Enhanced Coset Symmetries and Higher Derivative Corrections

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Abstract

After dimensional reduction to three dimensions, the lowest order effective actions for pure gravity, M-theory and the Bosonic string admit an enhanced symmetry group. In this paper we initiate study of how this enhancement is affected by the inclusion of higher derivative terms. In particular we show that the coefficients of the scalar fields associated to the Cartan subalgebra are given by weights of the enhanced symmetry group.

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1 Introduction

Classical supergravity actions have played a central role in theoretical high energy physics and string theory in particular. One unexpected development was the realization that these theories lead to enhanced symmetries when dimensionally reduced on a torus. This phenomenon was first observed in the dimensional reduction of four-dimensional Einstein gravity to three dimensions where the resulting theory was found to have an $SL(2)$ symmetry [1]. More generally, gravity reduced to three dimensions on an $n$-torus leads to a theory with an $SL(n+1)$ symmetry. The $SL(n)$ part of this symmetry is just the diffeomorphisms of the higher dimensional theory that are unbroken by the torus. The enhancement to $SL(n+1)$ only arises when one dualizes the rank two field strength of the graviphotons (the components of the metric that have an internal and space-time index) to obtain scalars in three dimensions. These scalars, together with those which arise directly from the metric, form a non-linear realization of $SL(n+1)$ with local subalgebra $SO(n+1)$.

The dimensional reduction of eleven dimensional supergravity, or the type II supergravities, on a torus to four and three dimensions leads to an $E_7$ [2] and $E_8$ [3] symmetry respectively. Furthermore, the effective theory associated with the Bosonic string, which consists of gravity, a scalar and a rank two gauge field, when dimensional reduced on a $n$ torus to three dimensions leads to an $O(n+1,n+1)$ symmetry. For other examples see [4].

In this paper we will only consider the Bosonic sector of the effective action. However it would be most interesting to extend our analysis to include Fermions. The dimensional reduction to three dimensions is special in that it is the first dimension in which all fields of the original theory become scalars after the appropriate dualizations. Thus the resulting theory only consists of scalars and, for all the theories mentioned above, is a non-linear realization of a group $G$ with local sub-algebra $H$. Since the action is only second order in space-time derivatives it is given by

$$S = \int d^3x \sqrt{-g} \left( R - \text{Tr}(g^{-1}D_\mu gg^{-1}D^\mu g) \right)$$  \hspace{1cm} (1)

where $g \in G$ and $D_\mu$ is the $H$-covariant derivative. This action has the manifest symmetry $g \rightarrow g_0 gh$ where $g_0 \in G$ is a constant transformation, (i.e. rigid transformation) and $h \in H$ is local. In fact in all the above cases the local subalgebra $H$ is the just the subalgebra of $G$ that is invariant under the Cartan involution. A generic theory, when dimensionally reduced, will not possess an enhanced symmetry algebra and so will not be expressible as a non-linear realization of the above form. Indeed, it requires a very precise field content with corresponding couplings to find such a symmetry [5, 6].
To show that the theories mentioned above, when dimensionally reduced to three dimensions, have the action of equation (1) is in general quite technically complicated. However, the above action has a particularly characteristic expression. Taking account of the local transformations the group element may be chosen to be

\[ g = e^{\sum_{\alpha>0} \chi_{\alpha} E_{\alpha}} e^{-\frac{1}{\sqrt{2}} \phi H} \]  

where \( H \) and \( E_{\alpha} \) are the Cartan subalgebra and positive root generators of \( G \) respectively. One then finds that the coset part of the action has the form

\[ \text{Tr}(g^{-1} D_\mu gg^{-1} D^\mu g) = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi + \frac{1}{2} \sum_{\alpha>0} e^{\sqrt{2} \phi \cdot \alpha} \partial_\mu \chi_{\alpha} \partial^\mu \chi_{\alpha} + \ldots \]  

where the ellipsis denotes higher order terms in \( \chi_{\alpha} \). Note that the roots of the algebra \( G \) show up as vectors that occur in the exponentials that multiply the kinetic fields \( \chi_{\alpha} \). These fields arise from the off-diagonal metric components and the full effective action can be complicated to evaluate exactly. On the other hand the fields \( \phi \) arise from the diagonal components of the metric (and dilaton if present) and their role in the dimensional reduction procedure is relatively easy to follow. As such it is straightforward to find out in which theories the roots of an algebra arise and in addition what algebra they belong to.

The IIA \([7, 8, 9]\) and IIB \([10, 11, 12]\) supergravity theories are the complete low energy effective actions for the IIA and IIB superstring theories. As a result, much of our understanding of non-perturbative effects of string theory has been derived from these theories. These theories are now believed to be different perturbative descriptions of M-theory and their dimensional reduction can be obtained considering the reduction of eleven-dimensional supergravity \([13]\).

Part of the symmetries that occur in the dimensional reduction are T-dualities \([14, 15, 16]\), which have been shown to be a symmetry of string theory at all orders of perturbation theory. However, all the symmetries found in the dimensional reduction of the supergravity theories, or more precisely a discrete subgroup thereof, have been conjectured to be symmetries of string theory and called U-duality \([17]\). In fact U-dualities can generated by T-duality together with a discrete version of the \( SL(2) \) symmetry of IIB supergravity \([10]\).

It has been conjectured that M-theory possess an infinite dimensional Kac-Moody symmetry, traces of which can be found in the type II supergravity theories \([18]\). In particular, one should consider the non-linear realization of \( E_{11} \) with the local subalgebra being the Cartan involution invariant subalgebra. From this viewpoint, the symmetries that arise upon dimensional
reduction are not accidents of dimensional reduction, as is commonly believed, but are remnants of the symmetries that occur in the $E_{11}$ non-linearly realized theory before dimensional reduction. A related approach [19], which shares some similar ideas, is based on $E_{10}$, a subalgebra of $E_{11}$. The difference in approach can be traced to the discovery of "cosmological billiards", or BKL behavior, that occurs when eleven-dimensional supergravity is considered in the region of a space-like singularity [20]. In a recent paper [21] higher derivative terms were also considered within the context of the BKL limit and it was found that the higher derivative terms thought to occur in M-theory lead to the appearance of roots of $E_{10}$ in the resulting cosmological billiards. However, it has been suggested that these should be thought of as weights and furthermore weights should also arise when considering the BKL limit of higher derivative terms in pure gravity [22].

Much of the discussion of these symmetries has been in the context of the lowest order, two derivative, effective action. An exception is [23] which considers how T-duality is altered by the leading order higher derivative corrections, although it does not consider the enhancement that occurs in three dimensions. Indeed to the best of our knowledge there has been little or no discussion as to whether or not the enhanced symmetries that occur in the dimensional reduction of the low energy effective actions extend to the higher derivative corrections. Therefore in this paper we begin a systematic study how the enhanced symmetry is affected by higher derivative terms.

In general higher derivative corrections are very complicated and only some specific terms are known [24]-[43], although the complete next-to-leading order result has recently been reported [44]. Hence the subject is not sufficiently advanced at this stage to determine whether or not the higher derivative terms, once dimensionally reduced to three dimensions, possess an enhanced non-linearly realized symmetry. However, for the generic higher derivative terms it is possible to derive the dependence of the diagonal components of the metric (and dilaton if present) in the dimensional reduction to three dimensions and then read off their coefficients. These coefficients form a constant vector. As explained above for the low energy effective action at lowest order in derivatives, one finds the roots of the enhanced symmetry algebra.

In this paper we will show that for gravity, M-theory and the Bosonic string one finds weights of the enhanced symmetry algebra, but only for certain classes of higher derivative terms. For higher derivative terms outside these classes the vectors that arise have no apparent interpretation in terms of objects associated with the algebra. However the class of higher derivative terms for which weights arise are just those that are expected on the basis of string theory arguments. Demanding the appearance of weights can
therefore be used to predict the correct class of higher derivative terms. The appearance of weights provides strong evidence for some enhanced symmetry structure in the higher derivative terms. On the other hand, the appearance of weights, as opposed to roots, is intriguing as the Cartan forms that are usually used to construct actions for non-linearly realized theories only contain the roots.

The rest of this paper is organized as follows. In section two we consider the case of pure gravity and recall the derivation of the $SL(n+1)$ symmetry for the reduction of gravity on an $n$ torus to three dimensions. We then introduce in detail a new technique for deriving the dependence of the higher order effective action on the diagonal components of the metric, \textit{i.e.} the fields associated with the Cartan subalgebra. We then show that higher derivative terms lead to weights of the extended $SL(n+1)$ symmetry appearing as the coefficients of the scalar fields which form the diagonal components of the internal metric. In the section three we repeat this analysis for M-theory. Here we find that the weights of $E_8$ appear only if the higher derivative terms have the form $(\hat{R})^{3k+1}$, for $k = 0, 1, 2, ...$, as is expected from quantum string calculations. In section four we consider the Bosonic string and our analysis shows that the weights of $O(n+1, n+1)$ appear only if the terms have a particular power of the dilaton which in turn corresponds to a particular genus contribution of a perturbative string calculation. In section four we discuss the effect of having more than one derivative acting on each field. In section five we give a conclusion. The paper contains also three appendices which discuss technical details that are used throughout the main text. In particular the first gives details of the dimensional reduction, the second recalls some facts about non-linear realizations and the third some of the required theory of group representations.

2 Pure Gravity

In this section we will discuss pure gravity, compactified on and $n$-torus from $D$-dimensions to 3-dimensions. Our main motivation is to develop the techniques that we will use later. We start with the dimensional reduction of the pure gravitational action

$$ S = \int d^D x \sqrt{-\hat{g} \hat{R}} $$

(4)

where we use a hat to denote $D$-dimensional quantities. We write the $D$-dimensional metric as

$$ ds^2 = e^{2\alpha} ds^2 + e^{2\beta} G_{ij} (dx^i + A^i_\mu dx^\mu)(dx^j + A^j_\mu dx^\mu) $$

(5)
and we refer the reader to Appendix A for details of our conventions and various formulae. Following these calculations we find that the Einstein-Hilbert action, dimensionally reduced to three dimensions in Einstein frame, is

\[
S = \int d^3x \sqrt{-g} \left( R - \frac{1}{4} e^{-\frac{n+1}{n} \alpha \rho} G_{ij} F^i_{\mu \nu} F^{j\mu \nu} - \text{Tr}(S_\mu S^\mu) - \frac{1}{2} \partial_\mu \rho \partial^\mu \rho \right) \quad (6)
\]

Here \( S^i_{\mu \nu} \) is the symmetric part of \( e^k_i \partial_\mu e^i_j \) where \( e^i_j \) is the vielbein for the internal metric \( G_{ij} \) and an over-line denotes a tangent frame index.

Here we see that the action has a manifest \( SL(n)/SO(n) \) symmetry that acts on the internal vielbein and rotates the graviphotons (we refer the reader to Appendix B for a review of \( G/H \) coset Lagrangians). In particular the action of \( SL(n) \) corresponds to diffeomorphisms that preserve the torus whereas the \( SO(n) \) generates local tangent frame rotations.

