ON THE CLASSIFICATION OF INCOMPRESSIBLE FLUIDS AND A MATHEMATICAL ANALYSIS OF THE EQUATIONS THAT GOVERN THEIR MOTION*

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Abstract. In the first part of the paper we provide a new classification of incompressible fluids characterized by a continuous monotone relation between the velocity gradient and the Cauchy stress. The considered class includes Euler fluids, Navier-Stokes fluids, classical power-law fluids as well as stress power-law fluids, and their various generalizations including the fluids that we refer to as activated fluids, namely fluids that behave as an Euler fluid prior activation and behave as a viscous fluid once activation takes place. We also present a classification concerning boundary conditions that are viewed as the constitutive relations on the boundary. In the second part of the paper, we develop a robust mathematical theory for activated Euler fluids associated with different types of the boundary conditions ranging from no-slip to free-slip and include Navier’s slip as well as stick-slip. Both steady and unsteady flows of such fluids in three-dimensional domains are analyzed.

Key words. implicit constitutive theory, generalized viscosity, generalized fluidity, stress power-law fluid, shear thinning/shear thickening fluids, activated fluids, activation criterion, boundary conditions, slip, activated boundary conditions, long-time and large-data existence theory, weak solution

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1. Introduction. The concept of a fluid defies precise definition as one can always come up with a counter-example to that definition that seems to fit in with our understanding of what constitutes a fluid. As Goodstein [35] appropriately remarks “Precisely what do we mean by the term liquid? Asking what is a liquid is like asking what is life; we usually know when we see it, but the existence of some doubtful cases make it hard to define precisely.” The concept of a fluid is treated as a primitive concept in mechanics, but unfortunately it does not meet the fundamental requirement of a primitive, that of being amenable to intuitive understanding. This makes the study under consideration that much more difficult as it is our intent to classify fluid bodies. In this study we shall consider a subclass of the idealization of a fluid, namely that of incompressible fluid bodies. While no material is truly incompressible, in many bodies the change of volume is sufficiently small to be ignorable. Our ambit will include at one extreme materials that could be viewed as incompressible Euler fluids and at the other extreme materials that offer so much resistance to flow that they are “rigid-like” in their response, with a whole host of “fluid-like” behavior exhibited by bodies whose response lie in between these two extremes, such as fluids exhibiting shear thinning/shear thickening, stress thinning/stress thickening, etc.

Before discussing the constitutive classification of fluid bodies, it would be useful to consider another type of classification that is used, namely that of flow classification with regard to the flows of a specific fluid, so that we do not confuse these two types of classifications. One of the most useful approximations and an integral part of fluid dynamics is the boundary layer approximation for the flow of a Navier-Stokes...
The main tenet of the approximation is the notion that for flows of a Navier-Stokes fluid past a solid boundary, at sufficiently high Reynolds number, the vorticity is confined to a thin region adjacent to the solid boundary. In this region, referred to as the boundary layer, the flow is dominated by the effects of viscosity while these effects fade away as one moves further away from the solid boundary. Sufficiently far from the boundary, the effects of viscosity are negligible and the equations governing the flow are identical to those for an Euler (ideal) fluid and one solves the problem by melding together the solution for the Euler fluid far from the boundary and the boundary layer approximation in a thin layer adjacent to the boundary. The reason for developing such an approach is the fact that in the “boundary layer region” an approximation is obtained for the Navier-Stokes equations that is more amenable to analysis than the fully non-linear equations.

It is important to bear in mind that boundary layer theory is an approximation of the Navier-Stokes equations in different parts of the flow domain, that which is immediately adjacent to the solid boundary and that which is away from the solid boundary. Great achievements in the field of aerodynamics are a testimony to the efficacy and usefulness of such an approximation with regard to solving analytically or computationally relevant problems in a particular geometrical setting. On the other hand, rigorous analysis of the Prandtl boundary layer equations is, despite significant effort, far from being satisfactory (see [4], [32, 34, 33], [45, 46], [53], [54], [64], [66, 67], [78, 79]). An alternative viewpoint for modeling the boundary layer phenomena might thus bring some new insight on this issue.

The boundary layer approximation is not a constitutive approximation based on different flow regimes though it seems to resemble such an approximation. That is, one does not assume different constitutive assumptions for different regions in the flow domain, based on some kinematical or other criterion, but based on the value for the Reynolds number one merely carries out an approximation of the Navier-Stokes equation in the flow domain. It is possible, for instance, to assign different constitutive relations, based on the shear rate, namely the fluid being an Euler fluid below a certain shear rate and a Navier-Stokes or a non-Newtonian fluid above the critical shear rate (such a classification is considered in section 2.5), or as another possibility a non-Newtonian fluid if the shear rate is below a certain value and a Navier-Stokes fluid above that shear rate, or any such assumption for the constitutive response of the material, and to solve the corresponding equations for the balance of linear momentum in the different flow domains. Such distinct constitutive responses below and above a certain kinematical criterion is akin to models for the inelastic response of bodies wherein below a certain value of the strain or stress, the body behaves as an elastic body while for values above the critical value the body responds in an inelastic manner, which in turn might lead to certain parts of a body to respond like an elastic body while other parts could be exhibiting inelastic response. To make matters clear, in a solid cylindrical body that is undergoing torsion, a yield condition based on the strain would lead to the body beyond a certain radius to respond inelastically while below that threshold for the radius it responds as an elastic body. In such an approach different constitutive relations are used in different domains while in the classical boundary layer theory one uses approximation of the equations of motion of a particular fluid.

In this paper, we adopt the approach of assuming different constitutive response relationships in different flow domains of the fluid, based on the value of the shear rate or the value of the shear stress. We consider the possibility that the character of the fluid changes when the certain “activation” criterion is reached. Here we consider
an “activation” criterion that is based on the shear rate or the shear stress, but it could be any other criterion, say for instance the level of the electrical field in an electrorheological fluid, or the temperature which changes the character of the material from a fluid to a gas or a fluid to a solid, etc. Such an approach also provides an alternative way to viewing the classical boundary layer approximation in that it allows the fluid to behave like an Euler fluid in a certain flow domain and a Navier-Stokes fluid elsewhere. Furthermore, based on other criteria such as the Reynolds number we can carry out further approximations with regard to governing equations in the different flow domains.

Within the context of the Navier-Stokes theory, boundary layers occur at flows at sufficiently high Reynolds numbers. However, in the case of some non-Newtonian fluids it is possible to have regions that are juxtaposed to a solid boundary where the vorticity is concentrated even in the case of creeping flow, i.e., flows wherein the inertial effect is neglected when the Reynolds number is zero (see Mansutti and Rajagopal [62], [71], [38]). Thus, boundary layers are connected with the nonlinearities in the governing equation and are not a consequence of just high Reynolds numbers. Boundary layers can also occur at high Reynolds number in non-Newtonian fluids of the differential type (see Mansutti et al. [61]) and of the integral type (see Rajagopal and Wineman [77]). It is also possible that in non-Newtonian fluids one can have multiple decks with dominance of different physical mechanisms in the different decks, and in these different layers one can have the effects of viscosity, elasticity, etc., being significant, the delineation once again being determined within the context of a specific governing equation (see Rajagopal et al. [75, 74]). On the other hand, we could have a more complicated situation wherein the flow is characterized by different constitutive equations in different domains, and in these different domains it might happen that one can further delineate different subregions.

In the first part of this study, we provide a systematic classification of the response of incompressible fluid-like materials ranging from the ideal Euler fluid to non-Newtonian fluids that exhibit shear thinning/shear thickening, stress thinning/stress thickening, as well as those responses where the constitutive character of the material changes due to a threshold based on a kinematical, thermal, stress or some other quantity (an example of the same is the Bingham fluid which does not flow below a certain value of the shear stress and starts to flow once the threshold is overcome) based on some criterion concerning the level of shear rate or shear stress. We also provide a systematic study of both activated and non-activated boundary conditions ranging from free-slip to no-slip. In carrying out our classification, we come across the delineation of a class of fluids that, to our best knowledge, seems to have not been studied by fluid dynamicists. This class of fluids is characterized by the following intriguing dichotomy: (i) when the shear rate is below a certain critical value the fluid behaves as the Euler fluid (i.e., there is no effect of the viscosity, the shear stress vanishes), on the other hand (ii) if the shear rate exceeds the critical value, dissipation starts to take place and fluid can respond as a shear (or stress) thinning or thickening fluid or as a Navier-Stokes fluid. Implicit constitutive theory, cf. [73] and also [72, 76], provides an elegant framework to express such responses involving the activation criterion in a compact and elegant manner that is also more suitable for further mathematical and computational analysis.

In the second part of the paper, we study the mathematical properties of three-dimensional internal flows in bounded smooth domains for fluids belonging to this new class. We subject such flows to different types of boundary conditions including no-slip, Navier-slip, free-slip and activated boundary conditions like stick-slip. For
this class of fluids and boundary conditions we prove the global-in-time existence of a weak solution in the sense of Leray to initial and boundary value problems.

2. Classification of incompressible fluids. Incompressible fluids are subject to the restriction on the admissible class of the velocity fields $\mathbf{v}$ of the form

$$\text{div} \mathbf{v} = 0,$$

which can be written alternatively as

$$(2.1) \quad \text{tr} \mathbf{D} = \mathbf{D} : \mathbf{I} = 0,$$

where $\mathbf{D}$ (sometimes denoted $\mathbf{D}\mathbf{v}$) stands for the symmetric part of the velocity gradient, i.e., $\mathbf{D} = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$, $\mathbf{v}$ being the velocity.

Due to this restriction, it is convenient to split the Cauchy stress tensor $\mathbf{T}$ into its traceless (deviatoric) part $\mathbf{S}$ and the mean normal stress, denoted by $m$ (more frequently expressed as $-p$), i.e.,

$$(2.2) \quad \mathbf{S} = \mathbf{T} - \frac{1}{3}(\text{tr} \mathbf{T}) \mathbf{I} \quad \text{and} \quad -p = m = \frac{1}{3} \text{tr} \mathbf{T}.$$ 

Hence

$$\mathbf{T} = m \mathbf{I} + \mathbf{S} = -p \mathbf{I} + \mathbf{S},$$

and, in virtue of (2.1), the stress power $\mathbf{T} : \mathbf{D}$ satisfies

$$\mathbf{T} : \mathbf{D} = \mathbf{S} : \mathbf{D}.$$ 

The main result of this section will be the classification of fluids using a simple framework that is characterized by a relation between $\mathbf{S}$ and $\mathbf{D}$, i.e., we are interested in materials whose response can be incorporated into the setting given by the implicit constitutive equation

$$(2.3) \quad \mathcal{G}(\mathbf{S}, \mathbf{D}) = \mathbf{O}.$$ 

The only restriction that we place is the requirement that response has to be monotone. For relevant discussion concerning non-monotone responses, we refer the reader to [57], [50], and [40].

The incompressible Navier-Stokes fluid is a special sub-class of (2.3) where the relation between $\mathbf{S}$ and $\mathbf{D}$ is linear. This can be written either as

$$(2.4) \quad \mathbf{S} = 2\nu_\ast \mathbf{D} \quad \text{with} \quad \nu_\ast > 0,$$

where $\nu_\ast$ is called the (shear) viscosity, or as

$$(2.5) \quad \mathbf{D} = \alpha_\ast \mathbf{S} := \frac{1}{2\nu_\ast} \mathbf{S} \quad \text{with} \quad \alpha_\ast > 0,$$

where the coefficient $\alpha_\ast$ is called the fluidity. Note that the stress power takes then the form

$$\mathbf{S} : \mathbf{D} = 2\nu_\ast |\mathbf{D}|^2 = \alpha_\ast |\mathbf{S}|^2.$$
There are two limiting cases when the stress power $S : D$ vanishes. Either

\[(2.6) \quad S = O,\]

which implies that the fluid under consideration is the incompressible Euler fluid \((T = -pI, \text{see } (2.2)), or\]

\[(2.7) \quad D = O \quad \text{for all admissible flows.}\]

The latter corresponds to the situation where the body admits merely rigid body motions. More precisely, the flows fulfilling (2.7) can be characterized through

\[\nu(t, x) = a(t) \times x + b(t) \quad \text{in all admissible flows}.\]

Response of models (2.4) (or (2.5)), (2.6), and (2.7) is shown in the Figure 2.1.

2.1. Classical power-law fluids. Classical power law fluids are described by

\[(2.8) \quad S = 2\tilde{\nu}|D|^{r-2}D,\]

which leads to

\[S : D = 2\tilde{\nu}|D|^r.\]

Since $|D|^{r-2}D$ should have meaning for $|D| \to 0$, we require a lower bound on $r$, namely

\[(2.9) \quad r > 1.\]

Otherwise, if $r = 1$ then $\lim_{|D| \to 0} D / |D|$ does not exist, and if $r < 1$ then $|S| \to +\infty$ and stress concentration occurs at points where $D$ vanishes (i.e., $S$ plays the role of penalty for point where $D$ could vanish).

In what follows we study power-law fluids with the power-law index satisfying (2.9) and we shall investigate the responses of these fluids for $r \to 1$ and $r \to \infty$. The latter corresponds to the case when the dual exponent $r' := r / (r - 1)$ tends to 1.

We introduce the generalized viscosity through

\[(2.10) \quad \nu_g(|D|) = \tilde{\nu}_s|D|^{r-2}.\]

In order to have the same units for $\nu_g$ as for the viscosity $\nu_s$ that appears in the formula for the Navier-Stokes fluid (see (2.4)), $D$ should scale as $d_s$ that has the unit
Thus, we replace (2.10) by
\[ \nu_g(|D|) = \nu_s \left( \frac{|D|}{d_s} \right)^{r-2} \]
where \([d_s] = s^{-1}\) and \([\nu_s] = \text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-1}\)
and we replace (2.8) by
\[ (2.11) \quad S = 2\nu_s \left( \frac{|D|}{d_s} \right)^{r-2} D \quad \text{with} \quad [d_s] = s^{-1} \quad \text{and} \quad [\nu_s] = \text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-1}. \]

Of course, \(\nu_s\) in (2.11) and \(\nu_*\) in (2.4) are different in general; they however have the same units.

On considering (2.11), we notice the following equivalence\(^1\)
\[ (2.12) \quad S = 2\nu_s \left( \frac{|D|}{d_s} \right)^{r-2} D \iff D = \frac{1}{2\nu_s} \left( \frac{|S|}{2\nu_* d_s} \right)^{\frac{2-r}{r}} S, \]
which gives rise to the following expressions for the generalized viscosity and generalized fluidity
\[ \nu_g(|D|) = \nu_s \left( \frac{|D|}{d_s} \right)^{r-2} \quad \text{and} \quad \alpha_g(|S|) = \frac{1}{2\nu_s} \left( \frac{|S|}{2\nu_* d_s} \right)^{\frac{2-r}{r}}. \]

It also allows us to express the stress power in the form \((r^r := r/(r-1))\)
\[
S : D = \left( \frac{1}{r} + \frac{1}{r^r} \right) S : D = \frac{1}{r} S : D + \frac{1}{r^r} S : D \\
= 2\nu_s d_s^2 \left( \frac{|D|}{d_s} \right)^r + \frac{1}{r^r} \left( \frac{|S|}{2\nu_* d_s} \right)^{r^r}.
\]

Summarizing,
\[ (2.13) \quad S = 2\nu_g(|D|^2) D = 2\nu_s \left( \frac{|D|}{d_s} \right)^{r-2} D \iff D = \alpha_g(|S|^2) S = \frac{1}{2\nu_*} \left( \frac{|S|}{2\nu_* d_s} \right)^{r^r} S, \]
emphasizing that the equivalence in (2.13) holds only if \(r \in (1, +\infty)\) (which is equivalent to \(r^r \in (1, +\infty)\)).

\(^1\)Indeed, starting for example from the formula on the left-hand side of (2.12), we conclude that
\[ |S| = \frac{2\nu_*}{d_s^{-2}} |D|^{r-1}, \]
which implies
\[ |D|^{2-r} = \left( \frac{d_s^{-2}}{2\nu_*} |S| \right)^{\frac{2-r}{r}}. \]

Hence
\[ D = \frac{1}{2\nu_*} \left( \frac{|D|}{d_s} \right)^{2-r} S = \left( \frac{1}{2\nu_*} \right)^{1+\frac{2-r}{r^r} \left( d_s^{-2} \right)^{\frac{2-r}{r}} + 1} |S|^{2-r} S = \frac{1}{2\nu_*} \left( \frac{|S|}{2\nu_* d_s} \right)^{\frac{2-r}{r^r}} S, \]
which leads to the formula on the right-hand side of (2.12).
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Generalizing the approach used in [60], we will investigate the limits of $S$ and $\nu_g$ as $D$ tends to zero or infinity, or vice versa, study limits of $D$ and $\alpha_g$ as $S$ vanishes or tends to infinity.

Letting $|D| \to 0+$ we obtain, starting from the formula on the left-hand side of (2.13),

$$
|S| \to 0 \quad \text{if } r > 1,
|S| \leq 2\nu_* d_* \quad \text{if } r = 1, \quad \text{(as } |D| \to 0+ )
|S| \to +\infty \quad \text{if } r < 1,
$$

and

$$
|\nu_g(|D|)| \to 0 \quad \text{if } r > 2,
\nu_g(|D|) = 2\nu_* \quad \text{if } r = 2, \quad \text{(as } |D| \to 0+ )
|\nu_g(|D|)| \to +\infty \quad \text{if } r < 2.
$$

(2.14)

Thus, we note that the Cauchy stress $T$ in the fluid tends to a purely spherical stress when $r$ is greater than 1 and the norm of $D$ tends to $0+$, or put differently, the constitutive relation for the fluid reduces to that for an Euler fluid.

Similarly, letting $|D| \to +\infty$ we have

$$
|S| \to +\infty \quad \text{if } r > 1,
|S| \leq 2\nu_* d_* \quad \text{if } r = 1, \quad \text{(as } |D| \to +\infty )
|S| \to 0 \quad \text{if } r < 1,
$$

and

$$
|\nu_g(|D|)| \to +\infty \quad \text{if } r > 2,
\nu_g(|D|) = 2\nu_* \quad \text{if } r = 2, \quad \text{(as } |D| \to +\infty )
|\nu_g(|D|)| \to 0 \quad \text{if } r < 2.
$$

(2.15)

In order to investigate the behavior of $D$ and fluidity in the limiting case, it is useful to employ the expression on the right-hand side of (2.13). Thus, for $|S| \to 0+$, we get

$$
|D| \to 0 \quad \text{if } r' > 1,
|D| \leq d_* \quad \text{if } r' = 1, \quad \text{(as } |S| \to 0+ )
|D| \to +\infty \quad \text{if } r' < 1,
$$

and

$$
|\alpha_g(|S|)| \to 0 \quad \text{if } r' > 2,
\alpha_g(|S|) = \frac{1}{2\nu_*} \quad \text{if } r' = 2, \quad \text{(as } |S| \to 0+ )
|\alpha_g(|S|)| \to +\infty \quad \text{if } r' < 2.
$$

(2.15)

Similarly, letting $|S| \to +\infty$ we have

$$
|D| \to +\infty \quad \text{if } r' > 1,
|D| \leq d_* \quad \text{if } r' = 1, \quad \text{(as } |S| \to +\infty )
|D| \to 0 \quad \text{if } r' < 1,
$$
Next, we study the response of the classical power-law fluid with regard to its dependence on the value of power-law index ($r \rightarrow 1+$ and $r' \rightarrow 1+$). Figure 2.2 illustrates behavior for both large $r$ and $r'$ approaching 1.

Letting $r \rightarrow 1+$ in (2.13), we observe that, for $D \neq O$, $S = 2\nu_* d_* |D|$ (and thus $|S| \leq 2\nu_* d_*$), while for $D = O$ and for any $A \in R_{sym}^{3 \times 3}$ such that $|A| \leq 2\nu_* d_*$ we can find a sequence $\{D_n\}_{n=1}^\infty$ converging to zero and

$$\lim_{n \rightarrow \infty} 2\nu_* d_* |D_n| = A.$$ 

Consequently, (2.13) for $r \rightarrow 1+$ approximates the response that could be referred to as rigid/free-flow like behavior:

$$\alpha_g(|S|) \rightarrow +\infty \quad \text{if} \quad r' > 2,$$
$$\alpha_g(|S|) = \frac{1}{2\nu_*} \quad \text{if} \quad r' = 2, \quad \text{(as} \quad |S| \rightarrow +\infty)$$
$$\alpha_g(|S|) \rightarrow 0 \quad \text{if} \quad r' < 2.$$

Instead of viewing (2.16) as multivalued response (both in the variables $D$ and $S$), it is possible to write (2.16) as a continuous graph over the Cartesian product $R_{sym}^{3 \times 3} \times R_{sym}^{3 \times 3}$ (see the framework (2.3)) defined through a (scalar) equation

$$(|S| - 2\nu_* d_*)^+ + |2\nu_* d_* D - |D||S| = 0.$$ 

For determining the behavior of (2.13) as $r \rightarrow +\infty$ we prefer to study (2.13)$_2$ for $r' \rightarrow 1+$ and analogous to the above consideration we observe that

$$\alpha_g(|S|) \rightarrow +\infty \quad \text{if} \quad r' > 2,$$
$$\alpha_g(|S|) = \frac{1}{2\nu_*} \quad \text{if} \quad r' = 2,$$
$$\alpha_g(|S|) \rightarrow 0 \quad \text{if} \quad r' < 2.$$

We can call this response Euler/rigid like response. We can again rewrite (2.18) as

$$(2\nu_* |D| - d_*)^+ + |2\nu_* S D - d_* S| = 0.$$
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The models (2.16) and (2.18) are examples of fluids described within the context of an activation criterion. More examples will be discussed in Subsections 2.4 and 2.5. The slash formalism \texttt{name1/name2} (that we will use also below) means that material behaves as \texttt{name1} before activation and as \texttt{name2} after the activation criterion is met.

