A global second-order Sobolev regularity for \( p \)-Laplacian type equations with variable coefficients in bounded domains

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Abstract
Let \( \Omega \subset \mathbb{R}^n \) be a bounded convex domain with \( n \geq 2 \). Suppose that \( A \) is uniformly elliptic and belongs to \( W^{1,n} \) when \( n \geq 3 \) or \( W^{1,q} \) for some \( q > 2 \) when \( n = 2 \). For \( 1 < p < \infty \), we establish a global second-order regularity estimate

\[
\| D[|Du|^{p-2}Du] \|_{L^2(\Omega)} + \| D[\langle ADu, Du \rangle^{\frac{p-2}{p}} ADu] \|_{L^2(\Omega)} \leq C \| f \|_{L^2(\Omega)}
\]

for the inhomogeneous \( p \)-Laplace type equation

\[
-\text{div}(\langle ADu, Du \rangle^{\frac{p-2}{p}} ADu) = f
\]

in \( \Omega \) with Dirichlet or Neumann homogeneous boundary condition. Similar result was also established for certain bounded Lipschitz domains whose boundary is weakly second-order differentiable and satisfies some smallness assumptions.

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1 Introduction

We first recall the $L^2$-integrability of second-order derivatives (also called the Hessian or Calderon-Zygmund estimate) for the Poisson equation in bounded convex domains and its extension to divergence type elliptic equations; see for example [1, 2, 10, 11, 15, 20] and references therein. For these and also their extension to non-smooth domains we refer to for example [1, 2, 10, 11, 15, 20] and references therein. To be precise, let $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ be any bounded convex domain. Given any $f \in L^2(\Omega)$, the uniqueness weak solution $u$ to the Poisson equation

$$\Delta u := -\text{div}(Du) = f \text{ in } \Omega$$

with Dirichlet homogeneous boundary condition $u|_{\partial \Omega} = 0$ satisfies

$$u \in W^{2,2}(\Omega) \quad \text{and} \quad \|D^2u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

(1.1)

for some constant $C$ depending on $n$. Note that (1.1) also holds whenever $f \in L^2(\Omega)$ with $\int_\Omega f(x) \, dx = 0$, and $u$ is any weak solution to $\Delta u = f$ in $\Omega$ with Neumann homogeneous boundary condition $Du \cdot \nu|_{\partial \Omega} = 0$. Here and below $\nu$ always denotes the outer normal to the boundary $\partial \Omega$. Regarding the Neumann homogeneous boundary condition, the condition $\int_\Omega f(x) \, dx = 0$ is necessary for the existence of weak solutions, and weak solutions are unique up to some additive constant.

Moreover, consider the inhomogeneous elliptic equation

$$\mathcal{L}_A u := -\text{div}(ADu) = f \text{ in } \Omega,$$

(1.2)

where and below we always suppose that $A : \Omega \rightarrow \mathbb{R}^{n \times n}$ is a symmetric matrix-valued function satisfying the elliptic condition

$$\frac{1}{L} |\xi|^2 \leq (A(x)\xi, \xi) \leq L |\xi|^2 \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^n$$

(1.3)

for some $L > 1$; for short we write $A \in \mathcal{E}_L(\Omega)$ below. Under the integrability condition for the distributional derivative $DA$

$$DA \in L^q(\Omega) \quad \text{for some } q \geq n \text{ when } n \geq 3 \text{ or } q > n \text{ when } n = 2,$$

(1.4)

it was shown in [15, Lemma 8.1] that (1.1) also holds, for some constant $C$ depending on $n, \Omega, q, L$ and $DA$, whenever $f \in L^2(\Omega)$ and $u$ is a weak solution to (1.2) with Dirichlet homogeneous boundary condition, or whenever $f \in L^2(\Omega)$ with $\int_\Omega f(x) \, dx = 0$ and $u$ is a weak solution to (1.2) with Neumann homogeneous boundary condition $ADu \cdot \nu|_{\partial \Omega} = 0$.

Recently, Cianchi-Maz’ya [8] proved a nonlinear version of the above Calderon-Zygmund estimate for the inhomogeneous $p$-Laplace equation

$$\Delta_p u := -\text{div}(|Du|^{p-2}Du) = f \text{ in } \Omega,$$

(1.5)

where $1 < p < \infty$. Indeed, they proved

$$|Du|^{p-2}Du \in W^{1,2}(\Omega) \quad \text{with the norm bound} \quad \|D[|Du|^{p-2}Du]\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

(1.6)

for some constant $C$ depending on $n, p, \Omega$, whenever $f \in L^2(\Omega)$ and $u$ is a generalized solution to (1.5) with $u|_{\partial \Omega} = 0$ or whenever $f \in L^2(\Omega)$ with $\int_\Omega f(x) \, dx = 0$ and $u$ is a generalized solution to (1.5) with $Du \cdot \nu|_{\partial \Omega} = 0$. Note that the datum $f$ is assumed to be merely
that A
for some suitably small constant
where C
> 0.
These results were extended by [4, 9] to the vector-valued
case.
As a consequence, it is natural to work with generalized solutions; for more about
generalized solutions see [7, 8]. These results were extended by [4, 9] to the vector-valued
result.
In this paper we consider the inhomogeneous p-Laplace type equation
\[ \mathcal{L}_{A,p}u := -\text{div}( (ADu, Du)^\frac{p-2}{2} ADu) = f \quad \text{in } \Omega, \]  
(1.7)
where the coefficient \( A \in \mathcal{E}_L(\Omega) \). Motivated by above Hessian or Calderon-Zygmund esti-
mates and their nonlinear version, it is natural to ask, under the condition (1.4), whether some
similar global second order regularity holds for generalized solutions to (1.7) with Dirichlet
or Neumann homogeneous boundary condition.

The main purpose of this paper is to answer the above question.

**Theorem 1.1** Let \( \Omega \) be a bounded convex domain of \( \mathbb{R}^n \) with \( n \geq 2 \). Let \( 1 < p < \infty \). Suppose
that \( A \in \mathcal{E}_L(\Omega) \) for some \( L > 1 \) and satisfies (1.4).

If \( f \in L^2(\Omega) \) and \( u \) is a generalized solution to (1.7) with \( u|_{\partial \Omega} = 0 \), or if \( f \in L^2(\Omega) \)
with \( \int_{\Omega} f(x) \, dx = 0 \) and \( u \) is a generalized solution to (1.7) with \( ADu \cdot v|_{\partial \Omega} = 0 \), then we have
\[ \begin{align*}
\langle ADu, Du \rangle^{\frac{p-2}{2}} ADu &\in W^{1,2}(\Omega) \quad \text{and} \quad |Du|^{p-2} Du \in W^{1,2}(\Omega) \quad \text{with norm bounds} \\
\|D[\langle ADu, Du \rangle^{\frac{p-2}{2}} ADu]\|_{L^2(\Omega)} + \|D[|Du|^{p-2} Du]\|_{L^2(\Omega)} &\leq C \|f\|_{L^2(\Omega)},
\end{align*} \]
(1.8)
where \( C > 0 \) is a constant independent of \( u \) and \( f \).

Moreover, beyond convex domains, Cianchi-Maz’ya [8] proved that the above estimate
(1.6) holds for the equation (1.5) in any bounded Lipschitz domain \( \Omega \) provided that
the boundary \( \partial \Omega \in W^2 L^{n-1,\infty} \) when \( n \geq 3 \) and \( \partial \Omega \in W^2 L^{1,\infty} \log L \) when \( n = 2 \) (1.9)
and that the weak second fundamental form \( \mathcal{B} \) on \( \partial \Omega \) satisfies
\[ \lim_{r \to 0} \Psi_{\mathcal{B}}(r) \leq c \]  
(1.10)
for some suitably small constant \( c = c(\text{Lip}_\Omega, d_\Omega, n, p) \), where
\[ \Psi_{\mathcal{B}}(r) := \begin{cases} 
\sup_{x \in \partial \Omega} \|\mathcal{B}\|_{L^{n-1,\infty}(\partial \Omega \cap B_r(x))} & \text{if } n \geq 3, \\
\sup_{x \in \partial \Omega} \|\mathcal{B}\|_{L^{1,\infty} \log L(\partial \Omega \cap B_r(x))} & \text{if } n = 2.
\end{cases} \]
(1.11)
Here \( d_\Omega \) is the diameter of \( \Omega \) and \( \text{Lip}_\Omega \) is the Lipschitz constant of \( \Omega \). Recall that \( \Omega \) is a
Lipschitz domain, if, in a neighborhood \( B(x, r) \cap \partial \Omega \) of each boundary point \( x \), \( \Omega \) is the
subgraph \( \Gamma(\phi) \) of a Lipschitz continuous function \( \phi : \mathbb{R}^{n-1} \to \mathbb{R} \) of \( (n - 1) \)-variables.
We say \( \partial \Omega \) satisfies (1.9) if each such \( \phi \) is twice weakly differentiable, and that its second-
order derivatives belong to either the weak Lebesgue space \( L^{n-1,\infty} \) if \( n \geq 3 \), or the weak
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Zygmund space $L^{1,\infty} \log L$ if $n = 2$. The assumption (1.9) guarantees that the weak second fundamental form $\mathcal{B}$ on $\partial \Omega$ belongs to the same weak type spaces with respect to the $(n - 1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ on $\partial \Omega$, and hence guarantees $\lim_{r \to 0} \Psi_{\mathcal{B}}(r) < \infty$. But (1.9) cannot give the smallness of $\Psi_{\mathcal{B}}(r)$, that is, (1.10). As revealed by Cianchi-Maz’ya, to get (1.1) in the case $p = 2$ and also (1.6) for $1 < p < \infty$, the smallness assumption (1.10) for $\lim_{r \to 0} \Psi_{\mathcal{B}}(r)$ is necessary and optimal; (1.10) cannot be weakened to $\lim_{r \to 0} \Psi_{\mathcal{B}}(r) < \infty$. For more details we refer to [8].

In this paper, we also show that the convexity assumption on $\Omega$ in Theorem 1.1 can be reduced to the smallness assumption (1.10) as in [8].

**Theorem 1.2** Let $\Omega$ be a bounded Lipschitz domain of $\mathbb{R}^n$ with $n \geq 2$ and satisfy (1.9). Let $1 < p < \infty$. Suppose that $A \in \mathcal{E}_L(\Omega)$ for some $L > 1$ and satisfies (1.4).

There exists a constant $\delta_\ast > 0$ depending on $n, p, L, d_\Omega$, $\text{Lip}_\Omega$ such that if $\lim_{r \to 0} \Psi_{\mathcal{B}}(r) \leq \delta_\ast$, then (1.8) holds whenever $f \in L^2(\Omega)$ and $u$ is a generalized solution to (1.7) with $u|_{\partial \Omega} = 0$ or whenever $f \in L^2(\Omega)$ with $\int_\Omega f(x) \, dx = 0$ and $u$ is a generalized solution to (1.7) with $A Du \cdot v|_{\partial \Omega} = 0$.

**Remark 1.3** In the statements of Theorems 1.1 and 1.2, the dependence of constant $C$ on the main parameters is unclear. Here we clarify them respectively.

(i) Regards of the constant $C$ in Theorem 1.1, we consider two cases.

- In the case $A \in \dot{W}^{1,q}(\Omega)$ for some $q > n \geq 2$, the constant $C$ in Theorem 1.1 depends only on $n, p, L, d_\Omega = \text{diam} \, \Omega$, $|\Omega|$ and $C_{\text{ext},q}(\Omega)\|DA\|_{L^q(\Omega)}$. In the proof, we extend $A \in \dot{W}^{1,q}(\Omega)$ to $\widetilde{A} \in \dot{W}^{1,q}(\mathbb{R}^n)$ so that

$$\|D\widetilde{A}\|_{L^q(\mathbb{R}^n)} \leq C_{\text{ext},q}(\Omega)\|DA\|_{L^q(\Omega)} + \|A\|_{L^q(\Omega)},$$

where $C_{\text{ext},q}(\Omega)$ is the norm of the extension operator.

- In the case $A \in \dot{W}^{1,n}(\Omega)$ with $n \geq 3$, we know that $\lim_{r \to 0} \Phi_{A,\Omega}(r) = 0$, where

$$\Phi_{A,\Omega}(r) := \sup_{x \in \Omega} \|DA\|_{L^n(B(x,r) \cap \Omega)}.$$

We extend $A \in \dot{W}^{1,n}(\Omega)$ to $\widetilde{A} \in \dot{W}^{1,n}(\mathbb{R}^n)$ so that

$$\Phi_{\widetilde{A},\Omega'}(r) \leq C_{\text{ext},n}(\Omega)\Phi_{A,\Omega}(C_{\text{ext},n}(\Omega)r) \quad \text{whenever } 0 < t < r,$$

where $C_{\text{ext},n}(\Omega)$ is a constant and $\Omega'$ is the $t$-neighbourhood of $\Omega$. To obtain (1.8) in Theorem 1.1, we require that $\Phi_{A,\Omega}(C_{\text{ext},n}(\Omega)r_A) \leq \delta_\ast / C_{\text{ext},n}(\Omega)$ for some sufficiently small $\delta_\ast > 0$ depending on $n, p, L$ and some $r_A \in (0, 1)$. The constant $C$ in Theorem 1.1 depends only on $n, p, L r_A, \text{Lip}_\Omega, d_\Omega$ and $|\Omega|$.

(ii) The constant $C$ in Theorem 1.2 not only depends on parameters as stated in (i) above in a similar way, but also on some constant caused by the condition (1.10), that is, it depends on $1/r_\Omega$, where $r_\Omega \in (0, 1)$ satisfies $\Psi_{\mathcal{B}}(r_\Omega) \leq \delta_\ast$ and $\delta_\ast$ is given in (1.10).

To prove Theorems 1.1 and 1.2, we consider some regularized equation of (1.7), that is, the equation

$$\mathcal{L}_{\epsilon, A, p} u = -\text{div} \left((|ADu|, Du) + \epsilon \frac{p-2}{2} |ADu|^p \right) = f \quad \text{in } \Omega \text{ with } u|_{\partial \Omega} = 0 \text{ or } ADu \cdot v|_{\partial \Omega} = 0,$$

where $\epsilon \in (0, 1]$ and $\Omega, A, f$ satisfies
(S1) \( A \in \mathcal{E}_L(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n) \) for some \( L \geq 1 \),
(S2) \( \Omega \) is a bounded smooth domain,
(S3) \( f \in C_c^\infty(\Omega) \) (and \( \int_\Omega f(x) \, dx = 0 \) in the case of Neumann homogeneous boundary condition).

Note that, under assumptions (S1)-(S3), weak solutions to the regularized equation (1.12) with Dirichlet or Neumann homogeneous boundary condition are always smooth in \( \bar{\Omega} \).

We establish the following global quantitative second-order regularity for the regularized equation (1.12) in Theorems 1.4 and 1.5. From them, via a standard approximation argument we conclude Theorems 1.1 and 1.2 (including Remark 1.3) respectively; we refer to Sect. 7 for details.

**Theorem 1.4** Let \( 1 < p < \infty, \epsilon \in (0, 1] \), and \( \Omega \) be a bounded convex domain. Suppose that \( \Omega \), \( A \) and \( f \) satisfy assumptions (S1)-(S3). Let \( u \) be any weak solution to (1.12).

(i) If \( \| DA \|_{L^p(\Omega)} \leq R_\varepsilon \) for some \( q > n \geq 2 \) and \( 0 < R_\varepsilon < \infty \), then

\[
\| D[(Du)^2 + \epsilon]^{\frac{p-2}{2}} Du \|_{L^2(\Omega)} + \| D[(A Du, Du) + \epsilon]^{\frac{p-2}{2}} ADu \|_{L^2(\Omega)} \leq C \| f \|_{L^2(\Omega)}
\]  

(1.13)

with the constant \( C \) depending only on \( n, p, q, L \) and \( R_\varepsilon \).

(ii) There exists a constant \( \delta_\varepsilon > 0 \) depending on \( n, p, L, d_\Omega \), \( \text{Lip}_\Omega \) such that if \( \Phi_{A, \Omega}(r_\varepsilon) \leq \delta_\varepsilon \) for some \( r_\varepsilon > 0 \), then (1.13) holds with the constant depending only on \( n, p, L \) and \( r_\varepsilon \).

**Theorem 1.5** Let \( 1 < p < \infty \) and \( \epsilon \in (0, 1] \). Suppose that \( \Omega \), \( A \) and \( f \) satisfy (S1)-(S3). Let \( u \) be a weak solution to (1.12).

(i) There exists a constant \( \delta_\varepsilon > 0 \) depending only on \( n, p, L, \text{Lip}_\Omega \) and \( d_\Omega \) such that if \( \Phi_{B}(r_\varepsilon) \leq \delta_\varepsilon \) for some \( 0 < r_\varepsilon < 1 \), and if \( \| DA \|_{L^q(\Omega)} \leq R_\varepsilon \) for some \( q > n \) and \( 0 < R_\varepsilon < \infty \), then (1.13) holds with the constant \( C \) depending only on \( n, p, L \) and \( R_\varepsilon \) and \( r_\varepsilon \).

(ii) There exist constants \( \delta_\varepsilon > 0 \) and \( \delta_\varepsilon' > 0 \), depending only on \( n, p, L, \text{Lip}_\Omega \) and \( d_\Omega \), such that if \( \Psi_B(r_\varepsilon) \leq \delta_\varepsilon \) for some \( 0 < r_\varepsilon < 1 \) and if \( \Phi_{A, \Omega}(r_\varepsilon) \leq \delta_\varepsilon' \) for some \( r_\varepsilon > 0 \), then (1.13) holds with the constant \( C \) depending only on \( n, p, L, r_\varepsilon \) and \( r_\varepsilon' \).

We prove Theorems 1.4 and 1.5 in Sect. 2 with the aid of key Lemma 2.1, Lemma 2.2, Lemma 2.3, Lemmas 2.5 and 2.6, whose proofs are postponed to Sects. 3, 4, 5 and 6 correspondingly.

The main novelty is that, instead of the Euclidean gradient \( D \), we consider the intrinsic (Riemannian) gradient \( \sqrt{A} D \), which allows us to combine the approach of Cianchi-Maz’ya for the equation \( \Delta_{p, \Omega} u = f \) and also the classical approach to Calderon-Zygmund estimates for the equation \( L_{A} u = f \). Moreover, we use different analytic properties of the boundary of domains when dealing with Dirichlet and Neumann homogeneous boundary condition; see key Lemma 2.1.

Finally, for reader’s convenience we explain some notations in the following Remark.