But wait there’s more! To see this we can add the Lagrange multiplier term associated to the Bianchi identity of the graviphotons

\[
S_{LM} = \frac{1}{2} \int d^3x e^{\mu \lambda} \chi_i \partial_\mu F^i_{\nu \lambda}
\]

\[
= -\frac{1}{2} \int d^3x e^{\mu \lambda} \partial_\mu \chi_i F^i_{\nu \lambda}
\]

\[
(7)
\]

We can then integrate out \( F^i_{\mu \nu} \) and arrive at a purely scalar Lagrangian

\[
S_{dual} = \int d^3x \sqrt{-g} \left( R - \frac{1}{2} e^{\frac{n+1}{n} \alpha \rho} G^{ij} \partial_\mu \chi_i \partial^\mu \chi_j - \text{Tr}(S_\mu S^\mu) - \frac{1}{2} \partial_\mu \rho \partial^\mu \rho \right) \quad (8)
\]

Let us consider the \( n+1 \)-dimensional vielbein

\[
\tilde{e}^J_I = \left( e^{\alpha \rho} e_i^0, e^{\alpha \rho} e_i^j, 0, e^{-\alpha \rho} e_i^0 \right)
\]

\[
(9)
\]

where \( I, J = 0, ..., n \). The symmetric part of \( e^K_J \partial_\mu e^K_I \) is

\[
\tilde{S}_\mu^J S^i_{\mu \nu} = \left( \frac{\alpha \partial_\mu \rho}{2} e^k_i \partial_\mu e^k_j \chi_k, \frac{1}{2} e^{\frac{n+1}{n} \alpha \rho} e^k_i \partial_\mu e^k_j \chi_k \right) \quad (10)
\]

Therefore we have

\[
\tilde{S}_\mu^J S^\mu_J = S^\mu_J S^\mu_J + \frac{1}{2} e^{\frac{n+1}{n} \alpha \rho} G^{ij} \partial_\mu \chi_i \partial^\mu \chi_j + \frac{1}{2} \partial_\mu \rho \partial^\mu \rho
\]

\[
(11)
\]

\[
6
\]
Hence, after dualization, the dimensionally reduced action can be written as

\[ S_{\text{dual}} = \int d^3 x \sqrt{-g} \left( R - \text{Tr}(\tilde{\mathcal{S}_\mu \tilde{\mathcal{S}}}^\mu) \right) \]  \hspace{1cm} (12)

This action is three-dimensional gravity coupled to an \( SL(n+1)/SO(n+1) \) coset, i.e. a non-linear realization of \( SL(n+1) \) with local subgroup \( SO(n+1) \).

Let us return to the dualized Lagrangian (8). We have already demonstrated that this Lagrangian can in fact be written as an \( SL(n+1)/SO(n+1) \) coset Lagrangian in terms \( \tilde{e}_I \). However we now wish to explain a simple procedure for identifying the enhancement of \( SL(n) \) to \( SL(n+1) \) that will be important for analyzing the more complicated higher derivative terms that are the main subject of this paper.

The dualized Lagrangian (8) has a manifest \( SL(n) \) symmetry. The vielbein \( e_I^J \) of the internal torus plays the role of the \( SL(n) \) group element with local \( SO(n) \) symmetry. As outlined in appendix B, this leads to the Cartan forms of \( SL(n) \) out of which the \( \text{Tr}(\mathcal{S}_\mu \mathcal{S}^\mu) \) term in the action is constructed. As such, we may write the vielbein as

\[ e = e_G \sum_{\alpha > 0} \lambda_\alpha E_\alpha \ e^{-\frac{1}{\sqrt{2}} \phi H} \]  \hspace{1cm} (13)

where \( H \) and \( E_\alpha \) are the Cartan sub-algebra and positive root generators of \( SL(n) \) respectively. When evaluating the Cartan forms it does not matter what representation of these generators we take as the calculation only depends on the structure constants of the Lie algebra. However, to recover the vielbein we need to consider the representation of \( SL(n) \) that has highest weight \( \lambda^{n-1} \) and also, as explained in appendix C, its dual which is a representation with highest weight \( \lambda^1 \). If we choose as our basis of states of this latter representation as \( <j, \lambda^1|, j = 1, \ldots, n \), then the vielbein on the torus is given by

\[ <j, \lambda^1|U(e) = \sum_i <i, \lambda^1|e_i^J \]  \hspace{1cm} (14)

where \( U \) is the representation. We observe that (see equation (124) in appendix C) this object transforms only under local rotations. Taking into account the transformation of the basis vectors, one sees that the vielbein transforms under local rotations on its upper over-lined index and rigid \( SL(n) \) transformations in the vector representation on its lower index. The metric on the torus is then given by

\[ G_{ij} = <i, \lambda^1|U(M)|j, \lambda^1> \]  \hspace{1cm} (15)
where, as before, \( \mathcal{M} = ee^\# \). Here the kets transform under the Cartan twisted \( \Delta^{n-1} \) representation, which is the \( \Delta^1 \) representation.

Similarly we find that the inverse metric is given by taking the \( SL(n) \) representation with highest weight \( \Delta^{n-1} \)

\[
G^{ij} = \langle i, \Delta^{n-1} | U(\mathcal{M}^{-1}) | j, \Delta^{n-1} \rangle
\]  

As explained in appendix C, we should take the bras to be the dual of the Cartan twisted \( \lambda^{n-1} \) representation, which is simply the \( \lambda^{n-1} \) representation.

Our goal here is to identify how the \( SL(n+1) \) roots arise in (8) from a more abstract point of view, which will generalize to the more complicated terms that occur in the higher derivative corrections. The main term in the action, apart from the usual coset action for \( SL(n) \), is \( G^{ij} \partial_\mu \chi_i \partial^{\mu} \chi_j \) which arises from the dual graviphotons. We will now see how this term leads to the enhanced \( SL(n+1) \) symmetry. In particular we wish to show how the positive roots of \( SL(n+1) \), which are not roots of \( SL(n) \), arise from this term.

To proceed it is helpful to introduce different notations for \( n \) and \( (n+1) \) dimensional vectors which arise in \( SL(n) \) and \( SL(n+1) \) respectively. We will denote the former using a bar and the later using an arrow. In particular we will write

\[
\vec{\phi} = (\rho, \phi)
\]  

The roots \( \alpha \) of \( SL(n) \) that appear in \( S_\mu \) can be lifted to roots of \( SL(n+1) \) simply by taking

\[
\vec{\alpha} = (0, \alpha)
\]  

To evaluate \( \partial_\mu \chi_i G^{ij} \partial_\mu \chi_j \) we note that \( \partial_\mu \chi_i \) transforms as a vector under the manifest \( SL(n) \) symmetry that is in the representation with highest weight \( \Delta^1 \). As explained in appendix C, we may write this term as

\[
< \chi | U(\mathcal{M}^{-1}) | \chi >
\]

In carrying out this step we have expressed \( | \chi > \) in terms of the basis states of the \( \Delta^1 \) representation, that is \( | \chi > = \sum_i | i, \Delta^1 > \). Also as we are only interested in the group theoretic structure of this term we have suppressed the presence of space-time derivatives and other unimportant factors. We have that

\[
\mathcal{M}^{-1} = e^{-\sum_{\alpha > 0} \chi_\alpha E_\alpha} e^{\sqrt{2} \phi \cdot H} e^{-\sum_{\alpha > 0} \chi_\alpha E_\alpha}
\]

and hence

\[
< \chi | U(\mathcal{M}^{-1}) | \chi > = \sum_{w \in \Delta^1} < w | e^{\sqrt{2} \phi \cdot H} | w > + \ldots
\]

\[
= \sum_{w \in \Delta^1} e^{\sqrt{2} \phi \cdot w} < w | w > + \ldots
\]
where \([\lambda^1]\) denotes the class of weights that appear in the representation whose highest weight is \(\lambda^1\) and the ellipsis denotes \(\chi_\alpha\)-dependent terms which also have exponentials containing the above weights. We have used a basis for the representation that consist of states which are eigenstates of \(H\), that is are labeled by the weights of the \(\lambda^1\) representation of \(SL(n)\).

We are interested in the vectors that occur together with \(\vec{\phi}\) in the exponential in the \(\frac{1}{2}e^{\frac{2\pi i}{n} \alpha^a G^{ij} \partial_\mu \chi_i \partial^\mu \chi_j}\) term of the action (8). We find the vectors

\[ \vec{W} = (\sqrt{\frac{2n+1}{n} \alpha}, [\lambda^1]) \]  

(21)

where \([\lambda^1]\) denotes all the weights that occur in the \(SL(n)\) representation with highest weight \(\lambda^1\). In particular these are of the form of \(\lambda^1\) minus a particular set of positive roots of \(SL(n)\). As explained in appendix C, the fundamental representation of \(SL(n)\) with highest weight \(\mu^k\) and lowest weight \(\mu^{-k}\) is related to the fundamental representation with highest weight \(\lambda^k\) by \(\mu^{-k} = -\lambda^k\). Hence \([\lambda^1]\) consists of the root string beginning with \(\lambda^1\) and ending with \(-\lambda^{n-1}\).

In the action of (8) we find the roots of \(SL(n)\) as the coefficients of \(\vec{\phi}\) in the exponentials form the \(Tr(S^i S^j)\) term. We also find the above set of vectors which arise from the \(\frac{1}{2}e^{\frac{2\pi i}{n} \alpha^a G^{ij} \partial_\mu \chi_i \partial^\mu \chi_j}\) term. The simplest way to realize that we just have the set of positive roots of \(SL(n+1)\) is to consider taking the lowest weight for \([\lambda^1]\) in (21) i.e replace \([\lambda^1]\) with \(-\lambda^{n-1}\). This is just the vector

\[ \vec{\alpha}_n = (\sqrt{\frac{2n+1}{n} \alpha}, -\lambda^{n-1}) \]  

(22)

whose scalar products are

\[
\begin{align*}
\vec{\alpha}_n \cdot \vec{\alpha}_n &= 2 \\
\vec{\alpha}_n \cdot \vec{\alpha}_{n-1} &= -1 \\
\vec{\alpha}_n \cdot \vec{\alpha}_a &= 0 \quad a < n - 1
\end{align*}
\]  

(23)

Here we recognize the Cartan matrix of \(SL(n+1)\). We note that this root “attaches” itself to the \((n-1)\)th node of the \(SL(n)\) Dynkin diagram. Thus the simple roots of \(SL(n+1)\) are given by

\[ \vec{\alpha}_a = (0, \alpha_a), \ a = 1, \ldots, n-1; \quad \vec{\alpha}_n = (\sqrt{\frac{2n+1}{n} \alpha}, -\lambda^{n-1}) \]  

(24)

It is straightforward to see that the other vectors in (21) are positive integer combinations of the above simple roots since, when selecting \(\vec{\alpha}_n\), we took the
lowest weight vector in the $SL(n)$ representation. Therefore by construction all other vectors are obtained from this one by the addition of the positive roots $\tilde{\alpha}_a$ of $SL(n)$.

We observe that the fundamental weights $\tilde{\lambda}^a$ of $SL(n+1)$ can be expressed in terms of the fundamental weights $\Delta^a$ of $SL(n)$ by [45]

$$\tilde{\lambda}^a = \left( \frac{\sqrt{2}}{n} a, \Delta^a \right) \quad a = 1, \ldots, n-1 \quad \tilde{\lambda}^n = (\sqrt{2}, 0)$$  \hspace{1cm} (25)

Taking the above expression for the roots of $SL(n+1)$ it is easy to verify that these obey the equation $\tilde{\alpha}_a \cdot \tilde{\lambda}^b = \delta^b_a$.

In summary we have observed that, by looking for roots in the exponential terms, we can see an enhanced $SL(n+1)/SO(n+1)$ coset structure in the dimensional reduction of the gravity when suitably dualized. This is an old result, but our purpose in this section has been to introduce a new technique for deriving this result that will prove very effective when analyzing the very complicated higher derivative terms.

