Attractor solutions in an interacting quintessence with varying-mass dark matter in Lyra’s geometry

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In background dynamics of a spatially flat FLRW model of the universe, we investigate an interacting scenario of quintessence scalar field as dark energy which interacts with pressure-less dust as dark matter, mass of which varies with time through scalar field in context of Lyra’s geometry. Mass of dark matter particles and the potential of the quintessence field are considered to be functions depending exponentially on scalar field. Cosmological evolution equations in this context are converted into an autonomous system of ordinary differential equations by suitable transformation of the cosmological variables into dimensionless dynamical variables and the nature of critical points are found by applying linear stability theory. We have obtained cosmologically interested critical points which describe the attractor solutions at late times. Some of which representing the accelerated scalar field-dark matter scaling solutions in attractor regime with similar order of energy densities can provide the possible solution of the coincidence problem.

PACS numbers: 95.36.+x, 95.35.+d, 98.80.-k, 98.80.Cq.
Keywords: Quintessence; Interaction; Varying mass Dark matter, Lyra’s manifold, Dynamical system analysis, Phase space, Stability

1. INTRODUCTION

In theory, present observed phenomena [1–3] of accelerating universe is realized either by introducing an exotic type matter source in the right hand side of Einstein’s field equations or by modifying the geometric part (left hand side) of field equations. This exotic type matter with huge negative pressure dubbed as Dark Energy (DE) which can provide the possible mechanism to overcome the gravitation pull and responsible for acceleration of the universe. Though, the nature of DE is completely unknown to us except its negative pressure, it is assumed that the DE to be the dominant part of the universe (as 70% of total energy density is contributed from DE). The simplest DE model is the Cosmological constant which with Cold Dark Matter (CDM) (is another mysterious component of the universe) constitutes $\Lambda$CDM model provides the mechanism best fitting the present observational data. Theoretically, it is suffering from two severe problems namely, ‘cosmological constant problem’ [4–6] and ‘coincidence problem’ [7]. Time varying DE models (dynamical DE models) such as Quintessence based on scalar field, one of most popular DE models, can solve the cosmological constant problem. Dynamical DE model based on scalar field has gained a great attention to cosmologists and there are huge scalar field DE models have been studied in the literature (see for a review [8]). Further, ‘Coincidence problem’ namely, ‘why the energy densities of DE and DM are similar today though they scale differently in their cosmic evolution’ can be solved by introducing an interaction term in dark sector. Also, since the components of DE and DM in dark sectors are the dominant source of the universe, interaction between them cannot be neglected, even there exists a small positive coupling in interaction which can support the observational constraints. In a cosmological model from the study of dynamical system, one can get ratio of energy density parameters for DE and DM to be constant depending on model parameters in attractor regime with $\omega_{\text{eff}} < -\frac{1}{3}$ can provide the mechanism to solve coincidence problem. Thus, the dynamics of DE models interacting with DM with dynamical system analysis have extensively been studied in the literature [9–24]. Recently, a detailed DE and interactions in dark sectors can be found in a review article, see for instance [25] and references therein.

An interacting DE model, where DE interacts with varying mass DM particles can also provide a mechanism to alleviate the coincidence problem. The variable mass particle scenario assumes that the mass of the CDM to be time dependent through the scalar field $\phi$. The mass of the DM particles as well as the potential of real scalar field

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can be chosen in various forms of functions on scalar field \( \phi \) such as either exponential or power-law form or both. Investigation of interacting quintessence with varying mass DM where mass of DM particles and potential of scalar field depending on exponential or power-law form of scalar field \( \phi \) have been reported in Refs. [26–28]. Further, an interacting phantom scenario with varying mass DM particles has been investigated in Ref. [29]. Recently, using Center Manifold Theory an interacting phantom with varying mass DM particles is investigated in Ref. [30].

Another approach for explaining the present acceleration of the universe is the introduction to modified gravity theory in cosmological study which arises either by modifying Hilbert action or by the Riemannian geometry. The latter is the Lyra’s geometry, first proposed by Lyra [31] in 1951, is based on the modification of geometric part of Einstein’s field equations by introducing a Gauge function into the structureless manifold removing the non-integrability condition of length of vector under parallel transport. As a result, a displacement vector field \( \beta \) arises naturally in this context. A constant displacement vector field can mimic the cosmological constant and provide the possible mechanism to describe late time acceleration without invoking the cosmological constant term in field equation while the time varying displacement field can give some interesting scenario with equation of state parameter \( \omega = +1 \) possessing the stiff matter alone. Lyra’s geometry is extensively studied [32, 33] in the context of cosmology with both the constant and time varying displacement vector field. In Ref. [34], the authors have shown that the displacement vector when interacts with pressureless DM can mimic the cosmological constant. However, in order to explain the inflation and late time acceleration scenario modification in gravity theories and introduction of various dark energy models based on scalar field, received a lot of interest now. Thus, the study of scalar field DE models would be of great interest in the context of Lyra’s geometry which may be relevant to late time acceleration models [35]. An interacting quintessence DE models has been investigated in Lyra’s manifold in Ref. [36].

Motivated from the above facts, in this paper, we shall investigate dynamical analysis of an interacting quintessence scalar field model which interacts with pressureless dark matter varying with time via a scalar field \( \phi \) in the framework of Lyra’s geometry where displacement vector field \( \beta \) is to be considered evolving with time through scale factor as \( \beta(t) \propto a^{-\gamma}(t) \). We also consider the potential term of the quintessence scalar field and mass of the DM particles dependent of scalar field \( \phi \) exponentially. We convert the cosmological evolution equations into a system of ordinary differential equations and linear stability theory is employed to study of nature of critical points. From the analysis, we obtain some cosmologically interesting crtical points which represent late time attractor solutions. Some of these attractors are scaling describing the accelerated evolution of universe at late times solving coincidence problem.

The paper is organised as follows: in section 2, we present the model of interacting quintessence in framework of Lyra’s geometry and formation of autonomous system. While section 3 comprises the autonomous system and section 4 shows cosmological implications of the model and finally a short discussion has been made to the last section 5.

