Stability of a Ramanujan Type Additive Functional Equation

Arun Kumar M and Sathya E

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Abstract

In this paper, the authors achieve the generalized Ulam - Hyers stability of a Ramanujan Type Additive Functional Equation in Paranormed Spaces and Modular spaces via classical Hyers Method.

Key words:  Additive Functional Equations, Ulam - Hyers Stability, Paranormed Spaces, Modular Spaces.

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1 Introduction

The stability of a functional equation initiated from a question raised by Ulam: when is it true that the solution of an equation differing slightly from a given one must of necessity be close to the solution of the given equation? (see [37]). The first answer (in the case of Cauchys functional equation in Banach spaces) to Ulams question was given by Hyers in [11].

Following his result, a abundant number of papers on the stability problems have been extensively available as generalizing Ulams problem and Hyers theorem in various directions; see for instance [8, 10, 28, 29, 31], and the references given there.

Notice that certain results on the stability of various several functional equations can be establish in [11, 41, 5, 6, 7, 12, 13, 14, 50, 32].

1Department of Mathematics, Government Arts College, Tiruvannamalai - 606 603, TamilNadu, India.
2Department of Mathematics, Shannuga Industries Arts and Science College, Tiruvannamalai, TamilNadu, India.
Emails: *1drarunmaths@gmail.com, 2sathya24mathematics@gmail.com
The well known **Cauchy - Additive Functional Equation** is

\[
R(x + y) = R(x) + R(y).
\]  

### 1.1 Ramanujan Numbers

The world of mathematics is renowned for a number of interesting and fascinating numbers. Now Ramanujan Number also makes such a place in the list. Ramanujan Numbers (preciously termed as Hardy-Ramanujan Numbers) are those numbers that are the smallest positive integers that can be represented or expressed as a sum of 2 positive integers in \( n \) ways. Now lets discuss the above ways in a mathematical way.

#### 1.2 Ramanujans 1 - Way Solution

Integers that are expressed as the sum of 2 cubes (in at least one way). Some of these numbers include:

\[
\{2, 9, 16, 28, 35, 54, 65, 72, 91, 126, 128, 133, 152, 189, 217, 224, 243, 250, 280, 341, 344, 351, 370, 407, 432, 468, 513, 520, 539, 559, 576, 637, 686, 728, 730, 737, \cdots \}.
\]

It is easy to verify that \( 2 = 1^3 + 1^3 \), \( 9 = 2^3 + 1^3 \), \( 16 = 2^3 + 2^3 \cdots \). Here all these numbers can be expressed as a sum of 2 cubes in a single way and so all these numbers from the above set can be expressed in this way.

The Ramanujans 1 - way solution can be converted to **Ramanujan Additive Functional Equation** of the form

\[
R(\alpha_1^{3}x_1 + \beta_1^{3}y_1) = \alpha_1^{3}R(x_1) + \beta_1^{3}R(y_1).
\]  

**Theorem 1.1** Assume \( W_1 \) and \( W_2 \) be real vector spaces. Suppose \( R : W_1 \to W_2 \) satisfies the functional equation \( [1] \) if and only of \( R : W_1 \to W_2 \) satisfies the functional equation \( [2] \).

Proof: With the help of oddness of \( R \) and additiveness the proof is trival.

In this paper, the authors achieve the generalized Ulam - Hyers stability of a Ramanujan Type Additive Functional Equation \( [2] \) in Paranormed Spaces and Modular spaces via classical Hyers Method.
2 Basic Concepts And Stability on Paranormed Spaces

Now, we give to adopt the usual terminologies, notations, definitions and properties of the theory of paranormed spaces given in [8, 9, 15, 18, 25, 33, 35].

**Definition 2.1** Let $X$ be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on $X$ such that

(P1) $P(0) = 0$;
(P2) $P(-x) = P(x)$;
(P3) $P(x + y) \leq P(x) + P(y)$ (triangle inequality);
(P4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair $(X, P)$ is called a **paranormed space** if $P$ is a paranorm on $X$.

**Definition 2.2** Let $X$ be a paranormed space and let $\{x_n\}$ be a sequence in $X$ then $\{x_n\}$ is called Cauchy if for any $\epsilon > 0$ if $P(x_n - x_m) \to 0$ for sufficiently large $m, n \in \mathbb{N}$.

**Definition 2.3** The paranorm is called **total** if, in addition, we have

(P5) $P(x) = 0$ implies $x = 0$.

**Definition 2.4** A **Fréchet space** is a total and complete paranormed space.

**Definition 2.5** A complete normed linear space is called **Banach space**.

3 Basic Concepts And Stability on Modular Spaces

Now, we give to adopt the usual terminologies, notations, definitions and properties of the theory of modular spaces given in [2, 19, 20, 21, 23, 22, 24, 26, 27, 36, 39].

