Gauge Enhanced Quantum Criticality

Between Grand Unifications:

Categorical Higher Symmetry Retraction

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Abstract

Prior work [1] shows that the Standard Model (SM) naturally arises near a gapless quantum critical region between Georgi-Glashow (GG) \(su(5)\) and Pati-Salam (PS) \(su(4) \times su(2) \times su(2)\) models of quantum vacua (in a phase diagram or moduli space), by implementing a modified \(so(10)\) Grand Unification (GUT) with a \(Spin(10)\) gauge group plus a new discrete Wess-Zumino Witten term matching a 4d nonperturbative global mixed gauge-gravity \(w_2w_3\) anomaly. In this work, we include Barr’s flipped \(u(5)\) model into the quantum landscape, showing these four GUT-like models arise near the quantum criticality near SM. The SM and GG models can have either 15 or 16 Weyl fermions per generation, while the PS, flipped \(u(5)\), and the modified \(so(10)\) have \(16n\) Weyl fermions. Highlights include: First, we find the precise GG or flipped \(u(5)\) gauge group requires to redefine a Lie group \(U(5)\hat{q} \equiv (SU(5) \times U(1))/\mathbb{Z}_{\hat{q}}\) with \(\hat{q} = 2\) or \(3\) (instead of non-isomorphic analog \(\hat{q} = 1\) or \(4\)), and different \(\hat{q}\) are related by multiple covering. Second, for \(16n\) Weyl fermions, we show that the GG and flipped \(u(5)\) are two different symmetry-breaking vacua of the same order parameter separated by a first-order Landau-Ginzburg transition. We also show that analogous 3+1d deconfined quantum criticalities, both between GG and PS, and between the flipped \(u(5)\) and PS, are beyond Landau-Ginzburg paradigm. Third, topological quantum criticality occurs by tuning between the \(15n\) vs \(16n\) scenarios. Fourth, we explore the generalized higher global symmetries in the SM and GUTs. Gauging the \(\mathbb{Z}_2\) flip symmetry between GG and flipped \(u(5)\) models, leads to a potential categorical higher symmetry that is a non-invertible global symmetry: within a gauge group \([U(1)_{X_1} \times \mathbb{Z}_{4_X} U(1)_{X_2}] \times \mathbb{Z}_{2}^{flip}\), the fusion rule of 2d topological surface operator splits. Even if the mixed anomaly between \(\mathbb{Z}_2\) flip symmetry and two \(U(1)\) magnetic 1-symmetries at IR is absent, the un-Higgs \(Spin(10)\) at UV retracts this categorical symmetry.

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Dedicate to Subir Sachdev (60), Xiao-Gang Wen (60), Edward Witten (70), and Shing-Tung Yau (72), anniversaries of various researchers mentioned in: Related presentation videos available online
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1 Introduction and Summary

1.1 Unity of Gauge Forces vs Many Dualities

One of the open unsolved problems in fundamental physics and high-energy physics (HEP) is:

“If the strong, electromagnetic, and weak forces of the Standard Model are unified at high energies, by which gauge group (of the gauged internal symmetry) is this unification governed?”

We may quote this perspective as “Unity of Gauge Forces.” It is conventional to regard our quantum vacuum in the 4-dimensional spacetime (denoted as 4d or 3+1d) governed by one of the candidate \( su(3) \times su(2) \times u(1) \) Standard Models (SMs) [2–5], while lifting towards one of some Grand Unification-like (GUT-like) structure [6–12] or String Theory at higher energy scales. This perspective may be schematically summarized as Fig. 1 (a).

![Figure 1: (a) “Unity of Gauge Forces” perspective seeks for a single unified dynamically gauged internal symmetry at high energy, which is a more kinematic or static issue of gauge theories towards higher energy. (b) Our “quantum competing criticality” perspective [1,13] suggests that the SM is a low energy quantum vacuum arising from the quantum competition of various neighbor GUT-like vacua (which can also appear at higher energy). Especially when there are constraints from topological terms or nonperturbative global anomalies, the SM arises near a gapless quantum critical region (schematically shown as the shaded gray area). The gapless quantum critical region induces new beyond-SM gapless modes, Dark Gauge forces, or excitations. This perspective is more on the dynamics, criticality, or phase transition issue of gauge theories.]

However, gauge symmetry is not a physical symmetry (unlike the global symmetry) but only a gauge redundancy to describe interactions between matters; thus the gauge group is not physical nor universal. Furthermore, it is widely known that there are many different dual descriptions of the same physical theories via different gauge groups. We may quote this perspective as “Many Dualities.”

This raises the conflict between the above two perspectives: How could we ask for the governing gauge group for the Unity of Gauge Forces, if gauge groups are not universal, and if there are Many Dualities of possible different gauge theory realizations of the same unification? Partly motivated to provide a resolution of this conceptual conflict, our prior work [1] initiates an alternative viewpoint: we propose that the SM vacuum may be a low energy quantum vacuum arising from the quantum competition of various neighbor higher-energy GUT or other unified theories’ vacua in an immense quantum phase diagram, schematically summarized as Fig. 1 (b). Here we highlight and summarize some of the viewpoints of Ref. [1]:

1Throughout our article, we denote \( n \)d for \( n \)-dimensional spacetime, or \( n+1 \)d as an \( n' \)-dimensional space and 1-dimensional time. We also denote the Lie algebra in the lower case such as the \( so(10) \) Lie algebra in the \( so(10) \) GUT [8], but denote their Lie group in the capital case such as Spin(10).
1. When we treat the internal symmetry as a global symmetry (or in the weakly gauged or ungauged limit), it is physically sensible to ask "what is the Unity of the Governing Internal Symmetry Group $G_\text{internal}$?" In this global symmetry limit, the immense quantum phase diagram in fact not only contains many different GUTs in the same Hilbert space with same 't Hooft anomalies [14] of their internal global symmetries, but also give rise to the SM near the quantum criticality between different GUTs. The quantum field theories (QFTs), sharing the same 't Hooft anomalies especially with the same global symmetries, are believed to live in the same phase diagram with the "same" Hilbert space, possibly by adding new degrees of freedom at the short-distance or the higher energy. These QFTs are deformable to each other via symmetry-preserving interactions — known as the deformation class of QFTs, particularly advocated by Seiberg [16]. The deformation class of quantum gravity theory is also proposed, by McNamara-Vafa [17]. The deformation class of the standard model is studied in [18,19].

2. Since the SM arises near the quantum criticality between different GUTs, it makes sense to study the emergent (gauged or global) symmetries and dualities of QFTs at this quantum criticality. We can further gauge the internal symmetry to be a gauge theory, thus we can study possible Many Dualities of these gauge theories.

3. The above two different viewpoints highlight the validity of Unity of Internal Symmetry and Many Dualities respectively, but now in the same quantum phase diagram and same framework (Fig. 1 (b)). Moreover, various GUTs may encounter stable gapless quantum critical regions (the shaded gray area in Fig. 1 (b)) to enter other neighbor GUTs, if we apply the idea that the quantum criticality is protected by the 't Hooft anomalies of some spacetime-internal symmetries:3

$$\bar{G} \equiv G_\text{spacetime} \ltimes N_\text{shared} G_\text{internal} \equiv \left( \frac{G_\text{spacetime} \ltimes G_\text{internal}}{N_\text{shared}} \right).$$  \hspace{1cm} (1.1)$$

The spacetime $G_\text{spacetime}$ and internal $G_\text{internal}$ symmetries may share a common normal subgroup $N_\text{shared}$. If there is a 't Hooft anomaly in $\bar{G}$ (here 4d, denoted as a one-higher dimensional invertible topological quantum field theory [iTQFT] defined on some $\bar{G}$-structure manifold in 5d with a partition function $\mathbb{Z}$), this iTQFT may be written schematically as

$$\mathbb{Z}[\mathcal{B}_{\bar{G}}] = \mathbb{Z}[\mathcal{B}_{G_{\text{GUT}_1}} \sim \mathcal{B}_{G_{\text{GUT}_2}} \sim \ldots],$$  \hspace{1cm} (1.2)$$

where $\mathcal{B}_{\bar{G}}$ is the background field coupling to the spacetime-internal symmetry $\bar{G}$.4 When we have the symmetry breaking from $\bar{G}$ to $G_{\text{GUT}_j}$, etc., we are only left with the valid background field $\mathcal{B}_{G_{\text{GUT}_j}}$ while other background fields must be turned off — thus the 't Hooft anomaly may be canceled to zero by this symmetry breaking. In contrast, if we preserve the full $\bar{G}$ (happening especially at the quantum critical

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3Let us clarify the terminology on criticalities vs phase transitions.

• The **criticality** means the system with gapless excitations (gapless thus critical, sometimes conformal) and with an infinite correlation length, it can be either (i) a **continuous phase transition** as an unstable critical point/line/etc. as an unstable renormalization group (RG) fixed point which has at least one relevant perturbation in the phase diagram, or (ii) a **critical phase** as a stable critical region controlled by a stable RG fixed point which does not have any relevant perturbation in the phase diagram.

• The **phase transition** [15] means the phase interface between two (or more) bulk phases in the phase diagram. The phase transition can be a **continuous phase transition** (second order or higher order, with gapless modes) or a **discontinuous phase transition** (first-order, without gapless modes, and with a finite correlation length).

In the phase diagram, the spacetime dimensionality of the phase interface is the same as that of the bulk phase. This is in contrast with the one-lower spacetime dimensional physical interface or physical boundary of the bulk phase.

4The notation $G_1 \ltimes N_\text{shared} G_2 \equiv G_1 \ltimes N_\text{shared} G_2 \ltimes N_\text{shared}$ means modding out their common normal subgroup $N_\text{shared}$. The $\ltimes$ is a semidirect product (as a generalization of direct product) to specify a particular group extension.
region in between GUT phases), and if we have other possibilities to cancel the ’t Hooft anomaly other than by symmetry breaking, then we must have no symmetric trivial gapped phase between neighbor GUT phases, which likely resulting in gapless quantum criticalities in many cases. In general, these phases are called nontrivial phases: (1) symmetry-preserving gapless, (2) symmetry-preserving gapped topologically order with a low energy TQFT, (3) symmetry-breaking gapless or gapped, or (4) their mixed combinations. In short, some nontrivial state of matter must be at the phase transition or the quantum criticality to match the ’t Hooft anomaly.

In fact, the deconfined quantum criticality (DQC) in 2+1d in condensed matter physics [20, 21] is the incarnation and reminiscence of this above idea of ’t Hooft anomaly protected quantum criticality (See Ref. [21] and Appendix C of Ref. [1] for a contemporary QFT overview of DQC). Recently the DQC phenomena in 3+1d are explored in [22–25], where Ref. [1] provides the first 3+1d DQC analog in the SM and beyond-the Standard Model (BSM) physics.

4. In particular, when the internal symmetry is treated as a global symmetry, Ref. [1] shows that between the neighbor GG SU(5) GUT and PS vacua: with this of a 5d bulk. However, Ref. [1] modifies the n = 1 mod 2 constraint field, then the generation number n times of 16 SM Weyl fermions with n GUT-Higgs field requires a partons) and their 4d WZW term alone, we just need to ensure the anomaly index from GUT-Higgs WZW

This mixed gauge-gravitational anomaly is tightly related to the new SU(2) anomaly [26] due to the mixed combinations. In short, some nontrivial state of matter must be at the phase transition or the quantum criticality to match the ’t Hooft anomaly.

\[ Z(M^5) = \exp(i\pi \int_{M^5} w_2 w_3) = \exp(i\pi \int_{M^5} w_2(TM) w_3(TM)) = \exp(i\pi \int_{M^5} w_2(V_{SO(10)}) w_3(V_{SO(10)})). \] (1.3)

This 4d ’t Hooft anomaly requires the spacetime-internal global symmetry (Spin × \mathbb{Z}_2 Spin(10)) on a 4-manifold \( M^4 \), captured by this 5d bulk invertible TQFT [26, 29] living on a 5-manifold \( M^5 \) with the anomaly-inflow bulk-boundary correspondence \( \partial M^5 = M^4 \).

This mixed gauge-gravitational anomaly is tightly related to the new SU(2) anomaly [26] due to the bundle constraint \( w_2 w_3(TM) = w_2 w_3(V_G) \) with \( G \) can be substituted by SO(3) \( \subset \) SO(10) related to the embedding SU(2) = Spin(3) \( \subset \) Spin(10). This mod 2 class \( w_2 w_3 \) global anomaly has been checked to be absent in the conventional so(10) GUT by Ref. [26, 29]; thus the conventional so(10) GUT is free from 4d anomaly classified by the \( d = 5 \)-th cobordism group defined in Freed-Hopkins [30],

\[ \Omega^d_G \equiv TP_d(G), \] (1.4)

here with \( d = 5, G = (\text{Spin} \times \mathbb{Z}_2 \text{Spin}(10)), \) and \( \Omega^5_{(\text{Spin} \times \mathbb{Z}_2 \text{Spin}(10))} = \mathbb{Z}_2 \) generated by \( \exp(i\pi \int_{M^5} w_2 w_3) \).

However, Ref. [1] modifies the so(10) GUT by appending a new 4d Wess-Zumino-Witten (WZW) term with this \( w_2 w_3 \) global anomaly in order to realize the SM vacuum as the quantum criticality phenomenon between the neighbor GG SU(5) GUT and PS vacua:

- On either GG SU(5) or PS sides of the quantum phases, the \( w_2 w_3 \) anomaly is matched by breaking the internal Spin(10) symmetry down to their GUT subgroups.
- But at the critical (gapless) region between GG and PS quantum phases, the full Spin(10) symmetry can be preserved, while the \( w_3 w_3 \) anomaly is matched by the BSM sector from the new 4d WZW topological term (constructed out of the GUT-Higgs fields or their fractionalized partons) living on a 4d boundary of a 5d bulk.
- Since the mod 2 class \( w_2 w_3 \) anomaly is matched by the sector of GUT-Higgs fields (or their fractionalized partons) and their 4d WZW term alone, we just need to ensure the anomaly index from GUT-Higgs WZW sector contributes 1 mod 2. If each generation of 16 SM Weyl fermions associates with its own GUT-Higgs field, then the generation number \( n \) times of 16 SM Weyl fermions with \( n \) GUT-Higgs field requires a constraint \( n = 1 \mod 2 \) to match the \( w_2 w_3 \) anomaly, where \( n = 3 \) generation indeed works. However, in

\(^5\)The \( w_j \) is the \( j \)-th Stiefel-Whitney (SW) characteristic class. The \( w_j(TM) \) is the SW class of spacetime tangent bundle \( TM \) of manifold \( M \). The \( w_j(V_G) \) is the SW class of the principal \( G \) bundle. The 5-manifold that detects \( w_2(TM) w_3(TM) \) is a Dold manifold \( \mathbb{C}F \times S^4 \) or a Wu manifold SU(3)/SO(3) which yields a path integral \( Z(M^5) = -1 \) [26–28].
general, we can just introduce a single or any odd number of GUT-Higgs field sectors (independent from the \( n = 3 \) of SM) to match the 1 mod 2 class of \( w_2w_3 \) anomaly.

- The dynamics of this modified \( so(10) \) GUT with WZW term can be fairly complicated, giving rise to many possible gapless phases or gapped TQFT phases at low energy, enumerated in [1].

In this present work, we continue developing from Ref. [1] to include another GUT model: the flipped \( su(5) \) model (originally proposed by Barr [10] and others [11]) and the left-right (LR) model [9] into our quantum landscape or quantum phase diagram, exploring further the neighbors of the critical region near SM. The GG model can have either choice of a gauge group of SU(5) or U(5), but Barr’s flipped model must require a U(5). We should emphasize that the U(5) Lie group of GG or Barr’s model requires a certain refined Lie group that we name \( U(5)_{\hat{q}=2} \), see Sec. 2 for details. In many cases, we do require the additional \( u(1) \) gauge sector in addition to the \( su(5) \) gauge sector of GG or Barr’s model, we thus call the corresponding models as the GG \( u(5) \) model and Barr’s flipped \( u(5) \) model.

The SM and GG \( su(5) \) models can have either choice of 15 or 16 Weyl fermions for each generation. In contrast, in order to be consistent with the SM data constraint, the PS, the GG \( u(5) \), the flipped \( u(5) \), and the modified \( so(10) \) have 16 Weyl fermions per generation. In this article, we mainly focus on the scenarios all with 16n Weyl fermions with \( n \) the number of generations.6

1.2 Outline: The plan of the article

In Sec. 2, we clarify the U(5) Lie group structure of the GG or flipped models. They should be both refined as \( U(5)_{\hat{q}} \) gauge theories, with \( \hat{q} = 2, 3 \), which is non-isomorphic to \( \hat{q} = 1, 4 \). For applications, there we point out group theoretical facts like \( Spin(10) \supset U(5)_{\hat{q}=2,3} \) but \( Spin(10) \not\supset U(5)_{\hat{q}=1,4} \).

In Sec. 3, we clarify various GUT models as different vacua or different phases of QFTs, and present their representations of SMs and five GUT-like models in a unified Table 1. Additional details of quantum numbers and representations of SMs and GUTs are provided in Appendix A.

In Sec. 4, we organize various SM or GUT models in a quantum landscape or in a quantum phase diagram. The parameter space of the quantum phase diagram can be specified for example via the GUT-Higgs condensation. Thus the parameter space is also a moduli space. In Sec. 4.1, we provide the embedding web by their internal symmetry groups. In Sec. 4.2, we derive a quantum phase diagram based on the mother effective field theory of a modified \( so(10) \) GUT of [1]. As a toy model, we clarify the quantum phase structures when the internal symmetries are treated as global symmetries in Sec. 4.2.1. Then we clarify the quantum phase structures when the internal symmetries are dynamically gauged in Sec. 4.2.2.

In Sec. 5, as the ordinary internal symmetry is dynamically gauged, the outcome gauge theory can have the generalized global symmetries [39]. We systematically explore these generalized global symmetries (the higher symmetries) of SM and GUT.

6Readers can find Ref. [31–34] and [29] for the systematical studies on the nonperturbative global anomalies of various SM and GUT via generalized cohomology or cobordism theories. In contrast to the scenarios of SM or GUT with 16n Weyl fermions, Ref. [35–37] considers the SM or GUT with 15n Weyl fermions and with a discrete variant of baryon minus lepton number \( B - L \) symmetry [38] preserved. Ref. [35–37] then suggests the missing 16th Weyl fermions can be substituted by additional symmetry-preserving 4d or 5d gapped topological quantum field theories (TQFTs), or by the symmetry-preserving 4d gapless interacting conformal field theories (CFTs), or other symmetry-breaking sectors (e.g., the right-handed neutrinos), to saturate a certain \( Z_{16} \) global anomaly. We will comment about the topological phase transitions between the 15n to 16n Weyl fermions at the very end in Sec. 7.
In Sec. 6, we study the potential non-invertible global symmetries [40–50] (also known as categorical symmetries [51–55]) of SM and GUT. We show that part of the gauge structure of the GG U(5) and the flipped U(5) gauge theories, with their $Z_2^{\text{flip}}$ symmetry gauged, contains a gauge sector $[(U(1)_{x_1} \times z_{4,x} U(1)_{x_2} \times Z_2^{\text{flip}})]$. There is a non-invertible global symmetry exhibited by the topological 2-surface operators as the gauge-invariant symmetry generators of their magnetic 1-symmetries. But we show that this categorical symmetry is retracted thus disappears, when we embed the theory in the modified $\text{so}(10)$ GUT of Spin(10).

In Sec. 7, we conclude and comment on future research directions.

In Appendix B, we provide various matrix representations of Lie algebras and Lie groups of SU(5), U(5)$_\tilde{q}$, SO(10), and Spin(10). Then we describe how they could embed each other properly. In Appendix C, we show the flipping isomorphism between two U(5)$_{\tilde{q}=2}$ (the GG’s and the flipped model’s) while both U(5)$_{\tilde{q}=2}$ can be embedded inside the Spin(10). Then we show that the intersection of two U(5)$_{\tilde{q}=2}$ contains the SM Lie groups, while the minimal Lie group union of two U(5)$_{\tilde{q}=2}$ is exactly the Spin(10).

2 Refined U(5)$_\tilde{q}$ gauge theory

Here we point out there are in fact different non-isomorphic versions of U(5) Lie groups (and their corresponding gauge theories) that we should refine and redefine them as several U(5)$_\tilde{q}$ with $\tilde{q} \in \mathbb{Z}$:

$$U(5)$_{\tilde{q}} \equiv \frac{\text{SU}(5) \times \text{U}(1)}{Z_5} \equiv \text{SU}(5) \times Z_{5,\tilde{q}} \text{U}(1) \equiv \{(g, e^{i\theta}) \in \text{SU}(5) \times \text{U}(1) | (e^{i\frac{2\pi n}{\tilde{q}}} \cdot 1, 1) \sim (\mathbb{1}, e^{i\frac{2\pi n q}{\tilde{q}}}), n \in \mathbb{Z}_5\}.$$  \hspace{1cm} (2.1)

We use two data $(g, e^{i\theta})$ to label the SU(5) × U(1) group elements respectively, while we identify $(e^{i\frac{2\pi n}{\tilde{q}}} \cdot 1, 1) \sim (\mathbb{1}, e^{i\frac{2\pi n q}{\tilde{q}}})$ for $n \in \mathbb{Z}_5$, with a rank-5 identity matrix $\mathbb{1}$. Different identifications of the generator of the center $Z(SU(5)) = \mathbb{Z}_5$ with the U(1) charge $\tilde{q}$ in principle give rise to different Lie groups.

All different U(5)$_{\tilde{q}}$ obey the group extension as the short exact sequence $1 \rightarrow SU(5) \rightarrow U(5)$_{\tilde{q}}$ $\rightarrow U(1)^{\prime} \rightarrow 1$ where $U(1)^{\prime} \equiv \frac{U(1)}{Z_{5\tilde{q}}}$ is related to modding out $Z_{5\tilde{q}}$ of the U(1) defined in U(5)$_{\tilde{q}}$ in (2.1), while different $\tilde{q}$ identifications specify different U(1) actions on the SU(5). But there are in fact the following group isomorphisms

$$U(5)$_{\tilde{q}} \cong U(5)_{-\tilde{q}} \cong U(5)_{5m \pm \tilde{q}}$$  \hspace{1cm} (2.2)

for any $m \in \mathbb{Z}$.\footnote{We can prove the group isomorphism by the following. For U(5)$_{\tilde{q}}$ and U(5)$_{-\tilde{q}}$ defined in (2.1), we can map $h_j = (g_j, e^{i\theta_j}) \in U(5)$_{\tilde{q}}$ and define the group homomorphism map $f(h_j) = (g_j, e^{-i\theta_j})$. Then we check that the group homomorphism $f(h_1) \cdot f(h_2) = f(h_1 \cdot h_2)$ is true: On the left-hand side, $f(h_1) \cdot f(h_2) = (g_1 g_2, e^{-i(\theta_1 + \theta_2)})$. On the right-hand side, $h_1 \cdot h_2 = (g_1 g_2, e^{i(\theta_1 + \theta_2)})$ thus $f(h_1 \cdot h_2) = (g_1 g_2, e^{-i(\theta_1 + \theta_2)})$. In addition, it is injective (one-to-one) and surjective (onto) thus a bijective group homomorphism. It would only be bijective if the map $f(h_j) = (g_j, e^{\pm i\theta_j})$, either a trivial identity map or the outer automorphism on the U(1) part; thus U(5)$_{\tilde{q}} \cong U(5)_{-\tilde{q}}$. The only exception of other isomorphism is when we shift $\tilde{q}$ to $5m + \tilde{q}$ (which does not modify the identification in (2.1)), thus we prove U(5)$_{\tilde{q}} \cong U(5)_{-\tilde{q}} \cong U(5)_{5m \pm \tilde{q}}$.}

Thus among general $\tilde{q} \in \mathbb{Z}$, we have three distinct non-isomorphic types of U(5)$_{\tilde{q}}$ group for any $m \in \mathbb{Z}$:

1. $U(5)_1 \cong U(5)_4 \cong U(5)_{5m+1} \cong U(5)_{5m-1}$.
2. $U(5)_2 \cong U(5)_3 \cong U(5)_{5m+2} \cong U(5)_{5m-2}$.
3. $U(5)_0 \cong U(5)_{5m} \cong SU(5) \times U(1)$.  \hspace{1cm} (2.3)

We emphasize the distinctions of these three different non-isomorphic U(5)$_{\tilde{q}}$ Lie groups (and their gauge theories) somehow seem not yet been carefully examined in the previous high-energy particle physics.
literature. Here we make attempts to address their Lie group differences in the context of gauge theories and GUTs. Several comments are in order:\(^8\)

1. U(5)\(\hat{q}\) as a \(k\)-sheeted covering space of U(5)\(\hat{k}\): If we compare the definition of U(5)\(\hat{q}\) and U(5)\(\hat{k}\), we find that from (2.1), the U(5)\(\hat{q}\) identifies \((e^{i\frac{2\pi}{k}}\mathbb{I}, 1) \sim (1, e^{i\frac{2\pi\hat{k}}{k}})\) while the U(5)\(\hat{k}\) identifies \((e^{i\frac{2\pi}{k}}\mathbb{I}, 1) \sim (1, e^{i\frac{2\pi\hat{k}}{k}})\). If the U(5)\(\hat{q}\) has the periodicity of U(1) as \(\theta \in [0, 2\pi)\), then the U(5)\(\hat{k}\) has the periodicity of U(1) as \(\theta \in [0, 2\frac{2\pi}{k})\). Similarly, if the U(5)\(\hat{q}\) has the periodicity of U(1) as \(\theta \in [0, 2\pi)\), then the U(5)\(\hat{k}\) has the periodicity of U(1) as \(\theta \in [0, 2\frac{2\pi}{k})\). So importantly,
   - the U(5)\(\hat{q}\) is a \(k\)-sheeted covering space of U(5)\(\hat{k}\).
   - the U(5)\(\hat{q}=1\) is a double covering space of U(5)\(\hat{k}=1\).
   - the U(5)\(\hat{q}=2\) is a double covering space of U(5)\(\hat{k}=4\).
   - the U(5)\(\hat{q}=1\) is a quadruple covering space of U(5)\(\hat{k}=4\), but which goes back to itself because of the isomorphism U(5)\(\hat{q}=1 \cong U(5)\hat{k}=4\).
   - the U(5)\(\hat{q}=2\) is a quadruple covering space of U(5)\(\hat{k}=3\), but which goes back to itself because of the isomorphism U(5)\(\hat{q}=2 \cong U(5)\hat{k}=3\).
   - the U(5)\(\hat{q}\) is a quadruple covering space of U(5)\(\hat{q}\) \(\cong U(5)\hat{q}\mod 5\), but which goes back to itself because of the isomorphism U(5)\(\hat{q}\) \(\cong U(5)\hat{q}\mod 5\).

2. It can be shown that

\[
\text{U(5)}\hat{q}=1,4 \subset \text{SO}(10) \text{ but } \text{U(5)}\hat{q}=1,4 \not\subset \text{Spin}(10)
\]

because the homotopy group maps between \(\pi_1(\text{U(5)}\hat{q}=1,4) = \mathbb{Z}\) to \(\pi_1(\text{SO}(10)) = \mathbb{Z}_2\) cannot be lifted to the SO(10)'s double-cover since Spin(10) has \(\pi_1(\text{Spin}(10)) = 0\), which violates the lifting criterion (See Proposition 1.33 of Hatcher [57]).