## 2. MODEL OF INTERACTING QUINTESSENCE WITH VARYING MASS DARK MATTER PARTICLES IN LYRA’S MANIFOLD AND FORMATION OF AUTONOMOUS SYSTEM FOR THIS MODEL

In this section we shall discuss the interacting DE model with varying mass dark matter particles in the context of Lyra’s manifold and then discuss the formation of autonomous system from the cosmological evolution equations after transforming the cosmological variables by a suitable choice of dynamical variables.

### 2.1. The model

In the context of Lyra’s geometry which is due to the modification of Riemannian geometry by introducing a Gauge function into the structureless manifold, a displacement vector \( \psi^\mu = g^{\mu\nu} \psi_\nu \) arises naturally, and as a result of which the modified field equation reduces to the form:

\[
G_{\mu\nu} + \frac{3}{2} \psi_\mu \psi_\nu - \frac{3}{4} g_{\mu\nu} \psi^\alpha \psi_\alpha = T_{\mu\nu},
\]

where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \) is the Einstein tensor and \( T_{\mu\nu} \) is the energy-momentum tensor for matter field with perfect fluid satisfying

\[
T_{\mu\nu} = (\rho + p) u_\mu u_\nu - pg_{\mu\nu},
\]
where \( u_\mu = (1, 0, 0, 0) \) is the co-moving four-velocity vector with norm \( u^\mu u_\mu = 1 \) and \( \rho(t) \) and \( p(t) \) are energy density and thermodynamic pressure of the matter field characterised by the equation of state parameter \( \omega = \frac{p(t)}{\rho(t)} \). Consider the displacement vector as:
\[
\psi_\mu = (\frac{\beta}{\sqrt{3}} \beta(t), 0, 0, 0),
\]
where \( \beta(t) \) is the time-like time varying displacement vector. Although, time-like constant displacement vector has been introduced in literature, we are interested here to study the model with time varying displacement vector field. Here the factor of \( \frac{\beta}{\sqrt{3}} \) is taken with \( \beta \) for mathematical simplicity only. Now we consider the metric in the spatially flat, homogeneous and isotropic FLRW model of the universe:
\[
ds^2 = -dt^2 + a^2(t) \left( dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right)
\]
which in the context of Lyra’s geometry, with Eqn.(1) and Eqn.(2) leads to the modified Friedmann equations and acceleration equation as: (using natural units \( (\kappa^2 = 8\pi G = c = 1) \))
\[
3H^2 - \beta^2(t) = \rho_{Total}
\]
and
\[
-2\dot{H} - \beta^2(t) - 3H^2 = p_{Total},
\]
where \( H = \frac{\dot{a}}{a} \) is Hubble parameter defined by function of scale factor \( a(t) \). Over-dot stands for the differentiation with respect to the cosmic time \( t \). Also, the parameters \( \theta \) and \( \phi \) considered here, describe the usual azimuthal and polar angles of spherical coordinates with \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi < 2\pi \). In the above equations \( \rho_{Total} \) and \( p_{Total} \) are the total effective energy density and pressure of all matter content in the universe. We consider the universe is dominated mostly by dark sector in which pressureless dust is considered as DM and quintessence scalar field as DE. As a result, in context of Lyra’s geometry which includes \( \beta(t) \) as displacement field and with DM and DE, contributes the total energy density of the universe, then one can rearrange the total energy density and total pressure as:
\[
\rho_{Total} = \rho_m + \rho_\phi + \beta^2(t)
\]
and
\[
p_{Total} = p_\phi + \beta^2(t)
\]
which obey the continuity equation as
\[
\dot{\rho}_{Total} + 3H(\rho_{Total} + p_{Total}) = 0
\]
which is expanded with Eqn. (6) and Eqn. (7) as
\[
\dot{\rho}_m + \dot{\rho}_\phi + 2\beta \dot{\beta} + 3H(\rho_m + \rho_\phi + \beta^2 + p_\phi + \beta^2) = 0
\]
Now if we specify the contriution of energy from dark sector as combination of energy densities from scalar field and from dark matter to the universe. Since the energy related to displacement field \( \beta(t) \) is assumed not to be interacting with others, we have:
\[
\dot{\rho}_i + 3H(\rho_i + p_i) + 2\beta \dot{\beta} + 3H(\beta^2 + \beta^2) = 0
\]
where \( \rho_i = \rho_\phi + \rho_m \) and \( p_i = p_\phi \)
Now Quintessence scalar field is governed by its energy density \( \rho_\phi \) and pressure \( p_\phi \) which are combined with potential and kinetic terms as follows:
\[
\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi)
\]
and
\[
p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi),
\]
where \( V(\phi) \) is the potential function of the scalar field and \( \frac{1}{2} \dot{\phi}^2 \) is the kinetic term of the scalar field. The usual equation of state parameter for scalar field is \( \omega_\phi = \frac{p_\phi}{\rho_\phi} \) and with the total energy density \( \rho_{Total} \) and pressure \( p_{Total} \) one obtains the total (effective) equation of state parameter for the model as
\[
\omega_{eff} = \frac{p_{Total}}{\rho_{Total}} = \frac{p_\phi + \beta^2}{\rho_m + \rho_\phi + \beta^2}.
\]
Considering there is no interaction between energy density of dark sector $\rho_i$ and displacement vector field $\beta(t)$, then the equation (10) can be splitted into

$$\dot{\beta}(t) + 3H\beta(t) = 0$$

(14)

from which one can observe that $\beta(t)$ evolves like a fluid with $\beta(t) \propto a^{-3}(t)$ and energy density $\rho_\beta$ and pressure density $p_\beta$ for the displacement field $\rho_\beta = p_\beta = \beta^2$ which corresponds to equation of state for this field $\omega_\beta = +1$ is like a stiff fluid. Now, after splitting the continuity equation related to displacement field $\beta$ part, the Eqn. (10) leads to the continuity equation of dark components in dark sector as:

$$\dot{\rho}_i + 3H(\rho_i + p_i) = 0$$

(15)

where, the subscripts $i \equiv m$ stands for DM component and $i \equiv \phi$ indicates the dark energy component respectively.

