Resonance beyond frequency-matching: multidimensional resonance

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Abstract

Resonance, conventionally defined as the oscillation of a system when the temporal frequency of an external stimulus matches a natural frequency of the system, is important in both fundamental physics and applied disciplines. However, the spatial character of oscillation is not considered in this definition. We reveal the creation of spatial resonance when the stimulus matches the space pattern of a normal mode in an oscillating system. The complete resonance, which we call multidimensional resonance, should be a combination of both the temporal and the spatial resonance. We further elucidate that the spin wave produced by multidimensional resonance drives considerably faster reversal of the vortex core in a magnetic nanodisc. Multidimensional resonance provides insight into the nature of wave dynamics and opens the door to novel applications.

1. Introduction

Resonance is a universal property of oscillations in both classical and quantum physics [1, 2]. It occurs at a wide range of scales, from subatomic particles [2, 3] to astronomical objects [4]. A thorough understanding of resonance is crucial for related fundamental researches [4–8] and applications [9–12]. The simplest resonance system, such as a pendulum, has one oscillating element and features a single inherent resonance frequency. More commonly, a resonator contains interacting elements and has multiple resonance frequencies. Each resonance frequency corresponds to a normal mode [1] that is characterized by unique spatial variation of the oscillation amplitude and phase. There have been extensive studies on the tunability of resonance frequency and the accessibility of local modes by geometrical means in artificial nanostructures [6, 7, 12, 13], while the spatial feature of the normal modes receives little attention.

To date, in the conventional definition of resonance, the only criterion is whether the temporal frequency of the external stimulus is equal to one of the resonance frequencies of the system, and the spatial character is ignored. Since frequency describes the periodic pattern of oscillation in the time domain, we choose to use a more specific term ‘temporal resonance’ for such defined phenomena. Here we reveal the generation of spatial resonance, i.e. resonance in the space domain, when we align the spatial distribution of the external stimulus with the space pattern of a normal mode. The complete resonance, which we call multidimensional resonance, must incorporate both temporal and spatial resonance. Temporal resonance alone has low capability of stimulating all modes but the fundamental one, as a result of neglecting the space variables. In contrast, multidimensional resonance is efficient for all normal modes and can be several orders stronger in magnitude.

We conduct analytical derivations and micromagnetic simulations (see appendix A) on the behavior of elastic waves in a two-dimensional circular membrane and spin waves in a ferromagnetic nanodisc, respectively. The interplay between spin waves and magnetization reversal in small magnetic structures is of particular interest to high-speed data storage devices [14–16] and logic circuits [17–20]. In ferromagnetic nanodiscs a unique spiral spin configuration called vortex state [21–23] is favorable due to the competition between
magnetostatic and exchange interactions. The magnetization at the vortex core can point either upwards (polarity $p = +1$) or downwards ($p = -1$); so, a magnetic vortex can be used as a data storage element carrying one bit of information. Three classes of spin wave modes, namely gyrotropic [24], azimuthal and radial [25–28], have been discovered in magnetic discs. We focus on the radial spin wave mode of a ferromagnetic disc because it is analogous to the radially symmetric mode [29] in a two-dimensional elastic membrane, thus facilitating direct comparison.

Our calculations demonstrate spatial resonance in both the magnetic and mechanical resonators under the space-pattern-matching condition. This suggests that spatial resonance does not depend on material and, similar to the temporal resonance, is a universal property of oscillation systems. We also show that multidimensional resonance drives markedly stronger radial spin waves than does temporal resonance alone and increases the core reversal speed by more than 500%. Additional results for mechanical and magnetic strings are presented in appendices B and C, respectively, as demonstrations of multidimensional resonance in spatially one-dimensional systems.

2. Normal modes and space pattern

Consider the forced small amplitude vibrations of a membrane stretched in a rigid circular frame; the equation of motion is given by [29, 30]

$$\frac{\partial^2 W}{\partial t^2} - a^2 \nabla^2 W = \frac{p(\rho, \varphi, t)}{\rho} = g(\rho) \cos m \varphi \sin kat.$$  \hspace{1cm} (1)

Here, $\nabla^2$ is the Laplace operator, $W(\rho, \varphi, t)$ is the vertical displacement of a membrane with radius $\rho_0$ and mass per unit area $\rho$, $a = \sqrt{T/\rho}$ is the propagation velocity of the transverse waves, and $T$ is the isotropic tension in the membrane with dimensions of force per unit length. To facilitate discussing the efficiency of different driving forces (both spatially and temporally), the external force acting normal to the membrane is assumed to have surface density $p(\rho, \varphi, t)$ taking the form of the right side of equation (1) and subject to normalization condition

$$\int_0^{2\pi} \int_0^{\rho_0} |g(\rho) \cos m \varphi| \rho d\rho d\varphi = c \pi \rho_0^2,$$

where $c$ is a constant.

