ON AN ANALOGUE OF A BRAUER THEOREM FOR FUSION CATEGORIES

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Abstract. In this paper we prove an analogue of Brauer’s theorem for faithful objects in fusion categories. Other notions, such as the order and the index associated to faithful objects of fusion categories are also discussed. We show that the index of a faithful simple object of a fusion categories coincides to the order of the universal grading group of the fusion category.

1. Introduction

In representation theory of finite groups a $\mathbb{C}G$-module $M$ is called faithful if its kernel $\ker_{\mathbb{C}G}(M)$ is trivial. A celebrated theorem of Brauer states that in this case any other representation of $\mathbb{C}G$ can be found as a constituent of at least one tensor power $M$. The goal of this paper is to generalize this result to fusion categories with commutative Grothendieck rings. We should also mention the fact that in the literature this theorem is also called Burnside-Brauer theorem.

Grothendieck rings of tensor and fusion categories were recently intensively studied by various authors. In this paper we give some new properties for the Grothendieck rings of the fusion categories generated by a single object. We say that an object $X \in \mathcal{O}(\mathcal{C})$ is faithful if the fusion subcategory $\mathcal{C}(X)$ generated by $X$ coincides with the whole category $\mathcal{C}$. Faithful objects were considered previously in the literature, see for example [17]. One of the main goals of this paper is to formulate a result analogue to Brauer’s theorem for faithful representations.

Recently, Brauer’s theorem for representations of finite groups was generalized to Hopf algebras by several authors ([13, 21, 3]). The starting point of all these generalizations was the results obtained by Rieffel in [22]. In all these references the notion of a faithful module is considered without defining the notion of kernel of a representation. More recently, the author defined the notion of left kernel and right kernel for modules over Hopf algebras.

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In this paper we propose a notion of kernel for objects in arbitrary fusion categories. We show that an analogue of Brauer’s theorem holds in the case of fusion categories with a commutative Grothendieck ring. In [17] the author has shown that if a fusion category has a faithful object then the universal grading group of \( C \) is cyclic. In the present paper we show that the order of this cyclic group equals the index of the object \( X \). Analogue to [13], one can define the index of any object of \( C \) as the index of the imprimitivity of the matrix associated to the operator \( l_X \) of left multiplication by \( [X] \) on the Grothendieck ring \( K_0(C) \). For more details see Section 4.

This paper is organized as follows. In Section 2 we recall the basics from the theory of fusion categories that are needed through the paper. In the next section we introduce the notion of kernel of an object in a fusion category and prove an analogue of Brauer’s theorem for fusion categories. We also introduce the notion of the center of a character and present some properties of the index of a character. In Section 4 we discuss the notion of index of an object and also introduce the related concept of center of an object of a fusion category. As an example, in Section 5 we revisit the case of semisimple Hopf algebras and we show how the new notion of kernel relates to the previous notion of left (right) kernel introduced by the author in [4]. Applications of Brauer’s theorem to the case of a modular fusion category \( C \) are presented in the last section of the paper.

All fusion categories and algebras in this paper are considered over an algebraically closed base field \( \kappa \) of characteristic zero.

2. Preliminaries

As usually, by a fusion category we mean a \( \kappa \)-linear semisimple rigid tensor category \( C \) with finitely many isomorphism classes of simple objects, finite dimensional spaces of morphisms, and such that each simple object \( S \) is scalar (that is, \( \text{End}_C(S) = \kappa \)) and the unit object \( 1 \) of \( C \) is simple. We refer the reader to [7] for basics on fusion categories. We denote by \( O(C) \) the class of all objects of \( C \) and by \( \text{Irr}(C) \) the set of isomorphism classes of simple objects of \( C \).

A fusion subcategory of a given category is a full monoidal replete subcategory which is also fusion category. In a fusion category \( C \), the left and right duals \( X^* \) and \( X^* \) of an object \( X \) are isomorphic (but it is still not known whether there always exists a sovereign structure, that is a natural monoidal isomorphism between the two duals). Recall that the Grothendieck group \( K_0(C) \) of a fusion category \( C \) is the free \( \mathbb{Z} \)-module with basis given by the isomorphism classes of
simple objects of $\mathcal{C}$. It is a well known result now (see also [7]) that $K_0(\mathcal{C})$ is a semisimple based ring with the ring structure endowed from the monoidal structure of $\mathcal{C}$. By abuse of notation, if no confusion, we sometimes may write $X \in K_0(\mathcal{C})$ instead of $[X] \in K_0(\mathcal{C})$.

2.1. Frobenius-Perron dimension of objects. If $A$ is a matrix with nonnegative entries then, by Frobenius-Perron theorem, the matrix $A$ has a positive eigenvalue $\lambda$ which has the biggest absolute value among all the other eigenvalues of $A$ (see [8]). The eigenspace corresponding to $\lambda$ has a unique vector with all entries positive. The eigenvalue $\lambda$ is called the principal value of $A$ and will also be denoted by $\text{FPdim}(A)$. The corresponding positive eigenvector of $\lambda$ is called the principal vector of $A$. Also the eigenspace of $A$ corresponding to $\lambda$ is called the principal eigenspace of the matrix $A$.

Let $\mathcal{C}$ be a fusion category and $K(\mathcal{C}) := k \otimes \mathbb{Z} K_0(\mathcal{C})$ be its Grothendieck ring. For an object $X \in \mathcal{O}(\mathcal{C})$ let $l_{[X]}$ be the operator of left multiplication by $[X]$ on $K(\mathcal{C})$. Also let $a(X)$ be the matrix associated to the operator $l_{[X]}$ with respect to the basis of $K(\mathcal{C})$ given by the simple objects of $\mathcal{C}$. Then by definition one has that $\text{FPdim}(X) := \text{FPdim}(a(X))$ (see also [7]). We denote by $\text{FPdim}(\cdot) : K_0(\mathcal{C}) \to k$ the unique ring homomorphism for which $\text{FPdim}(X) > 0$ for any simple object $X$ of $\mathcal{C}$, see [7]. We also denote by $X(\mathcal{C})$ the set of all unitary ring homomorphisms $\text{FPdim}(\cdot) : K_0(\mathcal{C}) \to k$.