**Definition 3.1** Let $X$ be a linear space over a field $K(R or C)$. We say that a generalized functional $\rho : X \to [0, \infty]$ is a modular if for any $x, y \in X$,.
(MS1) $\rho(x) = 0$ if and only if $x = 0$;
(MS2) $\rho(\alpha x) = \rho(x)$ for all scalar $\alpha$ with $|\alpha| = 1$;
(MS3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all scalar $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.
(MS4) If (MS3) is replaced by $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all scalar $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, then the functional $\rho$ is called a convex modular.

**Definition 3.2** A modular $\rho$ defines the following vector space:

$$X_\rho = \{ x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0 \},$$

and we say that $X_\rho$ is a modular space.

**Definition 3.3** Let $X_\rho$ be a modular space and let $\{x_n\}$ be a sequence in $X_\rho$ then $\{x_n\}$ is $\rho$–convergent to a point $x \in X_\rho$ and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n x) \to 0$ as $n \to \infty$.

**Definition 3.4** Let $X_\rho$ be a modular space and let $\{x_n\}$ be a sequence in $X_\rho$ then $\{x_n\}$ is called $\rho$–Cauchy if for any $\epsilon > 0$ one has $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$.

**Definition 3.5** Let $X_\rho$ be a modular space and let $\{x_n\}$ be a sequence in $X_\rho$. A subset $K \subseteq X_\rho$ is called $\rho$–complete if any $\rho$–Cauchy sequence is $\rho$–convergent to a point in $K$.

It is said that the modular $\rho$ has the Fatou property if and only if $\rho(x) \leq \liminf \rho(x_n)$ whenever the sequence $\{x_n\}$ is $\rho$–convergent to $x$ in modular space $X_\rho$.

**Theorem 3.6** In modular spaces,

1. if $x_n \xrightarrow{\rho} x$ and $a$ is a constant vector, then $x_n + a \xrightarrow{\rho} x + a$, and
2. if $x_n \xrightarrow{\rho} x$ and $y_n \xrightarrow{\rho} y$ then $\alpha x_n + \beta y_n \xrightarrow{\rho} \alpha x + \beta y$, where $\alpha + \beta \leq 1$ and $\alpha, \beta \geq 0$.

**Definition 3.7** A modular $\rho$ is said to satisfy the $\Delta_2$–condition if there exists $k > 0$ such that $\rho(Gx) \leq k \rho(x)$ for all $x \in X_\rho$.

**Remark 3.8** Suppose that $\rho$ is convex and satisfies $\Delta_2$–condition with $\Delta_2$–constant $k > 0$. If $k < \Gamma$, then $\rho(x) \leq k \rho(x) \leq k \frac{1}{\Gamma} \rho(x)$, which implies $\rho = 0$. Therefore, we must have the $\Delta_2$–constant $k \geq \Gamma$ if $\rho$ is convex modular.
On the other hand, many authors have investigated the stability using fixed point theorem of quasicontraction mappings in modular spaces without $\Delta_2-$ condition, which has been introduced by Khamsi [17]. Recently, the stability results of additive functional equations in modular spaces equipped with the Fatou property and $\Delta_2-$condition were investigated in H. M. Kim, H. Y. Shin [16] and Sadeghi [34] who used Khamsis fixed point theorem. Also the stability of quadratic functional equations in modular spaces satisfying the Fatou property without using the $\Delta_2-$condition was proved by Wongkum, Chaipunya and Kumam [38].

4 Stability Theorem: Paranormed Spaces

To prove stability results, in this section let us take $(\mathcal{N}, P)$ be a Fréchet space and $(\mathcal{M}, || \cdot ||)$ be a Banach space.

**Theorem 4.1** If $R : \mathcal{N} \to \mathcal{M}$ is a function satisfying the inequality

$$P \left( R(\alpha^3 x_1 + \beta^3 y_1) - \{\alpha^3 R(x_1) + \beta^3 R(y_1)\} \right) \leq B(x_1, y_1); \quad \forall \ x_1, y_1 \in \mathcal{N}. \quad (3)$$

Then there exists a unique additive mapping $R_{bn} : \mathcal{N} \to \mathcal{M}$ which satisfying the functional equation (2) and the inequality

$$P \left( R_{bn}(y) - R(y) \right) \leq \frac{1}{\Gamma} \sum_{a=\frac{1-b}{2}}^{\infty} \frac{1}{\Gamma^{ab}} B(\Gamma^{ab} y, \Gamma^{ab} y); \quad \forall \ y \in \mathcal{N}, \quad (4)$$

where $\Gamma = (\alpha^3 + \beta^3); \ b = \pm 1$ and $R_{bn}(y)$ is given by

$$P \left( \lim_{c \to \infty} \frac{R(\Gamma^{bc} y)}{\Gamma^{bc}} - R_{bn}(y) \right) \to 0; \quad \forall \ y \in \mathcal{N}. \quad (5)$$

