Necessary and sufficient conditions for the solvability of the Gauss variational problem for infinite dimensional vector measures

Natalia Zorii

Abstract

We continue our investigation of the Gauss variational problem for infinite dimensional vector measures associated with a condenser \((A_i)_{i \in I}\). It has been shown in Potential Anal., DOI:10.1007/s11118-012-9279-8 that, if some of the plates (say \(A_\ell\) for \(\ell \in \mathcal{L}\)) are noncompact then, in general, there exists a vector \(a = (a_i)_{i \in I}\), prescribing the total charges on \(A_i, i \in I\), such that the problem admits no solution. Then, what is a description of all the vectors \(a\) for which the Gauss variational problem is nevertheless solvable? Such a characterization is obtained for a positive definite kernel satisfying Fuglede’s condition of perfectness; it is given in terms of a solution to an auxiliary extremal problem intimately related to the operator of orthogonal projection onto the cone of all positive scalar measures supported by \(\bigcup_{\ell \in \mathcal{L}} A_\ell\). The results are illustrated by examples pertaining to the Riesz kernels.

Subject classification: 31C15.

Key words: infinite dimensional vector measure, external field, minimal energy problem

1 Introduction

The interest in minimal energy problems in the presence of an external field, which goes back to the pioneering works by Gauss [13] and Frostman [9], was initially motivated by their direct relations with the Dirichlet and balayage problems. A new impulse to this part of potential theory (which is often referred to as the Gauss variational problem) appeared in the 1980’s when Gonchar and Rakhmanov [14, 16], Mhaskar and Saff [23] efficiently applied logarithmic potentials with external fields in the investigation of orthogonal polynomials and rational approximations to analytic functions. E.g., the vector setting of the problem, earlier suggested by Ohtsuka [27, Section 2.9], has nowadays become particularly interesting in connection with Hermite–Padé rational approximations (see [1, 14, 15, 17, 25, 28]).

However, the potential-theoretical methods applied in these studies were mainly based on the vague (=weak*) topology, which made it possible to establish the existence of a solution only for vector measures of finite dimensions and compact support, though with a rather general matrix of interaction between their components. See, e.g., [27, Theorem 2.30].

In order to treat the Gauss variational problem for vector measures \(\mu = (\mu_i)_{i \in I}\) of infinite dimensions and/or noncompact support, associated with a condenser, in [36] we have suggested an approach based on introducing a metric structure on the class of all \(\mu\) with finite energy, which agrees properly with the vague topology, and also on establishing an infinite dimensional version of a completeness theorem (see below for details).

This enabled us to obtain sufficient conditions for the solvability of the problem (see [36, Theorem 8.1]); moreover, we have shown these sufficient conditions to be sharp by providing examples of the nonsolvability (see [36, Examples 8.1, 8.2]; see below for some details).

It is worth emphasizing that, as is seen from these examples, such a phenomenon of the nonsolvability occurs even for the Coulomb kernel \(|x - y|^{-1}\) in \(\mathbb{R}^3\) and a standard condenser of two oppositely signed plates, one of them being noncompact, which at first glance looks to be rather surprising because of an electrostatic interpretation of the problem.
In accordance with an electrostatic interpretation of a condenser, we assume that the inter- 
the product space \(\prod\) of all real-valued scalar Radon measures \(\nu\) on \(X\) equipped with the 
vague topology, i.e., the topology of pointwise convergence on the class \(C_0(X)\) of all continuous 
functions \(\varphi\) on \(X\) with compact support.\(^1\)

A kernel \(\kappa\) on \(X\) is meant to be an element from \(\Phi(X \times X)\), where \(\Phi(Y)\) consists of all lower 
semicontinuous functions \(\psi : Y \to (-\infty, \infty]\) such that \(\psi \geq 0\) unless \(Y\) is compact. Given 
\(\nu, \nu_1 \in \mathcal{M}\), the \textit{mutual energy} and the \textit{potential} relative to the kernel \(\kappa\) are defined by

\[
\kappa(\nu, \nu_1) := \int \kappa(x, y) \, d(\nu \otimes \nu_1)(x, y) \quad \text{and} \quad \kappa(\cdot, \nu) := \int \kappa(\cdot, y) \, d\nu(y),
\]

respectively. (When introducing a notation, we always tacitly assume the corresponding object 
on the right to be well defined — as a finite number or \(\pm \infty\).) For \(\nu = \nu_1\), \(\kappa(\nu, \nu_1)\) defines the 
energy of \(\nu\). Let \(\mathcal{E} = \mathcal{E}_\kappa\) consist of all \(\nu \in \mathcal{M}\) with \(-\infty < \kappa(\nu, \nu) < \infty\).

We shall mainly be concerned with a \textit{positive definite} kernel \(\kappa\), which by \(^1\) means that it is 
symmetric (i.e., \(\kappa(x, y) = \kappa(y, x)\) for all \(x, y \in X\)) and \(\kappa(\nu, \nu), \nu \in \mathcal{M}\), is nonnegative whenever 
defined. Then \(\mathcal{E}\) forms a pre-Hilbert space with the scalar product \(\kappa(\cdot, \cdot)\) and the seminorm 
\(\|\nu\| : = \sqrt{\kappa(\nu, \nu)}\). The topology on \(\mathcal{E}\) defined by this seminorm is called \textit{strong}. A 
positive definite kernel \(\kappa\) is \textit{strictly positive definite} if the seminorm \(\|\cdot\|\) is a norm.

Recall that a measure \(\nu \geq 0\) is \textit{concentrated} on a set \(E \subset X\) if \(E^c := X \setminus E\) is locally \(\nu\)-negligible; 
or equivalently, if \(E\) is \(\nu\)-measurable and \(\nu = \nu|_E\), where \(\nu|_E\) is the trace of \(\nu\) on \(E\) (see, e.g., 
[2][8]). Let \(\mathcal{M}^+_E\) be the convex cone of all nonnegative measures concentrated on \(E\), and let 
\(\mathcal{E}^+_E := \mathcal{M}^+_E \cap \mathcal{E}\). We also write \(\mathcal{M}^+_\infty := \mathcal{M}^+_1\) and \(\mathcal{E}^+_\infty := \mathcal{E}^+_1\).

The \textit{interior capacity} of a set \(E\) relative to the kernel \(\kappa\) is given by the formula\(^2\)

\[
C(E) := C_\kappa(E) := 1/\inf_{\nu \in \mathcal{E}^+_E} \kappa(\nu, \nu),
\]

where the infimum is taken over all \(\nu \in \mathcal{E}^+_E\) with \(\nu(E) = 1\).

We consider a countable, locally finite collection\(^3\) \(A = (A_i)_{i \in I}\) of closed sets, called plates, 
\(A_i \subset X\) with the sign \(+1\) or \(-1\) prescribed such that the oppositely signed sets are mutually 
disjoint. Let \(\mathcal{M}^+(A)\) stand for the Cartesian product \(\prod_{i \in I} \mathcal{M}^+(A_i)\); then \(\mu \in \mathcal{M}^+(A)\) is a 
(nonnegative) \textit{vector measure} \((\mu^i)_{i \in I}\) with the components \(\mu^i \in \mathcal{M}^+(A_i)\). The topology of 
the product space \(\prod_{i \in I} \mathcal{M}^+(A_i)\), where every \(\mathcal{M}^+(A_i)\) is endowed with the vague topology, is 
likewise called \textit{vague}. If \(\mu \in \mathcal{M}^+(A)\) and a vector-valued function \(u = (u_i)_{i \in I}\) with the \(\mu^i\)-
measurable components \(u_i : A_i \to [-\infty, \infty]\) are given, then we write \(\langle u, \mu \rangle := \sum_{i \in I} \int u_i \, d\mu^i\).\(^4\)

In accordance with an electrostatic interpretation of a condenser, we assume that the interaction 
between the charges lying on the conductors \(A_i, \ i \in I\), is characterized by the matrix 
\((\alpha_{i,j})_{i,j \in I}\), where \(\alpha_{ij} := \text{sign} A_i\). Given \(\mu, \mu_1 \in \mathcal{M}^+(A)\), we define the \textit{mutual energy}

\[
\kappa(\mu, \mu_1) := \sum_{i,j \in I} \alpha_{i,j} \kappa(\mu^i, \mu_1^j) \quad (1.1)
\]

\(^1\)When speaking of a continuous function, we understand that the values are \textit{finite real} numbers.

\(^2\)Throughout the paper, the infimum over the empty set is taken to be \(+\infty\).

\(^3\)If \(I\) is a singleton, then we preserve the normal fonts instead of the bold ones.

\(^4\)Here and in the sequel, an expression \(\sum_{i \in I} c_i\) is meant to be well defined provided that so is every 
summand \(c_i\) and the sum does not depend on the order of summation — though might be \(\pm \infty\). Then, by the 
Riemann series theorem, the sum is finite if and only if the series converges absolutely.
and the *vector potential* \( \kappa_\mu(x) \), \( x \in X \), as a vector-valued function with the components

\[
\kappa_\mu^i(x) := \sum_{j \in I} \alpha_i \alpha_j \kappa(x, \mu^j), \quad i \in I. \tag{1.2}
\]

For \( \mu = \mu_1 \), the mutual energy \( \kappa(\mu, \mu_1) \) defines the *energy* of \( \mu \). Let \( \mathcal{E}^+(A) \) consist of all \( \mu \in \mathfrak{M}^+(A) \) with \( -\infty < \kappa(\mu, \mu) < \infty \).

Fix a vector-valued function \( f = (f_i)_{i \in I} \), where each \( f_i : X \to [-\infty, \infty] \) is treated as an external field acting on the charges on \( A_i \). Then the *f-weighted vector potential* and the *f-weighted energy* of \( \mu \in \mathcal{E}^+(A) \) are defined by

\[
W_\mu := \kappa_\mu + f, \quad G_f(\mu) := \kappa(\mu, \mu) + 2(f, \mu), \tag{1.3}
\]

respectively. Throughout this paper, we assume that \( f_i(x) = \alpha_i \kappa(x, \chi) \) for all \( i \in I \), \( \tag{1.4} \)

where a (signed) measure \( \chi \in \mathcal{E} \) is given, and we also write \( G_\chi(\mu) := G_f(\mu) \).

Also fix a numerical vector \( a = (a_i)_{i \in I} \) with \( a_i > 0 \) for all \( i \in I \) such that

\[
|a| := \sum_{i \in I} a_i < \infty \tag{1.5}
\]

and a continuous function \( g \) on \( A := \bigcup_{i \in I} A_i \) with

\[
g_{\inf} := \inf_{x \in A} g(x) > 0. \tag{1.6}
\]

We are interested in the problem of minimizing \( G_\chi(\mu) \) over the class of all \( \mu \in \mathcal{E}^+(A) \) normalized by \( \langle g, \mu^I \rangle = a_i \) for all \( i \in I \), referred to as the *Gauss variational problem*.

The main question is whether minimizers \( \lambda \) in the problem exist. If the condenser \( A \) is finite and compact, the kernel \( \kappa \) is continuous on \( A_i \times \overline{A_j} \) whenever \( \alpha_i \neq \alpha_j \) and \( f_i \in \Phi(X) \) for all \( i \in I \), then the existence of those \( \lambda \) can easily be established by exploiting the vague topology only, since then the class of admissible vector measures is vaguely compact, while \( G_f(\mu) \) is vaguely lower semicontinuous. See [27, Theorem 2.30] (cf. also [14, 15, 25, 28] for the logarithmic kernel in the plane).

However, these arguments break down if any of the above-mentioned four assumptions is dropped, and then the problem on the existence of minimizers becomes rather nontrivial. In particular, the class of admissible vector measures is no longer vaguely compact if any of the \( A_i \) is noncompact. Another difficulty is that, under assumption \[15\], the *f-weighted energy* \( G_f(\mu) \) might not be vaguely lower semicontinuous.

For a positive definite kernel satisfying Fuglede’s condition of consistency between the strong and vague topology on \( \mathcal{E}^+ \) (see Definition 2.1 below), these difficulties have been overcome in [30] in the frame of an approach\[3\] based on the following crucial arguments.

---

3. The results obtained and the approach developed in [30] also hold provided that \( f_i \in \Phi(X) \) for all \( i \in I \) (which is, in particular, the case if each \( f_i \) is the potential of a Dirac measure). However, for a finer analysis to be carried out in the present study in order to establish necessary and sufficient conditions of the solvability, we need to assume the external field to be generated— in the sense of \[30\]— by a charge of finite energy.

6. For a background of this approach, see the pioneering work by Fuglede \[10\] (where \( I = \{1\} \), \( g = 1 \), and \( f = 0 \)) and also the author’s studies \[32\]–\[35\] (where \( I \) is finite).
The set $\mathcal{E}^+(A)$ has been shown to be a semimetric space with the semimetric
\[
\|\mu_1 - \mu_2\|_{\mathcal{E}^+(A)} := \left[ \sum_{i,j \in I} \alpha_i \alpha_j \kappa(\mu_i^1 - \mu_i^2, \mu_j^1 - \mu_j^2) \right]^{1/2},
\]
and one can define an inclusion $R$ of $\mathcal{E}^+(A)$ into $\mathcal{E}$ such that $\mathcal{E}^+(A)$ becomes isometric to its $R$-image, the latter being regarded as a semimetric subspace of the pre-Hilbert space $\mathcal{E}$. See [36, Theorem 3.1]; cf. also Theorem 4.11 below. We therefore call the topology of the semimetric space $\mathcal{E}^+(A)$ strong.

Another crucial fact is that if, in addition, the kernel $\kappa$ is bounded from above on the product of the oppositely signed plates, then the topological subspace of $\mathcal{E}^+(A)$ consisting of all $\mu$ with $\sum_{i \in I} \mu_i'(X) \leq b$ (where $b$ is given) is strongly complete. Moreover, the strong topology on this subspace is in agreement with the vague one in the sense that any strong Cauchy net converges strongly to each of its vague cluster points. See [36, Theorem 9.1]; cf. also Theorem 4.2 below.

All these enabled us to prove in [36] that, moreover, $g$ is bounded from above, then the Gauss variational problem is solvable provided that each $A_i$ either is compact or has finite interior capacity. See [36, Theorem 8.1]; cf. also Theorem 5.1 below. However, if some of the plates, say $A_\ell$ for $\ell \in L$, are noncompact and have infinite interior capacity then, in general, there exists a vector $a$ such that the problem admits no solution. See Examples 8.1 and 8.2 in [36], illustrating such a phenomenon of the nonsolvability (cf. also Example 5.1 below; see also [18, 26] for some related numerical experiments).

Then, for given $\kappa$, $A$, $g$, and $\chi$, what is a description of the set $S_\kappa(A, g, \chi)$ of all vectors $a$ for which the Gauss variational problem is nevertheless solvable?

In this paper, such a characterization is established; it is given in terms of a solution to an auxiliary extremal problem intimately related to the operator of orthogonal projection onto the cone of all positive scalar measures supported by $\bigcup_{\ell \in I} \alpha \kappa$. In the case where $L$ is a singleton, the description obtained is complete.

To give a hint how it looks like, we start with the following example, where $L = \{\ell\}$ while $\kappa$ is the Riesz kernel $\kappa_\alpha(x, y) = |x - y|^{\alpha - n}$ of order $\alpha \in (0, 2]$, $\alpha < n$, in $\mathbb{R}^n$, $n \geq 2$. For this kernel, the operator of orthogonal projection in $\mathcal{E}_\kappa$ onto $\mathcal{E}_\kappa^+$ is, in fact, the operator of Riesz balayage $\beta^\kappa_{\kappa, A_\ell}$ onto $A_\ell$, related to the notion of $\alpha$-Green function $g^\alpha_{\kappa, A_\ell}$ of $A_\ell$ by the formula
\[
g^\alpha_{\kappa, A_\ell}(x, y) := \kappa_\alpha(x, \varepsilon_y) - \kappa_\alpha(x, \beta^\kappa_{\kappa, A_\ell} \varepsilon_y),
\]
$\varepsilon_y$ being the unit Dirac measure at $y$ (see, e.g., [22, Chapter 4, Section 5]).