### 2.1 Higher Derivative Corrections

Let us now turn our attention to an effective action for gravity that includes higher derivative terms. One might have hoped that, after compactification to three dimension and dualization, the higher order terms can be written entirely in terms of the enhanced $SL(n+1)/SO(n+1)$ coset Cartan form $\tilde{S}_\mu$. However this cannot be the case. To see this we note that the form of the curvature components shows that a generic higher derivative term will be proportional to $e^{(2-l)\alpha^\rho}$ where $l$ counts the number of derivatives. At the lowest order $l = 2$ and this factor disappears, essentially because we are in Einstein frame. At a general order $l$ the action has the form

$$S_l = \int d^D x \sqrt{-\hat{g}(\hat{R})^l}$$  \hspace{1cm} (26)

where we have suppressed the possible combinations of Lorentz indices that can appear. Upon compactification this will lead to terms of the form $e^{(2-l)\alpha^\rho}(S_\mu S^\mu)^l$ with $l > 2$ and these cannot be written in terms of $\tilde{S}_\mu$ alone since the roots that appear in the latter, which contain $\rho$ contributions, also come with $\partial \chi$-type terms and hence are higher order in derivatives.

Therefore, since $\tilde{S}_\mu$ is determined entirely by the roots of $SL(n+1)$, when we evaluate the higher derivative terms we do not expect to find only the roots of $SL(n+1)$ appearing in the exponential terms, as we did for the lowest order terms. Arguably the next best thing is to hope that the exponential
terms will involve the weights of $SL(n+1)$. Indeed we will now show that this is the case.

For most of this paper we will assume that the higher derivative corrections only involve first order derivatives. It was shown in [23] that the next-to-leading order corrections in the effective action for gravity and the Bosonic String can always be made to be first order in derivatives by a field redefinition. Indeed one might expect that this is possible in general as a well-defined perturbation expansion of some underlying microscopic theory should not alter the degree of the equation of motion, which would then require that additional initial data must be specified on an initial value hypersurface. In other words we expect that in the underlying quantum theory the Cauchy problem does not have to be reformulated at every order in perturbation theory (at least for a certain choice of field variables). Furthermore the inclusion of higher order derivative terms would complicate the dualization of vector fields into scalars. In the presence of such terms one could proceed by simply using the first order dualization prescription however this would neglect many higher order contributions. Nevertheless in section five we show that, assuming the same dualization prescription that we use for second order equations of motion, the inclusion of terms involving more than one spacetime derivative does not affect our conclusions.

From the expressions in the appendix ones see that the higher derivative action, once compactified to three dimensions, will take the form

$$S = \int d^3x e^{2\alpha \rho} L(e^{(\beta-2\alpha)\rho} F^i, e^{-\alpha \rho} \partial \rho, e^{-\alpha \rho} S)$$  \hspace{1cm} (27)

where we have suppressed all Lorentz indices and dependence on the threedimensional metric for simplicity. Next we dualize this Lagrangian by adding the Lagrange multiplier term

$$S = \int d^3x e^{2\alpha \rho} L(e^{(\beta-2\alpha)\rho} F^i, e^{-\alpha \rho} \partial \rho, e^{-\alpha \rho} S) + \int d^3x \varepsilon^{i\rho\lambda} \chi_i \partial \rho, e^{-\alpha \rho} S)$$  \hspace{1cm} (28)

Integrating the Lagrange multiplier term by parts and solving for $F^i$ one finds

$$F^i = e^{-(\beta-2\alpha)\rho} f^i(e^{-\beta \rho} \partial \chi_i, e^{-\alpha \rho} \partial \rho, e^{-\alpha \rho} S)$$  \hspace{1cm} (29)

Substituting this back into the action leads to

$$S_{dual} = \int d^3x \sqrt{-g} e^{2\alpha \rho} L_{dual}(e^{-\beta \rho} \partial \chi_i, e^{-\alpha \rho} \partial \rho, e^{-\alpha \rho} S)$$  \hspace{1cm} (30)

From this we see that a generic higher derivative term (26) of order $l$ will lead to three-dimensional terms of the form

$$e^{2(1-r-s+\frac{1}{2})\alpha \rho} (\partial \rho)^{2r}(\partial \chi_i)^{2s}$$  \hspace{1cm} (31)
where the order is
\[ l = 2r + 2s + 2t \]  
(32)

We have used that fact that only even powers of \( \partial \rho, S \) and \( \partial \chi_i \) can appear. To see this we observe that since \( x^i \to -x^i \) is a symmetry of the action we must have that \( A^i_{\mu} \to -A^i_{\mu} \) is a symmetry and hence the dual action is an even function of \( \partial \chi_i \). We also note that if \( e \) is a choice of internal vielbein, \( i.e. \) an \( n \times n \) matrix with unit determinant, then so is \( e^{-1} \). Therefore if \( S \) is a solution to the equations of motion so is \( -S \) and hence the action must be an even function of \( S \). It then follows that the action must be an even function of \( \partial \rho \) since it must be even in derivatives.

We can now read off the vectors that appear in the exponentials of \( \vec{\phi} \). Using appendix C, and the analysis of the previous section, we find the vectors
\[ \vec{W}_{r,s,t} = \left( \sqrt{2} \left( 1 - \frac{l}{2} + \frac{n+1}{n} t \right) \alpha, s[\theta] + t[\Lambda^1] \right) \]  
(33)

We note that the coset representative \( S^* \) only contains positive \( SL(n) \) roots and so only the positive roots in the equivalence class \( [\theta] \) actually appear. As above \( [\Lambda] \) means the collection of weights that occur in the representation with highest weight \( \Lambda \). As a result we note that the subscripts \( (r,s,t) \) do not uniquely specify the vectors that arise. The highest weight for the adjoint representation is called \( \theta \) and in particular, for \( SL(n) \), \( \theta = \Lambda^1 + \Lambda^{n-1} \). These contributions arise from the factors of \( S_{\mu} \) while those proportional to \( t \) arise from the graviphoton.

At order \( l = 2 \), so that one of \( (r,s,t) \) equals two with the others vanishing, we recover the roots \( \alpha_a, a = 1, \ldots, n \) of \( SL(n+1) \) that we considered above. More generally we may write the vector (33) as
\[ \vec{W}_{r,s,t} = \vec{W}_{r,s,t}^H + \text{negative roots of the } SL(n) \text{ subalgebra} \]  
(34)

where
\[ \vec{W}_{r,s,t}^H = \left( \sqrt{2} \left( 1 - \frac{l}{2} + \frac{n+1}{n} t \right) \alpha, s[\theta] + t[\Lambda^1] \right) \]  
(35)

Recalling that \( \theta = \Lambda^1 + \Lambda^{n-1} \) one can show that
\[ \vec{W}_{r,s,t}^H = (1 - 2s - r)\tilde{\lambda}^n + (t + s)\tilde{\lambda}^1 + s\tilde{\lambda}^{n-1} \]  
(36)

It follows that the vectors that have arisen in the higher derivative terms belong to the weight lattice of \( SL(n+1) \).

Any weight can be expanded in terms of the fundamental weights \( \tilde{\lambda}^a \) as \( \tilde{\lambda} = \sum c_a \tilde{\lambda}^a \). We say that \( \tilde{\lambda} \) is dominant if and only if \( c_a \geq 0 \) for all \( a \). A central theorem of Lie algebras asserts that the finite dimensional
irreducible representations of a Lie group are in one-to-one correspondence with the dominant weights. Furthermore the highest weight of an irreducible representation must be dominant. The difference between any two weights of an irreducible representation always gives a root. One can define the highest weight to be the unique weight $\Lambda^H$ such that $\Lambda^H + \alpha$ is not a weight in the representation for any positive root $\alpha$. Similarly the lowest weight $\Lambda^L$ such that $\Lambda^L - \alpha$ is not a weight in the representation for any positive root $\alpha$.

We now wish to identify what representation of $SL(n+1)$ these weights belong to. If there they can be identified with the weights of an irreducible representation then the highest weight must be expressible as positive integers times the fundamental weights of $SL(n+1)$. Examining equation (36) ones see that we can only take $r = s = 0$ or $r = 1, s = 0$. Taking the former choice we claim that the highest weight is

$$\bar{\lambda}^H = \frac{l}{2} \bar{\lambda}^1 + \bar{\lambda}^n$$

(37)

To prove that this is indeed the highest weight of a representation we must show that this weight, minus any other weight, is a sum of positive roots. To this end we evaluate

$$\bar{\lambda}^H - \bar{W}_{r,s,t} = s(2\bar{\lambda}^n - \bar{\lambda}^n - 1) + r(\bar{\lambda}^n + \bar{\lambda}^1) + \text{positive roots of the } SL(n) \text{ subalgebra}$$

(38)

Lastly we can show by direct computation that $\bar{\lambda}^1 + \bar{\lambda}^n$ and $2\bar{\lambda}^n - \bar{\lambda}^n - 1$ have non-negative and integral innerproducts with the fundamental weights. This implies that they are positive roots. Therefore since $\bar{\lambda}^H - \bar{W}_{r,s,t}$ is a sum of positive roots we conclude that $\bar{\lambda}^H$ is indeed the highest weight. We note that this calculation also shows that the difference between any two weights that we find is indeed a root.

It is also instructive to identify the lowest weight $\bar{\lambda}^L$ that appears in the effective action, in the sense that $\bar{W}_{r,s,t} - \bar{\lambda}^L$ is a positive root for all $\bar{W}_{r,s,t}$. By a similar argument we find

$$\bar{\lambda}^L = (1 - \frac{l}{2})\bar{\lambda}^n$$

(39)

(recall that only the positive roots in the equivalence class $[\theta]$ appear). We recall that, for $SL(n+1)$, the lowest weight in the representation whose highest weight is $\bar{\lambda}^a$ is $-\bar{\lambda}^{n+1-a}$. Therefore the lowest weight in the representation whose highest weight is $\bar{\lambda}^H$ is actually $-\bar{\lambda}^1 - \frac{l}{2} \bar{\lambda}^n$. However we see that

$$\bar{\lambda}^L = -\bar{\lambda}^1 - \frac{l}{2} \bar{\lambda}^n + \bar{\theta}$$

(40)
where $\vec{\theta} = \vec{\lambda}^1 + \vec{\lambda}^n$ is the highest weight associated to the adjoint representation. This result is not surprising as one expects that the local $SO(n + 1)$ symmetry has been used to gauge away terms that are associated with the lowest weights. This is consistent with the lowest order Lagrangian which only contains positive roots because all the terms associated to the negative roots have been gauged away by $SO(n + 1)$.

In summary we have seen that the higher derivative corrections, which apparently cannot be written just in terms of an $SL(n + 1)/SO(n + 1)$ coset, are still organized by $SL(n + 1)$, in the sense that the weights of $SL(n + 1)$ appear in the exponential. Thus our result shows evidence for an $SL(n + 1)$ symmetry in a full theory of gravity and not just the low energy effective theory.

### 3 M-theory

Our final example is to consider the effective action of M-theory whose lowest order effective action is

$$S = \int d^{11}x \sqrt{-\hat{g}} \left( \hat{R} - \frac{1}{48} \hat{G}_{\mu\nu\lambda\rho} \hat{G}^{\mu\nu\lambda\rho} \right) + \ldots$$

(41)

where $G_{\mu\nu\lambda\rho}$ is a closed 4-form and the ellipsis denotes a Chern-Simons term that will not play a role here. Since we are interested in compactifying this action on an eight-torus our discussion also applies to the type II superstring theories. Note that here $D = 11$ and we wish to reduce to three-dimensions so that $n = 8$ and hence $\alpha = 2/3$.

