Exact relations for thermodynamics of heavy quarks

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We derive finite-temperature sum rules for excesses in internal energy and in (volume-integrated) pressure arising due to presence of heavy quarks in \( SU(N) \) gluon plasma. In the limit of zero temperature our formulae reduce to the Michael-Rothe sum rules. The excesses in energy and pressure of the gluon plasma are related to expectation values of certain gluon condensates, and, simultaneously, to the heavy quark potential. The sum rules lead to a known relation between the internal energy and the potential, and to a new expression for the excess in the pressure. The pressure appears in the free energy as a generalized force associated with variations of the spatial size of the heavy-quark system. We find that the excess in gluonic pressure around a heavy quarkonium is always negative. Finally, we derive an exact equation of state that provides a relationship between the gluonic energy and pressure of heavy quarks.

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Introduction. The most anticipated result of experiments on heavy-ion collisions is a possible creation of a deconfined state of matter in a form of quark-gluon plasma. Long time ago it was suggested that an enhanced dissociation of heavy quark-antiquark bound states could be a good signal of the deconfinement phase \[1\]. Theoretical efforts aimed to study properties of heavy quarkonia require a nonperturbative input in a form of quark-antiquark potentials at finite temperature \[2, 3\]. A powerful numerical tool to study such potentials is based on first-principle calculations of Polyakov loop correlators in lattice simulations of QCD \[3\]. The renormalized correlator of Polyakov loops provides us with the free energy, and – via thermodynamic relations – with internal energy and entropy of the heavy quarks \[4\]. Since the quarks are heavy particles they cannot be excited thermally, so that the term “free energy of heavy quarks” is usually understood as an excess in free energy of thermal gluons that appears due to presence of the strong chromoelectric fields of the heavy quarks \[4\].

It is clear that presence of heavy quarks affects thermodynamics of the quark-gluon plasma. Despite the fact that the thermodynamic effect of one quark-antiquark pair is not an extensive quantity, a multiple production of quarkonia may provide a contribution to the bulk properties of the plasma. In addition to the free energy, internal energy and entropy of the heavy quarks, we find that the quarks are also able to make contribution to the pressure of the system. The effect of pressure is not seen in a standard approach because the pressure is usually associated with variations of the volume of the system, which does not enter the excess in the free energy of a finite-sized quark pair. However, below we show that the variations in size of a quark system do couple to the pressure.

In the heavy-quark thermodynamics, the spatial extent of the quark pair can be treated as an external variable while the pressure enters a quantity that plays a role of the corresponding generalized force. From this point of view the renormalized quark potential can be associated with an excess in a generalized Helmholtz free energy.

We use a sum rule approach that is generally known as powerful analytical for investigation of various nonperturbative properties of QCD physics \[2\]. For example, in absence of the external sources the susceptibility of the trace of the energy-momentum tensor can be related via certain sum rules to the bulk viscosity in the quark-gluon plasma \[6\]. The susceptibilities of components of the energy-momentum tensor were shown to be related to (derivatives of) thermodynamic potentials \[7\].

We derive exact finite-temperature sum rules for excesses in internal energy and in (volume-integrated) pressure in the presence of external heavy quarks. In the limit of zero temperature certain combinations of these sum rules reduce to the well-known action and energy sum rules derived by Michael and Rothe in a lattice formulation of the theory. The Michael-Rothe equations associate the quark-antiquark potential to an excess in chromoelectric and chromomagnetic condensates. Historically, the first attempt to derive such sum rules was done in \[8\], but the derivation turns out to be correct only partly \[6\]. The rules were subsequently corrected \[10\] and extended \[11\], and an important role of the conformal anomaly was stressed \[12\]. A lattice check of the sum rules was done in Ref. \[13\]. Below we use the new finite-temperature sum rules to study the heavy-quark thermodynamics further.

Thermodynamics of \( SU(N) \) Yang-Mills theory. It is instructive to discuss first the thermodynamics of Yang-Mills theory without external quark sources. Despite the derivation is well known, we repeat it for the sake of further convenience.