**Example 1.1** Under these hypotheses, let the Euclidean distance between the oppositely signed plates be nonzero, $A_i \cap A_\ell = \emptyset$ for all $i \neq \ell$, $C_\kappa(A_i) > \delta > 0$ for all $i \in I$, and let $\chi \in \mathcal{E}_\kappa$ be compactly supported in $A^c$.

**Proposition 1.1** Then the set $S_\kappa(A, g, \chi)$ consists of all vectors $a = (a_i)_{i \in I}$ such that
\[
a_\ell \leq \langle g, \beta^\kappa_{\kappa, A_\ell}(\chi + \sum_{i \neq \ell} \alpha_i \sigma^i) \rangle,
\]
where $(\sigma^i)_{i \neq \ell}$ is a solution (it exists) to the problem of minimizing the $\alpha$-Green energy
\[
\left\| \chi + \sum_{i \neq \ell} \alpha_i \mu_i \right\|_{g^\alpha_{\kappa, A_\ell}}^2
\]
over all $(\mu^i)_{i \neq \ell}$ with the properties that $\mu^i$, $i \neq \ell$, is supported by $A_i$ and $\langle g, \mu^i \rangle = a_i$.

\[\text{Note that } \sum_{i \neq \ell} \alpha_i \sigma^i \text{ is a scalar Radon measure due to the local finiteness of } A, \text{ so that formula } 1.9 \text{ makes sense.}\]
The rest of the paper is organized as follows. In Sections 4 and 5 we summarize without proof some results from [36], necessary for the understanding of the subject matter. A description of the set $S_\kappa(A, g, \chi)$ is provided by Theorems 7.2 and 7.3, the main results of the study; it is given in terms of a solution $\tilde{\lambda}$ to an auxiliary extremal problem (see Section 5 for its formulation), while the existence of this $\tilde{\lambda}$ is guaranteed by Theorem 7.1. The proofs of Theorems 7.1, 7.2 and 7.3 are provided in Sections 10, 11 and 13, respectively; see also Sections 3, 8, 9 and 12 for some crucial auxiliary results. Finally, Section 7.2 contains some further examples with the Riesz kernels, illustrating the results obtained.

2 Preliminaries

We write $S(\nu)$ or $S_\nu$ for the support of $\nu \in \mathcal{M}$. A measure $\nu$ is called finite if $S_\nu$ is compact, and bounded if $|\nu|(X) < \infty$. Here $|\nu| := \nu^+ + \nu^-$, where $\nu^+$ and $\nu^-$ are respectively the positive and negative parts in the Hahn–Jordan decomposition of $\nu$.

In this section we assume the kernel $\kappa$ to be positive definite. Then $\mathcal{E}$ forms a pre-Hilbert space with the scalar product $\kappa(\nu, \nu_1)$ and the seminorm $\|\nu\| := \|\nu\|_\kappa := \sqrt{\kappa(\nu, \nu)}$, while the potential $\kappa(\cdot, \nu)$ of any $\nu \in \mathcal{E}$ is well defined and finite nearly everywhere (n.e.) in $X$, i.e. except for a subset with interior capacity zero; see [10]. In minimal energy problems, the following lemma from [10] is often useful.

**Lemma 2.1** Consider a convex set $\mathcal{H} \subset \mathcal{E}$ and assume there exists $\lambda_\mathcal{H} \in \mathcal{H}$ such that

$$\|\lambda_\mathcal{H}\| = \min_{\nu \in \mathcal{H}} \|\nu\|.$$

Then

$$\|\nu - \lambda_\mathcal{H}\|^2 \leq \|\nu\|^2 - \|\lambda_\mathcal{H}\|^2 \quad \text{for all } \nu \in \mathcal{H}.$$

2.1 Consistent and perfect kernels

In addition to the strong topology on $\mathcal{E}$, determined by the seminorm $\|\nu\|$, sometimes we shall consider the weak topology on $\mathcal{E}$, defined by means of the seminorms $\nu \mapsto |\kappa(\nu, \nu_1)|$, $\nu_1 \in \mathcal{E}$ (see, e.g., [10]). The Cauchy–Schwarz inequality

$$|\kappa(\nu, \nu_1)| \leq \|\nu\| \|\nu_1\|,$$

where $\nu, \nu_1 \in \mathcal{E}$, implies immediately that the strong topology on $\mathcal{E}$ is finer than the weak one.

In [10]–[11], Fuglede introduced the following two equivalent properties of consistency between the induced strong, weak, and vague topologies on $\mathcal{E}^+$:

(C$_1$) Every strong Cauchy net in $\mathcal{E}^+$ converges strongly to any of its vague cluster points;

(C$_2$) Every strongly bounded and vaguely convergent net in $\mathcal{E}^+$ converges weakly to the vague limit.

**Definition 2.1** Following Fuglede [10], we call a kernel $\kappa$ consistent if it satisfies either of the properties (C$_1$) and (C$_2$), and perfect if, in addition, it is strictly positive definite.

**Remark 2.1** One has to consider nets or filters in $\mathcal{M}^+$ instead of sequences, since the vague topology in general does not satisfy the first axiom of countability. We follow Moore’s and Smith’s theory of convergence, based on the concept of nets (see [24]: cf. also [8] Chapter 0 and [21] Chapter 2). However, if $X$ is metrizable and can be written as a countable union of compact sets, then $\mathcal{M}^+$ satisfies the first axiom of countability (see [10] Lemma 1.2.1) and the use of nets may be avoided.
Theorem 2.1 (Fuglede [10]) A kernel $\kappa$ is perfect if and only if $E^+$ is strongly complete and the strong topology on $E^+$ is finer than the vague one.

Remark 2.2 In $\mathbb{R}^n$, $n \geq 3$, the Newtonian kernel $|x - y|^{2-n}$ is perfect [4]. So are the Riesz kernels $|x - y|^{\alpha-n}$, $0 < \alpha < n$, in $\mathbb{R}^n$, $n \geq 2$ [5, 6], and the restriction of the logarithmic kernel $-\log |x - y|$ in $\mathbb{R}^2$ to the open unit disk [22]. Furthermore, if $D$ is an open set in $\mathbb{R}^n$, $n \geq 2$, and its generalized Green function $g_D$ exists (see, e.g., [20, Theorem 5.24]), then the kernel $g_D$ is perfect as well [7].

It is seen from Definition 2.1 and Theorem 2.1 that the concept of consistent or perfect kernels is an efficient tool in minimal energy problems over classes of nonnegative scalar Radon measures with finite energy. Indeed, Fuglede's theory of capacities of sets has been developed in [10] for exactly those kernels. The following fundamental result of this theory will often be used below.

Theorem 2.2 Assume that the kernel $\kappa$ is consistent. Then for any set $E \subset X$ with $C(E) < \infty$ there exists a measure $\theta_E \in E^+$ with the properties

$$\theta_E(X) = \|\theta_E\|^2 = C(E),$$

$$\kappa(x, \theta_E) \geq 1 \text{ n.e. in } E,$$

$$\kappa(x, \theta_E) \leq 1 \text{ for all } x \in S(\theta_E).$$

This $\theta_E$, called an interior equilibrium measure associated with $E$, is a solution to the problem of minimizing $\kappa(\nu, \nu)$ over the class $\Gamma_E$ of all $\nu \in E$ such that $\kappa(x, \nu) \geq 1 \text{n.e. in } E$, and it is determined uniquely up to a summand with seminorm zero.

Remark 2.3 In [35, 36] we have shown that the concept of consistent or perfect kernels is efficient, as well, in minimal energy problems over classes of vector measures of finite or infinite dimensions, associated with a condenser $A$. This is guaranteed by a theorem on the completeness of certain topological subspaces of the semimetric space $E^+(A)$ (see [35, Theorem 13.1] and [36, Theorem 9.1], cf. also Theorem 4.2 below; compare with Theorem 2.1).

3 Auxiliary results related to scalar measures

In the following Lemmas 3.1–3.5 the kernel is arbitrary (not necessarily positive definite).

Lemma 3.1 (Fuglede [10]) For any given $E \subset X$, it holds that $C(E) = 0 \iff E^+ = \emptyset$.

Lemma 3.2 If $\nu \in E^+_E$ is bounded, then any proposition holds $\nu$-almost everywhere ($\nu$-a.e.) in $X$, provided that it holds n.e. in $E$.

Proof. Indeed, then $E$ is $\nu$-integrable and hence, by [8 Proposition 4.14.1], any locally $\nu$-negligible subset of $E$ is $\nu$-negligible. Since, according to Lemma 3.1, a set of interior capacity zero is locally $\xi$-negligible for any $\xi \in E^+$, the lemma follows. }

\[\text{8}\]Compare with [33, Theorem 10.1], generalizing Theorem 2.2 to the interior capacities of condensers with finitely many plates.
3.1 On continuity of potentials

We shall need the following lemmas on continuity, the first being well known (see, e.g., [10]).

**Lemma 3.3** For any \( \psi \in \Phi(X) \) the map \( \nu \mapsto \langle \psi, \nu \rangle \) is vaguely lower semicontinuous on \( \mathcal{M}^+ \).

In particular, this implies that the potential \( \kappa(\cdot, \nu) \) of any \( \nu \in \mathcal{M}^+ \) belongs to \( \Phi(X) \).

**Lemma 3.4** Assume that \( \nu \in \mathcal{M}^+ \) is bounded, \( \kappa(x, y) \) is continuous for \( x \neq y \) and

\[
\sup_{x \in K, y \in S_{\nu}} \kappa(x, y) < \infty \quad \text{for every compact } K \subset S^c_{\nu}. \tag{3.1}
\]

Then the potential \( \kappa(\cdot, \nu) \) is continuous at every \( x_0 \notin S_{\nu} \).

**Proof.** Having fixed a point \( x_0 \notin S_{\nu} \) and its compact neighborhood \( V_{x_0} \) so that \( V_{x_0} \cap S_{\nu} = \emptyset \), we consider the function \( \kappa^*(x, y) \) on \( V_{x_0} \times S_{\nu} \) defined by

\[
\kappa^*(x, y) := -\kappa(x, y) + \sup_{x' \in V_{x_0}, y' \in S_{\nu}} \kappa(x', y'). \tag{3.2}
\]

Under the hypotheses of the lemma, \( \kappa^* \) is nonnegative and continuous; hence,

\[
\kappa^*(x, \nu) = \int \kappa^*(x, y) \, d\nu(y), \quad x \in V_{x_0},
\]

is lower semicontinuous as the potential of \( \nu \in \mathcal{M}^+ \) relative to the kernel \( \kappa^* \).

On the other hand, integrating (3.2) with respect to the (bounded) measure \( \nu \), we conclude from assumption (3.1) that \( \kappa^*(x, \nu), \ x \in V_{x_0} \), coincides up to a finite summand with the restriction of \(-\kappa(x, \nu)\) to \( V_{x_0} \). What has been shown just above therefore implies that \(-\kappa(\cdot, \nu)\) is lower semicontinuous at \( x_0 \), and the lemma follows. \( \square \)

**Definition 3.1** A kernel \( \kappa \) is said to possess the property \((\infty_X)\) if \( \kappa(\cdot, y) \to 0 \) as \( y \to \infty_X \) uniformly on compact sets; then for every \( \varepsilon > 0 \) and every compact \( K \subset X \) there exists a compact set \( K' \subset X \) such that \( |\kappa(x, y)| < \varepsilon \) for all \( x \in K \) and \( y \in X \setminus K' \).

For any \( b \in (0, \infty) \), write \( \mathcal{M}_b := \{ \nu \in \mathcal{M} : |\nu|(X) \leq b \} \).

**Lemma 3.5** Fix a closed set \( F \subset X \), a closed subset \( Q \) of \( F^c \), and \( b \in (0, \infty) \). If a kernel \( \kappa(x, y) \) is continuous for \( x \neq y \) and possesses the property \((\infty_X)\), then the mapping

\[
(x, \nu) \mapsto \kappa(x, \nu) \quad \text{on } Q \times (\mathcal{M}^+_F \cap \mathcal{M}_b)
\]

is continuous in the product topology, where \( Q \) and \( \mathcal{M}^+_F \cap \mathcal{M}_b \) are considered to be topological subspaces of \( X \) and \( \mathcal{M} \), respectively.

**Proof.** Fix \( x_0, x_s \in Q \) and \( \nu_0, \nu_s \in \mathcal{M}^+_F \cap \mathcal{M}_b \), \( s \in S \), such that \( (x_s, \nu_s) \to (x_0, \nu_0) \) (as \( s \) ranges along \( S \)) in the topology of the product space \( Q \times (\mathcal{M}^+_F \cap \mathcal{M}_b) \). We need to show that

\[
\kappa(x_0, \nu_0) = \lim_{s \to \infty} \kappa(x_s, \nu_s). \tag{3.3}
\]

\footnote{Compare with [10] Lemma 2.2.1, assertion (b).}
Due to the property \((\infty_X)\), for any \(\varepsilon > 0\) one can choose a compact neighborhood \(W_{x_0}\) of the point \(x_0\) and a compact neighborhood \(W\) of the set \(W_{x_0}\) so that \(W_{x_0} \cap F = \emptyset\) and
\[
|\kappa(x, y)| < \varepsilon b^{-1} \quad \text{for all} \quad (x, y) \in W_{x_0} \times W^c.
\]
(3.4)

Certainly, there is no loss of generality in assuming \(V := W \cap F \neq \emptyset\).

Let \(E^F\) and \(\partial_F E\) denote respectively the complement and the boundary of \(E\) relative to \(F\), where \(F\) is treated as a topological subspace of \(X\). Having observed that \(\kappa|_{W_{x_0} \times F}\) is continuous, we proceed by constructing a function \(\varphi \in C_0(W_{x_0} \times F)\) such that
\[
\varphi|_{W_{x_0} \times V} = \kappa|_{W_{x_0} \times V},
\]
(3.5)
\[
|\varphi(x, y)| \leq \varepsilon b^{-1} \quad \text{for all} \quad (x, y) \in W_{x_0} \times V^cF.
\]
(3.6)

To this end, consider a compact neighborhood \(V_s\) of \(V\) in \(F\) and write
\[
g := \begin{cases} \kappa & \text{on} \quad W_{x_0} \times \partial_F V, \\ 0 & \text{on} \quad W_{x_0} \times \partial_F V_s. \end{cases}
\]

Note that \(K := (W_{x_0} \times \partial_F V) \cup (W_{x_0} \times \partial_F V_s)\) is a compact subset of the Hausdorff and compact (hence, normal) space \(W_{x_0} \times V_s\), while the function \(g\) is continuous on \(K\). By using the Tietze–Urysohn extension theorem (see, e.g., [8, Chapter 0]), we deduce from relation (3.4) that there exist a continuous function \(\tilde{g} : W_{x_0} \times V_s \rightarrow [-\varepsilon b^{-1}, \varepsilon b^{-1}]\) such that \(\tilde{g}|_{K} = g|_{K}\). Hence, the function in question can be defined by means of the formula
\[
\varphi := \begin{cases} \kappa & \text{on} \quad W_{x_0} \times V, \\ \tilde{g} & \text{on} \quad W_{x_0} \times (V_s \setminus V), \\ 0 & \text{on} \quad W_{x_0} \times V^cF. \end{cases}
\]

Furthermore, since such a function \(\varphi\) is continuous on \(W_{x_0} \times F\) and has compact support, one can choose a compact neighborhood \(U_{x_0}\) of \(x_0\) in \(W_{x_0}\) so that
\[
|\varphi(x, y) - \varphi(x_0, y)| < \varepsilon b^{-1} \quad \text{for all} \quad (x, y) \in U_{x_0} \times F.
\]
(3.7)

Therefore, for any \(\nu \in \mathcal{M}_F^+ \cap \mathcal{M}_b\) and \(x \in U_{x_0}\) we get, due to relations (3.4)–(3.7),
\[
|\kappa(x, \nu|_{W^c})| \leq \varepsilon,
\]
(3.8)
\[
\kappa(x, \nu|_W) = \int \varphi(x, y) \, d(\nu - \nu|_{W^c})(y),
\]
(3.9)
\[
\left|\int \varphi(x, y) \, d(\nu|_{W^c})(y)\right| \leq \varepsilon,
\]
(3.10)
\[
\left|\int [\varphi(x, y) - \varphi(x_0, y)] \, d\nu(y)\right| \leq \varepsilon.
\]
(3.11)

Choose \(s_0 \in S\) so that for all \(s \in S\) that follow \(s_0\) there hold \(x_s \in U_{x_0}\) and
\[
\left|\int \varphi(x_0, y) \, d(\nu_s - \nu_0)(y)\right| < \varepsilon.
\]

Combined with relations (3.8)–(3.11), this shows that for all \(s \geq s_0\),
\[
|\kappa(x_s, \nu_s) - \kappa(x_0, \nu_0)| \leq |\kappa(x_s, \nu_s|_W) - \kappa(x_0, \nu_0|_W)| + 2\varepsilon
\]
\[
\leq \left|\int \varphi(x_s, y) \, d\nu_s(y) - \int \varphi(x_0, y) \, d\nu_0(y)\right| + 4\varepsilon
\]
\[
\leq \left|\int [\varphi(x_s, y) - \varphi(x_0, y)] \, d\nu_s(y)\right| + \left|\int \varphi(x_0, y) \, d(\nu_s - \nu_0)(y)\right| + 4\varepsilon
\]
\[
\leq \varepsilon + \varepsilon + 4\varepsilon = 6\varepsilon,
\]

which in view of the arbitrary choice of \(\varepsilon\) establishes (3.3). \(\square\)
3.2 Orthogonal projections in $\mathcal{E}$

Throughout this section we require the kernel $\kappa$ to be perfect (see Definition 2.1).

