Nanolensed Fast Radio Bursts

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Abstract

It is suggested that fast radio bursts can probe gravitational lensing by clumpy dark matter objects that range in mass from \(10^{-3} \, M_\odot\) to \(10^3 \, M_\odot\). They may provide a more sensitive probe than observations of lensings of objects in the Magellanic Clouds, and could find or rule out clumpy dark matter with an extended mass spectrum.

Key words: gravitational lensing: micro – techniques: miscellaneous

1. Introduction

Zheng et al. (2014) and Muñoz et al. (2016) have suggested that compact dark matter objects (MACHOs)\(^1\) of masses \(M \sim 30 \, M_\odot\) could be detected by microlensing background fast radio bursts (FRBs). Because compact objects in that mass range would cause time delays between respective arrivals of multiple FRB images of the order of 1 ms, the same order of the FRB duration itself, the multiple images could be resolved in time. Aside from dark matter, the stars and black holes that are known to exist in that same mass range constitute only a tiny optical depth (\(\ll 10^{-5}\)) to lensing, so we should not expect to detect any gravitational echoes of FRBs caused by known objects until hundreds of thousands of FRBs have been detected. Even at a detection rate of 20 per day, this would take many decades. So 30 \(M_\odot\) MACHOs could stand out above stars of similar mass even if they comprise only a small fraction of the dark matter.

On the other hand, the fraction of cosmic mass in familiar stars at \(\lesssim 1 \, M_\odot\) is of the order of \(10^{-2} \, \rho_c\) — far larger than that at \(\sim 30 \, M_\odot\). If the dark matter is in the form of (not-so-massive) compact halo objects (CHOs), \(M \lesssim 10 \, M_\odot\), then the delay is far less than 1 ms, and will go unnoticed in the inspection of an FRB light curve. However, the constraints from EROS and MACHO rule out the total being in any one decade in the \(10^{-7} \, M_\odot \lesssim M \lesssim M_\odot\) mass range, so the only possibility of having \(\Omega \gtrsim 0.2\) in CHO within this mass range is if it is spread out over many decades. Perhaps this is possible in some rollover scenarios for the early universe (Carr et al. 2016). In any case, there is motivation to search the \(M \lesssim M_\odot\) mass range further, to look for lensing manifestations of both known and hypothetical objects in the cosmic density range \(10^{-2} - 10^{-1} \, \rho_c\).

In this paper, it is suggested that lensing delays, \(t_{\text{delay}}\), of much less than 1 ms could be detected by nanolensing of FRBs. Because FRBs are detected at frequencies of nearly 1 GHz, lensing delays as small as 1 ns can affect the waveform at the detector. The effect is similar to “femtolensing” (Gould 1992), but is obviously sensitive to a different mass range of the lenses, and has different detection challenges. It is suggested here that nanolensing of FRBs is in principle detectable. At first, this claim might seem surprising because intergalactic dispersion smears out the arrival time of FRBs by a factor of \(10^5\) times their true physical durations of the order of 1 ms. The true pre-dispersion waveform is deduced by measuring the arrival time at different frequencies, which essentially measures the amount of dispersion that was introduced over the travel time of the pulse. However, the dispersion within each frequency bin may be more than the baseband period, and even more than the lensing delay \(t_{\text{delay}}\).

Similarly, as discussed below, delays associated with pulse broadening can exceed that due to the lens. Thus, it becomes questionable whether enough information can be recovered to allow the detection of lensing delays much shorter than the FRB, or detections even shorter than the scattering delay. Though this is not entirely obvious, we argue below that in general this information can be recovered.

2. Long Range Correlation in the Stokes Parameters

Suppose the received pulse is of the form

\[
E(t) = \int G(\nu') \exp[-i \nu' t] d\nu',
\]

where \(G\) is non-negligible over the frequency range \(\nu/2\pi \equiv \nu\) to \(\nu + \delta\nu\) of the detector, which, for FRB, is of the order of hundreds of MHz. For simplicity, we consider only one polarization but define \(E\) to be a complex number that contains phase information. We can assume that \(G\) is negligible over frequencies that differ from the detector’s frequency range by orders of magnitude, because any other possibility is unnecessary to build a radio pulse at frequencies of hundreds of megahertz that last over far longer timescales \(\delta t\) than \(\nu^{-1}\); i.e., because \(\delta t \gg \nu^{-1}\), it follows that \(\delta\nu\) is allowed by the uncertainty relation to be much less than \(\nu\), and therefore \(G\) need not extend in frequency beyond the intrinsic spectrum of the source.