It can be shown that the U(5)\(\hat{q}=2,3\) as a double cover version of U(5)\(\hat{q}=1,4\) satisfies:

\[
\text{U(5)}\hat{q}=2,3 \not\subset \text{SO}(10) \text{ but } \text{U(5)}\hat{q}=2,3 \subset \text{Spin}(10).
\]

Since \(Z(\text{Spin}(10)) = \mathbb{Z}_4\), \(Z(\text{SO}(10)) = \mathbb{Z}_2\), and \(Z\left(\frac{\text{SO}(10)}{\mathbb{Z}_2}\right) = 0\), while U(5)\(\hat{q}=2,3\) is a quadruple cover of itself; overall we have:\(^9\)

\[
\begin{array}{ccc}
\text{U(5)}\hat{q}=2,3 & \longrightarrow & \text{Spin}(10) \\
\downarrow & & \downarrow \\
\text{U(5)}\hat{q}=1,4 & \longrightarrow & \text{SO}(10) \\
\downarrow & & \downarrow \\
\text{U(5)}\hat{q}=2,3 & \longrightarrow & \frac{\text{SO}(10)}{\mathbb{Z}_2}
\end{array}
\]

3. Follow Atiyah-Bott-Shapiro [58], it is shown that the group homomorphism and the embedding SU(5) \(\hookrightarrow\) SO(10) can be lifted to SU(5) \(\hookrightarrow\) Spin(10). Similarly, the group homomorphism and the embedding U(5)\(\hat{q}=1\) \(\hookrightarrow\) SO(10) \(\times\) \(\frac{U(1)}{\mathbb{Z}_2}\) can be lifted to U(5)\(\hat{q}=1\) \(\hookrightarrow\) Spin\(^4\)(10) \(\cong\)

---

\(^8\)We provide the detailed mathematical proofs of many statement listed here in a separate work (jointly with Zheyan Wan et al) in [56].

\(^9\)Here “\(G_1 \hookrightarrow G_2\)” implies the inclusion thus also implies the group embedding “\(G_1 \subset G_2\).” The vertical arrow “\(\downarrow\)” implies the upper group \(G\) is a double cover of the lower group \(G\) such that we have a nontrivial extension \(1 \rightarrow \mathbb{Z}_2 \rightarrow \tilde{G} \rightarrow G \rightarrow 1\).
\textbf{Spin(10)×U(1)} \mathbb{Z}_2. We can double cover or half-cover of these results to obtain:

\[
\begin{array}{c}
\text{U}(5)_{\hat{q}=2,3} \xrightarrow{\sim} \text{Spin}(10) \times U(1) \\
\downarrow \\
\text{U}(5)_{\hat{q}=1,4} \xrightarrow{\sim} \text{Spin}^c(10) \equiv \frac{\text{Spin}(10) \times U(1)}{\mathbb{Z}_2} \\
\downarrow \\
\text{U}(5)_{\hat{q}=2,3} \xrightarrow{\sim} \frac{\text{Spin}(10) \times U(1)}{\mathbb{Z}_4}
\end{array}
\]

which says not only the embedding \( U(5)_{\hat{q}=1,4} \subset \text{Spin}^c(10) \), but also the embedding \( U(5)_{\hat{q}=2,3} \subset \text{Spin}(10) \times U(1) \) and \( U(5)_{\hat{q}=2,3} \subset \frac{\text{Spin}(10) \times U(1)}{\mathbb{Z}_4} \).

4. In a short summary, for our application on the SM and GUT physics, we shall particularly focus on these two results:

\[ U(5)_{\hat{q}=2,3} \subset \text{Spin}(10), \quad U(5)_{\hat{q}=2,3} \subset \frac{\text{Spin}(10) \times U(1)}{\mathbb{Z}_4}. \]

Here \( \frac{\text{Spin}(10) \times U(1)}{\mathbb{Z}_4} \) can be interpreted as \( \frac{\text{Spin}(10) \times U(1)}{\mathbb{Z}_4, X} \) with \( X \equiv 5(B - L) - 4Y \), including the baryon minus lepton number \( B - L \) and the electroweak hypercharge \( Y \), is a good global symmetry respected by SM and the \( su(5) \) GUT [38].

We provide a verification on (2.6) via exponential maps of the Lie algebras into these Lie groups embedding in Appendix B.

5. In order to study the \( U(5)_{\hat{q}} \) gauge theory, we should understand the allowed Wilson line operators and their endpoint particle charge representations (if the 1d line can be broken by the particle at open ends). In particular, when the matter field \( \psi \) is in the fundamental rep \( 5 \) of SU(5), we can ask which \( U(1) \) charge representation \( Q \) of \( \psi \) is allowed. Because of the identification \( (e^{i \frac{2\pi n}{5}} 1, 1) \sim (1, e^{i \frac{2\pi nQ}{5}}) \) (for \( n \in \mathbb{Z}_5 \)) must act on the \( \psi \) in \( (5, Q) \) in the same way, regardless of whether we consider

- \( (e^{i \frac{2\pi n}{5}} 1, 1) \) on \( \psi \) of \( (5, Q) \) which sends \( \psi \) to \( e^{i \frac{2\pi nQ}{5}} \psi \).

or consider

- \( (1, e^{i \frac{2\pi nQ}{5}}) \) on \( \psi \) of \( (5, Q) \) which gives \( e^{i \frac{2\pi nQ}{5}} \psi \).

The group element identification also means that the \( e^{i \frac{2\pi nQ}{5}} = e^{i \frac{2\pi nQ}{5}} \), which is true if \( \hat{q}Q = 1 \) mod 5. Thus we derive the relation between the Lie group \( U(5)_{\hat{q}} \) and its corresponding matter representation:

\[
\begin{array}{c|c|c}
\hat{q} & Q & \text{matter rep } (5, Q), (\bar{5}, -Q) \\
\hline
U(5)_{\hat{q}=1} & 1 & (5, 1), (\bar{5}, -1) \\
U(5)_{\hat{q}=2} & 2 & (5, 3), (\bar{5}, -3) \\
U(5)_{\hat{q}=3} & 3 & (5, 2), (\bar{5}, -2) \\
U(5)_{\hat{q}=4} & 4 & (5, 4), (\bar{5}, -4)
\end{array}
\]  \hspace{1cm} (2.7)

6. If we want to choose the appropriate \( U(5)_{\hat{q}} \) Lie group for the GG \( su(5) \) or the flipped \( su(5) \) models, we should consider the group contains \( (\bar{5}, -3) \) of \( su(5) \times u(1) \). This (2.7) means that the \( U(5)_{\hat{q}=2} \) is the correct choice. This matter representation \( (\bar{5}, -3) \) is naturally included in \( U(5)_{\hat{q}=2} \).

7. We can generalize the above discussions to \( U(N)_{\hat{q}} \) cases to find non-isomorphisms for some of \( (N, \hat{q}) \).
3 Various Grand Unification (GUT) Models as Vacuum Phases

In this article, we require the SM gauge group $G_{SM} = \frac{SU(3)_c \times SU(2)_L \times U(1)_Y}{Z_q}$ with $q = 6$, where the 16 Weyl fermions are in the following representation (see Fig. 2):

$$
\bar{d}_R \oplus l_L \oplus q_L \oplus \bar{u}_R \oplus \bar{e}_R \oplus \nu_R = (\mathbf{3}, \mathbf{1})_{2,L} \oplus (\mathbf{1}, \mathbf{2})_{-3,L} \oplus (\mathbf{3}, \mathbf{2})_{1,L} \oplus (\mathbf{3}, \mathbf{1})_{-4,L} \oplus (\mathbf{1}, \mathbf{1})_{6,L} \oplus (\mathbf{1}, \mathbf{1})_{0,L}, \quad (3.1)
$$

written all in the left-handed ($L$) Weyl basis. Here we use the $U(1)_Y$ hypercharge instead of HEP phenomenology $U(1)_Y$ hypercharge which is 1/6 of $U(1)_Y$'s.\textsuperscript{10} The $u$ and $d$ are the up and down quarks, the $\nu$ and $e$ are the neutrino and electron.\textsuperscript{11} The $q_L$ and $l_L$ are both of the $SU(2)_L$ doublets; the $q_L$ contains the $(u_L, d_L)$ while the $l_L$ contains the $(\mu_L, e_L)$. We use the $L$ and $R$ to specify the left/right-handed spacetime spinor of Spin(1,3). We use the $L$ and $R$ to specify the left or right internal spinor representation, such as $su(2)_L$ of the SM and the $su(2)_L \times su(2)_R$ of the Pati-Salam model. Conventionally, the spacetime $L$ and the internal $L$ are locked in the sense that the spacetime $L$-handed spinor is also the internal $SU(2)_L$ doublet, while the spacetime $R$-handed spinor is the $SU(2)_L$ singlet (or $SU(2)_R$ doublet in the PS model). But we can regard the spacetime $R$-handed anti-particle as the $L$-handed particle as written in (3.1).

![Figure 2: Standard Model with 16n Weyl fermions and their $su(3)_c \times su(2)_L \times u(1)_Y$ representation (rep): $d_R \oplus l_L \oplus q_L \oplus \bar{u}_R \oplus \bar{e}_R \oplus \nu_R = (\mathbf{3}, \mathbf{1})_{2,L} \oplus (\mathbf{1}, \mathbf{2})_{-3,L} \oplus (\mathbf{3}, \mathbf{2})_{1,L} \oplus (\mathbf{3}, \mathbf{1})_{-4,L} \oplus (\mathbf{1}, \mathbf{1})_{6,L} \oplus (\mathbf{1}, \mathbf{1})_{0,L}$.](image)

![Figure 3: Georgi-Glashow $u(5)$ GUT with 16n Weyl fermions and their $u(5)^{1st} = su(5)^{1st} \times u(1)_X = su(5)^{1st} \times u(1)_X$ rep: $(d_R \oplus l_L) \oplus (q_L \oplus \bar{u}_R \oplus \bar{e}_R) \oplus (\nu_R) = \mathbf{5}_{-3} \oplus \mathbf{10}_1 \oplus \mathbf{1}_5$.](image)

\textsuperscript{10}Namely, $q_{U(1)_Y} = \frac{1}{6} q_{U(1)_Y}$. If we use the hypercharge $U(1)_Y$, then we have instead: $(\mathbf{3}, \mathbf{1})_{\frac{1}{2},L} \oplus (\mathbf{1}, \mathbf{2})_{-\frac{1}{2},L} \oplus (\mathbf{3}, \mathbf{2})_{\frac{1}{2},L} \oplus (\mathbf{3}, \mathbf{1})_{-\frac{2}{3},L} \oplus (\mathbf{1}, \mathbf{1})_{1,L} \oplus (\mathbf{1}, \mathbf{1})_{0,L}$.

\textsuperscript{11}This matter content is for the first generation of quarks and leptons. We can replace these quarks to the charm $c$ and strange $s$ for the second generation, or to the top $t$ and bottom $b$ for the third generation. We can replace these leptons to the muon $\mu$ and tauon $\tau$ for the second and third generations.
Figure 4: Flipped $u(5)$ GUT with 16n Weyl fermions and their $u(5)^{2nd} = su(5)^{2nd} \times u(1)_X = su(5)^{1st} \times u(1)_{X_2}$ rep: $(u_R \oplus l_L) \oplus (q_L \oplus d_R \oplus \nu_R) \oplus (\bar{e}_R) = \bar{5} - 3 \oplus 10_1 \oplus 1_5$. Note that the charge of $X_1 = X \neq \chi = X_2$.

Figure 5: Pati-Salam $su(4) \times su(2)_L \times su(2)_R$ model and their rep: $(q_L \oplus l_L) \oplus (q_R \oplus l_R) = (u_L \oplus d_L \oplus \nu_L \oplus \nu_L) \oplus (u_R \oplus d_R \oplus \nu_R \oplus \bar{e}_R) = (4, 2, 1) \oplus (4, 1, 2)$.

Figure 6: Left-Right $su(3) \times su(2)_L \times su(2)_R \times u(1)_{B-L}$ model and their rep: $q_L \oplus l_L \oplus q_R \oplus l_R = (u_L \oplus d_L) \oplus (\nu_L \oplus e_L) \oplus (u_R \oplus d_R) \oplus (\nu_R \oplus \bar{e}_R) = (3, 2, 1)_{\frac{1}{6}} \oplus (1, 2, 1)_{\frac{1}{2}} \oplus (3, 1, 2)_{-\frac{1}{6}} \oplus (1, 1, 2)_{\frac{1}{2}}$.

Figure 7: The $so(10)$ GUT with 16 Weyl fermions in the rep $16^+$ of $G_{so(10)} \equiv Spin(10)$ gauge group.
3.1 Georgi-Glashow u(5) vs Flipped u(5) models

1. The su(5) or u(5) Grand Unification (su(5) or u(5) GUT): Georgi-Glashow (GG) [6] hypothesized that the three SM gauge interactions merged into a single electronuclear force at higher energy under a simple Lie algebra su(5), or precisely a Lie group \( G_{\text{GG}} = SU(5) \) gauge theory. The su(5) GUT works for 15n Weyl fermions, also for 16n Weyl fermions (i.e., 15 or 16 Weyl fermions per generation). The Weyl fermions are in the representation of \( u(5)^{1\text{st}} = su(5)^{1\text{st}} \times u(1)_X = su(5)^{1\text{st}} \times u(1)_X \), as (see Fig. 3):

\[
(\bar{d}_R \oplus l_L) \oplus (q_L \oplus \bar{u}_R \oplus \bar{e}_R) \oplus (\bar{\nu}_R) = 5_{-3} \oplus 10_1 \oplus 1_5,
\]

(3.2)

More precisely, they are in the representation of a refined U(5)\( \hat{q} = 2 \) group that we carefully define in Sec. 2:

\[
U(5)^{1\text{st}}_{\hat{q} = 2} \equiv \frac{SU(5)^{1\text{st}} \times \hat{q} = 2 \ U(1)_X}{Z_5} = \frac{SU(5)^{1\text{st}} \times \hat{q} = 2 \ U(1)_X}{Z_5}.
\]

(3.3)

The 16th Weyl fermion is an extra neutrino, sterile to the su(5) but not sterile to the \( u(1) \) gauge force. The corresponding \( U(1) \) is typically called the \( X \) as \( U(1)_X \equiv U(1)_X \equiv U(1)_{5(B-L)-4Y} \equiv U(1)_{5(B-L)-\frac{3}{2}Y} \) which we also call \( X_1 \) because this corresponds to the U(1) subgroup of the first type of u(5) GUT.

2. The Barr’s flipped su(5) or u(5) GUT [10]:

The Weyl fermions are also in the representation of \( u(5)^{2\text{nd}} = su(5)^{2\text{nd}} \times u(1)_\chi = su(5)^{2\text{nd}} \times u(1)_X \) (also precisely a refined U(5)\( \hat{q} = 2 \) group defined in Sec. 2) but arrange in a different manner (see Fig. 4):

\[
(\bar{u}_R \oplus l_L) \oplus (q_L \oplus \bar{d}_R \oplus \bar{\nu}_R) \oplus (\bar{e}_R) = 5_{-3} \oplus 10_1 \oplus 1_5.
\]

(3.4)

More precisely, they are in the representation of a refined U(5)\( \hat{q} = 2 \) group defined in Sec. 2:

\[
U(5)^{2\text{nd}}_{\hat{q} = 2} \equiv \frac{SU(5)^{2\text{nd}} \times \hat{q} = 2 \ U(1)_\chi}{Z_5} = \frac{SU(5)^{2\text{nd}} \times \hat{q} = 2 \ U(1)_X}{Z_5}.
\]

(3.5)

The corresponding \( U(1) \) is typically called the \( \chi \), as \( U(1)_\chi \equiv U(1)_X \) which we also call \( X_2 \) because this corresponds to the U(1) subgroup of the second type of the u(5) GUT (namely the flipped model [10]).

3.2 There are only two types of u(5) GUTs

Given the SM data and fermion representations, we can prove that there are only two types of u(5) GUTs (both embeddable inside a Spin(10) gauge group) inside the largest possible internal U(16) group.

We sketch the proof below. The normalizer of this SU(5) in U(16) is in fact

\[
N_{U(16)}(SU(5)) = U(5) \times U(1) \times U(1)
\]

(precisely we need \( U(5)_{\hat{q} = 2} \times U(1) \times U(1) \)). There are in fact the important four U(1) subgroups in \( U(5)_{\hat{q} = 2} \times U(1) \times U(1) \) listed below.

1. The \( U(1)_Y \) which in our convention is also generated by the 24th generator of the Lie algebra of u(5) (precisely there are \( U(1)_{Y_1} \equiv U(1)_{\tilde{Y}} \) or \( U(1)_{Y_2} \) depending on which u(5) GUT models we choose). This \( U(1)_Y \) is inside the SU(5). More precisely, the \( U(1)_{Y_1} \subset SU(5)^{1\text{st}} \) of the GG model, and \( U(1)_{Y_2} \subset SU(5)^{2\text{nd}} \) of the Barr’s flipped model.
2. The U(1)\(_X\) which in our convention is also generated by the 25th generator of the Lie algebra of \(u(5)\) (precisely there are U(1)\(_X_1\) or U(1)\(_X_2\) depending on the which \(u(5)\) GUT models we choose). This U(1)\(_X\) is not inside the SU(5), but inside the U(5) (precisely the U(5)\(_{\tilde{q}=2}\) in Sec. 2).

3. For two additional U(1)\(^2\) in the normalizer \(N_{U(16)}(SU(5)) = U(5) \times U(1) \times U(1)\), we can choose one U(1) to act on \(\bar{5}\) alone, another U(1) to act on 1 alone.

Naively, other than the two SM Weyl fermion representation combinations of the \(\bar{5} \oplus 10 \oplus 1\) of SU(5) given in (3.2) and (3.4), there shall be two more kinds of interpretations (thus totally four kinds):

\[
\begin{align*}
(\bar{u}_R \oplus l_L) \oplus (q_L \oplus \bar{d}_R \oplus \bar{\nu}_R) \oplus (\bar{\nu}_R) &= \bar{5} \oplus 10 \oplus 1, \\
(\bar{d}_R \oplus l_L) \oplus (q_L \oplus \bar{u}_R \oplus \bar{\nu}_R) \oplus (\bar{\nu}_R) &= \bar{5} \oplus 10 \oplus 1.
\end{align*}
\]

All these four arrangements can establish the embedding SU(5) \(\supset\) SU(3)\(_c\) \(\times\) SU(2)\(_L\). However, only the (3.2) leads to SU(5)\(_{\text{1st}}\) \(\supset\) G\(_{\text{SM}}\) and the (3.4) leads to U(5)\(_{\text{2nd}}\) \(\supset\) G\(_{\text{SM}}\).

The (3.6) and (3.7) both lead to unsuccessful embeddings:

\[
\begin{align*}
N_{U(16)}(SU(5))^{3\text{rd}} &= U(5)^{3\text{rd}} \times U(1) \times U(1) \not\supset G_{\text{SM}}, \quad \text{and} \\
N_{U(16)}(SU(5))^{4\text{th}} &= U(5)^{4\text{th}} \times U(1) \times U(1) \not\supset G_{\text{SM}},
\end{align*}
\]

because the linear combinations of their U(1) subgroups cannot give rise to the SM’s U(1)\(_Y\) or U(1)\(_{\tilde{Y}}\). This concludes our proof.

### 3.3 Pati-Salam \(su(4) \times su(2)_L \times su(2)_R\) model

Pati-Salam (PS) [7] hypothesized that the lepton carries the fourth color, extending SU(3) to SU(4). The PS also puts the left SU(2)\(_L\) and a hypothetical right SU(2)\(_R\) on the equal footing. The PS gauge Lie algebra is \(su(4) \times su(2)_L \times su(2)_R\), and the PS gauge Lie group is

\[G_{\text{PS}_{q'}} \equiv \frac{SU(4)_c \times (SU(2)_L \times SU(2)_R)}{Z_{q'}} = \frac{\text{Spin}(6) \times \text{Spin}(4)}{Z_{q'}}\]

with the mod \(q' = 1, 2\) depending on the global structure of Lie group. We require \(q' = 2\) in order to embed \(G_{\text{PS}_{q'}}\) into the Spin(10) group. The particle excitations of this PS with 16n Weyl fermions are constrained by the representation of \(G_{\text{PS}_{q'}}\) as (see Fig. 5):

\[
(q_L \oplus l_L) \oplus (q_R \oplus l_R) = (u_L \oplus d_L \oplus \nu_L \oplus \epsilon_L) \oplus (\bar{u}_R \oplus \bar{d}_R \oplus \bar{\nu}_R \oplus \bar{\epsilon}_R) = (4, 2, 1) \oplus (\bar{4}, 1, 2),
\]

written all in the left-handed (L) Weyl basis.\(^{12}\)

\(^{12}\)To be clear, we have the Weyl spacetime spinor \(2_L\) of Spin(1,3) for \((4, 2, 1) \oplus (\bar{4}, 1, 2)\) of \(su(4) \times su(2)_L \times su(2)_R\). Here we use the \(L\) and \(R\) to specify the left/right-handed spacetime spinor of Spin(1,3). We use the L and R to specify the left or right internal spinor representation of \(su(2)_L \times su(2)_R\).
3.4 The Left-Right $su(3) \times su(2)_L \times su(2)_R \times u(1)_{B-L}$ model

Two version of internal symmetry groups for Senjanovic-Mohapatra’s Left-Right (LR) model \[9\] are,

\[G_{LR,q'} \equiv \frac{SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}}{\mathbb{Z}_{3q'}}\]

with $q' = 1, 2$. The particle excitations of this LR model with $16n$ Weyl fermions are constrained by the representation of $G_{LR,q'}$ as (see Fig. 6):

\[q_L \oplus \ell_L \oplus q_R \oplus \bar{\ell}_R = (u_L \oplus d_L) \oplus (\nu_L \oplus e_L) \oplus (\bar{u}_R \oplus \bar{d}_R) \oplus (\bar{\nu}_R \oplus \bar{e}_R) = (3, 2, 1)_{\frac{1}{6}} \oplus (1, 2, 1)_{-\frac{1}{2}} \oplus (\bar{3}, 1, 2)_{\frac{1}{2}} \oplus (1, 1, 2)_{\frac{1}{2}}, \tag{3.9}\]

written all in the left-handed ($L$) Weyl basis.

In general, there is a QFT embedding, the PS model ($G_{PS,q'}$) ⊂ the LR model ($G_{LR,q'}$) ⊂ the SM ($G_{SM_{q=3q'}}$) for both $q' = 1, 2$ via the internal symmetry group embedding, see the details in Sec. 4.1.

3.5 The $so(10)$ Grand Unification and a DSpin-structure modification

Georgi and Fritzsch-Minkowski \[8\] hypothesized the $so(10)$ GUT (with a local Lie algebra $so(10)$) that quarks and leptons as spacetime Weyl fermions (each as a 2-component complex $2_L$ of $Spin(1, 3)$) become the irreducible 16-dimensional spinor representation of $Spin(10)$ (see Fig. 7):

\[16^+ \text{ of } G_{so(10)} \equiv Spin(10) \text{ gauge group.} \tag{3.10}\]

Thus, the $16n$ Weyl fermions can interact via the $Spin(10)$ gauge fields at a higher energy. In this case, the $16$th Weyl fermion, previously a sterile neutrino to the $SU(5)$, is no longer sterile to the $Spin(10)$ gauge fields; it also carries a charge 1, thus not sterile, under the gauged center subgroup $Z(Spin(10)) = \mathbb{Z}_4$.

In Ref. [1], there is a new sector involving either a new discrete torsion WZW term or a fermion parton theory. The new sector can be manifested via the new beyond-standard-model (BSM) Dirac fermions (each as a 4-component complex $2_L \oplus 2_R$ of $Spin(1, 3)$) in the 10-dimensional vector representation of $so(10)$ or $Spin(10)$:

\[10 \text{ of } G_{so(10)} \equiv Spin(10) \text{ gauge group.} \tag{3.11}\]

This BSM fermion is not compatible with the required spacetime-internal symmetry group structure ($Spin \times \mathbb{Z}^F_2 Spin(10)$) that is necessary to realize the $w_2 w_3$ anomaly. The incompatibility is due to the spin-charge relation (i.e., the spacetime spin and internal charge relation) imposed by ($Spin \times \mathbb{Z}^F_2 Spin(10)$) constrains that the fermions can be in the $16$ but not the $10$ of $Spin(10)$.

The remedy, provided by Ref. [1], introduces two different fermion parities, $\mathbb{Z}^F_2$ and $\mathbb{Z}^{F'}_2$, for the SM fermion $16$ and BSM fermion $10$ respectively, which together formally forms a double spin structure called DSpin that shares the same spacetime special orthogonal SO rotation:

\[\text{DSpin} \equiv (\mathbb{Z}^F_2 \times \mathbb{Z}^{F'}_2) \rtimes SO. \tag{3.12}\]

Here $Spin = \mathbb{Z}^F_2 \times SO$ and the $Spin' = \mathbb{Z}^{F'}_2 \times SO$ is another new copy of Spin structure. Thus, the modified $so(10)$ GUT in Ref. [1] asks for a new spacetime-internal structure:

\[G_{so(10)-GUT} \equiv (\text{DSpin} \times \mathbb{Z}^F_2 Spin(10)) \times \mathbb{Z}^{F'}_2 G_{\text{internal-fermionic-parton}}. \tag{3.13}\]
Figure 8: The modified $so(10)$ GUT [1] with new Dirac fermions in the rep $10$ of $G_{so(10)} \equiv \text{Spin}(10)$ gauge group. These fermions are called colorons ($c$) and flavorons ($f$). These fermions can be regarded as the fractionalizations of GUT-Higgs field (see $\Phi_{bi}$ in Sec. 4.2) in terms of the color-flavor separation, similar to the spin-charge separation [59–61] in condensed matter phenomenon. Their representation $c \oplus f \oplus c' \oplus f' = (3, 1)_{1,2} \oplus (1, 2)_{1,2} \oplus (\bar{3}, 1)_{1,2} \oplus (1, 2)_{1,-2}$ of $su(3)_c \times su(2)_L \times u(1)_Y \times u(1)_X$. They are $(c \oplus f) \oplus (c' \oplus f') = 5 \oplus \bar{5}$ of $su(5)$, and $10$ of $so(10)$.

where the internal symmetry of the fermionic parton theory $G_{\text{internal-fermionic-parton}}$ can be implemented via $U(1)'$ or $SU(2)'$, etc., see [1].

Several tables of representation data for the SM, the GG $su(5)$, the PS, and the $so(10)$ models, can already be found in Appendix A of our previous work [1]. In comparison, here we provide the flipped $su(5)$ or $u(5)$ model data in Appendix A.
### 3.6 Representations of SMs and Five GUT-like models in a unified Table

For readers’ convenience to check the quantum numbers of various elementary particles or field quanta of SMs and GUTs, we combine all the SM, the Georgi-Glashow (GG) or the flipped (su(5) or u(5)) models, the Pati-Salam (PS), the left-right (LR), and the so(10) models in a single Table 1. This Table 1 summarizes the more elaborated Appendix A of Ref. [1] and Appendix A of present article in a brief but unified way.

| SM fermion spinor field | SU(3) | SU(2)_L | SU(2)_R | U(1)_{\frac{u}{2}} | U(1)_{Y_1} | U(1)_{Y_R} | U(1)_{EM} | U(1)_{X_1} | Z_{4,X} | Z_{2}^F | U(1)_{X_2} | U(1)_{Y_2} | SU(5)^{1st} | SU(5)^{2nd} | G_{PS} | Spin(10) |
|-------------------------|-------|---------|---------|-------------------|-------------|-------------|-----------|-------------|--------|--------|-------------|-------------|--------------|--------------|-------|---------|
| u_l                     | 3     | 1       | 1/6     | 1                 | 1           | 4           | 2/3       | 1           | 1      | 1      | 1           | 1           |              |              | 4,    | 2,      |
| d_l                     | 3     | 1       | 1/6     | 1                 | -2          | -1/3        | 1         | 1           | 1      | 1      | 1           | 1           |              |              | 1,    |         |
| ν_l                     | 1     | 1       | -1/2    | -3               | 0           | 0           | -3        | 1           | 1      | -3     | -3          | 1           |              |              | 1/2   |         |
| e_l                     | 1     | 1       | -1/2    | -3               | -6          | -1          | -3        | 1           | 1      | 1      | -3          | -3          |              |              | 16    |         |
| u_R                     | 3     | 1       | -1/6    | -4               | -1          | -2/3        | 1         | 1           | 1      | 1      | 2           |              |              |              |        |         |
| d_R                     | 3     | 1       | -1/6    | 2                | -1          | 1/3         | -3        | 1           | 1      | 1      | -4          |              |              |              |        |         |
| ν_R = ν_L               | 1     | 1       | 1/2     | 0                | 3           | 0           | 5         | 1           | 1      | 1      | 1           |              |              |              |        |         |
| e_R = e_L               | 1     | 1       | 1/2     | 6                | 3           | 1           | 1         | 1           | 5      | 0      | 6           |              |              |              |        |         |

|                      | 1/6   | 4/3    | 1/6   | 2/3    | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
|                      | 1/6   | 1/3    | 1/6   | 2/3    | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
|                      | 1/6   | 2/3    | 1/6   | 2/3    | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |

Table 1: Here we list down the Weyl fermion’s representations in the $su(3) \times su(2)_L \times u(1)$ SM, the left-right $su(3) \times su(2)_L \times su(2)_R \times u(1)$, the Pati-Salam $su(4) \times su(2)_L \times su(2)_R$, the Georgi-Glashow $U(5)^{1st} = \frac{SU(5)^{1st} \times U(1)_{X_1=1}}{Z_5}$, the flipped $U(5)^{2nd} = \frac{SU(5)^{2nd} \times U(1)_{X_2=2}}{Z_5}$ model, and the so(10) GUT of Spin(10) group.