Fix $\nu \in \mathcal{E}$ and a closed set $F \subset X$ with $C(F) > 0$, and let $P_F$ be the operator of orthogonal projection in the pre-Hilbert space $\mathcal{E}$ onto the convex cone $\mathcal{E}_F^+$ (see [8] Section 1.12.3); note that $\mathcal{E}_F^+ \neq \emptyset$ due to Lemma 3.1. Then $P_F\nu$ is a measure in $\mathcal{E}_F^+$ such that

$$\|\nu - P_F\nu\| = \inf_{\omega \in \mathcal{E}_F^+} \|\nu - \omega\| =: g(\nu, \mathcal{E}_F^+).$$

Observe that $\mathcal{E}_F^+$, treated as a metric subspace of $\mathcal{E}$, is complete in consequence of Theorem 2.1 hence, according to [8] Theorem 1.12.3, $P_F\nu$ exists and is determined uniquely.

**Lemma 3.6** If $K$ ranges through the increasing filtering family $\{K\}_F$ of all compact subsets of $F$, then

$$P_K\nu \to P_F\nu \quad \text{strongly and vaguely.}$$

**Proof.** For each $K \in \{K\}_F$ one can certainly assume $C(K) > 0$, which causes no loss of generality because of $C(F) = \sup_{K \in \mathcal{E}} C(K)$ (see [10] p. 153); hence, the projection $P_K\nu$ exists (and is unique). We next observe that $g(\nu, \mathcal{E}_K^+)$ decreases as $K \uparrow F$ and

$$g(\nu, \mathcal{E}_F^+ \setminus \mathcal{E}_K^+) \leq \lim_{K \uparrow F} g(\nu, \mathcal{E}_K^+). \quad (3.12)$$

On the other hand, applying [10] Lemma 1.2.2 to the measure $P_F\nu \otimes P_F\nu$ and the function $\kappa$, as well as to $P_F\nu$ and each of $\kappa(\cdot, \nu^+)$ and $\kappa(\cdot, \nu^-)$, we obtain

$$\|P_F\nu\| = \lim_{K \uparrow F} \|(P_F\nu)|_K\| \quad \text{and} \quad \kappa(\nu, P_F\nu) = \lim_{K \uparrow F} \kappa(\nu, (P_F\nu)|_K),$$

therefore

$$g(\nu, \mathcal{E}_F^+) = \|\nu - P_F\nu\| = \lim_{K \uparrow F} \|\nu - (P_F\nu)|_K\| \geq \lim_{K \uparrow F} g(\nu, \mathcal{E}_K^+).$$

When combined with relation (3.12), this establishes the equality

$$g(\nu, \mathcal{E}_F^+) = \lim_{K \uparrow F} g(\nu, \mathcal{E}_K^+). \quad (3.13)$$

Fix $K, \hat{K} \in \{K\}_F$ such that $K \subset \hat{K}$. Applying Lemma 2.1 to the (convex) set

$$\nu - \mathcal{E}_K^+ := \{\nu - \omega : \omega \in \mathcal{E}_K^+\},$$

in view of $\nu - P_K\nu \in \nu - \mathcal{E}_K^+$ we get

$$\|P_{\hat{K}}\nu - P_K\nu\|^2 = \|\nu - (P_{\hat{K}}\nu) - (\nu - P_K\nu)\|^2 \leq g^2(\nu, \mathcal{E}_K^+) - g^2(\nu, \mathcal{E}_{\hat{K}}^+).$$

As $g(\nu, \mathcal{E}_K^+), K \in \{K\}_F$, is fundamental in $\mathbb{R}$ because of (3.13), the last relation shows that $(P_K\nu)_{K \in \{K\}_F}$ is a strong Cauchy net in $\mathcal{E}_F^+$. Since $\mathcal{E}_F^+$ is strongly complete, this net converges to some $\omega_0 \in \mathcal{E}_F^+$ strongly and, hence, weakly. Repeated application of (3.13) then yields

$$\lim_{K \uparrow F} g(\nu, \mathcal{E}_K^+) = \|\nu - P_K\nu\| = \|\nu - \omega_0\| = g(\nu, \mathcal{E}_F^+),$$

which due to the uniqueness statement gives $\omega_0 = P_F\nu$, and the lemma follows. \qed
As an immediate consequence of Lemmas 3.3 and 3.6, we get

\[
P_F \nu(X) \leq \liminf_{K \uparrow F} P_K \nu(X), \tag{3.14}\]
\[
\kappa(x, P_F \nu) \leq \liminf_{K \uparrow F} \kappa(x, P_K \nu) \text{ n.e. in } X. \tag{3.15}\]

**Lemma 3.7** It holds that

\[
\kappa(x, P_F \nu) \geq \kappa(x, \nu) \text{ n.e. in } F. \tag{3.16}\]

If, moreover, \(\kappa(x, y)\) is continuous for \(x \neq y\), \(\nu^+\) is bounded, \(S(\nu^+) \cap F = \emptyset\) and \(\sup_{x \in K, y \in S(\nu^+)} \kappa(x, y) < \infty\) for every compact \(K \subset F\), then

\[
\kappa(x, P_F \nu) \leq \kappa(x, \nu) \text{ for all } x \in S(P_F \nu), \tag{3.17}\]

and therefore

\[
\kappa(x, P_F \nu) = \kappa(x, \nu) \text{ n.e. in } S(P_F \nu). \tag{3.18}\]

**Proof.** According to [8, Proposition 1.12.4], \(P_F \nu\) is uniquely characterized by the relations

\[
\kappa(\nu - P_F \nu, \omega) \leq 0 \text{ for all } \omega \in E_F^+, \tag{3.19}\]
\[
\kappa(\nu - P_F \nu, P_F \nu) = 0, \tag{3.20}\]

We observe that assertion (3.16) can be obtained from inequality (3.19) with the help of arguments similar to those in [22, Proof of Theorem 4.16]. Indeed, for each \(\omega \in E_F^+\) the set

\[
E := \{x \in F : \kappa(x, P_F \nu) < \kappa(x, \nu)\}
\]

is \(\omega\)-measurable and, hence, one can consider \(\omega|_E\), the trace of \(\omega\) on \(E\). Since \(\omega|_E\) is an element of \(E_F^+\) as well, relation (3.19) gives

\[
\int \kappa(x, \nu - P_F \nu) \, d\omega|_E(x) \leq 0,
\]

which, however, is possible only provided that \(E\) is locally \(\omega\)-negligible. In view of the arbitrary choice of \(\omega \in E_F^+\), this together with Lemma 3.1 yields \(C(E) = 0\) as desired.

Let now all the hypotheses of the second part of the lemma be satisfied. It follows from inequality (3.15) that, while proving assertion (3.17), one can assume \(F\) to be compact. Then, by Lemma 3.2 relation (3.16) holds \(P_F \nu\)-a.e. in \(F\). This implies

\[
\kappa(x, \nu - P_F \nu) = 0 \text{ \(P_F \nu\)-a.e. in } F,
\]

for if not, we would arrive at a contradiction with (3.20) when integrating inequality (3.16) with respect to \(P_F \nu\). In turn, the last relation yields that for every \(x \in S(P_F \nu)\) one can choose a net \((x_s)_{s \in S} \subset F\) with the properties that \(x_s \to x\) and

\[
\kappa(x_s, \nu - P_F \nu) = 0 \text{ for all } s \in S.
\]

Hence, assertion (3.17) will be proved once we show that \(\kappa(x, \nu - P_F \nu)\) is upper semicontinuous on \(F\). As \(\kappa(x, \nu^+ - P_F \nu)\) is lower semicontinuous on \(F\), it is enough to establish the continuity of \(\kappa(x, \nu^+)\) on \(F\), but this is a direct consequence of Lemma 3.4. \(\square\)
Definition 3.2 (see, e.g., [22]) A kernel $\kappa$ satisfies the generalized maximum principle with a constant $h$ if for every finite $\nu \in M^+$ with the property

$$\sup_{x \in S(\nu)} \kappa(x, \nu) =: M < \infty$$

one has $\kappa(x, \nu) \leq hM$ for all $x \in X$.

Lemma 3.8 Suppose that the kernel $\kappa$ is $\geq 0$ and satisfies the generalized maximum principle with a constant $h$. If, moreover, all the assumptions of Lemma 3.7 hold true, then

$$P_{F\nu}(X) \leq h\nu^+(X).$$

Proof. It follows from inequality (3.14) that, while proving the lemma, one can assume $F$ to be compact; then $C(F) < \infty$ ($\kappa$ being strictly positive definite). Hence, according to Theorem 2.2, an equilibrium measure $\theta = \theta_{S(P_F\nu)}$ of $S(P_F\nu)$ exists and satisfies the relations

$$\kappa(x, \theta) \geq 1 \text{ n.e. in } S(P_F\nu), \quad (3.21)$$

$$\kappa(x, \theta) \leq 1 \text{ for all } x \in S(\theta). \quad (3.22)$$

Since then, due to the boundedness of both $P_F\nu$ and $\theta$, relations (3.18) and (3.21) hold true $\theta$-a.e. and $P_F\nu$-a.e., respectively, on account of $\kappa \geq 0$ we get

$$P_{F\nu}(X) \leq \int \kappa(x, \theta) dP_{F\nu}(x) = \int \kappa(x, \nu) d\theta(x) \leq \int \kappa(x, \theta) d\nu^+(x).$$

On the other hand, inequality (3.22) yields, by Definition 3.2

$$\kappa(x, \theta) \leq h \text{ for all } x \in X.$$

When substituted into the preceding relation, this yields the lemma. \qed

4 Vector measures associated with condensers; their energies and potentials. Strong completeness theorem

Let $I^+$ and $I^-$ be fixed countable, disjoint sets of indices $i \in \mathbb{N}$, where the latter is allowed to be empty, and let $I := I^+ \cup I^-$. Assume that to every $i \in I$ there corresponds a nonempty, closed set $A_i \subset X$.

Definition 4.1 A collection $A = (A_i)_{i \in I}$ is called an $(I^+, I^-)$-condenser (or simply a condenser) in $X$ if every compact subset of $X$ intersects with at most finitely many $A_i$ and

$$A_i \cap A_j = \emptyset \quad \text{for all } i \in I^+, \ j \in I^-.$$ \quad (4.1)

We call $A$ compact if so are all $A_i$, $i \in I$, and finite if $I$ is finite. The sets $A_i$, $i \in I^+$, and $A_j$, $j \in I^-$, are called the positive and negative plates, respectively. (Note that any two equally signed plates can intersect each other or even coincide.) For any $I_0 \subseteq I$, write

$$CI_0 := I \setminus I_0, \quad A_{I_0} := \bigcup_{i \in I_0} A_i, \quad A^+_I := A^+_I, \quad A^-_I := A^-_I, \quad A := A_I;$$

10 Then, obviously, $h \geq 1.$
then $A_i$ is closed ($A$ being locally finite), and it is compact if $A_i$, $i \in I_0$, are compact while $I_0$ is finite.

Given $A$, let $\mathcal{M}^+(A)$ consist of all (nonnegative) vector measures $\mu = (\mu^i)_{i \in I}$, where $\mu^i \in \mathcal{M}^+(A_i)$ for all $i \in I$. The product topology on $\mathcal{M}^+(A)$, where every $\mathcal{M}^+(A_i)$ is equipped with the vague topology, is likewise called vague.

In accordance with an electrostatic interpretation of a condenser, we assume that the law of interaction between the charges lying on the plates $A_i$, $i \in I$, is determined by the matrix $(\alpha_i, \alpha_j)_{i,j \in I}$, where $\alpha_i$ equals $+1$ if $i \in I^+$ and $-1$ otherwise. Given $\mu, \mu_1 \in \mathcal{M}^+(A)$, we then define the mutual energy $\kappa(\mu, \mu_1)$ and the vector potential $\kappa_\mu(x)$, $x \in X$, by relations (1.1) and (1.2), respectively. For $\mu = \mu_1$, the mutual energy $\kappa(\mu, \mu_1)$ defines the energy of $\mu$. Let $\mathcal{E}^+(A)$ consist of all $\mu \in \mathcal{M}^+(A)$ with $-\infty < \kappa(\mu, \mu) < \infty$.

From now on we shall always require the kernel $\kappa$ to be positive definite. Then, according to [36] Lemma 3.3, for $\mu \in \mathcal{M}^+(A)$ to have finite energy, it is sufficient that $\sum_{i \in I} \|\mu^i\| < \infty$.

In this paper we are concerned with minimal energy problems over subsets of $\mathcal{E}^+(A)$, to be treated in the frame of the approach developed in [36]. For the convenience of the reader, in this and the next sections we summarize without proof some results from [36], necessary for the understanding of the subject matter.

Due to the local finiteness of $A$, one can define the mapping $R: \mathcal{M}^+(A) \rightarrow \mathcal{M}$,

$$R\mu(\varphi) = \sum_{i \in I} \alpha_i \mu^i(\varphi) \quad \text{for all } \varphi \in C_0(X).$$

Such a mapping is, in general, non-injective; it is injective if and only if $A_i$, $i \in I$, are mutually disjoint. Also observe that $R\mu$ is a signed (scalar) measure; because of assumption (4.1), the positive and negative parts in the Hahn–Jordan decomposition of $R\mu$ are given by

$$R\mu^+ = \sum_{i \in I^+} \mu^i \quad \text{and} \quad R\mu^- = \sum_{i \in I^-} \mu^i. \quad \text{(4.2)}$$

**Lemma 4.1** For any $\psi \in \Phi(X)$ and $\mu \in \mathcal{M}^+(A)$, the value $\langle \psi, R\mu \rangle$ is finite if and only if $\sum_{i \in I} |\langle \psi, \mu^i \rangle| < \infty$, and then

$$\langle \psi, R\mu \rangle = \sum_{i \in I} \alpha_i \langle \psi, \mu^i \rangle.$$

**Corollary 4.1** $\mu \in \mathcal{M}^+(A)$ has finite energy if and only if so does its scalar $R$-image, $R\mu$.

**Corollary 4.2** For any $\mu, \mu_1 \in \mathcal{E}^+(A)$, it holds that

$$\kappa(\mu, \mu_1) = \kappa(R\mu, R\mu_1).$$

Furthermore, $\kappa(\mu, x)$ is well defined and finite n.e. in $X$ and has the components

$$\kappa^i(\mu, x) = \alpha_i \kappa(\mu, x), \quad i \in I. \quad \text{(4.3)}$$

**Theorem 4.1** $\mathcal{E}^+(A)$ forms a semimetric space with the semimetric $\|\mu_1 - \mu_2\|_{\mathcal{E}^+(A)}$, defined by (1.8), and such a space is isometric to its $R$-image, $R\mathcal{E}^+(A)$. That is,

$$\|\mu_1 - \mu_2\|_{\mathcal{E}^+(A)} = \|R\mu_1 - R\mu_2\|_{\mathcal{E}^+(A)} \quad \text{for all } \mu_1, \mu_2 \in \mathcal{E}^+(A). \quad \text{(4.4)}$$

The semimetric $\|\mu_1 - \mu_2\|_{\mathcal{E}^+(A)}$ is a metric if and only if the kernel $\kappa$ is strictly positive definite while all $A_i$, $i \in I$, are mutually disjoint.