**Remark 1.6** (i) Let \( m \geq 1 \) be an integer. The Lebesgue space \( L^q(\Omega, \mathbb{R}^m) \) consists of all measurable functions: \( v = (v_1, v_2, ..., v_m) : \Omega \rightarrow \mathbb{R}^m \) with

\[
\| v \|_{L^q(\Omega, \mathbb{R}^m)} := \left( \sum_{i=1}^m \int_\Omega |v_i|^q \, dx \right)^{\frac{1}{q}} < \infty
\]
for $1 \leq q < +\infty$. Let $k \geq 1$ be an integer. The Sobolev space $W^{k,q}(\Omega, \mathbb{R}^m)$ consists of all functions $v = (v_1, v_2, \ldots, v_m) \in L^q(\Omega, \mathbb{R}^m)$ such that for each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ with $1 \leq |\alpha| \leq k$, $D^\alpha v_i$ exists in the weak sense and belongs to $L^q(\Omega, \mathbb{R}^m)$ with any $1 \leq i \leq m$. Furthermore,

$$||v||_{W^{k,q}(\Omega, \mathbb{R}^m)} := \left( \sum_{i=1}^m \sum_{|\alpha|=0}^k \int_\Omega |D^\alpha v_i|^q \, dx \right)^{\frac{k}{q}}.$$ 

Throughout this paper, we omit the target space in the notions of Lebesgue space and Sobolev space for short. Namely, we write $L^q(\Omega)$ and $W^{k,q}(\Omega)$ instead of $L^q(\Omega, \mathbb{R}^m)$ and $W^{k,q}(\Omega, \mathbb{R}^m)$, respectively.

(ii) Let $M$ be a $n \times n$ symmetric matrix with $\langle M\xi, \xi \rangle \geq 0$ for all $\xi \in \mathbb{R}^n$. The square root $\sqrt{M}$ of $M$ is defined as below. Note that $M$ has nonnegative eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. By the Schur decomposition, one can write $M = O^T \text{diag}(\lambda_1, \ldots, \lambda_n) O$, where $\text{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ on the diagonal and $O$ is an orthogonal matrix. Since $O^T = O^{-1}$, and $\{\lambda_1, \ldots, \lambda_n\}$ are nonnegative, one further gets

$$M = O^T \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}) O O^T \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}) O.$$ 

We define the $n \times n$ symmetric matrix $\sqrt{M} := O^T \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}) O$ as the square root of $M$. Obviously, $M = \sqrt{M} \sqrt{M}$.

2 Proofs of Theorems 1.4 and 1.5

Let $1 < p < \infty$ and $\epsilon \in (0, 1]$. Suppose that $\Omega$, $A$ and $f$ satisfy (S1)-(S3). Let $u$ be a weak solution to (1.12). We are going to prove (1.13) and then Theorems 1.4 and 1.5. For simplicity of notation, we always write

$$F_{A,\epsilon}(Du) = \left(|\sqrt{AD}u|^2 + \epsilon\right)^{\frac{p-2}{2}} ADu \quad \text{and} \quad U_{A,\epsilon}(Du) = \left(|\sqrt{AD}u|^2 + \epsilon\right)^{\frac{p-2}{2}} |\sqrt{AD} u\sqrt{A}|,$$

and also

$$F_{\epsilon}(Du) = \left(|Du|^2 + \epsilon\right)^{\frac{p-2}{2}} Du \quad \text{and} \quad U_{\epsilon}(Du) = \left(|Du|^2 + \epsilon\right)^{\frac{p-2}{2}} |D^2 u|.$$ 

Then (1.13) reads as

$$||DF_{A,\epsilon}(Du)||_{L^2(\Omega)} + ||DF_{\epsilon}(Du)||_{L^2(\Omega)} \leq C ||f||_{L^2(\Omega)}. \quad (2.1)$$

Note that

$$|DF_{\epsilon}(Du)| \leq C(n, p)U_{\epsilon}(Du) \leq C(n, p, L)U_{A,\epsilon}(Du),$$

we only need to show

$$||DF_{A,\epsilon}(Du)||_{L^2(\Omega)} + ||U_{A,\epsilon}(Du)||_{L^2(\Omega)} \leq C ||f||_{L^2(\Omega)}. \quad (2.2)$$

We proceed as below to prove (2.2). Firstly, we establish the following fundamental inequality.
Lemma 2.1 One has
\[ f^2 = (L_{ε,A,p}u)^2 \geq \frac{1}{2} \min\{1, (p-1)^2\}[U_{A,ε}(Du)]^2 - C|DA|^2|V_{A,ε}(Du)|^2 + \mathbf{I}_i, \quad i = 1, 2, \]  
(2.3)
where
\[ \mathbf{I}_1 = \text{div} \left\{ \left( |ADu|^2 + ε \right)^{p-2} \left[ \text{tr}(AD^2u)ADu - AD^2uADu \right] \right\}, \]  
(2.4)
and
\[ \mathbf{I}_2 = \text{div} \left\{ \left( |ADu|^2 + ε \right)^{p-2} \left[ \text{div} (ADu)ADu - (ADu \cdot D)ADu \right] \right\}. \]  
(2.5)

Here \( C > 1 \) is a constant depending on \( n, p, L \).

We introduce different \( \mathbf{I}_1 \) and \( \mathbf{I}_2 \) as in (2.4), (2.5) and, later in Lemma 2.3 as we will use them for Dirichlet and Neumann homogeneous boundary problem respectively. This is crucial and also necessary for us to get Theorems 1.4 and 1.5 (and hence Theorems 1.1 and 1.2) under the merely regularity assumption \( DA \in L^q(Ω) \) as in (S2). See Remark 2.4(i) for detailed reasons.

We prove (2.3) with \( \mathbf{I}_2 \) given by (2.5) and \( \frac{1}{2} \) replaced by \( \frac{3}{4} \) via considering the intrinsic (Riemannian) gradient \( D_Au = √A Du \) and borrowing some ideas from Maz’ya-Cianchi [8, Lemma 3.1], see Lemma 3.1 for the details. The additional term \( C|DA|^2V_{A,ε}(Du)^2 \) in (2.3) appears in a natural way. One may use a similar argument to prove (2.3) with \( \mathbf{I}_1 \) given by (2.4). Instead, we bound the difference between (2.4) and (2.5) by
\[ δU_{A,ε}(Du) + \frac{1}{δ} C|DA|^2V_{A,ε}(Du)^2 \]  
for any \( δ > 0 \),

see Lemma 3.2 for details. This allows us to get (2.3) with \( \mathbf{I}_1 \) given by (2.4).

In the special case \( A = I_n \), our proof gives
\[ [\text{div}((|Du|^2 + ε)^{p-2}Du)]^2 \geq \min\{1, (p-1)^2\}(|Du|^2 + ε)^{p-2}|D^2u|^2 - \mathbf{I}, \]  
where
\[ \mathbf{I} = \text{div} \left\{ \left( |Du|^2 + ε \right)^{p-2} \left[ ΔuDu - D^2uDu \right] \right\}, \]

no matter which is given by (2.4) or (2.5); see Remark 3.6. This inequality with the coefficient \( \min\{1, (p-1)^2\} \) replaced by some \( κ > 0 \) was first proved by Maz’ya-Cianchi [8]. Recently, a simplified proof with the explicit constant \( \min\{1, (p-1)^2\} \) for this inequality is given by Balci-Cianchi-Diening-Maz’ya [4]. When \( 1 < p < 2 \), our argument also gives a explicit coefficient \( (p-1)^2 \); see Remark 3.6 for more details. Moreover, the key inequality (3.4) used by [8] and also here can be proved in a simple way; see Remark 3.5 and Lemma 3.3.

Multiplying both sides of (2.3) by some test functions and integrating, we could conclude the following Lemma. Note that the boundary term \( \mathbf{K}(φ) \) as in (2.7) below and the boundary term \( \mathbf{K}(φ) \) as in (2.8) below come, respectively, from (2.4) and from (2.5).

Lemma 2.2 For any \( φ \in C^∞(\mathbb{R}^n) \), one has
\[ \|φDF_{A,ε}(Du)\|^2_{L^2(Ω)} + \|U_{A,ε}(Du)\|^2_{L^2(Ω)} \leq C\|φ\|^2_{L^2(Ω)} + C_1\||DA|φF_{A,ε}(Du)|\|^2_{L^2(Ω)} + C_2\|Dφ|F_{A,ε}(Du)|\|^2_{L^2(Ω)} + \mathbf{K}(φ), \]  
(2.6)
where
\[
K(\phi) = -C \int_{\partial \Omega} \phi^2 \left( |\sqrt{A} Du|^2 + \epsilon \right)^{p-2} \left[ \text{tr}(AD^2 u) AD u - AD^2 u A D u \right] \cdot v d\mathcal{H}^{n-1}(x),
\]
(2.7)
or
\[
K(\phi) = -C \int_{\partial \Omega} \phi^2 \left( |\sqrt{A} Du|^2 + \epsilon \right)^{p-2} \left[ \text{div}(AD u) A D u - (AD u \cdot D) A D u \right] \cdot v d\mathcal{H}^{n-1}(x).
\]
(2.8)

Here, \( C \), \( C_1 \) and \( C_2 \) are positive constants depending only on \( n \), \( p \), \( L \).

Thanks to Lemma 2.2, to get (2.1) we only need to bound the last three terms in the right hand side of (2.6), that is,
\[
C_1 \| |DA| \|_{L^2(\Omega)} \leq C_2 \| |D\phi| \|_{L^2(\Omega)} \quad \text{and} \quad K(\phi).
\]
The boundary term \( K(\phi) \) is bounded as in Lemma 2.3.

**Lemma 2.3** (i) If \( \Omega \) is bounded smooth convex domain, then \( K(\phi) \leq 0 \) whenever \( \phi \in C_\infty(\mathbb{R}^n) \).

(ii) If \( \Omega \) is bounded smooth domain, then
\[
K(\phi) \leq C_\ast \Psi_g(r) \left[ \| \phi DF_{A,e}(Du) \|_{L^2(\Omega)}^2 + \| D\phi| F_{A,e}(Du) \|_{L^2(\Omega)}^2 \right]
\]
whenever \( \phi \in C_\infty(B(z,r)) \) with \( z \in \overline{\Omega} \) and \( 0 < r < 1 \). Here \( C_\ast > 0 \) is a constant depending only on \( n, L \).

See Sect. 5 for the proof of Lemma 2.3. Here are some necessary remarks for the proof.

**Remark 2.4** (i) When \( u \) satisfies Dirichlet homogeneous boundary condition, to get some suitable estimate of \( K(\phi) \) as above, we have to use \( K(\phi) \) as given in (2.7), which comes from (2.4). Otherwise, if we use the \( K(\phi) \) as given in (2.8), then \( DA|_{\partial \Omega} \) will appear in the upper bound of \( K(\phi) \). However, no assumption is made on \( DA|_{\partial \Omega} \) in this paper; \( DA \in L^q(\Omega) \) as in (S2) does not give any information of \( DA|_{\partial \Omega} \).

Similarly, when \( u \) satisfies Neumann homogeneous boundary condition, to get some suitable estimate of \( K(\phi) \) as above, we also have to use \( K(\phi) \) as given in (2.8), which comes from (2.5).

(ii) Besides (i), the proof of Lemma 2.3 (i) relies on the convex geometry; the proof of Lemma 2.3 (ii) relies on trace formula by Cianchi-Maz’ya [8]. Some careful calculations/observations are also necessary.

Concerning the term \( C_1 \| |DA| \phi F_{A,e}(Du) \|_{L^2(\Omega)}^2 \), we use the Gagliardo-Nirenberg-Sobolev inequality to get the following upper bound via the \( L^q \)-norm of \( DA \) with \( q > n \geq 2 \), and also local \( L^n \)-norm of \( DA \) with \( n \geq 3 \); for the proof see Sect. 6.

**Lemma 2.5** (i) Given \( q > n \), for any \( 0 < \eta < 1 \) we have
\[
\| |DA| \phi F_{A,e}(Du) \|_{L^2(\Omega)}^2 \leq \eta \| DA \|_{L^q(\Omega)}^2 \| DF_{A,e}(Du) \|_{L^2(\Omega)}^2
\]
\[
+ \frac{C}{\eta} \| DA \|_{L^q(\Omega)}^2 \| F_{A,e}(Du) \|_{L^1(\Omega)}^2.
\]
(ii) Given any \( \phi \in C^\infty_c(B(z, r)) \) for some \( z \in \overline{\Omega} \) and \( 0 < r < 1 \), we have

\[
\| DA|\phi| F_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 \leq C_2 \| DA\|_{L^n(\Omega \cap B(z, r))} \| \phi DF_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 + C_3 \| DA\|_{L^n(\Omega \cap B(z, r))} \| D\phi| F_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 + C \| \phi F_{A,\epsilon}(Du)\|_{L^1(\Omega)}^2.
\]

For the term \( C_2 \| D\phi| F_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 \), similarly to Lemma 2.5, one has the following.

**Lemma 2.6** For any \( \eta > 0 \) we have

\[
\| F_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 \leq \eta \| DF_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 + C \| F_{A,\epsilon}(Du)\|_{L^1(\Omega)}^2.
\]

Observe that the \( L^1 \)-norm of \( F_{A,\epsilon}(Du) \) or \( \phi F_{A,\epsilon}(Du) \) appears in Lemma 2.4 and Lemma 2.5. To handle them, we need the following \( L^1 \)-estimate by [7]:

\[
\|(Du)^2 + \epsilon\|^2 \| Du\|_{L^1(\Omega)} \leq C \| f\|_{L^1(\Omega)}^2. \tag{2.9}
\]

As a consequence, we have

**Corollary 2.7** We have \( \| F_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 \leq C \| f\|_{L^2(\Omega)}^2 \).

Now, with Lemma 2.2 to Lemma 2.5, and Corollary 2.7 in hand, we conclude (2.1) as below.

If \( \Omega \) is convex, Lemma 2.3 (i) gives \( K(\phi) \leq 0 \).

(i) If \( DA \in L^q(\Omega) \) with \( q > n \), we choose a test function \( \phi \) with \( \phi = 1 \) in \( \Omega \) so that \( C \| D\phi| F_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 = 0 \). Then (2.1) follows from Lemma 2.2 and Lemma 2.5 (i) with sufficiently small \( \eta > 0 \) so that the coefficient \( \eta \| DA\|_{L^q(\Omega)} \) is small. This gives Theorem 1.4(i).

(ii) If \( DA \in L^n(\Omega) \) with \( n \geq 3 \), we find small \( 0 \leq r_z \leq 1 \) so that the coefficient \( C_2 \| DA\|_{L^n(\Omega \cap B(z, r))} \) that appeared in Lemma 2.4(ii) is small. Then cover \( \Omega \) by a family of balls \( B_k \) with radius \( r_z/4 \leq r_k < r_z \), and denote by \( \{\phi_k\} \) be an associated partition of unity. Apply (2.6) to such such \( \phi_k \), and use Lemma 2.4(ii) to bound \( \| DA|\phi_k F_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 \). Summation over all \( \phi_k \) and using Lemma 2.5 we get the desired upper bound as in Theorem 1.4(ii). See Sect. 7 for details.

For general domains \( \Omega \), we use Lemma 2.3 (ii) to bound \( K(\phi) \). We find \( 0 < r_* \leq 1 \) so that the coefficient \( C_5 \| \psi_E(r)\|_{L^q(B(z, r))} \) appeared in Lemma 2.3 is sufficiently small. Then we cover \( \Omega \) by a family of balls \( B_k \) with radius \( r_/4 \leq r_k < r_* \), and denote by \( \{\psi_k\} \) be an associated partition of unity. Apply (2.6) to such such \( \psi_k \), and apply Lemma 2.5(ii) to bound \( \| DA|\psi_k F_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 \). Summation over all \( \psi_k \) we obtain \( L^2 \)-norm of \( DF_{A,\epsilon}(Du) \) is bounded by the summation of the \( L^2 \)-norm of \( f, |DA| F_{A,\epsilon}(Du) \), and \( F_{A,\epsilon}(Du) \), see Lemma 2.8 below.

(iii) If \( DA \in L^q(\Omega) \) with \( q > n \), then (2.1) follows from this and Lemma 2.3 (ii), Lemma 2.5 and Corollary 2.7. This gives Theorem 1.5(i).

(iv) If \( DA \in L^n(\Omega) \) with \( n \geq 3 \), we need to localize Lemma 2.8 via a unit of partition as in the proof of Theorem 1.4 (ii). Then using the argument therein, we derive (2.1). This gives Theorem 1.5(ii). See Sect. 2.1 for details.
2.1 Proof of Theorem 1.4

Suppose that $\Omega$ is convex. By Lemma 2.2(i), we have $K(\phi) \leq 0$. Thus (2.6) gives
\[
\|\phi DF_A,\epsilon(Du)\|_{L^2(\Omega)}^2 + \|\phi U_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 \\
\leq C\|\phi f\|_{L^2(\Omega)}^2 + C_1\|DA|\phi FA_{\epsilon}(Du)\|_{L^2(\Omega)}^2 + C_2\|D\phi FA_{\epsilon}(Du)\|_{L^2(\Omega)}^2.
\]

(2.10)

Proof of Theorem 1.4 (i). Choose $\phi \in C^c_c(\mathbb{R}^n)$ such that $\phi = 1$ in $\Omega$. Then $D\phi = 0$ in $\Omega$ and hence (2.10) reads as
\[
\|DF_A,\epsilon(Du)\|_{L^2(\Omega)}^2 + \|U_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 \leq C\|f\|_{L^2(\Omega)}^2 + C_1\|DA|FA_{\epsilon}(Du)\|_{L^2(\Omega)}^2.
\]

(2.11)

By Lemma 2.5 (i) and choosing $\eta$ sufficiently small such that $C_1\eta\|DA\|_{L^2(\Omega)}^2 \leq \frac{1}{4}$, we obtain
\[
C_1\|DA|FA_{\epsilon}(Du)\|_{L^2(\Omega)}^2 \leq \frac{1}{4}\|DF_A,\epsilon(Du)\|_{L^2(\Omega)}^2 + C\|DA\|_{L^2(\Omega)}^2\|FA_{\epsilon}(Du)\|_{L^2(\Omega)}^2.
\]

Thanks to this and $\|FA_{\epsilon}(Du)\|_{L^2(\Omega)}^2 \leq C\|f\|_{L^2(\Omega)}^2$ given in Corollary 2.7, from (2.11), we conclude
\[
\frac{3}{4}\|DF_A,\epsilon(Du)\|_{L^2(\Omega)}^2 + \|U_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 \leq C\|f\|_{L^2(\Omega)}^2,
\]

that is, (2.1) holds.