2.2. Generalized power-law fluids and stress power-law fluids. The formula (2.13) suggests the introduction of \textit{generalized power-law fluids} and \textit{generalized stress power-law fluids} by requiring that, for the former,

$$
S = 2\nu_g (|D|^2) D
$$

and, for the latter,

$$
D = \alpha_g (|S|^2) S
$$

where \(\nu_g\) and \(\alpha_g\) are non-negative continuous functions referred to as the \textit{generalized viscosity} and the \textit{generalized fluidity}. The quantities \(|D|^2 = \text{tr} D^2\) and \(|S|^2 = \text{tr} S^2\) representing the second invariants of \(D\) and \(S\), respectively, can be viewed as natural higher dimensional generalizations of the shear-rate and the shear stress, respectively.

We further introduce \textit{zero shear rate viscosity} as

$$
\nu_0 := \lim_{|D| \to 0} \nu_g (|D|^2)
$$

and \textit{zero shear stress fluidity} through

$$
\alpha_0 := \lim_{|S| \to 0} \alpha_g (|S|^2) .
$$

It follows from (2.14) that for classical power-law fluids the zero shear rate viscosity vanishes for \(r > 2\), is finite for \(r = 2\), and blows up if \(r \in (1, 2)\). A similar behavior can be inferred from (2.15) for the zero shear stress fluidity: for \(r' > 2\) (i.e., for \(r \in (1, 2)\)) \(\alpha_0\) is zero, for \(r = 2\) it is positive, and for \(r' \in (1, 2)\) (it means for \(r > 2\)) the generalized fluidity becomes singular in the vicinity of the origin.

Such behavior is not experimentally observed in any fluid; more frequently both the zero shear rate viscosity and zero shear stress fluidity are finite. The most popular generalizations of the classical power-law fluid that exhibit these features as \(|D| \to 0+\) (resp. \(|S| \to 0+\)) take the form

(2.20) \quad $S = 2\nu_* \left( \frac{1}{2} + \frac{1}{2} \frac{|D|^2}{d_*^2} \right)^{\frac{r-2}{2}} D$

and

(2.21) \quad $D = \alpha_* \left( \frac{1}{2} + \frac{1}{2} \frac{|S|^2}{(2\nu_* d_*)^2} \right)^{\frac{r'-2}{2}} S$.

We refer the reader to [57] for further details; here we emphasize two observations. First, although both (2.20) and (2.21) are invertible for \(r \in (1, +\infty)\), (2.21) is not an inverse of (2.20) and vice versa (compare it to (2.13)). Second, both formulas are defined for all \(r \in (-\infty, +\infty)\) and \(r' \in (-\infty, +\infty)\) respectively. The relationship \(r' = \frac{r-2}{2} \) is however understood as a relation valid for \(r > 1\).

If \(r < 1\), then \(S\) considered as a function of \(D\) given by (2.20) is not monotone; the same holds for (2.21) if \(r' < 1\). We refer the reader to [57] for more details and to [50] for further nontrivial extensions.
2.3. Fluids that can be viewed as a mixture of power-law fluids. It is natural to consider the possibility that the total response of the fluid-like material is given as the sum of particular responses (a simplified scenario for the mixtures) of the individual (in our case two) contributors, i.e.,

\[
S = S_1 + S_2.
\]

One may think of putting two dashpots into a parallel arrangement; the response of one dashpot captures behavior of a fluid A, and the response of another dashpot corresponds to a fluid B; see Figure 2.3. To illustrate the potential of this setting, we consider for illustration three examples:

(i) \(S_1 = 2\nu_* \left( \frac{\mathbf{D} \cdot \mathbf{D}}{\nu_*} \right)^{r-2} \mathbf{D} \) and \(S_2 = 2\tilde{\nu}_* \left( \frac{1}{2} + \frac{1}{2} \frac{\mathbf{D} \cdot \mathbf{D}}{\tilde{\nu}_*} \right)^{r/2} \mathbf{D} \) with \(r \in (1, 2), q > 2\);

(ii) \(S_1 = 2\nu_* \mathbf{D} \) and \(S_2 \) fulfills \(\mathbf{D} = \frac{1}{2\nu_*} \left( \frac{1}{2} + \frac{1}{2} \frac{\mathbf{S}_2^2}{(2\nu_* d_s)^2} \right)^{-2} \mathbf{S}_2 \) with \(r' \in (1, +\infty) \setminus \{2\}\);

(iii) \(S_1 \) responds as in (2.17) and \(S_2 \) responds as in (2.19).

The responses of fluids modeled by the constitutive expressions (i) and (ii) are shown in Figure 2.4 for some choice of parameters. Further examples will be provided in Subsection 2.5.

2.4. Fluids with bounded shear rate or bounded shear stress. We have seen in Subsection 2.1 that the models (2.16) and (2.18) exhibit an interesting feature, namely, the stress \(S\) is bounded as the symmetric part of the velocity gradient is varied, and vice versa. While there are several mathematical advantages to the stress \(S\) expressed as a function of \(\mathbf{D}\), as this would greatly simplify the number of equations that one needs to consider when one substitutes it into the balance of linear momentum (even an explicit expression of the symmetric part of the velocity gradient as a function of \(S\) might lead to a simplified structure for the equations) experiments on colloidal
fluids clearly show that a fully implicit theory is necessary (see for example [12, 39] and further references in [68]). When considering the models (2.20) and (2.21) with \( r = 1 \) and \( r' = 1 \) (and adjusting a scaling by factor \( \sqrt{2} \)), we obtain

\[
S = 2\nu_s \frac{D}{\sqrt{1 + \frac{D^2}{\sigma^2}}}
\]

and

\[
D = \alpha_s \frac{S}{\sqrt{1 + \frac{S^2}{(2\nu_s d_s)^2}}}
\]

and we observe that, in the case of (2.23),

\[
|S| \leq 2\nu_s d_s \quad \text{for all } D,
\]

and, in the case of (2.24),

\[
|D| \leq d_s \quad \text{for all } S.
\]

It is convenient to generalize (2.23) and (2.24) in the following manner: for parameters \( a, b \in (0, +\infty) \) consider

\[
S = 2\nu_s \frac{D}{\left(1 + \left(\frac{|D|}{d_s}\right)^a\right)^\frac{1}{a}}
\]

and

\[
D = \alpha_s \frac{S}{\left(1 + \left(\frac{|S|}{2\nu_s d_s}\right)^b\right)^\frac{1}{b}}.
\]

In both cases, it is worth studying the behavior of the fluids for large \( a \) and \( b \). When \( a \to +\infty \) in (2.25), the constitutive relation approximates the response of the activated fluid which behaves as the Euler fluid prior to the activation and the magnitude of the stress remains bounded; analogously, when \( b \to +\infty \) in (2.26), the constitutive relation approximates the response such that the magnitude of \( D \) remains bounded and the body admits merely rigid body motions till the activation takes place, see Figure 2.5.

**2.5. Activated fluids.** In this section we study two classes of fluids: the first class is activation based on the value of the stress (similar in character to a Bingham fluid) while the second class is activation based on the value of the shear rate.

The first class of fluids that are studied flow only if the generalized shear stress \( |S| = (\text{tr}S^2)^\frac{1}{2} \) exceeds a certain critical value \( \sigma_* \), referred to as the yield stress. Once the fluid flows, we assume the fluid behavior is described by the constitutive expression for a generalized power-law or a generalized stress power-law fluid. In the parts of the subdomain where \( |S| \) is below \( \sigma_* \) the fluid can only translate or rotate as a rigid body. Such responses are traditionally described (see [27]) through the dichotomy

\[
|S| \leq \sigma_* \iff D = 0,
\]

\[
|S| > \sigma_* \iff S = \sigma_* \frac{D}{|D|} + S_2 \quad \text{with } \left\{ \begin{array}{ll}
\text{either} & S_2 = 2\nu_s \left(\frac{|D|^2}{d_s}\right)D, \\
\text{or} & D = \alpha_s \left(\frac{|S|^2}{2\nu_s d_s}\right)S.
\end{array} \right.
\]
In the case of the stress $\mathbf{S}_2 = 2\nu_*\mathbf{D}$ we obtain the constitutive representation for the Bingham fluid (see Figure 2.6 on the left) and if $\mathbf{S}_2 = 2\nu_g (|\mathbf{D}|^2) \mathbf{D}$ then we obtain the constitutive representation for the Herschel-Bulkley fluid. It is worth of observing that (2.27) can be equivalently written within the context of the framework for implicit constitutive equations (2.3). Specifically, considering (2.27) with the expression $\mathbf{S}_2 = 2\nu_g (|\mathbf{D}|^2) \mathbf{D}$ the equivalent formulation can be expressed as

$$2\nu_g (|\mathbf{D}|^2) \mathbf{D} = \left(\frac{|\mathbf{S}| - \sigma_*}{|\mathbf{S}|}\right)^+ \mathbf{S},$$

where $(t)^+ = \max\{t, 0\}$ for $t \in \mathbb{R}$.

On the other hand, considering (2.27) with the expression $\mathbf{D} = \alpha_g (|\mathbf{S}_2|^2) \mathbf{S}_2$, the equivalent representation reads

$$\mathbf{D} = \alpha_g \left(|\mathbf{S} - \sigma_* \frac{\mathbf{D}}{|\mathbf{D}|}|^2\right) \left(\frac{|\mathbf{S}| - \sigma_*}{|\mathbf{S}|}\right)^+ \mathbf{S},$$

which is in our opinion worthy of detailed investigation.

The next class is a dual to (2.27) in the following sense. If the generalized shear rate $|\mathbf{D}|$ is below a critical value $\delta_*$, the flow is frictionless. Inner friction between the fluid layers becomes important only when $|\mathbf{D}|$ exceeds $\delta_*$. Then the fluid can flow as a Navier-Stokes fluid, or a power-law fluid (see Figure 2.6 on the right), or a generalized power-law fluid, or a generalized stress power-law fluid. To summarize,
analogous to (2.27), we can describe such a response through the relation
\[ |D| \leq \delta_\ast \iff S = 0, \]
\[ |D| > \delta_\ast \iff D = \delta_\ast \frac{S}{|S|} + \alpha_\ast (|S|^2) S. \]

It is not surprising that this relation can be written as an explicit relation for the stress \( S \) in terms of the symmetric part of the velocity gradient \( D \), namely
\[
\alpha_\ast (|S|^2) S = \frac{(|D| - \delta_\ast)^+}{|D|} D.
\]

If there is \( \nu_\ast \) such that
\[
D = \alpha_\ast (|S|^2) S \iff S = 2\nu_\ast (|D|^2) D
\]
then (2.28) can be written in the form ((2.29) describes the behavior of the fluid after activation)
\[
S = 2\nu_\ast (|D|^2) \frac{(|D| - \delta_\ast)^+}{|D|} D.
\]

Explicit equivalence holds for the Navier-Stokes fluid (see (2.5)) and for the standard power-law fluids (see (2.13)). Then we obtain the response
\[
S = 2\nu_\ast \frac{(|D| - \delta_\ast)^+}{|D|} D
\]
and
\[
S = 2\nu_\ast \left( \frac{|D|}{d_\ast} \right)^{r-2} \frac{(|D| - \delta_\ast)^+}{|D|} D,
\]
respectively (see Figure 2.6 on the right). We call the response (2.30) the Euler/Navier-Stokes fluid and the response (2.31) the Euler/power-law fluid. If
\[
S = \left( 2\nu_\ast + 2\tilde{\nu}_\ast \left( \frac{|D|}{d_\ast} \right)^{r-2} \right) \frac{(|D| - \delta_\ast)^+}{|D|} D,
\]
with \( r > 2 \) we call this response Euler/Ladyzhenskaya fluid, since O. A. Ladyzhenskaya was the first to consider the generalization of the Navier-Stokes constitutive equation to the form
\[
S = \left( 2\nu_\ast + 2\tilde{\nu}_\ast |D|^{r-2} \right) D
\]
and showed that unsteady internal flows of such fluids in a bounded smooth container admit unique weak solutions if \( r > \frac{d+2}{2} \) (or \( \frac{d+2}{2} \) in general dimension \( d \)); see [21, 20] for improvement of the uniqueness result to \( r \geq \frac{d+2}{2} \). Ladyzhenskaya used kinetic theory arguments to derive (2.32) with \( r = 4 \); see [47, 48, 49].
Simple shear flows of the Euler/Navier-Stokes fluid. For the sake of illustration of response of the fluid (2.30) we consider a simple shear flow of such a fluid. In order to characterize such fluids it is also useful to consider a modified model which we call the regularized Euler/Navier-Stokes fluid given by

\begin{equation}
S = 2\nu_s \left( \epsilon_* + \frac{(|D| - \delta_*)^+}{|D|} \right) D
\end{equation}

with an extra parameter \( \epsilon_* \geq 0 \). We will consider solutions of the balance equations (see (2.40), (2.41) below) for the response of the fluids described by (2.33) and the degenerate case (2.30)) in \( \mathbb{R}^2 \), with the velocity taking the form

\[ v(x, y) = (u(y), 0) \quad x, y \in \mathbb{R} \]

for some \( u : \mathbb{R} \to \mathbb{R} \) absolutely continuous on every compact interval in \( \mathbb{R} \). We note that now we deal with solutions of the governing equations in \( \mathbb{R}^2 \) so there are no boundary conditions involved. It is easy to check that if \( \epsilon_* > 0 \) then all such solutions of (2.40), (2.41), and (2.33) fulfill

\begin{align}
(2.34a) & \quad \left( \epsilon_* + \mathcal{H}(|u'| - \sqrt{2}\delta_*) \right) u'' = -2C \quad \text{a.e. in} \ \mathbb{R}, \\
(2.34b) & \quad p(x) = -2\nu_s C x + p_0,
\end{align}

with some \( C \in \mathbb{R}, \mathcal{H}(t) = 1 \) if \( t > 0 \) and \( \mathcal{H}(t) = 0 \) otherwise. The formulas (2.34) represent a generalization of the well-known equations for simple shear flows of the Navier-Stokes fluid, which is a special case with \( \delta_* = 0 \). In fact all the solutions of (2.34) take the form

\begin{align}
(2.35a) & \quad u(y) = \begin{cases} 
-\frac{C}{\epsilon_*} (y - y_0)^2 + u_0 & |y - y_0| \leq \frac{\sqrt{2\delta_*}}{2|C|}, \\
-\frac{C}{\epsilon_*} (y - y_0)^2 + \sqrt{2\delta_*} \left| \frac{y - y_0}{\epsilon_*} \right| - \epsilon_* \left( \frac{\sqrt{2\delta_*}}{\epsilon_*} \right)^2 + u_0 & |y - y_0| \geq \frac{\sqrt{2\delta_*}}{2|C|},
\end{cases} \\
(2.35b) & \quad p(x) = -2\nu_s C x + p_0,
\end{align}

with any \( C, y_0, u_0, p_0 \in \mathbb{R} \). In the interval \( \left\{ |y - y_0| \leq \frac{\sqrt{2\delta_*}}{2|C|} \right\} \) the fluid is in the regime below the “activation” threshold (where \( |u'| \leq \sqrt{2}\delta_* \)) with the viscosity \( \epsilon_*\nu_s \) while outside this interval the threshold is exceeded and the generalized viscosity has the value \( \nu_s \left( \epsilon_* + 1 - \frac{\sqrt{2\delta_*}}{|u'|} \right) \). Taking the limit \( \epsilon_* \to 0+ \) one obtains

\begin{align}
(2.36a) & \quad u(y) = -C \left( (y - y_0)^2 + \sqrt{2\delta_*} \frac{|y - y_0|}{C} \right) + u_0, \\
(2.36b) & \quad p(x) = -2\nu_s C x + p_0,
\end{align}

which indeed is a solution of the balance equations for the Euler/Navier-Stokes fluid (2.30) with any \( C, y_0, u_0, p_0 \in \mathbb{R} \). Now the flow exceeds the activation threshold everywhere except of \( y = y_0 \) where \( u'(y_0 \pm) = \mp \sqrt{2\delta_*} \frac{C}{\epsilon_*} \) and the shear rate jumps there by virtue of the vanishing viscosity. In Figure 2.7 we display a family of solutions (2.35a) for varying \( \epsilon_* \) and (2.36a) (matching \( \epsilon_* = 0 \)) for fixed values of \( C, y_0 \).

Apart of the family of solutions (2.36) the Euler/Navier-Stokes fluid (2.30) admits also the simple shear flows which do not exceed the threshold \( |u'| = \sqrt{2}\delta_* \), the viscosity is zero and therefore any admissible velocity profile is a solution. Such solutions are characterized by

\begin{align}
(2.37a) & \quad |u'| \leq \sqrt{2}\delta_*, \quad \text{a.e. in} \ \mathbb{R}, \\
(2.37b) & \quad p = p_0,
\end{align}
Fig. 2.7. Simple shear flows of the regularized Euler/Navier-Stokes fluid for various values of the added viscosity $\epsilon_*\nu_*$ and fixed $C > 0$. Circles mark the point of activation $y = y_0 \pm \sqrt{2}\delta_*^2$ where $|u'| = \sqrt{2}\delta_*$. The degenerate case $\epsilon_* = 0$ is activated everywhere, i.e., $|u'| \geq \sqrt{2}\delta_*$. Note that the velocity profiles are determined up to an additive constant (because no boundary conditions are enforced); here we take $u_0 = 0$.

Table 2.1
Summary of systematic classification of fluid-like response with the corresponding $|D|\cdot|S|$ diagrams

| Euler/rigid | Navier-Stokes/limiting shear-rate | fluid body allowed to move only rigidly |
|------------|----------------------------------|--------------------------------------|
| Euler/shear-thickening | shear-thickening | rigid/shear-thickening |
| Euler/Navier-Stokes | Navier-Stokes | rigid/Navier-Stokes |
| Euler/shear-thinning | shear-thinning | rigid/shear-thinning |
| Euler | limiting | rigid/free-flow |

$|D| \leq \delta_* \iff S = O$ no activation $|S| \leq \sigma_* \iff D = O$

with some locally Lipschitz continuous $u : \mathbb{R} \to \mathbb{R}$ and $p_0 \in \mathbb{R}$. In fact the families (2.36) and (2.37) are all possible weak solutions for a simple shear flow of the Euler/Navier-Stokes fluid.

2.6. Classification of incompressible fluids. The previous exposition should indicate the broad spectrum of fluid responses that can be described within the setting

$$b \left( |D|^2 \right) D = a \left( |S|^2 \right) S,$$

where $a$ and $b$ are continuous (not necessarily always differentiable) functions.

The following Table 2.1 summarizes these observations in a different way, paying attention to the broad range of models covered by (2.38). It includes the Euler (frictionless) fluid at one extreme and a fluid that only moves rigidly at the other
extreme and contains the responses ranging from the fluids enforcing the activation criterion $|\mathbf{D}| \leq \delta_* \iff \mathbf{S} = \mathbf{0}$ through non-activated fluids to the fluids that are governed by the activation criterion $|\mathbf{S}| \leq \sigma_* \iff \mathbf{D} = \mathbf{0}$. Vertically, the range of $r$ is iterated from top to bottom: $r = +\infty$, $r \in (2, +\infty)$, $r = 2$, $r \in (1, 2)$, and $r = 1$. Thus, at the bottom left corner we have perfectly frictionless Euler fluid and at the top right corner we have a fluid that can only undergo rigid motions. In the middle of the table the Navier-Stokes model is placed.

2.7. Activated boundary conditions. Boundary conditions can have as much impact on the nature of the flow as the constitutive equation for the fluid in the bulk. We illustrate it explicitly in this subsection. Here, for the sake of clarity, we restrict ourselves to internal flows, i.e., we assume that

\[(2.39) \quad \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on} \ \partial \Omega,\]

where $\mathbf{n} : \partial \Omega \to \mathbb{R}^d$ denotes the mapping that assigns the outward unit normal vector to any $x \in \partial \Omega$. The behavior of the fluid at the tangential direction near the boundary is described by the equations that reflect mutual interaction between the solid boundary and the fluid flowing adjacent to the boundary. These are constitutive equations which we shall concern ourselves with. In order to specify them, in the spirit of previous parts of the paper, we first recall some energy estimates.

Considering the constraint that the fluid can undergo only isochoric motions

\[(2.40) \quad \text{div} \mathbf{v} = 0 \quad \text{in} \ \Omega,\]

and assuming, for simplicity, that the density is uniform, i.e., $\rho \equiv \rho_* > 0$, motions of such a fluid are described by the balance equations for linear and angular momenta that take the form (see also (2.2))

\[(2.41) \quad \rho_* \left( \frac{\partial \mathbf{v}}{\partial t} + \sum_{k=1}^{d} v_k \frac{\partial \mathbf{v}}{\partial x_k} \right) = \text{div} \mathbf{S} - \nabla p \quad \text{in} \ \Omega,\]

\[\mathbf{S} = \mathbf{S}^\top \quad \text{in} \ \Omega.\]

Forming the scalar product of the first equation with $\mathbf{v}$ and integrating the result over $\Omega$, we arrive at

\[\frac{d}{dt} \int_{\Omega} \rho_* \frac{|\mathbf{v}|^2}{2} \, dx + \int_{\Omega} \text{div} \left( \rho_* \frac{|\mathbf{v}|^2}{2} \mathbf{v} \right) \, dx = \int_{\Omega} \text{div} (\mathbf{S} \mathbf{v}) \, dx - \int_{\Omega} \mathbf{S} : \mathbf{D} \, dx - \int_{\Omega} \text{div} (p \mathbf{v}) \, dx,\]

where we have used (2.40) twice. Gauss’ theorem and the requirement (2.39) then lead to

\[\frac{d}{dt} \int_{\Omega} \rho_* \frac{|\mathbf{v}|^2}{2} \, dx + \int_{\Omega} \mathbf{S} : \mathbf{D} \, dx + \int_{\partial \Omega} (-\mathbf{S}) : (\mathbf{v} \otimes \mathbf{n}) \, dS = 0,\]

where $\mathbf{a} \otimes \mathbf{b}$ denotes the second order tensor with the components $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$. In virtue of the symmetry of $\mathbf{S}$, see (2.41), we obtain

\[(-\mathbf{S}) : (\mathbf{v} \otimes \mathbf{n}) = (-\mathbf{S}) : (\mathbf{n} \otimes \mathbf{v}) = (-\mathbf{S} \mathbf{n}) : \mathbf{v}_\tau,\]

where $\mathbf{z}_\tau$ denotes the projection of $\mathbf{z} : \partial \Omega \to \mathbb{R}^d$ to the plane tangent to $\partial \Omega$ (at the point of $\partial \Omega$ under consideration). Finally, introducing the notation

\[\mathbf{s} := (-\mathbf{S} \mathbf{n})_\tau \quad (\text{projection of the normal traction to the tangent plane}),\]
we can rewrite (2.42) in the form
\[
\frac{d}{dt} \int_\Omega \rho \frac{|v|^2}{2} dx + \int_\Omega \mathbf{S} : \mathbf{D} dx + \int_{\partial \Omega} \mathbf{s} \cdot \mathbf{v}_r dS = 0.
\]

The discussion in Section 2 thus far has been focused on discussing models within the context of the framework \( \mathcal{G} (\mathbf{S}, \mathbf{D}) = \mathcal{O} \) (see (2.3)). In fact, the discussion concerned the restricted class of models of the form
\[
(2.43) \quad a (|\mathbf{D}|^2) \mathbf{D} = b (|\mathbf{S}|^2) \mathbf{S},
\]
where \( a \) and \( b \) were non-negative continuous (not necessarily everywhere differentiable) functions.

