Asynchronous Games on Petri Nets and ATL

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Abstract. We define a game on distributed Petri nets, where several players interact with each other, and with an environment. The players, or users, have perfect knowledge of the current state, and pursue a common goal. Such goal is expressed by Alternating-time Temporal Logic (ATL). The users have a winning strategy if they can cooperate to reach their goal, no matter how the environment behaves. We show that such a game can be translated into a game on concurrent game structures (introduced in order to give a semantics to ATL). We compare our game with the game on concurrent game structures and discuss the differences between the two approaches. Finally, we show that, when we consider memoryless strategies and a fragment of ATL, we can construct a concurrent game structure from the Petri net, such that an ATL formula is verified on the net if, and only if, it is verified on the game structure.

1 Introduction

We describe the interaction between a group of users and an environment through a game defined on distributed Petri nets \cite{5,14}, and played on their unfoldings. We assume that the users have full observability on the system, i.e.: they have perfect information on the structure of the system and of its current state. The users have a common goal, and each of them can control the occurrence of a subset of transitions. Some of the transitions belong to the environment, and are uncontrollable by the users. The environment is indifferent to the users’ goal. The users have a winning strategy if they can cooperate to reach their goal, no matter how the environment behaves.

We express the goal of the users with Alternating-time Temporal Logic (ATL). This logic was first defined in \cite{2} on concurrent game structures. We compare our game with the game on concurrent game structures and discuss the differences between the two approaches.

Finally, we show that, when we consider memoryless strategies and ATL without the next operator, we can construct a concurrent game structure from the Petri net, such that an ATL formula is verified on the net if, and only if, it is verified on the game structure.

The paper is structured as follows. In the next section, we recall the basic notions about Petri nets: elementary net systems, and their unfoldings, as well
as distributed net systems. In Sect. 3 we define the game on the unfolding of distributed net systems, and the notions of strategy for a user and of winning strategy for a set of cooperating users. In Sect. 4 we propose to specify the winning condition by ATL. After the syntax of ATL formulas, we define their semantics on the unfolding. In Sec. 5 we present the reduction of games on distributed nets into concurrent game structures; namely, in Sec. 5.1 we recall the definition of turn-based asynchronous game structures, a special case of concurrent game structures, and provide an intuition of ATL semantics on these structures. In Sec. 5.2 we show how to derive a turn-based asynchronous game structure from a Petri net game. In Sec. 5.3 we prove the strict relationship between the plays on the unfolding and the infinite fair computations on the turn-based asynchronous game structure. The main result is in Sec. 5.4, where we show that a set of memoryless winning strategies exists on the unfolding of a distributed net system for an ATL formula, without next operator, if and only if there exists a set of memoryless winning strategies on the associated game structure, for the same formula. Moreover, in the case of full memory strategies, we prove that if there are winning strategies on the net system, the same holds on the game structure. Sec. 6 concludes the paper with a discussion on related works and on some critical points, indicating possible developments of this work.

2 Petri nets

Petri nets were introduced by C.A. Petri as a formal tool to represent flows of information in distributed systems. In the last decades, several classes of nets have been defined and studied. In this work we use the class of elementary nets, as defined in [13].

Definition 1. A net is a triple $N = (P,T,F)$, where $P$ and $T$ are disjoint sets. The elements of $P$ are called places and are represented by circles, the elements of $T$ are called transitions and are represented by squares. $F$ is called flow relation, with $F \subseteq (P \times T) \cup (T \times P)$, and is represented by arcs.

For each element of the net $x \in P \cup T$, the pre-set of $x$ is the set $\mathbf{x} = \{y \in P \cup T \mid (y,x) \in F\}$, the post-set of $x$ is the set $x^* = \{y \in P \cup T \mid (x,y) \in F\}$.

We assume that each transition has non-empty pre-set and post-set: $\forall t \in T$, $\mathbf{t} \neq \emptyset$ and $t^* \neq \emptyset$.

Two transitions, $t_1$ and $t_2$, are independent if $(\mathbf{t}_1 \cup t_2^*)$ and $(t_1^* \cup \mathbf{t}_2)$ are disjoint. They are in conflict, denoted with $t_1 \# t_2$, if $\mathbf{t}_1^* \cap \mathbf{t}_2 \neq \emptyset$.

A net $N' = (P',T',F')$ is a subnet of $N = (P,T,F)$ if $P' \subseteq P$, $T' \subseteq T$, and $F'$ is $F$ restricted to the elements in $N'$.

Definition 2. An elementary net system is a quadruple $\Sigma = (P,T,F,m_0)$ consisting of a finite net $N = (P,T,F)$ and an initial marking $m_0 \subseteq P$. A marking is a subset of $P$ and represents a global state.

A transition $t$ is enabled at a marking $m$, denoted $m[t]$, if $\mathbf{t} \subseteq m \land t^* \cap m = \emptyset$. 2
A transition \( t \), enabled at \( m \), can occur (or fire) producing a new marking \( m' = t^* \cup (m \setminus t) \), denoted \( m[t]m' \). A marking \( m' \) is reachable from another marking \( m \), if there is a sequence \( t_1t_2\ldots t_n \) such that \( m[t_1]m_1[t_2]\ldots m_{n-1}[t_n]m' \); in this case, we write \( m' \in [m] \). The set of reachable markings is the set of markings reachable from the initial marking \( m_0 \), denoted \([m_0]\).

In a net system, two transitions, \( t_1 \) and \( t_2 \), are concurrent at a marking \( m \) if they are independent and both enabled at \( m \).

An elementary net system is contact-free iff \( \forall m \in [m_0], \forall t \in T \) if \( t^* \subseteq m \) then \( t^* \cap m = \emptyset \). In this paper we consider only contact-free elementary net systems.

The non sequential behaviour of contact-free elementary net systems can be recorded by occurrence nets, which are used to represent by a single object the set of potential histories of a net system. In the following, by \( F^* \) we denote the reflexive and transitive closure of \( F \).

Given two elements \( x, y \in P \cup T \) we write \( x \preceq y \), if there exist \( t_1, t_2 \in T : t_1 \neq t_2, t_1F^*x, t_2F^*y \land \exists p \in t_1 \cap t_2 \).

**Definition 3.** A net \( N = (B, E, F) \) is an occurrence net if

- for all \( b \in B \), \( |b| \leq 1 \)
- \( F^* \) is a partial order on \( B \cup E \)
- for all \( x \in B \cup E \), the set \( \{y \in B \cup E \mid yF^*x\} \) is finite
- for all \( x \in B \cup E \), \( x \preceq x \) does not hold

We will say that two elements \( x \) and \( y \), \( x \neq y \), of \( N \) are concurrent, and write \( x \concurrent y \), if they are not ordered by \( F^* \), and \( x \preceq y \) does not hold.

By \( \min(N) \) we will denote the set of minimal elements with respect to the partial order induced by \( F^* \).

A \( B \)-cut of \( N \) is a maximal set of pairwise concurrent elements of \( B \). \( B \)-cuts represent potential global states through which a process can go in a history of the system. By analogy with net systems, we will sometimes say that an event \( e \) of an occurrence net is enabled at a \( B \)-cut \( \gamma \), denoted \( \gamma[e] \), if \( e \subseteq \gamma \). We will denote by \( \gamma + e \) the \( B \)-cut \( (\gamma[e]) \cup e^* \). A \( B \)-cut is a deadlock cut if no event is enabled at it.

