A complementary proof of Baker's theorem of completely invariant components for transcendental entire functions

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Abstract

Baker in [1] proved that for transcendental entire functions there is at most one completely invariant component of the Fatou set. It was observed by Julien Duval that there is a missing case in Baker’s proof. In this article we follow Baker’s ideas and give some alternative arguments to establish the result.

1 Introduction

Let \( \mathcal{E} \) be the set of transcendental entire functions \( f : \mathbb{C} \to \mathbb{C} \). For \( f \in \mathcal{E} \), we write \( f^n = f \circ f^{n-1} \) for the \( n \)-th iterate of \( f \), \( n \in \mathbb{N} \), and \( f^0 = \text{Id} \) where the symbol \( \circ \) denotes composition. When \( f^n(z_0) = z_0 \), for some \( n \in \mathbb{N} \), the point \( z_0 \) is called a periodic point. If \( n \) is the minimal positive integer for which this equality holds, we say that \( z_0 \) has period \( n \). If \( n = 1 \), \( z_0 \) is called a fixed point. The classification of a periodic point \( z_0 \) of period \( n \) of \( f \in \mathcal{E} \) can be attracting, super-attracting, rationally indifferent, irrationally indifferent and repelling.

Given \( f \in \mathcal{E} \), the Fatou set \( \mathcal{F}(f) \) is defined as the set of all points \( z \in \mathbb{C} \) such that the sequence of iterates \( (f^n)_{n \in \mathbb{N}} \) forms a normal family in some neighborhood of \( z \). The Julia set, denoted by \( \mathcal{J}(f) \), is the complement of the Fatou set.

Some properties of the Julia and Fatou sets for functions in class \( \mathcal{E} \) are mentioned below:

(i) \( \mathcal{F}(f) \) is open, so \( \mathcal{J}(f) \) is closed.
(ii) \( \mathcal{J}(f) \) is perfect and non-empty.
(iii) The sets \( \mathcal{J}(f) \) and \( \mathcal{F}(f) \) are completely invariant under \( f \).
(iv) \( \mathcal{F}(f) = \mathcal{F}(f^n) \) and \( \mathcal{J}(f) = \mathcal{J}(f^n) \) for all \( n \in \mathbb{N} \).
(v) The repelling periodic points are dense in \( \mathcal{J}(f) \).

See [2], [5], [6] and [7] for definitions, proofs and more details concerning the Fatou and Julia sets.

We denote by \( \text{CV} \) the set of critical values and by \( \text{OV} \) the set of omitted values of a function \( f \in \mathcal{E} \).

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We recall that a Fatou component $G$ of $f$ is completely invariant if $f^{-1}(G) = G$. Also, for any two points $z_1, z_2 \in G$, there is a path contained in $G$ that joins the two points. For $f$ a transcendental entire function, due to Picard theorem, every completely invariant Fatou component of $f$ is unbounded. If $f$ has $k$ completely invariant components, $G_k$, with $k \in \mathbb{N}$, then for every point $z \in \mathfrak{F}(f)$ and any neighborhood $N_z$ of $z$, we have $G_k \cap N_z \neq \emptyset$.

**Observation 1.** Let $f \in \mathcal{E}$, and $G$ a completely invariant Fatou component of $f$. Let $w \in G$ a regular value of $f$ and $z(w)$ any pre-image point, then there exist an oriented curve $\Gamma \subset G$ beginning at $w$, such that: (i) intersects any neighborhood of infinity with $\Gamma \cap \Omega V = \emptyset$ and (ii) has a pre-image $\Gamma'$ beginning at $z(w)$, so $f(\Gamma') = \Gamma$.

The construction of the curve $\Gamma$ can be obtained by successive applications of a generalization of the Gross-star theorem, due to Kaplan [11] Theorem 3. In few words, Kaplan proves in particular, that for a (star) family of non intersecting bounded curves beginning at a regular value $w$, the pre-images based at any $z(w)$ exist and can be continued indefinitely, for almost all of the curves, see details in [11] (compare with Inversen’s Theorem in [3]).