In a manner similar to the class of models defined through (2.43), we could develop analogously the identical framework of relations linking \( \mathbf{s} \) and \( \mathbf{v}_r \), i.e., to consider various classes of boundary conditions that fit the form
\[
(2.44) \quad h (\mathbf{s}, \mathbf{v}_r) = 0,
\]
where we deal with a (monotone) continuous function \( h : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \), or a more restrictive class
\[
(2.44) \quad \tilde{a} (|\mathbf{v}_r|^2) \mathbf{v}_r = \tilde{b} (|\mathbf{s}|^2) \mathbf{s},
\]
where \( \tilde{a}, \tilde{b} : [0, +\infty) \rightarrow [0, +\infty) \) are non-negative continuous functions.

We shall not consider the problem within the context of such generality for the following two reasons: (i) we do not have enough experimental data that would support nonlinear relations between \( \mathbf{s} \) and \( \mathbf{v}_r \), and (ii) the extension of the framework developed to take into account such nonlinear relations is straightforward and follows in a manner similar to that discussed in Subsections 2.1–2.5.

In what follows, we restrict ourselves to both activated and non-activated responses that are (after activation) linear. In a manner similar to that in the introductory part of Section 2 where we considered the product \( \mathbf{S} : \mathbf{D} \), we notice here that the product \( \mathbf{s} \cdot \mathbf{v}_r \) vanishes if either
\[
(2.45) \quad s = 0 \quad \text{on } \partial \Omega,
\]
or
\[
(2.46) \quad \mathbf{v}_r = 0 \quad \text{on } \partial \Omega \quad \text{for all admissible flows}.
\]

The condition (2.45) is referred to as the free-slip condition condition, expressing the fact that the boundary exhibits no friction to the flow, in the sense that the shear stress vanishes, and fluid flows tangentially to the boundary. On the other hand, the condition (2.46), referred to as the no-slip boundary condition, requires that flows adhere to the boundary. It has the character of a boundary constraint.

A linear relation between \( \mathbf{s} \) and \( \mathbf{v}_r \) is known as the Navier-slip:
\[
(2.47) \quad \mathbf{s} = \gamma_* \mathbf{v}_r \quad \text{with } \gamma_* > 0.
\]

We consider two types of activated boundary conditions. First, the relation
\[
(2.48) \quad |s| \leq s_* \quad \iff \quad \mathbf{v}_r = 0,
\]

\[
|s| > s_* \quad \iff \quad s = s_* \frac{\mathbf{v}_r}{|\mathbf{v}_r|} + \gamma_* \mathbf{v}_r,
\]
Table 2.2
Classification of boundary activation of fluid response with the corresponding $|v_\tau| - |s|$ diagrams. The last row reflects the usage of the term slip in section 3 to describe a broad class of boundary conditions of the slip type.

| Boundary Condition   | Description                      | Equation           |
|----------------------|----------------------------------|--------------------|
| free-slip            | $s = 0$                          | $(2.45)$           |
| Navier-slip          | $|v_\tau| \leq v_*$ $\iff$ $s = 0$ | $(2.47)$           |
| no-slip/Navier-slip  | $|s| \leq s_*$ $\iff$ $v_\tau = 0$ | $(2.48)$ or $(2.49)$ |
| no-slip              |                                  | $(2.46)$           |

which has been coined as the stick/slip condition in the literature, but which we will refer to as the no-slip/Navier-slip condition for consistency; $s_*$ is the yield stress that is positive. It can be rewritten into the form $(2.44)$ through its equivalent characterization

$(2.49)$

$$\gamma_* v_\tau = \frac{(|s| - s_*)^+}{|s|} s.$$  

In analogy, the second type of activated condition is given through the description

$(2.50)$

$$|v_\tau| \leq v_* \iff s = 0,$$

$$|v_\tau| > v_* \iff v_\tau = v_* \frac{s}{|s|} + \frac{1}{\gamma_*} s,$$

where $v_* > 0$. The equivalent description of $(2.50)$, that can be referred to as the free-slip/Navier-slip condition, takes the form

$(2.51)$

$$s = \gamma_* \frac{(|v_\tau| - v_*)^+}{|v_\tau|} v_\tau.$$  

These responses are summarized in the Table 2.2.

Simple shear flows of the Navier-Stokes fluid and the Euler/Navier-Stokes fluid subject to activated boundary conditions. Finally, in order to emphasize the role of boundary conditions in determining the nature of the flow, we consider Poiseuille flow between two parallel plates located at $y = \pm L$. All types of boundary conditions listed in Table 2.2 will be considered. We can write boundary conditions $(2.49)$ and $(2.51)$ together as

$(2.52)$

$$\gamma_* \frac{(|v_\tau| - v_*)^+}{|v_\tau|} v_\tau = \frac{(|s| - s_*)^+}{|s|} s \quad \text{on} \partial(\mathbb{R} \times (-L, L))$$

requiring that at least one of $v_*$ and $s_*$ is zero. Let us consider a simple shear flow of the Euler/Navier-Stokes fluid $(2.36)$ in domain $\mathbb{R} \times (-L, L)$ for given $L > 0$. As shown in Section 2.5 simple shear flows of the Euler/Navier-Stokes fluid are in the
form (2.36) or (2.37). Let us assume symmetry \( y_0 = 0 \) in (2.36a); it will be obvious later that the converse is not possible. Normalizing (2.36a) to a given flow rate \( Q \in \mathbb{R} \) such that

\[
\int_{-L}^{L} u(u) dy = Q
\]

we obtain

\[
u(a) u = -C \left( y^2 + \sqrt{2} \delta_* \frac{|y|}{C} \right) + \frac{Q}{2L} + C \left( \frac{L^2}{3} + \frac{\sqrt{2} \delta_* L}{2|C|} \right),
\]

\[
u(b) p = -2 \nu_* C x + p_0
\]

being defined for any \( C \in \mathbb{R} \setminus \{0\} \). It requires a trivial, but tedious, computation to check that simple shear flow of Euler/Navier-Stokes fluid (2.54) solves the balance equations in \( \mathbb{R} \times (-L,L) \) together with boundary condition (2.52) on \( \{y = \pm L\} \) provided that

\[
u(c) C = \frac{3Q}{4L^2} \left[ \left( 1 - \frac{\sqrt{2} \delta_* L^2 + 2 \nu_* L}{|Q|} \right)^+ - \frac{3 \nu_*}{3 \nu_* + \gamma_* L} \left( 1 - \frac{\sqrt{2} \delta_* L^2 + 2 \nu_* L + \frac{2 \nu_* L^2}{|Q|}} |Q| \right) \right]^+
\]

and \( p_0 \in \mathbb{R} \) is arbitrary. If \( C \) given by formula (2.55) is zero, then all flows which fulfill (2.37), (2.52), and (2.53) are solutions; more precisely if \( C = 0 \) all the solutions are given by

\[
u(a) |u'| \leq \sqrt{2} \delta_* \quad \text{a.e. in } \mathbb{R},
\]

\[
u(b) \gamma_* |u| \leq \gamma_* v_* \quad \text{a.e. on } \{|y| = L\},
\]

\[
u(c) \int_{-L}^{L} u dy = Q,
\]

\[
u(d) p = p_0,
\]

with some Lipschitz continuous \( u : [-L,L] \to \mathbb{R} \) and \( p_0 \in \mathbb{R} \). Family (2.54), (2.55) and family (2.56) represent in fact all possible simple shear flow solutions of motions of the Euler/Navier-Stokes fluid subject to no-slip/Navier-slip or free-slip/Navier-slip boundary conditions (2.52). We summarize combinations of bulk and boundary activation criterions in Table 2.3.

3. Mathematical analysis of flows of activated Euler fluids. Long-time and large-data existence theory (within the context of weak solutions) for a broad class of fluids described by implicit constitutive relation (2.3) has been developed in [16, 17]. These works deal with internal flows of incompressible fluids with monotone responses, asymptotically behaving as \( |S| = \mathcal{O} \left( |D| r^{-1} \right) \) as \( |D| \to \infty \) or \( |D| = \mathcal{O} \left( \inf |S| r^{-1} \right) \) as \( |S| \to \infty \) with \( \frac{d}{2} \leq r < \infty \) (or \( \frac{2d}{d+2} < r < \infty \) in general dimension \( d \)). In order to get a pressure\(^3\) which is integrable over the space-time cylinder in the unsteady

\(^3\)In fact, in [16, 17] the results are established even in a more general setting replacing the Lebesgue spaces by the Orlicz spaces.

\(^3\)Subtle difference between thermodynamic pressure, the mean normal stress (the latter usually referred to as pressure in mathematical literature on incompressible fluids), and the Lagrange multiplier is not to be discussed in this paper and we refer interested reader to [59, 70]. Henceforth we refer to the Lagrange multiplier that enforces the incompressibility condition as the “pressure”.
Table 2.3

Solutions for simple shear flows of the Euler/Navier-Stokes fluid in combination with different activated and classical boundary conditions. The middle column contains $|v\tau|s$ diagrams of boundary response (on the left) and $|\mathcal{B}|\mathcal{S}$ diagrams of bulk response (on the right). The solid segments and the circles (colored red in the electronic version) mark the part of the response being attained in the specific case. Note that no-slip/Navier-slip (contrary to free-slip/Navier-slip) admits a mode with the activation threshold exceeded in the bulk with the boundary under activation threshold. Also note that the free-slip condition admits only Euler mode, frictionless solutions. For the Navier-Stokes limit just let $\delta_* = 0$.

|free-slip/Navier-slip: $\ (2.52)$ & $s_* = 0$ |
|---|
| $|Q| \leq \sqrt{2\delta_* L^2 + 2v_* L}$ | 
\[ C = \frac{\gamma_* L}{3s_* + \gamma_* L} \left( \frac{3\delta_* L^2 - 2v_* L}{\gamma_* L} \right) |Q| \]  

| $|Q| \geq \sqrt{2\delta_* L^2 + 2v_* L}$ | 
\[ C = \frac{\gamma_* L}{3s_* + \gamma_* L} \left( \frac{3\delta_* L^2 - 2v_* L}{\gamma_* L} \right) |Q| \]  

|  

|no-slip/Navier-slip: $\ (2.52)$ & $v_* = 0$ |
|---|
| $|Q| \leq \sqrt{2\delta_* L^2}$ | 
\[ C = \frac{3\delta_* L^2}{4L^2} |Q| \]  

| $|Q| \geq \sqrt{2\delta_* L^2 + \frac{2s_* L^2}{\delta_*}}$ | 
\[ C = \frac{3\delta_* L^2}{4L^2} |Q| + \frac{3s_* L^2}{\delta_*} |Q| \]  

|  

|free-slip: $\ (2.52)$ & $s_* = 0$ & $v_* \to \infty$ |
|---|
| $Q \in \mathbb{R}$ | 
\[ u(\pm L) = 0, \ (2.56c), (2.56d) \]  

|no-slip: $\ (2.52)$ & $v_* = 0$ & $s_* \to \infty$ |

|Navier-slip: $\ (2.52)$ & $v_* = 0$ & $s_* = 0$ |
|---|
| $|Q| \leq \sqrt{2\delta_* L^2}$ | 
\[ C = \frac{\gamma_* L}{3s_* + \gamma_* L} \left( \frac{3\delta_* L^2 - 2v_* L}{\gamma_* L} \right) |Q| \]  

| $|Q| \geq \sqrt{2\delta_* L^2}$ | 
\[ C = \frac{\gamma_* L}{3s_* + \gamma_* L} \left( \frac{3\delta_* L^2 - 2v_* L}{\gamma_* L} \right) |Q| \]  

\[ u(\pm L) = 0, \ (2.56c), (2.56d) \]
case, the theory is developed with a boundary condition allowing some kind of slip. The overview of the problem concerning the connection between the integrability of the pressure and a specific boundary condition is given in [29]; see also the original studies [43, 44]. Existence theory when the Navier-slip boundary condition is enforced has recently been extended to the stick/slip boundary condition in [19, 18]. The theory for unsteady flows subject to the no-slip boundary condition can be found in a more recent study [13], where the solenoidal Lipschitz approximations of solenoidal Bochner-Sobolev functions are constructed and analyzed (from the point of view of their mathematical properties on approximation parameters). With such constructions, the analysis of the problem can be performed without introducing the notion of pressure.

In this section we will provide an existence theory for steady and unsteady flows of activated Euler fluids considering various types of behavior after activation and various types of boundary conditions. More specifically, we will study the system

\begin{align}
(3.1a) \quad & \nabla \cdot \mathbf{v} = 0 \quad \text{in} \ (0, T) \times \Omega, \\
(3.1b) \quad & \frac{\partial \mathbf{v}}{\partial t} + \nabla (\mathbf{v} \cdot \mathbf{v}) - \nabla p + \mathbf{b} = -\nabla \cdot \mathbf{S} \quad \text{in} \ (0, T) \times \Omega, \\
(3.1c) \quad & \mathbf{S} = 2\nu \, (|\mathbf{D}| - \delta_s)^+ \, S(|\mathbf{D}|) \mathbf{D} / |\mathbf{D}| \quad \text{in} \ (0, T) \times \Omega, \\
(3.1d) \quad & \mathbf{h}(s, \mathbf{v}_r) = 0 \quad \text{on} \ (0, T) \times \partial \Omega, \\
(3.1e) \quad & \mathbf{v}(s, 0, \cdot) = \mathbf{v}_0 \quad \text{in} \ \Omega.
\end{align}

Here $S : [0, \infty) \rightarrow [0, \infty)$ is supposed to be of the following forms: either

\[ S \equiv 1 \]

giving the Euler/Navier-Stokes fluid (2.30), or

\[ S(d) = \left( \frac{d}{d^*} \right)^{-2} \quad \text{or} \quad S(d) = \left( A + \left( \frac{d}{d^*} \right)^2 \right)^{\frac{r-2}{2}}, \quad A > 0, \]

leading to the Euler/power-law fluid (2.31), or

\[ S(d) = 1 + A \left( \frac{d}{d^*} \right)^{r-2}, \quad r > 2, \quad A > 0, \]

leading to the Euler/Ladyzhenskaya fluid.

It is not difficult to verify (see Appendix B) that the graph $\mathcal{G} \subset \mathbb{R}^{3 \times 3}_{\text{sym}} \times \mathbb{R}^{3 \times 3}_{\text{sym}}$ defined through

\[ (\mathbf{S}, \mathbf{D}) \in \mathcal{G} \quad \text{if and only if} \quad \mathbf{S} \quad \text{and} \quad \mathbf{D} \quad \text{fulfill} \quad (3.1c) \]

is a maximal monotone $r$-graph, i.e., $\mathcal{G}$ has the following properties:

(\text{G1}) $\langle \mathbf{0}, \mathbf{D} \rangle \in \mathcal{G}$; 
(\text{G2}) $\langle \mathbf{S}_1 - \mathbf{S}_2, \mathbf{D}_1 - \mathbf{D}_2 \rangle \geq 0$ for all $\langle \mathbf{S}_1, \mathbf{D}_1 \rangle \in \mathcal{G}$ and $\langle \mathbf{S}_2, \mathbf{D}_2 \rangle \in \mathcal{G}$; 
(\text{G3}) if $\mathbf{S}, \mathbf{D} \in \mathbb{R}^{3 \times 3}_{\text{sym}}$ satisfy $\langle \mathbf{S} - \mathbf{S}, \mathbf{D} - \mathbf{D} \rangle \geq 0$ for all $\langle \mathbf{S}, \mathbf{D} \rangle \in \mathcal{G}$, then $\langle \mathbf{S}, \mathbf{D} \rangle \in \mathcal{G}$; 
(\text{G4}) there exist $r \in (1, \infty], \alpha, \beta \in (0, \infty)$ such that $\mathbf{S} : \mathbf{D} \geq \alpha \langle \mathbf{S} - \mathbf{S}', \mathbf{D} - \mathbf{D}' \rangle - \beta$ whenever $\langle \mathbf{S}, \mathbf{D} \rangle \in \mathcal{G}$ and $\frac{1}{r} + \frac{1}{\alpha} + \frac{1}{\beta} = 1$.

\footnote{We assume that density $\rho$ is constant and we replace $\frac{1}{\rho}$ merely by $p$ throughout the whole section. Note that such $p$, although customarily called “pressure” in mathematical fluid dynamics literature, is actually the mean normal stress scaled by the (constant) density.}
Suitable choices of the function \( h(s, v_\ast) \) cover the boundary conditions (2.45), (2.46), (2.47), (2.49), (2.51). Analogous to the above setting, we require that the graph \( B \subset \mathbb{R}^3 \times \mathbb{R}^3 \) defined through

\[(s, v) \in B \leftrightarrow s \text{ and } v \text{ fulfill } (3.1c)\]

is a maximal monotone 2-graph, i.e., \( B \) has the following properties:

(B1) \((0, 0) \in B\);
(B2) \((s_1 - s_2) \cdot (v_1 - v_2) \geq 0\) for all \((s_1, v_1) \in B\) and \((s_2, v_2) \in B\);
(B3) if \( s, v \in \mathbb{R}^3 \) satisfy \((s - \tilde{s}) \cdot (v - \tilde{v}) \geq 0\) for all \((\tilde{s}, \tilde{v}) \in B\), then \((s, v) \in B\);
(B4) there are \( \tilde{\alpha}, \tilde{\beta} \in (0, \infty) \) such that \( s \cdot v \geq \tilde{\alpha} (|s|^2 + |v|^2) - \tilde{\beta} \) for all \((s, v) \in B\). The requirement (B4) can be easily verified for the boundary conditions (2.47), (2.49) and (2.51).

Note that there is no boundary term in the weak formulation of the problem in the case of the free-slip condition (2.45); this condition does not invalidate the analysis. On the other hand the no-slip boundary condition (2.46) needs to be treated separately.

Apart from the general purpose of this paper we are further motivated to study the problem (3.1) for the following reasons.

1. The most studied systems of PDEs (partial differential equations) in fluid mechanics are the Euler equations (when \( S = 0 \), or \( \delta_* \to \infty \) in (3.1c)) and the Navier-Stokes equations (when \( S = 2\nu D \), or \( \delta_* = 0 \) and \( S \equiv 1 \) in (3.1c)). The system of PDEs considered here is placed between them, as \( \delta_* \in (0, \infty) \). While (3.1a)–(3.1c) can, particularly for \( \delta_* \) large, share several features associated with the physics of the Euler fluid (or the Euler equations), we will document that the mathematical properties of the flows described by (3.1) are similar to those described by the Navier-Stokes equations. This is important as recent achievements in the mathematical theory of the Euler equations considered in a reasonable physical setting show that the equations exhibit pathological solutions within the framework of weak solutions with bounded (kinetic) energy (see [23, 82]). Fluids described by (3.1c) seem to have been completely overlooked both in physics and mathematical fluid dynamics literature; this may well be due to the fact that such behavior has not been observed. Below, we will focus on filling this lacuna and on developing the mathematical foundations associated with the problem (3.1).

2. It is worth noticing that the activated Euler fluids characterized by (3.1c) represent the models dual to the Bingham fluids that are obtained by interchanging the role of \( D \) and \( S \) in (3.1c). A mathematical theory for Bingham fluids, in the spirit of the theory developed here, is given in [18, 19, 63], where the reader can also find more references concerning the earlier results on the analysis of flows of the Bingham fluids and their generalizations.

3. The set-up of the problem considered here will be also used to show how different types of boundary conditions can be treated (while restricting ourselves to internal flows). We will also focus on the relation between the considered boundary conditions and the properties of the mean normal stress \( p \).

4. Since the operator \( \text{div} S \) is elliptic and degenerates for \( |S| \leq \delta_* \), the theory presented below can be viewed as an approach for studying degenerate problems.
5. Finally, the constitutive relation (3.1c) is regularized by

\[ S'(\mathbf{D}) = \left( \epsilon |\mathbf{D}|^{q-2} + 2\nu_\ast \frac{|\mathbf{D} - \delta_\ast^+|}{|\mathbf{D}|} S(|\mathbf{D}|) \right) \mathbf{D} \]

with \( \epsilon > 0 \) and \( q \geq 2 \) large enough. This explicit regularization allows us to proceed explicitly in the subsequent analysis.