Proof of Theorem 1.4(ii). Since $DA \in L^q(\Omega)$, then there exists $r_{\omega} > 0$ depending on $n$ and $A$ such that
\[
C_1C_2\|DA\|_{L^2(\Omega)}^2 \leq \frac{1}{4}, \quad \forall x \in \overline{\Omega},
\]

where $C_2$ is as in Lemma 2.5 (ii). By this inequality and Lemma 2.5 (ii), for any $\phi \in C^c_c(B(x, r_{\omega}))$, one has
\[
C_1\|DA|\phi FA_{\epsilon}(Du)\|_{L^2(\Omega)}^2 \leq \frac{1}{4}\|DF_A,\epsilon(Du)\|_{L^2(\Omega)}^2 + \frac{C_3}{4C_2}\left[\|D\phi|FA_{\epsilon}(Du)\|_{L^2(\Omega)}^2 + C\|\phi FA_{\epsilon}(Du)\|_{L^2(\Omega)}^2\right].
\]

(2.12)

Inserting this inequality into (2.10) yields
\[
\frac{3}{4}\|DF_A,\epsilon(Du)\|_{L^2(\Omega)}^2 + \|U_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 \leq C\|f\|_{L^2(\Omega)}^2 + \left[C_2 + \frac{C_3}{4C_2}\right]\|D\phi|FA_{\epsilon}(Du)\|_{L^2(\Omega)}^2 + C\|\phi FA_{\epsilon}(Du)\|_{L^2(\Omega)}^2.
\]

(2.13)

Next let $\{B_{r_k}\}_{1 \leq k \leq N}$ be a covering of $\overline{\Omega}$ by balls $B_{r_k}$, with $r_{\omega}/4 \leq r_k \leq r_{\omega}$, such that either $B_{r_k}$ is center on $\partial\Omega$, or $B_{r_k} \subseteq \Omega$. Note that the covering can be chosen in the way that the multiplicity $N$ of overlapping among the balls $B_{r_k}$ depends only on $n$. Let $\{\phi_k\}_{k \in N}$ be a family of functions such that $\phi_k \in C^c_c(B_{r_k})$ and $|\nabla \phi_k| \leq C_4(r_{\omega})^{-1}$, and that $\{\phi_k^2\}_{1 \leq k \leq N}$ is a partition of unity associated with the covering $\{B_{r_k}\}_{k \in N}$. Thus $\sum_{k \in N}\phi_k^2 = 1$ in $\overline{\Omega}$.
applying inequality (2.13) with \( \phi = \phi_k \) for each \( k \), and summing the resulting inequalities, one obtains

\[
\frac{3}{4} \| D F_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2 + \| U_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2 \leq C \| f \|_{L^2(\Omega)}^2 + N C_4^2 \left[ C_2 + \frac{C_3}{4 C_4} \right] \| F_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2 + C \| F_{A, \varepsilon} (D u) \|_{L^1(\Omega)}^2. \quad (2.14)
\]

Choose \( \eta \) small enough so that

\[
N C_4^2 \left[ C_2 + \frac{C_3}{4 C_4} \right] \frac{1}{r_*^2} \eta \leq \frac{1}{4}.
\]

According to Lemma 2.6, we have

\[
\frac{1}{2} \| D F_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2 + \| U_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2 \leq C \| f \|_{L^2(\Omega)}^2 + C \| F_{A, \varepsilon} (D u) \|_{L^1(\Omega)}^2. \quad (2.15)
\]

As \( \| F_{\varepsilon} (AD) \|_{L^1(\Omega)}^2 \leq C \| f \|_{L^2(\Omega)}^2 \), we obtain the desired (2.1).

\[\square\]

### 2.2 Proof of Theorem 1.5

**Lemma 2.8**

\[
\frac{3}{4} \| D F_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2 + \| U_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2 \leq C \| f \|_{L^2(\Omega)}^2 + C_1 \| D A | F_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2. \quad (2.16)
\]

**Proof** Applying Lemma 2.2 and Lemma 2.3(ii), we have

\[
\| \phi D F_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2 + \| \phi U_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2 \leq C \| \phi f \|_{L^2(\Omega)}^2 + C_1 \| D A \phi F_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2
\]

\[
+ \left[ C_2 + C_1 \Psi_B (r) \right] \| D \phi F_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2 + C_1 \Psi_B (r) \| \phi F_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2. \quad (2.17)
\]

Next let \( \{ B_k \}_{1 \leq k \leq N} \) be a covering of \( \Omega \) by balls \( B_k \), with \( r_*/4 \leq r_k \leq r_* \), such that either \( B_k \) is centered on \( \partial \Omega \), or \( B_k \subseteq \Omega \). Note that the covering can be chosen in the way that the multiplicity \( N \) of overlapping among the balls \( B_k \) depends only on \( n \). Let \( \{ \psi_k \}_{k \in N} \) be a family of functions such that \( \psi_k \in C^\infty_c (B_k) \) and \( |D \psi_k| \leq C_4 (r_*)^{-1} \), and that \( \{ \psi_k \}_{1 \leq k \leq N} \) is a partition of unity associated with the covering \( \{ B_k \}_{k \in N} \). By applying inequality (2.13) with \( \phi = \psi_k \) for each \( k \), and summing the resulting inequalities, one obtains

\[
\| D F_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2 + \| U_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2 \leq C \| f \|_{L^2(\Omega)}^2 + C_1 \| D A | F_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2
\]

\[
+ N C_4^2 \left[ C_2 + C_1 \Psi_B (r_* \frac{1}{r_*}) \right] \frac{1}{r_*^2} \| F_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2 + C_1 \Psi_B (r_* \frac{1}{r_*}) \| F_{A, \varepsilon} (D u) \|_{L^2(\Omega)}^2. \quad (2.18)
\]

Choose \( \eta > 0 \) small enough such that

\[
\left\{ N C_4^2 \left[ C_2 + C_1 \Psi_B (r_* \frac{1}{r_*}) \right] \frac{1}{r_*^2} + C_1 \Psi_B (r_* \frac{1}{r_*}) \right\} \eta \leq \frac{1}{4}.
\]
Applying Lemma 2.6 we have
\[
\frac{3}{4} \| DF_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)} + \| U_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)} \\
\leq C \| f \|^2_{L^2 (\Omega)} + C_1 \| DA F_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)} + C \| F_{A,\epsilon} (Du) \|^2_{L^1 (\Omega)}.
\]
By Corollary 2.7 we further have (2.17) as desired. \( \square \)

**Proof of Theorem 1.5 (i)** Since (2.16) is similar to (2.11), by exactly the same argument as the proof of Theorem 1.4 (i), we obtain Theorem 1.5(i). \( \square \)

**Proof of Theorem 1.5 (ii)** It suffices to show that
\[
C_1 \| |D| F_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)} \leq \frac{1}{4} \| DF_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)} + C \| f \|^2_{L^2 (\Omega)}.
\]
To this end, let \( \{ B_k \}_{1 \leq k \leq N} \) be a covering of \( \overline{\Omega} \) as in the proof of Theorem 1.4 (ii) and correspondingly \( \{ \phi_k \}_{k \in \mathbb{N}} \) be therein. Write
\[
C_1 \| |D| F_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)} = \sum_k C_1 \| D F_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)}.
\]
Note that \( C_1 \| D F_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)} \) is bounded by (2.12) with \( \phi = \phi_k \). Thus
\[
C_1 \| |D| F_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)} \leq \frac{1}{4} \sum_k \| D F_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)} + \frac{C_3}{4C_\sharp} \left[ \sum_k \| D \phi_k F_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)} + C \sum_k \| \phi_k F_{A,\epsilon} (Du) \|^2_{L^1 (\Omega)} \right]
\]
\[
\leq \frac{1}{4} \| DF_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)} + \frac{C_3}{4C_\sharp} \left[ \frac{NC_4^2}{r_\sharp^2} \| F_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)} + C \| F_{A,\epsilon} (Du) \|^2_{L^1 (\Omega)} \right]. \tag{2.19}
\]
Choose \( \eta \) small enough so that
\[
\frac{C_3}{4C_\sharp} \frac{NC_4^2}{r_\sharp^2} \eta \leq \frac{1}{4}.
\]
Applying Lemma 2.6, we have
\[
C_1 \| |D| F_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)} \leq \frac{1}{2} \| DF_{A,\epsilon} (Du) \|^2_{L^2 (\Omega)} + C \| F_{A,\epsilon} (Du) \|^2_{L^1 (\Omega)}.
\]
By \( \| F_{\epsilon} (AD) \|^2_{L^1 (\Omega)} \leq C \| f \|^2_{L^2 (\Omega)} \), one has (2.18) as desired. \( \square \)

**3 Proof of Key Lemma 2.1**

In this section, we always let \( 1 < p < \infty \) and \( \epsilon \in (0, 1] \), and suppose that \( \Omega \) is a smooth domain and \( A \in E_L (\Omega) \cap C^\infty (\Omega) \). To get Lemma 2.1, it is suffices to prove the following two lemmas.
Lemma 3.1 For any $u \in C^3(\Omega)$, we have
\[
(L_{\epsilon, A, p} u)^2 \geq \frac{3}{4} \min\{1, (p-1)^2\} [U_{A, \epsilon}(Du)]^2 - C|DA|^2 |F_{A, \epsilon}(Du)|^2 + \text{div} \left\{ \left( |\sqrt{A} Du|^2 + \epsilon \right)^{p-2} \left[ \text{div} (ADu) ADu - (ADu \cdot DADu) \right] \right\} \text{ in } \Omega,
\]
where $C > 1$ is a constant depending on $n$, $p$ and $L$.

Lemma 3.2 For any $u \in C^3(\Omega)$ and any $\eta \in (0, 1)$, we have
\[
\left| \text{div} \left\{ \left( |\sqrt{A} Du|^2 + \epsilon \right)^{p-2} \left[ \text{div} (ADu) ADu - (ADu \cdot DADu) \right] \right\} - \text{div} \left\{ \left( |\sqrt{A} Du|^2 + \epsilon \right)^{p-2} \left[ \text{tr} (AD^2 u) ADu - AD^2 u ADu \right] \right\} \right| \leq \eta[U_{A, \epsilon}(Du)]^2 + \frac{C}{\eta} |DA|^2 |F_{A, \epsilon}(Du)|^2 \text{ in } \Omega,
\]
where $C > 1$ is a constant depending on $n$, $p$ and $L$.

Proof of Lemma 2.1 The inequality (2.3) with $I_2$ given by (2.5) follows from (3.1). Moreover the inequality (2.3) with $I_1$ given by (2.4) follows from (3.1) and (3.2) with $\eta = \frac{1}{4} \min\{1, (p-1)^2\}$.

To prove Lemmas 3.1 and 3.2, we need the following two lemmas.

Lemma 3.3 For any symmetric $n \times n$ matrix $M$ and any vector $\xi \in \mathbb{R}^n$ with $|\xi| \leq 1$, we have
\[
|M|^2 \geq 2|M\xi|^2 - |M\xi \cdot \xi|^2.
\]

Proof If $|\xi| = 1$, then $\xi = Oe_n$ for some orthogonal matrix $O$. Write $O^T MO = (m_{ij})_{1 \leq i, j \leq n}$. We have
\[
|M|^2 = |O^T MO|^2 = \sum_{1 \leq i, j \leq n} m_{ij}^2 \geq \sum_{i=1}^n m_{ii}^2 + \sum_{i=1}^n m_{in}^2 - m_{nn}^2.
\]
Then (3.3) follows from
\[
m_{nn} = O^T MOe_n \cdot e_n = MOe_n \cdot Oe_n = M\xi \cdot \xi,
\]
and
\[
\sum_{i=1}^n m_{ii}^2 = \sum_{i=1}^n m_{in}^2 = |O^T MOe_n|^2 = |M\xi|^2,
\]
where we note $O^T MOe_n = (m_{in})_{1 \leq i \leq n}$ and use symmetry.

Assume $|\xi| < 1$. If $|\xi| = 0$, then (3.3) holds trivially. It $0 < |\xi| < 1$, applying (3.3) to $\xi/|\xi|$, we have
\[
|M|^2 \geq 2\frac{|M\xi|^2}{|\xi|^2} \frac{|M\xi \cdot \xi|^2}{|\xi|^4} = \frac{|M\xi|^2}{|\xi|^2} |\xi|^2 - \frac{|M\xi \cdot \xi|^2}{|\xi|^4} |\xi|^4.
\]
Since the Cauchy-Schwarz inequality gives $|M\xi|^2|\xi|^2 \geq |M\xi \cdot \xi|^2$, by $|\xi| < 1$ and $|\xi| + 1/|\xi| \geq 2$ we have

$$|M|^2 \geq \frac{|M\xi|^2}{|\xi|^2} + |M\xi|^2|\xi|^2 - |M\xi \cdot \xi|^2 \geq 2|M\xi|^2 - |M\xi \cdot \xi|^2.$$ 

□

**Remark 3.4** After diagonalizing $M$, the inequality (3.3) reads as

$$\left( \sum_{i=1}^{n} \xi_i^2 \lambda_i \right)^2 - 2 \sum_{i=1}^{n} \xi_i^2 \lambda_i^2 + \sum_{i=1}^{n} \lambda_i^2 \geq 0, \quad \forall \xi \in \mathbb{R}^n \text{ with } |\xi| \leq 1, \quad (3.4)$$

which was proved by [8, Lemma 3.2] and used to prove Lemma 3.1 with $A = I_n$. The above is a much simpler proof to (3.4).

**Lemma 3.5** Assume that $M$ is an $n \times n$ symmetric matrix with $\frac{1}{L} \leq M\xi \cdot \xi \leq L$ for all $\xi \in \mathbb{R}^n$ with $|\xi| = 1$. Then $\frac{1}{\sqrt{L}} \leq \sqrt{M}\xi \cdot \xi \leq \sqrt{L}$ for all $\xi \in \mathbb{R}^n$ with $|\xi| = 1$. Moreover, for any $n \times n$ matrix $H$, we have

$$\frac{1}{\sqrt{L}} |H| \leq |\sqrt{M}H| \leq \sqrt{L} |H| \quad \text{and} \quad \frac{1}{\sqrt{L}} |H| \leq |H\sqrt{M}| \leq \sqrt{L} |H|. \quad (3.5)$$

**Proof of Lemma 3.5** We can find an orthogonal matrix $O$ such that $O^TMO = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ with $\frac{1}{L} \leq \lambda_1 \leq \cdots \leq \lambda_n \leq L$ and $O^T\sqrt{M}O = \text{diag}\{\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}\}$ with $\frac{1}{\sqrt{L}} \leq \sqrt{\lambda_1} \leq \cdots \leq \sqrt{\lambda_n} \leq \sqrt{L}$. Thus $\frac{1}{\sqrt{L}} \leq \sqrt{M}\xi \cdot \xi \leq \sqrt{L}$ for all $\xi \in \mathbb{R}^n$ with $|\xi| = 1$.

Given any matrix $H$, we have

$$|\sqrt{M}H|^2 = |O^T\sqrt{M}OM^T H|^2 = |\text{diag}(\lambda_1, \cdots, \lambda_n) O^T H|,$$

and hence

$$\frac{1}{L} |H|^2 \leq \lambda_1 |O^T H|^2 \leq |\sqrt{M}H|^2 \leq \lambda_n |O^T H|^2 \leq L |H|^2.$$ 

The same result holds for $|H\sqrt{M}|^2$. □

Below we prove Lemma 3.1 and Lemma 3.2.

**Proof of Lemma 3.1** For simplicity, we write

$$\Pi := [\mathcal{L}_{\epsilon, A, \rho} u]^2 - \text{div}\left( (|\sqrt{A}Du|^2 + \epsilon)^{p-2} \text{div} (ADu)ADu - (ADu \cdot D)ADu \right),$$

and then, (3.1) is equivalent to

$$\Pi \geq \frac{3}{4} \min\{1, (p-1)^2\} (|DAu|^2 + \epsilon)^{p-2} |D^2 Au|^2 - C(n, p, L)|DA|^2 |F_{\epsilon}(Du)|^2. \quad (3.6)$$

We prove (3.6) as below. For vectors $a, b \in \mathbb{R}^n$, we use $\langle a, b \rangle = a \cdot b$ to denote the usual inner product. We always use the Einstein summation convention, that is, $a_i b_i = a^l b_i = \sum_{i=1}^{n} a_i b_i$. For short, we always write $Du = (\partial_i v)_{1 \leq i \leq n} = (v_i)_{1 \leq i \leq n}$.

$$D_A v := \sqrt{A}Du, \quad D^2_A v := \sqrt{A}D^2 v \sqrt{A}, \quad (3.7)$$

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and
\[\xi \cdot Dv = \xi_i \partial_i v, \quad \langle (DA)\xi, \xi \rangle = \xi^T (DA)\xi = (a^{ij}_{k} \xi_i \xi_j)_{1 \leq k \leq n}.\]

Firstly, a direct calculation yields
\[
\mathcal{L}_{\epsilon, A, p} u = \text{div} \left( (|DAu|^2 + \epsilon)^{\frac{p-2}{2}} ADu \right) \\
= (|DAu|^2 + \epsilon)^{\frac{p-2}{2}} \text{div} (ADu) + (p-2)(|DAu|^2 + \epsilon)^{\frac{p-4}{2}} \left\{ D \frac{|DAu|^2}{2}, ADu \right\},
\]
and hence
\[
[\mathcal{L}_{\epsilon, A, p} u]^2 = (|DAu|^2 + \epsilon)^{p-2} [\text{div} (ADu)]^2 \\
+ 2(p-2)(|DAu|^2 + \epsilon)^{p-3} \text{div} (ADu) \left\{ D \frac{|DAu|^2}{2}, ADu \right\} \\
+ (p-2)^2 (|DAu|^2 + \epsilon)^{p-4} \left\{ D \frac{|DAu|^2}{2}, ADu \right\}^2.
\]
(3.8)