2.2. The symmetric associative bilinear form on $K_0(\mathcal{C})$. Let $\mathcal{C}$ be a fusion category and as above $K(\mathcal{C}) := C \otimes \mathbb{Z} K_0(\mathcal{C})$ be its Grothendieck ring.

It is well known that $K(\mathcal{C})$ is a semisimple $k$-algebra. Since $k$ is algebraically closed we may suppose that $K(\mathcal{C}) \simeq M_{n_0}(k) \times M_{n_1}(k) \cdots \times M_{n_s}(k)$

Let $E_0, E_1, \ldots, E_s$ be a complete set of primitive central idempotents of $K(\mathcal{C})$ and let $\mu_0, \ldots, \mu_s$ be the corresponding irreducible characters of $K_0(\mathcal{C})$. Without loss of generality we may assume that $E_0 = \frac{1}{\text{FPdim}(\mathcal{C})} \mathcal{R}_{\mathcal{C}}$ where $\mathcal{R}_{\mathcal{C}}$ is the virtual regular element of $\mathcal{C}$ defined by

$$\mathcal{R}_{\mathcal{C}} = \sum_{X \in \Lambda_{\mathcal{C}}} \text{FPdim}(X)[X].$$

Then the corresponding character of $E_0$ is $\mu_0 = \text{FPdim}(\cdot)$ given by $[X] \mapsto \text{FPdim}(X)$ and $n_0 = 1$.

Denote by $m_{\mathcal{C}}$ the $k$-bilinear form on the Grothendieck ring $K(\mathcal{C})$ defined on the generators $X, Y \in \text{Irr}(\mathcal{C})$ by $m_{\mathcal{C}}(X, Y) = \dim_k \text{Hom}(X, Y)$ and then extended linearly on both arguments.
The bilinear form $m_{\mathcal{C}}$ has the following properties:

1. symmetry: $m_{\mathcal{C}}(x, y) = m_{\mathcal{C}}(y, x)$;
2. adjunction property: $m_{\mathcal{C}}(x, yz) = m_{\mathcal{C}}(y^*, x) = m_{\mathcal{C}}(xz^*, y)$

Define also $B: K_0(\mathcal{C}) \otimes K_0(\mathcal{C}) \to \mathbb{k}$ by $B(x, y) := m_{\mathcal{C}}(1, xy)$. Then it is easy to check that $B$ is a symmetric associative nondegenerate bilinear form on $\mathcal{K}(\mathcal{C})$. It follows that there are scalars $b_i \in \mathbb{k}^*$ such that

\begin{equation}
B(x, y) = \sum_{i=0}^{s} b_i \mu_i(xy)
\end{equation}

Moreover the dual bases equation imply that

\begin{equation}
\sum_{i=1}^{s} \frac{1}{b_i} E_{ik}^i \otimes E_{kj}^i = \sum_{X \in \Lambda_{\mathcal{C}}} [X] \otimes [X^*]
\end{equation}

where $E_{ik}^i$ are the elementary matrix entries in $M_{n_i}(\mathbb{k})$. Note that $b_i \neq 0$ since the form is nondegenerate.

It follows from \cite{19} that $\frac{1}{b_i} = f_{\mu_i}$ is the formal codegree of the irreducible representation of $\mathcal{K}(\mathcal{C})$. Therefore $f_{\mu_i}$ is a real positive number.

Note that the above equalities also show that:

\begin{equation}
m_{\mathcal{C}}([X], [Y]) = B([X], [Y]^*) = \sum_{i=1}^{s} b_i \mu_i([X][Y]^*)
\end{equation}

for any $X, Y \in \mathcal{O}(\mathcal{C})$.

3. **Brauer’s theorem and faithful objects in fusion categories**

3.1. **Frobenius-Perron eigenvalues.** Let $\mathcal{C}$ be a fusion category with the Grothendieck ring $K_0(\mathcal{C})$ commutative. Let $\psi: \mathcal{K}(\mathcal{C}) \to \mathbb{k}$ be a linear character and $E \in \mathcal{K}(\mathcal{C})$ be its primitive central idempotent. Then since $E$ is an eigenvector for $l_{[X]}$ it follows that $|\psi([X])| \leq \text{FPdim}(X)$ for any object $X \in \mathcal{O}(\mathcal{C})$. Indeed $[X]E = \psi([X])E$ and therefore $E$ is an eigenvector for the operator $l_{[X]}$, left multiplication by $[X]$.

**Definition 3.1.** Define $\ker_{\mathcal{C}}(\psi)$ as the set of all simple objects $X$ such that $\psi([X]) = \text{FPdim}(X)$.

**Remark 3.2.** It is known that $\psi(X) \in \mathbb{Q}(\xi)$ for some root of unity $\xi$, see \cite{7} Corollary 8.53.

**Proposition 3.3.** Assume that the Grothendieck ring $K_0(\mathcal{C})$ is commutative. If $X$ and $Y$ have a principal eigenvector $V \in K_0(\mathcal{C})$ then any simple object of the fusion subcategory generated by $X$ and $Y$ have $V$ as a principal eigenvector.
Proof. Suppose that $V = \sum_{i=0}^{s} \mu_i(V) E_i$. Since $XV = \text{FPdim}(X)V$ it follows that $\text{FPdim}(X) = \mu_i(X)$ for all $i$ with $\mu_i(V) \neq 0$. Therefore it is enough to consider the case $V = E_i$ some idempotent of $K(C)$.

