OPTIMALITY CONDITIONS FOR $E$-DIFFERENTIABLE VECTOR OPTIMIZATION PROBLEMS WITH THE MULTIPLE INTERVAL-VALEUDED OBJECTIVE FUNCTION

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Abstract. In this paper, a nonconvex vector optimization problem with multiple interval-valued objective function and both inequality and equality constraints is considered. The functions constituting it are not necessarily differentiable, but they are $E$-differentiable. The so-called $E$-Karush-Kuhn-Tucker necessary optimality conditions are established for the considered $E$-differentiable vector optimization problem with the multiple interval-valued objective function. Also the sufficient optimality conditions are derived for such interval-valued vector optimization problems under appropriate (generalized) $E$-convexity hypotheses.

1. Introduction. In mathematical programming, we usually deal with real numbers which are assumed to be fixed. However, in many real-life situations, the coefficients of decision support models are not exactly known and, therefore, data suffer from inexactness. In other words, we often encounter cases where the information items can’t be determined with certainty. The interval-valued optimization problems are closely related to optimization problems with inexact data. According to the decision maker’s point of view under changeable conditions, we may replace the real numbers by the interval numbers to formulate optimization problems more appropriately. Therefore, the interval-valued optimization problems have been of much interest in recent past and thus explored the extent of optimality conditions and duality applicability in different areas (see, for example, [1], [2], [4], [5], [7], [13], [14], [15], [16], [17], [18], [19], [25], [26], [29], [30], [35], [36], and others).

Several generalizations of the definition of a convex function have been introduced to optimization theory in order to weaken an assumption of convexity for establishing optimality and duality results for new classes of nonconvex optimization problems, including vector optimization ones. Youness [32] brought forward

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the concepts of $E$-convex sets, $E$-convex functions and $E$-convex mathematical programming by relaxing the definition of convex sets and convex functions, discussed some of their basic properties and established some optimality results for optimization problems under $E$-convexity hypotheses. This kind of generalized convexity is based on the effect of an operator $E : \mathbb{R}^n \rightarrow \mathbb{R}^n$ on the sets and the domain of the definition of functions. Unfortunately, some of them turned out to be incorrect as it was noted by Yang \cite{31}. The initial results of Youness \cite{32} inspired a great deal of subsequent works, which greatly expanded the role of $E$-convexity in optimization theory (see, for instance, \cite{6}, \cite{8}, \cite{9}, \cite{11}, \cite{12}, \cite{22}, \cite{24}, \cite{27}, \cite{28}, \cite{33}, \cite{34}, and others). Youness \cite{33} established some properties of $E$-convex functions and some necessary and sufficient optimality criteria for nonlinear $E$-convex mathematical programming problems. Later, Syau and Lee \cite{28} presented the concept of $E$-quasiconvex functions and discussed some properties of $E$-convex and $E$-quasiconvex functions. Emam and Youness \cite{10} introduced a new class of $E$-convex sets and $E$-convex functions, which are called strongly $E$-convex sets and strongly $E$-convex functions, respectively, by taking the images of two points under an operator $E : \mathbb{R}^n \rightarrow \mathbb{R}^n$ besides the two points themselves. Youness \cite{34} gave a characterization of efficient solutions for multiobjective programming problems involving $E$-convex functions. Soleimani-damaneh \cite{27} established some properties of $E$-convex and generalized $E$-convex functions. Recently, Megahed et al. \cite{21} introduced a combined interactive approach for solving nonlinear generalized $E$-convex multiobjective programming problems.

In this paper, the class of $E$-differentiable multiobjective programming problems with multiple interval-valued objective functions and with both inequality and equality constraints is considered. The so-called $E$-Karush-Kuhn-Tucker necessary optimality conditions are established for such nondifferentiable vector optimization problems under the constraint qualification introduced in the paper. The case is illustrated in which these necessary optimality conditions are not fulfilled if the introduced constraint qualification is not satisfied. Under $E$-convexity and/or generalized $E$-convexity assumptions, the sufficient optimality conditions for so-called (weak) $LU$-$E$-Pareto optimality are also established for the considered $E$-differentiable interval-valued multiobjective programming problem. It is also illustrated that the optimality conditions established in the paper are also applicable for such vector optimization problems with multiple interval-valued objective functions for which the classical optimality conditions can be avoided. Namely, an example of a nondifferentiable vector optimization problem with the multiple interval-valued objective function is presented for which both the classical Karush-Kuhn-Tucker necessary optimality conditions and the sufficient optimality conditions for differentiable interval-valued multiobjective programming problems can not be applied. However, the $E$-Karush-Kuhn-Tucker necessary optimality conditions and the sufficient optimality conditions established in the paper may be used in such a case to analyze (weak) $LU$-$E$-Pareto optimality of feasible solutions.

2. Preliminaries. Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and $\mathbb{R}_+^n$ be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper.

For any vectors $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_n)^T$ in $\mathbb{R}^n$, we define:

1. $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;
2. $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, ..., n$;
Definition 2.2. Let \( R \) be a class of all closed and bounded intervals in \( R \). Throughout this paper, when we say that \( A \) is a closed interval, we mean that \( A \) is also bounded in \( R \). If \( A \) is a closed interval, we use the notation \( A = [a^L, a^U] \), where \( a^L \) and \( a^U \) mean the lower and upper bounds of \( A \), respectively. In other words, if \( A = [a^L, a^U] \in I (R) \), then \( A = [a^L, a^U] = \{ x \in R : a^L \leq x \leq a^U \} \). If \( a^L = a^U = a \), then \( A = [a, a] = a \) is a real number.

Let \( A = [a^L, a^U] \in I (R) \), \( B = [b^L, b^U] \in I (R) \). Then, by definition, we have:
1. \( A + B = \{ a + b : a \in A \text{ and } b \in B \} = [a^L + b^L, a^U + b^U] \),
2. \( -A = \{ -a : a \in A \} = [-a^U, -a^L] \),
3. \( A - B = A + (-B) = \{ a - b : a \in A \text{ and } b \in B \} = [a^L - b^U, a^U - b^L] \),
4. \( k + A = \{ k + a : a \in A \} = [k + a^L, k + a^U] \), where \( k \) is a real number,
5. \( kA = \{ ka^L, ka^U \} \) if \( k > 0 \), where \( k \) is a real number.

For more details on the topic of interval analysis, we refer to Moore [23] and Alefeld and Herzberger [3].

In interval mathematics, an order relation is often used to rank interval numbers and it implies that an interval number is better than another but not that one is larger than another.

For \( A = [a^L, a^U] \) and \( B = [b^L, b^U] \), we write
\[
A \leq_{LU} B \text{ if and only if } \begin{cases} a^L \leq b^L \\ a^U \leq b^U \end{cases}.
\]

It means that \( A \) is inferior to \( B \), or \( B \) is superior to \( A \). It is easy to see that \( \leq_{LU} \) is a partial ordering on \( I (R) \).

Further, we can write \( A <_{LU} B \) if and only if \( A \leq_{LU} B \) and \( A \neq B \). Equivalently, \( A <_{LU} B \) if and only if
\[
\begin{cases} a^L < b^L \\ a^U \leq b^U \end{cases} \quad \text{or} \quad \begin{cases} a^L \leq b^L \\ a^U < b^U \end{cases} \quad \text{or} \quad \begin{cases} a^L < b^L \\ a^U < b^U \end{cases}.
\]

Throughout this section, let \( X \) be a nonempty open subset of \( R^n \). A function \( \psi : X \to I (R) \) is called an interval-valued function if \( \psi(x) = [\psi^L(x), \psi^U(x)] \) with \( \psi^L, \psi^U : X \to R \) such that \( \psi^L(x) \leq \psi^U(x) \) for each \( x \in X \).

Now, we shall consider the differentiation of an interval-valued function. Namely, we use a very straightforward concept of differentiation introduced by Wu [29].

**Definition 2.1.** An interval-valued function \( f : X \to I (R) \) with \( f(x) = [f^L(x), f^U(x)] \) is called weakly differentiable at \( \bar{x} \in D \) if the real-valued functions \( f^L \) and \( f^U \) are differentiable at \( \bar{x} \) (in the usual sense).

We now give the definition of an \( E \)-differentiable function introduced by Megahed et al. [21].

**Definition 2.2.** Let \( f : R^n \to R \) be a (not necessarily) differentiable function on \( R^n \), \( \bar{x} \in R^n \) and \( E : R^n \to R^n \). It is said that \( f \) is \( E \)-differentiable at \( \bar{x} \) if and only if \( f \circ E \) is a differentiable function at \( \bar{x} \) (in the usual sense) and
\[
(f \circ E)(x) = (f \circ E)(\bar{x}) + \nabla (f \circ E)(\bar{x})(x - \bar{x}) + \theta(\bar{x}, x - \bar{x})\|x - \bar{x}\|,
\]
where \( \theta(\bar{x}, x - \bar{x}) \to 0 \) as \( x \to \bar{x} \).
Now, in a natural way, we extend the definition of $E$-differentiability to the case of an interval-valued function.

