HITCHIN’S PROJECTIVELY FLAT CONNECTION, TOEPLITZ OPERATORS AND THE ASYMPTOTIC EXPANSION OF TQFT CURVE OPERATORS

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Abstract. In this paper, we will provide a review of the geometric construction, proposed by Witten, of the SU(n) quantum representations of the mapping class groups which are part of the Reshetikhin-Turaev TQFT for the quantum group $U_q(sl(n, \mathbb{C}))$. In particular, we recall the differential geometric construction of Hitchin’s projectively flat connection in the bundle over Teichmüller space obtained by push-forward of the determinant line bundle over the moduli space of rank $n$, fixed determinant, semi-stable bundles fibering over Teichmüller space. We recall the relation between the Hitchin connection and Toeplitz operators which was first used by the first named author to prove the asymptotic faithfulness of the SU(n) quantum representations of the mapping class groups. We further review the construction of the formal Hitchin connection, and we discuss its relation to the full asymptotic expansion of the curve operators of Topological Quantum Field Theory. We then go on to identifying the first terms in the formal parallel transport of the Hitchin connection explicitly. This allows us to identify the first terms in the resulting star product on functions on the moduli space. This is seen to agree with the first term in the star-product on holonomy functions on these moduli spaces defined by Andersen, Mattes and Reshetikhin.

1. Introduction

Witten constructed, via path integral techniques, a quantization of Chern-Simons theory in 2 + 1 dimensions, and he argued in [Wi] that this produced a TQFT, indexed by a compact simple Lie group and an integer level $k$. For the group SU(n) and level $k$, let us denote this TQFT by $Z_k^{(n)}$. Witten argues in [Wi] that the theory $Z_k^{(2)}$ determines the Jones polynomial of a knot in $S^3$. Combinatorially, this theory was first constructed by Reshetikhin and Turaev, using representation theory of $U_q(sl(n, \mathbb{C}))$ at $q = e^{(2\pi i)/(k+n)}$, in [RT1] and [RT2]. Subsequently, the TQFT’s $Z_k^{(n)}$ were constructed using skein theory by Blanchet, Habegger, Masbaum and Vogel in [BHMV1], [BHMV2] and [B1].

The two-dimensional part of the TQFT $Z_k^{(n)}$ is a modular functor with a certain label set. For this TQFT, the label set $\Lambda_k^{(n)}$ is a finite subset (depending on $k$) of the set of finite dimensional irreducible representations of SU(n). We use the usual labeling of irreducible representations by Young diagrams, so in particular $\Box \in \Lambda_k^{(n)}$ is the defining representation of SU(n). Let further $\Lambda_0^{(d)} \in \Lambda_k^{(n)}$ be the Young diagram consisting of $d$ columns of length $k$. The label set is also equipped with an involution, which is simply induced by taking the dual representation. The trivial representation is a special element in the label set which is clearly preserved by the involution.
$Z_k^{(n)} : \begin{cases} \text{Category of (extended) closed} \\ \text{oriented surfaces} \\ \text{with } \Lambda_k^{(n)} \text{-labeled} \\ \text{marked points with} \\ \text{projective tangent} \\ \text{vectors} \end{cases} \rightarrow \begin{cases} \text{Category of finite} \\ \text{dimensional vector} \\ \text{spaces over } \mathbb{C} \end{cases}$

The three-dimensional part of $Z_k^{(n)}$ is an association of a vector, $Z_k^{(n)}(M, L, \lambda) \in Z_k^{(n)}(\partial M, \partial L, \partial \lambda)$, to any compact, oriented, framed 3–manifold $M$ together with an oriented, framed link $(L, \partial L) \subseteq (M, \partial M)$ and a $\Lambda_k^{(n)}$-labeling $\lambda : \pi_0(L) \rightarrow \Lambda_k^{(n)}$.

This association has to satisfy the Atiyah-Segal-Witten TQFT axioms (see e.g. [At], [Se] and [Wi]). For a more comprehensive presentation of the axioms, see Turaev’s book [T].

The geometric construction of these TQFTs was proposed by Witten in [Wi] where he derived, via the Hamiltonian approach to quantum Chern-Simons theory, that the geometric quantization of the moduli spaces of flat connections should give the two-dimensional part of the theory. Further, he proposed an alternative construction of the two-dimensional part of the theory via WZW-conformal field theory. This theory has been studied intensively. In particular, the work of Tsuchiya, Ueno and Yamada in [TUY] provided the major geometric constructions and results needed. In [BK], their results were used to show that the category of integrable highest weight modules of level $k$ for the affine Lie algebra associated to any simple Lie algebra is a modular tensor category. Further, in [BK], this result is combined with the work of Kazhdan and Lusztig [KL] and the work of Finkelberg [Fi] to argue that this category is isomorphic to the modular tensor category associated to the corresponding quantum group, from which Reshetikhin and Turaev constructed their TQFT. Unfortunately, these results do not allow one to conclude the validity of the geometric constructions of the two-dimensional part of the TQFT proposed by Witten. However, in joint work with Ueno, [AU1], [AU2], [AU3] and [AU4], we have given a proof, based mainly on the results of [TUY], that the TUY-construction of the WZW-conformal field theory, after twist by a fractional power of an abelian theory, satisfies all the axioms of a modular functor. Furthermore, we have proved that the full 2+1-dimensional TQFT resulting from this is isomorphic to the aforementioned one, constructed by BHMOV via skein theory. Combining this with the theorem of Laszlo [La1], which identifies (projectively) the representations of the mapping class groups obtained from the geometric quantization of the moduli space of flat connections with the ones obtained from the TUY-constructions, one gets a proof of the validity of the construction proposed by Witten in [Wi].
Part of this TQFT is the quantum SU\((n)\) representations of the mapping class groups. Namely, if \(\Sigma\) is a closed oriented surfaces of genus \(g\), \(\Gamma\) is the mapping class group of \(\Sigma\), and \(p\) is a point on \(\Sigma\), then the modular functor induces a representation

\[
Z_k^{(n,d)} : \Gamma \to \mathbb{P} \text{Aut}(Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}))
\]

For a general label of \(p\), we would need to choose a projective tangent vector \(v_p \in T_p \Sigma / \mathbb{R}^+\), and we would get a representation of the mapping class group of \((\Sigma, p, v_p)\). But for the special labels \(\lambda_0^{(d)}\), the dependence on \(v_p\) is trivial and in fact we get a representation of \(\Gamma\). Furthermore, the curve operators are also part of any TQFT: For \(\gamma \subseteq \Sigma - \{p\}\) an oriented simple closed curve and any \(\lambda \in \Lambda_k^{(n)}\), we have the operators

\[
Z_k^{(n,d)}(\gamma, \lambda) : Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) \to Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}),
\]

defined as

\[
Z_k^{(n,d)}(\gamma, \lambda) = Z_k^{(n,d)}(\Sigma \times I, \gamma \times \{\frac{1}{2}\} \coprod \{p\} \times I, \{\lambda, \lambda_0^{(d)}\})
\]

The curve operators are natural under the action of the mapping class group, meaning that following diagram,

\[
\begin{array}{ccc}
Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) & \overset{Z_k^{(n,d)}(\gamma, \lambda)}{\longrightarrow} & Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) \\
Z_k^{(n,d)}(\phi) \downarrow & & \downarrow Z_k^{(n,d)}(\phi) \\
Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) & \overset{Z_k^{(n,d)}(\phi(\gamma), \lambda)}{\longrightarrow} & Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)})
\end{array}
\]

is commutative for all \(\phi \in \Gamma\) and all labeled simple closed curves \((\gamma, \lambda) \subset \Sigma - \{p\}\).

For the curve operators, we can derive an explicit formula using factorization: Let \(\Sigma'\) be the surface obtained from cutting \(\Sigma\) along \(\gamma\) and identifying the two boundary components to two points, say \(\{p_+, p_-\}\). Here \(p_+\) is the point corresponding to the "left" side of \(\gamma\). For any label \(\mu \in \Lambda_k^{(n)}\), we get a labeling of the ordered points \((p_+, p_-)\) by the ordered pair of labels \((\mu, \mu^1)\).

Since \(Z_k^{(n)}\) is also a modular functor, one can factor the space \(Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)})\) as a direct sum, 'along' \(\gamma\), over \(\Lambda_k^{(n)}\). That is, we get an isomorphism

\[
Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)}) \cong \bigoplus_{\mu \in \Lambda_k^{(n)}} Z^{(k)}(\Sigma', p_+, p_-, p, \mu, \mu^1, \lambda_0^{(d)}).
\]

Strictly speaking, we need a base point on \(\gamma\) to induce tangent directions at \(p_\pm\). However, the corresponding subspaces of \(Z^{(k)}(\Sigma, p, \lambda_0^{(d)})\) do not depend on the
The choice of base point. The isomorphism (3) induces an isomorphism
\[ \text{End}(Z^k(\Sigma, p, \lambda_0^{(d)})) \cong \bigoplus_{\mu \in \Lambda_k^{(n)}} \text{End}(Z^k(\Sigma', p_+, p_-, \mu, \mu^\dagger, \lambda_0^{(d)})), \]
which also induces a direct sum decomposition of \( \text{End}(Z^k(\Sigma, p, \lambda_0^{(d)})) \), independent of the base point.

The TQFT axioms imply that the curve operator \( Z^k(\gamma, \lambda) \) is diagonal with respect to this direct sum decomposition along \( \gamma \). One has the formula
\[ Z^k(\gamma, \lambda) = \bigoplus_{\mu \in \Lambda_k^{(n)}} S_{\lambda, \mu}(S_{0, \mu})^{-1} \text{Id}_{Z^k(\Sigma', p_+, p_-, \mu, \mu^\dagger, \lambda_0^{(d)})}. \]

Here \( S_{\lambda, \mu} \) is the \( S \)-matrix of the theory \( Z^{(n)} \). See e.g. [B1] for a derivation of this formula.

Let us now briefly recall the geometric construction of the representations \( Z^{(n,d)}_k \) of the mapping class group, as proposed by Witten, using geometric quantization of moduli spaces.