Suppose that $X, Y \in \ker_C(\psi)$ and let
\begin{equation}
X \otimes Y = \bigoplus_{Z} N_{X,Y}^Z Z
\end{equation}
If $|\psi([X])| = \text{FPdim}(X)$ and $|\psi([Y])| = \text{FPdim}(Y)$ then
\[|\psi(X \otimes Y)| = |\bigoplus_{Z} N_{X,Y}^Z \psi(Z)| \leq \bigoplus_{Z} N_{X,Y}^Z \text{FPdim}(Z) = \text{FPdim}(X \otimes Y)\]
Since $\psi$ is an algebra homomorphism it follows that $\psi(X \otimes Y) = \text{FPdim}(X) \text{FPdim}(Y)$ and the previous inequality shows that $\psi(Z) = \text{FPdim}(Z)$ for all $Z$ with $N_{X,Y}^Z \neq 0$. □

Similarly one can prove the following:

**Corollary 3.5.** Assume that the Grothendieck ring $K_0(C)$ is commutative. If $[X]V = \xi \text{FPdim}([X])V$ for some object $X \in O(C)$ and some root of unity $\xi$ then $[Y]V = \xi \text{FPdim}([Y])V$ for any simple subobject $Y$ of $X$.

Proposition 3.3 implies:

**Lemma 3.6.** The set $\ker_C(\psi)$ is closed under tensor products and therefore it spans a fusion subcategory of $C$.

**Example 1.** Let $G$ be a finite group and $C = \text{Rep}(G)$. Then $K(C) = C(G)$, the character ring of $G$. Any ring homomorphism of $C(G)$ corresponds to a conjugacy class $X$ of $G$. It is given by $\hat{X}$ which is the characteristic function of $X$ on $G$. Then clearly $\ker_C(\hat{X}) = \ker_{kG^*}(g)$ where $g \in X$ is an arbitrary group element of $X$.

**Definition 3.7.** For any object $X \in O(C)$ define
\begin{equation}
(3.8) \quad \ker_C([X]) := \{ \psi \in \text{Irr}(K(C)) \mid [X] \in \ker_C(\psi) \}
\end{equation}

Note that for any object $X \in O(C)$ one has that $\text{FPdim}(\cdot) \in \ker_C(X)$.

We say that an object $X$ has a trivial kernel if $\ker_C([X]) = \{ \text{FPdim}(\cdot) \}$. Next theorem can be regarded as a generalization of Brauer’s Theorem.

**Theorem 3.9.** Let $C$ be a fusion category with commutative Grothendieck ring. If $X$ is an object of $C$ with trivial kernel then any other object of $C$ is a subobject of some tensor power of $X$.

**Proof.** Using Equation (2.2) the proof follows the lines of the proof from the classical case of group representations, see for example [11]...
Theorem 19.10. Note that if $\mathcal{K}(\mathcal{C})$ is a commutative ring then Equation (2.2) implies that

$$B(x, y) = \sum_{i=0}^{s} b_i \mu_i(x) \mu_i(y)$$

for any $x, y \in \mathcal{K}(\mathcal{C})$.

Suppose that

$$[X] = \sum_{i=0}^{s} \mu_i([X]) E_i$$

and let $Y \in \text{Irr}(\mathcal{C})$ with

$$[Y] = \sum_{i=0}^{s} \mu_i([Y]) E_i$$

Since $\ker\mathcal{C}(X) = \{\text{FPdim}()\}$ it follows that $\mu_i([X]) \neq \text{FPdim}(X)$ for $i \geq 1$. If $Y$ is not a direct summand in any tensor power of $X$ then by Equation (3.10) one has that:

$$\sum_{i=0}^{s} \mu_i([X])^{n} \mu_i([Y]) = 0.$$ 

for any $n \geq 1$. Denote by $\mathcal{A}_i$ the set of all indices $0 \leq j \leq r$ with $\mu_j([X]) = \mu_i([X])$. It follows that $\mathcal{A}_0 = \{0\}$.

Then by taking a van der Monde determinant, Equation (3.11) implies that each $\sum_{j \in \mathcal{A}_X} n_j \mu_j(Y) = 0$. Note that this is impossible for $i = 0$ since $n_0 \text{FPdim}(Y) \neq 0$. \hfill \Box

3.2. On the Grothendieck ring of $\mathcal{C}(X)$. Let $\mathcal{C}$ be a fusion category with commutative Grothendieck ring $\mathcal{K}(\mathcal{C})$. We define $\mathcal{D} := \mathcal{C}(X)$. Clearly $\mathcal{K}(\mathcal{D}) \subset \mathcal{K}(\mathcal{C})$ and therefore one can write the primitive central idempotent $\frac{r_D}{\text{FPdim}(\mathcal{D})}$ as sum of central primitive idempotents of $\mathcal{K}(\mathcal{C})$:

$$\frac{r_D}{\text{FPdim}(\mathcal{D})} = \sum_{j \in \mathcal{A}_X} E_j$$

for a subset $\mathcal{A}_X \subseteq \{0, 1, \cdots, s\}$.

**Proposition 3.12.** With the above notations one has that

$$\ker\mathcal{C}(X) = \{\mu_j \mid j \in \mathcal{A}_X\}$$

*Proof.* Let $F_0, F_1, \cdots, F_r$ be the primitive idempotents of $\mathcal{K}(\mathcal{D})$ and $\chi_0, \cdots, \chi_r$ be their associated irreducible characters. Moreover as above one may suppose that $F_0 = \frac{r_D}{\text{FPdim}(\mathcal{D})}$ and consequently $\chi_0 = \text{FPdim}()$. 

We look at the restrictions of the characters $\mu_i$ at $K(D)$. Then there is a surjective function $f : \{0, \ldots, s\} \to \{0, \ldots, r\}$ such that $\mu_j|_{K(D)} = \chi_{f(j)}$ for all $0 \leq j \leq s$.