**Definition 2.3.** Let $E : R^n \to R^n$. An interval-valued function $f : R^n \to I(R)$ with $f(x) = [f^L(x), f^U(x)]$ is called $E$-differentiable at a given point $\overline{x}$ if the real-valued functions $f^L \circ E$ and $f^U \circ E$ are differentiable at $\overline{x}$ (in the usual sense) and

$$
(f^L \circ E)(x) = (f^L \circ E)(\overline{x}) + \nabla (f^L \circ E)(\overline{x}) (x - \overline{x}) + \theta^L(\overline{x}, x - \overline{x}) \|x - \overline{x}\|, \quad (3)
$$

$$
(f^U \circ E)(x) = (f^U \circ E)(\overline{x}) + \nabla (f^U \circ E)(\overline{x}) (x - \overline{x}) + \theta^U(\overline{x}, x - \overline{x}) \|x - \overline{x}\|, \quad (4)
$$

where $\theta^L(\overline{x}, x - \overline{x}) \to 0, \theta^U(\overline{x}, x - \overline{x}) \to 0$ as $x \to \overline{x}$.

The definition of an $E$-convex set and the definition of an $E$-convex function were introduced by Youness [32]. Now, for convenience, we recall these definitions.

**Definition 2.4.** [32] A set $S \subseteq R^n$ is said to be $E$-convex (with respect to an operator $E : R^n \to R^n$) if and only if the relation

$$
E(u) + \lambda (E(x) - E(u)) \in S
$$

holds for all $x, u \in S$ and any $\lambda \in [0, 1]$.

Note that every convex set is $E$-convex (if $E$ is the identity map), but the converse is not true. If $S \subseteq R^n$ is an $E$-convex set, then $E(S) \subseteq S$. If $E(S)$ is a convex set and $E(S) \subseteq S$, then $S$ is $E$-convex (see, [33]).

**Definition 2.5.** [32] Let $E : R^n \to R^n$ and $S$ be a nonempty $E$-convex subset of $R^n$. A real-valued function $f : S \to R$ is said to be $E$-convex on $S$ if and only if the inequality

$$
f(\lambda E(x) + (1 - \lambda) E(u)) \leq \lambda f(E(x)) + (1 - \lambda) f(E(u))
$$

holds for all $x, u \in S$ and any $\lambda \in [0, 1]$.

Now, we extend this definition to the case of an interval-valued function.

**Definition 2.6.** Let $E : R^n \to R^n$ and $S$ be a nonempty $E$-convex subset of $R^n$. An interval-valued function $f : S \to I(R)$ is said to be $E$-convex on $S$ if and only if the inequality

$$
f(\lambda E(x) + (1 - \lambda) E(u)) \leq \lambda f(E(x)) + (1 - \lambda) f(E(u))
$$

holds for all $x, u \in S$ and any $\lambda \in [0, 1]$.

In other words, (6) is equivalent to the fact that the following inequalities

$$
f^L(\lambda E(x) + (1 - \lambda) E(u)) \leq \lambda f^L(E(x)) + (1 - \lambda) f^L(E(u)),
$$

$$
f^U(\lambda E(x) + (1 - \lambda) E(u)) \leq \lambda f^U(E(x)) + (1 - \lambda) f^U(E(u))
$$

hold for all $x, u \in S$ and any $\lambda \in [0, 1]$.

**Definition 2.7.** Let $E : R^n \to R^n$ and $S$ be a nonempty $E$-convex subset of $R^n$. An interval-valued function $f : S \to I(R)$ is said to be strictly $E$-convex on $S$ if and only if the inequality

$$
f(\lambda E(x) + (1 - \lambda) E(u)) < \lambda f(E(x)) + (1 - \lambda) f(E(u))
$$

holds for all $x, u \in S$ such that $E(x) \neq E(u)$ and any $\lambda \in (0, 1)$.
In other words, (9) is equivalent to the fact that at least one of the inequalities (7) and (8) is strict.

Let $f$ be a real-valued differentiable function on an $E$-convex set. The definition of a differentiable $E$-convex function was introduced by Piao et al. [24]. The next result characterizes an $E$-differentiable $E$-convex interval-valued function with respect to the gradients of its lower and upper functions.

**Proposition 1.** Let $E : R^n \rightarrow R^n$, $S$ be an $E$-convex subset of $R^n$, $f : S \rightarrow I(R)$ be an $E$-convex (strictly $E$-convex) function on $S$ and $u \in S$. Further, assume that $f$ is $E$-differentiable at $u$. Then, the inequalities

\[ f^L (E(x)) - f^L (E(u)) \geq \nabla f^L (E(u)) (E(x) - E(u)), \quad (> \text{ (10)}) \]
\[ f^U (E(x)) - f^U (E(u)) \geq \nabla f^U (E(u)) (E(x) - E(u)) \quad (> \text{ (11)}) \]

hold for all $x \in S$ ($E(x) \neq E(u)$).

**Proof.** Assume that $S$ is an $E$-convex set, $f : S \rightarrow R$ is an $E$-convex function on $S$ and $u \in S$. By Definition 2.6, it follows that the inequalities

\[ f^L (\lambda E(x) + (1 - \lambda) E(u)) \leq \lambda f^L (E(x)) + (1 - \lambda) f^L (E(u)), \]
\[ f^U (\lambda E(x) + (1 - \lambda) E(u)) \leq \lambda f^U (E(x)) + (1 - \lambda) f^U (E(u)) \]

hold for all $x, u \in S$ and any $\lambda \in [0, 1]$. Thus, the above inequalities yield, respectively,

\[ f^L (E(x)) - f^L (E(u)) \geq \frac{f^L (E(u) + \lambda (E(x) - E(u))) - f^L (E(u))}{\lambda}, \]
\[ f^U (E(x)) - f^U (E(u)) \geq \frac{f^U (E(u) + \lambda (E(x) - E(u))) - f^U (E(u))}{\lambda}. \]

Letting $\lambda \rightarrow 0$, we obtain the inequalities (10) and (11), respectively. \hfill \Box

Now, we define the concepts of generalized $E$-convexity for $E$-differentiable interval-valued functions.

**Definition 2.8.** Let $E : R^n \rightarrow R^n$, $S$ be a nonempty $E$-convex subset of $R^n$ and $f : S \rightarrow I(R)$ be an $E$-differentiable function at $u \in S$. Then $f$ is said to be a pseudo $E$-convex function at $u$ on $S$ if the relations

\[ (f^L \circ E) (x) < (f^L \circ E) (u) \implies \nabla (f^L \circ E) (u) (E(x) - E(u)) < 0, \quad (12) \]
\[ (f^U \circ E) (x) < (f^U \circ E) (u) \implies \nabla (f^U \circ E) (u) (E(x) - E(u)) < 0 \quad (13) \]

hold for all $x \in S$. If (12) and (13) are satisfied for each $u \in S$, then $f$ is said to be a pseudo $E$-convex function on $S$.

**Definition 2.9.** Let $E : R^n \rightarrow R^n$, $S$ be a nonempty $E$-convex subset of $R^n$ and $f : S \rightarrow I(R)$ be an $E$-differentiable function at $u \in S$. Then $f$ is said to be a strictly pseudo $E$-convex function at $u$ on $S$ if the relations

\[ (f^L \circ E) (x) \leq (f^L \circ E) (u) \implies \nabla (f^L \circ E) (u) (E(x) - E(u)) < 0, \quad (14) \]
\[ (f^U \circ E) (x) \leq (f^U \circ E) (u) \implies \nabla (f^U \circ E) (u) (E(x) - E(u)) < 0 \quad (15) \]

hold for all $x, u \in S$, $E(x) \neq E(u)$. If (14) and (15) are satisfied for each $u \in S$, $E(x) \neq E(u)$, then $f$ is said to be a strictly pseudo $E$-convex function on $S$. 
Definition 2.10. Let $E : R^n \rightarrow R^n$ and $S$ be a nonempty $E$-convex subset of $R^n$. A real-valued function $f : S \rightarrow R$ is said to be quasi $E$-convex at $u$ on $S$ if the relation
\[
 f (\lambda E (x) + (1 - \lambda) E (u)) \leq \max \{ (f \circ E) (x), (f \circ E) (u) \}
\]
holds for all $x, u \in S$ and any $\lambda \in [0, 1]$.

Definition 2.11. Let $E : R^n \rightarrow R^n$, $S$ be a nonempty $E$-convex subset of $R^n$ and $f : S \rightarrow I (R)$ be an $E$-differentiable function at $u \in S$. Then $f$ is said to be a quasi $E$-convex function at $u$ on $S$ if the relations
\[
 (f^L \circ E) (x) \leq (f^L \circ E) (u) \implies \nabla (f^L \circ E) (u) (E (x) - E (u)) \leq 0, \quad (16)
\]
\[
 (f^U \circ E) (x) \leq (f^U \circ E) (u) \implies \nabla (f^U \circ E) (u) (E (x) - E (u)) \leq 0 \quad (17)
\]
hold for all $x \in S$. If (16) and (17) are satisfied for each $u \in S$, then $f$ is said to be a quasi $E$-convex function on $S$.