The equation of state of the \( SU(N) \) Yang–Mills theory
can be determined from the energy momentum tensor
\[
\mathcal{T}_{\mu \nu} = 2 \text{Tr} \left[ G_{\mu \sigma} G_{\nu \rho} - \frac{1}{4} \eta_{\mu \nu} G_{\rho \sigma} G_{\rho \sigma} \right],
\]
(1)
where \( G_{\mu \nu} \equiv C_{\mu \nu}^{\rho} t^a = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + ig[A_{\mu}, A_{\nu}] \) is the field strength tensor of the gauge field \( A_{\mu} \equiv t^a A^a_{\mu} \), and \( t^a \) \((a = 1, \ldots, N^2 - 1)\) are the generators of the gauge group.

At a classical level the energy–momentum tensor is traceless because the bare Yang–Mills theory is a conformal theory. At the quantum level, however, the conformal invariance is broken due to a dimensional transmutation. As a consequence, the trace of the energy–momentum tensor exhibits a conformal anomaly,
\[
\mathcal{T}_{\mu \mu}(x) = \frac{\beta(g)}{2g} G^{\mu}_{\mu}(x) G^{\mu}_{\mu}(x) \equiv \frac{3 \beta(g)}{g} \mathcal{L}(x),
\]
(2)
where \( \mathcal{L} \) is the Euclidean Lagrangian of the theory:
\[
\mathcal{L} = \frac{1}{2} \text{Tr} C_{\mu \nu}^{2} - \frac{1}{2} (E^{2} + B^{2}),
\]
(3)
Here \( E^{2} \) and \( B^{2} \) denote, respectively, chromoelectric and chromomagnetic field strength tensors squared.

The Gell-Mann-Low \( \beta \)-function in Eq. (2) is
\[
\beta(g) = \frac{\partial g(\mu)}{\partial \ln \mu} = - g^{3} (b_{0} + b_{1} g^{2} + \ldots),
\]
(4)
where \( \mu \) is the renormalization scale at which the Yang–Mills coupling constant \( g \) is defined. One- and two-loop perturbative coefficients of the \( \beta \)-function [11] are
\[
b_{0} = \frac{11 N}{3(4 \pi)^{2}}, \quad b_{1} = \frac{34 N^{2}}{3(4 \pi)^{4}}.
\]
(5)

In the thermodynamic limit the energy of the system is \( E = \varepsilon V \) where \( \varepsilon \) is the energy density and \( V \) is the volume of the system (at the end of the calculation we set \( V \rightarrow \infty \)). The energy \( E \) and the pressure \( P \) are determined via the derivatives of the partition function \( Z = Z(V, T, g(\mu)) \) with respect to the temperature \( T \) and volume, respectively,
\[
E = T \frac{\partial \ln Z}{\partial \ln T}, \quad PV = T \frac{\partial \ln Z}{\partial \ln V}.
\]
(6)

The logarithm of the partition function is a dimensionless quantity proportional to the free energy \( F = fV \),
\[
\ln Z = - \frac{F}{T} = - f \frac{V}{T}.
\]
(7)
Thus, one gets on dimensional grounds
\[
\left( 3 \frac{\partial}{\partial \ln V} - \frac{\partial}{\partial \ln T} - \frac{\partial}{\partial \ln \mu} \right) \ln Z = 0.
\]
(8)
The free energy is a physical quantity, therefore its total derivative with respect to the scale \( \mu \) should vanish:
\[
\frac{d \ln Z}{d \ln \mu} = \left( \frac{\partial}{\partial \ln \mu} + \beta(g) \frac{\partial}{\partial g} \right) \ln Z = 0.
\]
(9)
Here we used Eq. (1) and followed partly Ref. [9]. The identity \( \partial \ln Z / \partial g = (V/T)(2C/g) \) and Eqs. (2), (3), (6), (8) give us the conformal anomaly
\[
\Theta \equiv E - 3 PV = \int d^{3} x \left\langle \frac{\beta(g)}{g} \left[ E^{2}(x) + B^{2}(x) \right] \right\rangle_{T},
\]
(10)
where we implemented the subtraction of the \( T = 0 \) contribution, \( \left\langle \ldots \right\rangle = \left\langle \ldots \right\rangle_{T = 0} \). Equation (10) is the spatial integral of the conformally-anomalous trace (2) of the energy-momentum tensor [11].