With the above notations it will be shown that $f^{-1}(0) = A_X$. Indeed, since $X$ is a faithful object of $D$ it follows that $\ker D(X) = \{\mu_0\}$. On the other hand note that $\chi_j([X]) = \chi_{f(j)}([X]) = \mu_{f(j)}([X])$. Thus $\chi_j \in \ker C(X)$ if and only if $\mu_{f(j)} \in \ker D(X)$, i.e. $f(j) = 0$. \qed

One can also formulate the converse of the Brauer’s theorem in categorical settings:

**Proposition 3.14.** If a fusion category $C$ with the Groethendieck ring $K(C)$ commutative is generated by a single object $X$ then $\ker C(X) = \{\text{FPdim}(X)\}$

*Proof.* Suppose that $\mu_i([X]) = \text{FPdim}(X)$ for some $1 \leq i \leq s$. Then Lemma [3.6] implies that $\mu_i(Y) = \text{FPdim}(Y)$ for any object $Y \in O(C)$ and therefore $\mu_i = \text{FPdim}(X)$. \qed

**Example 2.** Let $G$ be a finite group and $C = \text{Rep}(G)$. Then as in Example [1] one has $K(C) = C(G)$. It follows that $\chi \in \text{Irr}(G)$ is faithful in the above sense if and only if the kernel of $\chi$ is trivial, i.e. it is faithful in the classical sense.

4. **The Index of a Simple Object and its Center**

The second part of Frobenius-Perron theorem states that for a non-negative indecomposable matrix $A$ any other eigenvalue $\mu$ of $A$ such that $|\mu| = \text{FPdim}(A)$ has a unique (up to a scalar) corresponding eigenvector. The number of such eigenvalues $\mu$ with $|\mu| = \text{FPdim}(A)$ is called the *imprimitivity index* of $A$. Moreover, if $\xi$ is a primitive root of unity of order equal to the index $A$ then $\mu = \xi^i\text{FPdim}(A)$ for some $0 \leq i \leq \text{ind}(A) - 1$.

Let $C$ be a fusion category. One can define the *index* of an object as the index of the imprimitivity of the matrix $a(X)$ associated to the operator $l_{[X]}$ on $K(C)$ with respect to the canonical basis of $K(C)$ given by the simple objects of $C$.

Similarly to [13] one can show that an object $X \in O(C)$ is faithful if and only if the associated matrix $a(X)$ of $l_{[X]}$ is indecomposable. Recall [8] that a $n \times n$-matrix $A = (a_{ij})$ is called *decomposable* if it is possible to find a decomposition of $\{1, 2, \cdots, n\} = M \cup N$ of into disjoint nonempty sets $M$ and $N$ such that $a_{ij} = 0$ whenever $i \in M$ and $j \in N$. Otherwise the matrix $A$ is called *indecomposable*. 
Theorem 4.1. Let $\mathcal{C}$ be a fusion category. Suppose that a simple object $Y$ of $\mathcal{C}$ is a constituent of some tensor powers $X^\otimes m$ and $X^\otimes n$ of $X$ with $m, n \geq 0$. Then $m - n$ is divisible by $\text{ind}(X)$.

Proof. One may assume that $\mathcal{C} = \mathcal{C}(X)$. Let $\xi$ be a primitive root of unity of order $\text{index}(X)$. Let also $v \in K(\mathcal{C})$ be the unique (up to a scalar) eigenvector for the linear operator $l_{[X]}$ corresponding to the eigenvalue $\xi \text{FPdim}(X)$. It follows that for any other object $Z \in \mathcal{O}(\mathcal{C})$ one has that $v[Z]$ is also an eigenvector for $l_{[X]}$ corresponding to the same eigenvalue $\xi \text{FPdim}(X)$. Therefore, by the above discussion there is a scalar $\gamma([Z]) \in k$ such that $v[Z] = \gamma([Z])v$ for all $Z \in \mathcal{O}(\mathcal{C})$.

Clearly $\gamma : \mathcal{K}(\mathcal{C}) \to k$ is an algebra homomorphism. Since $\mathcal{K}(\mathcal{C})$ is a semisimple algebra it follows that $v$ is a central element of $\mathcal{K}(\mathcal{C})$. Therefore one can write that
$$v = \sum_{i \in A} v_i E_i,$$
for some subset $A \subseteq \{0, 1, \cdots, s\}$ and $v_i \neq 0$ for $i \in A$.

Since $v$ is unique up to a scalar it follows that $[X]E_i \neq 0$ for any $i \in A$. Therefore $A$ should be a set with one element. We may suppose without loss of generality that $A = \{1\}$. Therefore one can write $[X]^m E_1 = \xi^m \text{FPdim}(X)^m E_1$. Then Lemma 3.5 implies that $\mu_1([Y]) = \xi^m \text{FPdim}(Y)$ since $Y$ is a constituent of $X^\otimes m$. On the other hand the same lemma implies that $\mu_1([Y]) = \xi^n \text{FPdim}(Y)$ since $Y$ is also constituent of $X^\otimes n$. Thus $m - n$ is divisible by the index of $X$. \hfill \Box

Note that in [17, Theorem 4.1] it was proven that the universal grading group $U(\mathcal{C}(X))$ is cyclic. Next theorem gives a precise description for the order of this cyclic group. It can also be seen as a generalization of the result [13, Proposition 4.5].

Theorem 4.2. The universal grading group of $\mathcal{C}(X)$ is isomorphic to $\mathbb{Z}_n$ where $n = \text{ind}(X)$.