We assume from now on that the genus of the closed oriented surface \( \Sigma \) is at least two. Let \( M \) be the moduli space of flat \( SU(n) \) connections on \( \Sigma - p \) with holonomy around \( p \) equal to \( \exp(2\pi id/n) \text{Id} \in SU(n) \). When \( (n, d) \) are coprime, the moduli space is smooth. In all cases, the smooth part of the moduli space has a natural symplectic structure \( \omega \). There is a natural smooth symplectic action of the mapping class group \( \Gamma \) of \( \Sigma \) on \( M \). Moreover, there is a unique prequantum line bundle \((L, \nabla, \langle \cdot, \cdot \rangle)\) over \((M, \omega)\). The Teichmüller space \( T \) of complex structures on \( \Sigma \) naturally, and \( \Gamma \)-equivariantly, parametrizes Kähler structures on \((M, \omega)\). For \( \sigma \in T \), we denote by \( M_{\sigma} \) the manifold \((M, \omega)\) with its corresponding Kähler structure. The complex structure on \( M_{\sigma} \) and the connection \( \nabla \) in \( L \) induce the structure of a holomorphic line bundle on \( L \). This holomorphic line bundle is simply the determinant line bundle over the moduli space, and it is an ample generator of the Picard group \([DN]\).

By applying geometric quantization to the moduli space \( M \), one gets, for any positive integer \( k \), a certain finite rank bundle over Teichmüller space \( T \) which we will call the Verlinde bundle \( \mathcal{V}_k \) at level \( k \). The fiber of this bundle over a point \( \sigma \in T \) is \( \mathcal{V}_{k, \sigma} = H^0(M_{\sigma}, L^k) \). We observe that there is a natural Hermitian structure \( \langle \cdot, \cdot \rangle \) on \( H^0(M_{\sigma}, L^k) \) by restricting the \( L_2 \)-inner product on global \( L^k \) sections of \( L^k \) to \( H^0(M_{\sigma}, L^k) \).

The main result pertaining to this bundle is:

**Theorem 1** (Axelrod, Della Pietra and Witten; Hitchin). The projectivization of the bundle \( \mathcal{V}_k \) supports a natural flat \( \Gamma \)-invariant connection \( \hat{\nabla} \).

This is a result proved independently by Axelrod, Della Pietra and Witten [ADW] and by Hitchin [H]. In section 2 we review our differential geometric construction of the connection \( \hat{\nabla} \) in the general setting discussed in [A6]. We obtain as a corollary that the connection constructed by Axelrod, Della Pietra and Witten projectively agrees with Hitchin’s.

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1 The \( S \)-matrix is determined by the isomorphism that a modular functor induces from two different ways of gluing an annulus to obtain a torus. For its definition, see e.g. [MS], [SS], [Wa] or [BK] and references in there. It is also discussed in [AV3].
Definition 1. We denote by $Z_{k}^{(n,d)}$ the representation,

$$Z_{k}^{(n,d)} : \Gamma \to \mathbb{P} \text{Aut}(Z_{k}^{(n)}(\Sigma, p, \lambda_{0}^{(d)})),$$

obtained from the action of the mapping class group on the covariant constant sections of $\mathbb{P}(V_{k})$ over $\mathcal{T}$.

The projectively flat connection $\hat{\nabla}$ induces a flat connection $\hat{\nabla}^{e}$ in $\text{End}(V_{k})$. Let $\text{End}_{0}(V_{k})$ be the subbundle consisting of traceless endomorphisms. The connection $\hat{\nabla}^{e}$ also induces a connection in $\text{End}_{0}(V_{k})$, which is invariant under the action of $\Gamma$.

In [A3], we proved Theorem 2 (Andersen). Assume that $n$ and $d$ are coprime or that $(n, d) = (2, 0)$ when $g = 2$. Then, we have that

$$\bigcap_{k=1}^{\infty} \ker(Z_{k}^{(n,d)}) = \begin{cases} \{1, H\} & g = 2, n = 2 \text{ and } d = 0 \\ \{1\} & \text{otherwise}, \end{cases}$$

where $H$ is the hyperelliptic involution.

The main ingredient in the proof of this Theorem is the Toeplitz operators associated to smooth functions on $M$. For each $f \in C^{\infty}(M)$ and each point $\sigma \in \mathcal{T}$, we have the Toeplitz operator,

$$T_{f,\sigma}^{(k)} : H^{0}(M_{\sigma}, L_{\sigma}^{k}) \to H^{0}(M_{\sigma}, L_{\sigma}^{k}),$$

which is given by

$$T_{f,\sigma}^{(k)} = \pi_{\sigma}^{(k)}(fs)$$

for all $s \in H^{0}(M_{\sigma}, L_{\sigma}^{k})$. Here $\pi_{\sigma}^{(k)}$ is the orthogonal projection onto $H^{0}(M_{\sigma}, L_{\sigma}^{k})$ induced from the $L_{2}$-inner product on $C^{\infty}(M, L^{k})$. We get a smooth section of $\text{End}(V^{(k)})$,

$$T_{f}^{(k)} \in C^{\infty}(\mathcal{T}, \text{End}(V^{(k)})),$$

by letting $T_{f}^{(k)}(\sigma) = T_{f,\sigma}^{(k)}$ (see [A3]). See section 3 for further discussion of the Toeplitz operators and their connection to deformation quantization.

The sections $T_{f}^{(k)}$ of $\text{End}(V^{(k)})$ over $\mathcal{T}$ are not covariant constant with respect to Hitchin’s connection $\hat{\nabla}^{e}$. However, they are asymptotically as $k$ goes to infinity. This will be made precise when we discuss the formal Hitchin connection below.

As a further application of TQFT and the theory of Toeplitz operators together with the theory of coherent states, we recall the first authors solution to a problem in geometric group theory, which has been around for quite some time (see e.g. Problem (7.2) in Chapter 7, ”A short list of open questions”, of [BHV]): In [A8], Andersen proved that

Theorem 3 (Andersen). The mapping class group of a closed oriented surface, of genus at least two, does not have Kazhdan’s property (T).

Returning to the geometric construction of the Reshetikhin-Turaev TQFT, let us recall the geometric construction of the curve operators. First of all, the decomposition [3] is geometrically obtained as follows (see [A7] for the details):

One considers a one parameter family of complex structures $\sigma_{t} \in \mathcal{T}, t \in \mathbb{R}_{+}$, such that the corresponding family in the moduli space of curves converges in the Mumford-Deligne boundary to a nodal curve, which topologically corresponds to
shrinking \( \gamma \) to a point. By the results of \([A1]\), the corresponding sequence of complex structures on the moduli space \( M \) converges to a non-negative polarization on \( M \) whose isotropic foliation is spanned by the Hamiltonian vector fields associated to the holonomy functions of \( \gamma \). The main result of \([A7]\) is that the covariant constant sections of \( V^{(k)} \) along the family \( \sigma_t \) converges to distributions supported on the Bohr-Sommerfeld leaves of the limiting non-negative polarization as \( t \) goes to infinity. The direct sum of the geometric quantization of the level \( k \) Bohr-Sommerfeld levels of this non-negative polarization is precisely the left-hand side of (3). A sewing-construction, inspired by conformal field theory (see \([TUY]\)), is then applied to show that the resulting linear map from the right-hand side of (3) to the left-hand side is an isomorphism. This is described in detail in \([A7]\).

In \([A7]\), we further prove the following important asymptotic result. Let \( h_{\gamma,\lambda} \in C^\infty(M) \) be the holonomy function obtained by taking the trace in the representation \( \lambda \) of the holonomy around \( \gamma \).

**Theorem 4 (Andersen).** For any one-dimensional oriented submanifold \( \gamma \) and any labeling \( \lambda \) of the components of \( \gamma \), we have that

\[
\lim_{k \to \infty} \| Z^{(n,d)}_k(\gamma, \lambda) - T^{(k)}_{h_{\gamma,\lambda}} \| = 0.
\]

Let us here give the main idea behind the proof of Theorem 4 and refer to \([A7]\) for the details. One considers the explicit expression for the \( S \)-matrix, as given in formula (13.8.9) in Kac’s book \([Kac]\)

\[
S_{\lambda,\mu}/S_{0,0} = \lambda(e^{-2\pi i \frac{\hat{\nu}+\rho}{n}}),
\]

where \( \rho \) is half the sum of the positive roots and \( \hat{\nu} \) any element of \( \Lambda \) is the unique element of the Cartan subalgebra of the Lie algebra of \( SU(n) \) which is dual to \( \nu \) with respect to the Cartan-Killing form \((\cdot, \cdot)\).

From the expression (4), one sees that under the isomorphism \( \hat{\mu} \mapsto \mu \), the expression \( S_{\lambda,\mu}/S_{0,0} \) makes sense for any \( \hat{\mu} \) in the Cartan subalgebra of the Lie algebra of \( SU(n) \). Furthermore, one finds that the values of this sequence of functions (depending on \( k \)) is asymptotic to the values of the holonomy function \( h_{\gamma,\lambda} \) at the level \( k \) Bohr-Sommerfeld sets of the limiting non-negative polarizations discussed above (see \([A1]\)). From this, one can deduce Theorem 4. See again \([A7]\) for details.

Let us now consider the general setting treated in \([A6]\). Thus, we consider, as opposed to only considering the moduli spaces, a general prequantizable symplectic manifold \((M, \omega)\) with a prequantum line bundle \((L, (\cdot, \cdot), \nabla)\). We assume that \( T \) is a complex manifold which holomorphically and rigidly (see Definition \([D]\) parameterizes Kähler structures on \((M, \omega)\). Then, the following theorem, proved in \([A6]\), establishes the existence of the Hitchin connection under a mild cohomological condition.

**Theorem 5 (Andersen).** Suppose that \( I \) is a rigid family of Kähler structures on the compact, prequantizable symplectic manifold \((M, \omega)\) which satisfies that there exists an \( n \in \mathbb{Z} \) such that the first Chern class of \((M, \omega)\) is \( n[S] \in H^2(M, \mathbb{Z}) \) and \( H^1(M, \mathbb{R}) = 0 \). Then, the Hitchin connection \( \nabla \) in the trivial bundle \( \mathcal{H}^{(k)} = T \times C^\infty(M, \mathcal{L}^k) \) preserves the subbundle \( H^{(k)} \) with fibers \( H^0(M, \mathcal{L}^k) \). It is given by

\[
\hat{\nabla}_V = \hat{\nabla}_V^t + \frac{1}{4k + 2n} \{ \Delta_{G(V)} + 2\nabla_{G(V)}df + 4kV'[F] \},
\]
where \( ^\hat{} \nabla_t \) is the trivial connection in \( H^{(k)} \), and \( V \) is any smooth vector field on \( T \).

In section 4, we study the formal Hitchin connection which was introduced in [A6]. Let \( D(M) \) be the space of smooth differential operators on \( M \). Let \( C_h \) be the trivial \( C^\infty_c(M) \)-bundle over \( T \).