Now we consider the model of varying mass dark matter in the dark sector. For this scenario, dark matter particles depend on time ‘t’ through scalar field $\phi$ and its number density must obey the following conservation equation:

$$\dot{n}_m + 3Hn_m = 0.$$  

(16)

Since the DM particles $M_m(\phi)$ is a scalar field $\phi$ dependent function, then $\phi$ dependent energy density $\rho_m$ of DM is given by

$$\rho_m(\phi) = M_m(\phi)n_m$$

(17)

Using Eqn.(16), Eqn.(17) gives the evolution equation for $\rho_m(\phi)$:

$$\dot{\rho}_m + 3H\rho_m = \frac{M_m'(\phi)}{M_m(\phi)} \ddot{\phi}\rho_m$$

(18)

which is the modified conservation equation of DM, where $' \equiv \frac{d}{d\phi}$ stands for derivative with respect to scalar field $\phi$. Since the total energy density in the dark sector is conserved from Eqn.(15) conservation equation for DE can be obtained as

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -\frac{M_m'(\phi)}{M_m(\phi)} \ddot{\phi}\rho_m$$

(19)

Therefore, the study of the varying mass DM particles is equivalent to the study of the interacting DE-DM scenario by an appropriate interaction term. In this case, the term $\frac{M_m'(\phi)}{M_m(\phi)} \ddot{\phi}\rho_m$ plays a role of interaction term in dark sector and there happens energy exchange from DM to DE if $M_m'(\phi)\ddot{\phi} < 0$ while energy flow occurs in the direction of DM from DE if $M_m'(\phi)\ddot{\phi} > 0$. An equivalent way of writing the above conservation equations for DM (Eqn.(18)) and DE (19) in terms of effective equation of state for DM and DE as follows:

$$\dot{\rho}_m + 3H\rho_m(1 + \omega^{(m)}_{eff}) = 0$$

(20)

where $\omega^{(m)}_{eff} = -\frac{M_m'(\phi)}{M_m(\phi)} \ddot{\phi} 3H$ is the effective equation of state for DM, and

$$\dot{\rho}_\phi + 3H\rho_\phi(1 + \omega^{(\phi)}_{eff}) = 0$$

(21)

where $\omega^{(\phi)}_{eff} = \omega_{\phi} + \frac{M_m'(\phi)}{M_m(\phi)} \ddot{\phi} \frac{\rho_m}{\rho_\phi}$ is the effective equation of state for DE. The evolution equation of scalar field (from Eqn. (19), using Eqn. (11) and Eqn. (12)) takes the form

$$\ddot{\phi} + V' + 3H\dot{\phi} = -\frac{M_m'(\phi)}{M_m(\phi)} \rho_m$$

(22)

where $' \equiv \frac{d}{d\phi}$ represents the derivative with respect to scalar field $\phi$. 
2.2. Autonomous system and cosmological parameters

In the model of exponential variable mass particles of DM, we consider the mass of DM is function of scalar field $\phi$ as $M_m(\phi) = M_{m0} \exp\{-\alpha \phi\}$ and the potential of the scalar field $\phi$ as $V(\phi) = V_0 \exp\{\gamma \phi\}$, where $M_{m0}$ and $V_0$ are constant and $\alpha$ and $\gamma$ are constant parameter. For qualitative analysis, we choose the following dimensionless variables

$$x^2 = \frac{\dot{\phi}^2}{6H^2}, \quad y^2 = \frac{V}{3H^2}, \quad \text{and} \quad z^2 = \frac{\beta^2}{3H^2} \quad (23)$$

which are normalized over Hubble scale. Now with the dynamical variables (in equation (23)) and exponential potential and exponential mass function of DM, the evolution equations lead to the following autonomous system of ordinary differential equations:

$$\frac{dx}{dN} = \frac{3}{2} \left[-x (1 - x^2 + y^2 - z^2) - \sqrt{\frac{2}{3}} \gamma y^2 + \frac{2}{3} \alpha (1 - x^2 - y^2 - z^2) \right],$$

$$\frac{dy}{dN} = \frac{3}{2} y \left[1 + x^2 - y^2 + z^2 + \sqrt{\frac{2}{3}} \gamma x\right],$$

$$\frac{dz}{dN} = \frac{3}{2} z \left[-1 + x^2 - y^2 + z^2\right], \quad (24)$$

where $N = \ln a$ is the e-folding parameter taken to be independent variable.

Now we obtain the cosmological parameters in terms of dynamical variables as follows:

Density parameters for quintessence scalar field (DE), dark matter and for displacement vector field $\beta$ are

$$\Omega_\phi = x^2 + y^2, \quad (25)$$

$$\Omega_m = 1 - x^2 - y^2 - z^2, \quad (26)$$

and

$$\Omega_\beta = z^2. \quad (27)$$

Effective equation of state parameter for quintessence scalar field (DE):

$$\omega_{eff}^{(\phi)} = \frac{x^2 - y^2}{x^2 + y^2} - \frac{\sqrt{\frac{2}{3}} \alpha x (1 - x^2 - y^2 - z^2)}{x^2 + y^2}, \quad (28)$$

and the effective equation of state parameter for dark matter

$$\omega_{eff}^{(m)} = \sqrt{\frac{2}{3}} \alpha x. \quad (29)$$

The effective equation of state parameter for the model is

$$\omega_{eff} = x^2 - y^2 + z^2, \quad (30)$$

and the deceleration parameter for the model is

$$q = -1 + \frac{3}{2}(1 + \omega_{eff}) \quad (31)$$

which describes the condition for acceleration: $q < 0$ i.e. $\omega_{eff} < -\frac{1}{3}$ and for deceleration: $q > 0$ i.e. $\omega_{eff} > -\frac{1}{3}$. Friedmann equation (4) gives the constraint equation in terms of density parameter for DM is

$$\Omega_m = 1 - x^2 - y^2 - z^2 \quad (32)$$

by which due to the energy condition $0 \leq \Omega_m \leq 1$, we obtain the constraints for dynamical variables in the region:

$$0 \leq x^2 + y^2 + z^2 \leq 1 \quad (33)$$
3. PHASE SPACE ANALYSIS OF AUTONOMOUS SYSTEM (24):

The critical points for the system (24) are the following

- **I. Critical Point**: $A = (\sqrt[3]{\frac{2}{3}} \alpha, 0, 0)$
- **II. Critical Point**: $B = (\sqrt[3]{1 - \frac{2}{3} z_c^2}, 0, z_c)$
- **III. Critical Point**: $C = (-\sqrt[3]{1 - \frac{2}{3} z_c^2}, 0, z_c)$
- **IV. Critical Point**: $D = \left(-\frac{\gamma}{\sqrt[3]{6}}, \sqrt[3]{1 - \frac{2}{3} z_c^2}, 0\right)$
- **V. Critical Point**: $E = \left(-\frac{\gamma}{\sqrt[3]{6}}, -\sqrt[3]{1 - \frac{2}{3} z_c^2}, 0\right)$
- **VI. Critical Point**: $F = \left(-\frac{\sqrt[3]{6}}{2(\alpha + \gamma)}, \frac{\sqrt[3]{\alpha(\alpha + \gamma) + 2}}{(\alpha + \gamma)}, 0\right)$
- **VII. Critical Point**: $G = \left(-\frac{\sqrt[3]{6}}{2(\alpha + \gamma)}, -\frac{\alpha(\alpha + \gamma) + 2}{(\alpha + \gamma)}, 0\right)$

Critical points and their corresponding physical parameters are shown in the table (I).

