Surprising structures hiding in Penrose’s future null infinity

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Abstract
Since the late1950s, almost all discussions of asymptotically flat (Einstein–Maxwell) space-times have taken place in the context of Penrose’s null infinity, \( \mathcal{I}^+ \). In addition, almost all calculations have used the Bondi coordinate and tetrad systems. Beginning with a known asymptotically flat solution to the Einstein–Maxwell equations, we show first, that there are other natural coordinate systems, near \( \mathcal{I}^+ \), (analogous to light-cones in flat-space) that are based on (asymptotically) shear-free null geodesic congruences (analogous to the flat-space case). Using these new coordinates and their associated tetrad, we define the complex dipole moment, (the mass dipole plus \( i \) times angular momentum), from the \( l = 1 \) harmonic coefficient of a component of the asymptotic Weyl tensor. Second, from this definition, from the Bianchi identities and from the Bondi–Sachs mass and linear momentum, we show that there exists a large number of results—identifications and dynamics—identical to those of classical mechanics and electrodynamics. They include, among many others, \( \mathbf{P} = m \mathbf{v} + \cdots, \mathbf{L} = \mathbf{r} \times \mathbf{P} \), spin, Newton’s second law with the rocket force term (\( \dot{m} \mathbf{v} \)) and radiation reaction, angular momentum conservation and others. All these relations take place in the rather mysterious H-space rather than in space-time.

This leads to the enigma: ‘why do these well known relations of classical mechanics take place in H-space?’ and ‘What is the physical meaning of H-space?’

Keywords: null infinity, Bondi coordinates, stereographic coordinates

1. Introduction
The modern era of the study of gravitational radiation began in the 1950s with the pioneering work of Hermann Bondi [1]. This was quickly expanded by the major contributions of Rainer Sachs [2] and Roger Penrose [3–5, 9] among many others. After years of further developments, theoretical, numerical and observational, we had its culmination with the observation...
and analytic understanding of the collision and merger of the pair of black holes that produced the gravitational wave signal, GW105 [6], that was seen by LIGO in 2016. Gravitational wave theory now could play a major role in astrophysics and physics.

Bondi’s work began with integrating the Einstein equations in the asymptotic region—in the vicinity of future null infinity. This involved the important step of using special null surfaces as part of the coordinate system referred to as Bondi coordinates, \((r, u, \zeta, \overline{\zeta})\). (The Bondi coordinates are defined uniquely up to a group of transformations known as the BMS group. References [3, 7]) The idea of working near or even at infinity, though at the beginning slightly nebulous, was formalized (by Penrose [3]) by bringing infinity into a finite region of the space-time by its conformal compactification, (rescaling and contraction). Future null Infinity was then represented by a null three-surface in space-time (referred to as \(I^+\), vocalized by SCRI+). \(I^+\), a null 3-surface with the topology of \(S^2 \times \mathbb{R}\), (visualized as a light-cone at future null infinity, apex at time-like infinity) is coordinatized by the complex stereographic coordinates, \((\zeta = e^{i\phi} \cot \frac{\theta}{2}, \overline{\zeta} = e^{-i\phi} \cot \frac{\theta}{2})\), on the \(S^2\) part, (labeling the null generators) and with \(u\) on the \(\mathbb{R}\) part (labeling the cross-sections of \(I^+\)). The BMS group can be described as coordinate transformations among the coordinates \((u, \zeta, \overline{\zeta})\). In addition to the introduction of null surfaces, Bondi’s other insight was the realization that several components of the asymptotic Weyl tensor could be identified with the mass/energy and linear momentum of the source and their loss—in analogy, in Maxwell theory, to charge and charge conservation as integrals of the fields at infinity. These Weyl tensor components come from the harmonic components of the leading coefficients in the \(r^{-1}\) expansion in the spin-coefficient version of the Weyl tensor and are thus functions just on \(I^+\), i.e. are functions of \((u, \zeta, \overline{\zeta})\).

An important idea to recognize is that the leading far-field components of the Weyl tensor (in the spin-coefficient formulation [9]), depend very much on the choice of the tetrad and coordinates systems to be used on \(I^+\). In the past almost all Weyl tensor components were chosen in a Bondi system. However, in the present work, the major new ingredient that leads to our results is the introduction of totally new coordinates and tetrad systems on \(I^+\) and its neighborhood, (very different from Bondi’s) that very closely mimic certain natural coordinate systems on the Minkowski space \(I^+\). This allows us to describe other functions on \(I^+\) (arising from other Weyl tensor coefficients) that yield—with the Bianchi identities—a variety of additional physical quantities, such as angular momentum, center of mass, its position and velocity, their evolution, as well as force laws and electric and magnetic dipole moments and center of charge. In other words these results and relationships from classical mechanics simply appear as components of the Weyl tensor at infinity.

The prime idea involved in the choice of these systems is use of the special null surfaces that are associated with asymptotically shear-free null geodesic congruences—as directly opposed to Bondi’s null surfaces which do have non-vanishing asymptotic shear—the time-integral of the Bondi news function.

These special null surfaces, which are standard and easily understood surfaces in Minkowski space, are of two types. The first are the null cones (with generators automatically shear-free and twist-free) with apex on arbitrary time-like curves—a special case being time-like geodesics. The second type arise (formally) from complex light-cones with apex on complex world-lines [7, 10]. In this case, the associated real congruence, though shear-free, is now twisting. They will be first reviewed and described in section 2. Section 3 will be devoted to their generalization (in asymptotically flat spaces) to asymptotically shear-free congruences, i.e. their definition and construction. Though at first it appears that there is a serious impediment to their construction, it turns out that by a slight zigzag or maneuver, the impediment can be overcome and the construction can be completed in exact analogy to the flat-space case.
We will have coordinate systems on (asymptotically flat) \( \mathcal{I}^+ \), very closely matching those in Minkowski space. The asymptotic generators (the null geodesics) of the complex surfaces (by construction) will be real and asymptotically shear-free but, in general, they will be twisting. In sections 4 and 5 we will, by using earlier work, show how the asymptotic Weyl tensor components, expressed in the new coordinates with their associated tetrads, naturally yield a large number of functions determining the interior space time properties as mentioned earlier. In the discussion, the strange appearance of H-space-coordinates is addressed. We emphasize that though complex ideas are used, we are dealing with real space-times.

For close to 50 years the coordinatization of \( \mathcal{I}^+ \) by Bondi coordinates has been almost sacrosanct—nevertheless we present (what we believe is a strong argument) that other choices of coordinates on \( \mathcal{I}^+ \) have considerable value and their use should be seriously considered.

2. \( \mathcal{I}^+ \) of Minkowski space

The material of this section, namely an analytic description of shear-free null geodesic congruences in Minkowski space—with and without twist—though not new, does contain a new point of view. It is given in detail so that it can be compared with similar null geodesic congruences in asymptotically flat space-times. In that context it certainly appears new and useful.

Using standard Minkowski coordinates, \( x^a \), with metric \( \eta_{ab} \) and signature \((+,-,-,-)\), the family of null cones with apex at the origin is described parametrically, \((u, r, \zeta, \overline{\zeta})\), by

\[
x^a = u\delta^a + \hat{r}^a(\zeta, \overline{\zeta})
\]

with \( \hat{r}^a(\zeta, \overline{\zeta}) \) a null vector that sweeps out the null cone as \( \zeta = \cot \frac{\theta}{2} e^{i\phi} \) varies over the sphere of null directions, i.e.

\[
\hat{r}^a(\zeta, \overline{\zeta}) = \frac{\sqrt{2}}{2} \left( 1, \frac{\zeta + \overline{\zeta}}{1 + \zeta \overline{\zeta}}, \frac{-i(\zeta - \overline{\zeta})}{1 + \zeta \overline{\zeta}}, \frac{1 - \zeta \overline{\zeta}}{1 + \zeta \overline{\zeta}} \right) = \left( \frac{\sqrt{2}}{2}, \frac{1}{2} \mathbf{Y}_0^{(1)} \right)
\]

\[
r^a = \delta^a_0.
\]

The \( r \) is the affine parameter along the null generators of the cone and \( u \) the time at the spatial origin (or the retarded time labeling the cone itself).

