d-Branes in the Stream§

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Abstract

Evolution of extended data is considered in various flow problems, using Nambu brackets as a tool applicable to all cases. Extra dimensions, N-brackets, and extended structures are first employed to linearize the Euler equations. N-bracket induced evolutions of strings, membranes, and other d-branes are then discussed in detail.

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This talk consists of two more or less independent parts. The first part is a discussion of simple Euler fluid flow, in somewhat unconventional terms. The second part is a consideration of the motion of continuous collections of points (i.e. branes) in response to a given ambient flow field. Although treated independently, both considerations involve some common mathematical language: the use of Nambu brackets and extended data structures.

1 Linearizing the Euler-Monge equations

General results in $n$ dimensions  Encode the local velocity field for $n$-dimensional flow $u(x,t)$ into a non-local structure, after dimension-doubling. Define

$$U_k (x, t, a) \equiv \int \cdots \int dq_1 \cdots dq_n \delta (q_k - x_k) \left( e^{\alpha_k u_k (q,t)} - 1 \right) \times$$

$$\times \det \left( \frac{\partial}{\partial q_{1}} (\varepsilon (q_1 - x_1) e^{\alpha_1 u_1 (q,t)}) \quad \ldots \quad \frac{\partial}{\partial q_{n}} (\varepsilon (q_1 - x_1) e^{\alpha_1 u_1 (q,t)}) \right)$$

$$\vdots \quad \ldots \quad \vdots$$

$$\frac{\partial}{\partial q_{1}} (\varepsilon (q_n - x_n) e^{\alpha_n u_n (q,t)}) \quad \ldots \quad \frac{\partial}{\partial q_{n}} (\varepsilon (q_n - x_n) e^{\alpha_n u_n (q,t)}) \right) ,$$

where $\varepsilon (s) \equiv \pm \frac{1}{2} \quad \text{for} \quad s \geq 0$ . The local velocity data is given by the extra-dimension boundary values

$$u_k (x, t) = \lim_{a \to 0} U_k (x, t, a) .$$

Then with

$$\mathcal{M}_n [u] \equiv \frac{\partial}{\partial t} - \sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j} , \quad \mathcal{H}_n \equiv \frac{\partial}{\partial t} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j \partial a_j} ,$$

we have

$$\mathcal{M}_n [u] u_i (x, t) = 0$$

if and only if

$$\mathcal{H}_n U_k (x, t, a) = 0 .$$

The latter equations are solved by the heat kernel method, as given formally by

$$U_k (x, t, a) = e^{t \sum_{j=1}^{n} \partial^2 / \partial x_j \partial a_j} U_k (x, t = 0, a) .$$

Expansion and evaluation of the RHS yields the formal time power series solution in terms of the initial data. Restricting this series to the extra dimension boundary, $a = 0$, we obtain the time power series solution for $u_k (x, t)$ in terms of $u_k (x, t = 0)$ and its derivatives.

Dimension-doubling here is formally similar to point-splitting in Schrödinger quantum mechanics [1]. But it is nonetheless a bona fide doubling of dimensions, in the same sense as the Fourier transform of point-split wave-function bilinears in quantum mechanics yields a Wigner function on a phase-space whose dimension is twice that of the original wave-function configuration space.
Nambu brackets  Classically, N-brackets are defined as Jacobians.

\[
\{ A_1, \ldots, A_n \}_{q_1 \ldots q_n} \equiv \frac{\partial (A_1, \ldots, A_n)}{\partial (q_1, \ldots, q_n)} .
\]  (7)

These multi-linear generalizations of the Poisson bracket arise naturally in the previous construction, since

\[
\delta (q_k - x_k) \times \det (\cdots)_{\text{exclude kth row and kth column}} = \{ \varepsilon (q_1 - x_1) e^{a_1 u_1}, \ldots, \varepsilon (q_k - x_k), \ldots, \varepsilon (q_n - x_n) e^{a_n u_n} \}_{q_1 \ldots q_n},
\]  (8)

with the exponential missing from the kth entry in the bracket.

In the extra-dimension-boundary limit, the N-bracket becomes a Dirac-delta equating all components of x to those of q.

\[
\lim_{\alpha \to 0} \{ \varepsilon (q_1 - x_1) e^{a_1 u_1}, \ldots, \varepsilon (q_k - x_k), \ldots, \varepsilon (q_n - x_n) e^{a_n u_n} \}_{q_1 \ldots q_n} = \delta^{(n)} (q - x) .
\]  (9)

This ensures that U collapses down to u on the \( a \to 0 \) boundary.