The mapping $B : \mathcal{N}^2 \to [0, \infty)$ fullfilling the condition

$$\lim_{c \to \infty} \frac{1}{\Gamma^{bc}} B(\Gamma^{bc} x_1, \Gamma^{bc} y_1) = 0; \quad \forall \ x_1, y_1 \in \mathcal{N}. \quad (6)$$

Proof: Fixing $x_1 = y_1 = y$ in (3), we obtain

$$P \left( R(\Gamma y) - \Gamma R(y) \right) \leq B(y, y); \quad \forall \ y \in \mathcal{N}. \quad (7)$$

It follows from (7) and (P3) for a positive integer $c$, we arrive
\[
P \left( \frac{1}{\Gamma_c} R(\Gamma^c y) - R(y) \right) = P \left( \sum_{a=0}^{c-1} \frac{1}{\Gamma^{a+1}} \left[ \Gamma R(\Gamma^a y) - R(\Gamma^{a+1} y) \right] \right)
\leq \sum_{a=0}^{c-1} \frac{1}{\Gamma^{a+1}} P \left( \Gamma R(\Gamma^a y) - R(\Gamma^{a+1} y) \right)
\leq \frac{1}{\Gamma} \sum_{a=0}^{c-1} \frac{1}{\Gamma^a} B(\Gamma^a y, \Gamma^a y); \quad \forall \ y \in \mathcal{N}.
\]

(8)

It is ensure that the sequence
\[
\left\{ \frac{1}{\Gamma_c} R(\Gamma^c y) \right\}
\]
is a Cauchy sequence in \( M \). Since \( M \) is complete, there exists a limit function \( \mathcal{R}_n : \mathcal{N} \rightarrow M \) defined by
\[
P \left( \lim_{c \rightarrow \infty} \frac{R(\Gamma^c y)}{\Gamma^c} - \mathcal{R}_n(y) \right) \rightarrow 0; \quad y \in \mathcal{N}.
\]

(9)

Also, by continuity of multiplication, we have
\[
P \left( \lim_{c \rightarrow \infty} \frac{t_c R(\Gamma^c y)}{\Gamma^c} - t \mathcal{R}_n(y) \right) \rightarrow 0; \quad y \in \mathcal{N}.
\]

Indeed, suppose if we replace \( y \) as \( \Gamma^d y \) and divided by \( \Gamma^d \) in (8), we get
\[
P \left( \frac{1}{\Gamma_c \Gamma^d} R(\Gamma^c \Gamma^d y) - \frac{1}{\Gamma^d} R(\Gamma^d y) \right) = \frac{1}{\Gamma^d} P \left( \frac{1}{\Gamma_c} R(\Gamma^c \Gamma^d y) - R(\Gamma^d y) \right)
\leq \frac{1}{\Gamma} \sum_{a=0}^{c-1} \frac{1}{\Gamma^{a+d}} B(\Gamma^{a+d} y, \Gamma^{a+d} y)
\rightarrow 0 \text{ as } d \text{ to } \infty.
\]

(10)

Letting \( p \rightarrow \infty \) in (8) and using (9), we arrive (4). If we replacing \( (x_1, y_1) \) as \( (\Gamma^c x_1, \Gamma^c y_1) \) and divided by \( \Gamma^c \) in (3), we arrive
\[
P \left( \frac{1}{\Gamma^c} R \left( \alpha_1^3 \Gamma^c x_1 + \beta_1^3 \Gamma^c y_1 \right) - \alpha_1^3 R(\Gamma^c x_1) - \beta_1^3 R(\Gamma^c y_1) \right) \leq \frac{1}{\Gamma^c} B(\Gamma^c y_1, \Gamma^c y_2)
\]
for all \( x_1, y_1 \in \mathcal{N} \). Letting \( c \rightarrow \infty \) in the above inequality and using (9), (P5) and we see that \( \mathcal{R}_n(y) \) satisfies the Ramanujan Type additive functional equation (2). It is clear that the existence of \( \mathcal{R}_n(y) \) is unique.
Indeed, suppose that assume $R_1(y)$ be another additive mapping satisfying (2) and (9). So, one can easy to verify that for a positive integer $d$, we observe

$$R_1(\Gamma^d y) = \Gamma^d R_1(y); \quad R_n(\Gamma^d y) = \Gamma^d R_n(y)$$

(11)

and

$$R_1(y) = \frac{1}{\Gamma^d} R_1(\Gamma^d y); \quad R_n(y) = \frac{1}{\Gamma^d} R_n(\Gamma^d y)$$