An alternate interpretation of equation (1) is that it is the coordinate transformation between the \( x^a \) and the null coordinates \((u, r, \zeta, \overline{\zeta})\). We will refer to the coordinate transformation (and coordinates) given by equation (1), as well as its later generalization in section 3, as ‘static null coordinates’.

2.1. Aside

The full null tetrad, \( (\hat{I}, \hat{n}, \hat{m}, \hat{\overline{m}}) \), (with \( \hat{I} \cdot \hat{n} = -\hat{m} \cdot \hat{\overline{m}} = 1 \), all other products vanishing) associated with equation (2), is given by

\[
\hat{n}^a = \frac{\sqrt{2}}{2\delta}(1 + \zeta \overline{\zeta}, -(\zeta + \overline{\zeta}), i(\zeta - \overline{\zeta}), 1 - \zeta \overline{\zeta}),
\]

\[
\hat{m}^a = \delta^a_0 = \frac{\sqrt{2}}{2\delta}(0, 1 - \zeta^2, -i(1 + \zeta^2), 2\overline{\zeta}),
\]

\[
\overline{m}^a = \overline{\delta}^a_0 = \frac{\sqrt{2}}{2\delta}(0, 1 - \overline{\zeta}^2, i(1 + \overline{\zeta}^2), 2\zeta).
\]
\[ P = 1 + \zeta \bar{\zeta}. \]  

(5)

The metric tensor in these new coordinates, given by

\[ ds^2 = du^2 + 2udu - 4r^2P^{-2}d\zeta d\bar{\zeta} \]

(6)

can be conformally transformed (rescaled) by \( \Omega^2 = r^{-2} \), leading to

\[ d\tilde{s}^2 = \frac{1}{\tilde{\Omega}^2} du^2 + \Omega^2 d\tilde{u}^2 - 4P^{-2}d\tilde{\zeta} d\bar{\tilde{\zeta}}. \]

(7)

The surface defined by \( \Omega = 0 \), a null surface, is identified as the \( \mathcal{I}^+ \) of Minkowski space. It can be thought of as the intersection of the endpoints of the future null cones that have apex on the world-line \( x^a = ut^a \), with future null infinity. It is coordinatized by \( (u, \zeta, \bar{\zeta}) \) and is a special case of Bondi coordinates.

These null coordinates can be generalized to a new set, \( (t, r^a, \zeta^a, \bar{\zeta}^a) \), by basing them on null cones with apex on an arbitrary time-like world-line, \( x^a = \xi^a(t) \), by the parametric form, \( (t, \zeta^a, \bar{\zeta}^a), t \) a real parameter,

\[ x^a = \xi^a(t) + r^a(t) \zeta^a. \]

(8)

\( \tilde{r}^a \) is again a null vector sweeping out the null directions on the cone. By equating the right sides of equations (1) and (8), multiplying by the four null tetrad vectors associated with \( \tilde{t}^a(t, \zeta, \bar{\zeta}) \), equation (4), and passing to the limit \( r \to \infty \), (or \( \Omega = 0 \)) we find (after some effort) the relationship, (light-cone cuts) [11, 12],

\[ u = G_F(t, \zeta, \bar{\zeta}) \equiv \xi^a(t) \tilde{t}^a(t) = \frac{\sqrt{\bar{\Omega}}}{2} \xi^a(t) + \frac{1}{2} \zeta(t) \gamma^a_{\mu}(\zeta, \bar{\zeta}), \]

(9)

that describes, in Bondi coordinates, the intersection of the null cones, apex on \( \xi^a(t) \), with \( \mathcal{I}^+ \). Equation (9), thus defines a one real parameter, \( t \), family of “slicings” or cuts, \( (S^2) \) of \( \mathcal{I}^+ \).

At a point on \( \mathcal{I}^+ \), \( (u, \zeta, \bar{\zeta}) \), where the two null vectors \( \tilde{t}^a \) and \( \tilde{\xi}^a(t) \) meet, the null angle [3] (on their past light-cone) between them (stated in stereographic coordinates, \( L \) and \( \bar{L} \)), is given by

\[ L = \partial G_F, \quad \bar{L} = \bar{\partial} G_F \]

(10)

Asymptotically, the null vectors (and associated tetrads), \( \tilde{t}^a \) and \( \tilde{\xi}^a(t) \), are related by

\[ \tilde{t}^a = \tilde{t}^a + bm^a + m\tilde{t}^a, \]

\[ \tilde{\xi}^a = \tilde{\xi}^a + m^a, \]

\[ \tilde{m}^a = \tilde{m}^a, \]

\[ b = \frac{L}{r} + 0(r^{-2}). \]

(11)

This relationship is (for each value of \( \zeta, \bar{\zeta} \)) a special case of a local (tetrad) Lorentz transformation (referred to as a null rotation around \( \tilde{t}^a \)) parametrized by the value of the complex \( b \).

Comment: the description of \( \tilde{t}^a \) and \( \tilde{\xi}^a(t) \) via equation (2) is simply that of a null vector pointing in a spheres worth of directions which are given by the stereographic coordinates \( \zeta, \bar{\zeta} \). In equations (11) and (8) the same vectors are then used in a very different context: they are used to parametrically describe null geodesic congruences—as the tangent vectors to the null geodesics. In this context they define a field of null vectors—now with the change in notation \( \tilde{t}^a(t, \zeta) \) and \( \tilde{\xi}^a(t, \zeta) \) \( \Rightarrow \) \( (t^a(x^a) \) and \( \tilde{t}^a(x^a) \) ). We will not give the analytic description of \( t^a(x^a) \) and \( \tilde{t}^a(x^a) \) as it is complicated and not needed.
It is useful to distinguish between the null coordinates using equation (1), \((u, r, \zeta, \bar{\zeta})\), referred to as static null coordinates (based on a straight time-like line as apex) and those using equation (8), \((t, r^*, \zeta^*, \bar{\zeta}^*)\), referred to as dynamic null coordinates (based on an arbitrary time-like curve as apex).

An important point for us is to note that the cuts \(u = G_F(t, \zeta, \bar{\zeta})\) satisfy the (so-called) flat-space good cut equation,

\[
\partial^2 G_F = 0,
\]

namely the condition for the null normals to the ‘cuts’ to define null vectors that are shear-free [7, 3, 9]. In the following section dealing with asymptotically flat spaces, this equation will be generalized to the good cut equation,

\[
\partial^2 G = \sigma^0(G, \zeta, \bar{\zeta})
\]

where \(\sigma^0(u, \zeta, \bar{\zeta})\) is the asymptotic shear—the time integral of the Bondi news function.

