Recent Results on Domain Decomposition
Preconditioning for the High-frequency
Helmholtz Equation using Absorption

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Abstract In this paper we present an overview of recent progress on the development and analysis of domain decomposition preconditioners for discretised Helmholtz problems, where the preconditioner is constructed from the corresponding problem with added absorption. Our preconditioners incorporate local subproblems that can have various boundary conditions, and include the possibility of a global coarse mesh. While the rigorous analysis describes preconditioners for the Helmholtz problem with added absorption, this theory also informs the development of efficient multilevel solvers for the “pure” Helmholtz problem without absorption. For this case, 2D experiments for problems containing up to about 50 wavelengths are presented. The experiments show iteration counts of order about \( \mathcal{O}(n^{0.2}) \) and times (on a serial machine) of order about \( \mathcal{O}(n^\alpha) \), with \( \alpha \in [1.3, 1.4] \) for solving systems of dimension \( n \). This holds both in the pollution-free case corresponding to meshes with grid size \( \mathcal{O}(k^{-3/2}) \) (as the wavenumber \( k \) increases), and also for discretisations with a fixed number of grid points per wavelength, commonly used in applications. Parallelisation of the algorithms is also briefly discussed.

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1 Introduction

In this paper we describe recent work on the theory and implementation of domain decomposition methods for iterative solution of discretisations of the Helmholtz equation:

\[-(\Delta + k^2)u = f, \text{ in a domain } \Omega,\]  

where \(k(x) = \omega/c(x)\), with \(\omega\) denoting frequency and \(c\) denoting the speed of acoustic waves in \(\Omega\). Our motivation originates from applications in seismic imaging, but the methods developed are applicable more generally, e.g. to earthquake modelling or medical imaging. While practical imaging problems often involve the frequency domain reduction of the elastic wave equation or Maxwell’s equations, the scalar Helmholtz equation (1) is still an extremely relevant model problem which encapsulates many of the key difficulties of more complex problems.

We will focus here on solving (3) on a bounded domain \(\Omega\), subject to the first order absorbing (impedance) boundary condition:

\[\frac{\partial u}{\partial n} - ik u = g \text{ on } \Gamma = \partial \Omega,\]  

although the methods presented are more general.

The theoretical part of this paper is restricted to the case of \(k\) constant. However the methods proposed can be used in the variable \(k\) case, and preliminary experiments are done on this case in §5.3.

Important background for our investigation is the large body of work on “shifted Laplace” preconditioning for this problem, starting from [12] and including, for example [11] and recent work on deflation [29]. (A fuller survey is given in [17, 21] and elsewhere in this volume.) In those papers (multigrid) approximations of the solution operator for the perturbed problem

\[-(\Delta + (k^2 + i\epsilon))u = f, \text{ with } \frac{\partial u}{\partial n} - ik u = g \text{ on } \Gamma,\]  

(suitably discretised and with carefully tuned “absorption” parameter \(\epsilon > 0\)), were used as preconditioners for the iterative solution of (1). When \(k\) is variable, a slightly different shift strategy is appropriate (see §5.3).

One can see immediately the benefit of introducing \(\epsilon\) in (3): When \(k\) is constant the fundamental solution \(G_{k,0}\) of the operator in (3) (for example in 3D) satisfies, for fixed \(x \neq y\) with \(k|x-y| = O(1)\) and \(\epsilon \ll k^2\),

\[G_{k,\epsilon}(x,y) = G_{k,0}(x,y) \exp\left( -\frac{\epsilon}{2k} |x-y| \right) \left( 1 + O\left( \left( \frac{\epsilon}{k^2} \right)^2 k|x-y| \right) \right), \text{ as } k \to \infty.\]

Thus, the effect of introducing \(\epsilon\) is to exponentially damp the oscillations in the fundamental solution of problem (3), with the amount of damping proportional to \(\epsilon/k\). With slightly more analysis one can show that the weak form of problem (3) enjoys a coercivity property (with coercivity constant of order \(O(\epsilon/k^2)\) in the
This has the useful ramification that any finite element method for (3) is always well-posed (independent of mesh size) and enjoys a corresponding (albeit $\varepsilon-$ and $k-$ dependent) quasioptimality property. Therefore preconditioners constructed by applying local and coarse mesh solves applied to (3) are always well-defined; this is not true when $\varepsilon = 0$.

A natural question is then, how should one choose $\varepsilon$? To begin to investigate this question, we first introduce some notation. Let $A_\varepsilon$ denote the finite element approximation of (3) and write $A = A_0$. Then $A$ is the system matrix for problem (1), (2), which we want to solve.

Suppose an approximate inverse $B_\varepsilon^{-1}$ for $A_\varepsilon$ is constructed. Then a sufficient condition for $B_\varepsilon^{-1}$ to be a good preconditioner for $A$ is that $I - B_\varepsilon^{-1}A$ should be sufficiently small. Writing

$$I - B_\varepsilon^{-1}A = (I - B_\varepsilon^{-1}A_\varepsilon) + B_\varepsilon^{-1}A_\varepsilon(I - A_\varepsilon^{-1}A),$$

we see that a sufficient condition for the smallness of the term on the left-hand side is that

(i) $I - A_\varepsilon^{-1}A$ should be sufficiently small, and

(ii) $I - B_\varepsilon^{-1}A_\varepsilon$ should be sufficiently small.

At this stage, one might already guess that achieving both (i) and (ii) imposes somewhat contradictory requirements on $\varepsilon$. Indeed, on the one hand, (i) requires $\varepsilon$ to be sufficiently small (since the ideal preconditioner for $A$ is $A^{-1} = A_0^{-1}$). On the other hand, the larger $\varepsilon$ is, the less oscillatory the shifted problem is, and the easier it should be to construct a good approximation to $A_\varepsilon^{-1}$ for (ii).

Regarding (i): The spectral analysis in [14] of a 1-d finite-difference discretisation concluded that one needs $\varepsilon < k$ for the eigenvalues to be clustered around 1 (which partially achieves (i)). The analysis in [17] showed that, in both 2- and 3-d for a range of geometries and finite element discretisations, (i) is guaranteed if $\varepsilon/k \leq C_1$ for a small enough positive constant $C_1$, with numerical experiments indicating that this condition is sharp. Somewhat different investigations are contained in the references [12], [13], [11]. These performed spectral analyses that essentially aim to achieve (i) on a continuous level, and explored the best preconditioner of the form (3) for (1) in the 1D case with Dirichlet boundary conditions, based on the ansatz $k^2 + i\varepsilon = k^2(a + ib)$, where $a, b$ are to be chosen; related more general results are in [31]. (For more detail, see, e.g., the summary in [17] and other articles in this volume.)

Regarding (ii): several authors have considered the question of when multigrid converges (in a $k-$independent number of steps) when applied to the shifted problem $A_\varepsilon$, with the conclusion that one needs $\varepsilon \sim k^2$ [8], [2], [14]. Note that this question of convergence is not quite the same question as whether a multigrid approximation to $A_\varepsilon^{-1}$ is a good preconditioner for $A_\varepsilon$ (property (ii)) or for $A_0$ (the original problem), but these questions are investigated numerically in [8]. For classical Additive Schwarz domain decomposition preconditioners, it was shown in [21] that (ii) is guaranteed (under certain conditions on the coarse grid diameter) if $\varepsilon \sim k^2$ (resonating with the multigrid results). In fact [21] also provides $\varepsilon-$explicit estimates of the
rate of GMRES convergence when $A_\varepsilon$ is preconditioned by the Schwarz algorithm. Although these estimates degrade sharply when $\varepsilon$ is chosen less than $k^2$, numerical experiments in [21] indicate that improved estimates may be possible in the range $k \lesssim \varepsilon \lesssim k^2$.

The contradictory requirements that (i) requires $\varepsilon/k$ to be sufficiently small, and (ii) requires $\varepsilon \sim k^2$ (at least for classical Additive Schwarz domain decomposition preconditioners) motivate the question of whether new choices of $B_\varepsilon^{-1}$ can be devised that operate best when $\varepsilon$ is chosen in the range $k \lesssim \varepsilon \lesssim k^2$. Such choices should necessarily use components that are more suitable for “wave-like” problems, rather than the essentially “elliptic” technology of classical multigrid or classical domain decomposition. In fact our numerical experiments below indicate that, for the preconditioners studied here, the best choice of $\varepsilon$ varies, but is generally in the range $[k, k^{1.6}]$.