The correlation function \(\langle E^*(t) E(t + \delta) \rangle \equiv \int_{-\infty}^{+\infty} E^*(t) E(t + \delta) dt\) is then given by

\[
E^*(t) E(t + \Delta) = \int_{-\infty}^{+\infty} G^*(\nu') G(\nu') \exp[i \nu' t - i \nu'' t - i \nu'' \Delta] d\nu'' d\nu' = \int_{-\infty}^{+\infty} G^*(\nu') G(\nu') \exp[i \nu' t - i \nu'' t - i \nu'' \Delta] d[\nu' - \omega''] d\omega'' ,
\]

and

\[
\int E^*(t) E(t + \Delta) dt = 2\pi \int_{-\infty}^{+\infty} G^*(\nu') G(\nu') \times \exp[-i \nu' \Delta] d\nu'.
\]
Now, if
\[ \Delta \gg \omega^{-1}, \] (4)
wherever \( G(\omega') \) is non-negligible, then \( \langle E^*(t)E(t+\Delta) \rangle \) is exponentially small, whereas if the inverse is true, the right side of Equation (3) is approximately \( \int_{-\infty}^{\infty} |G(\omega)'|^2 d\omega' \).

As a function of \( \Delta \), then, \( \int E^*(t)E(t+\Delta)dt \) is sharply peaked around \( \Delta = 0 \) and this peak has a width of \( \sim 1/\omega \equiv 1/2\pi \nu \).

When \( \Delta = 0 \),
\[ 2\pi \int_{-\infty}^{\infty} G^*(\omega')G(\omega')\exp[-i\omega'\Delta]d\omega' = \int |E^2(t)|dt. \] (5)

On the other hand, if \( E(t) \) is of the form \( aF(t) + bF(t-\Delta) \), as it would be for lensed FRBs with \( \Delta_0 \) being the lensing delay between the two images, then \( G(\omega') = [aG_F + be^{i\Delta_0}G_F] \), where \( G_F(\omega') = \int_{-\infty}^{\infty} e^{i\omega't}F(t)dt \), so \( G^*(\omega')G(\omega') = [a^2 + b^2G_F^2 + 2ab\cos(\omega'\Delta_0)G_F^2]. \)

Clearly, there is a second peak at \( \Delta = \Delta_0 \gg 1/\omega \), and
\[ \int E^*(t)E(t+\Delta_0)dt \approx 2\pi ab \int |F^2(t)|dt. \] (6)

Said differently: the short term \( \omega' \)-periodicity in \( G^*(\omega')G(\omega') \), \( \cos(\omega'\Delta_0) \), that is associated with lensing delays, resonates with the \( e^{-i\omega'\Delta_0} \) factor in the integrand of Equation (3) when \( \Delta = \Delta_0 \), whereas, in the absence of lensing, the rapid variation of \( e^{-i\omega'\Delta_0} \) with \( \omega' \) causes self-cancellation of the integrand. We therefore note that another necessary condition for distinguishing lensed sources from unlensed ones is that \( G(\omega) \) be sufficiently broadband that the spread in frequencies of the signal, \( \delta\omega \), obeys
\[ \delta\omega \gg \Delta_0^{-1}. \] (7)

This condition, however, is invariably met by most astrophysical sources (except for line emitters).

Note that the right side of Equation (3) contains no phase information: if the power spectrum \( G^*(\omega')G(\omega') \) is known, then this uniquely specifies the left side. Other physical phenomena that cause delay in one light path relative to another, such as scattering, affect the phase of each Fourier component \( G(\omega') \), but their effect on the correlation function \( \langle E^*(t)E(t+\Delta) \rangle \) is expressed in Equation (3) through the rapid fluctuation of \( G^*(\omega')G(\omega') \).

Also, note this point, which will prove to be important below: when diffractive scintillation is discussed, although the \( e^{-i\omega'\Delta_0} \) causes near cancellation of the integral (3), when \( \Delta_0 \gg 1/\omega \), rapid fluctuation in \( G^*(\omega')G(\omega') \) with \( \omega' \), as is produced by scintillation, does not cause cancellation, even when the fluctuation scale \( \delta\omega' \) is much smaller than \( 1/\Delta_0 \) (i.e., the scattering delay is much larger than \( \Delta_0 \)), because \( G^*(\omega')G(\omega') \) is positive-definite.