For relevance and analogy with the following section, we point out that in this flat space discussion we could have taken the \(\zeta^i(\tau)\) to be a complex world-line [7, 10]. The \(L = \partial G_F\) and its complex conjugate, via equation (11), would still lead to \(\dot{r}^*\) being shear-free but now the null geodesic field would be twisting. The cuts however would be intrinsically complex and for equation (11) to make sense one must construct and use the real part of the cuts after taking the derivatives in equation (10). Though this construction is described in detail in the following section we remark that it is the analogue of the following: consider a complex parameter, \(z^*\), as an arbitrary (to be determined) complex world-line \([7, 10]\). The \(z^*\) real, \(\dot{z}^* = 0\), implying \(y = f(x)\). This is followed by substituting \(y = f(x)\) into \(L(x, y)\), leading to \(\Theta(x, y(x)) = \Theta(x, f(x))\).

The basic idea is that the real cuts are used after the differentiations. The imaginary part of the cuts contain the information of the twist and (see later) information about spin.

3. \(I^+\) of asymptotically flat space

Turning from Minkowski space to asymptotically flat spaces, we begin with \(I^+\) constructed from fixed but arbitrary Bondi coordinates, \((u, r, \zeta, \bar{\zeta})\), and Bondi tetrad, \((l^*, m^*, \bar{m}^*, n^*)\), with \(l^*\) tangent to the Bondi null surfaces, \(m^*, \bar{m}^*\), tangent to the Bondi cuts at \(I^+\) and \(n^*\) tangent to the \(I^+\) null generators, with \(n, m, \bar{m}\) parallel propagated down the null geodesics on \(u\). The radiation free-data is

\[
\sigma^0(u, \zeta, \bar{\zeta}) = \xi^0(u)Y^2_{2ij} + \ldots.
\]

We now mimic the construction of the shear-free congruences of the previous section.

It is known [7, 8] that the general regular solution of equation (13) depends on four complex parameters, \(z^*\), (defining H-space), written as

\[
u = G(z^*, \zeta, \bar{\zeta}).
\]

In other words each of the asymptotically shear-free cuts of \(I^+\) is labeled by the four H-space coordinates \(z^*\). They, in turn, can be taken as functions of the complex parameter \(\tau\) and written as \(z^* = \xi^0(\tau)\), i.e. as an arbitrary (to be determined) complex world-line in H-space. The solution (via coordinate conditions on the first four harmonics, \(l = 0, 1\)) can then be written in the form

\[
u = G^*(\tau, \zeta, \bar{\zeta}) \equiv G(\xi^0(\tau), \zeta, \bar{\zeta}) \equiv z^i l^i_a(\zeta, \bar{\zeta}) + \xi^0(z^*)Y^2_{2ij}(\zeta, \bar{\zeta}) + \ldots
\]
\[
\zeta^a = \xi^a(\tau) \tag{16}
\]
with the quadrupole term \(\xi^0\), arising from the data, \(\sigma^0 = \xi^0 Y^2_{ij}(\zeta, \bar{\zeta}) + \ldots\). From the freedom to rescale \(\tau\), i.e. \(\tau^* = F(\tau)\), we set \(\xi^0(\tau) = \tau\). This is the velocity normalization and the slow motion approximation, with \(\xi^a' \equiv v^a\), \(\xi^0' = \sqrt{1 + v^2} \approx 1\).

The asymptotically shear-free (ASF) cuts, equation (15), then become
\[
u = \frac{\tau}{\sqrt{2}} + \frac{1}{2} \xi^i(\tau) Y^0_{ij}(\zeta, \bar{\zeta}) + \xi^0(\xi^0(\tau)) Y^0_{ij}(\zeta, \bar{\zeta}) + \ldots \tag{17}
\]

(Aside: The \(\xi^0\) turn out to be the time-derivatives of the gravitational quadrupoles
\[
\xi^{ij} = \frac{\sqrt{6}}{24\pi} \left(Q^\mu_{ij\mu} + i Q^\mu_{ij\mu}\right).
\]

The idea is now to generalize the flat-space cuts, equation (9), to a one-parameter family of real cuts in the asymptotically flat situation, via equation (17). Unfortunately this does not work immediately since, in general, for arbitrary \(\sigma^0(u, \zeta, \bar{\zeta})\), the \(G^\mu(\tau, \zeta, \bar{\zeta})\) will be complex and there will essentially be no real cuts. Before we see a way around this problem we mention that IF the \(\sigma^0\) was of pure electric type \([13]\) then real cuts could be found and the situation would resemble equation (9), but modified by the \(\sigma^0\). For general type of \(\sigma^0\) the remark at the end of the previous section becomes relevant, i.e. it is the analogue of the present case.

The way around the problem of the complexity of the cuts, equation (15), is the following: treating \(\tau\) as complex, we first construct the null angles,
\[
L = \partial G(\xi^a, \zeta, \bar{\zeta}),
\]
\[
\tilde{L} = \overline{\partial G(\xi^a, \zeta, \bar{\zeta})},
\tag{18}
\]

(Aside: Recall that \(L = 0\) —both are equal for real \(G\). Using the holomorphic conjugate the congruence converges to a point in H-space but in general appears not to have any relationship to the congruence constructed from \(L\). How ever in the case of Minkowski space there is a close (albeit complicated) relationship between the complex conjugate and holomorphic construction \([10]\).

Next, for the reality conditions, using
\[
\tau = t + i \lambda, \tag{20}
\]
we decompose \(G^\mu(\tau, \zeta, \bar{\zeta})\) into its real and imaginary parts,
\[
G(\xi^a(\tau), \zeta, \bar{\zeta}) = G_R(\xi^a(t, \lambda), \zeta, \bar{\zeta}) + i G_I(\xi^a(t, \lambda), \zeta, \bar{\zeta}).
\]

By setting \(G_I \equiv G_I(\xi^a(t, \lambda), \zeta, \bar{\zeta}) = 0\) and solving it for \(\lambda\), i.e.
\[
\lambda = \Lambda(t, \zeta, \bar{\zeta}), \tag{22}
\]
and substituting \( \lambda \) into \( G_R \), we have the one real parameter family of real cuts,

\[
u_G = G_R(\xi^\mu(t, \Lambda), \xi^\nu(t, \Lambda), \zeta, \bar{\zeta}). \tag{23}\]

This construction can be done under fairly general conditions assuming \( \partial G / \partial \lambda \neq 0 \).

With the \( L \) and \( \bar{L} \), of equation (18) evaluated on the real cuts, the associated real but twisting asymptotically shear-free null vector field is,

\[
l^a = l^a - \frac{L}{r}m^a - \frac{\bar{L}}{r}m^a + 0(r^{-2}). \tag{24}\]

We have the situation in asymptotically flat spaces that is totally analogous to the situation we had in Minkowski space. We refer to the \( \mathbb{H} \)-space coordinates, \( \zeta^a \), as complex Minkowski-like coordinates, since in the flat-limit they are the complexified Minkowski coordinates. The \( \mathcal{I}^+ \) coordinates associated with these Minkowski-like coordinates i.e. the \( (t, \zeta, \bar{\zeta}) \), will be referred to as Minkowski-like cuts.