(12)

for all $y \in \mathcal{N}$. Now,

$$P (R_n(y) - R_1(y))$$

$$= P \left( \frac{1}{\Gamma^d} R_n(\Gamma^d y) - \frac{1}{\Gamma^d} R_1(\Gamma^d y) \right)$$

$$\leq P \left( \frac{1}{\Gamma^d} R_n(\Gamma^d y) - \frac{1}{\Gamma^d} R(\Gamma^d y) \right) + P \left( \frac{1}{\Gamma^d} R(\Gamma^d y) - \frac{1}{\Gamma^d} R_1(\Gamma^d y) \right)$$

$$\leq \frac{1}{\Gamma^d} P (R_n(\Gamma^d y) - R(\Gamma^d y)) + \frac{1}{\Gamma^d} P (R(\Gamma^d y) - R_1(\Gamma^d y))$$

$$\leq 2 \sum_{a=0}^{\infty} \frac{1}{\Gamma^a \Gamma^d} B(\Gamma^{ab} \Gamma^d y, \Gamma^{ab} \Gamma^d y)$$

$$\rightarrow 0 \text{ as } d \text{ to } \infty.$$  

(13)

for all $y \in \mathcal{N}$. Again, taking $y$ as $\frac{y}{\Gamma}$ in (7), we have

$$P \left( R(y) - \Gamma R \left( \frac{y}{\Gamma} \right) \right) \leq B \left( \frac{y}{\Gamma}, \frac{y}{\Gamma} \right); \quad \forall \ y \in \mathcal{N}.$$  

(14)

In widely for a positive integer $p$, we arrive

$$P \left( R(y) - \Gamma^c R \left( \frac{y}{\Gamma^c} \right) \right) \leq \frac{1}{\Gamma} \sum_{a=1}^{bc} \Gamma^a \Phi \left( \frac{y}{\Gamma^a}, \frac{y}{\Gamma^a} \right); \quad \forall \ y \in \mathcal{N}.$$  

(15)

The rest of proof is similar to that of previous one. This completes the proof of the theorem.
Corollary 4.2 If $R : \mathcal{N} \to \mathcal{M}$ is a function satisfying the inequality

$$P \left( R \left( a_1^3 x_1 + b_1^3 y_1 \right) - \left\{ a_1^3 R(x_1) + b_1^3 R(y_1) \right\} \right) \leq \left\{ \begin{array}{ll}
T, \\
T \left\{ P(x_1)^r + P(y_1)^r \right\}, \\
T \left\{ P(x_1)^{r_1} + P(y_1)^{r_2} \right\}, \\
T \ P(x_1)^r P(y_1)^r
\end{array} \right. \tag{16}$$

for all $x_1, y_1 \in \mathcal{N}$ and $T > 0$. Then there exists a unique additive mapping $\mathcal{R}_{bn} : \mathcal{N} \to \mathcal{M}$ which satisfying the functional equation (2) and the functional inequality

$$P \left( \mathcal{R}_{bn}(y) - R(y) \right) \leq \left\{ \begin{array}{ll}
\frac{T}{|\Gamma - 1|}; \\
\frac{2TP(y)^r}{|\Gamma - \Gamma^r|}; \\
\frac{TP(y)^{r_1}}{|\Gamma - \Gamma^{r_1}|} + \frac{TP(y)^{r_2}}{|\Gamma - \Gamma^{r_2}|}; \\
\frac{TP(y)\Gamma}{|\Gamma - \Gamma^\Gamma|};
\end{array} \right. \quad r \neq 1; \quad r_1, r_2 \neq 1; \quad 2r \neq 1; \tag{17}$$

for all $y \in \mathcal{N}$.

5 Stability Theorem: Without Using The $\Delta_2$– Condition

To prove stability results, in Sections 5 and 6, let us take $\mathcal{N}$ be an linear space and $\mathcal{M}_\rho$ be an $\rho$–complete convex modular space.