Domain decomposition methods offer the attractive feature that their coarse grid and local problems can be adapted to allow for “wave-like” behaviour. There is indeed a large literature on this (e.g. [4, 16, 18], but methods that combine many subdomains and coarse grids and include a convergence analysis are still missing. The paper [21] provides the first such rigorous analysis in the many subdomain case, and current work is focused on extending this to the case when wave-like components are inserted, such as using (optimised) impedance or PML conditions on the local solves.

Another class of preconditioners for Helmholtz problems of great recent interest is the “sweeping” preconditioner [10] and its related variants - e.g. [7], [30], [32]. In principle these methods require the direct solution of Helmholtz subproblems on strips of the domain. A method of expediting these inner solves with an additional domain decomposition and off-line computation of local inverses is presented in [33]. Related domain decomposition methods for these inner solves, using tuned absorption, and with applications to industrial problems, are explored in [1], [27], [28].

Finally it should be acknowledged that, while the reduction of the complicated question of the performance of $B_\varepsilon^{-1}$ as a preconditioner for $A$ into two digestible subproblems ((i) and (ii) above) is theoretically convenient, this approach is also very crude in several ways: Firstly the splitting of the problem into (i) and (ii) may not be optimal and secondly the overarching requirement that $\|I - B_\varepsilon^{-1}A\|$ should be small is far from necessary when assessing $B_\varepsilon^{-1}$ as a preconditioner for $A$: for example good GMRES convergence is still assured if the field of values of $B_\varepsilon^{-1}A$ is bounded away from the origin in the complex plane (in a suitable inner product) and that $B_\varepsilon^{-1}A$ is bounded from above in the corresponding norm. We use this in the theory below.
2 Domain Decomposition

To start, we denote the nodes of the finite element mesh as \( \{ x_j : j \in J^h \} \), for a suitable index set \( J^h \). These include nodes on the boundary \( \Gamma \) of \( \Omega \). The continuous piecewise linear finite element hat function basis is denoted \( \{ \phi_j : j \in J^h \} \). To define preconditioners, we choose a collection of \( N \) non-empty relatively open subsets \( \Omega_\ell \) of \( \Omega \), which form an overlapping cover of \( \Omega \). Each \( \Omega_\ell \) is assumed to consist of a union of elements of the finite element mesh, and the corresponding nodes on \( \Omega_\ell \) are denoted \( \{ x_j : j \in I_h(\Omega_\ell) \} \).

Now, for any \( j \in I_h(\Omega_\ell) \) and \( j' \in I_h \), we define the restriction matrix \( (R_\ell)_{j,j'} := \delta_{j,j'} \). The matrix \( A_{\ell} := R_\ell A \ell R_\ell^T \) is then just the minor of \( A \) corresponding to rows and columns taken from \( J^h(\Omega_\ell) \). This matrix corresponds to a discretisation (on the fine mesh) of the original problem \( (3) \) restricted to the local domain \( \Omega_\ell \), with a homogeneous Dirichlet condition at the interior boundary \( \partial \Omega_\ell \setminus \Gamma \) and impedance condition at the outer boundary \( \partial \Omega_\ell \cap \Gamma \) (when this is non-empty).

One-level domain decomposition methods are constructed from the inverses \( A_{\ell}^{-1} \).

More precisely,

\[
B_{\ell}^{-1} := \sum_\ell R_\ell A_\ell R_\ell^T,
\]

is the classical one-level Additive Schwarz approximation of \( A^{-1} \) with the subscript “local” indicating that the solves are on local subdomains \( \Omega_\ell \).

The overlapping subdomains are required to satisfy certain technical conditions concerning their shape and the size and uniformity of the overlap. Moreover, each point in the domain is allowed to lie only in a bounded number of overlapping subdomains as the mesh is refined. We do not repeat these conditions here but refer the interested reader to \([21, \S 3]\). The theorems presented in \([3]\) require these assumptions for their proof, as well as a quasi-uniformity assumption on the coarse mesh which is introduced next.

Two-level methods are obtained by adding a global coarse solve. We introduce a family of coarse simplicial meshes with nodes \( \{ x_j^H : j \in J^H \} \), where each coarse element is also assumed to consist of the union of a set of fine grid elements. The basis functions are taken to be the continuous \( P_1 \) hat functions on the coarse mesh, which we denote \( \{ \Phi_p^H : p \in J^H \} \). Then, introducing the fine-to-coarse restriction matrix \( (R_0)_{pj} := \phi_p^H(x_j^h) \), \( j \in J^h \), \( p \in J^H \), we can define the corresponding coarse mesh matrix \( A_{\ell,0} := R_0 A_\ell R_0^T \). Note that, due to the coercivity property for problem \( (3) \), both \( A_{\ell,0} \) and \( A_{\ell,\ell} \) are invertible for all mesh sizes \( h,H \) and all choices of \( \epsilon \neq 0 \).

The classical Additive Schwarz preconditioner is then

\[
B_{\ell,AS}^{-1} := R_0 A_{\ell,0}^{-1} R_0 + B_{\ell,AS,local}^{-1},
\]

(i.e. the sum of coarse solve and local solves) with \( B_{\ell,AS,local}^{-1} \) defined in \([3]\).
The theoretical results outlined in the next section concern the properties of $B_{e,AS}^{-1}$ as a preconditioner for $A_{e}$ (i.e. criterion (ii) in §1). The hypotheses for the theory involve conditions on $k, \varepsilon$ and $H$ (the coarse mesh diameter) as well as $H_{sub}$ (the maximum of the diameters of the local subdomains $\Omega_\ell$). This theory is verified by some of the numerical experiments in [21] and we do not repeat those here. Instead, in §5 below we focus in detail on the performance of (variants of) $B_{e,AS}^{-1}$ when used as preconditioners for the pure Helmholtz matrix $A$ (hence aiming to satisfy criteria (i) and (ii) of §1 simultaneously). The variants of (5) which we will consider include the Restricted, Hybrid and local impedance preconditioners. These are defined in §4.

First we give a summary of the theoretical results for (5). These are taken from [21]. The proofs are based on an analysis of projection operators onto subspaces with respect to the sesquilinear form which underlies the shifted problem (3). This type of analysis is well-known for coercive elliptic problems, but [21] was the first to devise such a theory for the high-frequency Helmholtz equation.

3 Main Theoretical Results

Here we describe the main results from [21], namely Theorems 5.6 and 5.8 in that reference. Since the systems arising from the discretisation of (3) are not Hermitian we need to use a general purpose solver. Here we used GMRES. Estimates of the condition number of the preconditioned matrix are not then enough to predict the convergence rate of GMRES. Instead one has to estimate either (i) the condition of the basis of eigenvectors of the system matrix, or (ii) bounds on its field of values. Here we take the second approach, making use of the classical theory of [9] (see also [3]). A brief summary of this theory is as follows.

Consider a nonsingular linear system $Cx = d$ in $\mathbb{C}^n$. Choose an initial guess $x^0$ for $x$, then introduce the residuals $r^0 = d - Cx^0$ and the usual Krylov spaces: $\mathcal{K}^m(C, r^0) := \text{span}\{C^j r^0 : j = 0, \ldots, m - 1\}$. Introduce a Hermitian positive definite matrix $D$ and the corresponding inner product on $\mathbb{C}^n$: $\langle V, W \rangle_D := W^* D V$, and let $\| \cdot \|_D$ denote the corresponding induced norm.

For $m \geq 1$, define $x^m$ to be the unique element of $\mathcal{K}^m$ satisfying the minimal residual property:

$$\|r^m\|_D := \|d - Cx^m\|_D = \min_{x \in \mathcal{K}^m(C, r^0)} \|d - Cx\|_D,$$

When $D = I$ this is just the usual GMRES algorithm, and we write $\| \cdot \| = \| \cdot \|_I$, but for more general $D$ it is the weighted GMRES method [15] in which case its implementation requires the application of the weighted Arnoldi process [22]. The reason for including weighted GMRES in the discussion will become clear later in this section.