Note the Nambu bracket is a derivation, and is linear in each argument. So another way to say the above is

\[
\delta (q_k - x_k) e^{a_k u_k} \det (\cdots)_{\text{exclude kth row and kth column}} = \{ \varepsilon (q_1 - x_1) e^{a_1 u_1}, \ldots, \varepsilon (q_n - x_n) e^{a_n u_n} \}_{q_1 \ldots q_n}
\]  (10)

A trivial example in one-dimension  As the simplest, local example, consider

\[
U (x, t, a) \equiv \frac{e^{au(x,t)} - 1}{a}, \quad u (x, t) = \frac{1}{a} \ln (1 + aU (x, t, a)) = \lim_{a \to 0} U (x, t, a) .
\]  (11)

Then

\[
\frac{\partial}{\partial t} U (x, t, a) = \frac{\partial^2}{\partial a \partial x} U (x, t, a)
\]  (12)

if and only if

\[
\frac{\partial}{\partial t} u (x, t) = u (x, t) \frac{\partial}{\partial x} u (x, t) .
\]  (13)

Moreover,

\[
U (x, t, a) = e^{\frac{a^2}{\alpha \beta x}} U (x, t = 0, a)
\]  (14)

immediately gives the time power series solution from initial data

\[
u (x, t) = \lim_{\alpha \to 0} e^{\frac{t^j}{\alpha}} \left( \frac{e^{u(x,t)} - 1}{a} \right) = \sum_{j=0}^{\infty} \frac{t^j}{(1+j)!} \frac{d^j}{dx^j} (u (x))^{1+j} .
\]  (15)

So, for example, an initial linear profile, \( u (x) = \alpha + \beta x \), evolves into \( u (x, t) = (\alpha + \beta x) / (1 - \beta t) \). For a given interval in \( u \), this steepens as time increases, for \( \beta > 0 \), and becomes vertical at \( t_{\text{break}} = 1/\beta \). Other simple breaking wave examples can be found in [P] (or at [http://curtright.com/waves.html]).
Two-dimensions and some non-locality  Define the extended data structures \( (U_1 \equiv U, \; U_2 \equiv V, \; \text{etc.}) \)

\[
U (x, y, t, a, b) \equiv \int dr \; \varepsilon (y - r) \; e^{au(x, r, t) + bv(x, r, t)} \frac{\partial u(x, r, t)}{\partial r},
\]

\[
V (x, y, t, a, b) \equiv \int dq \; \varepsilon (x - q) \; e^{au(q, y, t) + bv(q, y, t)} \frac{\partial v(q, y, t)}{\partial q} .
\]  \hspace{1cm} (16)

Then

\[
\mathcal{H}_2 U = 0 = \mathcal{H}_2 V
\]  \hspace{1cm} (17)

if and only if

\[
\mathcal{M}_2 u = 0 = \mathcal{M}_2 v .
\]  \hspace{1cm} (18)

Again, note the extra-dimension boundary limits

\[
u(x, y, t) = \lim_{a, b \to 0} U(x, y, t, a, b) , \quad v(x, y, t) = \lim_{a, b \to 0} V(x, y, t, a, b) .
\]  \hspace{1cm} (19)

The demonstration of the equivalence of (17) and (18) goes as follows. Direct calculation gives

\[
\mathcal{H}_2 U (x, y, t, a, b) = e^{au(x, y, t) + bv(x, y, t)} \mathcal{M}_2 u (x, y, t)
\]

\[
+ b \int dr \; \varepsilon (y - r) \; e^{au(x, r, t) + bv(x, r, t)} \left( \frac{\partial u(x, r, t)}{\partial r} \mathcal{M}_2 v(x, r, t) - \frac{\partial v(x, r, t)}{\partial r} \mathcal{M}_2 u(x, r, t) \right) .
\]

Similarly for \( V \). So then \( \mathcal{M}_2 u = 0 = \mathcal{M}_2 v \) clearly implies \( \mathcal{H}_2 U = 0 \). The converse follows by using the obvious limit

\[
\lim_{a, b \to 0} \mathcal{H}_2 U (x, y, t, a, b) = \mathcal{M}_2 u (x, y, t) ,
\]  \hspace{1cm} (20)

along with the similar relation for \( V \). QED

Three-dimensions and Poisson brackets  \( (U_1 \equiv U, \; U_2 \equiv V, \; U_3 = W, \; \text{etc.}) \)

\[
\mathcal{H}_3 U = \mathcal{H}_3 V = \mathcal{H}_3 W = 0
\]  \hspace{1cm} (21)