To construct $\Gamma$ we consider any neighborhood $N_1$ of $\infty$ and any pre-image $w_1$ of $w$ in $N_1$, then let $\tau_1$ any path contained in $G$ joining $w$ and $w_1$ (with that orientation). By Kaplan’s theorem, for any small enough neighborhood $B(w_1)$ of $w_1$, there is a curve $\tau'_1$ beginning at $w$ and ending in some $w'_1 \subset B(w_1)$ which has a pre-image $\tau_2$, beginning at $w'_1$.

Now, proceed inductively by choosing neighborhoods $N_i \subset N_{i-1}$, with $N_i$ converging to $\infty$ as $i$ tends to $\infty$ and choosing points $w_i \subset N_i$, with $w_i$ a pre-image of $w_{i-1}'$, also take paths $\tau_i$ joining $w_{i-1}'$ with $w_i$ and modify them to $\tau'_i$ accordingly to Kaplan’s theorem.

In conclusion $\Gamma := \bigcup_{i=1}^{\infty} \tau'_i$ is a curve satisfying (i) and (ii) and its closure $\overline{\Gamma}$ is a continuum in the sphere.

An important result related to completely invariant components of transcendental entire functions was given by Baker in [1], it is stated as follows.

**Theorem 1.1.** If $f \in \mathcal{E}$, then there is at most one completely invariant component of $\mathfrak{F}(f)$.

As it was mentioned in the abstract that there is a missing case in Baker’s proof, in this paper we follow Baker’s ideas and give some alternative arguments to solve the missing case.

It is interesting to note that a recent paper by Rempe and Sixmith [12], studies the connectivity of the pre-images of simply-connected domains of a transcendental entire function. The paper describes in detail the error in Baker’s proof and mentions Duval’s example, which is equivalent to the case of Figure 5 in this article. Also, they prove that if infinity is accessible from some Fatou component, then at least one of the pre-images of some component is disconnected. Since infinity is accessible in Baker domains, they conclude that if the function has two completely invariant Fatou components both components must be attracting or parabolic basins. In their article it is included a list of papers which use Baker’s result.

While we were making final corrections of this paper we got, by communication, the results obtained by Rempe and Sixmith in [12].
2 Proof of Theorem

The idea of the proof is by contradiction assuming that there are at least two completely invariant open components $G_1$ and $G_2$. We begin considering the cut system of Baker in Step 1, which is an open disc $D_1$ with boundary the simple curves $\hat{\gamma}_1$, $\beta_1, \gamma_2, \beta_2$, with the properties that $\beta_1 \subset G_1$ and $\beta_2 \subset G_2$ and such that $f(\hat{\gamma}_1) \subset \gamma$, $f(\hat{\gamma}_2) \subset \gamma$ are conformal injections, for $\gamma$ a segment with extremes at $G_1$ and $G_2$. In Step 2, we consider the image of the disc $f(D_1)$ which has to be bounded and state some of its properties. We proceed in Step 3 to extend the curves $f(\beta_1)$ and $f(\beta_2)$ to infinity as in the Observation 1, creating two unbounded curves $\Gamma \subset G_1$ and $\Theta \subset G_2$. Such curves can be very complicated inside $f(D_1)$, so we consider their intersection with the complement of $f(D_1)$ that we named $B$. Then, we studied their pre-images in the complement of $D_1$. By adding a certain path $\sigma$ (Case A) and $\Sigma$ (Case B) between those pre-images in the same component, we show that one of the regions $G_1$ or $G_2$ is disconnected, which is a contradiction. It is important for the proof, the cut system since it helps to have certain control on the pre-images of $\Gamma \cap B = \Gamma_0$ and $\Theta \cap B = \Theta_0$. The differences between Case A and Case B rely in the way the pre-images of $\Gamma_0$ and $\Theta_0$ intersect the cut system, as indicated in the Step 3.

Proof. Suppose that $\mathfrak{F}(f)$ has at least two mutually disjoint completely invariant domains $G_1$ and $G_2$.

Step 1. The Cut System and the cancelation procedure.