Theorem 5.1 Assume $\mathcal{M}_\rho$ satisfy the Fatou property. If $R : \mathcal{N} \to \mathcal{M}_\rho$ is a function satisfying the inequality

$$\rho \left( R \left( a_1^3 x_1 + b_1^3 y_1 \right) - \left\{ a_1^3 R(x_1) + b_1^3 R(y_1) \right\} \right) \leq B(x_1, y_1); \forall \ x_1, y_1 \in \mathcal{N}. \quad (18)$$

Then there exists a unique additive mapping $\mathcal{R}_n : \mathcal{N} \to \mathcal{M}_\rho$ which satisfying the functional equation (2) and the functional inequality

$$\rho \left( \mathcal{R}_n(y) - R(y) \right) \leq \frac{1}{\Gamma} \sum_{a=0}^{\infty} \frac{1}{\Gamma^a} B(\Gamma^a y, \Gamma^a y); \quad \forall \ y \in \mathcal{N},$$

(19)
where \( \Gamma = (\alpha^3 + \beta^3) \) and \( \mathcal{R}_n(y) \) is given by

\[
\lim_{c \to \infty} \rho \left( \frac{R(\Gamma^c y)}{\Gamma^c} - \mathcal{R}_n(y) \right) = 0; \quad \forall \ y \in N.
\]  

(20)

The mapping \( \mathcal{B} : N^2 \to [0, \infty) \) fulfilling the condition

\[
\lim_{c \to \infty} \frac{1}{\Gamma^c} \mathcal{B}(\Gamma^c x_1, \Gamma^c y_1) = 0; \quad \forall \ x_1, y_1 \in N.
\]  

(21)

Proof: Fixing \( x_1 = y_1 = y \) in (18), we obtain

\[
\rho \left( R(\Gamma y) - \Gamma R(y) \right) \leq \mathcal{B}(y, y); \quad \forall \ y \in N.
\]  

(22)

Without using the \( \Delta_2 \)-condition it follows from (22) and (MS3)x for a positive integer \( c \), we arrive

\[
\rho \left( \frac{1}{\Gamma^c} R(\Gamma^c y) - R(y) \right) = \rho \left( \sum_{a=0}^{c-1} \frac{1}{\Gamma^{a+1}} \left[ \Gamma R(\Gamma^a y) - R\left(\Gamma^{a+1} y\right) \right] \right)
\leq \sum_{a=0}^{c-1} \frac{1}{\Gamma^{a+1}} \rho \left( \Gamma R(\Gamma^a y) - R\left(\Gamma^{a+1} y\right) \right)
\leq \frac{1}{\Gamma} \sum_{a=0}^{c-1} \frac{1}{\Gamma^a} \mathcal{B}(\Gamma^a y, \Gamma^a y); \quad \forall \ y \in N.
\]  

(23)

It is ensure that the sequence

\[
\left\{ \frac{1}{\Gamma^c} R(\Gamma^c y) \right\}
\]

is a \( \rho \)-Cauchy sequence in \( M_{\rho} \). Since \( M_{\rho} \) is \( \rho \)-complete, there exists a \( \rho \)- limit function \( \mathcal{R}_1 : N \to M_{\rho} \) defined by

\[
\rho - \lim_{c \to \infty} \frac{R(\Gamma^c y)}{\Gamma^c} = \mathcal{R}_n(y) \ or \ \lim_{c \to \infty} \rho \left( \frac{R(\Gamma^c y)}{\Gamma^c} - \mathcal{R}_n(y) \right) = 0; \quad y \in N.
\]  

(24)
Indeed, suppose if we replace \( y \) as \( \Gamma^d y \) and divided by \( \Gamma^d \) in (23), we get
\[
\rho \left( \frac{1}{\Gamma^c \Gamma^d} R \left( \Gamma^c \Gamma^d y \right) - \frac{1}{\Gamma^d} R \left( \Gamma^d y \right) \right) = \frac{1}{\Gamma^d} \rho \left( \frac{1}{\Gamma^c} R \left( \Gamma^c \Gamma^d y \right) - R \left( \Gamma^d y \right) \right)
\]
\[
\leq \frac{1}{\Gamma} \sum_{a=0}^{c-1} \frac{1}{\Gamma^{a+d}} B \left( \Gamma^{a+d} y, \Gamma^{a+d} y \right)
\]
\[
\rightarrow 0 \quad \text{as} \quad d \quad \text{to} \quad \infty. \quad (25)
\]

It follows from the Fatou property that the inequality
\[
\rho \left( \mathcal{R}_n(y) - R(y) \right) \leq \liminf_{c \to \infty} \rho \left( \frac{R(\Gamma^c y)}{\Gamma^c} - \mathcal{R}_n(y) \right) \leq \frac{1}{\Gamma} \sum_{a=0}^{\infty} \frac{1}{\Gamma^a} B \left( \Gamma^a y, \Gamma^a y \right) ; y \in \mathcal{N}.
\]

