The global solvability of initial-boundary value problem for nondiagonal parabolic systems

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Abstract

In this paper we study the quasilinear nondiagonal parabolic type systems. We assume that the principal elliptic operator, which is part of the parabolic system, has a divergence structure. Under certain conditions it is proved the well-posedness of classical solutions, which exist globally in time.

1 Introduction

The main purpose of this paper is to present some techniques and results concerning global existence of classical solutions for nondiagonal parabolic systems. To be precise, let \((t, x) \in \mathbb{R} \times \mathbb{R}^d, (d \in \mathbb{N} \text{ fixed})\), be the points in the time-space domain. Throughout this paper \(\Omega \subset \mathbb{R}^d\) is an open bounded domain of class \(C^1\), \(\mathbf{n} = (n^1, \ldots, n^d)\) is the unitary normal vector field on \(\partial\Omega =: \Gamma\).

For \(T > 0\) and \(N \in \mathbb{N}\), we define \(Q_T := (0, T) \times \Omega\) and consider the vector function \(\mathbf{u} : \overline{Q_T} \to \mathbb{R}^N\), which is supposed to be governed by the following reaction-diffusion system

\[ \partial_t \mathbf{u}_\alpha(t, \mathbf{x}) + \text{div}_x \mathbf{f}_\alpha(t, \mathbf{x}, \mathbf{u}) = g_\alpha(t, \mathbf{x}), \quad (t, \mathbf{x}) \in Q_T, \quad (\alpha = 1, \ldots, N). \]

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where \( f_\alpha \) is a given flux defined by
\[
f^j_\alpha(x, u) := \varphi^j_\alpha(x, u) - A^{jk}_{\alpha\beta}(x, u) \frac{\partial u_\beta}{\partial x_k}, \quad (\alpha, \beta = 1, \ldots, N).
\]

Hereafter, the usual summation convention is used. Moreover, Greek and Latin indices ranges respectively from 1 to \( N \) and from 1 to \( d \). Although, we are not going to enter in physical details, we should mention that there are many physical applications of the above reaction-diffusion system, we list for instance: Flows in porous media, diffusion of polymers, population dynamics, reaction and diffusion in electrolysis, phase transitions, among others.

We shall assume
\[
A^{jk}_{\alpha\beta} \in C^2(\Omega \times \mathbb{R}^N), \quad 0 < \lambda_0 := \inf \{ A^{jk}_{\alpha\beta}(x, v) \xi_\alpha \xi_\beta \},
\]
where the infimum is taken over all \( \xi \in S^{(Nd) - 1} \), \( (S^{(Nd) - 1} \) denotes the unit sphere in \( \mathbb{R}^{Nd} \), and \( (x, v) \in \Omega \times \mathbb{R}^N \). Also
\[
\varphi^j_\alpha \in C^2(\Omega \times \mathbb{R}^N), \quad g_\alpha \in C^2(\Omega \times \mathbb{R}^N).
\]
The parabolic system (1.1) is supplemented with an initial-data
\[
u(0, x) = u_0(x) \in C(\Omega),
\]
and the following types of boundary-conditions on \( \Gamma_T = (0, T) \times \Gamma \): For some \( 0 \leq K \leq N, K \in \mathbb{N} \) be fixed, we set for \( x \in \partial \Omega \)
\[
\alpha = 1, \ldots, K, \quad (f^j_\alpha n^j)(t, x) = u_{ba}(t, x) \quad (\text{Flux condition}),
\]
\[
\alpha = K + 1, \ldots, N, \quad u_\alpha(t, x) = u_{ba}(t, x) \quad (\text{Dirichlet condition}),
\]
where \( u_{ba} \) is a given function, and \( \alpha = 1, \ldots, 0 \) or \( \alpha = N + 1, \ldots, N \), means clearly \( \alpha \equiv 0 \). The regularity of \( u_\alpha \) will be establish below, more precisely, see Theorem 2.1.1 in Section 2 (general theory), Theorem 3.9 in Section 3.2.1 for Dirichlet condition of three-phase capillary-flow type systems.

The local existence of unique classical solution to the parabolic system (1.1)–(1.5) might be proven either via fix point arguments in Hölder space [9], or in weight Hölder space [11], and also via semigroup theory in \( L^p \) space [5]. The important problem to answer is the question, whether this local solution can be continued to be a global solution. It cannot be expected that
it is possible in all circumstances, as certain counterexamples seen to indicate that solutions may start smoothly and even remain bounded, but develop a singularity after finite time, see [18, 17], and also [12], [13]. Although, in some papers, for instance in [4], [6], [8], [14] and [15] the global existence result is proved under some structural conditions. This information leads to the possibility to control some lower-order norms "a-priory".

In the first part of the paper, we introduce our strategy to study global solutions to nondiagonal parabolic systems. We show in Section 2 that, classical solutions exist globally in time provided their orbits are pre-compact in the space of bounded and uniformly continuous functions. First, we assume that $u_0 \in E$, with $E \subset C(\Omega)$, such that, in case of Dirichlet boundary condition, for each $x \in \partial \Omega$, $u_{0\alpha}(x) = u_{ba}(0, x)$, $\alpha = K + 1, \ldots, N$. In the papers [1, 2], it was proved the local existence and uniqueness of solution for (1.1), (1.4) and (1.5), with $u_0 \in E$. Also in the papers [1, 2], it was proved the continuous dependence of solutions of the initial data in $E$. Let us write $u(t, x; u_0)$ the local solution of the problem (1.1), (1.4), (1.5) and $[0, T_{\text{max}})$ the maximal interval of the existence of classical solution. If the set $D := \{u(t, x); t \in (\varepsilon, T_{\text{max}})\}$, 

where $\varepsilon < T_{\text{max}}$ is any fixed positive number, is pre-compact in $C(\overline{\Omega})$, that is to say, $D$ is compact in $C(\overline{\Omega})$, then the solution $u(t, x; u_0)$ is global, i.e. $T_{\text{max}} = \infty$, which is proved in Theorem 2.4. We highlight that, this result is established in a general context. Furthermore, if we have a priori estimate for $u(t, x; u_0)$ in $C^\gamma(\overline{\Omega})$, with $t \in (\varepsilon, T_{\text{max}})$, $\gamma \in (0, 1)$, then the local solution $u(t, x; u_0)$ is global, see Corollary 2.5 below.

Now, we recall an example from O. John and J. Stará [17]. For $d, N = 3$, $\kappa \in (0, 4)$, the real analytic function

$$u(t, x; u_0) = \frac{x}{\sqrt{\kappa(1 - t) + |x|^2}}$$  \hspace{1cm} (1.6)$$

is, for each $t \in [0, 1)$, a classical solution of the system (1.1), with real analytic function $A_{\alpha\beta}^j(u)$ (in a neighbourhood of $\overline{B(0, 1)}$), $\varphi_\alpha = g_\alpha = 0$, and respectively the initial and boundary data

$$u_0(x) = \frac{x}{\sqrt{\kappa + |x|^2}}, \quad u_b(t, x) = \frac{x}{\sqrt{\kappa(1 - t) + 1}} \quad \text{if } |x| = 1.$$  

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This function \((1.6)\) is bounded in \(Q_1\), but at time \(t = 1\), i.e. \(u(1, x; u_0)\) is not continuous at \(x = 0\). Therefore, this example indicates some sharpness for the result establish by Theorem 2.4. It is interesting to observe that, we have in this example from O. John and J. Stará [17], \(\lambda_0 \simeq 0.04 \lambda_1\), with \(\lambda_1\) defined as
\[
\lambda_1 := \sup \left\{ A^{jk}_{\alpha\beta}(x, v) \xi^j \epsilon^k \right\},
\]
where the supremum is taken over all \(\xi \in S^{(Nd)-1}\), and \((x, v) \in \overline{\Omega} \times \mathbb{R}^N\).

On the other hand, if \(\lambda_0 \geq 0.33 \lambda_1\), \((d = 3)\), usually called Cordes type conditions, then E. Kalita [19] proved that the solution \(u(t)\) is bounded in \(C^\gamma(\Omega)\). Therefore, applying Corollary 2.5 we obtain that the solution is global. This could be seen as a first application of our strategy to answer the question whenever the local solution can be continued to be a global solution. For instance, this strategy solves the usual examples that come from physics, where the matrix \(A\) is a perturbation of the identity, that is, \(A = \lambda I_d + \mu B\), where \(\lambda > 0\) is arbitrary and \(\mu\) is a sufficiently small parameter.