The $U(1)_{X_1}$ and $U(1)_{X_2}$ Lie algebra generators in (3.3) and (3.5) are also denoted as the 25th generators out of the 25 generators of the $u(5)^{1st}$ and $u(5)^{2nd}$ (the 1st for the GG, and the 2nd for the flipped model). The $U(1)_{Y_1}$ and $U(1)_{Y_2}$ Lie algebra generators are part of the $su(5)^{1st}$ and $su(5)^{2nd}$ — they are the 24th generators out of the 25 generators of the $u(5)^{1st}$ and $u(5)^{2nd}$. In short, we have these relations between different expressions:

\[
\begin{align*}
U(1)_{Y_1} & \equiv U(1)_{\tilde{Y}} \equiv U(1)_{6Y} \equiv U(1)_{T_{24}}. \\
U(1)_{Y_2} & \equiv U(1)_{T_{24}}. \\
U(1)_{X_1} & \equiv U(1)_{X} \equiv U(1)_{5(B-L)-4Y} \equiv U(1)_{5(B-L)-\frac{2}{3}Y_1} \equiv U(1)_{T_{25}}. \\
U(1)_{X_2} & \equiv U(1)_{\tilde{X}} \equiv U(1)_{T_{25}}. \\
U(1)_{X_1} \text{ and } U(1)_{X_2} & \text{ share the same normal subgroup } Z(\text{Spin}(10)) = Z_{4,X}.
\end{align*}
\]

(3.14)

Here the compactification size of $U(1)_{\tilde{Y}}$ is 1/6 of $U(1)_{Y}$, another way is identifying $\frac{U(1)_{\tilde{Y}}}{Z_6} = U(1)_{Y}$. So their charge quantizations are related via $q_{U(1)_{Y_1}} = q_{U(1)_{Y_2}} = 6q_{U(1)_{Y_1}}$. We shall simply denote their charge relation as $Y_1 \equiv \tilde{Y} \equiv 6Y$. 


These U(1) factors of $X_1, Y_1, X_2$ and $Y_2$ whose corresponding quantized charges span some lattice subspace of a two-dimensional $\mathbb{R}^2$-plane. The $X_1, Y_1, X_2$ and $Y_2$ charges have the following relations:

$$
\begin{align}
(X_1) & = \begin{pmatrix} 1 & 1 \\ 5 & 6 \\ -1 \\ 4 \\ -1 \end{pmatrix} \equiv M_{Z_{2}^{\text{flip}}} (X_2), \\
(Y_1) & = \begin{pmatrix} 1 & 1 \\ 5 & 6 \\ -1 \\ 4 \\ -1 \end{pmatrix} \equiv M_{Z_{2}^{\text{flip}}} (Y_2), \\
(X_2) & = \begin{pmatrix} 1 & 1 \\ 5 & 6 \\ -1 \\ 4 \\ -1 \end{pmatrix} \equiv M_{Z_{2}^{\text{flip}}} (X_1), \\
(Y_2) & = \begin{pmatrix} 1 & 1 \\ 5 & 6 \\ -1 \\ 4 \\ -1 \end{pmatrix} \equiv M_{Z_{2}^{\text{flip}}} (Y_1). \\
\end{align}
$$

(3.15)

- The (3.15) shows that $(X_1, Y_1)$ and $(X_2, Y_2)$ can be mapped onto each other via the $M_{Z_{2}^{\text{flip}}} \equiv \frac{1}{5} \begin{pmatrix} 1 & 4 \\ 6 & 1 \end{pmatrix}$ where $M_{Z_{2}^{\text{flip}}} = 1$, $\det(M_{Z_{2}^{\text{flip}}} = -1$, and $M_{Z_{2}^{\text{flip}}} = M_{Z_{2}^{\text{flip}}}^{-1}$ is itself a self-inverse. The $M_{Z_{2}^{\text{flip}}}$ is part of the $Z_2^{\text{flip}}$ symmetries transformation that swaps the GG $u(5)^{\text{1st}}$ model and the flipped $u(5)^{\text{2nd}}$ model.

- The (3.16) shows $M_{XY}M_{YX} = +1$, $\det(M_{XY}) = -2/3$, and $\det(M_{YX}) = -3/2$; the $M_{XY}$ and $M_{YX}$ are inverse with respect to each other.

- The two sets of charge lattices intersect at integer points that match the charge assignment of the SM Weyl fermions, as shown in Fig. 9.

\[\text{Figure 9: Charge lattices of } (X_1, Y_1) \text{ (in blue as in Georgi-Glashow [GG] Fig. 3) and } (X_2, Y_2) \text{ (in orange as in the flipped } u(5) \text{ GUT Fig. 4). The charge assignments of the SM Weyl fermions coincide with the intersection (also the common integer points) of the two sets of charge lattices. We mark the GG (3.2)'s } (d_R \oplus l_L) \oplus (q_L \oplus u_R \oplus \bar{e}_R) \oplus (\bar{v}_R) = 5 \oplus 3 \oplus 10_1 \oplus 1 \text{ of } su(5)^{\text{1st}} \times u(1)_{X_1} \text{ highlighted in the three vertical blue bars. We mark the (3.4)'s } (u_R \oplus l_L) \oplus (q_L \oplus d_R \oplus \bar{v}_R) \oplus (\bar{e}_R) = 5 \oplus 3 \oplus 10_1 \oplus 1 \text{ of } su(5)^{\text{2nd}} \times u(1)_{X_2} \text{ highlighted in the three tilted horizontal orange bars.}\]

\[\text{In Fig. 9, the } (X_1, Y_1) = (X_2, Y_2) = (5, 5) \text{ is also compatible with two integral charge lattices, but this particle carries a hypercharge } Y_1 \neq 5 \text{ and a net EM charge } \frac{5}{2} \text{ which we do not observe in the experiment. Other examples, such as the } (X_1, Y_1) = (5, -5) \text{ with } (X_2, Y_2) = (-3, 7) \text{ and } (X_1, Y_1) = (-3, 7) \text{ with } (X_2, Y_2) = (5, -5), \text{ also with nontrivial hypercharges or EM charges, that have no evidences yet for their real-world existence.}\]
Other than the above U(1) factors (i.e., $X_1, Y_1, X_2$ and $Y_2$), we can find the following three U(1) factors of $T_{3L}$, $T_{3R}$, and $\frac{B-L}{2}$. Some comments about these U(1) factors:

1. $T_{3L}$ is generated by the third Lie algebra generator of the SU(2)$_L$, while $T_{3R}$ is generated by the third Lie algebra generator of the SU(2)$_R$. We take its $T_{3L/R}$’s charge ($\pm \frac{1}{2}$) as the coefficient of its Lie algebra generator $\frac{1}{2}(0^- _1)$.

2. The $B - L$ is the baryon (B) minus lepton (L) number.

3. We have $T_{3L} + Y = Q_{EM}$, the Lie algebra linear combination of the third generator of SU(2)$_L$ and U(1)$_Y$ gives the U(1)$_{EM}$ charge $Q_{EM}$.

4. We have $T_{3R} + Y = \frac{B-L}{2}$, the Lie algebra linear combination of the third generator of SU(2)$_R$ and the U(1)$_Y$ gives the U(1)$_{B-L}$ charge. We choose the right-handed anti-particle to be in 2 of SU(2)$_R$ (so its right-handed particle to be in 2 of SU(2)$_R$) that makes a specific assignment on the ± sign of its $T_{3R}$ charge. So we have the formula, $T_{3L} - T_{3R} = Q_{EM} - \frac{B-L}{2}$.

5. We can introduce a new internal right hypercharge U(1)$_{Y_R}$ for SU(2)$_R$, as an analog of the internal left electroweak hypercharge U(1)$_Y = U(1)Y_L$ for SU(2)$_L$ sector, such that $-T_{3R} + Y_R = Q_{EM}$ and $-T_{3L} + Y_R = \frac{B-L}{2}$.

6. In fact, we can express all aforementioned U(1) in terms of some linear combinations of three independent generators $T_{3L}$, $T_{3R}$, and $\frac{B-L}{2}$. Here we list down the relations of their charges via the linear combinations of $T_{3L}$, $T_{3R}$, and $\frac{B-L}{2}$ charges:

   $$
   \begin{cases}
   -T_{3R} + \frac{B-L}{2} = -T_{3L} + Q_{EM} = Y_L \equiv Y \equiv \frac{1}{6}Y_1. \\
   T_{3L} + \frac{B-L}{2} = T_{3R} + Q_{EM} = Y_R \equiv \frac{1}{6}Y_R. \\
   T_{3L} - T_{3R} + \frac{B-L}{2} = Q_{EM}. \\
   4T_{3R} + 6(\frac{B-L}{2}) = X \equiv X_1 = 5(B - L) - 4Y = 5(B - L) - \frac{2}{3}Y_1.
   \end{cases}
   $$

   (3.17)

   Since the appropriate linear combination of $T_{3R}$ and $(\frac{B-L}{2})$ in (3.17) contains the $X_1$ and $Y_1$, thus which linear combination as in (3.15) contains also $X_2$ and $Y_2$.

7. The SM (Fig. 2) with a gauge group $G_{SM}$ contains the U(1) Lie algebra generators of $T_{3L}$ and $Y_1$, thus also $Q_{EM}$. The SM does not contain those of $T_{3R}$, $\frac{B-L}{2}$, $X_1$, $X_2$, or $Y_2$.

8. The GG (Fig. 3) su(5) GUT with a gauge group SU(5) contains the U(1) Lie algebra generators of $T_{3L}$ and $Y_1$, thus their linear combinations include $Q_{EM}$, but not the other U(1). The GG $u(5)$ GUT with a gauge group U(5)$_q=2$ contains the U(1) Lie algebra generators of $T_{3L}$, $Y_1$ and $X_1$, thus their linear combinations include everything else: $Q_{EM}$, $\frac{B-L}{2}$, $Y_2$ and $X_2$, also $T_{3R}$ and $Y_R$.

9. The flipped model (Fig. 4) su(5) with only a gauge group SU(5) contains the U(1) Lie algebra generators of $T_{3L}$ and $Y_2$, thus their linear combinations does not include $Q_{EM}$, nor the other U(1). Thus the flipped model with only a gauge group SU(5) is not enough to contain the SM gauge group $G_{SM}$.

   The flipped $u(5)$ GUT with a gauge group U(5)$_q=2$ contains the U(1) Lie algebra generators of $T_{3L}$, $Y_2$ and $X_2$, thus their linear combinations include everything else: $Q_{EM}$, $\frac{B-L}{2}$, $Y_1$ and $X_1$, also $T_{3R}$ and $Y_R$.

10. The PS model (Fig. 5) with a gauge group $G_{PS}$ contains the U(1) Lie algebra generators of $T_{3L}$, $T_{3R}$, and $\frac{B-L}{2}$. The LR model contains also all these generators. Since the linear combinations of these three generators give all the aforementioned U(1) Lie algebra generators in (3.17), the PS and LR models contain all these.

11. The so(10) GUT (Fig. 7) with a gauge group Spin(10) contains the U(1) Lie algebra generators of $T_{3L}$, $Y_1$ and $X_1$. But it contains not only the discrete $Z_{4,X}$ subgroup but also the continuous Lie group $U(1)_{X_1}$. Note that $Z(\text{Spin}(10)) = Z_{4,X_1}$.
4 Quantum Landscape as Quantum Phase Diagram (Moduli Space)

We present the internal symmetry group embedding of various SMs and GUTs in Sec. 4.1. Then we use this group embedding structure to explore a quantum phase diagram containing these SMs and GUTs, and their quantum criticalities (e.g., critical points or critical regions) in Sec. 4.2.

4.1 Embedding Web by Internal Symmetry Groups

For 16 Weyl fermion models, there is a maximal internal symmetry group $U(16)$ that rotates the 16 flavor of spacetime Weyl spinors in $2_L$. But this $U(16)$ again requires a refined definition, say $U(16)_{\hat{q}}$, as we did in Sec. 2. For our purpose, we just need some appropriate subgroup $\text{Spin}(10)$ or $\text{Spin}(10) \times Z_4 \times U(1)$ of $U(16)_{\hat{q}}$, say $\hat{q} = 1$, such that $U(16)_{\hat{q}} \supset \text{Spin}(10)$ or $U(16)_{\hat{q}} \supset \text{Spin}(10) \times Z_4 \times U(1)$. Then we can obtain the following embedding web in Fig. 10 and Fig. 11. The arrow direction $G_1 \rightarrow G_2$ implies that $G_1$ internal symmetry can be broken down to $G_2$ (for example via appropriate scalar Higgs condensation), this also implies the embedding $G_1 \supset G_2$ and the inclusion $G_1 \leftarrow G_2$.

![Diagram](image)

Figure 10: Internal symmetry group embedding web for the $so(10)$ GUT, the Georgi-Glashow (GG) and the flipped $u(5)$ models, and the Standard Model (SM).

Some comments are in order:

1. From (2.4), (2.5), and (2.6), we see that $\text{Spin}(10) \supset U(5)^{\hat{q}=2}$, and $\frac{\text{Spin}(10) \times U(1)}{Z_4} \supset U(5)^{\hat{q}=2}$. We provide a verification via exponential maps of the Lie algebras into these Lie groups embedding in Appendix B. Also there are two versions of $U(5)^{\hat{q}=2}$, the 1st for GG and the 2nd for the flipped model (see Sec. 3.2). Obviously, we also have $U(5)^{\hat{q}} \supset U(5)$ for both the 1st GG and the 2nd the flipped model.

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Figure 11: Follow Fig. 10, an internal symmetry group embedding web for the so(10) GUT, the GG and the flipped u(5) models, and the SM; here we include also the Pati-Salam (PS) and Left-Right (LR) models. We can use the GUT-Higgs condensation to trigger the different routes of the symmetry breaking patterns — we explain the realization of the vacuum expectation value \( \langle \Phi_{1st} \rangle \), \( \langle \Phi_{2nd} \rangle \), and \( \langle \Phi_{54} \rangle \) in Sec. 4.2. This figure generalizes the previous studies in [62] and in our prior work [1].

2. We can explicitly check that the intersection and the union of two Lie groups of GG and the flipped model:

\[
\begin{align*}
U(5)^{1st} \cap U(5)^{2nd} &= (SU(3)_c \times SU(2)_L) \times \mathbb{Z}_6 U(1)_{Y_1} \times \mathbb{Z}_6 U(1)_{Y_2}. \\
U(5)^{1st} \cup U(5)^{2nd} &= Spin(10). \\
SU(5)^{1st} \cap SU(5)^{2nd} &= (SU(3)_c \times SU(2)_L).
\end{align*}
\]

(4.1)

This check is presented in Appendix C.2 based on the Lie algebra data in Table 8, then we can verify the exponential maps of the Lie algebras into these Lie groups Also we show SU(5)^{1st} \supset G_{SM_6} but SU(5)^{2nd} \not\supset G_{SM_6}, but U(5)^{2nd} \supset G_{SM_6} and SU(5)^{2nd} \supset (SU(3)_c \times SU(2)_L) \times \mathbb{Z}_6 U(1)_{Y_2}.

3. Fig. 11 follows Fig. 10 by adding the PS and LR models. We have the PS model \supset the LR model \supset the SM for both \( q' = 1, 2 \) via the internal symmetry group embedding:

\[
G_{PS^{q'}} \supset G_{LR^{q'}} \supset G_{SM_2=3q'} \equiv \frac{SU(3)_c \times SU(2)_L \times U(1)_{Y}}{Z_{q'=3q'}}.
\]

(4.2)

Namely, when \( q' = 1 \), we have \( G_{PS_1} \supset G_{LR_1} \supset G_{SM_1} \). Furthermore, only when \( q' = 2 \), we can have the whole embedded into the Spin(10) for the so(10) GUT: Spin(10) \supset G_{PS_2} \supset G_{LR_2} \supset G_{SM_6}.
4.2 Quantum Phase Diagram and a Mother Effective Field Theory

Now we follow Ref. [1] on the proposed parent effective field theory (EFT) as a modified $so(10)$ GUT (with a Spin(10) gauge group) with a discrete torsion class of WZW term. In Ref. [1], we had proposed that Georgi-Glashow (GG) $su(5)$ and the Pati-Salam (PS) $su(4) \times su(2) \times su(2)$ models could manifest different low energy phases of the same parent EFT, but overall all of them share the same quantum phase diagram (see Figure 5 and 7 in [1]). By quantum phase diagram, we mean to find the governing zero-temperature quantum ground states in the parameter spaces by tuning the QFT coupling strengths.

In our present work, we further include the GG $u(5)$ and the flipped $u(5)$, and the Left-Right (LR) model into this parent EFT (see Fig. 11, and Fig. 12 below). The parent EFT is basically the same modified $so(10)$ GUT in Ref. [1] (except we need to refine the vev $\langle \Phi^{1st}_{45} \rangle$, $\langle \Phi^{2nd}_{45} \rangle$ and $\langle \Phi_{54} \rangle$ below in Sec. 4.2.1), thus we follow exactly the same notations and conventions there in [1]. This parent EFT contains the following actions:

\[ S_{\text{YM-Weyl}} = \int_{M^4} \text{Tr}(F \wedge *F) + d^4x (\gamma^i_1 (i\sigma^a D_{\mu,A}) \psi_L), \]  
\[ S_{\text{Higgs}} = \int_{M^4} d^4x (|D_{\mu,A} \Phi_R|^2 - U(\Phi_R)), \]  
\[ S_{\text{Yukawa}} = \int_{M^4} d^4x \left( \frac{1}{2} \phi^T \Phi^b_i \phi + \frac{1}{2} \sum_{a=1}^5 (\psi^T_1 i \sigma^2 (\phi_{2a-1} \Gamma_{2a-1} - i \phi_{2a} \Gamma_{2a}) \psi_L + \text{h.c.}) \right), \]  
\[ S^{\text{WZW}} = \frac{1}{\pi} \int_{M^5} B(\Phi_{54}) \wedge dC(\Phi_{45}) \big|_{M^4 = \partial M^5} = \pi \int_{M^5} B(\hat{\Phi}^{bi}) \wedge \delta C(\hat{\Phi}^{bi}) \big|_{M^4 = \partial M^5}. \]  

In summary, we have:

- the action of Yang-Mills field strength 2-form $F = dA - igA \wedge A$ and Weyl fermion $\psi_L$ coupling term $S_{\text{YM-Weyl}}$ in (4.3),
- the Higgs or GUT-Higgs $\Phi_R$ action $S_{\text{Higgs}}$ of some representation $R$ in (4.4),
- the Yukawa-like action (4.5) coupling between the fermion $\psi_L$ in 16 of Spin(10), the GUT-Higgs $\phi$ in the vector $10$ of Spin(10), and the $\Phi^{bi}_{ab}$ in the bivector $100 = 10 \times 10$ of Spin(10). The $\sigma^2$ matrix acts on the 2-component spacetime Weyl spinor $\psi_L$. The $\Gamma_a$ (with $a \in \{1,2,\ldots,10\}$) are ten rank-16 matrices satisfying $\{\Gamma_{a-1}, \Gamma_{a-1}\} = 2\delta_{ab}$, $\{\Gamma_{2a-1}, \Gamma_{2b}\} = 2\delta_{ab}$, $\{\Gamma_{2a-1}, \Gamma_{2b}\} = 0$ (for $a,b = 1, 2, \ldots, 5$). So far all the terms listed above, (4.3), (4.4), and (4.5), are within the framework of Landau-Ginzburg type of internal symmetry breaking via the Higgs field.
- the WZW term. In order to go beyond the Landau-Ginzburg paradigm to realize a richer quantum criticality between GG and PS models, we require to add a WZW term written on a 4d manifold $M^4$ as a 5d $M^5$’s boundary. The WZW term (4.6) is written in terms of $\mathbb{R}$-valued gauge fields, 2-form $B(\Phi_{54})$ and 2-form $C(\Phi_{45})$, or $\mathbb{Z}_2$-valued cohomology class fields $B(\hat{\Phi}^{bi})$ and $C(\hat{\Phi}^{bi})$. Ref. [63] verifies first that the specific $BdC$ term matches the $w_2(TM)w_3(TM)$ anomaly. Ref. [1] later constructs the $B(\hat{\Phi}^{bi}) \sim \delta C(\hat{\Phi}^{bi})$ out of the GUT-Higgs fields to matches the $w_2(TM)w_3(TM) = w_2(V_{SO(10)})w_3(V_{SO(10)})$ anomaly. An SO(10) real bivector field $\Phi^{bi}_{ab} \in \mathbb{R}$ is obtained from the tensor product of the two $\phi$, in the $10 \otimes 10 = 1_S \oplus 45_A \oplus 54_S$ of $so(10)$ also of Spin(10), with the anti-symmetric (A) and symmetric (S) parts of tensor product: $\Phi^{bi}_{ab} = \phi_a \phi_b$ includes

\[ \{ \begin{array}{c} \text{Tr} \Phi^{bi} = \sum_a \phi^a_{ab} \text{ gives } \Phi_R = \Phi_1 \text{ in } 1_S. \\ \hat{\Phi}^{bi} \equiv \hat{\Phi}^{bi}_{[ab]} = \frac{1}{2} (\Phi^{ab}_{ab} - \Phi^{ba}_{ba}) = \frac{1}{2} (\phi_a \phi_b - \phi_b \phi_a) = \frac{1}{2} (\phi_a, \phi_b) \text{ gives } \Phi_R = \Phi_{45} \text{ in } 45_A. \\ \hat{\Phi}^{bi} \equiv \hat{\Phi}^{bi}_{(ab)} = \frac{1}{2} (\Phi^{ab}_{ab} + \Phi^{ba}_{ba}) = \frac{1}{2} (\phi_a \phi_b + \phi_b \phi_a) = \frac{1}{2} (\phi_a, \phi_b) \text{ gives } \Phi_R = \Phi_{54} \text{ in } 54_S. \end{array} \]  

We construct this WZW term (4.6) under a precise constraint to match the mod 2 class of 4d global gauge-gravitational anomaly captured by the 5d $w_2w_3$ term.
Figure 12: The quantum phase diagram (also the moduli space of vacuum expectation values (vevs) of a set of scalar fields) is separated to eight octants. The colors of quantum phases are designed to match the colors in Fig. 2 to Fig. 7. Here the real parameter $r^R \in \mathbb{R}$ denotes the coefficient of the effective quadratic potential (4.18)'s $U(\Phi^R)$ of $\Phi$ field in the representation $R$. The corresponding Higgs $\Phi$ field condenses in the representation -$R$ if $r^R < 0$. The $\langle \Phi_{54} \rangle \neq 0$ condenses when $r_{54} < 0$. There are however two distinct $\langle \Phi_{45}^{1st} \rangle$ and $\langle \Phi_{45}^{2nd} \rangle$ condensations for $r_{45} < 0$; the two vevs are selected by $h^> 0$ and $h^< 0$ respectively. The quantum phase diagram contains the following phases in the eight octants:

- the so(10) GUT (with $\langle \Phi_{45}^{45} \rangle = \langle \Phi_{54} \rangle = 0$ in the first and fifth octants, labeled by (I) and (V)),
- the Georgi-Glashow u(5) GUT (GG, with $\langle \Phi_{45}^{1st} \rangle \neq 0$ but $\langle \Phi_{54} \rangle = 0$ in the second octant (II)),
- the flipped u(5) GUT (with $\langle \Phi_{45}^{2nd} \rangle \neq 0$ but $\langle \Phi_{54} \rangle = 0$ in the sixth octant (VI)),
- the su(4) $\times$ su(2)$_L$ $\times$ su(2)$_R$ Pati-Salam model (PS, with $\langle \Phi_{54} \rangle \neq 0$ but $\langle \Phi_{45} \rangle = 0$ in the fourth and eight octants, (IV) and (VIII)),
- the su(3) $\times$ su(2) $\times$ u(1) Standard Model (SM, with both $\langle \Phi_{45} \rangle \neq 0$ and $\langle \Phi_{54} \rangle \neq 0$, in the third and seventh octants, (III) and (VII)).
- the quantum critical region (around (0) in the white ball region) occurs if the criticality is enforced by the 4d boundary anomaly on of a 5d $w_2w_3$ invertible TQFT, and if we had added the WZW term into the modified so(10) GUT, and if the U(1)$_{\text{dark}}$ is deconfined, namely its fine structure constant $g'^2$ is below a certain critical value $g'^2_c$, and typically near the origin with small $r_{45}$ and $r_{54}$. The region outside (0) is shown in Fig. 13 (a), the region (0) inside is shown in Fig. 13 (b). We summarize the nature of the phase transitions in Sec. 7’s Table 2.
Figure 13: (a) If $U(1)^{\text{dark}}_{\text{gauge}}$ is confined, namely its fine structure constant $g'^2$ is above a certain critical value, then we call those eight octants as (I) to (VIII). (b) If $U(1)^{\text{dark}}_{\text{gauge}}$ is deconfined with $g'^2$ is below a certain critical value, then we call those eight octants as (I)' to (VIII)', where they are altogether shown as the white ball region (0) in Fig. 12, such that $(0)=(I)+(II)+(III)+(IV)+(V)+(VI)+(VII)+(VIII)$. We will also summarize the nature of the phase transitions in Sec. 7's Table 2.