**Definition 2.** A formal connection \( D \) is a connection in \( C_h \) over \( T \) of the form

\[
D_V f = V[f] + \tilde{D}(V)(f),
\]

where \( \tilde{D} \) is a smooth one-form on \( T \) with values in \( D_h(M) = D(M)[[h]] \), \( f \) is any smooth section of \( C_h \), \( V \) is any smooth vector field on \( T \) and \( V[f] \) is the derivative of \( f \) in the direction of \( V \).

Thus, a formal connection is given by a formal series of differential operators

\[
\tilde{D}(V) = \sum_{l=0}^\infty \tilde{D}^{(l)}(V) h^l.
\]

From Hitchin’s connection in \( H^{(k)} \), we get an induced connection \( \hat{\nabla}^{\epsilon} \) in the endomorphism bundle \( \text{End}(H^{(k)}) \). As previously mentioned, the Toeplitz operators are not covariant constant sections with respect to \( \hat{\nabla}^{\epsilon} \), but asymptotically in \( k \) they are. This follows from the properties of the formal Hitchin connection, which is the formal connection \( D \) defined through the following theorem (proved in [A6]).

**Theorem 6.** (Andersen) There is a unique formal connection \( D \) which satisfies that

\[
\hat{\nabla}^{\epsilon} (T^{(k)}_f) \sim T^{(k)}_{(D_V f)(1/(k+n/2))} \text{ for all smooth section } f \text{ of } C_h \text{ and all smooth vector fields on } T. \text{ Moreover,}
\]

\[
\tilde{D} = 0 \mod h.
\]

Here \( \sim \) means the following: For all \( L \in \mathbb{Z}_+ \) we have that

\[
\left\| \hat{\nabla}^{\epsilon}_V T^{(k)}_f - \left( T^{(k)}_{V[f]} + \sum_{l=1}^L T^{(k)}_{D^{(l)}_V f (1/(k+n/2))} \right) \right\| = O(k^{-(L+1)}),
\]

uniformly over compact subsets of \( T \), for all smooth maps \( f : T \to C^\infty(M) \).

Now fix an \( f \in C^\infty(M) \), which does not depend on \( \sigma \in T \), and notice how the fact that \( \tilde{D} = 0 \mod h \) implies that

\[
\left\| \hat{\nabla}^{\epsilon}_V T^{(k)}_f \right\| = O(k^{-1}).
\]

This expresses the fact that the Toeplitz operators are asymptotically flat with respect to the Hitchin connection.

We define a mapping class group equivariant formal trivialization of \( D \) as follows.

**Definition 3.** A formal trivialization of a formal connection \( D \) is a smooth map \( P : T \to D_h(M) \) which modulo \( h \) is the identity, for all \( \sigma \in T \), and which satisfies

\[ D_V(P(f)) = 0, \]

for all vector fields \( V \) on \( T \) and all \( f \in C^\infty_c(M) \). Such a formal trivialization is mapping class group equivariant if \( P(\phi(\sigma)) = \phi^* P(\sigma) \) for all \( \sigma \in T \) and \( \phi \in \Gamma \).
Since the only mapping class group invariant functions on the moduli space are the constant ones (see [Go1]), we see that in the case where $M$ is the moduli space, such a $P$, if it exists, must be unique up to multiplication by a formal constant.

Clearly if $D$ is not flat, such a formal trivialization cannot exist even locally on $T$. However, if $D$ is flat and its zero-order term is just given by the trivial connection in $C_h$, then a local formal trivialization exists, as proved in [A6].

Furthermore, it is proved in [A6] that flatness of the formal Hitchin connection is implied by projective flatness of the Hitchin connection. As was proved by Hitchin in [H], and stated above in Theorem 1, this is the case when $M$ is the moduli space.

Furthermore, the existence of a formal trivialization implies the existence of unique (up to formal scale) mapping class group equivariant formal trivialization, provided that $H^1(T, D(M)) = 0$. The first steps towards proving that this cohomology group vanishes have been taken in [AV1, AV2, AV3, Yi]. In this paper, we prove that

**Theorem 7.** The mapping class group equivariant formal trivialization of the formal Hitchin connection exists to first order, and we have the following explicit formula for the first order term of $P$

$$P^{(1)}(f) = \frac{1}{4} \Delta_\sigma(f) + i \nabla_{X''_F}(f),$$

where $X''_F$ denotes the $(0,1)$-part of the Hamiltonian vector field for the Ricci potential.

For the proof of the theorem, see section 4. We will make the following conjecture.

**Conjecture 1.** The mapping class group equivariant formal trivialization of the formal Hitchin connection exists, and for any one-dimensional oriented submanifold $\gamma$ and any labeling $\lambda$ of the components of $\gamma$, we have the full asymptotic expansion

$$Z_{k}(n,d)_{\gamma,\lambda} \sim T_{P(h_{\gamma,\lambda})},$$

which means that for all $L$ and all $\sigma \in T$, we have that

$$\|Z_{k}(n,d)_{\gamma,\lambda} - \sum_{l=0}^{L} T_{P^{(l)}(h_{\gamma,\lambda})} \frac{1}{(k + n/2)^l}\| = O(k^{L+1}).$$

It is very likely that the techniques used in [A7] to prove Theorem 4 can be used to prove this conjecture.

When we combine this conjecture with the asymptotics of the product of two Toeplitz operators (see Theorem 11), we get the full asymptotic expansion of the product of two curve operators:

$$Z_{k}^{(n,d)}(\gamma_1,\lambda_1) Z_{k}^{(n,d)}(\gamma_2,\lambda_2) \sim T^{(k)}_{P(h_{\gamma_1,\lambda_1})} *^{BT}_{\sigma} P(h_{\gamma_2,\lambda_2}),$$

where $*^{BT}_{\sigma}$ is very closely related to the Berezin-Toeplitz star product for the Kähler manifold $(M,\omega)$, as first defined in [BMS]. See section 3 for further details regarding this.

Suppose that we have a mapping class group equivariant formal trivialization $P$ of the formal Hitchin connection $D$. We can then define a new smooth family of star products parametrized by $T$ as follows:

$$f *_{\sigma} g = P_{\sigma}^{-1}(P_{\sigma}(f) *^{BT}_{\sigma} P_{\sigma}(g))$$
for all \( f, g \in C^\infty(M) \) and all \( \sigma \in T \). Using the fact that \( P \) is a trivialization, it is not hard to prove that \( \star_\sigma \) is independent of \( \sigma \), and we simply denote it \( \star \). The following theorem is proved in section 4.

**Theorem 8.** The star product \( \star \) has the form
\[
f \star g = fg - \frac{i}{2} \{ f, g \} + O(h^2).
\]

We observe that this formula for the first-order term of \( \star \) agrees with the first-order term of the star product constructed by Andersen, Mattes and Reshetikhin in [AMR2], when we apply the formula in Theorem 8 to two holonomy functions \( h_{\gamma_1, \lambda_1} \) and \( h_{\gamma_2, \lambda_2} \):
\[
h_{\gamma_1, \lambda_1} \star h_{\gamma_2, \lambda_2} = h_{\gamma_1, \gamma_2 \cup \lambda_1 \cup \lambda_2} - \frac{i}{2} h_{\{ \gamma_1, \gamma_2 \}, \lambda_1 \cup \lambda_2} + O(h^2).
\]

We recall that \( \{ \gamma_1, \gamma_2 \} \) is the Goldman bracket (see [Go2]) of the two simple closed curves \( \gamma_1 \) and \( \gamma_2 \).

A similar result was obtained for the abelian case, i.e. in the case where \( M \) is the moduli space of flat \( U(1) \)-connections, by the first author in [A2], where the agreement between the star product defined in differential geometric terms and the star product of Andersen, Mattes and Reshetikhin was proved to all orders.

We would finally also like to recall that the first named author has shown that the Nielsen-Thurston classification of mapping classes is determined by the Reshetikhin-Turaev TQFTs. We refer to [A5] for the full details of this.

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2. The Hitchin connection

In this section, we review our construction of the Hitchin connection using the global differential geometric setting of [A6]. This approach is close in spirit to Axelrod, Della Pietra and Witten’s in [ADW], however we do not use any infinite dimensional gauge theory. In fact, the setting is more general than the gauge theory setting in which Hitchin in [H] constructed his original connection. But when applied to the gauge theory situation, we get the corollary that Hitchin’s connection agrees with Axelrod, Della Pietra and Witten’s.

Hence, we start in the general setting and let \((M, \omega)\) be any compact symplectic manifold.

**Definition 4.** A prequantum line bundle \((\mathcal{L}, (\cdot, \cdot), \nabla)\) over the symplectic manifold \((M, \omega)\) consist of a complex line bundle \(\mathcal{L}\) with a Hermitian structure \((\cdot, \cdot)\) and a compatible connection \(\nabla\) whose curvature is
\[
F_\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = -i\omega(X, Y).
\]

We say that the symplectic manifold \((M, \omega)\) is prequantizable if there exist a prequantum line bundle over it.

Recall that the condition for the existence of a prequantum line bundle is that \(\frac{i}{2\pi} \in \text{Im}(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}))\). Furthermore, the inequivalent choices of prequantum line bundles (if they exist) are parametrized by \(H^1(M, U(1))\) (see e.g. [Wo]).
We shall assume that \((M, \omega)\) is prequantizable and fix a prequantum line bundle \((\mathcal{L}, \cdot, \cdot, \nabla)\).

Assume that \(T\) is a smooth manifold which smoothly parametrizes Kähler structures on \((M, \omega)\). This means that we have a smooth map \(I : T \to C^\infty(M, \text{End}(TM))\) such that \((M, \omega, I)\) is a Kähler manifold for each \(\sigma \in T\).

We will use the notation \(M_\sigma\) for the complex manifold \((M, I_\sigma)\). For each \(\sigma \in T\), we use \(I_\sigma\) to split the complexified tangent bundle \(TM_{\mathbb{C}}\) into the holomorphic and the anti-holomorphic parts. These we denote by

\[T_\sigma = E(I_\sigma, i) = \text{Im}(\text{Id} - iI_\sigma)\]

and

\[\bar{T}_\sigma = E(I_\sigma, -i) = \text{Im}(\text{Id} + iI_\sigma)\]

respectively.

The real Kähler-metric \(g_\sigma\) on \((M_\sigma, \omega)\), extended complex linearly to \(TM_{\mathbb{C}}\), is by definition

\[g_\sigma(X, Y) = \omega(X, I_\sigma Y),\]

where \(X, Y \in C^\infty(M, TM_{\mathbb{C}})\).

