Interrelations Between the Neutron’s Magnetic Interactions and the Magnetic Aharonov-Bohm Effect

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Abstract:

It is proved that the phase shift of a polarized neutron interacting with a spatially uniform time-dependent magnetic field, demonstrates the same physical principles as the magnetic Aharonov-Bohm effect. The crucial role of inert objects is explained, thereby proving the quantum mechanical nature of the effect. It is also proved that the nonsimply connectedness of the field-free region is not a profound property of the system and that it cannot be regarded as a sufficient condition for a nonzero phase shift.
1. Introduction

The interaction of the neutron’s magnetic moment with external magnetic field has been used for studying properties of magnetic materials, for understanding the nature of the neutron’s magnetic moment[1-3] and for an analysis of general physical principles[4,5]. In [5], an examination is carried out for a polarized neutron travelling through a time dependent, spatially uniform magnetic field of a solenoid. The field is parallel to the neutron’s magnetic dipole and to its velocity. These properties provide an environment where the neutron travels through a force-free and torque-free region. Thus, taking a classical point of view, one may conclude that the neutron can be regarded as a free particle. Nevertheless, the experiment shows that the neutron acquires a phase shift which affects its interference pattern.

The authors of [5] relate their experiment to a kind of the Aharonov-Bohm (AB) effect[6,7]. The fact that the neutron behaves as an inert object and remains in its quantum mechanical ground state has been analyzed recently[8]. It is proved there that this property is essential for the phase shift obtained. Indeed, if the neutron is replaced by a “classical neutron”, then the phase shift disappears.

The present work performs a further analysis of the neutron experiment [5]. It is proved that this experiment demonstrates the same physical principles as the electron interference experiment[9] showing the existence of the magnetic AB effect.

The discussion carried out below assumes that the nonrelativistic limit holds. Units where $\hbar = c = 1$ are used.

The physical elements of the polarized neutron experiment are summarized in section 2. A description of the magnetic AB effect is presented in
section 3. In section 4 it is proved that the two experiments demonstrate the same physical principles. In section 5 it is proved that topological features of the field-free region cannot be regarded as a necessary and sufficient condition for a nonzero phase shift. Concluding remarks are the contents of the last section.

2. The Polarized Neutron Experiment

Let us examine the physical properties of the neutron experiment [5]. The neutron’s nonrelativistic Lagrangian boils down to the following expression [8]

$$L = \frac{1}{2}mv^2 + m \cdot B$$  \hspace{1cm} (1)

where $m$ denotes the neutron’s magnetic moment and $B$ is the magnetic field of the solenoid. The first term of this Lagrangian is independent of the external magnetic field $B$ (because the neutron travels in a force-free and torque-free region) whereas the second term is proportional to this field. Therefore, the action and its associated phase vary due to the interaction of the external magnetic field $B$ with the neutron’s magnetic moment $m$.

Evidently, the moving neutron is a part of the entire system. The full picture is obtained after including the solenoid’s interaction with the moving neutron. The magnetic field $B$ of (1) is associated with the motion of charges along the solenoid’s wires. Thus, (1) is a part of a system which is a sum of 2-body interactions of the following kind

$$L = L_n + L_e + L_{ne},$$  \hspace{1cm} (2)

where $L_n$ and $L_e$ denote the single particle interactions of the neutron and the electron, respectively and $L_{ne}$ denotes the neutron-electron interaction.
The summation of $L_{ne}$ of (2) on all charges boils down to the last term of (3).

Obviously, the 2-body Lagrangian (2) yields a 2-body Hamiltonian whose Schrödinger equation that takes the form

$$ (H_n + H_e + H_{ne})\psi = i\hbar \frac{\partial \psi}{\partial t}. $$

In the neutron experiment [5], it is assumed that the neutron does not affect the solenoid’s state. The correctness of this assumption is proved here. Performing a Lorentz transformation on the magnetic field of a motionless neutron, one realizes that in the laboratory frame, the moving neutron has an electric field that acts on the solenoid’s current. However, it is easy to see that the solenoid’s current is not affected by this field. This point, which is generally taken for granted, is proved here because of its importance for the following discussion.

