace and diamond structures, we have $q_c = 4 + 2\sqrt{2} = 6.828427...$, which also corresponds to the critical number of states for the relevance of disorder in a ferromagnetic Potts model on a fully disordered diamond lattice [9].
For necklace and diamond lattices with \( b \) bonds and \( m \) branches, and a suitable two-letter substitution sequence with period \( \lambda_1 = b \), we can always write

\[
\Lambda_1 = 2C(x^*) = b^{D/(2-\alpha_u)},
\]

where \( D = \ln(mb)/\ln b \) is the fractal dimension of the lattice, \( \alpha_u \) is the specific heat critical exponent of the underlying uniform model, and

\[
\Lambda_2 = \lambda_2 C(x^*) = \lambda^\omega C(x^*) = b^\omega C(x^*),
\]

where the wandering exponent \( \omega \) comes from Eq. (5). We then have

\[
\Lambda_2 = b^{\omega-1+D/(2-\alpha_u)},
\]

that leads to an exact version of Luck’s criterion\[5\] for the relevance of geometric fluctuations,

\[
\omega > 1 - \frac{D}{2-\alpha_u}.
\]

For the \( q \)-state Potts model on both diamond and necklace lattices with \( b = 2 \) bonds and \( m = 2 \) branches, this inequality is equivalent to \( q > q_c = 4 + 2\sqrt{2} \), as we have pointed in the last paragraph. In this special case, it also corresponds to \( \alpha_u > 0 \), which is in agreement with the usual form of Harris criterion for the relevance of disorder\[9\]. It should be pointed out that, according to the usual expectations, given \( q, b \) and \( m \), the eigenvalues \( \Lambda_1 \) and \( \Lambda_2 \) assume the same values for both diamond and necklace structures.

We now turn to the investigation of the critical behavior when the symmetric fixed point is fully unstable. Numerically, we can see that, instead of a hypothetical aperiodic fixed point, there appears a two-cycle, with a saddle-point character, in parameter space. We then invoke scaling arguments to
where $f(x)$ is the reduced free energy per bond, $g(x)$ is a regular function, and $x''$ is a second iterate of the recursion relations. Note that the factor $b^{2D}$ has to be included because we need two iterates to go back to the vicinity of the initial point in parameter space. This equation leads to the asymptotic solution

$$f(x) \sim |x - x^*|^{2-\alpha} P \left( \frac{\ln |x - x^*|}{\ln \Lambda_{cyc}} \right),$$

where $x^*$ is one of the points of the two-cycle, $\Lambda_{cyc}$ is the largest eigenvalue of the linearization of the second iterate of the recursion relations about any one of the points of the cycle, $P(z)$ is an arbitrary function of period 1, and the specific heat critical exponent is given by

$$\alpha = 2 - 2 \frac{\ln b^D}{\ln \Lambda_{cyc}} = 2 - 2 \frac{\ln (mb)}{\ln \Lambda_{cyc}}.$$

Let us show some calculations for the $q$-state Potts model on a necklace lattice, with $b = 2$ bonds, $m$ branches, and a distribution of exchange interactions according to the period-doubling two-letter sequence: (i) for $q = 7$ and $m = 2$, the two-cycle is located at $(x_A^*, x_B^*) = (2.299..., 2.764...)$ and $(2.587..., 2.179...)$, with eigenvalues of the second iterate given by $\Lambda_1 = 3.993...$ and $\Lambda_2 = 0.985...$. From Eq. (18), we have $\alpha = -0.00226...$, to be compared with the positive value $\alpha_u = 0.010...$, for the uniform system; (ii) for $q = 25$ and $m = 2$, the two-cycle is located at $(x_A^*, x_B^*) = (2.634..., 15.308...)$ and $(6.246..., 1.957...)$, with eigenvalues of the second iterate given by $\Lambda_1 = 4.243...$ and $\Lambda_2 = 0.343...$. From Eq. (18), we have $\alpha = 0.08174...$, to be compared with $\alpha_u = 0.40456...$, for the corresponding uniform system. For other values of $m$, we find qualitatively similar results. The specific heat exponent is definitely depressed as compared to the values for the underlying uniform system. Also, given $q$, $b$, and $m$, the exponents assume the same values for diamond and necklace structures.

To check the validity of the scaling arguments, and the role of the two-cycle as the responsible for the new critical behavior, we have performed direct numerical analyses of the singularity of the thermodynamic free-energy. For the $q$-state Potts model on the $b = 2$ necklace hierarchical lattice, and with the period-doubling rule, we can write the free energy per bond as a
series expansion,

\[ f(x_A, x_B) = -k_B T \sum_{n=0}^{\infty} \frac{1}{(2m)^n} \left\{ \frac{1}{3m} \ln \left[ \left( x_A^{(n)} \right)^m + \left( x_B^{(n)} \right)^m + q - 2 \right] \right. \\
+ \left. \frac{1}{6m} \ln \left[ 2 \left( x_A^{(n)} \right)^m + q - 2 \right] \right\}. \quad (19) \]

