Effect of Self-Magnetic Fields on the Nonlinear Dynamics of Relativistic Electron Beam with Virtual Cathode

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The report is devoted to the results of the numerical study of the virtual cathode formation conditions in the relativistic electron beam under the influence of the self-magnetic and external axial magnetic fields. The azimuthal instability of the relativistic electron beam leading to the formation of the vortex electron structure in the system was found out. This instability is determined by the influence of the self-magnetic fields of the relativistic electron beam and it leads to the decrease of the critical value of the electron beam current (current when the non-stationary virtual cathode is formed in the drift space). The typical dependencies of the critical current on the external uniform magnetic field value were discovered. The effect of the beam thickness on the virtual cathode formation conditions was also analyzed.

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I. INTRODUCTION

The analysis of nonlinear microwave oscillations and complex structure formation mechanisms in spatially extended systems with intensive beams of charged particles in the regimes of the virtual cathode (VC) formation attracts great attention of scientific community\(^1\). It is well known\(^2,4,10\) that the systems with VC are characterized by the complex dynamics and can demonstrate a wide range of nonlinear phenomena, including dynamical chaos and pattern formation\(^11-14\). Microwave generators using electron beams with VC (vircators\(^1,15\)) are perspective devices of high-power microwave electronics for the generation of the impulses of wide-band microwave radiation due to its high output microwave power, a simple construction (particularly vircators can operate without external focusing magnetic field), a possibility of a simple frequency tuning and regime switching\(^3,5,16-18\). All these circumstances increase the fundamental and applied importance of studies of the nonlinear dynamics of the electron beams with VC.

The oscillating VC is known to appear in an electron beam when the beam current exceeds a certain critical value, \(I_{cr}\) (space charge limiting (SCL) current\(^1,19\)), and the beam space charge is strong enough in order to form a potential barrier (VC) which reflects the electrons back to the injection plane. The mechanisms of the VC formation have been investigated in detail in case of the one-dimensional (1D) electron beam motion (fully magnetized beam\(^7,12,20-22\), with the critical beam current value being analytically defined for this case in Ref.\(^19\).

However, the use of 1D-theory to study the vircator systems is inefficient in many cases because it ignores a lot of important factors in the VC behavior and does not agree often with the experimental results. So, the development and use of 2D and 3D models for the analysis of the dynamics of electron beam with VC have attracted the great interest of scientific community recently\(^10,23-25\). The important problem in this field is the analysis of the VC formation conditions and nonlinear dynamics of electron structures in relativistic electron beams (REB) in the presence of finite external magnetic field or even without external magnetic field which focuses the REB. In particular, the studies of REBs with overcritical currents are necessary for the analysis of contemporary high-power devices with VC – relativistic vircators\(^5\) and ion acceleration systems\(^3,26\).

Analyzing REBs, it is necessary to take into account effects being insignificant for weakly
relativistic beams, in particular, the influence of the self-magnetic field of the REB that effects considerably on the system dynamics in case of ultra-relativistic electron beams. Therefore, the 3D fully electromagnetic self-consistent model of REB dynamics is required for accurate and correct analysis of the VC formation in this case.

The present report deals with the 3D numerical electromagnetic study of the VC nonlinear dynamics of the annular REB in the presence of an external finite uniform axial magnetic field. The structure of this paper is the following. Section II contains the brief formalism describing a 3D mathematical model for the nonlinear interaction simulation of electron beam with overcritical current and electromagnetic fields. Section III deals with the nonlinear dynamics of REB with overcritical current. We analyze the influence of the external magnetic field on the VC formation conditions and structure formation in REB and discuss the azimuthal REB instability which leads to the formation of a vortex electron structure. In Section IV the influence of the beam thickness on the VC formation conditions is analyzed. The conclusions of this paper are summarized in Section V.

II. GENERAL FORMALISM

The model under study consists of finite-length cylindrical waveguide region (electron beam drift chamber) with length $L$, radius $R$ and grid electrodes at both ends. An axially-symmetrical monoenergetic annular electron beam with the current $I$, electron energy $W_e$, radius $R_b$ and thickness $d$ is injected through the left (entrance) electrode. Electrons can leave the waveguide region by escaping through the right (exit) grid or by touching the side wall of the drift chamber. In the present work the values of geometric parameters were chosen following: $L = 40\, \text{mm}$, $R = 10\, \text{mm}$, $R_b = 5\, \text{mm}$, $d = 1.5\, \text{mm}$ (except for the Section IV where the influence of the beam thickness on the VC formation conditions is analyzed).

