The gravity of light-waves

J.W. van Holten

Nikhef, Amsterdam
and
Leiden University
Netherlands

Abstract

Light waves carry along their own gravitational field; for simple plain electromagnetic waves the gravitational field takes the form of a $\text{pp}$-wave. I present the corresponding exact solution of the Einstein-Maxwell equations and discuss the dynamics of classical particles and quantum fields in this gravitational background.

* Lecture presented at the meeting Estate Quantistica 2018, Scalea (Italy)
1. Setting the stage

The gravitational properties of light waves have been studied extensively in the literature \cite{1}-\cite{11}. In this lecture I describe the exact solutions of Einstein-Maxwell equations discussed in \cite{5,6} and some applications.

The discussion concerns plain electromagnetic waves propagating in a fixed direction chosen to be the $z$-axis of the co-ordinate system. As they propagate at the universal speed $c$, taken to be unity: $c = 1$ in natural units, it is useful to introduce light-cone co-ordinates $u = t - z$, $v = t + z$. Then the electromagnetic waves to be discussed are described by a transverse vector potential

$$A_i(u) = \int \frac{dk}{2\pi} \left( a_i(k) \sin ku + b_i(k) \cos ku \right), \quad i = (x, y). \tag{1}$$

This expression explicitly makes use of the superposition principle for electromagnetic fields, guaranteed in Minkowski space by the linearity of Maxwell’s equations and well-established experimentally. The corresponding minkowskian energy-momentum tensor is

$$T_{\mu\nu} = F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{4} \eta_{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}, \tag{2}$$

the only non-vanishing component of which in light-cone co-ordinates is

$$T_{uu} = \frac{1}{2} \left( E^2 + B^2 \right). \tag{3}$$

Here the components of the transverse electric and magnetic fields are expressed in terms of the vector potential (1) by

$$E_i(u) = -\varepsilon_{ij}B_j(u) = A'_i(u). \tag{4}$$

the prime denoting a derivative w.r.t. $u$.

The same expression for light-waves also holds in general relativity, the corresponding special solution of the Einstein equations being described by the line element

$$ds^2 = -du dv - \Phi(u, x, y)du^2 + dx^2 + dy^2 \tag{5}$$

For this class of metrics \cite{12,13} the only non-vanishing components of the connection are

$$\Gamma_{uu}^v = \partial_u \Phi, \quad \Gamma_{iu}^v = 2\Gamma_{uu}^i = \partial_i \Phi, \tag{6}$$

and the complete Riemann tensor is given by the components

$$R_{uinj} = -\frac{1}{2} \partial_i \partial_j \Phi. \tag{7}$$

As a result the Ricci tensor is fully specified by

$$R_{uu} = -\frac{1}{2} \left( \partial_x^2 + \partial_y^2 \right) \Phi, \tag{8}$$

which matches the form of the energy-momentum tensor (3) and thus allows solutions of the Einstein equations specified by

$$\Phi = 2\pi G (x^2 + y^2) \left( E^2 + B^2 \right)(u) + \Phi_0(u, x^i), \tag{9}$$

with $\Phi_0$ representing a free gravitational wave of $pp$-type.
2. Geodesics

The motion of electrically neutral test particles in a light-wave (1) is described by the geodesics $X^\mu(\tau)$ of the $pp$-wave space-time (5). They are found by solving the geodesic equation

$$\ddot{X}^\mu + \Gamma^\mu_{\lambda\nu} \dot{X}^\lambda \dot{X}^\nu = 0,$$

(10)

the overdot denoting a derivative w.r.t. proper time $\tau$. The equation for the geodesic light-cone co-ordinate $U(\tau)$ is especially simple, as its momentum (representing a Killing vector) is conserved:

$$\dot{U} = \gamma = \text{constant}.\quad (11)$$

Another conservation law is found from the hamiltonian constraint obtained by substitution of the proper time in the line element:

$$-1 = -\dot{U} \dot{V} - \Phi(U, X^i)\dot{U}^2 + \dot{X}^{i2} \iff \frac{1}{\gamma^2} = \frac{1 - v^2}{(1 - v_z)^2} + \Phi,$$

(12)

where $v = dX/dT$ is the velocity in the observer frame. Finally, using (11) to substitute $U$ for $\tau$, the equations for the transverse co-ordinates become

$$\frac{d^2X^i}{dU^2} + \frac{1}{2} \frac{\partial \Phi}{\partial X^i} = 0.$$

(13)

For quadratic $pp$-waves $\Phi(u, x^i) = \kappa_{ij}(u) x^i x^j$ this takes the form of a parametric oscillator equation

$$\frac{d^2X^i}{dU^2} + \kappa_{ij}(U) X^j = 0.$$

(14)