For the reality considerations, we have two separate cases: (1) the ‘dynamic’ choice of an arbitrary \( \xi^a(t) \), in equation (17), to be determined by choosing it to be the complex center of mass or (2) by the choice of the ‘static’ Lorentzian-like \( \mathcal{I}^+ \) coordinates, i.e. by taking (a ‘straight’ \( \mathbb{H} \)-space world-line) \( \xi^a = \hat{\tau} \delta^a_0 \), for the cuts:

\[
u_G = \frac{\hat{\tau}}{\sqrt{2}} + \xi^a(\xi^0(\hat{\tau}))Y^2_{ij}(\zeta, \bar{\zeta}) + \ldots \tag{25}\]

If we assume that both \( \lambda \) is small and the slow motion approximation, these constructions can be explicitly carried out. They lead, via the assumed form of the asymptotic shear,

\[
\sigma^0 = \xi^a(u)Y^2_{ij}(\zeta, \bar{\zeta}) + \ldots \tag{26} \]

to the linearized expressions (that we will need),

no. 1, center of mass coordinates

\[
\lambda = \Lambda(t, \zeta, \bar{\zeta}) \equiv \frac{\sqrt{2}}{2} \xi^0(t)Y^0_{ij}(\zeta, \bar{\zeta}) - \sqrt{2}\xi^0(t)Y^0_{ij}(\zeta, \bar{\zeta}), \tag{27}\]

\[
u_G = G_R = \frac{t}{\sqrt{2}} - \frac{1}{2}\xi^0(t)Y^0_{ij}(\zeta, \bar{\zeta}) + \xi^0(t)Y^0_{ij}(\zeta, \bar{\zeta}), \tag{28}\]

\[
\xi^a(u) = \xi^0(u) + i\xi^0(u). \tag{29}\]

no. 2, ‘static’ Lorentzian-like coordinates

\[
\tau = \hat{\tau} + i\hat{\lambda} \tag{30}\]

\[
\zeta = \hat{\lambda}(\hat{\tau}, \zeta, \bar{\zeta}) \equiv -\sqrt{2}\xi^0(\hat{\tau})Y^0_{ij}(\zeta, \bar{\zeta}), \tag{31}\]

\[
u_G = G_R = \frac{\hat{\tau}}{\sqrt{2}} + \xi^0(\hat{\tau})Y^0_{ij}(\zeta, \bar{\zeta}). \tag{32}\]

We have here, equations (28) and (33), the coordinate transformation from Bondi coordinates, \( (u_R, \zeta, \bar{\zeta}) \) to the two real ASF coordinate systems on \( \mathcal{I}^+ \), namely \( (t, \zeta, \bar{\zeta}) \) and \( (\hat{\tau}, \zeta, \bar{\zeta}) \).
4. Review and further developments

As mentioned earlier, much of the material described here will involve functions or structures that ‘live’ on $I^+$ and originate with the leading Weyl and Maxwell tensor components. Our major interest will center on the asymptotic behavior, the physical meaning, the evolution and transformation properties of these tensors. Using Bondi coordinates and tetrad, the five complex self-dual NP components of the Weyl tensor and three complex Maxwell components are [4]:

\[
\begin{align*}
\Psi_0 &= -C_{abcd} m^a \Gamma^b m^d = -C_{1313}, \\
\Psi_1 &= -C_{abcd} n^a \Gamma^b m^d = -C_{1213}, \\
\Psi_2 &= -C_{abcd} m^a \rho^b m^d = -C_{1342}, \\
\Psi_3 &= -C_{abcd} n^a \rho^b m^d = -C_{1242}, \\
\Psi_4 &= -C_{abcd} n^a \rho^b \rho^d = -C_{2442}.
\end{align*}
\]

\[
\begin{align*}
\phi_0 &= F_{ab} m^b, \\
\phi_1 &= \frac{1}{2} F_{ab} (n^b + m^b m^a), \\
\phi_2 &= F_{ab} n^b m^a.
\end{align*}
\]

From the radial asymptotic Bianchi identities and Maxwell equations, we have their asymptotic behavior (the ‘peeling’ theorem) [4]:

\[
\begin{align*}
\Psi_0 &= \Psi_0^0 r^{-5} + O(r^{-6}), \\
\Psi_1 &= \Psi_1^0 r^{-4} + O(r^{-5}), \\
\Psi_2 &= \Psi_2^0 r^{-3} + O(r^{-4}), \\
\Psi_3 &= \Psi_3^0 r^{-2} + O(r^{-3}), \\
\Psi_4 &= \Psi_4^0 r^{-1} + O(r^{-2}).
\end{align*}
\]

\[
\begin{align*}
\phi_0 &= \phi_0^0 r^{-3} + O(r^{-4}), \\
\phi_1 &= \phi_1^0 r^{-2} + O(r^{-3}), \\
\phi_2 &= \phi_2^0 r^{-1} + O(r^{-2}).
\end{align*}
\]

with

\[
\begin{align*}
\Psi_n^0 &= \Psi_n^0 (u, \zeta, \overline{\zeta}), \\
\phi_n^0 &= \phi_n^0 (u, \zeta, \overline{\zeta}).
\end{align*}
\]

The non-radial Bianchi identities and Maxwell equations yield the evolution equations for these leading terms (our basic variables):

\[
\begin{align*}
\dot{\Psi}_2 &= -\partial \Psi_2^0 + \sigma^0 \Psi_3^0 + k \phi_2^0 \phi_2^0, \\
\dot{\Psi}_1 &= -\partial \Psi_2^0 + 2 \sigma^0 \Psi_3^0 + 2 k \phi_1^0 \phi_2^0.
\end{align*}
\]
\begin{align}
\dot{\Psi}_0^0 &= -\partial_0\Psi_0^0 + 3\sigma_0^0\Psi_2^0 + 3k\phi_0^0\phi_2^0, \quad (41) \\
k &= 2Gc^{-4}, \quad (42) \\
\dot{\phi}_1^0 &= -\partial_0\phi_2^0, \quad (43) \\
\dot{\phi}_0^0 &= -\partial_0\phi_1^0 + \sigma_0^0\phi_2^0. \quad (44)
\end{align}

The \textit{u}-derivative is denoted by the overdot.

After the final coordinate transformation to the ‘static’ Minkowski-like coordinates and ‘static’ Minkowski-like cuts, the equations (39)–(44) are seen to contain our classical (mechanical) equations of motion.

The quantity \(\sigma^0(u, \zeta, \bar{\zeta})\) (referred earlier to as the asymptotic shear), is the leading term in the shear of the geodesic congruence, \(l\); i.e.

\[
\sigma = r^{-2}\sigma^0(u, \zeta, \bar{\zeta}) + O(r^{-4}),
\]

while its first \textit{u}-derivative is the Bondi news function. We consider \(\sigma^0(u, \zeta, \bar{\zeta})\) as a free function but take it only up to the quadrupole terms, equation (26). It, as such, plays a significant role in what later follows. From the spin-coefficient equations one finds that

\begin{align}
\psi_3^0 &= \partial(\sigma^0), \\
\psi_4^0 &= -(\sigma^0)^-, \quad (45)
\end{align}

4.1. Physical identifications

From the definition of the \textit{mass aspect}, \(\Psi\), (real from field equations) by

\[
\Psi = \overline{\Psi} \equiv \psi_2^0 + \partial^2\sigma^0 + \sigma^0(\sigma^0), \quad (46)
\]

Bondi and Sachs defined the asymptotic mass, \(M_B\), and 3-momentum, \(P_B^3\) as the \(l = 0\) and \(l = 1\) harmonic coefficients of \(\Psi\). Specifically,

\textbf{Definition 1.}

\[
\Psi = \Psi^0 + \Psi^0Y_1^0 + \Psi^0Y_2^0 + .
\]

\[
\Psi^0 = -\frac{2\sqrt{2}G}{c^2}M_B \quad (47)
\]

\[
\Psi^1 = -\frac{6G}{c^2}P_B^3 \quad (48)
\]

By rewriting equation (39), replacing the \(\psi_2^0\) by \(\Psi\) via equation (46), we have

\[
\dot{\Psi} = (\sigma^0)(\sigma^0) + k\phi_0^0\phi_2^0.
\]

Immediately we have the Bondi mass/energy loss theorem:

\[
M_B = -\frac{c^2}{2\sqrt{2}G} \int ((\sigma^0)(\sigma^0) + k\phi_0^0\phi_2^0)d^2S < 0, \quad (50)
\]
the integral taken over the unit 2-sphere at constant \( u \). This relationship is at the basis of almost all the contemporary work on the detection of gravitational radiation.