In the second part of this paper, i.e. Section 3, we apply our strategy to show classical global solutions to nondiagonal parabolic systems, when the matrix \(A^{jk}_{\alpha\beta}\) is triangular (w.r.t. Greek indexes), this is the major structural assumption. The motivation to study such problems comes from three-phase capillary flow in porous media. Therefore, we consider the parabolic system
\[
(1.1)
\]
with \(N = 2\) and \(d \geq 1\), that is
\[
\begin{align*}
\partial_t u_1 + \text{div}_x \varphi_1(x, u) &= \text{div}_x \left( A_{1\beta}(x, u) \nabla u_\beta \right) + g_1(x, u), \\
\partial_t u_2 + \text{div}_x \varphi_2(x, u) &= \text{div}_x \left( A_{22}(x, u) \nabla u_2 \right) + g_2(x, u),
\end{align*}
\]
where \(A_{21}(x, u) = 0\) (major structural assumption). Others conditions have to be considered, for instance see (3.16)-(3.19), which are used to establish the positively invariant regions (maximum principle), but we stress the following:
\[
\partial A^{jk}_{22}/\partial u_1 \equiv 0.
\]

Applying a different technic focused in a priori estimates in Hölder spaces and the Leray-Schauder’s fixed-point theorem, H. Frid and V. Shelukhin [14] studied the homogeneous case \((f_\alpha = f_\alpha(u))\) of parabolic system \((1.7)\) in one dimension \((d = 1)\), with \(g_\alpha \equiv 0\). In that paper, under the main condition \(A_{21}(u) \equiv 0\), they proved existence and uniqueness of classical solution, see
Theorem 1.1 (flux type condition), and existence of a classical solution with boundary condition assumed in the $L^2$-sense, see Theorem 1.2 (Dirichlet condition). Albeit, they have not considered the condition (1.8), it seems to us that this condition has not been avoided.

Later S. Berres, R. Bürger and H. Frid [10] to a similar (now $N \times N$) parabolic system cited before (that is, in [14]), they showed existence and uniqueness of classical solution, see Theorem 1.1 (perturbed flux condition), and existence of a classical solution with boundary condition assumed in the $L^2$-sense, see Theorem 1.2 (flux condition). Moreover, H. Amann [4] showed that, it is sufficient to have an $L^\infty$ a priori bound with respect to $x$-variable and uniform Hölder continuity in time, with $\gamma > d/(d + 1)$, to guarantee global existence. We should mention that, the second part of our work is neither contained in [4] nor [10, 14].

2 General theory

2.1 Local well-posedness

Let us assume that the initial data of problem (1.1), (1.4), (1.5) belongs to the space $E$, which is constituted by vector functions

$$u_0(x) = (u_0^1(x), \ldots, u_0^N(x)),$$

such that $u_{0\alpha}(x)$ belongs to $C(\overline{\Omega})$. In case of Dirichlet boundary condition, we assume also the agreement condition that $u_{0\alpha}(x) = u_{0\alpha}(0, x)$, for $x \in \partial\Omega$, and $\alpha = K + 1, \ldots, N$ holds for $u_{\alpha}(t, x)$. Therefore, we impose no additional assumption in the case of flux (boundary) condition (1.5) for $u_{\alpha}(t, x)$, except the containment of $u_{0\alpha}(x)$ in space $C(\overline{\Omega})$. The local existence theorem of problem (1.1), (1.4), (1.5) with initial data in $E$ is proved in [1], see also [2].

In order to prove a local existence theorem with initial data in $E$, we need to use the estimates of linear parabolic systems in weighted Hölder classes, obtained by V. Belonosov and T. Zelenjak in one dimensional case [9], and simultaneously by V. Belonosov [7, 8], also V. Solonnikov and A. Khachatryan [25] for parabolic system in several space variables. Let us present these classes as they are related to the case under discussion. Let $f(t, x)$ be a real function defined in $Q_T$. Denote

$$\nabla_y^2 f = f(t, x) - f(t, y), \quad \nabla_t^d f = f(t, x) - f(\tau, x),$$
and suppose \( s \geq 0, \ s \leq r \leq s \). Given a function \( u(t, x) \) which is defined and continuous in \( Q_T := (0, T] \times \Omega \) together with its derivatives \( D_t^\mu D_x^\nu u \) of order \( 2\mu + |\nu| \leq s \), \( \nu = (\nu_1, \ldots, \nu_N) \), where \( \nu_\alpha, (\alpha = 1, \ldots, N) \) are nonnegative integers, with \( |\nu| = \sum_{\alpha=1}^N \nu_\alpha \). Then, we introduce the following semi-norms

\[
[u]_{Q_T}^{r,s;\cdot} = \sup \{ t^{-\frac{\nu}{2}} \frac{|\nabla_x D_t^\mu D_x^\nu u|}{|x-y|^{s-|\nu|}} \},
\]

and

\[
[u]_{Q_T}^{r,s;\cdot} = \sup \{ \theta^{-\frac{\nu}{2}} \frac{|\nabla_t D_t^\mu D_x^\nu u|}{|t-\tau|^{s-2\mu-|\nu|}} \},
\]

where \( \theta = \min\{\tau, t\} \). The supremum in the first semi-norm is taken over all points \((t, x) \neq (t, y)\) of \( Q_T \), and all \( \mu, \nu \) such that \( 2\mu + |\nu| = [s] \). The second supremum is taken over all points \((t, x) \neq (\tau, x)\) of \( Q_T \), and \( \mu, \nu \) such that \( 0 < s - 2\mu - |\nu| < 2 \). Here, \([s]\) denotes the largest integer which is smaller or equal to \( s \).

In addition, for \( k \geq 0 \) be an integer and each \( r \leq k \), we set

\[
|u|_{Q_T}^{r,k} = \sup \{ t^{\frac{k-r}{2}} |D_t^\mu D_x^\nu u| \},
\]

where the supremum is taken over all \((t, x) \in Q_T\) and all values of \( \mu, \nu \) such that, \( 0 < s - 2\mu - |\nu| = k \). Finally, we define

\[
[u]_{Q_T}^{r,s} := [u]_{Q_T}^{r,s;\cdot} + [u]_{Q_T}^{r,s;\cdot}.
\]

We observe that, the symbols \([u]_{Q_T}^{s} \) and \(|u|_{Q_T}^{s}\) will denote respectively the seminorms resulting from \([u]_{Q_T}^{r,s}\) and \(|u|_{Q_T}^{r,k}\), when \( r = s \) and \( r = k \).

Let \( H^s(Q_T) \) be the space of functions \( u(t, x) \) having continuous partial derivatives \( D_t^\mu D_x^\nu u \) of order \( 2\mu + |\nu| \leq s \) in \( Q_T \), and the finite norm

\[
\|u\|_{Q_T}^{s} = [u]_{Q_T}^{s} + \sum_{0 \leq k < s} |u|_{Q_T}^{k}.
\]

This space is also usually denoted by \( H^{s, s/2} \) or \( C^{s, s/2} \), but it is more convenient for us to use the abridged notation \( H^s \). We note that, if \( u \) does not depend on \( t \), then the quantity \( \|u\|_{Q_T}^{s} \) is converted into the norm \( \|u\|_{s}^{s} \) in the standard Hölder space \( C^s(\Omega) \).
Now suppose $s \geq 0$ and $r \leq s$. When $r$ is not integer, the space $H^s_r(Q_T)$ is the set of functions $u(t,x)$ having continuous derivatives $D^\mu_i D^\nu_j u$ of order $2\mu_i + |\nu_j| \leq s$ in $Q'_T$, and the finite norms

$$
\|u\|_{r,s}^Q = [u]_{r,s}^Q + \sum_{r<k<s} |u|_{r,s}^Q + \|u\|_{r}^Q,
$$

$$
\|u\|_{r,s}^{Q_T} = [u]_{r,s}^{Q_T} + \sum_{0\leq k<s} |u|_{r,s}^{Q_T},
$$

respectively when $r \geq 0$, and $r < 0$. Moreover, when $r$ is an integer the space $H^s_r(Q_T)$ is defined as the completion of $H^s(Q_T)$ with respect to the above norm.

Finally, the space $H^s_r(\Gamma_T)$ of functions defined on the lateral surface of the cylinder $Q_T$ is defined as the set of traces on $\Gamma_T$ of functions in $H^s_r(Q_T)$. The norm in this space is defined by the following quantity

$$
\|\varphi\|_{r,s}^{\Gamma_T} = \inf \|\Phi\|_{r,s}^{Q_T},
$$

where the infimum is taken over all $\Phi \in H^s_r(Q_T)$ coinciding with $\varphi$ on $\Gamma_T$.

Our first result is based on two theorems which are proven in [1] (see also [2]):

**Theorem 2.1.** (Local existence with initial data from $E$) Suppose that the problem (1.1), (1.4) and (1.5) satisfies assumptions (1.2), (1.3), with $u_0 \in E$ and $u_{0\alpha} \in H^{\gamma(\alpha)}_0(\Gamma_T)$,

$$
\gamma(\alpha) = \begin{cases}
1 + \gamma, & \alpha = 1, \ldots, K, \\
2 + \gamma, & \alpha = K + 1, \ldots, N,
\end{cases}
$$

with $0 < \gamma < 1$. Then, there exists a positive number $T$ such that, the problem (1.1), (1.4) and (1.5) has a unique classical solution

$$
u(t,x;u_0) \in H^{2+\gamma}_0(\overline{Q_T}).
$$

The positive constant $T$ depends on the value $\|u_0\|_0^\Omega$ and the modulus of continuity of the function $u_0(x)$. 