### 3.1. Function spaces

In what follows we assume that \( \Omega \subset \mathbb{R}^3 \) is a domain, i.e., a bounded open connected set. For \( 1 \leq p \leq \infty, k \in \mathbb{N} \), \( L^p(\Omega) \) and \( W^{k,p}(\Omega) \) denote the standard Lebesgue and Sobolev space respectively, i.e., spaces of measurable functions of finite norm

\[
\|f\|_{L^p(\Omega)} = \|f\|_{L^p,\Omega} = \left( \int_\Omega |f|^p \right)^{\frac{1}{p}},
\]

\[
\|f\|_{W^{k,p}(\Omega)} = \|f\|_{W^{k,p},\Omega} = \sum_{j=0}^k \|\nabla^j f\|_{L^p,\Omega}, \quad |\nabla^j f| = \left( \sum_{|\alpha|=j} |D^\alpha f|^2 \right)^{\frac{1}{2}}
\]

respectively. When there is no risk of confusion the subscript \( \Omega \) can be omitted. Often we will use symbol \( L^p(\Omega)_{\text{sym}} \) to denote functions with values in symmetric tensors with \( L^p \)-integrable components. Bold-face symbols \( W^{k,p} \) and \( L^p \) will denote vector-valued Sobolev and Lebesgue functions respectively. Parentheses \( \langle \cdot, \cdot \rangle_\Omega \) will denote duality pairing in \( L^p(\Omega) \) and \( L^{\frac{1}{p'}}(\Omega) \) including vector and tensor-valued case; the subscript \( \Omega \) will be typically dropped where is no danger of confusion. Analogously, angle brackets \( \langle \cdot, \cdot \rangle_{V^*,V} \) denote a duality pairing between spaces \( V^* \) and \( V \), where \( V^* \) denotes the dual of \( V \); the subscript can be omitted. For any vector-valued function \( \mathbf{v} \), the symmetric part of the gradient is defined through \( \mathbf{Dv} := \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^\top) \). We use notation \( L^p(0,T;X) \) and \( C^k(I;X) \) to denote Bochner spaces of functions with values in the Banach space \( X \) and \( k \)-times continuously differentiable functions on interval \( I \subset \mathbb{R} \) with values in \( X \) respectively; \( C^k_b(I;X) \) denotes functions from \( C^k(I;X) \) which are compactly supported in \( I \).

The function spaces relevant to the problems that are being investigated vary depending on the type of boundary conditions. Two cases are being considered separately. First, the case of the no-slip boundary condition (the no-slip case, in short) and then other boundary conditions that involve various kinds of slipping mechanisms (the slip case, in short), cf. Table 2.2.

#### 3.1.1. No-slip case

We consider the space of compactly supported smooth functions and its subspace of solenoidal functions:

\[ C_0^\infty := \{ \mathbf{v} : \Omega \to \mathbb{R}^3; \mathbf{v} \text{ smooth; supp } \mathbf{v} \subset \Omega \}, \quad C_{0,\text{div}}^\infty := \{ \mathbf{v} \in C_0^\infty; \text{div } \mathbf{v} = 0 \} \]

and their closures in \( L^p \)-norm, \( W^{1,p} \)-norm (with \( 1 < p < \infty \)) and \( W^{3,2} \)-norm:

\[
W_0^{1,p} := \overline{C_0^\infty ||_{W^{1,p}}}, \quad L_{0,\text{div}}^p := \overline{C_{0,\text{div}}^\infty ||_{L^p}}, \quad W_0^{1,p} := \overline{C_{0,\text{div}}^\infty ||_{W^{1,p}}}, \quad W_0^{3,2} := \overline{C_{0,\text{div}}^\infty ||_{W^{3,2}}}
\]
As a consequence of the Poincaré and Korn inequalities, see [2, Corollary 6.31], [25, Theorem 5.15], the following norms are equivalent on $W_0^{1,p}$ (and $W_0^{1,p}$) for $1 < p < \infty$:

\begin{equation}
\|Dv\|_p \leq \|\nabla v\|_p \leq \|v\|_{1,p} \leq C_p \|\nabla v\|_p \leq C_p C_K \|Dv\|_p \quad \text{for all } v \in W_0^{1,p},
\end{equation}

with $\|\nabla v\|_p := \|\nabla v\|_p$, $\|Dv\|_p := \|Dv\|_p$; the constant $C_p > 0$ that appears due to the Poincaré inequality depends on $p$ and $\Omega$, while the constant $C_K > 0$ that appears due to the Korn inequality depends only on $p$.

Note that for a domain $\Omega$ (without further regularity assumption on the smoothness of $\partial\Omega$) we have the embedding $W_0^{1,2} \hookrightarrow W_0^{1,\infty}(\Omega)^3$. If additionally $\Omega$ is a $C^{0,1}$ domain, i.e., $\Omega$ is a domain with Lipschitz boundary $\partial\Omega$, then the following characterization holds true:

\begin{equation}
W_0^{1,p} = \{ v \in W^{1,p}(\Omega); v = 0 \text{ on } \partial\Omega \text{ in the sense of traces} \},
\end{equation}

\begin{equation}
W_0^{1,p,\text{div}} = \{ v \in W_0^{1,p}; \text{div } v = 0 \};
\end{equation}

moreover we use (3.3) as a definition of $W_0^{1,\infty}$ and $W_0^{1,\infty,\text{div}}$ in the case that $p = \infty$ and $\Omega$ is a $C^{0,1}$ domain.

We can occasionally denote the norm on $(W_0^{1,p})^*$, the topological dual of $W_0^{1,p}$, by $\|\cdot\|_{-1,p^*}$.

3.1.2. Slip case. Here we assume $\Omega$ is a $C^{0,1}$ domain. We denote by $n : \partial\Omega \to \mathbb{R}^3$ the unit outer normal vector to $\partial\Omega$. The space of smooth vector-valued functions with vanishing normal component on the boundary and its solenoidal subspace are then introduced through:

\begin{equation}
C_n^\infty := \{ v : \Omega \to \mathbb{R}^3; v \text{ smooth}; v \cdot n = 0 \text{ on } \partial\Omega \},
\end{equation}

\begin{equation}
C_n^{\infty,\text{div}} := \{ v \in C_n^\infty; \text{div } v = 0 \}.
\end{equation}

Since $\partial\Omega$ is Lipschitz we can define the following spaces with vanishing normal trace:

\begin{equation}
W_n^{3,2} := \{ v \in W^{3,2}(\Omega)^3; v \cdot n = 0 \text{ on } \partial\Omega \},
\end{equation}

\begin{equation}
W_n^{3,2,\text{div}} := W_n^{3,2} \cap L_n^{2,\text{div}},
\end{equation}

and subsequently, for $1 < p \leq \infty$,

\begin{equation}
W_n^{1,p} := \overline{W_n^{3,2} \|\| W^{1,p} }, \quad W_n^{1,p,\text{div}} := \overline{W_n^{3,2,\text{div}} \|\| W^{1,p} }.
\end{equation}

The condition $v \cdot n = 0$ on $\partial\Omega$ for $\Omega$ bounded is sufficient for validity of the Poincaré inequality: for $1 < p < \infty$ there exists $C_p > 0$ depending on $p$ and $\Omega$ such that

\begin{equation}
\|\nabla v\|_p \leq \|v\|_{1,p} \leq C_p \|\nabla v\|_p \quad \text{for all } v \in W_n^{1,p}.
\end{equation}

\begin{footnote}
To verify it, assume for the sake of contradiction, that there is $\{ v_j \}_{j=1}^\infty \subset W_n^{1,p}$ with $\|\nabla v_j\|_p \to 0$ and $\|v_j\|_{1,p} = 1$. Relying on the compact Sobolev embedding, it follows that there is a (not relabeled) subsequence $\{ v_j \}_{j=1}^\infty$ which converges strongly in $L^p(\Omega)^3$ to some $v \in L^p(\Omega)^3$. This implies that $\{ v_j \}_{j=1}^\infty$ is a Cauchy sequence in $W_0^{1,p}$ and converges in $W_0^{1,p}$ to $v \in W_n^{1,p}$ with $\nabla v = 0$. Hence $v$ is constant and by virtue of the boundedness of $\Omega$ and the boundary condition it follows that $v = 0$, which is a contradiction.
\end{footnote}
For the steady problem we will consider two inequalities of the Korn type depending on whether the type of considered boundary conditions leads to the control of the trace of \( \mathbf{v} \) on the boundary or not.\(^6\) In the first case, it follows from [22, Lemma 1.11] and (3.4): for \( 1 < p < \infty \) there exists \( C_K > 0 \) depending on \( p \) and \( \Omega \) such that

\[
\|\nabla \mathbf{v}\|_p \leq C_K (\|\mathbf{Dv}\|_p + \|\mathbf{v}\|_{2,\partial\Omega}) \quad \text{for all } \mathbf{v} \in W^{1,p}_n \text{ with } \mathbf{v}_\tau \in L^2(\partial\Omega).
\]

The second situation when \( s = 0 \) on \( \partial\Omega \) requires us to rule out domains that admit nontrivial rigid motions. We say that \( \Omega \) is axisymmetric if there exists a rigid body motion tangential to boundary, i.e., there is \( \mathbf{v} \in W^{1,\infty}_n \) with \( \mathbf{Dv} = \mathbf{0} \) and \( \nabla \mathbf{v} \neq \mathbf{0} \) constant in \( \Omega \). In the other words, there is \( \mathbf{v} \in W^{1,\infty}_n \) of the form \( \mathbf{v}(x) = \mathbf{Q}(x - x_0) \) for some \( \mathbf{Q} \subset \mathbb{R}^{3 \times 3} \) non-zero skew-symmetric matrix and constant \( x_0 \in \mathbb{R}^3 \). From [9, Theorem 11, Remark 12] it follows that if \( \Omega \) is not axisymmetric and \( 1 < p < \infty \) there exists \( C_K > 0 \) depending on \( \Omega \) and \( p \) such that

\[
\|\nabla \mathbf{v}\|_p \leq C_K \|\mathbf{Dv}\|_p \quad \text{for all } \mathbf{v} \in W^{1,p}_n.
\]

For the unsteady case we will use the following Korn-type inequality:\(^7\)

\[
\|\nabla \mathbf{v}\|_P \leq C_K (\|\mathbf{Dv}\|_P + \|\mathbf{v}\|_1) \quad \text{for all } \mathbf{v} \in W^{1,p}_n.
\]

### 3.2. Analysis of steady flows.

In this section, we investigate internal flows that are independent of time. Under such circumstances, the governing system of equations takes the form

\[
\begin{align*}
(3.9a) & \quad \text{div } \mathbf{v} = 0 \quad \text{in } \Omega, \\
(3.9b) & \quad \text{div} (\mathbf{v} \otimes \mathbf{v} - \mathbf{S}) = -\nabla p + \mathbf{b} \quad \text{in } \Omega, \\
(3.9c) & \quad (\mathbf{S}, \mathbf{Dv}) \in \mathcal{G} \quad \text{in } \Omega, \\
(3.9d) & \quad \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega,
\end{align*}
\]

where \( \mathcal{G} \) is a maximal monotone \( r \)-graph, fulfilling (\( \mathcal{G}1 \))–(\( \mathcal{G}4 \)), of the form

\[
\mathcal{G} := \left\{ (\mathbf{S}, \mathbf{D}) \in \mathbb{R}^{3 \times 3}_{\text{sym}} \times \mathbb{R}^{3 \times 3}_{\text{sym}} : \mathbf{S} = 2\nu_s (|\mathbf{D}| - \delta_s)^+ \mathbf{S}(|\mathbf{D}|) \frac{\mathbf{D}}{|\mathbf{D}|} \right\}.
\]

We will distinguish two cases depending on the class of boundary conditions considered. The first case concerns the no-slip condition, i.e.,

\[
(3.11) \quad \mathbf{v}_\tau = 0 \quad \text{on } \partial\Omega.
\]

\(^6\)A priori estimates for a steady problem subject to a slip-type condition given by a maximal monotone \( 2 \)-graph with (\( B4 \)) will ensure control of \( \|\mathbf{v}_\tau\|_{2,\partial\Omega} \). On the other hand, under the free-slip condition (3.25) there is no a priori control over the tangential velocity \( \|\mathbf{v}_\tau\|_{2,\partial\Omega} \) and rigid motions, if admissible in \( W^{1,\infty}_n \), prevent one to obtain a steady solution.

\(^7\)This is a consequence of another Korn-type inequality:

\[
(3.7) \quad \|\nabla \mathbf{v}\|_P \leq C_K (\|\mathbf{Dv}\|_P + \|\mathbf{v}\|_P) \quad \text{for all } \mathbf{v} \in W^{1,P}_n(\Omega);
\]

see [56, Theorem 1.10]. To verify (3.8), assume that there is \( \{\mathbf{v}^j\}_{j=1}^\infty \subset W^{1,p}_n \) such that \( \|\nabla \mathbf{v}^j\|_P = 1 \) and \( \|\mathbf{Dv}^j\|_P + \|\mathbf{v}^j\|_1 \to 0 \). Poincaré inequality (3.4) implies that \( \{\mathbf{v}^j\}_{j=1}^\infty \) is bounded in \( W^{1,\infty}_n \). By virtue of its reflexivity there is a (not relabeled) subsequence such that \( \mathbf{v}^j \rightharpoonup \mathbf{v} \) weakly in \( W^{1,p}_n \) and by the Sobolev embedding \( \mathbf{v}^j \to \mathbf{v} \) in \( L^p \). On the other hand \( \mathbf{v}^j \to \mathbf{0} \) in \( L^1 \) hence by uniqueness of the limit we conclude \( \mathbf{v} = \mathbf{0} \). Summarizing, the right-hand side of (3.7) (with \( \mathbf{v}^j \) in place of \( \mathbf{v} \)) goes to zero but the left-hand side is equal to unity, which is a contradiction.

We can see that (3.8) in fact holds independently of the considered boundary condition, i.e., for all \( W^{1,p}_n \).
The second case includes all other boundary conditions involving tangential part of the normal traction; it refers to either

\[ s = 0 \quad \text{on } \partial \Omega \]

or

\[ (s, v_\tau) \in \mathcal{B} \quad \text{on } \partial \Omega, \]

where \( \mathcal{B} \) is a maximal monotone 2-graph fulfilling (B1)–(B4).

**3.2.1. No-slip case.** Let \( \Omega \subset \mathbb{R}^3 \) be a domain, \( r \geq \frac{6}{5} \), \( b \in (W^{1,r}_0)^* \) and \( \mathcal{G} \) be a maximal monotone \( r \)-graph specified in (3.10). We say that

\[ (v, S) \in W^{1,r}_{0,\text{div}} \times L^r(\Omega)^{3\times3}_{\text{sym}} \]

is a weak solution to (3.9), (3.10), (3.11) if

\[ (S, D\varphi) - (v \otimes v, \nabla \varphi) = (b, \varphi) \quad \text{for all } \varphi \in C^\infty_{0,\text{div}} \]

and

\[ (S, Dv) \in \mathcal{G} \quad \text{a.e. in } \Omega; \]

equivalently, we can require that (3.12) holds for all \( \varphi \in W^{1,\frac{3r}{r-6}}_{0,\text{div}} \cap W^{1,r}_{0,\text{div}}. \)

**Theorem 3.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a domain. Let \( r > \frac{6}{5} \), \( b \in (W^{1,r}_0)^* \) and \( \mathcal{G} \) be a maximal monotone \( r \)-graph of the form (3.10). Then there is a weak solution \( (v, S) \in W^{1,r}_{0,\text{div}} \times L^r(\Omega)^{3\times3}_{\text{sym}} \) to (3.9), (3.11) which fulfills (3.13) and

\[ (S, D\varphi) - (v \otimes v, D\varphi) = (b, \varphi) \quad \text{for all } \varphi \in \begin{cases} W^{1,\frac{3r}{r-6}}_{0,\text{div}} & \text{if } r \geq \frac{6}{5}, \\ W^{1,r}_{0,\text{div}} & \text{if } r \in \left(\frac{6}{5}, \frac{9}{5}\right). \end{cases} \]

In addition, if \( \Omega \) is a \( C^{0,1} \) domain then there is

\[ p \in \begin{cases} L^r(\Omega) & \text{if } r \geq \frac{6}{5}, \\ L^{\frac{3r}{r-6}}(\Omega) & \text{if } r \in \left(\frac{6}{5}, \frac{9}{5}\right), \end{cases} \]

such that

\[ (S, D\varphi) - (v \otimes v, \nabla \varphi) = (p, \text{div } \varphi) + (b, \varphi) \quad \text{for all } \varphi \in \begin{cases} W^{1,\frac{3r}{r-6}}_{0,\text{div}} & \text{if } r \geq \frac{6}{5}, \\ W^{1,r}_{0,\text{div}} & \text{if } r \in \left(\frac{6}{5}, \frac{9}{5}\right). \end{cases} \]

**Proof.** The case \( r \geq \frac{6}{5} \). Since \( W^{1,r}_{0,\text{div}} \) is separable, there is a countable basis denoted by \( \{\omega^r_n\}_{n=1}^\infty \). For \( N \in \mathbb{N} \) arbitrary, but fixed, we first look for the vector \( c^N = (c^N_1, \ldots, c^N_N) \in \mathbb{R}^N \) such that

\[ v^N(x) := \sum_{r=1}^N c^N_r \omega^r(x) \]
satisfies the system of $N$ nonlinear equations
\begin{equation}
(S^N, D\omega^r) - (v^N \otimes v^N, \nabla \omega^r) = \langle b, \omega^r \rangle, \quad r = 1, \ldots, N,
\end{equation}
where
\begin{equation}
S^N := S(Dv^N) := 2\nu_s \left( |Dv^N| - \delta_s \right) \nu_s \left( |Dv^N| \right) \frac{\partial_{\nu^N}}{\partial_{\nu^N}}.
\end{equation}

Introducing the (continuous) mapping $P^N: \mathbb{R}^N \to \mathbb{R}^N$ as
\begin{equation}
\left(P^N(c^r)\right) = \left(S^N, D\omega^r\right) - \langle b, v^N \rangle, \quad r = 1, \ldots, N,
\end{equation}
then
\begin{equation}
P^N(c^N) \cdot c^N = (S^N, Dv^N) - \langle b, v^N \rangle.
\end{equation}

It follows from (G4) and (3.18) that
\begin{equation}
P^N(c^N) \cdot c^N > 0 \quad \text{for} \quad |c^N| \text{ sufficiently large.}
\end{equation}

As a consequence of Brouwer’s fixed-point theorem (see [51, p. 53]), (3.20) implies the existence of $c^N$ fulfilling $P^N(c^N) = 0$, i.e., (3.17) holds, and, by (3.19),
\begin{equation}
(S^N, Dv^N) = \langle b, v^N \rangle.
\end{equation}

This together with (G4), (3.2) and Young’s inequality leads to
\begin{equation}
\|S^N\|_{r'} + \|\nabla v^N\|_r \leq c_1 \|b\|_{(W^1_{0,q})'} + c_2.
\end{equation}

This implies the existence of $v \in W^{1,r}_{0,\text{div}}$ and $S \in L'(\Omega)^{3 \times 3}_{\text{sym}}$ such that for suitable (not relabeled) subsequences
\begin{align}
&v^N \rightharpoonup v \quad \text{weakly in} \quad W^{1,r}_{0,\text{div}},
&\text{D}v^N \rightharpoonup \text{D}v \quad \text{weakly in} \quad L'(\Omega)^{3 \times 3},
&\text{S}^N \rightharpoonup \text{S} \quad \text{weakly in} \quad L'(\Omega)^{3 \times 3}_{\text{sym}},
\end{align}
as $N \to \infty$. Consequently, as $W^{1,q}_0(\Omega)$ is compactly embedded into $L^2(\Omega)$ for any $q > \frac{6}{5}$, we also have
\begin{equation}
v^N \to v \quad \text{strongly in} \quad L^2(\Omega)^3 \quad \text{as} \quad N \to \infty.
\end{equation}

Then, (3.17) leads to, for $r \geq \frac{9}{5}$,
\begin{equation}
\langle S, D\omega^s \rangle - (v \otimes v, \nabla \omega^s) = \langle b, \omega^s \rangle, \quad s = 1, 2, \ldots.
\end{equation}

Note that the restriction $r \geq \frac{9}{5}$ is due to the requirement that for $s \in \mathbb{N}$ arbitrary
\begin{equation}
\int_{\Omega} (v \otimes v) : \nabla \omega^s \, dx < \infty \quad \text{for} \quad v, \omega^s \in W^{1,r}_0.
\end{equation}

Hence (3.23) implies that
\begin{equation}
\langle S, D\omega \rangle - (v \otimes v, \nabla \omega) = \langle b, \omega \rangle \quad \text{for} \quad \omega \in W^{1,r}_{0,\text{div}},
\end{equation}
which completes the proof of (3.14) for \( r \geq \frac{9}{5} \). Taking \( \omega = v \) in (3.24) one obtains

\[(3.25) \quad (\mathbf{S}, \mathbf{D}v) = \langle b, v \rangle.\]

Taking the limit with \( N \to \infty \) in (3.21), we conclude from (3.25) and (3.22a) that

\[
\lim_{N \to \infty} (\mathbf{S}^N, \mathbf{D}v^N) = (\mathbf{S}, \mathbf{D}v).
\]

In virtue of the graph convergence lemma (see Lemma A.6 in Appendix) this implies together with (3.22) that \((\mathbf{S}, \mathbf{D}v) \in \mathcal{G}\), i.e., (3.13) holds.