We also observe that
\[
\text{div} \left[ (|DAu|^2 + \epsilon)^{p-2} [\text{div} (ADu) ADu - (ADu \cdot D) ADu] \right] \\
= (|DAu|^2 + \epsilon)^{p-2} \text{div} [\text{div} (ADu) ADu - (ADu \cdot D) ADu] \\
+ 2(p-2)(|DAu|^2 + \epsilon)^{p-3} \text{div} (ADu) \left\{ D \frac{|DAu|^2}{2}, ADu \right\} \\
- 2(p-2)(|DAu|^2 + \epsilon)^{p-3} \left\{ D \frac{|DAu|^2}{2}, ADu \right\}.
\]
(3.9)

where
\[
\text{div} [\text{div} (ADu) ADu - (ADu \cdot D) ADu] \\
= [\text{div} (ADu)]^2 + (ADu \cdot D) \text{div} (ADu) - (ADu \cdot D) \text{div} (ADu) \\
- \left( \partial_i (a^{js} u_s) \right) \cdot \left( \partial_j (a^{ik} u_k) \right) \\
= [\text{div} (ADu)]^2 - \left( \partial_i (a^{js} u_s) \right) \cdot \left( \partial_j (a^{ik} u_k) \right).
\]
(3.10)

Since
\[
-\left( \partial_i (a^{js} u_s) \right) \cdot \left( \partial_j (a^{ik} u_k) \right) = -a^{js}_i u_s a^{ik}_j u_k - 2a^{js}_i u_s \delta^{jk} a^{ik}_j u_k - a^{ik} a^{js} u_s u_k,
\]
and
\[
u^{jk} a^{js} u_s a^{ik} = \text{tr} (AD^2 u) AD^2 u \\
= \text{tr} (\sqrt{A} D^2 u AD^2 u \sqrt{A}) = \text{tr} (\sqrt{A} D^2 u \sqrt{A})^2 = \text{tr} (D^2 A u)^2 = |D^2 A u|^2,
\]
we have
\[
\text{div} [\text{div} (ADu) ADu - (ADu \cdot D) ADu] \\
= [\text{div} (ADu)]^2 - |D^2 A u|^2 - a^{js}_i u_s a^{ik}_j u_k - 2a^{js} u_s a^{ik}_j u_k.
\]
(3.11)
From (3.8), (3.9) and (3.11), we deduce that
\[
\Pi = (|D_A u|^2 + \epsilon)^{p-2} |D_A^2 u|^2 \\
+ 2(p - 2)(|D_A u|^2 + \epsilon)^{p-3} \left( D \frac{|D_A u|^2}{2}, (ADu \cdot D) ADu \right) \\
+ (p - 2)^2(|D_A u|^2 + \epsilon)^{p-4} \left( D \frac{|D_A u|^2}{2}, ADu \right)^2 \\
+ (|D_A u|^2 + \epsilon)^{p-2} [a^{ijs} u a^{ik} u_k + 2a^{ijs} u a^{ik} u_k] \\
=: \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4. \tag{3.12}
\]

Next we bound \( \Pi_2, \Pi_3 \) and \( \Pi_4 \) as below. To bound \( \Pi_2 \), write
\[
D \frac{|D_A u|^2}{2} = \frac{1}{2} D(ADu, Du) = D^2 u ADu + \frac{1}{2} (Du)^T (DA) Du. \tag{3.13}
\]

Then
\[
\left\{ D \frac{|D_A u|^2}{2}, ADu \right\} = \langle D^2 u ADu, ADu \rangle + \frac{1}{2} \langle (Du)^T (DA) Du, ADu \rangle \\
= (Du)^T D_A^2 u ADu + \frac{1}{2} (Du)^T [(ADu \cdot D) A] Du,
\]
that is,
\[
\left\{ D \frac{|\sqrt{AD} Du|^2}{2}, ADu \right\}^2 = [(Du)^T D_A^2 u ADu]^2 + \frac{1}{4} \langle (Du)^T [(ADu \cdot D) A] Du \rangle^2 \\
+ [(Du)^T D_A^2 u ADu] (Du)^T [(ADu \cdot D) A] Du.
\]

Multiplying both side by \((p - 2)^2\) and using Cauchy-Schwarz inequality, we obtain
\[
(p - 2)^2 \left\{ D \frac{|\sqrt{AD} Du|^2}{2}, ADu \right\}^2 \\
\geq (p - 2)^2 [(Du)^T D_A^2 u ADu]^2 - \eta |D_A^2 u|^2 |Du|^4 - \frac{C}{\eta} |DA|^2 |DA u|^4 |Du|^2. \tag{3.14}
\]

Multiplying both side by \(|D_A u|^2 + \epsilon\)^{p-4}, we obtain
\[
\Pi_2 \geq (p - 2)^2 (|D_A u|^2 + \epsilon)^{p-4} [(Du)^T D_A^2 u ADu]^2 - \eta \Pi_1 - \frac{C}{\eta} |DA|^2 |F_{A, \epsilon} (Du)|^2. \tag{3.15}
\]

To bound \( \Pi_3 \), by (3.13) and
\[
(ADu \cdot D) ADu = AD^2 u ADu + [(ADu \cdot D) A] Du,
\]
we have
\[
\left\{ D \frac{|\sqrt{AD} Du|^2}{2}, (ADu \cdot D) ADu \right\} \\
= \left\{ D^2 u ADu + \frac{1}{2} (Du)^T (DA) Du, AD^2 u ADu + [(ADu \cdot D) A] Du \right\} \\
= \langle D^2 u ADu, AD^2 u ADu \rangle + \langle D^2 u ADu, [(ADu \cdot D) A] Du \rangle \\
+ \frac{1}{2} \langle (Du)^T (DA) Du, AD^2 u ADu \rangle + \frac{1}{2} \langle (Du)^T (DA) Du, [(ADu \cdot D) A] Du \rangle.
\]
By the Cauchy-Schwarz inequality one gets
\[
2(p - 2) \left( \frac{|ADu|^2}{2} + (ADu \cdot D) ADu \right) \geq 2(p - 2)|DA2uDAu|^2 - \eta|DA^2u|^2|DAu|^2 - \frac{C}{\eta}|DA|^2|DAu|^2|Du|^2. \tag{3.16}
\]

Multiplying both sides by \((|DAu|^2 + \epsilon)^{p-3}\) we obtain
\[
\Pi_3 \geq (p - 2)(|DAu|^2 + \epsilon)^{p-3}|DA^2uDAu|^2 - \eta \Pi_1 - \frac{C}{\eta}|DA|^2|FA_{A, \epsilon}(Du)|^2. \tag{3.17}
\]

For \(\Pi_4\), we observe that
\[
a^{ij}_s u_s a^{ik}_j u_k \geq -|DA|^2|Du|^2.
\]
By Lemma 3.5 and the Cauchy-Schwarz inequality, one also has
\[
2a^{js}_i u_s a^{lk}_j u_k \geq -|AD^2u||DA||Du| \geq -L|DA^2u|^2|Du| \geq -\eta|DA^2u|^2 - \frac{C}{\eta}|DA|^2|Du|^2.
\]
Thus
\[
\Pi_4 \geq -\eta \Pi_1 - \frac{C}{\eta}|DA|^2|FA_{A, \epsilon}(Du)|^2.
\]

From the lower bound of \(\Pi_2\), \(\Pi_3\) and \(\Pi_4\), noting \((|DAu|^2 + \epsilon)^{p-2}|Du|^2 \leq L^2|FA_{A, \epsilon}(Du)|^2\), we obtain
\[
\Pi \geq (1 - 3\eta)(|DAu|^2 + \epsilon)^{p-2}|DA^2u|^2 + 2(p - 2)(|DAu|^2 + \epsilon)^{p-3}|DA^2uDAu|^2 \\
+ (p - 2)^2(|DAu|^2 + \epsilon)^{p-4}(|DAu|^TDA^2uDAu)^2 - \frac{C}{\eta}|DA|^2|FA_{A, \epsilon}(Du)|^2. \tag{3.18}
\]

If \(p \geq 2\), taking \(\eta = \frac{1}{16}\) in (3.18), noting that the second and third terms in the right hand side of (3.18) are nonnegative, we have (3.6).

Hence below we assume that \(1 < p < 2\). Then \(1 - p(2 - p) = (p - 1)^2 > 0\) and hence, \(1 > p(2 - p) > 0\). We split the coefficient \((1 - 3\eta)\) of \((|DAu|^2 + \epsilon)^{p-2}|DA^2u|^2\) in the right hand side of (3.18) as
\[
(1 - 3\eta) = [1 - p(2 - p) - 3\eta] + p(2 - p) \geq \frac{3}{4}(p - 1)^2 + p(2 - p) \tag{3.19}
\]
where we choose \(\eta = \frac{1}{16}(p - 1)^2\). Then,
\[
\Pi \geq \frac{3}{4}(p - 1)^2(|DAu|^2 + \epsilon)^{p-2}|DA^2u|^2 - \frac{C}{\eta}|DA|^2|FA_{A, \epsilon}(Du)|^2 \\
+ (2 - p)(|DAu|^2 + \epsilon)^{p-2}\left\{ p|DA^2u|^2 - 2\frac{|DA^2uDAu|^2}{|DAu|^2 + \epsilon} + (2 - p)\frac{|DAu|^TDA^2uDAu|^2}{(|DAu|^2 + \epsilon)^2} \right\}. \tag{3.20}
\]
Applying (3.3) to \(DA^2u\) and \(DAu/\sqrt{|DAu|^2 + \epsilon}\), and multiplying both sides by \(p\), one has
\[
p|DA^2u|^2 \geq 2p\frac{|DA^2uDAu|^2}{|DAu|^2 + \epsilon} - p\frac{|DAu|^TDA^2uDAu|^2}{(|DAu|^2 + \epsilon)^2} \\
= 2\frac{|DA^2uDAu|^2}{|DAu|^2 + \epsilon} + \left[(2p - 2)\frac{|DA^2uDAu|^2}{|DAu|^2 + \epsilon} - p\frac{|DAu|^TDA^2uDAu|^2}{(|DAu|^2 + \epsilon)^2}\right]. \tag{3.21}
\]
Since Cauchy-Schwarz inequality gives
\[
|D^2_A u D_A u|^2 \geq \frac{|(D_A u)^T D^2_A u D_A u|^2}{|D_A u|^2 + \epsilon},
\]
establishes
\[
p|D^2_A u|^2 \geq 2 \frac{|D^2_A u D_A u|^2}{|D_A u|^2 + \epsilon} - (2 - p) \frac{|(D_A u)^T D^2_A u D_A u|^2}{(|D_A u|^2 + \epsilon)^2}.
\] (3.22)

This inequality shows us the third term in the right hand side of (3.20) is nonnegative. Hence, (3.6) follows from (3.20).

\[\blacksquare\]

**Remark 3.6** In the case \( A = I_n \), we can take \( \eta = 0 \) in the above proof to get
\[
\Pi = [\text{div}(|Du|^2 + \epsilon) \frac{p}{p-2} Du^2 - \text{div}(|Du|^2 + \epsilon)^p - 2|\Delta u Du - D^2 u Du|)
\geq \min\{1, (p - 1)^2\}(|Du|^2 + \epsilon)^p - 2|D^2 u|^2.
\] (3.23)

Indeed, when \( A = I_n \), (3.12) becomes
\[
\Pi = (|Du|^2 + \epsilon)^p - 2|Du|^2 + 2(p - 2) \frac{|D^2 u Du|^2}{|Du|^2 + \epsilon} + (p - 2)^2 \frac{|(Du)^T D^2 u Du|^2}{(|Du|^2 + \epsilon)^2}.
\] (3.24)

If \( p \geq 2 \), one has \( \Pi \geq (|Du|^2 + \epsilon)^p - 2|D^2 u|^2 \). If \( 1 < p < 2 \), by splitting the coefficient 1 of \( |D^2 u|^2 \) as \((p - 1)^2 + p(2 - p)\) we have
\[
\Pi = (p - 1)^2 (|Du|^2 + \epsilon)^p - 2|D^2 u|^2
+ (2 - p)(|Du|^2 + \epsilon)^p - 2 \left[ p|D^2 u|^2 - 2 \frac{|D^2 u Du|^2}{|Du|^2 + \epsilon} + (2 - p) \frac{|(Du)^T D^2 u Du|^2}{(|Du|^2 + \epsilon)^2} \right].
\]

Applying Lemma 3.3 to \( D^2 u \) and \( Du/\sqrt{|Du|^2 + \epsilon} \), by an argument same as (3.22) we obtain
\[
p|D^2 u|^2 \geq 2 \frac{|D^2 u Du|^2}{|Du|^2 + \epsilon} - (2 - p) \frac{|(Du)^T D^2 u Du|^2}{(|Du|^2 + \epsilon)^2}.
\]

Hence \( \Pi \geq (p - 1)^2 (|Du|^2 + \epsilon)^p - 2|D^2 u|^2 \) as desired.

**Proof of Lemma 3.2** A direct calculation gives
\[
\text{div} \left\{ (|\sqrt{A} Du|^2 + \epsilon)^p - 2 [\text{div} (ADu) ADu - (ADu \cdot D) ADu] \right\}
- \text{div} \left\{ (|\sqrt{A} Du|^2 + \epsilon)^p - 2 [\text{tr}(AD^2 u) ADu - AD^2 u ADu] \right\}
= (|\sqrt{A} Du|^2 + \epsilon)^p - 2 \cdot \text{div} \{ [\text{div} (ADu) ADu - (ADu \cdot D) ADu] - [\text{tr}(AD^2 u) ADu - AD^2 u ADu] \}
+ D(|\sqrt{A} Du|^2 + \epsilon)^p - 2 \cdot \{ [\text{div} (ADu) ADu - (ADu \cdot D) ADu] - [\text{tr}(AD^2 u) ADu - AD^2 u ADu] \}
=: J_1 + J_2.
\]

It then suffices to prove that, for \( i = 2 \),
\[
J_i \geq - \eta [U_{A,\epsilon}(Du)]^2 - \frac{C}{\eta} |DA|^2 F_{A,\epsilon}(Du)^2.
\]
A direct calculation also yields
\[
\text{div} \left( (ADu) ADu - (ADu \cdot D) ADu \right) - \text{tr}(AD^2 u) ADu - AD^2 u ADu
= \left( \text{div} A \cdot Du \right) ADu - \left( (ADu \cdot D) A \right) Du,
\]
(3.25)
where \( \text{div} A \cdot Du = a^j_k u_k \). Since
\[
D(|DAu|^2 + 4) = (p - 2)(|DAu|^2 + 4)^{p-4}[2D^2 u ADu + (Du)^T (DA) Du],
\]
by the Cauchy-Schwarz inequality and Lemma 3.5 one has
\[
J_2 = (p - 2)(|DAu|^2 + 4)^{p-4}[2D^2 u ADu + (Du)^T (DA) Du]
\]
\[
\cdot \left[ (\text{div} A \cdot Du) ADu - \left( (ADu \cdot D) A \right) Du \right]
\]
\[
= (p - 2)(|DAu|^2 + 4)^{p-4}\{(2(\text{div} A \cdot Du)(ADu)^T D^2 u ADu - 2D^2 u ADu \cdot [(ADu \cdot D) A] Du
\]
\[
+ (\text{div} A \cdot Du)(Du)^T (DA) Du \cdot ADu - (Du)^T (DA) Du \cdot [(ADu \cdot D) A] Du \}
\]
\[
\geq \eta(|DAu|^2 + 4)^{p-2}|DAu|^2 - \frac{C}{\eta} |DA|^2 |DAu|^2 - |DA|^2 |DAu|^2
\]
\[
\geq -\eta|U_{A,e}(Du)|^2 - \frac{C}{\eta} |DA|^2 |F_{A,e}(Du)|^2,
\]
for any \( \eta \in (0, 1) \). Moreover, by the Cauchy-Schwarz inequality, one has
\[
\text{div} \left[ (\text{div} A \cdot Du) ADu - \left( (ADu \cdot D) A \right) Du \right]
\]
\[
= \langle D^2 u (\text{div} A), ADu \rangle + (\text{div} A \cdot Du)^2 + (\text{div} A \cdot Du) \text{tr}(AD^2 u)
\]
\[
\quad - a^{jk}_i u_{k,i} j_{l,i} - a^{ik}_j u_{k,i} a_{j,l} - a^{jk}_i u_{j,i} a_{j,l}
\]
\[
\geq -\eta|DAu|^2 - \frac{C}{\eta} |DA|^2 |A\text{Du}|^2.
\]
Thus
\[
J_1 \geq -\eta|U_{A,e}(Du)|^2 - \frac{C}{\eta} |DA|^2 |F_{A,e}(Du)|^2,
\]
as desired. \( \square \)

4 Proof of Key Lemma 2.2

In this section, we prove Lemma 2.2. We let \( 1 < p < \infty \) and \( \epsilon \in (0, 1) \) and suppose that \( \Omega, f \) and \( A \) satisfy assumptions (S1)–(S3). Let \( u \) be a weak solution to (1.12) with Dirichlet or Neumann homogeneous boundary condition.

Proof of Lemma 2.2 Recall that (2.3) gives
\[
\frac{1}{2} \min\{1, (p - 1)^2\}|U_{A,e}(Du)|^2 \leq f^2 + C|DA|^2 |F_{A,e}(Du)|^2 - I_i, \quad i = 1, 2.
\]
Since
\[
DF_{A,e}(Du) = (p - 2)(|DAu|^2 + \epsilon)^{p-4}[D^2 u ADu \otimes ADu + \frac{1}{2}(Du)^T (DA) Du \otimes ADu]
\]
\[
+ (|DAu|^2 + \epsilon)^{p-2}[(DA) Du + AD^2 u],
\]
by the Cauchy-Schwarz inequality one has
\[ |DF_{A,e}(Du)|^2 \leq L[1 + |p - 2||U_{A,e}(Du)|^2 + C|DA|^2|F_{A,e}(Du)|^2. \]

We then get
\[ [U_{A,e}(Du)]^2 + |DF_{A,e}(Du)|^2 \leq Cf^2 + C|DA|^2|F_{A,e}(Du)|^2 - CD_i, \quad i = 1, 2. \]

Multiplying both sides by \( \phi^2 \) for any \( \phi \in \mathcal{C}^\infty_0(\mathbb{R}^n) \), and integrating over \( \Omega \), we get
\[
\int_\Omega \phi^2 |DF_{A,e}(Du)|^2 dx + \int_\Omega \phi^2 |U_{A,e}(Du)|^2 dx \leq C \int_\Omega \phi^2 f^2 dx + C \int_\Omega \phi^2 |DA|^2 |F_e(ADu)|^2 dx + \tilde{K}_i(\phi), \quad i = 1, 2,
\]

where
\[
\tilde{K}_1(\phi) = -C \int_\Omega \phi^2 \text{div} \left\{ \left( \sqrt{ADu}^2 + \epsilon \right)^{p-2} [\text{tr}(AD^2u)ADu - AD^2uADu] \right\} dx,
\]

and
\[
\tilde{K}_2(\phi) = -C \int_\Omega \phi^2 \text{div} \left\{ \left( \sqrt{ADu}^2 + \epsilon \right)^{p-2} [\text{div}(ADu)ADu - (ADu \cdot D)ADu] \right\} dx.
\]

We use the divergence theorem for (4.2) to get
\[
\tilde{K}_1(\phi) = 2C \int_\Omega \phi D\phi \left( \sqrt{ADu}^2 + \epsilon \right)^{p-2} [\text{tr}(AD^2u)ADu - AD^2uADu] dx
\]
\[
- C \int_{\partial\Omega} \phi^2 (\sqrt{ADu}^2 + \epsilon)^{p-2} [\text{tr}(AD^2u)ADu - AD^2uADu] \cdot \nu d\mathcal{H}^{n-1}(x).
\]

Owing to Young’s inequality, one has
\[
2C \int_\Omega \phi D\phi \left( \sqrt{ADu}^2 + \epsilon \right)^{p-2} [\text{tr}(AD^2u)ADu - AD^2uADu] dx
\]
\[
\leq C \int_\Omega |\phi D\phi| \left( \sqrt{ADu}^2 + \epsilon \right)^{p-2} |AD^2u||ADu| dx
\]
\[
\leq \frac{1}{4} \int_\Omega \phi^2 \left( \sqrt{ADu}^2 + \epsilon \right)^{p-2} |AD^2u|^2 dx + C \int_\Omega |D\phi|^2 |ADu|^2 \left( \sqrt{ADu}^2 + \epsilon \right)^{p-2} dx
\]
\[
= \frac{1}{4} \int_\Omega \phi^2 |U_{A,e}(Du)|^2 dx + C \int_\Omega |D\phi|^2 |F_{A,e}(Du)|^2 dx.
\]

We therefore obtain (2.6) with \( K(\phi) \) given by (2.7).

