Analysis of Geometrically Nonlinear Euler-Bernoulli Beam using EFGM

D Paavani, M Aswathy*, C O Arun and I R Praveen Krishna
Department of Aerospace Engineering, Indian Institute of Space science and Technology, Trivandrum
*aswathymekkanakkil@gmail.com

Abstract. The current paper presents a meshless formulation for the analysis of geometrically nonlinear one-dimensional Euler-Bernoulli beams. Element-free Galerkin method (EFGM) is used as the meshless numerical tool and Newton-Raphson method as the iterative scheme. Numerical examples of beams with four different boundary conditions are solved, and the results are compared with that of the finite element method (FEM). EFGM results are found in good agreement with that of FEM. Also, it is found that EFGM eliminates the effect of membrane locking in thin beams.

1. Introduction
Finite element method (FEM) is the most widely used conventional numerical technique to solve complex problems in structural analysis. Generation of mesh in FEM is a complex task, and mesh dependency creates issues while dealing with problems with large deformation, problems with moving discontinuities like crack propagation, stress concentration problems etc. In this respect, meshless method can be regarded as the most viable alternative to FEM. Unlike FEM, which requires predefined elemental connectivity, meshless methods need only nodes arbitrarily distributed in the problem domain as well as in the boundary of the domain. Among different types of meshless techniques [1–7], Element-free Galerkin method (EFGM) becomes very popular due to its simplicity and comparability of the formulations with FEM. It is first proposed by Belytschko et al. [2, 8]. Moving least-squares (MLS) interpolants [9] are applied for the construction of shape functions.

Valencia et al. [10] carried out a study on one-dimensional axially loaded bar, and Valencia et al. [11] analysed the beam bending problems to understand the influence of the characteristic parameters on the EFGM solution. Though EFGM is very efficient, there exists issues in applying boundary conditions and point loads. Due to the loss of Kronecker-delta property, exact imposition of the essential boundary conditions is not possible. Several methods like Lagrange multipliers approach [2,12], penalty factor [13], coupled FEM [14], transformation method [15–17] etc. are used to overcome this drawback.

In 2D and 3D EFGM formulations, the application of point loads is somewhat complicated because of the character of MLS interpolants. A method is suggested by Pang [18] in which concentrated forces are transformed into distributed forces by Dirac function. After such a transformation, the concentrated load can be made analogous to distributed load and thus, increasing the scope of EFGM. Thin-walled beams may undergo large deflection with small strains and moderate rotations, and the analysis of such structural component requires the incorporation of von Kármán strains [19]. Analysis of geometrically nonlinear 1D Euler-Bernoulli beam, incorporating von Kármán strains, using EFGM is explored in the current study.

2. von Kármán strains
Thin-walled beams are always prone to large deformations with small strains and moderate rotations leading to non-linear kinematic relationships. Figure 1 depicts the displacement kinematics of 1D Euler-Bernoulli beam in
X-Z plane. Let it be subjected to an arbitrary loading \( q(x) \) in the positive \( Z \) direction and a moment \( M \) is applied as shown.

Let \( U, V \) and \( W \) be the displacements in \( X, Y \) and \( Z \) directions respectively, and which are functions of \( x \) and \( z \), given by

\[
\begin{align*}
U(x, z) &= u(x) + z\theta(x) \\
V(x, z) &= 0 \\
W(x, z) &= w(x)
\end{align*}
\]

where, \( u \) and \( w \) are the center-line displacements along \( x \) and \( z \) directions respectively, which are functions of \( x \) alone. From the Green-Lagrange strain definitions, the nonlinear strain (\( \varepsilon \)) - displacement (\( U, V \) and \( W \)) relation can be written as

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left( \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \right)
\]

Note that \( u_1 = U(x, z), u_2 = V(x, z), u_3 = W(x, z) \), \( x_1 = x \), \( x_2 = y \) and \( x_3 = z \). Substituting the displacements, the strain-displacement relations are obtained as

\[
\begin{align*}
\begin{bmatrix}
\varepsilon_{xx}^0 \\
\varepsilon_{xx}^1 \\
\gamma_{xz}
\end{bmatrix}
&= \begin{bmatrix}
\frac{d u}{d x} \\
\frac{d w}{d x} + z \frac{d^2 w}{d x^2} \\
\frac{d u}{d z} + \frac{d w}{d z} - \frac{d^2 w}{d x^2} \frac{d x}{d z}
\end{bmatrix} + \begin{bmatrix}
\frac{1}{2} \left( \frac{d u}{d x} \right)^2 + \frac{1}{2} \left( \frac{d w}{d x} \right)^2 \\
\frac{1}{2} \left( \frac{d u}{d z} \right)^2 + \frac{1}{2} \left( \frac{d w}{d z} \right)^2 \\
\frac{1}{2} \left( \frac{d w}{d x} \right)^2
\end{bmatrix}
\end{align*}
\]