3. $E$-differentiable interval-valued multiobjective programming. In this paper, consider the following (not necessarily differentiable) interval-valued multiobjective programming problem with both inequality and equality constraints:

\[
 \begin{align*}
 & \text{minimize} \quad f (x) = (f_1 (x), \ldots, f_p (x)) \\
 & \text{subject to} \quad g_j (x) \leq 0, \quad j \in J = \{1, \ldots, m\}, \\
 & \quad h_t (x) = 0, \quad t \in T = \{1, \ldots, s\}, \\
 \end{align*}
\]

(IVP)

where each objective function $f_k : R^n \rightarrow I (R)$, $k \in K = \{1, \ldots, p\}$ is an interval-valued function defined on $R^n$, each function $g_j : R^n \rightarrow R$, $j \in J$ and each function $h_t : R^n \rightarrow R$, $t \in T$, are real-valued functions defined on $R^n$.

For the purpose of simplifying our presentation, we will next introduce some notations which will be used frequently throughout this paper. We will write $g := (g_1, \ldots, g_m) : R^n \rightarrow R^m$ and $h := (h_1, \ldots, h_s) : R^n \rightarrow R^s$ for convenience. Let

$$
\Omega := \{ x \in R^n : g_j (x) \leq 0, \quad j \in J, \quad h_t (x) = 0, \quad t \in T \}
$$

be the set of all feasible solutions of (IVP).

For such interval-valued multicriterion optimization problems, Wu [30] proposed the following different concepts of (weak) Pareto optimal solutions in terms of a weak LU-Pareto (weakly LU-efficient) solution and a LU-Pareto (LU-efficient) solution in the following sense:

Definition 3.1. A feasible point $\bar{x}$ is said to be a weak LU-Pareto (weakly LU-efficient) solution of (IVP) if and only if there is no another feasible solution $x$ such that, for each $k \in K$,

$$
f_k (x) <_{LU} f_k (\bar{x}).
$$

Definition 3.2. A feasible point $\bar{x}$ is said to be a LU-Pareto (LU-efficient) solution of (IVP) if and only if there is no another feasible solution $x$ such that

$$
f (x) <_{LU} f (\bar{x}).
$$

Further, let $E : R^n \rightarrow R^n$ be a given one-to-one and onto operator. Throughout the paper, we shall assume that the functions constituting the considered interval-valued multiobjective programming problem (IVP) are $E$-differentiable.
Now, we prove the so-called $E$-Karush-Kuhn-Tucker necessary optimality conditions for the problem (IVP). Therefore, for the considered $E$-differentiable interval-valued multiobjective programming problem (IVP), we construct the following associated vector optimization problem (IVP) with the multiple interval-valued objective function:

$$
\begin{align*}
\text{minimize} & \quad f(E(x)) = (f_1(E(x)), \ldots, f_p(E(x))) \\
\text{subject to} & \quad g_j(E(x)) \leq 0, \quad j \in J = \{1, \ldots, m\}, \\
& \quad h_t(E(x)) = 0, \quad t \in T = \{1, \ldots, s\}. \\
\end{align*}
$$

We call the problem (IVP) an $E$-vector optimization problem with the multiple interval-valued objective function or an interval-valued $E$-vector optimization problem. Let

$$
\Omega_E := \{x \in \mathbb{R}^n : g_j(E(x)) \leq 0, \quad j \in J, \quad h_t(E(x)) = 0, \quad t \in T\}
$$

be the set of all feasible solutions of (IVP).

Now, we give the definitions of a weak Pareto optimal solution and a Pareto solution in terms of a weak $LU$-Pareto (weakly $LU$-efficient) solution and a $LU$-Pareto ($LU$-efficient) solution of the differentiable (in the usual sense) interval-valued $E$-vector optimization problem (IVP) which are, at the same time, (weak) $E$-Pareto optimal solutions in terms of a weak $LU$-$E$-Pareto (weakly $LU$-$E$-efficient) solution and a $LU$-$E$-Pareto ($LU$-$E$-efficient) solution of the considered $E$-differentiable interval-valued multiobjective programming problem (IVP).

**Definition 3.3.** A feasible point $E(\bar{x})$ is said to be a weak $LU$-$E$-Pareto (weakly $LU$-$E$-efficient) solution of (IVP) if and only if there is no another feasible solution $E(x)$ such that, for each $k \in K$,

$$
f_k(E(x)) <_{LU} f_k(E(\bar{x})).
$$

**Definition 3.4.** A feasible point $E(\bar{x})$ is said to be a $LU$-$E$-Pareto ($LU$-$E$-efficient) solution of (IVP) if and only if there is no another feasible solution $E(x)$ such that

$$
f(E(x)) <_{LU} f(E(\bar{x})).
$$

Before proving the optimality conditions for (weakly) $LU$-efficiency of the interval-valued $E$-vector optimization problem (IVP) defined above and, thus, for (weakly) $LU$-$E$-efficiency of the considered (not necessarily differentiable) multiobjective programming problem (IVP) with the multiple interval-valued objective function, we establish some useful results which show the equivalency between these interval-valued vector optimization problems.

**Lemma 3.5.** Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a one-to-one and onto operator and

$$
\Omega_E = \{z \in X : (g_j \circ E)(z) \leq 0, \quad j \in J, \quad (h_t \circ E)(z) = 0, \quad t \in T\}.
$$

Then $E(\Omega_E) = \Omega$.

**Proof.** We now assume that $x \in E(\Omega_E)$. By assumption, $E$ is a one-to-one and onto operator. Hence, by the definition of the set $\Omega_E$, there exists $z \in \Omega_E$ such that $x = E(z)$. By means of contradiction, suppose that $x \notin \Omega$. Then there exists at least one $j \in J$ such that $g_j(x) > 0$ or at least one $t \in T$ such that $h_t(x) \neq 0$. Thus, by $x = E(z)$, we have, for at least one $j \in J$, that $(g_j \circ E)(z) > 0$ or, for at least one $t \in T$, that $(h_t \circ E)(z) \neq 0$, which contradicting $z \in \Omega_E$. Thus, $E(\Omega_E) \subset \Omega$.

On the other hand, let $x \in \Omega$. We proceed by contradiction. Suppose that $x \notin E(\Omega_E)$. By assumption, this means that $E^{-1}(x) \notin \Omega_E$. By the definition of $\Omega_E$,
it follows that there exists at least one \( j \in J \) such that \((g_j \circ E) (E^{-1} (x)) > 0 \) or at least one \( t \in T \) such that \((h_t \circ E) (E^{-1} (x)) \neq 0 \). Therefore, there exists at least one \( j \in J \) such that \(g_j(x) > 0 \) or at least one \( t \in T \) such that \(h_t(x) \neq 0 \), contradicting \( x \in \Omega \). Thus, \( \Omega \subseteq E (\Omega_E) \).

Hence, by \( E (\Omega_E) \subseteq \Omega \) and \( \Omega \subseteq E (\Omega_E) \), we conclude that \( E (\Omega_E) = \Omega \).

Now, we prove the relationship between (weak) LU-Pareto optimal solutions in both interval-valued vector optimization problems (IVP) and (IVP\(_E\)).

**Lemma 3.6.** Let \( E : R^n \rightarrow R^n \) be a one-to-one and onto operator and \( \pi \in \Omega \) be a weak LU-Pareto solution (a LU-Pareto solution) of the considered multiobjective programming problem (IVP) with the multiple interval-valued objective function. Then, there exists \( \pi \in \Omega_E \) such that \( \pi = E (\pi) \) and \( \pi \) is a weak LU-Pareto solution (a LU-Pareto solution) of the E-vector optimization problem (IVP\(_E\)) with the multiple interval-valued objective function.