The divergence of a vector field \(X\) is the unique function \(\delta(X)\) determined by

\[\mathcal{L}_X \omega^m = \delta(X) \omega^m.\]

It can be calculated by the formula \(\delta(X) = \Lambda d(iX \omega)\), where \(\Lambda\) denotes contraction with the Kähler form. Eventhough the divergence only depends on the volume, which is independent of the of the particular Kähler structure, it can be expressed in terms of the Levi-Civita connection on \(M_\sigma\) by \(\delta(X) = \text{Tr} \nabla_\sigma X\).

Inspired by this expression, we define the divergence of a symmetric bivector field \(B \in C^\infty(M, S^2(TM_{\mathbb{C}}))\) by

\[\delta_\sigma(B) = \text{Tr} \nabla_\sigma B.\]

Notice that the divergence on bivector fields does depend on the point \(\sigma \in T\).

Suppose \(V\) is a vector field on \(T\). Then, we can differentiate \(I\) along \(V\) and we denote this derivative by \(V[I] : T \to C^\infty(M, \text{End}(TM_{\mathbb{C}}))\). Differentiating the equation \(I^2 = -\text{Id}\), we see that \(V[I]\) anti-commutes with \(I\). Hence, we get that

\[V[I]_\sigma \in C^\infty(M, (T^*_\sigma \otimes \bar{T}_\sigma) \oplus (\bar{T}^*_\sigma \otimes T_\sigma))\]

for each \(\sigma \in T\). Let

\[V[I]_\sigma = V[I]'_\sigma + V[I]''_\sigma\]

be the corresponding decomposition such that \(V[I]'_\sigma \in C^\infty(M, \bar{T}^*_\sigma \otimes T_\sigma)\) and \(V[I]''_\sigma \in C^\infty(M, T^*_\sigma \otimes \bar{T}_\sigma)\).

Now we will further assume that \(T\) is a complex manifold and that \(I\) is a holomorphic map from \(T\) to the space of all complex structures on \(M\). Concretely, this means that

\[V'[I]_\sigma = V[I]'_\sigma\]

\footnote{Here a smooth map from \(T\) to \(C^\infty(M, W)\), for any smooth vector bundle \(W\) over \(M\), means a smooth section of \(\pi_M^* W\) over \(T \times M\), where \(\pi_M\) is the projection onto \(M\). Likewise, a smooth \(p\)-form on \(T\) with values in \(C^\infty(M, W)\) is, by definition, a smooth section of \(\pi_M^* \Lambda^p(T) \otimes \pi_M^* W\) over \(T \times M\). We will also encounter the situation where we have a bundle \(W\) over \(T \times M\) and then we will talk about a smooth \(p\)-form on \(T\) with values in \(C^\infty(M, W_\sigma)\) and mean a smooth section of \(\pi_T^* \Lambda^p(T) \otimes W\) over \(T \times M\).}
and
\[ V''[I]_\sigma = V'[I]''_\sigma \]
for all \( \sigma \in T \), where \( V' \) means the \((1,0)\)-part of \( V \) and \( V'' \) means the \((0,1)\)-part of \( V \) over \( T \).

Let us define \( \tilde{G}(V) \in C^\infty(M,TM_C \otimes TM_C) \) by
\[ V[I] = \tilde{G}(V) \omega, \]
and define \( G(V) \in C^\infty(M,T_\sigma \otimes T_\sigma) \) such that
\[ \tilde{G}(V) = G(V) + \overline{G}(V) \]
for all real vector fields \( V \) on \( T \). We see that \( \tilde{G} \) and \( G \) are one-forms on \( T \) with values in \( C^\infty(M,TM_C \otimes TM_C) \) and \( C^\infty(M,T_\sigma \otimes T_\sigma) \), respectively. We observe that
\[ V''[I] = G(V)\omega, \]
and \( G(V) = G(V') \).

Using the relation (6), one checks that
\[ \tilde{G}(V) = -V[g^{-1}], \]
where \( g^{-1} \in C^\infty(M,S^2(TM)) \) is the symmetric bivector field obtained by raising both indices on the metric tensor. Clearly, this implies that \( \tilde{G} \) takes values in \( C^\infty(M,S^2(TM_C)) \) and therefore that \( G \) takes values in \( C^\infty(M,S^2(T_\sigma)) \).

On \( \mathcal{L}^k \), we have the smooth family of \( \bar{\partial} \)-operators \( \nabla^{0,1}_\sigma \) defined at \( \sigma \in T \) by
\[ \nabla^{0,1}_\sigma = \frac{1}{2}(1 + iI_\sigma)\nabla. \]
For every \( \sigma \in T \), we consider the finite-dimensional subspace of \( C^\infty(M,\mathcal{L}^k) \) given by
\[ H^{(k)}(\sigma) = H^0(M_\sigma,\mathcal{L}^k) = \{ s \in C^\infty(M,\mathcal{L}^k)|\nabla^{0,1}_\sigma s = 0 \}. \]
Let \( \tilde{\nabla}^t \) denote the trivial connection in the trivial bundle \( \mathcal{H}^{(k)} = T \times C^\infty(M,\mathcal{L}^k) \), and let \( \mathcal{D}(M,\mathcal{L}^k) \) denote the vector space of differential operators on \( C^\infty(M,\mathcal{L}^k) \). For any smooth one-form \( u \) on \( T \) with values in \( \mathcal{D}(M,\mathcal{L}^k) \), we have a connection \( \tilde{\nabla}^t \) in \( \mathcal{H}^{(k)} \) given by
\[ \tilde{\nabla}^t V = \tilde{\nabla}^t V - u(V) \]
for any vector field \( V \) on \( T \).

**Lemma 1.** The connection \( \tilde{\nabla}^t \) in \( \mathcal{H}^{(k)} \) preserves the subspaces \( H^{(k)}(\sigma) \subset C^\infty(M,\mathcal{L}^k) \), for all \( \sigma \in T \), if and only if
\[ \frac{i}{2}V[I]\nabla^{1,0}_\sigma s + \nabla^{0,1}_\sigma u(V)s = 0 \]
for all vector fields \( V \) on \( T \) and all smooth sections \( s \) of \( H^{(k)} \).

This result is not surprising. See [16] for a proof this lemma. Observe that if this condition holds, we can conclude that the collection of subspaces \( H^{(k)}(\sigma) \subset C^\infty(M,\mathcal{L}^k) \), for all \( \sigma \in T \), form a subbundle \( H^{(k)} \) of \( \mathcal{H}^{(k)} \).

We observe that \( u(V'') = 0 \) solves (8) along the anti-holomorphic directions on \( T \) since
\[ V''[I]\nabla^{1,0}_\sigma s = 0. \]
In other words, the \((0,1)\)-part of the trivial connection \( \tilde{\nabla}^t \) induces a \( \bar{\partial} \)-operator on \( H^{(k)} \) and hence makes it a holomorphic vector bundle over \( T \).
This is of course not in general the situation in the $(1,0)$ direction. Let us now consider a particular $u$ and prove that it solves (8) under certain conditions.

On the Kähler manifold $(M_\sigma, \omega)$, we have the Kähler metric and we have the Levi-Civita connection $\nabla$ in $T_\sigma$. We also have the Ricci potential $F_\sigma \in C_0^\infty(M, \mathbb{R})$. Here

$$C_0^\infty(M, \mathbb{R}) = \left\{ f \in C^\infty(M, \mathbb{R}) \mid \int_M f \omega^m = 0 \right\},$$

and the Ricci potential is the element of $F_\sigma \in C_0^\infty(M, \mathbb{R})$ which satisfies

$$\text{Ric}_\sigma = \text{Ric}_\sigma^H + 2i \partial_\sigma \bar{\partial}_\sigma F_\sigma,$$

where $\text{Ric}_\sigma \in \Omega^{1,1}(M_\sigma)$ is the Ricci form and $\text{Ric}_\sigma^H$ is its harmonic part. We see that we get in this way a smooth function $F : T \to C_0^\infty(M, \mathbb{R})$.

For any symmetric bivector field $B \in C^\infty(M, S^2(TM))$ we get a linear bundle map $B : TM^* \to TM$ given by contraction. In particular, for a smooth function $f$ on $M$, we get a vector field $Bdf \in C^\infty(M, TM)$.

We define the operator

$$\Delta_B : C^\infty(M, \mathcal{L}^k) \sum \nabla X \nabla Y - \nabla \nabla_{X, Y} C^\infty(M, TM \otimes \mathcal{L}^k) \xrightarrow{\nabla_{X, Y} \otimes \text{Id} + \text{Id} \otimes \nabla_{X, Y}} C^\infty(M, TM^* \otimes TM \otimes \mathcal{L}^k) \xrightarrow{\text{Tr}} C^\infty(M, \mathcal{L}^k).$$

Let’s give a more concise formula for this operator. Define the operator

$$\nabla_{X, Y} = \nabla X \nabla Y - \nabla \nabla_{X, Y},$$

which is tensorial and symmetric in the vector fields $X$ and $Y$. Thus, it can be evaluated on a symmetric bivector field and we have

$$\Delta_B = \nabla_{X, Y}^2 + \delta_{B}.\nabla.$$

Putting these constructions together, we consider, for some $n \in \mathbb{Z}$ such that $2k + n \neq 0$, the following operator

$$u(V) = \frac{1}{k + n/2} o(V) - V' [F],$$

where

$$o(V) = -\frac{1}{4} (\Delta_G(V) + 2 \nabla_G(V) df - 2n V' [F]).$$

The connection associated to this $u$ is denoted $\hat{\nabla}$, and we call it the Hitchin connection in $\mathcal{H}^{(k)}$.

**Definition 5.** We say that the complex family $I$ of Kähler structures on $(M, \omega)$ is **Rigid** if

$$\bar{\partial}_\sigma (G(V)_\sigma) = 0$$

for all vector fields $V$ on $T$ and all points $\sigma \in T$.

We will assume our holomorphic family $I$ is rigid.
Theorem 9 (Andersen). Suppose that $I$ is a rigid family of Kähler structures on the compact, prequantizable symplectic manifold $(M, \omega)$ which satisfies that there exists an $n \in \mathbb{Z}$ such that the first Chern class of $(M, \omega)$ is $n[\varpi_{\frac{n}{2}}] \in H^2(M, \mathbb{Z})$ and $H^1(M, \mathbb{R}) = 0$. Then $u$ given by (9) and (10) satisfies (8), for all $k$ such that $2k + n \neq 0$.

Hence, the Hitchin connection $\hat{\nabla}$ preserves the subbundle $H^{(k)}$ under the stated conditions. Theorem 9 is established in [A6] through the following three lemmas.