where, normal strain in the \( x \) direction is given by, \( \varepsilon_{xx} = \varepsilon_{xx}^0 + z \varepsilon_{xx}^1 \). \( \varepsilon_{xx}^0 \) is the axial strain and \( z \varepsilon_{xx}^1 \) at any section is the bending strain. \( \varepsilon_L \) and \( \varepsilon_{NL} \) denote the linear and nonlinear parts of strain. In \( \varepsilon_{NL} \), \( \frac{1}{2} \left( \frac{d w}{d x} \right)^2 \) is the most dominant term and so, all the other terms can be neglected. Also from the thin beam assumption, the shear strain can be neglected i.e., \( \gamma_{xz} = 0 \), and it implies \( \theta = -\frac{d w}{d z} \). Thus, the only strain experienced by the structure is the normal strain which can be written as

\[
\begin{align*}
\varepsilon_{xx} &= \frac{d u}{d x} - z \frac{d^2 w}{d x^2} + \frac{1}{2} \left( \frac{d w}{d x} \right)^2 \\
\varepsilon_{xx}^0 &= \frac{d u}{d x} + \frac{1}{2} \left( \frac{d w}{d x} \right)^2, \quad \varepsilon_{xx}^1 = -z \frac{d^2 w}{d x^2}
\end{align*}
\]

These strains are known as von Kármán strains.
3. EFGM formulation

The Galerkin weak form of principle of virtual displacement is used here to formulate the discretized set of equations.

3.1. Principle of virtual displacement

The principle of virtual displacement states that if a body is in equilibrium, the total virtual work done by both the internal and external forces in moving the body through the virtual displacement is zero [19].

\[ \delta W^e \equiv \delta W_i^e - \delta W_E^e = 0 \]

where \( \delta W_i^e \) is the virtual strain energy stored in the system and \( \delta W_E^e \) is the work done by externally applied forces.

\[
\begin{align*}
\delta W_i^e &= \int_V \delta \epsilon_{ij} \sigma_{ij} dV \\
\delta W_E^e &= \int_I q \delta w dx + \int_I f \delta u dx + \sum_{i=1}^{3n} Q_i^e \delta \Delta_i
\end{align*}
\]

Here, \( V \) is the volume of the body, \( I \) is the length of the beam, \( q \) is the distributed transverse load per unit length, \( f \) is the distributed axial load per unit length and \( Q_i \) are the nodal forces corresponding to the nodal displacements \( \Delta_i \). For a beam discretized to \( n \) nodes, \( \Delta = [u_1 \ u_2 \ ... \ u_n \ w_1 \ \theta_1 \ w_2 \ \theta_2 \ ... \ w_n \ \theta_n]^T \) where, \( u_i \), \( w_i \) and \( \theta_i \) are the respective axial displacement, transverse displacement and slope at the node \( i \). \( \delta \Delta_i \) are the generalized virtual displacements.

The virtual work statements for a Euler-Bernoulli beam incorporating von Kármán strains can be written in two sets of equations as,

\[
\int_I EA \frac{d^2 u}{dx^2} \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] dx - \int_I f u(x) dx - \sum_{i=1}^{n} Q_i \delta \Delta_i = 0
\]

\[
\int_I \left\{ EA \frac{d^2 w}{dx^2} \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] + EI \frac{d^4 w}{dx^4} \right\} dx - \int_I f w(x) dx - \sum_{i=n+1}^{3n} Q_i \delta \Delta_i = 0
\]

where, \( A \) is the cross-sectional area of the beam, \( I \) is the second moment about y-axis and \( E \) is the modulus of elasticity of the material.

3.2. MLS shape functions for axial and transverse displacements

MLS method is employed in EFGM to construct shape functions for the axial as well as transverse displacements.