**Proof.** Let \( \pi \in \Omega \) be a weak LU-Pareto solution for (IVP). Moreover, \( E : R^n \rightarrow R^n \) is assumed to be a one-to-one and onto operator. Hence, by Lemma 3.5, there exists \( \pi \in \Omega_E \) such that \( \pi = E (\pi) \). Now, we prove that \( \pi \) is a weak LU-Pareto solution of the interval-valued E-vector optimization problem (IVP\(_E\)). By means of contradiction, suppose that \( \pi \) is not a weak LU-Pareto solution of the problem (IVP\(_E\)). Then, by the definition of a weak LU-Pareto solution, there exists \( \tilde{\pi} \in \Omega_E \) such that \((f \circ E) (\tilde{\pi}) <_{LU} (f \circ E) (\pi)\). Hence, by the definition of the relation \(<_{LU}\), it follows that, for any \( k \in K \),

\[
\begin{align*}
(f_k^L (E (\tilde{\pi})) &< f_k^L (E (\pi)) \land f_k^U (E (\tilde{\pi})) \leq f_k^U (E (\pi))) \\
or \quad & (f_k^L (E (\tilde{\pi})) \leq f_k^L (E (\pi)) \land f_k^U (E (\tilde{\pi})) < f_k^U (E (\pi))) \\
or \quad & (f_k^L (E (\tilde{\pi})) < f_k^L (E (\pi)) \land f_k^U (E (\tilde{\pi})) < f_k^U (E (\pi))).
\end{align*}
\]

By Lemma 3.5, we have that there exists \( \tilde{x} \in \Omega \) such that \( \tilde{x} = E (\tilde{\pi}) \). Taking also that \( \pi = E (\pi) \), the above inequalities yield, respectively,

\[
\begin{align*}
(f_k^L (\tilde{x}) &< f_k^L (\pi) \land f_k^U (\tilde{x}) \leq f_k^U (\pi)) \\
or \quad & (f_k^L (\tilde{x}) \leq f_k^L (\pi) \land f_k^U (\tilde{x}) < f_k^U (\pi)) \quad \quad (18) \\
or \quad & (f_k^L (\tilde{x}) < f_k^L (\pi) \land f_k^U (\tilde{x}) < f_k^U (\pi)).
\end{align*}
\]

Hence, by the definition of the relation \(<_{LU}\), inequalities (18) imply that the inequality \( f (\tilde{x}) <_{LU} f (\pi) \) is fulfilled, which is a contradiction to weakly LU-efficiency of \( \pi \) for the problem (IVP). The proof for the case, in which \( \pi \in \Omega \) is a LU-Pareto solution of the problem (IVP), is analogous and, therefore, it is omitted.

**Lemma 3.7.** Let \( E : R^n \rightarrow R^n \) be a one-to-one and onto operator and let \( \pi \in \Omega_E \) be a weak LU-Pareto solution (a LU-Pareto solution) of the interval-valued E-vector optimization problem (IVP\(_E\)). Then \( E (\pi) \) is a weak LU-Pareto solution (a LU-Pareto solution) of the considered multiobjective programming problem (IVP) with the multiple interval-valued objective function.

**Proof.** Assume now that \( \pi \in \Omega_E \) is a weak LU-Pareto solution of the interval-valued E-vector optimization problem (IVP\(_E\)). Note that, by Lemma 3.5, it follows that \( E (\pi) \in \Omega \). We proceed by contradiction. Suppose, contrary to the result, that \( E (\pi) \) is not a weak LU-Pareto solution of the considered interval-valued vector optimization problem (IVP). Then, by Definition 3.3, there exists \( \bar{x} \in \Omega \) such that \( f_k (\bar{x}) <_{LU} f_k (E (\pi)) \) for each \( k \in K \). Hence, by Lemma 3.5, there exists \( \bar{\pi} \in \Omega_E \).
such that \( \bar{x} = E(\bar{z}) \). Thus, the inequality above implies that \((f_k \circ E)(\bar{z}) <_{LU} (f_k \circ E)(\bar{z})\) for each \( k \in K \), which is a contradiction to weakly \( LU \)-efficiency of \( \bar{z} \) for the interval-valued \( E \)-vector optimization problem (VP\(_E\)). The proof when it is assumed that \( \bar{z} \in \Omega_E \) is a \( LU \)-Pareto solution of the problem (VP\(_E\)) is similar and, therefore, it is omitted.

**Remark 1.** As it follows from Lemma 3.7, if \( \bar{z} \in \Omega_E \) is a weak \( LU \)-Pareto solution (a \( LU \)-Pareto solution) of the \( E \)-vector optimization problem (IVP\(_E\)), then \( E(\bar{z}) \) is a weak \( LU \)-Pareto solution (a \( LU \)-Pareto solution) of the considered multiobjective programming problem (IVP) with the multiple interval-valued objective function. Therefore, we call \( E(\bar{z}) \) a weak \( LU \)-\( E \)-Pareto solution (a \( LU \)-\( E \)-Pareto solution) of the \( E \)-differentiable interval-valued multiobjective programming problem (IVP).

As it follows from the above lemmas, there is an equivalence between the interval-valued vector optimization problems (IVP) and (IVP\(_E\)). This means that, if we prove optimality results for the differentiable interval-valued \( E \)-vector optimization problem (IVP\(_E\)), they will be applicable also for the original (not necessarily differentiable) multiobjective programming problem (IVP) with the multiple interval-valued objective function in which the involved functions are \( E \)-differentiable.

In order to prove the \( E \)-Karush-Kuhn-Tucker necessary optimality conditions for a weak \( LU \)-\( E \)-Pareto solution of (IVP), we introduce the so-called \( E \)-Kuhn-Tucker constraint qualification for \( E \)-differentiable optimization problems with both inequality and equality constraints.

**Definition 3.8.** Let \( E : R^n \to R^n \) and \( \bar{x} \in \Omega_E \) be given. Further, assume that the constraint functions \( g = (g_1, ..., g_m) \) and \( h = (h_1, ..., h_s) \) are \( E \)-differentiable at \( \bar{x} \).

It is said that the \( E \)-Kuhn-Tucker constraint qualification (E-CQ) is satisfied at \( \bar{x} \) if, for any \( d \in R^n \), \( d \neq 0 \), such that \( \nabla (g_j \circ E)(\bar{x})^T d \leq 0 \) for all \( j \in J(E(\bar{x})) := \{ j \in J : g_j(E(\bar{x})) = 0 \} \), and \( \nabla (h_k \circ E)(\bar{x})^T d = 0 \), \( t \in T \), there exist a function \( \varphi : [0, 1] \to R^n \), which is continuously differentiable at 0, and some real scalar \( \beta > 0 \) such that

\[
\varphi(0) = \bar{x}, \quad \varphi(\alpha) \in \Omega_E \text{ for all } \alpha \in [0, 1] \text{ and } \varphi'(0) = \beta d. \tag{19}
\]

Before we establish the \( E \)-Karush-Kuhn-Tucker necessary optimality conditions for the \( E \)-differentiable interval-valued multiobjective programming problem (IVP), we re-call the Motzkin’s theorem of the alternative.

**Theorem 3.9.** [20] (Motzkin’s theorem of the alternative). Let \( A, C \) and \( D \) be given matrices, with \( A \) being nonvacuous. Then either the system of inequalities

\[
Ax < 0, \quad Cx \leq 0, \quad Dx = 0
\]

has a solution \( x \), or the system

\[
A^T y_1 + C^T y_2 + D^T y_3 = 0, \quad y_1 \geq 0, \quad y_2 \geq 0
\]

has a solution \( y_1, y_2, \) and \( y_3 \).

In [30], Wu proved the Karush-Kuhn-Tucker necessary optimality conditions for a differentiable scalar optimization problem with the multiple interval-valued objective function under the Kuhn-Tucker constraint qualification. Now, we generalize this result to the case of an \( E \)-differentiable vector optimization problem with multiple interval-valued function and with both inequality and equality constraints.
Definition 2.3, \(E\) \(\beta >\)

Further, assume that the optimization problem (IVP) with the multiple interval-valued objective function.

Suppose, contrary to the result, that there exists any \(d \in \mathbb{R}^n\), \(d \neq 0\), satisfying the following system of inequalities:

\[
\sum_{k=1}^{p} \lambda_k^L \nabla (f_k^L \circ E)(\pi) + \sum_{k=1}^{p} \lambda_k^U \nabla (f_k^U \circ E)(\pi) + \sum_{j=1}^{m} \mu_j \nabla (g_j \circ E)(\pi) + \sum_{i=1}^{s} \xi_i \nabla (h_i \circ E)(\pi) = 0, \quad \mu_j (g_j \circ E)(\pi) = 0, \quad j \in J,
\]

\[
(\lambda^L, \lambda^U) \geq 0, \quad \mu \geq 0.
\]

**Proof.** Let \(\pi \in \Omega_E\) be a weak \(LU\)-Pareto solution of the \(E\)-vector optimization problem (IVP) with the multiple interval-valued objective function. Hence, by Lemma 3.7, \(E(\pi)\) is a weak \(LU\)-\(E\)-Pareto solution of the considered multicriteria optimization problem (IVP) with the multiple interval-valued objective function. Further, assume that the \(E\)-Kuhn-Tucker constraint qualification is satisfied at \(\pi\).