Manifestation of the WZW term in terms of a fermionic parton theory: Follow Ref. [1], we have another realization of WZW term (4.6) by integrating out some massive 5d fermionic parton theory ($|m| \gg 0$, we denote QED' for it is beyond the ordinary SM’s quantum electrodynamics QED sector):

$$S_{\text{QED}'}^{\text{WZW}}[\xi, \bar{\xi}, a, \Phi, B, C] = \int_{M^5} \tilde{\xi} (i \gamma^\mu D'_\mu - m - \tilde{\Phi}^b i \gamma^b) \xi \, d^5 x. \quad (4.8)$$

The covariant derivative $D'_\mu = \nabla_\mu - i a_\mu - ig A_\mu$ contains the minimal coupling of the fermionic parton $\xi$ to a new emergent dynamical $U(1)^{\text{dark}}_{\text{gauge}}$ field $a_\mu$, as well as the minimal coupling to the SO(10) gauge field $A_\mu$ (which is part of the Spin(10) gauge field). We may treat the gauge field $A_\mu$ as a background field for now, and discuss the dynamically gauged $A_\mu$ later in Sec. 4.2.2. The previously introduced two 2-form $\mathbb{R}$ gauge fields $\mathcal{B} = B_{\mu \nu} \, dx^\mu \wedge dx^\nu$ and $\mathcal{C} = C_{\mu \nu} \, dx^\mu \wedge dx^\nu$ couple to the 8-component Dirac fermionic parton $\xi$ (or doubled version of 4-component Dirac fermion) in 5d. While the 4d interface at $m = 0$ appears in between the 5d bulk $m > 0$ and $m < 0$ phases:

$$S_{\text{QED}'}^{\text{WZW}}[\xi, \bar{\xi}, a, \Phi] = \int_{M^4} \xi (i \gamma^\mu D'_\mu - \bar{\Phi}^b i \gamma^b) \xi \, d^4 x, \quad (4.9)$$

now with the 4-component Dirac fermionic parton $\xi$ in 4d. Some explanations below:

- In 4d, we can already define five gamma matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$, and $\gamma^{\text{FIVE}} \equiv (i \gamma^0 \gamma^1 \gamma^2 \gamma^3)$, all are rank-4 matrices.\(^{14}\) The 4d Dirac fermion $\xi$ in (4.9) is a 4-component complex fermion $2_L \oplus 2_R$ of Spin(1,3). The $\xi$ is also in the 10-dimensional vector representation of $so(10)$ or Spin(10). Namely, the 4d Dirac

\(^{14}\)Denote $\sigma^{\mu \nu} \equiv \sigma^\mu \otimes \sigma^\nu \otimes \cdots$ as the direct product of the standard Pauli matrices. Explicit matrix representation of 4d gamma matrices are

$$\gamma^0 = \sigma^{10}, \gamma^1 = i \sigma^{21}, \gamma^2 = i \sigma^{22}, \gamma^3 = i \sigma^{23}, \gamma^{\text{FIVE}} \equiv (i \gamma^0 \gamma^1 \gamma^2 \gamma^3) = -\sigma^{30}.$$
fermion $\xi$ in (4.9) is overall in the following rep:

$$\left(2_L \oplus 2_R\right) \otimes \text{Spin}(1,3) \times \left(10 \text{ of Spin}(10)\right).$$  \hfill (4.10)

- However, the 4d parton theory has some extra symmetry that are not presented in the original theory, including: (i) $U(1)': \xi \to e^{i\theta} \xi$, (ii) $Z_2^{CP'}: \xi(t,x) \to \gamma^\text{FIVE}_L \xi(t,-x)$, (iii) $Z_2^{T'}: \xi(t,x) \to K \gamma^0 \gamma^\text{FIVE}_L \xi(-t,x)$ with the complex conjugation operation $K$ sending $i \to -i$. Note that the CP' and T' symmetries are for partons from the fractionalization of the WZW term ((4.6) and (4.9)), which are unrelated to the CP and T symmetries of the chiral fermions in the GUTs. The massless Dirac fermion are two Weyl fermions: $\xi = \xi_L + \xi_R$.

- In 5d, we define five gamma matrices $\tilde{\gamma}^0, \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3$, and $\tilde{\gamma}^4$. However, the 5d gamma matrices have different matrix representations than the 4d gamma matrices — they are related by the dimensional reduction on the domain wall normal to the $x_1$ direction. By doubling the fermion content, we are able to introduce two more gamma matrices, denoted $\tilde{\gamma}^5$ and $\tilde{\gamma}^6$, such that all seven gamma matrices $\tilde{\gamma}^0, \ldots, \tilde{\gamma}^6$ are rank-8 matrices satisfying the Clifford algebra relation $\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\delta^{\mu\nu}$. So the 5d Dirac fermion $\xi$ in (4.8) is a doubled version of 4-component complex fermion (4 of Spin(1,4)), as the 8-component complex fermion. The $\xi$ is also in the 10-dimensional vector representation of so(10) or Spin(10). Namely, the 5d Dirac fermion $\xi$ in (4.8) is in the following rep:

$$2 \times (4 \text{ of Spin}(1,4)) \times (10 \text{ of Spin}(10)).$$  \hfill (4.11)

Notice that the 5d bulk fermions (4.11) have doubled components of the 4d interface fermions (4.10).

- For the 5d bulk theory, if we define the $m > 0$ as a trivial gapped vacuum (say at $x > 0$), then one can check that the $m < 0$ side (say at $x < 0$) might be a nontrivial gapped vacuum with a low energy invertible TQFT describing either gapped invertible topological order or gapped symmetry-protected topological states (SPTs) [64] in quantum matter. Indeed, the (4.6)'s partition function $\exp(i\sigma WZW)$ can match with $\exp(i\sigma \int w_2 w_3)$ in a closed 5d bulk without a boundary, thus it describes the invertible TQFT $w_2 w_3$. The WZW term (4.6) also gives a 4d interface description, that lives on a 4d boundary of a 5d bulk.

Below we explore quantum phases and their criticalities or phase transitions in Fig. 12 by two aspects: (1) when the internal symmetries are treated as global symmetries (as toy models) in Sec. 4.2.1, and (2) when the internal symmetries are dynamically gauged (as they are gauged in our real-world quantum vacuum) in Sec. 4.2.2.

\textsuperscript{15}In contrast to footnote 14, explicit matrix representations of 5d gamma matrices are

$$\tilde{\gamma}^0 = \sigma^{200}, \tilde{\gamma}^1 = i\sigma^{300}, \tilde{\gamma}^2 = i\sigma^{131}, \tilde{\gamma}^3 = i\sigma^{132}, \tilde{\gamma}^4 = i\sigma^{133}, \tilde{\gamma}^5 = i\sigma^{170}, \tilde{\gamma}^6 = i\sigma^{120}.$$
4.2.1 Internal symmetries treated as global symmetries

In the limit when we treat their internal symmetries as global symmetries (then the Yang-Mills gauge field $A$ is only a non-dynamical background field), the GG, the flipped, PS and LR models match the global gauge-gravitational $w_2 w_3 (TM) = w_2 w_3 (V_{so(10)})$ anomaly via the internal symmetry-breaking from Spin(10) to each individual subgroup (as the breaking pattern in Fig. 11). The corresponding QFTs in Fig. 11 again manifest different low energy phases of the same parent EFT, but overall all of them share the same quantum phase diagram. By sharing the same quantum phase diagram, we mean that they have the same Hilbert space and the same 't Hooft anomaly constraints at a deeper UV. In other words, they are in the same deformation class of QFTs, particularly advocated by Seiberg [16].

1. Starting from the $so(10)$ GUT (modified with WZW term or not), the condensation of $\Phi_{45}$ or/and $\Phi_{54}$ will drive the symmetry breaking transitions to various lower energy and lower internal symmetry phases, summarized in Fig. 11. In particular, if $\Phi$ condenses to a specific configuration $\langle \Phi \rangle$, the original Lie algebra $g_{so(10)}$ will be broken to its subalgebra that commutes with $\langle \Phi \rangle$, given by $g_{small} = \mathfrak{z}_{g_{large}}(\langle \Phi \rangle) \equiv \{ T \in g_{large} | [T, \langle \Phi \rangle] = 0 \}$. (4.12)

To realize the symmetry breaking from the Lie algebra $g_{so(10)} \equiv so(10)$,

$$
\begin{align*}
g_{so(10)} &\rightarrow g_{PS} \equiv su(4) \times su(2)_L \times su(2)_R, \\
g_{so(10)} &\rightarrow g_{GG} \equiv su(5)^{1st} \times u(1)_{X_1}, \\
g_{so(10)} &\rightarrow g_{flipped} \equiv su(5)^{2nd} \times u(1)_{X_2},
\end{align*}
$$

we must have the relations

$$
\begin{align*}
\mathfrak{z}_{so(10)}(\langle \Phi_{54} \rangle) &= g_{PS}, \\
\mathfrak{z}_{so(10)}(\langle \Phi_{45}^{1st} \rangle) &= g_{GG}, \\
\mathfrak{z}_{so(10)}(\langle \Phi_{45}^{2nd} \rangle) &= g_{flipped},
\end{align*}
$$

whose solutions read

$$
\begin{align*}
\langle \Phi_{54} \rangle &\propto \begin{pmatrix} -3 \\ -3 \\ 2 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\
\langle \Phi_{45}^{1st} \rangle &\propto \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\langle \Phi_{45}^{2nd} \rangle &\propto \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{align*}
$$

- The $\langle \Phi_{54} \rangle$ explicitly distinguishes the first four-dimensional subspace from the last six-dimensional subspace of the $so(10)$ vector, therefore breaking $g_{so(10)} = so(10)$ down to $g_{PS} = so(6) \times so(4)$.
- The $\langle \Phi_{45}^{1st} \rangle$ is proportional to the $u(1)_{X_1}$ generator, which effectively requires the unbroken generators

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16 In fact, many works on gapping the anomaly-free chiral fermions are based on the same logic: The chiral fermions that are free from 't Hooft anomalies of the chiral symmetry $G$, must be gappable without any chiral symmetry breaking in $G$, via the $G$-symmetry preserving interaction deformations. See a series of work along this direction and references therein: Fidkowski-Kitaev [65] in 0+1d, Wang-Wen [66,67] for gapping chiral fermions in 1+1d, You-He-Xu-Vishwanath [68,69] in 2+1d, and notable examples in 3+1d by Eichten-Preskill [70], Wen [71], You-BenTov-Xu [72,73], BenTov-Zee [74], Kikukawa [75], Wang-Wen [29], Razamat-Tong [76,77], Catterall et al [78,79], etc. The techniques of gapping chiral fermions can be used in gapping the mirror sector.
3. How many distinct phase transitions there are in Fig. 12? We can enumerate those occur in the Quantum criticalities and phase transitions due to GUT-Higgs, with or without WZW term:

- The $\Phi_{45}^{(2nd)}$ can be obtained via the $Z_2^{flip}$ transformation on the $\Phi_{45}^{(1st)}$, where $Z_2^{flip}$ is described in Sec. 3.6 and Appendix C.1. The $\Phi_{45}^{(2nd)}$ is proportional to the $u(1)_{X_2}$ generator, which effectively requires the unbroken generators to commute with $u(1)_{X_2}$ generator. This singles out $g_{GG} = j_{so(10)}(u(1)_{X_2}) = su(5)^{2nd} \times u(1)_{X_2}$.

- Using (4.15), one can further verify that

$$\delta_{PS}(\langle \Phi_{45}^{(1st)} \rangle) = \delta_{PS}(\langle \Phi_{45}^{(2nd)} \rangle) = \delta_{GG}(\langle \Phi_{54} \rangle) = \delta_{3g_{flipped}}(\langle \Phi_{54} \rangle) = g_{SM},$$

which explicitly confirms that the simultaneous condensation of $\Phi_{45}$ (any of $\Phi_{45}^{(1st)}$ and $\Phi_{45}^{(2nd)}$) and $\Phi_{54}$ indeed breaks the internal symmetry to $g_{SM}$. These results also agree with the representation data and branching rules listed in [80–82].

2. Quantum criticalities and phase transitions due to GUT-Higgs, with or without WZW term:

In Fig. 12 and its caption, we have enumerated all the ground states in all the eight octants of the phase diagram descended from the 4d parent theory. In particular, the 4d phases (in the bulk portion of the phase diagram) maintain regardless of whether we add the WZW term $S^{WZW}$ into the Landau-Ginzburg type of 4d parent theory of the action:

$$S_{YM-Weyl} + S_{Higgs} + S_{Yukawa}. \quad (4.17)$$

The phase transition (between the eight octants in Fig. 12) is triggered by the following GUT-Higgs potential $U(\Phi_R)$ appeared in (4.4), for example:

$$U(\Phi_R) = \left( r_{45}(\Phi_{45})^2 + \lambda_{45}(\Phi_{45})^4 \right) + \left( r_{54}(\Phi_{54})^2 + \lambda_{54}(\Phi_{54})^4 \right) + h \Phi_{45} \cdot \left( \langle \Phi_{45}^{(1st)} \rangle - \langle \Phi_{45}^{(2nd)} \rangle \right) + \ldots. \quad (4.18)$$

Although the 4d phases in all the eight octants are not sensitive to the WZW term, the quantum critical region (labeled as (0)) and the phase boundaries between the eight octants are highly sensitive to the WZW term. In those critical regions, we must examine the nature of criticality by looking into the full action

$$S_{YM-Weyl} + S_{Higgs} + S_{Yukawa} + S^{WZW} = S_{YM-Weyl} + S_{Higgs} + S_{Yukawa} + S^{WZW}_{QED^5_{2D}}. \quad (4.19)$$

In particular, in the right hand side of equality, we focus on a specific scenario that the low energy physics of WZW is manifested by the 5d bulk/4d interface QED' written as $S^{WZW}_{QED^5_{2D}}$ from (4.8) and (4.9), with deconfined emergent dark gauge fields of $U(1)'$ only near the critical region.

Below in Remark 4 and 5 respectively, we describe the Landau-Ginzburg type criticalities (not sensitive to WZW term), and the beyond Landau-Ginzburg type criticalities (sensitive to WZW term and anomaly constraints).

3. How many distinct phase transitions there are in Fig. 12? We can enumerate those occur in the $h > 0$ side with the four quadrants (I), (II), (III), and (IV) shown in Fig. 14. Then we can enumerate those occur in the $h < 0$ side with the last four quadrants (V), (VI), (VII), and (VIII) shown in Fig. 15.

In Fig. 14, there are four phase transitions between these four phases: (I)-(II), (II)-(III), (III)-(IV), and (IV)-(I). There are also another four phase transitions from either of these four phases into the quantum critical region in (0): Namely, (I)-(0), (II)-(0), (III)-(0), and (IV)-(0). So there are totally eight possible phase transitions in Fig. 14.

In Fig. 15, there are four phase transitions between these four phases: (V)-(IV), (VI)-(VII), (VII)-(VIII), and (VIII)-(V). There are also another four phase transitions from either of these four phases into the quantum critical region in (0): Namely, (V)-(0), (VI)-(0), (VII)-(0), and (VIII)-(0). So there are also totally eight possible phase transitions in Fig. 15.
Figure 14: A typical \( h > 0 \) slice of Fig. 12’s quantum phase diagram. The coordinates are potential tuning parameters \( r_{45} \) and \( r_{54} \) of (4.18).

Figure 15: A typical \( h < 0 \) slice of Fig. 12’s quantum phase diagram. The coordinates are potential tuning parameters \( r_{45} \) and \( r_{54} \) of (4.18).

Furthermore, the phase structures show many phases are the same: (I)=(V), (III)=(VII), and

\[ \begin{align*}
17 & \text{Here we extract } h \Phi_{45} \cdot (\langle \Phi_{45}^{1st} \rangle - \langle \Phi_{45}^{2nd} \rangle) \text{ from } h' \left( (\Phi_{45} - \langle \Phi_{45}^{1st} \rangle)^2 - (\Phi_{45} - \langle \Phi_{45}^{2nd} \rangle)^2 \right) + \ldots.
\end{align*} \]
(IV) = (VIII). The differences between (II) and (VI) are simply two different Landau-Ginzburg symmetry-breaking vacua. Thus all phase transitions in Fig. 14 have the exactly same nature as those in Fig. 15. There is only one more phase transition between (II) and (VI), that is not shown in Fig. 14 or Fig. 15. So totally there are nine distinct possible phase transitions in Fig. 12 that we will enumerate.

Even more precisely, the stable quantum critical region (0) can also have distinct symmetry breaking orders, so we can further precisely denote the dark gauge $U(1)'$-deconfined critical region (0) as

\[(0) = (I)' + (II)' + (III)' + (IV)' + (V)' + (VI)' + (VII)' + (VIII)'\].

Each of them has the symmetry breaking orders from the $U(1)'$-confined region (mentioned previously as (I) to (VIII)). Thus, we also have the identifications of some deconfined critical phases: $(I)' = (V)'$, $(III)' = (VII)'$, and $(IV)' = (VIII)'$.

Here are some terminologies for the type of phase transitions:

1. **Landau-Ginzburg** type: Based on the original symmetry group (kinematic) broken down to an unbroken symmetry group (dynamics) via a symmetry-breaking order parameter $\langle O_{LG} \rangle$.

2. **Beyond Landau-Ginzburg** type: Cannot be characterized via merely symmetry-breaking order parameters. e.g., phase transitions involving topological terms (e.g., WZW term), SPTs, intrinsic topological orders, or the ’t Hooft anomaly matching differently on two sides of phases, etc.

3. **Wilson-Fisher** type: Phase transition due to a scalar field condensation $\langle \Phi \rangle \neq 0$, especially via $r\Phi^2 + \Lambda \Phi^4$ type potential. However, Wilson-Fisher fixed point requires the beyond-mean-field RG correction to the Gaussian fixed point.

4. **Gross-Neveu type** type: Typically the Wilson-Fisher type with additional Yukawa coupling between fermions $\chi$ and scalars $\Phi$ as $\chi^\dagger \Phi \chi$, again a phase transition due to a condensation $\langle \Phi \rangle \neq 0$.

5. **Order of phase transition**:

   - For a minimal positive $N$, if the $N$th derivative of the free energy (of QFT) with respect to the driving parameter is discontinuous at the transition, it is called the $N$th-order phase transition.
   - First-order transition is also called a discontinuous transition, while the correlation length remains finite and no additional gapless excitations appear at the transition.
   - Second-order and higher-order is called a continuous transition, while the correlation length diverges to infinite and additional gapless excitations (thus called critical, sometimes described by conformal field theory) appear at the transition.

This above definition is applicable to Landau-Ginzburg as well as beyond Landau-Ginzburg paradigm.

If the transition happens to be within Landau-Ginzburg paradigm, then:

- First-order means the order parameter has a discontinuous jump at the transition.
- Second-order and higher-order means the order parameter changes continuous without a jump at the transition.

4. **Landau-Ginzburg type criticalities and phase transitions**:

   - **Phase transition between the octant (II) to (VI)**:

     This is the phase transition between the **GG $u(5)$ and flipped $u(5)$ GUTs**. In both the octant (II) and (VI), we have $r_{45} < 0$. The phase transition between (II) and (VI) is triggered by $h > 0$ and $h < 0$, causing $\langle \Phi_{45}^{1\text{st}} \rangle \neq 0$ or $\langle \Phi_{45}^{2\text{nd}} \rangle \neq 0$. In general, by tuning $h$, these condensations jump from zero to nonzero, thus it is the **first-order phase transition of traditional Landau-Ginzburg symmetry-breaking type with the order parameter discontinuity**. The WZW term does not play any role in the phase transition. This also means there is no critical gapless mode directly associated with this phase transition.

   - **Phase transition between the octant (II) to (III), similar to (VI) to (III)**:

     Here the correlation function for Landau-Ginzburg paradigm is typically the two-point correlator $\langle O(x_1)O(x_2) \rangle$ of local order parameters of individual spacetime points. In contrast, the correlation function for topological order beyond the Landau-Ginzburg paradigm is the correlator of strings or higher-dimensional extended operators.
This is the phase transition between the GG $u(5)$ GUT and SM (similarly, the transition between the flipped $u(5)$ GUT and SM). The phase transition is triggered by tuning $r_{54} > 0$ to $r_{54} < 0$ of $r\Phi^2 + \lambda\Phi^4$ in (4.18). It is the **continuous phase transition of Wilson-Fisher type Landau-Ginzburg symmetry-breaking type**.

- **Phase transition between the octant (IV) to (III):**
  
  This is the phase transition between the PS and SM. The phase transition is triggered by tuning $r_{45} > 0$ to $r_{45} < 0$ of $r\Phi^2 + \lambda\Phi^4$ in (4.18). It is again the **continuous phase transition of Wilson-Fisher type Landau-Ginzburg symmetry-breaking type**.

5. **Beyond Landau-Ginzburg type criticalities and phase transitions:** With the WZW term, the criticality between the GG and the PS, and the criticality between the flipped $u(5)$ and the PS, both of these criticalities are governed by the Beyond Landau-Ginzburg paradigm. The critical regions are drawn in the white region around the origin in Fig. 14 or Fig. 15.

- **Phase transition between the octant (I) to (II), similar to (I) to (VI):**
  
  The WZW term and the anomaly of 5d $w_{2w3}$ invertible TQFT play a crucial role in (I). The 4d phase transition from the modified $so(10)$ (I) to GG (II) (similarly, the $so(10)$ (I) to the flipped (VI)) is a boundary phase transition on the 5d bulk $w_{2w3}$. The phase transition is triggered not merely by tuning $r_{45} > 0$ to $r_{45} < 0$ of $r\Phi^2 + \lambda\Phi^4$ in (4.18), but also by the symmetry breaking to cancel the anomaly on the 4d boundary of 5d invertible TQFT (when entering from (I) to (II) or to (VI)). Overall, it is the **continuous phase transition of Wilson-Fisher type but Beyond-Landau-Ginzburg paradigm due to the anomaly matching via the symmetry breaking on the 4d boundary of 5d invertible TQFT**.

- **Phase transition between the octant (I) to (IV):**
  
  The WZW term and the anomaly of 5d $w_{2w3}$ invertible TQFT play a crucial role in (I). The 4d phase transition from the modified $so(10)$ (I) to PS (IV) is a boundary phase transition on the 5d bulk $w_{2w3}$. The phase transition is triggered not merely by tuning $r_{54} > 0$ to $r_{54} < 0$ of $r\Phi^2 + \lambda\Phi^4$ in (4.18), but also by the symmetry breaking to cancel the anomaly on the 4d boundary of 5d invertible TQFT (when entering from (I) to (IV)). Overall, it is the **continuous phase transition of Wilson-Fisher type but Beyond-Landau-Ginzburg paradigm due to the anomaly matching via the symmetry breaking on the 4d boundary of 5d invertible TQFT**.

- **Phase transition between the octant (II) to the critical region (0):**
  
  The WZW term and the anomaly of 5d $w_{2w3}$ invertible TQFT play a crucial role in the critical region (0). The 4d phase transitions from the either models of GUT/SM of (II), (III) and (IV) to the critical region (0) is a boundary phase transition on the 5d bulk $w_{2w3}$. These 4d phase transitions are Gross-Neveu type because we also have Yukawa-Higgs interactions $\chi^{\dagger}\Phi\chi$ in (4.9) (more than Wilson-Fisher of $r\Phi^2 + \lambda\Phi^4$). Moreover, there are deconfined dark gauge fields of $U(1)'$ in the critical region (0), but the $U(1)'$ is confined outside the critical region (0). Therefore, overall it is the **continuous phase transition of deconfined-confined QED'-Gross-Neveu type beyond-Landau-Ginzburg paradigm**. It is beyond Landau-Ginzburg also due to two effects (1) WZW term and (2) anomaly matching via the symmetry breaking on the 4d boundary of 5d invertible TQFT.

- **Phase transition between the octant (I) to the critical region (0) (or more precisely (I)'):**
  
  The WZW term and the anomaly of 5d $w_{2w3}$ invertible TQFT play a crucial role in both the critical region (I)' and the modified $so(10)$ GUT (I).

The 4d phase transitions from the (I) to the critical region (I)' is a boundary phase transition that the

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19 More precisely, we really mean to specify here the phase transitions from the $U(1)'$ confined phase of GUT/SM to the corresponding $U(1)'$ deconfined phase of GUT/SM, namely (II) to (II)', (III) to (III)', (IV) to (IV)', and (VI) to (VI)'. The critical region (0) actually has children sub-phases including those in Fig. 13 (b).
5d bulk $w_2w_3$ is always required since the Spin(10) is preserved throughout the transition (if we regard the Spin(10) global symmetry is realized locally onsite).

— If the deconfined dark gauge fields $U(1)'$ in the critical region (I)' becomes confined in the region (I), then overall it is the continuous deconfined-confined phase transition of QED$_4'$ Beyond-Landau-Ginzburg paradigm, without any global symmetry-breaking. The QED$_4'$ describes the $U(1)'$ dark gauge field coupled to fermionic partons. When the gauge coupling is strong enough, $g''^2 > g_c^2$, it is possible to drive a confinement transition, which gaps out all the fermionic partons and removes the $U(1)'$ photon from the low-energy spectrum. However, this nonperturbative nature of deconfined to confined phase transition of QED$_4'$ cannot be captured easily by perturbative renormalization group or Feynman diagram analysis.

— If the deconfined dark gauge fields $U(1)'$ in the critical region (I)' remains deconfined in the region (I), then overall there is no phase transition. The critical region (I)' and (I) are smoothly connected as the same critical region.

The critical region (0) is further broken down to the different symmetry-breaking orders from (I)' to (VIII)' shown in (4.20) with totally 5 refined phases. Hence there are more refined versions of phases transitions than what we had discussed above.

Overall, we need to enumerate all the possible phase transitions between these regions: the $U(1)_{\text{gauge}}^{\text{dark}}$-confined regions (I)'=(V)', (II)', (III)'=(VII)', (IV)'=(VIII)', and (VI)', and the $U(1)_{\text{gauge}}^{\text{dark}}$-deconfined critical regions (I)'=(V)', (II)', (III)'=(VII)', (IV)'=(VIII)', and (VI)'. We summarize the nature of these phase transitions in Table 2.

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Ref. [83] studies the deconfined to confined phase transition of QED$_4'$. Ref. [83] suggests that its nature is a Berezinskii-Kosterlitz-Thouless (BKT) phase transition, as an infinite order continuous phase transition. 

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| Properties of Quantum Phase Transition | Internal Global Symmetry |
|---------------------------------------|---------------------------|
| Order of transition | Landau-Ginzburg order parameter ($O_{LG}$) | Critical theory | Anomaly Sym. Preserved | Fermionic parton | Deconfined U(1)$_{gauge}$ | Beyond LGW |
| (I)-(II) so(10)-GG | Cont. Condense ($\Phi_{45}^{1st}$) \( \neq 0 \) | WF | SP to SB | gapped (confined) | No | Yes (anom.) |
| (I)-(VI) so(10)-flipped | Cont. Condense ($\Phi_{45}^{1st}$) \( \neq 0 \) | WF | SP to SB | gapped (confined) | No | Yes (anom.) |
| (I)-(IV) so(10)-PS | Cont. Condense ($\Phi_{54}^{1st}$) \( \neq 0 \) | WF | SP to SB | gapped (confined) | No | Yes (anom.) |
| (III)-(VI) flipped-SM | Cont. Condense ($\Phi_{54}^{1st}$) \( \neq 0 \) | WF | SB to SB | gapped (confined) | No | No |
| (IV)-(III) PS-SM | Cont. Condense ($\Phi_{45}^{1st}$) \( \neq 0 \) | WF | SB to SB | gapped (confined) | No | No |
| (I)-(III) so(10)-SM | Cont. Condense ($\Phi_{45}^{1st}$), ($\Phi_{54}^{1st}$) \( \neq 0 \) | WF | SP to SB | gapped (confined) | No | Yes (anom.) |
| (II)-(IV) flipped-PS | Cont. Swap ($\Phi_{45}^{1st}$) \( \neq 0 \) and ($\Phi_{54}^{1st}$) \( \neq 0 \) | WF | SB to SB | gapped (confined) | No | No |
| (II)-(VI) GG-flipped | 1st Swap ($\Phi_{45}^{1st}$) \( \neq 0 \) and ($\Phi_{54}^{1st}$) \( \neq 0 \) | No | SB to SB | gapped (confined) | No | No |
| (I')-(II') so(10)'-GG' | Cont. Condense ($\Phi_{45}^{1st}$) \( \neq 0 \) | QED$'_4$-GNY | SP to SB | gapless (deconfined) | Yes | Yes (an.decf.) |
| (I')-(VI') | Cont. Condense ($\Phi_{54}^{1st}$) \( \neq 0 \) | QED$'_4$-GNY | SP to SB | gapless (deconfined) | Yes | Yes (an.decf.) |
| (I')-(IV') so(10)'-PS' | Cont. Condense ($\Phi_{54}^{1st}$) \( \neq 0 \) | QED$'_4$-GNY | SP to SB | gapless (deconfined) | Yes | Yes (an.decf.) |
| (II')-(III') flipped-PS' | Cont. Condense ($\Phi_{54}^{1st}$) \( \neq 0 \) | QED$'_4$-GNY | SP to SB | gapless (deconfined) | Yes | Yes (an.decf.) |
| (IV')-(III') PS'-SM' | Cont. Condense ($\Phi_{45}^{1st}$) \( \neq 0 \) | QED$'_4$-GNY | SP to SB | gapless (deconfined) | Yes | Yes (an.decf.) |
| (I')-(III') so(10)'-SM' | Cont. Condense ($\Phi_{45}^{1st}$), ($\Phi_{54}^{1st}$) \( \neq 0 \) | QED$'_4$-GNY | SP to SB | gapless (deconfined) | Yes | Yes (an.decf.) |
| (II')-(IV') flipped-P'S | Cont. Swap ($\Phi_{45}^{1st}$) \( \neq 0 \) and ($\Phi_{54}^{1st}$) \( \neq 0 \) | QED$'_4$-GNY | SP to SB | gapless (deconfined) | Yes | Yes (an.decf.) |
| (II')-(VI') flipped-Gg' | 1st Swap ($\Phi_{45}^{1st}$) \( \neq 0 \) and ($\Phi_{54}^{1st}$) \( \neq 0 \) | No | SB to SB | gapped (Higgs SSB) | Yes | No |
| (I')-(I) so(10)'-so(10) | Cont. No | deconfined-confined | SP to SP | gapless (deconfined) | Yes | Yes (decf.) |
| (II')-(II) GG'-GG | Cont. No | deconfined-confined | SB to SB | gapless (deconfined) | Yes | Yes (decf.) |
| (III')-(III) SM'-SM | Cont. No | deconfined-confined | SB to SB | gapless (deconfined) | Yes | Yes (decf.) |
| (IV')-(IV) PS'-PS | Cont. No | deconfined-confined | SB to SB | gapless (deconfined) | Yes | Yes (decf.) |
| (VI')-(VI) flipped-flipped | Cont. No | deconfined-confined | SB to SB | gapless (deconfined) | Yes | Yes (decf.) |

Table 2: Here we summarize the properties of quantum phase transition in the limit when the internal symmetries is treated as ungauged global symmetries in Sec. 4.2.1. We abbreviate: Wilson-Fisher (WF), the QED$'_4$ with Gross-Neveu-Yukawa (QED$'_4$-GNY), Symmetry Preserved (SP), Symmetry Breaking (SB), and Landau-Ginzburg-Wilson (LGW). For beyond LGW, (anom.) means due to anomaly, (decf.) means due to deconfined-confined phase transition. (an.decf.) means due to both anomaly and deconfined-confined effects.
4.2.2 Internal symmetries are dynamically gauged

When the internal symmetries are treated as global symmetries in Sec. 4.2.1, the $U(1)^{\text{dark}}$ gauge is the only dynamical gauge sector, which becomes deconfined only emergent near the quantum critical region (the white region (0) in Fig. 12, in Fig. 13 (b), and in Fig. 14 and Fig. 15). The emergent deconfined $U(1)^{\text{dark}}$ gauge field near the quantum critical region is exactly the reason why we name the Gauged Enhanced Quantum Criticality beyond the Standard Model in our prior work [1].