Lemma 2. Assume that the first Chern class of $(M, \omega)$ is $n[\varpi_{\frac{n}{2}}] \in H^2(M, \mathbb{Z})$. For any $\sigma \in T$ and for any $G \in H^0(M_\sigma, S^2(T_\sigma))$, we have the following formula

$$\nabla^0_{\sigma}(\Delta_G(s) + 2\nabla_GdF_\sigma(s)) = -i(2k + n)\omega G\nabla(s) + 2i(kGdF_\sigma)\omega + i\delta_\sigma(G)\omega,$$

for all $s \in H^0(M_\sigma, \mathcal{L}^k)$.

Lemma 3. We have the following relation

$$4i\partial_\sigma(V''[F]) = 2(G(V)dF)_{\sigma}\omega + \delta_\sigma(G(V))_{\omega}\omega,$$

provided that $H^1(M, \mathbb{R}) = 0$.

Lemma 4. For any smooth vector field $V$ on $T$, we have that

$$2(V'[\text{Ric}])^{1, 1} = \partial(\delta(G(V))\omega).$$

Let us here recall how Lemma 3 is derived from Lemma 4. By the definition of the Ricci potential

$$\text{Ric} = \text{Ric}^H + 2i\partial\bar{\partial}F,$$

where $\text{Ric}^H = n\omega$ by the assumption $c_1(M, \omega) = n[\varpi_{\frac{n}{2}}]$. Hence

$$V'[\text{Ric}] = -dV'[I]dF + 2i\partial\bar{\partial}V'[F],$$

and therefore

$$4i\partial_\sigma(V''[F]) = 2(V'[\text{Ric}])^{1, 1} + 2\partial V''[I]dF.$$

From the above, we conclude that

$$(2(G(V)dF)_{\omega} + \delta(G(V))_{\omega} - 4i\partial\bar{\partial}V'[F])_{\sigma} \in \Omega^{0, 1}_\sigma(M)$$

is a $\partial_\sigma$-closed one-form on $M$. From Lemma 2 it follows that it is also $\partial_\sigma$-closed, whence it must be a closed one-form. Since we assume that $H^1(M, \mathbb{R}) = 0$, we see that it must be exact. But then it in fact vanishes since it is of type $(0, 1)$ on $M_\sigma$.

From the above we conclude that

$$u(V) = \frac{1}{k + n/2}o(V) - V'[F] = -\frac{1}{4k + 2n} \{\Delta_G(V) + 2\nabla_GdF + 4kV'[F]\}$$

solves (8). Thus we have established Theorem 9 and hence Theorem 5.

In [AGL] we use half-forms and the metaplectic correction to prove the existence of a Hitchin connection in the context of half-form quantization. The assumption that the first Chern class of $(M, \omega)$ is $n[\varpi_{\frac{n}{2}}] \in H^2(M, \mathbb{Z})$ is then just replaced by the vanishing of the second Stiefel-Whitney class of $M$ (see [AGL] for more details).

Suppose $\Gamma$ is a group which acts by bundle automorphisms of $\mathcal{L}$ over $M$ preserving both the Hermitian structure and the connection in $\mathcal{L}$. Then there is an induced action of $\Gamma$ on $(M, \omega)$. We will further assume that $\Gamma$ acts on $T$ and that $I$ is $\Gamma$-equivariant. In this case we immediately get the following invariance.
Lemma 5. The natural induced action of $\Gamma$ on $H^{(k)}$ preserves the subbundle $H^{(k)}$ and the Hitchin connection.

We are actually interested in the induced connection $\hat{\nabla}^c_V$ in the endomorphism bundle $\text{End}(H^{(k)})$. Suppose $\Phi$ is a section of $\text{End}(H^{(k)})$. Then for all sections $s$ of $H^{(k)}$ and all vector fields $V$ on $T$, we have that

$$(\hat{\nabla}^c_{V}\Phi)(s) = \hat{\nabla}_{V}\Phi(s) - \Phi(\hat{\nabla}_{V}(s)).$$

Assume now that we have extended $\Phi$ to a section of $\text{Hom}(H^{(k)}, H^{(k)})$ over $T$. Then

$$(12) \quad \hat{\nabla}^c_{V}\Phi = \hat{\nabla}^c_{V}^{\text{tr}}\Phi + [\Phi, u(V)],$$

where $\hat{\nabla}^c_{V}^{\text{tr}}$ is the trivial connection in the trivial bundle $\text{End}(H^{(k)})$ over $T$.

3. TOEPLITZ OPERATORS AND BEREZIN-TOEPLITZ DEFORMATION

We shall in this section discuss the Toeplitz operators and their asymptotics as the level $k$ goes to infinity. The properties we need can all be derived from the fundamental work of Boutet de Monvel and Sjöstrand. In [BdMS], they did a microlocal analysis of the Szegő projection which can be applied to the asymptotic analysis in the situation at hand, as it was done by Boutet de Monvel and Guillemin in [BdMG] (in fact in a much more general situation than the one we consider here) and others following them. In particular, the applications developed by Schlichenmaier and further by Karabegov and Schlichenmaier to the study of Toeplitz operators in the geometric quantization setting is what will interest us here. Let us first describe the basic setting.

For each $f \in C^\infty(M)$, we consider the prequantum operator, namely the differential operator $M_f^{(k)} : C^\infty(M, L^k) \to C^\infty(M, L^k)$ given by

$$M_f^{(k)}(s) = fs$$

for all $s \in H^0(M, L^k)$.

These operators act on $C^\infty(M, L^k)$ and therefore also on the bundle $H^{(k)}$, however, they do not preserve the subbundle $H^{(k)}$. In order to turn these operators into operators which acts on $H^{(k)}$ we need to consider the Hilbert space structure.

Integrating the inner product of two sections against the volume form associated to the symplectic form gives the pre-Hilbert space structure on $C^\infty(M, L^k)$

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_M \langle s_1, s_2 \rangle \omega^m.$$

We think of this as a pre-Hilbert space structure on the trivial bundle $H^{(k)}$ which of course is compatible with the trivial connection in this bundle. This pre-Hilbert space structure induces a Hermitian structure $\langle \cdot, \cdot \rangle$ on the finite rank subbundle $H^{(k)}$ of $H^{(k)}$. The Hermitian structure $\langle \cdot, \cdot \rangle$ on $H^{(k)}$ also induces the operator norm $\| \cdot \|$ on $\text{End}(H^{(k)})$.

Since $H^{(k)}_s$ is a finite dimensional subspace of $C^\infty(M, L^k) = H^{(k)}_s$ and therefore closed, we have the orthogonal projection $\pi^{(k)}_s : H^{(k)}_s \to H^{(k)}_s$. Since $H^{(k)}$ is a smooth subbundle of $H^{(k)}$, the projections $\pi^{(k)}_s$ form a smooth map $\pi^{(k)}$ from $T$ to the space of bounded operators on the $L_2$-completion of $C^\infty(M, L^k)$. The easiest way to see
this is to consider a local frame \((s_1, \ldots, s_{\text{Rank } H^k})\) of \(H^k\). Let \(h_{ij} = \langle s_i, s_j \rangle\), and let \(h_{ij}^{-1}\) be the inverse matrix of \(h_{ij}\). Then

\[
\pi^{(k)}_\sigma(s) = \sum_{i,j} \langle s, (s_i)_\sigma \rangle (h_{ij}^{-1})_{\sigma}(s_j)_{\sigma}.
\]

From these projections, we can construct the Toeplitz operators associated to any smooth function \(f \in C^\infty(M)\). It is the operator \(T_{f,\sigma}^{(k)} : \mathcal{H}^k_\sigma \rightarrow \mathcal{H}^k_\sigma\) defined by

\[
T_{f,\sigma}^{(k)}(s) = \pi^{(k)}_\sigma(f s)
\]

for any element \(s \in \mathcal{H}^k_\sigma\) and any point \(\sigma \in T\). We observe that the Toeplitz operators are smooth sections \(T_{f,\sigma}^{(k)}\) of the bundle \(\text{Hom}(\mathcal{H}^k, H^k)\) and restrict to smooth sections of \(\text{End}(H^k)\).

**Remark 1.** Similarly, for any Pseudo-differential operator \(A\) on \(M\) with coefficients in \(\mathcal{L}^k\) (which may even depend on \(\sigma \in T\)), we can consider the associated Toeplitz operator \(\pi^{(k)}A\) and think of it as a section of \(\text{Hom}(\mathcal{H}^k, H^k)\). However, whenever we consider asymptotic expansions of such or consider their operator norms, we implicitly restrict them to \(H^k\) and consider them as sections of \(\text{End}(H^k)\) or equivalently assume that they have been precomposed with \(\pi^{(k)}\).

Suppose that we have a smooth section \(X \in C^\infty(M, T_\sigma)\) of the holomorphic tangent bundle of \(M_\sigma\). We then claim that the operator \(\pi^{(k)}_\sigma \nabla_X\) is a zero-order Toeplitz operator. Supposing that \(s_1 \in C^\infty(M, \mathcal{L}^k)\) and \(s_2 \in H^0(M_\sigma, \mathcal{L}^k)\), we have that

\[
X(s_1, s_2) = \langle \nabla_X s_1, s_2 \rangle.
\]

Now, calculating the Lie derivative along \(X\) of \((s_1, s_2)\omega^m\) and using the above, one obtains after integration that

\[
\langle \nabla_X s_1, s_2 \rangle = -\langle \delta(X)s_1, s_2 \rangle,
\]

Thus

\[
\pi^{(k)}_\sigma \nabla_X = -T_{\delta(X)}^{(k)},
\]

as operators from \(C^\infty(M, \mathcal{L}^k)\) to \(H^0(M_\sigma, \mathcal{L}^k)\).

Iterating (14), we find for all \(X_1, X_2 \in C^\infty(M, T_\sigma)\) that

\[
\pi^{(k)}_\sigma \nabla_{X_1} \nabla_{X_2} = T_{\delta(X_2)\delta(X_1)+X_2[\delta(X_1)]}^{(k)},
\]

again as operators from \(C^\infty(M, \mathcal{L}^k)\) to \(H^0(M_\sigma, \mathcal{L}^k)\).