Now, we prove that there does not exist a vector \(d \in \mathbb{R}^n, d \neq 0\), satisfying the following system of inequalities:

\[
\nabla (f_k^L \circ E)(\pi)^T d < 0, \quad \nabla (f_k^U \circ E)(\pi)^T d < 0, \quad k \in K,
\]

\[
\nabla (g_j \circ E)(\pi)^T d \leq 0, \quad j \in J(E(\pi)),
\]

\[
\nabla (h_i \circ E)(\pi)^T d = 0, \quad t \in T.
\]

Suppose, contrary to the result, that there exists any \(d \in \mathbb{R}^n, d \neq 0\), satisfying (23), (24) and (25). By assumption, the \(E\)-Kuhn-Tucker constraint qualification is satisfied at \(\pi\). Hence, there exist a function \(\varphi : [0, 1] \rightarrow \mathbb{R}^n\) which is continuously differentiable at 0, and some real scalar \(\beta > 0\) such that (18) is satisfied. By assumption, the functions constituting (IVP) are \(E\)-differentiable. Hence, by Definition 2.3, \(f_k^L \circ E\) and \(f_k^U \circ E\), \(k \in K\), are differentiable at \(\pi\). Then, we can approximate the objective functions \(f_k^L \circ E\) and \(f_k^U \circ E\) linearly as follows:

\[
(f_k^L \circ E)(\varphi(\alpha)) = (f_k^L \circ E)(\pi) + \nabla (f_k^L \circ E)(\pi)^T (\varphi(\alpha) - \pi) + ||\varphi(\alpha) - \pi|| \theta_k^L(\pi, \varphi(\alpha) - \pi),
\]

and

\[
(f_k^U \circ E)(\varphi(\alpha)) = (f_k^U \circ E)(\pi) + \nabla (f_k^U \circ E)(\pi)^T (\varphi(\alpha) - \pi) + ||\varphi(\alpha) - \pi|| \theta_k^U(\pi, \varphi(\alpha) - \pi),
\]

where \(\theta_k^L(\pi, \varphi(\alpha) - \pi) \rightarrow 0\) and \(\theta_k^U(\pi, \varphi(\alpha) - \pi) \rightarrow 0\) as ||\varphi(\alpha) - \varphi(0)|| \rightarrow 0.

Thus, we re-write (26) and (27) as follows

\[
(f_k^L \circ E)(\varphi(\alpha)) = (f_k^L \circ E)(\pi) + \alpha \nabla (f_k^L \circ E)(\pi)^T \left(\frac{\varphi(\alpha) - \varphi(0)}{\alpha}\right) + ||\varphi(\alpha) - \varphi(0)|| \theta_k^L(\pi, \varphi(\alpha) - \varphi(0)),
\]

\[
(f_k^U \circ E)(\varphi(\alpha)) = (f_k^U \circ E)(\pi) + \alpha \nabla (f_k^U \circ E)(\pi)^T \left(\frac{\varphi(\alpha) - \varphi(0)}{\alpha}\right) + ||\varphi(\alpha) - \varphi(0)|| \theta_k^U(\pi, \varphi(\alpha) - \varphi(0)),
\]

where \(\theta_k^L(\pi, \varphi(\alpha) - \pi) \rightarrow 0\) and \(\theta_k^U(\pi, \varphi(\alpha) - \pi) \rightarrow 0\) as ||\varphi(\alpha) - \varphi(0)|| \rightarrow 0.
By the $E$-Kuhn-Tucker constraint qualification, it follows that $\varphi$ is a differentiable function at 0. Hence, we have

$$\varphi'(0) = \lim_{\alpha \to 0} \frac{\varphi(\alpha) - \varphi(0)}{\alpha}. \quad (30)$$

Also it follows from the $E$-Kuhn-Tucker constraint qualification, for any $d \in \mathbb{R}^n$, $d \neq 0$, there exists a real scalar $\beta > 0$ such that

$$\varphi'(0) = \beta d. \quad (31)$$

Combining (28)-(31) and $\|\varphi(\alpha) - \varphi(0)\| \to 0$ as $\alpha \to 0$, we get, for sufficiently small $\alpha$, respectively,

$$(f^L_k \circ E) (\varphi(\alpha)) = (f^L_k \circ E)(\pi) + \alpha \beta \nabla (f^L_k \circ E)(\pi) d, \quad (32)$$

$$(f^U_k \circ E) (\varphi(\alpha)) = (f^U_k \circ E)(\pi) + \alpha \beta \nabla (f^U_k \circ E)(\pi) d. \quad (33)$$

By assumption, (23) is satisfied for $d \in \mathbb{R}^n$, $d \neq 0$. Therefore, by $\alpha > 0$ and $\beta > 0$, (32) and (33) yield, respectively, that the inequalities

$$(f^L_k \circ E)(\varphi(\alpha)) < (f^L_k \circ E)(\pi), \quad k \in K,$$

$$(f^U_k \circ E)(\varphi(\alpha)) < (f^U_k \circ E)(\pi), \quad k \in K$$

hold for sufficiently small $\alpha$. This is a contradiction to the assumption that $\pi \in \Omega_E$ is a weak $LU$-Pareto solution of (IVP$_E$). This means that there does not exist any $d \in \mathbb{R}^n$ satisfying the system of inequalities (23)-(25). Hence, by Motzkin’s theorem of the alternative (see Theorem 3.9), there exist Lagrange multipliers $\lambda^L \in \mathbb{R}^p$, $\lambda^U \in \mathbb{R}^p$, $\overline{\lambda}_j$, $j \in J(E(\pi))$, and $\overline{\lambda} \in \mathbb{R}^s$ such that

$$\sum_{k=1}^p \lambda^L_k \nabla (f^L_k \circ E)(\pi) + \sum_{k=1}^p \lambda^U_k (f^U_k \circ E)(\pi) + \sum_{j \in J(E(\pi))} \overline{\lambda}_j \nabla (g_j \circ E)(\pi) + \sum_{t=1}^s \overline{\lambda}_t \nabla (h_t \circ E)(\pi) = 0. \quad (34)$$

If we set $\overline{\lambda}_j = 0$ for all $j \in J \setminus J(E(\pi))$, then (34) implies (20). Further, note that also the complementary slackness condition (21) is satisfied. Indeed, if $g_j(E(\pi)) < 0$, then $\overline{\lambda}_j = 0$ for $j \in J \setminus J(E(\pi))$. 

\hfill \Box

**Remark 2.** It is assumed in Theorem 3.10 that $\pi \in \Omega_E$ is a (weak) $LU$-Pareto solution of the differentiable interval-valued vector optimization problem (IVP$_E$). Then, the fact that $E(\pi)$ is a (weak) $LU$-E-Pareto solution of the considered $E$-differentiable multicriteria optimization problem (IVP) with the multiple interval-valued objective function follows directly from Lemma 3.7. Hence, the Karush-Kuhn-Tucker necessary optimality conditions established in Theorem 3.10 for the problem (IVP$_E$) are also applicable for the problem (IVP). In such a case, we call them the $E$-Karush-Kuhn-Tucker necessary optimality conditions for the considered $E$-differentiable multiobjective programming problem (IVP) with the multiple interval-valued objective function.

**Definition 3.11.** $(\pi, \lambda, \overline{\lambda}, \overline{\lambda}) \in \Omega_E \times \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^s$ is said to be a Karush-Kuhn-Tucker point for the $E$-vector optimization problem (IVP$_E$) with the multiple interval-valued objective if the Karush-Kuhn-Tucker necessary optimality conditions (20)-(22) are satisfied at $\pi$ with Lagrange multipliers $\lambda$, $\overline{\lambda}$, and $\overline{\lambda}$. 

In order to illustrate the above result, we present an example of such a nondifferentiable vector optimization problem with the multiple interval-valued objective function for which the $E$-Kuhn-Tucker constraint qualification is not satisfied. Note that in such a case Lagrange multipliers $\lambda^L$ and $\lambda^U$ corresponding to the objective functions $f^L$ and $f^U$ can be equal to 0.

**Example 3.12.** Consider the following nondifferentiable vector optimization problem with the multiple interval-valued objective function:

$$
\text{minimize } f(x) = (\lfloor f^L_1(x), f^U_1(x) \rfloor, \lfloor f^L_2(x), f^U_2(x) \rfloor) = \\
\left( \left[ 4x_1^2 + (\sqrt{2} - 3)^2, 4x_1^2 + (\sqrt{2} - 3)^2 + 1 \right], \\
\left[ 8x_1^2 + (\sqrt{2} - 2)^4 - 1, 8x_1^2 + (\sqrt{2} - 2)^4 \right] \right),
$$

$$
g_1(x) = 2x_1 - (1 - \sqrt{2})^3 \leq 0, \\
g_2(x) = -x_1 \leq 0.
$$

Note that the set of all feasible solutions in the considered vector optimization problem (IVP1) with the multiple interval-valued objective function is $\Omega = \{(x_1, x_2) \in R^2 : x_1 \geq 0 \land 2x_1 - (1 - \sqrt{2})^3 \leq 0 \}$. Further, note that some of the functions constituting (IVP1) are nondifferentiable at feasible solutions $(x_1, 0)$, where $x_1 \geq 0$.