Equation (10) is implemented in most first-principle determinations of the gluonic equation of state from lattice simulations of Yang-Mills theory (see, e.g., Refs. [15, 16]). Indeed, combining Eq. (7) with (6) one gets:
\[
E = F - T \frac{\partial F}{\partial T}, \quad P = \frac{F}{V} \equiv - f,
\]
(11)
so that \( \Theta = VT^{5} \frac{\partial P(T)}{\partial T} \). The left hand side of (10) becomes \( T^{5} \frac{g}{g^{2} + P(T)} \) while the right hand side can be determined numerically. Integration of this equation gives us full information on the thermodynamics of the theory.

Thus, in thermodynamic limit of a uniform system – such as the vacuum of \( SU(N) \) gauge theory – the energy density and pressure are extensive variables that are related to each other by thermodynamical identities. Equation (10) provides us with an additional relation between the energy and pressure of the system so that we have the complete system of two equations for two unknown quantities. However, the contributions to energy density and pressure coming from the external sources (for example, from heavy quark-antiquark systems) cannot, in general, be described by extensive variables. Thus, the standard relation between energy and pressure should not work in these systems by default, and we need to determine these quantities separately.

Fortunately, Eq. (10) can be extended further by introducing separate renormalization scales \( \mu_{s} \) and \( \mu_{t} \) in, respectively, temporal and spatial directions. The procedure is well-known in the lattice gauge theory [17, 18], and we reformulate it here in continuum terms for the future use. The analogues of Eq. (8) are:
\[
3 \frac{\partial \ln Z}{\partial \ln V} - \frac{\partial \ln Z}{\partial \ln \mu_{t}} = 0, \quad \frac{\partial \ln Z}{\partial \ln T} + \frac{\partial \ln Z}{\partial \ln \mu_{s}} = 0,
\]
(12)
while the requirement of the scale independence provides us with following counterparts of Eq. (9):
\[
\frac{d \ln Z}{d \ln \mu_{a}} = \frac{\partial \ln Z}{\partial \ln \mu_{a}} + \sum_{b=s,t} \frac{\partial g_{b}}{\partial \ln \mu_{a}} \frac{\partial \ln Z}{\partial g_{b}} = 0.
\]
(13)
Here \( a = s, t \), and we always imply silently that at the end of all differentiations we set \( \mu_{s} = \mu_{t} = \mu \). The spatial and temporal couplings are defined by the relations
\[
\frac{1}{g_{s}^{2}} = \frac{1}{g^{2}} \mu_{s}, \quad \frac{1}{g_{t}^{2}} = \frac{1}{g^{2}} \mu_{t},
\]
(14)
respectively. Then Eqs. (12), (13), (14) provide us with following expressions for the energy and pressure (19):

\[
\int d^3x \varepsilon = \int d^3x \left\{ \frac{1}{2} \left[ \bar{B}^2(x) - \bar{E}^2(x) \right] \right\}_T \tag{15}
\]

\[
+ \left\{ \beta(g) \left[ \bar{E}^2(x) + \bar{B}^2(x) \right] \right\}_T ,
\]

\[
3 \int d^3x P = \int d^3x \left\{ \left[ \frac{1}{2} \bar{B}^2(x) - \bar{E}^2(x) \right] \right\}_T \tag{16}
\]

\[
- \left\{ \beta_\sigma(g) \left[ \bar{E}^2(x) + \bar{B}^2(x) \right] \right\}_T ,
\]

where the generalizations of the \( \beta \)-functions (11) are:

\[
\beta_\sigma(g(\mu)) = \frac{\partial g(\mu, \mu_\sigma)}{\partial \ln \mu_\sigma} , \quad a = s, t . \tag{17}
\]

The first term in the curly brackets in each of Eqs. (15) and (16) is a classical contribution coming from the classical energy-momentum tensor (11). The second term in these expressions corresponds to the quantum correction related to the conformal trace anomaly (2). Thus, it is natural that the first terms cancel in the trace of the energy-momentum tensor, \( T_{\mu\nu} = \varepsilon - 3P \), while the second terms give us (10) because of the equality \( \beta_\varepsilon + \beta_\sigma = \beta \).

In a lattice regularization, relations (15) and (16) were analytically calculated long time ago (17). Explicit expressions of the specific \( \beta \)-functions (17) are given in (18).

**Gluon energy and pressure via gluon condensates in presence of heavy quarks and antiquarks.**

Now imagine that we have inserted a very heavy quark-antiquark pair in the thermal vacuum of Yang-Mills theory. Since the quark and the antiquark are very heavy they cannot be excited by the thermal fluctuations. Nevertheless, the gluon field of the pair affects the thermal fluctuations of gluons in the thermal bath, and, consequently, makes a contribution to energy density and pressure around the quark sources.