Proof. As we already mentioned above we know that $U(\mathcal{C}(X))$ is a cyclic group. By [9, Proposition 3.1] it is enough to show that one has $\text{Aut}_{\otimes}(\text{id}_{\mathcal{C}(X)}) \cong \mathbb{Z}_n$. Note that in order to give a $\phi \in \text{Aut}_{\otimes}(\text{id}_{\mathcal{C}(X)})$ it is enough to give for each $X \in \text{Irr}(\mathcal{C})$ a scalar $s(X)$ such that $\phi_X = s(X) \text{id}_X$. Then one can to define
$$\phi_Y = s(X)^m \text{id}_Y$$
for any object $Y \in \mathcal{O}(\mathcal{C}(X))$ with $Y$ a constituent of $X^\otimes m$. By previous theorem it follows that $\phi$ is well defined. Indeed if $Y$ is a direct summand in $X^\otimes m$ and $X^\otimes n$ then $m - n$ is divisible by $\text{ind}(X)$. Moreover
s(X) should be a root of unity of order \(\text{ind}(X)\) since \(X^\text{ind(X)}\) contains 1 and \(\phi_1 : 1 \to 1\) is identity. Conversely, any root of unity of order \(\text{ind}(X)\) determines a unique element \(\phi \in \text{Aut}_\otimes(\text{id}_C(X))\).

**Construction of the universal grading of** \(\mathcal{C}(X)\). Let \(\xi\) be a primitive root of unity of order \(\text{ind}(X)\). As in the proof of Theorem 4.1 let also \(E_1\) be the central idempotent corresponding to the eigenvalue \(\xi\text{FPdim}(X)\) where \(\xi\) is a primitive root of unity of order \(\text{ind}(X)\). Thus

\[ E_1[Y] = \xi^m\text{FPdim}(Y)E_1 \]

for all objects \(Y\) that are constituents of \(X \otimes^m\).

For all \(0 \leq i \leq n - 1\) define the full abelian subcategory \(\mathcal{D}_i\) of \(\mathcal{C}(X)\) by

\[ \mathcal{D}_i := \{ Y \in \text{Irr}(\mathcal{C}(X)) \mid \mu_1([Y]) = \xi^i\text{FPdim}([Y]) \}. \]

Then it can easily check that \(\mathcal{D}_i \otimes \mathcal{D}_j\) is mapped to \(\mathcal{D}_{i+j}\). This shows that the decomposition \(\mathcal{C} = \bigoplus_{i=0}^{n-1} \mathcal{D}_i\) is a grading of \(\mathcal{C}\). By the above arguments this grading should coincide with the universal grading of \(\mathcal{C}(X)\).

4.1. **The order of an object.** As explained in \([13, 17]\) for any object \(X \in \mathcal{O}(\mathcal{C})\) there is a smallest integer \(n \geq 0\) such that \(m_{\mathcal{C}}(1_C, X \otimes^m) > 0\), i.e \(X \otimes^m\) contains the unit object \(1_C\). Then \(n\) is called the order of \(X\) and denoted by \(o(X)\). This also shows that \(X^*\) is also subobject of some tensor power of \(X\). If \(X\) is a self dual object clearly its order is 2.

**Corollary 4.4.** The index of \([X]\) divides the order of \(X\).

**Proof.** If \(X^m\) contains the unit element then under the above grading it follows that \(\xi^m = 1\) and therefore \(\text{ind}(X)\mid \text{ord}(X)\). □

4.2. **The center of an object.**

**Definition 4.5.** Let \(\mathcal{C}\) be a fusion category over \(k\) with a commutative Groethendieck ring and \(X\) be an object of \(\mathcal{C}\). Define

\[ Z_{\mathcal{C}}(X) = \{ \mu_i \in \text{Irr}(\mathcal{K}(\mathcal{C})) \mid |\mu_i([X])| = \text{FPdim}(X) \} \]

Note that similarly to \([25, \text{Proposition 3.1}]\) or \([18, \text{Proposition 10}]\) one has that \(\psi([X^*]) = \psi([X])\) for any simple object \(X\) of \(\mathcal{C}\) and any character \(\psi : \mathcal{K}(\mathcal{C}) \to k\). This implies that for any simple object \(\ker_{\mathcal{C}}(X \otimes X^*) = Z_{\mathcal{C}}(X)\) and therefore

\[ \ker_{\mathcal{C}}(C_{ad}) = \bigcap_{X \in \text{Irr}(\mathcal{C})} Z_{\mathcal{C}}(X) \]
whre $C_{ad} := \bigoplus_{X \in \text{Irr}(C)} X \otimes X^*$. 

For any $0 \leq i \leq \text{ind}(X) - 1$ let $\mu_i$ be the linear character of $\mathcal{K}(C(X))$ corresponding to the unique (up to a scalar) eigenvector of $l_{[X]}$ with eigenvalue $\xi^i \text{FPdim}(X)$ in $\mathcal{K}(C(X))$.

**Corollary 4.7.** With the above notations one has that:

\[(4.8) \quad Z_{C(X)}(X) = \{\mu_0, \ldots, \mu_{\text{ind}(x)}\}.\]

### 5. Examples from semisimple Hopf algebras

Let $A$ be a Hopf algebra over a field $k$. We use the standard Hopf algebra notations that can be found for example in [14]. We denote by $\text{Rep}(A)$ the tensor category of finite dimensional representations of $A$. If $A$ is a semisimple Hopf algebra then it is known that $\text{Rep}(A)$ is a fusion category.

Let $M$ be a finite dimensional left $A$-module. Then the left kernel $\text{Lker}_A(M)$ of a finite dimensional representation $M$ of $A$ is defined by

\[(5.1) \quad \text{Lker}_A(M) = \{a \in A \mid a_1 \otimes a_2 m = a \otimes m \text{ for all } m \in M\}.\]

It follows (see [4]) that $\text{Lker}_A(M)$ is the largest left coideal subalgebra of $A$ that acts trivially on $M$. Recall that a left coideal is a subspace $L$ of $A$ with the property that $\Delta(L) \subseteq A \otimes L$. A *left coideal subalgebra* is a left coideal which also is a subalgebra of $A$.