**The case** \( r \in \left( \frac{6}{5}, \frac{9}{5} \right) \). In this case, we consider, for \( \epsilon > 0 \), the following approximating problem:

\[
- \text{div} (\mathbf{S} + \epsilon \mathbf{D}v - v \otimes v) = -\nabla p + b \quad \text{in } \Omega,
\]

\[
\text{div } v = 0 \quad \text{in } \Omega,
\]

\[
(\mathbf{S}, \mathbf{D}v) \in \mathcal{G} \quad \text{in } \Omega,
\]

\[
v = 0 \quad \text{on } \partial \Omega.
\]

For fixed \( \epsilon \), the existence of a weak solution to (3.26) follows from the above proof for the case \( r \geq \frac{9}{5} \). More precisely, following step-by-step the proof of existence via the Galerkin method used above we can show that, for \( \epsilon > 0 \) fixed, there is \((v^\epsilon, \mathbf{S}^\epsilon) \in W_{0, \text{div}}^{1, 2} \times L^r(\Omega)^{3 \times 3}_{\text{sym}}\) such that

\[
(3.27a) \quad (\mathbf{S}^\epsilon + \epsilon \mathbf{D}v^\epsilon - v^\epsilon \otimes v^\epsilon, \mathbf{D} \varphi) = \langle b, \varphi \rangle \quad \text{for all } \varphi \in W_{0, \text{div}}^{1, 2},
\]

\[
(3.27b) \quad (\mathbf{S}^\epsilon, \mathbf{D}v^\epsilon) \in \mathcal{G} \; \text{a.e. in } \Omega.
\]

Moreover, taking \( \varphi = v^\epsilon \) in (3.27a),

\[
(\mathbf{S}^\epsilon, \mathbf{D}v^\epsilon) + \epsilon \| \mathbf{D}v^\epsilon \|_2^2 = \langle b, v^\epsilon \rangle.
\]

The last identity together with the assumption \((\mathcal{G}4)\) implies the following estimate

\[
(3.28) \quad \| \mathbf{S}^\epsilon \|_{r'}^r + \| \mathbf{D}v^\epsilon \|_{r'}^r + \epsilon \| \mathbf{D}v^\epsilon \|_2^2 \leq \| b \|_{r'} \left( W_{0, \text{div}}^{1, 2} \right)^* + C.
\]

---

In order to verify that

\[
\limsup_{N \to \infty} \int_\Omega \mathbf{S}^N : \mathbf{D}v^N \leq \int_\Omega \mathbf{S} : \mathbf{D}v,
\]

one uses the identities

\[
\int_\Omega \mathbf{S}^N : \mathbf{D}v^N + \epsilon \int_\Omega |\mathbf{D}v^N|^2 = \langle b, v^N \rangle,
\]

\[
\int_\Omega \mathbf{S} : \mathbf{D}v + \epsilon \int_\Omega |\mathbf{D}v|^2 = \langle b, v \rangle,
\]

the weak lower semicontinuity of the \( L^2 \)-norm of \( \mathbf{D}v^N \), and the inequality

\[
\liminf_{n \to \infty} a_n + \limsup_{n \to \infty} b_n \leq \limsup_{n \to \infty} (a_n + b_n)
\]

applied to any sequences \( \{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \) with \( a_n \geq 0, b_n \geq 0 \) for all \( n \in \mathbb{N} \).
This implies the existence of \((v, S) \in W_{0, \text{div}}^{1,r} \times L^r(\Omega)^{3 \times 3}\) such that, for a suitable vanishing subsequence \(\{\epsilon_n\}_{n=1}^{\infty}\) and \((v^n, S^n) := (v^{\epsilon_n}, S^{\epsilon_n})\),

\[
\begin{align*}
(3.29a) & \quad S^n \to S \quad \text{weakly in } L^r(\Omega)^{3 \times 3}, \\
(3.29b) & \quad Dv^n \to Dv \quad \text{weakly in } L^r(\Omega)^{3 \times 3}, \\
(3.29c) & \quad v^n \to v \quad \text{strongly in } L^q(\Omega)^3 \text{ for all } q \in [1, \frac{3r}{3-r}).
\end{align*}
\]

The last piece of information provides the strong convergence of \(\{v^n\}_{n=1}^{\infty}\) in \(L^2(\Omega)^3\) provided that \(\frac{3r}{3} > 2\), which gives the bound stated in the formulation of the theorem, namely \(r > \frac{6}{7}\). Consequently, \(v\) and \(S\) fulfill (3.14). It remains to prove that \((S, Dv) \in C\text{ a.e. in } \Omega\). By the graph convergence lemma (Lemma A.6), it is enough to show that

\[
\limsup_{n \to \infty} (S^n, Dv^n) \leq (S, Dv).
\]

To prove it, we first subtract (3.14) from (3.27a) to obtain

\[
(3.30) \quad (S^n - S, D\varphi) + \epsilon_n (Dv^n, D\varphi) + (v \otimes v - v^n \otimes v^n, D\varphi) = 0
\]

for all \(\varphi \in W_{0, \text{div}}^{1,2} \cap W_{0, \text{div}}^{1,\frac{6}{3-r}}\).

Let \(B \subset \mathbb{R}^3\) be an arbitrary ball of radius \(R\) such that \(\frac{B}{2} \subset B \subset 2B \subset \Omega\) and \(\chi \in C_0^\infty(B)\) be such that \(\chi = 1\) in \(\frac{B}{2}\), \(\chi \leq 1\) in \(B\), and \(|\nabla \chi| \leq CR^{-1}\) in \(B\). Then we set

\[
u^n := \chi(v^n - v) - h^n,
\]

where \(h^n \in W_0^{1,r}(B)\) solves

\[
\text{div } h^n = \nabla \chi \cdot (v^n - v) \quad \text{in } B,
\]

which is solvable as the compatibility condition \(\int_B \nabla \chi \cdot (v^n - v) = 0\) is met. In fact, there is a continuous linear operator \(B : \{q \in L^p(B), \int_{\Omega} q = 0\} \to W_0^{1,p}(B) : g \mapsto u\) such that \(\text{div } u = g\), cf. Remark A.8 (ii). Consequently, \(u^n\) extended by zero in \(\Omega \setminus B\) fulfills \(\text{div } u^n = 0\) in \(\Omega\). Next we consider divergence-free Lipschitz approximations \(u^{n,k}\) to \(u^n\) from Lemma A.4. Taking \(\varphi := u^{n,k}\) in (3.30) and letting \(n \to \infty\), we conclude, using in particular the property (d) of Lemma A.4 and (3.28), that

\[
(3.31) \quad \lim_{n \to \infty} (S^n - S, Du^{n,k}) = 0.
\]

Using the properties of \(u^{n,k}\) (see Lemma A.4) and splitting the integral on the left-hand side of (3.31) into integrals over \(O^{n,k}\) and \(B \setminus O^{n,k}\), we conclude

\[
\lim_{n \to \infty} (S^n - S, Du^n)_{B \setminus O^{n,k}} \leq C2^{-k} \quad \text{for all } k \in \mathbb{N}.
\]

It follows from the definition of \(u^n\), the properties of the operator \(B\) and the compactness of \(v^n\) that

\[
\lim_{n \to \infty} \int_{B \setminus O^{n,k}} (S^n - S) : (Dv^n - Dv) \chi \leq C2^{-k} \quad \text{for arbitrary } k \in \mathbb{N},
\]
which implies, by applying the Hölder inequality, that
\[
\lim_{n \to \infty} \int_B |(S^n - S) : (Dv^n - Dv)|^{\frac{2}{\kappa}} \leq C 2^{-k} \quad \text{for arbitrary } k \in \mathbb{N}.
\]
This leads to
\[
\lim_{n \to \infty} \int_B |(S^n - S) : (Dv^n - Dv)|^{\frac{2}{\kappa}} \leq C 2^{-k} \quad \text{for arbitrary } k \in \mathbb{N}.
\]
Let us set \(g_n := |(S^n - S) : (Dv^n - Dv)|\). Clearly \(g_n \geq 0\) and \(g_n \to 0\) almost everywhere in \(\Omega\). But as \(B\) is arbitrary, we conclude that
\[
g_n \to 0 \quad \text{almost everywhere in } \Omega.
\]
(3.22)

Since \(\{g_n\}_{n=1}^{\infty}\) is bounded in \(L^1(\Omega)\) and has the pointwise limit (3.22), Corollary A.3, a consequence of the biting lemma (Lemma A.2), then implies existence of a subsequence \(\{g_{n_j}\}_{j=1}^{\infty}\) and a sequence of sets \(\{E_k\}_{k=1}^{\infty}\) with \(\Omega \supset E_1 \supset E_2 \supset \ldots, |E_k| \to 0\) such that for all \(k \in \mathbb{N}\)
\[
g_{n_j} \to 0 \quad \text{strongly in } L^1(\Omega \setminus E_k).
\]
From the definition of \(g_{n_j}\) we conclude that
\[
\limsup_{j \to \infty} \int_{\Omega \setminus E_k} (S^{n_j} - S) : (Dv^{n_j} - Dv) = 0,
\]
which implies as a consequence of (3.29a) and (3.29b) that for all \(k \in \mathbb{N}\)
\[
\limsup_{j \to \infty} \int_{\Omega \setminus E_k} S^{n_j} : Dv^{n_j} = \int_{\Omega \setminus E_k} S : Dv.
\]
Since \(|E_k| \to 0\), we can conclude from the graph convergence lemma (Lemma A.6) that
\[
(S, Dv) \in \mathcal{G} \quad \text{almost everywhere in } \Omega
\]
so that (3.13) holds and the first part of the theorem is proved.

**On the pressure.** Setting
\[
\langle F, \varphi \rangle := (S, D\varphi) - (v \otimes v, \nabla \varphi) - \langle b, \varphi \rangle
\]
we observe that
\[
\langle F, \varphi \rangle = 0 \quad \text{for all } \varphi \in C_0^\infty
\]
and
\[
(3.33) \quad F \in \left\{ \begin{array}{ll}
(W_0^{1,r})^* & \text{if } r \geq \frac{9}{5}, \\
(W_0^{1,\frac{9}{5} - \epsilon})^* & \text{if } r \in \left(\frac{9}{5}, \frac{9}{4}\right).
\end{array} \right.
\]
By the de Rham theorem, see [5, Theorem 2.1], there is \(p \in (C_0^\infty(\Omega))^*\) such that
\[
(3.34) \quad \langle F, \varphi \rangle = \langle -\nabla p, \varphi \rangle \quad \text{for all } \varphi \in C_0^\infty.
\]
Since \(\Omega\) is \(C^{0,1}\) domain, the Nečas theorem (see Lemma A.7, Remark A.9) together with (3.33) and (3.34) implies (3.15) and (3.16).
3.2.2. Slip case. In this part we replace the no-slip boundary condition either by

\begin{equation}
 v \cdot n = 0 \quad \text{and} \quad s = 0 \quad \text{on } \partial \Omega
\end{equation}

or by

\begin{equation}
 v \cdot n = 0 \quad \text{and} \quad (s, v_\tau) \in \mathcal{B} \quad \text{on } \partial \Omega,
\end{equation}

where \( \mathcal{B} \) fulfills the conditions \((B1)-(B4)\).

We prove the following result.

**Theorem 3.2.** Let \( \Omega \subset \mathbb{R}^3 \) be a \( C^{1,1} \) domain.\(^9\) Let further \( r > \frac{6}{5}, \ b \in (W^{1,r})^*, \) \( G \) be a maximal monotone \( r \)-graph of the form \((3.10)\).

(i) (Boundary condition \((3.36)\)) Let \( \mathcal{B} \) be a maximal monotone 2-graph. Then there is a weak solution

\[ (v, S, s) \in W^{1,r}_{n, \text{div}} \times L'^{(3\times 3)}_{\text{sym}} \times L^2(\partial \Omega)^3 \]

to \((3.9)\) and \((3.36)\) such that

\begin{align}
 (S, Dv) &\in G \ a.e. \ in \ \Omega, \\
 (s, v_\tau) &\in \mathcal{B} \ a.e. \ in \ \partial \Omega,
\end{align}

and

\begin{equation}
 (S, D\varphi) - (v \otimes v, D\varphi) + (s, \varphi)_{\partial \Omega} = \langle b, \varphi \rangle
\end{equation}

for all \( \varphi \in \left\{ W^{1,r}_{n, \text{div}} \ if \ r \geq \frac{9}{5}, \ \right. \]

\[ W^{1,\frac{6}{5}-\frac{r}{5}}_{n, \text{div}} \ if \ r \in \left( \frac{6}{5}, \frac{9}{5} \right). \]

(ii) (Boundary condition \((3.35)\)) Assume that \( \Omega \) is not axisymmetric. Then there is a weak solution

\[ (v, S) \in W^{1,r}_{n, \text{div}} \times L'^{(3\times 3)}_{\text{sym}} \]

to \((3.9)\) and \((3.35)\) such that \((3.37)\) and \((3.39)\) with \( s = 0 \) hold true.

In addition, there is

\begin{equation}
 p \in \left\{ L'^{\frac{r}{2r-6}}(\Omega) \ if \ r \geq \frac{9}{5}, \right. \\
 L^{\frac{2r}{3r-6}}(\Omega) \ if \ r \in \left( \frac{6}{5}, \frac{9}{5} \right), \quad \int_{\Omega} p \, dx = 0
\end{equation}

such that

\begin{equation}
 (S, D\varphi) - (v \otimes v, D\varphi) + (s, \varphi)_{\partial \Omega} = \langle p, \text{div } \varphi \rangle + \langle b, \varphi \rangle
\end{equation}

for all \( \varphi \in \left\{ W^{1,r}_{n, \text{div}} \ if \ r \geq \frac{9}{5}, \right. \]

\[ W^{1,\frac{6}{5}-\frac{r}{5}}_{n, \text{div}} \ if \ r \in \left( \frac{6}{5}, \frac{9}{5} \right). \]

\(^9\)In the case of the boundary condition \((3.35)\), the required regularity of the boundary can be weakened at the cost of losing the information concerning the pressure.
Proof. The proof of existence of \( \mathbf{v}, \mathbf{S}, \mathbf{s} \) with (3.37), (3.38), and (3.39) follows the same scheme as in the case of the no-slip boundary condition. The only differences are

(i) due to a different choice of the function spaces for the velocity as \( W^{1,r}_{0,\text{div}} \) is replaced by \( W^{1,r}_{n,\text{div}} \),

(ii) due to the presence of the term \( \int_{\partial \Omega} \mathbf{s} \cdot \varphi \) in the weak formulation of the balance of linear momentum,

(iii) and due to the necessity to verify validity of the boundary condition \( h(s, \mathbf{v}_\tau) = 0 \).

Note that in the case \( s = \mathbf{0} \) on \( \partial \Omega \), the last two differences disappear. In the case of the stick/slip boundary condition \( \mathbf{v}_\tau = \frac{1}{\gamma_s} \frac{(|\mathbf{s}| - \sigma_s) +}{|\mathbf{s}|} \mathbf{s} + \frac{\epsilon}{\gamma_s} \mathbf{s}, \quad \epsilon > 0. \)

Other boundary conditions of the type (3.36) employ similar approximations.

For \( r \geq \frac{\alpha}{\gamma_s} \), the proof then proceeds as in the case of no-slip boundary conditions up to the use of the appropriate form of Korn’s inequality (3.6) for boundary condition (3.35) assuming the domain is not axisymmetric, or (3.5) for boundary condition (3.36)) and the following points. The additional term \( \int_{\partial \Omega} \mathbf{s} \cdot \varphi \) in the balance of linear momentum is treated using the weak convergence \( \mathbf{s}^N \to \mathbf{s} \) in \( L^2(\partial \Omega) \). Since \( W^{1,r}(\Omega) \hookrightarrow W^{1,\frac{r}{2}}(\partial \Omega) \hookrightarrow L^2(\partial \Omega) \) provided \( r > \frac{\alpha}{2} \) and \( \Omega \) is a \( C^{0,1} \) domain, the space \( W^{1,r}_n \) is compactly embedded into \( L^2(\partial \Omega) \) even for \( r > \frac{\alpha}{2} \), and consequently \( \mathbf{v}^N_\tau \to \mathbf{v}_\tau \) strongly in \( L^2(\partial \Omega) \) and thus

\[
\int_{\partial \Omega} \mathbf{s} \cdot \mathbf{v}^N_\tau \to \int_{\partial \Omega} \mathbf{s} \cdot \mathbf{v}_\tau.
\]

This implies (see Lemma A.6) that \( (\mathbf{s}, \mathbf{v}_\tau) \in \mathcal{B} \) a.e. on \( \partial \Omega \).

For \( r \in \left( \frac{\alpha}{\gamma_s}, \frac{\alpha}{2} \right) \), the validity of \( (\mathbf{S}, \mathbf{D}\mathbf{v}) \in \mathcal{G} \) a.e. in \( \Omega \) can be established in the same way as in the no-slip case since the proof is based on local (interior) arguments. It remains to show that \( (\mathbf{s}, \mathbf{v}_\tau) \in \mathcal{B} \) a.e. on \( \partial \Omega \). For \( r > \frac{\alpha}{2} \), it follows from (3.42) and Lemma A.6. To use Lemma A.6 also for \( r \in \left( \frac{\alpha}{2}, \frac{\alpha}{\gamma_s} \right) \), it suffices to show that

\[
\limsup_{N \to \infty} \int_{\partial \Omega} \mathbf{s}^N \cdot \mathbf{v}^N \leq \int_{\partial \Omega} \mathbf{s} \cdot \mathbf{v}.
\]

However, for \( r > 1 \): \( W^{1,r}(\Omega) \hookrightarrow L^1(\partial \Omega) \), the strong convergence \( \mathbf{v}^N_\tau \to \mathbf{v}_\tau \) in \( L^1(\partial \Omega) \) together with Egorov’s theorem implies that for any \( \delta > 0 \) there is \( U_\delta \subset \partial \Omega \) such that \( |\partial \Omega \setminus U_\delta| < \delta \) and \( \mathbf{v}^N_\tau \to \mathbf{v}_\tau \) strongly in \( L^\infty(U_\delta)^3 \). Hence

\[
\limsup_{N \to \infty} \int_{U_\delta} \mathbf{s}^N \cdot \mathbf{v}^N_\tau \leq \int_{U_\delta} \mathbf{s} \cdot \mathbf{v}_\tau.
\]

Consequently, by the graph convergence lemma (Lemma A.6 in Appendix), \( (\mathbf{s}, \mathbf{v}_\tau) \in \mathcal{B} \) a.e. in \( U_\delta \). As \( \delta > 0 \) was arbitrary, we conclude that \( (\mathbf{s}, \mathbf{v}_\tau) \in \mathcal{B} \) a.e. on \( \partial \Omega \). The proof of the first part of the theorem is complete.

It remains to prove the existence of pressure (3.40) fulfilling (3.41). Let us define a linear functional \( \mathbf{F} \) through the relation

\[
\langle \mathbf{F}, \varphi \rangle := \langle \mathbf{b}, \varphi \rangle - \langle \mathbf{S} - \mathbf{v} \otimes \mathbf{v}, \mathbf{D}\varphi \rangle - \langle \mathbf{s}, \varphi \rangle_{\partial \Omega} \quad \text{for any } \varphi \in \mathcal{C}_0^\infty.
\]
From (3.39) we can see that \( F \in (W^{1,q}_n)^* \) where \( q = r \) if \( r \geq \frac{3}{2} \) and \( q = \frac{3r}{s - r} \) if \( r \in \left( \frac{5}{2}, \frac{9}{4} \right) \) and

\[
(3.43) \quad \langle F, \varphi \rangle = 0 \quad \text{for all} \quad \varphi \in W^{1,q}_{n,\text{div}}.
\]

Now consider a variational problem to find \( p \in L^q(\Omega) \) with \( \int_\Omega p = 0 \) such that

\[
(3.44) \quad (p, -\Delta \phi) = \langle F, \nabla \phi \rangle \quad \text{for all} \quad \phi \in W^{2,q}(\Omega) \quad \text{with} \quad \nabla \phi \in W^{1,q}_n.
\]

As a consequence of the \( C^{1,1} \) smoothness of the domain, one can employ Lemma A.11 to conclude that (3.44) is equivalent with the problem: find \( p \in L^q(\Omega) \) such that \( \int_\Omega p = 0 \) and

\[
(3.45) \quad (p, q) = \langle F, \nabla A^{-1} q \rangle \quad \text{for all} \quad q \in L^q(\Omega) \quad \text{with} \quad \int_\Omega q = 0
\]

where \( A^{-1} \) is the solution operator for the Neumann-Poisson problem (A.2). The problem (3.45) has a unique solution by virtue of Lemma A.11. Thus we have constructed \( p \) with properties (3.40). To verify (3.41), consider a test function \( \varphi \in W^{1,q}_n \).

With the Helmholtz decomposition (see Corollary A.12 in Appendix) \( \varphi = \nabla \phi + \varphi_0 \) with \( \nabla \phi \in W^{1,q}_n \) and \( \varphi_0 \in W^{1,q}_{n,\text{div}} \) we can immediately obtain, using (3.43) and (3.44), that

\[
\langle F, \varphi \rangle = \langle F, \nabla \phi \rangle + \langle F, \varphi_0 \rangle = (p, -\Delta \phi) = -(p, \nabla \varphi).
\]

This proves (3.41). The proof of Theorem 3.2 is thus complete.

\[\square\]

3.3. Analysis of unsteady flows. In this section, we investigate unsteady internal flows, i.e., flows governed by (3.1). Again, we treat separately two cases: the no-slip boundary condition and the boundary conditions allowing slip.

3.3.1. No-slip case. We first provide an existence result for the no-slip case, i.e., we investigate the system (3.1a)–(3.1d), (3.1f), and \( \mathbf{v}_\tau = 0 \) on \( (0, T) \times \partial \Omega \) as a special case of (3.1c).

**Theorem 3.3.** Let \( T \in (0, \infty), \ \Omega \subset \mathbb{R}^3 \) be a domain and \( Q := (0, T) \times \Omega \). Let \( r > \frac{3}{2}, \ b \in L^{r'}(0, T; (W^{1,r}_0)^* ) \) and \( \mathbf{v}_0 \in L^2_{n,\text{div}} \). Let further \( \mathcal{G} \subset \mathbb{R}_{\text{sym}}^3 \times \mathbb{R}_{\text{sym}}^3 \) be a maximal monotone \( r \)-graph of the form (3.10) fulfilling (G1)–(G4). Then there exists a pair \((\mathbf{v}, \mathbf{S})\):

\[
(3.46a) \quad \mathbf{v} \in L^\infty(0, T; L^2_{n,\text{div}}) \cap L^r(0, T; W^{1,r}_{0,\text{div}}),
\]

\[
(3.46b) \quad \mathbf{S} \in L^r(Q)^{3 \times 3}_{\text{sym}}
\]

satisfying

\[
(3.46c) \quad \lim_{t \to 0^+} \int_\Omega |\mathbf{v}(t, \cdot) - \mathbf{v}_0|^2 = 0,
\]

\[
(3.46d) \quad \int_Q \mathbf{S} : \mathbf{Dw} = \int_0^T \langle \mathbf{b}, \mathbf{w} \rangle + \int_Q \mathbf{v} \otimes \mathbf{v} : \mathbf{Dw} + \int_Q \mathbf{v} \cdot \frac{\partial \mathbf{w}}{\partial t} + \int_\Omega \mathbf{v}_0 \cdot \mathbf{w} (0, \cdot)
\]

for all \( \mathbf{w} \in C^\infty_0((0, T); W^{1,q}_{0,\text{div}}), \ q = \max \left\{ r, \frac{3r}{s - r} \right\}, \)

\[
(3.46e) \quad \mathbf{S} = 2\nu_* (|\mathbf{Dv}| - \delta_*)^+ S(|\mathbf{Dv}|) \frac{\mathbf{Dv}}{|\mathbf{Dv}|} \quad \text{almost everywhere in} \ Q.
\]
Moreover, the energy inequality holds:

\[
\int_{\Omega} \frac{|v(t, \cdot)|^2}{2} + \int_0^t \int_{\Omega} S : Dv \leq \int_{\Omega} \frac{|v_0|^2}{2} + \int_0^t \langle b, v \rangle
\]

for almost all \( t \in (0, T) \) and for \( t = T \);

if \( r \geq \frac{11}{5} \), (3.47) becomes equality.