The following theorem is then a simple generalisation of the classical convergence result stated (for $D = I$) in [3]. A proof is given in [21].
Theorem 1. Suppose $0 \not\in W_D(C)$. Then

$$\frac{\|r_m\|_D}{\|r_0\|_D} \leq \sin^m(\beta), \quad \text{where} \quad \cos(\beta) := \frac{\text{dist}(0, W_D(C))}{\|C\|_D}, \quad (6)$$

where $W_D(C)$ denotes the field of values (also called the numerical range of $C$) with respect to the inner product induced by $D$, i.e.

$$W_D(C) = \{(x, Cx)_D : x \in \mathbb{C}^n, \|x\|_D = 1\}.$$ 

This theorem shows that if the preconditioned matrix has a bounded norm, and has field of values bounded away from the origin, then GMRES will converge independently of all parameters which are not present in the bounds.

With this criterion for robust convergence in mind, the following results were proved in [21]. These results use the notation $A \lesssim B$ (equivalently $B \gtrsim A$) to mean that $A/B$ is bounded above by a constant independent of $k$, $\varepsilon$, and mesh diameters $h, H_{\text{sub}}, H$. We write $A \sim B$ when $A \lesssim B$ and $B \gtrsim A$. In all the theoretical results below $k$ is assumed constant.

In Theorem 2 and Corollary 1 below, the matrix $D_k$ which appears is the stiffness matrix arising from discretising the energy inner product for the Helmholtz equation using the finite element basis. More precisely, the Helmholtz energy inner product and associated norm are defined by

$$(v, w)_{1,k} := \int_{\Omega} \left( \nabla v \cdot \nabla w + k^2 v w \right) dx, \quad \text{and} \quad \|v\|_{1,k} = (v, v)_{1,k}^{1/2}. \quad (7)$$

For star-shaped Lipschitz domains, the norm $\|u\|_{1,k}$ of the solution $u$ of the Helmholtz boundary-value problems (1), (2) (or alternatively (3) in the case of absorption) can be estimated in terms of the data $f$ and $g$ (measured in suitable norms) with a constant that is independent of $k$ and $\varepsilon$ (provided $\varepsilon$ grows no faster than $O(k^2)$). This fact is the starting point (and a crucial building block) for the theory in [21]. If $\phi_i$ are the basis functions for the finite element space on the fine mesh, then the matrix $D_k$ is defined by

$$(D_k)_{\ell,m} = (\phi_\ell, \phi_m)_{1,k} \quad \text{for all} \quad \ell, m.$$ 

The matrix $D_k^{-1}$ appears as a weight in the result for right preconditioning in Theorem 3. These weights appear as artefacts of the method of analysis of the domain decomposition method which makes crucial use of the analysis of the Helmholtz equation in the energy norm. Fortunately, in practice, standard GMRES performs just as well as weighted GMRES (and is more efficient) - see Remark 1 below for more details.

Theorem 2 (Left preconditioning).

(i) $\|B_{\varepsilon,A}^{-1}A_{\varepsilon}\|_{D_k} \lesssim \left( \frac{k^2}{\varepsilon} \right)$ for all $H, H_{\text{sub}}$. 

Furthermore, there exists a constant $C_1$ such that
\[ |\langle V, B^{-1}_{\epsilon,AS}A\epsilon V \rangle_{D_k} | \gtrsim \left( \frac{\epsilon}{k^2} \right)^2 \|V\|^2_{D_k}, \quad \text{for all} \quad V \in \mathbb{C}^n, \]

when
\[
\max \left\{ kH_{\text{sub}}, kH \left( \frac{k^2}{\epsilon} \right)^2 \right\} \leq C_1 \left( \frac{\epsilon}{k^2} \right).
\]

(8)

This result contains a lot of information. In particular, if \( \epsilon \sim k^2 \) and \( kH, kH_{\text{sub}} \) are uniformly bounded, then (weighted) left-preconditioned GMRES applied to systems with matrix \( A_{\epsilon} \) will converge in a parameter-independent way. However when \( \epsilon/k^2 \to 0 \), the bounds degrade. Nevertheless, numerical experiments in [21] (in the regime \( H \sim H_{\text{sub}} \)) suggest there is some room to sharpen the theory: In particular, if \( \epsilon \sim k^2 \) the convergence of GMRES is parameter-independent even when \( kH \to \infty \) quite quickly (that is much coarser meshes than those predicted by the theory are possible). However if \( \epsilon \sim k \) then there appears not to be much scope to further reduce the coarse mesh diameter \( H \).

Combining Theorem 1 and Theorem 2 we obtain:

**Corollary 1 (GMRES convergence for left preconditioning).** Consider the weighted GMRES method where the residual is minimised in the norm induced by \( D_k \). Let \( r_m \) denote the \( m \)th iterate of GMRES applied to the system \( A_{\epsilon} \), left preconditioned with \( B^{-1}_{\epsilon,AS} \). Then
\[
\|r_m\|_{D_k} \|r_0\|_{D_k} \lesssim \left( 1 - \left( \frac{\epsilon}{k^2} \right)^6 \right)^{m/2},
\]

provided condition (8) holds.

Nowadays both left- and right- preconditioning play important roles in system solvers, and, in particular, right preconditioning is necessary if one wants to use Flexible GMRES (FGMRES) [26]. Fortunately Theorem 2 can be adapted to the case of right preconditioning as follows.

The first observation is that, for any \( n \times n \) complex matrix \( C \) (and working in the inner product \( \langle \cdot, \cdot \rangle_D \) induced by some SPD matrix \( D \)), we have, for any \( v \in \mathbb{C}^n \) and \( w := Dv \),
\[
\frac{\langle V, CV \rangle_D}{\langle V, V \rangle_D} = \frac{\langle W, C^*W \rangle_{D^{-1}}}{\langle W, W \rangle_{D^{-1}}},
\]

(10)

where \( C^* = C^\top \) denotes the Hermitian transpose of \( C \). Thus estimates for the distance of the field of values of \( C \) from the origin with respect to \( \langle \cdot, \cdot \rangle_D \) are equivalent to analogous estimates for the field of values of \( C^* \) with respect to \( \langle \cdot, \cdot \rangle_{D^{-1}} \).

The second observation is that Theorem 2 also holds for the adjoint of problem (3). In the adjoint case, the sign of \( \epsilon \) is reversed in the PDE and the boundary condition is replaced by \( \partial u/\partial n + iku = g \). In this case the estimates in Theorem 2 continue to hold, but with \( \epsilon \) replaced by \( |\epsilon| \). This is also proved in [21].

To handle the right-preconditioning case, we consider the field of values of the matrix \( A_{\epsilon}B^{-1}_{\epsilon,AS} \) in the inner product induced by \( D_k^{-1} \). By (10) these are provided by estimates of the field of values of \( B^{-\ast}_{\epsilon,AS}A_{\epsilon}^\ast \) in the inner product induced by \( D_k \). The
latter are provided directly by the (the extended version of) Theorem 2. The required estimates for the norm of $A_{\varepsilon}B_{\varepsilon,AS}^{-1}$ are obtained by a similar argument.

The result (from [21]) is as follows.

**Theorem 3 (right preconditioning).** With the same notation as in Theorem 2 we have

$$(i) \quad \|A_{\varepsilon}B_{\varepsilon,AS}^{-1}\|_{D_{k}^{-1}} \lesssim \left( \frac{k^2}{\varepsilon} \right) \quad \text{for all} \quad H, H_{\text{sub}}.$$

Furthermore, provided condition (8) holds,

$$(ii) \quad |\langle V, A_{\varepsilon}B_{\varepsilon,AS}^{-1}V \rangle_{D_{k}^{-1}}| \gtrsim \left( \frac{\varepsilon}{k^2} \right)^2 \|V\|_{D_k}^2, \quad \text{for all} \quad V \in \mathbb{C}^n.$$

**Remark 1.** As described earlier, the estimates above are in the weighted inner products induced by $D_k$ and $D_{k}^{-1}$. It would be inconvenient to have to implement GMRES with these weights, especially the second one. It is thus an interesting question whether the use of weighted GMRES is necessary in practice for these problems. We investigated both standard and weighted GMRES (in the case of left preconditioning and with weight $D_k$) for a range of problems (some covered by the theory, some not). In practice there was little difference between the two methods. Therefore, the numerical experiments reported here use standard GMRES.

**Remark 2.** The theorems in [21] also allowed general parameter $\delta > 0$ which described the amount of overlap between subdomains, and included the dependence on $\delta$ explicitly in the estimates. We suppressed this here in order to make the exposition simpler.