When internal symmetries are dynamically gauged as they are in our quantum vacuum, then internal symmetry groups become gauge groups. We have additional gauge sectors such as $\text{Spin}(10)$, $U(5)^{\hat{q}=2}$, $\text{SU}(4) \times U(1)^{X_1}$, $\text{SU}(3) \times U(1)^{X_2}$, etc. We can ask the dynamical fates of these gauge theories: confined or deconfined? When the number of fermions are comparably small as in our quantum vacuum, the RG beta function computation shows the asymptotic freedom at UV and the confinement at IR for a non-abelian gauge theory. Other abelian gauge sectors, such as $U(1)^{EM}$, can stay deconfined. We summarize their dynamics in Table 3.

| Internal Symmetry Gauged | confined sectors | deconfined sectors |
|--------------------------|------------------|-------------------|
| (I)=(V) | $\text{so}(10)$ | $U(1)^{X_1}$ |
| (II) | $\text{su}(5)$ | $U(1)^{X_1}$ |
| (III)=(VII) | $\text{su}(3) \times \text{su}(2)$ | $U(1)^{EM}$ |
| (IV)=(VIII) | $\text{su}(4) \times \text{su}(2)^L \times \text{su}(2)^R$ | $U(1)^{X_1} \times U(1)^{dark}$ |
| (VI) | $\text{su}(5)$ | $U(1)^{EM}$ |
| (I)'=(V)' | $\text{so}(10)$ | $U(1)^{dark}$ |
| (II)' | $\text{su}(5)$ | $U(1)^{dark}$ |
| (III')=(VII)' | $\text{su}(3) \times \text{su}(2)$ | $U(1)^{EM} \times U(1)^{dark}$ |
| (IV')=(VIII)' | $\text{su}(4) \times \text{su}(2)^L \times \text{su}(2)^R$ | $U(1)^{dark}$ |
| (VI)' | $\text{su}(5)$ | $U(1)^{dark}$ |

Table 3: When internal symmetries are dynamically gauged as in our quantum vacuum, we enlist the confined gauge groups and deconfined gauge groups for each of the phases in Fig. 14 and Fig. 15. For simplicity, we only write down the Lie algebra in the lower case. The full Lie group in the capital case can be read from Fig. 11. The $U(1)^{dark}$ only emerges and deconfines in the quantum critical region (0), including (I)' to (VIII)'. The $U(1)^{dark}$ disappears and confines in the (I) to (VIII).

When internal symmetries are dynamically gauged, the phase transitions described in Table 2 also change or upgrade. For example, the Wilson-Fisher transitions of scalar fields become the Anderson-Higgs transitions of scalar fields interacting with gauge fields. Some of the QED' transitions also need to take into account of the nonabelian gauge fields from Spin(10) or its subgroups, which lead to various QCD' transitions.

Moreover, once the ordinary internal symmetry (i.e., the 0-symmetry) is dynamically gauged, the outcome gauge theory can have extended objects (e.g., Wilson or 't Hooft 1d lines) which are the charged objects of the generalized global symmetries [39] (i.e., here the 1-symmetry). We systematically explore these generalized global symmetries of SM and GUT in the next Section 5.
5 Higher Symmetries of Standard Models and Grand Unifications

Here we point out that once the internal symmetry of SM and GUT \((G_{SM_q=1,2,3,6}, G_{GG}, G_{PS_{d'=1,2}}, G_{so(10)}, \ldots)\) are dynamically gauged (as they are in the dynamical gauge theories), there are dynamical Wilson or 't Hooft lines as charged objects charged under the generalized global symmetries \([39]\). Pioneer works \([84,85]\) have pointed out the Wilson line spectrum differences between different versions of \(G_{SM_q}\) for \(q = 1, 2, 3, 6\). Other prior works also study the higher symmetries for these \(G_{SM_q}\) \([86–88]\). In comparison, our present work will include not only the higher symmetries of SM\(_q\) in \([86–88]\), but also other pertinent GUT models.

Recall \([39]\), when the **charged objects** are 1d line operators (say along a closed curve \(\gamma^1\)), the **charge operators** (also known as the **symmetry generators** or **symmetry defects**) are codimension-2 topological operators (as 2d surface operators say on a closed surface \(\Sigma^2\), in a 4d spacetime). As an example, when the gauge group is abelian, the path integral expectation value of the link configuration between the 1d line and 2d surface operators with a linking number \(\text{Lk}(\Sigma^2, \gamma^1)\) evaluated on a closed 4-manifold \(M^4\) is schematically given by:

\[
\langle \exp(i\theta \oint_{\Sigma^2} B_{\text{charge}}) \cdot \exp(\int q \oint_{\gamma^1} A_{\text{charged}}) \rangle = e^{i q \theta \text{Lk}(\Sigma^2, \gamma^1)} \cdot \langle \exp(\int q \oint_{\gamma^1} A_{\text{charged}}) \rangle |_{M^4}.
\]

The expectation value is topological independent from the continuous deformation of the codimension-2 topological operator of \(B\), as long as it does not cross the charged object of \(A\). Hence this explains the meaning of the name **topological operator**: the system and the topological operator of \(B\) do not have to be gapped, but its correlator is the same independent from the topological deformation. However, we need to include \(\langle \exp(\int q \oint_{\gamma^1} A_{\text{charged}}) \rangle \) on the right hand side whenever this operator \(A\) is non-topological. If \(A\) describes the 1d Wilson line (or denoted as 1-line), the topological 2d surface operator of \(B\) is often called the topological Gukov-Witten 2-surface operator \([89,90]\). See further in-depth discussions on the topological operators in \([48]\).

This expression is also the analogous Ward identity for the **generalized 1-form global symmetry**, or simply denoted as the **1-symmetry**. If we normalize the expectation value properly, the proportionality \((\propto)\) becomes the equality \((=)\) to the statistical Berry phase \(e^{i q \theta \text{Lk}(\Sigma^2, \gamma^1)}\). The abelian phase \(e^{i q \theta}\) means that we are focusing on the abelian 1-symmetry:

- When it is an abelian \(U(1)\) 1-symmetry denoted as \(U(1)_{[1]}\), then \(q \in \mathbb{Z}\) and \(\theta \in [0, 2\pi)\).
- When it is an abelian \(Z_N\) 1-symmetry denoted as \(Z_{N,[1]}\), then \(q \in \mathbb{Z}_N\) and \(\theta = \frac{2\pi k}{\sqrt{N}}\) with \(k \in \mathbb{Z}_N\).

For a gauge theory of gauge group \(G_g\), the electric 1-symmetry is associated with the unbroken center subgroup \(Z(G_g)\), the magnetic 1-symmetry is associated with the unbroken Pontryagin dual group of the first homotopy group: \(\pi_1(G_g)^\vee \equiv \text{Hom}(\pi_1(G_g), U(1))\). Here are some familiar examples (results summarized in Table 4):

1. For a 4d pure \(U(1)\) gauge theory, we have the electric 1-symmetry \(U(1)_{[1]}^e\) and the magnetic 1-symmetry \(U(1)_{[1]}^m\), whose 1-symmetry measurements are characterized by the following two expectation values:

\[
\begin{align*}
\langle \exp(i \theta_e \oint_{\Sigma^2} \star dA_{\text{charge}-U(1)_{[1]}^e}) \cdot \exp(\int q_e \oint_{\gamma^1} A_{\text{charged}-U(1)_{[1]}^e}) \rangle &= e^{i q_e \theta_e \text{Lk}(\Sigma^2, \gamma^1)} \cdot \langle \exp(\int q_e \oint_{\gamma^1} A_{\text{charged}}) \rangle |_{M^4}.
\end{align*}
\]

\[
\begin{align*}
\langle \exp(i \theta_m \oint_{\Sigma^2} dA_{\text{charge}-U(1)_{[1]}^m}) \cdot \exp(\int q_m \oint_{\gamma^1} V_{\text{charged}-U(1)_{[1]}^m}) \rangle &= e^{i q_m \theta_m \text{Lk}(\Sigma^2, \gamma^1)} \cdot \langle \exp(\int q_m \oint_{\gamma^1} V_{\text{charged}}) \rangle |_{M^4}.
\end{align*}
\]

The Wilson line of a 1-form gauge field \(A\) is the \(U(1)_{[1]}^e\) electric charged object, and 't Hooft line’s of a dual 1-form gauge field \(V\) is the \(U(1)_{[1]}^m\) magnetic charged object; they are related by the Hodge dual \(\star\) as
2. For a 4d pure SU(N) gauge theory, we have the electric 1-symmetry \( Z_{N,[1]}^e \) characterized by

\[
\langle \exp\left(\frac{2\pi i}{N} \int_{\Sigma^2} A \right) \rangle = \exp\left(\frac{i2\pi}{N} \text{Lk}(\Sigma^2, \gamma^1) \right) \cdot \langle \exp\left(\frac{2\pi i}{N} \int_{\gamma^1} a \right) \rangle,
\]

where gauge field \( a \) is the Lie algebra \( \text{su}(N) \) valued. The \( \exp(\frac{2\pi i}{N} a) \) specifies a SU(N) group element where \( P \) is the path ordering. \( Tr \) is the trace in the fundamental representation for \( R \). The \( A \in H^2(M^4, \mathbb{Z}_N) \) as a \( \mathbb{Z}_N \)-cohomology class, tightly related to the generalized second Stiefel-Whitney class \( w_2(V_{PSU(N)}) \in H^2(M, \mathbb{Z}_N) \), as the obstruction of promoting the PSU(N) bundle to SU(N) bundle. This becomes obvious when we promote the SU(N) gauge theory to a U(N) gauge theory with additional constraints, here and below following [39]. The U(N) gauge theory also has the benefits to go to the PSU(N) gauge theory. In the U(N) gauge theory, we introduce this constraint to the path integral,

\[
\int \text{[D}A\text{]} \ldots \exp\left(\frac{i2\pi}{N} \int_{M^4} A \right)
\]

with the gauge bundle constraint \( c_1 = w_2(V_{PSU(N)}) = B \) mod \( N \) where the first Chern class \( c_1 = c_1(V_{U(N)}) \in \mathbb{Z} \) is from the U(1) part of U(N). By staring at the two expressions in (5.2) and (5.3), it becomes also clear that an open boundary of the magnetic 2-surface \( w_2(V_{PSU(N)}) = B \) gives rise to 1d object closely related to the improperly quantized electric Wilson 1-line of \( a \). Vice versa, an open boundary of the electric 2-surface \( \Lambda \) gives rise to 1d object closely related to the improperly quantized magnetic ’t Hooft 1-line of PSU(N) gauge theory.

3. Thus, for a 4d pure PSU(N) gauge theory, we have the magnetic 1-symmetry \( Z_{N,[1]}^m \) characterized by the magnetic 2-surface operator \( \exp\left(\frac{2\pi i}{N} \int_{\Sigma^2} B \right) \) linking with the magnetic ’t Hooft 1-line of PSU(N) gauge theory.

4. For a 4d pure U(N) gauge theory (or the refined U(N)\(_\hat{q}\) gauge theory discussed in Sec. 2), the electric 1-symmetry is given by the center \( Z(U(N)) = U(1) \), while the magnetic 1-symmetry is given by the Pontryagin dual group of the first homotopy group \( \pi_1(U(N))' = \text{Hom}(\pi_1(U(N)), U(1)) = \text{Hom}(\mathbb{Z}, U(1)) = U(1) \). This means that:

- 4d pure U(N) gauge theory without matter kinematically has \( U(1)_{[1]}^e \) and \( U(1)_{[1]}^m \) 1-symmetries.
- 4d pure U(N) (say the refined U(N)\(_\hat{q}=1\)) gauge theory with the gauge-charged matter in the fundamental of SU(N) and in the unit charge of U(1) written as the \( (N, 1) \) representation, kinematically reduces the electric 1-symmetry to none — because the charged Wilson line of \( U(1)_{[1]}^e \) of the earlier pure U(N) gauge theory now becomes breakable with two open ends attached the gauge-charged matter \( (N, 1) \) and \( (\bar{N}, -1) \), which nullify \( U(1)_{[1]}^m \) to zero. But the magnetic 1-symmetry \( U(1)_{[1]}^m \) maintains.

Now we are ready to provide some systematic applications to SM or GUT (results summarized in Table 4):\(^{21}\)

\(^{21}\)Part of these results presented here follow Section 1.5 of [86] and some unpublished notes of the first author (J. Wang) with Miguel Montero [87]. JW thanks Miguel Montero on the related discussions and collaborations.
Higher symmetries of 4d pure gauge theories

| QFT       | $Z(G_g)$ | $\pi_1(G_g)$, $\pi_1(G_g)^\vee$ | 1-form $e$ sym $G_{e}[1]$ | 1-form $m$ sym $G_{m}[1]$ |
|-----------|----------|--------------------------------|---------------------------|---------------------------|
| U(1)      | U(1)     | Z, U(1)                        | U(1)$_e^{[1]}$            | U(1)$_m^{[1]}$            |
| SU(N)     | $Z_N$    | 0, 0                           | $Z_N^{e[1]}$              | 0                         |
| PSU(N)    | 0        | $Z_N, Z_N$                     | $Z_N^{m[1]}$              |                           |
| U(N)      | U(1)     | Z, U(1)                        | U(1)$_e^{[1]}$            | U(1)$_m^{[1]}$            |

Higher symmetries of 4d SMs or GUTs with SM matters

| QFT       | $Z(G_g)$ | $\pi_1(G_g)$, $\pi_1(G_g)^\vee$ | 1-form $e$ sym $G_{e}[1]$ | 1-form $m$ sym $G_{m}[1]$ |
|-----------|----------|--------------------------------|---------------------------|---------------------------|
| $G_{SM_q}$ | $SU(3)\times SU(2)\times U(1)_{Y}$ | $Z_{6/q} \times U(1)$ | Z, U(1) | $Z_{6/q}[1]$ | $U(1)$$_m^{[1]}$ |
| $G_{SM_6}$ | $SU(3)\times SU(2)\times U(1)_{Y}$ | $Z_6$ | U(1) | Z, U(1) | 0 | $U(1)$$_m^{[1]}$ |
| SU(5) (GG or flipped) | $Z_5$ | U(1) | Z, U(1) | 0 | 0 |
| U(5)$_q$ (GG or flipped) | U(1) | Z, U(1) | U(1)$_m^{[1]}$ | 0 |
| $G_{PS_{q'}}$ | $SU(4)\times SU(2)\times SU(2)$_R | $Z_q$ | $Z_q \times Z_2, Z_q'$ | $Z_{2,q'}^{[1]}$ | $Z_{q'}^{m[1]}$ |
| $G_{PS_2}$ | $SU(4)\times SU(2)\times SU(2)$_R | $Z_2$ | $Z_2 \times Z_2, Z_2, Z_2$ | 0 | $Z_{2}^{m[1]}$ |
| Spin(10) | $Z_4$ | 0, 0 | 0 | 0 |

Table 4: For an internal symmetry $G_{internal} = G_g$ as a gauge group, we list down its center subgroup $Z(G_g)$, its first homotopy group $\pi_1(G_g)$ and its Pontryagin dual $\pi_1(G_g)^\vee$. We also list down the 1-form $e$ sym $G_{e}[1]$ and 1-form $m$ sym $G_{m}[1]$ (without matter for the pure gauge theory, and with SM matter for the SMs and GUTs). For $SM_q$, there is a choice of $q = 1, 2, 3, 6$. Here the SM matters are the 15 of 16 left-handed ($L$) Weyl fermions: $(\mathbf{3, 1})_{2,L} \oplus (\mathbf{1, 2})_{-3,L} \oplus (\mathbf{3, 2})_{1,L} \oplus (\mathbf{3, 1})_{-4,L} \oplus (\mathbf{1, 1})_{6,L} \oplus (\mathbf{1, 1})_{0,L}$ of $G_{SM_q} \equiv \frac{SU(3)\times SU(2)\times U(1)}{Z_6}$; or $\mathbf{5} \oplus 10 \oplus 1$ of SU(5); or the $(\mathbf{4, 2, 1}) \oplus (\mathbf{4, 1, 2})$ of the $G_{PS_{q'}}$ with $q' = 1, 2$; or the $\mathbf{16}$ of the Spin(10).

1. Higher Symmetry for $G_{SM_q}$ gauge theory with $q = 1, 2, 3, 6$:
   The electric 1-symmetry is related to the center $Z(G_{SM_q}) = Z_{6/q} \times U(1)$, while the magnetic 1-symmetry is related to the Pontryagin dual group of homotopy group $\pi_1(G_{SM_q})^\vee = U(1)$. This means:
   - without the SM fermionic matter of quarks and leptons, we have the corresponding $Z_{6/q[1]} \times U(1)_{e[1]}$ and $U(1)$_m$[1]$ 1-symmetries.
   - with the SM fermionic matter for $G_{SM_q}$, we are left with no electric 1-symmetry, because the gauged charge matter $(\mathbf{3, 2})_{1,L}$ explicitly can open up thus break the minimal charged object Wilson line of $U(1)$_e$[1]$ with two open ends. But the magnetic 1-symmetry $U(1)$_m$[1]$ maintains.
   - with the SM fermionic matter for $G_{SM_q}$, we are left with electric 1-symmetry $Z_{6/q[1]}$ and magnetic 1-symmetry $U(1)$_m$[1]$.

2. Higher Symmetry for SU(5) of the Georgi-Glashow (GG) or the Barr’s flipped $su(5)$ models:
   - without the SM fermionic matter, the center $Z(SU(5)) = Z_5$ gives rise to the electric 1-symmetry $Z_5^{e[1]}$, while $\pi_1(SU(5)) = 0$ gives no magnetic 1-symmetry.
   - with SM fermionic matter such as $\mathbf{5}$ of SU(5) breaks the electric 1-symmetry to none.

3. Higher Symmetry for U(5)$_q=2$ of the GG or the flipped $su(5)$ models:
   - without the SM fermionic matter, the center $Z(U(5)) = U(1)$ gives rise to the electric 1-symmetry $U(1)^{e[1]}$, while $\pi_1(U(5))^\vee = U(1)$ gives the magnetic 1-symmetry $U(1)^{m[1]}$.
   - with SM fermionic matter such as $\mathbf{5} - 3$ of U(5)$_q=2$ breaks the electric 1-symmetry to none. But the magnetic 1-symmetry $U(1)^{m[1]}$ maintains.

4. Higher Symmetry for Pati-Salam $G_{PS_{q'}}$ gauge theory with $q' = 1, 2$:
• without the SM fermionic matter, the center $Z(G_{PS, q'}) = Z_4 \times Z_{q'}$ gives rise to the electric 1-symmetry $(Z_4 \times Z_{q'} (Z_2 \times Z_2))^{q'}$, while the $\pi_1(G_{PS, q'}) = Z_{q'}$ gives the magnetic 1-symmetry $Z_{q'}^{m}$.  
• with the matter $(4, 2, 1) \oplus (\bar{4}, 1, 2)$, the electric 1-symmetry becomes $Z_4^{m} / q'$, while the magnetic 1-symmetry remains.

5. Higher Symmetry for the so(10) GUT and Spin(10) gauge group:  
• without any SM matter (no 16 of Spin(10)), the center $Z(Spin(10)) = Z_4$ gives rise to the electric 1-symmetry $Z_4^{m}$, while $\pi_1(Spin(10))^{\vee} = 0$ gives no magnetic 1-symmetry.  
• with 16 of Spin(10), there are no electric nor magnetic 1-symmetries left.

6 Categorical Symmetry and Its Retraction

Table 4 summarizes various higher symmetries of SMs and GUTs. In particular, for those embeddable into the Spin(10) group (e.g., only $G_{SM}$ with $q = 6$ and $G_{PS, q'}$ with $q' = 2$, listed in Fig. 10 and Fig. 11), we only have 1-form symmetries for them:

- $G_{SM, 6}$: $U(1)^{m}$
- GG su(5) GUT with SU(5): none;
- GG or flipped $u(5)$ GUT with $U(5)_{q=2}$: $U(1)^{m}$
- PS model with $G_{PS, 2}$: $Z_{2}^{m}$
- so(10) or modified so(10) GUT with Spin(10): none. (6.1)

Some curious facts are:

1. All electric 1-symmetries are broken by gauged charged fermionic matter. We are only left with either magnetic 1-symmetries or none in (6.1).

2. Regardless of which lower energy GUTs that we start with, when we approach to the (modified) so(10) GUT as a mother unified EFT at the deeper UV, all 1-symmetries are gone.

In Sec. 5, we had investigated the higher symmetries that are invertible global symmetries. By invertible global symmetries, we mean that the fusion algebras of symmetry generators (i.e., charge operators) follow the group law. For any symmetry generator (say $U_1, U_2, \ldots$) of an invertible global symmetry, its fusion algebra is a binary operation (say “×” for the fusion) which must obey:

1. the closure,
2. the associativity $U_1 \times (U_2 \times U_3) = (U_1 \times U_2) \times U_3$,
3. the identity operator 1 existence, so $U \times 1 = 1 \times U = U$,
4. the inverse operator operator $U^{-1}$ existence so that $U \times U^{-1} = 1$.

In this Section 6, we investigate any potential non-invertible global symmetries in the SM or GUT models. The non-invertible global symmetries correspond to the fusion algebras of symmetry generators (i.e., charge operators) do not follow the group law. In particular, we will search for the existence of a symmetry generator $U$ such that the fusion rule of $U$ with any operator $U'$ can never produce only the identity operator. Namely

\begin{equation}
\text{generally } U \times U' = \sum U_j \\
\text{or at most } U \times U' = 1 + \ldots
\end{equation}

(6.2)
where the \( \ldots \) may be nonzero, including other operators (e.g., \( U'' \), etc.) The formal form of fusion algebra sometimes uses “\( \otimes \)” for the fusion, and uses “\( \oplus \)” for the splitting on the right hand side. Here we simply uses the product “\( \times \)” and the sum “\( + \)” because these relations in (6.2) hold in the correlation function computation, both in the QFT path integral formulation or in the quantum matter lattice regularization formulation. Namely, we indeed have this relation holds in the expectation value form

\[
\langle U \times U' \rangle = \langle \sum_j U_j \rangle. \tag{6.3}
\]

Non-invertible global symmetries have appeared long ago in the 2-dimensional CFTs [40–45]. They also appeared recently under the name of algebraic higher symmetry or categorical symmetry [53, 54], and fusion category symmetry [51, 55].\(^\text{22}\) From now on, follow the recent development [46–50, 52], we shall call these types of symmetries as non-invertible global symmetries or categorical symmetries interchangeably.

### 6.1 Potential Non-Invertible Categorical Symmetry induced by \( \mathbb{Z}_2^{\text{flip}} \)

Given the SM’s \( SU(3)_c \times SU(2)_L \), there are actually two ways to embed it inside an \( SU(5) \). The first one is the Georgi-Glashow (GG) \( SU(5) \) that we denoted \( SU(5)^\text{1st} \); the second one is the Barr’s flipped \( SU(5) \) that we denoted \( SU(5)^\text{2nd} \). We can define a \( U(5)^\text{1st} \) \( \widetilde{\pi} \equiv SU(5)^\text{1st} \times U(1)_{\hat{q}=2} = SU(5)^\text{1st} \times U(1)_{X, \hat{q}=2} = SU(5)^\text{1st} \times U(1)_{X, \hat{q}=2} \).

We denote the GG’s \( U(5) \) as the first kind \( U(5)^\text{1st} \) \( \hat{q} = 2 \) embedded inside the Spin(10), where

\[
U(5)^\text{1st} \equiv SU(5)^\text{1st} \times U(1)_{\hat{q}=2} \equiv SU(5)^\text{1st} \times U(1)_{X, \hat{q}=2} = \frac{SU(5)^\text{1st} \times U(1)_{X, \hat{q}=2}}{\mathbb{Z}_5}. \tag{6.3}
\]

We denote \( U(1)_{X} \) also the \( U(1)_{X_1} \) which is also generated by the 25th Lie algebra generator \( T_{25} \) of this \( U(5)^\text{1st} \). The \( \hat{q} = 2 \) specifies the identification between the \( SU(5) \)’s center and the normal subgroup of \( U(1) \) via our definition in (2.1).

We denote the Barr’s flipped \( U(5) \) as the second kind \( U(5)^\text{2nd} \) \( \hat{q} = 2 \) embedded inside the Spin(10), where

\[
U(5)^\text{2nd} \equiv SU(5)^\text{1st} \times U(1)_{\hat{q}=2} \equiv SU(5)^\text{1st} \times U(1)_{X, \hat{q}=2} = \frac{SU(5)^\text{1st} \times U(1)_{X_2, \hat{q}=2}}{\mathbb{Z}_5}. \tag{6.3}
\]

We denote \( U(1)_{X} \) also the \( U(1)_{X_2} \) which is also generated by the 25th Lie algebra generator \( T_{25} \) of this \( U(5)^\text{2nd} \).