How to take into account the presence of the heavy quarks on thermodynamics of gluons? An obvious way is to evaluate the gluon condensates in Eqs. (15) and (16) with insertions of quark creation operators.

A static quark at the position \( \bar{x} \) is created with the help of the gauge-invariant Polyakov loop operator

\[
L(\bar{x}) = \frac{1}{N} \text{Tr} \expq{ig \int_0^{1/T} dt A_\mu(\bar{x}, t)} \tag{18}
\]

where the operator \( \mathcal{P} \) implies the path ordering of the integration which goes along the straight line parallel to compactified (temperature) direction of the Euclidean space-time. The temperature is introduced via the imaginary time formalism. The length of the compactified direction is \( 1/T \). The path of the integration in Eq. (18) is closed via the periodic boundary condition imposed in the compactified direction. The conjugated operator \( L^\dagger(\bar{x}) \) creates an antiquark at the spatial point \( \bar{x} \).

A system \( Q = Q_1 \ldots Q_{N_Q} \bar{Q}_1 \ldots \bar{Q}_{N_{\bar{Q}}} \), consisting of \( N_Q \) quarks located at spatial positions \( \bar{x}_i \) and \( N_{\bar{Q}} \) antiquarks at points \( \bar{y}_k \), is created by the operator

\[
L_Q(\{ \bar{R}_i \}) = L(\bar{x}_1) \ldots L(\bar{x}_{N_Q}) \cdot L^\dagger(\bar{y}_1) \ldots L^\dagger(\bar{y}_{N_{\bar{Q}}}) , \tag{19}
\]

where \( \{ \bar{R}_i \} \equiv \{ \bar{x}_1, \ldots, \bar{x}_{N_Q}, \bar{y}_1, \ldots, \bar{y}_{N_{\bar{Q}}} \} \). Since generalizations to various representations is straightforward, we consider here the color-averaged operators (19) only.

According to Eqs. (15) and (16), in order to estimate the effect of the heavy (anti)quark system on the thermodynamics of gluons one should evaluate the chromoelectric and chromomagnetic condensates in the presence of the heavy quarks. Since Eqs. (15) and (16) are linear in \( \delta \), we can replace both (20) and (21) do not depend on the spatial size of the system. These quantities are finite but, in general, nonzero. We used the symbol \( \mathcal{V} \) (reminiscent of \( V \cdot P \)) to denote the integral over the spatial volume, \( \int d^3x \). In order to avoid an abuse of notations, hereafter we remove the \( \delta \)-signs in front of excesses in thermodynamic variables, so that \( E_Q \equiv \delta E_Q \), \( \mathcal{V}_Q \equiv \delta \mathcal{V}_Q \) etc.

The contribution of the quark system \( Q \) into both gluon condensates \( \mathcal{O} = \int d^3x \bar{E}^2 \) and \( \mathcal{O} = \int d^3x \bar{B}^2 \) is:

\[
\langle \mathcal{O} \rangle_{Q,T} = \frac{\langle \mathcal{O} L_Q \rangle_T}{\langle L_Q \rangle_T} - \langle \mathcal{O} \rangle_T , \tag{22}
\]

where the product \( L_Q \) of the Polyakov loops is defined in Eq. (19). The right hand side of Eq. (22) is independent on the imaginary time.

Substituting (15) and (16) into Eqs. (20) and (21) we get the excesses in the pressure and the energy of the system due to the presence of the heavy (anti)quarks

\[
E_Q(T) = \int d^3x \left\{ \frac{1}{2} \left[ \bar{B}^2(x) - \bar{E}^2(x) \right] \right\}_{Q,T} \tag{23}
\]

\[
+ \left\{ \beta(g) \left[ \bar{E}^2(x) + \bar{B}^2(x) \right] \right\}_{Q,T} ,
\]

\[
3 \mathcal{V}_Q(T) = \int d^3x \left\{ \frac{1}{2} \left[ \bar{B}^2(x) - \bar{E}^2(x) \right] \right\}_{Q,T} \tag{24}
\]

\[
- \left\{ \beta(g) \left[ \bar{E}^2(x) + \bar{B}^2(x) \right] \right\}_{Q,T} .
\]
Notice that the expectation value $\langle O \rangle_{Q,T}$ in Eqs. (15), (16) gets automatically replaced by $\langle O \rangle_{Q,T}$ in Eqs. (23), (24).