### 4 Variants of the Preconditioners

In this section we describe the variants of the classical Additive Schwarz method defined in (5) which are investigated in the numerical experiments which follow.

The first variant which we consider is the Restrictive Additive Schwarz (RAS) preconditioner, which is well-known in the literature [5], [23]. Here, to define the local operator, for each $j \in \mathcal{I}^h$, choose a single $\ell = \ell(j)$ with the property that $x_j \in \Omega_{\ell(j)}$. Then the action of the local contribution, for each vector of fine grid freedoms $v$, is:

$$(B_{\varepsilon,\text{RAS,local}}^{-1})_j = \left( R_{\ell(j)}^T A_{\varepsilon,\ell(j)}^{-1} R_{\ell(j)} v \right)_j, \quad \text{for each} \quad j \in \mathcal{I}^h. \quad (11)$$

We denote this one level preconditioner as RAS1. (We shall in fact use a slight variation on this - as described precisely in [5].)

From this we could build the RAS preconditioner (in analogy to the standard Additive Schwarz method):
\[ B_{\epsilon,\text{RAS}}^{-1} = R_{0}\epsilon^{-1}A_{0}^{-1}R_{0} + B_{\epsilon,\text{RAS,local}}^{-1} \tag{12} \]

However we shall not use this directly in the following. Rather, instead of doing all the local and coarse grid problems independently (and thus potentially in parallel), we first do a coarse solve and then perform the local solves on the residual of the coarse solve. This was first introduced in [24]. As described in [20], this method is closely related to the deflation method [25], which has been used recently to good effect in the context of shifted Laplacian combined with multigrid [29]. The Hybrid RAS (HRAS) preconditioner then takes the form

\[ B_{\epsilon,\text{HRAS}}^{-1} := R_{0}\epsilon^{-1}A_{0}^{-1}R_{0} + P_{0}^{T}\left( B_{\epsilon,\text{RAS,local}}^{-1} \right) P_{0} \tag{13} \]

where

\[ P_{0} = I - A R_{0}\epsilon^{-1}A_{0}^{-1}R_{0}. \]

Remembering that the local solves in \( B_{\epsilon,\text{RAS,local}}^{-1} \) are solutions of local problems with a Dirichlet condition on interior boundaries of subdomains, and noting that these are not expected to perform well for genuine wave propagation (i.e. \( \epsilon \) small and \( k \) large), we also consider the use of impedance boundary conditions on the local solves. Let \( A_{\epsilon,\text{Imp},\ell} \) be the stiffness matrix arising from the solution of (3) restricted to \( \Omega_{\ell} \), where the impedance condition \( \partial u / \partial n - iku \) is imposed on the boundary \( \partial \Omega_{\ell} \), and dealt with in the finite element method as a natural boundary condition. This can be used as a local operator in the HRAS operator (13). The one-level variant is

\[ (B_{\epsilon,\text{Imp,HRAS}}^{-1}v)_{j} = \left( \tilde{R}_{\ell}(j)^{T}A_{\epsilon,\text{Imp},\ell}(j)\tilde{R}_{\ell}(j)v \right)_{j} \quad \text{for each} \quad j \in \mathcal{J}^{h}, \tag{14} \]

Here (noting that the local impedance condition is handled as a natural boundary condition on \( \Omega_{\ell} \)), \( \tilde{R}_{\ell} \) denotes the restriction operator \( (\tilde{R}_{\ell})_{j,j'} = \delta_{j,j'} \), (as before) \( j' \) ranges over all \( \mathcal{J}^{h} \), but now \( j \) runs over all indices such that \( x_{j} \in \overline{\Omega}_{\ell} \).

The hybrid two-level variant is

\[ B_{\epsilon,\text{Imp,HRAS}}^{-1} := R_{0}\epsilon^{-1}A_{0}^{-1}R_{0} + P_{0}^{T}\left( B_{\epsilon,\text{Imp,HRAS,local}}^{-1} \right) P_{0} \tag{15} \]

We refer to these as the one- and two-level ImpHRAS preconditioners.

In the following section we will concentrate on illustrating the use of the four preconditioners defined in (11), (13), (14) and (15) for solving various problems with system matrix \( A \) (i.e. the discretisation of (3) with \( \epsilon = 0 \)). In our discussion and in the tables below we will use the following notation for the preconditioners:

\[ (11) \quad \text{RAS1}, \quad (13) \quad \text{HRAS}, \quad (14) \quad \text{ImpRAS1}, \quad (15) \quad \text{ImpHRAS}. \tag{16} \]
5 Numerical Experiments

Our numerical experiments concern the solution of (3) on the unit square, with \( \eta = k \) and \( \varepsilon = 0 \), discretised by the continuous piecewise linear finite element method on a uniform triangular mesh. Thus, the problem being solved here is the “pure Helmholtz” problem without absorption and can be completely specified by the fine mesh diameter, here denoted \( h_{\text{prob}} \). In [21] we also computed iteration numbers for solving (3) with \( \varepsilon > 0 \), thus an additional parameter \( \varepsilon_{\text{prob}} \) was needed to specify the problem being solved. Here we restrict to the case \( \varepsilon_{\text{prob}} = 0 \). For the solver we shall use domain decomposition preconditioners built from various approximate inverses for (3). The choice of \( \varepsilon > 0 \) which is used to build the preconditioner is denoted \( \varepsilon_{\text{prec}} \).

The experiments in §5.1 will be concerned with the case when the fine grid diameter is \( h_{\text{prob}} \sim k^{-3/2} \). This is the discretisation level generally believed to be necessary to remove the pollution effect: roughly speaking the relative error obtained with this choice of \( h_{\text{prob}} \) is not expected to grow as \( k \to \infty \). (However there is no proof of this except in the 1D case: See, e.g., the literature reviews in [17, Remark 4.2] and [19, §1.2.2].)

However the case of a fixed number of grid points per wavelength (\( h_{\text{prob}} \sim k^{-1} \)) is also frequently used in practice (especially in 3D) and provides sufficient accuracy in a limited frequency range. This regime is often studied in papers about Helmholtz solvers and so we include a substantial subsection (§5.2) on results for this case, which was not specifically discussed in [21]. Nevertheless the question of preconditioning the problem defined by \( h_{\text{prob}} \sim k^{-1} \) and \( \varepsilon_{\text{prob}} \sim k \) did arise in [21], as an “inner problem” in the multilevel solution of the problem with \( h_{\text{prob}} \sim k^{-3/2} \), \( \varepsilon_{\text{prob}} = 0 \). (This is discussed again in §5.1 below.)

Interestingly, it turns out that the asymptotics (as \( k \) increases) of the solvers in each of the two cases \( h_{\text{prob}} \sim k^{-3/2} \) and \( h_{\text{prob}} \sim k^{-1} \) (both with \( \varepsilon_{\text{prob}} = 0 \)) are somewhat different from each other and the best methods for one case are not necessarily the best for the other.

In the general theory given in §3 coarse grid size \( H \) and subdomain size \( H_{\text{sub}} \) are permitted to be unrelated. In our experiments here we construct local subdomains by first choosing a coarse grid and then taking each of the elements of the coarse grid and extending them to obtain an overlapping cover of subdomains with overlap parameter \( \delta \). This is chosen as large as possible, but with the restriction no two extended subdomains can touch unless they came from touching elements of the original coarse grid. In the literature this is called generous overlap and \( H_{\text{sub}} \sim H \). Thus our preconditioners are completely determined by specifying the values of \( H \) and \( \varepsilon \). In the case of constant \( k \), we denote these by

\[
H_{\text{prec}} \quad \text{and} \quad \varepsilon_{\text{prec}}. \tag{17}
\]

We also have to specify how the RAS subdomains (recall (11)) are defined. Actually in our implementation involves a slight variation on (11) as follows. Our RAS subdomains are the original elements of the coarse grid (before extension). These
overlap, but only at the edges of the coarse grid. Each node of the fine grid lies in a unique RAS subdomain except for nodes on the coarse grid edges. At these nodes the RAS operator (11) is extended so that it performs averaging of the contributions from all relevant subdomains at all such edge nodes.

When designing good domain decomposition methods we should be aware of cost. In the classical context (which we adopt here) where coarse grid and local problems are linked, a large-sized coarse grid problem will imply small-sized local problems and vice-versa. Coarse grids which are very fine and very coarse can both lead to very good methods in terms of iteration numbers, but not necessarily optimal in terms of time.