Let \( \Gamma \) be the set of \( B \)-cuts of \( N \). A partial order on \( \Gamma \) can be defined as follows: let \( \gamma_1, \gamma_2 \) be two \( B \)-cuts. We say \( \gamma_1 < \gamma_2 \) iff

1. \( \forall y \in \gamma_2 \exists x \in \gamma_1 :xF^*y \)
2. \( \forall x \in \gamma_1 \exists y \in \gamma_2 :xF^*y \)
3. \( \exists x \in \gamma_1, \exists y \in \gamma_2 :xF^*y \)

In words, \( \gamma_1 < \gamma_2 \) if any condition in the second \( B \)-cut is or follows a condition of the first \( B \)-cut and any condition in the first \( B \)-cut is or comes before a condition of the second \( B \)-cut (and there exists at least one condition coming before).

A sequence of \( B \)-cuts, \( \gamma_0\gamma_1\ldots \gamma_i \ldots \) is increasing if \( \gamma_i < \gamma_{i+1} \) for all \( i \geq 0 \). A cut \( \gamma \) is compatible with an increasing sequence of \( B \)-cuts \( \delta \) iff there are two cuts \( \gamma_i, \gamma_{i+1} \in \delta \) such that \( \gamma_i \leq \gamma \leq \gamma_{i+1} \).
Given an increasing sequence of B-cuts \( \delta \), we define a refinement \( \delta' \) of \( \delta \) as an increasing sequence of B-cuts such that for each \( \gamma \) B-cut in \( \delta \), \( \gamma \) is also a B-cut in \( \delta' \). A maximal refinement \( \delta' \) is an increasing sequence of cuts such that there is no \( \gamma \notin \delta' \) compatible with \( \delta' \).

We say that an event \( e \in E \) precedes a B-cut \( \gamma \), and write \( e < \gamma \), iff there is \( y \in \gamma \) such that \( eF^+ y \). In this case, each element of \( \gamma \) either follows \( e \) or is concurrent with \( e \) in the partial order induced by the occurrence net.

The next definitions are adapted from [8].

**Definition 4.** A branching process of a contact-free elementary net system \( \Sigma = (P, T, F, m_0) \) is an occurrence net \( N = (B, E, F) \), together with a labelling function \( \mu : B \cup E \to P \cup T \), such that

- \( \mu(B) \subseteq P \) and \( \mu(E) \subseteq T \)
- for all \( e \in E \), the restriction of \( \mu \) to \( e \) is a bijection between \( e \) and \( \mu(e) \); the same holds for \( e^* \)
- the restriction of \( \mu \) to \( \min(N) \) is a bijection between \( \min(N) \) and \( m_0 \)
- for all \( e_1, e_2 \in E \), if \( e_1 = e_2 \) and \( \mu(e_1) = \mu(e_2) \), then \( e_1 = e_2 \)

A run of \( \Sigma \) is a branching process \((N, \mu)\) such that there is no pair of elements \( x, y \) in \( N \) such that \( x \sim y \).

For \( \gamma \) a B-cut of \( N \), the set \( \{ \mu(b) \mid b \in \gamma \} \) is a reachable marking of \( \Sigma \) ([S]), and we refer to it as the marking corresponding to \( \gamma \).

Let \((N_1, \mu_1)\) and \((N_2, \mu_2)\) be two branching processes of \( \Sigma \), where \( N_i = (B_i, E_i, F_i), i = 1, 2 \). We say that \((N_1, \mu_1)\) is a prefix of \((N_2, \mu_2)\) if \( N_1 \) is a subnet of \( N_2 \), and \( \mu_2|_{B_1 \cup E_1} = \mu_1 \). For any contact-free elementary net system \( \Sigma \), there exists a unique, up to isomorphism, maximal branching process of \( \Sigma \). We will call it the unfolding of \( \Sigma \), and denote it by \( \text{unf}(\Sigma) \).

A run of \( \Sigma \) describes a particular history of \( \Sigma \), in which conflicts have been solved. Any run of \( \Sigma \) is a prefix of the unfolding \( \text{unf}(\Sigma) \); we also say that it is a run on \( \text{unf}(\Sigma) \).

In this paper we are interested in Petri nets modelling systems in which several users interact with one another, and with an environment. Each user controls a subset of transitions, deciding whether to fire them or not when they are enabled.

We also assume that choices among transitions are local; this means that every choice is completely determined either by the environment or by one of the users.

As a formal setting, we refer to the so-called distributed net systems, as introduced and studied in [5] and in [14].

**Definition 5.** A distributed net system over a set \( L \) of locations is an elementary net system \( \Sigma = (P, T, F, m_0) \) together with a map

\[ \alpha : (P \cup T) \to L \]

such that for every \( p \in P, t \in T, \) if \( p \in *t \), then \( \alpha(p) = \alpha(t) \).
From now on, we will equivalently denote a distributed net system as a pair 
\( \langle \Sigma, \alpha \rangle \) or with \( \Sigma = (P, T, F, m_0, \alpha) \).

Let \( \langle \Sigma, \alpha \rangle \) be a distributed net system, we associate every location with one of the agents interacting on \( \Sigma \). Specifically, if we are considering a system with \( k \) users, then \( L = \{ \text{env}, u_1, \ldots, u_k \} \), i.e. the distributed net system has \( k + 1 \)
locations, representing the environment (\( \text{env} \)) and the \( k \) users (\( u_i, i \in \{1, \ldots, k\} \)); we denote with \( T_{u_i} \) the subset of transitions belonging to location \( u_i \). We assume that each user controls all transitions in its location.

The notions of unfolding and run apply in the obvious way to distributed net systems. We will use \( E_{u_i} \) to denote the set of events in the unfolding controllable by user \( u_i \) (occurrences of transitions that belong to location \( u_i \)), and \( E_{nc} = E \setminus \bigcup_{i \in \{1, \ldots, k\}} E_{u_i} \) to denote uncontrollable events. Uncontrollable transitions are meant to represent actions performed by the environment.

Example 1. The net in Fig. 1 is a distributed net system with three locations, where two users (represented with different tones of grey) interact with an environment (in white in the picture). Fig. 2 represents its unfolding.

3 An asynchronous game played on unfoldings

Let \( \Sigma = (P, T, F, m_0, \alpha) \) be a distributed net system. We define a game on \( \text{UNF}(\Sigma) \), adapting some of the ideas introduced in [4] and [1].

Definition 6. Let \( \rho = (B_\rho, E_\rho, F_\rho, \mu_\rho) \) be a run on \( \text{UNF}(\Sigma) \) and \( \delta = \gamma_0, \gamma_1, \cdots, \gamma_i, \cdots \) an increasing sequence of \( B \)-cuts. The pair \( (\rho, \delta) \) is a play iff:

1. for each uncontrollable event \( e \) in \( \text{UNF}(\Sigma) \), the net obtained by adding \( e \) and its postconditions to \( \rho \) is not a run of \( \text{UNF}(\Sigma) \);
2. If $\rho$ is finite, for each event $e$ in the unfolding controllable by some user, the net obtained by adding $e$ and its postconditions to $\rho$ is not a run of $\text{UNF}(\Sigma)$;
3. $\forall e \in E_\rho$ there is a B-cut $\gamma_i \in \delta$ such that $e < \gamma_i$.