- Critical point $A$ corresponds to matter-scalar field scaling solution and it exists for $-\sqrt[3]{\frac{3}{2}} \leq \alpha \leq \sqrt[3]{\frac{3}{2}}$ in the phase space $x - y - z$. The ratio of energy densities DE and DM is $r = \frac{\Omega_\phi}{\Omega_m} = \frac{2\alpha^2}{3 - 2\alpha^2}$. Contribution of energy density due to displacement field $\beta$ of Lyra’s manifold to the point is absent (since, the dimensionless density parameter associated with time dependent displacement vector is zero, i.e., $\Omega_\beta = z^2 = 0$). However, the critical point becomes completely DE dominated ($\Omega_\phi \approx 1$) for $\alpha \rightarrow \pm \sqrt[3]{\frac{3}{2}}$, on the other hand it will be of DM dominated ($\Omega_m \approx 1$) for $\alpha \rightarrow 0$. There exists always decelerating universe near the critical point $A$ since $q \geq \frac{1}{2}$ always (see Table I). Since the effective equation of state for both DE and DM are same, i.e., $\omega_{eff} = \omega_{eff}^{(m)} = \frac{2\alpha^2}{3}$ in the evolution of the critical point, DE and DM behaves similarly as both the fluid show the nature of dust like fluid for $\alpha = 0$ and of stiff fluid for $\alpha = \pm \sqrt[3]{\frac{3}{2}}$ for the point. Eigenvalues of linearised Jacobian matrix are: $\{\lambda_1(A) = \gamma \alpha + \frac{3}{2} + \alpha^2, \lambda_2(A) = \alpha^2 - \frac{3}{2}, \lambda_3(A) = \alpha^2 - \frac{3}{2}\}$. The point is stable for $(-\sqrt[3]{\frac{3}{2}} \leq \alpha < 0$ and $\gamma > -\frac{2\alpha^2 - 3}{2\alpha})$ or $\left(0 < \alpha < \sqrt[3]{\frac{3}{2}} \text{ and } \gamma < -\frac{2\alpha^2 - 3}{2\alpha}\right)$. Therefore, the critical point $A$ corresponds to late time decelerated matter-scalar field scaling attractor solution under the above parameter region (see fig. 1(a)). Physically accepted scenario can be achieved by the critical point only when it comes into complete DM dominance for $\alpha = 0$ in its evolution. For this case, the point shows decelerated dust dominated ($\Omega_m = 1, \Omega_\phi = 0, \omega_{eff} = 0, q = \frac{1}{2}$) saddle-like transient nature (one eigenvalue is positive) and that can be an intermediate phase of the evolution of the universe (see fig. 3(a),3(c)).

- Set of critical points $B$ is a non-isolated set which exists for all parameters $\alpha$ and $\gamma$ in $0 \leq z_c \leq 1$ in the phase space. The set is dominated by both the scalar field (kinetic part) DE and the Lyra’s manifold displacement vector $\beta$-fluid and the ratio of dimensionless density parameters for scalar field and for displacement vector $\beta$ is $\Omega_c = \frac{1 - z_c^2}{z_c^2}$. The set will be completely DE dominated solution when $z_c = 0$. On the other hand it will be completely $\beta-$ fluid dominated for $z_c = 1$. Scalar field DE behaves as stiff fluid in nature since $\omega_{eff}^{(0)} = 1$ for the set $B$. There exists an ever decelerating universe near the set since $\omega_{eff} = 1$ and $q = 2$ (see Table I). Eigenvalues of linearised Jacobian matrix for the set of point $B$ are: $\left\{\lambda_1(B) = 3 + \frac{\sqrt[3]{6}}{2\sqrt{1 - \frac{2}{3} z_c^2}}, \lambda_2(B) = 3 - \sqrt[3]{6}\alpha \sqrt{1 - \frac{2}{3} z_c^2}, \lambda_3(B) = 0\right\}$. One vanishing eigenvalue indicates that the non-isolated set of points is of non-hyperbolic type. Also, from dynamical system perspective, a non-isolated set with exactly one vanishing eigenvalue associated with each point on the set is called normally hyperbolic set [37, 38] and the stability of this set can be determined by finding the nature of remaining non-zero eigenvalues. Thus, the condition for the set to be stable is $0 \leq z_c < 1$, $\alpha > \sqrt[3]{\frac{3}{2} \sqrt{1 - z_c^2}}$ and $\gamma < -\sqrt[3]{\frac{1}{6} \sqrt{1 - z_c^2}}$. See fig. 2(a),
where $B$ is a stable attractor solution in $x-z$ plane. Now kinetic part of scalar field dominated solution is achieved only when $z_c = 0$, then the set becomes a single point with coordinate $(1,0,0)$ which is source for $\gamma > -\sqrt{6}$ and $\alpha < \sqrt{\frac{2}{3}}$. On the other hand, for $z_c = 1$, the set becomes a point with coordinate $(0,0,1)$ which shows the complete density parameter of displacement vector of Lyra’s manifold dominated solution ($\Omega_B = 1$) and corresponds to a decelerated past attractor (source) solution in the phase space (see figs. 3(a), 3(c), 4(a) and 4(c)). Therefore, the set can describe the late time decelerated stable attractor when either it is dominated by kinetic part of the scalar field in the phase space or by $\beta$ displacement vector density parameter or both.