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Theorem 2.2. (The continuous dependence of solutions) Suppose that all conditions of Theorem 2.1 are satisfied, and \( u(t, x; u_0) \) is the classical solution to problem (1.1)–(1.5) in \( Q_T \) given by Theorem 2.1. Then, there exists a positive number \( \delta > 0 \), such that, if \( v_0 \in E, v_{ba} \in H^0(\alpha) \), satisfy
\[
\|u_0 - v_0\|_0^\Omega + \|u_{ba} - v_{ba}\|_{0,\gamma(\alpha)}^{\Gamma_T} < \delta
\]
and also the compatibility conditions, then there exists a classical solution \( v(t, x; v_0) \) to problem (1.1)–(1.5) in \( Q_T \), with the initial and boundary data respectively \( v_0 \) and \( v_b \). Moreover, it follows that
\[
\|u(t, x; u_0) - v(t, x; v_0)\|_{0,2+\gamma}^{Q_T} \leq C \|u_0 - v_0\|_0^\Omega + \|u_{ba} - v_{ba}\|_{0,\gamma(\alpha)}^{\Gamma_T}, \quad (2.9)
\]
where the constant \( C \) does not depend on \( v_0 \) and \( v_b \).

We remark that, the proofs of the above theorems are slight modifications to that ones, respectively given for Theorem 1 and Theorem 2 in [1].

Remark 2.3. 1. First, the example recalled from [17] shows that, we may not neglect the dependence of \( T > 0 \) on the modulus of continuity of \( u_0 \).

2. We can consider the problem (1.1), (1.5) and the initial condition \( u_0(x) = u(\tau, x) \), with \( u_{0\alpha}(x) = u_{ba}(\tau, x) \) for \( \alpha = K + 1, \ldots, N \), and each \( x \in \partial \Omega \), such that, the existence time interval does not depend on \( \tau \in [0, T] \).

2.2 Global solutions

To prove the global solvability of the problem (1.1), (1.4) and (1.5) in the space \( H^{2+\gamma}(Q_T) \), we use the result from Section 2.1 related to time local classical solvability of problem (1.1), (1.4) and (1.5) in \( H^{2+\gamma}(Q_T) \). Denote by \([0, T_{\text{max}})\) the maximal existence interval for the smooth solution \( u(t, x) \). Then, for all \( t \in (\varepsilon, T_{\text{max}}) \), we have \( u(t) \in C^{2+\gamma}(\Omega) \), and \( |u| \leq M \).

Theorem 2.4. If \( \mathcal{D} \) is compact in \( C(\overline{\Omega}) \), then \( T_{\text{max}} = \infty \) or equivalently, for any \( T > 0 \) the problem (1.1), (1.4) and (1.5) has global solution in \( Q_T \).

Proof. First, let us suppose that \( T_{\text{max}} < \infty \). Then, we set \( u_n(x) = u(t_n, x) \), with \( t_n \to T_{\text{max}} \) (monotonically crescent). Since \( \mathcal{D} \) is compact by hypothesis, hence the sequence contains a subsequence \( u_{n_k}(x) \), which converges in \( C(\overline{\Omega}) \). Denote this subsequence also by \( u_n(x) \), and
\[
u_n(x) \to \bar{u}_0(x) \quad \text{in} \quad C(\overline{\Omega}).\]
From Theorem 2.1 there exists $\sigma > 0$, such that the problem (1.1)–(1.5) has a classical solution $u(t, x; \tilde{u}_0)$ if $t \in [T_{\text{max}}, T_{\text{max}} + \sigma]$. Moreover, it follows from Theorem 2.2 that, for $n$ sufficiently large the classical solution of (1.1), (1.4) and (1.5) (i.e. with initial data $u_n(x) = u(t_n, x)$) exists if $t \in [t_n, t_n + \sigma]$. Therefore, the function

$$u(t, x) = \begin{cases} u(t, x; u_0) & \text{if } t \in [0, t_n], \\ u(t, x; u_n) & \text{if } t \in [t_n, t_n + \sigma], \end{cases}$$

is a classical solution of (1.1)–(1.5). Consequently, for $n$ sufficiently large, $t_n + \sigma > T_{\text{max}}$, which is a contradiction. $\square$

**Corollary 2.5.** If $\gamma > 0$ and $\Omega$ is bounded set in $C^\gamma(\Omega)$, then the problem (1.1), (1.4) and (1.5) has global solution in $Q_T$.

For each $\varepsilon > 0$, we denote hereafter $Q_\varepsilon := (\varepsilon, T) \times \Omega$, also $\Gamma_\varepsilon := (\varepsilon, T) \times \Gamma$.

Then, another consequence of Theorem 2.4 is the following

**Proposition 2.6.** Let $T > 0$ be arbitrary and conditions (1.2), (1.3) hold. Assume that $u_\alpha(t, x)$ is a classical solution for the initial-boundary value problem (1.1), (1.4) and (1.5) in $Q_T$, with $u_\alpha \in C(Q_T)$. Then, there exists $\gamma \in (0, 1)$, such that, the unique classical solution exists globally in time in $Q_T$, and for each $\varepsilon > 0$, $u_\alpha \in C^{2+\gamma, 1+\frac{\gamma}{2}}(Q_\varepsilon)$.

**Proof.** The proof follows from Theorem 2.4 and the definition of the weighted spaces. $\square$

### 3 Three-phase capillary-flow type systems

In this part of the article, we establish the existence (global) and uniqueness theorem for the IBVP (1.1)–(1.5), when $N = 2$ and the system admits some additional conditions, to be precisely below. Here to avoid more technicality, we assume zero-flux boundary condition. This system is motivated by one-dimensional three-phase capillary flow through porous medium (e.g. oil, water and gas), which is related for instance to planning operation of oil wells. Here, we are not going to enter in more physical details. The interesting reader is addressed to H. Frid and V. Shelukhin [14], in order to obtain more information about the laws of multi-phase flows in a porous medium, and also applications.
First, we define for $\eta \in \mathbb{R}^d$

$$
\Lambda_{\alpha\beta}(x, u, \eta) := A^{jk}_{\alpha\beta}(x, u) \eta^j \eta^k,
$$

$$
\Lambda^u_{\alpha\beta}(x, u, \eta) := A^{jk}_{\alpha\beta}(x, u) \eta^j \eta^k.
$$

(3.10)

Then, we have the following

**Definition 3.1.** The family $\{A^{jk}_{\alpha\beta}\}$ is called **normally elliptic** on $Q_T$, when

i) For each $(t, x) \in \overline{Q}_T$, and all $\eta \in S^{d-1}$

$$
\sigma(\Lambda_{\alpha\beta}(x, u, \eta)) \subset [Re(z) > 0] \equiv \{z \in \mathbb{C} : Re(z) > 0\},
$$

(3.11)

where $u = u(t, x)$, and $\sigma(M)$ denotes the spectrum of the matrix $M$.

ii) For each $r \in \Gamma$, $\xi \in T_r\Gamma$, $|\xi| = 1$ and $\omega \in [Re(z) \geq 0]$, with $\omega \neq 0$, zero is the unique exponentially decaying solution of the BVP

$$
[\omega + \Lambda_{\alpha\beta}(x, u, \eta + n \, i\partial_t)]u = 0 \quad \text{on } \Gamma_T,
$$

$$
\Lambda^u_{\alpha\beta}(x, u, \eta + n \, i\partial_t)u_0 = 0 \quad \text{on } \Gamma,
$$

(3.12)

where $i$ is the imaginary number. Moreover, the system (1.1) is said parabolic in Petrovskii-sense and admits the Lopatinski’s compatibility condition, when the family $\{A^{jk}_{\alpha\beta}\}$ respectively satisfies (3.11) and (3.12) (see, for instance, §8 Chapter VII in Ladyzenskaja, Solonikov and Ural’ceva [20]).