if and only if

\[
\mathcal{M}_3 u = \mathcal{M}_3 v = \mathcal{M}_3 w = 0 ,
\]  \hspace{1cm} (22)

where

\[
U (x, y, z, t, a, b, c) \equiv \int dr \; \varepsilon (y - r) \; e^{au + bw + cw} \frac{\partial u(x, r, z, t)}{\partial r}
\]

\[
- c \int \int dr ds \; \varepsilon (y - r) \; \varepsilon (z - s) \; e^{au + bw + cw} \{u, w\}_rw (x, r, s, t) ,
\]  \hspace{1cm} (23)

\[
V (x, y, z, t, a, b, c) \equiv \int ds \; \varepsilon (z - s) \; e^{au + bw + cw} \frac{\partial v(x, y, s, t)}{\partial s}
\]

\[
- a \int \int dq ds \; \varepsilon (x - q) \; \varepsilon (z - s) \; e^{au + bw + cw} \{v, u\}_sq (q, y, s, t) ,
\]  \hspace{1cm} (24)
\[
W(x, y, z, t, a, b, c) \equiv \int dq \varepsilon(x - q) e^{au + bv + cw} \frac{\partial w(q, y, z, t)}{\partial q} - b \int dq dr \varepsilon(x - q) \varepsilon(y - r) e^{au + bv + cw} \{w, v\}_{qr}(q, r, z, t),
\]

and where the Poisson bracket is given as usual by

\[
\{u, v\}_{rs} = \frac{\partial u}{\partial r} \frac{\partial v}{\partial s} - \frac{\partial u}{\partial s} \frac{\partial v}{\partial r}.
\]

Once again the proof of the equivalence of (21) and (22) is given by direct calculation. Higher dimensions lead to higher rank Nambu brackets, as expressed by the general \( n \) result quoted above.

## 2 Nambu evolution of d-branes

Consider flow-driven motion in a local velocity field \( u[x] \). For points moving along with the flow\(^1\)

\[
dx^i = u^i[x] d\tau.
\]

This deceptively simple description is general enough to incorporate the phase-space evolution of any Hamiltonian dynamical system, upon identifying pairs of variables as canonically conjugate, say \((x_j, x_{j+1}) \rightarrow (x_j, p_j)\).

Is there an elegant way to describe the motion of an extended, continuous collection of points moving with the flow? For example, consider (vortex) strings, (surface interface) membranes, or higher dimensional manifolds of points immersed in fluids flowing in higher dimensions. The latter higher dimensional generalization is most important for dynamics in phase-space, which for a system with \( N \) degrees of freedom is \( 2N \) dimensional. That is to say, if we parameterize the extended set of points for a given \( \tau \) as

\[
x(\tau, \alpha_1, \alpha_2, \cdots, \alpha_d),
\]

then we have a d-dimensional data-brane, or “d-brane” for short. What is the most economical way to describe the collective evolution of this extended set of data?

A suggested answer to both these rhetorical questions is as follows. If the ambient velocity field is given by a Nambu bracket, as

\[
u^i[x] = \{x^i, I_1, \cdots, I_{n-1}\},
\]

where the \( I_s \) are “flow invariants,” then we have satisfied a sufficient condition to obtain an exact \( n \)-form, \( \Omega = d\Lambda \), that provides a minimal coupling of the d-brane to the velocity field, for the case \( d = n - 2 \). The individual points are governed by Nambu dynamics \[^9\] in this case, with

\[
\frac{dx^i}{d\tau} = \{x^i, I_1, \cdots, I_{n-1}\}.
\]

\(^1\)There is one caveat to keep in mind here. It is tempting to think of \( \tau \) as conventional time \( t \), defined universally. But this need not be so. In fact, in some situations it can be misleading to think of \( \tau \) this way. For streamline fluid flow, e.g., \( \tau \) could be allowed to vary from one streamline to the next such that evolution along each streamline is governed by that streamline’s own individual clock. In this sense, \( \tau \) is more like proper time in relativity. In the context of Nambu mechanics in phase-space, \( \tau \) is generally related to \( t \) by a dynamical scale factor, and therefore cannot be simply identified with universal time across all dynamical sectors \[^2\]. That is to say, \( \tau \) may be trajectory dependent. For this and other reasons to become apparent, we will call \( \tau \) “Nambu time.”
Examples of such dynamical systems in phase-space are abundantly provided by superintegrable systems. An $N$ degree-of-freedom system, with phase-space of dimension $2N$, is \textit{maximally superintegrable} and described as above when $n = 2N$ with a corresponding set of $2N - 1$ invariants, $I_i$. In this $2N$ phase-space, there can be no more than $2N - 1$ invariants. These ideas are extensively discussed in [2].