The overall EMF force of the electric field of the moving neutron is obtained by integrating this field along the solenoid’s wires. The magnetic field of a magnetic dipole is[10]

$$ B = \frac{[3(m \cdot r)r - r^2m]}{r^5}. $$

The electric field of the moving neutron is obtained from a Lorentz transformation of the magnetic field of a motionless neutron[11]

$$ E = -v \times B. $$

Let us use cylindrical coordinates. The $z$-axis is chosen along the solenoid’s axis. Hence, in this experiment, the neutron’s magnetic dipole is in the $z$-direction, too. The interesting quantity is the component of the electric field of the moving neutron (3) which is parallel to the direction of the solenoid’s electric current, namely the $\phi$-direction. Since the neutron’s velocity $v$ is
parallel to the z-axis, the relevant magnetic field component of (3) is \( B_r \). Now, for an infinite solenoid, one performs an integration on the cylindrical surface \( S \) of the solenoid, uses (3), Gauss theorem and Maxwell equation and obtains

\[
\int_S E_\phi ds = -v \int_S B_r ds = -v \int_V \nabla \cdot B d^3 r = 0.
\]  

(6)

This result proves that the solenoid’s time-depending electric current is not affected by the electric field of the moving neutron.

It is proved in [8] that an essential element of the experiment is the fact that the neutron’s internal state remains constant throughout the entire experiment. Indeed, it is shown there that if the neutron is replaced by a “classical neutron” whose self energy may change, then the action is independent of the magnetic field and a null phase shift is obtained. This point emphasizes the quantum mechanical nature of the experiment.

The experimental setup of the neutron-electron interaction has the following properties:

1. The moving particle travels in a force-free region.

2. The overall force exerted by the neutron on the solenoid’s charges vanishes.

3. The change of the action and of the corresponding phase emerge from the interaction of the neutron’s magnetic dipole \( m \) with the solenoid’s magnetic field \( B \), namely, \( m \cdot B \).

4. The neutron remains in its quantum mechanical ground state throughout the experiment.

5. If the neutron is replaced by a “classical neutron” whose self energy can be changed then the action becomes independent of the magnetic field
and the phase shift disappears.

These items are called hereinafter properties 1,...,5, respectively.

The physics examined here is the phase shift and the corresponding interference pattern. Examining these phenomena, one realizes that property 1 above is not essential for the phase shift effect. It is used only as a convincing proof of the quantum mechanical nature of the results. Further aspects of the force-free region are discussed in the last section of this work.

In the rest of this work it is proved that the same kind of 2-body interaction as well as properties 1-5 are found in an experiment[9] which demonstrates the magnetic AB effect[6,7]. In other words, it is proved below that the neutron's interference experiment[5] confirms the same physical principles as standard experiments proving the validity of the magnetic AB effect. For this end, let us start with a brief description of this effect.

3. The Magnetic Aharonov-Bohm Effect

Consider an infinitely long cylindrically shaped permanent magnet which is fixed in the laboratory and its axis coincides with the z-axis. Let $R_m$ and $\Phi$ denote the radius of the magnet’s cross section and its magnetic flux, respectively. An electron moves in the positive direction of the $y$-axis. At a point $y = -Y_0$, the electron’s wave function is split into two subpackets, $\psi_L$ and $\psi_R$, which continue to move parallel to the $y$-axis along the lines $x = \pm a$, $z = 0$ and pass on the left and right hand sides of the magnet, respectively. Thus, the single particle wave function is

$$\psi_e = \psi_L + \psi_R. \quad (7)$$
Later, the two subpackets interfere on a screen $S$ (see fig. 1). In some cases below, this electron is called the travelling electron.

Let us analyze the influence of the permanent magnet on the electron’s interference pattern. The electron’s nonrelativistic Lagrangian is (see [11], p. 46)

$$L = \frac{1}{2}mv^2 - eV + e\mathbf{v} \cdot \mathbf{A}, \quad (8)$$

where $\mathbf{v}$ denotes the electron’s velocity and $V, \mathbf{A}$ denote the electromagnetic scalar and vector potentials, respectively.