For a stretched membrane initially at rest, the initial conditions are $W(\rho, \varphi, 0) = 0$ and $[\partial W/\partial t]_{t=0} = 0$, whereas the boundary conditions are $W(\rho_0, \varphi, t) = 0$ and $[\partial W/\partial \rho]_{\rho=\rho_0} = 0$. The solution to equation (1) satisfying these initial-boundary value conditions is obtained by separation of variables, with the result

$$W(\rho, \varphi, t) = \frac{\cos m \varphi}{a} \sum_{n=1}^{\infty} \frac{J_m(k_n^{(m)} \rho)}{k_n^{(m)}} I_n(t),$$  \hspace{1cm} (2)

in which

$$I_n = \frac{2}{\rho_0^2 |J_{m+1}(k_1^{(m)} \rho_0)|^2} \int_0^{\rho_0} g(\rho) J_m(k_n^{(m)} \rho) \rho d\rho$$  \hspace{1cm} (3)

and

$$L_n(t) = \int_0^t \sin kat \sin k_n^{(m)} a(t - \tau) d\tau.$$  \hspace{1cm} (4)

Here, $J_n(x)$ is the Bessel function of the first kind of order $n$, $k_n^{(m)} = x_n^{(m)} / \rho_0$, and $x_n^{(m)}$ is the $n$th non-negative root of $J_m(x)$. Expression (2) shows that the stimulated vibration is a superposition of the normal modes $J_n(k_n^{(m)} \rho) \cos m \varphi$, as indicated by spatial pattern index $(n, m)$, where $n = 1, 2, \ldots, \infty$ and $m = 0, 1, \ldots, \infty$. The normal modes are determined by the resonator's geometry and boundary conditions. $J_m(x)$ changes in sign whenever $x$ moves across a node at $x_n^{(m)}$, resulting in phase reversal. The angular factor of the normal mode, $\cos m \varphi$, has opposite signs on either side of the nodal lines at $\varphi = \pm \pi/2m, \pm \pi/2m, \ldots, \pm (2m - 1)\pi/2m$. Thus, space can be partitioned into phase zones by nodal lines, forming a space pattern that is unique to the index $(n, m)$, see figure 1(a).

Now, we turn to micromagnetic simulations on the radial spin wave mode of a 300 nm diameter ($R = 150$ nm) and 5 nm thick ($L = 5$ nm) permalloy (Ni$_{80}$Fe$_{20}$) nanodisc. In the static state, the disc forms a vortex state (figure 2(a)) with core polarity +1. We apply a sinc function field, expressed as

$$H_{\text{Sinc}} = A \sin(2\pi \nu_c (t - t_0)) / 2\pi \nu_c (t - t_0)$$

with $A = 50$ Oe, $\nu_c = 100$ GHz and $t_0 = 1$ ns, over the entire disc to excite spin waves of all frequencies up to 100 GHz. The resulting temporal oscillation of the z-component magnetization averaged over the whole disc, $\langle m_z \rangle$, is given in figure 2(b). Next, we perform fast Fourier transform (FFT) of $\langle m_z \rangle$ to obtain the amplitude spectrum in the frequency domain [27, 31, 32]. Seven primary peaks corresponding to radial modes ($n = 1, 2, \ldots, 7$) appear at resonant frequencies of $\nu_n = 6.8, 9.8, 12.6, 15.9, 19.6, 24.0$ and 29.0 GHz (figure 2(c)). These agree well with the experiment in [28]. For the first three radial modes, the experimental results, for this sample aspect ratio $L/R = 0.033$, are about 20% lower than ours as
expected, because saturation magnetization of their samples is 20% smaller than the normal value of 800 KA m\(^{-1}\) used in our numerical calculations. For higher modes, direct comparison between samples with same aspect ratio is inappropriate because the magnetostatic energy remains dominant for the micron sized samples in the experiment while both magnetostatic and exchange energy are significant for our nanometer scale sample. In the FFT-amplitude spatial distribution diagrams for the first four normal modes (figure 2(d) upper panels), we see clear quantization of the spin wave, and the number of nodes along radial direction corresponds to the mode index \(n\). In the FFT phase diagrams for these modes (figure 2(d) lower panels), we observe high spatial uniformity for \(n = 1\) but phase discontinuity of \(\pi\) across nodal lines for \(n > 1\). Therefore, the entire sample can be partitioned into ring-shaped phase zones similar to the case of the membrane.