We use the divergence theorem for (4.3) to get
\[
\tilde{K}_2(\phi) = 2C \int_\Omega \phi D\phi \left( \sqrt{ADu}^2 + \epsilon \right)^{p-2} [\text{tr}(ADu)ADu - (ADu \cdot D)ADu] dx
\]
\[
- C \int_{\partial\Omega} \phi^2 \left( \sqrt{ADu}^2 + \epsilon \right)^{p-2} [\text{div}(ADu)ADu - (ADu \cdot D)ADu] \cdot \nu d\mathcal{H}^{n-1}(x).
\]
Owing to Young’s inequality, one has
\[
2C \int_{\Omega} \phi D\phi \left( |\sqrt{A}Du|^2 + \varepsilon \right)^{p-2} [\text{div} (ADu) ADu - (ADu \cdot D) ADu] dx \\
\leq C \int_{\Omega} |\phi D\phi| \left( |\sqrt{A}Du|^2 + \varepsilon \right)^{p-2} |\nabla (ADu)| |ADu| dx \\
\leq \frac{1}{4} \int_{\Omega} \phi^2 (|\sqrt{A}Du|^2 + \varepsilon)_{p-2} |AD^2u|^2 dx \\
+ C \int_{\Omega} \left( |D\phi|^2 |ADu|^2 + |\phi|^2 |DA|^2 |Du|^2 \right) \left( |\sqrt{A}Du|^2 + \varepsilon \right)^{p-2} dx \\
\leq \frac{1}{4} \int_{\Omega} \phi^2 (UA_{A,e}(Du))^2 dx \\
+ C \int_{\Omega} |D\phi|^2 |FA_{A,e}(Du)|^2 dx + C \int_{\Omega} |\phi|^2 |DA|^2 |FA_{A,e}(Du)|^2 dx. \tag{4.4}
\]
We therefore obtain (2.6) with \( K(\phi) \) given by (2.8). \( \square \)

5 Proof of Key Lemma 2.3

Given any bounded smooth domain \( \Omega \subset \mathbb{R}^n \), recall that \( \nu \) denotes the unit outer normal vector to the boundary \( \partial \Omega \). Below we write \( T(\partial \Omega) \) the tangential space of \( \partial \Omega \), and let \( \{\tau_1, \ldots, \tau_{n-1}\} \) be an normalized orthogonal basis of \( T(\partial \Omega) \) so that \( \{\tau_1, \ldots, \tau_{n-1}, \nu\} \) has the same orientation as \( \{e_1, \ldots, e_n\} \). We use \( \text{div}_T \) and \( \nabla_T \) to denote the divergence and the gradient operator on \( \partial \Omega \). We also recall that \( B \) is the second fundamental form of \( \partial \Omega \), which is given by
\[
B(\xi, \eta) = -\frac{\partial}{\partial \xi} \cdot \eta = -\sum_{k=1}^{n-1} \left( \xi \cdot \tau_k \right) \frac{\partial \nu}{\partial \tau_k} \cdot \eta = -\sum_{i,k=1}^{n-1} \left( \frac{\partial \nu}{\partial \tau_k} \cdot \tau_i \right) (\xi \cdot \tau_k)(\eta \cdot \tau_i), \quad \forall \xi, \eta \in T(\partial \Omega),
\]
where \( \frac{\partial}{\partial \xi} \) denote the derivative along the direction \( \xi \). We use \( |B| \) to denote the norm of \( B \), that is,
\[
|B| = \sup_{|\xi|,|\eta| \leq 1} B(\xi, \eta).
\]
The trace of \( B \) is given by
\[
\text{tr} B = \sum_{i=1}^{n-1} B(\tau_i, \tau_i) = -\sum_{i=1}^{n-1} \frac{\partial \nu}{\partial \tau_i} \cdot \tau_i.
\]

To prove Lemma 2.3, we need a series of lemmas. Firstly, we need the following result to bound the term in \( K(\phi) \) as in (2.7). The proof of Lemma 5.1 is postponed to Sect. 5.1.

**Lemma 5.1** Assume that \( u \in C^2(\overline{\Omega}) \) and \( u = 0 \) on \( \partial \Omega \). We have
\[
-\left[ \text{tr} (AD^2u) ADu - AD^2u ADu \right] \cdot \nu \leq C |B|^2 |ADu|^2 \quad \text{on} \ \partial \Omega \tag{5.1}
\]
with the constant \( C \) depending on \( L \); moreover if \( \Omega \) is convex, we have
\[
-\left[ \text{tr} (AD^2u) ADu - AD^2u ADu \right] \cdot \nu \leq 0 \quad \text{on} \ \partial \Omega. \tag{5.2}
\]
We also need the following result to bound the term in $K(\phi)$ as in (2.8). The proof of Lemma 5.2 is postponed to Sect. 5.1.

**Lemma 5.2** Assume that $u \in C^2(\Omega)$ and $ADu \cdot v = 0$ on $\partial \Omega$. We have

$$-\left[ \text{div}(ADu)ADu - (ADu \cdot D)ADu \right] \cdot v \leq |B||ADu|^2 \text{ on } \partial \Omega.$$  

Moreover, if $\Omega$ is convex, we have

$$-\left[ \text{div}(ADu)ADu - (ADu \cdot D)ADu \right] \cdot v \leq 0 \text{ on } \partial \Omega.$$

The following trace inequality was proved by Cianchi-Maz’ya [8], and will be used to prove Lemma 2.3 (ii).

**Lemma 5.3** For any $x \in \partial \Omega$ and $0 < r < 1$, when $n \geq 3$, one has

$$\int_{\partial \Omega \cap B_r(x)} v^2 |B| d\mathcal{H}^{n-1}(y) \leq C_{\Psi} \int_{\Omega \cap B_r(x)} |\nabla v|^2 d\mathcal{H}^n(x). \quad (5.3)$$

Now, we are ready to prove Lemma 2.3.

**Proof of Lemma 2.3 (i).** If $\phi$ is supported in $\Omega$, then $K(\phi) = 0$. Assume that the support of $\phi$ has non-empty intersection with $\partial \Omega$. Since $\Omega$ is convex, when $u = 0$ on $\partial \Omega$, applying Lemma 5.1, we have

$$-\left[ \text{tr}(AD^2u)ADu - AD^2uADu \right] \cdot v \leq 0.$$  

Hence, for the $K(\phi)$ given by (2.7), we have

$$K(\phi) = -C \int_{\partial \Omega} \phi^2 \left( |\sqrt{ADu}|^2 + \epsilon \right)^{p-2} [\text{tr}(AD^2u)ADu - AD^2uADu] \cdot v d\mathcal{H}^{n-1}(x) \leq 0.$$

When $ADu \cdot v = 0$ on $\partial \Omega$, in view of Lemma 5.2 we know

$$-\left[ \text{div}(ADu)ADu - (ADu \cdot D)ADu \right] \cdot v \leq 0.$$  

Hence, for the $K(\phi)$ given by (2.8), we have

$$K(\phi) = -C \int_{\partial \Omega} \phi^2 \left( |\sqrt{ADu}|^2 + \epsilon \right)^{p-2} [\text{div}(ADu)ADu - (ADu \cdot D)ADu] \cdot v d\mathcal{H}^{n-1}(x) \leq 0.$$

\[\square\]

**Proof of Lemma 2.3 (ii).** When $u = 0$ on $\partial \Omega$, taking advantage of Lemma 5.1, we have

$$-\left[ \text{tr}(AD^2u)ADu - AD^2uADu \right] \cdot v \leq |B||ADu|^2.$$  

Hence, for the $K(\phi)$ given by (2.7), we have

$$K(\phi) = -C \int_{\partial \Omega} \phi^2 \left( |\sqrt{ADu}|^2 + \epsilon \right)^{p-2} [\text{tr}(AD^2u)ADu - AD^2uADu] \cdot v d\mathcal{H}^{n-1}(x) \leq C \int_{\Omega} \phi^2 |F_{A,\epsilon}(Du)| |B| d\mathcal{H}^{n-1}(x).$$

When $ADu \cdot v = 0$ on $\partial \Omega$, applying Lemma 5.2, one has

$$-\left[ \text{div}(ADu)ADu - (ADu \cdot D)ADu \right] \cdot v \leq |B||ADu|^2.$$
Hence, for the \( K(\phi) \) given by (2.8), we have
\[
K(\phi) = -C \int_{\partial \Omega} \phi^2 \left( |\sqrt{ADu}|^2 + \epsilon \right)^{p-2} \left[ \text{div}(ADu)ADu - (ADu \cdot D)ADu \right] \cdot \nu d\mathcal{H}^{n-1}(x)
\]
\[
\leq C \int_{\partial \Omega} \phi^2 |F_{A,\epsilon}(Du)|^2 |B|d\mathcal{H}^{n-1}(x).
\]
Finally, in light of Lemma 5.3 and letting \( v = \phi F_{\epsilon}(ADu) \), in both cases we have
\[
|K(\phi)| \leq C \int_{\partial \Omega} \phi^2 |F_{A,\epsilon}(Du)|^2 |B|d\mathcal{H}^{n-1}(x)
\]
\[
\leq C \Psi_B(r)\|\phi DF_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 + C \Psi_B(r)\|D\phi F_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2.
\]
\( \square \)

5.1 Proofs of Lemma 5.2 and Lemma 5.1

To see Lemma 5.2 and Lemma 5.1, we need the following Lemma, For its proof, we refer to for example [10, (3.1.1.6)]. Here we omit the details.

\textbf{Lemma 5.4} Assume that \( u \in C^2(\overline{\Omega}) \). Then on \( \partial \Omega \)
\[
[\text{div}(ADu)ADu - (ADu \cdot D)ADu] \cdot \nu =
\]
\[
= \text{div}_T((ADu \cdot \nu)(ADu)_T) - (\text{tr} B)(ADu \cdot \nu)^2
\]
\[
- \mathcal{B}((ADu)_T, (ADu)_T) - 2(ADu)_T \cdot \nabla_T(ADu \cdot \nu).
\]
\quad (5.4)

Above and below, we write
\[
(ADu)_\nu = \nu \cdot (ADu) \text{ and } (ADu)_T = ADu - (ADu)_\nu \nu.
\]

\textbf{Proof of Lemma 5.2} Since \( ADu \cdot \nu = 0 \) on \( \partial \Omega \), we have \( \nabla_T(ADu \cdot \nu) = 0 \) and hence,
\[
\text{div}_T((ADu \cdot \nu)(ADu)_T) = 0 \text{ on } \partial \Omega.
\]
We therefore obtain
\[
-[\text{div}(ADu)ADu - (ADu \cdot D)ADu] \cdot \nu = \mathcal{B}((ADu)_T, (ADu)_T) \text{ on } \partial \Omega.
\]
Thus
\[
-[\text{div}(ADu)ADu - (ADu \cdot D)ADu] \cdot \nu \leq |\mathcal{B}| |ADu|^2 \text{ on } \partial \Omega.
\]
If \( \Omega \) is convex, we have \( -\mathcal{B} \geq 0 \), and hence
\[
-[\text{div}(ADu)ADu - (ADu \cdot D)ADu] \cdot \nu \leq 0 \text{ on } \partial \Omega.
\]
\( \square \)

To prove Lemma 5.1 we need the following auxiliary lemma.

\textbf{Lemma 5.5} Under the Dirichlet homogeneous boundary condition \( u = 0 \) on \( \partial \Omega \), we have
\[
-[\text{tr}(AD^2u)ADu - AD^2uADu] \cdot \nu = [A_{\nu,\nu} \text{tr}(B_{T,T}) + \mathcal{B}(A_{T,\nu}, A_{T,\nu})] \left( \frac{\partial u}{\partial \nu} \right)^2 \text{ on } \partial \Omega,
\]
\quad (5.5)
where and below we always set
\[
A_{\nu, \nu} := \langle A\nu, \nu \rangle, \quad A_{T, \nu} := A\nu - (A_{\nu, \nu})\nu, \quad A_{T, T} = (\langle A\tau_i, \tau_j \rangle)_{1 \leq i, j \leq n-1}.
\]
Proof Recall that (3.25) gives
\[
[\text{tr}(AD^2u)ADu - AD^2uADu] \cdot v = [\text{div}(ADu)ADu - (ADu \cdot D)ADu] \cdot v
- \{[(\text{div}A) \cdot Du]ADu - [(ADu \cdot D)A]Du\} \cdot v.
\] (5.6)

It then suffices to prove that
\[
[\text{div}(ADu)ADu - (ADu \cdot D)ADu] \cdot v
= - \left[ A_{v,v} \text{tr}(B(T_{T,T}) + B(T_{T,v}, A_{T,v})) \left( \frac{\partial u}{\partial v} \right)^2 \right.
+ \sum_{k=1}^{n-1} \left\{ A_{v,v} \left( \frac{\partial A}{\partial \tau_k} v \cdot \tau_k \right) - (A_{T,v} \cdot \tau_k) \left( \frac{\partial A}{\partial \tau_k} v \cdot v \right) \right\} \left( \frac{\partial u}{\partial v} \right)^2 ,
\] (5.7)

and
\[
\{[(\text{div}A) \cdot Du]ADu - [(ADu \cdot D)A]Du\} \cdot v
= \sum_{k=1}^{n-1} \left\{ A_{v,v} \left( \frac{\partial A}{\partial \tau_k} v \cdot \tau_k \right) - (A_{T,v} \cdot \tau_k) \left( \frac{\partial A}{\partial \tau_k} v \cdot v \right) \right\} \left( \frac{\partial u}{\partial v} \right)^2 .
\] (5.8)

We show (5.7) by considering all terms in its right side in order. Note that \( u = 0 \) on \( \partial \Omega \), which implies that \( \nabla_T u = 0 \), we have \( Du|_{\partial \Omega} = \frac{\partial u}{\partial v} v \),
\[
(ADu)_v = (ADu, v) = (Av, Du) = A_{v,T} \cdot \nabla_T u + A_{v,v} \frac{\partial u}{\partial v} = A_{v,v} \frac{\partial u}{\partial v} ,
\] (5.9)

and
\[
(ADu)_T = ADu - (ADu)_v = A_{T,T} \cdot \nabla_T u + A_{T,v} \frac{\partial u}{\partial v} = A_{T,v} \frac{\partial u}{\partial v} .
\]

Thus,
\[
\text{div}_T(ADu \cdot v(ADu)_T) = \text{div}_T(\left( A_{v,v} \left( \frac{\partial u}{\partial v} \right)^2 A_{T,v} \right))
= A_{v,v} \left( \frac{\partial u}{\partial v} \right)^2 \text{div}_T A_{T,v} + A_{T,v} \cdot \nabla_T (A_{v,v} \left( \frac{\partial u}{\partial v} \right)^2 ) ,
\]
\[
- (\text{tr} B)(ADu \cdot v)^2 = -A_{v,v}^2 \left( \frac{\partial u}{\partial v} \right)^2 \text{tr} B, -B(((ADu)_T, (ADu)_T)
= - \left( \frac{\partial u}{\partial v} \right)^2 B(A_{T,v}, A_{T,v}).
\]

Moreover,
\[
-2(ADu)_T \cdot \nabla_T (ADu \cdot v) = -2 \left( \frac{\partial u}{\partial v} A_{T,v} \right) \cdot \nabla_T (A_{v,v} \frac{\partial u}{\partial v})
= -2 A_{T,v} \cdot \nabla_T (A_{v,v} \left( \frac{\partial u}{\partial v} \right)^2 ) + 2 A_{T,v} \cdot A_{v,v} \nabla_T (\left( \frac{\partial u}{\partial v} \right)^2 )
= -2 A_{T,v} \cdot \nabla_T (A_{v,v} \left( \frac{\partial u}{\partial v} \right)^2 ) + A_{T,v} \cdot A_{v,v} \nabla_T (\left( \frac{\partial u}{\partial v} \right)^2 )
= -A_{T,v} \cdot \nabla_T (A_{v,v} \left( \frac{\partial u}{\partial v} \right)^2 ) - \left( \frac{\partial u}{\partial v} \right)^2 A_{T,v} \cdot \nabla_T A_{v,v} .
\]

Combining them together, we obtain
\[
\text{div}(ADu) ADu \cdot v = \left((ADu \cdot D)ADu\right) \cdot v
\]
\[
= \left(\frac{\partial u}{\partial v}\right)^2 \left[-A_{v,v}^2 \text{tr}(B) - \mathcal{B}(A_{T,v}, A_{T,v}) + A_{v,v} \text{div}_T A_{T,v} - A_{T,v} \cdot \nabla_T A_{v,v}\right]. \quad (5.10)
\]

Note that by \(A_{T,v} = Av - A_{v,v} \cdot v\), we have
\[
A_{v,v} \text{div}_T (A_{T,v}) = A_{v,v} \sum_{k=1}^{n-1} \left(\frac{\partial}{\partial \tau_k} A_{T,v}\right) \cdot \tau_k
\]
\[
= A_{v,v} \sum_{k=1}^{n-1} \left(\frac{\partial}{\partial \tau_k} \left(Av - A_{v,v} \cdot v\right)\right) \cdot \tau_k
\]
\[
= A_{v,v} \left[\sum_{k=1}^{n-1} \left(\frac{\partial}{\partial \tau_k} (Av) \cdot \tau_k + \frac{\partial A}{\partial \tau_k} v \cdot \tau_k - A_{v,v} \frac{\partial v}{\partial \tau_k} \cdot \tau_k\right)\right]
\]
\[
= A_{v,v} \left[Av \text{tr} B + \sum_{i,k=1}^{n-1} \frac{\partial v}{\partial \tau_k} (A_{T,T})_{i,k} \cdot \tau_i + \sum_{k=1}^{n-1} \frac{\partial A}{\partial \tau_k} v \cdot \tau_k\right]
\]
\[
= A_{v,v}^2 \text{tr} B - A_{v,v} \text{tr}(\mathcal{B}A_{T,T}) + A_{v,v} \sum_{k=1}^{n-1} \frac{\partial A}{\partial \tau_k} v \cdot \tau_k. \quad (5.11)
\]

On the other hand, by \(A_{v,v} = v \cdot Av\), we also have
\[
\frac{\partial}{\partial \tau_k} A_{v,v} = 2A \frac{\partial v}{\partial \tau_k} \cdot v + \frac{\partial A}{\partial \tau_k} v \cdot v = 2 \frac{\partial v}{\partial \tau_k} \cdot Av + \frac{\partial A}{\partial \tau_k} v \cdot v.
\]

Since \(\frac{\partial v}{\partial \tau_k} \cdot v = 0\) and \(A_{T,v} = Av - A_{v,v} \cdot v\), we have
\[
\frac{\partial}{\partial \tau_k} A_{v,v} = 2 \frac{\partial v}{\partial \tau_k} \cdot A_{T,v} + \frac{\partial A}{\partial \tau_k} v \cdot v, \quad (5.12)
\]
which directly yields
\[
-A_{T,v} \cdot \nabla_T A_{v,v} = -\sum_{k=1}^{n-1} (A_{T,v} \cdot \tau_k) \frac{\partial A_{v,v}}{\partial \tau_k} = 2\mathcal{B}(A_{T,v}, A_{T,v}) - \sum_{k=1}^{n-1} \left(\frac{\partial A}{\partial \tau_k} v \cdot \tau_k\right) (A_{T,v} \cdot \tau_k). \quad (5.13)
\]

Plugging the above identity and (5.11) into (5.10), we get the desired identity (5.7).

