String corrections to gauge couplings
from a field theory approach\textsuperscript{1}

D.M. Ghilencea\textsuperscript{a} and S. Groot Nibbelink\textsuperscript{b,†}

\textsuperscript{a}DAMTP, CMS, University of Cambridge
Wilberforce Road, Cambridge, CB3 0WA, United Kingdom.

\textsuperscript{b}Department of Physics and Astronomy, University of Victoria,
PO Box 3055 STN CSC, Victoria, BC, V8W 3P6 Canada,
\textsuperscript{†}CITA National Fellow.

Abstract

An effective field theory approach is introduced to compute one-loop radiative corrections to the gauge couplings due to Kaluza-Klein states associated with a two-torus compactification. The results are compared with those of the string in the field theory “limit” \( \alpha' \to 0 \). The whole \( U \) and the leading \( T \) moduli dependence of the gauge dependent part of the string corrections to the gauge couplings can be recovered using the effective field theory approach.

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1 Introduction.

String theory provides a consistent framework for investigating the physics of very high energies in general, and that of additional compact dimensions in particular. The physics of extra dimensions can be studied with some success on effective field theory grounds as well. Establishing an exact link of the string results with those of the effective field theory is however a difficult task. The latter theory may reproduce some results from string theory in the limiting case of an infinite string scale $M_S$ (zero slope $\alpha' \propto 1/M_S^2$). It is this point of view that we adopt in our effective field theory analysis below, while seeking the exact relationship with string results for the problem of radiative corrections to the gauge couplings. However, the results we present are also relevant for models with “large” extra dimensions, without any reference to string theory.

Upon compactification, provided some conditions are fulfilled, remnant effects of the extra dimensions may be present and affect the low energy physics, such as the gauge couplings of the theory. For our discussion below we consider the example of a two-torus compactification. The model we address is an $\mathcal{N} = 1$ supersymmetric orbifold with one $\mathcal{N} = 2$ sub-sector (e.g. $\mathbb{Z}_4$ orbifold). This two dimensional sub-sector (“bulk”), if charged under the gauge group of the theory, may bring one-loop corrections to the gauge couplings of the theory. At the string level these will be due to Kaluza-Klein and winding modes excitations with respect to the additional compact dimensions. These corrections are well-known in string case and were investigated in [1], see also e.g. [2], [3].

An effective field theory approach to investigating such corrections to the gauge couplings due to the two extra dimensions is also possible. Such an approach can only account for the effects of Kaluza-Klein states: the winding states effects cannot be described by a field theory approach. However, in the limit of a large compactification radius (compared to the ultraviolet length scale $1/\Lambda$), one would hope that the effects of winding states are minimised. The reason for this would be that the above condition corresponds on the string side to the limit $\alpha' \rightarrow 0$ in which case the mass of the winding states is significantly larger than that of the momentum modes. Given the infinite number of these states it is not actually clear that they decouple in the low energy limit. Indeed, the density of momentum and winding modes is $\rho = R + 1/R$, thus winding states may still affect the gauge couplings even if $R$ is very large [4] in string units. Comparing the effective field theory result (due to effects of momentum modes only) with that of the string (in its $\alpha' \rightarrow 0$ limit) will allow us understand whether the winding states have any UV effect on the gauge couplings. Previous effective field theory approaches [5] to such radiative corrections were restricted to understanding the UV behaviour of the couplings in the presence of extra dimensions. The exact link with string theory was recently addressed in [6].

We thus consider an $\mathcal{N} = 1$ orbifold with a $\mathcal{N} = 2$ sub-sector, and attempt to keep a general
approach, without unnecessary model dependence. To begin with we note that the radiative corrections to the “bare” or string coupling $\alpha_u$, induced at one-loop level in the two-torus compactification are usually written as

$$\alpha_i^{-1}(Q) = \alpha_u^{-1} + \frac{b_i}{2\pi} \ln \frac{M_S}{Q} + \Delta_i + \cdots,$$

(1)

Here $Q$ is a low energy scale, $b_i$ the $\mathcal{N} = 1$ beta function coefficient, and $M_S$ is the ultraviolet scale (string scale). The first logarithm in eq. (1) accounts for the one loop effects of the massless states of the model. These states can include for example “twisted” string states (candidates for the MSSM-like matter fields), and also gauge bosons’ contributions. Computing their overall radiative effect on the gauge couplings requires an infrared (IR) and ultraviolet (UV) regularisation. The second term $\Delta_i$ is due to massive Kaluza-Klein and, in string case winding states as well. The dots stand for higher order corrections and for mixing effects between the massless and $\mathcal{N} = 2$ massive states sectors.