In other words, a play is a run, weakly fair with respect to uncontrollable transitions, together with an increasing sequence of B-cuts, which can be seen as a potential record of the play as observed by an external entity. In a play, the users have weaker fairness constraints than the environment: they are not forced to fire any enabled event, if some uncontrollable event is enabled; if the only enabled events are controllable by some users, then one of them has to fire an event, i.e. a play can be finite only if it ends in a deadlock state. An example of play on the unfolding in Fig. 2 is represented in Fig. 3, where thick lines show the B-cuts. For every pair of cuts $\gamma_i, \gamma_{i+1}$ in the play, there may be many events

![Fig. 2. Unfolding of the net in Fig. 1](image1)

![Fig. 3. A play on the unfolding](image2)
in between, that are concurrent or sequential with each other.

The winning condition for a team of users is a set of plays. We are particularly interested in the case in which the winning plays satisfy a certain property, such as the reachability or the avoidance of a target place.

The behaviour of each user during the play can be determined by a strategy.

**Definition 7.** Let $\Gamma$ be the set of B-cuts on the unfolding and $T_u$ be the set of transitions controllable by user $u$. A strategy for a user $u$ is a function $f_u : \Gamma \to 2^{T_u}$ such that for every $t \in T_u$ and, for every $\gamma \in \Gamma$, if $t \in f_u(\gamma)$, then there is an event $e$ enabled in $\gamma$ and such that $\mu(e) = t$.

A strategy $f_u$ is memoryless if for every pair of cuts $\gamma_1, \gamma_2$, if $\mu(\gamma_1) = \mu(\gamma_2)$, then $f_u(\gamma_1) = f_u(\gamma_2)$. In this case, we can equivalently define a strategy as a function $f_u : Q \to 2^{T_u}$, where $Q$ is the set of reachable markings of the net.

Whenever an event $e \in E_u$ occurs, we say that the user $u$ made a move in the game.

**Definition 8.** A user $u$ is finally postponed in a play $(\rho, \delta)$, if there is a cut $\gamma$ in $\delta$ such that $f_u(\gamma) \neq \emptyset$ for every $\gamma_j \geq \gamma$ compatible with $\delta$, and $u$ did not make any move after $\gamma$.

From the set of all the plays, we are interested only in those in which the users follow their strategy.

**Definition 9.** Let $S \subseteq \{u_1, ..., u_k\}$ be a set of users. A play $(\rho, \delta)$ is consistent with a set of strategies $F_S$ iff:

1. For every user $u \in S$, for every event $e \in E_u \cap E_\rho$, there is a cut $\gamma_i \in \delta$ such that $\mu(e) \in f_u(\gamma_i)$, $f_u \in F_S$, and $e$ is the only event between $\gamma_i$ and $\gamma_{i+1}$;
2. There is no user $u \in S$ finally postponed.

A set of strategies $F_S$ is winning for the users in $S$ iff there is at least a play consistent with $F_S$, and the users win every play that is consistent with $F_S$, whatever the other agents behave.

We will restrict our attention to the case where all the users cooperate to reach the same goal against the environment.

### 4 Logical specification of the users’ goal

In Sect. 3, we have defined the winning condition as a set of plays. Such a set can be specified in different ways. In this section, we propose to use Alternating-time Temporal Logic (ATL), introduced in [2].

ATL was introduced as a generalization of CTL, with a more flexible set of path quantifiers. In [2], ATL is interpreted over concurrent game structures (cgs), a generalization of Kripke models; we will formally introduce those structures in Sect. 5.1. Intuitively, a cgs models a system where several players interact. Quantifiers allow to specify that a formula holds if a subset of players has a
strategy which guarantees to reach a given goal. In this section we will define a semantics of ATL on the unfoldings of distributed net systems.

The elements that characterize an ATL formula are the following: a set $P$ of elementary propositions; a finite set of players; the symbols $\lor$ and $\neg$ interpreted in the usual manner; path quantifiers $\langle\langle A \rangle\rangle$, where $A$ is a subset of players, and some temporal operators.

Similarly to LTL and CTL, in ATL the temporal operators are $X$ (next), $G$ (always) and $U$ (until). In addition to these, we will use the operator $F$ (eventually), which can be derived from the ‘until’ operator: for every ATL formula $\phi$, $\langle\langle A \rangle\rangle F\phi$ is equivalent to $\langle\langle A \rangle\rangle U(true, \phi)$.

The set $\Phi$ of ATL formulas is defined as follows: Let $p \in P$, $\phi_1, \phi_2 \in \Phi$.

1. $p$ is an ATL formula;
2. $\neg \phi_1, \phi_1 \lor \phi_2$ are ATL formulas;
3. $\langle\langle A \rangle\rangle X\phi_1, \langle\langle A \rangle\rangle G\phi_1, \langle\langle A \rangle\rangle U(\phi_1, \phi_2)$ are ATL formulas, where $A$ is a set of players.

Let $\Sigma = (P,T,F,m_0,\alpha)$ be a distributed net system. Let $N = \text{unf}(\Sigma) = (B,E,F,\mu)$ be its unfolding, and $\gamma$ a B-cut of $N$.

We first define the validity of a formula in a B-cut, and write $\gamma \models \phi$ to denote that formula $\phi$ holds (or is satisfied) in cut $\gamma$.

Let $\phi_1$ and $\phi_2$ be two ATL formulas and $p \in P$. The formula $p$ holds in a B-cut $\gamma$ iff there is a $b$ in $\gamma$ such that $\mu(b) = p$.

The formula $\neg \phi_1$ is true in $\gamma$ iff $\phi_1$ is false in $\gamma$; $\phi_1 \lor \phi_2$ is true in $\gamma$ iff $\phi_1$ is true in $\gamma$ or $\phi_2$ is true in $\gamma$.

Consider now $S \subseteq \{u_1, \ldots, u_k\}$:

- $\langle\langle S \rangle\rangle X\phi_1$ is satisfied in a B-cut $\gamma$ iff there exists a set $F_S$ of strategies, one for each player in $S$, such that for every event $e$ that can occur in $\gamma$, consistent with the strategies, the cut $\gamma + e$ satisfies $\phi_1$.
- $\langle\langle S \rangle\rangle G\phi_1$ is satisfied in a B-cut $\gamma$ iff there exists a set $F_S$ of strategies, one for each player in $S$, such that for every play $\rho', \delta'$ that starts in $\gamma$, consistent with the strategies, in every cut $\gamma'$, compatible with $\delta'$, $\phi_1$ holds.
- $\langle\langle S \rangle\rangle U(\phi_2, \phi_1)$ is satisfied in a B-cut $\gamma$ iff there exists a set $F_S$ of strategies, one for each player in $S$, such that for every play $\rho', \delta'$ that starts in $\gamma$, consistent with the strategies:
  - there is a cut $\gamma'$ compatible with $\delta'$ such that $\phi_1$ is satisfied in $\gamma'$;
  - for every B-cut $\gamma'' < \gamma'$ compatible with $\delta'$, $\phi_2$ is satisfied.

An ATL formula is satisfied on $N$ iff it is satisfied in the initial cut $\gamma_0$ of $N$. If a formula $\phi$ is satisfied on $N$, we write

$$N \models \phi$$

For an ATL formula in the form $\langle\langle S \rangle\rangle \eta$, given a set of strategies $F_S$, we can define a winning play for the players in $S$. Let $(\rho, \delta)$ be a play on the unfolding, consistent with $F_S$. 

8
Given an ATL formula $\langle \langle S \rangle \rangle X \phi_1$, $(\rho, \delta)$ is a winning play iff for every cut $\gamma$ reachable from $\gamma_0$ after the occurrence of a single event and compatible with $\delta$, $\gamma$ satisfies $\phi_1$.