In addition, if \( \Omega \) is a \( C^{0,1} \) domain with sufficiently small Lipschitz constant (smallness depending only on \( r \)) or \( \Omega \) is any \( C^1 \) domain, then there are \( P^1 \in L^\infty(0, T; L^6(\Omega)) \), \( P^1(t, \cdot) \) harmonic in \( \Omega \) for almost every \( t \in (0, T) \) and \( p^2 \in L^q(Q) \) with \( q = \max\{r, \frac{5r}{5r-6}\} \) such that

\[
\int_Q S : Dw = \int_0^T \langle b, \omega \rangle + \int_Q v \otimes v : Dw + \int_Q v \cdot \frac{\partial v}{\partial t} + \int_{\Omega} v_0 \cdot \omega(0, \cdot)
\]

\[
- \int_Q P^1 \text{div} \frac{\partial v}{\partial t} + \int_Q p^2 \text{div} \omega
\]

for all \( \omega \in C^\infty_c([0, T); W^{1,q}_0) \), \( q = \max\{r, \frac{5r}{5r-6}\} \).

Functions \( P^1 \) and \( p^2 \) can be chosen such that \( \int_\Omega P^1(t, \cdot) = \int_\Omega P^2(t, \cdot) = 0 \) for almost every \( t \in (0, T) \). If, in addition, \( \Omega \) is a \( C^{1,1} \) domain then it holds \( \nabla P^1 \in L^\infty(0, T; L^2(\Omega)) \cap L^\infty(Q) \).

Remark 3.4. (1) We could define the weak solution to the problem considered differently. We could say that \( v \) is a weak solution to the problem if \( v \) fulfills (3.46a), (3.46c) and

\[
\int \frac{2\nu_+ (|Dv| - \delta_+)}{|Dv|} S(|Dv|) \frac{ Dw}{|Dv|} : Dw = \int_0^T \langle b, w \rangle + \int_Q v \otimes v : Dw
\]

\[
+ \int_Q v \cdot \frac{\partial w}{\partial t} + \int_{\Omega} v_0 \cdot w(0, \cdot)
\]

for all \( w \in C^\infty_c([0, T); W^{1,q}_{0,\text{div}}) \), \( q = \max\{r, \frac{5r}{5r-6}\} \).

(2) It holds \( v \in C(0, T; L^2(\Omega)^3) \) if \( r \geq \frac{11}{5} \) and \( v \in C(0, T; L^2_{\text{weak}}(\Omega)^3) \) if \( r \in \left( \frac{6}{5}, \frac{11}{5} \right) \).

Proof of Theorem 3.3. We shall distinguish two cases (that can be also identified via behavior of the total dissipation of energy with respect to scaling invariance of the governing equations, see [58]): the subcritical/critical case \( r \geq \frac{11}{5} \) and the supercritical case \( r \in \left( \frac{6}{5}, \frac{11}{5} \right) \). The problem can be analyzed in an arbitrary spatial dimension \( d \); then the supercritical case corresponds to \( r \in \left( \frac{2d}{d+2}, 1 + \frac{2d}{d+2} \right) \) and the subcritical/critical case to \( r \geq 1 + \frac{2d}{d+2} \). Note that the case \( r = 2 \) (including the Euler/Navier-Stokes fluid) belongs to the supercritical case in any spatial dimension \( d > 2 \).

The case \( r \geq \frac{11}{5} \). Step 1. Galerkin approximations. We first construct a finite-dimensional approximation to the problem by the Galerkin method. To proceed, we consider an auxiliary eigenvalue problem to find \( \lambda \in \mathbb{R} \) and \( \omega \in W^{3,2}_{0,\text{div}} \mapsto W^{1,\infty}(\Omega)^3 \) satisfying

\[
\langle \omega, \varphi \rangle = \lambda \langle \omega, \varphi \rangle \text{ for all } \varphi \in W^{3,2}_{0,\text{div}},
\]

where \( \langle \cdot, \cdot \rangle \) is a scalar product in \( L^2(\Omega)^3 \) and \( \langle \cdot, \cdot \rangle \) is a scalar product in \( W^{3,2}_{0,\text{div}} \), i.e.,

\[
\langle \omega, \varphi \rangle := (\nabla^3 \omega, \nabla^3 \varphi) + \langle \omega, \varphi \rangle. \]

It is known, see for example [56, Appendix A.4],
that there exist eigenvalues \( \{\lambda_m\}_{m=1}^{\infty} \) and corresponding eigenfunctions \( \{\omega^m\}_{m=1}^{\infty} \) for the eigenvalue problem (3.49) such that \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots, \lambda_m \to \infty \) as \( m \to \infty \), \( (\omega^m, \omega^n) = \delta_{mn} \). Furthermore, the mappings \( P^N : W^{3,2}_{0,\text{div}} \to H^N := \text{span}\{\omega^1, \omega^2, \ldots, \omega^N\} \) defined by \( P^N v := \sum_{i=1}^{N} (v, \omega^i) \omega^i \) are continuous orthonormal projectors in \( L^2(\Omega)^3 \), \( W^{3,2}_{0,\text{div}} \) and \( (W^{3,2}_{0,\text{div}})^* \), in particular

\[
\|P^N\|_{L^2(\Omega)^3} \leq 1, \quad \|P^N\|_{L(W^{3,2}_{0,\text{div}})} \leq 1, \quad \|P^N\|_{L((W^{3,2}_{0,\text{div}})^*)} \leq 1.
\]

Galerkin approximations \( \mathbf{v}^N(t) \in H^N \) of the form \( \mathbf{v}^N(t,x) = \sum_{j=1}^{N} c_j^N(t) \omega^j(x) \) are introduced in such a way that the coefficients \( c^N = (c_1^N, c_2^N, \ldots, c_N^N) \) fulfill

\[
\left( \frac{d\mathbf{v}^N}{dt}, \omega^j \right) - \left( \mathbf{v}^N \otimes \mathbf{v}^N, \nabla \omega^j \right) + (\mathbf{S}(\mathbf{D} \mathbf{v}^N), \mathbf{D} \omega^j) = (P^N \mathbf{b}, \omega^j) \quad j = 1, 2, \ldots, N,
\]

where

\[
\mathbf{S}(\mathbf{D}) := 2\nu \left( |\mathbf{D}| - \delta_s \right)^+ S(|\mathbf{D}|) \frac{\mathbf{D}}{\|\mathbf{D}\|}.
\]

Since the mappings \( z \mapsto z \otimes z \) and \( z \mapsto \mathbf{S}(\mathbf{D} \mathbf{z}) \) are continuous, the Carathéodory theory for systems of ordinary differential equations implies local existence of a solution \( c^N \) solving (3.51). Global existence then follows from the fact that

\[
\sup_{t \in (0,T)} |c^N(t)|_{\mathbb{R}^N} < \infty.
\]

This piece of information is a simple consequence of the orthogonality of the basis \( \{\omega^j\}_{j=1}^{\infty} \) and a priori estimates that will follow, see (3.54) below.

**Step 2. Uniform estimates and their consequences.** Multiplying (3.51) by \( c_j^N(t) \), taking the sum over \( j = 1, 2, \ldots, N \), using the fact \( (z \otimes z, \nabla z) = 0 \) for \( z \) with \( \text{div} z = 0, z \cdot n = 0 \) on \( \partial \Omega \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\mathbf{v}^N\|_2^2 + (\mathbf{S}(\mathbf{D} \mathbf{v}^N), \mathbf{D} \mathbf{v}^N) = (P^N \mathbf{b}, \mathbf{v}^N).
\]

Since \( \mathcal{G} \) is an \( r \)-graph fulfilling (G4), we conclude that for all \( t \in (0,T] \)

\[
\|\mathbf{v}^N(t)\|_2^2 + \alpha \int_0^t \left( \|\mathbf{S}^R\|_r^R + \|\mathbf{D} \mathbf{v}^N\|_r^R \right) \leq C \left( \beta, \|v_0\|_2^2, \|\mathbf{b}\|_{L^r((0,T),W^{1,r}_{0,\text{div}})} \right).
\]

Using the orthogonality of \( \{\omega^j\}_{j=1}^{N} \) in \( L^2(\Omega)^3 \), this, in particular, implies that

\[
\sup_{t \in (0,T)} |c^N(t)|_{\mathbb{R}^N} < \infty
\]

so that the proof of global-in-time existence of \( c^N : [0,T] \to \mathbb{R}^N \) is complete.

Furthermore, Korn’s inequality, see (3.2), the interpolation inequality

\[
\|u\|_q \leq \|u\|_2^{(1-\lambda)q} \|u\|_2^{\lambda q} \quad \text{with } \lambda q = \frac{3r(q-2)}{5r-6},
\]

that is, for all \( u \in L^q(\Omega) \),
and the embedding $W_0^{1,r} \hookrightarrow L^{\frac{2r}{3r+1}}(\Omega)^3$ together with (3.53) imply

$$\int_0^T \left\| \frac{dW}{dt} \right\|_{L^{\frac{2r}{3r+1}}(\Omega)^3} \leq C \left( \beta, \|v_0\|_2, \|b\|_{L^r(0,T;W_0^{1,r})^3} \right).$$

Finally, since for all $\varphi \in L^s(0,T;W_0^{3,2})$

$$\int_0^T \left( \frac{dW}{dt}, \varphi \right) = \int_0^T \left( \frac{dW}{dt}, \Pi^N \varphi \right),$$

it follows from (3.51), (3.53), the fact that $2r' = \frac{2r}{3r+1} \leq \frac{5r}{7} \Leftrightarrow r \geq \frac{4}{3}$, and (3.50), that

$$\left\| \frac{dW}{dt} \right\|_{L^r(0,T;W_0^{3,2})^3} \geq \sup_{\|\varphi\|_{L^r(0,T;W_0^{3,2})} \leq 1} \left| \int_0^T \left( \frac{dW}{dt}, \varphi \right) \right| \leq C \left( \beta, \|v_0\|_2, \|b\|_{L^r(0,T;W_0^{1,r})^3} \right).$$

Consequently, there are (not relabeled) subsequences so that

$$v^N \rightharpoonup v \quad \text{*-weakly in} \quad L^\infty(0,T;L^2(\Omega)^3),$$

$$Dv^N \rightharpoonup Dv \quad \text{weakly in} \quad L^r(0,T;L^r(\Omega)^{3\times3}),$$

$$\nabla v^N \rightharpoonup \nabla v \quad \text{weakly in} \quad L^r(0,T;L^r(\Omega)^{3\times3}),$$

$$S^N \rightharpoonup S \quad \text{weakly in} \quad L^r(0,T;L^r(\Omega)^{3\times3}),$$

$$\partial_t v^N \rightharpoonup \partial_t v \quad \text{weakly in} \quad \left( L^r(0,T;W_0^{3,2})^* \right),$$

(3.55) $v^N \rightarrow v$ strongly in $L^q(0,T;L^q(\Omega)^3)$ for all $q \in [1, \frac{5r}{3})$,

where the last limit (3.55) follows from the Aubin-Lions compactness lemma applied to $W_0^{3,2} \hookrightarrow W_0^{1,r} \hookrightarrow \hookrightarrow L^r_{n,div} \hookrightarrow (W_0^{3,2})^*$. Finally, letting $N \rightarrow \infty$ in (3.51) for $j \in \mathbb{N}$ arbitrary but fixed, one concludes that $(v, S)$ satisfy

$$\int_0^T \left( \frac{\partial v}{\partial t}, \omega \right) - (v \otimes v, \nabla \omega) + (S, D\omega) \right) \phi(t)dt = (b, \omega \phi)$$

valid for all $\phi \in C^\infty_0(-\infty, \infty)$ and $\omega \in W_0^{3,2}$. Since the space $W_0^{3,2}$ is dense in $W_0^{1,r}$ as $r \geq \frac{4}{3}$, and functions of the form $\phi(t)\omega(x)$ are dense in $L^r(0,T;W_0^{1,r})$, and finally

$$\left| \int_0^T (v \otimes v, \nabla \psi) \right| \leq \left( \int_0^T \|v\|^2_{L^2} \right)^{\frac{1}{2}} \left( \int_0^T \|\nabla \psi\|^2_{L^2} \right)^{\frac{1}{2}} \leq C \left( \int_0^T \|v\|^2_{L^2} \right)^{\frac{1}{2}} \left( \int_0^T \|\nabla \psi\|^2_{L^2} \right)^{\frac{1}{2}} < +\infty,$$
we deduce that \((v, S)\) satisfies

\[
\int_0^T \langle \partial_t v, \omega \rangle + \int_Q S : D\omega = \langle b, \omega \rangle + \int_Q (v \otimes v, D\omega) \quad \text{for all } \omega \in L^r(0, T; W^{1,r}_{0, \text{div}}). \tag{3.56}
\]

This implies that \(\partial_t v \in L^r(0, T; (W^{1,r}_{0, \text{div}})^*)\). Inserting \(\omega := v\) into (3.56), we obtain the energy equality (3.47). It remains to show (3.46c).

**Step 3. Attainment of the constitutive equation.** To prove (3.46c), we wish to use the graph convergence lemma (see Lemma A.6 in Appendix). To apply this lemma, we need to show that

\[
\limsup_{N \to \infty} \int_Q S_N : Dv_N \leq \int_Q S : Dv.
\]

However (3.55) implies, in particular, that

\[
v_N(t) \to v(t) \quad \text{in } L^2(\Omega)^3 \quad \text{for almost all } t \in (0, T].
\]

Integrating (3.52) from 0 to such \(t\)'s, and letting \(N \to \infty\), one concludes

\[
\frac{1}{2} \|v(t)\|^2 + \limsup_{N \to \infty} \int_0^t \int_{\Omega} S_N : Dv_N = \langle b, v \rangle + \frac{1}{2} \|v_0\|^2.
\]

By comparing this identity with (3.47) (which is an equality as \(r \geq \frac{11}{5}\)), we conclude

\[
\limsup_{N \to \infty} \int_Q S_N : Dv_N = \int_Q S : Dv.
\]

The graph convergence lemma (Lemma A.6) then implies that \(S\) and \(Dv\) fulfill (3.46c). The proof for \(r \geq \frac{11}{5}\) is thus complete.

**The case \(r \in \left(\frac{6}{5}, \frac{11}{5}\right)\). Step 1. Approximations and their validity.** For \(\epsilon > 0\), we look for \((v^\epsilon, S^\epsilon)\) such that

\[
\begin{align*}
\text{(3.57a)} & \quad v^\epsilon \in L^\infty(0, T; L^2_{0, \text{div}}) \cap L^{\frac{11}{5}}(0, T; W^{1,\frac{11}{5}}_{0, \text{div}}), \\
\text{(3.57b)} & \quad \partial_t v^\epsilon \in L^{\frac{11}{5}}(0, T; (W^{1,\frac{11}{5}}_{0, \text{div}})^*), \\
\text{(3.57c)} & \quad S^\epsilon \in L^r(Q)^{3 \times 3}_{\text{sym}}
\end{align*}
\]

satisfy

\[
\begin{align*}
\int_Q S^\epsilon : D\varphi + \epsilon \int_Q |Dv^\epsilon|^2 + & \int_Q Dv^\epsilon : D\varphi \\
= & \langle b, \varphi \rangle + \int_Q (v^\epsilon \otimes v^\epsilon) : D\varphi + \int_Q v^\epsilon \cdot \frac{\partial \varphi}{\partial t} + \int_{\Omega} v_0 \cdot \varphi(0, \cdot) \\
\quad \text{for all } \varphi \in L^{\frac{11}{5}}(0, T; W^{1,\frac{11}{5}}_{0, \text{div}}) \text{ with } \varphi(T, \cdot) = 0 \tag{3.58}
\end{align*}
\]

and

\[
\text{(3.59) } S^\epsilon = 2\nu_\ast (|Dv^\epsilon| - \delta_\ast)^+ S(|Dv^\epsilon|) \frac{Dv^\epsilon}{|Dv^\epsilon|} \quad \text{almost everywhere in } Q.
\]
The existence of \((v^\epsilon, S^\epsilon)\) fulfilling (3.57)–(3.59) for arbitrary but fixed \(\epsilon > 0\) can be proved in the same way as the existence of a weak solution to the problem for the case \(r \geq \frac{1}{5}\). In addition, by taking \(\varphi := v^\epsilon\), we have

\[
(3.60) \quad \frac{1}{2} \|v^\epsilon(t)\|^2 - \frac{1}{2} \|v_0\|^2 + \epsilon \int_0^t \|Dv^\epsilon\|^2 \, dt + \int_0^t \int_\Omega \sigma : Dv^\epsilon = \langle b, v^\epsilon \chi_{(0,t) \times \Omega} \rangle
\]

for almost all \(t \in (0, T)\) (and for \(t = T\)) where \(Dv^\epsilon\) and \(S^\epsilon\) satisfy (3.59).

**Step 2. Estimates uniform with respect to \(\epsilon\) and their consequences.** Since \(S : D \geq \alpha (|D|^\tau + |S|^\tau) - \beta\) for all \((S, D)\) fulfilling (3.59), i.e., \((S, D) \in \mathcal{G}\), the energy identity (3.60) implies that

\[
\begin{align*}
& (3.61a) \quad \{v^\epsilon; \epsilon > 0\} \text{ is bounded in } L^\infty(0, T; L^2(\Omega)^3), \\
& (3.61b) \quad \{v^\epsilon; \epsilon > 0\} \text{ is bounded in } L^r(0, T; W^{1,r}_{0, \text{div}}), \\
& (3.61c) \quad \{Dv^\epsilon; \epsilon > 0\} \text{ is bounded in } L^r(0, T; L^r(\Omega)^{3 \times 3}), \\
& (3.61d) \quad \{\epsilon \frac{\partial}{\tau} Dv^\epsilon; \epsilon > 0\} \text{ is bounded in } L^\infty(0, T; L^\infty(\Omega)^3), \\
& (3.61e) \quad \{S^\epsilon; \epsilon > 0\} \text{ is bounded in } L^r(0, T; L^r(\Omega)^{3 \times 3}).
\end{align*}
\]

Using the fact that

\[
\int_0^T \langle \partial_t v^\epsilon, \varphi \rangle = -\int_Q v^\epsilon \cdot \partial_t \varphi - \int_\Omega v_0 \cdot \varphi(0, \cdot)
\]

for all \(\varphi \in L^\infty(0, T; W^{1,\infty}_{0, \text{div}})\) with \(\varphi(T, \cdot) = 0\) we conclude from (3.58) and (3.61) that

\[
\{\partial_t v^\epsilon; \epsilon > 0\} \text{ is bounded in } \left(L^{\frac{5r}{3r-\epsilon}}(0, T; W^{1,\frac{5r}{3r-\epsilon}}_0(\Omega))\right)^*.
\]

These estimates together with Korn’s inequality (3.2) and the Aubin-Lions lemma lead to the existence of \(v\) and \(S\) such that for suitable sequences \((v^m, S^m) := (v^\epsilon^m, S^\epsilon^m)\) and \(m \to \infty\) the following convergences hold:

\[
\begin{align*}
& v^m \rightharpoonup^* v \quad \text{\(\Rightarrow\) weakly in } L^\infty(0, T; L^2(\Omega)^3), \\
& v^m \rightharpoonup v \quad \text{weakly in } L^r(0, T; W^{1,r}_{0, \text{div}}), \\
& S^m \rightharpoonup S \quad \text{weakly in } L^r(0, T; L^r(\Omega)^{3 \times 3}), \\
& S^m \rightharpoonup S \quad \text{strongly in } L^q(0, T; L^q(\Omega)^3) \text{ for all } q \in [1, \frac{5r}{r-\epsilon}).
\end{align*}
\]

Also (3.61d) implies that

\[
(3.64) \quad \epsilon_m \int_Q |Dv^m|^\frac{3}{\tau} \, |Dv^m| : D\varphi \leq \epsilon_m \left( \int_Q |Dv^m|^\frac{\tau}{3} \right)^\frac{3}{\tau} \|D\varphi\|_{L^\infty, Q} \xrightarrow{m \to \infty} 0.
\]

Let us consider (3.58) with \(\varphi \in L^q(0, T; W^{1,q}_{0, \text{div}}), q = \frac{5r}{r-\epsilon}, \varphi(T, \cdot) = 0\) and let \(m \to \infty\). Then we easily arrive to (3.46d).