Equations (23) and (24) can already be used for a numerical estimation of the contribution of the heavy (anti)quark systems into the gluon thermodynamics. However, there is an analytical way to proceed further.

Generalization of Michael-Rothe sum rules. Consider a quark $Q$ and an antiquark, $\bar{Q}$, separated by the distance $R$. The color-averaged $Q\bar{Q}$ potential $V_{Q\bar{Q}}$ is given by the expectation value (20):

$$e^{-V_{Q\bar{Q}}/T} = (L(0)L(\bar{R})).$$

(25)

Its important to notice that the thermodynamics of heavy quarks is often discussed with the help of the renormalization Polyakov loops $L^{ren}(\bar{R})$ that are multiplied by a perimeter/length-dependent factor $Z$:

$$L(\bar{R}) \rightarrow L^{ren}(\bar{R}) = [Z(g)]^{\mu/\gamma} \cdot L(\bar{R}).$$

(26)

However, at this stage the renormalization is irrelevant for the determination of both the energy (23) and the pressure (24), because Eq. (22) – used both in (23) and in (24) – is invariant under the renormalization shift (20).

The contribution of the finite-sized heavy-quark system into the thermodynamics is not an extensive quantity. Thus, the heavy quark potential (23) is a function of four dimensionfull parameters: the temperature $T$, the distance between the quark and antiquark $R$, and the spatial and temporal renormalization scales $\mu_s$ and $\mu_t$. In the presence of the heavy quarks we rewrite the dimensional identities (19) and the scale-independence requirements (12) using the quantity $\ln(L(0)L(\bar{R})) \equiv -V_{Q\bar{Q}}/T$ instead of $\ln Z$. The first pair of equations is:

$$\left( \frac{\partial}{\partial \ln R} - \frac{\partial}{\partial \ln \mu_s} \right) \frac{V_{Q\bar{Q}}(R,T;\mu_s,\mu_t)}{T} = 0,$$

(27)

while the second pair gives us

$$T^2 \frac{\partial}{\partial T} \frac{V_{Q\bar{Q}}}{T} = -\int d^3x \left\{ \frac{1}{2} \left[ \vec{B}^2(x) - \vec{E}^2(x) \right] \right\}_{Q\bar{Q},T} + \left\{ \frac{\beta_1(g)}{g} \left[ \vec{E}^2(x) + \vec{B}^2(x) \right] \right\}_{Q\bar{Q},T},$$

(29)

These sum rules is a generalization of already known low-energy relations that were derived in $T=0$ lattice gauge theory in Refs. [8, 12]. For example, subtracting Eq. (29) from Eq. (30) and using (18) one gets:

$$\left( 1 + R \frac{\partial}{\partial R} - T \frac{\partial}{\partial T} \right) V_{Q\bar{Q}} = \frac{2\beta_1(g)}{g} \int d^3x \mathcal{L}(x).$$

(31)

This relation is nothing but a natural finite-temperature extension of a well-known “action sum rule”:

$$V_{Q\bar{Q}} + R \frac{\partial V_{Q\bar{Q}}}{\partial R} = \frac{2\beta_1(g)}{g} \int d^3x \mathcal{L}(x)_{Q\bar{Q}(T=0)}.$$  

(32)

Moreover, a zero-temperature limit of (29) gives us a continuum version of the so-called “energy sum rule”:

$$V_{Q\bar{Q}} = \int d^3x \left\{ \frac{1}{2} \left[ \vec{B}^2(x) - \vec{E}^2(x) \right] \right\}_{Q\bar{Q}(T=0)} + \left\{ \frac{\beta_1(g)}{g} \left[ \vec{E}^2(x) + \vec{B}^2(x) \right] \right\}_{Q\bar{Q}(T=0)}.$$  

(33)

Let us mention that we have three pairs of equations that look similar but have very different meanings:

i) The pair (15) and (16) is an equality, that relates the energy and pressure of the gluonic fields to the expectation values of the gluon condensates. This pair is a continuum version of the corresponding lattice formulae derived in Ref. 17.

ii) The pair (23) and (24) is a natural definition of the contribution of heavy quarks to the energy (15) and pressure (16) of the gluons.

iii) Finally, the pair (29) and (30) describes the new finite-temperature sum rules.