For light-like geodesics the equations are essentially the same, except that the hamiltonian constraint is replaced by

$$\frac{1 - v^2}{(1 - v_z)^2} + \Phi = 0.$$

(15)

Note that in Minkowski space, where $\Phi = 0$, this reduces to $v^2 = c^2 = 1$. These equations take a specially simple form for circularly polarized light waves sharply peaked around a central frequency

$$A_x(u) = \int \frac{dk}{2\pi} a(k) \cos ku, \quad A_y(u) = \int \frac{dk}{2\pi} a(k) \sin ku,$$

(16)

where the domain of $a(k)$ is centered around the value $k_0$ with width $\Delta k$ and central amplitude $a_0$. Then

$$\mu^2 \equiv 2\pi G \left( E^2 + B^2 \right) = 2G \int dk k^2 a^2(k) \sim G \Delta k k_0^2 a_0^2.$$

(17)

and therefore

$$\Phi = \mu^2 \left( x^2 + y^2 \right), \quad \mu^2 = 4\pi G \int \frac{dk}{2\pi} k^2 a^2(k).$$

(18)

Then equation (14) reduces to a simple harmonic oscillator equation with angular frequency $\mu$ in the $U$-domain.
3. Field theory

In the previous sector we studied the equation of motion of test particles, supposed to have negligible back reaction on the gravitational field described by the metric (5). Similarly one can study the dynamics of fields in this background space-time in the limit in which the fields are weak enough that their gravitational back reaction can be neglected. First we consider a scalar field \( \Psi(x) \) described by the Klein-Gordon equation

\[
(-\Box_{pp} + m^2) \Psi = 0, \quad \Box_{pp} = -4\partial_u \partial_v + 4\Phi(u, x^i) \partial_v^2 + \partial_x^2 + \partial_y^2.
\]  

(19)

It is convenient to consider the Fourier expansion w.r.t. the light-cone variables \((u, v)\):

\[
\Psi(u, v, x^i) = \frac{1}{2\pi} \int ds dq \psi(s, q, x^i) e^{-i(su + qv)}.
\]  

(20)

Note that

\[
su + qv = Et - pz, \quad E = s + q, \quad p = s - q.
\]  

(21)

Then the amplitudes \(\psi\) satisfy the equation

\[
\left[ \partial_x^2 + \partial_y^2 + 4sq - 4q^2\Phi(-i\partial_s, x^i) - m^2 \right] \psi = 0.
\]  

(22)

This equation can be solved explicity for the circularly polarized wave packets which lead to the simple quadratic amplitude \((18)\). Then

\[
(4sq - m^2) \psi = (-\partial_x^2 - \partial_y^2 + 4\mu^2q^2(x^2 + y^2)) \psi.
\]  

(23)

The right-hand side describes a couple of quantum oscillators with frequency \(\omega = 2\mu|q|\) possessing an eigenvalue spectrum

\[
2\mu|q| (n_x + n_y + 1) \equiv 4\sigma|q|, \quad n_i = 0, 1, 2, ...
\]  

(24)

Thus equation \((23)\) reduces to

\[
4sq - 4\sigma|q| = m^2 \quad \text{or} \quad \begin{cases} (E - \sigma)^2 = (p - \sigma)^2 + m^2, & q > 0; \\ (E + \sigma)^2 = (p + \sigma)^2 + m^2, & q < 0. \end{cases}
\]  

(25)

The final result for the scalar field then becomes

\[
\Psi(u, v, x^i) = \frac{1}{2\pi} \int_0^\infty dq \sqrt{q} \sum_{n_i=0}^\infty \left( a_{n_i}(q) e^{-iqv-i\left(\frac{w^2}{4\sigma}+\sigma\right)u} + a^*_{n_i}(q) e^{iqv+i\left(\frac{w^2}{4\sigma}+\sigma\right)u} \right)
\]

\[
\times \sqrt{\frac{2\mu}{\pi}} \prod_{j=x,y} \left[ \frac{H_{n_j}(\xi_j)}{\sqrt{2^{n_j}n_j!}} e^{-\xi_j^2} \right], \quad \xi_j = \sqrt{2\mu q} x_j.
\]  

(26)
4. Electromagnetic fluctuations in a light-wave background

On top of an electromagnetic wave described by equation (1) there can be fluctuations of the electromagnetic field. The general form of the Maxwell field then is of the form

$$A_\mu(u, v, x^i) = \delta_\mu^i A^\text{wave}_i(u) + a_\mu(u, v, x^i).$$

(27)