3. Spatial resonance and multidimensional resonance

We first theoretically establish the spatial resonance in the elastic membrane. Because expression (3) means the projection of \(g(\rho)\) on each mode shape \(J_n(k_{n}^{(m)} \rho)\) and because \(g(\rho)\) denotes the radial distribution of the driving force, \(I_n\) implies space-domain resonance in the radial dimension, when \(g(\rho)\) matches the space pattern of \(J_n(k_{n}^{(m)} \rho)\). Similarly, expression (4) for \(L_n(t)\) is the convolution of the temporal variations of the driving force and the generated wavelet (i.e., a response to stimulation in the time domain) and implies time-domain resonance. A higher degree of similarity between the response and stimulation as functions of time gives rise to a larger convolution \(L_n(t)\), which is given by

\[
L_n(t) = \begin{cases} 
\frac{k \sin k_{n}^{(m)} at - k_{n}^{(m)} \sin kat}{a(k + k_{n}^{(m)}) (k - k_{n}^{(m)})} \sin k_{n}^{(m)} at & \text{if } k \neq k_{n}^{(m)}, \\
- \frac{t \cos k_{n}^{(m)} at}{2} + \frac{\sin k_{n}^{(m)} at}{2k_{n}^{(m)} a} & \text{if } k = k_{n}^{(m)};
\end{cases}
\]

(5)

thus, only temporal resonance causes a time accumulation effect.

For a spatially uniform stimulus with conventional (temporal) resonance, i.e., \(m = 0\) and \(p(\rho, \varphi, t)/\hat{p} = c \sin k_{n}^{(0)} at\), we have \(I_n = 2c/|x_n^{(0)}| J_0(k_{n}^{(0)} \rho)\). Then, the energy in the membrane grows as

\[E = \int_0^{2\pi} \int_0^{\rho_{\text{max}}} \hat{p} (\partial W / \partial \theta)^2 \rho d\rho d\varphi = \hat{E} [2/|x_n^{(0)}|^2 + \text{HFOTs}],\]

(6)

where \(\hat{E} = (\pi \rho_{\text{max}} c^2/8)^{1/2} (1 - \cos 2k_{n}^{(0)} at)\), and the high frequency oscillating terms (HFOTs) determine fine structure of the energy curve with no influence on its general trend. Table 1 shows that the energy growth rate quickly drops for larger \(n_0\) as \(E/\hat{E} \approx 4/|n_0 - 1/4|\) because the stimulus, which is uniform in space and has the spatial pattern index \((1, 0)\), is spatially off-resonance with pattern \((n_0, 0)\) when \(n_0 \neq 1\); and is farther away from spatially resonant for larger \(n_0\) although they are all temporally resonant.

For a radially symmetric stimulus taking the spatial resonance into consideration, for example, \(m = 0\) and \(p(\rho, \varphi, t)/\hat{p} = (c/s_{n_0}) h_0(k_{n_0}^{(0)} \rho) \sin k_{n_0}^{(0)} at\), the normalization condition demands

\[s_{n_0} = 2|x_n^{(0)}|^{-2} \sum_{i=1}^{n_0} (-1)^{i-1} h_i, \]

where \(h_i = x_n^{(0)} j_0(x_i^{(0)}) - x_{i-1}^{(0)} j_0(x_{i-1}^{(0)})\) (see figure 1(b) for \(n_0 = 2\)). Then, \(I_n\)
equals $c/s_n$, if $n = n_0$ and 0 if $n \neq n_0$, which shows that a stimulus tends to generate a wave similar to itself and that the normal modes are mutually orthogonal. Hence, the energy in the membrane grows as

$$E = E'[x^{(0)}]/s_n^2 + HFOTs. \quad (7)$$

Table 1 shows that the energy grows at similar rates for different $n_0$ (limiting at $E = 1.388\tilde{E}$ for large $n_0$). In this case, the stimulus has spatial pattern index $(n_0,0)$ and is on-resonance with space pattern $(n_0, 0)$. The energy growth expressed by equation (7) is at least one order of magnitude larger than that expressed by equation (6), since the latter case is spatially off-resonant for $n_0 > 1$. 