2 One-loop corrections from the string.

We first review some details of the string calculation for the one-loop corrections $\Delta_i$ to the gauge couplings [1]. At string level there exists an additional, gauge independent (universal) correction to the gauge couplings, which was “absorbed” into the (re)definition of the “bare” or string coupling $\alpha_u$. This ensures that at the string level this coupling is invariant under the symmetry $SL(2,\mathbb{Z})_T \times SL(2,\mathbb{Z})_U \times \mathbb{Z}_2^{T+U}$ of the (heterotic) string [2].

The separation of the radiative effects on the gauge couplings into massless and massive modes contributions in [1] in the string calculation is not imposed by a string symmetry or principle. In general this is done because string calculations only compute the corrections due to massive modes alone. The expression for $\Delta_i$ in string case (hereafter denoted $\Delta^H_i$) is [1 8]

$$\Delta^H_i = \frac{b_i}{4\pi} \int_{\Gamma} \frac{d\tau_1 d\tau_2}{\tau_2} (Z_{torus} - 1)$$

(2)

Here $b_i$ is the beta function coefficient associated with the $\mathcal{N} = 2$ sub-sector of the $\mathcal{N} = 1$ orbifold. $Z_{torus}$ is the two-torus string partition function. The modulus $\tau = \tau_1 + i\tau_2$ of the world sheet torus is integrated over the fundamental domain $\Gamma = \{\tau_2 > 0, |\tau_1| < 1/2, |\tau| > 1\}$. As the massless states contribution is added separately in eq. (1), the massless (“zero”) mode contribution has been subtracted out in (2) and accounted for by “-1”. One can show that the integral over $Z_{torus}$ alone is modular invariant but is (infrared) divergent. However, by subtracting the massless contribution the result is (made) finite, but is not modular invariant anymore due to this last contribution. Briefly, modular invariance does not ensure a finite string
result. Finally, there seems to be no reason to integrate this massless modes contribution “-1” over the fundamental domain \( \Gamma \), except to make the string result finite. Quantitatively, \( \Delta_i^H \) can be written as \[ \Delta_i^H = \frac{b_i}{4\pi} \int_{\Gamma} \frac{d\tau}{\tau_2} \left[ \frac{T_2}{\tau_2} \sum_A e^{-2\pi i T A} \exp \left[ -\frac{\pi T_2}{\tau_2} U_2 \left| A^{(\tau)} \right|^2 \right] - 1 \right]; \quad A = \begin{pmatrix} n_1 \\ n_2 \\ p_1 \\ p_2 \end{pmatrix} \] (3)

where integers \( p_{1,2} \) are (Poisson re-summed) Kaluza-Klein levels and \( n_{1,2} \) are winding modes.

The moduli fields \( T = T_1 + iT_2, U = U_1 + iU_2 \) can be expressed in terms of the anti-symmetric tensor background \( B_{ij} = B\epsilon_{ij} \) and the radii \( R_{1,2} \) and angle \( \theta \) of the two-torus as \( T = 2[B + iR_1 R_2 \sin \theta/(2\alpha')] \), \( U = R_2/R_1 \exp(i\theta) \).

The integral in eq.(3) can be written as a sum over the “orbits” of the modular group \( SL(2,\mathbb{Z}) \). This is just a sum over classes of matrices \( A \) with entries integer numbers labelling (Poisson re-summed) Kaluza-Klein and winding levels, giving \[ \Delta_i^H = \frac{b_i}{4\pi} \left[ J^{(A=0)} + J^{(\det A=0)} + J^{(\det A\neq0)} + \int_{\Gamma} \frac{d\tau_1 d\tau_2}{\tau_2} (-1)_{\text{reg}} \right] \] (4)

The subscript “reg” indicates that the splitting of the integrals in eqs.(2), (4) only makes sense in the presence of an infrared regulator. For example, each of the separate terms can be multiplied by \( R(\tau_2) = (1 - \exp(-N/\tau_2)) \) with \( N \to \infty \). This regularisation breaks modular invariance, but other IR regularisation schemes, which are modular invariant \[ \] can be used.

The three (regularised) contributions in eq.(4) are classified in function of the matrices \( A \) and correspond to: the zero orbit \( J^{(A=0)} \), the degenerate orbit \( J^{(\det A=0)} \), and the non-degenerate orbit \( J^{(\det A\neq0)} \). \( J^{(A=0)} \) is due to infinitely many original Kaluza-Klein modes. \( J^{(\det A=0)} \) is due to a mixing of momentum and winding modes (if \( n_1 = p_1 = 0 \), or momentum modes alone (if \( n_{1,2} = 0 \)) or winding modes alone (if \( p_{1,2} = 0 \)). In the limit of an infinite string scale or \( \alpha' \to 0 \) the momentum modes contribution is dominant. \( J^{(\det A\neq0)} \) is due to a mixing of momentum and winding modes and is vanishing in the limit of an infinite string scale or \( \alpha' \to 0 \) when winding modes’ effects are suppressed. Quantitatively, one has