The above assumes that the lensing delay \( \Delta_0 \) is well-defined to better than \( 1/\omega \) for values of \( \omega \) with a significant spectral component. If \( \Delta_0 \) is by hypothesis less than the duration of the undispersed pulse, \( \tau \), which is of the order of 1 ms, then the spread in the lensing delays is only of the order of \( (R/c)/(2\Gamma) \Delta_0 \ll (R/c)/(2\Gamma)\tau \), where \( R \) and \( c \) are respectively the apparent source radius and the Einstein radius of the lens. (That the delay is only second-order in \( (R/c)/(2\Gamma) \) follows from Fermat’s principle, namely that the propagation time is a local minimum relative to all other neighboring trajectories, e.g., Narayan & Bartleman 1996). If FRBs are at cosmological distances, then, as the apparent source size \( R_0 \) is limited to the FRB pulse width \((\sim 10^{-3}c) \), and \( R_E^2 \) is of the order of \( c^2H^{-1}\Delta_0 \), it follows that the spread in delays, which is of the order of \( (R/c)/(2\Gamma) \Delta_0 \), is indeed much less than 1 ns.

Similarly, the difference in the dispersion time \( t_d \sim 1 \) along the two light paths, if the intergalactic medium is smooth, of the order of \( 0.01t_d \), where \( \theta d \), the angular separation of the images, obeys \( \theta d^2/H \sim 2\Delta_0 \). If \( \Delta_0 \ll 1 \) ms, then \( \theta d < 10^{-11}. \) Conceivably, the intervening medium is clumpy on a much smaller scale than \( c/H \), so the difference in dispersions — in this case governed by the sharp density gradients of the clumps — is larger. This would not destroy the effect; the delay is nonetheless well-defined. It would be due to dispersion differences rather than the lens, but the existence of the multiple paths — and hence the existence of the lens — would be demonstrated.

A related question is whether the delay is constant over the course of the FRB. While the apparent source size of an FRB is only about 1 milli-lightsecond, the bulk Lorentz factor \( \Gamma \) of the emitting region must be at least of the order of \( 10^{3.5} \) (Lyubarsky 2014), so the emitting region may move relative to the observer with an apparent velocity of \( \geq 10^{3.5}c \). Now, for a source at a cosmological distance \( c/H \), the Einstein radius of a lens of mass \( m \) is \((c/H)(2Gm/c^3)^{1/2} \approx (10^{16}(m/M_\odot))^{1/2} \) cm. So the apparent time to cross the Einstein radius is \( \sim 10^5(3.5/\Gamma)[m/M_\odot]^{1/2} \) s, and over the 1 ms duration the change in the source position could be at most \( 10^{-5}(3.5/\Gamma)[m/M_\odot]^{-1/2}R_E \). The delay in the change \( \Delta_0 \) due to the motion of the emitting region would then be less than \( 10^{-5}(3.5/\Gamma)[m/M_\odot]^{-1/2}\Delta_0 \approx 10^{-15}[\Gamma /10^{-5}]^2 \) s, which, for imaginable values of \( \Gamma \), is much less than the baseline period.

The importance of condition (3) over the full frequency range of significant \( G(\omega) \), together with the broadband requirement (7) on \( G \), is emphasized. The effect suggested here is not apparent in any formalism that assumes a monochromatic wave because adding a monochromatic wave to a delayed version of itself merely shifts the phase, and changes the amplitude by a modest factor (e.g., Deguchi & Watson 1986, Stanek et al. 2016). This is insufficient to reveal to the observer whether or not a delay has occurred.

The diffraction width over a Hubble distance \( c/H \) is the diffraction angle \( \theta d \approx \lambda/R_E \) times \( c/H \). This is small relative to the deflection length, which is of the order of \( R_E \approx (2Gm/c^2)^{1/2} \) if
\[ \lambda \ll 2Gm/c^2, \] (8)
and for \( \lambda \sim 10^2 \) cm, this imposes the condition \( m \gg 10^{-3} M_\odot \). If this condition is not met, diffraction overwhelms the effect of the lens and there are no lensing effects to speak of.