An “ideal” situation may be when all sub-problems are “load balanced”. Let \( h_{\text{prob}} \) be the fine grid diameter and let \( H_{\text{prec}} \) be the coarse grid diameter, so that in \( \mathbb{R}^d \), the dimension of the coarse grid problem is \( \mathcal{O}(H_{\text{prec}}^{-d}) \), while the dimension of the local problems are \( \mathcal{O}((H_{\text{prec}}/h_{\text{prob}})^d) \). Then the classical domain decomposition method is load-balanced when \( H_{\text{prec}} \sim h_{\text{prob}}^{1/2} \). If generous overlap is used, then a slightly smaller \( H_{\text{prec}} \) will give us load balancing. For example, in the pollution-free case \( h_{\text{prob}} = k^{-3/2} \), the domain decomposition will be load-balanced at about \( H_{\text{prec}} = k^{-0.8} \). While load balancing occurs at about \( H_{\text{prec}} \sim k^{-0.6} \) when we are taking a fixed number of points per wavelength (\( h_{\text{prob}} \sim k^{-1} \)). We use these estimates as a guide in the experiments below.

In all the experiments below the stopping tolerance for GMRES was that the relative residual should be reduced by \( 10^{-6} \).

In the experiments below, the system being solved is always the pure Helmholtz system \( Au = f \). In the results given in Tables 1-3 the right hand side vector \( f \) was chosen so that the finite element solution is an approximation of a plane wave (see [21 §6.2]). For the rest of the experiments \( f = 1 \) was used.

5.1 Pollution-free systems (\( h_{\text{prob}} \sim k^{-3/2} \))

The timings given in Tables[1] below were for implementation on a serial workstation with Intel Xeon E5-2630L CPUs with 48GB RAM. The later experiments were on a multiprocessor, described in [5.2].

The performance of GMRES for this case is investigated in detail in [21]. There we first studied the performance of domain decomposition preconditioners for systems with absorption (i.e. we set \( \varepsilon_{\text{prob}} = \varepsilon > 0 \) and we studied the performance of \( B^{-1}_\varepsilon \) as a preconditioner for \( A_\varepsilon \)). With respect to that question we found that:

(i) the performance of the solvers reflected the theory given in [3];
(ii) There was little difference between left- and right-preconditioning;
(iii) There was little difference between the performance of standard GMRES and GMRES which minimised the residual in the weighted norm (in the case of left preconditioning) induced by \( D_\varepsilon \) (see Remark [1] at the end of [3]).
(iv) There was a marked superiority for HRAS over several other variants of Additive Schwarz:
(v) If $H_{\text{prec}}$ is small enough ($H_{\text{prec}} \sim k^{-1}$ is sufficient), then $B^{-1}_e$ is a good preconditioner for $A_e$ even for rather small $\varepsilon$ (in fact, even $\varepsilon = 1$ gives acceptable results for HRAS);
(vi) If $H_{\text{prec}}$ is small enough then it makes little difference whether the local problems have Dirichlet or impedance boundary conditions;
(vii) For larger $H_{\text{prec}}$. Dirichlet local problems perform very badly, while impedance local problems work well for large enough $H_{\text{prec}}$. In this case the coarse grid solver can be switched off without degrading the convergence of GMRES.

Based on these observations, the discussion in [21] then turned to the more important question of the solution of problems without absorption (i.e. $\varepsilon_{\text{prob}} = 0$). The discussion in the rest of this subsection is an expansion of the discussion in [21].

We compare HRAS (Hybrid Restricted Additive Schwarz with Dirichlet local problems), as defined in (13) with ImpHRAS (Hybrid RAS with Impedance local problems), as defined in (15). In these experiments, $h_{\text{prob}} = k^{-3/2}$ and in Table 1 below we give the number of GMRES iterations (with # denoting iteration count) for each of these two methods for various choices of $H_{\text{prec}}$ and $\varepsilon_{\text{prec}}$. In Table 1, the headline figure for each case is the iteration number for the Hybrid method (13) or (15), while as a subscript we give the iteration count for the corresponding one level methods (omitting the coarse grid solve), given respectively by (11) and (14). We include iteration numbers for the three cases $\varepsilon_{\text{prec}} = k, k^{1/2}, k^2$. The optimal choice turns out to be around $\varepsilon_{\text{prec}} \in [k, k^{1/2}]$, while $\varepsilon_{\text{prec}} = k^2$ is provided for comparison. Data for a larger range of $\varepsilon_{\text{prec}}$ and $H_{\text{prec}}$ is given in [21]. A * in the tables means the iteration did not converge after 200 iterations.

| $H_{\text{prec}} \sim k^{-1}$, $\varepsilon_{\text{prec}} = k$ | $H_{\text{prec}} \sim k^{-1}$, $\varepsilon_{\text{prec}} = k^{1/2}$ | $H_{\text{prec}} \sim k^{-1}$, $\varepsilon_{\text{prec}} = k^2$ |
|------------|----------------|----------------|
| $k$ | # HRAS | # ImpHRAS | $k$ | # HRAS | # ImpHRAS | $k$ | # HRAS | # ImpHRAS |
| 20 | 1292 | 1705 | 20 | 1392 | 1805 | 20 | 3793 | 3413 |
| 40 | 18, 21 | 18, 21 | 40 | 18, 21 | 18, 21 | 40 | 63, 56 |
| 60 | 25, 27 | 25, 27 | 60 | 25, 27 | 25, 27 | 60 | 86, 78 |
| 80 | 33, 35 | 33, 35 | 80 | 33, 35 | 33, 35 | 80 | 101, 101 |
| 100 | 43, 45 | 43, 45 | 100 | 43, 45 | 43, 45 | 100 | 136, 123 |

| $H_{\text{prec}} \sim k^{-3/2}$, $\varepsilon_{\text{prec}} = k$ | $H_{\text{prec}} \sim k^{-3/2}$, $\varepsilon_{\text{prec}} = k^{1/2}$ | $H_{\text{prec}} \sim k^{-3/2}$, $\varepsilon_{\text{prec}} = k^2$ |
|----------------|----------------|----------------|
| $k$ | # HRAS | # ImpHRAS | $k$ | # HRAS | # ImpHRAS | $k$ | # HRAS | # ImpHRAS |
| 20 | 513 | 2612 | 20 | 483 | 2632 | 20 | 394 | 3942 |
| 40 | 1253 | 1253 | 40 | 1145 | 4841 | 40 | 814 | 736 |
| 60 | 693 | 693 | 60 | * | 693 | 693 | 60 | 1130 | 1049 |
| 80 | 743 | 743 | 80 | * | 743 | 743 | 80 | 135121 | 126111 |
| 100 | 845 | 845 | 100 | * | 845 | 845 | 100 | 156141 | 148131 |

Table 1 Comparison of HRAS and ImpHRAS for the problem with $h_{\text{prob}} \sim k^{-3/2}$, $\varepsilon_{\text{prob}} = 0$, using various choices of $H_{\text{prec}}$ and $\varepsilon_{\text{prec}}$. 

Based on the results in Table 1, we can make the following observations:

(i) When \( H_{\text{prec}} \sim k^{-1} \), the coarse grid is sufficiently fine and does a good job. Using the data for \( H_{\text{prec}} \sim k^{-1} \) and \( \varepsilon_{\text{prec}} \sim k^{1.2} \) we observe that we have \( \#HRAS \sim k^{0.71} \). Since we are here solving problems of size \( n \sim k^{3} \), this is equivalent to \( \#HRAS \sim n^{0.24} \). (Throughout the paper, rates of growth are obtained by linear least squares fits to the relevant log-log data.) Note that when \( H_{\text{prec}} \sim k^{-1} \), there is little difference between HRAS and ImpHRAS, i.e. it does not matter here whether the local problems have Dirichlet or Impedance condition. This preconditioner has a competitive performance as \( n \) increases, but it incorporates an expensive coarse grid solve of size \( H_{\text{prec}}^{-2} \sim k^{2} \) and it does not work without the coarse solve.

(ii) When \( H_{\text{prec}} \sim k^{-0.6} \) the local problems are rather large (size \( \sim k^{9/5} \)) the ImpHRAS method works reasonably well with a slow growth of iteration count with respect to \( k \) (although higher actual iterations), while HRAS is not usable. Moreover in the case of ImpHRAS, the coarse grid solve has almost no effect and can be neglected.