There is a magnetic 1-symmetry \( U(1)^{m_{X_1}} \) from the GG’s \( \pi_1(U(5)^\text{1st} \hat{q}=2) = \pi_1(U(1)_{X_1}) = U(1) \). There is a magnetic 1-symmetry \( U(1)^{m_{X_2}} \) from the Barr’s flipped model’s \( \pi_1(U(5)^\text{2nd} \hat{q}=2) = \pi_1(U(1)_{X_2}) = U(1) \). There is a \( \mathbb{Z}_2^{\text{flip}} \) transformation, swapping between \( U(5)^\text{1st} \) and \( U(5)^\text{2nd} \). Naively if we study a gauge theory including the union of the gauge group \( U(5)^\text{1st} \) and \( U(5)^\text{2nd} \) denoted as \( U(5)^\text{1st} \cup U(5)^\text{2nd} \) together with the outer automorphism exchanging these two \( U(5)s \), as \( U(5)^\text{1st} \cup U(5)^\text{2nd} \times \mathbb{Z}_2^{\text{flip}} \), then we expect to find a potential categorical symmetry for this 4d gauge theory. However, such a potential categorical symmetry is not realized, for various reasons that we explore in this section.

\(^{22}\)Note that Wen et al’s usage of categorical symmetry [91] is different from other research groups’ usage of categorical symmetry. Instead, Wen et al’s usage of algebraic higher symmetry [53] is the same as other research groups’ usage of categorical symmetry.
6.2 Categorical symmetry retraction from two $U(5)_{q=2}$ to Spin(10)

1. Need Spin(10) to contain both of the two $U(5)_{q=2}$: Dynamically gauging the union $(U(5)^{1\text{st}})_{q=2} \cup (U(5)^{2\text{nd}})_{q=2}$ already inevitably brings us to the full gauge group Spin(10) (checked in Appendix C and (C.6)). Thus, needless to say whether we gauge the $Z_2^{\text{flip}}$ or not, when the two $U(5)$s are gauged, we are already at the full Spin(10). For a pure Spin(10) gauge theory without matter fields, there is only a 1-form electric global symmetry $Z_{4,[1]}$, and no categorical symmetry. For the Spin(10) gauge theory with fermions in the 16, there are neither higher symmetries (according to Table 4) nor categorical symmetries.

2. GUT-Higgs scale eliminates some electric 1-symmetries: We hope to rationalize and characterize why 1-symmetries are gone at the deeper UV in the Spin(10) gauge group, but there seems to have 1-symmetries in the $U(5)^{1\text{st}}_{q=2}$ and $U(5)^{2\text{nd}}_{q=2}$ gauge theories. The closer look at the Higgs mechanism from the $so(10)$ GUT model with Spin(10) gauge group to the $U(5)$ gauge theories, we already add GUT-Higgs with the rep 45 of Spin(10), so we can Higgs Spin(10) down to $U(5)$. But this rep 45 has the branching rule from Spin(10) to $SU(5)$ as $45 \sim 1 \oplus 10 \oplus \overline{10} \oplus 24$ or to the $U(5)_{q=2}$ $45 \sim 1_0 \oplus 10_4 \oplus \overline{10}_4 \oplus 24_0$. These branching rules tell us that the electric 1-symmetries, if any, are broken at the GUT-Higgs scale. Thus, any electric 1-symmetries (such as the $Z_{2,q,[1]}$ for the pure SU(5) gauge theory, or the $U(1)^{q}_f$ for the pure U(5) gauge theory) could be emergent at much lower energy at IR far below the GUT-Higgs scale.

How are the two magnetic 1-symmetries $U(1)^{mX}_1$ and $U(1)^{mX}_2$ disappear when we go to deeper energy from two $U(5)_{q=2}$ to the Spin(10)? After all the GUT-Higgs are in the rep of the electric sector not the magnetic sector, so the GUT-Higgs does not remove the magnetic 1-symmetries. In order to understand how magnetic 1-symmetries disappear or are removed, we study a toy model involving only the two crucial U(1) gauge sector: the $(U(1)_{X_1} \times z_{4,X} U(1)_{X_2})$ gauge theory.

6.3 Categorical symmetries in a toy model $[(U(1)_{X_1} \times z_{4,X} U(1)_{X_2}) \rtimes Z_2^{\text{flip}}]$ gauge theory

We introduce a toy model of $U(1)_{X_1} \times z_{4,X} U(1)_{X_2}$ gauge theory, in terms of two $U(1)$ sectors: $U(1)_{X_1} \equiv U(1)^{1\text{st}}_{X}$ and $U(1)_{X_2} \equiv U(1)^{2\text{nd}}_{X}$, while $U(1)_{Y_1} \equiv U(1)^{1\text{st}}_{Y}$ and $U(1)_{Y_2} \equiv U(1)^{2\text{nd}}_{Y_{24}}$, we have the following spacetime-internal structures with the spacetime Spin group:

$$
\text{Spin} \times Z_2^F (U(1)_{X_1} \times z_{4,X} U(1)_{X_2}) = \left\{
\begin{array}{l}
\text{Spin} \times Z_2^F (U(1)_{Y_1} \times z_5 U(1)_{X_1}) = (\text{Spin} \times Z_2^F U(1)_{Y_1}) \times z_5 U(1)_{X_1} \supset (\text{Spin} \times Z_2^F Z_{4,X_1}) \times z_5 U(1)_{Y_1},
\\
\text{Spin} \times Z_2^F (U(1)_{Y_2} \times z_5 U(1)_{X_2}) = (\text{Spin} \times Z_2^F U(1)_{Y_2}) \times z_5 U(1)_{Y_2} \supset (\text{Spin} \times Z_2^F Z_{4,X_2}) \times z_5 U(1)_{Y_2}.
\end{array}\right.
$$

(Dynamically gauge the internal symmetry group $(U(1)_{X_1} \times z_{4,X} U(1)_{X_2}) = (U(1)_{Y_1} \times z_5 U(1)_{X_1}) = (U(1)_{Y_2} \times z_5 U(1)_{X_2})$, we obtain their gauge theory.

There is also a $Z_2^{\text{flip}}$ as an outer automorphism of $(U(1)_{X_1} \times z_{4,X} U(1)_{X_2})$ exchanging the two $U(1)$ subgroups, which we can define on this spacetime-internal structures

$$
\text{Spin} \times Z_2^F ((U(1)_{X_1} \times z_{4,X} U(1)_{X_2}) \rtimes Z_2^{\text{flip}}).
$$

Several comments about their higher symmetries and categorical symmetries, and their potential obstructions:

1. Higher symmetries without gauge charged matter:
For a pure 4d $G_g = (U(1)_{X_1} \times Z_{4,X} \ U(1)_{X_2})$ gauge theory without gauge charged matter, we have the center
\[
Z(G_g) = (U(1)_{X_1} \times Z_{4,X} \ U(1)_{X_2})
\]

itself inducing the product of the two U(1) electric 1-symmetries, denoted as $U(1)^{e_{X_1}}_{[1]} \times Z_{4,X}^{e_{X_2}} U(1)^{e_{X_2}}_{[1]}$ modding out the shared common $Z_{4,X}$ 1-symmetry. We have
\[
\pi_1(G_g)^\vee = \text{Hom}(\pi_1(G_g), U(1)) = \text{Hom}(Z_{X_1} \times Z_{X_2}, U(1)) = U(1)_{X_1} \times U(1)_{X_2}
\]

inducing the magnetic 1-symmetry $U(1)^{m_{X_1}}_{[1]} \times U(1)^{m_{X_2}}_{[1]}$.

2. $\left((U(1)_{X_1} \times Z_{4,X} \ U(1)_{X_2}) \rtimes Z_2^{\text{flip}}\right)$ gauge theory:

If we further gauge the $Z_2^{\text{flip}}$ symmetry, we get a nonabelian gauge theory of a gauge group
\[
\left((U(1)_{X_1} \times Z_{4,X} \ U(1)_{X_2}) \rtimes Z_2^{\text{flip}}\right).
\]

If such a 4d gauge theory $\left((U(1)_{X_1} \times Z_{4,X} \ U(1)_{X_2}) \rtimes Z_2^{\text{flip}}\right)$ exists as part of the GUT or BSM physics, we have a potential categorical symmetry that we elaborate below in the Remark 5.

3. Electric higher symmetries:

For a 4d $G_g = (U(1)_{X_1} \times Z_{4,X} \ U(1)_{X_2})$ gauge theory with SM gauged electrically charged matter, the 1-form $e$ symmetry $U(1)^{e_{X_1}}_{[1]} \times Z_{4,X}^{e_{X_2}} U(1)^{e_{X_2}}_{[1]}$ is explicitly broken to only a subgroup $U(1)^e_{[1]}$. Furthermore, the remained $U(1)^e_{[1]}$ is gone when embedding $(U(1)_{X_1} \times Z_{4,X} \ U(1)_{X_2})$ into the U(5)$_{\tilde q=2}$ subgroup. Thus, we are not interested in pursuing the electric higher symmetries further since they are all broken at the GG and flipped $su(5)$ models with matter.

4. Magnetic higher symmetries:

For a 4d $G_g = (U(1)_{X_1} \times Z_{4,X} \ U(1)_{X_2})$ gauge theory with SM gauged electrically charged matter, but we are still left with two 1-form $m$ symmetries preserved: $U(1)^{m_{X_1}}_{[1]} \times U(1)^{m_{X_2}}_{[1]}$, together with the SO structure and the $Z_2^{\text{flip}}$ symmetry. The full spacetime-internal symmetry for this gauge theory requires at least the structure:
\[
\text{SO} \times \left((U(1)^{m_{X_1}}_{[1]} \times U(1)^{m_{X_2}}_{[1]} \rtimes Z_2^{\text{flip}}\right)
\]

where the $Z_2^{\text{flip}}$ belongs to an outer automorphism of $U(1)^{m_{X_1}}_{[1]} \times U(1)^{m_{X_2}}_{[1]}$ exchanging the two $U(1)_{[1]}$.

If we gauge $Z_2^{\text{flip}}$ to obtain the 4d gauge theory $\left((U(1)_{X_1} \times Z_{4,X} \ U(1)_{X_2}) \rtimes Z_2^{\text{flip}}\right)$, naively we only require the spacetime-internal symmetry structure:
\[
\text{SO} \times \left((U(1)^{m_{X_1}}_{[1]} \times U(1)^{m_{X_2}}_{[1]}\right).
\]

But there is a potential categorical symmetry that we elaborate below in the Remark 5.

5. Potential categorical symmetry in $\left((U(1)_{X_1} \times Z_{4,X} \ U(1)_{X_2}) \rtimes Z_2^{\text{flip}}\right)$ gauge theory:

Just like $O(2) = U(1) \times Z_2$ or $U(1) \times S_N$ gauge theories has categorical symmetries [46, 48], we can look at the potential categorical symmetry of the $\left((U(1)_{X_1} \times Z_{4,X} \ U(1)_{X_2}) \rtimes Z_2^{\text{flip}}\right)$ gauge theory. As mentioned

\[23\text{Follow the discussion in [48], a similar example is gauging the } Z_2 \text{ outer automorphism exchanging the two } E_8 \text{ in an } (E_8 \times E_8) \text{ gauge theory. The outcome is a disconnected compact Lie group gauge theory}
\]

\[(E_8 \times E_8) \rtimes Z_2\]

referred to as the gauge symmetry of the $(E_8 \times E_8)$ heterotic string theory [92]. But the center $Z(E_8) = 0$ and the $\pi_1(E_8)^\vee = 0$, so there are no higher nor categorical symmetries in this gauge theory.
since the electric higher symmetries are explicitly broken by gauged charged matter, we focus on the potential categorical symmetry involving magnetic higher symmetries. Each of the $U(1)_{X_1}$ and $U(1)_{X_2}$ has their own 1d ’t Hooft line operators and 2d magnetic topological surface operators:

$$T_{q_m j}^{X_j} \equiv \exp(i q_m \oint_{\gamma_j} V_{X_j}), \quad U_{\theta_m j}^{X_j} \equiv \exp(i \theta_m \oint_{\Sigma^2} \ast dV_{X_j}),$$

for $j = 1, 2$ along some closed 1-curve and some closed 2-surfaces $\Sigma^2$. They are labeled by magnetic charges $q_m$ and magnetic angles $\theta_m \in [0, 2\pi)$. But once the $Z^\text{flip}_2$ is dynamically gauged, these operators are not gauge invariant. Instead, the gauge invariant ’t Hooft line is \[24\]

$$T_{q_m}^{\text{flip}} \equiv T_{q_m}^{X_1} + T_{q_m}^{X_2} \equiv \exp(i q_m \oint_{\gamma_1} V_{X_1}) + \exp(i q_m \oint_{\gamma_2} V_{X_2}). \quad (6.6)$$

The gauge invariant topological magnetic surface operator is

$$U_{\theta_m}^{\text{flip}} \equiv U_{\theta_m}^{X_1} + U_{\theta_m}^{X_2} \equiv \exp(i \theta_m \oint_{\Sigma^2} \ast dV_{X_1}) + \exp(i \theta_m \oint_{\Sigma^2} \ast dV_{X_2}). \quad (6.7)$$

Under the $Z^\text{flip}_2$ gauged transformation, the operators labeled by $X_1$ and $X_2$ are swapped. The ’t Hooft lines in (6.6) are non-topological, in the sense that their deformations introduce propagations of dual photons. The fusions of these non-topological operators may introduce short-distance singularities and other non-universal contributions in the operator product expansion (OPE). Nevertheless, there is still a fusion rule following from the topological contribution

$$T_{q_m}^{\text{flip}} \times T_{q'_m}^{\text{flip}} = T_{q_m+q'_m}^{\text{flip}} + T_{q_m}^{X_1} \times T_{q'_m}^{X_2} + T_{q'_m}^{X_1} \times T_{q_m}^{X_2}. \quad (6.8)$$

The topological magnetic 2-surface operators are topological of quantum dimension 2, with the following fusion rule:

$$U_{\theta_m}^{\text{flip}} \times U_{\theta'_m}^{\text{flip}} = \exp(i (\theta_m + \theta'_m) \oint_{\Sigma^2} \ast dV_{X_1}) + \exp(i (\theta_m + \theta'_m) \oint_{\Sigma^2} \ast dV_{X_2})$$

$$+ \exp(i \theta_m \oint_{\Sigma^2} \ast dV_{X_1}) \exp(i \theta'_m \oint_{\Sigma^2} \ast dV_{X_2}) + \exp(i \theta'_m \oint_{\Sigma^2} \ast dV_{X_1}) \exp(i \theta_m \oint_{\Sigma^2} \ast dV_{X_2})$$

$$= U_{\theta_m+\theta'_m}^{\text{flip}} + (U_{\theta_m}^{X_1} \times U_{\theta'_m}^{X_2} + U_{\theta_m}^{X_2} \times U_{\theta'_m}^{X_1}). \quad (6.9)$$

The fusion rule splits, which indicates the magnetic 2-surface operators are non-invertible. Only the first operator $U_{\theta_m+\theta'_m}^{\text{flip}}$ of quantum-dimension 2 is the same type of magnetic 2-surface operator (6.7) that we start with. The remained term is a more general type of 2-surface operator

$$U_{\theta_m,\theta'_m}^{\text{flip}} \equiv U_{\theta_m,\theta'_m}^{\text{flip}} \equiv (U_{\theta_m}^{X_1} \times U_{\theta'_m}^{X_2} + U_{\theta'_m}^{X_1} \times U_{\theta_m}^{X_2}) \quad (6.10)$$

carrying the dependence of $(\theta_m, \theta'_m)$ that still gauge invariant under the $Z^\text{flip}_2$’s swapping $X_1 \leftrightarrow X_2$. The earlier topological surface operator in (6.7) is the special kind of the more general case:

$$U_{\theta_m}^{\text{flip}} \equiv U_{\theta_m,0}^{\text{flip}}, \quad U_{\theta'_m}^{\text{flip}} \equiv U_{0,\theta'_m}^{\text{flip}}, \quad U_{\text{flip}}^{\text{flip}} \equiv U_{\text{flip}}^{\text{flip}} \equiv U_{\text{flip}}^{\text{flip}}.$$
The fusion rule between two such general operators also splits
\[ U^{\text{flip}}_{\theta_m, \theta'_{m}} \times U^{\text{flip}}_{\theta_m, \theta'_{m}} = (U_{\theta_m+\theta_m} \times U_{\theta_m+\theta_m}) + (U_{\theta_m+\theta_m} \times U_{\theta_m+\theta_m}) + (U_{\theta_m+\theta_m} \times U_{\theta_m+\theta_m}) = U^{\text{flip}}_{\theta_m+\theta_m, \theta'_{m}} + U^{\text{flip}}_{\theta_m+\theta_m, \theta'_{m}} + U^{\text{flip}}_{\theta_m+\theta_m, \theta'_{m}} \] (6.11)

The global symmetry generated by (6.7) is thus non-invertible. It is a non-invertible global symmetry, or a categorical symmetry.

6. Categorical symmetry retraction: Where does the categorical symmetry (6.11) of the \([(U(1)_{X_1} \times \mathbb{Z}_{4,x} U(1)_{X_2}) \times \mathbb{Z}_2^{\text{flip}}]\) gauge theory go in the end, after embedding into the GUTs?

(a) Follow Sec. 6.2, the categorical symmetry is retracted when we embed \(U(1)_{X_1} \subset U(5)^{1\text{st}}\) and \(U(1)_{X_2} \subset U(5)^{2\text{nd}}\). Because the union of gauge group, either \("(U(5)^{1\text{st}} \cup U(5)^{2\text{nd}})"\) or \("(U(5)^{1\text{st}} \cup U(5)^{2\text{nd}}) \times \mathbb{Z}_2^{\text{flip}}\)\), require to gauge the full Spin(10) group with Weyl fermions in the 16, which has no higher symmetries nor category symmetries.

(b) Another possible explanation at IR of categorical symmetry breaking, is checking the possible mixed anomaly between the magnetic 1-form symmetries \((U(1)^{m_{X_1}} \times U(1)^{m_{X_2}})\) and the \(\mathbb{Z}_2^{\text{flip}}\)-symmetry. If there is any such mixed anomaly, when the \(\mathbb{Z}_2^{\text{flip}}\)-symmetry is dynamically gauged, then at least part of the magnetic 1-symmetry needs to be broken. If so, the categorical symmetry must also be broken.

For example, based on the calculation in [27, 86], the 5th bordism group gives:

\[ \Omega_5^{SO \times Z_2 \times BU(1)^{m_{X_1}} \times BU(1)^{m_{X_2}}} = \mathbb{Z}_2^5, \]

generated by \(w_2 w_3, \tau_5^{m_{X_1}} = Sq^2 \tau_3, \tau_5^{m_{X_2}} = Sq^2 \tau_3, a^5, aw_2^2\). (6.12)

\[ \Omega_5^{SO \times Z_2 \times BU(1)^{m_{X_1}} \times BU(1)^{m_{X_2}}} = \mathbb{Z}_2^{11}, \]

generated by \(w_2 w_3, a^5, a^2 w_1^2, aw_1^4, aw_2^2, a^2 \tau_3^{m_{X_1}}, a^2 \tau_3^{m_{X_2}}, w_1^2 \tau_3^{m_{X_1}}, w_1^2 \tau_3^{m_{X_2}}, \tau_5^{m_{X_1}}, \tau_5^{m_{X_2}}\). (6.13)

The particular cobordism classification of 4d anomalies that fits our theory is this [93]:

\[ \Omega_5^{SO \times Z_2 \times BU(1)^{m_{X_1}} \times BU(1)^{m_{X_2}}} = \mathbb{Z}_2^{11}. \] (6.14)

However, the twisted cobordism calculation is more difficult, Ref. [93] predicts that the \(\Omega_5^{SO \times Z_2 \times BU(1)^{m_{X_1}} \times BU(1)^{m_{X_2}}} \) is either the same as \(\mathbb{Z}_2^5\) in (6.12), or as \(\mathbb{Z}_2^4\) where the two generators \(\tau_5^{m_{X_1}} = Sq^2 \tau_3\) and \(\tau_5^{m_{X_2}} = Sq^2 \tau_3\) in \(\Omega_5^{SO \times Z_2 \times BU(1)^{m_{X_1}} \times BU(1)^{m_{X_2}}} \) reduce to a single one \(\tau_5^{m_{X_1}} + \tau_5^{m_{X_2}}\) in \(\Omega_5^{SO \times Z_2 \times BU(1)^{m_{X_1}} \times BU(1)^{m_{X_2}}} \). Thus we can use the data from (6.12) and (6.13) to deduce whether our \((U(1)^{m_{X_1}} \times U(1)^{m_{X_2}}) \times \mathbb{Z}_2^{\text{flip}}\) symmetry has any 't Hooft anomaly or not. Several comments are in order:

i. Here \(w_j \equiv w_j(TM)\) is the \(j\)-th Stiefel-Whitney (SW) characteristic class of spacetime tangent bundle \(TM\) of manifold \(M\).

ii. The \(\tilde{\tau}_3\) is the generator of \(H^3(B^2U(1)) = Z\) and \(\tau_3 = (\tilde{\tau}_3 \mod 2)\) is the generator of \(H^3(B^2U(1), \mathbb{Z}_2) = \mathbb{Z}_2\). The \(Sq^2\) is from the Steenrod square, here mapping the 3rd cohomology group to the 5th cohomology group.

iii. The upper labels \(m_{X_1}\) and \(m_{X_2}\) are for specifying magnetic 1-symmetries from either sector of \((U(1)^{m_{X_1}} \times U(1)^{m_{X_2}})\).
iv. The $a$ is the generator of $H^1(B\mathbb{Z}_2^{\text{flip}}, \mathbb{Z}_2) = \mathbb{Z}_2$.

v. In particular, the classification in (6.13) indicates the $a^2\tau_3^m\chi_j = \text{Sq}^1(a\tau_3^m\chi_j) = w_1a\tau_3^m\chi_j$ term, which specifies the potential mixed anomalies in 4d between the $\mathbb{Z}_2^{\text{flip}}$, symmetry and any of the magnetic $U(1)$ 1-symmetries (for both $j = 1, 2$).

vi. To check whether this $a^2\tau_3^m\chi_j$ anomaly is present in our theory, we can couple the magnetic 1-symmetry $U(1)^{m\chi_j}$ to the 2-form magnetic background $B^{m\chi_j}$ field, and couple the $\mathbb{Z}_2^{\text{flip}}$ symmetry to the 1-form or 1-cochain gauge field $a$. Suppose in the presence of $B^{m\chi_j}$ field, under the larger gauge transformation of $\mathbb{Z}_2^{\text{flip}}$, symmetry, the partition function is not fully invariant but obtains only a $(-1)$ phase, then it signals that the theory has a mod 2 global anomaly.

vii. Under the $\mathbb{Z}_2^{\text{flip}}$ symmetry, the $U(1)_{X_1}$ and $U(1)_{X_2}$ are swapped. Thus their associated gauged charges of the quarks and leptons are also swapped. Out of the 16 Weyl fermions per generation, the conventional left-handed Weyl spinors $(u_L, d_L, \nu_L, \epsilon_L)$, those also coupled to the $SU(2)_L$ maintain their $U(1)_{X_1} = U(1)_{X_2}$ gauged charges; but only the conventional right-handed Weyl spinors $(\bar{u}_R, \bar{d}_R, \bar{\nu}_R, \bar{\epsilon}_R$, those can be coupled to the $SU(2)_R$, which we also flip them to the left-handed particle as the right-handed anti-particle) swap their $U(1)_{X_1}$ and $U(1)_{X_2}$ gauged charges:

| $U(1)_{X_1}$ | $U(1)_{1st}$ | $U(1)_{X_2}$ | $U(1)_{2nd}$ |
|--------------|---------------|---------------|---------------|
| $\bar{u}_R$  | 1             | 1             |              |
| $\bar{d}_R$  | -3 3          |              | 1            |
| $\bar{\nu}_R = \nu_L$ | 5         | 1             |              |
| $\bar{\epsilon}_R = \epsilon_L^+$ | 1             | 5             |              |

(6.15)

Although we do not yet know the full classification of anomalies from (6.13), but we have enough informations to deduce that actually our $[(U(1)_{X_1} \times_{\mathbb{Z}_2^{\text{flip}}} U(1)_{X_2}) \times \mathbb{Z}_2^{\text{flip}}]$ gauge theory is:

**mixed anomaly free** within $\mathbb{Z}_2^{\text{flip}}$, and magnetic 1-symmetries $(U(1)^{m\chi_j}_{[1]} \times U(1)^{m\chi_j}_{[1]}) \times \mathbb{Z}_2^{\text{flip}}$.

The reasoning is that the 4d anomaly is captured by the large gauge transformation of $\mathbb{Z}_2^{\text{flip}}$-background field $a \in H^1(B\mathbb{Z}_2^{\text{flip}}, \mathbb{Z}_2) = \mathbb{Z}_2$ and the $U(1)^{m\chi_j}_{[1]}$-background field $\tau_3 \in H^1(B^2U(1), \mathbb{Z}_2) = \mathbb{Z}_2$. To trigger a non-vanishing 4d anomaly, we must turn on the 3-dimensional background field $\tau_3$, while the possible anomalies term (involving 1-symmetries) based on dimensional analysis counting into a 5d iTQFT can be: $a^2\tau_3^m\chi_j$ or $w_1a\tau_3^m\chi_j$ or $\tau_5^m\chi_j = \text{Sq}^2\tau_3^m\chi_j$. Only $a^2\tau_3^m\chi_j$ or $w_1a\tau_3^m\chi_j$ or some linear form of $a$ as $\bar{w}a\tau_3^m\chi_j$ (where $\bar{w}$ can be some twisted 1-cochain specified by the structure of (6.13)) involve the desired mixed anomalies. But the anomaly coefficients seem to be zero by any consideration. Given this data in (6.15), these anomaly coefficients suggest that it is highly impossible to have any mod 2 global anomaly.

viii. Thus we propose the categorical symmetry (6.11) of the $[(U(1)_{X_1} \times_{\mathbb{Z}_2^{\text{flip}}} U(1)_{X_2}) \times \mathbb{Z}_2^{\text{flip}}]$ gauge theory actually is sensible and well-defined kinematically, as long as we do not embed the theory into the full combined theory of GG and flipped $su(5)$ GUTs. The categorical symmetry in (6.11) is only retracted after embedding into the GUTs.

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25If the anomaly coefficient depends on the number of fermions are swapped under $\mathbb{Z}_2^{\text{flip}}$, then it is the two fermions swapped with the other two fermions in (6.15), then the anomaly coefficient seems to be 0 mod 2. If the anomaly coefficient depends on their $U(1)_{X_1} - U(1)_{X_2}$ charge differences, we still have $((3 - 1) + (1 - (-3)) + (1 - 5) + (5 - 1)) = 0$ and $((-3 - 1)^2 + (1 - (-3))^2 + (1 - 5)^2 + (5 - 1)^2) = 4^2 = 0 \mod 64$. 

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7 Conclusion and Future Directions

Here are some concluding remarks and future directions:

1. **Three generations of SM quarks and leptons**: Our work so far had focused on 1 generation of SM quarks and leptons. We can ask whether the number \(n\) of generations of quarks and leptons modify any story we had discussed previously.

In the conventional \(so(10)\) GUT without a WZW term, there is no constraint on \(n\) generation since we can take \(n\) copies of theories.

In the modified \(so(10)\) GUT with a WZW term, the \(mod\ 2\) class \(w_2w_3\) anomaly of (1.3) is matched by the sector of GUT-Higgs fields (or their fractionalized partons) and their 4d WZW term alone, we just need to ensure the anomaly index from GUT-Higgs WZW sector contributes 1 \(mod\ 2\).

- If each generation of 16 SM Weyl fermions associates with its own GUT-Higgs field, then the generation number \(n\) times of 16 SM Weyl fermions with \(n\) GUT-Higgs WZW sector requires a constraint \(n = 1 \ mod\ 2\) to match the \(w_2w_3\) anomaly, where \(n = 3\) generation indeed works.

- However, in general, we can just introduce a single (or any odd number) of GUT-Higgs WZW sector to match the \(1 \ mod\ 2\) class of \(w_2w_3\) anomaly. After all, we may only need a single GUT-Higgs to achieve the gauge symmetry-breaking from Spin(10) to other subgroups, regardless of the number of \(n\). In this case, our discussion on quantum criticalities can be applied to any \(n\) of SM or GUT.

Of course, an open question is which model fits the best to the HEP phenomenology and experiment.