Given an ATL formula $\langle \langle S \rangle \rangle G \phi_1$, $(\rho, \delta)$ is a winning play iff every cut compatible with $\delta$ satisfies $\phi_1$.

Given an ATL formula $\langle \langle S \rangle \rangle U(\phi_1, \phi_2)$, $(\rho, \delta)$ is a winning play iff for every $\delta'$ refinement of $\delta$ there is a cut $\gamma$ compatible with $\delta'$ in which $\phi_2$ is satisfied, and in all the cuts $\gamma' < \gamma$ such that $\gamma'$ is compatible with $\delta'$, $\phi_1$ is satisfied.

**Example 2.** Consider the net in Fig. 1 and its unfolding in Fig. 2. We want to verify whether the two users controlling the gray components can force the system to reach $p_6$ infinitely often. We identify with $u_1$ the user on the light gray component and with $u_2$ the user on the dark gray. We can express this goal with the formula:

$$\langle \langle u_1, u_2 \rangle \rangle G \langle \langle u_1, u_2 \rangle \rangle F p_6.$$  

(1)

This formula is satisfied on the unfolding. To see that, consider the strategy $f_{u_1} : \Gamma \rightarrow 2^{\mathcal{F}_{v_1}}$, defined in this way: for every cut $\gamma$ such that $\mu(\gamma) = \{p_0, p_4, p_8\}$, $f_{u_1}(\gamma) = \{t_0\}$, for every cut $\gamma$ such that $\mu(\gamma) = \{p_0, p_5, p_8\}$, $f_{u_1}(\gamma) = \{t_1\}$, otherwise $f_{u_1}(\gamma) = \emptyset$; and the strategy $f_{u_2}$ defined as $f_{u_2}(\gamma) = \{t_6, t_7\}$ if $\mu(\gamma) = \{p_6, p_8\}$, $f_{u_2}(\gamma) = \{t_7\}$ if $\mu(\gamma) = \{p_0, p_{10}\}$, $f_{u_2}(\gamma) = \emptyset$ otherwise. $F_{u_1, u_2} = \{f_{u_2}, f_{u_1}\}$ is a set of winning strategies for the users. Indeed, the users have a strategy to reach always cuts in which they have a strategy to reach a place labelled as $p_6$.

We could also consider a safety goal, for example we could ask that the play never reaches a state in which $p_{10}$ is true. This can be expressed with the formula:

$$\langle \langle u_1, u_2 \rangle \rangle G \neg p_{10}.$$  

(2)

Since the users must respect a fairness constraint, even if this is weaker than the one imposed to the environment, they cannot just decide not to fire any transition and avoid in this way the unsafe place. Nevertheless, they have winning strategies: $f'_{u_1}(\gamma) = \{t_1, t_2\}$ if $\mu(\gamma) \in \{p_0, p_4, p_8\}, \{p_0, p_5, p_8\}$, $f'_{u_1}(\gamma) = \emptyset$ otherwise; $f'_{u_2}(\gamma) = \{t_7\}$ if $\mu(\gamma) = \{p_6, p_8\}$, $f'_{u_2}(\gamma) = \emptyset$ otherwise.

### 5 Reduction of games on Petri nets to concurrent game structures

In [2], the authors present many model checking algorithms to decide whether an ATL formula is satisfied on a concurrent game structure. In many cases those algorithms can be exploited also for the game on Petri nets. In particular, we show that when we consider memoryless strategies and ATL formulas without the next ($X$) operator, we can reduce a game on a net into a turn-based asynchronous game structure with fairness constraints.
5.1 Turn-based asynchronous game structures

We first recall from [2] the definition of turn-based asynchronous game structure, which is a special case of a concurrent game structure.

**Definition 10.** A turn-based asynchronous game structure is a tuple \( G = \langle n, Q, \Pi, w, d, \tau \rangle \), where

- \( n \geq 2 \) is the number of players. Every player is identified with a number \( 1, \ldots, n \). Player \( n \) represents the scheduler. At every turn, the scheduler selects one of the other players.
- \( Q \) is a finite set of states.
- \( \Pi \) is a set of propositions.
- \( \forall q \in Q, w(q) \subseteq \Pi \) is the set of propositions that are true in \( q \).
- \( \forall a \in \{1, \ldots, n\}, q \in Q, d_a(q) \in \mathbb{N} \) is the number of moves available for the player \( a \) in the state \( q \). Every move is identified with a number \( 1, \ldots, d_a(q) \).
  - For every state \( q \in Q, d_a(q) = n - 1 \).
- For every state \( q \in Q, D(q) \) is the set \{1,...,d_1(q)\} x ... x \{1,...,d_n(q)\} of move vectors.
- \( \tau \) is the transition function. For every \( q \in Q \) and \( \langle j_1, \ldots, j_n \rangle \) vector move, \( \tau(q, j_1, \ldots, j_n) \in Q \) is the state where the system is if from the state \( q \), every player \( a \in \{1, \ldots, n\} \) chooses move \( j_a \). For all move vectors \( \langle j_1, \ldots, j_n \rangle \), \( \langle j_1', \ldots, j_n' \rangle \), if \( j_n = j_n' = a \) and \( j_a = j_a' \), then \( \tau(q, j_1, \ldots, j_n) = \tau(q, j_1', \ldots, j_n') \), for every \( q \in Q \).

Given an initial state, an infinite computation \( \lambda \) on a turn-based asynchronous game structure is an infinite sequence of states such that the successor of each element is fully determined by the moves chosen in the previous state.

We denote with \( Q^+ \) the set of all the finite prefixes of the computations \( \lambda \) in the concurrent game structure and with \( \lambda_i \) the prefix of \( \lambda \) such that \( |\lambda_i| = i \). A strategy for a player \( a \) is a function \( f_a : Q^+ \rightarrow \mathbb{N} \) such that \( f_a(\lambda_i) \leq d_a(q) \), where \( q \) is the last element of \( \lambda_i \). The strategy is memoryless if, and only if, for every pair of prefixes \( \lambda_i, \lambda_j \) ending with the same state, \( f_a(\lambda_i) = f_a(\lambda_j) \); a memoryless strategy can be equivalently defined as a function \( f : Q \rightarrow \mathbb{N} \). We say that a computation \( \lambda \) follows a strategy \( f_a \) iff, for every prefix \( \lambda_i \) of a computation, player \( a \) chooses the move \( f_a(\lambda_i) \).

Let \( A \) be a set of players, and \( F_A \) be a set of strategies, one for each player in \( A \). We denote with out(\( q, F_A \)) the set of computations starting from \( q \) and such that the players in \( A \) follow the strategies in \( F_A \).

On turn-based asynchronous game structures, the syntax of ATL is equivalent to the one defined in Sect. 4 for distributed net systems. We provide an intuition of the semantics, the formal definition is in [2].

Let \( \langle (A) \eta \rangle \) be an ATL formula. We can evaluate it by considering a game in which players in \( A \) are against the others. An infinite computation \( \lambda \) is winning for players in \( A \) iff the computation satisfies the formula \( \eta \), read as a linear temporal formula, with outermost operator \( X, G \) or \( U \). The formula \( \langle (A) \eta \rangle \) is satisfied in a state \( q \) of the game structure iff there exists a set of strategies \( F_A \).
such that the players in $A$ win all the computations in $\text{out}(q, F_A)$. In this case, we say that $F_A$ is a winning set of strategies.