Let $E : R^2 \rightarrow R^2$ be defined as follows: $E(x_1, x_2) = \left( \frac{1}{2}x_1, x_2^2 \right)$. Then, it can be shown by Definition 2.3 that the objective function $f$ is $E$-differentiable at $\pi = (0, 1)$ and, by Definition 2.2, the constraints $g_1$ and $g_2$ are $E$-differentiable at $\pi = (0, 1)$.

Indeed, we have that the functions $(f^L_l \circ E)(x) = x_1^2 + (x_2 - 3)^2$, $(f^U_l \circ E)(x) = x_1^2 + (x_2 - 3)^2 + 1$, $(f^L_2 \circ E)(x) = 2x_1^2 + (x_2 - 2)^4 - 1$, $(f^U_2 \circ E)(x) = 2x_1^2 + (x_2 - 2)^4 + 1$, $(g_1 \circ E)(x) = x_1 - (1 - x_2)^3$, $(g_2 \circ E)(x) = -\frac{1}{2}x_1$ are differentiable at $\pi = (0, 1)$ and $\nabla (f^L_l \circ E)(\pi) = [0, -4]^T$, $\nabla (f^U_l \circ E)(\pi) = [0, -4]^T$, $\nabla (f^L_2 \circ E)(\pi) = [0, -4]^T$, $\nabla (f^U_2 \circ E)(\pi) = [0, -4]^T$, $\nabla (g_1 \circ E)(\pi) = [1, 0]^T$, $\nabla (g_2 \circ E)(\pi) = [-\frac{1}{2}, 0]^T$. Also it can be established by Definition 3.2 that $\pi = (0, 1) \in \Omega$ is a $LU-E$-Pareto solution of (IVP1). However, note that the $E$-Karush-Kuhn-Tucker necessary optimality conditions are not satisfied at $\pi = (0, 1)$.

Namely, by (20), it follows that $-4 \left( \lambda^L_1 + \lambda^U_1 + \lambda^L_2 + \lambda^U_2 \right) = 0$, which is not possible. This follows from the fact that the $E$-Kuhn-Tucker constraint qualification is not fulfilled at $\pi = (0, 1)$. Indeed, for any function $\varphi : [0, 1] \rightarrow R^2$, which is continuously differentiable at 0, satisfying $\varphi(0) = \pi$, $\varphi(\alpha) \in \Omega$ for all $\alpha \in [0, 1]$, the condition that there exists a scalar $\beta > 0$ such that $\varphi'(0) = \beta d$ is not satisfied. Namely, if we set, for example, $\varphi(\alpha) = (1 - \alpha)d$, where $d = (0, 1)$, then we have $\varphi(\alpha) = (1 - \alpha)(0, 1) = (0, 1 - \alpha) \in \Omega$ for any $\alpha \in [0, 1]$ and $\nabla (g_2 \circ E)(\pi)d^T = [1, 0][0, 1]^T = 0 \leq 0$, $\nabla (g_2 \circ E)(\pi)d^T = 0$. However, it is not difficult to note that the condition $\varphi'(0) = \beta d$ is not satisfied for each $\beta > 0$.

Now, we prove the sufficiency of the $E$-Karush-Kuhn-Tucker necessary optimality conditions for the $E$-differentiable interval-valued vector optimization problem (IVP) under appropriate $E$-convexity hypotheses.

**Theorem 3.13.** Let $E : R^n \rightarrow R^m$ be a given one-to-one and onto operator and $(\bar{\pi}, \bar{\lambda}, \bar{\pi}, \bar{\lambda}) \in \Omega_\pi \times R^p \times R^m \times R^m$ be a Karush-Kuhn-Tucker point of the $E$-vector optimization problem (IVP$_E$) with the multiple interval-valued objective function,
where \( \bar{\lambda} = \left( \bar{\lambda}^L, \bar{\lambda}^U \right) > 0 \). Let \( T^+ \left( E(\bar{x}) \right) = \{ t \in T : \bar{\xi}_t > 0 \} \) and \( T^- \left( E(\bar{x}) \right) = \{ t \in T : \bar{\xi}_t < 0 \} \). Furthermore, assume that the following hypotheses are fulfilled:

a) the objective function \( f \) is an \( E \)-differentiable \( E \)-convex interval-valued function at \( \bar{x} \) on \( \Omega_E \),

b) each inequality constraint function \( g_j, j \in J \left( E(\bar{x}) \right) \), is \( E \)-differentiable \( E \)-convex at \( \bar{x} \) on \( \Omega_E \),

c) each equality constraint function \( h_t, t \in T^+ \left( E(\bar{x}) \right) \), is \( E \)-differentiable \( E \)-convex at \( \bar{x} \) on \( \Omega_E \),

d) each function \( -h_t, t \in T^- \left( E(\bar{x}) \right) \), is \( E \)-differentiable \( E \)-convex at \( \bar{x} \) on \( \Omega_E \).

Then \( \bar{x} \) is a \( LU \)-Pareto solution of the problem \( (IVP_E) \) and, thus, \( E(\bar{x}) \) is a \( LU \)-\( E \)-Pareto solution of the problem \( (IVP) \).

**Proof.** By assumption, \( (\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi}) \in \Omega_E \times R^{2p} \times R^m \times R^r \) is a Karush-Kuhn-Tucker point of \( (IVP_E) \). Then, by Definition 3.11, the \( E \)-Karush-Kuhn-Tucker necessary optimality conditions (20)-(22) are satisfied at \( \bar{x} \) with Lagrange multipliers \( \bar{\lambda} \in R^{2p} \), \( \bar{\mu} \in R^m \), and \( \bar{\xi} \in R^r \). We proceed by contradiction. Suppose, contrary to the result, that \( \bar{x} \) is not a \( LU \)-Pareto solution of the problem \( (IVP_E) \). Hence, by Definition 3.3, there exists another \( \bar{x} \in \Omega_E \) such that

\[
\begin{align*}
f(E(\bar{x})) <_{LU} f(E(\bar{\mu})).
\end{align*}
\]

Hence, by the definition of the \( <_{LU} \) relation and (35), we have

\[
\begin{align*}
(f^L(E(\bar{x})) &< f^L(E(\bar{x})) \land f^U(E(\bar{x})) \leq f^U(E(\bar{x}))) \\
or \quad (f^L(E(\bar{x})) \leq f^L(E(\bar{x})) \land f^U(E(\bar{x})) < f^U(E(\bar{x}))) \\
or \quad (f^L(E(\bar{x})) < f^L(E(\bar{x})) \land f^U(E(\bar{x})) < f^U(E(\bar{x}))).
\end{align*}
\]

Using hypotheses a)-d), by Proposition 1 and Definition 2.5, the following inequalities

\[
\begin{align*}
f^L_k(E(\bar{x})) - f^L_k(E(\bar{x})) &\geq \nabla f^L_k(E(\bar{x}))(E(\bar{x}) - E(\bar{x})), \quad k \in K, \\
(f^U_k(E(\bar{x})) - f^U_k(E(\bar{x}))) &\geq \nabla f^U_k(E(\bar{x}))(E(\bar{x}) - E(\bar{x})), \quad k \in K, \\
g_j(E(\bar{x})) - g_j(E(\bar{x})) &\geq \nabla g_j(E(\bar{x}))(E(\bar{x}) - E(\bar{x})), \quad j \in J(\bar{x}), \\
h_t(E(\bar{x})) - h_t(E(\bar{x})) &\geq \nabla h_t(E(\bar{x}))(E(\bar{x}) - E(\bar{x})), \quad t \in T^+(\bar{x}), \\
h_t(E(\bar{x})) - h_t(E(\bar{x})) &\geq \nabla h_t(E(\bar{x}))(E(\bar{x}) - E(\bar{x})), \quad t \in T^-(\bar{x})
\end{align*}
\]

hold, respectively. Combining (36)-(38), then multiplying the resulting inequalities by the corresponding Lagrange multipliers and adding both their sides, we get

\[
\begin{align*}
\sum_{k=1}^p \bar{\lambda}^L_k \nabla \left( f^L_k \circ E \right)(\bar{x}) + \sum_{k=1}^p \bar{\lambda}^U_k \nabla \left( f^U_k \circ E \right)(\bar{x}) (E(\bar{x}) - E(\bar{x})) < 0.
\end{align*}
\]

Multiplying inequalities (39)-(41) by the corresponding Lagrange multipliers, respectively, we obtain

\[
\begin{align*}
\bar{\mu}_j g_j(E(\bar{x})) - \bar{\mu}_j g_j(E(\bar{x})) &\geq \bar{\mu}_j \nabla g_j(E(\bar{x}))(E(\bar{x}) - E(\bar{x})), \quad j \in J(\bar{x}), \\
\bar{\xi}_t h_t(E(\bar{x})) - \bar{\xi}_t h_t(E(\bar{x})) &\geq \bar{\xi}_t \nabla h_t(E(\bar{x}))(E(\bar{x}) - E(\bar{x})), \quad t \in T^+(\bar{x}), \\
\bar{\xi}_t h_t(E(\bar{x})) - \bar{\xi}_t h_t(E(\bar{x})) &\geq \bar{\xi}_t \nabla h_t(E(\bar{x}))(E(\bar{x}) - E(\bar{x})), \quad t \in T^-(\bar{x}).
\end{align*}
\]