- Set of critical point $C$ is also a non-isolated set which exists for all $\alpha$ and $\gamma$ in $0 \leq z_c \leq 1$ and behaves similarly almost with the set $B$ in the phase space. The set is also dominated by kinetic part of scalar field DE and displacement field $\beta$ with ratio of energy densities $\Omega_\phi / \Omega_\beta = \frac{1-\alpha^2}{z_c^2}$. This set is always decelerating in nature as it has positive deceleration value with stiff matter: $\omega_{\epsilon ff} = 1, q = 2$. Eigenvalues for this set of points are: \( \{ \lambda_{1(C)} = 3 + \sqrt{6}\alpha\sqrt{1-\frac{\alpha^2}{3}}, \quad \lambda_{2(C)} = 3 - \sqrt{6}\alpha\sqrt{1-\frac{\alpha^2}{3}}, \quad \lambda_{3(C)} = 0 \} \). This non-isolated set of critical points like $B$ with exactly one vanishing eigenvalue is normally hyperbolic set and the condition for the stability of this set is: $0 \leq z_c < 1, \alpha < -\sqrt{\frac{2}{3}} \sqrt{1-\frac{\alpha^2}{3}}$ and $\gamma > \sqrt{6}\sqrt{1-\frac{\alpha^2}{3}}$. Fig. 2(b) shows the stability of $C$ in $x-z$ plane. Therefore, the set $B$, the normally hyperbolic set $C$ describes the late time decelerated stable attractor dominated either by scalar field for $z_c = 0$ or by energy density of displacement vector for $z_c = 1$. Further, it represents the solution dominated by kinetic part of scalar field ($\Omega_\phi = 1$) for $z_c = 0$ and for that it becomes a point with co-ordinate (-1,0,0) which is a source (past attractor) for $\alpha > -\sqrt{\frac{2}{3}}$ and $\gamma < \sqrt{6}$ (because this non-hyperbolic set has 2D unstable submanifold associated with two non-zero positive eigenvalues). On the other hand for $z_c = 1$, the set becomes a point with coordinate $(0,0,1)$ which corresponds to a decelerated solution dominated by density parameter of displacement vector of Lyra’s manifold and a past attractor (source) in the phase space since there exists 2D unstable submanifold associated with two non-zero positive eigenvalues (see figs. 3(a),3(c), 4(a) and 4(c)).

- Critical points $D$ and $E$ are same in all respect and they exist for the restriction: $-\sqrt{6} \leq \gamma \leq \sqrt{6}$ in phase space. Both the points represent the scalar field (DE) dominated solutions (since $\Omega_\phi = 1, \Omega_m = 0, \Omega_\beta = 0$ see Table I) completely. DE behaves as any perfect fluid model (since the effective equation of state for DE is $\omega_{\epsilon ff} = \frac{\alpha^2}{3} - 1$) and depending upon $\gamma$, it behaves as quintessence or cosmological constant like fluid. Specially, DE behaves as quintessence for $0 < \gamma^2 < 2$ and as cosmological constant for $\gamma = 0$. DE behaves as any exotic type fluid for $2 < \gamma^2 < 3$ but can never behave as phantom like fluid. There exists an accelerating universe near the points in the interval $-\sqrt{2} < \gamma < \sqrt{2}$. Eigenvalues of linearized Jacobian matrix for both the points are: \( \{ \lambda_{1(D,E)} = \gamma \alpha + \gamma^2 - 3, \quad \lambda_{2(D,E)} = \frac{\gamma^2}{2} - 3, \quad \lambda_{3(D,E)} = \frac{\gamma^2}{2} - 3 \} \). The points are hyperbolic in nature since all of the eigenvalues are non-zero. They can be non-hyperbolic for $\gamma = \pm \sqrt{6}$. The conditions for stability for these hyperbolic points are:

(i) $\alpha \leq -\frac{1}{\sqrt{2}}$ and $-\sqrt{\frac{\alpha^2+12}{2}} - \frac{\alpha}{2} < \gamma < \sqrt{6}$, or

(ii) $-\sqrt{\frac{2}{3}} < \alpha \leq \frac{1}{\sqrt{2}}$ and $-\sqrt{\frac{\alpha^2+12}{2}} - \frac{\alpha}{2} < \gamma < \sqrt{\frac{\alpha^2+12}{2}} - \frac{\alpha}{2}$, or

(iii) $\alpha > \frac{1}{\sqrt{2}}$ and $-\sqrt{6} < \gamma < \sqrt{\frac{\alpha^2+12}{2}} - \frac{\alpha}{2}$.

See figs. 3(a), 3(c), where $D$ and $E$ are stable attractor solutions in late times. Now the points show late time attractors in quintessence era (with $-1 < \omega_{\epsilon ff} < -\frac{1}{3}$) for the following restrictions:

(i) $\alpha \leq -\frac{1}{\sqrt{2}}$ and $(-\frac{\alpha}{2} - \sqrt{\frac{\alpha^2+12}{2}} < \gamma < 0$ or $0 < \gamma < \sqrt{2})$, or

(ii) $-\frac{1}{\sqrt{2}} < \alpha \leq \frac{1}{\sqrt{2}}$ and $(-\sqrt{2} < \gamma < 0$ or $0 < \gamma < \sqrt{2})$, or

(iii) $\alpha > \frac{1}{\sqrt{2}}$ and $(-\sqrt{2} < \gamma < 0$ or $0 < \gamma < \frac{\alpha}{2} + \sqrt{\frac{\alpha^2+12}{2}}$).

Further, the points are stable solutions attracted by cosmological constant (with $\omega_{\epsilon ff} = q = -1$) for $\gamma = 0$. That is for constant potential of scalar field ($\gamma = 0$), the points can describe the late time accelerated de Sitter attractor solutions with $\Omega_\phi = 1, \Omega_m = 0, \Omega_\beta = 0, \omega_{\epsilon ff} = -1, q = -1$ and for this case scalar field DE behaves as cosmological constant like fluid $\omega_{\epsilon ff}^{(\phi)} = -1$. So, one can conclude that the scalar field dominated solutions, namely, critical points $D$ and $E$ are physically interesting in late time as they correspond to accelerated stable attractors in quintessence era for a non-constant potential ($\gamma \neq 0$) and the accelerated de Sitter universe for
constant potential of the scalar field.