Table 1. Membrane energy growth for different stimulus in expressions (6), (7), (10) and (11).

| $n_0$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | $\infty$ |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|----------|
| $E/\tilde{E}$ in (6) | 0.692 | 0.131 | 0.0534 | 0.0288 | 0.0179 | 0.0122 | 0.0089 | 0.00674 | 0.00529 | 0       |
| $E/\tilde{E}$ in (7) | 1.445 | 1.404 | 1.396 | 1.393 | 1.392 | 1.391 | 1.390 | 1.389 | 1.389 | 1.388    |
| $E/\tilde{E}$ in (10) | 3.710 | 2.838 | 2.615 | 2.515 | 2.457 | 2.420 | 2.395 | 2.376 | 2.361 | 2.250     |
| $E/\tilde{E}$ in (11) | 1.535 | 1.634 | 1.664 | 1.678 | 1.687 | 1.692 | 1.696 | 1.699 | 1.701 | 1.712     |

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It is technically difficult to set the driving force varying as a Bessel function in the radial dimension. For ease of implementation, we employ a delta-function-like stimulus as follows

$$\frac{p_r(\rho, \varphi, t)}{\dot{\rho}} = \frac{c_{\rho_0}^2}{s_{n_0} \left| X_{n_0}^{(m)}(x) \right|^2} \sum_{i=1}^{n_0} h_i \cos \left( \frac{\rho - \rho_i}{\rho} \right) \sin k_{n_0}^{(m)} \delta \text{at},$$  \hspace{1cm} (8)

where $\rho_i = \rho_0 \left( X_{i}^{(m)} / X_{n_0}^{(m)} \right)$, $X_{i}^{(m)}$ is the $i$th extremum of $f_m(x)$, and the coefficient comes from the normalization condition. The radially symmetric case of this stimulus has the same space pattern (n0, 0) as the preceding example, except that the force is concentrated on each mode extremum and is expected to be more effective (see figure 1(c) for $n_0 = 2$). Now we have

$$I_n = \frac{2c_{\rho_0}}{s_{n_0} \left| X_{n_0}^{(m)}(x) \right|^2} \sum_{i=1}^{n_0} h_i f_0 \left( X_{i}^{(m)} / X_{n_0}^{(m)} \right),$$  \hspace{1cm} (9)

and

$$E = \bar{E} \left[ \frac{\sum_{i=1}^{n_0} h_i f_0 \left( X_{i}^{(m)} / X_{n_0}^{(m)} \right)}{\bar{f}(x_{n_0}) \sum_{i=1}^{n_0} (-1)^{i-1} h_i} \right]^2 + \text{HFOTs}.  \hspace{1cm} (10)

Referring to table 1, the stimulus with $n_0 = 1$ in this case (i.e., a point stimulus concentrated on the membrane center) is the most effective. And as expected the energy growth rate is enhanced by 62%–157% for different $n_0$ comparing to the case expressed by equation (7).

In the case of a magnetic nanodisc, we choose the $n = 2$ radial mode to demonstrate the spatial resonance of the magnetic vortex by numerical calculations. We apply a harmonic field tuned to the second modal frequency, i.e., $\nu = \nu_2$, with a small oscillation amplitude of 10 Oe for 10 ns. The spatial distribution of the field phase aligns (partially or fully) with the $n = 2$ mode. The FFT amplitude and phase images for the $n = 2$ mode after application of the resonant frequency field is shown in figure 3. The nodal line at $\rho = 87.5$ nm partitions the sample into two phase zones: $Z_i$ for $\rho \leq 87.5$ nm and $Z_i$ for $\rho > 87.5$ nm. Figures 3(a) and (b) show the FFT amplitude and phase diagrams when the field is localized in $Z_1$, and $Z_2$, respectively. The FFT amplitude has a slightly higher value in figure 3(b), which can be explained by the larger area of $Z_2$ than $Z_1$. The FFT phase variations along $\rho$ are opposite. As the external field reverses its oscillation phase, the resultant FFT phase images are reversed, whereas the amplitude profile remains intact (figure 3(c)). The same effect exists for an elastic membrane. Refer to expressions (2) and (9), the factor $I_n$ in the vibration $W(\rho, \varphi, t)$ changes in sign whenever the stimulus moves across a node, resulting in a phase reverse. We can thus interpret the small oscillation amplitude generated by a uniform global field (which produces temporal resonance only) in figure 3(d) as a result of destructive addition of oscillations in figures 3(a) and (b). In sharp contrast, we observe a much larger oscillation amplitude in figure 3(e) when the field distribution matches the space pattern of the mode, thus satisfying the spatial resonance condition, and the resultant oscillation corresponds to constructive addition of oscillations in figures 3(a) and (c). This reasoning is also applicable to explaining why multidimensional resonance is far more energetic than temporal resonance for the membrane, as shown in table 1.