**Definition 2.** Though it has been a controversial subject and there is no general agreement, we adopt the definition (which comes from linear theory) of the complex mass dipole moment, \((D^l_{\text{complex}}) = D^l_{\text{(mass)}} + ic^{-1}J^l\), as the \( l = 1 \) harmonic component of \( \Psi^0_1 \),

\[
\Psi^0_1 = -6\sqrt{2}Gc^{-2}(D^l_{\text{(mass)}} + ic^{-1}J^l)Y^1_{1i} + .... \tag{51}
\]

\( D^l \) the mass dipole and \( J^l \), the total angular momentum, as ‘seen’ at null infinity. The main defense of this definition is that it works extremely well.

**Definition 3.** Our identification—which is standard—for the complex E and M dipole, (electric and magnetic dipoles, \((D^l_{\text{Elec}} + iD^l_{\text{Mag}}})\) as the \( l = 1 \) harmonic component of \( \phi^0_0 \) is:

\[
\phi^0_0 = 2(D^l_{\text{Elec}} + iD^l_{\text{Mag}})Y^1_{1i}, \tag{52}
\]

\[
D^l_{\text{ElecM}} = (D^l_{\text{Elec}} + iD^l_{\text{Mag}}) = q\xi^i. \tag{53}
\]

Later we will connect these three physical identifications with the complex center of mass. Actually, for the general situation there is the independent complex center of charge. Here, however for simplicity, we assume that they coincide. This is not necessary but is a simplifying restriction [7].

**Comment.** For later use we note that from the asymptotic Maxwell equations, equations (43) and (44), we have that [7]

\[
\phi^0_0 = q + \sqrt{2}q\xi^iY^0_{1i} + Q_1 + .... \tag{54}
\]

with the \( Q_s \) representing known quadrupole terms and \( q \) the Coulomb charge.

**Comment.** The indices \( 0,i,j,k, ... \) have direct geometric meaning coming from the position or tangent vectors in H-space. They also can be interpreted as vectors in a representation space of the Lorentz group that has its origin via a subgroup of the BMS group acting on \( I^+ \). [7]

### 4.2. Modus operandi

#### 4.2.1. Review and perspective.
Before starting the detailed discussion of our main results—which are out of the mainstream, are complicated and are not easy to follow (involving many different transformations, Clebsch–Gordan expansions, new variables, ...)—we thought that a brief review and an attempt at giving perspective would be of value even though it causes a slight redundancy.

We recall from section 2 that we are concerned with three types of null geodesic congruences in Minkowski space: (1) those based on the null generators of light cones with apex on a time-like geodesic, (2) Those based on the light-cones with apex on an arbitrarily time-like world-line, and (3) those real congruences that are constructed from the light-cones of ‘complex’ world-lines in complex Minkowski space. All three congruences are shear-free (which is fundamental for us) while the first two are twist free, the third has non-vanishing twist. All three can be constructed as the null normals to a one parameter family of slices or cuts of \( I^+ \). The cuts, described in Bondi coordinates, are given by a function of the form \( u = G(t, \zeta, \bar{\zeta}) \) with our three cases expressed by,
**case 1.** \( u = \frac{t}{\sqrt{2}} \)  

**case 2a.** \( u = \xi^a(t) I_a(\zeta, \overline{\zeta}) = \frac{\sqrt{2}}{2} \xi^0(t) + \frac{1}{2} \xi^i(t) Y^0_i(\zeta, \overline{\zeta}), \)  

\[
\eta_{ab} \xi^a \xi^b = 1, \quad \xi^a \text{ real} \tag{57}
\]

**case 2b.** \( u = \xi^a I_a(\zeta, \overline{\zeta}) \)  

\[
\eta_{ab} \xi^a \xi^b = 1, \quad \xi^a(\tau), \; \tau \text{ and } u \text{ complex} \tag{59}
\]

All three cases satisfy the flat-space good cut equation, \( u = G(\tau, \zeta, \overline{\zeta}), \; \delta^2 G = 0, \) i.e. the shear-free condition.

Case 1, a special case of two or three, is intended to represent ‘static’ Minkowski-like null coordinates while cases case 2a and 2b. Represent general null coordinates in real and complex Minkowski space. For case 2b, we first construct the real congruence and then restrict the \( \tau \) and \( u \) to be real. The world-line, \( \xi^a(t), \) eventually—in the generalization to the asymptotically flat situation—becomes our complex center of mass line (see definition 2) on which the complex Minkowski space. For the asymptotically flat space-times, we mirror the flat-case construction looking for slicing (cuts) whose normal null geodesic congruences are now asymptotically shear-free (ASF), i.e. satisfy the good cut equation, \( \delta^2 G = \sigma^0 = \xi^i(t) Y^2_i(\zeta, \overline{\zeta}) \) + ... . The solutions all have a form similar to the flat cases but modified by the \( \sigma^0, \) i.e.

\[
u = \zeta^a I_a(\zeta, \overline{\zeta}) + \xi^i(t) Y^0_i(\zeta, \overline{\zeta}) + ..., \tag{60}
\]

with

**case 1.** \( \zeta^a = \overline{\tau} \delta^a_0 \)  

**cases 2a. and b.** \( \zeta^a = \xi^a(\tau) = (\tau, \; \xi^i(\tau)) \)  

(If \( \sigma^0 \) is of ‘electric’ type the associated congruence is asymptotically twist-free, i.e. case 2a.) At this stage \( \xi^i(\tau) \) is an arbitrary complex three-vector in H-space—to be determined.

The following—intended to clarify our procedure—is almost a perfect analogue of our planned construction:

**Aside.** Consider a charge distribution, an origin and the associated electric dipole \( \vec{D}_E. \) A shift of origin by \( \vec{R}, \) so that \( \vec{\tilde{\tau}} = \vec{\tau} - \vec{R}, \) leads to the dipole transformation, \( \vec{D}^*_E = \vec{D}_E - q\vec{R}. \) Setting \( \vec{D}^*_E = 0, \) defines the center of charge by \( \vec{R} = \vec{D}_E / q \) or expresses \( \vec{D}_E = q\vec{R}. \) Formally this can be generalized to the complex center of charge, by including the magnetic dipole, \( \vec{D}_M, \) via the complex dipole moment, \( \vec{D}_C = \vec{D}_E + i\vec{D}_M. \) The position of the complex center of charge then is defined by \( \vec{R}_{C} = \vec{D}_C / q \) and the complex dipole moment by \( \vec{D}_C = q\vec{R}_{C}. \)

For our construction we start with definition 2, the complex mass dipole (sitting in the far field Weyl tensor) with Bondi slicings (i.e. functions of \( u \)). Then transform it to the case 2 slicing (now function of the constant \( \tau \) slicing ). Setting the transformed complex mass dipole to zero then determines the complex center of mass position, \( \xi^a(\tau), \) the analogue of \( \vec{R}_{C}. \) This in turn allows us to express the original (in the Bondi frame) complex mass dipole in terms...
of the $\xi^i$. Finally we transform the dipole to the ‘static’ Lorentzian-like slicing, i.e. case 1, and in the process convert $u$ and $\tau$ to the real. Using the asymptotic Bianchi identities and the Bondi–Sachs energy-momentum definitions, we find most of the standard classical mechanics basic relationships, simply sitting and waiting to be discovered, in the Weyl tensor.