All of our numerical checks fully confirm the scaling results. For example, for \( m = 2 \) and \( q = 100 \), with \( J_A/J_B = 5 \), a numerical analysis of the specific heat divergence at the critical temperature leads to \( \alpha = 0.27 \pm 0.03 \), to be compared with the scaling prediction from the two-cycle, \( \alpha = 0.27204... \) (which is definitely different from the value for the uniform system, \( \alpha_u = 0.64846... \)). Within the error estimates, we always obtain the same numerical results for all values of the ratio \( J_A/J_B \). Again, given \( q, b, \) and \( m \), and for all values of the ratio \( J_A/J_B \), the exponents assume the same values for diamond and necklace structures. From the numerical calculations, and particularly for bigger values of \( q \), we have been able to detect a log-periodic oscillatory behavior of the thermodynamic functions. The period of oscillation is compatible with Eq. (17), which has also been confirmed in the independent calculations of Andrade[11].

3 Directed polymers with aperiodic interactions

Along the lines of a paper by Mukherji and Bhattacharjee[6], we now consider a model of two interacting directed polymers on a diamond lattice (see Fig. 3). The polymers start at one end of the lattice and meet at the other end. There is an attractive interaction if a bond of the lattice is shared by two polymers.

Consider a diamond lattice \((b = 2 \) bonds and \( m \) branches\), with a layered distribution of interactions, according to the two-letter period-doubling sequence given by Eq. (1). It is easy to write the recursion relations

\[ y'_A = \frac{1}{m} y_A y_B + \frac{m-1}{m}, \quad (20) \]

and

\[ y'_B = \frac{1}{m} y_A^2 + \frac{m-1}{m}, \quad (21) \]
where \( y_{A,B} = \exp(\beta v_{A,B}) \), and \( v_{A,B} > 0 \) is the interaction energy at bonds of types \( A \) and \( B \), respectively. For \( y_A = y_B = y \), we recover the recursion relation for the uniform model,

\[
y' = \frac{1}{m} y^2 + \frac{m-1}{m}. \tag{22}
\]

Besides the trivial fixed points, \( y^* = 1 \) and \( \infty \), associated with zero and infinite temperatures, there is a nontrivial fixed point, \( y^* = m - 1 \), which becomes physically acceptable, and is associated with a binding-unbinding transition, for \( m > 2 \) (there is no phase transition on the simple diamond lattice with \( m = 2 \) branches).

The recursion relations (20) and (21) are so simple, that it easy to see that, for \( m > 2 \), there is no physically acceptable nontrivial fixed points except the symmetric fixed point \( y_A^* = y_B^* = y^* \). The linearization of the recursion relations in the neighborhood of this symmetric fixed point leads to matrix form

\[
\begin{pmatrix}
\Delta y_A' \\
\Delta y_B'
\end{pmatrix} = \frac{y^*}{m} \begin{pmatrix}
1 & 1 \\
2 & 0
\end{pmatrix} \begin{pmatrix}
\Delta y_A \\
\Delta y_B
\end{pmatrix}, \tag{23}
\]

with eigenvalues \( \Lambda_1 = 2y^*/m = 2(m-1)/m \), and \( \Lambda_2 = -y^*/m = -(m-1)/m \). Therefore, if \( m > 2 \), we have \( \Lambda_1 > 1 \) and \( |\Lambda_2| < 1 \), which shows that the geometric fluctuations are completely irrelevant in this case.

We now turn to a more interesting case. Consider a diamond lattice with \( b = 3 \) bonds and \( m \) branches (in Fig. 3, we sketch a diamond lattice with \( b = 3 \) bonds and \( m = 2 \) branches). Suppose that the (layered) interactions are chosen according to the period-3 two-letter sequence, \( A \rightarrow ABB \) and \( B \rightarrow AAA \). The substitution matrix is characterized by the eigenvalues \( \lambda_1 = b = 3 \) and \( \lambda_2 = -2 \), with the wandering exponent \( \omega = \ln 2/\ln 3 = 0.630092\ldots \). The new recursion relations are given by

\[
y_A' = \frac{1}{m} y_A y_B^2 + \frac{m-1}{m}, \tag{24}
\]
There is just a single nontrivial fixed point, for \( m > 3 \), at a symmetric location,
\[
y_A^* = y_B^* = y^* = \frac{-1}{2} + \frac{1}{2} \sqrt{4m - 3}.
\]
(26)

The linearization in the neighborhood of this fixed point leads to the matrix form
\[
\begin{pmatrix}
\Delta y'_A \\
\Delta y'_B
\end{pmatrix} = \left( \frac{y^*}{m} \right)^2 \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} \Delta y_A \\ \Delta y_B \end{pmatrix},
\]
(27)
with the eigenvalues
\[
\Lambda_1 = 3 \frac{y^{*2}}{m} = \frac{3}{2m} \left[ 2m - 1 - \sqrt{4m - 3} \right],
\]
(28)
and
\[
\Lambda_2 = -2 \frac{y^{*2}}{m} = -\frac{1}{m} \left[ 2m - 1 - \sqrt{4m - 3} \right].
\]
(29)

