ON COMPLETENESS AND DYNAMICS OF COMPACT BRINKMANN SPACETIMES

LILIA MEHIDI AND ABDELGHANI ZEGHIB

ABSTRACT. Brinkmann Lorentz manifolds are those admitting an isotropic parallel vector field. We prove geodesic completeness of the compact and also compactly homogeneous Brinkmann spaces. We also prove, partially, that their parallel vector field generates an equicontinuous flow.

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1. INTRODUCTION

There is a lack of completeness and a lack of compactness of Lorentz structures in comparison to the Riemannian ones! Compact Riemannian manifolds are complete and have a compact isometry group, but compact Lorentz manifolds are (generically) incomplete and some have a non-compact isometry group! This is mainly due to existence of degenerate objects for Lorentz metrics. So Brinkmann Lorentz manifolds appear as a consecration of this phenomenon, since they are defined by having an isotropic parallel vector field. We will however see here, strikingly, that Brinkmann manifolds have a Riemannian like behaviour: they are complete and their isometry group tends to be compact!

Our work generalizes T. Leistner and D. Schliebner’s completeness result of a spacial class of Brinkmann spaces, those with abelian holonomy [27], also called pp-waves. From the dynamical point of view, we are studying here the opposite situation to the one recently considered by C. Frances who got a quite precise description of Lorentz manifolds having an isometry group of exponential growth [17].

1.1. Completeness. Generic Lorentz metrics on compact manifolds are thought of to be incomplete! The historical example is the Clifton-Pohl torus \( \mathbb{R}^2 - \{(0,0)\} \) endowed with the metric \( \frac{dx dy}{x^2 + y^2} \) (see [13, 35, 36, 39, 29] for various results on completeness of Lorentz surfaces).

In the sequel, we will mention known completeness results (essentially all, to our best knowledge). One observes here that all these results assume some symmetry or a high local symmetry hypothesis.

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Homogeneous case. The “oldest” one is perhaps Marsden’s result [28] stating completeness of compact homogeneous pseudo-Riemannian manifolds. Let us observe here that, contrary to the Riemannian case, indefinite homogeneous pseudo-Riemannian (non-compact) manifolds fail to be complete in general, unless in some classes, for example symmetric spaces and left invariant metrics on 2-step nilpotent groups, as this was proved by Guediri [23].

Example 1.1. \( U = \{(x, y) \in \mathbb{R}^2, y > 0\}, g = 2dx dy. \) The subgroup of \( O(1, 1) \times \mathbb{R}^2 \) consisting of Lorentzian affine transformations of the form \((x, y) \mapsto (ax + b, \alpha^{-1}y), \alpha > 0, b \in \mathbb{R}, \) acts transitively on \( U, \) which is then homogeneous. It is however incomplete since it is a proper subset of \((\mathbb{R}^2, 2dx dy).\) Observe that this has a Brinkmann structure, with parallel null vector field \( V = \partial_y. \) Its Lorentzian metric is homogeneous, but it is not homogeneous for the Brinkmann structure, i.e. the group of isometries does not preserve the parallel vector field.

Constant curvature case. The most striking result is Carrière’s Theorem [12] on completeness of compact flat Lorentz manifolds, and its generalization to the constant (sectional) curvature case by Klingler [26].

Among locally homogeneous spaces, those of constant curvature can be interpreted as having a maximal local isometry group. There are however examples of non-complete locally compact homogeneous spaces, e.g. a quotient \( SL(2, \mathbb{R})/\Gamma, \) where \( \Gamma \) is a co-compact lattice, and \( SL(2, \mathbb{R}) \) is endowed with a generic left invariant Lorentz metric [7].

Timelike Killing field. Other results in the compact case assume existence of a few symmetry, e.g. a Killing field, say \( V, \) but with a given constant causal character. For example, the Euler vector field whose flow is the homothetic action on \( \mathbb{R}^2, \) determines a Killing field on the Clifton-Pohl torus, but with a varying causal character. Romero and Sanchez [37] proved completeness when \( V \) is (everywhere) timelike i.e. of negative square length, actually merely assuming \( V \) being a conformal Killing field.

Lightlike Killing field? Taking a product of a non-complete Lorentz manifold with the circle (with a Riemannian metric) shows that existence of spacelike Killing fields does not imply completeness. But, what about (everywhere) isotropic Killing fields? Again, this does not lead to completeness as shown by the following 3-dimensional variant of the Clifton-Pohl torus. Consider on \((\mathbb{R}^2 - \{(0, 0)\}) \times \mathbb{R}, \) the metric \( \frac{2du + dx + dy + dz}{z^2}, \) and take the quotient by the group generated by \((x, y, z) \mapsto (2x, 2y, z) \) and \((x, y, z) \mapsto (x, y, z + 1). \) This is a 3-dimensional torus, on which \( \frac{2du + dx + dy + dz}{z^2} \) is a null Killing field. There is also a (non-compact) 3-dimensional Lorentz homogeneous space, called in [14] Lorentz-SOL geometry, which is incomplete although having a null Killing field.

Parallel fields. Now, what about existence of a parallel field? Recall that a vector field \( V \) is parallel if its covariant derivative vanishes, i.e. \( \nabla_X V = 0, \) for any vector field \( X, \) where \( \nabla \) is the Levi-Civita connection of the Lorentz metric \( g. \) Equivalently, \( V \) is invariant under parallel transport: for a smooth curve \( c : [0, 1] \to M, \) \( \tau_c(V(c(0))) = V(c(1)), \) where \( \tau_c \) denotes the parallel transport \( T_{c(0)}M \to T_{c(1)}M. \) Again, the question makes sense only in the lightlike case. This is exactly what our main result answers, since by definition, Brinkmann spaces are those admitting a null parallel field.

Theorem 1.2. A compactly homogeneous Brinkmann spacetime is complete. (Compactly homogeneous means there is a compact subset whose iterates by the isometry group cover all the space).

Here, homogeneity is understood in the sense of the Brinkmann structure, i.e. the group of isometries of the Lorentz manifold which preserve the parallel null vector field \( V \) acts transitively. If we assume homogeneity for the metric only, Example 1.1 shows that we can have incomplete examples.
Some comments are in order:

As principal corollaries, compact as well as homogeneous Brinkmann spaces are complete.

**pp-waves.** Let \((M, g, V)\) be a Brinkmann Lorentz manifold. Since \(V\) is parallel, its orthogonal distribution \(V^\perp\) is invariant by the Levi-Civita connection and hence is integrable and has totally geodesic (and degenerate) leaves. We will always note the so defined foliation \(\mathcal{F}\). Each leaf of \(\mathcal{F}\) inherits an induced connection. Then, pp-waves are defined by the fact that all the \(\mathcal{F}\)-leaves are flat. These spacetimes are very important to General Relativity. As previously mentioned, completeness of compact pp-waves was proved by Leistner and Schliebner in [27]. It is this result that motivates our present work.

Observe that Brinkmann class is more flexible than the pp-waves one. For instance, the product of a Brinkmann space with a Riemannian space is still Brinkmann. One needs taking the product with a flat Riemannian manifold in the pp-wave case. We will also see in par. 2.2 a construction of a Brinkmann structure on \(M^*\), the bundle of orthonormal frames of \(V^\perp\). The so-constructed structure is never a pp-wave (say for \(\dim M > 4\)).

**Ehlers-Kundt conjecture.** It is somewhat unrelated to our subject since it concerns non-compact spaces, still it is a completeness question on Brinkmann spaces, which aims to characterize complete Ricci-flat (global) pp-waves. A pp-wave metric can be written locally in the following special form on \(\mathbb{R}^2 \times \mathbb{R}^n\): 
\[
g = 2du(dv + H(u, z)du) + \text{euc}_{\mathbb{R}^n},
\]
where \((u, v) \in \mathbb{R}^2, z \in \mathbb{R}^n\) and \(\text{euc}_{\mathbb{R}^n}\) denotes the Euclidean metric \(\Sigma(dz^2)^2\). The parallel field is \(\nabla\) (this derives from the fact that \(H\) does not depend on \(v\)). The Ehlers-Kundt conjecture says that in the global case, with \(n = 2\), i.e. \(H\) is defined globally on \(\mathbb{R} \times \mathbb{R}^2\), if \(g\) is Ricci-flat, then \(g\) is complete exactly when \(H\) is quadratic on \(z\), in which case \(g\) is said to be a plane wave (see [15] for the most recent results on the subject). Observe that Ricci-flatness is equivalent to harmonicity with respect to \(z \in \mathbb{R}^2\) of \(H\). This fact together with the reduction of the geodesic equation to a mechanical system, that will be discussed below, show the post-Newtonian character of pp-waves, which applies also to Brinkmann spaces. The latter spaces also admit special but more complicated local charts for their metrics, see par. 2.1.2.

**Weakly Brinkmann.** A weakening of the Brinkmann condition will be to assume existence of a parallel direction field \(V\), i.e., existence of a 1-dimensional sub-bundle \(E \subset TM\) invariant under parallel transport. In a same manner, one can define a weak version of pp-waves. An example given in [27] shows that they can be non-complete (although compact).

It is also asked in this paper if the leaves of the lightlike geodesic foliation \(\mathcal{F}\) (tangent to \(V^\perp\)) are complete. Actually, in our current work on the Brinkmann case, we give a “synthetic” proof of completeness of \(\mathcal{F}\) (Section 3), which seems to somehow generalize to the weak Brinkmann case, that is, the \(\mathcal{F}\)-leaves would be complete as soon as the (1-dimensional) \(V\)-leaves are. More abstractly, let us say that the Brinkmann class which generalizes naturally pp-waves, is in its turn part of the wider class of Kundt spacetimes, defined as those having a codimension one lightlike geodesic foliation \(\mathcal{F}\) (see for instance [6]). It is worthwhile to bring out situations, in the Kundt situation, where one can expect such \(\mathcal{F}\) to be complete.

**An associated mechanical system.** Other works that should be quoted here are [8] and [9]. They relate completeness of pp-waves to that of a second order differential equation on a Riemannian manifold \((M_0, g)\), defined by a general class of forces:
\[
\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = F_{(\gamma(t), t)} \dot{\gamma}(t) + X_{(\gamma(t), t)},
\]
where \(\nabla\) denotes the Levi-Civita connection of \(g\), \(F\) a \((1, 1)\)-smooth tensor field and \(X\) a smooth vector field on \(M_0 \times \mathbb{R}\). Observe that this is a linear perturbation of the geodesic equation of \((M_0, g)\). Sufficient conditions on \(F\) and \(X\) for the trajectories to be complete
are given in [9], assuming \((M_0, g)\) complete, and also in [38], both in the smooth case and in low regularity cases (assuming the metric functions of distributional nature).

In the general Brinkmann case, one meets, in adapted local coordinates, equations of the form:

\[
\nabla^t_\gamma(t) \dot{\gamma}(t) = F_{\gamma(t),t}(\gamma) + X_{\gamma(t),t}.
\]

Now, \(\nabla^t\) is the Levi-Civita connection or a (locally defined) Riemannian metric \(g_t\), varying with time. So our situation is noticeably different from the one investigated in [9], first because there is no globally defined Riemannian metric on \(M\), secondly because of the time dependency of the connection involved in Equation (2).

In addition to the previous comments, let us mention here that in dimension 3, all Brinkmann manifolds are pp-waves. Their completeness in the "roughly homogeneous" case, i.e. when the isometry group acts transitively, without preserving the parallel vector field, is investigated in the literature, for instance in [19]. There are maximal non-complete homogeneous examples, that is they admit no embedding in a larger space. Let us mention that the so-called Walker spaces, in some references, are generalizations of Brinkmann spaces in a general pseudo-Riemannian context.

1.2. Reduction to an (almost) locally homogeneous situation. Beyond completeness, we want to understand how Brinkmann spaces are made up? In Section 6, inspired by [17], we will prove the following result which will be next used to study the dynamics of \(V\).

**Theorem 1.3.** Let \((M, g, V)\) be a compact Brinkmann spacetime. Then \(M\) admits a core \(N\), a closed subset of \(M\), invariant under a finite index subgroup of \(\text{Iso}(M, g, V)\) such that

1. Either \(N\) is a locally homogeneous Lorentzian Brinkmann closed submanifold of \(M\); more precisely, a lift \(\tilde{N}\) in the universal cover \(\tilde{M}\) is the orbit of a finite index subgroup of \(G = \text{Iso}(\tilde{U}, \tilde{g}, \tilde{V})\), where \(\tilde{U}\) is an open subset of \(\tilde{M}\) containing \(\tilde{N}\), and invariant under all local isometries of \((\tilde{M}, \tilde{g}, \tilde{V})\).

2. Or \(N\) is a locally co-homogeneity one Lorentzian Brinkmann closed submanifold of \(M\), with boundary, which is a trivial fibration over an interval. The fibers are lightlike totally geodesic and locally homogeneous. More precisely, in the universal cover \(\tilde{M}\), the lift of the fibers of \(\tilde{N}\) are the orbits of a finite index subgroup of \(G = \text{Iso}^0(\tilde{U}, \tilde{g}, \tilde{V})\).

In each case, the core \(N\) or its (codimension 1) fibers have the form \(\Gamma \setminus G/I\), where \(I\) is a closed subgroup in \(G\) and \(\Gamma\) is a discrete sub-group of \(G\) acting properly and freely on \(G/I\).

Actually, in each one of the previous cases, there exists an open subset of \(M\) which is trivially fibred by submanifolds like \(N\).

1.3. Dynamics. Besides completeness issues, the other interesting topic on Lorentz manifolds is to understand when their isometry group is non-compact (assuming the manifold compact), see for instance [46, 21, 1, 2, 42, 44, 43, 33, 16]. The more recent article [17] studies actions of discrete groups with exponential growth, a situation somehow opposite to ours here. Indeed, we want to know here whether \(V\) is "essential" or not? In other words, we ask the question if the flow of \(V\) is equicontinuous (or not), or equivalently, if it preserves (or not) an auxiliary Riemannian metric. If it does not preserve such a Riemannian metric, then it is really of Lorentzian nature. Actually, one may ask such a question for a parallel vector field, say \(W\), not necessarily null as in the Brinkmann case.

- In the case where \(W\) is timelike, it is obviously equicontinuous: just apply a “wick rotation” to the Lorentz metric \(g\) to make it Riemannian; precisely, keep \(g\) on \(W^\perp\) and multiply it by \(-1\) on \(\mathbb{R}W\).
- There are interesting non-equicontinuous examples in the case where \(W\) is spacelike.
For instance, a hyperbolic matrix $A \in SL(2, \mathbb{Z})$ preserves a flat Lorentz metric on $\mathbb{T}^2$. Its suspension is a parallel spacelike flow on a flat Lorentz 3-manifold $\mathbb{T}^3$. It is Anosov and so far away from being equicontinuous.

- For lightlike parallel vector fields, we will prove, partially, that their flows are equicontinuous. For this, we will refer to the second situation in Theorem 1.3 as the degenerate case. Also, in order to state our next result, notice that the foliation $\mathcal{F}$ determined by $V_\perp$ is defined by a non-singular closed 1-form. Hence, it is either minimal, i.e. all its leaves are dense, or all its leaves are closed.

**Theorem 1.4.** Let $(M, g, V)$ be a compact Brinkmann spacetime. Then, the flow of $V$ is equicontinuous, that is its closure in $\text{Iso}(M, g)$ is compact, in each of the two following cases:

- the foliation determined by $V_\perp$ is not minimal;
- the degenerate case of the reduction Theorem 1.3, i.e. when the core $N$ is not locally homogeneous, but rather has local co-homogeneity one.

1.3.1. Dynamics in the locally homogeneous case. We think that this equicontinuity result extends to the general case of compact Brinkmann spacetimes. Our results reduce the proof to the locally homogeneous case, that is $M$ has the form $\Gamma \setminus G/I$ where:

- $I$ is a closed subgroup in $G$ (in fact contained in the nil-radical of $G$),
- The $G$-action on $G/I$ preserves a Lorentz metric $\hat{g}$.
- $\Gamma$ is a discrete sub-group of $G$ acting properly freely and co-compactly on $G/I$.
- $Z$ is a central 1-parameter subgroup of $G$ defining a parallel vector field $\hat{V}$ on $G/I$.

The question, which we guess has a positive answer, is whether the $Z$-action on $M$ lies in a compact torus $T \subset \text{Iso}(M, g)$?

This would allow one to give a somewhat exact description of $M$. One starts with a compact manifold $M$ with a toral action, and sees if a 1-parameter group $Z \subset T$ can be parallel for some Lorentzian metric?

It could appear paradoxical to be able to handle equicontinuity in the degenerate case, but not in the locally homogeneous case! The reason is that this symplectic algebraic form $\Gamma \setminus G/I$ hides arithmetic and dynamical formidable difficulties due to the discreteness of $\Gamma$ and the non-compactness of $I$.

1.3.2. Cahen-Wallach spaces. They are (indecomposable) Brinkmann (globally) symmetric spaces. More precisely, they are global metrics on $\mathbb{R}^2 \times \mathbb{R}^n$ given by a formula: $g = 2du(dv + H(z)du) + euc_{\mathbb{R}^n}$, where $(u, v) \in \mathbb{R}^2, z \in \mathbb{R}^n, euc_{\mathbb{R}^n}$ is the Euclidean metric $\Sigma((dz)_z)^2$, and $H$ is a quadratic form on $z$. Their discrete groups $\Gamma$ giving rise to compact quotients are investigated by I. Kath and M. Olbrich in [25]. It is proved in their cases, but after a long algebraic preparation, that their parallel vector field is in fact periodic, that is it corresponds to a $S^1$-action.