We can include some fairness constraints to the game structure, in order to ignore some computations.

**Definition 11.** A fairness constraint is a pair $(a, c)$, where $a$ is a player, and $c$ is a function that for every state $q \in Q$ selects a subset of moves available for $a$ in $q$.

Let $\lambda$ be an infinite computation and $q_i$ its $i$-th element. A fairness constraint $(a, c)$ is enabled in $q_i$ if $c(q_i) \neq \emptyset$; $(a, c)$ is taken in $q_i$ if there is a vector move $(j_1, \ldots, j_n)$ with $j_a \in c(q_i)$ and $q_{i+1} = \tau(q_i, (j_1, \ldots, j_n))$. If $a < n$ we also require that $j_n = a$.

A computation $\lambda$ is weakly fair with respect to a fairness constraint $(a, c)$ if $(a, c)$ is not enabled in infinitely many positions of $\lambda$ or if it is taken infinitely many times in $\lambda$.

5.2 Construction of a turn-based asynchronous game structure from a Petri net game

Given a distributed net system $\Sigma$ with $k+1$ locations, we construct a turn-based asynchronous game structure $G_\Sigma$ in this way:

- Every location is represented by a player in $G_\Sigma$. For every $i \in \{1, \ldots, k\}$, we identify the user $u_i$ on the net as player $i$, and the environment as player $k+1$. In addition, we insert a fictitious player identified with $k+2$ that has the role of the scheduler.
- The reachable markings are all and only the states in $G_\Sigma$. We denote the set of all the states with $Q$, and with $q_0$ the initial marking of $\Sigma$ and the corresponding state in $G_\Sigma$.
- We identify the set of propositions as the set $P$ of places: every place can be interpreted as a proposition, that is true if the place is part of the current marking, false otherwise,
- According to the previous point, for every state $q \in Q$, the set of propositions that are true in $q$ is $w(q) = \{p \in P : p \in q\}$.
- On the net, for every marking $q$, for every user $a$, $a$ can decide to fire one of the transitions in $T_a$ that are enabled in $q$ or not to move. Then, denoting with $r(q, a)$ the number of transitions that are enabled in $q$ and belong to $T_a$, the player $a$ has $r(q, a) + 1$ different possible behaviours in every cut associated with the marking $q$. This is translated on $G_\Sigma$ by putting $d_a(q) = r(q, a) + 1$ for every $q \in Q$ and a user.
- $d_{k+1}(q)$ is equal to the number of uncontrollable transitions enabled in $q$. If there are no uncontrollable transitions enabled in $q$, then $d_{k+1}(q) = 1$ and the move remains in the same state.

For the sake of simplicity, we will refer to each move that changes the state of the system with the name of its associated transition on the distributed
net system; we will refer to the move that leaves the system in the same state with $\emptyset$.

Finally, $d_{k+2}(q) = k + 1$ for every $q \in Q$.

The vector $\langle q, j_1, ..., j_{k+2} \rangle$ collects a move of the players in $q$.

- For every $q \in Q$, $\tau(q, j_1, ..., j_{k+2})$ is the state corresponding to the marking that is reached if, starting from the state $q$ in the net, we execute the transition in $j_{k+2}$.

We represent the fairness constraints on the net with weak fairness constraints on $G^\Sigma$. We ask that there is no user with a finally postponed move by asking that the scheduler is fair, and no player is neglected forever. Formally, we ask that $\langle k + 2, c_j \rangle$ is a weak fairness constraint for every $j \in \{1, ..., k + 1\}$ and $c_j$ is a function such that for every $q \in Q$, $c_j(q) = \{j\}$.

In addition, we need to add constraints to guarantee that the plays are weakly fair with respect to uncontrollable transitions. For every state $q$, if $q$ enables uncontrollable transitions on the net, then for every transition $t$ enabled in $q$ we consider the subset $t\#(q)$ of uncontrollable transitions in conflict with $t$ and enabled in $q$, and we define a weak constraint in the state $q$ in this way: $\langle k + 1, c_t \rangle$, where $c_t(q) = t\#(q) \cup \{t\}$.

Finally, we need to guarantee that, if we are not in a deadlock state, but no uncontrollable transition is enabled, the users cannot block the system. In order to address this problem, we add a set of fairness constraints for the users in this way: for each $q$, in which only users’ transitions are enabled, and for each player $a_1, ..., a_l$ with enabled transitions in $q$, we define the fairness constraint $\langle a_i, c_q \rangle$, with $i \in \{1, ..., l\}$, such that $c_q(q)$ is the set of all the moves that do not keep the concurrent game structure in the same state, and $c_q(q') = \emptyset$ for every $q' \neq q$.

In the special case of a single user playing against the environment, we can avoid this last set of constraints and just remove the possibility for the user to stay in the same state, if only controllable transitions are enabled in that state.

5.3 Relation between the plays in the two models

Let $\Sigma = (P, T, F, q_0, \alpha)$ be a distributed net system, and $G^\Sigma$ the associated game structure, as defined in Sect. 5.2.

We now prove some propositions that show the relations between the plays on the unfolding and the infinite fair computations on the concurrent game structure. This will be helpful to find the relation between the existence of winning strategies in the two models.

Let $\lambda = q_0 q_1 \cdots$ be a fair computation on $G^\Sigma$. For every pair of consecutive states $q_i, q_{i+1}$, such that $q_i \neq q_{i+1}$, the move on the concurrent game structure is associated, by construction, with an enabled transition in $\Sigma$.

Given $\lambda$, we construct a run $\rho$ on the unfolding of $\Sigma$, starting from the initial cut and adding the events associated with the moves that bring from a state to the next one in the same order that the states have in $\lambda$. At the same time we construct an increasing sequence $\delta$ of cuts: initially, $\delta = \gamma_0$; after an event $e$ is added to the run, we add to $\delta$ the cut $\gamma = \gamma' \setminus e \cup e^*$, where $\gamma'$ is the last cut added in $\delta$ before $e$. The pair derived in this way will be denoted by $(\rho, \delta) [\lambda]$. 


Proposition 1. The pair \((\rho, \delta)|\lambda\) is a play on \(\text{unf}(\Sigma)\).

Proof. The sequence of cuts \(\delta\) satisfies condition 3 of Definition 6 by construction.

We need to prove that there is no uncontrollable event \(e \notin \rho\) such that \(\rho \cup \{e\}\) is a run on the unfolding. By contradiction, assume that we can find such an event \(e\). Then, there is \(n \in \mathbb{N}\) such that, on \(G_\Sigma\), the move \(\mu(e) = t\) is enabled in every state \(q_m \in \lambda: m > n\), but is never chosen. The reason cannot be that the environment player is never selected by the scheduler, otherwise, the constraint \(\langle k + 2, c_{k+1}\rangle\) would not be satisfied, and \(\lambda\) would not be a fair computation. Hence the move \(t\) is available in every state \(q_m\) for the player \(k + 1\), but he never chooses it. Then the computation does not respect the fairness constraint \(\langle k + 1, c_{k+2}\rangle\), where, for each \(q \in Q, c_{t}(q)\) is the set \(\{t\} \cup t\#(q)\): by construction, if a move associated with \(t\) or with one of the transitions in conflict with \(t\) occurs in \(q_m\), then \(t\) would not be available in \(q_{m+1}\).