New sum rules and exact relations for the heavy-quark thermodynamics. A comparison of the exact relations (23), (24) with new sum rules (29), (30) shows that the excess in the energy (20) and the (volume-integrated) excess in the pressure (21) due to the presence of the heavy quark-antiquark pair are, respectively:

$$E_{Q\bar{Q}}(R,T) = V_{Q\bar{Q}} - T \frac{\partial V_{Q\bar{Q}}}{\partial T} \equiv -T^2 \frac{\partial}{\partial T} \frac{V_{Q\bar{Q}}}{T},$$

(34)

$$\mathcal{V}_{Q\bar{Q}}(R,T) = -\frac{R}{3} \frac{\partial V_{Q\bar{Q}}}{\partial R}.$$  

(35)

These relations are exact.

From Eqs. (34) and (35) we get a Maxwell-type relation between the excesses in the energy and pressure:

$$R \left( \frac{\partial E_{Q\bar{Q}}}{\partial R} \right)_T = 3T^2 \left( \frac{\partial \mathcal{V}_{Q\bar{Q}}}{\partial T} \right)_R,$$  

(36)

where the subscripts mean that the derivatives are taken at the fixed temperature $T$ and the fixed interquark distance $R$, respectively. Equation (36) is an exact equation of state for heavy quarks.
A comparison of the energy excess (34) with the corresponding thermodynamical relation [the first formula in (11)] would lead us to conclude that the heavy quark potential $V_{Q\bar{Q}}$ may be interpreted as up to a renormalization constant (25) as an excess in the Helmholtz free energy $F_{Q\bar{Q}}$ due to the presence of the heavy quarks. However, the excess in the pressure (35) has nothing to do with its analogue in the thermodynamical limit (the second formula in (11)), making the free energy interpretation of the heavy quark potential $V_{Q\bar{Q}}$ obscure.

Nevertheless, one can still interpret $V_{Q\bar{Q}}$ as an excess in the Helmholtz free energy of gluons due to the presence of the heavy (anti)quarks. Indeed, besides the temperature $T$, the heavy quark thermodynamics has the additional external variable $R$ but lacks the usual volume variable $V$. In this case the usual fundamental thermodynamic relation, $dE = TdS - PdV$, should be written as follows

$$dE_{Q\bar{Q}} = TdS_{Q\bar{Q}} - X_{Q\bar{Q}}dR,$$

where $S_{Q\bar{Q}}$ is the excess in the entropy and $X_{Q\bar{Q}}$ is the generalized force associated with the distance $R$ between the quark and antiquark. From Eq. (34) one gets:

$$S_{Q\bar{Q}} = -\left(\frac{\partial V_{Q\bar{Q}}}{\partial T}\right)_{R},$$

$$X_{Q\bar{Q}} = -\left(\frac{\partial V_{Q\bar{Q}}}{\partial R}\right)_{T} = -\frac{3}{R}V_{Q\bar{Q}}.$$

so that the generalized force $X_{Q\bar{Q}}$ is related to the (integrated) pressure (35). Then the differential of the excess in the Helmholtz free energy $F_{Q\bar{Q}} \equiv E_{Q\bar{Q}} - TS_{Q\bar{Q}}$ is:

$$dF_{Q\bar{Q}} = -S_{Q\bar{Q}}dT - X_{Q\bar{Q}}dR.$$  

(40)

This formula is consistent with other relations: the pair of relations (40), (35) restores the energy relation (34), while the pair (40), (39) gives us back the pressure excess (35).

The excess in the free energy $F_{Q\bar{Q}}$ can be easily obtained by an integration of Eq. (40):

$$F_{Q\bar{Q}}(R, T) = V_{Q\bar{Q}}(R, T).$$

(41)

A free integration constant – which inevitably appears in any integration – is to be interpreted as the renormalization quark-antiquark potential (29). We have just shown that Eq. (11) – despite its standard appearance – has somewhat nonstandard interpretation because it involves the generalized fundamental thermodynamic relation (37) that includes the generalized force (39).