Take a value $\alpha$ in $G_1$ such that $f(z) = \alpha$ has infinitely many simple roots $z_i$ ($f'(z) = 0$ at only countably many $z$ so we have to avoid only countably many choices of $\alpha$). All $z_i$ are in $G_1$. Similarly take $\beta$ in $G_2$ such that $f(z) = \beta$ has infinitely many simple roots $z'_i$ in $G_2$. By Gross’ star theorem [10] we can continue all the regular branches $g_i$ of $f^{-1}$ such that $g_i(\alpha) = z_i$, along almost every ray to $\infty$ without meeting any singularity (even algebraic). Thus we can move $\beta \in G_2$ slightly if necessary so that all $g_i$ continue to $\beta$ analytically along the line $\gamma$, which joins $\alpha$ and $\beta$. The images $g_i(\gamma)$ are disjoint curves joining $z_i$ to $z'_i$. Denote $g_i(\gamma) = \gamma_i$. Note that $\gamma_i$ is oriented from $z_i$ to $z'_i$, see Figure 1.

![Figure 1: The images $g_i(\gamma) = \gamma_i$](image)

The branches $f^{-1}$ are univalent so $\gamma_i$ are disjoint simple arcs. Different $\gamma_i$ are disjoint since $\gamma_i$ meets $\gamma_j$ at say $w_0$ only if two different branches of $f^{-1}$ become equal with values $w_0$ which can occur only if $f^{-1}$ has branch point at $f(w_0)$ in $\gamma_i$, but this does not occur.
Take $\gamma_1$ and $\gamma_2$. Since $G_1$ is a domain we can join $z_1$ to $z_2$ by an arc $\delta_1$ in $G_1$ and similarly $z'_1$ to $z'_2$ by an arc $\delta_2$ in $G_2$. If $\delta_2$ is oriented from $z'_1$ to $z'_2$, let $p'$ be the point where, for the last time, $\gamma_1$ meets $\delta_2$ and $q'$ be the point where, for the first time, $\gamma_2$ meets $\delta_2$. If $\delta_1$ is oriented from $z_1$ to $z_2$, let $p$ be the point where, for the last time, $\gamma_1$ meets $\delta_1$ and $q$ be the point where, for the first time, $\gamma_2$ meets $\delta_1$, these might look like Figure 2.

![Figure 2](image)

Figure 2: The points $p$, $p'$, $q$, $q'$ and the curves $\delta_1$, $\delta_2$, $\gamma_1$ and $\gamma_1$

Now we denote by $\beta_1$ the part of $\delta_1$ which joins the points $p$ and $q$, by $\beta_2$ the part of $\delta_2$ which joins the points $p'$ and $q'$, by $\hat{\gamma}_1$ the part of $\gamma_1$ which joins the points $p$ and $p'$, oriented from $p$ to $p'$, and by $\hat{\gamma}_2$ the part of $\gamma_2$ which joins the points $q$ and $q'$, oriented from $q$ to $q'$. Then $\hat{\gamma}_1\beta_2\hat{\gamma}_2^{-1}\beta_1^{-1}$ is a simple closed curve with an interior $D_1$, see Figure 3.

![Figure 3](image)

Figure 3: The arcs $\beta_1$, $\beta_2$, $\hat{\gamma}_1$, $\hat{\gamma}_2$ and $D_1$

**Step 2. The map on the Disc $D_1$.**

Recall that the disc $D_1$ has boundary $\beta_1 \cup \beta_2 \cup \hat{\gamma}_1 \cup \hat{\gamma}_2$, the end points of the curve $\beta_1$ are the points $p$ and $q$, the end points of the curve $\beta_2$ are the points $p'$ and $q'$, the end points of the curve $\hat{\gamma}_1$ are the points $p$ and $p'$ and the end points of the curve $\hat{\gamma}_2$ are the points $q$ and $q'$, see Figure 3. The function $f$ maps $\hat{\gamma}_i$ injectively into the cut $\gamma_i$ for $i = 1, 2$, and we consider $f(\beta_1)$ and $f(\beta_2)$ two non intersecting curves (with possible self intersections) with ends at $f(p)$, $f(q)$ and $f(p')$, $f(q')$ respectively.
A natural question arises: Where is mapped $D_1$ under $f$?