Because of the linearity of Maxwell’s equations the field equations for the wave background and the fluctuations separate. The fluctuating field equations in the gravitational pp-wave background are derived from the action

$$S = \int dudvdxdy \left[ (\partial_u a_v - \partial_v a_u)^2 + (\partial_u a_i - \partial_i a_u) (\partial_v a_i - \partial_i a_v) - \Phi (\partial_i a_i - \partial_i a_v)^2 - \frac{1}{8} (\partial_i a_j - \partial_j a_i)^2 \right],$$

(28)

and read

$$\frac{\delta S}{\delta a_u} = 4\partial_v \partial_u a_v - \Delta_\perp a_v - 2\partial_v \left( \partial_v a_u + \partial_u a_v - \frac{1}{2} \partial_i a_i \right) = 0,$$

$$\frac{\delta S}{\delta a_v} = 4\partial_v \partial_u a_u - \Delta_\perp a_u - 2\partial_u \left( \partial_v a_u + \partial_u a_v - \frac{1}{2} \partial_i a_i \right) + 2\partial_i \left[ \Phi (\partial_i a_v - \partial_v a_i) \right] = 0,$$

$$\frac{\delta S}{\delta a_i} = -2\partial_v \partial_u a_i + \frac{1}{2} \Delta_\perp a_i + \partial_i \left( \partial_v a_u + \partial_u a_v - \frac{1}{2} \partial_j a_j \right) - 2\partial_v \left[ \Phi (\partial_i a_v - \partial_v a_i) \right] = 0,$$

(29)

where $\Delta_\perp = \partial_x^2 + \partial_y^2$. As the fluctuating field equations possess their own gauge invariance they can be restricted without loss of generality by the constraint

$$\partial_v a_u + \partial_u a_v - \frac{1}{2} \partial_i a_i = 0.$$ 

(30)

However, this does not yet exhaust the freedom to make gauge transformations, as the condition (30) is respected by special gauge transformations

$$a'_\mu = a_\mu + \partial_\mu \alpha, \quad \text{with} \quad (4\partial_u \partial_v - \Delta_\perp) \alpha = 0.$$ 

(31)

As can be seen from the first equation (29) these transformations can be used to eliminate the component $a_v$ by taking

$$\partial_v \alpha = -a_v \Rightarrow a'_v = 0.$$ 

(32)

We are then left with a fluctuating field component $a_u$ restricted by (30):

$$\partial_v a_u = \frac{1}{2} \partial_i a_i.$$ 

(33)
implying \( a_u \) to satisfy the Gauss law constraint

\[
\partial_i [\partial_t a_u - 2 (\partial_u - \Phi \partial_v) a_i] = 0. \tag{34}
\]

The only remaining dynamical degrees of freedom are now the transverse components \( a_i \) which are solutions of the Klein-Gordon type of equations

\[
\left( -2\partial_u \partial_v + 2\Phi \partial_v^2 + \frac{1}{2} \Delta_\perp \right) a_i = 0. \tag{35}
\]

For \( pp \)-backgrounds of the special form (18) these solutions take the form (26) with \( m^2 = 0 \).

In the full theory also the gravitational field must fluctuate in a corresponding fashion. In the limit where the fluctuations are due to irreducible quantum noise, a corresponding quantum effect must be present in the space-time curvature. In view of the result [9] for the photon fluctuations in the light-beam itself these are expected to take the form of associated spin-0 graviton excitations.

References

[1] R. C. Tolman, P. Ehrenfest, and B. Podolsky, Phys. Rev. 37 (1931), 602
[2] W. B. Bonnor, Comm. Math. Phys. 13 (1969), 163
[3] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, *Exact solutions of Einstein’s field equations* (Cambridge Univ. Press, 2003)
[4] J.W. van Holten, Fortschr. Phys. 45 (1997), 439
[5] J.W. van Holten, Lect. Notes Phys. 541 (2000), 365
[6] J.W. van Holten, Fortschr. Phys. 59 (2011), 284
[7] A. Fuster and J.W. van Holten, Phys. Rev. D72 (2005) 024011
[8] G. Brodin, D. Eriksson, and M. Marklund, Phys. Rev. D 74 (2006), 124028
[9] G. Sparano, G. Vilasi and S. Vilasi, Class. Quant. Grav. 28 (2011), 195014
[10] D. Rätzel, M. Wilkens and R. Menzel, New J. Phys. 18 (2016), 023009
[11] D. Lynden-Bell and J. Bičak, Phys. Rev. D 96 (2017), 104053
[12] H.W. Brinkmann, Proc. Nat. Ac. Sci. 9 (1923), 1
[13] O. Baldwin and G. Jeffery, Proc. Roy. Soc. A111 (1926), 95