Figure 4 shows the growth in total energy ($\Delta E$) of the seven radial modes of our magnetic nanodisc under temporal and multidimensional resonance conditions. The external field oscillation amplitude is small (10 Oe) in all cases to suppress nonlinear dynamics. To make the results comparable, the summation of the magnetic flux (absolute value) in all phase zones is the same in the micromagnetic simulations. For the temporal resonance in figure 4(a), we observe that the fundamental mode ($n = 1$) accumulates energy at a rate of more than one order higher than the other modes. The general trend is that for odd $n$, the rate dramatically decreases as $n$ increases, similar to the temporal resonance of the membrane shown in table 1. For even $n$, the corresponding energy is close to zero. However, multidimensional resonance produces strikingly different results in figure 4(b). Here, the external field matches the space pattern of the corresponding radial mode, but the field amplitude is uniformly distributed over the entire disc. The sample increases in energy at a faster rate for higher modes, so the fundamental mode has the lowest efficiency. The system gains 3.5 times more energy for mode $n = 7$ than $n = 1$. Furthermore, modes $n = 2$ to 7 increase in energy much faster than their temporal resonance counterparts shown in figure 4(a). The ratio of the two energy gain values is a staggering 1856 for $n = 7$. The fundamental mode has no difference in energy growth because it is the only mode with a globally uniform phase. The system receives an additional energy boost when the spatial variation of the external field matches both the space pattern and the mode amplitude profile, as shown in figure 4(c). The energy growth rates are 4%–103% higher than those in figure 4(b). Finally, we
observe the highest rate (19%–34% higher for all modes than their counterparts in figure 4(c)) when we concentrate the field at the antinodes (figure 4(d)).

4. Ultrafast reversal of the magnetic vortex core

Recent studies have shown that the resonant radial mode is capable of reversing the core polarity of a magnetic vortex through a novel mechanism [33–36], wherein the core switching is attributable to the breathing nature of spin wave, instead of the well-known vortex–antivortex annihilation process found in the gyrotropic- and azimuthal-mode-driven core reversal [14, 37–42]. Because multidimensional resonance excites a strikingly stronger spin wave than does temporal resonance, we expect multidimensional resonance to facilitate a much
faster vortex core reversal. Figure 5 displays the core reversal time $t_{SW}$ and threshold field $H_{thr}$ versus the radial mode index. When studying the reversal time, the field amplitude is uniformly distributed at 300 Oe within all phase zones. The reversal time $t_{SW}$ is 588 ps for $n = 1$, decreases dramatically to 288 ps for $n = 2$, and continues to decrease at slower pace for higher modes until reaching 93 ps for $n = 7$ (figure 6), which marks a 532% faster reversal speed than for the fundamental mode. These results are in sharp contrast to references [34, 36], where the authors produced temporal resonance only and the reversal time increases quickly for higher modes. In addition to achieving sub-100 picosecond reversal, multidimensional resonance greatly lowers the threshold field for core reversal. Figure 5 shows that the threshold $H_{thr}$ is 285 Oe for $n = 1$ and steadily decreases to 55 Oe at $n = 5$. The unexpected, slightly higher values at $n = 6$ and 7 are attributable to the reduced accuracy of the numerical calculation for higher modes, in which the ring-shaped phase zones appear more zigzag-like in the micromagnetic simulation.

5. Spatial resonance in the angular dimension

In the above study on the membrane, for simplicity, we postulate that the stimulus is on-resonance in the angular dimension and discuss only the radial dimension. The factor $\cos m\varphi$ in (1) and (2) implies

![Figure 5](image1.png)

Figure 5. The core reversal time $t_{SW}$ and threshold field $H_{thr}$ versus the radial mode index. When studying the reversal time, the field amplitude is uniformly distributed at 300 Oe in all phase zones.