Observe that $f(D_1)$ can be either unbounded or bounded. If $f(D_1)$ is unbounded, so there is a pole in $D_1$. Thus we ruled out this case. Necessarily $f(D_1)$ must be bounded and $f(\beta_1)$ or $f(\beta_2)$ need not be closed curves. This is the missing case in Baker’s proof.

Remember that, the orientations of $\gamma_1$ and $\gamma_2$ are given by the chosen orientation in $\gamma$ as in Step 1 above. Two main possibilities arises when we consider the orientation of $\gamma_1$ together with the order of the set of points $\{p, p'\}$ and the orientation of $\gamma_2$ together with the order of the set of points $\{q, q'\}$. Let us define $a < b$ for $a, b$ points in an oriented curve $\gamma(t)$, if $\gamma(t_1) = a$ and $\gamma(t_2) = b$ and $t_1 < t_2$. The possibilities are: (a) $\gamma_1$ and $\gamma_2$ preserve the same order, that is, if $p < p'$, then $q < q'$, see Figure 4 or (b) $\gamma_1$ and $\gamma_2$ reverse the order, that is, $p < p'$ but $q > q'$ or $q < q'$ but $p > p'$, see Figure 5.

On the other hand, the curves $f(\beta_i)$ has winding number either $+1$, $0$ or $-1$ with respect to the points $\alpha$ and $\beta$. So, several possibilities occurs for the topology of $f(D_1)$ accordingly to how the intervals $f(\hat{\gamma}_1)$ and $f(\hat{\gamma}_2)$ are placed in the cut $\gamma$. In Figure 6 there are two examples, one when $f(\hat{\gamma}_1) \cap f(\hat{\gamma}_2) \neq \emptyset$ and the other when $f(\hat{\gamma}_1) \cap f(\hat{\gamma}_2) = \emptyset$. 

![Figure 4](image1.png)

**Figure 4:** (a) $\gamma_1$ and $\gamma_2$ have the same orientations

![Figure 5](image2.png)

**Figure 5:** (b) $\gamma_1$ and $\gamma_2$ have opposite orientations
Step 3. Unbounded curves and their pre-images.

From now on, we will assume without lost of generality that \( f(\beta_2) \) surrounds \( f(\beta_1) \), it may look like Figures 5 or 6. Also we assume that \( \gamma_1, \gamma_2 \) and \( \gamma \) are compatibly oriented, as in Step 1.

For \( x, w \in \mathbb{C} \), we denote by \( \overline{pq} \) the oriented segment from \( p \) to \( w \) and by \( T_x(\tau) \) the tangent at \( x \) of some parametrization of a curve \( \tau \).

The step consists of considering certain unbounded curves on the regions \( G_1 \) and \( G_2 \) and their pre-images. We recall that \( \beta_1 \subset G_1 \) and \( \beta_2 \subset G_2 \). The curve \( f(\beta_1) \) has end points at \( f(p) \) and \( f(q) \) in \( \gamma \), and the curve \( f(\beta_2) \) has end points at \( f(p') \) and \( f(q') \) in \( \gamma \). So we define their pre-images on \( \gamma_1 \) and on \( \gamma_2 \) as follows: \( f^{-1}(f(q')) \cap \gamma_1 = p'_1 \), \( f^{-1}(f(q')) \cap \gamma_2 = q'_1 \), \( f^{-1}(f(p')) \cap \gamma_1 = p' \), \( f^{-1}(f(p')) \cap \gamma_2 = q' \), \( f^{-1}(f(p)) \cap \gamma_1 = p \), \( f^{-1}(f(p)) \cap \gamma_2 = q \), \( f^{-1}(f(q)) \cap \gamma_1 = p_1 \), \( f^{-1}(f(q)) \cap \gamma_2 = q_1 \), see Figure 7 as an example. The point \( p'_1 \) is the beginning of another pre-image of \( f(\beta_1) \), and \( p_1 \) is the end point of some pre-image of \( f(\beta_2) \) and \( q_1 \) is the beginning of another pre-image of \( f(\beta_2) \).