In particular, a simple case of normally elliptic family occurs when the family $\{A^{jk}_{\alpha\beta}\}$ is triangular w.r.t. the Greek indexes, for instance upper triangular. Hence in this case, $\Lambda$ and $\Lambda^u$ are also upper triangular and conditions (3.11), (3.12) holds true if, and only if, there exists $\mu_0 > 0$, such that

$$
\Lambda_{\alpha\alpha}(x, u, \eta) \geq \mu_0,
$$

which is satisfied in our case, since from assumption (12) we have

$$
\Lambda_{\alpha\alpha}(x, u, \eta) = A^{jk}_{\alpha\alpha}(x, u) \eta^j \eta^k \geq \lambda_0 > 0.
$$

Therefore, from this point we shall consider

$$
A^{jk}_{21}(x, u) \equiv 0.
$$

(3.13)
Now, we consider the following domain

\[ B_\Delta := \{ (u_1, u_2) \in \mathbb{R}^2 : 0 \leq u_1, u_2 \leq 1, \ u_1 + u_2 \leq 1 \}, \quad (3.14) \]

which is motivated by the applications. Then, we assume

\[ u_0(x), u_b(t, x) \in B_\Delta \quad \text{for each } x \in \overline{\Omega} \text{ and } t \geq 0. \quad (3.15) \]

Following Serre’s book [22], Vol 2 (Chapter 1), we may have the triangle \( B_\Delta \) written also by the intersection of the following functions

\[ G_1(u) = -u_1, \quad G_2(u) = -u_2, \quad G_3(u) = -1 + u_1 + u_2, \]

that is,

\[ B_\Delta \equiv \bigcap_{h=1}^3 \{ u \in \mathbb{R}^2 : G_h(u) \leq 0 \}, \quad \partial B_\Delta = \{ u \in \overline{B_\Delta} : G_h(u) = 0, h = 1, 2, 3 \}. \]

We seek for functions \( u(t, x) \) solutions of the IBVP (1.1)–(1.5), such that \( u(t, x) \in B_\Delta \), which is satisfied under some additional hypotheses on \( \varphi^{j}_\alpha \), \( A^{j}_{\alpha\beta} \) and \( g_\alpha \), which is to say, for each \( x \in \overline{\Omega} \), and all \( u \in \partial B_\Delta \), \( (h = 1, 2, 3, \text{here no summation on indices } h) \)

\[ \frac{\partial \varphi^{j}_\alpha(x, u)}{\partial u_\beta} \frac{\partial G_h(u)}{\partial u_\alpha} = \lambda^j_h(x, u) \frac{\partial G_h(u)}{\partial u_\beta}, \quad (3.16) \]

\[ A^{j}_{\alpha\beta}(x, u) \frac{\partial G_h(u)}{\partial u_\alpha} = \mu^{j}_{h}(x, u) \frac{\partial G_h(u)}{\partial u_\beta}, \quad (3.17) \]

\[ (g_\alpha(x, u) - \gamma_\alpha(x, u)) \frac{\partial G_h(u)}{\partial u_\alpha} \leq 0, \quad (3.18) \]

\[ (\varphi^{j}_\alpha n^j)|_{\{u_1=0\}} = \left( (\varphi^{j}_1 + \varphi^{j}_2) n^j \right)|_{\{u_1+u_2=0\}} \equiv 0, \quad (3.19) \]

for some functions \( \lambda^j_h \) and \( \mu^{j}_{h} \), where \( \gamma_\alpha(x, v) := \partial_{x_j}\varphi^{j}_\alpha(x, v) \). Remark that, for \( h = 1, 2, 3, \mu^{j}_{h} \eta^j \eta^k > 0 \).

Due to assumptions (3.16)–(3.19) we can adapt the theory of (positively) invariant regions for nonlinear parabolic systems developed by Chuey, Conley and Smoller [11], see also the cited book of Serre and the Smoller’s book [23].
One remarks that, conditions (3.16), (3.17) holds true for instance in three-phase capillary flow in porous media (black oil model), further in this case, (3.18) is trivially satisfied.

Under the above considerations, let us assume that the solution \( u_\alpha(t, x) \) of the system (1.1)-(1.5) exists for each \( t \in [0, T_{\text{max}}) \), with \( T_{\text{max}} < \infty \). If we show that \( u_\alpha(t, x) \in C^\gamma(Q_{T_{\text{max}}}), \gamma \in (0, 1) \), then from Corollary 2.5 it follows that \( T_{\text{max}} = \infty \), which is a contradiction. Hence \( T_{\text{max}} = \infty \) and we have classical global existence theorem. In fact, to establish a global result we need first a uniform estimate, which will be given in the next section.

### 3.1 Positively invariant regions

We consider the parabolic system (1.7) with the initial-boundary data (1.4) and (1.5). Let \( v = (v_1, v_2) \) be a constant vector, such that \( v_1, v_2 > 0 \) and \( v_1 + v_2 < 1 \). For each \( \epsilon > 0 \), we regard the following approximated system

\[
\partial_t u_\epsilon^\alpha + \text{div}_x \varphi_\alpha(x, u^\epsilon) = \text{div}_x (A_{\alpha\beta}(x, u^\epsilon) \nabla u_\beta^\epsilon) + g_\epsilon^\alpha(x, u^\epsilon),
\]

with the initial condition in \( \Omega \)

\[
u^\epsilon(0, x) = u_0^\epsilon(x),
\]

and the boundary conditions on \( \Gamma_T \)

\[
(\varphi^i_\alpha(x, u^\epsilon) - A^{j^i}_{\alpha\beta}(x, u^\epsilon) \frac{\partial u_\beta^\epsilon}{\partial x_k}) n^j = \epsilon \ A^{j^k}_{\alpha\beta}(x, u^\epsilon) n^j \eta^k (u_\beta^\epsilon - u_b^\epsilon), \quad \text{or} \quad u_\alpha^\epsilon = u_b^\epsilon,
\]

where \( g_\epsilon^\alpha(x, u^\epsilon) = g_\alpha(x, u^\epsilon) + \epsilon(v_\alpha - u_\alpha^\epsilon) \), \( u_0^\epsilon := (1 - \epsilon) \ (u_0 + \epsilon/2) \) and similarly \( u_b^\epsilon := (1 - \epsilon) \ (u_b + \epsilon/2) \).

Let \( T > 0 \) be given. In order to show the existence of positively invariant regions, we assume that, for each \( \epsilon > 0 \), there exists \( u_\epsilon^\alpha \in C^{2,1}(Q_T) \) the unique solution to (3.20)-(3.22), see Proposition 2.6. Then, we have the following

**Lemma 3.2.** Let \( T > 0 \) be given. If \( u_0, u_b \) take values in \( B_\Delta \), then the unique solution \( u^\epsilon \) of (3.20)-(3.22) take values in the interior of \( B_\Delta \), i.e. \( u^\epsilon(t, x) \in \text{int} B_\Delta \), for each \( (t, x) \in Q_T \).
Proof. For simplicity, we drop throughout the proof the superscript \( \epsilon \), whenever it is not strictly necessarily. First, let us denote \( Z_h(t, x) = G_h(u(t, x)) \), \((h = 1, 2, 3)\). Since \( u_0^h \in \text{int}B_\Delta \), we have
\[
\sup_{x \in \Omega} Z_h(0, x) < 0, \quad (h = 1, 2, 3).
\]

By contradiction, suppose that there exists \( t_0 > 0 \), the first time, such that
\[
Z_h(t_0, x_0) := \sup_{x \in \Omega} Z_h(t_0, x) = 0,
\]
for some \( h = 1, 2 \) or 3. That is to say, \( u(t_0, x_0) \in \partial B_\Delta \), where \( x_0 \in \overline{\Omega} \) and
\[
\sup_{x \in \Omega} Z_h(t, x) < 0 \quad \text{for all} \quad 0 \leq t < t_0.
\]

If \( x_0 \in \Omega \), then it follows by (3.20) at \((t_0, x_0)\) that
\[
\partial_t u_\alpha + \frac{\partial \varphi^j_{\alpha}}{\partial u_\beta} \frac{\partial u_\beta}{\partial x_j} + \gamma_\alpha = \frac{\partial}{\partial x_j} \left( A_{\alpha\beta}^{jk}(x, u) \frac{\partial u_\beta}{\partial x_k} \right) + g_\alpha + \epsilon \left( v_\alpha - u_\alpha \right).
\]

Now, we multiply the above equation by \( \partial G_h/\partial u_\alpha \) and applying conditions (3.16) and (3.17), we have
\[
\partial_t Z_h + \lambda^j_h \frac{\partial Z_h}{\partial x_j} + \mu^j_h \frac{\partial^2 Z_h}{\partial x_j \partial x_k} = \frac{\partial G_h}{\partial u_\alpha} (g_\alpha - \gamma_\alpha) + \epsilon \frac{\partial G_h}{\partial u_\alpha} (v_\alpha - u_\alpha). \tag{3.23}
\]

By assumption of the contradiction, we must have at \((t_0, x_0)\)
\[
\partial_t Z_h \geq 0, \quad \nabla_x Z_h = 0, \quad \Delta_x Z_h \leq 0,
\]
hence by (3.23), using (3.18) and since \( \mu^j_h \eta^i \eta^k > 0 \), we obtain the following contradiction
\[
0 \leq \partial_t Z_h \leq \epsilon \frac{\partial G_h}{\partial u_\alpha} (v_\alpha - u_\alpha) < 0.
\]