By \(D_T u = 0\) on \(\partial \Omega\) we have
\[
(\text{div}A) \cdot Du = (\text{div}A) \cdot \frac{\partial u}{\partial v} = \left[\left(\frac{\partial A}{\partial v}\right) v, v\right] \frac{\partial u}{\partial v} \text{ on } \partial \Omega.
\]

By \(ADu \cdot v = A_{v,v} \frac{\partial u}{\partial v}\) on \(\partial \Omega\), we obtain
\[
[(\text{div}A) \cdot Du]ADu \cdot v = \left[\left(\frac{\partial A}{\partial v}\right) v, v\right] A_{v,v} \left(\frac{\partial u}{\partial v}\right)^2 \text{ on } \partial \Omega. \quad (5.14)
\]

Moreover, by \(D_T u = 0\) on \(\partial \Omega\), we have
\[
(ADu \cdot D)A = A_{v,v} \frac{\partial u}{\partial v} \frac{\partial A}{\partial v} + \sum_{k=1}^{n-1} \frac{\partial u}{\partial v} (A_{T,v} \cdot \tau_k) \frac{\partial A}{\partial \tau_k} \text{ on } \partial \Omega,
\]
\[(ADu \cdot D)A Du = A_{v,v} \left( \frac{\partial u}{\partial v} \right)^2 \frac{\partial A}{\partial v} v + \sum_{k=1}^{n-1} \left( \frac{\partial u}{\partial v} \right)^2 (A_{T,v} \cdot \tau_k) \frac{\partial A}{\partial \tau_k} v, \]

and

\[(ADu \cdot D)A Du = A_{v,v} \left( \frac{\partial u}{\partial v} \right)^2 \left( \frac{\partial A}{\partial v} v, v \right) + \sum_{k=1}^{n-1} \left( \frac{\partial u}{\partial v} \right)^2 (A_{T,v} \cdot \tau_k) \left( \frac{\partial A}{\partial \tau_k} v, v \right). \]

From the above identity and (5.14) we conclude (5.8).

Now we are ready to prove Lemma 5.1.

**Proof of Lemma 5.1** Recall that

\[-\text{tr}(AD^2u ADu - AD^2u ADu) \cdot v = \left[ A_{v,v} \text{tr}(BA_{T,T}) + B(A_{T,v}, A_{T,v}) \right] \left( \frac{\partial u}{\partial v} \right)^2. \]

We obviously have

\[-[\text{tr}(AD^2u ADu - AD^2u ADu) \cdot v \leq C|\mathcal{B}||ADu|^2 \]

for some constant C depending on L, that is, (5.2) holds.

Moreover, if Ω is convex, we know that \( -\mathcal{B} \geq 0 \), and hence \( \mathcal{B}(A_{T,v}, A_{T,v}) \leq 0 \). Since \( \mathcal{B} \geq 0, \) we have \( A_{v,v} \geq 0 \) and \( \sqrt{\mathcal{A}}B\sqrt{\mathcal{A}} \geq 0 \), and hence \( \text{tr} \left( \sqrt{\mathcal{A}}B\sqrt{\mathcal{A}} \right) \geq 0 \). Note that

\[\text{tr}(B_{n-1} A_{T,T}) = \text{tr}(BA) = \text{tr} \left( \sqrt{\mathcal{A}}B\sqrt{\mathcal{A}} \right),\]

and that \( A_{v,v} \text{tr}(BA_{T,T}) = A_{v,v} \text{tr}(B_{n-1} A_{T,T}) \). We conclude that \( A_{v,v} \text{tr}(BA_{T,T}) \leq 0 \). Thus by (5.15) we have

\[-[\text{tr}(AD^2u ADu - AD^2u ADu) \cdot v \leq 0, \]

that is, (5.1) holds.

\[\square\]

**6 Proofs of key Lemmas 2.5 and 2.6**

Recall the following Garliardo-Nirenberg-Sobolev inequality:

**Lemma 6.1** For any \( 1 \leq s < \frac{n+2}{2} \) and \( \theta = \frac{2n}{n+2} - \frac{s}{s-1} \), or \( s = \frac{n+2}{n-2} \) with \( n \geq 3 \) and \( \theta = 1 \), we have

\[\|v - v_\Omega\|_{L^s(\Omega)}^2 \leq C\|Du\|_{L^q(\Omega)}^{2\theta} \|v\|_{L^1(\Omega)}^{2(1-\theta)} \quad \forall v \in W^{1,2}(\Omega), \]

where \( v_\Omega = \frac{1}{|\Omega|} \int_\Omega v dx \), and \( C > 0 \) is a constant depends on \( s, n \) and \( \text{Lip}_\Omega \).

**Proof of Lemma 2.5** (i) Let \( s = \frac{2q}{q-2} \), that is, \( q = \frac{2s}{s-2} \). Note that \( q > n \geq 2 \) implies that \( 2 < s = \frac{2q}{q-2} < \frac{2n}{n-2} \). By Hölder’s inequality, one has

\[\square\]
\[
\int_{\Omega} |DA|^2 |F_{A,\epsilon}(Du)|^2 \, dx \leq \left( \int_{\Omega \cap B(x,r)} |DA|^\frac{2n}{n+2} \, dx \right)^{\frac{n}{n+2}} \left( \int_{\Omega} |F_{A,\epsilon}(Du)|^s \, dx \right)^{\frac{2}{s}}
\]
\[
= \|DA\|_{L^q(\Omega \cap B(x,r))}^2 \|F_{A,\epsilon}(Du)\|_{L^1(\Omega)}^2.
\]
Write \( \theta = \frac{2n}{n+2} \frac{s-1}{s} \) and note \( 0 < \theta < 1 \). By the Garliardo-Nirenberg-Sobolev inequality we have
\[
\|F_{A,\epsilon}(ADu) - [F_{A,\epsilon}(ADu)]_\Omega\|_{L^1(\Omega)}^2 \leq C \|DF_{A,\epsilon}(ADu)\|_{L^2(\Omega)}^{2\theta} \|F_{A,\epsilon}(ADu)\|_{L^1(\Omega)}^{2(1-\theta)}.
\]
Applying Young’s inequality, for any \( \eta \in (0, 1) \) we have
\[
\|F_{A,\epsilon}(ADu) - [F_{A,\epsilon}(ADu)]_\Omega\|_{L^1(\Omega)}^2 \leq \eta \|DF_{A,\epsilon}(ADu)\|_{L^2(\Omega)}^2 + \frac{C}{\eta} \|F_{A,\epsilon}(ADu)\|_{L^1(\Omega)}^2.
\]
Thus
\[
\|F_{A,\epsilon}(Du)\|_{L^1(\Omega)}^2 \leq \eta \|DF_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 + \frac{C}{\eta} \|F_{A,\epsilon}(Du)\|_{L^1(\Omega)}^2.
\]
(ii) For any \( \phi \in C_c^\infty(B(z, r)) \), by Hölder’s inequality, we have
\[
\int_{\Omega} |DA|^2 |F_{A,\epsilon}(Du)|^2 \phi^2 \, dx \leq \|DA\|_{L^q(\Omega \cap B(z, r))}^2 \|\phi F_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2.
\]
Recall that the Sobolev imbedding gives
\[
\|\phi F_{A,\epsilon}(Du) - [\phi F_{A,\epsilon}(Du)]_\Omega\|_{L^{2n/(n+2)}(\Omega)} \leq C \|D[\phi F_{A,\epsilon}(Du)]\|_{L^2(\Omega)}.
\]
We have
\[
\|\phi F_{A,\epsilon}(Du)\|_{L^{2n/(n+2)}(\Omega)}^2 \leq C \|D[\phi F_{A,\epsilon}(Du)]\|_{L^2(\Omega)}^2 + \|\phi F_{A,\epsilon}(Du)\|_{L^1(\Omega)}^2
\]
\[
\leq C \|\phi DF_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 + C \|D\phi\|_{L^2(\Omega)}^2 \|DF_{A,\epsilon}(Du)\|_{L^2(\Omega)}^2 + \|\phi F_{A,\epsilon}(Du)\|_{L^1(\Omega)}^2.
\]
\[\square\]
Below we prove Lemma 2.6.

**Proof of Lemma 2.6** Applying Lemma 6.1 for \( v = F_{A,\epsilon}(Du) \) with \( s = 2 \) and \( \theta = \frac{n}{n+1} \), in light of Young’s inequality we obtain
\[
\|F_{A,\epsilon}(Du) - [F_{A,\epsilon}(Du)]_\Omega\|_{L^2(\Omega)} \leq \eta \|DF_{A,\epsilon}(Du)\|_{L^2(\Omega)} + \frac{C}{\eta} \|F_{A,\epsilon}(Du)\|_{L^1(\Omega)}
\]
for any \( \eta \in (0, 1) \). This then gives the desired inequality. \[\square\]

## 7 Proofs of Theorems 1.1 and 1.2

To prove Theorems 1.1 and 1.2, we need the following auxiliary lemmas.

The first is the following extension lemma, whose proof is given in the “Appendix”, where we adapt some arguments of Sobolev extension operator by Jones and also [13].

**Lemma 7.1** Suppose that \( \Omega \) is a bounded Lipschitz domain. Let \( A \in \mathcal{E}_L(\Omega) \) satisfy (1.4). There exists a family \( \{A^\epsilon\}_{\epsilon \in (0, 1)} \subset \mathcal{E}_L(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n) \) such that

(i) if \( DA \in L^q(\Omega) \) with \( q > n \geq 2 \), we have \( A^\epsilon \to A \in W^{1, q}(\Omega) \) and \( \|DA^\epsilon\|_{L^q(\mathbb{R}^n)} \leq C \|DA\|_{L^q(\Omega)} \)

\( \square \)
(ii) if \( DA \in L^n(\Omega) \) with \( n \geq 2 \), we have \( A^c \to A \in W^{1,n}(\Omega) \), \( \| DA^c \|_{L^n(\mathbb{R}^n)} < C \| DA \|_{L^n(\Omega)} \) and
\[
\Phi_{A^c,\Omega}(r) := \sup_{x \in \Omega_r} \| DA^c \|_{L^n(B(x,r) \cap \Omega')} \leq C \Phi_A(Cr), \quad \forall 0 < t, \epsilon < r < \text{diam } \Omega
\]
for some constant \( C \) depending on \( n \) and \( \Omega \). Here and below
\[
\Omega' := \{ x \in \mathbb{R}^n, \text{ dist } (x, \Omega) < t \}.
\]

Next we recall the following two approximation results of domains; see for example [8].

**Lemma 7.2** Given any bounded convex domain \( \Omega \) in \( \mathbb{R}^n \), there is a sequence \( \{\Omega_k\} \) of smooth bounded convex domains in \( \mathbb{R}^n \) such that
\[
\Omega \Subset \Omega_k, \lim_{k \to \infty} |\Omega_k \setminus \Omega| = 0 \quad \text{and} \quad \lim_{k \to \infty} d_H(\Omega_k, \Omega) = 0. \tag{7.1}
\]

**Lemma 7.3** Given any Lipschitz domain \( \Omega \) satisfying (1.9), there is a sequence \( \{\Omega_k\} \) of smooth domains in \( \mathbb{R}^n \) such that
\[
\Omega \Subset \Omega_k, \lim_{k \to \infty} |\Omega_k \setminus \Omega| = 0 \quad \text{and} \quad \lim_{k \to \infty} d_H(\Omega_k, \Omega) = 0, \quad \Phi_{\Omega_k}(r) \leq C \Phi_{\Omega}(Cr), \tag{7.2}
\]
where \( C \) is a constant depending only in \( n \) and \( \Omega \).

Moreover the following two approximate solutions to certain equations are needed. Recall that notions \( F_{A,e}(Du) \) and \( F_e(Du) \) in Sect. 2, we write \( F_A(Du) = F_{A,0}(Du) \) and \( F(Du) = F_0(Du) \).

**Lemma 7.4** Let \( \Omega_\infty = \Omega \) and \( \{\Omega_k\}_{k \in \mathbb{N}} \) be as Lemma 7.2, or be as in Lemma 7.3. Let \( g \in C_c^\infty(\Omega), A \in C^\infty(\mathbb{R}^n) \cap \mathcal{E}_L(\mathbb{R}^n) \) and \( \epsilon \in (0, 1] \). For \( k \in \mathbb{N} \cup \{\infty\} \), denote by \( v_k \in W^{1,p}(\Omega_k) \) the weak solution to \( \mathcal{L}_{A,e,p}v_k = g \in \Omega_k \) with Dirichlet or Neumann homogenous boundary condition. Then \( F_{A,e}(Du_k) \to F_{A,e}(Du_\infty) \) and \( F_e(Dv_k) \to F_e(Dv_\infty) \) in \( \mathbb{R}^n \) as \( k \to \infty \).

**Lemma 7.5** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain, \( g \in C_c^\infty(\Omega) \) and \( \{A^\epsilon\}_{\epsilon \in [0,1]} \subset \mathcal{E}_L(\Omega) \) with \( A^\epsilon \to A^0 \) in as \( \epsilon \to 0 \). For \( \epsilon \in [0, 1] \), denote by \( v_\epsilon \in W^{1,p}(\Omega) \) be a weak solution to \( \mathcal{L}_{A^\epsilon,p}v_\epsilon = g \in \Omega \) with Dirichlet or Neumann homogeneous boundary condition. Then \( F_{A^\epsilon}(Du_\epsilon) \to F_{A^0}(Du_0) \) and \( F_e(Dv_\epsilon) \to F(Dv_\epsilon) \) almost everywhere in \( \Omega \) as \( \epsilon \to 0^+ \).

**Proof of Theorem 1.1** We only consider the case with Dirichlet homogeneous boundary condition; the case with Neumann homogeneous boundary condition follows exactly the same argument. Let \( \Omega \) be a bounded convex domain, \( A \in \mathcal{E}_L(\Omega) \) satisfying (1.4), and \( f \in L^2(\Omega) \). Let \( u \in W^{1,2}_0(\Omega) \) be the unique generalized solution to \( \mathcal{L}_{A,p}u = f \) in \( \Omega \) with Dirichlet 0-boundary. We prove (1.8) as below. Note that (1.8) reads as
\[
F_A(Du), F(Du) \in W^{1,2}(\Omega) \text{ with } \| DF_A(Du) \|_{L^2(\Omega)} + \| DF(Du) \|_{L^2(\Omega)} \leq C \| f \|_{L^2(\Omega)} \tag{7.3}
\]
We choose \( \{f^\ell\}_{\ell \in \mathbb{N}} \subset C_c^\infty(\Omega) \) so that \( \| f^\ell \|_{L^2(\Omega)} \leq 2 \| f \|_{L^2(\Omega)} \) for all \( \ell \) and \( \| f^\ell \|_{L^2(\Omega)} \to 0 \) as \( \ell \to \infty \). Let \( \{A^\epsilon\}_{\epsilon \in (0,1]} \) be as in Lemma 7.1 and \( \{\Omega_k\}_{k \in \mathbb{N}} \) be as in Lemma 7.4. Given any \( \ell \in \mathbb{N}, \epsilon \in (0, 1) \) and \( k \in \mathbb{N} \), let \( u_{(\ell,\epsilon,k)} \) be the smooth solution to the problem
\[
\mathcal{L}_{A^\epsilon,p}u_{(\ell,\epsilon,k)} = f^\ell \quad \text{in } \Omega_k; \quad u_{(\ell,\epsilon,k)} = 0 \quad \text{on } \partial \Omega_k. \tag{7.4}
\]
Observe that by Lemma 7.1 and Lemma 7.4, we have

\( \Box \) Springer
(i) If $DA \in L^q(\Omega)$ for some $q > n \geq 2$, writing $R_\varepsilon = C_{ext,q}(\Omega)\|A\|_{L^q(\Omega)}$, we have 
$\|DA\|_{L^q(\Omega)} \leq R_\varepsilon$ for all $k$.