(iii) In all cases the best choice of absorption parameter \( \varepsilon_{\text{prec}} \) seems to be about \( \varepsilon_{\text{prec}} \sim k^{\beta} \) with \( \beta \) close to 1.2. We note that the choice \( \varepsilon_{\text{prec}} \sim k^{2} \) is remarkably inferior. A more extensive study of the variation of iteration numbers with respect to \( \varepsilon_{\text{prec}} \) and \( H_{\text{prec}} \) is given in [21].

These observations led to the formulation of an inner-outer strategy for problems with \( h_{\text{prob}} \sim k^{-3/2} \), with the outer iteration having preconditioner specified by \( H_{\text{prec}} = k^{-1} \) and \( \varepsilon_{\text{prec}} = k^{1.2} \). This “outer preconditioner” is a discretisation of (3) with \( h_{\text{prob}} \sim k^{-1} \) and \( \varepsilon_{\text{prob}} \sim k^{1.2} \), which is to be solved by a preconditioned inner iteration. So, as a precursor to formulating the inner-outer method, we study iteration counts for typical instances of this inner iteration. Here are some sample results with \( h_{\text{prob}} = \pi/5k \sim k^{-1} \), \( \varepsilon_{\text{prob}} = k^{1.2} \) using ImpHRAS as a preconditioner, with \( H_{\text{prec}} \sim k^{-1/2} \) and \( \varepsilon_{\text{prec}} = k^{1.2} \):

| \( k \) | \#ImpHRAS |
|-------|-----------|
| 20    | 1416      |
| 40    | 2123      |
| 60    | 2830      |
| 80    | 3235      |
| 100   | 3638      |
| 120   | 3938      |
| 140   | 4343      |

Table 2 Iteration numbers for ImpHRAS with \( \varepsilon_{\text{prob}} = k^{1.2} = \varepsilon_{\text{prec}}, h_{\text{prob}} = \pi/5k \) and \( H_{\text{prec}} \sim k^{-1/2} \)

We see from Table 2 that, even without the coarse solve, the iteration numbers grow slowly, and even seem to be slowing down as \( k \) increases. Extrapolation using the last five entries of Table 2 (without the coarse solve) indicates that \#ImpHRAS grows with approximately \( O(k^{0.38}) = O(n^{0.19}) \), where \( n \) is the size of the systems being solved in Table 2.
Therefore in [21] we proposed an inner-outer FGMRES iteration using (as the outer solver) HRAS with $H_{\text{prec}} = k^{-1}$ and (as the inner solver) ImpRAS1 with $H_{\text{prec}} = k^{-1/2}$. This method solves a system of dimension $O(k^3)$ by solving $O(k^2 + k)$ independent subdomain problems of dimension $O(k^{1/2} \times k^{1/2}) = O(k)$ and was found to have competitive properties.

In particular the subproblems are sufficiently small as to be very efficiently solved by a sparse direct solver. (Here we use umfpack included in the scipy sparse matrix package.) In this regard, an interesting observation is that, while positive definite systems coming from 2D finite element approximations of elliptic problems are often reported to be solvable by sparse direct solvers in optimal time ($O(n)$), for dimension $n$ up to about $10^5$, this appears not to be the case for the indefinite systems encountered here. In our experience the computation time for the sub-systems encountered here grows slightly faster than linearly with respect to dimension $n$.

The following table gives some sample results for the composite inner/outer algorithm with $\varepsilon_{\text{prec}} = k^\beta$ (for both inner and outer iterations, for various $\beta$) and an inner tolerance $\tau = 0.5$ (found in [21] to be empirically best). The numbers in bold font denote the number of outer (respectively inner) iterations, while the smaller font numbers underneath denote the total time in seconds [with an average time for each outer iteration in square brackets]. (Other choices of inner tolerance are explored in [21].) Recall that the outer tolerance is $10^{-3}$.

![Table 3](image)

Table 3 GMRES iteration counts and timings for the inner-outer algorithm with $\varepsilon_{\text{prob}} = 0$, $h_{\text{prob}} = k^{-3/2}$, $H_{\text{prec}} = k^{-1}$ in the outer iteration, $H_{\text{prec}} = k^{-1/2}$ in the inner iteration and $\varepsilon_{\text{prec}} = k^\beta$ in both inner and outer iterations

with $\beta \in [1, 1.2]$. Using the data in the column headed $\beta = 1$ (and remembering that we are here solving systems of dimension $n = k^3$), the outer iteration count grows with about $O(k^{0.53}) \approx O(n^{0.18})$, while the time per iteration is about $O(n^{1.11})$ and the total time is $O(n^{1.43})$. To give an idea of the size of the systems being solved, when $k = 100$, $n = 1,002,001$.

An interesting observation in Table [3] is the relative insensitivity of the results to the choice of $\beta$ in the range $\beta \in [0, 1.6]$, and the very poor performance of $\beta = 2$. Thus for this method the choice of absorption $\varepsilon_{\text{prec}} = k^2$ is a relatively poor one, while in fact the choice $\varepsilon_{\text{prec}} = 1 = k^0$ is quite competitive. This is quite different
to the experience reported using multigrid shifted Laplacian preconditioners. Note also that the number of inner iterations decreases as we read the rows of Table 3 from left to right, because increasing $\beta$ means putting more absorption into the preconditioner and hence makes the inner problem easier to solve.

The remainder of the experiments in the paper were done on a linux cluster of 130 nodes. Each node consists of 2 CPUs (Intel Xeon E5-2660 v2 @ 2.20GHz) with 10 cores: in total 20 cores and 64GB RAM on each node. The nodes are connected with 4x QDR Infiniband networks. This cluster was used in serial mode except for the modest parallel experiment in Table 5 in which up to 10 of the 130 nodes were used.

### 5.2 10 grid-points per wavelength ($h \sim k^{-1}$)

#### 5.2.1 Experiments with ImpRAS1 and ImpHRAS

In this section we consider the discretisation of (3) with $\varepsilon = 0$ and $h = \pi/5k$ (i.e. 10 grid points per wavelength). In this case the domain decomposition is load-balanced at about $H = k^{-0.6}$ and so we investigated the performance of preconditioned GMRES only for $H = k^{-\alpha}$, with $\alpha$ in the range $[0.4, 0.8]$. We found, for all choices of $\alpha$, the method HRAS not to be effective (with or without coarse grid solve), and so we focused attention on ImpHRAS and its one-level variant ImpRAS1.

Sample results for ImpRAS1 (top) and ImpHRAS (bottom) are given in Table 4. Here $T$ denotes the timing for the total solve process, while $T_i$ denotes the time per iteration. Here the cost of the coarse grid solve is relatively small and the time per iteration for ImpHRAS is almost the same as that for ImpRAS1. Overall ImpRAS1 is slightly quicker than ImpHRAS: Using the last 6 entries of each column for ImpHRAS with $H = k^{-0.4}$, #GMRES is growing with order $O(n^{0.18})$, while the total time is growing with order $O(n^{1.5})$.