The CHIME array is expected to detect \( \sim 20 \) FRBs per day (V. Kaspi 2017, private communication), hence a sample set of many thousands will be collected over several years. Self-correlation can pick out multiple images even with large brightness contrasts, so the nanolensing cross section for any distant source is \( \pi R_E^2 \), where \( R_E \) is the Einstein radius. The optical depth for nanolensing from FRBs of high-redshift sources is of order \( \Omega_\Omega \), where \( \Omega_\Omega \) is the fraction of the critical density that is in the form of compact objects that play the role of the lens. So for sample sets exceeding \( 10^2 \) in number, the method can be checked with known objects such as stars. The method in principle can limit \( \Omega_\Omega \) to \( \leq 10^{-2} \) in the sub-\( M_\odot \) mass range. This would be
enough to rule out (or confirm) the possibility that the dark matter is in clumps, even with an extended mass spectrum. Fast Fourier Transform radio telescope arrays (Tegmark & Zaldarriaga 2008) typically correlate signals over the different antennas and thus establish the direction of any source. Clearly, the phase information at each antenna must be accurate to a time resolution of much better than $v^{-1}$ and stored long enough to compute the correlation. For any direction (other than the zenith), a given signal appears delayed in some antennas relative to others. A lensed FRB, on the other hand, creates a long-term time correlation within any given single antenna. It is important that the design of the telescope include this possibility—i.e., that phase information be stored long enough and then searched for self-time-correlation within individual antennas.

3. The Effects of an Inhomogeneous Plasma on the Signal

FRB are strongly scattered (Masui et al. 2015) and this scattering can cause arrival delays of the order of 1 ms, which is by hypothesis larger than the gravitational lens delay. It should be emphasized that Equation (1), which refers to the received pulse at the detector, does not deny this possibility. The signal, until it arrives at the detector, can experience whatever processing one would care to imagine by the medium through which it propagates, including delays larger than the delay associated with the gravitational lens. What is assumed in the later analysis is simply that whatever delays are induced by the medium are the same for both of the gravitationally lensed images following the lens. Careful consideration suggests that this can be the case.

Observationally, interstellar refractive scintillation changes the observed intensity over a timescale of hours, diffractive scintillation of pulsars over a timescale of minutes, and interplanetary scintillation over a timescale of seconds. This is not fast enough to influence one gravitationally lensed image relative to another in our context, where the delay is much less than 1 ms. Note that intensity changes are caused by the relative changes in phase over different nearby paths. The spatial separation of gravitationally lensed images would be much less than that for multiple images associated with scintillation, hence the timescale for a change of relative phase would be slower than that for scintillation. Moreover, the following argument suggests that even the absolute rate of change of phase by interstellar turbulence is too slow to matter here.

The absolute change in phase due to a medium of fluctuating electron density $\langle N^2 \rangle^{1/2}$, with a cell size of $a$ and a thickness $l$ is, for a given ray path, is

$$\phi = \lambda r_e [l/a]^{1/2} \langle N^2 \rangle^{1/2},$$

(9) where $r_e$ is the classical radius of the electron (Alurkar 1997). So the absolute rate of change of the phase is

$$d\phi/dt = \lambda r_e [l/a]^{1/2} \langle N \rangle [d\ln N/dt],$$

(10) where $d\ln N/dt$ can be written as $\nabla \cdot v$. For multi-scale turbulence, each scale $a$ has an associated velocity $v(a)$, and $\nabla \cdot v = v(a)/c_s$, where $c_s$ is the sound velocity. For a sound wave $c_s = 1$, whereas for a non-compressible motion, such as pure shear, $c_s = 0$. For Kolmogorov turbulence, $v$ is $a^{1/3}$ so $\lambda r_e [l/a]^{1/2} \langle N \rangle [d\ln N/dt] \propto a^{1/6}$. But $\epsilon$ scales as $v^2/c_s^2$, where $c_s$ is the sound velocity, and this is $\epsilon \propto a^{2/3}$. So the quantity $\lambda r_e [l/a]^{1/2} \langle N \rangle [d\ln N/dt] \propto a^{1/2}$ decreases with $a$, and it is the largest scales that dominate the absolute rate of change of phase.