As already noted, the preferred continuous collection of points, i.e. the minimally coupled d-brane, is of dimension $d = n - 2$ when the velocity field is given by the above Nambu bracket. The explicit $n$-form that provides the minimal coupling of the brane to the velocity field is then

$$\Omega = \varepsilon_{i_1 \ldots i_n} \left( dx^{i_1} - u^{i_1} [x] \right) \wedge \cdots \wedge \left( dx^{i_n} - u^{i_n} [x] \right)$$

$$= \varepsilon_{i_1 \ldots i_n} \left( dx^{i_1} \wedge \cdots \wedge dx^{i_n} - n \varepsilon_{i_1 \ldots i_n} u^{i_1} [x] \right) \wedge \left( dx^{i_2} \cdots \wedge dx^{i_n} \right)$$

$$= n! \left( dx^1 \wedge \cdots \wedge dx^n - d\tau \wedge dI_1 \wedge \cdots \wedge dI_{n-1} \right). \tag{31}$$

The transition from the first to the second line here is trivial, while the detailed steps leading from the second to the third lines are given below, in [18]. Even without detailed index manipulations, however, the form of the third line is transparent given (29) and the form of the second line, and vice versa.

For instance, to go from the second to the third lines, first note that $\Omega$ is first-order in $u$. Then, since $u^i$ is multi-linear and totally antisymmetric in the first partial $x^j \neq i$ derivatives of the $I$s, the terms linear in $u^i$ must sum to give $dI_1 \wedge \cdots \wedge dI_{n-1}$. The only thing that is perhaps not obvious is the relative coefficient between the two exact $n$-forms that appear in the third line of (31). This relative coefficient is carefully and directly determined below, in [18]. An alternate albeit indirect verification of the relative coefficient, as written in (31), was first given in [14] through Tahktajan’s demonstration that the obvious extrema of the integrals of the first and second lines of (31) are also extrema of integrals of the third line. This test would fail for other relative coefficients.

Further consideration of such integrals is warranted. We want $\Omega$ to be exact \footnote{A necessary condition for $\Omega$ to be exact is that it be closed. This implies the flow should be divergenceless, $\nabla \cdot u = 0$, as $\varepsilon_{i_1 \ldots i_n} \partial_{x^{i_1}} u^{i_1} [x] \wedge dx^{i_2} \cdots \wedge dx^{i_n} = \varepsilon_{i_1 \ldots i_n} \partial_{x^{i_1}} u^{i_1} dx^{i_2} \wedge dx^{i_3} \cdots \wedge dx^{i_n} = \frac{1}{n!} \left( \partial_{u^j} \varepsilon_{i_1 \ldots i_n} dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_n} \right).$} to trivially reduce $\int_{M_n} \Omega$ to $\int_{\partial M_n} \Lambda$, i.e. to a “boundary action,” that describes evolution of the d-brane. That action will be manifestly extremal on the solution set defined by (27), given the multilinear form of $\Omega$ in terms of those first order equations of motion. The action that governs the evolution of the d-brane (for $n = 3$, see [3], and for general $n$, see [14], especially Remark 9) is

$$A = \int_{M_n} \Omega = n! \int_{\partial M_n} \left( x^1 \wedge dx^2 \wedge \cdots \wedge dx^n + I_1 d\tau \wedge dI_2 \wedge \cdots \wedge dI_{n-1} \right). \tag{32}$$

The latter form of the action is an integral over the $n - 1$ dimensional world-volume swept out by the evolving d-brane, confirming that at any instant the d-brane is of dimension $n - 2$. Parameterizing the world-volume by coordinates $\alpha_1, \ldots, \alpha_{n-2}$, as well as $\tau$, as already chosen, the action is

$$A = \int_{\partial M_n} d\tau d\alpha_1 \cdots d\alpha_{n-2} \left[ (n-1)! \varepsilon_{i_1 \ldots i_n} x^{i_1} \partial_{\tau} x^{i_2} \partial_{\alpha_1} x^{i_3} \cdots \partial_{\alpha_{n-2}} x^{i_n} + (n-2)! n \varepsilon^{i_1 \ldots i_{n-1}} I_{j_1} \partial_{\tau} I_{j_2} \cdots \partial_{\alpha_{n-2}} I_{j_{n-1}} \right]. \tag{33}$$

NB In this last expression, all $i$s are summed from 1 to $n$, but all $j$s are summed from 1 to $n - 1$.