For \( 3 < m < 3 + \sqrt{5} \), it is easy to show that \( \Lambda_1 > 1 \), and \( |\Lambda_2| < 1 \). As in the case of the simple diamond lattice with \( b = 2 \) bonds, geometric fluctuations are irrelevant and the critical behavior is identical to the uniform case. However, for \( m > 3 + \sqrt{5} = 5.236068... \), we have \( |\Lambda_2| > 1 \), and the symmetric fixed point becomes fully unstable. For example, for \( m = 5 \), we have \( y_A^* = y_B^* = y^* = 1.561552... \), with eigenvalues \( \Lambda_1 = 2.140568... \) and \( \Lambda_2 = 0.951363... < 1 \). For \( m = 6 \), however, we have \( y_A^* = y_B^* = y^* = 1.791287... \), with eigenvalues \( \Lambda_1 = 2.573958... \) and \( \Lambda_2 = 1.143981... > 1 \). As in the case of the Potts model, there is then a cycle-two, of hyperbolic character, in parameter space. It is easy to locate this cycle at \((y_A^*, y_B^*) = (1.419001..., 2.267305...)\) and \((2.049103..., 1.309541...)\), with eigenvalues of the second iterate given by \( \Lambda_1 = 2.624300... > 1 \) and \( \Lambda_2 = 0.772598... < 1 \).

4 Conclusions

In conclusion, we have taken advantage of suitable diamond and necklace hierarchical lattices, and used a distribution of (layered) interactions according to a class of two-letter substitution rules, to perform an exact analysis...
of the effects of geometric fluctuations on the critical behavior of ferromagnetic \( q \)-state Potts models. We derived an exact expression for an analog of Luck’s criterion on a hierarchical lattice. Also, due to the simplicity of the formulation, we have been able to show that the novel critical behavior under the influence of relevant geometric fluctuations is governed by a two-cycle attractor in parameter space. This new universality class can be described by scaling arguments, that are fully supported by direct thermodynamic calculations for the singularity of the free energy (as usual, critical exponents do not depend on details of the systems, as the relative strength of the interactions). It is not difficult to extend these calculations for an Ising model in the presence of direct and staggered fields\[12\], to analyze the effects of aperiodicity on the tricritical behavior in a mixed-spin Ising system\[13\], and to consider some noncommutative sequences\[14\].

We have also reported some results for the critical behavior of a model of two directed polymers on a diamond lattice with a (layered) distribution of aperiodic interactions (according to suitable two-letter substitution rules). The exact recursion relations in parameter space turn out to be very simple, but with a similar structure as in the case of the aperiodic Potts model. We have presented some examples to show that, for relevant geometric fluctuations, a symmetric fixed point in parameter space becomes fully unstable and there is again a novel cycle-two attractor, of saddle-point character, that governs a new class of (aperiodic) critical behavior.

This work has been supported by the Brazilian agencies FAPESP and CNPq.

References

[1] S. T. R. Pinho, T. A. S. Haddad, and S. R. Salinas, Braz. J. Phys. 27, 567 (1997); Physica A 257, 515 (1998).

[2] S. T. R. Pinho, T. A. S. Haddad, and S. R. Salinas, Phys. Rev. E 61, 3330 (1998).

[3] S. T. R. Pinho, T. A. S. Haddad, and S. R. Salinas, Braz. J. Phys. 30, 741 (2000).

[4] A. C. N. de Magalhães, S. R. Salinas, and C. Tsallis, J. Phys. A: Math. Gen. 31, L567 (1998).

[5] J. M. Luck, Europhys. Lett. 24, 359 (1993); J. Stat. Phys. 72, 417 (1993).
[6] S. Mukherji and S. M. Bhattacharjee, Phys. Rev. E 52, 1930 (1995).

[7] S. M. Bhattacharjee and S. Mukherji, Phys. Rev. Lett. 70, 49 (1993); S. Mukherji and S. M. Bhattacharjee, Phys. Rev. E 48, 3483 (1993).

[8] J. M. Luck, C. Godrèche, A. Janner, and T. Janssen, J. Phys. A: Math. Gen. 26, 1951 (1993).

[9] B. Derrida and E. Gardner, J. Phys. A: Math. Gen. 17, 3223 (1984).

[10] B. Derrida, “Pure and random models of statistical mechanics on hierarchical lattices”, in “Phénomènes critiques, systèmes aléatoires, théories de jauge”, edited by K. Osterwalder and R. Stora, Les Houches, 1984, session XLIII, Elsevier, Amsterdam, p. 989.

[11] R. F. S. Andrade, Phys. Rev. E 59, 150 (1999); Phys. Rev. E 61, 7196 (2000).

[12] Angsula Ghosh, T. A. S. Haddad, and S. R. Salinas, Int. J. Mod. Phys. B 14, 1473 (2000).

[13] T. A. S. Haddad, Angsula Ghosh, and S. R. Salinas, Phys. Rev. E 62, 7773 (2000).

[14] S. R. Salinas and W. F. Wrezisnki, Physica A 297, 434 (2001).