4.2.2. Execution. Our operation now consists of taking the Weyl tensor components (mainly $\Psi^0_0$ and $\Psi^0_1$) and transferring them from the Bondi tetrad $(l^a, m^a, n^a, \bar{n}^a)$ to the dynamic Minkowski-like tetrad $(l^a, m^a, \bar{m}^a, n^a)$ via

\begin{align*}
l^a &= l^a + b m^a + \bar{b} m^a + 0(r^{-2}), \\
m^a &= m^a + b n^a, \\
\bar{n}^a &= n^a, \\
b &= -\frac{L}{r} + 0(r^{-2}).
\end{align*}

and

\begin{align*}
\Psi^0_0 &= \Psi^0_0 - 4L\Psi^0_1 + 6L^2\Psi^0_2 - 4L^3\Psi^0_3 + L^4\Psi^0_4, \\
\Psi^0_1 &= \Psi^0_1 - 3L\Psi^0_2 + 3L^2\Psi^0_3 - L^3\Psi^0_4, \\
\Psi^0_2 &= \Psi^0_2 - 2L\Psi^0_3 + L^2\Psi^0_4, \\
\Psi^0_3 &= \Psi^0_3 - L\Psi^0_4, \\
\Psi^0_4 &= \Psi^0_4.
\end{align*}

The $L$ and its complex conjugate, $L$, are determined by equation (18) with $G$ given by, equation (17)

\[ G = \frac{\tau}{\sqrt{2}} - \frac{1}{2} \xi(\tau)Y^0_0(\zeta, \bar{\zeta}) + \xi^u(\xi(\tau))Y^u_0(\zeta, \bar{\zeta}) + .. \]

We perform this tetrad rotation twice: first with the dynamic Minkowski-like coordinates, with the $\xi^a(\tau)$ to be determined by the center of mass condition and then again for the ‘static’ Minkowski-like coordinates, $\xi^a = \tau \delta^a_0$, done to put the final results in a ‘static’ Lorentzian-like frame.

In addition, in each case, we must change the $I^+$ coordinates from Bondi to the $(\bar{\tau}, \zeta, \bar{\zeta})$ slicing via equation (15).

Aside. Eventually the $(\bar{\tau}, \zeta, \bar{\zeta})$ will be changed to the real $I^+$ coordinates $(t, \zeta, \bar{\zeta})$.

These transformations will be implemented in several stages.

Stage I. We go from the Bondi coordinates to the dynamic Minkowski-like situation via equation (17). After both the tetrad and coordinate change, for constant $\tau$, we concentrate on the $l = 1$ component, i.e. $\Psi^0_1$ of $\Psi^0_1$. Using the definition of the complex mass dipole moment, the complex center of mass ‘position’ is determined (defined) by setting the $\Psi^0_1$ equal to zero. This allows us to determine the three components of $\xi^i$, (the center of mass), with $\xi^0 = \tau$.

Stage II. With the $\xi^a(\tau)$ now known we can go back and find the $l = 1$ spherical harmonic component of the Bondi $\Psi^0_1(u)$ in terms of the complex center of mass. Using, in the Bondi frame, the evolution equations, equations (39)–(44), with the just found center of
mass position, $\xi^a(u)$, many of our mechanical equations are obtained but now expressed as functions of our $u$.

**Stage III.** With all these relations, including the complex center of mass and the dynamics—expressed in the Bondi frame—we do the transformation back to the Minkowski-like system but now to the ‘static’ frame to obtain our final results. The ‘static frame’ mimics the ordinary flat-space Lorentzian frame.

The calculations involved in these three stages were rather involved. They included the coordinate and tetrad transformations between the Bondi frame and the Minkowski-like frames several times, they often involved Taylor expansions up to the quadrupole terms and the frequent use of Clebsch–Gordan expansion of the spherical harmonics products. Since much of this has been completed and appeared in published and refereed literature, [7], we will not redo them but simply refer to those results for our present use.

5. Results

5.1. Stage I

We begin with an asymptotically flat space-time in a Bondi coordinate and tetrad system (previous section) and perform the null rotation, equation (63), to the null vector field

\[
\begin{align*}
I^a &= f^a + bm^a + \tilde{b}m^a + 0(r^{-2}), \\
m^a &= m^a + bm^a, \\
n^a &= n^a,
\end{align*}
\]

\[b = \frac{L}{r} + 0(r^{-2}).\]  

(70)

with $L = \partial G(\xi(\tau), \zeta, \bar{\zeta})$ and its complex conjugate, $\bar{L} = \overline{G}$, which eventually—when we go to real slicings—yields the real null vector field. The $G$ is given by equation (69),

\[u = G = \frac{\tau}{\sqrt{2}} - \frac{1}{2} \xi'(\tau)Y^0_0(\zeta, \bar{\zeta}) + \xi^0(\xi(\tau))Y^2_0(\zeta, \bar{\zeta}) + \ldots\]  

(71)

with $\xi^0(\tau)$ to be now determined. The relevant (for us) Weyl tensor component, $\Psi^0$, transforms, equation (65), as

\[\Psi^a = \Psi^0_0 - 3L\Psi^0_2 + 3L^2\Psi^0_3 - L^3\Psi^0_4.\]

Considering $L$ and $\sigma^0$ as first order and $M_B$ in equation (46), as zero order, with the mass aspect $\Psi$,

\[
\begin{align*}
\Psi &= \Psi^0_1 + \Psi^0_1 Y^0_0 + \Psi^0_2 Y^0_2 + \ldots, \\
\Psi^0 &= -\frac{2\sqrt{2}G}{c^2} M_B \\
\Psi' &= -\frac{6G}{c^3} P^0_0
\end{align*}
\]

we have

\[\Psi^a = \Psi^0_1 - 3L(\Psi - \bar{\partial}^3\sigma^0).\]  

(73)
Our procedure for finding the complex center of mass now centers on equation (73). The right-side, which is a function of both $u$ and $\tau$, is transformed to a function of only $\tau$ via equation (71). All the variables on the right side are then expanded in spherical harmonics and simplified by Clebsch–Gordon expansions. We separate out and set to zero the three $l = 1$ harmonics on the right side, i.e. we force $\Psi^1_{\tau}=0$ for constant $\tau$ slices. This is a lengthy and difficult task and approximations are needed.

From this result there are two things that could be done: (1) from the three $l = 1$ terms we could solve for the three $\xi^i(\tau)$ or, as we do, (2) go backwards and solve for the original $\Psi^0_0$ but now in terms of the center of mass coordinates, $\xi_i(\tau)$.

After considerable work we have $\Psi^0_1$, found first as a function of $\tau$ and then finally, via the (approximate) inverse function to equation (71), i.e.

$$\frac{\tau}{\sqrt{2}} = u + 1 \xi(u) Y_{11}^i(\zeta, \bar{\zeta}) + \xi^j(\xi^i(u)) Y_{0}^j(\zeta, \bar{\zeta}) + ...$$

then expressed as a function of $u$. After its spherical harmonic expansion, we finally have the $l = 1$ harmonic coefficient, $\Psi^0_1$—(definition 2), of the complex mass dipole, namely

$$\Psi^0_1 = -\frac{6\sqrt{2}G}{c} M_B \xi^i + \frac{6\sqrt{2}G}{c^3} P_{B}^{k} \epsilon_{ikj} - \frac{576G}{5c^3} P_{B}^{k} \epsilon_{ikj} + i \frac{16\sqrt{2}}{5c} \xi_{j}^{\alpha} \epsilon_{jk}$$

The Bondi–Sachs mass–momentum, definition 1, has already been used.