Indeed, suppose for instance \( Z_1(t_0, x_0) = 0 \), hence \( u_1(t_0, x_0) = 0 \) and then, we have
\[
0 \leq \partial_t Z_1 \leq \epsilon (-1)(v_1 - 0) < 0.
\]
Analogous result when $h = 2$. For $h = 3$, we have $u_1(t_0, x_0) + u_2(t_0, x_0) = 1$, therefore

$$0 \leq \partial_t Z_3 \leq \epsilon (v_1 + v_2 - 1) < 0.$$ 

It remains to show the case $x_0 \in \Gamma$, and let us first consider the simple case of Dirichlet boundary condition. Indeed, the cases $h = 1, 2$ are trivial, which is to say, we obtain the contradiction $u_{\alpha x} < 0$. For $h = 3$, it follows that

$$0 = (1 - \epsilon)(u_{b1} + \epsilon/2) - u_1 + (1 - \epsilon)(u_{b1} + \epsilon/2) - u_2 = (u_{b1} + u_{b2} - 1) + (\epsilon - \epsilon(u_{b1} + u_{b2})) - \epsilon^2.$$ 

It is enough to consider the limit possibilities, that is, $u_{b1} + u_{b2} = 0$ and $u_{b1} + u_{b2} = 1$, and the result follows. Finally, we consider the flux condition. Recall that, the flux condition is given by

$$\left( \varphi'_\alpha(x, u) - A_{\alpha\beta}^j(x, u) \frac{\partial u_\beta}{\partial x_k} \right) n^j = \epsilon A_{\alpha\beta}^j(x, u) n^j \eta^k (u_\beta - u_{b\beta}).$$

Thus multiplying the above equation by $\partial G_{h}/\partial u_\alpha$, we obtain

$$\mu_h^{jk} n^j \frac{\partial Z_h}{\partial x_k} = \epsilon \mu_h^{jk} n^j \eta^k \frac{\partial G_h}{\partial u_\beta} (u_{b\beta} - u_\beta),$$

where we have used (3.19). From a similar argument used before for Dirichlet condition, we derive the following contradiction

$$0 \leq \mu_h^{jk} n^j \frac{\partial Z_h}{\partial x_k} < 0,$$

which finish the proof. 

\textbf{Corollary 3.3.} \textit{Since the estimates in Lemma 3.2 for $u'(t, x)$ depends continuously on $\epsilon > 0$, therefore passing to the limit as $\epsilon$ goes to 0+, we obtain that $u(t, x) \in B_{\Delta}$.}

\textbf{3.2 Main results}

In this section we consider the problem (3.20)-(3.22) with $\epsilon = 0$, that is

$$\partial_t u_\alpha + \text{div}_x \varphi_\alpha(x, u) = \text{div}_x \left( A_{\alpha\beta}(x, u) \nabla_x u_\beta \right) + g_\alpha(x, u), \quad (3.24)$$
the initial condition in $\Omega$

$$u(0, x) = u_0(x), \quad (3.25)$$

and the boundary conditions on $\Gamma_T$

$$(\varphi^j_\alpha(x, u) - A^{jk}_{\alpha\beta}(x, u) \frac{\partial u_\beta}{\partial x_k}) n^j = 0, \quad \text{or} \quad u_\alpha = u_{b\alpha}. \quad (3.26)$$

Under conditions (3.15)–(3.19), we proved that $u \in B_\Delta$. To prove the global solvability of problem (3.24)–(3.26), we use first Theorem 2.1 about time-local classical solvability of this problem in $H_0^{2+\gamma}(Q_T)$. Denote by $[0, T_{\max})$ the maximal existence interval of solution from Theorem 2.1. Let us assume that $T_{\max} < \infty$. If we prove that $u_\alpha \in C^\gamma(Q_{T_{\max}})$ and $\|u(t)\|_\gamma^\Omega \leq C$, with $t \in [\varepsilon, T_{\max})$, it follows from the proof of Theorem 2.1 that there exists $\delta > 0$, such that, the solution $u(t, x)$ exists on $[0, T_{\max} + \delta)$, which is a contradiction of the definition of $T_{\max}$.

Let us denote by $V_{2,0}^{1,0}(Q_T)$ the space of functions with finite norm

$$\sup_{[0, T]} \|u(t, \cdot)\|_{2, \Omega} + \|\nabla x u\|_{2, Q_T}.$$ 

It shall be understood that, the vector-value function $u$ belongs to $V_{2,0}^{1,0}(Q_T)$, if $u_\alpha \in V_{2,0}^{1,0}(Q_T)$ for $\alpha = 1, 2$.

**Lemma 3.4.** Consider the initial-boundary value problem (3.24)–(3.26), and assume the conditions (3.13)–(3.19). If $u_0 \in E$ and $u_{b} \in H_0^{2+\gamma}(\Gamma_T)$, with $\gamma \in (0, 1)$, then $u_\alpha \in V_{2,0}^{1,0}(Q_{T_{\max}})$, $\alpha = 1, 2$.

**Proof.** 1. If $u_{b\alpha} = 0$ or we have zero-flux boundary condition, then multiplying (3.24) by $u_\alpha$ and integrating in $\Omega$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega (u_\alpha)^2 dx + \int_\Omega A^{jk}_{\alpha\beta} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\alpha}{\partial x_j} dx = \int_\Omega \varphi^j_\alpha \frac{\partial u_\alpha}{\partial x_j} dx + \int_\Omega g_\alpha u_\alpha dx.$$ 

Therefore, from (1.2) and $u_\alpha \in B_\Delta$, there exists a positive constant $C_1$, such that

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |u_\alpha|^2 dx + \frac{\lambda_0}{2} \int_\Omega |\nabla x u_\alpha|^2 dx \leq C_1.$$

Then, we have for each $t \in [0, T_{\max}]$

$$\int_\Omega |u_\alpha(t)|^2 dx + \lambda_0 \int_{Q_{T_{\max}}} |\nabla x u_\alpha(t)|^2 dx \leq 2C_1 + \int_\Omega |u_0\alpha|^2 dx =: C_2, \quad (3.27)$$

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from which it follows that \( u_\alpha \in V_2^{1,0}(Q_{T_{\max}}) \). Observe that \( C_2 \) does not depend on \( t < T_{\max} \).

2. Now, if \( u_{b\alpha} \neq 0 \), then let us consider the system
\[
\begin{align*}
\partial_t v &= \text{div}_x(\nabla_x v) \quad \text{in } Q_{T_{\max}}; \\
v &= u_0 \quad \text{on } \Gamma_{T_{\max}}; \\
v|_{t=0} &= v_0 \quad \text{in } \Omega,
\end{align*}
\]
where \( v_0 \) is a function in \( C(\bar{\Omega}) \), satisfying the compatibility condition
\[
v_{0\alpha}(x) = u_{b\alpha}(0, x),\]
for \( x \in \partial\Omega \). It is clear that, \( v \in H^{2+\gamma}_0(Q_{T_{\max}}) \), and making \( \tilde{u} = u - v \), we apply the same argument as in item 1, multiplying now by \( \tilde{u} \).

Before we pass to consider separated the Dirichlet and flux conditions, let us consider another important restriction, which will be used in particular for \( d \geq 2 \) with the Dirichlet boundary condition, that is
\[
\partial \varphi^j_2 / \partial u_1 \equiv 0. \tag{3.28}
\]

### 3.2.1 Dirichlet condition

First, let us consider Dirichlet boundary condition in (3.26), i.e. for \( \alpha = 1, 2 \). To begin, we work with equation (3.24), when \( \alpha = 2 \).

**Lemma 3.5.** Let \( 0 < \varepsilon < T_{\max} \) be fixed and consider for \( \alpha = 2 \) the initial-boundary value problem (3.24)–(3.26) (Dirichlet condition), with \( u_{02} \in E \) and \( u_{b2} \in H^{2+\gamma}_0(\Gamma_T) \), \( \gamma \in (0, 1) \). Assume the conditions (3.13), (3.15)–(3.19). Then, there exists \( \gamma_1 \in (0, \gamma) \), such that
\[
u_2 \in C^{2+\gamma_1,1+\gamma_1/2}(Q_{T_{\max}} \cup \Gamma_{T_{\max}}^\varepsilon) \quad \text{and} \quad |u_2|_{Q_{T_{\max}}^\varepsilon} \leq C_2,
\]
where \( C_2 \) is independent of \( t < T_{\max} \).