The external uniform magnetic field with induction $B_z = B_0$ is applied along the longitudinal axis of waveguide. The electron beam source is supposed to be magnetically unshielded in the considered model. This assumption means that the external magnetic field in the drift tube is equal to the magnetic field in the electron source region, therefore, the electron beam doesn’t acquire azimuthal velocity components at the injection plane (in accordance with Busch’s theorem\textsuperscript{27}). Such magnetic field distribution is typical for many devices of the
high-power electronics, particularly for magnetically isolated diodes that forms high-current REB\textsuperscript{27}.

Time-dependent fully 3D electromagnetic model of REB dynamics based on solving the self-consistent set of Maxwell equations and equations of charged particles motion accompanied by corresponding initial and boundary conditions (particle-in-cell method)\textsuperscript{28,29} is used in the present paper. The main equations are written as:

\[
\begin{align*}
\mathbf{rot} \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}, \\
\mathbf{rot} \mathbf{H} &= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j},
\end{align*}
\]

\[
\begin{align*}
\frac{d\mathbf{p}_i}{dt} &= \mathbf{E}_i + [\mathbf{p}_i, \mathbf{B}_i]/\gamma_i, \\
\frac{d\mathbf{r}_i}{dt} &= \mathbf{p}_i/\gamma_i, \\
i &= 1 \ldots N,
\end{align*}
\]

where \( \mathbf{E} \) and \( \mathbf{H} \) are the electric and magnetic intensities, \( \rho \) and \( \mathbf{j} \) are the charge and current densities, \( \mathbf{r}, \mathbf{p}, \gamma = (1 - (v/c)^2)^{-1/2}, \mathbf{v} \) are the radius vector, impulse, relativistic factor and velocity of the charged particles, correspondingly. The subscript \( i \) denotes the number of particle and \( N \) is the full number of particles used to simulate the charged particles beam.

The numerical simulation 3D scheme is based on 2.5D model developed in our previous work\textsuperscript{30}. The equations of charged particles motion (2) are used for electron beam simulation and solved numerically by means of Bóris algorithm\textsuperscript{31}. The longitudinal \( v_z \), radial \( v_r \) and azimuthal \( v_\theta \) velocity components are calculated with the help of this algorithm on each time step.

The electromagnetic fields in the drift chamber of REB are obtained by means of the numerical solution of the Maxwell’s equations (1) in cylindrical geometry on the shifted spatio-temporal meshes with constant spatial longitudinal, \( \Delta z \), radial, \( \Delta r \), and azimuthal, \( \Delta \theta \), steps and time step, \( \Delta t \).\textsuperscript{28,29} The values of steps of spatio-temporal meshes is picked out from the Courant-Friedrichs-Lewy condition.\textsuperscript{28,32} The every field component is calculated on the own mesh (see\textsuperscript{28,29}). The space charge and current densities on the meshes are calculated using a bilinear weighing procedure.\textsuperscript{28} To model the electromagnetic power output we use the approach\textsuperscript{30} based on the filling the section of electrodynamical system \( (L < z < 1.2L) \) with the conducting medium with the conductance \( \sigma \).
III. NONLINEAR DYNAMICS OF RELATIVISTIC ELECTRON BEAM WITH OVERCRITICAL CURRENT

A. Virtual cathode formation conditions

Let us consider the results of numerical simulation of VC formation features in the annular REB in the presence of the external magnetic field. Here we study the influence of the magnetic field on the critical current of REB (i.e., VC formation conditions) as well as on the nonlinear dynamics of the electron structures in the REB. External magnetic field, $B_0$, varies within the range $[0, 40]$ kGs.

![Amplitude spectrum of oscillations of self-electric field longitudinal component $E_z$ at the VC region for $I = 17$ kA, $B_0 = 0$](image)

FIG. 1. Amplitude spectrum of oscillations of self-electric field longitudinal component $E_z$ at the VC region for $I = 17$ kA, $B_0 = 0$

VC in the electron beam with overcritical current is characterized by the complex non-stationary spatial-temporal oscillations that leads to the excitation of vircator’s electrodynamic system, and, consequently, to the generation of high-power microwave radiation in such system. Fig. 1 demonstrates the numerically obtained typical amplitude spectrum of oscillations of self-electric field longitudinal component at the VC region in the considered vircator system. The carried out electrodynamic analysis has shown that frequencies of spectral components in Fig. 1 are determined by the corresponding eigenmodes of the electrodynamic system formed by the drift chamber walls. REB critical current is supposed to correspond to such value of the beam current when the spectral components appear at the output spectrum of vircator system and electrons start to reflect back to the injection plane $z = 0$. 