\[ J^{(A=0)} = \int_{\Gamma} \frac{d\tau_1 d\tau_2}{\tau_2} T_2 = \frac{\pi}{3} T_2; \] \[ J^{(\det A=0)} = \ln N - \ln \left[ 4\pi e^{-2\gamma} T_2 U_2 |\eta(U)|^4 \right] + O \left( \left( T_2 U_2 / N \right)^{1/2} \right) \] \[ J^{(\det A\neq0)} = -\ln \prod_{n_1=1}^{\infty} \left| 1 - e^{2\pi i n_1 T} \right|^4, \quad J^{(\det A\neq0)} \to 0 \text{ if } \alpha' \to 0 \] (7)

\[ \int_{\Gamma} \frac{d\tau_1 d\tau_2}{\tau_2} (-1)_{\text{reg.}} = -\ln N + \ln 3\sqrt{3} e^{-1-\gamma}/2, \] (8)

The term \( O((T_2 U_2 / N)^{1/2}) \) is just a correction depending on the infrared regulator and vanishes in the limit of removing it, \( N \to \infty \). Adding together the above equations, one obtains the final
\[ \Delta_i^H = -\frac{b_i}{4\pi} \ln \left\{ C_{\text{reg}} U_2 |\eta(U)|^4 T_2 |\eta(T)|^4 \right\} + \mathcal{O}((T_2 U_2/N)^{\frac{3}{2}}), \]

where \( C_{\text{reg}} = 8\pi e^{1-\gamma}/(3\sqrt{3}) \) is a regularisation scheme dependent constant (for the DR scheme \[10\], \( C_{\text{reg}} = 4\pi e^{1-\gamma} \)).

### 3 One-loop corrections from a field theory approach.

A field theory approach to computing the effects of the two extra dimensions of the two-torus on the gauge couplings can only sum the effects due to Kaluza-Klein states (no winding modes). The calculation we present is also relevant in the context of models with “large” compactification radii (relative to the UV cut-off length) and without any reference to the string. As the massless states’ effects are already accounted for in eq.\[1\] the one-loop radiative corrections to the gauge couplings due to massive modes alone is (hereafter denoted \( \Delta_i^* \))

\[ \Delta_i^* = \frac{1}{4\pi} \sum_i T(R_i) \sum_{m_1, m_2 \in \mathbb{Z}} \int_0^{\infty} \frac{dt}{\xi} e^{-\pi \frac{t M_{m_1, m_2}^2}{\mu^2}}, \]

The result of summing a finite or infinite tower of Kaluza-Klein states to the gauge couplings is divergent. Since the integral above is divergent in the UV \((t \to 0)\), we introduced a UV dimensionless (proper time) regulator \( \xi \to 0 \) as the lower limit of the integral, and an arbitrary (finite) mass scale \( \mu \). A “prime” on the sum indicates that it does not include the effects of the massless state \((m_1, m_2) \neq (0, 0)\), which is already present in \[1\]. Finally, the mass of the Kaluza-Klein states is \[11\]

\[ M_{m_1, m_2}^2 = \frac{1}{\sin^2 \theta} \left[ \frac{m_1^2}{R_1^2} + \frac{m_2^2}{R_2^2} - \frac{2 m_1 m_2 \cos \theta}{R_1 R_2} \right] = \frac{|m_2 - U m_1|^2}{(\mu^2 T_2(\mu))} U_2^2, \]

where \( T(\mu) \equiv iT_2(\mu) = i \mu^2 R_1 R_2 \sin \theta \) and the complex moduli \( U \) is identical to that in the string case, \( U = R_2/R_1 \exp(i\theta) \). For the particular case of an orthogonal torus \( \theta = \pi/2 \) one recovers a more familiar mass formula: \( M_{m_1, m_2}^2 = \frac{m_1^2}{R_1^2} + \frac{m_2^2}{R_2^2} \), and if the radii are equal one obtains that \( M_{m_1, m_2}^2 = \frac{(m_1^2 + m_2^2)}{R_2^2} \).

The calculation of the integral for \( \Delta_i^* \) is rather technical, so we only quote the final result \[6\]

\[ \Delta_i^* = -\frac{b_i}{4\pi} \ln \left[ 4\pi e^{-\gamma} e^{-T_2^*} T_2^* U_2 |\eta(U)|^4 \right], \]

\[ T_2^* \equiv \Lambda^2 R_1 R_2 \sin \theta, \quad \Lambda^2 \equiv \mu^2 / \xi, \quad \max\{1/(R_2 \sin \theta), 1/R_1\} \ll \Lambda \]

where \( b_i \) is a sum over the beta function contributions of those states with a Kaluza-Klein tower and \( \Lambda \) is the UV scale associated with the UV regulator \( \xi \). This result for \( \Delta_i^* \) holds true only in
the limit of “removing” the dependence on the regulator $\xi \to 0$, when additional corrections in $\xi$ vanish. This condition is converted into bounds on the mass scales of the theory presented in (13). Notice that $R_2 \sin \theta$ plays the role of an effective radius.