In the experiment discussed here, the scalar potential of the magnet $V$ vanishes. The magnetic field is confined to the inner part of the magnet, namely to a region where $r < R_m$. Therefore, the electron travels in a force free region and the first term of (8) is a constant of the motion. Using cylindrical coordinates, one finds that the components of $\mathbf{A}$ at $r > R_m$ are

$$A_r = A_z = 0, \quad A_\phi = \frac{\Phi}{2\pi r} \quad (9)$$

where $\Phi$ denotes the solenoid’s magnetic flux. (The validity of this relation is easily verified. Using the cylindrical symmetry of the magnet, one takes the integral along a circle of radius $r$: $\oint \mathbf{A} \cdot d\mathbf{l} = \int \text{curl} \mathbf{A} \cdot d\mathbf{s} = \oint \mathbf{B} \cdot d\mathbf{s} = \Phi$.) Hence, the action difference $\Delta I$ between $\psi_L$ and $\psi_R$, associated with the permanent magnet, is obtained from the substitution of (9) into the last term of the Lagrangian (8). A straightforward calculation yields (see [6], p. 487)

$$\Delta I = e \int_{-\infty}^{\infty} \mathbf{v} \cdot \mathbf{A}(x = a) dt - e \int_{-\infty}^{\infty} \mathbf{v} \cdot \mathbf{A}(x = -a) dt $$

$$= e \int_{-\infty}^{\infty} [A_y(x = a) - A_y(x = -a)] dy$$

$$= e\Phi. \quad (10)$$

This outcome is associated with a phase shift that affects the interference pattern of the electron. The quantum mechanical foundations of the results
are explained in the following section, where the role of the quantized state of the magnet is emphasized.

4. The Analogy Between the Polarized Neutron Experiment and the Magnetic Aharonov-Bohm Effect

Let us analyze the magnetic AB experiment whose principles are described above. The travelling electron interacts with the magnet which is made of neutral atoms, each of which has an intrinsic magnetic moment. Here, the 2-body interaction takes the form of (2), where the subscript $n$ is replaced by $A$ which denotes a magnetic atom. The corresponding 2-body Hamiltonian and the Schrodinger equation take the form of (3). Thus, one realizes that the underlying 2-body interaction of the neutron experiment [5] is the same as that of the magnetic AB effect[6,7,9]. It is proved later that the 2-body interaction $m \cdot B$ of (1) equals the corresponding term $ev \cdot A$ of (8).

On top of that, it is shown here that the experimental setup designed for measuring the magnetic AB effect is endowed with the five properties 1...5 of the neutron experiment. In order to do that, one should find the correspondence between elements of the neutron experiment[5] and those of the magnetic AB effect. Each experiment consists of a single particle (a neutron in [5] and an electron in the magnetic AB experiment) interacting with a multitude of other particles (the electrons which make the solenoid’s current and its magnetic field in [5], and the magnetized atoms which make the permanent magnet in the magnetic AB case). The linearity of electrodynamics enables one to write the overall interaction as a sum of two body interactions.
Thus, the neutron interference experiment[5] is based on the interaction of the neutron with an electron moving along the solenoid’s wire. Similarly, in the magnetic AB effect, the travelling electron interacts with a magnetized neutral atom.

The foregoing discussion shows that the two experiments are based on a two body interaction of an electron with an electrically neutral particle having a nonvanishing magnetic moment. It follows that the two experiments have an intrinsic similarity. In the following analysis, subscripts $e$ and $m$ denote quantities pertaining to the electron and the magnetic particle, respectively.

Let us prove that the magnetic AB experiment satisfies properties 1-5.

Evidently, the travelling electron moves in a field-free region. Hence, properties 1,2 hold.

In order to prove property 3, one has to compare the neutron’s Lagrangian (1) with the electron’s Lagrangian (8). It turns out that property 3 depends on the validity of the following relation

$$B_e \cdot m = e v_e \cdot A_m$$

where $e$ is the electronic charge ($e$ has a negative numerical value). This relation is proved by means of a direct calculation. The origin of coordinates is at the location of the magnet whose moment is in the $z$-direction and the electron moves in the $y$-direction. The left hand side of (11), is calculated first. Let $R_s$ denote the solenoid’s radius and the electron is at a point $x = R_s$, $y = 0$, $z = Z_0$. At the origin, the $x$-component of the electric field of the electron is $-e \sin \theta \cos \phi/r^2$. Using $B = v \times E$ (which is an analogue of (5)), and the $y$-direction of the electron’s velocity, one finds that the left hand side of (11) equals $mev \sin \theta \cos \phi/r^2$ (where $m$ denotes the strength of the magnetic dipole and $\cos \phi = 1$). Now, let us turn to the right hand side of (11). Using spherical polar coordinates, one finds that the components of
the vector potential of a magnetic dipole whose moment is in the \(z\)-direction are\[10\] \(A_r = A_\theta = 0, \ A_\phi = m \sin \theta / r^2\). Since the electron’s velocity \(v_y = v \cos \phi\), one finds that the right hand side of (11) equals \(mev \sin \theta \cos \phi / r^2\), too. Hence, relation (11) holds.