Upon dimensional reduction to three-dimensions the 4-form $\hat{G}$ leads to two the dynamical fields

$$\hat{G}_{\mu \nu ij} = F_{\mu \nu ij}, \quad \hat{G}_{\mu ijk} = \partial_{\mu} \chi_{ijk}$$

(42)

Therefore, using the results for pure gravity, a generic higher derivative term, one reduced to three dimensions, will be of the form

$$S = \int d^3x e^{2\alpha \rho} L(e^{(\beta - 2\alpha)\rho} F^i, e^{-\alpha \rho} \partial \rho, e^{-\alpha \rho} S, e^{-(2\alpha + 2 \beta)\rho} F_{ij}, e^{-(\alpha + 3 \beta)\rho} \partial \chi_{ijk})$$

(43)

Just as before we must introduce Lagrange multipliers for the two types of 2-form fields strengths

$$S = \int d^3x e^{2\alpha \rho} L(e^{(\beta - 2\alpha)\rho} F^i, e^{-\alpha \rho} \partial \rho, e^{-\alpha \rho} S, e^{-(2\alpha + 2 \beta)\rho} F_{ij}, e^{-(\alpha + 3 \beta)\rho} \partial \chi_{ijk})$$

$$+ \int d^3x e^{2\alpha \rho} \chi_{ij} \partial_{\mu} F_{\nu \lambda} + \int d^3x e^{2\alpha \rho} \chi_{ij} \partial_{\mu} F_{\nu \lambda}$$

(44)
where \( F^i_{\mu\nu} \) are the graviphoton field strengths. Integrating the Lagrange multiplier terms by parts and solving for \( F^i_{\mu\nu} \) and \( F_{\mu\nu ij} \) one finds

\[
F^i_{\mu\nu} = e^{-(\beta-2\alpha}\rho} f^i_{\mu\nu} \left( e^{-\beta\rho}\partial\chi_i, e^{-\alpha\rho}\partial\rho, e^{-\alpha\rho}\partial S, e^{-(\alpha+3\beta)\rho}\partial\chi_{ijk}, e^{2\beta\rho}\partial\chi_{ij} \right)
\]
\[
F_{\mu\nu ij} = e^{(2\alpha+2\beta)\rho} f_{\mu\nu ij} \left( e^{-\beta\rho}\partial\chi_i, e^{-\alpha\rho}\partial\rho, e^{-\alpha\rho}\partial S, e^{-(\alpha+3\beta)\rho}\partial\chi_{ijk}, e^{2\beta\rho}\partial\chi_{ij} \right)
\]

Therefore the dual Lagrangian takes the form

\[
S_{\text{dual}} = \int d^3x e^{2\alpha\rho} L(e^{-\beta\rho}\partial\chi_i, e^{-\alpha\rho}\partial\rho, e^{-\alpha\rho}\partial S, e^{-(\alpha+3\beta)\rho}\partial\chi_{ijk}, e^{2\beta\rho}\partial\chi_{ij})
\]

A generic term will therefore be

\[
e^{2\left(1-r-s+\frac{t}{2}-\frac{p_1}{2}+\frac{p_2}{2}\right)\alpha\rho}(\partial\rho)^{2r}(S)^{2s}(\partial\chi_i)^{2t}(\partial\chi_{ijk})^{2p_1}(\partial\chi_{ij})^{2p_2}
\]

where the order is \( l = 2r + 2s + 2t + 2p_1 + 2p_2 \).

The vectors that we encounter will be 8-dimensional so we take

\[\vec{\phi} = (\rho, \phi)\]

and in this way we find

\[W_{r,s,t,p_1,p_2} = (w, s[\lambda^1] + t[\lambda^3] + p_1[\lambda^6])\]

with

\[w = \frac{2\sqrt{2}}{3} \left(1 - \frac{l}{2} + \frac{9}{8}t + \frac{3}{8}p_1 + \frac{6}{8}p_2\right)\]

Here \([\theta]\) is one of the positive roots that appears as a weight in the adjoint representation and \([\Lambda^a]\) is one of the \(SL(8)\) weights that appears in the representation whose highest weight is \(\lambda^a\). Note that \(W_{r,s,t,p_1,p_2}\) is not uniquely determined by specifying the positive integers \((r, s, t, p_1, p_2)\) due to the different choice of representative of \([\Lambda^a]\).

In \(W_{r,s,t,p_1,p_2}\) the weights \([\lambda^1]\) arise just as they did for the case of pure gravity. However we should comment on why \([\lambda^3]\) and \([\lambda^6]\) have appeared. These also arise in the same manner that \([\lambda^1]\) does, only rather than coming from evaluating \(\partial_\mu\chi_i G^{ij} \partial_\mu\chi_j\) they arise from evaluating terms involving \(\partial_\mu\chi_{ijk} G^{il} G^{jm} G^{kn} \partial_\nu\chi_{lmn}\) and \(\partial_\mu\chi_{ij} G_{ik} G_{jl} \partial_\nu\chi_{kl}\) respectively. The discussion that we had for pure gravity still applies except that we must change the representation that the fields transform under. In the former case, since \(\chi_{ijk}\) transforms in the 3-fold anti-symmetric representation we find the weights \([\lambda^3]\) of this representation. For \(\partial_\mu\chi_{ij} G_{ik} G_{jl} \partial_\nu\chi_{kl}\) one first notes that the \(\epsilon_{ijklmnpq}\) symbol can be used to lower the indices so that \(\chi_{klmpq} = \epsilon_{ijklmpq}\chi_{ij}\)
transforms in the 6-fold anti-symmetric representation. Hence it appears with the weights $[λ^6]$.

At lowest order $l = 2$ and the action is given by (41). Upon dimensional reduction and dualization the vectors that appear are

$$
\begin{align*}
\vec{W}_{0,1,0,0} & = (0, \alpha) \\
\vec{W}_{0,0,1,0} & = \left(\frac{3\sqrt{2}}{4}, [λ^1]\right) \\
\vec{W}_{0,0,0,1} & = \left(\frac{\sqrt{2}}{4}, [λ^3]\right) \\
\vec{W}_{0,0,0,0} & = \left(\frac{2\sqrt{2}}{4}, [λ^6]\right)
\end{align*}
$$

(51)

where we have used the fact that $n = 8$. The first line just consists of the roots of $SL(n)$ whereas the second line gives the same roots that we saw for pure gravity and which enhance $SL(n)$ to $SL(n + 1)$ however this root is no longer simple in this case. In particular we consider the third line and take the lowest weight in the representation with weights $[λ^3]$

$$
\vec{α}_8 = \left(\frac{\sqrt{2}}{4}, -λ^5\right)
$$

(52)

It is straightforward to see that $\vec{α}_8 \cdot \vec{α}_8 = 2$ and $\vec{α}_8 \cdot \vec{α}_5 = -1$ with the other innerproducts vanishing. The other choices of $W_{0,0,0,1,0}$ are obtained from $\vec{α}_8$ by adding a positive root from $SL(n)$ and hence these are positive but not simple roots. One also sees that none of the choices for $W_{0,0,0,1}$ and $W_{0,1,0,0,0}$ are simple. Thus we have recovered the root diagram for $E_8$. To summarize these roots are

$$
\vec{α}_a = (0, \alpha_a) , a = 1, ..., 7 \quad \vec{α}_8 = \left(\frac{\sqrt{2}}{4}, -λ^5\right)
$$

(53)

The fundamental weights can be calculated to be

$$
\begin{align*}
\vec{λ}^a & = \left(\frac{3\sqrt{2}}{4}a, \alpha_a\right), a = 1, 2, 3, 4 \\
\vec{λ}^a & = \left(\frac{5\sqrt{2}}{4}(8-a), \alpha_a\right), a = 5, 6, 7 \\
\vec{λ}^8 & = (2\sqrt{2}, 0)
\end{align*}
$$

(54)
and hence one finds that
\[ \tilde{W}_{r,s,t,p_1,p_2} = \left( \frac{1}{3} - \frac{l}{6} - s - p_1 - p_2 \right) \tilde{\lambda}^8 + (s + t) \tilde{\lambda}^1 + p_1 \tilde{\lambda}^3 + p_2 \tilde{\lambda}^6 + s \tilde{\lambda}^7 + \text{negative roots of the } SL(8) \text{ subalgebra} \]

(55)

Therefore we see that the condition for \( \tilde{W}_{r,s,t,p_1,p_2} \) to be a weight is
\[ \frac{1}{3} - \frac{l}{6} \in \mathbb{Z} \iff l = 6g + 2 \]

(56)

for some \( g = 0, 1, 2, \ldots \). Thus the conjecture that the higher derivative terms involve the weights of \( E_8 \) implies that they can only come at orders \( l = 2, 8, 14, \ldots \), i.e. \( R, R^4, R^7, \ldots \). This result is well known from quantum considerations of type IIA string theory however here we see that it is also a consequence of the existence of an enhanced coset structure arising in three dimensions.

It is instructive to identify the highest and lowest weights that appear at a fixed order \( l \). By inspection one sees that a reasonable guess for highest weight is
\[ \tilde{\lambda}^H = -\frac{1}{6}(l - 2)\tilde{\lambda}^8 + \frac{l}{2} \tilde{\lambda}^1 \]

(57)

To prove this we proceed as we did for pure gravity and note that
\[ \tilde{\lambda}^H - \tilde{W}_{r,s,t,p_1,p_2} = s(\tilde{\lambda}^8 - \tilde{\lambda}^7) + r \tilde{\lambda}^1 + p_1(\tilde{\lambda}^8 + \tilde{\lambda}^1 - \tilde{\lambda}^3) + p_2(\tilde{\lambda}^8 + \tilde{\lambda}^1 - \tilde{\lambda}^3) + \text{positive roots of the } SL(8) \text{ subalgebra} \]

(58)

One can show by explicit calculations that \( \tilde{\lambda}^8 - \tilde{\lambda}^7, \tilde{\lambda}^1, \tilde{\lambda}^8 + \tilde{\lambda}^1 - \tilde{\lambda}^3 \) and \( \tilde{\lambda}^8 + \tilde{\lambda}^1 - \tilde{\lambda}^3 \) are all positive roots of \( E_8 \). A similar calculation shows that the lowest weight, in the sense that \( \tilde{W}_{r,s,t,p_1,p_2} - \tilde{\lambda}^L \) is a positive root for all \( \tilde{W}_{r,s,t,p_1,p_2} \), is
\[ \tilde{\lambda}^L = -\frac{1}{6}(l - 2)\tilde{\lambda}^8 \]

(59)

We observe that these are not dominant weights and therefore they cannot be the highest weights of a finite dimensional irreducible representation of \( E_8 \). Note that \( E_8 \) is special in that the weight lattice and root lattice are the same. Curiously one finds that \( \tilde{\lambda}^H \) and \( \tilde{\lambda}^L \) are positive and negative roots respectively. We also see that
\[ \tilde{\lambda}^L = \tilde{\lambda}^H - \frac{l}{2} \theta \]

(60)
where $\vec{\theta} = \vec{\lambda}_1$ is the highest weight of the adjoint representation of $E_8$. As with pure gravity we expect that some weights have been gauged away using the local symmetry. Precisely how this happens is not clear to us and the pattern of weights that we find deserves further study.