1.3.3. Flat case. Let us first note that if a Lorentz manifold $(M, g)$ has $d > 1$, linearly independent null parallel vector fields, then its universal cover splits as $\tilde{M} = \tilde{N} \times \mathbb{R}^d$, where $\tilde{N}$ is Riemannian, $\mathbb{R}^d$ is flat Lorentzian, and $\pi_1(M)$ acts by translation along the factor $\mathbb{R}^d$. One can then find a null parallel vector field whose closure in $\text{Iso}(M, g))$ is a torus of dimension $d$, or maybe more.

Consider now the case where $M$ is flat with exactly one (up to scaling) null parallel vector field. Then, by Carriére completeness Theorem, $M$ is a quotient of the Minkowski space $\text{Mink}^{1,n}$ by a discrete group $\Gamma$. Many results are known regarding $\Gamma$. In particular, $\Gamma$ is solvable, in fact it has a “crystallographic hull” $L$ containing it, a solvable connected Lie $L$, acting simply transitively and isometrically on $\text{Mink}^{1,n}$ [18, 20, 22]. One can deduce from this that if $M$ has exactly one null parallel field, then as in the Cahen-Wallach case, the so defined flow is periodic.
1.3.4. A non-periodic example. Consider on $\tilde{M} = (I \times \mathbb{R}) \times \mathbb{R}^n$, parameterized by $(u, v, z^1, \ldots, z^n)$, a metric of the form: $g = 2dudv + \alpha_{ij}(u)dz^i dz^j$ where $\alpha : u \in I \rightarrow (\alpha_{ij})_{i,j}(u)$ is a curve of positive definite matrices, and $I$ is either an interval or the circle $\mathbb{S}^1$.

These are pp-waves and non-flat for a generic $\alpha$. They are even plane waves (written in Rosen coordinates), see [4] p. 6.

The abelian group $\mathbb{R}^{n+1}$ acts isometrically, trivially on $u$ and by translation on the coordinates $(v, z^1, \ldots, z^n)$. This action is transitive on the $u$-levels. Let $\Lambda$ a be a lattice in $\mathbb{R}^{n+1}$ with respect to which the $v$-axis is irrational, that is the translation along $v$ determines a minimal linear flow on $\mathbb{R}^{n+1}/\Lambda$. The quotient $\tilde{M}/\Lambda$ has the topology of $I \times \mathbb{T}^{n+1}$, on which the flow of $\frac{\partial}{\partial v}$ acts minimally on the $\mathbb{T}^{n+1}$-factor. It will be noted $M(\alpha, \Lambda)$. In the case $I = \mathbb{S}^1$, one gets compact pp-waves with topology $\mathbb{T}^{n+2}$, for which closures of the parallel flow are given by a $\mathbb{T}^{n+1}$-action.

In fact, it turns out that any Brinkmann spacetime as in the situation of Theorem 1.4, is built up by pieces of the form $M(\alpha, \Lambda)$. (This may be extracted from our last Sections 7, 8 and 9. Details will be published elsewhere).

1.4. Organization of the article.

- Section 2.1 contains some generalities about Brinkmann spacetimes. In Sections 2.2 and 2.3, we give some properties on the local geometry of Brinkmann spacetimes; these properties are interesting on their own, but they will also be needed in the development of the rest of the article. In particular, we introduce in Section 2.2 some principal bundle over a Brinkmann spacetime $(M, V)$ with the property that the induced action of the subgroup of $\text{Iso}(M)$ preserving $V$ has unipotent isotropy; this feature will serve in the study of the $V$-dynamics on $M$, carried out in Sections 7 and 8. Among submanifolds of a pseudo-Riemannian manifold, totally geodesic ones are fundamental. In Section 2.3, we prove the existence of many such submanifolds in a Brinkmann spacetime.

- We will then use all this in Section 3 to give a synthetic proof of completeness through the lightlike geodesic foliation $F$ orthogonal to $V$.

- Section 4 gives the geodesic equations on a Brinkmann spacetime, and Section 5 focuses on the proof of Theorem 1.2 on the completeness of compactly homogeneous Brinkmann spacetimes, through an analysis of the geodesic equation.

- In Section 6, we prove Theorem 1.3 on the existence of a core $N$, and Section 7 gives more details on the structure of this core.

- Sections 8 and 9 contain the proof Theorem 1.4. Section 8 seems to be the most technical. One deals with a homogeneous lightlike space $G/I$, and a discrete subgroup $\Gamma \subset G$ acting properly co-compactly. A proven approach to such a problem is to find a kind of connected envelope $H$ containing $\Gamma$ and still acting properly. The difficulty in implementing this idea lies on one hand in the possible existence of a semi-simple compact factor of $G$, and on the other hand the fact that the radical of $G$ is not nilpotent. What helps us here is that $V$ defines a 1-dimensional transversally Riemannian foliation, say $\bar{V}$. As this is the case of any transversally Riemannian foliation, the closure of the $\bar{V}$-leaves defines a transversally Riemannian foliation (possibly singular) $\bar{\nabla}$. By a result of Y. Carrière [10, 11], because $\dim V = 1$, $\bar{\nabla}$ has toral leaves. The algebraic richness of our situation allows us to find such a syndetic hull $H$.

2. BRINKMANN GEOMETRY

2.1. Preliminaries and local coordinates. A Lorentzian manifold is called a Brinkmann manifold if it admits a parallel null vector field. In this section, we derive some interesting properties on its local geometry.
Throughout this section, \((M, g, V)\) is a Brinkmann manifold of dimension \(n + 1\), with \(g\) its Lorentzian metric and \(V\) its parallel null vector field, which we assume to be complete. Denote by \(V\) the 1-dimensional foliation defined by \(V\), and by \(\mathcal{F}\) the foliation of codimension 1 defined by the parallel distribution \(V^\perp\).

### 2.1. Transverse Riemannian structure

The leaves of \(\mathcal{F}\) are lightlike submanifolds of \(M\) foliated by the 1-dimensional foliation of \(V\). Since \(V\) is parallel, \(\mathcal{F}\) is a geodesic foliation. Therefore, the foliation of \(V\) along any leaf \(F\) of \(\mathcal{F}\) admits a transverse Riemannian structure invariant by the local flow of any vector field tangent to \(V\). Locally, it is given by any \((n - 1)\)-submanifold \(S\) contained in \(F\) and transverse to \(V\), endowed with the Riemannian metric induced by \(g\).

#### 2.1.2. Local coordinates

This paragraph gives classical ways in Brinkmann geometry for defining local coordinates.

**Construction of a Brinkmann chart.**

**Fact 2.1.** \((M, g)\) is a Brinkmann spacetime of dimension \(n + 1\) if and only if there is a globally defined vector field \(V\) on \(M\), such that any point \(p \in M\) admits a coordinate chart \((u, v, x^1, \ldots, x^{n-1})\) in which the metric takes the form

\[
g = 2dudv + H(u, v)du^2 + W_i(u, v)du^i + g_{ij}(x, u)dx^i dx^j,
\]

with \(V = \partial_v\). Such a coordinate chart is called a Brinkmann chart (see for instance [3], paragraph 4).

**Proof.** A Brinkmann chart is defined as follows. Consider \(F_0\) the leaf of \(\mathcal{F}\) containing \(p\), and take any \(n\)-submanifold \(\Omega\) in \(M\) through \(p\), transverse to \(\mathcal{F}\) and \(V\). The foliation \(\mathcal{F}\) induces an \((n - 1)\)-dimensional foliation on \(\Omega\). Consider a vector field \(Z\) tangent to \(\Omega\) such that \(g(Z, V) = 1\) (such a vector field exists since \(\Omega\) is transverse to \(\mathcal{F}\)).

The local flow of \(Z\) maps a leaf of \(\mathcal{F}\) into a leaf of \(\mathcal{F}\). This is a consequence of a general fact: if \(\alpha\) is a closed 1-form such that \(\alpha(Z) = 1\), then the flow of \(Z\) leaves invariant the distribution \(\ker \alpha\). Here, take \(\alpha := g(V, \cdot)\) which is closed since \(V\) is parallel.

Denote by \(\phi^s\) (resp. \(\psi^s\)) the local flow of \(V\) (resp. \(Z\)).

Define local coordinates \((x^1, \ldots, x^{n-1})\) on \(\Omega_0 := \Omega \cap F_0\) such that \(x(p) = 0\), and extend them on a neighbourhood \(O\) of \(\Omega_0\) in \(\Omega\) by the flow of \(Z\). This defines a local chart \(f = (u, v, x^1, \ldots, x^{n-1})\) on \(\Omega\) by setting for every \(q \in O\), \(u(q)\) to be the unique element in \(\mathbb{R}\) such that \(\psi^{-u(q)}(q) \in \Omega_0\), and \(x^i(q) = x^i(\psi^{-u(q)}(q))\).

Finally, we define a local chart in a neighborhood of \(p\) by extending the other coordinates by the flow of \(V\). This gives a diffeomorphism \(\tilde{f} = (u, v, x^1, \ldots, x^{n-1})\) from a neighborhood \(U\) of \(p\) into an open subset of \(\mathbb{R}^{n+1}\), by setting for every \(q \in U\), \(f(q) = (u(q), v(q), x^1(q), \ldots, x^{n-1}(q))\), where \(\phi_{-u(q)}(q) \in \Omega\), \(\psi_{-u(q)}(q) \in F_0\) and \(x^i = x^i(\phi_{-u(q)} \circ \psi_{-u(q)}(q))\).

This determines \(n + 1\) commuting vector fields \(\{Z, V, E_1, \ldots, E_{n-1}\}\) on \(U\) such that \(Z = \tilde{f}^{-1}_s(\partial_s), V = \tilde{f}^{-1}_s(\partial_v)\) and \(E_i = \tilde{f}^{-1}_s(\partial_{x^i})\) for all \(i = 1, \ldots, n - 1\). Since \(V\) is parallel, \(E_1, \ldots, E_{n-1}\) are everywhere tangent to \(\mathcal{F}\), and \(g(Z, V) = 1\). So in these coordinates, the metric has the given form, and the leaves of the distribution \(V^\perp\) are given by \(u \equiv \text{constant}\), with \(u = 0\) on \(F_0\).

Conversely, one can see by a computation of Christoffel symbols that in the local coordinates, \(\partial_v\) is isotropic and parallel for a metric of the announced form.

**Another local coordinate system: Rosen coordinates.**

**Fact 2.2.** \((M, g)\) is a Brinkmann spacetime of dimension \(n + 1\) if and only if there is a globally defined vector field \(V\) on \(M\), such that any point \(p \in M\) admits a coordinate chart \((u, v, x^1, \ldots, x^{n-1})\) in which the metric takes the form

\[
g = 2dudv + g_{ij}(x, u)dx^i dx^j,
\]
with \( V = \partial_v \). These coordinates are referred to in [4] as Rosen coordinates.

**Proof.** Rosen coordinates are defined as follows. Consider again \( F_0 \) a leaf of \( \mathcal{F} \), and take any \((n - 1)\)-submanifold \( \Omega_0 \) in \( F_0 \) transverse to \( V \). The induced metric on \( \Omega_0 \) is a riemannian metric. Denote by \( Z \) the null vector field along \( \Omega_0 \) which is orthogonal to \( \Omega_0 \) and transverse to \( V \), such that \( g(Z, V) = 1 \). Such a vector field is uniquely defined. Indeed, for every \( x \in \Omega_0, T_x\Omega_0 \) is spacelike of dimension \( n - 1 \), hence \( T_x\Omega_0^\perp \) is Lorentzian of dimension 2. Thus \( T_x\Omega_0^\perp \) contains a second isotropic direction other than that of \( V \). Therefore, \( Z \) is uniquely defined when adding the assumption that \( g(Z, V) = 1 \). Now, denote by \( \Omega \) the hypersurface transverse to \( V \) such that \( \Omega \cap F_0 = \Omega_0 \), obtained by taking for every \( x \in \Omega_0 \) the \((n)\) geodesic with initial velocity \( Z_x \). Choosing \( \Omega_0 \) smaller, if necessary, these geodesics define a local flow on \( \Omega \); we denote again by \( Z \) its infinitesimal generator. Denote by \( \phi^\nu \) (resp. \( \psi^\nu \)) the local flow of \( V \) (resp. \( Z \)).

For \( q \in \Omega \), define \( u(q) \) to be the unique real such that \( \phi^{u(q)}(q) \in \Omega_0 \). Take some local coordinates \((x^1, \ldots, x^{n-1})\) in \( \Omega_0 \) and extend them in a neighborhood of \( \Omega_0 \) of \( \Omega \) by setting for \( q \in \Omega \), \( x(q) = x(q_0) \), where \( q_0 = \phi^{u(q)}(q) \). We obtain local coordinates \((u, x^1, \ldots, x^{n-1})\) on \( \Omega \). Finally, define a local chart in a neighborhood of \( p \) by extending the latter coordinates by the flow of \( V \). This gives a diffeomorphism \( f = (u, v, x^1, \ldots, x^{n-1}) \) from a neighborhood \( U \) of \( p \) onto an open subset of \( \mathbb{R}^{n+1} \), by setting for \( q \in U \), \( f(q) = (u(q), v(q), x^1(q), \ldots, x^{n-1}(q)) \), where \( \phi^{-v(q)}(q) \in \Omega, \psi^{-u(q)}(q) \in F_0 \) and \( x(q) = x(\phi^{-v(q)} \circ \psi^{-u(q)}(q)) \).

Again by taking \( \Omega_0 \) smaller, we can assume that the flows of \( V \) and \( Z \) are defined for \(|v|, |u| < \epsilon \), for some \( \epsilon > 0 \). If we take \( \Omega_0 \) to be a metric ball of radius \( r \) (with respect to the induced Riemannian metric), this defines a differomorphism of a neighborhood of \( p \) onto a set \( B_{n-1}(0, r) \times I \times J \), where \( B_{n-1}(0, r) \) is the open ball of center 0 and radius \( r \) in \( \mathbb{R}^{n-1} \), and \( I \) and \( J \) are open intervals of \( \mathbb{R} \). We claim that the metric in these coordinates has the given form:

- \( g(\partial_v, \partial_u) = 1 \) and \( g(\partial_u, \partial_u) = 0 \): since \( V \) acts isometrically, the orbits of \( \partial_u \) are null geodesics, hence \( g(\partial_v, \partial_u) \) is constant (Clairaut’s constant) equal to 1, and \( g(\partial_u, \partial_u) = 0 \);
- \( g(\partial_u, \partial_x) = 0 \): as in the construction of the Brinkmann chart, the flow of \( \partial_u \) leaves invariant the distribution tangent to \( \mathcal{F} \), so \( \partial_x, i = 1, \ldots, n - 1 \), are everywhere tangent to \( \mathcal{F} \);
- \( g(\partial_u, \partial_x) = 0 \): fix a geodesic \( \gamma \) tangent to \( \partial_u \); since the flows of \( \partial_u \) and \( \partial_x \) commute, and the orbits of \( \partial_u \) are geodesics, then \( \partial_x \) is a Jacobi field along \( \gamma \), which is orthogonal to \( \gamma \) on \( F_0 \), hence everywhere orthogonal to it.

Conversely, one can check by a computation of Christoffel symbols that in the local coordinates, \( \partial_v \) is isotropic and parallel for a metric of the announced form.

\[ \square \]

2.2. The \( O(n) \)-principal bundle \( M^\ast \) over \( M \). Let \((M, g, V)\) be a Brinkmann spacetime of dimension \( n + 1 \), with \( V \) its parallel null vector field. Define the vector bundle

\[ E := V^\perp / V \rightarrow M, \]

which is equipped with a positive definite metric induced by \( g \),

\[ g_E([X], [Y]) = g(X, Y). \]

A linear orthonormal frame \( r_p \) of \( E \) at a point \( p \in M \) is an ordered orthonormal basis \( r_p = (e_1 + V, \ldots, e_{n-1} + V) \) of the vector space \( E_p \) for the Riemannian metric \( g_E \). So \((e_1, \ldots, e_{n-1})\) is a \( g \)-orthonormal family of vectors contained in \( V^\perp \). Let \( M^\ast \) be the set of all orthonormal frames \( r \) of \( E \) at all points of \( M \), and denote by \( \pi \) the natural projection which maps \( r \) to \( x \). This is a \( O(n-1) \)-principal bundle. For each \( r \in M^\ast \), let \( G_r \) be the subspace of \( T_r M^\ast \) consisting of vectors tangent to the fibre through \( r \). The Levi-Civita connection associated to \( g \) gives a connection on \( M^\ast \) i.e. a horizontal \( O(n-1) \)-invariant distribution \( \mathcal{H} \) such that \( T_r M^\ast = \mathcal{H}_r + G_r \) (direct sum) for every \( r \in M^\ast \). Denote by \( \omega \)
the connection form: it is a \( \sigma(n-1) \)-valued 1-form on \( M^* \), and by \( \sigma : \sigma(n-1) \to \chi(M^*) \)
the Lie algebra homomorphism that maps \( \alpha \in \sigma(n-1) \) to the fundamental vector field on \( M^* \) associated to \( \alpha \).