**Step 3. Attainment of the constitutive equation (3.46c).** In order to apply the graph convergence lemma (Lemma A.6) we need to proceed in a more subtle way as
\( w := v \) is not an admissible test function in (3.46d). For \( u^m := v^m - v \), the following identity holds
\[
- \int_Q (v^m - v) \cdot \partial_t w + \int_Q (S^m - S) : Dw + \epsilon_m \int_Q |Dv^m|^\frac{5}{2} Dv^m : Dw
\]
for all \( w \in C^\infty([0,T]; C_0^{\infty}) \)
\( (3.65) \)
and
\[
\begin{align*}
&u^m \rightharpoonup 0 \quad \text{*-weakly in } L^\infty(0,T; L^2_{n, \text{div}}), \\
&u^m \to 0 \quad \text{weakly in } L'(0,T; W^{1, r}_{0, \text{div}}), \\
&u^m \to 0 \quad \text{strongly in } L^q(0,T; L^q(\Omega)^3) \text{ for all } q \in [1, \frac{5}{2}).
\end{align*}
\]
We also observe that besides (3.64)
\[
(S^m - S) := H^m_1 \to 0 \quad \text{weakly in } L'^r(Q),
\]
\[
\left( \epsilon_m |Dv^m|^\frac{5}{2} Dv^m + (v \otimes v - v^m \otimes v^m) \right) := H^m_2 \to 0 \quad \text{strongly in } L^\sigma(Q) \text{ for some } \sigma \in (1, \frac{5}{2})
\]
and we rewrite (3.65) as
\[
\int_Q u^m \cdot \partial_t w = \int_Q H^m_1 : Dw + \int_Q H^m_2 : Dw.
\]
Let \( Q_0 \subset Q \) be any parabolic cylinder. Take \( \zeta \in C_0^\infty(\frac{1}{3}Q_0) \) such that
\[
\chi_{\frac{1}{3}Q_0} \leq \zeta \leq \chi_{\frac{2}{3}Q_0}.
\]
Then applying the Lipschitz truncation (Lemma A.5) we conclude, using the above convergences, that
\[
\limsup_{m \to \infty} \left| \int_{\frac{1}{3}Q_0 \setminus Q_0} (S^m - S) : (Dv^m - Dw) \right| \leq C2^{-k}.
\]
This, together with the property (h) of the truncation lemma (Lemma A.5) and Hölder’s inequality, implies that
\[
\limsup_{m \to \infty} \int_{\frac{1}{3}Q_0} \left| (S^m - S) : (Dv^m - Dw) \right|^\frac{5}{2} \leq C2^{-k}.
\]
Set \( g^m := |(S^m - S) : (Dv^m - Dw)| \). Clearly \( g^m \geq 0 \) and \( g^m \to 0 \) almost everywhere in \( \frac{1}{3}Q_0 \). But as \( Q_0 \) is arbitrary, we conclude that
\( (3.66) \)
\[
g^m \to 0 \quad \text{almost everywhere in } Q.
\]
Since \( \{g^m\}_{m=1}^\infty \) is bounded in \( L^1(Q) \) and has pointwise limit (3.66), Corollary A.3, a consequence of the biting lemma (Lemma A.2), ensures the existence of a subsequence \( \{g^{m_k}\}_{k=1}^\infty \) and a sequence of sets \( \{E_k\}_{k=1}^\infty \) with \( Q \supset E_1 \supset E_2 \supset \ldots \), \(|E_k| \to 0 \) such that for all \( k \in \mathbb{N} \)
\[
g^{m_k} \to 0 \quad \text{strongly in } L^1(Q \setminus E_k).
\]
From the definition of $g^m$, we conclude that
\[
\limsup_{j \to 0} \int_{Q \setminus E_k} (S^{m_j} - S) : (Dv^{m_j} - Dv) = 0,
\]
which implies with the help of (3.62) and (3.63) that for all $k \in \mathbb{N}$
\[
\limsup_{j \to 0} \int_{Q \setminus E_k} S^{m_j} : Dv^{m_j} = \int_{Q \setminus E_k} S : Dv.
\]
Since $|E_k| \to 0$, we can conclude from the graph convergence lemma (Lemma A.6) that
\[
(S, Dv) \in \mathcal{G} \text{ almost everywhere in } Q
\]
so that (3.46c) holds.

The energy inequality and the initial condition. Since $(S, Dv) \in \mathcal{G}$ and $\mathcal{G}$ is monotone, we first observe that $g^m = (S^m - S) : (Dv^m - Dv) \geq 0$. It thus follows from (3.66), Fatou's lemma applied to the functions $\{g^m\}_{m=1}^\infty$, that are non-negative, and from the weak convergences (3.62) and (3.63) that
\[
\int_Q S : Dv \leq \liminf_{n \to \infty} \int_Q S^m : Dv^m.
\]
It is then easy to conclude the energy inequality (3.47) from (3.60).

Attainment of the initial condition (3.46c), which is proved with the help of the energy inequality (3.47), is standard and we omit it; see [58, sections B.3.8–10]. Thus the first part of the theorem is proved.

On the pressure. Let us consider for fixed $t \in (0, T)$ the functionals
\[
(F^1(t), \varphi) := \int_\Omega (v(t, \cdot) - v_0) \cdot \varphi,
\]
\[
(F^2(t), \varphi) := \int_\Omega \int_0^t (S - v \otimes v) : D\varphi - \int_0^t b, \varphi)
\]
for $\varphi \in W^{1,q}_0$ with $q := \max\{r, \frac{m}{m-1}\}$. Clearly $F^1(t), F^2(t) \in (W^{1,q}_0)^*$ for almost every $t \in (0, T)$. Testing (3.46d) by $w^j \in \mathcal{C}_0^\infty([0, T]; \mathcal{C}_{0, \text{div}}^\infty)$ such that $w^j \to w$ and
\[
w(s, x) = \begin{cases} \varphi(x) & s \in [0, t), \\ 0 & s \in [t, T) \end{cases}
\]
with arbitrary $\varphi \in \mathcal{C}_{0, \text{div}}^\infty$ and comparing with (3.67) we obtain
\[
\langle (F^1 + F^2)(t), \varphi \rangle = 0 \quad \text{for all } \varphi \in \mathcal{C}_{0, \text{div}}^\infty \text{ and a.e. } t \in (0, T).
\]
Now consider the Stokes problems
\[
\begin{align*}
&-\Delta U^1 + \nabla P^1 = F^1, \quad \text{div } U^1 = 0 \quad \text{in } Q, \quad U^1 = 0 \quad \text{on } (0, T) \times \partial \Omega, \\
&-\Delta U^2 + \nabla P^2 = F^2, \quad \text{div } U^2 = 0 \quad \text{in } Q, \quad U^2 = 0 \quad \text{on } (0, T) \times \partial \Omega.
\end{align*}
\]
By virtue of the assumptions on the domain, we conclude from (A.4) of Lemma A.13 that
\[
\|\nabla U^1(t)\|_6 + \|P^1(t)\|_6 \leq C\|v(t) - v_0\|_{-1,6} \leq C\|v(t) - v_0\|_2,
\]
\[
\|\nabla U^2(t)\|_{q'} + \|P^2(t)\|_{q'} \leq C\|F^2(t)\|_{-1,q'}
\]
\[
\leq C \int_0^t \|S - v \otimes v\|_{q'} + C \int_0^t \|b\|_{-1,q'},
\]
which leads to $P^1 \in L^\infty(0, T; L^6)$ and $p^2 \equiv \partial_t P^2 \in L^\prime (Q)$. Testing (3.69a) with $\nabla \phi$ for $\phi \in C_0^\infty(\Omega)$ we get
\[
\left(\nabla U^1(t), \nabla \phi \right) - (P^1(t), \Delta \phi) = (v(t) - v_0, \nabla \phi) = -(\text{div}(v(t) - v_0), \phi) = 0
\]
so that $(P^1(t), \Delta \phi) = 0$ for all $\phi \in C_0^\infty(\Omega)$ and Weyl’s lemma (cf. [81, Lemma 2], [36, Chapter 10]) yields that $P^1(t)$ is harmonic.

Testing (3.69) by $\varphi \in W^{1,q}_{0, \text{div}}$ we obtain, by using (3.68),
\[
\left(\nabla (U^1(t) + U^2(t)), \nabla \varphi \right) = 0 \quad \text{for all } \varphi \in W^{1,q}_{0, \text{div}} \text{ and a.e. } t \in (0, T),
\]
which shows together with $U^1(t) + U^2(t) \in W^{1,q'}_{0, \text{div}}$ that $U^1 + U^2 = 0$. Now we are in a position to sum up (3.69), test by $\partial_t v$ with $u \in C_0^\infty([0, T) \times \Omega)$, integrate over $Q$, and use the facts shown above and $P^2(0) = 0$ to obtain (3.48).

Furthermore, when $\Omega$ is a $C^{1,1}$ domain, (A.5) from Lemma A.13 yields
\[
\|\nabla^2 U^1(t)\|_2 + \|\nabla P^1(t)\|_2 \leq C\|v(t) - v_0\|_2,
\]
\[
\|\nabla^2 U^1(t)\|_{\frac{q}{q'}} + \|\nabla P^1(t)\|_{\frac{q}{q'}} \leq C\|v(t) - v_0\|_{\frac{q}{q'}}
\]
so that $\text{ess sup}_{t \in (0, T)} \|\nabla P^1(t)\|_2 \leq C \text{ess sup}_{t \in (0, T)} \|v(t) - v_0\|_2$ and $\int_0^T \|\nabla P^1\|_{\frac{q}{q'}} \leq C \int_0^T \|v - v_0\|_{\frac{q}{q'}} \leq C$ and the proof is complete.

**3.3.2. Slip case.** Here we consider the boundary condition
\[
(3.70) \quad v \cdot n = 0 \quad \text{and} \quad s = 0 \quad \text{on } (0, T) \times \partial \Omega
\]
or the boundary condition
\[
(3.71) \quad v \cdot n = 0 \quad \text{and} \quad (s, v_\tau) \in \mathcal{B} \quad \text{on } (0, T) \times \partial \Omega
\]
where $\mathcal{B}$ fulfills (B1)–(B4). The following result holds.

**Theorem 3.5.** Let $T \in (0, \infty)$, $\Omega \subset \mathbb{R}^3$ be a $C^{0,1}$ domain, and $Q := (0, T) \times \Omega$. Let $r > \frac{6}{5}$, $b \in L^{r'}(0, T; (W^{1,r}_n)*)$, and $v_0 \in L^2_{\text{div}}$. Let $\mathcal{G} \subset \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3}$ be a maximal monotone r-graph of the form (3.10) fulfilling (G1)–(G4).

(i) (Boundary condition (3.71)) Let $\mathcal{B} \subset \mathbb{R}^{3 \times 3}$ be a maximal monotone 2-graph fulfilling (B1)–(B4). Then there exists a triplet $(v, S, s)$ satisfying
\[
(3.72a) \quad v \in L^{\infty}(0, T; L^\infty_{n, \text{div}}) \cap L^r(0, T; W^{1,r}_{n, \text{div}}),
\]
\[
(3.72b) \quad S \in L^{r'}(Q; \mathbb{R}^{3 \times 3})
\]
\[
(3.72c) \quad s \in L^2((0, T) \times \partial \Omega)^3,
\]
and

\[ (3.72d) \quad \lim_{t \to 0^+} \int_{\Omega} |v(t, \cdot) - v_0|^2 = 0, \]

\[ (3.72e) \quad \int_Q S : Dw + \int_{0,T} \int_{\partial\Omega} s \cdot w = \int_0^T \langle b, w \rangle + \int_Q v \otimes v : Dw + \int_Q v \cdot \frac{\partial w}{\partial t} \]

\[ + \int_{\partial\Omega} v_0 \cdot w(0, \cdot) \quad \text{for all } w \in C^0_0([0, T); W^{1,q}_{n,\text{div}}), \]

\[ q = \max \{ r, \frac{5r}{5r-6} \}. \]

\[ (3.72f) \quad S = 2\nu_a (|Dv| - \delta_+)^+ S(|Dv|) \frac{Dv}{|Dv|} \quad \text{almost everywhere in } Q, \]

\[ (3.73) \quad (s, w_\tau) \in B \quad \text{almost everywhere in } (0, T) \times \partial\Omega. \]

(ii) (Boundary condition (3.70)) There exists a couple \((v, S)\) satisfying (3.72a), (3.72b), (3.72d), (3.72f), and (3.72e) with \(s = 0\).

Moreover, the following energy inequality holds:

\[ (3.74) \quad \int_{\Omega} \frac{|v(t, \cdot)|^2}{2} + \int_0^t \int_{\Omega} S : Dw + \int_0^t \int_{\partial\Omega} s \cdot v \leq \int_{\Omega} \frac{|v_0|^2}{2} + \int_0^t \langle b, v \rangle \]

for almost all \(t \in (0, T)\) and \(t = T\);

if \(r \geq \frac{11}{5}\), then (3.74) becomes an equality.

In addition, if \(\Omega\) is a \(C^{1,1}\) domain, then there is \(p \in L^q(Q)\) with \(q = \max \{ r, \frac{5r}{5r-6} \}\) such that \(\int_{\Omega} p(t, \cdot) = 0\) for almost every \(t \in (0, T)\) and

\[ \int_Q S : Dw + \int_{0,T} \int_{\partial\Omega} s \cdot w = \int_0^T \langle b, w \rangle + \int_Q v \otimes v : Dw + \int_Q v \cdot \frac{\partial w}{\partial t} + \int_{\partial\Omega} v_0 \cdot w(0, \cdot) \]

\[ + \int_Q p \text{div } w \quad \text{for all } w \in C^0_0([0, T); W^{1,q}_{n,\text{div}}). \]

Remark 3.6. (1) In the case of the boundary condition (3.70) we could define the weak solution to the problem considered differently. We could say that \(v\) is a weak solution to the problem if \(v\) fulfills (3.72a), (3.72d), and

\[ \int_Q 2\nu_a (|Dv| - \delta_+)^+ S(|Dv|) \frac{Dv}{|Dv|} : Dw = \int_0^T \langle b, w \rangle + \int_Q v \otimes v : Dw \]

\[ + \int_Q v \cdot \frac{\partial w}{\partial t} + \int_{\partial\Omega} v_0 \cdot w(0, \cdot) \quad \text{for all } w \in C^0_0([0, T); W^{1,q}_{n,\text{div}}), q = \max \{ r, \frac{5r}{5r-6} \}. \]

(2) It holds \(v \in C(0, T; L^2(\Omega)^3)\) if \(r \geq \frac{11}{5}\) and \(v \in C(0, T; L^2_{\text{weak}}(\Omega)^3)\) if \(r \in \left( \frac{6}{5}, \frac{11}{5} \right)\).

Proof of Theorem 3.5. We focus only on the details in which the proof differs from the proof of Theorem 3.3. Note however that a remarkable difference concerns the pressure: for the no-slip boundary condition the pressure is not integrable up to the boundary; here, for \(C^{1,1}\) domains, we establish the existence of the pressure
belonging to $L^s(Q)$ for some $s > 1$. This concerns in particular the no-slip/Navier-slip boundary condition which "approximates" well the no-slip boundary condition and in addition its mathematical theory admits integrable pressure.

Regarding the case $r \geq \frac{11}{5}$, the main departures from the problem with the no-slip boundary condition is due to the choice of function spaces and due to the formulation of the eigenvalue problem that generates the basis for Galerkin approximations. Here, we look for $\lambda \in \mathbb{R}$ and $\omega \in W^{3,2}_{n,\text{div}} \rightarrow W^{1,\infty}(\Omega)^3$ satisfying

$$\langle \omega, \varphi \rangle = \lambda \langle \omega, \varphi \rangle \quad \text{for all } \varphi \in W^{3,2}_{n,\text{div}},$$

where $\langle \cdot, \cdot \rangle$ is again the scalar product in $L^2(\Omega)^3$ and $\langle \cdot, \cdot \rangle$ is a scalar product in $W^{3,2}_{n,\text{div}}$ defined through $\langle \omega, \varphi \rangle := (\nabla^1 \omega, \nabla^3 \varphi) + \langle \omega, \varphi \rangle + (\omega_\tau, \varphi_\tau)_{\partial \Omega}$. The properties of the eigenfunctions are the same as in (3.49) and consequently, for the free-slip boundary condition (3.70) there is no other change in the proof.

If the other slipping conditions are considered, then we regularize the boundary conditions as in the time independent case. Independent of the approximation parameter, we, in addition to standard uniform estimates, know that $\{s^n\}_{n=1}^\infty$ and $\{v^n\}_{n=1}^\infty$ are bounded in $L^2(0, T; L^2(\partial \Omega)^3)$. Furthermore, as $W^{1, r}(\Omega)$ compactly embeds into $W^{\frac{5}{3} - q}(\Omega)$ for all $q < r$, we conclude that

$$v^N_\tau \rightarrow v_\tau \quad \text{strongly in } L^r(0, T; L^1(\partial \Omega)^3).$$

Then (up to a subsequence which we do not relabel)

$$v^N_\tau \rightarrow v_\tau \quad \text{a.e. on } (0, T) \times \partial \Omega$$

and by Egorov’s theorem, for any $\delta > 0$,

$$v^N_\tau \rightarrow v_\tau \quad \text{strongly in } L^\infty(\mathcal{U}_\delta)$$

where $\mathcal{U}_\delta \subset (0, T) \times \partial \Omega$ is such that $|(0, T) \times \partial \Omega \setminus \mathcal{U}_\delta| < \delta$. The last convergence implies that

$$\lim \sup_{N \to \infty} \int_{\mathcal{U}_\delta} s^N \cdot v^N_\tau = \int_{\mathcal{U}_\delta} s \cdot v_\tau.$$

Consequently, by Lemma A.6, $(s, v_\tau) \in B$ a.e. on $\mathcal{U}_\delta$. This is true for all $r > 1$ and gives (3.73).

If $r \in \left(\frac{6}{5}, \frac{9}{5}\right)$, the proof of $(S, Dv) \in G$ is carried out as in the no-slip case, as the proof is based on local analysis in the interior of $\Omega$.

Finally, we reconstruct the pressure. We set $p = p_1 + p_2$ where $p_1 \in L^{\frac{5}{3} - r}(Q)$ solves

$$(p_1, -\Delta z) = (v \otimes v, \nabla^2 z) \quad \text{for all } z \in W^{2, \frac{5}{3} - r} \text{ with } \nabla z \in W^{1, \frac{5}{3} - r}_n,$$

$$\int_\Omega p_1(t) = 0 \quad \text{for a.e. } t \in (0, T)$$

and $p_2 \in L^{r'}(Q)$ solves

$$(p_2, -\Delta z) = (b, \nabla z) - (S, D\nabla z) - (s, \nabla z)_{\partial \Omega} \quad \text{for all } z \in W^{2, \frac{5}{3} - r}, \nabla z \in W^{1, \frac{5}{3} - r}_n,$$

$$\int_\Omega p_2(t) = 0 \quad \text{for a.e. } t \in (0, T).$$
Note that this is a well-posed definition because of the $C^{1,1}$ regularity of the domain $\Omega$ and Lemma A.11. Now consider a test function $\varphi \in L^q(0,T; W^{1,q}_n)$ and its Helmholtz decomposition using Corollary A.12:

$$\varphi = \nabla \phi + \varphi_0 \quad \text{with} \quad \nabla \phi \in L^q(0,T; W^{1,q}_n), \varphi_0 \in L^q(0,T; W^{1,q}_{n,\text{div}}).$$

Then we have

$$\left\langle \frac{\partial v}{\partial t}, \varphi \right\rangle - (p, \text{div} \varphi) = \left\langle \frac{\partial v}{\partial t}, \nabla \phi + \varphi_0 \right\rangle - (p, \text{div}(\nabla \phi + \varphi_0)) = \left\langle \frac{\partial v}{\partial t}, \varphi_0 \right\rangle + (p_1 + p_2, -\Delta \phi)$$

$$= (v \otimes v, D\varphi_0) - (S, D\varphi_0) - (s, \varphi_0)_{\partial \Omega} + (b, \varphi_0) + (v \otimes v, \nabla^2 \phi) + (b, \nabla \phi) - (S, D\nabla \phi) - (s, \nabla \phi)_{\partial \Omega}$$

$$= (v \otimes v, D\varphi) - (S, D\varphi) - (s, \varphi)_{\partial \Omega} + (b, \varphi)$$

for a.a. $t \in (0,T)$. Thus the theorem is proven.

4. **Concluding remarks.** We have classified incompressible fluids that span the gamut from Euler fluids – Navier-Stokes fluid – power-law fluids – generalized power-law fluids – stress power-law fluids – to fluids that only undergo rigid motions, that can change their constitutive character due to an activation criterion based on the value of the norm of the symmetric part of the velocity gradient or the shear stress. In the process we came across constitutive relations that have hitherto been unrecognized but could possibly be useful. In the course of our investigation we have delineated how an Euler fluid is different from a fluid that behaves like an Euler fluid prior activation and behaves like a viscous fluid when the activation criterion takes place. The latter fluid would lead to governing equations that imbed the boundary layer equations as a special case, the philosophy behind the development of the boundary layer equations and the equations governing the activated fluid being totally different. We have touched upon one important aspect in this study, namely the tremendously different properties that are exhibited by the Euler fluids and the activated Euler fluids. It is known that while the Euler fluid exhibits pathological features (such as existence of a nontrivial solution to internal flows with zero initial data and vanishing external body forces), we have shown that the new class of activated Euler fluids admits a weak solution that might be even unique in its dependence of what kind of response occurs after activation.

A classification similar to that presented here for incompressible fluids can be carried out within the context of compressible fluids, where however the framework is more complicated as there are two type of viscosities (bulk and shear) and corresponding fluidities. This issues will be addressed in a subsequent study.

**Appendix A. Auxiliary convergence tools.** In this section, we state, without proofs, several characterizations of weak compactness in $L^1$. Then, following [13], we summarize several properties of refined (divergence-free) Lipschitz approximations of (divergence-free) Sobolev and Bochner-Sobolev functions. Next, we present a convergence lemma (proved recently in [18]) regarding stability of maximal monotone constitutive equations (maximal monotone $r$-graphs) with respect to weakly converging sequences. Finally, we close this section by the Nečas theorem and Sobolev regularity results for the Neumann-Poisson problem and the Stokes system.

In the following lemma, several assertions characterizing weak compactness in $L^1$, namely the Dunford-Pettis criterion (ii), uniform integrability (iii), and the de la
Vallé-Poussin criterion (iv), are provided. The exact statement is taken from [28, p. 21, Theorem 10].

**Lemma A.1** (Characterization of weak compactness in $L^1$). Let $Q \subset \mathbb{R}^M$ be a bounded measurable set and $V \subset L^1(Q)$. Then the following conditions are equivalent:

(i) any sequence $\{v_n\}_{n=1}^{\infty} \subset V$ contains a subsequence weakly converging in $L^1(Q)$;

(ii) for any $\epsilon > 0$ there exists $K > 0$ such that for all $v \in V$

$$\int_{\{|v| \geq K\}} |v(y)| dy \leq \epsilon;$$

(iii) for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $v \in V$ and for any measurable set $M \subset Q$ such that $|M| < \delta$

$$\int_M |v(y)| dy < \epsilon;$$

(iv) there exists a nonnegative function $\Phi \in C([0, \infty))$ fulfilling

$$\lim_{z \to \infty} \frac{\Phi(z)}{z} = \infty,$$

such that

$$\sup_{v \in V} \int_Q \Phi(|v(y)|) dy < \infty.$$

Since $L^1$ is not reflexive, weak precompactness does not follow from boundedness. Instead bounded sequences in $L^1$ can exhibit local concentrations weakly converging only in the space of measures. The next lemma ensures that these concentrations are located on arbitrarily small sets and when removed (by “biting”), bounded sets are $L^1$-weak precompact on the complement (“unbitten” part). See original reference [15] and also [8] for a simple proof and other references.