2. **What is mass?**

We come back to a philosophical or metaphysical question: What is mass? From a modern quantum matter perspective, we can address this issue with a physical and mathematical answer. The mass more generally describes a massive or gapped energy spectrum – gapped with respect to the ground state(s). Here the mass is defined as the correlation function (of the corresponding operators/excitations/states) decaying exponentially. Then we can further address what are the known mechanisms for providing a massive or gapped spectrum.

- Traditionally, the free/single-particle/mean-field mechanism includes Anderson-Higgs \([94–96]\) or chiral symmetry breaking \([97]\) with a bilinear quadratic mass term turned on in a Lagrangian or Hamiltonian theory.

- Modern quantum systems provide other many-body interacting mechanisms to generate a non-quadratic non-mean-field mass, see Table 5.

In our work \([1]\), and in Ultra Unification \([35–37]\), these many-body interacting mechanisms ((4)-(6)) are used for the new BSM physics.

3. **What is criticality? What is a phase transition?**

Following footnote 2, now we can revisit the terminology on criticalities vs phase transitions with examples in Fig. 12-Fig. 15.

- The criticality means the system with gapless excitations (gapless thus critical, sometimes conformal) and with an infinite correlation length, it can be either (i) a continuous phase transition (many examples in Table 2) as an unstable critical point/line/etc. as an unstable renormalization group (RG) fixed point which has at least one relevant perturbation in the phase diagram, or (ii) a critical phase (the white region (0) in Fig. 12-Fig. 15) as a stable critical region controlled by a stable RG fixed point which does not have any relevant perturbation in the phase diagram.
Table 5: Some mechanisms to generate the mass or energy gap, both in the traditional free/single-particle/mean-field level or the quantum many-body interacting systems. Chiral SB for the chiral symmetry breaking. The s confinement is the smooth confinement without symmetry breaking. ✔ for Yes and ✗ for no. The symmetry-extension mechanism and its short exact sequence or fibration \( K \to \tilde{G} \to G \) to construct \( G \)-symmetric gapped phase, are summarized in [100]. Symmetry-extension method extends the \( G \)-symmetry to \( \tilde{G} \)-symmetry so to trivialize the anomaly by a gapped system, which can subsequently produce Symmetric Gapped Topological Order (TO) if we gauge \( K \) while still preserves \( G \) in a spacetime dimension \( d \geq 3 \).

- The phase transition [15] means the phase interface between two (or more) bulk phases in the phase diagram. The phase transition can be a continuous phase transition (second order or higher order, with gapless modes; many examples in Table 2) or a discontinuous phase transition (first-order, without gapless modes, and with a finite correlation length; e.g., (II)-(VI) and (II)'-(VI)' Fig. 12-13).

4. Boundary criticality or phase transition: The modified \( so(10) \) GUT with a 4d WZW term lives on a boundary of 5d invertible TQFT \( w_2 w_3(TM) = w_2 w_3(V_{SO(10)}) \) of (1.3). When the internal Spin(10) symmetry is treated as a global symmetry (not yet dynamically gauged), the modern condensed matter viewpoint is that

(1) If we impose any regularization (e.g., lattice) such that the internal Spin(10) symmetry acts onsite (i.e., the global symmetry acts on a local site), then we must realize the 4d criticality as a boundary criticality on the 4d boundary of the 5d bulk invertible TQFT. Thus the ’t Hooft anomaly in 4d is manifested by the boundary physics of 5d invertible TQFT via the anomaly inflow [101]. The internal onsite Spin(10) symmetry can be dynamically gauged if we gauge altogether the full 4d boundary-5d bulk coupled system (e.g., the long-range entangled gauged boundary-bulk coupled systems in Sec. 7 of [102]).
In contrast, if we are allowed to realize the internal Spin(10) symmetry acts non-onsite, then the ’t Hooft anomaly in 4d is realized as the obstruction to gauge this non-onsite symmetry in 4d.

Since we aim to gauge the internal symmetry at the end, we shall take the viewpoint (1), and we can interpret our criticalities or phase transitions in Table 2 as 4d boundary criticalities or phase transitions of a 5d gapped bulk.

5. **Bulk criticality or phase transition:** When the internal Spin(10) symmetry is dynamically gauged, the 5d bulk turns from a gapped (1.3) to a gapless system (because at the Gaussian fixed point where the pure Yang-Mills kinetic term (dA)^2 in 5d is marginal, the A^2 dA and A^4 become weak and irrelevant at IR, hence the confined gauge fields in 4d becomes deconfined and gapless in the 5d bulk). The purple regions of the phases with the gauged Spin(10) in Fig. 10 (in Fig. 14 and Fig. 15) thus have a 5d gapless bulk.

- The Spin(10) preserving system (purple regions) can be regarded as the GUT-Higgs field disordered phase, which has the GUT-Higgs field disordered both in the 4d boundary and 5d bulk.
- The Spin(10) breaking system (regions other than purple) can be regarded as the GUT-Higgs field ordered phase on the 4d boundary, but the GUT-Higgs field can be either ordered (breaking) or disordered (preserving) in the 5d bulk (e.g., see a recent study [103] in condensed matter, and references therein).

So the 5d bulk phases (in Fig. 14 and Fig. 15) moving from the purple region to non-purple regions, can go through either —

- 5d bulk phase transition: a disordered (Spin(10) preserving and gauge boson gapless) phase to an ordered (Spin(10) breaking to subgroups and gauge boson partially gapped) phase, which happens together with the boundary disorder-order transition. The presence of Spin(10) anomaly on the 4d boundary (and the SPT order in the 5d bulk) could modify the critical exponents and scaling dimensions of the critical GUT-Higgs fields at the transition, which was discussed as gapless SPT states in condensed matter literature [104–111].

- 5d bulk no phase transition: maintain a disordered (Spin(10) preserving and gauge boson gapless) phase. In this case, the disorder-order transition is only a boundary transition that happens only in the 4d boundary [103].

The 5d bulk phases (in Fig. 14 and Fig. 15) moving between different non-purple regions, can introduce another possibility —

- 5d bulk phase transition: between different ordered (Spin(10) breaking to subgroups and gauge boson partially gapped) phases.

6. **16n vs 15n Weyl fermions, and topological criticality or topological phase transition:**

The SM and GG su(5) models can have either choice of 15 or 16 Weyl fermions for each generation. In contrast, in order to be consistent with the SM data constraint, the PS, the GG u(5), the flipped u(5), and the modified so(10) have 16 Weyl fermions per generation. Thus, in all the figures from Fig. 12-Fig. 13 and Fig. 14-Fig. 15, we see that the hyperplane/line set at r_45 = 0 separates one side r_45 > 0 which has models with 16n Weyl fermions, while the other side r_45 < 0 can have models with either choice of 15n or 16n Weyl fermions (except the flipped u(5) can only have 16n Weyl fermions).

Overall, we could view the QFTs are governed by a deformation class of QFTs by replacing (1.3) to

\[
Z_{5d-iTQFT}^{(p,\nu)} \equiv \exp(i \pi \cdot p \cdot \int_{M_5} w_2 w_3) \cdot \exp\left(\frac{2\pi i}{16} \cdot \nu \cdot \eta(PD(A_{Z_4}) \mod 2)\right)_{M_5},
\]

with \( p \in \mathbb{Z}_2 \), a 4d Atiyah-Patodi-Singer \( \eta \) invariant \( \equiv \eta_{Pin^+} \in \mathbb{Z}_{16} \), \( \nu \in \mathbb{Z}_{16} \). (7.1)

The QFTs with the ungauged internal global symmetries can be deformed to each other, as long as they are matched by the 4d boundary ’t Hooft anomaly of the 5d cobordism class (7.1): First, a potential global \( \mathbb{Z}_2 \) anomaly, the \( w_2 w_3 \) anomaly for our 4d WZW term. Second, the \( \mathbb{Z}_{16} \) global anomaly captured...
by a 5d version of Atiyah-Patodi-Singer (APS) eta invariant for the Spin × Z_{2}^{\nu} Z_{4, X}-structure from TP_{5}(\text{Spin} \times Z_{2}^{\nu} Z_{4, X}) = Z_{16}. We can write that 5d APS invariant in terms of the 4d APS invariant of \text{Pin}^{+}-structure from TP_{4}(\text{Pin}^{+}) = Z_{16}. The \mathcal{A}_{Z_{4}} \in H^{1}(M, Z_{4, X}) is a cohomology class discrete gauge field of the Z_{4, X}-symmetry. The two combined invertible TQFTs, labeled by \nu \in Z_{16}, have a partition function \mathcal{Z}^{(p, \nu)}_{5d\text{-TQFT}} on M^{5}.

- On the GG u(5) model, if we are able to break down the continuous U(1)_{X_{1}} variant of the baryon minus lepton number \chi_{1} \equiv 5(B - L) - \frac{2}{3}Y_{1} symmetry down to a discrete variant of \mathbb{Z}_{4, X_{1}} = Z_{4, X} around the hyperplane at \nu_{45} = 0, then the Z_{4, X}-symmetry gives rise to a Z_{16} class global anomaly [32,34], such that we can introduce new sectors in addition to 15n Weyl fermions to go beyond the SM: via adding gapped TQFTs or gapless CFTs to cancel the global anomaly (known as Ultra Unification [35–37]).

- On the flipped u(5) model, there is also a continuous U(1)_{X_{2}} symmetry, where \chi_{2} \equiv \frac{1}{5}X_{1} + \frac{4}{5}Y_{1} = (B - L) + \frac{2}{5}Y_{1}, but this U(1)_{X_{2}} cannot be broken down to \mathbb{Z}_{4, X_{2}} = Z_{4, X} if we want to preserve the SM’s U(1)_{Y_{1}}. So the Z_{16} class global anomaly of Z_{4, X}-symmetry does not occur to constrain the flipped u(5) model.

So over the 8 octants in Fig. 12-Fig. 15, only the regions of the blue (II) and the red (III)=(VII) can have the 15n Weyl fermion scenario with new BSM TQFT/CFT sectors. Also their U(1)_{\nu}^{' \text{dark}} anti-deconfined criticality as well as topological criticality simultaneously.

An interesting future direction is that: Under what mechanism without fine-tuning, can the 16n to 15n Weyl fermion topological phase transition (involving with 4d TQFTs/CFTs on the 15n fermion side) occur as the same phase transition simultaneously as those phase transitions described previously in Table 2?

7. Completeness Hypothesis vs Absence of Global Symmetries vs Absence of Topological Operators:

We have found that the potential categorical symmetry in the low energy sector of \bigl[U(1)_{X_{1}} \times Z_{4, X}, U(1)_{X_{2}}\bigr] \rtimes Z_{2}^{\text{flip}}-gauge theory (presumably if we Higgs down the so(10) GUT’s Spin(10) down to this restricted subgroup of the two U(5) GUTs: GG and flipped models). But the categorical symmetry in fact disappears to none, even when we try to embed this \bigl[(U(1)_{X_{1}} \times Z_{4, X, u} U(1)_{X_{2}}) \rtimes Z_{2}^{\text{flip}}\bigr] to the dynamically gauged union of \bigl[U(5)^{1^{\text{st}}}_{q=2} \cup U(5)^{2^{\text{nd}}}_{q=2}\bigr], which already inevitably requires the full gauge group Spin(10) (see (C.6)).

Although our present work only studies QFT and GUT models (not quantum gravity), we find the absence of generalized global symmetries (neither higher symmetries nor categorical symmetries) in the UV model. It will be interesting to know whether this phenomenon is relatable to several conjectures on the universal features of quantum gravity (QG):

- **Completeness Hypothesis** about the spectrum [112–114]: every representation of any gauge group must be occupied by particle states.

- **Absence of Global Symmetries** [115–120]: All global symmetries (including the generalization of higher symmetries of extended objects, or non-invertible categorical symmetries, etc.) in QG must be either explicitly broken or dynamically gauged.

- **Absence of Topological Operators**: Take examples in 4d (or general dd) following [48,121],
  - the completeness of gauge theory particle state spectrum is equivalent to no topological Gukov-Witten 2-surface (or more general (d−2)d) operators (if and only if Wilson 1-line operators are endable with the 0d particles);
  - the completeness of twist vortices (cosmic strings, or string states) is equivalent to no topological Wilson 1-line operators (if and only if Gukov-Witten 2-surface operators [or generally (d−2)d] are endable with the twist vortices [or generally (d−3)d]).
  - the completeness of magnetic monopole is equivalent to no topological magnetic 2-surface operators...
(if and only if ’t Hooft 1-line operators [or generally \((d-3)d\)] are endable with the \(0d\) magnetic monopoles [or generally \((d-4)d\)].

— the completeness of magnetic twist vortices is equivalent to no topological ’t Hooft 1-line operators [or generally \((d-3)d\)] (if and only if magnetic 2-surface operators are endable with the \(1d\) magnetic twist vortices).

It will be illuminating to learn whether the absence of higher and categorical symmetries in our model has implications on the completion of the spectrum. It will be interesting to understand how our model on the modified \(so(10)\) GUT has any interpretations/connections to the above QG conjectures.

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A Quantum Numbers and Representations of SMs and GUTs

Follow the setup of Appendix A of Ref. [1], here we summarize the representations of “elementary” chiral fermionic particles of quarks and leptons of SMs and GUTs in Tables. In particular, we had already discussed the representations of the GG \(su(5)\) GUT and the PS model, and their embedding into the \(so(10)\) GUT in Appendix A.1 and A.2 of Ref. [1]; so we shall skip those. We focus on the flipped \(su(5)\) GUT.

**Spacetime symmetry representation** Weyl fermions are spacetime Weyl spinors, which we prefer to write all Weyl fermions as \(2_L\) of \(\text{Spin}(1,3) = \text{SL}(2, \mathbb{C})\) with a complex representation in the 4d Lorentz signature. On the other hand, the Weyl spinor is \(2_L\) of \(\text{Spin}(4) = \text{SU}(2)_L \times \text{SU}(2)_R\) with a pseudoreal representation in the 4d Euclidean signature.

**Internal symmetry representation** Below we provide Table 6 to organize the internal symmetry representations of particle contents of the flipped \(su(5)\) GUT with the \(U(5)_q^{2\text{nd}}\) gauge group, embedding into the \(so(10)\) GUT with the \(\text{Spin}(10)\) gauge group.

A.1 Embed the SM into the flipped \(u(5)\) GUT with \(U(5)_q^{2\text{nd}}\), then into the \(so(10)\) GUT

Let us compare the GG and the flipped \(su(5)\) GUT embedding:
1. The gauge theory embedding \( so(10) \) GUT \( \supset G_{SM}^{GUT} \) \( \supset G_{SM} \) the \( su(5) \) GUT \( \supset \) the SM only contains the \( G_{SM} \) via an internal symmetry group embedding:

\[
\text{Spin}(10) \supset U(5) \supset G_{SM} = SU(3) \times SU(2) L \times U(1) Y \equiv \frac{SU(3) \times SU(2) L \times U(1) Y}{\mathbb{Z}_6}.
\]  

(A.1)

The representations of quarks and leptons for these models are organized in Table 6 of Ref. [1]. Here the \( U(5) \) contains the SM \( U(1) X \) \( \equiv U(1)_{X} \) subgroup.

2. The gauge theory embedding \( so(10) \) GUT \( \supset U(5) \supset G_{SM} \) the SM, only contains the \( G_{SM} \) via an internal symmetry group embedding:

\[
\text{Spin}(10) \supset U(5) \supset (SU(3) \times SU(2) L) \times \mathbb{Z}_6 \equiv U(5) \supset G_{SM} = \frac{SU(3) \times SU(2) L}{\mathbb{Z}_6}.
\]  

(A.2)

In contrast, if we break \( U(5) \) to \( SU(5) \) the route does not contain \( G_{SM} \):

\[
\text{Spin}(10) \supset U(5) \supset SU(5) \supset (SU(3) \times SU(2) L) \times \mathbb{Z}_6 \equiv SU(5) \supset G_{SM} = \frac{SU(3) \times SU(2) L}{\mathbb{Z}_6}.
\]  

(A.3)

The representations of quarks and leptons for these models are organized in Table 6. Here the \( U(5) \) contains the SM \( U(1) X \) subgroup.

| SM fermion spinor field | SU(3) | SU(2) | U(1) \( x_2 \) | U(1) \( y_2 \) | U(1) \( y_1 \) = \( SU(5) \cap SU(3) \times SU(2) L \) | U(1) \( EM \) | U(1) \( B-L \) | U(1) \( X_1 \) | \( Z_5, X \) | \( Z_4, X \) | \( Z_2 \) | SU(5) |
|------------------------|-------|-------|----------------|----------------|-----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------|
| \( \bar{u}_R \)        | 3     | 1     | -3            | 2              | -4              | \( \frac{3}{2} \) | 1/3         | 1           | 1           | 1           | 1           | 5     |
| \( l_L \)              | 1     | 2     | -3            | -3             | -3              | 0 or \(-1\)     | -1          | -3          | -3          | 1           | 1           | 10    |
| \( q_L \)              | 3     | 2     | 1             | 1              | 1 \( \equiv \frac{2}{3} \) | 1/3           | 1           | 1           | 1           | 1           | 1           | 1     |
| \( d_R \)              | 3     | 1     | 1             | -4             | 2               | \( \frac{1}{3} \) | -1/3        | -3          | -3          | 1           | 1           | 1     |
| \( \bar{u}_R = \nu_L \)| 1     | 1     | 1             | 6              | 0               | 0             | 1           | 5           | 0           | 1           | 1           | 1     |
| \( e_R = e_L \)        | 1     | 1     | 5             | 0              | 6               | 1             | 1           | 1           | 1           | 1           | 1           | 1     |

Table 6: The \( G_{SM} \) cannot be embedded into \( SU(5) \) given the way their representations are chosen as (3.4). The \( G_{SM} \) can be embedded into the flipped \( u(5) \) GUT only if we include the full \( U(5) \) gauge group. Then we can embed this into the \( so(10) \) GUT with a Spin(10) gauge group. The \( SU(5) \) \( \supset U(1)_Y \) or \( U(1)_{Y_1} \), but we have \( U(5) \supset U(1)_Y \) or \( U(1)_{Y_1} \).

B Embedding

B.1 Embedding Strategy

The strategy to embed SU(5) in Spin(10) is to first embed SU(5) in SO(10) by identifying \( \mathbb{C}^5 \) with \( \mathbb{R}^{10} \) via the complex-to-real mapping \( x + iy \mapsto (x, y) \), and then lift the embedding from SO(10) to Spin(10) (from \( r \) to \( r' \) in the following diagram).
Furthermore, we ask “how to embed U(5), or more precisely U(5)_{\hat{q}} (several non-isomorphic versions $U(5)_{\hat{q}} \equiv \frac{SU(5) \times U(1)}{Z_5}$ defined in (2.1), via the identification $U(5)_{\hat{q}} \equiv \{(g, e^{i\theta}) \in SU(5) \times U(1) | (e^{i \frac{2\pi n}{\hat{q}}} l, 1) \sim (l, e^{i \frac{2\pi n q}{\hat{q}}}), n \in Z_5\}$, into SO(10) or Spin(10)?” As we shall show, it turns out that $U(5)_{\hat{q}} \subset SO(10)$ for $\hat{q} = 1, 4$, while $U(5)_{\hat{q}} \subset Spin(10)$ for $\hat{q} = 2, 3$.

1. Here we carefully distinguish several different $U(1)$: (i) the $U(1)$ in $U(5)_{\hat{q}} \equiv \frac{SU(5) \times U(1)}{Z_5}$, (ii) the $\tilde{U}(1)$ as the center of $U(5)_{\hat{q}}$ such that $\tilde{U}(1) = Z(U(5)_{\hat{q}}) \supset Z(SU(5)) = Z_5$, (iii) the $U(1)'$ as the quotient $\frac{U(5)_{\hat{q}}}{SU(5)}$. All these three groups are all isomorphic to any ordinary $S^1$, but they have different roles that we explain below.

2. • The “$\hookrightarrow$” means the inclusion, the former group can be embedded into the later group. The “$\rightarrow$” means the group homomorphism.

• Here $SU(5) \hookrightarrow U(5)_{\hat{q}} \xrightarrow{\det} U(1)'$ is part of the short exact sequence $1 \rightarrow SU(5) \rightarrow U(5)_{\hat{q}} \xrightarrow{\det} U(1)' \rightarrow 1$ where $U(1)' \equiv \frac{U(1)}{Z_{5\hat{q}}}$ is related to modding out $Z_{5\hat{q}}$ of the $U(1)$ defined in $U(5)_{\hat{q}}$ in (2.1). The inclusion $SU(5) \hookrightarrow U(5)_{\hat{q}}$ indicates that $SU(5)$ is a normal subgroup of $U(5)_{\hat{q}}$, while the $U(1)'$ is a quotient group of $U(5)_{\hat{q}}$. So

$$U(1)' = \frac{U(5)_{\hat{q}}}{SU(5)} = \frac{U(1)}{Z_{5\hat{q}}}.$$

• Here $Z_2 \hookrightarrow Spin(10) \xrightarrow{p} SO(10)$ is part of the short exact sequence $1 \rightarrow Z_2 \rightarrow Spin(10) \rightarrow SO(10) \rightarrow 1$, so $Spin(10)/SO(10) = Z_2$.

• Here $Z_5 \xrightarrow{e^{2\pi i n} / 5} \tilde{U}(1) \xrightarrow{\hat{z}^{5\hat{q}}} U(1)'$ is part of the short exact sequence $1 \rightarrow Z_5 \rightarrow \tilde{U}(1) \xrightarrow{\hat{z}^{5\hat{q}}} U(1)' \rightarrow 1$.

• There are no short exact sequences in this (B.1), other than the above relations that we report.

3. For $Z_5 \xrightarrow{z = e^{2\pi i n} / 5} \tilde{U}(1) \xrightarrow{\hat{z}^{5\hat{q}}} U(1)'$, the first map $Z_5 \xrightarrow{z = e^{2\pi i n} / 5} \tilde{U}(1)$ inclusion specifies the identification of the center $Z(SU(5)) = Z_5 \equiv \{z = e^{2\pi i n} | n \in \{0, 1, 2, 3, 4\}\}$ and its map to the total group $\tilde{U}(1) \equiv \{z = e^{i\theta} | \theta \in [0, 2\pi)\}$. The second map $\tilde{U}(1) \xrightarrow{\hat{z}^{5\hat{q}}} U(1)' = \frac{U(5)_{\hat{q}}}{SU(5)} = \frac{U(1)}{Z_{5\hat{q}}}$ says that the $z \in \tilde{U}(1)$ maps to the $\hat{z}^{5\hat{q}} \in U(1)'$, thus also says that the $e^{2\pi i \frac{n}{\hat{q}}} \in \tilde{U}(1)$ becomes the kernel of the map $\tilde{U}(1) \xrightarrow{\hat{z}^{5\hat{q}}} U(1)$ that maps to the identity $e^{2\pi i \hat{q}} = 1 \in U(1)$.

All these facts coincide with our definition of $U(5)_{\hat{q}}$ in (2.1), which also justifies (B.1)’s second line: $SU(5) \hookrightarrow U(5)_{\hat{q}} \xrightarrow{\det} U(1)'$.

4. The $U(5)_{\hat{q}} \xrightarrow{\sim} SO(10)$ and the $U(5)_{\hat{q}} \xrightarrow{\hat{z}'} Spin(10)$ only imply that the maps are group homomorphisms. The map may or may not be inclusion or embedding, which depends on the value of $\hat{q}$.
If \( U(5)_q \subset SO(10) \) (in particular \( q = 1 \)), then a non-trivial fact is that the lifting is only possible, if the lifted \( U(5)_{q'} \subset Spin(10) \) has the SU(5) fundamental representation carries an \( q' = 2q \mod 5 \) charge under \( U(1) \). Because the isomorphism (2.2), we can deduce that

\[
U(5)_{q=1,4} \subset SO(10) \text{ while } U(5)_{q=2,3} \subset Spin(10).
\]

In the following subsections, we provide some nontrivial checks of these facts.

### B.2 Definition of Lie Groups

#### B.2.1 Lie algebra and Lie Group of \( U(5)_q \)

Let \( \{ |\psi_i \rangle | i = 1, 2, \cdots, 5 \} \) be a set of orthonormal basis of \( \mathbb{C}^5 \), i.e. \( \langle \psi_i | \psi_j \rangle = \delta_{ij} \) given the dual basis \( \langle \psi_i | = | \psi_i \rangle \). The U(5) group is the group of isometries of \( \mathbb{C}^5 \), which is canonically identified with the group of \( 5 \times 5 \) unitary matrices. They are generated by 25 generators \( T_a \) \( (a = 1, 2, \cdots, 25) \):

\[
U \in U(5) : U = \exp \left( \sum_{a=1}^{25} i \theta_a T_a \right). \tag{B.2}
\]

The generators \( T_a \) are basis of \( 5 \times 5 \) Hermitian matrices, and can be chosen as

\[
\begin{align*}
T_1 &= |\psi_1 \rangle \langle \psi_2 | + |\psi_2 \rangle \langle \psi_1 |, \\
T_2 &= |\psi_1 \rangle \langle \psi_3 | + |\psi_3 \rangle \langle \psi_1 |, \\
T_3 &= |\psi_1 \rangle \langle \psi_4 | + |\psi_4 \rangle \langle \psi_1 |, \\
T_4 &= |\psi_1 \rangle \langle \psi_5 | + |\psi_5 \rangle \langle \psi_1 |, \\
T_5 &= |\psi_2 \rangle \langle \psi_3 | + |\psi_3 \rangle \langle \psi_2 |, \\
T_6 &= |\psi_2 \rangle \langle \psi_4 | + |\psi_4 \rangle \langle \psi_2 |, \\
T_7 &= |\psi_2 \rangle \langle \psi_5 | + |\psi_5 \rangle \langle \psi_2 |, \\
T_8 &= |\psi_3 \rangle \langle \psi_4 | + |\psi_4 \rangle \langle \psi_3 |, \\
T_9 &= |\psi_3 \rangle \langle \psi_5 | + |\psi_5 \rangle \langle \psi_3 |, \\
T_{10} &= |\psi_4 \rangle \langle \psi_5 | + |\psi_5 \rangle \langle \psi_4 |, \\
T_{21} &= |\psi_1 \rangle \langle \psi_1 | - |\psi_2 \rangle \langle \psi_2 |, \\
T_{22} &= 2|\psi_3 \rangle \langle \psi_3 | - |\psi_4 \rangle \langle \psi_4 | - |\psi_5 \rangle \langle \psi_5 |, \\
T_{23} &= |\psi_4 \rangle \langle \psi_4 | - |\psi_5 \rangle \langle \psi_5 |, \\
T_{24} &= -3|\psi_1 \rangle \langle \psi_1 | - 3|\psi_2 \rangle \langle \psi_2 | + 2|\psi_3 \rangle \langle \psi_3 | + 2|\psi_4 \rangle \langle \psi_4 | + 2|\psi_5 \rangle \langle \psi_5 |, \\
T_{25} &= \hat{q}(|\psi_1 \rangle \langle \psi_1 | + |\psi_2 \rangle \langle \psi_2 | + |\psi_3 \rangle \langle \psi_3 | + |\psi_4 \rangle \langle \psi_4 | + |\psi_5 \rangle \langle \psi_5 |).
\end{align*}
\]

In particular, the last five generators \( T_{21}, T_{22}, \cdots, T_{25} \) are the Cartan generators. Among them, the last one \( T_{25} \) is the U(1) subgroup generator. The overall coefficient \( \hat{q} = 1, 2, 3, 4 \) in \( T_{25} \) labels the U(1) charge carried by the SU(5) fundamental representation, which is to be determined later.