Let \( \phi^i \) be the 1-parameter group generated by \( V \). The flow of \( V \) induces an action on \( L(M) \), the bundle of orthonormal frames at points of \( M \), and since \( V \) acts isometrically, this induces an action on the linear frames of \( E \). Therefore, for each \( t \), \( \phi^t \) induces a transformation \( \psi^t \) of \( M^* \). Thus we obtain a global 1-parameter group of transformations \( \psi^t \) of \( M^* \), which induces a vector field on \( M^* \) that we will denote by \( V^* \).

**Fact 2.3.** If \( V \) is parallel then \( V^* \) is horizontal, i.e. it is the horizontal lift of \( V \).

**Proof.** Fix an orthonormal frame \( r_p \) of \( E_p \) for a point \( p \in M \), and consider the orbit \( \phi_p(t) \) of \( V \) such that \( \phi_p(0) = p \). Write \( r_p = (X_1(p) + V, \ldots, X_{n-1}(p) + V) \). For every \( i = 1, \ldots, n-1 \), consider a curve tangent to \( F \) whose tangent vector at \( p \) is \( X_i(p) \), and denote again by \( X_i \) its tangent vector field. Extend \( X_i \) by the flow of \( V \). We obtain a vector field \( X_i \) tangent to \( F \) such that \( [X_i, V] = 0 \). Since \( V \) is parallel, then \( X_i \) is the parallel transport of \( X_i(p) \) along \( \phi_p(t) \). It follows that the curve \( c(t) = (\phi^t_p)_o(r_p) \) in \( M^* \) is exactly the horizontal lift of \( \phi_p(t) \) such that \( c(0) = r_p \), and this proves the fact. \( \square \)

2.2.1. A Brinkmann spacetime structure on \( M^* \).

**Fact 2.4.** There is a natural \( O(n-1) \)-invariant Lorentzian metric \( g^* \) on \( M^* \) for which \((M^*, g^*, V^*)\) is a Brinkmann spacetime.

We can define a \( O(n-1) \)-invariant Lorentzian metric on \( M^* \) in the following way: let \( h_0 \) be the positive definite inner product on \( \sigma(n-1) \) given by the Killing form. And set for \( X, Y \in T_r M^* \):

\[
g^*(X, Y) := g_\pi(r)(d_r \pi(X), d_r \pi(Y)) + h_0(\omega(X), \omega(Y)).
\]

So

- for every \( r \in M^* \), \( d_r \pi : \mathcal{H}_r \to T_r M^* \) is a linear isometry.
- the vertical and the horizontal subspaces of \( T_r M^* \) are orthogonal for every \( r \in M^* \).

**Fact 2.5.** \( V^* \) is a null Killing vector field for \((M^*, g^*)\) that preserves the distribution \( \mathcal{H} \).

**Proof.** Here, we only need that \( V \) is a Killing vector field. First, observe that the distribution \( \mathcal{H} \) is invariant by the flow of \( V^* \). Indeed, let \( X \in T_{r_0} M^* \) be a horizontal tangent vector at \( r_0 \in M^* \). And let \( \gamma(s) \) be a horizontal curve in \( M^* \) tangent to \( X \) at \( r_0 \). It is a horizontal lift of the curve \( \gamma(s) = \pi(r(s)) \) on \( M \). For each \( t \), consider the curve \( \psi^t(r(s)) \) defined by the action of the flow of \( V^* \) on \( r(s) \).

Since \( r(s) \) is a parallel frame field along \( \gamma(s) \), and \( V \) is a Killing vector field, then every curve \( \psi^t(r(s)) \), for a fixed \( t \), is a parallel frame field along \( \phi^t(\gamma(s)) \). It follows that for each \( t \), \( \psi^t(r(s)) \) is a horizontal lift of \( \phi^t(\gamma(s)) \). Hence \( \psi^t(X) = \frac{d}{ds} \psi^t(r(s)) \) is a horizontal vector.

Now, write \( \phi_\alpha \circ ds = ds \circ \psi_\alpha \). Again, since \( V \) is a Killing field and \( d\pi \circ \mathcal{H}_r \to T_r M \) is an isometry, then \( g^*(X, X) = g^*(\psi^t_\alpha(X), \psi^t_\alpha(X)) \) for every horizontal vector \( X \).

On the other hand, \( V^* \) is \( O(n-1) \)-invariant, so the flow \( \psi^t \) commutes with \( R_{\sigma} \) for every \( A \in O(n-1) \). It follows that for every \( t \), \( s \in \mathbb{R} \), and every \( \alpha \in \sigma(n-1) \), \( \psi^t \circ R_{\exp(sA)} = R_{\exp(sA)} \circ \psi^t \). Thus, taking the derivative with respect to \( s \) at \( s = 0 \) yields \( \psi^t_\alpha(\sigma(\alpha)(r)) = \sigma(\alpha)(\psi^t(r)) \) for every \( \alpha \in \sigma(n-1) \). In consequence, if \( X \in T_r M^* \) then \( \psi^t_\alpha(X) \) is \( \omega \)-constant. This proves the fact. \( \square \)

**Fact 2.6.** \( V^* \) is parallel for the Levi-Civita connection induced by \( g^* \).

This will be proved by finding Brinkmann local coordinates for \( g^* \), and concluding by Fact 2.1.
Brinkmann local charts on $M^*$. Let $(u,v,x_1,\ldots,x^{n-1})$ be a Rosen local coordinate system in a coordinate neighborhood $U$ of $M$, in which the metric $g$ reads
\[ 2dudv + g_{ij}(x,u)dx^idx^j, \quad V = \partial_u. \]

Define a local section $s : U \to M^*$ that maps $p \in U$ to the frame field $([\frac{\partial}{\partial u}]_p, \ldots, [\frac{\partial}{\partial x^r}]_p)$. It is a horizontal section along the integral curves of $V$ in $U$.

Every frame $r$ of $E$ at $p \in U$ can be written uniquely in the form $r = (r_1,\ldots,r_{n-1})$ with $r_j = \sum_{i=1}^n A_{ij}[\frac{\partial}{\partial x^i}]_p = s(p).A$, where $A = (A_{ij})$ is a matrix in $O(n-1)$. We get a diffeomorphism
\[ F : U \times O(n-1) \to \pi^{-1}(U) \subset M^* \]
\[ (p,A) \mapsto s(p).A \]

Every vector field $X$ on $U$ induces a vector field $\tilde{X}$ on $U \times O(n-1)$ in a natural manner. This vector field is $O(n-1)$-invariant and projects onto $X$ via $\pi$. On $U \times O(n-1)$, the flow of $\tilde{X}$ sends $(p,A)$ to $(f^t(p),A)$, where $f^t$ is the flow of $X$ on $U$.

Now, take local coordinates $(y_1,\ldots,y_{N-1})$ on $O(n-1)$. Again, the vector fields $[\frac{\partial}{\partial y^i}]_p$ on $O(n-1)$ induce vector fields on $U \times O(n-1)$.

**Fact 2.7.** We have $F(\tilde{V}) = V^*$, i.e. $F \circ \tilde{\phi}^t = \psi^t \circ F$, where $\tilde{\phi}^t$ is the flow of $\tilde{V}$.

This can be seen using the fact that $\psi^t \circ s = s \circ \phi^t$ (by definition of Rosen coordinates on $U$), and that the vector field $V^*$ is $O(n-1)$-invariant as the horizontal lift of $V$. An equivalent way to saying this is the following: since the section $s$ is horizontal along integral curves of $V$, and the distribution $H$ is $O(n-1)$-invariant, then $F_s(\tilde{V})$ is a horizontal vector field that projects on $V$ via $\pi$. By unicity of the horizontal lift, $F_s(\tilde{V}) = V^*$.

Henceforth, we denote by the same symbols the vector fields on $U$ and $O(n-1)$, and the induced vector fields on $U \times O(n-1)$.

The vector fields $F_s(\partial_u), F_s(\partial_u), F_s(\frac{\partial}{\partial y^i}), F_s(\frac{\partial}{\partial y^i})$ are invariant by the flow of $V^*$ since $F(\partial_u) = V^*$. It follows by Fact 2.5 that all the scalar products $F^*g^r(X,Y)$ where $X$ and $Y$ are in the list $\{\partial_u, \partial_{v}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$ are $v$-independent. And the metric expresses as
\[ F^*g^r = 2dudv + (G_{ij}(u,x,y)dx^idx^j + H_{ij}(u,x,y)dx^idy^j + K_{ij}(u,x,y)dy^idy^j) + (W_i(u,x,y)dudx^i + Z_i(u,x,y)dudy^i). \]

This is a Brinkmann chart with the Riemannian part given by $(G_{ij}(u,x,y)dx^idx^j + H_{ij}(u,x,y)dx^idy^j + K_{ij}(u,x,y)dy^idy^j)$ and $\partial_u$ the parallel null vector field, which proves Fact 2.6.

**A parallelism on $M^*$.** We define a parallelism on $M^*$ using the connection on it induced by the Levi-Civita connection of $g$. We have $T_rM^* = H_r + G_r$ for every $r \in M^*$.

Take $Y_1,\ldots,Y_{N-1}$ a basis of the Lie algebra $o(n)$, and consider $\sigma(Y_1),\ldots,\sigma(Y_{N-1})$ the associated vertical vector fields on $M^*$.

Define horizontal vector fields $X_1,\ldots,X_{n-1}$ on $M^*$ as follows: for every $r = (r_1,\ldots,r_{n-1}) \in M^*$, take $X_r$ to be the unique tangent vector at $r$ such that $d_r\pi(X_r) = r_i$.

The vector fields $(\sigma(Y_1),\ldots,\sigma(Y_{N-1}), X_1,\ldots,X_{n-1})$ define a parallelism on $M^*$.

An isometry of $M$ clearly induces a bundle automorphism of the principal bundle $M^*$. Let $f \in \text{Iso}(M,g)$ and denote by $f^*$ its lift to $M^*$. 
Fact 2.8. $f^*$ preserves the parallelism on $M^*$.

Proof. Since $f^*$ commutes with the $O(n - 1)$-action on $M^*$, it preserves the vertical vector fields $\sigma(Y_i)$ (as this has already been seen in the proof of Fact 2.5 for the flow of $V^*$).

To see that $f^*$ preserves the horizontal vector fields $X_i$, write $\pi \circ f^* = f \circ \pi$. This gives $d_{f(r)}\pi \circ d_{f^*(X_i)} = d_{\pi(r)}\pi \circ d_{f(X_i)} = d_{\pi(r)}f(r)$. Hence $d_{\pi(r)}(d_{f^*(X_i)}) = r_i^\prime$, with $r^\prime = f^*(r) = (df(r_1), \ldots, df(r_{n-1}))$. So $X_i(f^*(r)) = d_{f^*(X_i)}(r_i) = 0$.

It follows that $f^*$ acts isometrically on $(M^*, g^*)$ (if $r$ is an orthonormal frame on $M$, then the frame on $T_rM^*$ given by the parallelism defined above is $g^*$-orthonormal, so $f^*$ maps orthonormal frames to orthonormal frames).

Corollary 2.9. The lift of an isometry of $M$ to $M^*$ acts freely on $M^*$.

Let $f \in \text{Isom}(M, g)$ and denote by $f^*$ its lift to $M^*$. Since $f^*$ preserves a parallelism on $M^*$, then if $f^*(r) = r$ for some $r \in M^*$, then $d_r f^* = I$. But $f^*$ acts isometrically on $(M^*, g^*)$; this yields $f^* = I$.

2.3. Totally geodesic surfaces in a Brinkmann spacetime. Totally geodesic submanifolds appear only in special contexts. A generic pseudo Riemannian manifold does not admit any geodesic submanifold of dimension $\geq 1$. In a Brinkmann spacetime, we will find many such submanifolds.

Let $(M, g, V)$ be a Brinkmann spacetime, with $V$ its parallel null vector field. Consider the associated Grassmannian bundle $Gr_2(M)$, obtained by replacing each fiber $T_xM$ by $Gr(r(2), T_xM)$. Define a sub-bundle $X$ of $Gr_2(M)$ in the following way: for each $x \in M$, the fibre at $x$ is

$$X_x := \{p_x \in Gr_2(M)_x, V \in p_x\}.$$ 

Proposition 2.10. For any $p_x \in X$, there is a totally geodesic flat surface through $x$ in $M$ whose tangent space at $x$ is $p_x$.

Proof. Let $p \in X$ with $\pi(p) = x$: we have to prove that $\exp_x(p) \in X$ is totally geodesic. Write $p = \text{Vect}(Q, V)$. Let $\gamma$ be the geodesic in $M$ tangent to $Q$, and denote again by $Q$ its tangent vector field. Take the image of $\gamma$ by the flow of $V$. This defines in a neighborhood of $x$ a surface $S_p$ tangent to $p$ at $x$, equipped with two commuting vector fields $V$ and $Q$. Since $V$ is parallel, we have $\nabla_V Q = \nabla_Q V = 0$; furthermore, $\nabla_Q Q = 0$ (for $V$ acts isometrically on $M$). So $V$ and $Q$ are two parallel vector fields along any curve contained in $S_p$, which proves that the latter is totally geodesic in $M$. Observe that $S_p$ is exactly $\exp_x(p)$ in a neighborhood of $x$, since radial geodesics tangent to $p_x$ are contained in $S_p$.

Finally, it is a well known fact that a surface admitting a parallel vector field is actually flat. \eq

2.3.1. Flat bands in $(M, g, V)$.

Definition 2.11 (flat bands). Let $S$ be a surface in $M$ saturated by $V$. We say that $(S, V)$ is

(1) a degenerate flat band if $S$ is isometric to $\mathbb{R} \times I$, $d(y^2) = (x, y) \in \mathbb{R} \times I$, where $I$ is an open interval of $\mathbb{R}$,

(2) a Lorentzian flat band if $S$ is isometric to $\mathbb{R} \times I$, $d(x^2 - dy^2) = (x, y) \in \mathbb{R} \times I$, where $I$ is an open interval of $\mathbb{R}$.

Let $(S, V)$ be a flat band with $V$ its parallel null vector field. The foliation associated to $V$ admits a transverse Riemannian metric invariant by the action of $V$ (in fact by any vector field tangent to $V$ in the case of a degenerate band). Fix an orientation on $S$, together with a leaf $l_0$ of $V$.

1) Suppose $S$ is a degenerate flat band. We define the forward (resp. backward) length of the band to be the length of (any) maximal geodesic $\gamma : [0, T] \rightarrow S$ transverse to $V$ such that $\gamma(0) \in l_0$, and $(V_\gamma, \dot{\gamma})$ is positively (resp. negatively) oriented. We denote them $l^+(S)$
and $l^-(S)$ respectively.

2) If $S$ is a Lorentzian flat band, $\omega = \langle V, \cdot \rangle$ is a closed 1-form that defines the foliation of $V$. Integrating $\omega$ along curves transverse to $V$ defines a transverse Riemannian structure.

Let $\gamma$ be a maximal geodesic transverse to $V$ defined on $[0, T]$, such that $\gamma(0) \in I_0$, and set $l(S) := \int_0^T \langle V, \dot{\gamma} \rangle dt$. We define the forward (resp. backward) length of the band to be $l(S)$ if $\langle V, \dot{\gamma} \rangle$ is positively (resp. negatively) oriented. We denote them again $l^+(S)$ and $l^-(S)$ respectively.

We say that $S$ is an infinite flat band if both $l^+(S)$ and $l^-(S)$ are infinite.

**Fact 2.12.** Let $p \in X$ and let $S$ be the totally geodesic (maximal) surface in $M$ tangent to $p$. Then $\tilde{S}$, the universal cover of $S$, is a flat band.

This follows from the following lemma.

**Lemma 2.13.** Let $S$ be a simply connected surface, equipped with a flat and torsion free connection $\nabla$. Suppose that $S$ admits a complete parallel vector field $V$. Then there is a diffeomorphism $\phi : S \to \mathbb{R} \times I$, $(x, y) \in \mathbb{R} \times I$ $(I$ is an interval of $\mathbb{R})$, where $\mathbb{R} \times I$ is equipped with its affine structure, and $\phi_* V = \partial_x$.

**Proof.** The surface $S$ admits a $(\text{Aff}(\mathbb{R}^2), \mathbb{R})$-structure. We shall prove that the developing map $D : S \to \mathbb{R}^2$, which is a $(\text{Aff}(\mathbb{R}^2), \mathbb{R})$-local diffeomorphism, is actually a global diffeomorphism onto its image.

- $D$ is a diffeomorphism on the orbits of $V$: since $V$ parallel and complete, an orbit of $V$ is a complete geodesic. So $D$ sends such an orbit on a geodesic of the affine space $\mathbb{R}^2$, i.e. on a line segment in $\mathbb{R}^2$. But we know that a local diffeomorphism on a manifold of dimension 1 is actually a global diffeomorphism, so $D$ sends an orbit of $V$ diffeomorphically onto a line (a complete geodesic) in $\mathbb{R}^2$. In particular, $D$ sends the distribution defined by $V$ on a 1-dimentional distribution of $\mathbb{R}^2$ of constant direction. So we can suppose that $D$ sends $V$ on the constant vector field $e_1 = \partial_x$.