**Lemma A.2** (Biting lemma). Let $Q \subset \mathbb{R}^M$ be bounded and measurable. Let $\{v_n\}_{n=1}^{\infty}$ be a sequence bounded in $L^1(Q)$. Then there exist a subsequence $\{v_{n_j}\}_{j=1}^{\infty} \subset \{v_n\}_{n=1}^{\infty}$, a function $v \in L^1(Q)$, and a sequence of measurable sets $\{E_k\}_{k=1}^{\infty}$, $Q \supset E_1 \supset E_2 \supset \ldots$, $|E_k| \to 0$ such that for all $k \in \mathbb{N}$

$$v_{n_j} \rightharpoonup v \quad \text{weakly in } L^1(Q \setminus E_k) \quad \text{as } j \to \infty.$$

In the following corollary of the preceding lemmas we establish strong convergence in $L^1$ up to arbitrarily small sets for a pointwise null sequence bounded in $L^1$.

**Corollary A.3.** Let the assumptions of Lemma A.2 be fulfilled. Furthermore, assume that

$$v_n \to 0 \quad \text{a.e. in } Q \quad \text{as } n \to \infty.$$

Then for the sequences $\{v_{n_j}\}_{j=1}^{\infty}$ and $\{E_k\}_{k=1}^{\infty}$ from Lemma A.2 and for every $k \in \mathbb{N}$

$$v_{n_j} \to 0 \quad \text{strongly in } L^1(Q \setminus E_k) \quad \text{as } j \to \infty.$$
Proof. Let \( k \in \mathbb{N} \) be fixed. The sequence \( \{v_n\}_{j=1}^\infty \) provided by the biting lemma A.2 is weakly compact in \( L^1(Q \setminus E_k) \) and by the lemma A.1, (ii), \( \{v_n\}_{j=1}^\infty \) is uniformly continuous with respect to the Lebesgue measure on \( Q \setminus E_k \). By the Vitali convergence theorem, the assertions follows. \( \Box \)

Lipschitz approximations of solenoidal Bochner-Sobolev functions is another useful tool needed in the analysis of isochoric flows. There are several variants: Acerbi and Fusco survey the basic properties of Lipschitz approximations of Sobolev functions in [1]; further refinements have been put into place, see [30, 24]. The extension to evolutionary problems goes back to [41, 42, 20]. Further extensions have been established in [17, 13].

We first state the version [13, Theorem 4.2], which is suitable for analysis of steady problems.

**Lemma A.4** (Divergence-free Lipschitz truncation of Sobolev functions). Let \( B \subset \mathbb{R}^3 \) be an arbitrary ball. Let \( r \in (1, \infty) \). Let \( \{u^m\}_{m=1}^\infty \subset W_{0,\text{div}}^{1,r}(B) \) be weakly converging to zero in \( W_{0,\text{div}}^{1,r}(B) \).

Then there is a double sequence \( \{\lambda_{m,k}\}_{m,k=1}^\infty \subset (0, \infty) \) with

(a) \( 2^{-k} \leq \lambda_{m,k} \leq 2^{2^{k+1}} \),

a double sequence of functions \( \{u^{m,k}\}_{m,k=1}^\infty \), a double sequence \( \{O^{m,k}\}_{m,k=1}^\infty \) of measurable subsets of \( 2B \), a constant \( C > 0 \), and \( k_0 \in \mathbb{N} \) such that for all \( k \geq k_0 \) it holds:

(b) \( u^{m,k} \in W_{0,\text{div}}^{1,\infty}(2B) \) and \( u^{m,k} = u^m \) in \( 2B \setminus O^{m,k} \) for all \( m \in \mathbb{N} \),

(c) \( \|\nabla u^{m,k}\|_{L^\infty(2B)} \leq C\lambda_{m,k} \) for all \( m \in \mathbb{N} \),

(d) \( u^{m,k} \to 0 \) strongly in \( L^\infty(2B) \) as \( m \to \infty \),

(e) \( \nabla u^{m,k} \overset{\ast}{\rightharpoonup} 0 \) weakly-* in \( L^\infty(2B) \) as \( m \to \infty \),

(f) \( (\lambda_{m,k})^r|O^{m,k}| \leq C2^{-k}\|\nabla u^m\|^r_{r} \) for all \( m \in \mathbb{N} \).

Next we will formulate the assertion suitable for analysis of time-dependent problems; the presented version is taken from [13].

**Lemma A.5** (Divergence-free Lipschitz truncation of Bochner-Sobolev functions). Let \( Q_0 = I_0 \times B_0 \subset \mathbb{R} \times \mathbb{R}^3 \) be a space-time cylinder. Let \( 1 < r < \infty \) with \( r, r' > \sigma > 1 \), \( \frac{1}{r} + \frac{1}{r'} = 1 \). Assume that there are sequences of functions \( \{u^m\}_{m=1}^\infty \) and \( \{H^m\}_{m=1}^\infty \) such that

\[
\frac{\partial u^m}{\partial t} = -\text{div} \ H^m \quad \text{a.e. in } Q_0,
\]

\[
\frac{\partial u^m}{\partial t} = -\text{div} \ H^m \quad \text{in the sense of distributions } \left( C_{0,\text{div}}^{\infty}(Q_0) \right)^*,
\]

\[
u^m \rightharpoonup 0 \quad \text{weakly in } L^r(I_0, W^{1,r}(B_0)),
\]

\[
u^m \to 0 \quad \text{strongly in } L^\sigma(Q_0),
\]

and \( H^m = H^m_1 + H^m_2 \) satisfies

\[
H^m_1 \to 0 \quad \text{weakly in } L^{r'}(Q_0),
\]

\[
H^m_2 \to 0 \quad \text{strongly in } L^\sigma(Q_0).
\]

Then there is a double sequence \( \{\lambda^{m,k}\}_{m,k=1}^\infty \subset (0, \infty) \) with

(a) \( 2^{-k} \leq \lambda^{m,k} \leq 2^{2^{k+1}} \),

a double sequence of functions \( \{u^{m,k}\}_{m,k=1}^\infty \subset L^1(Q_0)^3 \), a double sequence \( \{O^{m,k}\}_{m,k=1}^\infty \) of measurable subsets of \( Q_0, C > 0 \), and \( k_0 \in \mathbb{N} \) such that for all \( k \geq k_0 \):
can also be rephrased as the following:
is closely related to the results known as the Lions lim sup \( u(d) \) by Bogovski˘i [B].

The operator \( \nabla \) of Lemma (g) is usually called the Bogovski˘i operator due to the explicit construction by Bogovski˘i [10, 11]. We refer the reader to [5], where these relations are discussed in detail.

Moreover, if in addition \( \{u(m)\} \) is bounded in \( L^\infty(I_0, L^6(B_0)) \) then

\[
\limsup_{m \to \infty} \int_{I_0} |\nabla u(m, f, r)| dx dt \leq C \limsup_{m \to \infty} (\lambda(m, f, r) r |O_m|).
\]

Another tool that we use is a simple lemma concerning the stability of the constitutive equations (represented as maximal monotone \( r \)-graphs) with respect to weakly converging sequences; see [19] for a short proof.

**Lemma A.6 (Graph convergence lemma).** Let \( D \subset \mathbb{R}^M \) be an arbitrary measurable set and let a graph \( \mathcal{G} \) fulfill the assumptions \((\mathcal{G}2)\) and \((\mathcal{G}3)\) on page 21. Assume that, for some \( r \in (1, \infty) \),

\[
(S^n, D^n) \in \mathcal{G} \quad \text{almost everywhere in } D,
\]

\[
D^n \rightharpoonup D \quad \text{weakly in } L^r(D)^{d \times d},
\]

\[
S^n \rightharpoonup S \quad \text{weakly in } L^\infty_{r^*}(D)^{d \times d},
\]

\[
\limsup_{n \to \infty} \int_D S^n : D^n \leq \int_D S : D.
\]

Then

\[
(S, D) \in \mathcal{G} \quad \text{almost everywhere in } D.
\]

Next, we state a theorem due to Nečas [65]; the following version is from [5, Corollary 2.5 ii)].

**Lemma A.7 (Nečas theorem).** Let \( \Omega \subset \mathbb{R}^M \) be a domain of class \( C^{0,1} \). Let \( r \in (1, \infty) \). Then there exists \( \beta > 0 \) such that

\[
\|\nabla \varphi\|_{(W^{1,r})^*} := \sup_{\varphi \in W^{1,r}} \frac{(q, \nabla \varphi)}{\|\nabla \varphi\|_r} \geq \beta \|q\|_r \quad \text{for all } q \in L^r(\Omega) \text{ with } \int_\Omega q = 0.
\]

**Remark A.8.** Lemma A.7 is closely related to the results known as the Lions lemma (coined in [55]), the Babuška-Aziz inequality, or the Ladyzhenskaya-Babuška-Brezzi condition (see [7, 14]). If we set \( L^p := \{q \in L^p(\Omega), \int_\Omega q = 0\} \), the statement of Lemma A.7 can also be rephrased as the following:

(i) the gradient operator \( \nabla : L^0_{r^*} \to (W^{1,r})^* \) is injective with closed range,

(ii) the divergence operator \( \text{div} : W^{1,r}_0 \to L^0_{r^*} \) is surjective and has a continuous right inverse, i.e., there is a bounded linear operator \( B : L^0_{r^*} \to W^{1,r}_0 \) such that \( \text{div} B \) is identity on \( L^0_{r^*} \).

The operator \( B \) is usually called the Bogovski˘i operator due to the explicit construction by Bogovski˘i [10, 11]. We refer the reader to [5], where these relations are discussed in detail.
Remark A.9. It is shown in [5, Proposition 2.10 ii)] that for the validity of the estimate of Lemma A.7 it is sufficient to assume a priori that \( q \in (C^\infty_0(\Omega))^* \), \( \int_\Omega q = 0 \), and \( \nabla q \in (W_0^{1,r})^* \). Then, provided that \( \Omega \) is Lipschitz, \( q \in L^r(\Omega) \) and the estimate (A.1) holds.

Remark A.10. The Lipschitz condition on \( \Omega \) in Lemma A.7 can be weakened, see for example [25].

Now we mention few regularity results for the Neumann problem and the Stokes system.

Lemma A.11 \((W^{2,q}\text{-regularity of Neumann-Poisson problem})\). Let \( \Omega \) be a domain of class \( C^{1,1} \). Let \( 1 < q < \infty \) be given. Then there exists \( C > 0 \) such that for every \( f \in L^q(\Omega) \) with \( \int_\Omega f = 0 \) there is a weak solution \( u \in W^{1,q}(\Omega) \) of the problem

\[
\begin{align*}
(A.2a) & \quad -\Delta u = f \quad \text{in } \Omega, \\
(A.2b) & \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega, \\
(A.2c) & \quad \int_\Omega u = 0
\end{align*}
\]

fulfilling \( u \in W^{2,q}(\Omega) \), \( \nabla u \in W_n^{1,q} \), and

\[ ||\nabla^2 u||_q \leq C ||f||_q. \]

Proof of Lemma A.11 is outlined in [6, Remark 3.2] and invokes [3, 37, 52]. As a consequence of the lemma, we get the following result concerning the Helmholtz decomposition for functions from \( W_n^{1,q} \).

Corollary A.12 \((\text{Helmholtz decomposition})\). Let \( \Omega \) be a domain of class \( C^{1,1} \). Let \( 1 < q < \infty \) be given. Then there exists \( C > 0 \) such that the following holds. For every \( \varphi \in W_n^{1,q} \) there exists a couple \((\phi, \varphi_0)\) fulfilling

\[
\begin{align*}
\phi & \in W^{2,q}(\Omega), \\
\nabla \phi & \in W_n^{1,q}, \\
\varphi_0 & \in W_n^{1,\text{div}}, \\
\varphi & = \nabla \phi + \varphi_0, \\
||\nabla^2 \phi||_q + ||\nabla \varphi_0||_q & \leq C ||\nabla \varphi||_q.
\end{align*}
\]

The following lemma contains certain regularity results for the Stokes system, see [31, Theorem 2.1], [5, Proposition 4.3].

Lemma A.13 \((\text{Regularity of the Stokes system})\). Let \( \Omega \subset \mathbb{R}^M \) be a domain and \( 1 < q < \infty \) be given.

If \( \Omega \) is of class \( C^{0,1} \) with sufficiently small Lipschitz constant \( \lambda > 0 \) \((\text{i.e., } \lambda \leq L_0 \text{ with } L_0 > 0 \text{ depending only on } M \text{ and } q)\) or \( \Omega \) is of class \( C^1 \) then there exists \( C_0 > 0 \) \((\text{depending on } M, \, q)\) such that for every \( b \in (W_0^{1,q})^* \) there is a unique weak solution \((v, p) \in W_n^{1,q} \times L^q(\Omega)\) of the problem

\[
\begin{align*}
(A.3a) & \quad -\Delta v + \nabla p = b \quad \text{in } \Omega, \\
(A.3b) & \quad \text{div } v = 0 \quad \text{in } \Omega, \\
(A.3c) & \quad v = 0 \quad \text{on } \partial \Omega
\end{align*}
\]

and the following estimate holds true

\[
\|\nabla v\|_q + ||p||_q \leq C_0 ||b||_{(W_0^{1,q})^*}.
\]
Furthermore, if $\Omega$ is of class $C^{1,1}$ then there exists $C_1 > 0$ (depending on $\Omega, M, q$) such that for every $b \in L^q(\Omega)^3$ the unique weak solution $(v, p) \in W_0^{1, q} \times L^q(\Omega)$ of the problem (A.3) fulfills additionally $v \in W^{2, q}, p \in W^{1, q}(\Omega)$, and admits the estimate
\[(A.5) \quad \|\nabla^2 v\|_q + \|\nabla p\|_q \leq C_1 \|b\|_q.
\]

Proof. The first part of the lemma is exactly the statement [31, Theorem 2.1]. This statement guarantees existence of unique $(v, p) \in W_0^{1, q} \times L^q(\Omega)$ and gives (A.4) under the aforementioned conditions.

The second part, i.e., the inclusions $v \in W^{2, q}, p \in W^{1, q}(\Omega)$ and the estimate (A.5) follow from [5, Proposition 4.3]. Remark 4.4 therein warns that Proposition 4.3 ibid.

Appendix B. Examples of maximal monotone graphs. Let us consider a graph $G \subset \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3}$ characterized by the relationship
\[(B.1) \quad (S, D) \in G \iff S = \left( \frac{|D| - \delta_s}{|D|} \right) S(|D|) D
\]
with
\[(B.2a) \quad \text{either} \quad S(d) = (1 + d^2)^{r-2},
\]
\[(B.2b) \quad \text{or} \quad S(d) = 1 + d^{r-2}.
\]
We will prove the following statement.

**Lemma B.1.** The graph $G$ characterized by (B.1) and (B.2a) with some $\delta_s \geq 0$ and $r \in (1, \infty)$ is a maximal monotone $r$-graph fulfilling (G1)-(G4).

**Proof.** (i). Clearly $(O, O) \in G$.

(ii). Let $S = \frac{(|D| - \delta_s)^+}{|D|} (1 + |D|^2)^{r-2} D$ and $D_s \equiv D_2 + s(D_1 - D_2)$ for any $D_1, D_2 \in \mathbb{R}^{3 \times 3}$. Then
\[
(S(D_1) - S(D_2)) : (D_1 - D_2)
\]
\[
= (D_1 - D_2) : \int_0^1 \frac{d}{ds} \left[ \frac{(|D_s| - \delta_s)^+}{|D_s|} (1 + |D_s|^2)^{\frac{r-2}{2}} D_s \right] ds
\]
\[
= |D_1 - D_2|^2 \int_0^1 \left( \frac{|D_s| - \delta_s}{|D_s|} \right)^+ (1 + |D_s|^2)^{\frac{r-2}{2}} ds
\]
\[
+ \int_0^1 (D_s : (D_1 - D_2))^2 \left\{ H \left( |D_s| - \delta_s \right) \frac{(1 + |D_s|^2)^{\frac{r-2}{2}}}{|D_s|^2} + (r - 2) \frac{(|D_s| - \delta_s)^+}{|D_s|} (1 + |D_s|^2)^{\frac{r-4}{2}} - \frac{(|D_s| - \delta_s)^+}{|D_s|^3} (1 + |D_s|^2)^{\frac{r-2}{2}} \right\} ds.
\]
Since $H \left( |D_s| - \delta_s \right) - \frac{(|D_s| - \delta_s)^+}{|D_s|} = \frac{\delta_s}{|D_s|}$ if $|D_s| > \delta_s$ (otherwise it is zero), we observe that for $r \geq 2$
\[
(S(D_1) - S(D_2)) : (D_1 - D_2) \geq 0.
\]
If \( r \in (1, 2) \), then the property (G2) follows as well from the fact

\[
|D_1 - D_2|^2 \int_0^1 \frac{(|D_s| - \delta_s)^+}{|D_s|} (1 + |D_s|^2)^{\frac{\nu}{2}} \, ds
\]

\[
- (2 - r) \int_0^1 (D_s : (D_1 - D_2))^2 \frac{(|D_s| - \delta_s)^+}{|D_s|} (1 + |D_s|^2)^{\frac{\nu}{2}} \, ds
\]

\[
\geq |D_1 - D_2|^2 \int_0^1 \frac{(|D_s| - \delta_s)^+}{|D_s|} (1 + |D_s|^2)^{\frac{\nu}{2}} (1 + |D_s|^2 - (2 - r)|D_s|^2) \, ds
\]

\[
\geq (r - 1)|D_1 - D_2|^2 \int_0^1 \frac{(|D_s| - \delta_s)^+}{|D_s|} (1 + |D_s|^2)^{\frac{\nu}{2}} \, ds \geq 0.
\]

This also implies the monotone property for the graph with \( S(d) = d^{r-2} \) and \( r > 1 \). Consequently, the same is true for \( S(d) = 1 + d^{r-2} \).

(iii). In order to show that the graph is a maximal monotone graph, we note that the assumption: \( (S, D) \in \mathbb{R}^{3\times 3}_{\text{sym}} \times \mathbb{R}^{3\times 3}_{\text{sym}} \)

\[
(S - \tilde{S}, D - \tilde{D}) \geq 0 \quad \text{for all } (\tilde{S}, \tilde{D}) \in \mathcal{G}
\]

implies, using (B.1) and (B.2), that

\[
\left( S - \frac{(D - \delta_s)^+}{|D|} S(|D||D) : (D - \tilde{D}) \right) \geq 0 \quad \text{for all } \tilde{D} \in \mathbb{R}^{3\times 3}_{\text{sym}}.
\]

Taking \( \tilde{D} = D \pm \lambda A, A \) arbitrary, \( \lambda > 0 \), we conclude from (B.3) that

\[
\exists A : \left( S - \frac{(D - \delta_s)^+}{|D|} S(|D||D) : (D - \tilde{D}) \right) \geq 0.
\]

Letting \( \lambda \to 0^+ \), we finally obtain (using continuity of the involved functions)

\[
\left( S - \frac{(D - \delta_s)^+}{|D|} S(|D||D) : A \right) = 0 \quad \text{for all } A \in \mathbb{R}^{3\times 3}_{\text{sym}}.
\]

Hence \( S \) and \( D \) fulfill the right-hand side of (B.1) and thus \( (S, D) \in \mathcal{G} \).

(iv). Assume that \( \delta_0 > 0 \). For \( d \geq 0 \) we have

\[
\min \left\{ \frac{1(1 + (2\delta_s)^2)^{\frac{\nu}{2}}}{(2\delta_s)^{r-2}} \right\} H(d - 2\delta_s)d^{r-1} \leq (1 + d^2)^{\frac{\nu}{2}} d \leq (1 + d^r)^{\frac{\nu}{2}}
\]

\[
(2\delta_s)^{-r-1} \leq (\max\{d, 2\delta_s\})^{-r-1},
\]

\[
d^{q-1} \leq (2\delta_s)^{r-2}(\max\{d, 2\delta_s\})^{q-1} + (2\delta_s)^{-q}(\max\{d, 2\delta_s\})^{q-1}
\]

where \( q = \max\{r, 2\} \), \( H(t) = 1 \) for \( t > 0 \), \( H(t) = 0 \) otherwise. Let us define

\[
q := \begin{cases} r & \text{case (B.2a)}, \\ \max\{r, 2\} & \text{case (B.2b)}. \end{cases}
\]
Due to (B.4) we have, for the both cases in (B.2),

\[(B.5) \quad C_1(\delta_*, r) H(|D| - 2\delta_*) |D|^{q-1} \leq S(|D|)|D| \leq C_2(\delta_*, r) \left( \max\{|D|, 2\delta_*\} \right)^{q-1}\]

with certain $C_1(\delta_*, r), C_2(\delta_*, r) > 0$ independent of $D$. Notice also that

\[(B.6) \quad \frac{1}{2} H(|D| - 2\delta_*) \leq \frac{(|D| - \delta_*)^+}{|D|} \leq 1.\]

Now define $S = \frac{(|D| - \delta_*)^+}{|D|} S(|D|)|D|$ and observe that with the help of the right-wing inequalities of (B.5) and (B.6) we obtain

\[|D|^q + |S|^q \leq |D|^q + \left( C_2(\delta_*, r) \left( \max\{|D|, 2\delta_*\} \right)^{q-1} \right)^q \leq C_3(\delta_*, r) \left( \max\{|D|, 2\delta_*\} \right)^q\]

with certain $C_3(\delta_*, r) > 0$ independent of $D$ and $S$. Hence, with the help of the left-wing inequalities of (B.5) and (B.6),

\[C_3(\delta_*, r)^{-1} \left( |D|^q + |S|^q \right) - (2\delta_*)^q \leq \left( \max\{|D|, 2\delta_*\} \right)^q - (2\delta_*)^q \leq H(|D| - 2\delta_*) |D|^q \leq 2C_1(\delta_*, r)^{-1} \frac{(|D| - \delta_*)^+}{|D|} S(|D|)|D|^2 \leq 2C_1(\delta_*, r)^{-1} S : D\]

which is the last property (G4). We leave the case $\delta_* = 0$ as an exercise. \[\square\]

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