In the interstellar medium we can take $l$ to be 1 Kpc, and $a$ to be maybe 100 pc. $N$ is at most $1/cm^2$, $\epsilon \approx 1$, and $v(100 \text{ pc}) \leq 10^{23} \text{ cm s}^{-1}$. So for $\lambda = 30 \text{ cm}$,

$$d\phi/dt = \lambda r_e [l/a]^{1/2} \langle N \rangle [d\ln N/dt] \leq 10^{-10.5 + 6.5} \text{ s}^{-1} \approx 10^{-4} \text{ s}^{-1}. \quad (11)$$

This gives a timescale of at least $10^4$ s for an absolute change of phase, i.e., several hours, similar to the observed timescale of variation of BL Lac objects, which is attributed to refractive scintillation. Note that the relative change of phase between two nearby paths might be much smaller than the absolute change of either, but could not be much greater.

The motion of the solar system and Earth is of the order of $v_E = 30 \text{ km s}^{-1}$, and the rate of change of phase for a given scale $a$ in a slab of thickness $l$ due to this motion is

$$d\phi/dt = \lambda r_e [l/a]^{1/2} v_E \cdot \nabla N \sim \lambda r_e [l/a]^{1/2} \delta N(a),$$

(12) where $\delta N(a) = a \nabla N$. While the factor $[l/a]^{1/2}$ can increase with decreasing $a$ as fast as $a^{-1/2}$, the amplitude of density fluctuations, as argued above, probably decreases as least as fast as $a^{-2/3}$, so even here $d\phi/dt$ is expected to decrease with decreasing $a$. So purely temporal separation of the two gravitationally lensed images would not lead to a significant phase change of one relative to the other. The angular separation, in the case of diffractive scintillation, is discussed below.

The above discussion has assumed that each image of the gravitational lens follows a well-defined path to the observer. If the source is small enough to undergo diffractive scintillation, this is not the case because each image is in fact the contribution of many different paths. The smallest-scale density fluctuations, which have the broadest diffraction peaks ($\propto \lambda/a$), contribute in greater numbers ($\propto a^{-2}$) to the signal at the detector, so even if the amplitude of the density fluctuations decreases as $a^{-2}$, the larger number of cells contributing makes the smaller $a$ dominate the overall phase change in time. Empirically, high-latitude pulsars within our Galaxy scintillate over a timescale of minutes, say $10^2$ s, and this is generally interpreted as the timescale over which the pulsar moves across one cell in the turbulence and on to the next (e.g., Narayan 1992), so its line of sight now passes through the latter. Assuming that the pulsar moves across the sky at a speed of the order of $2 \times 10^6 \text{ km s}^{-1}$, the smallest scale that causes significant phase change is then of the order of $2 \times 10^9 \text{ cm}$. This is a very rough estimate, and it provides a rough estimate of the angular distance, $2 \times 10^6/d$ cm in the observer’s sky, over which the brightness change induced by scintillation is correlated. Here, $d$ is the thickness of the interstellar disk $\sim 10^{20.5} \text{ cm}$.

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5 Or at least buffered long enough to determine whether an FRB has occurred, and then, if so, to store the data with full phase information for further analysis.
A remaining question is whether the scattering of the FRB in the host galaxy raises the effective transverse size of the source, \( \gamma \), enough to cause significant dispersion in the gravitational delay. The scattering delay for FRB 110523 at 800 MHz is \( \delta t = 1.66 \text{ ms} \) (Masui et al. 2015) and also an implied delay, given a scintillation decorrelation bandwidth of \( \sim 1 \) MHz, a delay of \( \sim 1 \mu \text{s} \), which is typical of the scattering delay in our own Galaxy. The delay \( \delta t \) due to scattering is given by

\[
\delta t = (y/2c^2)(l/s)^{1/2}, \tag{13}
\]

where \( s \) is the scattering length and \( y \) is the perpendicular distance between the scattering site and the line of sight. The largest value of \( y \) for a given delay is obtained assuming that there is a single scattering (\( l = s \)) over the path length out of the disk of the host galaxy. Taking this length to be \( l = 163 \text{ pc} \), which is a typical path length through a spiral galactic disk at a typical angle, the light travel time is \( l/c = 1.66 \times 10^{10} \text{ s} \), so the scattering angle due to the \( \mu \text{s} \) delay would be \( \theta \sim y/s = (2c\delta t/s)^{1/2} = 1.5 \times 10^{-8} \), whence \( y \lesssim 10^{13} \text{ cm} \). The 1.66 ms delay is much larger than the usual scattering delay for propagation through a galactic disk, and, as argued by Katz (2016), this suggests that the scattering occurs much closer to the source than \( l \). Denote this separation \( y_l, \eta \ll 1 \).