- For any maximal geodesic $\gamma$ transverse to $V$, the open subset $U$ of $S$ obtained by taking the image of $\gamma$ by the flow of $V$ is an injective domain for the developing map, hence isomorphic to the band $\mathbb{R} \times I_0 \subset \mathbb{R}^2$, where $I_0$ is an open interval of $\mathbb{R}$; here again, $D$ sends $\gamma$ into a line segment in $\mathbb{R}^2$, hence sends $U$ bijectively to the saturation of that segment by the flow of $e_1$, i.e. to a band $\mathbb{R} \times I_0 \subset \mathbb{R}^2$. Another parametrization of $\gamma$ defines an isomorphism onto another band $\mathbb{R}^2 \times I_1$ of $\mathbb{R}^2$, affinely equivalent to $\mathbb{R}^2 \times I_0$.

- If $U_1$ and $U_2$ are two open sets in $S$ as before, then $D(U_1 \cap U_2) = \mathbb{R} \times I_{1,2} \subset \mathbb{R} \times \mathbb{R}$, where $I_{1,2}$ is an open interval of $\mathbb{R}$.

The space of the leaves of $\mathcal{V}$ is a manifold of dimension 1 diffeomorphic to $\mathbb{R}$. An atlas for this space is given by a countable set of geodesics $(\gamma_i)_{i \in \mathbb{Z}}$ transverse to $V$, such that $\gamma_i \cap \gamma_j \neq \emptyset$ if and only if $|i - j| = 1$. Denote by $U_i$ the open set defined by taking the image of $\gamma_i$ by the flow of $V$. We get an atlas $(U_i, D(U_i))$ on $S$ such that for any $i, j \in \mathbb{Z}$, $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| = 1$, and

$\forall (x, y) \in D_{i+1}(U_i \cap U_{i+1}), D_i \circ D_{i+1}^{-1}(x, y) = (x + \alpha_i, \lambda_i y + \beta_i), \lambda_i > 0, \alpha_i, \beta_i \in \mathbb{R}$. So $S$ is diffeomorphic under $D$ to the quotient of $\prod_{i \in \mathbb{Z}} \mathbb{R} \times I_i$ by $(x, y) \in C_{i+1} \sim f_i(x, y) \in C_i$, where $C_i := D_i(U_i \cap U_{i+1})$ and $f_i := D_i \circ D_{i+1}^{-1}$. Such a manifold is isomorphic to $\mathbb{R} \times I$, equipped with its affine structure, where $I$ is an open interval of $\mathbb{R}$ (that could be either $[0, \infty)$ or $[0, +\infty)$, or $[0, 1]$).

**Proof of Fact 2.12.** It follows in particular from the previous lemma that a geodesic in $\tilde{S}$ transverse to $V$ cuts all the leaves of $\mathcal{V}$.

The surface $\tilde{S}$ can be either Lorentzian or degenerate; denote its metric by $\tilde{g}$. In the first case, take $\gamma$ a null geodesic transverse to $V$ such that $\tilde{g}(\gamma, V) = 1$. It cuts all the leaves of $\mathcal{V}$. So the developing map in the previous lemma sends $\tilde{S}$ diffeomorphically onto a band $\mathbb{R} \times I \subset \mathbb{R}^2$, such that $D_* V = \partial_x$ and $(D^{-1})^* y = 2 dx dy$.
In the second case, take a geodesic $\gamma$ transverse to $V$ such that $\bar{g}(\dot{\gamma}, \dot{\gamma}) = 1$. Again, $\gamma$ cuts all the leaves of $\mathcal{V}$. Then $D$ sends $\bar{S}$ diffeomorphically onto a band $\mathbb{R} \times I \subset \mathbb{R}^2$, such that $D_sV = \partial_s$ and $(D^{-1})^*g = dg^2$.

□

3. A SYNTHETIC PROOF OF COMPLETENESS OF THE LIGHTLIKE GEODESIC FOLIATION

In this section, we use the equivalence between the existence of $d$-dimensional totally geodesic submanifolds in $M$ and the integrability of some distribution on the Grassman bundle $Gr_2(M)$ over $M$ (which is an extension of the notion of the geodesic flow on the projective bundle) to define a 2-dimensional foliation on some sub-bundle of $Gr_2(M)$. We will see that some properties of this foliation allow to study completeness properties of $M$.

Denote by $\pi : Gr_2(M) \to M$ the natural projection. The affine connection on $TM$ defines a connection on $Gr_2(M)$, so that the tangent bundle decomposes $TGr_2(M) = H \oplus Vr$, where $Vr$ is the (canonical) vertical sub-bundle and $H$ is the horizontal sub-bundle (determined by the connection).

The geodesic plane field on $Gr_2(M)$. The geodesic plane field $\tau$ on $Gr_2(M)$ is defined for any $x \in M$, $p \in Gr_2(M)_x$ by the unique horizontal subspace $\tau_p \subset H_x$ (of dimension 2) which projects on $p$ via $d_p\pi$. This is equivalent to defining $\tau_p$ as follows: consider a curve $x(t)$ in $M$ such that $x(0) \in P$, and parallel transport $\tilde{P}$ along $x(t)$. This defines a horizontal curve $\alpha(t) = (x(t), p(t))$ in $Gr_2(M)$, whose infinitesimal generator at 0 gives a vector in $H_p$ that projects into $p$. The subspace $\tau_p$ is the one obtained by considering all such curves in $M$ tangent to $p$ at 0.

This defines a 2-dimensional horizontal distribution $\tau$ on $Gr_2(M)$. A curve $\alpha(t) = (x(t), P(t), t \in I)$, is tangent to $\tau$ if:

- $x'(t) \in p$ for all $t \in I$.
- $p(t)$ is parallel along $x(t)$.

The leaves of $\tau$. A leaf $F$ of $\tau$ is a submanifold of dimension 2 of $Gr_2(M)$ that projects into a plane field along a surface $S$ in $M$, which is parallel along any curve contained in $S$. This is equivalent to saying that there exists a surface $S$ in $M$, with $F = \{x \mapsto T_xS, x \in S\}$, and $S$ is totally geodesic.

Let $p \in Gr_2(M)_x$. It results from the above that there exists a leaf of $\tau$ through $p$ if and only if $p$ is the tangent space at $x$ of a totally geodesic surface in $M$ through $x$, which in turn is equivalent to $\exp_x(p)$ being a totally geodesic submanifold of $M$ in a neighborhood of $x$.

A domain of integrability of the geodesic plane field. Consider the sub-bundle $X$ defined in Section 2.3, and let $p \in X$. Since $V$ is parallel, if there exists a leaf of $\tau$ through $p$, then it is entirely contained in $X$. Regarding Proposition 2.10 and what is said above, we have the following fact.

Fact 3.1. The distribution $\gamma_X$ is integrable, and a leaf of $\tau$ through $p \in X$ projects under $\pi$ on a totally geodesic flat surface in $M$ foliated by $\mathcal{V}$.

Notation: We denote by $\mathcal{G}$ the 2-dimensional foliation on $X$ tangent to $\tau$.

Completeness of the leaves of $\mathcal{F}$. Question: do the leaves of $X$ project onto complete surfaces in $M$?

Let $(M, g, V)$ be a compact Brinkmann spacetime. The sub-bundle $X$ is compact, as a bundle over a compact manifold with compact fibre (the fibre of $X$ at $x$ is isomorphic to the projective space of $T_xM/V$, hence to $\mathbb{R}P^{n-1}$).

We will see that in this case, the answer is yes when the projection is a surface contained in a leaf of $\mathcal{F}$. For this, observe that for $p \in X$, the surface $S_p$ tangent to $p$ is contained in
Proof. Let $\text{Fact 3.2.}$ Let $Y \in \mathcal{V}$ projects to $V$ transverse to $Y$ such that every $\tilde{p}$ universal cover $W$ can define $M$ projects into a surface in $\tilde{V}$ transverse to $\tilde{W}\gamma$. Then $Y$ is a closed submanifold of $X$ (hence compact), which is $G$-invariant. A leaf of $Y$ projects into a surface in $M$ contained in a leaf of $\mathcal{F}$.

The flow of $V$ induces a flow on $Gr_2(M)$; denote by $\tilde{V}$ the infinitesimal generator of this flow. Since $V$ is parallel, $\tilde{V}$ is also the unique horizontal vector field on $X$ which projects to $V$ under $d\pi$.

**Fact 3.2.** Let $S'$ be a leaf of $\mathcal{G}$ and denote by $Y$ its projection on $M$. If $S'$ is contained in $Y$, then $\tilde{S}$, the universal cover of $S$, is an infinite flat band.

**Proof.** Let $W$ be a smooth vector field on $Y$ such that for any $p \in Y$, $W_p \in T_pG$ and $W$ is transverse to $\tilde{V}$ (to be more accurate, it is a direction field that we can always construct, and a vector field up to a double cover). By definition, $g(d_p\pi(W), V) = 0$. Since $Y$ is compact, $W$ is complete. Furthermore, there is $\alpha > 0$ such that $g(d_p\pi(W), d_p\pi(W)) \geq \alpha > 0$ for every $p \in Y$.

Let $S'$ be a leaf of $\mathcal{G}$. Since $\pi$ restricted to $S'$ is a diffeomorphism onto its image, we can define $W_S := d\pi(W_{S'})$. Denote by $\tilde{W}_S$ (resp. $\tilde{V}$) the lift of $W_S$ (resp. $V$) to the universal cover $\tilde{S}$. So $\tilde{S}$ is a flat band foliated by $\tilde{V}$, and equipped with a vector field $\tilde{W}_S$ transverse to $\tilde{V}$. To see that it is an infinite band, let $c$ be an integral curve of $\tilde{W}_S$ defined on $[0, T]$, it cuts all the leaves of $\tilde{V}$. Set $l(c) = \int_0^T (g(c(t))(\tilde{W}_S, \tilde{W}_S))^{1/2} dt$ the length of $c$. We have $l(c) \geq \alpha^{1/2} \cdot T$. Since $W$ is complete, we have $l(c) = \infty$. \hfill $\square$

**Corollary 3.3.** The leaves of $\mathcal{F}$ are complete.

**Proof.** Let $S$ be as in the proof of the previous fact. Let $\gamma$ be a maximal geodesic in $\tilde{S}$ defined on $[0, T]$, transverse to $\tilde{V}$, with unit tangent vector. By the previous fact, the completeness of $\gamma$ is equivalent to the fact that $\gamma$ cuts all the leaves of $\tilde{V}$, and this is true since otherwise it would accumulate on a leaf of $\tilde{V}$, which is impossible. \hfill $\square$

### 4. The Geodesic Equation

Let $\gamma$ be a geodesic in $M$. Fix Rosen local coordinates $(u, v, x^1, ..., x^{n-1}) \in I \times J \times B_{n-1}(0, 1)$, the geodesic equations read:

\begin{align}
\ddot{x}^k + \Gamma^k_{ij}(x, u) \dot{x}^i \dot{x}^j + \Gamma^k_{iu}(x, u) \dot{x}^i \dot{u} &= 0 \quad \forall k \in \{1, ..., n-1\} \\
\ddot{u} + \Gamma^u_{ij}(x, u) \dot{x}^i \dot{x}^j + \Gamma^u_{iu}(x, u) \dot{x}^i \dot{u} &= 0,
\end{align}

where the $\Gamma^u_{ij}$'s are the Christoffel symbols of the Levi-Civita connection of the metric.

Since $V$ is parallel, $g(\dot{\gamma}, V)$ is constant along $\gamma$. We distinguish two different situations: either $g(\dot{\gamma}, V) = 0$ or $g(\dot{\gamma}, V) \neq 0$. In the first case, $\gamma$ is contained in a leaf of $\mathcal{F}$ and $u = u_0$ is constant along $\gamma$. In the second case, one can suppose without loss of generality that $g(\dot{\gamma}, V) = 1$; this yields $\dot{u} = 1$ hence $u = t$ (up to translation of the geodesic parameter).
We see that when $\gamma$ is tangent to $\mathcal{F}$, the coefficients in the geodesic equations are time independent, and the previous equations read
\[
\ddot{x}^k + \Gamma^k_{ij}(x, u_0) \dot{x}^i \dot{x}^j = 0 \quad \forall k \in \{1, ..., n\},
\]
\[
\ddot{v} + \Gamma^i_{ij}(x, u_0) \dot{x}^i \dot{x}^j = 0.
\]
They are autonomous equations.

Denote by $h_t$ the Riemannian metric induced by $g$ on each slice $B_t := B_{n-1}(0,1) \times \{0\} \times \{t\}$. The equation on $x(t)$ can be written in the following way:
\[
(6) \quad \nabla^t_{\dot{x}(t)} \dot{x}(t) = A(t, x) \cdot \dot{x}(t) + B(t, x),
\]
where $\nabla^t$ is the Levi-Civita connection of the Riemannian metric $h_t, A(t, x) \in GL_{n-1}(\mathbb{R})$ and $B(t, x) \in \mathbb{R}^{n-1}$ are given by
\[
A_{ik}(t, x) = \begin{cases} -\Gamma^k_{iu}(t, x) & \text{if } \gamma \text{ is not tangent to } \mathcal{F} \\ 0 & \text{otherwise,} \end{cases}
\]
and
\[
B_{ik}(t, x) = \begin{cases} -\Gamma^k_{u \alpha}(t, x) & \text{if } \gamma \text{ is not tangent to } \mathcal{F} \\ 0 & \text{otherwise.} \end{cases}
\]

5. ANALYSIS OF THE GEODESIC EQUATION

**Definition 5.1.** A compactly homogeneous Brinkmann manifold is a Brinkmann manifold such that there is a compact subset whose iterates by the isometry group cover all the space.

Let $(M, g, V)$ be a compactly homogeneous Brinkmann manifold.

Let $H$ be a smooth $(n-1)$-dimensional distribution on $M$, such that for every $p \in M$, $H_p$ is a non-degenerate Riemannian subspace of $T_p M$ tangent to $\mathcal{F}$. For every $p \in M$, there exists an open neighborhood $O_p$ of 0 in $H_p$ such that the exponential map $\exp_p : H_p \subset T_p M \to M$ restricted to $H_p$ is a diffeomorphism from $O_p$ onto its image in $M$. Set $S_p := \exp_p O_p$, it is an $(n-1)$-submanifold through $p$.

For every $p \in M$, consider a normal coordinate system $(x^1, ..., x^{n-1})$ on $S_p$. If $\rho = \sum(x^i)^2$, then for $\delta > 0$ sufficiently small, $B_p(\delta) = \{ q \in S_p, \rho(q) < \delta \}$ is a normal neighborhood of $p$ in $S_p$ diffeomorphic to an open ball in $\mathbb{R}^{n-1}$.

**Fact 5.2.** Since $M$ is compactly homogeneous, a continuity argument shows that there exists $r_0 > 0$ such that for any $p \in M$, we can take $B_p$ above to be the open ball of radius $r_0$ and center $p$ in $S_p$.

Again, by compact homogeneity of $M$, there exists a positive constant, which we denote again by $r_0$, such that every point $p \in M$ admits a Rosen coordinate chart $(u, v, x^1, ..., x^{n-1}) : U \subset M \to I \times J \times B_{n-1}(0, r_0)$ defined on an open neighborhood $U$ of $p$, such that $U \cap S_p = B_p(r_0)$ is a normal neighborhood of $p$ in $S_p$, and $|I| = |J| = r_0$. Henceforth, to each point $p \in M$ we associate a Rosen coordinate chart as before. We can assume WLOG $r_0 = 1$.

For every $p \in M$, we have a diffeomorphism $f_p : I \times J \times B_{n-1}(0, 1) \to U$ into a neighborhood $U$ of $p$, given by $f_p(u, v, x) = \psi_u \circ \phi_v(\exp_p(\sum_{i=1}^{n-1} x_i e_i))$, $e_i = \frac{\partial}{\partial x^i}$, with $f(0, 0, 0) = p$. Denote by $h^p$ the pull-back of the metric $g_{U}$ by $f_p$, and by $h^p_U$ the Riemannian metric induced by $h^p$ on each slice $\{u\} \times \{0\} \times B_{n-1}(0, 1)$.

Define for $a, b, c \in \{1, ..., n-1, r, v\}$, the map $F_{(a, b, c)} : M \times I \times J \times B_{n-1}(0, 1) \to \mathbb{R}, F(p, u, v, x) = (\Gamma^c_{ab})_p(u, x)$, where $(\Gamma^c_{ab})_p$ is the Christoffel symbol associated to the
metric $f_p^*g$ of $I \times J \times B_{n-1}(0,1)$, the pull-back of $g$ by the diffeomorphism $f_p$. Since $f_p$ depends smoothly on $p$, $F_{(a,b,c)}$ is a smooth map for every $a, b, c \in \{1, \ldots, n-1, u, v\}$.