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Fig. 2a shows the dependencies of REB critical current, $I_{cr}$, on the external magnetic field value, $B_0$, for different beam energies, $W_e$. The insert panel in Fig. 2a demonstrates the corresponding critical current values for weakly relativistic electron beam with the initial electron energy $W_e = 79$ keV (see Ref. 25, 34). Analyzing this Figure, one can see that the curves $I_{cr}(B_0)$ for REB have characteristic feature in comparison with the weakly relativistic case when the critical current decreases monotonously approaching the minimal constant value with the growth of $B_0$.

![Graph showing dependencies of REB critical current on the external magnetic field induction value for different electron beam initial energy values $W_e$. Curve 1 corresponds to 480 keV, 2 - 600 keV, 3 - 850 keV, and 4 - 1 MeV. The insert panel demonstrates the dependency of the beam critical current on the magnetic field $B_0$ in the weakly relativistic case (beam energy $W_e = 79$ keV). (b) Illustration of the typical REB curve $I_{cr}(B_0)$ with denoted character values of the external magnetic field, where $B_{ch}^i$ is the value of the magnetic field when the critical beam current reaches the first minimum; $B_{max}^i$ is the magnetic field when the critical beam current reaches the local maximum and $B_{min}^i$ is the magnetic field when the curve $I_{cr}(B_0)$ saturates.]

The typical shape of REB critical current dependency on external magnetic field value is shown in Fig. 2b. As one can see from Fig. 2b, the REB critical current increases with
the growth of induction, $B_0$, within the range $B < B_{i\text{max}}^i$. With further growth of magnetic field for $B_0 > B_{i\text{max}}^i$ the REB critical current decreases and for $B_0 \sim B_{i\text{min}}^i$ the saturation at the constant level is observed. Electron beam appears to be fully magnetized in the case of the strong external magnetic field $B_0 > B_{i\text{min}}^i$ and moves generally one-dimensionally. Bogdankevich and Ruhadze were shown\textsuperscript{19} that the analytical expression for the critical current value in the case of 1D motion of electron beam has the following form:

$$I_{SCL} = \frac{c^3}{\eta} \frac{\left(\frac{\gamma_0^{2/3}}{3} - 1\right)^{3/2}}{d/R_b + 2 \ln(R/R_b)}, \quad \gamma_0 = \frac{1}{\sqrt{1 - \left(\frac{v_0}{c}\right)^2}}, \quad (3)$$

where $R$ is the radius of cylindrical drift tube, $R_b$, $d$ are the radius and thickness of annular REB, $\gamma_0$ and $v_0$ are the relativistic factor and velocity of REB electrons at injection plane, $c$ is the speed of light, and $\eta$ is the specific electronic charge. Table\textsuperscript{1} demonstrates the comparison of REB critical current values obtained analytically using relation (3) with the results of numerical simulation in the case of the strongly magnetized beam ($B_0 = 40$ kG) for the different values of the electron energy $W_e$. Analyzing Table\textsuperscript{1} one can see that the numerical results for REB critical currents agree accurately with the values obtained analytically, and, therefore, we can conclude that the developed numerical 3D fully electromagnetic model gives correct results. Note also, that the numerically obtained critical current value exceeds slightly the analytical results. It’s a consequence of the 2D dynamics of the electron beam occurring in the system even in the case of the strong external magnetic field, e.g. the cyclotron rotation of electrons, pulsations of the beam boundary et al. These 2D effects reduce the charge density in the VC area and, consequently, enlarge weakly the critical current value in comparison with 1D estimation (3) which doesn’t take into account these effects.