The implied scattering angle is then \( 1.4 \times 10^{-6.5} \eta^{1/2} \), and \( y \sim 2.3 \times 10^{14} \eta^{1/2} \text{ cm} \). Assuming the FRB is at a distance of 1 Gpc, the Einstein radius for a lens of mass \( m_l \) is \( R_E = 10^{16.25}(m_l/M_\odot)^{1/2} \text{ cm} \). The dispersion in the gravitational lens delay is \( \frac{y}{R_E} \sim 2 \lesssim 10^{-4} \eta(m_l/M_\odot)^{-1} \), so, as the gravitational delay is of the order of \( 10^{-5} (m_l/M_\odot) \text{ s} \), its dispersion is then much less than 1 ns.

**Scattering Delay.** Finally, it could be asked whether gravitational delays, which occur in a very small minority, \( \lesssim 10^{-2} \), of cases, could be detected above the scattering delay caused by density fluctuations in the interstellar plasma, which, at high Galactic latitude, are typically 1 \( \mu \text{s} \) at GHz frequency (Cordes & Rickett 1998). Such scattering delay \( \Delta_\gamma \) would thus be larger than the gravitational lensing delay \( \Delta_l \), when the gravitational lens mass is less than 0.1 \( M_\odot \). As the scattering delay is distributed over a wide range of values \( \{ \Delta_\gamma \} \) (comparable in width to the mean \( \Delta_\gamma \)), the arrival time of each image, in the case of a gravitational lens, is accordingly spread over a larger range of \( \Delta \) than their separation \( \Delta_l \), and the question is whether this would make them difficult to distinguish (as one might suppose) from the single peak one would obtain in the absence of a gravitational lens. To address this question, first note the ordering of timescales in the problem: \( 2\pi v \gg \omega' \gg 1/\tau \gg 1/t_{\text{sc}} \), where \( 2\pi v \) is the characteristic carrier frequency of the FRB at which the FRB is detected, \( \omega' \sim 1 \text{ MHz} \) (\( \Delta_\gamma \sim 10^{-6} \text{ s} \)) is the frequency interval over which the scintillation pattern decorrelates (typical delay in travel time associated with the interstellar diffraction), \( \tau \sim 10^{-3} \text{ s} \) is the duration of the FRB, and \( t_{\text{sc}} \gtrsim 10^6 \text{ s} \) is the effective scintillation timescale for point sources due to Galactic density fluctuations. In the absence of gravitational lens delays, the spectrum of a scintillating source can be written as

\[
G_F(\omega') = \int \int g(\phi_x, \phi_y, \omega') e^{i\phi(\theta_\gamma, \theta_\Omega, \omega')} d\theta_\gamma d\theta_\Omega, \tag{14}
\]

where \( \theta_\gamma \) and \( \theta_\Omega \) are the two local sky coordinates in the vicinity of the source direction, and \( g(\phi_x, \phi_y, \omega') \) is the spectrum before passing through the scattering screen weighted by the probability that light is scattered back into the line of sight from this direction. When there is a gravitationally lensed signal, the delayed image can be written as

\[
G_d(\omega') = \int \int g_d(\phi_x, \phi_y, \omega') e^{i\phi(\theta_\gamma, \theta_\Omega, \omega')} d\theta_\gamma d\theta_\Omega, \tag{15}
\]

Here, \( \phi_d(\theta_\gamma, \theta_\Omega, \omega') \) is the phase of the delayed image in a given pixel. Since the scattering screen is assumed to be stationary, and the scattering angle is very small, \( \phi_d(\theta_\gamma, \theta_\Omega, \omega') \) can be taken to be equal to \( \phi(\theta_\gamma, \theta_\Omega, \omega') \), whereas the function \( g \) changes with the angle of incidence on the scattering screen. The total spectrum can then be written as

\[
G(\omega') = \int \int [(age^{i\phi(\theta_\gamma, \theta_\Omega, \omega')} + bg_d e^{i\phi(\theta_\gamma, \theta_\Omega, \omega')}) \times \exp[i(\omega \Delta_t)] d\theta_\gamma d\theta_\Omega, \tag{16}
\]