**Lemma 5.3.** Let $I$ and $J$ be intervals in $\mathbb{R}$ and consider the space $D = I \times J \times B_{n-1}(0,1)$ equipped with a metric $g = du dv + g_{ij}(u,x)dx^i dx^j (u,v,x) \in D$. For every $u \in I$, the slice $\{u\} \times \{0\} \times B_{n-1}(0,1)$ inherits a Riemannian metric $h_u$ that depends smoothly on $u$. Let $c(t) = (u(t), v(t), x(t))$ be a geodesic in $D$. There exists $\epsilon > 0$ and $C > 0$ such that for any initial condition $(x(0), \dot{x}(0))$ with $x(0) = 0$ and $h_0(\dot{x}(0), \dot{x}(0)) \geq 1$, $x(t)$ exists on $[0, \|x(0)\|_0]$.

For all $t \in [0, \|x(0)\|_0]$, \[ \|\dot{x}(t)\|_0 \leq 1 + Ct, \]

and \[ \|\dot{x}(t)\|_t \leq 1 + Ct, \]

where $\|\cdot\|_u$ is the norm for the metric $h_u$.

**Proof.** Consider in the ball $B_{n-1}(0,1)$ of $\mathbb{R}^{n-1}$:

\[ \dot{x}^k = -\Gamma^j_i(t,x)\dot{x}^i\dot{x}^j + A_{ik}(t,x)\dot{x}^i, \]

for $k = 1, \ldots, n-1$, with initial condition $x(0) = 0$, $\dot{x}(0) \in \mathbb{R}^{n-1}$ such that $h_0(\dot{x}(0), \dot{x}(0)) \geq 1$. And let $y(t) := x(\lambda^{-1}t)$, with $\lambda := \|\dot{x}(0)\|$, then $y(t)$ satisfies the equation

\[ \dot{y}^k(t) = -\Gamma^j_i(\lambda^{-1}t,y)\dot{y}^i\dot{y}^j + \lambda^{-1}A_{ik}(\lambda^{-1}t,y)\dot{y}^i, \]

with initial conditions $y(0) = 0$ and $h_0(\dot{y}(0), \dot{y}(0)) = 1$.

Write $\Gamma^k_{ij}(\lambda^{-1}t,y) = \Gamma^k_{ij}(0,y) + \lambda^{-1}tF^k_{ij}(\lambda^{-1}t,y)$, where $F^k_{ij}$ are continuous and $t \in [0,1]$.

Consider:

\[ \frac{\partial}{\partial t} h_0(\dot{y}, \dot{y}) = 2h_0(\nabla^0\dot{y}, \dot{y}). \]

But

\[ \nabla^0\dot{y} = (\dot{y}^k(t) + \Gamma^k_{ij}(0,y)\dot{y}^i\dot{y}^j)\partial_k, \text{ with } \partial_k = \frac{\partial}{\partial y^k}. \]

Thus, from Equation (8),

\[ \nabla^0\dot{y} = \frac{1}{\lambda} (tF^k_{ij}(\lambda^{-1}t,y)\dot{y}^i\dot{y}^j + A_{ik}(\lambda^{-1}t,y)\dot{y}^i)\partial_k \]

hence:

\[ \frac{\partial}{\partial t} h_0(\dot{y}, \dot{y}) = \frac{2}{\lambda} h_0((tF^k_{ij}(\lambda^{-1}t,y)\dot{y}^i\dot{y}^j + A_{ik}(\lambda^{-1}t,y)\dot{y}^i)\partial_k, \dot{y}). \]

There exists $\epsilon > 0$ such that for any initial condition $(y(0), \dot{y}(0))$, with $y(0) = 0$ and $h_0(\dot{y}(0), \dot{y}(0)) = 1$, $y(t)$ is defined on $[0, \epsilon]$ and $\dot{y}(t)$ has a uniformly bounded $h_0$ norm.

Now using Equation (9), we get a finer control of the $h_0$ norm of $\dot{y}(t)$ for $t \in [0, \epsilon]$. A continuity argument on the coefficients involved in the equation ensures the existence of constants $\epsilon' > 0$ and $r_0 > 0$ such that all the coefficients are bounded on $[0, \epsilon'] \times I \times B_{n-1}(0, r_0)$.

We denote again $\epsilon$ instead of $\epsilon'$.

Therefore there is a constant $C$ such that:

\[ 2h_0((tF^k_{ij}(\lambda^{-1}t,y)\dot{y}^i\dot{y}^j + A_{ik}(\lambda^{-1}t,y)\dot{y}^i, \dot{y}) \leq C, t \in [0, \epsilon]. \]

Hence
\[ \frac{\partial}{\partial t} h_0(y, y) \leq \frac{C}{\lambda}, \ t \in [0, \epsilon]. \]

Thus
\[ h_0(\dot{y}(t), \dot{y}(t)) \leq \frac{C}{\lambda} t + 1, \ t \in [0, \epsilon] \] (since \( h_0(\dot{y}(0), \dot{y}(0)) = 1 \)).

Hence:
\[
\sqrt{\frac{h_0(\dot{x}(t), \dot{x}(t))}{h_0(\dot{x}(0), \dot{x}(0))}} \leq \sqrt{1 + Ct} \leq 1 + Ct, \ t \in [0, \epsilon],
\]
which gives the first inequality.

Let us now compare compare the \( h_t \) and the \( h_0 \)-norms of \( \dot{x}(t) \), for \( t \in [0, \epsilon / |x(0)|] \).

Consider the function \( f(t, u) = \frac{h_t(u, u)}{h_0(u, u)} \). Observe that \( f(t, \alpha u) = f(t, u) \), and hence we can assume \( h_0(u, u) = 1 \). By smoothness and compactness, \( |f_s(t) - f_s(0)| = |f_s(t) - 1| \) is bounded by some constant, say the same \( C \) as above. So
\[
\sqrt{\frac{h_t(\dot{x}(t), \dot{x}(t))}{h_0(\dot{x}(0), \dot{x}(0))}} \leq \sqrt{1 + Ct} \leq 1 + Ct.
\]

Let \( \gamma \) be a geodesic of \( M \) (if \( \gamma \) is not tangent to \( F \), we parameterize in such a way that \( g(\gamma, V) = 1 \)), such that \( \gamma(0) = p \). Let \( f_p : I \times J \times B_{n-1}(0, 1) \to U(p) \) be a Rosen coordinate chart at \( p_0 \). Consider Equations (3), (4) and (5) with initial conditions \( x(0) = 0 \), \( \dot{x}(0) \in \mathbb{R}^{n-1}, v(0) = 0 \), \( \bar{v}(0) \in \mathbb{R} \) and \( u(0) = 0, \bar{u}(0) = 1 \).

**Lemma 5.4.** If the solution \( x(t) \) is defined on \([0, T]\), then \( \gamma(t) \) is defined on \([0, T']\), with \( T' := \text{inf}(T, |I|) \), and \( |I| \) is the length of \( I \).

**Proof.** Observation: actually, it is more precise than that. Denote by \( s \) the unique real number such that \( \phi^{-s}(\gamma(T')) \in U_p \), has \( \nu \)-coordinate zero, where \( \phi^s \) is the flow of \( V \). Then \( \phi^{-s}(\gamma(T')) \) has coordinates \((\dot{u}(T'), \dot{v}(T'), \dot{x}(T'))\).

It follows from Equation (5) on \( u(t) \) that either \( u(t) = 0 \) or \( u(t) = \bar{v} \). Consider the interval \([0, T]\) on which \( x(t) \) is defined. Then \( u(t) \) is defined on the same interval, and \( u(T) = \bar{u}(0) = T \). Set \( T' = \text{inf}(T, |I|) \). Then on the interval \([0, T']\), both the \( x(t) \) and the \( u(t) \) components are defined.

On the other hand, it appears from Equation (4) that \( v(t) \) is defined on the same interval as \( x(t) \). However, \( v(t) \) may well leave the interval \( J \) before the \( x(t) \) component reaches \( x(T') \), so that \( \gamma(t) \) is not defined on \([0, T']\). To see that \( \gamma \) is defined on \([0, T']\), we use the fact that the flow of \( V \) acts isometrically on \( M \).

Let \( A := \{ t \in [0, T] \mid \text{such that the lemma -together with the observation- is true} \} \). Set \( \alpha = \sup A \), and suppose \( \alpha < T \).

Set \( q := \gamma(\alpha) \) and \( s := v(\alpha) \). Consider \( \gamma_1 = \phi^{-s} \circ \gamma \), it is a geodesic through \( \phi^{-s}(q) \in U(p) \) with \( \dot{\gamma}_1(\alpha) = \phi^{-s}(\dot{\gamma}(\alpha)) \). By definition of \( \alpha \), \( \phi^{-s}(\dot{\gamma}(\alpha)) \) has \( \nu \)-component equal to \( \dot{x}(\alpha) \), and \( \nu \)-component equal to \( \dot{v}(\alpha) \). So \( \gamma_1 \) satisfies Equations (3) and (4) on the local chart \( I \times J \times B_{n-1}(0, 1) \) at \( p \), with initial conditions \( x_1(\alpha) = x(\alpha), \dot{x}_1(\alpha) = \dot{x}(\alpha) \) for \( x_1(t) \), and \( v_1(\alpha) = 0, \bar{v}_1(\alpha) = \dot{v}(\alpha) \) for \( v_1(t) \). It follows that \( x_1(t) = x|_{[a, T']} \) for \( t \in [a, T'] \), and \( v_1(t) = v(t) - v(\alpha) \). And there exists \( \alpha' \in [a, T'] \) such that \( v_1(\alpha') \leq \bar{v}(\alpha) \), so that \( x_1(t) \) and \( v_1(t) \) are both defined on \([a, \alpha']\). Thus \( \gamma_1 \) is defined on \([a, \alpha']\), and then \( \gamma \) is defined on \([0, \alpha]\). Denote by \( s' \) the unique real number such that \( \phi^{-s'}(\gamma(\alpha')) \in U_p \) has zero \( \nu \)-coordinate. By construction of Rosen coordinates, \( \phi^{-s'}_s(\dot{\gamma}(\alpha')) = \phi^{-s'+s}_s(\dot{\gamma}_1(\alpha')) \) has the same coordinates as \( \dot{\gamma}_1(\alpha') \), which are given by
(\dot{\alpha}', \ddot{\alpha}', \dot{\alpha}'). This yields \alpha' \in A with \alpha' > \alpha. This contradicts the assumption on \alpha. Hence \alpha = T', which ends the proof.

\[ \text{Proof.} \]

Step 1: Fix \( p \in M \) and consider \( (u, v, x^1, \ldots, x^{n-1}) : U \subset M \rightarrow I \times \mathbb{R}^{n-1} \) the associated Rosen coordinate chart defined on an open neighborhood \( U \) of \( p. \) The geodesic equation gives in the ball \( B_n(0, 1) \) of \( \mathbb{R}^{n-1}; \)

\[ \ddot{x}^k = -(\Gamma^k_{ij})_p(t, x)\dot{x}^i \dot{x}^j + (A_{ik})_p(t, x)\dot{x}^i, \]

for \( k = 1, \ldots, n - 1, \) with initial condition \( x(0) = 0, \dot{x}(0) \in \mathbb{R}^{n-1}. \)

Denote by \( \| \cdot \|_p \) the norm associated to \( h^p_0. \)

- There exists \( \epsilon > 0 \) such that for any initial condition \( x(0) = 0, \| x(0) \|_p \leq 1, \)

the solution \( x(t) \) is defined on \([0, \epsilon]. \) Since \( M \) is compactly homogeneous, the choice of \( \epsilon \) is uniform with respect to \( p, \) for then, the subset of vectors \( X \) in \( T_0\mathbb{R}^{n-1} \) such that \( h^p_0(X, X) \leq 1 \) for some \( p \in M, \) is a compact set in \( T_0\mathbb{R}^{n-1}. \)

- Now suppose \( \| \dot{x}(0) \|_p \geq 1. \) By Lemma 5.3, there exists \( \epsilon' > 0 \) and \( C > 0 \) such that for any initial condition \((0, x(0), \dot{x}(0))\) with \( x(0) = 0 \) and \( h_0^p(\dot{x}(0), \dot{x}(0)) \geq 1, \)

\( x(t) \) exists on \([0, \epsilon'|x(0)|] \) and

\[ \frac{\| x(t) \|_p}{\| x(0) \|_p} \leq 1 + Ct, \quad t \in [0, \frac{\epsilon'}{\| x(0) \|_p}], \]

and

\[ \sqrt{h_0^p(\dot{x}(t), \dot{x}(t))} \leq 1 + Ct. \]

If we look again at the proof of Lemma 5.3, it follows from the compact homogeneity of \( M \) and the smoothness of the maps \( F_{(\alpha, h, \epsilon)} \) defined at the beginning of this section that the choices of \( \epsilon' \) and \( C \) are actually uniform with respect to \( p \in M. \)

So on the one hand, for all \( p \in M, \| \dot{x}(t) \|_p \leq \| \dot{x}(0) \|_p(1 + Ct), \) for all \( t \in [0, \frac{\epsilon'}{\| x(0) \|_p}]. \)

More precisely, on this interval, we have

\[ \| \dot{x}(t) \|_p \leq \| \dot{x}(0) \|_p(1 + Ct) = \| \dot{x}(0) \|_p + C \| \dot{x}(0) \|_p \frac{\epsilon'}{\| x(0) \|_p} = \| \dot{x}(0) \|_p + \epsilon' C. \]

On the other hand, for all \( p \in M \) and all \( t \in [0, \frac{\epsilon'}{\| x(0) \|_p}], \)

\[ \sqrt{h_0^p(\dot{x}(t), \dot{x}(t))} \leq \sqrt{h_0^p(\dot{x}(0), \dot{x}(0))} + C \sqrt{\frac{h_0^p(\dot{x}(0), \dot{x}(0))}{h_0^p(\dot{x}(0), \dot{x}(0))}} \]

\[ \leq \sqrt{h_0^p(\dot{x}(0), \dot{x}(0))} + \epsilon' C (1 + \frac{\epsilon' C}{\| x(0) \|_p}). \]

Step 2: Now, let \( \gamma \) be a geodesic in \( M \) (if \( \gamma \) is not tangent to \( F, \)) we parameterize in such a way that \( g(\gamma, V) = 1 \) such that \( \gamma(0) = p_0. \) Write Equation (3) on \((I \times J \times B_{n-1}(0,1), f_{p_0})\) with initial conditions \( x_0(0) = 0, \dot{x}_0(0) \in \mathbb{R}^{n-1}. \) By Step 1, the solution is defined on \([0, s_0], \) with \( s_0 = \inf(\epsilon', \| x_0(0) \|_p). \) By Lemma 5.4, \( \gamma \) is defined on \([0, s_0'], \)

with \( s_0' = \inf([I], s_0), \) Set \( p_1 = \gamma(s_0'). \)

Next, write Equation (3) on \((I \times J \times B_{n-1}(0,1), f_{p_1})\) with initial conditions \( x_1(0) = 0, \)

and \( \dot{x}_1(0) \in \mathbb{R}^{n-1} \) the \( x_1 \)-component of \( \dot{\gamma}(s_0') \) in the local chart. The solution is defined
on $[0, s_1]$, with $s_1 = \inf(\epsilon, \frac{C'}{\|\tau(0)\|_{s_0}^2})$, and by Lemma 5.4, $\gamma$ is defined on $[0, s'_0 + s'_1]$, with $s'_1 = \inf(\|I\|, s_1)$.

Notation: we denote by $\| \cdot \|_1$ the $h^0_{r_i}$-norm, and by $\| \cdot \|'_1$ the $h^1_{r_i}$-norm.

Recall that the fact that $V$ is parallel implies that any two $(n-1)$-submanifolds through $p_1$ tangent to $F$ and transverse to $V$ are locally isometric. It follows that $\| \dot{x}_1(0) \|_1 = \| x_0(s'_0) \|_{I_0}$.

We get a sequence $\gamma(t_n)$, with $t_n = \sum_{i=0}^{n} s'_i$, such that $s'_i$ is

- either equal to $\epsilon$ or to $\|I\|$ for infinitely many $i$,

or

- $s'_i = \frac{\|\tau(0)\|}{\|\tau_i(0)\|}$, for all but a finite number of $i$, with $\| \dot{x}_{i+1}(0) \|_{i+1} = \| \dot{x}_i(s'_i) \|'_i$ for all $i$. Furthermore, on the one hand, $\| \dot{x}_i(s'_i) \|'_i \leq \| \dot{x}(0) \|_i + \epsilon'C$ and on the other hand

$$\| \dot{x}_{i+1}(0) \|_{i+1} \leq \| \dot{x}(s'_i) \|_i + \epsilon'C(1 + \frac{\epsilon'C}{\| \tau_0(0) \|_i})$$

It appears that if the series $\sum_{i=0}^{\epsilon'} \frac{\|\tau_i(0)\|}{\|\tau_i(0)\|}$ is convergent, then the sequence $\| \dot{x}_i(0) \|_i$ is $(2\epsilon C + \epsilon'^2 C^2 K)$-dense in the interval $[\| \dot{x}(0) \|, +\infty[$, hence $\sum_{i=0}^{\epsilon'} \frac{\|\tau_i(0)\|}{\|\tau_i(0)\|} = \infty$, which contradicts the assumption.

So in all cases, $\Sigma s'_i = \infty$, and $\gamma$ is then complete. 