The equations of motion that follow from extremizing this action are then projected versions of the original (27), namely

$$\varepsilon_{i_1 \ldots i_n} \left( \partial_{x^{i_2}} u^{i_2} - u^{i_2} \right) \partial_{\alpha_1} x^{i_3} \cdots \partial_{\alpha_{n-2}} x^{i_n} = 0. \tag{34}$$
This projected form of the equations of motion (27) allows for arbitrary reparameterizations of the \((n-2)\)-brane surface, at any given \(\tau\), including \(\tau\)-dependent choices for the \(\alpha_j\). Through this reparameterization freedom, as emphasized by Regge et al. [7], and subsequently by Tahktajan [14] (especially Remark 8), there exists a \(\tau\)-dependent parameterization such that (27) is fully recovered. However, in more geometrical/physical terms, the evolution of a point according to (27) is fully recovered just by considering the (34)-driven evolution of \(n-1\) intersecting branes, appropriately configured to intersect at the point in question. This is most easily visualized for the string case, with \(n=3\). A point defined by the intersection of two appropriately chosen strings, with each string moving according to (34), will itself evolve according to (27).

What is most interesting in all this is the incorporation of interactions through the use of Nambu brackets, particularly for the case of even dimensional phase-space d-branes for superintegrable systems. This elegant geometrical characterization of integrable systems with a maximal number of invariants has not been fully appreciated previously, I believe. Also, this novel geometrical view holds considerable promise for understanding the quantization of such maximally superintegrable systems – promise made all the more possible by recent work on the quantization of the Nambu bracket using both traditional Hilbert space operator and non-Abelian deformation methods of quantization [2].

Specification of the dynamics through the \(n\)-form \(\Omega\) does not by itself endow the brane with any inherent stability or cohesion. That is, the brane is tensionless so far as the above action dictates. The data comprising the brane evolves smoothly and remains contiguous without tearing only through the courtesy of the driving velocity field. If the latter is smooth enough, then so is the evolution of the extended data set. It may move forward along with the ambient flow and give the appearance of stability, much as a thin sheet of smoke, dust, or snow blown by the wind may stay together if the driving flow of air allows it. Moreover, for a \(\tau\)-independent but otherwise arbitrarily chosen parameterization, the d-brane responds similarly to an untethered, tensionless, “teflon-coated” sail (for the \(n=4\) case, with more challenging mental pictures for generalizations to higher \(n\)). Only the flow component perpendicular to the surface has any apparent effect on it. This would suggest that any shear in the ambient flow will cause all impacted sections of the surface to rotate and become parallel to \(\mathbf{u}\) (i.e. tangential). Once parallel to \(\mathbf{u}\), the surface sections will cease being moved extrinsically by the velocity field, although intrinsic \(\tau\)-dependent reparameterizations can be used to maintain the tangential motion inherent in (27). At least, that is the case for motion due to the action of the \(n\)-form alone.

For classification purposes, it is sufficient to consider just the free kinetic portion of the \(n\)-form (Hopf term).

\[
\omega = \varepsilon_{i_1 \cdots i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n} = n! \left( dx^1 \wedge \cdots \wedge dx^n \right).
\]  
(35)

This alone is exact, so we obtain only a boundary action from its integral.

\[
\int_{M_n} \omega = \int_{\partial M_n} \varepsilon_{i_1 \cdots i_n} x^{i_1} dx^{i_2} \wedge \cdots \wedge dx^{i_n} = (n-1)! \int d\tau d\alpha_1 \cdots d\alpha_{n-2} \varepsilon_{i_1 \cdots i_n} x^{i_1} \partial_\tau x^{i_2} \partial_{\alpha_1} x^{i_3} \cdots \partial_{\alpha_{n-2}} x^{i_n},
\]  
(36)

where in the last step we have again used the parameterization of the \((n-1)\)-dimensional world-volume swept out by the evolving \((n-2)\)-brane. We have singled out one of the parameters as the Nambu time, \(\tau\). So the free term may be used to classify the various extended structures, as for
the expected exact 2-form, well-known from Hamiltonian dynamics. So the action is the usual

\[ \int \omega = \int_{\partial M_2} \varepsilon_{ij} x^i \, dx^j = \int d\tau \varepsilon_{ij} x^i \partial_\tau x^j \] point particle, \hfill (37)

\[ \int \omega = \int_{\partial M_3} \varepsilon_{ijk} x^i \, dx^j \wedge dx^k = 2 \int d\tau d\alpha \varepsilon_{ijk} x^i \partial_\tau x^j \partial_\alpha x^k \] string, \hfill (38)