Considering that many of the quadrupole terms involve high time derivatives (effectively high powers of $c^{-1}$) and considering quadrupole–quadrupole interactions as weak, we approximate $\Psi^0_1$ simply as

$$\Psi^0_1(u) = -\frac{6\sqrt{2}G}{c^2} M_B \xi^i + \frac{6\sqrt{2}G}{c^3} P_{B}^{k} \epsilon_{ikj}...$$ (74)

5.2. Stage 2

Equating our definition 2

$$\Psi^0_1 = -6\sqrt{2}Gc^{-2}(D^\prime_{(\text{mass})} + ic^{-1}J^\prime)...$$ (75)

of the complex mass dipole with equation (74), using

$$\xi^i = \xi^i_R + i\xi^i_I,$$ (76)

we obtain our

Result:1—Dipole and angular momentum

$$D^\prime_{(\text{mass})} = M_B \xi^i_R - c^{-1}P_{B}^{k} \epsilon_{ikj} +...,$$ (77)

$$J^\prime = cM_B \xi^i_I + P_{B}^{k} \epsilon_{ikj} +...$$ (78)

or

$$\vec{D}_{(\text{mass})} = M_B \vec{r} + c^{-2}M_B^{-1} \vec{P}_{B} \times \vec{S}.$$ (79)
The first term in $D^{\text{mass}}_i$ is the standard dipole definition while the second term is identical to a dipole term in the relativistic angular-momentum tensor $[14, 16]$. The expression for angular momentum has (1), the intrinsic spin $\vec{S}$ (the same as for the Kerr metric) $[7]$ and (2) the conventional orbital angular momentum term.

Continuing, we substitute equation (75) into the evolutionary Bianchi Identity, equation (40),

$$\dot{\Psi}^0_1 = - \partial \Psi^0_2 + 2 \sigma^0 \Psi^0_3 + 2 k \phi^0_1 \phi^0_2.$$  

(83)

Using definition 1 and equation (54), we find, directly from the real part (with no manipulation or derivation), that

Result: 2—Kinematic linear momentum

$$P^i_B = M_B \dot{\xi}^i_R - \frac{2 q^2}{3 c^3} \dot{\xi}^i_R + \text{H.O.}$$  

(84)

$\text{H.O.} = \text{quadrupole and higher order terms.}$  

(85)

We obtain, for the Bondi-Sachs momentum, the familiar kinematic $M_B \vec{v}$ term and the term familiar from electrodynamics, the radiation reaction contribution to the linear momentum.

From the imaginary part of the Bianchi identity we have the angular momentum loss equation;

Result: 3—Angular momentum conservation

$$J^i = - \frac{2 q^2}{3 c^3} \dot{\xi}^i_R + \frac{2 q^2}{3 c^3} (\dot{\xi}^j_R \ddot{\xi}^k_R + \dot{\xi}^j_R \ddot{\xi}^k_R) \epsilon_{kji} + \text{H.O.}$$  

(86)

There are several things to note: (1) the first term on the right side can be moved to the left, which simply changes the definition of $J^i$ to

$$J^i = J^i + \frac{2 q^2}{3 c^3} \dot{\xi}^i_R = \frac{2 q^2}{3 c^3} (\dot{\xi}^j_R \ddot{\xi}^k_R + \dot{\xi}^j_R \dddot{\xi}^k_R) \epsilon_{kji},$$  

(87)

i.e. adding an electromagnetic part to the spin term $\vec{S}$, and (2) we have exactly the Landau and Lifschitz $[15]$ expression as the special case of equation (87 with the $\xi^i_R$ terms omitted.

Finally substituting the relevant terms into the evolutionary Bianchi Identity, equation (39),

$$\dot{\Psi}^0_2 = - \partial \Psi^0_3 + \sigma^0 \Psi^0_4 + k \phi^0_1 \phi^0_2,$$  

(88a)

we have, first for the $l = 0$ harmonic coefficient, the (Bondi) mass loss expression but now including the well known (classical) electromagnetic energy losses, i.e.

Result: 4—Energy loss

$$M_B = - \frac{G}{3 c^3} (Q^{\text{Mass}}_{\text{Max}} Q^{\text{Mass}}_{\text{Max}} + Q^{\text{Spin}}_{\text{Spin}} Q^{\text{Spin}}_{\text{Spin}}) - \frac{4 q^2}{3 c^3} (\ddot{\xi}^j_R \ddot{\xi}^j_R + \dddot{\xi}^j_R \dddot{\xi}^j_R)$$  

(89)
\[-\frac{4}{45c^7}(Q_E^{jk}Q^j_E^{\cdot\cdot} + Q_M^{jk}Q^j_M^{\cdot\cdot}). \tag{90}\]

The first term is the standard Bondi quadrupole mass loss (now including the spin-quadrupole contribution to the loss—maybe new), the second and third terms are the classical E and M dipole and quadrupole energy loss—including the correct numerical factors. Note that this is sitting in the Bianchi Identities—with no derivation—arising simply from the Ricci tensor expressed via the Maxwell stress tensor.

The \(l = 1\) terms lead to the momentum loss expression,

Result: 5—Newton’s second law

\[\dot{P}_B^i = F^i_{\text{recoil}} \tag{91}\]

where \(F^i_{\text{recoil}}\) is composed of many non-linear radiation terms involving the time derivatives of the gravitational quadrupole and the E and M dipole and quadrupole moments. These terms are known and given [7] but not relevant to us now. Instead we substitute equation (84) into equation (91) leading to Newton’s second law;

\[M_B\ddot{\xi}_R^i = F^i \equiv M_B\dot{\xi}_R^i + 2\frac{q^2}{3c^3}\xi^{\cdot\cdot}_{R}^i + F^i_{\text{recoil}}. \tag{92}\]

Result: 6—Rocket force and radiation reaction force

We find this rather astonishing—we have exactly the classical mechanics standard rocket mass loss expression and the classical radiation reaction term. The last term is just the momentum recoil force—also known explicitly.

In the context of these results we mention two further automatic results:

1. From earlier results,

\[\xi_R^i = \text{center of mass position}\]
\[S^i = Mc\xi_R^i = \text{Spin—Angular momentum}\]
\[D_M^i = q\xi_R^i = \text{Magnetic dipole Moment}\]

and the classical classical gyromagnetic ratio \(\gamma = \frac{q}{2Mc}g\), we see

\[\gamma = \frac{D_M^i}{L_{\text{spinang.mom}}} = \frac{q\xi_R^i}{M_Bc\xi_R^i} = \frac{q}{M_Bc}, \tag{93}\]

and discover the Dirac value of the \(g\)-factor to be \(g = 2\).

2. In classical relativistic mechanics [16] one has the definition of the relativistic angular momentum tensor, \(M^{ab}\),

\[M^{ab} = L^{ab} + S^{ab}\]
\[L^{ab} = 2MX^{[\alpha}V^{\beta]}\]
\[S^{ab} = -\gamma^{abcd}S^c_{\cdot\cdot\cdot\cdot}V_d, \quad S^c_{\cdot\cdot\cdot\cdot}V^c = 0\]

so that

\[M^i = L^i + S^i = M(X^iV^j - V^iX^j) - \epsilon^{ijk}(S^c_{\cdot\cdot\cdot\cdot}V_0 - V_kS^0_c) \tag{94}\]
Using our results, \( S' = cS' = Mc^j_1 \), \( S_0 = 0 \), \( V_0 \sim 1 \), \( V_k \sim 0 \) and multiplying by \( \epsilon_{ijk} \), we have agreement with our equation (78).