Using the \( E \)-Karush-Kuhn-Tucker necessary optimality condition (21) together with \( \bar{x} \in \Omega_E \) and \( \bar{x} \in \Omega_E \), we get, respectively,

\[
\begin{align*}
\bar{\mu}_j \nabla g_j(E(\bar{x}))(E(\bar{x}) - E(\bar{x})) &\leq 0, \quad j \in J(\bar{x}), \\
\end{align*}
\]
Let Theorem 3.14. \[\xi_t \nabla h_t \left( E(\xi) \right) \left( E(\xi) - E(\tilde{\xi}) \right) \leq 0, \quad t \in T^+(E(\xi)), \] \[\xi_t \nabla h_t \left( E(\xi) \right) \left( E(\tilde{\xi}) - E(\xi) \right) \leq 0, \quad t \in T^-(E(\xi)). \] Adding both sides of the above inequalities, by (42), we obtain that the inequality
\[\sum_{k=1}^{p} \xi_k^L \nabla \left( f_k^L \circ E \right) \left( \xi \right) + \sum_{k=1}^{p} \tilde{\xi}_k^U \nabla \left( f_k^U \circ E \right) \left( \xi \right) + \sum_{j=1}^{m} \tilde{\eta}_j \nabla g_j \left( E(\xi) \right) + \sum_{t=1}^{s} \tilde{\mu}_t \nabla h_t \left( E(\xi) \right) \left( E(\tilde{\xi}) - E(\xi) \right) < 0 \] holds, which is a contradiction to the \( E \)-Karush-Kuhn-Tucker necessary optimality condition (20). Hence, \( \xi \) is a \( LU \)-Pareto solution of \( (IVP_E) \). Thus, by Lemma 3.7, it follows directly that \( E(\xi) \) is a \( LU-E \)-Pareto solution of the considered multicriteria optimization problem \( (IVP) \) with the multiple interval-valued objective function. Then, the proof of this theorem is completed. \( \Box \)

Remark 3. As it follows from the proof of Theorem 3.13, the sufficient conditions are also satisfied if all or some of the functions \( g_j, j \in J \left( E(\xi) \right) \), \( h_t, t \in T^+ \left( E(\xi) \right) \), \(-h_t, t \in T^- \left( E(\xi) \right) \), are \( E \)-differentiable quasi \( E \)-convex functions at \( \xi \) on \( \Omega_E \).

**Theorem 3.14.** Let \( E : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a given one-to-one and onto operator and \((\tilde{\xi}, \tilde{\eta}, \tilde{\mu}, \tilde{\lambda}) \in \Omega_E \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \) be a Karush-Kuhn-Tucker point of the interval-valued \( E \)-vector optimization problem \((IVP_E)\). Further, assume that the following hypotheses are fulfilled:

a) the objective function \( f \) is an \( E \)-differentiable strictly \( E \)-convex interval-valued function at \( \xi \) on \( \Omega_E \),
b) each inequality constraint function \( g_j, j \in J \left( E(\xi) \right) \), is an \( E \)-differentiable \( E \)-convex function at \( \xi \) on \( \Omega_E \),
c) each equality constraint function \( h_t, t \in T^+ \left( E(\xi) \right) \), is an \( E \)-differentiable \( E \)-convex function at \( \xi \) on \( \Omega_E \),
d) each function \(-h_t, t \in T^- \left( E(\xi) \right) \), is an \( E \)-differentiable \( E \)-convex function at \( \xi \) on \( \Omega_E \).

Then \( \xi \) is a weak \( LU \)-Pareto solution of the problem \((IVP_E)\) and, thus, \( E(\xi) \) is a weak \( LU-E \)-Pareto solution of the considered interval-valued multicriteria optimization problem \((IVP)\).

Now, under the concepts of generalized \( E \)-convexity, we prove the sufficient optimality conditions for a feasible solution to be a weak \( LU-E \)-Pareto solution of the problem \((IVP)\).

**Theorem 3.15.** Let \( E : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a given one-to-one and onto operator and \((\tilde{\xi}, \tilde{\eta}, \tilde{\mu}, \tilde{\lambda}) \in \Omega_E \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \) be a Karush-Kuhn-Tucker point of the \( E \)-vector optimization problem \((IVP_E)\) with the multiple interval-valued objective function. Let \( T^+ \left( \xi \right) = \left\{ t \in T : \tilde{\xi}_t > 0 \right\} \) and \( T^- \left( \xi \right) = \left\{ t \in T : \tilde{\xi}_t < 0 \right\} \). Furthermore, assume the following hypotheses:

a) the objective function \( f \) is an \( E \)-differentiable strictly pseudo \( E \)-convex interval-valued function at \( \xi \) on \( \Omega_E \),
b) each inequality constraint function \( g_j, j \in J \left( E(\xi) \right) \), is \( E \)-differentiable quasi \( E \)-convex at \( \xi \) on \( \Omega_E \),
c) each equality constraint function \( h_t, t \in T^+ \left( E(\xi) \right) \), is \( E \)-differentiable quasi \( E \)-convex at \( \xi \) on \( \Omega_E \),
d) each function \(-h_t, \ t \in T^- (E(\bar{x}))\), is \(E\)-differentiable quasi \(E\)-convex at \(\bar{x}\) on \(\Omega_E\).

Then \(\bar{x}\) is a weak LU-Pareto solution of the problem (IVP\(_E\)) and, thus, \(E(\bar{x})\) is a weak LU-\(E\)-Pareto solution of the (IVP).

Proof. By assumption, \((\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi}) \in \Omega_E \times R^{2p} \times R^m \times R^s\) is a Karush-Kuhn-Tucker point of (IVP\(_E\)). Then, by Definition 3.11, the \(E\)-Karush-Kuhn-Tucker necessary optimality conditions (20)-(22) are satisfied at \(\bar{x}\) with Lagrange multipliers \(\bar{\lambda} \in R^{2p}, \bar{\mu} \in R^m,\) and \(\bar{\xi} \in R^s\). We proceed by contradiction. Suppose, contrary to the result, that \(\bar{x}\) is not a weak \(LU\)-Pareto solution of (IVP\(_E\)). Hence, by Definition 3.3, there exists another \(\bar{x} \in \Omega_E\) such that

\[
f_k(E(\bar{x})) <_{LU} f_k(E(\bar{\pi})), \quad k \in K.
\] (49)

Thus, by the definition of the <\(_{LU}\) relation, for each \(k \in K\), we have

\[
(f^L_k(E(\bar{x})) < f^L_k(E(\bar{\pi})) \land f^U_k(E(\bar{x})) \leq f^U_k(E(\bar{\pi}))),
\]

or

\[
(f^U_k(E(\bar{x})) \leq f^U_k(E(\bar{\pi})) \land f^L_k(E(\bar{x})) < f^L_k(E(\bar{\pi}))),
\]

or

\[
(f^L_k(E(\bar{x})) < f^L_k(E(\bar{\pi})) \land f^U_k(E(\bar{x})) < f^U_k(E(\bar{\pi}))).
\]

By hypothesis a), the objective function \(f\) is an \(E\)-differentiable strictly pseudo \(E\)-convex interval-valued function at \(\bar{x}\) on \(\Omega_E\). Then, (50) gives

\[
\nabla (f^U_k \circ E) (\bar{x}) (E(\bar{x}) - E(\bar{\pi})) < 0, \quad k \in K,
\] (51)

\[
\nabla (f^L_k \circ E) (\bar{x}) (E(\bar{x}) - E(\bar{\pi})) < 0, \quad k \in K.
\] (52)

By the \(E\)-Karush-Kuhn-Tucker necessary optimality condition (22), inequalities (51) and (52) yield

\[
\left[ \sum_{k=1}^{p} \bar{\lambda}_k \nabla (f^L_k \circ E) (\bar{x}) + \sum_{k=1}^{p} \bar{\lambda}_k \nabla (f^U_k \circ E) (\bar{x}) \right] (E(\bar{x}) - E(\bar{\pi})) < 0.
\] (53)

Since \(\bar{x} \in \Omega_E\), by the definition of the indices set \(J(E(\bar{x}))\), we have

\[
g_j(E(\bar{x})) - g_j(E(\bar{\pi})) \leq 0, \quad j \in J(E(\bar{x})).
\]

From the assumption, each \(g_j, \ j \in J(E(\bar{x}))\), is an \(E\)-differentiable quasi \(E\)-convex function at \(\bar{x}\) on \(\Omega_E\). Then, by Definition 2.11, we get

\[
\nabla g_j (E(\bar{x})) (E(\bar{x}) - E(\bar{\pi})) \leq 0, \quad j \in J(E(\bar{x})).
\] (54)