The excess in the pressure due to the presence of the heavy quark pairs is always negative (35) because the $V_{Q\bar{Q}}$ potential is a concave function of the distance (22). Thus, the presence of the quarkonium states should decrease the pressure in the quark-gluon plasma. A gas of weakly-interacting heavy-quark bound states with the density $\rho$ should produce the total pressure deficit

$$\Delta P_{Q\bar{Q}}(R, T) = -\frac{\rho(R, T)R}{3} \frac{\partial V_{Q\bar{Q}}(R, T)}{\partial R} < 0,$$

(42)

(we assumed that all $Q\bar{Q}$-states have the same size $R$).

It is interesting to notice that even at zero temperature the integrated pressure (35) is not a positively defined quantity. This fact does not contradict the common wisdom. For example, a similar feature characterizes the well-known Casimir effect: the vacuum between two perfectly conducting metallic plates has a negative pressure, negative free energy and negative internal energy (24). Less exotic example of a similar system at nonzero temperature is a solution of ethanol in water, that demonstrates in normal conditions an effect of volume contraction (i.e., negative excess of volume). The maximal volume contraction is reached at approximately 40% fraction of ethanol, thus mimicking, in very loose terms, a noticeable (excess of) negative pressure.

It was noticed recently (25) that the contribution of the heavy quarks to the specific heat of the gluon plasma becomes negative at high enough temperatures. It was concluded that the heavy quark potential does not behave as a true thermodynamic free energy (27) indicating that one should call $V_{Q\bar{Q}}$ as “heavy quark interaction energy”. This conclusion is based on the numerical data for the renormalized heavy quark potential obtained in $SU(3)$ lattice gauge theory (4, 26) and supported by a perturbative calculation (27). We certainly agree with the fact (25) that one should not consider the quark contribution as the thermodynamic quantity on its own, so that the identification (11) – which we still claim to be correct taking into account its derivation in our paper – should be understood as “the excess in the free due to heavy quarks is equal to the (renormalized) heavy-quark potential” (following, e.g., (4)). Indeed, there is no ab initio restriction on the sign of the excess in any thermodynamic quantity due to presence of external particles. For example, it is well-known that adding salt to water reduces the specific heat of the water, exactly as heavy quarks should do with the plasma at very high temperatures. Nevertheless, we do not attribute a negative specific heat to the salt thus avoiding a thermodynamical puzzle.

**Multiquarks and a single quark.** A generalization of our results to multiquark systems is quite simple. Instead of the $Q\bar{Q}$ potential $V_{Q\bar{Q}}$ one should use its multiquark counterpart $V_Q$ defined via the multiquark loop (15) as follows:

$$e^{-V_Q((\vec{R})), T} = \langle L_Q((\vec{R})) \rangle.$$  

(43)

In addition, the derivatives with respect to the $Q\bar{Q}$ dis-
tance should be replaced by the following scale derivative:
\[ R \frac{\partial}{\partial R} V_{Q\bar{Q}}(R,T) = \frac{\partial}{\partial \lambda} V_Q(\{\lambda \bar{R}_i\},T) \bigg|_{\lambda=1}. \]  

(44)

According to Eqs. (35), (43) and (44), a single quark \( Q \) cannot affect the pressure of the system even in the deconfinement phase. Then we get the following nonobvious relation [an analogue of Eq. (30)] for \( Q = Q \):
\[ \left\langle \left( \frac{1}{2} \frac{\beta_s(g)}{g} \right) \int d^3 x B^2(x) \right\rangle_{Q,T} = \left\langle \left( \frac{1}{2} + \frac{\beta_s(g)}{g} \right) \int d^3 x \bar{E}^2(x) \right\rangle_{Q,T}, \]

(45)

which leads to an analogue of (24) for a single quark:
\[ V_Q - T \frac{\partial V_Q}{\partial T} = \left\langle \frac{2 \beta_s(g)}{g} \int d^3 x \mathcal{L}(x) \right\rangle_{Q,T}. \]

(46)

This is yet another new sum rule, that reduces to the known relation (32) in the limit \( T \to 0 \) (to this end one should notice that \( \partial V_Q/\partial R \equiv 0 \)).

A few phenomenological examples. The excesses in free energy (41), entropy (38) and internal energy (34) were thoroughly studied in lattice simulations of SU(3) Yang-Mills theory [4, 22]. Here we discuss briefly the new quantity, the (volume-integrated) excess in the pressure (35) caused by the presence of the heavy quark-antiquark pair.