The U(1) subgroup contains a \( Z_5 \) subgroup that can be shared with the center of SU(5),

\[
Z_5 = \left\{ e^{\frac{2\pi i}{5} T_{25}} \right\} = \left\{ e^{\frac{2\pi i n}{5} T_{25}} | n = 0, 1, 2, 3, 4 \right\}. \tag{B.4}
\]

Assigning \( T_{25} \) with different charge numbers \( \hat{q} \) will lead to different embeddings of the \( Z_5 \) subgroup in SU(5), which lead to different U(5)\( _{\hat{q}} = SU(5) \times Z_5, \hat{q} \) U(1) group structures,

\[
e^{\frac{2\pi i}{5} T_{25}} = \begin{cases} 
 e^{\frac{2\pi i}{5}(T_{24} - 5T_{21})} & \hat{q} = 1, \\
 e^{\frac{2\pi i}{5}T_{24}} & \hat{q} = 2, \\
 e^{-\frac{2\pi i}{5}T_{24}} & \hat{q} = 3, \\
 e^{-\frac{2\pi i}{5}(T_{24} - 5T_{21})} & \hat{q} = 4.
\end{cases} \tag{B.5}
\]
The difference will become important when embedding $U(5)q$ into $SO(10)$ or $Spin(10)$.

B.2.2 Lie algebra and Lie Group of $SO(10)$

Let $\{ |e_i| : i = 1, 2, \cdots, 10 \}$ be a set of orthonormal basis of $\mathbb{R}^{10}$, i.e. $\langle e_i | e_j \rangle = \delta_{ij}$ given by the dual basis $|e_i|^T = |e_i\rangle$. The $SO(10)$ group is the group of rotations in $\mathbb{R}^{10}$, which is canonically identified with the group of $10 \times 10$ orthogonal matrices. They are generated by 45 generators:

$$O \in SO(10) : O = \exp \left( \sum_{\{i,j\} \subset \{1, \cdots, 10\}} i \theta_{i\wedge j} e_{i\wedge j} \right).$$

(B.6)

The generators $e_{i\wedge j}$ are basis of $10 \times 10$ pure-imaginary anti-symmetric Hermitian matrices, given by

$$e_{i\wedge j} = i (|e_i\rangle \langle e_j| - |e_j\rangle \langle e_i|)$$

(B.7)

for all 2-subsets $\{i, j\}$ in $\{1, 2, \cdots, 10\}$. Note that the basis is anti-symmetric $e_{i\wedge j} = -e_{j\wedge i}$ by definition.

B.2.3 Lie algebra and Lie Group of $Spin(10)$

The construction of the $Spin(10)$ group starts with the Clifford algebra $\text{Cl}(10) = M_{16}(\mathbb{H})$, which is isomorphic to the algebra of $16 \times 16$ matrices over the quaternion field $\mathbb{H}$. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_{10}$ be the generators of $\text{Cl}(10)$, satisfying $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$. For the purpose of spelling out the matrix representations, these $16 \times 16$ quaternion matrices can be written as $32 \times 32$ complex matrices, by embedding $M_{16}(\mathbb{H})$ into $M_{32}(\mathbb{C})$. One explicit choice of the complex matrix representation can be

$$\Gamma_1 \cong \sigma^{11000}, \quad \Gamma_2 \cong \sigma^{30100},$$
$$\Gamma_3 \cong \sigma^{13000}, \quad \Gamma_4 \cong \sigma^{30300},$$
$$\Gamma_5 \cong \sigma^{12012}, \quad \Gamma_6 \cong \sigma^{30221},$$
$$\Gamma_7 \cong \sigma^{12020}, \quad \Gamma_8 \cong \sigma^{30202},$$
$$\Gamma_9 \cong \sigma^{12032}, \quad \Gamma_{10} \cong \sigma^{30223},$$

(B.8)

where $\sigma^{\mu \nu \cdots} \equiv \sigma^\mu \otimes \sigma^\nu \otimes \cdots$ denotes the direct product of Pauli matrices. The symbol $\cong$ (reads “can be represented as”) indicates that the equality is only a basis dependent statement.

The $Spin(10)$ group is a Lie group, whose Lie algebra corresponds to the grade-2 subspace of $\text{Cl}(10)$, which is spanned by 45 basis elements (as Lie group generators),

$$\Gamma_{i\wedge j} = \frac{i}{2} [\Gamma_i, \Gamma_j],$$

(B.9)

such that the Lie group elements are generated by

$$O \in Spin(10) : O = \exp \left( \sum_{\{i,j\} \subset \{1, \cdots, 10\}} \frac{i \theta_{i\wedge j}}{2} \Gamma_{i\wedge j} \right).$$

(B.10)

The Spin group is fully contained in the even-graded subspace $\text{Cl}^{\text{even}} = \text{Cl}^0 \oplus \text{Cl}^2 \oplus \text{Cl}^4 \oplus \cdots$ of the associated Clifford algebra. For $\text{Cl}(10) = M_{16}(\mathbb{H})$, the even-graded subspace $\text{Cl}^{\text{even}}(10) = M_{16}(\mathbb{C}) \oplus M_{16}(\mathbb{C})$ splits into two independent algebras of $16 \times 16$ matrices over the complex field $\mathbb{C}$. The two $M_{16}(\mathbb{C})$ subspaces are specified by the following projection operators (projected down from $M_{32}(\mathbb{C})$),

$$P_\pm = \frac{1 \pm i \prod_{j=1}^{10} \Gamma_j}{2}.$$  

(B.11)
This means that the irreducible fundamental representation of Spin(10) is only 16-dimensional, even though the associated Cl(10) requires a 32-dimensional representation. By choosing one of the $M_{16}(\mathbb{C})$ subspaces (say the subspace of that survives the $P_+$ projection), $\Gamma_{i\lambda j}$ can be represented as $16 \times 16$ matrices as follows (substituting Eqn. (B.8) into Eqn. (B.9) followed by the projection $\Gamma_{i\lambda j} \rightarrow P_+\Gamma_{i\lambda j}P_+$)

\[
\begin{align*}
\Gamma_{1\lambda 2} &\doteq \sigma^{1100}, & \Gamma_{1\lambda 3} &\doteq \sigma^{2000}, & \Gamma_{1\lambda 4} &\doteq \sigma^{1300}, & \Gamma_{1\lambda 5} &\doteq -\sigma^{2012}, & \Gamma_{1\lambda 6} &\doteq \sigma^{1221}, \\
\Gamma_{1\lambda 7} &\doteq -\sigma^{3020}, & \Gamma_{1\lambda 8} &\doteq \sigma^{1202}, & \Gamma_{1\lambda 9} &\doteq -\sigma^{3032}, & \Gamma_{1\lambda 10} &\doteq \sigma^{1223}, & \Gamma_{2\lambda 3} &\doteq -\sigma^{3100}, \\
\Gamma_{2\lambda 4} &\doteq \sigma^{2020}, & \Gamma_{2\lambda 5} &\doteq -\sigma^{2112}, & \Gamma_{2\lambda 6} &\doteq -\sigma^{0321}, & \Gamma_{2\lambda 7} &\doteq -\sigma^{2120}, & \Gamma_{2\lambda 8} &\doteq -\sigma^{0302}, \\
\Gamma_{2\lambda 9} &\doteq -\sigma^{2132}, & \Gamma_{2\lambda 10} &\doteq -\sigma^{0323}, & \Gamma_{3\lambda 4} &\doteq \sigma^{3200}, & \Gamma_{3\lambda 5} &\doteq \sigma^{3212}, & \Gamma_{3\lambda 6} &\doteq \sigma^{3221}, \\
\Gamma_{3\lambda 7} &\doteq \sigma^{1020}, & \Gamma_{3\lambda 8} &\doteq \sigma^{3202}, & \Gamma_{3\lambda 9} &\doteq \sigma^{1032}, & \Gamma_{3\lambda 10} &\doteq \sigma^{3223}, & \Gamma_{4\lambda 5} &\doteq -\sigma^{3212}, \\
\Gamma_{4\lambda 6} &\doteq \sigma^{0121}, & \Gamma_{4\lambda 7} &\doteq -\sigma^{3220}, & \Gamma_{4\lambda 8} &\doteq \sigma^{0102}, & \Gamma_{4\lambda 9} &\doteq -\sigma^{3232}, & \Gamma_{4\lambda 10} &\doteq \sigma^{0123}, \\
\Gamma_{5\lambda 6} &\doteq \sigma^{2233}, & \Gamma_{5\lambda 7} &\doteq -\sigma^{0032}, & \Gamma_{5\lambda 8} &\doteq -\sigma^{2210}, & \Gamma_{5\lambda 9} &\doteq \sigma^{0020}, & \Gamma_{5\lambda 10} &\doteq -\sigma^{2231}, \\
\Gamma_{6\lambda 7} &\doteq -\sigma^{2201}, & \Gamma_{6\lambda 8} &\doteq \sigma^{0002}, & \Gamma_{6\lambda 9} &\doteq -\sigma^{2213}, & \Gamma_{6\lambda 10} &\doteq \sigma^{0002}, & \Gamma_{7\lambda 8} &\doteq \sigma^{2222}, \\
\Gamma_{7\lambda 9} &\doteq -\sigma^{0012}, & \Gamma_{7\lambda 10} &\doteq \sigma^{2203}, & \Gamma_{8\lambda 9} &\doteq -\sigma^{2230}, & \Gamma_{8\lambda 10} &\doteq -\sigma^{0021}, & \Gamma_{9\lambda 10} &\doteq \sigma^{2211}.
\end{align*}
\]

The representation space is $\mathbb{C}^{16}$.

### B.3 Embedding and Projection Maps

As laid out in Eqn. (B.1), some of the $U(5)_{\hat{q}}$ group can be embedded into SO(10) by an embedding map $r$, and the Spin(10) group can be projected to SO(10) by a projection map $p$. This potentially enables us to lift the embedding $r$ to $r'$, then $r'$ will tell us how to embed which $U(5)_{\hat{q}}$ in Spin(10). However, we will show that the lifting is not always possible. It crucially depends on the choice of the charge number $\hat{q}$ or $\hat{q'}$.

#### B.3.1 Embedding Map $r$

The embedding map $r : \mathbb{C}^5 \rightarrow \mathbb{R}^{10}$ sends the representation space $\mathbb{C}^5$ to $\mathbb{R}^{10}$ by

\[
\begin{align*}
\sum_{i=1}^{5} z_i |\psi_i \rangle &\mapsto \sum_{i=1}^{5} (\text{Re} z_i |e_{2i-1}\rangle + \text{Im} z_i |e_{2i}\rangle).
\end{align*}
\]

It induces a functorial map $r : U(5) \rightarrow SO(10)$ by

\[
\begin{align*}
\begin{array}{ccc}
U(5) & \xrightarrow{r} & SO(10) \\
\mathbb{C}^5 & \xrightarrow{r} & \mathbb{R}^{10}
\end{array}
\end{align*}
\]

which is the embedding map $r$. The embedding map between Lie groups also implies the embedding map of the corresponding Lie algebras, as $r : u(5) \rightarrow so(10)$, which allows us to establish the relation among
generators:

\[
\begin{align*}
    r(T_1) &= e_{1A4} - e_{2A3}, \\
    r(T_2) &= e_{1A6} - e_{2A5}, \\
    r(T_3) &= e_{1A8} - e_{2A7}, \\
    r(T_4) &= e_{1A10} - e_{2A9}, \\
    r(T_5) &= e_{3A6} - e_{4A5}, \\
    r(T_6) &= e_{3A8} - e_{4A7}, \\
    r(T_7) &= e_{3A10} - e_{4A9}, \\
    r(T_8) &= e_{5A8} - e_{6A7}, \\
    r(T_9) &= e_{5A10} - e_{6A9}, \\
    r(T_{10}) &= e_{7A10} - e_{8A9}, \\
    r(T_{21}) &= e_{1A2} - e_{3A4}, \\
    r(T_{22}) &= 2e_{5A6} - e_{7A8} - e_{9A10}, \\
    r(T_{23}) &= e_{7A8} - e_{9A10}, \\
    r(T_{24}) &= -3e_{1A2} - 3e_{3A4} + 2e_{5A6} + 2e_{7A8} + 2e_{9A10}, \\
    r(T_{25}) &= \hat{q}(e_{1A2} + e_{3A4} + e_{5A6} + e_{7A8} + e_{9A10}).
\end{align*}
\]

(B.15)

Given that \( r \) is a linear map (i.e. \( r(\theta_a T_a) = \theta_a r(T_a) \)), Eqn. (B.15) automatically specifies the map \( r \) for any element in the Lie algebra.

### B.3.2 Projection Map \( p \)

The projection map \( p : \text{Spin}(10) \to \text{SO}(10) \) is defined by the short exact sequence

\[
1 \to \mathbb{Z}_2 \to \text{Spin}(10) \xrightarrow{p} \text{SO}(10) \to 1.
\]

(B.16)

By comparing Eqn. (B.10) and Eqn. (B.6), the projection map simply identifies the Lie group generators

\[
p\left(\frac{\Gamma_{i\wedge j}}{2}\right) = e_{i\wedge j}
\]

(B.17)

for all pairs \( \{i, j\} \subset \{1, 2, \cdots, 10\} \). The mapping \( p : \text{spin}(10) \xrightarrow{\sim} \mathfrak{so}(10) \) is one-to-one (hence invertible) on the Lie algebra level (but not on the Lie group level), which allows us to define \( p^{-1} : \mathfrak{so}(10) \xrightarrow{\sim} \text{spin}(10) \),

\[
p^{-1}(e_{i\wedge j}) = \frac{\Gamma_{i\wedge j}}{2}.
\]

(B.18)

### B.3.3 Embedding Map \( r' \)

The invertibility of \( p \) on the Lie algebra level allows us to lift the map \( r \) to \( r' \) on the Lie algebra level by defining \( r' = p^{-1} \circ r \), i.e. \( r'(T_i) = p^{-1}(r(T_i)) \).

\[
\begin{CD}
    \text{u}(5) @>r'>> \text{spin}(10) @>p\leftarrow>> \mathfrak{so}(10)
\end{CD}
\]

(B.19)
Using Eqn. (B.15) and Eqn. (B.18), we obtain

\[
\begin{align*}
r'(T_1) &= \frac{1}{7} (\Gamma_{1\Lambda 4} - \Gamma_{2\Lambda 3}), \\
r'(T_2) &= \frac{1}{7} (\Gamma_{1\Lambda 6} - \Gamma_{2\Lambda 5}), \\
r'(T_3) &= \frac{1}{7} (\Gamma_{1\Lambda 8} - \Gamma_{2\Lambda 7}), \\
r'(T_4) &= \frac{1}{7} (\Gamma_{1\Lambda 10} - \Gamma_{2\Lambda 9}), \\
r'(T_5) &= \frac{1}{7} (\Gamma_{3\Lambda 6} - \Gamma_{4\Lambda 5}), \\
r'(T_6) &= \frac{1}{7} (\Gamma_{3\Lambda 8} - \Gamma_{4\Lambda 7}), \\
r'(T_7) &= \frac{1}{7} (\Gamma_{3\Lambda 10} - \Gamma_{4\Lambda 9}), \\
r'(T_8) &= \frac{1}{7} (\Gamma_{3\Lambda 8} - \Gamma_{4\Lambda 7}), \\
r'(T_9) &= \frac{1}{7} (\Gamma_{5\Lambda 10} - \Gamma_{6\Lambda 9}), \\
r'(T_{10}) &= \frac{1}{2} (\Gamma_{7\Lambda 10} - \Gamma_{8\Lambda 9}), \\
\end{align*}
\]

(B.20)

which applies to the whole Lie algebra given that \( r' \) is a linear map (i.e. \( r'(\theta a T_0) = \theta a r'(T_0) \)). However, there could be obstruction to further lift the Lie algebra embedding \( r' \) to the Lie group level. The key relies on whether the \( \mathbb{Z}_5 \) subgroup shared between \( SU(5) \) and \( U(1) \) can be consistently defined after lifting \( U(5)q \) to Spin(10). This depends on the choice of the charge number \( q \).

### B.4 Obstruction to Lifting: Compatibility of \( \mathbb{Z}_5 \) Center

Recall Eqn. (B.5) that the charge number \( q \) affects how the \( \mathbb{Z}_5 \) generator \( e^{\frac{2\pi i}{5} T_{25}} \in U(1) \) is identified with a group element inside \( SU(5) \). The key is to check if any of these identifications will be broken under the embedding map \( r' \). Since Eqn. (B.5) only involves three generators \( T_{21}, T_{24}, \) and \( T_{25} \), we will only focus on their images under the \( r' \) map. Further more, because \( T_{21}, T_{24}, \) and \( T_{25} \) are commuting Cartan generators (so will be \( r'(T_{21}), r'(T_{24}) \) and \( r'(T_{25}) \)), we can find a good basis of \( \mathbb{C}^{16} \) to simultaneously diagonalize \( r'(T_{21}) \), \( r'(T_{24}) \) and \( r'(T_{25}) \). By explicit calculation (by substituting Eqn. (B.12) to Eqn. (B.20) and diagonalize the 16 \( \times \) 16 matrices), the eigenvalues of \( r'(T_{21}), r'(T_{24}) \) and \( r'(T_{25}) \) in their common eigenbasis are listed in Table 7.

Because of the group isomorphism given in (2.2), we will only have to check two cases, \( q = 1 \) and \( q = 2 \):

- For \( q = 1 \), \( e^{\frac{2\pi i}{5} T_{25}} \) is identified with \( e^{\frac{2\pi i}{5} (T_{24} - 5 T_{21})/2} \) in \( U(5) \), but \( e^{\frac{2\pi i}{5} r'(T_{25})} \) can not be identified with \( e^{\frac{2\pi i}{5} r'(T_{24} - 5 T_{21})/2} \) in Spin(10), because

\[
r'(T_{24} - 5 T_{21})/2 - r'(T_{25}|_{q=1}) \equiv \text{diag}(\frac{5}{2}, -\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, -\frac{5}{2}, -\frac{5}{2}, -\frac{5}{2}, \frac{5}{2}, -\frac{5}{2}, -\frac{5}{2}, -\frac{5}{2}, \frac{5}{2}, -\frac{5}{2}, -\frac{5}{2}, -\frac{5}{2}) \mod 5
\]

\[
\equiv \frac{5}{2} \mathbb{I},
\]

(B.21)

meaning \( e^{\frac{2\pi i}{5} r'(T_{24} - 5 T_{21})/2} = -e^{\frac{2\pi i}{5} r'(T_{25})} \), which posts an obstruction to lift \( r' \) out of the exponent to the Lie group level.
Table 7: Eigenvalues of \( r'(T_{21}) \), \( r'(T_{24}) \) and \( r'(T_{25}) \) in their common eigenbasis.

| SU(5) | \( r'(T_{21}) \) | \( r'(T_{24}) \) | \( r'(T_{25}) \) |
|-------|-------------------|-------------------|-------------------|
| 5     | -1                | -3                | -3\( \hat{q}/2 \) |
|       | 1                 | -3                | -3\( \hat{q}/2 \) |
| 10    | 0                 | 2                 | -3\( \hat{q}/2 \) |
|       | 0                 | 2                 | -3\( \hat{q}/2 \) |
| 1     | 0                 | 0                 | 5\( \hat{q}/2 \)  |

- For \( \hat{q} = 2 \), \( e^{\frac{2\pi i}{5}T_{25}} \) is identified with \( e^{\frac{2\pi i}{5}T_{24}} \) in U(5), while \( e^{\frac{2\pi i}{5}r'(T_{25})} \) can also be identified with \( e^{\frac{2\pi i}{5}r'(T_{24})} \) in Spin(10) consistently, because

  \[
  r'(T_{24}) - r'(T_{25}|_{\hat{q}=2}) \equiv \text{diag}(0, 0, 5, 5, 5, -5, -5, 0, 0, 0, 0, 0, 5, -5) \mod 5 = 0, \tag{B.22}
  \]

  meaning \( e^{\frac{2\pi i}{5}r'(T_{24})} = e^{\frac{2\pi i}{5}r'(T_{25})} \), which posts no obstruction to lift \( r' \) out of the exponent to the Lie group level.

In conclusion, the \( \mathbb{Z}_5 \) center can be compatibly defined and shared between SU(5) and U(1), if and only if \( \hat{q} = 2 \). With \( \hat{q} = 2 \), there is no obstruction to further lift \( r' \) to the Lie group level and hence the embedding map \( r': U(5) \rightarrow \text{Spin}(10) \) is well-defined.

\section{C Flipping}

\subsection{C.1 Flipping Isomorphism}

The standard GG U(5) and flipped U(5) (denoted as U(5)\textsubscript{1st} and U(5)\textsubscript{2nd} in the following) can be both embedded in the same Spin(10) group as long as their charge numbers are taken to be the same even integer. We will implicitly assume the \( q = 2 \) case in the following discussion. U(5)\textsubscript{1st} and U(5)\textsubscript{2nd} are related by the flipping isomorphism, denoted as \( f \), which is an outer automorphism of U(5) and also an inner automorphism of SO(10) as well as Spin(10), as shown in the diagram below.
The flipping isomorphism $f$ can be specified as an inner automorphism of $f : SO(10) \to SO(10)$, such that for $O \in SO(10): f(O) = F^{-1}OF$ with $F = e^{i\pi e_{2\Lambda^4}}$. This can also be interpreted as a basis transformation of $\mathbb{R}^{10}$,

$$f(|e_i\rangle) = F|e_i\rangle = \begin{cases} -|e_i\rangle & i \in \{2, 4\} \\ |e_i\rangle & i \notin \{2, 4\} \end{cases} = (-1)^{\delta_{i\in\{2,4\}}} |e_i\rangle,$$  \hfill (C.2)

where $\delta_{i\in\{2,4\}}$ is the Kronecker delta symbol that equals 1 when $i \in \{2, 4\}$ and equals 0 when $i \notin \{2, 4\}$. The flipping isomorphism is a duality, as $F^2 = 1$, or $f \circ f = \text{id}$.

The flipping isomorphism can be lifted to $f : \text{Spin}(10) \to \text{Spin}(10)$, such that for $O \in \text{Spin}(10): f(O) = F^{-1}OF$ with $F = \pm e^{i(\pi/2)\Gamma_{2\Lambda^4}} = \pm i\Gamma_{2\Lambda^4} = \mp \Gamma_2 \Gamma_4$. The $\pm$ sign ambiguity arise from the $\mathbb{Z}_2$ subgroup freedom when embedding $SO(10)$ in $\text{Spin}(10)$. Nevertheless, this sign ambiguity does not affect the definition of the flipping isomorphism $f(O)$ because $F$ always appears twice. The flipping isomorphism can also be translated to an inner automorphism of $\text{Cl}(10)$, as

$$f(\Gamma_i) = F^{-1} \Gamma_i F = (-1)^{\delta_{i\in\{2,4\}}} \Gamma_i$$ \hfill (C.3)

which applies to the $\text{Spin}(10)$ generators (as grade-2 elements of $\text{Cl}(10)$) as

$$f(\Gamma_{i\wedge j}) = F^{-1} \Gamma_{i\wedge j} F = (-1)^{\delta_{i\in\{2,4\}} + \delta_{j\in\{2,4\}}} \Gamma_{i\wedge j}.$$ \hfill (C.4)

Again, one can see $f \circ f = \text{id}$.

Given the embedding map $r_1' : U(5)_{1\text{st}} \to \text{Spin}(10)$ in Eqn. (B.20) and the flipping isomorphism $f$, the flipped embedding map $r_2' : U(5)_{2\text{nd}} \to \text{Spin}(10)$ can be defined as $r_2' = f \circ r_1'$, i.e. $r_2'(T_i) = f(r_1'(T_i))$, which enables us to compare the two embeddings $r_1'$ and $r_2'$ as in Table 8.

### C.2 Intersection and Join

Treat $u(5)_{1\text{st}}$ and $u(5)_{2\text{nd}}$ as Lie subalgebra of $\text{spin}(10)$. Their intersection is (see the highlighted rows in Table 8)

$$u(5)_{1\text{st}} \cap u(5)_{2\text{nd}} = \text{span}\{T_1, T_8, T_9, T_{10}, T_{11}, T_{18}, T_{19}, T_{20}, T_{21}, T_{22}, T_{23}, T_{24}, T_{25}\}$$

$$= \text{span}\{T_8, T_9, T_{10}, T_{18}, T_{19}, T_{20}, T_{22}, T_{23}\} \oplus \text{span}\{T_1, T_{11}, T_{21}\}$$

$$= \text{su}(3) \oplus \text{su}(2) \oplus \text{u}(1)_Y \oplus \text{u}(1)_X.$$ \hfill (C.5)

The join of $u(5)_{1\text{st}}$ and $u(5)_{2\text{nd}}$ (the $u(5)_{1\text{st}}$ and $u(5)_{2\text{nd}}$ together generate the minimal Lie algebra as their union, which should be understood as the minimal vector space generated by both Lie algebras together.
On the Lie group level, we have

\[
\begin{array}{c}
T_i \\
\hline
T_1 & r'_1(T_i) \text{ (as } r'_1 : u(5)_{1st} \to \text{spin}(10)) \\
T_2 & -r'_1(T_2) \text{ (as } r'_2 : u(5)_{2nd} \to \text{spin}(10)) \\
T_3 & \end{array}
\]

Table 8: Embeddings of \(u(5)_{1st}\) and \(u(5)_{2nd}\) (both of \(q = 2\)) in \(\text{spin}(10)\).

closed under any of their Lie brackets) is

\[
\begin{aligned}
\text{u}(5)_{1st} \cup \text{u}(5)_{2nd} &= (\text{u}(5)_{1st} \cap \text{u}(5)_{2nd}) \cup \text{span}\{\Gamma_i\}_{i \in \{1, 2, 3, 4\}, j \in \{5, 6, 7, 8, 9, 10\}} \\
&= \text{span}\{\Gamma_{i,j}\}_{i,j \in \{1, 2, 3, 4\}} \\
&= \text{spin}(10). \\
\end{aligned}
\]

On the Lie group level, we have

\[
\begin{array}{c}
\text{Spin}(10) \\
\hline
\text{U}(5)_{1st} & r'_1 \\
\hline
\text{U}(5)_{2nd} & r'_2 \\
\hline
\text{G}_\text{SM} \times \mathbb{Z}_5 \times \text{U}(1)_X
\end{array}
\]

where \(G_\text{SM} = (\text{SU}(3) \times \text{SU}(2)) \times \mathbb{Z}_6 \times \text{U}(1)_Y\). We have \(G_\text{SM} \times \mathbb{Z}_5 \times \text{U}(1)_X\) because that the \(\text{U}(1)_Y \times \mathbb{Z}_5 \times \text{U}(1)_X\) structure has a shared \(\mathbb{Z}_5 = \mathbb{Z}_{5,X} = \mathbb{Z}_{5,Y}\), see Table 1.
C.3 Charge Lattice

Define the $U(1)$ charges $Y_1 := r'_1(T_{24})$, $X_1 := r'_1(T_{25})$ in $U(5)_{1st}$, and $Y_2 := r'_2(T_{24})$, $X_2 := r'_2(T_{25})$ in $U(5)_{2nd}$. Based on Table 8, these charges are related by the flipping isomorphism $f$:

\[
\begin{pmatrix}
X_1 \\
Y_1
\end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & 4 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} X_2 \\
Y_2
\end{pmatrix},
\begin{pmatrix}
X_2 \\
Y_2
\end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & 4 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\
Y_1
\end{pmatrix}.
\]  
(C.8)

See also the discussions in Sec. 3.6.

The two sets of charge lattices intersect at points that matches the charge assignment of fundamental fermions in the SM, as shown in Fig. 9. Moreover, $U(1)_{X_1}$ and $U(1)_{X_2}$ shares a $\mathbb{Z}_4$ subgroup, because

\[
X_1 - X_2 = r'_1(T_{25}|_{q=2}) - r'_2(T_{25}|_{q=2})
\approx \text{diag}(0, 0, -4, -4, 4, 4, 4, 0, 0, 0, 0, 0, -4, 4)
\mod 4 = 0,
\]  
(C.9)

meaning that $e^{2\pi i} X_1 = e^{2\pi i} X_2$, which generates a $\mathbb{Z}_4$ group

\[
\mathbb{Z}_4 = \langle e^{2\pi i} X_1 \rangle = \{e^{2\pi i m} X_1 | m = 0, 1, 2, 3\}.
\]  
(C.10)

This is also the $\mathbb{Z}_4$ center of Spin(10).

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