(ii) If $DA \in L^n(\Omega)$ with $n \geq 3$, let $\delta_\varepsilon$ be as in Theorem 1.4 (ii). Then $\Phi_{A,\Omega}(r_\varepsilon) < \delta_\varepsilon/C_{ext,n}(\Omega)$ for some $r_\varepsilon > 0$. Let $r_\varepsilon = r_\varepsilon/C_{ext,n}(\Omega)$. Then $\Phi_{A^\varepsilon,\Omega}(r_\varepsilon) < \delta_\varepsilon$ whenever $\varepsilon < r_\varepsilon$ and $k$ large such that $d_{H}(\Omega_k, \Omega) \leq r_\varepsilon$.

Thanks to this, for all $\ell \in \mathbb{N}$, for all sufficiently small $\varepsilon > 0$ and all sufficiently large $k$, we apply Theorem 1.4 to $u_{(\ell,\varepsilon, k)}$ so to obtain

$$
\|DF_{A^\varepsilon,\varepsilon}(Du_{(\ell,\varepsilon, k)})\|_{L^2(\Omega)} + \|DF_{\varepsilon}(Du_{(\ell,\varepsilon, k)})\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} \leq 2C\|f\|_{L^2(\Omega)}.
$$

(7.5)

where the constant $C$ is as determined by Theorem 1.4, in particular, independent of $\varepsilon, \ell, k$.

From this, we conclude the desired result (7.3) by sending $k \to \infty$, $\varepsilon \to 0$, and $\ell \to \infty$ in order. The details are given as below.

**Send** $k \to \infty$. Fix any $\ell \in \mathbb{N}$ and any sufficiently small $\varepsilon > 0$, from (7.5) one deduces that $F_{A^\varepsilon,\varepsilon}(Du_{(\ell,\varepsilon, k)}) \in W^{1,2}(\Omega)$ and $F_{\varepsilon}(Du_{(\ell,\varepsilon, k)}) \in W^{1,2}(\Omega)$, both of which are uniform in all sufficiently large $k$. By the compactness of Sobolev space, we know that $F_{A^\varepsilon,\varepsilon}(Du_{(\ell,\varepsilon, k)})$ converges to some function $G \in W^{1,2}(\Omega)$ and $F_{\varepsilon}(Du_{(\ell,\varepsilon, k)})$ converges to $\tilde{G}$ in $L^2(\Omega)$ and weakly in $W^{1,2}(\Omega)$, and

$$
\|DG\|_{L^2(\Omega)} + \|\tilde{D}\tilde{G}\|_{L^2(\Omega)} \leq \liminf_{k \to \infty}[\|DF_{A^\varepsilon,\varepsilon}(Du_{(\ell,\varepsilon, k)})\|_{L^2(\Omega)} + \|DF_{\varepsilon}(Du_{(\ell,\varepsilon, k)})\|_{L^2(\Omega)}] \leq 2C\|f\|_{L^2(\Omega)}.
$$

On the other hand, denote by $u_{(\ell,\varepsilon)} \in W^{1,p}(\Omega)$ the weak solution to the equation $L_{A^\varepsilon,\varepsilon,p}u_{(\ell,\varepsilon)} = f^\ell$ in $\Omega$ with Dirichlet homogeneous boundary condition. By Lemma 7.4 one has $F_{A^\varepsilon,\varepsilon}(Du_{(\ell,\varepsilon, k)}) \to F_{A^\varepsilon,\varepsilon}(Du_{(\ell,\varepsilon)})$ almost everywhere in $\Omega$ as $k \to \infty$, and also $F_{\varepsilon}(Du_{(\ell,\varepsilon, k)}) \to F_{\varepsilon}(Du_{(\ell,\varepsilon)})$ almost everywhere in $\Omega$ as $k \to \infty$. Thus $G = F_{A^\varepsilon,\varepsilon}(Du_{(\ell,\varepsilon)}) \in W^{1,2}(\Omega)$ and $\tilde{G} = F_{A^\varepsilon,\varepsilon}(Du_{(\ell,\varepsilon)}) \in W^{1,2}(\Omega)$, and hence

$$
\|DF_{A^\varepsilon,\varepsilon}(Du_{(\ell,\varepsilon)})\|_{L^2(\Omega)} + \|DF(Du_{(\ell,\varepsilon)})\|_{L^2(\Omega)} \leq 2C\|f\|_{L^2(\Omega)}.
$$

(7.6)

**Send** $\varepsilon \to \infty$. Given any sufficiently large $\ell$, denote by $u_{(\ell)} \in W^{1,p}(\Omega)$ the unique weak solution to the equation $L_{A,\varepsilon}u_{(\ell)} = f^\ell$ in $\Omega$ with Dirichlet homogeneous boundary condition. By Lemma 7.5, $F_{A^\varepsilon,\varepsilon}(Du_{(\ell,\varepsilon)}) \to F_{A^\varepsilon}(Du_{(\ell)})$ and $F_{\varepsilon}(Du_{(\ell,\varepsilon)}) \to F(Du_{(\ell)})$ almost everywhere as $\varepsilon \to 0$ (up to some subsequence) Thanks to this, (7.6) and the compactness of Sobolev space $W^{1,2}(\Omega)$, we know that $F_{A^\varepsilon,\varepsilon}(Du_{(\ell,\varepsilon)})$ converges to some function $F_{A}(Du_{(\ell)})$ and $F_{\varepsilon}(Du_{(\ell,\varepsilon)})$ converges to $F(Du_{(\ell)})$ in $L^2(\Omega)$ and weakly in $W^{1,2}(\Omega)$, and

$$
\|DF_{A}(Du_{(\ell)})\|_{L^2(\Omega)} + \|DF(Du_{(\ell)})\|_{L^2(\Omega)} \leq \liminf_{\varepsilon \to 0}[\|DF_{A^\varepsilon}(Du_{(\ell,\varepsilon)})\|_{L^2(\Omega)} + \|DF(Du_{(\ell,\varepsilon)})\|_{L^2(\Omega)}] \leq 2C\|f\|_{L^2(\Omega)}.
$$

(7.7)

**Send** $\ell \to \infty$. Sending $\ell \to \infty$, by [7], we have $u_{(\ell)} \to u$ in $W^{1,p}(\Omega)$. This yields that $F_{A^\varepsilon,\varepsilon}(Du_{(\ell)}) \to F_{A}(Du_{(\ell)})$ almost everywhere as $\varepsilon \to 0$ (up to some subsequence) and $F_{\varepsilon}(Du_{(\ell,\varepsilon)}) \to F(Du_{(\ell)})$ almost everywhere as $\ell \to 0$. By this, (7.7) and the compactness of Sobolev space, $F_{A^\varepsilon,\varepsilon}(Du_{(\ell,\varepsilon)})$ converges to some function $F_{A}(Du_{(\ell)})$ and $F_{\varepsilon}(Du_{(\ell,\varepsilon)})$ converges to $F(Du_{(\ell)})$ in $L^2(\Omega)$ and weakly in $W^{1,2}(\Omega)$, and

$$
\|DF_{A}(Du)\|_{L^2(\Omega)} + \|DF(Du)\|_{L^2(\Omega)} \leq \liminf_{\ell \to \infty}[\|DF_{A}(Du_{(\ell)})\|_{L^2(\Omega)} + \|DF(Du_{(\ell)})\|_{L^2(\Omega)}] \leq 2C\|f\|_{L^2(\Omega)}
$$

as desired. \qed
Proof of Theorem 1.2. Given any \( \Omega \) satisfying (1.9), let \( \Omega_k \) be as in Lemma 7.3. Let \( \delta_* \) be as in Theorem 1.6 and \( C_* \) be as in Lemma 7.3. If \( \Psi_{\Omega}(r_*) \leq \delta_*/C_* \), then \( \Psi_{\Omega_k}(r_*) \leq \delta_* \). Following the proof for the procedure of Theorem 1.1, we will get Theorem 1.2. We omit the details.

\[ \square \]

7.1 Proofs of Lemmas 7.4 and 7.5

Proof of Lemma 7.4. Case 1. Under Dirichlet homogeneous boundary condition. It suffices to prove \( v_k \to v_\infty \) in \( C^{1,\alpha}(\Omega_\infty) \) as \( \epsilon \to 0 \). Firstly, we show that, for \( k \in \mathbb{N} \),

\[
\int_{\Omega_k} |\sqrt{A^*} Dv_k|^p \, dx \leq C(n, p, L, \Omega_1) \int_{\Omega} (|g|^p + 1) \, dx. \tag{7.8}
\]

Indeed, since \( v_k \in W^{1,p}_0(\Omega_k) \) is a weak solution to \( L_{A', \epsilon, \rho} v_k = g \) in \( \Omega_k \) we obtain

\[
\int_{\Omega_k} \left( \sqrt{A} \right)^{-1} F_{A, \epsilon}(Dv_k) \cdot \sqrt{A} Dv_k \, dx = \int_{\Omega_k} F_{A, \epsilon}(Dv_k) \cdot Dv_k \, dx = \int_{\Omega_k} g v_k \, dx = \int_{\Omega_\infty} g v_k \, dx. \tag{7.9}
\]

Observe that

\[
|\sqrt{A^*} Dv_k|^p \leq \left( \sqrt{A} \right)^{-1} F_{A, \epsilon}(Dv_k) \cdot \sqrt{A} Dv_k + C(p)\epsilon^p \quad \forall \epsilon \in (0, 1).
\]

Thanks to this, applying Young’s inequality with \( \eta \in (0, 1) \) we have

\[
\int_{\Omega_k} |\sqrt{A^*} Dv_k|^p \, dx \leq \frac{4}{\eta} \int_{\Omega} |g|^p \, dx + \eta \int_{\Omega_k} |v_k|^p \, dx + C(p, \Omega_1). \tag{7.10}
\]

Since \( v_k \in W^{1,p}_0(\Omega_1) \), where \( v_k \) is extended to the Lipschitz domain \( \Omega_1 \) by setting \( v_k = 0 \) on \( \Omega_1 \setminus \Omega_k \), by Poincaré’s inequality we have

\[
\int_{\Omega_k} |v_k|^p \, dx = \int_{\Omega_1} |v_k|^p \, dx \leq C(n, p, \Omega_1) \int_{\Omega_1} |Dv_k|^p \, dx = C(n, p, \Omega_1) \int_{\Omega_k} |Dv_k|^p \, dx. \tag{7.11}
\]

Since \( \frac{1}{L} \leq \sqrt{A^*} \leq L \), choosing \( \eta \) smooth enough so that

\[
C(n, p, \Omega_1)\eta \int_{\Omega_k} |Dv_k|^p \, dx \leq \frac{1}{2} \int_{\Omega_k} |\sqrt{A^*} Dv_k|^p \, dx.
\]

From this and (7.10), we conclude (7.8).

On the other hand, it is well-known that, there exists \( \alpha > 0 \) such that for any smooth subdomain \( U \subset \Omega_\infty \), \( v_k \in C^{1,\alpha}(\overline{U}) \) uniformly in all \( k \in \mathbb{N} \). Moreover, observe that \( \Omega_k \) satisfies the regular condition uniformly in \( k \), that is,

\[
|B(x, r) \setminus \Omega| \geq c|B(x, r)| \quad \forall x \in \partial \Omega_k, \forall k \in \mathbb{N}
\]

for some constant \( c > 0 \). There exists some \( \beta > 0 \) such that \( v_k \in C^{0,\beta}(\Omega_k) \) with the norm \( \sup_{k \in \mathbb{N}} \|v_k\|_{C^{0,\beta}(\Omega_k)} < \infty \); see [14, 15, 24]. Since \( v_k|_{\partial \Omega_k} = 0 \) and \( d_H(\Omega, \Omega_k) \to 0 \) we know that \( v_k|_{\partial \Omega} \to 0 \) uniformly as \( k \to \infty \). Thus, we can find a function \( v \in C^{0,\beta}(\overline{\Omega}) \) and \( Dv_k \to Dv \) in \( C^{0,\alpha}(\Omega_\infty) \) as \( k \to \infty \) (up to some subsequence). Consequently, one has \( v_\infty|_{\partial \Omega} = 0 \) and \( v \in C^{0,\beta}(\overline{\Omega}) \), and moreover,
$\frac{1}{r^n} \int_{B(x,r) \cap \Omega} |v| \, dx = 0$ as $r \to 0$ for all $x \in \partial \Omega$. By [21] and $v \in W^{1,p}(\Omega_{\infty})$ one conclude $v \in W^{1,p}_{0}(\Omega)$.

Next, as $k \to \infty$, since $Dv_k \to Dv$ in $C^{0,\alpha}(\Omega_{\infty})$, we have $F_{A,e}(Dv_k) \to F_{A,e}(Dv)$ in $C^{0,\alpha}(\Omega_{\infty})$. Thus for any $\phi \in C_c^{\infty}(\Omega_{\infty})$, it follows that

$$\int_{\Omega_{\infty}} F_{A^e,\epsilon}(Dv) \cdot D\phi \, dx = \lim_{k \to \infty} \int_{\Omega_{\infty}} F_{A^e,\epsilon}(Dv_k) \cdot D\phi \, dx.$$

Observe that (7.8) implies that $F_{A^e,\epsilon}(Dv_k) \in L^{p'}(\Omega_{\infty})$ uniformly in $k \in \mathbb{N}$. By a density argument, one has

$$\int_{\Omega_{\infty}} F_{A^e,\epsilon}(Dv) \cdot D\phi \, dx = \int_{\Omega_{\infty}} g \phi \, dx \quad \text{for all} \quad \phi \in W^{1,p'}_{0}(\Omega_{\infty}),$$

that is, $L^{A^e,\epsilon,\rho}v = g$ in $\Omega_{\infty}$ in weak sense. By the uniqueness of solutions to the equation $L^{A^e,\epsilon,\rho}v = g$ in $\Omega_{\infty}$ with Dirichlet 0-boundary, we have $v = v_{\infty}$ as desired.

**Case 2.** Under Neumann homogeneous boundary condition. In this case we may assume in addition that $\int_{\Omega_k} v_k \, dx = 0$. It suffices to prove $v_k \to v_{\infty}$ in $C^{1,\alpha}(\Omega_{\infty})$ as $\epsilon \to 0$. Firstly we show that (7.8) also holds with some constant $C$ independent of $k$. The proof is very similar to the case with Dirichlet homogeneous boundary condition. We sketch it. First, since $v_k \in W^{1,p}(\Omega)$ is a weak solution to $L^{A^e,\rho}v_k = g$ in $\Omega_k$ with Neumann homogeneous boundary condition, one also has (7.9), and then gets (7.10). Thanks to the assumption $\int_{\Omega_k} v_k \, dx = 0$ in this case, we could apply the Sobolev-Poincaré inequality to get (7.11) with the constant uniformly in $k$, where note that $\partial \Omega_k$ are uniform Lipschitz and has uniform bounded diameters. We then choose small $\eta$ to get the desired result.

Next, it is well-known that, there exists $\alpha > 0$ such that for any smooth subdomain $U \Subset \Omega_{\infty}$, $v_k \in C^{1,\alpha}(\bar{U})$ uniformly in $k \in \mathbb{N}$. Noting the assumption $\int_{\Omega_k} v_k \, dx = 0$ in this case, we can find a function $v \in C^{1,\alpha}(\Omega_{\infty})$ such that $v_k \to v$ and $Dv_k \to Dv$ in $C^{0,\alpha}(\Omega_{\infty})$ as $k \to \infty$ (up to some subsequence). In particular, $F_{A^e,\epsilon}(Dv_k) \to F_{A^e,\epsilon}(Dv)$ in $C^{0,\alpha}(\Omega_{\infty})$ as $k \to \infty$. In particular $F_{A^e,\epsilon}(Dv_k) \to F_{A^e,\epsilon}(Dv)$ in $C^{0,\alpha}(\Omega_{\infty})$.

Moreover, given any $\phi \in W^{1,\infty}(\Omega_{\infty})$ we extend it to a function $\tilde{\phi} \in W^{1,\infty}(\mathbb{R}^n)$. Let $U_m \Subset U_{m+1} \Subset \Omega_{\infty}$ with $d_H(\Omega_{\infty}, U_m) \to 0$ as $m \to \infty$. One has

$$\int_{\Omega_{\infty}} F_{A^e,\epsilon}(Dv) \cdot D\phi \, dx = \lim_{m \to \infty} \int_{U_m} F_{A^e,\epsilon}(Dv) \cdot D\phi \, dx \quad \text{as} \quad m \to \infty.$$

Observe that

$$\int_{\Omega_k} F_{A^e,\epsilon}(Dv_k) \cdot D\tilde{\phi} \, dx = \int_{\Omega_k} g \tilde{\phi} \, dx = \int_{\Omega_{\infty}} g \phi \, dx.$$

Since $\|\sqrt{A}Dv_k\|_{L^{p}(\Omega_k)} \leq C \|g\|_{L^{p'}(\Omega_{\infty})}^{\rho/p}$ for all $k \in \mathbb{N}$, one has
\[ \int_{\Omega_k \setminus U_m} F_{A^\epsilon, \epsilon}(Dv_k) \cdot D\phi \, dx \leq \|D\phi\|_{L^\infty(\mathbb{R}^n)} \|F_{A^\epsilon, \epsilon}(Dv_k)\|_{L^1(\Omega_k \setminus U_m)} \]

\[ \leq \|D\phi\|_{L^\infty(\mathbb{R}^n)} \|F_{A^\epsilon, \epsilon}(Dv_k)\|_{L^{p'}(\Omega_k)} |\Omega_k \setminus U_m|^{1/p} \to 0 \]

as \( k \to 0 \). We therefore get

\[ \int_{\Omega_\infty} F_{A^\epsilon, \epsilon}(Dv) \cdot D\phi \, dx = \int_{\Omega_\infty} g \phi \, dx. \]

Since \( W^{1, \infty}(\Omega_\infty) \) is dense in \( W^{1, p'}(\Omega_\infty) \), and \( F_{A, \epsilon}(Dv_k) \in W^{1, p}(\Omega_\infty) \) uniformly in \( k \), we know that this holds for all \( \phi \in W^{1, p}(\Omega) \). Thus \( v \) is a weak solution to \( L_{A, \epsilon, p} v = g \) in \( \Omega_\infty \) with Neumann homogeneous boundary condition.