We note that there is a separate argument for the condition that $l = 6g + 2$ based on the fact that the $M$-theory effective action is the lift to eleven dimensions of the type IIA effective action. In particular consider a higher derivative term in the $M$-theory effective action of the form

$$S_l \sim \int d^{11}x \sqrt{-\hat{g}(\hat{R})}$$  \hspace{1cm} (61)$$

The type IIA metric, in string frame, is related to the 11-dimensional metric through

$$d\hat{s}^2 = e^{-2\Phi/3}d\tilde{s}^2 + e^{4\Phi/3}(dx^{11} + A_\mu dx^\mu)^2$$  \hspace{1cm} (62)$$

where $\Phi$ is the dilaton. Therefore, in ten-dimensions,

$$S_l \sim \int d^{10}x \sqrt{\hat{g}} e^{-8\Phi/3}(e^{2\Phi/3}(R) + \ldots)^{\frac{1}{2}}$$

$$\sim \int d^{10}x \sqrt{\hat{g}} e^{-(8-l)\Phi/3}(\hat{R}^{\frac{1}{2}} + \ldots)$$  \hspace{1cm} (63)$$

However from string perturbation theory we recall that $g_s = e^\Phi$ is the string coupling and that the action is an expansion in $e^{2(g-1)\Phi}$ with $g = 0, 1, 2, \ldots$. Thus we see that $-(8-l)/3 = 2(g-1)$ \emph{i.e.} $l = 6g + 2$.

Finally we should comment on other terms that are known to arise in the $M$-theory effective action and which do not naively take the form that we have considered. At lowest order there is a Chern-Simons term that only involves the three-form gauge field

$$S_{CS} = \int \hat{G} \wedge \hat{G} \wedge \hat{C}_3$$  \hspace{1cm} (64)$$

Upon reduction to three-dimensions, and integration by parts, this leads to a term the form

$$S_{CS} = \int d^3 x \ e^{\mu\nu\lambda} e^{i_1 \ldots i_8} F_{\mu\nu i_{12}} \chi_{i_3 i_4 i_5} \partial_\lambda \chi_{i_6 i_7 i_8}$$  \hspace{1cm} (65)$$

The effect of this term is to alter the dualization formulae by adding a $\chi \partial \chi$ term. In particular it leads to higher order $\chi \partial \chi$ terms which we have consistently neglected throughout this paper.

At the next-to-leading order, \emph{i.e.} $l = 8$, there is an anomaly term of the form \cite{30}

$$S_{I_8} = \int I_8(\hat{R}) \wedge \hat{C}_3$$  \hspace{1cm} (66)$$

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where \( I_8(R) \) is an eight-dimensional topological class. Upon reduction to three-dimensions this leads to terms of the form

\[
S_{I_8} = \int d^3 x e^{2\alpha \rho} \Omega_{I_8} \left( e^{(\beta-2\alpha)\rho} F^i, e^{-\alpha \rho} \partial \chi_{ij}, e^{-\alpha \rho} \partial \rho, e^{-\alpha \rho} S \right) e^{-(\alpha+3\beta)\rho} \chi_{ijk}
\]

\[
+ e^{2\alpha \rho} \Omega_{I_8}' \left( e^{(\beta-2\alpha)\rho} F^i, e^{-\alpha \rho} \partial \chi_{ij}, e^{-\alpha \rho} \partial \rho, e^{-\alpha \rho} S \right) e^{-(2\alpha+2\beta)\rho} A_{ij}
\]

(67)

where we have allowed for the fact that the curvature terms involve higher order derivatives acting on the fields. Some terms, namely those which contain an undifferentiated \( \chi_{ijk} \), play a role similar to the Chern-Simons term and introduce higher order \( \chi \partial \chi \) corrections into the dualization formulae which we neglect. We believe that the remaining terms are those that contain an extra spacetime derivative and can be rearranged using integrated by parts into the form

\[
S_{I_8} = \int d^3 x e^{2\alpha \rho} \Omega_{I_8} \left( e^{(\beta-2\alpha)\rho} F^i, e^{-\alpha \rho} \partial \chi_{ij}, e^{-\alpha \rho} \partial \rho, e^{-\alpha \rho} S \right) e^{-(\alpha+3\beta)\rho} \partial \chi_{ijk}
\]

\[
+ e^{2\alpha \rho} \Omega_{I_8}' \left( e^{(\beta-2\alpha)\rho} F^i, e^{-\alpha \rho} \partial \chi_{ij}, e^{-\alpha \rho} \partial \rho, e^{-\alpha \rho} S \right) e^{-(2\alpha+2\beta)\rho} F_{ij}
\]

(68)

Indeed this must be the case if this correction preserves the second order nature of the equations of motion. However these terms are of the type that we have already considered in (43), although it is important to note that any internal indices are not contracted using the metric but rather an \( \epsilon \)-symbol and this will affect the calculation of the weights. In any case we believe that the vectors which arise as coefficients of the scalar fields in these terms are consistent with the pattern of \( E_8 \) weights that we found above.

## 4 The Bosonic String

Our next example is the Bosonic String in \( D \)-dimensions. At lowest order in derivatives the effective action is

\[
S = \int d^D x \sqrt{-g} \left( \hat{R} - \frac{1}{2} \partial \mu \phi \partial^\mu \phi - \frac{1}{12} e^{\sqrt{\frac{8}{D-2}} \phi} \hat{H}_{\mu\nu\lambda} \hat{H}^{\mu\nu\lambda} \right)
\]

(69)

where \( \hat{H} \) is a closed 3-form and the scalar \( \phi \) is related to the string theory dilaton \( \Phi \) by \( \phi = -\sqrt{8/(D-2)} \Phi \). Since we will reduce to three-dimensions we will take \( n = D - 3 \) and hence \( \sqrt{8/(D-2)} = \sqrt{8/(n+1)} \). Upon dimensional reduction to three-dimensions the three form \( \hat{H} \) leads to the dynamical fields

\[
\hat{H}_{\mu
u} = G_{\mu
u} \quad \hat{H}_{\mu ij} = \partial_{\mu} \chi_{ij}
\]

(70)
A generic higher derivative term, one reduced to three-dimensions, will be of the form

$$S = \int d^3x e^{2\alpha \rho} L(e^{(\beta - 2\alpha)\rho} F^i, e^{-\alpha \rho} \partial \rho, e^{-\alpha \rho} S, e^{-(2\alpha + \beta)\rho} G^i, e^{-(\alpha + 2\beta)\rho} \partial \chi_{ij}, \phi)$$

(71)

Before proceeding to dualize the electromagnetic field strengths as we did above we note that the most general higher derivative action will contain arbitrary exponentials of $\phi$ that multiply the electromagnetic field strengths. This complicates the dualization and we have been unable to find tractable formulae for the general term in this case, as we did for pure gravity and M-theory. Therefore in this section we will simply use the dualization formulae that arise at the lowest order. In particular after reducing (69) to three dimensions we add the terms

$$\int d^3x \epsilon_{\mu \nu \lambda} \chi_i \partial_\mu F^i_{\nu \lambda} + \int d^3x \epsilon_{\mu \nu \lambda} \psi^i \partial_\mu G^i_{\nu \lambda}$$

(72)

where $F^i_{\mu \nu}$ is the graviphoton field strength. Integrating out the field strengths $F^i_{\mu \nu}$ and $G^i_{\mu \nu}$ we find

$$F^i_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \lambda} G_{ij} \epsilon^{2(\alpha - \beta)\rho} \partial^\lambda \chi_j$$

$$G^i_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \lambda} G_{ij} \epsilon^{2(\alpha + \beta)\rho} e^{-\sqrt{\frac{1}{n+1}} \phi} \partial^\lambda \psi^i$$

(73)

In general these equations will be corrected by higher order terms. Therefore if we continue by substituting these lowest order expressions into the full effective action we will in fact be neglecting many additional terms. However the terms that we find in this manner would still be present even if we could somehow perform a complete treatment. Therefore we will proceed with the knowledge that our results only capture a subset of all the terms that appear. However we do not believe that our conclusions would be altered by a more complete treatment.

To continue we observe that a generic term in the effective action will be of the form

$$e^{u \phi} e^{2(1 - r - s) - \frac{2n+1}{n} t - \frac{n-2}{n} q_1 - \frac{2n-1}{n} q_2) \alpha \rho} (\partial \rho)^{2r} (S)^{2s} (F^i)^{2t} (\partial \chi_{ij})^{2q_1} (G^i)^{2q_2} (\partial \phi)^{2v}$$

(74)

After dualization, using the lowest order expressions (73) this term takes the form

$$e^{\left(u - 2\sqrt{\frac{1}{n+1}} q_2 \right) \phi} e^{2w \alpha \rho} (\partial \rho)^{2r} (S)^{2s} (\partial \chi_{ij})^{2t} (\partial \chi_{ij})^{2q_1} (\partial \psi^i)^{2q_2} (\partial \phi)^{2v}$$

(75)
where
\[ w = 1 - r - s + \frac{1}{n} t - \frac{n - 2}{n} q_1 - \frac{1}{n} q_2 \]  
(76)
and the order is \( l = 2r + 2s + 2t + 2q_1 + 2q_2 + 2v \).

Note that since we also have the a dependence on the dilaton the vectors that we find will have two additional entries as compared to the weights of \( SL(n) \). Therefore in this section we define
\[ \vec{\phi} = (\phi, \rho, \phi) \]  
(77)
and hence with this notation we obtain
\[ \vec{W}_{r,s,t,q_1,q_2} = \left( \frac{u}{\sqrt{2}} - \frac{4 q_2}{\sqrt{n+1}}, \frac{u}{\sqrt{2}} \left( \frac{1}{2} - \frac{n+1}{n} t + \frac{2}{n} q_1 + \frac{n-1}{n} q_2 \right) \alpha, \right. \]
\[ \left. s[\theta] + t[\lambda^1] + q_1[\lambda^2] + q_2[\lambda^{n-1}] \right) \]  
(78)
Once again \( [\lambda] \) represents any of the weights that appear in a representation whose highest weight is \( \Lambda \) and \( \theta = \lambda^1 + \lambda^{n-1} \) is the highest weight of the adjoint representation. We should comment on why \( [\lambda^2] \) and \( [\lambda^{n-1}] \) appear. However these arise in the same way that \( [\lambda^3] \) and \( [\lambda^6] \) did in the previous section on M-theory.

At order \( l = 2 \) find
\[ \vec{W}_{0,0,0,0} = (0, 0, [\theta]) \]
\[ \vec{W}_{0,0,1,0} = (0, \sqrt{2} \frac{n+1}{n} \alpha, [\lambda^1]) \]
\[ \vec{W}_{0,0,0,1} = \left( \sqrt{\frac{4}{n+1}}, 2 \sqrt{2} \frac{1}{n} \alpha, [\lambda^2] \right) \]
\[ \vec{W}_{0,0,0,0} = \left( - \sqrt{\frac{4}{n+1}}, \sqrt{2} \frac{n-1}{n} \alpha, [\lambda^{n-1}] \right) \]  
(79)
where we have used the fact that at lowest order \( u = 0, \sqrt{\frac{8}{n+1}} \). In particular
\[ \vec{\alpha}_a = (0, 0, [\alpha]) \quad a = 1, \ldots, n-1 \]
\[ \vec{\alpha}_n = \left( \sqrt{\frac{4}{n+1}}, 2 \sqrt{2} \frac{1}{n} \alpha, -\lambda^{n-2} \right) \]
\[ \vec{\alpha}_{n+1} = \left( - \sqrt{\frac{4}{n+1}}, \sqrt{2} \frac{n-1}{n} \alpha, -\lambda^1 \right) \]  
(80)
One can show that $\vec{W}_{0,0,1,0,0}$ is not simple, i.e. it can be expressed as a linear combination of the $\vec{\alpha}_a \, a = 1, \ldots, n + 1$ with integer coefficients. The roots $\vec{\alpha}_a \, a = 1, \ldots, n - 1$ are those of $SL(n)$. In addition we find $\vec{\alpha}_n$ and $\vec{\alpha}_{n+1}$. Each has length $\sqrt{2}$ and the only other non-vanishing innerproducts are $\vec{\alpha}_n \cdot \vec{\alpha}_{n-2} = \vec{\alpha}_{n+1} \cdot \vec{\alpha}_1 = -1$. Thus we have recovered the simple roots of $O(n + 1, n + 1)$.