Thus, we arrive (19). If we replacing \((x_1, y_1)\) as \((\Gamma^c x_1, \Gamma^c y_1, \ldots)\) and divided by \(\Gamma^c\) in (18), we arrive
\[
\rho \left( \frac{1}{\Gamma^c} \left\{ R \left( \alpha_1^3 \Gamma^c x_1 + \beta_1^3 \Gamma^c y_1 \right) - \alpha_1^3 R \left( \Gamma^c x_1 \right) - \beta_1^3 R \left( \Gamma^c y_1 \right) \right\} \right) \leq \frac{1}{\Gamma^c} B \left( \Gamma^c y_1, \Gamma^c y_2 \right)
\]
for all \(x_1, y_1 \in \mathcal{N}\). By convexity of \(\rho\) that
\[
\rho \left( \frac{1}{4} \mathcal{R}_n \left( \alpha_1^3 x_1 + \beta_1^3 y_1 \right) - \frac{1}{4} \alpha_1^3 \mathcal{R}_n \left( x_1 \right) - \frac{1}{4} \beta_1^3 \mathcal{R}_n \left( y_1 \right) \right)
\]
\[
\leq \frac{1}{4} \rho \left( \mathcal{R}_n \left( \alpha_1^3 x_1 + \beta_1^3 y_1 \right) - \frac{1}{\Gamma^c} R \left( \alpha_1^3 \Gamma^c x_1 + \beta_1^3 \Gamma^c y_1 \right) \right)
\]
\[
+ \frac{1}{4} \rho \left( -\mathcal{R}_n \left( \alpha_1^3 x_1 \right) + \frac{1}{\Gamma^c} R \left( \alpha_1^3 \Gamma^c x_1 \right) \right) + \frac{1}{4} \rho \left( -\mathcal{R}_n \left( \beta_1^3 y_1 \right) + \frac{1}{\Gamma^c} R \left( \beta_1^3 \Gamma^c y_1 \right) \right)
\]
\[
+ \frac{1}{4} \rho \left( \frac{1}{\Gamma^c} R \left( \alpha_1^3 \Gamma^c x_1 + \beta_1^3 \Gamma^c y_1 \right) - \frac{1}{\Gamma^c} R \left( \alpha_1^3 \Gamma^c x_1 \right) - \frac{1}{\Gamma^c} R \left( \beta_1^3 \Gamma^c y_1 \right) \right)
\]
for all \(x_1, y_1 \in \mathcal{N}\). Taking \(p\) tends to infinity in the above inequality, we arrive
\[
\rho \left( \frac{1}{4} \mathcal{R}_n \left( \alpha_1^3 x_1 + \beta_1^3 y_1 \right) - \frac{1}{4} \alpha_1^3 \mathcal{R}_n \left( x_1 \right) - \frac{1}{4} \beta_1^3 \mathcal{R}_n \left( y_1 \right) \right) = 0
\]
for all \(x_1, y_1 \in \mathcal{N}\). Thus, \(\mathcal{R}_n(y)\) satisfies the Ramanujan Type additive functional equation (2). It is clear that the existence of \(\mathcal{R}_n(y)\) is unique.

Indeed, suppose that assume \(\mathcal{R}_1(y)\) be another additive mapping satisfying (2)
and (24). So, one can easily verify that for a positive integer \(d\), we observe
\[
\mathcal{R}_1(\Gamma^d y) = \Gamma^d \mathcal{R}_1(y); \quad \mathcal{R}_n(\Gamma^d y) = \Gamma^d \mathcal{R}_n(y)
\] (26)
and
\[
\mathcal{R}_1(y) = \frac{1}{\Gamma^d} \mathcal{R}_1(\Gamma^d y); \quad \mathcal{R}_n(y) = \frac{1}{\Gamma^d} \mathcal{R}_n(\Gamma^d y)
\] (27)
for all \(y \in \mathcal{N}\). Now,
\[
\begin{align*}
\rho \left( \frac{1}{2} \mathcal{R}_n(y) - \frac{1}{2} \mathcal{R}_1(y) \right) &
\leq \frac{1}{2} \rho \left( \frac{1}{\Gamma^d} \mathcal{R}_n(\Gamma^d y) - \frac{1}{\Gamma^d} \mathcal{R}_1(\Gamma^d y) \right) \\
&\leq \frac{1}{2} \rho \left( \frac{1}{\Gamma^d} \mathcal{R}_n(\Gamma^d y) - \frac{1}{\Gamma^d} \mathcal{R}(\Gamma^d y) \right) + \frac{1}{2} \rho \left( \frac{1}{\Gamma^d} \mathcal{R}(\Gamma^d y) - \frac{1}{\Gamma^d} \mathcal{R}_1(\Gamma^d y) \right) \\
&\leq \frac{1}{2 \Gamma^d} \rho \left( \mathcal{R}_n(\Gamma^d y) - \mathcal{R}(\Gamma^d y) \right) + \frac{1}{2 \Gamma^d} \rho \left( \mathcal{R}(\Gamma^d y) - \mathcal{R}_1(\Gamma^d y) \right) \\
&\leq \frac{1}{\Gamma} \sum_{a=0}^{\infty} \frac{1}{\Gamma^a \Gamma^d} B \left( \Gamma^a \Gamma^d y, \Gamma^a \Gamma^d y \right) \\
&\to 0 \quad \text{as} \quad d \to \infty.
\end{align*}
\] (28)
for all \(y \in \mathcal{N}\).