3.2.1. Axial Displacement

The MLS approximate of \( u(x) \) is \( u^h(x) \) defined by

\[
u^h(x) = p^T(x) a(x) = \sum_{i=1}^{n} p_i(x) a_i(x)
\]

where, \( p^T(x) = (p_0(x), p_1(x), ..., p_m(x)) = (1, x, x^2, ..., x^m) \) is the basis vector of order \( m \), \( a^T(x) = \{a_0(x), a_1(x), ..., a_m(x)\} \) is the coefficient vector, which is also a function of \( x \). From the basic definition of moving least squares method, the error norm is defined as

\[
J(x) = \sum_{i=1}^{n} w_{li} \left[ p^T(x_i) a(x) - \hat{u}_i \right]^2
\]

where, \( x_i \) is the coordinate of the node, \( \hat{u}_i \) is nodal displacement parameters given by \( u(x_i) \), \( w_{li} \) is the weight at point \( x \) corresponding to node \( i \). In matrix notation, equation (10) can be written as

\[
J(x) = (P^T a(x) - \hat{u})^T W (P^T a(x) - \hat{u})
\]

where, \( \hat{u} = [\hat{u}_1, \hat{u}_2, ..., \hat{u}_n] \), \( W = \text{diag}(w_{l1}(x), w_{l2}(x), ..., w_{ln}(x)) \) and \( P = [p^T(x_1), p^T(x_2), ..., p^T(x_n)] \).

The error norm can be minimized by setting the partial derivatives \( \nabla J = 0 \), where \( \nabla = \left[ \frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, ... \frac{\partial}{\partial a_n} \right]^T \). This results in equation of the form

\[
A(x) a(x) = C(x) d
\]
where, $A(x) = P^TWP$, $C(x) = [P^T W]$ and, $d = [\hat{u}]^T$.

In terms of shape functions, the approximate $u^h(x)$ is written as

$$u^h(x) = \sum_{i=1}^{n} \psi_{i\ell}(x) \hat{u}_i$$  \hspace{1cm} (13)

where,

$$\psi_{i\ell}(x) = p(x)A^{-1}(x)P^TW$$  \hspace{1cm} (14)

3.2.2. Transverse Displacement. This formulation is adopted from [16]. The MLS approximate of $w(x)$ is $w^h(x)$ defined by

$$w^h(x) = p^T(x)a(x) + P^T_w a(x) = \sum_{i=1}^{n} p_i(x) a_i(x) + \frac{\partial p_i(x)}{\partial x} a_i(x)$$  \hspace{1cm} (15)

where $P^T_w = \left( \frac{\partial p_i(x)}{\partial x}, \frac{\partial p_i(x)}{\partial x}, \ldots, \frac{\partial p_i(x)}{\partial x} \right)$ is the derivate of basis with respect to $x$. In matrix notation, the error norm can be written as

$$J(x) = [P a(x) - \hat{w}]^T W [P a(x) - \hat{w}] + [P_w a(x) - \hat{\theta}]^T W [P_w a(x) - \hat{\theta}]$$ \hspace{1cm} (16)

$$J(x) = \left\{ \begin{array}{c} P \end{array} \right\} a(x) - \left\{ \begin{array}{c} \hat{w} \end{array} \right\}^T W \left\{ \begin{array}{c} P \end{array} \right\} a(x) - \left\{ \begin{array}{c} \hat{\theta} \end{array} \right\}$$ \hspace{1cm} (17)

After minimization of error norm, the approximation for $w^h(x)$ is written as

$$w^h(x) = \sum_{i=1}^{n} \phi_{i\ell}(x) \hat{w}_i + \phi_{i\ell}(x) \hat{\theta}_i$$ \hspace{1cm} (18)

where,

$$\phi_{i\ell}(x) = p(x)A^{-1}(x)P^TW = p(x)A^{-1}(x)C_w(x)$$ \hspace{1cm} (19)

$$\phi_{i\ell}(x) = p(x)A^{-1}(x)P^T_w W = p(x)A^{-1}(x)C_w(x)$$ \hspace{1cm} (20)

Substituting the MLS approximations in equation (8), we obtain

$$\sum_{j=1}^{n} \left[ K_{ij}^{11} \Delta_j \right] + \sum_{j=m+1}^{n} \left[ K_{ij}^{12} \Delta_j \right] - F_i^1 = 0 \hspace{1cm} (i = 1,2\ldots n)$$ \hspace{1cm} (21a)

$$\sum_{j=1}^{n} \left[ K_{ij}^{11} \Delta_j \right] + \sum_{j=m+1}^{n} \left[ K_{ij}^{12} \Delta_j \right] - F_i^2 = 0 \hspace{1cm} (i = n+1, n+2\ldots 3n)$$ \hspace{1cm} (21b)