To proceed we note that the fundamental weights are

$$\vec{\lambda}^a = \left( \frac{a - 1}{\sqrt{n + 1}}, \sqrt{2} \alpha \frac{(n + a)}{n}, \Delta^a \right), \quad a = 1, \ldots, n - 2$$

$$\vec{\lambda}^{n-1} = \left( \frac{1}{2} \frac{n - 3}{\sqrt{n + 1}}, \sqrt{2} \alpha \frac{(n - 1)}{n}, \Delta^{n-1} \right)$$

$$\vec{\lambda}^n = \left( \frac{1}{2} \frac{n - 1}{\sqrt{n + 1}}, \sqrt{2} \alpha, 0 \right)$$

$$\vec{\lambda}^{n+1} = \left( - \frac{1}{\sqrt{n + 1}}, \sqrt{2} \alpha, 0 \right)$$

(81)

It follows that

$$\vec{W}_{r,s,t,q_1,q_2} = (s + t)\vec{\lambda}^1 + q_1 \vec{\lambda}^2 + (s + q_2)\vec{\lambda}^{n-1} + w_n \vec{\lambda}^n + w_{n+1} \vec{\lambda}^{n+1}$$

+ negative roots of the $SL(n)$ subalgebra

(82)

where

$$w_n = \frac{\sqrt{2} \sqrt{n + 1} u + 2 - l - s(n + 1) - 4 q_1 - q_2 (n + 5)}{n + 1}$$

$$w_{n+1} = 1 - \frac{l}{2} - 2 s - q_1 - w_n$$

(83)

Thus we see that $\vec{W}_{r,s,t,q_1,q_2}$ will be a weight if and only if $w_n, w_{n+1} \in \mathbb{Z}$. We note that $w_{n+1}$ is an integer if and only if $w_n$ is an integer so that we arrive at the condition

$$\frac{\sqrt{2} \sqrt{n + 1} u + 2 - l - s(n + 1) - 12 q_1 - q_2 (n - 3)}{n + 1} \in \mathbb{Z}$$

(84)

To illuminate the physical content of this condition we observe that the string metric $\hat{g}_{\hat{\mu} \hat{\nu}}$ is related to the Einstein metric through $\hat{g}_{\hat{\mu} \hat{\nu}}^E = e^{-\frac{2}{1 + \pi} \Phi} \hat{g}_{\hat{\mu} \hat{\nu}}$.
with \( \phi = -\sqrt{\frac{8}{n+1}} \Phi \). Thus in string frame the general term (74) comes from a term of the form

\[
e^{2(g-1)\Phi} \sqrt{-\hat{g}^S (\hat{R})^{r+s+t} (\hat{H})^{2q_1+2q_2} (\partial \Phi)^{2v}}
\]

with

\[
g - 1 = \frac{2(r + s + t) + 6q_1 + 6q_2 + 2v - (n + 3) - \sqrt{2\sqrt{n+1}u}}{n + 1}
\]

Such a term arises in string perturbation theory at genus \( g \) and hence we must identify \( g \) with a non-negative integer. Solving for \( u \) as a function of \( r, s, t, q_1, q_2, v \) and \( g \) we can substitute back into (84) and find

\[
\vec{W}_{r,s,t,q_1,q_2} = (s + t)\vec{\lambda}^1 + q_1\vec{\lambda}^2 + (s + q_2)\vec{\lambda}^{n-1} - (g + s + q_2)\vec{\lambda}^n + (1 + g + q_2 - q_1 - s - \frac{l}{2})\vec{\lambda}^{n+1} + \text{negative roots of the } SL(n) \text{ subalgebra}
\]

Thus the condition that we find weights of the enhanced coset is equivalent to the statement that the higher order terms arise from string perturbation theory on a genus \( g \) surface. We note that all these weights have a negative coefficient of \( \vec{\lambda}^n \) when \( l > 2 \) and hence cannot be dominant. Therefore they cannot be identified with the highest weights of a finite-dimensional irreducible representation of \( O(n+1, n+1) \). Although we recall that we have only used the lowest order dualization procedure and hence that we have neglected many terms and their associated weights.

We would like to comment further on the nature of the terms that we have ignored by taking the lowest order dualization formulae. Since the higher order corrections to the dualization formulae can be viewed as a power series in \( e^{2\Phi} \) their inclusion will only lead to corrections in the dualized Lagrangian which are also a power series in \( e^{2\Phi} \). Thus we expect that they will also be consistent with the condition that the vectors that we obtain are weights of the \( O(n+1, n+1) \) symmetry if and only if they arise from a given order of string perturbation theory.

5 Weights of Terms With Two or More Space-time Derivatives Acting on a Field.

Although we have indicated that one might not expect to find higher derivative terms that have more than one spacetime derivative acting on a field,
we will now examine what weights correspond to such contributions. Let us first consider the case of pure gravity. Examining (98) of appendix A we find that in the Riemann tensor there are terms that have more than one spacetime derivative acting on a field, i.e. $\partial F^i$, $(\partial S)$ and $(\partial^2 \rho)$. Rather than compute the weights of such terms directly, it is simpler to compute the change of weight when we swap the term $F^i F^i$ by $\partial^2$. This preserves the number of spacetime derivatives. Of course the latter factor acts on some other fields but for the purpose of deriving the change in the weight one does not have to know which factor it acts on. The term $F^i F^i$ leads to the weight $\left(\sqrt{2} \alpha n, -\lambda n - 1\right)$ while $\partial^2$ leads to $(-\sqrt{2} \alpha, 0)$. Hence swapping the former term by the latter leads to the change in weight

$$\Delta W = \left(-\sqrt{2} \alpha n + 1, \lambda n - 1\right) + \text{negative root of the } SL(n) \text{ subalgebra}$$

Thus $\Delta W$ is a negative $SL(n+1)$ root. Since the higher derivative term that includes $F^i F^i$ leads to an $SL(n+1)$ weight, we must conclude that including the above term also leads to $SL(n+1)$ weights. Thus all terms that arise in a correction constructed from the Riemann tensor lead to weights of $SL(n+1)$.

We now consider the case of M-theory. So far we have only considered higher derivative terms that are polynomial in the Riemann tensor and the four form field strength. The second derivatives that were discussed above for gravity also arise in M-theory. However the $SL(9)$ algebra of pure gravity compactified from eleven dimensions to three dimensions is a subalgebra of $E_8$. In particular one can show that, in terms of $E_8$ roots

$$\Delta W = -3\alpha_8 + \text{negative root of the } SL(8) \text{ subalgebra}$$

Thus this change will take weights of $E_8$ to other weights of $E_8$. However, the higher derivative terms that arise from the quantum corrections also involve derivatives of the four form field strength. We will now discuss these terms. Rather than work out all such terms and their corresponding weights it is simpler to work out what is the change of the weight when we replace the term $F^i F^i$ by $\partial^2$. As remarked above, the latter factor acts on some other field strength, but for the purpose of deriving the change in the weight one does not have to know which factor it acts on. The $F^i F^i$ factor gives rise to the $e^{-2(\alpha + 3\beta)\rho} e^{\sqrt{2} \lambda \cdot \phi} = e^{-2\alpha_8} e^{\sqrt{2} \lambda \cdot \phi}$ while $\partial^2$ just comes with
\[ e^{-2\alpha\rho}. \] Hence the change in weight due to replacing the former factor by the latter factor is \[ \Delta W^{(1)} = \left( -\frac{\sqrt{2}}{4}, -[\lambda^3] \right) \]

\[ = \left( -\frac{\sqrt{2}}{4}, \Delta^5 \right) + \text{negative root of the } SL(8) \text{ subalgebra} \]

\[ = -\vec{\alpha}_8 + \text{negative root of the } SL(8) \text{ subalgebra} \]

(90)

Since this is a negative \( E_8 \) root we see that the replacement will indeed change a weight into a weight.

One can also find the change in the weight that is induced by swapping the term \( F_{\mu\nu ij}F^{\mu\nu ij} \) with \( \partial^2 \). Using the same techniques one finds that the change is given by

\[ \Delta W^{(2)} = \left( -\frac{2\sqrt{2}}{4}, -[\lambda^6] \right) \]

\[ = \left( -\frac{2\sqrt{2}}{4}, \Delta^2 \right) + \text{negative root of the } SL(8) \text{ subalgebra} \]

\[ = \vec{\lambda}^2 - \vec{\lambda}_8 + \text{negative root of the } SL(8) \text{ subalgebra} \]

\[ = -\vec{\alpha}_8 + \text{negative root of the } SL(8) \text{ subalgebra} \]

(91)

and again we find that the replacement takes one weight to another.

Hence if we include these terms where two or more spacetime derivatives act on the fields then we find that they also lead to weights of \( E_8 \). In particular the replacement \( F^2 \to \partial^2 \) always changes the weight by a negative root of the enhanced symmetry algebra.

It is important to note that in carrying out these steps we have assumed that the dualisation is the same as that found in the rest of the paper. In particular this is obtained by considering only terms that have only a single derivative acting on each field. It is not clear to us how the dualization procedure can be extended to include terms with additional derivatives. It is likely that the dualization procedure will be changed by the presence of these new terms, however, it will still contain the original dualization prescription as the local piece.

6 Conclusions

In this paper we have considered the higher derivative terms that arise for pure gravity, M-theory and the Bosonic string when dimensionally reduced
to three dimensions. We have derived the general dependence of the terms in the effective action on the diagonal metric components (and dilaton when present) and shown that these are given by the weights of the enhanced symmetry group. More precisely the diagonal metric components are associated with the Cartan subalgebra of the symmetry group $G$ whereas the off diagonal components give rise to scalars $\chi_\alpha$ that are associated to raising operators $E_\alpha$ in the Lie algebra. For the lowest order term in the effective action this procedure leads to the positive roots and hence uniquely identifies the enhanced symmetry algebra.

In more detail we found that in the case of gravity, dimensionally reduced on a $n$-torus to three dimensions, all the higher derivative terms lead to the appearance of weights of $SL(n+1)$. For the M-theory we find that only higher derivative terms of the form $R^{3k+1} + \ldots$ lead to weights of $E_8$ while other higher derivative terms the vectors that arise have no interpretation in terms of the $E_8$ Lie algebra. For the Bosonic string we found that the weights of $O(n+1, n+1)$ occur only when the terms in the effective action can be identified as arising at a particular genus in the string perturbation expansion. Thus the higher derivative terms for which weights appear are just those that are expected based on arguments using the underlying string theory. Alternatively one could view the appearance of weights as a way of predicting which higher derivative terms should arise.

The appearance of weights for precisely for those higher derivative terms that are expected to arise from the underlying quantum theory provides strong evidence for the existence of some form of the enhanced symmetry in the fundamental theory. If we had found that the vectors contained within the higher derivative terms had no interpretation in terms of the symmetry algebra of the low energy effective action then one would be led to conclude that the U-duality conjectures [17] were incorrect. The same would apply to the $E_{11}$ conjecture [18] for the eleven dimensional theory as its dimensional reduction must possess the residual symmetry. Thus the appearance of weights can be seen as support for both of these conjectures. However, it might also tell us something about how one should think about of these conjectures beyond the low energy theory. Fortunately, in the analysis carried out in this paper one is discussing only a finite dimensional subalgebra of the full $E_{11}$ symmetry and so the questions raised by this paper may be easier to resolve.