---

11This implies in particular that $\mathcal{E}^+(A)$ forms a convex cone.

12We say that a collection of propositions $P_i(x)$, $i \in I$, subsists n.e. in $X$ if this is the case for each $P_i(x)$. 

---

12
Corollary 4.3 Given $A$, define $\mathcal{E}_A$ as the set of all $\nu$ such that $\nu^+ \in \mathcal{E}^+_{A^+}$ and $\nu^- \in \mathcal{E}^+_{A^-}$ and equip it with the semimetric structure induced from $\mathcal{E}$. Then

$$\mathcal{E}_A = R\mathcal{E}^+(A),$$

(4.5)

so that, by Theorem 4.1, the semimetric spaces $\mathcal{E}_A$ and $\mathcal{E}^+(A)$ are isometric.

Proof. Indeed, $R\mathcal{E}^+(A) \subset \mathcal{E}_A$ can be obtained directly from relation (4.2) and Corollary 4.1. To prove the converse inclusion, fix $\nu \in \mathcal{E}_A$ and define $\mu^i := |\nu|_{A_i}$ and

$$\mu^{i+1} := \left| |\nu| - \sum_{k=1}^i \mu^k \right|_{A_{i+1}}$$

for all $i = 1, 2, \ldots$.

Then $\mu^i \in \mathcal{M}^+(A_i)$ and $\nu = R\mu$, where $\mu := (\mu^i)_{i \in I}$. Since $\kappa(\nu, \nu)$ is finite, repeated application of Corollary 4.1 shows that $\mu \in \mathcal{E}^+(A)$, and identity (4.5) follows. \qed

Because of (1.4), the topology on $\mathcal{E}^+(A)$ defined by the semimetric $||\mu_1 - \mu_2||_{\mathcal{E}^+(A)}$ is called strong. Two elements of $\mathcal{E}^+(A)$, $\mu_1$ and $\mu_2$, are equivalent if $||\mu_1 - \mu_2||_{\mathcal{E}^+(A)} = 0$. The equivalence in $\mathcal{E}^+(A)$ implies $R\mu_1 = R\mu_2$ provided that the kernel $\kappa$ is strictly positive definite, and it implies $\mu_1 = \mu_2$ if, moreover, all the $A_i$ are mutually disjoint.

Corollary 4.4 If $\mu_1$ and $\mu_2$ are equivalent in $\mathcal{E}^+(A)$, then $\kappa_{\mu_1}(x) = \kappa_{\mu_2}(x)$ n.e. in $X$.

For any $b \in (0, \infty)$, let $\mathcal{M}^+(A)$ consist of all $\mu \in \mathcal{M}^+(A)$ with

$$|R\mu|(X) = (R\mu^+ + R\mu^-)(X) = \sum_{i \in I} \mu^i(X) \leq b.$$

This class is vaguely bounded and closed; hence, by [36] Lemma 3.1, it is vaguely compact.

The following theorem on the strong completeness of $\mathcal{E}_b^+(A) := \mathcal{E}^+(A) \cap \mathcal{M}^+(A)$, treated as a topological subspace of $\mathcal{E}^+(A)$, is crucial in our approach (compare with Theorem 2.1).

Theorem 4.2 Assume that the kernel $\kappa$ is consistent and bounded on $A^+ \times A^-$.\(^\text{13}\) Then the following two assertions hold:

(i) The semimetric space $\mathcal{E}_b^+(A)$ is complete. In more detail, if $(\mu_s)_{s \in S} \subset \mathcal{E}_b^+(A)$ is a strongly Cauchy net and $\mu$ is one of its vague cluster points, then $\mu \in \mathcal{E}_b^+(A)$ and $\mu_s \to \mu$ strongly, i.e.

$$\lim_{s \in S} \|\mu_s - \mu\|_{\mathcal{E}^+(A)} = 0.$$

(ii) If the kernel $\kappa$ is strictly positive definite (hence, perfect), while all $A_i$, $i \in I$, are mutually disjoint, then the strong topology on $\mathcal{E}_b^+(A)$ is finer than the vague one. In more detail, if $(\mu_s)_{s \in S} \subset \mathcal{E}_b^+(A)$ converges strongly to $\mu_0 \in \mathcal{E}^+(A)$, then actually $\mu_0 \in \mathcal{E}_b^+(A)$ and $\mu_s \to \mu_0$ vaguely.

\(^\text{13}\) In view of the fact that the semimetric space $\mathcal{E}_b^+(A)$ is isometric to its $R$-image, Theorem 4.2 has singled out a strongly complete topological subspace of the pre-Hilbert space $\mathcal{E}$, whose elements are signed Radon measures. This is of independent interest since, according to a well-known counterexample by Cartan [4], the whole space $\mathcal{E}$ is strongly incomplete even for the Newtonian kernel $|x - y|^{2-n}$ in $\mathbb{R}^n$, $n \geq 3$.

\(^\text{14}\) For the Riesz kernels $\kappa_\alpha$ of order $\alpha \in (0, n)$ in $\mathbb{R}^n$, $n \geq 2$, Theorem 4.2 remains valid without assuming $\kappa_\alpha$ to be bounded on $A^+ \times A^-$; cf. [14] Theorem 1 and [36] Remark 9.2. The proof is based on Deny’s theorem stating that the pre-Hilbert space $\mathcal{E}_{\kappa_\alpha}$ can be completed by making use of distributions of finite energy.
5 Gauss variational problem

Fix \( \chi \in \mathcal{E} \) and define \( f = (f_i)_{i \in I} \) where \( f_i(x) := \alpha_i \kappa(x, \chi) \) for all \( i \in I \); such an \( f \) is well defined and finite n.e. in \( X \), while all its components are universally measurable, i.e. \( \nu \)-measurable for every \( \nu \in \mathcal{M}^+ \). We treat each \( f_i \) as an external field acting on the charges on \( A_i \) and then we define the \( f \)-weighted vector potential \( W_\mu = (W^i_\mu)_{i \in I} \) and the \( f \)-weighted energy \( G_\chi(\mu) := G_f(\mu) \) of \( \mu \in \mathcal{E}^+(A) \) by relations (1.3) and (1.4), respectively.

Given \( \mu \in \mathcal{E}^+(A) \), application of Lemma 4.1 to each of \( \kappa(\cdot, \chi^+) \) and \( \kappa(\cdot, \chi^-) \) gives

\[
\langle f, \mu \rangle = \kappa(\chi, R\mu).
\]

Combined with (4.3), this implies that \( G_\chi(\mu) \) is finite and can be written in the form

\[
G_\chi(\mu) = \|R\mu\|^2 + 2\kappa(\chi, R\mu) = -\|\chi\|^2 + \|\chi + R\mu\|^2. \tag{5.1}
\]

Fix a vector \( a = (a_i)_{i \in I} \) with \( a_i > 0 \) for all \( i \in I \) and a continuous function \( g \) on \( A \) satisfying conditions (1.6) and (1.7). Also having fixed \( J \) such that \( I^+ \subseteq J \subseteq I \), we write

\[
\mathcal{E}^+(A, a, g, J) := \left\{ \mu \in \mathcal{E}^+(A) : \langle g, \mu^i \rangle = a_i \quad \text{for all} \quad i \in J \right\}
\]

and further we consider

\[
G_\chi(A, a, g, J) := \inf_{\mu \in \mathcal{E}^+(A, a, g, J)} G_\chi(\mu).
\]

Then, because of (5.1),

\[
G_\chi(A, a, g, J) = -\|\chi\|^2 + \inf_{\mu \in \mathcal{E}^+(A, a, g, J)} \|\chi + R\mu\|^2. \tag{5.2}
\]

In the case \( J = I \), the symbol \( J \) in these and other notations will often be dropped. That is, we write \( \mathcal{E}^+(A, a, g) := \mathcal{E}^+(A, a, g, I) \), \( G_\chi(A, a, g) := G_\chi(A, a, g, I) \), and so on.

Below we use the following terminology. Given \( A, a, g, \chi, \) and \( J \), the \( (A, a, g, \chi, J) \)-problem is that on the existence of \( \lambda \in \mathcal{E}^+(A, a, g, J) \) with

\[
G_\chi(\lambda) = \inf_{\mu \in \mathcal{E}^+(A, a, g, J)} G_\chi(\mu).
\]

The \( (A, a, g, \chi, J) \)-problem is said to be solvable if the class \( \mathcal{E}_\chi(A, a, g, J) \) of all its minimizers is nonempty. The \( (A, a, g, \chi, I) \)-problem (or shortly the \( (A, a, g, \chi) \)-problem), main in the present study, is referred to as the Gauss variational problem. A minimizer in the \( (A, a, g, \chi) \)-problem will often be denoted simply by \( \lambda \) (with the tilde dropped).

To make these problems well defined, we shall always suppose that

\[
G_\chi(A, a, g) < \infty; \tag{5.3}
\]

then for every \( J \) one has

\[
-\infty < G_\chi(A, a, g, J) < \infty,
\]

where the inequality on the right is an obvious consequence of assumption (5.3), while that on the left follows from (5.2) due to the positive definiteness of the kernel.

\^5Necessary and/or sufficient conditions for assumption (5.3) to hold can be found in [36] (see Lemmas 6.2 and 6.3 therein). In particular, then necessarily \( C(A_i) > 0 \) for all \( i \in I \). See also Remark 6.2 below.
In addition to all the assumptions stated above, in the rest of the paper we shall always require that the kernel \( \kappa \) is consistent and bounded on \( A^+ \times A^- \), i.e.

\[
\sup_{(x,y) \in A^+ \times A^-} \kappa(x, y) < \infty,
\]

while for every \( i \in I \) either \( g|_{A_i} \) is bounded or there exist \( r_i \in (1, \infty) \) and \( \nu_i \in \mathcal{E} \) such that

\[
g|_{A_i}^{r_i}(x) \leq \kappa(x, \nu_i) \quad \text{n.e. in } X.
\]

Then sufficient conditions for the solvability of the Gauss variational problem are given by the following theorem (see \cite{36}, Theorem 8.1).

**Theorem 5.1** Under the above-mentioned hypotheses, the \((A, a, g, \chi)\)-problem is solvable if each \( A_i \), \( i \in I \), either is compact or has finite interior capacity.

Moreover, this theorem is sharp; that is, if at least one of the plates is noncompact and has infinite interior capacity, then the \((A, a, g, \chi)\)-problem, in general, admits no solution. This can be illustrated by the following example (cf. \cite{36} Proposition 8.1)\(^{16}\)

**Example 5.1** In \( \mathbb{R}^n \), \( n \geq 2 \), consider the Riesz kernel \( \kappa_n(x, y) = |x - y|^{n-2} \) of order \( n \), where \( \alpha \in (0, 2] \) and \( \alpha < n \). Assume \( I^+ = \{1\}, I^- = \{2\} \), \( A_1 \) and \( A_2 \) are closed disjoint sets with \( C_{\kappa_n}(A_i) \neq 0 \), \( i = 1, 2 \), and let \( A_1 \) be compact while \( A_2 \) connected. If, moreover, \( \chi \in \mathcal{E}^+ \) is compactly supported in \( A_2 \) and \( a_2 = a_1 + \chi(\mathbb{R}^n) \), then the corresponding \((A, a, g, \chi)\)-problem has no solution if and only if \( C_{\kappa_n}(A_2) = 0 \) though \( A_2 \) is \( \alpha \)-thin at \( \infty \in \mathbb{R}^n \).\(^{17}\)

**Problem 5.1** Given \( \kappa, A = (A_i)_{i \in I} \), \( g \), and \( \chi \), what is a description of the set \( \mathcal{S}_n(A, g, \chi) \) of all vectors \( a = (a_i)_{i \in I} \) for which the \((A, a, g, \chi)\)-problem is nevertheless solvable?

### 6 Standing assumptions

In addition to the standing assumptions stated in Section 5 in the rest of the paper we assume that \( X \) is noncompact (hence, \( \kappa \geq 0 \)). This involves no loss of generality, since for a compact \( X \) the Gauss variational problem is always solvable due to Theorem 5.1.

We also require that the kernel \( \kappa \) is strictly positive definite (hence, perfect), continuous for \( x \neq y \), possesses the property \((\infty, \chi)\) and satisfies the generalized maximum principle with a constant \( h \geq 1 \) (see Definitions 3.1 and 3.2). Furthermore, assume that the measure \( \chi \in \mathcal{E} \), generating the external field by means of \((1.5)\), is bounded and satisfies the assumptions

\[
S(\chi^+) \cap A^- = \emptyset, \quad S(\chi^-) \cap A^+ = \emptyset.
\]

**Remark 6.1** Then for each \( i \in I \) the \( i \)-component of the external field, \( f_i = \alpha_i \kappa(\cdot, \chi) \), is lower semicontinuous on \( A_i \), which is seen from Lemma 3.4.

---

\(^{16}\) See also \cite{36} Example 8.2 pertaining to the Coulomb kernel \(|x - y|^{-1}\) in \( \mathbb{R}^3 \), and also \cite{18, 20} for some related numerical experiments.

\(^{17}\) A closed set \( F \subset \mathbb{R}^n \) is \( \alpha \)-thin at \( \infty \in \mathbb{R}^n \) if \( F^* \), the inverse of \( F \) relative to the unit sphere, is \( \alpha \)-thin at \( x = 0 \), or equivalently \cite{22} Theorem 5.10, if either \( F \) is bounded or \( x = 0 \) is an \( \alpha \)-irregular point for \( F^* \). See \cite{30} for \( \alpha = 2 \), such a definition is due to M. Brelot (see \cite{3}; cf. also \cite{12, 19}). We refer to \cite{29} for an example of a set of infinite Newtonian capacity, though 2-thin at \( \infty \in \mathbb{R}^n \) (cf. also Example 7.2 in Section 7.2 below).
Remark 6.2 Suppose for a moment that \( \chi \in \mathcal{E} \) is compactly supported in \( A^c \). It is seen from [36] Lemma 6.3] that, under the above hypotheses on the kernel \( \kappa \), assumption [36] would hold automatically if one would require \( C(A_i) > \delta > 0 \) for all \( i \in I \).

Remark 6.3 Note that the standing assumptions on a kernel are not too restrictive. In particular, in \( \mathbb{R}^n, n \geq 3 \), they all are satisfied by the Newtonian kernel \( |x-y|^{\alpha-n} \) with \( \alpha \in (0,n), \) or Green kernels \( g_D, \) provided that the Euclidean distance between \( A^+ \) and \( A^- \) is nonzero. Here \( D \subset \mathbb{R}^n \) is a regular domain (in the sense of the solvability of the classical Dirichlet problem) and \( g_D \) its Green function.

7 Description of the set \( \mathcal{S}_\kappa(A, g, \chi) \)

Recall that we are working under the standing assumptions stated in Sections 5 and 6.

Before having formulated an answer to Problem 5.1, we observe that in the case \( J \neq I \) the auxiliary \((A, a, g, \chi, J)\)-problem can equivalently be formulated in terms of orthogonal projections \( P_{A_C} \) onto the convex cone \( \mathcal{E}^+_{A_C} \) in the sense of the following lemma.