It should be noted that the effects of the decrease of the space charge limiting current in comparison with 1D electrostatic case\textsuperscript{3}\textsuperscript{15} due to the electromagnetic transients in the injected currents defined by the inductive voltage, $LdI/dt$, and the power loss in the drift chamber through the open boundary have been reported earlier\textsuperscript{35,36}. However, these phenomena do not play the significant role for the system under consideration. Actually, the inductive transient effect is pronounced only for the short drift chambers ($W/L > 1$, where $W = 2R_b$ is the beam diameter, and $L$ is the drift chamber length)\textsuperscript{35}, while for the considered system $W/L = 0.25$. The power loss is also negligible in our simulations, so its influence on the critical current value is insignificant in the considered case\textsuperscript{36}.\textsuperscript{7}
TABLE I. Critical current values obtained analytically using relation (3) \( I_{\text{an}}^{\text{cr}} \) (third column) and numerically \( I_{\text{num}}^{\text{cr}} \) (fourth column) for the different values of the beam energy \( W_e \) (first column); second column shows corresponding values of \( v_0/c \); \( B_0 = 40 \text{ kG} \).

| \( W_e \) keV | \( v_0/c \) | \( I_{\text{an}}^{\text{cr}} \) kA | \( I_{\text{num}}^{\text{cr}} \) kA |
|---------------|-------------|--------------|--------------|
| 480           | 0.8571      | 4.82         | 4.93         |
| 600           | 0.8882      | 6.45         | 6.61         |
| 850           | 0.9270      | 10.01        | 10.12        |
| 1000          | 0.9412      | 12.11        | 12.23        |

\( ^a \) Analytical results
\( ^b \) Numerical results

Thereby, the carried out analysis has shown (Fig. 2a) that REB critical current curves have local maxima at the certain external magnetic field value \( B_0 = B_{\text{max}}^{i} \) dependent on the beam energy \( W_e \). Such behavior is not observed in the weakly relativistic electron beams and develops with the electron beam energy growth.

B. Structure formation in REB with overcritical current

The observed behavior of the REB critical current \( I_{\text{cr}}(B_0) \), (see Fig. 2a) is determined by the influence of the self-magnetic fields of REB on the space charge dynamics and electron structures formation. Therefore, this effect is more pronounced in the case of the weak external magnetic field and high energy of the injected electron beam \( W_e > 600 \text{ keV} \), when the influence of the self-magnetic fields of REB is stronger. Let us consider the effect of electron structure formation in the presence of the REB self-magnetic field in detail.

Figures 3–5 show the typical phase portraits of the electron beam which are presented by the projections of the instantaneous positions (black dots in figures) of charged particles of the beam at the longitudinal \( (z, r) \) and transverse planes \( (r, \theta) \) for the beam currents being close to the critical values \( I_{\text{cr}}(B_0) \) (\( I \gtrsim I_{\text{cr}}(B_0) \)) and the different characteristic values of the external magnetic field, \( B_0 \). Fig. 3 corresponds to the case \( B_0 < B_{\text{max}}^{i} \) when the external magnetic field value is less than magnetic field \( B_{\text{max}}^{i} \) corresponding to the local maximum on curve 1 in Fig. 2a, Fig. 4 — to the case of \( B_0 \sim B_{\text{max}}^{i} \) and Fig. 5 — to the case when
the magnetic field corresponds to the range of saturation of $I_{cr}(B_0)$-curve on the constant level (i.e., $B_0 \gg B_{\text{min}}^1$).

**FIG. 3.** Projections of the instantaneous positions of the electron beam charged particles at the plane $(z,r)$ (left figures) and plane $(r,\theta)$ for $z_s = 6\text{ mm}$ (right figures) at the consecutive time points $t_1$ and $t_2$ ($t_2 - t_1 = 0.1\text{ ns}$) for $B_0 = 3\text{ kGs}$, $I = 7.5\text{ kA}$; $W_e = 480\text{ keV}$. Only particles behind the projection plane ($z > z_s$) are shown in Figures. The vertical dashed line with coordinate $z_s$ in figures $a$ and $c$ denotes the projection plane for figures $b$ and $d$)

The space charge dynamics of the ultra-relativistic electron beam at the VC region differs considerably in comparison with the weakly relativistic case for the weak external magnetic field $0 \leq B_0 \leq B_{\text{max}}^i$ (see Fig. 2b). This major difference is determined by the effect of the instability of REB found for the wide range of beam currents in the case when $0 \leq B_0 \leq B_{\text{max}}^i$. The observed instability, the so-called azimuthal instability of REB, leads to the axial symmetry loss of the electron beam dynamics.