- Finally, the existence for the dark matter-scalar field scaling solutions represented by critical points $F$ and $G$ are following:
  
  (i) $\alpha < -\sqrt{\frac{2}{3}}\gamma$ and $\gamma \leq -\sqrt{\frac{\alpha^2+12}{2}} - \frac{2}{\alpha}$, or
  
  (ii) $\alpha = -\sqrt{\frac{2}{3}}\gamma$ and $\left( \gamma \leq -\frac{2}{3} \text{ or } \gamma = \sqrt{6} \right)$, or
  
  (iii) $\frac{2}{3} < \alpha < 0$ and $\left( \gamma \leq -\sqrt{\frac{\alpha^2+12}{2}} - \frac{2}{\alpha} \text{ or } \sqrt{\frac{\alpha^2+12}{2}} - \frac{2}{\alpha} \leq \gamma \leq -\frac{2}{2\alpha} \right)$, or
  
  (iv) $\alpha = 0$ and $\left( \gamma \leq -\sqrt{3} \text{ or } \gamma \geq \sqrt{3} \right)$, or
  
  (v) $0 < \alpha < \sqrt{\frac{3}{2}}$ and $\left( \frac{-2\alpha^2-3}{2\alpha} \leq \gamma \leq -\sqrt{\frac{\alpha^2+12}{2}} - \frac{2}{\alpha} \text{ or } \gamma \geq \sqrt{\frac{\alpha^2+12}{2}} - \frac{2}{\alpha} \right)$, or
  
  (vi) $\alpha = \sqrt{\frac{2}{3}}\gamma$ and $\left( \gamma = -\sqrt{6} \text{ or } \gamma \geq \sqrt{\frac{3}{2}} \right)$, or
  
  (vii) $\alpha > \sqrt{\frac{2}{3}}\gamma$ and $\gamma \geq \sqrt{\frac{\alpha^2+12}{2}} - \frac{2}{\alpha}$.

The ratio of density parameters for DE and DM is $r = \frac{\Omega_{\phi}}{\Omega_m} = \frac{\alpha(a+\gamma)+3}{(a+\gamma)-3}$. Associated DE with the points behaves as any perfect fluid nature which depending on parameters $\alpha$ and $\gamma$, can behave either as quintessence, cosmological constant or a phantom fluid. However, under the existence criteria of the critical points the DE can only mimic with quintessence ($-1 < \omega_{\phi}^{(\phi)} < -\frac{1}{3}$) in certain parameter space. In the same parameter space, the condition for acceleration of the universe is achieved (since the evolution of effective equation of state for DE and effective equation of state for the model are same for the points $\omega_{\phi}^{(\phi)} = \omega_{\phi}^{(\phi)} = -\frac{\alpha}{\gamma+\alpha}$); either for (i) $(\alpha < -\sqrt{\frac{2}{3}}\gamma$ and $2\alpha \leq \gamma \leq -\sqrt{\frac{\alpha^2+12}{2}} - \frac{2}{\alpha}$), or (ii) $(\alpha > \sqrt{\frac{2}{3}}\gamma$ and $\sqrt{\frac{\alpha^2+12}{2}} - \frac{2}{\alpha} \leq \gamma < 2\alpha$).

Eigenvalues of linearized Jacobian matrix for both the points $F$ and $G$ are:

$$\lambda_1(F,G) = -\sqrt{\frac{3(\gamma+2\alpha)}{2}(\gamma+\alpha)} - \frac{\alpha}{2(\gamma+\alpha)}.$$

$$\lambda_2(F,G) = -\frac{3(\gamma+2\alpha)}{2} - \frac{\alpha}{2(\gamma+\alpha)}.$$

$$\lambda_3(F,G) = -\frac{3(\gamma+2\alpha)}{2} - \frac{\alpha}{2(\gamma+\alpha)}.$$

Conditions for stability of the hyperbolic type (since all the eigenvalues are non-vanishing) critical points $F$ and $G$ are:

(i) $\alpha < -\sqrt{\frac{2}{3}}\gamma$ and $\gamma < -\sqrt{\frac{\alpha^2+12}{2}} - \frac{2}{\alpha}$, or

(ii) $\sqrt{\frac{2}{3}}\gamma < \alpha < 0$ and $\left( \gamma \leq -\sqrt{\frac{\alpha^2+12}{2}} - \frac{2}{\alpha} \text{ or } \sqrt{\frac{\alpha^2+12}{2}} - \frac{2}{\alpha} \leq \gamma \leq -\frac{2}{2\alpha} \right)$, or

(iii) $\alpha = 0$ and $\left( \gamma < -\sqrt{3} \text{ or } \gamma \geq \sqrt{3} \right)$, or

(iv) $0 < \alpha < \sqrt{\frac{3}{2}}\gamma$ and $\left( \frac{-2\alpha^2-3}{2\alpha} < \gamma < -\sqrt{\frac{\alpha^2+12}{2}} - \frac{2}{\alpha} \text{ or } \gamma \geq \sqrt{\frac{\alpha^2+12}{2}} - \frac{2}{\alpha} \right)$, or

(v) $\alpha \geq \sqrt{\frac{3}{2}}\gamma$ and $\gamma \geq \sqrt{\frac{\alpha^2+12}{2}} - \frac{2}{\alpha}$. Both the critical points are stable solutions in quintessence era for the following parameter restrictions:

(i) $\alpha < -\sqrt{\frac{1}{3}}\gamma$ and $2\alpha < \gamma < -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2+12}{2}}$, or

(ii) $\alpha > \sqrt{\frac{1}{3}}\gamma$ and $-\frac{3}{2} + \sqrt{\frac{\alpha^2+12}{2}} < \gamma < 2\alpha$. Universe near the critical points (with existence) can never be attracted with cosmological constant or in phantom regime. Therefore, the critical points $F$ and $G$ represent the accelerated scaling attractors in quintessence era with $\omega_{\phi}^{(m)} = \omega_{\phi}^{(\phi)} = -\frac{\alpha}{\gamma+\alpha}$ (see figs. 4(a), 4(c)).