We calculate the adjoint of \(\nabla_X\) for any complex vector field \(X \in C^\infty(M, TM_C)\). For \(s_1, s_2 \in C^\infty(M, \mathcal{L}^k)\), we have that

\[
\bar{X}(s_1, s_2) = \langle \nabla_{\bar{X}} s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle.
\]

Computing the Lie derivative along \(\bar{X}\) of \((s_1, s_2)\omega^m\) and integrating, we get that

\[
\langle \nabla_{\bar{X}} s_1, s_2 \rangle + \langle (\nabla_{\bar{X}})^* s_1, s_2 \rangle = -\langle \delta(\bar{X})s_1, s_2 \rangle.
\]

Hence, we see that

\[
(\nabla_{\bar{X}})^* = -\nabla_{\bar{X}} - \delta(\bar{X})
\]
as operators on $C^{\infty}(M, \mathcal{L}^k)$. In particular, if $X \in C^{\infty}(M, T_\sigma)$ is a section of the holomorphic tangent bundle, we see that
\begin{equation}
\pi^{(k)}(\nabla X)^*\pi^{(k)} = -T^{(k)}_{\delta(X)}|_{H^0(M_\sigma, \mathcal{L}^k)},
\end{equation}
again as operators on $H^0(M_\sigma, \mathcal{L}^k)$.

The product of two Toeplitz operators associated to two smooth functions will in general not be the Toeplitz operator associated to a smooth function again. But, by the results of Schlichenmaier [Sch], there is an asymptotic expansion of the product in terms of such Toeplitz operators on a compact Kähler manifold.

**Theorem 10** (Schlichenmaier). For any pair of smooth functions $f_1, f_2 \in C^{\infty}(M)$, we have an asymptotic expansion
\[ T^{(k)}_{f_1, \sigma} T^{(k)}_{f_2, \sigma} \sim \sum_{l=0}^\infty c^{(l)}_{\sigma}(f_1, f_2)_{(f_1, f_2), \sigma} k^{-l}, \]
where $c^{(l)}_{\sigma}(f_1, f_2) \in C^{\infty}(M)$ which correspond to the expansion of the product in $1/(k + n/2)$ (where $n$ is some fixed integer):
\[ T^{(k)}_{f_1, \sigma} T^{(k)}_{f_2, \sigma} \sim \sum_{l=0}^\infty c^{(l)}_{\sigma}(f_1, f_2)_{(f_1, f_2), \sigma} (k + n/2)^{-l}. \]

For future reference, we note that the first three coefficients are given by $c^{(0)}_{\sigma}(f_1, f_2) = c^{(0)}_{\sigma}(f_1, f_2)$, $c^{(1)}_{\sigma}(f_1, f_2) = c^{(1)}_{\sigma}(f_1, f_2)$, and $c^{(2)}_{\sigma}(f_1, f_2) = c^{(2)}_{\sigma}(f_1, f_2) + \frac{1}{2} c^{(1)}_{\sigma}(f_1, f_2)$.

**Remark 2.** It will be useful for us to define new coefficients $\tilde{c}^{(l)}_{\sigma}(f, g) \in C^{\infty}(M)$ which correspond to the expansion of the product in $1/(k + n/2)$ (where $n$ is some fixed integer):
\[ T^{(k)}_{f_1, \sigma} T^{(k)}_{f_2, \sigma} \sim \sum_{l=0}^\infty c^{(l)}_{\sigma}(f_1, f_2)_{(f_1, f_2), \sigma} (k + n/2)^{-l}. \]
For future reference, we note that the first three coefficients are given by $c^{(0)}_{\sigma}(f_1, f_2) = c^{(0)}_{\sigma}(f_1, f_2)$, $c^{(1)}_{\sigma}(f_1, f_2) = c^{(1)}_{\sigma}(f_1, f_2)$, and $c^{(2)}_{\sigma}(f_1, f_2) = c^{(2)}_{\sigma}(f_1, f_2) + \frac{1}{2} c^{(1)}_{\sigma}(f_1, f_2)$.

Theorem 10 is proved in [Sch] where it is also proved that the formal generating series for the $c^{(l)}_{\sigma}(f_1, f_2)$'s gives a formal deformation quantization of symplectic manifold $(M, \omega)$.

We recall the definition of a formal deformation quantization. Introduce the space of formal functions $C^\infty_h(M) = C^\infty(M)[[h]]$ as the space of formal power series in the variable $h$ with coefficients in $C^\infty(M)$. Let $C_h = \mathbb{C}[[h]]$ denote the formal constants.

**Definition 6.** A deformation quantization of $(M, \omega)$ is an associative product $\star$ on $C^\infty_h(M)$ which respects the $C^\infty_h$-module structure. For $f, g \in C^\infty(M)$, it is defined as
\[ f \star g = \sum_{l=0}^\infty c^{(l)}(f, g) h^l, \]
through a sequence of bilinear operators
\[ c^{(l)} : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M), \]
which must satisfy
\[ c^{(0)}(f, g) = fg \quad \text{and} \quad c^{(1)}(f, g) - c^{(1)}(g, f) = -i \{f, g\}. \]
The deformation quantization is said to be differential if the operators \( c^{(l)} \) are bidifferential operators. Considering the symplectic action of \( \Gamma \) on \((M,\omega)\), we say that a star product is \( \Gamma \)-invariant if

\[
\gamma^*(f \ast g) = \gamma^*(f) \ast \gamma^*(g)
\]

for all \( f, g \in C^\infty(M) \) and all \( \gamma \in \Gamma \).

**Theorem 11** (Karabegov & Schlichenmaier). The product \( \ast_{BT}^{\sigma} \) given by

\[
f \ast_{BT}^{\sigma} g = \sum_{l=0}^\infty c^{(l)}(f, g) h^l,
\]

where \( f, g \in C^\infty(M) \) and \( c^{(l)}(f, g) \) are determined by Theorem 10, is a differentiable deformation quantization of \((M,\omega)\).

**Definition 7.** The Berezin-Toeplitz deformation quantization of the compact Kähler manifold \((M_\sigma,\omega)\) is the product \( \ast_{BT}^{\sigma} \).

**Remark 3.** Let \( \Gamma_\sigma \) be the \( \sigma \)-stabilizer subgroup of \( \Gamma \). For any element \( \gamma \in \Gamma_\sigma \), we have that

\[
\gamma^*(T_{f,\sigma}^{(k)}) = T_{\gamma^*f,\sigma}^{(k)}.
\]

This implies the invariance of \( \ast_{BT}^{\sigma} \) under the \( \sigma \)-stabilizer \( \Gamma_\sigma \).

**Remark 4.** Using the coefficients from Remark 3, we define a new star product by

\[
f \tilde{\ast}_{BT}^{\sigma} g = \sum_{l=0}^\infty c^{(l)}(f, g) h^l.
\]

Then

\[
f \tilde{\ast}_{BT}^{\sigma} g = ((f \circ \phi^{-1}) \ast_{BT}^{\sigma} (g \circ \phi^{-1})) \circ \phi
\]

for all \( f, g \in C^\infty_h(M) \), where \( \phi(h) = \frac{2h}{2 + nh} \).

4. THE FORMAL HITCHIN CONNECTION

In this section, we study the the formal Hitchin. We assume the conditions on \((M,\omega)\) and \( I \) of Theorem 10 thus providing us with a Hitchin connection \( \hat{\nabla} \) in \( H^{(k)} \) over \( T \) and the associated connection \( \hat{\nabla}^e \) in \( \text{End}(H^{(k)}) \).

Recall from the introduction the definition of a formal connection in the trivial bundle of formal functions. Theorem 6 establishes the existence of a unique formal Hitchin connection, expressing asymptotically the interplay between the Hitchin connection and the Toeplitz operators.

We want to give an explicit formula for the formal Hitchin connection in terms of the star product \( \ast_{BT}^{\sigma} \). We recall that in the proof of Theorem 10 given in [A6], it is shown that the formal Hitchin connection is given by

\[
\hat{D}(V)(f) = -V[F]f + V[F] \ast_{BT}^{\sigma} f + h(E(V)(f) - H(V) \ast_{BT}^{\sigma} f),
\]

where \( E \) is the one-form on \( T \) with values in \( D(M) \) such that

\[
T_{E(V)f}^{(k)} = \pi^{(k)} o(V)^* f \pi^{(k)} + \pi^{(k)} f o(V) \pi^{(k)},
\]

and \( H \) is the one form on \( T \) with values in \( C^\infty(M) \) such that \( H(V) = E(V)(1) \). Thus, we must find an explicit expression for the operator \( E(V) \).

The following lemmas will prove helpful.
Lemma 6. The adjoint of $\Delta_B$ is given by

$$\Delta_B^* = \Delta_B,$$

for any (complex) symmetric bivector field $B \in C^\infty(M, S^2(TM_C))$.

Proof. First, we write $B = \sum^R_x X_r \otimes Y_r$. Then

$$\Delta_B = \sum^R \nabla_{X_r} \nabla_{Y_r} + \nabla_{\delta(X_r)Y_r}.$$

Now, using (19), we get that

$$\nabla_{(X_r)Y_r}^* = (\nabla_{X_r})^*(\nabla_{Y_r})^* = (\nabla_{X_r} + \delta(\bar{Y}_r)) (\nabla_{\bar{X}_r} + \delta(\bar{X}_r))$$

$$= \nabla_{\bar{Y}_r} \nabla_{\bar{X}_r} + \nabla_{\bar{Y}_r} \delta(X_r) + \delta(\bar{Y}_r) \nabla_{\bar{X}_r} + \delta(\bar{Y}_r) \delta(X_r),$$

and

$$(\nabla_{\delta(X_r)Y_r})^* = -\nabla_{\delta(\bar{X}_r)\bar{Y}_r} - \delta(\bar{X}_r) \bar{Y}_r$$

$$= -\delta(X_r) \nabla_{\bar{Y}_r} - \bar{Y}_r [\delta(\bar{X}_r)] - \delta(\bar{X}_r) \delta(\bar{Y}_r)$$

$$= -\nabla_{\bar{Y}_r} \delta(X_r) - \delta(\bar{X}_r) \delta(\bar{Y}_r),$$

so we conclude that

$$\Delta_B^* = \sum^R \nabla_{\bar{Y}_r} \nabla_{\bar{X}_r} + \delta(\bar{Y}_r) \nabla_{\bar{X}_r} = \Delta_B,$$

since $B$ is symmetric. \qed

Lemma 7. The operator $\Delta_B$ satisfies

$$\pi^{(k)} \Delta_B s = 0,$$

for any section $s \in C^\infty(M, \mathcal{L}^k)$ and any symmetric bivector field $B$.