\[ \int \omega = \int_{\partial M_4} \varepsilon_{ijk} x^i \, dx^j \wedge dx^k \wedge dx^l = 6 \int d\tau d\alpha d\beta \varepsilon_{ijk} x^i \partial_\tau x^j \partial_\alpha x^k \partial_\beta x^l \] membrane, \hfill (39)

\[ \int \omega = (2N)! \int_{M_{2N}} dx^1 \wedge dp^1 \wedge dx^2 \wedge dp^2 \wedge \cdots \wedge dx^N \wedge dp^N = 2^N (2N - 1)!! \int_{M_{2N}} (\omega_2)^N \] maximal phase-space d-brane,

where in the next to last line, \( \omega_2 = \sum_j dx^j \wedge dp^j \) is the canonical two-form on the phase-space.

Take the simplest case of a particle in phase-space undergoing Hamiltonian flow, with \( n = 2 \). The phase-space velocity field in this case is given by

\[ u^x = \frac{\partial}{\partial p} H, \quad u^p = -\frac{\partial}{\partial x} H, \quad \frac{\partial}{\partial x} u^x + \frac{\partial}{\partial p} u^p = 0 . \] \hfill (40)

The additional term is then

\[ \varepsilon_{i_1 \cdots i_n} u^{i_1} [x] \, d\tau \wedge dx^{i_2} \cdots \wedge dx^{i_n} \rightarrow 2 \int d\tau \wedge (u^x dp - u^p dx) = d\tau \wedge (\partial_p H dp + \partial_x H dx) = d\tau \wedge dH , \] \hfill (41)

the expected exact 2-form, well-known from Hamiltonian dynamics. So the action is the usual

\[ A = \int M_2 \Omega = \int_{\partial M_2} x dp - pdx + 2H d\tau = 2 \int_{\partial M_2} H d\tau - pdx . \] \hfill (42)

The method always works for the point-particle case when the flow is Hamiltonian.

Take the second simplest case of a “vortex string” (see Regge et al. \[7\]), with \( n = 3 \).

\[ \varepsilon_{i_1 \cdots i_n} u^{i_1} [x] \, d\tau \wedge dx^{i_2} \cdots \wedge dx^{i_n} \rightarrow 3 \int d\tau \wedge (u^x dy \wedge dz + u^y dz \wedge dx + u^z dx \wedge dy) . \] \hfill (43)

When the velocity field is given by a Nambu bracket, with invariant generating entries that we shall designate \( H \) and \( L \), we obtain:

\[ u^x = \{ x, H, L \} = \partial_y H \partial_z L - \partial_z H \partial_y L , \]

\[ u^y = \{ y, H, L \} = \partial_x H \partial_z L - \partial_z H \partial_x L , \]

\[ u^z = \{ z, H, L \} = \partial_x H \partial_y L - \partial_y H \partial_x L . \] \hfill (44)

For the velocity field around a vortex, these two invariants may in fact be written in terms of the Clebsch potentials \[10, 11\]. In any case,

\[ u^x dy \wedge dz + u^y dz \wedge dx + u^z dx \wedge dy = (\partial_y H \partial_z L - \partial_z H \partial_y L) dy \wedge dz + (\partial_x H \partial_z L - \partial_z H \partial_x L) dz \wedge dx + (\partial_x H \partial_y L - \partial_y H \partial_x L) dx \wedge dy \]

\[ = (\partial_x L dx + \partial_y H dy + \partial_z H dz) \wedge (\partial_x L dx + \partial_y L dy + \partial_z L dz) = dH \wedge dL . \] \hfill (45)
Thus the interaction with the velocity field becomes
\[ \varepsilon_{i_1 \cdots i_n} u^{i_1} \left[ x \right] d\tau \wedge dx^{i_2} \cdots \wedge dx^{i_n} \quad (n-3) \quad 2 \quad d\tau \wedge dH \wedge dL , \]  \[ \int_{M_3} \Omega = 6 \int_{M_3} dx \wedge dy \wedge dz - d\tau \wedge dH \wedge dL = 6 \int_{\partial M_3} xdy \wedge dz - H dL \wedge d\tau . \]  