**Corollary 5.2** Let \(\mathcal{N}\) be a normed space with norm \(\| \cdot \|\) and \(\mathcal{M}_\rho\) satisfy the Fatou property. If \(\mathcal{R} : \mathcal{N} \to \mathcal{M}_\rho\) is a function satisfying the inequality
\[
\rho \left( R \left( a_1^3 x_1 + b_1^3 y_1 \right) - \left\{ a_1^3 R(x_1) + b_1^3 R(y_1) \right\} \right) \leq \begin{cases} 
\mathcal{T}, \\
\mathcal{T} \left\{ \|x_1\|^r + \|y_1\|^r \right\}, \\
\mathcal{T} \left\{ \|x_1\|^{r_1} + \|y_1\|^{r_2} \right\}, \\
\mathcal{T} \|x_1\|^r \|y_1\|^{r'} \end{cases}
\] (29)
for all \(x_1, y_1 \in \mathcal{N}\) and \(\mathcal{T} > 0\). Then there exists a unique additive mapping \(\mathcal{R}_n : \mathcal{N} \to \mathcal{M}_\rho\) which satisfies the functional equation (2) and the functional inequality...
\[
\rho \left( R_n(y) - R(y) \right) \leq \begin{cases} 
\frac{T}{(\Gamma - r)}; & r < 1; \\
\frac{2T||y||^2}{(\Gamma - r)^2}; & r_1 < 1; \\
\frac{T||y||^r}{(\Gamma - r_1)}; & r_1 < 1; \\
\frac{T||y||^r}{(\Gamma - r_2)}; & r_2 < 1; \\
\frac{T||y||^2}{(\Gamma - r_2)}; & 2r < 1;
\end{cases}
\tag{30}
\]

for all \( y \in \mathcal{N} \).

6. Stability Theorem: Using The \( \Delta_2 \)– Condition

**Theorem 6.1** Assume \( M_\rho \) satisfy the Fatou property. If \( R : \mathcal{N} \to M_\rho \) is a function satisfying the inequality

\[
\rho \left( R (a_1^3 x_1 + b_1^3 y_1) - \{ a_1^3 R(x_1) + b_1^3 R(y_1) \} \right) \leq \mathcal{B}(x_1, y_1); \quad \forall \ x_1, y_1 \in \mathcal{N}. \quad (31)
\]

Then there exists a unique additive mapping \( R_n : \mathcal{N} \to M_\rho \) which satisfying the functional equation (2) and the functional inequality

\[
\rho \left( R(y) - R(y) \right) \leq \frac{1}{k} \sum_{a=1}^{\infty} \left( k^2 \Gamma \right)^a \mathcal{B} \left( \frac{y}{\Gamma^a}; \frac{y}{\Gamma^a} \right); \quad \forall \ y \in \mathcal{N}, \quad (32)
\]

where \( \Gamma = (a_1^3 + b_1^3) \) and \( R_n(y) \) is given by

\[
\lim_{c \to \infty} \rho \left( \Gamma^c R \left( \frac{y}{\Gamma^c} \right) - R_n(y) \right) = 0; \quad \forall \ y \in \mathcal{N}. \quad (33)
\]

The mapping \( \mathcal{B} : \mathcal{N}^2 \to [0, \infty) \) fullfilling the condition

\[
\lim_{c \to \infty} k^c \mathcal{B} \left( \frac{x_1}{\Gamma^c}; \frac{y_1}{\Gamma^c} \right) = 0; \quad \forall \ x_1, y_1 \in \mathcal{N}. \quad (34)
\]

Proof: Fixing \( x_1 = y_1 = y \) and again replace \( y = \frac{y}{\Gamma} \) in (31), we reach

\[
\rho \left( R(y) - \Gamma R \left( \frac{y}{\Gamma} \right) \right) \leq \mathcal{B} \left( \frac{y}{\Gamma}; \frac{y}{\Gamma} \right); \quad \forall \ y \in \mathcal{N}. \quad (35)
\]
Using the $\Delta_2$—condition it follows from (35) and the convexity of the modular $\rho$ that,

$$\rho \left( R \left( y \right) - \Gamma R \left( \frac{y}{\Gamma} \right) \right) \leq \frac{k}{\Gamma} \mathcal{B} \left( \frac{y}{\Gamma^a}, y \Gamma^a \right); \quad \forall \ y \in \mathcal{N}. \quad (36)$$