**Proof.** First, from Lemma 3.4 we have \( u_2 \in V_2^{1,0}(Q_{T_{\max}}) \). The thesis follows applying Theorem 10.1 in Chapter III (linear theory) from Ladyzenskaja, Solonikov and Ural’ceva [20], with
\[
a_{jk}(t, x) = A_{jk}^\varepsilon(x, u_1(t, x), u_2(t, x)), \quad f_j(t, x) = \varphi^j_2(x, u_1(t, x), u_2(t, x)),
\]
also
\[ f(t, x) = g_2(x, u_1(t, x), u_2(t, x)), \quad a_j = b_j = a = 0. \]

Since \( u \in B_\Delta \), and \( f, f_j \) are uniformly bounded, for each \( d \geq 1 \), there exist positive \( q \) and \( r \), such that
\[ \frac{1}{r} + \frac{d}{2q} = 1 - \kappa_1, \quad \kappa_1 \in (0, 1) \]
and \( \|f^2_j, f\|_{q,r,Q_{T_{\max}}} \leq C \), where \( C \) is a positive constant independent of \( t < T_{\max} \).

**Proposition 3.6.** Under conditions of Lemma 3.5, and also condition (1.8), then for \( t < T_{\max} \) and \( j = 1, \ldots, d \)
\[ \int_{\varepsilon}^{t} \int_{\Omega} |\partial_{x_j} u_2(\tau, x)|^4 \, d\tau \, dx \leq C, \]
where \( C \) does not dependent on \( t \).

**Proof.**

1. First, we consider without loss of generality that, \( u_{2\rho} = 0 \). Otherwise, we proceed exactly as in the proof of Lemma 3.4. For \( \rho > 0 \) and any \( x_0 \in \Omega \), let \( \Omega_{2\rho} = B_{2\rho}(x_0) \cap \Omega \). Let \( 0 \leq \zeta(t, x) \leq 1 \) be a smooth function, such that, for each \( t \in (\varepsilon, T_{\max}) \), \( 0 < \varepsilon < T_{\max} \), \( \zeta(t, x) \equiv 0 \) in \( (\varepsilon, T_{\max}) \times (\Omega \setminus B_{2\rho}) \cup ((0, \varepsilon) \times \Omega) \) and \( \zeta(t, x) \equiv 1 \) in \( (\varepsilon, T_{\max}) \times \Omega_{\rho} \), with the obvious notation.

2. For \( \alpha = 2 \), we multiply equation (3.24) by
\[ \frac{\partial}{\partial x_l}(\frac{\partial}{\partial x_l} u_2 \zeta^2), \]
and integrating in \( (\varepsilon, t) \times \Omega_{2\rho} \), it follows that
\[ 0 = \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} \left( - \partial_t u_2 + \frac{\partial}{\partial x_j}(A_{jk}^{22} \frac{\partial u_2}{\partial x_k} - \varphi_j^2) + g_2 \right) \frac{\partial}{\partial x_l}(\frac{\partial}{\partial x_l} u_2 \zeta^2) \, dx \, dt \]
\[ = \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} u_{2x_l} u_{2x_1} \zeta^2 \, dx \, dt \]
\[ + \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} \frac{\partial}{\partial x_l}(A_{jk}^{22} u_{2x_k}) \frac{\partial}{\partial x_j}(u_{2x_1} \zeta^2) \, dx \, dt \]
\[ - \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} \frac{\partial}{\partial x_l}(\varphi_j^2) \frac{\partial}{\partial x_j}(u_{2x_1} \zeta^2) \, dx \, dt + \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} g_2 \frac{\partial}{\partial x_l}(u_{2x_1} \zeta^2) \, dx \, dt \]
\[ =: I_1 + I_2 - I_3 + I_4. \]
Then, we have

\[ I_1 = \left( \frac{1}{2} \int_{\Omega_{2\rho}} u_{2x_i}^2 \, dx \right)^t - \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} u_{2x_i}^2 \, dx dt. \]  

(3.29)

Moreover, from \( I_2 \) we obtain

\[ I_2 = \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} A_{22}^{jk} \frac{\partial^2 u_2}{\partial x_k \partial x_l} \frac{\partial^2 u_2}{\partial x_l \partial x_j} \zeta^2 \, dx dt + \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} A_{22}^{jk} \frac{\partial^2 u_2}{\partial x_k \partial x_l} \, u_{2x_i}^2 \zeta \zeta_j \, dx dt \]

\[ + \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} \frac{\partial A_{22}^{jk}}{\partial u_2} \, u_{2x_i}^2 \frac{\partial^2 u_2}{\partial x_l \partial x_j} \zeta^2 \, dx dt \]

\[ + \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} \frac{\partial A_{22}^{jk}}{\partial u_2} \, u_{2x_k} \, u_{2x_k} \, u_{2x_i} \zeta \zeta_j \, dx dt \]

\[ + \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} \frac{\partial A_{22}^{jk}}{\partial x_l} \, u_{2x_k} \frac{\partial^2 u_2}{\partial x_l \partial x_j} \zeta^2 \, dx dt + \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} \frac{\partial A_{22}^{jk}}{\partial x_l} \, u_{2x_k} \, u_{2x_i} \, u_{2x_i} \, \zeta \zeta_j \, dx dt \]

\[ =: J_1 + K_1 + K_2 + K_3 + K_4 + K_5. \]

Denoting by \( D_x^2 u_2 \) the Hessian of the function \( u_2 \), we consider the following estimates:

\[ J_1 \geq \lambda_0^2 \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} |D_x^2 u_2|^2 \zeta^2 \, dx dt, \]

\[ |K_1| \leq \varepsilon_1 \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} |D_x^2 u_2|^2 \zeta^2 \, dx dt + C_1, \]

\[ |K_2| \leq \varepsilon_2 \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} |D_x^2 u_2|^2 \zeta^2 \, dx dt + C_2 \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} |\nabla_x u_2|^4 \, dx dt, \]

\[ |K_3| \leq \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} |\nabla_x u_2|^4 \zeta^2 \, dx dt + C_3, \]

\[ |K_4| \leq \varepsilon_4 \int_{\varepsilon}^{t} \int_{\Omega_{2\rho}} |D_x^2 u_2|^2 \zeta^2 \, dx dt + C_4, \quad |K_5| \leq C_5, \]

where we have used Lemma 3.4, the condition (1.2) and the generalized
Young’s inequality. Therefore, from the above estimates

\[ I_1 + J_1 \leq |I_3| + |I_4| + \sum_{i=1}^{5} |K_i| \leq |I_3| + |I_4| \]

\[ + \varepsilon_6 \int_t^t |D_x^2 u_2|^2 \, \zeta^2 \, dxdt + C_6 \int_t^t |\nabla_x u_2|^4 \, \zeta^2 \, dxdt + C_7. \]

(3.30)

Now, we proceed to estimate \( I_3 \), first

\[ I_3 = \int_t^t \int_{\Omega_2^p} \left( \frac{\partial^2 \varphi_2^{j}}{\partial u_1} u_{1x_i} \frac{\partial^2 u_2}{\partial x_i \partial x_j} \zeta^2 + \frac{\partial \varphi_2^{j}}{\partial u_1} u_{1x_i} u_{2x_i} 2\zeta \zeta_x \right) \]

\[ + \frac{\partial \varphi_2^{j}}{\partial u_2} u_{2x_i} \frac{\partial^2 u_2}{\partial x_i \partial x_j} \zeta^2 + \frac{\partial \varphi_2^{j}}{\partial u_2} u_{2x_i} u_{2x_i} 2\zeta \zeta_x \]

\[ + \frac{\partial \varphi_2^{j}}{\partial x_i} \frac{\partial^2 u_2}{\partial x_i \partial x_j} \zeta^2 + \frac{\partial \varphi_2^{j}}{\partial x_i} u_{2x_i} 2\zeta \zeta_x \right) \right) dxdt, \]

then we consider the following estimates:

\[ \int_t^t \int_{\Omega_2^p} \frac{\partial \varphi_2^{j}}{\partial u_1} u_{1x_i} \frac{\partial^2 u_2}{\partial x_i \partial x_j} \zeta^2 \, dxdt \leq \varepsilon_8 \int_t^t \int_{\Omega_2^p} |D_x^2 u_2|^2 \, \zeta^2 \, dxdt + C_8, \]

\[ \int_t^t \int_{\Omega_2^p} \frac{\partial \varphi_2^{j}}{\partial u_2} u_{2x_i} \frac{\partial^2 u_2}{\partial x_i \partial x_j} \zeta^2 \, dxdt \leq \varepsilon_9 \int_t^t \int_{\Omega_2^p} |D_x^2 u_2|^2 \, \zeta^2 \, dxdt + C_9, \]

\[ \int_t^t \int_{\Omega_2^p} \frac{\partial \varphi_2^{j}}{\partial x_i} \frac{\partial^2 u_2}{\partial x_i \partial x_j} \zeta^2 \, dxdt \leq \varepsilon_{10} \int_t^t \int_{\Omega_2^p} |D_x^2 u_2|^2 \, \zeta^2 \, dxdt + C_{10}, \]

\[ \int_t^t \int_{\Omega_2^p} \left( \frac{\partial \varphi_2^{j}}{\partial u_1} u_{1x_i} + \frac{\partial \varphi_2^{j}}{\partial u_2} u_{2x_i} \right) 2\zeta \zeta_x \, dxdt \leq C_{11}. \]