Next, consider the details of the general case. We will use the generalized Kronecker symbol and its trace (both \( i \)s and \( j \)s are summed from 1 to \( n \) here).
\[ \varepsilon_{i_1 \cdots i_n} \varepsilon_{j_1 \cdots j_n} = \delta^{i_1 \cdots i_n}_{j_1 \cdots j_n} = \delta^{i_1}_{j_1} \times \delta^{i_2 i_3 \cdots i_n}_{j_1 j_2 j_3 \cdots j_n} - \delta^{i_1}_{j_2} \times \delta^{i_2 i_3 \cdots i_n}_{j_1 j_2 j_3 \cdots j_n} - \quad (n \text{ terms}), \]  \[ \delta^{i_2 \cdots i_n}_{j_2 \cdots j_n} = \delta^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_n} = \delta^{i_1}_{j_1} \times \delta^{i_2 \cdots i_n}_{j_1 j_2 \cdots j_n} . \]  

Note from the examples the point is always to make an exact form out of
\[ \varepsilon_{i_1 \cdots i_n} u^{i_1} \left[ x \right] dx^{i_2} \cdots \wedge dx^{i_n} . \]  

But when the velocity field is given by an \( n \)-bracket, \( u^{i_1} = \{ x^{i_1}, I_1, \cdots, I_{n-1} \} \) this becomes
\[ \varepsilon_{i_1 \cdots i_n} \{ x^{i_1}, I_1, \cdots, I_{n-1} \} dx^{i_2} \wedge \cdots \wedge dx^{i_n} = \varepsilon_{i_1 \cdots i_n} \varepsilon_{j_1 \cdots j_n} \partial_{j_1} x^{i_1} \partial_{j_2} I_1 \cdots \partial_{j_n} I_{n-1} dx^{i_2} \wedge \cdots \wedge dx^{i_n} = \varepsilon_{i_1 \cdots i_n} \varepsilon_{j_1 \cdots j_n} \delta^{i_1}_{j_1} \partial_{j_2} I_1 \cdots \partial_{j_n} I_{n-1} dx^{i_2} \wedge \cdots \wedge dx^{i_n} = \delta^{i_2 \cdots i_n}_{j_2 \cdots j_n} \partial_{j_1} I_1 \cdots \partial_{j_n} I_{n-1} dx^{i_2} \wedge \cdots \wedge dx^{i_n} = (n-1)! \partial_{i_1} I_1 dx^{i_2} \wedge \cdots \wedge dx^{i_n} = (n-1)! dI_1 \wedge \cdots \wedge dI_{n-1} . \]  

That’s all there is to it. In summary, there are two steps needed:
\[ \varepsilon_{i_1 \cdots i_n} u^{i_1} \left[ x \right] dx^{i_2} \cdots \wedge dx^{i_n} = \varepsilon_{i_1 \cdots i_n} \{ x^{i_1}, I_1, \cdots, I_{n-1} \} dx^{i_2} \wedge \cdots \wedge dx^{i_n} = (n-1)! dI_1 \wedge \cdots \wedge dI_{n-1} . \]  

Thus, for such velocity fields given by Nambu brackets, we recover (31).

A more complete recount of the history of the ambient “flowing fluid” picture of the classical dynamics of extended objects would take note of the following. Following the seminal paper of Nambu, the classical theory in the general case was worked out in various stages, by Estabrook, by Lund, Rasetti, and Regge, by Tahktajan, and by Matsuo and Shibusa. While extended objects were not explicitly discussed, this picture was nevertheless initiated, formally, as a generalization of the usual Hamiltonian 2-form framework to higher-dimensional forms by Estabrook in the context of Nambu mechanics [9]. A physical interpretation as an extended object moving in a three-dimensional fluid was given by Regge et al. [7], for the case of vortex strings, but without using N-brackets. The whole business was mathematically codified for general \( n \) by Tahktajan [12], using N-brackets, but without any additional physical interpretation, and with some unnecessary complication involving an extended phase-space picture. More recently, Matsuo and Shibusa [8] have developed further the ideas of Regge et al. to higher \( n \), and Pioline [12] has given a summary of past developments, with emphasis on strings and membranes, while stressing the case of 3-forms. Even more recently, Curtright and Zachos [2] have discussed at length the Nambu formalism in
conventional phase-space contexts for superintegrable systems, have emphasized the differences between even and odd N-brackets, have pointed out the particularly nice properties of implementing Nambu mechanics with 4-brackets for models with \( su(2) \) invariance, and have carefully discussed the quantization of the formalism using conventional Hilbert space and/or deformation quantization methods. Finally, in a conference talk reviewing [2], Zachos has discussed 2-branes using 4-brackets [12].

Only classical considerations were made above. However, the resulting mental picture evoked by the formalism is somewhere intermediate between the classical Hamiltonian evolution of ideal points, and the quantum evolution of globally well-behaved distributions (Wigner functions, say, defined throughout the full phase space) with support of codimension zero. Here we are led to envision distributions of data on surfaces of non-zero codimension, evolved quite elegantly through the use of a single Nambu bracket in such a way that the implicit evolution of ideal points comprising the surface is not necessarily just uniform time evolution, but rather evolution involving a range of time scales dictated by the breadth of the extended distribution in the various invariants.