Given \( A \) and \( J \neq I \), define \( a_j := (a_i)_{i \in J} \) and \( A_j := (A_i)_{i \in J} \), the latter being thought of as an \((I^+, I^- \cap J)\)-condenser. For any \( \mu = (\mu_i)_{i \in J} \in 2M^+(A) \), also write \( \mu_j := (\mu_i)^{+, J} \).

With these assumptions and notations, consider the minimum energy problem

\[
\inf_{\nu \in \mathcal{E}^+(A, a, g, J)} \| \chi + R\nu - P_{A_C \mu}(\chi + R\nu) \|^2. \tag{7.1}
\]

Lemma 7.1 For \( \tilde{\chi} \) to solve the \((A, a, g, \chi, J)\)-problem with \( J \neq I \), it is necessary and sufficient that there exist a solution \( \sigma \) to the problem (7.1) such that

\[
\tilde{\chi}_J = \sigma \quad \text{and} \quad R\tilde{\chi}_C = P_{A_C}(\chi + R\sigma). \tag{7.2}
\]

Proof. Fix \( \mu \in \mathcal{E}^+(A, a, g, J) \) and let \( \omega = (\omega_i)^{+, J} \) be one of the \( R \)-preimages of \( P_{A_C \mu}(\chi + R\mu_j) \). Consider \( \mu' \) such that \( \mu'_J = \mu_j \) and \( \mu'_C = \omega \). Then, by Corollaries 4.1 and 4.3 \( \mu' \in \mathcal{E}^+(A, a, g, J) \); actually, \( \mu' \in \mathcal{E}^+(A, a, g, J) \). Therefore, using representation (5.1), we get

\[
G_\chi(\mu) = -\|\chi\|^2 + \|\chi + R\mu\|^2 \geq -\|\chi\|^2 + \|\chi + R\mu - P_{A_C \mu}(\chi + R\mu)\|^2 = G_\chi(\mu') \geq G_\chi(A, a, g, J),
\]

where the first inequality is an equality if and only if \( R\mu = P_{A_C \mu}(\chi + R\mu) \). In view of the arbitrary choice of \( \mu \in \mathcal{E}^+(A, a, g, J) \), this yields the lemma. \( \square \)

7.1 Main results

Let \( L \) consist of all \( \ell \in I \) such that \( C(A_\ell) = \infty \). Assume \( L \neq \emptyset \), for otherwise Problem 5.1 has already been solved by Theorem 5.1. In all that follows, we also suppose that \( L \subseteq I^- \).

A description of the set \( \mathcal{S}_\kappa(A, g, \chi) \), required in Problem 5.1, is given by Theorems 7.2 and [36] the main results of this study. It is formulated in terms of a solution to the auxiliary \((A, a, g, \chi, C\ell_L)\)-problem, while the solvability of the latter is guarantied by the following theorem with \( J = C\ell_L \).

\footnote{Note that \( \mathcal{E}^+_{A_C \mu} \neq \emptyset \) due to the fact that \( C(A_i) > 0 \) for all \( i \in I \) (see the footnote to formula [5.3]), so that for any \( \nu \in \mathcal{E} \) an orthogonal projection \( P_{A_C \mu} \nu \) exists and is determined uniquely (see Section 3.2).}
Theorem 7.1 Given $J$, where $I^+ \subseteq J \subseteq I$, assume that
\[ C(A_j) < \infty \quad \text{for all } j \in J. \] (7.3)
Then the $(A, a, g, \chi, J)$-problem is solvable, and the class $\mathfrak{S}_\chi(A, a, g, J)$ of all its solutions is vaguely compact.

Fix a solution $\tilde{\lambda}$ to the auxiliary $(A, a, g, \chi, CL)$-problem. Then, equivalently, $\tilde{\lambda}_{CL}$ gives a solution to the minimum energy problem (7.1) with $J = CL$, while $\tilde{\lambda}_L$ is one of the $R$-pre-images of $P_{A_L}(\chi + R\tilde{\lambda}_{CL})$; see Lemma 7.1.

Theorem 7.2 Consider $a = (a_i)_{i \in I}$ such that
\[ a_\ell \geq \langle g, \tilde{\lambda}^\ell \rangle \quad \text{for all } \ell \in L. \] (7.4)
Then
\[ G_\chi(A, a, g) = G_\chi(A, a, g, CL), \] (7.5)
while the main $(A, a, g, \chi)$-problem is solvable if and only if all the inequalities (7.4) are equalities.

In the case where $L$ is a singleton, Theorem 7.2 can be refined as follows to establish a complete description of the set $S_\kappa(A, g, \chi)$.

Theorem 7.3 Let $L = \{\ell\}$ and assume, moreover, that
\[ A_i \cap A_\ell = \emptyset \quad \text{for all } i \neq \ell. \] (7.6)
Then the set $S_\kappa(A, g, \chi)$ consists of all vectors $a = (a_i)_{i \in I}$ with the property
\[ a_\ell \leq \langle g, \tilde{\lambda}^\ell \rangle \]
or, equivalently,
\[ a_\ell \leq \left\langle g, P_{A_\ell} \left( \chi + \sum_{i \neq \ell} a_i \tilde{\lambda}^i \right) \right\rangle =: \Sigma_\ell. \] (7.7)

Remark 7.1 Theorem [7.2] makes sense, since, under its hypotheses, the value $\Sigma_\ell = \langle g, \tilde{\lambda}^\ell \rangle$ does not depend on the choice of $\tilde{\lambda} \in \mathfrak{S}_\chi(A, a, g, I \setminus \{\ell\})$ (see Section 13).

See Sections 10, 11, and 13 for the proofs of Theorems 7.1, 7.2, 7.3. Combining Theorem 7.2 with Lemma 8.1 for $J = CL$ (see Section 8 below), we arrive at the following corollary.

Corollary 7.1 Assume $L$ is finite and $g_{\sup} := \sup_{x \in A} g(x) < \infty$. Then the $(A, a, g, \chi)$-problem is nonsolvable for every $a$ with
\[ a_\ell > hg_{\sup} \left[ \chi^+(X) + 2|a_{CL}g_{\min}^{-1} \right] \quad \text{for all } \ell \in L, \]
h being the constant in the generalized maximum principle.

\[ \text{19} \text{The values } \langle g, \tilde{\lambda}^\ell \rangle, \ell \in L, \text{ can be shown to be finite (see Section 11), so that inequalities (7.4) make sense.} \]
7.2 Examples

In this section, we withdraw all the standing assumptions stated in Sections 5 and 6. To illustrate the results obtained, we consider the Riesz kernel $\kappa_\alpha(x, y) = |x - y|^{\alpha - n}$ of order $\alpha \in (0, 2], \alpha < n$, in $\mathbb{R}^n$, $n \geq 2$, and we assume that $A_\ell$, where $\ell \in I^-$, is the only plate with infinite interior capacity. For this kernel, the operator of orthogonal projection $P_{A_\ell}$ is, in fact, the operator of Riesz balayage $\beta^{\kappa_\alpha}_{A_\ell}$ onto $A_\ell$, while (see, e.g., [22])

$$\|\nu - \beta^{\kappa_\alpha}_{A_\ell} \nu\|_{\kappa_\alpha} = \|\nu\|_{g^{\kappa_\alpha}_{A_\ell}}$$

for all $\nu \in \mathcal{E}_{\kappa_\alpha}$, where $g^{\kappa_\alpha}_{A_\ell}$ is the $\alpha$-Green function of the open set $A_\ell$.

Hence, the auxiliary problem (7.1) with $J = I \setminus \{\ell\}$ can be rewritten as

$$\inf \|\chi + R\nu\|_{g^{\kappa_\alpha}_{A_\ell}},$$

the infimum being taken over all $\nu \in \mathcal{E}^+_{{\kappa_\alpha}}(A_{I\setminus\{\ell\}})$ such that $(g, \nu^i) = a_i$ for all $i \neq \ell$, while $\Sigma_\ell$ from formula (7.7) now takes the form

$$\Sigma_\ell = \langle g, \beta^{\kappa_\alpha}_{A_\ell} (\chi + R\sigma) \rangle,$$

where $\sigma$ is a solution to the minimum $\alpha$-Green energy problem (7.8).

Combined with Theorem 7.3, this yields Proposition 1.1 (see Example 1.1), providing a complete description of the set $S_{\kappa_\alpha}(A, g, \chi)$. See also Examples 7.1–7.3 below, where such a description takes a much simpler form, given in terms of the geometry of the plate $A_\ell$.

In each of the following examples, let $C_{\kappa_\alpha}(A_i) > \delta > 0$ for all $i \in I$ and let $\chi \in \mathcal{E}_{\kappa_\alpha}$ be compactly supported in $A_\ell$. We also require that $\chi \geq 0$, dist $(A^+, A^-) > 0$, $I^- = L = \{\ell\}$, $A_\ell^c$ is connected, and $g = 1$.

**Example 7.1** Under the hypotheses stated above, assume moreover that

$$2a_\ell \geq |a| + \chi(\mathbb{R}^n).$$

**Proposition 7.1** Then the Gauss variational problem (relative to $\kappa_\alpha$, $A$, $a$, $\chi \geq 0$, and $g = 1$) is solvable if and only if $A_\ell$ is not $\alpha$-thin at $\infty_{\mathbb{R}^n}$, while

$$2a_\ell = |a| + \chi(\mathbb{R}^n).$$

**Proof.** It is known from the Riesz balayage theory (see, e.g., [22]) that, for any bounded positive measure $\nu \in \mathcal{E}_{\kappa_\alpha}(A_\ell)$, it holds $\beta^{\kappa_\alpha}_{A_\ell} \nu(\mathbb{R}^n) \leq \nu(\mathbb{R}^n)$, while according to [30] Theorem 4 the inequality here is an equality if and only if $A_\ell$ is not $\alpha$-thin at $\infty_{\mathbb{R}^n}$. Combining this for $\nu = \chi + R\sigma$ with Theorem 7.3, where $\Sigma_\ell$ is now given by

$$\Sigma_\ell = \beta^{\kappa_\alpha}_{A_\ell} (\chi + R\sigma)(\mathbb{R}^n),$$

and taking assumption (7.9) into account, we obtain the proposition. \[\square\]

---

\[\text{20} \text{Then assumption } (5.3) \text{ holds automatically (cf. Remark 6.2).} \]
Example 7.2 Consider the Coulomb kernel $\kappa_2(x, y) = |x - y|^{-1}$ in $\mathbb{R}^3$ and let $A_k$ be a rotational body consisting of all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that $q \leq x_1 < \infty$ and $0 \leq x_2^2 + x_3^2 \leq \rho(x_1)$, where $q \in \mathbb{R}$ and $\rho(x_1)$ approaches 0 as $x_1 \to \infty$. We focus with the following three cases:

\[
\rho(r) = r^{-s}, \quad \text{where } s \in [0, \infty), \quad (7.11)
\]
\[
\rho(r) = \exp(-r^s), \quad \text{where } s \in (0, 1], \quad (7.12)
\]
\[
\rho(r) = \exp(-r^s), \quad \text{where } s > 1. \quad (7.13)
\]

Then $A_k$ is not 2-thin at $\infty_{\mathbb{R}^3}$ in case (7.11), has finite (Newtonian) capacity in case (7.12), and it is 2-thin at $\infty_{\mathbb{R}^3}$ though $C_{\kappa_2}(A_k) = \infty$ in case (7.13); see [29]. Hence, if $a$ satisfies condition (7.9), then, by Proposition 7.1, the Gauss variational problem is nonsolvable in case (7.11), while in case (7.12) it admits a solution if and only if (7.10) additionally holds. Also observe that, by Theorem 5.1 in case (7.13) the problem is solvable for any $a$.

Example 7.3 Consider $A = (A_1, A_2)$ with $\alpha_1 = +1$ and $\alpha_2 = -1$, and let $\chi = 0$. Then, in consequence of Theorem 7.3 the set $S_{\kappa_2}(A, g, \chi)$ forms the cone in $\mathbb{R}^2$ defined by

\[a_2/a_1 \leq (\beta_2^{\alpha_2}\theta_{A_1})(A_2), \quad (7.14)\]

where $\theta_{A_1}$ is obtained from the equilibrium measure of $A_1$ relative to the kernel $g_{\kappa_2}$ when divided by $C_{\kappa_2}(A_1)$. Note that $\theta_{A_1}(A_1) = 1$; hence, by [29, Theorem 4], the value on the right in relation (7.14) is equal to 1 if $A_2$ is not $\alpha$-thin at $\infty_{\mathbb{R}^3}$, while otherwise it is $< 1$.

8 On the admissible measures in the auxiliary $(A, a, g, J, \chi)$-problem

For any $b \in (0, \infty)$ and $J, I^+ \subseteq J \subseteq I$, write $\mathcal{E}_b^+(A, a, g, J) := \mathcal{E}^+(A, a, g, J) \cap \mathcal{E}_b^+(A)$, where $\mathcal{E}_b^+(A)$ has been defined in Section 4.

Lemma 8.1 The value $G_b(A, a, g, J)$ remains unchanged if the class $\mathcal{E}^+(A, a, g, J)$ in its definition is replaced by $\mathcal{E}_b^+(A, a, g, J)$, where

\[H := h\left[\chi^+(X) + 2|a_j|g_{\inf}^{-1}\right]. \quad (8.1)\]

Proof. Observe that $H$ is finite, which is clear from assumptions (1.6), (1.7) and the boundedness of $\chi$. Also note that, for any $\mu = (\mu^i)_{i \in J} \in \mathcal{E}^+(A, a, g, J)$,

\[\mu^j(X) \leq a_jg_{\inf}^{-1} < \infty \quad \text{for all } j \in J. \quad (8.2)\]

Hence, if $J = I$, then $\mathcal{E}^+(A, a, g, I) \subset \mathcal{E}_H^+(A, a, g, I)$ and the lemma is obvious. Therefore, assume $J \neq I$. Then, as is clear from the proof of Lemma 7.1, $G_b(A, a, g, J)$ remains unchanged if the class $\mathcal{E}^+(A, a, g, J)$ in its definition is replaced by its subclass consisting of all $\mu \in \mathcal{E}^+(A, a, g, J)$ with the additional property that $R\mu_{C,J} = P_{A_{C,J}}(\chi + R\mu_{J})$. Thus, the lemma will follow once we prove

\[R\mu_{J}(X) + P_{A_{C,J}}(\chi + R\mu_{J})(X) \leq H, \quad (8.3)\]

but this is a direct consequence of relation (8.2) and Lemma 8.8 \qed

Corollary 8.1 If $\hat{\lambda}$ is a minimizer in the $(A, a, g, \chi, J)$-problem, then $\hat{\lambda} \in \mathcal{E}_H^+(A, a, g, J)$.

Proof. For $J \neq I$ this follows from relations (7.2) and (8.3), while otherwise it is obvious. \qed
9 Extremal measures

Definition 9.1 We call a net $(\mu_s)_{s \in S}$ minimizing in the $(A, a, g, \chi, J)$-problem if

\[ (\mu_s)_{s \in S} \subset E_H^+(A, a, g, J), \]

$H$ being defined by (8.1), and furthermore

\[ \lim_{s \in S} G_\chi(\mu_s) = G_\chi(A, a, g, J). \]  \hspace{1cm} (9.1)

Let $M_\chi(A, a, g, J) \subset S$ consist of all those nets; this class is nonempty in consequence of assumption (5.3) and Lemma 8.1. We denote by $M_\chi(A, a, g, J)$ the union of the vague cluster sets of $(\mu_s)_{s \in S}$, where $(\mu_s)_{s \in S}$ ranges over $M_\chi(A, a, g, J)$.

Remark 9.1 Taking a subnet if necessary, we shall always assume $(\mu_s)_{s \in S} \in M_\chi(A, a, g, J)$ to be strongly bounded. Then so are $(R\mu^+_s)_{s \in S}, (R\mu^-_s)_{s \in S}$ and $(\mu^i_s)_{s \in S}$ for all $i \in I$; that is,

\[ \sup_{s \in S, i \in I} \left\{ \|\mu^i_s\|, \|R\mu^+_s\|, \|R\mu^-_s\| \right\} < \infty. \]  \hspace{1cm} (9.2)

Indeed, this can be obtained from $\kappa \geq 0$ and $\kappa(R\mu^+_s, R\mu^-_s) \leq M \leq \infty$ for all $s \in S$, the latter being a consequence of $|R\mu_s|\langle X \rangle \leq H$ and assumption (5.4).