6. PROOF OF THEOREM 1.3, AN ADAPTED CARTAN STRUCTURE

The aim of the present section is to prove the existence of a core as in Theorem 1.3, the algebraic description which completes the proof, will be given in the next section. The exposition is largely inspired by the Sections 4 and 5 of [17], and also by [32] which provides a natural and efficient approach to Gromov’s theory on rigid transformation groups (see [21]) via Cartan connections.

6.1. Null frames. Let $(E, \langle . , . \rangle)$ be a linear Lorentz space of dimension $n + 1$. A frame $(e_0, \ldots, e_n)$ is said to be orthonormal if all the vectors $e_i$ are orthogonal and $1 = -\langle e_0, e_0 \rangle = \langle e_1, e_1 \rangle = \ldots = \langle e_n, e_n \rangle$. This frame is said to be null if all Lorentz products vanish but $1 = \langle e_0, e_n \rangle = \langle e_1, e_1 \rangle = \ldots = \langle e_{n-1}, e_{n-1} \rangle$ (in particular $e_0$ and $e_n$ are isotropic). There is a simple correspondence between orthonormal and null frames, mainly, if $(e_0, \ldots, e_n)$ is orthonormal, then $(\frac{e_0}{\sqrt{2}}, e_2, \ldots, e_{n-1}, \frac{e_n}{\sqrt{2}})$ is a null frame.

Let $(M, g)$ be a Lorentz manifold. Classical connections, as well as Cartan connections, are usually developed on the orthonormal frame bundle, but there is no harm to consider instead null frames. So, let $\tilde{M}$ be this space of null frames (in all tangent spaces of $M$). This is a $O(1, n)$-principal bundle ($n + 1 = \dim M$).

6.2. Adapted Cartan structure. Let $V$ be a (non-singular) null vector field on $M$ and consider $V \subset \tilde{M}$ be those null frames starting with $V$. This is a $H$-principal bundle, where $H$ the subgroup of $O(1, n)$ preserving an isotropic vector.

Consider the Lorentz quadratic form $Q = x_0 x_n + x_1^2 + \ldots + x_{n-1}^2$, for which the canonical basis is null. Elements of the orthogonal group $O(Q)$ preserving $e_0$ have matrices of the form:

$$
\begin{pmatrix}
1 & \lambda_1 & \cdots & \lambda_{n-1} & * \\
0 & \alpha_{1,1} & \cdots & \alpha_{1,n-1} & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \alpha_{n-1,1} & \cdots & \alpha_{n-1,n-1} & * \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}
$$

where $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{R}^{n-1}$ and $\alpha = (\alpha_{ij})_{1 \leq i,j \leq n-1} \in SO(n-1)$, and the $*$-entries are completely determined by $\lambda$ and $\alpha$.

It follows in particular that $H$ is isomorphic to $SO(n-1) \times \mathbb{R}^{n-1}$, which is the Euclidean group, $\text{Euc}_{n-1}$, the (affine) isometry group of the Euclidean space $\mathbb{R}^{n-1}$.
The Levi-Civita connection associated to \( g \) gives rise to a bundle connection on \( \tilde{M} \), i.e. a horizontal \( O(1, n) \)-invariant distribution \( \mathcal{H} \). This yields a “tautological” parallelism of \( T\tilde{M} \). From the point of view of Cartan connections, this parallelism is expressed as a vectorial differential 1-form \( \omega \) which establishes, for any \( \hat{x} \in \tilde{M} \), an isomorphism \( \omega\hat{x} : T_{\hat{x}}\tilde{M} \to \mathfrak{p}\mathfrak{o} \), where \( \mathfrak{p}\mathfrak{o} \) is the Lie algebra of the Poincaré group \( \text{Poi}_{1,n} = O(1, n) \rtimes \mathbb{R}^{n+1} \).

Furthermore, \( \omega \) is \( O(1, n) \)-equivariant, and sends the tangent space of fibers identically to \( \mathfrak{o}(1, n) \subset \mathfrak{p}\mathfrak{o} \). We also have that the horizontal \( \mathcal{H} \) is sent by \( \omega \) to \( \mathbb{R}^{n+1} \), see \([32, 17]\).

**Fact 6.1.** The horizontal \( \mathcal{H} \) is tangent to \( \hat{V} \) iff \( V \) is parallel, iff, \( \omega \) sends \( \hat{V} \) to the Lie subalgebra \( \mathfrak{h} \ltimes \mathbb{R}^{n+1} \subset \mathfrak{p}\mathfrak{o} \).

**Proof.** Let \( X \in \mathcal{H}_u, u \in \hat{V} \), and \( \gamma \) a curve in \( M \) such that \( \gamma(0) = \pi(u) \) and \( \gamma'(0) = d_u\pi(X) \). The horizontal lift of \( \gamma \) starting at \( u \) is the curve \( u(t) \) is a parallel frame field along \( \gamma \) with \( u(0) = u \) and \( u'(0) = X \). Then \( V \) is parallel along \( \gamma \) if and only if the horizontal curve \( u(t) \in \hat{V} \) for every \( t \). It follows that if \( V \) is parallel, then \( X \in T_u\tilde{V} \), for every \( X \in \mathcal{H}_u \). Now, suppose that the horizontal distribution \( \mathcal{H} \) is tangent to \( \hat{V} \), then for every curve \( \gamma \), the horizontal lift \( u(t) \) starting at \( u \in \hat{V} \) is everywhere tangent to \( \hat{V} \), and hence contained in \( \hat{V} \), proving that \( V \) is parallel.

On the other hand, \( \omega^{-1}_u(\{0\} \times \mathbb{R}^{n+1}) \) is a horizontal subspace of \( T_u\tilde{M} \) of dimension \( n + 1 \), hence \( \omega^{-1}_u(\{0\} \times \mathbb{R}^{n+1}) = \mathcal{H}_u \). Knowing that \( \hat{V} \) is an \( H \)-principal bundle on \( M \), it follows that \( \omega \) sends \( \hat{V} \) to \( \mathfrak{h} \ltimes \mathbb{R}^{n+1} \) if and only if \( \mathcal{H} \) is tangent to \( \hat{V} \). \( \square \)

Henceforth, we will assume \( V \) a parallel vector field.

In a different but related meaner, parallel submanifolds of \( \tilde{M} \) are defined in \([32]\) by the fact that \( \omega \) send all their tangent spaces to the same subspace of \( \mathfrak{p}\mathfrak{o} \). This is the case of \( \hat{V} \).

In fact \((\hat{V}, M, \omega|_{\hat{V}})\) is a Cartan geometry, say, it is a sub Cartan geometry of \((\tilde{M}, M, \omega)\) in a natural sense.

6.2.1. **Curvatures.** The curvature of the Cartan connection \( \omega \) is defined by the Cartan-Maurer formula \( \Omega = d\omega + 1/2[\omega, \omega] \). Observe that the curvature of \( \hat{V} \) is just the restriction of that of \( \tilde{M} \).

Using the parallelism, the \( \tilde{M} \) curvature is encoded in a \( O(1, n) \)-equivariant map

\[
\kappa : \tilde{M} \to \mathcal{W}_0 = \text{Hom}(\Lambda^2(\mathfrak{p}\mathfrak{o}/\mathfrak{o}(1, n)), \mathfrak{p}\mathfrak{o}) = \text{Hom}(\Lambda^2\mathbb{R}^{n+1}, \mathfrak{p}\mathfrak{o})
\]

The associated curvature map \( \kappa_{\hat{V}} \) for \( \hat{V} \) is just the restriction of \( \kappa \). It is \( H \)-equivariant and takes values in \( \mathcal{W}_{\hat{V}}^0 = \text{Hom}(\Lambda^2\mathbb{R}^{n+1}, \mathfrak{h} \ltimes \mathbb{R}^{n+1}) \).

Similarly, the parallelism allows one to express the differential \( d\kappa \) as a vectorial map \( D^1\kappa : \tilde{M} \to \text{Hom}(\mathfrak{p}\mathfrak{o}, \mathcal{W}_0) = \mathcal{W}_1 \), and in the same way, we have \( D^i\kappa_{\hat{V}} : \tilde{M} \to \text{Hom}(\mathfrak{h} \ltimes \mathbb{R}^{n+1}, \mathcal{W}_{\hat{V}}^i) = \mathcal{W}_1^\hat{V} \). One defines inductively \( D^i, D^i\kappa, \mathcal{W}_i, \mathcal{W}_i^\hat{V} \), for \( i > 1 \).

To put all these curvatures together, for any \( l \), consider \( \kappa_l : M \to \mathcal{W}_0 \times \mathcal{W}_1 \times \ldots \times \mathcal{W}_l = \mathcal{Y}_l \), and analogously, \( \kappa_{\hat{V}}^l : \tilde{M} \to \mathcal{W}_{\hat{V}}^0 \times \mathcal{W}_{\hat{V}}^1 \times \ldots \times \mathcal{W}_{\hat{V}}^l = \mathcal{Y}_{\hat{V}}^l \).

6.2.2. **Pseudo-group (resp. pseudo-algebra) of local (resp. infinitesimal) isometries.** A local isometry \((M, g)\) is a triplet \((U_1, U_2, f)\) where \( U_1 \) and \( U_2 \) are open in \( M \) and \( f \) is an isometry \( U_1 \to U_2 \) (for the restricted metrics). Here, we will restrict ourselves to local isometries preserving the vector field \( V \). Define a local Killing field as a pair \((U, X)\) where \( U \) is open and \( X \) is a Killing field defined on \( U \). Again, we will assume that \( X \) commutes with \( V \).

Let \( \mathcal{G} \) be the collection of all these local isometries, and \( \mathfrak{g} \) that of all these local Killing fields. Let also \( \mathcal{G}^0 \subset \mathcal{G} \) be the subset of local isometries given by composition of local flows of local Killing fields. It plays the role of the identity component of \( \mathcal{G} \).
It is quite delicate, and this is not our purpose here, to formulate all these concepts (of pseudo-groups and pseudo-Lie algebras), but let us observe that it is straightforward to define orbits of $\hat{\mathcal{G}}$ and also of $\mathcal{G}^0$ (see for instance [24] and related references for a rigorous treatment of pseudo-groups). One can in particular see that $\mathcal{G}^0$ has connected orbits. One can also use $\hat{\mathcal{G}}$ to define locally homogeneous spaces by the fact that $\hat{\mathcal{G}}$ has one orbit.

Observe here that $\hat{\mathcal{G}}$ acts locally on $\hat{M}$, by preserving $\hat{V}$, the Cartan connection $\omega$, its restriction $\omega|_C$, and therefore all the previously defined curvature maps.

6.3. A closed partition $\mathcal{P}_l$ of $M$. Fix $l$ (to be chosen later, big enough), then the $\kappa_l^{\hat{V}}$-levels determine a partition, say $\mathcal{P}_l$, into closed subsets. This projects to a partition $\mathcal{P}_l$ of $M$, but no longer by closed subsets, a priori. If we want the $\mathcal{P}_l$-parts to be closed in $M$, we have to be sure that the $H$-saturation of a $\mathcal{P}_l$-part is closed in $\hat{M}$. But this is nothing but the $\kappa_l^{\hat{V}}$-inverse image of an $H$-orbit in $\mathcal{Y}_l^{\hat{V}}$. This is guaranteed by:

**Fact 6.2.** The $H$-orbits in $\mathcal{Y}_l^{\hat{V}}$ are closed.

**Proof.** Remember $H$ is a semi-direct product $SO(n-1) \ltimes \mathbb{R}^{n-1}$. It is enough to show closeness of $N$-orbits, where $N = \mathbb{R}^{n-1}$ is the nil-radical of $H$. Let us now observe that the $N$-representation in $\mathcal{Y}_l^{\hat{V}}$ is unipotent. This can be shown explicitly, or deduced from the fact that this is nothing but the restriction to $N$ of the $\mathcal{O}(1,n)$-representation in $\mathcal{Y}_l$. Now, $N$ is unipotent in $\mathcal{O}(1,n)$ and hence, because $\mathcal{O}(1,n)$ is semi-simple, for any representation of $\mathcal{O}(1,n)$, $N$ acts unipotently.

Finally, we use Kostant-Rosenlicht theorem which says that unipotent groups have closed orbits ([5, Proposition 4.10]).

6.3.1. $\mathcal{P}_l$ is somewhere a trivial fibration. Let $\hat{U}$ be the (open) subset of $\hat{M}$ where $\hat{f} = \kappa_l^{\hat{V}}$ has a maximal rank. Consider $\hat{f}(\hat{M} - \hat{U})$ as a subset of $\mathcal{Y}_l^{\hat{V}}$. Since $\hat{f}$ is $H$-equivariant and $M$ is compact, $\hat{f}(\hat{M} - \hat{U})$ can be written as the $H$-saturation of a compact subset $K \subset \mathcal{Y}_l^{\hat{V}}$.

Now $\mathcal{Y}_l^{\hat{V}}$ admits a $H$-equivariant stratification $\mathcal{Y}_l^{\hat{V}} = Y_0 \supset Y_1 \supset \ldots \supset Y_k$, such that the $H$-action of each $Z_i = Y_i - Y_{i+1}$ defines a fibration. Let $m$ be the smallest integer such that $A = \hat{f}(\hat{U}) \cap Z_m \neq \emptyset$. In particular $\hat{f}^{-1}(A)$ is open in $\hat{U}$.

Let $K' = K \cap Z_m$. This is closed in $Z_m$ and hence its saturation $B = H.K$ is also closed in $Z_m$, since the $H$-orbits determine a fibration. By Sard Theorem, $B$ has a vanishing Hausdorff measure of exponent the maximal rank of $\hat{f}$. In particular $B \cap A$ has empty interior in $A$, and thus $\hat{U}' = \hat{f}^{-1}(A - B)$ is open, and is mapped by $\hat{f}$ into the regular values set of $\hat{f}$.

Now, saturate everything by the $H$-action, that is instead of considering levels $\hat{f}^{-1}(z)$, consider inverse images $\hat{f}^{-1}(Hz)$. Since $Hz$ is closed in $\mathcal{Y}_l^{\hat{V}}$, $\hat{f}^{-1}(Hz)$ projects onto closed submanifolds in $M$.

Project everything on $M$ and get an open set $U' \subset M$ with a submersion $f : U' \to Z_m/H$. Thus, on $U'$, we have a partition into closed (in $M$) submanifolds (not necessarily connected) given by the levels of a global submersion.

Let $K$ be the so defined foliation, i.e. with leaves the connected components of the $f$-levels.

Let $\tau$ be a small transversal to a leaf $C$, so that $f|_{\tau}$ is injective. Let $U''$ be the saturation of $\tau$ by $K$. Now, $f|_{U''}$ is a submersion with connected levels. We already know that these levels are compact. It turns out that this implies that $f(U'') : U'' \to f(U'')$ is a bundle map (see [40]). In conclusion, we have proved:

**Fact 6.3.** There is an open subset $B \subset M$ where the $\mathcal{P}_l$-classes are closed submanifolds of $M$ and form a trivial fibration.
6.4. Reduction. Let us now take $l = \dim \mathbb{O}(1, n)$ and denote $\mathcal{P}_1$ simply by $\mathcal{P}$ (actually it is enough to take $l = \dim H = \dim SO(n - 1) + (n - 1)$). From [32], we have

**Proposition 6.4.** [32] On the open set $B$ defined in Fact 6.3, the $\mathcal{G}^0$-orbits coincide with the fibers. In particular each fiber is locally homogeneous.

Furthermore, a finite index sub-pseudo-group of $\mathcal{G}$ preserves $B$ and has the same orbits as $\mathcal{G}^0$.

6.4.1. Existence of a core $N$, Proof of Theorem 1.3. If some fiber in Proposition 6.4 is somewhere non-degenerate, i.e. the restriction of the metric to it is of Lorentzian type, then it is everywhere non-degenerate by local-homogeneity. Take $N$ to be this fiber.

If all fibers are degenerate, then, consider a transversal (to the fibration) curve $c$, and let $N$ be its saturation (by the fibration). So $N$ is a Lorentzian manifold with boundary, diffeomorphic to a product $F \times [0, 1]$. The local isometry pseudo-group $\mathcal{G}^0$ preserves and acts transitively on the lightlike geodesic factors $F \times \{u\}$.

7. Further results on the core $N$

Our goal is to study the $V$-dynamics on $M$, by first replacing $M$ by $M^*$ (as introduced in par. 2.2), and then replacing $M$ (which is in fact $M^*$) by its core $N$.

This section is devoted to justify these reductions and to prove the global algebraic structure of the core $N$ stated in Theorem 1.3.

7.1. Passing to $M^*$ and getting a unipotent isotropy. Consider $M^*$ the Steifel manifold of orthonormal frames of $V^\perp / \mathbb{R}V$, as described in Section 2.2. We can do for it the same analysis as that for $M$ from the point of view of its pseudo-Lie algebra of local Killing fields. We will however here just lift the $\mathfrak{g}$-action to it (where $\mathfrak{g}$ is the pseudo-Lie algebra of local Killing fields of $M$).

**Fact 7.1.** Let $B^*$ be the inverse image of $B$ (defined in Proposition 6.4) in $M^*$ endowed with the local $\mathcal{G}^0$-action. Then the $\mathcal{G}^0$ orbits in $B^*$ are closed submanifolds in $M^*$, and so they form a fibration.