For a subspace $S$ of $A$ we denote by $S^+$ the subspace $S \cap \ker \epsilon$. If $L$ is a normal coideal subalgebra then it is known that $AL^+$ is a Hopf ideal and therefore $A/AL^+$ is a quotient of Hopf algebra. Via the canonical projection $\pi_L : A \to A//L$ one can view $\text{Rep}(A//L)$ as a fusion subcategory of $\text{Rep}(A)$.

Brauer’s theorem for Hopf algebras, see [31 Theorem 4.2.1] states that $\text{Rep}(A//\text{Lker}_A(M))$ is precisely the fusion subcategory of $\text{Rep}(A)$ generated by the object $M$.

**Proposition 5.2.**

1. Suppose that $\mathcal{D}$ and $\mathcal{E}$ are fusion subcategories of a fusion category $\mathcal{C}$. If $r_D r_E = \text{FPdim}(D)r_E$ then $\mathcal{D} \subseteq \mathcal{E}$.

2. Let $L$ and $K$ be two normal coideal subalgebras of a Hopf algebra $A$. If $AL^+ \subseteq AK^+$ then $L \subseteq K$.

**Proof.** i) The right coset equivalence $r_E^C$ on the set of isomorphism classes of simple objects $\text{Irr}(C)$ (see [3]) implies that any element of $\mathcal{D}$ is equivalent to the unit $1_C$ and therefore it is contained in $\mathcal{E}$. 
ii) If $AL^+ \subseteq AK^+$ then one has a canonical Hopf projection $\pi_{K,L} : A/AL^+ \rightarrow A/AK^+$. Moreover the composition of the Hopf projections $A \xrightarrow{\pi_L} A/AL^+ \xrightarrow{\pi_{K,L}} A/AK^+$ coincides to $\pi_K$. Then following $[23]$ one has that $L = A^{\text{cot}} \subseteq A^{\text{cot},K,L} \circ \pi_L = A^{\text{cot}K} = K$. □

5.1. Central idempotents of the character algebra $C(A)$. Let $A$ be a semisimple Hopf algebra and $D(A)$ be its Drinfeld double. Then $A$ is a $D(A)$-module (see $[26]$) via the scion

$$f \triangleleft a \circ b = (a_1bS(a_2)) \leftarrow S^{-1}f$$

for all $f \in A^*$ and $a, b \in A$.

For a decomposition $A = V \oplus W$ of $A$ as a direct sum of two $D(A)$-submodules define a linear functional $p_V \in A^*$ such that $p_V(v) = \epsilon(v)$ and $p_V(w) = 0$ for all $v \in V$ and all $w \in W$.

Let now

$$A = V_1 \oplus V_2 \cdots \oplus V_s$$

be a decomposition of $A$ as sum of simple $D(A)$-modules. Note that the simple $D(A)$-submodules of $A$ are the minimal normal left coideals of $A$. Recall that a left coideal is called normal if it closed under the left adjoint action i.e. $x_MS(x_2) \subseteq M$ for any $x \in M$.

By $[5$, Theorem 15.3$]$ it follows that $\{p_{V_i}\}$ is a complete set of central primitive orthogonal idempotents of $C(A)$. Recall that the character algebra $C(A)$ is defined as $K_0(\text{Rep}(A)) \otimes \mathbb{C}$.

**Theorem 5.5.** Let $A$ be a Hopf algebra with $C(A)$ a commutative ring (e.g. quasitriangular). Then one has that:

$$\ker_{\text{Rep}(A)}(M) = \{\mu_V \mid V \subset \text{LKer}_A(M)\}$$

**Proof.** Note that if $C(A)$ is a commutative ring the $[5$, Theorem 15.3$]$ implies that a linear basis of $C(A)$ is given by $p_V$ with $V \subset A$ a simple $D(A)$-submodule. Denote by $\mu_V : C(A) \rightarrow k$ the corresponding algebra homomorphism. Also note that any normal left coideal subalgebra $L$ of $A$ can be regarded as a $D(A)$-submodule of $A$.

Suppose now that

$$\chi_M = \sum_{V \subset A} \mu_V(\chi_M)p_V$$

Then it follows that $\chi_M(v) = \epsilon(v)\mu_{V_i}(\chi_M)$ for any $v \in V_i$. 

Suppose now that \( V \subset \text{LKer}_A(M) \). Then \( vm = \epsilon(v)m \) for all \( m \in M \) since any element of the left kernel \( \text{LKer}_A(M) \) acts trivially on \( M \). Therefore \( \chi_M(v) = \epsilon(v)\chi_M(1) \) for any \( v \in V \). This implies that \( \chi_M p_V = \chi_M(1)p_V \) and therefore \( \mu_V(\chi_M) = \chi_M(1) \). Hence \( \mu_V \in \ker_{\text{Rep}(A)}(\chi_M) \).

Recall that the regular character of a semisimple Hopf algebra \( A \) coincides to the integral on the dual with the value at 1 equal to the dimension of \( A \). Using this, by Lemma 3.12 one has that \( \ker_{\text{Rep}(A)}(\chi_M) = \{ \mu_V \mid V \subseteq AL^+ \} \) where \( t \in (A/\text{LKer}_A(M))^* \) is the nonzero integral with \( t(1) = 1 \). Let \( \pi : A \to A/\text{L} \) be the canonical Hopf projection. Note that \( t = p_V \) if and only if \( t(\pi(x)) = \epsilon(x) \) for any \( x \in V \). Indeed on the \( D(A) \) submodule complement of \( V \) one has that both \( tp_V \) and \( p_V \) vanishes.

On the other hand on \( x \in V \) one has that \( p_V t(x) = t(\pi(x)) \) while \( p_V(x) = \epsilon(x) \).