Finally, we have to show that point 2 of definition 6 holds: by contradiction, we assume that \(\rho\) is finite, and there is an event \(e \in E_a\), controllable by user \(a\), such that \(\rho \cup \{e\}\) is a run. If \(\rho\) is finite, there is a position \(i\) in \(\lambda\) such that \(q_i = q_{j+1}\) for every \(j \geq i\). In the state \(q_j\), there is no uncontrollable enabled transition, otherwise point 1 of the definition would not be respected. Then there must be a fairness constraint \(\langle a, c_{q_j}\rangle\) not respected by \(\lambda\). Hence, if \(\lambda\) is fair, there cannot be controllable enabled transitions in \(q_j\), and \(q_j\) must be a deadlock in the distributed net system.

Hence \((\rho, \delta)|\lambda\) constructed in this way is a play on the unfolding. \(\square\)

Given a computation \(\lambda = q_0q_1...q_n...\) (finite or infinite) we can construct a new sequence \(\pi(\lambda)\) in which all the consecutive states \(q_i = q_{i+1}\) has been identified. Given an infinite sequence \(\lambda, \pi(\lambda)\) can be also infinite, if there is no state \(q_i \in \lambda\) such that for each \(q_j: j > i q_j = q_{j+1}\), or finite otherwise.

In general, a game on the unfolding is not equivalent to a game on the concurrent game structure. This is mainly due to the fact that infinite sequences of states on the asynchronous game structure are not equivalent to sequences of cuts on the unfolding: (1) the cuts do not report the presence of moves that do not change the state of the system, while the sequences of states do; (2) on the unfolding of distributed net systems, the memory is ‘stored’ in the cuts, from which a user can determine which set of events fired, but not their total order, while a sequence of events gives full information about the order.

In order to address point (1), we consider only ATL formulas that do not use the \(X\) (next) operator. This is due to the fact that, given two infinite computations \(\lambda_1\) and \(\lambda_2\), if \(\pi(\lambda_1) = \pi(\lambda_2)\), then the players cannot distinguish on the system whether \(\lambda_1\) or \(\lambda_2\) occurred, hence we want that \(\lambda_1\) is a winning play for the users iff \(\lambda_2\) is.

Lemma 1. Let \(\Sigma\) be a distributed net system, \(\psi\) be an ATL formula that does not use the operator \(X\) and \(\lambda_1\) and \(\lambda_2\) two infinite weakly fair computations on \(G_\Sigma\) such that \(\pi(\lambda_1) = \pi(\lambda_2)\). Then if there is a winning strategy for the users, \(\lambda_1\) is a winning play for the users iff \(\lambda_2\) is.
Proof. If the operator $X$ is not allowed, then the validity of $\psi$ depends only on the sequence of distinct states in computations; since, by hypothesis, $\pi(\lambda_1) = \pi(\lambda_2)$, the thesis follows. \qed

Every play on the unfolding can be associated with a set of infinite fair computations on the turn-based asynchronous game structure: let $(\rho, \delta)$ be a play on the unfolding; as first step, we consider all the possible sequentializations included between the initial cut $\gamma_0$ and the next cut $\gamma_1 \in \delta$. For each of these linearizations we can find a prefix of a computation on the concurrent game structure by executing on it, from the state corresponding to the initial marking, all the events in the order given by the linearization; then, we extend all these prefixes by considering the successive pairs of consecutive cuts, and the sequentializations of the events between them. If the play on the distributed net system reaches a deadlock, then the only possibility in the associated concurrent game structure is to execute the transition that remains in the same state infinitely often.

In this way, we obtain a set of infinite computations associated with a play $(\rho, \delta)$, denoted by $\Lambda(\rho, \delta)$.

**Proposition 2.** Let $(\rho, \delta)$ be a play on $\text{unf}(\Sigma)$. For every computation $\lambda \in \Lambda(\rho, \delta)$ on $G_\Sigma$, there is at least a computation $\lambda' : \pi(\lambda') = \pi(\lambda)$ satisfying the fairness constraints defined in Section 5.2.

Proof. The first set of constraints guarantees that no player is neglected forever by the scheduler. If $\rho$ ends with a deadlock, then it is easy to see that every associated computation is fair on $G_\Sigma$, because the only available move for all players after a finite number of steps is the one that remains in the same state. For the same reason, when $\rho$ is finite, also the other constraints are respected: after a finite number of states there is no enabled transition, therefore those constraints are always taken when the system reaches the state corresponding to the deadlock.

We now consider the case in which $\rho$ is infinite. Since a play on the unfolding must be fair with respect to uncontrollable transitions, in the computation $\lambda$ the fairness constraint $\langle k+2, c_{k+1} \rangle$ must be respected. If $\rho$ has infinitely many events belonging to location $u_i$, then the fairness constraint $\langle k+2, c_i \rangle$ is also satisfied by construction; otherwise, since in case of an infinite play, the users have always the possibility not to move in the system, we can construct a sequence $\lambda' : \pi(\lambda') = \pi(\lambda)$ with some repeated states, that represent points of the sequence in which the user was selected by the scheduler in $G_\Sigma$, but the state of the system did not change.

The set of fairness constraints for the environment $k + 1$ must be satisfied: by contradiction, we assume that there is a function $c_{t\#}$, such that $\langle k+1, c_{t\#} \rangle$ is not satisfied in $\lambda'$; this means that $\langle k + 1, c_{t\#} \rangle$ is not enabled in a finite number of states in $\lambda$, but it is taken only a finite number of times. This means that there is an uncontrollable event $e \notin \rho$ in the net that is enabled in all the cuts compatible with $\delta$, except for a finite set; since $\rho$ is infinite, there must be a cut $\gamma \in \delta$ such that $e$ is enabled in $\gamma$ and in all the cuts $\gamma_j \in \rho : \gamma_j > \gamma$; hence $\rho \cup \{e\}$ is a run on $\text{unf}(\Sigma)$. This is in contradiction with the hypothesis that $\rho$ is a play.
Finally, the set of constraints for the users are also satisfied: consider a fairness constraint referred to a user \( a \) in a state \( q \); since there is no position \( i \) in \( \lambda \) such that \( q_j = q \) for every \( j \geq i \), there are infinitely many positions in which the constraint is not enabled.

**Lemma 2.** Let \( \psi \equiv \langle \langle \text{users} \rangle \rangle \eta \) be an ATL formula where each path quantifier is the set of all the users and without the \( X \) operator:

1. if \((\rho, \delta)\) is a winning play for the users on the unfolding, then any infinite fair computation \( \lambda \in A(\rho, \delta) \) is winning for the users;
2. if \( \lambda \) is an infinite fair computation starting in \( q_0 \) that is winning for the users, the derived play \((\rho, \delta)[\lambda]\) on the unfolding is winning for the users.

**Proof.** In order to prove this lemma, we have to consider \( G \) and \( U \) as outermost operator in \( \eta \).

- Let \( \psi \equiv \langle \langle \text{users} \rangle \rangle G\phi \), \((\rho, \delta)\) a winning play for the users and \( \lambda \in A(\rho, \delta) \). By construction, \( \lambda \) is a sequence of states associated with the markings of a refinement \( \delta' \) of \( \delta \), in the same order as in \( \delta' \). Since \( \phi \) must be satisfied in every cut \( \gamma \in \delta' \), and for every \( \gamma \) the set of propositions that are true in \( \gamma \) and in \( q = \mu(\gamma) \) is the same, it must also be satisfied in each state \( q \in \lambda \). Viceversa, let \( \lambda \) be a winning computation for the users and \((\rho, \delta)[\lambda]\) the associated play. By construction, \( i \)-th element in \( \delta \) is associated with the \( i \)-th state in \( \pi(\lambda) \) for each \( i \in \mathbb{N} \), hence if \( \phi \) is verified in each state in \( \lambda \), it must be satisfied in each cut in \( \delta \). Since all the cuts compatible with \( \delta \) also belong to it by construction, then \((\rho, \delta)[\lambda]\) is winning for the users.