![Figure 6](image2.png)

Figure 6. Snapshot images of the temporal evolution of the $n = 7$ radial mode with corresponding $m_z$ profiles. The external field satisfies both the temporal and spatial resonance conditions, with the field amplitude uniformly distributed in all seven phase zones at 300 Oe.
space-domain resonance in the angular dimension; thus, each space dimension should be given an index, and together these indices constitute the spatial pattern index, which completely describes the spatial character.

For a radially asymmetric stimulus with spatial resonance, for example, $p(\rho, \varphi, t)/\bar{p} = (\epsilon/\beta) J_n(k_\rho^{(m)} \rho) \cos m \varphi \sin k_\delta^{(m)} t$, the normalization condition demands $s_0^{(m)} = \frac{4}{\pi \rho_0^2} \int_0^{\rho_0} |J_n(k_\rho^{(m)} \rho)| \rho \mathrm{d}\rho$. Hence, the energy in the membrane grows as

$$E = \bar{E}^{(m)} \left[ J_{m+1}(x_n^{(m)}) / \sqrt{2} x_n^{(m)} \right]^2 + \text{HFOTs},$$

where $\bar{E}^{(m)} = (\pi \rho_0^2 \beta \epsilon^2 / 8)^2 (1 - \cos 2k_\rho^{(m)} \delta t)$. In this case, the stimulus has spatial pattern index $(n_0, m)$ and is on-resonance with space pattern $(n_0, m)$. For $m = 1$, the energy grows at similar rates for different $n_0$ (limiting at $E = 1.712 \bar{E}^{(1)}$ for large $n_0$) but is distinct from the radially symmetric case in that the coefficient $E/\bar{E}^{(1)}$ increases with $n_0$ (see table 1). In passing, it is interesting to notice one feature of a radially asymmetric mode, namely that the membrane center remains still at all times.

In the analytical theory of thin magnetic nanodiscs [25], the Bessel function $J_1(k_\rho^{(1)} \rho)$ is the approximate wave function of the radial mode. One key difference between $J_1(k_\rho^{(1)} \rho)$ and the micromagnetic simulation result is that $J_1(k_\rho^{(1)} \rho)$ is zero but the numerical solution is significant at the boundary. The small deviation of the mode shape in the analytical and numerical solutions produces a large difference in spatial resonance. The analytical solution predicts only a slightly higher energy growth rate for larger $n_0$, as given in table 1. For comparison, the numerical solution presents a much more energetic spatial resonance for larger $n$, as shown in figure 4(c). These results highlight the sensitivity of the spatial resonance to the mode shape, which can be tuned by factors such as sample geometry and composition and boundary conditions.

### 6. Discussion

The circular symmetry of the membrane (and the ferromagnetic nanodisc) is responsible for the non-periodic space pattern in the radial dimension. For a rectangular membrane [29] or the strings (see appendices B and C), the geometrical symmetry results in a periodic space pattern for the oscillation modes. Similarly, frequency, i.e. the temporal pattern of oscillation, can also be tuned by confinement of wave in the time domain, or even the geometry of spacetime [43]. Pure periodic oscillation exists only when the oscillation exists for infinitely long time within a flat spacetime. Therefore, the essence of both temporal and spatial resonance is the pattern-matching. Additionally, concentration of stimulus at the mode extremes can considerably enhance the resonance effect, as shown in table 1 and figure 4.

In summary, our calculations on elastic membranes and magnetic nano-discs demonstrate multidimensional resonance wherein the external stimulus aligns with not only the temporal frequency but also the space pattern of a normal mode. For a spatially uniform stimulus, the external stimuli acting on zones of opposite phases are destructive. Therefore, the conventional (temporal) resonance is inefficient at exciting all modes but the fundamental mode. In contrast, multidimensional resonance is efficient for all modes and creates a much stronger oscillation because the external stimulus reverses its direction in adjoining zones; thus, all stimuli are constructive. Because the manifestation of wave interference does not depend on matter, multidimensional resonance is expected to be a universal property of oscillation systems. Our research reshapes the theory on resonance and opens a new arena wherein the spatial character of oscillation modes plays a key role. To realize multidimensional resonance in experiments, one could utilize advanced lithographic techniques to fabricate space-pattern-matched stimuli (e.g. electrical transducers, wave guides, microcavities), in order to excite specific modes in mesoscopic systems. We note that there is little knowledge on both the practical and scientific applications of the multidimensional resonant higher order modes so far, despite the demonstration of ultrafast reversal of magnetic vortex in the present work. Future studies on ways of tailoring the mode shape may lead to novel technological innovations that exploit the nature of multidimensional resonance.