Since \( \pi(x_1) t(\pi(x_2)) = t(\pi(x)) \pi(1) \) one obtains that \( t(\pi(x)) = \epsilon(x) \) for any \( x \in V \) if and only if \( V^+ \subseteq L^+ \). The implies that

\[
\ker_{\text{Rep}(A)}(\chi_M) = \{ \mu_V \mid V^+ \subseteq AL^+ \}
\]

Let \( <V> \) denote the subalgebra of \( A \) generated by \( V \) inside \( A \). Note that \( <V> \) is also a left normal coideal subalgebra of \( A \). In the situation of Equation (5.7) one has \( A < V >^+ \subseteq AL^+ \) and therefore \( <V> \subseteq L \) by Proposition 5.2.

Following [4] from the previous theorem one gets that:

**Corollary 5.8.** Let \( C = \text{Rep}(A) \) be the category of finite dimensional representations of a semisimple Hopf algebra \( A \) and \( M \) a finite dimensional representation of \( A \). Then \( M \) as an object of the fusion category \( \text{Rep}(A) \) is faithful if and only if \( \text{LKer}_A(M) = k \).

**Remark 5.9.** Remark that the proof of the previous theorem shows that

\[
\ker_{\text{Rep}(H)}(\chi_M) \supseteq \{ \mu_V \mid V \subseteq \text{LKer}_H(M) \}
\]

even when the character ring \( C(H) \) is noncommutative.

### 5.2. On the centre of a representation of a Hopf algebra

Let \( A \) be a semisimple Hopf algebra and \( \pi : A \to \text{End}_k(M) \) be a finite dimensional representation of \( A \). Recall that as in [17] one can define a notion of center by

\[
\text{LZ}(\pi) := \text{LKer}_A(\pi \otimes \pi^*)
\]
ON AN ANALOGUE OF A BRAUER THEOREM FOR FUSION CATEGORIES

Then as in [17, Proposition 3.5] $LZ(\pi)$ is the largest left coideal subalgebra of $A$ for which $\pi(a) = \lambda(a)\pi(1)$ for a linear character $\lambda : A \to k$.

Note that similarly to the definition of the left kernel, see [4], one can also define $LZ(\pi)$ as the set of all elements $a \in A$ for which $\sum a_1 \otimes a_2 m = \sum a_1 \lambda(a_2) \otimes m$ for some scalars $\lambda(a_2) \in k$.

For a semisimple Hopf algebra we denote by $K(A)$ the largest central Hopf subalgebra of $A$. Moreover by [9, Theorem 3.8] one has that $K(A) = kU(A)$ where $U(A)$ is the universal grading group of $A$. We also denote by $A_{ad}$ the left adjoint $A$-module. Thus $A_{ad} = A$ as vector spaces and the module structure is given by $a.b = a_1bS(a_2)$ for all $a,b \in A$. Since $A_{ad}$ generates the fusion subcategory $\text{Rep}(A//K(A)$ (see [2, Proposition 18]) it follows that $L\text{Ker}_A(A_{ad}) = K(A)$. On the other hand it is not difficult to check by definition that $L\text{Ker}_A(A_{ad})$ is the largest left coideal subalgebra contained in the center of $A$. Thus, Brauer’s theorem for Hopf algebras implies that $K(A)$ is also the largest central left (right) coideal subalgebra of $A$.

**Theorem 5.11.** Let $A$ be a semisimple Hopf algebra with a commutative character ring and $\chi$ be the character of a finite dimensional representation $M$ of $A$. With the above notations one has the following

$$LZ(\pi)/L\text{Ker}(\pi) \subseteq K(A/L\text{Ker}(\pi)).$$

Moreover $K(A/L\text{Ker}(\pi))$ is the Hopf algebra of a cyclic group of order equal to the index of $\pi$. If $\pi$ is an irreducible character then one has equality above.

**Proof.** Let $L := L\text{Ker}_A(\pi)$. By Brauer’s theorem, any irreducible representation of $A//L$ is a constituent of some tensor power. One has that $LZ(\pi)//L$ is a left coideal subalgebra contained in the center of $A//L$ since any of its elements act as scalars on any irreducible representation of $A//L$. \qed

Note that the above result generalizes [10, Lemma 2.27] from group representations to Hopf algebras representations.

6. THE CASE OF A MODULAR FUSION CATEGORY

In this section we apply our previous results to modular fusion categories, (see [1, 24]). Recall that a braided tensor category $\mathcal{C}$ is a tensor category equipped for all $X, Y \in \mathcal{O}(\mathcal{C})$ with natural isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$ satisfying the hexagon axiom, see for example [11, 12].

A twist on a braided fusion category $\mathcal{C}$ is a natural automorphism $\theta : \text{id}_\mathcal{C} \to \text{id}_\mathcal{C}$ satisfying $\theta_1 = \text{id}_1$ and $\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y)c_{Y,X}c_{X,Y}$. A
braided fusion category is called \textit{premodular} or \textit{ribbon} if it has a twist satisfying \( \theta_X^* = \theta_X^* \) for all \( X \in \mathcal{O}(\mathcal{C}) \).

Recall that the entries of the \( S \)-matrix, \( S = \{ s_{X,Y} \} \) of a premodular category are defined as the quantum trace \( s_{X,Y} := \text{tr}_q(c_{Y,X}c_{X,Y}) \), see \cite{24}. Then it follows from \cite{16} (see also \cite{9, Lemma 6.5}) that \( |s_{X,Y}| \leq \text{FPdim}(X)\text{FPdim}(Y) \) and \( s_{X,Y} = \text{FPdim}(X)\text{FPdim}(Y) \) if and only if
\begin{equation}
(6.1) \quad c_{X,Y}c_{Y,X} = \text{id}_{XY}.
\end{equation}

In the situation of Equation (6.1) we say that \( X \) and \( Y \) \textit{centralize each other}. Moreover the \textit{centralizer} \( \mathcal{D}' \) of a fusion subcategory \( \mathcal{D} \) is defined as the full fusion subcategory of \( \mathcal{D} \) generated by all the objects of \( \mathcal{C} \) that centralize any object of \( \mathcal{D} \). We say that two objects \( X \) and \( Y \) \textit{projectively centralize} each other if \( c_{X,Y}c_{Y,X} = \omega \text{FPdim}(X)\text{FPdim}(Y) \) for a root of unity \( \omega \in \mathbb{k} \).