This azimuthal instability of initially axially symmetrical beam in cylindrical drift chamber facilitates the formation of the vortex electron structure in the drift space. This instability is caused by the influence of the self-magnetic fields of the REB (see Fig. 3). Actually, there is the intensive transverse beam current in the system as a result of the beam divergence due to Coulomb’s repulsion forces action in the presence of the weak external magnetic field which does not restrain these forces. This transverse current results in the
FIG. 4. Similar to Fig. 3 but for the following parameters: $B_0 = 5$ kGs, $I = 9$ kA; $W_e = 480$ keV; $t_2 - t_1 = 0.2$ ns

appearance of the longitudinal self-magnetic field, $B_s^z$, that causes the azimuthal Lorentz force action on the electrons moving in the transverse direction. Therefore, these electrons get the azimuthal velocities and REB starts to rotate around symmetry axis of the drift space, $r = 0$. Centrifugal force acts on rotating electrons and, consequently, the vortex electron structure is formed in the beam that results in the strong azimuthal asymmetry of the REB. The formed vortex structure rotates in the drift tube as can be clearly seen by comparing the configurational portraits in Fig. 3a and c for two consecutive time moments. The physical mechanism of the azimuthal instability development has similar features with the well-known convective instability of the electron beam in the finite external magnetic field that arises due to the non-uniform distribution of the beam electrons density or velocity along the radial direction. However, the azimuthal instability in the considered case arises due to interaction of the beam with the self-magnetic, rather than external magnetic field.

The azimuthal instability leads to the decrease of the REB critical current due to the decrement of the longitudinal velocity, $v_z$, and the increase of the velocity spread of electrons in the region of the formation of the vortex electron structure. As a consequence, VC is formed in the region of the vortex structure onset, where the REB space charge density is
FIG. 5. Similar to Fig. 3 but for the following parameters: $B_0 = 30 \text{kGs}$, $I = 10 \text{kA}$; $W_e = 480 \text{keV}$; $t_2 - t_1 = 0.2 \text{ns}$.

maximal due to a large number of electrons with low energy.

Note, generally the vortex structure may be formed in the system without the VC onset. The azimuthal instability is developed in the REB when its current is greater than a certain threshold value. So, if the REB current exceeds this threshold value and it is less than the critical current (when the VC is observed in the system) only the vortex structure is formed in the system.

With the growth of the external magnetic field value the azimuthal instability becomes to be suppressed by the focusing force of the external magnetic field which is directed oppositely to the centrifugal force. Fig. 4 corresponding to the case of $B_0 \sim B_{max}^1$ (see Fig. 2a) demonstrates clearly this effect. Actually, the rotating vortex electron structure (see Fig. 3b and d) disappears in the drift space and the electron beam fills the whole space without the essential azimuthal inhomogeneity. The space charge density decreases and, as a consequence, the critical current for the VC formation increases in this case. So, the suppression of the REB azimuthal instability in the applied external magnetic field leads to the growth of the REB critical current with the increase of the external magnetic field $B_0$.

Note, for the ultra-relativistic electron beams with energy $W_e > 0.7 \text{MeV}$ this increase starts simultaneously with the growth of the magnetic field $B_0 > 0$ (curves 3 and 4 in Fig. 2a),
whereas for the beams with the less energy \((W_e \approx (0.4 \div 0.7) \text{ MeV})\) the increase of the REB critical current starts for the values of the external magnetic field \(B_0 \geq B_{ch}^i\) (see curves 1 and 2 in Fig. 2a).

A further growth of the magnetic field \((B_0 > B_{max}^i)\) leads to the decrease of the REB critical current value and its saturation at constant level for \(B_0 \sim B_{min}^i\) similarly to the weakly relativistic case\(^{25,34,40}\). Such behavior of \(I_{cr}(B_0)\) is determined by the mechanism of the charged particle transversal dynamics limitation by means of the external magnetic field (see Fig. 5). As a consequence, the space charge density of the REB also increases with the growth of the external magnetic field, and the critical beam current decreases. When the value of the external magnetic field induction equals to \(B_{min}^i\) the REB transversal dynamics appears to be completely suppressed. Fig. 5 shows that the transversal beam dynamics is not observed in the system in the case of the strong magnetic field \(B_0 \geq B_{min}^i\).