**Proof.** For $x \in M$, let $\mathcal{G}^0_x$ denote its stabilizer in $\mathcal{G}^0$. Its (faithful) representation in $T_x M$ makes it a subgroup $I_x$ of $H = SO(n - 1) \ltimes \mathbb{R}^{n-1}$. A standard fact from Gromov’s rigid transformation groups Theory [21], or its Cartan connections variant states that $I_x$ is algebraic. From this, one infers that, up to a finite index, $I_x$ has the form $K \ltimes \mathbb{R}^k$, where $K$ is a closed subgroup of $SO(n - 1)$. Next, one can check that if $x^* \in M^*$ projects on $x$, then the $\mathcal{G}^0_x x^*$ fibers over $\mathcal{G}^0_x x$ with fiber $K$. This implies in particular that the $\mathcal{G}^0$ orbits on $M^*$ are closed. □

The interest of working on $M^*$ lies in the following fact brought up in par. 2.2:

**Fact 7.2.** The $\mathcal{G}^0$-action on $M^*$ has unipotent isotropy.

**Proof.** The elements of the orthogonal group preserving an isotropic vector are given by $SO(n - 1) \ltimes \mathbb{R}^{n-1}$, and it follows from Corollary 2.9 that the isotropy group of the $\mathcal{G}^0_x$-action on $M^*$ lies in the $\mathbb{R}^{n-1}$ factor, which is unipotent in $O(1, n)$. □

7.2. Equicontinuity on $N$ implies equicontinuity on $M$.

**Fact 7.3.** If the flow of $V$ acts equicontinuously on the core $N$, then it acts equicontinuously on $M$.

**Proof.** The fact is actually true for any isometric flow $\phi^t$ acting on a compact Lorentz manifold $(M, g)$: if it preserves a closed Lorentzian submanifold $N$ and acts equicontinuously on it, then it acts equicontinuously on $M$. Indeed, the hypothesis means that $\phi^t$ preserves a Riemannian metric $h$ on $TN$. Construct a Riemannian metric $h'$ on $TM$ along $N$ given as
\[ h' = h \oplus g_{T^\perp N}. \] Observe here that we used the fact that \( g \) is spacelike on \( T^\perp N \). Now, it is known that for flows preserving affine connections, equicontinuity along a closed invariant subset (not necessarily a submanifold) implies everywhere equicontinuity (see for instance [42]).

\[ \square \]

**Fact 7.4.** In the local co-homogeneity one case, if the flow of \( V \) acts equicontinuously on a fiber of \( N \), then it acts equicontinuously on \( N \) (and hence on \( M \)).

**Proof.** Let \( F \) be such a fiber. If \( \phi^t \) preserves a Riemannian metric \( h \), then it preserves \( E \) the orthogonal of \( V \) with respect to \( h \). This \( E \) is spacelike (with respect to \( g \)). Consider \( E^\perp \), its orthogonal with respect to \( g \). It has dimension 2 and contains \( V \). There is a well defined isotropic vector field \( U \) in \( E^\perp \) such that \( g(U, V) = 1 \). Since \( \phi^t \) preserves \( V \), it preserves \( U \) too. Thus, \( \phi^t \) acts equicontinuously on \( TN|_F \). As in the previous proof, this implies that \( \phi^t \) is equicontinuous on \( N \). \[ \square \]

### 7.3. Completeness in the universal cover \( \tilde{M} \).

#### 7.3.1. Summarizing up.

The previous developments justify to replace \( M \) by \( M^* \) and then \( M^* \) by its core, and so, henceforth, we will assume that our initial Brinkmann space \((M, g, V)\) is such that:

- Either \( M \) is locally homogeneous or has a local co-homogeneity one type.
- The isotropy is unipotent.

The remaining part of the section is devoted to the proof of the following result.

**Proposition 7.5.** Let \( G \) be the identity component of the isometry group \( \text{Isom}(\tilde{M}, \tilde{g}, \tilde{V}) \) of the universal cover, endowed with the lift of the Brinkmann structure. Then, either \( M \) has the form \( \Gamma \setminus G/I \), or it is a trivial fibration \( M \to [0, 1] \), and each fiber has the form \( \Gamma \setminus G/I_u \), where \( I_u \) depends on the parameter \( u \in [0, 1] \).

Remember the notations of the objects in \((M, g)\): \( V \) the Brinkmann parallel vector field, \( V \) the 1-foliation that it determines, \( F \) the codimension 1 lightlike geodesic foliation tangent to \( V^\perp \). We denote by \( \tilde{V}, \tilde{V}^\perp, \tilde{F} \) the lift of the corresponding objects to \( \tilde{M} \). If \( F \) is a leaf of \( F \), then \( \tilde{F} \) is a connected component of its lift in \( \tilde{M} \).

The universal cover \( \tilde{M} \) is either locally homogeneous or topologically of the form \( \tilde{F} \times [0, 1] \), and each \( \tilde{F} \times \{t\} \) is locally homogeneous, and obviously simply connected. In both cases, the whole \( \tilde{M} \) or the factors \( \tilde{F} \times \{t\} \) (real) analytic. A classical result [31] says, in this case, that any locally defined vector field \( X \) on an open subset \( U \) of \( M \) extends coherently to \( \tilde{M} \). So, the (global) Killing algebra \( \mathfrak{g} \) of \( \tilde{M} \) acts either transitively or with codimension one orbits.

Let us prove that this infinitesimal \( \mathfrak{g} \)-action is complete, i.e. that any Killing field \( X \in \mathfrak{g} \) has a complete flow, or alternatively, the simply connected “abstract” group \( G \) with Lie algebra \( \mathfrak{g} \) acts (globally) on \( \tilde{M} \).

Let us start treating the co-homogeneity one case, and show afterwards how to adapt the proof to the homogeneous case.

#### 7.3.2. The co-homogeneity one case.

**Fact 7.6.** Let \( \tilde{p} \in \tilde{F} \) and \( i \subseteq \mathfrak{g} \) its stabilizer algebra. Then, \( i \) is abelian and acts unipotently on \( T_{\tilde{p}} \tilde{M} \).

**Proof.** The group \( I \) determined by \( i \) is contained in the subgroup of the orthogonal group \((T_{\tilde{p}} \tilde{M}, \tilde{g}_{\tilde{p}})\) preserving \( \tilde{V}(\tilde{p}) \) and acting trivially on \( \tilde{V}(\tilde{p})^\perp / \mathbb{R} \tilde{V}(\tilde{p}) \) since preserving a parallelism. This group is abelian and acts unipotently. \[ \square \]

**Fact 7.7.** Let \( \mathfrak{j} \) be the Lie subalgebra preserving the 1-leaf \( \tilde{V}(\tilde{p}) \). Then \( \mathfrak{j} = \mathfrak{i} \oplus \mathfrak{z} \), where \( \mathfrak{z} = \mathbb{R} \tilde{V} \) (which is contained in the center of \( \mathfrak{g} \)). Furthermore, \( \mathfrak{j} \) is an ideal of \( \mathfrak{g} \).
Proof. $j$ contains $i \oplus j$ and $i$ has codimension 1 in $j$, so we have equality. The Lie algebra $\mathfrak{g}$ acts, locally, on the quotient space $\tilde{F}/\tilde{V}$ by preserving a parallelism. The Lie algebra $j$ acts by fixing the point $\tilde{V}(\tilde{p})$ of $\tilde{F}/\tilde{V}$ and hence acts trivially on it (an automorphism of a parallelism is trivial if it has a fixed point). It follows that $j$ is exactly the kernel of the $\mathfrak{g}$-action on $\tilde{F}/\tilde{V}$, and hence it is normal. □

Corollary 7.8. $j$ is contained in the nil-radical of $\mathfrak{g}$.

Proof. Because $j$ is an abelian ideal. □

Fact 7.9. Let $G$ be the simply connected Lie group determined by $\mathfrak{g}$, $I$ and $J$ its subgroups tangent to $i$ and $j$ respectively. Then $I$ and $J$ are closed in $G$ (and are simply connected). Furthermore, $\tilde{F}$ is isomorphic to $G/I$.

Proof. This follows from the completeness of $F$ and thus $\tilde{F}$ (by Theorem 1.2 or even its partial version in Section 3). □

7.3.3. The locally homogeneous case. Now, in the locally homogeneous case, consider $\mathfrak{h}$ the codimension 1 subalgebra preserving a leaf $\tilde{F}$ of $\tilde{F}$. It preserves in fact individually these leaves, and is thus an ideal. Its associated subgroup $H$ in $G$ is closed, and as in the previous case, the isotropy group is closed in $H$ and hence in $G$. As in the previous case, we can take the quotient $G/I$ and use completeness of $M$ to deduce that $\tilde{M}$ is isomorphic to $G/I$. □

8. The degenerate case, the $\nabla$-foliation on a leaf $F$

We investigate here the degenerate case, that is where $M$ has local co-homogeneity one. Thus, in particular, $F$ has all its leaves compact.

In all the section, we fix a leaf $F$ of $\mathcal{F}$.

8.1. The cocktail of geometries on $F$. We will exploit existence of many compatible geometric structures on the compact manifold $F$, which we list here:

- A lightlike metric induced from $(M, g)$.
- A connection induced from $M$, since $F$ is geodesic.
- The parallel vector field $V$.
- The 1-foliation $\mathcal{V}$ determined by $V$. It is transversally Riemannian, as it is the case of the characteristic (null) foliation on any lightlike geodesic submanifold in a Lorentz manifold (see for instance [45]).
- The (pseudo) Lie algebra $\mathfrak{g}$ acting transitively on $F$ by preserving all these structures.
- It is a standard technique in transversally Riemannian foliation theory to lift the foliation to the Steifel space of transversal orthonormal frames, in order to get a transversally parallelizable foliation. In our situation, we already replaced $M$ by $M^*$, without loosing its Brinkmann nature, essentially for this aim (par. 7.3.1). So the $\mathcal{V}$-foliation on $F$ is transversally parallelizable.

8.2. Dynamics of the closure foliation $(\overline{F, \mathcal{V}})$, the holonomy subgroup $\Gamma^0$. One fundamental result in Riemannian foliation theory is that taking the closure of a Riemannian foliation $\mathcal{V}$, gives rise to a singular Riemannian foliation $\overline{\mathcal{V}}$, which is actually regular in the transversally parallelizable case [30]. Carrière’s Theorem, [10, 11], applies since $\dim \mathcal{V} = 1$, and says that the $\overline{\mathcal{V}}$-leaves are tori. The goal of this section is to show in our rich situation that, essentially, $\overline{\mathcal{V}}$ is given by an isometric action of a torus.
8.2.1. Notations. Fix \( p \in F \in M \), so \( F \) is the \( F(p) \)-leaf. Consider a lift \( \tilde{p} \in \tilde{M} \) of \( p \) and let \( \tilde{F} = \tilde{F}(\tilde{p}) \).

Consider now the following objects:

- The foliation \( \tilde{\mathcal{V}} \) of \( F \) given by the closure of the \( \mathcal{V} \)-leaves, and let \( \tilde{\mathcal{V}} \) be its lift to \( \tilde{F} \).
- \( \mathcal{I} \), the stabilizer of \( \tilde{p} \) in \( G \), then \( \tilde{F} \sim G/I \).
- \( \mathcal{J} \), the stabilizer of \( \tilde{\mathcal{V}}(\tilde{p}) \) in \( G \). Remember that \( J \) is normal in \( G \) and is furthermore abelian and contained in the nil-radical of \( G \) (par. 7.3.2).
- \( \pi : G \to Q = G/J \), the quotient map. In fact, \( Q \) is identified to the quotient space \( \tilde{F}/\tilde{\mathcal{V}} \).
- \( \Gamma = \pi_1(F) \subset G \). This also equals \( \pi_1(M) \) since \( M \) is a product of \( F \) by an interval (Proposition 7.5).
- \( L \), the closure in \( Q \) of \( \pi(\Gamma) \), and \( L^0 \) its identity component.

Set \( P = \pi^{-1}(L) \) and \( P^0 = \pi^{-1}(L^0) \). Then \( P^0 \) is the identity component of \( P \), and it contains \( J \). Furthermore, \( \pi(\Gamma) \cap L^0 \) is dense in \( L^0 \).

\( \Gamma^0 = \Gamma \cap P^0 \). As \( L^0 \) is normal in \( L \), \( P^0 \) and \( \Gamma^0 \) are normal in \( P \) and \( \Gamma \) respectively.

8.2.2. Algebraic description. Let us give an algebraic description, i.e. by means of \( G \), of all the present quantities:

- First, \( \tilde{F} = G/I \).
- For \( x \in G \), the \( \mathcal{V} \)-leaf of \( xI \) is \( ZxI \), where \( Z \) is the simply connected Lie group determined by \( J = \mathbb{R} \), and we have \( ZxI = xZI = xJ = Jx \). So it corresponds to the subset \( Jx \) of \( G \).
- The quotient leaf-space \( \tilde{F}/\tilde{\mathcal{V}} = G/J = Q \).
- The orbit \( P(xI) \) corresponds to the subset of \( G \): \( P \cdot xI = P(xIx^{-1})x = P \cdot x \), since \( xIx^{-1} \subset JxJ^{-1} = J \subset P \).
- Similarly \( P^0 \cdot xI = P^0x \subset G \).
- The \( J \)-saturation of a leaf \( \tilde{\mathcal{V}}(xI) \) corresponds to \( \Gamma JxI = \Gamma Jx \), and its closure is \( \Gamma Jx = \Gamma Jx = P \cdot x \).
- The connected component of \( xI \) in \( \Gamma(\mathcal{V}(xI)) \) corresponds to \( P^0 \cdot x = \Gamma^0Jx \). In other words, the connected component of the lift in \( \tilde{F} \) of the closure of any \( \mathcal{V} \)-leaf in \( F \), has the form \( P^0 \cdot x \) (as a subset in \( G \) or as a \( P \)-orbit of \( xI \in G/I \)). Therefore, the closure foliation \( \tilde{\mathcal{V}} \) corresponds to the \( P^0 \)-action.

8.2.3. Structure of \( L^0 \). Observe that: \( L^0 = 1 \iff P^0 = J \iff \Gamma \cap J \neq 1 \), and all these are equivalent to the fact that all \( \mathcal{V} \) leaves in \( F \) are periodic, with the same period. This periodic case is trivial with respect to our considerations here and so henceforth, we will assume \( L^0 \neq 1 \).

**Fact 8.1.** \( \Gamma^0 \) is abelian. Furthermore, assume we are not in the periodic case, then \( \pi(\Gamma^0) \) is dense in \( L^0 \), and \( \pi : P^0 \to L^0 \) is injective on \( \Gamma^0 \), and hence \( L^0 \) is abelian.

**Proof.** This follows from Carrière’s Theorem [10, 11] on closure of orbits of transversally Riemannian flows (i.e. foliations of dimension 1). With our notation, it says that a leaf of \( \mathcal{V} \) is a torus, on which \( \mathcal{V} \) is diffeomorphic to a minimal (i.e. having dense leaves) linear foliation (of dimension 1). We are here in a transversally parallelizable case, where all \( \mathcal{V} \)-leaves are diffeomorphic. They are all given by \( P^0 \)-orbits. Such an orbit is of the form \( \Gamma^0 \setminus P^0/I \). Since the \( \mathcal{V} \)-leaf is a torus, and \( \Gamma^0 \) is the deck group of its cover \( P^0/I \), then \( \Gamma^0 \) is abelian. Now, \( L^0 \) is the quotient space of the \( \mathcal{V} \)-foliation when lifted to the universal cover. Since \( \mathcal{V} \) is minimal (in the \( \mathcal{V} \)-leaf), then \( \Gamma^0 \) is dense in \( L^0 \).

Finally, besides the period case, if we assume \( \Gamma^0 \cap J = 1 \), then \( \Gamma^0 \) injects in \( L^0 \), and \( L^0 \) is therefore abelian. 

Remark 8.2. The quotient space \( \tilde{F}/\tilde{V} \) is identified to the quotient group \( Q = G/J \). Using the language of transversally Riemannian foliation [30], one says that \( V \) is a transversally Lie foliation with structural group \( Q \), which means that \( V \) has a transversally geometric structure modeled on \((Q,Q)\) (where the group \( Q \) acts by left translation on the space \( Q \)).

In general, one reduces the study of transversally Riemannian foliations, first to the transversally parallelizable case. Then, for closures of leaves, or say when there is a dense leaf, one proves that the transversally parallelizable foliation is indeed a transversally Lie foliation. Actually, Carrière’s Theorem [10, 11] was proved for transversally Lie foliations.

Here, we have a richer situation, where the ambient manifold itself \( F \) has a (local) geometric structure of type \((G,G/I)\), which induces a transversally Lie foliation structure of type \((Q,Q)\) for \( V \).

8.3. The syndetic hull \( H(G^0, P^0) \).

Proposition 8.3. Let \( C^0 \) be the identity component of the centralizer \( Z(G^0) \) of \( G^0 \) in \( P^0 \). Then:
- \( C^0 \) is transversal to \( J \) (in \( P^0 \)) and hence acts transitively on the \( \tilde{V} \)-leaves,
- \( C^0 \) is abelian and contains \( \Gamma^0 \).