where, $\Delta$ is the nodal parameter vector given by $\Delta = [\hat{u}_1 \hat{u}_2 \ldots \hat{u}_n \hat{\theta}_1 \hat{\theta}_2 \ldots \hat{\theta}_n]^T$ and

$$K_{ij}^{11} = EA \int_{x}^{} \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} dx,$$

$$K_{ij}^{12} = 0.5EA \int_{x}^{} \left( \frac{d\psi_i}{dx} \right) \frac{d\phi_j}{dx} dx,$$

$$K_{ij}^{21} = EA \int_{x}^{} \left( \frac{d\phi_i}{dx} \right) \frac{d\psi_j}{dx} dx,$$

$$K_{ij}^{22} = EI \int_{x}^{} \frac{d^2\phi_i}{dx^2} \frac{d^2\phi_j}{dx^2} dx + 0.5EA \int_{x}^{} \left[ \left( \frac{d\psi_i}{dx} \right)^2 \right] \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx$$

$$F_i^1 = \int_{x}^{} f \psi_i dx + Q_i$$

$$F_i^2 = \int_{x}^{} q \phi_i dx + Q_i$$

\hspace{1cm} (22)
The stiffness and force are arranged in the matrix form as,

\[
K = \begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}, \quad F = \begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
\]  

(23)

In the nonlinear case, the axial and transverse displacements are coupled and this coupling appears in the \(K_{12}\) and \(K_{21}\) terms.

The enforcement of essential boundary conditions is achieved here by the transformation method [17]. Thus, the force balance can be expressed as

\[
\hat{K}(\hat{\Delta})\hat{\Delta} = \hat{F}
\]

(24)

in which, \(\hat{K} = \Lambda^{-T}K\Lambda^{-1}\) and \(\hat{F} = \Lambda^{-T}F\), where \(\Lambda\) is transformation matrix.

In equation (24), stiffness matrix is a function of displacement, containing terms like \(\frac{dw}{dx}\). This is to be solved iteratively by substituting the displacement and its derivatives from the previous iteration. In this study, Newton-Raphson method is used owing to its fast convergence [19].

4. Numerical examples

Beams with different boundary conditions are considered which are already solved using FEM by Reddy [19]. A beam of length 0.1m having area of cross-section 1cm\(^2\) with Young’s modulus \(E = 30 \times 10^8 N/m^2\), subjected to a uniformly distributed load \(q_0 N/m\) is considered. The beam is discretized using 20 uniformly distributed nodes. EFGM scaling parameter is taken as \(d_{max} = 3\), a quadratic polynomial basis, an exponential weight function and 4 point Gauss-quadrature scheme for numerical integration are utilized. Beams with four different boundary conditions are studied. For each case, the load \(q_0\) is varied from 0 to 2kN/m in 10 load steps and variation of deflection with load is plotted. For all the cases, the results obtained are validated with those obtained from FEM [19].

4.1. Pinned-Pinned beam

The geometric boundary conditions are

\[
\begin{align*}
u(0) = w(0) = u(L) = 0 \\
\frac{dw}{dx}\bigg|_{x=L} = 0
\end{align*}
\]

(25)

Figure 2. Load Vs Mid-point deflection of a pinned-pinned beam.
The nonlinear effect can be clearly witnessed in the Load-Deflection plot shown in figure 2. It is evidently a strain stiffening problem. Figure 3 shows the plot of deflection, axial deformation and slope along the length of the beam for a pinned-pinned beam. Due to symmetry, only half of the span is considered. The number of iterations required for each load step to converge is less than 5, since Newton-Raphson has fast convergence. The FEM results are almost coincident with the EFGM results.

4.2. Clamped-Clamped beam

The geometric boundary conditions are

\[ u(0) = w(0) = \frac{dw}{dx}\bigg|_{x=0} = u\left(\frac{L}{2}\right) = \frac{dw}{dx}\bigg|_{x=\frac{L}{2}} = 0 \]  

(26)

![Diagram of Load Vs Mid-point deflection of a clamped-clamped beam.](image)
This is also a strain stiffening problem. The number of iterations required for each load step to converge is less than 10. Evidently, the deflection in this case is less than that of pinned-pinned case. The FEM results are quite close to EFGM results. Though no axial force is applied, due to the constrained boundaries, axial displacement and strains exist.

4.3. Simply supported beam
The geometric boundary conditions are

\[ w(0) = \mu \left( \frac{L}{2} \right) = \frac{dw}{dx} \bigg|_{x = \frac{L}{2}} = 0 \] (27)

Figure 6 shows the load-deflection behaviour and figure 7 shows axial deformation, transverse deflection and rotation along the length of a beam with roller constraints on both the ends using EFGM, FEM without reduced integration, and FEM with reduced integration.