For brevity, we define \( I_1 \) as the interval \( \overline{p'p'_1} \) and \( I_2 \) as the interval \( \overline{q'_1q'} \). Thus \( I_1 \) and \( I_2 \) are pre-images of the interval \( I_0 = \overline{f(p')f(q')} \) in \( \gamma_1 \) and \( \gamma_2 \) respectively. We have two situations.

(i) Let us consider an unbounded oriented curve \( \Gamma \subset G_1 \) beginning at \( f(q) \), and an unbounded oriented curve \( \Theta \subset G_2 \) beginning at \( f(q') \), as in the Observation 1 in Section 1, see for instance Figure 8. More conveniently, we are interested in the piece of such curves complementary to \( f(D_1) \). Denote by \( B \) the complement of \( f(D_1) \) in the sphere and let \( \Gamma_0 = \Gamma \cap B \) and \( \Theta_0 = \Theta \cap B \).

(ii) The curves \( \Gamma \) and \( \Theta \) may oscillate and may intersect the interval \( I_0 \) in many points, so in this case \( \Gamma_0 \) and \( \Theta_0 \) are a union of curves beginning at points in \( I_0 \). By applying the Kaplan’s theorem to each of these curves, we consider their pre-images beginning at points in \( I_1 \), denoted by \( \Gamma_1 \) and \( \Theta_1 \) respectively and pre-images beginning at \( I_2 \), denoted by \( \Gamma_2 \) and
\[ f(q') \]
\[ f(p') \]
\[ f(q) \]
\[ f(p) \]
\[ f(\alpha) \]
\[ f(\beta) \]
\[ f(\gamma) \]

Figure 7: The pre-images of \( f(p), f(q), f(p') \) and \( f(q') \)

\[ \Gamma \]
\[ \Theta \]

Figure 8: The curves \( \Gamma \) and \( \Theta \)

\[ \Theta_2 \]

respectively, none of these curves intersects \( D_1 \) or more generally \( f^{-1}(D_1) \), it may look like Figure 8. If \( N_\infty \) is any neighborhood of infinity, we have \( \Gamma_i \cap N_\infty \neq \emptyset \) also \( \Theta_i \cap N_\infty \neq \emptyset \), \( i = 0, 1, 2 \), they are unbounded.

\[ \Theta_2 \]

Figure 9: The curves \( \Gamma_1, \Theta_1, \Gamma_2 \) and \( \Theta_2 \)
We have now two cases, either (A) the intersection of the set \( \Gamma_1 \) or \( \Theta_1 \) with \( I_1 \) is finite, consequently the same for \( \Gamma_2 \) or \( \Theta_2 \), or (B) the intersection of both sets is not finite.

Case A. Assume without loss of generality that \( \Gamma_i \) intersects \( I_i \) in a finite set, \( i = 1, 2 \). We consider the component of \( \Gamma_1 \) which is unbounded and denote it by \( \Gamma'_1 \), similarly we have an unbounded component \( \Gamma'_2 \). Both curves are in \( G_1 \) and recall that their closure in the sphere is a continua that contains infinity. Let us denote \( x_1 = \Gamma'_1 \cap I_1 \) and \( x_2 = \Gamma'_2 \cap I_2 \), observe that the pairs \((T_{x_1}(I_1), T_{x_1}({\gamma}')_1)\) and \((T_{x_2}(I_2), T_{x_2}({\gamma}')_2)\) are sent conformally by \( f' \) to the corresponding pair \((T_{f(x_1)}(I_0), T_{f(x_1)}({\gamma}_0))\). Under such conditions, for any path \( \sigma \) joining \( x_1 \) with \( x_2 \) which does not intersect \( \Theta_1 \) nor \( \Theta_2 \), then the curve \(-\Gamma'_1 \cup \sigma \cup \Gamma'_2\) disconnects \( \Theta_1 \) from \( \Theta_2 \), it may look like Figure 10. Therefore, if \( \sigma \in G_1 \), then \( G_2 \) is disconnected.