We also note that the existence of the enhanced symmetry in three dimensions is responsible for the fact that, upon further dimensional reduction to two-dimensions, the symmetry of the lowest order effective action becomes infinite-dimensional (for example see [46, 47, 48, 49]). Thus our results also support the conjectures that, at least in two-dimensions, the full quantum
theory possess an infinite-dimensional symmetry.

This paper has been somewhat phenomenological in nature. In particular we have observed a pattern in the higher derivative terms which is associated to the enhanced symmetry group. However we have not explained why this occurs. Therefore it is of interest to understand the meaning of our results and in particular demonstrate that the enhanced group really is a symmetry. The most naive expectation is that the higher derivative terms can be expressed in terms of the group element of the non-linear realization that appears at lowest order. However at first sight this would not seem to be the case. It could be that the enhanced symmetry does not act simply once higher derivative terms are included and in particular requires an order by order modification of the group action. It is also possible that the symmetry algebra becomes enlarged in the presence of the higher derivative terms to incorporate the new representations that arise. We hope to report on these points in the near future.

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Appendix A

Our compactification ansatz is

$$ds^2 = e^{2\alpha} ds^2 + e^{2\beta} G_{ij}(dx^i + A_i^\mu dx^\mu)(dx^j + A_j^\mu dx^\mu) \quad (92)$$

where a hat denotes a $D$-dimensional quantity. In this appendix we will quote results for a general compactification from $D$-dimensions to $d$-dimensions. We will use the indices $\mu, \nu = 0, 1, 2, ..., d - 1$ and $i, j = d + 1, ..., D - 1$ and an overlined index refers to the tangent frame. The internal metric $G_{ij}$ is constrained to have unit determinant (so that $\rho$ alone determines the volume of the internal torus). We will also use $n = D - d$ to denote the dimension of the internal torus.
A vielbein frame for this compactification is
\[ \hat{e}^\mu = e^{\alpha \rho} e^{\rho}_\mu dx^\mu \]
\[ \hat{e}_j = e^{\beta \rho} e^{\rho}_j (dx^j + A^j_\mu dx^\mu) \] (93)

where \( e^{\rho}_\mu \) and \( e^{\rho}_j \) are vielbein frames for \( g_{\mu \nu} \) and \( G_{ij} \) respectively.

The spin connection is
\[ \omega^{\hat{T}}_j = -e^{-\alpha \rho} Q^{T}_j e^{\hat{T}} \]
\[ \omega^{\hat{\nu}}_j = \beta e^{-\alpha \rho} \partial_{\nu} \rho e^{\hat{\nu}} + e^{-\alpha \rho} S^{\hat{\nu}}_j \hat{e}^{\hat{\nu}} \]
\[ \omega^{\hat{\nu}}_T = \omega^{\hat{\nu}}_T + \alpha \hat{\nu} \partial_{\mu} e^{-\alpha \rho} e^{\hat{\nu}} - \alpha \hat{\nu} \partial_{\rho} e^{-\alpha \rho} e^{\hat{\nu}} - \frac{1}{2} e^{(\beta - 2 \alpha) \rho} F^{T}_{\hat{\nu} \hat{\nu}} e^{\hat{T}} \] (94)

Here we have introduced
\[ F^{i}_{\mu \nu} = \partial_{\mu} A^i_\nu - \partial_{\nu} A^i_\mu \] (95)

and
\[ e^{\rho}_T \partial_{\mu} e^{\rho}_k = S^{\rho} T_{\mu \nu} + Q^{\rho} T_{\mu \nu} \] (96)

where \( Q^{\rho T}_{\mu} = S^{\rho T}_{\mu} \) = 0. This split leads to the identities
\[ \partial_{\mu} Q_{\nu} - \partial_{\nu} Q_{\mu} - [Q_{\mu}, Q_{\nu}] - [S_{\mu}, S_{\nu}] = 0 \]
\[ \partial_{\mu} S_{\nu} - \partial_{\nu} S_{\mu} - [Q_{\mu}, S_{\nu}] + [Q_{\nu}, S_{\mu}] = 0 \] (97)

We can now calculate the Riemann curvature terms to be
\[ \hat{R}^{\hat{T} T}_{\mu \nu \lambda \rho} = e^{-2 \alpha \rho} \left( S^{\hat{T} T}_{\mu \nu} S^{\hat{T} T}_{\lambda \rho} - S^{\hat{T} T}_{\mu \nu} S^{\hat{T} T}_{\rho \lambda} + \beta \partial^{\hat{T} T} \rho (S^{\hat{T} T}_{\mu \nu} S^{\hat{T} T}_{\lambda \rho} - S^{\hat{T} T}_{\mu \nu} S^{\hat{T} T}_{\rho \lambda} + S^{\hat{T} T}_{\rho \lambda} S^{\hat{T} T}_{\mu \nu}) \right) + \beta^2 (\partial \rho)^2 (\delta^\rho_{\mu \nu} \delta^\rho_{\lambda \rho} - \delta^\rho_{\mu \rho} \delta^\rho_{\lambda \nu}) \]
\[ \hat{R}^{\hat{T} T}_{\mu \nu \rho \lambda} = \frac{1}{2} e^{(\beta - 2 \alpha) \rho} \left( S^{\hat{T} T}_{\mu \nu} F^{\hat{T} T}_{\rho \lambda} - S^{\hat{T} T}_{\mu \rho} F^{\hat{T} T}_{\nu \lambda} - \beta \partial^{\hat{T} T} \rho F^{\hat{T} T}_{\mu \nu} \delta^{\hat{T} T}_{\rho \lambda} - \beta \partial^{\hat{T} T} \rho F^{\hat{T} T}_{\mu \rho} \delta^{\hat{T} T}_{\nu \lambda} \right) \]
\[ \hat{R}^{\hat{T} T}_{\mu \nu \rho \lambda} = 2 e^{-2 \alpha \rho} \left( - \nabla^{\hat{T} T} Q^{\hat{T} T}_{\mu \nu} + (Q^{\hat{T} T}_{\mu \nu} Q^{\hat{T} T}_{\rho \lambda}) + \frac{1}{4} e^{2(\beta - \alpha) \rho} (F^{\hat{T} T}_{\mu \nu} F^{\hat{T} T}_{\rho \lambda}) \right) \]
\[ \hat{R}^{\hat{T} T}_{\mu \nu \rho \lambda} = e^{-2 \alpha \rho} \left( (2 \alpha \beta - \beta^2) \partial_{\mu} \rho \partial_{\rho} \partial_{\lambda} \delta^{\hat{T} T}_{\nu \lambda} - \beta \partial_{\rho} \partial_{\lambda} \partial_{\mu} \delta^{\hat{T} T}_{\nu \lambda} - \alpha \beta (\partial \rho)^2 \eta^{\hat{T} T}_{\mu \nu} \delta^{\hat{T} T}_{\rho \lambda} - \alpha \beta \partial^{\hat{T} T} \rho S^{\hat{T} T}_{\mu \nu} \delta^{\hat{T} T}_{\rho \lambda} \right) + \frac{1}{4} e^{2(\beta - \alpha) \rho} (F^{\hat{T} T}_{\mu \nu} F^{\hat{T} T}_{\rho \lambda}) - \nabla^{\hat{T} T} S^{\hat{T} T}_{\mu \nu} + (\alpha - \beta) (\partial_{\nu} \rho S^{\hat{T} T}_{\mu \lambda} + \partial_{\lambda} \rho S^{\hat{T} T}_{\mu \nu}) - \alpha \partial^{\hat{T} T} \rho S^{\hat{T} T}_{\mu \nu} - (S^{\hat{T} T}_{\mu \nu} Q^{\hat{T} T}_{\rho \lambda}) \right) \]
\[ \hat{R}_{\lambda\rho\mu\nu} = \frac{1}{2} e^{(3-3\alpha)\rho} \left( \epsilon_{ij} \nabla_i F^j_{\lambda\rho} - (\beta - \alpha) (2\partial_{\lambda\rho} F_{\mu\nu} + \partial_{\mu\nu} F_{\lambda\rho} + \partial_{\lambda\rho} F_{\mu\nu}) + \alpha \partial F^\rho_{\lambda\rho\mu\nu} - F_{\lambda\rho\mu\nu} \right) \]

\[ \hat{R}_{\lambda\rho\mu\nu} = e^{-2\alpha\rho} R_{\lambda\rho\mu\nu} - \frac{1}{2} e^{2(\beta-2\alpha)\rho} \left( F^i_{\lambda\rho} F^i_{\mu\nu\rho} - \frac{1}{4} e^{2(\beta-2\alpha)\rho} \left( F^i_{\lambda\rho} F^i_{\mu\nu\rho} - F^i_{\lambda\rho} F^i_{\mu\nu\rho} \right) \right) \]

From these expressions we find

\[ \hat{R} = e^{-2\alpha\rho} \left( R - \frac{1}{4} e^{2(\beta-2\alpha)\rho} G^i_{ij} F^i_{\mu\nu} F^j_{\rho\sigma} - \frac{1}{2} S_{ij} S_{\mu\nu} - \gamma^2 (\partial\rho)^2 \right) \]

where

\[ \gamma^2 = (d-1)(d-2)\alpha^2 + (2dn - 4n)\alpha \beta + n(n+1)\beta^2 \]

To reduce to Einstein frame we require that

\[ \beta = -\left( \frac{d-2}{D-d} \right) \alpha \]

Finally we fix \( \alpha \) by taking the standard normalization for the kinetic energy of a scalar field, \( \gamma^2 = 1/2 \). In this paper we are interested in taking \( d = 3 \) in which case these formulae simplify to

\[ \alpha^2 = \frac{1}{2} \frac{n}{n+1} \]

\[ \beta = -\frac{1}{n} \alpha \]
Appendix B

Let review some elementary facts about $G/H$ cosets. We assume that $G$ is a finite dimensional semi-simple Lie algebra. It consists of the elements $H$, which form the Cartan subalgebra, and $E_\alpha$ subject to the relations

$$[H, E_\alpha] = \alpha E_\alpha$$

(104)

In this appendix we use an underline to denote quantities such as roots vectors, however, in the body of the paper such quantities have underlines only if they belong to manifest $SL(n)$ symmetry associated with the torus reduction and arrows if they belong to the group associated to the enhanced coset symmetry. The roots $\alpha$ can be split into positive and negative roots, which we denote by $\alpha > 0$ and $\alpha < 0$ respectively.

Such algebras admit the Cartan involution

$$\tau : (E_\alpha, E_{-\alpha}, H) \rightarrow -(E_{-\alpha}, E_\alpha, H)$$

(105)

where $\alpha > 0$. This is a group automorphism and so obeys $\tau(g_1 g_2) = \tau(g_1) \tau(g_2)$ for any two group elements $g_1$ and $g_2$. This allows us to construct a generalized transpose $A^\# = -\tau(A)$. We then take $H$ to be the sub-algebra invariant under the Cartan involution. It terms of the $\#$ operation it consist of the elements of the algebra for which $A^\# = -A$, or in terms of group elements those that obey $h^\# = h^{-1}$.