![Figure 5](image1)

**Figure 5.** Deflection plot for a beam with clamped ends.

![Figure 6](image2)

**Figure 6.** Load Vs Mid-point deflection of a simply supported beam.
A simply supported beam does not have any end conditions on $u$. Only symmetric boundary is applied at the center. Therefore, the beam does not experience any axial strain. \( \Rightarrow e_{xx}^{\text{Ax}} = \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 = 0 \). In the case of FEM, for example, if $u$ is interpolated linearly and $w$ is interpolated using hermite functions, the order of $\frac{du}{dx}$ and $\left( \frac{dw}{dx} \right)^2$ is quite different. This introduces spurious constraints in the system. As a result, the element stiffness matrix becomes more stiff. This is known as membrane locking. In order to satisfy this constraint, a reduced integration is used. However, in the case of EFGM, the order of the polynomial is certainly higher. From figures 6 and 7 it is clear that the results with reduced integration in FEM matches EFGM results and which should match with linear analytical solution. So, it can be concluded that the issue of membrane locking does not exist in EFGM.

4.4. Cantilever beam

The geometric boundary conditions are

\[
u(0) = w(0) = \left. \frac{dw}{dx} \right|_{x=0} = 0
\]

(28)

Figure 7. Symmetric Deflection plot for a simply supported beam.

Figure 8. Load Vs Tip deflection of a cantilever beam.
In the case of cantilever beam also, the axial strain is zero, since only one end is constrained. Figures 8 and 9 again show that EFGM formulations are free from membrane locking. Figures 6 and 8 depict that von Kármán strains do not induce nonlinearity if the center line axial strain is zero.

5. Conclusions

In the present study, EFGM formulation for a geometrically nonlinear 1D Euler-Bernoulli beam is explained. Nonlinearity is introduced in the form of von Kármán strains. Numerical examples of Euler-Bernoulli beams with different boundary conditions are solved. The results are compared with those obtained from the converged FEM results. It is also found that the EFGM formulations are free of membrane locking. The reason for the same can be attributed to the fact that field approximation in EFGM is governed by the continuity of the weight function.

6. References

[1] Gingold R A and Monaghan J J 1977 Monthly Notices of the Royal Astronomical Society 181 375–389
[2] Belytschko T, Lu Y Y and Gu L 1994 International Journal for Numerical Methods in Engineering 37 229–256
[3] Atluri S N and Zhu T 1998 Computational Mechanics 22 117–127
[4] Liu W K, Jun S, Li S, Adee J and Belytschko T 1995 International Journal for Numerical Methods in Engineering 38 1655–1679
[5] Ohtake E, Igelsohn S, Zienkiewicz O and Taylor R 1996 International Journal for Numerical Methods in Engineering 39 3839–3866
[6] Liu G R and Karamanlidis D 2003 Applied Mechanics Reviews 56 B17
[7] Liu G R and Gu Y T 2005 An introduction to meshfree methods and their programming (Springer Science & Business Media)
[8] Dolbow J and Belytschko T 1998 Archives of Computational Methods in Engineering 5 207–241
[9] Lancaster P and Salkauskas K 1981 Mathematics of Computation 37 141–158
[10] Valencia O, Gómez-Escalonilla F and Díez J L 2008 Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science 222 1621–1633
[11] Valencia O, Gómez-Escalonilla F and López-Díez J 2009 Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science 223 1579–1590
[12] Lu Y Y, Belytschko T and Gu L 1994 Computer Methods in Applied Mechanics and Engineering 113 397–414
[13] Liu G R and Yang K Y 1998 Proc. 3rd HPC Asia vol 98 pp 715–721
[14] Kronauer Y and Belytschko T 1996 Computer Methods in Applied Mechanics and Engineering 131 133–145
[15] Rao B N and Rahman S 2000 Computational Mechanics 26 398–408
[16] Dodagoudar G R, Rao B N and Sunitha N V 2015 International Journal of Geotechnical Engineering 9 298–306
[17] Arun C O, Rao B N and Sivakumar S M 2009 J. Struct. Eng 182–194
[18] Pang Z H 2000 Communications in Numerical Methods in Engineering 16 335–341
[19] Reddy J N 2014 An Introduction to Nonlinear Finite Element Analysis: with applications to heat transfer, fluid mechanics, and solid mechanics (OUP Oxford)