Definition 9.2 We call $\tilde{\gamma} = (\tilde{\gamma}^i)_{i \in I} \in E_\chi^+(A)$ extremal in the $(A, a, g, \chi, J)$-problem if there exists $(\mu_s)_{s \in S} \in M_\chi(A, a, g, J)$ that converges to $\tilde{\gamma}$ both strongly and vaguely; such a net $(\mu_s)_{s \in S}$ is said to generate $\tilde{\gamma}$. The class of all those $\tilde{\gamma}$ will be denoted by $E_\chi(A, a, g, J)$\hspace{1cm} (9.3)

It follows from Lemma 3.3 with $\psi = g$ that, for any $\tilde{\gamma} \in E_\chi(A, a, g, J)$,

\[ (g, \tilde{\gamma}^i) \leq a_j \text{ for all } j \in J. \]  \hspace{1cm} (9.3)

Also observe that

\[ E_\chi(A, a, g, J) \subset E_\chi(A, a, g, J), \]  \hspace{1cm} (9.4)

since, because of Corollary 8.1 each $\lambda \in E_\chi(A, a, g, J)$ (provided exists) can be treated as an extremal measure generated by the constant sequence $(\lambda)_{n \in N} \in M_\chi(A, a, g, J)$.

Lemma 9.1 Furthermore, the following assertions hold true:

\[ \begin{align*}
(i) & \text{ From every minimizing net one can select a subnet generating an extremal measure; hence, } E_\chi(A, a, g, J) \text{ is nonempty. Moreover, } \\
& E_\chi(A, a, g, J) = M_\chi(A, a, g, J). \\
(ii) & \text{ For any } \tilde{\gamma}_1, \tilde{\gamma}_2 \in E_\chi(A, a, g, J), \text{ it holds that } \|\tilde{\gamma}_1 - \tilde{\gamma}_2\|_{E_\chi(A)} = 0. \text{ Hence, } R\tilde{\gamma}_1 = R\tilde{\gamma}_2, \\
& \text{while } \tilde{\gamma}_1 = \tilde{\gamma}_2 \text{ if and only if all } A_i, i \in I, \text{ are mutually disjoint.} \\
(iii) & E_\chi(A, a, g, J) \text{ is vaguely compact.}
\end{align*} \]

\[ \text{In the case } J = I, \text{ the tilde in the notation of an extremal measure will often be dropped.} \]
Proof. Choose arbitrary \((\mu_s)_{s \in S}\) and \((\xi_t)_{t \in T}\) in \(M_\chi(A, a, g, J)\). Then
\[
\lim_{(s, t) \in S \times T} \|\mu_s - \xi_t\|_{E^+(A)} = 0,
\] (9.6)
where \(S \times T\) is the directed product of the directed sets \(S\) and \(T\) (see, e.g., [21, Chapter 2, Section 3]). Since \(E^+_H(A, a, g, J)\) is convex (cf. the footnote to Corollary 4.1), we get
\[
4G_\chi(A, a, g, J) \leq 4G_\chi\left(\frac{\mu_s + \xi_t}{2}\right) = \|R\mu_s + R\xi_t\|^2 + 4\kappa(\chi, R\mu_s + R\xi_t).
\]

On the other hand, applying the parallelogram identity in the pre-Hilbert space \(E\) to \(R\mu_s\) and \(R\xi_t\) and then adding and subtracting \(4\kappa(\chi, R\mu_s + R\xi_t)\), we obtain
\[
\|R\mu_s - R\xi_t\|^2 = -\|R\mu_s + R\xi_t\|^2 - 4\kappa(\chi, R\mu_s + R\xi_t) + 2G_\chi(\mu_s) + 2G_\chi(\xi_t).
\]

When combined with the preceding relation, this gives
\[
0 \leq \|R\mu_s - R\xi_t\|^2 \leq -4G_\chi(A, a, g, J) + 2G_\chi(\mu_s) + 2G_\chi(\xi_t),
\]
which yields (9.6) when combined with (9.1).

Relation (9.6) implies that \((\mu_s)_{s \in S}\) is strongly fundamental. By Theorem 1.2 with \(b = H\), for every vague cluster point \(\mu\) of \((\mu_s)_{s \in S}\) (it exists) we therefore get \(\mu_s \to \mu\) strongly. This shows that \(\mu\) is an extremal measure; actually, \(M_\chi(A, a, g, J) \subset E_\chi(A, a, g, J)\). Since the converse inclusion is obvious, relation (9.5) follows.

Having thus proved assertion (i), we proceed by verifying (ii). Choose \((\mu_s)_{s \in S}\) and \((\xi_t)_{t \in T}\) generating \(\tilde{\gamma}_1\) and \(\tilde{\gamma}_2\), respectively. Then application of (9.6) shows that \((\mu_s)_{s \in S}\) converges to both \(\tilde{\gamma}_1\) and \(\tilde{\gamma}_2\) strongly, hence \(\|\tilde{\gamma}_1 - \tilde{\gamma}_2\|_{E^+(A)} = 0\), and consequently \(\|R\tilde{\gamma}_1 - R\tilde{\gamma}_2\| = 0\) by (4.3). In view of the strict positive definiteness of the kernel, assertion (ii) follows.

To establish assertion (iii), fix \((\tilde{\gamma}_s)_{s \in S} \in E_\chi(A, a, g, J)\); then \((\tilde{\gamma}_s)_{s \in S} \subset E^+_H(A)\). Since the class \(M_\chi(A)\) is vaguely compact, there exists a vague cluster point \(\tilde{\gamma}_0\) of \((\tilde{\gamma}_s)_{s \in S}\). Let \((\tilde{\gamma}_t)_{t \in T}\) be a subnet of \((\tilde{\gamma}_s)_{s \in S}\) that converges vaguely to \(\tilde{\gamma}_0\); then for every \(t \in T\) one can choose a net \((\xi_t)_{s \in S_t} \in M_\chi(A, a, g, J)\) converging vaguely to \(\tilde{\gamma}_t\). Consider the Cartesian product \(\prod_{s \in S_t} S_t\) — that is, the collection of all functions \(\beta\) on \(T\) with \(\beta(t) \in S_t\), and let \(D\) denote the directed product \(T \times \prod_{t \in T} S_t\). For every \((t, \beta) \in D\), write \(\mu_{(t, \beta)} := \mu_{(t)}\). Then the theorem on iterated limits from [21, Chapter 2, Section 4] implies that the net \(\mu_{(t, \beta)}\), \((t, \beta) \in D\), belongs to \(M_\chi(A, a, g, J)\) and converges vaguely to \(\tilde{\gamma}_0\); thus, \(\tilde{\gamma}_0 \in M_\chi(A, a, g, J)\). Combined with (9.3), this yields assertion (iii).

**Corollary 9.1** Any two minimizers \(\lambda_1, \lambda_2 \in E_\chi(A, a, g, J)\) (provided exist) are equivalent in \(E^+(A)\). Hence, \(R\lambda_1 = R\lambda_2\), while \(\lambda_1 = \lambda_2\) if and only if all \(A_i\), \(i \in I\), are mutually disjoint.

**Proof.** This is a direct consequence of inclusion (9.4) and assertion (ii) of Lemma 9.1. \(\square\)

**Corollary 9.2** For every extremal measure \(\tilde{\gamma} \in E_\chi(A, a, g, J)\) it holds that
\[
G_\chi(\tilde{\gamma}) = G_\chi(A, a, g, J).
\] (9.7)

**Proof.** Applying (5.1) to \(\tilde{\gamma} \in E_\chi(A, a, g, J)\) and each of \(\mu_s, s \in S\), where \((\mu_s)_{s \in S}\) is a net generating \(\tilde{\gamma}\), and using the fact that \(\mu_s \to \tilde{\gamma}\) strongly, we get
\[
G_\chi(\tilde{\gamma}) = \|\chi + R\tilde{\gamma}\|^2 - \|\chi\|^2 = \lim_{s \in S} \left[\|\chi + R\mu_s\|^2 - \|\chi\|^2\right] = \lim_{s \in S} G_\chi(\mu_s),
\]
which yields (9.7) when combined with (9.1). \(\square\)
Corollary 9.3 Let $J = I$. If the $(\mathcal{A}, a, g, \chi)$-problem is solvable, then the class $\mathcal{E}_\chi(\mathcal{A}, a, g)$ of all its solutions is compact in the vague topology; moreover,

\[ \mathcal{E}_\chi(\mathcal{A}, a, g) = \mathcal{E}_\chi(\mathcal{A}, a, g). \] (9.8)

Proof. Due to assertion (iii) of Lemma 9.1, the corollary will follow once we prove (9.8). As is seen from relations (9.4) and (9.7), to this end it is enough to show that, for any given $\gamma \in \mathcal{E}_\chi(\mathcal{A}, a, g)$,

\[ \langle g, \gamma^i \rangle = a_i \quad \text{for all } i \in I. \] (9.9)

Because of assertion (ii) of Lemma 9.1 for any $\lambda \in \mathcal{E}_\chi(\mathcal{A}, a, g)$ one has $R\gamma = R\lambda$, so $|R\gamma| = |R\lambda|$, hence $\langle g, |R\gamma| \rangle = \langle g, |R\lambda| \rangle$ and, as an application of Lemma 4.1 with $\psi = g$,

\[ \sum_{i \in I} \langle g, \gamma^i \rangle = \sum_{i \in I} \langle g, \lambda^i \rangle = |a|, \]

$|a|$ being finite by assumption (1.6). In view of relation (9.3), this yields (9.9). \(\square\)

10 Proof of Theorem 7.1

Given $J$, where $I^+ \subseteq J \subseteq I$, fix an extremal measure $\tilde{\gamma} \in \mathcal{E}_\chi(\mathcal{A}, a, g, J)$; it exists according to assertion (i) of Lemma 9.1. A part of the proof of Theorem 7.1 is given as the following lemma, to be needed also in the sequel.

Lemma 10.1 If $C(A_j) < \infty$ for some $j \in J$, then

\[ \langle g, \tilde{\gamma}^j \rangle = a_j. \] (10.1)

Proof. We treat $A_j$ as a locally compact space with the topology induced from $X$. Given a set $E \subset A_j$, let $\chi_E$ denote its characteristic function and $\theta_E \in \mathcal{E}(E)$ the interior equilibrium measure associated with $E$ (as the kernel is perfect, the existence of $\theta_E$ is guaranteed by Theorem 2.2). Also write $E^c := E^{\chi \gamma} := A_j \setminus E$, and let $\{K_j\}_{A_j}$ be the increasing filtering family of all compact subsets $K_j$ of $A_j$.

We then observe that there is no loss of generality in assuming $g|A_j$ to satisfy condition (5.5) with some $r_j \in (1, \infty)$ and $\nu_j \in \mathcal{E}$. Indeed, otherwise $g|A_j$ has to be bounded from above (say by $M$), which combined with relation (2.2) for $E = A_j$ results in inequality (5.5) with $\nu_j := M^{\gamma \gamma \theta_{A_j}}$, $r_j \in (1, \infty)$ being arbitrary.

To establish (10.1), choose a (strongly bounded) net $(\mu_s)_{s \in S} \in \mathcal{M}(\mathcal{A}, a, g, J)$ generating $\tilde{\gamma}$.

Since $g\chi_{K_j}$ is upper semicontinuous on $A_j$ while $\mu_s^j \rightarrow \tilde{\gamma}^j$ vaguely, we get

\[ \langle g\chi_{K_j}, \tilde{\gamma}^j \rangle \geq \limsup_{s \in S} \langle g\chi_{K_j}, \mu_s^j \rangle \quad \text{for every } K_j \in \{K_j\}_{A_j}. \]

On the other hand, application of Lemma 1.2.2 from [10] yields

\[ \langle g, \tilde{\gamma}^j \rangle = \lim_{K_j \uparrow A_j} \langle g\chi_{K_j}, \tilde{\gamma}^j \rangle, \]

which together with the preceding inequality and relation (9.3) gives

\[ a_j \geq \langle g, \tilde{\gamma}^j \rangle \geq \limsup_{(s, K_j) \in S \times \{K_j\}_{A_j}} \langle g\chi_{K_j}, \mu_s^j \rangle = a_j - \liminf_{(s, K_j) \in S \times \{K_j\}_{A_j}} \langle g\chi_{K_j}, \mu_s^j \rangle. \]
Proof of Theorem 7.2

Observe that \( \langle \lambda, \rangle \) the lemma follows.

For any \( K_j, \tilde{K}_j \in \{ K_j \} \), such that \( K_j \subset \tilde{K}_j \), consider \( \theta K_j \) and \( \theta \tilde{K}_j \), the interior equilibrium measures associated with \( K_j \) and \( \tilde{K}_j \), respectively. Then \( \theta \tilde{K}_j \) minimizes the energy in the class \( \Gamma_{\tilde{K}_j} \), while \( \theta K_j \) belongs to \( \Gamma_{K_j} \); see Theorem 2.2. By Lemma 2.1 with \( H = \Gamma_{\tilde{K}_j} \) and \( \nu = \theta K_j \), this yields

\[
\| \theta K_j - \theta \tilde{K}_j \|^2 \leq \| \theta K_j \|^2 - \| \theta \tilde{K}_j \|^2.
\]

But, as is seen from (2.1), the net \( \| \theta K_j \| \), \( K_j \in \{ K_j \} \), is bounded and nonincreasing, and so it is fundamental in \( \mathbb{R} \). The preceding inequality then yields that the net \( \langle \theta K_j \rangle_{K_j \in \{ K_j \}} \) is strongly fundamental in \( E^+ \). Since, clearly, it converges vaguely to zero, the property (C) (see Section 2.1) implies that zero is also its strong limit; hence,

\[
\lim_{K_j \uparrow A_j} \| \theta K_j \| = 0.
\] (10.3)

Write \( q_j := r_j (r_j - 1)^{-1} \), where \( r_j \in (1, \infty) \) is the number involved in condition (5.5). Combining relations (10.5) and (2.2), the latter with \( E = K_j^c \), shows that the inequality

\[
g(x) \chi_{K_j^c}(x) \leq \kappa(x, \nu_j)^{1/r} \kappa(x, \theta K_j)\]

subsists a.e. in \( A_j \), and hence \( \mu^j \)-a.e. in \( X \) by Lemma 3.2. Having integrated this relation with respect to \( \mu^j \), we then apply the Hölder and, subsequently, the Cauchy–Schwarz inequalities to the integrals on the right. This gives

\[
\langle g \chi_{K_j^c}, \mu^j \rangle \leq \left[ \int \kappa(x, \nu_j) d\mu^j(x) \right]^{1/r_j} \left[ \int \kappa(x, \theta K_j) d\mu^j(x) \right]^{1/q_j} \leq \| \nu_j \|^{1/r_j} \| \theta K_j \|^2 \| \mu^j \|.
\]

Taking limits here along \( S \times \{ K_j \} \) and using relations (9.2) and (10.3), we get (10.2) as desired.

Let now assumption (7.3) in Theorem 7.1 hold. Then, in consequence of Lemma 10.1 the extremal measure \( \gamma \) belongs to \( E^+(A, a, g, J) \). Because of (7.7), this yields that, actually, \( \gamma \) is a solution to the \( (A, a, g, \chi, J) \)-problem, and the statement on the solvability follows.

It has thus been proved that \( \mathcal{E}_\chi(A, a, g, J) \subset \mathcal{E}_\chi(A, a, g, J) \). Since the converse also holds according to inclusion (9.1), these two classes have to be equal. On account of assertion (iii) of Lemma 9.1 this implies the vague compactness of \( \mathcal{E}_\chi(A, a, g, J) \). \( \square \)

11 Proof of Theorem 7.2

Recall that \( L \subseteq I^- \) consists of \( \ell \in I \) with \( C(A_\ell) = \infty \). Fix \( \hat{\lambda} \in \mathcal{E}_\chi(A, a, g, CL) \) (it exists due to Theorem 7.1 with \( J = CL \)) and assume \( \alpha_\ell, \ell \in L, \) to satisfy relation (7.4).