### 4. COSMOLOGICAL IMPLICATIONS

In the context of Lyra’s geometry, interacting quintessence with varying mass dark matter, when potential of scalar field and mass of dark matter vary exponentially with scalar field $\phi$, exhibits some cosmologically interesting scenarios are the following:

Late time accelerated scalar field dominated attractor solutions are achieved by the critical point $D$ and $E$. Depending upon some restrictions on the parameters $\alpha$ and $\gamma$, evolution of the universe is attracted in quintessence era for non-constant (i.e., for $\gamma \neq 0$) potential of scalar field. Critical points $D$ and $E$ also correspond to de Sitter solutions in late times where accelerated universe driven by constant potential of scalar field ($\gamma = 0$) is attracted by cosmological constant ($\Omega_{\phi} = 1, \omega_{\phi}^{(\phi)} = q = -1$). Note that the critical points with $\Omega_{\phi} = 1$ cannot solve the coincidence problem. Figure (3) for parameter values $\alpha = 0$ and $\gamma = 0.1$ shows the phase portrait of the system.
TABLE I: The Critical Points and the corresponding physical parameters are presented.

| Critical Points | $\Omega_m$ | $\Omega_\phi$ | $\Omega_\beta$ | $\omega_{eff}(m)$ | $\omega_{eff}(\phi)$ | $\omega_{eff}$ | $q$ |
|----------------|------------|---------------|----------------|-------------------|-------------------|--------------|-----|
| $A$            | $1 - \frac{2\alpha^2}{3}$ | $\frac{2\alpha^2}{3}$ | $0$ | $\frac{2\alpha^2}{3}$ | $\alpha^2 + \frac{1}{2}$ | $\alpha^2 + \frac{1}{2}$ | $\alpha^2 + \frac{1}{2}$ |
| $B$            | $0$ | $1 - \frac{2\beta^2}{3}$ | $z^2$ | $1$ | $1$ | $1$ | $0$ |
| $C$            | $0$ | $1 - \frac{2\gamma^2}{3}$ | $z^2$ | $1$ | $1$ | $1$ | $0$ |
| $D$            | $0$ | $1$ | $0$ | $\frac{2}{3} - 1$ | $\frac{2}{3} - 1$ | $\frac{2}{3} - 1$ | $0$ |
| $E$            | $0$ | $1$ | $0$ | $\frac{2}{3} - 1$ | $\frac{2}{3} - 1$ | $\frac{2}{3} - 1$ | $0$ |
| $F$            | $\frac{\alpha(\alpha + \gamma)^2}{(\alpha + \gamma)^3}$ | $\frac{\alpha(\alpha + \gamma)^2}{(\alpha + \gamma)^3}$ | $0$ | $-\frac{2\alpha}{(\alpha + \gamma)}$ | $-\frac{2\alpha}{(\alpha + \gamma)}$ | $-\frac{2\alpha}{(\alpha + \gamma)}$ | $0$ |
| $G$            | $\frac{\alpha(\alpha + \gamma)^2}{(\alpha + \gamma)^3}$ | $\frac{\alpha(\alpha + \gamma)^2}{(\alpha + \gamma)^3}$ | $0$ | $-\frac{2\alpha}{(\alpha + \gamma)}$ | $-\frac{2\alpha}{(\alpha + \gamma)}$ | $-\frac{2\alpha}{(\alpha + \gamma)}$ | $0$ |

![Diagram](image)

FIG. 1: The figure is plotted with $\alpha = 1$ and $\gamma = -3$. Panel (a) shows the phase projection on $x - z$ plane, where the scaling solution $A$ is a stable attractor. While panel (b) shows the time evolution of the cosmological parameters with initial conditions: $x[0] = 0.816$, $y[0] = 0.001$, $z[0] = 0.001$ where late time scaling attractor approaches in decelerated era.

(24), where in fig.3(a) and fig.3(c) the critical points $D$ and $E$ are the DE dominated stable attractors in late times connecting through a matter dominated saddle point $A$. Also, the sets described by $B$ and $C$ are unstable source. Time evolution of different cosmological parameters (density parameters: $\Omega_\phi$, $\Omega_m$ and $\Omega_\beta$; effective equation of state parameter for DE and DM: $\omega_{eff}(m)$ and $\omega_{eff}(\phi)$; effective equation for the model: $\omega_{eff}$) are shown in the fig.3(b), where the late time accelerated universe evolves in quintessence era and ultimately it is attracted by cosmological constant.

Late time scaling attractor solutions also obtained which are very much interesting from the cosmological point of view. First, we have found non-isolated set of points namely, set $B$ and set $C$ which represent the scalar field DE-time varying displacement vector of Lyra’s manifold scaling solutions in the phase space. Each set is non-hyperbolic with exactly one vanishing eigenvalue leading to a normally hyperbolic set. From the stability analysis, we found that the sets $B$ and $C$ correspond to the late time stable attractor solutions in the phase space. Expansion of the universe on the sets is always decelerating ($\omega_{eff} = 1$, $q = 2$). The sets represent the kinetic energy of scalar field dominated universe (for $z_c = 0$, $\Omega_\phi = 1$) which is always decelerating in nature, with DE mimicking a stiff fluid ($\omega_{eff}(\phi) = 1$). Also, there exists density parameter of Lyra’s manifold displacement vector dominated solution corresponding to a decelerated universe for $z_c = 1$. Depending upon some parameter ($\alpha$ and $\gamma$) restrictions, these sets describe the late time stable attractor. Figure (2) is the projection of phase portrait on $x - z$ plane for different parameter values showing that the set $B$ and $C$ are stable attractors. In fig.2(a), for $\alpha = 30$ and $\gamma = -60$ the DE-displacement field scaling solution $B$ (with green coloured arc) is stable attractor while the set $C$ with blue coloured arc is unstable source. On the other hand, fig.2(b) shows that the scaling solution $C$ is stable attractor and set $B$ is unstable source.
for $\alpha = -30$ and $\gamma = 60$.

Dark matter-dark energy scaling solution described by the point $A$ corresponds to a decelerated universe in its cosmic evolution when total energy density is contributed from matter part and kinetic energy of the scalar field, energy due to displacement field is absent here. When the parameter $\alpha \to \pm \sqrt{3}$, DE comes into dominance over DM to the total energy content of the universe, where DE behaves as a stiff fluid. On the other hand, for $\alpha = 0$, the critical point $A$ will become completely DM (dust) dominated and a decelerated evolution of the universe can be achieved. The point will be saddle-like solution and has a transient nature which is much important in an interacting scenario to describe intermediate phase of the universe. Interestingly, for non-constant potential ($\alpha \neq 0$), the point can describe a late time stable attractor with $0 < \Omega_\phi < 1$ corresponding to a decelerated universe which cannot solve the coincidence problem. Figure (1) shows the projection of phase portrait on $x-z$ plane and evolution of cosmological parameters for $\alpha = 1$ and $\gamma = -3$ of the system (24). Fig.1(a) shows that the point $A$ describes the late time attractor. In fig.1(b), the time evolution of cosmological parameters show that the universe is attracted in decelerated era.