In the deconfinement phase the data for the heavy-quark potential can be described by the relation (28):
\[ V_{Q\bar{Q}}(R,T) = -\frac{e(T)}{RT} \frac{T}{d(T)} e^{-\mu(T) R}, \quad [T \geq T_c], \]

(47)

where \( e(T) \) and \( d(T) \) are dimensionless temperature-dependent parameters, and \( \mu(T) \) is the screening mass. Using Eq. (35) we find that in the deconfinement phase the absolute value of the \( Q\bar{Q} \)-induced pressure is a decreasing function of the distance \( R \):
\[ V_{Q\bar{Q}}(R,T \geq T_c) = -\frac{d + \mu(T) R}{3} V_{Q\bar{Q}}(R,T) < 0. \]

(48)

At large separations the pressure excess vanishes due to the screening.

In the confinement phase the finite-\( T \) potential can be described at large distances (\( RT \gg 1 \)) by the formula (28):
\[ V_{Q\bar{Q}} = V_0 + \sigma(T) R + C T \ln(2RT), \quad [T < T_c]. \]

(49)

where \( V_0 \) and \( C \) are certain constants. One gets
\[ V_{Q\bar{Q}}(R,T < T_c) = -\frac{C T + \sigma R}{3}, \]

so that at large distances the pressure is a linear function of the separation between the quark and antiquark.

At zero temperature both the short and long distances can be described by the Cornell potential
\[ V_{Q\bar{Q}}(R,T = 0) = -\frac{\alpha}{R} + \sigma R, \]

(51)

with \( \alpha \approx 0.471 \) and \( \sigma \approx 0.192 \text{GeV}^2 \) (one can take the parameters, e.g., from the heavy quarkonia fits [29]). The integrated pressure,
\[ V_{Q\bar{Q}}(R,T = 0) = -\frac{\alpha}{3R} - \frac{\sigma R}{3}, \]

(52)

takes its maximum \( V_{Q\bar{Q}}^{\text{max}} = -2\sqrt{\alpha/3} \simeq -200 \text{MeV} \) at the distance \( R_{\text{max}}^{\text{max}} = \sqrt{\alpha/\sigma} \simeq 0.31 \text{fm} \). In the confinement phase the integrated excess in the gluon pressure, Eqs. (30) and (32), decreases linearly with the increase in the \( Q\bar{Q} \) separation \( R \) at large enough \( R \). Physically, this effect emerges due to fluctuations of the QCD string that is spanned between the quark and antiquark. In turn, the fluctuations of the string cause the backreaction of the gluons in a close proximity to the pair. In fact, it is well known that the fluctuations of the QCD string rise with the distance between its ends \( R \), leading to a gradual widening of the string (43), (44). The author is grateful to Stam Nicolis for useful discussions.

Summarizing, we have derived finite-temperature sum rules for the internal energy (30) and pressure (31) of heavy quarks systems. The Michael-Rothe sum rules for action (32) and energy (33) are recovered automatically in the zero-temperature limit. It turns out that the new sum rules provide us with a known relation (41) for the internal energy \( E \) of the quarks (29).

At the same time, these sum rules lead to the new expression (35) for the spatial integral (20) of the excess in the gluonic pressure, \( V_Q \). The pressure excess in a heavy quarkonia is always negative similarly to the Casimir pressure in certain systems. The excess is caused by interactions between the heavy quarks because an isolated (anti)quark does not affect the pressure at all.

The variation in internal energy (57) and in free energy (40) can be expressed via the entropy \( S_{Q\bar{Q}} \), Eq. (35), and the new generalized force, \( X_{Q\bar{Q}} \propto V_Q \), Eq. (39). The force \( X_{Q\bar{Q}} \) comes in conjunction with the spatial size of the heavy quark system \( R \), similarly to the entropy-temperature pair.

We derived the exact equation of state (36) that relates the excess in the gluonic energy to the excess in the gluonic pressure around the heavy-quark systems. The generalization to multiquark systems is straightforward [43, 44].

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[20] We use the symbol \( V \) both for the volume (which always appears without a subscript) and for the multiquark potential \( V_Q \) (which always comes with a subscript).

[21] The excess in any thermodynamical quantity due to presence of a finite number of finitely-separated quarks is not an extensive property of the system. Due to the effect of screening (i.e., due to the finite mass gap) the finite-volume corrections to these quantities should behave as \( C_1 \exp\{-|C_2|V^{1/3}\} \). Thus, the volume is a thermodynamically irrelevant variable for large enough systems.

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