Observe that \( \langle g, \hat{\lambda} \rangle \) is finite for each \( i \in I, \) so that inequalities (7.4) make sense. Indeed, since \( \hat{\lambda}(X) \leq H < \infty \) according to Corollary 8.1, one can assume \( g \) to satisfy condition (15.5), for
if not, then \( g \) has to be bounded from above and the finiteness of \( \langle g, \tilde{\lambda}^i \rangle \) is obvious. By the HÖlder and Cauchy–Schwarz inequalities, we then obtain

\[
\langle g, \tilde{\lambda}^i \rangle \leq \left[ \int \kappa(x, \nu_i) \, d\tilde{\lambda}^i(x) \right]^{1/r_i} \left[ \int 1 \, d\tilde{\lambda}^i \right]^{1/q_i} \leq \|\nu_i\|^{1/r_i} \|	ilde{\lambda}^i\|^{1/r_i} H^{1/q_i} < \infty
\]
as desired. Here, \( r_i \) and \( \nu_i \) are the same as in condition (5.3) and \( q_i := r_i (r_i - 1)^{-1} \).

To establish the theorem, for each \( \ell \in L \) we choose probability measures \( \tau_n^\ell \in \mathcal{E}^+(A_{\ell}) \), \( n \in \mathbb{N} \), such that \( \tau_n^\ell \to 0 \) vaguely as \( n \to \infty \) and

\[
\|\tau_n^\ell\| \leq n^{-1}.
\] (11.1)

Such \( \tau_n^\ell \) exist due to the fact that \( C(A_{\ell} \setminus K) = \infty \) for any compact \( K \), which in turn follows from \( C(A_{\ell}) = \infty \) because of the strict positive definiteness of the kernel.

Further, for each \( n \in \mathbb{N} \) define \( \hat{\tau}_n = (\hat{\tau}_n^i)_{i \in I} \), where \( \hat{\tau}_n^i := 0 \) for all \( i \in CL \) and

\[
\hat{\tau}_n^i := \frac{[a_i - \langle g, \tilde{\lambda}^i \rangle] \tau_n^i}{\langle g, \tau_n^i \rangle}
\] otherwise.

Since \( \langle g, \tau_n^i \rangle \geq g_{\text{inf}} > 0 \) for all \( \ell \in L \) because of assumption (1.7), we have

\[
\hat{\tau}_n \to 0 \quad \text{vaguely (as } n \to \infty \text{)}
\] (11.2)

and also, due to relations (11.3) and (11.1),

\[
\sum_{i \in I} \|\hat{\tau}_n^i\| \leq |a_L| g_{\text{inf}}^{-1} n^{-1} < \infty, \quad n \in \mathbb{N}.
\] (11.3)

Estimate (11.3) yields that \( \hat{\tau}_n \in \mathcal{E}^+(A) \) for all \( n \in \mathbb{N} \) (see Section 4); hence, in view of the convexity of \( \mathcal{E}^+(A) \),

\[
\mu_n := \tilde{\lambda} + \hat{\tau}_n \in \mathcal{E}^+(A, a, g), \quad n \in \mathbb{N},
\] (11.4)

and consequently, by (5.1),

\[
G_{\chi}(A, a, g) \leq G_{\chi}(\mu_n) = -\|\chi\|^2 + \|\chi + R\mu_n\|^2
\]
\[
= -\|\chi\|^2 + \|\chi + R\tilde{\lambda} + R\hat{\tau}_n\|^2 \leq G_{\chi}(\tilde{\lambda}) + c(n),
\] (11.5)

where \( c(n) := \|R\hat{\tau}_n\|(\|R\hat{\tau}_n\| + 2\|\chi + R\tilde{\lambda}\|) \). But

\[
G_{\chi}(\tilde{\lambda}) = G_{\chi}(A, a, g, CL) \leq G_{\chi}(A, a, g),
\] (11.6)

while \( c(n) \to 0 \) as \( n \to \infty \), the latter being a consequence of estimate (11.3). Combining relations (11.3) and (11.6) and then letting \( n \to \infty \), we thus get

\[
G_{\chi}(\tilde{\lambda}) = G_{\chi}(A, a, g, CL) = G_{\chi}(A, a, g) = \lim_{n \to \infty} G_{\chi}(\mu_n).
\]

This establishes assertion (7.3) and, on account of inclusion (11.4), also the fact that

\[
(\mu_n)_{n \in \mathbb{N}} \in \mathcal{M}_{\chi}(A, a, g).
\]

As \( \mu_n \to \tilde{\lambda} \) vaguely because of relation (11.2), the last inclusion yields \( \tilde{\lambda} \in \mathcal{M}_{\chi}(A, a, g) \) and hence, by (9.5) with \( J = I \),

\[
\tilde{\lambda} \in \mathcal{E}_{\chi}(A, a, g).
\]

Using Corollary 9.3, we then see that the main \( (A, a, g, \chi) \)-problem is solvable if and only if \( \tilde{\lambda} \) serves as one of its minimizers, which in turn, due to (9.7) with \( J = I \), holds if and only if

\[
\tilde{\lambda} \in \mathcal{E}^+(A, a, g).
\]

Since the latter is equivalent to the requirement that all the inequalities (7.4) are equalities, the proof is complete. \( \square \)
12 Description of the \( f \)-weighted extremal potentials in the Gauss variational problem

It is seen from (4.3) that, for any \( \mu \in E^+(A) \), the \( f \)-weighted vector potential \( W_\mu = (W^i_\mu)_{i \in I} \) is finite n.e. in \( X \) and its components can be written in the form

\[
W^i_\mu(x) = \alpha_i \kappa(x, \chi + R\mu) \quad \text{n.e. in } X.
\] (12.1)

Therefore, for any \( \mu, \mu_1 \in E^+(A) \),

\[
\langle W_\mu, \mu_1 \rangle = \sum_{i \in I} \langle W^i_\mu, \mu^i_1 \rangle = \kappa(\chi + R\mu, R\mu_1),
\] (12.2)

which follows from Lemma 4.1 when applied to \( R\mu_1 \) and each of the functions \( \kappa(\cdot, \chi^+ + R\mu^+) \) and \( \kappa(\cdot, \chi^- + R\mu^-) \).

Let \( \gamma \) be extremal in the Gauss variational problem. Taking (5.1) and (9.7) into account, we then conclude from (12.2) with \( \mu = \mu_1 = \gamma \) that

\[
\sum_{i \in I} \langle W^i_\gamma, \gamma^i \rangle = \frac{1}{2} \left[ \|\gamma\|_{E^+(A)}^2 + G_\chi(A, a, g) \right],
\] (12.3)

where the expression on the right (hence, also that on the left) does not depend on the choice of \( \gamma \) in consequence of assertion (ii) of Lemma 9.1.

The proof of Theorem 7.3, to be given in Section 13, is based on a description of the \( f \)-weighted extremal potential \( W_\gamma = (W^i_\gamma)_{i \in I} \) provided as follows.

**Theorem 12.1** Given \( \gamma \in E_\chi(A, a, g) \), it holds that\(^{23}\)

\[
a_i W^i_\gamma(x) \geq \langle W^i_\gamma, \gamma^i \rangle g(x) \quad \text{n.e. in } A_i, \quad i \in I.
\] (12.4)

Relations (12.3) and (12.4) characterize uniquely the vector \( \langle W^i_\gamma, \gamma^i \rangle, i \in I \), in the sense that, if these two relations are satisfied by some \( \eta_i \) in place of \( \langle W^i_\gamma, \gamma^i \rangle \), then

\[
\eta_i = \langle W^i_\gamma, \gamma^i \rangle \quad \text{for all } i \in I.
\] (12.5)

**Proof.** Having fixed \( i \in I \), we start with the observation that, for any \( \xi \in E \), \( \langle W^i_\mu, \xi \rangle \) is a strongly continuous function of \( \mu \in E^+(A) \). Indeed, this is clear from representation (12.1) in view of the isometry between \( E^+(A) \) and \( R\mathbb{E}^+(A) \).

Choose a net \( (\mu_s)_{s \in S} \in M_\chi(A, a, g) \) generating the extremal measure \( \gamma \). Then the above observation can slightly be generalized in the following way:

\[
\langle W^i_\gamma, \gamma^i \rangle = \lim_{s \in S} \langle W^i_{\mu_s}, \mu^i_s \rangle.
\] (12.6)

Indeed, according to relation (12.2), the net \( (\mu^i_s)_{s \in S} \) is strongly bounded (say by \( M \)). Since it converges to \( \gamma^i \) vaguely, the property \((C_2)\) (see Section 2.1) yields that \( \mu^i_s \to \gamma^i \) weakly; hence, for every \( \varepsilon > 0 \),

\[
|\kappa(\chi + R\gamma, \mu^i_s - \gamma^i)| < \varepsilon
\]

\(^{22}\)Cf. also Corollary 12.1 below.

\(^{23}\)Recall that \( C(A_i) > 0 \) for all \( i \in I \) (see the footnote to assumption (5.3)).
whenever \( s \in S \) is large enough. Furthermore, by the Cauchy–Schwarz inequality,
\[
|\kappa(\chi + R\gamma, \mu_s^i) - \kappa(\chi + R\gamma_s, \mu_s^i)| \leq ||\mu_s^i|| \|R\gamma - R\mu_s\| \leq M\|R\gamma - R\mu_s\|.
\]
As \( R\mu_s \rightarrow R\gamma \) strongly, the last two relations combined give
\[
\kappa(\chi + R\gamma, \gamma^i) = \lim_{s \in S} \kappa(\chi + R\mu_s, \mu_s^i),
\]
which is equivalent to (12.6) because of representation (12.1).

In order to establish assertion (12.4), we now assume, on the contrary, that there is a set \( E \subset A_i \) of interior capacity nonzero with the property
\[
a_i W^i_\gamma(x) < \langle W^i_\gamma, \gamma^i \rangle g(x) \quad \text{for all} \quad x \in E.
\]
Having chosen \( \omega \in \mathcal{E}^*_\mathcal{F} \) so that \( (g, \omega) = a_i \) (such a measure exists), we then obtain
\[
\langle W^i_\gamma, \omega \rangle < \langle W^i_\gamma, \gamma^i \rangle.
\]

To get a contradiction, for all \( r \in (0, 1) \) and \( s \in S \) we define \( \hat{\mu}_s = (\hat{\mu}_s^k)_{k \in I} \), where
\[
\hat{\mu}_s^k := \begin{cases} 
\mu_s^i - r(\mu_s^i - \omega) & \text{if } k = i, \\
\mu_s^k & \text{otherwise.}
\end{cases}
\]
Since then
\[
R\hat{\mu}_s = -\alpha_i r(\mu_s^i - \omega) + R\mu_s,
\]
Corollary 4.1 implies \( \hat{\mu}_s \in \mathcal{E}^+(A) \). Actually, \( \hat{\mu}_s \in \mathcal{E}^+(A, a, g) \) for all \( s \in S \), which in view of relations (5.1), (9.2), and (12.1) yields
\[
G_\chi(A, a, g) \leq G_\chi(\hat{\mu}_s) = \|R\hat{\mu}_s\|^2 + 2\kappa(\chi, R\hat{\mu}_s)
\]
\[
= \|R\mu_s\|^2 - 2\alpha_i r\kappa(R\mu_s, \mu_s^i - \omega) + r^2\|\mu_s^i - \omega\|^2 + 2\kappa(\chi, R\mu_s) - 2\alpha_i r\kappa(\chi, \mu_s^i - \omega)
\]
\[
\leq G_\chi(\mu_s) - 2r \langle W^i_\gamma, \mu_s^i - \omega \rangle + r^2 M_1,
\]
\( M_1 \) being positive. Passing here to the limit as \( s \) ranges along \( S \), on account of (9.1), (12.6) and the continuity of \( \langle W^i_\mu, \omega \rangle \) relative to \( \mu \in \mathcal{E}^+(A) \) we get
\[
0 \leq -2r \langle W^i_\gamma, \gamma^i - \omega \rangle + r^2 M_1,
\]
which leads to \( \langle W^i_\gamma, \gamma^i - \omega \rangle \leq 0 \) by letting \( r \to 0 \). This contradicts to inequality (12.5).

To complete the proof, assume now relations (12.3) and (12.4) to hold also with some \( \eta_i, i \in I \), in place of \( \langle W^i_\gamma, \gamma^i \rangle \). Using the fact that the union of two universally measurable sets with interior capacity zero has interior capacity zero as well (see [10]), for each \( i \in I \) we then have
\[
a_i W^i_\gamma(x) \geq \max \left\{ \eta_i, \langle W^i_\gamma, \gamma^i \rangle \right\} g(x) \quad \text{n.e. in } A_i,
\]
which according to Lemma 5.2 also holds \( \mu_s^i \)-a.e. in \( X \) for each \( s \in S \). Having integrated this relation with respect to \( \mu_s^i \) and then summing up the inequalities obtained over all \( i \in I \), on account of (12.2) and (12.3) we get
\[
\kappa(\chi + R\gamma, R\mu_s) = \sum_{i \in I} \langle W^i_\gamma, \mu_s^i \rangle \geq \sum_{i \in I} \max \left\{ \eta_i, \langle W^i_\gamma, \gamma^i \rangle \right\} \geq \sum_{i \in I} \eta_i
\]
\[
= \frac{1}{2} \left[ \|\gamma\|_{\mathcal{E}^+(A)} + G(A, a, g) \right] = \sum_{i \in I} \langle W^i_\gamma, \gamma^i \rangle = \kappa(\chi + R\gamma, R\gamma),
\]
which in view of the strong (hence, weak) convergence of \( (R\mu_s)_{s \in S} \) to \( R\gamma \) proves (12.5). \( \square \)
Corollary 12.1 For any $\gamma, \gamma_1 \in \mathcal{E}_\lambda(A, a, g)$, it holds that
$$\langle W^i_{\gamma_1}, \gamma_1 \rangle = \langle W^i_{\gamma}, \gamma \rangle$$ for all $i \in I$.

Proof. Indeed, then $\|\gamma - \gamma_1\|_{\mathcal{E}^+(A)} = 0$ by assertion (ii) of Lemma 9.1, which in view of Corollary 12.4 yields $W_{\gamma} = W_{\gamma}$ n.e. in $X$. Combining this with relation (12.3), we get
$$a_i W^i_{\gamma}(x) \geq \langle W^i_{\gamma}, \gamma \rangle g(x) \text{ n.e. in } A_i, \quad i \in I,$$
which establishes the corollary due to the uniqueness statement in Theorem 12.1. $\square$

13 Proof of Theorem 7.3

Fix $\hat{\lambda} \in \mathcal{E}_\lambda(A, a, g, CL)$; it exists according to Theorem 7.1 with $J = CL = I \setminus \{\ell\}$. In consequence of Theorem 7.2, Theorem 7.3 will be proved once we establish the existence of a solution to the (main) $(\mathcal{A}, a, g, \chi)$-problem under the hypothesis
$$a_\ell < \langle g, \hat{\lambda} \rangle.$$ (13.1)

(Observe that the value on the right does not depend on the choice of $\hat{\lambda}$, which is clear from assumption (7.6) in view of $R\lambda_1 = R\lambda_2$ for all $\lambda_1, \lambda_2 \in \mathcal{E}_\lambda(A, a, g, CL)$; see Corollary 9.1.)