Proof. Again, we write $B = \sum^R X_r \otimes Y_r$ and recall from (15) that

$$\pi^{(k)} \nabla_{X_r} \nabla_{Y_r} s = \pi^{(k)} (\delta(X_r) \delta(Y_r) + Y_r [\delta(X_r)]) s.$$

On the other hand, we have that

$$\pi^{(k)} \nabla_{\delta(X_r)Y_r} s = -\pi^{(k)} (\delta(X_r) Y_r) s = -\pi^{(k)} (\delta(X_r) \delta(Y_r) + Y_r [\delta(X_r)]) s,$$

and it follows immediately that

$$\pi^{(k)} \Delta_B s = \pi^{(k)} \sum^R (\nabla_{X_r} \nabla_{Y_r} + \nabla_{\delta(X_r)Y_r}) s = 0,$$

which proves the lemma. \qed

Finally, it will prove useful to observe that

$$\delta(Bdf) = \Delta_B(f),$$

for any function $f$ and any bivector field $B$.

Now, the adjoint of $o(V)$ is given by

$$o(V)^* = -\frac{1}{4} (\Delta_{\bar{G}(V)} - 2\nabla_{\bar{G}(V)}dF - 2\Delta_{\bar{G}(V)}(F) - 2nV''[F]),$$
where we used (20). Furthermore, we observe that \( o(V)^* \) differentiates in anti-holomorphic directions only, which implies that
\[
\pi^{(k)} o(V)^* f \pi^{(k)} = \pi^{(k)} o(V)^* (f) \pi^{(k)}
\]
\[
= -\frac{1}{4} \pi^{(k)} (\Delta_{\bar{G}(V)} (f) - 2 \nabla_{\bar{G}(V) df}(f) - 2 \Delta_{\bar{G}(V)} (F) f - 2 n V'' [F] f) \pi^{(k)}.
\]
This gives an explicit formula for the first term of (19).

To determine the second term of (17), we observe that
\[
\Delta_{\bar{G}(V)} f s = f \Delta_{\bar{G}(V)} s + \Delta_{\bar{G}(V)} (f) s + 2 \nabla_{\bar{G}(V) df} s.
\]
Projecting both sides onto the holomorphic sections and applying Lemma 7 and the formula (20), we get that
\[
\pi^{(k)} f \Delta_{\bar{G}(V)} = -\pi^{(k)} (\Delta_{\bar{G}(V)} (f) + 2 \nabla_{\bar{G}(V) df}) = \pi^{(k)} \Delta_{\bar{G}(V)} (f).
\]
Furtermore, observe that
\[
\pi^{(k)} f \nabla_{\bar{G}(V) df} = \pi^{(k)} (\nabla_{\bar{G}(V) df} f - \nabla_{\bar{G}(V) df}(f))
\]
\[
= -\pi^{(k)} (\nabla_{\bar{G}(V) df}(f) + \Delta_{\bar{G}(V)} (f) f),
\]
where we once again used (20) for the last equality. Thus, we get that
\[
\pi^{(k)} f o(V) \pi^{(k)} = -\frac{1}{4} \pi^{(k)} (\Delta_{\bar{G}(V)} (f) - 2 \nabla_{\bar{G}(V) df}(f) - 2 \Delta_{\bar{G}(V)} (F) f - 2 n V' [F] f) \pi^{(k)},
\]
which gives an explicit formula for the second term of (19). Finally, we can conclude that
\[
E(V)(f) = -\frac{1}{4} (\Delta_{\bar{G}(V)} (f) - 2 \nabla_{\bar{G}(V) df}(f) - 2 \Delta_{\bar{G}(V)} (F) f - 2 n V [F] f),
\]
satisfies (19) and hence (18). Also, we note that
\[
H(V) = E(V)(1) = \frac{1}{2} (\Delta_{\bar{G}(V)} (F) + n V [F]).
\]

Summarizing the above, we have proved the following

**Theorem 12.** The formal Hitchin connection is given by
\[
D_V f = V[f] - \frac{1}{4} h \Delta_{\bar{G}(V)} (f) + \frac{1}{2} h \nabla_{\bar{G}(V) df}(f) + V[F] \hat{\star}^{BT} f - V[F] f
\]
\[
- \frac{1}{2} h (\Delta_{\bar{G}(V)} (F) \hat{\star}^{BT} f + n V[F] \hat{\star}^{BT} f - \Delta_{\bar{G}(V)} (F) f - n V[F] f)
\]
for any vector field \( V \) and any section \( f \) of \( C_h \).

The next lemma is also proved in [A6], and it follows basically from the fact that
\[
\hat{\nabla}_V^e (T^{(k)}_f T^{(k)}_g) = \hat{\nabla}_V^e (T^{(k)}_f T^{(k)}_g) + T^{(k)}_f \hat{\nabla}_V^e (T^{(k)}_g).
\]
We have

**Lemma 8.** The formal operator \( D_V \) is a derivation for \( \hat{\star}^{BT}_\sigma \) for each \( \sigma \in T \), i.e.
\[
D_V (f \hat{\star}^{BT}_\sigma g) = D_V (f) \hat{\star}^{BT}_\sigma g + f \hat{\star}^{BT}_\sigma D_V (g)
\]
for all \( f, g \in C^\infty (M) \).
If the Hitchin connection is projectively flat, then the induced connection in the endomorphism bundle is flat and hence so is the formal Hitchin connection by Proposition 3 of [A6].

Recall from Definition 3 in the introduction the definition of a formal trivialization. As mentioned there, such a formal trivialization will not exist even locally on $T$, if $D$ is not flat. However, if $D$ is flat, then we have the following result.

**Proposition 1.** Assume that $D$ is flat and that $\tilde{D} = 0 \mod h$. Then locally around any point in $T$, there exists a formal trivialization. If $H^1(T, \mathbb{R}) = 0$, then there exists a formal trivialization defined globally on $T$. If further $H^1_1(T, D(M)) = 0$, then we can construct $P$ such that it is $\Gamma$-equivariant.

In this proposition, $H^1_1(T, D(M))$ simply refers to the $\Gamma$-equivariant first de Rham cohomology of $T$ with coefficients in the real $\Gamma$-vector space $D(M)$.

Now suppose we have a formal trivialization $P$ of the formal Hitchin connection $D$. We can then define a new smooth family of star products, parametrized by $T$, by

$$f \star_{\sigma} g = P^{-1}_{\sigma}(P_{\sigma}(f) \star^{BT}_{\sigma} P_{\sigma}(g))$$

for all $f, g \in C^\infty(M)$ and all $\sigma \in T$. Using the fact that $P$ is a trivialization, it is not hard to prove.

**Proposition 2.** The star products $\star_{\sigma}$ are independent of $\sigma \in T$.

Then, we have the following which is proved in [Ar].

**Theorem 13 (Andersen).** Assume that the formal Hitchin connection $D$ is flat and

$$H^1_1(T, D(M)) = 0,$$

then there is a $\Gamma$-invariant trivialization $P$ of $D$ and the star product

$$f \star g = P^{-1}_{\sigma}(P_{\sigma}(f) \star^{BT}_{\sigma} P_{\sigma}(g))$$

is independent of $\sigma \in T$ and $\Gamma$-invariant. If $H^1_1(T, C^\infty(M)) = 0$ and the commutant of $\Gamma$ in $D(M)$ is trivial, then a $\Gamma$-invariant differential star product on $M$ is unique.

We calculate the first term of the equivariant formal trivialization of the formal Hitchin connection. Let $f$ be any function on $M$ and suppose that $P(f) = \sum I \hat{f}^I h^I$ is parallel with respect to the formal Hitchin connection. Thus, we have that

$$0 = D_V P(f) = h(V[\hat{f}] \hat{f} - \frac{1}{4} \Delta_{\tilde{G}(V)}(\hat{f}) + \frac{1}{2} \nabla_{\tilde{G}(V)} dF(\hat{f}) + c^{(1)}(V[F], \hat{f})) + O(h^2).$$

But $\hat{f}_0 = f$, and so we get in particular

$$0 = V[\hat{f}] - \frac{1}{4} \Delta_{\tilde{G}(V)}(f) + \frac{1}{2} \nabla_{\tilde{G}(V)} dF(f) + c^{(1)}(V[F], f).$$

By the results of [KS], $\star^{BT}$ is a differential star product with separation of variables, in the sense that it only differentiates in holomorphic directions in the first entry and antiholomorphic directions in the second. As argued in [Kar], all such star products have the same first order coefficient, namely

$$c^{(1)}(f_1, f_2) = -g(\partial f_1, \bar{\partial} f_2) = i \nabla_{X^r_{f_1}} (f_2)$$

(21)
for any functions $f_1, f_2 \in C^\infty(M)$. From this, it is easily seen that

$$V[c^{(1)}](f_1, f_2) = \frac{1}{2} \delta f_1 \hat{G}(V) df_2 - \frac{1}{2} \delta G(V) df_1 \tilde{f_2}. $$

Applying this to (21), we see that

$$(23) \quad V[\hat{f}] = \frac{1}{4} \Delta \hat{G}(V)(f) - V[c^{(1)}](F, f).$$

But the variation of the Laplace-Beltrami operator is given by

$$V[\Delta] = V[\delta(g^{-1}df)] = \delta(V[g^{-1}]df) = -\Delta \hat{G}(V)f,$$

and so we conclude that

$$(24) \quad V[\hat{f}] = -V[\frac{1}{4} \Delta f + c^{(1)}(F, f)].$$

We have thus proved

**Proposition 3.** When it exists, the equivariant formal trivialization of the formal Hitchin connection has form

$$P = Id - h\left(\frac{1}{4} \Delta + i \nabla_{X_p} f\right) + O(h^2).$$

Using this proposition, one easily calculates that

$$P(f_1) \star^{BT} P(f_2) = f_1 f_2 - h\left(\frac{1}{4} \Delta f_1 + \frac{1}{4} \Delta f_2 + i \nabla_{X_p} f_1 + i \nabla_{X_p} f_2\right)$$

$$+ h c^{(1)}(f_1, f_2) + O(h^2).$$

Finally, using the explicit formula (22) for $c^{(1)}$, we get that

$$P^{-1}(P(f_1) \star^{BT} P(f_2)) = f_1 f_2 - h g(\partial f_1, \bar{\partial} f_2) + \frac{1}{2} h g(df_1, df_2) + O(h^2)$$

$$= f_1 f_2 - h \left(\frac{1}{2} g(\partial f_1, \bar{\partial} f_2) - g(\bar{\partial} f_1, \partial f_2)\right) + O(h^2)$$

$$= f_1 f_2 - ih \left\{ f_1, f_2 \right\} + O(h^2).$$

This proves Theorem $^8$

**References**

[A1] J.E. Andersen, *New polarizations on the moduli space and the Thurston compactification of Teichmuller spaces*, International Journal of Mathematics, 9, No.1 (1998), 1–45.