Finally, we have obtained late time accelerated dark matter-dark energy scaling attractor solutions namely, the critical point $F$ and $G$. Energy contribution from displacement field is absent to the total energy distribution of the universe ($\Omega_\beta = 0$). Stability analysis confirms that the points can be solutions at late times and can provide interesting dynamics of late phase evolution of the universe with $\omega_{eff} < -\frac{1}{3}$ satisfying constant ratio of density parameters for DE and DM in the evolution of the universe as $r = \frac{\Omega_\phi}{\Omega_m} = \frac{\alpha(\alpha + \gamma) + 3}{\gamma(\alpha + \gamma) - 3}$, as a result of which the coincidence problem could be alliaviated. Therefore, one can conclude that the points correspond to late time accelerated evolution of the universe attracted in quintessence era with $0 < \Omega_\phi < 1$ alleviating the coincidence problem. Fig. 4(b) for $\alpha = 1$ and $\gamma = 1.92$ with proper initial conditions shows that the late-time accelerated evolution of the universe is attracted in quintessence era satisfying $\omega_{eff} = \omega_{eff}^{(\phi)}$ in the attractor regime solving the coincidence problem. It is worthy to note that the fig. 4(a) and fig.4(c) for $\alpha = 1$ and $\gamma = 1.92$ show that the physical parameter values: $\Omega_\phi \approx 0.7$, $\Omega_m \approx 0.3$, $\omega_{eff} \approx -0.34$, $\omega_{eff}^{(m)} \approx -0.34$ can support the present evolution of the universe in quintessence era.
FIG. 3: The figure is plotted for parameters $\alpha = 0, \gamma = 0.1$. Panel (a) shows the phase space projection on $x - y$ plane where the scalar field dominated solutions $D$ and $E$ are stable attractors. In panel (b), with the initial conditions: $x[0] = -0.041, y[0] = 0.787, z[0] = 0.2$, the time evolution of physical parameters show that the late time attractor is approaching in accelerated era with cosmological constant. Finally, 3D plot in panel (c) shows that the set of points (an arc of circle) with green coloured and red coloured arc denoting $B$ and $C$ respectively are source, where as the point $A$ is saddle and $D$ and $E$ are late time attractors.

5. DISCUSSIONS WITH CONCLUDING REMARKS

An interacting DE scenario is investigated in the background of spatially flat FLRW universe where quintessence scalar field is taken as the model of DE with self interacting potential in the form of exponential as $V(\phi) = V_0 \exp(\gamma \phi)$ and pressureless dust is taken as DM, mass of which varies with time through a scalar field $\phi$ in the sense that decaying of DM particles reproduces the scalar field DE. The mass of DM particles is considered here to be dependent on an exponential of scalar field $\phi$ as $M_m(\phi) = M_0 \exp(-\alpha \phi)$. Additionally, a modification on geometric part of the Einstein’s equations is also considered and due to Lyra’s geometry a time varying displacement vector field which varies with scale factor as $\beta(t) \propto a^{-3}(t)$ arises naturally into the evolution equations. The Modified Friedmann equation, acceleration equation, conservation equations of varying mass DM and DE with displacement field obtained in this context are very much complicated to solve them analytically. Therefore, Dynamical systems tools have
been undertaken to study the complex model qualitatively. For that purpose, we converted the equations into an autonomous system of non-linear ODEs by suitable choice of dimensionless variables which are normalised over Hubble scale. From the dynamical analysis of the model, we have found some cosmologically interesting scenario as follows:

DM-scalar field scaling solution namely the critical point $A$ exists for parameter values $\alpha$ and depending on this the point can represent late time attractor, or saddle like intermediate phase of the universe, which is always decelerating in nature. In particular, it can be scalar field dominated solution for $\alpha = \pm \sqrt{\frac{3}{2}}$ where scalar field behaves as stiff matter. On the other hand, the point describes a dust dominated decelerated universe and has the transient nature (since the critical point is saddle like in nature) in its evolution for $\alpha = 0$ where scalar field behaves as dust matter.

From the dynamical point of view, there are two critical points $B$ and $C$ depict the similar evolutionary scheme of the universe. The points are normally hyperbolic set. From the analysis, we observe that the sets correspond to
scalar field- displacement field scaling solution, where depending the coordinate value \(z_c\) as \(z_c = 1\), the total energy density is contributed by the displacement field \(\beta(t)\) due to Lyra’s manifold. We obtained an ever decelerating universe near the sets. Depending upon some parameter restrictions, the sets can represent late time stable attractor with \(\beta\) energy dominated decelerated universe which has been discussed in earlier section.

Completely scalar field dominated solutions namely, critical points \(D\) and \(E\) exist in the parameter space \(-\sqrt{6} \leq \gamma \leq \sqrt{6}\). In this interval the points are same in all respect. They exhibit the late time accelerated universe which is attracted either in quintessence era or by cosmological constant. Late time accelerated evolution of the universe is attracted by cosmological constant for \(\gamma = 0\), interestingly, here the expansion of the universe is exponential and it represents the late time de Sitter solution.

Finally, the scalar field- matter scaling solutions are achieved namely, critical points \(F\) and \(G\) which corresponding to late time accelerated evolution of the universe attracted in quintessence era with having similar energy densities of DE and DM in its evolution which can solve the coincidence problem.

Therefore, one can conclude that the study of interacting scenario with varying mass DM particles in context of Lyra’s manifold, scalar field dominated late time accelerated attractor can be obtained as well as the decelerated scalar field-displacement field scaling attractor solution are also obtained and both the models do not have mechanisms to solve the coincidence problem. While for the accelerated scalar field-DM scaling attractor solutions, one acquires the possible mechanism to alleviate the coincidence problem because in this attractor regime both scalar field and DM scale in a similar order.

Acknowledgments

The author Goutam Mandal is grateful to UGC, govt. of India for giving research fellowship with Award letter No.F.82-1/2018(SA-III) for Ph.D.

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