Let $\{K_\ell\}_{A_\ell}$ denote the increasing filtering family of all compact subsets $K_\ell$ of $A_\ell$. For each $K_\ell \in \{K_\ell\}_{A_\ell}$, define the $(I^+, I^-)$-condenser $A_{K_\ell} = (A^K_\ell)_{i \in \ell}$ by
$$A^K_\ell := A_i \quad \text{for all } i \neq \ell, \quad A^K_\ell := K_\ell.$$

Lemma 13.1 There holds the following statement on continuity:
$$G_\chi(A, a, g) = \lim_{K_\ell \uparrow A_\ell} G_\chi(A_{K_\ell}, a, g).$$

Proof. Fix $\mu \in \mathcal{E}^+(A, a, g)$. For every $K_\ell \in \{K_\ell\}_{A_\ell}$, consider $\mu^K_\ell := \mu^K\big|_{K_\ell}$, the trace of $\mu^K$ on $K_\ell$. Applying [10 Lemma 1.2.2], we then obtain [24]
$$\langle g, \mu^K_\ell \rangle = \lim_{K_\ell \uparrow A_\ell} \langle g, \mu^K_\ell \rangle, \quad (13.2)$$
$$\kappa(\chi, \mu^K_\ell) = \lim_{K_\ell \uparrow A_\ell} \kappa(\chi, \mu^K_\ell), \quad (13.3)$$
$$\kappa(\mu^K_\ell, \mu^K_\ell) = \lim_{K_\ell \uparrow A_\ell} \kappa(\mu^K_\ell, \mu^K_\ell), \quad (13.4)$$
$$\kappa(\mu^K_\ell, R\mu_{CL}) = \lim_{K_\ell \uparrow A_\ell} \kappa(\mu^K_\ell, R\mu_{CL}). \quad (13.5)$$

Choose a compact set $K^{0}_\ell \subset A_\ell$ so that $\langle g, \mu^K_\ell \rangle > 0$, which is possible due to (13.2), and for all $K_\ell \in \{K_\ell\}_{A_\ell}$ that follow $K^{0}_\ell$ define $\hat{\mu}_{A_{K_\ell}} = (\hat{\mu}^i_{A_{K_\ell}})_{i \in I}$, where
$$\hat{\mu}^i_{A_{K_\ell}} := \mu^K_i \quad \text{for all } i \neq \ell, \quad \hat{\mu}^i_{A_{K_\ell}} := \frac{a_\ell}{\langle g, \mu^K_i \rangle} \mu^K_i.$$ (13.6)

Since then
$$R\hat{\mu}_{A_{K_\ell}} = R\mu_{CL} - \frac{a_\ell}{\langle g, \mu^K_i \rangle} \mu^K_i,$$

\footnote{See Section 1 for the notation $\mu_{ij}$.}
Lemma 13.2

For this reason, in consequence of relation (5.3) and Lemma 13.1. Then, according to Theorem 5.1, in the rest of the proof we assume

\[ G(x) = \|R \mu_{CL}\|^2 - 2\kappa(\mu^e, R \mu_{CL}) + \|\mu\|^2 + 2\kappa(\chi, R \mu_{CL}) - 2\kappa(\chi, \mu^e) \]

\[ = \|R \mu_{CL}\|^2 - 2\kappa(\chi, R \mu_{CL}) + \lim_{K_l \uparrow A_l} \left\{ \|\mu_{A_{K_l}}\|^2 - 2\kappa(\mu_{A_{K_l}}, R \mu_{CL}) - 2\kappa(\chi, \mu^e_{A_{K_l}}) \right\} \]

which in view of the arbitrary choice of \( \mu \in E^+(A, \alpha, g) \) proves

\[ G(x)(A, \alpha, g) \geq \lim_{K_l \uparrow A_l} G(x)(A_{K_l}, \alpha, g). \]

Since the converse inequality is obvious, the lemma is established.

In the rest of the proof we assume \( G(x)(A_{K_l}, \alpha, g) < \infty \) for all \( K_l \in \{K_l\}_{A_l} \), which involves no loss of generality because of condition (5.3) and Lemma 13.1. Then, according to Theorem 5.1 for every \( A_{K_l} \), there exists \( \lambda_{A_{K_l}} \in \Theta_\chi(A_{K_l}, \alpha, g) \), while in consequence of Lemma 13.1 these minimizers form a net minimizing in the \( (A, \alpha, g, \chi) \)-problem, i.e.

\[ \lambda_{A_{K_l}} \in \prod_{K_l \in \{K_l\}_{A_l}} \in M_\chi(A, \alpha, g). \]

Fix a (particular) extremal measure \( \gamma \in E_\chi(A, \alpha, g) \) that is a vague cluster point of the net \( \lambda_{A_{K_l}} \in \{K_l\}_{A_l} \); it exists due to inclusion (13.7) and assertion (i) of Lemma 9.1.

Lemma 13.2

For this (particular) extremal measure \( \gamma \), it holds that

\[ a_i W_{\gamma}^i(x) \leq \langle W_{\gamma}, \gamma^i \rangle_{g}(x) \quad \text{for all} \quad x \in S(\gamma^i), \quad i \in I, \]

and therefore, in consequence of relation (12.4),

\[ a_i W_{\gamma}^i(x) = \langle W_{\gamma}, \gamma^i \rangle_{g}(x) \quad \text{n.e. in} \quad S(\gamma^i), \quad i \in I. \]

Proof. Fix \( i \in I \) and \( x_0 \in S(\gamma^i) \). Passing to a subnet if necessary, assume \( \lambda_{A_{K_l}} \in \{K_l\}_{A_l} \) to converge vaguely to \( \gamma \). Then one can find \( K_l \in S(\gamma^i) \) such that \( K_l \uparrow A_l \).

Note that, under the standing assumptions (cf. also Remark 6.1, [26, Theorem 7.2]) is applicable to each of \( \lambda_{A_{K_l}}, K_l \in \{K_l\}_{A_l} \). This gives

\[ a_i W_{A_{K_l}}^i(\gamma^i) \leq \langle W_{A_{K_l}}^i, \lambda_{A_{K_l}}^i \rangle_{g}(\gamma^i). \]

Another observation is that, according to (12.9) with \( \lambda_{A_{K_l}} \in \{K_l\}_{A_l} \) instead of \( \lambda_{A_{K_l}} \),

\[ \langle W_{A_{K_l}}^i, \gamma^i \rangle = \lim_{K_l \uparrow A_l} \langle W_{A_{K_l}}^i, \lambda_{A_{K_l}}^i \rangle. \]

Having substituted (12.4) with \( \mu = \lambda_{A_{K_l}} \) into the left-hand side of inequality (13.7), we pass to the limit as \( K_l \uparrow A_l \). In view of (13.11) and the lower semicontinuity of \( f_i = a_i \kappa(x, \chi) \) on \( A_i \) (see Remark 6.1), we then see that assertion (13.8) will be proved once we establish

\[ \kappa(x_0, R \gamma^-) = \lim_{K_l \uparrow A_l} \kappa(\gamma^i, R \lambda^i_{A_{K_l}}) \quad \text{and} \quad \kappa(x_0, R \gamma^+) \leq \lim_{K_l \uparrow A_l} \kappa(\gamma^i, R \lambda^i_{A_{K_l}}), \]

but these two follow directly from Lemma 8.3 and the lower semicontinuity of the mapping \( (x, \nu) \mapsto \kappa(x, \nu) \) on \( X \times M^+(X) \) (see [10, Lemma 2.2.1]), respectively.

\[ \square \]
Applying Lemma 10.1 to \( \gamma \), we get \( \langle g, \gamma^i \rangle = a_i \) for all \( i \neq \ell \), so that
\[
\gamma \in \mathcal{E}^+(A, a, g, CL). \tag{13.12}
\]
We next proceed to show that, due to the additional requirement (13.1), it also holds
\[
\langle g, \gamma^\ell \rangle = a_\ell. \tag{13.13}
\]

**Lemma 13.3** It is true that
\[
\langle W^\ell, \gamma^\ell \rangle \neq 0. \tag{13.14}
\]

**Proof.** Assume, on the contrary, that
\[
\langle W^\ell, \gamma^\ell \rangle = 0. \tag{13.14}
\]
Then, according to (12.2) with \( \mu = \mu_1 = \gamma \),
\[
\sum_{j \neq \ell} \langle W^j, \gamma^j \rangle = \sum_{i \in I} \langle W^j, \gamma^i \rangle = \kappa(\chi + R\gamma, R\gamma). \tag{13.15}
\]
As all the coordinates of the given \( \tilde{\lambda} \in \mathfrak{G}_\chi(A, a, g, CL) \) are bounded by Corollary 8.1, for each \( i \in I \) relation (12.4) holds \( \tilde{\lambda} \)-a.e. in \( X \); that is,
\[
a_i W^j(x) \geq \langle W^j, \gamma^j \rangle g(x) \quad \tilde{\lambda} \text{-a.e. in } X, \quad i \in I.
\]
Having integrated this relation, divided by \( a_i \), with respect to \( \tilde{\lambda} \) and then summing up the inequalities obtained over all \( i \in I \), on account of equalities (13.14), (13.15) and \( \langle g, \lambda^i \rangle = a_i \) for all \( i \neq \ell \) we get
\[
\sum_{i \neq \ell} \langle W^i, \lambda^i \rangle \geq \sum_{i \neq \ell} \frac{\langle W^j, \gamma^j \rangle}{a_i} \langle g, \lambda^i \rangle = \sum_{j \neq \ell} \langle W^j, \gamma^j \rangle = \kappa(\chi + R\gamma, R\gamma),
\]
which because of (12.2) with \( \mu = \gamma \) and \( \mu_1 = \tilde{\lambda} \) can be rewritten in the form
\[
\kappa(\chi + R\gamma, R\gamma) \leq \kappa(\chi + R\gamma, R\tilde{\lambda})
\]
or, equivalently,
\[
\|\chi + R\gamma\| \leq \kappa(\chi + R\gamma, \chi + R\tilde{\lambda}).
\]
Applying the Cauchy–Schwarz inequality and, subsequently, identity (5.1), we then obtain
\[
G_\chi(\gamma) \leq G_\chi(A, a, g, CL),
\]
which in view of inclusion (13.12) yields \( \gamma \in \mathfrak{G}_\chi(A, a, g, CL) \).
By relation (13.12) and the observation following it, we thus get
\[
\langle g, \gamma^\ell \rangle > a_\ell.
\]
Since this contradicts to relation (9.3), the lemma is established. \( \square \)

Now, having integrated (13.9) for \( i = \ell \) with respect to the (bounded) measure \( \gamma^\ell \), on account of Lemma 13.3, we obtain (13.13), which together with inclusion (13.12) shows that, actually, \( \gamma \in \mathcal{E}^+(A, a, g) \). Combined with Corollary 9.2, this proves that \( \gamma \) is, in fact, a solution to the \( (A, a, g, \chi) \)-problem, and Theorem 7.3 follows. \( \square \)

**Acknowledgments** The author is very grateful to Professor W.L. Wendland for the careful reading of the manuscript and his suggestions to improve it. A part of this research was done during the author’s visit to the DFG Cluster of Excellence Simulation Technologies at the University of Stuttgart during February of 2012, and the author acknowledges this institution for the support and the excellent working conditions.
References

[1] Aptekarev, A.I., Lysov, V.G.: Systems of Markov functions generated by graphs and the asymptotics of their Hermite–Padé approximants. Sb. Math. 201, 183–234 (2010)

[2] Bourbaki, N.: Elements of Mathematics, Integration, chapters 1–6. Springer, Berlin (2004)

[3] Brelot, M.: On Topologies and Boundaries in Potential Theory. Lectures Notes in Math., vol. 175. Springer, Berlin (1971)

[4] Cartan, H.: Théorie du potentiel Newtonien: énergie, capacité, suites de potentiels. Bull. Soc. Math. Fr. 73, 74–106 (1945)

[5] Deny, J.: Les potentiels d’énergie finite. Acta Math. 82, 107–183 (1950)

[6] Deny, J.: Sur la définition de l’énergie en théorie du potentiel. Ann. Inst. Fourier Grenoble 2, 83–99 (1950)

[7] Edwards, R.: Cartan’s balayage theory for hyperbolic Riemann surfaces. Ann. Inst. Fourier 8, 263–272 (1958)

[8] Edwards, R.: Functional Analysis. Theory and Applications. Holt, Rinehart and Winston, New York (1965)

[9] Frostman, O.: Potentiel d’équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. Comm. Sém. Math. Univ. Lund 3, 1–118 (1935)

[10] Fuglede, B.: On the theory of potentials in locally compact spaces. Acta Math. 103, 139–215 (1960)

[11] Fuglede, B.: Caractérisation des noyaux consistants en théorie du potentiel. Comptes Rendus 255, 241–243 (1962)

[12] Fuglede, B.: Asymptotic paths for subharmonic functions and polygonal connectedness of fine domains. In: Lectures Notes in Math., vol. 814, 97–115. Springer, Berlin (1980)

[13] Gauss, C.F.: Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstoßungs–Kräfte (1839). Werke 5, 197–244 (1867)

[14] Gonchar, A.A., Rakhmanov, E.A.: On convergence of simultaneous Padé approximants for systems of functions of Markov type. Proc. Steklov Inst. Math. 157, 31–50 (1983)

[15] Gonchar, A.A., Rakhmanov, E.A.: On the equilibrium problem for vector potentials. Russ. Math. Surv. 40(4), 183–184 (1985)

[16] Gonchar, A.A., Rakhmanov, E.A.: Equilibrium measure and the distribution of zeros of extremal polynomials. Math. USSR-Sb. 53, 119–130 (1986)

[17] Gonchar, A.A., Rakhmanov, E.A., Sorokin, V.N.: Hermite–Padé approximants for systems of Markov-type functions. Sb. Math. 188, 671–696 (1997)

[18] Harbrecht, H., Wendland, W.L., Zorii, N.: Riesz minimal energy problems on $C^{k-1,1}$-manifolds. Preprint Series Stuttgart Research Centre for Simulation Technology (2012)

[19] Hayman, W.K.: Subharmonic Functions, vol. 2. Academic Press, London (1989)
[20] Hayman, W.K., Kennedy, P.B.: Subharmonic Functions, vol. 1. Academic Press, London (1976)
[21] Kelley, J.L.: General Topology. Princeton, New York (1957)
[22] Landkof, N.S.: Foundations of Modern Potential Theory. Springer, Berlin (1972)
[23] Mhaskar, H.N., Saff, E.B.: Extremal problems for polynomials with exponential weights. Trans. Amer. Math. Soc. 285, 204–234 (1984)
[24] Moore, E.H., Smith, H.L.: A general theory of limits. Amer. J. Math. 44, 102–121 (1922)
[25] Nikishin, E.M., Sorokin, V.N.: Rational Approximations and Orthogonality. Translations of Mathematical Monographs, vol. 44. Amer. Math. Soc., Providence (1991)
[26] Of, G., Wendland, W.L., Zorii, N.: On the numerical solution of minimal energy problems. Complex Variables and Elliptic Equations 55, 991–1012 (2010)
[27] Ohtsuka, M.: On potentials in locally compact spaces. J. Sci. Hiroshima Univ. Ser. A-1 25, 135–352 (1961)
[28] Saff, E.B., Totik, V.: Logarithmic Potentials with External Fields. Springer, Berlin (1997)
[29] Zorii, N.: An extremal problem on the minimum of energy for space condensers. Ukr. Math. J. 38, 365–370 (1986)
[30] Zorii, N.: A problem of minimum energy for space condensers and Riesz kernels. Ukr. Math. J. 41, 29–36 (1989)
[31] Zorii, N.: A noncompact variational problem in Riesz potential theory. I. Ukr. Math. J. 47, 1541–1553 (1995)
[32] Zorii, N.: Equilibrium potentials with external fields. Ukr. Math. J. 55, 1423–1444 (2003)
[33] Zorii, N.: Equilibrium problems for potentials with external fields. Ukr. Math. J. 55, 1588–1618 (2003)
[34] Zorii, N.: Necessary and sufficient conditions for the solvability of the Gauss variational problem. Ukr. Math. J. 57, 70–99 (2005)
[35] Zorii, N.: Interior capacities of condensers in locally compact spaces. Potential Anal. 35, 103–143 (2011), DOI:10.1007/s11118-010-9204-y
[36] Zorii, N.: Equilibrium problems for infinite dimensional vector potentials with external fields. Potential Anal., DOI:10.1007/s11118-012-9279-8