4 A comparison with string theory corrections.

We can now compare the field theory result (12) with the heterotic string expression (9) for the gauge thresholds. For this comparison to make sense we note that one should actually compare the field theory results with the limit $\alpha' \to 0$ of the string results (9), when the effects of the winding modes, not included in the field theory calculation, are suppressed.

We observe that the $U$ dependence of the two equations is identical. As both the effective field theory and the string theory mass spectra are $SL(2, \mathbb{Z})_U$ symmetric, and $U$ is by definition $\alpha'$ independent (i.e. scale independent), it may not be surprising that the $U$ dependence of the final string result can be entirely recovered on effective field theory grounds.

Regarding the $T$ dependent part of the two results (9), (12) we note the following. The field theory UV cut-off in (12) can be identified with the string scale: $\Lambda^2 \equiv \mu^2/\xi \to 1/\alpha'$. Removing the regulator, $\xi \to 0$ on the field theory side in (10) corresponds to an infinite string scale or $\alpha' \to 0$. In this limit eq. (9) (see also (5), (6)) has quadratic and logarithmic divergences in the string scale similar to those in eq. (12). However, the coefficient of the quadratic divergence is different: $T_2^* \Delta^*_i$ and $\left(\pi/3\right) T_2 \Delta^H_i$. In the field theory case this coefficient is regulator dependent, thus it can be chosen such that this difference is avoided. Such choice provides the appropriate definition for the UV cut-off scale of an effective field theory, compatible with the modular invariance of its string embedding. We conclude that the full UV structure of the string thresholds in the limit $\alpha' \to 0$ can be obtained on pure field theory grounds, except that the coefficient of the quadratically divergent part is arbitrary in the field theory approach.

To explain how the factor $\pi/3$ arises in the heterotic string, and its link with winding modes effects, one should analyse its origin in eq. (5). This factor is consequence of the integration over the fundamental domain $\Gamma$, and is thus an effect of modular invariance symmetry and indirectly, of the winding modes. As this symmetry is not present in the effective field theory, the factor $\pi/3$ cannot be recovered. Thus winding modes indirectly control the coefficient of the term quadratic in string scale ($T_2 \propto M_2^2$), therefore they do have an UV role, even in the limit $\alpha' \to 0$.

We would like to end with an additional remark on the field theory limit $\alpha' \to 0$ of the string result, eq. (9). A closer look at the final (infrared regularised) string result of eq. (9) shows that higher corrections due to the infrared regularisation $\mathcal{O}\left((U_2 T_2/N)^{1/2}\right)$ are discarded in the final result for $\Delta^H_i$ when $N \to \infty$. This is true if $T_2$ is finite. However, in the field theory limit of
$\alpha' \to 0$ or equivalently $T_2 \to \infty$ (with $T_2$ expressed in string units) the term $O((U_2 T_2/N)^{1/2})$ is not vanishing. Therefore this term should be kept in the field theory limit of the string result $\Delta_i^H$. This issue requires an investigation at the string level, to clarify if any string symmetry left after compactification can still impose the order to take the limits of removing the infrared regulator $N \to \infty$ and the field theory limit $T_2 \to \infty$ on the string result. This is relevant because these two limits do not commute. For further discussions on this and its relationship to the field theory approach see recent results in [12].

5 Conclusions.

We provided an effective field theory approach to computing the corrections to the gauge couplings due to Kaluza-Klein states and a comparison with the string result which also includes the effects of the winding states.

While the heterotic string calculation is well known and studied in the literature, we attempted to emphasize some points which are usually overlooked: one is that that the finiteness of the string threshold correction $\Delta_i^H$ is not due to modular invariance. It is the massless modes’ contribution which is introduced to “make” the result finite, by subtracting the (infrared) divergence due to the massive momentum and winding modes. The integration of the massless modes contribution (non-modular invariant) over the fundamental domain remains a procedure to make the string result finite, but is not required or supported by a string symmetry. One should clarify at the string level if any string symmetry can dictate the order to take the limits of removing the string infrared regulator and the field theory limit, since these two limits do not commute.

The effective field theory approach to computing corrections to the gauge couplings due to the two extra dimensions of the torus provides results similar to those of the string in the limit of an infinite string scale. The full dependence on the shape moduli ($U$) is recovered. In addition, the UV divergences of the string result in the limit $\alpha' \to 0$ were reproduced by the field theory calculation. The coefficient in front of the leading UV divergences in string theory (in the limit $\alpha' \to 0$) turns out to be equal to $\pi/3$ as a result of modular invariance symmetry, while at field theory level this coefficient is regularisation scheme dependent.

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