Thus, by the \(E\)-Karush-Kuhn-Tucker necessary optimality condition (22), (54) gives

\[
\sum_{j \in J(E(\bar{x}))} \bar{\mu}_j \nabla g_j (E(\bar{x})) (E(\bar{x}) - E(\bar{\pi})) \leq 0.
\]

Hence, taking into account that \(\bar{\mu}_j = 0, \ j \notin J(E(\bar{x}))\), we have

\[
\sum_{j=1}^{m} \bar{\mu}_j \nabla g_j (E(\bar{x})) (E(\bar{x}) - E(\bar{\pi})) \leq 0.
\] (55)

By \(\bar{x} \in \Omega_E, \bar{\pi} \in \Omega_E\), it follows that

\[
h_t(E(\bar{x})) - h_j(E(\bar{\pi})) = 0, \ t \in T^+(E(\bar{\pi})),
\] (56)

\[
h_t(E(\bar{x})) - (-h_j(E(\bar{\pi}))) = 0, \ t \in T^-(E(\bar{\pi})).
\] (57)
Since all equality constraint functions \( h_t, \, t \in T^+ (E (\bar{x})) \) and \(-h_t, \, t \in T^- (E (\bar{x}))\) are \( E \)-differentiable quasi \( E \)-convex at \( \bar{x} \) on \( \Omega_E \), therefore, by Definition 2.11, (56) and (57) give, respectively,
\[
\nabla h_t (E (\bar{x})) (E (\bar{x}) - E (\bar{x})) \leq 0, \, t \in T^+ (E (\bar{x})), \tag{58}
\]
\[
- \nabla h_t (E (\bar{x})) (E (\bar{x}) - E (\bar{x})) \leq 0, \, t \in T^- (E (\bar{x})). \tag{59}
\]
Thus, (58) and (59) yield
\[
\sum_{t \in T^+ (E (\bar{x}))} \bar{\xi}_t \nabla h_t (E (\bar{x})) + \sum_{t \in T^- (E (\bar{x}))} \bar{\xi}_t \nabla h_t (E (\bar{x})) (E (\bar{x}) - E (\bar{x})) \leq 0.
\]
Hence, taking into account \( \bar{\xi}_t = 0, \, t \notin T^+ (E (\bar{x})) \cup T^- (E (\bar{x})) \), we have
\[
\sum_{t=1}^s \bar{\xi}_t \nabla h_t (E (\bar{x})) (E (\bar{x}) - E (\bar{x})) \leq 0. \tag{60}
\]
Combining (53), (55), and (60), we get that the following inequality
\[
\sum_{k=1}^p \bar{\lambda}_k \nabla (f_k^L \circ E) (\bar{x}) + \sum_{k=1}^p \bar{\lambda}_k \nabla (f_k^U \circ E) (\bar{x}) + \sum_{j=1}^m \bar{\mu}_j \nabla g_j (E (\bar{x})) + \sum_{t=1}^s \bar{\xi}_t \nabla h_t (E (\bar{x})) (E (\bar{x}) - E (\bar{x})) < 0
\]
holds, which is a contradiction to the \( E \)-Karush-Kuhn-Tucker necessary optimality condition (20). Hence, \( \bar{x} \) is a weak \( LU \)-Pareto solution of the problem (IVP\(_E\)). Then, by Lemma 3.7, it follows directly that \( E (\bar{x}) \) is a weak \( LU \)-E-Pareto solution of the considered multicriteria optimization problem (IVP) with the multiple interval-valued objective function. Thus, the proof of this theorem is completed. \( \Box \)

Now, we illustrate the optimality results established in the paper by the example of a nonconvex nondifferentiable interval-valued vector optimization problem in which the involved functions are \( E \)-differentiable.

**Example 3.16.** Consider the following nonconvex nondifferentiable vector optimization problem with a multiple interval-valued objective function
\[
f(x) = \left( \left[ \frac{1}{2}, 1 \right] (x_1^2 + x_2), \left[ \frac{1}{2}, 1 \right] (x_1^2 + x_2) \right) \rightarrow \min
\]
\[
g_1(x) = x_1^2 + x_1 + x_2 \leq 1,
\]
\[
g_2(x) = -x_1 \leq 0,
\]
\[
g_3(x) = -x_2 \leq 0.
\]
(IVP2)

Note that \( \Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_1 + x_2 \leq 1 \land -x_1 \leq 0 \land -x_2 \leq 0 \right\} \) and \( \bar{x} = (0, 0) \) is a feasible solution of the problem (IVP2). Let \( E : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a mapping defined by \( E (x_1, x_2) = (x_1^2, x_2) \). Note that \( E \) is a one-to-one and surjective map. Further, note that all functions constituting (IVP2) are \( E \)-differentiable at \( \bar{x} = (0, 0) \), although some of them are not differentiable (in the usual sense) at this point. Therefore, for the considered nondifferentiable interval-valued multiobjective
programming problem (IVP2), we construct its associated interval-valued \( E \)-vector optimization problem (IVP2\(_E\)) as follows:

\[
\begin{align*}
    f(E(x)) &= \left( \left[ \frac{1}{2} \right], 1 \right] (x_1 + x_2) \quad \text{min} \\
    g_1(E(x)) &= x_1 + x_1^3 + x_2 \leq 1, \\
    g_2(E(x)) &= -x_3^7 \leq 0, \\
    g_3(E(x)) &= -x_2 \leq 0.
\end{align*}
\]

(IVP2\(_E\))

It can be shown that the \( E \)-Karush-Kuhn-Tucker necessary optimality conditions (20)-(22) are fulfilled at \( \pi = (0,0) \) with Lagrange multipliers \( \lambda^L = \left( \frac{1}{2}, \frac{1}{2} \right) \), \( \lambda^U = \left( \frac{1}{2}, \frac{1}{2} \right) \), and \( \pi = \left( \frac{1}{2}, 1, \frac{1}{2} \right) \). Further, it can be proved that \( f \) is an \( E \)-differentiable \( E \)-convex function at \( \pi \) on \( \Omega_E \), the constraint functions \( g_1 \) and \( g_3 \) are \( E \)-differentiable \( E \)-convex at \( \pi \) on \( \Omega_E \), and the constraint function \( g_2 \) is \( E \)-differentiable quasi \( E \)-convex at \( \pi \) on \( \Omega_E \). Hence, by Theorem 3.13 (see also Remark 3), \( \pi = (0,0) \) is a \( LU \)-Pareto solution of the interval-valued \( E \)-vector optimization problem. Thus, \( E(\pi) \in \Omega \) is a \( LU-E \)-Pareto solution of the considered nondifferentiable vector optimization problem (IVP2) with the multiple interval-valued objective function.

Remark 4. Note that the classic Karush-Kuhn-Tucker necessary optimality conditions for differentiable interval-valued multiobjective problems are not applicable for the vector optimization problem (IVP2) with a multiple interval-valued objective function considered in Example 3.16. This is a consequence of the fact that both objective functions \( f_1, f_2 \) and the constraint function \( g_1 \) are not differentiable at \( \pi = (0,0) \). However, they are \( E \)-differentiable at this point. Therefore, in order to overcome this difficulty, we apply the \( E \)-Karush-Kuhn-Tucker necessary optimality conditions (20)-(22) established in the paper for \( E \)-differentiable interval-valued vector optimization problems. As it follows even from Example 3.16, the \( E \)-Karush-Kuhn-Tucker necessary optimality conditions (20)-(22) are useful for a larger class of vector optimization problems with multiple interval-valued objective functions than the classical Karush-Kuhn-Tucker necessary optimality conditions which are previously established in the literature for smooth interval-valued multiobjective programming problems (see, for example, [13], [25], [26], [29], [30], and others).

4. Concluding remarks. In this paper, an \( E \)-differentiable multiobjective programming problem with multiple interval-valued objective function and both inequality and equality constraints has been considered in which the involved functions are not necessarily differentiable in the classical sense. The so-called \( E \)-Karush-Kuhn-Tucker necessary optimality conditions have been established for the considered \( E \)-differentiable interval-valued vector optimization problem under the introduced \( E \)-Kuhn-Tucker constraint qualification. These necessary optimality conditions have been established to be useful even for such vector optimization problems with multiple interval-valued objective functions in which the involved functions are not differentiable in the classic sense. The result has been illustrated that, if the \( E \)-Kuhn-Tucker constraint qualification is not satisfied, the \( E \)-Karush-Kuhn-Tucker necessary optimality conditions fail in such a case. Further, the sufficient optimality conditions have been established for the considered \( E \)-differentiable multiobjective programming problem with multiple interval-valued objective function under appropriate \( E \)-convexity and/or generalized \( E \)-convexity hypotheses. Also it has been illustrated that these sufficient optimality conditions can be applied also
for such nonconvex vector optimization problems with multiple interval-valued objective functions in which the involved functions are not necessarily differentiable in the classical sense.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar optimality results for other classes of interval-valued optimization problems. We shall investigate these questions in our subsequent papers.

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