That is to say, the complete extended object evolves smoothly and uniformly under the passage of a single Nambu time, \( \tau \). But each individual point comprising the object may evolve according to its own time scale such that a spectrum of conventional individual particle times \( t \) pass during the evolution. Aside from being at least implicit in the references cited earlier, the only similar classical precursor of such collectively evolved data, as far as we know, arises in the formalism of the phase-space surface-of-section used in the exploration of chaos and to test for integrability. In that formalism, the points making-up the initial surface are evolved individually, for example by Hamiltonian methods (see, for example, [6]). When following individual trajectories of points initially on the surface of section, the times required to return to the initial surface may vary between the various trajectories. Under Nambu time evolution, the analogue of the surface of section would evolve as a coherent whole for each Nambu time increment.

The quantization of the d-brane dynamics described here is under investigation. Early results on quantization issues for 1-brane vortices are described in [7]. A follow-up infusion of general, more rigorous results for quantized 2-branes of vorticity is given in [5]. Still more recent results for 1-branes are in [11]. None of these other studies of the quantization of vorticity and 1-branes makes particular use of quantum Nambu brackets, however. Such usage seems far less problematic today than it did, say, one year ago due to the developments in [2]. When N-brackets are quantized, they have many surprising features, not the least of which is that the time-scale becomes dynamical in all but one known special case, when applied to the evolution of point-particle Wigner functions. This is one feature that naturally associates quantum N-brackets to the evolution of d-branes described above, and strongly suggests that the latter can provide a convenient bridge between classical and quantum behavior. A full report on this subject will have to be given elsewhere.

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References

[1] T L Curtright and D B Fairlie, “Extra Dimensions and Nonlinear Equations” J Math Phys 44 (2003) 2692-2703 [math-ph/0207008];
T L Curtright and D B Fairlie, “Morphing Quantum Mechanics and Fluid Dynamics” J Phys A (2003) to appear [math-ph/0303003]

[2] T L Curtright and C K Zachos, “Classical and quantum Nambu mechanics” Phys Rev D (2003) to appear [hep-th/0212267];
T L Curtright and C K Zachos, “Deformation Quantization of Superintegrable Systems and Nambu Mechanics” N J Phys 4 (2002) 83 [hep-th/0205063].

[3] F B Estabrook, “Comments on Generalized Hamiltonian Dynamics” Phys Rev D8 (1973) 2740-2743.

[4] U R Fischer, “Motion of Quantized Vortices as Elementary Objects” Ann. Phys. (N.Y.) 278 (1999) 62-85 [cond-mat/9907457].

[5] G A Goldin, R Menikoff, and D H Sharp, “Quantum Vortex Configurations in Three Dimensions” Phys Rev Lett 67 (1991) 3499-3502.

[6] M C Gutzwiller, Chaos in Classical and Quantum Mechanics, Springer-Verlag, 1990.

[7] M Rasetti and T Regge, “Vortices in He II, current algebras, and quantum knots” Physica 80A (1975) 217-233;
F Lund and T Regge, “Unified approach to strings and vortices with soliton solutions” Phys Rev D14 (1976) 1524-1535.

[8] Y Matsuo and Y Shibusa, “Volume preserving diffeomorphism and noncommutative branes” JHEP 02 (2001) 006 [hep-th/0010040].

[9] Y Nambu, “Generalized Hamiltonian Dynamics” Phys Rev D7 (1973) 2405-2412.

[10] Y Nambu, “Hamilton-Jacobi Formalism for Strings” Phys Lett B92 (1980) 327-330.

[11] Y A Rylov, “Hydrodynamic equations for incompressible inviscid fluid in terms of Clebsch potentials” [physics/0303065].

[12] B Pioline, “Comments on the Topological Open Membrane” Phys Rev D66 (2002) 025010 [hep-th/0201257].

[13] A D Speliotopoulos, “A topological string: the Rasetti-Regge Lagrangian, topological quantum field theory, and vortices in quantum fluids” J Phys A35 (2002) 8859-8866.

[14] L Takhtajan, “On foundation of the generalized Nambu mechanics” Comm Math Phys 160 (1994) 295–315 [hep-th/9301111].

[15] C K Zachos, “Membranes and Consistent Quantization of Nambu Mechanics” talk at the 8th Wigner Symposium, NYC, May 2003 [hep-th/0306222].