This calculation proves that the two body interaction of these experiments are the same. Hence, property 3 holds also for the magnetic AB effect. (As a matter of fact, relation (11) can be regarded as an extension of a well known charge-potential relation of electrostatics of two charges \(q_1V_2 = q_2V_1\).)

The validity of property 4, namely the assumption that the permanent magnet remains in its ground state throughout the experiment, is generally taken for granted. This approach is justified here by an order of magnitude evaluation of the interactions involved. Here the interaction of a magnetic atom with its neighbours has to be compared with its interaction with the magnetic field \(B = v \times E\) of the travelling electron. The inter-atomic distance is of the order of \(10^{-8}\) cm whereas the distance between the travelling electron and the magnet is about \(10^{-4}\) cm[9]. Hence, a comparison with atomic field shows that the distance-depending factor of the magnetic field of the travelling electron is weaker by \(10^{-8}\). Therefore, since the transition probability is proportional to the square of the ratio of the interactions[12], one concludes that the transition probability is less than \(10^{-16}\). In [9], the number of magnetic atoms is about \(10^{12}\). For this reason, the permanent magnet is regarded as an inert object whose state is not affected by the fields of the travelling electron. It follows that property 4 is confirmed for the magnetic AB effect. (It is interesting to note that in an evaluation of the magnetic AB effect, the Lagrangian of the magnet’s constituents has to be added to (8). Its omission is justified only after it is proved that it behaves as an inert object throughout the experiment.)
The validity of property 5 is examined in a system where the magnet is replaced by a classical device. This device is a cylindrical solenoid which consists of 2 helixes, each of which is a pipe made of an insulating material. A charged liquid flows frictionlessly along the pipes. The electric field of this charge is screened by a static charge of opposite sign which is spread uniformly on the outer side of the pipes. The charged fluid ascends in one pipe and descends in the other. Thus, the mean current flows in the \((x, y)\) plane. Let \(j\) denote the solenoid’s current \((j = nI\) where \(n\) denotes the number of loops per unit length and \(I\) is the ordinary electric current). Hence, the solenoid’s magnetic flux is

\[
\Phi = 4\pi^2 R_s^2 j, \tag{12}
\]

where \(R_s\) denotes the solenoid’s radius. The Lagrangian of the system is (see [11], p. 46)

\[
L = \frac{1}{2} MV^2 + \frac{1}{2} mv^2 + ev \cdot A \tag{13}
\]

where \(M\) and \(V\) denote the mass and velocity of the charged liquid, respectively and the other terms refer to the travelling electron. (In this experiment the electric field of the solenoid vanishes at the outer region of the pipes and the term depending on the scalar potential is deleted from (13). Similarly, motionless parts of the solenoid are also omitted from this expression.)

Let us use (13) and calculate the action for an electron passing on the right hand side of the solenoid along a line \(x = a, \ -\infty < y < \infty, \ z = 0\). As mentioned above, the second term is independent of the magnetic field. The contribution of the last term is one half of (10), namely

\[
\Delta I_1 = e\Phi/2. \tag{14}
\]

The quantity \(\frac{1}{2} MV^2\) of (13) changes during the experiment, due to the force exerted by the electric field of the travelling electron on the charged liq-
uid. Let \( \Delta W = M(V^2 - V_0^2)/2 \) denote the energy variation. Using cylindrical coordinates and Maxwell equations, one integrates the power associated with the electric field of the travelling electron and finds

\[
\Delta W = \int j E \cdot dl \, dz \, dt = j \int (\text{curl} E) \cdot ds \, dz \, dt = -j \int B_z \, ds \, dz
\]

(15)

where \( dl \) is a line element in the \( \phi \)-direction and \( B_z \) is the \( z \)-component of the magnetic field of the travelling electron. This electron moves in the \( y \)-direction. Hence,

\[
B_z = (v \times E)_z = \frac{vae}{(a^2 + y^2 + z^2)^{3/2}}.
\]

(16)

The calculation is carried out first for a very thin solenoid. In this case, \( B_z \) of the travelling electron is assumed to be uniform at a cross section of the solenoid \( z = \text{const.} \) and the integration on \( ds \) boils down to a multiplication by \( \pi R_s^2 \). Substituting (16) into (15) and performing the integration on \( z \)[14], one obtains

\[
\Delta W = -2\pi R_s^2 vae j \int A dy = -2\pi R_s^2 vae j = -\frac{e \Phi}{2}.
\]

(17)

Integrating (17) on the time, one finds the variation of the action \( \Delta I_2 \) which emerges from the first term of (13). Using \( v \, dt = dy \) and (12), one obtains

\[
\Delta I_2 = -2\pi R_s^2 e j \int_{-\infty}^{\infty} \frac{a \, dy}{a^2 + y^2} = -2\pi R_s^2 e j = -\frac{e \Phi}{2}.
\]

(18)

This result is independent of the impact parameter \( a \). Hence, it also holds for any solenoid, since the latter can be regarded as an assembly of very thin solenoids.