The mentioned difference in the curves behavior in Fig. 2 (cf. curves 1 and 2 with curves 3 and 4) for the case of weak external magnetic fields \(0 \leq B_0 \leq B_{ch}^i\) is the consequence of the competition of two described above physical mechanisms occurring in the system with the growth of the external magnetic field. The first mechanism (the suppression of the azimuthal instability of the REB) leads to the critical current value increase. The second one, which is connected with the charged particle transversal dynamics limitation in external focusing magnetic field, conducts on the contrary to the decrease of critical current. For the electron beams with energy \(W_e \approx (0.4 \div 0.7) \text{ MeV}\) (see curves 1 and 2 in Fig. 2a) the both mechanisms have an important role: the second mechanism determines the behavior of the dependencies for small values of external magnetic field \(0 \leq B_0 \leq B_{ch}^i\), and the first one begins to dominate for \(B_{ch}^i \leq B_0 \leq B_{max}^i\). In the case of ultra-relativistic electron beams with energy \(W_e > 0.7 \text{ MeV}\) when self-magnetic fields are much stronger, the formation of VC is determined mainly by the mechanism of vortex electron structure formation, whereas the suppression of REB transversal dynamics with the growth of external magnetic field has a minor role. So, the first physical mechanism dominates the second one considerably. As a consequence the critical current dependencies in this case (see curves 3 and 4 in Fig. 2a) demonstrate the immediate increase with the external magnetic field growth \((0 \leq B_0 \leq B_{max}^i)\).

Taking into account that the external magnetic field \(B_0 \geq B_{min}^i\) is strong enough to neglect the influence of the self-magnetic fields, the value \(B_{min}^i\) may be easily estimated
analytically. Let the REB with current $I_{cr}$ has the radius $R_b$ at the injection plane and the character radius $R_{VC}$ — at the VC region ($R_{VC} > R_b$ as a result of Coulomb’s repulsion of electrons of beam; the value of $R_{VC}$ is taken from the simulation). Moving in the system between the points with radii $R_b$ and $R_{VC}$ in the presence of the external magnetic field, the electrons acquire the angular momentum. This momentum is proportional to the difference of induction fluxes across the cross-sections of the REB at the points with radii $R_b$ and $R_{VC}$, respectively:

\[ R_{VC}^2 \frac{d\theta}{dt} = \frac{\pi \eta B_0}{2 \pi \gamma_0} \left( R_{VC}^2 - R_b^2 \right), \]  

(4)

where $d\theta/dt$ is the azimuthal velocity of electrons. The motion of electrons of REB is determined by the action of centrifugal force $F_c = \gamma_0 m_e r \left( \frac{d\theta}{dt} \right)^2$, Coulomb’s repulsion force $F_k = -eE_r$ and Lorentz force $F_L = -er(\frac{d\theta}{dt})B$ (here $e$ and $m_e$ are the charge and the mass of the electron, respectively, $r$ is the radial coordinate of electron, $E_r$ is the radial component of the space charge field intensity). Then, one can write the motion equation for the boundary electron of the beam, taking into account the above, the relation (4) and equation $d^2 r/dt^2 = (2\eta V_0/\gamma_0)d^2 r/dz^2$:

\[ \frac{d^2 r}{dz^2} + \frac{\eta B_0^2}{8 V_0 \gamma_0} R_{VC} \left[ 1 - \left( \frac{R_b}{R_{VC}} \right)^4 \right] - \frac{I \sqrt{\gamma_0}}{4 \pi \varepsilon_0 \sqrt{2\eta V_0^{3/2} R_{VC}}} = 0, \]  

(5)

where $V_0$ is the accelerating voltage.

Eq. (5) implies that there is such external magnetic field, $B_{min}^i$, for which REB keeps the constant radius in the system. Actually, if we put $d^2 r/dz^2 = 0$ in Eq. (5) (it means the lack of the acceleration in the radial direction), we obtain the quadratic equation for $B_0$. The solution of this equation gives the required value of the external magnetic field $B_{min}^i$ when the dependency $I_{cr}(B_0)$ saturates (see Fig. 2):

\[ B_{min}^i = R_{VC} \sqrt{\frac{2I \gamma_0^{3/2}}{\pi \varepsilon_0 \eta^{3/2} \sqrt{V_0 (R_{VC}^4 - R_b^4)}}}. \]  

(6)