\[
= \int \int [(age^{i\phi(\theta_\gamma, \theta_\Omega, \omega')} + bg_d e^{i\phi(\theta_\gamma, \theta_\Omega, \omega')} \exp[i(\omega \Delta_t)]) + br(\phi_x, \phi_y, \omega') e^{i\phi(\theta_\gamma, \theta_\Omega, \omega')} \exp[i(\omega \Delta_t)]) d\theta_\gamma d\theta_\Omega, \tag{17}
\]

where \( r(\phi_x, \phi_y, \omega') \equiv g_d(\phi_x, \phi_y, \omega') - g(\phi_x, \phi_y, \omega') \), and finally

\[
G(\omega') = (a + b \exp[i(\omega \Delta_t)])G_F + b \exp[i(\omega \Delta_t)]R, \tag{18}
\]

where \( R(\omega') \equiv \int \int r(\phi_x, \phi_y, \omega') e^{-i\phi(\theta_\gamma, \theta_\Omega, \omega')} d\theta_\gamma d\theta_\Omega \) and

\[
G^*_G = (a^2 + b^2)G^*_F G_F + 2ab G^*_F G_F \cos(\omega \Delta_t) + b^2(G^*_F R + R^* G_F) + b^2 R^* R + ab(G^*_F R e^{-i\omega \Delta_t}) + R^* G_F e^{-i\omega \Delta_t}). \tag{19}
\]

When \( R \ll G_F \), the last three terms of Equation (19) can be neglected. As each pixel on the sky sees the same gravitational lens delay, \( [a + b \exp[i(\omega \Delta_t)] \) factor can be taken out of the integral over \( d\theta_\gamma d\theta_\Omega \) in Equation (17), so \( G(\omega') \) assumes the same form as before. As discussed above, the fact that integrating over different light paths in the \( \theta_\gamma, \theta_\Omega \) plane introduces rapid decorrelation in \( G(\omega')G(\omega') \) with changing \( \omega' \)—the frequency decorrelation scale associated with the scintillation (\( \sim 1 \text{ MHz} \) for the Galactic disk)—does not greatly diminish the long-term correlation produced by the gravitational lens. It is true that the fluctuations in \( G(\omega')G(\omega') \) may introduce Fourier components of this function at scales \( \Delta \sim \Delta_\gamma \gg 1/2\pi v \), leading to some long-term correlations at \( \Delta \sim \Delta_\gamma \), but these correlations, should they exist, would be much weaker than that produced by the gravitational lens, because only a tiny fraction of the signal is subject to any particular value within the wide range of scattering delays \( \Delta_\gamma \), in contrast to the gravitational lens delay, to which the entire second image is subjected. This is seen by writing \( I(\omega') = G(\omega')G(\omega') \approx \langle I \rangle + \delta I(\omega') \), where \( \langle I \rangle \approx \int E^*(t')E(t'')dt' \) and \( \langle \delta I \rangle = 0 \). We can then write the Fourier
transform variable of $I$ as

$$T(\Delta) \equiv \int_{-\infty}^{\infty} I(\omega') e^{i\omega' \Delta} d\omega'. = \langle I \rangle \delta(\Delta) + \int_{-\infty}^{\infty} \delta I(\omega') e^{i\omega' \Delta} d\omega'. \quad (20)$$

It is convenient at this point to consider a finite frequency interval $[\omega_1, \omega'_2]$ over which the source spectrum is more or less constant, and consider discrete Fourier modes of the variable $I(\omega') = \sum_i a_i \cos(\omega' \Delta_i) + \sum_i b_i \sin(\omega' \Delta_i)$, where the values of $\Delta_i$ are separated by $2\pi / [\omega'_2 - \omega'_1]$. The larger frequency $\omega'_2$ can be taken to be of the order of the carrier frequency $2\pi v$. Then, $a_0 = \langle I \rangle$ and $\delta I(\omega') = \sum_i a_i \cos(\omega' \Delta_i)$. Assuming the square of the mean intensity of a scintillating object $\langle I \rangle^2 = a_0^2$ is comparable to the variance $\sum_{i \neq 0} a_i^2$, it follows that $a_0$ exceeds $a_{i \neq 0}$ by roughly the factor $N^{1/2}$, where $N$ is the number of modes that have significantly non-zero values for $a_i$. As $N$ is of order $10^{10}$, the delay $\Delta_i$ is of the order of the reciprocal of the scale of variation of $\delta I$ with frequency, it follows that $N \gg 1$ and that $a_0 \gg a_{i \neq 0}$. It then follows that the correlation caused by the gravitational lens would easily stand out, even if somehow weakened considerably, above the spectrum of delays caused by the density fluctuations in the intervening plasma.