In Table 5 we give preliminary timing results for a parallel implementation of the ImpRAS1 method. The implementation is in python and is based on numpy and scipy with the mpi4py library used for message passing. The problem is run on $P = M^2$ processes, where $M^2$ is the number of subdomains in the preconditioner. Processes are mapped onto $M$ cluster nodes with $M$ processes running on each node. The column labelled $P$ is the number of processors, which coincides with the number of subdomains. The column labelled $n_{loc}$ gives the dimension of the local problem being solved on each processor. Note that $n_{loc}$ grows with about $k^{1.2}$ while $P$ grows with about $k^{0.8}$ in this implementation. $T$ is the serial time, $T_{par}$ is the parallel time and $S = T/T_{par}$. Based on the last 6 entries of the column $T_{par}$, the parallel solve time is growing with about $O(k^{2.1}) = O(n^{1.05})$ where $n$ is the system dimension.


| $k$ |  $n$ | $H = k^{-0.5}$ | $H = k^{-0.4}$ | $G$ | $T$ | $T_0$ | $T_1$ | $T_0$ |
|-----|------|----------------|----------------|-----|-----|------|------|------|
| 60  | 35   | 6.83           | 0.15           | 20  | 4.67 | 0.16 |
| 80  | 39   | 13.01          | 0.27           | 23  | 9.21 | 0.30 |
| 100 | 43   | 24.21          | 0.47           | 25  | 18.8 | 0.59 |
| 120 | 45   | 37.10          | 0.69           | 29  | 29.5 | 0.83 |
| 140 | 49   | 63.85          | 1.12           | 28  | 43.3 | 1.27 |
| 160 | 51   | 84.65          | 1.43           | 33  | 67.1 | 1.73 |
| 180 | 54   | 113.86         | 1.85           | 32  | 91.0 | 2.43 |
| 200 | 57   | 159.67         | 2.47           | 30  | 114.2| 3.26 |
| 220 | 59   | 190.50         | 2.86           | 34  | 160.4| 4.11 |
| 240 | 61   | 249.48         | 3.64           | 35  | 203.3| 5.12 |
| 260 | 66   | 323.79         | 4.43           | 35  | 262.7| 6.67 |
| 280 | 70   | 390.81         | 5.07           | 39  | 354.6| 8.17 |
| 300 | 78   | 459.72         | 6.13           | 38  | 420.1| 9.98 |

Table 4: Performance of ImpRAS1 (top) and ImpHRAS (bottom) with $\epsilon_{\text{prob}} = 0$, $\epsilon_{\text{prec}} = k$ and $h = \pi/5k$, for $H_{\text{prec}} = k^{-0.5}, k^{-0.4}$.

### 5.2.2 A multilevel version of ImpRAS1

From Table 4 we see that the case $H = k^{-0.4}$ provides a solver with remarkably stable iteration counts, having almost no growth with respect to $k$. However (although the coarse grid component of the preconditioner can be neglected), the local systems to be solved at each iteration are relatively large, being of dimension $\mathcal{O}(k^{0.6})^2 = \mathcal{O}(k^{1.2})$. We therefore consider inner-outer iterative methods where these large local problems are resolved by an inner GMRES preconditioned with an ImpRAS1 preconditioner based on decompositon of the local domains of diameter $k^{-0.4}$ into much smaller domains of diameter $(k^{-0.4})^2 = k^{-0.8}$. (Such inner-outer methods are also investigated in different ways in [33] and [28].) The local prob-
Table 5  Parallel performance of ImpRAS1 with $\varepsilon_{\text{prob}} = 0$, $\varepsilon_{\text{prec}} = k$ and $h = \pi/5k$, for $H_{\text{prec}} = k^{-0.4}$. Relative speedup $S$ is shown for comparison of total time $T_{\text{par}}$ on $P$ processes with serial implementation time $T$.

| $k$ | $P = M^2$ | $n_{\text{loc}}$ | $\#\text{GMRES}$ | $T$ | $T_{\text{par}}$ | $S$ |
|-----|-----------|------------------|------------------|-----|------------------|-----|
| 60  | 25        | 1444             | 20               | 4.67| 0.38             | 12.25|
| 80  | 36        | 1764             | 23               | 9.21| 0.51             | 17.97|
| 100 | 36        | 2916             | 25               | 18.8| 1.02             | 18.54|
| 120 | 49        | 2916             | 29               | 29.50| 1.15              | 25.62|
| 140 | 49        | 3969             | 28               | 43.31| 1.62             | 26.66|
| 160 | 64        | 3969             | 33               | 67.15| 1.93             | 34.76|
| 180 | 64        | 5041             | 32               | 91.01| 2.37             | 38.43|
| 200 | 64        | 6241             | 30               | 114.27| 3.05             | 37.43|
| 220 | 81        | 6084             | 34               | 160.46| 3.24             | 49.53|
| 240 | 81        | 6889             | 35               | 203.30| 4.14             | 49.11|
| 260 | 81        | 8281             | 35               | 262.77| 5.34             | 49.23|
| 280 | 100       | 8100             | 39               | 354.60| 5.71             | 62.15|
| 300 | 100       | 9025             | 38               | 420.12| 6.73             | 62.43|

lems to be solved then are of dimension $O((k^{0.2})^2) = O(k^{0.4})$ and there are $O(k^{1.6})$ of them to solve at each iteration.

The inclusion of this method in the present paper is rather tentative, because (for the range of $k$ considered), breaking up the local problems of size $O(k^{1.2})$ into smaller subproblems is not competitive time-wise with the direct solver in 2D. The times of this multilevel variant are far inferior to those reported in Table 4. However even though the inner tolerance is set quite large at 0.5, the (outer) iteration numbers are remarkably unaffected (sample results are given in Table 6). In this table the outer tolerance is (as before) relative residual reduction of $10^{-6}$. Similar results (although slightly inferior) are obtained with $\varepsilon_{\text{prec}} = k$, in which case the inner iterations are also almost identical with those reported in Table 4 for ImpRAS1 in the case $H_{\text{prec}} = k^{-0.4}$.

Since the action of this preconditioner involves the solution of $O(k^{1.6})$ independent local systems of dimension only $O(k^{0.4})$, this method has strong parallel potential and is also worth investigating in 3D, where the direct solvers are less competitive.

5.3 Variable wave speed ($h \sim \omega^{-3/2}$)

In this subsection we give some initial results on the performance of our algorithms when applied to problems with variable wave speed. A more detailed investigation of this problem is one of our next priorities and the discussion here should be regarded as somewhat preliminary.

Domain decomposition methods have the advantage that the subdomains (and possibly the coarse mesh) can be chosen to resolve jumps in the wave speed, if the
wave speed is geometrically simple enough. At present the variable speed case is not covered by any theory, so this section is necessarily experimental.

We consider the analogue of the problem (3) with $k = \omega / c$ where $\omega$ is the angular frequency and $c = c(x)$ is the spatially dependent wave speed. For the preconditioners we consider approximate inverses of problems with variable absorption of the form:

$$-\Delta u - (1 + i\rho) \left( \frac{\omega}{c} \right)^2 u = f, \quad \text{on} \quad \Omega, \quad (18)$$

on a bounded domain $\Omega$ with impedance boundary condition

$$\frac{\partial u}{\partial n} - i \left( \frac{\omega}{c} \right) u = g \quad \text{on} \quad \Gamma \quad (19)$$

where $\rho = \rho_{\text{prec}} \geq 0$ is a parameter to be chosen. Thus when $c$ is constant, and $k := \omega / c$, the perturbed wavenumber is $k^2 + i\rho k^2$ and so the choice $\epsilon = k^\beta$ in (3) corresponds to the choice $\rho = k^{\beta - 2}$ in (18). On the other hand when $c$ is variable, the amount of absorption added is proportional to $(\omega / c)^2$ so more absorption is effectively added where $c$ is relatively small and less is added when $c$ is relatively large. We do not insert any absorption into the boundary condition (19).

We consider a test problem where $\Omega$ is the unit square. An internal square $\Omega_1$ of side length $1/3$ is placed inside $\Omega$ and the wave speed is taken to have value $c^*$ in the inner square and value 1 in $\Omega_2 := \Omega \setminus \Omega_1$. The square $\Omega_1$ is either placed in the centre of $\Omega$ (this is the case “discontinuity resolved”, where the coarse grid described below will resolve the interface) or at a position a few fine grid elements to the north and west of centre, with the distance moved in the directions north and west equal to the size of the overlap of the subdomains. In the latter case the coarse grid passes through the interface (and this is called “discontinuity unresolved” below). We perform experiments with $c^*$ both bigger than 1 and less than 1 with the latter case expected to be hardest.

| $\kappa$ | $\beta$ | 1.2 | 1.6 |
|----------|---------|-----|-----|
| 100      | 26(6)   | 31(4) |     |
| 120      | 31(6)   | 36(4) |     |
| 140      | 29(6)   | 35(4) |     |
| 160      | 33(7)   | 39(5) |     |
| 180      | 33(7)   | 38(5) |     |
| 200      | 32(7)   | 39(5) |     |
| 220      | 35(8)   | 42(5) |     |
| 240      | 35(8)   | 42(5) |     |
| 260      | 34(8)   | 42(5) |     |
| 280      | 39(9)   | 45(6) |     |
| 300      | 39(9)   | 45(6) |     |

Table 6 Sample iteration counts for the inner-outer ImpRAS1 preconditioner $\epsilon_{\text{prob}} = 0$, $h_{\text{prob}} = \pi/5 k$, $\epsilon_{\text{prec}} = k^\beta$, $H_{\text{prec}} = k^{-0.4}$ (for the outer iteration) and $H_{\text{prec}} = k^{-0.8}$ (for the inner iteration).
The problem is discretised by a uniform fine grid with $h_{\text{prob}} \sim \omega^{-3/2}$ and with the fine grid resolving the interface $\Gamma_{1,2}$ between $\Omega_1$ and $\Omega_2$. No absorption is added to the problem to be solved, i.e. $\rho_{\text{prob}} = 0$.