![Figure 10: The curves \( \Gamma_0, \Theta_0 \) and the points \( z_i \) and \( w_j, i = 0, 1, 2 \).](image)

Case B. In this case the intersection \( I_1 \cap \Gamma_1 \) is an infinite collection of points \( \{x_i^j\} \), equally \( I_1 \cap \Theta_1 \) is an infinite collection of points \( \{w_i^j\} \) for \( i = 1, 2 \) and \( j \in \mathbb{N} \). Being \( I_1 \) and \( I_2 \) compact, the sequence \( \{x_i^j\} \) has at least an accumulation point, say \( x_i \), and let \( w_i \) be an accumulation point for the sequence \( \{w_i^j\}, i = 1, 2 \). Again we have two situations, either (1) at least one of the points \( x_1 \) or \( w_1 \) is in the Fatou set, or (2) both points \( x_1 \) and \( w_1 \) are in the Julia set.

(1) Assume without loss of generality that \( x_1 \) and so \( x_2 \) are in the Fatou set. Consider the closure on the sphere \( \overline{\Gamma}_i \) of \( \Gamma_i \), \( i = 0, 1, 2 \). Let \( \sigma \) be a path between \( x_1 \) and \( x_2 \) that does not intersect \( \Theta_i \), \( i = 1, 2 \). As in Case A, the pairs \((T_{x_1}(I_1), T_{x_1}(\overline{\Gamma}_1))\) and \((T_{x_2}(I_2), T_{x_2}(\overline{\Gamma}_2))\) are sent conformally by \( f' \) to the corresponding pair \((T_{f(x_1)}(I_0), T_{f(x_1)}(\overline{\Gamma}_0))\), where \((T_{x_i}(I_i), T_{x_i}(\overline{\Gamma}_i))\) means \( \lim_{x'_i \to x_i} (T_{x_i}(I_i), T_{x_i}(\overline{\Gamma}_i)) \), \( i = 0, 1, 2 \).

Under such conditions, for any path \( \sigma \) joining \( x_1 \) with \( x_2 \) which does not intersect neither \( \Theta_1 \) nor \( \Theta_2 \), the set \( \overline{\Gamma}_1 \cup \sigma \cup \overline{\Gamma}_2 \) disconnects \( \Theta_1 \) from \( \Theta_2 \). Therefore, if \( \sigma \in G_1 \), then \( G_2 \) is disconnected.
(2) Assume that $x_i$ and $w_i$ are in the Julia set, $i = 1, 2$. Consider paths $\sigma_j$ between $x_1^j$ and $x_2^j$, $j \in \mathbb{N}$ and let $\Sigma = \bigcup_j \sigma_j$ be the closure of the union of all the paths $\{\sigma_j\}$ in the sphere. Also, the pairs $(T_{x_1}(I_1), T_{x_1}(\Gamma_1))$ and $(T_{x_2}(I_2), T_{x_2}(\Gamma_2))$ are sent conformally by $f'$ to the corresponding pair $(T_{f(x_1)}(I_0), T_{f(x_1)}(\Gamma_0))$. As explained in (ii), the intersection of the sets $\overline{\Gamma_1}, \overline{\Gamma_2}, \overline{\Theta_1}$ and $\overline{\Theta_2}$ with $D_1$ is empty.

Observe that the set $\Sigma \cup \overline{\Gamma_1} \cup \overline{\Gamma_2}$ disconnects $G_2$, since in any neighborhood of $x_1$ and $x_2$ there are points that belong to $G_2$. Now, if all $\sigma_j \in G_1$, then $\Sigma \cap \overline{G_2} = \emptyset$ and $\Sigma \cup \overline{\Gamma_1} \cup \overline{\Gamma_2}$ is disjoint of $G_2$, therefore in this case $G_2$ is disconnected.

In all these cases $G_2$ is disconnected which is a contradiction. Thus we have finished the proof of the Theorem 1.1.

**Remark.** This above proof applies also to the case of a transcendental meromorphic map with a finite number of poles, see [4], since we can choose a disc $D_1$ without poles exactly as in the proof of the Theorem and this case proceeds as above.

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