To get \( v = v_\infty \), it then suffices to show that \( \int_{\Omega_\infty} v_\infty \, dx = 0 \). Indeed, note that

\[ \int_{\Omega_\infty} v_\infty \, dx = \lim_{k \to \infty} \int_{\Omega_\infty} v_k \, dx = \lim_{k \to \infty} \left[ \int_{\Omega_k} v_k \, dx + \int_{\Omega_k \setminus \Omega_\infty} v_k \, dx \right]. \]

by \( \int_{\Omega_k} v_k \, dx = 0 \) and Hölder’s inequality, one has

\[ \left| \int_{\Omega_\infty} v_\infty \, dx \right| = \lim_{k \to \infty} \left| \int_{\Omega_k \setminus \Omega_\infty} v_k \, dx \right| \leq \liminf_{k \to \infty} \|v_k\|_{L^p(\Omega_k)} |\Omega_k \setminus \Omega_\infty|^{1/p}. \]

By \( \int_{\Omega_k} v_k \, dx = 0 \) again and the Sobolev-Poincaré inequality, one has

\[ \|v_k\|_{L^p(\Omega_k)} \leq C \|Dv_k\|_{L^p(\Omega_k)} \leq C \|g\|_{L^{p'/p}(\Omega_\infty)}. \]

Since \( |\Omega_k \setminus \Omega_\infty| \to 0 \) as \( k \to \infty \), we have \( \int_{\Omega_k} v_k \, dx = 0 \) as desired. \( \square \)

**Proof of Lemma 7.5** It suffices to prove \( \sqrt{A^\epsilon} Dv_\epsilon \to \sqrt{A} Dv_0 \) in \( L^p(\Omega) \) as \( \epsilon \to 0 \).

In the case of Dirichlet homogeneous boundary condition, similarly to the proof of Lemma 7.4, for any \( \epsilon \in (0, 1) \), one has

\[ \int_{\Omega} |\sqrt{A^\epsilon} Dv_\epsilon|^p \, dx \leq C(n, p, L, \Omega) \int_{\Omega} (|\epsilon|^p + 1) \, dx. \] (7.12)

Since \( v_\epsilon - v_0 \in W^{1, p}_0(\Omega) \), one has

\[ \int_{\Omega} F_{A^\epsilon, \epsilon}(Dv_\epsilon) \cdot (Dv_\epsilon - Dv_0) \, dx = \int_{\Omega} g(v_\epsilon - v_0) \, dx, \] (7.13)

and using \( v_\epsilon - v_0 \) to replace \( v_0 \) one also has

\[ \int_{\Omega} F_{A^\epsilon_0, \epsilon}(Dv_\epsilon) \cdot (Dv_\epsilon - Dv_0) \, dx = \int_{\Omega} g(v_\epsilon - v_0) \, dx. \] (7.14)

In the case of Neumann homogeneous boundary condition, we may further assume that \( \int_{\Omega} v_\epsilon \, dx = 0 \) for \( \epsilon \in [0, 1] \). Similarly to the proof of Lemma 7.4, one also has (7.12), then (7.13) and (7.14).

From (7.13) and (7.14), it follows that

\[ \int_{\Omega} \left( \sqrt{A^\epsilon} \right)^{-1} F_{A^\epsilon, \epsilon}(Dv_\epsilon) \cdot \left( \sqrt{A^\epsilon} Dv_\epsilon - \sqrt{A} Dv_0 \right) \, dx \]

\[ = \int_{\Omega} \left( \sqrt{A^0} \right)^{-1} F_{A^0, \epsilon}(Dv_0) \cdot \left( \sqrt{A^0} Dv_\epsilon - \sqrt{A^0} Dv_0 \right) \, dx, \]

and hence

\[ \square \]
\begin{align*}
\int_{\Omega} \left( \sqrt{A^\epsilon} \right)^{-1} F_{A^\epsilon,\epsilon}(Dv_\epsilon) \cdot \left( \sqrt{A^\epsilon} Dv_\epsilon - \sqrt{A^0} Dv_0 \right) \, dx \\
= \int_{\Omega} \left( \sqrt{A^\epsilon} \right)^{-1} F_{A^\epsilon,\epsilon}(Dv_\epsilon) \cdot \left( \sqrt{A^\epsilon} Dv_\epsilon - \sqrt{A^0} Dv_0 \right) \, dx \\
+ \int_{\Omega} \left( \sqrt{A^0} \right)^{-1} F_{A^0}(Dv_0) \cdot \left( \sqrt{A^0} Dv_\epsilon - \sqrt{A^0} Dv_0 \right) \, dx.
\end{align*}

Moreover, adding both sides with

\[- \int_{\Omega} \left( \sqrt{A^0} \right)^{-1} F_{A^0,\epsilon}(Dv_0) \left( \sqrt{A^\epsilon} Dv_\epsilon - \sqrt{A^0} Dv_0 \right) \, dx,\]

we further have

\begin{align*}
I &= \int_{\Omega} \left[ \left( \sqrt{A^\epsilon} \right)^{-1} F_{A^\epsilon,\epsilon}(Dv_\epsilon) - \left( \sqrt{A^0} \right)^{-1} F_{A^0,\epsilon}(Dv_0) \right] \cdot \left( \sqrt{A^\epsilon} Dv_\epsilon - \sqrt{A^0} Dv_0 \right) \, dx \\
&= \int_{\Omega} \left( \sqrt{A^\epsilon} \right)^{-1} F_{A^\epsilon,\epsilon}(Dv_\epsilon) \cdot \left( \sqrt{A^\epsilon} Dv_\epsilon - \sqrt{A^0} Dv_0 \right) \, dx \\
&\quad + \int_{\Omega} \left[ \left( \sqrt{A^0} \right)^{-1} F_{A^0,0}(Dv_0) - \left( \sqrt{A^0} \right)^{-1} F_{A^0,\epsilon}(Dv_0) \right] \cdot \left( \sqrt{A^0} Dv_\epsilon - \sqrt{A^0} Dv_0 \right) \, dx \\
&\quad + \int_{\Omega} \left( \sqrt{A^0} \right)^{-1} F_{A^0,\epsilon}(Dv_0) \cdot \left( \sqrt{A^0} Dv_\epsilon - \sqrt{A^\epsilon} Dv_\epsilon \right) \, dx \\
&= J_1 + J_2 + J_3. \tag{7.15}
\end{align*}

Now we show that \( I \to 0 \) as \( \epsilon \to 0 \). Indeed, by Hölder’s inequality and (7.12),

\[|J_1| \leq \left\| \left( \sqrt{A^\epsilon} \right)^{-1} F_{A^\epsilon,\epsilon}(Dv_\epsilon) \right\|_{L^{p/\epsilon}(\Omega)} \left\| \left( \sqrt{A^\epsilon} Dv_\epsilon - \sqrt{A^0} Dv_0 \right) \right\|_{L^p(\Omega)} \]

\[\leq C \left[ 1 + \left\| g \right\|_{L^{p/\epsilon}(\Omega)} \right] \left\| \left( \sqrt{A^\epsilon} - \sqrt{A^0} \right) Dv_0 \right\|_{L^p(\Omega)}.\]

Thanks to \( A^\epsilon \to A^0 \) almost everywhere as \( \epsilon \to 0 \) and \( A^\epsilon \in E_L(\Omega) \), we deduce \( J_1 \to 0 \) as \( \epsilon \to 0 \).

By Hölder’s inequality and (7.12), one has

\[|J_2| \leq C \left\| g \right\|_{L^{p/\epsilon}(\Omega)} \left\| \left( \sqrt{A^0} \right)^{-1} F_{A^0,0}(Dv_0) - \left( \sqrt{A^0} \right)^{-1} F_{A^0,\epsilon}(Dv_0) \right\|_{L^{p/\epsilon}(\Omega)}. \tag{7.16}\]

Observing

\[|\left( \sqrt{A^0} \right)^{-1} F_{A^0,\epsilon}(Dv_0)|^{p'} \leq |\sqrt{A^0} Dv_0|^p + 1 \tag{7.17}\]

and \( F_{A^0,\epsilon}(Dv_0) \to F_{A^0,0}(Dv_0) \) almost everywhere as \( \epsilon \to 0 \), thanks to (7.12) we have \( J_2 \to 0 \) as \( \epsilon \to 0 \). By Hölder’s inequality again, one has
\[ |J_3| = \int_{\Omega} \left( |\sqrt{A^0} Dv_0|^2 + \epsilon \right)^{p/2} \left( (\sqrt{A^0} - \sqrt{A^\epsilon}) \sqrt{A^0} Dv_0 \right) \cdot Dv_\epsilon \, dx \]
\[ \leq \| |\sqrt{A^0} Dv_0|^2 + \epsilon \|_{L^p(\Omega)}^{p/2} \| (\sqrt{A^0} - \sqrt{A^\epsilon}) \sqrt{A^0} Dv_0\|_{L^p(\Omega)} \| Dv_\epsilon\|_{L^p(\Omega)}. \]

By (7.12) and (7.17), noting \( A^\epsilon, A \in \mathcal{E}_L(\mathbb{R}^n) \) and \( A^\epsilon \to A \) almost everywhere as \( \epsilon \to 0 \), we obtain that \( J_3 \to 0 \) as \( \epsilon \to 0 \) as desired.

On the other hand, recall that

\[ (|\xi|^2 + |\eta|^2 + \epsilon)^{p/2} |\xi - \eta|^2 \leq C(p)(||\xi|^2 + \epsilon||^{p/2} \xi - (|\eta|^2 + \epsilon)^{p/2} \eta) \cdot (\xi - \eta) \quad \forall \xi, \eta \in \mathbb{R}^n. \]

Applying this to \( \xi = \sqrt{A^\epsilon} Dv_\epsilon \) and \( \eta = \sqrt{A} Dv \) we have

\[ \int_{\Omega} \left( |\sqrt{A^\epsilon} Dv_\epsilon|^2 + |\sqrt{A} Dv|^2 + \epsilon \right)^{p/2} \left| \sqrt{A^\epsilon} Dv_\epsilon - \sqrt{A} Dv \right|^2 \, dx \leq I \to 0. \]

If \( p \geq 2 \), this obviously yields \( \| \sqrt{A^\epsilon} Dv_\epsilon - \sqrt{A} Dv \|_{L^p(\Omega)} \to 0 \). If \( 1 < p < 2 \), by Holder’s inequality,

\[ \int_{\Omega} |\sqrt{A^\epsilon} Dv_\epsilon - \sqrt{A} Dv|^p \, dx \]
\[ \leq \left( \int_{\Omega} (|\sqrt{A^\epsilon} Dv_\epsilon|^2 + |\sqrt{A} Dv|^2 + \epsilon)^{p/2} |\sqrt{A^\epsilon} Dv_\epsilon - \sqrt{A} Dv|^2 \, dx \right)^{p/2} \]
\[ \left( \int_{\Omega} (|\sqrt{A^\epsilon} Dv_\epsilon|^2 + |\sqrt{A} Dv|^2 + \epsilon)^{p/2} \, dx \right)^{2-p}, \quad (7.18) \]

which converges to 0 as \( \epsilon \to 0 \).

By this, noting \( A^\epsilon, A \in \mathcal{E}_L(\mathbb{R}^n) \) and \( A^\epsilon \to A \) almost everywhere, write

\[ |Dv_\epsilon - Dv| \leq L |\sqrt{A^\epsilon} Dv_\epsilon - \sqrt{A} Dv| + L \left| \sqrt{A^\epsilon} - \sqrt{A} \right| Dv|, \]

one has \( \|Dv_\epsilon - Dv\|_{L^p(\Omega)} \to 0 \).

\[ \square \]

**Appendix A: Proof of Lemma 7.1**

To prove Lemma 7.1, given any bounded uniform domain \( \Omega \), below we briefly recall the construction of the extension operator \( \Lambda : \dot{W}^{1,q}(\Omega) \to \dot{W}^{1,q}(\mathbb{R}^n) \) by Jones [12] (see also [13]). For \( 1 \leq q < \infty \), denote by \( \dot{W}^{1,q}(\Omega) \) the homogeneous Sobolev space in any domain \( \Omega \subset \mathbb{R}^n \), that is, the collection of all function \( v \in L^q_{\text{loc}}(\Omega) \) with its distributional derivative \( Dv \in L^q(\Omega) \).

Recall that \( \Omega \) is an \( \epsilon_0 \)-uniform domain for some \( \epsilon_0 > 0 \) if for any \( x, y \in \Omega \) one can find a rectifiable curve \( \gamma : [0, T] \to \Omega \) joining \( x, y \) so that

\[ T = \ell(\gamma) \leq \frac{1}{\epsilon_0} |x - y| \quad \text{and dist} (\gamma(t), \partial \Omega) \geq \epsilon_0 \min\{t, T - t\} \quad \forall t \in [0, T], \]

where \( C \) is a constant. Note that \( |\partial \Omega| = 0 \). It is well-known that Lipschitz domains are always \( \epsilon_0 \)-uniform domains, where \( \epsilon_0 \) depends on Lipschitz constant of \( \Omega \). In the case \( \Omega \) is convex, \( \epsilon_0 \) depends on \( \text{diam} \Omega \) and \( |\Omega| \).

Denote by \( W_1 = \{S_j\} \) the Whitney decomposition of \( \Omega \) and \( W_2 = \{Q_j\} \) as the Whitney decomposition of \( (\Omega)^6 \) as [13, Section 2]. Set also \( W_3 = \{Q \in W_2, \ell(Q) \leq \frac{\epsilon_0}{100} \text{diam} \Omega \} \) as [13, Section 2]. By Jones and also [13], any cube \( Q \in W_3 \) has a reflection cube \( Q^* \in W_1 \).
such that \( \ell(Q) \leq \ell(Q^*) \leq 4\ell(Q) \) and hence \( \text{dist}(Q^*, Q) \leq C\ell(Q) \) for some constant \( C \geq 1 \) depending only on \( e_0 \) and \( n \). For any \( Q \in W_2 \setminus W_3 \) we just write \( Q^* = \Omega \).

Let \( \{Q_\ell\}_{\ell \in W_2} \) be a partition of unit associated to \( W_2 \) so that \( \text{supp}\varphi_Q \subset \frac{17}{16}Q \). The extension operator is then defined by

\[
\Lambda v(x) = \begin{cases} 
\sum_{Q \in W_2} \left( \int_{Q^*} v \, dz \right) \varphi_Q & \forall x \in (\Omega)^c \\
\liminf_{\epsilon \to 0} \int_{B(x, \epsilon) \cap \Omega} v \, dz & \forall x \in \partial \Omega \\
v(x) & \forall x \in \Omega.
\end{cases}
\]

Such extension operator is a slight modification of that in [13] and also [12].

By essentially the argument of Jones [12] (see also [13]), for \( 1 \leq q < \infty \) one has that \( \Lambda : W^{1,q}(\Omega) \to W^{1,q}(\Omega) \) is a linear bounded extension operator, that is, for any \( \nu \in W^{1,q}(\Omega) \) we have \( \Lambda \nu \in \tilde{W}^{1,q}(\mathbb{R}^n) \) so that \( \Lambda \nu |_{\Omega} = \nu \) and \( \|D\Lambda \nu\|_{L^q(\mathbb{R}^n)} \leq C\|D\nu\|_{L^q(\Omega)} \) for some \( C \) depending on \( n, e_0 \) and \( q \).

Moreover, by the arguments in [13], for any \( x \in \Omega^c \) and \( r \leq \frac{e_0}{16} \text{diam} \Omega \), one has \( \|D\Lambda \nu\|_{L^q(B(x, r))} \leq C\|D\nu\|_{L^q(\Omega)^c} \). In fact, the choice of \( r \) implies that \( B(x, r) \cap \Omega = \emptyset \) for any \( Q \in W_2 \setminus W_3 \) and hence one only need to bound \( H_{1,1} \) in [13, P.1422] and \( H_{1,2} = 0 \) and \( H_2 = 0 \) in [13, P.1422]. Thus \( \|D\Lambda v\|_{L^q(B(x, r))} \leq C\|Dv\|_{L^q(\Omega)^c} \). Moreover, for any \( x \notin \Omega \), denote by \( \tilde{x} \in \partial \Omega \) is the nearest point of \( x \). If \( \text{dist} (x, \partial \Omega) < r < \text{diam} \Omega \), one has

\[
\|D\Lambda v\|_{L^q(B(x, r))} \leq \|D\Lambda v\|_{L^q(B(\tilde{x}, 2r))} \leq C\|Dv\|_{L^q(\Omega)^c}.
\]

**Proof of Lemma 7.1.** Let \( A = (a_{ij}) \in \mathcal{E}_L(\Omega) \) with \( DA \in L^q(\Omega) \) with \( q \geq n \). Write \( \tilde{A} = (\Lambda a_{ij}) \). By the boundedness of \( \Lambda \), we have \( \|D\tilde{A}\|_{L^q(\mathbb{R}^n)} < C\|DA\|_{L^q(\Omega)} \). Noting

\[
\langle \tilde{A}(\tilde{x})\xi, \xi \rangle \geq \sum_{Q \in W_2} \left( \int_{Q^*} \langle A(z)\xi, \xi \rangle \, dz \right) \varphi_Q(x)
\]

we know that \( \tilde{A} \in \mathcal{E}_L(\mathbb{R}^n) \). Moreover in the case \( \|DA\|_{L^q(\Omega)} < \infty \), we have \( \Phi_{\tilde{A}}(\Omega_\eta, r) \leq C\Phi_A(\Omega_\eta, r) \) whenever \( x \in \Omega_\eta \) and \( 0 < \eta < r < \text{diam} \Omega \).

For \( \epsilon > 0 \), \( A^\epsilon = \tilde{A} * \eta_\epsilon \), where \( \eta_\epsilon \) is the standard smooth mollifier. Since \( \langle \tilde{A} * \eta(x)\xi, \xi \rangle = \langle \tilde{A}\xi, \xi \rangle * \eta(x) \), we know that \( A^\epsilon \in \mathcal{E}_L(\mathbb{R}^n) \). Moreover, in the case \( \|DA\|_{L^q(\Omega)} < \infty \), we have \( \|A^\epsilon * \eta\|_{L^q(B(x, r))} \leq \|\tilde{A}\|_{L^q(B(\tilde{x}, r))} \). For any \( x \in \Omega_\eta \) and \( 0 < \epsilon \leq \eta < r < \text{diam} \Omega \), we know that \( \Phi_{A^\epsilon}(\Omega_\eta, r) \leq \Phi_{\tilde{A}}(\Omega_{\eta + \eta}, r + \epsilon) \), and hence \( \Phi_{A^\epsilon}(\Omega_\eta, r) \leq C\Phi_A(\Omega, Cr) \) as desired. \( \square \)

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**Declarations**

**Conflict of interest** The authors state that there is no conflict of interest.
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