- \( \psi \equiv \langle \langle \text{users} \rangle \rangle U(\phi_1, \phi_2) \), \((\rho, \delta)\) be a winning play for the users, and \( \lambda \in A(\rho, \delta) \). By construction, the sequence of states in \( \lambda \) is associated with a maximal refinement \( \delta' \) of \( \delta \). By definition, there must be a cut \( \gamma \) compatible with \( \delta' \) in which \( \phi_2 \) is verified. By definition of maximal refinement, \( \gamma \in \delta' \), therefore there is a state \( q \in \lambda \) in which \( \phi_2 \) is verified. For each cut \( \gamma' \in \delta' : \gamma' < \gamma \), \( \phi_1 \) is satisfied. By construction, each state \( q' \in \lambda \) preceding \( q \) is associated with one of these cuts \( \gamma' \), hence in each of them \( \phi_1 \) is satisfied. Viceversa, let \( \lambda \) be an infinite fair computation winning for the users, and \((\rho, \delta)\) the associated play on the unfolding. Let \( q_i \) be the state in which \( \phi_2 \) is true for the first time; all the states \( q_j \in \lambda : i < j \) satisfy \( \phi_1 \). By construction, each cut in \( \delta \) can be associated with a state in \( \lambda \), and the order in which the cuts appear in \( \delta \) is the same in which the associated states are in \( \lambda \), hence there must be a cut \( \gamma \in \delta \) associated with the state \( q_i \), in which \( \phi_2 \) is verified and such that for all the cuts \( \gamma' \in \delta : \gamma' < \gamma \), \( \phi_1 \) is verified in \( \gamma' \). By construction, \( \delta \) cannot be refined further, hence the previous fact is enough to state that \((\rho, \delta)\) is winning for the users on the net.

**5.4 Relation between winning strategies in the two models**

We now restrict ourselves to the case of memoryless strategies, formulas without \( X \) operator and in which in all the quantifiers there is the set of all the users.
Theorem 1. Let $\psi$ be an ATL formula as described above, $\Sigma = (P,T,F,q_0,\alpha)$ be a distributed net system, UNF$(\Sigma)$ its unfolding, and $G_{\Sigma}$ the associated game structure. A set of memoryless winning strategies exists on UNF$(\Sigma)$ for $\psi$ if a set of memoryless winning strategies exists on $G_{\Sigma}$ from $q_0$ for $\psi$.

Proof. As first step, we show that if there is a set of memoryless winning strategies on UNF$(\Sigma)$ for $\psi$, then there is a set of memoryless winning strategies for the users from $q_0$ on $G_{\Sigma}$.

Let $A$ be the set of users on the net, and $F_A' = \{ f'_a : Q \rightarrow 2^T, a \in A \}$ a set of memoryless winning strategies, one for each user on UNF$(\Sigma)$, where $Q$ is the set of reachable markings of $\Sigma$. We define a set of strategies $F_A$ for the users on the concurrent game structure in this way:

1. If $f'_a(q) \neq \emptyset$, and $t$ a transition $t \in f'_a(q)$ arbitrarily chosen, then $f_a(q) = t$.
2. $f_a(q) = \emptyset$ otherwise.

We prove that such a strategy is winning in $G_{\Sigma}$.

Let out$(q_0,F_A)$ be the set of fair computations starting from $q_0$ that the users enforce when they follow the strategies in $F_A$. For any $\lambda \in \text{out}(q_0,F_A)$, we show that the play $(\rho,\delta)[\lambda]$ satisfies the condition of Definition 9 and therefore it is consistent with the strategies in $F_A'$: (1) let $e \in E_a \cap \rho$ be an event controllable by user $a$. By construction, $\delta$ is a maximal refinement, hence there must be two cuts $\gamma_j$ and $\gamma_{j+1}$ such that $e$ is the only event between them. By construction, if $e$ was added to the run after $\gamma_j$, there must be a state $q \in \lambda : \mu(\gamma_j) = q$ such that $f_a(q) = \mu(e)$. By construction of the strategy, $\mu(e) \in f'_a(q)$. (2) By contradiction, assume that there is a user $a$ finally postponed; then there is a cut $\gamma$ such that for each cut $\gamma_j > \gamma$, $f'_a(\mu(\gamma_j)) \neq \emptyset$. By construction, there must be a state $q_i \in \lambda : \mu(\gamma) = q_i$ such that $f_a(q_j) \neq \emptyset$ for each $q_j, j > i$; then $\lambda$ cannot be fair with respect to $(k+2,c_a)$.

This run is a play, and it is consistent with the strategy by construction, hence it is a winning play for the users and it respects the property expressed by $\psi$. Hence also $\lambda$ satisfies it on the concurrent game structure.

As second step, we want to show that if $F_A$ is a set of memoryless winning strategies for the users on $G_{\Sigma}$ with initial state $q_0$, then there is a set of memoryless winning strategies for the users on UNF$(\Sigma)$.

Let $F_A = \{ f_a : a \in A \}$ be a set of winning strategies on $G_{\Sigma}$. We construct a family of strategies on UNF$(\Sigma)$ by putting for every user $a$ and for every state $q \in Q$, $f'_a(q) = \{ f_a(q) \}$. We want to show that $F_A' = \{ f'_a : a \in A \}$ is a set of winning strategies for the users.

Let $(\rho,\delta)$ be a play consistent with $F_A'$. By contradiction, we assume that the formula $\psi$ is not satisfied on UNF$(\Sigma)$, and that the play $(\rho,\delta)$ testifies it. Then there must be a maximal refinement $\delta'$ of $\delta$, such that $(\rho,\delta')$ does not satisfy the formula. $\delta'$ is associated with an infinite sequence of states $\lambda$ on $G_{\Sigma}$. If $\lambda$ is a fair computation on $G_{\Sigma}$, then it is also a computation consistent with the strategies: every time that there is a user’s move, it occurs in a state corresponding to a marking in which that transition is enabled. An event associated to this transition must be chosen in UNF$(\Sigma)$ in a cut corresponding to that marking, and
therefore it is chosen also by the state corresponding to that marking in the concurrent game structure by construction. If \( \lambda \) is not fair, we know by Proposition 2 that we can construct a fair computation \( \lambda' \) such that \( \pi(\lambda) = \pi(\lambda') \). In addition, from the proof of Proposition 2 we know that the only reason why \( \lambda \) can be unfair is that there are users that are finally neglected during the play. This cannot happen to the users that have their strategy finally not-empty, otherwise the play on the net would not be consistent with their strategy (by definition the strategies select only enabled events). Then, for every user \( a \) neglected by the scheduler, there is an infinite number of states in the sequence in which \( f_a \) selects only the transition that keeps the system in the same state. We add a copy of these states in the position coming immediately after them. This computation \( \lambda' \) is fair with respect to the user, therefore it is a play consistent with the strategy on the concurrent game structure and by hypothesis is winning. By construction \( \pi(\lambda) = \pi(\lambda') \), hence Lemma 1 guarantees that also \( \lambda \) respect the ATL formula. \( \square \)

We do not know if the equivalence holds also for full memory strategies. We can prove that in case of winning strategies on the distributed net system, there is a strategy on the turn-based asynchronous game structure, but we could not prove anything for the opposite direction.