Now suppose that $R$ cannot be neglected because the delayed image makes an angle with the original image that is comparable to or larger than the angular decorrelation scale of the scintillation pattern. Then, the term $ab(G^*_F \Re e^{i\omega' \Delta} + R^* G_F e^{-i\omega' \Delta})$ must be comparable to $2ab G^*_F G_F \cos(\omega' \Delta)$ and in fact, the ensemble average of $ab(G^*_F \Re e^{i\omega' \Delta} + R^* G_F e^{-i\omega' \Delta})$ is given by

$$\langle ab(G^*_F \Re e^{i\omega' \Delta} + R^* G_F e^{-i\omega' \Delta}) \rangle = 2ab \eta(\omega') \langle G^*_F G_F \rangle \cos(\omega' \Delta), \quad (21)$$

where $\eta(\omega') = \langle 1 + (G^*(\omega')G_{\omega}(\omega')) / \langle G^*_F G_F \rangle \rangle$. Thus, if $G^*_F$ is completely uncorrelated with $G$, then $\eta = -1$ and the fifth term on the right side of Equation (19) exactly cancels the second term, so there is no coherent interference between the two images. For any particular event, of course, the value of $R$ is generally different from $-1$ and Equation (19) has a component proportional to $e^{i\omega' \Delta}$ of order $abG^*_F$, but the integration of $G(\omega'G(\omega')$ over a range of $\omega'$ that is large compared to the correlation frequency interval $\omega'_c$, may be nearly equivalent to an ensemble average. In any case, if there is even a small residual correlation, then the correlation $\langle E^*(t)E(t+\Delta) \rangle$, albeit weakened, remains non-zero, and is possibly detectable. Thus, the marginal case where the angle between the images just happens to be not too much larger that the angular decorrelation scale of $G(\omega)$ might still allow for a detection of the time correlation of the Stokes parameters of the FRB signal over $\Delta t$.

The angle $\theta_c = I_{sc} \cdot d\theta / dt$ over which scintillation patterns are decorrelated by $1/e$ in the Galactic disk can be determined by measuring both the proper angular motion $d\theta / dt$ of the pulsar and the correlation timescale $I_{sc}$ for the scintillation pattern. For the high-latitude pulsars 1115 + 5030, 1239 + 2453, and 1509 + 5531, $\theta_c$ is determined in this way to be $5 \times 10^{-12}$, $8 \times 10^{-12}$, and $2 \times 10^{-12}$, respectively (Cordes & Rickett 1998). This is comparable to the angular separation between the two images of a source at a distance of 1 Gpc, lensed by an object of solar mass, so for lenses much more massive than this, the correlation over $\Delta t$ would be weakened. More distant sources allow somewhat higher lens masses.

4. Summary

We have suggested that nanolensing of FRBs—i.e., gravitational lensing delays of $10^{-8} \leq \Delta \leq 10^{-7}$ s—can probe the distribution of mass in compact objects in the individual mass range $10^{-3} - 10^2 M_\odot$ with unprecedented sensitivity. The technique could not only set limits on dark matter, but apparently investigate the mass distribution of brown dwarfs, if it is not exceeded by that of dark matter. The angular image separation $10^{-11.5} (m/M_\odot)^{1/2}$ due to the gravitational lens may, for $m \gtrsim M_\odot$, be comparable to the correlation angle of the scintillation pattern of a point source due to small-scale plasma density fluctuations in our Galaxy. Because of the phase decorrelation of the two images in this situation, the coherence of Stokes parameters over the gravitational lens delay $\Delta t$ is diminished. We have argued, however, that even at $m \sim M_\odot$, this is not necessarily lethal.

Leaving aside the history of ideas about lensing, the technique proposed here has nothing to do with FRBs per se. Any radio source at cosmological distances would work as well, provided it is compact enough that its gravitational lens delay is well-defined to better than the period of the carrier wave $2\pi/\omega$.

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