A premodular category \( \mathcal{C} \) is called \textit{modular} if the above \( S \)-matrix is nondegenerate. Recall also that a fusion subcategory \( \mathcal{D} \) of a tensor modular category \( \mathcal{C} \) is called \textit{nondegenerate} if \( \mathcal{D} \cap \mathcal{D}' = \text{Vec} \).

Now let \( \mathcal{C} \) be a modular fusion category. For any \( X \in \text{Irr}(\mathcal{C}) \) the assignment \( s_X([Y]) = \frac{s_{X,Y}}{s_{X,1}} \) extends linearly to a ring homomorphism \( \mathcal{K}(\mathcal{C}) \to \mathbb{k} \), see \cite{11 Theorem 3.1.1}. Conversely, any such ring homomorphism is of the type \( s_X \) for a simple object \( X \) of \( \mathcal{C} \). Thus in the modular case one has
\begin{equation}
(6.2) \quad X(\mathcal{C}) = \{ s_X \mid X \in \text{Irr}(\mathcal{C}) \}.
\end{equation}

In this situation it follows that
\[ \ker_{\mathcal{C}}(s_X) = \{ Y \in \text{Irr}(\mathcal{C}) \mid Y \text{ centralizes } X \} = \mathcal{C}(X)'. \]

and
\[ \mathcal{Z}_{\mathcal{C}}(s_X) = \{ Y \in \text{Irr}(\mathcal{C}) \mid Y \text{ projectively centralizes } X \} \]

By \cite{9 Proposition 6.7} it follows that
\[ \mathcal{Z}_{\mathcal{C}}(s_X) = (\mathcal{C}(X)')^c = (\mathcal{C}(X)_{ad})'. \]

Therefore it follows from Theorem \ref{3.9} that in a modular category a simple object \( X \) of \( \mathcal{C} \) is \textit{faithful} if and only if \( X \) does not centralize any other object than the trivial object.

For \( S \subseteq X(\mathcal{C}) \) we define \( \tilde{S} \) as the subset of simple objects of \( \mathcal{C} \) such that \( s_X \in S \).

**Corollary 6.3.** Suppose that \( \mathcal{C} \) is a single generated braided fusion category. Then \( \mathcal{Z}_{\mathcal{C}}(X) := \langle Y \mid s_Y \in \mathcal{Z}_{\mathcal{C}}(X) \rangle \) coincides to group of invertible objects of \( \mathcal{C} \).
Remark 6.4. It follows from [9, Lemma 6.1] that the simple objects $X_i$ that projectively centralize any other object of $\mathcal{C}$ are precisely the invertible objects of $\mathcal{C}$. Thus one can write that:

\begin{equation}
\bigcap_{i \in I} \mathcal{Z}_C(X_i) = X(\mathcal{C}).
\end{equation}

6.1. Relation with quantum doubles and kernel of the adjoint object. Let $\mathcal{C}$ be a spherical fusion category. Then it is known that $\mathcal{Z}(\mathcal{C})$ is a modular tensor category see [15, Theorem 1.2] or [7, Theorem 2.15] with $\text{FPdim}(\mathcal{Z}(\mathcal{C})) = \text{FPdim}(\mathcal{C})^2$. Moreover the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ induces a surjective ring homomorphism $F_* : \mathcal{K}(\mathcal{Z}(\mathcal{C})) \to \mathcal{K}(\mathcal{C})$, see [7].

Now let $E \in \text{Irr}(\mathcal{K}(\mathcal{C}))$. Then $\mathcal{K}(\mathcal{Z}(\mathcal{C}))$ acts irreducibly on $E$ via $F_*$. Let $\hat{E} \in \text{Irr}(\mathcal{K}(\mathcal{Z}(\mathcal{C})))$ be the corresponding 1-dimensional representation of $\mathcal{K}(\mathcal{Z}(\mathcal{C}))$. Thus by Equation (6.2) there exists a unique $A_E \in \text{Irr}(\mathcal{Z}(\mathcal{C}))$ such that $\hat{E} = s_{A_E}$ and in addition

\begin{equation}
f_{\hat{E}} = \frac{\text{dim} \mathcal{C}}{\text{dim} A_E^2}.
\end{equation}

Moreover by [20, Theorem 2.13] we have

\begin{equation}
\text{dim} A_E = \frac{\text{dim} \mathcal{C}}{f_{\hat{E}}}
\end{equation}

and

\begin{equation}
[I(1) : A_E] = \text{dim} E
\end{equation}

Write $C_{ad} := \bigoplus_{X \in \text{Irr}(\mathcal{C})} X \otimes X^*$

**Proposition 6.7.** Let $\mathcal{C}$ be a spherical fusion category. With the above notations it follows that

\begin{equation}
\ker_{\mathcal{C}}(C_{ad}) = \{ F_*(s_A) \mid A \in \text{Inv}(\mathcal{Z}(\mathcal{C})), [I(1) : A] > 0 \}
\end{equation}

where $\text{Inv}(\mathcal{Z}(\mathcal{C}))$ is the set of isomorphism classes of invertible objects of $\mathcal{Z}(\mathcal{C})$.

**Proof.** Using Equation (2.3) it follows that $C_{ad} = \sum_{i=0}^r f_i E_i$. Moreover there is a simple object $A \in \text{Irr}(\mathcal{Z}(\mathcal{C}))$ such that $\mu_i = F_*(s_A)$. Moreover by Equation (6.6) one has that $f_i = \frac{\text{dim} \mathcal{C}}{\text{dim} A}$. Thus $f_i = \text{FPdim}(\mathcal{C})$ if and only if $\text{dim} A = 1$, i.e $A$ is an invertible object of $\mathcal{Z}(\mathcal{C})$. \qed
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