**Proposition 3.** Let \( F_A' = \{ f_a' : \Gamma \rightarrow 2^{T\Sigma} : a \in \{1,...,k\} \} \) be a set of winning strategies for the users on \( \text{UNF}(\Sigma) \) for the ATL formula \( \langle\langle A\rangle\rangle \eta \). Then there is a winning strategy for the users from \( q_0 \) on \( G_\Sigma \) for the same formula.

**Proof.** Let \( \gamma \) be a cut on the unfolding. The set of events that occurred from the initial cut to \( \gamma \) is uniquely determined. We consider all the sequentializations of markings generated by firing the transitions in this set respecting the partial order imposed by the unfolding. By construction, each of these sequences can be reproduced on \( G_\Sigma \) starting from \( q_0 \) on the cgs. We denote with \( A(\gamma) \) the set of finite sequences obtained in this way.

For each player \( a \in \{1,...,k\} \) on \( G_\Sigma \), we define a strategy \( f_a \) in this way:

1. If \( f_a'(\gamma) \neq \emptyset \), then \( f_a(\lambda) = t \) for each \( \lambda \in A(\gamma) \), where \( t \in f_a'(\gamma) \).
2. For each pair \( \lambda, \lambda' \) such that \( \pi(\lambda) = \pi(\lambda') \), \( f_a(\lambda) = f_a(\lambda') \).
3. \( f_a(\lambda) = \emptyset \) otherwise.

We prove that such a strategy is winning in \( G_\Sigma \). Let \( \text{out}(q_0, F_A) \) the set of fair computations starting from \( q_0 \) that the players in \( A \) enforce when they follow the strategies in \( F_A \). For any \( \lambda \in \text{out}(q_0, F_A) \), we consider the play \( (\rho, \delta)\vert\lambda \) on the unfolding of the distributed net system. We have to show that it is consistent with the strategies: the first point of Definition 9 is satisfied by construction; by contradiction, we assume that the second point is not respected. By construction, if there is a user \( a \) for which \( f_a'(\gamma) \neq \emptyset \) for every \( \gamma > \gamma' : \gamma' \in \rho \), then there is also an index \( l \) such that for every \( m > l \) for every prefix \( \lambda_m \) of \( \lambda \), \( f_a(\lambda) \neq \emptyset \). If no event belonging to user \( a \) occurs in \( \lambda \) after the prefix \( \lambda_l \), then player \( a \) has been finally neglected by the scheduler, and \( \lambda \) cannot be a fair computation. Hence \( (\rho, \delta)\vert\lambda \) is a consistent play, and by hypothesis it is winning for the users. Then by Lemma 2 \( \lambda \) is also winning for the users on \( G_\Sigma \). \( \square \)
6 Related works and conclusions

In this work we have defined a game for a group of users and an environment on the unfolding of distributed net systems, by adapting the idea of game on the unfolding defined in [1] for two players, a sequential user and the environment.

While in [1] the goal of the user was to force the occurrence of a target transition, here we specify the possible goals of the users with the temporal logic ATL, initially introduced in [2] on concurrent game structures. With respect to [1], we have also modified some of the fairness constraints in order to put it in relation with the game developed in [2], and we have shown that in the case of memoryless strategies, the users have a winning strategy on the unfolding if, and only if, they have a winning strategy on a derived concurrent game structure.

We now discuss related works and some critical issues in our approach.

After its introduction in [2], ATL was studied and expanded by several authors. In particular, [6] deals with ATL with partial observability and a particular set of fairness constraints, defined as unconditional. For this case, a model-checker of ATL is also available ([12]).

Temporal logics are usually model checked on Kripke models, but some authors also defined procedures to verify formulas directly on Petri nets: [9] and [7] provide model checking algorithms, respectively for LTL and CTL, based on the unfolding of Petri nets; the tool described in [15] exploits algorithms previously developed in order to verify LTL, CTL, and other properties on Petri nets.

Finally, in the last years, other authors defined games on Petri nets in order to study reachability or safety properties on partially controllable concurrent systems. In the game developed in [10], the players are the tokens of the net, and the goal is the avoidance of a certain place. In [11], the goal is the reachability of a target marking, and the moves of the players are strictly alternating on the net. Closer to the game introduced here, in [3] the authors present a two-player game for the reachability of a target transition developed on the case graph of the net; in this game, every player can fire enabled transitions without any alternating rule.

With respect to critical issues in our approach, note that in Def. 9 we require that each event controllable by a user occurring in a consistent play is immediately preceded and followed by a cut; in this way we are able to specify the exact moment in which the event occurred in the play, even if other events may be concurrent with it.

For many goals expressed with ATL this does not create any problem, however there are cases in which these constraints lead to unrealistic scenarios, as illustrated in Ex. 3.

Example 3. The distributed net system in Fig. 4 has two locations: the environment, represented in white and a user $u$, in grey. Let $\psi$ be the ATL formula

$$\psi \equiv (\langle u \rangle) F ((p_0 \land p_3) \lor (p_1 \land p_4)).$$

Consider the strategy $f : Q \rightarrow 2^{T_u}$ defined in this way: $f(\{p_0, p_2\}) = \{t_3\}$, $f(\{p_1, p_2\}) = \{t_2\}$. In each play consistent with the strategy there must be
either an occurrence of $t_2$ or an occurrence of $t_3$; if not, $u$ would be finally postponed, since, when $p_2$ is marked, his strategy is never empty. Then the user $u$ has a winning strategy, because whenever a transition in $T_u$ fires, the formula will be satisfied. This means that there must be a point of the play in which the user is able to make his move before the environment. In a concurrent system this is unrealistic: the two disjoint component of the net system (the left component formed by $t_0$ and $t_1$ and the right formed by $t_2, t_3, t_4$ and $t_5$) behave independently, and there is no reason to impose interleaving between the left component and the transitions of the user.

In addition to that, if we consider the formula:

$$\langle\langle u \rangle\rangle F(p_0 \land p_3),$$ \hspace{1cm} (4)

the user does not have any winning strategy; the strategy with $f(\{p_0, p_2\}) = \{t_3\}$, $f(q) = \emptyset$ for every $q \neq \{p_0, p_2\}$ is not winning, because the move of the user is not finally postponed, since in the marking $\{p_1, p_2\}$ the strategy is empty, and $t_0 t_1$ repeated infinitely often would be a valid execution for a consistent play. Filling the strategy by putting $f(\{p_1, p_2\}) = \{t_3\}$ would not make the strategy winning, because, even if $t_3$ must be in the run, the user cannot be sure that $\{p_3, p_0\}$ is reached. A valid execution would be $t_0, t_3, t_5, t_1, ...$ that never crosses the desired places together; the user cannot impose any order to the transitions of the environment.

Starting from these considerations, we plan to study the existence of a strategy directly on structures that explicitly represent concurrency, such as the unfolding of a net, without forcing sequential executions; on these structures, we will also study formulas with the ‘next’ operator. Another goal is to consider partial observability of the users, dropping the assumption that the users always know the global state of the system. This will help to have a more realistic representation of interactions on concurrent systems.

Finally, we plan to investigate further whether ATL can be successfully used to study properties of concurrent systems, and possibly consider an extension of it to define properties on the nets.
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