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Appendix A. Methods of numeric simulations

Our micromagnetic simulations are conducted using LLG Micromagnetics Simulator code, which numerically solves the Landau–Lifshitz–Gilbert equation for the dynamic magnetization process. The total energy includes contributions from the Zeeman, exchange and demagnetization energy. The magnetic nanodisc and nanowire are composed of permalloy (Ni_{80}Fe_{20}). Parameters used for permalloy: saturation magnetization $M_s = 8.0 \times 10^5$ A m$^{-1}$, exchange stiffness constant $A_{an} = 1.3 \times 10^{-11}$ J m$^{-1}$, Gilbert damping constant $\alpha = 0.01$, and zero magnetocrystalline anisotropy. In the simulations, the mesh cell size is $2.5 \times 2.5 \times 5.0$ nm$^3$ for the nanodisc and $2.5 \times 2.5 \times 2.5$ nm$^3$ for the nanowire.

Appendix B. Multidimensional resonance in an elastic string

The forced small amplitude vibration of a stretched finite string is the solution of partial differential equation

$$\frac{\partial^2 W}{\partial t^2} - a_0^2 \frac{\partial^2 W}{\partial x^2} = \frac{f^s(x, t)}{\rho^s} = g(x) \sin k_0 t,$$

under initial conditions $W(x, 0) = 0$ and $[\partial W / \partial t]_{t=0} = 0$ and boundary conditions $W(0, t) = W(x_l, t) = 0$ and $[\partial W / \partial t]_{x=x_0}$, $x_0$ is the length of the string, $\rho^s$ is its mass per unit length, $a_0 = \sqrt{T_0 / \rho^s}$, $T_0$ is the tension in the string, and $f^s(x, t)$ denotes the normal density of the external driving force which satisfies normalization condition $\int_{x_0}^{x_l} g(x) \, dx = c_0$. The solution is well-known:

$$W(x, t) = \sum_{n=1}^{\infty} \frac{2}{a_0^2 x_0} \sin \left( \frac{n \pi x}{x_0} \right) \int_0^{x_0} \sin \left( \frac{n \pi x_0}{x_l} \right) \sin \left( \frac{n \pi x_0}{x_0} \right) \sin \left( \frac{n \pi a_0 t}{x_0} \right) \, dx.$$

For a stimulus with temporal resonance but is uniform in space, i.e., $f^s(x, t) / \rho^s = (c / x_0) \sin (n_0 \pi a_0 t / x_0)$, the energy in the string grows as $E = E^0 / n_0^2 + \text{HFOs for odd } n_0 \text{ and null for even } n_0$, where $E^0 = (\rho^s c / \pi x_0)^2 (1 - \cos 2n_0 \pi a_0 t / x_0)$. The energy, whereas similar to the two-dimensional-space case in growing proportionally to $1 / n_0^2$ for odd $n_0$ does not grow with time for even $n_0$. Only when $n_0 = 1$ does spatial resonance condition satisfy; otherwise, the external force acting on adjoining zones are destructive and cancel each other exactly for even $n_0$, or remain one zone for odd $n_0$ (i.e., $\int_0^{x_0} g(x) \sin n_0 \pi a_0 \, dx = 0$ for even $n_0$ if $g(x)$ is a constant).

For stimulus with spatial and temporal resonance but uniform in each space zone, i.e.

$$\frac{f^s(x, t)}{\rho^s} = \frac{4c}{\pi} \sum_{k=1}^{\infty} \sin \left( \frac{n_0 \pi (2k - 1)}{x_0} \right) \sin \left( \frac{n_0 \pi a_0 t}{x_0} \right),$$

energy in the string grows as $E = E^* + \text{HFOs}$. If the preceding stimulus of complete resonance goes a step further from spatial pattern-matching to profile-matching, such that $f^s(x, t) / \rho^s = (c / \pi) \sin (n_0 \pi a_0 t / x_0)$ then the energy grows as $E = \pi^2 E^* / 64$. Finally, calculations show that if the stimulus in each space zone is concentrated on each mode extremum, the energy grows as $E = \pi^2 E^* / 4$. The foregoing three cases are complete resonance; their energy growth does not depend on $n_0$. And obviously, the string receives additional energy boost when greater proportion of the force acts on the mass points with larger oscillation displacement.