Definition 8.4 (Syndetic hull). As \( C^0 \) is a connected abelian Lie group and \( \Gamma^0 \subset C^0 \) is discrete, there exists a unique subgroup of \( C^0 \) containing \( \Gamma^0 \) and in which \( \Gamma^0 \) is a lattice. Since it was defined by means of \( P^0 \), it will be denoted by \( H(G^0, P^0) \) and called the syndetic hull of \( G^0 \) in \( P^0 \).

Remark 8.5. In the nilpotent case, there is a construction of a (unique) Malcev envelope which associates to a discrete subgroup (say of finite type) a syndetic hull where it is a lattice [34]. In our case here, \( P^0 \) is solvable, a situation where the construction does not extend. Actually, contrary to “easy life” in semi-simple or nilpotent groups, in solvable groups and worse, in semi-direct products of type compact by solvable, there is no way to fill in discrete sub-groups, even Borel density fails for lattices (compare with [41])!

Proof. We have an exact sequence \( 1 \rightarrow J \rightarrow P^0 \rightarrow L^0 \rightarrow 1 \). Since \( J \) is abelian, the \( P^0 \)-action by conjugacy on \( J \) reduces to a \( L^0 \)-action (on \( J \)). For \( l \in L^0 \), let \( Z_J(l) \) be the fixed point set of its action on \( J \). The trivial factor of the \( L^0 \) representation is thus \( T = \cap_{l \in L^0} Z_J(l) \). So \( T \) is the intersection of \( J \) with the center of \( P^0 \).

For a generic \( l \), \( Z_J(l) = T \). More precisely, there is a finite set of (proper and closed) subgroups \( L_{l1}^0, \ldots, L_{lk}^0 \), \( k \) such that if \( l \notin \cap l_i \), then \( Z_J(l) = T \).

Similarly, one defines \( Z_J(p) \) for \( p \in P^0 \), specially for \( p = \gamma \in G^0 \). Since \( \gamma(G^0) \) is dense in \( L^0 \), for \( \gamma \) “generic”, \( Z_J(\gamma) = T \).

By taking the quotient \( P^0/T \), we can assume \( T = 1 \) (observe, however, that by doing this, i.e. passing to the quotient space, \( G^0 \) could become non-discrete).

Let \( \gamma \) be generic and assume it belongs to two one parameter groups \( h_1(t) \) and \( h_2(t) \) having the same projection in \( L^0 \), that is \( \pi(h_1(t)) = \pi(h_2(t)) \). On one hand, \( \pi(h_1(t)h_2^{-1}(t)) = 1 \) and thus belongs to \( J \), and on the other hand, \( h_1(t)h_2^{-1}(t) \) commutes with \( \gamma \). Hence \( h_1(t) = h_2(t) \).

Let now \( c \in P^0 \) commuting with \( \gamma \). For any one parameter group \( h_1(t) \) containing \( \gamma \), \( h_2(t) = Ad_c(h_1(t)) \) is another one having the same projection in \( L^0 \) (since \( L^0 \) is abelian and hence the \( Ad_c \) -action on it is trivial). This implies that \( h_1 = h_2 \), that is \( c \) commutes with any element \( h_1(t) \).

From all this, we infer that if \( \gamma \) and \( \gamma' \) are two generic elements of \( G^0 \), then any one parameter groups \( h(t) \) and \( h'(t) \) containing \( \gamma \) and \( \gamma' \), respectively, commute.

Regarding existence of such one parameter groups \( h(t) \), let \( K \) be a one parameter group in (the abelian) \( L^0 \) containing \( \pi(\gamma) \), and consider \( \pi^{-1}(K) \), which is connected (since \( J \) is connected), and contains \( \gamma \). So \( \pi^{-1}(K) \) is a semi-direct product \( \mathbb{R} \ltimes \mathbb{R}^k \) (\( L^0 \cong \mathbb{R}^k \)). Let us refer to Lemma 8.6 below, for the proof that generic elements of such a semi-direct product can be reached by one parameter groups.
Now, let $\gamma_1, \ldots, \gamma_d$ be generic elements of $\Gamma^0$ contained in one parameter groups $h_1(t), \ldots, h_d(t)$, such that the derivatives $h'_1(0), \ldots, h'_d(0)$ generate a subspace of the Lie algebra of $\Gamma^0$ of dimension $d = \dim L^0$. They determine an abelian group $A$ transversal to $J$ and centralizing $\Gamma^0$.

Let us prove that $A$ contains $\Gamma^0$. To be precise on the significance of genericity of elements of $\Gamma^0$, let us denote by $U$ the open dense set $L^0 - \cup_i L^1_i$, where the $L^1_i$ are the subgroups defined above (where the centralizer does not achieve the minimal dimension). Then, $\gamma$ is generic if $\pi(\gamma) \in U$.

Therefore, $B = \pi^{-1}(U) \cap \Gamma^0$ is contained in $A$.

The subgroup $D$ generated by $B$ is contained in $A \cap \Gamma^0$ and the projection $\pi$ sends $D$ to the subgroup generated by $\pi(\Gamma^0) \cap U$. But the last subgroup equals $\pi(\Gamma^0)$, since $\pi(\Gamma^0)$ is dense (in $L^0$) and $U$ is open (say, if $X$ is a dense subgroup of $\mathbb{R}^d$ and $U$ is open in $\mathbb{R}^d$), then for $a, b \in X \cap U$ close, the translated $X \cap U - b$ is a neighbourhood of 0 in $X$, and hence $X \cap U$ generates $X$).

It follows that $\pi$ sends $D$ surjectively onto $\pi(\Gamma^0)$. But $\pi$ maps bijectively $\Gamma^0$ to $\pi(\Gamma^0)$, and $D \subset \Gamma^0$ which implies that $D = \Gamma^0$, that is $\Gamma^0 \subset A$.

Now, if the trivial factor $T$ was not trivial, then $C^0$, the identity component of the centralizer of $\Gamma^0$ is exactly $AT$. It satisfies all the claimed properties. $\square$

In order to finish the proof, we need:

**Lemma 8.6.** Let $K$ be a semi-direct product $\mathbb{R} \ltimes \mathbb{R}^d$. Then, any generic element of $S$ belongs to a one parameter group. More precisely, assume that $\mathbb{R}$ acts on $\mathbb{R}^d$ via a representation $t \rightarrow e^{ta}$ with $a \in \mathfrak{gl}(d, \mathbb{R})$. Let $\lambda_1, \ldots, \lambda_l$ be the purely imaginary eigenvalues of $a$. If $t \notin \cup_i \frac{2\pi}{\lambda_i} \sqrt{-1}\mathbb{Z}$, then, for any $v \in \mathbb{R}^d$, $(t, v)$ belongs to some one parameter group of $K$.

**Proof.** The Lie algebra $\mathfrak{t}$ of $K$, is generated by one element $Y$ and $\mathbb{R}^d$, with non-vanishing brackets $[Y, u] = a(u)$, for $u \in \mathbb{R}^d$. It embeds in $\mathfrak{aff}(\mathbb{R}^d) = \mathfrak{gl}(d, \mathbb{R}) \ltimes \mathbb{R}^d$, the Lie algebra of the affine group $\text{Aff}(\mathbb{R}^d)$, by sending $Y$ to $a$ and $\mathbb{R}^d$ to $\mathbb{R}^d$. Let $K'$ be the Lie subgroup of $\text{Aff}(\mathbb{R}^d)$ determined by this embedding. Then, $K$ is isomorphic to $K'$, unless $e^{ta}$ is periodic: there exists $t_0 \neq 0$ such that $e^{ta} = 1$. In this case, $K$ will be the (cyclic) universal cover of $K'$. Such $K$ is a non-linear group, i.e. cannot be injectively embedded in any $\text{GL}(n, \mathbb{R})$. Their prototype is the Euclidean group $\text{Euc}_2 = \text{SO}(2) \ltimes \mathbb{R}^2$. Actually, once treating $K'$, we can lift to $K$, and hence we will assume we are not in this periodic case.

Consider now the map $\phi : (t, x) \in \mathbb{R} \times \mathbb{R}^d \rightarrow \phi^t(x) = e^{ta}x + (e^{ta}u - u) \in \mathbb{R}^d$. One checks that $t \rightarrow \phi^t$ is a one parameter group in $\text{Aff}(\mathbb{R}^d)$. Its infinitesimal generator is the vector field $x \rightarrow X(a) = a(x) + a(u)$. Accordingly, the vector field $x \rightarrow a(x) + u$ has a flow $(t, x) \rightarrow e^{ta}x + a^{-1}(e^{ta}u - u)$, where here $a^{-1}(e^{ta} - 1)$ is understood to be equal to $B_a(t) = 1 + \frac{ta}{2!} + \frac{ta^2}{3!} + \cdots$. Let us see when $B_a(t)$ is not surjective. By means of a Jordan decomposition, we reduce the question to the case where $a$ has a unique (complex) eigenvalue $\lambda$, that is $(a - \lambda 1)d = 0$.

First, if $\lambda = 0$, then $a$ is nilpotent and $B_a(t)$ is surjective for any $t$.

If $\lambda \neq 0$, then $a^{-1}$ exists and $B_a(t) = a^{-1}(e^{ta} - 1)$. Thus if $\det B_a(t) = 0$, then $e^{\lambda t} = 1$; in particular, $\lambda \in \sqrt{-1}\mathbb{R}$ and $t \in \frac{2\pi}{\lambda} \sqrt{-1}\mathbb{Z}$.

In sum, let $\lambda_1, \ldots, \lambda_l$ be the purely imaginary eigenvalues of $a$. If $t \notin \cup_i \frac{2\pi}{\lambda_i} \sqrt{-1}\mathbb{Z}$, then, for any $v \in \mathbb{R}^d$, $(t, v)$ belongs to some one parameter group. $\square$

### 8.4. $\Gamma^0$ seen in the identity component of its centralizer in $G$.

**Proposition 8.7.** Remember the definition of $C^0$ (in Proposition 8.3) and that of the syn
detic hull $H(\Gamma^0, \mathbb{R}^d)$. Define the following subgroups of $G$:

- $C^{00}$: the identity component of the centralizer of $\Gamma^0$ in $G$.
- $C^{000}$: the identity component of the center of $C^{00}$. 


Then \( C^{000} \) contains \( H(\Gamma^0, P^0) \) and in particular acts transitively on the \( \tilde{V} \)-leaves.

In fact, \( H(\Gamma^0, P^0) \) is also the syndetic hull of \( \Gamma^0 \) in \( C^{000} \).

**Proof.** If \( g \in G \) centralizes \( \Gamma^0 \), then it normalizes \( J\Gamma^0 \) (since \( J \) is normal) and hence normalizes \( P^0 = J\Gamma^0 \). It follows that the centralizer of \( \Gamma^0 \), and hence also its identity component \( C_{\Gamma^0} \), normalizes \( P^0 \). But then, \( C^{000} \) preserves \( C_{\Gamma^0} \) too, since the latter is defined as the identity component of \( Z(\Gamma^0) \) in \( P^0 \). It also preserves the syndetic hull \( H(\Gamma^0, P^0) \).

But, since it centralizes the lattice \( \Gamma^0 \) in \( H(\Gamma^0, P^0) \), \( C^{000} \) acts trivially on \( H(\Gamma^0, P^0) \). That is \( H(\Gamma^0, P^0) \subset C^{000} \) and hence \( H(\Gamma^0, P^0) = H(\Gamma^0, C^{000}) \).

\[ \square \]

9. **End of the Proof of Theorem 1.4**

So far, we fixed a leaf \( F \) and considered its \( \mathcal{V} \) and \( \tilde{\mathcal{V}} \)-foliations. Now, we consider a small curve \( p : u \in [0, 1] \to p_u \) transversal to \( F \). This leaf \( F_u = F(p_u) \) comes with its associated foliation \( \tilde{\mathcal{V}} \). There is, a priori, no obvious continuity or even semi-continuity of \( \tilde{\mathcal{V}} \). In other words, \( \mathcal{V} \) is a foliation on each leaf but not a foliation of \( F \).

We have in particular groups \( I_u \), the stabilizers of \( \tilde{p}_u \), and \( J_u \) the stabilizers of \( \tilde{V}(\tilde{p}_u) \).

These two groups depend continuously on \( u \).

There is also \( P^0_u \) and \( \Gamma^0_u \) associated to \( F_u \). Again, a priori, they satisfy no obvious continuity, or even semi-continuity on \( u \).

**Fact 9.1.** Let \( u_1, u_2 \) be two parameters \( \in [0, 1] \). \( P^0_{u_1}, P^0_{u_2}, \Gamma^0_{u_1} \) and \( \Gamma^0_{u_2} \) their associated groups. If \( \Gamma^0_{u_1} = \Gamma^0_{u_2} = \Gamma^0 \), then \( \Gamma^0 \) has the same syndetic hull in \( P^0_{u_1} \) and \( P^0_{u_2} \). In particular this syndetic hull acts transitively on the \( \tilde{\mathcal{V}} \)-leaves of both \( F_{u_1} \) and \( F_{u_2} \).

**Proof.** Apply Proposition 8.7 and observe that \( C^{000} \) is the same for \( u_1 \) and \( u_2 \) (it is defined by means of \( \Gamma^0 \) only).

\[ \square \]

**Fact 9.2.** We can assume that for a dense set \( D \subset [0, 1] \) of parameters \( u \), \( \Gamma^0_u \) is a constant \( \Gamma^0 \).

**Proof.** Let \( B(e, n) \) be the ball of radius \( n \) and centered at the neutral element of \( \Gamma = \pi_1(M) \), with respect to a word metric given by some generating set. Let \( X_n = \{ u \in [0, 1] \mid \Gamma^n_u \cap B(e, n) \) generates \( \Gamma^0_u \} \). For \( n \in \mathbb{N} \) fixed, the subsets \( \Gamma^n_u \cap B(e, n), u \in [0, 1] \), of \( B(e, n) \) are in finite number. By Baire’s Theorem, there exists \( n_0 \) such that \( X_{n_0} \) contains a non-trivial interval of parameters. The map \( u \in [0, 1] \to \Gamma^n_u \cap B(e, n_0) \), from \([0, 1]\) into subsets of \( B(e, n_0) \), has a finite image. So there is a level whose closure contains an interval. We will assume it is \([0, 1]\) itself.

\[ \square \]

**Corollary 9.3.** All the syndetic hulls \( H(\Gamma^0, P_u) \) coincide, for \( u \in D \subset [0, 1] \) a dense set of parameters, say \( H(\Gamma^0, P_u) = H \), for \( u \in D \). In particular \( H \) acts transitively on \( \tilde{\mathcal{V}} \)-leaves of all points \( \tilde{p}_u \) \( u \in [0, 1] \).

**Proof.** The previous fact implies constancy \( H = H(\Gamma^0, P_u) \), for \( u \) is a dense subset \( D \subset [0, 1] \). It then follows that \( H \) acts transitively on a dense set of \( \tilde{\mathcal{V}} \)-leaves. This extends to all leaves.

\[ \square \]

**9.1. End of the proof of Theorem 1.4.** Consider the cover \( M' = \tilde{M} / \Gamma^0 \). Then \( H \) acts on it since it centralizes \( \Gamma^0 \). The \( \mathcal{F}' \)-leaves of \( M' \) have the form \( \Gamma^0 \setminus G/I \). As a class in \( G \), a point \( \Gamma^0 x I \) has an \( H \)-orbit \( H \Gamma^0 x I = H x I \). This is a torus \( \Gamma^0 \setminus H \) in \( M' \). Taking \( H Z \) (which is still abelian since \( Z \) is contained in the center of \( G \)) instead of \( H \) if necessary, we may assume that \( Z \subset H \).

Let \( \tau \) be the image of the transversal curve \( u \to p_u \) considered above, and \( \tau' \) a lift in \( M' \). The orbit \( T' = H \tau' \) is topologically a product of a torus by an interval. It embeds to a submanifold \( T \) in \( M \). It is Lorentzian and \( V \)-invariant, in fact \( \tilde{\mathcal{V}} \)-invariant. The \( V \)-action
on $T$ commutes with the $H$-one. So, the $V$-action on each torus is conjugate to a linear one and hence equicontinuous. With respect to Facts 7.3 and 7.4, this $T$ can be thought of as the core $N$: equicontinuity of $V$ on a torus in $T$ implies equicontinuity on $T$, and then equicontinuity on $M$ (since $T$ is timelike).

□

Remark 9.4. Carrière’s Theorem says that $V$ restricted to a $\mathcal{V}$-leaf is diffeomorphic to the foliation determined by a minimal linear flow on the torus. In this general context, the transversally Riemannian foliation $\mathcal{V}$ is given without parametrization. All the investigation in last sections aimed to check that in our parametric setting, the vector field $V$ itself is (smoothly) conjugate to a linear one.

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ÉCOLE NORMALE SUPÉRIEURE DE LYON
46, ALLÉE D’ITALIE 69364 LYON CEDEX 07, FRANCE
Email address: lilia.mehidi@ens-lyon.fr

UMPA, CNRS, ÉCOLE NORMALE SUPÉRIEURE DE LYON
46, ALLÉE D’ITALIE 69364 LYON CEDEX 07, FRANCE
Email address: abdelghani.zeghib@ens-lyon.fr
http://www.umpa.ens-lyon.fr/~zeghib/