Hence we conclude that

\[ |I_3| \leq \varepsilon_{12} \int_t^t \int_{\Omega_2^p} |D_x^2 u_2|^2 \, \zeta^2 \, dxdt + C_{12}. \]

(3.31)

Finally, we easily have

\[ |I_4| \leq C_{13}. \]

(3.32)
Taking $\varepsilon_6 + \varepsilon_{12} \leq \lambda_0^2/4$, we obtain from (3.29)–(3.32)

\[
\left( \frac{1}{2} \int_{\Omega_{2\rho}} u_{2x_1}^2 \zeta^2 \, dx \right) \bigg|_\varepsilon^t + \frac{\lambda_0^2}{4} \int_{\varepsilon}^t \int_{\Omega_{2\rho}} |D_x^2 u_2|^2 \zeta^2 \, dx \, dt \leq C_{14} \int_{\varepsilon}^t \int_{\Omega_{2\rho}} |\nabla_x u_2|^4 \zeta^2 \, dx \, dt + C_{15}.
\]

(3.33)

3. Now, let us consider Lemma 5.4 in Chapter II, from Ladyzenskaja, Solonikov and Ural’ceva [20]. Taking $s = 1$, we have

\[
\int_{\Omega_{2\rho}} |\nabla_x u_2|^4 \zeta^2 \, dx \leq 16 \text{osc}^2[u_2, \Omega_{2\rho}] \int_{\Omega_{2\rho}} C(|D_x^2 u_2|^2 \zeta^2 + |\nabla_x u_2|^2) \, dx.
\]

(3.34)

We recall that, $\text{osc}[u(x); \Omega]$ is the oscillation of $u(x)$ in $\Omega$, which means the difference between $\text{ess sup}_{\Omega} u(x)$ and $\text{ess inf}_{\Omega} u(x)$, therefore, it follows from Lemma 3.6 that,

\[
16 \text{osc}^2[u_2, \Omega_{2\rho}] \leq C_{16} \rho^{2\gamma_1}.
\]

Then, for $\rho \leq \rho_0$, such that, $\rho_0^{2\gamma_1} C_{16} C_{14} C \leq \lambda_0^2/8$, we obtain from (3.33),

\[
\left( \int_{\Omega_{2\rho}} u_{2x_1}^2 \zeta^2 \, dx \right) \bigg|_\varepsilon^t + \int_{\varepsilon}^t \int_{\Omega_{2\rho}} |D_x^2 u_2|^2 \zeta^2 \, dx \, dt \leq C_{17}
\]

and

\[
\max_{t_1 \in (\varepsilon, t)} \int_{\Omega_{2\rho}} u_{2x_1}^2 \zeta^2 \, dx(t_1) + \int_{\varepsilon}^t \int_{\Omega_{2\rho}} |D_x^2 u_2|^2 \zeta^2 \, dx \, dt + \int_{\varepsilon}^t \int_{\Omega_{2\rho}} |\nabla_x u_2|^4 \zeta^2 \, dx \, dt \leq C_{18}.
\]

From this estimate and equation (3.24) ($\alpha = 2$), we obtain

\[
\int_{\varepsilon}^t \int_{\Omega_{2\rho}} (u_{2t})^2 \zeta^2 \, dx \, dt \leq C_{19}.
\]

Consequently, it follows from the above estimates

\[
\max_{t_1 \in (\varepsilon, t)} \int_{\Omega_{\rho}} u_{2x_1}^2 \zeta^2 \, dx(t_1) + \int_{\varepsilon}^t \int_{\Omega_{\rho}} |D_x^2 u_2|^2 \zeta^2 \, dx \, dt
\]

\[
+ \int_{\varepsilon}^t \int_{\Omega_{\rho}} |\nabla_x u_2|^4 \zeta^2 \, dx \, dt + \int_{\varepsilon}^t \int_{\Omega_{2\rho}} (u_{2t})^2 \zeta^2 \, dx \, dt \leq C_{20}.
\]

(3.35)
4. Finally, the thesis of the Lemma is proved from estimate (3.35) and a standard argument of partition of unity subordinated to a finite local cover of \( \Omega \), since \( \bar{\Omega} \) is a compact subset of \( \mathbb{R}^d \).

**Lemma 3.7.** For \( d \geq 2, j = 1, \ldots, d \), under conditions of Lemma 3.5 and also conditions (1.8), (3.28), then for \( t < T_{\text{max}} \)

\[
\int_{\varepsilon}^{t} \int_{\Omega} |\partial_{x_j} u_2(\tau, x)|^{2s+4} d\tau \, dx \leq C,
\]

where \( C \) does not dependent on \( t \), and \( s \) is any non-negative integer, such that, \( s > (d - 2)/2 \).

**Proof.** Let us consider the second equation of our system, which is to say

\[
\begin{align*}
\partial_t u_2 &= \partial_{x_j} \left( A^{jk}_{22}(x, u_2) \partial_{x_k} u_2 - \varphi_2^j(x, u_2) \right) + g_2(x, u_1, u_2), \\
u_2|_{\Gamma} &= u_{b_2}, \\
u_2(0) &= u_{0_2}.
\end{align*}
\]

This problem satisfies the conditions (3.1)-(3.6) in Section 3 of Chapter V (non-linear theory) from Ladyzenskaja, Solonikov and Ural’ceva [20]. In this book is proved in Section 4 of Chapter V, the estimates (4.10) for any \( s > 0 \):

\[
\max_{\varepsilon \leq t \leq T} \int_{\Omega} |u_2|^{2s+2} dx + \int_{0}^{T} \int_{\Omega} |u_2|^{2s+4} dx dt \leq \text{Const}.
\]

The thesis of this lemma follows from the above estimate.

Now, from the above estimates obtained for \( u_2 \), we are going to consider equation (3.24), when \( \alpha = 1 \). Then, similarly to Lemma 3.5 we have the following

**Lemma 3.8.** Let \( 0 < \varepsilon < T_{\text{max}} \) be fixed and consider for \( \alpha = 1 \) the Dirichlet problem in (3.24)-(3.26), with \( u_{0_1} \in E \) and \( u_{b_1} \in H^2_0(\Gamma_T) \), \( \gamma \in (0,1) \). Assume the conditions (1.8), (3.28) for \( d \geq 2 \), (3.13), (3.15)-(3.19). Then, there exists \( \gamma_2 > 0 \), such that

\[
|u_1|_{2+\gamma_2}^{Q_1} \leq C,
\]

where \( C \) is a positive constant independent of \( t \), with \( t \in (\varepsilon, T_{\text{max}}) \).
Proof. Again the result follows applying Theorem 10.1 in Chapter III from Ladyzenskaja, Solonikov and Ural’ceva [20], with
\[
a_{jk}(t, x) = A^{jk}_{11}(x, u_1(t, x), u_2(t, x)),
\]
\[
f_j(t, x) = A^{jk}_{12}(x, u_1(t, x), u_2(t, x)) \partial_{x_k} u_2 + \varphi_j^1(x, u_1(t, x), u_2(t, x)),
\]
and
\[
f(t, x) = g_1(x, u_1(t, x), u_2(t, x)), \quad a_j = b_j = a = 0.
\]
Since \( u \in B_\Delta \), and considering the higher estimates of \( \partial_{x_k} u_2 \) obtained from Lemma 3.6 (in particular for \( d = 1 \)), Lemma 3.7 (\( d > 1 \)), we may consider \( f_j, f^2 \) uniformly bounded, i.e. \( \| f_j^2, f \|_{q, r, Q_1} \leq C \), with \( q = s + 2, \ r = s + 2 \), where \( C \) is a positive constant independent of \( t < T_{\text{max}} \). Moreover, we have
\[
\frac{1}{r} + \frac{d}{2q} = 1 - \kappa_1 \in (0, 1),
\]
where
\[
\kappa_1 = 1 - \frac{1}{s + 2} - \frac{d}{2s + 4} > 0
\]
and therefore we obtain (3.36).

Theorem 3.9. Let any \( T > 0 \) be given and consider \( u_0 \in E, \ u_b \in H^{2+\gamma}_0(\Gamma_T) \), with \( \gamma \in (0, 1) \). Assume the conditions (\( 1.8 \)), (\( 3.13 \)), (\( 3.15 \)) – (\( 3.19 \)), (\( 3.28 \)) for \( d \geq 2 \). Then, the initial-boundary value problem (\( 3.24 \)) – (\( 3.26 \)), with Dirichlet condition, has a unique global solution \( u \in H^{2+\gamma}_0(Q_T) \). Moreover, for each \( (t, x) \in Q_T \), \( u(t, x) \in B_\Delta \).