We apply the inner-outer algorithm as described in §5.1 (see Table 3) for this problem. The outer solver is HRAS with $H_{\text{prec}} \sim k^{-1}$ while the inner solver is ImpRAS1 with $H_{\text{prec}} \sim k^{-1/2}$. For both inner and outer solvers we set $\rho_{\text{prec}} = \omega^{\beta - 2}$.

In all cases generous overlap is used and the RAS domains are determined by the coarse grid as described in the introductory paragraphs to this section. The coarse grid for the outer solve consists of uniform triangles of diameter $\sim k^{-1}$ which are chosen to resolve the square $\Omega_1$ when it is placed in the centre, and do not resolve it when the square is moved. Numerical results, comparing the cases $c^* = 1.5, 1.0, 0.66$ are given in Table 7. In each row, for each value of $\beta$, the three figures indicate the number of outer HRAS iterations, the number of inner ImpRAS1 iterations (in brackets) and the total time on a serial machine. The outer tolerance is set at $10^{-6}$ while the inner tolerance is set at $0.5$.

The times for $\beta = 1.6$ grow with about $\mathcal{O}(n^{1.4})$ in the case $c^* = 1.5$ and $c^* = 1$ (rather similar to the performance observed in Table 3). The actual times in the case $c^* = 0.66$ are considerably worse (which is to be expected as smaller $c^*$ implies larger effective frequency on that domain. But the rate of growth of time with $n$ is not affected very much, being about $\mathcal{O}(n^{1.5})$ in Table 3. The case $c^* = 1.5$ seems a little easier to solve than the case $c^* = 1$. There is not much difference in any case between the resolved and the unresolved cases.

| $\omega$, $\beta$ | 1.0 | 1.2 | 1.6 | 1.8 |
|-------------------|-----|-----|-----|-----|
| 10 | 19(1) 0.71 | 19(1) 0.55 | 20(1) 0.53 | 21(1) 0.54 |
| 20 | 20(2) 3.25 | 20(2) 3.22 | 27(1) 3.65 | 30(1) 3.84 |
| 40 | 22(3) 50.09 | 23(3) 50.55 | 29(2) 54.04 | 44(1) 62.99 |
| 60 | 25(4) 356.71 | 26(4) 358.10 | 35(2) 381.06 | 57(1) 445.19 |
| 80 | 29(5) 1244.13 | 29(4) 1240.80 | 40(2) 1394.72 | 66(1) 1606.64 |
| 100 | 35(6) 3479.95 | 35(5) 3697.02 | 45(2) 3820.97 | 78(1) 4309.29 |
| $c^* = 1.5$, discontinuity unresolved |
| $\omega$, $\beta$ | 1.0 | 1.2 | 1.6 | 1.8 |
|-------------------|-----|-----|-----|-----|
| 10 | 18(1) 0.70 | 19(1) 0.56 | 20(1) 0.53 | 21(1) 0.54 |
| 20 | 20(2) 3.26 | 20(2) 3.25 | 27(1) 3.65 | 30(1) 3.87 |
| 40 | 22(3) 50.80 | 23(3) 51.30 | 29(2) 54.56 | 44(1) 63.76 |
| 60 | 25(4) 363.19 | 26(4) 364.96 | 35(2) 387.40 | 58(1) 454.04 |
| 80 | 30(5) 1273.11 | 30(4) 1347.66 | 40(2) 1417.61 | 66(1) 1623.74 |
| 100 | 35(6) 3545.44 | 35(5) 3541.62 | 45(2) 3660.37 | 78(1) 4042.62 |

Table 7 Performance of the inner-outer algorithm described in §5.3. Discontinuous wave speed, $c^* = 1.5$. 
In this paper we considered the construction of preconditioners for the Helmholtz equation (without or with absorption) by using domain decomposition methods applied to the corresponding problem with absorption. These methods are related to the shifted Laplacian multigrid methods, but the relative simplicity of the method considered here permits rigorous analysis of the convergence of GMRES through estimates of the field of values of the preconditioned problem. The flexibility of the domain decomposition approach also allows for the insertion of sub-solvers which are appropriate for high frequency Helmholtz problems, such as replacing Dirichlet local problems with impedance (or PML) local problems.

For the analysis, two theoretical subproblems are identified: (i) What range of absorption is permitted, so that the problem with absorption remains an optimal preconditioner for the problem without absorption? and (ii) What range of absorp-

Table 8 Performance of the inner-outer algorithm described in §5.3 Continuous wave speed $c^*=1$

| $\omega/\beta$ | 1.0 | 1.2 | 1.6 | 1.8 |
|----------------|-----|-----|-----|-----|
| 10             | 18(1) | 0.70 | 18(1) | 0.54 | 19(1) | 0.51 | 21(1) | 0.54 |
| 20             | 19(2) | 3.12 | 19(2) | 3.16 | 25(1) | 3.43 | 29(1) | 3.73 |
| 40             | 22(3) | 48.76 | 22(3) | 48.58 | 28(2) | 51.79 | 45(1) | 62.22 |
| 60             | 28(5) | 353.26 | 28(4) | 352.74 | 35(2) | 368.99 | 56(1) | 429.53 |
| 80             | 36(5) | 1235.44 | 35(5) | 1244.01 | 42(2) | 1361.78 | 66(1) | 1476.53 |
| 100            | 45(7) | 3487.02 | 44(6) | 3693.13 | 49(2) | 3728.06 | 79(1) | 4179.60 |

Table 9 Performance of the inner-outer algorithm described in §5.3 Discontinuous wave speed, $c^*=0.66$

| $\omega/\beta$ | 1.0 | 1.2 | 1.6 | 1.8 |
|----------------|-----|-----|-----|-----|
| 10             | 19(1) | 0.73 | 20(1) | 0.58 | 21(1) | 0.54 | 23(1) | 0.58 |
| 20             | 22(2) | 3.38 | 22(2) | 3.43 | 28(1) | 3.68 | 33(1) | 4.03 |
| 40             | 31(4) | 55.03 | 32(3) | 55.22 | 37(2) | 57.46 | 54(1) | 68.06 |
| 60             | 48(5) | 418.78 | 48(4) | 415.63 | 54(2) | 426.52 | 79(1) | 502.58 |
| 80             | 85(7) | 1709.73 | 78(5) | 1628.70 | 74(2) | 1630.55 | 108(1) | 1925.38 |
| 100            | 124(8) | 4881.62 | 115(7) | 4853.22 | 93(2) | 4448.73 | 134(1) | 5151.77 |

6 Summary
tion is needed so that the domain decomposition method performs optimally as a preconditioner for the problem with absorption?

The ranges that result from studying problems (i) and (ii) separately have been analysed, and this analysis is reviewed in the paper (§113). Since these ranges are disjoint, the best methods are obtained by using a combination of insight provided by the rigorous analysis and by numerical experimentation. The best methods involve careful tuning of the absorption parameter, the choice of coarse grid and the choice of boundary condition on the subdomains.

Of those methods studied, the best (in terms of computation time on a serial machine) differ, depending on the level of resolution of the underlying finite element grid. For problems with constant wave speed and with mesh diameter \( h \sim k^{-3/2} \) (so chosen to resolve the pollution effect), a multilevel method with serial time complexity \( O(n^\alpha) \) with \( \alpha \in [1.3, 1.4] \) is presented, where \( n \sim k^3 \) is the dimension of the system being solved (§5.1). In this method a two level preconditioner with a fairly fine coarse grid is used, and the coarse grid problem is resolved by an inner iteration with a further one-level preconditioner with impedance local solves.

For discretisations involving a fixed number of grid points per wavelength, similar time complexity is achieved by highly parallelisable one-level methods using impedance local solves on relatively large subdomains.

We also illustrate the method when it is applied to a model problem with jumping wave speed (§5.3). A preliminary parallel experiment is also given.

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