We consider elements of the group which depend on space-time to transform as

$$g \rightarrow g_0 g h$$

(106)

where $g_0 \in G$ is a rigid, that is constant group element and $h \in H$ is space-time dependent. Using this latter local invariance one can take an element $g \in G$ to be of the form

$$g = e^{\sum_{\alpha > 0} \chi_\alpha E_\alpha} e^{-\frac{1}{\sqrt{2}} \phi \cdot H}$$

(107)

To construct the dynamics it is useful to use the Cartan forms

$$V_\mu = g^{-1} \partial_\mu g = -\frac{1}{\sqrt{2}} \partial_\mu \phi \cdot H + \sum_{\alpha > 0} e^{\frac{1}{\sqrt{2}} \phi \cdot h} \partial_\mu \chi_\alpha E_\alpha$$

(108)

up to higher order terms in $\chi_\alpha$. Under the transformation (106) the Cartan forms transform as $V_\mu \rightarrow h^{-1} V_\mu h + h^{-1} \partial_\mu h$. Using the Cartan involution we can construct the objects

$$S_\mu = \frac{1}{2} V_\mu + \frac{1}{2} \tau(V_\mu), \quad \omega_\mu = \frac{1}{2} (V_\mu - \tau(V_\mu))$$

(109)
which transform as

\[ S_\mu \rightarrow h^{-1} S_\mu h, \quad \omega_\mu \rightarrow h^{-1} \omega_\mu h + h^{-1} \partial_\mu h \]  \hspace{1cm} (110)

Under the # operation we find that \( \omega^#_\mu = \omega_\mu \) and \( S^#_\mu = -S_\mu \) and hence they lie in the subalgebra \( H \) and its complement \( G - H \) respectively. The first such quantity is given by

\[ S_\mu = -\frac{1}{\sqrt{2}} \partial_\mu \phi \cdot H + \frac{1}{2} \sum_{\alpha > 0} e^{\frac{1}{\sqrt{2}} \alpha_\mu} \partial_\mu \chi_\alpha (E_\alpha + E^#_\alpha) \]  \hspace{1cm} (111)

We can then construct an invariant Lagrangian by taking

\[ L = -\text{Tr}(S_\mu S^\mu) \]
\[ = -\frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} \sum_{\alpha > 0} e^{\frac{1}{\sqrt{2}} \alpha_\mu} \partial_\mu \chi_\alpha \partial^\mu \chi_\alpha \]  \hspace{1cm} (112)

up to higher order terms in \( \chi_\alpha \).

There is an alternative way to construct the same Lagrangian. This time one starts with

\[ M = gg^# \]
\[ = e^{\sum_{\alpha > 0} \chi_\alpha E_\alpha} e^{-\sqrt{2} \phi H} e^{\sum_{\alpha > 0} \chi_\alpha E^#_\alpha} \]
\[ M^{-1} = e^{-\sum_{\alpha > 0} \chi_\alpha E^#_\alpha} e^{\sqrt{2} \phi H} e^{-\sum_{\alpha > 0} \chi_\alpha E_\alpha} \]  \hspace{1cm} (113)

Under \( g \rightarrow g_0 gh \) we see that \( M \rightarrow g_0 M g_0^# \) since the action of # in the Lie algebra lifts to \( h^# = h^{-1} \) for \( h \in H \). Thus another possible invariant Lagrangian is

\[ L = -\frac{1}{4} \text{Tr}(\partial_\mu M \partial^\mu M^{-1}) \]  \hspace{1cm} (114)

but in fact this is the same as (112).

When \( G = SL(n) \), # is simply the transpose so that \( H = SO(n) \) and \( S_\mu \) is the symmetric part of \( g^{-1} \partial_\mu g \).

**Appendix C**

In this appendix we will give an account of certain aspects of the theory of group representations that are required in this paper. We recall that a representation \( R \) of a group \( G \) consists of a vector space \( V \) and a set of operators \( U(g) \), \( \forall g \in G \) which act on \( V \), namely \( |\psi> \rightarrow U(g)|\psi> \) such that
\[ U(g_1,g_2) = U(g_1)U(g_2) \]. We will take the algebra \( G \) to be a finite dimensional semi-simple and simply laced. The states in the representation can be chosen so as to be eigenstates of \( H \). The eigenvalues being called the weights. It can be shown that the weights of \( G \) belong to the dual lattice to the lattice of roots, \( \text{i.e.} \) a weight \( w \) satisfies

\[ w \cdot \alpha \in \mathbb{Z} \tag{115} \]

for the simple roots \( \alpha_n \). The representations of interest to us are finite dimensional and so must have a highest weight \( \Lambda \) which is the one such that \( \Lambda + \alpha_n \) is not a weight for all simple roots \( \alpha_n \). The representations will also have a lowest root denoted \( \mu \). Of particular interest are the fundamental representations which are those whose highest weights \( \Lambda^a \) obey the relation

\[ \Lambda^a \cdot \alpha_b = \delta^a_b \tag{116} \]

for all simple roots \( \alpha_n \). The roots are themselves weights and these correspond to the adjoint representation, whose highest weight we will denote by \( \theta \).

For \( SL(n) \), \( \text{i.e.} \) \( A_{n-1} \), the fundamental weights \( \Lambda^a \) satisfy

\[ \Lambda^a \cdot \Lambda^b = a(n-b)/n \tag{117} \]

for \( b \geq a \). The representation with highest weight \( \Lambda^{n-k} \) is realized on a tensor with \( k \) totally anti-symmetrized superscript indices, \( \text{i.e.} \) \( T^{i_1 \ldots i_k} = T[i_1 \ldots i_k] \). Using the group invariant epsilon symbol \( \epsilon^{i_1 \ldots i_n} \), this representation is equivalent to taking a tensor with \( n-k \) lowered indices.

Given any simple root one may carry out its Weyl reflection on any weight

\[ S_\alpha(w) = w - (\alpha \cdot w)\alpha \tag{118} \]

The collection of all such reflections is called the Weyl group and it can be shown that any member of it can be written in terms of a product of Weyl reflections in the simple roots. Although the precise decomposition of a given element of the Weyl group is not unique its length is defined to be the smallest number of simple root reflections required. However, there does exist a unique Weyl reflection, denoted \( W_0 \), that has the longest length. This element obeys \( W_0^2 = 1 \), takes the positive simple roots to negative simple roots and its length is the same as the number of positive roots. As a result, \( -W_0 \) exchanges the positive simple roots with each other and, as Weyl transformations preserve the scalar product, it must also preserve the Cartan matrix. Consequently, it must lead to an automorphism of the Dynkin diagram. Given any representation of \( G \) the highest and lowest weights are related by

\[ \mu = W_0\Lambda \tag{119} \]
Given the definition of the fundamental weights and carrying out a Weyl transformation $W_0$, we may conclude that the negative of the highest and lowest weights of a given fundamental representation are the lowest and highest representation of one of the other fundamental representations. It is always the case that the two representations have the same dimension. However it can happen that a fundamental representation is self-dual.

For $SL(n)$ $W_0 = (S_{\alpha_1} \ldots S_{\alpha_{n-1}})(S_{\alpha_1} \ldots S_{\alpha_{n-2}}) \ldots (S_{\alpha_1} S_{\alpha_2})S_{\alpha_1}$ and one finds that, in this case,

$$W_0 \Delta_{n-k} = \mu_{n-k} \iff W_0 \mu_{n-k} = \Delta_{n-k} = -\mu_k.$$  (120)

This result also follows from the above remarks on $W_0$ as it must take a fundamental representation to a fundamental representation and correspond to an automorphism of the Dynkin diagram which in this case is just takes the nodes $k$ to $n-k$.

Given a representation acting on $|\chi > \in V$ we may consider the dual representation $R_D$ that is carried by the space of linear functionals, denoted $V^*$, acting on $V$. The group action is defined by

$$<\chi_D| \rightarrow <\chi_D|U(g^{-1}), \forall g \in G, <\chi_D| \in V^*$$  (121)

We note that $<\chi_D|\chi >$ is $G$-invariant. Since the linear functionals carry a representation we may also choose a basis them that is labeled by the weights. It is easy to see that a linear functional with a weight $\mu$ only has a non-zero result on a state with weight $-\mu$. A little further thought allows one to conclude that if the representation $R$ has highest and lowest weight $\Delta$ and $\mu$ respectively then the dual representation has a highest weight $-\mu$ and lowest weight $-\Delta$. Indeed the dual representation has the same dimension as the original representation. For the case of $SL(n)$, i.e. $A_{n-1}$, if the representation $R$ is the fundamental representation with highest weight $\Delta^k$ then it follows from equation (121) that the dual representation is the fundamental representation with highest weight $\Delta^{n-k}$. Thus the representations carried by $T_i^{1 \ldots (n-k)}$ is dual to the representation carried by $T_i^{1 \ldots k}$ or in that latter case carried by $T_{i_1 \ldots i_{(n-k)}}$ as might be expected.

Given the representation $R$ and any automorphism of the group $\tau$ (i.e. $\tau(g_1 g_2) = \tau(g_1)\tau(g_2)$) we may also define a twisted representation $R_T$ on the same vector space $V$ by

$$|\phi_T > \rightarrow U(\tau(g))|\phi_T > \forall g \in G, |\phi_T > \in V$$  (122)

The case of interest to us in this paper is when we take the automorphism to be the Cartan involution which we also denoted by $\tau$. It is easy to see that
if the representation $R$ has highest and lowest weight $\lambda$ and $\mu$ respectively then the dual representation has a highest weight $-\mu$ and lowest weight $-\lambda$ and so the twisted representation is isomorphic to the dual representation.

In the first systematic account of the theory of non-linear realizations [50] it was explained that one can convert a linear realization into a non-linear realization. In particular, given a non-linear realization with group element $g$ which transforms as $g \rightarrow g_0gh$ and any linear representation $R$ we find that

$$U(g^{-1})|\psi > \rightarrow U(h^{-1})U(g^{-1})|\psi >$$  \hspace{1cm} (123)

We now want to generalize this construction to the situation we will encounter in this paper. Given the dual representation $R_D$ (121) we find that

$$<\chi_D|U(g) \rightarrow <\chi_D|U(g)U(h)$$ \hspace{1cm} (124)

and if we take the Cartan twisted representation of equation (121) then

$$U(g^\#)|\phi_T > \rightarrow U(h^{-1})U(g^\#)|\phi_T >$$ \hspace{1cm} (125)

It follows that if we consider any representation $|\chi>$, then $<\chi_D|U(gg^\#)|\chi_T >$ is inert under local and rigid transformations. In this latter expression $|\chi_T>$ and $<\chi_D|$ are the Cartan twisted and dual representations derived from $|\chi>$ respectively. We note that the twisted and dual representations are actually isomorphic to each other. As we have seen, this quantity plays a central role in this paper. Similarly one can show that

$$<(\chi_T)_D|U(\mathcal{M}^{-1})|\chi >$$ \hspace{1cm} (126)

where $\mathcal{M} = gg^\#$, is invariant under local and rigid transformations.

We note that the two representations appearing in equation (126) are isomorphic. If we take as a basis of this representation to be given by the states $|I, \Delta>$ we may write $|\chi>$ as $|\chi> = \sum_I \chi_I |I, \Delta>$, and similarly for $<(\chi_T)_D|$ then the above expression becomes

$$<(\chi_T)_D|U(\mathcal{M}^{-1})|\chi >= \sum_{IJ} \chi_I <I, \Delta|U(\mathcal{M}^{-1})|J, \Delta > \chi_J$$ \hspace{1cm} (127)

Indeed, if we consider the group to be $SL(n)$ and take $|\chi>$ to be the representation with highest weight $\lambda^1$, that is in the vector representation, then $<(\chi_T)_D|$ is also an $SL(n)$ vector. As a result, we find that

$$<(\chi_T)_D|U(\mathcal{M}^{-1})|\chi >= \sum_{IJ} \chi_I g^{IJ} \chi_J$$ \hspace{1cm} (128)

as in this case $g^{IJ} = <I, \lambda^1|U(\mathcal{M}^{-1})|J, \lambda^1 >$ is the metric on the torus as we have shown in section 2.
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