[A2] J.E. Andersen, *Geometric quantization and deformation quantization of abelian moduli spaces*, Commun. Math. Phys. 255 (2005), 727–745.

[A3] J. E. Andersen, *Asymptotic faithfulness of the quantum SU(n) representations of the mapping class groups*. Annals of Mathematics, 163 (2006), 347–368.

[A4] J.E. Andersen, *Asymptotic faithfulness of the quantum SU(n) representations of the mapping class groups in the singular case*, In preparation.

[A5] J.E. Andersen, *The Nielsen-Thurston classification of mapping classes is determined by TQFT*, [math.QA/0605036](http://arxiv.org/abs/math.QA/0605036).

[A6] J.E. Andersen, *Hitchin’s connection, Toeplitz operators and symmetry invariant deformation quantization*, [math.DG/0611126](http://arxiv.org/abs/math.DG/0611126).

[A7] J.E. Andersen, *Asymptotic in Teichmuller space of the Hitchin connection*, In preparation.

[A8] J.E. Andersen, *Mapping Class Groups do not have Kazhdan’s Property (T)*, [math.QA/0706.2184](http://arxiv.org/abs/math.QA/0706.2184).

[AC] J.E. Andersen & M. Christ, *Asymptotic expansion of the Szegő kernel on singular algebraic varieties*, In preparation.
[AGL] J.E. Andersen, M. Lauritsen & N. L. Gammelgaard, Hitchin’s Connection in Half-form Quantization, arXiv:0711.3995.

[AMU] J.E. Andersen, G. Masbaum & K. Ueno, Topological quantum field theory and the Nielsen-Thurston classification of $M(0, 4)$, Math. Proc. Cambridge Philos. Soc. 141 (2006), no. 3, 477–488.

[AMR1] J.E. Andersen, J. Mattes & N. Reshetikhin, The Poisson Structure on the Moduli Space of Flat Connections and Chord Diagrams, Topology 35, pp.1069–1083 (1996).

[AMR2] J.E. Andersen, J. Mattes & N. Reshetikhin, Quantization of the Algebra of Chord Diagrams. Math. Proc. Camb. Phil. Soc. 124 pp.451–467 (1998).

[AU1] J.E. Andersen & K. Ueno, Abelian Conformal Field theories and Determinant Bundles, International Journal of Mathematics. 18, (2007) 919–993.

[AU2] J.E. Andersen & K. Ueno, Constructing modular functors from conformal field theories, Journal of Knot theory and its Ramifications. 16 2 (2007), 127–202.

[AU3] J.E. Andersen & K. Ueno, Modular functors are determined by their genus zero data, math.QA/0611087.

[AU4] J.E. Andersen & K. Ueno, Construction of the Reshetikhin-Turaev TQFT from conformal field theory, In preparation.

[AV1] J.E. Andersen & R. Villemoes, Degree One Cohomology with Twisted Coefficients of the Mapping Class Group, arXiv:0710.2203v1.

[AV2] J.E. Andersen & R. Villemoes, The first cohomology of the mapping class group with coefficients in algebraic functions on the SL(2, C) moduli space, arXiv:0802.4372v1.

[AV3] J.E. Andersen & R. Villemoes, Cohomology of mapping class groups and the abelian moduli space, In preparation.

[At] M. Atiyah, The Jones-Witten invariants of knots. Séminaire Bourbaki, Vol. 1989/90. Astérisque No. 189-190 (1990), Exp. No. 715, 7–16.

[AB] M. Atiyah & R. Bott, The Yang-Mills equations over Riemann surfaces. Phil. Trans. R. Soc. Lond., Vol. A308 (1982) 523–615.

[ADW] S. Axelrod, S. Della Pietra, E. Witten, Geometric quantization of Chern-Simons gauge theory, J.Diff.Geom. 33 (1991) 787–902.

[BC] S. Bleiler & A. Casson, Automorphisms of surfaces after Nielsen and Thurston, Cambridge University Press, 1988.

[BMS] M. Bordeman, E. Meinrenken & M. Schlichenmaier, Toeplitz quantization of Kähler manifolds and $gl(N)$, $N \to \infty$ limit, Comm. Math. Phys. 165 (1994), 281–296.

[BdMQ] L. Boutet de Monvel & V. Guillemin, The spectral theory of Toeplitz operators, Annals of Math. Studies 99, Princeton University Press, Princeton.

[BdMS] L. Boutet de Monvel & J. Sjöstrand, Sur la singularité des noyaux de Bergmann et de Szegő, Astrélique 34-35 (1976), 123–164.

[DN] J.-M. Drezet & M.S. Narasimhan, Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques, Invent. math. 97 (1989) 53–94.

[Fal] G. Faltings, Stable $G$-bundles and projective connections, J.Alg.Geom. 2 (1993) 507–568.

[FLP] A. Fathi, F. Laudenbach & V. Poénaru, Travaux de Thurston sur les surfaces, Astérisque 66–67 (1991/1979).

[Fi] M. Finkelberg, An equivalence of fusion categories, Geom. Funct. Anal. 6 (1996), 249–267.

[Fr] D.S. Freed, Classical Chern-Simons Theory, Part I, Adv. Math. 113 (1995), 237–303.

[FWW] M. H. Freedman, K. Walker & Z. Wang, Quantum SU(2) faithfully detects mapping class groups modulo center, Geom. Topol. 6 (2002), 523–539.
V. V. Fock & A. A. Rosly, Flat connections and polyubles. Teoret. Mat. Fiz. 95 (1993), no. 2, 228–238; translation in Theoret. and Math. Phys. 95 (1993), no. 2, 526–534

V. V. Fock & A. A. Rosly, Moduli space of flat connections as a Poisson manifold. Advances in quantum field theory and statistical mechanics: 2nd Italian-Russian collaboration (Como, 1996). Internat. J. Modern Phys. B 11 (1997), no. 26-27, 3195–3206.

B. Van Geemen & A. J. De Jong, On Hitchin’s connection. J. of Amer. Math. Soc., 11 (1998), 189–228.

W. M. Goldman, Ergodic theory on moduli spaces. Ann. of Math. (2) 146 (1997), no. 3, 475–507.

W. M. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface group representations. Invent. Math. 85 (1986), no. 2, 263–302.

S. Gutt & J. Rawnsley, Equivalence of star products on a symplectic manifold. J. of Geom. Phys., 29 (1999), 347–392.

N. Hitchin, Flat connections and geometric quantization. Comm. Math. Phys., 131 (1990) 347–380.

A. V. Karabegov & M. Schlichenmaier, Identification of Berezin-Toeplitz deformation quantization. J. Reine Angew. Math. 540 (2001), 49–76.

A. V. Karabegov, Deformation Quantization with Separation of Variables on a Kähler Manifold. Comm. Math. Phys. 180 (1996) (3), 745—755.

V. G. Kac, Infinite dimensional Lie algebras, Third Edition, Cambridge University Press, (1995).

D. Kazhdan, Connection of the dual space of a group with the structure of its closed subgroups. Funct. Anal. Appl. 1 (1967), 64–65.

D. Kazhdan & G. Lustzig, Tensor structures arising from affine Lie algebras I. J. AMS, 6 (1993), 905–947; II J. AMS, 6 (1993), 949–1011; III J. AMS, 7 (1994), 335–381; IV, J. AMS, 7 (1994), 383–453.

Y. Laszlo, Hitchin’s and WZW connections are the same. J. Diff. Geom. 49 (1998), no. 3, 547–576.

G. Masbaum, Quantum representations of mapping class groups. Low-dimensional topology (Funchal, 1998), 137–139, Contemp. Math., 233, Amer. Math. Soc., Providence, RI, 1999.

G. Masbaum. Quantum representations of mapping class groups. In: Groupes et Géométrie (Journée annuelle 2003 de la SMF). pages 19–36

G. Moore and N. Seiberg, Classical and quantum conformal field theory, Comm. Math. Phys. 123 (1989), 177–254.

M.S. Narasimhan and C.S. Seshadri, Holomorphic vector bundles on a compact Riemann surface, Math. Ann. 155 (1964) 69–80.

M.S. Narasimhan and C.S. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, Ann. Math. 82 (1965) 540–67.

T.R. Ramadas, Chern-Simons gauge theory and projectively flat vector bundles on $M_8$. Comm. Math. Phys. 126 (1990), no. 2, 421–426.

T.R. Ramadas, I.M. Singer and J. Weitsman, Some Comments on Chern – Simons Gauge Theory, Comm. Math. Phys. 126 (1989) 409-420.

N. Reshetikhin & V. Turaev, Ribbon graphs and their invariants derived from quantum groups. Comm. Math. Phys. 127 (1990), 1–26.

N. Reshetikhin & V. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, Invent. Math. 103 (1991), 547–597.

J. Roberts, Irreducibility of some quantum representations of mapping class groups. J. Knot Theory and its Ramifications 10 (2001) 763 – 767.

M. Schlichenmaier, Berezin-Toeplitz quantization and conformal field theory. Thesis.

M. Schlichenmaier, Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization. In Conference Moshé Flato 1999, Vol II (Dijon), 289–306, Math. Phys. Stud., 22, Kluwer Acad. Publ., Dordrecht, (2000), 289–306.

M. Schlichenmaier, Berezin-Toeplitz quantization and Berezin transform. In Long time behaviour of classical and quantum systems (Bologna, 1999), Ser. Concr. Appl. Math., 1, World Sci. Publishing, River Edge, NJ, (2001), 271–287.

G. Segal, The Definition of Conformal Field Theory, Oxford University Preprint (1992).
[Th] W. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. 19 (1988), 417–431.

[TUY] A. Tsuchiya, K. Ueno & Y. Yamada, *Conformal Field Theory on Universal Family of Stable Curves with Gauge Symmetries*, Advanced Studies in Pure Mathematics, 19 (1989), 459–566.

[T] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, 1994. x+588 pp. ISBN: 3-11-013704-6

[Tuyn] Tuynman, G.M., *Quantization: Towards a comparison between methods*, J. Math. Phys. 28 (1987), 2829–2840.

[Vi] R. Villemoes, *The mapping class group orbit of a multicurve*, arXiv:0802.3000v2

[Wa] K. Walker, *On Witten’s 3-manifold invariants*, Preliminary version # 2, Preprint 1991.

[Wi] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys 121 (1989) 351–98.

[Wo] N.J. Woodhouse, *Geometric Quantization*, Oxford University Press, Oxford (1992).

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