Appendix C. Multidimensional resonance in a magnetic nanowire

The magnetic nanowire in the model is 300 nm long along $x$, 2.5 nm wide along $y$ and 2.5 nm thick along $z$. In the initial state, the magnetization is along $+x$ because of the large shape anisotropy of the wire. In addition, we apply a pinning field $H_{pin}$ of the order of 1.5 kOe along $+x$ at the two ends to achieve a fixed boundary condition, as shown in the inset of figure C1.

To study the normal mode, we apply a sinc function field along $z$ in the form of $H_{inc} = A_0 \sin \left[ 2\pi \nu (\nu - \nu_0) / 2\pi \nu (\nu - \nu_0) \right]$, with $A_0 = 50$ Oe, $\nu = 100$ GHz and $t_0 = 1$ ns, over the entire wire to stimulate a spin wave. FFT analysis on the subsequent temporal oscillation of the averaged $m_z$ over the whole wire yields the resultant FFT amplitude in the frequency domain, as shown in figure C1. The four peaks at frequencies $\nu_e = 14.17, 14.91, 16.41$ and 18.65 GHz correspond to normal modes with indices of $n = 1, 3, 5, 7$, respectively. Because the globally uniform field is unable to excite even-index modes, we use a spatial resonance field to selectively drive the individual modes. For instance, to excite the $n = 2$ mode, the external field is in the form of $H_{inc}$ within $[-150 \text{ nm}, 0 \text{ nm}]$ and $[0 \text{ nm}, 150 \text{ nm}]$.

\footnote{We used LLG Micromagnetic Simulator ver. 3.14d developed by Scheinfein to carry out micromagnetic simulations.}
−*H*_{sinc}, within [0 nm, 150 nm]. The resultant FFT amplitude spectra for *n* = 2, 4, and 6 obtained by this method are given in figure C1, which shows the primary resonance peaks at *ν*₂ = 14.45 GHz, *ν*₄ = 15.56 GHz and *ν*₆ = 17.44 GHz, respectively. The spatial resonance field for *n* = 2 is also capable of exciting the *n* = 6 mode; thus, a secondary peak at *ν*₆ = 17.44 GHz appears in the corresponding FFT spectrum in figure C1.

Figure C2 presents the spatial distribution of the FFT amplitude and phase diagrams for the seven normal modes. We observe the clear standing wave nature of a one-dimensional oscillation system with a fixed boundary. For each mode, the nodes partition the space into zones of alternating phases, defining a unique space pattern.

Figure C3 displays the increase in total energy (*ΔE*) of the seven normal modes under temporal and multidimensional resonance conditions. The oscillation amplitude of the external field is small (10 Oe) to suppress nonlinear dynamics. To make the results comparable, we keep the summation of the magnetic flux (absolute value) in all phase zones the same in the simulations. The temporal resonance in figure C3(a) shows the highest energy growth rate at *n* = 1 and a much lower rate as *n* increases, with the exception of *n* = 2. The resonance frequency at *n* = 2 is very close to that at *n* = 1. As a result, the spin wave excited by a globally uniform field at frequency *ν*₂ resembles the fundamental mode rather than the *n* = 2 mode, as displayed in the inset of figure C3(a). We observe significant enhancement in growth of *ΔE* under multidimensional resonance
for all modes except the fundamental mode, as shown in figure C3(b). The spatial distribution of the external field matches the space pattern of the corresponding normal mode, but the field amplitude is uniform in each phase zone. The fundamental mode is the only mode that has a uniform phase distribution; therefore, multidimensional resonance and temporal resonance produce the same result at $n = 1$. The growth rate of $\Delta E$ increases by approximately 50% for all modes when the external field aligns with not only the space pattern but also the mode amplitude profile, as shown in figure C3(c). In figure C3(d), we observe the greatest energy growth rate when the external field is concentrated at the antinodes. The rate is approximately 100% higher for $n = 1$ and 67% higher for $n \geq 2$ than the results in figure C3(c).

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