Adding (14) and (18), one obtains a null result. This outcome proves
property 5 for the magnetic AB effect.

5. Topological Features and the Magnetic AB Effect

The foregoing discussion casts a new light on the topological features of the magnetic AB effect[6,7], where it is required that the travelling electron should move in a nonsimply connected field-free region. The proof of item 5 shows that the effect disappears if the magnet is replaced by an equivalent classical device (which conserves the nonsimply connectedness of the field-free region). At this point one may conclude that the nonsimply connectedness is at most a necessary (but not sufficient) condition for the magnetic AB effect.

This matter can be analyzed from another point of view. A decomposition of the system’s interaction into a sum of two body interactions shows that the nonsimply connectedness is just a mathematical feature found in one and only one of two alternative calculations. Indeed, relation (11) shows that the interaction of the travelling electron with the vector potential of the magnetic atom \( ev_e \cdot A_m \) can be replaced by the interaction of the magnetic atom with the magnetic field of the travelling electron \( B_e \cdot m \). A summation of the vector potential of all magnetic atoms at the position \( r(e) \) of the travelling electron, yields the electron’s interaction with the vector potential of the entire magnet (19).

\[
L_{int} = ev_e \cdot [(\Sigma A_{(m)i}(r(e)))] = ev_e \cdot A.
\]

In this picture, one obtains an electron moving in a field-free region which is multiply connected. (Here and below, the subscripts \( m, e \) are put in brackets, in order to be distinguished from the summation index \( i \).)

On the other hand, one may perform the summation on the magnetic
atoms $m_i$ interacting with the magnetic field of the travelling electron and find for the same physical experiment

$$L_{int} = \sum B_e(r_{m_i}) \cdot m_i. \quad (20)$$

Here the magnetic moment of each atom interacts with the local magnetic field of the travelling electron $B_e(r_{m_i})$ and no multiply connected field-free region exists because the magnetic field of the travelling electron is nonzero at the location of the relevant magnetic atoms.

Evidently, the 2 pictures describing the interaction of the travelling electron with the source of the magnetic field are equivalent and one may use either of them. Since the topological field-free property does not exist in the second picture, one concludes that in the magnetic AB effect, topology has no profound physical meaning.

6. Conclusions

It is proved in this work that the polarized neutron experiment[5] and the one demonstrating the magnetic AB effect[9] are based on the same physical principles. Indeed, the two body interaction of each of them is an interaction of a charge (an electron) with a neutral particle having an intrinsic magnetic moment (a neutron in [5] and a magnetic atom in [9]). Furthermore, the main features of the experimental setup of the two experiments are the same.

It is further proved here that the inert nature of the magnetic neutral component is vital for the nonvanishing phase shift obtained. This point is a manifestation of the quantum mechanical nature of the effect demonstrated by the two experiments. Indeed, it is shown in [8] that if the neutron is replaced by a classical device then the effect disappears. An analogous prop-
erty holds for the magnetic AB effect, as shown in the proof of item 5 in the last part of section 4. In section 5 it is proved that topological features of the field-free region do not have a fundamental physical significance.

It can be concluded that the quantum mechanical foundations of the polarized neutron experiment and of the magnetic AB effect are based on the structure of their magnetic dipole constituents that behave as inert objects. These objects stay in their quantum mechanical state throughout the experiment and yield a nonzero phase shift that affects the interference pattern.
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Figure captions

Fig. 1:

The magnetic AB effect. Two subpackets of an electron, $\psi_L$ and $\psi_R$, move in the $(x,y)$ plane. The subpackets move parallel to the $y$-axis and pass on either side of a magnet. The axis of the infinitely long magnet coincides with the $z$-axis and its magnetic field $B$ is confined to the magnet’s inner part. Later, the subpackets interfere on the screen $S$. 