Generalizing for a positive integer $c > 0$, we obtain

$$\rho \left( \Gamma^c R \left( \frac{y}{\Gamma^c} \right) - \Gamma^d R \left( \frac{y}{\Gamma^d} \right) \right) \leq \frac{1}{k} \left( \frac{\Gamma}{\Gamma^a} \right)^d \sum_{a=d+1}^{c} \left( \frac{k}{\Gamma^a} \right)^a \mathcal{B} \left( \frac{y}{\Gamma^a}, y \Gamma^a \right); \quad \forall \ y \in \mathcal{N}. \quad (37)$$

So, for all $c, d \geq 0$ with $c \geq d$, we have

$$\rho \left( \Gamma^c R \left( \frac{y}{\Gamma^c} \right) - \Gamma^d R \left( \frac{y}{\Gamma^d} \right) \right) \leq \frac{1}{k} \left( \frac{\Gamma}{\Gamma^a} \right)^d \sum_{a=d+1}^{c} \left( \frac{k}{\Gamma^a} \right)^a \mathcal{B} \left( \frac{y}{\Gamma^a}, y \Gamma^a \right); \quad \forall \ y \in \mathcal{N}. \quad (38)$$

Thus the sequence $\{ \Gamma^c R \left( \frac{y}{\Gamma^c} \right) \}$ is a $\rho$—Cauchy sequence in $\mathcal{M}_\rho$. Since $\mathcal{M}_\rho$ is $\rho$—complete, there exists a $\rho$—limit function $\mathcal{R}_n : \mathcal{N} \rightarrow \mathcal{M}_\rho$ defined by

$$\rho - \lim_{c \rightarrow \infty} \Gamma^c R \left( \frac{y}{\Gamma^c} \right) = \mathcal{R}_n(y) \quad \text{or} \quad \lim_{c \rightarrow \infty} \rho \left( \Gamma^c R \left( \frac{y}{\Gamma^c} \right) - \mathcal{R}(y) \right) = 0; \ y \in \mathcal{N}. \quad (39)$$

Letting $d = 0$ and $c \rightarrow \infty$ in (38) and using (39), we arrive (32). The rest of proof is similar to that of Theorem 5.1. This completes the proof of the theorem.

**Corollary 6.2** Let $\mathcal{N}$ be a normed space with norm $|| \cdot ||$ and $\mathcal{M}_\rho$ satisfy the Fatou property. If $R : \mathcal{N} \rightarrow \mathcal{M}_\rho$ is a function satisfying the inequality

$$\rho \left( R \left( a_1^3 x_1 + b_1^3 y_1 \right) - \{ a_1^3 R (x_1) + b_1^3 R (y_1) \} \right) \leq \begin{cases} \mathcal{T}, \\ \mathcal{T} \{ ||x_1||^r + ||y_1||^r \}, \\ \mathcal{T} \{ ||x_1||^{r_1} + ||y_1||^{r_2} \}, \\ \mathcal{T} \ ||x_1||^{r_1} ||y_1||^{r_2}, \\ \mathcal{T} \ ||x_1||^{r_1} ||y_1||^{r_2} \end{cases} \quad (40)$$

for all $x_1, y_1 \in \mathcal{N}$ and $\mathcal{T} > 0$. Then there exists a unique additive mapping $\mathcal{R}_n : \mathcal{N} \rightarrow \mathcal{M}_\rho$ which satisfying the functional equation (2) and the functional inequality
\[ \rho(\mathcal{R}_n(y) - R(y)) \leq \begin{cases} \frac{T}{\Gamma - k^2}, & 2Tk \leq \frac{T}{\Gamma^1 + r_2} - k^2; \\ \frac{T}{\Gamma^1 + r_1^2} - k^2; & 2r; r_1; r_2; 2r; r_1 + r_2 > \log_2 \frac{k^2}{\Gamma}. \end{cases} \] (41)

for all \( y \in \mathcal{N} \) with \( r; r_1; r_2; 2r; r_1 + r_2 > \log_2 \frac{k^2}{\Gamma} \).

7 Conclusion

In this paper with the help of classical Hyers Method, we analyze the generalized Ulam-Hyers stability of a Ramanujan Type Additive Functional Equation in Paranormed Spaces and Modular spaces. The stability results in these two spaces are varying due to their respective definitions.

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