Then from

\[
M^0_i = L^0_i + S^0_i = 2MX^0_i \xi - \xi^0_{ijk} S^j V_k
\]

we have agreement with our equation (77), i.e. with our \( P \times S \) term.

The relativistic angular momentum tensor (unrelated to physical space-time) is also sitting quietly and unobserved in our Weyl tensor.

5.3. Stage 3

Going from the Bondi slicings of stage 2 to the ‘static’ real Minkowski-like slicing of stage 3 is easy. From equation (61), we see that the \( l = 1 \) harmonics are now missing from the coordinate transformation

\[
\tilde{\tau} = \tilde{t} + \tilde{\lambda}
\]

\[
\tilde{\lambda} = \tilde{\lambda}(\tilde{t}, \zeta, \bar{\zeta}) \equiv -\sqrt{2} \xi^j_{\tilde{t}}(\tilde{t}) Y^0_{2j}(\zeta, \bar{\zeta}),
\]

\[
u_R = \frac{i}{\sqrt{2}} + \xi^j_{\tilde{t}}(\tilde{t}) Y^0_{2j}(\zeta, \bar{\zeta})...
\]

From our approximations, i.e. disregarding the quadrupoles, we can simply replace our \( u \) with \( \tilde{t} \), so that all our results, 1–6, remain unchanged. We emphasize that these results are for real time and real coordinates even though most of the calculations involved complex variables.

6. Discussion

Several remarks are in order.

1. The present work is the third in a developing series of papers dealing with unusual structures hiding on \( I^+ \). In the earlier work [7, 14] the coordinates as well as the time (for the classical equations of mechanics), were, in general complex—here they are real. Furthermore, the \( I^+ \) cuts were complex and were open to the perplexing question ‘What was the significance of the complex Bondi slicing? Why was their use important?’ Here we have the answer. They were not important. The important observation is that here we are using real slicings (‘static’ Minkowski-like) that mimic (or are as close to) real flat-Lorentzian slicing as possible—but yield the same results as in the Bondi slicing. This does clear away one of the enigmas associated with the earlier work [14]—why was Bondi slicing important? It was not. Since H-space has such an attractive mathematical structure (holomorphic), the issue of why are we going to the real slices and real coordinates rather than using the complex slicings of H-space has been brought up by a referee. The basic answer is that we are trying to find—or hoping to find—the physical meaning of H-space, which so resembles complex Minkowski space. It appears natural to see if it has some real aspects. It was surprising to see how close to a real appearing physical space-time we could get—with the real part related to classical mechanics and the imaginary part to spin.
2. It is the collection of results, equations (77), (78), (84), (86), (89) and (92), mimicking or imitating classical mechanics, that constitute our main contribution—in conjunction with our novel ASF Minkowski-like slicings. Nevertheless, there still remains the major enigma, ‘Why and how is it that the (now) real H-space coordinates play so accurately the role of space-time coordinates. Though they appear to be space-time equations of motion there is no physical space-time that is associated with the equations. They (as we said earlier) take place on the strange H-space labeled by the H-space coordinates. Is this just a giant coincidence? We find that difficult to believe. What possible meaning can one give to them—and to the H-space? What is the physical meaning of the decomposition, \( \xi^i(\tau) = \xi^i_{\text{R}}(\tau) + i\zeta^i(\tau) \). Why should the imaginary part of the H-space coordinate be related to spin? Is there something here of real significance? We do not know—but it is suggestive. We also observe the curious result—in our construction—that when we have spin-angular momentum, we also have the case of the twisting shear-free congruence. Twist seems happily to be related to spin.

3. There is the suggestion that H-space might play the role of some sort of observation or optical image space. Due to curvature affects one can not look straight back and expect to see distant objects along the line of sight. Is this a manifestation of this image forming? This is pure speculation and maybe not to be taken seriously.

4. Among the six results of the previous section, all but result \#3 are identical to standard classical (mechanical and E and M) equations. The special case of result \#3 where spin is omitted is exactly given in [15]. We have not been able to find any reference that contains our complete result. A related question is—if our conservation of angular momentum result is new, can we consider it to be a prediction.

5. We point out the utter simplicity of the origin of the radiation reaction term in the expression for \( P_1^\theta \). In equation (40)

\[
\dot{\psi}_1^\theta = -\ddot{\psi}_2^\theta + 2\sigma^0\psi_3^\theta + 2k\dot{\phi}_1^\theta\phi_2^\theta,
\]

the \( P_1^\theta \) is sitting in the \( \ddot{\psi}_2^\theta \), the \( q \) sits in the \( \phi_1^\theta \), while \( \ddot{\psi}_1^\theta \) is in the \( \phi_2^\theta \). The numerical factors are there. The radiation reaction term is sitting there—no assumptions, no derivation, it is waiting to be observed in the equation.

6. We emphasize several things;

a. We are using only the standard Einstein equations coupled in the standard way to the standard Maxwell equations. Furthermore, we are using the standard asymptotically flat solutions of Bondi, Sachs, Penrose and Newman–Unti—with nothing further added. We note that in this standard framework we search for shear-free null geodesic congruences via the so-called ‘good cut equation’—our main research tool. From the null surfaces associated with these congruences and their intersections with the asymptotically flat \( I^+ \), we construct -(mimicking the virtually identical flat-space construction) the one-parameter family of real cuts.

b. On these ASF cuts, using the tetrad adapted to the cuts (different from the Bondi tetrad), we defined the complex center of mass in terms of a Weyl tensor component. From that definition and that of the Bondi-Sachs mass and momentum, all of our results followed. No formal lengthy derivations were needed; the results were just sitting there in the Weyl tensor and Bianchi Identities.

c. The subtlety of some of the results were rather surprising: e.g. finding the rocket force, the Abraham–Lorentz radiation reaction force as well as the electromagnetic energy loss from both the dipole and quadrupole radiation (both of electric and magnetic type, and
all with correct coefficients) and the angular momentum loss. Some of the results clearly arise from the inclusion of the Maxwell stress-energy tensor in the Einstein–Maxwell equations. There however was no matter stress tensor used.

7. Though it is not obvious where to go next, we notice that there is a chance to expand on Penrose’s asymptotic twistor theory via twistor curves on the new family of ASF shear-free cuts of $I^+$. In the past they were defined on the associated H-space. Now it could possibly be done on the ASF cuts of the physical space itself. This must be investigated.

8. The natural question also arises—can our method of obtaining equations of motion be extended and applied to a two-body problem? At the moment we see possibilities but not good ones.

9. A mea culpa: we have not yet explored how the BMS group interacts with the present results. On the basis of a cursory study we believe that at the level of our approximations all our results remain unchanged. At a higher level there will certainly be many changes. For example the time translation of case 1. \[ \tau^a = \tilde{\tau} \delta^a_0, \] equation (7) will almost certainly not be preserved. Also, we see no difficulty in preserving our reality conditions under the BMS action or when using higher order terms.

10. And finally we fully acknowledge that we have no good idea for the physical meaning of the H-space and its coordinates except that they mirror the flat space ASF cuts—why should their real parts be appropriate for the description of the center-of-mass motion or why are their imaginary parts associated with spin? Does the H-space metric play any role? Is all this an empty accident? or is there something profound? It is sufficiently crazy and far-off the mainstream that it might well be profound. In this context, we note that instead of looking at the leading Weyl tensor terms, we could—getting the same results—just as well have used weighted integrals (powers of $r$ and spherical harmonics) over the ‘shear-free’ spheres (cuts) at $I^+$ that so resemble Minkowski space light-cone cuts. This seems to help pick out our Lorentzian-like mechanical results.

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