Fig. 6 demonstrates analytical curve (6) (solid line) and the numerically calculated values (points) of the external magnetic field $B_{min}^i$ for the different beam energies $W_e$. One can see that relation (6) describes the results of the numerical simulation correctly and gives the accurate values of the magnetic field $B_{min}^i$ being close to the corresponding numerical results. The value $B_{min}^i$ increases with the growth of the beam energy, $W_e$, due to the following reason. We would remind that $B_{min}$ is such external magnetic field value for
which the transversal dynamics of electron beam is completely suppressed in the system. Obviously, the electron inertia increases with the growth of the beam energy, $W_e$, due to the electron velocities increase and, as a consequence, the relativistic growth of the electron mass is observed. As a result, the greater external magnetic field value is required for the electrons in the system to be focused (the limitation of the beam transversal dynamics), therefore, $B_{\text{min}}$ increases monotonously with the growth of the REB energy. The behavior of the dependency $B_{\text{min}}(W_e)$ on the energy $W_e$ is qualitatively similar to the weakly and strong relativistic cases (see Fig. 6).

IV. EFFECT OF THE ELECTRON BEAM THICKNESS ON THE VIRTUAL CATHODE FORMATION CONDITIONS

In the previous sections we have analyzed the dynamics of a thin annular REB with a fixed thickness, $d$ (see Section II). Let us consider here the results of numerical simulation of the influence of the beam thickness, $d$, on the critical current of REB as well as on the dynamics of the vortex structure in the system.

Fig. 7 illustrates the dependency of the REB critical current on the beam thickness $d$. One can see that the critical current value decreases monotonously with the growth of $d$. More homogeneous filling of the drift tube by the electron beam with the increasing of the beam thickness results in greater potential sagging in the beam drift space. As a
FIG. 7. Dependency of the REB critical current on the normalized beam thickness $d/R_b$ for the external magnetic field induction value $B_0 = 3 \text{kGs}$ and the electron beam initial energy value $W_e = 480 \text{keV}$.

Consequently, the beam critical current decreases with the growth of the beam thickness that agrees qualitatively with the analytical dependency of the critical current value on the beam thickness in the case of 1D motion of electron beam [3]–[9].

FIG. 8. Projections of the instantaneous positions of charged particles of the solid electron beam at the transverse plane $(r, \theta)$ for $z_s = 9.5 \text{mm}$ at the consecutive time points $t_1$ and $t_2$ ($t_2 - t_1 = 0.2 \text{ns}$) for $B_0 = 3 \text{kGs}$ (upper figures) and $B_0 = 30 \text{kGs}$ (lower figures); $I = 7.5 \text{kA}$, $W_e = 480 \text{keV}$. Only particles behind the projection plane ($z > z_s$) are shown in Figures.
Fig. 8 shows the typical projections of the instantaneous positions of charged particles of the solid REB at the transverse plane \((r, \theta)\) for the beam current being greater the critical value and the different characteristic values of the external magnetic field, \(B_0\). Analyzing these figures, one can see that the vortex structure in the solid REB is formed (see Fig. 8a, b) similarly to the case of the annular electron beam in the weak external magnetic fields (see Section III B; compare Fig. 8a, b and Fig. 3). Let us note that the obtained results for the solid REB agree qualitatively with the experimental investigation of the intense REB carried out in the Ref.\textsuperscript{41} where the tracings of witness place damage patterns (see Fig. 6 (left panel) in Ref.\textsuperscript{41}) demonstrate the presence of the vortex structure in the solid beam in the case of weak external focusing magnetic field. The same result was obtained for different beam thicknesses. In particular, with increase of external magnetic field we observe the suppression of the vortex pattern formation (see Fig. 8c, d) which is also in good agreement with an experiment (see Fig. 6 (right panel) in Ref.\textsuperscript{41}).

V. CONCLUSION

Summarizing the obtained results, it is obvious that the VC formation and electron structure dynamics in the REB differ considerably in the cases of the weakly and ultra relativistic beams. We shown that the nonlinear dynamics of the REB in the weak external magnetic fields is defined by the self-magnetic field of REB which leads to the azimuthal instability and the vortex electron structure formation. Such behavior reduces the value of the REB critical current. The growth of the external magnetic field causes the suppression of the azimuthal instability and, consequently, the critical current increases. In the case of the strong external magnetic field the critical current value decreases and saturates similarly to the weakly relativistic case\textsuperscript{25,34}. The beam thickness does not effect considerably on the process of the vortex structure formation, however, the growth of the beam thickness leads to the decrease of the beam critical current.

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