Proof. First, we have from the previous results that, \( u_\alpha \in C^\gamma(\overline{Q_T}) \) (\( \alpha = 1, 2 \)) for some \( \gamma > 0 \). Therefore, applying Corollary 2.5 we obtain the global classical solution \( u \) for the initial-boundary value problem (\( 3.24 \)) – (\( 3.26 \)), with Dirichlet condition. Finally, \( u(t, x) \in B_\Delta \), for each \( (t, x) \in Q_T \), follows from Corollary 3.3.

\[\square\]

3.2.2 Flux condition

Now, let us consider flux-boundary condition in (\( 3.26 \)), for \( \alpha = 1, 2 \). Analogously, we begin establishing estimates for \( u_2 \). Then, we have the following
Lemma 3.10. Let $0 < \varepsilon < T_{\text{max}}$ be fixed and consider for $\alpha = 2$ the initial-boundary value problem (3.24)–(3.26) (flux condition), with $u_{02} \in E$. Assume the conditions (1.8), (3.28), (3.13), (3.15)–(3.19). Then, there exists $\gamma_1 > 0$, such that

$$u_2 \in C^{2+\gamma_1,1+\gamma_1/2}(Q_{T_{\text{max}}}^\varepsilon \cup \Gamma_{T_{\text{max}}}^\varepsilon) \quad \text{and} \quad |u_2|_{\gamma_1}^{Q_{T_{\text{max}}}^\varepsilon} \leq C_2,$$

where $C_2$ is independent of $t < T_{\text{max}}$.

**Proof.** 1. First, the interior estimates in $C^{2+\gamma_1,1+\gamma_1/2}(Q_{T_{\text{max}}}^\varepsilon)$, for any $Q_{T_{\text{max}}}^\varepsilon \subset \subset Q_{T_{\text{max}}}^\varepsilon$, follow applying Theorem 10.1 in Chapter III (linear theory) from Ladyzenskaja, Solonikov and Ural’ceva [20], with

$$a_{jk}(t, x) = A_{jk}^2(x, u_2(t, x)), \quad f_j(t, x) = \varphi_j^2(x, u_2(t, x)),$$

also

$$f(t, x) = g_2(x, u_1(t, x), u_2(t, x)), \quad a_j = b_j = a = 0.$$

Since $u \in B_\Delta$, and $f, f_j$ are uniformly bounded, for each $d \geq 1$, there exist positive $q$ and $r$, such that

$$\frac{1}{r} + \frac{d}{2q} = 1 - \kappa_1, \quad \kappa_1 \in (0, 1)$$

and $\|f_j^2, f\|_{q,r,Q_{T_{\text{max}}}^\varepsilon} \leq C$, where $C$ is a positive constant independent of $t < T_{\text{max}}$.

2. Now, to derive the estimates closely to the boundary, we may apply the non-linear theory from Ladyzenskaja, Solonikov and Ural’ceva [20] developed in Chapter V, Section 7. More precisely, the equation for $u_2$ is in the same form of (7.1)-(7.3), and admits the conditions (7.4)-(7.6). Therefore, it follows from Theorem 7.1 in that book, the thesis of the lemma.

□

Lemma 3.11. Under conditions of Lemma 3.10, there exists $M_1 > 0$, such that

$$\sup_{Q_{T_{\text{max}}}^\varepsilon} |\nabla_x u_2(t, x)| \leq M_1.$$
Proof. The thesis of the Lemma follows directly from Theorem 3 in the paper of A.I. Nazarov, N.N. Uraltseva [21]. Also, from Theorem 4 in that paper, there exists $M_{1+\gamma} > 0$, such that
\[
\|u_2(t, x)\|_{1+\gamma}^{Q_{T_{\text{max}}}} \leq M_{1+\gamma}.
\]

Now, let us rewrite the second equation of our system, which is to say
\[
\partial_t u_2 + \partial_{x_j} \varphi_2^j + \partial_{u_2} \varphi_2^j \partial_{x_j} u_2 = A_{22}^{kj}(x, u_2) \partial_{x_k x_j}^2 u_2
\]
\[
+ \partial_{u_2} (A_{22}^{kj}(x, u_2)) \partial_{x_k} u_2 \partial_{x_j} u_2 + \partial_{x_j} (A_{22}^{kj}(x, u_2)) \partial_{x_k} u_2
\]
\[
+ g_2(x, u_1, u_2), \quad (t, x) \in Q_{T_{\text{max}}}.
\]

Supplemented with the boundary condition
\[
\left( A_{22}^{kj}(x, u_2) \partial_{x_k} u_2 + \varphi_2^j(x, u_2) \right) \cos(n, x_j) = 0, \quad (t, x) \in \Gamma_{T_{\text{max}}}.
\]

Also we have $u_2(\varepsilon, \cdot) \in C^{2+\gamma}(\Omega)$. From the above estimates, Lemma 3.10 and Lemma 3.11 we obtain
\[
|\partial_{u_2} (A_{22}^{kj}) \partial_{x_k} u_2 \partial_{x_j} u_2 + \partial_{x_j} (A_{22}^{kj}) \partial_{x_k} u_2 + g_2 - \partial_{x_j} \varphi_2^j - \partial_{u_2} \varphi_2^j \partial_{x_j} u_2|^{Q_{T_{\text{max}}}} \leq M_2,
\]
and
\[
|A_{22}(x, u_2)^{\gamma}|_{0}^{Q_{T_{\text{max}}}} \leq M_3.
\]

Moreover, it is possibly to apply the $W^{1,2}_p$–estimates for linear parabolic problem with flux boundary conditions (V.A. Solonikov [24], Amann [3])
\[
\|u_2\|_{W^{1,2}_p(Q_{T_{\text{max}}})} \leq M_4
\]
for any $p > 1$, where the positive constant $M_4$ depend on $p$.

Lemma 3.12. Let $0 < \varepsilon < T_{\text{max}}$ be fixed and consider for $\alpha = 1$ the initial-boundary value problem (3.24)–(3.26) (flux condition), with $u_{0,1} \in E$. Assume the conditions (1.8), (3.28), (3.13), (3.15)–(3.19). Then, there exists $\gamma_2 > 0$, such that
\[
|u_1|^{Q_{T_{\text{max}}}} \leq C,
\]
where $C$ is a positive constant independent of $t$, with $t \in (\varepsilon, T_{\text{max}})$. 

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Proof. 1. Again, the interior estimates in $C^{\gamma}(Q^T_{T_{\text{max}}})$, $(Q^T_{T_{\text{max}}} \subset \subset Q^T_{T_{\text{max}}})$, follow applying Theorem 10.1 (linear theory) in Chapter III from Ladyzenskaja, Solonikov and Ural’ceva [20], with

$$a_{jk}(t,x) = A^{jk}_{11}(x,u_1(t,x),u_2(t,x)),$$

$$f_j(t,x) = \phi_j^1(x,u_1(t,x),u_2(t,x)) - A^{jk}_{12}(x,u_1(t,x),u_2(t,x)) \partial_{x_k} u_2(t,x),$$

also

$$f(t,x) = g_2(x,u_1(t,x),u_2(t,x)), \quad a_j = b_j = a = 0.$$

Since $u \in B_\Delta$, and $f, f_j$ are uniformly bounded, for each $d$, there exist positive $q$ and $r$, such that

$$\frac{1}{r} + \frac{d}{2q} = 1 - \kappa_1, \quad \kappa_1 \in (0,1)$$

and $\|f_j^2,f\|_{q,r,Q_{T_{\text{max}}}} \leq C$, where $C$ is a positive constant independent of $t < T_{\text{max}}$.

2. Let us observe that, the estimate closely to the boundary does not follow in the same way to $u_1$, that is to say, it is not possible to apply the same strategy as done for $u_2$ in Lemma 3.10. Therefore, we proceed to derive the estimates closely to the boundary, applying the non-linear theory from Ladyzenskaja, Solonikov and Ural’ceva [20] developed in Section 7 of Chapter V. More precisely, the equation for $u_1$ is in the same form of (7.1)-(7.3), and admits the conditions (7.4)-(7.6) from that book. Then, from Theorem 7.1 in that book, it follows the thesis of the lemma.

Theorem 3.13. Let any $T > 0$ be given and consider $u_0 \in E$. Assume the conditions (1.8), (3.28), (3.13), (3.15)-(3.19). Then, the initial-boundary value problem (3.24)-(3.26), with flux condition, has a unique global solution $u \in H^{2+\gamma}_0(Q_T)$. Moreover, for each $(t,x) \in Q_T$, $u(t,x) \in B_\Delta$.

Proof. First, we have from the above lemmas that, $u_\alpha \in C^\gamma(\overline{Q}_T)$ ($\alpha = 1,2$) for some $\gamma > 0$. Therefore, from Corollary 2.5 we obtain the global classical solution $u$ for the problem (1.1)-(1.5).

Remark 3.14. Clearly, it remains to consider Dirichlet boundary condition for $\alpha = 1$, flux-condition for $\alpha = 2$, and vice-versa. These type of mixed boundary conditions follow from suitable consideration of the above results.
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